Li filtrations of SUSY vertex algebras

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Abstract
Any vertex algebra has a canonical decreasing filtration, called the Li filtration, whose associated graded space has a natural structure of a vertex Poisson algebra. In this note, we introduce an analogous filtration for any SUSY vertex algebra, which was introduced by Heluani and Kac as a superfield formalism for supersymmetric vertex algebras. We prove that the associated graded superspace of our filtration has a structure of a SUSY vertex Poisson algebra. We also introduce and discuss related notions, such as Zhu’s $C_2$-Poisson superalgebras, associated superschemes, and singular supports, for SUSY vertex algebras.

Keywords Vertex algebras · SUSY vertex algebras · Li filtration · Vertex Poisson algebras · Associated schemes

Mathematics Subject Classification 17B69 · 17B63 · 17B65 · 17B68

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0 Introduction

**Li filtration**

A basic philosophy in the theory of vertex algebras is that they are quantized objects. This phrase can be understood in various ways, and one of them is given by the work of Li [22]. He introduced a canonical decreasing filtration for an arbitrary vertex algebra $V$, nowadays called the *Li filtration* of $V$. The associated graded space has a natural structure of a vertex Poisson algebra [12, Chap. 16], i.e., a combined structure of a commutative vertex algebra and a vertex Lie algebra. A vertex Poisson algebra can be considered as a Poisson object in the world of vertex algebras, and in this sense, the existence of the Li filtration shows that we can view any vertex algebra as a quantization of the associated graded vertex Poisson algebra.

For explicitness, let us recall the definition of the Li filtration (see also Sect. 3.2). Let $V$ be a vertex algebra, and for $a \in V$ we denote by $a^{(n)} \in \text{End } V$ the $(n)$-operator in the expansion $Y(a, z) = \sum_{n \in \mathbb{Z}} z^{-n-1} a^{(n)}$ of the state-field correspondence. Then, the Li filtration of $V$ is a decreasing sequence of subspaces

$$V = E_0(V) \supset E_1(V) \supset \cdots \supset E_n(V) \supset \cdots$$
consisting of

\[ E_n(V) := \text{span} \left\{ a_{1}^{1-k_1} \cdots a_{r}^{1-k_r} b \left| \begin{array}{c}
 r \in \mathbb{Z}_{>0}, \ a^i, b \in V, \ k_i \in \mathbb{N} \\
 \text{satisfying} \ k_1 + \cdots + k_r \geq n
\end{array} \right. \right\}. \ (0.0.1) \]

Besides the philosophical reason explained above, there has been increasing interest in the Li filtration and the associated graded vertex Poisson algebra. In particular, there are currently many studies with a geometric viewpoint where one sees the spectrum of the associated graded vertex Poisson algebra as a vertex algebra (a.k.a. chiral, or semi-infinite) analogue of a Poisson scheme. Such geometric studies have a strong point that the complicated structure of the original vertex algebra can be treated with less complexity by using Poisson geometry arguments or by making appropriate analogue of them. Among various studies, let us here name the work of Arakawa and Moreau [6], where the notion of chiral symplectic cores is introduced and studied as a vertex Poisson analogue of symplectic leaves in Poisson varieties. Also, relationship between the associated scheme and its arc space (see Sect. 4.1) has found importance recently in connection with chiral homology [11] and 4d/2d duality [4].

**SUSY vertex algebras**

This note is written under the influence of those Poisson-geometric studies of vertex algebras. We focus on SUSY vertex algebras introduced by Heluani and Kac [16]. It is a superfield formulation of vertex algebras equipped with supersymmetry. The detailed explanation will be given in Sect. 2, but here let us give a brief account. There are two classes of structures, \( N_W = N \) SUSY vertex algebras and \( N_K = N \) SUSY vertex algebras, for a positive integer \( N \). Both structures are given by a linear superspace \( V \), an element \( |0⟩ \in V \) and a linear operator \( Y(\cdot, Z) \) on \( V \) whose value for \( a \in V \) is taken in a superfield of the form

\[ Y(a, Z) = \sum_{(j|J)} Z^{-1-j|J} a_{(j|J)}, \quad Z^{j|J} := z^j \zeta^{j_1} \cdots \zeta^{j_r}, \quad a_{(j|J)} \in \text{End} V. \quad (0.0.2) \]

Here \( Z = (z, \xi^1, \ldots, \xi^N) \) denotes a supervariable with an even \( z \) and odd \( \xi^i \)'s, and the index \( (j|J) \) runs over integers \( j \in \mathbb{Z} \) and ordered subsets \( J = \{ j_1 < \cdots < j_r \} \) of the ordered set \( [N] := \{ 1 < \cdots < N \} \). \( V \) is called the superspace of states, \( |0⟩ \) is called the vacuum, and \( Y(\cdot, Z) \) is called the state-superfield correspondence. These should satisfy the vacuum axiom and the locality axiom, similarly as in the non-SUSY case, i.e., the case of ordinary vertex (super)algebras.

The difference between \( N_W = N \) and \( N_K = N \) cases lies in the axiom of translation invariance. An \( N_W = N \) SUSY vertex algebra is equipped with an even endomorphism \( T \) and odd endomorphism \( S^i \)'s for \( i \in [N] \) which satisfy

\[ [T, Y(a, Z)] = \partial_z Y(a, Z), \quad [S^i, Y(a, Z)] = \partial_{\xi^i} Y(a, Z). \]

Here \([\cdot, \cdot]\) denotes the supercommutator (see Sect. 1.1 for a formal definition). On the other hand, an \( N_K = N \) SUSY vertex algebra is equipped with odd operations \( S^i_K \)'s
for $i \in [N]$ satisfying

$$[S^i_K, Y(a, Z)] = D^i_Z Y(a, Z), \quad D^i_Z : = \partial_{\xi^i} + \xi^i \partial_z.$$ 

In this $N_K = N$ case, we automatically have $(S^i_K)^2 = \cdots = (S^N_K)^2$ corresponding to $(D^i_Z)^2 = \partial_z$, and setting $T : = (S^i_K)^2$, we have $[T, Y(a, Z)] = \partial_z Y(a, Z)$.

Similarly as the superfield formalism in supersymmetric quantum field theories, a SUSY vertex algebra has a compact description of the structure. Aside from such formal convenience, it has also much importance. Here we name only a few studies which have geometric flavor: Heluani’s construction [14] of chiral algebras over super curves and superconformal curves, his other work [15] of SUSY structure on the chiral de Rham complex of a Calabi–Yau manifold, and the study of super-modular properties of SUSY conformal blocks for $NW = 1$ case by Heluani and Van Ekeren [17].

**Summary of results**

Now we can explain the main theme of this note. For any $NW = N$ or $NK = N$ SUSY vertex algebra $V$, we introduce a canonical decreasing filtration

$$V = E_0(V) \supset E_1(V) \supset \cdots \supset E_n(V) \supset \cdots$$

where, using the $(j|J)$-operator in (0.0.2), we set

$$E_n(V) : = \text{span}\left\{ a^{(-1-k_1|K_1)} \cdots a^{(-1-k_r|K_r)} b \mid r \in \mathbb{Z}_{\geq 0}, \ a^i, b \in V, k_i \in \mathbb{N}, K_i \subset [N] \right\}. \quad (0.0.3)$$

We call it the Li filtration of $V$ (Definitions 3.2.2 and 3.3.2).

Our definition is just neglecting the odd index $J$ and using the non-SUSY case definition (0.0.1). See also Remark 3.2.3 for another origin of the definition. Despite its simple-mindedness, we need some careful arguments on SUSY vertex algebras to obtain the following SUSY analogue of Li’s theorem in [22, Theorem 2.12].

**Theorem 1** (Theorems 3.2.7 and 3.3.6) Let $V$ be an $NW = N$ (resp. $NK = N$) SUSY vertex algebra, and $E_n = E_n(V)$ be the linear sub-superspaces of $V$ in (0.0.3). Then, the associated graded space

$$\text{gr}_E V : = \bigoplus_{n \in \mathbb{N}} E_n/E_{n+1}$$

has the following structure of an $NW = N$ (resp. $NK = N$) SUSY vertex Poisson algebra (see Definitions 3.1.6 and 3.1.17). Let $a \in E_r$ and $b \in E_s$.

- The commutative multiplication $\cdot$ is

$$(a + E_{r+1}) \cdot (b + E_{s+1}) : = a^{(-1-|N]} b + E_{r+s+1}.$$

- The unit is $1 : = |0| + E_1$. 

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The even operator $\partial$ is $\partial(a + E_{r+1}) := Ta + E_{r+2}$, and the odd operator $\delta^i$ for $i \in [N]$ is $\delta^i(a + E_{r+1}) := S^i a + E_{r+2}$ (resp. the odd operator $\delta^i$ is $\delta^i(a + E_{r+1}) := S^i a + E_{r+2}$).

The SUSY vertex Lie structure $Y_-$ is

$$Y_-(a + E_{r+1}, Z)(b + E_{s+1}) := \sum_{(j|J), j \geq 0} Z^{1-j}[N\backslash J](a_{(j|J)}b + E_{r+s-j+1}).$$

Once we have obtained the Li filtration of SUSY vertex algebras, we can discuss various related notions, which are already established in the non-SUSY case, and can expect to develop a Poisson-geometric theory of SUSY vertex algebras. In this note, as a first step toward such a geometric theory, we study a SUSY analogue of associated schemes and singular supports introduced by Arakawa [2]. Our definitions and the arguments are straightforward analogues of loc. cit. As a result, we have the following SUSY analogue of the equivalence between the $C_2$-cofiniteness and the lisse condition [2, Theorem 3.3.3].

**Theorem 2** (Theorem 4.2.9) Let $V$ be an $N_W = N$ or $N_K = N$ SUSY vertex algebra which is finitely strongly generated and has a lower-bounded grading (see Assumption 4.2.3 for the details). Then, we have

$$V \text{ is } C_2\text{-cofinite (Definition 4.1.1)} \iff V \text{ is lisse (Definition 4.2.8)}.$$

**Organization**

Let us explain the organization of this note. Section 1 is a preliminary part. Section 1.1 summarizes the super notation used throughout this note. Section 1.2 is also a notational preliminary, focusing on the derivations and the de Rham complex in the super setting. We remark that there are several notations for the super de Rham complex in the literature, and the aim is to choose and fix one of them. Section 1.3 is largely based on recollections, but has some new terminologies and lemmas. There we give SUSY analogue of the infinite jet algebras (or arc algebras), calling them the superjet algebras and the superconformal jet algebras, corresponding to $N_W = N$ and $N_K = N$, respectively. They will be the foundation of our study of commutative SUSY vertex algebras and SUSY vertex Poisson algebras in the following sections.

A large part of Sect. 2 gives recollection on the theory of SUSY vertex algebras established in [16]. After introducing the superfield notation in Sect. 2.1, we explain the definition and basic properties of SUSY vertex algebras in Sects. 2.2 and 2.3, dividing the $N_W = N$ case and $N_K = N$ case, respectively. We also give SUSY analogue of Borcherds’ iterate formula (Lemmas 2.2.10 and 2.3.13), which will be used in the proof of Theorem 1. Section 2.4 starts our original study. There we study commutative SUSY vertex algebras and show that the superjet algebra (resp. the superconformal algebra) has a natural structure of a commutative $N_W = N$ (resp. $N_K = N$) SUSY vertex algebra.

Section 3 is the main body of this note, introducing the Li filtration of SUSY vertex algebras and showing Theorem 1. As a preliminary, we give in Sect. 3.1 an exposition
of SUSY vertex Poisson algebras. They are introduced in [16], using the language of SUSY Lie conformal algebras. For the following analysis, we give a restatement in terms of SUSY vertex Lie algebras. We also introduce the level 0 SUSY vertex algebra structure on the superjet and superconformal jet algebras (Propositions 3.1.9 and 3.1.18), which will be the foundation of the study of associated superschemes and singular supports in Sect. 4. The $N_W = N$ case of Theorem 1 is shown in Sect. 3.2, and the $N_K = N$ case is in Sect. 3.3. The strategy is just mimicking Li’s argument for the non-SUSY case [22, Sect. 2], which is based on Borcherds’ commutator formula and iterate formula. Although the SUSY analogues of commutator formula are already given in [16], we cannot find the SUSY iterate formula in the literature, which is the reason for preparing Lemmas 2.2.10 and 2.3.13.

Section 4 is an application of the previous Theorem 1, and more or less a straightforward SUSY analogue of Arakawa’s work [2], as mentioned above. In Sect. 4.1, we introduce the $C_2$-Poisson superalgebra (Proposition 4.1.2) and the associated superscheme (Definition 4.1.8) of an $N_W = N$ or $N_K = N$ SUSY vertex algebra and study their basic properties (e.g., Proposition 4.1.5). One remark is that there appears Poisson superalgebras of parity $N \mod 2$ (Definition 3.1.7). Section 4.2 gives a study of singular supports, and shows Theorem 2.

**Notation and terminology**

Here we list the notations and terminology used throughout in this paper.

- For a set $S$, we denote by $|S| = \#S$ its cardinality, and for $r, s \in S$, we denote by $\delta_{r,s}$ the Kronecker delta.
- We denote by $\mathbb{N} = \mathbb{Z}_{\geq 0} : = \{0, 1, 2, \ldots \}$ the set of nonnegative integers.
- For $\alpha, \beta \in \mathbb{C}$, we denote $\alpha \geq \beta$ if $\alpha - \beta \in \mathbb{R}_{\geq 0}$.
- The binomial coefficient is defined to be $\binom{n}{m} : = \frac{1}{m!} n(n-1) \cdots (n-m+1)$ for $m \in \mathbb{N}$ and some indeterminate or number $n$.
- A ring or an algebra means a unital and associative one unless otherwise stated. A ring action on its module is denoted by the period “.”, i.e., for a left module $M$ over a ring $R$, we denote by $r.m$ the action of $r \in R$ on $m \in M$.
- For a linear (super)space $V$ over a field and a subset $S \subset V$, we denote by $\text{span} S$ the subspace of $V$ linearly spanned of $S$.
- For a category $\mathcal{C}$, we denote $X \in \mathcal{C}$ to mean that $X$ is an object of $\mathcal{C}$.
- Sets denotes the category of sets.

1 **Super preliminaries**

1.1 **Super notation**

We start with delivering notations for super objects. We follow [24, Chap. 3] and [20, 1.1] with slight modifications on symbols.

