1. Preliminaries

Recall that a Drinfel’d algebra (or a quasi-bialgebra in the original terminology of [1]) is an object $A = (V, \cdot, \Delta, \Phi)$, where $(V, \cdot, \Delta)$ is an associative, not necessarily coassociative, unital and counital $k$-bialgebra, $\Phi$ is an invertible element of $V^{\otimes 3}$, and the usual coassociativity property is replaced by the condition which we shall refer to as quasi-coassociativity:

$$\tag{1} (\mathbb{1} \otimes \Delta) \Delta \cdot \Phi = \Phi \cdot (\Delta \otimes \mathbb{1}) \Delta,$$

where we use the dot $\cdot$ to indicate both the (associative) multiplication on $V$ and the induced multiplication on $V^{\otimes 3}$. Moreover, the validity of the following “pentagon identity” is required:

$$(\mathbb{1}^2 \otimes \Delta)(\Phi) \cdot (\Delta \otimes \mathbb{1}^2)(\Phi) = (1 \otimes \Phi) \cdot (\mathbb{1} \otimes \Delta \otimes \mathbb{1})(\Phi) \cdot (\Phi \otimes 1),$$

where $1 \in V$ being the unit element and $\mathbb{1}$, the identity map on $V$. If $\epsilon : V \to k$ ($k$ being the ground field) is the counit of the coalgebra $(V, \Delta)$ then, by definition, $(\epsilon \otimes \mathbb{1}) \Delta = (\mathbb{1} \otimes \epsilon) \Delta = \mathbb{1}$. We have a natural splitting $V = \overline{V} \oplus k$, $\overline{V} := \text{Ker}(\epsilon)$, given by the embedding $k \to V$, $k \ni c \mapsto c \cdot 1 \in V$.

For a $(V, \cdot)$-bimodule $N$, recall the following generalization of the $M$-construction of [8, par. 3] introduced in [4]. Let $F^* = \bigoplus_{n \geq 0} F^n$ be the free unitary nonassociative $k$-algebra generated by $N$, graded by the length of words. The space $F^n$ is the direct sum of copies of $N^{\otimes n}$ over the set $\text{Br}_n$ of full bracketings of $n$ symbols, $F^n = \bigoplus_{\alpha \in \text{Br}_n} N^\otimes_{\alpha}$. For example, $F^0 = k$, $F^1 = N$, $F^2 = N^{\otimes 2}$, $F^3 = N^{\otimes 3} \oplus N^{\otimes 3}$, etc. The algebra $F^*$ admits a natural left action, $(a, f) \mapsto a \bullet f$, of the algebra $(V, \cdot)$ given by the rules:

(i) on $F^0 = k$, the action is given by the augmentation $\epsilon$,
(ii) on $F^1 = N$, the action is given by the left action of $V$ on $N$ and
(iii) $a \bullet (f \ast g) = \sum (\Delta'(a) \bullet f) \ast (\Delta''(a) \bullet g)$.

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where $\ast$ stands for the multiplication in $F^*$ and we use the Sweedler notation $\Delta(a) = \sum \Delta'(a) \otimes \Delta''(a)$. The right action $(f,b) \mapsto f \cdot b$ is defined by similar rules. These actions define on $F^*$ the structure of a $(V, \cdot)$-bimodule.

Let $\sim$ be the relation on $F^* \ast$-multiplicatively generated by the expressions of the form

$$
\sum \left( (\Phi_1 \cdot x) \ast \left( (\Phi_2 \cdot y) \ast (\Phi_3 \cdot z) \right) \right) \sim \sum \left( (x \cdot \Phi_1) \ast (y \cdot \Phi_2) \right) \ast (z \cdot \Phi_3),
$$

where $\Phi = \sum \Phi_1 \otimes \Phi_2 \otimes \Phi_3$ and $x, y, z \in F^*$. Put $\mathcal{O}(N) := F/\sim$. Just as in [3, Proposition 3.2] one proves that the $\cdot$-action induces on $\mathcal{O}(N)$ the structure of a $(V, \cdot)$-bimodule (denoted again by $\cdot$) and that $\ast$ induces on $\mathcal{O}(N)$ a nonassociative multiplication denoted by $\circ$. The operations are related by

$$
a \cdot (f \circ g) = \sum (\Delta'(a) \cdot f) \circ (\Delta''(a) \cdot g) \text{ and } (f \circ g) \cdot b = \sum (f \cdot \Delta'(b)) \circ (g \cdot \Delta''(b)),
$$

for $a, b \in V$ and $f, g \in \mathcal{O}(N)$. The multiplication $\circ$ is quasi-associative in the sense

$$
(2) \quad \sum (\Phi_1 \cdot x) \circ (\left( (\Phi_2 \cdot y) \circ (\Phi_3 \cdot z) \right) = \sum (x \cdot \Phi_1) \circ (y \cdot \Phi_2) \circ (z \cdot \Phi_3)
$$

The construction described above is functorial in the sense that any $(V, \cdot)$-bimodule map $f : N' \to N''$ induces a natural $(V, \cdot)$-linear algebra homomorphism $\mathcal{O}(f) : \mathcal{O}(N') \to \mathcal{O}(N'')$. As it was shown in [3], for any $(V, \cdot)$-bimodule $N$ there exists a natural homomorphism of $k$-modules $J = J(N) : \mathcal{O}(N) \to \mathcal{O}(N)$. If $f : N' \to N''$ is as above then $J(N'') \circ \mathcal{O}(f) = \mathcal{O}(f) \circ J(N')$.

Since the defining relations (2) are homogeneous with respect to length, the grading of $F^*$ induces on $\mathcal{O}(N)$ the grading $\mathcal{O}^*(N) = \bigoplus_{i \geq 0} \mathcal{O}^i(N)$. If $N$ itself is a graded vector space, we have also the obvious second grading, $\mathcal{O}(N) = \bigoplus_j \mathcal{O}(N)^j$, which coincides with the first grading if $N$ is concentrated in degree 1.

Let $\text{Der}_V^\mathcal{O}(\mathcal{O}(N))$ denote the set of $(V, \cdot)$-linear derivations of degree $n$ (relative to the second grading) of the (nonassociative) graded algebra $\mathcal{O}(N)^\ast$. One sees immediately that there is an one-to-one correspondence between the elements $\theta \in \text{Der}_V^\mathcal{O}(\mathcal{O}(N))$ and $(V, \cdot)$-linear homogeneous degree $n$ maps $f : N^\ast \to \mathcal{O}(N)^\ast$.

