Kernel-based method for joint independence of functional variables

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Abstract. This work investigates the problem of testing whether $d$ functional random variables are jointly independent using a modified estimator of the $d$-variable Hilbert Schmidt Independence Criterion ($d$HSIC) which generalizes HSIC for the case where $d \geq 2$. We then get asymptotic normality of this estimator both under joint independence hypothesis and under the alternative hypothesis. A simulation study shows good performance of the proposed test on finite sample.

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1. INTRODUCTION

The question of relationships between variables has always been a major concern in statistical analysis. This is why methods providing answers to this type of problems, such as regression methods or independence tests, occupy an important place in the statistical literature. However, there exists just a few works that deal with independence testing in the context of functional data analysis, despite the importance of this field over the last two decades. Kokoszka et al. (2008) and Aghoukeng Jiofack and Nkiet (2010) introduced non-correlation tests between two functional variables, whereas Górecki et al. (2020), Lai et al. (2021) and Lauman et al. (2021) recently proposed tests for independence. Kernel-based methods, that use the distance between the kernel embeddings of probability measures in a reproducing kernel Hilbert space (RKHS), were also introduced in Gretton et al. (2005, 2007). For the case where more than two variables are considered, pairwise independence
tests can be obtained by performing the aforementioned tests on each pair, but it is not sufficient for some statistical problems such as those arising in causal inference where models often assume the existence of jointly independent noise variables, or in independent component analysis. So, it can be more relevant to consider testing for joint independence of several random variables rather than testing pairwise independence. Recently, Pfister et al. (2018) proposed kernel-based methods for testing for joint independence of \( d \) random variables valued into metric spaces, with \( d \geq 2 \). For doing that, they introduced the \( d \)-variable Hilbert-Schmidt independence criterion (\( d \)HSIC) as a measure of joint independence that extends the Hilbert-Schmidt independence criterion (HSIC) of Gretton et al. (2005) and contains it as a special case. For testing for joint independence they considered, as test statistic, a consistent estimator of \( d \)HSIC and derived its asymptotic distribution under null hypothesis. However, as it is the case for most measures introduced in kernel-based methods such as maximal mean discrepancy (MMD) (Gretton et al. (2012)), generalized maximal mean discrepancy (GMMD) (Balogoun et al. (2021,2022)) and HSIC, this asymptotic distribution is an infinite sum of distributions and, therefore, can not be used for performing the test. That is why Pfister et al. (2018) proposed three approaches based on \( d \)HSIC: a permutation test, a bootstrap test and a gamma approximation of the limiting distribution. An alternative approach can be tackled as it was done in Makigusa and Naito (2020) and in Balogoun et al. (2021) for the MMD and the GMMD. It consists in constructing a test statistic from an appropriate modification of a naive estimator in order to yield asymptotic normality both under the null hypothesis and under the alternative. In this paper, such an approach is adopted for the \( d \)HSIC. It allows us to propose a new test for joint independence of several random variables valued into metric spaces, including functional variables. The rest of the paper is organized as follows. The \( d \)HSIC is recalled in Section 2, and Section 3 is devoted to its estimation by a modification of a naive estimator, and to the main results. A simulation study on functional data that allows to compare the proposed test to that of Pfister et al. (2018) is given in Section 4. All the proofs are postponed in Section 5.

2. \( d \)HSIC AND JOINT INDEPENDENCE PROPERTY

For an integer \( d \geq 2 \), let \( X^{(1)}, \ldots, X^{(d)} \) be random variables defined on a probability space \((\Omega, \mathcal{F}, P)\) and with values in compact metric spaces.
Then, we consider the random variable
\[ X = (X^{(1)}, \cdots, X^{(d)}) \]
with values in \( \mathcal{X} := \mathcal{X}_1 \times \cdots \times \mathcal{X}_d \). For \( \ell \in \{1, \cdots, d\} \), let us introduce a reproducing kernel Hilbert space (RKHS) \( \mathcal{H}_\ell \) of functions from \( \mathcal{X}_\ell \) to \( \mathbb{R} \) with associated kernel \( K_\ell : \mathcal{X}_\ell^2 \to \mathbb{R} \). Each \( K_\ell \) is a symmetric function such that, for any \( f \in \mathcal{H}_\ell \) and any \( t \in \mathcal{X}_\ell \), one has \( K_\ell(t, \cdot) \in \mathcal{X}_\ell \) and \( f(t) = \langle K_\ell(t, \cdot), f \rangle_{\mathcal{X}_\ell} \) (see Berlinet and Thomas-Agnan (2004)), where \( \langle \cdot, \cdot \rangle_{\mathcal{X}_\ell} \) denotes the inner product of \( \mathcal{X}_\ell \). Denoting by \( f_1 \otimes \cdots \otimes f_d \) the tensor products of functions defined by:
\[
(f_1 \otimes \cdots \otimes f_d)(x_1, \cdots, x_d) = \prod_{\ell=1}^d f_\ell(x_\ell),
\]
we consider \( \mathcal{H} := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_d \) the completed of the vector space spanned by the set \( \{f_1 \otimes \cdots \otimes f_d / f_\ell \in \mathcal{H}_\ell, \ell = 1, \cdots, d\} \). It is known that \( \mathcal{H} \) is a RKHS associated with the kernel \( K \) satisfying:
\[
K(x, y) = \prod_{\ell=1}^d K_\ell(x_\ell, y_\ell)
\]
for any \( x = (x_1, \cdots, x_d) \in \mathcal{X} \) and \( y = (y_1, \cdots, y_d) \in \mathcal{X} \). Throughout this paper, we make the following assumption:
\[
(\mathcal{A}_1) : \|K_\ell\|_\infty := \sup_{(t, s) \in \mathcal{X}_\ell^2} K_\ell(t, s) < +\infty, \; \ell = 1, \cdots, d;
\]
it implies that \( \|K\|_\infty < +\infty \) and ensures the existence of the kernel mean embeddings \( m_\ell = \mathbb{E}(K_{\ell}(X^{(\ell)}, \cdot)), \ell = 1, \cdots, d \), and \( m = \mathbb{E}(K(X, \cdot)) \). The random variables \( X^{(1)}, \cdots, X^{(d)} \) are said to be jointly independent if
\[
\mathbb{P}_X = \mathbb{P}_{X^{(1)}} \otimes \mathbb{P}_{X^{(2)}} \otimes \cdots \otimes \mathbb{P}_{X^{(d)}},
\]
where \( \mathbb{P}_X \) (resp. \( \mathbb{P}_{X^{(j)}} \)) denotes the distribution of \( X \) (resp. \( X^{(j)} \)). For measuring this property, Pfister et al. (2018) introduced the \( d \)-variable Hilbert-Schimdt independence criterion (dHSIC) defined as
\[
dHSIC(X) = \left\| \mathbb{E}(K(X, \cdot)) - \bigotimes_{\ell=1}^d m_\ell \right\|_{\mathcal{H}}^2 = \left\| \mathbb{E}\left( \prod_{\ell=1}^d K_\ell(X^{(\ell)}, \cdot) \right) - \bigotimes_{\ell=1}^d m_\ell \right\|_{\mathcal{H}}^2.
\]
where $\otimes_{\ell=1}^{d} m_\ell = m_1 \otimes m_2 \otimes \cdots \otimes m_d$. It is known that if the kernels $K_\ell$ are characteristic ones, then $d$HSIC fully characterizes joint independence since the equality $P_X = P_{X^{(1)}} \otimes P_{X^{(2)}} \otimes \cdots \otimes P_{X^{(d)}}$ is equivalent to $d$HSIC$(X) = 0$ (see Pfister et al. (2018)). So, in order to test for joint independence, that is testing for the hypothesis $H_0: P_X = P_{X^{(1)}} \otimes P_{X^{(2)}} \otimes \cdots \otimes P_{X^{(d)}}$, a consistent estimator of $d$HSIC$(X)$ can be used as test statistic. Let $\{X_i\}_{1 \leq i \leq n}$ be an i.i.d. sample of $X$, with $X_i = (X_i^{(1)}, \cdots, X_i^{(d)})$, Pfister et al. (2018) introduced the statistic $\hat{d}$HSIC$_n$ given by

