THE BLASCHKE CONJECTURE AND GREAT CIRCLE
FIBRATIONS OF SPHERES

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Abstract. We construct an explicit diffeomorphism taking any "nondegenerate" fibration of a sphere by great circles into the Hopf fibration. The diffeomorphism is a local differential invariant, algebraic in derivatives.

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1. INTRODUCTION

"Notice that the classification of fibrations of spheres by great circles is an interesting but almost untouched subject..." Arthur L. Besse [11] pg. 135.

In studying the Blaschke conjecture [11] and in the theory of elliptic partial differential equations [11], one encounters fibrations of the standard round sphere by great circles. The best known example is the Hopf fibration

\[ S^1 \rightarrow S^{2n+1} \]

\[ \mathbb{C}P^n \]

given by the circle action

\[ e^{i\theta} (z_0, \ldots, z_n) = (e^{i\theta} z_0, \ldots, e^{i\theta} z_n) \]

on the unit sphere inside \( \mathbb{C}^{n+1} \).

Yang’s conjecture [16] states that for any smooth great circle fibration of a sphere, there is a diffeomorphism of the sphere carrying it to the Hopf fibration. Sato proved Yang’s conjecture for PL-homeomorphism rather than diffeomorphism [14]. Yang [16, 17] stated that Sato’s proof is flawed, and proved his conjecture for spheres of dimension at least seven, missing the 3-sphere and 5-sphere, although he claimed that his proof works on the 5-sphere modulo the Poincaré conjecture. Ballmann and Grove (personal communication from Grove) stated that Yang’s proof is flawed. Gluck & Warner [6] proved Yang’s conjecture for the 3-sphere. In an earlier version of this paper, I attempted to prove Yang’s conjecture in any dimension. Yang’s proof of his conjecture is quite difficult, involving a mixture of differential geometry and surgery, and employing the \( h \) and \( s \) cobordism theorems and the signature theorem; the diffeomorphism is not explicit. For great circle fibrations which are, in a suitable sense, nondegenerate, I will give an explicit diffeomorphism, which can be written in algebraic functions of the first and second derivatives of the functions cutting out the great circle fibration, in any local coordinates. Our diffeomorphism is “linear” on each fiber, i.e. a linear fractional transformation of each circle. The nondegeneracy is satisfied by the Hopf fibration, and by every great circle fibration near the Hopf fibration, but I do not know if there are degenerate great circle fibrations.

A Blaschke manifold is a Riemannian manifold whose injectivity radius equals its diameter. Every Blaschke manifold has the cohomology of a unique compact rank one symmetric space, its model. The Blaschke conjecture claims that every

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1Gluck, Warner and Yang [6] is a nice introduction to great sphere fibrations of spheres.
Blaschke manifold is isometric to its model, up to a constant rescaling. The conjecture above has the corollary that every Blaschke manifold modelled on a complex projective space is homeomorphic to its model. Yang [16] claims diffeomorphic, but I cannot see how to prove such a result, even after a careful reading of Yang’s paper, nor can Karsten Grove, and even modulo Yang’s conjecture. An abridged version of the paper you have before you has appeared in the American Journal of Mathematics, with an attempt at a proof of Yang’s conjecture. In 2016, I rewrote parts of this paper to clarify the proofs, correct errors discovered by Karsten Grove and Werner Ballmann, and to improve the citations. Below I give credit to Reznikov for discovery of invariants of great circle fibrations, and I remove the claim that every Blaschke manifold modelled on a complex projective space is diffeomorphic, or even homeomorphic, to its model.

2. Overview

Section 3 presents a review of some results on the topology of circle fibrations of spheres, which are intended for the reader’s interest, but are not employed subsequently.

The long section 4 develops a description of a great circle fibration in terms of local data, following Cartan’s method of the moving frame. This associates to each great circle fibration of a sphere a principal bundle over the sphere, with a collection of differential forms on it representing the local data of the great circle fibration. The idea is to successively determine subbundles of this bundle, by algebraic equations among those differential forms. This process is also a part of Cartan’s method. In order to make it work, one needs to find algebraic relations among the coefficients of the various differential forms, and then show that the functions expressing these relations (called the torsion functions), which are differential invariants of the great circle fibration, satisfy regularity hypotheses. These regularity hypotheses are strong enough that the subset on which the torsion satisfies some algebraic condition is a submanifold. Recall that the original bundle is a principal bundle—we will see that these coefficients vary according to an action of the structure group of that bundle. This will force the subbundle on which the torsion satisfies an appropriately chosen algebraic equation to be a principal subbundle.

The first torsion to show up is essentially an endomorphism of the normal bundle of each great circle. (It is an endomorphism twisted by a real line bundle, but this is irrelevant.) Subsection 4.6 shows that this endomorphism satisfies an ordinary differential equation as we move along the great circles (using the Levi–Civita connection to differentiate). This ordinary differential equation is explicitly integrated, and we find that consistent initial data for it which will remain finite as we tranverse the great circle consists in endomorphisms with no real eigenvalues.

This leads to a digression: section 4.7 shows that an endomorphism of a vector space which has no real eigenvalues determines invariantly a complex structure which commutes with it. Applied to the endomorphism of the normal bundle, we obtain an almost complex structure on the base manifold of the great circle fibration. In subsection 4.8 on page 18, I change bases for the differential forms, splitting forms into complex linear and conjugate linear parts, and find that the Cayley transform of our endomorphism is a complex linear matrix with eigenvalues in the unit disk. The structure group of our principal bundle acts on this matrix,
moving its spectrum around. Section 4.10 on page 24 finds that one can normalize it to have vanishing trace.

In section 4.10 on page 24, an elementary step enables reduction of the structure group of our original principal bundle to a group \( \Gamma_0 \). This group is the same group which appears as structure group for the Hopf fibration. But this is exactly the isotropy group of a point of the base manifold in the Hopf fibration, signalling an end to the method of the moving frame, since there can in general be no further reduction, as the Hopf fibration admits no invariant subbundle contained in this one.

Section 5 on page 26 shows that the structure equations satisfied by our differential forms now appear similar to those of complex projective space. Section 6 on page 28 proves that the Hopf fibration is precisely the great circle fibration with maximal symmetry group.

Section 7 on page 29 turns to another description of a great circle fibration: every great circle in a sphere spans a 2-plane in the ambient vector space containing the sphere, so a family of great circles is a family of 2-planes. More precisely, the base of a great circle fibration is an embedded compact connected submanifold of the Grassmannian of 2-planes. Section 8 on page 29 characterizes the submanifolds of the Grassmannian which represent great circle fibrations as being precisely those which are compact, connected, and elliptic. Ellipticity is a local condition on an immersed submanifold of the Grassmannian; ellipticity is a first order partial differential inequality. I believe that this inequality satisfies the \( h \)-principle of Gromov; I explain in section 10 on page 34 why one is apparently unable to use Gromov’s techniques to prove the \( h \)-principle. The space of all great circle fibrations, viewed as the space of all compact, connected, elliptic submanifolds of the Grassmannian, is straightforward to parameterize locally, and shows itself as an infinite dimensional manifold. Its topology is unknown; for example it is not known to be connected.

In section 11 on page 35, the geometry of the principal bundle we have constructed is used to determine an osculating complex structure at each point of the base manifold of the great circle fibration. This is a complex structure whose associated Hopf fibration has base manifold sitting inside the Grassmannian just touching the base manifold of our given great circle fibration, and is in some sense the best approximating Hopf fibration.

All of the theory developed in this article is based on the conviction that great circle fibrations provide a very natural mechanism for deforming complex geometry. This article is a contribution to the microlocal theory of such deformations. So a great circle fibration of a sphere should be thought of as a kind of nonlinear complex structure on the vector space containing the sphere. The base of the great circle fibration is a kind of deformed complex projective space. The next step is to define complex hyperplanes in that space. We do this by looking at hyperspheres in our sphere, and asking for the family of great circles from our fibration which lie entirely inside the hypersphere. We prove that this “hyperplane” is a smooth submanifold in the base manifold, of real codimension 2. It is not an almost complex submanifold in general.

Hajime Sato [14] attempted to prove part of the topological Blaschke conjecture using a map, which was perhaps not well defined [16]. Therefore the next step is to define Sato’s map. This map embeds the base of a great circle fibration into a complex projective space of much higher dimension. A diffeomorphism from the
base manifold to a complex projective space is constructed out of the Sato map, essentially using linear projections, assuming a nondegeneracy condition. We carry over this map into an isomorphism of the given great circle fibration with the Hopf fibration, achieving our main result.

The analogy between great circle fibrations of spheres and complex structures of vector spaces is significantly strengthened in section 16 on page 46. We describe the notion of twisted complex structure, or nonlinear $J$. To each great circle fibration, we assign a nonlinear $J$, using local invariants. To each nonlinear $J$, we associate a great circle fibration. However, the two concepts are not equivalent: rather the set of nonlinear $J$ form an infinite dimensional fiber bundle with contractible fibers over the infinite dimensional manifold of great circle fibrations. There is a canonical section of this bundle. This picture tells us that these nonlinear $J$ are really superfluous, and that the real object of our theory, generalizing the concept of complex structure, is the great circle fibration.

3. Elementary Topology

A sphere of even dimension does not admit a circle fibration. A circle fibration determines a unit vector field up to sign, and therefore determines a unit vector field on a 2-1 cover. Because spheres of dimension two or more are simply connected, this determines two unit vector fields, and therefore shows the vanishing of the Euler characteristic. The Euler characteristic of a sphere is zero just when the dimension of the sphere is odd. Therefore we restrict our attention to the odd dimensional spheres.

Fix a smooth foliation of $S^{2n+1}$ by great circles. The foliation is a principal circle bundle, say $S^1 \to S^{2n+1} \to X$, since the fibers can be consistently oriented, and then we can apply rotations by various angles (measured in the usual Riemannian geometry on the sphere $S^{2n+1}$) to implement a circle action. (We will see this from another point of view below.) The compactness of $S^{2n+1}$ forces $X$ to be compact.

Using the orientations of the circles and of the sphere, we have a quotient orientation on $X$. Chasing through the relevant exact sequences [4] p. 230, [10] p. 134, the homotopy groups, Whitehead products and cohomology ring of $X$ (with coefficients in any ring; see McCleary [10], page 134) are those of $\mathbb{C}\mathbb{P}^n$. (These authors carry out their calculations with the hypotheses that the base manifold $X$ is $\mathbb{C}\mathbb{P}^n$ and that the fibration is the Hopf fibration. However, they do not use these hypotheses. All they require is that the total space be a homotopy sphere, and that the fiber be a circle.) Turning to characteristic classes [10] p. 199, follow the Leray spectral sequence to see that the Chern class of the bundle $S^{2n+1} \to X$ (i.e. the transgression of the $S^1$ generating class), which we write as $-[H]$, generates the cohomology. Keeping track of signs (using the Hopf fibration as our guide), find that $[H]^n$ is Poincaré dual to the fundamental class $[X]$.

4. The Moving Frame

4.1. Structure equations of a flat projective structure. For my purposes, the sphere $S^{2n+1}$ and the real projective space $\mathbb{R}\mathbb{P}^{2n+1}$ are equally reasonable spaces to work on, since a great circle fibration of a sphere is the same thing as a fibration of the [antipodal quotient] projective space by projective lines: choose the sphere. The group of symmetries of a fiber bundle is always infinite dimensional, but in our case we want the concept of great circle to be preserved.
Theorem 1. Every invertible (not necessarily continuous) map of real projective space of dimension 2 or more to itself taking projective lines to projective lines is the action of a projective linear transformation.

Proof. The result is well known for projective planes \([8]\) p. 48 theorem 3.13. Assume by induction that \(n \geq 3\) and that we have proven the result for all dimensions less than \(n\). Take an invertible map \(f : \mathbb{RP}^n \to \mathbb{RP}^n\) taking lines to lines. Recall that a set of points of projective space is in general position if any \(k\) of them lie in a unique projective subspace of dimension \(k-1\) for an integer \(k\) with \(1 \leq k \leq n\). Note that the projective general linear group acts transitively on \((n+2)\)-tuples of points in general position. Take \(n+2\) vectors \(v_0, v_1, \ldots, v_n, v_{n+1}\) in \(\mathbb{R}^{n+1}\) whose images in \(\mathbb{RP}^n\) are in general position. For example, we can assume that \(v_0 = e_0, \ldots, v_n = e_n\) are the standard basis vectors and that \(v_{n+1} = \sum e_i\). Composing \(f\) with a projective linear transformation, we can assume that \(f\) fixes the images of these points in projective space. So \(f\) leaves invariant the projective subspaces of all dimensions spanned by any subset of these points. By induction on dimension, \(f\) is projective linear on each of those projective subspaces. In particular, any \(n\) of our points lie in a unique projective subspace of dimension \(n-1\), on which we find that \(f\) is projective linear for a linear transformation with the various \(v_i\) as eigenvectors:

\[
f \left[ \sum_{i \neq j} a_i e_i \right] = \sum_{i \neq j} a_i \lambda_i e_i
\]

for some numbers \(\lambda_i\). Allowing \(a_k = 0\) for some \(k\), we find that the eigenvalue \(\lambda_i\) is independent of \(j\). By the same token

\[
f \left[ a e_0 + \sum_j e_j \right] = \left[ a \lambda_0 e_0 + \mu \sum_j e_j \right],
\]

for some eigenvalue \(\mu \neq 0\). Setting \(a = -1\),

\[
f \left[ \sum_{j>0} e_j \right] = [(-\lambda_0) e_0 + \mu e_1 + \cdots + \mu e_n].
\]

Linearity also gives

\[
f \left[ \sum_{j>0} e_j \right] = [\lambda_1 e_1 + \cdots + \lambda_1 e_n].
\]

Therefore all eigenvalues are equal, i.e. \(f\) fixes each \([e_i]\) in projective space, and \(f\) fixes all points on all of the projective subspaces spanned by any set of these \([e_i]\), in particular all points of all coordinate hyperplanes are fixed. Any point of projective space is the intersection point of a pair of lines through points of coordinate hyperplanes. \(\square\)

Corollary 1. Any continuous bijection of a sphere taking great circles to great circles is the action of a linear transformation.

Proof. Any such map of the sphere takes antipodal points to antipodal points, since these are intersection points of great circles. So it quotients to a map of real projective space taking lines to lines, a projective linear automorphism. After composing with a linear transformation, we arrange that this projective transformation is the
identity, so our continuous bijection induces the trivial projective automorphism, so each point is preserved or reflected to its antipode. By continuity, our bijection is either the identity or minus the identity. □

Let $V = \mathbb{R}^{2n+2}$ have basis $e_0, \ldots, e_{2n+1}$. Let $G = \text{SL}(V)$ act on the sphere $S^{2n+1} = (V \backslash 0) / \mathbb{R}^+$. Let $\mathfrak{g}$ be the Lie algebra of $G$. Write $[v] = \mathbb{R}^+ v$ for the ray through a vector $v \in V \backslash 0$. Let $G_0$ be the stabilizer of $[e_0]$, i.e. the group of matrices of the form

$$
\begin{pmatrix}
ge_0^0 & g_0^\nu \\
0 & g_\mu^\nu
\end{pmatrix}
$$

with real entries satisfying $g_0^0 > 0$, $g_0^0 \det g_\mu^\nu = 1$, and Greek indices $\mu, \nu = 1, \ldots, 2n+1$. For reference, our index conventions in this paper are:

- $\mu, \nu, \sigma = 1, \ldots, 2n+1$
- $i, j, k = 2, \ldots, 2n+1$
- $p, q, r = 1, \ldots, n$
- $P, Q, R = 0, \ldots, n$.

Let $\mathfrak{g}_0$ be the Lie algebra of $G_0$. The sphere $S^{2n+1}$ is the quotient $G / G_0$ via the right action of $G_0$. More concretely, the map $G \to S^{2n+1}$ is the map $g \mapsto [ge_0]$.

We will follow Élie Cartan’s method of the moving frame [3]. The left invariant Maurer–Cartan 1-form

$$
\omega = g^{-1} dg \in \Omega^1(G) \otimes \mathfrak{g}
$$

satisfies $d\omega = -\omega \wedge \omega$. Our subgroup $G_0$ acts on the right on $G$, thus not preserving the Maurer–Cartan 1-form, but instead if $R_{g_0}$ is the right action on $G$ of an element $g_0 \in G_0$ then

$$
R^*_{g_0} \omega = \text{Ad}_{g_0}^{-1} \omega.
$$

Let us divide $\omega$ into 1-forms according to

$$
\omega = \begin{pmatrix}
\omega_0^0 & \omega_\mu^\nu \\
\omega_\mu^0 & \omega_\mu^\nu
\end{pmatrix}.
$$

The 1-forms $\omega_\mu^\nu$, which we will write as $\omega^\mu$, are semibasic for the fiber bundle map $G \to S^{2n+1}$, which we see because they are linearly independent on $G$ but vanish on $G_0$ and therefore on the left translates of $G_0$. Their exterior derivatives are

$$
d\omega^\mu = -\gamma^\mu_\nu \wedge \omega^\nu
$$

where we define

$$
\gamma^\mu_\nu := \omega^\mu_{\nu} - \delta^\mu_\nu \omega^0_0.
$$

Their exterior derivatives are

$$
d\gamma^\mu_\nu = -\gamma^\mu_\sigma \wedge \gamma^\sigma_\nu + (\delta^\mu_\nu \omega_\sigma + \omega_\nu \delta^\mu_\sigma) \wedge \omega^\sigma
$$

where we write $\omega_\mu$ for $\omega_0^\mu$. Our structure equations can now be rewritten as

$$
d\omega^\mu = -\gamma^\mu_\nu \wedge \omega^\nu
$$

$$
d\gamma^\mu_\nu = -\gamma^\mu_\sigma \wedge \gamma^\sigma_\nu + (\delta^\mu_\nu \omega_\sigma + \omega_\nu \delta^\mu_\sigma) \wedge \omega^\sigma
$$

$$
d\omega_\mu = \gamma^\nu_\mu \wedge \omega_\nu.
$$

These are the structure equations of a flat projective structure [9].
4.2. **Canonically defined vector bundles on a manifold with flat projective structure.** It is not essential to work out the theory of invariantly defined vector bundles on the sphere determined by the projective structure, but it makes clearer the interpretation of the invariantly defined vector bundles which we will produce from a great circle fibration in subsection 5.1 on page 26.

We can now take any representation of the group $G_0$, say $\rho: G_0 \to \text{GL}(W)$, and use it to define a vector bundle $\tilde{W} \to S^{2n+1}$ by

$$\tilde{W} = G \times^{G_0} W = (G \times W)/G_0$$

where the quotient is taken by the $G_0$ action

$$(g, w)g_0 = \left( gg_0, \rho(g_0)^{-1}w \right)$$

for

$$g_0 \in G_0, g \in G, w \in W.$$  

A section of $\tilde{W} \to S^{2n+1}$ is precisely a $G_0$-equivariant map $f: G \to W$. If we pick a basis $w_\alpha$ of $W$, $f$ has components $f^\alpha$. Write $\rho: g_0 \to \text{gl}(W)$ for the Lie algebra morphism induced by our morphism $\rho: G_0 \to \text{GL}(W)$ of Lie groups. The differential of $f^\alpha$ is

$$df^\alpha + \rho\left(\omega^0_\alpha \omega^0_\nu \omega^0_\mu\right) f^\beta = f^\alpha_{\mu} \omega^\mu$$

for some functions $f^\alpha_{\mu}$ on $G$, or in other words

$$df + \rho(g_0^{-1}dg_0)f = \nabla f \omega$$

where

$$\nabla f: G \to e_0^* \otimes W$$

is the covariant derivative of the section $f$. We will say that $\rho$ solders the bundle $\tilde{W}$. Since all of the Lie groups I will employ in this article are connected, the Lie algebra representation will suffice for our purposes to identify the group representation, and we will usually only indicate the Lie algebra representation, saying that the bundle is soldered by the expression

$$\rho\left(\omega^0_\alpha \omega^0_\nu \omega^0_\mu\right)$$.

