Gravity and Form Scattering and Renormalisation of Gravity in Six and Eight Dimensions

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Abstract

We calculate one-loop scattering amplitudes for gravitons and two-forms in dimensions greater than four. The string based Kawai-Lewellen-Tye relationships allow gravitons and two-forms to be treated in a unified manner. We use the results to determine the ultra-violet infinities present in these amplitudes and show how these determine the renormalised one-loop action in six and eight dimensions.

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1 Introduction

Quantum gravity [1] has proven a difficult theory to fit into the context of quantum field theory. Due to the dimensionful nature of its coupling constant,

\[ [\kappa^2] = (D - 2) \]  

any renormalisation of the theory must involve the introduction of new operators rather than a redefinition of the coupling constant. With increasing loop order increasingly higher dimension operators may appear and we obtain a theory described by an infinite set of operators which lacks predictive power. The only escape from such a scenario is if this process truncates after a finite number of loops and we call such a theory finite. The most natural assumption is that additional symmetries will be needed to forbid the presence of the potential counterterms. The search for a finite theory has led physicists in many diverse direction with mostly negative results. The sole spectacular candidate of a finite theory, including gravity, lies in superstring theory [2]. Although superstring theory is thought to be finite, the other issue, namely the determination of ultra-violet infinities in other theories has proved to be a very difficult problem with few concrete results. Unless a finite field theory of gravity can be constructed, gravity must be regarded as a low-energy effective theory of a more fundamental theory such as string theory. In this case the low-energy effective action will play the role of the counterterm action and by studying this we may hope to learn of the symmetries and properties of the fundamental theory.

In general, in \( D \)-dimensions, at \( L \) loops counterterms such as

\[ \nabla^n R^m \]  

appear where \( n + 2m = (D - 2)L + 2 \) and we have suppressed the indices on the Riemann tensor \( R_{abcd} \). We use forms of dimensional regularisation to evaluate the ultra-violet structure of a theory. (And thus only obtaining divergences in even dimensions.) There are two aspects to determining the counterterms. Firstly one can determine the possible counterterms consistent with the symmetries and secondly one must determine their coefficient by specific calculations.

At one-loop for \( D = 4 \), pure Einstein gravity is actually finite [3, 4], although matter coupled to gravity is not [5, 6]. Although matter coupled to gravity is ultra-violet divergent, the divergences do not appear in one-loop amplitudes with only external gravitons. Beyond one-loop it has been shown that pure gravity has a two-loop infinity, as first calculated by Goroff and Sagnotti [7] and later Van De Ven [8]. Matter in general does not improve renormalisability, however, special combinations can lead to cancellation of infinities. The best understood example of this are theories with supersymmetry which have much better ultra-violet properties. For example, \( N = 4 \) super-Yang-Mills is a finite theory [9] in \( D = 4 \) and supergravity theories are two-loop finite [10] in \( D = 4 \).

In this paper we calculate divergences appearing in amplitudes in dimensions higher than four at one-loop and examine the effect of matter upon the infinities which appear and examine whether there exist simplifying combinations of matter. We calculate amplitudes with mixtures of gravitons and antisymmetric two-forms and we determine the divergences appearing in physical on-shell amplitudes for which many specialised calculational techniques exist. String theory via the relations
first written down by Kawai, Lewellen and Tye [11] for tree amplitudes and later further developed for loop amplitudes [12] also allows, in some cases, the relatively easy computation of amplitudes involving gravity from amplitudes which involve gauge particles. (Alternative approaches involve the calculation of off-shell functions typically with a smaller number of legs.) We restrict ourselves to four-point amplitudes thus effectively only being sensitive to counterterms up to $\partial^n R^4$. We present particular helicity amplitudes which exhibit divergences in all (even) dimensions greater than four, thus indicating one-loop counterterms are always necessary (in even dimensions). We also use the divergences to evaluate the form of the counterterms in $D = 6$ and $D = 8$. The $D = 6$ one-loop result has been previously calculated as a precursor to calculating the two-loop $D = 4$ infinity since both of these have the same $R^3$ structure. In $D = 8$ we have evaluated the exact counterterm structure for comparison to that found in supersymmetric theories. For matter coupled gravity, the amplitudes with only external gravitons do not completely determine the counterterms which depend exclusively on the Riemann tensor and so we also evaluate amplitudes which are mixtures of gravitons and antisymmetric two-forms to enable us to fix the counterterms containing the Ricci tensor.

2 Organisation of the Amplitudes

2.1 Basic Theory

We consider the calculation of amplitudes with gravity minimally coupled to a variety of matter. For example the coupling to a complex scalar and a two-form is given by

$$S = \int d^D x \sqrt{|g|} \left[ \frac{2}{\kappa^2} R + \nabla_a \phi^* \nabla^a \phi + \frac{1}{6} F_{bcd} F^{bcd} \right]$$

(2.1)

where

$$F_{abc} = \nabla_a B_{bc} + \nabla_b B_{ca} + \nabla_c B_{ab}$$

(2.2)

and $B_{ab}$ is the two-form field which is antisymmetric. The field strength, $F_{abc}$, is invariant under

$$B_{ab} \rightarrow B_{ab} + \nabla_a \eta_b - \nabla_b \eta_a$$

(2.3)

We investigate the computation of scattering amplitudes in this theory focusing upon four-point on-shell one-loop amplitudes. In a gauge or gravitational theory smaller point amplitudes vanish on-shell and so the four-point amplitudes are the first non-trivial amplitudes. However, as we shall see they contain a great deal of information regarding the quantum theory.

We calculate amplitudes for dimensions $D > 4$ however, we can simplify the four-point case by using the four momenta to define a four dimensional hyper-plane in $D$ dimensions. With respect to this hyper-plane many of the well developed four dimensional organisational [13, 14] techniques can be applied to these calculations. One of the most useful techniques is that of spinor helicity which, unfortunately, does not easily generalise to $D > 4$. However, with respect to the four dimensional hyper-plane it can still prove a useful technique which we now describe.
2.2 $D > 4$ Spinor Helicity

In four dimensional gauge theory calculations, it is extremely useful to organise amplitudes according to the helicity of the external gluon or quark (or even scalar).

Furthermore one can use spinor helicity techniques [15, 13] where the polarisation vector of a gluon is realised as combinations of four dimensional Weyl spinors $|k^{\pm}\rangle$,

$$
\epsilon_{\mu}^{+}(k; q) = \frac{\langle q^- | \gamma_{\mu} | k^- \rangle}{\sqrt{2 \langle q k \rangle}}
\quad \epsilon_{\mu}^{-}(k; q) = \frac{\langle q^+ | \gamma_{\mu} | k^+ \rangle}{\sqrt{2 \langle k q \rangle}}
$$

where $k$ is the gluon momentum and $q$ is an arbitrary null ‘reference momentum’ which drops out of the final gauge-invariant amplitudes. The plus and minus labels on the polarization vectors refer to the gluon helicities and we use the notation $\langle ij \rangle \equiv \langle k_i^- | k_j^+ \rangle$, $[ij] \equiv \langle k_i^+ | k_j^- \rangle$. These spinor products are anti-symmetric and satisfy $\langle ij \rangle [ji] = 2k_i \cdot k_j \equiv s_{ij}$. For four-point amplitudes we use the usual Mandelstam variables $s = s_{12}$, $t = s_{14}$ and $u = s_{13}$.

Although spinor helicity is a four dimensional concept it can be used in higher dimensions. First consider the polarisation tensors for a $D$-dimensional vector particle. When considering four-point amplitudes, momentum conservation implies the first four dimensions can be defined so that the momenta of the scattered particles lie exclusively in this four dimensional hyper-plane. Defining

$$
x^a = (x^\mu; x^I)
$$

where $x^\mu$ denotes the coordinates of the four dimensional hyper-plane and $x^I$ are the remaining $(D - 4)$. The coordinates are chosen so

$$
k_i^I = 0
$$

for the four external momenta, $k_i$. Using this frame we can choose the helicity vectors $\epsilon_a$ to be of two types: $\epsilon_a^\pm$ and $\epsilon_a^I$ [16]

$$
\epsilon_a^\pm = (\epsilon_\mu^\pm; 0)
\quad \epsilon_a^I = (0; 0, \ldots, 0, 1, 0, \ldots, 0)
$$

which provide $(D - 2)$ independent polarisation vectors. These satisfy

$$
\epsilon^\pm \cdot \epsilon^I = 0, \quad k_i \cdot \epsilon^I = 0, \quad \epsilon^I \cdot \epsilon^J = -\delta^{IJ}
$$

We use the above polarisations vectors in $D$ dimensions to construct the graviton polarisation tensors, which are required to be symmetric, transverse and traceless.

For the four dimensional case there are only two graviton helicities whose polarisation tensors can be constructed from direct products of polarisations vectors [17, 18],

$$
\epsilon_{ab}^{++} = \epsilon_a^+ \epsilon_b^+
\quad \epsilon_{ab}^{--} = \epsilon_a^- \epsilon_b^-
$$

$\epsilon_{ab}$
In $D > 4$ the additional polarisation tensors may also be constructed from the polarisation vectors

\[ e^{+I}_{ab} = \frac{1}{\sqrt{2}} \left( \epsilon^{+a}_{a} \epsilon^{I}_{b} + \epsilon^{+a}_{b} \epsilon^{I}_{a} \right) \quad [D - 4] \]

\[ e^{-I}_{ab} = \frac{1}{\sqrt{2}} \left( \epsilon^{-a}_{a} \epsilon^{I}_{b} + \epsilon^{-a}_{b} \epsilon^{I}_{a} \right) \quad [D - 4] \]

\[ e^{IJ}_{ab} = \frac{1}{\sqrt{2}} \left( \epsilon^{I}_{a} \epsilon^{J}_{b} + \epsilon^{J}_{a} \epsilon^{I}_{b} \right) \quad [(D - 4)(D - 5)/2] \]

\[ e^{IJ}_{ab} = \frac{1}{\sqrt{3}} \left[ \epsilon^{I}_{a} \epsilon^{J}_{b} - \frac{1}{2} \left( \epsilon^{+a}_{a} \epsilon^{-a}_{b} + \epsilon^{-a}_{a} \epsilon^{+a}_{b} \right) \right] \quad [D - 4] \]

where $I \neq J$. The figures in square brackets refer to the number of independent such polarisations. Together with $\epsilon^{++}$ and $\epsilon^{--}$ they provide the necessary $(D - 2)(D - 1)/2 - 1$ polarisations.

We can also use spinor helicity techniques for the polarisation tensors of the antisymmetric two-form. In this case the polarisation tensors for the two-form, $B_{ab}$, must be transverse and antisymmetric. These can also be constructed from the polarisation vectors

\[ e^{+-}_{ab} = \frac{1}{\sqrt{2}} \left( \epsilon^{+a}_{a} \epsilon^{-a}_{b} - \epsilon^{-a}_{a} \epsilon^{+a}_{b} \right) \quad [1] \]

\[ e^{+I}_{ab} = \frac{1}{\sqrt{2}} \left( \epsilon^{+a}_{a} \epsilon^{I}_{b} - \epsilon^{I}_{a} \epsilon^{+a}_{b} \right) \quad [D - 4] \]

\[ e^{-I}_{ab} = \frac{1}{\sqrt{2}} \left( \epsilon^{-a}_{a} \epsilon^{I}_{b} - \epsilon^{I}_{a} \epsilon^{-a}_{b} \right) \quad [D - 4] \]

\[ e^{IJ}_{ab} = \frac{1}{\sqrt{2}} \left( \epsilon^{I}_{a} \epsilon^{J}_{b} - \epsilon^{J}_{a} \epsilon^{I}_{b} \right) \quad [(D - 4)(D - 5)/2] \]

providing $(D - 2)(D - 3)/2$ independent polarisations.

### 3 Kawai-Lewellen-Tye Relationships

The Kawai-Lewellen-Tye (KLT) [11] relationships express closed string tree amplitudes as sums of products of open string tree amplitudes. Heuristically there is a very obvious relationship between the amplitudes of closed and open strings since an open string amplitude may be written [2]

\[ A_{\text{open}} \sim \int dxK \quad (3.1) \]

where $K$ is a “kinematic factor” and a closed string amplitude may be written

\[ A_{\text{closed}} \sim \int d^{2}zK_{l} \times K_{r} \quad (3.2) \]

where the $K_{l}$ and $K_{r}$ are individually the kinematic factors for a open string theory. This heuristic argument suggests a relationship, however the suggested relationship is weaker than that contained in the KLT relations. (The proof is far from trivial.) For four and five-point amplitudes the KLT-relationship is

\[
M_{4}^{\text{tree}}(1, 2, 3, 4) = -\frac{i s_{12}}{4} A_{4}^{\text{tree}}(1, 2, 3, 4) A_{4}^{\text{tree}}(1, 2, 4, 3) \\
M_{5}^{\text{tree}}(1, 2, 3, 4, 5) = \frac{i s_{12} s_{24}}{8} A_{5}^{\text{tree}}(1, 2, 3, 4, 5) A_{5}^{\text{tree}}(2, 1, 4, 3, 5) + \frac{i s_{13} s_{24}}{8} A_{5}^{\text{tree}}(1, 3, 2, 4, 5) A_{5}^{\text{tree}}(3, 1, 4, 2, 5) 
\]

(3.3)
where \( M_4(1, 2, 3, 4) \) is a closed string amplitude and \( A_4(1, 2, 3, 4) \) are color-stripped open string partial amplitudes. These exact relationships between open and closed string tree amplitudes becomes, in the infinite string tension limit, a relationship between the field theory amplitudes for massless particles.