\[
\hat{d}$HSIC$_n = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \prod_{\ell=1}^{d} K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \right) + \frac{1}{n^{2d}} \sum_{i_1=1}^{n} \cdots \sum_{i_{2d}=1}^{n} \prod_{\ell=1}^{d} K_\ell(X_{i_{2\ell-1}}^{(\ell)}, X_{i_{2\ell}}^{(\ell)}) \\
- \frac{2}{n^{d+1}} \sum_{i_1=1}^{n} \cdots \sum_{i_{d+1}=1}^{n} \prod_{\ell=1}^{d} K_\ell(X_{i_{2\ell}}^{(\ell)}, X_{i_{2\ell+1}}^{(\ell)})
\] (3)

if $n \geq 2d$, and $\hat{d}$HSIC$_n = 0$ if $n \in \{1, \cdots, 2d - 1\}$. They proved that, under $H_0$, the sequence $n \hat{d}$HSIC$_n$ converges in distribution, as $n \to +\infty$, to $\left( \sum_{m=1}^{+\infty} \lambda_m Z_m^2 \right)$, where $(\lambda_m)_{m\geq1}$ is an appropriate sequence of positive real numbers and $(Z_m)_{m\geq1}$ is a sequence of independent standard normal random variables. This limiting distribution cannot be used for performing the test because it is an infinite sum of distributions. That is why Pfister et al. (2018) introduced three approach for this test based on the above estimator: a permutation test, a bootstrap analogue and a procedure based on a gamma approximation. We propose in this paper another estimator for which we can obtain asymptotic normality under the null hypothesis. The main advantage is that it permits to avoid the use of permutation or bootstrap methods for achieving the test, so leading to a faster procedure.

3. MODIFIED ESTIMATOR AND ASYMPTOTIC NORMALITY

A natural approach for estimating $d$HSIC$(X)$ consists in replacing in (2) each
expectation by its empirical counterpart. This leads to the naive estimator
\[
\tilde{D}_n = \left\| \frac{1}{n} \sum_{i=1}^{n} K(X_i, \cdot) - \otimes_{\ell=1}^{d} \hat{m}_\ell \right\|^2_n
\]
\[
= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \prod_{\ell=1}^{d} K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \right) + \frac{1}{n^{2d}} \prod_{\ell=1}^{d} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \right)
\]
\[
- \frac{2}{n^{d+1}} \sum_{i=1}^{n} \prod_{\ell=1}^{d} \left( \sum_{j=1}^{n} K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \right),
\]
(4)

where \( \hat{m}_\ell = \frac{1}{n} \sum_{i=1}^{n} K_\ell(X_i^{(\ell)}, \cdot) \). Instead, we adopt an approach introduced in Makigusa and Naito (2020), and also used in Balogoun et al. (2021), consisting in introducing a modification in (4) in order to get a test statistic for which asymptotic normality can be obtained both under \( H_0 \) and under \( H_1 \).

For \( \gamma \in [0, 1] \), let \( (w_{i,n}(\gamma))_{1 \leq i \leq n} \) be a sequence of positive numbers satisfying:

\( (\mathcal{A}_2) \): There exists a strictly positive real number \( \tau \) and an integer \( n_0 \) such that for all \( n > n_0 \):
\[
n \left| \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) - 1 \right| \leq \tau.
\]

\( (\mathcal{A}_3) \): There exists \( C > 0 \) such that \( \max_{1 \leq i \leq n} w_{i,n}(\gamma) < C \) for all \( n \in \mathbb{N}^* \) and all \( \gamma \in [0, 1] \).

\( (\mathcal{A}_4) \): For any \( \gamma \in [0, 1] \), \( \lim_{n \to +\infty} \frac{1}{n} \sum_{i=1}^{n} w_{i,n}^2(\gamma) = w^2(\gamma) > 1 \).

An example of such sequence was given in Ahmad (1993) as \( w_{i,n}(\gamma) = 1 + (-1)^i \gamma \). For this example, one has \( C = 2 \), \( w^2(\gamma) = 1 + \gamma^2 \) and \( \tau \) is any positive real number. Another example is \( w_{i,n}(\gamma) = 1 + \sin(i\pi\gamma) \) which corresponds to \( \tau = 1/|\sin(\pi\gamma/2)| \), \( C = 2 \) and \( w^2(\gamma) = 3/2 \).