If $N = X \oplus Y$, then $\mathcal{O}(X \oplus Y)$ is naturally bigraded, $\mathcal{O}^*(X \oplus Y) = \bigoplus_{i,j \geq 0} \mathcal{O}^{ij}(X \oplus Y)$, the bigrading being defined by saying that a monomial $w$ belongs to $\mathcal{O}^{ij}(X \oplus Y)$ if there are exactly $i$ (resp. $j$) occurrences of the elements of $X$ (resp. $Y$) in $w$. If $X, Y$ are graded vector spaces then there is a second bigrading $\mathcal{O}(X \oplus Y)^{\ast, \ast} = \bigoplus_{i,j} \mathcal{O}(X \oplus Y)^{ij}$ just as above.

Let $(\mathcal{B}_s(V), d_B)$ be the (two-sided) normalized bar resolution of the algebra $(V, \cdot)$ (see [2, Chapter X]), but considered with the opposite grading. This means that $\mathcal{B}_s(V)$ is the graded $(V, \cdot)$-bimodule, $\mathcal{B}_s(V) = \bigoplus_{n \leq 1} \mathcal{B}_n(V)$, where $\mathcal{B}_1(V) := V$ with the $(V, \cdot)$-bimodule structure induced by the multiplication $\cdot$, $\mathcal{B}_0(V) := V \otimes V$ (the free $(V, \cdot)$-bimodule on $k$), and for $n \leq -1, \mathcal{B}_n(V)$ is the free $(V, \cdot)$-bimodule on $V^{\otimes (n-1)}$, i.e. the vector space $V \otimes V^{\otimes (n-1)} \otimes V$ with
the action of \((V, \cdot)\) given by
\[
u \cdot (a_0 \otimes \cdots \otimes a_{-n+1}) := (u \cdot a_0 \otimes \cdots \otimes a_{-n+1}) \quad \text{and} \quad (a_0 \otimes \cdots \otimes a_{-n+1}) \cdot w := (a_0 \otimes \cdots \otimes a_{-n+1} \cdot w)
\]
for \(u, v, a_0, a_{-n+1} \in V\) and \(a_1, \ldots, a_{-n} \in V\). If we use the more compact notation (though a nonstandard one), writing \((a_0 \cdots | a_{-n+1})\) instead of \(a_0 \otimes \cdots \otimes a_{-n+1}\), the differential \(d_B : B_n(V) \to B_{n+1}(V)\) is, for \(n \leq 0\), defined as
\[
d_B(a_0 \cdots | a_{-n+1}) := \sum_{0 \leq i \leq -n} (-1)^i (a_0 \cdots | a_i \cdot a_{i+1} \cdots | a_{-n+1}).
\]

Here, as is usual in this context, we make no distinction between the elements of \(V/k \cdot 1\) and their representatives in \(V\). We use the same convention throughout all the paper. Notice that the differential \(d_B\) is a \((V, \cdot)\)-bimodule map.

Put \(\mathcal{O}(V, B_\ast(V)) := \mathcal{O}(\uparrow V \oplus \uparrow B_\ast(V))\), where \(\uparrow\) denotes, as usual, the suspension of a graded vector space and \(V\) is interpreted as a graded vector space concentrated in degree zero. Let \(\text{Der}^i_V(\mathcal{O}(V, B_\ast(V)))\) denote, for each \(i\), the space of degree \(i\) derivations of the algebra \(\mathcal{O}(V, B_\ast(V))\) which are also \((V, \cdot)\)-linear maps. Let us define the derivation \(D_{-1} \in \text{Der}^1_V(\mathcal{O}(V, B_\ast(V)))\) by \(D_{-1}|_{B_n(V)} := \uparrow d_B \downarrow \) and \(D_{-1}|_{\mathcal{O}} := 0\). Clearly \(D_{-1}(\mathcal{O}(V, B_\ast(V))^{i,j}) \subset \mathcal{O}(V, B_\ast(V))^{i,j+1}\) and \(D_{-1}(\mathcal{O}^{i,j}(V, B_\ast(V))) \subset \mathcal{O}^{i,j}(V, B_\ast(V))\) for any \(i, j \geq 0\). We also see immediately that \(D_{-1}^2 = 0\).

Let us consider, for any \(n \geq 1\), the complex \((\mathcal{O}^{n-1,1}(V, B_\ast(V)), D_{-1})\), i.e. the complex
\[
0 \leftarrow \mathcal{O}^{n-1,1}(V, V) \xrightarrow{D_{-1}} \mathcal{O}^{n-1,1}(V, B_0(V)) \xrightarrow{D_{-1}} \mathcal{O}^{n-1,1}(V, B_{-1}(V)) \leftarrow \cdots
\]

**Lemma 1.1.** The complex \((\mathcal{O}^{n-1,1}(V, B_\ast(V)), D_{-1})\) is acyclic, for any \(n \geq 1\).

**Proof.** We have the decomposition
\[
\mathcal{O}^{n-1,1}(V, B_\ast(V)) = \bigoplus_{1 \leq i \leq n} \mathcal{O}^{n-1,1}_i(V, B_\ast(V)),
\]
where \(\mathcal{O}^{n-1,1}_i(V, B_\ast(V))\) denotes the subspace of \(\mathcal{O}^{n-1,1}(V, B_\ast(V))\) spanned by monomials having an element of \(B_\ast(V)\) at the \(i\)-th place. The differential \(D_{-1}\) obviously respects this decomposition and the canonical isomorphism \(J\) of \([\mathcal{I}]\) mentioned above identifies \(\mathcal{O}^{n-1,1}_i(V, B_\ast(V))\) to \(V^{\otimes (i-1)} \otimes B_\ast(V) \otimes V^{\otimes (n-i)}\). Under this identification the differential \(D_{-1}\) coincides with \(\mathbb{I}^{\otimes (i-1)} \otimes d_B \otimes \mathbb{I}^{\otimes (n-i)}\) and the rest follows from the Künneth formula and the acyclicity of \((B_\ast(V), d_B)\). Q.E.D.

