RENORMALISATION GROUP AIDED
FINITE TEMPERATURE REDUCTION
OF QUANTUM FIELD THEORIES

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Abstract

Dimensional reduction of finite temperature quantum field theories can be improved with help of continuous renormalisation group steps. The method is applied to the integration of the lowest non-static \((n = \pm 1)\) modes of the finite temperature \(\Phi^4\)-theory. A second, physically important application is the integration of the Debye-screened \(A_0(x)\) static scalar potential in the gauged \(SU(2)\) Higgs model.
1 Introduction

Thermal field theories at finite temperature can be equivalently formulated with help of countably infinite number of three-dimensional fields representing each quantum field (Matsubara decomposition). In the three dimensional formulation each set of representatives can be classified at tree level as belonging to the group of massive \( (m \sim T) \) or massless (static) components. (Fermions are represented exclusively by massive modes.) The reduction program consists of integrating out the massive components, their role being reduced in this way to influence (sometimes essentially) the interaction between static modes. The fluctuations of these latter are widely accepted to govern temperature driven phase transitions in quantum field theory.

The usual procedure of perturbative reduction \cite{1, 2} is of restricted validity. Especially detailed quantitative non-universal characteristics \((T_c, \text{ order parameter discontinuity, surface tension, etc.)} might be very sensitive to the correct dependence of the couplings of the reduced theory on the temperature and on the parameters of the original \((T = 0)\) theory. For this one should achieve the most faithful mapping of the full finite temperature system onto the effective theory. In case of the QCD deconfining transition an answer has been attempted to this problem by matching numerically several observables of the full finite temperature theory to the values simulated with the three-dimensional effective theory \cite{3, 4, 5, 6}.

The use of an ”exact” Renormalisation Group transformation for the local part of the action has been practised in critical phenomena already for two decades for the determination of the RG-flow and of the fixed point effective potential \cite{7, 8, 9, 10, 11, 12, 13}. To our knowledge, however, the investigation of its usefulness from the point of view of the dimensional reduction appears for the first time in the present paper.

We shall test its applicability by completing the reduction in models where, after a partial (perturbatively performed) reduction only finite number of three-dimensional fields are left, but for some reason still a well-defined gap persists between their respective thermal masses. Intuitively, perturbative integration should hold for the most massive (highest) Matsubara modes, but considerable improvement is expected to result from a more accurate non-perturbative integration over modes, lying the closest to the lightest ones.

The first question we should like to clarify is to what extent it is possible
to take into account higher loop effects in the reduction, which reflect the interaction between the non-static modes. Our proposal is to integrate out perturbatively only the modes with $|n| < n_0$, and proceed to a second stage of integration with the couplings already modified in the first step. Actually, the application of the RG-approach in the second stage allows the realisation of a continuous feedback of the modified couplings on the process of lowering the momentum scale for the fields to be integrated out. As a testing ground for the procedure outlined above, we shall investigate the finite temperature 1-component $\Phi^4$ theory with the choice $n_0 = 1$.

A similar situation is arrived at in the gauged SU(2) Higgs model, after having integrated out perturbatively all non-static modes. The phenomenon of Debye-screening creates a new distinct mass scale ($m \sim gT$), separating the $A_0(x)$ static modes from the rest [14, 15, 16]. It is then natural to continue with a second (hierarchical) integration of the 3 $A_0$-field components, and apply MC-simulation methods to the resulting three-dimensional gauge+Higgs effective system. Propositions for perturbative realisation of this program have been already published [17, 18].

In this paper we shall expose and test the application of the "exact" RG-transformation to situations corresponding to the two problems shortly outlined above. We are going to use a local version of this transformation, therefore effects of the field renormalisation will not be considered in the present paper.

In Section 2 the general strategy of the application of Renormalisation Group motivated reduction will be discussed in detail. Thorough qualitative description of some intrinsic features of the procedure will be given, providing insight into the temperature range where the proposed RG aided reduction is expected to work. In Section 3 we introduce our test-models mentioned above (the lengthy, but interesting exercise of partial perturbative reduction of the $\Phi^4$-theory to the $n_0 = 1$ level is relegated to the Appendix). In Section 4 the RG aided reduction will be realised on these systems under the assumption that the local potential density of the resulting effective theory can be well truncated to the linear combination of a finite number of lowest dimensional operators. The arising finite coupled set of differential infrared RG-flow equations will be carefully integrated, and the resulting couplings compared with the characteristics of the fully perturbative (sometimes higher-loop) reductions. The summary of our results with a detailed discussion of the qualitative features of the final light theory is presented in
Section 5.

2 The Strategy of the Renormalisation Group Aided Reduction

The generic situation one deals with is a three-dimensional theory (arrived at in previous – possibly perturbative – reduction steps) with one light \( (\phi) \) and one heavy \( (\Phi) \) field in interaction. At momentum scale \( \Lambda \) their interaction potential is known

\[ U_\Lambda(\Phi, \phi) = V(\Phi, \phi, \Lambda, T) \]  

and in particular one has an explicit expression for the corresponding mass-terms:

\[
\begin{align*}
\frac{\partial^2 V}{\partial \phi^2} |_{\Phi=\phi=0} &= m^2 + a_\phi T^2 - b_\phi \Lambda T \equiv M^2_\phi(\Lambda, T), \\
\frac{\partial^2 V}{\partial \Phi^2} |_{\Phi=\phi=0} &= m^2 + a_\Phi T^2 - b_\Phi \Lambda T \equiv M^2_\Phi(\Lambda, T).
\end{align*}
\]

These masses are renormalised from the four-dimensional \( (T = 0) \) point of view, but contain explicitly the induced three-dimensional "counterterm"-contribution depending linearly on the cut-off. If the reduction is realised with higher loop accuracy the linear cut-off dependence might be modulated by logarithmic terms.

By assumption there is a relation between the values of the two bare masses at high momentum:

\[ M^2_\phi(\Lambda, T) < M^2_\Phi(\Lambda, T). \]  

\( (m^2 \) is the mass parameter of the original theory renormalised by a condition imposed at \( T = 0. \) \)

The integration of the heavy field \( \Phi \) will be performed in infinitesimal steps, integrating in each step over its Fourier components lying in the thin layer \( k \leq |p| \leq k + \Delta k \equiv K, \Delta k \to 0, \) and feeding back the resulting modified couplings into the next integration step. For an infinitesimally thin layer the 1-loop contribution will be dominant. It has to be evaluated in a background of the fields \( \phi_L(x), \Phi_L(x) \) composed exclusively of Fourier components of momenta \( |p| < k. \) For the computation of the potential energy density \( \phi_L, \Phi_L \) can be set constant.
The matrix characterising the high spatial frequency fluctuations $\Phi_H(x)$ at the Gaussian (1-loop) level is given by

$$U''_K(\Phi_L, \phi_L) \equiv \frac{\partial^2 U_K}{\partial \Phi \partial \phi} \bigg|_{p=K} (\Phi_L, \phi_L) = M^2_\Phi(K, T) + c_1(K, T)\Phi^2_L + c_2(K, T)\phi^2_L + \mathcal{O}(\Phi^4_L, \Phi^2_L, \phi^4_L)$$  \hspace{1cm} (4)

The change in the potential energy is governed by the familiar differential equation

$$k \frac{\partial U_k}{\partial k} = -\frac{1}{4\pi^2} k^3 \ln(k^2 + U''_k(\Phi_L, \phi_L)).$$  \hspace{1cm} (5)

This equation should be integrated "downwards" on the momentum scale $k$, with the initial conditions (1),(2) fixed at $k = \Lambda$.

For small field values of $\Phi_L, \phi_L$ the quadratic form (4) is well approximated by the terms written out explicitly. Near the cut-off $\Lambda$ the large negative contribution to $M^2_\Phi(\Lambda, T)$, linear in $\Lambda$ is compensated by adding in the argument of the logarithm to it $k^2 = \Lambda^2$ and the contribution to $U_k$ is real. In case $M^2_\Phi(k, T)$ would not run appropriately with $k$, the argument of the logarithm on the right hand side of (5) would change sign at a certain scale $k_\Phi(T)$, which would mean that a consistent integration of $\Phi$ at that temperature would prove impossible.

Since $M^2_\Phi(k = 0, T)$ is the renormalised mass of the heavy field, for sufficiently high temperature one can expect with certainty it to be positive. Then the reduction can be performed meaningfully. There is a threshold temperature $T_\Phi$ at which $M^2_\Phi$ crosses zero first as the temperature is being lowered. It presents a limit in temperature to the consistent reduction with the present simple formulation of the Renormalisation Group idea. It could well be that by appropriate fine-tuning of the cut-off procedure one could avoid the reduced potential becoming imaginary, in the same way as it has been demonstrated for the effective potential by [11, 19]. However, this would make the actual procedure rather complicated and it seems to us not necessary for the purposes of the reduction on the basis of the following simple argument.

We would like to study the immediate neighbourhood of the transition temperature $T_c$. Therefore the reduction procedure is useful only if one finds $T_c > T_\Phi$. It is plausible that the heavy-light mass-inequality (3) will persists all the way when rolling down on the momentum scale $k$:

$$M^2_\Phi(k, T) < M^2_\Phi(k, T).$$  \hspace{1cm} (6)
Then repeating the above logical steps for the light degree of freedom we are led to the conclusion that the instability of the constant background field $\phi_L$, signalling the phase transition, will indeed occur at higher temperature, than $T_\Phi$.

3 Finite temperature effective models with heavy-light field hierarchy

The challenge we face is to find the simplest representation of the electroweak theory at finite temperature containing the smallest number of degrees of freedom, without loosing any essential aspect of the phase transition. The perturbative integration over $n \neq 0$ Matsubara modes seems to be intuitively correct since their mass scale $2\pi T$ is much larger than the scale characterising nonperturbative quantities, like magnetic screening. This latter is characterized by the scale $g^2 T$ \[20\], and for small $g$ it is even smaller than the ”electric” Debye-screening mass-scale $g T$.

