ERGODIC SOLENOIDAL HOMOLOGY

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ABSTRACT. We define generalized currents associated with immersions of abstract solenoids with a transversal measure. We realize geometrically the full real homology of a compact manifold with these generalized currents, and more precisely with immersions of minimal uniquely ergodic solenoids. This makes precise and geometric De Rham’s realization of the real homology by only using a restricted geometric subclass of currents. These generalized currents do extend Ruelle-Sullivan and Schwartzman currents. We extend Schwartzman theory beyond dimension 1 and provide a unified treatment of Ruelle-Sullivan and Schwartzman theories via Birkhoff’s ergodic theorem for the class of immersions of controlled solenoids. We develop some intersection theory of these new generalized currents that explains why the realization theorem cannot be achieved only with Ruelle-Sullivan currents.

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2000 Mathematics Subject Classification. Primary: 37A99. Secondary: 58A25, 57R95, 55N45.

Key words and phrases. Real homology, Ruelle-Sullivan current, Schwartzman current, solenoid, ergodic theory.

First author supported through MCyT grant MTM2004-07090-C03-01 (Spain) and NSF grant DMS-0111298 (US). Second author supported by CNRS (UMR 7539) and NSF grant DMS-0202494.
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1. Introduction

We consider a smooth compact oriented manifold $M$ of dimension $n \geq 1$. Any closed oriented submanifold $N \subset M$ of dimension $0 \leq k \leq n$ determines a homology class in $H_k(M, \mathbb{Z})$. This homology class in $H_k(M, \mathbb{R})$, as dual of De Rham cohomology, is explicitly given by integration of the restriction to $N$ of differential $k$-forms on $M$. Unfortunately, because of topological reasons dating back to Thom [Th], not all integer homology classes in $H_k(M, \mathbb{Z})$ can be realized in such a way by a compact submanifold. Geometrically, we can realize any class in $H_k(M, \mathbb{Z})$ by topological $k$-chains. The real homology $H_k(M, \mathbb{R})$ classes are only realized by formal combinations with real coefficients of $k$-cells. This is not satisfactory for various reasons. In particular, for diverse purposes it is important to have an explicit realization, as geometric as possible, of real homology classes.

The first contribution in this direction came in 1957 from the work of S. Schwartzman [Sc]. Schwartzman showed how, by a limiting procedure, one-dimensional curves embedded in $M$ can define a real homology class in $H_1(M, \mathbb{R})$. More precisely, he proved that this happens for almost all curves solutions to a differential equation admitting an invariant ergodic probability measure. Schwartzman’s idea is very natural. It consists on integrating 1-forms over large pieces of the parametrized curve and normalizing this integral by the length of the parametrization. Under suitable conditions, the limit exists and defines an element of the dual of $H^1(M, \mathbb{R})$, i.e. an element of $H_1(M, \mathbb{R})$. This procedure is equivalent to the more geometric one of closing large pieces of the curve by relatively short closing paths. The closed curve obtained defines an integer homology class. The normalization by the length of the parameter range provides a class in $H_k(M, \mathbb{R})$. Under suitable hypothesis, there exists a unique limit in real homology when the pieces exhaust the parametrized curve, and this limit is independent of the closing procedure. In sections 9, 10 and 11 we study the different aspects of the Schwartzman procedure, that we extend to higher dimension.

Later in 1975, D. Ruelle and D. Sullivan [RS] defined, for arbitrary dimension $0 \leq k \leq n$, geometric currents by using oriented $k$-laminations embedded in $M$ and endowed with a transversal measure. They applied their results to Axiom A diffeomorphisms. In a later article Sullivan [Su] extended further these results and their applications. The point of view of Ruelle and Sullivan is also based on duality. The observation is that $k$-forms can be integrated on each leaf of the lamination and then all over the lamination using the transversal measure. This makes sense locally in each flow-box, and then it can be extended globally by using a partition of unity. The result only depends on the cohomology class of the $k$-form. In section 7 we review and extend Ruelle-Sullivan theory.
It is natural to ask whether it is possible to realize every real homology class using a Ruelle-Sullivan current. A first result, with a precedent in [HM], confirms that this is not the case (see section 7).

**Theorem 1.1.** Homology classes with non-zero self-intersection cannot be represented by Ruelle-Sullivan currents.

More precisely, for each Ruelle-Sullivan lamination with a non-atomic transversal measure, we can construct a smooth \((n-k)\)-form which provides the dual in De Rham cohomology. Using it, we prove that the self-intersection of a Ruelle-Sullivan current is 0, therefore it is not possible to represent a real homology class in \(H^k(M,\mathbb{R})\) with non-zero self-intersection. This obstruction only exists when \(n-k\) is even. This may be the historical reason behind the lack of results on the representation of an arbitrary homology class by Ruelle-Sullivan currents.

Therefore, in order to hope to represent every real homology class we must first enlarge the class of Ruelle-Sullivan currents. This is done by considering immersions of abstract oriented solenoids. We define a \(k\)-solenoid to be a Hausdorff compact space foliated by \(k\)-dimensional leaves with finite dimensional transversal structure (see the precise definition in section 2). For these oriented solenoids we can consider \(k\)-forms that we can integrate provided that we are given a transversal measure invariant by the holonomy group. We define an immersion of a solenoid \(S\) into \(M\) to be a regular map \(f : S \to M\) that is an immersion in each leaf. If the solenoid \(S\) is endowed with a transversal measure \(\mu\), then any smooth \(k\)-form in \(M\) can be pulled back to \(S\) by \(f\) and integrated. The resulting numerical value only depends on the cohomology class of the \(k\)-form. Therefore we have defined a current that we denote by \([f,S_\mu] \in H^k(M,\mathbb{R})\) and that we call a generalized current. Using these generalized currents, the above mentioned obstruction disappears. Our main result is:

**Theorem 1.2. (Realization Theorem)** Every real homology class in \(H^k(M,\mathbb{R})\) can be realized by a generalized current \([f,S_\mu]\) where \(S_\mu\) is an oriented, minimal, uniquely ergodic solenoid.

Minimal and uniquely ergodic solenoids are defined later on. This result gives a geometric version and makes precise De Rham’s realization theorem of homology classes by abstract currents, i.e. forms with coefficients distributions.

But the space of solenoids is large, and we would like to realize the real homology classes by a minimal class of solenoids enjoying good properties. We are first naturally led to topological minimality. As we prove in section 2 the spaces of \(k\)-solenoids is inductive and therefore there are always minimal \(k\)-solenoids. However, the transversal structure and the holonomy group of minimal solenoids can have a rich structure. In particular, such a solenoid may have many different transversal measures, each one yielding a different generalized current for the same immersion \(f\). Also when we push
Schwartzman ideas beyond 1-homology for some nice classes of solenoids, we see that in general, even when the immersion is an embedding, the generalized current does not necessarily coincide with the Schwartzman homology class of the immersion of each leaf (actually not even this Schwartzman class needs to be well defined). Indeed the classical literature lacks of information about the precise relation between Ruelle-Sullivan and Schwartzman currents. One would naturally expect that there is some relation between the generalized currents and the Schwartzman current (if defined) of the leaves of the lamination. We study this problem in section 8 for 1-dimensional currents and in section 11 in general. The main result is that there is such relation for the class of minimal, ergodic and controlled solenoids (see definition in section 11) for which the transversal structure is well behaved. A controlled solenoid has a trapping region (see definition 11.9), and the holonomy group is generated by a single map. Then the bridge between generalized currents and Schwartzman currents of the leaves is provided by Birkhoff’s ergodic theorem as explained in sections 10 and 11.

**Theorem 1.3.** Let $S_\mu$ be a controlled and minimal solenoid endowed with an ergodic transversal measure $\mu$. Let $f : S_\mu \to M$ be an immersion of $S_\mu$ into $M$. Then for $\mu$-almost all leaves $l \subset S_\mu$, the Schwartzman homology class of $f(l) \subset M$ is well defined and coincides with the generalized current $[f, S_\mu]$.

We are particularly interested in uniquely ergodic solenoids, with only one ergodic transversal measure. As is well known, in this situation we have uniform convergence of Birkhoff’s sums, which implies the stronger result:

**Theorem 1.4.** Let $S_\mu$ be a minimal and uniquely ergodic solenoid. Let $f : S_\mu \to M$ be an immersion of $S_\mu$ into $M$. Then for all leaves $l \subset S_\mu$, the Schwartzman homology class of $f(l) \subset M$ is well defined and coincides with the generalized current $[f, S_\mu]$.

We also make a thorough study of Riemannian solenoids. We identify transversal measures with the class of measures that desintegrate as volume along leaves (daval measures), we define Schwartzman measures as limits of $k$-volume normalized measures along leaves, and we prove a canonical decomposition of measures into a daval measure and a singular part, corresponding to the classical Lebesgue decomposition on manifolds (see section 6).

Numerous arguments and auxiliary results in this article are part of the folklore or have appeared earlier in the literature on foliations or laminated spaces. We do not give systematic references. The reader can consult the monograph [CC] for a comprehensive bibliography and classical results.

**Acknowledgements.** The authors are grateful to Alberto Candel, Etienne Ghys and Jaume Amorós for their comments and interest on this work. In particular, Etienne Ghys early pointed out on the impossibility of realization in general of integer homology classes by embedded manifolds.
The first author wishes to acknowledge Universidad Complutense de Madrid and Institute for Advanced Study at Princeton for their hospitality and for providing excellent working conditions. The second author thanks Jean Bourgain and the IAS at Princeton for their hospitality and facilitating the collaboration of both authors.
2. Minimal solenoids

We first define abstract solenoids, which are the main tools in this article.

**Definition 2.1.** Let \(0 \leq r, s \leq \omega\), and let \(k, l \geq 0\) be two integers. A foliated manifold (of dimension \(k+l\), with \(k\)-dimensional leaves, of regularity \(C^{r,s}\)) is a smooth manifold \(W\) of dimension \(k+l\) endowed with an atlas \(\mathcal{A} = \{(U_i, \varphi_i)\}\) whose changes of charts

\[
\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j),
\]

are of the form \(\varphi_{ij}(x, y) = (X_{ij}(x,y), Y_{ij}(y))\), where \(Y_{ij}(y)\) is of class \(C^s\) and \(X_{ij}(x,y)\) is of class \(C^{r,s}\).

A flow-box for \(W\) is a pair \((U, \varphi)\) consisting of an open subset \(U \subset W\) and a map \(\varphi : U \to \mathbb{R}^k \times \mathbb{R}^l\) such that \(\mathcal{A} \cup \{(U, \varphi)\}\) is still an atlas for \(W\).

Clearly an open subset of a foliated manifold is also a foliated manifold.

Given two foliated manifolds \(W_1, W_2\) of dimension \(k+l\), with \(k\)-dimensional leaves, and of regularity \(C^{r,s}\), a regular map \(f : W_1 \to W_2\) is a continuous map which is locally, in flow-boxes, of the form \(f(x,y) = (X(x,y), Y(y))\), where \(Y\) is of class \(C^s\) and \(X\) is of class \(C^{r,s}\).

A diffeomorphism \(\phi : W_1 \to W_2\) is a homeomorphism such that \(\phi\) and \(\phi^{-1}\) are both regular maps.

**Definition 2.2.** \((k\text{-solenoid})\) Let \(0 \leq r, s \leq \omega\), and let \(k, l \geq 0\) be two integers. A \(k\)-solenoid of dimension \(k\), of class \(C^{r,s}\) and transversal dimension \(l\) is a pair \((S, W)\) where \(W\) is a foliated manifold and \(S \subset W\) is a compact subspace which is a collection of leaves.

Two \(k\)-solenoids \((S, W_1)\) and \((S, W_2)\) are equivalent if there are open subsets \(U_1 \subset W_1, U_2 \subset W_2\) with \(S \subset U_1\) and \(S \subset U_2\), and a diffeomorphism \(f : U_1 \to U_2\) such that \(f\) is the identity on \(S\).

A \(k\)-solenoid of class \(C^{r,s}\) and transversal dimension \(l\) (or just a \(k\)-solenoid, or a solenoid) is an equivalence class of \(k\)-solenoids.

We usually denote a solenoid by \(S\), without making explicit mention of \(W\). We shall say that \(W\) defines the solenoid structure of \(S\).

**Definition 2.3.** \((\text{Flow-box})\) Let \(S\) be a solenoid. A flow-box for \(S\) is a pair \((U, \varphi)\) formed by an open subset \(U \subset S\) and a homeomorphism

\[
\varphi : U \to D^k \times K(U),
\]

where \(D^k\) is the \(k\)-dimensional open ball and \(K(U) \subset \mathbb{R}^l\), such that there exists a foliated manifold \(W\) defining the solenoid structure of \(S\), \(S \subset W\), and a flow-box \(\tilde{\varphi} : \tilde{U} \to \mathbb{R}^k \times \mathbb{R}^l\) for \(W\), with \(U = \tilde{U} \cap S\), \(\varphi(U) = D^k \times K(U) \subset \mathbb{R}^k \times \mathbb{R}^l\) and \(\varphi = \tilde{\varphi}|_U\).
The set $K(U)$ is the transversal space of the flow-box. The dimension $l$ is the transversal dimension.

As $S$ is locally compact, any point of $S$ is contained in a flow-box $U$ whose closure $\overline{U}$ is contained in a bigger flow-box. For such flow-box, $\overline{U} \cong \overline{D}^k \times K(U)$, where $\overline{D}^k$ is the closed unit ball, $K(U)$ is some compact subspace of $\mathbb{R}_l$, and $U = D^k \times K(U) \subset \overline{D}^k \times K(U)$. We might call these flow-boxes good. All flow-boxes that we shall use are of this type so we shall not append any appellative to them.

We refer to $k$ as the dimension of the solenoid and we write $k = \dim S$.

Note that, contrary to manifolds, this dimension in general does not coincide with the topological dimension of $S$. The local structure and compactness imply that solenoids are metrizable topological spaces. The Hausdorff dimension of the transversals $K(U)$ is well defined and the Hausdorff dimension of $S$ is well defined, and equal to $
abla^{-1}_H S = k + \max \dim H K(U_i) \leq k + l < +\infty$.

When the transversals of flow-boxes $K(U) \subset \mathbb{R}_l$ are open sets of $\mathbb{R}_l$ we talk about full transversals. In this case the solenoid structure is a $(k + l)$-dimensional compact manifold foliated by $k$-dimensional leaves.

Remark 2.4. The definition of solenoid admits various generalizations. We could focus on intrinsic changes of charts in $S$ with some transverse Whitney regularity but without requiring a local diffeomorphism extension. Such a definition would be more general, but it is not necessary for our purposes. The present definition balances generality and simplicity.

Another alternative generalization would be to avoid any restrictive transversal assumption beyond continuity, and allow for transversals of flow-boxes any topological space $K(U)$. But a fruitful point of view is to regard the theory of solenoids as a generalization of the classical theory of manifolds. Therefore it is natural to restrict the definition only allowing finite dimensional transversal spaces. For an alternative approach see [MS].

Definition 2.5. (Diffeomorphisms of solenoids) Let $S_1$ and $S_2$ be two $k$-solenoids of class $C^{r,s}$ with the same transversal dimension. A $C^{r,s}$-diffeomorphism $f : S_1 \to S_2$ is the restriction of a $C^{r,s}$-diffeomorphism $\hat{f} : W_1 \to W_2$ of two foliated manifolds defining the solenoid structures of $S_1$ and $S_2$, respectively.

A homeomorphism of solenoids is a diffeomorphism of class $C^{0,0}$.

This defines the category of smooth solenoids of a given regularity. Note that we have the subcategory of smooth solenoids with full transversals, and we have a forgetting functor into the category of smooth manifolds.
Definition 2.6. (Leaf) A leaf of a $k$-solenoid $S$ is a leaf $l$ of any foliated manifold $W$ inducing the solenoid structure of $S$, such that $l \subset S$. Note that this notion is independent of $W$.

Note that $S \subset W$ is the union of a collection of leaves. Therefore, for a leaf $l$ of $W$ either $l \subset S$ or $l \cap S = \emptyset$.

Observe that when the transversals of flow-boxes $K(U)$ are totally disconnected then the leaf-equivalence coincides with path connection equivalence, and the leaves are the path connected components of $S$.

Definition 2.7. (Oriented solenoid) An oriented solenoid is a solenoid $S$ such that there is a foliated manifold $W \supset S$ inducing the solenoid structure of $S$, where $W$ has oriented leaves.

It is easy to see that $S$ is oriented if and only if there is an orientation for the leaves of $S$ such that there is a covering by flow-boxes which preserve the orientation of the leaves.

Notice that we do not require $W$ oriented. For example, we can foliate a Möbius strip and create an oriented solenoid.

Definition 2.8. We define $S_{k,l}^{r,s}$ to be the space of $C^{r,s}$ $k$-solenoids with transversal dimension $l$.

Proposition 2.9. Let $S_0 \in S_{k,l}^{r,s}$ be a solenoid. A non-empty compact subset $S$ of $S_0$ which is a union of leaves is a $k$-solenoid of class $C^{r,s}$ and transversal dimension $l$.

Proof. Let $W$ be a $C^{r,s}$-foliated manifold defining the solenoid structure of $S_0$. Then $S \subset W$ and $W$ defines a $C^{r,s}$-solenoid structure for $S$.

Note that the flow-boxes of $S_0$ give, by restriction to $S$, flow-boxes for $S$.

Corollary 2.10. Connected components of solenoids $S_{k,l}^{r,s}$ are in $S_{k,l}^{r,s}$.

Theorem 2.11. The space $(S_{k,l}^{r,s}, \subset)$ ordered by inclusion is an inductive set.

Proof. Let $(S_n) \subset S_{k,l}^{r,s}$ be a nested sequence of solenoids, $S_{n+1} \subset S_n$. Define

$$S_\infty = \bigcap_n S_n.$$ 

Then $S_\infty$ is a non-empty compact subset of $S_1$ as intersection of a nested sequence of such sets. It is also a union of leaves since each $S_n$ is so. Therefore by proposition 2.9 it is an element of $S_{k,l}^{r,s}$.

Corollary 2.12. The space $S_{k,l}^{r,s}$ has minimal elements.
**Proposition 2.13.** If $S \in S_{k,l}^{r,s}$ is minimal then $S$ is connected and all leaves of $S$ are dense.

**Proof.** Each connected component of $S$ is a solenoid, thus by minimality $S$ must be connected.

Also the closure $\overline{L}$ of any leaf $L \subset S$ is a non-empty compact set union of leaves. Thus it is a solenoid and by minimality we must have $\overline{L} = S$. $\square$

3. **Topological transversal structure of solenoids**

**Definition 3.1. (Transversal)** Let $S$ be a $k$-solenoid. A local transversal at a point $p \in S$ is a subset $T$ of $S$ with $p \in T$, such that there is a flow-box $(U, \varphi)$ of $S$ with $U$ a neighborhood of $p$ containing $T$ and such that

$$\varphi(T) = \{0\} \times K(U).$$

A transversal $T$ of $S$ is a compact subset of $S$ such that for each $p \in T$ there is an open neighborhood $V$ of $p$ such that $V \cap T$ is a local transversal at $p$.

If $S$ is a $k$-solenoid of class $C^{r,s}$, then any transversal $T$ inherits an $l$-dimensional $C^s$-Whitney structure.

We clearly have:

**Proposition 3.2.** The union of two disjoint transversals is a transversal.

**Definition 3.3.** A transversal $T$ of $S$ is a global transversal if all leaves intersect $T$.

The next proposition is clear.

**Proposition 3.4.** The union of two disjoint transversals, one of them global, is a global transversal.

**Proposition 3.5.** If $S$ is minimal then all transversals are global. Moreover, if $S$ is minimal then any local transversal intersects all leaves of $S$.

**Proof.** It is enough to see the second statement, since it implies the first. Let $U$ be a flow-box and $T = \varphi^{-1}(\{0\} \times K(U))$ a local transversal. By proposition 2.13 all leaves intersect $U$ and therefore they intersect $T$. $\square$

Observe that the definition of solenoid with regular transverse structure says that $S$ is always embedded in a $(k+l)$-dimensional manifold $W$. Therefore $S \subset W$ has an interior and a boundary relative to $W$. These sets do not depend on the choice of $W$.

**Definition 3.6. (Proper interior and boundary)** Let $S$ be a $k$-solenoid. Let $W$ be a foliated manifold defining the solenoid structure of $S$. The proper interior of $S$ is the interior of $S$ as a subset of $W$, considered as a $(k+l)$-dimensional manifold.

The proper boundary of $S$ is defined as the complement in $S$ of the proper interior.
Let $\hat{\varphi} : \hat{U} \to \mathbb{R}^k \times \mathbb{R}^l$ be a flow-box for $W$ such that $U = \hat{U} \cap S$ and $\varphi = \hat{\varphi}|_U : U \to D^k \times K(U)$ is a flow-box for $S$. Then $K(U) \subset \mathbb{R}^l$. The proper interior, resp. the proper boundary, of $S$, intersected with $U$, consists of the collection of leaves $\varphi^{-1}(D^k \times \{y\})$, where $y \in K(U)$ is in the interior, resp. boundary, of $K(U) \subset \mathbb{R}^l$.

Note that the proper boundary of a solenoid that is a foliation of a manifold is empty. We prove the converse.

**Proposition 3.7.** If the proper boundary of $S$ is non-empty then it is a sub-solenoid of $S$.

**Proof.** The proper boundary is a compact subset of $S$ and a union of leaves. The result follows from proposition 2.9. □

**Theorem 3.8.** Let $S \in S$ be a minimal solenoid. If $S$ is not the foliation of a manifold then $S$ has empty proper interior, i.e. $K(U) \subset \mathbb{R}^l$ has empty interior for any flow-box $(U, \varphi)$.

**Proof.** The proper boundary of $S$ is non-empty because otherwise, for each flow-box $U$, $K(U) \subset \mathbb{R}^l$ is an open set. Thus $S$ would be an open subset of $W$, where $W$ is a foliated manifold defining the solenoid structure of $S$, and so $S$ is itself a foliated $(k + l)$-manifold. This contradicts the assumption.

Now by minimality the proper boundary must coincide with $S$ and the proper interior is empty. □

Solenoids with a one dimensional transversal play a prominent role in the sequel. We have for these the following structure theorem.

**Theorem 3.9.** (Minimal solenoids with a 1-dimensional transversal). Let $S \in S$ be a minimal $k$-solenoid which admits a 1-dimensional transversal $T$.

Then we have two cases:

(1) $T$ is a finite union of circles, and $S$ is a 1-dimensional foliation of a connected manifold of dimension $k + 1$.

(2) $T$ is totally disconnected, in which case we have two further possibilities:
   (a) $T$ is a finite set and $S$ is a connected manifold of dimension $k$,
   (b) $T$ is a Cantor set.

**Proof.** We define the proper interior of $T$ as the intersection of the proper interior of $S$ with $T$. Now we have two cases.

If the proper interior of $T$ is non-empty, then the proper interior of $S$ is non-empty. Then the complement of the proper interior of $S$, if non-empty, is a sub-solenoid of $S$, contradicting minimality. Thus the proper interior of $S$ is all $S$, so the proper interior of $T$ is the whole of $T$. This means that any point $p \in T$ has a neighbourhood (in
homeomorphic to an interval. Therefore $T$ is a topological compact 1-dimensional manifold, thus a finite union of circles. This ends the first case.

If the proper interior of $T$ is empty, then $T$ is totally disconnected. In this case, if $T$ has an isolated point $p$, then $S$ has only one leaf because by minimality any other leaf must accumulate the leaf containing $p$, and this is only possible if it coincides with it. Then $S$ is a $k$-dimensional connected manifold. If $T$ has no isolated points, then $T$ is non-empty, perfect, compact and totally disconnected, i.e. it is a Cantor set.

\[ \square \]

4. Holonomy, Poincaré return and suspension

**Definition 4.1. (Holonomy)** Given two points $p_1$ and $p_2$ in the same leaf, two local transversals $T_1$ and $T_2$, at $p_1$ and $p_2$ respectively, and a path $\gamma : [0, 1] \to S$, contained in the leaf with endpoints $\gamma(0) = p_1$ and $\gamma(1) = p_2$, we define a germ of a map (the holonomy map)

$$ h_\gamma : (T_1, p_1) \to (T_2, p_2), $$

by lifting $\gamma$ to nearby leaves.

We denote by $\text{Hol}_S(T_1, T_2)$ the set of germs of holonomy maps from $T_1$ to $T_2$.

The following result is clear.

**Proposition 4.2.** If $T_1$ and $T_2$ are global transversals then the sets of holonomy maps from $T_1$ to $T_2$ is non-empty. In particular, if $S$ is minimal the set of holonomy maps between two arbitrary local transversals is non-empty.

**Definition 4.3. (Holonomy pseudo-group)** The holonomy pseudo-group of a local transversal $T$ is the pseudo-group of holonomy maps from $T$ into itself. We denote it by $\text{Hol}_S(T) = \text{Hol}_S(T, T)$.

The holonomy pseudo-group of $S$ is the pseudo-group of all holonomy maps. We denote it by $\text{Hol}_S$,

$$ \text{Hol}_S = \bigcup_{T_1, T_2} \text{Hol}_S(T_1, T_2). $$

The following proposition is obvious.

**Proposition 4.4.** The orbit of a point $x \in S$ by the holonomy pseudo-group coincides with the leaf containing $x$.

Therefore, a solenoid $S$ is minimal if and only if the action of the holonomy pseudo-group is minimal, i.e. all orbits are dense.

The Poincaré return map construction reduces sometimes the holonomy to a classical dynamical system.
Definition 4.5. (Poincaré return map) Let $S$ be an oriented minimal 1-solenoid and $T$ be a local transversal. Then the holonomy return map is well defined for all points in $T$ and defines the Poincaré return map

$$R_T : T \to T.$$ 

The return map is well defined because in minimal solenoids “half” leaves are dense.

Lemma 4.6. Let $S$ be a minimal 1-solenoid. Let $p_0 \in S$ and let $l_0 \subset S$ be the leaf containing $p_0$. The point $p_0$ divides the leaf $l_0$ into two connected components. They are both dense in $S$.

Proof. Consider one connected component of $l_0 - \{p_0\}$, and let $C$ be its accumulation set. Then $C$ is non-empty, by compactness of $S$, and it is compact, as a closed subset of the compact solenoid $S$. It is also a union of leaves because if $l \subset S$ is a leaf, then $C \cap l$ is open in $l$ as is seen in flow-boxes, and also $C \cap l$ is closed in $l$. Therefore by connectedness of $l$, $C \cap l$ is empty or $l \subset C$.

We conclude that $C$ is a sub-solenoid, and by minimality we have $C = S$. 

When $S$ admits a global transversal (in particular when $S$ is minimal and admits a transversal) and the Poincaré return map is well defined, we have that it is continuous (without any assumption on minimality of $S$).

Proposition 4.7. Let $S$ be an oriented 1-solenoid and let $T$ be a global transversal such that the Poincaré return map $R_T$ is well defined. Then the holonomy return map $R_T$ is continuous. If the Poincaré return map for the reversed orientation of $S$ is also well defined, then $R_T$ is a homeomorphism of $T$. Moreover, if $S$ is a solenoid of class $C^{r,s}$ then $R_T$ is a $C^s$-diffeomorphism.

Proof. The map $R_T$ is continuous because the inverse image of an open set is clearly open.

If the Poincaré return map $R_T^r$ for the same transversal obtained for the reverse orientation of $S$ is also well defined, then $R_T$ is bijective because by construction its inverse is $R_T^r$. Hence $R_T$ is a homeomorphism of $T$. Moreover, letting $W$ be a foliated manifold defining the solenoid structure of $S$, $T$ is a subset of an open manifold $U$ of dimension $l$, and the map $R_T$ extends as a homeomorphism $U_1 \to U_2$, where $U_1, U_2$ are neighborhoods of $T$ (at least locally). If the transversal regularity of $S$ is $C^s$ then the local extension of $R_T$ is a $C^s$-diffeomorphism. 

When $T$ is only a local transversal then in general $R_T$ is not continuous. Nevertheless the discontinuities of $R_T$ are well controlled in practice and are innocuous when we deal with measure theoretic properties of $R_T$, as for example in section 10.

The suspension construction reverses Poincaré construction of the first return map.
Definition 4.8. (Suspension construction) Let $X \subset \mathbb{R}^l$ be a compact set and let $f : X \to X$ be a homeomorphism which has a $C^s$-diffeomorphism extension to a neighborhood of $X$ in $\mathbb{R}^l$. The suspension of $f$ is the oriented 1-solenoid $S_f$ defined by the suspension construction

$$S_f = ([0, 1] \times X) / (0,x) \sim (1,f(x)) .$$

The solenoid $S_f$ has regularity $C^{ω,s}$.

The transversal $T = \{0\} \times X$ is a global transversal and the associated Poincaré return map $R_T : T \to T$ is well defined and equal to $f$.

In particular, the theory of dynamical systems for $X \subset \mathbb{R}^l$ and diffeomorphisms $f : X \to X$ (extending to a neighborhood of $X$) is contained in the theory of transversal structures of solenoids.

5. MEASURABLE TRANSVERSAL STRUCTURE OF SOLENOIDs

Definition 5.1. (Transversal measure) Let $S$ be a $k$-solenoid. A transversal measure $μ = (μ_T)$ for $S$ associates to any local transversal $T$ a locally finite measure $μ_T$ supported on $T$, which are invariant by the holonomy pseudogroup. More precisely, if $T_1$ and $T_2$ are two transversals and $h : V \subset T_1 \to T_2$ is a holonomy map, then

$$h_∗(μ_{T_1}|_V) = μ_{T_2}|_{h(V)} .$$

We assume that a transversal measure $μ$ is non-trivial, i.e. for some $T$, $μ_T$ is non-zero.

We denote by $S_μ$ a $k$-solenoid $S$ endowed with a transversal measure $μ = (μ_T)$. We refer to $S_μ$ as a measured solenoid.

Observe that for any transversal measure $μ = (μ_T)$ the scalar multiple $cμ = (cμ_T)$, where $c > 0$, is also a transversal measure. Notice that there is no natural scalar normalization of transversal measures.

Definition 5.2. (Support of a transversal measure) Let $μ = (μ_T)$ be a transversal measure. We define the support of $μ$ by

$$\text{supp } μ = \bigcup_T \text{supp } μ_T ,$$

where the union runs over all local transversals $T$ of $S$.

Proposition 5.3. The support of a transversal measure $μ$ is a sub-solenoid of $S$.

Proof. For any flow-box $U$, $\text{supp } μ \cap U$ is closed in $U$, since $\text{supp } μ_{K(U)}$ is closed in $K(U)$. Hence, $\text{supp } μ$ is closed in $S$. Also, locally in flow-boxes $\text{supp } μ$ contains full leaves of $U$. Therefore a leaf of $S$ is either disjoint from $\text{supp } μ$ or contained in
supp $\mu$. Also supp $\mu$ is non-empty because $\mu$ is non-trivial. We conclude that supp $\mu$ is a sub-solenoid.

Definition 5.4. (Transverse ergodicity) A transversal measure $\mu = (\mu_T)$ on a solenoid $S$ is ergodic if for any Borel set $A \subset T$ invariant by the pseudo-group of holonomy maps on $T$, we have

$$\mu_T(A) = 0 \text{ or } \mu_T(A) = \mu_T(T).$$

We say that $S_\mu$ is an ergodic solenoid.

Definition 5.5. (Transverse unique ergodicity) Let $S$ be a $k$-solenoid. The solenoid $S$ is transversally uniquely ergodic, or a uniquely ergodic solenoid, if $S$ has a unique (up to scalars) transversal measure $\mu$ and moreover $\text{supp } \mu = S$.

Observe that in order to define these notions we only need continuous transversals. These ergodic notions are intrinsic and purely topological, i.e. if $S_1$ and $S_2$ are two homeomorphic solenoids by a homeomorphism $h: S_1 \to S_2$, then $S_1$ is uniquely ergodic if and only if $S_2$ is. If $S_{1,\mu_1}$ and $S_{2,\mu_2}$ are homeomorphic and $\mu_2 = h_\ast \mu_1$ via the homeomorphism $h: S_1 \to S_2$, then $S_{1,\mu_1}$ is ergodic if and only if $S_{2,\mu_2}$ is.

These notions of ergodicity generalize the classical ones and do exactly correspond to the classical notions in the situation described by the next theorem.

Theorem 5.6. Let $S$ be an oriented 1-solenoid. Let $T$ be a global transversal such that the Poincaré return map $R_T : T \to T$ is well defined.

Then the solenoid $S$ is minimal, resp. ergodic, uniquely ergodic, if and only if $R_T$ is minimal, resp. ergodic, uniquely ergodic.

Proof. We have by proposition 4.7 that $R_T$ is continuous. A leaf of $S$ is dense if and only if its intersection with $T$ is a dense orbit of $R_T$, hence the equivalence of minimality.