The \( M_n \)'s are the amplitudes in a gravity theory and the \( A_n \)'s are the color-ordered partial amplitudes in a gauge theory. The full gauge theory amplitude is obtained by multiplying the \( A_n \) by color-traces \([13, 19, 20]\)

\[
A_n^{\text{tree}}(1, 2, \ldots, n) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{Tr} \left( T^{\alpha_1} \ldots T^{\alpha_n} \right) A_n^{\text{tree}}(\sigma(1), \ldots, \sigma(n))
\]

(3.4)

where \( S_n/Z_n \) is the set of all permutations, but with cyclic rotations removed, and \( g \) is the gauge theory coupling constant. The \( T^\alpha \) are fundamental representation matrices for the Yang-Mills gauge group \( SU(N_c) \), normalized so that \( \text{Tr}(T^\alpha T^\beta) = \delta^{\alpha\beta} \). For states coupling with the strength of gravity, the full amplitude including the gravitational coupling constant is,

\[
M_n^{\text{tree}}(1, \ldots, n) = \kappa^{n-2} M_n^{\text{tree}}(1, \ldots, n)
\]

(3.5)

Consider the case where the massless open string states are vector bosons described by polarisation vectors \( \epsilon_i \). Then the open string amplitudes will be

\[
A_4^{\text{tree}}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)
\]

(3.6)

Using the KLT relationship with two such tree amplitudes, \( A_4^{\text{tree}}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \) and \( A_4^{\text{tree}}(\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3, \bar{\epsilon}_4) \), we form the combination

\[
M_4^{\text{tree}, P}(\epsilon_1, \bar{\epsilon}_1; \epsilon_2, \bar{\epsilon}_2; \epsilon_3, \bar{\epsilon}_3; \epsilon_4, \bar{\epsilon}_4) = -\frac{i S_{12}}{4} A_4^{\text{tree}}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) A_4^{\text{tree}}(\bar{\epsilon}_1, \bar{\epsilon}_2, \bar{\epsilon}_3, \bar{\epsilon}_4)
\]

(3.7)

which we will refer to as a primitive amplitude. This primitive amplitude corresponds to the scattering of massless states described by polarisation tensors \( \epsilon_i^{a b} = \epsilon_i^a \bar{\epsilon}_i^b \), which in general will not be irreducible states but will be a combination of the polarisation tensors of a graviton, a two-form and a scalar

\[
\epsilon_i^{a b} = \left[ \frac{1}{2}(\epsilon_i^a \bar{\epsilon}_i^b + \epsilon_i^b \bar{\epsilon}_i^a) - \eta^{a b} D \epsilon_i \cdot \epsilon_i \right] + \frac{1}{2}(\epsilon_i^a \bar{\epsilon}_i^b - \epsilon_i^b \bar{\epsilon}_i^a) + \eta^{a b} D \epsilon_i \cdot \epsilon_i
\]

(3.8)

Consequently the scattering amplitudes of irreducible states such as the graviton will be a linear combination of these primitive amplitudes.

We can also use the KLT for states without polarisation tensors, i.e. scalars. Specifically we can calculate the amplitude for two scalars and two gravitons where the graviton polarisation tensors are

\[
\epsilon_2^{a b} = \frac{1}{\sqrt{2}} \left( \epsilon_2^a \bar{\epsilon}_2^a + \epsilon_2^b \bar{\epsilon}_2^b \right) \quad \epsilon_3^{a b} = \frac{1}{\sqrt{2}} \left( \epsilon_3^a \bar{\epsilon}_3^a + \epsilon_3^b \bar{\epsilon}_3^b \right)
\]

where (for simplicity) \( \epsilon_i \cdot \bar{\epsilon}_i = 0 \), from the primitive amplitudes involving two scalars and two non-trivial polarisations

\[
M_4^{\text{tree}}(s_1; 2g; 3g; 4_s) = \frac{1}{2} \left[ M_4^{\text{tree}, P}(s_1; \epsilon_2, \bar{\epsilon}_2; \epsilon_3, \bar{\epsilon}_3; s_4) + M_4^{\text{tree}, P}(s_1; \epsilon_2, \bar{\epsilon}_2; \epsilon_3, \bar{\epsilon}_3; s_4) \right. \\
+ \left. M_4^{\text{tree}, P}(s_1; \bar{\epsilon}_2, \epsilon_2; \bar{\epsilon}_3, \epsilon_3; s_4) + M_4^{\text{tree}, P}(s_1; \bar{\epsilon}_2, \epsilon_2; \bar{\epsilon}_3, \epsilon_3; s_4) \right]
\]

(3.9)
where the primitive amplitudes may be calculated using the KLT relations. (Formalisms where one need not symmetrise between left and right helicities also exist for gravity [21].) Of the four terms in this expression there is a doubling up to give two separate terms because of a total symmetry between left and right,

\[ M_{\text{tree}}^4(s_1; 2_g; 3_g; 4_s) = M_{\text{tree}}^4(s_1; 2_g; 3_g; 4_s) \]

(3.10)

Primitive amplitudes can also generate amplitudes with external two-forms by antisymmetrising. For example for the graviton scattering we have

\[ M_{\text{tree}}^4(s_1; 2_g; 3_g; 4_s) = M_{\text{tree}}^4(s_1; 2_g; 3_g; 4_s) + M_{\text{tree}}^4(s_1; 2_g; 3_g; 4_s) \]

(3.11)

while for the antisymmetric tensor

\[ M_{\text{tree}}^4(s_1; 2_B; 3_B; 4_s) = M_{\text{tree}}^4(s_1; 2_B; 3_B; 4_s) - M_{\text{tree}}^4(s_1; 2_B; 3_B; 4_s) \]

(3.12)

As we can see, encompassed in the primitive amplitudes are the contributions corresponding to a variety of Feynman diagrams. Symmetrising or antisymmetrising projects to two rather different subsets of these. Diagrammatically

\[ M_{\text{tree}}^4(s; 2_g; 3_g; 4_s) + M_{\text{tree}}^4(s; 2_g; 3_g; 4_s) = \]

\[ M_{\text{tree}}^4(s; 2_B; 3_B; 4_s) - M_{\text{tree}}^4(s; 2_B; 3_B; 4_s) = \]

4 One-Loop Amplitudes

There are a variety of techniques for calculating on-shell loop amplitudes, often more efficient than a Feynman diagram approach. In our calculations, we use two quite different alternates to Feynman diagrams.

4.1 Cutkosky Cutting Technique

The optical theorem leads to the Cutkosky cutting rules [22] in field theory and it is possible to use these rules to determine amplitudes provided one evaluates the cuts to “all orders in \( \epsilon \)” [23, 20, 16, 24]. (This is within the context of dimensional regularisation where amplitudes are evaluated in \( D = 2N - 2\epsilon \).) These all-\( \epsilon \) results allow a complete reconstruction of the amplitude for a range of dimensions.
The cuts of a loop amplitude can be expressed in terms of amplitudes containing fewer loops. For example, the two-particle cut of a one-loop four-point amplitude in the $s$-channel, as shown in figure 4.1, can be expressed as a product of tree amplitudes

$$-i \text{ Disc } M_4^{1\text{-loop}}(1, 2, 3, 4) \big|_{s\text{-cut}} = \int dLIPS \sum_{\text{internal states,s}} M_4^{\text{tree}}(-L_s^1, 1, 2, L_s^3) M_4^{\text{tree}}(-L_s^3, 3, 4, L_s^1)$$  \hspace{1cm} (4.1)

where the $dLIPS$ denotes integrating over the exchange momenta $L_i$ subject to on-shell constraints and where $L_3 = L_1 - k_1 - k_2$ and the sum runs over all states crossing the cut. The right-hand-side can be rewritten as the cut of a covariant integral

$$\sum_{\text{internal states,s}} \int \frac{d^D L_1}{(2\pi)^D} \frac{i}{L_1^2} M_4^{\text{tree}}(-L_s^1, 1, 2, L_s^3) \times \frac{i}{L_3^2} M_4^{\text{tree}}(-L_s^3, 3, 4, L_s^1) \bigg|_{s\text{-cut}}$$  \hspace{1cm} (4.2)

We label $D$-dimensional momenta with capital letters and four-dimensional components with lower case letters. We apply the on-shell conditions, $L_1^2 = L_3^2 = 0$, to the amplitudes appearing in the cut even though the loop momentum is unrestricted; only functions with a cut in the given channel under consideration are determined in this way. By evaluating expressions with the correct cut in all channels the full amplitude is determined.

![Figure 4.1: The s-channel cut](image)

When evaluating graviton amplitudes in this way, the the KLT expressions may be used to replace the graviton tree amplitudes appearing in the cuts with products of gauge theory amplitudes. As an example, consider the specific case of a four graviton amplitude where all four external (outgoing) states have the polarisation tensor $\epsilon_{ab}^{++}$. Consider the one-loop amplitude where a complex scalar circulates in the loop. This amplitude has non-zero cuts in all three channels however, if we evaluate the $s$-channel the others may be obtained by symmetry.

The tree amplitudes we need are for two gravitons and two scalars, and these may be determined using the KLT relationships from gauge theory partial tree amplitudes with two external complex scalar legs and two gluons. This partial amplitude is

$$A_4^{\text{tree}}(-L_s^1, 1^+, 2^+, L_s^3) = -i \frac{\mu^2 \[12\]}{\langle 12 \rangle (\ell_1 - k_1)^2 - \mu^2}$$  \hspace{1cm} (4.3)

where we split the momenta into their four dimensional components and $(D - 4)$-dimensional components, $L_1 = \ell_1 + \mu_1$. Since the external momenta are purely four dimensional, $\mu_1 = \mu_3 \equiv \mu$. 
The overall factor of $\mu^2$ appearing in these tree amplitudes indicates that they vanish in the four-dimensional limit, in accord with a supersymmetry Ward identity [25]. Calculating the gravity amplitude,

$$M_4^{\text{tree}}(-L_1^s, 1_g^{++}, 2_g^{++}, L_3^s) = \frac{i}{4} A_4^{\text{tree}}(-L_1^s, 1^+, 2^+, L_3^s) A_4^{\text{tree}}(-L_1^s, 2^+, 1^+, L_3^s)$$

$$= \frac{i}{4} \frac{(\mu^2)^2}{(12)^2} \frac{|12|^2}{[(\ell_1 - k_1)^2 - \mu^2] [((\ell_1 - k_2)^2 - \mu^2]}$$

$$= -\frac{i}{4} \left( \frac{\mu^2}{\langle 12 \rangle} \right)^2 \times \left[ \frac{1}{(\ell_1 - k_1)^2 - \mu^2} + \frac{1}{(\ell_1 - k_2)^2 - \mu^2} \right]$$

where we have used the fact that $s + ((\ell_1 - k_1)^2 - \mu^2) + ((\ell_1 - k_2)^2 - \mu^2) = 0$.

Thus we have

$$\frac{i}{L_1^4} M_4^{\text{tree}}(-L_1, 1_g^{++}, 2_g^{++}, L_3^s) \times \frac{i}{L_3^4} M_4^{\text{tree}}(-L_3^s, 3_g^{++}, 4_g^{++}, L_1^s)$$

$$= \frac{1}{16} \frac{1}{L_1^4 L_3^4} \left( \frac{(\mu^2)^2}{\langle 12 \rangle \langle 34 \rangle} \right)^2 \left[ \frac{1}{(\ell_1 - k_1)^2 - \mu^2} + \frac{1}{(\ell_1 - k_2)^2 - \mu^2} \right] \left[ \frac{1}{(\ell_1 - k_3)^2 - \mu^2} + \frac{1}{(\ell_1 - k_4)^2 - \mu^2} \right]$$

$$= \frac{1}{16} \left( \frac{\mu^2}{\langle 12 \rangle \langle 34 \rangle} \right)^2 \left[ \frac{1}{(\ell_1 - k_1)^2 - \mu^2} + \frac{1}{(\ell_1 - k_2)^2 - \mu^2} \right] \left[ \frac{1}{(\ell_1 - k_3)^2 - \mu^2} + \frac{1}{(\ell_1 - k_4)^2 - \mu^2} \right]$$

In this expression there is an overall factor which does not depend upon the loop momentum, this multiplies an expression which is the product of four propagators with a factor of $(\mu^2)^4$ in the numerator. The four terms corresponds to the four different orderings of the legs 1234 which have a s-cut. This means

$$\int \frac{d^D L_1}{(2\pi)^D} \frac{i}{L_1^2} M_4^{\text{tree}}(-L_1, 1_g^{++}, 2_g^{++}, L_3) \times \frac{i}{L_3^2} M_4^{\text{tree}}(-L_3, 3_g^{++}, 4_g^{++}, L_1) \bigg|_{s\text{-cut}}$$

$$= \frac{2i}{16(4\pi)^{D/2}} \left( \frac{\mu^2}{\langle 12 \rangle \langle 34 \rangle} \right)^2 \times \left( I_{1234}^{D}[(\mu^2)^4] + I_{1243}^{D}[(\mu^2)^4] \right) \bigg|_{s\text{-cut}}$$

where

$$I_{1234}^{D}[X] \equiv -i(4\pi)^{D/2} \int \frac{d^D L}{(2\pi)^D} \frac{L^2(L - k_1)^2(L - k_1 - k_2)^2(L - k_1 - k_2 - k_3)^2}{X}$$

and where the terms have doubled up since $I_{1234}^{D}[(\mu^2)^4] = I_{1243}^{D}[(\mu^2)^4]$. This expression, by construction has the correct s-cut. The t and u channel cuts, in this case, can be obtained by relabeling and a combined expression can be formed by noting

$$\frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle} = \frac{[13][24]}{\langle 13 \rangle \langle 24 \rangle} = \frac{[14][23]}{\langle 14 \rangle \langle 23 \rangle} = \frac{-st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

which leads us to an expression which has the correct cuts, to all orders in $\epsilon$,

$$M^{1\text{-loop}}(1_g^{++}, 2_g^{++}, 3_g^{++}, 4_g^{++}) = \frac{2i}{16(4\pi)^{D/2}} \left( \frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2$$

$$\times \left( I_{1234}^{D}[(\mu^2)^4] + I_{1243}^{D}[(\mu^2)^4] + I_{1243}^{D}[(\mu^2)^4] \right)$$
This integral $I_{1234}^D[(\mu^2)^4]$ can be converted to a “shifted box integral” \[40\]

\[
I_{1234}^D[(\mu^2)^4] = \frac{(D-4)(D-2)(D)(D+2)}{16} I_{1234}^{D+8}
\]

This form of the amplitude is valid for all dimensions $D \geq 4$. In even dimensions, for example $D = 6, 8, 10, 12$, the shifted box integral is ultra-violet infinite,

\[
\begin{align*}
I_{1234}^{D=6}[(\mu^2)^4] \bigg|_{1/\epsilon} &= -\frac{2\epsilon.2.4.6}{16} \times \frac{1}{2\epsilon} \frac{2s^2 + st + 2t^2}{2520} \\
I_{1234}^{D=8}[(\mu^2)^4] \bigg|_{1/\epsilon} &= \frac{2.4.6.8}{16} \times \frac{1}{6\epsilon} \frac{u(3s^2 - 2st + 3t^2)}{30240} \\
I_{1234}^{D=10}[(\mu^2)^4] \bigg|_{1/\epsilon} &= \frac{6.8.10.12}{16} \times \frac{1}{120\epsilon} \frac{(s^4 - 4s^2t + 6st^2 - 8t^3 + 10t^4)}{4324320} \\
I_{1234}^{D=12}[(\mu^2)^4] \bigg|_{1/\epsilon} &= \frac{8.10.12.14}{16} \times \frac{1}{720\epsilon} \frac{(60s^6 + 10s^4t + 4s^2t^2 + 3s^3t^2 + 3s^3t^3 + 10s^4t^2 + 10st^5 + 60t^6)}{108972864000}
\end{align*}
\]

which produce infinities in the amplitude $\mathcal{M}^{1\text{-loop}}(1^{++}, 2^{++}, 3^{++}, 4^{++})$

\[
\begin{align*}
D = 6 : & \quad \frac{i\kappa^4}{8\epsilon(4\pi)^3} \left( \frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 \frac{s,t}{504} \\
D = 8 : & \quad \frac{i\kappa^4}{8\epsilon(4\pi)^4} \left( \frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 \frac{(s^2 + t^2 + u^2)^2}{15120} \\
D = 10 : & \quad \frac{i\kappa^4}{8\epsilon(4\pi)^5} \left( \frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 \frac{stu(t^2 + u^2 + s^2)}{31680} \\
D = 12 : & \quad \frac{i\kappa^4}{8\epsilon(4\pi)^6} \left( \frac{st}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right)^2 \frac{1}{4612608} \left( s^2 + t^2 + u^2 \right)^3 + \frac{109.8(stu)^2}{75}
\end{align*}
\]

This amplitude is finite in $D = 4$ but in all even dimension $D > 4$ it has non-vanishing ultra-violet infinities indication that the subtraction of ultra-violet infinities will require the introduction of counterterms in all even dimension larger than four thus precluding any possibility of a “magic” dimension where all infinities cancel.