**Notation 1.1.1** Here is the first set of notations and terminologies:

- We denote by $\mathbb{Z}_2 : = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ the parity or the supergrading.
• A super abelian group or a supermodule means a \(\mathbb{Z}_2\)-graded module

\[ M = M_0 \oplus M_1 \]

For each \(\gamma \in \mathbb{Z}_2\), an element \(v \in M_\gamma\) is called of pure parity, and for such \(v\), we denote \(p(v) = \gamma\). We denote by \(M_{\text{pure}} = M_0 \sqcup M_1\) the set of elements of pure parity. An element \(v \in M_0\) is called even, and \(v \in M_1\) is called odd. If \(M = M_0\), then \(M\) is called purely even, and if \(M = M_1\), then \(M\) is called purely odd.

• For supermodules \(M\) and \(N\), we denote by \(\text{Hom}(M, N)\) the set of all homomorphisms of underlying modules. This set is naturally a module equipped with \(\mathbb{Z}_2\)-grading, denoted as

\[ \text{Hom}(M, N) = \text{Hom}(M, N)_0 \oplus \text{Hom}(M, N)_1 \]

Here \(\text{Hom}(M, N)_0\) consists of module homomorphisms \(f\) which preserve parity, i.e., \(f(M_0) \subset N_0\) and \(f(M_1) \subset N_1\). Similarly, \(\text{Hom}(M, N)_1\) consists of module homomorphisms which exchange parity. We call an element of \(\text{Hom}(M, N)_0\) an even homomorphism of supermodules and call an element of \(\text{Hom}(M, N)_1\) an odd homomorphism of supermodules. We also call an element of \(\text{Hom}(M, N)_p\) a homomorphism of supermodules of parity \(p\).

• We denote by \(\text{SAb}\) the category of all supermodules whose morphism set for \(M, N \in \text{SAb}\) is

\[ \text{Hom}_{\text{SAb}}(M, N) = \text{Hom}(M, N)_0 \oplus \text{Hom}(M, N)_1. \]

Thus, a morphism in \(\text{SAb}\) is noting but an even homomorphism of supermodules in the last item. Note also that the category \(\text{SAb}\) has a natural super enhancement \(\text{SAb}'\), where the morphism supermodule is given by \(\text{Hom}_{\text{SAb}'}(M, N) = \text{Hom}(M, N)_0 \oplus \text{Hom}(M, N)_1\).

• We denote by \(\Pi: \text{SAb} \rightarrow \text{SAb}\) the parity change functor. Thus, we have

\[ (\Pi M)_\bar{\gamma} = M_{\bar{\gamma}}, \quad (\Pi M)_{\bar{\gamma}} = M_{\bar{\gamma}} \]  

(1.1.1)

for a supermodule \(M \in \text{SAb}\). The parity functor is involutive: \(\Pi^2 = \text{id}\).

• \(\text{SAb}\) is a symmetric monoidal category with monoidal structure \(\otimes\) given by the standard graded tensor product and unit object being \(\mathbb{Z}\). The symmetry transformation is given by

\[ M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto (-1)^{p(m)p(n)} n \otimes m. \]  

(1.1.2)

We always consider \(\text{SAb}\) as the category equipped with this symmetric monoidal structure.

**Remark 1.1.2** (After the referees’ comments) Here and hereafter, we distinguish the words *morphism* and *homomorphism*. The word *morphism* is used to mean an even homomorphism of super objects, i.e., a homomorphism-preserving parity. The word *homomorphism* means an even or odd homomorphism of super objects.
Notation 1.1.3  We turn to ring structures in super setting.

- A superring $R$ is an associative ring object in the monoidal category $(\text{SAb}, \otimes, \mathbb{Z})$. The supercommutator in $R$ is defined by

$$[r, s] := rs - (-1)^{p(r)p(s)}sr$$  \hfill (1.1.3)

for $r, s \in R$ of pure parity, and extended by linearity. It satisfies the super Jacobi rule

$$(-1)^{p(r)p(t)}[r, [s, t]] + (-1)^{p(s)p(r)}[s, [t, r]] + (-1)^{p(t)p(s)}[t, [r, s]] = 0 \quad (r, s, t \in R_{\text{pure}}).$$

- A commutative superring is a commutative associative ring object in $\text{SAb}$, i.e., a superring whose supercommutator always vanishes. A morphism $R \to S$ of commutative superrings is a morphism in $\text{SAb}$ respecting the ring structures. We denote by $\text{SCom}$ the resulting category of commutative superrings. In the literature, the word super-commutative ring is also used instead of our commutative superring.

- Similarly as $\text{SAb}$, the category $\text{SCom}$ has a natural super enhancement $\text{SCom}^s$, where the morphism supermodule for superrings $R$ and $S$ is given by

$$\text{Hom}_{\text{SCom}}(R, S) := \text{Hom}_{\text{Com}}(R, S)_{\overline{0}} \oplus \text{Hom}_{\text{Com}}(R, S)_{\overline{1}},$$

where $\text{Hom}_{\text{Com}}(R, S)_p$ with $p = \overline{0}$ (resp. $p = \overline{1}$) denotes the module of homomorphisms $R \to S$ of underlying commutative rings which preserve (resp. exchange) parity. We call an element of each module a homomorphism of commutative superrings of parity $p$. In particular, a morphism in $\text{SCom}$ is an even homomorphism of commutative superrings, and we have

$$\text{Hom}_{\text{SCom}}(R, S) = \text{Hom}_{\text{Com}}(R, S)_{\overline{0}}.$$

- Given a morphism $R \to A$ of commutative superrings, we call $A$ a commutative superalgebra over $R$ or an commutative $R$-superalgebra. Commutative $R$-superalgebras form a category, denoted by $\text{SCom} R$. This category also has a natural super enhancement $\text{SCom}^s R$, and the morphism supermodule for $R$-superalgebras $A$ and $B$ is given by

$$\text{Hom}_{\text{SCom}}(A, B) := \text{Hom}_{\text{Com}_R}(A, B)_{\overline{0}} \oplus \text{Hom}_{\text{Com}_R}(A, B)_{\overline{1}}.$$

Here $\text{Hom}_{\text{Com}_R}(A, B)_p$ with $p = \overline{0}$ (resp. $p = \overline{1}$) denotes the module of homomorphisms of commutative $R$-algebras preserving (resp. exchanging) parity. We call an element of each module a homomorphism of commutative $R$-superrings of parity $p$.

Hereafter we always assume:
In every commutative superring $R$, the element 2 is invertible.

As a consequence, if $r \in R_1$, then $r^2 = \frac{1}{2}[r, r] = 0$, so each odd element is nilpotent.

**Notation** Next we introduce terminologies for modules over a commutative superring $R$.

- A **left $R$-supermodule** $L$ is a supermodule with a left module structure over the underlying ring of $R$ such that the module structure $R \times L \to L, r \otimes l \mapsto r.l$ is a morphism in $S\text{Ab}$.

- A **right $R$-supermodule** $M$ is similarly defined and can be regarded as a left $R$-supermodule by $r.m := (-1)^p(r)p(m)r$ for pure elements $r \in R$ and $m \in M$. Conversely, a left $R$-supermodule is regarded as a right $R$-supermodule by the same correspondence. Henceforth we call a left $R$-supermodule just by an $R$-supermodule.

- $R$-supermodules form a category $\text{SMod } R$ with morphisms being parity-preserving homomorphisms of underlying $R$-modules. The category $\text{SMod } R$ inherits the symmetric monoidal structure $\otimes$ from that of $S\text{Ab}$, where the unit object is $R$ equipped with the natural $R$-supermodule structure.

- If $R = k$ is a field, then a $k$-supermodule is called a **linear $k$-superspace**.

Let $R$ be a commutative superring. The category $\text{SMod } R$ of $R$-supermodules has a natural super enhancement $\text{SMod}^s R$, similarly as the categories $S\text{Ab}$, $S\text{Com}$ and $S\text{Com } R$. The morphism supermodule of $R$-supermodules $M$ and $N$ is given by

$$\text{Hom}_{\text{SMod}^s R}(M, N) := \text{Hom}_{\text{Mod } R}(M, N)_{\overline{0}} \oplus \text{Hom}_{\text{Mod } R}(M, N)_{\overline{1}}.$$  \hspace{1cm} (1.1.4)

Here $\text{Hom}_{\text{Mod } R}(M, N)$ denotes the module of homomorphisms $M \to N$ of underlying $R$-modules (we regard $R$ as an algebra, forgetting the super structure). The subset $\text{Hom}_{\text{Mod } R}(M, N)_{p}$ with $p = \overline{0}$ (resp. $p = \overline{1}$) consists of parity-preserving (resp. exchanging) maps. We call an element of each submodule an even (resp. odd) $R$-homomorphism of $R$-supermodules. We also call it an $R$-homomorphism of parity $p$.

As before, a morphism in $\text{SMod } R$ is an even $R$-homomorphism of $R$-supermodules, and we have

$$\text{Hom}_{\text{SMod } R}(M, N) = \text{Hom}_{\text{Mod } R}(M, N)_{\overline{0}}.$$  \hspace{1cm} (1.1.5)

Let us keep the notation $R$, $M$ and $N$. Then the set $\text{Hom}_{\text{Mod } R}(M, N)$ has a natural $R$-supermodule structure whose supermodule structure is given by (1.1.4), and $R$-action is given by $(r.f)(m) := r.f(m)$ for $r \in R, f \in \text{Hom}_{\text{Mod } R}(M, N)$ and $m \in M$. The obtained $R$-supermodule $\text{Hom}_{\text{Mod } R}(M, N)$ is actually an internal hom in the symmetric monoidal category $(\text{SMod } R, \otimes R, R)$. To stress this point, we denote

$$\text{Hom}_{\text{SMod } R}(M, N) := \text{Hom}_{\text{Mod } R}(M, N) \in \text{SMod } R.$$  \hspace{1cm} (1.1.5)

The definition of internal hom claims that we have

$$\text{Hom}_{\text{SMod } R}(L, \text{Hom}_{\text{SMod } R}(M, N)) \cong \text{Hom}_{\text{SMod } R}(L \otimes R M, N).$$
for any $L, M, N \in \text{SMod} R$.

For later use, let us explain in detail that the parity change functor $\Pi$ on $\text{SAb}$ naturally lifts to $\text{SMod} R$. For $M \in \text{SMod} R$, we define $\Pi M$ as an object of $\text{SAb}$ by (1.1.1) and equip it with the $R$-supermodule structure

$$r.(\Pi m) := (-1)^{p(r)} \Pi (r.m) \quad (r \in R_{\text{pure}}, \ m \in M),$$

(1.1.6)

where $\Pi m \in \Pi M$ is the element corresponding to $m \in M$. For an $R$-morphism $f \in \text{Hom}_{\text{SMod} R}(M, N)$, we define $f^\Pi : \Pi M \to \Pi N$ to be the morphism which agrees with $f$ as a map of sets. Here the symbol $f^\Pi$ is borrowed from [24, Chap. 3, Sect. 1.5].

Next, for each $f \in \text{Hom}_{\text{SMod} R}(M, N)$, we define

$$\Pi f \in \text{Hom}_{\text{SMod} R}(M, \Pi N), \quad f^\Pi \in \text{Hom}_{\text{SMod} R}(\Pi M, N)$$

by

$$(\Pi f)(m) := \Pi (f(m)), \quad (f^\Pi)(\Pi m) := f(m) \quad (m \in M).$$

(1.1.7)

Then we have $f^\Pi = \Pi f^\Pi$. More generally, we have the following isomorphisms of $R$-supermodules.

$$\text{Hom}_{\text{SMod} R}(\Pi M, N) \xleftrightarrow{f^\Pi} \text{Hom}_{\text{SMod} R}(M, \Pi N) \xleftrightarrow{f} \text{Hom}_{\text{SMod} R}(\Pi M, N),$$

(1.1.8)

### 1.2 Super derivations and the de Rham complex

Next, we introduce derivations in super setting.

**Definition 1.2.1** (c.f. [24, Chap. 3, Sect. 2.8]) Let $R \to A$ be a morphism of commutative superrings, and $M$ be an $A$-supermodule.

1. An $R$-derivation $D : A \to M$ of parity $q \in \mathbb{Z}_2$ is an $R$-homomorphism $D \in \text{Hom}_{\text{Mod} R}(A, M)_q$ of parity $q$ (where $A$ and $M$ are regarded as $R$-supermodules) such that $D(r) = 0$ for $r \in R$, and that the super Leibniz rule holds for $a, b \in A$ of pure parity:

$$D(ab) = D(a).b + (-1)^{p(a)q} a.D(b).$$

An $R$-derivation of parity $\overline{0}$ (resp. of parity $\overline{1}$) is also called an even $R$-derivation (resp. an odd $R$-derivation).

2. The set of $R$-derivations $A \to M$ of parity $p$ is denoted by $\text{Der}_R(A, M)_p$, which is a module. Then we define a supermodule $\text{Der}_R(A, M)$ by

$$\text{Der}_R(A, M) := \text{Der}_R(A, M)_{\overline{0}} \oplus \text{Der}_R(A, M)_{\overline{1}}.$$
Note that, for an even \( R \)-derivation \( D: A \to M \), the maps \( \Pi D: A \to \Pi M \) and \( D\Pi: \Pi A \to M \) given in (1.1.7) are odd \( R \)-derivations. Conversely, if \( D \) is an odd \( R \)-derivation, then \( \Pi D \) and \( D\Pi \) are even \( R \)-derivations. These are checked by using the \( R \)-supermodule structure (1.1.6) of \( \Pi M \).

**Notation 1.2.2** In the case \( M = A \), we denote

\[
\mathfrak{X}^1_{A/R} = \text{Der}_R(A) : = \text{Der}_R(A, A),
\]

which is a Lie superalgebra in terms of the supercommutator (1.1.3).

The supermodule \( \text{Der}_R(A, M) \) is an \( A \)-supermodule, since for \( a, b, c \in A \) and \( D \in \text{Der}_R(A, M) \) of pure parity, we have \( p(a) + p(D) = p(aD) \) and

\[
(aD)(bc) = a(D(b).c + (-1)^{p(b)p(D)}b.D(c)) = (aD)(b).c + (-1)^{p(b)p(aD)}b.(aD)(c).
\]

In particular, we have an auto-functor

\[
\text{Der}_R(A, \cdot): \text{SMod} \ A \to \text{SMod} \ A.
\]

As in the non-super case [26, Tag 00RM, 10.131 Differentials], this functor is represented by the supermodule of Kähler differentials. In the super case, there are two versions (\( \Omega^1_{A/R, \text{ev}} \) and \( \Omega^1_{A/R, \text{od}} \) given below). For the definition, recall that a free supermodule over a commutative superring \( R \) is a supermodule which is a module over the underlying algebra of \( R \) with basis of pure parity. See [24, Chap. 3, Sect. 1.6] for details.