For example, we define $\mathcal{O} (-1)$ to be the bundle $\mathcal{O} (-1) = \tilde{W}$ where $W = \langle e_0 \rangle \subset V$ is the span of $e_0$ in $V$. It is not difficult to see that if we were to quotiente our sphere down to the underlying real projective space, then this bundle $\mathcal{O} (-1)$ would become the algebraic geometer’s usual universal line bundle. Since $G_0$ preserves an orientation in this line $\langle e_0 \rangle$, the bundle $\mathcal{O} (-1)$ has oriented fibers. Write any section $\sigma$ of $\mathcal{O} (-1)$ (perhaps only defined on an open subset of the sphere) as

$$\sigma (g [e_0]) = f(g)e_0.$$  

Then $f: G \to \mathbb{R}$ and $df + \omega^0_\mu f = f_{\mu} \omega^\mu$ for some functions $f_{\mu}: G \to \mathbb{R}$. So $\mathcal{O} (-1)$ is soldered by $\omega^0_\mu$.

Define $\mathcal{O} (1) := \mathcal{O} (-1)^*$ and similarly

$$\mathcal{O} (p) := \mathcal{O} (1)^{\otimes p} = (\langle e_0 \rangle^*)^{\otimes p}.$$  

Then $\mathcal{O} (p)$ is soldered by $-p\omega^0_\mu$. 


We will also want to consider the bundle $\tilde{V}$ soldered by
\[
\left( \begin{array}{cc}
\omega^0_0 & \omega^0_\nu \\
0 & \omega^\nu_\nu
\end{array} \right)
\]
(i.e. $\rho$ is the identity). This is a trivial bundle, since any element $v \in V$ gives rise to a global section $f_v(g) = g^{-1}v$ of $\tilde{V} \rightarrow S^{2n+1}$. However, there is no trivialization invariant under $\text{SL}(V)$. Therefore we will prefer to consider $\tilde{V}$ as a vector bundle. The sections of this bundle correspond to functions $f : G \rightarrow V$ so that in terms of our usual basis of $V$
\[
d \left( \begin{array}{c} f^0 \\
f^\mu
\end{array} \right) + \left( \begin{array}{cc}
\omega^0_0 & \omega^0_\nu \\
0 & \omega^\nu_\nu
\end{array} \right) \left( \begin{array}{c} f^0 \\
f^\nu
\end{array} \right) = \left( \begin{array}{c} f^0 \\
f^\nu
\end{array} \right) \omega^\nu.
\]
Take a fixed vector $v \in V$ and let $f = g^{-1}v$, so that $df = -\omega f$ or
\[
d \left( \begin{array}{c} f^0 \\
f^\mu
\end{array} \right) + \left( \begin{array}{cc}
\omega^0_0 & \omega^0_\nu \\
0 & \omega^\nu_\nu
\end{array} \right) \left( \begin{array}{c} f^0 \\
f^\nu
\end{array} \right) = 0.
\]
We see the covariant derivatives when we write it as
\[
d \left( \begin{array}{c} f^0 \\
f^\mu
\end{array} \right) + \left( \begin{array}{cc}
\omega^0_0 & \omega^0_\nu \\
0 & \omega^\nu_\nu
\end{array} \right) \left( \begin{array}{c} f^0 \\
f^\nu
\end{array} \right) = - \left( \begin{array}{c} 0 \\
\delta^\nu_\mu
\end{array} \right) \omega^\nu.
\]
If a vector bundle $\tilde{W}$ is soldered by $\rho^I_J$ then its dual $\tilde{W}^* = \tilde{W}^*$ is soldered by $-\rho^I_J$, i.e. the negative transpose.

When we add two representations, say $U$ and $W$, with bases $u_\alpha$ and $w_M$ we obtain a representation $U \oplus W$ with basis $z_I$ where $I$ runs over first the $\alpha$ indices and then the $M$ indices. The matrix elements of the Lie algebra representation (or the Lie group) on the sum are
\[
\rho^I_J = \rho^I_J \delta^I_M \delta^M_J + \delta^I_I \delta^M_J \rho^M_N.
\]
Similarly under tensor product, the new index $I$ runs over pairs $(\alpha, M)$ and gives matrix elements of the Lie algebra representation (not the same as the group representation)
\[
\rho^I_J = \rho^I_J \delta^M_N + \delta^I_J \rho^M_N
\]
where
\[
I = (\alpha, M), \ J = (\beta, N).
\]
For a quotient representation, $W/U$ where $U$ is an invariant subspace of $W$, we have indices $\alpha, \beta$ for $U$, $M, N$ for $W/U$, and $I, J$ for $W$, and the relation
\[
(\rho^I_J) = \left( \begin{array}{cc}
\rho^\alpha_\beta & \rho^\alpha_N \\
0 & \rho^M_N
\end{array} \right)
\]
so that we can dig out the matrix elements of the quotient from those of the original representation.

Lemma 1. The tangent bundle of the sphere is soldered by $\gamma^I_\nu$ and is canonically and $\text{SL}(V)$-equivariantly isomorphic to
\[
\mathcal{O}(1) \otimes \left( V/\mathcal{O}(-1) \right).
\]
Proof. Let's write \( \langle e_0 \rangle \) for the line in \( V \) through \( e_0 \). The tangent space to the sphere \( S^{2n+1} \) at \( [e_0] \) is given by

\[
0 \to T_{[e_0]}[e_0] = \langle e_0 \rangle \to T_{e_0}V = V \to T_{[e_0]}S^{2n+1} = V/\langle e_0 \rangle \to 0.
\]

But if we change the choice of the point \( e_0 \), by a positive multiple, then we have to rescale \( V \) by this multiple, and rescale \( \langle e_0 \rangle \) by the same multiple, while we don’t rescale the sphere at all. Therefore this description of the tangent space to the sphere is certainly not \( G_0 \)-equivariant, and it will only become scale invariant if we rewrite it as

\[
0 \to \langle e_0 \rangle^* \otimes \langle e_0 \rangle \to \langle e_0 \rangle^* \otimes V \to T_{[e_0]}S^{2n+1} \to 0
\]

with the first map just an inclusion, and the second map defined by

\[
\xi \otimes v \mapsto \xi(e_0) v/\langle e_0 \rangle \in V/\langle e_0 \rangle = T_{[e_0]}S^{2n+1}.
\]

We have to check that this map is \( G_0 \)-equivariant, invariant under positive rescaling, because this cancels out from each factor. Under linear transformations \( g_0 \in G_0 \), which take \( e_0 \) to \( g_0 e_0 \) with \( g_0^0 > 0 \),

\[
\xi \otimes v \mapsto \xi(g_0 e_0) v/\langle e_0 \rangle = \xi(e_0) g_0 v/\langle e_0 \rangle
\]

we see that the map is \( G_0 \)-equivariant, so that

\[
T_{[e_0]}S^{2n+1} = (\langle e_0 \rangle^* \otimes V)/(\langle e_0 \rangle^* \otimes \langle e_0 \rangle) = \langle e_0 \rangle^* \otimes (V/\langle e_0 \rangle)
\]

and therefore

\[
TS^{2n+1} = \langle e_0 \rangle^* \otimes \left( \tilde{V}/\langle e_0 \rangle \right) = O(1) \otimes (\tilde{V}/O(-1)) = O(1) \otimes \tilde{V}/O(0).
\]

Finally, we will consider the soldering of the tangent bundle. The representation \( \omega_\mu^\nu \) solders \( \tilde{V}/O(-1) \) and \( -\omega_0^0 \) solders \( O(1) \). Therefore the tensor product

\[
O(1) \otimes (\tilde{V}/O(-1))
\]

is soldered by

\[
(-\omega_0^0) \delta_\mu^\nu + \delta_0^0 \omega_\mu^\nu = \omega_\mu^\nu - \delta_\nu^\mu \omega_0^0.
\]

□

Corollary 2.

\[
T^*S^{2n+1} = O(0)^\perp \subset O(-1) \otimes \tilde{V}^*
\]

is soldered by \( -\gamma_\mu^\nu \). while

\[
\text{Det} TS^{2n+1} = O(-(2n + 2))
\]

is soldered by \( -(2n + 2) \omega_0^0 \).
4.3. Structure equations of a geodesic foliation in a flat projective structure. Let the Roman indices \(i, j, k, l\) run from 2 to \(2n + 1\). Inside \(G\) we have a subgroup \(G_{\text{circle}}\) consisting of matrices of the form

\[
\begin{pmatrix}
g_0^0 & g_1^0 & g_0^0 \\
g_0^1 & g_1^1 & g_0^1 \\
0 & 0 & g_2^0
\end{pmatrix}
\]

with \(\det = 1\) and

\[
\det \begin{pmatrix} g_0^0 & g_1^0 \\ g_0^1 & g_1^1 \end{pmatrix} > 0.
\]

This is the subgroup of all elements of \(G\) which preserve the oriented plane spanned by \(e_0\) and \(e_1\). Under the map \(G \to S^{2n+1}\), this subgroup projects to the oriented great circle which is the image of the \(e_0, e_1\) plane. This subgroup satisfies \(\omega^i = \gamma_1^i = 0\) as do all of its left translates, by left invariance of the Maurer–Cartan 1-form. Thus the geodesics (the great circles) are the curves in the sphere which are the projections of the integral manifolds of \(\omega^i = \gamma_1^i = 0\) in \(G\). The manifold \(G/G_{\text{circle}}\) is the manifold of all oriented great circles on the sphere \(S^{2n+1}\).

Similarly, any pointed great circle on the sphere is the projection of a left translate of the subgroup \(G_1\) of matrices of the form

\[
\begin{pmatrix}
g_0^0 & g_1^0 & g_0^0 \\
g_0^1 & g_1^1 & g_0^1 \\
0 & 0 & g_2^0
\end{pmatrix}
\]

where \(g_0^0, g_1^1 > 0\) and \(g_0^0 g_1^1 \det (g_0^i) = 1\) (the subgroup of \(G\) preserving not only the oriented \(e_0, e_1\) plane, but also fixing the point \(e_0\), up to positive factor). The left translates of \(G_1\) are precisely the leaves of the foliation of \(G\) given by the equations \(\omega^i = \omega^j = \gamma_1^i = 0\). Hence the quotient space \(G/G_1\) is the space of pointed great circles.

Starting with a foliation \(F\) by curves, look inside \(G\) and consider the subbundle \(B_1\) whose fiber above any point \(x \in S^{2n+1}\) consists of the linear maps \(g: V \to \mathbb{R}^{2n+2}\) which identify our point \(x\) of the sphere, i.e. a ray in \(V\), with a given ray in \(\mathbb{R}^{2n+2}\), say the ray through \(e_0\), and which identify the tangent line to the \(F\) curve through \(x\) with a given 2-plane in \(\mathbb{R}^{2n+2}\), say the span of \(e_0, e_1\). The map \(B_1 \to S^{2n+1}\) is the pullback

\[
\begin{pmatrix}
B_1 \\
S^{2n+1}
\end{pmatrix} \to \begin{pmatrix}
G \\
G/G_1
\end{pmatrix}
\]

above the map \(S^{2n+1} \to G/G_1\) (taking any point \(p \in S^{2n+1}\) to the great circle which is tangent at \(p\) to the leaf of \(F\) through \(p\)) and so \(B_1 \to S^{2n+1}\) is a smooth right principal \(G_1\) bundle. The fibers of the bundle \(B_1\) are cut out by the equations \(\omega^i = \omega^j = \gamma_1^i = 0\). But the \(\omega^i, \omega^j\) are linearly independent 1-forms on the bundle \(B_1\) (since we have made no restriction on motions in the base manifold, the sphere \(S^{2n+1}\)). Therefore on this bundle \(B_1\) \(\gamma_1^i = t_i^1 \omega^1 + t_i^j \omega^j\) for some functions \(t_i^1\) and \(t_i^j\). The equation \(\omega^i = 0\) cuts out a foliation \(F_1\) of \(B_1\), by the Frobenius theorem. Indeed the leaves of \(F_1\) are precisely the preimages in \(B_1\) of the leaves of \(F\) down on the sphere. To have \(F\) constitute a foliation by geodesics, we will need each
leaf of $F_1$ to sit inside a right translate of a subgroup of $G$ satisfying $\omega^i = \gamma^i_1 = 0$. Therefore we need $t^i_1 = 0$, and henceforth we will assume this; i.e. on $B_1$:

\begin{equation}
\gamma^i_1 = t^j_i \omega^j.
\end{equation}

The Reznikov invariant is $t = (t^i_j)$ \cite{13} p. 89.

Differentiating this last equation gives an expression for the derivatives of the Reznikov invariant

\[
\nabla t^i_j = dt^i_j - t^i_j \omega^1_j + \gamma^i_k t^k_j - t^i_k \gamma^k_j - \delta^i_j \omega_1
\]

and we can calculate that

\begin{equation}
\nabla t^i_j = -t^i_k t^k_j \omega^1 + t^i_j \omega^k
\end{equation}

where $t^i_jk = t^i_{kj}$.

Note that this “covariant derivative” is determined not with a connection on the tangent bundle, but using the flat projective connection on the sphere.

**Lemma 2.** Suppose that $S^{2n+1} \rightarrow X^{2n}$ is a great circle fibration. The tangent bundle to $X^{2n}$ is soldered by

\[
\gamma^i_j = \omega^i_j - \delta^i_j \omega^0_0.
\]

**Proof.** The proof is essentially the same as that of lemma 1 on page 9 except that we use the representation

\[
(c_0^* \otimes V) / (c_0^* \otimes \langle c_0, e_1 \rangle)
\]

where $\langle c_0, e_1 \rangle$ is the span of $c_0, e_1$. \qed

**4.4. Structure equations of the Hopf fibration.** To fix the Hopf fibration as well as the flat projective structure, transformations must take complex lines to complex lines, since the Hopf fibration on the sphere is the quotient (by rescaling by positive numbers) of the fibration of $\mathbb{C}^{n+1}\setminus 0$ into complex lines through the origin.

The circle fibers can be oriented by using the natural orientation on complex lines, and we will take them to be thus oriented.

**Lemma 3.** Let $V$ be a complex vector space of dimension at least two. Every invertible real linear map of $V$ which takes complex lines to complex lines, preserving the natural orientation of complex lines, is complex linear.

The result is not true for $V$ of one complex dimension.

**Proof.** Take $c_1, c_2 \in V$ any two vectors which are linearly independent over the complex numbers. Then $gc_1, gc_2$ must still be linearly independent over the complex numbers, because $c_1, c_2$ belong to distinct complex lines, so $g$ must take these to distinct complex lines. Take $h$ any complex linear transformation taking $gc_1 \mapsto e_1, gc_2 \mapsto e_2$. To show that $g$ is complex linear on the complex 2-plane spanned by $c_1, c_2$ it suffices to show that $hg$ is. So without loss of generality, we can assume $ge_1 = e_1$ and $ge_2 = e_2$. Consequently there must be real constants $a_j, b_j$ so that

\[
g\sqrt{-1}e_j = a_je_j + b_je_j.
\]

The map $g$ also must preserve the complex line spanned by $e_1 + e_2$, which forces $a_1 = a_2, b_1 = b_2$. Preserving the complex line spanned by $e_1 + \sqrt{-1}e_2$ forces $a_1 = 0, b_1 = 1$. This makes $g$ complex linear on the 2-plane spanned by $e_1, e_2$.

Since $c_1$ and $c_2$ are arbitrary this proves the lemma. \qed
So the group of symmetries of the Hopf fibration as a geodesic foliation of a flat projective structure is the group

\[ \Gamma = \text{SL}(2n + 2, \mathbb{R}) \cap \text{GL}(n + 1, \mathbb{C}). \]

The subgroup preserving the point \( e_0 \) up to positive rescaling (i.e. fixing the north pole of the sphere) is the group \( \Gamma_0 \) of complex matrices of the form

\[
\begin{pmatrix}
g_0^0 & g_0^p \\
g_0^q & g_q^q
\end{pmatrix}
\]

where the indices \( p, q \) here run from 1 to \( n \), with \( g_0^0 > 0 \), the \( g_0^q \) and \( g_q^p \) are complex numbers, and

\[ |g_0^0 \det(g_q^p)|^2 = 1 \]

(the real linear determinant must be 1). The Hopf fibration is represented by the fiber bundle

\[ \Gamma_0 \longrightarrow \Gamma \]

\[ S^{2n+1} \]

given by the obvious right action of \( \Gamma_0 \) on \( \Gamma \). The group \( \Gamma \) is a subgroup of the group \( G = \text{SL}(2n + 2, \mathbb{R}) \) that we encountered previously. Let

\[
J_0 = \begin{pmatrix}
0 & -1 \\
1 & 0 \\
& & \ddots \\
& & & 0 & -1 \\
& & & 1 & 0
\end{pmatrix}
\]

be the usual complex structure on \( \mathbb{R}^{2n+2} = \mathbb{C}^{n+1} \), and let

\[
K_0 = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
& & \ddots \\
& & & 0 & 1 \\
& & & 1 & 0
\end{pmatrix}
\]

be the usual complex conjugation. Taking the Maurer–Cartan 1-form \( \omega \) from \( G \), we can split it into

\[ \omega = \Omega^{1,0} + \Omega^{0,1} K_0 \]

by

\[ \Omega^{1,0} = \frac{1}{2}(\omega - J_0 \omega J_0) \]

\[ \Omega^{0,1} = \frac{1}{2}(\omega + J_0 \omega J_0) K_0. \]

We can now write these in complex components, since each of \( \Omega^{1,0} \) and \( \Omega^{0,1} \) is a matrix built out of \( 2 \times 2 \) blocks like

\[
\begin{pmatrix}
a & -b \\
b & a
\end{pmatrix}
\]
which can be identified with the complex number $a + b\sqrt{-1}$. Write out

$$\Omega^{1,0} = \begin{pmatrix} \Omega^0_0 & \Omega^0_q \\ \Omega^p_0 & \Omega^p_q \end{pmatrix},$$

$$\Omega^{0,1} = \begin{pmatrix} \Omega^0_0 & \Omega^0_q \\ \Omega^p_0 & \Omega^p_q \end{pmatrix}.$$

To be more explicit,

$$\Omega^P_Q = \frac{1}{2} \left( \omega^P_{2Q} + \omega^P_{2Q+1} \right) + \frac{\sqrt{-1}}{2} \left( \omega^P_{2Q+1} - \omega^P_{2Q} \right)$$

for $P, Q = 0,\ldots,n$. Since the real trace of $\omega$ vanishes, so does $\Omega^P_Q + \Omega^P_Q$. We will write $\Omega^P_Q$ for the conjugate of $\Omega^P_Q$ and $\Omega^P_Q$ for the conjugate of $\Omega^P_Q$. The structure equations of $G$ can now be written in this notation as

$$d\Omega^P_Q = -\Omega^P_R \wedge \Omega^R_Q - \Omega^P_Q \wedge \Omega^R_Q$$

$$d\Omega^P_Q = -\Omega^P_R \wedge \Omega^R_Q - \Omega^P_Q \wedge \Omega^R_Q.$$

The Hopf fibration satisfies the equations $\Omega^P_Q = 0$. These imply

$$\omega_1 = -\omega^1$$

$$\omega_{2P} = \gamma^1_{2P+1}$$

$$\omega_{2P+1} = -\gamma^1_{2P}$$

$$\gamma^1_1 = 0$$

$$\gamma^1_{2P+1} = -\omega_{2P+1}$$

$$\gamma^1_{2P} = \omega_{2P}$$

$$\gamma^2_{2Q+1} = -2^{2P}$$

$$\gamma^2_{2Q} = -2^{2P+1}$$

So for the Hopf fibration, the invariant $t$ is

$$(t^j_i) = \begin{pmatrix} 0 & -1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & \cdots & 1 \end{pmatrix}.$$\]

4.5. Structure equations of the standard round metric on the sphere. It is helpful to compare the structure equations of a geodesic foliation on the sphere to the equations we obtain with the standard metric in place. We obtain the structure equations of $\text{SO}(2n+2)$ from those of $\text{GL}(2n+2, \mathbb{R})$ by imposing the relations

$$\omega_\mu = -\omega^{\mu'}$$

$$\gamma^\mu_\nu = -\gamma^{\nu}_{\mu'}.$$

(3)
In other words, $\text{SO}(2n+2)$ is the connected subgroup of $\text{GL}(2n+2, \mathbb{R})$ of largest dimension on which these equations are satisfied.

**Lemma 4.** The integral manifolds of the equations on the facing page are precisely the left translates of $\text{SO}(2n+2)$ inside $\text{GL}(2n+2, \mathbb{R})$.

