WEITZENBÖCK DERIVATIONS
OF FREE METABELIAN ASSOCIATIVE ALGEBRAS

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ABSTRACT. By the classical theorem of Weitzenböck the algebra of constants $K[X_d]^\delta$ of a nonzero locally nilpotent linear derivation $\delta$ of the polynomial algebra $K[X_d] = K[x_1, \ldots, x_d]$ in several variables over a field $K$ of characteristic 0 is finitely generated. As a noncommutative generalization one considers the algebra of constants $F_d(\mathfrak{G})^\delta$ of a locally nilpotent linear derivation $\delta$ of a finitely generated relatively free algebra $F_d(\mathfrak{G})$ in a variety $\mathfrak{G}$ of unitary associative algebras over $K$. It is known that $F_d(\mathfrak{G})^\delta$ is finitely generated if and only if $\mathfrak{G}$ satisfies a polynomial identity which does not hold for the algebra $U_2(K)$ of $2 \times 2$ upper triangular matrices. Hence the free metabelian associative algebra $F_d = F_d(\mathfrak{G}) = F_d(\mathfrak{G}_{U_2}) = F_d(\text{var}(U_2(K)))$ is a crucial object to study. We show that the vector space of the constants $(F_d)^\delta$ in the commutator ideal $F_d^\delta$ is a finitely generated $K[U_d, V_d]^\delta$-module, where $\delta$ acts on $U_d$ and $V_d$ in the same way as on $X_d$. For small $d$, we calculate the Hilbert series of $(F_d^\delta)$ and find the generators of the $K[U_d, V_d]^\delta$-module $(F_d^\delta)$$. This gives also an (infinite) set of generators of the algebra $F_d^\delta$.

1. INTRODUCTION

Recall that a derivation of an algebra $R$ over a field $K$ is a linear operator $\delta : R \to R$ such that $\delta(uv) = \delta(u)v + u\delta(v)$ for all $u, v \in R$. In this paper we fix a field $K$ of characteristic 0, an integer $d \geq 2$ and a set of variables $X_d = \{x_1, \ldots, x_d\}$. For the polynomial algebra $K[X_d] = K[x_1, \ldots, x_d]$ in $d$ variables, every mapping $\delta : X_d \to K[X_d]$ can be extended to a derivation of $K[X_d]$ which we shall denote by the same symbol $\delta$. We assume that $\delta$ is a Weitzenböck derivation, i.e., acts as a nonzero nilpotent linear operator of the vector space $KX_d$ with basis $X_d$. Up to a change of the basis of $KX_d$, the derivation $\delta$ is determined by its Jordan normal form $J(\delta)$ with Jordan cells $J_1, \ldots, J_s$ with zero diagonals. Hence the essentially different Weitzenböck derivations are in a one-to-one correspondence with the partition $(p_1 + 1, \ldots, p_s + 1)$ of $d$, where $p_1 \geq \cdots \geq p_s \geq 0$, $(p_1 + 1) + \cdots + (p_s + 1) = d$, and the correspondence is given in terms of the size $(p_i + 1) \times (p_i + 1)$ of the Jordan cells $J_i$ of $J(\delta)$, $i = 1, \ldots, s$. We shall denote the derivation corresponding to this partition...
by \(\delta(p_1, \ldots, p_s)\). The algebra of constants of \(\delta\) 

\[ K[X_d]^{\delta} = \ker \delta = \{ u \in K[X_d] \mid \delta(u) = 0 \} \]

can be studied also with methods of classical invariant theory because it coincides with the algebra of invariants \(K[X_d]^{UT_2(K)}\) of the unitriangular group \(UT_2(K) \cong \{ \exp(\alpha \delta) \mid \alpha \in K \}\). The classical theorem of Weitzenböck [W] states that for any Weitzenböck derivation \(\delta\) the algebra of constants \(K[X_d]^{\delta}\) is finitely generated, see the book by Nowicki [N] for more details.

The polynomial algebra \(K[X_d]\) is free in the class of all commutative algebras. As a noncommutative generalization one considers the relatively free algebra \(F_d(\mathfrak{A})\) in a variety \(\mathfrak{A}\) of unitary associative algebras over \(K\), see e.g., [DD] for a background on varieties of algebras. As in the polynomial case, \(F_d(\mathfrak{A})\) is freely generated by the set \(X_d\) and every map \(X_d \to F_d(\mathfrak{A})\) can be extended to a derivation of \(F_d(\mathfrak{A})\). Again, we shall call the derivations \(\delta\) which act as nilpotent linear operators of the vector space \(K X_d\) Weitzenböck derivations and shall denote them \(\delta(p_1, \ldots, p_s)\).

In the theory of varieties of unitary associative algebras there is a dichotomy. Either all polynomial identities of the variety \(\mathfrak{A}\) follow from the metabelian identity 

\[ [x_1, x_2][x_3, x_4] = 0 \] (which is equivalent to the condition that \(\mathfrak{A}\) contains the algebra \(U_2(K)\) of \(2 \times 2\) upper triangular matrices), or \(\mathfrak{A}\) satisfies an Engel polynomial identity 

\[ [x_2, x_1, \ldots, x_{n}] = x_2 \text{ad}^n x_1 = 0. \] Drensky and Gupta [DG] studied Weitzenböck derivations \(\delta\) acting on \(F_d(\mathfrak{A})\). In particular, if \(U_2(K) \in \mathfrak{A}\), then the algebra of constants \(F_d(\mathfrak{A})^{\delta}\) is not finitely generated. If \(U_2(K)\) does not belong to \(\mathfrak{A}\), a result of Drensky [D2] gives that the algebra \(F_d(\mathfrak{A})^{\delta}\) is finitely generated. Hence the free metabelian associative algebra 

\[ F_d = F_d(\mathfrak{M}) = F_d(\mathfrak{U}_2(\mathfrak{A})) = F_d(\text{var}(U_2(K))) \]

is a crucial object in the study of Weitzenböck derivations.

The present paper may be considered as a continuation of our recent paper [DDF], where we have studied the algebra of constants \((L_d/L_d')^{\delta}\) of the Weitzenböck derivation \(\delta\) acting on the free metabelian Lie algebra \(L_d/L_d'\), where \(L_d\) is the free \(d\)-generated Lie algebra. As in the Lie case we show that the algebra of constants \(F_d^{\delta}\) when \(\delta\) acts on the free metabelian associative algebra \(F_d = F_d(\text{var}(U_2(K)))\) is very close to finitely generated. In the Lie case the commutator ideal \(L_d'/L_d''\) has a natural structure of a \(K[X_d]\)-module. In [DDF] we have seen that the vector space of constants \((L_d'/L_d'')^{\delta}\) is a finitely generated \(K[X_d]^{\delta}\)-module. In the associative case the commutator ideal \(F_d^{\delta}\) is a \(K[X_d]\)-bimodule. We prove that the vector space of constants \((F_d^{\delta})^{\delta}\) is a finitely generated \(K[U_d, V_d]\)-\(\delta\)-module, where \(\delta\) acts on the variables \(U_d\) and \(V_d\) in the same way as on the variables \(X_d\). Then, using methods of [BBD] we give an algorithm how to calculate the Hilbert series of \(F_d^{\delta}\) and calculate it for small \(d\). If the Jordan form of \(\delta\) contains a \(1 \times 1\) cell, we may assume that \(\delta\) acts as a nilpotent linear operator on \(K X_{d-1}\) and \(\delta(x_d) = 0\). In this situation we express the Hilbert (or Poincaré) series of \(F_d^{\delta}\) and the generators of the \(K[U_d, V_d]\)\(\delta\)-module \((F_d^{\delta})^{\delta}\) in terms of the Hilbert series of \(F_{d-1}^{\delta}\) and the generators of the \(K[U_{d-1}, V_{d-1}]\)\(\delta\)-module \((F_{d-1}^{\delta})^{\delta}\). Finally, we find the generators of the \(K[U_d, V_d]\)\(\delta\)-module \((F_d^{\delta})^{\delta}\) for \(d \leq 3\) which gives also an explicit (infinite) set of generators of the algebra \(F_d^{\delta}\).
2. Finite Generation

Let $A_d = K\langle X_d \rangle$ be the free associative algebra of rank $d$ with free generating set $X_d = \{x_1, \ldots, x_d\}$, and let $A'_d$ be its commutator ideal, i.e., the ideal generated by all commutators $[f, g] = fg - gf$, $f, g \in A_d$. The metabelian variety $\mathfrak{M}$ consists of all associative algebras satisfying the polynomial identity $[x_1, x_2][x_3, x_4] = 0$. The free metabelian algebra $F_d = F_d(\mathfrak{M})$ of rank $d$ is isomorphic to the factor-algebra $A/(A')^2$. Clearly, $K[X_d] \cong A_d/A'_d \cong F_d/F'_d$. We assume that all Lie commutators are left normed and,

$$[x_1, \ldots, x_{n-1}, x_n] = [[x_1, \ldots, x_{n-1}], x_n] = [x_1, \ldots, x_{n-1}] \text{ad} x_n.$$  

It is well known, see [DT], that the commutator ideal $F'_d$ of $F_d$ has a basis consisting of all

$$x_1^{a_1} \cdots x_d^{a_d} [x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_n}], \quad a_i \geq 0, 1 \leq j_i \leq d, j_1 > j_2 \leq j_3 \leq \cdots \leq j_n.$$  

The metabelian identity implies the identity

$$x_{i_1(1)} \cdots x_{i_m(n)} [x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_n}] = x_{i_1} \cdots x_{i_m} [x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_n}],$$  

where $\rho$ and $\sigma$ are arbitrary permutations of $1, \ldots, m$ and $3, \ldots, n$, respectively. This identity allows to define an action of the polynomial algebra $K[U_d, V_d]$ on $F'_d$, where $U_d = \{u_1, \ldots, u_d\}$ and $V_d = \{v_1, \ldots, v_d\}$ are two sets of commuting variables: If $f \in F'_d$, then

$$fu_i = x_i f, \quadfv_i = f \text{ad} x_i, \quad i = 1, \ldots, d.$$  

In this way, $F'_d$ of $F_d$ is a $K[U_d, V_d]$-module (or a $K[X_d]$-bimodule).