For the ergodicity, observe that we have a correspondence between measures on $T$ invariant by $R_T$ and transversal measures for $S$. Each transversal measure for $S$, locally defines a measure on $T$, hence defines a measure on $T$. Conversely, given a measure $\nu$ on $T$, we can transport $\nu$ in order to define a measure in each local transversal $T'$ in the following way. We can define a map $R_{T',T} : T' \to T$ of first impact on $T$ by following leaves of $S$ from $T'$ in the positive orientation. By the global character of the transversal this map is well defined. By construction $R_{T',T}$ is injective. So we can define $\mu_{T'} = R_{T',T}^\ast \nu_{R_{T',T}(T')}$. Then $(\mu_{T'})$ defines a transversal measure. The equivalence of unique ergodicity follows. Also $\nu$ is ergodic if and only $(\mu_{T'})$ is ergodic because any decomposition of $\nu = \nu_1 + \nu_2$ induces a decomposition of $(\mu_{T'})$ by the transversal measures corresponding to the decomposing measures. □
When we have an ergodic oriented 1-solenoid $S_\mu$ and $T$ is a local transversal, then the Poincaré return map is well defined $\mu_T$-almost everywhere and $\mu_T$ is ergodic.

**Proposition 5.7.** Let $S$ be an oriented 1-solenoid and let $T$ be a local transversal of $S$. Let $\mu$ be an ergodic transversal measure for $S$. Then the Poincaré return map $R_T$ is well defined for $\mu_T$-almost all points of $T$ and $\mu_T$ is an ergodic measure of $R_T$.

**Proof.** Let $A_T \subset T$ be the set of wandering points of $T$, i.e. those points whose positive half leaves through them never meet $T$ again. Clearly $A_T$ is a Borel set. If $\mu_T(A_T) \neq 0$ we can decompose $\mu_T$ by decomposing $\mu_T|_{A_T}$ and transporting the decomposition (back and forward) by the holonomy in order to decompose the transversal measure. Therefore $\mu_T(A_T) = 0$. As before, a decomposition of $\mu_T$ into invariant measures by $R_T$ yields a decomposition of the transversal measure $\mu$ invariant by holonomy. □

Recall that a dynamical system is minimal when all orbits are dense, and that uniquely ergodic dynamical systems are minimal. We have the same result for uniquely ergodic solenoids.

**Proposition 5.8.** An oriented uniquely ergodic 1-solenoid $S$ is minimal.

**Proof.** If $S$ has a non-dense leaf $l \subset S$, we can consider a local transversal $T_0$ such that $T_0 \cap \bar{l} \neq \emptyset, T_0$. Let $(l_n)$ be an exhaustion of $l$ by compact subsets. Let $\mu_n$ be the atomic probability measure on $T_0$ equidistributed on the intersection of $l_n$ with $T_0$. Any limit measure $\mu_{n_k} \to \nu$ is a probability measure on $T_0$ with supp $\nu \subset T_0 \cap \bar{l}$. It follows easily that $\nu$ is invariant by the holonomy on $T_0$. Transporting by the holonomy, $\nu$ defines a transversal measure $\mu = (\mu_T)$ (up to normalization, in each transversal it is also a limit of counting measures). But this contradicts unique ergodicity since supp $\mu \neq S$. □

Given a measured solenoid $S_\mu$ we can talk about “$\mu$-almost all leaves” with respect to the transversal measure. More precisely, a Borel set of leaves has zero $\mu$-measure if the intersection of this set with any local transversal $T$ is a set of $\mu_T$-measure zero.

Now Poincaré recurrence theorem for classical dynamical systems translates as:

**Proposition 5.9.** (Poincaré recurrence) Let $S_\mu$ be an ergodic oriented 1-solenoid with supp $\mu = S$. Then $\mu$-almost all leaves are dense and accumulate themselves.

**Proof.** For each local transversal $T$ we know by proposition 5.7 that the Poincaré return map $R_T$ is defined for $\mu_T$-almost every point and leaves invariant $\mu_T$. Therefore by Poincaré recurrence theorem, $\mu_T$-almost every point has a dense orbit by $R_T$ in supp $\mu_T = T$.

Observe that $S_\mu$ ergodic implies that $S$ is connected (otherwise we may decompose the invariant measure by restricting it to each connected component).
By compactness we can choose a finite number of local transversals $T_i = \varphi^{-1}(\{0\} \times K(U_i))$ with flow-boxes $\{U_i\}$ covering $S$. We can assume that we have that $U_i \cap U_j$ is a flow-box if non-empty. This and connectedness of $S$ imply that any Borel set of leaves that has either total or zero measure in a flow-box $U_i$, has the same property in $S$.

Now, the set of leaves $S_i$ that are non-dense in a given flow-box $U_i$ is of zero $\mu$-measure in $U_i$ (by Poincaré recurrence theorem applied to $R_{T_i}$). By the above, $S_i$ is of zero $\mu$-measure in $S$. Finally the set of non-dense leaves in $S$ is the finite union of the $S_i$, therefore is a set of leaves of zero $\mu$-measure. \hfill $\Box$

**Definition 5.10.** We denote by $\mathcal{M}_T(S)$ the space of transversal positive measures on the solenoid $S$ equipped with the topology generated by weak convergence in each local transversal. We denote by $\overline{\mathcal{M}}_T(S)$ the quotient of $\mathcal{M}_T(S)$ by positive scalar multiplication.

**Proposition 5.11.** The space $\mathcal{M}_T(S)$ is a cone in the vector space of all transversal signed measures $\mathcal{V}_T(S)$. Extremal measures of $\mathcal{M}_T(S)$ correspond to ergodic transversal measures.

*Proof.* Only the last part needs a proof. If $(\mu_T)$ is not ergodic, then there exists a local transversal $T_0$ and two disjoint Borel set $A, B \subset T_0$ invariant by holonomy with $\mu_{T_0}(A) \neq 0$, $\mu_{T_0}(B) \neq 0$ and $\mu_{T_0}(A) + \mu_{T_0}(B) = \mu_{T_0}(T_0)$. Let $S_A \subset S$, resp. $S_B \subset S$, be the union of leaves of the solenoid intersecting $A$, resp. $B$. These are Borel subsets of $S$. Let $(\mu_{T|T \cap S_A})$ and $(\mu_{T|T \cap S_B})$ be the transversal measures conditional to $T \cap S_A$, resp. $T \cap S_B$. Then

$$(\mu_T) = (\mu_{T|T \cap S_A}) + (\mu_{T|T \cap S_B}),$$

and $(\mu_T)$ is not extremal. \hfill $\Box$

**Corollary 5.12.** If $\mathcal{M}_T(S)$ is non-empty then $\mathcal{M}_T(S)$ contains ergodic measures.

*Proof.* The proof follows from the application of Krein-Milman theorem after the identification of $\overline{\mathcal{M}}_T(S)$ to a compact convex set of a locally convex topological vector space given below by theorem 6.8. \hfill $\Box$

### 6. Riemannian solenoids

All measures considered are Borel measures and all limits of measures are understood in the weak-* sense. We denote by $\mathcal{M}(S)$ the space of probability measures supported on $S$.

**Definition 6.1. (Riemannian solenoid)** Let $S$ be a $k$-solenoid of class $C^{r,s}$ with $r \geq 1$. A Riemannian structure on $S$ is a Riemannian structure on the leaves of $S$ such that there is a foliated manifold $W$ defining the solenoid structure of $S$ and a metric $g_W$ on the leaves of $W$ of class $C^{r-1,s}$ such that $g = g_W|_S$. 
As for compact manifolds, any solenoid can be endowed with a Riemannian structure. In the rest of this section $S$ denotes a Riemannian solenoid.

Note that a Riemannian structure defines a $k$-volume on leaves of $S$.

**Definition 6.2. (Flow group)** We define the flow group $G^0_S$ of a Riemannian $k$-solenoid $S$ as the group of $k$-volume preserving homeomorphisms of $S$ isotopic to the identity in the group of homeomorphisms of $S$. The full group consisting of $k$-volume preserving homeomorphisms of $S$ (non necessarily isotopic to the identity) is denoted by $G_S$.

**Definition 6.3. (daval measurers)** Let $\mu$ be a measure supported on $S$. The measure $\mu$ is a daval measure if it desintegrates as volume along leaves of $S$, i.e. for any flow-box $(U, \varphi)$ with local transversal $T = \varphi^{-1}((0) \times K(U))$, we have a measure $\mu_{U,T}$ supported on $T$ such that for any Borel set $A \subset U$

$$\mu(A) = \int_T \text{Vol}_k(A_y) \, d\mu_{U,T}(y),$$

where $A_y = A \cap \varphi^{-1}(D^k \times \{y\}) \subset U$.

We denote by $\mathcal{M}_{\mathcal{E}}(S) \subset \mathcal{M}(S)$ the space of probability daval measures.

It follows from this definition that the measures $(\mu_{U,T})$ do indeed define a transversal measure as we prove in the next proposition.

**Proposition 6.4.** Let $\mu$ be a daval measure on $S$. Then we have the following properties:

(i) For a local transversal $T$, the measures $\mu_{U,T}$ do not depend on $U$. So they define a unique measure $\mu_T$ supported on $T$

(ii) The measures $(\mu_T)$ are uniquely determined by $\mu$.

(iii) The measures $(\mu_T)$ are locally finite.

(iv) The measures $(\mu_T)$ are invariant by holonomy and therefore define a transversal measure.

**Proof.** For (i) and (ii) notice that for any Riemannian metric $g$ we have, denoting by $B^g_\epsilon(y)$ the Riemannian ball of radius $\epsilon$ around $y$ in its leaf,

$$\lim_{\epsilon \to 0} \frac{\text{Vol}_k(B^g_\epsilon(y))}{\epsilon^k} = c(k),$$

where $c(k)$ is a constant only depending on $k$. Therefore by dominated convergence we have for any Borel set $C \subset T$

$$\mu_{U,T}(C) = \lim_{\epsilon \to 0} \int_T \frac{\text{Vol}_k(B^g_\epsilon(y))}{c(k)\epsilon^k} \, d\mu_{U,T}(y) = \lim_{\epsilon \to 0} \frac{\mu(V_\epsilon(C))}{c(k)\epsilon^k},$$
where $V_{\epsilon}(C)$ denotes the $\epsilon$-neighborhood of $C$ along leaves. The last limit is clearly independent of $U$, thus $\mu_{U,T}$ is independent of $U$ as claimed, and $\mu_T$ is uniquely determined by $\mu$.

For (iii) observe that for each flow-box $U$ we have that

$$y \mapsto \text{Vol}_k(L_y),$$

$L_y = \varphi^{-1}(D^k \times \{y\})$, is continuous on $y \in T$, therefore for any compact subset $C \subset T$ exists $\epsilon_0 > 0$ such that for all $y \in C$, $\text{Vol}_k(L_y) \geq \epsilon_0$.

Let $V = \varphi^{-1}(D^k \times C)$, then we have

$$\mu(V) = \int_C \text{Vol}_k(L_y) \, d\mu_T(y) \geq \epsilon_0 \, \mu_T(C),$$

therefore $\mu_T(C) < +\infty$.

Regarding (iv), consider a flow-box $(U, \varphi)$ and two local transversals $T_1$ and $T_2$ in $U$ of the form $T_i = \varphi^{-1}(\{x_i\} \times K(U))$, $i = 1, 2$, $x_i \in D^k$. These transversals are associated to flow-boxes $(U, \varphi_i)$ with the same domain $U$. There is a local holonomy map in $U$, $h : T_1 \to T_2$. For any Borel set $A \subset U$, we have by definition

$$\int_{T_1} \text{Vol}_k(A_y) \, d\mu_{U,T_1}(y) = \mu(A) = \int_{T_2} \text{Vol}_k(A_{y'}) \, d\mu_{U,T_2}(y').$$

On the other hand, the change of variables, $y' = h(y)$, gives

$$\int_{T_1} \text{Vol}_k(A_y) \, d\mu_{U,T_1}(y) = \int_{T_2} \text{Vol}_k(A_{y'}) \, dh \ast \mu_{U,T_1}(y').$$

Thus for any Borel set $A \subset U$,

$$\int_{T_2} \text{Vol}_k(A_{y'}) \, d\mu_{U,T_2}(y') = \int_{T_2} \text{Vol}_k(A_{y'}) \, dh \ast \mu_{U,T_1}(y').$$

And taking horizontal Borel sets, this implies

$$\mu_{U,T_2} = h \ast \mu_{U,T_1}.$$
From this it follows that Riemannian solenoids do not necessarily have daval measures (i.e. $\mathcal{M}_L(S)$ can be empty), because there are solenoids that do not admit transversal measures (see [Pl] for interesting examples).

Proposition 6.5. The space of probability daval measures $\mathcal{M}_L(S)$ is a compact convex set in the vector space of signed measures $\mathcal{V}(S)$.

Proof. The convexity is clear, and by compactness of $\mathcal{M}(S)$ we only need to show that $\mathcal{M}_L(S)$ is closed which follows from the more precise lemma that follows.

Lemma 6.6. Let $(\mu_n)$ be a sequence of measures on $S$ that desintegrate as volume on leaves in a flow-box $U$. Then any limit $\mu$ desintegrates as volume on leaves in $U$ and the transversal measure is the limit of the transversal measures.

Proof. We assume that $\mu_n \to \mu$. Given the transversal $T$, the tranversal measures $(\mu_n, T)$ are locally finite by proposition 6.4. Moreover, formula (1) shows that they are uniformly locally finite. Extract (with a diagonal process) a converging subsequence $\mu_{n_k}$ to a locally finite measure $\mu_T$. For any vertically compactly supported continuous function $\varphi$ defined on $U$ and depending only on $y \in T$,

$$\int_U \varphi \ d\mu_{n_k} = \int_T \varphi \text{Vol}_k(L_y) \ d\mu_{n_k,T}(y),$$

with $L_y = \varphi^{-1}(D^k \times \{y\})$. Passing to the limit $k \to +\infty$,

$$\int_U \varphi \ d\mu = \int_T \varphi \text{Vol}_k(L_y) \ d\mu_T(y).$$

Therefore the limit measure $\mu$ desintegrates as volume on leaves in $U$ with transversal measure $\mu_T$. Since $\mu_T$ is uniquely determined by $\mu$ (by proposition 6.4), the only limit of the transversal measures is $\mu_T$. □

Theorem 6.7. A finite measure $\mu$ on $S$ is $G^0_S$-invariant if and only if $\mu$ is a daval measure.

Proof. If $\mu$ desintegrates as volume along leaves, then it is clearly invariant by a transformation in $G^0_S$ close to the identity as is seen in each flow-box. Then it is $G^0_S$-invariant since a neighborhood of the identity in $G^0_S$ generates $G^0_S$.

Conversely, assume that $\mu$ is a $G^0_S$-invariant finite measure. We must prove that in any flow-box $(U, \varphi)$ we have $\mu = \text{Vol}_k \times \mu_{K(U)}$. We can find a map $h : D^k \times K(U) \to \mathbb{R}^k \times K(U)$ of class $C^{r,s}$, preserving leaves, and such that it takes the $k$-volume for the Riemannian metric to the Lebesgue measure on $\mathbb{R}^k$. On $h(D^k \times K(U))$, we still denote by $\mu$ the corresponding measure. We can desintegrate $\mu = \{\nu_y\} \times \eta$, where $\eta$ is supported on $K(U)$ and $\nu_y$ is a measure on each horizontal leaf, parametrized by...
\( y \in K(U) \) (see [Bon, Di]), i.e.

\[
\int_U \varphi \, d\mu = \int_{K(U)} \left( \int_{L_y} \varphi \, d\nu_y \right) \, d\eta(y).
\]

The group \( G^0_S \) in this chart contains all small translations. Each translation must leave invariant \( \eta \)-almost all measures \( \nu_y \). Therefore a countable number of translations leave invariant \( \eta \)-almost all measures \( \nu_y \). Now observe that if \( \tau_n \) are translations leaving invariant \( \nu_y \), and \( \tau_n \to \tau \), then \( \tau \) leaves \( \nu_y \) invariant. Thus taking a countable and dense set of translations of fixed small displacement, and taking limits, it follows that all small translations leave invariant \( \nu_y \) for \( \eta \)-almost all \( y \). By Haar theorem these measures are proportional to the Lebesgue measure, \( \nu_y = c(y) \text{Vol}_k \). We have that \( c \in L^1(K(U), \eta) \) by applying (2) with \( \varphi \) being the characteristic function of a sub-flow-box with horizontal leaves being balls of fixed \( k \)-volume. We define the transversal measure \( \mu_{K(U)} \) as

\[ d\mu_{K(U)} = c \, d\eta. \]

Therefore \( \mu = \text{Vol}_k \times \mu_{K(U)} \) on \( U \), hence \( \mu \) is a daval measure.

**Theorem 6.8. (Tranverse measures of the Riemannian solenoid)** There is a one-to-one correspondence between transversal measures \( (\mu_T) \) and finite daval measures \( \mu \). Furthermore, there is an isomorphism

\[ \overline{\mathcal{M}}_T(S) \cong \mathcal{M}_E(S). \]

**Proof.** The open sets inside flow-boxes form a basis for the Borel \( \sigma \)-algebra, and the formula

\[ \mu(A) = \int_T \text{Vol}_k(A_y) \, d\mu_T(y), \]

for \( A \) in a flow-box \( U \) with local transversal \( T \), is compatible for different flow-boxes. So it defines a measure \( \mu \). This measure is finite because by compactness we can cover \( S \) by a finite number of flow-boxes with finite mass. By construction, \( \mu \) is a daval measure. The converse was proved earlier in proposition 6.4.

This correspondence is clearly a topological isomorphism.

**Definition 6.9. (Volume of measured solenoids)** For a measured Riemannian solenoid \( S_\mu \) we define the volume measure of \( S \) as the unique probability measure (also denoted by \( \mu \)) associated to the transversal measure \( \mu = (\mu_T) \) by theorem 6.8.

For uniquely ergodic Riemannian solenoids \( S \), this volume measure is uniquely determined by the Riemannian structure (as for a compact Riemannian manifold). We observe that, contrary to what happens with compact manifolds, there is no canonical total mass normalization of the volume of the solenoid depending only on the Riemannian metric. This is the reason why we normalize \( \mu \) to be a probability measure.
Definition 6.10. (Controlled growth solenoids) Let $S$ be a Riemannian solenoid. Fix a leaf $l \subset S$ and an exhaustion $(C_n)$ by subsets of $l$. For a flow-box $(U, \varphi)$ write
\[ C_n \cap U = A_n \cup B_n, \]
where $A_n$ is composed by all full disks $L_y = \varphi^{-1}(D^k \times \{y\})$ contained in $C_n$, and $B_n$ contains those connected components $B$ of $C_n \cap U$ such that $B \neq L_y \cap U$ for any $y$. The solenoid $S$ has controlled growth with respect to $l$ and $(C_n)$ if for any flow-box $U$ in a finite covering of $S$
\[ \lim_{n \to +\infty} \frac{\operatorname{Vol}_k(B_n)}{\operatorname{Vol}_k(A_n)} = 0. \]

The solenoid $S$ has controlled growth if $S$ contains a leaf $l$ and an exhaustion $(C_n)$ such that $S$ has controlled growth with respect to $l$ and $(C_n)$.

For a Riemannian solenoid $S$, it is natural to consider the exhaustion by balls $B(x_0, R_n)$ in a leaf $l$ centered at a point $x_0 \in l$ and with $R_n \to +\infty$, and test the controlled growth condition with respect to such exhaustions.

The controlled growth condition depends a priori on the Riemannian metric. As we see next, it guarantees the existence of daval measures, hence the existence of transversal measures on $S$. Indeed the measures we construct are Schwartzman measures defined as:

Definition 6.11. (Schwartzman limits and measures) We say that a measure $\mu$ is a Schwartzman measure if it is obtained as the Schwartzman limit
\[ \mu = \lim_{n \to +\infty} \mu_n, \]
where the measures $(\mu_n)$ are the normalized $k$-volume of the exhaustion $(C_n)$ with uniformly bounded total mass. We denote by $\mathcal{M}_S(S)$ the space of probability Schwartzman measures.

Compactness of probability measures show:

Proposition 6.12. There are always Schwartzman measures on $S$, $\mathcal{M}_S(S) \neq \emptyset$.

Theorem 6.13. If $S$ is a solenoid with controlled growth, then any Schwartzman measure is a daval measure,
\[ \mathcal{M}_S(S) \subset \mathcal{M}_L(S). \]
In particular, $\mathcal{M}_L(S) \neq \emptyset$ and $S$ admits transversal measures.

Proof. Let $\mu_n \to \mu$ be a Schwartzman limit as in definition 6.11. For any flow-box $U$ we prove that $\mu$ desintegrates as volume on leaves of $U$. Since $S$ has controlled
growth, pick a leaf and an exhaustion which satisfy the controlled growth condition. Let
\[ C_n \cap U = A_n \cup B_n, \]
be the decomposition for \( C_n \cap U \) described before. The set \( A_n \) is composed of a finite number of horizontal disks. We define a new measure \( \nu_n \) with support in \( U \) which is the restriction of \( \mu_n \) to \( A_n \), i.e. it is proportional to the \( k \)-volume on horizontal disks. The measure \( \nu_n \) desintegrates as volume on leaves in \( U \). The transversal measure is a finite sum of Dirac measures. Moreover the controlled growth condition implies that \((\nu_n)\) and \((\mu_n|U)\) must converge to the same limit. But we know that \( M_L(S) \) is closed, thus the limit measure \( \mu|U \) desintegrates on leaves in \( U \). So \( \mu \) is a daval measure. \( \square \)

**Corollary 6.14.** The volume \( \mu \) of a uniquely ergodic solenoid with controlled growth is the unique Schwartzman measure. Therefore there is only one Schwartzman limit
\[ \mu = \lim_{n \to +\infty} \mu_n, \]
which is independent of the leaf and the exhaustion.

**Proof.** There are always Schwartzman limits. Theorem 6.13 shows that any such limit \( \mu \) desintegrates as volume on leaves. Thus the measure \( \mu \) defines the unique (up to scalars) transversal measure \( (\mu_T) \). But, conversely, the transversal measure determines the measure \( \mu \) uniquely. Therefore there is only possible limit \( \mu \), which is the volume of the uniquely ergodic solenoid. \( \square \)

Following the proof of theorem 6.13 we can be more precise. We first define irregular measures. These are measures which have no mass that desintegrates as volume along leaves.

**Definition 6.15.** Let \( \mu \) be a measure supported on \( S \). We say that \( \mu \) is irregular if for any Borel set \( A \subset S \) and for any non-zero measure \( \nu \in M_L(S) \) we do not have
\[ \nu|_A \leq \mu|_A. \]

**Theorem 6.16.** Let \( \mu \) be any measure supported on \( S \). There is a unique canonical decomposition of \( \mu \) into a regular part \( \mu_r \in M_L(S) \) and an irregular part \( \mu_i \),
\[ \mu = \mu_r + \mu_i. \]

We can define the regular part by
\[ \mu_r(A) = \sup_{\nu} \nu(A) \leq \mu(A), \]
for any Borel set \( A \subset S \), where the supremum runs over all measures \( \nu \in M_L(S) \), with \( \nu|_A \leq \mu|_A \) (if no such measure exists then \( \mu_r(A) = 0 \)).
Note that this theorem corresponds to the decomposition of any measure on a regular manifold into an absolutely continuous part with respect to a Lebesgue measure and a singular part. Indeed, it generalizes that decomposition to solenoids, since this theorem reduces to the classical result when the solenoid is a manifold.

**Proof.** Consider all measures \( \nu \in \mathcal{M}_L(S) \) such that \( \nu \leq \mu \). We define \( \mu_r = \sup \nu \). Considering a countable basis \((A_i)\) for the Borel \(\sigma\)-algebra and extracting a triangular subsequence, we can find a sequence of such measures \((\nu_n)\) such that \( \nu_n(A_i) \to \mu_r(A_i) \), for all \( i \), i.e. \( \nu_n \to \mu_r \). Since \( \mathcal{M}_L(S) \) is closed it follows that \( \mu_r \in \mathcal{M}_L(S) \). By construction, \( \mu - \mu_r \) is a positive measure and irregular. Moreover the decomposition

\[
\mu = \mu_r + \mu_i
\]

is unique, because for another decomposition

\[
\mu = \nu_r + \nu_i,
\]

we have by construction of \( \mu_r \),

\[
\nu_r \leq \mu_r.
\]

Therefore

\[
\nu_i = (\mu_r - \nu_r) + \mu_i,
\]

and \( \mu_i \) being positive this implies that

\[
0 \leq \mu_r - \nu_r \leq \nu_i.
\]

By definition of irregularity of \( \nu_i \), this is only possible if \( \mu_r = \nu_r \), then also \( \mu_i = \nu_i \), and the decomposition is unique. \( \square \)

7. **Generalized Ruelle-Sullivan currents**

We fix in this section a \( C^\infty \) manifold \( M \) of dimension \( n \).

**Definition 7.1. (Immersion and embedding of solenoids)** Let \( S \) be a \( k \)-solenoid of class \( C^{r,s} \) with \( r \geq 1 \). An immersion

\[
f : S \to M
\]

is a regular map (that is, it has an extension \( \hat{f} : W \to M \) of class \( C^{r,s} \), where \( W \) is a foliated manifold which defines the solenoid structure of \( S \)), such that the differential restricted to the tangent spaces of leaves has rank \( k \) at every point of \( S \). We denote by \((f,S)\) an immersed solenoid.

Let \( r, s \geq 1 \). A transversally immersed solenoid \((f,S)\) is a regular map \( f : S \to M \) such that it admits an extension \( \hat{f} : W \to M \) which is an embedding (of a \((k+l)\)-dimensional manifold into an \( n \)-dimensional one) of class \( C^{r,s} \), such that the images of the leaves intersect transversally in \( M \).

An embedded solenoid \((f,S)\) is a transversally immersed solenoid of class \( C^{r,s} \), with \( r, s \geq 1 \), with injective \( f \), that is, the leaves do not intersect or self-intersect.
Note that under a transversal immersion, resp. an embedding, \( f : S \to M \), the images of the leaves are immersed, resp. injectively immersed, submanifolds.

A foliation of \( M \) can be considered as a solenoid, and the identity map is an embedding.

**Definition 7.2. (Generalized currents)** Let \( S \) be an oriented \( k \)-solenoid of class \( C^{r,s} \), \( r \geq 1 \), endowed with a transversal measure \((\mu_T)\). An immersion \( f : S \to M \) defines a real homology class \([f, S_\mu] \in H_k(M, \mathbb{R})\) by duality with differential forms as follows.

Let \( \omega \) be an \( k \)-differential form in \( M \). The pull-back \( f^* \omega \) defines a \( k \)-differential form on the leaves of \( S \). Let \( S = \bigcup S_i \) be a measurable partition such that each \( S_i \) is contained in a flow-box \( U_i \). We define

\[
\langle [f, S_\mu], \omega \rangle = \sum_i \int_{K(U_i)} \left( \int_{L_y \cap S_i} f^* \omega \right) d\mu_{K(U_i)}(y),
\]

where \( L_y \) denotes the horizontal disk of the flow-box.

Note that this definition does not depend on the measurable partition (given two partitions consider the common refinement). If the support of \( f^* \omega \) is contained in a flow-box \( U \) then

\[
\langle [f, S_\mu], \omega \rangle = \int_{K(U)} \left( \int_{L_y} f^* \omega \right) d\mu_{K(U)}(y).
\]

In general, take a partition of unity \( \{\rho_i\} \) subordinated to the covering \( \{U_i\} \), then

\[
\langle [f, S_\mu], \omega \rangle = \sum_i \int_{K(U_i)} \left( \int_{L_y} \rho_i f^* \omega \right) d\mu_{K(U_i)}(y).
\]

Also for any exact differential \( \omega = d\alpha \) we have

\[
\langle [f, S_\mu], d\alpha \rangle = \sum_i \int_{K(U_i)} \left( \int_{L_y} \rho_i f^* d\alpha \right) d\mu_{K(U_i)}(y)
= \sum_i \int_{K(U_i)} \left( \int_{L_y} d(\rho_i f^* \alpha) \right) d\mu_{K(U_i)}(y)
- \sum_i \int_{K(U_i)} \left( \int_{L_y} d\rho_i \wedge f^* \alpha \right) d\mu_{K(U_i)}(y) = 0.
\]

The first term vanishes using Stokes in each leaf (the form \( \rho_i f^* \alpha \) is compactly supported on \( U_i \)), and the second term vanishes because \( \sum_i d\rho_i \equiv 0 \). Therefore \([f, S_\mu]\) is a well defined homology class of degree \( k \).
In their original article [RS], Ruelle and Sullivan defined this notion for the restricted class of solenoids embedded in $M$.

We can define for the solenoid $S$ homology groups $H_p(S)$ (whose construction is recalled in Appendix B). If $S$ is an oriented $k$-solenoid and $\mu$ is a transversal measure, then there is an associated fundamental class $[S_\mu] \in H_k(S)$. The generalized current is the push-forward of this fundamental class by $f$.

**Proposition 7.3.** Let $S_\mu$ be an oriented measured $k$-solenoid. If $f : S \to M$ is an immersion, we have

$$f_*[S_\mu] = [f, S_\mu] \in H_k(M, \mathbb{R}).$$

**Proof.** The equality $\langle [f, S_\mu], \omega \rangle = \langle [S_\mu], f^*\omega \rangle$ is clear for any $\omega \in \Omega^k(M)$ (see the construction of the fundamental class in definition B.1). The result follows. □

From now on, we shall consider a $C^\infty$ compact and oriented manifold $M$ of dimension $n$. Let $(f, S_\mu)$ be an oriented measured $k$-solenoid immersed in $M$. We aim to construct a $(n-k)$-form representing $[f, S_\mu]^* \in H^{n-k}(M, \mathbb{R})$, the dual of $[f, S_\mu]$ under the Poincaré duality isomorphism $H_k(M, \mathbb{R}) \cong H^{n-k}(M, \mathbb{R})$.

Fix an accessory Riemannian metric $g$ on $M$. This endows $S$ with a solenoid Riemannian metric $f^*g$. We can define the normal bundle

$$\pi : \nu_f \to S,$$

which is an oriented bundle of rank $n-k$, since both $S$ and $M$ are oriented. The total space $\nu_f$ is a (non-compact) $n$-solenoid whose leaves are the preimages by $\pi$ of the leaves of $S$.

By Appendix B there is a Thom form $\Phi \in \Omega^{n-k}_{cv}(\nu_f)$ for the normal bundle. This is a closed $(n-k)$-form on the total space of the bundle $\nu_f$, with vertical compact support, and satisfying that

$$\int_{\nu_{f,p}} \Phi = 1,$$

for all $p \in S$, where $\nu_{f,p} = \pi^{-1}(p)$. Denote by $\nu_r \subset \nu$ the disc bundle formed by normal vectors of norm at most $r$ at each point of $S$. By compactness of $S$, there is an $r_0 > 0$ such that $\Phi$ has compact support on $\nu_{r_0}$.

For any $\lambda > 0$, let $T_\lambda : \nu_f \to \nu_f$ be the map which is multiplication by $\lambda^{-1}$ in the fibers. Then set

$$\Phi_r = T_{r/r_0}^* \Phi,$$
for any $r > 0$. So $\Phi_r$ is a closed $(n - k)$-form, supported in $\nu_r$, and satisfying

$$\int_{\nu_f} \Phi_r = 1,$$

for all $p \in S$. Hence it is a Thom form for the bundle $\nu_f$ as well. By Appendix B $[\Phi_r] = [\Phi]$ in $H^{n-k}_{cv}(\nu_f)$, i.e. $\Phi_r - \Phi = d\beta$, with $\beta \in \Omega^{n-k-1}_{cv}(\nu_f)$.

Using the exponential map and the immersion $f$, we have a map $j : \nu_f \to M$, given as $j(p, v) = \exp_f(p)(v)$, which is a regular map from the $\nu_f$ (as an $n$-solenoid) to $M$. By compactness of $S$, there are $r_1, r_2 > 0$ such that for any disc $D$ of radius $r_2$ contained in a leaf of $S$, the map $j$ restricted to $\pi^{-1}(D) \cap \nu_{r_1}$ is a diffeomorphism onto an open subset of $M$. Let us now define a push-forward map $j_* : \Omega^p_{cv}(\nu_{r_1}) \to \Omega^p(M)$.

Consider first a flow-box $U \cong D^k \times K(U)$ for $S$, where the leaves of the flow-box are contained in discs of radius $r_2$. Then

$$\pi^{-1}(U) \cap \nu_{r_1} \cong D^{n-k}_{r_1} \times D^k \times K(U),$$

where $D^{n-k}_{r_1}$ denotes the disc of radius $r > 0$ in $\mathbb{R}^{n-k}$. Let $\alpha \in \Omega^p_{cv}(\nu_{r_1})$ with support in $\pi^{-1}(U) \cap \nu_{r_1}$. Then we define

$$j_*\alpha := \int_{K(U)} ((j_y)_*(\alpha|_{D^{n-k}_{r_1} \times D^k \times \{y\}}))d\mu_{K(U)}(y),$$

where $j_y$ is the restriction of $j$ to $D^{n-k}_{r_1} \times D^k \times \{y\} \subset \pi^{-1}(U) \cap \nu_{r_1}$, which is a diffeomorphism onto its image in $M$. This is the average of the push-forwards of $\alpha$ restricted to the leaves of $\nu_f$, using the transversal measure.