The effective three-dimensional theory has been derived on the 1-loop level in different renormalisation schemes by different groups \[18, 21\]. Following \[21\] we use the form obtained with momentum cut-off regularisation in the scheme where the location and the value of the non-trivial minimum of the potential energy density of the effective model is kept at the values fixed on the classical level \[22\]. Its expression is the following:

$$S[\bar{A}_i, \bar{A}_0, \phi] = \int d^3x[L_{\text{kin}} + U(\bar{A}_0, \phi)],$$  

(7)

where

$$L_{\text{kin}} = \frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} [\left( \partial_i + i \bar{g} A_i \right) \phi]^2 + \frac{1}{2} [\left( \partial_i + i \bar{g} \bar{A}_i \right) \phi] + \frac{1}{2} (\partial_i \bar{A}_0 + \bar{g} \epsilon^{abc} \bar{A}_i^b \bar{A}_0^c)^2,$$

$$U_{\text{dim4}}(\bar{A}_0, \phi) = \frac{1}{2} m^2 \phi^+ \phi + \frac{1}{2} m_D^2 \bar{A}_0^2 + \frac{1}{8} \bar{g}^2 \bar{A}_0^2 \phi^+ \phi + \frac{\bar{g}^2}{24} (\phi^+ \phi)^2 + \frac{17 \bar{g}^4}{192 \pi^2} (\bar{A}_0^2)^2 - \frac{5 \bar{g}^2}{4 \pi^2} \bar{A}_0^2 + \left( \frac{9 \bar{g}^2}{4} + \lambda \right) \frac{1}{4 \pi^2} \phi^+ \phi \Lambda,$$

(8)

$$m^2_\phi = \hat{m}^2 + \left( \frac{3}{16} \bar{g}^2 + \frac{\bar{\lambda}}{12} \right) T, \quad m_D^2 = \frac{5}{6} \bar{g}^2 T + \frac{m^2 \bar{g}^2}{8 \pi^2 T},$$

(9)

$$\hat{m}^2 = m^2 (1 - \frac{1}{32 \pi^2} (\frac{9}{2} \bar{g}^2 + \lambda) \ln \frac{3 \bar{g}^2 \mu^2}{47} + \lambda \ln \frac{\lambda \bar{g}^2}{3 \mu^2}) - \frac{1}{128 \pi^2} (45 \bar{g}^2 + 20 \lambda + \frac{27 \bar{g}^4}{\lambda}),$$

6
\[
\hat{\lambda} = \lambda - \frac{3}{8\pi^2} (g^4 \left( \frac{3}{8} \ln \frac{g^2\lambda^2}{4T^2} - \frac{3}{2} \ln \frac{g^2\lambda^2}{\sqrt{2}T^2} \right) + \frac{\lambda^2}{4} \ln \frac{\lambda^2}{4T^2} + 3 \left( \frac{3g^2}{4} + \frac{1}{6} \right)^2 \ln \frac{3g^2\lambda^2}{4T^2} ) - \\
- \frac{9}{16\pi^2} \left( \frac{9g^4}{16} + \frac{3g^2\lambda}{4} + \frac{\lambda^2}{3} \right).
\] (10)

Perturbative investigations of the phase transition have shown that near the transition point the screening mass of the \(A_0 (\sim gT)\) quanta is considerably larger than the masses of the Higgs and of the magnetic gauge field fluctuations. For this reason we assume (and shall test subsequently the consistency of the assumption) that in this model the \(A_0\) multiplet represents the heavy fields which can be integrated out in some careful procedure.

An analogous situation can be created by construction also within the familiar 1-component \(\Phi^4\)-model, if instead of integrating over all non-static fields at once, as a first step only the \(|n| \geq 2\) modes are integrated out. As a result an effective theory with a light \((M \sim \lambda T)\) and a heavy \((M \sim 2\pi T)\) field arises. (One should note the difference in the size of the heavy and light mass-scales between the two models).

We expect the subsequent integration of the heavy field will display also some features of the 2-loop reduction, but not any really dramatic deviation from the standard 1-step reduction. Applying a non-perturbative integration technique to the heavy component one can test the stability of the standard reduction (in the weak coupling regime) against this expectation.

The derivation of the effective heavy-light theory in the \(\Phi^4\)-theory represents a technically quite involved perturbative procedure. Since in the main text we wish to concentrate on features of the non-perturbative reduction, we describe details of the perturbative part of the reduction in Appendix A. The result of this (partial) reduction is given by the following effective action:

\[
S_{3D}[\phi_0, \phi_1] = \int d^3x \left\{ \frac{1}{2} (\partial_i \phi_0)^2 + |\partial_i \phi_1|^2 + \frac{1}{2} (\phi_0^2 + 2|\phi_1|^2)(m^2 + \frac{13\lambda T}{24} - \frac{3\lambda T}{4\pi^2}) \\
+ (2\pi T)^2 |\phi_1|^2 + \frac{\lambda}{2\pi} \phi_0^4 + 12(1 + a\lambda) \phi_0^2 |\phi_1|^2 + 6(1 + b\lambda)|\phi_1|^4 \right\} + U_6[\phi_0, \phi_1],
\] (11)

where

\[
\phi_0 = \sqrt{\beta} \Phi_0, \quad \phi_1 = \sqrt{\beta} \Phi_1, \quad \bar{\lambda} = \lambda T,
\] (12)

and the numerical constants (see Appendix A)

\[
a = 3.56 \times 10^{-3}, \quad b = 3.17 \times 10^{-3}.
\] (13)
The normalisation condition required at this level to ensure the 4-dimensional ultraviolet finiteness of the theory is of the following form:

$$\frac{\partial^4 U_{3D}}{\partial \phi_0^4}|_{\phi_1=0} = \lambda.$$  

(14)

This condition explains the presence of finite ($O(\lambda^2)$) corrections to the terms $|\phi_1|^4$ and $|\phi_1|^2\phi_0^2$ above.

Also the derivation of the sixth-power contribution to the potential is given in Appendix A. It is of the form:

$$\Delta U_6 = \Delta U_6^{(1)} + \Delta U_6^{(2)}$$  

(15)

with the following explicit expressions for the terms on the right hand side:

$$\Delta U_6^{(1)} = -T^2 \lambda^2 \left( \frac{1}{4} \phi_0^2 |\phi_1|^4 \frac{1}{m^2 + \omega_n^2} + \frac{1}{36} |\phi_1|^6 \frac{1}{m^2 + \omega_n^2} \right),$$

$$\Delta U_6^{(2)} = 5.40183 \times 10^{-6} M^6 + 2.7566 \times 10^{-5} \lambda^2 \phi_0^6 |\phi_1|^2 M^2$$

$$+ 6.6586 \times 10^{-6} \lambda^2 |\phi_1|^4 M^2 - 1.6991 \times 10^{-5} \lambda^3 \phi_0^2 |\phi_1|^4$$  

(16)

($M^2 = m^2 + \tilde{\lambda}(\phi_0^2 + 2|\phi_1|^2)/2, \omega_n = 2\pi n T$).

This contribution is usually neglected, since it is suppressed at high temperature by $\Phi^2/T^2$ relative to the quartic terms, when $|\Phi| \ll T$. It might prove interesting to include it into the non-perturbative treatment of the $\phi_1$-integration, at least to check the validity of the truncation at the quartic term.

In the next section the evolution equations for the potential arising from the non-perturbative integration over subsequent momentum layers of the $\phi_1$ field will be worked out explicitly. In the approximation, when the $\phi_0$-potential of the resulting theory is projected onto a quartic polynomial, these equations can be exactly mapped onto the flow equations of the Higgs-potential of the electroweak theory induced by the $A_0$-integration. This allows a uniform formal treatment of the two physically different heavy-light systems from the point of view of the reduction.

4 Non-perturbative integration of heavy fields (finite polynomial approximation)

The detailed analysis of the non-perturbative integration will be illustrated with the $\phi_0 - \phi_1$ system. At the end of the section we describe the corre-
sponding results for the effective theory of the electroweak phase transition.

The first step of the procedure consists of the application of two subsequent Hubbard-Stratonovich transformations, through which the operators $\sim |\phi_1|^6$ and $\sim |\phi_1|^4$ can be expressed in a form apparently quadratic in $\phi_1$ with help of two auxiliary Gaussian variables $(\chi_1, \chi_2)$. Next, the integration in an infinitesimal momentum layer over the high spatial frequency part of the $\phi_1$-field is performed in the background of the slowly varying fields $\phi_{0L}, \phi_{1L}, \chi_{1L}, \chi_{2L}$. The accuracy of the result can be optimised by choosing appropriate $\phi_0, \phi_1$ dependence for the auxiliary fields $\chi_1$ and $\chi_2$.

The result of the integration is of non-polynomial nature. Due to the large mass of the $n = 1$ mode the interaction is short ranged and screens the infrared behaviour. In other words, the scale dependence is given by the ultraviolet scaling relations for all length scales. This allows to restrict the solution for up to the sixth order terms of the local potential and to expand the renormalization group equation in the interaction, \textbf{[28]}. By this argument throughout this paper we approximate the full potential energy density by a finite order polynomial (namely, up to $\phi^4$ or $\phi^6$). The result of an infinitesimal step can be expressed then in terms of a set of coupled first order non-linear differential equations. They describe the flow of the theory in a restricted coupling space as the momentum scale bounding the support of the heavy field ($\phi_1$) is lowered.