Now we need an embedding of $F_d$ into a wreath product. The construction is a partial case of the construction of Lewin [L] used in [DG] and is similar to the construction of Shmel’kin [Sh] in the case of free metabelian algebras (as used in [DDF]). Let $Y_d = \{y_1, \ldots, y_d\}$ be a set of commuting variables and let

$$M_d = \bigoplus_{i=1}^{d} a_i K[U_d, V'_d] = A_d K'[U_d, V'_d]$$

be the free $K[U_d, V'_d]$-module of rank $d$ generated by $A_d = \{a_1, \ldots, a_d\}$, where $V'_d = \{v'_1, \ldots, v'_d\}$. We equip $M_d$ with trivial multiplication $M_d \cdot M_d = 0$ and with a structure of a free $K[Y_d]$-bimodule with the action of $K[Y_d]$ defined by

$$y_j a_i = a_i u_j, \quad a_i y_j = a_i v'_j, \quad i, j = 1, \ldots, d.$$  

Then the algebra $W_d = K[Y_d] \triangleright M_d$ satisfies the metabelian identity and hence belongs to $\mathfrak{M}$. The following proposition is a partial case of [L].

**Proposition 2.1.** The mapping $\iota: x_j \mapsto y_j + a_j$, $j = 1, \ldots, d$ defines an embedding $\iota$ of $F_d$ into $W_d$.

If

$$w = \sum f_{ij}(X_d)[x_i, x_j] g_{ij}(\text{ad} X_d), \quad f_{ij}(X_d), g_{ij}(X_d) \in K[X_d],$$

then

$$\iota(w) = \sum (a_i v_j - a_j v_i) f_{ij}(U_d) g_{ij}(V_d),$$

where $V_d = \{v_1, \ldots, v_d\}$, $v_i = v'_i - u_i$, $i = 1, \ldots, d$. We may replace the variables $V'_d$ with $V_d$. In this way $M_d$ becomes a free $K[U_d, V_d]$-module. To simplify the notation we shall omit $\iota$ and shall consider $F_d$ as a subalgebra of $W_d$. Since the
action of $K[U_d, V_d]$ on $F_d'$ agrees with its action on $M_d$, we shall also think that $F_d'$ is a $K[U_d, V_d]$-submodule of $M_d$. If $\delta$ is a Weitzenb"{o}ck derivation of $F_d$, such that

$$\delta(x_j) = \sum_{i=1}^{d} \alpha_{ij} x_i, \quad \alpha_{ij} \in K, \quad i, j = 1, \ldots, d,$$

we define an action of $\delta$ on $W_d$ assuming that

$$\delta(a_j) = \sum_{i=1}^{d} \alpha_{ij} a_i, \quad \delta(y_j) = \sum_{i=1}^{d} \alpha_{ij} y_i,$$

$$\delta(u_j) = \sum_{i=1}^{d} \alpha_{ij} u_i, \quad \delta(v_j) = \sum_{i=1}^{d} \alpha_{ij} v_i, \quad j = 1, \ldots, d.$$

Obviously, if $w \in F_d$, then $\imath(\delta(w)) = \delta(\imath(w))$. It is clear that the vector space $M_d^\delta$ of the constants of $\delta$ in the $K[U_d, V_d]$-module $M_d$ is a $K[U_d, V_d]^\delta$-module. The following lemma is a partial case of [D2, Proposition 3].

**Lemma 2.2.** The vector space $M_d^\delta$ is a finitely generated $K[U_d, V_d]^\delta$-module.

The next theorem is a direct consequence of the lemma.

**Theorem 2.3.** Let $\delta$ be a Weitzenb"{o}ck derivation of the free metabelian associative algebra $F_d$. Then the vector space $(F_d')^\delta$ of the constants of $\delta$ in the commutator ideal $F_d'$ of $F_d$ is a finitely generated $K[U_d, V_d]^\delta$-module.

**Proof.** By Lemma 2.2 the $K[U_d, V_d]^\delta$-module $M_d^\delta$ is finitely generated. By the theorem of Weitzenb"{o}ck [W] the algebra $K[U_d, V_d]^\delta$ is also finitely generated. Hence all $K[U_d, V_d]^\delta$-submodules of $M_d^\delta$, including $(F_d')^\delta$, are also finitely generated. $\square$

### 3. Hilbert series

The free metabelian associative algebra $F_d = F_d(\mathfrak{H})$ is a graded vector space. If $F_d^{(n)}$ is the homogeneous component of degree $n$ of $F_d$, then the Hilbert series of $F_d$ is the formal power series

$$H(F_d, z) = \sum_{n \geq 0} \dim F_d^{(n)} z^n.$$

The algebra $F_d$ is also multigraded, with a $\mathbb{Z}^d$-grading which counts the degree of each variable $x_j$ in the monomials in $F_d$. If $F_d^{(n_1, \ldots, n_d)}$ is the multihomogeneous component of degree $(n_1, \ldots, n_d)$, then the corresponding Hilbert series of $F_d$ is given by

$$H(F_d, z_1, \ldots, z_d) = \sum_{n_j \geq 0} \dim F_d^{(n_1, \ldots, n_d)} z_1^{n_1} \cdots z_d^{n_d}.$$

If $\delta$ is a Weitzenb"{o}ck derivation of $F_d$, the algebra of constants $F_d^\delta$ is also graded and its Hilbert series is

$$H(F_d^\delta, z) = \sum_{n \geq 0} \dim (F_d^\delta)^{(n)} z^n.$$

The Hilbert series of $F_d(\mathfrak{H})$ is well known. We shall give the proof for self-containedness.
Lemma 3.1. The Hilbert series of the free metabelian associative algebra $F_d$ is

$$H(F_d, z_1, \ldots, z_d) = \prod_{j=1}^{d} \frac{1}{1 - z_j} \left( 2 + (z_1 + \cdots + z_d - 1) \prod_{j=1}^{d} \frac{1}{1 - z_j} \right).$$

Proof. The Hilbert series of a unitary relatively free algebra $F_d(\mathfrak{M})$ can be expressed in terms of the Hilbert series of the subalgebra $B_d(\mathfrak{M})$ of the so-called proper (or commutator) polynomials, see, e.g. [D1]:

$$H(F_d(\mathfrak{M}), z_1, \ldots, z_d) = H(K[X_d], z_1, \ldots, z_d)H(B_d(\mathfrak{M}), z_1, \ldots, z_d)$$

$$= \prod_{i=1}^{d} \frac{1}{1 - z_i} H(B_d(\mathfrak{M}), z_1, \ldots, z_d).$$

For the metabelian variety $\mathfrak{M}$ the algebra of proper polynomials is spanned by the base field $K$ and the commutator ideal $L'_d/L''_d$ of the free metabelian Lie algebra. The Hilbert series of $L'_d/L''_d$ is

$$H(L'_d/L''_d, z_1, \ldots, z_d) = 1 + (z_1 + \cdots + z_d - 1) \prod_{j=1}^{d} \frac{1}{1 - z_j}.$$

Hence

$$H(F_d(\mathfrak{M}), z_1, \ldots, z_d) = \prod_{i=1}^{d} \frac{1}{1 - z_i} (1 + H(L'_d/L''_d, z_1, \ldots, z_d))$$

which gives immediately the result. $\square$

Starting from the Hilbert series of the algebra $F_d = F_d(\mathfrak{M})$ and the sizes of the Jordan cells of the Weitzenböck derivation $\delta$, there are many ways to compute the Hilbert series of the algebra of constants $F^\delta_d$, see the comments in [BBD] and [DDF]. We shall sketch the way used in [DDF] in the Lie case.