We then propose to estimate \( d\text{HSIC}(X) \) by a modification of (4) given by
\[
\hat{D}_{\gamma,n} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \prod_{\ell=1}^{d} K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \right) + \frac{1}{n^{2d}} \prod_{\ell=1}^{d} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \right)
\]
\[
- \frac{2}{n^{d+1}} \sum_{i=1}^{n} w_{i,n}(\gamma) \prod_{\ell=1}^{d} \left( \sum_{j=1}^{n} K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \right),
\]
(5)
and we take this estimator as test statistic for testing for $\mathcal{H}_0$ against $\mathcal{H}_1$. For achieving this test, the asymptotic distribution under $\mathcal{H}_0$ of the test statistic is needed. We will obtain an asymptotic normality result which is usable both under $\mathcal{H}_0$ and under $\mathcal{H}_1$, so permitting to perform the test. For any $(k, q) \in \mathbb{N}^*$ such that $q \geq k$, we put $\nu^q_k = \bigotimes_{\ell=k}^q m_\ell$. Then, considering the functions $U$ and $V$ from $\mathcal{X}$ to $\mathbb{R}$ defined on any $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathcal{X}$ by

$$U(x) = \left\langle K_1(x^{(1)}, \cdot) \otimes \nu^d_2 \right. \right.$$  

$$+ \sum_{\ell=2}^{d-1} \nu^{\ell-1}_1 \otimes K_\ell(x^{(\ell)}, \cdot) \otimes \nu^d_{\ell+1} + \nu^{d-1}_1 \otimes K_d(x^{(d)}, \cdot) - d \nu^d_1, \nu^d_1 - m \right\rangle_{\mathcal{H}}$$  

$$+ \left\langle K(x, \cdot) - m, m \right\rangle_{\mathcal{H}}$$

and $V(x) = \left\langle K(x, \cdot) - m, \nu^d_1 \right\rangle_{\mathcal{H}}$, we have:

**Theorem 1** Assume that $(\mathcal{A}_1)$ to $(\mathcal{A}_4)$ hold. Then as $n \to +\infty$, one has

$$\sqrt{n} \left( \hat{D}_{\gamma,n} - d\text{HSIC}(X) \right) \xrightarrow{p} \mathcal{N}(0, \sigma^2_{\gamma}),$$

where:

$$\sigma^2_{\gamma} = 4\text{Var} (U(X_1)) + 4w^2(\gamma)\text{Var} (V(X_1)) - 8\text{Cov} (U(X_1), V(X_1)).$$

This theorem gives asymptotic normality in the general case. The particular case where $\mathcal{H}_0$ holds can then be deduced. Indeed, in this case one has $\nu^d_1 = \bigotimes_{\ell=1}^d m_\ell = m$ and, therefore, $U = V$. Since $d\text{HSIC}(X) = 0$ it follows that

$$\sqrt{n} \hat{D}_{\gamma,n} \xrightarrow{p} \mathcal{N}(0, \sigma^2_{\gamma}),$$

where:

$$\sigma^2_{\gamma} = 4(w^2(\gamma) - 1)\text{Var} (V(X_1)).$$

This asymptotic variance in unknown since it depends on the distribution of $X$ which is unknown. So, in order to perform the test, we have to seek a consistent estimator of it. The following proposition gives such an estimator:
Proposition 1 Assume that \((\mathcal{A}_1)\) to \((\mathcal{A}_4)\) hold, and put \(\hat{\sigma}_\gamma^2 = 4(w(\gamma)^2 - 1)\hat{\alpha}^2_n\), where \[
\hat{\alpha}^2_n = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \left[ \prod_{\ell=1}^{d} K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \right] \right)^2 - \left( \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \prod_{\ell=1}^{d} K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \right] \right)^2.
\]
Then, under \(\mathcal{H}_0\), \(\hat{\sigma}_\gamma^2\) converges in probability to \(\sigma_\gamma^2\) as \(n \to +\infty\).

We deduce from the previous theorem and proposition that, under the hypothesis \(\mathcal{H}_0\), \(\sqrt{n} \hat{\sigma}_\gamma^{-1} \hat{D}_{\gamma,n} \xrightarrow{d} \mathcal{N}(0,1)\) as \(n \to +\infty\). This allows to perform the test: for a given significant level \(\alpha \in ]0,1[\), one has to reject \(\mathcal{H}_0\) if \(|\hat{D}_{\gamma,n}| > n^{-1/2} \hat{\sigma}_\gamma \Phi^{-1}(1-\alpha/2)\) where \(\Phi\) is the cumulative distribution function of the standard normal distribution.

Remark 1. This test can easily be applied on functional data. Indeed, suppose that each \(X^{(\ell)}_i\) is a random functions belonging to \(L^2([0,1])\) and observed on points \(t^{(\ell)}_1, \ldots, t^{(\ell)}_{r^{(\ell)}}\) of a fine grid in \([0,1]\) such that \(t^{(\ell)}_1 = 0\) and \(t^{(\ell)}_{r^{(\ell)}} = 1\). If gaussian kernels are used, one has

\[
K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) = \exp \left( -\eta_\ell^2 \|X_i - X_j\|^2 \right) = \exp \left( -\eta_\ell^2 \int_0^1 \left( X_i^{(\ell)}(t) - X_j^{(\ell)}(t) \right)^2 dt \right),
\]

where \(\eta_\ell > 0\), and this term can be approximated by using trapezoidal rule:

\[
K_\ell(X_i^{(\ell)}, X_j^{(\ell)}) \simeq \exp \left( -\eta_\ell^2 \sum_{m=1}^{r^{(\ell)}-1} \left( \frac{t^{(\ell)}_{m+1} - t^{(\ell)}_m}{2} \left( X_i^{(\ell)}(t^{(\ell)}_m) - X_j^{(\ell)}(t^{(\ell)}_m) \right)^2 \right. \right.
\left. + \left. \left( X_i^{(\ell)}(t^{(\ell)}_m) - X_j^{(\ell)}(t^{(\ell)}_{m+1}) \right)^2 \right) \right). \tag{6}
\]

Then, \(\hat{D}_{\gamma,n}\) and \(\hat{\alpha}^2_n\) are to be computed by using \((6)\).