2. Properties of \(\text{Der}^1_V(\mathcal{O}(V, B_\ast(V)))\)

Let \(C = (C, \cdot, 1_C)\) be a unital associative algebra and let \(C \xleftarrow{c} (\mathcal{R}, d_\mathcal{R}), (\mathcal{R}, d_\mathcal{R}) = R_0 \xleftarrow{d_R} R_1 \xleftarrow{d_R} \cdots\), be a complex of free \(C\)-bimodules (we consider \(C\) as a \(C\)-bimodule with the bimodule
structure induced by the multiplication). Similarly, let $D \leftarrow^n (S, d_S)$ with $(S, d_S) = S_0 \leftarrow^1 S_1 \leftarrow^2 \cdots$, be an acyclic complex of $C$-bimodules. To simplify the notation, we write sometimes $R_{-1}$ (resp. $S_{-1}$, resp. $d_R$, resp. $d_S$) instead of $C$ (resp. $D$, resp. $\epsilon$, resp. $\eta$). Let

$$Z := \{ f = (f_i)_{i \geq -1}; \ f_i : R_i \to S_i \text{ a } C \text{-bimodule map and } f_i \circ d_R = d_S \circ f_{i+1} \text{ for any } i \geq -1 \}.$$ 

Let us define, for a sequence $\chi = (\chi_i)_{i \geq -1}$ of $C$-bimodule maps $\chi_i : R_i \to S_{i+1}$, $\nabla(\chi) = (\nabla(\chi_i)_{i \geq -1} \in Z$ by $\nabla(\chi_i) := d_S \circ \chi_i + \chi_{i-1} \circ d_R$. Let $B := \text{Im}(\nabla) \subset Z$. For a $C$-bimodule $M$ let $M_I$ denote the set of invariant elements of $M$, $M_I := \{ x \in M; cx = xc \text{ for any } c \in C \}$.

**Lemma 2.1.** Under the notation above, the correspondence $Z \ni f = (f_i)_{i \geq -1} \mapsto f_{-1}(1_C) \in D_I$ induces an isomorphism $\Omega : Z/B \cong D_I/\eta(S_{0I})$. Moreover, if $f_{-1}(1_C) = \eta(h)$ for some $h \in S_{0I}$ then $f = \nabla(\chi)$ for some $\chi = (\chi_i)_{i \geq -1}$ with $\chi_{-1}(1) = h$.

**Proof.** We show first that $\Omega$ is well-defined. If $f = \nabla(\chi)$ then $f_{-1} = \eta \circ \chi_{-1}$, therefore $f_{-1}(1_C) = \eta(h)$ with $h := \chi_{-1}(1_C) \in S_{0I}$ and $\Omega(f) = 0$.

Let us prove that $\Omega$ is an epimorphism. For $z \in D_I$ define a $C$-bimodule map $f_{-1} : C \to D$ by $f_{-1}(c) := cz$ ($=zc$) for $c \in C$. Because $(R, d_R)$ is free and $(S, d_S)$ is acyclic, $f_{-1}$ lifts to some $f = (f_i)_{i \geq -1} \in Z$ by standard homological arguments [3, Theorem III.6.1].

It remains to prove that $\Omega$ is a monomorphism. For $f = (f_i)_{i \geq -1} \in Z$, $\Omega(f) = 0$ means that $f_{-1}(1_C) = \eta(h)$ for some $h \in S_{0I}$. The $C$-bimodule map $\chi_{-1} : C \to S_0$ defined by $\chi_{-1}(c) := ch$ ($=hc$) for $c \in C$ clearly satisfies $f_{-1} = \eta \circ \chi_{-1}$. A standard homological argument (see again [3, Theorem III.6.1]) then enables one to extend $\chi_{-1}$ to a ‘contracting homotopy’ $\chi = (\chi_i)_{i \geq -1}$ with $f = \nabla(\chi)$.

**Q.E.D.**

**Definition 2.2.** For $n \geq 2$ and $k \geq 0$ let $J_k(n)$ be the subspace of $\text{Der}^{n-1-k}_V(\mathcal{O}(V, B_c(V)))$ consisting of derivations $\theta$ satisfying

(i) $\theta((\mathcal{O}(V, B_c(V)))^{i,j}) \subset \mathcal{O}(V, B_c(V))^{i+n-1,j-k}$,

(ii) $\theta((\mathcal{O}^i(V, B_c(V)))) \subset \mathcal{O}^{i+n-1,j}(V, B_c(V))$,

(iii) $[D_{-1}, \theta] = 0$ if $k = 0$ and $\theta|_{\mathcal{O}_V} = 0$ if $k \geq 1$.

Let us observe that, for $\theta \in J_k(n)$, $\theta|_{\mathcal{O}_V} = 0$. This follows from item (i) of the definition above. Observe also that $J_*(n)$ is invariant under the differential $\nabla$ defined by $\nabla(\theta) := [D_{-1}, \theta]$, $\nabla(J_k(n)) \subset J_{k-1}(n)$ for $k \geq 1$ and $\nabla(J_0(n)) = 0$.

**Proposition 2.3.** $H_{\geq 1}(J_*(n), \nabla) = 0$ while

$$H_0(J_*(n), \nabla) = \mathcal{O}^{n-1,1}(V, B_1(V))_I \oplus \mathcal{O}^n(V)_I.$$
PROOF. Let \( k > 0 \) and let \( \theta \in J_k(n) \). As \( \theta|_V = 0 \), \( \theta \) is given by its restriction to \( B_*(V) \), namely by a sequence of \((V,\cdot\cdot\cdot)\)-bimodule maps \( \theta_i : B_i(V) \to \bigodot^{n-1,1}(V, B_{i-k}(V)) \), \( i \leq 1 \). Suppose that \( \theta \) is a \( \nabla \)-cocycle, i.e. that \( \nabla(\theta) = 0 \). This means that the diagram

\[
\begin{array}{cccc}
0 & B_1(V) = V & B_0(V) & B_{-1}(V) & \cdots \\
\downarrow \theta_1 & d_B & \downarrow \theta_0 & \downarrow \theta_{-1} & \downarrow D_{-1} \\
0 & \bigodot^{n-1,1}(V, B_{1-k}(V)) & \bigodot^{n-1,1}(V, B_{-1-k}(V)) & \bigodot^{n-1,1}(V, B_{-1-k}(V)) & \cdots \\
\end{array}
\]

is commutative. Since \( \theta_1 = 0 \) by item (iii) of Definition \[2.2\], Lemma \[2.1\] (with \( C = (V,\cdot,1_V) \) and \( D = \bigodot^{n-1,1}(V, B_{1-k}(V)) \)) gives a sequence \( \chi_i : B_i(V) \to \bigodot^{n-1,1}(V, B_{i-1-k}(V)) \) of \((V,\cdot\cdot\cdot)\)-bimodule maps, \( i \leq 1 \), such that \( \theta_i = D_{-1} \circ \chi_i + \chi_{i+1} \circ d_B \). We can, moreover, suppose that \( \chi_1 = 0 \), thus the sequence \((\chi_i)_{i\leq 1}\) determines a derivation \( \chi \in J_{k+1}(n) \) with \( \nabla(\chi) = \theta \). This proves \( H_k(J_*(n), \nabla) = 0 \) for \( k \geq 1 \).

