Quantum Yang–Mills theory can be rewritten in terms of gauge-invariant variables: it has the form of the so-called BF gravity, with an additional ‘æther’ term. The BF gravity based on the gauge group $SU(N)$ is actually a theory of high spin fields (up to $J = N$) with high local symmetry mixing up fields with different spins — as in supergravity but without fermions. As $N \to \infty$, one gets a theory with an infinite tower of spins related by local symmetry, similar to what one has in string theory. We thus outline a way of deriving a string theory from the local Yang–Mills theory in the large $N$ limit.
lously addressed before. The main finding of this paper is that dual gauge invariance translates into an exciting new symmetry, namely a local symmetry under mixing fields with different spins.

This symmetry can be paralleled to supergravity which is invariant under local rotations of fields carrying integer and half-integer spins, and to string theory where an infinite tower of spins is related by an infinite-dimensional algebra. In contrast to the former theory, we have only boson fields, and the number of higher spins can be arbitrary. In contrast to the latter theory, we can have a finite number of higher spins. Only in the limit \( N \to \infty \) of the \( SU(N) \) gauge group the symmetry relates an infinite tower of spins. However, it is only in the large \( N \) limit that the Yang–Mills theory is expected to be equivalent to string theory.

The paper is organized as follows. In the next section we explain why the ‘mixed’ term of the first order formalism is diffeomorphism-invariant. In Sec. 3 we write down the \( SU(2) \) Yang–Mills theory in \( d = 4 \) in terms of gauge-invariant variables. In Sec. 4 we concentrate on \( d = 3 \) theories where we are able to go further than in \( d = 4 \). In Sec. 5 we briefly recall the solution for the \( SU(2) \) case. In Sec. 6 we proceed to higher \( SU(N) \) groups and introduce gauge-invariant variables. These turn out to be fields carrying spin from zero to \( N \); each spin appears twice, except the ‘edge’ spins 0, 1, \( N - 1 \) and \( N \), which appear only once. We point out the transformation of those spins through one another, which leaves the BF action invariant. Finally, we speculate that the noninvariant ‘æther’ term lifts the degeneracy of spins and gives rise to a finite string slope \( \alpha' \).

2. Hidden diffeomorphism invariance of Yang–Mills theory in flat space

The key observation is the following. Let us rewrite the partition function \( Z \) over dual field strength variables \( A_\mu \) and \( G_{\mu\nu} \):

\[
Z = \int D A_\mu \exp \int d^4x \left( -\frac{1}{2g^2} \text{Tr} F_{\mu\nu} F_{\mu\nu} \right)
\]

\[
+ \int DA_\mu DG_{\mu\nu} \exp \int d^4x \left( -\frac{g^2}{2} \text{Tr} G_{\mu\nu} G_{\mu\nu} \right)
\]

\[
+ \frac{i}{2} \epsilon^{\alpha\beta\mu\nu} \text{Tr} \left( G_{\alpha\beta} F_{\mu\nu} \right),
\]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu A_\nu] \) is the standard Yang–Mills field strength and \( \epsilon^{\alpha\beta\mu\nu} \) is the antisymmetric tensor. To avoid possible confusion we write down explicitly all indices. To be specific, the gauge group is \( SU(N) \) with \( N^2 - 1 \) generators \( t^a \), \( \text{Tr} t^a t^b = \delta^{ab}/2 \).

Both terms in Eq. (1) are invariant under the \((N^2 - 1)\)-function gauge transformation

\[
\delta A_\mu = [D_\mu, \alpha],
\]

\[
\delta G_{\mu\nu} = [G_{\mu\nu}, \alpha],
\]

(2)

where \( D_\mu = \partial_\mu - iA_\mu t^a \) is the Yang–Mills covariant derivative, \( [D_\mu D_\nu] = -if_{\mu\nu} \).

Due to the Bianchi identity, \( \epsilon^{\mu\nu\rho\sigma} [D_\mu F_{\rho\sigma}] = 0 \), the second (mixed) term in Eq. (1) is, in addition, invariant under the \( 4 \cdot (N^2 - 1) \)-function ‘dual’ gauge transformation,

\[
\delta A_\mu = 0,
\]

\[
\delta G_{\mu\nu} = [D_\mu, \beta] - [D_\nu, \beta],
\]

(3)

Taking a particular combination of the functions in Eqs. (2), (3),

\[
\alpha = v^\mu A_\mu,
\]

\[
\beta = v^\lambda G_{\lambda\mu},
\]

(4)

leads to the transformation

\[
\delta G_{\mu\nu} = -G_{\lambda\nu} \partial_\mu v^\lambda - G_{\mu\lambda} \partial_\nu v^\lambda - \partial_\lambda G_{\mu\nu} v^\lambda,
\]

(5)

being the known transformation of a (covariant) tensor under general coordinate transformation. Therefore, the ‘mixed’ term is diffeomorphism-invariant, and is known as BF gravity.\(^1\) It defines a topological field theory of Schwarz type. Moreover, it is invariant not under four but under as much as \( 4 \cdot (N^2 - 1) \) local transformations; four diffeomorphisms are but their small subset. We shall see later on that the additional local transformations mix up fields with different spins.

The first term in Eq. (1) is not invariant under the dual gauge transformation (3), therefore it is not invariant under diffeomorphisms. For that reason, we call it the ‘æther’ term: it distinguishes the Yang–Mills theory from a non-propagating topological BF gravity represented by the second (mixed) term in the action (1).

3. \( SU(2), d = 4 \) BF gravity in a basis-independent formulation

In the first-order formalism, the integral (1) over the Yang–Mills connection \( A_\mu \) is Gaussian, and one can integrate it out. This was done many years ago (4), \( B \), but in contrast to that work we wish to write down the result of the integration in an explicitly gauge-invariant way. This was performed some time ago by Ganor and Sonnenschein (8). It has been shown that the resulting theory contains the Einstein–Hilbert action for the metric tensor and an additional 5-component self-dual field

\(^1\)With our notations it would be more appropriate to call it ‘GF gravity’ but we follow the tradition.
interacting with the metric. The final action of Ref. [5] is very lengthy, and its symmetry under a 12-function transformation has not been discussed. However, an important finding of Ref. [5] is the way one constructs the metric tensor out of the dual field strength \( G_{\alpha \beta} \).

In this section, we write down the result of the \( A_\mu \) integration in a compact form which makes clear the 12-function symmetry of the BF action. From the point of view of the Yang–Mills theory, it solves the problem of reformulating it in terms of local gauge-invariant variables. From the point of view of BF gravity, we rewrite it in a basis-invariant formalism.

The Gaussian integration over \( A_\mu \) in Eq. (1) is equivalent to the saddle-point approximation. The saddle point (which we denote by \( \tilde{A}_\mu \)) is found from varying the ‘mixed’ term in \( A_\mu \):

\[
\epsilon^{i \alpha \beta} \left[ D_\mu (\tilde{A}) G_{\alpha \beta} \right] = 0. \tag{6}
\]

We need to solve this equation with respect to the saddle-point YM connection \( \tilde{A}_\mu \) and to substitute it back into the BF action

\[
S_2 = \frac{i}{2} \int d^4 x \epsilon^{\alpha \beta \mu \nu} \text{Tr} \ G_{\alpha \beta} F_{\mu \nu}(\tilde{A}). \tag{7}
\]

The goal is to write down the result for \( S_2 \) through gauge-invariant combinations made of the dual field strength \( G_{\alpha \beta} \) and to reveal its 12-function symmetry.