Let $\delta = \delta(p_1, \ldots, p_s)$ be a Weitzenböck derivation of $F_d$. We assume that $X_d$ is a Jordan basis for the action of $\delta$ on the vector space $KX_d$. If $Y_i = \{x_j, x_{j+1}, \ldots, x_{j+p_i}\}$ is the part of the basis $X_d$ corresponding to the $i$-th Jordan cell of $\delta$, we have an action of the linear automorphism $\exp(\delta)$ on the vector space $KY_i$ which is the same as its action on the vector space of the binary forms of degree $p_i$. We extend this action to the action of $GL_2(K)$. In this way $KX_d$ is a direct sum of the $GL_2(K)$-module of the binary forms of degree $p_i$, $i = 1, \ldots, s$. Then we extend this action of $GL_2(K)$ diagonally on the whole $F_d$. The basis $X_d = \bigcup_{i=1}^{s} Y_i$ consists of eigenvectors of the diagonal subgroup of $GL_2(K)$. If $g = \xi_1 e_{11} + \xi_2 e_{22}$, $\xi_1, \xi_2 \in K^*$, is a diagonal matrix, then $g(x_{j+k}) = \xi_1^{p_i-k} \xi_2^{k} x_{j+k}$, $k = 0, 1, \ldots, p_i$. This defines a bigrading on $F_d$ assuming that the bidegree of $x_{j+k}$ is $(p_i - k, k)$.

Now $F_d$ is a direct sum of irreducible polynomial $GL_2(K)$-submodules. The irreducible polynomial $GL_2(K)$-modules are indexed by partitions $\lambda = (\lambda_1, \lambda_2)$. If $W = W(\lambda)$ is an irreducible component of $F_d$, it contains a unique (up to a multiplicative constant) nonzero element $w$ of bidegree $(\lambda_1, \lambda_2)$. It is a $\delta$-constant and the algebra $F^\delta_d$ is spanned by these vectors $w$. We express the Hilbert series $H(F_d, z_1, \ldots, z_d)$ as a bigraded vector space. For this purpose we replace in the Hilbert series $H(F_d, z_1, \ldots, z_d)$ the variables $z_j, z_{j+1}, \ldots, z_{j+p_i-1}, z_{j+p_i}$ corresponding to each set
\[ Y_i = \{ x_j, x_{j+1}, \ldots, x_{j+p_i-1}, x_{j+p_i} \} \] by \( t_{1}^{p_i} z, t_{1}^{p_i-1} t_{2} z, \ldots, t_{1} t_{2}^{p_i-1} z, t_{2}^{p_i} z \), respectively, and obtain the Hilbert series

\[ H_{GL^2}(F_d, t_1, t_2, z) = H(F_d, t_{1}^{p_1} z, t_{1}^{p_1-1} t_{2} z, \ldots, t_{1} t_{2}^{p_1-1} z, t_{2}^{p_1} z, t_{1}^{p_2} z, t_{1}^{p_2-1} t_{2} z, \ldots, t_{2}^{p_2} z). \]

The variable \( z \) gives the total degree and \( t_1, t_2 \) count the bidegree induced by the action of the diagonal subgroup of \( GL^2(K) \): The coefficient of \( t_{1}^{n_1} t_{2}^{n_2} z^n \) in \( H_{GL^2}(F_d, t_1, t_2, z) \) is equal to the dimension of the elements of \( F_d \) which are linear combinations of products of length \( n \) in the variables \( X_d \) and are of bidegree \((n_1, n_2)\).

The Hilbert series \( H_{GL^2}(F_d, t_1, t_2, z) \) plays the role of the character of the \( GL^2(K) \)-module \( F_d \). If we present it as an infinite linear combination of Schur functions

\[ H_{GL^2}(F_d, t_1, t_2, z) = \sum_{n \geq 0} \sum_{(\lambda_1, \lambda_2)} m(\lambda_1, \lambda_2, n) S_{(\lambda_1, \lambda_2)}(t_1, t_2) z^n, \]

then the multiplicity \( m(\lambda_1, \lambda_2, n) \) is equal to the multiplicity of the irreducible \( GL^2(K) \)-module \( W(\lambda_1, \lambda_2) \) in the homogeneous component \( F_d^{(n)} \) of total degree \( n \) of \( F_d \). This implies that the bigraded Hilbert series of the algebra \( F_d^\delta \) is

\[ H_{GL^2}(F_d^\delta, t_1, t_2, z) = \sum_{n \geq 0} \sum_{(\lambda_1, \lambda_2)} m(\lambda_1, \lambda_2, n) t_{1}^{\lambda_1} t_{2}^{\lambda_2} z^n. \]

The Hilbert series of \( F^\delta_d \) as a \( Z \)-graded vector space is

\[ H(F_d^\delta, z) = \sum_{n \geq 0} (F_d^\delta)^{(n)} z^n = H_{GL^2}(F_d^\delta, 1, 1, z). \]

The Hilbert series \( H_{GL^2}(F_d, t_1, t_2, z) \) is a so called nice rational symmetric function in \( t_1, t_2 \), i.e., its denominator is a product of binomials of the form \( 1 - t_{1}^{a_1} t_{2}^{a_2} z^n \). This allows to use the same method as in \( [\text{DDF}] \). It is an improvement (given in \( [\text{BBD}] \)) of the method of Elliott \([\text{E}]\) and its further development by McMahon \([\text{Mc}]\), the so called partition analysis or \( \Omega \)-calculus. If we already have a function \( f(t_1, t_2, z) \) it is much easier to check (even by hand) whether it is equal to the Hilbert series \( H_{GL^2}(F_d, t_1, t_2, z) \). It is known that \( f(t_1, t_2, z) = H_{GL^2}(F_d, t_1, t_2, z) \) if and only if

\[ H_{GL^2}(F_d, t_1, t_2, z) = \frac{1}{t_1-t_2} (t_1 f(t_1, t_2, z) - t_2 f(t_2, t_1, z)). \]

Now we shall give the Hilbert series of the subalgebras of constants of Weitzenböck derivations of free metabelian associative algebras with small number of generators. In some of the cases we give both Hilbert series, as graded and bigraded vector spaces, because we shall use the results in the last section of our paper. We do not give results for derivations with a one-dimensional Jordan cell because we shall handle them in the next section.

**Example 3.2.** Let \( \delta = \delta(p_1, \ldots, p_s) \) be the Weitzenböck derivation acting on \( KX_d \) with Jordan cells of size \( p_1 + 1, \ldots, p_s + 1 \). Let \( \delta \) act on the vector spaces \( KU_d \) and \( KV_d \) in the same way as on \( KX_d \). We extend the action of \( \delta \) on the free metabelian associative algebra \( F_d \) generated by \( X_d \) and on the polynomial algebra \( K[U_d, V_d] \) of \( 2d \) variables. Then the Hilbert series of the algebras of constants \( F^\delta_d \) and \( K[U_d, V_d]^\delta \) are:

\[ d = 2, \ \delta = \delta(1): \]

\[ H_{GL^2}(F^\delta_2, t_1, t_2, z) = \frac{1}{1 - t_{1} z} + \frac{t_{1} t_{2} z^2}{(1 - t_{1} z)^2(1 - t_{1} t_{2} z^2)}. \]
\[ H(F^8_2, z) = \frac{1}{1 - z} + \frac{z^2}{(1 - z)^2(1 - z^2)}; \]
\[ H_{GL2}(K[U_2, V_2]^\delta, t_1, t_2, z) = \frac{1}{(1 - t_1 z)^2(1 - t_1 t_2 z^2)}; \]
\[ H(K[U_2, V_2]^\delta, z) = \frac{1}{(1 - z)^2(1 - z^2)}; \]
\[ d = 3, \delta = \delta(2): \]
\[ H_{GL2}(F^8_3, t_1, t_2, z) = \frac{1}{(1 - t_1 z)(1 - t_1 t_2 z^2)(1 - t_1^2 z^2)} + \frac{t_1^3 t_2 (1 + t_1 t_2 z)(1 + t_1 t_2 z + t_1 t_2^2 z^2) z^2}{(1 - t_1^2 z^2)^3(1 - t_1^2 t_2^2 z^4)^3}; \]
\[ H(F^8_3, z) = \frac{1}{(1 - z)(1 - z^2)} + \frac{z^2(1 + z)(1 + 2 z - z^2)}{(1 - z)^2(1 - z^2)^3}; \]
\[ H_{GL2}(K[U_3, V_3]^\delta, t_1, t_2, z) = \frac{1 + t_1^3 t_2 z^2}{(1 - t_1^2 z^2)^2(1 - t_1^2 t_2^2 z^4)^3}; \]
\[ H(K[U_3, V_3]^\delta, z) = \frac{1 + z^2}{(1 - z)^2(1 - z^2)^3}; \]
\[ d = 4, \delta = \delta(3): \]
\[ H(F^8_4, z) = \frac{1 - z + z^2}{(1 - z)^2(1 - z^2)} + \frac{z^2 p(z)}{(1 - z)^4(1 - z^2)^2(1 - z^4)^3}, \]
\[ p(z) = 2 + z + 3 z^2 + 4 z^3 - 6 z^4 - 13 z^5 + 13 z^6 - 14 z^7 + 2 z^8 + 9 z^9 - 5 z^{10} + 4 z^{11} + 2 z^{12}; \]
\[ H(K[U_4, V_4]^\delta, z) = \frac{1 - 2 z + 4 z^2 - 3 z^4 - 3 z^8 + 4 z^{10} - 2 z^{11} + z^{12}}{(1 - z)^4(1 - z^2)^2(1 - z^4)^3}; \]
\[ d = 4, \delta = \delta(1,1): \]
\[ H(F^8_4, z) = \frac{1}{(1 - z)^2(1 - z^2)} + \frac{z^2(4 + 2 z + z^2 - 22 z^3 + 9 z^4 + 10 z^5 - 3 z^6 - 2 z^7 + z^8)}{(1 - z)^4(1 - z^2)^5}, \]
\[ H(K[U_4, V_4]^\delta, z) = \frac{1 + z^2 - 4 z^3 + z^4 + z^6}{(1 - z)^4(1 - z^2)^5}; \]
\[ d = 5, \delta = \delta(4): \]
\[ H(F^8_5, z) = \frac{1 - z + z^2}{(1 - z)^2(1 - z^2)(1 - z^3)} + \frac{z^2 q(z)}{(1 - z)^5(1 - z^2)^3(1 - z^3)^3}, \]
\[ q(z) = 2 + 4 z + 5 z^2 - 6 z^3 - 15 z^4 + 11 z^5 - 10 z^6 + 3 z^7 + 5 z^8 + 2 z^9 + z^{10} - 5 z^{11} + 4 z^{12} - z^{13}; \]
\[ H(K[U_5, V_5]^\delta, z) = \frac{1 - 3 z + 6 z^2 - 7 z^3 + 3 z^4 + 2 z^5 + z^6 - 9 z^7 + 8 z^8 - 3 z^9 + z^{10}}{(1 - z)^5(1 - z^2)^3(1 - z^3)^3}; \]
\[ d = 5, \delta = \delta(2,1): \]
\[ H(F^8_5, z) = \frac{1 + z^2}{(1 - z)^2(1 - z^2)(1 - z^3)} + \frac{z^2 r(z)}{(1 - z)^5(1 - z^2)^3(1 - z^3)^3}, \]
\[ r(z) = 4 + 8 z + 8 z^2 - 9 z^3 - 20 z^4 - 5 z^5 - 2 z^6 + 9 z^7 + 7 z^8 - 2 z^{11} + 3 z^{12} - z^{13}; \]
\[ H(K[U_5, V_5]^{\delta}, z) = \frac{1 + 5z^2 + z^3 - 4z^4 - 3z^5 - 3z^6 - 4z^7 + z^8 + 5z^9 + z^{11}}{(1 - z)^6(1 - z^2)^3(1 - z^3)^3}. \]

