ON WELL-POSEDNESS OF A DISPERSIVE SYSTEM OF THE
WHITHAM–BOUSSINESQ TYPE

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Abstract. The initial-value problem for a particular bidirectional Whitham system modelling surface water waves is under consideration. This system was recently introduced in [4]. It is numerically shown to be stable and a good approximation to the incompressible Euler equations. Here we prove local in time well-posedness. Our proof relies on an energy method and a compactness argument. In addition some numerical experiments, supporting the validity of the system as an asymptotic model for water waves, are carried out.

1. Introduction

We regard the Cauchy problem for the system that in non-dimensional variables has the form

\[ \eta_t = -v_x - i \tanh D(\eta v), \]
\[ v_t = -i \tanh D\eta - i \tanh D v^2/2 \]

where \( D = -i \partial_x \) and so \( \tanh D \) is a bounded self-adjoint operator in \( L^2(\mathbb{R}) \). The system models the two-dimensional water wave problem for an inviscid incompressible flow. As usual \( \eta \) denotes the surface elevation. Its dual variable \( v \) roughly speaking has the meaning of the surface fluid velocity.

Equations (1.1)-(1.2) appeared in literature recently as an alternative to other linearly fully dispersive models able to describe two wave propagation [4]. Those models capture many interesting features of the full water waves problem and are in a good agreement with experiments [1]. As to well-posedness, the existing results for them are not satisfactory. For example, the system regarded in [5] is locally well posed if only an additional non-physical condition \( \eta \geq C > 0 \) is imposed. This system is probably ill-posed for large data if one removes the assumption \( \eta > 0 \). An heuristic argument is given in [9]. This is not a problem for System (1.1)-(1.2).

Another important property of System (1.1)-(1.2) is its Hamiltonian structure. Indeed, regarding the functional

\[ \mathcal{H}(\eta, v) = \frac{1}{2} \int_{\mathbb{R}} \left( \eta^2 + v \frac{D}{\tanh D} v + \eta v^2 \right) dx \]

Equations (1.1)-(1.2) can be rewritten in the form

\[ \partial_t (\eta, v)^T = J \nabla \mathcal{H}(\eta, v) \]

with the skew-adjoint matrix

\[ J = \begin{pmatrix} 0 & -i \tanh D \\ -i \tanh D & 0 \end{pmatrix} \]

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In particular, $H$ is a conserved quantity. Thus Equations (1.1)-(1.2) provide an example of a nonlinear Hamiltonian system that is locally well posed.

In some sense (1.1)-(1.2) can be regarded as a regularization of the system introduced in [6]. Indeed, if one formally admits that $\tanh D \sim D$ for small frequencies, then substituting $D$ instead of $\tanh D$ to the nonlinear part of System (1.1)-(1.2) one arrives to the system regarded in [6]. Such approximation is in line with the long wave framework, when we keep all dispersive terms in the linear part and exactly first dispersive term untouched in the nonlinear part. That is formally justified due to smallness of regarded water waves. Changing variables and admitting $\tanh D \sim D$ in nonlinear part, as explained in [4], one can arrive to the system studied in [5]. It is also worth to notice that for the system regarded in [6] the Benjamin–Feir instability of periodic travelling waves is proved. If one in addition formally discard the term $\eta \partial_x u$ in the system given in [6], then a new alternative system turns out to be locally well-posed and features wave breaking [7].

In addition it is worth to notice that System (1.1)-(1.2) outperforms other bidirectional Whitham models both in the sense of numerical stability and accuracy of approximation of Euler equations [4]. This is might not be surprising since in the nonlinear part of Equations (1.1)-(1.2) we have a bounded operator. However, if one tries to diagonalise the system then one will encounter a fractional derivative $|D|^{1/2}$ both in the linear and nonlinear parts. So further considerations turn out to be not completely straightforward.

Finally, let us formulate the main result. We stick to the usual notations of Sobolev spaces $H^s = H^s(\mathbb{R})$ with the norm defined via Fourier transform.