**Proof.** The Lie algebra of $\text{SO}(2n+2)$ satisfies these equations, so all of $\text{SO}(2n+2)$, as do all left translates of $\text{SO}(2n+2)$ by left invariance of the equations. The left translates form a foliation of $\text{GL}(2n+2, \mathbb{R})$ and by the Frobenius theorem, they are all of the integral manifolds. \hfill $\square$

Consider the equations

$$
\begin{align*}
\omega_1 &= -\omega^1 \\
\omega_i &= \omega^i = 0 \text{ for } i > 1 \\
\gamma &= 0.
\end{align*}
$$

These are also the equations of a subgroup—in this case the circle subgroup of $\text{SO}(2n+2)$ which turns the $e_0, e_1$ plane and fixes the perpendicular directions.

**Lemma 5.** The integral curves of equation 4 are precisely the left translates of this circle subgroup of $\text{SO}(2n+2)$. In particular these integral curves are compact and

$$
\int \omega^1 = 2\pi
$$

when integrating over any of the integral curves.

4.6. **Following the invariants around the circles.** Consider the behaviour of our invariant $t$ on a circle subgroup left translate. Imposing the relations satisfied by a circle subgroup translate given in equation 4 we find that, if $t = (t_j)$ is treated as a matrix then

$$
dt = -(I + t^2) \omega^1.
$$

Such a circle subgroup translate lives inside our bundle $B_1$ above each of our great circles, by construction of $B_1$, since above each circle on the sphere, $B_1$ contains a left translate of the subgroup $G_1$ which contains a circle subgroup. Writing $\omega^1 = d\theta$ we have the ordinary differential equation

$$
\frac{dt}{d\theta} = -(I + t^2).
$$

Consider first how solutions of this ordinary differential equation behave if $t$ is just a complex number. The solutions are $t = -\tan(\theta + c)$, except for the two singular solutions $t = \pm \sqrt{-1}$. So all solutions have period $\pi$. Indeed on the Riemann sphere, this equation is a rotation, and has the two exceptional points $\pm \sqrt{-1}$ as rotational fixed points.
To have our complex number $t$ remain finite, it can never be real-valued, since it would then go to infinity in time at most $\pi/2$ in one direction or the other. For a matrix $t$ the solution is

$$t(\theta) = (t(0) - \tan(\theta) I) (I + \tan(\theta) t(0))^{-1}$$

and this tells us that $t$ has no real spectrum. Since $t$ is a real matrix, the eigenvalues and eigenspaces of $t$ come in conjugate pairs. Since the equation is invariant under change of linear coordinates, the eigenspaces will remain invariant under the flow, while the eigenvalues will change. For any initial conditions $t(0)$ with no real eigenvalues, we find that $t$ will remain defined as a function of $\theta$ for any positive or negative $\theta$. For generic initial conditions, for example for $t$ diagonalizable at $\theta = 0$, we find that $t$ is $\pi$ periodic. Therefore $T$ is also $\pi$ periodic for any initial conditions with no real eigenvalues.

Notice that we have not so far invoked the hypothesis that the sphere has odd dimension. In fact, this is an immediate consequence of the Reznikov invariant having no real eigenvalues, since $t$ is a square matrix whose size is one less than the dimension of the sphere. If $t$ had odd size, then its characteristic polynomial would have odd degree, and so would have a real root—hence a real eigenvector.

4.7. **Linear transformations without real eigenvalues.** Take any linear transformation $T: V \to V$ of a real vector space $V$, with no real eigenvalues. We will also write $V$ as $V_\mathbb{R}$ to emphasize that we are studying its real points, and write $V_\mathbb{C}$ for $V \otimes \mathbb{C}$. We naturally have $V_\mathbb{C}$ split into generalized eigenspaces of $T$, say

$$E_\lambda T = \{v \in V_\mathbb{C} \mid (T - \lambda I)^k v = 0, \text{ for some } k > 0\}.$$

Since $T$ is real, $\overline{E_\lambda T} = E_\lambda T$. Pick the eigenspaces $E_\lambda T$ of $T$ where the eigenvalues $\lambda$ have positive imaginary parts, and define a subspace

$$V_{1,0}^T := \bigoplus_{\Im \lambda > 0} E_\lambda T \subset V_\mathbb{C}.$$

Let $V^{0,1}$ be the conjugate of $V^{1,0}$, i.e. the sum of the eigenspaces whose eigenvalues have negative imaginary part. Note that $V^{1,0}$ and $V^{0,1}$ are complex subspaces of $V_\mathbb{C}$ and are complementary:

$$V^{1,0} \cap V^{0,1} = 0, V_\mathbb{C} = V^{1,0} \oplus V^{0,1}.$$

Define a linear transformation $J: V_\mathbb{C} \to V_\mathbb{C}$ by letting $J$ act via $\sqrt{-1}$ on $V^{1,0}$ and by $-\sqrt{-1}$ on $V^{0,1}$. This $J$ is real, i.e. acts as a real linear transformation on $V_\mathbb{R}$, since $J$ is conjugation invariant, with $J^2 = -I$, since this equation holds on $V^{1,0}$ and on $V^{0,1}$ by construction. Write $J$ as $J_T$.

**Lemma 6.** Let $H_V$ be the set of linear transformations of a finite dimensional real vector space $V$ which have no real eigenvalues (think of it as a “generalized upper half plane”), and $\mathcal{J}_V$ be the subset of real linear transformations $J$ which satisfy $J^2 = -I$ (the complex structures). Then the map

$$T \in H_V \mapsto J_T \in \mathcal{J}_V$$

is a smooth fiber bundle, with the inclusion $\mathcal{J}_V \subset H_V$ as a section. This fiber bundle is $\text{GL}(V)$-equivariant: $J_{gTg^{-1}} = gJ_Tg^{-1}$ for any $g \in \text{GL}(V)$. Its fiber above any point $J$ consists precisely of the $J$-complex linear maps $T: V \to V$, i.e. $TJ = JT$, all of whose eigenvalues on $(J = \sqrt{-1}) \subset V_\mathbb{C}$ have positive imaginary part. If $c \neq 0$ is a real number, then $J_{T+cI} = J_T$ and $J_{cT} = \text{sign}(c)J_T$. 

Proof. Notice that if $V$ has odd dimension, then (looking at the characteristic polynomial, which is of odd degree) we find $H_V$ is empty. Similarly, taking determinant, we find that $\mathcal{J}_V$ is empty. So we can assume $V$ has even dimension.

The equivariance of $T \mapsto J_T$ under $\text{GL}(V)$ is elementary.

First, we construct the space $Z \subset H_V \times \mathcal{J}_V$ which consists of pairs $(T, J)$ satisfying

$$TJ = JT, J^2 = -I$$

and with $T$ having no real eigenvalues. We have maps

$$
\begin{array}{ccc}
Z & \xrightarrow{TJ = JT, J^2 = -I} & H_V \times \mathcal{J}_V \\
\downarrow & & \downarrow \\
H_V & & \mathcal{J}_V 
\end{array}
$$

given by taking a pair $(T, J)$ and either forgetting $J$ or forgetting $T$. Now given any $T \in H_V$, we know how to construct a $J = J_T$ commuting with it so that $J^2 = -I$, i.e. the map $T \mapsto J_T$ has graph lying in $Z$.

On the other hand, if we take any $T$ with no real eigenvalues, i.e. $T \in H_V$, then we can also have constructed a $J$ commuting with it in other ways; in fact if we just put any collection of eigenspaces of $T$ together and call their sum $V^{1,0}$ and call the sum of their conjugates $V^{0,1}$, so that every eigenspace is in one or the other, then we can define $J$ to be $\sqrt{-1}$ on $V^{1,0}$ and $-\sqrt{-1}$ on $V^{0,1}$. Because $T$ has no real eigenvalues, this procedure unambiguously determines $J$, up to the choices of which eigenspaces of $T$ go to $V^{1,0}$ and which go to $V^{0,1}$.

Conversely, if we pick a pair $(T, J)$ in $Z$, the fact that $T$ and $J$ commute ensures that each one leaves the eigenspaces of the other invariant. Looking at the minimal polynomial of $J$, $x^2 + 1$, we see that $J$ has exactly two eigenspaces, and we call them $V^{1,0}$ (the $\sqrt{-1}$-eigenspace) and $V^{0,1}$ (the $-\sqrt{-1}$-eigenspace). The map $T$ takes $V^{1,0}$ to itself, and takes $V^{0,1}$ to itself, because $T$ commutes with $J$, and $T$ is complex linear on each. Decomposing $V^{1,0}$ into complex eigenspaces of $T$, and decomposing $V^{0,1}$ into the conjugates of those eigenspaces, we find $V_\mathbb{C} = V^{1,0} \oplus V^{0,1}$ decomposed into conjugate eigenspaces of $T$. Because $T$ is real, its eigenspaces are conjugate, with half of them in $V^{1,0}$ and half in $V^{0,1}$. Therefore $J$ comes about from $T$ by the construction outlined in the last paragraph.

Thus the map $Z \to H_V$ has finitely many points in each stalk—each point corresponds to a choice of which eigenspaces of $T$ go into $V^{1,0}$. We want to show that the points which make up the graph of $T \mapsto J_T$ form a smooth subvariety of $Z$. We have only to show that they are transverse to the fibers of the map

$$H_V \times \mathcal{J}_V \to H_V$$

given by $(T, J) \mapsto T$.

The algebraic equations cutting out $Z$ are $TJ = JT$ and $J^2 = -I$. Differentiating these in motions up the fiber, we find that the equations of a vertical tangent vector are $TJ = JT$ and $JJ + JJ = 0$. The first equation tells us that $J$ preserves the eigenspaces of $T$, while the second tells us that $J$ swaps the eigenspaces of $J$. But the eigenspaces of $T$ are entirely contained inside those of $J$. Therefore $J = 0$, and there are no vertical tangent vectors. This shows that $Z$ is smooth and transverse to the fibers of $H_V \times \mathcal{J}_V \to \mathcal{J}_V$. So the map $T \mapsto J_T$ is smooth, being
just a single branch of \( Z \), and \( J_T = T \) for any \( T \) with \( T^2 = -I \). Hence this map \( T \mapsto J_T \) is a smooth surjection \( H_V \to \mathcal{J}_V \).

To find the rank of the map \( T \mapsto J_T \) differentiate the equations \( J^2 = -I \) and \( JT = TJ \) to find

\[
JJ + J\dot{J} = 0 \quad \text{and} \quad JT + J\dot{T} = \dot{T}J + TJ.
\]

The kernel of the derivative of \( T \mapsto J_T \) is the set of \( \dot{T} \)s satisfying \( J\dot{T} = \dot{T}J \), i.e. the \( J \) complex linear maps. Since the real general linear group acts transitively on complex structures, we find that the space of such \( \dot{T} \)s has dimension independent of \( J \). Indeed if \( \dim_{\mathbb{R}} V = 2n \), then the derivative of the map \( T \mapsto JT \) has kernel of dimension \( 2n^2 \), and so fibers of dimension also \( 2n^2 \). The dimension of the base is also \( 2n^2 \), so the map \( T \mapsto J_T \) is a smooth submersion.

Next we want to show that this map is locally trivial, so that it will be a fiber bundle. First, we note once again that \( \mathcal{J}_V \) is a homogeneous space under the action of \( \text{GL}(V) \). So if we pick a particular \( J_0 \in \mathcal{J}_V \) then we have the fiber bundle

\[
\text{GL}(V, J_0) \to \text{GL}(V) \to \mathcal{J}_V.
\]

Every homogeneous space is the base spaces of a fiber bundle, with total space given by the transitively acting group. Thus this bundle is locally trivial, i.e. to every sufficiently small open set \( U \subset \mathcal{J}_V \) we can associate a map \( U \to \text{GL}(V) \), say \( g(J) \), so that if \( g = g(J) \) then \( gJg^{-1} = J_0 \).

We map the fiber \( H_{V,J_0} \) above \( J_0 \) to the fiber \( H_{V,J} \) above \( J \) by

\[
T_0 \mapsto T = T(J,T_0) = gT_0g^{-1}.
\]

By \( \text{GL}(V) \)-equivariance, this is a local trivialization of \( H_V \to \mathcal{J}_V \), so that this map is a fiber bundle.

The reader who wishes to understand this lemma might work out the whole story in matrices for \( V = \mathbb{R}^2 \). This complex structure is \( J \) due to Reznikov \[13\] p. 89.

4.8. **Back to the great circle fibration.** Inside \( \text{SO}(2n+2) \) we do not have enough room to put the Reznikov invariant \( t \) into a normal form. If we return to \( G = \text{SL}(2n+2, \mathbb{R}) \) we have enough room to arrange by moving up and down the fibers of \( B_1 \) that at least the associated linear map \( J_t \) is normalized:

\[
J_t = J_0 = \begin{pmatrix}
0 & -1 & 0 & \cdots \\
1 & 0 & 0 & \\
0 & 1 & 0 & \\
& & & \ddots
\end{pmatrix}.
\]

There is a subbundle \( B_2 \subset B_1 \) on which these equations hold. This bundle \( B_2 \) is a principal \( G_2 \) subbundle, where \( G_2 \) is the group of matrices of the form

\[
\begin{pmatrix}
g_0 & g_0^1 & g_0^2 \\
g_1^0 & g_1 & g_1^2 \\
g_2^0 & 0 & g_2
\end{pmatrix}
\]

with \( g_0^0, g_1^1 > 0 \) and \( g_0^0g_1^1 \det(g_i^j) = 1 \) and where \( (g_i^j) \) commutes with \( J \).
This matrix $t^i_j$ is now $J_0$ complex linear, so that we can say that the 1-forms
\[
\omega_1^{2p} + \sqrt{-1}\omega_1^{2p+1}
\]
amer complex multiples of the 1-forms
\[
\omega_0^{2p} + \sqrt{-1}\omega_0^{2p+1}.
\]
Written in terms of the complex $\Omega$ notation,
\[
\omega_1^{2p} + \sqrt{-1}\omega_1^{2p+1} = \Omega_0^p + \sqrt{-1}\Omega_0^p
\]
and
\[
\omega_0^{2p} + \sqrt{-1}\omega_0^{2p+1} = \Omega_0^p + \sqrt{-1}\Omega_0^p
\]
so that the equation $\gamma_1^j = t^i_j\omega^j$ can be written as
\[
\Omega_0^p + \sqrt{-1}\Omega_0^p = t^p_q (\Omega_0^q + \sqrt{-1}\Omega_0^q)
\]
with $p, q = 1, \ldots, n$, and $t^p_q$ a complex matrix whose eigenvalues lie in the upper half plane.

Let’s write linear fractional transformations using the notation\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} z = \frac{az + b}{cz + d}.
\]
Define the Cayley map
\[
C(z) = \begin{bmatrix}
\sqrt{-1} & 1 \\
1 & \sqrt{-1}
\end{bmatrix} z.
\]
Solving equation 5 for $\Omega_0^p$ in terms of $\Omega_0^q$, we find $\Omega_0^p = s^p_q \Omega_0^q$ expressed in terms of the Cayley mapped matrix $s = C(t)$. Henceforth we employ $s$ instead of $t$ and call $s$ the Reznikov invariant. The condition that $t$ have all of its eigenvalues in the upper half plane is equivalent to its image $s = C(t)$ under the Cayley map having all of its eigenvalues inside the unit disk. In particular, for the Hopf fibration, on the subbundle $\Gamma \subset B_2$ we find $s = 0$.

Consider the ordinary differential equation
\[
\frac{dt}{d\theta} = -(1 + t^2).
\]
If we set $s = C(t)$, then
\[
\frac{ds}{d\theta} = -2\sqrt{-1}s
\]
so that $s(\theta) = s(0)e^{-2\sqrt{-1}\theta}$ evolves by rotation. We will see that the Reznikov invariant is a section of a complex vector bundle over the base manifold of the circle fibration (i.e. the manifold parameterizing the circles). For the Hopf fibration, this base manifold is $\mathbb{CP}^n$.

The restriction of structure group imposed by the equation $J_t = J_0$ requires $\Omega_0^p$ to be semibasic as well, for $p, q = 1, \ldots, n$. Recall that the 1-forms $\omega^1, \omega^i$ form a basis for the semibasic 1-forms. The 1-forms $\Omega_0^p$ are semibasic, but (even together with their conjugates) they do not span the semibasic 1-forms—only the $\omega^i (i = 2, \ldots, 2n + 1)$ are multiples of them, while $\omega^1$ is not.
Taking the exterior derivative of both sides of the equation $\Omega_{0}^{p} = s^{p}_{q} \Omega_{0}^{q}$,

$$0 = \left( ds_{q}^{p} + \Omega_{q}^{r}s_{q}^{r} - s_{r}^{p}\Omega_{q}^{r} - \Omega_{0}^{r}s_{q}^{p} + \left( \Omega_{0}^{0} - \Omega_{0}^{0} \right) s_{q}^{p} + s_{r}^{p}s_{q}^{r}\Omega_{0}^{0} \right) \wedge \Omega_{0}^{q}$$

$$+ \left( \Omega_{q}^{p} - s_{r}^{p}\Omega_{q}^{r}s_{q}^{l} \right) \wedge \Omega_{0}^{q}.$$

By Cartan's lemma, this implies that there are constants

$$s_{q}^{p} = s_{r}^{p}, s_{rq}^{p} = s_{q}^{p}, s_{qmr}^{p} = s_{r}^{p}$$

so that

$$ds_{q}^{p} + \Omega_{q}^{r}s_{q}^{r} - s_{r}^{p}\Omega_{q}^{r} - \Omega_{0}^{r}s_{q}^{p} + \left( \Omega_{0}^{0} - \Omega_{0}^{0} \right) s_{q}^{p} + s_{r}^{p}s_{q}^{r}\Omega_{0}^{0} = s_{q}^{p}\Omega_{0}^{q} + s_{r}^{p}\Omega_{0}^{r}$$

$$\Omega_{q}^{p} - s_{r}^{p}\Omega_{q}^{r}s_{q}^{l} = s_{q}^{p}\Omega_{0}^{q} + s_{r}^{p}\Omega_{0}^{r}.$$  \hspace{1cm} (6)

We will need to see that it is possible to solve for $\Omega_{q}^{p}$. Consider the operation on matrices $M \mapsto M - sMs$. We need to show that it is invertible, as long as the matrix $s$ has all of its eigenvalues in the unit disk. The kernel of this operation consists in matrices $M$ satisfying $M = sMs$. This implies that $sMs = s^{2}M^{2}$ etc. so that $M = s^{k}M^{k}$ for all positive integers $k$. But sufficiently high powers of $s$ are strictly contracting, since all eigenvalues of $s$ are in the unit disk, so $M = 0$. The equation 7 can be solved for $\Omega_{q}^{p}$ as a complex linear combination of the 1-forms $\Omega_{r}^{q}$ and $\Omega_{0}^{r}$.

The Reznikov invariant $s$ transforms under motions through the fibers of the bundle $B_{2}$ (via the action of the structure group $G_{2}$) in a very complicated action, which consists of conjugations via the $\Omega_{q}^{p}$ “variables,” and, as we will see, linear fractional transformations via the $\Omega_{0}^{0} - \Omega_{0}^{0}$ and $\Omega_{0}^{0}, \Omega_{0}^{0}$ “variables.”

4.9. **Linear fractional transformations acting on matrices.** Recall that the linear fractional transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az + b}{cz + d}$$

constitute an action of $\mathbb{P} SL(2, \mathbb{C}) = SL(2, \mathbb{C}) / \pm 1$ on the Riemann sphere,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P} SL(2, \mathbb{C}), \ z \in \mathbb{C} \cup \infty.$$  \hspace{1cm} (7)

The infinitesimal action is

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} z = (b + 2az - cz^{2}) \frac{\partial}{\partial z}$$

for

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \in \mathfrak{sl}(2, \mathbb{C}).$$  \hspace{1cm} (8)

The subgroup of $\mathbb{P} SL(2, \mathbb{C})$ which preserves the unit disk $D$, call it $\text{Aut} D$, is the quotient modulo $\pm 1$ of the group of matrices of the form

$$\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in SL(2, \mathbb{C})$$
subject to $|a|^2 - |b|^2 = 1$. The Lie algebra of $\text{Aut} \, D$ consists of the matrices of the form

$$
\begin{pmatrix}
\sqrt{-1}a & b \\
\bar{b} & -\sqrt{-1}a
\end{pmatrix}
$$

where $a \in \mathbb{R}, b \in \mathbb{C}$. The Cayley map identifies the upper half plane with the unit disk, identifying $\text{Aut} \, D$ with $\text{SL}(2, \mathbb{R})$. There is a (unique up to conjugation) connected 2-dimensional subgroup of $\text{SL}(2, \mathbb{R})$; it is nilpotent and consists of the matrices of the form

$$
\begin{pmatrix}
a & b \\
0 & 1/a
\end{pmatrix} \in \text{SL}(2, \mathbb{R})
$$

with $a, b \in \mathbb{R}$ and $a > 0$. The quotient of this group by $\pm 1$ is the unique connected 2-dimensional subgroup of $\text{PSL}(2, \mathbb{R})$. Such a matrix acts on the element $\sqrt{-1}$ in the upper half plane by

$$
\begin{pmatrix}
a & b \\
0 & 1/a
\end{pmatrix} \sqrt{-1} = a^2 \sqrt{-1} + ab,
$$

a transitive action.