### 3.1. Gauge-invariant variables

First of all, we need a convenient parametrization of \( G_{\alpha \beta} \), which have 6 \( \cdot \) 3 degrees of freedom (dof’s), out of which 18 \(-\) 3 = 15 are gauge-invariant. Our main variable will be an antisymmetric tensor \( T^i_{\alpha \beta} = - T^i_{\beta \alpha} \) (the Greek indices run from 1 to 4 whereas the Latin ones run from 1 to 3). Given \( T \), one constructs the quantity

\[
(\sqrt{g})^3 = \frac{1}{48} \left( \epsilon_{ijk} T^i_{\alpha \beta} T^j_{\gamma \delta} T^k_{\epsilon \mu} \right) (\epsilon_{\lambda \mu \nu} T^\lambda_{\epsilon \mu} T^\mu_{\gamma \delta} T^\nu_{\lambda \beta}) \times \epsilon^{\alpha \beta \gamma \delta \mu \nu} \epsilon^{\epsilon \lambda \mu \nu}.
\tag{8}
\]

With its aid we construct the contravariant antisymmetric tensor

\[
T^{i \mu \nu} \equiv \frac{1}{2 \sqrt{g}} \epsilon^{\mu \nu \alpha \beta} T^i_{\alpha \beta} \tag{9}
\]

and require the orthonormalization condition,

\[
T^i_{\alpha \beta} T^j_{\alpha \beta} = \delta^{ij}. \tag{10}
\]

This condition is ‘dimensionless’ in \( T \), therefore it imposes 5 rather than 6 constraints on the covariant tensor \( T^i_{\alpha \beta} \), which thus carries \( 18 - 5 = 13 \) dof’s. The general solution to Eq. (10) is given by

\[
T^i_{\alpha \beta} = \eta^i_{AB} e^A_\alpha e^B_\beta \tag{11}
\]

where \( e^A_\alpha \) can be called a tetrad; \( \eta^i_{AB} \) is the ’t Hooft symbol whose algebra is given in [6]. All algebraic statements of this section can be verified by exploiting the \( \eta \)-symbol algebra. There are 16 dof’s in the tetrad, however three rotations under one of the \( SO(3) \) subgroups of the \( SO(4) \) Euclidean group do not enter into the combination \{11\}, therefore the r.h.s. of Eq. (11) carries, as it should, 13 dof’s.

We next introduce the metric tensor,

\[
g_{\mu \nu} \equiv \frac{1}{6} \epsilon^{ijk} T^i_{\mu \alpha} T^j_{\alpha \beta} T^k_{\beta \nu} = \epsilon^A_\mu \epsilon^A_\nu. \tag{12}
\]

It explains the previous notation: Eq. (5) is consistent with the determinant of this metric tensor.

Finally, we parametrize the dual field strength as

\[
G^a_{\alpha \beta} = d^a_{\mu} T^i_{\alpha \beta} = d^a_{\mu} \eta^i_{AB} e^A_\alpha e^B_\beta \tag{13}
\]

where the new variable \( d^a_i \) (we shall call it a triad) is subject to the normalization constraint \( \det d^a_i = 1 \) and therefore contains 8 dof’s. In fact, the combination \{13\} is invariant under simultaneous \( SO(3) \) rotations of \( T^i \) and \( d_i \), therefore the r.h.s. of Eq. (13) contains \( 8 + 3 - 18 \) dof’s, as does the l.h.s. Thus, Eq. (13) is a complete parametrization of \( G^a_{\alpha \beta} \).

It is now clear how to organize the 15 gauge-invariant variables made of \( G^a_{\alpha \beta} \). These are the 5 dof’s contained in the symmetric \( 3 \times 3 \) tensor

\[
h_{ij} = \frac{d^a_i}{d^a_j}, \quad \det h = 1, \tag{14}
\]

and 13 dof’s of \( T^i_{\alpha \beta} \). However, \( h_{ij} \) and \( T^i_{\alpha \beta} \) will always enter contracted in \( i, j \) (as follows from Eq. \{13\}), so that the dof’s associated with the simultaneous \( SO(3) \) rotation will drop out. In other words, one can choose \( h_{ij} \) to be diagonal and containing only 2 dof’s.

### 3.2. Christoffel symbols, covariant derivative, Riemann tensor

We are now prepared to solve the saddle-point equation \{10\} and to express the BF action \{11\} in a nice geometric way. We substitute \{12\} into Eq. \{10\} and rewrite it as

\[
0 = \frac{1}{2} \epsilon^{i \mu \alpha \beta} D_{\mu} (\tilde{A}) (d^a_i T^i_{\alpha \beta}) = \frac{1}{\sqrt{g}} D_{\mu} (d^a_i \sqrt{g} T^{i \lambda \mu}) = \frac{1}{\sqrt{g}} \partial_{\mu} (\sqrt{g} T^{i \lambda \mu}) d^a_i + T^{i \lambda \mu} D_{\mu} d^a_i. \tag{15}
\]

The action of the YM covariant derivative on the triad can be decomposed in the triad again:

\[
D^a_{\mu} d^b_i = \gamma^a_{\mu \nu} a^a_{\nu} \tag{16}
\]

which serves as a definition of the ‘minor’ Christoffel symbol \( \gamma^a_{\mu \nu} \) (not to be confused with the ordinary Christoffel symbol in \( d = 4 \)). With its help we define the ‘minor’ covariant derivative,

\[
(\nabla_{\mu})^a_i \equiv \partial_{\mu} \delta^a_i + \gamma^a_{\mu \nu} \tag{17}
\]

and the ‘minor’ Riemann tensor,

\[
R^a_{i \mu \nu} = [\nabla_{\mu} \nabla_{\nu}]^a_i = \partial_{\nu} \gamma^a_{\mu \iota} - \partial_{\mu} \gamma^a_{\nu \iota} + \gamma^a_{\mu \kappa} \gamma^\kappa_{\nu \iota} - \gamma^a_{\nu \kappa} \gamma^\kappa_{\mu \iota}. \tag{18}
\]
The saddle point equation (15) can be compactly written as
\[
\left(\nabla_\mu\right)^2(\sqrt{g} T^{i\lambda\mu}) = 0, \quad \text{or}
\]
\[
T_{\kappa\lambda;\mu} + T_{\lambda\mu;\kappa} + T_{\mu\kappa;\lambda} = 0,
\]
meaning that the antisymmetric tensor \(T^{i\lambda\mu}\) is ‘covariantly constant’. Another consequence of Eqs. (15), (16) is that the symmetric tensor \(h_{ij}\) is covariantly constant, too:
\[
h_{ik;\mu} \overset{\text{def}}{=} \partial_\mu h_{ik} - \gamma^j_{\mu i} h_{kj} - \gamma^j_{\mu k} h_{ij} = 0.
\]
The ‘minor’ Christoffel symbol can be found explicitly; it consists of symmetric and antisymmetric parts:
\[
\gamma^j_{\mu i} = \frac{1}{2} h^{ji}(\partial_\mu h_{ni} + \epsilon_{njk} S^k_\mu),
\]
\[
S^k_\mu = T^{k\beta}_\nu T^{\nu\mu\beta} g^{\alpha\beta} h_{i\mu} \partial_\lambda (g h_{i\mu}),
\]
where we have used contravariant upper indices to denote the inverse matrices \(h^{in}\), \(g^{\alpha\beta}\).

Given the Christoffel symbol, one may return to Eq. (15) and find the saddle-point YM field \(\bar{A}_\mu\); it coincides with the old result of Refs. [4, 5]. However, we do not need an explicit form of \(\bar{A}_\mu\) to find the action (6) at the saddle point.

