Dbar-approach to coupled nonlocal NLS equation and general nonlocal reduction

Xueru Wang | Junyi Zhu

School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, Henan, People’s Republic of China

Correspondence
Junyi Zhu, School of Mathematics and Statistics, Zhengzhou University, 100 Kexue Road, Zhengzhou, Henan 450001, People’s Republic of China.
Email: jyzhu@zzu.edu.cn

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Abstract
The coupled nonlocal nonlinear Schrödinger (NLS) equation is studied by virtue of the $2 \times 2$ Dbar-problem. Two spectral transform matrices are introduced to define two associated Dbar-problems. The relations between the coupled nonlocal NLS potential and the solution of the Dbar-problem are constructed. The spatial transform method is extended to obtain the coupled nonlocal NLS equation and its conservation laws. The general nonlocal reduction of the coupled nonlocal NLS equation to the nonlocal NLS equation is discussed in detail. The explicit solutions are derived.

KEYWORDS
Dbar-problem, dressing method, general nonlocal reduction, nonlocal NLS equation

1 | INTRODUCTION

By virtue of a novel left-right Riemann–Hilbert problem, the inverse scattering transform of nonlocal nonlinear Schrödinger (nNLS) equation and a circumstantial comparison with the classical NLS equation are given in Refs. 1 and 2. The inverse scattering transform for the nNLS equation with nonzero boundary conditions at infinity is presented in Refs. 3 and 4. Under the $PT$-symmetric transformation, coupled nonlocal NLS equation and general vector nonlocal NLS equation are discussed in Refs. 5 and 6. Alice–Bob systems are introduced in Refs. 7 and 8. The long-time behavior of the nonlocal NLS equation was considered in Ref. 9. In particular, the nNLS equation admits both bright and dark solitons. 10 The higher order rational solitons of the nNLS equation are given in Refs. 11 and 12. Rogue waves 13,14 in the nonlocal $PT$-symmetric NLS equation are given in Refs. 15, 16. The multilinear form and some self-similar solutions are investigated in
Ref. 17. Discrete nonlocal NLS equation was presented in Refs. 10, 18, 19, and 20. The reverse-time nNLS equations are discussed in Refs. 21–26. A nonlocal derivative NLS equation is introduced. 27 Transformations between nonlocal and local integrable equations are presented in Ref. 28. Nonlocal reductions for nonlocal integrable equations are investigate in Refs. 29–31. In addition, the higher order rational solitons of the nNLS equation are also derived in Ref. 32, and a discrete PT-symmetric nonlocal NLS equation is discussed in Ref. 33.

̄(Dbar)-problem is an effective tool to study nonlinear evolution equations and to give their explicit solutions. 34–41 The Dbar-problem can be regarded as a generalization of the Riemann–Hilbert problem. For the Riemann–Hilbert problem, one needs to investigate the analytic regions of the eigenfunction for certain linear spectral problem associated with a nonlinear integrable system, where the boundary contour of the analytic regions is determined by the so-called dispersion relation. It can be regarded as a bridge to connect the physical space with the spectral space. For the Dbar-problem, one only needs to carry out the discussion in the spectral space, where the spectral transform matrix is curbed by certain dispersion relation, see (11) and (12). The Dbar dressing method and the certain symmetry conditions will establish the relation between the potential function of the nonlinear integrable system and the Dbar data. The Dbar approach with the spectral transform matrix containing impulses is equivalent to the inverse scattering transform in the reflectionless case, and the explicit solutions of the nonlinear integrable system can be obtained.

Recently, we extended the Dbar approach to study the NLS equation with nonzero boundary condition. 42 The existing research methods to nonlocal integrable equations are mainly the inverse scattering method (the Riemann–Hilbert problem) and the Darboux transformation. The Dbar-problem to investigate the nonlocal integrable equation is still an open problem. In this paper, we extend the Dbar-approach to investigate the coupled nonlocal NLS (cnNLS) equation

\[ iq_t(x, t) = q_{xx}(x, t) - 2\sigma q^2(x, t)\tilde{q}(-x, t), \]
\[ i\tilde{q}_t(x, t) = \tilde{q}_{xx}(x, t) - 2\sigma \tilde{q}^2(x, t)q(-x, t), \quad \sigma = \mp 1. \]

(1)

It is noted that, for the cnNLS equation (1), if \( \{q(x, t), \tilde{q}(x, t)\} \) is a set of solution so is \( \{q(x, -t), \tilde{q}(x, -t)\} \), and so is \( \{\tilde{q}(x, -t), q(-x, t)\} \). In addition, if let \( V(x, t) = -2\sigma q(x, t)\tilde{q}(-x, t) \) and \( \tilde{V}(x, t) = -2\sigma \tilde{q}(x, t)q(-x, t) \), then \( \tilde{V}(x, t) = \tilde{V}(-x, t) \).

Equation (1) reduces to the nNLS equation

\[ iq_t(x, t) = q_{xx}(x, t) - 2\sigma q^2(x, t)q(-x, t). \]

(2)

if \( q(x, t) = \tilde{q}(x, t) \). We note that the cnNLS equation (1) is derived from a 2×2 matrix linear problem, so it is different from the multicomponent or vector ones. 5,6

It is known that the relation between the NLS potential and the solution of the Dbar problem is established by the Dbar dressing method, among which a spectral transform matrix is introduced. The explicit solution can be given by choosing the spectral transform matrix with certain scattering data, which are called the Dbar data. While for cnNLS equation, we have to define two different spectral transform matrices \( R(k; x, t) \) and \( \tilde{R}(k; x, t) \), which give two associated Dbar problems \( \tilde{\psi}(k; x, t) = \psi(k; x, t)R(k; x, t) \) and \( \tilde{\psi}(k; x, t) = \tilde{\psi}(k; x, t)\tilde{R}(k; x, t) \). With the Dbar-approach to the cnNLS equation with \( \sigma = -1 \), we show a simple and clear picture about the reconstruction of the cnNLS potential about the scattering data that are equivalent to the Dbar data given by \( \{\lambda_i, d_i\}_{i=1}^N \) and \( \{k_j, \epsilon_j\}_{j=1}^N \) in the spectral transform matrices \( R(k; x, t) \) and \( \tilde{R}(k; x, t) \). For the first
spectral transform matrix, we have one set of representations

\[ q(x, t) = 2i \sum_{l=1}^{N} d_l e^{-2i\theta(-\lambda_l; x, t)} \psi_{11}(-\lambda_l; x, t), \]

\[ r(x, t) = -2i \sum_{j=1}^{N} c_j e^{2i\theta(x, k_j; x, t)} \psi_{22}(k_j; x, t), \]

and for the second spectral transform matrix, we obtain another set of representations

\[ \hat{q}(x, t) = -2i \sum_{j=1}^{N} \hat{c}_j e^{-2i\theta(-\hat{k}_j; x, t)} \hat{\psi}_{11}(-\hat{k}_j; x, t), \]

\[ \hat{r}(x, t) = 2i \sum_{l=1}^{N} \hat{d}_l e^{2i\theta(\lambda_l; x, t)} \hat{\psi}_{22}(\lambda_l; x, t), \]

where \( \theta(k; x, t) = kx - 2k^2 t \). It is noted that the eigenfunctions admit the following symmetry conditions

\[ \hat{\psi}_{11}(k; x, t) = \overline{\psi_{22}(-k; -x, t)}, \quad \hat{\psi}_{12}(k; x, t) = -\sigma \overline{\psi_{21}(-k; -x, t)}, \]

\[ \hat{\psi}_{21}(k; x, t) = -\sigma \overline{\psi_{12}(-k; -x, t)}, \quad \hat{\psi}_{22}(k; x, t) = \psi_{11}(-k; -x, t). \]

Then we find \( r(x, t) = \sigma \overline{\hat{q}(-x, t)} \) and \( \hat{r}(x, t) = \sigma \overline{q(-x, t)} \). In addition, we extended the spatial transform method\(^{43}\) to find the cnNLS equation and its conservation laws.