We shall skip the explicit computations of the Hilbert series of \( F_d^{\delta} \) and \( K[U_d, V_d]^{\delta} \). But we can check the result in the cases \( d = 2 \) and \( d = 3 \) using the equality

\[ H_{GL_2}(F_d, t_1, t_2, z) = \frac{1}{t_1 - t_2} (t_1 H_{GL_2}(F_d^{\delta}, t_1, t_2, z) - t_2 H_{GL_2}(F_d^{\delta}, t_2, t_1, z)). \]

For example, for \( d = 3 \), \( \delta = \delta(2) \) we have \( H(F_3, z_1, z_2, z_3) \)

\[ H_{GL_2}(F_3, t_1, t_2, z) = \frac{2}{(1 - t_1^2 z)(1 - t_2^2 z)(1 - t_3^2 z)} \left( \frac{2}{(1 - t_1^2 z)(1 - t_2^2 z)(1 - t_3^2 z)} + \frac{(t_1^2 + t_2^2 + t_3^2) - 1}{(1 - t_1^2 z)^2(1 - t_2^2 z)^2(1 - t_3^2 z)^2} \right), \]

\[ H_{GL_2}(F_3^{\delta}, t_1, t_2, z) = \frac{1}{1 - (t_1^2 t_2 z)} f_1(t_1, t_2) + \frac{t_1 t_2 z^2}{(1 - t_1^2 z)(1 - t_2^2 z)} f_2(t_1, t_2) + \frac{(t_1 t_2^3)(t_1 + t_2^3) z^3}{(1 - t_1^2 z)(1 - t_2^2 z)^2} f_3(t_1, t_2), \]

\[ f_1(t_1, t_2) = \frac{1}{1 - t_1^2 z^2}, \quad f_2(t_1, t_2) = \frac{t_2^2}{(1 - t_1^2 z)^2}, \quad f_3(t_1, t_2) = \frac{t_1^2}{(1 - t_1^2 z)^2}, \]

\[ g_1(t_1, t_2) = \frac{1}{t_1 - t_2} (t_1 f_1(t_1, t_2) - t_2 f_1(t_2, t_1)) = \frac{1 + t_1 t_2 z}{(1 - t_1^2 z)(1 - t_2^2 z)}, \]

\[ g_2(t_1, t_2) = \frac{1}{t_1 - t_2} (t_1 f_2(t_1, t_2) - t_2 f_2(t_2, t_1)) = \frac{(t_1 + t_2)^2 - t_1 t_2 (1 + t_1 t_2 z)^2}{(1 - t_1^2 z)^2(1 - t_2^2 z)^2}, \]

\[ g_3(t_1, t_2) = \frac{1}{t_1 - t_2} (t_1 f_3(t_1, t_2) - t_2 f_3(t_2, t_1)) = \frac{(t_1 + t_2)(1 - (t_1 t_2 z)^2)}{(1 - t_1^2 z)^2(1 - t_2^2 z)^2}. \]

Now direct computations give that

\[ \frac{1}{t_1 - t_2} (t_1 H_{GL_2}(F_3^{\delta}, t_1, t_2, z) - t_2 H_{GL_2}(F_3^{\delta}, t_2, t_1, z)) \]

\[ = g_1(t_1, t_2) + \frac{t_1 t_2 z^2 g_2(t_1, t_2)}{1 - (t_1 t_2 z)^2} + \frac{(t_1 t_2 z^2)^2 g_3(t_1, t_2)}{(1 - t_1 t_2 z)(1 - (t_1 t_2 z)^2)} + \frac{(t_1 t_2^3)(t_1 + t_2^3) z^3 g_3(t_1, t_2)}{(1 - t_1 t_2 z)(1 - (t_1 t_2 z)^2)^2} = H_{GL_2}(F_3, t_1, t_2, z). \]

4. Derivations with one-dimensional Jordan cell

In this section we study the Weitzenböck derivations \( \delta \) with the property that their actions on the vector space \( KX_d \) have a Jordan form with a \( 1 \times 1 \) cell. We may assume that \( \delta \) acts as a nilpotent linear operator on \( KX_{d-1} \) and \( \delta(x_d) = 0 \). Also, \( \delta \) acts on \( KU_d \) and \( KV_d \) in a similar way: the vector subspaces \( KU_{d-1} \) and \( KV_{d-1} \) are \( \delta \)-invariant, and \( \delta(u_d) = \delta(v_d) = 0 \). Let \( \omega(K[V_d-1]) \) denote the augmentation ideal of the polynomial algebra \( K[V_d-1] \). As a \( K[U_{d-1}, V_{d-1}] \)-module the ideal

\[ K[U_{d-1}, V_{d-1}] = K[U_{d-1}] \otimes_K \omega(K[V_d-1]) \]

of \( K[U_{d-1}, V_{d-1}] \) is generated by \( V_{d-1} \). By \[22\] Proposition 3 \( K[U_{d-1}, V_{d-1}]^{\delta} \) is a finitely generated \( K[U_{d-1}, V_{d-1}]^{\delta} \)-module.
Lemma 4.1. The Hilbert series of \((F_d')^\delta\), \((F_{d-1}')^\delta\) and \(K[U_{d-1}, V_{d-1}]^\delta\) are related by

\[
H_{GL_2}((F_d')^\delta, t_1, t_2, z) = \frac{1}{(1-z)^2} \left( H_{GL_2}((F_{d-1}')^\delta, t_1, t_2, z) + z H_{GL_2}(K[U_{d-1}, V_{d-1}]_{\omega}, t_1, t_2, z) \right).
\]

Proof. If \(\delta\) act on \(KU_{d-1}\) and \(KV_{d-1}\) as \(\delta = \delta(p_1, \ldots, p_{s-1})\), then it acts on \(KU_d\) and \(KV_d\) as \(\delta(p_1, \ldots, p_{s-1}, 0)\). By Lemma 3.1 the Hilbert series of the commutator ideal of \(F_d\) is

\[
H(F_d', z_1, \ldots, z_d) = \prod_{j=1}^d \frac{1}{1 - z_j} H(L_d'/L_d'', z_1, \ldots, z_d),
\]

where \(L_d'/L_d''\) is the commutator ideal of the free metabelian Lie algebra. Following the procedure described in Section 3 we replace its variables \(z_j\) with \(t_1^{q_j} t_2^{r_j} z\), where the nonnegative integers \(q_j, r_j\) depend on the size of the corresponding Jordan cell and the position of the variable \(x_j\) in the Jordan basis of \(KX_d\). In particular, we have to replace the variable \(z_d\) with \(z\). Hence

\[
H_{GL_2}(F_d', t_1, t_2, z) = \frac{1}{1-z} \prod_{j=1}^{d-1} \frac{1}{1-t_1^{q_j} t_2^{r_j} z} H_{GL_2}(L_d'/L_d'', t_1, t_2, z).
\]

We know from [DDF] Lemma 4.1 that

\[
H_{GL_2}(L_d'/L_d'', t_1, t_2, z) = \frac{1}{1-z} \left( H_{GL_2}(L_{d-1}'/L_{d-1}'', t_1, t_2, z) + z H_{GL_2}(\omega(K[V_{d-1}]), t_1, t_2, z) \right).
\]