### 3.3. Action in terms of gauge-invariant variables

In order to find the Yang–Mills field strength \(F_{\mu\nu}\) at the saddle point we consider the double commutator of YM covariant derivatives,
\[
[D_\mu [D_\nu d_i]] = [D_\mu, [D_\nu d_i]] = d_j (\partial_\mu \gamma^j_{\nu i} + \gamma^j_{\mu k} \gamma^k_{\nu i}),
\]
and subtract the same commutator with \((\mu\nu)\) interchanged:
\[
[D_\mu [D_\nu d_i]] - [D_\nu [D_\mu d_i]] = - d_j [D_\mu D_\nu] = i [d_i, F_{\mu\nu}(\bar{A})] = d_j R^i_{j\mu\nu}.
\]
Hence we see that the YM curvature at the saddle point is expressed via the ‘minor’ Riemann tensor, Eq. (18).

Explicitly,
\[
F^a_{\mu\nu}(\bar{A}) = \frac{2}{\epsilon} e^{abc} d^a_i d^b_j R^i_{j\mu\nu}
\]
where the inverse triad, \(d^a_i d^b_j = \delta^a_i \epsilon_{ab}\), has been used; \(d^a_i = h^{im} d_m^i\). We put Eq. (23) into the action (6) and get finally
\[
S_2 = i \frac{2}{\epsilon} \int d^4 x \sqrt{g} R^i_{j\mu\nu} T^{i\mu\nu} \epsilon_{ijk} h^{jk}.
\]

This is the \(SU(2)\), \(d = 4\) BF gravity action in the gauge-invariant or basis-independent formulation, since it is expressed in terms of the gauge-invariant variables \(T\) and \(h\). We notice that it is covariant with respect to both Greek and Latin indices.

To get the full YM action in a gauge-invariant form one has to add the first (‘æther’) term of Eq. (1),
\[
S_1 = \frac{g^2}{4} \int d^4 x T^{i}_{\mu\nu} h_{ij} T^j_{\nu\mu},
\]
which is gauge- but not diffeomorphism-invariant.

In the particular case when \(h_{ij} = \delta_{ij}\), the action (24) can be rewritten in terms of the 4-dimensional metric \(g_{\mu\nu}\) being a particular combination of \(T^{i}_{\mu\nu}\), see Eq. (12). In this case the BF action (24) becomes the usual Einstein–Hilbert action,
\[
S_2 \bigg|_{h_{ij} = \delta_{ij}} = \frac{i}{2} \int d^4 x \sqrt{g} R,
\]
where \(R\) is the standard scalar curvature made of \(g_{\mu\nu}\).

Another particular case is an arbitrary (but constant) field \(h_{ij}\) and a conformally flat metric, \(g_{\mu\nu} = \Phi g_{\mu\nu}\). In this case (being of relevance to instantons) the BF action (23) is
\[
S_2 \bigg|_{\text{conf.flat}} = \frac{i}{2} \int d^4 x \left(-\partial^2 \Phi + \frac{1}{2\Phi} \partial_\lambda \Phi \partial^\lambda \Phi\right) \times (h_{ij} h_{jj} - 2 h_{jj} h_{ij}).
\]

Our choice of the gauge-invariant variables \(T, h\) is not imperative. For example, one can use the 15 variables
\[
W_{\alpha\beta\gamma\delta} \overset{\text{def}}{=} G^a_{\alpha\beta} G^a_{\gamma\delta} = T^{i}_{\alpha\beta} h_{ij} T^j_{\gamma\delta},
\]
or some other set of 15 variables, depending on what properties of the theory one wishes to fix upon.

### 3.4. Gauge-invariant perturbation theory

A seeming paradox is that the Yang–Mills theory has gluon degrees of freedom at short distances, whereas in a gauge-invariant formulation there is no place for explicitly colour degrees of freedom. We shall show now that Eqs. (23), (24) possess two transversely polarized gluons (times 3 colours). This is the correct gauge-invariant content of the perturbation theory at zero order.

Since \(S_1\) is proportional to the coupling constant and \(S_2\) is not, the zero order corresponds to \(S_2 = 0\), i.e., to the ‘minor’ Riemann tensor \(R^i_{j\mu\nu} = 0\), that is to the flat dual space. It implies that the ‘minor’ Christoffel symbol \(\gamma^i_{\mu j}\) is a “pure gauge”,
\[
\gamma^i_{\mu j} = (O^{-1})^i_{k} \partial_\mu O^k_j, \quad \det O \neq 0.
\]

Indeed, in this case the Riemann tensor is zero:
\[
(\nabla_\mu)^j_{\alpha} c^j = \partial_\mu c^j + \partial^i_{\alpha j} c^j = (O^{-1})^i_{k} \partial_\mu (O^k_j c^j);
\]
\[
R^i_{j\mu\nu} c^j = [(\nabla_\mu)^i_{\alpha} (\nabla_\nu)^j = (\mu \leftrightarrow \nu)] c^j = (O^{-1})^i_{k} \partial_\mu \partial_\nu (O^k_j c^j) - (\mu \leftrightarrow \nu) = 0
\]
for any vector \(c^j\), therefore \(R^i_{j\mu\nu} = 0\).
We substitute Eq. (31) into Eq. (23) and get
\[ 0 = h_{ik;} = \partial_{\mu} \left[ \left( O^{-1} \right)^{\mu}_{i} \partial_{\nu} \left( O^{-1} \right)^{\nu}_{k} \right] O^{\mu}_{i} O^{\nu}_{k} \]
(32)
meaning that \( h_{pq} = O^{p}_{i} O^{q}_{i} D_{ij} \) where \( D_{ij} \) is a constant matrix. Next, we substitute Eq. (30) into Eq. (19) and obtain
\[ \partial_{\nu} \left( O^{\nu}_{i} T^{\mu}_{j} \right) + \partial_{\lambda} \left( O^{\nu}_{i} T^{\mu}_{j \lambda} \right) + \partial_{\mu} \left( O^{\nu}_{i} T^{\mu}_{j \lambda} \right) = 0, \]
(33)
whose general solution is \( O^{\nu}_{i} T^{\mu}_{j} = \partial_{\nu} B_{i}^{\lambda} - \partial_{\lambda} B_{i}^{\nu} \). The first term in the action \((1)\) is then
\[ G^{a}_{\mu \nu} G^{a}_{\mu \nu} = h_{pq} T^{p}_{\mu \nu} T^{q}_{\mu \nu} = D_{ij} \left( O^{p}_{i} T^{q}_{\mu \nu} \right) \]
\[ = D_{ij} \left( \partial_{\nu} B_{i}^{\lambda} - \partial_{\lambda} B_{i}^{\nu} \right) \left( \partial_{\nu} B_{j}^{\lambda} - \partial_{\lambda} B_{j}^{\nu} \right), \]
(34)
\( D_{ij} \) is a constant matrix and can be set to be \( \delta_{ij} \) by a linear transformation of the three vector fields \( B_{i}^{\lambda} \); therefore, we obtain the Lagrangian of three massless gauge fields. It is an expected result.

In Ref. [1], the correct renormalization of the gauge coupling constant has been demonstrated in the first-order formalism. It would be most instructive to follow how “11/3” of the Yang–Mills \( \beta \) function arises in the gauge-invariant formulation.