Now in general, consider a covering $\{U_i\}$ of $S$ by flow-boxes such that the leaves of the flow-boxes $U_i$ are contained in discs of radius $r_2$. Then, for any form $\alpha \in \Omega^p_{cv}(\nu_{r_1})$, we decompose $\alpha = \sum \alpha_i$ with $\alpha_i$ supported in $\pi^{-1}(U_i) \cap \nu_{r_1}$. Define

$$j_*\alpha := \sum j_*\alpha_i.$$

This does not depend on the chosen cover.

**Proposition 7.4.** There is a well defined push-forward linear map

$$j_* : \Omega^p_{cv}(\nu_{r_1}) \to \Omega^p(M),$$

such that $dj_*\alpha = j_*d\alpha$, and $j_*(\alpha \wedge \beta) = j_*\alpha \wedge j_*\beta$, for $\alpha, \beta \in \Omega^p_{cv}(\nu_{r_1})$.

**Proof.** $j_*d\alpha = d j_*\alpha$ holds in flow-boxes, hence it holds globally. The other assertion is analogous. \qed
When $M$ is a compact and oriented $n$-manifold, the generalized current $[f, S_\mu] \in H_k(M, \mathbb{R})$ gives an element

$$[f, S_\mu]^* \in H^{n-k}(M, \mathbb{R}),$$

under the Poincaré duality isomorphism $H_k(M, \mathbb{R}) \cong H^{n-k}(M, \mathbb{R})$.

We can construct a form representing the dual of the generalized current.

**Proposition 7.5.** Let $M$ be a compact oriented manifold. Let $(f, S_\mu)$ be a oriented measured solenoid immersed in $M$. Let $\Phi_r$ be the Thom form of the normal bundle $\nu_f$ supported on $\nu_r$, for $0 < r < r_1$. Then $j_*\Phi_r$ is a closed $(n-k)$-form representing the dual of the generalized current,

$$[j_*\Phi_r] = [f, S_\mu]^*.$$

**Proof.** As $\Phi_r$ is a closed form, we have

$$d j_*\Phi_r = j_*d \Phi_r = 0,$$

for $0 < r \leq r_1$, so the class $[j_*\Phi_r] \in H^{n-k}(M, \mathbb{R})$ is well-defined.

Now let $r, s$ such that $0 < r \leq s < r_1$. Then $[\Phi_r] = [\Phi_s]$ in $H^{n-k}_{cv}(\nu_f)$, so there is a vertically compactly supported $(n-k-1)$-form $\eta$ with

$$(3) \quad \Phi_r - \Phi_s = d\eta.$$

Let $r_3 > 0$ be such that $\eta$ has support on $\nu_{r_3}$. We can define a smooth map $F$ which is the identity on $\nu_s$, which sends $\nu_{r_3}$ into $\nu_{r_1}$ and it is the identity on $\nu_f - \nu_{2r_3}$. Pulling back (3) with $F$, we get

$$\Phi_r - \Phi_s = d(F^*\eta),$$

where $F^*\eta \in \Omega^{n-k-1}_{cv}(\nu_{r_1})$. We can apply $j_*$ to this equality to get

$$j_*\Phi_r - j_*\Phi_s = d j_* (F^*\eta),$$

and hence $[j_*\Phi_r] = [j_*\Phi_s]$ in $H^{n-k}(M, \mathbb{R})$.

Now we want to prove that $[j_*\Phi_r]$ coincides with the dual of the generalized current $[f, S_\mu]^*$. Let $\beta$ be any $k$-form in $\Omega^k(S)$. Consider a cover $\{U_i\}$ of $S$ by flow-boxes such that the leaves of each flow-box are contained in discs of radius $r_2$, and let $\{\rho_i\}$ be a partition of unity subordinated to this cover. Let $\Phi_i = \rho_i \Phi$, which is supported on $\pi^{-1}(U_i) \cap \nu_f$, and

$$\Phi_{r,i} = \rho_i \Phi_r = T^*_{r/r_0} \Phi_i.$$
supported on $\pi^{-1}(U_i) \cap \nu_r$. For $0 < \epsilon \leq r_1$, we have

$$
\int_M j_* \Phi_{\epsilon,i} \wedge \beta = \int_M \left( \int_{K(U_i)} (j_{i,y})_* \Phi_{\epsilon,i|A_y} \right) d\mu_{K(U_i)}(y) \wedge \beta 
= \int_{K(U_i)} \left( \int_M (j_{i,y})_* \Phi_{\epsilon,i} \wedge \beta \right) d\mu_{K(U_i)}(y) 
= \int_{K(U_i)} \Phi_{\epsilon,i} \wedge j_{i,y}^* \beta d\mu_{K(U_i)}(y),
$$

where $A_{y}^{\epsilon} = D_{\epsilon}^{n-k} \times D^{k} \times \{y\} \subset \pi^{-1}(U_i)$ for $y \in K(U_i)$, and $j_{i,y} = j|_{A_y^{\epsilon}}$.

In coordinates $(v_1, \ldots, v_{n-k}, x_1, \ldots, x_k, y)$ for $\pi^{-1}(U_i) \cong \mathbb{R}^{n-k} \times D^{k} \times K(U_i)$, we can write

$$
\Phi = \Phi(v, x, y) = g_0 dv_1 \wedge \cdots \wedge dv_{n-k} + \sum_{|I| > 0} g_{IJ} dx_I \wedge dv_J,
$$

where $g_0, g_{IJ}$ are functions, and $I = \{i_1, \ldots, i_a\} \subset \{1, \ldots, n-k\}$ and $J = \{j_1, \ldots, j_b\} \subset \{1, \ldots, k\}$ multi-indices with $|I| = a$, $|J| = b$, $a+b = n-k$. Pulling-back via $T = T_{\epsilon/r_0}$, we get

$$
\Phi_{\epsilon} = \left( \frac{\epsilon}{r_0} \right)^{-n-k} \left( g_0 \circ T \right) dv_1 \wedge \cdots \wedge dv_{n-k} + \sum_{|I| > 0} \left( \frac{\epsilon}{r_0} \right)^{|I|} (g_{IJ} \circ T) dx_I \wedge dv_J
= \left( \frac{\epsilon}{r_0} \right)^{-n-k} \left( (g_0 \circ T) dv_1 \wedge \cdots \wedge dv_{n-k} + O(\epsilon) \right),
$$

since $|g_{IJ} \circ T|$ are uniformly bounded. Note that the support of $\Phi_{\epsilon|\mathbb{R}^{n-k} \times D^{k} \times \{y\}}$ is inside $D_{\epsilon}^{n-k} \times D^{k} \times \{y\}$.

Also write

$$
j_{i,y}^* \beta(v, x) = h_0(x, y) dx_1 \wedge \cdots \wedge dx_k + \sum_{|J| > 0} h_{IJ}(x, y) dx_I \wedge dv_J + O(|v|),
$$

and note that $f^* \beta|_{D^{k} \times \{y\}} = h_0(x, y) dx_1 \wedge \cdots \wedge dx_k$. 
So
\[
\int_{A_y} \Phi_{\epsilon,i} \wedge j_{i,y}^* \beta = \int_{\mathbb{R}^{n-k} \times D_k} \rho_i \Phi_{\epsilon} \wedge j_{i,y}^* \beta
\]
\[
= \left( \frac{\epsilon}{r_0} \right)^{-(n-k)} \left( \int_{\mathbb{R}^{n-k} \times D_k} \rho_i (g_0 \circ T) dv_1 \wedge \cdots \wedge dv_{n-k} \wedge j_{i,y}^* \beta + O(\epsilon^{n-k+1}) \right)
\]
\[
= \left( \frac{\epsilon}{r_0} \right)^{-(n-k)} \left( \int_{\mathbb{R}^{n-k} \times D_k} \rho_i (g_0 \circ T) dv_1 \wedge \cdots \wedge dv_{n-k} \wedge (h_0 dx_1 \wedge \cdots \wedge dx_k + O(|v|)) + O(\epsilon) \right)
\]
\[
= \left( \frac{\epsilon}{r_0} \right)^{-(n-k)} \left( \int_{\mathbb{R}^{n-k} \times D_k} \rho_i h_0 (g_0 \circ T) dv_1 \wedge \cdots \wedge dv_{n-k} \wedge dx_1 \wedge \cdots \wedge dx_k + O(\epsilon) \right)
\]
\[
= \int_{D^k} \rho_i h_0 dx_1 \wedge \cdots \wedge dx_k + O(\epsilon)
\]
\[
= \int_{D^k \times \{y\}} \rho_i f^* \beta_{D^k \times \{y\}} + O(\epsilon)
\]
The second equality holds since $|\rho_i| \leq 1$, $|j_{i,y}^* \beta|$ is uniformly bounded, and the support of $\rho_i (g_1 \circ T) dx_1 \wedge dv_1 \wedge j_{i,y}^* \beta$ is contained inside $D_\epsilon^{n-k} \times D_k$, which has volume $O(\epsilon^{n-k})$.

In the fourth line we use that $|v| \leq \epsilon$ and
\[
\left( \frac{\epsilon}{r_0} \right)^{-(n-k)} \int_{\mathbb{R}^{n-k}} (g_0 \circ T) dv_1 \wedge \cdots \wedge dv_{n-k} = \int_{\nu_{f,p}} T^* \Phi = 1.
\]
The same equality is used in the fifth line.

Adding over all $i$, we get
\[
\langle [j_* \Phi], [\beta] \rangle = \int_M j_* \Phi \wedge \beta = \sum_i \int_M j_* \Phi_{\epsilon,i} \wedge \beta
\]
\[
= \sum_i \int_{K(U_i)} \left( \int_{A_y} \Phi_{\epsilon,i} \wedge j_{i,y}^* \beta \right) d\mu_{K(U_i)}(y)
\]
\[
= \sum_i \int_{K(U_i)} \left( \int_{D^k \times \{y\}} \rho_i f^* \beta_{D^k \times \{y\}} + O(\epsilon) \right) d\mu_{K(U_i)}(y)
\]
\[
= \langle [f_S \mu], [\beta] \rangle + O(\epsilon).
\]
Taking $\epsilon \to 0$, we get that
\[
[j_* \Phi] = \lim_{\epsilon \to 0} [j_* \Phi] = [f_S \mu]^*,
\]
for all $0 < r < r_1$.

For $M$ non-compact, we have the isomorphism $H_k(M, \mathbb{R}) \cong H^{n-k}_c(M, \mathbb{R})$, where $H_c^*(M, \mathbb{R})$ denotes compactly supported cohomology of $M$. Then the generalized
current \([f, S_\mu]\) of an immersed oriented measured solenoid \((f, S_\mu)\) gives an element

\[ [f, S_\mu]^* \in H_c^{n-k}(M, \mathbb{R}) . \]

The construction of the proof of proposition 7.5 gives a smooth compactly supported form \(j_\ast \Phi_r\) on \(M\), for \(r\) small enough, with

\[ [j_\ast \Phi_r] = [f, S_\mu]^* \in H_c^{n-k}(M, \mathbb{R}) . \]

For \(M\) non-oriented, let \(\sigma\) be the local system defining the orientation of \(M\). Let \((f, S_\mu)\) be an immersed oriented measured solenoid. Then both \([f, S_\mu]^*\) and \([j_\ast \Phi_r]\) are classes which correspond under the isomorphism

\[ H_k(M, \mathbb{R}) \cong H_c^{n-k}(M, \sigma) . \]

The same proof shows that they are equal.

**Theorem 7.6. (Self-intersection of embedded solenoids)** Let \(M\) be a compact, oriented, smooth manifold. Let \((f, S_\mu)\) be an embedded oriented solenoid, such that the transversal measures \((\mu_T)\) have no atoms. Then we have

\[ [f, S_\mu]^* \cup [f, S_\mu]^* = 0 \]

in \(H^{2(n-k)}(M, \mathbb{R})\).

**Proof.** If \(n - k > k\) then \(2(n-k) > n\), therefore the self-intersection is 0 by degree reasons. So we may assume \(n - k \leq k\).

Let \(\beta\) be any closed \((n - 2(n-k))\)-form on \(M\). We must prove that

\[ \langle [f, S_\mu]^* \cup [f, S_\mu]^* \cup [\beta], [M] \rangle = 0 , \]

where \([M]\) is the fundamental class of \(M\). By proposition 7.5

\[ \langle [f, S_\mu]^* \cup [f, S_\mu]^* \cup [\beta], [M] \rangle = \langle [f, S_\mu], j_\ast \Phi_\epsilon \wedge \beta \rangle , \]

for \(\epsilon > 0\) small enough.

Consider a covering of \(f(S) \subset M\) by open sets \(U_i \subset M\) and another covering of \(f(S)\) by open sets \(V_i \subset M\) such that the closure of \(V_i\) is contained in \(U_i\). We may assume that the covering is chosen so that \(\{V_i = f^{-1}(V_i)\}\) satisfies the properties needed for computing \(j_\ast \Phi_\epsilon\) locally (the auxiliary Riemannian structure is used). Let \(\{\rho_i\}\) be a partition of unity of \(S\) subordinated to \(\{V_i\}\) and decompose \(\Phi_\epsilon = \sum \Phi_{\epsilon,i}\) with \(\Phi_{\epsilon,i} = \rho_i \Phi_\epsilon\). We take \(\epsilon > 0\) small enough so that \(j(supp \Phi_{\epsilon,i}) \subset U_i\). Then

\[ \langle [f, S_\mu]^* \cup [f, S_\mu]^* \cup [\beta], [M] \rangle = \langle [f, S_\mu], j_\ast \Phi_\epsilon \wedge \beta \rangle = \sum_i \langle [f, S_\mu], j_\ast \Phi_{\epsilon,i} \wedge \beta \rangle . \]
As $f$ is an embedding, we may suppose the open sets $U_i = f^{-1}(\hat{U}_i)$ are flow-boxes of $S$. Therefore

$$\langle [f, S]_\mu, j_* \Phi_{\epsilon, i} \wedge \beta \rangle = \int_{K(U_i)} \left( \int_{L_y} f^*(j_* \Phi_{\epsilon, i} \wedge \beta) \right) d\mu_{K(U_i)}(y).$$

We may compute

$$\int_{L_y} f^*(j_* \Phi_{\epsilon, i} \wedge \beta) = \int_{f(L_y)} \left( \int_{K(V_i)} (j_{i, z})_* \Phi_{\epsilon, i} \wedge \beta d\mu_{K(V_i)}(z) \right)$$

$$= \int_{K(V_i)} \left( \int_{f(L_y)} (j_{i, z})_* \Phi_{\epsilon, i} \wedge \beta \right) d\mu_{K(V_i)}(z).$$

Note that $(j_{i, z})_* \Phi_{\epsilon, i}|_{f(L_y)}$ consists of restricting the form $\Phi_{\epsilon, i}$ to $\pi^{-1}(L_z)$, the normal bundle over the leaf $L_z$, then sending it to $M$ via $j$, and finally restricting to the leaf $f(L_y)$.

Since $f$ is an embedding, we may suppose that in a local chart $f : U_i = D^k \times K(U_i) \to \hat{U}_i \subset M$ is the restriction of a map (that we denote with the same letter) $f : D^k \times B \to \hat{U}_i$, where $B \subset \mathbb{R}^l$ is open and $K(U_i) \subset B$, which in suitable coordinates for $M$ is written as $f(x, y) = (x, y, 0)$. The map $j$ extends to a map from the normal bundle to the horizontal foliation of $D^k \times B$, as $j : D^n_{\epsilon} \times D^k \times B \to M$,

$$j(v, x, z) = (x_1, \ldots, x_k, z_1 + v_1, \ldots, z_l + v_l, v_{l+1}, \ldots, v_{n-k}) + O(|v|^2).$$

Using the formula of $\Phi_{\epsilon}$ given in [4], we have

$$(j_{i, z})_* \Phi_{\epsilon}(x, y) = \sum_{|I|+|J|=n-k} \left( \frac{\epsilon}{r_0} \right)^{|I|-n} (g_{IJ} \circ T)(x, y - z) dx_I \wedge dy_J + O(|y - z|).$$

We restrict to $L_y$, and multiply by $\beta$, to get

$$((j_{i, z})_* \Phi_{\epsilon, i} \wedge \beta)|_{L_y} = \sum_{|I|=n-k} (\rho_l \cdot (g_{I0} \circ T))(x, y - z) dx_I \wedge \beta + O(|y - z|),$$

which is bounded by a universal constant.

Hence

$$|\langle [f, S]_\mu, j_* \Phi_{\epsilon, i} \wedge \beta \rangle| \leq C_0 \mu_{K(U_i)}(K(U_i)) \mu_{K(V_i)}(K(V_i)) \leq C_0 \mu_{K(U_i)}(K(U_i))^2,$$

where $C_0$ is a constant that is valid for any refinement of the covering $\{U_i\}$. So

$$|\langle [f, S]_\mu, j_* \Phi_{\epsilon} \wedge \beta \rangle| \leq C_0 \sum \mu_{K(U_i)}(K(U_i))^2.$$
Observe that \( \mu_K(U_i) \leq C_1 \mu(U_i) \) and that \( \sum_i \mu(U_i) \leq C_2 \) for some positive constants \( C_1 \) and \( C_2 \) independent of the refinements of the covering. Therefore,
\[
| \langle [f, j_* \Phi_\epsilon \wedge \beta], j^* \Phi \rangle | \leq C_0 (\max_i \mu_K(U_i)) C_1 \sum \mu(U_i) \\
\leq C_0 C_1 C_2 \max_i \mu_K(U_i) .
\]

When we refine the covering, if the transversal measures have no atoms, we get that \( \max_i \mu_K(U_i) \to 0 \) and then
\[
\langle [f, j_* \Phi_\epsilon \wedge \beta], j^* \Phi \rangle = 0,
\]
as required. \( \square \)

Note that for a compact solenoid, atoms of transversal measure \( s \) must give compact leaves (contained in the support of the atomic part), since otherwise at the accumulation set of the leaf we would have a transversal \( T \) with \( \mu_T \) not locally finite. In particular if \( S \) is a minimal solenoid which is not a \( k \)-manifold, then all transversal measures have no atoms. Therefore, the existence of transversal measures with atomic part is equivalent to the existence of compact leaves. This observation gives the following corollary.

**Corollary 7.7.** Let \( M \) be a compact, oriented, smooth manifold. Let \( (f, S) \) be an embedded oriented solenoid, such that \( S \) has no compact leaves. Then for any transversal measure \( \mu \), we have
\[
[f, S] \star \cup [f, S] \star = 0
\]
in \( H^{2(n-k)}(M, \mathbb{R}) \).

We conclude this section observing that if we want to represent a homology class \( a \in H_k(M, \mathbb{R}) \) by an immersed solenoid in an \( n \)-dimensional manifold \( M \) and \( a \cup a \neq 0 \), then the solenoid cannot be embedded. Note that when \( n - k \) is odd, there is no obstruction. We shall see later that if \( n - k \) is odd then we can always obtain a transversally immersed solenoid representing \( a \), and that if \( n - k \) is even then we can obtain an immersed solenoid.

### 8. Schwartzman clusters and asymptotic cycles

Let \( M \) be a compact \( C^\infty \) Riemannian manifold. Observe that since \( H_1(M, \mathbb{R}) \) is a finite dimensional real vector space, it comes equipped with a unique topological vector space structure.

The map \( \gamma \mapsto [\gamma] \) that associates to each loop its homology class in \( H_1(M, \mathbb{Z}) \subset H_1(M, \mathbb{R}) \) is continuous when the space of loops is endowed with the Hausdorff topology. Therefore, by compactness, oriented rectifiable loops in \( M \) of uniformly bounded length define a bounded set in \( H_1(M, \mathbb{R}) \).
We have a more precise quantitative version of this result.

**Lemma 8.1.** Let \((\gamma_n)\) be a sequence of oriented rectifiable loops in \(M\), and \((t_n)\) be a sequence with \(t_n > 0\) and \(t_n \to +\infty\). If

\[
\lim_{n \to +\infty} \frac{l(\gamma_n)}{t_n} = 0,
\]

then in \(H_1(M, \mathbb{R})\) we have

\[
\lim_{n \to +\infty} \frac{[\gamma_n]}{t_n} = 0.
\]

**Proof.** Via the map

\[
\omega \mapsto \int_\gamma \omega,
\]

each loop \(\gamma\) defines a linear map \(L_\gamma\) on \(H^1(M, \mathbb{R})\) that only depends on the homology class of \(\gamma\). We can extend this map to \(\mathbb{R} \otimes H^1(M, \mathbb{Z})\) by

\[
c \otimes \gamma \mapsto c \cdot L_\gamma.
\]

We have the isomorphism

\[
H^1(M, \mathbb{R}) = \mathbb{R} \otimes H^1(M, \mathbb{Z}) \cong (H^1(M, \mathbb{R}))^*.
\]

The Riemannian metric gives a \(C^0\)-norm on forms. We consider the norm in \(H^1(M, \mathbb{R})\) given as

\[
||[\omega]||_{C^0} = \min_{\omega \in [\omega]} ||\omega||,
\]

and the associated operator norm in \(H^1(M, \mathbb{R}) \cong (H^1(M, \mathbb{R}))^*\).

We have

\[
|L_\gamma([\omega])| = \left| \int_\gamma \omega \right| \leq l(\gamma) ||\omega||_{C^0} \leq l(\gamma) ||[\omega]||_{C^0},
\]

so

\[
||L_\gamma|| \leq l(\gamma).
\]

Hence \(l(\gamma_n)/t_n \to 0\) implies \(L_{\gamma_n}/t_n \to 0\) which is equivalent to \([\gamma_n]/t_n \to 0\). \(\square\)

**Definition 8.2.** (Schwartzman asymptotic 1-cycles) Let \(c\) be a parametrized continuous curve \(c: \mathbb{R} \to M\) defining an immersion of \(\mathbb{R}\). For \(s, t \in \mathbb{R}, s < t\), we choose a rectifiable oriented curve \(\gamma_{s,t}\) joining \(c(s)\) to \(c(t)\) such that

\[
\lim_{t \to +\infty, s \to -\infty} \frac{l(\gamma_{s,t})}{t - s} = 0.
\]
The parametrized curve $c$ is a Schwartzman asymptotic 1-cycle if the juxtaposition of $c|_{[s,t]}$ and $\gamma_{s,t}$, denoted $c_{s,t}$ (which is a 1-cycle), defines a homology class $[c_{s,t}] \in H_1(M, \mathbb{Z})$ such that the limit

$$\lim_{t \to +\infty} \lim_{s \to -\infty} \frac{[c_{s,t}]}{t - s} \in H_1(M, \mathbb{R})$$

exists.

We define the Schwartzman asymptotic homology class as

$$[c] := \lim_{t \to +\infty} \lim_{s \to -\infty} \frac{[c_{s,t}]}{t - s}.$$ 

Thanks to lemma 8.1 this definition does not depend on the choice of the closing curves $(\gamma_{s,t})$. If we take another choice $(\gamma'_{s,t})$, then as homology classes,

$$[c_{s,t}] = [c'_{s,t}] + [\gamma'_{s,t} - \gamma_{s,t}],$$

and

$$\frac{l(\gamma'_{s,t} - \gamma_{s,t})}{t - s} = \frac{l(\gamma'_{s,t})}{t - s} + \frac{l(\gamma_{s,t})}{t - s} \to 0,$$

as $t \to \infty$, $s \to -\infty$. By lemma 8.1

$$\lim_{t \to +\infty} \lim_{s \to -\infty} \frac{[\gamma_{s,t} - \gamma'_{s,t}]}{t - s} = 0,$$

thus

$$[c] = \lim_{t \to +\infty} \lim_{s \to -\infty} \frac{[c_{s,t}]}{t - s} = \lim_{t \to +\infty} \lim_{s \to -\infty} \frac{[c'_{s,t}]}{t - s}.$$ 

Note that we do not assume that $c(\mathbb{R})$ is an embedding of $\mathbb{R}$, i.e. $c(\mathbb{R})$ could be a loop. In that case, the Schwartzman asymptotic homology class coincides with a scalar multiple (the scalar depending on the parametrization) of the integer homology class $[c(\mathbb{R})]$. This shows that the Schwartzman homology class is a generalization to the case of immersions $c : \mathbb{R} \to M$. More precisely we have:

**Proposition 8.3.** If $c : \mathbb{R} \to M$ is a loop then it is a Schwartzman asymptotic 1-cycle and the Schwartzman asymptotic homology class is a scalar multiple of the homology class of the loop $[c(\mathbb{R})] \in H_1(M, \mathbb{Z})$.

If $c : \mathbb{R} \to M$ is a rectifiable loop with its arc-length parametrization, and $l(c)$ is the length of the loop $c$, then

$$[c] = \frac{1}{l(c)} [c(\mathbb{R})].$$
Proof. Let \( t_0 > 0 \) be the minimal period of the map \( c : \mathbb{R} \to M \). Then

\[
[c_{s,t}] = \left[ \frac{t - s}{t_0} \right] [c(\mathbb{R})] + O(1).
\]

Then

\[
\lim_{t \to +\infty} \frac{[c_{s,t}]}{t - s} = \frac{1}{t_0} [c(\mathbb{R})].
\]

When \( c : \mathbb{R} \to M \) is the arc-length parametrization of a rectifiable loop, the period \( t_0 \) coincides with the length of the loop. \( \square \)

We will assume also in the definition of Schwartzman asymptotic 1-cycle that we choose \( (\gamma_{s,t}) \) such that \( l(\gamma_{s,t})/(t - s) \to 0 \) uniformly and separately on \( s \) and \( t \) when \( t \to +\infty \) and \( s \to -\infty \). For simplicity we can decide to choose always \( \gamma_{s,t} \) with uniformly bounded length, and even with \( \{\gamma_{s,t}; s < t\} \) contained in a compact subset of the space of continua of \( M \). Then the uniform boundedness will hold for any Riemannian metric and the notions defined will not depend on the Riemannian structure.

**Definition 8.4. (Positive and negative asymptotic cycles)** Under the assumptions of definition 8.2, if the limit

\[
\lim_{t \to +\infty} \frac{[c_{s,t}]}{t - s} \in H_1(M, \mathbb{R})
\]

exists then it does not depend on \( s \), and we say that the parametrized curve \( c \) defines a positive asymptotic cycle. The positive Schwartzman homology class is defined as

\[
[c_+] = \lim_{t \to +\infty} \frac{[c_{s,t}]}{t - s}.
\]

The definition of negative asymptotic cycle and negative Schwartzman homology class is the same but taking \( s \to -\infty \),

\[
[c_-] = \lim_{s \to -\infty} \frac{[c_{s,t}]}{t - s}.
\]

The independence of the limit (6) on \( s \) follows from

\[
\lim_{t \to +\infty} \frac{[c_{s,t}]}{t - s'} = \lim_{t \to +\infty} \frac{[c_{s,s}]}{t - s} + \frac{[c_{s',s}]}{t - s'} + O(1) \cdot \frac{t - s}{t - s'} = \lim_{t \to +\infty} \frac{[c_{s,t}]}{t - s}.
\]

**Proposition 8.5.** A parametrized curve \( c \) is a Schwartzman asymptotic 1-cycle if and only if it is both a positive and a negative asymptotic cycle and

\[
[c_+] = [c_-].
\]

In that case we have

\[
[c] = [c_+] = [c_-].
\]
Proof. If $c$ is a Schwartzman asymptotic 1-cycle, then for $t \to +\infty$ take $s \to -\infty$ very slowly, say satisfying the relation $t = s^2 I(c_{[s,0]})$, which defines $s = s(t) < 0$ uniquely as a function of $t > 0$. Then

\[
[c] = \lim_{s \to -\infty} \frac{[c_{s,t}]}{t-s} = \lim_{t \to +\infty} \frac{[c_{s,0}] + [c_{0,t}] + O(1)}{t-s}
\]

\[
= \lim_{t \to +\infty} \left( \frac{[c_{s,0}] + O(1)}{t} + \frac{[c_{0,t}]}{t} \right) \frac{t}{t-s} = \lim_{t \to +\infty} \frac{[c_{0,t}]}{t},
\]

since $\frac{t}{t-s} \to 1$ because $\frac{s}{t} \to 0$, and $\frac{|c_{s,0}|}{t} \to 0$ by lemma 8.1. So $c$ is a positive asymptotic cycle and $[c] = [c+]$. Analogously, $c$ is a negative asymptotic cycle and $[c] = [c-]$. Conversely, assume that $c$ is a positive and negative asymptotic cycle with $[c+] = [c-]$. For $t$ large we have

\[
\frac{[c_{0,t}]}{t} = [c+] + o(1).
\]

For $-s$ large we have

\[
\frac{[c_{s,0}]}{-s} = [c-] + o(1).
\]

Now

\[
\frac{[c_{s,t}]}{t-s} = \frac{-s}{t-s} \cdot \frac{[c_{s,0}]}{-s} + \frac{t}{t-s} \cdot \frac{[c_{0,t}]}{t} + \frac{O(1)}{t-s} = \frac{-s}{t-s} [c+] + \frac{t}{t-s} [c-] + o(1).
\]

As $[c+] = [c-]$, we get that this limit exists and equals $[c] = [c+] = [c-]$. □

**Definition 8.6. (Schwartzman clusters)** Under the assumptions of definition 8.2, we can consider, regardless of whether (5) exists or not, all possible limits

\[
(7) \lim_{n \to +\infty} \frac{[c_{s_n,t_n}]}{t_n - s_n} \in H_1(M, \mathbb{R}),
\]

with $t_n \to +\infty$ and $s_n \to -\infty$, that is, the derived set of $([c_{s,t}]/(t-s))_{t \to \infty, s \to -\infty}$. The limits (7) are called Schwartzman asymptotic homology classes of $c$, and they form the Schwartzman cluster of $c$

\[
C(c) \subset H_1(M, \mathbb{R}).
\]

A Schwartzman asymptotic homology class \((7)\) is balanced when the two limits

\[
\lim_{n \to +\infty} \frac{[c_{0,t_n}]}{t_n} \in H_1(M, \mathbb{R}),
\]

and

\[
\lim_{n \to +\infty} \frac{[c_{s_n,0}]}{-s_n} \in H_1(M, \mathbb{R}),
\]
do exist in $H_1(M, \mathbb{R})$. We denote by $\mathcal{C}_b(c) \subset \mathcal{C}(c) \subset H_1(M, \mathbb{R})$ the set of those balanced Schwartzman asymptotic homology classes. The set $\mathcal{C}_b(c)$ is named the balanced Schwartzman cluster.

We define also the positive and negative Schwartzman clusters, $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$, by taking only limits $t_n \to +\infty$ and $s_n \to -\infty$ respectively.

**Proposition 8.7.** The Schwartzman clusters $\mathcal{C}(c)$, $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$ are closed subsets of $H_1(M, \mathbb{R})$.

If $\{[c_{s,t}]/(t - s) ; s < t\}$ is bounded in $H_1(M, \mathbb{R})$, then the Schwartzman clusters $\mathcal{C}(c)$, $\mathcal{C}_+(c)$ and $\mathcal{C}_-(c)$ are non-empty, compact and connected subsets of $H_1(M, \mathbb{R})$.

**Proof.** The Schwartzman cluster $\mathcal{C}(c)$ is the derived set of $([c_{s,t}]/(t - s))_{t \to \infty, s \to -\infty}$, in $H_1(M, \mathbb{R})$, hence closed.

Under the boundedness assumption, non-emptiness and compactness follow. Also the oscillation of $([c_{s,t}])_{s,t}$ is bounded by the size of $[\gamma_{s,t}]$. Therefore the magnitude of the oscillation of $([c_{s,t}]/(t - s))_{s,t}$ tends to 0 as $t \to \infty, s \to -\infty$. This forces the derived set to be connected under the boundedness assumption, since it is $\epsilon$-connected for each $\epsilon > 0$. (A compact metric space is $\epsilon$-connected for all $\epsilon > 0$ if and only if it is connected.)

Also $\mathcal{C}_+(c)$, resp. $\mathcal{C}_-(c)$, is closed because it is the derived set of $([c_{0,t}]/t)_{t \to \infty}$, resp. $([c_{s,0}]/(-s))_{s \to -\infty}$, in $H_1(M, \mathbb{R})$. Non-emptiness, compactness and connectedness under the boundedness assumption follow for the cluster sets $\mathcal{C}_\pm(c)$ in the same way as for $\mathcal{C}(c)$. $\square$

Note that all these cluster sets may be empty if the parametrization is too fast.

The balanced Schwartzman cluster $\mathcal{C}_b(c)$ does not need to be closed, as shown in the following counter-example.

