Ehresman Semigroups Whose Categories are EI and Their Representation Theory

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Abstract

We study simple and projective modules of a certain class of Ehresmann semigroups, a well-studied generalization of inverse semigroups. Let \( S \) be a finite right (left) restriction Ehresmann semigroup whose \( \tilde{H}_E(e) \)-class is a group for every projection \( e \in E \). This means that its corresponding Ehresmann category is an EI-category, that is, every endomorphism is an isomorphism. We show that the collection of finite Ehresmann semigroups whose categories are EI is a pseudovariety and we show in the infinite case, that the collection of Ehresmann semigroups whose categories have endomorphism monoids each having one idempotent is a quasivariety. We prove that the simple modules of the semigroup algebra \( \mathbb{k}S \) (over any field \( \mathbb{k} \)) are induced Schützenberger modules of the maximal subgroups of \( S \). Moreover, we show that over fields with good characteristic the indecomposable projective modules can be described in a similar way but using generalized Green’s relations instead of the standard ones. As a natural example we consider the monoid \( \mathcal{PT}_n \) of all partial functions on an \( n \)-set. Over the field of complex numbers, we give a natural description of its indecomposable projective modules and obtain a formula for their dimension. Moreover, we find certain zero entries in its Cartan matrix.

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1 Introduction

A semigroup $S$ is called inverse if every element $a \in S$ has a unique inverse, that is, an element $b \in S$ such that $aba = a$ and $bab = b$. Inverse semigroups are one of the central objects of study in semigroup theory (see [18]) and in particular their representation theory is well-studied (see [35, Part IV]). There are several generalizations of inverse semigroups that keep some of their properties and structure. In this paper we discuss representations of a generalization called Ehresmann semigroups. Let $E$ be a subsemilattice of a semigroup $S$ (i.e., a commutative subsemigroup of idempotents). Define two equivalence relations $\tilde{L}_E$ and $\tilde{R}_E$ in the following way: Elements $a, b \in S$ satisfy $a \tilde{L}_E b$ ($a \tilde{R}_E b$) if $a$ and $b$ have the same set of right (respectively, left) identities from $E$. We also define $\tilde{H}_E = \tilde{R}_E \cap \tilde{L}_E$. Assume that every $\tilde{L}_E$ and $\tilde{R}_E$ class contains precisely one idempotent from $E$ denoted $a^*$ and $a^+$ respectively. The semigroup $S$ is called Ehresmann (or $E$-Ehresmann if the set $E$ is to be emphasized) if $\tilde{L}_E$ is a right congruence and $\tilde{R}_E$ is a left congruence. An Ehresmann semigroup is called right restriction if the identity $ea = a(ea)^*$ holds for every $e \in E$ and $a \in S$. The main class of semigroups we consider in this paper are right restriction Ehresmann semigroups with the additional property that the $\tilde{H}_E$-class of every $e \in E$ is a unipotent monoid (i.e, a monoid with unique idempotent). This is a generalization of a class known in the literature as weakly ample semigroups (see [8, Section 4]). A natural example of such a semigroup is the monoid $\mathcal{PT}_X$ of all partial functions on a set $X$. In Section 3 we prove that this class is a quasivariety in signature $(2,1,1)$ and in the finite case it is a pseudovariety. We also discuss cases when such semigroups are embeddable in $\mathcal{PT}_X$ as a bi-unary semigroup.

Let $k$ be a field. Given a $k$-algebra $A$, it is of interest to describe important modules related to $A$ and to calculate their dimension. For a classical example consider the $\mathbb{C}$-algebra of the symmetric group $S_n$. Its simple modules can be described by means of Young tabloids and its dimension is described by the hook-length formula. Let $S$ be a finite right restriction Ehresmann semigroup such that the $\tilde{H}_E$-class of every projective $e \in E$ is a group. We call such
semigroups right restriction EI-Ehresmann because the corresponding Ehresmann category is an EI-category (i.e., every endomorphism is an isomorphism).

In Section 4 we show that the simple modules of the semigroup algebra \( kS \) (over any field \( k \)) are given by induced Schützenberger modules of the maximal subgroups of \( S \). Unlike inverse semigroups, Ehresmann semigroups algebras need not be semisimple, even over fields of “good” characteristic. Therefore, an Ehresmann semigroup has (non-semisimple) projective modules. Another goal of this paper is to describe the indecomposable projective modules of finite right restriction EI-Ehresmann semigroups. For this we need a construction which replaces Green’s \( \mathcal{L} \)-relation by \( \tilde{\mathcal{L}}_E \)-classes as the basis of induction. Let \( S \) be a finite semigroup and let \( E \subseteq S \) be some subset of idempotents. Choose some \( e \in E \) and let \( \tilde{\mathcal{L}}_E(e) \) be its \( \tilde{\mathcal{L}}_E \)-class. In Section 5 we characterize certain cases where \( G_e \) (the maximal subgroup with unit element \( e \)) acts on the right of \( \tilde{\mathcal{L}}_E(e) \). In particular we prove that \( G_e \) acts on the right of \( \tilde{\mathcal{L}}_E(e) \) for every Ehresmann semigroup. If we denote by \( k\tilde{\mathcal{L}}_E(e) \) the \( k \)-vector space with basis \( \tilde{\mathcal{L}}_E(e) \), this implies that \( k\tilde{\mathcal{L}}_E(e) \otimes_{kG_e} V \) is a \( kS \)-module for every \( kG_e \)-module \( V \).

A very successful way to study algebras of inverse semigroups or their generalizations is to relate them to algebras of some associated category (or a “partial semigroup”). This is often done with some appropriate Möbius function as in [10, 11, 15, 26, 29, 33, 34, 37]. In particular, the second author has proved in [30, 31] the following theorem. Let \( S \) be a finite right restriction Ehresmann semigroup. Then the semigroup algebra \( kS \) (over any field \( k \)) is isomorphic to the category algebra \( kC \) for the associated Ehresmann category \( C \). This result has led to several applications in the representation theory of monoids of partial functions [28, 32]. Now assume again that \( S \) is a finite right restriction EI-Ehresmann semigroup and also assume that the order of every subgroup of \( S \) is invertible in \( k \). In Section 6 we use the isomorphism between the semigroup algebra and the corresponding category algebra mentioned above to prove that any module of the form \( k\tilde{\mathcal{L}}_E(e) \otimes_{kG_e} V \) (where \( V \) is a simple \( kG_e \)-module) is an indecomposable projective module of \( kS \). Moreover, any indecomposable projective module is of this form. This gives a purely semigroup theoretic construction and it is very similar to other known constructions in the representation theory of semigroups [7, 23]. We also give a formula for the dimension of this module over an algebraically closed field. As an application, we consider the monoid \( PT_n \) of all partial functions on the set \( \{1, \ldots, n\} \) in the case \( k = \mathbb{C} \). It was
already proved that the simple and indecomposable injective modules of \(\mathbb{C}PT_n\) can be described using induced and co-induced representations respectively (essentially this is a consequence of [22, Theorem 4.4]). In Section 7 we complete this picture by giving a description of the indecomposable projective modules of \(\mathbb{C}PT_n\) along with the natural epimorphism onto their simple images. The general formula for the dimension of the indecomposable projective modules boils down in this case to a combinatorial formula that sums up certain Kostka numbers. We also draw some conclusions regarding the Cartan matrix of \(\mathbb{C}PT_n\) by showing that certain entries of it are zero. In the appendix of this paper we give a counter-example to the above isomorphism between the algebras of an Ehremsann semigroup and the associated category in the case where \(S\) is not right (or left) restriction hence showing that this requirement cannot be omitted. We also give an example of an EI-Ehresmann semigroup that is neither left nor right restriction but its algebra is isomorphic to the algebra of the corresponding Ehremsmann category.

2 Preliminaries

2.1 Semigroups

Let \(S\) be a semigroup and let \(S^1 = S \cup \{1\}\) be the monoid formed by adjoining a formal unit element. We denote by \(\mathcal{H}, \mathcal{L}, \mathcal{R}\) and \(\mathcal{J}\) the usual Green’s equivalence relations:

\[
\begin{align*}
    a \mathcal{R} b & \iff aS^1 = bS^1 \\
    a \mathcal{L} b & \iff S^1 a = S^1 b \\
    a \mathcal{J} b & \iff S^1 aS^1 = S^1 bS^1
\end{align*}
\]

and \(\mathcal{H} = \mathcal{R} \cap \mathcal{L}\). We denote by \(E(S)\) the set of all idempotents of \(S\). It is well-known that if \(e \in E(S)\) then its \(\mathcal{H}\)-class forms a group that we will denote by \(G_e\). This is the maximal subgroup of \(S\) with unit element \(e\). An element \(a \in S\) is called regular if there exists \(b \in S\) such that \(aba = a\). Now assume that \(S\) is finite. A \(\mathcal{J}\)-class \(J\) is called regular if it contains a regular element. It is known that in this case all its elements are regular and each \(\mathcal{R}\) and \(\mathcal{L}\) class contains an idempotent. Moreover, if \(J\) is a regular \(\mathcal{J}\)-class then for every two idempotents \(e_1, e_2 \in J\) we have \(G_{e_1} \simeq G_{e_2}\). So we can denote this group by \(G_J\)
the unique maximal group associated with \( J \). Let \( J \) be a regular \( J \)-class and fix some idempotent \( e \in J \). Clearly, the \( L(e) \) and \( R(e) \) classes are partitioned into \( H \)-classes. Choose representatives \( \{ \lambda_b \mid b \in B \} \) of the \( H \)-classes of \( R(e) \) and \( \{ \rho_a \mid a \in A \} \) for the \( H \)-classes of \( L(e) \). We denote by \( P^J \) the sandwich matrix of \( J \) relative to these representatives. This is the \( B \times A \) matrix over \( G_e \cup \{0\} \) defined by

\[
P^J_{b,a} = \begin{cases} 
\lambda_b \rho_a & \lambda_b \rho_a \in G_e \\
0 & \text{otherwise.}
\end{cases}
\]

Another important fact is that \( \lambda_b \rho_a \in G_e \) if and only if \( R(\rho_a) \cap L(\lambda_b) \) contains an idempotent. A deeper explanation on the definition of the sandwich matrix and its usefulness can be found in [13, Chapter 3] or [24, Section 4.13]. The sandwich matrix of \( J \) is not unique but it can be shown that properties like right invertibility (over some group algebra of \( G_e \)) does not depend on the choice of representatives or the idempotent \( e \in J \). We denote by \( B_X \) the monoid of all binary relations on a set \( X \) with composition as operation and by \( PT_X \) its submonoid of all partial functions. The submonoid of all total functions is denoted \( T_X \) and \( IS_X \) will be our notation for the submonoid consists of all injective partial functions. In most cases the set \( X \) will be finite so we will write \( B_n, PT_n, T_n \) and \( IS_n \) where \( |X| = n \). We remark that we compose binary relations (and in particular partial functions) from right to left in this paper. A standard textbook for elementary semigroup theory is [13]. For a text devoted to finite semigroups see [24].

### 2.2 Ehresmann semigroups and categories

Let \( S \) be a semigroup and let \( E \subseteq S \) be a subset of idempotents. We define two preorders \( \leq_{L_E} \) and \( \leq_{R_E} \) on \( S \) by

\[
a \leq_{L_E} b \iff (\forall e \in E \quad be = b \Rightarrow ae = a)
\]

\[
a \leq_{R_E} b \iff (\forall e \in E \quad eb = b \Rightarrow ea = a).
\]

The corresponding equivalence relations are denoted \( \tilde{L}_E \) and \( \tilde{R}_E \).

\[
a \tilde{L}_E b \iff (\forall e \in E \quad be = b \Leftrightarrow ae = a)
\]

\[
a \tilde{R}_E b \iff (\forall e \in E \quad eb = b \Leftrightarrow ea = a).
\]
Moreover, we define \( \overline{H}_E = \overline{L}_E \cap \overline{R}_E \). We remark that \( \mathcal{L} \subseteq \overline{L}_E, \mathcal{R} \subseteq \overline{R}_E \) and \( \mathcal{H} \subseteq \overline{H}_E \). A subset \( E \subseteq S \) of idempotents is called a subsemilattice if it is a commutative subsemigroup. It is well known that any commutative semigroup of idempotents has the structure of a semilattice (i.e., a poset where every two elements have a meet) if one defines \( a \leq b \) whenever \( ab = ba = a \). A semigroup \( S \) with a subsemilattice \( E \subseteq S \) is called right \( E \)-Ehresmann if every \( \overline{L}_E \)-class contains a unique idempotent from \( E \) and \( \overline{L}_E \) is a right congruence. We denote the unique idempotent in the \( \overline{L}_E \)-class of \( a \) by \( a^* \). Note that \( a^* \) is the unique minimal element \( e \in E \) such that \( ae = a \). It is well known that \( \overline{L}_E \) is a right congruence if and only if the identity \( (ab)^* = (a^*b)^* \) holds for every \( a, b \in S \).

