Three Dimensional Black Hole Coupled to the Born-Infeld Electrodynamics.

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Abstract: A nonlinear charged version of the (2+1)-anti de Sitter black hole solution is derived. The source to the Einstein equations is a Born-Infeld electromagnetic field, which in the weak field limit becomes the (2+1)-Maxwell field. The obtained Einstein-Born-Infeld solution for certain range of the parameters (mass, charge, cosmological and the Born-Infeld constants) represent a static circularly symmetric black hole. Although the covariant metric components and the electric field do not exhibit a singular behavior at \( r = 0 \) the curvature invariants are singular at that point.

Keywords: 2+1 dimensions, Born-Infeld black hole

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Einstein gravity in (2+1)-dimensions has been intensively studied in this decade [1,2], largely because of the existence of black holes solutions in (2+1)-anti de Sitter spacetimes [3,4], which possess certain features inherent to the (3+1)-black holes. Moreover, it is believed that (2+1)-gravity will provide new insights towards a better understanding of the physically relevant (3+1)-gravity. Nevertheless, to our knowledge, the existing solutions in (2+1)-Einstein theory, do not consider equations of motion which are non-linear in the Maxwell field. Since non-linear electromagnetic Lagrangians, in particular the Born-Infeld Lagrangian [5], arise in open string theory (the low-energy effective action for a constant electromagnetic field is precisely the Born-Infeld action) [6,7], they deserve a special attention. In this context, string theory has emerged as the most promising candidate for the consistent quantization of gravity. In particular, as we mentioned above, the open string theory has Born-Infeld coupled vector fields, but it is not clear that this remains the case after compactification to three dimensional space with negative cosmological constant \( \Lambda \). On the other hand, the Born-Infeld electrodynamics is free from some singularities appearing in the classical theory of electromagnetic field and one may guess that string theories with Born-Infeld type effective actions should also be free from physical singularities.

It should be mentioned that Born-Infeld theory has recently attracted considerable interest in various contexts [8,9]. Among other notable features, string theory has become a theory that gives interesting answers towards other fields, such as the physics of black holes, cosmology, etc.

In (2+1)-dimensions, electromagnetic theories can be constructed from Lagrangians (without higher derivatives) depending upon a single (non-vanishing) invariant \( F = \frac{1}{4} F_{ab} F^{ab} \), which can be expressed in terms of the electric (vector) and magnetic (scalar) fields: in a lorentzian frame, for an observer moving with the 3-velocity \( v^a \), the electric and the magnetic fields are correspondingly defined as

\[
E_a = F_{ab} v^b, \quad B = \frac{1}{2} \epsilon_{abc} F^{bc} v^a,
\]

where latin indices run the values 0, 1, 2 and \( \epsilon_{abc} \) is the totally anti-symmetric Levi-Civita symbol with \( \epsilon_{012} = 1 \), usually the \( v^a \) is oriented along the time coordinate, i.e, \( v^a = \delta^a_t \), with such a choice

\[
E_a = F_{a0}, \quad B = F_{12}.
\]

Thus the invariant can be expressed by \( F \equiv \frac{1}{4} F^{ab} F_{ab} = \frac{1}{2} (B^2 - E^2) \).

In general one can construct a \((2+1)-\)Einstein theory coupled with nonlinear electrodynamics starting from the action

\[
S = \int \sqrt{-g} \left( \frac{1}{16\pi} (R - 2\Lambda) + L(F) \right) d^3x,
\]

with the electromagnetic Lagrangian \( L(F) \) unspecified explicitly at this stage; physically one requires the Lagrangian to coincide with the Maxwell one at small values of the electromagnetic fields, \( L(F)_{\text{Maxwell}} = -F/4\pi \). We are using units in which \( c = G = 1 \), because of the ambiguity in the definition of the gravitational constant (there is not newtonian gravitational limit in (2+1)-dimensions) we prefer to maintain the factor \( 1/16\pi \) in the action to keep the parallelism with (3+1)-gravity. Varying this action with respect to gravitational field gives the Einstein equations

\[
G_{ab} + \Lambda g_{ab} = 8\pi T_{ab},
\]

\[
T_{ab} = g_{ab} L(F) - F_{ac} F^c_b L_{,F}^b.
\]
while the variation with respect to the electromagnetic potential \( A_a \) entering in \( F_{ab} = A_{b,a} - A_{a,b} \), yields the electromagnetic field equations

\[
∇_a \left( F^{ab} L_\rho \right) = 0.
\]  

(6)