It is remarked that the choice of the parameters \( \{\lambda_l, d_l\}_{l=1}^{N} \) and \( \{k_j, c_j\}_{j=1}^{N} \) for obtaining the explicit solutions of the cnNLS equation (1) is more free. To construct the solution of nNLS equation (2), one needs to consider the reduction and to introduce some constraint conditions on the parameters to make sure that \( \hat{q}(x, t) = q(x, t) \). We note that the current nonlocal reductions are usually to construct the first few solutions (\( N = 1, 2, 3 \)) for nonlocal equation, but very few investigations for the general nonlocal reduction are presented. Here, we express the solution with two sets of special determinants of symmetry matrices and give a full discussion of the general nonlocal reduction for cnNLS equation. We show that the constraint conditions are \( N = \hat{N}, \ k_j = ib_j, \ \lambda_j = i\eta_j \) are imaginary numbers, and \( \prod_{1 \leq m < m' \leq N} (\eta_m - \eta_{m'}) \) as well as

\[ |c_j|^2 = \frac{\prod_{l=1}^{N} (\eta_l - b_j)^2}{\prod_{s=1, s \neq j}^{N} (b_s - b_j)^2}, \quad |d_j|^2 = \frac{\prod_{l=1}^{N} (\eta_j - b_l)^2}{\prod_{s=1, s \neq j}^{N} (\eta_s - \eta_j)^2}. \]

The outline of this paper is as follows. In Section 2, we introduce two local Dbar problems. In Section 3, we derive the focusing/defocusing cnNLS equation and its conservation laws. In Section 4, we present the explicit solutions for the focusing cnNLS equation. In Section 5, we discuss the nonlocal reductions to the nNLS equation in detail.
2 | DOUBLE DBAR-PROBLEMS AND DRESSING METHOD

Consider the first local Dbar-problem

\[ \bar{\delta}\psi(k) = \psi(k)R(k), \tag{7} \]

with the normalization condition

\[ \psi(k) \to I, \quad k \to \infty, \tag{8} \]

where \( R(k) \) is the spectral transform matrix. The Dbar-problem (7) and (8) equivalent to the following integral equation:

\[ \psi(k) = I + \psi(k)R(k)C_k, \tag{9} \]

where the Cauchy–Green operate in complex plane is defined as

\[ \psi(k)R(k)C_k = \frac{1}{2\pi i} \iint \frac{dz \wedge d\bar{z}}{z - k} \psi(k)R(k). \tag{10} \]

The aim of dressing method is construct the relation between the cnNLS potential and the solution of the Dbar-problem. To this end, a good way is to construct the cnNLS equation and its Lax pair from the Dbar-problem. It is noted that the Dbar-problem is defined in the spectral space, whereas the cnNLS equation is in the physical space. Thus, we need to introduce the physical variables \( x, t \) into the function \( \psi(k) \), which can be done by extending the spectral transform matrix to be the form \( R(k; x, t) \), and letting

\[ R_x(k; x, t) = -ik[\sigma_3, R(k; x, t)], \tag{11} \]

\[ R_t(k; x, t) = 2ik^2[\sigma_3, R(k; x, t)]. \tag{12} \]

We note that the solution of the system (11) and (12) is not unique. Under the dressing procedure, \(^{37–39} \) we find that

\[ \psi_x(k; x, t) = -ik[\sigma_3, \psi(k; x, t)] + Q(x, t)\psi(k; x, t), \tag{13} \]

\[ Q(x, t) = -i[\sigma_3, \langle \psi(k; x, t)R(k; x, t) \rangle], \]

and

\[ \psi_t(k; x, t) = 2ik^2[\sigma_3, \psi(k; x, t)] - 2kQ(x, t)\psi(k; x, t) \]

\[ + i\sigma_3[Q^2(x, t) - Q_x(x, t)]\psi(k; x, t). \tag{14} \]
where
\[ \langle \psi(k; x, t)R(k; x, t) \rangle = \frac{1}{2\pi i} \oint \psi(k; x, t)R(k; x, t)dk \wedge d\bar{k}. \] (15)

For the cnNLS equation, we need to consider the second local Dbar problem
\[ \bar{\delta}\hat{\psi}(k; x, t) = \hat{\psi}(k; x, t)\hat{R}(k; x, t), \]
\[ \hat{\psi}(k; x, t) \to I, \quad k \to \infty, \] (16)
where the new spectral transform matrix \( \hat{R}(k; x, t) \) is another solution of the evolution system (11) and (12). Then we have
\[ \hat{\psi}(k; x, t) = I + \hat{\psi}(k; x, t)\hat{R}(k; x, t)C_k. \] (17)
A similar procedure gives another potential \( \hat{Q}(x, t) \)
\[ \hat{Q}(x, t) = -i[\sigma_3, \langle \hat{\psi}(k; x, t)\hat{R}(k; x, t) \rangle], \] (18)
and the another linear spectral system
\[ \hat{\psi}_x(k; x, t) = -ik[\sigma_3, \hat{\psi}(k; x, t)] + \hat{Q}(x, t)\hat{\psi}(k; x, t), \] (19)
and
\[ \hat{\psi}_t(k; x, t) = 2ik^2[\sigma_3, \hat{\psi}(k; x, t)] - 2k\hat{Q}(x, t)\hat{\psi}(k; x, t) \]
\[ + i\sigma_3[\hat{Q}^2(x, t) - \hat{Q}_x(x, t)]\hat{\psi}(k; x, t). \] (20)

In addition, to get the cnNLS equation, one also needs to introduce a symmetry condition about the two potentials
\[ \hat{Q}(x, t) = -\Lambda\overline{Q(-x, t)}\Lambda^{-1}, \quad \Lambda = \begin{pmatrix} 0 & 1 \\ -\sigma & 0 \end{pmatrix}, \] (21)
then the symmetry condition of the eigenfunction takes the following form
\[ \hat{\psi}(k; x, t) = \Lambda\overline{\psi(-k; -x, t)}\Lambda^{-1}. \] (22)
Thus, we have
\[ Q = \begin{pmatrix} 0 & q(x, t) \\ \sigma\overline{q(-x, t)} & 0 \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} 0 & \hat{q}(x, t) \\ \sigma\overline{q(-x, t)} & 0 \end{pmatrix}. \] (23)
\[ \dot{\psi}_{11}(k; x, t) = \overline{\psi}_{22}(-\bar{k}; -x, t), \quad \dot{\psi}_{12}(k; x, t) = -\sigma \overline{\psi}_{21}(-\bar{k}; -x, t), \]
\[ \dot{\psi}_{21}(k; x, t) = -\sigma \overline{\psi}_{12}(-\bar{k}; -x, t), \quad \dot{\psi}_{22}(k; x, t) = \overline{\psi}_{11}(-\bar{k}; -x, t). \]  

(24)

3  THE CNNLS EQUATION AND CONSERVATION LAWS

From (8), we know that \( \psi(k; x, t) \) has the following asymptotic behaviors

\[ \psi(k; x, t) = \sum_{n=0}^{\infty} \frac{a_n(x, t)}{k^n}, \quad k \to \infty, \]  

(25)

where \( a_0(x, t) = I \). In view of the temporal linear spectral problem (14), we get

\[ \psi_t(k; x, t) + 2ik^2 \psi(k; x, t) \sigma_3 = T(k; x, t) \psi(k; x, t), \]  

(26)

where

\[ T(k; x, t) = 2ik^2 \sigma_3 - 2kQ(x) + i\sigma_3(Q^2(x) - Q_x(x)). \]  

(27)

Let the trace of the spectral transform matrix \( R(k; x, t) \) is zero, then the Dbar problem (7) implies \( \bar{\partial} \det \psi(k; x, t) = 0 \), and further \( \det \psi(k; x, t) = 1 \) in view of the asymptotic behaviors (25). Since \( \psi^{-1} = \sigma_2 \psi^T \sigma_2 \), Equation (26) can be rewritten as

\[ T(k; x, t) = \psi_t(k; x, t) \sigma_2 \psi^T(k; x, t) \sigma_2 + 2k^2 \Psi(k; x, t) \sigma_2, \]  

(28)

where

\[ \Psi = \psi \sigma_1 \psi^T. \]  

(29)

It is noted that

\[ \Psi(k; x, t) = \sum_{n=0}^{\infty} \frac{\Psi_n(x, t)}{k^n}, \quad k \to \infty, \]  

(30)

where

\[ \Psi_0(x, t) = \sigma_1, \quad \Psi_n(x, t) = \sum_{m=0}^{n} a_m \sigma_1 a^T_{n-m}, \quad (n \geq 1). \]  

(31)

Substituting the expansions (25) and (30) into (28), and considering the \( O(k^{-1}) \) items, we obtain

\[ a_{1,t}(x, t) = -2\Psi_3(x, t) \sigma_2. \]  

(32)
In view of the spatial linear spectral problem (13), $\Psi$ satisfies

$$\Psi_x(k; x, t) = -ik[\sigma_3 \Psi(k; x, t) + \Psi(k; x, t)\sigma_3] + Q\Psi(k; x, t) + \Psi(k; x, t)Q^T,$$