**Counter-example 8.8.** We consider the torus $M = \mathbb{T}^2$. We identify $H_1(M, \mathbb{R}) \cong \mathbb{R}^2$, with $H_1(M, \mathbb{Z})$ corresponding to the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Consider a line $l$ in $H_1(M, \mathbb{R}^2)$ of irrational slope passing through the origin, $y = \sqrt{2} x$ for example. We can find a sequence of pairs of points $(a_n, b_n) \in \mathbb{Z}^2 \times \mathbb{Z}^2$ in the open lower half plane $H_l$ determined by the line $l$, such that the sequence of segments $[a_n, b_n]$ do converge to the line $l$, and the middle point $(a_n + b_n)/2 \to 0$ (this is an easy exercise in diophantine approximation). We assume that the first coordinate of $b_n$ tends to $+\infty$, and the first coordinate of $a_n$ tends to $-\infty$. Now we can construct a parametrized curve $c$ on $\mathbb{T}^2$ such that for all $n \geq 1$ there are an infinite number of times $t_{n,i} \to +\infty$ with
Thus in homology the curve \( c \) oscillates wildly. We can adjust the velocity of the parametrization so that
\[
-\frac{\sum_{i} \sum \frac{c_{s_{n_{i}},0}}{t_{n_{i}} - s_{n_{i}}}}{\sum_{i} \sum \frac{a_{n} - b_{n}}{t_{n_{i}} - s_{n_{i}}} + O(1)} = \frac{a_{n} + b_{n}}{2},
\]
when \( i \to +\infty \), and the two ends balance each other. We have great freedom in constructing \( c \), so that we may arrange to have always \( [c_{s,t}] \subset H_{i} \). Then we get that 0 \( \in C(c) \) and all \( (a_{n} + b_{n})/2 \in C_{0}(c) \) but 0 \( \notin C_{1}(c) \).

We have that \( c \) is a Schwartzman asymptotic 1-cycle (resp. positive, negative) if and only if \( C(c) \) (resp. \( C_{+}(c), C_{-}(c) \)) is reduced to one point. In that case the Schwartzman asymptotic 1-cycle is balanced. The next result generalizes proposition 8.5. We need first a definition.

**Definition 8.9.** Let \( A, B \subset V \) be subsets of a real vector space \( V \). For \( a, b \in V \) the segment \([a, b] \subset V \) is the convex hull of \( \{a, b\} \) in \( V \). The additive hull of \( A \) and \( B \) is
\[
A \hat{+} B = \bigcup_{a \in A, b \in B} [a, b].
\]

**Proposition 8.10.** The Schwartzman balanced cluster \( C_{0}(c) \) is contained in the additive hull of \( C_{+}(c) \) and \( C_{-}(c) \)
\[
C_{0}(c) \subset C_{+}(c) \hat{+} C_{-}(c).
\]
Moreover, for each \( a \in C_{+}(c) \) and \( b \in C_{-}(c) \), we have
\[
C_{0}(c) \cap [a, b] \neq \emptyset.
\]

**Proof.** Let \( x \in C_{0}(c) \),
\[
x = \lim_{n \to +\infty} \frac{[c_{s_{n},t_{n}}]}{t_{n} - s_{n}}.
\]
We write
\[
\frac{[c_{s_{n},t_{n}}]}{t_{n} - s_{n}} = \frac{[c_{s_{n},0}]}{-s_{n}} \cdot \frac{-s_{n}}{t_{n} - s_{n}} + \frac{[c_{0,t_{n}}]}{t_{n}} \cdot \frac{t_{n}}{t_{n} - s_{n}} + O(1),
\]
and the first statement follows.

For the second, consider
\[
a = \lim_{n \to +\infty} \frac{[c_{0,t_{n}}]}{t_{n}} \in C_{+}(c),
\]
and
\[
b = \lim_{n \to +\infty} \frac{[c_{s_{n},0}]}{-s_{n}} \in C_{-}(c).
\]
Then taking any accumulation point \( \tau \in [0, 1] \) of the sequence \((t_n/(t_n - s_n))_n \subset [0, 1]\) and taking subsequences in the above formulas, we get a balanced Schwartzman homology class

\[
c = \tau a + (1 - \tau) b \in \mathcal{C}_b(c).
\]

**Corollary 8.11.** If \( \mathcal{C}_+(c) \) and \( \mathcal{C}_-(c) \) are non-empty, then \( \mathcal{C}_b(c) \) is non-empty, and therefore \( \mathcal{C}(c) \) is also non-empty.

Note that we can have \( \mathcal{C}_+(c) = \mathcal{C}_-(c) = \emptyset \) (then \( \mathcal{C}_b(c) = \emptyset \)) but \( \mathcal{C}(c) \neq \emptyset \) (modify appropriately counter-example \[8.8\]).

There is one situation where we can assert that the balanced Schwartzman cluster set is closed.

**Proposition 8.12.** If \( B = \{ [c_s,t]/(t - s); s < t \} \subset H_1(M, \mathbb{R}) \) is a bounded set, then \( \mathcal{C}(c), \mathcal{C}_+(c), \mathcal{C}_-(c) \) and \( \mathcal{C}_b(c) \) are all compact sets. More precisely, they are all contained in the convex hull of \( \overline{B} \).

**Proof.** Obviously \( \mathcal{C}(c) \), \( \mathcal{C}_+(c) \) and \( \mathcal{C}_-(c) \) are bounded as cluster sets of bounded sets, hence compact by proposition \[8.7\].

In order to prove that \( \mathcal{C}_b(c) \) is bounded, we observe that the additive hull of bounded sets is bounded, therefore boundedness follows from proposition \[8.10\]. We show that \( \mathcal{C}_b(c) \) is closed. Since \( \mathcal{C}_b(c) \subset \mathcal{C}(c) \) and \( \mathcal{C}(c) \) is closed, any accumulation point \( x \) of \( \mathcal{C}_b(c) \) is in \( \mathcal{C}(c) \). Let

\[
x = \lim_{n \to +\infty} \frac{[c_{s_n,t_n}]}{t_n - s_n},
\]

and write as before

\[
\frac{[c_{s_n,t_n}]}{t_n - s_n} = \frac{[c_{s_n,0}]}{t_n - s_n} \cdot \frac{-s_n}{t_n - s_n} + \frac{[c_{0,t_n}]}{t_n} \cdot \frac{t_n}{t_n - s_n} + o(1).
\]

Note that \( ([c_{s_n,0}]/(-s_n))_n \) and \( ([c_{0,t_n}]/t_n)_n \) stay bounded. Therefore we can extract converging subsequences and also for the sequence \((t_n/(t_n - s_n))_n \subset [0, 1]\). The limit along these subsequences \( t_{n_k} \to +\infty \) and \( s_{n_k} \to -\infty \) give the same Schwartzman homology class \( x \) which turns out to be balanced.

The final statement follows from the above proofs. \( \square \)

The situation described in proposition \[8.12\] is indeed quite natural. It arises each time that \( M \) is a Riemannian manifold and \( c \) is an arc-length parametrization of a rectifiable curve. In the following proposition we make use of the natural norm \( \| \cdot \| \) in the homology of a Riemannian manifold defined in \textbf{Appendix A}.
Proposition 8.13. Let $M$ be a Riemannian manifold and denote by $\| \cdot \|$ the norm in homology. If $c$ is a rectifiable curve parametrized by arc-length then the cluster sets $C(c)$, $C_+(c)$, $C_-(c)$ and $C_b(c)$ are compact subsets of $\overline{B}(0,1)$, the closed ball of radius 1 for the norm in homology.

So $C(c)$ and $C_\pm(c)$ are non-empty, compact and connected, and $C_b(c)$ is non-empty and compact.

Proof. Observe that we have

$$l(c_{s,t}) = l(c|_{s,t}) + l(\gamma_{s,t}) = t - s + l(\gamma_{s,t}).$$

Thus

$$l([c_{s,t}]) \leq t - s + l(\gamma_{s,t}).$$

By theorem A.4

$$\| [c_{s,t}] \| \leq t - s + l(\gamma_{s,t}),$$

and

$$\left\| \frac{[c_{s,t}]}{t - s} \right\| \leq 1 + \frac{l(\gamma_{s,t})}{t - s}.$$ 

Since $\frac{l(\gamma_{s,t})}{t - s} \to 0$ uniformly, we get that $B = \{ [c_{s,t}] / (t - s); s < t \} \subset H_1(M, \mathbb{R})$ is a bounded set.

By proposition 8.7, $C(c)$ and $C_\pm(c)$ are non-empty, compact and connected. By corollary 8.11 $C_b(c)$ is non-empty and by proposition 8.12 it is compact. \qed

Obviously the previous notions depend heavily on the parametrization. For a non-parametrized curve we can also define Schwartzman cluster sets.

Definition 8.14. For a non-parametrized oriented curve $c \subset M$, we define the Schwartzman cluster $C(c)$ as the union of the Schwartzman clusters for all orientation preserving parametrizations of $c$. We define the positive $C_+(c)$, resp. negative $C_-(c)$, Schwartzman cluster set as the union of all positive, resp. negative, Schwartzman cluster sets for all orientation preserving parametrizations.

Proposition 8.15. For an oriented curve $c \subset M$ the Schwartzman clusters $C(c)$, $C_+(c)$ and $C_-(c)$ are non-empty closed cones of $H_1(M, \mathbb{R})$. These cones are degenerate (i.e. reduced to $\{0\}$) if and only if $\{ [c_{s,t}]; s < t \}$ is a bounded subset of $H_1(M, \mathbb{Z})$.

Proof. We can choose the closing curves $\gamma_{s,t}$ only depending on $c(s)$ and $c(t)$ and not on the parameter values $s$ and $t$, nor on the parametrization. Then the integer homology class $[c_{s,t}]$ only depends on the points $c(s)$ and $c(t)$ and not on the parametrization. Therefore, we can adjust the speed of the parametrization so that $[c_{s,t}] / (t - s)$ remains in a ball centered at 0. This shows that $C(c)$ is not empty. Adjusting the speed of the parametrization we equally get that it contains elements that are not 0, provided that the set $\{ [c_{s,t}]; s < t \}$ is not bounded in $H_1(M, \mathbb{Z})$. Certainly, if $\{ [c_{s,t}]; s < t \}$ is
bounded, all the cluster sets are reduced to \{0\}. Observe also that if \( a \in \mathcal{C}(c) \) then any multiple \( \lambda a, \lambda > 0 \), belongs to \( \mathcal{C}(c) \), by considering the new parametrization with velocity multiplied by \( \lambda \). So \( \mathcal{C}(c) \) is a cone in \( H_1(M, \mathbb{R}) \).

Now we prove that \( \mathcal{C}(c) \) is closed. Let \( a_n \in \mathcal{C}(c) \) with \( a_n \to a \in H_1(M, \mathbb{R}) \). For each \( n \) we can choose a parametrization of \( c \), say \( c^{(n)} = \tilde{c} \circ \psi_n \) (here \( \tilde{c} \) is a fixed parametrization and \( \psi_n \) is an orientation preserving homeomorphism of \( \mathbb{R} \)), and parameters \( s_n \) and \( t_n \) such that \( \|c^{(n)}_{s_n,t_n} - a\| \leq 1/n \) (considering any fixed norm in \( H_1(M, \mathbb{R}) \)). For each \( n \) we can choose \( t_n \) as large as we like, and \( s_n \) negative as we like. Choose them inductively such that \( (t_n) \) and \( (\psi_n(t_n)) \) are both increasing sequences converging to \( +\infty \), and \( (s_n) \) and \( (\psi_n(s_n)) \) are both decreasing sequences converging to \( -\infty \). Construct a homeomorphism \( \psi \) of \( \mathbb{R} \) with \( \psi(t_n) = \psi_n(t_n) \) and \( \psi(s_n) = \psi_n(s_n) \).

It is clear that \( a \) is obtained as Schwartzman limit for the parametrization \( \tilde{c} \circ \psi \) at parameters \( s_n, t_n \).

The proofs for \( \mathcal{C}_+(c) \) and \( \mathcal{C}_-(c) \) are similar. \( \square \)

**Remark 8.16.** The image of these cluster sets in the projective space \( \mathbb{P}H_1(M, \mathbb{R}) \) is not necessarily connected: On the torus \( M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2 \), choose a curve in \( \mathbb{R}^2 \) that oscillates between the half \( y \)-axis \( \{ y > 0 \} \) and the half \( x \)-axis \( \{ x > 0 \} \), remaining in a small neighborhood of these axes and being unbounded for \( t \to +\infty \), and being bounded when \( s \to -\infty \). Then its Schwartzman cluster consists is two lines through \( 0 \) in \( H_1(\mathbb{T}^2, \mathbb{R}) \cong \mathbb{R}^2 \), and its projection in the projective space consists of two distinct points.

**Remark 8.17.** Let \( c \) be a parametrized Schwartzman asymptotic 1-cycle, and consider the unparametrized oriented curve defined by \( c \), denoted by \( \tilde{c} \). Assume that the asymptotic Schwartzman homology class is \( a = [c] \neq 0 \). Then

\[
\mathcal{C}_\pm(\tilde{c}) = \mathcal{C}(\tilde{c}) = \mathbb{R}_{\geq 0} \cdot a,
\]

as a subset of \( H_1(M, \mathbb{R}) \). This follows since any parametrization of \( \tilde{c} \) is of the form \( c' = c \circ \psi \), where \( \psi: \mathbb{R} \to \mathbb{R} \) is a positively oriented homeomorphism of \( \mathbb{R} \). Then

\[
\frac{c'_{s,t}}{t-s} = \frac{c_{\psi(s),\psi(t)}}{\psi(t)-\psi(s)} \cdot \frac{\psi(t)-\psi(s)}{t-s}.
\]

The first term in the right hand side tends to \( a \) when \( t \to +\infty, s \to -\infty \). If the left hand side is to converge, then the second term in the right hand side stays bounded. After extracting a subsequence, it converges to some \( \lambda a \).

We define now the notion of asymptotically homotopic curves.

**Definition 8.18. (Asymptotic homotopy)** Let \( c_0, c_1: \mathbb{R} \to M \) be two parametrized curves. They are asymptotically homotopic if there exists a continuous family \( c_u \),
$u \in [0, 1]$, interpolating between $c_0$ and $c_1$, such that
\[ c : \mathbb{R} \times [0, 1] \to M, \; c(t, u) = c_u(t), \]
satisfies that $\delta_t(u) = c(t, u)$, $u \in [0, 1]$ is rectifiable with
\[ l(\delta_t) = o(|t|). \]

Two oriented curves are asymptotically homotopic if they have orientation preserving parametrizations that are asymptotically homotopic.

**Proposition 8.19.** If $c_0$ and $c_1$ are asymptotically homotopic parametrized curves then their cluster sets coincide:
\[ C_{\pm}(c_0) = C_{\pm}(c_1), \]
\[ C_b(c_0) = C_b(c_1), \]
\[ C(c_0) = C(c_1). \]

If $c_0$ and $c_1$ are asymptotically homotopic oriented curves then their clusters sets coincide:
\[ C_{\pm}(c_0) = C_{\pm}(c_1), \]
\[ C(c_0) = C(c_1). \]

**Proof.** For parametrized curves we have
\[ [c_{0,s,t}] = [c_{1,s,t}] + [\delta_s - \gamma_{1,s,t} - \delta_t + \gamma_{0,s,t}]. \]
The length of the displacement by the homotopy is bounded by (9), so
\[ l(\delta_s - \gamma_{1,s,t} - \delta_t + \gamma_{0,s,t}) = l(\gamma_{1,s,t}) + l(\gamma_{0,s,t}) + o(|t| + |s|), \]
thus
\[ \frac{[c_{0,s,t}]}{t - s} = \frac{[c_{1,s,t}]}{t - s} + o(1). \]

For non-parametrized curves, the homotopy between two particular parametrizations yields a one-to-one correspondence between points in the curves
\[ c_0(t) \mapsto c_1(t). \]

Using this correspondence, we have a correspondence of pairs of points $(a, b) = (c_0(s), c_0(t))$ with pairs of points $(a', b') = (c_1(s), c_1(t))$. Thus if the sequence of pairs of points $(a_n, b_n)$ gives a cluster value for $c_0$, then the corresponding sequence $(a'_n, b'_n)$ gives a proportional cluster value, since (with obvious notation)
\[ [c_{0,a_n,b_n}] = [c_{1,a'_n,b'_n}] + O(1). \]
So we can always normalize the speed of the parametrization of $c_1$ in order to assure that the limit value is the same. This proves that the clusters sets coincide. □
9. Calibrating functions

Let $M$ be a $C^\infty$ smooth compact manifold. We define now the notion of calibrating function.

Let $\pi : \tilde{M} \to M$ be the universal cover of $M$ and let $\Gamma$ be the group of deck transformations of the cover.

Fix a point $\tilde{x}_0 \in \tilde{M}$ and $x_0 = \pi(\tilde{x}_0)$. There is a faithful and transitive action of $\Gamma$ in the fiber $\pi^{-1}(x_0)$ induced by the action of $\Gamma$ in $\tilde{M}$, and we have a group isomorphism $\Gamma \cong \pi_1(M, x_0)$. Thus from the group homomorphism

$$\pi_1(M, x_0) \to H_1(M, \mathbb{Z}) ,$$

we get a group homomorphism

$$\rho : \Gamma \to H_1(M, \mathbb{Z}) .$$

**Definition 9.1. (Calibrating function)** A map $\Phi : \tilde{M} \to H_1(M, \mathbb{R})$ is a calibrating function if the diagram

$$\begin{array}{ccc}
\Gamma & \cong & \pi_1(M, x_0) \\
\rho \downarrow & & \downarrow \Phi \\
H_1(M, \mathbb{Z}) & \to & H_1(M, \mathbb{R}) \\
\end{array}$$

is commutative and $\Phi$ is equivariant for the action of $\Gamma$ on $\tilde{M}$, i.e. for any $g \in \Gamma$ and $\tilde{x} \in \tilde{M}$,

$$\Phi(g \cdot \tilde{x}) = \Phi(\tilde{x}) + \rho(g) .$$

If $\tilde{x}_0 \in \tilde{M}$ we say that the calibrating function $\Phi$ is associated to $\tilde{x}_0$ if $\Phi(\tilde{x}_0) = 0$.

**Proposition 9.2.** There are smooth calibrating functions associated to any point $\tilde{x}_0 \in \tilde{M}$.

**Proof.** Fix a smooth non-negative function $\varphi : \tilde{M} \to \mathbb{R}$ with compact support $K = \overline{U}$ with $U = \{ \varphi > 0 \}$ such that $\pi(U) = M$. Moreover, we can request that $U \cap \pi^{-1}(x_0) = \{ \tilde{x}_0 \}$.

For any $g_0 \in \Gamma$, define $\varphi_{g_0}(\tilde{x}) = \varphi(g_0^{-1} \cdot \tilde{x})$. The support of $\varphi_{g_0}$ is $g_0 K$, and $(g_0 K)_{g_0 \in \Gamma}$ is a locally finite covering of $\tilde{M}$, as follows from the compactness of $K$. Set

$$\psi_{g_0}(\tilde{x}) := \frac{\varphi_{g_0}(\tilde{x})}{\sum_{g \in \Gamma} \varphi_g(\tilde{x})} .$$

Then $\psi_{g_0}(\tilde{x}) = \psi_e(g_0^{-1} \cdot \tilde{x})$ and

$$\sum_{g \in \Gamma} \psi_g \equiv 1 .$$
Also $\psi_{g_0}$ has compact support $g_0 K$, and it is a smooth function since the denominator is strictly positive (because $\pi(U) = M$) and it is at each point a finite sum of smooth functions.

We define the map

$$\Phi : \tilde{M} \to H_1(M, \mathbb{R}) ,$$

by

$$\Phi(\tilde{x}) = \sum_{g \in \Gamma} \psi_y(\tilde{x}) \rho(g) .$$

We check that $\Phi$ is a calibrating function:

$$\Phi(g \cdot \tilde{x}) = \sum_{h \in \Gamma} \psi_h(g \cdot \tilde{x}) \rho(h)$$

$$= \sum_{h \in \Gamma} \psi_{g^{-1}h}(\tilde{x}) \left( \rho(g) + \rho(g^{-1}h) \right)$$

$$= \sum_{h' \in \Gamma} \psi_{h'}(\tilde{x}) \rho(g) + \sum_{h' \in \Gamma} \psi_{h'}(\tilde{x}) \rho(h')$$

$$= \rho(g) + \Phi(\tilde{x}) .$$

Notice that by construction $\Phi(\tilde{x}_0) = 0$. □

We note also that choosing a function $\phi$ of rapid decay, we may do a similar construction, as long as $\sum_{g \in \Gamma} \phi_g$ is summable (we may need to add a translation to $\Phi$ in order to ensure $\Phi(\tilde{x}_0) = 0$).

Observe that the calibrating property implies that for a curve $\gamma : [a, b] \to M$, the quantity $\Phi(\tilde{\gamma}(b)) - \Phi(\tilde{\gamma}(a))$ does not depend on the lift $\tilde{\gamma}$ of $\gamma$, because for another choice $\tilde{\gamma}'$, we would have for some $g \in \Gamma$,

$$\tilde{\gamma}'(a) = g \cdot \tilde{\gamma}(a) ,$$

and

$$\tilde{\gamma}'(b) = g \cdot \tilde{\gamma}(b) .$$

Therefore

$$\Phi(\tilde{\gamma}'(b)) - \Phi(\tilde{\gamma}'(a)) = \Phi(g \cdot \tilde{\gamma}(b)) - \Phi(g \cdot \tilde{\gamma}(a)) = \Phi(\tilde{\gamma}(b)) - \Phi(\tilde{\gamma}(a)) .$$

This justifies the next definition.

**Definition 9.3.** Given a calibrating function $\Phi$, for any curve $\gamma : [a, b] \to M$, we define $\Phi(\gamma) := \Phi(\tilde{\gamma}(b)) - \Phi(\tilde{\gamma}(a))$ for any lift $\tilde{\gamma}$ of $\gamma$.

**Proposition 9.4.** For any loop $\gamma \subset M$ we have

$$\Phi(\gamma) = [\gamma] \in H_1(M, \mathbb{Z}) .$$
Proof. Modifying $\gamma$, but without changing its endpoints nor $\Phi(\gamma)$ nor $[\gamma]$, we can assume that $x_0 \in \gamma$. Since $\Gamma \cong \pi_1(M, x_0)$, let $h_0 \in \Gamma$ be the element corresponding to $\gamma$. Then $\gamma$ lifts to a curve joining $\tilde{x}_0$ to $h_0 \cdot \tilde{x}_0$, and
\[
\Phi(\gamma) = \Phi(h_0 \cdot \tilde{x}_0) = \Phi(\tilde{x}_0) = \rho(h_0) = [\gamma] \in H_1(M, \mathbb{Z}).
\]
\qed

Proposition 9.5. We assume that $M$ is endowed with a Riemannian metric and that the calibrating function $\Phi$ is smooth. Then for any rectifiable curve $\gamma$ we have
\[
|\Phi(\gamma)| \leq C \cdot l(\gamma),
\]
where $l(\gamma)$ is the length of $\gamma$, and $C > 0$ is a positive constant depending only on the metric.

Proof. The calibrating function $\Phi$ is a smooth function on $\tilde{M}$ and $\Gamma$-equivariant, hence it is bounded as well as its derivatives. The result follows. \qed

Example 9.6. For $M = \mathbb{T}$, $\tilde{M} = \mathbb{R}$, $H_1(M, \mathbb{Z}) = \mathbb{Z} \subset \mathbb{R} = H_1(M, \mathbb{R})$, $\Gamma = \mathbb{Z}$ and $\rho : \Gamma \to H_1(M, \mathbb{Z})$ is given (with these identifications) by $\rho(n) = n$. We can take $\varphi(x) = |1 - x|$, for $x \in [-1, 1]$, and $\varphi(x) = 0$ elsewhere. Then
\[
\sum_{n=-\infty}^{\infty} \varphi(x - n) = 1,
\]
and
\[
\psi_n(x) = \varphi_n(x) = \varphi(x - n).
\]
Therefore we get the calibrating function
\[
\Phi(x) = \sum_{n=-\infty}^{\infty} \varphi(x - n) n = x.
\]
It is a smooth calibrating function (despite that $\varphi$ is not).

A similar construction works for higher dimensional tori.

Proposition 9.7. Let $c : \mathbb{R} \to M$ be a $C^1$ curve. Consider two sequences $(s_n)$ and $(t_n)$ such that $s_n < t_n$, $s_n \to -\infty$, and $t_n \to +\infty$.

Then the following conditions are equivalent:

1. The limit
\[
[c] = \lim_{n \to +\infty} \frac{[c_{s_n}, t_n]}{t_n - s_n} \in H_1(M, \mathbb{R})
\]
exists.
The limit
\[ [c]_\Phi = \lim_{n \to \infty} \frac{\Phi[c|s_n,t_n]}{t_n - s_n} \in H_1(M, \mathbb{R}) \]
exists.

(3) For any closed 1-form \( \alpha \in \Omega^1(M) \), the limit
\[ [c](\alpha) = \lim_{n \to \infty} \frac{1}{t_n - s_n} \int_{c([s_n,t_n])} \alpha \]
exists.

(4) For any cohomology class \([\alpha] \in H^1(M, \mathbb{R})\), the limit
\[ [c][\alpha] = \lim_{n \to \infty} \frac{1}{t_n - s_n} \int_{c([s_n,t_n])} \alpha \]
extists, and does not depend on the closed 1-form \( \alpha \in \Omega^1(M) \) representing the cohomology class.

(5) For any continuous map \( f : M \to \mathbb{T} \), let \( \tilde{f} \circ c : \mathbb{R} \to \mathbb{R} \) be a lift of \( f \circ c \), the limit
\[ \rho(f) = \lim_{n \to +\infty} \frac{\tilde{f} \circ c(t_n) - \tilde{f} \circ c(s_n)}{t_n - s_n} \]
extists.

(6) For any (two-sided, embedded, transversally oriented) hypersurface \( H \subset M \) such that all intersections \( c(\mathbb{R}) \cap H \) are transverse, the limit
\[ [c] \cdot [H] = \lim_{n \to \infty} \frac{\# \{ u \in [s_n,t_n] | c(u) \in H \}}{t_n - s_n} \]
exists. The notation \# means a signed count of intersection points.

When these conditions hold, we have \([c] = [c]_\Phi\) for any calibrating function \( \Phi \).

If \( \alpha \in \Omega^1(M) \) is a closed form, then \([c](\alpha) = [c][\alpha] = \langle [c], [\alpha] \rangle\). If \( f : M \to \mathbb{T} \) is a continuous map and \( a = f^*[dx] \in H^1(M, \mathbb{Z}) \) is the pull-back of the generator \( [dx] \in H^1(\mathbb{T}, \mathbb{Z}) \), and \( H \) is a hypersurface such that \([H] \) is the Poincaré dual of \( a \), then \( \langle [c], [a] \rangle = \rho(f) = [c] \cdot [H] \).

**Proof.** The equivalence of (1) and (2) follows from the properties of \( \Phi \). Let \( c : \mathbb{R} \to M \) be a curve. Then
\[ \Phi(c|s_n,t_n) = \Phi([c_{s_n,t_n}]) - \Phi(\gamma_{s_n,t_n}) = [c_{s_n,t_n}] + O(l(\gamma_{s_n,t_n})) \).

Dividing by \( t_n - s_n \) and passing to the limit the equivalence of (1) and (2) follows.
We prove that (1) is equivalent to (3). First note that
\[
\left| \int_{\gamma_{s_n, t_n}} \alpha \right| \leq C l(\gamma_{s_n, t_n}) \|\alpha\|_{C^0}. \]

We have when \( t_n - s_n \to +\infty \),
\[
\frac{1}{t_n - s_n} \int_{c([s_n, t_n])} \alpha = \frac{1}{t_n - s_n} \int_{c_{s_n, t_n}} \alpha + O \left( \frac{l(\gamma_{s_n, t_n})}{t_n - s_n} \right) = \frac{[c_{s_n, t_n}](\alpha)}{t_n - s_n} + o(1),
\]
and the equivalence of (1) and (3) results.

The equivalence of (3) and (4) results from the fact that the limit
\[
[c](\alpha) = \lim_{n \to \infty} \frac{1}{t_n - s_n} \int_{c([s_n, t_n])} \alpha
\]
does not depend on the representative of the cohomology class \( a = [\alpha] \). If \( \beta = \alpha + d\phi \), with \( \phi : M \to \mathbb{R} \) smooth, then \([c](\alpha) = [c](\beta)\) since
\[
[c](d\phi) = \lim_{n \to \infty} \frac{1}{t_n - s_n} \int_{c([s_n, t_n])} d\phi = \lim_{n \to \infty} \frac{\phi(c(t_n)) - \phi(c(s_n))}{t_n - s_n} \to 0,
\]
since \( \phi \) is bounded. Also \([c][\alpha] = [c](\alpha)\).

We turn now to (4) implies (5). First note that there is an identification \( H^1(M, \mathbb{Z}) \cong [M, K(\mathbb{Z}, 1)] = [M, \mathbb{T}] \), where any cohomology class \( [\alpha] \in H^1(M, \mathbb{Z}) \) is associated to a (homotopy class of a) map \( f : M \to \mathbb{T} \) such that \([\alpha] = f^*[\mathbb{T}]\), where \([\mathbb{T}] \in H^1(\mathbb{T}, \mathbb{Z})\) is the fundamental class. To prove (5), assume first that \( f \) is smooth. With the identification \( \mathbb{T} = \mathbb{R}/\mathbb{Z} \), the class \( f^*(dx) = df \in \Omega^1(M) \) represents \([\alpha]\). Therefore
\[
\frac{\bar{f} \circ c(t_n) - \bar{f} \circ c(s_n)}{t_n - s_n} = \frac{1}{t_n - s_n} \int_{[s_n, t_n]} d(f \circ c) = \frac{1}{t_n - s_n} \int_{[s_n, t_n]} (df)(c') = \frac{1}{t_n - s_n} \int_{c([s_n, t_n])} df,
\]
and from the existence of the limit in (4) we get the limit in (5) that we identify as
\[
\rho(f) = [c][df].
\]
If \( f \) is only continuous, we approximate it by a smooth function, which does not change the limit in (5).

Conversely, if (5) holds, then any integer cohomology class admits a representative of the form \( \alpha = df \), where \( f : M \to \mathbb{T} \) is a smooth map. Then using (10) we have
\[
\frac{1}{t_n - s_n} \int_{c([s_n, t_n])} \alpha \to \rho(f).
\]
So the limit in (4) exists for \( \alpha = df \). This implies that the limit in (4) exists for any closed \( \alpha \in \Omega^1(M) \), since \( H^1(M, \mathbb{Z}) \) spans \( H^1(M, \mathbb{R}) \).

We check the equivalence of (5) and (6). First, let us see that (6) implies (5). As before, it is enough to prove (5) for a smooth map \( f : M \to \mathbb{T} \). Let \( x_0 \in \mathbb{T} \) be a regular value of \( f \), so that \( H = f^{-1}(x_0) \subset M \) is a smooth (two-sided) hypersurface. Then \([H]\) represents the Poincaré dual of \([df]\) \( \in H^1(M, \mathbb{Z}) \). Choose \( x_0 \) such that it is also a regular value of \( f \circ c \), so all the intersections of \( c(\mathbb{R}) \) with \( H \) are transverse. Now for any \( s < t \),

\[
[c_s,t] : [H] = \#c([s,t]) \cap H + \#\gamma_{s,t} \cap H,
\]

where \# denotes signed count of intersection points (we may assume that all intersections of \( \gamma_{s,t} \) and \( H \) are transverse, by a small perturbation of \( \gamma_{s,t} \); also we do not count the extremes of \( \gamma_{s,t} \) in \( \#\gamma_{s,t} \cap H \) in case that either \( c(s) \in H \) or \( c(t) \in H \).

Now

\[
\#c([s,t]) \cap H = [\tilde{f} \circ c(t)] + [-\tilde{f} \circ c(s)] = \tilde{f} \circ c(t) - \tilde{f} \circ c(s) + O(1),
\]

where \([\cdot] \) denotes the integer part, and \( \#\gamma_{s,t} \cap H \) is bounded by the total variation of \( \tilde{f} \circ \gamma_{s,t} \), which is bounded by the maximum of \( df \) times the total length of \( \gamma_{s,t} \), which is \( o(t - s) \) by assumption. Hence

\[
\lim_{n \to +\infty} \frac{\tilde{f} \circ c(t_n) - \tilde{f} \circ c(s_n)}{t_n - s_n} = \lim_{n \to +\infty} \frac{\#c([s_n,t_n]) \cap H}{t_n - s_n}
\]

exists.