4. SIMULATIONS

In this section, we present a simulation study made in order to evaluate the finite sample performance of the proposed test and compare it to the test of Pfister et al. (2018) based on a permutation method using \(\hat{d}\text{HSIC}_n\). For convenience, we denote our test as \(T_1\), and the test of Pfister et al.
(2018) as T2. We computed empirical sizes and powers through Monte Carlo simulations. We considered the case where \( d = 3, X_1 = X_2 = X_3 = L^2([0,1]) \), and used the following models for generating functional data:

\[
X^{(1)}(t) = \sqrt{2} \sum_{k=1}^{50} \alpha_k \cos(k\pi t), \quad X^{(2)}(t) = \sqrt{2} \sum_{k=1}^{50} \beta_k \cos(k\pi t) + \lambda f(X^{(1)}(t)),
\]

\[
X^{(3)}(t) = \sqrt{2} \sum_{k=1}^{50} \xi_k \cos(k\pi t) + \lambda f(X^{(1)}(t)) + \lambda f(X^{(2)}(t)),
\]

where the \( \alpha_k \)'s, the \( \beta_k \)'s and the \( \xi_k \)'s are independent and i.i.d. having the uniform distribution on \([0,1]\), \( f \) is a given function and \( \lambda \) is a real number giving the level of dependence between the functional variables. Indeed, \( H_0 \) holds in case \( \lambda = 0 \), and the dependence level increases with \( \lambda \). Empirical sizes and powers were computed on the basis of 200 independent replicates. For each of them, we generated samples of size \( n = 100 \) of the above processes in discretized versions on equispaced values \( t_1, \ldots, t_{51} \) in \([0,1]\), where \( t_j = (j-1)/50, j = 1, \ldots, 51 \). For performing our method, we took \( \gamma = 0.32 \) and used the gaussian kernels

\[
K_1(x, y) = K_2(x, y) = K_3(x, y) = \exp\left(-150 \int_0^1 (x(t) - y(t))^2 dt\right).
\]

The test statistics given in (3) and (5) were computed by approximating integrals involved in the above kernels by using the trapezoidal rule, as indicated in (6). The significance level was taken as \( \alpha = 0.05 \). The method T2 was used with 100 permutations. Table 1 reports the obtained results. They are close to the nominal size for both methods when \( \lambda = 0 \). For \( \lambda = 0.25 \), T2 gives slightly larger powers than T1, but the differences are low. For \( m = 0.5,1 \), the two methods give maximal power, except for the case where \( f(x) = \sin(x) \) for which the powers are surprisingly low. This highlights the interest of the proposed test: it is powerful enough and is fast compared to the method of Pfister et al. (2018) based on permutations which leads to very high computation times.

5. PROOFS

5.1. PRELIMINARY RESULTS

We first give some technical lemmas which will be useful for proving the main theorems.
Lemma 1 As $n \to +\infty$, we have

$$\sqrt{n} \left( \bigotimes_{\ell=1}^{d} \hat{m}_{\ell} - \bigotimes_{\ell=1}^{d} m_{\ell} \right) \overset{\mathcal{D}}{\to} \mathcal{W},$$

(7)

where $\mathcal{W}$ is a random variable having the normal distribution in $\mathcal{H}$ with mean 0 and covariance operator equal to that of

$$K_1(X^{(1)}, \cdot) \otimes \bigotimes_{\ell=2}^{d} m_{\ell} + \sum_{\ell=2}^{d-1} \left( \bigotimes_{k=1}^{\ell-1} m_k \otimes K_\ell(X^{(\ell)}, \cdot) \otimes \bigotimes_{k=\ell+1}^{d} m_k \right) + \bigotimes_{\ell=1}^{d-1} m_{\ell} \otimes K_d(X^{(d)}, \cdot).$$

Proof. Let us consider the random variables valued into $\mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_d$ defined as

$$Z = \left( K_1(X^{(1)}, \cdot), K_2(X^{(2)}, \cdot), \ldots, K_d(X^{(d)}, \cdot) \right),$$

$$Z_i = \left( K_1(X^{(1)}_i, \cdot), K_2(X^{(2)}_i, \cdot), \ldots, K_d(X^{(d)}_i, \cdot) \right), i = 1, \ldots, n,$$
with mean equal to $\mu = (m_1, m_2, \ldots, m_d)$. For each $\ell \in \{1, \cdots, d\}$, we have
\[ \sqrt{n}(\hat{m}_\ell - m_\ell) = \pi_\ell(\hat{H}_n), \]
where $\hat{H}_n = \sqrt{n}\left(\frac{1}{n} \sum_{i=1}^n Z_i - \mu\right)$ and $\pi_\ell$ is the canonical projection:
\[ \pi_\ell : (f_1, f_2, \ldots, f_d) \in H_1 \times H_2 \times \cdots \times H_d \mapsto f_\ell \in H_\ell. \]

Then, from the decomposition
\[
\sqrt{n}\left( \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right) = \left( \sqrt{n}(\hat{m}_1 - m_1) \right) \otimes \bigotimes_{\ell=2}^d \hat{m}_\ell \\
+ \sum_{\ell=2}^{d-1} \left( \bigotimes_{k=1}^{\ell-1} m_k \otimes \left( \sqrt{n}(\hat{m}_\ell - m_\ell) \right) \otimes \bigotimes_{k=\ell+1}^d \hat{m}_k \right) \\
+ \bigotimes_{\ell=1}^d m_\ell \otimes \left( \sqrt{n}(\hat{m}_d - m_d) \right)
\]
we get
\[
\sqrt{n}\left( \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right) = \hat{\Phi}_n(\hat{H}_n), \tag{8}
\]
where $\hat{\Phi}_n$ is the random operator from $H_1 \times H_2 \times \cdots \times H_d$ to $\mathcal{H}$ such that
\[
\hat{\Phi}_n(T) = \pi_1(T) \otimes \bigotimes_{\ell=2}^d \hat{m}_\ell + \sum_{\ell=2}^{d-1} \left( \bigotimes_{k=1}^{\ell-1} m_k \otimes \pi_\ell(T) \otimes \bigotimes_{k=\ell+1}^d \hat{m}_k \right) + \bigotimes_{\ell=1}^d m_\ell \otimes \pi_d(T).
\]