(33)

which can be rewritten as

$$\Psi_x^{[d]}(k; x, t) = -2ik\sigma_3\Psi^{[d]}(k; x, t) + Q\Psi^{[o]}(k; x, t) + \Psi^{[o]}(k; x, t)Q^T,$$

(34)

and

$$\Psi_x^{[o]}(k; x, t) = Q\Psi^{[d]}(k; x, t) + \Psi^{[d]}(k; x, t)Q^T,$$

(35)

where $\Psi^{[d]}$ and $\Psi^{[o]}$ denote the diagonal part and off-diagonal part of the matrix $\Psi$. Substituting the expansion (30) into (34) and (35)

$$\Psi_1^{[d]} = -i\sigma_3 Q\sigma_1,$$

$$\Psi_{n+1}^{[d]} = \frac{i}{2}\sigma_3 \Psi_n^{[d]} - \frac{i}{2} \sigma_3 \left( Q\Psi_n^{[o]} + \Psi_n^{[o]} Q^T \right),$$

(36)

$$\Psi_n^{[o]} = \left( Q\Psi_n^{[d]} + \Psi_n^{[d]} Q^T \right).$$

For simplicity, here and after we omit the variables $(x, t)$. From the above recurrent formula, we find

$$\Psi_1^{[d]} = -Q\sigma_2, \quad \Psi_1^{[o]} = 0,$$

$$\Psi_2^{[d]} = \frac{1}{2}Q_x \sigma_1, \quad \Psi_2^{[o]} = \frac{1}{2}Q^2 \sigma_1,$$

$$\Psi_3^{[d]} = \frac{1}{4}Q_{xx} \sigma_2 - \frac{1}{2}Q^2 \sigma_2, \quad \Psi_3^{[o]} = \frac{1}{4}(QQ_x - Q_x Q) \sigma_2,$$

$$\Psi_4^{[d]} = \left( -\frac{1}{8}Q_{xxx} + \frac{3}{4}Q^2 Q_x \right) \sigma_1,$$

$$\Psi_4^{[o]} = -\frac{1}{8}(QQ_{xx} + Q_{xx} Q - Q_x^2 - 3Q^4) \sigma_1.$$

(37)

From the off-diagonal part of Equation (32), we obtain the nonlinear equation

$$i\sigma_3 Q_t - Q_{xx} + 2Q^2 Q = 0,$$

(38)

which implies the nonlocal NLS equation (1). From the diagonal part of (32), we find the first conservation law

$$i(qr)_t = (q_x r - qr_x)_x, \quad r = \sigma q(-x, t).$$

(39)

Similarly, the $O(k^{-2})$ terms in the expansion of (28) have the following form

$$a_{2,t} + a_{1,t} \sigma_2 a_1^T \sigma_2 + 2\Psi_4 \sigma_2 = 0,$$

(40)
The off-diagonal part of Equation (40) also implies Equation (38), and the diagonal part gives the second conservation law

$$i(q_r)_t = (q_x r_x + q^2 r^2 - q r_{xx})_x, \quad r = \sigma \bar{q}(-x, t). \quad (41)$$

The more conservation laws of the cnNLS equation can be derived similarly from (28). The same results can be derived from the second linear system (19) and (20).

## 4 THE SOLUTIONS OF COUPLED NONLOCAL FOCUSING NLS EQUATION

In this section, we give the explicit solutions of cnNLS equation in the case \( \sigma = -1 \).

According to the above symmetry conditions, we let the first spectral transform matrix \( R(k; x, t) \) has the following form:

$$R(k; x, t) = \pi \begin{pmatrix} 0 & \sum_{j=1}^{N} \tilde{d}_j e^{-2i\vartheta(k;x,t)} \delta(k + \tilde{\lambda}_j) \\ \sum_{j=1}^{N} c_j e^{2i\vartheta(k;x,t)} \delta(k - k_j) & 0 \end{pmatrix}, \quad (42)$$

and take the second spectral transform matrix be of the form

$$\hat{R}(k; x, t) = -\pi \begin{pmatrix} 0 & \sum_{j=1}^{N} \tilde{c}_j e^{-2i\vartheta(k;x,t)} \delta(k + \bar{k}_j) \\ \sum_{j=1}^{N} d_j e^{2i\vartheta(k;x,t)} \delta(k - \lambda_j) & 0 \end{pmatrix}, \quad (43)$$

where

$$\vartheta(k; x, t) = kx - 2k^2 t. \quad (44)$$

From Equations (42) and (13), we get

$$q(x, t) = 2i \sum_{l=1}^{N} \tilde{d}_l e^{-2i\vartheta(-\tilde{\lambda}_l;x,t)} \psi_{11}(-\tilde{\lambda}_l; x, t),$$

$$r(x, t) = -2i \sum_{j=1}^{N} c_j e^{2i\vartheta(x,k_j;x,t)} \psi_{22}(k_j; x, t). \quad (45)$$
Equations (43) and (18) imply another representations of the solution of the cnNLS equation

\[
\hat{q}(x, t) = -2i \sum_{j=1}^{N} \hat{c}_j e^{-2i\theta(-\hat{k}_j; x, t)} \hat{\psi}_{11}(-\hat{k}_j; x, t), \\
\hat{r}(x, t) = 2i \sum_{l=1}^{N} \hat{d}_l e^{2i\theta(\hat{\lambda}_l; x, t)} \hat{\psi}_{22}(\hat{\lambda}_l; x, t).
\] (46)

Using the symmetry condition (24) and \(\theta(-\hat{k}; -x, t) = \theta(k; x, t)\), we find that \(r(x, t) = -\hat{q}(-x, t)\) and \(\hat{r}(x, t) = -\hat{q}(-x, t)\).

Substituting (42) into (9), we obtain

\[
\psi(k; x, t) = I + \left( \sum_{j=1}^{N} \frac{g_j}{k - k_j} \psi[2](k_j; x, t), \sum_{l=1}^{N} \frac{\hat{h}_l}{k + \hat{\lambda}_l} \psi[1](-\hat{\lambda}_l; x, t) \right),
\] (47)

where \([j]\) denote the \(j\)th column of \(\psi(k; x, t)\). Similarly, from (43) and (17), we have

\[
\hat{\psi}(k; x, t) = I - \left( \sum_{l=1}^{N} \frac{h_l}{k - \hat{\lambda}_l} \psi[2](\hat{\lambda}_l; x, t), \sum_{j=1}^{N} \frac{\hat{g}_j}{k + \hat{k}_j} \psi[1](-\hat{k}_j; x, t) \right).
\] (48)

Here we have used the following notations:

\[
g_j = c_j e^{2i\theta(k_j; x, t)}, \quad \hat{g}_j = \bar{c}_j e^{-2i\theta(-\hat{k}_j; x, t)}, \\
h_l = d_l e^{2i\theta(\hat{\lambda}_l; x, t)}, \quad \hat{h}_l = \bar{d}_l e^{-2i\theta(-\hat{\lambda}_l; x, t)}.
\] (49)

We note that \(\hat{g}_j(x) = \overline{g_j(-x)}, \hat{h}_l(x) = \overline{h_l(-x)}\), and Equations (47) and (48) are equivalent to each other in views of the symmetry condition (22).

From (48) and (47), we find that \(\hat{\psi}_{11}(-\hat{k}_j; x, t)\) and \(\psi_{11}(-\hat{\lambda}_l; x, t)\) admit the following linear system:

\[
\Omega \hat{G} \hat{\psi}_{11} = \bar{E}^T, \quad \hat{\Omega} \hat{H} \psi_{11} = \bar{E}^T,
\] (50)

where \(E = (1, 1, \ldots, 1)_N, \bar{E} = (1, 1, \ldots, 1)_{\bar{N}}\) and

\[
\hat{\psi}_{11} = (\hat{\psi}_{11}(-\hat{k}_1; x, t), \ldots, \hat{\psi}_{11}(-\hat{k}_N; x, t))^T, \\
\psi_{11} = (\psi_{11}(-\hat{\lambda}_1; x, t), \ldots, \psi_{11}(-\hat{\lambda}_N; x, t))^T, \\
\hat{G} = \text{diag}(\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_N), \quad \hat{H} = \text{diag}(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_N), \\
G = \text{diag}(g_1, g_2, \ldots, g_N), \quad H = \text{diag}(h_1, h_2, \ldots, h_N),
\] (51)