Dually, we can consider semigroups for which every \( \overline{R}_E \) class contains a unique idempotent. We denote the unique idempotent in the \( \overline{R}_E \) class of \( a \) by \( a^+ \). Such semigroup is called left \( E \)-Ehresmann if \( \overline{R}_E \) is a left congruence, or equivalently if \( (ab)^+ = (ab^+)^+ \) for every \( a, b \in S \). A semigroup \( S \) with a subsemilattice \( E \subseteq S \) is called \( E \)-Ehresmann if it is both left and right \( E \)-Ehresmann. The semilattice \( E \) is also called the set of projections of \( S \).

It is known that \( E \)-Ehresmann semigroups form precisely the variety of \((2, 1, 1)\)-algebras (where \( ^+ \) and \( ^* \) are the unary operations) subject to the identities:

\[
\begin{align*}
x^+x &= x, & (x^+y^+)^+ &= x^+y^+, \\
x^+y^+ &= y^+x^+, & x^+(xy)^+ &= (xy)^+ = (xy^+)^+, \\
xx^* &= x, & (x^*y^*)^+ &= x^*y^+, \\
x^*y^* &= y^*x^*, & (xy)^*y^* &= (xy)^* = (x^*y)^*, \\
x(yz) &= (xy)z, & (x^+)^* &= x^+, & (x^*)^+ &= x^* \\
\end{align*}
\]

Therefore, we can drop the explicit mention of the set \( E \) and consider an Ehresmann semigroup \( S \) to be a bi-uniary semigroup where the set of projections \( E \) is determined implicitly by the unary operations:

\[ E = \{a^* \mid a \in S\} = \{a^+ \mid a \in S\}. \]

A right (left) Ehresmann semigroup \( S \) is called right (respectively, left) restric-
tion if the “right ample” (respectively, “left ample”) identity $ea = a(ea)^*$ (respectively, $ae = (ae)^+a$) holds for every $a \in S$ and $e \in E$.

For every Ehresmann semigroup $S$, we can associate a category $\mathcal{C}(S) = \mathcal{C}$ in the following way. The object set of $\mathcal{C}(S)$ is the set $E$ of projections and the morphisms of $\mathcal{C}(S)$ are in one-to-one correspondence with elements of $S$. For every $a \in S$, we associate a morphism $C(a) \in \mathcal{C}^1$ with domain $a^+$ and range $a^*$. If the range of $C(a)$ is the domain of $C(b)$ the composition $C(a) \cdot C(b)$ is defined to be $C(ab)$. The convention is to compose morphisms in Ehresmann categories “from left to right”. However, for us it will be more convenient to use composition “from right to left”. In this case, we will associate to every $a \in S$ a morphism $C(a) \in \mathcal{C}^1$ with domain $a^*$ and range $a^+$ and if the range of $C(a)$ is the domain of $C(b)$ the composition $C(b) \cdot C(a)$ is defined to be $C(ba)$. For more facts and proofs on Ehresmann semigroups the reader is referred to [8, 9].

2.3 Algebras and representations

Let $A$ be a $k$-algebra where $k$ is a field. Unless stated otherwise, we assume that algebras are unital and finite dimensional. Likewise, when we say that $M$ is a module over $A$ (or an $A$-module) we mean that $M$ is a finite dimensional left module over $A$. We will denote the set of simple or irreducible modules of $A$ by $\text{Irr } A$. Recall that two idempotents $e, f \in A$ are called orthogonal if $ef = fe = 0$. A non-zero idempotent $p \in A$ is called primitive if it is not a sum of two non-zero orthogonal idempotents. A complete set of primitive orthogonal idempotents is a set $\{p_1, \ldots, p_k\}$ of primitive, pairwise orthogonal idempotents such that $p_1 + \ldots + p_k = 1_A$. Recall that an $A$-module $P$ is called projective if $\text{Hom}_A(P, -)$ is an exact functor, or equivalently, if $P$ is a direct summand of a free module $A^k$ for some $k \in \mathbb{N}$. In particular, any module of the form $Ae$ for an idempotent $e \in A$ is projective. It is well known that if $\{p_1, \ldots, p_k\}$ is a complete set of primitive orthogonal idempotents then $Ap_1, \ldots, Ap_k$ are all indecomposable projective modules and every indecomposable projective module $P$ is isomorphic to some $Ap_i$. Moreover, it is well known that every simple module $S$ is the maximal semisimple image of some indecomposable projective module $Ap_i$. In this case $Ap_i$ is the projective cover of $S$. In general $Ap_i$ might be isomorphic to $Ap_j$ for $i \neq j$. In this case we say that the idempotents $p_i$ and $p_j$ are equivalent. Other basic facts about algebras and their representations can be found in [2]. We will be interested in semigroup algebras and category algebras. The semigroup algebra $kS$ of a finite semigroup $S$ is defined in the following way. It is a vector
space over $k$ with basis the elements of $S$, that is, it consists of all formal linear combinations:
\[ \{ k_1 s_1 + \ldots + k_n s_n \mid k_i \in k, s_i \in S \} . \]

The multiplication in $kS$ is the linear extension of the semigroup multiplication.

The category algebra $kC$ of a finite category $C$ is defined in the following way. It is a vector space over $k$ with basis the morphisms of $C$, that is, it consists of all formal linear combinations:
\[ \{ k_1 m_1 + \ldots + k_n m_n \mid k_i \in k, m_i \in C \} . \]

The multiplication in $kC$ is the linear extension of the following:
\[ m' \cdot m = \begin{cases} m'm & \text{if } m' \cdot m \text{ is defined} \\ 0 & \text{otherwise} \end{cases} \]

### 2.4 Representations of finite Semigroups and Groups

Let $S$ be a finite semigroup, let $J$ be regular $J$-class and fix some idempotent $e \in J$. Let $k$ be a field. The semigroup $S$ acts by partial functions on $L(e)$, the $L$-class of $e$, according to
\[ s \cdot x = \begin{cases} sx & sx \in L(e) \\ \text{undefined} & \text{otherwise} \end{cases} \]

for $s \in S$ and $x \in L(e)$. Moreover, the group $G_e \simeq G_J$ acts on $L(e)$ by right multiplication. Therefore, $kL(e)$ is a $kS - kG_J$ bi-module (where $kL(e)$ is the set of all formal linear combinations of elements of $L(e)$). Dually, $kR(e)$ is a $kG_J - kS$ bi-module. Let $V$ be a $kG_J$-module, we define the induced Schützenberger modules corresponding to $J$ and $V$ as follows:
\[ \text{Ind}_{G_J}(V) = kL(e) \bigotimes_{kG_J} V \]

It can be shown that this module does not depend on the specific choice of $e \in J$. This justify the notation $\text{Ind}_{G_J}(V)$. If $V \in \text{Irr} kG_J$ then the maximal semisimple image of $\text{Ind}_{G_J}(V)$ is a simple $kS$-module. Moreover, the Clifford-Munn-Ponizovskii theorem says that this induces a one-to-one correspondence between simple modules of $kS$ and pairs $(J, V)$ where $J$ is regular $J$ class and
Let $G$ be a finite group. The well known Maschke’s theorem says that $kG$ is semisimple if and only if the order of $G$ is invertible in $k$. Let $H \subseteq G$ be a subgroup and let $U$ a $kH$-module. We denote by $\text{Ind}^G_H U$ the induced module, which is a $kG$-module defined by

$$\text{Ind}^G_H U = kG \otimes_{kH} U.$$ 

Assume that $G$ acts by permutations on some finite set $X$. Denote by $kX$ the vector space of all linear combinations of elements of $X$. $kX$ is a $kG$-module in the natural way. A module of this form is called a permutation module. It is well known that if $X_1, \ldots, X_r$ are the orbits of this action then $kX = kX_1 \oplus \cdots \oplus kX_r$. Now assume that $G$ acts transitively on $X$ and let $K$ be the stabilizer of some $x \in X$. It is well known that $kX \simeq \text{Ind}^G_K \text{tr}_K$ (where $\text{tr}_K$ is the trivial module of $kK$), no matter which $x \in X$ is chosen.

We recall some elementary facts regarding the representation theory of the symmetric group $S_n$ in the case $k = \mathbb{C}$. More details can be found in [14, 25]. Recall that an integer composition of $k$, denoted $\lambda \vdash k$, is a tuple $\lambda = [\lambda_1, \ldots, \lambda_r]$ of positive integers such that $\lambda_1 + \cdots + \lambda_r = k$ while an integer partition of $k$, denoted $\lambda \vdash k$, is an integer composition such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$. If $\lambda = [\lambda_1, \ldots, \lambda_r] \vdash k$ is a composition we denote $|\lambda| = r$. It is well known that irreducible modules of $S_k$ are indexed by integer partitions of $k$. We denote the irreducible module associated to the partition $\lambda$ (also called its Specht module) by $S^\lambda$. An explicit description of $S^\lambda$ can be found in [25, Section 2.3].

We can associate to any partition $\lambda = [\lambda_1, \ldots, \lambda_r] \vdash k$ a graphical description called a Young diagram, which is a table with $\lambda_i$ boxes in its $i$-th row. For instance the Young diagram associated to the partition $[3, 3, 2, 1]$ of 9 is:

$$
\begin{array}{c|c|c|c}
\hline
& & & \\
\hline
& & & \\
\hline
& & & \\
\hline
& & & \\
\hline
\end{array}
$$

We identify the two notions and regard integer partition and Young diagram
as synonyms. A Young tableau is a Young diagram whose boxes are filled with numbers. We call the original diagram the shape of the tableau. Let \( t \) be a Young tableau with \( k \) boxes such that the number of boxes with entry \( i \) is \( \mu_i \). The content of \( t \) is the composition \( \mu = [\mu_1, \ldots, \mu_1] \). We say that a Young tableau is semi-standard if its columns are increasing and its rows are non-decreasing. The Kostka number \( K_{\lambda \mu} \) is the number of semistandard Young tableaux with shape \( \lambda \) and content \( \mu \) (see [25, Section 2.11]).

For every composition \( \lambda = [\lambda_1, \ldots, \lambda_r] \vdash k \) we can associate the subgroup \( S_{\lambda} = S_{\lambda_1} \times \cdots \times S_{\lambda_r} \) of \( S_k \), called the Young subgroup corresponding to \( \lambda \). Let \( \mu = [\mu_1, \ldots, \mu_r] \vdash k \) and denote by \( tr_\mu \) the trivial module of the Young subgroup \( S_\mu \). The module \( \text{Ind}_{S_\mu}^{S_k} tr_\mu \) is called the Young module corresponding to \( \mu \). For every partition \( \lambda \vdash k \) the multiplicity of \( S^\lambda \) in the decomposition of \( \text{Ind}_{S_\mu}^{S_k} tr_\mu \) is precisely \( K_{\lambda \mu} \) - the Kostka number of \( \lambda, \mu \).

3 Some Classes of Ehresmann semigroups

By an endomorphism of a category \( C \) we mean a morphism \( f \) whose domain and range are the same object. For an object \( c \) of \( C \) we call \( C(c, c) \), the endomorphism monoid at \( c \).

**Definition 3.1.** A category is called an **EI-category** if every endomorphism is an isomorphism, i.e, the endomorphism monoids are groups.

There is a lot in the literature about representations of EI-categories and their applications (for few examples see [19, 21, 36, 38]).

**Definition 3.2.** Let \( S \) be an Ehresmann semigroup. We call \( S \) an **EI-Ehresmann** semigroup if \( \tilde{H}_E(e) \) is a group for every projection \( e \in E \).

Note that \( a \in \tilde{H}_E(e) \) if and only if \( C(a) \) is an endomorphism of \( e \) in the associated Ehresmann category \( C(S) \). Therefore \( S \) is EI-Ehresmann if and only if the associated Ehresmann category \( C(S) \) is an EI-category.