Cutting techniques can also be used to provide exact expressions for the amplitudes involving two-forms. In Appendix B we demonstrate the computations leading to the expressions,

\[
\begin{align*}
\mathcal{M}^{1\text{-loop}}(1_I^J, 2_g^J, 3^K, 4^L) &= + \frac{i\kappa^4}{16(4\pi)^{D/2}} \left\{ I_{1234}^{D+8}(s, t) + I_{1243}^{D+8}(s, u) + I_{1423}^{D+8}(t, u) \right. \\
&\quad \left. - \left( I_3^{D+6}(s) + I_3^{D+6}(t) \right) + \frac{1}{8} \left( I_2^{D+4}(s) + I_2^{D+4}(t) \right) \right\} \\
\mathcal{M}^{1\text{-loop}}(1_I^J, 2_g^J, 3^L_g, 4_B^L) &= - \frac{i\kappa^4}{16(4\pi)^{D/2}} \left\{ \frac{1}{2} I_3^{D+6}(s) - \frac{1}{8} I_2^{D+4}(s) \right\} \\
\mathcal{M}^{1\text{-loop}}(1_B^J, 2_B^J, 3^L_B, 4_B^L) &= + \frac{i\kappa^4}{16(4\pi)^{D/2}} \left\{ \frac{1}{2} \left( I_2^{D+4}(s) + I_2^{D+4}(u) \right) \right\}
\end{align*}
\]

where $I_2^D(s)$ and $I_2^D(s)$ denote triangle and bubble integrals.
4.2 String Based Rules

The Bern-Kosower rules for evaluating QCD amplitudes [26] arose from the low-energy limit of string theory amplitudes. In conventional field theory they have been shown to be related to mixed gauge choices [27] and also to the “World-line formalism” [28]. The derivation of these rules and details of their validity and application will not be repeated here since several reviews are available [29, 20].

Since String theory exists most naturally in $D = 10$ or $D = 26$, the rules may be trivially adapted to $D \leq 10$, although the World-line formalism would suggest they are valid for all dimensions $D$.

The initial step in the rules is to draw all labeled $\phi^3$ diagrams, excluding tadpoles. The contribution from each labeled $n$-point $\phi^3$-like diagram with $n_l$ legs attached to the loop is

$$D = i \frac{(-\kappa)^n}{(4\pi)^{D/2}} \Gamma(n_l - D/2) \prod_{\ell=1}^{1} dx_{i_\ell} \prod_{i<j}^{n} \exp \left[ \frac{k_i \cdot k_j G^{ij}_B}{\pi} \right] \exp \left[ (k_i \cdot \epsilon_j - k_j \cdot \epsilon_i) \tilde{G}^{ij}_B - \epsilon_i \cdot \epsilon_j \tilde{G}^{ij}_B \right]$$

$$\times \exp \left[ (k_i \cdot \bar{\epsilon}_j - k_j \cdot \bar{\epsilon}_i) \tilde{G}^{ij}_B - \epsilon_i \cdot \epsilon_j \tilde{G}^{ij}_B \right] \left[ -(\epsilon_i \cdot \epsilon_j + \epsilon_j \cdot \epsilon_i) H^{ij}_B \right]_{\text{multi-linear}}$$

where the ordering of the loop parameter integrals corresponds to the ordering of the $n_l$ lines attached to the loop, $x_{ij} \equiv x_i - x_j$. The $x_{lm}$ are related to ordinary Feynman parameters by $x_{im} = \sum_{j=1}^{n} a_j$.

This expression corresponds to the expression one obtains in a Feynman diagram calculation after evaluating the vertex algebra and carrying out the loop momentum integral. The string based rules are algebraic rules for determining $K_{\text{red}}$ - the “reduced kinematic expression”, diagram by diagram from an overall kinematic expression

$$K = \int \prod_{i=1}^{n} dx_i \bar{dx}_i \prod_{i<j}^{n} \exp \left[ \frac{k_i \cdot k_j G^{ij}_B}{\pi} \right] \exp \left[ (k_i \cdot \epsilon_j - k_j \cdot \epsilon_i) \tilde{G}^{ij}_B - \epsilon_i \cdot \epsilon_j \tilde{G}^{ij}_B \right]$$

where the ‘multi-linear’ indicates that only the terms linear in all $\epsilon_i$ and $\bar{\epsilon}_i$ are included. The graviton polarization tensor is reconstructed from the $\epsilon_i^a \bar{\epsilon}_i^b$ as before. Although the above expression contains much information in string theory, when one takes the infinite string tension limit [26, 20] it should merely be regarded as a function which contains all the information necessary to generate $K_{\text{red}}$ for all graphs. The utility of the string based method partially lies in this compact representation (which is valid for arbitrary numbers of legs). The existence of an overall function which reduces to the Feynman parameter polynomial for each diagram is one of the most useful features of the string based rules.

As an example of the string based technique we can look at the four-point amplitude $\mathcal{M}^{1\text{-loop}}(1_g^-, 2_g^+, 3_g^+, 4_g^+)$ with a complex scalar circulating in the loop. This choice of helicity simplifies the kinematic expression considerably and we can deduce that the amplitude is given by (This was first calculated using the string-based technique of Bern and Kosower [26, 27] applied
to quantum gravity calculations [30, 31])

\[
\mathcal{M}^{1\text{-loop}}(g^-, g^+, g^+, g^+) = \frac{2i\kappa^4}{16} \left( \begin{array}{c} 2 \left[ 4 \right]^2 s^2 t \\ 12 \end{array} \right) \left( \begin{array}{c} 3 \left[ 4 \right] \left[ 4 \right] [41] \\ 23 \end{array} \right) \right)^2 \left( I_s^{(a)} + I_s^{(b)} + I_s^{(c)} + I_s^{(d)} + I_s^{(e)} \right)
\]

where

\[
I_s^{(a)} = \frac{\Gamma(4-D/2)}{(4\pi)^{D/2}} \int_0^1 \prod_{i=1}^4 da_i \delta \left( 1 - \sum_{i=1}^4 a_i \right) \frac{(a_1 + a_2)a_3^2 a_4^2}{(-s a_1 a_3 - ta_2 a_4)^{4-D/2}}
\]

\[
I_s^{(b)} = \frac{\Gamma(4-D/2)}{(4\pi)^{D/2}} \int_0^1 \prod_{i=1}^4 da_i \delta \left( 1 - \sum_{i=1}^4 a_i \right) \frac{(a_1 + a_2)a_3^2 a_4^2}{(-u a_1 a_3 - ta_2 a_4)^{4-D/2}}
\]

\[
I_s^{(c)} = \frac{\Gamma(4-D/2)}{(4\pi)^{D/2}} \int_0^1 \prod_{i=1}^4 da_i \delta \left( 1 - \sum_{i=1}^4 a_i \right) \frac{(a_1 + a_2)(a_3 + a_4)a_2^2 a_4^2}{(-s a_1 a_3 - u a_2 a_4)^{4-D/2}}
\]

\[
I_s^{(d)} = \frac{\Gamma(3-D/2)}{(4\pi)^{D/2}} \frac{1}{s} \int_0^1 \prod_{i=2}^4 da_i \delta \left( 1 - \sum_{i=2}^4 a_i \right) \frac{(a_2 a_3 a_4)^2}{(-s a_2 a_3)^{3-D/2}}
\]

\[
I_s^{(e)} = - \frac{\Gamma(3-D/2)}{(4\pi)^{D/2}} \frac{1}{u} \int_0^1 \prod_{i=2}^4 da_i \delta \left( 1 - \sum_{i=2}^4 a_i \right) \frac{(a_2 a_3 a_4)^2}{(-u a_2 a_3)^{3-D/2}}
\]

These yield a finite result in \(D = 4\), however in higher dimensions they give rise to ultra-violet infinities

\[
D = 6 : \quad \frac{i\kappa^4}{8(4\pi)^2} \left( \begin{array}{c} 2 \left[ 4 \right]^2 s^2 t \\ 12 \end{array} \right) \left( \begin{array}{c} 3 \left[ 4 \right] \left[ 4 \right] [41] \\ 23 \end{array} \right) \right)^2 \times \left( \begin{array}{c} t \\ 5040s t u \end{array} \right)
\]

\[
D = 8 : \quad \frac{i\kappa^4}{8(4\pi)^2} \left( \begin{array}{c} 2 \left[ 4 \right]^2 s^2 t \\ 12 \end{array} \right) \left( \begin{array}{c} 3 \left[ 4 \right] \left[ 4 \right] [41] \\ 23 \end{array} \right) \right)^2 \times (0)
\]

\[
D = 10 : \quad \frac{i\kappa^4}{8(4\pi)^3} \left( \begin{array}{c} 2 \left[ 4 \right]^2 s^2 t \\ 12 \end{array} \right) \left( \begin{array}{c} 3 \left[ 4 \right] \left[ 4 \right] [41] \\ 23 \end{array} \right) \right)^2 \times (0)
\]

\[
D = 12 : \quad \frac{i\kappa^4}{8(4\pi)^4} \left( \begin{array}{c} 2 \left[ 4 \right]^2 s^2 t \\ 12 \end{array} \right) \left( \begin{array}{c} 3 \left[ 4 \right] \left[ 4 \right] [41] \\ 23 \end{array} \right) \right)^2 \times \left( \begin{array}{c} t \\ 3783 t 800 \end{array} \right)
\]

Again we see the presence of ultra-violet infinities in higher dimensional one-loop amplitudes however, in this case they vanish for \(D = 8\) and \(D = 10\) indicating that the counterterms must have a form which does not contribute to this amplitude. In fact, as we see later, all of the possible counterterms which are consistent with the symmetries of gravity in \(D = 8\) have vanishing contributions for this particular helicity configuration.

Including the two examples we have just calculated, there are sixty-nine independent, non-vanishing helicity configurations for four external gravitons in \(D > 4\) dimensions. (Amplitudes not listed are either zero to all orders or obtainable from the list by relabeling or complex conjugation.) The tree amplitudes of these are listed in Appendix A. The string based rules may be used to calculate the loop amplitude for any of these. The ultra-violet infinities in \(D = 6, 8, 10\) for the first thirty-one of these amplitudes is given in Appendix C. This subset of the amplitudes provides more than sufficient information to determine the counterterms necessary to cancel the infinities in four graviton amplitudes. In the following sections we shall detail this process for \(D = 6, 8\).
The string based rules can be applied to determine the contributions to amplitudes for particle types other than that of scalars circulating in the loop. This corresponds to applying different algebraic rules in determining $K_{\text{red}}$. This will allow us to determine infinities in the amplitudes and hence the counterterms induced by other particle types.

5 Counterterms

In this section we enumerate the possible independent counterterms in six and eight dimensions and show how the results of the one-loop amplitude calculations determine the various coefficients.

5.1 Symmetries

In general, graviton scattering amplitudes, in $D$ dimensions at $L$ loops, are rendered ultra-violet finite by the introduction of counterterms of the form

$$\nabla^n R^m$$

where $n + 2m = (D - 2)L + 2$ and we have suppressed the indices on $R$. $R$ may stand either for the Riemann tensor, $R_{abcd}$, the Ricci tensor $R_{ab} \equiv g^{cd} R_{acbd}$ or the curvature scalar $R \equiv g^{ab} R_{ab}$. Although, there are a large number of tensor structures which may appear, fortunately, the symmetries of the Riemann tensor reduce these considerably. Firstly, there are the basic symmetries of $R_{abcd}$

$$R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}$$

and the cyclic symmetry,

$$R_{abcd} + R_{acdb} + R_{adbc} = 0$$

Secondly, we have the Bianchi identity for $\nabla_e R_{abcd}$,

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0$$

There are also “derivative symmetries” which involve two covariant derivatives,

$$\nabla_e \nabla_f R_{abcd} - \nabla_f \nabla_e R_{abcd} = R_{aej} R_{gbdf} + R_{bfj} R_{agcd} + R_{cej} R_{abgd} + R_{dfj} R_{abcg}$$

$$\nabla^2 R_{abcd} = 2 R_{ace} R^d_{bfj} - 2 R^j_{bce} R^e_{daf} - R^e_{dab} R_{ce} + R^e_{cab} R_{de}$$

$$+ \nabla_c \nabla_a R_{bd} - \nabla_c \nabla_b R_{ad} - \nabla_d \nabla_a R_{bc} + \nabla_d \nabla_b R_{ac}$$

These symmetries will be used to determine the minimal set of inequivalent counterterms.
5.2 Graviton and Two-Form Scattering in $D = 6$

By power counting the possible counterterms in $D = 6$ are of the form $R^3$ or $\nabla^2 R^2$. The independent terms involving $R_{abcd}$, $R_{ab}$ and $R$ are [32, 33],

\[
\begin{align*}
T_1 &= \nabla_a R \nabla^a R \\
T_2 &= \nabla_a R_{bc} \nabla^a R^{bc} \\
T_3 &= \nabla_e R_{abcd} \nabla^e R^{abcd} \\
T_4 &= \nabla_e R_{ab} \nabla^b R^{ac} \\
T_5 &= R^3 \\
T_6 &= R R_{ab} R^{ab} \\
T_7 &= R R_{abcd} R^{abcd} \\
T_8 &= R_{abcd} R^{abce} R^{de} \\
T_9 &= R_{abcd} R^{ac} R^{bd} \\
T_{10} &= R_a^b R_b^c R_c^a \\
T_{11} &= R_{cd}^e R_{ef}^c R_{ab}^{ef} \\
T_{12} &= R_{abcd} R^{a} R^{c} R^{bedf} 
\end{align*}
\]

(For $D = 4$ only ten of this set are independent.) For the case of pure gravity, the counterterm structure can be represented as a single counterterm with a numerical coefficient. This numerical coefficient has been calculated previously [34]. We review the argument leading to the conclusion that a single counterterm is sufficient. When matter is coupled to gravity this conclusion no longer follows.