Considering the operator $\Phi$ from $H_1 \times H_2 \times \cdots \times H_d$ to $\mathcal{H}$ such that
\[
\Phi(T) = \pi_1(T) \otimes \bigotimes_{\ell=2}^d m_\ell + \sum_{\ell=2}^{d-1} \left( \bigotimes_{k=1}^{\ell-1} m_k \otimes \pi_\ell(T) \otimes \bigotimes_{k=\ell+1}^d m_k \right) + \bigotimes_{\ell=1}^d m_\ell \otimes \pi_d(T), \tag{9}
\]
we have
\[
\left\| \Phi_n(\hat{H}_n) - \Phi(\hat{H}_n) \right\|_{\mathcal{H}} \leq \left\| \pi_1(\hat{H}_n) \otimes \left( \bigotimes_{\ell=2}^d m_\ell \otimes \hat{m}_\ell - \bigotimes_{\ell=2}^d m_\ell \right) \right\|_{\mathcal{H}}
\[+ \sum_{\ell=2}^{d-1} \left\| \bigotimes_{p=1}^{\ell-1} m_\ell \otimes \pi_\ell(\hat{H}_n) \otimes \left( \bigotimes_{p=\ell+1}^d \hat{m}_p - \bigotimes_{p=\ell+1}^d m_p \right) \right\|_{\mathcal{H}}
\[+ \sum_{\ell=2}^{d-1} \left\| \bigotimes_{p=1}^{\ell-1} m_\ell \otimes \pi_\ell(\hat{H}_n) \otimes \left( \bigotimes_{p=\ell+1}^d \hat{m}_p - \bigotimes_{p=\ell+1}^d m_p \right) \right\|_{\mathcal{H}}(10)
\]
The central limit theorem ensures that \( \hat{H}_n \) converges in distribution, as \( n \to +\infty \), to a random variable \( H \) having a centered normal distribution in \( \mathcal{H} \) with covariance operator equal to that of \( Z \). Since \( \pi_\ell \) is continuous, we deduce that \( \pi_\ell(\hat{H}_n) \) converges in distribution, as \( n \to +\infty \), to \( \pi_\ell(H) \). On the other hand, from the law of large numbers each \( \hat{m}_\ell \) converges almost surely, as \( n \to +\infty \), to \( m_\ell \). Then, for any \( k \in \{1, \ldots, d-1\} \), we deduce from the inequality
\[
\left\| \bigotimes_{\ell=k}^d \hat{m}_\ell - \bigotimes_{\ell=k}^d m_\ell \right\|_{\mathcal{H}} = \left\| (\hat{m}_k - m_k) \otimes \bigotimes_{\ell=k+1}^d \hat{m}_\ell \right\|
\[+ \sum_{\ell=k+1}^{d-1} \left( \bigotimes_{j=1}^{\ell-1} m_j \otimes (\hat{m}_\ell - m_\ell) \otimes \bigotimes_{j=\ell+1}^d \hat{m}_j \right)
\[+ \bigotimes_{\ell=1}^{d-1} m_\ell \otimes (\hat{m}_d - m_d) \right\|_{\mathcal{H}}
\[\leq \left\| \hat{m}_k - m_k \right\|_{\mathcal{H}_k} \prod_{\ell=k+1}^d \left\| \hat{m}_\ell \right\|_{\mathcal{H}_\ell}
\[+ \sum_{\ell=k+1}^{d-1} \left( \prod_{j=1}^{\ell-1} \left\| m_j \right\|_{\mathcal{H}_j} \left\| \hat{m}_\ell - m_\ell \right\|_{\mathcal{H}_\ell} \prod_{j=\ell+1}^d \left\| \hat{m}_j \right\|_{\mathcal{H}_j} \right)
\[+ \left( \prod_{\ell=1}^{d-1} \left\| m_\ell \right\|_{\mathcal{H}_\ell} \right) \left\| \hat{m}_d - m_d \right\|_{\mathcal{H}_d}
\]
that \( \| \bigotimes_{k=1}^{d} \hat{m}_k - \bigotimes_{k=1}^{d} m_k \|_H \) converges almost surely, as \( n \to +\infty \), to 0. Therefore, from (10) we deduce that \( \Phi_n(\hat{H}_n) - \Phi(\hat{H}_n) \) converges in probability, as \( n \to +\infty \), to 0; consequently, \( \sqrt{n} \left( \bigotimes_{k=1}^{d} \hat{m}_k - \bigotimes_{k=1}^{d} m_k \right) = \Phi(\hat{H}_n) + o_P(1) \) and, since \( \Phi \) is continuous, Slutsky’s theorem implies the convergence in distribution, as \( n \to +\infty \), to \( \mathcal{W} = \Phi(H) \). This random variable has a centered normal distribution in \( \mathcal{H} \) with covariance operator equal to that of

\[
\Phi(Z) = K_1(X^{(1)}, \cdot) \otimes \bigotimes_{\ell=2}^{d} m_\ell + \sum_{\ell=2}^{d-1} \left( \bigotimes_{k=1}^{\ell-1} m_k \otimes K_\ell(X^{(\ell)}, \cdot) \otimes \bigotimes_{k=\ell+1}^{d} m_k \right) \\
+ \bigotimes_{\ell=1}^{d-1} m_\ell \otimes K_d(X^{(d)}, \cdot). 
\]

\[\square\]

**Lemma 2** Let \( (Z_i)_{1 \leq i \leq n} \) be an i.i.d. sample of a random variable \( Z \) valued into a Hilbert space \( H \) and admitting a mean \( m_Z \). If there exists a real number \( M > 0 \) such that \( \|Z\| \leq M \), then the statistic

\[
z_n^2 = \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, \overline{Z}_n \rangle^2 - \left( \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, \overline{Z}_n \rangle \right)^2, 
\]

where \( \overline{Z}_n = \frac{1}{n} \sum_{i=1}^{n} Z_i \), is a consistent estimator of \( \text{Var} \left( \langle Z, m_Z \rangle \right) \).

**Proof.** We have

\[
\frac{1}{n} \sum_{i=1}^{n} \langle Z_i, \overline{Z}_n \rangle^2 - \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, m_Z \rangle^2 = \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, \overline{Z}_n - m_Z \rangle^2 + \frac{n}{n} \sum_{i=1}^{n} \langle Z_i, m_Z \rangle \langle Z_i, \overline{Z}_n - m_Z \rangle;
\]

and using the Cauchy-Schwarz inequality, we get the inequalities

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, \overline{Z}_n - m_Z \rangle \right| \leq M^2 \| \overline{Z}_n - m_Z \|^2
\]

and

\[
\left| \frac{2}{n} \sum_{i=1}^{n} \langle Z_i, m_Z \rangle \langle Z_i, \overline{Z}_n - m_Z \rangle \right| \leq 2M^2 \| m_Z \| \| \overline{Z}_n - m_Z \|
\]