The effective light ($\phi_0$) theory is arrived at when the above bound reaches zero, that is the $\phi_1$ integration being completed. It is worthwhile to emphasize that $\phi_1$ does not fully vanish from the theory. Is present as a constant background, the integration of the $\phi_0$-theory (for instance the calculation of its effective potential) should be performed in such background. This procedure clearly will react back on the final $\phi_1$ dependence of this two-variable function. It might look just a curiosity, that our two-step integration over the non-static modes allows us searching for a non-trivial minimum in a two-variable ($\phi_0, \phi_1$) domain. Usually it is correct to assume that the absolute minimum lies for any temperature in the $\phi_1 = 0$ slice. Still, it could occur in some segment of the coupling space, that a ground state which breaks invariance under $\tau$-translation is realised.
4.1 The generalised Hubbard-Stratonovich transformation for the $\Phi^4$-theory

In the following analysis the projection of the potential energy density will be restricted to dimension-6 operators. The starting action at momentum scale $K$ is parametrized in the form:

$$S[\phi_0, \phi_1]_K = \int d^4x \left[ \frac{1}{2} (\partial_i \phi_0)^2 + |\partial_i \phi_1|^2 + \frac{1}{2} A_1(K) \phi_0^2 + \frac{1}{24} A_2(K) \phi_0^4 + (2\pi T)^2 + B_1(K))|\phi_1|^2 + \frac{1}{24} B_2(K)|\phi_1|^4 + C(K)\phi_0^6|\phi_1|^2 + D_1(K)\phi_0^4 + D_2(K)\phi_0^2|\phi_1|^4 + D_3(K)\phi_0^2|\phi_1|^4 + D_4(K)|\phi_1|^6 \right].$$  (17)

A two-step generalisation of the Hubbard-Stratonovich transformation will be used, based on the following simple identities:

$$\int_{C-i\infty}^{C+i\infty} d\chi_1 \exp \left[ \frac{1}{2} \chi_1^2 - \xi_1 \chi_1 |\phi_1|^2 - \xi_2 \chi_1 |\phi_1|^4 + \frac{1}{2} \xi_1^2 |\phi_1|^4 \right] \sim \exp (-\xi_1 \xi_2 |\phi_1|^4 + O(|\phi_1|^8)), \quad \xi_1 \xi_2 = D_4, \quad \frac{1}{2} \xi_3^2 = D_3 \phi_0^2 + \frac{1}{12} B_2 + \xi_2 \chi_1 - \frac{1}{2} \xi_1^2 \quad \text{(19)}$$

Requiring the relations

$$\xi_1 \xi_2 = D_4, \quad \frac{1}{2} \xi_3^2 = D_3 \phi_0^2 + \frac{1}{12} B_2 + \xi_2 \chi_1 - \frac{1}{2} \xi_1^2 \quad \text{(19)}$$

one can use the equalities (18) for rewriting the the action in extended form (possible terms of $O(|\phi_1|^8)$ are omitted):

$$S[\phi_0, \phi_1]_K = \int d^4x \left[ \frac{1}{2} (\partial_i \phi_0)^2 + |\partial_i \phi_1|^2 + \frac{1}{2} A_1(K) \phi_0^2 + \frac{1}{24} A_2(K) \phi_0^4 + D_1(K)\phi_0^4 - \frac{1}{2} (\chi_1^2 + \chi_2^2) \right] \left[ (2\pi T)^2 + B_1(K) + \xi_1 \chi_1 + \xi_3 \chi_2 + C(K)\phi_0^6 + D(K)\phi_0^4 |\phi_1|^2 \right].$$  (20)

For the $\chi$-integrals we follow the usual approximation of saddle point dominance. The long wavelength parts of both $\phi_0$ and $\phi_1$ are replaced by constants, which enables one to perform the integration over the short wavelength part of $\phi_1$ easily:

$$\frac{1}{2\pi^2} \int_{k}^{K} dp \, p^2 \log(p^2 + (2\pi T)^2 + B_1 + \tilde{\chi}_1 + \tilde{\chi}_2 + C\phi_0^2 + D\phi_0^4).$$  (21)

For the scaled quantities

$$\tilde{\chi}_1 = \xi_1 \chi_1, \quad \tilde{\chi}_2 = \xi_3 \chi_2 \quad \text{(22)}$$
one gets the following gap equations:

\[ \tilde{\chi}_1 = (\xi_1^2 + D_4 \tilde{\chi}_2/\xi_2^3)(|\phi_1|^2 + I_1(\Delta)), \quad \tilde{\chi}_2 = \xi_2^3(|\phi_1|^2 + I_1(\Delta)). \] (23)

Here the notation \( \Delta \equiv \tilde{\chi}_1 + \tilde{\chi}_2 + B_1 + C\phi_0^2 + D_2\phi_0^4 \) and the integrals

\[ I_n(\Delta) = \frac{1}{2\pi^2} \int_k^K \frac{p^2 dp}{(p^2 + (2\pi T)^2 + \Delta)^n} \] (24)

have been introduced.

In the classical extended action (20), evaluated with the long wavelength parts of the fields the saddle point contribution can be shown to differ from the classical expression only in terms proportional to higher powers of \( I_1 \). They give negligible contribution to the couplings at scale \( k \), when the interval \((k, K)\) becomes infinitesimal.

Therefore, the running of the couplings comes exclusively from (21). Its field-dependent part can be expanded into powers of \(|\phi_1|^2\) and \(\phi_0^2\):

\[ \Delta U = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (\tilde{\chi}_1 + \tilde{\chi}_2 + C\phi_0^2 + D_2\phi_0^4)^n I_n(B_1). \] (25)

Since each term is multiplied by an integral over the infinitesimal range \((k, K)\), for the differential running that part of \(\tilde{\chi}_1 + \tilde{\chi}_2\) which is proportional to another similar integral \(I_n\) does not play any role. Combining the two equations of (23) and the definitions of \(\xi_1^2, \xi_2^3\), one can show that

\[ \tilde{\chi}_1 + \tilde{\chi}_2 = 2D_3\phi_0^2|\phi_1|^2 + 3D_4|\phi_4|^4 + \frac{1}{6}B_2|\phi_1|^4 + O(|\phi_1|^6) \] (26)

is the relevant (classical) value to be used for the derivation of the differential running equations of the various couplings. The \(O(|\phi_1|^6)\) contribution would produce a linearly divergent term in the running of \(D_4\). It should be omitted by the neglect of the irrelevant \(|\phi_1|^8\) term. The full set of evolution equations appears in Appendix B. Here we discuss analytically the evolution of dimension 2 and 4 operators.

When one projects the potential onto a fourth order polynomial, it is sufficient to keep only the first two terms of (23). The most convenient is to introduce dimensionless variables by scaling all three-dimensional couplings by appropriate powers of \(2\pi T\):

\[ x = \frac{k}{2\pi T}, \quad a_1 = \frac{A_1}{(2\pi T)^2}, \quad b_1 = \frac{B_1}{(2\pi T)^2}, \]
\[ a_2 = \frac{A_2}{2\pi T}, \quad b_2 = \frac{B_2}{2\pi T}, \quad c = \frac{C}{2\pi T}. \] (27)
Then the following simple set of coupled first order differential equations arises:

\[
\begin{align*}
\frac{da}{dx} &= -\frac{c}{\pi^2} x^2 + 1 + b_1, \\
\frac{db}{dx} &= -\frac{b_2}{12\pi^2} x^2 + 1 + b_1, \\
\frac{dc}{dx} &= \frac{b_2 c}{12\pi^2} (x^2 + 1 + b_1)^2, \\
\frac{da_2}{dx} &= \frac{6c^2}{\pi^2} x^2 (x^2 + 1 + b_1)^2, \\
\frac{db_2}{dx} &= 6\frac{c^2}{\pi^2} x^2 (x^2 + 1 + b_1)^2.
\end{align*}
\] (28)

Comparing the right hand sides one easily finds some natural relationships between the evolution of various couplings:

\[
c = Q_1 b_2, \quad a_2 = 72Q_1^2 b_2 + Q_2, \quad a_1 = 12Q_1 b_1 + Q_3,
\] (29)

where \(Q_i\) are integration constants. These relations imply that the true task is the solution of the coupled evolution equations of \(b_1\) and \(b_2\).

In the weak coupling regime, where also \(b_1\) can be assumed in perturbative sense to be small, in the first approximation \(b_1\) can be neglected in the denominator of the right hand side of the equation for \(b_2\) and can be represented by an expansion of the denominator up to linear terms on the right hand side of the equation of \(b_1\) (One works to quadratic order on the right hand sides of (28)). Then, it is easy to find explicit analytic solutions:

\[
\begin{align*}
b_2(x) &= \frac{24\pi^2(1+x^2)}{x + (1+x^2)(24\pi^2Q_4 - \tan^{-1}x)}, \\
b_1(x) &= \frac{1+x^2}{x + (1+x^2)(24\pi^2Q_4 - \tan^{-1}x)}[Q_5 + 2(\tan^{-1}x - x)].
\end{align*}
\] (30)

The integration constants can be found from the initial conditions which are the couplings at momentum scale \(\Lambda\), arising from the 1-loop integration over the \(|n| \geq 2\) modes:

\[
\begin{align*}
a_1(x_{\Lambda}) &= b_1(x_{\Lambda}) = \frac{m^2}{(2\pi T)^2} + \frac{13\lambda}{96\pi^2} - \frac{3\lambda \Lambda}{10\pi^4 T}, \\
a_2(x_{\Lambda}) &= \frac{\Lambda}{2\pi}, \quad b_2(x_{\Lambda}) = \frac{\Lambda}{2\pi} (1 + b\lambda), \quad c(x_{\Lambda}) = \frac{\Lambda}{4\pi} (1 + a\lambda).
\end{align*}
\] (31)

The constants \(Q_i\) are found with straightforward algebra when equating (31) to the large \(x_{\Lambda}\) limiting values of the corresponding functions in (29) and (30).