For pure gravity the equation of motion is

\[ R_{ab} = 0 \] (5.7)

Hence terms involving the Ricci tensor or curvature scalar

\[ R_{ab} X^{ab} , \quad R X = R_{ab} (g^{ab} X) \] (5.8)

will not contribute to the $S$-matrix and such terms can be discarded when calculating the counterterms. If calculating an off-shell object, such counterterms can, and do, appear. Ignoring such terms leaves us with three tensors - $T_3$, $T_{11}$ and $T_{12}$. The term $T_3$

\[ \nabla_e R_{abcd} \nabla^e R^{abcd} = - R_{abcd} \nabla^2 R^{abcd} \] (5.9)

can be rearranged using the identity in eq. (5.5) into terms involving the Ricci tensor plus cubic terms in the Riemann tensor. Thus for pure gravity this term is equivalent to a combination of $T_{11}$ and $T_{12}$ and can thus be eliminated from the list of inequivalent counterterms.

In six dimensions the scalar topological density can be written

\[
\delta_{[mnprqs]}^{abcd} R_{ab}^{mn} R^{pq}_{cd} R^{rs}_{ef} \] (5.10)
which implies the combination
\[ \sum_{i=5}^{12} a_i T_i = 0 \quad (5.11) \]
is topological for some coefficients \(a_i\). Hence for pure gravity amplitudes we can replace \(T_{12}\) for \(T_{11}\) (or vice versa). Thus we are led to the fact that the counterterm can be taken as a single tensor with a coefficient. This argument also applies to the two-loop case of pure gravity in \(D = 4\) [7]. In both this calculation and that of \(D = 6\) pure gravity the counterterm was chosen to be
\[ c \int d^6 x R^{ab}_{\ c d} R^{cd}_{\ e f} R^{e f}_{\ ab} \quad (5.12) \]

For our case we are considering gravity amplitudes with scalar loops. For gravity coupled to matter the field equation is
\[ R_{ab} - \frac{\kappa^2}{4(D-2)} \left[ g_{ab} T_c^c - (D-2) T_{ab} \right] = 0 \quad (5.13) \]
so counterterms involving the Ricci tensor can no longer trivially be dropped. However we shall always be able to make the replacement
\[ R_{ab} X^{ab} \longrightarrow \frac{\kappa^2}{4(D-2)} \left[ g_{ab} T_c^c - (D-2) T_{ab} \right] X^{ab} \quad (5.14) \]
without changing the \(S\)-matrix. The right-hand-side involves at least two matter fields and thus does not contribute to pure graviton amplitudes, but may contribute to amplitudes involving two gravitons and two matter fields. Thus we may still neglect counterterms involving the Ricci tensor provided we are restricting attention to external gravitons. (This is similar to the situation in \(D = 4\).) Thus we are led to the same conclusion as for pure gravity in that the infinities can be renormalised by a single counterterm.

Knowing the counterterm is unique we can fix \(c\) from a single amplitude - providing the amplitude is non-zero for that term. Either of the amplitudes we presented earlier, \(\mathcal{M}_{1\text{-loop}}(1^{-}, 2^{++}, 3^{++}, 4^{++})\) and \(\mathcal{M}_{1\text{-loop}}(1^{++}, 2^{++}, 3^{++}, 4^{++})\), would be sufficient to determine the coefficient. Thus from either of these amplitudes we can confirm the non-vanishing of the counterterm and extract the coefficient (The value we obtain matches that of all the amplitudes we calculate in Appendix C.)
\[ c = -\frac{1}{(4\pi)^3\epsilon} \times \frac{1}{15120} \quad (5.15) \]

This counterterm will make amplitudes with a complex scalar loop and external gravitons finite. As we shall see later the pure gravity case will simply be \(9/2\) times this. Multiplying this by a factor of \(9/2\) does indeed give the previously calculated result.

When considering amplitudes other than pure graviton scattering the single counterterm above will not be sufficient to cancel the infinities. To fully determine all the coefficients it would be necessary to compute six-point amplitudes involving, for example, six \(B_{ab}\) or scalar fields since terms such as \(T_5\) can be replaced by tensors involving six matter fields. Alternatively one could say these terms are unnecessary to cancel the infinities in four-point amplitudes. However, some of
the counterterms involving the Ricci tensor will need to be introduced to cancel infinities in four-point amplitudes where some of the external states are matter states. For example, if we consider amplitudes with two external two-forms and two external gravitons (still with a scalar loop). This is computationally fairly straightforward since within string theory, the graviton and antisymmetric two-form are very closely tied together. Using string based rules this means that the amplitudes involving two-forms are very closely related to the amplitudes involving gravitons - the amplitude is formed from the same primitive amplitudes but with different signs. From a more traditional field theory view it would also be relatively easy. The antisymmetric tensor does not couple to the scalar field directly so the Feynman graphs are of the form,

As we can see from this diagram, this is equivalent to probing the off-shell graviton three-point function. In six dimensions if we trace the equations of motion eq. (5.13)

\[
g^{ab} R_{ab} = \frac{\kappa^2}{4(D-2)} g^{ab} \left[ g_{ab} T_c - (D - 2) T_{ab} \right]
\]

\[
= g^{ab} \left( \frac{\kappa}{2} \right)^2 \left[ \frac{2}{3(D-2)} g_{ab} F_{cde} F^{cde} - F_{acd} F_{bc} - 2 \partial_a \phi \partial_b \phi \right]
\]

\[
= \left( \frac{\kappa}{2} \right)^2 (-2 \partial_a \phi \partial_b \phi)
\]

the two-form is eliminated. Hence a counterterm involving the curvature scalar \( R \) can be replaced by a counterterm quadratic in the scalar field and such counterterms will not contribute to amplitudes with external two-forms. The counterterms which will give non-vanishing contributions to amplitudes with two gravitons and two 2-forms are \( T_3, T_8, T_11 \) and \( T_{12} \). This set of four tensors are not independent and we can use the previous argument for eliminating \( T_3 \) and \( T_{12} \) from the minimal set of tensors leaving the two counterterms \( T_8 \) and \( T_{11} \). The coefficient of \( T_{11} \) is fixed by the amplitudes with four external gravitons. The counterterm

\[
T_8 = R_{abcd} R^{abce} R^{de}
\]

is equivalent, for these amplitudes, to the tensor

\[
\left( \frac{\kappa}{2} \right)^2 R_{abcd} R^{abce} R^{de}
\]

The infinities in a sufficiently large set of two graviton and two 2-form amplitudes are given in the Appendix D. Both \( T_8 \) and \( T_{11} \) contribute to these amplitudes and the following combination of these counterterms is needed to cancel both these infinities and those of the four graviton amplitudes

\[
- \frac{2}{(4\pi)^3} \left\{ \frac{1}{30240} R_{cd} R^{cd}_{ef} R^{ef}_{ab} + \frac{1}{1008} \left( \frac{\kappa}{2} \right)^2 R_{abcd} R^{abce} \left[ \frac{1}{6} g_{de} F_{fg} F^{fg} - F_{df} F_{ef} \right] \right\}
\]

(5.19)
or if we wish to expressly write this as $R^3$ terms,

$$-\frac{2}{(4\pi)^3\epsilon} \left\{ \frac{1}{30240} R_{cd}^{ab} R_{e f}^{c d} R_{ab}^{ef} + \frac{1}{1008} R_{abcd} R_{bcde} R_{e}^{cd} \right\} \quad (5.20)$$

We can further probe the counterterm action by calculating amplitudes with four external two-forms. Only the tensors

\[ T_1 = \nabla_a R \nabla^a R \quad T_2 = \nabla_a R_{bc} \nabla^a R^{bc} \]

will contribute to the scattering of four matter states. Due to the fact that in $D = 6$ the equation of motion for $R$ does not depend upon the two-form, only $T_2$ contributes to that of four two-form states. Replacing $R_{ab}$ using the equation of motion, $T_2$ is equivalent to

\[ \left( \frac{\kappa}{2} \right)^4 \partial_a \left( F_{bde} F_c^{de} \right) \partial^a \left( F_{gh}^b F^{cgh} \right) - \frac{1}{6} \partial_a \left( F_{bde} F^{bde} \right) \partial^a \left( F_{fgh} F^{fgh} \right) \]

or equivalently

\[ -\frac{2}{(4\pi)^3\epsilon} \left\{ \frac{1}{30240} \nabla_a R_{bc} \nabla^a R^{bc} \right\} \quad (5.24) \]

In Appendix D the infinities for a set of four 2-forms amplitudes are presented. (The tensors $T_8$ and $T_{11}$ do not contribute to these amplitudes.) Canceling these divergences fixes the coefficient of $T_2$,

\[ -\frac{2}{(4\pi)^3\epsilon} \left\{ \frac{1}{30240} \nabla_a R_{bc} \nabla^a R^{bc} \right\} \quad (5.25) \]

Thus we can conclude that the counterterms necessary to make amplitudes with external gravitons or two-forms finite is

\[ -\frac{2}{(4\pi)^3\epsilon} \left\{ \frac{1}{30240} R_{cd}^{ab} R_{e f}^{c d} R_{ab}^{ef} + \frac{1}{1008} R_{abcd} R_{bcde} R_{e}^{cd} + \frac{1}{1680} \nabla_a R_{bc} \nabla^a R^{bc} \right\} \quad (5.25) \]

This is the counterterm generated by a complex scalar loop. In the following section we examine the effects of having more complicated particle types circulating in the loop.

### 5.3 Counterterms Generated by More General Matter in $D = 6$

We have considered the counterterms generated by a single complex scalar. The results for more general matter combinations are very closely related to the complex scalar case. If we have minimally coupled matter the resultant counterterm is

\[ \mathcal{L} = \frac{(N_B - N_F)}{2} \times \mathcal{L}_{\text{Scalar}} \]

where $N_B$ is the number of bosonic degrees of freedom and $N_F$ is the number of fermionic degrees of freedom.

Our argument leading to this simple result is actually rather complicated and uses the string based rules for graviton scattering. These algebraic rules are for generating Feynman parameter
integrals as discussed in section 4.2. The rules can generate the contributions for different matter combinations.

These rules are based upon string theory amplitudes in $D = 10$. In $D = 10$ language there are three underlying types of contributions which we label $[S]$, $[V]$ and $[F]$. In terms of particle content these correspond to the contributions from the $1$, $8_v$ and $8_s/c$ representations of $SO(8)$. For a closed string, which has left and right moving quanta, we have the option of different $SO(8)$ representations for left and right. So the rules for gravity generate contributions corresponding to the product of these representations. In terms of particle content in $D = 10$, this corresponds to

$$ [S; S] = 1 \otimes 1 \equiv \phi $$

$$ [V; S] = 1 \otimes 8_v \equiv A_a $$

$$ [V; V] = 8_v \otimes 8_v = 1 + 28 + 35 \equiv \phi + A_{ab} + g_{ab} $$

$$ [S; F] = 1 \otimes 8_s = 8_s \equiv \lambda $$

$$ [V; F] = 8_v \otimes 8_s = 8_c + 56_s \equiv \psi_a + \lambda $$

$$ [F; F] = 8_s \otimes 8_s = 1 + 28 + 35 \equiv \phi + A_{ab} + A_{abcd}^{SD} $$

$$ [F; \bar{F}] = 8_s \otimes \bar{8}_s = 8_c + 56_s \equiv A_a + A_{abc} $$

where $A_{abcd}^{SD}$ is a self-dual four-form field and $\phi$ is a real scalar field.

In $D = 10$ “knowing” the contributions from the above combinations of matter does not actually allow us to determine the contribution due to a single particle type - the five contributions $[S; S]$, $[V; S]$, $[V; V]$, $[F; F]$ and $[F; \bar{F}]$ cannot be disentangled to determine the contributions from the six individual particles - $\phi$, $g_{ab}$, $A_a$, $A_{ab}$, $A_{abc}$ and $A_{abcd}^{SD}$. However, in $D < 10$ the five basic combinations may be sufficient to determine the contributions from all the particle types. For example, in $D = 4$ the antisymmetric tensors will all reduce to combinations of three basic particles - $\phi$, $A_a$ and $g_{ab}$ and in this case there is enough information to (over)determine the three basic particle types.

In $D = 6$ it transpires there is just enough information to determine the contributions from the five basic bosonic particle types. The string contributions, for the bosonic terms, will be

$$ [S; S] = 1 \otimes 1 \equiv \phi $$

$$ [V; S] = 1 \otimes 8_v \equiv A_a + \sum_{i=1}^4 \phi^i $$

$$ [V; V] = 8_v \otimes 8_v \equiv \sum_{i=1}^{17} \phi^i + A_{ab} + g_{ab} + \sum_{i=1}^8 A_i^a $$

$$ [S; F] = 1 \otimes 8_s = \sum_{i=1}^2 \phi^i + \sum_{i=1}^4 A_i^a + \sum_{i=1}^6 A_i^{ab} $$

$$ [V; F] = 8_v \otimes 8_s = 8_c + \sum_{i=1}^6 A_i^{ab} + \sum_{i=1}^7 A_i^a + \sum_{i=1}^8 A_i^{abc} $$

$$ [F; F] = 8_s \otimes 8_s = A_{abc} + \sum_{i=1}^6 A_i^{ab} + \sum_{i=1}^7 A_i^a + \sum_{i=1}^8 \phi^i $$

In this expression the combinations $[F; F]$ and $[F; \bar{F}]$ correspond to identical sets of fields in $D = 6$. This is because the two type II supergravities are dual when compactified to $D < 10$ [35]. This
then leaves us with four independent pieces of information. This system is solvable because the three-form $A_{abc}$ is dual to a vector $A_a$ in $D = 6$ which means we have only four independent field types - $\phi$, $A_a$, $A_{ab}$ and $g_{ab}$. The fermionic terms easily allow us to determine the contribution from the spinor $\lambda$ and gravitino $\psi_a$.

This tells us that if we can determine the string based contributions we can solve to obtain the individual particle types. The explicit results of calculations can be summarised for the ultra-violet infinities,

$$
\begin{align*}
[V; S] &= + 8 \{S; S\} \\
[V; S] &= + 64 \{S; S\} \\
[F; F] &= [F; F] = + 64 \{S; S\} \\
[S; F] &= - 8 \{S; S\} \\
[V; F] &= - 64 \{S; S\}
\end{align*}
$$

(5.29)

The solution to this system of equations is that the infinities in the one-loop amplitude from a set of particles, $P$, is given by

$$
M^{1-\text{loop}}_P \bigg|_{1/\epsilon} = \frac{(N_B - N_F)}{2} M^{1-\text{loop}}_{\text{scalar}} \bigg|_{1/\epsilon}
$$

(5.30)

This relationship then obviously extends to the counterterm Lagrangian.