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from which we conclude that
\[ \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, Z_n \rangle^2 - \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, m_Z \rangle^2 = o_P(1) \]
since \( \| Z_n - m_Z \| = o_P(1) \). The law of large numbers and Slutsky’s theorem allow to conclude that the sequence \( \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, Z_n \rangle^2 \) converges in probability to \( \mathbb{E} \left( \langle Z, m_Z \rangle^2 \right) \) as \( n \to +\infty \). Similarly, from
\[ \left| \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, Z_n \rangle - \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, m_Z \rangle \right| \leq M \| Z_n - m_Z \|, \]
we get \( \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, Z_n \rangle - \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, m_Z \rangle = o_P(1) \), therefore, and \( \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, Z_n \rangle \) converges in probability to \( \mathbb{E} \left( \langle Z, m_Z \rangle \right) \) as \( n \to +\infty \) just like \( \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, m_Z \rangle \). So, \( \hat{s}_n^2 \) is a consistent estimator of \( \text{Var} \left( \langle Z, m_Z \rangle \right) \). \( \square \)

5.2. PROOF OF THEOREM \( \Box \)

Using \( \Box \) and the reproducing properties of \( K \), it is easy to see that
\[ \hat{D}_{\gamma,n} = \left\| \frac{1}{n} \sum_{i=1}^{n} K(X_i, \cdot) \right\|^2_{\mathcal{H}} + \left\| \bigotimes_{\ell=1}^{d} \hat{m}_{\ell} \right\|^2_{\mathcal{H}} - \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \left\langle K(X_i, \cdot), \bigotimes_{\ell=1}^{d} \hat{m}_{\ell} \right\rangle_{\mathcal{H}} \]
\[ = \left\| \hat{m} - m \right\|^2_{\mathcal{H}} - \left\| m \right\|^2_{\mathcal{H}} + 2\langle \hat{m}, m \rangle_{\mathcal{H}} + \left\| \bigotimes_{\ell=1}^{d} \hat{m}_{\ell} - \bigotimes_{\ell=1}^{d} m_{\ell} \right\|^2_{\mathcal{H}} - \left\| \bigotimes_{\ell=1}^{d} m_{\ell} \right\|^2_{\mathcal{H}} \]
\[ + 2\left\langle \bigotimes_{\ell=1}^{d} \hat{m}_{\ell}, \bigotimes_{\ell=1}^{d} m_{\ell} \right\rangle_{\mathcal{H}} - \frac{2}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \left\langle K(X_i, \cdot), \bigotimes_{\ell=1}^{d} \hat{m}_{\ell} \right\rangle_{\mathcal{H}}, \]
where \( \hat{m} = \frac{1}{n} \sum_{i=1}^{n} K(X_i, \cdot) \). Since \( d\text{HSIC}(X) = \left\| m \right\|^2_{\mathcal{H}} + \left\| \bigotimes_{\ell=1}^{d} m_{\ell} \right\|^2_{\mathcal{H}} + \)
\[ 2 \left\langle m, \bigotimes_{\ell=1}^d m_\ell \right\rangle \] and

\[ \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \left\langle K(X_i, \cdot), \bigotimes_{\ell=1}^d \hat{m}_\ell \right\rangle_H = \left\langle \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) K(X_i, \cdot), \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right\rangle_H \]

\[ + \left\langle \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) K(X_i, \cdot), \bigotimes_{\ell=1}^d m_\ell \right\rangle_H \]

\[ + \left\langle \hat{m} - m, \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right\rangle_H \]

\[ + \left\langle m, \bigotimes_{\ell=1}^d \hat{m}_\ell \right\rangle_H - \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \left\langle m, \bigotimes_{\ell=1}^d m_\ell \right\rangle_H \]

\[ + \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \left\langle m, \bigotimes_{\ell=1}^d m_\ell \right\rangle_H \]

it follows

\[ \sqrt{n} \left( \hat{D}_{\gamma,n} - d\text{HSIC}(X) \right) = A_n + B_n + C_n, \]

where

\[ A_n = \frac{1}{\sqrt{n}} \left\| \sqrt{n} (\hat{m} - m) \right\|_H^2 + \frac{1}{\sqrt{n}} \left\| \sqrt{n} \left( \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right) \right\|_H^2 \]

\[ - 2 \left\langle \hat{m} - m, \sqrt{n} \left( \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right) \right\rangle_H, \]

\[ B_n = -2 \left\langle \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) K(X_i, \cdot), \sqrt{n} \left( \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right) \right\rangle_H \]

\[ - 2 \sqrt{n} \left( \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \right) \left\langle m, \bigotimes_{\ell=1}^d m_\ell \right\rangle_H \]
and

\[ C_n = \sqrt{n} \left( -\|m\|_\mathcal{H}^2 + 2\langle \hat{m}, m \rangle_\mathcal{H} - \| \bigotimes_{\ell=1}^d m_\ell \|_\mathcal{H}^2 + 2\langle \bigotimes_{\ell=1}^d \hat{m}_\ell, \bigotimes_{\ell=1}^d m_\ell \rangle_\mathcal{H} \right) - 2\left( \frac{1}{n} \sum_{i=1}^n w_{i,n}(\gamma) K(X_i, \cdot) \bigotimes_{\ell=1}^d m_\ell \right)_\mathcal{H} - 2\langle m, \bigotimes_{\ell=1}^d \hat{m}_\ell \rangle_\mathcal{H}
\]

\[ + \frac{2}{n} \sum_{i=1}^n w_{i,n}(\gamma) \langle m, \bigotimes_{\ell=1}^d m_\ell \rangle_\mathcal{H} - \|m\|_\mathcal{H}^2 - \| \bigotimes_{\ell=1}^d m_\ell \|_\mathcal{H}^2 + 2\langle m, \bigotimes_{\ell=1}^d m_\ell \rangle_\mathcal{H} \]

\[ = \sqrt{n} \left( 2\langle \hat{m} - m, m \rangle_\mathcal{H} \right) - \frac{2}{n} \sum_{i=1}^n w_{i,n}(\gamma) \langle K(X_i, \cdot) - m, \bigotimes_{\ell=1}^d m_\ell \rangle_\mathcal{H}
\]

\[ - 2\langle \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell, \bigotimes_{\ell=1}^d m_\ell \rangle_\mathcal{H} \right). \quad (12) \]