A derivation \( \theta \in J_0(n) \) is given by two independent data: by the restriction \( \theta_V := \theta|_V : V \to \bigodot^{n}(V) \) and by the restriction \( \theta_{B_*(V)} := \theta|_{B_*(V)} : B_*(V) \to \bigodot^{n-1,1}(V, B_*(V)) \). As \( D_{-1}|_V = 0 \), the condition \( \nabla(\theta) = [D_{-1}, \theta] = 0 \) imposes no restrictions on \( \theta_V \) and, because \( \nabla(\chi)|_V = 0 \) for any \( \chi \in J_1(n) \), the contribution of \( \theta_V \) to \( H_0(J_*(n), \nabla) \) is parametrized by \( \theta_V|_V \), i.e. by an element of \( \bigodot^n(V)_I \). This explains the second summand in \((3)\).

The restriction \( \theta_{B_*(V)} \) is in fact a sequence \( \theta_i : B_i(V) \to \bigodot^{n-1,1}(V, B_i(V)) \), \( i \leq 1 \), of \((V,\cdot\cdot\cdot)\)-bimodule maps and the condition \( \nabla(\theta) = 0 \) means that the diagram

\[
\begin{array}{cccc}
0 & B_1(V) = V & B_0(V) & B_{-1}(V) & \cdots \\
\downarrow \theta_1 & d_B & \downarrow \theta_0 & \downarrow \theta_{-1} & \downarrow D_{-1} \\
0 & \bigodot^{n-1,1}(V, B_1(V)) & \bigodot^{n-1,1}(V, B_0(V)) & \bigodot^{n-1,1}(V, B_{-1}(V)) & \cdots \\
\end{array}
\]

is commutative. Similarly as above, a derivation \( \chi \in J_1(n) \) is given by a sequence \( \chi_i : B_i(V) \to \bigodot^{n-1,1}(V, B_{i-1}(V)) \), \( i \leq 1 \), of \((V,\cdot\cdot\cdot)\)-linear maps. The condition \( \nabla(\chi) = \theta \) then means that \( \theta_i = D_{-1} \circ \chi_i + \chi_{i+1} \circ d_B \), especially, \( \theta_1 = D_{-1} \circ \chi_1 \). This last equation implies, since \( \chi_1 = 0 \) by (iii) of Definition \[2.2\], that \( \nabla(\chi) = \theta \) forces \( \theta_1|_V = 0 \). On the other hand, if \( \theta_1|_V = 0 \) then Lemma \[2.3\] gives a \( \chi \in J_1(n) \) with \( \theta = \nabla(\chi) \) and we conclude that the contribution of \( \theta_{B_*(V)} \) to \( H_0(J_*(n), \nabla) \) is parametrized by \( \theta_{B_*(V)}|_V \in \bigodot^{n-1,1}(V, B_1(V)) \) which is the first summand of \((3)\).

Q.E.D.
Let us recall that a (right) differential graded (dg) comp algebra (or a nonunital operad in the terminology of [3]) is a bigraded differential space \( L = (L_*(\cdot), \nabla) \), \( L_*(\cdot) = \bigoplus_{k \geq 0, n \geq 2} L_k(n) \), \( \nabla(L_k(n)) \subset L_{k-1}(n) \), together with a system of bilinear operations

\[
\circ_i : L_p(a) \otimes L_q(b) \to L_{p+q}(a+b-1)
\]

given for any \( 1 \leq i \leq b \) such that, for \( \phi \in L_p(a) \), \( \psi \in L_q(b) \) and \( \nu \in L_r(c) \),

\[
\phi \circ_i (\psi \circ_j \nu) = \begin{cases} 
(-1)^{p \cdot q} \cdot \psi \circ_{j+a-1} (\phi \circ_i \nu), & \text{for } 1 \leq i \leq j - 1, \\
(\phi \circ_{i-j+1} \psi) \circ_j \nu, & \text{for } j \leq i \leq b + j - 1, \\
(-1)^{p \cdot q} \cdot \psi \circ_j (\phi \circ_{i-b+1} \nu), & \text{for } i \geq j + b.
\end{cases}
\]

We suppose, moreover, that for any \( \phi \in L_p(a) \), and \( \psi \in L_q(b) \), \( 1 \leq i \leq b \),

\[
\nabla(\phi \circ_i \psi) = \nabla(\phi) \circ_i \psi + (-1)^p \cdot \phi \circ_i \nabla(\psi).
\]

Any dg comp algebra determines a nonsymmetric (unital) operad in the monoidal category of differential graded spaces (see [1] for the terminology). To be more precise, let \( L = (L_*(\cdot), \circ_i, \nabla) \) be a dg comp algebra as above and let us define the bigraded vector space \( \mathcal{L}_*(\cdot) = \bigoplus_{k \geq 0, n \geq 1} \mathcal{L}_k(n) \) by \( \mathcal{L}_*(n) := L_*(n) \) for \( n \geq 2 \) and \( \mathcal{L}_*(1) = \mathcal{L}_0(1) := \text{Span}(1_L) \), where \( 1_L \) is a degree zero generator. Let us extend the definition of structure maps \( \circ_i \) to \( \mathcal{L} \) by putting \( f \circ_i 1_L := f \) and \( 1_L \circ_i g := g \), for \( f \in \mathcal{L}_*(m) \), \( g \in \mathcal{L}_*(n) \) and \( 1 \leq i \leq n \). Let us extend the differential \( \nabla \) by \( \nabla(1_L) := 0 \). In [3] we proved the following proposition.

**Proposition 2.4.** The composition maps \( \gamma : \mathcal{L}_*(a) \otimes \mathcal{L}_*(n_1) \otimes \cdots \otimes \mathcal{L}_*(n_a) \to \mathcal{L}_*(n_1 + \cdots + n_a) \) given by

\[
\gamma(\phi; \nu_1, \ldots, \nu_a) := \nu_1 \circ_1 (\nu_2 \circ_2 (\cdots \circ_{a-1} (\nu_a \circ_a \nu)))
\]

for \( \phi \in \mathcal{L}_*(a) \) and \( \nu_i \in \mathcal{L}_*(n_i) \), \( 1 \leq i \leq a \), define on \( \mathcal{L}_*(\cdot) \) a structure of a nonsymmetric differential graded operad in the monoidal category of differential graded vector spaces.