In four dimensions a similar string based argument holds, however, a more elegant supersymmetry argument can be used to achieve equivalent results. In four dimensions the helicity amplitude with all-plus helicities can be shown to vanish in any supersymmetric theory

$$
M^{1-\text{loop,susy multiplet}}(1^{++}, 2^{++}, 3^{++}, 4^{++}) = 0
$$

(5.31)

This applies to all supersymmetries $N \geq 1$. Since $N = 1$ multiplets are actually rather simple this relationship easily allows one to deduce for all particle types

$$
M^{1-\text{loop}}_P(1^{++}, 2^{++}, 3^{++}, 4^{++}) = \frac{(N_B - N_F)}{2} M^{1-\text{loop}}_{\text{scalar}}(1^{++}, 2^{++}, 3^{++}, 4^{++})
$$

(5.32)

This relationship is true of entire amplitudes and not merely the infinities.

## 5.4 Graviton Scattering in $D = 8$

For $D = 8$ the possible counterterms are of the form $\nabla^4 R^2$, $\nabla^2 R^3$ and $R^4$. As we shall see, for external graviton amplitudes the set of inequivalent counterterms can be constructed entirely using the $R^4$ counterterms.

First, recall that in purely graviton amplitudes terms involving the Ricci tensor and curvature scalar do not contribute leaving us with terms involving the Riemann tensor only. Consider the terms quadratic in the Riemann tensor. There are various possibilities for the indices of these tensors but we can organise these into three types depending on how many contractions the Riemann tensors have with each other. Representatives of the three types are,

$$
\begin{align*}
\nabla^c \nabla^d R_{abcd} &\nabla^e \nabla^f R^{abcdef} \\
\nabla^c \nabla^d R_{abcd} &\nabla^f \nabla^e R^{abce} \\
\nabla^c \nabla^d R_{abcd} &\nabla^e \nabla^f R^{abcd}
\end{align*}
$$

(5.33)
We have chosen the representatives such that there are no contractions between the derivatives and
the tensor they act upon. For such terms we can use the Bianchi identity
\[ \nabla^a R_{abcd} = -\nabla_c R_{a b d}^a - \nabla_d R_{a b c}^a = \nabla_c R_{bd} - \nabla_d R_{bc} \] (5.34)
to equate this to Ricci tensors which we discard. The order of derivatives can also be changed - but
at the expense of \( \nabla^2 R^3 \) terms eq. (5.5). Thus the generic terms are equivalent to the representative
terms given. We now show that these can be eliminated from the list of counterterms in favour of
\( \nabla^2 R^3 \) and \( R^4 \) terms. Using the antisymmetry of the \( e f \) indices the first term can be rewritten
\[ \nabla_e \nabla_f R_{abcd} \nabla^c \nabla^d R^{abef} = \frac{1}{2} (\nabla_e \nabla_f R_{abcd} - \nabla_f \nabla_e R_{abcd}) \nabla^c \nabla^d R^{abef} \equiv \nabla^2 R^3 \] (5.35)
For terms of second and third type we can commute \( \nabla \) (at the cost of creating \( \nabla^2 R^3 \) terms) and
integrate by parts to bring the contracted derivatives together. Acting on a Riemann tensor, equation (5.5) shows that such terms are equivalent to \( \nabla^2 R^3 \).

Turning to the \( \nabla^2 R^3 \) terms, there are four tensors involving the Riemann tensor. The normal
form of these is \[33\],
\begin{align*}
S_1 &= R_{abcd} \nabla_e R_f^a g^c \nabla^e R^{fbdg} \\
S_2 &= R_{abcd} \nabla^c R_{efg} a^d R^{efgb} \\
S_3 &= R_{abcd} \nabla^b R_{efg} a^d R^{efgc} \\
S_4 &= R_{abcd} R^{a e f g} \nabla^d R^{b c e f}
\end{align*}
(5.36)
In manipulating these terms, commuting derivatives will produce terms involving the Ricci or \( R^4 \) so
this can be done at will. Taking the \( S_1 \) first, integrating by parts yields
\[ S_1 = R_{abcd} \nabla_e R_f^a g^c \nabla^e R^{fbdg} = -\nabla_e R_{abcd} R_f^a g^c \nabla^e R^{fbdg} - R_{abcd} R_f^a g^c \nabla^2 R^{fbdg} \] (5.37)
so that
\[ S_1 = -\frac{1}{2} R_{abcd} R_f^a g^c \nabla^2 R^{fbdg} \] (5.38)
which is equivalent to \( R^4 \) terms. The second term
\[ S_2 = R_{abcd} \nabla^c R_{efg} a^d R^{efgb} \equiv -R_{abcd} R_{efg} a^c \nabla^d R^{efgb} = -\frac{1}{2} R_{abcd} R_{efg} a^c (\nabla^d \nabla^e - \nabla^e \nabla^d) R^{efgb} \] (5.39)
which is equivalent to \( R^4 \). For \( S_3 \) we expand the first Riemann tensor using its cyclic symmetry
\[ S_3 = R_{abcd} \nabla^b R_{efg} a^d R^{efgc} \]
\[ = -(R_{abcd} + R_{adbc}) \nabla^b R_{efg} a^d R^{efgc} \] (5.40)
By relabeling this is equal to
\[ R_{abcd} \nabla^b R_{efg} a^d (\nabla^d R^{efgc} - \nabla^c R^{efgd}) = R_{abcd} \nabla^b R_{efg} a^d (\nabla^d R^{efgc} + \nabla^c R^{efgd}) \]
\[ = -R_{abcd} \nabla^b R_{efg} a^d g R^{efcd} \] (5.41)
Taking the middle term out and noting that it multiplies a term which is antisymmetric under exchange of \( ab \)

\[
S_3 = -\nabla^b R_{efg}^a R_{abcd} \nabla^g R^{efcd}
\]

\[
= -\frac{1}{2} (\nabla^b R_{efg}^a - \nabla^a R_{efg}^b) R_{abcd} \nabla^g R^{efcd}
\]

\[
= +\frac{1}{2} \nabla_g R_{ef}^a b R_{abcd} \nabla^g R^{efcd}
\]

\[
= +\frac{1}{2} R_{abcd} \nabla_g R_{ef}^a b \nabla^g R^{efcd}
\]

Integrating by parts

\[
= -\frac{1}{2} \nabla_g R_{abcd} R_{ef}^a b \nabla^g R^{efcd} - \frac{1}{2} R_{abcd} R_{ef}^a b \nabla^2 R^{efcd}
\]

Taking the first term and integrating by parts with respect to the second \( g \)

\[
-\frac{1}{2} \nabla_g R_{abcd} R_{ef}^a b \nabla^g R^{efcd} = \frac{1}{2} \nabla_g R_{abcd} \nabla^g R_{ef}^a b \nabla^2 R^{efcd} + \frac{1}{2} \nabla^2 R_{abcd} R_{ef}^a b \nabla^{efcd}
\]

The leading term is merely a relabeling of the original so that

\[
-\frac{1}{2} \nabla_g R_{abcd} R_{ef}^a b \nabla^g R^{efcd} = \frac{1}{4} \nabla^2 R_{abcd} R_{ef}^a b \nabla^{efcd}
\]

and so

\[
S_3 = \frac{1}{4} \nabla^2 R_{abcd} R_{ef}^a b \nabla^{efcd} - \frac{1}{2} R_{abcd} R_{ef}^a b \nabla^2 R^{efcd}
\]

Since the \( \nabla^2 R_{abcd} \) leads to a combination of \( R^2 \) tensors and derivatives acting upon Ricci tensors then the term \( S_3 \), for external graviton states, is equivalent to \( R^4 \) tensors.

For the last tensor,

\[
S_4 = R_{abcd} R_{efg}^a \nabla^g \nabla^d R^{becf} \equiv -\nabla^g R_{abcd} R_{efg}^a \nabla^d R^{becf}
\]

\[
= -\frac{1}{2} \nabla^g R_{abcd} R_{efg}^a (\nabla^d R^{becf} - \nabla^c R^{bedf})
\]

\[
= -\frac{1}{2} \nabla^g R_{abcd} R_{efg}^a (\nabla^d R^{becf} + \nabla^c R^{bedf})
\]

\[
= +\frac{1}{2} \nabla^g R_{abcd} R_{efg}^a \nabla^f R^{bedc}
\]

\[
= -\frac{1}{2} S_2
\]

hence we can drop \( S_4 \) also. Thus we are led to the conclusion that, for pure graviton amplitudes, infinities can be removed by the introduction of purely \( R^4 \) counterterms.

From [33] the general \( R^4 \) counterterm is

\[
\frac{i}{(4\pi)^4 \epsilon} \left[ a_1 T_1 + a_2 T_2 + a_3 T_3 + a_4 T_4 + a_5 T_5 + a_6 T_6 + a_7 T_7 \right]
\]

(5.48)
where

\[ T_1 = (R_{abcd} R^{abcd})^2 \]
\[ T_2 = R_{abcd} R^{abc} \epsilon_{R fgh} \epsilon_{R} \]
\[ T_3 = R^{ab}_{\quad cd} R^{cde}_{\quad f} R_{gh}^f R_{gh}^{ab} \]
\[ T_4 = R_{abcd} R^{def}_{\quad e} R_{gh}^e R_{def}^{gh} \]
\[ T_5 = R_{abcd} R^{ef}_{\quad e} R_{g}^e R_{h}^{ef} \]
\[ T_6 = R_{abcd} R^{ac}_{\quad e} R_{g}^e R_{h}^{ac} R_{gh}^{ab} \]
\[ T_7 = R_{abcd} R^{ae}_{\quad e} R_{g}^e R_{h}^{ae} R_{gh}^{ab} \] (5.49)

Additionally the combination

\[ -\frac{T_1}{16} + T_2 - \frac{T_3}{8} - T_4 + 2T_5 - T_6 + 2T_7 \] (5.50)

vanishes on-shell due to it being proportional to the Euler form

\[ E \sim \epsilon_{a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8} \epsilon_{b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8} R^{a_1 a_2}_{\quad b_1 b_2} R^{a_3 a_4}_{\quad b_3 b_4} R^{a_5 a_6}_{\quad b_5 b_6} R^{a_7 a_8}_{\quad b_7 b_8} \] (5.51)

The \( R^4 \) tensors are an interesting set. In \( D = 4 \) they degenerate into two independent tensors. One of these, the famous “Bel-Robinson” tensor [36] was shown to be consistent with supersymmetry and thus became a candidate counterterm for supergravity theories [37]. In higher dimensions the Bel-Robinson tensor extends to a two-parameter set [38]. For maximal supergravity theories the uniqueness of the \( R^4 \) tensor extends to higher dimensions and is often written

\[ t_8 t_8 R^4 \] (5.52)

where \( t_8 \) may be found in ref [2] eq. (9.A.18). In \( D = 8 \), in \( N = 2 \) supergravity theory the four-point amplitudes are exactly proportional to this tensor [39, 40] and this tensor appears in the low-energy effective action of string theory [41]. For \( N = 1 \) supergravity there is an additional combination consistent with supersymmetry which appears if we calculate the \( N = 1 \) supergravity counterterms [42]. It is interesting to calculate the counterterms for simple gravity as a probe for the symmetries of the gravitational theory.

Calculation with a general counterterm gives, for example,

\[ M_1^{\text{counter}}(1_g^{++}, 2_g^{++}, 3_g^{++}, 4_g^{++}) = 8 (8a_1 + 2a_2 + 4a_3 + a_6) \frac{(s^2 + t^2 + u^2)^2}{s^2 t^2 u^2} \times K_1 \]
\[ M_{17}^{\text{counter}}(1_g^{++}, 2_g^{++}, 3_g^{++}, 4_g^{++}) = -\frac{8}{8} \left[ (4a_2 - a_5 + 2a_6) (t^2 + u^2) + 2 (4a_2 - a_5 - a_7) tu \right] \times K_{17} \] (5.53)

Clearly in this case it is not sufficient to look at amplitudes where the external polarisations are four dimensional. However, just from the \( M_1^{\text{loop}}(1_g^{++}, 2_g^{++}, 3_g^{++}, 4_g^{++}) \) we can clearly see that the counterterm does not vanish - although we can only impose a single relationship between the coefficients of the six counterterms.

In Appendix C we calculate the infinities present in a sufficiently large class to determine the coefficients of the \( T_i \) and the \( D = 8 \) counterterm for a real scalar loop is

\[ a_1 = \frac{11}{29030400} \quad a_2 = \frac{1}{362880} \quad a_3 = \frac{1}{14515200} \quad a_4 = 0 \quad a_5 = 0 \quad a_6 = -\frac{1}{1814400} \quad a_7 = \frac{1}{453600} \] (5.54)
The coefficient $a_4$ has been set to zero by choice but $a_5 = 0$ is non-trivial.