**Definition 1.2.3** Let \( \varphi: R \to A \) be a morphism of commutative superrings. Consider the free \( A \)-supermodules

\[
E : = \bigoplus_{r \in R_\text{pure}} A[r] \oplus \bigoplus_{a, b \in A_\text{pure}} A[(a, b)] \oplus \bigoplus_{f, g \in A_\text{pure}} A[(f, g)], \quad F : = \bigoplus_{a \in A_\text{pure}} A[a],
\]

where \( R_\text{pure} := R_\Sigma \sqcup R_\Pi \) and \( A_\text{pure} := A_\Sigma \sqcup A_\Pi \) as sets (see Notation 1.1.1), and the parity is defined by \( p([(a, b)]) := p(a) + p(b), \) \( p([r]) := p(r) \) and \( p([a]) := p(a) \).

We define two morphisms of \( A \)-supermodules

\[
\psi_\text{ev}, \psi_\text{od}: E \longrightarrow F
\]

by the following rules:

\[
[r] \longmapsto [\varphi(r)], \quad [(a, b)] \longmapsto [a + b] - [a] - [b], \quad ([f, g]) \longmapsto \begin{cases} [fg] - f[g] - (-1)^{p(f)p(g)}g[f] & \text{(for } \psi_\text{ev}) \\ [fg] - (-1)^{p(f)}f[g] - (-1)^{p(f)p(g)}g[f] & \text{(for } \psi_\text{od}) \end{cases} \quad \text{(1.2.1)}
\]
Then we define $A$-supermodules

$$\Omega^1_{A/R, ev} := \text{Cok } \psi_{ev}, \quad \Omega^1_{A/R, od} := \text{Cok } \psi_{od},$$

and define maps

$$d_{ev} : A \to \Omega^1_{A/R, ev}, \quad d_{od} : A \to \Omega^1_{A/R, od},$$

to be the ones sending $a$ to the class of $[a]$. We call $\Omega^1_{A/R, ev}$ and $\Omega^1_{A/R, od}$ the $A$-supermodules of Kähler differentials of $A$ over $R$ and call $d_{ev}$ and $d_{od}$ the universal differentials.

The module $\Omega^1_{A/R, ev}$ (resp. $\Omega^1_{A/R, od}$) is spanned by elements of the form $a.d_{ev}b$ (resp. $a.d_{od}b$) with $a, b \in A$, respectively. The rules (1.2.1) yield

$$d_{ev}(r) = 0, \quad d_{ev}(a + b) = d_{ev}a + d_{ev}b, \quad d_{ev}(ab) = d_{ev}a.b + a.d_{ev}b,$$

$$d_{od}(r) = 0, \quad d_{od}(a + b) = d_{od}a + d_{od}b, \quad d_{od}(ab) = d_{od}a.b + (−1)^{p(a)}a.d_{od}b.$$

By Definition 1.2.1, $d_{ev}$ and $d_{od}$ are an even and odd $R$-derivation, respectively. We also have the natural isomorphism of $A$-supermodules

$$\Omega^1_{A/R, od} \xrightarrow{\sim} \Pi \Omega^1_{A/R, ev}, \quad a.d_{ev}b \xrightarrow{\sim} \Pi (a.d_{od}b) \quad (a, b \in A) \quad (1.2.2)$$

Here we used the symbol $\Pi(a)$ for $a \in A$ in the same meaning as in (1.1.6) and (1.1.7). Then, by (1.1.8), we have an isomorphism $\text{Hom}_{SMod R}(A, \Omega^1_{A/R, ev}) \cong \text{Hom}_{SMod R}(A, \Omega^1_{A/R, od})$, which corresponds to the identity $d_{od} = \Pi d_{ev}$.

The following statement is an analogue of the universality of the module of Kähler differentials in the non-super case [26, Tag 00RM, Lemma 10.131.3].

**Lemma 1.2.4** Let $R \to A$ be a morphism of commutative superrings. Then we have isomorphisms of $A$-supermodules

$$\text{Hom}_{SMod A}(\Omega^1_{A/R, od}, M) \xleftarrow{\phi \Pi} \text{Hom}_{SMod A}(\Omega^1_{A/R, ev}, M) \xrightarrow{\phi} \text{Der}_R(A, M),$$

which are functorial with respect to $M \in SMod A$.

**Proof** The left isomorphism is given by (1.1.8) and (1.2.2). As for the right isomorphism, we can easily find that the map $\phi \mapsto \phi \circ d_{ev}$ is an injective morphism of $A$-supermodules. Given any $D \in \text{Der}_R(A, M)$, we have the induced map $[D] : \bigoplus_{a \in A_{pure}} A[a] \to M$, where we used the symbols in Definition 1.2.3. Then, the condition for $D$ to be an even or odd $R$-derivation is equivalent to the one that $[D]$ annihilates the image of the map (1.2.1). □
Notation Hereafter, if no confusion may arise, we simply denote
\[(\Omega^1_{A/R}, d) : = (\Omega^1_{A/R, ev}, d_{ev}).\]

Let us give another description of the \(A\)-supermodule \(\Omega^1_{A/R}\) as the first-order neighborhood of the diagonal \(\Delta_{A/R} \subset \text{Spec} A \times_{\text{Spec} R} \text{Spec} A = \text{Spec} A \otimes_R A\), which is also an analogous statement as the non-super case.

Lemma 1.2.5 (c.f. [26, Tag 00RM, Lemma 10.131.13]) Let \(R \to A\) be a morphism of commutative superrings, and \(I : = \text{Ker}(A \otimes_R A \to A)\) be the kernel of the multiplication map. Then, the map
\[\Omega^1_{A/R} \to I/I^2, \quad adb \mapsto (a \otimes b - ab \otimes 1 \mod I^2)\]
is an isomorphism of \(A\)-supermodules.

Proof The argument in the non-super case works with minor changes. \(\square\)

Finally we explain the de Rham complex in super setting. See also [24, Chap. 3, Sects. 2.5 and 4.4].

Notation 1.2.6 Let \(R\) be a commutative superring.

- For an \(R\)-supermodule \(M\), we denote by \(\text{Sym}^\bullet_R M\) the symmetric superalgebra of \(M\) over \(R\). It is a commutative \(R\)-superalgebra equipped with \(\mathbb{N}\)-grading expressed by \(\bullet\).
- Let \(\Omega^1_{A/R}\) be the supermodule of Kähler differentials for a commutative \(R\)-superalgebra \(A\). The de Rham complex of \(A\) over \(R\) is the complex defined to be
\[\Omega^\bullet_{A/R} : = (\text{Sym}^\bullet_A(\Pi\Omega^1_{A/R}), d_{dR}),\]
where the de Rham differential \(d_{dR} : \Omega^p_{A/R} \to \Omega^{p+1}_{A/R}\) is induced by the odd universal differential
\[\Pi d_{ev} = d_{od} : A \to \Pi\Omega^1_{A/R, ev} = \Omega^1_{A/R, od}\]
in a similar fashion as in the non-super case. If we focus on the \(A\)-superalgebra structure, we denote \(\Omega_{A/R}\) and call it the de Rham superalgebra of \(A\) over \(R\). We remind the reader to distinguish the superalgebra \(\Omega_{A/R}\) with the supermodule of Kähler differentials \(\Omega^1_{A/R}\).

Using the terminology of [7, 1.1.16], we can say that \(\text{Sym}^\bullet_R M\) is an \(\mathbb{N}\)-graded superalgebra, and \(\Omega^\bullet_{A/R}\) is a dg \(A\)-superalgebra. Henceforth, we also call \(\Omega^\bullet_{A/R}\) the de Rham dg superalgebra of \(A\) over \(R\).

The de Rham superalgebra \(\Omega^\bullet_{A/R}\) enjoys the universal property in Fact 1.2.8.
Notation 1.2.7 Let $\Lambda_R[\zeta]$ be the exterior algebra of one variable $\zeta$ over $R$. In other words, it is a commutative $R$-superalgebra consisting of the elements of the form $r + s\zeta$ for $r, s \in R$ with $\zeta$ an odd element, and the multiplication is given by $(r + s\zeta)(r' + s'\zeta) = rr' + (rs' + (-1)^{(p(s)+1)p(r')}s)r'\zeta$.

Fact 1.2.8 [20, Sect. 2.2] For a morphism $R \to A$ of commutative superrings, the de Rham superalgebra $\Omega^1_{A/R}$ represents the functor $\text{Hom}_{\text{SCom}^R}(A, \cdot \otimes_R \Lambda_R[\zeta]) : \text{SCom}^R \to \text{Sets}$. In other words, there is a functorial bijection

$$\text{Hom}_{\text{SCom}^R}(\Omega^1_{A/R}, B) = \text{Hom}_{\text{SCom}^R}(A, B \otimes_R \Lambda_R[\zeta]).$$

The construction $A \mapsto \Omega^1_{A/R}$ is functorial, so we can introduce:

Notation 1.2.9 Following [20], we call the functor the de Rham spectrum functor and denote $S(A) := \Omega^1_{A/R}$.

We can apply the functor $S$ repeatedly, and the superalgebra $S^N(A)$ thus obtained for $N \in \mathbb{Z}_{>0}$ is equipped with the de Rham differentials $d_{\text{dR}}^i$ for $i = 1, \ldots, N$. These are composable, and satisfy the anti-commutation relation $[d_{\text{dR}}^i, d_{\text{dR}}^j] = 0$. For later reference, we introduce:

Notation 1.2.10 For an ordered set $J = \{j_1 < \cdots < j_r\} \subseteq \{1 < 2 < \cdots < N\}$, we define $d_{\text{dR}}^J : A \to S^N(A)$ by $d_{\text{dR}}^J := d_{\text{dR}}^{j_1} \cdots d_{\text{dR}}^{j_r}$.

We have an obvious multi-variable analogue of Fact 1.2.8.

Lemma 1.2.11 Let $\Lambda_R[\zeta^1, \ldots, \zeta^N]$ be exterior algebra of $N$ variables over $R$. Then the $N$-times application of the de Rham spectrum functor $S^N(A)$ represents the functor $\text{Hom}_{\text{SCom}^R}(A, ? \otimes_R \Lambda_R[\zeta^1, \ldots, \zeta^N])$. In other words, we have a functorial bijection

$$\text{Hom}_{\text{SCom}^R}(S^N(A), B) \xrightarrow{\sim} \text{Hom}_{\text{SCom}^R}(A, B \otimes_R \Lambda_R[\zeta^1, \ldots, \zeta^N]).$$

1.3 Superjet and superconformal jet algebras

We give a brief recollection on infinite jet algebras or arc algebras, which we mean the spectrum of infinite jet affine schemes in the sense of [10, Sects. 2 and 3], [27] and [7, 2.3.2, 2.3.3]. We present it in super setting, following [20, Sect. 4.2]. We also recall the notion of superconformal vector fields or SUSY structure on super curves. In this subsection, the base field $k$ is assumed to be of characteristic 0.

1.3.1 Superjet algebras

As before, let $R$ be a commutative superring, and denote by $\text{SCom}^R$ the category of commutative $R$-superalgebras. Recall that in Definition 1.2.3 we used free $R$-supermodules. In the next Definition 1.3.1, we use the polynomial $R$-superalgebra.
$R[x_j \mid j \in J]$, where the generators $x_j$ are of pure parity and commutative. It can also be called the free commutative $R$-superalgebra generated by $x_j$'s. We also use the set $A_{\text{pure}} := A \mathbb{F} \sqcup A \mathbb{T}$ of elements of pure parity of a commutative $R$-superalgebra $A$.

**Definition 1.3.1** [27, Definition 1.3] Let $A \in \text{SCom } R$, and $f : R \to A$ be the structure morphism. For $m \in \mathbb{N} \sqcup \{\infty\}$, define a commutative $A$-superalgebra $HS_A^m/R$ to be the quotient of the polynomial $A$-superalgebra $A[x^{(i)} \mid x \in A_{\text{pure}}, i = 1, \ldots, m]$ by the ideal $I$ generated by the following terms:

$$
\begin{align*}
(f(r))^{(i)} & \quad (r \in R, i = 1, \ldots, m), \\
(x + y)^{(i)} - x^{(i)} - y^{(i)} & \quad (x, y \in A, i = 1, \ldots, m), \\
(xy)^{(i)} - \sum_{j+k=i} x^{(j)} y^{(k)} & \quad (x, y \in A, i = 0, \ldots, m),
\end{align*}
$$

where we denote $x^{(0)} := x$ for $x \in A$, and if $m = \infty$ we interpret the range “$i = 1, \ldots, m$” as $i \in \mathbb{Z}_{>0}$. We call $HS_A^m/R$ the $(A$-superalgebra of) Hasse–Schmidt derivations. We also define $R$-homomorphisms $d_i$ ($i = 0, \ldots, m$) by

$$
d_i : A \longrightarrow HS_A^m/R, \quad x \longmapsto (x^{(i)} \mod I),
$$

and call them the universal derivations.

**Remark 1.3.2** A few remarks are in order.

1. We have $HS_A^0/R \cong A$, and $HS_A^1/R \cong \text{Sym}_A \Omega^1_{A/R}$, the symmetric $A$-superalgebra of the module $\Omega^1_{A/R}$ of Kähler differentials, forgetting the $\mathbb{N}$-grading in Notation 1.2.6.
2. $HS_A^m/R$ is a commutative $R$-superalgebra by the structure morphism $f : R \to A$.
3. $d_1$ an even derivation of the $R$-superalgebra $A$ (Definition 1.2.1). If the base field $k$ is of characteristic 0, then $d_n = \frac{1}{m!} d_1^n$ for $n \in \mathbb{N}$.

The Hasse–Schmidt derivations $HS_A^m/R$ represent the functor of $m$-th jets, explained in Fact 1.3.3.

**Notation** Let $R$ be a commutative superring, and $z$ be an even indeterminate.

- We regard the formal power series ring $R[z]$ as

$$
R[z] = \lim_{\leftarrow m \in \mathbb{N}} R[z]/(z^{m+1}),
$$

and as a complete topological superring with respect to the $z$-adic topology.

- We denote by $\text{TopSCom } R$ the category of topological commutative $R$-superalgebras and continuous morphisms. In particular, we have $R[z] \in \text{TopSCom } R$.  

\[ \subseteq \text{ Springer} \]
Fact 1.3.3 [27, Corollary 1.8] For $A \in \text{SCom}_R$ and $m \in \mathbb{N} \cup \{\infty\}$, the Hasse–Schmidt derivations $\text{HS}_m^{A/R}$ represent the functor $\text{SCom}_R \to \text{Sets}$ given by

$$
\begin{cases}
\text{Hom}_{\text{SCom}_R}(A, \otimes_R R[z]/(z^{m+1})) & (m \in \mathbb{N}) \\
\text{Hom}_{\text{TopSCom}_R}(A, \otimes_R R[z]) & (m = \infty)
\end{cases}
$$

More precisely, we have a functorial bijection

$$
\text{Hom}_{\text{SCom}_R}(\text{HS}_m^{A/R}, B) \longrightarrow \text{Hom}_{\text{SCom}_R}(A, B[z]/(z^{m+1})),
$$

in the case $m \in \mathbb{N}$, and a similar one given by $\phi \mapsto (x \mapsto \sum_{i=0}^{m} z^i \phi(d_i x))$ in the case $m = \infty$.