**Theorem 1.1.** For any $\eta_0 \in H^{1/2}(\mathbb{R})$ and $v_0 \in H^1(\mathbb{R})$ there exists a positive time $T > 0$ depending only on the norm $\|\eta_0\|_{H^{1/2}} + \|v_0\|_{H^1}$ such that there exists unique solution $(\eta, v) \in C([0, T]; H^{1/2}(\mathbb{R}) \times H^1(\mathbb{R}))$ of System (1.1)-(1.2) with the initial data $(\eta_0, v_0)$. Moreover, it depends continuously on the initial data.

In the following section a priori bound is established. The complete proof of the existence would result from a standard compactness argument implemented on a regularised version of the system. In the third section, we derive an estimate for the difference of two solutions. With this estimate in hand, one can prove the uniqueness as well the continuity of the flow map. In the end, the relevance of (1.1)-(1.2) as an asymptotic model for water waves is supported by numerical calculations. The latter demonstrate a good agreement with the Euler equations.

## 2. A PRIORI ESTIMATE

Introduce a functional of the form

$$B(\eta, v) = \frac{1}{2} \int_\mathbb{R} (|\eta| |D| \eta + |v| |D|^2 v) \, dx$$

(2.1)

and a norm $E(\eta, v)$ of the view

$$E^2(\eta, v) = \frac{1}{2} \|\eta\|_{L_2}^2 + \frac{1}{2} \|v\|_{L_2}^2 + B(\eta, v)$$

(2.2)

that is obviously equivalent to $\|\eta\|_{H^{1/2}} + \|v\|_{H^1}$. Here the pair $\eta(x, t), v(x, t)$ represents a possible solution of System (1.1)-(1.2).

**Lemma 2.1** (A priori estimate). Suppose $\eta(t) \in H^{1/2}(\mathbb{R})$ and $v(t) \in H^1(\mathbb{R})$ solving System (1.1)-(1.2) are defined on some interval including zero. Then there exist constants
\[ E(t) \leq \frac{E_0 e^{Ct}}{1 - E_0 (e^{Ct} - 1)} \]

for any \( t \in [0, T) \). Here \( E(t) \) stands for \( E(\eta(t), v(t)) \) and \( E_0 = E(0) \).

**Proof.** Firstly, calculate the obvious derivative

\[ \frac{1}{2} \frac{d}{dt} \| \eta \|^2_{L^2} = \int \eta \eta_t = - \int \eta v_x - i \int \eta \tanh D(\eta v) \leq \| \eta \|_{L^2} \| \partial_x v \|_{L^2} + \| \eta \|^2_{L^2} \| v \|_{L^\infty} \]

that follows from Hölder’s inequality and boundedness of operator \( \tanh D \) in \( L^2(\mathbb{R}) \). Similarly

\[ \frac{1}{2} \frac{d}{dt} \| v \|^2_{L^2} \leq \| \eta \|_{L^2} \| v \|_{L^2} + \| \eta \|^2_{L^2} \| v \|_{L^\infty}. \]

Hence derivative of the first two terms in \((2.2)\) is bounded as

\[ \frac{1}{2} \frac{d}{dt} (\| \eta \|^2_{L^2} + \| v \|^2_{L^2}) \leq (\| \eta \|_{L^2} + \| \eta \|^2_{L^2} + \| \eta \|^2_{L^2} \| v \|_{H^1}). \quad (2.3) \]

Differentiating \((2.1)\) with respect to \( t \) obtain

\[ \frac{d}{dt} B(\eta, v) = \int \eta |D| \eta_t + v |D|^2 v \| \, dx = \]

\[ = \int \left[ -\eta v \right] |\partial_x v - i \eta |D| \tanh D(\eta v) - iv |D|^2 \tanh D \eta - iv |D|^2 \tanh D \frac{v^2}{2} \right] \, dx \quad (2.4) \]