The couplings reflecting the effect of \(\phi_1\)-fluctuations are given by \(a_1(0), b_1(0), c(0)\). The couplings \(a_i, c\) determine the effective fluctuation dynamics of the
static modes ($\phi_0$). $b_i$ characterize the effective potential of constant $\phi_1$ configurations. The assumption of the smallness of $B_2$ and $C$ which was essential for the consistent truncation of the infinite set of RG-equations to (28) is ensured by restricting $\lambda$ to small values. Then, it is sufficient to give the first few terms of the power series of different couplings with respect to $\lambda$:

\[
\begin{align*}
b_2(0) &\approx \frac{3\lambda}{2\pi} (1 + \lambda(b - \frac{1}{32\pi^2})), \quad c(0) \approx \frac{\lambda}{4\pi} (1 + \lambda(a - \frac{1}{32\pi^2})), \\
a_2(0) &\approx \frac{\lambda}{2\pi} (1 - \frac{3\lambda}{32\pi^2}), \\
b_1(0) &\approx \frac{m^2}{(2\pi T)^2} (1 - \frac{\lambda}{32\pi^2} (1 - \lambda(\frac{1}{32\pi^2} - b))) + \frac{7\lambda}{96\pi^2} - \frac{\lambda^2}{16\pi^2} (\frac{7}{192\pi^2} + \frac{1}{8\pi^2} + b) \\
&\quad - \frac{\Lambda}{8\pi^4 T} (1 - \frac{\lambda}{2}(b + \frac{1}{16\pi^2})), \\
a_1(0) &\approx \frac{m^2}{(2\pi T)^2} (1 - \frac{\lambda}{16\pi^2} (1 + \lambda(a - \frac{1}{32\pi^2}))) + \frac{\lambda}{96\pi^2} - \frac{\lambda^2}{8\pi^2} (\frac{7}{192\pi^2} + \frac{1}{8\pi^2} + a) \\
&\quad + \frac{\Lambda}{8\pi^4 T} (-\frac{1}{2} + \lambda(a + \frac{1}{16\pi^2})).
\end{align*}
\] (32)

### 4.2 Justification of the truncation

The solution of the approximate renormalization group equations (28) is based on the assumption $1 + x^2 > |b_1|$ which amounts to the inequalities

\[
\frac{m^2}{(2\pi T)^2} + \frac{13\lambda}{96\pi^2} - \frac{3\lambda\Lambda}{16\pi^4 T} < 1 + \frac{\Lambda^2}{(2\pi T)^2},
\] (33)

or

\[
\frac{m^2}{(2\pi T)^2} + \frac{13\lambda}{96\pi^2} - \frac{3\lambda\Lambda}{16\pi^4 T} > -1 - \frac{\Lambda^2}{(2\pi T)^2}.
\] (34)

(33) can be taken for granted since its violation requires

\[
0 < T^2 \left(4\pi^2 - \frac{13\lambda}{24} - \frac{9\lambda^2}{56\pi^4}\right) < m^2,
\] (35)

and $\Lambda = O(\lambda T)$.

(34) represents a more stringent condition,

\[
m^2 > -T^2 \left(4\pi^2 - \frac{13\lambda}{24} - \frac{9\lambda^2}{56\pi^4}\right).
\] (36)

For $m^2 = -O((2\pi T)^2)$ the $\phi_1$ system becomes critical for $\phi_0 = 0$. But this infrared instability is misleading, it comes from the parametrization of the
effective theory only. The coupling constants $a_1$, $a_2$ and $c$ were obtained by expanding the effective potential around $\phi_0 = 0$, so they are relevant in describing the fluctuations in the symmetrical phase or in the vicinity of the phase transition. For deeply broken symmetry systems the mass term $C\phi_0^2$ stabilizes the $\phi_1$ system. In this case the usefulness of the integration of $\phi_1$ alone is questionable and the $\phi_0 - \phi_1$ systems should be treated together.

These arguments demonstrate that the numerical solution of the original equation (28) would not provide in the weak coupling regime quantitatively different information on the reduced action compared to the approximation where the denominator is expanded in powers of $b_1$.

4.3 Comparison with results of simplified treatments of the $\phi_1$-integration

The coefficients $a_i$ appearing in (32) can be compared with less sophisticated ways of integrating out $\phi_1$.

The simplest is the 1-loop integration with constant $\phi_0$ background, which is a version of the hierarchical integration advocated previously within the 3D effective theory of the electroweak phase transition [17]. The quadratic form in the Fourier-space is simply determined by

$$k^2 + m^2 + (2\pi T)^2 + \frac{13 \lambda T^2}{24} + \frac{\bar{\lambda} \phi_0^2}{2} (1 + a \lambda),$$

which leads to the following contribution to the effective $\phi_0$-theory:

$$\frac{\bar{\lambda} \Lambda T}{2\pi^2} (1 + a \lambda) \frac{1}{2} \phi_0^2 - \frac{T}{6\pi} (m^2 + (2\pi T)^2 + \frac{13 \lambda T^2}{24} + \frac{\bar{\lambda} \phi_0^2}{2} (1 + a \lambda))^{3/2}. \quad (38)$$

For $\bar{\lambda} \phi_0^2 << T^2$ one expands this expression in power series with respect to $\lambda$ up to terms $\mathcal{O}(\lambda^2)$, what is equivalent to retain terms only up to $\mathcal{O}(\phi_0^4)$. In the expressions below also the leading $\mathcal{O}(m^2/T^2)$ corrections are included. The results for the coefficients $a_i$ are the following:

\begin{align*}
a_1(0) &= \frac{m^2}{(2\pi T)^2} (1 - \frac{\lambda}{16\pi^2} (1 + a \lambda) + \frac{13 \lambda^2}{3072\pi^4}) + \frac{\lambda}{96\pi^2} (a + \frac{13}{192\pi^2}) - \frac{\lambda}{8\pi^2} \frac{T}{T} (1 - a \lambda), \\
a_2(0) &= \frac{\lambda}{2\pi} (1 - \frac{3}{16\pi^2} \lambda (1 - \frac{m^2}{8\pi^2 T^2})).
\end{align*} \quad (39)

14
One instantly recognizes that the difference between (32) and (39) of the resulting couplings starts at $O(\lambda^2)$.

One might improve upon the simplest 1-loop hierarchical integration by applying first the HS-transformation, as described above. Essentially this amounts to the evaluation of $I_n$ in (24) with $k = 0$, $K = \Lambda$, and keeping only the $\phi_0$ background different from zero. The result is

$$a_1(0) = \frac{m^2}{(2\pi T)^2} \left(1 - \frac{\lambda}{16\pi^2} (1 + a\lambda) + \frac{13\lambda^2}{3072\pi^2}\right) + \frac{\lambda}{96\pi^2} \left(a + \frac{1}{192\pi^2}\right)$$

$$- \frac{\lambda}{8\pi^2 T} \left(\frac{1}{2} - \lambda(a - \frac{1}{32\pi^2})\right),$$

while the expression of $a_2(0)$ agrees with the 1-loop (and also the RG) result to $O(\lambda^2)$.

The change in the shape of the effective $\Phi_0 - \Phi_1$ potential depends on the renormalisation conditions. Since they were imposed at an intermediate stage of the integration, it is difficult to compare the above result with, for instance, an $O(\lambda^2)$ evaluation of the effective potential without reduction. At present the best one can do is to evaluate the phase transition temperature a la Landau from the vanishing of the finite part of $a_1(0)$ in the different approximations. The original estimate of Linde and Kirzhnitz [23] receives an $O(\lambda^0)$ correction as follows:

$$-\frac{\lambda}{24m^2} T_c^2 = 1 + 0.0596\lambda (RG), 1 + 0.1187\lambda (pert.), 1 + 0.0427\lambda (saddle),$$

as one calculates it from eqs. (32, 39, 40), respectively. This can be compared with the critical temperature calculated in the same way from the effective static $\Phi_0$ theory arrived at with the reduction peformed on 2-loop level [24]: $1 + 0.0462\lambda$. This shows that the saddle point integration in the present context is ”equivalent” to the 2-loop calculation, while the non-perturbative integration ”sums up” the two- and higher loop contributions into a reasonably nearby estimate.

### 4.4 Non-perturbative $A_0$-integration in the effective electroweak theory

The integration of the non-static modes has provided a new distinct screening scale, the Debye-mass of $A_0$, (3). This mass has been shown in perturbative
treatments to be consistently larger, near the phase transition point, than the effective masses of the magnetic vector- and Higgs-field fluctuations. Based on this observation a treatment fully analogous to that of $\Phi_1$ can be envisaged also for $A_0$.

For the simplicity of presentation we restrict the potential energy density to its quartic projection. In Appendix B we show, that the results are stable, when the projection is extended to dimension 6 operators.

Then the Lagrangian of the theory at scale $K$ can be parametrized as

$$L = \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} (D_i \phi)^\dagger (D_i \phi) + \frac{1}{2} (D_i A_0)^2 + \frac{1}{2} A_1 (K) \phi^\dagger \phi + \frac{1}{24} A_2 (K) (\phi^\dagger \phi)^2 + \frac{1}{2} B_1 (K) A_0^2 + \frac{1}{2} B_2 (K) (A_0^2)^2 + C (K) \phi^\dagger \phi A_0^2. \quad (42)$$

After the HS-transformation is applied to the quartic $A_0$-potential, the Lagrangian density gets the form:

$$L = \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} (D_i \phi)^\dagger (D_i \phi) + \frac{1}{2} (D_i A_0)^2 + \frac{1}{2} A_1 \phi^\dagger \phi + \frac{1}{24} A_2 (\phi^\dagger \phi)^2 - \frac{1}{8 B_2} \chi^2 + \frac{1}{2} (B_1 + \chi) A_0^2 + C \phi^\dagger \phi A_0^2. \quad (43)$$

The integration over the Fourier components of $A_0$ belonging to the $(k, K)$-layer contributes to the potential energy density

$$\frac{3}{4 \pi^2} \int_k^K dpp^2 \log (p^2 + B_1 + \chi + 2 C \phi^\dagger \phi). \quad (44)$$

The omission of the magnetic vector potential from the background, in which this expression has been evaluated corresponds to neglecting wave function renormalisation effects.

The optimisation of the HS-transformation is achieved through the choice of $\chi$ according to the gap-equation:

$$- \frac{\chi}{4 B_2} + \frac{1}{2} A_0^2 + \frac{3}{4 \pi^2} \int_k^K dpp^2 \frac{1}{p^2 + B_1 + \chi + 2 C \phi^\dagger \phi} = 0. \quad (45)$$

It has the following iterative solution to quadratic order in the couplings:

$$\chi_0 = 2 B_2 (A_0^2 + 3 I_1), \quad \chi_1 = -4 B_2 I_2 (2 C \phi^\dagger \phi + 2 B_2 (A_0^2 + 3 I_1)), \quad (46)$$

where the notation

$$I_n = \frac{1}{2 \pi^2} \int_k^K dpp^2 \frac{1}{(p^2 + B_1)^n} \quad (47)$$
has been used.