Combining Lemma 1.2.11 and Fact 1.3.3, we have Proposition 1.3.5. We use the following notations.

Notation 1.3.4 Let $N \in \mathbb{Z}_{>0}$, and $k$ be a field of characteristic 0. We denote by $[N]$ the ordered set $\{1 < 2 < \cdots < n\}$.

- for $A \in \text{SCom}_k$ we denote

$$
A_\infty := \text{HS}_\infty^{A/k}
$$

and call it the (infinite) jet algebra of $A$.

- Let $Z = (z, \zeta^1, \ldots, \zeta^N)$ be a set of indeterminates with $z$ even and $\zeta^j$ odd which are (super-) commutative with each other. We call $Z$ an $1|N$-supervariable. For $j \in \mathbb{Z}$ and an ordered subset $J = \{j_1 < \cdots < j_r\} \subset [N]$, we denote

$$
Z^{j|J} := z^{j_1} \zeta^{j_1} \cdots z^{j_r} \zeta^{j_r}.
$$

- Let $m := (Z)$ be the ideal of the polynomial superalgebra $k[Z] \in \text{SCom}_k$ generated by $z$ and $\zeta^i$’s. The natural projections $k[Z]/m^{n+1} \to k[Z]/m^n$ form a filtered system, and we denote

$$
O = O^{1|N} := k[Z] = \lim_{\leftarrow n \in \mathbb{N}} k[Z]/m^{n+1},
$$

regarding it as a complete topological $k$-superalgebra by $m$-adic topology.

Proposition 1.3.5 Let $N \in \mathbb{Z}_{>0}$, $k$ be a field of characteristic 0, and $Z = (z, \zeta^1, \ldots, \zeta^N)$ be a supervariable. For $A \in \text{SCom}_k$, the commutative superalgebra

$$
A^O = A^k[Z] := S^N(A_\infty)
$$
represents the functor $\text{Hom}_{\text{TopSCom}_k}(A, \cdot \otimes_k O) : \text{SCom}_k \to \text{Sets}$. More precisely, we have a functorial bijection

$$\text{Hom}_{\text{SCom}_k}(A^O, B) \xrightarrow{\sim} \text{Hom}_{\text{TopSCom}_k}(A, B \otimes_k O), \quad \phi \mapsto \sum_{n \in \mathbb{N}, J \subset [N]} Z^n J \circ d_{n|J}, \quad (1.3.2)$$

where $d_{n|J} : A \to A^O$ is the composition

$$d_{n|J} := d_{dR} \circ d_n \quad (1.3.3)$$

of the universal differentials $d_n : A \to A_\infty$ and the product of the de Rham differentials $d_{dR}^J$ in Notation 1.2.10. We call $S^N(A_\infty)$ the (infinite) $1|N$-superjet algebra of $A$.

Lemma 1.3.6 describes a natural basis of the superalgebra $A^O$. We omit the proof.

Lemma 1.3.6 In the situation of Proposition 1.3.5, we denote

$$a^{[n|J]} := d_{n|J}(a) \in A^O \quad (a \in A, n \in \mathbb{N}, J \subset [N]).$$

Then $A^O$ is linearly spanned by the elements of the form

$$d_1^{[n_1|J_1]} \cdots d_r^{[n_r|J_r]} \quad (a_i \in A, n_i \in \mathbb{N}, J_i \subset [N]).$$

Let us denote $e_i := \{i\} \subset [N]$ for each $i \in [N]$. Then, the operations $d_{1|0}$ and $d_{0|e_i}$ on $A^O$ are even and odd derivations, respectively. Let us record this observation as:

Lemma 1.3.7 The superalgebra $A^O$ has an even derivation $T$ and odd derivations $S^i$ for $i \in [N]$ determined by

$$T(a^{[n|J]}) := (n + 1)a^{[n+1|J]}, \quad S^i(a^{[n|J]}) := \sigma(e_i, J)a^{[n|J \setminus e_i]}.$$ 

Here $\sigma(e_i, J) \in \{ \pm 1, 0 \}$ is determined by the following rule: If $i \in J$, then $\sigma(e_i, J) := 0$. Otherwise, let $e_i \cup J$ be the reordered set of the union, and determine $\sigma(e_i, J)$ by $\zeta^i \zeta^J = \sigma(e_i, J)\zeta^{e_i \cup J}$.

For later reference, let us introduce the notion of differential superalgebras.

Definition 1.3.8 The following notions are defined over some base superring $R$, but we suppress it.

1. A differential superalgebra is a pair $(A, D)$ of a commutative superalgebra $A$ and a derivation on $A$. A morphism $(A, D) \to (B, D')$ of differential superalgebras is naturally defined.
2. A subset $I \subset A$ generates $(A, D)$ if every element of $A$ can be written as a polynomial of $D' a$ with $r \in \mathbb{N}$ and $a \in A$.

We have a multi-derivation analogue of these notions: A tuple $(A, D_i)$ of a commutative superalgebra $A$ and derivations $D_i$'s is also called a differential superalgebra. Then, by Lemmas 1.3.7 and 1.3.6, we have:
Lemma 1.3.9 For any \( A \in SCom k \), \((A^O, T, S^i)\) is a differential superalgebra, and it is generated by \( A \), where we identify \( a \in A \) with \( a_{\{0|N\}} \in A^O \).

Now the following assertion is easily shown:

Lemma 1.3.10 Let \( A, B \in SCom k \), and assume the following conditions.

- \( B \) contains \( A \) as a sub-superalgebra.
- \( B \) has an even derivation \( \partial \) and \( N \) odd derivations \( \partial^i \) for \( i \in [N] \).
- The differential superalgebra \((B, \partial, \partial^i)\) is generated by \( A \).

Then the injection \( A \hookrightarrow B \) can be extended to a surjective morphism \((A^O, T, S^i) \twoheadrightarrow (B, \partial, \partial^i)\) of differential superalgebras.

Finally, let us introduce finite superjet algebras.

Proposition 1.3.11 For a supervariable \( Z = (z, \zeta^1, \ldots, \zeta^N) \) and \( m \in \mathbb{N} \), we denote
\[
O_m = O_m^{1|N} := k[Z]/(z^{m+1}).
\]
Then, for \( A \in SCom k \), the commutative superalgebra
\[
A^{O_m} = k[Z]/(z^{m+1}):=S^N(\text{HS}_m^A/k)
\]
satisfies the universal property
\[
\text{Hom}_{SCom}(A^{O_m}, B) \sim \text{Hom}_{SCom}(A, B \otimes_k O_m),
\]
\[
\phi \mapsto \sum_{0 \leq n \leq m, J \subset [N]} Z^n|J \phi \circ d_{n|J},
\]
where \( d_{n|J} \) is given by the same formula as (1.3.3). We call \( A^{O_m} \) the \( m \)-th superjet algebra of \( A \).

Corresponding to the truncation map \( O \rightarrow O_m, z^n \mapsto 0 \) for \( n \geq m + 1 \), we have a morphism of commutative superalgebras \( \text{HS}_m^A/k \rightarrow \text{HS}_A^{\infty}/k = A^{\infty} \), which induces another morphism
\[
A^{O_m} \longrightarrow A^O.
\] (1.3.4)
We call it the projection of superjet algebras.

1.3.2 Superconformal jet algebras

Let us recollect the super Riemann surface structure or the SUSY structure on the superdisk \( O = O^{1|N} \). We denote the topological Lie superalgebra of \( k \)-derivations on \( O \) (Notation 1.2.2) as
\[
\text{Der} O := \text{Der}_k(O) = \mathfrak{X}^{1}_{O/k}.
\] (1.3.5)
By [12, Sect. 6.2] and [14, Sect. 3.1.2], the set of automorphisms

\[ \text{Aut } O : = \text{Aut}_{\text{TopSCom}} k (O) \]

is a group superscheme over \( k \) (we refer to [20, Sect. 1] for the basics of the theory of superschemes, some of which is recalled in Notation 4.1.7), and the set of \( k \)-valued points is described as

\[
(Aut O)(k) = \left\{ \right. \\
Z = (z, \xi^i) \mapsto \left( \sum_{n+|J| \geq 1} a_{n,J} Z^n J, \sum_{n+|J| \geq 1} b_{n,J} Z^n J \right) \left| \begin{bmatrix} a_{1,0} & a_{0,\xi} \\ b_{1,0} & b_{0,\xi} \end{bmatrix} \right| \\
\in \text{GL}_k (1|N) \left. \right\}. 
\]

The topological Lie superalgebra of Aut \( O \) is denoted by

\[ \text{Der}_0 O : = \text{Lie}(\text{Aut } O), \]

which is linearly compact and has a topological basis \( \{ z^n \partial_z, z^n \partial_{\xi^i} \mid n \geq 1, i \in [N] \} \cup \{ z^m \xi^i \partial_z, z^m \xi \partial_{\xi^i} \mid m \geq 0, i \in [N] \} \). It is actually a Lie sub-superalgebra of Der \( O \) given in (1.3.5).

**Definition 1.3.12** Consider the even differential 1-form on \( O \)

\[ \omega_Z : = dz + \sum_{i=1}^{N} \xi^i d\xi^i \in (\Omega^1_{A/\mathbb{R}})_T, \]

where \( d = d_{\text{ev}} \) denotes the even universal differential (Notation 1.2.6). We define Aut\textsuperscript{sc} \( O \) to be the subgroup of Aut \( O \) consisting of automorphisms preserving \( \omega_Z \) up to multiplication by a function. Its element is called a superconformal transformation of \( \mathbb{D} \). The topological Lie superalgebra of Aut\textsuperscript{sc} \( O \) is denoted by Der\textsuperscript{sc} \( O : = \text{Lie Aut}^{\text{sc}} O \)

and its element is called a superconformal vector field.

Let us recall basic properties of the Lie superalgebra Der\textsuperscript{sc} \( O \), following [14, Sect. 3.1.3] and [16, Example 2.12]. For \( i \in [N] \), we define an odd derivation \( D^i_Z \) of \( O \) by

\[ D^i_Z : = \partial_{\xi^i} + \xi \partial_{\xi^i}. \] (1.3.6)

Then, Der\textsuperscript{sc} \( O \) is equal to the Lie subalgebra of Der\( O \)

\[ \text{Der}^{\text{sc}}_0 O = \left\{ W \in \text{Der}_0 O \mid [W, D^i_Z] = f D^i_Z, \exists f \in O \right\}. \]
Also, $\text{Der}_{0}^{\text{sc}} O$ consists of the derivations of the form

$$D^{f} = f \partial_{z} + \frac{1}{2}(-1)^{p(f)} \sum_{i=1}^{N}(D^{f}_{z}f)D^{i}_{z}$$

with $f \in O$ of pure parity. In particular, it contains

$$l_{n} := -z^{n+1} \partial_{z} - \frac{n+1}{2}z^{n} \sum_{i=1}^{N} \xi^{i} \partial_{\xi^{i}} \quad (n \in \mathbb{N}),$$

$$g_{r}^{j} := -z^{r+\frac{1}{2}}(\partial_{\xi^{j}} - \xi^{j} \partial_{z}) + (r + \frac{1}{2})z^{r-\frac{1}{2}} \xi^{j} \sum_{i=1}^{N} \xi^{i} \partial_{\xi^{i}} \quad (r \in \mathbb{N} + \frac{1}{2}, j \in [N]).$$

In the case $N = 1$, the elements $l_{n}$ ($n \in \mathbb{N}$) and $g_{r} := g_{r}^{1}$ ($r \in \frac{1}{2} + \mathbb{N}$) are topological generators of $\text{Der}_{0}^{\text{sc}} O$, subject to the relations

$$[l_{m}, l_{n}] = (m - n)l_{m+n}, \quad [l_{n}, g_{r}] = (\frac{n}{2} - r)g_{n+r}, \quad [g_{r}, g_{s}] = 2l_{r+s}.$$ 

Thus, in the case $N = 1$, $\text{Der}_{0}^{\text{sc}} O$ is a Lie sub-superalgebra of the Neveu–Schwarz algebra of central charge 0 (see (2.3.9)).

Recall that the superjet algebra $A^{O}$ enjoys a universal property:

$$\text{Hom}_{\text{SCom}}(A^{O}, k) \rightarrow \text{Hom}_{\text{SCom}}(A, O).$$

Now we would like to find the odd derivations $S^{i}_{K}$ on $A^{O}$ which corresponds to the action of $D^{i}_{z} = \partial_{\xi^{i}} + \xi^{i} \partial_{z}$ on $O$. Let $\phi(Z) = \sum_{(\{J\}, n \geq 0)} Z^{n}_{\{J\}} \phi_{n, \{J\}}$ be the series corresponding to an element $\phi \in \text{Hom}_{\text{SCom}}(A, O)$. Then, we have

$$\left(D^{i}_{z}\phi\right)(Z) = \sum_{(\{J\}, n \geq 0, i \notin J)} Z^{n}_{\{J\}} \sigma(e_{i}, J)\phi_{n, \{J\} \cup \{e_{i}\}}$$

$$+ \sum_{(\{J\}, n \geq 0, i \in J)} Z^{n}_{\{J\}}(n + 1)\sigma(e_{i}, J \setminus e_{i})\phi_{n + 1, \{J\} \setminus e_{i}},$$

where we used the same symbols as in Lemma 1.3.7. This calculation indicates that the desired $S^{i}_{K}$ is given by

$$S^{i}_{K}(a^{\{n\} \cup \{e_{i}\}}) := \begin{cases} \sigma(e_{i}, J)a^{\{n\} \cup \{e_{i}\}} & (i \notin J) \\ (n + 1)\sigma(e_{i}, J \setminus e_{i})a^{\{n + 1\} \setminus e_{i}} & (i \in J) \end{cases}. \quad (1.3.7)$$

Using this action twice, we see that the square $T_{K} = (S^{i}_{K})^{2}$ is independent of $i$, and its action is given by

$$T_{K}(a^{\{n\} \cup \{e_{i}\}}) = (n + 1)a^{\{n + 1\} \cup \{e_{i}\}}. \quad (1.3.8)$$
Then, we immediately have:

**Lemma 1.3.13** The operators $S^i_K$’s and $T_K$ form a Lie superalgebra with commutation relation

$$[S^i_K, S^j_K] = 2\delta_{i,j} T_K, \quad [T_K, S^i_K] = 0.$$  

Recalling Definition 1.3.8, we find that $(A^O, S^i_K)$ is a differential superalgebra for any $A \in \text{SCom}_k$. Consulting Definition 1.3.12, we may name:

**Definition 1.3.14** For $A \in \text{SCom}_k$, the differential superalgebra $(A^O, S^i_K)$ is called the superconformal jet algebra of $A$, and denoted by $A^{Osc}$.