### 3.5. More general relativity

As discussed in Sec. 2, the BF action \( S_{2} \) is invariant under 12-function dual gauge transformations, 4 of which are the general coordinate transformations or diffeomorphisms. In this subsection we describe how this 12-function symmetry is translated after one integrates out the YM connection and arrives at the gauge-invariant action \((2)\).

The dual gauge transformation \((2)\) can be written as
\[ \delta G^{a}_{\mu \nu} = \partial_{\mu} b^{ab}_{\nu} - \partial_{\nu} b^{ab}_{\mu}, \]
(35)
We decompose 12 functions \( b^{ab}_{\mu} \) in the triad basis,
\[ b^{ab}_{\mu} = z_{i}^{a} d_{i}^{b}, \]
(36)
where \( z_{i}^{a} \) is another set of 12 arbitrary infinitesimal functions. Putting Eq. (34) into Eq. (35) and using Eq. (36), we obtain the variation
\[ \delta G^{a}_{\nu \mu} = d_{i}^{a} p_{i}^{\nu \mu}, \]
(37)
\[ p_{i}^{\nu \mu} = \left( \nabla_{\mu} \right) z_{i}^{\nu} - \left( \nabla_{\nu} \right) z_{i}^{\mu}, \]
(38)
where \( \nabla_{\mu} \) is the covariant derivative \((17)\). Importantly, we have excluded the YM connection \( A_{\mu} \) from the variation by using the saddle-point equation \((16)\). The variation \((17)\) is written in terms of the variables entering into \( S_{2} \) after the Gaussian integration over \( A_{\mu} \) is performed. Therefore, \( S_{2} \) in the resulting form of Eq. (25) is, by construction, invariant under the 12-function variation \((17)\).

We need to derive the transformation laws for the gauge-invariant quantities \( g_{\mu \nu}, T_{\mu \nu}^{a}, h_{ij} \), that follow from Eq. (37). First of all, we find the transformation law for the metric tensor \((12)\) which can be written as
\[ g_{\mu \nu} = \frac{1}{6} \epsilon^{\alpha \beta \rho \sigma} G_{\mu \alpha} G_{\nu \beta} G_{\rho \sigma}, \]
(39)
The variation of the metric tensor under the transformation \((37)\) is
\[ \delta g_{\mu \nu} = \frac{1}{2} \left( g_{\mu \nu} p_{i}^{\mu \nu} + g_{\nu \mu} p_{i}^{\nu \mu} - \frac{1}{3} g_{\mu \nu} p_{i}^{\alpha \beta} T^{i \alpha \beta} \right), \]
(40)
\[ \frac{\delta g}{g} = \frac{1}{3} h^{i \alpha \beta} T^{i \alpha \beta}. \]
(41)
The variation of the covariant tensor \( T_{\mu \nu}^{a} \) is
\[ \delta T_{\mu \nu}^{a} = p_{i}^{a \mu \nu} - Q_{i}^{a \mu \nu}, \]
(42)
\[ Q_{i}^{a \mu \nu} = \frac{1}{4} \left( \delta_{i}^{\alpha \beta \mu} - \frac{1}{4} \delta_{i}^{\alpha \beta \mu} \right) p_{i \alpha \beta \mu \nu} T^{a \alpha \beta \mu \nu}, \]
(43)
The variation of the contravariant tensor \( T^{i \mu \nu} \) can be found from
\[ \delta T^{i \mu \nu} = \frac{\epsilon_{\mu \nu \alpha \beta \lambda}}{2 \sqrt{g}} \left( T^{i \alpha \beta \lambda} - \frac{1}{2} \frac{\delta g}{g} T_{\lambda}^{i \alpha \beta} \right), \]
(44)
which supports the self-duality property, Eq. (1).

Finally, we find the transformation of \( h_{ij} \) to be
\[ \delta h_{ij} = Q_{j}^{k} h_{ik} + h_{ik} Q_{j}^{k}, \]
(45)
which supports \( \det h=1 \) under the variation. It should be noted that the variations \((12), (14), (15)\) are written up to possible \( SO(3) \) rotations in the Latin indices.

A special 4-function subclass of transformations are diffeomorphisms. This particular set of transformations is obtained by choosing in Eq. (38)
\[ z_{i}^{\mu} = T_{\mu \lambda}^{i} v^{\lambda}, \]
(46)
where \( v^{\lambda} \) is the infinitesimal displacement vector, \( x^{\lambda} \rightarrow x^{\lambda} + v^{\lambda}(x) \). It corresponds to taking
\[ p_{i}^{a \mu \nu} = \partial_{\mu} v^{\lambda} T_{\nu \lambda}^{i} - \partial_{\nu} v^{\lambda} T_{\mu \lambda}^{i} - v^{\lambda} \left( \nabla_{\lambda} \right) T_{\mu \nu}^{i}. \]
(47)
It is a matter of simple algebra to verify that on this subclass the variation of the metric tensor \((40)\) becomes
\[ \delta g_{\mu \nu} \mid_{\text{diff}} = -g_{\mu \lambda} \partial_{\nu} v^{\lambda} - g_{\nu \mu} \partial_{\nu} v^{\lambda} - \partial_{\lambda} g_{\mu \nu} v^{\lambda}, \]
(48)
which is the usual transformation under diffeomorphisms. Similarly, one finds that under general coordinate transformations \( T_{\mu \nu}^{a} \) transforms as a covariant tensor, while \( h_{ij} \) is a world scalar,
\[ \delta h_{ij} \mid_{\text{diff}} = -\partial_{\lambda} h_{ij} v^{\lambda}. \]
(49)

In general, however, the BF action \((25)\) is invariant not only under 4-function diffeomorphisms but under full 12-function transformations described above. The additional 8-function transformations mix, in a nonlinear way, the fields \( h_{ij} \) and \( T_{\mu \nu}^{a} \), i.e., world scalars with covariant tensors. In other words, the BF action has a
large local symmetry which mixes fields with different spin content.

This symmetry is, of course, a consequence of the invariance of the BF action under the dual gauge transformations \( \delta e \): it reveals itself when one integrates out the Yang–Mills connection \( A_i \). The number of free functions determining the dual gauge transformation is \( 4 \cdot (N^2 - 1) \) for the \( SU(N) \) gauge group; for \( SU(2) \) it is 12. For higher groups there will be more degrees of freedom in the symmetry. Simultaneously, for higher groups the BF action will involve higher spin fields, and the invariance under dual gauge transformation will be translated, after excluding the YM connection, into the invariance under mixing higher spins.

So far we have not developed the BF theory for higher groups in \( d = 4 \) but only in \( d = 3 \) where the formalism is simpler. Therefore, in the rest of the paper we concentrate on the gauge-invariant formulation of the Yang–Mills theory in \( d = 3 \). We shall show that when one integrates out the Yang–Mills connection from the BF action, one obtains a theory of fields carrying spin up to \( J = N \) (for the \( SU(N) \) gauge group), and that the invariance under dual gauge transformations is translated into a local symmetry which mixes all those spins.