Note that \( i |D|^2 \tanh D = \partial_x |D| \tanh D| \) and so combining the first and the third integral in \((2.4)\) gives

\[ - \int \eta |D| \partial_x v - \int v \partial_x |D|| \tanh D | \eta = \int v \partial_x |D|(1 - | \tanh D|) \eta \leq C \| \eta \|_{L^2} \| v \|_{L^2} \]

since operator \( \partial_x |D| (1 - | \tanh D|) \) is obviously bounded. Again applying \( i |D| \tanh D = \partial_x (| \tanh D| - 1) + \partial_x \) to the second part of the Integral \((2.4)\) obtain

\[ -i \int \eta |D| \tanh D(\eta v) = \int \eta \partial_x (1 - | \tanh D|)(\eta v) - \int \eta \partial_x (\eta v) \]

where the first integral is bounded by \( \| \eta \|^2_{L^2} \| v \|_{L^\infty} \) up to a constant. The second integral

\[ - \int \eta \partial_x (\eta v) = \frac{1}{2} \int v \partial_x |D|^2 = - \frac{1}{2} \int \eta^2 \partial_x v \leq \frac{1}{2} \| \eta^2 \|_{L^2} \| \partial_x v \|_{L^2} \leq C \| \eta \|^2_{H^{1/2}} \| v \|_{H^1} \]

where the \( L^4 \)-norm was controlled by \( H^{1/2} \)-norm as in Theorem 3.3 of the book by Linares and Ponce [10]. Noticing again \( i |D|^2 \tanh D = iD|D|(1 \tanh D| - 1) + \partial_x |D| \) one can treat the last part of \((2.4)\) as

\[ -i \int v |D|^2 \tanh D \frac{v^2}{2} = i \int v |D|(1 - \tanh D|) v^2 / 2 - \int v \partial_x |D| v \leq \]

\[ \leq C \| v \|^2_{L^2} \| v \|_{L^\infty} + \| \partial_x v \|^2_{L^2} \| v \|_{L^\infty} \leq C \| v \|^3_{H^1}. \]

Thus combining all these inequalities in Identity \((2.4)\) one arrives to

\[ \frac{d}{dt} B(\eta, v) \leq C \left( \| \eta \|_{L^2} \| v \|_{L^2} + \| \eta \|^2_{L^2} \| v \|_{H^1} + \| \eta \|^2_{H^{1/2}} \| v \|_{H^1} + \| v \|^3_{H^1} \right) \]

that is together with \((2.2)\) and \((2.3)\) results in

\[ \frac{d}{dt} E \leq C (E + E^2) \quad (2.5) \]
where equivalence of $E(\eta, v)$ to $\|\eta\|_{H^{1/2}} + \|v\|_{H^1}$ was used. Integration of (2.5) proves the lemma.

\[ \square \]

3. Uniqueness

Suppose on some time interval we have two solution pairs $\eta_1$, $v_1$ and $\eta_2$, $v_2$ of System (1.1)-(1.2) with the same initial data. Introduce functions $\theta = \eta_1 - \eta_2$, $w = v_1 - v_2$ and $\zeta = (\eta_1 + \eta_2)/2$, $u = (v_1 + v_2)/2$. Then $\theta$ and $w$ satisfy the following system

\begin{align*}
\theta_t &= -w_x - i \tanh D(u\theta + \zeta w), \quad (3.1) \\
\theta_x &= -i \tanh D\theta - i \tanh D(uw) \quad (3.2)
\end{align*}

with zero initial data. The idea is to obtain an estimate for this system similar to the priori bound given in the above lemma. For this purpose one calculates derivative of the square norm $E^2(\theta, w)$. Calculations are similar

\[ \frac{1}{2} \frac{d}{dt} \left( \|\theta\|_{L^2}^2 + \|w\|_{L^2}^2 \right) \leq \sqrt{2} \|\theta\|_{L^2} \|w\|_{H^1} + (\|\zeta\|_{L^2} + \|u\|_{H^1}) (\|\theta\|_{L^2} + \|w\|_{H^1})^2 \quad (3.3) \]

and for the derivative of the rest part of $E^2$ obtain

\[ \frac{d}{dt} B(\theta, w) = \int_{\mathbb{R}} \left[ -\theta |D| \partial_x w - \right. \]

\[ \left. -i\theta |D| \tanh D(u\theta + \zeta w) - i w |D|^2 \tanh D\theta - iw |D|^2 \tanh D(uw) \right] dx. \quad (3.4) \]