Under the Cayley map, this 2-dimensional subgroup gets mapped to a subgroup $N$ of $\text{Aut} \, D$. One can readily conjugate with the Cayley mapping to calculate that the elements of $N$ are precisely the elements of $\text{PSL}(2, \mathbb{C})$ of the form

$$(8) \quad \begin{pmatrix} a + \frac{1}{a} + \sqrt{-1}b & b + \sqrt{-1}(a - \frac{1}{a}) \\
b - \sqrt{-1}(a - \frac{1}{a}) & a + \frac{1}{a} - \sqrt{-1}b \end{pmatrix}$$

where $a, b \in \mathbb{R}$ and $a > 0$. This group $N$ acts transitively on the unit disk $D$, because its conjugate under the Cayley map acts transitively on the upper half plane. In particular, the Lie algebra of $N$, call it $n$, consists of matrices of the form

$$
\begin{pmatrix}
\sqrt{-1} (Q + \bar{Q}) & Q \\
\bar{Q} & -\sqrt{-1} (Q + \bar{Q})
\end{pmatrix}
$$

where $Q$ can be any complex number. The infinitesimal action on the unit disk of such a Lie algebra element, say $M$, is

$$
Mz = (Q - \sqrt{-1} (Q + \bar{Q}) z - \bar{Q}z^2) \frac{\partial}{\partial z}.
$$

A matrix can be plugged in to a linear fractional transformation as long as its spectrum lies in the domain where the linear fractional transformation is finite. In particular, for $s$ a complex matrix whose spectrum lies inside the unit disk, all elements of $\text{Aut} \, D$ can act on $s$. A 1-parameter family of motions of a matrix $s$ by elements of $N$ is the same as an ordinary differential equation like

$$
\frac{ds}{dt} = Q - \sqrt{-1} (Q + \bar{Q}) s - \bar{Q}s^2
$$

where the $Q(t)$ is any smooth complex-valued function of a real variable $t$.

We will henceforth orient the group $N$ using the orientation of the Lie algebra given by the usual orientation of the $Q$ complex plane. Write the Maurer–Cartan
1-form on $N$ as

$$\left( -\frac{\sqrt{-1}}{2} (\psi + \bar{\psi}) \quad \psi \quad \frac{\sqrt{-1}}{2} (\psi + \bar{\psi}) \right)$$

Reconsidering our equation 6 on page 20, we now see that this is precisely the sort of motion that $s$ undergoes when we move up the fibers of the bundle $B_2$, at least as long as the motion is only in the $\omega_1, \omega_1', \omega_0'$ directions. Indeed equation 6 on page 20 tells us that

$$ds^p_q = -\Omega_0^0 \delta^p_q + \left( \Omega_0^0 - \Omega_0^0 \right) s^p_q + \Omega_0^0 s^p_r s^r_q \quad (\text{mod } \Omega_0^r, \Omega_r^s).$$

Converting this into $\omega$ notation:

$$\Omega_0^0 = \frac{1}{2} \omega_1 + \frac{\sqrt{-1}}{2} \gamma_1$$

$$\Omega_0^0 - \Omega_0^0 = -\sqrt{-1} \omega_1$$

modulo semibasic terms. This implies that

$$ds^p_q = -\left( \frac{1}{2} \omega_1 + \frac{\sqrt{-1}}{2} \gamma_1 \right) \delta^p_q - \sqrt{-1} \omega_1 s^p_q + s^p_r s^r_q \left( \frac{1}{2} \omega_1 - \frac{\sqrt{-1}}{2} \gamma_1 \right) \quad (\text{mod } \omega^1, \Omega_0^r, \Omega_r^s).$$

Therefore we can set $\psi = -\Omega_0^0$ and find that any motion through the leaves of the foliation $\omega^1 = \Omega_0^r = \Omega_r^s = 0$ effects an action of $N$ on $s^p_q$. Any element of $N$ arises in this manner, since $N$ is a connected Lie group, as is obvious from equation 8 on the previous page.

**Lemma 7.** Suppose that $s$ is a square matrix with complex entries, and that the spectrum of $s$ is contained in the unit disk. Then there is a unique linear fractional transformation belonging to the group $N$, say

$$g = \left[ \begin{array}{cc} a + \frac{1}{a} + \sqrt{-1} b & b + \sqrt{-1} (a - \frac{1}{a}) \\ b - \sqrt{-1} (a - \frac{1}{a}) & a + \frac{1}{a} - \sqrt{-1} b \end{array} \right]$$

so that $g(s)$ has trace zero. This transformation $g$ depends analytically on $s$.

**Proof.** Assume $s$ is an $n \times n$ matrix. First, suppose that $s$ has trace zero, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then infinitesimal motions under the group $N$ affect the trace by

$$\frac{1}{n} ds^p_q = \psi - \frac{1}{n} s^p_q s^q_p.$$ 

The first term has larger complex coefficient than the second, since having spectrum in the unit disk forces

$$\frac{1}{n} s^p_q s^q_p = \frac{1}{n} \lambda_p^2 = \text{average squared eigenvalue}$$

inside the unit disk. Therefore the differential $ds^p_q$ as a linear map from the tangent space of $N$ to $\mathbb{C}$ is orientation preserving, and has full rank, at every zero of $s^p_q$. In particular, all zeros of $s^p_q$ are nondegenerate and positively oriented, and the set of elements $g \in N$ at which $\text{tr}(g(s)) = 0$ is discrete, for any matrix $s$ with spectrum in the unit disk.

Since $N$ acts transitively and via isometries of the hyperbolic metric on the unit disk, we can arrange that the center of mass of the spectrum, in the hyperbolic metric, sits wherever we like. The problem of arranging a vanishing trace is that
we have to get the center of mass in the Euclidean metric to vanish, and it is not obvious that this is possible.

We will analyze the behavior of elements of \( N \) “near infinity,” i.e. far away from the identity element. The points of the spectrum retain their hyperbolic distances under maps from \( N \), because the elements of \( N \) are hyperbolic isometries, but the points of the spectrum become very close in Euclidean norm if any one of them approaches the boundary of the unit disk, since near the boundary of the disk, the hyperbolic balls of fixed radius are contained in Euclidean balls of very small radius. Hence we can easily control the average of the eigenvalues of \( s \), to get the average of the eigenvalues close to any number \( e^{i\theta} \) on the boundary of the unit disk, using linear fractional transformations from \( N \).

Using the coordinates \( a, b \) for the group \( N \), we can see that elements of \( N \) with \( a \) close to zero (but positive) are close to the constant map \( z \mapsto -\sqrt{-1} \) given by the matrix

\[
\begin{bmatrix}
1 & -\sqrt{-1} \\
\sqrt{-1} & 1
\end{bmatrix}
\]

This matrix is not an invertible matrix, and obviously the map it generates is not either. Nonetheless, applying elements of \( N \) near \( a = 0 \) to our matrix \( s \), we can arrange that the entire spectrum of \( s \) lies close to \( -\sqrt{-1} \).

Consider the half circle

\[
a = r \cos \theta, \quad b = r \sin \theta
\]

for values of \( \theta \) from a little above \(-\pi/2\) to a little less than \(\pi/2\). We find that for \( r \) large positive number, the origin of the unit disk is taken by this linear fractional transformation to

\[
\frac{r^2 \sin \theta \cos \theta - \sqrt{-1} (1 - r^2 \cos^2 \theta)}{(1 + r^2 \cos \theta) - \sqrt{-1}r^2 \sin \theta \cos \theta}
\]

which, for large \( r \), is close to \( \sqrt{-1} \) for all values of \( \theta \). Now try the path \( a = 1/\sqrt{r}, b = \sqrt{r}t, -\sqrt{r} \leq t \leq \sqrt{r} \). The origin is taken to the point

\[
\frac{t + (\frac{1}{r} - 1) \sqrt{-1}}{1 + \frac{1}{r} - t\sqrt{-1}}
\]

which, for large \( r \), is very close to the point

\[
\frac{t - \sqrt{-1}}{1 - t\sqrt{-1}}
\]

which traverses the unit circle as \( t \) runs from \(-\infty\) to \(\infty\).
Consider any 1-parameter family \( g_\theta \) of linear fractional transformations from \( N \) which stays close to the contour
\[
a = r \cos \theta, b = r \sin \theta, \quad -\pi/2 + \varepsilon < \theta < \pi/2 - \varepsilon
\]
(some small \( \varepsilon \)) and then approaches a vertical line in the \( a, b \) plane, close to, but just to the right of \( a = 0 \), say at \( a = 1/\sqrt{r} \), completing a loop. Applying \( g_\theta \) to a matrix \( s \), we find that (if the radius \( r \) of the half circle in our contour is set large enough) the average of the eigenvalues of \( g_s \) stays close to the average. This ensures that there must be a zero of the average of the eigenvalues of \( g(s) \) for some value of \( a, b \) inside the semicircle of radius \( r \), since otherwise the winding number of the average eigenvalue would be unchanged as we shrunk the contour down to a point.

The winding number of the average eigenvalue around the loop is 1, and the zeros of \( \text{tr}(g(s)) \) are positively oriented, so there can only be one such zero, and it must depend analytically on \( s \) by the implicit function theorem. \( \square \)

For example, consider the transformations \( g \) which take \( s = 0 \) to a traceless matrix. They must look like
\[
g(0) = \begin{bmatrix}
a + \frac{1}{a} + \sqrt{-1}b & -b + \sqrt{-1} (a - \frac{1}{a}) \\
-b - \sqrt{-1} (a - \frac{1}{a}) & a + \frac{1}{a} - \sqrt{-1}b
\end{bmatrix}
\]
\[
= \left( -b + \sqrt{-1} \left( a - \frac{1}{a} \right) \right) \left( a + \frac{1}{a} - \sqrt{-1}b \right)^{-1} I
\]
which can not have vanishing trace unless \( a = 1 \) and \( b = 0 \), in which case \( g \) represents the identity transformation.

4.10. \textbf{Reducing the structure group.} We have seen that we can arrange \( s^p = 0 \), and that this occurs on a subbundle of \( B_2 \), call it \( B_3 \). Plugging into equation \ref{eq:6} on page \ref{page20} and taking trace, we find that there are some functions \( s^0_q, s^0_r, s^p_{qr} \) and \( s^p_{qr} \) satisfying
\[
s^p_{qr} - s^p_{u} s^u_{qr} s^\ell_q = s^p_{r} - s^p_{u} s^u_{r} s^\ell
\]
so that
\[
\begin{pmatrix}
\Omega^0_q \\
\Omega^0_r \\
\Omega^0_{qr}
\end{pmatrix} = \begin{pmatrix}
s^0_q \\
s^0_r \\
s^p_{qr}
\end{pmatrix} \begin{pmatrix}
\Omega^0_q \\
\Omega^0_r \\
\Omega^0_{qr}
\end{pmatrix}
\]
with \( s^0_p = 0 \).

\textbf{Corollary 3.} \textit{Inside our bundle \( B_2 \) there is a smaller bundle \( B_3 \) of points at which the trace \( s^p \) vanishes. The bundle \( B_3 \) is a principal \( G_3 \) bundle where \( G_3 \) is the group of matrices of the form}
\[
\begin{pmatrix}
g^0_0 & 0 & g^0_1 \\
0 & g^0_0 & g^1_2 \\
0 & 0 & g^2_3
\end{pmatrix}
\]
\textit{where \( g^0_0 > 0 \), \( (g^0_0)^2 \det(g^1_3) = 1 \) and \( (g^1_j) \) commutes with \( J \).}
Proof. This is the only closed Lie subgroup of $G_2$ with the required Lie algebra. □

Differentiating the equation for $\Omega^0_0$ in equations \(^9\) on the preceding page we find that modulo semibasic terms:

$$ds^0_q = -\Omega^0_0 + s^0_p \Omega^0_p + s^0_q \Omega^0_q \quad (\text{mod } \Omega^0_0, \Omega^0_1)$$

so that as we move up the fibers of $B_3$, we can arrange that $s^0_q$ vanish on a subbundle; call it $B \subset B_3$.

**Corollary 4.** Inside our bundle $B_3 \to S^{2n+1}$ there is a smaller bundle $B \to S^{2n+1}$ of points at which the functions $s^0_p$ vanish. The bundle $B$ is a principal $\Gamma_1$ bundle where $\Gamma_1$ is the group of complex matrices of the form

$$\begin{pmatrix} g^0_0 & 0 \\ 0 & g^0_q \end{pmatrix}$$

where $g^0_0 > 0$ and $(g^0_q)^2 |\det (g^0_q)|^2 = 1$. (This group is identified with a subgroup of $G_3$ in the manner outlined in subsection \(^4.4\) on page \(^12\).)

Proof. This is the Lie subgroup of $G_3$ satisfying $\Omega^0_0 = 0$. □

Henceforth, we will forget all of the other $B_j$ bundles, and work exclusively with the bundle $B$.

Our equations \(^9\) on the preceding page simplify to

\begin{equation}
\begin{pmatrix} \Omega^0_0 \\ \Omega^0_0 \\ \Omega^0_q \\ \Omega^0_q \\ \Omega^0_r \end{pmatrix} = \begin{pmatrix} s^0_p & 0 \\ 0 & s^0_p \\ s^0_q & 0 \\ s^0_q & s^0_p \\ s^0_p & s^0_q \end{pmatrix} \begin{pmatrix} \Omega^0_0 \\ \Omega^0_0 \\ \Omega^0_q \\ \Omega^0_q \end{pmatrix}
\end{equation}

with $s^0_p = 0$. But $\Omega^0_q$ is semibasic on $B$, and in fact from the same equations, we see that

$$\Omega^0_p = s^0_{pq} \Omega^0_q + s^0_{qp} \Omega^0_q$$

for some functions $s^0_{pq}, s^0_{qp}$ on the bundle $B$, with symmetries in the lower indices.

The invariants $s^0_q$ are related to these by, setting:

$$\nabla s^0_q = ds^0_q + \left(2\Omega^0_0 - \Omega^0_0\right) s^0_q - s^0_p \Omega^0_q + \Omega^0_0 s^0_p$$

and calculating:

$$\nabla s^0_q = s^0_{qr} \Omega^0_r + s^0_{qr} \Omega^0_r$$

with $s^0_{qr} = s^0_{rq}$. If we define

$$\nabla s^p := ds^p_q + \Omega^0_r s^p_r - s^p_p \Omega^0_q + \left(\Omega^0_0 - \Omega^0_0\right) s^0_q$$

we find

$$\nabla s^p = \left(s^p_{qr} - s^p_{rq} \Omega^0_q\right) \Omega^0_r + \left(-s^p_{tq} \Omega^0_q + \Omega^0_0 \Omega^0_r\right) s^0_{qr}$$

Differentiating the last of our structure equations we obtain the covariant derivatives

$$\nabla s^p_{qt} = ds^p_{qt} - s^p_{tq} \Omega^0_q + \Omega^0_p s^p_r - s^p_p \Omega^0_q + \Omega^0_0 s^p_{qt} - s^p_{tq} \Omega^0_r$$

$$\nabla s^p_{qt} = ds^p_{qt} + \Omega^0_r s^p_{qt} - s^p_p \Omega^0_q + \Omega^0_0 s^p_{qt} - s^p_{tq} \Omega^0_r$$

and we find these satisfy

$$\nabla s^p_{qt} + \tau^p_{qt} = s^p_{qtu} \Omega^0_0 + s^p_{qtu} \Omega^0_0$$

$$\nabla s^p_{qt} + \tau^p_{qt} = s^p_{qtu} \Omega^0_0 + s^p_{qtu} \Omega^0_0$$
where the functions \( s' \) on the right hand sides are symmetric in all lower indices, and the \( \tau \) 1-forms are given by

\[
\tau^p_{\bar{q}t} = \left( s^0_{q\bar{t}} \delta^p_u - s^0_{q\bar{t}} s_{\bar{u}q} s^0_{\bar{t}u} \right) \Omega^p_0 + \left( s^p_{q\bar{t}} s_{\bar{u}q} s^0_{\bar{t}u} + s^p_{q\bar{t}} s_{\bar{u}q} s^0_{\bar{t}u} \right) \Omega^q_0
\]

\[
\tau^p_{\bar{t}q} = \left( s^0_{q\bar{t}} \delta^p_u - s^p_{q\bar{t}} s_{\bar{u}q} s^0_{\bar{t}u} \right) \Omega^p_0 - s^p_{q\bar{t}} s_{\bar{u}q} s^0_{\bar{t}u} \Omega^q_0
\]

Applying the Cartan–Kähler theorem, these structure equations are involutive with general solution depending on \( 2n \) functions of \( 2n \) variables. We will see this from another point of view below. Since the equations are involutive, there is no need to proceed further along the path of the method of the moving frame; no further local invariants will appear, except for covariant derivatives of the invariants we have already found.

5. **Analogy with Complex Projective Structures**

Let us once again (for the last time) consider another notation: define

\[
\Omega^p := \Omega^p_0
\]

\[
\Gamma^p_q := \Omega^p_q - \delta^p_q \Omega^0_q
\]

\[
\Omega_p := \Omega^p_p.
\]

Calculating exterior derivatives using the equations we have derived so far gives:

\[
d\Omega^p = -\Gamma^p_q \wedge \Omega^q - \left( s^p_{q\bar{t}} s^0_{\bar{t}u} + s^p_{q\bar{t}} s^0_{\bar{t}u} \right) \Omega^p_0 \wedge \Omega^q_0 - s^0_{q\bar{t}} s^0_{\bar{t}u} \Omega^p_0 \wedge \Omega^q_0
\]

and

\[
d\Gamma^p_q = -\Gamma^0_q \wedge \Gamma^p_q + \left( \delta^p_q \Omega^0_q + \Omega^0_q \delta^p_q \right) \wedge \Omega^0_q
\]

\[
- \left( s^0_{q\bar{t}} s^0_{\bar{t}u} + s^p_{q\bar{t}} s^0_{\bar{t}u} - s^p_{q\bar{t}} s^0_{\bar{t}u} - \delta^p_q s^0_{q\bar{t}} s^0_{\bar{t}u} - \delta^p_q s^0_{q\bar{t}} s^0_{\bar{t}u} \right) \Omega^p_0 \wedge \Omega^q_0
\]

and

\[
d\Omega_p = \Gamma^0_q \wedge \Omega_q - \left( s^0_{q\bar{t}} s^0_{\bar{t}u} + s^0_{q\bar{t}} s^0_{\bar{t}u} \right) \Omega^p_0 \wedge \Omega^q_0
\]

\[
- \left( s^0_{q\bar{t}} s^0_{\bar{t}u} + s^0_{q\bar{t}} s^0_{\bar{t}u} - s^0_{q\bar{t}} s^0_{\bar{t}u} - s^0_{q\bar{t}} s^0_{\bar{t}u} \right) \Omega^p_0 \wedge \Omega^q_0
\]

and

\[
- s^0_{q\bar{t}} s^0_{\bar{t}u} \Omega^p_0 \wedge \Omega^q_0
\]

Modulo the various \( s \) functions, i.e. the torsion, these are the equations of a flat holomorphic projective structure on the base manifold \( S^{2d+1} \rightarrow X \) of our great circle fibration. However, when we include the \( s \) functions, we find that they are not the structure equations of a projective structure at all—in fact the base manifold is only equipped with an almost complex structure, which we will soon see.