We are denoting the derivative of \( L(F) \) with respect to \( F \) by \( L_\rho \). In what follows we shall restrict ourselves to the study of the Born-Infeld nonlinear electrodynamics with Lagrangian

\[
L(F) = -\frac{b^2}{4π} \left( \sqrt{1 + 2\frac{F}{b^2}} - 1 \right),
\]  

(7)

where the constant \( b \) is the Born-Infeld parameter. Notice that this Lagrangian reduce to the Maxwell one in the limit when \( b^2 \to \infty \), \( L(F)_{\text{Maxwell}} = -F/4\pi \). Therefore the field equations of the Einstein-Born-Infeld theory amount to

\[
G_{ab} + Λ g_{ab} = 2 \left( \frac{F_{ac}F^c_b}{\sqrt{1 + 2F/b^2}} - b^2 g_{ab}(\sqrt{1 + 2F/b^2} - 1) \right),
\]  

(8)

together with the electromagnetic field equations

\[
∇_a \left( \frac{F^{ab}}{\sqrt{1 + 2F/b^2}} \right) = 0.
\]  

(9)

As a concrete solution of Einstein-Born-Infeld dynamical equations we present a static self-consistent solution. To derive it, we consider a (2+1)-static circularly metric of the form

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2dΩ^2,
\]  

(10)

where \( f(r) \) is an unknown function of the variable \( r \). We restrict the electric field to be

\[
F_{ab} = E(r) \left( δ_a^0 δ_b^0 - δ_a^r δ_b^r \right).
\]  

(11)

The invariant then is given by

\[
2F = -E^2(r).
\]  

(12)

Substituting (11) and (12) into the electromagnetic field equations (3) we arrive at

\[
∂_r \left( \frac{rE(r)}{\sqrt{1 - E^2/b^2}} \right) = 0,
\]  

(13)

which integrates as

\[
E(r) = \frac{q}{\sqrt{r^2 + q^2/b^2}},
\]  

(14)

where \( q \) is an integration constant having the meaning of the charge, as one could expect. In the Maxwell limit, we obtain from the last expression the right \( E = q/r \) in (2+1)-dimensions. The Born-Infeld field is characterized by a charge density distribution \( ρ_e \), which can be evaluated from the Maxwell equations \( \text{div} \vec{E} = \nabla \vec{E} = 2π\rho_e \), which in the considered case amount to

\[
\text{div} \vec{E} = \frac{1}{r} \frac{d}{dr} \left( rE(r) \right) = 2π\rho_e,
\]  

(15)

substituting here \( E(r) \) from (14) we obtain

\[
\rho_e = \frac{q^2}{2πr(r^2 + q^2/b^2)^{3/2}},
\]  

(16)

where \( r_0 = q/b \). It is easy to verify that the surface integral of \( ρ_e \) is equal to \( q \), in effect

\[
\int_0^∞ ρ_e dA = \int_0^∞ \frac{dr}{r^2 + q^2/b^2} = q. 
\]  

(17)

We would like to point out the regular behavior of the vector \( \vec{E} \) of the electric field and the surface charge distribution \( ρ_e \), the same regular behavior one encounters for the static spherically symmetric electric field in (3+1)-Born-Infeld theory.

As far as the Einstein equation are concerned, the \( R_{tt} \) and \( R_{tm} \) components yield respectively the equations

\[
f_{rr} + \frac{f_r}{r} = -4Λ - \frac{4q^2}{r^2 + q^2/b^2}
\]  

\[
-8b^2 \left( \frac{r}{\sqrt{r^2 + q^2/b^2}} - 1 \right),
\]  

(18)

\[
f_r = -2Λr - \frac{4q^2}{\sqrt{r^2 + q^2/b^2}}
\]  

\[
-4b^2 r \left( \frac{r}{\sqrt{r^2 + q^2/b^2}} - 1 \right).
\]  

(19)

The general integral of this system is given by

\[
f(r) = -M - (Λ - 2b^2)r^2 - 2b^2 r \sqrt{r^2 + q^2/b^2}
\]  

\[
-2q^2 ln(r + \sqrt{r^2 + q^2/b^2}).
\]  

(20)

From this last expression one sees that there is a contribution of the Born-Infeld field to the term with the cosmological constant.