In this case it seems to be more convenient to scale dimensionfull quantities by appropriate powers of $m_D = \sqrt{5/6} g T$. On this scale we expect $b_1 \equiv B_1/m_D^2 = 1 + \mathcal{O}(g)$. With this assumption we break up $b_1$ into $1 + \bar{b}_1$. $\bar{b}_1$ is expected to sum up higher order contributions, but not to exceed the order of magnitude of $g$. Repeating the same steps as in case of the $\phi^4$ theory one arrives at the following set of renormalisation group equations:

\[
\begin{align*}
\frac{da_1}{dx} &= -\frac{3c}{\pi^2} \frac{x^2}{x^2 + 1 + b_1}, \\
\frac{db_1}{dx} &= -\frac{3b_2}{\pi^2} \frac{x^2}{x^2 + 1 + b_1}, \\
\frac{da_2}{dx} &= \frac{36c^2}{\pi^2} \frac{x^2}{(x^2 + 1 + b_1)^2}, \\
\frac{db_2}{dx} &= \frac{3b_2^2}{\pi^2} \frac{x^2}{(x^2 + 1 + b_1)^2}, \\
\frac{dc}{dx} &= \frac{3b_2 c}{\pi^2} \frac{x^2}{(x^2 + 1 + b_1)^2}.
\end{align*}
\] (48)

Like in the previous model, again some natural relations can be found between the quadratic couplings, as well as the quartic ones:

\[
\begin{align*}
c &= Q_1 b_2, \\
a_1 &= Q_1 b_1 + Q_2, \\
a_2 &= 12Q_1^2 b_2 + Q_3,
\end{align*}
\] (49)

therefore the problem is again reduced to the solution of the coupled set of the two equations for $\bar{b}_1$ and $b_2$. With the replacement

\[
b_2 = \frac{\bar{b}_2}{36}
\] (50)

in (48) one finds a form of the differential equations for $\bar{b}_1, \bar{b}_2$ which is identical to (28). It has the same solution with some constants of integrations, which can be determined from the initial conditions set at $x_\Lambda = \Lambda/m_D$:

\[
\begin{align*}
a_1(x_\Lambda) &= \frac{6}{5} \left( \frac{m_D^2}{(g T)^2} + \left( \frac{3}{16} + \frac{\lambda}{12g^2} \right) - \sqrt{\frac{5}{6} \left( \frac{9}{4} + \frac{\lambda}{g^2} \frac{2\pi^2}{252} \right)} \right), \\
\bar{b}_1(x_\Lambda) &= -\frac{\sqrt{\frac{5}{6}} g x_\Lambda}{\sqrt{252}}, \\
a_2(x_\Lambda) &= \sqrt{\frac{\Lambda}{g}}, \\
c(x_\Lambda) &= \sqrt{\frac{6}{5}} g, \\
b_2(x_\Lambda) &= \sqrt{\frac{5}{5}} g x_\Lambda.
\end{align*}
\] (51)

The analytic solutions are given with help of (30), if on the basis of the assumption $\bar{b}_1 << 1$ the $A_0$-propagators in (18) are expanded in powers of $\bar{b}_1$, keeping on the right hand sides terms up to quadratic order in the couplings.

After some ugly algebra one arrives at the following expressions for the effective couplings arising from the RG-improved integration over $A_0$:

\[
b_2(0) - b_2(x_\Lambda) = -\frac{289g^6}{1024\pi^3} \left( \frac{1}{1 + \sqrt{\frac{17g^2}{128\pi^2}}} \right)
\]
\[ c(0) - c(x_\Lambda) = -\frac{51g^4}{2560\pi^4} \frac{1}{1 + \frac{5}{128}\sqrt{\frac{27}{17}g^3}} \]

\[ a_2(0) - a_2(x_\Lambda) = -\frac{27g^2}{160\pi} \frac{1}{1 + \frac{5}{128}\sqrt{\frac{27}{17}g^3}} \]

\[ \bar{b}_1(0) - \bar{b}_1(x_\Lambda) = \sqrt{\frac{6}{5} \frac{3g}{32\pi^2}} \left[(1 + \sqrt{\frac{6}{5} \frac{5g}{8\pi}}) x_\Lambda - \frac{\pi}{2} (1 + \sqrt{\frac{6}{5} \frac{5g}{8\pi}})\right] \frac{1}{1 + \frac{5}{128}\sqrt{\frac{27}{17}g^3}} \]

\[ a_1(0) - a_1(x_\Lambda) = \sqrt{\frac{6}{5} \frac{3g}{8\pi^2}} \left[(1 + \sqrt{\frac{6}{5} \frac{5g}{8\pi}}) x_\Lambda - \frac{\pi}{2} (1 + \sqrt{\frac{6}{5} \frac{5g}{8\pi}})\right] \frac{1}{1 + \frac{5}{128}\sqrt{\frac{27}{17}g^3}} \] (52)

In all couplings in the physical \( g \)-region (\( \sim 2/3 \)) the common denominator can be omitted since it gives \( \sim 1/1000 \) relative correction.

The expression of \( \bar{b}_1 \) receives \( \mathcal{O}(g^3) \) corrections, justifying the starting assumption of \( \bar{b}_1 \ll 1 \). In the common square bracket of the expressions of \( a_1(0) - a_1(x_\Lambda) \) and \( \bar{b}_1(0) - \bar{b}_1(x_\Lambda) \) the \( \mathcal{O}(g) \) corrections are of cca. 15% relative importance in the interesting \( g \)-range.

For comparison we quote the corrections to the potential energy density from the one-step 1-loop integration of \( A_0 \) [17]:

\[ (a_1(0) - a_1(x_\Lambda))[1 - \text{loop}] = -\sqrt{\frac{6}{5} \frac{3g}{16\pi}} + \sqrt{\frac{6}{5} \frac{3g}{8\pi^2}} x_\Lambda, \]

\[ (a_2(0) - a_2(x_\Lambda))[1 - \text{loop}] = -\frac{27g^2}{160\pi}. \] (53)

One recognizes that the leading corrections to the scaled mass parameter, as calculated with both methods, coincide. The non-trivial result is the next term, which appears only in (52) and mimicks the effect of a reduction on the 2-loop level. We conclude that the improved integration leads to higher order differences in the couplings. They might influence the result of the numerical simulation of the resulting gauged Higgs system appreciably, since their contribution represents a 10-15% correction relative to the 1-loop \( A_0 \)-integration.

### 5 Discussion and conclusions

The derivation of an effective theory for three dimensional static fields is presented in this paper by means of different approximations. The idea of using effective models originates from combining different methods in dealing with different degrees of freedom.

At high temperature the natural strategy is to eliminate first the non-static modes with the simplest one-loop approximation. The more delicate
problem of infrared sensitive static modes is dealt with in the effective model where the dynamics of the non-static modes is incorporated into the effective vertices. Even the simplest leading order perturbative solution of the effective theory represents a resummation of the thermal mass.

This well-known scheme is a rudimentary application of the scheme of the renormalization group where the contributions of the modes which have been eliminated are taken into account in the next elimination step. In order to improve the effective model we pursued more systematically the renormalisation group strategy in the elimination of some massive non-static modes in the 1-component scalar model and of the $A_0$ components of the vector fields in the SU(2) Higgs model.

In the scalar model the $n > 1$ Matsubara modes were integrated out in the first step in the one-loop, independent mode approximation resulting in an effective theory for the $\phi_1$ and $\phi_0$ modes. By this we ensure that in the second step when the $n = 1$ modes are eliminated the interactions between the $n > 1$ and the $n = 1$ modes are retained. Since $\phi_1$ is the lightest heavy mode we performed a partial resummation of the perturbation expansion during its elimination. This consisted of applying a saddle point approximation for a Hubbard-Stratonovich auxiliary field representing nonlinearities of the interactions of the $\phi_1$ field, and solving the renormalization group equations for the relevant operators.

Higher loop contributions in the solution (30) are generated by irrelevant, higher order operators when projected back onto the relevant ones during the one-loop elimination step. For the solution of the renormalization group equation such UV divergence structure is expected which is different from that of a systematic multi-loop elimination. The difference between the one-loop and the renormalization group solution manifests itself in the higher loop contribution, $\frac{\lambda_0^2 \Lambda^{2/3}}{(2\pi)^6}$, in (32).

In the framework of the renormalized perturbation expansion all UV divergent contributions are treated as small perturbations because they are multiplied by a positive power of the coupling constant. Our renormalization group refers to the bare theory and does not distinguish between the renormalized part of the lagrangian and the counterterms. Thus it is not obvious to what extent the results obtained with the help of this equation would correspond to a partial resummation of renormalized graphs.

It turns out that the correspondence between the partial resummation of the renormalized perturbation expansion and the solution of the renormal-
The renormalization group equation can actually be maintained. To see this we start with
the observation that the scheme of the renormalized perturbation expansion
could be violated if the inequality
\[
\frac{3\lambda T}{4\pi^4} > |m^2 + 13\lambda T^2/24| \tag{54}
\]
is fulfilled, i.e. when the mass counterterm is larger than the finite part. For
\( m^2 < 0 \) and \( \lambda \approx 0 \) this gives
\[
x = \frac{\Lambda}{2\pi T} > -\frac{2\pi m^2}{3\lambda T^2} \tag{55}
\]
Since \( x^2 > x > 1 \) one can treat the counterterm as small beside the kinetic
energy and the expansion in the counterterms and the finite parts is justified
by the same stroke. (This result holds in four dimensions, too, since the mass
counterterm, though being quadratically divergent, is suppressed by \( \lambda \) when
compared with \( \Lambda^2 \).)

According to the argument presented above the UV divergent parts can
be treated perturbatively for \( \lambda \ll 1 \). Since \( \lambda \) has only finite renormaliza-
tion this inequality remains valid in the UV regime. Still, the ”small” UV
divergent part of the solution of the renormalization group equation makes
the renormalization of the theory more involved. If the effective theory is
derived and solved in the n-loop approximation then the general proof of
the renormalizability allows to eliminate the cut-off. When the derivation
and the solution of the effective theory are based on different approxima-
tions then the divergent parts of the bare coupling constants of the effective
theory do not cancel the divergences in the solution. In this manner unless
the derivation and the solution of the effective theory match the resulting
model is, strictly speaking, non-renormalizable.