5.1. **Invariantly defined vector bundles on the base of a great circle fibration.**
Lemma 8. Suppose that $S^{2n+1} \to X^{2n}$ is our great circle fibration. The bundle $B \to X$ is a principal $\Gamma_0$ bundle where (as in subsection 4.4 on page 12) $\Gamma_0$ is the group of matrices of the form
\[
\begin{pmatrix}
g_0^0 & g_0^q \\
g_0^q & g_0^p
\end{pmatrix}
\]
where all entries are complex numbers and $|g_0^0 \det (g_0^p)|^2 = 1$. The representation
\[
\begin{pmatrix}
\Gamma_p^q & 0 \\
0 & \Gamma_{\bar{q}}^\bar{p}
\end{pmatrix}
\]
solders the tangent bundle of $X^{2n}$. There is an invariantly defined almost complex structure on $X$ whose holomorphic tangent space $T^{1,0}X$ is soldered by $\Gamma_p^q$. The 1-form $\Omega_0^0$ solders the principal bundle $S^{2n+1} \to X$.

Proof. We need to show that the fibers of $B \to X$ are connected. But this factors as $B \to S^{2n+1} \to X$ so that the fibers of the first map are copies of $\Gamma_1$, which is connected, and the fibers of the second map are circles, hence connected.

The 1-forms $\Omega_p^q$ are semibasic for the projection to $X$. On the fibers of $B \to X$ the structure equations reduce to the structure equations of $\Gamma_0$ and its left translates. As before, this shows by connectedness of $\Gamma_0$ and of the fibers of $B \to X$ that the fibers are in fact left translates of $\Gamma_0$. Therefore $B \to X$ is a principal right $\Gamma_0$ bundle.

The soldering forms of the tangent bundle of $X$ are unchanged from lemma 2 on page 12 but rewritten in complex notation. The almost complex structure on $X$ is immediately visible from the structure equations, since the $\Gamma_p^q$ 1-forms solder in a complex representation.

I have not explained what it means to solder a principal bundle out of another one, but it should be obvious. The soldering of $\Omega_0^0$ is just the quotient by $\Gamma_1$ of the soldering of $\Gamma_0$, which is just $S^{2n+1} \to X$. 

\[\blacksquare\]

The base manifold $X$ is analogous to a complex projective space, and the circle fibration $S^{2n+1} \to X$ analogous to the Hopf fibration. Therefore henceforth we will refer to the complex line bundle soldered (on the bundle $B \to X$) by $\Omega_0^0$ as $\mathcal{O}(-1)$, and similarly define the line bundles
\[
\mathcal{O}(p) := \mathcal{O}(-1)^{\otimes (-p)}.
\]
Be careful to note that these are bundles on $X$ and are complex line bundles. We will never again refer to the similarly named real line bundles on $S^{2n+1}$, which were introduced simply to encourage an analogy between the soldering of these $\mathcal{O}(p) \to X$ bundles with the similarly named bundles on projective spaces.

From here on, we will use the notation $\tilde{W}$ for any complex representation $W$ of $\Gamma_0$ to mean the vector bundle $\tilde{W} \to X$ given as
\[
\tilde{W} = B \times_{\Gamma_0} W = (B \times W) / \Gamma_0.
\]
constructed in the same manner as in subsection 4.2 on page 8. For example, we will introduce the complex vector bundle $\tilde{V}$ out of the complex representation of $\Gamma_0$ given by the identity representation (recall that elements of $\Gamma_0$ are complex matrices).

Lemma 9.

\[
T^{1,0}X = \mathcal{O}(1) \otimes (\tilde{V}/\mathcal{O}(-1)).
\]
Proof. This is immediate from working out the soldering in 1-forms.

The equations satisfied on the bundle $B$ by a section of $\tilde{V}$ are

$$
\begin{pmatrix}
\varepsilon
\end{pmatrix}
= 
\begin{pmatrix}
\Omega_0^0 & \Omega_0^1 & \Omega_0^2 & \Omega_0^3 \\
\Omega_0^0 & \Omega_0^1 & \Omega_0^2 & \Omega_0^3 \\
\Omega_0^0 & \Omega_0^1 & \Omega_0^2 & \Omega_0^3 \\
\Omega_0^0 & \Omega_0^1 & \Omega_0^2 & \Omega_0^3
\end{pmatrix}
\begin{pmatrix}
\varepsilon
\end{pmatrix}.
$$

Plugging in our equations on $\Omega_0^0$ and $\Omega_0^q$ we might first get the impression that $\varepsilon$ is a holomorphic section of some line bundle. But the $\Omega_q^0$ factor is not semibasic, so this is not actually soldering any line bundle.

6. Homogeneous great circle fibrations

A symmetry of a great circle fibration which preserves the flat projective structure on the sphere acts as a symmetry of the right principal bundle $B \to X$. Suppose that the symmetry group acts transitively on $B$. Then the invariantly defined functions $s_0^q, s_p^q$ etc. are all constant on $B$. Plugging this hypothesis into our equations for covariant derivatives, we find that all of the invariants vanish, and applying the Frobenius theorem, along with simple connectivity of $X$, we see directly that the great circle fibration is a Hopf fibration.

**Theorem 2.** The symmetry group of a great circle fibration acts transitively on the adapted frame bundle precisely if the fibration is a Hopf fibration.

**Theorem 3.** The symmetry group of a great circle fibration injects into the bundle $B$, so that it is always of dimension at most that of $B$, i.e.

$$\dim \text{Aut} \leq 2(n + 1)^2 - 1.$$

Equality occurs only for the Hopf fibration.

**Proof.** If $\phi : B \to B$ is a symmetry, then by definition it preserves great circles, so $\phi = g \in \text{SL}(V)$. If we arrange that the identity element $I \in \text{SL}(V)$ belongs to $B$, then $gI = g \in B$, so in fact the diffeomorphism $\phi$ is an element of $B$. Therefore the symmetry group is actually a subgroup of $\text{SL}(V)$ sitting inside $B$. The symmetry group is closed, since the condition of being a symmetry is a closed condition. Therefore it is a Lie subgroup of $\text{SL}(V)$ lying inside $B$.

If the symmetry group has the same dimension as $B$, then it is an open subset of $B$, say $U$, and a closed subgroup of $\text{SL}(V)$. But $B \subset \text{SL}(V)$ is also closed, so $U$ is both open and closed in $B$, and therefore a union of path components of $B$. But $B$ is connected, since it is a bundle

$$
\begin{array}{ccc}
\Gamma_1 & \longrightarrow & B \\
\downarrow & & \\
S^{2n+1} & \end{array}
$$

with connected fibers and base. Therefore the symmetry group is precisely $B$. This forces all of the invariant functions $s$ on $B$ to be constants, and then the structure equations force them to vanish. \qed
7. Embedding into the Grassmannian of oriented 2-planes

Recall that $G = SL(V)$ and $G_{\text{circle}}$ is the subgroup preserving the oriented great circle on $S^{2n+1}$ passing from $e_0$ through to $e_1$. The space $G/G_{\text{circle}}$ is naturally identified with the space of all oriented great circles on the sphere, or with the space of oriented 2-planes in $V$:

$$G/G_{\text{circle}} = \tilde{\text{Gr}}^2 V.$$

We find that $\Gamma_0 \subset G_{\text{circle}}$ so that a fiber bundle $G/\Gamma_0 \to G/G_{\text{circle}} = \tilde{\text{Gr}}^2 V$ is defined, with fibers being homogeneous spaces of $G_{\text{circle}}$-equivariantly diffeomorphic to $G_{\text{circle}}/\Gamma_0$ which is the space of all complex structures on $V$ which have the oriented 2-plane $\langle e_0, e_1 \rangle$ as a complex line.

We can map $X = B/\Gamma_0 \to G/\Gamma_0 \to G/G_{\text{circle}} = \tilde{\text{Gr}}^2 V$.

**Lemma 10.** This map $X \to \tilde{\text{Gr}}^2 V$ is an embedding.

**Proof.** If two points $x, y \in X$ get mapped to the same place, then they correspond to the same great circle, since $X$ parameterizes the great circles of our fibration, and $\tilde{\text{Gr}}^2 V$ is parameterizing all great circles. So the map is injective.

We have to differentiate to see that the map is an immersion. Pull back a local coframing by 1-forms on $\tilde{\text{Gr}}^2 V$ to $X$ and show that it contains a coframing for $X$. But then it is sufficient to pull the coframing back to $B$ and show that on $B$ every semibasic 1-form for the bundle map $B \to X$ can be expressed as a linear combination (with real functions as coefficients) of the 1-forms from the coframing $\tilde{\text{Gr}}^2 V$.

The semibasic 1-forms for the map $G \to \tilde{\text{Gr}}^2 V$ are precisely those complimentary to the Lie algebra of $G_{\text{circle}}$, i.e. they are the 1-forms $\Omega^0_p, \Omega^p_0$ and their complex conjugates. Therefore any coframing on $\tilde{\text{Gr}}^2 V$ pulls back to $G$ to be a combination of these 1-forms, and conversely they are combinations of the 1-forms from the coframing. But pulling back to $B$, we find only the relations $\Omega^p_0 = s^q_p \Omega^q_0$. The remaining semibasic 1-forms $\Omega^p_0$ span the semibasic 1-forms on $X$. Therefore the map is an immersion. An injective immersion of a compact manifold is an embedding. \hfill $\square$

A consequence of the proof is that these $s^p_q$ invariants can be expressed in terms of the first derivative of the embedding $X \to \tilde{\text{Gr}}^2 V$. The embedding determines the great circle fibration. This embedding is a more useful way to examine great circle fibrations than looking directly at the sphere, because a perturbation of great circles on the sphere can never be local, while it can be local on $X$, as a small motion of $X$ inside the Grassmannian.

8. Characterizing the submanifolds of the Grassmannian which represent great circle fibrations

Consider an immersed submanifold $\iota: X \hookrightarrow \tilde{\text{Gr}}^2 V$ in the Grassmannian of oriented 2-planes of a vector space $V$. Assume that $\dim V = 2n + 2$ and $\dim X = 2n$. Our
next problem is to characterize when $X$ represents a great circle fibration. The tangent spaces to $\tilde{\text{Gr}}_V$ are canonically identified with

$$T_p\tilde{\text{Gr}}_V \cong P^* \otimes V/P.$$ 

So $T_pX$ is identified with a linear subspace

$$T_pX \cong U_p \subset P^* \otimes V/P.$$ 

We have a map $\alpha_P: P \to U_p^* \otimes V/P$ defined by

$$\alpha_P(p)(u) := u(p)$$ for $p \in P$ and $u \in U_p$.

Note that for each $p \in P$ the map $\alpha_p \in U_p^* \otimes V/P$ is a linear transformation between vector spaces of the same dimension. For each $p \in P$ we can define the polynomial

$$\xi_P(p) := \det(\alpha(p)) \in \text{Det}(U_p)^* \otimes \text{Det}(V/P).$$

which is a polynomial not valued in real numbers, but in the one dimensional vector space $\text{Det}(U_p)^* \otimes \text{Det}(V/P)$. We define the characteristic variety $\Xi_P$ to be the projective variety in the projective line $\mathbb{CP}(P \otimes \mathbb{C})$ cut out by $\xi_P$.

An immersed submanifold $\iota: X^{2n} \hookrightarrow \tilde{\text{Gr}}_V$ (where $\dim V = 2n + 2$) is elliptic at a point $P \in X$ if the characteristic variety $\Xi_P$ has no real points.

We have $\mathbb{RP}(P) \subset \mathbb{CP}(P)$ a real circle on a real sphere, cutting it into two halves. The orientation of $P$ chooses one of these halves: $P$ being oriented orients $\mathbb{RP}(P)$ and $\mathbb{CP}(P)$ is oriented by its complex structure. This coorients $\mathbb{RP}(P)$ and picks a half of $\mathbb{CP}(P) - \mathbb{RP}(P)$. The characteristic variety does intersect $\mathbb{RP}(P)$, so consists in a finite set of points lying on that half, and their conjugates on the other half. The linear fractional transformations we used are just real reparameterizations of $P$. These enable us to move these points of the characteristic variety by isometries of the Poincaré metric. We choose to move the points to obtain vanishing trace of the Reznikov $s^0_p$ invariant. This picks a specific choice of linear map from the family parameterized by $P$.

The Grassmannian $\tilde{\text{Gr}}_V$ is a homogeneous $G = \text{SL}(V)$ space, and we can write $\tilde{\text{Gr}}_V = G/G_{\text{circle}}$. Consider the pullback bundle

$$\begin{array}{ccc}
\iota^*G & \longrightarrow & G \\
\downarrow & & \downarrow \\
X & \longrightarrow & \tilde{\text{Gr}}_V
\end{array}$$

which is a principal $G_{\text{circle}}$ bundle. On $G$ we have our old Maurer–Cartan 1-forms

$$\begin{pmatrix}
\omega_0^0 & \omega_0^1 & \omega_0^2 \\
\omega_1^0 & \omega_1^1 & \omega_1^2 \\
\omega_2^0 & \omega_2^1 & \omega_2^2
\end{pmatrix}. $$

The group $G_{\text{circle}}$ is precisely the connected subgroup of $G$ satisfying $\omega_0^0 = \omega_0^1 = 0$. Therefore the 1-forms $\omega_0^i, \omega_1^i$ span the semibasic 1-forms for the map $G \to \tilde{\text{Gr}}_V$. On the pullback bundle over $X$, these 1-forms are no longer independent, since there are fewer degrees of freedom along $X$, in fact half as many, since $\dim X = \frac{1}{2} \dim \tilde{\text{Gr}}_V$. 

We need to consider how to express the $G$-invariant identification

$$T_p \tilde{\text{Gr}}^2_V \cong P^* \otimes V/P$$

in terms of these 1-forms. Recall how the identification is constructed: take any family of 2-planes $P(t) \in \tilde{\text{Gr}}^2_V$, and any family of linear maps $\phi(t) : V \to W$ for some fixed vector space $W$ of dimension $\dim W = \dim V - \dim P(t)$, with the maps $\phi(t)$ chosen so that $\ker \phi(t) = P(t)$. Write $\tilde{\phi}(t)$ for the induced map

$$\tilde{\phi}(t) : v + P \in V/P(t) \to \phi(t)(v) \in W$$

which is defined because $\ker \phi(t) = P(t)$. Then identify

$$P'(t) \sim \tilde{\phi}(t)^{-1} \phi'(t)|_P : P \to V/P.$$ 

This is well defined because if $\psi(t)$ is any other choice of maps replacing $\phi(t)$, with the same kernel,

$$0 \longrightarrow P(t) \longrightarrow V \xrightarrow{\psi(t)} U \longrightarrow 0$$

then differentiating the equation

$$\psi(t) = \tilde{\psi}(t)\phi(t)^{-1}\phi(t)$$

shows that

$$\tilde{\phi}(t)^{-1} \phi'(t)|_P = \tilde{\psi}(t)^{-1} \psi'(t)|_P$$

so that the resulting map in $P(t)^* \otimes V/P(t)$ is independent of the choice of map $\phi(t)$.

So far this only defines a map

$$T_p \tilde{\text{Gr}}^2_V \to P^* \otimes V/P.$$ 

We need to pick some local coordinates on $\tilde{\text{Gr}}^2_V$, and we will take them as follows: for any 2-plane $P_0 \in \tilde{\text{Gr}}^2_V$, we take coordinates $x^1, x^2, y^1, \ldots, y^{2n}$ on $V$ so that $P_0$ is cut out by $y = 0$. Then 2-planes near $P_0$ are cut out by equations like $y = px$ where $p$ is a $2n \times 2$ matrix. These $p$ are our local coordinates on $\tilde{\text{Gr}}^2_V$, and one can easily compute the transformations of coordinates if we change the choice of $P_0$ and the choice of coordinates $x, y$. In these coordinates, we can take $\phi_P(x, y) = px - y$ as our map, with kernel $P$. Then we find that for any family $P(t)$ with $P(0) = P_0$ we have $p(0) = 0$ and

$$\tilde{\phi}(0)^{-1}\phi'(0)x = -p'(0)x$$

or

$$\tilde{\phi}(0)^{-1}\phi'(0) = -p'(0).$$

Therefore this map

$$T_p \tilde{\text{Gr}}^2_V \to P^* \otimes V/P$$

is an isomorphism. Equivariance under linear transformations of $V$ is obvious.

Returning to the 1-forms $\omega^i_0$ and $\omega^i_1$, recall that the fiber of the bundle $G \to \tilde{\text{Gr}}^2_V$ over a point $P \in \tilde{\text{Gr}}^2_V$ consists precisely of the elements of $G$ taking the plane $\langle e_0, e_1 \rangle$ to the plane $P$, preserving orientation. Therefore we can define a map

$$\phi_g : V \to V/\langle e_0, e_1 \rangle$$
with kernel $P$ by

$$\phi_g(v) := g^{-1}v + \langle e_0, e_1 \rangle .$$

We find

$$\tilde{\phi}^{-1} d\phi = -dg g^{-1} + P$$

or

$$\bar{g}^{-1} (\tilde{\phi}^{-1} d\phi|_P) g = -\omega|_{\langle e_0, e_1 \rangle} + \langle e_0, e_1 \rangle
\begin{align*}
\quad = - (\omega_0^i + \omega_1^i) .
\end{align*}$$

Perhaps a little more concretely, if $Q \in TG$ is a tangent vector, and $a \in \langle e_0, e_1 \rangle$, we have

$$\bar{g}^{-1} (\tilde{\phi}^{-1} d\phi(Q)|_P) ga = - (\omega_0^i(Q)a^0 + \omega_1^i(Q)a^1) .$$

**Lemma 11.** If $X \subset \tilde{Gr}_V^2$ is the base manifold of a great circle fibration, then $X$ is elliptic at every point.

**Proof.** Pick a point $P \in X$ and a point $g \in B$ which is taken by $B \rightarrow X$ to $P$. As expressed in terms of $\omega_0^i, \omega_1^i$ above, we found earlier that a great circle fibration satisfies $\omega^i = t_j^i \omega_0^j$, with $t_j^i$ a real $2n \times 2n$ matrix with no real eigenvalues. For each vector $u \in T_PX$ we can write its components in the coframe $\omega_0^i$ as $u^i$. (The reader who is keeping track of what spaces we are working in will be puzzled to read that the $\omega_0^i$ are a coframe on $X$. What we mean of course is that since the $\omega_0^i$ are semibasic, at each point $g \in B$ we can form the coframe $\omega_0^i$ on $T_PX$ from which these $\omega_0^i$ are pulled back. These $\omega_0^i \in \Lambda^1(T_PX)$ change as we move up the fibers of $B \rightarrow X$.)

Each point $a^0 e_0 + a^1 e_1 \in \langle e_0, e_1 \rangle$ is carried by $g$ to an element of $P$ and we identify these. We find that in terms of equation (12) the map $\alpha_P$ defined in equation (11 on page 30) is given, up to factors of $g$ and $g^{-1}$ (which won’t affect the vanishing of the relevant determinant) by

$$\alpha \left( a^0 e_0 + a^1 e_1 \right)(u) = a^0 u^i + t_j^i a^1 u^i = \left( a^0 \delta^i_j + a^1 t_j^i \right) u^i .$$

Therefore the polynomial $\xi_P$ is

$$\xi_P \left( a^0, a^1 \right) = \det \left( a^0 \delta^i_j + a^1 t_j^i \right) .$$

If $a^1 = 0$ then a zero of $\xi_P$ can only occur at $a^0 = 0$. Therefore, taking $a^1 \neq 0$, we find that

$$\xi_P = (a^1)^{2n} \det \left( t_j^i + \frac{a^0}{a^1} \delta^i_j \right) .$$

A real zero of $\xi_P$ therefore corresponds precisely to a real eigenvalue of $t_j^i$. Consequently, if $X$ arises from a great circle fibration, then $X$ is elliptic.

We will now take an elliptic immersed submanifold $X \subset \tilde{Gr}_V^2$ and apply the method of the moving frame to calculate its structure equations.

**Lemma 12.** Suppose that $\iota : X \rightarrow \tilde{Gr}_V^2$ is an immersion of an elliptic submanifold. Then there is an invariantly defined principal right $\Gamma_0$ subbundle $B \rightarrow X$ and a $\Gamma_0$-equivariant map

$$\begin{array}{ccc}
B & \longrightarrow & \SL(V) \\
\downarrow & & \downarrow \\
X & \longrightarrow & \Gr_V^2
\end{array}$$
which satisfies the structure equations of a great circle fibration of the sphere $S^{2n+1}$, and $X$ is locally a great circle fibration.