Here the symmetry matrices \(\hat{\Omega}\) and \(\Omega\) are given by the symmetry matrix \(\hat{\Omega}\) and \(\Omega\) are given by

\[
\hat{\Omega}_{N \times \bar{N}} = \hat{H}^{-1} + \bar{\Lambda} \bar{\Lambda}^T, \quad \Omega_{N \times N} = \hat{G}^{-1} + \Lambda^T H \Lambda,
\] (52)
with $\Lambda$ is the $N \times N$ Cauchy type matrix with $\Lambda_{ij} = \frac{1}{\lambda_i + k_j}$. It is noted that, for some scattering data $S = \{\{c, k\}_{j=1}^N, \{d, \lambda\}_{i=1}^\tilde{N}\}$, the symmetry matrices $\tilde{\Omega}$ and $\Omega$ may degenerate in $x$-$t$ plane $\mathbb{E}^2$ at certain point sets denoted by $\tilde{P}_S$ and $P_S$. Beyond these point sets, we find the explicit solution for the cnNLS equation (1)

$$q(x, t) = -2i \frac{\det \tilde{\Omega}^a}{\det \tilde{\Omega}}, \quad \tilde{q}(x, t) = 2i \frac{\det \Omega^a}{\det \Omega}, \quad \mathbb{E}^2 \setminus (\tilde{P}_S \cup P_S),$$

(53)

where the symmetry matrix $\tilde{\Omega}$ and $\Omega$ are given by

$$\tilde{\Omega}^a = \begin{pmatrix} 0 & \tilde{E} \\ \tilde{E}^T & \tilde{\Omega} \end{pmatrix}, \quad \Omega^a = \begin{pmatrix} 0 & E \\ E^T & \Omega \end{pmatrix}.$$

(54)

In general, $q(x, t)$ and $\tilde{q}(x, t)$ have some singularities at the points in $\tilde{P}_S$ and $P_S$, respectively. It is important to express the solution of the cnNLS equation in the form (53) and (54). Because it makes us possible and easier to give the general nonlocal reduction of the cnNLS equation, which will be discussed in the next section.

For $N = \tilde{N} = 1$,

$$q(x, t) = \frac{2i}{\hat{n}_1^{-1} + \frac{g_1}{(\hat{\lambda}_1 + k_1)^2}},$$

$$\tilde{q}(x, t) = \frac{-2i}{\hat{g}_1^{-1} + \frac{\hat{n}_1}{(\hat{\lambda}_1 + \hat{k}_1)^2}},$$

(55)

where $\hat{g}_j, \hat{n}_j$ are defined in (49). If $k_1$ and $\lambda_1$ are chosen to be imaginary numbers, then the wave trains of $q(x, t)$ and $\tilde{q}(x, t)$ are on different lines, and the point sets $\tilde{P}_S$ and $P_S$ are located at the peaks (see Figure 1).
Figure 2 shows the amplitude of $q(x, t)$ and $\hat{q}(x, t)$ in (56), (57), and (53) with $\lambda_1 = 0.5i, \lambda_2 = 0.6i, k_1 = -2i, c_1 = e^{1+\pi/3}, d_1 = e^{1.5+\pi/4}, d_2 = e^{1.2+\pi/6}$.

For $N = 1, \tilde{N} = 2$, we have the solution (53) with

$$\det \tilde{\Omega}^a = -\left[\hat{h}_1^{-1} + \hat{h}_2^{-1} + \frac{g_1(\bar{\lambda}_2 - \bar{\lambda}_1)^2}{(\bar{\lambda}_1 + k_1)^2(\bar{\lambda}_2 + k_1)^2}\right],$$

$$\det \tilde{\Omega} = \hat{h}_1^{-1}\hat{h}_2^{-1} + \frac{g_1\hat{h}_1^{-1}}{\bar{\lambda}_2 + k_1} + \frac{g_1\hat{h}_2^{-1}}{\bar{\lambda}_1 + k_1},$$

and $\det \Omega^a = -1$,

$$\det \Omega = \hat{g}_1^{-1} + \frac{h_1}{(\lambda_1 + k_1)^2} + \frac{h_2}{(\lambda_2 + \bar{k}_1)^2}.$$  \hfill (57)

Figure 2 shows the amplitude of $q(x, t)$ and $\hat{q}(x, t)$ in (56), (57), and (53) with $\lambda_1, \lambda_2,$ and $k_1$ are imaginary numbers. One kind of the point sets $\tilde{P}_S$ and $P_S$ is shown.

Similarly, for $N = \tilde{N} = 2$, we obtain the solution (53) with

$$\det \tilde{\Omega}^a = -\left[\hat{h}_1^{-1} + \hat{h}_2^{-1} + \frac{g_1(\bar{\lambda}_2 - \bar{\lambda}_1)^2}{(\bar{\lambda}_1 + k_1)^2(\bar{\lambda}_2 + k_1)^2} + \frac{g_2(\bar{\lambda}_2 - \bar{\lambda}_1)^2}{(\bar{\lambda}_1 + k_2)^2(\bar{\lambda}_2 + k_2)^2}\right],$$

$$\det \tilde{\Omega} = \hat{h}_1^{-1}\hat{h}_2^{-1} + g_1g_2\frac{(\bar{\lambda}_2 - \bar{\lambda}_1)(k_2 - k_1)^2}{\prod_{j=1}(\bar{\lambda}_j + k_l)^2} + \frac{g_1\hat{h}_1^{-1}}{(\bar{\lambda}_2 + k_1)^2} + \frac{g_2\hat{h}_2^{-1}}{(\bar{\lambda}_1 + k_1)^2},$$

$$\det \Omega = \hat{g}_1^{-1} + \frac{h_1}{(\lambda_1 + k_1)^2} + \frac{h_2}{(\lambda_2 + \bar{k}_1)^2} + \frac{g_1\hat{h}_1^{-1}}{(\bar{\lambda}_1 + k_1)^2} + \frac{g_2\hat{h}_2^{-1}}{(\bar{\lambda}_1 + k_2)^2}.$$  \hfill (58)
FIGURE 3  The amplitude of $q(x, t)$ and its density in (58) and (53) with $\lambda_1 = 0.8i, \lambda_2 = 1.2i, k_1 = -i, k_2 = -0.5i, c_1 = c_2 = 1, d_1 = d_2 = 1$.

FIGURE 4  The amplitude of $\hat{q}(x, t)$ and its density in (59) and (53) with $\lambda_1 = 0.8i, \lambda_2 = 1.2i, k_1 = -i, k_2 = -0.5i, c_1 = c_2 = 1, d_1 = d_2 = 1$.

and

$$
\det \Omega^a = -\left[ \frac{1}{g_1^{-1} + g_2^{-1}} + \frac{h_1(k_2 - k_1)^2}{(\lambda_1 + k_1)(\lambda_1 + k_2)^2} + \frac{h_2(k_2 - k_1)^2}{(\lambda_2 + k_1)(\lambda_2 + k_2)^2} \right],
$$

$$
\det \Omega = \frac{1}{g_1^{-1} g_2^{-1}} + h_1h_2 \frac{(\lambda_2 - \lambda_1)^2(k_2 - k_1)^2}{\prod_{j=1}^{2}(\lambda_j + k_j)^2}
+ \frac{h_1g_1^{-1}}{(\lambda_1 + k_2)^2} + \frac{h_1g_2^{-1}}{(\lambda_1 + k_1)^2} + \frac{h_2g_1^{-1}}{(\lambda_2 + k_2)^2} + \frac{h_2g_2^{-1}}{(\lambda_2 + k_1)^2}.
$$

(59)

Figures 3 and 4 show the amplitude of $q(x, t)$ and $\hat{q}(x, t)$ in (58),(59), and (53) with $\lambda_1, \lambda_2$ and $k_1, k_2$ are imaginary numbers. The point sets $\tilde{P}_S$ and $P_S$ can be found in “smiling faces.”
From the above figures, we find that the two wave trains of \( q(x, t) \) and \( \tilde{q}(x, t) \) locate on different lines that are determined by the Dbar data \( \{\lambda_l, d_l\} \) and \( \{k_j, c_j\} \). So, we can carefully choose certain Dbar data to ensure \( q(x, t) = \tilde{q}(x, t) \).