In this section we deal also with infinite semigroups. For this case we need a slightly more general definition.

**Definition 3.3.** A monoid is called **unipotent** if the identity is its only idempotent.
Definition 3.4. Let $S$ be an Ehresmann semigroup. We call $S$ an *EU-Ehresmann* semigroup if $\tilde{H}_E(e)$ is a unipotent for every projection $e \in E$.

Note that $S$ is EU-Ehresmann if and only if the endomorphism monoids of $C(S)$ are unipotent (the “EU” stands for “endomorphism unipotent”).

If $S$ is finite, the notions of EU-Ehresmann and EI-Ehresmann coincide because a finite monoid is unipotent if and only if it is a group. In the following sections we consider only finite semigroups so we will use “EI-Ehresmann” to keep our terminology compatible with the literature on EI-categories. However in this section only we want to discuss infinite semigroups as well so we will use “EU-Ehresmann”.

3.1 EU-Ehresmann semigroups

We start with some characterizations of EU-Ehresmann semigroups. Recall that the *sandwich set* of two idempotents $f_1, f_2 \in S$ is defined by

$$S(f_1, f_2) = \{ h \in E(S) \mid f_2 hf_1 = h, \quad f_1 hf_2 = f_1 f_2 \}.$$  

Let $V(s)$ be the set of inverses of an element $s \in S$. It is well known that the sandwich set is non-empty if and only if $f_1 f_2$ is a regular element of $S$. Furthermore, in this case,

$$S(f_1, f_2) = \{ h \in V(f_1 f_2) \cap E(S) \mid f_2 hf_1 = h \}.$$  

In particular, if $f_1$ and $f_2$ commute, then $S(f_1, f_2)$ is non-empty since in this case, $f_1 f_2$ is an idempotent. See [10, Chapter 2.5] for details.

Lemma 3.5. Let $S$ be an Ehresmann semigroup, The following are equivalent.

1. $S$ is an EU-Ehresmann semigroup.
2. Every idempotent $f \in E(S)$ satisfies $f^* f f^+ = f^* f^+$.  
3. The projection $f^* f^+$ is an inverse of $f$ for every idempotent $f \in E(S)$.
4. Every idempotent $f \in E(S)$ satisfies $f \in S(f^*, f^+)$.  

Proof. (1 $\implies$ 2) Observe that $f^* f f^+$ is an idempotent since

$$f^* f f^+ \cdot f^* f f^+ = f^* f f^* f f^+ f^+ = f^* f f f^+ = f^* f f^+.$$  

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Moreover,
\[
(f^* f f^*)^+ = \left( f^* (f f^*)^+ \right)^+ = f^* (f f^+) = f^* f^+ + \]
where the first and third equalities follow from the left congruence identity \((ab)^+ = (ab^+)\). Likewise
\[
(f^* f f^+) = \left( (f^* f)^+ \right) = (f f)^+ f^+ = f^* f^+
\]
where the first and third equalities follow from the right congruence identity \((ab)^+ = (a^* b)^*\). Therefore \(f^* f f^+\) is \(\tilde{H}_E\)-equivalent to \(f^* f^+\).

The unipotency of \(\tilde{H}_E\) \((f^* f)^+\) implies that \(f^* f f^+ = f^* f^+\).

\([2 \implies 1]\) Let \(e \in E\) and let \(f \in E(S)\) be an idempotent such that \(f \tilde{H}_E e\) so \(f^+ = f^* = e\). The assumption \(f^* f f^+ = f^* f^+\) now reduces to
\[
e f e = e e = e
\]

and clearly \(e f e = f^+ f f^* = f\) so \(f = e\) as required.

\([2 \iff 3 \iff 4]\) Immediate from the properties of sandwich sets noted before this Lemma.

\[\square\]

We can interpret the last result in the language of universal algebra, namely (pseudo)varieties and quasivarieties. A variety of universal algebras, is a collection of universal algebras of the same type closed under direct product, subalgebra and homomorphic image. A pseudovariety is a collection of finite universal algebras of the same type closed under direct product of two members, subalgebra and homomorphic image. By a Theorem of Birkhoff, varieties are precisely the collections of algebras that can be defined by identities. By a Theorem of Reiterman, pseudovarieties are precisely the collections of finite algebras that can be defined by profinite identities, that is, by identities over free profinite algebras. We refer to [1] for details. A quasivariety is a collection of algebras of the same type that are closed under subalgebras, direct product and ultraproducts. Equivalently it is a class of all algebras of a set of quasiidentities, that is, implications of the form
\[
s_1 = t_1 \land \ldots \land s_n = t_n \rightarrow s = t
\]
where \( s, s_1, \ldots, s_n, t, t_1, \ldots, t_n \) are terms built up from variables using the operation symbols of the specified type. See [3] for details.

The equivalence \( 1 \iff 2 \) of Lemma 3.5 immediately implies the following corollary in the language of quasivarieties.

**Corollary 3.6.** The class of all EU-Ehresmann semigroups is a quasivariety of bi-unary semigroups defined by the identities of Ehresmann semigroups \((EH)\) (see in Section 2.2) and the quasoidentity

\[
f^2 = f \rightarrow f^*ff^+ = f^*f^+.
\]

Another corollary regards the finite case. If \( x \) is an element of a free profinite semigroup, then \( x^\omega \) is the unique idempotent in the closed subsemigroup generated by \( x \). In particular, if \( S \) is a finite semigroup, then \( x^\omega \) is the unique idempotent in the subsemigroup generated by \( x \).

**Corollary 3.7.** The class of all finite EU-Ehresmann (or EI-Ehresmann) semigroups is a pseudovariety of finite bi-unary semigroups defined by the identities of Ehresmann semigroups \((EH)\) and the profinite identity

\[
(x^\omega)^* x^\omega (x^\omega)^+ = (x^\omega)^* (x^\omega)^+.
\]

**Proof.** Let \( S \) be a finite Ehresmann semigroup. If \( S \) is EU-Ehresmann then the required profinite identity holds by Lemma 3.5 because \( x^\omega \) is an idempotent for every \( x \in S \). In the other direction, take \( f \in E(S) \). Then \( f^\omega = f \) so the assumption yields

\[
f^*ff^+ = f^*f^+
\]

which implies that \( S \) is EU-Ehresmann by Lemma 3.5. \( \square \)

Let \( V \) be a pseudovariety of finite groups and let \( \mathbf{V} \) be the pseudovariety of finite semigroups whose subgroups belong to \( V \). Note that if \( S \) is a finite EU-Ehresmann semigroups then every \( \mathcal{H}_E(e) \)-class is a group and in particular, \( \mathcal{H}_E(e) = \mathcal{H}(e) \) for every \( e \in E \). Therefore the pseudovariety of all finite Ehresmann semigroups whose \( \mathcal{H}_E \)-classes (or equivalently, the endomorphism groups in the category \( \mathbf{C}(S) \)) belongs to \( V \) is precisely the intersection between \( \mathbf{V} \) and the pseudovariety of Corollary 3.7.

**Example 3.8.** Consider the pseudovariety of all Ehresmann semigroups such that the corresponding Ehresmann category \( \mathbf{C}(S) \) is locally trivial (i.e, the only
endomorphisms are the identity functions). Some natural monoids which belong to this pseudovariety are discussed in [28]. In this case the \( H \)-classes are trivial so the subgroups of \( S \) are trivial. Such semigroups are called aperiodic (or \( H \)-trivial). This pseudovariety can be described by the Ehresmann identities (EH), and the profinite identities

\[
(x^\omega)^* x^\omega (x^\omega)^+ = (x^\omega)^* (x^\omega)^+ , \quad x^\omega x = x^\omega
\]

where the second one is for the aperiodicity of \( S \).

### 3.2 Right restriction EU-Ehresmann semigroups

Now we turn to the main class of semigroups that we want to discuss in this paper: Semigroups which are both EU-Ehresmann and right restriction. We start with the following lemma.

**Lemma 3.9.** Let \( S \) be a right restriction Ehresmann semigroup. Then \( f^+ \leq f^* \) for every idempotent \( f \in E(S) \).

**Proof.** Let \( f \in E(S) \) be an idempotent. According to the right congruence identity

\[
f^* f^+ = (f^* f^+)^+ = (f^* f)^+
\]

and according to the right ample identity

\[
(f^* f)^+ = (f (f^*)^+)^+ .
\]

Now, by the left congruence identity

\[
(f (f^*)^+)^+. = (f (f^*)^+)^+ = (f f^*)^+ = f^+
\]

so \( f^* f^+ = f^+ \) and thus \( f^+ \leq f^* \) as required. \( \square \)

**Lemma 3.10.** Let \( S \) be a right restriction Ehresmann semigroup. The following are equivalent.

1. \( S \) is an EU-Ehresmann semigroup.

2. Every idempotent \( f \in E(S) \) satisfies \( f f^+ = f^+ \).

\(^1\)We thank Professor Victoria Gould for this proof.
3. Every regular element \( a \in S \) satisfies \( a R a^+ \).

Proof. (1 \( \Rightarrow \) 2) If \( S \) is EU-Ehresmann then \( f^* f f^+ = f^* f^+ \) by Lemma 3.5

Lemma 3.9 implies that

\[ f^* f = f^* f^+ f = f^+ f = f \]

so we establish \( ff^+ = f^+ \) as required.

(2 \( \Rightarrow \) 3) If \( a \) is regular then \( a R f \) for some idempotent \( f \in E(S) \) and clearly \( f^+ = a^+ \). Now \( ff^+ = f^+ \) implies \( f R f^+ \) hence \( a R a^+ \) as required.

(3 \( \Rightarrow \) 1) Every idempotent \( f \in E(S) \) is regular, so \( f R f^+ \) which says that \( fx = f^+ \) for some \( x \in S \). Now

\[ ff^+ = ffx = fx = f^+ \]

so \( f^* ff^+ = f^* f^+ \) hence \( S \) is EU-Ehresmann by Lemma 3.5.

Remark 3.11. A sandwich set \( S(f_1, f_2) \), if not empty, is a rectangular band ([13, Proposition 2.5.3]), i.e., a subsemigroup satisfying \( ghg = g \) for every \( g, h \in S(f_1, f_2) \). The property \( ff^+ = f^+ \) implies that \( S(f^*, f^+) \) is a right zero semigroup (a rectangular band with one \( R \)-class). Indeed if \( h \in S(f^*, f^+) \) then

\[ fh = ff^+ hf^* = f^+ hf^* = h \]

and hence

\[ f = fhf = hf \]

so every element of \( S(f^*, f^+) \) is \( R \)-equivalent to \( f \).

Example 3.12. The following example, taken from [30, Example 5.12], shows that not every EU-Ehresmann semigroup is right or left restriction, so right restriction EU-Ehresmann semigroups form a proper subclass of EU-Ehresmann semigroups.

Recall that \( T_n \) is the monoid of all total functions on an \( n \)-element set. Denote by \( id \) the identity function and by \( k \) the constant functions that sends every element to \( k \). Define \( S \) to be the subsemigroup of \( T_2^{op} \times T_2 \) containing the six elements

\( (1, 1), (2, 1), (1, 2), (2, 2), (1, id), (id, 1) \)
It can be checked that $S$ is an $E$-Ehresmann semigroup with

$$E = \{(1, 1), (1, \text{id}), (\text{id}, 1)\}$$

as its set of projections. The corresponding Ehresmann category $C$ is given by the following drawing. Recall that composition is right to left.

Clearly, $C$ is an EI-category (in fact, it is even locally trivial) so $S$ is an EI-Ehresmann semigroup. However, it is easy to see that $(2, 2)^+ = (1, \text{id})$ and $(2, 2)^* = (\text{id}, 1)$ but $(2, 2)$ is not $C$-related to $(1, \text{id})$ and not $R$-related to $(\text{id}, 1)$ so $S$ is neither left nor right restriction. Note also that the sandwich set

$$S((\text{id}, 1), (1, \text{id})) = \{(1, 1), (2, 1), (1, 2), (2, 2)\}$$

is a rectangular band but not a right/\left zero semigroup.

As in the case of general EU-Ehresmann semigroups, we have two immediate corollaries of Lemma 3.10.