Thus

\[
H_{GL_2}(F_d', t_1, t_2, z) = \frac{1}{(1-z)^2} \left( H_{GL_2}(F_{d-1}', t_1, t_2, z) + z H_{GL_2}(K[U_{d-1}], t_1, t_2, z) H_{GL_2}(\omega(K[V_{d-1}]), t_1, t_2, z) \right).
\]

Using the fact that the formal power series \(f(t_1, t_2, z)\) is the Hilbert series of the \(\delta\)-constants of the \(\mathbb{Z}\)-graded \(GL_2(K)\)-module \(W\) if and only if

\[
H_{GL_2}(W, t_1, t_2, z) = \frac{1}{t_1 - t_2}(t_1 f(t_1, t_2, z) - t_2 f(t_2, t_1, z)),
\]

we obtain that the equality

\[
H_{GL_2}(F_d') = \frac{1}{(1-z)^2} (H_{GL_2}(F_{d-1}') + z H_{GL_2}(K[U_{d-1}, V_{d-1}]_{\omega}))
\]

is equivalent to the desired equality

\[
H_{GL_2}((F_d')^\delta, t_1, t_2, z) = \frac{1}{(1-z)^2} \left( H_{GL_2}((F_{d-1}')^\delta, t_1, t_2, z) + z H_{GL_2}(K[U_{d-1}, V_{d-1}]^\delta, t_1, t_2, z) \right).
\]
Clearly this implies that
\[
H((F'_{d-1})^\delta, z) = \frac{1}{(1-z)^2}H((F'_{d-1})^\delta, z) + zH(K[U_{d-1}, V_{d-1}]_\omega^\delta, z).
\]

Recall that \( F'_d \) is a \( K[U_d, V_d] \)-module. For every monomial
\[
u_{j_1} \cdots \nu_{j_m} \in K[U_{d-1}], v_{j_1} \cdots v_{j_m} \in \omega(K[V_{d-1}]), m, n \geq 1,
\]
we define a \( K \)-linear map \( \pi : K[U_{d-1}, V_{d-1}]_\omega \to F'_d \) by
\[
\pi(v_{j_1} \cdots v_{j_m}) = \sum_{k=1}^n [x_{d, j_k}] v_{j_1} \cdots v_{j_{k-1}} v_{j_k+1} \cdots v_{j_m},
\]
\[
\pi(u_{i_1} \cdots u_{i_m} v_{j_1} \cdots v_{j_m}) = \pi(v_{j_1} \cdots v_{j_m}) u_{i_1} \cdots u_{i_m},
\]
and
\[
\pi(u_{i_1} \cdots u_{i_m}) = 0.
\]

Lemma 4.2. (i) \([DDF]\) The map \( \pi \) satisfies the equality
\[
\pi(v_1 v_2) = \pi(v_1) v_2 + \pi(v_2) v_1, \quad v_1, v_2 \in \omega(K[V_{d-1}]^\delta).
\]

(ii) The derivation \( \delta \) and the map \( \pi \) commute.

Proof. The case (i) was already proved in \([DDF]\). It is sufficient to prove the case (ii) when \( w = uv \in K[U_{d-1}, V_{d-1}]_\omega \). Let \( u \in K[U_{d-1}] \) and \( v \in \omega(K[V_{d-1}]) \) be monomials. The map \( \pi \) sends \( v \) to the commutator ideal \( L'_d/L'_d \) of the free metabelian Lie algebra \( L'_d/L'_d \). In \([DDF]\) we have seen that \( \delta(\pi(v)) = \pi(\delta(v)) \).

Now
\[
\delta(\pi(u)) = \delta(\pi(v)) = \delta(\pi(v)u) = \delta(\pi(v))u + \pi(v)\delta(u) = \pi(\delta(v))u + \pi(v)\delta(u) = \pi(\delta(v)u + v\delta(u)) = \pi(\delta(vu)) = \pi(\delta(w))
\]
and this establishes (ii).

The next theorem and its corollary are the main results of the section.

Theorem 4.3. Let the Weitzenböck derivation \( \delta \) acting on the vector space \( KX_d \) have a Jordan form with a \( 1 \times 1 \) cell. Let \( \delta \) act as a nilpotent linear operator on \( KX_{d-1} \) and \( \delta(x_d) = 0 \), with a similar action on \( KU_d \) and \( KV_d \). Let \( \{d_i \mid i \in I\}, \{g_j \in K[U_{d-1}, V_{d-1}]_\omega \mid j \in J\} \) be, respectively, homogeneous vector space bases of the \( K[U_{d-1}, V_{d-1}]^\delta \)-module \( (F'_{d-1})^\delta \) and of \( K[U_{d-1}, V_{d-1}]_\omega^\delta \) with respect to both \( \mathbb{Z} \)- and \( \mathbb{Z}^2 \)-gradings. Then \( (F'_d)^\delta \) has a basis
\[
\{d_i u_{i_1} v_{i_2}, \pi(g_j) u_{i_1} v_{i_2} \mid i \in I, j \in J, m, n \geq 0\}.
\]

Proof. The Hilbert series of \( (F'_{d-1})^\delta \) and \( K[U_{d-1}, V_{d-1}]_\omega^\delta \) are equal, respectively, to the generating functions of their bases. Hence
\[
H_{GL_2}((F'_{d-1})^\delta, t_1, t_2, z) = \sum_{i \in I} t_1^{q_i} t_2^{r_i} z^{m_i},
\]
where \( d_i \) is of bidegree \( (q_i, r_i) \) and of total degree \( m_i \). Since \( u_{d} \) and \( v_{d} \) are of bidegree \( (0, 0) \) and of total degree 1, the generating function of the set \( D = \{d_i u_{i_1} v_{i_2} \mid i \in I, n \geq 0\} \) is
\[
G(D, t_1, t_2, z) = \sum_{m, n \geq 0} \sum_{i \in I} t_1^{q_i} t_2^{r_i} z^{m_i} z^{m+n} = \frac{1}{(1-z)^2} H_{GL_2}((F'_{d-1})^\delta, t_1, t_2, z).
\]
The map $\pi$ sends the monomials of $K[U_{d-1}, V_{d-1}]$ to linear combinations of commutators with an extra variable $x_d$ in the beginning of each commutator. Hence, if the Hilbert series of $K[U_{d-1}, V_{d-1}]$ is
\[ H_{GL_2}(K[U_{d-1}, V_{d-1}], t_1, t_2, z) = \sum_{j \in J} t_1^{k_j} t_2^{l_j} z^{n_j}, \]
where the bidegree of $g_j$ is $(k_j, l_j)$ and its total degree is $n_j$, then the generating function of the set $E = \{ \pi(g_j)u_{d}^{m_j}v_{d}^{n_j} \mid j \in J, m', n' \geq 0 \}$ is
\[ G(E, t_1, t_2, z) = \sum_{m, n \geq 0} \sum_{j \in J} t_1^{k_j} t_2^{l_j} z^{n_j+1} z^{m+n} = \frac{z}{1 - t_1 z} H_{GL_1}(K[U_{d-1}, V_{d-1}], t_1, t_2, z). \]
Hence, by Lemma 4.1
\[ H_{GL_2}((F')^\delta, t_1, t_2, z) = G(D, t_1, t_2, z) + G(E, t_1, t_2, z). \]
Since both sets $D$ and $E$ are contained in $(F')^\delta$, we shall conclude that $D \cup E$ is a basis of $(F')^\delta$ if we show that the elements of $D \cup E$ are linearly independent. For this purpose it is more convenient to work in the wreath product $W_d$.

The elements $d_i$ belong to $F'_{d-1} \subset W_{d-1}$ and hence are of the form
\[ d_i = \sum_{k=1}^{d-1} a_k w_{ki}(U_{d-1}, V_{d-1}), \quad w_{ki}(U_{d-1}, V_{d-1}) \in K[U_{d-1}, V_{d-1}]. \]
Therefore
\[ d_i u_{d}^{m} v_{d}^{n} = \sum_{k=1}^{d-1} a_k w_{ki}(U_{d-1}, V_{d-1}) u_{d}^{m} v_{d}^{n}. \]
On the other hand, the elements $\pi(g_j) u_{d}^{m'} v_{d}^{n'}$ are of the form
\[ \pi(g_j) u_{d}^{m'} v_{d}^{n'} = \sum_{k=1}^{d-1} [x_d, x_k] h_{kj}(U_{d-1}, V_{d-1}) u_{d}^{m'} v_{d}^{n'} \]
\[ = \sum_{k=1}^{d-1} (a_d v_k - a_k v_d) h_{kj}(U_{d-1}, V_{d-1}) u_{d}^{m'} v_{d}^{n'}, \]
where $h_{kj}(U_{d-1}, V_{d-1}) \in K[U_{d-1}, V_{d-1}]$. Let
\[ w = \sum \xi_{imn} d_i u_{d}^{m} v_{d}^{n} + \sum \eta_{im'n'} \pi(g_j) u_{d}^{m'} v_{d}^{n'} = 0 \]
for some $\xi_{imn}, \eta_{im'n'} \in K$. Clearly, we may assume that the elements $d_i u_{d}^{m} v_{d}^{n}$, $\pi(g_j) u_{d}^{m'} v_{d}^{n'}$ are homogeneous with respect to each of the groups of variables $U_{d-1}$, $A_{d-1} \cup V_{d-1}$, $\{ u_d \}$, and $\{ a_d, v_d \}$. It follows from the definition of $\pi$ that
\[ \pi(v_{j_1} \cdots v_{j_q}) = \sum_{k=1}^{q} [x_d, x_{j_k}] v_{j_1} \cdots v_{j_{k-1}} v_{j_k+1} \cdots v_{j_q} \]
\[ = \sum_{k=1}^{q} (a_d v_{j_k} - a_{j_k} v_d) v_{j_1} \cdots v_{j_{k-1}} v_{j_k+1} \cdots v_{j_q} \]
\[ = qa_d v_{j_1} \cdots v_{j_q} - \sum_{k=1}^{q} a_{j_k} v_{j_1} \cdots v_{j_{k-1}} v_{j_k+1} \cdots v_{j_q} v_d. \]
Hence, if \( g_j(U_{d-1}, V_{d-1}) \) is homogeneous of degree \( q \) with respect to \( V_{d-1} \), then

\[
\pi(g_j) u_{d}^{m'} v_{d}^{n'} = \left( qa_d g_j(U_{d-1}, V_{d-1}) - \sum_{k=1}^{d-1} a_k h_{kj}(U_{d-1}, V_{d-1}) v_d \right) u_{d}^{m'} v_{d}^{n'}.
\]