By Lemma 1.3.6, we have:

**Lemma 1.3.15** For any $A \in \text{SCom}_k$, the superconformal jet algebra $A^{Osc}$ is generated by $A$ as a differential superalgebra.

Then we have the following analogous of Lemma 1.3.10:

**Lemma 1.3.16** Let $A, B \in \text{SCom}_k$, and assume the following conditions.

- $B$ contains $A$ as a linear sub-superalgebra.
- $B$ has $N$ odd derivations $\partial_i$ $(i \in [N])$ which form the Lie superalgebra in Lemma 1.3.13.
- The differential superalgebra $(B, \partial_i)$ is generated by $A$.

Then the injection $A \hookrightarrow B$ can be extended to a surjective morphism $(A^O, S^i_K) \twoheadrightarrow (B, \partial_i)$ of differential superalgebras.

# 2 SUSY vertex algebras

This section gives a recollection of SUSY vertex algebras introduced in [16]. We also give a few preliminary observations on commutative SUSY vertex algebras. We will work over a fixed field $k$ of characteristic 0. Also we fix a positive integer $N$ and denote the ordered set $\{1 < \cdots < N\}$ by $[N]$.

## 2.1 Superfields

Here we explain the formalism of superfields used throughout the rest of this note. The original reference is [16, Sect. 2.6].

Let us continue to use Notation 1.3.4. So $Z = (z, \xi^1, \ldots, \xi^N)$ is a $1|N$-supervariable with even $z$ and odd $\xi^i$’s, and denote

$$Z^{j\in J} := z^j \xi^J = z^j \xi^{j_1} \xi^{j_2} \cdots \quad (j \in \mathbb{Z}, \ J = \{j_1, j_2, \ldots\} \subset [N]).$$

We also use a simplified notation

$$Z^{j\in [N]} := Z^{j\in [N]} = z^j \xi^1 \cdots \xi^N.$$
Below we will repeatedly use the sign notation given in [16, Sect. 3.1.1]. For disjoint ordered sets $I, J \subset [N]$, we define $\sigma(I, J) \in \{\pm 1\}$ by

$$\zeta^I \zeta^J = \sigma(I, J)\zeta^{I\cup J}, \quad (2.1.1)$$

where $I \cup J$ denotes the re-ordered set. If $I \cap J \neq \emptyset$, then we define $\sigma(I, J) := 0$. Also, for $i \in [N]$ and $J \subset [N]$, we use the following abbreviations.

$$e_i := \{i\}, \quad N \setminus J := [N] \setminus J. \quad (2.1.2)$$

For example, we denote $\sigma(N \setminus J, e_i) := \sigma([N] \setminus J, \{i\})$.

For a linear superspace $V$, we define $V[Z]$ and $V[Z^{\pm 1}]$ to be the linear spaces of series with coefficients in $V$:

$$V[Z] := \left\{ \sum_{n \in \mathbb{N}, J \subset [N]} Z^n|_{v_{n|J} \in V} \right\}, \quad V[Z^{\pm 1}] := \left\{ \sum_{j \in \mathbb{Z}, J \subset [N]} Z^j|_{v_{j|J} \in V} \right\}.$$

Here the summations are possibly infinite, and the index $J$ runs over ordered subset in $[N]$. These are linear superspaces by setting the parity as $p(Z^j|_{v}) := |J| + p(v) \mod 2$ for $v \in V$ of pure parity. We also denote

$$V(\langle Z \rangle) := \left\{ \sum_{j \in \mathbb{Z}, J \subset [N]} Z^j|_{v_{j|J} \in V[Z^{\pm 1}]} \mid v_{j|J} = 0 \forall j \ll 0 \right\}.$$

Let $\text{End}(V) := \text{Hom}(V, V)$ be the internal end in the category of linear superspaces (see (1.1.5)). In particular, it is a linear superspace. An $\text{End}(V)$-valued superfield of $1|N$-supervariable $Z$ is a series of the form

$$a(Z) = \sum_{(j|J)} Z^{-1 - j|N \setminus J} a_{(j|J)} \quad (a_{(j|J)} \in \text{End}(V)), \quad (2.1.3)$$

where the indices run over the range $j \in \mathbb{Z}, J \subset [N]$, and $N \setminus J := [N] \setminus J$, such that for each $v \in V$ we have $a(Z)v \in V(\langle Z \rangle)$. We denote the linear superspace of $\text{End}(V)$-valued superfields of supervariable $Z$ by

$$\text{SF}(V, Z) = \text{SF}(V).$$

Next, we recall the locality of superfields [16, Sect. 3.1.2]. Let us given two (super-)commuting $1|N$-supervariables $Z = (z, \xi^1, \ldots, \xi^N)$ and $W = (w, \omega^1, \ldots, \omega^N)$, and let $A$ be a superalgebra. An $A$-valued formal distribution in two variables $Z$ and $W$ is a (possibly infinite) series of the form

$$a(Z, W) = \sum_{(i|I), (j|J)} Z^i|I W^j|J a_{i|I, j|J} \quad (a_{i|I, j|J} \in A)$$
where the running index means $i, j \in \mathbb{Z}$ and $I, J \subset [N]$, and we wrote $Z_i^I W_j^J = z^i w^j \xi^I \omega^J$. We denote by $A[Z^{\pm 1}, W^{\pm 1}]$ the linear superspace of $A$-valued formal distributions. A formal distribution $a(Z, W) \in A[Z^{\pm 1}, W^{\pm 1}]$ is called local if there exists $n \in \mathbb{N}$ such that

$$(z - w)^n a(Z, W) = 0.$$  

(2.1.4)

Note that the left-hand side is well-defined in $A[Z^{\pm 1}, W^{\pm 1}]$.

2.2 $N_W = N$ SUSY vertex algebras

In this subsection, we cite from [16, Sect. 3] the notion of $N_W = N$ SUSY vertex algebras. We continue to use the notation in Sect. 2.1. In particular, $Z = (z, \xi^I) = (z, \xi^1, \ldots, \xi^N)$, and $W = (w, \omega^i)$ are $1|N$-supervariables.

2.2.1 Definition and basic properties

**Definition 2.2.1** [16, Definition 3.3.1] An $N_W = N$ SUSY vertex algebra ($N_W = N$ SUSY VA for short) is the data

$$(V, |0\rangle, T, S^i, Y)$$

consisting of

- a linear superspace $V = V_{\overline{0}} \oplus V_{\overline{1}}$, called the state superspace,
- an even vector $|0\rangle \in V_{\overline{0}}$, called the vacuum,
- an even operator $T \in \text{End}(V)_{\overline{0}}$,
- odd operators $S^i \in \text{End}(V)_{\overline{1}}$ for $i \in [N]$, and
- an even linear map $Y(\cdot, Z) : V \to SF(V, Z)$, called the state-superfield correspondence,

which satisfies the following axioms.

- **Vacuum axiom:** For any $a \in V$ and $i \in [N]$, we have
  $$Y(a, Z)|0\rangle = a + O(Z), \quad T|0\rangle = S^i|0\rangle = 0,$$
  (2.2.1)
  where $O(Z)$ is an element of $V[Z]$ which vanishes at $Z = 0$ (i.e., $z = \xi^i = 0$).

- **Translation invariance:** For each $a \in V$, we have
  $$[T, Y(a, Z)] = \partial_z Y(a, Z), \quad [S^i, Y(a, Z)] = \partial_{\xi^i} Y(a, Z) \quad (i \in [N]),$$
  (2.2.2)
  where we denoted $\partial_z : = \frac{\partial}{\partial z}$ and $\partial_{\xi^i} : = \frac{\partial}{\partial \xi^i}$, and used the supercommutator (1.1.3).

- **Locality axiom:** For any $a, b \in V$, there exists an $n \in \mathbb{N}$ such that
  $$(z - w)^n [Y(a, Z), Y(b, W)] = 0.$$  
  (2.2.3)

The identity is regarded as that of $(\text{End } V)[Z^{\pm 1}, W^{\pm 1}]$ (see (2.1.4)).

We abbreviate $V = (V, |0\rangle, T, S^i, Y)$ if no confusion may arise.
For $a \in V$, $j \in \mathbb{Z}$ and an ordered subset $J \subset [N]$, we define the Fourier mode or the $(j|J)$-operator $a_{(j|J)} \in \text{End}(V)$ by the expansion

$$Y(a, Z) = \sum_{(j|J)} Z^{-1-j|N\setminus J} a_{(j|J)}, \quad (2.2.4)$$

where we used similar symbols as in (2.1.3). We also denote $a_{(j|N)} := a_{(j|[N])}$.

Using $(j|J)$-operators, the vacuum axiom (2.2.1) is equivalent to

$$a_{(-1|N)}(0) = a, \quad a_{(-1|J)}(0) = 0 \quad (J \subset [N]), \quad a_{(n|J)}(0) = 0 \quad (n \in \mathbb{N}, J \subset [N]), \quad (2.2.5)$$

and by [16, (3.3.1.5)], the translation invariance (2.2.2) is equivalent to

$$[T, a_{(j|J)}] = -ja_{(j-1|J)}, \quad [S^i, a_{(j|J)}] = \begin{cases} \sigma(N\setminus J, e_i)a_{(j|J\setminus e_i)} & (i \in J) \\ 0 & (i \notin J) \end{cases}, \quad (2.2.6)$$

where we used the sign $\sigma$ in (2.1.1) and the abbreviations (2.1.2).

As in the non-SUSY case, the locality axiom (2.2.3) implies operator product expansion (OPE for short). To explain it, let us cite some notation on formal delta functions for $N_W = N$ SUSY VAs from [16, Sect. 2.3]. Let $Z = (z, \zeta^i)$ and $W = (w, \omega^j)$ be two commuting $1|N$-supervariables. We denote by $k((z, w))$ the field of fractions of $k[[z, w]]$, and set $k((Z, W)) := k((z, w)) \otimes k[\zeta^i, \omega^j]$. Then, we denote by

$$i_{z,w} : k((Z, W)) \hookrightarrow k((Z)\,(W)), \quad i_{w,z} : k((Z, W)) \hookrightarrow k((W)\,(Z)) \quad (2.2.7)$$

the embeddings obtained by the expansions with respect to $\frac{z}{w}$ (in the domain $|z| > |w|$), and with respect to $\frac{w}{z}$ (in the domain $|z| < |w|$), respectively. For $j \in \mathbb{Z}$ and an ordered subset $J = \{j_1 < \cdots < j_r\} \subset [N]$, we set

$$(Z - W)^{j|J} := (z - w)^j(\zeta - \omega)^J = (z - w)^j(\zeta^{j_1} - \omega^{j_1}) \cdots (\zeta^{j_r} - \omega^{j_r}). \quad (2.2.8)$$

The formal $\delta$-function for $N_W = N$ case is given by

$$\delta(Z, W) := (i_{z,w} - i_{w,z})(Z - W)^{-1|N} = (i_{z,w} - i_{w,z}) \frac{(\zeta - \omega)^N}{z - w}$$

$$= (i_{z,w} - i_{w,z}) \frac{(\zeta^1 - \omega^1) \cdots (\zeta^N - \omega^N)}{z - w}. \quad (2.2.9)$$

For $n \in \mathbb{N}$ and $J = \{j_1 < \cdots < j_r\} \subset [N]$, let

$$\partial^n_{(j|J)} := \partial^n_w \partial^n_{\omega^{j_1}} \cdots \partial^n_{\omega^{j_r}}, \quad \partial^n_{(n|J)} := (-1)^{\frac{n+1}{2}} \frac{1}{j!} \partial^n_{(j|J)}. \quad (2.2.10)$$
Then, by \[16\], Sect. 3.1.4 (4), we have

\[ \partial^{(n|J)}_{\mathcal{W}} \delta(Z, W) = (i_{z,w} - i_{w,z})(Z - W)^{-1-j|N\setminus J}. \] (2.2.11)

Finally, for \( f(Z) = \sum_{(j|J)} Z^{j|J} f_{j|J} \in \mathcal{V}[Z^{\pm 1}] \), we denote

\[ \text{res}_Z f(Z) := f_{-1|N}. \] (2.2.12)

Then, by \[16\], Lemma 3.1.3, any local distribution \( a(Z, W) \) is decomposed in the following finite sum.

\[ a(Z, W) = \sum_{(j|J), j \geq 0} \left( \partial^{(j|J)}_{Z} \delta(Z, W) \right) c_{j|J}(W), \quad c_{j|J}(W) := \text{res}_Z(Z - W)^{j|J} a(Z, W). \] (2.2.13)

Now, let \( \mathcal{V} \) be an \( \mathcal{N}_\mathcal{W} = \mathcal{N} \) SUSY VA, and \( a, b \in \mathcal{V} \) of pure parity. Then the locality axiom (2.2.3) and the expansion (2.2.13) imply the following OPE formula \[16\], Theorem 3.3.8 (5):

\[ [Y(a, Z), Y(b, W)] = \sum_{(j|J), j \geq 0} \left( \partial^{(j|J)}_{Z} \delta(Z, W) \right) Y(a_{j|J} b, W). \] (2.2.14)

As in the non-SUSY case, an \( \mathcal{N}_\mathcal{W} = \mathcal{N} \) SUSY VA \( \mathcal{V} \) is called strongly generated by a subset \( \{a^i | i \in I\} \subset \mathcal{V} \) if \( \mathcal{V} \) is spanned by the elements of the form

\[ a^{i_1}_{(-n_1|J_1)} \cdots a^{i_r}_{(-n_r|J_r)} |0 \] with \( r \in \mathbb{N}, n_i \in \mathbb{Z}_{>0} \) and \( J_i \subset [N] \).

We also have the notion of a module over an \( \mathcal{N}_\mathcal{W} = \mathcal{N} \) SUSY VA \( \mathcal{V} \). We omit the detailed definition and only give the notation: Let \( M \) be a \( \mathcal{V} \)-module. For \( a \in \mathcal{V} \) and \( m \in M \), we denote the \( \mathcal{V} \)-action on \( M \) by

\[ Y^M(a, Z)m = \sum_{(j|J)} Z^{-1-j|N\setminus J} a^M_{j|J} m \in M((Z)). \] (2.2.15)

### 2.2.2 Conformal \( \mathcal{N}_\mathcal{W} = \mathcal{N} \) SUSY vertex algebras

We cite from \[16\], Sect. 5 some examples of \( \mathcal{N}_\mathcal{W} = \mathcal{N} \) SUSY vertex algebras. To explain those examples, we use the \( \Lambda \)-bracket introduced by \[16\], which efficiently encodes OPE (2.2.14).