**Corollary 3.13.** The class of all right restriction EU-Ehresmann semigroups is a quasivariety of bi-unary semigroups defined by the identities of Ehresmann semigroups $\text{EH}$, the right ample identity $ea = a(ea)^*$ and the quasiidentity

$$f^2 = f \rightarrow ff^+ = f^+.$$

**Corollary 3.14.** The class of all finite right restriction EU-Ehresmann (or EI-Ehresmann) semigroups is a pseudovariety of finite bi-unary semigroups defined by the identities of Ehresmann semigroups $\text{EH}$, the right ample identity $ea = a(ea)^*$ and the profinite identity

$$x^\omega (x^\omega)^+ = (x^\omega)^+.$$
The most natural example of a right restriction EU-Ehresmann semigroup is the monoid $\mathcal{PT}_X$ of all partial functions on a set $X$. In Section 7 we will consider at length its representation theory in the case where $X$ is finite. Let $A \subseteq X$ and denote by $\text{id}_A$ the partial identity function on $A$. It is clear that $E = \{\text{id}_A \mid A \subseteq X\}$ is a subsemilattice of $\mathcal{PT}_X$ and it is well known that $\mathcal{PT}_X$ is a right restriction $E$-Ehresmann semigroup. Let $t : A \to B$ be a partial function. We denote by $\text{dom}(t) \subseteq A$ and $\text{im}(t) \subseteq B$ the domain and image of $t$ and it is clear that $t^+ = 1_{\text{im}(t)}$ and $t^* = 1_{\text{dom}(t)}$. It is also clear that an idempotent $f \in E(\mathcal{PT}_X)$ must satisfy $f|_{\text{im}(f)} = 1_{\text{im}(f)}$ so the only idempotent $f \in \mathcal{PT}_X$ with $\text{dom}(f) = \text{im}(f)$ is $f = 1_{\text{dom}(f)}$. Therefore, $\mathcal{PT}_X$ is indeed an EU-Ehresmann semigroup. Since right restriction EU-Ehresmann semigroups form a quasivariety, it is clear that any $(2,1,1)$-subalgebra of $\mathcal{PT}_X$ is also a right restriction EU-Ehresmann semigroup. We now show that at least for regular semigroups the converse also holds.

**Proposition 3.15.** Let $S$ be a regular right restriction EU-Ehresmann semigroup. Then there is an embedding of bi-unary semigroups $\Phi : S \to \mathcal{PT}_S$.

**Proof.** The “Cayley theorem” for right restriction semigroups (see [8, Theorem 6.2]) says that there is a semigroup monomorphism $\Phi : S \to \mathcal{PT}_S$ such that $\Phi(a^+) = \Phi(a)^*$ for every $a \in S$. It remains to show that $\Phi(a^+) = \Phi(a)^+$ as well. First note that

$$\Phi(a^+) = \Phi\left((a^+)^*\right) = \Phi\left(a^+\right)^*$$

so $\Phi(a^+)$ is a partial identity of $\mathcal{PT}_X$. Lemma 3.10 and the fact that $S$ is regular implies that $a \mathcal{R} a^+$ for every $a \in S$. Therefore $\Phi(a) \mathcal{R} \Phi(a^+)$ in $\mathcal{PT}_X$. Since $\Phi(a)^+$ is the only partial identity which is $\mathcal{R}$ equivalent to $\Phi(a)$ in $\mathcal{PT}_X$ we must conclude that $\Phi(a^+) = \Phi(a)^+$ as required.

We leave open the question of whether every right restriction EU-Ehresmann semigroup is embeddable in $\mathcal{PT}_X$ for some $X$ as a bi-unary semigroup.

### 3.3 EU-restriction semigroups

If a semigroup $S$ is both EU-Ehresmann and restriction (i.e., left and right restriction), the situation is reduced to a familiar one. A restriction semigroup $S$ with the property that $E = E(S)$ is called weakly ample (see [8, Section 4]).
Lemma 3.16. Let $S$ be a restriction semigroup. Then $S$ is EU-Ehresmann if and only if $S$ is weakly ample.

Proof. Clearly $E = E(S)$ implies that $e$ is the unique idempotent of $\tilde{H}_E(e)$ for every $e \in E$. In the other direction Lemma 3.9 and its dual implies that for every idempotent $f \in E(S)$ we have $f^+ \leq f^*$ and $f^* \leq f^+$ hence $f^+ = f^*$. This says that every idempotent $f$ is $\tilde{H}_E$-equivalent to some projection $e \in E$ and being EU-Ehresmann implies $f = e$. \hfill $\square$

4 Simple modules of finite right restriction EI-Ehresmann semigroups

From now on we discuss only finite semigroups so we switch our terminology and talk about EI-Ehresmann semigroups instead of EU-Ehresmann. The focus will be on the case of right restriction EI-Ehresmann semigroups. In this section we would like to describe the simple modules of algebras of such semigroups and in certain cases also the indecomposable injective ones. We remark that certain observations on the semisimple image of such semigroup algebras can be found in [30, Proposition 5.17].

Lemma 4.1. Let $S$ be a right restriction EI-Ehresmann semigroup and let $e \in E$. Then $e$ is the only idempotent in $\mathcal{L}(e)$.

Proof. Assume $f \in \mathcal{L}(e)$ is an idempotent so $ef = e$ and $fe = f$. Lemma 3.10 implies that $ff^+ = f^+$ and clearly $f^+ f = f$. Now,

$$
e = ef = ef^+ f = f^+ ef = f^+ e$$

$$f^+ = ff^+ = fef^+ = f f^+ e = f^+ e$$

so $e = f^+$. Therefore $f \mathcal{L} e$ and $f \mathcal{R} f^+ = e$ so $f \mathcal{H} e$ which implies $f = e$. \hfill $\square$

Proposition 4.2. Let $S$ be a finite right restriction EI-semigroup. Let $J$ be a regular $\mathcal{J}$-class of $S$. Then the sandwich matrix of $J$ is left invertible over $\mathbb{k}G_J$ for every field $\mathbb{k}$.

Proof. Let $E_J = E \cap J = \{e_1, \ldots, e_k\}$ be the set of projections which belongs to $J$. An $\mathcal{R}$-class cannot have two projections and a regular $\mathcal{R}$-class has at
least one by Lemma 3.10 so $J$ has precisely $k$ $R$-classes. Moreover, $e_1, \ldots, e_k$ are also in different $L$-classes. Fix $e = e_1$ and choose representatives $\rho_1, \ldots, \rho_k$ for the $H$-classes of $L(e)$ and $\lambda_1, \ldots, \lambda_l$ for the $H$-classes of $R(e)$ (note that $l \geq k$). Since projections of $E_J$ are in different $L$ and $R$-classes, we can order the representatives such that $e_i \in R(\rho_i) \cap L(\lambda_i)$ for $i \neq j$ and $1 \leq i, j \leq k$ since $e_j$ is the unique idempotent in $L(\lambda_j)$ (note that there can be idempotents in $R(\rho_i) \cap L(\lambda_j)$ for $j > k$). Therefore, the sandwich matrix is a $l \times k$ matrix whose upper $k \times k$ block is a diagonal matrix and all the entries on the main diagonal are non-zero. Therefore, the sandwich matrix is left invertible over $kG_J$.

If the sandwich matrix of a regular $J$-class $J$ is left invertible then the module induced from a left Schützenberger module of $J$ and $V \in \text{Irr}_{kG_J}$ is simple, see [35, Corollary 4.22] and [35, Lemma 5.20]. This fact and the Clifford-Munn-Ponizovskii theorem yields the following corollary.

**Theorem 4.3.** Let $S$ be a finite right restriction EI-Ehresmann semigroup and let $k$ be a field. Let $J$ be a regular $J$-class and let $V \in \text{Irr}_{kG_J}$. Then

$$\text{Ind}_{G_J}(V) = kL(e) \bigotimes_{kG_J} V$$

(for some idempotent $e \in J$) is a simple module of $kS$. In fact,

$$\{ \text{Ind}_{G_J}(V) \mid J \text{ is a regular } J\text{-class}, \ V \in \text{Irr}_{kG_J} \}$$

is a list of all the simple modules of $kS$ up to isomorphism with no isomorphic copies.

**Remark 4.4.** Assume in addition that $S$ is regular and that the orders of all subgroups of $S$ are invertible in $k$. Then the left invertibility of the sandwich matrices implies that the indecomposable injective modules of $S$ are given by the co-induced modules for regular $J$ classes and $V \in \text{Irr}_G$ (see [22, Theorem 4.4]). The co-induced module is define by

$$\text{Coind}_{G_J}(V) = \text{Hom}_{kG_J}(kR(e), V)$$

(for some idempotent $e \in J$). In this case $\text{Coind}_{G_J}(V)$ is the injective envelope of $\text{Ind}_{G_J}(V)$. In addition, it follows that in this case the global dimension of the algebra of $S$ is bounded by one less than the longest chain of $J$-classes of
S. The above results on indecomposable injective modules are true even if we replace the requirement of regularity by being left Fountain, see [23, Theorem 4.8].

5 Construction of $\tilde{L}_E$ - Modules

Our next goal is to obtain a description of indecomposable projective modules of finite right restriction EI-Ehresmann semigroups. The description involves an induction using the $\tilde{L}_E$ relation instead of $L$. In this intermediate section we consider several cases where such construction yields a well defined $kS$-module structure.

Let $S$ be a fixed semigroup and let $E \subseteq S$ be a subset of idempotents.

Lemma 5.1. Let $a \in S$ and denote by $\tilde{L}_E(a)$ the $\tilde{L}_E$-class of $a$. Then $S$ acts by partial functions on $\tilde{L}_E(a)$ according to

$$s \cdot x = \begin{cases} sx & \text{if } sx \in \tilde{L}_E(a) \\ \text{undefined} & \text{otherwise} \end{cases}$$

for $s \in S$ and $x \in \tilde{L}_E(a)$.

Proof. The set $L_1 = \{ x \in S \mid x \leq_{L_E} a \}$ is a left ideal of $S$. For if $x \in L_1$ and $s \in S$ then

$$ae = a \implies xe = x \implies sxe = sx$$

for every $e \in E$, so $sx \in L_1$ as well. The set $L_2 = \{ x \in S \mid x \not\leq_{L_E} a \}$ is also a left ideal of $S$. Indeed, assume $x \in L_2$ but $sx\tilde{L}_Ea$ for some $s \in S$. Then

$$xe = x \implies sxe = sx \implies ae = a$$

for every $e \in E$, so $x\tilde{L}_Ea$, a contradiction. Therefore, $S$ acts by left multiplication on $L_1$ and $L_2$. It is now clear that the required action is the Rees quotient $L_1/L_2$. \hfill \Box

Now, choose some projection $e \in E$. Recall that we denote the group $\mathcal{H}$-class of $e$ by $G_e$. In general, right multiplication does not induce a right action of the group $G_e$ on $\tilde{L}_E(e)$. For instance consider the following example.
Example 5.2. Let $S = PT_2$ and recall that we compose from right to left. Choose $E = \{ \text{id} = \text{id}_{\{1,2\}}, \text{id}_{\{1\}}, \text{id}_{\emptyset} \}$ and $e = \text{id}$ is the identity function. We have $\text{id}_{\{2\}} \bar{L}_E \text{id}$ but we can take the transposition $(12) \in G_e = S_2$ and acting on the right we obtain

$$\text{id}_{\{2\}}(12) = \begin{pmatrix} 1 & 2 \\ 2 & \emptyset \end{pmatrix}$$

which is not in the $\bar{L}_E$-class of id.

We want to consider cases where the group $G_e$ acts on the right of $\bar{L}_E(e)$ by right multiplication.

Definition 5.3. A semigroup $S$ is called right Fountain if every $\bar{L}_E(S)$ class contains an idempotent, where $E(S)$ is the set of all idempotents of $S$.

Proposition 5.4 ([24 Corollary 3.4]). Let $S$ be a finite right Fountain semigroup and let $e \in E(S)$ be an idempotent. Then $G_e$ acts on the right of $\bar{L}_E(S)(e)$.

Another case is where $\bar{L}_E$ is a right congruence.

Lemma 5.5. Let $S$ be a semigroup such that $\bar{L}_E$ is a right congruence then $G_e$ acts on the right of $\bar{L}_E(e)$ by right multiplication for every $e \in E$.