4. First-order formalism in \( d = 3 \)

In Euclidean \( d = 3 \) dimensions the Yang–Mills partition function in the first-order formalism reads:

\[
Z = \int DA_i^a \exp \left( -\frac{1}{4g^2} \int d^3x \, F_{ij}^a F_{ij}^a \right)
\]

\[
\int De_i^a \int DA_i \exp \left( -\frac{g^2}{2} e_i^a e_i^a + i \frac{2}{3} e^{ijk} F_{ij}^a e_j^a \right).
\]

(50)

We use the Latin indices \( i, j, k \ldots = 1, 2, 3 \) to denote spatial directions. The quantities \( e_i^a \) are analogues of the dual field strength \( G_{\mu
u}^a \) of \( d = 4 \); in three dimensions they are vectors. Similarly to \( d = 4 \), in addition to invariance under gauge transformations,

\[
\delta A_i^a = D_i^a \alpha^b, \quad \delta e_i^a = f^{abc} e_j^b \alpha^c,
\]

(51)

the second term in Eq. (50) is invariant under the local \((N^2 - 1)\)-function dual gauge transformation

\[
\delta A_i^a = 0, \quad \delta e_i^a = D_i^a (A) \beta^b.
\]

(52)

As in \( d = 4 \) (see section 2), there is a combination of the transformation functions \( \alpha, \beta \) such that the ‘bein’ \( e_i^a \) transforms as a vector under general coordinate transformations. Therefore, it is guaranteed that the ‘mixed’ term in Eq. (51) is diffeomorphism-invariant. Moreover, along with the 3-function diffeomorphisms, there is an additional local \((N^2 - 1 - 3)\)-function symmetry. It has the form of the dual gauge transformation (52) if one uses the standard form of the BF action but becomes something extremely interesting when one integrates out the YM connection \( A_i^a \) and writes down the ‘mixed’ term of the action in a basis-independent form.

The integration over \( A_i^a \) is Gaussian. The saddle point \( \bar{A}_i^a \) is found from

\[
e^{ij} D_i^a (\bar{A}) e_j^b = 0.
\]

(53)

In the simplest case of the \( SU(2) \) gauge group this equation can be solved in a nice way \([2, 3, 14]\), and we remind it in the next section.

5. \( SU(2), d = 3 \)YM theory in

gauge-invariant terms

A general solution of the saddle-point Eq. (53) in the case of the \( SU(2) \) gauge group is given by

\[
D_i^a (\bar{A}) e_j^b = \Gamma_{ij}^k e_k^a, \quad \Gamma_{ij}^k = \Gamma_{ji}^k.
\]

(54)

Following Lunev \([3]\), we call \( e_i^a \) a dreibein and construct the metric tensor

\[
g_{ij} = e_i^a e_j^a, \quad g^{ij} = e_i^a e_j^a, \quad e_i^a e^b_i = \delta^{ab}.
\]

(55)

Taking \( \partial_i g_{ij} \) and using Eq. (54), one finds that \( \Gamma_{ij}^k \) is the standard Christoffel symbol,

\[
\Gamma_{ij,k} = \frac{1}{2} (\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}), \quad \Gamma_{ij}^k = g^{kl} \Gamma_{ij,l}^l.
\]

(56)

whereas the saddle-point \( \bar{A}_i^a \) is the standard spin connection made of the dreibein,

\[
\bar{A}_i^a = -\frac{1}{4} e^{abc} \omega_i^{bc}, \quad \omega_i^{bc} = \frac{1}{2} \left[ e^{bk} (\partial_i e_k^c - \partial_k e_i^c) - e^{dl} e^{cm} e_i^a \partial_l e_m^c \right] - (b \leftrightarrow c).
\]

(57)

We recall the covariant derivative in curved space, \( d = 3 \),

\[
(\nabla_i)_{ij}^k = \partial_i \delta^k_j + \Gamma_{ij}^k,
\]

(58)

and build the Riemann tensor which appears to be related to the YM field strength at the saddle point:

\[
[\nabla_i \nabla_j]_k = R_{ij}^k + \epsilon^{abc} F_{ij}^a (\bar{A}) e^{bk} e_j^c.
\]

(59)

The two terms in the action (14) become

\[
e^{ij} F_{ij}^a (\bar{A}) e_j^b = \sqrt{g} R_{ij}^k g^{il} = \sqrt{g} R.
\]

(60)

Thus, the YM partition function can be rewritten in terms of the local gauge-invariant variables \( g_{ij} \) being the metric of the dual space \([13, 14]\):

\[
Z = \int Dg_{ij} g^{-\frac{d}{2}} \exp \int d^3x \left( -\frac{g^2}{2} g_{ii} + \frac{i}{2} \sqrt{g} R \right).
\]

(61)
The second term is the Einstein–Hilbert action, while the first term is not diffeomorphism-invariant, so we call it the ‘æther’ term. The functional integration measure in Eq. (61) is obtained as follows [14, 15]: first, one divides the integration over 9 components of the dreibein $e^a_i$ into integration over three rotations of the dreibein (since the action is gauge-invariant, one can always normalize the integral over three Euler angles to unity and cancel it out) and over six components of $g_{ij}$,

$$d^{(9)}e^a_i = d^{(3)}O^{ab} d^{(6)}g_{ij} g^{-2}.$$  \hspace{1cm} (62)

Second, there is another factor of $(\det e^a_i)^{-3/2} = g^{-4}$ arising from the Gaussian integration over $A^a_i$.

The ‘æther’ term distinguishes the YM theory from the topological non-propagating 3d Einstein gravity [16]; in particular, it is responsible for the propagation of transverse gluons at short distances [14, 15].

The partition function (61) is the desired formulation of the $SU(2)$ YM theory in terms of gauge-invariant variables. We now generalize it to higher gauge groups.

6. $SU(N)$, $d = 3$ BF gravity in a basis-independent formulation

As in the $SU(2)$ case, we wish to integrate out the YM connection $A^a_i$ whose saddle-point value is determined by Eq. (63) and express the result in terms of gauge-invariant combinations of $e^a_i$. In total, the quantities $e^a_i$ carry $3 \cdot (N^2 - 1)$ degrees of freedom, of which $2 \cdot (N^2 - 1)$ are gauge-invariant or, in other words, basis-independent. In the $SU(2)$ case these $2 \cdot 3 = 6$ variables are the components of the metric of the dual space; they can be decomposed into spin 0 (1 dof) and spin 2 (5 dof’s) fields, $1 + 5 = 6$.

For $SU(3)$ one needs $2 \cdot 8 = 16$ dof’s; these will be the 6 components of the metric tensor $g_{ij}$, plus 10 components of spin 1 (3 dof’s) and spin 3 (7 dof’s) fields, put together into a symmetric tensor $h_{ijk}$.

For $SU(4)$ one needs $2 \cdot 15 = 30$ dof’s; these will be the previous 16, plus new ones in the form of spin 2 (5 dof’s) and spin 4 (9 dof’s) fields.

The general pattern is that for the $SU(N)$ gauge group one adds new spin $N$ and spin $N-2$ fields to the ‘previous’ fields of the $SU(N-1)$ group. All in all, one has for the $SU(N)$ two copies of spin $2, 3 \ldots N - 2$ and one copy of the ‘edge’ spins 0, 1, $N - 1$ and $N$; they sum up into the needed $2 \cdot (N^2 - 1)$ dof’s.

The invariance of the BF term under the $(N^2 - 1)$-function dual gauge transformation (52) translates into an $(N^2 - 1)$-function local symmetry which mixes fields with different spins. Below we sketch the derivation of the appropriate action.

6.1. The $(N^2 - 1)$-bein

It will be convenient to introduce the dual field strength as a matrix, $e_i = e^a_i t^a$, where $t^a$ are $N^2 - 1$ generators of $SU(N)$, $\text{Tr} (t^a t^b) = \frac{1}{2} \delta^{ab}$, and to rewrite the saddle-point Eq. (53) in the matrix form,

$$e^{ijk} [D_i (A) e_j] = 0, \quad D_i = \partial_i - i A_i^a t^a.$$  \hspace{1cm} (63)

As noticed in Ref. [17], Eq. (54) is not a general solution of the saddle-point equation (53) because the basis $e_i$ is not complete at $N > 2$. Therefore, first of all we have to choose the basis vielbein, call it $e^a_i$, where both indices run from 1 to $N^2 - 1$. We shall use the traceless Hermitian matrices $e_i = e_i^a t^a$.