This can be illustrated by the inspection of (32) and (38). The one-loop
integration of \( \phi_1 \) generates ultraviolet divergences in (38) which are exactly
cancelled by subsequent one-loop integration of \( \phi_0 \). But when, inconsistently,
tree-level approximation is used for \( \phi_0 \) then the effective potential remains
cut-off dependent. In an analogous fashion the ”bare” mass, \( a_1(0) \) for \( \phi_0 \) in
(32) was obtained by the renormalisation group improved perturbation ex-
pansion and its cut-off dependence will not be compensated for by its straight
or any improved one-loop integration. In the same time the mass of \( \phi_1 \), \( b_1(0) \),
displays different ultraviolet divergences which remain unchanged after the
integration over $\phi_0$. At the end one finds different cut-off dependence for the masses of $\phi_0$, $\phi_1$ and these infrared quantities can not be kept simultaneously finite.

A pragmatic way out from this problem is offered by the careful examination of the cut-off dependences. If the derivation and the solution differ in higher order of the perturbation expansion then the mismatch is $O(\lambda^n \Lambda T)$, $n > 1$. The cut-off should be large enough compared to the light mass scale, $m_\ell = \sqrt{m^2 + \frac{13\Delta^2 T^2}{24}}$, and small enough to keep this mismatch smaller than the characteristic heavy mass square, $m_h = 2\pi T$. If the expected scaling behaviour can be established for

$$m_\ell < \Lambda < \frac{m_h^2}{\lambda \pi T},$$

then the given solution of the effective theory is acceptable in the scaling window (56).

It might happen that in the relevant cut-off range only a somewhat more modest approach can be followed, corresponding to a phenomenological scaling law for the mass squares,

$$m^2(k = 0, T, \Lambda) = m^2(T) + T\Lambda f(T/\Lambda).$$

If such relation is found to be fulfilled over an extended cut-off range then one might conjecture that it belongs to a certain renormalizable solution of the full theory.

This incertitude in the use of the divergent parts of the effective theory is also a major problem in extracting physical results for the finite temperature phase transition of the SU(2) Higgs model when its 3 dimensional effective representation is being used, for instance, in the calculation of $T_c$. The unscaled form determined in this paper looks like

$$S_{eff} = \int d^3x [\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} (D_i \phi) (D_i \phi) \frac{1}{2} + m^2_3(T) \phi^\dagger \phi + \frac{1}{24} \lambda_3(T) (\phi^\dagger \phi)^2],$$

$$m^2_2(T) = m^2 + \left( \frac{3}{16} g^2 + \frac{1}{12} \lambda - \frac{3g^2}{16\pi} \sqrt{\frac{5}{6}} (1 + \sqrt{\frac{6}{5}} \frac{g}{8\pi}) \frac{1}{1 + \sqrt{\frac{5}{24}}} \right) T^2$$

$$- \frac{\Delta T}{2\pi} \left[ \frac{9}{4} g^2 + \lambda - \frac{3}{4} g^2 (1 + \sqrt{\frac{6}{5}} \frac{g}{8\pi}) \frac{1}{1 + \sqrt{\frac{5}{24}}} \right],$$

$$\lambda_3(T) = \left[ \hat{\lambda} - \sqrt{\frac{5}{6}} \frac{27g^2}{160\pi} \right] \frac{1}{1 + \sqrt{\frac{5}{24}}} T.$$ (58)
It is very important for non-perturbative integrations of this effective model to test, how well the expected cut-off dependence (displayed in the second line of the expression of \(m_2^3\)) fits the actually observed scaling behaviour. The theoretical inaccuracy of the result is assessed by discussing the sensitivity of the critical data \((T_c, \text{etc.})\) to the \(O(g^4T^2)\) and \(O(g^3\Lambda T)\) terms in \(m_2^3(T)\).

**Appendix A**

1-loop integration of the \(|n| \geq 2\) Matsubara modes

The 1-loop integration over the \(n \neq 0, \pm 1\) Matsubara modes with space-independent, \(\tau\)-dependent background

\[
\Phi(x, \tau) = \Phi(\tau) + \phi(x, \tau) \\
\Phi(\tau) = \Phi_0 + \Phi_1 \exp(i\omega \tau) + \Phi_1^* \exp(-i\omega \tau), \quad \omega = 2\pi T, \\
\phi(x, \tau) = \sum_{n \neq 0, \pm 1} \phi_n(x) \exp(i\omega n \tau).
\] (A1)

starts by writing explicitly the action up to terms quadratic in the \(|n| \geq 2\) Matsubara modes:

\[
S = \beta V \left[ \frac{1}{2} m_2^2 (\Phi_0^2 + 2|\Phi_1|^2) + \frac{1}{24} \lambda (\Phi_0^4 + 12\Phi_0^2|\Phi_1|^2 + 6|\Phi_1|^4) \right] \\
+ \beta \frac{1}{2} m_2^2 + \frac{1}{4} (\Phi_0^2 + 2|\Phi_1|^2) \sum_{k,n} \phi_{k,n}^* \phi_{k,n} \\
+ \beta \phi_0^2 \sum_{k,n} (\Phi_1^* \phi_{k,n} \phi_{k,n+1} + \Phi_1 \phi_{k,n} \phi_{k,n+1}) \\
+ \beta \frac{1}{2} \phi_0 \sum_{k,n} (\Phi_2^2 \phi_{k,n} \phi_{k,n+2} + \Phi_1^2 \phi_{k,n} \phi_{k,n+2}) \\
+ \frac{\beta \lambda \sqrt{V}}{6} [3\phi_0 \Phi_1^2 \phi_{0,-2} + 3\phi_0 \Phi_1^2 \phi_{0,2} + \Phi_1^3 \phi_{0,-3} + \Phi_1^3 \phi_{0,3}] + \mathcal{O}(\phi^3).
\] (A2)

From the primed sums the modes \(n = 0, \pm 1\) are always omitted.

The matrix of the quadratic form splits up into two identical disjoint blocs for fixed \(k\), one for Matsubara modes \(n \geq 2\) and the other for \(n \leq -2\):

\[
M(k) = \begin{pmatrix}
\frac{k^2 + \omega_1^2 + M^2}{2} & \lambda \Phi_1^* \Phi_0 & \frac{1}{2} \Phi_1^2 & 0 & 0 & \cdots \\
\lambda \Phi_1 \Phi_0 & k^2 + \omega_1^2 + M^2 & \lambda \Phi_1^* \Phi_0 & \cdots & 0 & \cdots \\
\frac{1}{2} \Phi_1^2 & \lambda \Phi_1 \Phi_0 & k^2 + \omega_1^2 + M^2 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},
\] (A3)

where \(M^2 = m_2^2 + \frac{\lambda}{2} (\Phi_0^2 + 2|\Phi_1|^2)\).
The contribution to the potential part of the effective action is of the form
\[ \sum_k \log \det M(k) - \beta N_n^* M(0)^{-1}_{nm} N_m. \]
(A4)

with
\[ N_n = \lambda \sqrt{V} \begin{pmatrix} \frac{1}{2} \Phi_0 \Phi_1^* \\ \frac{1}{6} \Phi_1^3 \\ 0 \\ \vdots \end{pmatrix}. \]
(A5)

The 0 argument of \( M \) in the second term of (A4) tells that the inverse of the matrix (A3) should be evaluated at zero spatial momentum. Since we are interested in the reduced action up to terms of \( \mathcal{O}(\Phi^6) \), it is sufficient to keep only the diagonal terms in this inverse matrix
\[ M(0)^{-1} \sim \begin{pmatrix} (m^2 + \omega_n^2)^{-1} & 0 & \cdots \\ 0 & (m^2 + \omega_n^2)^{-1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \]
(A6)

The contribution of this term to the sixth power part of the potential is calculated readily:
\[ \Delta U_6^{(1)} = -\beta V \lambda^2 \left( \frac{1}{4} \Phi_0^2 |\Phi_1|^4 \frac{1}{m^2 + \omega_n^2} + \frac{1}{36} |\Phi_1|^6 \frac{1}{m^2 + \omega_n^2} \right). \]
(A7)

The expression of the first term in (A4), expanded up to sixth power terms in \( \Phi_0 \) and \( \Phi_1 \) reads
\[ \sum_k \log \det M(k) \simeq \sum_{k,n} \left( \frac{M_0^2}{(k^2 + \omega_n^2)(k^2 + \omega_{n+1}^2)} - \frac{M_0^4}{4(k^2 + \omega_n^2)^2(k^2 + \omega_{n+2}^2)} - \frac{M_0^6}{3(k^2 + \omega_n^2)^3(k^2 + \omega_{n+1}^2)(k^2 + \omega_{n+2}^2)} \right) \\
- \lambda^2 \Phi_0^2 |\Phi_1|^2 \left( \frac{1}{(k^2 + \omega_n^2)^2(k^2 + \omega_{n+1}^2)} - \frac{1}{(k^2 + \omega_n^2)^2(k^2 + \omega_{n+2}^2)} \right) \\
+ \lambda^2 \Phi_0^2 |\Phi_1|^2 M^2 \left( \frac{1}{(k^2 + \omega_n^2)^2(k^2 + \omega_{n+1}^2)^2} + \frac{1}{(k^2 + \omega_n^2)^2(k^2 + \omega_{n+2}^2)^2} \right) \\
+ \frac{\lambda^2}{4} |\Phi_1|^4 M^2 \left( \frac{1}{(k^2 + \omega_n^2)^2(k^2 + \omega_{n+2}^2)} + \frac{1}{(k^2 + \omega_n^2)(k^2 + \omega_{n+1}^2)} \right) \right). \]
(A8)