Proof. Take some $x \in \bar{L}_E(e)$ and $g \in G_e$. Then $x \bar{L}_E e$ so the right congruence property implies $xg \bar{L}_E eg$. Now $g \bar{H} e$ implies $g \bar{L}_E e$ and $eg = g$ so we have also $xg \bar{L}_E e$ as required. \hfill \Box

Example 5.6. Another trivial case is where $S$ is an $\bar{H}$-trivial semigroup hence $G_e = \{ e \}$ and clearly acts trivially on $\bar{L}_E(e)$.

Recall that $k\bar{L}_E(e)$ is the $k$-vector space with basis the elements of $\bar{L}_E(e)$.

The following is an immediate corollary of Lemma 5.1.

Corollary 5.7. If $G_e$ acts on the right of $\bar{L}_E(e)$ as in the above cases, then $k\bar{L}_E(e)$ is a $kS - kG_e$ bimodule. Therefore, if $V$ is a $kG_e$-module, the tensor product $k\bar{L}_E(e) \bigotimes_{kG_e} V$ is a $kS$-module.

Remark 5.8. As mentioned above, if $\bar{L}_E$ is replaced by the standard Green’s $L$ class then the above module is the induced Schützenberger module of $J$ and $V$ (where $e$ belongs to the $J$-class $J$).
6 Projective modules of right restriction EI-Ehresmann semigroups

In this section we would like to study projective modules of finite Ehresmann and right restriction semigroups. The main result will be in the case of right restriction EI-Ehresmann semigroups. We start with some lemmas that will be of later use.

Lemma 6.1. Let $S$ be an Ehresmann and right restriction semigroup and let $s, m \in S$. Then,

\[
 s \cdot m \in \overline{E}(m) \iff m^+ \leq s^*
\]

(where $\leq$ is the natural partial order on a subsemilattice).

Proof. First note that

\[
 s \cdot m \in \overline{E}(m) \iff (sm)^* = m^*.
\]

If $(sm)^* = m^*$ then by the right ample identity

\[
 s^* m = m(s^* m)^*
\]

and by the right congruence identity

\[
 m(s^* m)^* = m(sm)^* = mm^* = m.
\]

Since $m^+$ is the minimal projection which is a left identity of $m$ we obtain $m^+ \leq s^*$. In the other direction, assume $m^+ \leq s^*$ which implies $s^* m = m$. By the right ample and right congruence identities we have

\[
 m = s^* m = m(s^* m)^* = m(sm)^*
\]

which implies that $m^* \leq (sm)^*$. However, $smm^* = sm$ so $(sm)^* \leq m^*$ hence $(sm)^* = m^*$ as required.

Definition 6.2. Let $\leq$ be a poset on a set $S$ and let $s \in S$. The principal down ideal generated by $s$ is the set $s \downarrow = \{x \in S \mid x \leq s\}$.

Now consider the poset $\leq_I$ on an Ehresmann semigroup $S$. Recall that on the set of projections $E$, the poset $\leq_I$ coincide with the natural semilattice structure
denoted by $\leq$. Therefore

$$s \downarrow = \{ x \in S \mid x \leq_l s \}$$

and

$$s^* \downarrow = \{ e \in E \mid e \leq s^* \}$$

for every $s \in S$.

The following lemma is well-known and we prove it for the sake of completeness.

**Lemma 6.3.** Let $S$ be an Ehresmann semigroup and let $s \in S$. The function $F : s^* \downarrow \to s \downarrow$ defined by

$$f(e) = se$$

is an isomorphism of posets with inverse $G : s \downarrow \to s^* \downarrow$ defined by $G(x) = x^*$.

**Proof.** First note that $e_1 \leq e_2$ implies $se_1 \leq l se_2$ since $se_2 e_1 = se_1$ so $F$ is a homomorphism of posets. Now, $x_1 \leq_l x_2$ says that $x_1 = x_2 x_1^*$. By the right congruence identity

$$x_2^* x_1^* = (x_2^* x_1^*)^* = (x_2 x_1^*)^* = x_1^*$$

so

$$G(x_1) = x_1^* \leq x_2^* = G(x_2)$$

hence $G$ is also a homomorphism of posets. Finally, we show that $F$ and $G$ are inverses of each other:

$$G(F(e)) = G(se) = (se)^* = s^* e = e$$

since $e \leq s^*$ and

$$F(G(x)) = F(x^*) = sx^* = x$$

since $x \leq_l s$ so $x = sx^*$. 

Lawson [17] proved that the category of all Ehresmann semigroups is isomorphic to the category of all Ehresmann categories (with some appropriate choice of morphisms). In fact, the function $C$ described in Section 2.2 is the object part of this isomorphism. Another link between certain Ehresmann semigroups and their corresponding categories appear when we consider their algebras as seen in the following theorem.
**Theorem 6.4** ([31, Theorem 1.5]). Let \( k \) be any commutative unital ring, let \( S \) be a finite Ehresmann and right restriction semigroup and let \( C \) be the corresponding Ehresmann category. There is an isomorphism of \( k \)-algebras \( kS \cong kC \). Explicit isomorphisms \( \varphi : kS \to kC \), \( \psi : kC \to kS \) are defined (on basis elements) by

\[
\varphi(s) = \sum_{t \leq l} C(t)
\]

\[
\psi(C(x)) = \sum_{y \leq l} \mu(y, x) y
\]

where \( \mu \) is the Möbius function of the poset \( \leq_l \) (see [27, Chapter 3]).

**Remark 6.5.** This theorem is a generalization of several results due to Solomon [26], Steinberg [33], Guo and Chen [11] and the second author [29]. It was further generalized by Wang [37] to the class of \( P \)-Ehresmann and right \( P \)-restriction semigroups. \( P \)-Ehresmann and \( P \)-restriction semigroups were introduced by Jones in [10].

We want to use the isomorphism of Theorem 6.4 in order to study the projective modules of \( kS \) where \( S \) is an Ehresmann and right restriction semigroup. Let \( k \) be a field, and let \( e \in E \) be a projection. Recall that we consider the composition in the Ehresmann category \( C \) “from right to left” so a morphism \( C(a) \) has domain \( a^* \) and range \( a^+ \). The idempotent \( e \) is an object of \( C \) with identity morphism \( C(e) \). Consider the set \( C \cdot C(e) \) of all the morphisms whose domain is \( e \). The category \( C \) acts on the left of \( C \cdot C(e) \) by concatenation of morphisms. More precisely, recall that an action of \( C \) is a functor \( F \) from \( C \) to the category of sets and functions. Here, for every object \( c \) of \( C \) we set \( F(c) = C(e, c) \) to be the set of all morphisms with domain \( e \) and range \( c \). For every morphism \( C(m) \in C(c_1, c_2) \) the morphism \( F(C(m)) : C(e, c_1) \to C(e, c_2) \) is defined by left composition by \( C(m) \). This action yields a \( kC \) module denoted by \( kC \cdot C(e) \) whose elements are linear combinations of morphisms in \( C \cdot C(e) \). Again, we can describe this module as a functor from \( C \) to the category of vector spaces and linear transformations, but we prefer the description via a category algebra. It is clear that \( kC \cdot C(e) \) is a projective module since \( C(e) \) is an idempotent of \( kC \). Since \( \varphi \) of Theorem 6.4 is an isomorphism, we can view \( kC \cdot C(e) \) also as a \( kS \)-module whose action \( \star \) is defined by

\[
s \star C(m) = \varphi(s) \cdot C(m) = \left( \sum_{t \leq l} C(t) \right) \cdot C(m)
\]
where \( s, m \in S \) and \( \cdot \) is the action of \( kC \) on \( kC \cdot C(e) \). However, this description is not natural from the semigroup point of view, so we would like to describe this module using the structure of the semigroup itself. Let \( \tilde{L}_E(e) \) be the \( \tilde{C}_E \)-class of \( e \). Observe that \( m\tilde{L}_E(e) \) if and only if the domain of \( C(m) \) is \( e \). Therefore, the sets \( \tilde{L}_E(e) \) and \( C \cdot C(e) \) are in one-to-one correspondence. Furthermore we claim the following.

**Proposition 6.6.** There is an isomorphism of \( kS \)-modules

\[
\Phi : k\tilde{L}_E(e) \to kC \cdot C(e)
\]

defined (on basis elements) by

\[
\Phi(m) = C(m)
\]

for every \( m \in S \).

**Proof.** It is clear that \( \Phi \) is an isomorphism of \( k \)-vector spaces. For every \( s \in S \) it is left to prove that

\[
\Phi(s \cdot m) = s \star \Phi(m)
\]

where

\[
s \star \Phi(m) = \varphi(s)\Phi(m) = \sum_{r \leq s^*} C(r) \cdot C(m).
\]

First we prove that \( s \cdot m = 0 \iff \sum_{r \leq s^*} C(r) \cdot C(m) = 0 \). According to Lemma 6.1 \( s \cdot m = 0 \) implies \( m^+ \not\leq s^* \) so by Lemma 6.3 there is no element \( t \leq s \) such that \( t^* = m^+ \) hence

\[
\sum_{r \leq s} C(r) \cdot C(m) = 0.
\]

The converse implication follows in the same manner. Now, assume \( s \cdot m \neq 0 \) so \( m^+ \leq s^* \). By Lemma 6.3 there exists a unique \( t \leq s \) with \( t^* = m^+ \). In other words, there is a unique \( t \leq s \) such that \( C(t) \cdot C(m) \neq 0 \). Therefore,

\[
s \star \Phi(m) = \sum_{r \leq s} C(r) \cdot C(m) = C(t) \cdot C(m) = C(tm).
\]

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Note that $tm = st^*m = sm^+m = sm$ hence

$$\Phi(sm) = C(sm) = C(tm) = s \star \Phi(m)$$

as required. \qed

Let $G_e$ be the group $\mathcal{H}$-class of $e$ and take some $g \in G_e$. According to Lemma [3.6] $G_e$ acts on the right of $\tilde{L}_E(e)$ since $\tilde{L}_E$ is a right congruence. It is also clear that the domain and range of $C(g)$ is $e$, so $G$ acts on the right of $C \cdot C(e)$ as well. Moreover, for every $m \in \tilde{L}_E(e)$ we have

$$C(mg) = C(m)C(g)$$

so $\Phi$ is not only a $\mathbb{k}S$-modules isomorphism but a $\mathbb{k}S - \mathbb{k}G_e$ bimodule isomorphism. Therefore, we obtain the following corollary.

**Corollary 6.7.** Let $S$ be a finite Ehresmann and right restriction semigroup and let $e \in E$. Denote by $G_e$ the group $\mathcal{H}$-class of $e$ and let $V$ be any $G_e$-module. Then there is an isomorphism of $\mathbb{k}S$ modules

$$\mathbb{k}\tilde{L}_E(e) \otimes_{\mathbb{k}G_e} V \simeq \mathbb{k}C \cdot C(e) \otimes_{\mathbb{k}G_e} V.$$  

In particular, if $p \in \mathbb{k}G_e$ is a primitive idempotent of $\mathbb{k}G_e$, and $C(p)$ is the corresponding linear combination of morphisms, there is an isomorphism

$$\mathbb{k}CC(p) = \mathbb{k}CC(e)C(p) \simeq \mathbb{k}C \cdot C(e) \otimes_{\mathbb{k}G_e} \mathbb{k}G_e C(p) \simeq \mathbb{k}\tilde{L}_E(e) \otimes_{\mathbb{k}G_e} \mathbb{k}G_e p.$$  

Clearly, $\mathbb{k}CC(p)$ is a projective module since $C(p)$ is an idempotent. So this gives a semigroup theoretic description of certain projective modules of $S$. However, there is no reason for the indecomposable projective modules to be of this form. The situation is much better if we consider (finite) right restriction EI-Ehresmann semigroups. In this case the corresponding Ehresmann category is an EI-category, i.e., its endomorphism monoids are actually groups.

A proof for the following lemma can be found in [21, Lemma 9.31] or [38, Corollary 4.5].

**Lemma 6.8.** Let $C$ be a finite EI-category and given some object $e$ let $G_e = C(e,e)$ to be its automorphism group and $P_e = \{p_{e_1}^1, \ldots, p_{e_m}^c\}$ to be a complete
set of primitive orthogonal idempotents for $\mathbb{k}G_e$. Then

The set

$$\bigcup_e P_e$$

(where the union is taken over all objects of $\mathcal{C}$) is a complete set of primitive orthogonal idempotents for $\mathbb{k}\mathcal{C}$.

From now on we assume that the order of every (maximal) subgroup of $S$ is invertible in $\mathbb{k}$. This implies that $\mathbb{k}G_e$ is semisimple for every $e \in E$ by Maschke’s theorem.