5 | REDUCTION TO THE NONLOCAL FOCUSING NLS EQUATION

In general, \( q(x, t) \) and \( \tilde{q}(x, t) \) given in (53) are not equal to each other. From (49), we find that \( q(x, t) = \tilde{q}(x, t) \) implies the following conditions \( k_j = -\tilde{k}_j, \lambda_l = -\tilde{\lambda}_l \) and \( N = \tilde{N} \), as well as \( |c_j|, |d_l| \) dependent on \( \{k_j, \lambda_l\} \). To find the constraint conditions on \( |c_j| \) and \( |d_l| \) for \( q(x, t) = \tilde{q}(x, t) \), we take \( k_j = ib_j, \lambda_l = i\eta_l \), with \( b_j < 0 < \eta_j, j, l = 1, 2, \ldots, N \). It is remarked that

\[
\begin{align*}
g_j &= c_j e^{2i\theta(ib_j)}, \quad \tilde{g}_j^{-1} = \frac{1}{c_j} e^{2i\theta(ib_j)}, \\
h_j &= d_j e^{2i\theta(i\eta_j)}, \quad \tilde{h}_j^{-1} = \frac{1}{d_j} e^{2i\theta(i\eta_j)}.
\end{align*}
\]

The matrices \( \Omega \) and \( \tilde{\Omega} \) in (54) are symmetry matrix and the matrix \( K = i\Lambda = \left(\frac{1}{\eta_j - b_l}\right) \) is a Cauchy matrix. Using the Cauchy–Binet formula, we give the determinants in (53) as the following form Refs. 38 and 39:

\[
\det \tilde{\Omega}^a = \sum_{\sigma=1}^{N} (-1)^{\sigma} \prod_{l,s,n_\sigma} \frac{\tilde{h}_{n_\sigma}^{-1} g_s}{(\eta_l - b_s)^2} \prod_{l<s'} (\eta_l - \eta_{l'})^2 (b_{s'} - b_s)^2,
\]

\[
\det \tilde{\Omega} = \prod_{j=1}^{N} \tilde{h}_j^{-1} + \sum_{\sigma=1}^{N} (-1)^{\sigma} \prod_{l,m,n_\sigma} \frac{\tilde{h}_{n_\sigma}^{-1} g_m}{(\eta_l - b_m)^2} \prod_{l<s'} (\eta_l - \eta_{l'})^2 (b_{m'} - b_m)^2,
\]

and

\[
\det \Omega^a = \sum_{\sigma=1}^{N} (-1)^{\sigma} \prod_{l,s,n_\sigma} \frac{h_{s} \tilde{g}_{n_\sigma}^{-1}}{(\eta_s - b_l)^2} \prod_{l<s'} (\eta_s - \eta_{s'})^2 (b_{l'} - b_l)^2,
\]

\[
\det \Omega = \prod_{j=1}^{N} g_j^{-1} + \sum_{\sigma=1}^{N} (-1)^{\sigma} \prod_{l,m,n_\sigma} \frac{h_{m} \tilde{g}_{n_\sigma}^{-1}}{(\eta_m - b_l)^2} \prod_{l<s'} (\eta_m - \eta_{m'})^2 (b_{m'} - b_l)^2,
\]

where \( \sum_{\sigma=12} \) denotes the summation for indices \( 1 \leq j_1 \leq j_2 \leq \cdots \leq j_\sigma \leq N, 1 \leq r_2 \leq \cdots \leq r_\sigma \leq N \), and summation \( \sum_{\sigma=11} \) for \( 1 \leq j_1 \leq j_2 \leq \cdots \leq j_\sigma \leq N, 1 \leq r_1 \leq r_2 \leq \cdots \leq r_\sigma \leq N \).

If the determinants in (53) admit

\[
\det \tilde{\Omega}^a = (-1)^{N-1} \det \Omega^a, \quad \det \tilde{\Omega} = (-1)^N \det \Omega,
\]
which means that \( \det \tilde{\Omega}^a \det \Omega + \det \Omega^a \det \tilde{\Omega} = 0 \), then \( q(x, t) = \tilde{q}(x, t) \), and the cnNLS equation (1) reduces to the nNLS equation (2). In addition, from (61)–(63), the constraint conditions about \( |c_j|, |d_j| \) on \( \eta_j \) and \( b_j \) can be derived.

For \( N = \tilde{N} = 1 \), We have, from (61), (62), \( \det \tilde{\Omega}^a = \det \Omega^a = -1 \) and

\[
\det \Omega = \hat{h}_1^{-1} - \frac{1}{(\eta_1 - b_1)^2} g_1, \quad \det \tilde{\Omega} = \hat{g}_1^{-1} - \frac{1}{(\eta_1 - b_1)^2} h_1,
\]

which can also be obtained from (55). Then \( \det \tilde{\Omega} = - \det \Omega \) implies that

\[
|c_1| = |d_1| = \eta_1 - b_1,
\]

in terms of (60). In this case, we take \( c_1 = e^{2(\tau_1+i\alpha_1)}, d_1 = e^{2(\tau_1+i\beta_1)} \), then the solution of nNLS equation takes the form of

\[
q(x, t) = i \frac{e^{2\tilde{\tau}e^{-2it}} - e^{2\tilde{\tau}e^{-2it}}}{\cosh[2(\tilde{X} - X)] - \cos[2(\tilde{T} - T)]},
\]

where \( \eta_1 - b_1 = e^{2\tau_1} \) and

\[
\tilde{X} = \eta_1 x + \tau_1, \quad X = b_1 x + \tau_1, \quad \tilde{T} = 2\eta_1^2 t + \beta_1, \quad T = 2b_1^2 t + \alpha_1.
\]

This solution is singular at the points (see Figure 5)

\[
x = 0, \quad t = \frac{n\pi + \alpha_1 - \beta_1}{2(\eta_1^2 - b_1^2)}, \quad n \in \mathbb{Z}.
\]

In particular, if \( b_1 = -\eta_1 \), then \( \tilde{T} - T = \beta_1 - \alpha_1 \). Furthermore, if \( \cos[2(\tilde{T} - T)] = 1 \), \( (\beta_1 = \alpha_1 \) or \( \beta_1 = \alpha_1 + \pi \)), solution of the nNLS equation (2) reduces to

\[
q(x, t) = -2i\eta_1 e^{-2it} \text{csch}(2\eta_1 x),
\]

which is singular at the line \( x = 0 \).
In the case $b_1 = -\eta_1$ and $|\cos[2(\bar{T} - T)]| < 1$, (66) gives the stationary soliton solution of the nNLS equation (2) (see Figure 6). In the case $b_1 = -\eta_1$ and $\cos[2(\bar{T} - T)] = -1$ or $\beta_1 = \alpha_1 \pm \pi/2$, (66) yields

$$q(x, t) = 2i\eta_1 e^{-2i\bar{T}} \text{sech}(2\eta_1 x),$$

which means that the two distributions $R(k, x, t)$ and $\hat{R}(k; x, t)$ are equal. Hence, the nNLS equation (2) reduces to the NLS equation.

For $N = \bar{N} = 2$, using (61) and (62) or equivalent (58) and (59), we find, from $\det \tilde{\Omega}^a = - \det \Omega^a$, that

$$|c_j| = \frac{(\eta_1 - b_j)(\eta_2 - b_j)}{|\eta_2 - \eta_1|}, \quad |d_j| = \frac{(\eta_j - b_1)(\eta_j - b_2)}{|b_2 - b_1|}, \quad j = 1, 2,$$

and $\det \tilde{\Omega} = \det \Omega$ implies that $|c_1 c_2| = |d_1 d_2|$ and

$$\frac{|c_1|^2}{|d_1|^2} = \frac{(\eta_2 - b_1)^2}{(\eta_1 - b_2)^2} = \frac{|d_2|^2}{|c_2|^2}, \quad \frac{|c_2|^2}{|d_1|^2} = \frac{(\eta_2 - b_2)^2}{(\eta_1 - b_1)^2} = \frac{|d_2|^2}{|c_1|^2}.$$  

(72)

In addition, $|c_1 c_2| = |d_1 d_2|$ and (71) give $|\eta_2 - \eta_1| = |b_2 - b_1|$.