In the $SU(2)$ case we take $e_j = e_i$, with $I = i = 1, 2, 3$ and define the metric tensor

$$g_{ij} = \text{Tr} \{e_i e_j\}, \quad g_{ij} g^{jk} = \delta_i^k.$$  \hspace{1cm} (64)

For $SU(3)$ we build a quadratic expression in $e_i$ which is a traceless (both in matrix and gravity senses) rank-2 tensor

$$e_{\{i_1 i_2\}} = \frac{1}{2} \left( e_{i_1} e_{i_2} - \frac{1}{3} g_{i_1 i_2} (e_k e_k) g^{k_1 k_2} \right).$$  \hspace{1cm} (65)

The symbol $\{ \ldots \}$ denotes a sum of all permutations of matrices inside the curly brackets. There are only 5 independent components of $e_{\{i_1 i_2\}}$ since $g^{i_1 i_2} e_{\{i_1 i_2\}} = 0$. At $N = 2$ $e_{\{i_1 i_2\}}$ is zero. Thus, the next five components of $e_I$ are $e_I = e_{\{i_1 i_2\}}$ with $I = i_1 i_2 = 4, 5, 6, 7, 8$. We introduce the gauge-invariant symmetric rank-3 tensor

$$h_{ijk} = \frac{1}{3} \text{Tr} \{e_i e_j e_k\} = \text{Tr} e_I (e_I e_I);$$  \hspace{1cm} (66)

it describes spin 1 and spin 3 fields and possesses 10 dof’s. The spin 1 component is cut out by the contraction, $h_i = h_{ijk} g^{jk}$.

For $SU(4)$ we need further components of $e_I$, namely the irreducible rank-3 tensor cubic in $e_i$, call it $e_{\{i_1 i_2 i_3\}}$:

$$e_{\{i_1 i_2 i_3\}} = \frac{1}{3!} \left( e_{i_1} e_{i_2} e_{i_3} - \frac{1}{5} g_{i_1 i_2} (e_k e_k) g^{k_1 k_2} \right.$$

$$- \frac{1}{5} g_{i_1 i_3} (e_k e_k e_k) g^{k_1 k_3} - \frac{1}{5} g_{i_2 i_3} (e_k e_k e_k) g^{k_2 k_3}$$

$$\left. - \frac{1}{5} (h_{i_1 i_2 i_3} - \frac{1}{5} g_{i_1 i_2} h_{i_3} - \frac{1}{5} g_{i_1 i_3} h_{i_2} - \frac{1}{5} g_{i_2 i_3} h_{i_1}) \right).$$  \hspace{1cm} (67)

It has the following properties: a) $e_{\{i_1 i_2 i_3\}}$ is a Hermitian and traceless matrix, symmetric under permutations of any indices, b) it has only 7 independent components since $g^{i_1 i_2} e_{\{i_1 i_2 i_3\}} = 0$, c) at $N = 3$ it is automatically zero owing to an identity valid for any Hermitian $3 \times 3$ matrix,

$$M^3 = M^2 \text{Tr} (M) + M \frac{1}{2} [\text{Tr} (M^2) - (\text{Tr} M)^2]$$

$$+ 1 \text{ det}(M).$$  \hspace{1cm} (68)

Therefore, for $N \geq 4$ we take $e_I = e_{\{i_1 i_2 i_3\}}$ with $I = \{i_1 i_2 i_3\} = 9, 10, \ldots, 15$.

Using the above recipe for constructing irreducible tensors one can iteratively build higher-rank tensors.
suitable for higher groups and thus higher components of the \((N^2 - 1)\)-bein \(e_I\):

\[
e_I = (e_i, e_{i_1i_2}, e_{i_1i_2i_3}, \ldots)
\]  

(69)

For the general \(SU(N)\) group the total number of independent components of \(e_I\) is, as it should be, \(\sum_{j=1}^{N-1}(2J + 1) = N^2 - 1\). Eq. (69) reminds the Burnside basis for \(SU(N)\).

## 6.2. \(SU(N)\) metric tensor

Having built the \((N^2 - 1)\)-bein, we introduce a general- ized \(SU(N)\) metric tensor which is a real and symmetric \((N^2 - 1) \times (N^2 - 1)\) matrix:

\[
g_{IJ} = \text{Tr} \{e_I e_J\}
\]  

(70)

It is, generally, not degenerate, therefore one can also introduce the ‘contravariant’ metric tensor \(g^{IK}\), such that \(g_{IJ} g^{JK} = \delta^I_J\). It should be noted that the \(\delta\)- symbol here is not the usual Kronecker delta. Although its left-upper-corner component is the usual \(\delta^I_J\), another diagonal component is

\[
\delta^{(k_1k_2)}_{(i_1i_2)} = \frac{1}{2} \left( \delta_{i_1}^k g_{k_2}^i + \delta_{i_2}^k g_{k_1}^i - \frac{2}{g} g_{k_1k_2} g_{i_1i_2} \right).
\]  

(71)

Let us consider the covariant \(SU(N)\) metric tensor \(g_{IJ}\) in more detail. \(I, J\) are multi-indices running \(I = (i_1, \{i_1i_2\}, \{i_1i_2i_3\}, \ldots), J = (j, \{j_1j_2\}, \{j_1j_2j_3\}, \ldots)\). We denote

\[
p_{i_1i_2j_3j_4} = \frac{1}{12} \text{Tr} \{e_{i_1} e_{j_3} e_{j_4} e_{j_2}\} = \frac{1}{3} \text{Tr} \{e_{i_1} e_{j_2}\} \{e_{j_3} e_{j_4}\},
\]

\[
q_{i_1i_2j_3j_2} = \frac{1}{12} \text{Tr} \{e_{i_1} e_{i_2}\} \{e_{j_3} e_{j_2}\}.
\]  

(72)

We list the first few components of \(g_{IJ}\):

\[
g_{i_1j_2} = \delta_{i_1j_2}, \quad g_{i_1j_2j_3} = \frac{1}{2} g_{i_1} g_{j_2j_3},
\]

\[
g_{i_1j_2j_3j_4} = \frac{1}{3} g_{i_1} (g_{j_2j_3j_4} + g_{j_2j_4j_3} - g_{j_2j_3j_4}),
\]

\[
g^{i_1j_2j_3j_4} = \frac{1}{2} g_{i_1} \delta_{i_1j_2j_3j_4} - \frac{1}{3} g^{i_1j_2j_3j_4}
\]

\[
+ \frac{1}{3} g^{i_1j_2j_3j_4} g_{i_1} g_{i_1j_2j_3j_4} - \frac{1}{3} g_{i_1j_2j_3j_4} g_{i_1j_2} g_{i_1j_3j_4},
\]  

(73)

and so on.