**Theorem 6.9.** Let $S$ be a finite right restriction EI-Ehresmann semigroup. Let $e \in E$ and $G_e = C(e, e)$ be as above and let $V \in \text{Irr} \mathbb{k}G_e$. Then

$$\mathbb{k}\tilde{L}_E(e) \otimes_{\mathbb{k}G_e} V$$

is an indecomposable projective module of $\mathbb{k}S$. Moreover, every indecomposable projective module of $\mathbb{k}S$ is isomorphic to a module of this form for an appropriate choice of a projection $e \in E$ and a simple $G_e$-module $V$.

*Proof.* This is immediate from Corollary 6.7 and the fact that if $\mathbb{k}G_e$ is semisimple then $V \simeq \mathbb{k}G_e p$ where $p$ is some primitive idempotent of $\mathbb{k}G_e$ which is also a primitive idempotent of $\mathbb{k}\mathcal{C}$ by Lemma 6.8. \hfill \Box

**Remark 6.10.** A similar result for indecomposable projective modules of a certain type of right Fountain monoids is obtained in [23, Theorem 4.7].

**Isomorphic copies of indecomposable projective modules**

The set

$$\{\mathbb{k}\tilde{L}_E(e) \otimes_{\mathbb{k}G_e} V \mid e \in E, \ V \in \text{Irr} G_e\}$$

usually contains isomorphic modules. We want to obtain a list of all indecomposable projective modules up to isomorphism but without isomorphic copies. From the categorical point of view we can say that

$$\mathbb{k}\tilde{L}_E(e_1) \otimes_{\mathbb{k}G_{e_1}} V_1 \simeq \mathbb{k}\tilde{L}_E(e_2) \otimes_{\mathbb{k}G_{e_2}} V_2$$
if and only if $e_1$ is isomorphic to $e_2$ (as objects in a category) and $V_1 \simeq V_2$ as $G_{e_1}$-modules (note that $G_{e_1} \simeq G_{e_2}$) - see [38] Corollary 4.2 and Proposition 4.3]. From the semigroup point of view we can say even more. We start with two lemmas that will make the situation more transparent.

**Lemma 6.11 ([30] Corollary 5.5]).** Let $S$ be a finite Ehresmann semigroup and let $e, f \in E$ be two projections. Then $e$ and $f$ are isomorphic (as objects in the corresponding Ehresmann category) if and only if $e J f$ in $S$.

**Lemma 6.12.** Let $S$ be a finite EI-Ehresmann semigroup. Then in every regular $J$-class of $S$ there is a projection $e \in E$.

**Proof.** Let $J$ be a regular $J$-class and choose an idempotent $f \in J$. Lemma 3.5 says that $f^* f^+$ is an inverse of $f$. Therefore $f J f^* f^+$ so $e = f^* f^+$ is a projection in $J$. \[\square\]

**Remark 6.13.** The converse of Lemma 6.12 is not true. Recall that $B_2$ is the monoid of all binary relations on the set $\{1, 2\}$, i.e., subsets of $\{1, 2\}^2$. Choose $E = \{\text{id}, \text{id}_{\{1\}}, \text{id}_{\{2\}}, \text{id}_\emptyset\}$ to be the set of all partial identities. It is well known that $B_2$ is Ehresmann with $a^+ = 1_{\text{im}(a)}$ and $a^* = 1_{\text{dom}(a)}$ for every $a \in B_2$ (if we compose relations from right to left). Consider the submonoid $M \subseteq B_2$ of the 12 relations whose sizes are 0, 1, 2 or 4 (as subsets of $\{1, 2\}^2$). It is a routine matter to verify that this is indeed a submonoid of $B_2$. Since $E \subseteq M$ it is clear that $M$ is also an Ehresmann semigroup. It is also routine to check that $M$ has three $J$-classes: One of the two invertible elements, one with the zero element and another with the other 9 elements. Therefore, every $J$-class is regular and contains a projection. On the other hand, choose the idempotent $f = \{(1, 1), (2, 2), (1, 2), (2, 1)\} \in M$. Since $\text{dom}(f) = \text{im}(f) = \{1, 2\}$ it is clear that the morphism $C(f)$ is a non-invertible endomorphism of the object $\text{id} \in E$ in the corresponding Ehresmann category. Therefore, the category is not an EI-category and $M$ is not EI-Ehresmann.

According to Lemma 6.11 and Lemma 6.12 choosing one object from every isomorphism class of objects is precisely like choosing one projection from every regular $J$-class. So we can fix a set of representative projections $I = \{e_1, \ldots, e_k\}$, one from every regular $J$-class whose corresponding maximal groups are $G_{J_1}, \ldots, G_{J_k}$.

Now,

$$\left\{ k\tilde{\mathcal{L}}_{E(e_i)} \bigotimes_{k G_{J_i}} V \mid i = 1, \ldots, k \quad V \in \text{Irr} k G_{J_i} \right\}$$

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is a list of all the indecomposable projective modules (up to isomorphism) without isomorphic copies. This is an explicit correspondence between indecomposable projective modules of \( S \) and pairs \((J, V)\) where \( J \) is a regular \( J \)-class and \( V \in \text{Irr} kG_J \). From this point of view, the correspondence between a simple module and its projective cover is clear. It is easy to see that for every representative projection \( e \in I \) there is a \( kS \)-module epimorphism

\[
\psi : k\tilde{L}_E(e) \rightarrow kL(e)
\]

defined on basis elements by

\[
\psi(m) = \begin{cases} 
m & m \in L(e) \\
0 & m \notin L(e).
\end{cases}
\]

This induces an \( kS \)-module epimorphism

\[
\psi \otimes \text{id} : k\tilde{L}_E(e) \otimes_{kG_J} V \rightarrow kL(e) \otimes_{kG_J} V
\]

for every \( V \in \text{Irr} kG_J \) (where \( J \) is the regular \( J \)-class of \( e \)). This is an explicit description of the epimorphism from an indecomposable projective module and its simple image.

**Dimension of indecomposable projective modules** Next we want to describe the dimension of the projective modules. Recall that we assume that the order of every group being discussed is invertible in \( k \) hence its algebra is semisimple and here we also assume that \( k \) is algebraically closed. Recall that if \( V \) is a \( kG \)-module then the dual \( D(V) = \text{Hom}_k(V, k) \) is a right \( kG \)-module. We will use a well-known fact whose proof we will briefly sketch.

**Proposition 6.14.** Let \( G \) be a finite group, let \( M \) be a right \( kG \)-module and let \( V \in \text{Irr} kG \). The dimension of \( M \otimes_{kG} V \) (as a \( k \)-vector space) is the multiplicity of \( D(V) \) in the decomposition of \( M \) into simple (right) \( kG \)-modules.

*Proof.* Let \( V_1, \ldots, V_r \) be the simple \( kG \)-modules up to isomorphism. Fix a set \( \{p_1, \ldots, p_r\} \) of primitive orthogonal idempotents such that \( V_i \cong kGp_i \). It is easy to check that

\[
p_i kG \otimes_{kG} kGp_j = p_i kGp_j
\]
as $k$-vector spaces. Moreover, since $k$ is algebraically closed, the Wedderburn-Artin theorem implies that

$$p_i kG p_j \simeq \begin{cases} k & i = j \\ 0 & \text{otherwise} \end{cases}$$

hence,

$$\dim p_i kG p_j = \begin{cases} 1 & i = j \\ 0 & \text{otherwise}. \end{cases}$$

Now, consider the decomposition of $M$ into a sum of simple right $kG$-modules

$$M \simeq \bigoplus_{i=1}^r n_i p_i kG$$

and let $V \simeq kG p_i$ for some $i$. It is clear that

$$\dim M \otimes_{CG} kG p_i = \dim \bigoplus_{i=1}^r n_i p_i kG p_i = n_i.$$ 

So the dimension is indeed the multiplicity of $D(V) \simeq p_i kG$ in the decomposition of $M$ as required. 

As an immediate corollary of Theorem 6.9 and Proposition 6.14 we obtain the following result.

**Corollary 6.15.** Let $S$ be a finite right restriction EI-Ehresmann semigroup. Let $J$ be a regular $J$-class and choose a projection $e \in J$ (such exists by Lemma 6.12). Let $V \in \text{Irr} kG_J$ where $G_J$ is the maximal subgroup associated with $J$. Assume also that $k$ is algebraically closed and the order of every subgroup of $S$ is invertible in $k$. Then the dimension of $kL_E(e) \otimes_{CG_J} V$ as a $k$-vector space is the multiplicity of $D(V)$ in the decomposition of $kL_E(e)$ as a right $G_J$-module.

## 7 The monoid of partial functions

Recall that $\mathcal{P}T_n$ denotes the monoid of all partial functions on the set $\{1, \ldots, n\}$. In this section we apply the results of Section 6 in this specific case. For this
section we fix $k = \mathbb{C}$, the field of complex numbers. Let $A \subseteq \{1, \ldots, n\}$ and denote by $id_A$ the partial identity function on $A$. As already mentioned in Section 3.2, $PT_n$ is a right restriction EI-Ehresmann semigroup with

$$E = \{id_A \mid A \subseteq \{1, \ldots, n\}\}$$

as a subsemilattice of projections. Recall that we are composing functions from right to left. We denote the corresponding Ehresmann category by $E_n$, and it can be described in the following way. The objects of $E_n$ are subsets of $\{1, \ldots, n\}$ and for every $A, B \subseteq \{1, \ldots, n\}$ the hom-set $E_n(A, B)$ contains all the (total) onto functions from $A$ to $B$. The endomorphism monoid of an object $A$ is the symmetric group $S_A$ so the category $E_n$ is indeed an EI-category. Let $f : A \rightarrow B$ be a partial function. We denote by $\text{dom}(f) \subseteq A$ and $\text{im}(f) \subseteq B$ the domain and image of $f$. Recall that the (set theoretic) kernel of $f$ is the equivalence relation on $\text{dom}(f)$ defined by $a_1 \sim a_2$ if $f(a_1) = f(a_2)$. We denote it by $\ker(f).

Consider two partial functions $f_1, f_2 \in PT_n$. It is easy to see that $f_1 \sim L f_2$ if and only if $\text{dom}(f_1) = \text{dom}(f_2)$. It is well known that $f_1$ and $f_2$ are $J$-equivalent if $|\text{im}(f_1)| = |\text{im}(f_2)|$ and $L$-equivalent if $\text{dom}(f_1) = \text{dom}(f_2)$ and $\ker(f_1) = \ker(f_2)$. All the $J$-classes of $PT_n$ are regular and we choose a representative projection $id_k = id_{\{1, \ldots, k\}} \in E$ for every $J$-class. The group corresponding to the $J$-class of $id_k$ is the symmetric group $S_k$. Therefore, the maximal subgroups of $PT_k$ are $S_k$ for $0 \leq k \leq n$ (note that $S_0 \simeq S_1$). It is well known that simple modules of $\mathbb{C}S_k$ are indexed by partitions of $k$ (or Young diagrams). Therefore, the simple modules of $\mathbb{C}PT_n$ can be indexed by partitions $\lambda \vdash k$ for $0 \leq k \leq n$. We denote the simple module of $S_k$ corresponding to some partition $\lambda \vdash k$ by $S^\lambda$. According to Theorem 4.3 all simple modules of $\mathbb{C}PT_n$ are of the form $\mathbb{C}L(id_k) \otimes S^\lambda$ for $0 \leq k \leq n$ and $\lambda \vdash k$ (this is a known fact, see [6, Section 11.3], and this is also a corollary of [22, Theorem 4.4]). Theorem 6.9 yields the following result.

**Theorem 7.1.** The modules $\mathbb{C}L(id_k) \otimes S^\lambda$ for $0 \leq k \leq n$ and $\lambda \vdash k$ form a list of all the indecomposable modules of $\mathbb{C}PT_n$ up to isomorphism. (Note that $\tilde{L}(id_k)$ contains all the partial functions $f \in PT_n$ with $\text{dom}(f) = \{1, \ldots, k\}$).