**Notation 2.2.2** \[16\], Sect. 3.2.1 Let \( \mathcal{V} \) be an \( \mathcal{N}_\mathcal{W} = \mathcal{N} \) SUSY VA.

- We denote the commutative superalgebra freely generated by even \( \lambda \) and odd \( \chi^i \) \( (i \in [N]) \) as

\[ \mathcal{L}_\mathcal{W} = k[\Lambda] = k[\lambda, \chi, \ldots, \chi^N]. \]
• For $m \in \mathbb{N}$ and $M = \{m_1 < \cdots < m_r\} \subset [N]$, we denote $\Lambda^m|_M := \lambda^m \chi^M = \lambda^{m_1} \chi^{m_1} \cdots \chi^{m_r}$.

• Finally, for $a, b \in V$, we define $[a_{\Lambda} b] \in \mathcal{L}_W \otimes_k V$ by

$$[a_{\Lambda} b] = \sum_{(j, J), j \geq 0} \sigma(J, N \setminus J)(-1)^{(j + 1)(\frac{1}{2})} \frac{1}{j!} \Lambda^j J a_{(j, J)} b, \quad \Lambda^j J := \lambda^j \chi^j \cdots \chi^j.$$

We call $[a_{\Lambda} b]$ the $\Lambda$-bracket.

In the rest of this Sect. 2.2.2, we assume the base field $k = \mathbb{C}$.

**Example 2.2.3** [16, Example 5.1] $V(\mathcal{W}_N)$ is an $N_W = N$ SUSY vertex algebra strongly generated by an element $L$ of parity $N \mod 2$ and elements $Q^i, i \in [N]$, of parity $N + 1 \mod 2$ whose OPEs are

$$[L_{\Lambda} L] = (T + 2\lambda) L, \quad [Q^i_{\Lambda} Q^j] = (S^i + \chi^i) Q^j - \chi^j Q^i, \quad [L_{\Lambda} Q^j] = (T + \lambda) Q^j + (-1)^N \chi^j L. \quad (2.2.16)$$

If $N = 1, 2$, then it admits a central extension. For $N = 1$, it is given by

$$[L_{\Lambda} L] = (T + 2\lambda) L, \quad [Q_{\Lambda} Q] = SQ + \frac{\lambda \chi}{3} C, \quad [L_{\Lambda} Q] = (T + \lambda) Q - \chi L + \frac{\lambda^2}{6} C. \quad (2.2.17)$$

For $N = 2$, we have a central extension

$$[L_{\Lambda} L] = (T + 2\lambda) L, \quad [L_{\Lambda} Q^i] = (T + \lambda) Q^i + \chi^i L, \quad [Q^i_{\Lambda} Q^j] = S^i Q^j, \quad [Q^1_{\Lambda} Q^2] = (S^1 + \chi^1) Q^2 - \chi^2 Q^1 + \frac{\lambda}{6} C. \quad (2.2.18)$$

$V(\mathcal{W}_N)$ is an $N_W = N$ SUSY analogue of Virasoro vertex algebra. Recall that an even vertex algebra is called conformal if it has an element which generates Virasoro vertex algebra. The following is its $N_W = N$ SUSY analogue.

**Definition 2.2.4** (First part of [16, Definition 5.2]) An $N_W = N$ SUSY VA $(V, |0\rangle, T, S^i, Y)$ over $\mathbb{C}$ is called **conformal** if it has elements $\nu, \tau^1, \ldots, \tau^N$ satisfying the following conditions.

• The superfields $L(Z) := Y(\nu, Z)$ and $Q^i(Z) := Y(\tau^i, Z)$ satisfy the OPEs (2.2.16) (or the central extension (2.2.17), (2.2.18)).

• We have $\nu_{(0|0)} = 2T$ and $\tau^i_{(0|0)} = S^i$.

• The operator $\nu_{(1|0)}$ is semisimple and the eigenvalues $d$ are bounded below, i.e., there is a finite subset $\{d_1, \ldots, d_s\} \subset \mathbb{C}$ such that $d \in d_i + \mathbb{R}_{\geq 0}$ for some $i$.

We call the elements $\nu$ and $\tau^i$ the **conformal elements**.

For later use, let us also cite:
Definition 2.2.5 (Second part of [16, Definition 5.2]) An \( N_W = N \) SUSY VA \( V \) is called strongly conformal if it is conformal and the operators \( v_{(1|0)} \) and \( \sum_{i=1}^N \sigma(e_i, N \backslash e_i)\tau_i^{(0|e_i)} \) have integer eigenvalues. We call the eigenvalue \( \Delta \) of \( v_{(1|0)} \) the conformal weight.

As mentioned in [16] and extensively studied in [14], for a strongly conformal \( N_W = N \) SUSY VA \( V \), we can construct a vector bundle on an arbitrary super Riemann surface whose fiber is \( V \).

2.2.3 Identities in \( N_W = N \) SUSY vertex algebras

In this part, we explain basic identities valid in \( N_W = N \) SUSY vertex algebras, in particular, we give analogue for Borcherds’ commutator formula (Fact 2.2.9) and iterate formula (Lemma 2.2.10), which will be used for the discussion of Li filtration in Sects. 3.2 and 3.3.

Throughout this subsection, \( V = (V, |0\rangle, T, S^i, Y) \) denotes an \( N_W = N \) SUSY VA. For a supervariable \( Z = (z, \xi^i) \), we denote
\[
Z\nabla := zT + \sum_{i=1}^N \xi^i S^i.
\]

We also use the formal exponential \( e^X = \exp X = \sum_{n \in \mathbb{N}} \frac{1}{n!} X^n \).

Using these relations and following the argument of the non-SUSY case in [12, 3.1.1–3.1.7], we can show the following statements.

Fact 2.2.6 [16, Proposition 3.3.6 (1), Theorem 3.3.8 (3), (4), Corollary 3.3.9] For \( a \in V \) and \( i \in [N] \), we have the following.

1. \( Y(a, Z)|0\rangle = e^{Z\nabla} a = e^{zT} (1 + \xi^1 S^1) \cdots (1 + \xi^N S^N) a \).
2. Using the supercommutator, we have
\[
Y(T a, Z) = \partial_z Y(a, Z) = [T, Y(a, Z)], \quad Y(S^i a, Z) = \partial_{\xi^i} Y(a, Z)
\]
\[
= [S^i, Y(a, Z)].
\]

In terms of the Fourier modes, we have
\[
(T a)_{(j|J)} = -j a_{(j-1|J)}, \quad (S^i a)_{(j|J)} = \sigma(e_i, N \backslash J) a_{(j|J \backslash e_i)}.
\]

Next, we recall that operators \( T \) and \( S^i \) can be regarded as even and odd derivations, respectively.

Fact 2.2.7 [16, Corollary 3.3.9] For \( a, b \in V \) of pure parity, \( j \in \mathbb{Z}, J \subset [N] \) and \( i \in [N] \), we have
\[
T(a_{(j|J)} b) = (T a)_{(j|J)} b + a_{(j|J)}(T b),
\]
\[
S^i(a_{(j|J)} b) = (-1)^{N \backslash J}((S^i a)_{(j|J)} b + (-1)^{p(a)} a_{(j|J)}(S^i b))
\]
with \((-1)^{N \backslash J} := (-1)^{N - |J|} \).
Let us also recall the skew-symmetry of the state-superfield correspondence.

**Fact 2.2.8** [16, Proposition 3.3.12] For \( a, b \in V \) of pure parity, we have

\[
Y(a, Z)b = (-1)^{p(a)p(b)}e^{Z\nabla}Y(b, -Z)a.
\]

Now, recall Borcherds’ commutator formula in the even case [22, (2.1)], [12, Sect. 3.3.10, p. 56]:

\[
[a(l), b(m)] = \sum_{j \in \mathbb{N}} \binom{l}{j} (a(j)b)(l+m-j).
\]

We have the following \( N_W = N \) SUSY analogue.

**Fact 2.2.9** [16, Proposition 3.3.18, (3.3.4.13)] For \( a, b \in V, l \in \mathbb{Z} \) and \( L \subset [N] \), we have

\[
[a(l|L), Y(b, W)] = \sum_{(j|J), j \geq 0} (-1)^{|J||L|+|J|N+|L|N} \delta_{W}^{(j|J)} W^{l|L} Y(a(j|J), W)
\]

with \( \delta_{W}^{(j|J)} \) given in (2.2.10). In terms of Fourier modes, we have

\[
[a(l|L), b(m|M)] = \sum_{(j|J), j \geq 0, J \supset L \cap M} (-1)^{\alpha \bar{\sigma}} \cdot \binom{l}{j} (a(j|J)b)(l+m-j|M \cup (L \setminus J)),
\]

where we denoted

\[
\alpha : = (p(a) + N - |L|)(N - |M|) + (|L| - |J|)(n - |J|),
\]

\[
\bar{\sigma} : = \sigma(J, N \setminus J) \sigma(L, N \setminus L) \sigma(J, L \setminus J) \sigma(L \setminus J, N \setminus M \setminus (L \setminus J)).
\]

Note that the \( J = 1 \) part of the last formula coincides with the even case formula (2.2.21).

Finally, let us discuss an \( N_W = N \) SUSY analogue of Borcherds’ iterate formula. The even case is

\[
(a(k)b)(l) = \sum_{j \geq 0} (-1)^j \binom{k}{j} (a(k-j)b(l+j) - (-1)^k b(k+l-j)a(j))
\]

for \( a, b \) of an even vertex algebra and \( k, l \in \mathbb{Z} \) (see [22, (2.2)] and [12, Sect. 3.3.10, p. 56]). Here and hereafter, we denote \( a(k)b(l)u : = a(k)(b(l)u) \) for \( u \) of a vertex algebra module, and in a similar way for the super case.
Lemma 2.2.10 Let $V$ be an $N_W = N$ SUSY VA and $M$ be a $V$-module. For $a, b \in V$ of pure parity, $u \in M$, $k, l \in \mathbb{Z}$ and $K, L \subset [N]$, we have

$$(a_{(k|K)} b_{(l|L)}) u = \sum_{j \geq 0, J \subset K} (-1)^{j + a} \tilde{\sigma} \cdot \binom{k}{j} \left( (-1)^{(p(a)+N-|J|)|N\backslash J'} a_{(k-j|J)} b_{(l+j|J')} u - (-1)^{k+p(a)p(b)+(p(b)+N-|J|)|N\backslash J'|} b_{(k+l-j|J')} a_{(j|J)} u \right)$$

with $J' := L \cup (K \setminus J)$ and

$$\alpha := |K \setminus J| (1 + |N \setminus J|), \quad \tilde{\sigma} := \sigma(J, K \setminus J) \sigma(J, N \setminus J) \sigma(L, K \setminus J) \sigma(J', N \setminus J').$$

Proof Let $a, b \in V$ be elements of pure parity and recall the OPE (2.2.14):

$$[Y(a, Z), Y(b, W)] = \sum_{(j|J), j \geq 0} (\partial^{|J|}_Z \delta(Z, W)) Y(a_{(j|J)} b_{(J)}, W).$$

Following the argument in the even case [12, Sect. 3.3.10], we multiply both sides of this equality by some $f(Z, W) \in k[Z^{1|J}, W^{1|J}, (Z - W)^{-1}|J \subset [N]]$. Then, using (2.2.11), we have

$$\text{res}_W \text{res}_Z f(Z, W) a(Z) b(W) - (-1)^{p(a)p(b)} \text{res}_W \text{res}_Z f(Z, W) b(W) a(Z)$$

$$= \text{res}_W \text{res}_Z -W \sum_{(j|J), j \geq 0} f(Z, W)(Z - W)^{-1-|N\backslash J|} Y(a_{(j|J)} b_{(J)}, W).$$

(2.2.24)

To show Lemma 2.2.10, we set $f(Z, W) := W^{n|L}(Z - W)^{k|K}$. Let us calculate the first term in the left-hand side. We have

$$\text{res}_Z (Z - W)^{k|K} a(Z) = \text{res}_Z (z - w)^{k} (\xi - \omega)^{K} \sum_{(j|J)} z^{-1-j} \xi^{N\backslash J} a_{(j|J)}$$

$$= \text{res}_Z \left[ (z - w)^{k} \cdot \sum_{M \subset K} (-1)^{|K\setminus M|} \sigma(M, K \setminus M) \xi^{M \omega} K\setminus M \cdot \sum_{(j|J)} z^{-1-j} \xi^{N\backslash J} a_{(j|J)} \right]$$

$$= \text{res}_Z \left[ \sum_{M \subset K} \sum_{(j|J)} (-1)^{\beta} \cdot (z - w)^{k} z^{-1-j} \xi^{M \cup (N\backslash J)} \omega K\setminus M a_{(j|J)} \right]$$
with $\beta := |K \setminus M| + |K \setminus M| |N \setminus J|$ and $\tau := \sigma (M, K \setminus M) \sigma (M, N \setminus J)$. By definition of $\text{res}_Z$, the terms with $J = M$ survive, and

$$\text{res}_Z (Z - W)^{k|K} a(Z) = \sum_{(j|J), \ j \geq 0, \ J \subseteq K} (-1)^{\beta + k-j} \tau \cdot \binom{k}{j} W^{k-j} \omega^{K \setminus J} a_{(j|J)}.$$  

Then we have

$$\text{res}_W \text{res}_Z W^{|L|} (Z - W)^{k|K} a(Z) b(W)$$

$$= \text{res}_W \sum_{(j|J), \ j \geq 0, \ J \subseteq K} (-1)^{\beta + k-j} \tau \cdot \binom{k}{j} W^{|L|} W^{k-j} \omega^{K \setminus J} a_{(j|J)} \sum_{(p|P)} W^{-1-p|N \setminus P} b_{(p|P)}$$

$$= \sum_{(j|J), \ j \geq 0, \ J \subseteq K} (-1)^{k-j+\beta} \cdot \binom{k}{j} (-1)^{(p(a)+N-|J|)|N \setminus J|} a_{(j|J)} b_{(l+k-j|J')} \quad (2.2.25)$$

with $J' := L \cup (K \setminus J)$ and

$$\beta = |K \setminus J| + |K \setminus J| |N \setminus J| = \alpha, \quad \nu := \tau \cdot \sigma (L, K \setminus J) \sigma (J', N \setminus J') = \tilde{\sigma}.$$  

Replacing $(j, k-j) \mapsto (k-j, j)$, we have $k-j + \alpha \mapsto j + \alpha$, and (2.2.25) is equal to the first term in the right-hand side of the statement. By a similar calculation, we see the second terms in (2.2.24) and the statement coincide. For the right-hand side of (2.2.24), we have by (2.2.11) that $\text{res}_{Z-W} \partial_{W}^{(j|J)} \delta(Z, W) = \delta_{j, 0} \delta_{J, \theta}$, which yields

$$\text{res}_W \text{res}_{Z-W} \sum_{(j|J)} W^{|L|} (Z - W)^{k|K} (Z - W)^{-1-j|1-J} Y(a_{(j|J)} b, W) = (a_{(k|K)} b)_{(l|L)}.$$  

Hence, we have the consequence. $\square$

### 2.3 $N_K = N$ SUSY vertex algebras

In this subsection, we recall the notion of $N_K = N$ SUSY vertex algebras introduced in [16, Sect. 4]. Throughout this subsection, we fix a positive integer $N$.