For convenience, we let $c_j = |c_j|e^{2i\alpha_j}, d_j = |d_j|e^{2i\beta_j}$ and

$$g_j = |c_j|e^{2\delta_j}, \quad \hat{g}_j^{-1} = \frac{1}{|c_j|}e^{2\delta_j},$$

$$h_j = |d_j|e^{2\delta_j}, \quad \hat{h}_j^{-1} = \frac{1}{|d_j|}e^{2\delta_j},$$

(73)

where

$$\delta_j = -X_j + iT_j, \quad X_j = b_j x, \quad T_j = 2b_j^2 t + \alpha_j,$$

$$\hat{\delta}_j = -\bar{X}_j + i\bar{T}_j, \quad \bar{X}_j = \eta_j x, \quad \bar{T}_j = 2\eta_j^2 t + \beta_j,,$$

(74)
Then we find that solution of nNLSequation (2) is

\[ q(x, t) = -2i \frac{M^a}{M}, \]  

with

\[ M^a = |c_2|e^{2\delta_1} + |c_1|e^{2\delta_2} - |d_2|e^{2\delta_1} - |d_1|e^{2\delta_2}, \]  

\[ M = e^{2(\delta_1 + \delta_2)} + e^{2(\delta_1 + \delta_2)} - p_1 \left( e^{2(\delta_1 + \delta_2)} + e^{2(\delta_2 + \delta_1)} \right) - p_2 \left( e^{2(\delta_1 + \delta_1)} + e^{2(\delta_2 + \delta_2)} \right). \]  

where \( p_j \) are constants and satisfy the following relations:

\[ p_1 = \frac{|d_1c_1|}{(\eta_1 - b_1)^2} = \frac{|d_2c_2|}{(\eta_2 - b_2)^2}, \]  

\[ p_2 = \frac{|d_1c_2|}{(\eta_1 - b_2)^2} = \frac{|d_2c_1|}{(\eta_2 - b_1)^2}. \]  

We note that

\[ p_2 - p_1 = \frac{b_2 - b_1}{\eta_2 - \eta_1} = \text{sgn}(b_2 - b_1), \]  

if \( 0 < \eta_1 < \eta_2 \) in view of \( |\eta_2 - \eta_1| = |b_2 - b_1| \). The solution (75) with \( \eta_2 - \eta_1 = b_2 - b_1 \) is shown in Figure 7, and \( \eta_2 - \eta_1 = b_1 - b_2 \) is shown in Figure 8.

In particularly, if we take \( \beta_j = \alpha_j \) and \( b_j = -\eta_j, (j = 1, 2) \), then (75) reduces to

\[ q(x, t) = \frac{-2i \left( m_1 e^{-2it_1} \sinh 2X_2 + m_2 e^{-2it_2} \sinh 2X_1 \right)}{\cosh 2(X_2 + X_1) - p \cosh 2(X_2 - X_1) - (p - 1) \cos 2(T_2 - T_1)}, \]
FIGURE 8  The amplitude and its density of $q(x, t)$ in (83) with $\eta_1 = 0.5, \eta_2 = 1, b_1 = -0.8, b_2 = -1.3, \alpha_j = 0, \beta_j = 0, (j = 1, 2)$

FIGURE 9  The amplitude and its density of $q(x, t)$ in (81) with $\eta_1 = 0.5, \eta_2 = 1, \alpha_j = 0, \beta_j = 0, (j = 1, 2)$

where

$$m_1 = \frac{2\eta_1(\eta_1 + \eta_2)}{\eta_2 - \eta_1}, \quad m_2 = \frac{2\eta_2(\eta_1 + \eta_2)}{\eta_2 - \eta_1}, \quad p = \frac{(\eta_2 + \eta_1)^2}{(\eta_2 - \eta_1)^2} > 1.$$

and $\tilde{X}_j$ and $\tilde{T}_j$ are defined in (73). The solution (81) has some singularities, and is shown in Figure 9.

Now, if we take $b_1 = -\eta_2, b_2 = -\eta_1$ and $\beta_1 = \alpha_2 + \pi/2, \beta_2 = \alpha_1 + \pi/2$, then (75) reduces to a regular solution

$$q(x, t) = \frac{2i(m_1 e^{-2i\tilde{T}_1} \cosh 2\tilde{X}_2 + m_2 e^{-2i\tilde{T}_2} \cosh 2\tilde{X}_1)}{\cosh 2(\tilde{X}_2 + \tilde{X}_1) + (p_1 + 1) \cosh 2(\tilde{X}_2 - \tilde{X}_1) + p_1 \cos 2(T_2 - T_1)}, \quad (81)$$
The amplitude and its density of $q(x, t)$ in \((83)\) with $\eta_1 = 0.5, \eta_2 = 1, \alpha_j = 0, \beta_j = \pi/2, (j = 1, 2)$ where

$$m_1 = \frac{2\eta_1(\eta_1 + \eta_2)}{\eta_2 - \eta_1}, \quad m_2 = \frac{2\eta_2(\eta_1 + \eta_2)}{\eta_2 - \eta_1}, \quad p_1 = \frac{4\eta_1\eta_2}{(\eta_2 - \eta_1)^2}. \quad (82)$$

This solution is shown in Figure 10.

Furthermore, For $N = \tilde{N} = 3$, one may find, from \((61)\) and \((62)\), that

$$\det \tilde{\Omega}^a = - \sum_{1 \leq s < s' \leq 3} \tilde{h}_s^{-1} \tilde{h}_{s'}^{-1} + \sum_{1 \leq l < l' \leq 3} \sum_{s = 1}^3 \frac{(\eta_l - \eta_{l'})^2}{(\eta_l - b_s)^2(\eta_{l'} - b_s)^2} g_s \tilde{h}_n^{-1}$$

$$- \sum_{1 \leq s < s' \leq 3} \frac{(b_s - b_{s'})^2}{\prod_{1 \leq l \leq 3} (\eta_l - b_s)(\eta_l - b_{s'})} g_s g_{s'}, \quad n = \{1, 2, 3\} \{l, l'\}, \quad (83)$$

$$\det \Omega^a = - \sum_{1 \leq s < s' \leq 3} \tilde{g}_s^{-1} \tilde{g}_{s'}^{-1} + \sum_{1 \leq l < l' \leq 3} \sum_{s = 1}^3 \frac{(b_s - b_{s'})^2}{(\eta_l - b_s)^2(\eta_l - b_{s'})^2} h_l \tilde{g}_n^{-1}$$

$$- \sum_{1 \leq s < s' \leq 3} \frac{(\eta_s - \eta_{s'})^2}{\prod_{1 \leq l \leq 3} (\eta_l - b_s)(\eta_l - b_{s'})} h_s h_{s'}, \quad n = \{1, 2, 3\} \{s, s'\}. \quad (84)$$

Then $\det \Omega^a = \det \tilde{\Omega}^a$ implies that

$$|d_s d_{s'}|^2 = \frac{\prod_{l=1}^3 (\eta_l - b_l)^2(\eta_{s'} - b_l)^2}{(\eta_s - \eta_{s'})^2 \prod_{1 \leq l < l' \leq 3} (b_{l'} - b_l)^2},$$

$$|c_s c_{s'}|^2 = \frac{\prod_{l=1}^3 (\eta_l - b_l)^2(\eta_l - b_{s'})^2}{(b_{s'} - b_s)^2 \prod_{1 \leq l < l' \leq 3} (\eta_l - \eta_{l'})^2}. \quad (85)$$
In addition, we also have

\[
\det \tilde{\Omega} = \prod_{m=1}^{3} \hat{h}_m^{-1} - \prod_{m=1}^{3} \frac{g_m h_n^{-1} h_n^{-1}}{(\eta_l - b_m)^2} + \sum_{3(l,m)} \frac{(\eta_l - \eta_{l'})^2(b_{m'} - b_m)^2}{\prod'(\eta - b)} g_m g_{m'} h_n^{-1} \tag{86}
\]

\[
\prod_{1 \leq l < l' \leq 3} (\eta_l - \eta_{l'})^2 \prod_{1 \leq m < m' \leq 3} (b_{m'} - b_m)^2 \prod_{l, m=1}^{3} \frac{g_m}{(\eta_l - b_m)^2}.
\]

\[
\det \Omega = \prod_{m=1}^{3} \tilde{g}_m^{-1} - \prod_{m=1}^{3} \frac{h_m \tilde{g}_n^{-1} \tilde{g}_n^{-1}}{(\eta_l - b_m)^2} + \sum_{3(l,m)} \frac{(\eta_l - \eta_{m'})^2(b_{l'} - b_l)^2}{\prod(\eta - b)} h_m h_{m'} \tilde{g}_n^{-1} \tag{87}
\]

\[
- \prod_{1 \leq m < m' \leq 3} (\eta_l - \eta_{m'})^2 \prod_{1 \leq l < l' \leq 3} (b_{l'} - b_l)^2 \prod_{l, m=1}^{3} \frac{h_m}{(\eta_l - b_m)^2}.
\]

where the indices are defined

\[
3(l, m) := 1 \leq l < l' \leq 3, 1 \leq m < m' \leq 3,
\]

\[
n = \{1, 2, 3\}\backslash\{l, l'\}, \quad l, l' \in \{1, 2, 3\}, (l < l')
\]

\[
n_1, n_2 \in \{1, 2, 3\}\backslash\{l\}, \quad 1 \leq l \leq 3, (n_1 \neq n_2),
\]

and the product \(\prod^*(\eta - b)\) denotes

\[
\prod^*(\eta - b) = (\eta_m - b_l)^2(\eta_m - b_{l'})^2(\eta_{m'} - b_l)^2(\eta_{m'} - b_{l'})^2. \tag{89}
\]