Conversely, if (5) holds, consider a two-sided embedded topological hypersurface \( H \subset M \). Then there is a collar \( [0,1] \times H \) embedded in \( M \) such that \( H \) is identified with \( \{\frac{1}{2}\} \times H \). There exists a continuous map \( f : M \to \mathbb{T} \) such that \( H = f^{-1}(x_0) \) for \( x_0 = \frac{1}{2} \in \mathbb{T} \), constructed by sending \( [0,1] \times H \to [0,1] \to \mathbb{T} \) and collapsing the complement of \( [0,1] \times H \) to 0.

Now if all intersections of \( c(\mathbb{R}) \) and \( H \) are transverse, that means that for any \( t \in \mathbb{R} \) such that \( c(t) \in H \), we have that \( c(t - \epsilon) \) and \( c(t + \epsilon) \) are at opposite sides of the collar, for \( \epsilon > 0 \) small (the sign of the intersection point is given by the direction of the crossing). So \( f(c(s)) \) crosses \( x_0 \) increasingly or decreasingly (according to the sign of the intersection). Hence

\[
\frac{\#\{u \in [s_n,t_n] \mid c(u) \in H\}}{t_n - s_n} = \frac{\tilde{f} \circ c(t_n) - \tilde{f} \circ c(s_n)}{t_n - s_n} + o(1).
\]

The required limit exists. \( \square \)

Remark 9.8. Proposition 9.7 holds if we only assume the curve \( c \) to be rectifiable.
Corollary 9.9. Let $c : \mathbb{R} \to M$ be a $C^1$ curve. The following conditions are equivalent:

1. The curve $c$ is a Schwartzman asymptotic cycle.

2. The limit
$$\lim_{t \to +\infty} \lim_{s \to -\infty} \frac{\Phi(c|_{[s,t]})}{t-s} \in H_1(M, \mathbb{R})$$
de exists.

3. For any closed 1-form $\alpha \in \Omega^1(M)$, the limit
$$\lim_{t \to +\infty} \lim_{s \to -\infty} \frac{1}{t-s} \int_{c([s,t])} \alpha$$
de exists.

4. For any cohomology class $[\alpha] \in H^1(M, \mathbb{R})$, the limit
$$[c][\alpha] = \lim_{t \to +\infty} \lim_{s \to -\infty} \frac{1}{t-s} \int_{c([s,t])} \alpha$$
de exists, and does not depend on the closed 1-form $\alpha \in \Omega^1(M)$ representing the cohomology class.

5. For any continuous map $f : M \to \mathbb{T}$, let $\hat{f} \circ c : \mathbb{R} \to \mathbb{R}$ be a lift of $f \circ c$, we have that the limit
$$\lim_{t \to +\infty} \lim_{s \to -\infty} \frac{\hat{f} \circ c(t) - \hat{f} \circ c(s)}{t-s}$$
de exists.

6. For a (two-sided, embedded, transversally oriented) hypersurface $H \subset M$ such that all intersections $c(\mathbb{R}) \cap H$ are transverse, the limit
$$\lim_{t \to +\infty} \lim_{s \to -\infty} \frac{\#\{u \in [s,t]|c(u) \in H\}}{t-s}$$
de exists.

When $c$ is a Schwartzman asymptotic cycle, we have $[c] = [c]_{\Phi}$ for any calibrating function $\Phi$. If $\alpha \in \Omega^1(M)$ is a closed form then
$$[c](\alpha) = [c][\alpha] = \langle [\alpha], [c] \rangle.$$ 
If $f : M \to \mathbb{T}$ and $a = f^*[dx] \in H^1(M, \mathbb{Z})$, where $[dx] \in H^1(\mathbb{T}, \mathbb{Z})$ is the generator, and $H \subset M$ is a hypersurface such that $[H]$ is the Poincaré dual of $a$, then we have
$$\langle [c], [\alpha] \rangle = \rho(f) = [c] \cdot [H].$$
10. Schwartzman 1-dimensional cycles

We assume that $M$ is a compact $C^\infty$ Riemannian manifold, with Riemannian metric $g$.

**Definition 10.1. (Schwartzman representation of homology classes)** Let $(f, S)$ be an immersion in $M$ of an oriented 1-solenoid $S$. Then $S$ is a Riemannian solenoid with the pull-back metric $f^*g$.

1. If $S$ is endowed with a transversal measure $\mu = (\mu_T) \in M_T(S)$, the immersed solenoid $(f, S_\mu)$ represents an homology class $a \in H_1(M, \mathbb{R})$ if for $\mu_T$-almost all leaves $c : \mathbb{R} \to S$, parametrized positively and by arc-length, we have that $f \circ c$ is a Schwartzman asymptotic 1-cycle with $[f \circ c] = a$.

2. The immersed solenoid $(f, S)$ fully represents an homology class $a \in H_1(M, \mathbb{R})$ if for all leaves $c : \mathbb{R} \to S$, parametrized positively and by arc-length, we have that $f \circ c$ is a Schwartzman asymptotic 1-cycle with $[f \circ c] = a$.

Note that if $(f, S)$ fully represents an homology class $a \in H_1(M, \mathbb{R})$, then for all oriented leaves $c \subset S$, we have that $f \circ c$ is a Schwartzman asymptotic cycle and $C_+(f \circ c) = C_-(f \circ c) = C(f \circ c) = \mathbb{R}_{\geq 0} \cdot a \subset H_1(M, \mathbb{R})$, by remark 8.17.

Observe that contrary to what happens with Ruelle-Sullivan cycles, we can have an immersed solenoid fully representing an homology class without the need of a transversal measure on $S$.

**Definition 10.2. (Cluster of an immersed solenoid)** Let $(f, S)$ be an immersion in $M$ of an oriented 1-solenoid $S$. The homology cluster of $(f, S)$, denoted by $C(f, S) \subset H_1(M, \mathbb{R})$, is defined as the derived set of $\{((f \circ c)_s)/(t-s))_{c,t \to \infty, s \to -\infty}$, taken over all images of orientation preserving parametrizations $c$ of all leaves of $S$, and $t \to +\infty$ and $s \to -\infty$. Analogously, we define the corresponding positive and negative clusters.

The Riemannian cluster of $(f, S)$, denoted by $C^g(f, S)$, is defined in a similar way, using arc-length orientation preserving parametrizations. Analogously, we define the positive, negative and balanced Riemannian clusters.

As in section 8, we can prove with arguments analogous to those of propositions 8.13 and 8.15:

**Proposition 10.3.** The homology clusters $C(f, S)$, $C_{\pm}(f, S)$ are non-empty, closed cones of $H_1(M, \mathbb{R})$. If these cones are non-degenerate, their images in $PH_1(M, \mathbb{R})$ are non-empty and compact sets.

The Riemannian homology clusters $C^g(f, S)$, $C^g_{\pm}(f, S)$ are non-empty, compact and connected subsets of $H_1(M, \mathbb{R})$. 
The following proposition is clear, and gives the relationship with the clusters of
the images by \( f \) of the leaves of \( S \).

**Proposition 10.4.** Let \((f, S)\) be an immersion in \( M \) of an oriented 1-solenoid \( S \). We have
\[
\bigcup_{c \in S} C(f \circ c) \subset C(f, S),
\]
where the union runs over all parametrizations of leaves of \( S \). We also have
\[
\bigcup_{c \in S} C_\pm(f \circ c) \subset C_\pm(f, S),
\]
and
\[
\bigcup_{c \in S} C_b(f \circ c) \subset C_b(f, S).
\]
And similarly for all Riemanniann clusters with \( C_*(f \circ c) \) denoting the Schwartzman clusters for the arc-length parametrization.

We recall that given an immersion \((f, S)\) of an oriented 1-solenoid, \( S \) becomes a
Riemannnian solenoid and theorem 6.8 gives a one-to-one correspondence between the
space of transversal measures (up to scalar normalization) and the space of daval
measures,
\[
\mathcal{M}_T(S) \cong \overline{\mathcal{M}_E(S)}.
\]
Moreover, in the case of 1-solenoids that we consider here, they do satisfy the
controlled growth condition of definition 6.10. Therefore all Schwartzman measures
desintegrate as length on leaves by theorem 6.13.

Giving any transversal measure \( \mu \) we can consider the associated generalized current
\([f, S_\mu] \).

**Definition 10.5.** We define the Ruelle-Sullivan map
\[
\Psi : \mathcal{M}_T(S) \to H_1(M, \mathbb{R})
\]
by
\[
\mu \mapsto \Psi(\mu) = [f, S_\mu].
\]

The Ruelle-Sullivan cluster cone of \((f, S)\) is the image of \( \Psi \)
\[
C_{RS}(f, S) = \Psi(\mathcal{M}_T(S)) = \{[f, S_\mu]; \mu \in \mathcal{M}_T(S)\} \subset H_1(M, \mathbb{R}).
\]

The Ruelle-Sullivan cluster set is
\[
\mathbb{P}C_{RS}(f, S) \cong \{[f, S_\mu]; \mu \in \mathcal{M}_L(S)\} \subset H_1(M, \mathbb{R}),
\]
i.e. using transversal measures which are normalized (using the Riemannnian metric
of \( M \)).
Proposition 10.6. Let $\mathcal{V}_T(S)$ be the set of all signed measures, with finite absolute measure and invariant by holonomy, on the solenoid $S$. The Ruelle-Sullivan map $\Psi$ extended by linearity to $\mathcal{V}_T(S)$ is a linear continuous operator,

$$\Psi : \mathcal{V}_T(S) \to H_1(M, \mathbb{R}).$$

Proof. Coming back to the definition of generalized current, it is clear that $\mu \mapsto \langle f, S_\mu \rangle$ is linear in flow-boxes, therefore globally. It is also continuous because if $\mu_n \to \mu$, then $[f, S_{\mu_n}] \to [f, S_\mu]$ as can be seen in a fixed flow-box covering of $S$. □

Corollary 10.7. The Ruelle-Sullivan cluster $C_{RS}(f, S)$ is a non-empty, convex, compact cone of $H_1(M, \mathbb{R})$. Extremal points of the convex set $C_{RS}(f, S)$ come from the generalized currents of ergodic measures in $\mathcal{M}_E(S)$.

Proof. Since $\mathcal{M}_E(S)$ is non-empty, convex and compact set, its image by the continuous linear map $\Psi$ is also a non-empty, convex and compact set. Any extremal point of $C_{RS}(f, S)$ must have an extremal point of $\mathcal{M}_E(S)$ in its pre-image, and these are the ergodic measures in $\mathcal{M}_E(S)$ (according to the identification of $\mathcal{M}_E(S)$ to $\mathcal{M}_T(S)$ and by proposition 5.11). □

It is natural to investigate the relation between the Schwartzman cluster and the Ruelle-Sullivan cluster.

Theorem 10.8. Let $S$ be a 1-solenoid. For any immersion $f : S \to M$ we have

$$\bigcup_{c \subset S} C(f \circ c) \subset C_{RS}(f, S).$$

Proof. It is enough to prove the theorem for minimal solenoids, since each leaf $c \subset S$ is contained in a minimal solenoid $S_0 \subset S$, and

$$C(f \circ c) \subset C_{RS}(f, S_0) \subset C_{RS}(f, S).$$

The last inclusion holds because if $\mu$ is a transversal measure for $S_0$, then it defines a transversal measure $\mu'$ for $S$, which is clearly invariant by holonomy. Now the generalized currents coincide, $[f, S_{\mu'}] = [f, S_{0,\mu}]$, as can be seen by in a fixed flow-box covering of $S$.

The statement for minimal solenoids follows from theorem 10.9 below. □

Theorem 10.9. Let $S$ be a minimal 1-solenoid. For any immersion $f : S \to M$ we have

$$C(f, S) \subset C_{RS}(f, S).$$

Proof. Consider an element $a \in C(f, S)$ obtained as limit of a sequence $([f \circ c_n]_{s_n, t_n})$, where $c_n$ is an positively oriented parametrized leaf of $S$ and $s_n < t_n$, $s_n \to -\infty$, $t_n \to \infty$. The points $(c_n(t_n))$ must accumulate a point $x \in S$, and taking a subsequence, we can assume they converge to it. Choose a small local transversal $T$ of $S$ at this
point, such that \( f(T) \subset B \) where \( B \subset M \) is a contractible ball in \( M \). By definition \[4.5\) the return map \( R_T : T \to T \) is well defined.

Note that we may assume that \( \bar{T} \subset T' \), where \( T' \) is also a local transversal. By compactness of \( \bar{T} \), the return time for \( R_{T'} : T' \to T' \) of any leaf, measured with the arc-length parametrization, for any \( x \in \bar{T} \), is universally bounded. Therefore we can adjust the sequences \((s_n)\) and \((t_n)\) such that \( c_n(s_n) \in \bar{T} \) and \( c_n(t_n) \in \bar{T} \), by changing each term by an amount \( O(1) \). Now, after further taking a subsequence, we can arrange that \( c_n(s_n), c_n(t_n) \in T \).

Taking again a subsequence if necessary we can assume that we have a Schwartzman limit of the measures \( \mu_n \) which correspond to the arc-length on \( c_n([s_n, t_n]) \) normalized with total mass 1. The limit measure \( \mu \) desintegrates on leaves because of theorem \[6.13\), so it defines a transversal measure \( \mu \).

The transversal measures corresponding to \( \mu_n \) are atomic, supported on \( T \cap c_n([s_n, t_n]) \), assigning the weight \( l([x, R_T(x)]) \) to each point in \( T \cap c_n([s_n, t_n]) \). The transversal measure corresponding to \( \mu \) is its normalized limit. For each 1-cohomology class, we may choose a closed 1-form \( \omega \) representing it and vanishing on \( B \) (this is so because \( H^1(M, B) = H^1(M) \), since \( B \) is contractible). Assume that we have constructed \([(f \circ c_n)_{s_n, t_n}] \) by using \( \gamma_{n,s_n,t_n} \) inside \( B \). So

\[
\langle [f, S_{\mu_n}], \omega \rangle = \int_S f^* \omega \, d\mu_n = \int_{f \circ c_n([s_n, t_n])} \omega = \langle [(f \circ c_n)_{s_n, t_n}], [\omega] \rangle,
\]

thus

\[
\langle [f, S_{\mu}], [\omega] \rangle = \lim_{n \to \infty} \frac{1}{t_n - s_n} \langle [f, S_{\mu_n}], \omega \rangle = \lim_{n \to \infty} \langle \left( f \circ c_n \right)_{s_n, t_n}, [\omega] \rangle = \langle a, [\omega] \rangle.
\]

Thus the generalized current of the limit measure coincides with the Schwartzman limit.

We use the notation \( \partial^* C \) for the extremal points of a compact convex set \( C \). For the converse result, we have:

**Theorem 10.10.** Let \( S \) be a minimal solenoid and an immersion \( f : S \to M \). We have

\[
\partial^* \mathcal{C}_{RS}(f, S) \subset \bigcup_{c \subset S} \mathcal{C}(f \circ c) \subset \mathcal{C}(f, S).
\]

**Proof.** We have seen that the points in \( \partial^* \mathcal{C}_{RS}(f, S) \) come from ergodic measures in \( \mathcal{M}_\ell(S) \) by the Ruelle-Sullivan map. Therefore it is enough to prove the following theorem that shows that the Schwartzman cluster of almost all leaves is reduced to the generalized current for an ergodic 1-solenoid.

**Theorem 10.11.** Let \( S \) be a minimal 1-solenoid endowed with an ergodic measure \( \mu \in \mathcal{M}_\ell(S) \). Consider an immersion \( f : S \to M \). Then for \( \mu \)-almost all leaves \( c \subset S \)
we have that $f \circ c$ is a Schwartzman asymptotic 1-cycle and

$$[f \circ c] = [f, S_\mu] \in H_1(M, \mathbb{R}).$$

Therefore the immersion $(f, S_\mu)$ represents its generalized current.

In particular, this homology class is independent of the metric $g$ on $M$ up to a scalar factor.

**Proof.** The proof is an application of Birkhoff’s ergodic theorem. Choose a small local transversal $T$ such that $f(T) \subset B$, where $B \subset M$ is a small contractible ball. Consider the associated Poincaré first return map $R_T : T \to T$. Denote by $\mu_T$ the transversal measure supported on $T$.

For each $x \in T$ we consider $\varphi_T(x)$ to be the homology class in $M$ of the loop image by $f$ of the leaf $[x, R_T(x)]$ closed by a segment in $B$ joining $x$ with $R_T(x)$. In this way we have defined a measurable map

$$\varphi_T : T \to H_1(M, \mathbb{Z}).$$

Also for $x \in S$, we denote by $l_T(x)$ the length of the leaf joining $x$ with its first impact on $T$ (which is $R_T(x)$ for $x \in T$). We have then an upper semi-continuous map

$$l_T : S \to \mathbb{R}_+.\]$$

Therefore $l_T$ is bounded by compactness of $S$. In particular, $l_T$ is bounded on $T$ and therefore in $L^1(T, \mu_T)$. The boundedness of $l_T$ implies also the boundedness of $\varphi_T$ by lemma 8.1.

Consider $x_0 \in T$ and its return points $x_i = R_T^n(x_0)$. Let $0 < t_1 < t_2 < t_3 < \ldots$ be the times of return for the positive arc-length parametrization. We have

$$t_{i+1} - t_i = l_T(x_i).$$

Therefore

$$t_n = \sum_{i=0}^{n-1} (t_{i+1} - t_i) = \sum_{i=0}^{n-1} l_T \circ R_T^n(x_0),$$

and by Birkhoff’s ergodic theorem

$$\lim_{n \to +\infty} \frac{1}{n} t_n = \int_T l_T(x) \, d\mu_T(x) = \mu(S) = 1.$$

Now observe that, by contracting $B$, we have

$$[f \circ c_{0,t_n}] = [f \circ c_{0,t_1}] + [f \circ c_{t_1,t_2}] + \ldots + [f \circ c_{t_{n-1},t_n}]$$

$$= \varphi_T(x_0) + \varphi_T \circ R_T(x_0) + \ldots + \varphi_T \circ R_T^{n-1}(x_0).$$

We recognize a Birkhoff’s sum and by Birkhoff’s ergodic theorem we get the limit

$$\lim_{n \to +\infty} \frac{1}{n} [f \circ c_{0,t_n}] = \int_T \varphi_T(x) \, d\mu_T(x) \in H_1(M, \mathbb{R}).$$
Finally, putting these results together,
\[
\lim_{n \to +\infty} \frac{1}{t_n} [f \circ c_0, t_n] = \lim_{n \to +\infty} \frac{[f \circ c_0, t_n]}{t_n/n} = \int_T \varphi_T(x) d\mu_T(x) = \int_T \varphi_T(x) d\mu_T(x).
\]

Let us see that this equals the generalized current. Take a closed 1-form $\omega \in \Omega^1(M)$, which we can assume to vanish on $B$. Then
\[
\langle [f, S_\mu], \omega \rangle = \int_T \left( \int_{[x, R_T(x)]} f^* \omega \right) d\mu_T(x) = \int_T \langle \varphi_T(x), \omega \rangle d\mu_T(x),
\]
and so
\[
[f, S_\mu] = \int_T \varphi_T(x) d\mu_T(x).
\]

Observe that so far we have only proved that $\mathcal{C}_g^0(f \circ c) = \{[f, S_\mu]\}$ for almost all leaves $c \subset S$. Considering the reverse orientation, the result follows for the negative clusters, and finally for the whole cluster of almost all leaves.

The last statement follows since $[f, S_\mu]$ only depends on $\mu \in \mathcal{M}_T(S)$, which is independent of the metric up to scalar factor, thanks to the isomorphism of theorem 6.8.

Therefore for a minimal oriented ergodic 1-solenoid, the generalized current coincides with the Schwartzman asymptotic homology class of almost all leaves. It is natural to ask when this holds for all leaves, i.e. when the solenoid fully represents the generalized current. This indeed happens when the solenoid $S$ is uniquely ergodic (recall that unique ergodicity implies that all orbits are dense and therefore minimality).

**Theorem 10.12.** Let $S$ be a uniquely ergodic oriented 1-solenoid, and let $\mathcal{M}_\mathcal{L}(S) = \{\mu\}$. Let $f : S \to M$ be an immersion. Then for each leaf $c \subset S$ we have that $f \circ c$ is a Schwartzman asymptotic cycle with
\[
[f \circ c] = [f, S_\mu] \in H_1(M, \mathbb{R}),
\]
and we have
\[
\mathcal{C}_g^0(f \circ c) = \mathcal{C}_g^0(f, S) = \mathbb{P} \mathcal{C}_{RS}(f, S) = \{[f, S_\mu]\} \subset H_1(M, \mathbb{R}).
\]

Therefore $(f, S)$ fully represents its generalized current $[f, S_\mu]$.

### 11. Schwartzman $k$-dimensional cycles

We study in this section how to extend Schwartzman theory to $k$-dimensional submanifolds of $M$. We assume that $M$ is a compact $C^\infty$ Riemannian manifold.

Given an immersion $c : N \to M$ from an oriented smooth manifold $N$ of dimension $k \geq 1$, it is natural to consider exhaustions $(U_n)$ of $N$ with $U_n \subset N$ being
\(k\)-dimensional compact submanifolds with boundary \(\partial U_n\). We close \(U_n\) with a \(k\)-dimensional oriented manifold \(\Gamma_n\) with boundary \(\partial \Gamma_n = -\partial U_n\) (that is, \(\partial U_n\) with opposite orientation, so that \(N_n = U_n \cup \Gamma_n\) is a \(k\)-dimensional compact oriented manifold without boundary), in such a way that \(c_{U_n}\) extends to a piecewise smooth map \(c_n : N_n \to M\). We may consider the associated homology class \([c_n(N_n)] \in H_k(M, \mathbb{Z})\). By analogy with section 8 we consider

\[
\frac{1}{t_n} [c_n(N_n)] \in H_k(M, \mathbb{R}),
\]

for increasing sequences \((t_n), t_n > 0,\) and \(t_n \to +\infty,\) and look for sufficient conditions for (11) to have limits in \(H_k(M, \mathbb{R})\). Lemma 8.1 extends to higher dimension to show that, as long as we keep control of the \(k\)-volume of \(c_n(\Gamma_n)\), the limit is independent of the closing procedure.

**Lemma 11.1.** Let \((\Gamma_n)\) be a sequence of closed (i.e. compact without boundary) \(k\)-dimensional oriented manifolds with piecewise smooth maps \(c_n : \Gamma_n \to M\), and let \((t_n)\) be a sequence with \(t_n > 0\) and \(t_n \to +\infty\). If

\[
\lim_{n \to +\infty} \frac{\text{Vol}_k(c_n(\Gamma_n))}{t_n} = 0,
\]

then in \(H_k(M, \mathbb{R})\) we have

\[
\lim_{n \to +\infty} \frac{[c_n(\Gamma_n)]}{t_n} = 0.
\]

The proof follows the same lines as the proof of lemma 8.1 We define now \(k\)-dimensional Schwartzman asymptotic cycles.

**Definition 11.2.** (Schwartzman asymptotic \(k\)-cycles and clusters) Let \(c : N \to M\) be an immersion from a \(k\)-dimensional oriented manifold \(N\) into \(M\). For all increasing sequences \((t_n), t_n \to +\infty,\) and exhaustions \((U_n)\) of \(N\) by \(k\)-dimensional compact submanifolds with boundary, we consider all possible Schwartzman limits

\[
\lim_{n \to +\infty} \frac{[c_n(N_n)]}{t_n} \in H_k(M, \mathbb{R}),
\]

where \(N_n = U_n \cup \Gamma_n\) is a closed oriented manifold with

\[
\frac{\text{Vol}_k(c_n(\Gamma_n))}{t_n} \to 0.
\]

Each such limit is called a Schwartzman asymptotic \(k\)-cycle. These limits form the Schwartzman cluster \(\mathcal{C}(c, N) \subset H_k(M, \mathbb{R})\) of \(N\).

Observe that a Schwartzman limit does not depend on the choice of the sequence \((\Gamma_n)\), as long as it satisfies (12). Note that this condition is independent of the particular Riemannian metric chosen for \(M\).

As in dimension 1 we have
Proposition 11.3. The Schwartzman cluster \( C(c, N) \) is a closed cone of \( H_k(M, \mathbb{R}) \).

The Riemannian structure on \( M \) induces a Riemannian structure on \( N \) by pulling back by \( c \). We define the Riemannian exhaustions \( (U_n) \) of \( N \) as exhaustions of the form

\[ U_n = \bar{B}(x_0, R_n), \]

i.e. the \( U_n \) are Riemannian (closed) balls in \( N \) centered at a base point \( x_0 \in N \) and \( R_n \to +\infty \). If the \( R_n \) are generic, then the boundary of \( U_n \) is smooth.

We define the Riemannian Schwartzman cluster of \( N \) as follows. It plays the role of the balanced Riemannian cluster of section 8 for dimension 1.

Definition 11.4. The Riemann-Schwartzman cluster of \( (c, N) \), \( C^g(c, N) \), is the set of all limits, for all Riemannian exhaustions \( (U_n) \),

\[
\lim_{n \to +\infty} \frac{1}{\text{Vol}_k(c_n(N_n))}[c_n(N_n)] \in H_k(M, \mathbb{R}),
\]

such that \( N_n = U_n \cup \Gamma_n \) and

\[
\frac{\text{Vol}_k(c_n(\Gamma_n))}{\text{Vol}_k(c_n(N_n))} \to 0.
\]

All such limits are called Riemann-Schwartzman asymptotic \( k \)-cycles.

Definition 11.5. The immersed manifold \( (c, N) \) represents an homology class \( a \in H_k(M, \mathbb{R}) \) if the Riemann-Schwartzman cluster \( C^g(c, N) \) contains only \( a \),

\[ C^g(c, N) = \{a\}. \]

We denote \([c, N] = a\), and call it the Schwartzman homology class of \((c, N)\).

Now we can define the notion of representation of homology classes by immersed solenoids extending definition 10.1 to higher dimension.

Definition 11.6. (Schwartzman representation of homology classes) Let \((f, S)\) be an immersion in \( M \) of an oriented \( k \)-solenoid \( S \). Then \( S \) is a Riemannian solenoid with the pull-back metric \( f^*g \).

1. If \( S \) is endowed with a transversal measure \( \mu = (\mu_T) \in M_\mathbb{T}(S) \), the immersed solenoid \((f, S_\mu)\) represents an homology class \( a \in H_1(M, \mathbb{R}) \) if \( (\mu_T) \)-almost all leaves \( l \subset S \), we have that \((f, l)\) is a Riemann-Schwartzman asymptotic \( k \)-cycle with \([f, l] = a\).

2. The immersed solenoid \((f, S)\) fully represents an homology class \( a \in H_1(M, \mathbb{R}) \) if for all leaves \( l \subset S \), we have that \((f, l)\) is a Riemann-Schwartzman asymptotic \( k \)-cycle with \([f, l] = a\).
Definition 11.7. (Equivalent exhaustions) Two exhaustions \((U_n)\) and \((\hat{U}_n)\) are equivalent if
\[
\frac{\text{Vol}_k(U_n - \hat{U}_n) + \text{Vol}_k(\hat{U}_n - U_n)}{\text{Vol}_k(U_n)} \to 0.
\]

Note that if two exhaustions \((U_n)\) and \((\hat{U}_n)\) are equivalent, then
\[
\frac{\text{Vol}_k(\hat{U}_n)}{\text{Vol}_k(U_n)} \to 1.
\]
Moreover, if \(N_n = U_n \cup \Gamma_n\) are closings satisfying \([13]\), then we may close \(\hat{U}_n\) as follows: after slightly modifying \(\hat{U}_n\) so that \(U_n\) and \(\hat{U}_n\) have boundaries intersecting transversally, we glue \(F_1 = U_n - \hat{U}_n\) to \(\hat{U}_n\) along \(F_1 \cap \partial \hat{U}_n\), then we glue a copy of \(F_2 = \hat{U}_n - U_n\) (with reversed orientation) to \(\hat{U}_n\) along \(F_2 \cap \partial \hat{U}_n\). The boundary of \(\hat{U}_n \cup F_1 \cup F_2\) is homeomorphic to \(\partial U_n\), so we may glue \(\Gamma_n\) to it, to get \(\hat{N}_n = \hat{U}_n \cup F_1 \cup F_2 \cup \Gamma_n\). Note that
\[
\text{Vol}_k(\hat{N}_n) = \text{Vol}_k(N_n) + 2 \text{Vol}_k(\hat{U}_n - U_n) \approx \text{Vol}_k(N_n).
\]
Define \(\hat{c}_n\) by \(\hat{c}_n|_{F_1} = c|_{(U_n - \hat{U}_n)}\), \(\hat{c}_n|_{F_2} = c|_{(\hat{U}_n - U_n)}\) and \(\hat{c}_n|_{\Gamma_n} = c_n|_{\Gamma_n}\). Then
\[
[c_n(N_n)] = [\hat{c}_n(\hat{N}_n)],
\]
so both exhaustions define the same Schwartzman asymptotic \(k\)-cycles.

Definition 11.8. (Controlled solenoid) Let \(V \subset S\) be an open subset of a solenoid \(S\). We say that \(S\) is controlled by \(V\) if for any Riemann exhaustion \((U_n)\) of any leaf of \(S\) there is an equivalent exhaustion \((\hat{U}_n)\) such that for all \(n\) we have \(\partial \hat{U}_n \subset V\).

Definition 11.9. (Trapping region) An open subset \(W \subset S\) of a solenoid \(S\) is a trapping region if there exists a continuous map \(\pi : S \to \mathbb{T}\) such that

1. For some \(0 < \epsilon_0 < 1/2\), \(W = \pi^{-1}((\epsilon_0, \epsilon_0))\).
2. There is a global transversal \(T \subset \pi^{-1}(\{0\})\).
3. Each connected component of \(\pi^{-1}(\{0\})\) intersects \(T\) in exactly one point.
4. \(0\) is a regular value for \(\pi\), that is, \(\pi\) is smooth in a neighborhood of \(\pi^{-1}(\{0\})\) and it \(d\pi\) is surjective at each point of \(\pi^{-1}(\{0\})\) (the differential \(d\pi\) is understood leaf-wise).
5. For each connected component \(L\) of \(\pi^{-1}(\mathbb{T} - \{0\})\) we have \(\overline{L} \cap T = \{x, y\}\), where \(\{x\} \in \overline{L} \cap T \cap \pi^{-1}((\epsilon_0, 0))\) and \(\{y\} \in \overline{L} \cap T \cap \pi^{-1}((0, \epsilon_0))\). We define \(R_T : T \to T\) by \(R_T(x) = y\).

Let \(C_x\) be the (unique) component of \(\pi^{-1}(\{0\})\) through \(x \in T\). By (4), \(C_x\) is a smooth \((k - 1)\)-dimensional manifold. By (5), there is no holonomy in \(\pi^{-1}((\epsilon_0, \epsilon_0))\),
so \( C_x \) is a compact submanifold. Let \( L_x \) be the connected component of \( \pi^{-1}(\mathbb{T} - \{0\}) \) with \( \overline{T}_x \cap T = \{x, y\} \). This is a compact manifold with boundary \( \partial \overline{T}_x = C_x \cup C_y = C_x \cup C_{R_T(x)}. \)

**Proposition 11.10.** If \( S \) has a trapping region \( W \) with global transversal \( T \), then holonomy group of \( T \) is generated by the map \( R_T \).

**Proof.** If \( \gamma \) is a path with endpoints in \( T \), we may homotop it so that each time it traverses \( \pi^{-1}(\{0\}) \), it does it through \( T \). Then we may split \( \gamma \) into sub-paths such that each path has endpoints in \( T \) and no other points in \( \pi^{-1}(\{0\}) \). Each of this sub-paths therefore lies in some \( \overline{T}_x \) and has holonomy \( R_T, R_T^{-1} \) or the identity. The result follows. \( \Box \)

**Theorem 11.11.** A solenoid \( S \) with a trapping region \( W \) is controlled by \( W \).

**Proof.** Fix a base point \( y_0 \in S \) and a exhaustion \((U_n)\) of the leaf \( l \) through \( y_0 \) of the form \( U_n = \overline{B}(y_0, R_n), R_n \to +\infty \). Consider \( x_0 \in T \) so that \( y_0 \in \overline{T}_{x_0} \). The leaf \( l \) is the infinite union \( l = \bigcup_{n \in \mathbb{Z}} \overline{T}_{R_T^n(x_0)} \).

If \( R_T^n(x_0) = x_0 \) for some \( n \geq 1 \) then \( l \) is a compact manifold. Then for some \( N \), we have \( \hat{U}_N = l \), so the controlled condition of definition \([11.8]\) is satisfied for \( l \).

Assume that \( R_T(x_0) \neq x_0 \). Then \( l \) is a non-compact manifold. For integers \( a < b \), denote

\[
\hat{U}_{a,b} := \bigcup_{k=a}^{b-1} \overline{T}_{R_T^k(x_0)} \, .
\]

This is a manifold with boundary \( \partial \hat{U}_{a,b} = C_{R_T^a(x_0)} \cup C_{R_T^b(x_0)} \).