In the linear dependence between the elements \( d_i u_{d}^{m'} v_{d}^{n'} \) and \( \pi(g_j) u_{d}^{m'} v_{d}^{n'} \) we replace these elements with their expressions in \( W_d \) and obtain

\[
w = \sum \xi_{imn} \sum_{k=1}^{d-1} a_k w_{ki}(U_{d-1}, V_{d-1}) u_{d}^{m'} v_{d}^{n'}
+ \sum \eta_{jm'n'} \left( qa_d g_j(U_{d-1}, V_{d-1}) - \sum_{k=1}^{d-1} a_k h_{kj}(U_{d-1}, V_{d-1}) v_d \right) u_{d}^{m'} v_{d}^{n'} = 0.
\]

Hence \( m' = m \), \( n' = n - 1 \), and, canceling \( u_{d}^{m'} \) and \( v_{d}^{n-1} \), we obtain

\[
w = qa_d \sum \eta_{jm,n-1} g_j(U_{d-1}, V_{d-1}) - \sum \eta_{jm,n-1} \sum_{k=1}^{d-1} a_k h_{kj}(U_{d-1}, V_{d-1}) v_d
+ \sum \xi_{imn} \sum_{k=1}^{d-1} a_k w_{ki}(U_{d-1}, V_{d-1}) v_d = 0.
\]

Since the elements \( g_j \) are linearly independent in \( K[U_{d-1}, V_{d-1}]_{\delta} \), comparing the coefficient of \( w \) we conclude that \( \eta_{jm,n-1} = 0 \). Then, using that the elements \( d_i \) are linearly independent in \( (F_{d-1})_{\delta} \), we derive that \( \xi_{imn} = 0 \). Hence the set \( D \cup E \) is a basis of \( (F_{d-1})_{\delta} \).

Every polynomial \( f(U_{d-1}, V_{d-1}) \in K[U_{d-1}, V_{d-1}]_{\delta} \) can be written in the form

\[
f(U_{d-1}, V_{d-1}) = f'(U_{d-1}) + f''(U_{d-1}, V_{d-1}),
\]

where \( f'(U_{d-1}) \) does not depend on \( V_{d-1} \) and every monomial of \( f''(U_{d-1}, V_{d-1}) \) contains a variable from \( V_{d-1} \). Since \( \delta(f'(U_{d-1})) \) and \( \delta(f''(U_{d-1}, V_{d-1})) \) preserve these properties, both \( f'(U_{d-1}) \) and \( f''(U_{d-1}, V_{d-1}) \) belong to \( K[U_{d-1}, V_{d-1}]_{\delta} \). Hence we may fix a system of generators of the algebra \( K[U_{d-1}, V_{d-1}]_{\delta} \)

\[
\{ e_1(U_{d-1}), \ldots, e_k(U_{d-1}), f_1(U_{d-1}, V_{d-1}), \ldots, f_l(U_{d-1}, V_{d-1}) \},
\]

where \( e_i(U_{d-1}) \in K[U_{d-1}]_{\delta} \) and all monomials of \( f_j(U_{d-1}, V_{d-1}) \) depend on \( V_{d-1} \), i.e., \( f_j(U_{d-1}, V_{d-1}) \) belongs to \( K[U_{d-1}, V_{d-1}]_{\delta} \). Every element of \( K[U_{d-1}, V_{d-1}]_{\delta} \) is a linear combination of products \( e_1^{a_1} \cdots e_k^{a_k} f_1^{b_1} \cdots f_l^{b_l} \), where \( b_1 + \cdots + b_l > 0 \).

Hence the set

\[
\{ f_1(U_{d-1}, V_{d-1}), \ldots, f_l(U_{d-1}, V_{d-1}) \}
\]

the \( K[U_{d-1}, V_{d-1}]_{\delta} \)-module \( K[U_{d-1}, V_{d-1}]_{\delta} \). We also fix a set \( \{ c_1, \ldots, c_m \} \) of generators of the \( K[U_{d-1}, V_{d-1}]_{\delta} \)-module \( (F_{d-1})_{\delta} \). Without loss of generality we may assume that the polynomials in these systems are homogeneous. Our purpose is to find a generating set of the \( K[U_{d-1}, V_{d-1}]_{\delta} \)-module \( (F_{d-1})_{\delta} \).

**Corollary 4.4.** Let \( X_d \) be a Jordan basis of the Weitzenböck derivation \( \delta \) and let \( \delta \) have a \( 1 \times 1 \) Jordan cell corresponding to \( x_d \). Let \( \delta \) act in the same way on \( KU_d \) and \( KV_d \). Let \( \{ c_1, \ldots, c_m \} \) be a homogeneous generating set of the \( K[U_{d-1}, V_{d-1}]_{\delta} \)-module \( (F_{d-1})_{\delta} \). Let \( \{ c_1, \ldots, c_k, f_1, \ldots, f_l \} \) be a generating set of the algebra \( K[U_{d-1}, V_{d-1}]_{\delta} \),
where \( e_1, \ldots, e_k \) do not depend on \( V_{d-1} \) and every monomial of \( f_1, \ldots, f_i \) depends on \( V_{d-1} \). Then the \( K[U_d, V_d]^{\delta} \)-module \((F')^\delta_d\) is generated by the set
\[
\{c_1, \ldots, c_m\} \cup \{\pi(f_1), \ldots, \pi(f_i)\}.
\]

**Proof.** Clearly, the \( K[U_{d-1}, V_{d-1}]^{\delta} \)-module \((F_{d-1}')^\delta\) is spanned by the elements \( c_i e_1^{a_1} \cdots e_k^{a_k} f_1^{b_1} \cdots f_i^{b_i} \). In particular, in this way we obtain all elements \( d_i \) from the basis of the vector space \((F_{d-1}')^\delta\). Since \( u_d, v_d \in K[U_d, V_d]^{\delta}\), we obtain also all elements \( d_i u_d^n v_d^m \). Similarly, the \( K[U_{d-1}, V_{d-1}]^{\delta} \)-module \( K[U_{d-1}, V_{d-1}]^{\delta}\) is spanned by the products \( e_1^{a_1} \cdots e_k^{a_k} f_1^{b_1} \cdots f_i^{b_i} \) which satisfy the property \( b_1 + \cdots + b_i > 0 \).

By Lemma 4.2
\[
\pi(e_1^{a_1} \cdots e_k^{a_k} f_1^{b_1} \cdots f_i^{b_i}) = \sum_{j=1}^i \pi(f_j) e_1^{a_1} \cdots e_k^{a_k} f_1^{b_1} \cdots f_j^{b_j-1} \cdots f_i^{b_i}.
\]

Since \( K[U_d, V_d]^{\delta} = (K[U_{d-1}, V_{d-1}]^{\delta})[u_d, v_d] \), the elements \( \pi(g_j) u_d^n v_d^m \) belong to the \( K[U_{d-1}, V_{d-1}]^{\delta}\)-module generated by \( \pi(f_1), \ldots, \pi(f_i) \). Hence \( \{c_1, \ldots, c_m\} \cup \{\pi(f_1), \ldots, \pi(f_i)\} \) generate the \( K[U_d, V_d]^{\delta}\)-module \((F')^\delta_d\).

**5. Generating sets for small number of generators**

In this section we shall find the generators of the \( K[U_d, V_d]^{\delta}\)-module \((F')^\delta_d\) for \( d \leq 3 \).