The first and the third integral in (3.4) together are estimated exactly as the corresponding part in (2.4) by $\|\theta\|_{L^2} \|w\|_{L^2}$ up to some constant. Similarly also estimate the fourth integral in (3.4) by $\|u\|_{H^1} \|w\|_{H^1}^2$ up to a constant. Due to identity $i |D| \tanh D = \partial_x (| \tanh D | - 1) + \partial_x$, in stead of regarding the second integral in (3.4) it is enough to estimate the following integral

\[ \int |\theta \partial_x (\zeta w)| \leq \|\partial_x^{1/2} \theta\|_{L^2} \|\partial_x^{1/2} (\zeta w)\|_{L^2} \leq \]

\[ \leq C \|\partial_x^{1/2} \theta\|_{L^2} (\|\partial_x^{1/2} \zeta\|_{L^2} \|w\|_{L^\infty} + \|\zeta\|_{L^\infty} \|\partial_x^{1/2} w\|_{L^4}) \leq C \|\zeta\|_{H^{1/2}} \|\theta\|_{H^{1/2}} \|w\|_{H^1} \]

which finishes the estimation of Derivative (3.4). Firstly, the fractional Leibniz rule was used here, that was derived by Kenig, Ponce, and Vega [8]. For the exact form we apply, one can look on page 52 of the book by Linares and Ponce [10]. Secondly, $L_1$-norms were estimated via $H^{1/2}$-norms.

The resulting inequality has the form

\[ \frac{d}{dt} E(\theta, w) \leq C \left( \|\zeta\|_{H^{1/2}} + \|u\|_{H^1} \right) E(\theta, w). \quad (3.5) \]

Taking into account boundedness of the norm $\|\zeta\|_{H^{1/2}} + \|u\|_{H^1}$ on the regarded time interval one can deduce uniqueness from the obtained inequality (3.5).

4. Computation of solitary waves

In this section we calculate numerically solitary waves corresponding to the Whitham–Boussinesq system (1.1)-(1.2) and compare them with the Euler solitary waves. We also regard evolution of Euler solitary waves with respect to System (1.1)-(1.2). This comparison supports relevance of System (1.1)-(1.2) for water waves theory. It is just an additional justification to what have been done in [4].
For notational convenience, we use the same notations \( \eta, v \) for solitary waves profiles corresponding to (1.1)-(1.2). In other words, we write \( \eta(x,t) = \eta(x - ct) \) and \( v(x,t) = v(x - ct) \). Here \( c \) stands for a Froude number coinciding with the speed of a soliton in our non-dimensional framework. The corresponding solitary waves system has the view

\[
\begin{align*}
    c\eta &= v + \mathcal{K}(\eta v), \\
    cv &= \mathcal{K}\eta + \mathcal{K}v^2/2
\end{align*}
\]

where \( \mathcal{K} = \tanh D/D \) is a bounded self-adjoint operator in \( L_2(\mathbb{R}) \). A simple heuristic analysis shows that solutions of System (4.1)-(4.2) are smooth and exist for any \( c > 1 \). Indeed, expressing \( \eta \) via \( v \) by (4.2) and substituting to (4.1) one obtains

\[
v = \frac{1}{c^2} \mathcal{K}v + \frac{1}{2c} \mathcal{K}v^2 + \frac{1}{c} \mathcal{K}^2(v\mathcal{K}^{-1}v) - \frac{1}{2c^2} \mathcal{K}^2v^3.
\]

Clearly, operator \( \mathcal{K}^{-1} - |D| \) is bounded and the operator \( \mathcal{K} \) improves the smoothness of its operand by one order. So if one takes \( v \in H^1 \) and substitute it to the right part of the last identity then one obviously gets \( v \in H^2 \), which results in the fact that both solutions \( v \) and \( \eta \) are infinitely smooth and there is no restriction on their amplitudes.
A use of the Petviashvili iteration method is made to calculate solitary waves [2]. Applicability of the method is out of scope of this note. The essence of the method is to split the linear $L$ and the nonlinear $N$ parts as follows

$$L(\eta, v) = \left( \begin{array}{c} c \\ -\mathcal{K} \end{array} \right) \left( \begin{array}{c} \eta \\ v \end{array} \right), \quad N(\eta, v) = \left( \begin{array}{c} \mathcal{K}(\eta v) \\ \mathcal{K}v^2/2 \end{array} \right)$$