#### 2.3.1 Definition and basic properties

**Notation 2.3.1** [16, 4.1] For a $1|N$-supervariable $Z = (z, \xi^1, \ldots, \xi^N)$ and $i \in [N]$, we define odd derivations $D^i_Z$ and $\bar{D}^i_Z$ by

$$D^i_Z := \partial_{\xi^i} + \xi^i \partial_z, \quad \bar{D}^i_Z := \partial_{\xi^i} - \xi^i \partial_z.$$  

We have $(D^i_Z)^2 = \partial_z$ and $(\bar{D}^i_Z)^2 = -\partial_z$.  

\begin{center}
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\end{center}
Definition 2.3.2 [16, Definition 4.13] An $N_K = N$ SUSY vertex algebra is defined to be the data $(V, |0\rangle, S_K, Y)$ consisting of

- a linear superspace $V = V_0 \oplus V_1$, called the state superspace,
- an even element $|0\rangle \in V_0$, called the vacuum,
- $N$ odd operators $S_K^i \in \text{End}(V)_\tau (i \in [N]),$
- an even linear map $\mathcal{Y}(\cdot, Z) : V \rightarrow SF(V, Z)$, called the state-superfield correspondence,

which satisfies the following axioms.

1. Vacuum axiom: $\mathcal{Y}(a, Z)|0\rangle = a + O(Z)$ for all $a \in V$, and $S|0\rangle = 0$.
2. Translation invariance: For all $a \in V$, we have
   $$[S_K^i, \mathcal{Y}(a, Z)] = \overline{D}_Z \mathcal{Y}(a, Z).$$

3. Locality axiom: For all $a, b \in V$, there is an $n \in \mathbb{N}$ such that $(z - w)^n[Y(a, Z), Y(b, W)] = 0$.

We abbreviate $(V, |0\rangle, S_K^i, \mathcal{Y})$ to $V$ if no confusion may arise. Also, we often abbreviate the word “vertex algebra” to VA.

As in the $N_W = N$ case, we define the Fourier mode or the $(j|J)$-operator $a_{(j|J)} : \text{End}(V)$ for $a \in V$, $j \in \mathbb{Z}$ and $J \subset [N]$ by the expansion (2.2.4). Then, the vacuum axiom is equivalent to the same formula as in (2.2.5), and by [16, (4.6.1)], the translation invariance (2.3.1) is equivalent to

$$[S_K^i, a_{(j|J)}] = \begin{cases} \sigma(N \setminus J, e_i) a_{(j|J \setminus e_i)} & (i \in J) \\ -\sigma(N \setminus (J \cup e_i), e_i) j a_{(j-1|J \cup e_i)} & (i \notin J) \end{cases}. \quad (2.3.2)$$

We have $[S_K^i, [S_K^i, a_{(j|J)}]] = -ja_{(j-1|J)}$ for every $i \in [N]$, and the super Jacobi identity implies that

$$T := \frac{1}{2} [S_K^i, S_K^i] = (S_K^i)^2 \in \text{End}(V_0)$$

gives a well-defined even operator which satisfies

$$[T, a_{(j|J)}] = -ja_{(j-1|J)} \quad (j \in \mathbb{Z}, J \subset [N]), \quad \text{i.e.,} [T, \mathcal{Y}(a, Z)] = \partial_z \mathcal{Y}(a, Z).$$

The operators $S_K^i$’s and $T$ satisfy the commutation relation

$$[S_K^i, S_K^j] = 2\delta_{i,j}T, \quad [T, S_K^i] = 0,$$

which is the Lie superalgebra in Lemma 1.3.13.

Now let us explain the OPE formula for $N_K = N$ SUSY VAs, following [16, Sects. 4.1 and 4.2]. It involves a new delta function which is different from the $N_W = N$

\footnote{There is a typo in the second half of [16, (4.6.1)].}
case (2.2.9). Let \( Z = (z, \zeta^1, \ldots, \zeta^N) \) and \( W = (w, \omega^1, \ldots, \omega^N) \) be two commuting \( 1|N \)-supervariables, and let \( i_{z,w} : k((Z, W)) \leftrightarrow k((Z))((W)) \) be the embeddings in (2.2.7). For \( j \in \mathbb{Z} \) and \( J \subset [N] \), we set

\[
(Z - W)_K^{j|J} := \left( z - w - \sum_{i=1}^N \zeta^i \omega^i \right)^j (\zeta - \omega)^J.
\]  

(2.3.4)

Then the formal \( \delta \)-function for the \( NK = N \) case is defined to be

\[
\delta_K(Z, W) := (i_{z,w} - i_{w,z}) (Z - W)^{-1|N}_K = (i_{z,w} - i_{w,z}) \frac{\zeta - \omega}{z - w - \sum_{i=1}^N \zeta^i \omega^i} = (i_{z,w} - i_{w,z}) \frac{z - w}{z - w} = (i_{z,w} - i_{w,z}) \zeta - \omega,
\]

where in the last equality we used \( (Z - W)^{-1|0}_K = \frac{1}{z - w - \sum_{i=1}^N \zeta^i \omega^i} = \frac{1 + \sum_{i=1}^N \zeta^i \omega^i}{z - w} \).

The binomial (2.3.4) behaves with respect to the odd derivations \( D^i_W := \partial_{\zeta^i} + \zeta^i \partial_w \) in the same way as in the even case. For \( n \in \mathbb{N} \) and \( J = \{ j_1 < \cdots < j_r \} \subset [N] \), let

\[
D_W^{(n|J)} := \partial_w^n D_{W_1}^{j_1} \cdots D_{W_r}^{j_r}, \quad D_W^{(n|J)} := (-1)^{(r+1)} \frac{1}{n!} D_W^{(n|J)}.
\]  

(2.3.5)

Then, by [16, Lemma 4.1], we have

\[
D_W^{(n|J)} \delta_K(Z, W) = (i_{z,w} - i_{w,z})(Z - W)^{-1-j|N\setminus J}_K.
\]  

(2.3.6)

Also, by [16, Lemma 4.4], any local distribution \( a(Z, W) \) is decomposed as

\[
a(Z, W) = \sum_{(j|J), j \geq 0} (D_W^{(j|J)} \delta_K(Z, W)) c_{j|J}(W), \quad c_{j|J}(W) := \text{res}_Z (Z - W)^{j|J}_K a(Z, W),
\]  

(2.3.7)

where \( \text{res}_Z \) is defined in (2.2.12).

Now, by [16, Theorem 4.16 (4)], the OPE formula for an \( NK = N \) SUSY VA is

\[
[Y(a, Z), Y(b, W)] = \sum_{(j|J), j \geq 0} (D_Z^{(j|J)} \delta_K(Z, W)) Y(a_{(j|J)b, W}).
\]  

(2.3.8)

We also have the notion of a module \( M \) over an \( NK = N \) SUSY VA \( V \). We will use a similar notation for the \( V \)-action on \( M \) as in the \( NW = N \) case (2.2.15).

2.3.2 Conformal \( NK = N \) SUSY vertex algebras

Here we give some examples of \( NK = N \) SUSY vertex algebras. We use the following \( \Lambda \)-bracket to encode OPE (2.3.8), which is different from the one for \( NW = N \) case (Notation 2.2.2).
**Notation 2.3.3** [16, (3.3.4.6)] Let $V$ be an $N_K = N$ SUSY VA. We define the $\Lambda$-bracket $\{a_\Lambda b\}$ for $a, b \in V$ as follows.

- Let $\mathcal{L}_K$ be the (non-commutative) superalgebra generated by even $\lambda$ and odd $\chi^i$ ($i \in [N]$) subject to the commutation relation
  \[
  [\lambda, \chi^i] = 0, \quad [\chi^i, \chi^j] = -2\lambda\delta_{i,j}.
  \]
- For $m \in \mathbb{N}$ and $M = \{m_1 < \cdots < m_r\} \subset [N]$, we denote $\Lambda^{m|\bar{M}} := \lambda^m\chi^\bar{M} = \lambda^m\chi^{m_1}\cdots\chi^{m_r}$.
- Finally, for $a, b \in V$, we define $[a_\Lambda b] \in \mathcal{L}_K \otimes_k V$ by
  \[
  [a_\Lambda b] := \sum_{(m|\bar{M}), m \geq 0} \sigma(m, N\setminus M)(-1)^{(|M|+1)} \frac{1}{m!} \Lambda^{m|\bar{M}} a_{(m|\bar{M})} b.
  \]

Let us start with a standard example of $N_K = 1$ SUSY VA, which comes from an $N = 1$ superconformal vertex superalgebra. In the $N_K = 1$ case, a supervariable is denoted by $Z = (z, \zeta)$, and a superfield is expressed as:

\[
Y(a, Z) = \sum_{j \in \mathbb{N}, J = 0,1} Z^{-1-j|1-J} a_{(j|J)} = \sum_{j \in \mathbb{N}, J = 0,1} z^{-1-j} \zeta^{1-J} a_{(j|J)}
= \sum_{j \in \mathbb{N}} z^{-1-j} a_{(j|1)} + \sum_{j \in \mathbb{N}} z^{-1-j} \zeta a_{(j|0)}.
\]

Also, we denote the odd operator by $S_K$, and its square by $T$.

**Example 2.3.4** Let $V$ be an $N = 1$ superconformal vertex superalgebra in the sense of [18, p. 180]. Thus, it is a vertex superalgebra $(V, |0\rangle, T, Y^{cl})$ over $\mathbb{C}$ together with an odd element $\tau \in V$ whose Fourier modes in $Y^{cl}(\tau, z) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} z^{-r - \frac{3}{2}} G_r$ satisfy the following conditions.

1. $G_{\frac{1}{2}} \tau = 2v$ with $v$ a Virasoro element, i.e., an even element $v \in V$ whose Fourier modes in $Y^{cl}(v, z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n$ satisfy $L_{-1} = T, L_0 \tau = \frac{3}{2} \tau$ and $L_0$ being diagonalizable.
2. $G_{\frac{3}{2}} \tau = \frac{3}{2} c|0\rangle$ for some $c \in \mathbb{C}$.
3. $G_{r} \tau = 0$ for $r > \frac{3}{2}$.

We call $\tau$ the Neveu–Schwarz element of $V$ (which is called the $N = 1$ superconformal element in [18, Definition 5.9]).

The above Fourier modes form the Neveu–Schwarz algebra of central charge $c$. It is a Lie superalgebra with central extension which is generated by odd $G_r$ ($r \in \mathbb{Z} + \frac{1}{2}$) and even $L_n$ ($n \in \mathbb{Z}$) subject to the following commutation relations.

\[
\begin{align*}
[G_r, G_s] &= 2L_{r+s} + \frac{1}{3}(r^2 - \frac{1}{4})\delta_{r+s,0}c, \quad [G_r, L_n] = (r - \frac{n}{2})G_{n+r},
[L_m, L_n] &= (m - n)L_{m+n} + \delta_{m+n,0} m^3 m^{-m} c.
\end{align*}
\]
Such $V$ has a structure $(|0\rangle, S_K, Y)$ of $N_K = 1$ SUSY VA. The vacuum $|0\rangle$ is the same one, the state-superfield correspondence $Y$ is given by

$$Y(a, Z) := Y^{cl}(a, z) + \zeta Y^{cl}(G - \frac{1}{2}a, z)$$

for $a \in V$, and the odd operator is given by $S_K := G - \frac{1}{2} = \tau(0|1)$. 

**Example 2.3.5** As a special case in Example 2.3.4, we have the Neveu–Schwarz SUSY vertex algebra whose state superspace $V$ is the linear superspace of polynomials $\mathbb{C}[S_K^n \tau \mid n \in \mathbb{N}]$ and the vacuum is given by $|0\rangle = 1$. The state-superfield correspondence for the Neveu–Schwarz element $\tau$ is

$$Y(\tau, Z) := \sum_{r \in \mathbb{Z} + \frac{1}{2}} z^{-r - \frac{3}{2}} G_r + 2 \sum_{n \in \mathbb{Z}} z^{-n - 1} \zeta L_n,$$

and the Fourier modes

$$\tau_{(j|1)} = G_{j - \frac{1}{2}}, \quad \tau_{(j|0)} = 2L_{j - 1} \quad (j \in \mathbb{Z})$$

satisfy the commutation relations (2.3.9). We call $\nu := \frac{1}{2} S_K \tau \in V$ the Virasoro element of $V$. Using the $\Lambda$-bracket (Notation 2.3.3), the OPE of $Y(\tau, Z)$ is summarized as

$$[\tau_{\Lambda} \tau] = (2T + \chi S_K + 3\lambda)\tau + \frac{c}{3} \lambda^2 \chi |0\rangle.$$

For later reference, we give some formulas.

$$\tau = \tau_{(-1|1)}|0\rangle = G - \frac{1}{2}|0\rangle, \quad \nu = \frac{1}{2}(S_K \tau)_{(-1|1)}|0\rangle = \frac{1}{2} \tau_{(-1|0)}|0\rangle = L_{-2}|0\rangle.$$  

(2.3.11)

In the latter calculation, we used the recursion (2.3.12) given below.

Example 2.3.4 can be extended to higher $N_K$ cases.

**Definition 2.3.6** (First part of [16, Definition 5.6]) For $N \in \{1, 2, 3, 4\}$, an $N_K = N$ SUSY VA $(V, |0\rangle, S_K, Y)$ is called conformal if it has an element $\tau \in V$ of parity $N \text{ mod } 2$, satisfying the following conditions.