**Dimension of indecomposable projective modules** Our goal now is to obtain a formula for the dimension (as a $\mathbb{C}$-vector space) of the projective module $\mathbb{C}L(id_k) \otimes S^\lambda$ where $\lambda \vdash k$. By Corollary 6.15 we need to consider the
Let $f : \{1, \ldots, k\} \to \{b_1, \ldots, b_r\}$ be an onto function. We associate with $f$ a composition $\mu_f = [\mu_1, \ldots, \mu_r] \triangleright k$ where $\mu_i$ is the number of elements in the preimage $f^{-1}(b_i)$. For instance, if $f : \{1, 2, 3, 4, 5, 6\} \to \{2, 3, 6\}$ is defined by $f = (\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 2 & 6 & 3 & 6 \end{array})$, then $\mu_f = [2, 1, 3] \triangleright 6$. It is clear that two functions $f_1, f_2 \in \tilde{L}_E(id)\lambda$ are in the same orbit (under the right action of $S_k$) if $\text{im}(f_1) = \text{im}(f_2)$ and $\mu_{f_1} = \mu_{f_2}$. Therefore, we can index the orbits by pairs $(B, \mu)$ where $B \subseteq \{1, \ldots, n\}$ and $\mu \triangleright k$ is a composition with $|\mu| = |B|$. Denote such an orbit by $O(B, \mu)$. It is also easy to check that if $f \in O(B, \mu)$ with $\mu = [\mu_1, \ldots, \mu_r]$ then the stabilizer of $f$ is isomorphic to $S_\mu = S_{\mu_1} \times \ldots \times S_{\mu_r}$ (the Young subgroup corresponding to $\mu$).

Now, as a $\mathbb{C}$-vector space we have a decomposition

$$
\mathbb{C}\tilde{L}_E(id)\lambda \simeq \bigoplus_{(B, \mu)} O(B, \mu) \otimes S_\lambda
$$

where the sum is over all pairs $(B, \mu)$ where $B \subseteq \{1, \ldots, n\}$ and $\mu \triangleright k$ is a composition with $|\mu| = |B|$. This is clearly isomorphic to

$$
\bigoplus \left( \mathbb{C}O(B, \mu) \otimes S_\lambda \right).
$$

Since $S_k$ acts transitively on $O(B, \mu)$ and since $S_\mu$ is the stabilizer of this action it is known that $\mathbb{C}O(B, \mu) \simeq \text{Ind}_{S_\mu}^{S_k} \text{tr}_\mu$ where $\text{tr}_\mu$ is the trivial module of $S_\mu$. This is the Young module corresponding to $\mu$. The multiplicity of $S_\lambda$ in the decomposition of $\text{Ind}_{S_\mu}^{S_k} \text{tr}_\mu$ (or $D(S_\lambda)$ if we consider the right module) is the Kostka number $K_{\lambda\mu}$. Therefore,

$$
\dim \mathbb{C}O(B, \mu) \otimes S_\lambda = K_{\lambda\mu}
$$

and

$$
\dim \bigoplus \left( \mathbb{C}O(B, \mu) \otimes S_\lambda \right) = \sum K_{\lambda\mu}
$$
where again the sum is over all pairs \((B, \mu)\) where \(B \subseteq \{1, \ldots, n\}\) and \(\mu \vdash k\) is a composition with \(|\mu| = |B|\). Since \(K_{\lambda \mu}\) does not depend on \(B\) this equals

\[
\sum_{l=1}^{k} \sum_{\substack{\mu \vdash k \\
|\mu| = l}} \binom{n}{l} K_{\lambda \mu}.
\]

So we have obtained the following result.

**Proposition 7.2.** The dimension of the indecomposable projective module \(\mathbb{C}L_E(\text{id}_k) \otimes S^\lambda\) is given by the formula

\[
\sum_{l=1}^{k} \sum_{\substack{\mu \vdash k \\
|\mu| = l}} \binom{n}{l} K_{\lambda \mu}.
\]

**The Cartan matrix of \(\mathbb{C}PT_n\)**

**Definition 7.3.** Let \(A\) be a \(\mathbb{C}\)-algebra. Let \(\{S(1), \ldots, S(r)\}\) be the irreducible representations of \(A\) up to isomorphism and let \(\{P(1), \ldots, P(r)\}\) be the indecomposable projective modules of \(A\) ordered such that \(P(i)\) is the projective cover of \(S(i)\). The **Cartan matrix** of \(A\) is the \(r \times r\) matrix whose \((i,j)\) entry is the number of times that \(S(i)\) appears as a Jordan-Hölder factor in \(P(j)\).

We want to show that certain entries in the Cartan matrix of \(\mathbb{C}PT_n\) equal 0. The proof is similar to the proof of Proposition 7.2. The irreducible representations of \(\mathbb{C}PT_n\) (and hence the rows and columns of the Cartan matrix) are indexed by Young diagrams with \(k\) boxes for \(0 \leq k \leq n\). For every Young diagram \(\alpha \vdash k\) we denote by \(S(\alpha)\) and \(P(\alpha)\) the associated irreducible and indecomposable projective module respectively. We think of the Cartan matrix as an \((n+1) \times (n+1)\) block matrix where the \((i,j)\) block contains pairs \((\alpha, \beta)\) of Young diagrams such that \(\alpha \vdash (i-1)\) and \(\beta \vdash (j-1)\). Denote by \(E(k,r)\) the set of all onto functions \(f : \{1, \ldots, k\} \to \{1, \ldots, r\}\). It is clear that \(\mathbb{C}E(k,r)\) has a natural structure of \(\mathbb{C}S_r - \mathbb{C}S_k\) bi-module. The following result regarding the Cartan matrix was obtained in [32, Corollary 4.2] (this is a specific case of a known formula for the Cartan matrix of any EI-category algebra).

**Proposition 7.4.** Let \(\alpha \vdash r\) and \(\beta \vdash k\) be two Young diagrams. The number of times that \(S(\alpha)\) appears as a Jordan-Hölder factor of \(P(\beta)\) is the number of times that \(S^\alpha\) appears as an irreducible constituent in the \(\mathbb{C}S_r\) module \(\mathbb{C}E(k,r) \bigotimes_{\mathbb{C}S_k} S^\beta\).
Remark 7.5. It is proved in [32] that the Cartan matrix of $\mathbb{C}\mathcal{PT}_n$ is block upper unitriangular. Moreover, a combinatorial description for the first and second block superdiagonals is given there.

**Proposition 7.6.** Let $\beta \vdash k$ be a Young diagram such that $r < |\beta|$ then $\mathbb{C}E(k,r) \bigotimes_{\mathbb{C}S_k} S^\beta = 0$.

**Proof.** According to Proposition 6.14

\[
\dim \mathbb{C}E(k,r) \bigotimes_{\mathbb{C}S_k} S^\beta
\]

is the multiplicity of $D(S^\beta)$ in the right $\mathbb{C}S_k$-module $\mathbb{C}E(k,r)$. This is a permutation module, induced from the right action of $S_k$ on $E(k,r)$. It is clear that two functions $f_1, f_2 \in E(k,r)$ are in the same orbit if and only if $\mu_{f_1} = \mu_{f_2}$ (again, $\mu_f = [\mu_1, \ldots, \mu_r] \vdash k$ where $\mu_i = |f^{-1}(i)|$). Therefore, we can index the orbits by compositions, $\mu \vdash k$ with $|\mu| = r$. Denote such an orbit by $O_n$. As $\mathbb{C}$-Vector spaces

\[
\mathbb{C}E(k,r) \bigotimes_{\mathbb{C}S_k} S^\beta \simeq \bigoplus_{\mu \vdash k \atop |\mu| = r} O_n \bigotimes_{\mathbb{C}S_k} S^\beta.
\]

It is clear that the stabilizer of $f \in O_n$ is $S_\mu \simeq S_{\mu_1} \times \cdots \times S_{\mu_r}$. As we have already seen, this implies that

\[
\dim O_n \bigotimes_{\mathbb{C}S_k} S^\beta = \dim \text{Ind}_{S_\mu}^{\mathbb{C}S_k} \text{tr}_{\mu} \bigotimes_{\mathbb{C}S_k} S^\beta = K_{\beta \mu}.
\]

The assumption $|\mu| = r < |\beta|$ implies that $K_{\beta \mu} = 0$ since there can be no semistandard Young tableau with shape $\beta$ and content $\mu$ for $|\mu| < |\beta|$. □

Proposition 7.6 immediately implies the following corollary:

**Proposition 7.7.** Let $\alpha \vdash r$ and $\beta \vdash k$ two Young diagrams such that $r < |\beta|$. Then the $(\alpha, \beta)$ entry in the Cartan matrix of $\mathbb{C}\mathcal{PT}_n$ is 0.
Remark 7.8. This result is a generalization of some known facts about the projective modules of $\mathbb{C} \mathcal{PT}_n$. Consider the case where $\beta = [1^k]$ for $1 \leq k \leq n$. Proposition 7.7 (with the fact that the Cartan matrix is unitriangular) says that the only irreducible constituent in $P(\beta)$ is $S(\beta)$. Therefore $S(\beta) \simeq P(\beta)$ which is a known fact about $\mathcal{PT}_n$ (see [32, Lemma 6.7]). In the case where $\beta = [2, 1^{k-2}]$ for some $2 \leq k \leq n$, Proposition 7.7 is precisely [32, Lemma 6.4] which is a key step in the proof that the global dimension of $\mathbb{C} \mathcal{PT}_n$ is $n - 1$.

Appendix: Ehresmann semigroups and categories - two counterexamples

The isomorphism between the algebras of a right restriction Ehresmann semigroup and the corresponding category (Theorem 6.4) was presented in [30] without the requirement of $S$ being right restriction. However, Shoufeng Wang has found out that the proof implicitly assumes it. In fact, the functions $\psi$ and $\varphi$ of Theorem 6.4 are $k$-algebra homomorphisms if and only if $S$ is right restriction [37, Lemma 4.3]. This led to a correction [31], but it was not clear whether being right restriction is necessary for the conclusion of Theorem 6.4 to be true. In this appendix we would like to show that this requirement can not be omitted.

A counterexample is given by the partition monoid which arises in the study of partition algebras (see [12]). On the other hand, we show that the algebra of the semigroup in Example 3.12 is isomorphic to the corresponding category algebra despite being neither left nor right restriction.

The partition monoid $\mathcal{P}_n$ for $n \in \mathbb{N}$ can be described in the following way. Define $n = \{1, \ldots, n\}$ and $n' = \{1', \ldots, n'\}$. An element $\alpha \in \mathcal{P}_n$ is a partition of the set $n \cup n'$. For every $x, y \in n \cup n'$ we write $x \sim_\alpha y$ when $x$ and $y$ are in the same partition class of $\alpha$. We refer the reader to [12] for the description of the operation of $\mathcal{P}_n$ which is best pictured with certain diagrams.

For any partition $\alpha \in \mathcal{P}_n$, we follow [4] and define two equivalence relations on $n$ by

$$\text{ker}(\alpha) = \{(i, j) \in n \times n \mid i \sim_\alpha j\}$$

$$\text{coker}(\alpha) = \{(i, j) \in n \times n \mid i' \sim_\alpha j'\}.$$ 

Note that there are two types of equivalence classes in $\text{ker}(\alpha)$. A kernel class $K$ is in the domain of $\alpha$ if there exists $j' \in n'$ such that $j' \sim_\alpha i$ for some
(and hence all) $i \in K$. Likewise a cokernel class $K'$ is in the codomain of $\alpha$ if there exists $i \in n$ such that $i \sim_{\alpha} j'$ for some (all) $j' \in K'$. Note also that the number of domain classes in $\ker(\alpha)$ is equal to the number of codomain classes in $\coker(\alpha)$. For every equivalence relation $\sigma$ on $n$ we define a partition $e_{\sigma} \in P_n$ in the following way. Two elements $x, y \in n \cup n'$ will be in the same $e_{\sigma}$-class if $(i, j) \in \sigma$ where $x = i$ or $x = i'$ and $y = j$ or $y = j'$. It is clear that $e_{\sigma}$ is an idempotent with $\ker(e_{\sigma}) = \coker(e_{\sigma}) = \sigma$. Moreover, $e_{\sigma_1} e_{\sigma_2} = e_{\sigma_3}$ where $\sigma_3$ is the minimal equivalence relation such that $\sigma_1, \sigma_2 \subseteq \sigma_3$ (i.e., it is the join in the lattice of equivalence relations on $n$). Therefore, the set

$$E = \{e_{\sigma} \mid \sigma \text{ is an equivalence relation on } n\}$$

is a subsemilattice of $P_n$ (note that $e_{\sigma_1} \leq e_{\sigma_2}$ if $\sigma_2 \subseteq \sigma_1$). For every $\alpha \in P_n$ it is clear that $e_{\ker(\alpha)} (e_{\coker(\alpha)})$ is the minimal element of $E$ such that $e_{\ker(\alpha)} \alpha = \alpha$ ($\alpha = \alpha e_{\coker(\alpha)}$) hence we can write

$$\alpha^+ = e_{\ker(\alpha)}, \quad \alpha^* = e_{\coker(\alpha)}.$$

It is also possible to check that the identities

$$(\alpha \beta)^+ = (\alpha \beta^+)^+, \quad (\alpha \beta)^* = (\alpha^* \beta)^*$$

are satisfied for every $\alpha, \beta \in P_n$. Therefore, $P_n$ is an Ehresmann semigroup and there exists a corresponding Ehresmann category $C_n$. This fact was first noted by East and Gray [5]. In this specific case it will be more convenient to compose morphisms in a category “from left to right” because it fits better the standard description of the partition monoid. The objects of $C_n$ are in one-to-one correspondence with equivalence relations (or partitions) on $n$ and for every $\alpha \in P_n$ there is a corresponding morphism $C(\alpha)$ with domain $\ker(\alpha)$ and range $\coker(\alpha)$. According to [12], the algebra $\mathbb{C}P_n$ is not semisimple (for $n > 1$). In order to complete our counterexample it is enough to show that $\mathbb{C}C_n$ is semisimple, hence $\mathbb{k}P_n \not\cong \mathbb{k}C_n$ for $\mathbb{k} = \mathbb{C}$. Denote by $1_n$ the identity relation on $n$. Note that $e_{1_n}$ is the identity of $P_n$ which will also be denoted by $1_n$ for the sake of simplicity.