Proof. Consider the bundle

\[ \iota^*G \rightarrow G \]

\[ X \rightarrow \text{Gr}_V. \]

As before we will write $\phi_g(v) = g^{-1}v + \langle e_0, e_1 \rangle$ giving a map

$\phi_g : V \rightarrow V/\langle e_0, e_1 \rangle$

with kernel $P = g\langle e_0, e_1 \rangle$. Again we have the equation

$-g^{-1}(\partial^{-1}d\phi|_\mu)ga = \omega^0_0a^0 + \omega^1_1a^1$

for $a \in \langle e_0, e_1 \rangle$. We know that the $V/\langle e_0, e_1 \rangle$ valued 1-form $\omega^0_0a^0 + \omega^1_1a^1$ is a coframing on $T_PX$ for each $a \neq 0$, precisely expressing ellipticity. So $\omega^1_1$ is a coframing, as is $\omega^0_0$. Consequently there is an invertible linear transformation $t^i_j$ so that $\omega^1_1 = t^i_j\omega^0_0$. We must have

$\omega^0_0a^0 + \omega^1_1a^1 = (a^0\delta^i_j + a^1t^i_j)\omega^0_0$

also a coframing, as long as $a \neq 0$. This says precisely that $t^i_j$ has no real eigenvalues.

After this, we have obtained exactly the same structure equations as in equation 1 on page 12. Then we repeat the entire development of those structure equations, identically.

Given the resulting structure equations, we wish to construct a local great circle fibration out of a portion of $X$. Consider the circle bundle $\Sigma \rightarrow X$ consisting of pairs $(P, [v])$ where $P \in X$ and $v \in P \subset V$ with $v \neq 0$ and as usual $[v]$ means $v$ up to positive rescaling. Consider the map $\Phi : \Sigma \rightarrow S^{2n+1}$ given by $\Phi(x, [v]) = [v]$. We wish to show that $\Phi$ is a local diffeomorphism. This $\Sigma$ is the principal circle bundle associated to the complex line bundle $O(-1)_X$. The isotropy group of a point of $\Sigma$ inside the structure group of $B \rightarrow X$ is precisely the group $\Gamma_1$. We have maps

\[ B \rightarrow G \]

\[ \Sigma \rightarrow S^{2n+1} \]

\[ X. \]

Working out the semibasic 1-forms for the maps $G \rightarrow S^{2n+1}$ and $B \rightarrow \Sigma$, we find they are identical. Therefore the map is a local diffeomorphism. The fibers of $B \rightarrow \Sigma$ are contained in the left translates of the group $G_{\text{circle}}$ by structure equations, so these fibers sit in great circles on the sphere.

Taking any matrix $t^i_j$ with no real eigenvalues, build the subspace of linear maps of the form

$a \in \mathbb{R}^2 \mapsto (a^0\delta^i_j + a^1t^i_j)u^j$
for \( u \in \mathbb{R}^{2n} \) and you have a linear subspace inside \( \mathbb{R}^{2n} \otimes \mathbb{R}^{2n} \), which you think of as a subspace of some tangent space to a Grassmannian. Then you can take any submanifold of the Grassmannian with that tangent space at that point, and you have (at least near this point of the submanifold) an elliptic submanifold. So there are lots of elliptic submanifolds, locally.

**Proposition 1.** A submanifold of the Grassmannian is the base of a great circle fibration precisely if it is elliptic, compact and connected.

*Proof.* We have seen that the base of a great circle fibration is elliptic, compact and connected. Let \( X \subset \widetilde{Gr}_V^{2} \) be elliptic, compact and connected. The map \( \Phi: \Sigma \to S^{2n+1} \) from lemma 12 on page 32 is a local diffeomorphism, taking fibers of \( \Sigma \to X \) to great circles. The space \( \Sigma \) is the total space of circle bundle over \( X \), so compact. Because the sphere \( S^{2n+1} \) is simply connected, this forces \( \Phi \) to be a diffeomorphism. The map \( S^{2n+1} \to X \) is therefore defined, and satisfies our structure equations, so is a global great circle fibration. \( \square \)

**Corollary 5.** Given a great circle fibration \( S^{2n+1} \to X \), every \( C^1 \) small motion of \( X \) inside \( \widetilde{Gr}_V^{2} \) is the base of a great circle fibration. So there are lots of great circle fibrations, and they admit lots of deformations. The normal bundle \( \nu X \) of \( X \) inside \( \widetilde{Gr}_V^{2} \) has fibers

\[
\nu_P X = \Lambda^{0,1}(P) \otimes \mathbb{C} V/P
\]

for \( P \in X \) (so \( P \subset V \) a 2-plane) where the relevant complex structure on \( P \) and \( V/P \) is \( J_P \). So \( \nu X = T^{0,1} X \). The great circle fibrations near a given great circle fibration \( S^{2n+1} \to X \) are "parameterized" by sections of \( \nu X \) close to the zero section (for example, by using a Riemannian metric on \( \widetilde{Gr}_V^{2} \)). Hence the general great circle fibration depends on \( 2n \) real functions of \( 2n \) real variables.

**Corollary 6.** The space of \( C^k \) great circle fibrations is an infinite dimensional manifold, for \( k > 1 \).

9. A Taste of Elliptic Partial Differential Equations

Given a real surface \( C \) and an immersion \( \phi: C \to V \) to a vector space \( V \), construct the map \( \phi_1: C \to \widetilde{Gr}_V^{2} \) giving the tangent plane. Pick a great circle fibration \( X \subset \widetilde{Gr}_V^{2} \). If the image of \( \phi_1 \) is always in \( X \subset \widetilde{Gr}_V^{2} \), call \( \phi \) an immersed \( X \)-curve. The \( X \)-curves are the solutions of an involutive elliptic first order system of 2 partial differential equations for \( 2n \) functions of \( 2n \) variables. Another approach: consider pseudocomplex manifolds, as in section 16 on page 46.

10. Inapplicability of Methods to Prove the \( h \)-Principle

The relation of ellipticity for a submanifold of the Grassmannian is open but not ample in the sense of Gromov’s theory of convex integration. This makes it unlikely that convex integration can be applied here.

Because the relation is open, it is microflexible. I believe that some deformations of elliptic submanifolds nowhere parallel to a given hinge (see section 15 on page 41) are not microcompressible (see page 81 of Gromov [7]); if so this makes the sheaf of elliptic submanifolds with given hinge not flexible, and then the same is true for
the sheaf of all elliptic submanifolds. So probably we can not apply the method of sheaves.

The ellipticity relation is not the complement of a hypersurface in the 1-jet bundle, so we can not apply the method of elimination of singularities. It is not defined by a differential operator, so we can not apply the method of inversion of differential operators.

11. The osculating complex structure

Recall that $J^0_0$ is our fixed complex structure on $V$. The space of all complex structures on $V$ with the standard orientation is

$$\mathcal{J}_V = G/\Gamma = \text{SL}(V) / (\text{SL}(V) \cap \text{GL}(V, J_0)).$$

Now consider an elliptic submanifold $X \subset \tilde{\text{Gr}}_V^2$. Since $B \subset G$ we can map

$$B \to G \to G/\Gamma = \mathcal{J}_V.$$  

In fact, since $\Gamma_0 \subset \Gamma$, we can map

$$X = B/\Gamma_0 \to G/\Gamma_0 \to G/\Gamma = \mathcal{J}_V.$$  

We can obviously do better: the space $G/\Gamma_0$ is naturally identified with the space of all pairs $(J, P)$ where $J$ is a complex structure on $V$, and $P$ is a real 2-plane which is a $J$-complex line, i.e. $JP = P$. Then we can map

$$X = B/\Gamma_0 \to G/\Gamma_0 \to G/\Gamma = \mathcal{J}_V.$$  

We will write for $P \in X$ the corresponding point in $\mathcal{J}_V$ (i.e. complex structure on $V$) as $JP$. So $P \subset V$ is a $JP$ complex line.

The bundle $\tilde{V} \to X$ is actually pulled back from $\mathcal{J}_V$, as the isotropy groups are

$$\begin{array}{ccc}
B_{\Gamma_0} & \longrightarrow & G^V \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathcal{J}_V.
\end{array}$$

Since $\Gamma_0 \subset \Gamma \subset G$, and $V$ is a $G$ module, so a $\Gamma$ module, we find that $\tilde{V}$ is defined as a bundle on $\mathcal{J}_V$. The bundle $\text{Det}(\tilde{V})$ is defined to be the complex determinant line bundle, which is defined because $\Gamma$ is a complex Lie group. This $\text{Det}(\tilde{V})$ is soldered by

$$A = \sqrt{-1} (\Omega^0_0 + \Omega^p_p).$$

Differentiating, we find the curvature of $\text{Det}(\tilde{V})$ is

$$F = dA = -\sqrt{-1} \Omega^0_0 \wedge \Omega^0_0 - \sqrt{-1} \Omega^p_0 \wedge \Omega^0_0 - \sqrt{-1} \Omega^0_0 \wedge \Omega^p_0 - \sqrt{-1} \Omega^0_0 \wedge \Omega^p_0$$

which is a $(1, 1)$ form on $\mathcal{J}_V$, invariant under complex conjugation, i.e. $\bar{F} = F$.

For each $J \in \mathcal{J}_V$, let $X_J \subset \tilde{\text{Gr}}_V^2$ be the base manifold of the associated Hopf fibration, i.e. $X_J$ is the set of $J$ complex lines, suitably oriented.
Lemma 13. Recall that there is a canonical identification \( \tilde{\text{Gr}}_V^2 \cong P^* \otimes V/P \). Suppose that \( X_J \subset \tilde{\text{Gr}}_V^2 \) is the base of a Hopf fibration. Then \( T_P X_J \subset T_P \tilde{\text{Gr}}_V^2 \) is identified with the subset of \( J \) linear maps. The map \( X_J \to \mathcal{J}_V \) constructed above is constant, mapping to \( J \).

Proof. By \( SL(V) \)-invariance, we only have to prove the result for the standard complex structure \( J \) on \( V = \mathbb{C}^{n+1} \). The first result is a calculation, while the second is immediate from the structure equations:

\[
\Omega^p_Q = \Omega^0_p = \Omega^p_q = 0
\]
on \( X_J \), and the fact that \( X_J \) is connected. □

Take a great circle fibration \( S^{2n+1} \to X \) and a point \( P \in X \), and the complex structure \( J_P \). The osculating Hopf fibration to \( X \) at \( P \) is \( S^{2n+1} \to X_{J_P} \). Note that \( P \in X_{J_P} \cap X \subset \tilde{\text{Gr}}_V^2 \).

Lemma 14. The invariant \((s^p_q)\) vanishes at a point \( P \in X \) on an elliptic submanifold \( X \subset \tilde{\text{Gr}}_V^2 \) precisely when the osculating complex structure \( X_{J_P} \) at \( P \) is tangent to \( X \) inside \( \tilde{\text{Gr}}_V^2 \).

Proof. We showed already that this invariant is determined by the 1-jet of the immersion \( X \to \tilde{\text{Gr}}_V^2 \). □

Lemma 15. An elliptic submanifold \( X \subset \tilde{\text{Gr}}_V^2 \) is locally the base manifold of a Hopf fibration \( X_J \) precisely if all of the \( s \) invariants vanish:

\[
s^p_q = s^p_r = s^p_{qr} = s^0_q = s^0_p = s^0_{pq} = 0
\]

which happens precisely if the map \( X \to \mathcal{J}_V \) constructed above is constant.

Proof. The 1-forms which are semibasic for the map \( G \to \mathcal{J}_V \) are precisely the \( \Omega \) with no bars, i.e. \( \Omega^P_Q \), so the \( \Omega^0_Q \) are not semibasic. This means that we pull them back when we differentiate the map \( X \to \mathcal{J}_V \), and nothing else. Hence the map \( X \to \mathcal{J}_V \) has vanishing differential, and consequently is locally constant, precisely when all of these invariants vanish. Conversely, if they all vanish, then our structure equations become the same as for the Hopf fibration, and the result follows by the Frobenius theorem. □

Corollary 7. A submanifold \( X \subset \tilde{\text{Gr}}_V^2 \) is the base of a Hopf fibration precisely when it is compact and connected and all of the \( s \) invariants vanish.

We have an inclusion map

\[
\mathcal{J}_V \to \text{GCF}(V)
\]

(where \( \text{GCF}(V) \) means the space of great circle fibrations of the sphere \((V \setminus 0)/\mathbb{R}^+ \), and \( \mathcal{J}_V \) is the space of complex structures) given by mapping a complex structure to its Hopf fibration. Conversely, if we pick any nonzero vector \( v \in V \), we can associate to any great circle fibration \( \pi: S^{2n+1} \to X \) the osculating complex structure
$J_v^X$ to $X$ at the point $x = \pi(v)$. The diagram

\[
\begin{array}{ccc}
GCF(V) & \xrightarrow{J_v} & J_v \\
\downarrow & & \downarrow \text{id} \\
\mathcal{J}_V & & \mathcal{J}_V
\end{array}
\]

which sheds some light on the algebraic topology of the space of great circle fibrations.

12. Recognizing the Hopf fibration

Given a great circle fibration, if its Reznikov invariant $s_q^p$ vanishes with its first derivative at a point, then the structure equations show that $s_0^0$ also vanishes there. Consider what if the Reznikov invariant vanishes everywhere. The equations $s_0^0 = s_r^0 = 0$ for all $p, q, r$ form an involutive system, with general solution depending on functions of $n$ variables. In particular, there are real analytic elliptic immersed submanifolds $X \subset \Gr^2_\mathbb{R} V$ which have vanishing Reznikov invariant. Nothing is known about them. Previous versions of this paper wrongly stated that vanishing of the Reznikov invariant everywhere forces the great circle fibration to be a Hopf fibration.

13. “Hyperplanes”

Given a real valued 1-form $\xi \in V^*$ on $V$ we can define a section of the bundle $\tilde{V}^* \to X$ by setting $\sigma_\xi(x)$ to be $\xi$. This determines a function $f : B \to V^*$ by

$$f(g) = \xi g.$$ 

This function satisfies

$$0 = df = \left( \begin{array}{cccc} f_0 & f_0^0 & \Omega_0^0 & \Omega_0^0 \\ f_0^0 & f_0^0 & \Omega_0^0 & \Omega_0^0 \\ f_0^p & f_0^p & \Omega_0^p & \Omega_0^p \\ f_0^p & f_0^p & \Omega_0^p & \Omega_0^p \end{array} \right) \left( \begin{array}{c} f_0 \\ f_0 \\ f_0^p \\ f_0^p \end{array} \right).$$

Because $\xi$ is real,

$$f_0 = \bar{f}_0 \text{ and } f_\bar{p} = f_p.$$

The bundle $\mathcal{O}(1) \to X$ is a quotient bundle of $\tilde{V}$, and these sections determine sections of the quotient which are just the functions $f_0$. They satisfy

$$df_0 - \Omega_0^0 f_0 = f_0^p \Omega_0^p + f_\bar{p} \Omega_0^\bar{p}. $$

In particular, they are not holomorphic unless certain $s$ invariants vanish.

Suppose that $\xi \neq 0$. Consider the locus $(f_0 = 0)$ inside $X$. (Of course, we should write it as something like $(\sigma_0 = 0)$ since $f_0$ is really a function on $B$, not on $X$.) At each point of this set,

$$df_0 = f_q \Omega^q + f_\bar{q}^s \Omega^\bar{q}. $$

This can’t vanish, since $f_0 = 0$ forces some $f_q \neq 0$. Therefore the “hyperplanes” determined by vanishing of these sections of $\mathcal{O}(1)$ are smooth submanifolds of $X$. They are the analogues of complex hyperplanes. We will see that they are connected.
14. The Sato map

Sato [14] constructed a map from the base $X$ of a great circle fibration into a complex projective space. Yang [16] pointed out that this map was not well defined. We will now present a very similar (but obviously well defined) map, which we will call the Sato map.

To motivate it, consider any complex structure $J$ on our vector space $V$. As we have remarked above, this $J$ by definition is a linear map $J : V \to V$ satisfying $J^2 = -1$, and therefore $J$ has two eigenspaces $V^{1,0}, V^{0,1} \subset V_C = V \otimes \mathbb{C}$, where $V^{1,0}$ is the eigenspace with eigenvalue $\sqrt{-1}$, and $V^{0,1}$ is the eigenspace with eigenvalue $-\sqrt{-1}$. We thereby obtain two complex projective subspaces $\mathbb{CP}(V^{1,0}), \mathbb{CP}(V^{0,1}) \subset \mathbb{CP}(V_C)$.

We can identify $V$ with $V^{1,0}$ by $v \in V \mapsto v - \sqrt{-1} Jv \in V^{1,0}$ and projectivize this map to identify $\mathbb{CP}(V) = \mathbb{CP}(V^{1,0})$ (where the left hand side means the space of $J$ complex lines in $V$). The copies of $\mathbb{CP}^n$ inside $\mathbb{CP}(V_C)$ that occur this way are precisely those with no real points, not intersecting $\mathbb{RP}(V)$. So the generic $\mathbb{CP}^n \subset \mathbb{CP}(V_C)$ occurs in this way. We will now imitate this story in the context of a general great circle fibration.

Consider a great circle fibration $S^{2n+1} \to X$. Take any $x \in X$. This $x$ may be identified with an oriented 2-plane $x \subset V$ since $X \subset \widetilde{\text{Gr}}_V$. There is a complex structure $J_x$ on $V$ for which $x$ is a complex line, defined in section 11 on page 35. Map $x \to V_C$ by $\sigma : v \in x \mapsto v - \sqrt{-1} J_x v \in V_C$.

Since the fiber of $\mathcal{O}(-1) \to X$ above $x$ is just $x \subset V$ itself, the Sato map $\sigma$ is defined on the total space of $\mathcal{O}(-1)$.

Lemma 16. The Sato map satisfies $\sigma(J_x v) = \sqrt{-1} \sigma(v)$ for any $v \in x \in X$.

Proof.

\[
\sigma(J_x v) = J_x v - \sqrt{-1} J_x J_x v = \sqrt{-1} (v - \sqrt{-1} J_x v).
\]

As a consequence, $\sigma$ takes the $J_x$ complex line $x$ to a complex line $\sigma(x)$ in $V_C$.

Define the Sato map $\sigma : X \to \mathbb{CP}(V_C)$ to be the map that assigns to $x$ the line $\sigma(x)$. This gives a morphism of bundles

\[
\mathcal{O}(-1)_X \xrightarrow{\sigma} \mathcal{O}(-1)_{\mathbb{CP}(V_C)}
\]

\[
X \xrightarrow{\sigma} \mathbb{CP}(V_C)
\]

which is complex linear on the fibers and $\text{SL}(V)$-equivariant. The action of $\text{SL}(V)$ on $\mathbb{CP}(V_C)$ is by biholomorphisms (indeed, by projective automorphisms).

Lemma 17. The Sato map $X \to \mathbb{CP}(V_C)$ is injective.
Proof. Suppose that \( \sigma(x) = \sigma(y) \). Then for any \( v \in x \) and \( w \in y \) we must find \( \sigma(v) \) and \( \sigma(w) \) on the same complex line in \( V_\mathbb{C} \). Therefore,

\[
\sigma(v) = v - \sqrt{-1} J_x v = (a + b \sqrt{-1}) (w - \sqrt{-1} J_y w) = (a + bJ_y) w + \sqrt{-1} (b - aJ_y) w
\]

which implies that

\[
v = (a + bJ_y) w \in y.
\]

But the fiber above \( y \) is \( J_y \) invariant, so \( v \) and \( w \) lie on the same great circle fiber, and therefore \( x = y \). \( \square \)

Let us look at the locus of points \( z \in V_\mathbb{C} \) so that \( z \wedge \bar{z} = 0 \in \Lambda^{1,1}(V_\mathbb{C}) \). This projects to a variety \( \Delta \subset \mathbb{C}P(V_\mathbb{C}) \). We can parameterize this variety as follows: a vector \( z \in V_\mathbb{C} \) satisfies \( z \wedge \bar{z} \) precisely when it has the form \( z = v + \sqrt{-1} w \) with \( v \) and \( w \) belonging to the same real line in \( V \). Multiplying by a suitable complex number, we can get \( z = v \in V \). So we can identify each element of \( \Delta \) with a vector in \( V \) up to real multiples: \( \Delta = \mathbb{R}P(V) \) i.e. \( \Delta \) consists in the real points of \( \mathbb{C}P(V_\mathbb{C}) \).