and so System (4.1)-(4.2) can be rewritten as $L(\eta, v) = N(\eta, v)$. Clearly, the operator $L$ is invertible if and only if $c^2 > 1$. The Petviashvili iterative scheme is defined by

$$\left( \eta_{n+1}, v_{n+1} \right) = S_n^2 L^{-1}(N(\eta_n, v_n))$$

where $S_n$ is a stabilisation factor computed by

$$S_n = \frac{\int (\eta_n, v_n) L(\eta_n, v_n) dx}{\int (\eta_n, v_n) N(\eta_n, v_n) dx}.$$ 

An analogous splitting is applied to the Babenko equation describing Euler gravity solitary surface waves [2]. This is implemented in the code [3]. For time evolution performance of System (1.1)-(1.2), it is treated by the numerical scheme thoroughly described in [4].

For comparison with the fully nonlinear model we introduce the relative difference between waves $\eta_1$ and $\eta_2$ as

$$d(\eta_1, \eta_2) = \frac{\|\eta_1 - \eta_2\|_{L^2}}{\|\eta_1\|_{L^2}}. \quad (4.3)$$

As is pointed out above, solutions of System (4.1)-(4.2) are defined for any Froude number $c > 1$, whereas for the fully nonlinear solitary waves it does not exceed $c = 1.29421$. In Figure 1 solitons for different models are compared. On the left picture one can see the dependence of amplitude $a = \eta(0)$ on speed $c$. The black line corresponds to the Whitham–Boussinesq model and the dashed line to the full Euler model. On the right picture one can see the dependence on speed of the relative difference $\epsilon(c) = d(\eta_0, \eta)$, where Euler $\eta_0$ and Whitham $\eta$ solitons correspond to the same speed $c$. It is worth to notice that even for solitary waves with amplitude of order $a = 0.4$ the error of approximation does not exceed 10%. It approaches zero when amplitudes are taken small.

In Figure 2 approximation of relatively high solitary waves is examined. On the left picture solitons corresponding to $c = 1.25$ for different models are represented. The dashed line is for the Euler solitary wave $\eta_0(x)$. The latter is taken as an initial condition for numerical integration of System (1.1)-(1.2). Thus one can look at the time evolution of the fully nonlinear solitary wave with respect to the approximate model System (1.1)-(1.2) on the right picture in Figure 2. The shot is taken at the moment $t = 112$. The corresponding initial data has the form

$$\eta(x, 0) = \eta_0(x), \quad v(x, 0) = \mathcal{K}(u_1 + u_2 \partial_x \eta_0)$$

where elevation $\eta_0$, horizontal $u_1$ and vertical $u_2$ velocities are associated the Euler solitary wave moving with the speed $c = 1.25$ (the dashed line on the picture). One can see that the initial wave is diminishing leaving a dispersive tail behind. It is worth to notice that after some time this leading wave turns out to be a solitary solution of (4.1)-(4.2). More precisely, if one excludes the tail from the solution $\eta(x, t)$ then at the moment $t = 10$ minutes (according to our nondimensional settings) we have the difference $d(\eta_s, \eta) = 2.2 \cdot 10^{-5}$. Here $\eta_s$ is the solution of (4.1)-(4.2) corresponding to the Froude number $c = 1.22957$. This allows us to make a conjecture about asymptotic stability of solitary waves for the regarded model (1.1)-(1.2).
5. Conclusions

The dispersive Boussinesq system (1.1)-(1.2) was derived using Hamiltonian perturbation theory by Dinvay, Dutykh and Kalisch [4]. In the current paper this system has been proved to be locally well-posed. Its accuracy as of an asymptotic model was tested with solitary waves, the latter admit a complete characterization via speed-amplitude relation.

There are many possibilities for further study of System (1.1)-(1.2). First, it is desirable to prove rigorously consistency. Second, it is of interest to check if the model features modulational instability and wave breaking. Third, it would be interesting to try to extend the local result of the paper to a global well-posednes and possibly to prove asymptotic stability of solitary waves.

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