**Example 5.1.** Let \( d = 2, \delta = \delta(1) \), and let \( \delta(x_1) = 0, \delta(x_2) = x_1 \). It is well known, see e.g., [N], that \( K[U_2, V_2]^{\delta}\) is generated by the algebraically independent polynomials \( u_1, v_1 \) and \( u_1 v_2 - u_2 v_1 \). Hence
\[
H_{GL_2}(K[U_2, V_2]^{\delta}, t_1, t_2, z) = \frac{1}{1-t_1 z^2(1-t_1 t_2 z^2)},
\]
as indicated in Example 3.2. The same example gives that
\[
H_{GL_2}((F')^{\delta_2}, t_1, t_2, z) = \frac{t_1 t_2 z^2}{1-t_1 z^2(1-t_1 t_2 z^2)}.
\]

It is easy to see that the element \([x_2, x_1]\) is belong to \((F')^\delta_2\) and is of bidegree \((1, 1)\). Since the Hilbert series of \( K[U_2, V_2]^{\delta}\)-submodule of \((F')^\delta_2\) generated by \([x_2, x_1]\) is equal to the Hilbert series of the whole module \((F')^\delta_2\), we conclude that \((F')^\delta_2\) is a free cyclic \( K[U_2, V_2]^{\delta}\)-module generated by \([x_2, x_1]\). As a vector space \((F')^\delta_2\) is spanned by the elements \( x_1^k, [x_2, x_1]u_1 v_1^m(u_1 v_2 - u_2 v_1)^n, k, l, m, n \geq 0 \). Recall that the action of \( u_1 v_2 - u_2 v_1 \) on the commutator ideal \( F_{2}' \) is given by
\[
w(u_1 v_2 - u_2 v_1) = x_1[w, x_2] - x_2[w, x_1], \quad w \in F_{2}'.
\]

Knowing the basis of \( F_{2}' \) it is easy to derive that the algebra \((F_{2}')^{\delta}\) is generated by the infinite set
\[
\{x_1, [x_1, x_2](u_1 v_2 - u_2 v_1)^n \mid n \geq 0\}.
\]

**Example 5.2.** Let \( d = 3 \) and let the Jordan normal form of \( \delta \) have two cells, of size \( 2 \times 2 \) and \( 1 \times 1 \), respectively. Hence \( \delta = \delta(1, 0) \) in our notation and we may apply Corollary 4.3. By Example 5.1 for \( d = 2 \) and \( \delta = \delta(1) \), the algebra \( K[U_2, V_2]^{\delta}\) is generated by \( u_1, v_1 \) and \( u_1 v_2 - u_2 v_1 \). The \( K[U_2, V_2]^{\delta}\)-module \((F')^\delta_3\) is generated by the single element \([x_2, x_1]\). In the notation of Corollary 4.3
\[
e_1 = u_1, \quad f_1 = v_1, \quad f_2 = u_1 v_2 - u_2 v_1, \quad c_1 = [x_2, x_1].
\]
Hence the $K[U_3, V_3]^\delta$-module $(F'_3)^\delta$ is generated by $c_1 = [x_2, x_1]$, $\pi(f_1) = [x_3, x_1]$, and

$$\pi(f_2) = \pi(u_1v_2 - u_2v_1) = \pi(v_2)u_1 - \pi(v_1)u_2 = x_1[x_3, x_2] - x_2[x_3, x_1].$$

These three elements satisfy the relation

$$c_1u_1v_2 - \pi(f_1)(u_1v_2 - u_2v_1) + \pi(f_2)v_1$$

$$x_1[[x_2, x_1], x_3] - (x_1[x_3, x_1]x_2 - x_2[x_3, x_1]) + [x_1x_3, x_2] - x_2[x_3, x_1] = 0.$$ This result agrees with the Hilbert series

$$H_{GL_2}((F'_3)^\delta(1, 0), t_1, t_2, z) = \frac{t_1z^2 + t_1t_2z^2 + t_1t_2z^3 - t_1^2t_2z^4}{(1 - z)(1 - t_1z)^2(1 - t_1t_2z^2)}$$ of $K[U_3, V_3]^\delta$-module $(F'_3)^\delta$. (The summands $t_1z^2$, $t_1t_2z^2$, $t_1t_2z^3$ in the nominator correspond to the three generators $c_1$, $\pi(f_1)$, $\pi(f_2)$, and $-t_1^2t_2z^4$ corresponds to the relation.)

**Example 5.3.** Let $d = 3$, $\delta = \delta(2)$, and let $\delta(x_1) = 0$, $\delta(x_2) = x_1$, $\delta(x_3) = x_2$. The generators of $K[U_3, V_3]^\delta$ are given in [N]. In our notation they are

$$f_1 = u_1, \quad f_2 = v_1, \quad f_3 = u_2^2 - 2u_1u_3, \quad f_4 = u_3^2 - 2v_1v_3, \quad f_5 = u_1v_3 - u_2v_2 + u_3v_1, \quad f_6 = u_1v_2 - u_2v_1.$$ There are two more generators in [N],

$$f_7 = 2u_1^2v_3 - 2u_1u_2v_2 + u_2^2v_1, \quad f_8 = u_1v_2^2 - 2v_1u_2v_2 + 2u_3v_1^2,$$

but they can be expressed by the other ones:

$$f_7 = f_2f_5 + 2f_1f_5, \quad f_8 = f_1f_4 + 2f_1f_6.$$ The generators $f_1, f_2, f_3, f_4, f_5, f_6$ of $K[U_3, V_3]^\delta$ satisfy the defining relation

$$f_6^2 = f_1^2f_4 + f_2^2f_3 + 2f_1f_2f_5.$$ We easily conclude from the Hilbert series

$$H_{GL_2}(K[U_3, V_3]^\delta, t_1, t_2, z) = \frac{1 + t_1^3t_2z^2}{(1 - t_1z)^2(1 - t_1t_2z^2)}$$ that $K[U_3, V_3]^\delta$ is a free $K[f_1, f_2, f_3, f_4, f_5]$-module with generators 1 and $f_6$, and that the algebra $K[U_3, V_3]^\delta$ has the presentation

$$K[U_3, V_3]^\delta \cong K[f_1, f_2, f_3, f_4, f_5 | f_6^2 = f_1^2f_4 + f_2^2f_3 + 2f_1f_2f_5].$$ In particular, as a vector space $K[U_3, V_3]^\delta$ has a basis

$$\{f_1q_1, f_2^2f_3q_2, f_3f_4q_3, f_5^2f_6q_4, f_4f_5f_6^2q_5 | q_1, q_2, q_3, q_4, q_5 \geq 0\}.$$ By Example 3.2 the Hilbert series of $(F'_3)^\delta$ is

$$H_{GL_2}((F'_3)^\delta, t_1, t_2, z) = \frac{(1 + t_1t_2z + t_2^2z - t_1^2t_2z^2)(t_1^2t_2z^2 + t_1t_2t_2z^3)}{(1 - t_1z)^2(1 - t_1t_2z^2)^3},$$

$$= t_1^2t_2z^2(1 + 2t_1^2z) + t_1^4t_2^2z^3(2t_1^2 + t_1t_2) + \cdots$$ This suggests that the $K[U_3, V_3]^\delta$-module $(F'_3)^\delta$ has a generator $c_1$ of degree 2 and bidegree $(3, 1)$. It together with $c_1f_1$ and $c_1f_2$ give the contribution $t_1^2t_2z^2(1 + 2t_1^2z)$. We also expect three generators $c_2$, $c_3$ and $c_4$ of degree 3 and bidegree $(4, 2), (4, 2)$
and (3, 3), respectively. By easy calculations we have found the explicit form of
\(c_1, c_2, c_3, c_4\):
\[
c_1 = [x_2, x_1], \quad c_2 = [x_3, x_1]v_1 - [x_2, x_1]v_2 = [x_3, x_1, x_1] - [x_2, x_1, x_2],
\]
\[
c_3 = [x_3, x_1]u_1 - [x_2, x_1]u_2 = x_1[x_3, x_1] - x_2[x_2, x_1],
\]
\[
c_4 = [x_3, x_2]u_1 - [x_3, x_1]u_2 + [x_2, x_1]u_3 = x_1[x_3, x_2] - x_2[x_3, x_1] + x_3[x_2, x_1].
\]
For example, \(c_4\) is a linear combination of all elements of degree 3 and bidegree (3, 3) of the form:
\[
c_4 = \gamma_1 x_1[x_3, x_2] + \gamma_2 x_2[x_3, x_1] + \gamma_3 x_3[x_2, x_1], \quad \gamma_1, \gamma_2, \gamma_3 \in K,
\]
and the condition \(\delta(c_4) = 0\) gives
\[
0 = \gamma_1 x_1[x_3, x_1] + \gamma_2 (x_1[x_3, x_1] + x_2[x_2, x_1]) + \gamma_3 x_2[x_2, x_1]
= (\gamma_1 + \gamma_2) x_1[x_3, x_1] + (\gamma_2 + \gamma_3) x_2[x_2, x_1].
\]
Hence
\[
\gamma_1 + \gamma_2 = \gamma_2 + \gamma_3 = 0
\]
and, up to a multiplicative constant, the only solution is
\[
\gamma_1 = 1, \quad \gamma_2 = -1, \quad \gamma_3 = 1.
\]
The Hilbert series of the free \(K[U_3, V_3]^{GL}(3, 1)\)-module generated by four elements of bidegree (3, 1), (4, 2), (4, 2), and (3, 3) is
\[
H_{GL_2}(t_1, t_2, z) = \frac{t_1^3 t_2 z^2 (1 + (2 t_1 + t_2) t_2 z) (1 + t_1^3 t_2 z^2)}{(1 - t_1 z)^2 (1 - t_1^3 t_2^2 z^2)^3}.
\]
Hence
\[
H_{GL_2}(t_1, t_2, z) - H_{GL_2}((F'_3)^{GL}, t_1, t_2, z) = (t_1^2 - t_2^2) t_1^3 t_2 z^4 + \cdots
\]
which suggests that there is a relation of bidegree (6, 2) and a generator of bidegree (4, 4). Continuing in the same way, we have found one more generator
\[
c_5 = [x_3, x_1]u_3 v_1 - [x_3, x_1]u_1 v_3 + [x_3, x_2]u_1 v_2 - [x_3, x_2]u_2 v_1 - [x_2, x_1] u_3 v_2 + [x_2, x_1] u_2 v_3
\]
of bidegree (4, 4) and the relations
\[
R_1(6, 2) : c_1 f_6 = c_3 f_2 - c_2 f_1,
\]
\[
R_2(7, 3) : c_2 f_6 = c_4 f_2^2 - c_1 (f_1 f_4 + f_2 f_5),
\]
\[
R_3(7, 3) : c_3 f_6 = c_4 f_1 f_2 + c_1 (f_1 f_5 + f_2 f_3),
\]
\[
R_4(6, 4) : c_4 f_6 = c_2 f_3 + c_4 f_5 + c_5 f_1,
\]
\[
R_5(6, 4) : c_5 f_2 = c_2 f_5 + c_3 f_4,
\]
\[
R_6(7, 5) : c_5 f_6 = c_1 (f_3 f_4 - f_2^2) + c_4 (f_1 f_4 + f_2 f_5).
\]
The above relations show that \(c_j f_6, j = 1, \ldots, 5,\) and \(c_5 f_2\) can be replaced with a linear combination of other generators. Hence the \(K[U_3, V_3]^{GL}\)-module generated by \(c_1, \ldots, c_5\) is spanned by
\[
E = \{c_j f_{i_1}^{m_1} f_{i_2}^{n_1} f_{i_3}^{j_1} f_{i_4}^{j_2} f_{i_5}^{j_3} \mid m_j, n_j, p_j, q_j, r_j \geq 0, j = 1, 2, 3, 4\}
\]
\[
\cup \{c_5 f_{i_1}^{m_1} f_{i_2}^{n_1} f_{i_3}^{j_1} f_{i_4}^{j_2} f_{i_5}^{j_3} \mid m, p, q, r \geq 0\}.
\]
It is easy to check that the generating function of the set $E$ is equal to the Hilbert series of $(F_3)^5$. Hence, if we show that the elements of $E$ are linearly independent, we shall conclude that the $K[U_3,V_3]^5$-module $(F_3)^5$ is generated by $c_1, \ldots, c_5$. Let