First, using Cauchy-Schwartz inequality, we get

\[ |A_n| \leq \frac{1}{\sqrt{n}} \| \sqrt{n} (\hat{m} - m) \|_\mathcal{H}^2 + \frac{1}{\sqrt{n}} \left\| \sqrt{n} \left( \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right) \right\|_\mathcal{H}^2 \]

\[ + 2\|m\|_\mathcal{H} \left\| \sqrt{n} \left( \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right) \right\|_\mathcal{H}. \]

The law of large numbers ensures that \( \| \hat{m} - m \|_\mathcal{H} \to 0 \) as \( n \to +\infty \). Then, from Lemma 1 we deduce that \( \frac{1}{\sqrt{n}} \left\| \sqrt{n} \left( \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right) \right\|_\mathcal{H}^2 \) and \( \| \hat{m} - m \|_\mathcal{H} \left\| \sqrt{n} \left( \bigotimes_{\ell=1}^d \hat{m}_\ell - \bigotimes_{\ell=1}^d m_\ell \right) \right\|_\mathcal{H} \) converge in probability to 0 as \( n \to +\infty \).

The central limit theorem gives the convergence in distribution of \( \hat{m} - m \), as \( n \to +\infty \), to a random variable having a normal distribution in \( \mathcal{H} \), hence \( \frac{1}{\sqrt{n}} \left\| \sqrt{n} (\hat{m} - m) \right\|_\mathcal{H}^2 \) converges in probability to 0 as \( n \to +\infty \). Then, from the preceding inequality we get \( A_n = o_P(1) \). Secondly, another use of Cauchy-
Schwartz inequality gives for \( n \) large enough

\[
|B_n| \leq \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) K(X_i, \cdot) \right\|_{\mathcal{H}} \left\| \sqrt{n}\left( \bigotimes_{\ell=1}^{d} \hat{m}_\ell - \bigotimes_{\ell=1}^{d} m_\ell \right) \right\|_{\mathcal{H}} \\
+ 2\sqrt{n}\left| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \right| \left\| m \right\|_{\mathcal{H}} \left\| \bigotimes_{\ell=1}^{d} m_\ell \right\|_{\mathcal{H}} \\
\leq \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) K(X_i, \cdot) \right\|_{\mathcal{H}} \left\| \sqrt{n}\left( \bigotimes_{\ell=1}^{d} \hat{m}_\ell - \bigotimes_{\ell=1}^{d} m_\ell \right) \right\|_{\mathcal{H}} \\
+ \frac{2\tau}{\sqrt{n}} \left\| m \right\|_{\mathcal{H}} \left\| \bigotimes_{\ell=1}^{d} m_\ell \right\|_{\mathcal{H}}
\tag{13}
\]

since, from (\( \mathcal{A}_2 \)), we have \( n \left| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) \right| \leq \tau \). Lemma 1 in Manfoumbi Djonguet et al. (2022) ensures that \( \left\| \frac{1}{n} \sum_{i=1}^{n} (w_{i,n}(\gamma) - 1) K(X_i, \cdot) \right\|_{\mathcal{H}} = o_P(1) \), and since the sequence \( \left\| \sqrt{n}(\bigotimes_{\ell=1}^{d} \hat{m}_\ell - \bigotimes_{\ell=1}^{d} m_\ell) \right\|_{\mathcal{H}} \) converges in distribution, as \( n \to +\infty \), to an appropriate random variable (see Lemma 1), we deduce from (13) that \( B_n = o_P(1) \). Thirdly, reporting (8) in (12) gives

\[
C_n = \sqrt{n}\left\{ \frac{2}{n} \sum_{i=1}^{n} \left\langle K(X_i, \cdot) - m, m \right\rangle_{\mathcal{H}} - \frac{2}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \left\langle K(X_i, \cdot) - m, \bigotimes_{\ell=1}^{d} m_\ell \right\rangle_{\mathcal{H}} \right\} \\
+ 2\left\langle \Phi_n(\hat{H}_n) - \Phi(\hat{H}_n), \bigotimes_{\ell=1}^{d} m_\ell - m \right\rangle_{\mathcal{H}} + 2\left\langle \Phi(\hat{H}_n), \bigotimes_{\ell=1}^{d} m_\ell - m \right\rangle_{\mathcal{H}}
\tag{14}
\]

From

\[
\left\langle \Phi_n(\hat{H}_n) - \Phi(\hat{H}_n), \bigotimes_{\ell=1}^{d} m_\ell - m \right\rangle_{\mathcal{H}} \leq \left\| \Phi_n(\hat{H}_n) - \Phi(\hat{H}_n) \right\|_{\mathcal{H}} \left\| \bigotimes_{\ell=1}^{d} m_\ell - m \right\|_{\mathcal{H}}
\]

and the convergence in probability of \( \Phi_n(\hat{H}_n) - \Phi(\hat{H}_n) \) to 0 as \( n \to +\infty \) (see Lemma 1), we deduce that

\[
\left\langle \Phi_n(\hat{H}_n) - \Phi(\hat{H}_n), \bigotimes_{\ell=1}^{d} m_\ell - m \right\rangle_{\mathcal{H}} = o_P(1). \tag{15}
\]
We will now decompose the term \( \langle \Phi(\hat{H}_n), \bigotimes_{\ell=1}^{d} m_\ell - m \rangle \) using (9). For ease of notation we set \( \nu_k^q = \bigotimes_{\ell=k}^{d} m_\ell \), and we have

\[
\pi_1(\hat{H}_n) \otimes \bigotimes_{\ell=2}^{d} m_\ell = \sqrt{n}(\hat{m}_1 - m_1) \otimes \nu_2^q
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( K_1(X_i^{(1)}, \cdot) - m_1 \right) \otimes \nu_2^d
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( K_1(X_i^{(1)}, \cdot) \otimes \nu_2^d - \bigotimes_{\ell=1}^{d} m_\ell \right),
\]

\[
\bigotimes_{k=1}^{\ell-1} m_k \otimes \pi_\ell(\hat{H}_n) \otimes \bigotimes_{k=\ell+1}^{d} m_k = \nu_1^{\ell-1} \otimes \sqrt{n}(\hat{m}_\ell - m_\ell) \otimes \nu_\ell^{d+1}
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( K_\ell(X_i^{(\ell)}, \cdot) - m_\ell \right) \otimes \nu_\ell^{d+1}
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \nu_1^{\ell-1} \otimes K_\ell(X_i^{(\ell)}, \cdot) \otimes \nu_\ell^{d+1} - \bigotimes_{\ell=1}^{d} m_\ell \right),
\]

and

\[
\bigotimes_{\ell=1}^{d-1} m_\ell \otimes \pi_d(\hat{H}_n) = \nu_1^{d-1} \otimes (\sqrt{n}(\hat{m}_d - m_d))
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( K_d(X_i^{(d)}, \cdot) - m_d \right)
\]

\[
= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \nu_1^{d-1} \otimes K_d(X_i^{(d)}, \cdot) - \bigotimes_{\ell=1}^{d} m_\ell \right).
\]