Let \( N \) be a (graded) \( (V, \cdot) \)-bimodule, let \( d : N \to N \) be a \( (V, \cdot) \)-linear differential and define \( X_k(n) := \{ \theta \in \text{Der}_V^k(\mathcal{O}(N)); \theta(N) \subset \mathcal{O}^n(N) \} \). For \( \omega \in X_k(m) \), \( \theta \in X_k(n) \) and \( 1 \leq i \leq n \) let \( \omega_N := \omega|_N : N \to \mathcal{O}^n(N) \) and \( \theta_N := \theta|_N : N \to \mathcal{O}^n(N) \) be the restrictions. Let then \( \omega \circ_i \theta \in X_k(m+n-1) \) be a derivation defined by \( (\omega \circ_i \theta)|_N := (\mathbb{I} \circ^{(i-1)} \circ \omega_N \circ \mathbb{I} \circ^{(n-i)}) \circ \theta_N. \)

Let us extend the differential \( d \) to a derivation \( D \) of \( \mathcal{O}(N) \) and define \( \nabla(\theta) := [D, \theta]. \)

**Lemma 2.5.** The object \( X_*(\cdot) = (X_*(\cdot), \circ_i, \nabla) \) constructed above is a differential graded comp algebra.

**Proof.** Let \( \phi \in X_p(a), \psi \in X_q(b) \) and \( \nu \in X_r(c) \). The composition \( \phi \circ_i (\psi \circ_j \nu) \) is, by definition, given by its restriction \( [\phi \circ_i (\psi \circ_j \nu)]_N \) to \( N \) as

\[
[\phi \circ_i (\psi \circ_j \nu)]_N = (\mathbb{I} \circ^{(i-1)} \circ \phi_N \circ \mathbb{I} \circ^{(b+c-i-1)}) \circ (\psi \circ_j \nu)_N.
\]
with \((\psi \circ_j \nu)_N = (\mathbb{I}^{\circ(j-1)} \circ \psi_N \circ \mathbb{I}^{\circ(c-j)}) \circ \nu_N\). This implies that
\[
[\phi \circ_i (\psi \circ_j \nu)]_N = (\mathbb{I}^{\circ(i-1)} \circ \phi_N \circ \mathbb{I}^{\circ(b+c-i-1)}) \circ (\mathbb{I}^{\circ(j-1)} \circ \psi_N \circ \mathbb{I}^{\circ(c-j)}) \circ \nu_N.
\]
For \(i \leq j - 1\) we have (taking into the account that \(\text{Im}(\phi_N) \subset \mathcal{O}^a(N)\) and \(\text{Im}(\psi_N) \subset \mathcal{O}^b(N)\))
\[
(\mathbb{I}^{\circ(i-1)} \circ \phi_N \circ \mathbb{I}^{\circ(b+c-i-1)}) \circ (\mathbb{I}^{\circ(j-1)} \circ \psi_N \circ \mathbb{I}^{\circ(c-j)}) = (-1)^{pq} \cdot (\mathbb{I}^{\circ(j+a-2)} \circ \psi_N \circ \mathbb{I}^{\circ(c-j)}) \circ (\mathbb{I}^{\circ(i-1)} \circ \phi_N \circ \mathbb{I}^{\circ(c-i)}),
\]
which means that \([\phi \circ_i (\psi \circ_j \nu)]_N = (-1)^{pq} \cdot [\psi \circ_{j+a-1} (\phi \circ_i \nu)]_N\). This is the axiom \([\mathbb{I}]\) for \(i \leq j - 1\).

Similarly, for \(j \leq i \leq b + j - 1\) we have
\[
(\mathbb{I}^{\circ(i-1)} \circ \phi_N \circ \mathbb{I}^{\circ(b+c-i-1)}) \circ (\mathbb{I}^{\circ(j-1)} \circ \psi_N \circ \mathbb{I}^{\circ(c-j)}) = \mathbb{I}^{\circ(j-1)} \circ [(\mathbb{I}^{\circ(i-j)} \circ \phi_N \circ \mathbb{I}^{\circ(b+i-j-1)}) \circ \psi_N] \circ \mathbb{I}^{\circ(c-j)}
\]
which means that \([\phi \circ_i (\psi \circ_j \nu)]_N = [(\phi \circ_{i-j+1} \psi) \circ_j \nu]_N\). This is the axiom \([\mathbb{I}]\) for \(j \leq i \leq b + j - 1\).
The discussion of the remaining case \(i \geq j + b\) is similar.

Q.E.D.

Let us consider the special case of the construction above with \(N := \uparrow V \oplus \uparrow B_*(V)\) and \(d := 0 \oplus \uparrow d_g \downarrow\).

**Lemma 2.6.** The bigraded subspace \(J_*(\bullet)\) of \(X_*(\bullet)\) introduced in Definition 2.2 is closed under the operations \(\circ_i\) and the differential \(\nabla\).

The proof of the lemma is a straightforward verification. The lemma says that the dg comp algebra structure on \(X_*(\bullet)\) restricts to a dg comp algebra structure \((J_*(\bullet), \circ_i, \nabla)\) on \(J_*(\bullet)\).

3. More about Der\(_{V_+}\)(\(\mathcal{O}(V, B_*(V))\))

Let \(J_*(\bullet) = (J_*(\bullet), \circ_i, \nabla)\) and \(D_{-1} \in \text{Der}_{V_+}(\mathcal{O}(V, B_*(V)))\) be as in the previous section.

**Definition 3.1.** An infinitesimal deformation of \(D_{-1}\) is an element \(D_0 \in J_0(2)\) such that \(\nabla(D_0) = 0\). An integration of an infinitesimal deformation \(D_0\) is a sequence \(\bar{D} = \{D_i \in J_i(i+2); \ i \geq 1\}\) such that \(D := D_{-1} + D_0 + D_1 + \cdots\) satisfies \([D, D] = 0\).