An important question is that of the number of independent degrees of freedom encoded in various components of \(g_{IJ}\). The symmetric tensor \(g_{ij}\) contains 6 dof’s; it can be decomposed into spin 2 and spin 0 fields. For \(SU(2)\), six is exactly the number of gauge-invariant dof’s. Next, \(g_{i_1j_2j_3}\) has 10 dof’s which can be viewed as those belonging to spin 3 and spin 1 fields. The latter is represented by the vector field \(h_i\), the former is represented by the symmetric and traceless rank-3 tensor written in the last line of Eq. (67). The \((6 + 10) = 16\) dof’s of \(g_{ij}\) and \(g_{i_1j_2j_3}\) together compose the needed \(2(3^2 - 1) = 16\) gauge-invariant dof’s of the \(SU(3)\) group. It means, in particular, that the components \(g_{i_1j_2j_3}^{i_1j_2j_3}\) are not independent variables but are algebraically expressible through the tensors \(g_{ij}\) and \(h_{ijk}\).

Let us consider the \(SU(4)\) case. One has to add the components \(g_{i_1j_2j_3j_4}\) which are, generally speaking, a mixture of spins 4, 2 and 0. However, spin 0 is in fact absent in this tensor. To see it, we contract it with the combination

\[
\frac{1}{3} (g^{i_1j_2j_3j_4} + g^{i_2j_1j_3j_4} + g^{i_3j_1j_2j_4} - g^{i_1j_2j_3j_4} - g^{i_2j_1j_3j_4} - g^{i_3j_1j_2j_4})
\]  

(74)

which cuts out the spin 0 component of the rank-4 tensor. This contraction is zero, demonstrating that spin 0 is absent. Therefore, \(g_{i_1j_2j_3j_4}\) can be decomposed into spin 4 and spin 2 components only and thus carries \((2 \cdot 4 + 1) + (2 \cdot 2 + 1) = 9 + 5 = 14\) dof’s. These 14 add up with the previous 16 to give exactly 30 dof’s coinciding with the number of gauge-invariant combinations for the \(SU(4)\) group. Acting in the same fashion, one can verify that the component \(g_{i_1j_2j_3j_4}\) arising for groups \(SU(5)\) (and higher) has only spins 5 and 3 but not spin 1 and thus contains \(11 + 7 = 18\) dof’s which, together with the previous 30 give the 48 gauge-invariant variables of the \(SU(5)\) group.

It means that the first line in the general metric tensor, namely \(g_{ij}\) with \(i, j = 1, 2, 3, J = 1 \ldots N^2 - 1\), contains exactly \(2 \cdot (N^2 - 2)\) dof’s, i.e., the needed amount; all the rest components are algebraically expressible in terms of the first line (or the first column, \(g_{ij}\)).

## 6.3. \(SU(N)\) generalization of the Einstein-Hilbert action

We are now equipped for seeking a solution of the saddle-point Eq. (63) in the form generalizing Eq. (54) to an arbitrary gauge group

\[
[D_l e_j] = \Gamma^l_{ij} e_k + \Gamma^l_{ij} e_{k_{k_1k_2}} + \Gamma^l_{ij} e_{k_{k_1k_2k_3}}
\]

(75)

It solves Eq. (63) provided the Christoffel symbols are symmetric, \(\Gamma^l_{ij} = \Gamma^l_{ji}\). It is important that \(N^2 - 1\) components of \(e_K\) form a complete set of Hermitian and traceless \(N \times N\) matrices, therefore Eq. (75) gives a general solution of Eq. (63).

By considering derivatives of the metric tensor \(g_{ij}\) and using Eq. (64), one can express all \(\Gamma^l_{ij}\) in terms of the gauge-invariant (i.e., basis-independent) variables \(g_{ij}\) and their derivatives. Moreover, since all higher components of \(e_I\) are explicitly constructed from the first three \(e_i\’s\) it is possible to generalize Eq. (63) introducing the generalized Christoffel symbols \(\Gamma^l_{ij}\) such that

\[
[D_l e_j] = \Gamma^l_{ij} e_K.
\]  

(76)

The generalized Christoffel symbols are found from the condition that the covariant derivative of \(g_{IJ}\) is zero,

\[
g_{IJ;k} = \partial_k g_{IJ} - \Gamma^L_{kJL} g_{IL} - \Gamma^L_{kIL} g_{IL} = 0,
\]  

(77)
which expresses the $\Gamma$’s through the derivatives of the metric tensor. This equation follows from considering the derivative $\partial_\mu \Tr\{e_I e_J\} = \Tr\{[D_\mu e_I] e_J\} + \Tr\{e_I [D_\mu e_J]\}$ and using Eq. (76).

We next consider the double commutator,

$$[D_m [D_n e_I]] = [D_m, \Gamma^K_n] e_K = \partial_m \Gamma^K_n e_K + \Gamma^K_m \Gamma^K_n e_J. \tag{78}$$

We interchange $(mn)$ and subtract one from another:

$$[D_m [D_n e_I]] - [D_n [D_m e_I]] = -[e_I [D_m D_n]] = i [F_{mn} e_I]\]
$$

$$= (\partial_m \Gamma^J_n - \partial_n \Gamma^J_m + \Gamma^J_m \Gamma^K_n - \Gamma^K_m \Gamma^J_n) e_J$$

$$= [\nabla_m \nabla_n] e_I = \Gamma^K_I e_K, \tag{79}$$

which serves as a definition of the generalized Riemann tensor and simultaneously expresses the Yang–Mills field strength $F_{mn}$ at the saddle point in terms of the Riemann tensor. In the component form Eq. (73) is

$$- f^{abc} F^a_{mn} e^b = R^I_{mn} e^c. \tag{80}$$

We contract this equation with the contravariant vielbein $e^d_I$, $e^b I e^d = \delta^{bd}$,

$$- f^{abc} F^a_{mn} e^d = R^I_{mn} e^c e^d, \tag{81}$$

and then with $f^{cde}$,

$$N F_{mn} = f^{cde} e^c e^d R^I_{mn}. \tag{82}$$

The action density of the mixed term in Eq. (81) is

$$\mathcal{L} = e^{mnp} F^a_{mn} e^a = \frac{1}{N} \epsilon^{mnp} f^{cde} e^c e^d e^p R^I_{mn}, \tag{83}$$

so that the BF action becomes an $SU(N)$ generalization of the Einstein–Hilbert action,

$$S_2 = \frac{i}{2} \int d^3 x \mathcal{L}$$

$$= \frac{1}{N} \int d^3 x \Tr\{[e_I e_J] e_p\} R^I_{mn} e^{mnp}. \tag{84}$$

The first factor here is gauge-invariant (i.e., basis-independent) and is an algebraic combination of the components of the metric tensor $g_{ij}$. The Riemann tensor is also made of $g_{ij}$ and its derivatives and is thus gauge-invariant as well.

In the $SU(2)$ case when $I = i = 1, 2, 3$ and $J = j = 1, 2, 3$, Eq. (84) can be simplified since

$$\Tr\{[e_i e_j] e_p\} = \frac{i}{2} \epsilon^{ijp} g^{ij} \sqrt{g}, \tag{85}$$

so that we obtain the standard Einstein–Hilbert action,

$$S_2 \big|_{SU(2)} = \frac{i}{2} \int d^3 x \sqrt{g} R. \tag{86}$$

Finally, the first term in Eq. (80) is simply

$$S_1 = -\frac{g_A^2}{2} \int d^3 x g_{ii}. \tag{87}$$

just as in the $SU(2)$ case. We note that the relative strength of the two terms is $g_A^2 N$, which has a finite limit at $N \to \infty$.

Eqs. (84), (87) solve, in a somewhat symbolic form, the problem of rewriting an arbitrary Yang–Mills theory in $d = 3$ in terms of 2 · $(N^2 - 1)$ gauge-invariant variables contained in the metric tensor $g_{ij}$. However, there remains an algebraic problem of expressing explicitly all quantities described in this subsection through $g_{ij}$ and its derivatives.