Direct evaluation of the contributions proportional to \( \Phi^2 \) leads to
\[ \Delta U_2 = \beta V \frac{1}{2} \lambda (\Phi_0^2 + 2 |\Phi_1|^2) \left( \frac{\Lambda^2}{16 \pi^2} + \frac{13T^2}{24} - \frac{3 \Lambda T}{4 \pi^2} \right). \]
(A9)
For the evaluation of the fourth and sixth power contributions from \( \text{[A3]} \) it is very convenient to make use of the "mixed" or "Saclay"-representation of the n-sums \( [25] \):

\[
\Delta U_4 = \frac{\beta V}{2} \int \frac{d^3k}{(2\pi)^3} \int_0^\beta d\tau (-\frac{M^4}{2} - \lambda \Phi_0^2 |\Phi_1|^2 e^{i\omega \tau} - \frac{\lambda^2}{4} |\Phi_1|^4 e^{2i\omega \tau}) G'(\tau)^2,
\]

\[
\Delta U_6^{(2)} = \beta V \int \frac{d^3k}{(2\pi)^3} \int_0^\beta d\tau_1 G'(\tau_1) \int_0^\beta d\tau_2 G'(\tau_1 + \tau_2) G'(\tau_2) \times \left[ \frac{\lambda^2}{3} + \lambda^2 \Phi_0^2 |\Phi_1|^2 M^2 e^{i\omega \tau_1} + \frac{\lambda^2}{4} |\Phi_1|^4 M^2 e^{2i\omega \tau_1} - \frac{\lambda^2}{16} |\Phi_1|^4 e^{i\omega (\tau_1 + 2\tau_2)} \right].
\]

(A10)

Here the mixed finite temperature propagator is of the form

\[
G'(\tau) = T \sum_{j \neq 0, \pm 1} e^{-i j \omega \tau} \frac{1}{k^2 + \omega^2} \equiv G(\tau, k) - \frac{T}{k^2 + \omega^2} (e^{i\omega \tau} + e^{-i\omega \tau})
\]

\[
G(\tau, k) = \frac{1}{2\pi} \sum_{s = \pm} f_s(k) e^{-sk\tau}, \quad 0 \leq \tau \leq \beta,
\]

\[
f_- = n_k, \quad f_+ = 1 + n_k, \quad n_k = (e^{\beta k} - 1)^{-1}.
\]

(A11)

We elaborate on the computation of the quartic part, where the following relation, valid for any \( l \neq 0 \) can be exploited:

\[
\int_0^\beta d\tau e^{i\omega \tau} G'(\tau)^2 = \frac{1}{k^2 (k^2 + \omega^2)} - \frac{2T}{k^2 (k^2 + \omega^2)} (\frac{1}{k^2 + (l-1)^2 \omega^2} + \frac{1}{k^2 + (l+1)^2 \omega^2})
\]

\[
- \frac{2T}{k^2 (k^2 + \omega^2)} + \frac{2T}{k^2 (k^2 + \omega^2)} (\delta_{l,1} + \delta_{l,-1}) + \frac{\beta}{(k^2 + \omega^2)^2} (\delta_{l,2} + \delta_{l,-2}).
\]

(A12)

Upon the application of \( \text{[A12]} \) in the expression of \( \Delta U_4 \) one arrives at the following representation containing only 1-variable integrals:

\[
\Delta U_4 = -\frac{\lambda^2}{32 \pi^2} (\Phi_0^2 + 2 |\Phi_1|^2)^2 f dx x^2 \left[ \frac{1}{4x(x^2 + 4\pi^2)} (1 + 2 \frac{1}{e^{x^2} - 1}) - \frac{1}{x} \right]
\]

\[
- \beta V \frac{\lambda^2}{4\pi^2} \Phi_0^2 |\Phi_1|^2 f dx x^2 \left[ \frac{1}{4x(x^2 + 4\pi^2)} (1 + 2 \frac{1}{e^{x^2} - 1}) - \frac{2}{x^2 (x^2 + 4\pi^2)} - \frac{2}{(x^2 + 4\pi^2)^2 \pi^2} \right]
\]

\[
- \frac{\lambda^2}{16 \pi^2} |\Phi_1|^4 f dx x^2 \left[ \frac{1}{4x(x^2 + 4\pi^2)} (1 + 2 \frac{1}{e^{x^2} - 1}) - \frac{1}{x^2(x^2 + 16\pi^2)} \right]
\]

\[
- \frac{1}{x^2} \frac{1}{(x^2 + 4\pi^2)^2} \frac{1}{(x^2 + 16\pi^2)} - \frac{1}{x^2} \frac{1}{(x^2 + 4\pi^2)^2} \frac{1}{(x^2 + 36\pi^2)}.
\]

(A13)

Here the notation \( x = \beta k \) has been introduced, and for the divergent integrals a sharp cut-off \( \Lambda \beta \) is understood.
From the large $x$ behavior of the integrand one easily extracts the divergent piece of $\Delta U_4$: 

$$\Delta U_{4,\text{div}} = -\beta V \frac{\lambda^2}{128 \pi^2} \ln \frac{A}{\mu} [\Phi_0^4 + 12 \Phi_0^2|\Phi_1|^2 + 6|\Phi_1|^4]$$

$$= \frac{V}{24} \int_0^\beta d\tau \Phi(\tau)^4 (-\frac{3 \lambda^2}{16 \pi^2} \ln A).$$

(A14)

This result reproduces the usual 1-loop counterterm of the $\phi^4$-theory correctly.

In (A13) there are 3 integrals which can be evaluated only numerically:

$$\frac{1}{2\pi^2} \int_0^\beta d\tau \left[ \frac{e^{\tau} + 1}{4 \tau} + \frac{e^{\tau}}{2(e^{\tau} - 1)} \right] = \frac{1}{8\pi^2} \ln \frac{A}{\pi} - 1.4133 \times 10^{-2},$$

$$\frac{1}{8\pi^2} \int_0^\infty dx \frac{x}{x^2 + 4\pi^2} e^{\frac{1}{x}} = 1.7121 \times 10^{-3},$$

$$\frac{1}{8\pi^2} \int_0^\infty dx \frac{x}{x^2 + 4\pi^2} e^{\frac{1}{x}} = 4.8875 \times 10^{-4}.$$

(A15)

The logarithmic divergence of the first integral of (A13) contributes to (A14), and is absorbed eventually by the counterterm defined through the renormalisation condition:

$$\frac{\partial^4 U_4}{\partial \Phi_0^4}|_{\Phi_1=0} = \lambda_R.$$

(A16)

This leads to the appearance of finite corrections to the couplings of the other quartic terms of the potential, which are of $O(\lambda_R)$ relative to their tree-level values:

$$\Delta U_4 = \frac{\lambda_R \beta}{24} \int d^3 x [\Phi_0^4(x) + 12(1 + a \lambda_R)\Phi_0^2|\Phi_1|^2 + 6(1 + b \lambda_R)|\Phi_1|^4],$$

$$a = 3.56 \times 10^{-3}, \quad b = 3.17 \times 10^{-3}.$$

(A17)

The evaluation of $\Delta U_0^{(2)}$ is a quite tedious procedure. The easiest way to work out the complicated algebraic expressions of its integrand by making use of a symbolic programming package (MATHEMATICA). It is worthwhile to emphasize the importance of the correct implementation of the periodicity of $G(\tau_1 + \tau_2)$ on the square-interval $0 \leq \tau_1 \leq \beta, \ 0 \leq \tau_2 \leq \beta$. The result of the $\tau$-integrations could be expressed in analytic form, though its cumbersome form does not give any valuable insight. The radial $k$-integration has to be performed numerically (similarly to the case of $\Delta U_4$, given in (A13)). The integrand is finite both in the infrared and the ultraviolet domain, though separate terms might display singularities.
The final expression with explicit numerical coefficients looks like

\[
\Delta U_6 = \frac{dV}{dx}[5.40183 \times 10^{-6} M^6 + 2.7566 \times 10^{-5} \lambda^2 \Phi^2 |\Phi_1|^2 M^2 \\
+ 6.6586 \times 10^{-6} \lambda^2 |\Phi_1|^4 M^2 - 1.6991 \times 10^{-5} \lambda^3 \Phi_0^2 |\Phi_1|^4].
\] (A18)

This form also provides \(O\left(\frac{m^2}{T^2}\right)\) corrections to the coefficients of lower dimensional operators. All coefficients of dimension-6 operators are \(O(\lambda^3)\). In addition the numerical coefficients are 2 orders of magnitude smaller than in front of the \(O(\lambda^2)\) terms in \(\Delta U_4\). For this reason the starting values of these terms one can rightfully represented by (A7).

Appendix B

1. Flow-equations for the couplings of all dimension 6 non-derivative operators of the \(\Phi^4\)-theory upon integration over \(\Phi_4\)

The couplings appearing in the parametrisation of the action at scale \(K\) (17) will flow following the equations, which are derived identically to (28), just including all operators up to dimension 6:

\[
\frac{da_{1}}{dx} = -\frac{c_{1}}{\pi^{2}} \frac{x^2}{x^2+1+b_{1}}, \quad \frac{db_{1}}{dx} = -\frac{b_{1}}{12\pi^{2}} \frac{x^2}{x^2+1+b_{1}}, \\
\frac{dc}{dx} = \frac{b_{2}c}{12\pi^{2}} \frac{x^2}{(x^2+1+b_{1})^2}, \quad \frac{dD_{2}}{dx} = \frac{D_{2}}{\pi^{2}} \frac{x^2}{x^2+1+b_{1}}, \\
\frac{da_{2}}{dx} = \frac{6c^{2}}{\pi^{2}} \frac{x^2}{(x^2+1+b_{1})^2} - \frac{12D_{2}}{\pi^{2}} \frac{x^2}{x^2+1+b_{1}}, \\
\frac{db_{2}}{dx} = \frac{b_{2}^{2}}{12\pi^{2}} \frac{x^2}{(x^2+1+b_{1})^2} - \frac{18D_{2}}{\pi^{2}} \frac{x^2}{x^2+1+b_{1}}, \\
\frac{dD_{1}}{dx} = \frac{D_{2}c}{2\pi^{2}} \frac{x^2}{(x^2+1+b_{1})^2} - \frac{c^{3}}{6\pi^{2}} \frac{x^2}{(x^2+1+b_{1})^3}, \\
\frac{dD_{3}}{dx} = \frac{(b_{2}D_{3}+cD_{1})}{6\pi^{2}} \frac{x^2}{(x^2+1+b_{1})^2} - \frac{b_{2}c^{2}}{12\pi^{2}} \frac{x^2}{(x^2+1+b_{1})^3}, \\
\frac{dD_{4}}{dx} = \frac{(b_{2}D_{3}+3CD_{1})}{4\pi^{2}} \frac{x^2}{(x^2+1+b_{1})^2} - \frac{b_{2}c^{3}}{12\pi^{2}} \frac{x^2}{(x^2+1+b_{1})^3}.
\] (B1)