Let \(K_n\) be, for \(n \geq 2\), the Stasheff associahedron \([7]\). It is an \((n - 2)\)-dimensional cellular complex whose \(i\)-dimensional cells are indexed by the set \(\text{Br}_n(i)\) of all (meaningful) insertions of \((n - i - 2)\) pairs of brackets between \(n\) symbols, with suitably defined incidence maps. There is, for any \(a, b \geq 2\), \(0 \leq i \leq a - 2\), \(0 \leq j \leq b - 2\) and \(1 \leq t \leq b\), a map
\[
(-, -)_t : \text{Br}_a(i) \times \text{Br}_b(j) \to \text{Br}_{a+b-1}(i + j), \quad u \times v \mapsto (u, v)_t,
\]
where \((u, v)_t\) is given by the insertion of \((u)\) at the \(t\)-th place in \(v\). This map defines, for \(a, b \geq 2\) and \(1 \leq t \leq b\), the inclusions \(\iota_t : K_a \times K_b \hookrightarrow \partial K_{a+b-1}\). It is well-known that the sequence \(\{K_n\}_{n \geq 1}\) form a topological operad, see again [7].

Let \(CC_i(K_n)\) denote the set of \(i\)-dimensional oriented cellular chains with coefficients in \(k\) and let \(d_C : CC_i(K_n) \to CC_{i-1}(K_n)\) be the cellular differential. For \(s \in CC_p(K_a)\) and \(t \in CC_q(K_b)\), \(p, q \geq 0, a, b \geq 2\) and \(1 \leq i \leq b\), let \(s \times t \in CC_{p+q}(K_a \times K_b)\) denote the cellular cross product and put
\[
s \circ_i t := (\iota_i)_*(s \times t) \in CC_{p+q}(K_{a+b-1}).
\]

**Proposition 3.2.** The cellular chain complex \((CC_*(K_\ast), d_C)\) together with operations \(\circ_i\) introduced above forms a differential graded comp algebra.

The simplicial version of this proposition was proved in [4], the proof of the cellular version is similar. The dg comp algebra structure of Proposition 3.2 reflects the topological operad structure of \(\{K_n\}_{n \geq 2}\) mentioned above.

Let \(c_n\) be, for \(n \geq 0\), the unique top dimensional cell of \(K_{n+2}\), i.e. the unique element of \(Br_n(n+2)\) corresponding to the insertion of no pairs of brackets between \((n+2)\) symbols. Let us define \(e_n \in CC_n(K_{n+2})\) as \(e_n := 1 \cdot c_n\).

The following proposition was proved in [3], see also the comments below.

**Proposition 3.3.** The graded comp algebra \(CC_*(K_\ast) = (CC_*(K_\ast), \circ_i)\) is a free graded comp algebra on the set \(\{e_0, e_1, \ldots\}\).

Let us recall that the freeness in the proposition above means that for any graded comp algebra \(L_\ast(\ast) = (L_\ast(\ast), \circ_i)\) and for any sequence \(\alpha_n \in L_{n}(n+2), n \geq 0\), there exists a unique graded comp algebra map \(f : CC_*(K_\ast) \to L_\ast(\ast)\) such that \(f(e_n) = \alpha_n, i \geq 0\).

The proof of Proposition 3.3 is based on the following observation. There is a description of the free graded comp algebra (= free nonsymmetric nonunital operad) on a given set in terms of oriented planar trees. The free comp algebra \(\mathcal{F}(e_0, e_1, \ldots)\) on the set \(\{e_0, e_1, \ldots\}\) has \(\mathcal{F}(e_0, e_1, \ldots)(n)\) = the vector space spanned by oriented connected planar trees with \(n\) input edges. Each such a tree \(T\) then determines an element of \(Br_i(n)\) where \(i = \) the number of vertices of \(T\), i.e. a cell of \(CC_{n-i-2}(K_n)\). This correspondence defines a map \(\mathcal{F}(e_0, e_1, \ldots)(n) \to CC_*(K_n)\) which induces the requisite isomorphism of operads.

Let \(L = (L_\ast(\ast), \circ_i, \nabla)\) be a dg comp algebra. Let us define, for \(\phi \in L_p(a)\) and \(\psi \in L_q(b)\),
\[
\phi \circ \psi := \sum_{1 \leq i \leq b} (-1)^{(a+1)(i+q+1)} \cdot \phi \circ_i \psi \quad \text{and} \quad [\phi, \psi] := \phi \circ \psi - (-1)^{(a+p+1)(b+q+17)} \cdot \psi \circ \phi.
\]

In [3] we proved the following proposition.
**Proposition 3.4.** The operation \([-,-]\) introduced above endows \(L^* := \bigoplus_{a-p-1=*} L_p(a)\) with a structure of a differential graded (dg) Lie algebra, \(L = (L^*, [-,-], \nabla)\).

The construction above thus defines a functor from the category of dg comp algebras to the category of dg Lie algebras. Let us observe that for the comp algebras \(X_*(\cdot)\) and \(J_*(\cdot)\) this structure coincides with the Lie algebra structure induced by the graded commutator of derivations. We can also easily prove that the elements \(\{e_n\}_{n \geq 0}\) satisfy, for each \(m \geq 0\),

\[
d_C(e_m) + \frac{1}{2} \sum_{i+j=n} [e_i, e_j] = 0
\]

in the dg Lie algebra \(CC(K)^* = (CC(K)^*, [-,-], d_C)\).

**Proposition 3.5.** There exists an one-to-one correspondence between integrations of an infinitesimal deformation \(D_0\) in the sense of Definition 3.4 and dg comp algebra homomorphisms \(m : CC_*(K_\cdot) \rightarrow J_*(\cdot)\) with \(m(e_0) = D_0\).

**Proof.** Let us suppose we have a map \(m : CC_*(K_\cdot) \rightarrow J_*(\cdot)\) with \(m(e_0) = D_0\) and define, for \(n \geq 1\), \(D_n := m(e_n)\). We must prove that the derivation \(D := D_1 + D_0 + D_1 + \cdots\) satisfies \([D,D] = 0\). This condition means that

\[
\nabla(D_m) + \frac{1}{2} \sum_{i+j=n-1} [D_i, D_j] = 0
\]

for any \(m \geq 0\), which is exactly what we get applying on (5) the Lie algebra homomorphism \(m\).

On the other hand, suppose we have an integration \(\{D_n\}_{n \geq 1}\). The freeness of the graded comp algebra \(CC_*(K_\cdot)\) (Proposition 3.3) ensures the existence of a graded comp algebra map \(m : CC_*(K_\cdot) \rightarrow J_*(\cdot)\) with \(m(e_n) = D_n\) for \(n \geq 0\). We must verify that this unique map commutes with the differentials, i.e. that \(m(d_C(s)) = \nabla(m(s))\) for any \(s \in CC_*(K_\cdot)\). Because of the freeness, it is enough to verify the last condition for \(s \in \{e_n\}_{n \geq 0}\), i.e. to verify that \(m(d_C(e_n)) = \nabla(m(e_n))\) for \(n \geq 0\). Expanding \(d_C(e_n)\) using (3) we see that this follows from (6) and from the fact that the map \(m\) is a homomorphism of graded Lie algebras.