Thus \(\det \Omega = -\det \tilde{\Omega}\) yields

\[
|c_1 c_2 c_3|^2 = |d_1 d_2 d_3|^2 = \frac{\prod_{l, m=1}^{3} (\eta_m - b_l)^2}{\prod_{1 \leq m < m' \leq 3} (\eta_m - \eta_{m'})^2 \prod_{1 \leq l < l' \leq 3} (b_{l'} - b_l)^2}. \tag{90}
\]

From (85) and (90), we get the constraint conditions about \(|c_j|\) and \(|d_j|\), \((1 \leq j \leq 3)\)

\[
|c_j|^2 = \frac{\prod_{l=1}^{3} (\eta_l - b_j)^2}{\prod_{s=1, s \neq j}^{3} (b_s - b_j)^2}, \quad |d_j|^2 = \frac{\prod_{l=1}^{3} (\eta_l - b_j)^2}{\prod_{s=1, s \neq j}^{3} (\eta_s - \eta_j)^2}, \tag{91}
\]

and

\[
\prod_{1 \leq m < m' \leq 3} (\eta_m - \eta_{m'})^2 = \prod_{1 \leq l < l' \leq 3} (b_{l'} - b_l)^2. \tag{92}
\]

We note that Equation (85) is obtained from the first summation and the third summation in (83) and (84), and Equation (90) is derived from the first product and the forth product in (86) and (87). If choose \(s = n\) and \(l = n\) in the second summation in (83) and (84), we also have some equations about \(|c_s|\), \(|d_l|\) and \(\{\eta_j\}, \{b_m\}\), which can also be obtained from (91) and (92), just like (72). It is also true for the results obtained from the second summation and the third summation.
in (86) and (87). Hence, for \( N = 3 \), under the constrain condition (91) and (92), the solution of nNLS equation can be constructed, where

\[
\det \Omega = \frac{3}{\sum_{1 \leq s < s' \leq 3} \sum_{l=1}^{3} \frac{(\eta_i - \eta_l')^2}{(\eta_i - b_s)(\eta_l' - b_s)} g_s \hat{h}_l^{-1} n - \sum_{1 \leq s < s' \leq 3} \hat{h}_s^{-1} \hat{h}_{s'}^{-1} - \sum_{1 \leq s < s' \leq 3} \hat{g}_s^{-1} \hat{g}_{s'}^{-1}, \quad n = \{1, 2, 3\} \setminus \{l, l'\},
\]

and

\[
\det \hat{\Omega} = \prod_{m=1}^{3} \hat{h}_m^{-1} - \prod_{m=1}^{3} \hat{g}_m^{-1} - \sum_{j, m=1}^{3} \frac{g_m \hat{h}_{n_1}^{-1} \hat{h}_{n_2}^{-1}}{(\eta_i - b_m)^2} + \sum_{3(l,m)} \frac{(\eta_i - \eta_l')^2(b_{m'} - b_m)^2}{\prod (\eta - b)} g_m g_{m'} \hat{h}_n^{-1}.
\]

Here the summation indices and the product are defined in (88) and (89).

It is remarked that the constraint conditions (91) and (92) for \( N = N = 3 \) can be extended to general \( N = N \), just by change 3 to \( N \) in (91) and (92), which can be derived similarly from the first and last terms in (61) and (62) by virtue of the condition (63). If fact, the first term of \( \det \Omega \) takes the form

\[
- \sum_{1 \leq s < s' \leq 3} \hat{h}_s^{-1} \hat{h}_{s'}^{-1} \prod_{l < l'} \frac{(\eta_i - \eta_l')^2}{(\eta_i - b_s)(\eta_l' - b_s)} (b_{s'} - b_s)^2 = - \prod_{n_1} \hat{h}_n^{-1}
\]

\[
= - \sum_{1 \leq s_1 < \cdots < s_{N-1} \leq N} \hat{h}_{s_1}^{-1} \cdots \hat{h}_{s_{N-1}}^{-1} = - \sum_{j=1}^{N} \prod_{s=1}^{N} \hat{h}_s^{-1},
\]

and the last term is

\[
(-1)^N \sum_{N \leq l, s, n \leq N} \frac{\hat{h}_{n_1}^{-1} \hat{g}_s}{(\eta_i - b_s)^2} \prod_{l < l'} (\eta_i - \eta_l')^2 (b_{s'} - b_s)^2
= (-1)^N \sum_{j=1}^{N} \prod_{1 \leq l, s, l' \leq N} (\eta_l - \eta_l') \prod_{s < s'} (b_{s'} - b_s)^2 \prod_{l,s=1}^{N} \frac{g_s}{(\eta_l - b_s)^2}.
\]

The first term and the last term of \( \det \Omega \) are

\[
- \sum_{1 \leq s < s' \leq 3} \frac{h_s \hat{g}_{n_1}^{-1}}{(\eta_s - b_i)^2} \prod_{l < l'} (\eta_s - \eta_l')^2 (b_{l'} - b_l)^2 = - \sum_{j=1}^{N} \prod_{s=1}^{N} \hat{g}_s^{-1},
\]

\[
\text{(97)}
\]
\begin{align}
(-1)^N \sum_{N_{12}, s, n_N} \frac{h_s g_n^{-1}}{(\eta_s - b_l)^2} \prod_{l < l'} (\eta_s - \eta_{s'})^2 (b_{l'} - b_l)^2 \\
= (-1)^N \sum_{j=1}^N \prod_{s < s', s' \neq j} (\eta_s - \eta_{s'})^2 \prod_{1 \leq l < l' \leq N} (b_{l'} - b_l)^2 \prod_{l, s = 1, s \neq j}^N \frac{h_s}{(\eta_s - b_l)^2}.
\end{align}

(98)

By the condition \(\det \tilde{\Omega} = (-1)^{N-1} \det \Omega\), Equations (95) and (98) imply that

\begin{equation}
\prod_{s=1, s \neq j}^N |d_s|^2 = \frac{\prod_{l=1, s \neq j}^N (\eta_s - b_l)^2}{\prod_{s < s', s' \neq j} (\eta_s - \eta_{s'})^2 \prod_{s < s', s' \neq j} (b_{s'} - b_s)^2}.
\end{equation}

(99)

and Equations (96) and (97) give

\begin{equation}
\prod_{s=1, s \neq j}^N |c_s|^2 = \frac{\prod_{l=1, s \neq j}^N (\eta_l - b_s)^2}{\prod_{1 \leq l < l' \leq N} (\eta_l - \eta_{l'})^2 \prod_{s < s', s' \neq j} (b_{s'} - b_s)^2}.
\end{equation}

(100)

Similarly, the first term and the last term of \(\det \tilde{\Omega}\) and \(\det \Omega\) are

\begin{align}
\det \tilde{\Omega} : & \prod_{m=1}^N h_m^{-1}, \quad (-1)^N \prod_{1 \leq l, m \leq N} \frac{g_m}{(\eta_l - b_m)^2} \prod_{1 \leq l < l' \leq N} (\eta_l - \eta_{l'})^2 (b_{l'} - b_m)^2; \\
\det \Omega : & \prod_{m=1}^N g_m^{-1}, \quad (-1)^N \prod_{1 \leq l, m \leq N} \frac{h_m}{(\eta_l - b_m)^2} \prod_{1 \leq l < l' \leq N} (\eta_m - \eta_{m'})^2 (b_{l'} - b_m)^2.
\end{align}

(101)

which yields

\begin{equation}
\prod_{s=1}^N |d_s|^2 = \prod_{s=1}^N |c_s|^2 = \frac{\prod_{l=1}^N (\eta_s - b_l)^2}{\prod_{1 \leq s < s' \leq N} (\eta_s - \eta_{s'})^2 \prod_{1 \leq l < l' \leq N} (b_{l'} - b_l)^2}.
\end{equation}

(102)

in terms of \(\det \tilde{\Omega} = (-1)^N \det \Omega\). From (99),(100), and (102), we obtain

\begin{align}
|c_j|^2 = \frac{\prod_{l=1}^N (\eta_l - b_l)^2}{\prod_{s=1, s \neq j}^N (b_s - b_j)^2}, \quad |d_j|^2 = \frac{\prod_{l=1}^N (\eta_l - b_l)^2}{\prod_{s=1, s \neq j}^N (\eta_s - \eta_j)^2},
\end{align}

(103)

and

\begin{equation}
\prod_{1 \leq m < m' \leq N} (\eta_m - \eta_{m'})^2 = \prod_{1 \leq l < l' \leq N} (b_{l'} - b_l)^2.
\end{equation}

(104)
6 | CONCLUSIONS AND DISCUSSIONS

In this paper, we extended the Dbar-problem to discuss the cnNLS equation. In this approach, we introduced two spectral transform matrices to define two Dbar-problems. The general nonlocal relations between the cnNLS potential and the solutions of the Dbar-problems were established by Dbar-dressing method. Two sets of Dbar data were used to construct the special explicit solution of the focusing cnNLS equation. By using the Cauchy–Binet formula, we expanded the determinant solution, which was used to discuss nonlocal reductions of the cnNLS equation in detail.