$$
\sum_{j=1}^{5} c_j s_j = 0,
$$

where $s_j$ are polynomials in $f_1, f_2, f_3, f_4, f_5$, $j = 1, \ldots, 5$, and $s_5$ does not depend on $f_2 = v_1$. We shall show that this implies that $s_j = 0$, $j = 1, \ldots, 5$. The bidegrees of the polynomials $f_1, f_2, f_3, f_4, f_5$ consist of pairs of even numbers which implies that we can rewrite the equation above as

$$
c_1 s_1 + c_4 s_4 = 0, \quad c_2 s_2 + c_3 s_3 + c_5 s_5 = 0,
$$

since the only bidegrees consisting of odd numbers are the bidegrees of $c_1$ and $c_4$, which are $(3, 1)$ and $(3, 3)$, respectively. We shall work in the abelian wreath product $W_3$ and shall denote by $w_i$ the coordinate of $a_i$ of $w \in W_3$. First we consider the equation $c_1 s_1 + c_4 s_4 = 0$. The three coordinates $w_i$ of

$$
w = c_1 s_1 + c_4 s_4 = c_1 u_1 + c_2 u_2 + c_3 u_3 + c_4 u_4 + c_5 u_5 = 0
$$

define a linear homogeneous system

$$
w_i = 0, \quad i = 1, 2, 3,
$$

with unknowns $s_1$ and $s_4$. Since $w_3 = (u_1 v_1 - u_2 v_1) s_4 = 0$, then $s_4 = 0$ and thus $s_1 = 0$. Now let us consider

$$
w = c_2 s_2 + c_3 s_3 + c_5 s_5 = 0.
$$

We may assume that not all monomials of $s_2, s_3, s_5$ depend on $v_2$. We substitute $v_2 = 0$ in the wreath product. Then $f_1, f_2, f_3, f_4, f_5$ become

$$
\tilde{f}_1 = u_1, \quad \tilde{f}_2 = v_1, \quad \tilde{f}_3 = u_2^2 - 2u_1 u_3, \quad \tilde{f}_4 = -2v_1 v_3, \quad \tilde{f}_5 = u_1 v_3 + u_3 v_1
$$

and the generators $c_2, c_3$ and $c_5$ become

$$
\tilde{c}_2 = (-a_1 v_3 + a_3 v_1) v_1, \quad \tilde{c}_3 = -a_1 u_1 v_3 - a_2 u_2 v_1 + a_3 u_3 v_1, \quad \tilde{c}_5 = a_1 (u_1 v_3 - u_3 v_1) v_3 + a_2 u_2 v_1 v_3 + a_3 (u_1 v_3 - u_3 v_1) v_1
$$

in $W_3$. Also, at least one of the polynomials $\tilde{s}_2, \tilde{s}_3, \tilde{s}_5$ is different from 0. The equality $\tilde{w} = \tilde{c}_2 \tilde{s}_2 + \tilde{c}_3 \tilde{s}_3 + \tilde{c}_5 \tilde{s}_5 = 0$ gives the equalities of the coordinates

$$
\tilde{w}_1 = -v_3 (v_1 \tilde{s}_2 + u_1 \tilde{s}_3 - (u_1 v_3 - u_3 v_1) \tilde{s}_5) = 0, \quad \tilde{w}_2 = -u_2 v_1 (\tilde{s}_3 - 2v_3 \tilde{s}_5) = 0, \quad \tilde{w}_3 = (v_1 \tilde{s}_2 + u_1 \tilde{s}_3 - (u_1 v_3 - u_3 v_1) \tilde{s}_5) = 0.
$$

We consider these three equalities as a homogeneous system with unknowns $\tilde{s}_2, \tilde{s}_3, \tilde{s}_5$. Its only solution is

$$
\tilde{s}_2 = \frac{(u_1 v_3 + u_3 v_1) \tilde{s}_5}{v_1}, \quad \tilde{s}_3 = 2v_3 \tilde{s}_5.
$$

Since we work with polynomials $\tilde{s}_2, \tilde{s}_3, \tilde{s}_5$, we conclude that $v_1$ divides $\tilde{s}_5$ which contradicts with the assumption that $s_5$ does not depend on $v_1$. Hence $\tilde{s}_2 = \tilde{s}_3 = \tilde{s}_5 = 0$. Since $f_1, f_2, f_3, f_4,$ and $f_5$ are algebraically independent in $K[U_3, v_1, v_3]$, we obtain that $s_2 = s_3 = s_5 = 0$. This completes the proof that the $K[U_3,V_3]^5$-module $(F_3)^5$ is generated by $c_1, \ldots, c_5$. 

Now we want to find generators of the algebra $F_3^3$. We start with $x_1$ and $x_2^2 - 2x_1x_3$ which generate $F_3^3$ modulo the ideal $(F_3^3)$. We want to lift them to elements in $F_3^3$. Obviously, $x_1 \in F_3^3$. It is easy to check that the element $x_2^2 - (x_1x_3 + x_3x_1)$ belongs to $F_3^3$ and acts by multiplication on $F_3^3$ in the same way as $x_2^2 - 2x_1x_3$. Hence $x_2^2 - (x_1x_3 + x_3x_1)$ is the lifting of $x_2^2 - 2x_1x_3$ which we are searching for.

Then we shall take a subset of $E$ which, together with $x_1$ and $x_2^2 - 2x_1x_3$, generates $F_3^3$. If $w \in F_3^3$, then $w_1 = x_1w$, $w_2 = (x^2 - (x_1x_3 + x_3x_1))w$ and we can remove the elements of $E$

$$c_j f_1^{m_j} f_2^{n_j} f_3^{p_j} f_4^{q_j} f_5^{r_j}, \quad j = 1, 2, 3, 4,$$

which contain $f_1$ and $f_3$. Hence we may assume that $m_j, p_j, m, p = 0$. Also, $w_2 = wx_1 - x_1w$ and we may consider only those elements of $E$ with $n_j = 0$. Similarly,

$$w_2 = w(x_2^2 - (x_1x_3 + x_3x_1)) - (x_2^2 - (x_1x_3 + x_3x_1))w + 2wf_5$$

and we assume that $q_j = q = 0$. Hence, as a vector space

$$(F_3^3) = \sum_{j=1}^{5} K[x_1, x_2^2 - 2x_1x_3] \langle c_j f_5^q \rangle K[x_1, x_2^2 - 2x_1x_3].$$

As a consequence we obtain that the algebra $F_3^3$ is generated by

$$\{x_1, x_2^2 - (x_1x_3 + x_3x_1), c_j f_5^q \mid j = 1, \ldots, 5, p_j \geq 0\}.$$

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