Given \( U_n \), pick the maximum \( b \geq 1 \) and minimum \( a \leq 0 \) such that \( \hat{U}_{a,b} \subset U_n \), and denote \( \hat{U}_n = \hat{U}_{a,b} \) for such \( a \) and \( b \). Clearly \( \partial \hat{U}_n \subset W \). Let us see that \( (U_n) \) and \( (\hat{U}_n) \) are equivalent exhaustions, i.e. that

\[
\frac{\text{Vol}_k(U_n - \hat{U}_n)}{\text{Vol}_k(U_n)} \to 0 \, .
\]

Let \( b' \geq 1 \) the minimum and \( a' \leq 0 \) the maximum such that \( U_n \subset \hat{U}_{a',b'} \). Let us prove that

\[
\text{Vol}_k(\hat{U}_{a',b'} - \hat{U}_{a,b})
\]

is bounded. This clearly implies the result.
Take $y \in \overline{T}_{R^{b'-1}_{x}(x_0)} \cap U_n$. Then $d(y_0,y) \leq R_n$. By compactness of $T$, there is a lower bound $c_0 > 0$ for the distance from $C_x$ to $C_{R_T(x)}$ in $L_x$, for all $x \in T$. Taking the geodesic path from $y_0$ to $y$, we see that there are points in $y_i \in \overline{T}_{R^{b'-1}_{x}(x_0)}$ with $d(y_0,y_i) \leq R_n - (i-2)c_0$, for $2 \leq i \leq b'$.

As $\overline{T}_{R^{b}_{T}(x_0)}$ is not totally contained in $U_n$, we may take $z \in \overline{T}_{R^{b}_{T}(x_0)} - U_n$, so $d(y_0, z) > R_n$. Both $z$ and $y_{b'-b}$ are on the same leaf $\overline{T}_{R^{b}_{T}(x_0)}$. By compactness of $T$, the diameter for a leaf $\overline{T}_x$ is bounded above by some $c_1 > 0$, for all $x \in T$. So

$$R_n - (b' - b - 2)c_0 \geq d(y_0, y_{b'-b}) \geq d(y_0, z) - d(y_{b'-b}, z) > R_n - c_1,$$

hence

$$b' - b < \frac{c_1}{c_0} + 2.$$

Analogously,

$$a - a' < \frac{c_1}{c_0} + 2.$$

Again by compactness of $T$, the $k$-volumes of $\overline{T}_x$ are uniformly bounded by some $c_2 > 0$, for all $x \in T$. So

$$\text{Vol}_k(\hat{U}_{a',b'} - \hat{U}_{a,b}) \leq (b' - b + a - a')c_2 < 2 \left(\frac{c_1}{c_0} + 2\right)c_2,$$

concluding the proof. \hfill \Box

**Theorem 11.12.** Let $S$ be a minimal solenoid endowed with a transversal ergodic measure $\mu \in \mathcal{M}_C(S)$ and with a trapping region $W \subset S$. Consider an immersion $f : S \rightarrow M$ such that $f(W)$ is contained in a contractible ball in $M$. Then $(f, S_\mu)$ represents its generalized current $[f, S_\mu]$, i.e. for $\mu_T$-almost all leaves $l \subset S$,

$$[f, l] = [f, S_\mu] \in H_k(M, \mathbb{R}).$$

If $S_\mu$ is uniquely ergodic, then $(f, S_\mu)$ fully represents its generalized current.

In particular, this homology class is independent of the metric $g$ on $M$ up to a scalar factor.

**Proof.** We define a map $\varphi_T : T \rightarrow H_k(M, \mathbb{Z})$ as follows: given $x \in T$, consider $f(\overline{T}_x)$. Since $\partial f(\overline{T}_x)$ is contained in a contractible ball $B$ of $M$, we can close $f(L_x)$ locally as $N_x = f(\overline{\Gamma}_x) \cup \Gamma_x$ and define an homology class $\varphi_T(x) = [N_x] \in H_k(M, \mathbb{Z})$. This is independent of the choice of the closing. This map $\varphi_T$ is measurable and bounded in $H_k(M, \mathbb{Z})$ since the $k$-volume of $\Gamma_x$ may be chosen uniformly bounded. Also we can define a map $l_T : T \rightarrow \mathbb{R}_+$ by $l_T(x) = \text{Vol}_k(\overline{T}_x)$. It is also a measurable and bounded map.

We have seen that every Riemann exhaustion $(U_n)$ is equivalent to an exhaustion $(\hat{U}_n)$ with $\partial \hat{U}_n \subset W$. Note also that we can saturate the exhaustion $(\hat{U}_n)$ into
$(\hat{U}_{n,m})_{n \leq 0 \leq m}$, with $\hat{U}_{n,m}$ defined in (15), where $\partial \hat{U}_{n,m} = C_{R_T^k(x_0)} \cup C_{R_T(x_0)}$, and $x_0 \in T$ is a base point. Since $f(W)$ is contained in a contractible ball $B$ of $M$, we can always close $f(\hat{U}_{n,m})$, with a closing inside $B$, to get $N_{n,m}$ defining an homology class $[N_{n,m}] \in H_k(M, \mathbb{Z})$. Moreover we have

$$[N_{n,m}] = \sum_{i=n}^{m-1} \varphi_T(R_T^i(x_0)).$$

Thus by ergodicity of $\mu$ and Birkhoff’s ergodic theorem, we have that for $\mu_T$-almost all $x_0 \in T$,

$$\frac{1}{m-n} [N_{n,m}] \to \int_T \varphi_T d\mu_T.$$

Also

$$\text{Vol}_k(\hat{U}_{n,m}) = \sum_{i=n}^{m-1} l_T(R_T^i(x_0)),$$

where $\text{Vol}_k(N_{n,m})$ differs from $\text{Vol}_k(\hat{U}_{n,m})$ by a bounded quantity due to the closings. By Birkhoff’s ergodic theorem, for $\mu_T$-almost all $x_0 \in T$,

$$\frac{1}{m-n} \text{Vol}_k f(\hat{U}_{n,m}) \to \int_T l_T d\mu_T = \mu(S) = 1.$$

Thus we conclude that for $\mu_T$-almost $x_0 \in T$,

$$\frac{1}{\text{Vol}_k(N_{n,m})} [N_{n,m}] \to \int_T \varphi_T d\mu_T,$$

It is easy to see as in theorem 10.11 that $\int_T \varphi_T d\mu_T$ is the generalized current $[f, \mu]$. □

12. Realization of $H_1(M, \mathbb{R})$

Let $M$ be a $C^\infty$ smooth compact Riemannian manifold. Given a real 1-homology class $a \in H_1(M, \mathbb{R})$, we want to construct an immersion $(f, S)$ in $M$ of a uniquely ergodic solenoid $S_\mu$ fully representing $a$ up to scalar factor (see definition 10.1), and with generalized current $[f, S_\mu] = a$. By theorem 10.12 it is enough to construct an immersed, oriented, uniquely ergodic 1-solenoid $(f, S_\mu)$ with $[f, S_\mu]$ equals to a positive multiple $\lambda a$ of $a$, since in this case $[f, S_{\lambda^{-1} \mu}] = a$. If $\mu$ is the normalized ergodic measure, theorem 10.12 implies that $(f, S)$ fully represents $\lambda a$. Moreover, if we change the Riemannian metric of $M$, the property of fully representing $a$ up to positive scalar factor is preserved (although the scalar factor may vary).

In some situations (depending on the dimension) we will achieve an embedding. Actually the 1-solenoid $S$ that we will construct is independent of $a$ and of $M$, and moreover it has a 1-dimensional transversal structure.
Let \( h : \mathbb{T} \to \mathbb{T} \) be a diffeomorphism with an irrational rotation number (and therefore uniquely ergodic), which is a Denjoy counter-example, i.e. has the unique invariant probability measure supported in a Cantor set \( K \subset \mathbb{T} \). Let \( \mu_K \) denote the invariant probability measure. For the original construction of Denjoy counter-examples see [De]. Actually \( h \) can be taken to be of class \( C^{2-\epsilon} \) with \( \epsilon > 0 \) (see [Her]).

The suspension of \( h \),

\[
S_h = ([0,1] \times \mathbb{T})/(0,x)\sim(1,h(x))
\]

is \( C^{2-\epsilon} \)-diffeomorphic to the 2-torus \( T^2 \). More explicitly, the diffeomorphism is as follows: take \( c > 0 \) small, let \( h_t, t \in [0,c], \) be a (smooth) isotopy from \( \text{id} \) to \( h \), then we define the diffeomorphism \( H : T^2 \to X \) by

\[
H(x,t) = \begin{cases} 
(t, h^{-1}(h_t(x))), & \text{for } t \in [0,c], \\
(t, x), & \text{for } t \in [c,1].
\end{cases}
\]

Note that \( S_h \) is foliated by the horizontal leaves, so \( T^2 \) is foliated accordingly. It can be considered also as a 1-solenoid of class \( C^{\omega,2-\epsilon} \).

The sub-solenoid

\[
S = ([0,1] \times K)/\sim \subset S_h
\]

is an oriented 1-solenoid of class \( C^{\omega,2-\epsilon} \), with transversal \( T = \{0\} \times T \cap S = \{0\} \times K \). The holonomy is given by the map \( h \), which is uniquely ergodic. Moreover, the associated transversal measure is \( \mu_K \) on the transversal \( K \cong \{0\} \times K \). So \( S \) is an oriented, uniquely ergodic 1-solenoid.

Using the diffeomorphism \( H \), we may see the solenoid \( S \) inside the 2-torus, \( S \subset T^2 \), consisting of the paths \( (t,x), x \in K, t \in [c,1], \) together with the paths \( (t, h_t(x)), x \in K, t \in [0,c] \). The embedding \( S \hookrightarrow T^2 \) is of class \( C^{\omega,2-\epsilon} \), so we shall think of \( S \) as an oriented 1-solenoid of regularity \( C^{\omega,2-\epsilon} \).

**Figure 1.** The 1-solenoid \( S \).

**Theorem 12.1.** Let \( M \) be a compact smooth manifold, and let \( a \in H_1(M,\mathbb{R}) \) be a non-zero 1-homology class. If \( \dim M \geq 3 \) then (a positive multiple of) \( a \) can be fully...
represented by an embedding (of class $C^{\infty, 2-\epsilon}$) of the (oriented, uniquely ergodic) 1-solenoid $S$ into $M$. If $\dim M = 2$ then (a positive multiple of) $a$ can be fully represented by a transversal immersion of $S$ into $M$.

Proof. Let $C_1, \ldots, C_b$ be (integral) 1-cycles which form a basis of the (real) 1-homology of $M$. Switch orientations and reorder the cycles if necessary so that there are real numbers $\lambda_1, \ldots, \lambda_r > 0$ such that

$$a = \lambda_1 C_1 + \cdots + \lambda_r C_r.$$ 

By dividing by $\sum \lambda_i$ if necessary, we can assume that $\sum \lambda_i = 1$.

Consider the solenoid $S$ constructed above and partition the cantor set $K$ into $r$ disjoint compact subsets $K_1, \ldots, K_r$ in cyclic order, each of which with

$$\mu_K(K_i) = \lambda_i.$$ 

Consider the transversal $T = \{0\} \times \mathbb{T}$ in $S_h$. We consider angles $\tau_1, \tau_2, \ldots, \tau_n \in \mathbb{T}$ in the same cyclic order as the $K_i$, such that $K_i$ is contained in the open subset $U_i \subset T$ with boundary points $\tau_i$ and $\tau_{i+1}$ (denoting $\tau_{n+1} = \tau_1$). We may assume that $\tau_1 = 0$. Remove the segments $[c, 1] \times \{t\}$ from $S_h$ to get the open 2-manifold

$$U = S_h - \bigcup_i ([c, 1] \times \{\tau_i\}).$$

By construction, our solenoid $S \subset U$.

Suppose that $\dim M \geq 3$. Then we can $C^\infty$-smoothly embed $F : U \rightarrow M$ as follows: suppose that all cycles $C_i$ share a common base-point $p_0 \in M$ (and are otherwise disjoint to each other). Then embed the central part $(0, c) \times \mathbb{T} \subset U$ in a small ball $B$ around $p_0$ and embed each of the $[c, 1] \times U_i$ in $M - B$ in such a way that if we contract $B$ to $p_0$ then the images of $[c, 1] \times \{t\}, t \in U_i$, represent cycles homologous to $C_i$.

The embedding of $f$ of $S$ into $M$ is defined as the composition $S \hookrightarrow U \xrightarrow{F} M$. This is an embedding according to definition 7.1. By theorem 10.12 as $S$ is uniquely ergodic, to prove that $(f, S)$ fully represents $a$, it is enough to see that $[f, S] = a$. 

Figure 2. The open manifold $U$. 

By construction, our solenoid $S \subset U$. 

Suppose that $\dim M \geq 3$. Then we can $C^\infty$-smoothly embed $F : U \rightarrow M$ as follows: suppose that all cycles $C_i$ share a common base-point $p_0 \in M$ (and are otherwise disjoint to each other). Then embed the central part $(0, c) \times \mathbb{T} \subset U$ in a small ball $B$ around $p_0$ and embed each of the $[c, 1] \times U_i$ in $M - B$ in such a way that if we contract $B$ to $p_0$ then the images of $[c, 1] \times \{t\}, t \in U_i$, represent cycles homologous to $C_i$.

The embedding of $f$ of $S$ into $M$ is defined as the composition $S \hookrightarrow U \xrightarrow{F} M$. This is an embedding according to definition 7.1. By theorem 10.12 as $S$ is uniquely ergodic, to prove that $(f, S)$ fully represents $a$, it is enough to see that $[f, S] = a$. 

By construction, our solenoid $S \subset U$.

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The embedding of $f$ of $S$ into $M$ is defined as the composition $S \hookrightarrow U \xrightarrow{F} M$. This is an embedding according to definition 7.1. By theorem 10.12 as $S$ is uniquely ergodic, to prove that $(f, S)$ fully represents $a$, it is enough to see that $[f, S] = a$. 

By construction, our solenoid $S \subset U$.

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The embedding of $f$ of $S$ into $M$ is defined as the composition $S \hookrightarrow U \xrightarrow{F} M$. This is an embedding according to definition 7.1. By theorem 10.12 as $S$ is uniquely ergodic, to prove that $(f, S)$ fully represents $a$, it is enough to see that $[f, S] = a$. 

By construction, our solenoid $S \subset U$.

Suppose that $\dim M \geq 3$. Then we can $C^\infty$-smoothly embed $F : U \rightarrow M$ as follows: suppose that all cycles $C_i$ share a common base-point $p_0 \in M$ (and are otherwise disjoint to each other). Then embed the central part $(0, c) \times \mathbb{T} \subset U$ in a small ball $B$ around $p_0$ and embed each of the $[c, 1] \times U_i$ in $M - B$ in such a way that if we contract $B$ to $p_0$ then the images of $[c, 1] \times \{t\}, t \in U_i$, represent cycles homologous to $C_i$.

The embedding of $f$ of $S$ into $M$ is defined as the composition $S \hookrightarrow U \xrightarrow{F} M$. This is an embedding according to definition 7.1. By theorem 10.12 as $S$ is uniquely ergodic, to prove that $(f, S)$ fully represents $a$, it is enough to see that $[f, S] = a$. 

By construction, our solenoid $S \subset U$.
Let $\alpha$ be any closed 1-form on $M$. Since $H^1(M) = H^1(M, B)$, we may assume that $\alpha$ vanishes on $B$. We cover the solenoid $S$ by the flow-boxes $((0, c) \times \mathbb{T}) \cap S$ and $[c, 1] \times K_i, i = 1, \ldots, r$. As $f^* \alpha$ vanishes in the first flow-box, we have

$$\langle [f, S_\mu], [\alpha] \rangle = \sum_{i=1}^r \int_{K_i} \left( \int_{[c, 1]} f^* \alpha \right) d\mu_{K_i}(y) = \sum_{i=1}^r \int_{K_i} \langle C_i, [\alpha] \rangle d\mu_{K_i}(y)$$

$$= \sum_{i=1}^r \langle C_i, [\alpha] \rangle \mu(K_i) = \sum_{i=1}^r \lambda_i \langle C_i, [\alpha] \rangle = \langle a, [\alpha] \rangle,$$

proving that $[f, S_\mu] = a$. Now the result follows from theorem 10.12.

Now suppose that $\dim M = 2$. Let us do the appropriate modifications to the previous construction. Choose cycles $C_i$ sharing a common base-point $p_0 \in M$, and such that their intersections (and self-intersections) away from $p_0$ are transversal. Changing $C_i$ by $2C_i$ if necessary, we suppose that going around $C_i$ does not change the orientation (that is, the normal bundle to $C_i$ is oriented). From the manifold $U$ in Figure 2, remove $[0, c] \times \{\tau\}$ to get the open 2-manifold

$$V = \left( (0, c) \times (0, 1) \right) \cup \bigcup_{i} \left( [c, 1] \times U_i \right).$$

![Figure 3. The open manifold V](image)

The manifold $V$ can be immersed into the surface $M$, $F : V \to M$, in such a way that $(0, c) \times (0, 1)$ is sent to a ball $B$ around $p_0$, $[c, 1] \times U_i$ are sent to $M - B$, the images of $[c, 1] \times \{t\}$, $t \in U_i$, represent cycles homologous to $C_i$ if we contract $B$ to a point, and the intersections and self-intersections of horizontal leaves are always transverse.

Note that the solenoid $S$ is not contained in $V$, since we have removed $[0, c] \times \tau_1$ from $U$. So we cannot define an immersion $f : S \to M$ by restricting that of $F$. To define $f$ in $S \cap ((0, c) \times \mathbb{T})$, we need to explicit out our isotopy $h_t$. Consider $h : \mathbb{T} \to \mathbb{T}$ and lift it to $\tilde{h} : \mathbb{R} \to \mathbb{R}$ with $r := \tilde{h}(0) \in (0, 1)$. Consider a smooth
function $\rho : \mathbb{R} \to [0, 1]$, with $\rho(t) = 1$ for $t \leq 0$, $\rho(t) = 0$ for $t \geq c$, and $\rho'(t) < 0$ for $t \in (0, c)$. Then we can define

$$h_t(x) = \tilde{h}(\tilde{h}^{-1}(x)\rho(t) + x(1 - \rho(t))) \mod \mathbb{Z}.$$  

Define the immersion $f : S \to M$ as follows: $f$ equals $F$ for $(t, x) \in [c, 1] \times K \subset V$. For $(t, \tilde{h}^{-1}(h_t(x))) \in S \cap ([0, c] \times \mathbb{T})$, we set

$$f(t, h^{-1}(h_t(x))) = \begin{cases} F(t, (\tilde{h}^{-1}(x) + 1)(1 - \rho(t))) \cdot x, & x \in K \cap (0, r), \\ F(\tilde{h}^{-1}(h_t(x))\rho(t) + x(1 - \rho(t))), & x \in K \cap (r, 1). \end{cases}$$  

It is easily checked that $f$ sends $S \cap ([0, c] \times \mathbb{T})$ into the ball $B$ and the intersections of the leaves in this portion of the solenoid are transverse.

The proof that $(f, S)$ fully represents a positive multiple of $a$ and that the generalized current $[f, S]\mu = a$ goes as before. \hfill \Box

**Remark 12.2.** We do not need $M$ compact for the above construction to work. If $M$ is non-compact, take integer 1-cycles $C_1, C_2, \ldots$ (possibly infinitely many) which form a basis of $H_1(M, \mathbb{R})$. Then for any $a \in H_1(M, \mathbb{R})$ there exist an integer $r \geq 1$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ with $a = \sum \lambda_i C_i$. The construction of theorem [12.1] works.

The solenoid $S$ is oriented, regardless of $M$ being oriented or not.

### 13. Realization of $H_k(M, \mathbb{R})$

Let $M$ be a smooth compact oriented Riemannian $C^\infty$ manifold and let $a \in H_k(M, \mathbb{R})$ be a non-zero real $k$-homology class. We are going to generalize the construction of section [12] to obtain uniquely ergodic $k$-solenoids $(f, S)$ with a 1-dimensional transversal structure, immersed in $M$ and fully representing a positive multiple of $a$.

By theorem [11.12] it is enough to produce a uniquely ergodic oriented $k$-solenoid $S$, with a trapping region $W \subset S$ and an immersion $f : S \to M$ such that $f(W)$ is sent to a contractible ball in $M$, and with associated generalized current $[f, S]\mu = a$. Then, for the normalized (uniquely ergodic) measure $\nu = \mu/\mu(S)$, it is $[f, S]\nu = a/\mu(S)$, and so $(f, S)$ fully represents (by theorem [11.12]) the class $a/\mu(S)$.

Note also that if we change the Riemannian metric of $M$, then the solenoid $(f, S)$ will still fully represent a positive multiple of $a$ (although the scalar factor may change).

To start with, fix a collection of compact $k$-dimensional smooth oriented manifolds $S_1, \ldots, S_r$ and positive numbers $\lambda_1, \ldots, \lambda_r > 0$ such that $\sum \lambda_i = 1$. Let $h : \mathbb{T} \to \mathbb{T}$ be a diffeomorphism of the circle which is a Denjoy counter-example with an irrational rotation number and of class $C^{2-\epsilon}$, for some $\epsilon > 0$. Hence $h$ is uniquely ergodic. Let $\mu_K$ be the unique invariant probability measure, which is supported in a Cantor set.
is contained in the open subset $U$ of the $\tau$-dimensional open manifold $K \subset \mathbb{T}$. Partition the Cantor set $K$ into $r$ disjoint compact subsets $K_1, \ldots, K_r$ in cyclic order, each of which with $\mu_K(K_i) = \lambda_i$.

We fix two points on each manifold $S_i$, and remove two small balls, $D_i^+$ and $D_i^-$, around them. Denote $S_i' = S_i - (D_i^+ \cup D_i^-)$, so that $S_i'$ is a manifold with oriented boundary $\partial S_i' = \partial D_i^+ \cup \partial D_i^-$. Fix two diffeomorphisms: $\partial D_i^+ \cong S^{k-1}$, orientation preserving, and $\partial D_i^- \cong S^{k-1}$, orientation reversing. There are inclusions

$$A_\pm := \bigsqcup (\partial D_i^\pm \times K_i) \xrightarrow{i_\pm} S^{k-1} \times S^1,$$

with image $S^{k-1} \times K \subset S^{k-1} \times S^1$. Define

$$S = \bigsqcup (S_i' \times K_i)/_{x \sim i_+^{-1}(\text{id} \times h)i_-(x), x \in A_-}.$$

This is an oriented $k$-solenoid of class $C^{\infty,2-\epsilon}$, with 1-dimensional transversal dimension. As $S^{k-1} \times K \subset S$ in an obvious way, fixing a point $p \in S^{k-1}$, we have a global transversal $T = \{p\} \times K \subset S^{k-1} \times K \subset S$. Identifying $T \cong K$, the holonomy pseudo-group is generated by $h : K \to K$. Hence $S$ is uniquely ergodic. Let $\mu$ denote the tranversal measure corresponding to $\mu_K$.

We want to give an alternative description of $S$. Fix an isotopy $h_t$, $t \in [0,1]$, from id to $h$. Define

$$W' := \{(t, x, h^{-1}(h_t(y))) ; t \in [0,1], x \in S^{k-1}, y \in K\} \subset [0,1] \times S^{k-1} \times S^1.$$

Then

$$S = \left( \bigsqcup (S_i' \times K_i) \sqcup W' \right)/_{x \sim (0,1_{-\epsilon}(x)), x \in \partial D_i^- \times K_i}.$$

Strictly speaking, we should say that they are diffeomorphic, but we shall fix an identification. We define a map $\pi : S \to \mathbb{T}$ by

$$\pi(t, x, h^{-1}(h_t(y))) = t - \frac{1}{2}, \quad (t, x, h^{-1}(h_t(y))) \in W'$$

$$\pi(p) = \frac{1}{2}, \quad p \in S - W'$$

Then $W = \text{Int}(W') = \pi^{-1}(-\frac{1}{2}, \frac{1}{2})$ is a trapping region as in definition 11.9.

Consider angles $\tau_1, \tau_2, \ldots, \tau_n \in \mathbb{T}$ in the same cyclic order as the $K_i$, such that $K_i$ is contained in the open subset $U_i \subset T$ with boundary points $\tau_i$ and $\tau_{i+1}$ (denoting $\tau_{n+1} = \tau_1$). We may assume that $\tau_1 = 0$. Then solenoid $S$ sits inside the $(k+1)$-dimensional open manifold

$$X = \bigsqcup (S_i' \times U_i) \sqcup ([0,1] \times S^{k-1} \times S^1)/_{x \sim (0,1_{-\epsilon}(x)), x \in \partial D_i^- \times U_i}.$$

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as the collection of points \((x, y), x \in S'_i, y \in K_i, \) together with the points \((t, x, h^{-1}(h_t(y))), x \in S^{k-1}, y \in K, t \in [0, 1].\)

**Figure 4.** The manifold \(X.\)

**Remark 13.1.** The 1-solenoid constructed in section 12 corresponds to the case \(S_i = S^1, i = 1, \ldots, r.\)

**Theorem 13.2.** Let \(M\) be a compact oriented smooth Riemannian manifold of dimension \(n,\) and let \(a \in H_k(M, \mathbb{R})\) be a non-zero real \(k\)-homology class.

1. If \(n \geq 2k + 1\) then (a positive multiple of) \(a\) can be fully represented by an embedding of a uniquely ergodic oriented \(k\)-solenoid \(S\) into \(M.\)
2. If \(n - k\) is odd then (a positive multiple of) \(a\) can be fully represented by a transversally immersion of a uniquely ergodic oriented \(k\)-solenoid \(S\) into \(M.\)
3. In other cases (a positive multiple of) \(a\) can be fully represented by an immersed uniquely ergodic oriented \(k\)-solenoid \(S\) in \(M.\)

**Proof.** By a theorem of Thom [Th], if \(a \in H_k(M, \mathbb{Z})\) then there exists \(N \gg 0\) such that \(Na\) is represented by a smooth submanifold of \(M.\) This submanifold is oriented because it represents a non-zero homology class. Moreover, if \(n \geq 2k + 1\) or \(n - k\) is odd then it can be arranged that the normal bundle to the submanifold is trivial [Th].

Take a collection \(C_1, \ldots, C_{b_k} \in H_k(M, \mathbb{Z})\) which are a basis of \(H_k(M, \mathbb{Q})\) and such that \(C_i\) is represented by a smooth submanifold \(S_i \subset M\) (with trivial normal bundle if \(n \geq 2k + 1\) or \(n - k\) is odd). They can be assumed to be in general position. After switching the orientations of \(C_i\) if necessary, reordering the cycles and multiplying \(a\) by a suitable positive real number, we may suppose that

\[
a = \lambda_1 C_1 + \ldots + \lambda_r C_r,
\]

for some \(r \geq 1, \lambda_i > 0, 1 \leq i \leq r,\) and \(\sum \lambda_i = 1.\) We construct the solenoid \(S\) with the procedure above starting with the manifolds \(S_i\) and coefficients \(\lambda_i.\) This is a uniquely ergodic \(k\)-solenoid with a 1-dimensional transversal structure, and a trapping region \(W \subset S.\)
Now we want to define an immersion $f : S \to M$, and to prove that $(f, S)$ fully represents $a$. We have the following cases:

1. $n \geq 2k + 1$. The general position property on the $S_i$ implies that all $S_i$ are disjoint submanifolds of $M$. As the normal bundle to $S_i$ is trivial and $U_i$ is an interval, we can embedded $S_i \times U_i$ in a small neighbourhood of $S_i$.

Fix a base point $p_0 \in M$ off all $S_i$. Take a small box $B \subset M$ around $p_0$ of the form $B = [0, 1] \times D^{n-1}$, where $D^{n-1}$ is the open $(n-1)$-dimensional ball. Consider a circle $S^1 \subset D^{k+1} \subset D^{n-1}$ and let $D^k \times S^1 \subset D^{k+1} \subset D^{n-1}$ be a tubular neighbourhood of it, with boundary $S^{k-1} \times S^1$.

For each $i = 1, \ldots, r$, fix $x_i \in U_i$, and consider two paths in $M - \text{Int}(B)$, $\gamma_i^\pm$, where $\gamma_i^-$ goes from the point $(0, x_i) \in \{0\} \times U_i \subset \{0\} \times S^1 \subset \{0\} \times D^{n-1} \subset B$ to the point $(p^-_i, x_i) \in S_i \times U_i$, and $\gamma_i^+$ goes from $(1, x_i) \in \{1\} \times U_i \subset \{1\} \times S^1 \subset \{1\} \times D^{n-1} \subset B$ to $(p^+_i, x_i) \in S_i \times U_i$. We arrange that $\gamma_i^\pm$ are transverse to $S_i \times U_i$ at $(p^+_i, x_i)$ and are disjoint from all $S_j$ otherwise.

We thicken $\gamma_i^\pm$ to immersions $\gamma_i^\pm \times D^k \times U_i$ into $M - \text{Int}(B)$ such that one extreme goes to $D^k \times U_i$ and the other goes to either $D^k \times U_i \times \{0\} \subset D^k \times S^1 \times \{0\} \subset D^{n-1} \times \{0\} \subset B$ for $\gamma_i^-$, or $D^k \times U_i \times \{1\} \subset D^k \times S^1 \times \{1\} \subset D^{n-1} \times \{1\} \subset B$ for $\gamma_i^+$. It is possible to do this in such a way that the $U_i$ directions match, since $n \geq k + 2$.

Recall that $S'_i = S_i - (D^+_i \cup D^-_i)$, and set

$$S''_i = S'_i \cup \gamma_i^\pm \times S^{k-1},$$

which is diffeomorphic to $S'_i$ (to be rigorous, we should smooth out corners). Then

$$U := \bigcup((S''_i \times U_i) \cup (\gamma_i^\pm \times S^{k-1} \times U_i) \cup ([0, 1] \times S^{k-1} \times S^1))$$

is a $(k + 1)$-dimensional open manifold embedded in $M$. The manifold $U$ is foliated as follows: $S'_i \times U_i$ is foliated by $S''_i \times \{y\}$, for $y \in U_i$, and $[0, 1] \times S^{k-1} \times S^1$ is foliated by

$$L_y = \{(t, x, h^{-1}(ht(y))) \mid t \in [0, 1], x \in S^{k-1}\},$$

for $y \in S^1$. Clearly the solenoid $S$ is a subsolenoid of $U$, $S \subset U$. Restricting the embedding $F : U \to M$ to $S$ we get an embedding $f : S \to M$.

By construction $f(W) \subset \text{Int}(B)$, i.e. the image of the trapping region is contained in a contractible ball.

2. $n - k > 1$ and odd. As the codimension is odd, the general position property implies that the submanifolds $S_i$ and $S_j$, $i \neq j$, intersect transversally. This is argued as follows: represent $S_i$ as the preimage of a regular value $v_i$ of a smooth application $f_i : M \to S^{n-k}$. Then consider $f_i \times f_j : M \to S^{n-k} \times S^{n-k}$ and take a regular value $(v'_i, v'_j)$ of this map near $(v_i, v_j)$.
The same construction as in (1) works now, with the modification that we have to allow intersections of different leaves, but we may take them to be always transversal. So we get a transversal immersion \( f : S \to M \).

(3) \( n - k \) even. The embedded submanifolds \( S_i \subset M \) may have a non-trivial normal bundle. Moreover, they may intersect each other. The intersections are generically transversal along \( S_i \cap S_j \), but not everywhere transversal.

Take a generic section \( s_i \) of the normal bundle to \( S_i \), so that the image of the section intersects transversally the zero section. We have maps \( f_i : S_i \times (\varepsilon, \varepsilon) \to M \) defined by \( f_i(x,t) = \exp_x(ts_i(x)) \), where \( \exp \) is the exponential mapping (with respect to a Riemannian metric on \( M \)), which is an immersion for each \( t \) fixed, for \( \varepsilon > 0 \) small enough. With these maps, the previous construction gives a solenoid immersion \( f : S \to M \).

(4) \( n - k = 1 \). The submanifolds \( S_i \) have trivial normal bundle and they intersect each other transversally. We cannot avoid that the paths \( \gamma_i^\pm \) intersect other \( S_j \), but we arrange these intersections to be transverse. This produces a transversal immersion \( f \) of the region \( S - W \) of the solenoid into \( M - \text{Int}(B) \).

We have to modify the previous construction of the immersion of \( W \) into \( B \), as the codimension one does not leave enough room for it to work. Consider the box \( B = [0,1] \times D^{n-1} \) and remove the axis \( A = [0,1] \times \{0\} \). Use polar coordinates to identify \( B - A = [0,1] \times S^{k-1} \times (0,1) \), where the third coordinate corresponds to the radius. By construction, \( W' \subset S \) embeds into \( C = [0,l] \times S^{k-1} \times S^1 \), as the set of points \( (t,x,h^{-1}(h_t(y))), t \in [0,1], x \in S^{k-1} \) and \( y \in K \). We remove \( D = [0,1] \times S^{k-1} \times \tau_1 \) from \( C \), so that \( C - D = [0,1] \times S^{k-1} \times (0,1) \). Then \( W' \) immerses into \( C - D \), by using the process at the end of the proof of theorem [12.1] (now there is an extra factor \( S^{k-1} \) which plays no role). This is a transversal immersion.