Having explicitly the metric one easily calculates the curvature tensor components:

\[
R_{0110} = 2b^2 - Λ - \frac{2b^2r}{\sqrt{r^2 + q^2/b^2}},
\]  

(21)

\[
R_{0202} = f(r) \left( Λ r^2 - 2b^2r^2 + b^2r \sqrt{r^2 + q^2/b^2} \right).
\]  

(22)
and

\[ R_{1221} = f(r)^{-1} \left( \Lambda r^2 - 2b^2 r^2 + 2b^2 r \sqrt{r^2 + \frac{q^2}{b^2}} \right). \]  

(23)

It is surprising that the covariant metric and curvature components do not exhibit a singular behavior in neighborhood of the origin at \( r = 0 \). (3+1)-static and axially symmetric black holes do not behave in such a manner. Nevertheless this solution is singular at \( r = 0 \) in the sense that its invariant characteristics such as the Ricci scalar and the Ricci square blow up at \( r = 0 \), (the Riemann tensor squared is not necessarily evaluated since in (2+1)-dimension the Riemann tensor is given in terms of the Ricci tensor, curvature scalar and the metric tensor).

Additionally in (2+1)-dimension one considers the behavior of the invariant \( \det(R_{ab})/\det(g_{ab}) \) [3], thus one has to evaluate the invariants

\[ R, \quad R_{ab}R^{ab}, \quad \frac{\det(R_{ab})}{\det(g_{ab})} \]

at critical points. In our case these three invariants are given as

\[ R = 6\Lambda - 12b^2 + \frac{4(2q^4 + 5q^2b^2r^2 + 3b^4r^4)}{b^2r(r^2 + q^2/b^2)^{3/2}}, \]  

(24)

\[ R_{ab}R^{ab} = 3(4b^2 - 2\Lambda) - \frac{8b^2(4b^2 - 2\Lambda)}{r} \left[ \frac{3r^2 + 2q^2/b^2}{\sqrt{r^2 + q^2/b^2}} \right] \]

\[ + 8b^4 \left[ 2 + \frac{r^2}{r^2 + q^2/b^2} + \frac{3(r^2 + q^2/b^2)^2}{r^2} \right], \]  

(25)

and

\[ \frac{\det R_{ab}}{\det g_{ab}} = \left( \frac{2\Lambda - 4b^2}{r^2 + \frac{4b^2}{r}r^2 + \frac{q^2}{b^2}} \right)^2 \times \]

\[ \left( -2\Lambda + 4b^2 - \frac{2b^2(2r^2 + q^2/b^2)}{r\sqrt{r^2 + q^2/b^2}} \right)^2. \]  

(26)

Since these scalars go to infinity at \( r = 0 \), we conclude that they are singular at this point.

This solution is a black hole. To establish this assertion one has to demonstrate the existence of horizons, which require the vanishing of the \( g_{tt} \) component, i.e., \( f(r) = 0 \).

The roots of this equation give the location of the horizons (inner and outer in our case). Since this equation is a transcendental one, we are not able, as it is usual for charged (2+1)-black holes, to express the roots analytically, even for the charged static BTZ solution, the roots are expressed in terms of the Lambert \( W(x) \) function. To overcome this difficulty we study the extreme case, in which the derivative of \( \partial_r(f(r)) = 0 \) gives

\[ r_{extr} = \frac{2qb}{\sqrt{\Lambda^2 - 4b^2}} > 0 \]  

(27)

for \( \Lambda < 0 \). Now entering \( r_{extr} \) into \( f(r) = 0 \) one obtains a relation between mass, charge, cosmological constant and the Born-Infeld parameter, which can be solved explicitly for the mass—the extreme one—

\[ M_{extr} = -2q^2 \ln \left( \frac{q}{b} \sqrt{\frac{\Lambda - 4b^2}{\Lambda}} \right). \]  

(28)

We have an extreme black hole if \( \Lambda < 0, M_{extr} > 0 \) and \( q^2 < b^2 \), this last constraint arises from \( \Lambda < 4b^2q^2/(q^2 - b^2) \) when one demands \( M_{extr} > 0 \). Fixing the values of the \( M_{extr} \) for given values of \( q, b \) and \( \Lambda \) one has a black hole solution with inner and outer horizons when \( M > M_{extr} \). For \( M < M_{extr} \) one has a soliton solution, i.e., there are no horizons at all and we have a naked singularity (see FIG. 1). It may occur that for certain values of the parameters there is only one positive root of the equation \( f(r) = 0 \) – the horizon \( r_h > 0 \) – in such case one has also a black hole solution. A similar analysis can be carried out for other \( r_{extr} \) which arises for \( \Lambda > 0 \), this important case will be treated elsewhere.