We have also calculated the $R^4$ counterterms generated by other types of matter circulating within the loop. Unlike the $D = 6$ case, the counterterms from different matter combinations are not simply related. We present the coefficients of the $R^4$ counterterms necessary to eliminate all divergence in four external graviton amplitudes in table 5.1. We have included the counterterms where supersymmetric multiplets circulate for comparison. (These have been presented previously in ref [42].) In $D = 8$ the spinor has eight degrees of freedom and both $N = 1$ and $N = 2$ supergravity exist where the $N = 2$ is the reduction of $D = 10$, $N = 2$ supergravity [43, 44]. For $N = 1$ there is the graviton multiple and the matter multiplet. We have chosen to present the combination of the two multiplets corresponding to the reduction of $D = 10$, $N = 1$ supergravity (denoted $N = 1^*$). This prior to reduction has particle content, in representations of $SO(8)$, $\mathbf{8}_c \otimes (\mathbf{8}_c \oplus \mathbf{8}_v)$. We have chosen to give the combined contribution of a graviton and antisymmetric tensor in the loop because this combination arise most naturally in superstring inspired theories. (And in fact it is difficult to separate the two contributions in string theory.) In general one can rearrange the counterterms by addition of the Gauss term and we have used this freedom to set $a_4 = 0$. For the $N = 2$ case the counterterm can be simplified to

$$-\frac{1}{64} (T_4 - 4T_7)$$

(5.55)

| Matter | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ | $a_7$ |
|--------|-------|-------|-------|-------|-------|-------|-------|
| $\phi$ | $\frac{11}{29030400}$ | $\frac{1}{362880}$ | $\frac{1}{14515200}$ | 0 | 0 | $-\frac{1}{1814400}$ | $\frac{1}{453600}$ |
| $\lambda$ | $\frac{89}{14515200}$ | $\frac{41}{725760}$ | $\frac{-23}{907200}$ | 0 | $\frac{1}{20160}$ | $\frac{13}{453600}$ | $\frac{223}{1814400}$ |
| $A_a$ | $\frac{-949}{77414400}$ | $\frac{29}{322560}$ | $\frac{-389}{3570200}$ | 0 | $\frac{-1}{161280}$ | $\frac{13}{1612800}$ | $\frac{31}{345600}$ |
| $B_{ab} + g_{ab}$ | $\frac{3799}{11612160}$ | $-\frac{841}{145152}$ | $\frac{2939}{8806080}$ | 0 | $\frac{-221}{40320}$ | $\frac{5251}{725760}$ | $\frac{5779}{362880}$ |
| $\psi_a$ | $\frac{103}{141720}$ | $-\frac{2867}{725760}$ | $+\frac{457}{725760}$ | 0 | $\frac{-31}{40320}$ | $\frac{139}{453600}$ | $+\frac{2143}{362880}$ |
| $N = 2$ | $\frac{1}{1024}$ | $-\frac{1}{64}$ | $\frac{1}{512}$ | 0 | $-\frac{1}{32}$ | $\frac{1}{64}$ | $\frac{1}{32}$ |
| $N = 1^*$ | $\frac{-13}{737280}$ | $\frac{7}{46080}$ | $-\frac{-13}{368640}$ | 0 | $\frac{1}{23040}$ | $-\frac{1}{46080}$ | $\frac{1}{4608}$ |

Table 5.1: Counterterms in $D = 8$ for Various Matter Contents within the Loop

The form of the counterterms,

$$\nabla^n R^m$$

where $n + 2m = (D - 2)L + 2$

(5.56)

is symmetric under $(D - 2) \leftrightarrow L$. Which means for example the form of the counterterms at $D = 8, L = 1$ is the same as that for $D = 4, L = 3$ (up to dimensional dependent degeneracies).
It could be hoped that studying counterterms in $D = 8, L = 1$ will provide information about the $D = 4, L = 3$ case. In fact for the $D = 6, L = 1$ case there does appear to be a correlation of information - the $D = 6, L = 1$ counterterms vanish for a supersymmetric theory as is the case for $D = 4, L = 2$. For the $D = 4, L = 3$ case the situation is far from clear. The unitarity based results of ref. [12] indicate that for maximal supergravity the three-loop amplitude in $D = 4$ is finite. This conclusion has been supported by some field theoretical evidence [45]. We can examine the $D = 8, L = 1$ counterterms to see if any understanding of the $D = 4, L = 3$ result can be obtained. The $D = 8$ counterterms are written in terms of the six independent $T_i$ with non-zero coefficients. The combination compatible with maximal supersymmetry appears to be unique [12, 38]. If we were to write this complete tensor in $D = 4$ it is conceivable that it could vanish in which case the vanishing of the infinity of the counterterm would be a residual effect of reduction - analogous to the arguments of ref [45]. However the $D = 8$ combination reduces to a non-vanishing tensor - as evidence by the fact that the amplitude $M^{\text{counter}}(1_g^-, 2_g^-, 3_g^{++}, 4_g^{++})$ receives non-zero contributions from this tensor so that the vanishing in the $D = 4, L = 3$ infinity remains a puzzle from this viewpoint.

6 Conclusions

In this paper we have used an extension of four dimensional helicity to organise the scattering amplitudes for theories involving gravity. This allows scattering amplitudes to be split into minimal physical pieces which are generally simpler than the full amplitude. For many purposes, such as determining the coefficients of counterterms, we need only the results for a few helicity amplitudes. The individual helicity amplitudes are physical and can be useful for testing hypothesis and so we have included in our appendices rather more calculations than we needed so that they may serve as a database for others.

We have used the infinities in the physical four-point amplitudes to determine the counterterm structure in $D = 6, 8$. In the $D = 6$ case the counterterm was proportional to $N_B - N_F$ and hence vanishes in a theory with equal numbers of bosonic and fermionic degrees of freedom such as a supersymmetric theory. In contrast, the situation in $D = 8$ is quite different. The counterterms induced by different particles are different and although the $N = 2$ supersymmetric combinations are relatively simple they do not vanish.

Our investigations give no indications that a finite field theory of gravity is possible. However our calculations should provide indicators of the form of the low-energy effective action of the fundamental theory of which gravity is merely the low-energy limit.
Appendix A: Graviton Tree Amplitudes

In this appendix we present all the partial tree amplitudes for four graviton scattering for dimension $D \geq 4$ in table A.1. The partial amplitudes are given by

$$M_{\text{tree}}^i = i \kappa^2 K_i \times F_i$$

where the $K_i$ are products of spinor helicity factors. When evaluating the tree amplitude only the modulus of $K_i$ is relevant, however, the complex phases are needed when trees are interfered with loops. The full form of the $K_i$ is given in table A.2 and these will also be used for the loop calculations. For dimensions less than eight not all the tree amplitudes exist, for example in $D = 4$ only $M_1, M_2$ and $M_3$ exist. The minimal dimension that an amplitude exists in we call $D_M$ and this is also given in table A.1.

In general amplitudes are polynomials in $\epsilon \cdot k$ and $\epsilon \cdot \epsilon'$. In choosing the spinor helicity factors we generally evaluate these from the highest order terms in $\epsilon \cdot k$. For four-point amplitudes, if we choose the spinor helicity reference momentum, $q_i$, of an external states to be the external momentum of another external state, this highest term has a unique form. For example, for $M_2(1^{−−}_g, 2^{++}_g, 3^{++}_g, 4^{++}_g)$ we could choose the spinor helicity for the $\epsilon_{-}^{-}a_{b}^{-}$ to be $k_4$, that is $\epsilon_{-}^{-}a_{b}^{-}(k_1; k_4)$. This means that

$$\epsilon^{-}(k_1; k_4) \cdot k_4 = 0$$

$$\epsilon^{-}(k_1; k_4) \cdot k_3 = - \epsilon^{-}(k_1; k_4) \cdot k_2$$

and the leading term can be reduced to having a factor of $\epsilon_{-}^{-} \cdot k_2$. This means the leading polynomial in $M_2(1^{−−}_g, 2^{++}_g, 3^{++}_g, 4^{++}_g)$, if we choose the four reference momenta $(q_1, q_2, q_3, q_4) = (k_4, k_1, k_1, k_1)$ will have a factor of

$$\left( \epsilon_{-}^{-1} \cdot k_2 \epsilon_{2}^{-} \cdot k_3 \epsilon_{3}^{-} \cdot k_2 \epsilon_{4}^{-} \cdot k_2 \right)^2$$

This is the K-factor for $M_2$ which can be reduced to spinor products as given in table A.2. For amplitudes such as $M_{20}(1^{−−}_g, 2^{++}_g, 3^{IJ}_g, 4^{IJ}_g)$, where $\epsilon_4 \cdot k_i = 0$ for all external momenta $k_i$, the highest order term (after chooses, for example, $(q_1, q_2) = (k_2, k_1)$) will be

$$\left( \epsilon_{-}^{-1} \cdot k_3 \epsilon_{2}^{+} \cdot k_3 \right)^2$$

which is the $K$-factor. The $K$ factors are dependent on the choice of reference momenta although the combination $K_i \times F_i$ is not.
| Amplitude                | $D_M$ | $F_I$ | $16|K_{ij}|$ | Amplitude                | $D_M$ | $F_I$ | $16|K_{ij}|$ |
|-------------------------|-------|-------|-------------|-------------------------|-------|-------|-------------|
| M1 ($^{++}, +, +, +, +$) | 4     | 0     | $s^2 u^2$   | M36 ($^{++}, -I, +I, IJ$) | 5     | 0     | $4su$      |
| M2 ($^{--}, +, +, +, +$) | 4     | 0     | $s^2 u^2$   | M37 ($^{++}, +, +I, JJ$)  | 6     | 0     | $4su$      |
| M3 ($^{--}, +, +, +, +$) | 4     | 0     | $s^2 u^2$   | M38 ($^{--}, +I, IJ$)     | 6     | 0     | $4su$      |
| M4 ($^{++}, +, +I, I)$   | 5     | 0     | $2su^2$     | M39 ($^{++}, -I, JJ$)     | 6     | 0     | $s^2 u^2$  |
| M5 ($^{--}, +, +I, I)$   | 5     | 0     | $2su^2$     | M40 ($^{+I}, +J, IJ$)     | 6     | 0     | $8s$       |
| M6 ($^{--}, +I, +I$)    | 5     | 0     | $2u^3$      | M41 ($^{--}, +I, IJ, I$)  | 6     | 0     | $8su$      |
| M7 ($^{++}, +I, +I, I$) | 5     | 0     | $2s^2 u$    | M42 ($^{++}, +I, IJ, KK$) | 7     | 0     | $8s$       |
| M8 ($^{--}, ++, I, IJ$) | 5     | 0     | $2u^3$      | M43 ($^{++}, +I, IJ, KK$) | 7     | 0     | $8su$      |
| M9 ($^{+I}, +I, +I$)    | 5     | 0     | $4tu$       | M44 ($^{++}, IJ, IJ, I$)  | 6     | 0     | $8su$      |
| M10 ($^{--}, +I, +I, I$) | 5     | 0     | $4tu$       | M45 ($^{++}, IJ, IJ, IJ$) | 7     | 0     | $8su$      |
| M11 ($^{--}, +I, +I, I$) | 5     | 0     | $4tu$       | M46 ($^{++}, IJ, IJ, IJ$) | 7     | 0     | $8su$      |
| M12 ($^{++}, +I, +I, IJ$) | 6     | 0     | $4su$       | M47 ($^{++}, IJ, IJ, IJ, LL$) | 8     | 0     | $16$       |
| M13 ($^{--}, +I, +I, IJ$) | 6     | 0     | $4su$       | M48 ($^{++}, +, IJ, IJ$)  | 5     | 0     | $4s^2$     |
| M14 ($^{--}, +I, +I, IJ$) | 6     | 0     | $4su$       | M49 ($^{++}, ++, IJ, IJ$)  | 5     | 0     | $4s^2$     |
| M15 ($^{--}, +I, +I, IJ$) | 6     | 0     | $4su$       | M50 ($^{++}, +, IJ, IJ$)  | 5     | 0     | $4s^2$     |
| M16 ($^{++}, +I, +I, IJ$) | 6     | 0     | $4su$       | M51 ($^{++}, +, IJ, IJ$)  | 5     | 0     | $4s^2$     |
| M17 ($^{++}, I, +I, IJ$) | 6     | 0     | $4su$       | M52 ($^{++}, I, IJ, IJ$)  | 5     | 0     | $8s$       |
| M18 ($^{--}, +I, IJ$)   | 6     | 0     | $2u^2$      | M53 ($^{--}, I, IJ, IJ$)  | 5     | 0     | $8s$       |
| M19 ($^{--}, +I, IJ$)   | 6     | 0     | $2u^2$      | M54 ($^{--}, I, IJ, IJ$)  | 5     | 0     | $8s$       |
| M20 ($^{--}, +I, IJ$)   | 6     | 0     | $2u^2$      | M55 ($^{--}, I, IJ, IJ$)  | 5     | 0     | $8s$       |
| M21 ($^{++}, I, IJ$)    | 6     | 0     | $2u^2$      | M56 ($^{++}, I, IJ, IJ$)  | 5     | 0     | $8s$       |
| M22 ($^{++}, I, IJ$)    | 6     | 0     | $2u^2$      | M57 ($^{++}, I, IJ, IJ$)  | 5     | 0     | $8s$       |
| M23 ($^{++}, I, IJ, IJ$) | 7     | 0     | $8su$      | M58 ($^{++}, I, IJ, IJ$)  | 7     | 0     | $8s$       |
| M24 ($^{++}, I, IJ, IJ$) | 7     | 0     | $8su$      | M59 ($^{++}, I, IJ, IJ$)  | 7     | 0     | $8s$       |
| M25 ($^{++}, I, IJ, IJ$) | 7     | 0     | $8su$      | M60 ($^{++}, I, IJ, IJ$)  | 7     | 0     | $8s$       |
| M26 ($^{++}, I, IJ, IJ$) | 7     | 0     | $8su$      | M61 ($^{++}, I, IJ, IJ$)  | 7     | 0     | $8s$       |
| M27 ($^{++}, I, IJ, IJ$) | 7     | 0     | $8su$      | M62 ($^{++}, I, IJ, IJ$)  | 7     | 0     | $8s$       |
| M28 ($^{++}, I, IJ, IJ$) | 6     | 0     | $8su$      | M63 ($^{++}, I, IJ, IJ$)  | 6     | 0     | $8su$      |
| M29 ($^{++}, I, IJ, IJ$) | 6     | 0     | $8su$      | M64 ($^{++}, I, IJ, IJ$)  | 6     | 0     | $8su$      |
| M30 ($^{++}, I, IJ, IJ$) | 6     | 0     | $8su$      | M65 ($^{++}, I, IJ, IJ$)  | 6     | 0     | $8su$      |
| M31 ($^{++}, I, IJ, IJ$) | 6     | 0     | $8su$      | M66 ($^{++}, I, IJ, IJ$)  | 6     | 0     | $8su$      |
| M32 ($^{++}, I, IJ, IJ$) | 6     | 0     | $8su$      | M67 ($^{++}, I, IJ, IJ$)  | 6     | 0     | $8su$      |
| M33 ($^{++}, I, IJ, IJ$) | 6     | 0     | $8su$      | M68 ($^{++}, I, IJ, IJ$)  | 6     | 0     | $8su$      |
| M34 ($^{++}, I, IJ, IJ$) | 6     | 0     | $8su$      | M69 ($^{++}, I, IJ, IJ$)  | 6     | 0     | $8su$      |
| M35 ($^{++}, I, IJ, IJ$) | 6     | 0     | $8su$      | M70 ($^{++}, I, IJ, IJ$)  | 6     | 0     | $8su$      |