**Q.E.D.**

**Proposition 3.6.** An infinitesimal deformation \(D_0 \in \mathcal{J}_0(2)\) can be integrated if and only if \([D_0, D_0]|_V = [D_0, D_0]|_{\mathcal{G}_1(V)} = 0\).

**Proof.** Standard obstruction theory. Let us suppose that we have an integration \(\tilde{D} = \{D_i\}_{i \geq 1}\). Condition (6) with \(m = 1\) means that

\[
\nabla(D_1) + \frac{1}{2}[D_0, D_0] = 0.
\]

Because \(D_1 \in \mathcal{J}_1(3)\), \(D_1|_V = D_1|_{\mathcal{G}_1(V)} = 0\) by (iii) of Definition 2.2, therefore \(\nabla(D_1)|_V = \nabla(D_1)|_{\mathcal{G}_1(V)} = 0\) and (7) implies that \([D_0, D_0]|_V = [D_0, D_0]|_{\mathcal{G}_1(V)} = 0\).
On the other hand, let us suppose that \([D_0, D_0]|_V = [D_0, D_0]|_{\mathfrak{g}(V)} = 0\). By the description of \(H_0(J_*(3), \nabla)\) as it is given in Proposition 2.3 we see that the homology class of \([D_0, D_0] \in J_0(3)\) is zero, therefore there exists some \(D_1 \in J_1(3)\) such that \(\nabla(D_1) + \frac{1}{2}[D_0, D_0] = 0\).

Let us suppose that we have already constructed a sequence \(D_i \in J_i(i + 2), 1 \leq i \leq N, \) such that basic equation (3) holds for any \(m \leq N\). The element \(\frac{1}{2} \sum_{i+j=N} D_i D_j \in J_N(N + 3)\) is a \(\nabla\)-cycle (this follows from the definition of \(\nabla(-)\) as \([D_{-1}, -]\) and the Jacobi identity) and the triviality of \(H_N(J_*(N + 3))\) (again Proposition 2.3) gives some \(D_{N+1} \in J_{N+1}(N + 3)\) which satisfies (3) for \(m = N + 1\). The induction may go on.

Q.E.D.

Let \(G\) be the subgroup of \(\text{Aut}(\mathfrak{g}(V, \mathcal{B}_*(V))\) consisting of automorphisms of the form \(g = \mathbb{1} + \phi_{\geq 2},\) where \(\phi_{\geq 2}\) is a \((V, \cdot)\)-linear map satisfying \(\phi_{\geq 2}(\mathfrak{g}-j(V, \mathcal{B}_*(V))) \subset \mathfrak{g}^{-j+2j}(V, \mathcal{B}_*(V))\) for any \(i, j\).

Let us observe that \(G\) naturally acts on the set of integrations of a fixed infinitesimal deformation \(D_0\). To see this, let \(\tilde{D} = \{D_i\}_{i \geq 1}\) be such an integration and let us denote, as usual, \(D := D_{-1} + D_0 + D_1 + \cdots\). Then \(g^{-1}Dg\) is, for \(g \in G\), clearly also a \((V, \cdot)\)-linear derivation from \(\text{Der}_1^1(\mathfrak{g}(V, \mathcal{B}_*(V))\) and \((g^{-1}Dg)(\mathfrak{g}-i, j(V, \mathcal{B}_*(V))) \subset (g^{i+j}(V, \mathcal{B}_*(V))\). We may thus decompose \(g^{-1}Dg = g^{-1}Dg = \sum_{k \geq 1} D'_k\) with \(D'_k(\mathfrak{g}-i, j(V, \mathcal{B}_*(V))) \subset (g^{i+j+k+1}(V, \mathcal{B}_*(V))\). We observe that \(D'_{-1} = D_{-1}, D'_0 = D_0\) and that, from degree reasons, \(D'_k(\mathfrak{g}(V, \mathcal{B}_*(V))) \subset (g^{i+j+k+1}(V, \mathcal{B}_*(V))\) if \(k \geq 1\). This means that \(D'_k \in J_k(k + 2)\) for \(k \geq 1\). The equation \([g^{-1}Dg, g^{-1}Dg] = 0\) is immediate, therefore the correspondence \((g, \{D_i\}_{i \geq 1}) \rightarrow \{D'_i\}_{i \geq 1}\) defines the requisite action. The following proposition shows that this action is transitive.

**Proposition 3.7.** Let \(\tilde{D}' = \{D'_i\}_{i \geq 1}\) and \(\tilde{D}'' = \{D''_i\}_{i \geq 1}\) be two integrations of an infinitesimal deformation \(D_0\). If we denote \(D' := D_{-1} + D_0 + \sum_{i \geq 1} D'_i\) and \(D'' := D_{-1} + D_0 + \sum_{i \geq 1} D''_i\), then \(D' = g^{-1}D''g\) for some \(g \in G\).

**Proof.** Again standard obstruction theory. As we already observed, for any \(g \in G\), \(g^{-1}D''g\) decomposes as \(g^{-1}D''g = D_{-1} + D_0 + \sum_{i \geq 1} g^{-1}D''g_i\) with some \(\{g^{-1}D''g_i\}_i \in J_i(i + 2)\). Let us suppose that we have already constructed some \(g_N \in G, N \geq 1,\) such that \(\{g_N^{-1}D''g_N\}_i = D'_i\) for \(1 \leq i \leq N\). We have

\[
\nabla(D'_{N+1}) - \frac{1}{2} \sum_{i+j=N} D'_i D'_j = 0
\]

and, similarly,

\[
\nabla(\{g_N^{-1}D''g_N\}_{N+1}) - \frac{1}{2} \sum_{i+j=N} \{g_N^{-1}D''g_N\}_i, \{g_N^{-1}D''g_N\}_j = 0.
\]