### 6.4. Local symmetry mixing fields with different spins

We will find here a whole new class of local transformations mixing fields with different spins. In contrast to supergravity, only boson fields are involved, unless one considers the generalization of the BF action to incorporate supersymmetric fermions [18].

This new symmetry originates, of course, from the symmetry of the mixed term in the first-order formalism (or of BF gravity) under dual gauge transformations [22] and reveals itself when one integrates out the YM connection $A_I$. We decompose the infinitesimal matrix of the dual gauge transformation $\beta$ in the $(N^2 - 1)$-bein $e_K$, $\beta = y^K e_K$, and use the saddle-point equation (76) to rewrite the transformation in terms of covariant derivatives of the $N^2 - 1$ infinitesimal functions $y^K$.

The variation of the vielbein under which the action is invariant is

$$\delta e_i = [D_i (\bar{A})] \beta = (\partial_i y^K) e_K + y^K \Gamma^L_{ik} e_L$$

$$= e_L (\nabla_i)^L y^K. \tag{88}$$

The variation of the $(ij)$ component of the metric tensor (80) is, consequently,

$$\delta g_{ij} = \delta \Tr\{e_i e_j\} = g_{iL} (\nabla_j)^L y^K + g_{Lj} (\nabla_i)^L y^K$$

$$= g_{iK} \partial_j y^K + g_{Kj} \partial_i y^K + (g_{iL} \Gamma^L_{jk} + g_{Lj} \Gamma^L_{ik}) y^K. \tag{89}$$

Similar variations can be found for the other components of the generalized metric tensor $g_{ij}$.

In the particular case of diffeomorphisms one takes only three functions $y^K(x)$ with $K = k = 1, 2, 3$ and puts the rest $N^2 - 4$ functions to be zero. Using Eq. (77) one sees that in this case the variation (89) becomes the usual general coordinate transformation of the metric tensor,

$$\delta g_{ij} = g_{ik} \partial_j y^K + g_{kj} \partial_i y^K + \partial_k g_{ij} y^K. \tag{90}$$

In the $SU(2)$ case this is the only symmetry we have; the BF action (84) can be written as $\int \sqrt{g} R$, and its only local symmetry is the invariance under diffeomorphisms.

For higher groups there is an additional symmetry. In the general case when all $N^2 - 1$ functions $y^K(x)$ are nonzero the last term in Eq. (89) is a combination of the derivatives of the generalized metric tensor components
$g_{IJ}$ which, are, in turn, algebraic combinations of the $2 \cdot (N^2 - 1)$ independent variables $g_{ij}$.

We thus see that the BF action (84) is invariant under $N^2 - 1$ local transformations which mix, in a nonlinear way, various components of the metric tensor $g_{ij}$. For example, in case the gauge group is $SU(3)$, the transformation (89) mixes the gauge-invariant fields $g_{ij}$ and $h_{ijk}$ carrying spins 0, 1, 2 and 3. In case of the $SU(N)$ gauge group the transformation (89) mixes fields with spin from 0 to $N$. At $N \to \infty$ the BF action (84) has an infinite-dimensional local symmetry and mixes an infinite tower of spins, with all (integer) spins twice degenerate, except the lowest spins 0 and 1.

It reminds the symmetry of string theory. It would be interesting to reformulate the material of this section directly in the $N \to \infty$ limit and to reveal this symmetry explicitly. It must be similar but not identical to the Virasoro algebra whose realization is physical only in 26 dimensions. Neither is it the $W_{1+\infty}$ algebra introduced in connection with the inclusion of higher spins into general relativity in Ref. [10], since the spin content of the $SU(N)$ Yang–Mills theory is different.

We notice that the ‘$\alpha$-term’ [83] is the ‘dilaton’ term; in $d = 2 + 1$ the dimension of the gauge coupling constant $g_3^2$ is that of a mass. Therefore, one expects that the role of this term is to lift the degeneracy of the otherwise massless fields and to provide the string with a finite slope $\alpha' = O \left( (g_3^2 N)^{-2} \right)$.

7. Conclusions

Using the first-order formalism as a starting point, we have reformulated the $SU(N)$ Yang–Mills theory in $d = 3$ and the $SU(2)$ theory in $d = 4$ in terms of local gauge-invariant variables. In all cases these variables are either identical or closely related to the metric of the ‘colour dual space’. The Yang–Mills action universally contains two terms: one is a generalization of the Einstein–Hilbert action and possesses large local symmetry which includes invariance under diffeomorphisms, the other (‘$\alpha$-term’) does not have this symmetry but is simple. The ‘$\alpha$-term’ distinguishes Yang–Mills theory from non-propagating topological BF gravity.

BF gravity based on a gauge group $SU(N)$ is known to possess invariance under dual gauge transformations characterized by $N^2 - 1$ functions in $d = 3$ and by $4 \cdot (N^2 - 1)$ functions in $d = 4$. This symmetry is apparent when BF theory is presented in terms of the YM connection but is not so apparent when one integrates over the connection (which is possible since the integral is Gaussian) and writes down the result of the integration in basis-independent variables, e.g., the metric tensor. We have shown that in such a case the original invariance under dual gauge transformations does not disappear but manifests itself as a symmetry under (generally, a nonlinear one) mixing fields with different spins. The higher is the gauge group, the higher spins transform through one another. At $N \to \infty$, an infinite tower of spins are related by symmetry transformation. Since it is the kind of symmetry known in string theory, and some kind of string is expected to be equivalent to the Yang–Mills theory in the large $N$ limit, it is tempting to use this formalism as a starting point for deriving a string from a local field theory.

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References

[1] A.M. Polyakov, Nucl. Phys. B164, 171 (1980).
[2] Yu.M. Makeenko and A.A. Migdal, Phys. Lett. B88, 135 (1979).
[3] A.M. Polyakov, Yad. Fiz. 64, 594 (2001) [Phys. Atom. Nucl. 64, 540 (2001)], hep-th/0006132.
[4] S. Deser and C. Teitelboim, Phys. Rev. D13, 1592 (1976).
[5] M.B. Halpern, Phys. Rev. D16, 1798 (1977).
[6] F.A. Lunev, Phys. Lett. B295, 99 (1992).
[7] O. Ganor and J. Sonnenschein, Int. J. Mod. Phys. A11, 5701 (1996).
[8] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209, 129 (1991).
[9] J. Baez, e-print gr-qc/9905087.
[10] G. ’t Hooft, Phys. Rev. D14, 3432 (1976).
[11] M. Martellini and M. Zeni, Phys. Lett. B401, 62 (1997).
[12] F.A. Lunev, Mod. Phys. Lett. A9, 2281 (1994); e-print hep-th/9503133.
[13] R. Anishetty, P. Majumdar and H.S. Sharatchandra, Phys. Lett. B478, 373 (2000).
[14] D. Diakonov and V. Petrov, Phys. Lett. B493, 169 (2000).
[15] D. Diakonov and V. Petrov, Zh. Eksp. Teor. Fiz. 91, 1012 (2000) [J. Exp. Theor. Phys. 91, 873 (2000)].
[16] E. Witten, Nucl. Phys. B311, 46 (1988/89).
[17] P.E. Haagensen and K. Johnson, Nucl. Phys. B439, 597 (1995).
[18] R. Schiappa, Nucl. Phys. B517, 462 (1998).
[19] E. Bergshoeff, B. DeWitt and M.A. Vasiliev, Nucl. Phys. B366, 315 (1991).