There is one extra detail that we should be careful about. When connecting \( p_i^\pm \) with the two faces of \( B \), the orientations of the \( U_i \) should match. This happens because the normal bundle to \( S_i \) is trivial, and in this case \( S_i \times U_i \) is (diffeomorphic to) the normal bundle to \( S_i \).

To prove that \( (f, S) \) fully represents (a positive multiple of) \( a \), we use theorem [11.12]. The solenoid \( S \) has a trapping region \( W \), and \( f(W) \subset \text{Int}(B) \), a contractible ball in \( M \). So we only need to see that \( [f, S_\mu] = a \). Recall that the associated transversal measure is \( \mu_K \) on the transversal \( K \). Let \( \alpha \) be any closed 1-form on \( M \). Since \( H^1(M) = H^1(M,B) \), we may assume that \( \alpha \) vanishes on \( B \). We cover the solenoid \( S \) by the flow-boxes \( S_i'' \times K_i, i = 1, \ldots, r \), and \( W' \) (where the form \( \alpha \)
vanishes. Thus

\[ \langle [f, S_\mu], [\alpha] \rangle = \sum_{i=1}^r \int_{K_i} \left( \int_{S_{ii}} f^* \alpha \right) d\mu_{K_i}(y) = \sum_{i=1}^r \int_{K_i} \langle C_{ii}, [\alpha] \rangle d\mu_{K_i}(y) = \sum_{i=1}^r \langle C_{ii}, [\alpha] \rangle \mu(K_i) = \sum_{i=1}^r \lambda_i \langle C_{ii}, [\alpha] \rangle = \langle a, [\alpha] \rangle, \]

proving that \([f, S_\mu] = a.\]

\[\square\]

**Remark 13.3.** A similar comment to that of remark 12.2 applies to the present situation, that is, the compactness of \(M\) is not necessary.

If \(M\) is non-orientable, we may consider its oriented double cover \(\pi : \tilde{M} \to M.\) For any non-zero \(a \in H_k(M, \mathbb{R})\), there exists \(\tilde{a} \in H_k(\tilde{M}, \mathbb{R})\) with \(\pi_*(\tilde{a}) = a.\) We may construct an oriented uniquely-ergodic \(k\)-solenoid \((f, S)\) immersed in \(\tilde{M}\) fully representing \(\lambda \tilde{a},\) for some \(\lambda > 0.\) Then \((\pi \circ f, S)\) is immersed in \(M\) and fully represents \(\lambda a.\)

### 14. Homotopy of solenoids

**Definition 14.1. (Solenoid with boundary)** Let \(0 \leq r, s \leq \omega,\) and let \(k, l \geq 0\) be two integers. A foliated manifold with boundary (of dimension \(k + l\), with \(k\)-dimensional leaves, of class \(C^{r,s}\)) is a smooth manifold \(W\) with boundary, of dimension \(k + l\), endowed with an atlas \(\{(U_i, \varphi_i)\}\) of charts

\[ \varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^{k+l}_+ = \{(x_1, \ldots, x_k, y_1, \ldots, y_l ; x_1 \geq 0) \}, \]

whose changes of charts are of the form \(\varphi_i \circ \varphi_j^{-1}(x, y) = (X_{ij}(x, y), Y_{ij}(y)),\) where \(Y_{ij}(y)\) is of class \(C^s\) and \(X_{ij}(x, y)\) is of class \(C^{r,s}.\)

A pre-solenoid with boundary is a pair \((S, W)\) where \(W\) is a foliated manifold with boundary and \(S \subset W\) is a compact subspace which is a collection of leaves.

Two pre-solenoids with boundary \((S, W_1)\) and \((S, W_2)\) are equivalent if there are open subsets \(U_1 \subset W_1, U_2 \subset W_2\) with \(S \subset U_1\) and \(S \subset U_2\), and a diffeomorphism \(f : U_1 \to U_2\) (preserving leaves, of class \(C^{r,s}\)) which is the identity on \(S.\)

A \(k\)-solenoid with boundary is an equivalence class of pre-solenoids with boundary.

Note that any manifold with boundary is a solenoid with boundary.

The boundary of a \(k\)-solenoid with boundary \(S\) is the \((k - 1)\)-solenoid (without boundary) \(\partial S\) defined by the foliated manifold \(\partial W,\) where \(W\) is a foliated manifold with boundary defining the solenoid structure of \(S.\)

A \(k\)-solenoid with boundary \(S\) has two types of flow-boxes. If \(p \in S - \partial S\) is an interior point, then there is a flow-box \((U, \varphi)\) with \(p \in U,\) of the form \(\varphi : U \to \mathbb{R}^{k+l}_+,\)
$D^k \times K(U)$. If $p \in \partial S$ is a boundary point, then there is a flow-box $(U, \varphi)$ with $p \in U$ such that $\varphi$ is a homeomorphism

$$\varphi : U \to D^k_+ \times K(U),$$

where $D^k_+ = \{(x_1, \ldots, x_k) \in D^k; x_1 \geq 0\}$, and $K(U) \subset \mathbb{R}^l$, $\varphi(p) = (0, \ldots, 0, y_0)$, for some $y_0 \in K$. Note that writing

$$U' = \partial S \cap U = \varphi^{-1}(D^{k-1} \times K(U)),$$

where $D^{k-1} = \{(0, x_2, \ldots, x_k) \in D^k \subset D^k_+\}$, $(U', \varphi|_{U'})$ is a flow-box for $\partial S$. Therefore, if $T$ is a transversal for $\partial S$, then it is also transversal for $S$.

For a solenoid with boundary $S$ there is also a well-defined notion of holonomy pseudo-group. If $T$ is a local transversal for $\partial S$, and $h : T \to T$ is a holonomy map for $\partial S$ defined by a path in $\partial S$, then $h$ lies in the holonomy pseudo-group of $S$. So

$$\text{Hol}_{\partial S}(T) \subset \text{Hol}_S(T),$$

but they are in general not equal. In particular, if $S$ is connected with non-empty boundary then

$$\mathcal{M}_T(S) \subset \mathcal{M}_T(\partial S).$$

That is, if $\mu = (\mu_T)$ is a transversal measure for $S$, then it yields a transversal measure for $\partial S$, by considering only those transversals $T$ which are transversals for $\partial S$. We denote this transversal measure by $\mu$ again.

If $S$ comes equipped with an orientation, then $\partial S$ has a natural induced orientation. Note that any leaf $l \subset S$ is a manifold with boundary and each connected component of $\partial l$ is a leaf of $\partial S$.

**Theorem 14.2. (Stokes theorem)** Let $(f, S_\mu)$ be an oriented $(k+1)$-solenoid with boundary, endowed with a transversal measure, and immersed into a smooth manifold $M$. Let $\omega$ be a $k$-form on $M$. Then

$$\langle [f, S_\mu], d\omega \rangle = \langle [f|_{\partial S}, \partial S_\mu], \omega \rangle.$$

**Proof.** Let $\{U_i\}$ be a covering of $S$ by flow-boxes, and let $\{\rho_i\}$ be a partition of unity subordinated to it. Adding up the equalities

$$\int_{K(U_i)} \left( \int_{L_y} d\rho_i \wedge f^* \omega \right) d\mu_{K(U_i)}(y) + \int_{K(U_i)} \left( \int_{L_y} \rho_i f^* d\omega \right) d\mu_{K(U_i)}(y)$$

$$= \int_{K(U_i)} \left( \int_{L_y} d\rho_i f^* \omega \right) d\mu_{K(U_i)}(y) = \int_{K(U_i)} \left( \int_{\partial L_y} \rho_i f^* \omega \right) d\mu_{K(U_i)}(y),$$

$$\ldots$$
for all $i$, and using that $\sum d\rho_i \equiv 0$, we get
\[
\langle [f, S_\mu], d\omega \rangle = \sum_i \int_{K(U_i)} \left( \int_{L_y} \rho_i f^* d\omega \right) d\mu_{K(U_i)}(y) = \langle [f|_{\partial S}, \partial S_\mu], \omega \rangle.
\]
\[\square\]

Let $S$ be a $k$-solenoid of class $C^{r,s}$. We give $S \times I = S \times [0,1]$ a natural $(k+1)$-solenoid with boundary structure of the same class, by taking a foliated manifold $W \supset S$ defining the solenoid structure of $S$, and foliating $W \times I$ with the leaves $l \times I$, $l \subset W$ being a leaf of $W$. Then $S \times I \subset W \times I$ is a $(k+1)$-solenoid with boundary. The boundary of $S \times I$ is
\[\partial (S \times I) = (S \times \{0\}) \sqcup (S \times \{1\}).\]

If $S$ is oriented then $S \times I$ is naturally oriented and its boundary consists of $S \times \{0\} \cong S$ with orientation reversed, and $S \times \{1\} \cong S$ with orientation preserved.

Moreover if $T$ is a transversal for $S$, then $T' = T \times \{0\}$ is a transversal for $S' = S \times I$. The following is immediate.

**Lemma 14.3.** There is an identification of the holonomies of $S$ and $S \times I$. More precisely, under the identification $T \cong T' = T \times \{0\}$,
\[
\text{Hol}_S(T) = \text{Hol}_{S \times I}(T').
\]

In particular,
\[
\mathcal{M}_T(S) = \mathcal{M}_T(S \times I).
\]

**Definition 14.4. (Equivalence of immersions)** Two solenoid immersions $(f_0, S_0)$ and $(f_1, S_1)$ of class $C^{r,s}$ in $M$ are immersed equivalent if there is a $C^{r,s}$-diffeomorphism $h : S_0 \to S_1$ such that
\[f_0 = f_1 \circ h .\]

Two measured solenoid immersions are immersed equivalent if $h$ can be chosen to preserve the transversal measures.

**Definition 14.5. (Homotopy of immersions)** Let $S$ be a $k$-solenoid of class $C^{r,s}$ with $r \geq 1$. A homotopy between immersions $f_0 : S \to M$ and $f_1 : S \to M$ is an immersion of solenoids $f : S \times I \to M$ such that $f_0(x) = f(x,0)$ and $f_1(x) = f(x,1)$.

**Definition 14.6. (Cobordism of solenoids)** Let $S_0$ and $S_1$ be two $C^{r,s}$-solenoids. A cobordism of solenoids $S$ is a $(k+1)$-solenoid with boundary $\partial S = S_0 \sqcup S_1$.

If $S_0$ and $S_1$ are oriented, then an oriented cobordism is a cobordism $S$ which is an oriented solenoid such that it induces the given orientation on $S_1$ and the reversed orientation on $S_0$. 
If $S_0$ and $S_1$ have transversal measures $\mu_0$ and $\mu_1$, respectively, then a measured cobordism is a cobordism $S$ endowed with a transversal measure $\mu$ inducing the measures $\mu_0$ and $\mu_1$ on $S_0$ and $S_1$, respectively.

**Definition 14.7. (Homology equivalence)** Let $(f_0, S_0)$ and $(f_1, S_1)$ be two immersed solenoids in $M$. We say that they are homology equivalent if there exists a cobordism of solenoids $S$ between $S_0$ and $S_1$ and a solenoid immersion $f : S \to M$ with $f|_{S_0} = f_0$, $f|_{S_1} = f_1$. We call $(f, S)$ a homology between $(f_0, S_0)$ and $(f_1, S_1)$.

Let $(f_0, S_0, \mu_0)$ and $(f_1, S_1, \mu_1)$ be two immersed oriented measured solenoids. They are homology equivalent if there exists an immersed oriented measured solenoid $(f, S, \mu)$ such that $(f, S)$ is a homology between $(f_0, S_0)$ and $(f_1, S_1)$ and $S, \mu$ is a measured oriented cobordism from $S_0$ to $S_1$.

Clearly two homotopic immersions of a solenoid give homology equivalent immersions.

**Theorem 14.8.** Suppose that two oriented measured solenoids $(f_0, S_0, \mu_0)$ and $(f_1, S_1, \mu_1)$ immersed in $M$ are homology equivalent. Then the generalized currents coincide

\[ [f_0, S_0, \mu_0] = [f_1, S_1, \mu_1]. \]

The same happens if they are immersed equivalent.

**Proof.** In the first case, let $\omega$ be a closed $k$-form on $M$, then Stokes’ theorem gives

\[ \langle [f_1, S_1, \mu_1], \omega \rangle - \langle [f_0, S_0, \mu_0], \omega \rangle = \langle [f|_{\partial S}, \partial S, \mu], \omega \rangle = \langle [f, S, \mu], d\omega \rangle = 0. \]

In the second case, $f_0 = f_1 \circ h$ implies that the actions of the generalized currents over a closed form on $M$ coincide, since the pull-back of the form to the solenoids agree through the diffeomorphism $h$, and the integrals over the transversal measure gives the same numbers, since the measures correspond by $h$. □

**Remark 14.9.** In both definitions 14.6 and 14.5 we do not need to require that $f$ be an immersion. Actually, the generalized current $[f, S, \mu]$ makes sense for any measured solenoid $S, \mu$ and any regular map $f : S \to M$, of class $C^{r,s}$ with $r \geq 1$. Theorem 14.8 still holds with these extended notions.

15. **Intersection theory of solenoids**

Let $M$ be a smooth $C^\infty$ oriented manifold.

**Definition 15.1. (Transverse intersection)** Let $(f_1, S_1)$, $(f_2, S_2)$ be two immersed solenoids in $M$. We say that they intersect transversally if, for every $p_1 \in S_1$, $p_2 \in S_2$ such that $f_1(p_1) = f_2(p_2)$, the images of the leaves through $p_1$ and $p_2$ intersect transversally.
If two immersed solenoids \((f_1, S_1), (f_2, S_2)\), of dimensions \(k_1, k_2\) respectively, intersect transversally. We define the intersection of immersed solenoids \((f, S)\) as defined by

\[
S = \{(p_1, p_2) \in S_1 \times S_2 \mid f_1(p_1) = f_2(p_2)\}.
\]

and by the map \(f : S \to M\) given by

\[
f(p_1, p_2) = f_1(p_1) = f_2(p_2), \quad (p_1, p_2) \in S.
\]

We will see that \(S\), the intersection solenoid, is indeed a solenoid. Also the intersection \((f, S)\) of the two immersed solenoids \((f_1, S_1), (f_2, S_2)\) is an immersed solenoid. In order to prove this, we consider the intersection of the product solenoid \((f_1 \times f_2, S_1 \times S_2)\) in \(M \times M\) with the diagonal \(\Delta \subset M \times M\). So we have to analyze first the case of the intersection of an immersed solenoid with a submanifold. The notion of transverse intersection given in definition \[15.1\] applies to this case (a submanifold is an embedded solenoid).

**Lemma 15.2.** Let \((f, S)\) be an immersed \(k\)-solenoid in \(M\) intersecting transversally an embedded closed submanifold \(N \subset M\) of codimension \(q\). Suppose that \(S' = f^{-1}(N) \subset S\) is non-empty, then \((f|_{S'}, S')\) is an immersed \((k-q)\)-solenoid in \(N\).

If \(S\) and \(N\) are oriented, so is \(S'\).

If \(S\) has a transversal measure \(\mu\), then \(S'\) inherits a natural transversal measure, also denoted by \(\mu\).

**Proof.** First of all, note that \(S'\) is a compact and Hausdorff space.

Let \(W\) be a foliated manifold defining the solenoid structure of \(S\) such that there is a smooth map \(\hat{f} : W \to M\) of class \(C^{r,s}\), extending \(f : S \to M\), which is an immersion on leaves. By definition, for any leaf \(l \subset S\), \(f(l)\) is transverse to \(N\). Thus reducing \(W\) if necessary, the same transversality property occurs for any leaf of \(W\). The transversality of the leaves implies that the map \(\hat{f} : W \to M\) is transversal to the submanifold \(N \subset M\), meaning that for any \(p \in W\) such that \(\hat{f}(p) \in N\),

\[
d\hat{f}(p)(T_pW) + T_{\hat{f}(p)}N = T_{\hat{f}(p)}M.
\]

This implies that \(W' = \hat{f}^{-1}(N)\) is a submanifold of \(W\) of codimension \(q\) (in particular, \(k - q \geq 0\)). Moreover, it is foliated by the connected components \(l'\) of \(l \cap \hat{f}^{-1}(N) = (\hat{f}_l)^{-1}(N)\), where \(l\) are the leaves of \(W\). By transversality of \(\hat{f}\) along the leaves, \(l'\) is a \((k-q)\)-dimensional submanifold of \(l\). So \(W'\) is a foliated manifold with leaves of dimension \(k - q\). This gives the required solenoid structure to \(S' = S \cap \hat{f}^{-1}(N) = f^{-1}(N)\).

Clearly, \(f|_{S'} : S' \to N\) is an immersion (of class \(C^{r,s}\)) since \(\hat{f}|_{W'} : W' \to N\) is a smooth map which is an immersion on leaves.
If \( S \) and \( N \) are oriented, then each intersection \( l' = l \cap \hat{f}^{-1}(N) \) is also oriented (using that \( M \) is oriented as well). Therefore the leaves of \( S' \) are oriented, and hence \( S' \) is an oriented solenoid.

Let \( p \in S' \) and let \( U \cong D^k \times K(U) \) be a flow-box for \( S \) around \( p \). We can take \( U \) small enough so that \( f(U) \) is contained in a chart of \( M \) in which \( N \) is defined by functions \( x_1 = \ldots = x_q = 0 \). By the transversality property, the differentials \( dx_1, \ldots, dx_q \) are linearly independent on each leaf \( f(D^k \times \{y\}), y \in K(U) \). Therefore, \( x_1, \ldots, x_q \) can be completed to a set of functions \( x_1, \ldots, x_k \) such that \( dx_1, \ldots, dx_k \) are a basis of the cotangent space for each leaf (reducing \( U \) if necessary). Thus the pull-back of \( x = (x_1, \ldots, x_k) \) to \( U \) give coordinates functions so that, using the coordinate \( y \in K(U) \) for the transversal direction, \( (x,y) \) are coordinates for \( U \), and \( f^{-1}(N) \) is defined as \( x_1 = \ldots = x_q = 0 \). This means that

\[
S' \cap U \cong \{(0, \ldots, 0, x_{q+1}, \ldots, x_k, y) \in D^k \times K(U) \} \cong D^{k-q} \times K(U).
\]

Therefore any local transversal \( T \) for \( S' \) is a local transversal for \( S \), and any holonomy map for \( S' \) is a holonomy map for \( S \). So

\[
\text{Hol}_{S'}(T) \subset \text{Hol}_S(T).
\]

Hence a transversal measure for \( S \) gives a transversal measure for \( S' \).

Now we can address the general case.

**Proposition 15.3.** Suppose that \((f_1, S_1), (f_2, S_2)\) are two immersed solenoids in \( M \) intersecting transversally, and let \( S \) be its intersection solenoid defined in \((10)\) and let \( f \) be the map \((17)\). If \( S \neq \emptyset \), then \((f, S)\) is an immersed solenoid of dimension \( k = k_1 + k_2 - n \) (in particular, \( k \) is a non-negative number).

If \( S_1 \) and \( S_2 \) are both oriented, then \( S \) is also oriented.

If \( S_1 \) and \( S_2 \) are endowed with transversal measures \( \mu_1 \) and \( \mu_2 \) respectively, then \( S \) has an induced measure \( \mu \).

**Proof.** The product \( S_1 \times S_2 \) is a \((k_1 + k_2)\)-solenoid and

\[
F = f_1 \times f_2 : S_1 \times S_2 \to M \times M
\]

is an immersion. Let \( \Delta \subset M \times M \) be the diagonal. There is an identification (as sets)

\[
S = (S_1 \times S_2) \cap F^{-1}(\Delta).
\]

The condition that \((S_1, f_1), (S_2, f_2)\) intersect transversally can be translated into that \((F, S_1 \times S_2)\) and \( \Delta \) intersect transversally in \( M \times M \).

Therefore applying lemma 15.2, \((S, F|_S)\) is an immersed \( k \)-solenoid, where \( F|_S : S \to \Delta \) is defined as \( F(x_1, x_2) = f_1(x_1) \). Using the diffeomorphism \( M \cong \Delta, x \mapsto (x, x) \), \( F|_S \) corresponds to \( f : S \to M \). So \((S, f)\) is an immersed \( k \)-solenoid.
If \( S_1 \) and \( S_2 \) are both oriented, then \( S_1 \times S_2 \) is also oriented. By lemma 15.2, \( S \) inherits an orientation.

If \( S_1 \) and \( S_2 \) are endowed with transversal measures \( \mu_1 \) and \( \mu_2 \), then \( S_1 \times S_2 \) has a product transversal measure \( \mu \). For any local transversals \( T_1 \) and \( T_2 \) to \( S_1 \) and \( S_2 \), respectively, \( T = T_1 \times T_2 \) is a local transversal to \( S_1 \times S_2 \) (and conversely). We define

\[
\mu_T = \mu_{1,T_1} \times \mu_{2,T_2}.
\]

Now lemma 15.2 applies to give the transversal measure for \( S \). Note that the local transversals to \( S \) are of the form \( T_1 \times T_2 \), for some local transversals \( T_1 \) and \( T_2 \) to \( S_1 \) and \( S_2 \).

**Remark 15.4.** If \( k_1 + k_2 = n \) then \( S \) is a 0-solenoid. For a 0-solenoid \( S \), an orientation is a continuous assignment \( \epsilon : S \to \{ \pm 1 \} \) of sign to each point of \( S \).

Note also that for a 0-solenoid \( S \), \( T = S \) is a transversal and a transversal measure is a Borel measure on \( S \).

Let \((f_1, S_1), (f_2, S_2)\) be two immersed solenoids in \( M \) intersecting transversally, with \((f, S)\) its intersection solenoid. Let \( p = (p_1, p_2) \in S \). Then we can choose flow-boxes \( U_1 = D^{k_1} \times K(U_1) \) for \( S_1 \) around \( p_1 \) with coordinates \((x_1, \ldots, x_{k_1}, y)\), and \( U_2 = D^{k_2} \times K(U_2) \) for \( S_2 \) around \( p_2 \) with coordinates \((x_1, \ldots, x_{k_2}, z)\), and coordinates for \( M \) around \( f(p) \), such that

\[
\begin{align*}
    f_1(x, y) &= (x_1, \ldots, x_{k_1+k_2-n}, x_{k_1+k_2-n+1}, \ldots, x_{k_1}, B_1(x, y), \ldots, B_{n-k_1}(x, y)), \\
    f_2(x, z) &= (x_1, \ldots, x_{k_1+k_2-n}, C_1(x, z), \ldots, C_{n-k_2}(x, z), x_{k_1+k_2-n+1}, \ldots, x_{k_2}).
\end{align*}
\]

Then \( S \) is defined locally as \( D^{k_1+k_2-n} \times K(U_1) \times K(U_2) \) with coordinates \((x_1, \ldots, x_{k_1+k_2-n}, y, z)\) and

\[
\begin{align*}
    f(x_1, \ldots, x_{k_1+k_2-n}, y, z) &= (x_1, \ldots, x_{k_1+k_2-n}, C_1(x, z), \ldots, C_{n-k_2}(x, z), B_1(x, y), \ldots, B_{n-k_1}(x, y)).
\end{align*}
\]

**Theorem 15.5.** Let \((f, S_\mu)\) be an oriented measured \( k \)-solenoid immersed in \( M \) intersecting transversally a closed subvariety \( i : N \hookrightarrow M \) of codimension \( q \), such that \( S' = f^{-1}(N) \subset S \) is non-empty. Consider the oriented measured \((k - q)\)-solenoid immersed in \( N \), \((f', S'_\mu)\), where \( f' = f|_{S'} \). Then, under the restriction map

\[
i^* : H^*_{c}(M) \to H^*_{c}(N),
\]

the dual of the generalized current \([f, S'_\mu]^*\) maps to \([f', S'_\mu]^*\).

**Proof.** Let \( U \subset M \) be a tubular neighbourhood of \( N \) with projection \( \pi : U \to N \). Note that \( U \) is diffeomorphic to the unit disc bundle of the normal bundle of \( N \) in \( M \). Let \( \tau \) be a Thom form for \( N \subset M \), that is a closed form \( \tau \in \Omega^q(M) \) supported in
Proof. \[ \text{Note that } \beta \text{ of the generalized currents satisfy Theorem 15.6.} \]

Suppose that \( f \) intersecting transversally, and let \( \pi \) smaller if necessary, we can arrange that \( \pi \) the support of \( f \) on projecting in the normal directions along the leaves. Then \( (19) \) under Poincaré duality is the map \( \beta \) which sends \( \langle f, S_\mu \rangle, \tilde{\beta} \rangle = \langle [f', S'_\mu], \beta \rangle. \)

Take a covering of \( S \) by flow-boxes \( U_i \cong D^k \times K(U_i) \cong D^q \times D^{k-q} \times K(U_i) \) so that \( U_i' = U_i \cap S' \) is given by \( x_1 = \ldots = x_q = 0. \) Making the tubular neighborhood \( U \supset N \) smaller if necessary, we can arrange that \( f^{-1}(U) \cap U_i \) is contained in \( D^q \times D^{k-q} \times K(U_i), \) for some \( r < 1. \) It is easy to construct a map \( \bar{\pi} : f^{-1}(U) \to f^{-1}(N) \) which consists on projecting in the normal directions along the leaves. Then \( f \circ \bar{\pi} \) and \( \pi \circ f \) are homotopic.

Let \( S_i' \) be a measurable partition of \( S' \) with \( S_i' \subset U_i'. \) We may assume that \( S_i = \bar{\pi}^{-1}(S_i') \) is contained in \( U_i. \) The sets \( S_i \) form a measurable partition containing \( f^{-1}(U), \) the support of \( f^*\tilde{\beta} = f^*(\pi^*\beta \wedge \tau). \) Then

\[
\langle [f, S_\mu], \tilde{\beta} \rangle = \sum_i \int_{K(U_i)} \left( \int_{S_i \cap (D^k \times \{y\})} f^*(\pi^*\beta \wedge \tau) \right) d\mu_{K(U_i)}(y) \\
= \sum_i \int_{K(U_i)} \left( \int_{S_i \cap (D^k \times \{y\})} \bar{\pi}^*f^*\beta \wedge f^*\tau \right) d\mu_{K(U_i)}(y) \\
= \sum_i \int_{K(U_i)} \left( \int_{S_i \cap (D^{k-q} \times \{y\})} f^*\beta \right) \left( \int_{f(D^q)} \tau \right) d\mu_{K(U_i)}(y) \\
= \sum_i \int_{K(U_i)} \left( \int_{S_i' \cap (D^{k-q} \times \{y\})} f^*\beta \right) d\mu_K(U_i)(y) \\
= \langle [f', S'_\mu], \beta \rangle. 
\]

\( \square \)

**Theorem 15.6.** Suppose that \( (f_1, S_{1\mu_1}), (f_2, S_{2\mu_2}) \) are two immersed solenoids in \( M \) intersecting transversally, and let \( (f, S_\mu) \) be its intersection solenoid. Then the duals of the generalized currents satisfy

\[ [f, S_\mu]^* = [f_1, S_{1\mu_1}]^* \cup [f_2, S_{2\mu_2}]^*. \]

**Proof.** Note that \( [f_1, S_1]^* \in H_c^{n-k_1}(M) \) and \( [f_2, S_2]^* \in H_c^{n-k_2}(M), \) so \( [f_1, S_1]^* \cup [f_2, S_2]^* \) and \( [f, S]^* \) both live in

\[ H_c^{n-k_1+n-k_2}(M) = H_c^{n-k}(M). \]
Consider the immersed solenoid \((F, S_1 \times S_2)\), where \(F = f_1 \times f_2 : S_1 \times S_2 \to M \times M\) and \(S_1 \times S_2\) has the transversal measure \(\mu\) given by [13]. Let us see that the following equality, involving the respective generalized currents,

\[
[F, (S_1 \times S_2)_{\mu}] = [f_1, S_{1,\mu_1}] \otimes [f_2, S_{2,\mu_2}] \in H_{k_1+k_2}(M \times M)
\]

holds. We prove this by applying both sides to \((k_1+k_2)\)-cohomology classes in \(M \times M\). Using the Künneth decomposition it is enough to evaluate on a form \(\beta = p_1^* \beta_1 \wedge p_2^* \beta_2\), where \(\beta_1, \beta_2 \in H^*(M)\) are closed forms and \(p_1, p_2 : M \times M \to M\) are the two projections. Let \(\{U_i\}, \{V_j\}\) be open covers of \(S_1, S_2\) respectively, by flow-boxes, and let \(\{\rho_{1,i}\}, \{\rho_{2,j}\}\) be partitions of unity subordinated to such covers. Then

\[
\langle [F, (S_1 \times S_2)_{\mu}], \beta \rangle = \\
\sum_{i,j} \int_{K(U_i) \times K(V_j)} (\int_{L_y \times L_z} (p_1^* \rho_{1,i}) (p_2^* \rho_{2,j}) F^*(p_1^* \beta_1 \wedge p_2^* \beta_2)) \, d\mu_{K(U_i) \times K(V_j)}(y,z) \\
= \sum_{i,j} \int_{K(U_i) \times K(V_j)} (\int_{L_y \times L_z} p_1^*(\rho_{1,i} f_1^* \beta_1) \wedge p_2^*(\rho_{2,j} f_2^* \beta_2)) \, d\mu_{1,K(U_i)}(y) d\mu_{2,K(V_j)}(z) \\
= \left( \sum_i \int_{K(U_i)} \left( \int_{L_y} \rho_{1,i} f_1^* \beta_1 \right) d\mu_{1,K(U_i)}(y) \right) \left( \sum_j \int_{K(V_j)} \left( \int_{L_z} \rho_{1,j} f_2^* \beta_2 \right) d\mu_{2,K(V_j)}(y) \right) \\
= \langle [f_1, S_{1,\mu_1}], \beta_1 \rangle \langle [f_2, S_{2,\mu_2}], \beta_2 \rangle,
\]

as required.

Now we are ready to prove the statement of the theorem. Let \(\varphi : M \to \Delta\) be the natural diffeomorphism of \(M\) with the diagonal \(\Delta \subset M \times M\), and let \(i : \Delta \hookrightarrow M \times M\) be the inclusion. Then, using theorem [15,6]

\[
[f, S_{\mu}]^* = [\varphi \circ f, S_{\mu}]^* = i^*([F, (S_1 \times S_2)_{\mu}]^*) = \\
i^*([f_1, S_{1,\mu_1}]^* \otimes [f_2, S_{2,\mu_2}]^*) = [f_1, S_{1,\mu_1}]^* \cup [f_2, S_{2,\mu_2}]^*.
\]

Let us look more closely to the case where \(k_1 + k_2 = n\). We assume that \((f_1, S_{1,\mu_1})\) and \((f_2, S_{2,\mu_2})\) are two oriented immersed measured solenoids of dimensions \(k_1, k_2\) respectively, which intersect transversally. Let \((f, S_{\mu})\) is the intersection 0-solenoid of \((f_1, S_{1,\mu_1})\) and \((f_2, S_{2,\mu_2})\).

**Definition 15.7. (Intersection index)** At each point \(x = (x_1, x_2) \in S\), the intersection index \(\epsilon(x_1, x_2) \in \{\pm 1\}\) is the sign of the intersection of the leaf of \(S_1\) through \(x_1\) with the leaf of \(S_2\) through \(x_2\). The continuous function \(\epsilon : S \to \{\pm 1\}\) gives the orientation of \(S\).
Recall that the 0-solenoid \((f, S_\mu)\) comes equipped with a natural measure \(\mu\) (for a 0-solenoid the notions of measure and transversal measure coincide). If \(x = (x_1, x_2) \in S\), then locally around \(x\), \(S\) is homeomorphic to \(T = T_1 \times T_2\), where \(T_1\) and \(T_2\) are small local transversals of \(S_1\) and \(S_2\) at \(x_1\) and \(x_2\), respectively. The measure \(\mu_T\) is the product measure \(\mu_{1,T_1} \times \mu_{2,T_2}\).

**Definition 15.8. (Intersection measure)** The intersection measure is the transversal measure \(\mu\) of the intersection solenoid \((f, S_\mu)\), induced by those of \((f_1, S_{1,\mu_1})\) and \((f_2, S_{2,\mu_2})\).