Table A.1: Graviton Tree Amplitudes
| $K_i$ | Value                                      | $K_i$ | Value                                      |
|-------|--------------------------------------------|-------|--------------------------------------------|
| $16K_1$ | $\left(\frac{s^2tu}{(12)(23)(34)(41)}\right)^2$ | $4K_{36}$ | $-\frac{st}{u}\frac{(12)^2[13]^2}{(41)^2}$ |
| $16K_2$ | $(12)[2\ 4\ [2\ 3]]^4/t^2$                | $K_{37}$ | $K_{16}$                                   |
| $16K_3$ | $\left(\frac{st(12)^4}{(12)(23)(34)(41)}\right)^2$ | $K_{38}$ | $K_{35}$                                   |
| $8K_4$  | $-u^2\left(\frac{1[2\ 3]}{(1)}\right)^2$  | $K_{39}$ | $K_{36}$                                   |
| $8K_5$  | $-\left(\frac{(12)^4[2\ 3]}{(14)}\right)^2$ | $K_{40}$ | $K_{21}$                                   |
| $8K_6$  | $-\frac{(12)^4[3\ 4]^2}{2K_{41}}$          | $K_{42}$ | $K_{21}$                                   |
| $8K_7$  | $-\frac{(12)^4[1\ 3]^2(41)^2/t}{2K_{43}}$  | $K_{44}$ | $K_{41}$                                   |
| $8K_8$  | $-\frac{(13)^4[2\ 3]^2[2\ 4]^2/t}{2K_{45}}$| $K_{46}$ | $K_{41}$                                   |
| $K_9$   | $\sqrt{K_1}$                               | $2K_{44}$ | $(\frac{(42)(2\ 1)}{(4\ 1)}\right)^2$   |
| $4K_{10}$ | $(12)[2\ 3\ [2\ 4]]^2/s$                | $K_{47}$ | $1$                                        |
| $K_{11}$ | $\sqrt{K_3}$                               | $K_{48}$ | $K_{31}$                                   |
| $K_{12}$ | $\sqrt{K_3}$                               | $K_{49}$ | $K_{31}$                                   |
| $K_{13}$ | $\sqrt{K_3}$                               | $K_{50}$ | $K_{31}$                                   |
| $K_{14}$ | $\sqrt{K_3}$                               | $K_{51}$ | $K_{31}$                                   |
| $4K_{15}$ | $-\left(\frac{(13)^4st}{(1)(2)(2\ 3)(3\ 4)(4\ 1)}\right)^2$ | $K_{52}$ | $K_{21}$                                   |
| $4K_{16}$ | $-\left[\frac{1\ 2\ 2\ 3]}{[1\ 4]}\right]^2$ | $K_{53}$ | $K_{41}$                                   |
| $4K_{17}$ | $-\left(\frac{(4\ 2)(2\ 1)[2\ 3]}{(14)}\right)^2$ | $K_{54}$ | $K_{41}$                                   |
| $4K_{18}$ | $\left(2\ 3\ [2\ 4]^2[2\ 4]^2/s\right)$  | $K_{55}$ | $K_{41}$                                   |
| $K_{19}$ | $\left(\frac{2\ 3\ 3\ 1]}{2\ 1}\right)^2$ | $K_{56}$ | $K_{21}$                                   |
| $4K_{20}$ | $(2\ 3\ [2\ 3\ 3\ 1]\ 2\ 4]^2/s^2$    | $K_{57}$ | $K_{41}$                                   |
| $2K_{21}$ | $-\frac{(12)^2}{2K_{42}}$                | $K_{58}$ | $K_{21}$                                   |
| $2K_{22}$ | $-\frac{(2\ 3\ [3\ 1]\ 2\ 4]^2/s}{2K_{43}}$ | $K_{59}$ | $K_{41}$                                   |
| $K_{23}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{60}$ | $1$                                        |
| $K_{24}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{61}$ | $1$                                        |
| $K_{25}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{62}$ | $1$                                        |
| $K_{26}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{63}$ | $1$                                        |
| $K_{27}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{64}$ | $1$                                        |
| $K_{28}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{65}$ | $1$                                        |
| $K_{29}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{66}$ | $1$                                        |
| $K_{30}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{67}$ | $1$                                        |
| $K_{31}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{68}$ | $1$                                        |
| $K_{32}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{69}$ | $1$                                        |
| $K_{33}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{70}$ | $1$                                        |
| $K_{34}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{71}$ | $1$                                        |
| $K_{35}$ | $\left(\frac{(2\ 3\ [2\ 1]}{2\ 1}\right)^2$ | $K_{72}$ | $1$                                        |

Table A.2: The K factors for the Graviton Amplitudes
Appendix B: Examples of the Cutkosky Cutting Technique

Here we demonstrate the steps necessary to evaluate the all-\( \epsilon \) form of the one-loop amplitudes which involve both gravitons and two-forms. This also illustrates the links between the graviton and form scattering.

To evaluate the two-particle cuts we need the tree amplitudes for two external particles, with momenta in four dimensions, and for two internal particles with momenta in \( D \) dimensions, where \( D = 2N - 2\epsilon \). Since we are examining amplitudes with complex scalar loops these two internal particles should be complex scalars. The KLT relationships can be used to determine these “primitive amplitudes” from the Yang-Mills amplitudes,

\[
M^{\text{tree}}_4(1, 2, 3, 4) = -\frac{i s_{12}}{4} A^{\text{tree}}_4(1, 2, 3, 4) A^{\text{tree}}_4(1, 2, 4, 3)
\]

The Yang-Mills amplitude we shall need is

\[
A^{\text{tree}}_4(1_s, 2^I, 3^J, 4_s) = -2i \left( \frac{L^I L^J}{\Box_{23}} - \frac{\Box_{23}}{2s_{23}} \right)
\]

where \( \Box_{23} \equiv (L_1 - L_2)^2 \) is the propagator from leg two to leg three in a clockwise manner. From these we can deduce that

\[
M^{\text{tree}}(s; g^{IJ}; g^{KL}; s) = \frac{1}{\sqrt{2}} \left( M^{\text{tree}, P}(s; I, J; K, L; s) + M^{\text{tree}, P}(s; I, J; L, K; s) \right)
+ M^{\text{tree}, P}(s; J, I; K, L; s) + M^{\text{tree}, P}(s; J, I; L, K; s))
\]

\[
= is \left( A^{\text{tree}}(s, 1^I, 2^K, s) A^{\text{tree}}(s, 2^J, 1^L, s) + A^{\text{tree}}(s, 1^J, 2^K, s) A^{\text{tree}}(s, 2^I, 1^L, s) \right)
\]

\[
= is \left( \frac{L^I L^L}{\Box_{12} \Box_{21}} + \frac{L^J L^L}{\Box_{12} \Box_{21}} \right)
= 2is \left( \frac{L^I L^L}{\Box_{12} \Box_{21}} + \frac{L^J L^L}{\Box_{12} \Box_{21}} \right)
\]

which will prove to be an extremely useful form when cutting. We have used the identity

\[
\frac{1}{\Box_{12} \Box_{21}} + \frac{s_{22}}{s_{22} \Box_{21}} + \frac{1}{\Box_{12} s_{12}} = 0
\]

which follows from \( s_{12} + \Box_{12} + \Box_{21} = 0 \), which is true since the tree amplitude is fully on-shell. Similarly,

\[
M^{\text{tree}}(s; g^{IJ}; g^{JK}; s) = -2i L^I L^J L^K \left( \frac{1}{\Box_{12}} + \frac{1}{\Box_{21}} \right) - \frac{i L^J L^K}{2}
\]

\[
M^{\text{tree}}(s; g^{IJ}; g^{IJ}; s) = -2i L^I L^J L^I \left( \frac{1}{\Box_{12}} + \frac{1}{\Box_{21}} \right) - \frac{i}{2} \left( L^I L^I + L^J L^J \right) + \frac{i}{4s} \Box_{12} \Box_{21}
\]

Amplitudes involving forms are obtained from the same primitive amplitudes but with appropriate minus signs. For Amplitudes with one \( g \) and one \( B \) the tree amplitude vanishes

\[
M^{\text{tree}}(s; g^{IJ}; B^{KL}; s) = M^{\text{tree}}(s; g^{IJ}; B^{IK}; s) = M^{\text{tree}}(s; g^{IJ}; B^{IJ}; s) = 0
\]
For amplitudes with two 2-forms we obtain:

\[
M_{\text{tree}}(s; B_{IJ}; B_{KL}; s) = \frac{1}{\sqrt{2}} \left( M_{\text{tree}}^P(s; I, J; K, L; s) - M_{\text{tree}}^P(s; J, I; K, L; s) \right) - \frac{1}{\sqrt{2}} \left( M_{\text{tree}}^P(s; I, J; L, K; s) + M_{\text{tree}}^P(s; J, I; L, K; s) \right) = 0
\]

\[
M_{\text{tree}}(s; B_{IJ}; B_{IK}; s) = \frac{1}{\sqrt{2}} \left( M_{\text{tree}}^P(s; I, J; I, K; s) - M_{\text{tree}}^P(s; J, I; I, K; s) \right) - \frac{1}{\sqrt{2}} \left( M_{\text{tree}}^P(s; I, J; K, I; s) + M_{\text{tree}}^P(s; J, I; K, I; s) \right) = \left( A_{\text{tree}}(s, 1^I, 2^I, s) A_{\text{tree}}(s, 2^J, 1^K, s) - A_{\text{tree}}(s, 1^I, 2^K, s) A_{\text{tree}}(s, 2^J, 1^I, s) \right) = -\frac{i L^I L^J}{2}
\]

\[
M_{\text{tree}}(s; B_{IJ}; B_{IJ}; s) = \frac{1}{\sqrt{2}} \left( M_{\text{tree}}^P(s; I, J; I, J; s) - M_{\text{tree}}^P(s; J, I; I, J; s) \right) - \frac{1}{\sqrt{2}} \left( M_{\text{tree}}^P(s; I, J; J, I; s) + M_{\text{tree}}^P(s; J, I; J, I; s) \right) = \left( A_{\text{tree}}(s, 1^I, 2^I, s) A_{\text{tree}}(s, 2^J, 1^J, s) - A_{\text{tree}}(s, 1^J, 2^J, s) A_{\text{tree}}(s, 2^J, 1^J, s) \right) = -\frac{i}{2} (L^I L^J + L^J L^I) + \frac{i}{4 s} \frac{\Box_{12}}{\Box_{21}}
\]

We now have the building blocks necessary to evaluate the cuts in examples of one-loop amplitudes.

Example 1: \( M^{1-\text{loop}}(g_{1}^{IJ}, g_{2}^{IK}; B_{3}^{JJ}, B_{4}^{LK}) \)

A four-point amplitude will have, in general, three cuts - in the \( s \), \( t \) and \( u \) invariants. For this amplitude, to all orders in \( \epsilon \) the \( t \) and \( u \) cuts vanish identically, since the tree amplitudes for these cuts vanish, and the amplitude only has a \( s \)-cut as given in eq. (4.2)

\[
\sum_{\text{internal states},s} \int \frac{d^{4}L_{1}}{2\pi^{4}} \int_{L_{1}}^{i} M_{4}^{\text{tree}}(-L_{1}^{s}, 1^{s}, 2^{s}, L_{3}^{s}) \times \frac{i}{L_{3}^{s}} M_{4}^{\text{tree}}(-L_{3}^{s}, 3^{s}, 4^{s}, L_{1}^{s}) \bigg|_{s-\text{cut}}
\]

Manipulating the tree amplitudes within this cut

\[
M_{\text{tree}}^{s}(s_{1}, g_{1}^{IJ}; g_{2}^{IK}; s_{1}) \times M_{\text{tree}}^{s}(s_{2}; B_{3}^{JJ}, B_{4}^{LK} s_{1}) = i^{2} \left( -2L^{J} L^{J} L^{K} L^{K} \left[ \frac{1}{\Box_{12}} + \frac{1}{\Box_{21}} \right] - \frac{L^{J} L^{K}}{2} \right) \times \left( -\frac{L^{J} L^{K}}{2s} \right)
\]

\[
= -\frac{L^{J} L^{J} L^{J} L^{K} L^{J} L^{K} L^{J} L^{K}}{s_{12}} - \frac{L^{J} L^{J} L^{J} L^{K} L^{J} L^{K} L^{J} L^{K}}{s_{21}} - \frac{L^{J} L^{J} L^{J} L^{K}}{4s}
\]

and inserting this product into the cut, after adding the two propagators, these three terms can be recognised as the cuts of two triangle integrals and a bubble integral,

\[
-\frac{i}{(4\pi)^{D/2} s_{12}} I_{3}^{D} [L^{J} L^{J} L^{J} L^{K} L^{J} L^{K}] - \frac{i}{(4\pi)^{D/2} s_{21}} I_{3}^{D} [L^{J} L^{J} L^{J} L^{K} L^{J} L^{K}] + \frac{i}{(4\pi)^{D/2} 4s_{12}} I_{2}^{D} [L^{J} L^{J} L^{J} L^{K}]
\]
The effect of the internal momentum is to “shift” the dimension of the integral, for example,
\[ I^D[L^I L^I] = \frac{1}{2} I^{D+2}[1], \quad I^D[L^I L^I L^J L^J] = \frac{1}{4} I^{D+4}[1], \quad etc \]
and this integral is equal to
\[ \frac{i}{(4\pi)^{D/2}} \left( -\frac{1}{4} I^{D+6}_3(s) + \frac{1}{16} I^{D+4}_2(s) \right) \]
where we have chosen to indicate the momentum invariant upon which the integrals depend. This expression has the correct value for all the cuts of the amplitude to all orders in \( \epsilon \) so that
\[ M^{1-\text{loop}}(g_1^{IJ}; g_2^{IK}; B_3^{LJ}, B_4^{LK}) = \frac{i}{(4\pi)^{D/2}} \left( -\frac{1}{4} I^{D+6}_3(s) + \frac{1}{16} I^{D+4}_2(s) \right) \]
The infinities in this amplitude match those of table D.2 (after dividing by two to get the contribution from a real scalar.) although the full one-loop amplitude contains much more information than merely the ultra-violet infinities.

**Example 2:** \( M^{1-\text{loop}}(B_1^{IJ}; B_2^{IK}; B_3^{LJ}, B_4^{LK}) \)

For this amplitude the \( t \)-cut is identically zero leaving \( s \) and \( u \) cuts. Firstly the \( s \)-cut gives
\[ M^{\text{tree}}(s_{t_1}; B_1^{IJ}; B_2^{IK}; s_{t_3}) \times M^{\text{tree}}(s_{t_2}; B_3^{LJ}, B_4^{LK}; s_{t_1}) = \frac{iL^I L^K}{2} \times \frac{iL^J L^K}{2} \]
After inserting this into the cut, we find that the \( s \)-cut will be the cut of the bubble integral
\[ \frac{i}{4(4\pi)^{D/2}} I^{D}_2 [L^I L^K L^J L^K] \]
Again the internal momentum leads to a shifted integral,
\[ \frac{i}{16(4\pi)^{D/2}} I^{D+4}_2(s) \]
The \( u \)-cut is identical, after substituting \( s \to u \), giving the total amplitude as
\[ M^{1-\text{loop}}(B_1^{IJ}; B_2^{IK}; B_3^{LJ}, B_4^{LK}) = \frac{i}{16(4\pi)^{D/2}} \left( I^{D+4}_2(s) + I^{D+4}_2(u) \right) \]
whose infinities match those contained in table D.1.