**FIG. 1. Behavior of \(-g_{tt}\) for different values of \( \Lambda < 0 \).**

At infinity, for weak electromagnetic field our solution asymptotically behaves as the charged BTZ [44]:

\[ ds^2 = - \left( -M + \frac{r^2}{l^2} - q^2lnr \right) dt^2 \]

\[ + \frac{dr^2}{(-M + \frac{r^2}{l^2} - q^2lnr)} + r^2d\phi^2, \]  

(29)

and

\[ F_{ab} = \frac{q}{r} \left( \delta_a^t \delta^b_t - \delta_a^t \delta^b_t \right). \]  

(30)

with mass \( M \), cosmological constant \( \Lambda = -1/l^2 \), and charge \( q \). Hence for \( \Lambda = -1/l^2 \) our solution at infinity behaves as anti-de Sitter spacetime. As can be seen directly from this BTZ metric, it is singular at \( r = 0 \). If
one require additionally the vanishing of the cosmological constant one arrives at a solution reported in [13].

As regards the analytical extension of our solution, one has to follow step by step the procedure presented in the standard textbooks (for instance [1]) to determine the Kruskal-Szekeres coordinates. First one has to integrate for the tortoise $r_*$ coordinate: $r_*= \int 1/f(r)dr$, which in our case has no expression in terms of elementary functions, next one defines the null coordinate $u$ and $v$ by $u= t-r_*$, $v= t-r_*$; in these coordinates the studied metric acquires the form

$$ds^2 = -f(r)du dv + r^2 d\Omega^2, \quad (31)$$

where $r$ has to be interpreted as function of $u$ and $v$, $r_*= r_*(u,v)$. Further, one introduces null Kruskal-Szekeres coordinates $U= e^{-\alpha u}$ and $V= e^{\beta V}$ where $\alpha$ and $\beta$ to be chosen appropriately, finally one introduces the Kruskal-Szekeres coordinates $T= (U + V)/2$, $X= (V-U)/2$, arriving at the Kruskal extension.

If one were interested in the thermodynamics of the obtained solution one would to evaluate the temperature of the black hole, which is given in terms of its surface gravity by [13]

$$k_B T_H = \frac{\hbar}{2\pi} k. \quad (32)$$

In general, for a spherically symmetric (and for circularly symmetric in (2+1)-dimensions) system the surface gravity can be computed via (for our signature)

$$k = - \lim_{r \rightarrow r_+} \left[ \frac{1}{2} \frac{\partial_r g_{tt}}{\sqrt{-g_{tt}g_{rr}}} \right], \quad (33)$$

where $r_+$ is the outermost horizon. For our solution we have from (31), (32) and (33) that

$$k_B T = \frac{\hbar}{4\pi} \left( -2(\Lambda - 2b^2)r_+ - 4b^2 \sqrt{r_+^2 + q^2/b^2} \right). \quad (34)$$

Since in our case there is no an analytical expression of $r_+$ in terms of elementary functions, one can not give a parameter dependent expression of (34). It is easy to check that when $q = 0$, $T$ in (34) reduces to the BTZ temperature. In the extremal case [27], the temperature vanishes in (34). The entropy can be trivially obtained using the entropy formula $S = 4\pi r_+$. Other thermodynamic quantities such as heat capacity and chemical potential can be computed as in [3]. We recall that most of these quantities in the literature are evaluated for metrics given in terms of polynomial functions.

Notice that the four dimensional Einstein-Born-Infeld counterpart– the Kottler-Born-Infeld black hole [13,21]– can be given by the metric (14) with

$$f(r) = 1 - 2M/r - (\Lambda/3 - 2\ell^2/3)r^2 - \frac{2}{3} b^2 \sqrt{r^4 + q^2/b^2} - 2q^2/3r \int \frac{dr}{\sqrt{r^4 + q^2/b^2}},$$

where now $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$. As in the (2+1)-case, here there is also a contribution to the cosmological constant term of the nonlinear field. The corresponding electric field is given by

$$E(r) = \frac{q}{\sqrt{r^4 + q^2/b^2}}. \quad (35)$$

Notice that the electric field in this case is regular everywhere. This gravitational field asymptotically behaves as the Kottler charged solution, with the structural function and electromagnetic field of the form

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2 + \frac{q^2}{r^2} + O \left( \frac{1}{r^3} \right),$$

$$E(r) = \frac{q}{r^2} + O \left( \frac{1}{r^3} \right).$$

By cancelling $\Lambda$ one obtains an asymptotically flat solution.

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