By the induction, the second terms of the above equations are the same, therefore \(\nabla(D'_{N+1}) = \nabla(\{g_N^{-1}D''g_N\}_{N+1})\) which means that \(D'_{N+1} - \{g_N^{-1}D''g_N\}_{N+1} \in J_{N+1}(N + 3)\) is a cycle. The triviality of \(H_{N+1}(J_*(N + 3))\) (Proposition 2.3) gives some \(\phi \in J_{N+2}(N + 3)\) such that \(D'_{N+1} - \{g_N^{-1}D''g_N\}_{N+1} = \nabla(\phi)\). The element \(\exp(\phi) \in G\) is of the form \(\mathbb{1} + \phi + \phi_{\geq N+3}\) with \(
\phi_{\geq N+3}(\mathfrak{g}^{i+j}(V, \mathcal{B}_*(V))) \subset \mathfrak{g}^{i+j+N+3}(V, \mathcal{B}_*(V)),
\)
therefore $g_{N+1} := \exp(\phi)g_N$ satisfies $\{g_{N+1}^{-1}D''g_{N+1}\}_i = \{g_N^{-1}D''g_N\}_i = D'_i$ for $1 \leq i \leq N$ and $\{g_{N+1}^{-1}D''g_{N+1}\}_{N+1} = \{g_N^{-1}D''g_N\}_{N+1} + \nabla(\phi) = D'_{N+1}$ and the induction goes on. The prounipotency of the group $G$ assures that the sequence $\{g_N\}_{N \geq 1}$ converges to some $g \in G$ as required.

4. APPLICATIONS TO DRINFEŁ’D ALGEBRAS

In [4] we introduced two $(V, \cdot)$-linear ‘coactions’ $\lambda : \mathcal{B}_s(V) \to V \odot \mathcal{B}_s(V)$ and $\rho : \mathcal{B}_s(V) \to \mathcal{B}_s(V) \odot V$ as

$$
\lambda(a_0 | \cdots | a_{-n+1}) := \sum \Delta'(a_0) \cdots \Delta'(a_{-n+1}) \odot (\Delta''(a_0) | \cdots | \Delta''(a_{-n+1})), \quad \text{and}
$$

$$
\rho(a_0 | \cdots | a_{-n+1}) := \sum (\Delta'(a_0) | \cdots | \Delta'(a_{-n+1})) \odot \Delta''(a_0) \cdots \Delta''(a_{-n+1}).
$$

Let us define a derivation $D_0 \in J_0(2)$ by

$$
D_0|_{\mathcal{B}_s(V)} := (\uparrow \odot \uparrow)(\lambda + \rho)(\downarrow) \quad \text{and} \quad D_0|_V := (\uparrow \odot \uparrow)(\Delta)(\downarrow).
$$

**Proposition 4.1.** The derivation $D_0$ defined above is an integrable infinitesimal deformation of $D_{-1}$.

**Proof.** To prove that $D_0$ is an infinitesimal deformation of $D_{-1}$ means to show that $\nabla(D_0) = [D_{-1}, D_0] = 0$. This was done in [4].

The integrability of $D_0$ means, by Proposition 3.6, that $[D_0, D_0]|_V = [D_0, D_0]|_{\mathcal{B}_s(V)} = 0$. For $v \in V$ we have

$$
[D_0, D_0](\uparrow v) = (D_0 \circ D_0)(\uparrow v) = D_0(\uparrow \odot \uparrow)(\Delta)(\uparrow v) =
$$

$$
= [((\uparrow \odot \uparrow)(\Delta) \circ (\uparrow))(\Delta) - (\uparrow \odot (\uparrow \odot \uparrow)(\Delta))](\uparrow v) =
$$

$$
= (\uparrow \circ \uparrow \odot \downarrow \downarrow) \circ [(\Delta \odot \mathbb{1}) \circ (\Delta) - (\mathbb{1} \odot \Delta)(\uparrow v)]
$$

which is zero by the quasi-coassociativity (1) and by (2).

Similarly, for $(v) \in \mathcal{B}_1(V)$ we have

$$
D_0^2 = (\uparrow \circ \uparrow \circ \uparrow)[(\Delta \odot \mathbb{1}) \lambda - (\mathbb{1} \circ \lambda) \lambda - (\mathbb{1} \odot \rho) \lambda + (\lambda \circ \mathbb{1}) \rho + (\rho \circ \mathbb{1}) \rho - (\mathbb{1} \circ \Delta) \rho](\downarrow (v))
$$

and (1), (2) again imply that this is zero.

**Q.E.D.**

**Definition 4.2.** An integration of the infinitesimal deformation $D_0$ above is called a homotopy comodule structure.
Let $\mathcal{O}'(V, \mathcal{B}_s(V)) = \mathcal{O}^{*+}(V, \mathcal{B}_s(V))$ denote the submodule of $\mathcal{O}^{*+}(V, \mathcal{B}_s(V))$ with precisely one factor of $\mathcal{B}_s(V)$. Let $C^n(A)$ be the set of all degree $n$ homogeneous maps $f : \mathcal{O}'(V, \mathcal{B}_s(V)) \to \mathcal{O}^{*}(\mathcal{V})$ which are both $\mathcal{O}^{*}(\mathcal{V})$ and $(V, \cdot)$-linear. Let us define also a degree one derivation $d_C$ on $\mathcal{O}^{*}(\mathcal{V})$ by $d_C|_{\mathcal{V}} := (\uparrow \circ \uparrow)(\Delta)(\uparrow)$.

Let $\{D_i\}_{i \geq 1}$ be a homotopy comodule structure in the sense of Definition 4.2 and let $D := D_{-1} + D_0 + D_1 + \cdots$. Define a degree one endomorphism $d$ of $C^*(A)$ by $d(f) := f \circ D + (-1)^n d_C \circ f$. It is easy to show that $d$ is a differential and, following [4], we define the cohomology of our Drinfel’d algebra $A$ by $H^*(A) := H^*(C^*(A), d)$.

**Proposition 4.3.** The definition of the cohomology of a Drinfel’d algebra does not depend on the particular choice of a homotopy comodule structure.

**Proof.** Let $\{D'_i\}_{i \geq 1}$ and $\{D''_i\}_{i \geq 1}$ be two homotopy comodule structures, $D' := D_{-1} + D_0 + D'_1 + \cdots$ and $D'' := D_{-1} + D_0 + D''_1 + \cdots$. Let $d'(f) := f \circ D' + (-1)^n d_C \circ f$ and $d''(f) := f \circ D'' + (-1)^n d_C \circ f$. Proposition 3.7 then gives some $g \in G$ such that $D' \circ g = g \circ D''$. We see immediately that the map $\Psi : (C^*(A), d') \to (C^*(A), d'')$ defined by $\Psi(f) := f \circ g$ is an isomorphism of complexes.

Q.E.D.

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