**Example 3:** \( M^{1-\text{loop}}(g_1^{IJ}; g_2^{IK}; g_3^{KL}, g_4^{JL}) \)

This amplitude has cuts in all three channels, the simplest being the \( u \) channel where the cut is
\[ M^{\text{tree}}(s_{t_1}; g_1^{IJ}; g_3^{KL}; s_{t_2}) \times M^{\text{tree}}(s_{t_3}; g_2^{IK}; g_4^{JL}; s_{t_1}) \]
\[ = -2iL^I L^J L^K L^L \left( \frac{1}{\Box_{13}} + \frac{1}{\Box_{31}} \right) \times -2iL^I L^J L^K L^L \left( \frac{1}{\Box_{24}} + \frac{1}{\Box_{42}} \right) \]
which is the cut of
\[ \frac{i}{2(4\pi)^{D/2}} \left( I^{D+8}_{1243}(s, u) + I^{D+8}_{1423}(t, u) \right) \]
The $s$-cut is then

$$\text{is}\left(2\frac{L^J LI LL^K}{\Box_{12\Box_{21}}} - \frac{L^J LK}{2s}\right) \times \text{is}\left(2\frac{L^J LI LL^K}{\Box_{34\Box_{43}}} - \frac{L^J LK}{2s}\right)$$

which will be the cut of

$$\frac{i}{(4\pi)^{D/2}} \left( \frac{1}{2} \left( I_{1234}^{D+8}(s, u) + I_{1234}^{D+8}(s, t) \right) - \frac{1}{2} I_{3}^{D+6}(s) + \frac{1}{16} I_{2}^{D+4}(s) \right)$$

The $t$-cut can be obtained from the $s$ by relabeling. Putting the cuts together we find the entire amplitude is

$$M^{1-\text{loop}}(g_1^{IJ}, g_2^{IK}, g_3^{KL}, g_4^{IL}) = \frac{i}{(4\pi)^{D/2}} \left( \frac{1}{2} \left( I_{1234}^{D+8}(s, t) + I_{1234}^{D+8}(s, u) + I_{1423}^{D+8}(t, u) \right) 
- \frac{1}{2} \left( I_{3}^{D+6}(s) + I_{3}^{D+6}(t) \right) + \frac{1}{16} \left( I_{2}^{D+4}(s) + I_{2}^{D+4}(t) \right) \right)$$

**Appendix C: Infinities in One-Loop Four Graviton Amplitudes**

We have calculated the infinities in the partial amplitudes for four graviton scattering in $D = 6, 8, 10$ for real scalar loops. In $D = 6$ a single, non-zero infinity will be enough to specify the coefficient of the single counterterm required to make the four graviton amplitudes finite. In $D = 8$ the first ten amplitudes are sufficient to fix the coefficients of the six $R^4$ tensors which can appear. The infinities in table C.1 over-specify the system considerable and all the infinities match the counterterms precisely.

The infinities in the loop amplitudes are

$$M_i^{1-\text{loop}}(1g, 2g, 3g, 4g) \bigg|_{1/\epsilon} = \frac{iK_4^4}{(4\pi)^{D/2}\epsilon} \times K_i \times F_i^{1-\text{loop}}$$

where the $F_i^{1-\text{loop}}$ are given in the following table C.1, labeled by their dimension.
| Amplitude         | $D_M$ | $D = 6$                                      | $D = 8$                                      | $D = 10$                                      | $16|K_i|$ |
|-------------------|-------|---------------------------------------------|---------------------------------------------|---------------------------------------------|--------|
| M1(;++;++;++;+++) | 4     | $t$                                        | $(s^2+t^2+u^2)^t$                           | $(s^2+t^2+u^2)^t$                           | $s^2u^2$ |
|                   |       | $\frac{313600}{u}$                         | $15120u^2$                                  | $\frac{313600}{u}$                         |        |
| M2(;++;++;++;+++) | 4     | $\frac{3}{604800}$                        | 0                                           | 0                                           | $s^2u^2$ |
|                   |       |                                            |                                              |                                              |        |
| M3(;++;++;++;+++) | 4     | 0                                          | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $s^4$  |
|                   |       |                                             |                                              |                                              |        |
| M4(;++;++;++;+++) | 5     | $\frac{t}{604800}$                        | $(t^2+3u^2+2u^2)$                           | $10t^4+46tu^4+69tu^2+46u^4+10u^4$           | $2su^2$ |
|                   |       |                                             |                                              |                                              |        |
| M5(;++;++;++;+++) | 5     | 0                                          | $\frac{s}{604800}$                         | $\frac{s}{604800}$                         | $2su^2$ |
|                   |       |                                             |                                              |                                              |        |
| M6(;++;++;++;+++) | 5     | 0                                          | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $2su^2$ |
|                   |       |                                             |                                              |                                              |        |
| M7(;++;++;++;+++) | 5     | $\frac{t}{10080s}$                        | 0                                           | $\frac{1}{109560}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M8(;++;++;++;+++) | 5     | 0                                          | $\frac{t}{10080s}$                         | $\frac{1}{109560}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M9(;++;++;++;+++) | 5     | $\frac{t}{10080s}$                        | $\frac{1}{10080s}$                         | $\frac{1}{109560}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M10(;++;++;++;+++) | 5     | 0                                           | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M11(;++;++;++;+++) | 5     | 0                                           | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M12(;++;++;++;+++) | 6     | $\frac{t}{10080s}$                        | $\frac{1}{10080s}$                         | $\frac{1}{109560}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M13(;++;++;++;+++) | 6     | 0                                           | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M14(;++;++;++;+++) | 6     | 0                                           | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M15(;++;++;++;+++) | 6     | 0                                           | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M16(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M17(;++;++;++;+++) | 6     | 0                                           | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M18(;++;++;++;+++) | 6     | 0                                           | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M19(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M20(;++;++;++;+++) | 6     | 0                                           | $\frac{1}{604800}$                         | $\frac{1}{604800}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M21(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M22(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M23(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M24(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M25(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M26(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M27(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M28(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M29(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M30(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |
| M31(;++;++;++;+++) | 6     | $\frac{t}{20160s}$                        | $\frac{1}{20160s}$                         | $\frac{1}{20160s}$                         | $2s^2u$ |
|                   |       |                                             |                                              |                                              |        |

Table C.1: Infinities in the Graviton One-Loop Amplitudes due to a Circulating Real Scalar
Appendix D: Infinities in One-Loop Graviton 2-Form Amplitudes and Four 2-Form Amplitudes

Here we present sufficient one-loop two graviton two 2-form amplitudes and four 2-form amplitudes to determine the counterterms described in the main text.

The tables give both the tree amplitudes, which are of the form

\[ \mathcal{M}_\text{tree}(1, 2, 3, 4) = i\kappa^2 K \times F\text{tree} \]

and the infinities in the one-loop amplitudes

\[ \mathcal{M}_\text{1-loop}(1, 2, 3, 4)|_{1/\epsilon} = \frac{i\kappa^4}{(4\pi)^D/2\epsilon} \times K \times F_{1\text{-loop}} \]

where $|K|$, $F\text{tree}$ and $F_{1\text{-loop}}$ are presented in tables D.1 and D.2.

| Amplitude | $D_M$ | $F_{1\text{-loop}}, D = 6$ | $F_{1\text{-loop}}, D = 8$ | $F\text{tree}$ | $16|K|$ |
|-----------|------|-----------------|-----------------|------------|-------------|
| $M(1^+_{B} \cdot 2^+_{B} \cdot 3^+_{B} \cdot 4^+_{B})$ | 5 | $\frac{s^2 + st + t^2}{241920}$ | $\frac{s^2 + st + t^2}{2240}$ | 0 | 4 | 8 | $\frac{84}{7}$ |
| $M(1^-_{B} \cdot 2^+_{B} \cdot 3^+_{B} \cdot 4^+_{B})$ | 5 | 0 | 0 | 0 | 4 | $\frac{84}{7}$ |
| $M(1^-_{B} \cdot 2^-_{B} \cdot 3^+_{B} \cdot 4^+_{B})$ | 5 | $\frac{s}{241920}$ | $\frac{s}{2240}$ | $\frac{s}{2240}$ | $\frac{1}{2}$ | $\frac{84}{7}$ |
| $M(1^-_{B} \cdot 2^+_{B} \cdot 3^-_{B} \cdot 4^+_{B})$ | 6 | $\frac{s^2 + st + t^2}{241920}$ | $\frac{s^2 + st + t^2}{2240}$ | 0 | 4 | $\frac{84}{7}$ |
| $M(1^-_{B} \cdot 2^-_{B} \cdot 3^-_{B} \cdot 4^+_{B})$ | 6 | 0 | 0 | 0 | 4 | $\frac{84}{7}$ |
| $M(1^+_{B} \cdot 2^+_{B} \cdot 3^+_{B} \cdot 4^+_{B})$ | 6 | $\frac{s^2 + st + t^2}{241920}$ | $\frac{s^2 + st + t^2}{2240}$ | $\frac{s^2 + st + t^2}{2240}$ | $\frac{1}{2}$ | $\frac{84}{7}$ |
| $M(1^-_{B} \cdot 2^-_{B} \cdot 3^+_{B} \cdot 4^+_{B})$ | 6 | 0 | 0 | 0 | 4 | $\frac{84}{7}$ |

Table D.1: The Tree Amplitudes and One-Loop Infinities for the Four 2-Form Amplitudes
| Amplitude                  | $F^{1\text{-loop}}, D = 6$ | $F^{1\text{-loop}}, D = 8$ | $F^{\text{tree}}$ | $16|K|$ |
|---------------------------|-----------------------------|-----------------------------|-------------------|-------|
| $M(g^+,g^+,B^+,B^+)$      | $0$                         | $\frac{s^2}{201600}$       | $0$               | $2su^2$ |
| $M(g^-,g^+,B^+,B^+)$      | $0$                         | $\frac{s}{201600}$         | $0$               | $2s^3$  |
| $M(g^-,g^+,B^+,B^+)$      | $0$                         | $0$                         | $0$               | $2su^2$ |
| $M(g^+,g^+,B^+,B^-)$      | $-\frac{t}{100800}$        | $-\frac{t}{100800}$        | $0$               | $2s^2u$ |
| $M(g^-,g^+,B^-,B^+)$      | $0$                         | $\frac{t}{364800}$         | $\frac{2}{su}$   | $2u^3$  |
| $M(g^+,g^+,B^{IJ},B^{IJ})$| $-\frac{t^2+5ut+3u^2}{201600}$ | $\frac{2t^2+5ut+2u^2}{201600}$ | $0$               | $4s^2$  |
| $M(g^{+I},g^{+I},B^{+J},B^{+J})$ | $-\frac{s^2}{134400}$ | $\frac{s^2}{672000}$ | $0$               | $4su$  |
| $M(g^{-I},g^{+I},B^{+J},B^{+J})$ | $0$ | $0$ | $0$ | $4tu$  |
| $M(g^{+I},g^{+J},B^{+I},B^{+J})$ | $0$ | $0$ | $0$ | $4st$  |
| $M(g^{-I},g^{+J},B^{+I},B^{+J})$ | $0$ | $0$ | $0$ | $4st$  |
| $M(g^{-I},g^{-I},B^{+J},B^{+J})$ | $-\frac{s}{134400}$ | $\frac{s}{672000}$ | $-\frac{1}{2s}$ | $4s^2$  |
| $M(g^{-I},g^{+J},B^{-I},B^{+J})$ | $0$ | $0$ | $0$ | $-\frac{1}{2u}$ | $4u^2$  |
| $M(g^{-I},g^{-I},B^{-J},B^{+J})$ | $\frac{s}{134400}$ | $\frac{s}{672000}$ | $\frac{s-ut}{2s}$ | $4u^2$  |
| $M(g^{+I},g^{+I},B^{+J},B^{IJ})$ | $\frac{2t\sqrt{2}}{134400}$ | $\frac{s\sqrt{2}}{201600}$ | $0$ | $4su$  |
| $M(g^{+I},g^{+I},B^{JK},B^{JK})$ | $s(t^2+3ut+2u^2)$ | $403200$ | $0$ | $8s$  |
| $M(g^{+I},g^{+I},B^{IJ},B^{IJ})$ | $-\frac{tu}{201600}$ | $\frac{s(4u^2+7ut+4t^2)}{403200}$ | $0$ | $8s$  |
| $M(g^{+K},g^{+K},B^{+I},B^{+I})$ | $\bullet$ | $\frac{1}{201600}$ | $s^3$ | $0$ | $8s$  |
| $M(g^{-},g^{+I},B^{+J},B^{IJ})$ | $0$ | $0$ | $0$ | $-\frac{s\sqrt{2}}{2su}$ | $4su$  |
| $M(g^{+I},g^{-I},B^{JK},B^{JK})$ | $\bullet$ | $\frac{s}{201600}$ | $-\frac{1}{2}$ | $8tu$  |
| $M(g^{+I},g^{-I},B^{IJ},B^{IJ})$ | $\frac{s^2}{134400}$ | $\frac{s^3}{201600}$ | $\frac{(t^2+u^2)}{4tu}$ | $8tu$  |
| $M(g^{+K},g^{+K},B^{+I},B^{I})$ | $\bullet$ | $\frac{1}{134400}$ | $-\frac{1}{7}$ | $8tu$  |
| $M(g^{+},g^{-I},B^{-J},B^{IJ})$ | $\frac{tu\sqrt{2}}{134400}$ | $\frac{s\sqrt{2}}{201600}$ | $0$ | $2ts^2$  |
| $M(g^{+},g^{-I},B^{+J},B^{IJ})$ | $\frac{s^3}{403200}$ | $0$ | $8s$ | $0$ |
| $M(g^{+I},g^{+I},B^{+J},B^{IK})$ | $\bullet$ | $\frac{403200}{s^3}$ | $0$ | $8u$  |
| $M(g^{+I},g^{+I},B^{+J},B^{IK})$ | $\bullet$ | $\frac{403200}{s^3}$ | $0$ | $8u$  |
| $M(g^{+I},g^{+K},B^{+I},B^{+I})$ | $\bullet$ | $\frac{403200}{s^3}$ | $0$ | $8s$  |
| $M(g^{+I},g^{+K},B^{+I},B^{+I})$ | $\bullet$ | $\frac{403200}{s^3}$ | $0$ | $8s$  |
| $M(g^{+I},g^{+K},B^{KL},B^{KL})$ | $\bullet$ | $\frac{s}{806400}$ | $s^4$ | $-\frac{u}{8}$ | $16$ |
| $M(g^{+},g^{++},B^{IJ},B^{IJ})$ | $0$ | $0$ | $\frac{s}{tu}$ | $4tu^2$  |
| $M(g^{+I},g^{KL},B^{JK},B^{LI})$ | $\bullet$ | $0$ | $-\frac{3}{8}$ | $16$ |
| $M(g^{+I},g^{IJ},B^{KL},B^{KL})$ | $\bullet$ | $\frac{(2t^2+ut+2u^2)x^2}{806400}$ | $\frac{tu}{16}$ | $16$ |
| $M(g^{+I},g^{IJ},B^{KL},B^{KL})$ | $\bullet$ | $0$ | $\frac{ts}{4u}$ | $16$ |

Table D.2: The Tree Amplitudes and One-Loop Infinities for the Two Graviton Two 2-Form Amplitudes
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