**Proposition A.1.** Let $\mathbb{k}$ be a field.
1. The following equality holds
\[ \ker C_n \mathbf{1}_n \ker C_n = \ker C_n. \]

2. There is an isomorphism of $k$-algebras
\[ \mathbf{1}_n \ker C_n \mathbf{1}_n \simeq k \mathcal{I} \mathcal{S}_n \]
(where $\mathcal{I} \mathcal{S}_n$ is the symmetric inverse monoid).

Proof. 1. Let $\alpha \in \mathcal{P}_n$. Denote by $K_1, \ldots, K_r$ the kernel classes of the domain and by $K_{r+1}, \ldots, K_s$ the other kernel classes. Likewise, denote by $K'_1, \ldots, K'_r$ the cokernel classes of the codomain and by $K'_{r+1}, \ldots, K'_s$ the other cokernel classes ($s \neq s'$ in general). We order the classes such that $i \sim \alpha j$ for $i \in K_l$ and $j \in K'_l$ (for $l \leq r$). Define a partition $\alpha_1 \in \mathcal{P}_n$ by
\[ \alpha_1 = \{ K_1 \cup \{ 1' \}, \ldots, K_r \cup \{ r' \}, K_{r+1}, \ldots, K_s, \{ (r+1)' \}, \ldots, \{ n' \} \} \]
and another partition $\alpha_2 \in \mathcal{P}_n$ by
\[ \alpha_2 = \{ \{ 1 \} \cup K'_1, \ldots, \{ r \} \cup K'_r, \{ r+1 \}, \ldots, \{ n \}, K'_{r+1}, \ldots, K'_s \}. \]
First note that $\alpha = \alpha_1 \alpha_2 = \alpha_1 \mathbf{1}_n \alpha_2$ and moreover,
\[ \ker(\alpha_1) = \ker(\alpha) \]
\[ \coker(\alpha_1) = \mathbf{1}_n = \ker(\alpha_2) \]
\[ \coker(\alpha_2) = \coker(\alpha). \]
This implies that
\[ C(\alpha) = C(\alpha_1)C(\mathbf{1}_n)C(\alpha_2) \]
since the composition of morphisms on the right hand side is defined. Therefore, $\ker C_n \mathbf{1}_n \ker C_n = \ker C_n$ as required.

2. It is clear that $C(\alpha) \in \mathbf{1}_n \mathcal{C}_n \mathbf{1}_n$ if and only if $\ker(\alpha) = \coker(\alpha) = \mathbf{1}_n$ which implies that $\alpha \in \mathcal{I} \mathcal{S}_n$ (according to the natural embedding of $\mathcal{I} \mathcal{S}_n$ in $\mathcal{P}_n$, see [4, Section 3]). The claim follows immediately. \qed
Corollary A.2. \( kC_n \) is a semisimple algebra if and only if \( n! \) is invertible in \( k \).

Proof. It is well known (see [20, Theorem 2.8.7]) that for any algebra \( A \) and an idempotent \( e \in A \), if \( A = AeA \) then the algebras \( A \) and \( eAe \) are Morita equivalent. In our case, we take \( A = kC_n \) and \( e = 1_n \) so Proposition A.1 implies that \( kC_n \) is Morita equivalent to \( k\mathcal{I}S_n \). It is well known that \( k\mathcal{I}S_n \) is a semisimple algebra if and only if \( n! \) is invertible in \( k \) ([35, Corollary 9.7]) hence the same holds for \( kC_n \) as well.

Corollary A.2 for \( k = \mathbb{C} \) settles the counterexample which is the main goal of this section.

Finally, we would like to give an example of an EI-Ehresmann semigroup that is not left nor right restriction, but still has an algebra isomorphic to the algebra of its category. Consider Example 3.12 given above. The category \( C \) is a poset. That is, for any two objects \( a, b \) there is at most one morphism in the union of hom-sets \( C(a, b) \cup C(b, a) \). Therefore, its algebra is the incidence algebra of this poset. In this case the poset is the linear poset \( (\text{id}, 1) < (1, 1) < (1, \text{id}) \) so in this case the incidence algebra (over a field \( k \)) is the algebra \( UT_3(k) \) of all upper triangular matrices over \( k \). Now, consider the map \( \Phi : S \to UT_3(k) \) defined by

\[
\begin{align*}
(1, \text{id}) &\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & (\text{id}, 1) &\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
(1, 1) &\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & (2, 1) &\rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
(1, 2) &\rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & (2, 2) &\rightarrow \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.
\end{align*}
\]

It is not difficult to check that \( \Phi \) is a one-to-one semigroup homomorphism and that its image is a basis of \( UT_3(k) \). Therefore \( \Phi \) extends to an isomorphism \( \Phi : kS \to UT_3(k) \). This implies that \( kS \simeq kC \) in this case.

We leave as an open problem to determine necessary and sufficient conditions for an EI-Ehresmann finite semigroup to have an algebra isomorphic to that of its associated category algebra. It may be true that every finite EI-Ehresmann semigroup has an algebra isomorphic to the algebra of its category.
References

[1] Jorge Almeida. *Finite semigroups and universal algebra*, volume 3 of *Series in Algebra*. World Scientific Publishing Co. Inc., River Edge, NJ, 1994. Translated from the 1992 Portuguese original and revised by the author.

[2] Ibrahim Assem, Daniel Simson, and Andrzej Skowroński. *Elements of the representation theory of associative algebras. Vol. 1*, volume 65 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2006. Techniques of representation theory.

[3] Stanley Burris and H. P. Sankappanavar. *A course in universal algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.

[4] James East. On the singular part of the partition monoid. *Internat. J. Algebra Comput.*, 21(1-2):147–178, 2011.

[5] James East and Robert D. Gray. Ehresmann theory and partition monoids. in preparation.

[6] Olexandr Ganyushkin and Volodymyr Mazorchuk. *Classical finite transformation semigroups*, volume 9 of *Algebra and Applications*. Springer-Verlag London Ltd., London, 2009. An introduction.

[7] Olexandr Ganyushkin, Volodymyr Mazorchuk, and Benjamin Steinberg. On the irreducible representations of a finite semigroup. *Proc. Amer. Math. Soc.*, 137(11):3585–3592, 2009.

[8] Victoria Gould. Notes on restriction semigroups and related structures; formerly (weakly) left E-ample semigroups, 2010.

[9] Victoria Gould. Restriction and Ehresmann semigroups. In *Proceedings of the International Conference on Algebra 2010*, pages 265–288. World Sci. Publ., Hackensack, NJ, 2012.

[10] Xiaojiang Guo. A note on locally inverse semigroup algebras. *Int. J. Math. Math. Sci.*, 2008. Art. ID 576061, 5.

[11] Xiaojiang Guo and Lin Chen. Semigroup algebras of finite ample semigroups. *Proc. Roy. Soc. Edinburgh Sect. A*, 142(2):371–389, 2012.
[12] Tom Halverson and Arun Ram. Partition algebras. *European J. Combin.*, 26(6):869–921, 2005.

[13] John M. Howie. *Fundamentals of semigroup theory*, volume 12 of *London Mathematical Society Monographs. New Series*. The Clarendon Press Oxford University Press, New York, 1995. Oxford Science Publications.

[14] Gordon James and Adalbert Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.

[15] Yingdan Ji and Yanfeng Luo. Locally adequate semigroup algebras. *Open Math.*, 14:29–48, 2016.

[16] Peter R. Jones. A common framework for restriction semigroups and regular $*$-semigroups. *J. Pure Appl. Algebra*, 216(3):618–632, 2012.

[17] M. V. Lawson. Semigroups and ordered categories. I. The reduced case. *J. Algebra*, 141(2):422–462, 1991.

[18] Mark V. Lawson. *Inverse semigroups*. World Scientific Publishing Co., Inc., River Edge, NJ, 1998. The theory of partial symmetries.

[19] Liping Li. A characterization of finite EI categories with hereditary category algebras. *J. Algebra*, 345:213–241, 2011.

[20] Markus Linckelmann. *The block theory of finite group algebras. Vol. I*, volume 91 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2018.

[21] Wolfgang Lück. *Transformation groups and algebraic K-theory*, volume 1408 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989. Mathematika Gottingensis.

[22] Stuart Margolis and Benjamin Steinberg. The quiver of an algebra associated to the Mantaci-Reutenauer descent algebra and the homology of regular semigroups. *Algebr. Represent. Theory*, 14(1):131–159, 2011.

[23] Stuart Margolis and Benjamin Steinberg. Projective indecomposable modules and quivers for monoid algebras. *Comm. Algebra*, 46(12):5116–5135, 2018.
[24] John Rhodes and Benjamin Steinberg. *The q-theory of finite semigroups*. Springer Monographs in Mathematics. Springer, New York, 2009.

[25] Bruce E. Sagan. *The symmetric group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001. Representations, combinatorial algorithms, and symmetric functions.

[26] Louis Solomon. The Burnside algebra of a finite group. *J. Combinatorial Theory*, 2:603–615, 1967.

[27] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[28] Itamar Stein. Representation theory of order-related monoids of partial functions as locally trivial category algebras. *Algebras and Representation Theory*. to appear.

[29] Itamar Stein. The representation theory of the monoid of all partial functions on a set and related monoids as EI-category algebras. *J. Algebra*, 450:549–569, 2016.

[30] Itamar Stein. Algebras of Ehresmann semigroups and categories. *Semigroup Forum*, 95(3):509–526, 2017.

[31] Itamar Stein. Erratum to: Algebras of Ehresmann semigroups and categories. *Semigroup Forum*, 96(3):603–607, 2018.

[32] Itamar Stein. The global dimension of the algebra of the monoid of all partial functions on an $n$-set as the algebra of the EI-category of epimorphisms between subsets. *J. Pure Appl. Algebra*, 223(8):3515–3536, 2019.

[33] Benjamin Steinberg. Möbius functions and semigroup representation theory. *J. Combin. Theory Ser. A*, 113(5):866–881, 2006.

[34] Benjamin Steinberg. Möbius functions and semigroup representation theory. II. Character formulas and multiplicities. *Adv. Math.*, 217(4):1521–1557, 2008.

[35] Benjamin Steinberg. *Representation theory of finite monoids*. Universitext. Springer, Cham, 2016.
[36] Tammo tom Dieck. *Transformation groups*, volume 8 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.

[37] Shoufeng Wang. On algebras of $P$-Ehresmann semigroups and their associate partial semigroups. *Semigroup Forum*, 95(3):569–588, 2017.

[38] Peter Webb. An introduction to the representations and cohomology of categories. In *Group representation theory*, pages 149–173. EPFL Press, Lausanne, 2007.