Recently, an integrable space-time shifted nonlocal NLS equations was introduced in Ref. 44. We note that the present approach can also be used to study the new type of integrable shifted nonlocal equations. This can be done by introducing the symmetry condition \( \hat{Q}(x, t) = -\Lambda Q(x_0 - x, t)\Lambda^{-1} \). The detail discussions will be presented in separate paper.

DECLARATION OF INTEREST

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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REFERENCES

1. Ablowitz MJ, Musslimani ZH. Integrable nonlocal nonlinear Schrödinger equation. Phys Rev Lett. 2013;110:064105.
2. Ablowitz MJ, Musslimani ZH. Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation. Nonlinearity. 2016;29:915-946.
3. Ablowitz MJ, Luo XD, Musslimani ZH. Inverse scattering transform for the nonlocal nonlinear Schrödinger equation with nonzero boundary conditions. J Math Phys. 2018;59:011501.
4. Feng BF, Luo XD, Ablowitz MJ, Musslimani ZH. General soliton solution to a nonlocal nonlinear Schrödinger equation with zero and nonzero boundary conditions. Nonlinearity. 2018;31:5385-5409.
5. Yan ZY. Nonlocal general vector nonlinear Schrödinger equations: integrability, PT symmetrability, and solutions. Appl Math Lett. 2016;62:101-109.
6. Yu FJ, Fan R. Nonstandard bilinearization and interaction phenomenon for PT-symmetric coupled nonlocal nonlinear Schrödinger equations. Appl Math Lett. 2020;103:106209.
7. Lou SY, Qiao ZJ. Alice-Bob peakon systems. Chin Phys Lett. 2017;34:100201.
8. Lou SY. Alice-bob systems, \( \hat{P}-\hat{T}-\hat{C} \) symmetry invariant and symmetry breaking soliton solutions. J Math Phys. 2018;59:083507.
9. Rybalko Y, Shepelsky D. Long-time asymptotics for the integrable nonlocal nonlinear Schrödinger equation. J Math Phys. 2019;60:031504.
10. Sarma AK, Mira MA, Musslimani ZH, Christodoulides DN. Continuous and discrete Schrödinger systems with parity-time-symmetric nonlinearities. Phys Rev E. 2014;89:052918.
11. Wen XY, Yan ZY, Yang YQ. Dynamics of higher-order rational solitons for the nonlocal nonlinear Schrödinger equation with the self-induced parity-time-symmetric potential. Chaos. 2016;26:063123.
12. Zhang GQ, Yan ZY, Chen Y. Novel higher-order rational solitons and dynamics of the defocusing integrable nonlocal nonlinear Schrödinger equation via the determinants. Appl Math Lett. 2017;69:113-120.
13. Chen JB, Pelinovsky DE, White RE. Rogue waves on the double-periodic background in the focusing nonlinear Schrödinger equation. Phys Rev E. 2019;100:052219.
14. Chen JB, Pelinovsky DE, White RE. Periodic standing waves in the focusing nonlinear Schrödinger equation: Rogue waves and modulation instability. Physica D. 2020;405:132378.
15. Yang B, Yang JK. Nonlinear evolution equations associated with ‘energy-dependent Schrödinger potentials’. *Lett Math Phys*. 2019;109:945-973.
16. Yang B, Yang JK. On general rogue waves in the parity-time-symmetric nonlinear Schrödinger equation. *J Math Anal Appl*. 2020;487:124023.
17. Yan ZY. Integrable pt-symmetric local and nonlocal vector nonlinear Schrödinger equations: a unified two-parameter model. *Appl Math Lett*. 2016;62:101-109.
18. Ablowitz M, Musslimani Z. Integrable discrete PT symmetric model. *Phys Rev E*. 2014;90:032912.
19. Ablowitz MJ, Luo XD, Musslimani ZH. Discrete nonlocal nonlinear Schrödinger systems: integrability, inverse scattering and solitons. *Nonlinearity*. 2020;33:3653-3707.
20. Ma LY, Zhu ZN. Nonlocal nonlinear Schrödinger equation and its discrete version: soliton solutions and gauge equivalence. *J Math Phys*. 2016;57:083507.
21. Ablowitz MJ, Musslimani ZH. Integrable nonlocal nonlinear equations. *Stud Appl Math*. 2017;140:178-201.
22. Ma WX. Inverse scattering for nonlocal reverse-time nonlinear Schrödinger equations. *Appl Math Lett*. 2020;102:106161.
23. Ye RS, Zhang Y. General soliton solutions to a reverse-time nonlocal nonlinear Schrödinger equation. *Stud Appl Math*. 2020;145:197-216.
24. Ma WX. Inverse scattering and soliton solutions of nonlocal reverse-space-time nonlinear Schrödinger equations. *Proc Amer Math Soc*. 2021;149:251-263.
25. Ma WX, Huang YH, Wang FD. Inverse scattering transforms and soliton solutions of nonlocal reverse-space nonlinear Schrödinger hierarchies. *Stud Appl Math*. 2020;145:563-585.
26. Zhou ZX. Darboux transformations and global solutions for a nonlocal derivative nonlinear Schrödinger equation. *Commun Nonlinear Sci Numer Simul*. 2018;62:480-488.
27. Yang B, Yang JK. Transformations between nonlocal and local integrable equations. *Phys Lett A*. 2018;383:328-337.
28. Gürses M, Pekcan A. Nonlocal nonlinear Schrödinger equations and their soliton solutions. *J Math Phys*. 2018;59:051501.
29. Gürses M, Pekcan A. Nonlocal modified KdV equations and their soliton solutions by Hirota method. *Commun Nonlinear Sci Numer Simul*. 2019;67:427-448.
30. Gürses M, Pekcan A. (2+1)-dimensional local and nonlocal reductions of the negative AKNS system: soliton solutions. *Commun Nonlinear Sci Numer Simul*. 2019;71:161-173.
31. Xu T, Meng DX. Rational solitons in the parity-time-symmetric nonlocal nonlinear Schrödinger model. *J Phys Soc Jpn*. 2016;85:124001.
32. Xu T, Li HJ, Zhang HJ, Li M, Lan S. Darboux transformation and analytic solutions of the discrete PT-symmetric nonlinear Schrödinger equation. *Appl Math Lett*. 2017;63:88-94.
33. Beals R, Coifman RR. The D-bar approach to inverse scattering and nonlinear evolutions. *Physica D*. 1986;18:242-249.
34. Jaulent M, Manna M, Alonso LM. $\tilde{\delta}$ equations in the theory of integrable systems. *Inverse Probl*. 1988;4:123-150.
35. Bogdanov LV, Manakov SV. The non-local delta problem and (2+1)-dimensional soliton equations. *J Phys A: Math Gen*. 1988;21:1537-1544.
36. Doktorov EV, Leble SB. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Springer; 2007.
37. Zhu JY, Geng XG. A hierarchy of coupled evolution equations with self-consistent sources and the dressing method. *J Phys A: Math Gen*. 2013;46:035204.
38. Zhu JY, Geng XG. The AB equations and the Dbar-dressing method in semi-characteristic coordinates. *Math Phys Anal Geo*. 2014;17:49-65.
39. Kuang YH, Zhu JY. The higher-order soliton solutions for the coupled Sasa-Satsuma system via the $\tilde{\delta}$-dressing method. *Appl Math Lett*. 2017;66:47-53.
40. Zhu JY, Zhou SS, Qiao ZJ. Forced (2+1)-dimensional discrete three-wave equation. *Commun Theor Phys*. 2020;72:015004.
41. Zhu JY, Jiang XL, Wang XR. Dbar dressing method to nonlinear Schrödinger equation with nonzero boundary conditions. arXiv: 2011.09028.
43. Jaulent M, Manna M. The spatial transform method: \( \partial \) derivation of the AKNS hierarchy. Phys Lett A. 1987;117:62-66.

44. Ablowitz MJ, Musslimani ZM. Integrable space-time shifted nonlocal nonlinear equations. Phys Lett A. 2021;409:127516.

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