**Lemma 18.** The space \( S_V = \mathbb{C}P(V_\mathbb{C}) \setminus \mathbb{R}P(V) \) is canonically identified with the space of pairs \( (P, j) \) where \( P \subset V \) is a 2-plane, and \( j: P \to P \) is a complex structure.

**Proof.** Membership in \( \mathbb{R}P(V) \) is precisely given by the equation \( z \wedge \bar{z} = 0 \) and therefore the solutions in \( V_\mathbb{C} \) consist precisely in vectors \( z = v + \sqrt{-1} w \) so that \( v \wedge w = 0 \), i.e. \( v, w \in V \) belong to the same real line. Therefore away from \( \mathbb{R}P(V) \), points \( [v + \sqrt{-1} w] \in \mathbb{C}P(V_\mathbb{C}) \setminus \mathbb{R}P(V) \) map to 2-planes \( P = [v \wedge w] \). To a point \( [v + \sqrt{-1} w] \in S_V = \mathbb{C}P(V_\mathbb{C}) \setminus \mathbb{R}P(V) \), attach the complex structure \( j \) which maps \( jv = \bar{w}, jw = v \). On the other hand, given an oriented 2-plane \( P \) with complex structure \( j \) and a point \( v \in P \setminus 0 \), map it to \( \mathbb{C}P(v - \sqrt{-1} jv) \). This gives continuous maps in each direction between \( S_V \) and the space of 2-planes with complex structures, and these maps are easily seen to be inverses of one another. Since the space of 2-planes with complex structures is a homogeneous space of \( SL(V) \), and the maps are \( SL(V) \)-invariant, both sides are homogeneous spaces and these maps are equivariant diffeomorphisms. \( \square \)

Each fiber of \( S_V \to \tilde{Gr}^2_V \) is thus the set of complex structures on a given oriented 2-plane \( P \), forming a copy of the hyperbolic plane. We can look at the same 2-plane \( P \) with the opposite orientation, and find another such hyperbolic plane. These two planes are glued together inside \( \mathbb{C}P(V_\mathbb{C}) \) along the real locus of points of the projectivization of \( P \). In other words, the points of these two hyperbolic planes are the complex points of the projectivization of \( P \). Thus the fibers of \( S_V \to \tilde{Gr}^2_V \) are open subsets ("halves") of the complex points of real projective lines in \( \mathbb{C}P(V_\mathbb{C}) \).

We can write out the structure equations for \( G \to S_V \): the 1-forms \( \Omega^0_0, \Omega^p_0, \Omega^p_0 \) are semibasic (because they are independent on \( G \) but vanish on the structure group) and they satisfy

\[
(13) \quad d \begin{pmatrix} \Omega^p_0 \\ \Omega^p_0 \\ \Omega^0_0 \end{pmatrix} = - \begin{pmatrix} \Omega^p_0 - \delta^p_0 \Omega^0_0 & \Omega^p_0 & 0 \\
\Omega^p_0 & \Omega^0_0 - \delta^p_0 \Omega^0_0 & 0 \\ \Omega^0_0 & \Omega^0_0 & \Omega^0_0 - \Omega^0_0 \end{pmatrix} \wedge \begin{pmatrix} \Omega^p_0 \\ \Omega^p_0 \\ \Omega^0_0 \end{pmatrix} + \Omega^p_0 \wedge \begin{pmatrix} \Omega^p_0 \\ \Omega^0_0 \end{pmatrix}.
\]
From this expression, we find that the 1-forms \( \Omega^p_0, \Omega^q_0, \Omega^0_0 \) vary in a complex linear representation under the action of the structure group. Declaring them to be \((1,0)\)-forms for an almost complex structure, the torsion consists of \((1,1)\) forms, so the almost complex structure is a complex structure: \( S_V \) is a complex manifold.

**Lemma 19.** The complex structure on \( S_V \) given by the structure equations 13 on the preceding page is the same as that given by the embedding \( S_V \subset \mathbb{CP}(V_\mathbb{C}) \).

*Proof.* The generic \( \mathbb{CP}^n \subset \mathbb{CP}(V_\mathbb{C}) \) lies in \( S_V \) and occurs as the base of a Hopf fibration. From the structure equations of a Hopf fibration we see that these are complex submanifolds in that complex structure. But they are obviously complex submanifolds in the \( \mathbb{CP}(V_\mathbb{C}) \) complex structure. The complex structures on these \( \mathbb{CP}^n \) submanifolds agree, as we can see directly looking at the construction of the Hopf fibration. Since the tangent planes to these \( \mathbb{CP}^n \) form an open subset of \( n \) dimensional complex tangent planes, they force both complex structures on \( S_V \) to agree. \( \square \)

The two zeroes in these structure equations, in the expressions for \( d\Omega^p_0 \) and \( d\Omega^q_0 \), show that this manifold is fibered by complex curves, defined by the equations \( \Omega^p_0 = \Omega^q_0 = 0 \) and these are of course the fibers of \( S_V \to \tilde{\text{Gr}}_V \) since the 1-forms \( \Omega^p_0, \Omega^q_0 \) are semibasic for this map. Note \( S_V \to \tilde{\text{Gr}}_V \) is not a holomorphic map for any complex structure on \( \tilde{\text{Gr}}_V \), because the relevant representation of the structure group does not preserve a complex structure. The torsion terms up on \( S_V \) reorganize into the soldering 1-forms on \( \tilde{\text{Gr}}_V \).

**Corollary 8.** The Sato map \( \sigma: X \to S_V \subset \mathbb{CP}(V_\mathbb{C}) \) is an embedding. The complex line bundle \( \mathcal{O}(-1)_X \) is the pullback of the complex line bundle \( \mathcal{O}(-1)_{\mathbb{CP}(V_\mathbb{C})} \) by \( \sigma \). The image of the Sato map determines the great circle fibration \( S^{2n+1} \to X \).

*Proof.* The map \( X \to \tilde{\text{Gr}}_V \) is an embedding, and the map \( S_V \to \tilde{\text{Gr}}_V \) is an \( \text{SL}(V) \)-equivariant fiber bundle. \( \square \)

**Corollary 9.** The “hyperplanes” in \( X \) which were discussed in section 13 on page 37 are the intersections of \( X \) with complex hyperplanes in \( \mathbb{CP}(V_\mathbb{C}) \).

*Proof.* This is just saying that since \( \mathcal{O}(-1)_X \) is the pullback of \( \mathcal{O}(-1)_{\mathbb{CP}(V_\mathbb{C})} \), therefore the dual of the pullback, i.e. \( \mathcal{O}(1)_X \), is the pullback of the dual. \( \square \)

Consider the requirement that the 1-forms used to cut out the hyperplanes are real valued. Take a real valued 1-form \( \xi \in V^* \) and split it into complex linear and conjugate linear parts on \( V_\mathbb{C} \) by

\[
\xi^{1,0}(v + \sqrt{-1}w) = \xi(v) + \sqrt{-1}\xi(w)
\]

and

\[
\xi^{0,1}(v + \sqrt{-1}w) = \xi(v) - \sqrt{-1}\xi(w).
\]

Then the \( \xi^{1,0} \) part cuts out a hyperplane of \( v + \sqrt{-1}w \) so that \( \xi(v) + \sqrt{-1}\xi(w) = 0 \), i.e. where both \( \xi(v) \) and \( \xi(w) \) vanish.

The Sato map is given by taking elements of \( V_\mathbb{C}^* \) and using them as sections of \( \mathcal{O}(-1)_X \) and then using that to make an embedding

\[
P \mapsto [\sigma_0(P) : \cdots : \sigma_{2n}(P)].
\]
15. Mapping to projective space

Two great circle fibrations $X, X' \subset \Gr^2$ are parallel at a point $x \in X \cap X'$ if they induce the same complex structure on the 2-plane $P = P_x \subset V$ corresponding to the point $x \in \Gr^2$. If a point $x \in X$ is associated to a 2-plane $P \subset V$, then the differential of the map $X \to \Gr^2$ is

$$T_x X \to T_{P_x} \Gr^2 = P^* \otimes (V/P),$$

and becomes complex linear for unique complex structures on $T_x X$, $P$ and $V/P$, determined algebraically from the linear map

$$T_x X \to T_{P_x} \Gr^2,$$

i.e. from the tangent spaces of $X$. Any complex structure on $P$ arises this way, for a suitable choice of tangent space by $\SL(V)$-equivariance. Hence if we can slightly perturb the tangent planes of two great circle fibrations at their intersection points, then they will be nowhere parallel. Another way to phrase parallelism: points of parallelism are intersection points in the image of the Sato map inside $\mathbb{CP}(V_C)$.

Parallelism of great circle fibrations $X, X'$ at a point $x$ clearly implies that the osculating complex structures $J_{1,x}, J_{2,x}$ agree on the 2-plane $P = P_x$, and so

$$0 = \det (J_{1,x} - J_{2,x}).$$

In particular, if we can find some constant complex structure $J \in V^* \otimes V$ whose Hopf fibration base $X_J = \mathbb{CP}^n_x \subset \Gr^2$ satisfies

$$0 \neq \det (J - J_x)$$

for all $J_x \in K$, i.e. for all osculating complex structures to $X$ then $X_J$ and $X$ are nowhere parallel.

Take a great circle fibration with base space $X \subset \Gr^2$. Let $\mathbb{CP}^n = \mathbb{CP}^n_J$ be the Hopf fibration base of some complex structure $J$. By transversality, after a $C^2$-small perturbation of $X \subset \Gr^2$, we can arrange that $X$ has finitely many intersections with $\mathbb{CP}^n \subset \Gr^2$. We can also arrange that these intersection points $x \in X \cap \mathbb{CP}^n$ are transverse, and have “generic” tangent space $T_x X$. In particular, achieving generic tangent spaces $T_x X$ at intersection points $x \in X \cap \mathbb{CP}^n$ ensures that $X$ and $\mathbb{CP}^n$ are nowhere parallel, i.e. $X$ and $\mathbb{CP}^n$ are disjoint inside $\mathbb{CP}(V_C)$.

Take two complex structures $J$ and $J'$ on $V$, whose associated Hopf fibration bases are nowhere parallel to $X$. It is easy to check that small perturbation of $J'$ among complex structures will ensure that $J'$ and $X$ are pairwise nowhere parallel and that $J - J'$ is invertible. Define a map

$$f = f_{J,J',X}: v \in V \setminus 0 \mapsto (J' - J)^{-1} (J_x - J') v \in V \setminus 0,$$

where $v \in S^{2n+1} \mapsto x \in X$ is the bundle map of our great circle fibration, with osculating complex structure $J_x$.

The reader can easily check the following linear algebra fact: for any 3 complex structures $J_0, J, J'$ on the same vector space $V$, with $J' - J$ invertible, and $J_0$ and $J'$ nowhere parallel, the linear equations

$$0 = v_0 + v + v',$$

$$0 = J_0 v_0 + J v + J' v',$$
determine any two of \(v, v'\) as real linear functions of \(v_0\). From the form of these equations, if we replace \(v_0\) by \(J_0v_0\), and so on, the solutions change to

\[
0 = J_0v_0 + Jv + J'v', \\
0 = J_0J_0v_0 + JJv + J'J'v',
\]

i.e. the solution for a given \(v_0\) gives a complex linear map

\[
v_0 \in (V, J_0) \mapsto (v, v') \in (V, J) \times (V, J').
\]

Moreover, this solution is given by

\[
v = (J' - J)^{-1}(J_0 - J')v_0, \\
v' = -v_0 - v.
\]

In other words, if \(X = X_{J_0}\) is a Hopf fibration base, then the linear algebra tells us that the associated map \(v = f(v_0)\) is complex linear on each fiber of the Hopf fibration. Indeed in this simple case, \(f\) is a real linear automorphism of \(V\).

**Lemma 20.** Take any great circle fibration \(X \subset \tilde{G}_V^2\). Then there are complex structures \(J, J'\) for which \(J - J'\) is invertible and for which \(X\) is nowhere parallel to the Hopf fibration base \(X_{J'} = \mathbb{CP}^n_{J'}\), and for which the associated smooth map \(f\) above takes \(V \setminus 0\) to \(V \setminus 0\), taking each 2-plane \(x = \langle v, J_xv \rangle \subset V\) associated to a point \(x \in X\) to a \(J\)-linear 2-plane \(\langle w, Jw \rangle\) where \(w = f(v)\). The map \(f\) is linear on these planes. If \(f\) is a local diffeomorphism of \(V \setminus 0\) then it descends to a diffeomorphism \(X \to X_{J'} = \mathbb{CP}^n_{J'}\) to a Hopf fibration, linear on the fibers of the Hopf fibration.

**Proof.** Take a vector \(v \in V \setminus 0\) and let \(x = \langle v, J_xv \rangle \in X\) be the associated oriented 2-plane. If \(f(v) = 0\) then one easily checks that \(J_xv = Jv\) and so \(X\) and \(X_{J'}\) are parallel at \(x\). Consequently \(f : V \setminus 0 \to V \setminus 0\) is a smooth map, equivariant under rescaling.

Applying the linear algebra of the osculating complex structures to \(X\) to the equations above, we find that \(f\) is complex linear on each 2-plane \(x \in X\). Once we see that \(f\) is a local diffeomorphism, rescaling invariance allows us to treat \(f\) as a smooth local diffeomorphism of the sphere, so a diffeomorphism. \(\square\)

In order to see whether \(f\) is a local diffeomorphism, we need to see whether, for a suitable choice of \(J'\), \(J' (v_0) w_0 \neq 0\) for all \(w_0\). For each vector \(v_0 \neq 0\), we let \(J_{v_0}\) be \(J_x\) for the associated \(x \in X\), and we let \(A_{v_0} \in V^* \otimes V\) be the linear map

\[
A_{v_0} w = J_{v_0} w + \frac{d}{dt} \bigg|_{t=0} J_{v_0 + tw} v_0.
\]

Calculate

\[
f'(v_0) w = (A_{v_0} - J') w.
\]

Consider the map

\[
v_0 \in S^{2n+1} \to A_{v_0} \in V^* \otimes V.
\]

If we can choose \(J'\) so that additionally

\[
0 \neq \det J' - A_{v_0},
\]

for any \(v_0 \in V \setminus 0\), i.e. so that \(f'(v_0)\) has no kernel then \(f\) is a diffeomorphism.
Consider again the Hopf fibration, and its base manifold $X_J$. The base sits inside $\mathbb{C}P(V_C)$ as a linear subspace, cut out by the equation $Jz = \sqrt{-1}z$. So it is a linear projective subspace,

$$X_J = \mathbb{C}P^n \subset \mathbb{C}P^{2n+1} = \mathbb{C}P(V_C).$$

If we pick another linear $\mathbb{C}P^n$ subspace, call it $\mathbb{C}P^n_0$, which is not parallel to $X_J$, we can look at the family of all linear $\mathbb{C}P^{n+1}$ subspaces containing $\mathbb{C}P^n_0$. These $\mathbb{C}P^{n+1}$s will each intersect $\mathbb{C}P^n$ in a unique point transversely. We will call the $\mathbb{C}P^n_0$ a hinge.

On $S_V$, the 1-forms $\Omega^p_0, \Omega^\bar{p}_0, \Omega^0_0$ form a complex linear coframing, and if this coframing comes from a point of the adapted bundle $B \to X_J$, then the tangent space of $X_J$ is given by $\Omega^p_0 = \Omega^{\bar{p}}_0 = 0$. The $\mathbb{C}P^{n+1}$s striking this $X_J = \mathbb{C}P^n$ transversely at this point are given by a set of complex linear equations

$$\Omega^p_0 = a^p_\bar{q}_0 \Omega^\bar{q}_0 + a^p_0 \Omega^0_0.$$

The intersection of a $\mathbb{C}P^{n+1}$ with the submanifold $X_J$ inside $\mathbb{C}P(V_C)$ is described in our bundle $B$ by the equations

$$0 = \Omega^p_0 = \Omega^\bar{p}_0 = \Omega^0_0$$

cutting out the $\mathbb{C}P^{n+1}$ together with the equations

$$0 = \Omega^p_q = \Omega^\bar{p}_0 = \Omega^0_p = \Omega^0_0$$

cutting out the $X_J = \mathbb{C}P^n$.

Repeat this story using a general great circle fibration $S^{2n+1} \to X$; a tangent space to $X \subset S_V$ is given in an adapted coframe by

$$\Omega^p_0 = s^p_\bar{q}_0 \Omega^\bar{q}_0$$
$$\Omega^0_0 = s^0_\bar{q}_0 \Omega^\bar{q}_0$$

while the osculating Hopf fibration $X_J$ (for $J = J_P$, osculating at point $P \in X$) satisfies similar equations $0 = \Omega^p_0 = \Omega^\bar{p}_0$. It is easy to calculate that the common tangent vectors in these tangent spaces are precisely those given by $v \cdot \Omega^p_0 = v^p$ where $v^p$ is any eigenvector with eigenvalue 1 of the matrix

$$\begin{pmatrix} a^p_\bar{q}s^\bar{q}_p + a^p_0 s^0_0 \\ a^p_\bar{q}s^\bar{q}_q + a^p_0 s^0_q \end{pmatrix}. $$
If the spectrum of this matrix lies inside the unit disk at some point of the bundle $B$, then the same is true for any point in the same fiber over $X$, and we say that the $\mathbb{CP}^{n+1}$ is strongly transverse to $X$ at such a point.

Returning to the Hopf fibration $X_J = \mathbb{CP}^n \subset \mathbb{CP}^{n+1}$, take any other $\mathbb{CP}^n_0 \subset \mathbb{CP}^{n+1}$ so that $\mathbb{CP}^n_0$ is not parallel to $X_J$, i.e. so that $\mathbb{CP}^n_0 \cup X_J$ is not contained in any $\mathbb{CP}^{n+1}$. Call such a $\mathbb{CP}^n_0$ a hinge. Take all $\mathbb{CP}^{n+1}$ in $\mathbb{CP}(V_C)$ containing $\mathbb{CP}^n_0$. If each of these strikes $X_J$ at a single point transversely, we can identify $X_J$ with the projective quotient $\mathbb{CP}(V_C)/\mathbb{CP}^n_0 = \mathbb{CP}^n$, i.e. with the projectivized holomorphic normal bundle at any point of $\mathbb{CP}^n_0$.

The same procedure works for a general great circle fibration $S^{2n+1} \to X$, under some additional hypotheses. A linear subspace $\mathbb{CP}^n \subset \mathbb{CP}(V_C) = \mathbb{CP}^{2n+1}$ is parallel to $X$ at a point $P \in X$ if it is parallel to the osculating Hopf fibration $X_J$ for $J = J_P$. Conversely, a linear subspace $\mathbb{CP}^n \subset \mathbb{CP}(V_C)$ is called a hinge for $X$ if first it is nowhere parallel to $X$ and second it does not intersect $X$ and third every $\mathbb{CP}^{n+1}$ containing that $\mathbb{CP}^n$ intersects $X$ transversely and positively.

**Lemma 21.** Take $S^{2n+1} \to X$ a great circle fibration, and stick $X$ into $\mathbb{CP}(V_C)$ via the Sato map. Take a hinge $\mathbb{CP}^n_0$ for $X$. The family of $\mathbb{CP}^{n+1}$ linear subspaces containing that hinge is diffeomorphic to $\mathbb{CP}^n$. There is a diffeomorphism $X \to \mathbb{CP}^n$ given by taking a point $P \in X$ to the unique $\mathbb{CP}^{n+1}$ linear space containing $\mathbb{CP}^n_0$ which strikes $X$ at $P$.

**Proof.** Consider the manifold $Y \cong \mathbb{CP}^n$ of all $\mathbb{CP}^{n+1}$ containing the hinge $\mathbb{CP}^n_0$, and the incidence correspondence $Z$ of all pairs $(P, \Pi)$ with $\Pi \in Y$ and $P \in X$ and $X_J \subset \Pi$. By positivity of intersections, $Z$ is a submanifold of $X \times Y$ of the same dimension $(2n)$ as both $X$ and $Y$, and $X$ and $Y$ are compact, and so $Z$ is too. Again by positivity of intersections, $Z \to Y$ and $W \to X$ are covering maps, preserving orientation. But $X$ and $Y = \mathbb{CP}^n$ are both simply connected, so $Z \to Y$ and $Z \to X$ are diffeomorphisms. 