2. Flow-equations for the couplings of all dimension 6 non-derivative operators of the electroweak theory upon integration over \(A_0\)
The extended Lagrangian density is parametrised as
\[
L = \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} (D_i \phi \dagger D_i \phi) + \frac{1}{2} (D_i A_0) + \frac{1}{2} A_1 (K) \phi \dagger \phi + \frac{1}{2} A_2 (K) (\phi \dagger \phi)^2 \\
+ \frac{1}{2} B_1 (K) A_0^2 + \frac{1}{2} B_2 (K) (A_0^2) + C(K) \phi \dagger \phi A_0^4 + D(K) (\phi \dagger \phi)^3 \\
+ E(K) (\phi \dagger \phi)^2 A_0^2 + F(K) (A_0^2) (\phi \dagger \phi) + G(K) (A_0^2)^3. \tag{B2}
\]

The starting \((K = \Lambda)\) value of the couplings of the new (dimension 6) operators has been determined in \([17]\):
\[
D = \frac{\zeta(3)}{1024 \pi^4} (\frac{3g^6}{16} - \frac{3g^4}{8} - \frac{\lambda^2}{4} + \frac{5\lambda^3}{27}), \quad E = -\frac{\zeta(3)g^2}{1024 \pi^4} (\frac{109g^4}{16} + \frac{47\lambda^2}{6} + \frac{5\lambda^2}{9}), \\
F = -\frac{g^6}{64\pi^4}, \quad G = 0. \tag{B3}
\]

One notices that these couplings are \(O(g^6)\), (here the counting \(\lambda \sim g^2\) is the convenient choice). It is assumed that this classification will not change to the end of the \(A_0\)-integration. Even more, we assume, that to accuracy \(O(g^6)\) \(G(k) = 0\). All this has to be checked for selfconsistency at the end.

With these assumptions a single Hubbard-Stratonovich transformation is sufficient to reach an extended Lagrangian, formally quadratic in \(A_0\):
\[
L = \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} (D_i \phi \dagger D_i \phi) + \frac{1}{2} (D_i A_0) + \frac{1}{2} A_1 (K) \phi \dagger \phi + \frac{1}{2} A_2 (K) (\phi \dagger \phi)^2 \\
+ \frac{1}{2} [B_1 (K) + 2\sqrt{a} \chi + 2C(K) \phi \dagger \phi + 2E(K) (\phi \dagger \phi)^2] A_0^2 \\
+ D(K) (\phi \dagger \phi)^3 - \frac{1}{2} \chi^2 \tag{B4}
\]
with
\[
a = B_2 + 2F \phi \dagger \phi. \tag{B5}
\]

After integrating out the high frequency part of \(A_0\) in constant \(\chi, A_0, \phi\) background one finds the following gap equation for \(\chi = 2\sqrt{a} \chi^2\):
\[
\bar{\chi} = 2a A_0^2 + 3a I_1 (\Delta), \\
I_n (\Delta) = \frac{1}{4\pi^2} \int_{k}^{K} \frac{dp^2}{(p^2 + \Delta)^2}, \quad \Delta = B_1 + \bar{\chi} + 2C \phi \dagger \phi + 2E (\phi \dagger \phi)^2. \tag{B6}
\]

Repeating the arguments, already presented in the main text, we find the following contribution to the potential energy density of the reduced theory:
\[
\Delta U = \frac{3}{4 \pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(2B_2 A_0^2 + 4FA_0^2 \phi \dagger \phi + 2C \phi \dagger \phi + 2E (\phi \dagger \phi)^2\right)^n I_n (B_1) \tag{B7}
\]
For the flow-equations truncated at dimension 6 operators one truncates this series at the third term. The following coupled set of equations is arrived at:

\[ \frac{dA_1}{dk} = -\frac{3C}{\pi^2} \frac{k^2}{k^2 + B_1}, \quad \frac{dA_2}{dk} = -\frac{3B_2}{\pi^2} \frac{k^2}{k^2 + B_1}, \]

\[ \frac{dC}{dk} = -\frac{3E}{\pi^2} \frac{k^2}{k^2 + B_1} + \frac{3B_2C}{\pi^2} \frac{k^2}{(k^2 + B_1)^2}, \quad \frac{dB_1}{dk} = -\frac{3B_2^2}{\pi^2} \frac{k^2}{(k^2 + B_1)^2}, \]

\[ \frac{dD}{dk} = -\frac{2C^3}{\pi^2} \frac{k^2}{(k^2 + B_1)^2} + \frac{3CF}{\pi^2} \frac{k^2}{(k^2 + B_1)^2}, \quad \frac{dB_2}{dk} = -\frac{3B_2^2}{\pi^2} \frac{k^2}{(k^2 + B_1)^2}, \]

\[ \frac{dE}{dk} = -\frac{6B_2C^2}{\pi^2} \frac{k^2}{(k^2 + B_1)^2} + \frac{3B_2E}{\pi^2} \frac{k^2}{(k^2 + B_1)^2}, \quad \frac{dF}{dk} = -\frac{6B_2^2C}{\pi^2} \frac{k^2}{(k^2 + B_1)^2} + \frac{6B_2E}{\pi^2} \frac{k^2}{(k^2 + B_1)^2}, \]

\[ \frac{dG}{dk} = -\frac{2B_3^3}{\pi^2} \frac{k^2}{(k^2 + B_1)^2}. \]  

(B8)

At this stage we use the knowledge of the order of magnitude of the different couplings partly based on the solution of the set of equation truncated at dimension 4 operators (see main text), partly (for the new operators) by assumptions, whose consistency should be checked at the end of the calculation:

\[ A_1 \sim \mathcal{O}(g^2), \quad B_1 - m_D^2 \sim \mathcal{O}(g^4), \quad A_2 \sim \mathcal{O}(g^2), \]

\[ B_2 \sim \mathcal{O}(g^4), \quad C \sim \mathcal{O}(g^2), \quad D, E, F \sim \mathcal{O}(g^6). \]  

(B9)

One instantly recognizes, that the right hand side of the equation for \( G \) is \( \sim \mathcal{O}(g^{12}) \), therefore the assumption made for it is self-consistent. Similarly the right hand sides of the equations for \( E \) and \( F \) also vanish to \( \mathcal{O}(g^6) \), therefore they stay with their starting \( \mathcal{O}(g^6) \) values. Consistently omitting on the right hand sides all terms whose order of magnitude is smaller than \( \mathcal{O}(g^6) \), one obtains the following simplified set of equations:

\[ \frac{dA_1}{dk} = -\frac{3C}{\pi^2} \frac{k^2}{k^2 + B_1}, \quad \frac{dB_1}{dk} = -\frac{3B_2}{\pi^2} \frac{k^2}{k^2 + B_1}, \]

\[ \frac{dC}{dk} = -\frac{3E}{\pi^2} \frac{k^2}{k^2 + B_1} + \frac{3B_2C}{\pi^2} \frac{k^2}{(k^2 + B_1)^2}, \quad \frac{dB_2}{dk} = -\frac{3B_2^2}{\pi^2} \frac{k^2}{(k^2 + B_1)^2}, \]

\[ \frac{dD}{dk} = -\frac{2C^3}{\pi^2} \frac{k^2}{(k^2 + B_1)^2} \]  

with \( E, F \) being constants.

28
It is important to notice, that the equations for $B_1$ and $B_2$ are the same as before, and the solution \( (30) \) is accurate to $O(g^6)$. Then, one easily casts the equation of $C(k)$ into the form:

$$
\frac{d}{dk} \left( \frac{C}{B_2} \right) = -\frac{3F}{\pi^2} \frac{k^2}{k^2 + B_1(k) B_2(k)}.
$$

(B11)

Its solution is simply

$$
C(k) = Q_1 B_2(k) + \frac{3F}{\pi^2} B_2(k) \int_k^\Lambda \frac{p^2}{p^2 + B_1(p) B_2(p)} \frac{1}{B_2(p)}.
$$

(B12)

The value of $Q_1$ is determined from the values taken at $\Lambda$, therefore $Q_1$ equals to its value determined before.

Similarly, one transforms the equation of $A_1$ and $A_2$ into

$$
\frac{dA_2}{dk} = 12 \frac{dB_2}{dk} \left( Q_1 + \frac{3F B_2}{\pi^2} \int_k^\Lambda \frac{dp^2}{p^2 + B_1(p) B_2(p)} \frac{1}{B_2(p)} \right) - \frac{18E}{\pi^2} \frac{k^2}{k^2 + B_1},
$$

$$
\frac{dA_1}{dk} = \frac{dB_1}{dk} \left( Q_1 + \frac{3F}{\pi^2} \int_k^\Lambda \frac{dp^2}{p^2 + B_1(p) B_2(p)} \frac{1}{B_2(p)} \right).
$$

(B13)

The running of these three couplings is influenced by the dimension 6 operators only additively as it is shown by the next explicit formulæ. These equalities are $O(g^6)$ accurate solutions of the above flow equations. They demonstrate explicitly the stability of the $O(g^4)$ solution appearing in (52):

$$
C(0) - C(\Lambda) = Q_1 (B_2(0) - B_2(\Lambda)) + \frac{3F B_2(0)}{\pi^2} \int_0^\Lambda \frac{dp^2}{p^2 + B_1(p) B_2(p)} \frac{1}{B_2(p)},
$$

$$
A_1(0) - A_1(\Lambda) = Q_1 (B_1(0) - B_1(\Lambda)) - \frac{3F B_2(0)}{\pi^2} \int_0^\Lambda \frac{dp^2}{p^2 + B_1(p) B_2(p)} \frac{B_1(0) - B_1(p)}{B_2(p)},
$$

$$
A_2(0) - A_2(\Lambda) = 12 Q_1^2 (B_2(0) - B_1(\Lambda)) - \frac{18E}{\pi^2} \int_0^\Lambda \frac{dp^2}{p^2 + B_1(p)}.
$$

(B14)

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