Hence

\[
\Phi(\hat{H}_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ K_1(X_i^{(1)}, \cdot) \otimes \nu_2^d + \sum_{\ell=2}^{d-1} \left( \nu_1^{\ell-1} \otimes K_\ell(X_i^{(\ell)}, \cdot) \otimes \nu_\ell^{d+1} + \nu_1^{d-1} \otimes K_d(X_i^{(d)}, \cdot) - \bigotimes_{\ell=1}^{d} m_\ell \right) \right\}.
\]
and from (14) and (15) it follows that $C_n = D_n + o_P(1)$, where

$$\begin{align*}
D_n &= \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \left\{ \left\langle K(X_i, \cdot) - m, m \right\rangle_{\mathcal{H}} - w_{i,n}(\gamma) \left( \frac{d}{\ell} \right)^{\ell-1} \left\langle K(X_i^{(\ell)}, \cdot) - m, \right\rangle_{\mathcal{H}} \\
&+ \left\langle K_1(X_i^{(1)}, \cdot) \otimes \nu_2^d + \sum_{\ell=2}^{d-1} \left( \nu_1^{\ell-1} \otimes K_\ell(X_i^{(\ell)}, \cdot) \otimes \nu_2^d \right) + \left( \nu_1^{d-1} \otimes K_d(X_i^{(d)}, \cdot) - d \otimes m, \right)_{\mathcal{H}} \right\} + o_P(1) \\
&= \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathcal{U}(X_i) - w_{i,n}(\gamma) \mathcal{V}(X_i) \right) + o_P(1).
\end{align*}$$

Finally, $\sqrt{n} \left( \hat{D}_{\gamma,n} - d\text{HSIC}(X) \right) = E_n + o_P(1)$, where $E_n = \frac{2}{\sqrt{n}} \sum_{i=1}^{n} \left( \mathcal{U}(X_i) - w_{i,n}(\gamma) \mathcal{V}(X_i) \right)$ and, consequently, $\sqrt{n} \left( \hat{D}_{\gamma,n} - d\text{HSIC}(X) \right)$ has the same limiting distribution than $E_n$; it remains to derive this latter. Let us set

$$s_{n,\gamma}^2 = \sum_{i=1}^{n} \text{Var} \left( \mathcal{U}(X_i) - w_{i,n}(\gamma) \mathcal{V}(X_i) \right);$$

by similar arguments as in the proof of Theorem 1 in Makigusa and Naito (2020) we obtain that, for any $\varepsilon > 0$,

$$s_{n,\gamma}^{-2} \sum_{i=1}^{n} \int_{\left\{ x : \mathcal{U}(x) - w_{i,n}(\gamma) \mathcal{V}(x) \right\} > \varepsilon s_{n,\gamma}} \left( \mathcal{U}(x) - w_{i,n}(\gamma) \mathcal{V}(x) \right)^2 d\mathbb{P}_X(x)$$

converges to 0 as $n \to +\infty$. Then, by Section 1.9.3 in Serfling (1980),

$$\sqrt{n} s_{n,\gamma}^{-1} E_n \overset{P}{\to} N(0, 1).$$

However,

$$\left( \frac{s_{n,\gamma}}{\sqrt{n}} \right)^2 = \text{Var} \left( \mathcal{U}(X_1) \right) + \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) \text{Var} \left( \mathcal{V}(X_1) \right) - 2 \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) \text{Cov} \left( \mathcal{U}(X_1), \mathcal{V}(X_1) \right);$$

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from \((\mathcal{A}_2)\) and \((\mathcal{A}_4)\),
\[
\lim_{n \to +\infty} \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) = w^2(\gamma) \quad \text{and} \quad \lim_{n \to +\infty} \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) = 1.
\]
Thus
\[
\lim_{n \to +\infty} \left( \frac{1}{n} \sum_{i=1}^{n} w_{i,n}(\gamma) \right) = 1.
\]

5.3. PROOF OF PROPOSITION 1

Putting \(Z = K(X, \cdot)\), \(Z_i = K(X_i, \cdot)\) and \(Z_n = \frac{1}{n} \sum_{i=1}^{n} Z_i\), we have
\[
\|Z\|_H = \sqrt{K(X, X)} = \sqrt{\prod_{\ell=1}^{d} K_{\ell}(X^{(\ell)}, X^{(\ell)})} \leq \prod_{\ell=1}^{d} \|K_{\ell}\|_{\infty}^{1/2},
\]
\[
\langle Z_i, Z_n \rangle_H = \frac{1}{n} \sum_{j=1}^{n} \langle Z_i, Z_j \rangle_H = \frac{1}{n} \sum_{j=1}^{n} K(X_i, X_j) = \frac{1}{n} \sum_{j=1}^{n} \left( \prod_{\ell=1}^{d} K_{\ell}(X_i^{(\ell)}, X_j^{(\ell)}) \right)
\]
and
\[
\frac{1}{n} \sum_{i=1}^{n} \langle Z_i, Z_n \rangle^2 - \left( \frac{1}{n} \sum_{i=1}^{n} \langle Z_i, Z_n \rangle \right)^2 = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{n} \sum_{j=1}^{n} \left[ \prod_{\ell=1}^{d} K_{\ell}(X_i^{(\ell)}, X_j^{(\ell)}) \right] \right)^2
\]
\[
- \left( \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[ \prod_{\ell=1}^{d} K_{\ell}(X_i^{(\ell)}, X_j^{(\ell)}) \right] \right)^2
\]
\[
= \hat{\alpha}_n^2.
\]
By applying Lemma 2 we obtain the convergence in probability of \(\hat{\alpha}_n^2\) to \(\text{Var}((K(X, \cdot), m)_H) = \text{Var}(V(X_1))\) as \(n \to +\infty\) and, therefore, that of \(\hat{\sigma}_\gamma^2\) to \(\sigma_\gamma^2\).

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