**Definition 15.9. (Intersection pairing)** We define the intersection pairing as the real number

\[
(f_1, S_{1,\mu_1}) \cdot (f_2, S_{2,\mu_2}) = \int_S \epsilon \, d\mu.
\]

**Theorem 15.10.** If \((f_1, S_{1,\mu_1})\) and \((f_2, S_{2,\mu_2})\) are two oriented immersed measured solenoids of dimensions \(k_1, k_2\) respectively, which intersect transversally, such that \(k_1 + k_2 = n\). Then

\[
(f_1, S_{1,\mu_1}) \cdot (f_2, S_{2,\mu_2}) = [f_1, S_{1,\mu_1}]^* \cdot [f_2, S_{2,\mu_2}]^*.
\]

**Proof.** By theorem 15.6,

\[
[f_1, S_{1,\mu_1}]^* \cup [f_2, S_{2,\mu_2}]^* = [f, S_\mu]^* \in H^n_c(M, \mathbb{R}).
\]

The intersection product \([f_1, S_{1,\mu_1}]^* \cdot [f_2, S_{2,\mu_2}]^*\) is obtained by evaluating this cup product on the element \(1 \in H^0(M, \mathbb{R})\), i.e.

\[
[f_1, S_{1,\mu_1}]^* \cdot [f_2, S_{2,\mu_2}]^* = \langle [f, S_\mu], 1 \rangle = \int_S f^*(1) d\mu(x) = \int_S \epsilon d\mu,
\]

since the pull-back of a function gets multiplied by the orientation of \(S\), which is the function \(\epsilon\). \(\square\)

When the solenoids are uniquely ergodic we can recover this intersection index by a natural limiting procedure.

**Theorem 15.11.** Let \((f_1, S_{1,\mu_1})\) and \((f_2, S_{2,\mu_2})\) be two immersed, oriented, uniquely ergodic solenoids transversally intersecting. Let \(l_1 \subset S_1\) and \(l_2 \subset S_2\) be two arbitrary leaves. Choose two base points \(x_1 \in l_1\) and \(x_2 \in l_2\), and fix Riemannian exhaustions \((l_{1,n})\) and \((l_{2,n})\). Define

\[
(f_1, l_{1,n}) \cdot (f_2, l_{2,n}) = \frac{1}{M_n} \sum_{p=(p_1,p_2) \in l_{1,n} \times l_{2,n}, f_1(p_1) = f_2(p_2)} \epsilon(p),
\]

where \(M_n = \text{Vol}_{k_1}(l_{1,n}) \cdot \text{Vol}_{k_2}(l_{2,n})\).
Then
\[
\lim_{n \to +\infty} (f_1, l_1, n) \cdot (f_2, l_2, n) = (f_1, S_1, \mu_1) \cdot (f_2, S_2, \mu_2).
\]
In particular, the limit exists and is independent of the choices of \(l_1, l_2, x_1, x_2\) and the radius of the Riemannian exhaustions.

**Proof.** The key observation is that because of the unique ergodicity, the atomic transversal measures associated to the normalized k-volume of the Riemannian exhaustions (name them \(\mu_{1,n}\) and \(\mu_{2,n}\)) are converging to \(\mu_1\) and \(\mu_2\), respectively. In particular, in each local flow-box we have
\[
\mu_{1,n} \times \mu_{2,n} \to \mu_1 \times \mu_2 = \mu.
\]
Therefore the average defining \((f_1, l_1, n) \cdot (f_2, l_2, n)\) converges to the integral defining \((f_1, S_1, \mu_1) \cdot (f_2, S_2, \mu_2)\) since \(\epsilon\) is a continuous and integrable function (indeed bounded by 1).

\(\Box\)

**Remark 15.12.** The previous theorem and proof work in the same form for ergodic solenoids, provided that we know that the Schwartzman limit measure for almost all leaves is the given ergodic measure. This is simple to prove for ergodic solenoids with trappings regions mapping to a contractible ball in \(M\) (cf. theorem [11.12]).

16. Almost everywhere transversality

The intersection theory developed in section [15] is not fully satisfactory since we do have examples of solenoids (e.g. foliations) which do not intersect transversally, and cannot even be perturbed to it. However, a weaker notion is enough to develop intersection theory for solenoids. Indeed, the intersection pairing can also be defined for oriented, measured solenoids \((f_1, S_1, \mu_1)\) and \((f_2, S_2, \mu_2)\) immersed in an oriented \(n\)-manifold \(M\), with \(k_1 + k_2 = n, k_1 = \dim S_1, k_2 = \dim S_2\), which intersect transversally almost everywhere. Let us first define this notion.

**Definition 16.1. (Almost everywhere transversality)** Let \((f_1, S_1, \mu_1)\) and \((f_2, S_2, \mu_2)\) be two measured immersed oriented solenoids. They intersect almost everywhere transversally if the set
\[
F = \{(p_1, p_2) \in S_1 \times S_2; \quad f_1(p_1) = f_2(p_2), df_1(p_1)(T_{p_1}S_1) + df_2(p_2)(T_{p_2}S_2) \neq T_{f_1(p_1)}M\} \subset S_1 \times S_2
\]
of non-transversal intersection points is null-transverse in \(S_1 \times S_2\) (with the natural product transversal measure \(\mu\)), i.e. if the set of leaves of \(S_1 \times S_2\) intersecting \(F\) has zero \(\mu\)-measure.

We recall that a set \(F \subset S_\mu\) in a measured solenoid is null-transverse if for any local transversal \(T\), the set of leaves passing through \(F\) intersects \(T\) is a set of zero \(\mu_T\)-measure.
Observe that when $\mathcal{N} \subset \mathcal{M}$ is a closed submanifold of codimension $k$ and $(f, S_\mu)$ is a measured immersed oriented $k$-solenoid and $\mathcal{N} \subset \mathcal{M}$, then they intersect almost everywhere transversally if and only if the subspace of non-transversal intersection points

$$F = \{ p \in S ; f(p) \in \mathcal{N}, df(p)(T_p S) + T_{f(p)} \mathcal{N} \neq T_{f(p)} \mathcal{M} \} \subset S$$

is null-transverse.

Then it is useful to translate to $S_1 \times S_2$ the meaning of almost everywhere transversality. We have the following straightforward lemma.

**Lemma 16.2.** The solenoids $(f_1, S_{1,\mu_1})$ and $(f_2, S_{2,\mu_2})$ are almost everywhere transversal if and only if $(f_1 \times f_2, (S_1 \times S_2)_{\mu})$ and the diagonal $\Delta \subset \mathcal{M} \times \mathcal{M}$ intersect almost everywhere transversally.

Let $(f, S_\mu)$ be an immersed solenoid intersecting transversally almost everywhere a closed submanifold $\mathcal{N} \subset \mathcal{M}$. Write $S' = f^{-1}(\mathcal{N})$ and let $F \subset S'$ be the subset of non-transversal points. Note that $F$ is closed, hence $S_{\text{reg}}' = S' - F$ is open in $S'$. Moreover, $S_{\text{reg}}'$ consists of the transversal intersections, so the intersection index $\epsilon : S_{\text{reg}}' \to \{ \pm 1 \}$ is well defined and continuous. We define the intersection number as

$$\int_{S' - F} \epsilon(x)d\mu(x).$$

**Theorem 16.3.** Suppose that $(f, S_\mu)$ and $\mathcal{N} \subset \mathcal{M}$ intersect almost everywhere transversally. Then

$$[f, S_\mu]^* \cdot [\mathcal{N}] = \int_{S' - F} \epsilon \ d\mu.$$

**Proof.** Fix an accessory Riemannian metric on $\mathcal{M}$. Let $(U_n)$ be a nested sequence of open neighbourhoods of $F$ in $S'$ such that $\bigcap_{n \geq 1} U_n = F$. Then

$$\int_{S' - U_n} \epsilon \ d\mu \to \int_{S' - F} \epsilon \ d\mu.$$

In $S' - U_n$ the angle of intersection between $f(S)$ and $\mathcal{N}$ is bounded below, so there is a small $\rho > 0$ (depending on $n$) such that if $U_\rho$ is the $\rho$-tubular neighbourhood of $\mathcal{N}$ in $\mathcal{M}$, then for each intersection point $x \in S' - U_n$, there is a (topological) disc $D_x$ contained in a local leaf through $x$, which is exactly the path component of $f^{-1}(U_\rho)$ through $x$.

Let $\tau_\rho$ be a Thom form for $\mathcal{N} \subset \mathcal{M}$, that is a closed $k$-form supported in $U_\rho$, whose integral in the normal space to $\mathcal{N}$ is one. Then $\int_{D_x} \tau_\rho = 1$ for any $x \in S' - U_n$. So

$$\int_{S' - U_n} \epsilon \ d\mu = \int_{A_n} f^* \tau_\rho,$$
where $A_n$ consists of those discs $D_x$ with $x \in S' - U_n$, and $B_n = f^{-1}(U_\rho) - A_n$, so that $f^{-1}(U_\rho) = A_n \cup B_n \subset S$. Note that $B_n$ is contained in a neighbourhood of $F$ slightly larger than $U_n$ (say $U_{n-1}$, taking $\rho$ small enough).

On the other hand,

$$[f, S_\mu]^*[N] = \langle [f, S_\mu], [\tau_\rho] \rangle = \int_{S_\mu} f^* \tau_\rho = \int_{A_n} f^* \tau_\rho + \int_{B_n} f^* \tau_\rho = \int_{S' - U_n} \epsilon \, d\mu + \int_{B_n} f^* \tau_\rho.$$  

Note that $\int_{B_n} f^* \tau_\rho$ is independent of $\rho$ small enough.

So we need to see that

$$\int_{B_n} f^* \tau_\rho \to 0,$$

when $n \to \infty$. As $B_n \subset U_{n-1}$ and $U_{n-1}$ has very small transversal measure, everything reduces to bound uniformly

$$\int_{L_y} f^* \tau_\rho,$$

for any leaf $L_y \subset B_n$. If $L_y$ is contained in a flow-box, then

$$\int_{L_y} f^* \tau_\rho = [L_y, \partial L_y] : [N]$$

is the intersection number of $(L_y, \partial L_y)$ with $N$. This is well-defined because the boundary of $L_y$ does not intersect $N$ (as $\partial L_y \subset \partial B_n$, we have that $\partial L_y \subset \partial f^{-1}(U_\rho)$, so $\partial L_y \cap N = \emptyset$). Moreover, by compactness of $S$, the quantity (20) must be bounded uniformly. If $L_y$ is not contained in a flow-box, it may be pieced into several components (the number of them is uniformly bounded). Then there are contributions of the boundaries of the components to (20), but they cancel each other after addition. \[\Box\]

Consider now two immersed measured oriented solenoids $(f_1, S_1, \mu_1), (f_2, S_2, \mu_2)$ intersecting almost everywhere transversally. Let $F \subset S_1 \times S_2$ be the subspace of non-transversal intersection points, which has null-transversal measure in $S_1 \times S_2$. Set $S = (S_1 \times S_2) \cap f^{-1}(\Delta)$. Then there is an intersection index $\epsilon(x)$ for each $x \in S - F$ and an intersection measure $\mu$ on $S - F$. We define the intersection product as

$$\int_{S - F} \epsilon \, d\mu.$$ 

Theorem 16.3 implies that

$$[f_1, S_{1, \mu_1}] \cdot [f_2, S_{2, \mu_2}] = \int_{S - F} \epsilon(x) \, d\mu(x).$$

**Remark 16.4.** Let $(f, S_\mu)$ be an embedded $k$-solenoid in a $n$-manifold with $n = 2k$. Assume that the transversal measure has no atoms. Then the set of non-transversal points $F \subset S \times S$ is the diagonal $\Delta_S \subset S \times S$. As the transversal measures of $S$ have
no atoms, \( F \) has null-transversal measure, so \( S \) intersects itself almost everywhere transversally. Moreover

\[
S' = (S \times S) \cap f^{-1}(\Delta) = \Delta_S,
\]

because \( f \) is injective. Therefore

\[
[f, S_\mu] \cdot [f, S_\mu] = \int_{\Delta S - F} \epsilon \ d\mu = 0.
\]

This gives another proof of theorem 7.6 in the case \( n = 2k \).

Moreover, we can always homotop solenoids so that they become almost everywhere transverse.

**Theorem 16.5.** Let \( (f, S_\mu) \) be an immersed \( k \)-solenoid in \( M \), and let \( N \subset M \) be a closed submanifold of codimension \( k \). Then there exists a homotopic immersion \( f_1 : S \to M \) such that \( (f_1, S_\mu) \) intersects \( N \) almost everywhere transversally.

**Proof.** Let \( U = D^k \times K(U) \) be a flow-box for \( S \), and \((u_1, \ldots, u_n)\) coordinates on an open subset \( W \subset M \) containing \( f(U) \) such that \( N \) is defined as \( u_{k+1} = \cdots = u_n = 0 \) in such coordinates. Take a smooth function \( \rho \) on \( S \) with \( \rho|_U > 0 \) and \( \rho|_{S - U} \equiv 0 \).

In these coordinates, the immersion \( f : U = D^k \times K(U) \to W \) is written as \( f(x, y) = (f_1(x, y), \ldots, f_n(x, y)) \). Take \( \epsilon > 0 \) small so that \( f(x, y) + v \in W \), for any \( v \in \mathbb{R}^n \) with \( |v| < \epsilon \). For each \( y \in K(U) \), consider the map

\[
\bar{f}_y : D^k \to \mathbb{R}^k, \\
x \mapsto (f_1(x, y), \ldots, f_k(x, y)).
\]

and the map \( \tilde{f}_y = f_y / \rho \).

We say that \( a \in D^k_\epsilon \) is a regular value for \( \tilde{f}_y \) if \( \det(df_y(x)) \neq 0 \) for each \( x \in D^k_\epsilon \) with \( \tilde{f}_y(x) = a \). Consider the set

\[
A := \{(y, a) \in K(U) \times D^k_\epsilon ; a \text{ is a regular value for } \tilde{f}_y\}
\]

is open. The set \( A_y = (\{y\} \times D^k_\epsilon) \cap A \) is the set of regular values of \( \tilde{f}_y \) on \( D^k_\epsilon \). By Sard’s theorem, this is of full measure in \( D^k_\epsilon \).

Let \( \nu \) be the product measure in \( K(U) \times D^k_\epsilon \), i.e. \( \nu = \mu_{K(U)} \times \lambda \), where \( \lambda \) is the standard Lebesgue measure in \( D^k_\epsilon \). Then the set \( A \) is a set of full \( \nu \)-measure. This implies that for \( \lambda \)-almost every \( a \in D^k_\epsilon \), the set

\[
C_a = \{y \in K(U) ; a \text{ is not a regular value for } \tilde{f}_y\} \subset K(U)
\]

is of zero \( \mu_{K(U)} \)-measure. Fix one such \( a \in D^k_\epsilon \). Then the map

\[
\tilde{f} : U = D^k \times K(U) \to W \\
\tilde{f}(x, y) = (f_1(x, y) - a_1 \rho(x, y), \ldots, f_k(x, y) - a_k \rho(x, y), f_{k+1}(x, y), \ldots, f_n(x, y))
\]
is transverse to \( N \) for all \( y \in K(U) - C_n \): take \( x \in D^k \) such that \( \tilde{f}(x, y) \in N \). Then \( \tilde{f}_y(x) = a \), so \( \det(d\tilde{f}_y(x)) \neq 0 \). But setting \( \tilde{f}_y(x) = (f_1(x, y) - a_1\rho(x, y), \ldots, f_k(x, y) - a_k\rho(x, y)) \), we have
\[
\det(d\tilde{f}_y(x)) = \det(d\tilde{f}_y(x)) - a d_x\rho = \rho(x, y) d\tilde{f}_y(x).
\]
So \( \det(d\tilde{f}_y(x)) \neq 0 \) (since \( \rho(x, y) > 0 \)) and \( \tilde{f} \) is transverse to \( N \) at \((x, y)\).

We extend \( \tilde{f} \) to the rest of \( S \) by setting \( \tilde{f}|_{S-U} = f|_{S-U} \). Then \( \tilde{f} : S \to M \) is an immersion homotopic to \( f \) and close to it, and it satisfies that it is almost everywhere transversal to \( N \) over \( U \).

Fixing a finite covering of \( S \) by flow-boxes, and repeating this process, we obtain an immersion homotopic to \( f \) and almost everywhere transversal to \( N \).

\textbf{Corollary 16.6.} Let \((f_1, S_{1,\mu_1})\) and \((f_2, S_{2,\mu_2})\) be two immersed solenoids in \( M \), with \( k_1 + k_2 = n \). Then we may homotop the immersion \( f_1 : S_1 \to M \) so that they intersect almost everywhere transversally.

\textbf{Proof.} Consider the solenoid \( S = S_1 \times S_2 \), and flow-boxes \( U_1 \) for \( S_1 \) and \( U_2 \) for \( S_2 \).

Fix a smooth function \( \rho_1 \) on \( S_1 \) such that \( \rho_1|_{U_1} > 0 \) and \( \rho_1|_{S_1-U_1} \equiv 0 \). We have no need of introducing a smooth function for \( S_2 \) since we shall only perturb \( S_1 \).

Using natural coordinates \((z_1, \ldots, z_n, w_1, \ldots, w_n)\) for \( M \times M \), the diagonal is defined by the equations \( z_1 - w_1 = 0, \ldots, z_n - w_n = 0 \). The immersion \( F : S_1 \times S_2 \to M \times M \) is given in coordinates as
\[
F(x_1, y_1, x_2, y_2) = (f_1(x_1, y_1), f_2(x_2, y_2)),
\]
for \((x_1, y_1) \in U_1 = D^{k_1} \times K(U_1), (x_2, y_2) \in U_2 = D^{k_2} \times K(U_2)\). So the map (21) is in this situation
\[
(x_1, x_2) \mapsto f_1(x_1, y_1) - f_2(x_2, y_2).
\]

As in theorem 16.5 we find \( a \in \mathbb{R}^n \) small enough which is a regular value of (22) divided by \( \rho_1(x_1, y_1) \), for \( \mu_1 \)-almost every \((y_1, y_2) \in K(U_1) \times K(U_2)\).

The map \( \tilde{f}_1 : U_1 \to M \) defined as \( \tilde{f}_1(x_1, y_1) = f_1(x_1, y_1) - a\rho_1(x_1, y_1) \) is almost everywhere transversal to \( U_2 \) along \( U_1 \). The rest of the argument is analogous to that at the end of the proof of theorem 16.5.

Observe that the intersection pairing can be defined for solenoids endowed with a transversal signed measure (resp. complex measure). The space of such solenoids is a real (resp. complex) vector space, where the sum of immersed solenoids is their disjoint union, and the zero is the empty set. With this vector space structure, we have

\textbf{Proposition 16.7.} The intersection pairing is bilinear and graded commutative.
This is a consequence of theorem 15.10. Now using theorem 14.8 we get Proposition 16.8. The intersection pairing is invariant by cobordism of immersed solenoids.

Definition 16.9. We define the space $S_k(M, \mathbb{R})$ of immersed measured solenoids modulo cobordism.

Theorem 16.10. The intersection pairing of solenoids,

$$S_k(M, \mathbb{R}) \times S_{n-k}(M, \mathbb{R}) \to \mathbb{R},$$

is well-defined, bilinear and graded-commutative. It factors, via the Ruelle-Sullivan map, through real homology.

Whenever two solenoids intersect almost everywhere transversally, the intersection pairing is defined geometrically. This extends the case of manifolds and homology with integer coefficients.
Appendix A. Norm on the homology

Let $M$ be a compact $C^\infty$ Riemannian manifold. For each $a \in H_1(M, \mathbb{Z})$ we define

$$l(a) = \inf_{[\gamma]=a} l(\gamma),$$

where $\gamma$ runs over all closed loops in $M$ with homology class $a$ and $l(\gamma)$ is the length of $\gamma$,

$$l(\gamma) = \int_\gamma ds_g.$$

By application of Ascoli-Arzela it is classical to get

**Proposition A.1.** For each $a \in H_1(M, \mathbb{Z})$ there exists a minimizing geodesic loop $\gamma$ with $[\gamma] = a$ such that

$$l(\gamma) = l(a).$$

Note that the minimizing property implies the geodesic character of the loop. We also have

**Proposition A.2.** There exists a universal constant $C_0 = C_0(M) > 0$ only depending on $M$, such that for $a, b \in H_1(M, \mathbb{Z})$ and $n \in \mathbb{Z}$, we have

$$l(n \cdot a) \leq |n| \ l(a),$$

and

$$l(a + b) \leq l(a) + l(b) + C_0.$$  

(We can take for $C_0$ twice the diameter of $M$.)

**Proof.** Given a loop $\gamma$, the loop $n\gamma$ obtained from $\gamma$ running through it $n$ times (in the direction compatible the sign of $n$) satisfies

$$[n\gamma] = n \ [\gamma],$$

and

$$l(n\gamma) = |n| l(\gamma).$$

Therefore

$$l(n \cdot a) \leq l(n\gamma) = |n| l(\gamma),$$

and we get the first inequality taking the infimum over $\gamma$.

Let $C_0$ be twice the diameter of $M$. Any two points of $M$ can be joined by an arc of length smaller than or equal to $C_0/2$. Given two loops $\alpha$ and $\beta$ with $[\alpha] = a$ and $[\beta] = b$, we can construct a loop $\gamma$ with $[\gamma] = a + b$ by picking a point in $\alpha$ and another point in $\beta$ and joining them by a minimizing arc which pastes together $\alpha$ and $\beta$ running through it back and forth. This new loop satisfies

$$l(\gamma) = l(\alpha) + l(\beta) + C_0,$$
therefore
\[ l(a + b) \leq l(\alpha) + l(\beta) + C_0. \]
and the second inequality follows. \(\square\)

**Remark A.3.** It is not true that \(l(n \cdot a) = nl(\gamma)\) if \(l(a) = l(\gamma)\). To see this take a surface \(M\) of genus \(g \geq 2\) and two elements \(e_1, e_2 \in H_1(M, \mathbb{Z})\) such that
\[ l(e_1) + l(e_2) < l(e_1 + e_2). \]
(For instance we can take \(M\) to be the connected sum of a large sphere with two small 2-tori at antipodal points, and let \(e_1, e_2\) be simple closed curves, non-trivial in homology, inside each of the two tori.) Let \(a = e_1 + e_2\). Then
\[ l(n \cdot a) = l(n \cdot (e_1 + e_2)) \leq nl(e_1) + n l(e_2) + C_0, \]
we get for \(n\) large
\[ l(n \cdot a) < nl(a). \]

**Theorem A.4. (Norm in homology)** Let \(a \in H_1(M, \mathbb{Z})\). The limit
\[ ||a|| = \lim_{n \to +\infty} \frac{l(n \cdot a)}{n}, \]
exists and is finite. It satisfies the properties

(i) For \(a \in H_1(M, \mathbb{Z})\), we have \(||a|| = 0\) if and only if \(a\) is torsion.

(ii) For \(a \in H_1(M, \mathbb{Z})\) and \(n \in \mathbb{Z}\), we have \(||n \cdot a|| = |n| ||a||\).

(iii) For \(a, b \in H_1(M, \mathbb{Z})\), we have
\[ ||a + b|| \leq ||a|| + ||b||. \]

(iv) \(||a|| \leq l(a)\).

**Proof.** Let \(u_n = l(n \cdot a) + C_0\). By the properties proved before, the sequence \((u_n)\) is sub-additive
\[ u_{n+m} \leq u_n + u_m, \]
therefore
\[ \limsup_{n \to +\infty} \frac{u_n}{n} = \liminf_{n \to +\infty} \frac{u_n}{n}. \]
Moreover, we have also
\[ \frac{u_n}{n} \leq l(a) < +\infty, \]
thus the limit exists and is finite. Property (iv) holds.

Property (ii) follows from
\[ ||n \cdot a|| = \lim_{m \to \infty} \frac{l(mn \cdot a)}{m} = |n| \lim_{m \to \infty} \frac{l(m|n| \cdot a)}{m|n|} = |n| ||a||. \]

Property (iii) follows from
\[ l(n \cdot (a + b)) \leq l(n \cdot a) + l(n \cdot b) + C_0 \leq n l(a) + n l(b) + C_0, \]
dividing by \( n \) and passing to the limit.

Let us check property (i). If \( a \) is torsion then \( n \cdot a = 0 \), so \( ||a|| = \frac{1}{n}||n \cdot a|| = 0 \). If \( a \) is not torsion, then there exists a smooth map \( \phi : M \to S^1 \) which corresponds to an element \([\phi] \in H^1(M, \mathbb{Z})\) with \( m = \langle [\phi], a \rangle > 0 \). Then for any loop \( \gamma : [0, 1] \to M \) representing \( n \cdot a, n > 0 \), we take \( \phi \circ \gamma \) and lift it to a map \( \tilde{\gamma} : [0, 1] \to \mathbb{R} \). Thus

\[
\tilde{\gamma}(1) - \tilde{\gamma}(0) = \langle [\phi], n \cdot a \rangle = mn.
\]

Now let \( C \) be an upper bound for \( |d\phi| \). Then

\[
mn = |\tilde{\gamma}(1) - \tilde{\gamma}(0)| = l(\phi \circ \gamma) \leq C \cdot l(\gamma),
\]

so \( l(\gamma) \geq mn/C \), hence \( l(n \cdot a) \geq mn/C \) and \( ||a|| \geq m/C \). \( \square \)

Now we can define a norm in \( H^1(M, \mathbb{Q}) = \mathbb{Q} \otimes H^1(M, \mathbb{Z}) \) by

\[
||\lambda \otimes a|| = |\lambda| \cdot ||a||,
\]

and extend it by continuity to \( H^1(M, \mathbb{R}) = \mathbb{R} \otimes H^1(M, \mathbb{Z}) \).

### Appendix B. De Rham cohomology of solenoids

In this appendix, we present the definition of the De Rham cohomology groups for solenoids. The general theory for foliated spaces from [MS, chapter 3] can be applied to our solenoids. In [MS], the required regularity is of class \( C^{\infty,0} \), but it is easy to see that the arguments extends to the case of regularity of class \( C^{1,0} \).

Let \( S \) be a \( k \)-solenoid of class \( C^{r,s} \) with \( r \geq 1 \). The space of \( p \)-forms \( \Omega^p(S) \) consists of \( p \)-forms on leaves \( \alpha \), such that \( \alpha \) and \( d\alpha \) are of class \( C^{1,0} \). Note that the exterior differential

\[
d : \Omega^p(X) \to \Omega^{p+1}(X)
\]

is the differential in the leaf directions. We can define the De Rham cohomology groups of \( S \) as usual,

\[
H^p_{DR}(S) := \frac{\ker(d : \Omega^p(S) \to \Omega^{p+1}(S))}{\operatorname{im}(d : \Omega^{p-1}(S) \to \Omega^p(S))}.
\]

The natural topology of the spaces \( \Omega^p(X) \) gives a topology on \( H^p_{DR}(S) \), so this is a topological vector space, which is in general non-Hausdorff. Quotienting by \( \{0\} \), the closure of zero, we get a Hausdorff space

\[
\hat{H}^p_{DR}(S) = \frac{H^p_{DR}(S)}{\{0\}} = \frac{\ker(d : \Omega^p(S) \to \Omega^{p+1}(S))}{\operatorname{im}(d : \Omega^{p-1}(S) \to \Omega^p(S))}.
\]

We define the solenoid homology as

\[
H_k(S) := \operatorname{Hom}_{cont}(H^k_{DR}(S), \mathbb{R}) = \operatorname{Hom}_{cont}(\hat{H}^p_{DR}(S), \mathbb{R}),
\]
the continuous homomorphisms from the cohomology to \( \mathbb{R} \). For a manifold \( M \), \( H^k_{DR}(M) \) and \( H_k(M) \) are equal to the usual cohomology and homology with real coefficients.

**Definition B.1. (Fundamental class)** Let \( S \) be an oriented \( k \)-solenoid with a transversal measure \( \mu = (\mu_T) \). Then there is a well-defined map giving by integration of \( k \)-forms

\[
\int_{S_\mu} (\cdot) : \Omega^k(S) \to \mathbb{R},
\]

by assigning to any \( \alpha \in \Omega^k(S) \) the number

\[
\int_{S_\mu} \alpha := \sum_i \int_{K(U_i)} \left( \int_{L_y \cap S_i} \alpha(x, y) dx \right) d\mu_{K(U_i)}(y),
\]

where \( S_i \) is a finite measurable partition of \( S \) subordinated to a cover \( \{U_i\} \) by flow-boxes. It is easy to see, as in section 7, that \( \int_{S_\mu} d\beta = 0 \) for any \( \beta \in \Omega^{k-1}(S) \). Hence \( \int_{S_\mu} \) gives a well-defined map

\[
H^k_{DR}(S) \to \mathbb{R}.
\]

Moreover, this is a continuous linear map, hence it defines an element

\[
[S_\mu] \in H_k(S).
\]

We shall call \([S_\mu]\) the fundamental class of \( S_\mu \).

The following result is in \[MS\] theorem 4.27.

**Theorem B.2.** Let \( S \) be a compact, oriented \( k \)-solenoid. Then the map

\[
\mathcal{V}_T(S) \to H_k(S),
\]

which sends \( \mu \mapsto [S_\mu] \), is an isomorphism from the space of all signed transversal measures to the \( k \)-homology of \( S \).

The set of transversal measures \( \mathcal{M}_T(S) \) is a cone, which generates \( \mathcal{V}_T(S) \). Its extremal points are the ergodic transversal measures. These ergodic measures are linearly independent. Therefore, the dimension of \( H_k(S) \) coincides with the number of ergodic transversal measures of \( S \). Hence, if \( S \) is uniquely ergodic, then \( H_k(S) \cong \mathbb{R} \), and \( S \) has a unique fundamental class (up to scalar factor). The uniquely ergodic solenoids are the natural extension of compact manifolds without boundary. For a compact, oriented, uniquely ergodic \( k \)-solenoid \( S \), there is a Poincaré duality pairing,

\[
H^d_{DR}(S) \otimes H^{k-d}_{DR}(S) \to H^k_{DR}(S) \xrightarrow{\int_{S_\mu}} \mathbb{R},
\]

where \( \mu \) is the transversal measure (unique up to scalar).
A real vector bundle of rank $m$ over a solenoid $S$ is defined as follows. A rank $m$ vector bundle over a pre-solenoid $(S, W)$ is a rank $m$ vector bundle $\pi: E_W \to W$ whose transition functions

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \to \text{GL}(m, \mathbb{R})$$

are of class $C^{r,s}$. We denote $E = \pi^{-1}(S)$, so that there is a map $\pi: E \to S$. Let $(S, W_1)$ and $(S, W_2)$ be two equivalent pre-solenoids, with $f: U_1 \to U_2$ a diffeomorphism of class $C^{r,s}$, $S \subset U_1 \subset W_1$, $S \subset U_1 \subset W_1$ and $f|_S = \text{id}$, then we say that $\pi_1: E_{W_1} \to W_1$ and $\pi_2: E_{W_2} \to W_2$ are equivalent if $\pi_1^{-1}(S) = \pi_2^{-1}(S) = E$ and there exists a vector bundle isomorphism $\hat{f}: E_{W_1} \to E_{W_2}$ covering $f$ such that $\hat{f}$ is the identity on $E$. Finally a vector bundle $\pi: E \to S$ over $S$ is defined as an equivalence class of such vector bundles $E_W \to W$ by the above equivalence relation.

Note that the total space $E$ of a rank $m$ vector bundle over a $k$-solenoid $S$ inherits the structure of a $(k + m)$-solenoid (although non-compact).

A vector bundle $E \to S$ is oriented if each fiber $E_p = \pi^{-1}(p)$ has an orientation in a continuous manner. This is equivalent to ask that there exist a representative $E_W \to W$ (where $W$ is a foliated manifold defining the solenoid structure of $S$) which is an oriented vector bundle over the $(k + l)$-dimensional manifold $W$.

Let $S$ be a solenoid of class $C^{r,s}$ with $r \geq 1$, and let $E \to S$ be a vector bundle. We may define forms on the total space $E$. A form $\alpha \in \Omega^p(E)$ is of vertical compact support if the restriction to each fiber is of compact support. The space of such forms is denoted by $\Omega^p_{cv}(E)$. Note that this condition is preserved under differentials, so it makes sense to talk about the cohomology with vertical compact supports,

$$H^*_{cv}(E) = \frac{\ker(d: \Omega^p_{cv}(E) \to \Omega^{p+1}_{cv}(E))}{\text{im}(d: \Omega^{p-1}_{cv}(E) \to \Omega^p_{cv}(E))}.$$ 

**Definition B.3. (Thom form)** A Thom form for an oriented vector bundle $E \to S$ of rank $m$ over a solenoid $S$ is an $m$-form

$$\Phi \in \Omega^m_{cv}(E),$$

such that $d\Phi = 0$ and $\Phi|_{E_p}$ has integral 1 for each $p \in S$ (the integral is well-defined since $E$ is oriented).

By the results of [MS], Thom forms exist. They represent a unique class in $H^m_{cv}(E)$, i.e. if $\Phi_1$ and $\Phi_2$ are two Thom forms, then there is a $\beta \in \Omega^{m-1}_{cv}(E)$ such that $\Phi_2 - \Phi_1 = d\beta$. Moreover, the map

$$H^k(S) \to H^{m+k}_{cv}(E)$$

given by

$$[\alpha] \mapsto [\Phi \wedge \pi^*\alpha],$$

is an isomorphism.
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