**Lemma 22.** If a great circle fibration admits a hinge then it admits one of the form $\mathbb{CP}^n_0 = X_J$ for some complex structure $J$ on $V$.

**Proof.** By dimension count, we see that these $X_J \subset \mathbb{CP}(V_C)$ (which are in one to one correspondence with complex structures $J$ on $V$) have the same dimension as the space of all $\mathbb{CP}^n$ linear subspaces in $\mathbb{CP}(V_C)$, so they form an open subset. Indeed the $X_J$ subspaces are precisely those $\mathbb{CP}^n$ linear subspaces with no real points on them.
Now that we can map the base of our great circle fibration to a complex projective space, we have to map the great circles to the fibers of the Hopf fibration. Consider first how to do this for a pair Hopf fibrations, given by complex structures $J_1$ and $J_2$. We take any other Hopf fibration, given by a complex structure $J_0$, so that if the associated fibrations are $S^{2n+1} \to X_k$ for $k = 0, 1, 2$, then $X_0 \subset \mathbb{CP}(V_C)$ will not be parallel to $X_1$ or to $X_2$, i.e. $X_0$ is a hinge for $X_1$ and also for $X_2$. Of course, these $X_k$ are all $\mathbb{CP}^n$ linear subspaces inside $\mathbb{CP}^{2n+1} = \mathbb{CP}(V_C)$ given by linear equations 

$$X_k = (J_k = \sqrt{-1}).$$

Our map $X_1 \to X_2$ is constructed by taking all linear $\mathbb{CP}^{n+1}$ subspaces containing $X_0$ and matching the point $\mathbb{CP}^{n+1} \cap X_1$ to the point $\mathbb{CP}^{n+1} \cap X_2$.

First, we need to lift the entire picture up to a linear picture in $V_C$. Then we will see what is happening in $V$ itself. Up in $V_C$ we have 3 complex linear subspaces $V_0, V_1, V_2$, all isomorphic to $\mathbb{C}^{n+1}$, which are just the preimages of the $X_0, X_1, X_2$ linear subspaces in $\mathbb{CP}(V_C)$. Another way to say this:

$$V_k = \{v \mid J_kv = \sqrt{-1}v\}.$$

Since the $V_0$ has no vectors in common with $V_1$ or with $V_2$, every vector in $V_C$ can be written as a combination $w = w_0 + w_1$ of vectors from $V_0$ and $V_1$. The vectors in $V_k$ are of the form 

$$v - \sqrt{-1}J_kv$$

for vectors $v \in V$. The equation 

$$v_2 - \sqrt{-1}J_2v_2 = v_0 - \sqrt{-1}J_0v_0 + v_1 - \sqrt{-1}J_1v_1$$

for $v_0, v_1, v_2 \in V$ breaks into real and imaginary parts: 

$$v_2 = v_0 + v_1$$

$$J_2v_2 = J_0v_0 + J_1v_1$$

and has the solution 

$$v_0 = (J_2 - J_0)^{-1}(J_1 - J_2)v_1$$

$$v_2 = v_0 + v_1$$

so taking $v_1 \in V \to v_2 \in V$ by a linear isomorphism, taking the $\sqrt{-1}$ eigenspace of $J_1$ to that of $J_2$. The equation $J_2v_2 = J_0v_0 + J_1v_1$ ensures us that this map takes $J_1v_1$ to $J_2v_2$. So it is a complex linear map 

$$(V, J_1) \to (V, J_2).$$

By construction it matches the 2-planes $P_1$ and $P_2$ in $V$, just looking back at the construction of the Sato map.

As before, we can apply the same idea to a great circle fibration as follows: we take $S^{2n+1} \to X$ our great circle fibration, and $S^{2n+1} \to X_2$ a Hopf fibration, given by a complex structure $J_2$. Now we pick another Hopf fibration $S^{2n+1} \to X_0$ given by a complex structure $J_0$, so that $X_0$ is nowhere parallel to $X$ and to $X_2$. For each 2-plane $P \in X$ we take the osculating complex structure $J_P$ and use the above process to produce a complex linear map $(V, J_P) \to (V, J_2)$. This will map $P$ to a 2-plane $P_2$ complex linearly: 

$$(V, P, J_P) \to (V, P_2, J_2).$$
Hence it identifies the great circle fibrations. We have proven:

**Theorem 4.** If a great circle fibration $S^{2n+1} \to X$ admits a hinge, then the base manifold $X$ is diffeomorphic to $\mathbb{CP}^n$ by a diffeomorphism which lifts to a diffeomorphism $S^{2n+1} \to S^{2n+1}$ linear on the fibers. For each choice of hinge $X_{J_0} \subset \mathbb{CP}(V_C)$ for $X$ we obtain an isomorphism

$$
\begin{array}{ccc}
S^{2n+1} & \longrightarrow & S^{2n+1} \\
\downarrow & & \downarrow \\
X & \longrightarrow & \mathbb{CP}^n
\end{array}
$$

identifying the great circle fibration with a Hopf fibration. This isomorphism depends smoothly on the choice of hinge.

**Corollary 10.** Given any two great circle fibrations $S^{2n+1} \to X_0$ and $S^{2n+1} \to X_1$, both admitting a hinge, the embedded submanifolds $X_0, X_1 \subset \widetilde{\text{Gr}}^2 V$ are homotopic inside $\widetilde{\text{Gr}}^2 V$.

**Proof.** We pick a hinge for each $X_j$ and an orientation preserving linear isomorphism identifying the hinges. The linear isomorphism lies on a path inside the general linear group. So we can assume that the two hinges are the same. Deform $X_0$ to $X_1$ along the straight lines contained in each $\mathbb{CP}^{n+1} \setminus \mathbb{CP}^n = C^{n+1}$. Then take the image in $\widetilde{\text{Gr}}^2 V$. \hfill \Box

16. **Nonlinear $J$: twisted complex structures**

A great circle fibration $S^{2n+1} \to X$ determines a 2-plane $P$ through each point $v \in V \setminus 0$, and a complex structure $J_P : V \to V$ for which $P$ is a complex line. Define the map $J_X : V \setminus 0 \to V \setminus 0$ by $J_X v = J_P v$. Clearly $J_X$ is a smooth map, and satisfies $J_X^2 = -1$. The map $J_X$ is linear precisely if the great circle fibration is a Hopf fibration.

A **twisted complex structure** is a map $J : V \setminus 0 \to V \setminus 0$ which

1. is continuous,
2. satisfies $J^2 = -1$
3. leaves invariant each 2-plane span $\langle v, Jv \rangle$ and
4. is linear on those 2-planes.

A real vector space $V$ equipped with a twisted complex structure is a **twisted complex vector space**. (Twisted complex structures were discovered simultaneously by the author [12] and by Jean-Claude Sikorav [15].)

Extend our twisted complex structures from $V \setminus 0$ to $V$ by defining $J 0 = 0$, so that $J : V \to V$ is a homeomorphism, but not differentiable at 0 unless $J$ is linear, i.e. a complex structure. A twisted complex structure is smooth if $J$ is smooth on $V \setminus 0$. Note that a twisted complex structure extends to 0 to be smooth at 0 just when it is linear.

Given a smooth or topological twisted complex structure $J$, define a smooth or topological great circle fibration by taking $X(J) \subset \widetilde{\text{Gr}}^2 V$ to be the set of oriented 2-planes of the form $\langle v, J_X v \rangle$ for $v \in V$, oriented by setting $v \wedge J_X v$ to be positive, taking the great circles in $S^{2n+1} = (V \setminus 0)/\mathbb{R}^+$ to be the quotients by $\mathbb{R}^+$ of the $J$-invariant 2-planes.
Lemma 23. For any smooth great circle fibration $S^{2n+1} \to X$, $X(J_X) = X$.

Proof. The 2-planes which are $J_X$-invariant are precisely those belonging to $X$, with
the required orientation. So it is the same submanifold of the Grassmannian. □

A twisted complex structure $J$ is a pseudocomplex structure if it satisfies $J = J_X$
for $S^{2n+1} \to X$ a great circle fibration.

Lemma 24. The generic smooth twisted complex structure $J$ is not pseudocomplex,
i.e. $J_{X(J)} \neq J$.

Proof. Let $X = X(J)$ and $J' = J_{X(J)}$. Clearly $J'$ leaves a 2-plane invariant precisely
when $J$ does. These 2-planes have the same orientation, so they determine
the same great circle fibration, $X$. Each of $J$ and $J'$ also determine sections of
$S_V \to \tilde{\text{Gr}}_V$. Recall that $S_V$ is the space of pairs $(P, j)$ so that $P \subset V$ is a 2-plane and
$j: P \to P$ is a complex structure on $P$. Define $\sigma_J: X \to S_V$ by $\sigma_J(P) = (P, J|_P)$
restricting $J$ to $P$. But we can vary $J$ to an arbitrary section of $S_V \to \tilde{\text{Gr}}_V$ over
the same $X$, while $J'$ is fixed by the choice of $X$. □

Corollary 11. The space of smooth twisted complex structures retracts to the space
of great circle fibrations, i.e. to the space of pseudocomplex structures.

Proof. Indeed, using the notation of the preceding lemma, $J = J'$ precisely when
the sections $X \to S_V|_X$ agree. Note that these are disk bundles, so the space of $J$
with fixed $X(J)$ is contractible. Or, more canonically, just take $J \to X(J)$. □

If vector spaces $V_0, V_1$ have twisted complex structures $J_0, J_1$, then the sum
of these is $V = V_0 \oplus V_1$ with twisted complex structure

$$ J(v_0, v_1) = (J_0v_0, J_1v_1). $$

An obvious result:

Proposition 2. The sum of twisted complex vector spaces is twisted complex. The
sum is smooth just when one of the summands is linear, i.e. a complex structure.

It appears that Yang’s proof of the topological Blaschke conjecture for Blaschke
manifolds with the cohomology of complex projective spaces involves proving that
every such Blaschke manifold is the base manifold of a sum of two great circle
fibrations.

17. The space $M$

Consider the homogeneous $\text{SL}(V)$ space $M_V$ which consists of choices of pairs
$(P, J)$ of 2-plane $P \subset V$ and complex structure $J: V \to V$ so that $P$ is a complex
$J$-line. We have described a map $X \to M_V$ from the base manifold $X$ of any great
circle fibration $S^{2n+1} \to X$. Indeed the structure group of our bundle $B \to X$ is
precisely the isotropy group of a point of $M_V$. There are obvious maps

$$ M_V \xrightarrow{\text{Gr}_V} \tilde{\text{Gr}}_V \xrightarrow{f_V}. $$
The fibers of the map $M_{V} \to \mathcal{J}_{V}$ are copies of $\mathbb{CP}^{n}$. From the structure equations, we see that $M_{V}$ is a complex manifold. In fact it is possible to see this from a different point of view: since the complex structures on $V$ are identified with the complex linear subspaces $W \subset V$ with no real points, i.e. $W \cap V = 0$, this is an open subset of the Grassmannian $\text{Gr}_{\mathbb{C},2n+2}^{n+1}$. Above $\text{Gr}_{\mathbb{C},2n+2}^{n+1}$ we have the universal bundle $\mathcal{U} \to \text{Gr}_{\mathbb{C},2n+2}^{n+1}$ whose fiber above $W \in \text{Gr}_{\mathbb{C},2n+2}^{n+1}$ is $W$ itself. Projectivizing this bundle, we have the bundle

$$\mathbb{CP}(\mathcal{U}) \to \text{Gr}_{\mathbb{C},2n+2}^{n+1}$$

of complex projective spaces. A choice of $(P,J) \in M_{V}$ is precisely a choice of complex subspace $W \in \text{Gr}_{\mathbb{C},2n+2}^{n+1}$ not containing any real vectors: $W \cap V = 0$, and a choice of complex line inside $W$: the line consisting of the vectors

$$v - \sqrt{-1}Jv$$

for $v \in P$. Consequently, $M_{V}$ is just the pullback

$$
\begin{array}{ccc}
M_{V} & \longrightarrow & \mathbb{CP}(\mathcal{U}) \\
\downarrow & & \downarrow \\
\mathcal{J}_{V} & \longrightarrow & \text{Gr}_{\mathbb{C},2n+2}^{n+1}.
\end{array}
$$

We can also identify $M_{V}$ with the homogeneous space

$$M_{V} = \text{SL}(V) / \Gamma_{0}.$$

Since a great circle fibration

$$S^{1} \longrightarrow S^{2n+1} \longrightarrow X^{2n}$$

gives rise to a right principle $\Gamma_{0}$ bundle

$$\Gamma_{0} \longrightarrow B \longrightarrow X$$

with $B \subset \text{SL}(V)$, we have

$$
\begin{array}{ccc}
\Gamma_{0} & \longrightarrow & B \\
\downarrow & & \downarrow \\
X & \longrightarrow & \text{SL}(V).
\end{array}
$$

Consider also the map

$$X \to M_{V} \to \mathcal{J}_{V} \subset \text{Gr}_{\mathbb{C},2n+2}^{n+1}.$$

Picking a hinge, i.e. a choice of complex subspace $W_{0} \in \text{Gr}_{\mathbb{C},2n+2}^{n+1} \cap \mathcal{J}_{V}$ which is transverse to every complex subspace $W \in \text{Gr}_{\mathbb{C},2n+2}^{n+1} \cap \mathcal{J}_{V}$ arising as osculating complex structure to $X$, we can then trivialize the universal bundle over the part of
Gr$_C^{n+1,2n+2}$ consisting of subspaces transverse to $W_0$. We do that by taking complex coordinates $z,w$ on $V_C$ so that the complex $n + 1$ plane $z = 0$ is the hinge. Then all of the other complex $n + 1$ planes have the form $w = pz$ so that $p = (p_Q^U)$ is our Plücker coordinate system. Given two $n + 1$ planes, say with Plücker coordinates $p_0$ and $p_1$, use the map 

$$(z,p_0 z) \mapsto (z,p_1 z)$$

to identify them. This is a complex linear map, so it identifies complex lines with complex lines.

On $X \subset M$, map 

$$(z,pz) \in X \mapsto (z,0) \in X_0$$

where $X_0$ is the subspace associated to some Hopf fibration. This map takes complex lines to complex lines, and therefore identifies the complex structures on the 2-planes belonging to $X$ with those on $X_0$.

18. Further remarks

For applications to pseudoholomorphic curves, one would like to define a concept of totally real subspace $R \subset V$, $\dim_R R = n+1$. This should be precisely a subspace of $V$ which, thought of as a great $n$ sphere in $S^{2n+1}$, has no great circles in it from our fibration. Generic $R$ should have this property.

The general story of great sphere fibrations (the topological Blaschke theory) can probably be studied as follows: each Hopf fibration

$$S^k \longrightarrow S^N \quad \downarrow$$

$$X_{\text{Hopf}}$$

has a symmetry group $\Gamma \subset \text{SL}(N + 1, \mathbb{R})$, and a point of $X_{\text{Hopf}}$ has an isotropy subgroup $\Gamma_0 \subset \Gamma$. This gives an embedding of homogeneous $\text{SL}(N + 1, \mathbb{R})$ spaces

$$X_{\text{Hopf}} = \Gamma/\Gamma_0 \subset \text{SL}(N + 1, \mathbb{R})/\Gamma_0.$$ 

Then $\text{SL}(N + 1, \mathbb{R})$ acts on this picture to move the embedded submanifold around in a fibration, with base $\text{SL}(N + 1, \mathbb{R})/\Gamma$.

$$\Gamma/\Gamma_0 \longrightarrow \text{SL}(N + 1, \mathbb{R})/\Gamma_0 \quad \downarrow$$

$$\text{SL}(N + 1, \mathbb{R})/\Gamma.$$
Now any great sphere fibration

\[ S^k \longrightarrow S^N \]

\[ \downarrow \]

\[ X \]

probably gives, via moving frame calculations similar to those above, an embedding

\[ X \subset \text{SL}(N + 1, \mathbb{R}) / \Gamma_0 \]

which is “close” to vertical. Locally trivializing the fiber bundle

\[ \Gamma / \Gamma_0 \longrightarrow \text{SL}(N + 1, \mathbb{R}) / \Gamma_0 \]

\[ \downarrow \]

\[ \text{SL}(N + 1, \mathbb{R}) / \Gamma \]

in some “nice” way, one will probably find that this gives an explicit diffeomorphism

\[ X \rightarrow X_{\text{Hopf}}. \]

Taking \( M \) a manifold and \( SM = (TM \setminus 0) / \mathbb{R}^+ \), the tangent sphere bundle, one could consider a great sphere fibration of each sphere in \( SM \). Call this a Blaschke system. We can interpret such a system as a first order system of partial differential equations, so that a solution to such an equation is an immersed submanifold of \( M \) whose tangent spaces project via \( TM \setminus 0 \rightarrow SM \) to be great spheres belonging to our fibration. If there are involutive differential equations constructible in this way, with a suitable notion of taming, then this will provide a theory of pseudoquaternionic curves and pseudo-octave curves, and perhaps a theory of Gromov–Witten invariants for hyper-Kähler and octavic spaces.

For applications to elliptic partial differential equations, the most important result one would like to prove is probably the existence of a taming symplectic structure. Here that means an element \( \omega \in \Lambda^2(V^*) \) so that \( \omega > 0 \) on each 2-plane in \( X \). In local complex coordinates \( z, w \) on \( V \), we can write 2-planes as

\[ dw^i = p^i dz + q^i d\bar{z} \]

and then the symplectic form

\[ \omega = \frac{\sqrt{-1}}{2} \left( dz \wedge d\bar{z} + dw^i \wedge dw^\bar{i} \right) \]

becomes on that 2-plane

\[ \omega = \frac{\sqrt{-1}}{2} (1 + |p|^2 - |q|^2) dz \wedge d\bar{z}. \]

Therefore the null 2-planes for \( \omega \) form a real hyperquadric in \( \tilde{\text{Gr}}_V^2 \). When we look up in \( S_V \) we find two more dimensions to the space, and to the subspace of null 2-planes. It is still a real hyperquadric. The problem is then to show that this hypersurface does not intersect the base manifold of a great circle fibration, after suitable linear transformation.

In studying families of great circle fibrations, it would be helpful to have a retraction from the space of great circle fibrations to the space of complex structures, i.e. Hopf fibrations. Such a retraction is most likely to be found using a parabolic heat flow.
Right principal bundle | Semibasic 1-forms | Structure group
---|---|---
$B \rightarrow S^{2n+1}$ | $\Omega^p_0, \Omega^0_0 + \Omega^0_0$ | $\Gamma_0 = \left\{ \begin{pmatrix} g^0_0 & g^0_q \\ 0 & g^p_q \end{pmatrix}; g^0_0 \in \mathbb{R} \right\}$

| $B \rightarrow X$ | $\Omega^p_0$ | $\Gamma_1 = \begin{pmatrix} g^0_0 & g^0_q \\ 0 & g^p_q \end{pmatrix}$ |

| $G \rightarrow S^{2n+1}$ | $\Omega^p_0, \Omega^0_0$ | $G_0 = \begin{pmatrix} g^0_1 & g^0_j \\ 0 & g^j_1 \end{pmatrix}$ |

| $G \rightarrow \tilde{\mathrm{Gr}}_V$ | $\Omega^p_0, \Omega^p_0$ | $G_{circle} = \begin{pmatrix} g^0_0 & g^0_j \\ 0 & g^j_1 \end{pmatrix}$ |

| $G \rightarrow \mathcal{F}_V$ | $\Omega^0_0, \Omega^p_0, \Omega^p_0, \Omega^p_q$ | $GL(V, J_0)$ |

| $G \rightarrow S_V$ | $\Omega^0_0, \Omega^p_0, \Omega^p_0$ | $\begin{pmatrix} g^0_0 & -g^1_0 \\ g^1_0 & g^0_0 \\ 0 & g^j_1 \end{pmatrix}$ |

| $G \rightarrow M_V$ | $\Omega^p_0, \Omega^p_0, \Omega^0_0, \Omega^0_0, \Omega^0_q$ | $\Gamma_0$ |

Table 1. Actual structure groups are intersections of the above ones with $G = SL(V), V = \mathbb{R}^{2n+2}$.

| $\mu, \nu, \sigma = 1, \ldots, 2n+1$ |
| $i, j, k = 2, \ldots, 2n+1$ |
| $p, q, r = 1, \ldots, n$ |
| $P, Q, R = 0, \ldots, n$. |

Table 2. Index conventions

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