- Using Notation 2.3.3 of the $\Lambda$ bracket, the OPE of $Y(\tau, Z)$ is given by

  $$[\tau_{\Lambda} \tau] = (2T + (4 - N)\lambda + \sum_{i=1}^{N} \chi^i S_K^i)\tau + \begin{cases} \frac{c}{3} \lambda^3 \chi^N |0\rangle & (N \leq 3) \\ \lambda c |0\rangle & (N = 4) \end{cases}.$$  

- Using (2.1.2), we have

  $$\tau_{(0|0)} = 2T, \quad \tau_{(0|e_i)} = \sigma(N \backslash e_i, e_i) S_K^i.$$  

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• The operator $\tau_{(1|0)}$ is semisimple, and the eigenvalues $d$ are bounded below, i.e., there is a finite subset $\{d_1, \ldots, d_s\} \subset \mathbb{C}$ such that $d \in d_i + \mathbb{R}_{\geq 0}$ for some $i$.

We call $\tau$ the conformal element.

**Remark** The condition $N \leq 4$ comes from the fact that there is no central extension of the Lie superalgebra corresponding to the $\Lambda$-bracket $[\tau_\Lambda \tau] = (2t + (4 - N)\lambda + \sum_{i=1}^{N} \chi^i S_K) \tau$.

For later use, let us also cite:

**Definition 2.3.7** (Second part of [16, Definition 5.6]) An $N_K = N$ SUSY VA $V$ is called strongly conformal if it is conformal and satisfies the following conditions.

• $\tau_{(1|0)}$ has integer eigenvalues.
• If $N = 2$, then the operator $\sqrt{-1} \tau_{(0|N)}$ has integer eigenvalues.

As in the $N_W = N$ case (Definition 2.2.5), the strongly conformal condition permits us to construct a vector bundle on an arbitrary super Riemann surface whose fiber is $V$. See [14] for the detail.

### 2.3.3 Identities in $N_K = N$ SUSY vertex algebras

We explain basic identities for the $N_K = N$ case. Throughout this part, $V = (V, |0\rangle, S_K, Y)$ denotes an $N_K = N$ SUSY VA, and

$$Z \nabla := z T + \sum_{i=1}^{N} \xi_i S_K^i$$

for the operators $T = (S_K^i)^2$ and $S_K^i$ of $V$.

The following is an $N_K = N$ analogue of Fact 2.2.6.

**Fact 2.3.8** [16, Proposition 4.15 (1), Theorem 4.16 (3), Corollary 4.18] For $a \in V$, we have:

1. $Y(a, Z)|0\rangle = e^{Z \nabla} a = e^{z T} (1 + \xi S_K^1) \cdots (1 + \xi N S_K^N) a$.
2. $Y(T a, Z) = \partial_z Y(a, Z)$ and $Y(S_K^i a, Z) = D_z^i Y(a, Z) = (\partial_{z_i} + \xi_i \partial_z) Y(a, Z)$. In terms of the Fourier modes, we have

$$(Ta)_{(j|J)} = -ja_{(j-1|J)};$$

$$(S_K^i a)_{(j|J)} = \begin{cases} \sigma (N \setminus J, e_i) a_{(j|J \setminus e_i)} & (i \in J) \\ -j \sigma (N \setminus (J \cup e_i), e_i) a_{(j-1|J \cup e_i)} & (i \notin J). \end{cases}$$

**Remark 2.3.9** Some comments are in order.
(1) As noted in [16, Remark 4.17] and [15, Remark 3.3], we have

\[ Y(Ta, Z) = \partial_z Y(a, Z) = [T, Y(a, Z)], \quad Y(S_K^i a, Z) = D_Z^{i} Y(a, Z) \neq D_Z^{\overline{i}} Y(a, Z) = [S_K^i, Y(a, Z)]. \]

The inequality \( Y(S_K^i a, Z) \neq [S_K^i, Y(a, Z)] \) should be contrasted with the \( N_W = N \) case (2.2.19).

(2) By the equivalent form (2.2.5) of the vacuum axiom, we can recover the odd operators \( S_K^i \) from \( Y \) in the way

\[ S_K^i a = (S_K^i a)_{(−1|N)}0 = a_{(−1|N \setminus \{i\})0}. \]

Thus, the operators \( S_K^i \) can be omitted from Definition 2.3.2. Similarly, \( T \) is recovered as

\[ Ta = (Ta)_{(−1|N)}0 = a_{(−2|N)}0, \]

which is a reminiscent of the even case formula \( Ta = a_{(−2)}0 \).

Next we recall that the odd operator \( S_K^i \) can be regarded as a derivation.

**Fact 2.3.10** [16, Corollary 4.18] For \( a, b \in V \) of pure parity and \( i \in [N] \), we have

\[ S_K^i (a(j|j)b) = (-1)^{N\setminus J} ((S_K^i a)(j|j)b + (-1)^p(a) a(j|j)(S_K^i b)) \]

with \((-1)^{N\setminus J} := (-1)^{N − |J|} \).

Let us also recall the skew-symmetry of the state-superfield correspondence.

**Fact 2.3.11** [16, Proposition 3.3.12, Sect. 4.8] For \( a, b \in V \) of pure parity, we have

\[ Y(a, Z)b = (-1)^p(a)p(b) e^{Z\nabla} Y(b, −Z)a. \]

Next, we have the following \( N_K = N \) SUSY analogue of Borcherds’ commutator formula \([a(l), b(m)] = \sum_{j \in \mathbb{N}} \binom{j}{l} (a(j)b)(l+m−j)\) in (2.2.21).

**Fact 2.3.12** [16, Proposition 3.3.18, (4.3.3), Sect. 4.10] For \( a, b \in V, l \in \mathbb{Z} \) and \( L \subset [N] \), we have

\[ [a(l|L), Y(b, W)] = \sum_{(j|J), j \geq 0} (-1)^{|J||L|+|J||N|+|L|N} D_W^{(j|J)} W^{l|L} Y(a(j|J), W), \]

where \( D_W^{(j|J)} \) is given in (2.3.5). In terms of Fourier modes, we have

\[ [a(l|L), b(m|M)] = \sum_{(j|J), j \geq 0, M \cap (J \Delta L) = \emptyset} (-1)^{\alpha + \beta + \gamma} \cdot \frac{(l)}{j!} (a(j|J)b)(l+m−j−(#(J\setminus L)|M \cup (J \Delta L))) \]

(2.3.14)
where we denoted \((l)_n^1 := l(l - 1) \cdots (l - n + 1)\) for \(n \in \mathbb{N}\), \(J \Delta L := (J \setminus L) \cup (L \setminus J)\) and

\[
\alpha := (p(a) + N - |L|)(N - |M|), \quad \beta := |J| (N - |L|) + |L| N + \frac{|J \cap L|}{2} + \frac{|J| + 1}{2},
\]

\[
\tilde{\sigma} := \sigma(J \Delta L, N \setminus (M \cup (J \Delta L))) \sigma(J, N \setminus J) \sigma(L, N \setminus L) \sigma(J \setminus L, J \cap L) \sigma(J \cap L, L \setminus J) \sigma(J \setminus L, L \setminus J).
\]

In the \(N_K = 1\) case, (2.3.14) reduces to

\[
[a_{l(0)}, b_{m(0)}] = \sum_{j \in \mathbb{N}, \ j = 0, 1} (-1)^{(l_j^2)} \binom{l_j + J}{j} (a_{(j|J)} b_{(l+m-j-J|J)}) \times (-1)^p(a+1),
\]

\[
[a_{l(0)}, b_{m(1)}] = \sum_{j \in \mathbb{N}, \ j = 0, 1} (-1)^{(l_j^2)} \binom{l_j + J}{j} (a_{(j|J)} b_{(l+m-j-J|1)})
\]

\[
[a_{l(1)}, b_{m(0)}] = \sum_{j \in \mathbb{N}, \ j = 0, 1} (-1)^{J+1} \binom{l_j}{j} (a_{(j|J)} b_{(l+m-j|1-J)}) \times (-1)^p(a),
\]

\[
[a_{l(1)}, b_{m(1)}] = \sum_{j \in \mathbb{N}, \ j = 0, 1} (-1)^{J+1} \binom{l_j}{j} (a_{(j|J)} b_{(l+m-j|1)})
\]

for \(a, b \in V\) of pure parity and \(m, n \in \mathbb{Z}\). Note that the \(J = 1\) part of \([a_{l(1)}, b_{m(1)}]\) coincides with the even case formula (2.2.21).

Finally, let us discuss an \(N_K = N\) SUSY analogue of Borcherds’ iterate formula. Recall the even case formula (2.2.23): \((a_{(k)} b_{(l)}) = \sum_{j \geq 0} (-1)^{j} (a_{(k-j)} b_{(l+j)}) - (-1)^k b_{(k+l-j)} a_{(j)}\).

**Lemma 2.3.13** Let \(V\) be an \(N_K = N\) SUSY VA, and \(W\) be a \(V\)-module. For \(a, b \in V\) of pure parity, \(w \in W\), \(k, l \in \mathbb{Z}\) and \(K, L \subset [N]\), we have

\[
(a_{(k|K)} b_{(l|L)})w = \sum_{(j|J), \ j \geq 0, M \subset J \cap K} (-1)^j \tilde{\sigma} \cdot \binom{k}{j, |J \setminus M|} \left( (-1)^{(p(a)+N-|J|)/|N\setminus L'|} a_{(k-j-\#(J \setminus M)|J)} b_{(l+j|L')} w 
- (-1)^k + p(a)p(b) + (p(b)+N-|J|)/|N\setminus L'| b_{(l+k-j-\#(J \setminus M)|L')} a_{(j|J)} w \right),
\]
where \( \binom{k}{i,j} = \binom{k}{i} \binom{k-i}{j} \), \( L' = L \cup (J \setminus M) \cup (K \setminus M) \) and

\[
\alpha = \left( \binom{|J \setminus M| + 1}{2} + |J \setminus M| \right) + |J \setminus M|(|M| + |N \setminus J|) + |K \setminus M| \left( 1 + |N \setminus J| + |N \setminus J| |N \setminus L| \right),
\]

\[
\tilde{\sigma} = \sigma(M, K \setminus M) \sigma(J \setminus M, M) \sigma(J, N \setminus J) \sigma(J \setminus M, K \setminus M) \sigma(L, (J \setminus M) \cup (K \setminus M)) \sigma(L', N \setminus L').
\]

**Proof** The strategy is the same as in Lemma 2.2.10. Recall the OPE for the \( N_K = N \) case (2.3.8):

\[
[Y(a, Z), Y(b, W)] = \sum_{(j|J), j \geq 0} (D_Z^{j|j} \delta(Z, W)) Y(a_{(j|j)} b, W).
\]

For each \( f \in k[Z^{1|J}, W^{1|J}], (Z - W)^{-1|J} | J \subset [N] \), it yields the equality

\[
\text{res}_W \text{res}_Z f(Z, W)a(Z)b(W) - (-1)^{p(a)p(b)} \text{res}_W \text{res}_Z f(Z, W)b(W)a(Z)
\]

\[
= \text{res}_W \text{res}_{(Z - W)_K} \sum_{(j|J), j \geq 0} f(Z, W)(Z - W)^{-1-j|N \setminus J} Y(a_{(j|j)} b, W),
\]

(2.3.15)

where \( \text{res}_{(Z - W)_K} \) means \( \text{res}_{(z - w - \sum_{j=1}^{N} \xi_j^{j} \omega^j, \zeta - \omega)} \). We apply this equality for

\[
f(Z, W) := W^{i|L}(Z - W)^{k|K} = W^{i|L}(z - w - \sum_{i=1}^{N} \xi_i^{j} \omega^j)k(\zeta - \omega)^K.
\]

The first term in the left-hand side of (2.3.15) is

\[
\text{res}_{Z}(Z - W)^{k|K} a(Z)
\]

\[
= \text{res}_{Z}(z - w - \sum_{i=1}^{N} \xi_i^{j} \omega^j)k(\zeta - \omega)^K \sum_{(j|J)} z^{-1-j} \xi_N^{j\setminus J} a_{(j|j)}
\]

\[
= \text{res}_{Z} \left[ \sum_{I \subset [N]} (-1)^{(|I|+1)} \binom{k}{|I|} (z - w)^{k-|I|} \zeta_I^{\omega^I} \sum_{M \subset K} (-1)^{|K \setminus M|} \sigma(M, K \setminus M) \zeta_M^{M \setminus K \setminus M} \sum_{(j|J)} z^{-1-j} \xi_N^{j\setminus J} a_{(j|j)} \right]
\]

\[
= \text{res}_{Z} \left[ \sum_{I \subset [N]} \sum_{M \subset K} \sum_{(j|J)} (-1)^{\beta} \tau \left( \binom{k}{|I|} (z - w)^{k-|I|} \zeta_I^{\omega^I} \sum_{(j|I \cup M \cup (N \setminus J)} \omega^{I \cup (N \setminus J)} a_{(j|j)} \right) \right]
\]

with \( \beta = \binom{|I|+1}{2} + |K \setminus M| + |I| \cdot |M| + (|I| + |K \setminus M|) |N \setminus J| \) and \( \tau := \sigma(M, K \setminus M) \sigma(I, M) \sigma(I \cup M, N \setminus J) \sigma(I, K \setminus M) \). By definition of \( \text{res}_Z \), the terms with \( J = I \cup M \)
survive, and

\[
\text{res}_Z(Z - W)_K^{k|L} \cdot a(Z)
\]

\[
= \sum_{(j|J) \subseteq J \cap K} \sum_{M \subseteq J \cap K} (-1)^\gamma \left( k - |J\setminus M| \right) w^{k - |J\setminus M| - j} \omega(J\setminus M) \cup (K\setminus M) a(j|J),
\]

\[
\gamma := \beta + k - |J\setminus M| - j = k - j + \left( \frac{|J\setminus M|}{2} \right) + |J\setminus M|(|M| + |N\setminus J|)
\]

\[
+ |K\setminus M| (1 + |N\setminus J|),
\]

\[
\tau = \sigma(M, K\setminus M) \sigma(J\setminus M, M) \sigma(J, N\setminus J) \sigma(J\setminus M, K\setminus M).
\]

Then we have

\[
\text{res}_W \text{res}_Z W^{l|L} (Z - W)_K^{k|L} \cdot a(Z) b(W)
\]

\[
= \text{res}_W \sum_{(j|J) \subseteq J \cap K} \sum_{M \subseteq J \cap K} (-1)^\gamma \tau(j, |J\setminus M|) W^{l|L} w^{k - j - |J\setminus M|} \omega(J\setminus M) \cup (K\setminus M) a(j|J)
\]

\[
= \sum_{(j|J) \subseteq J \cap K} \sum_{M \subseteq J \cap K} (-1)^\gamma \left( N\setminus J \setminus N\setminus L' \right) \left( k - j \right) \left( |J\setminus M| \right)
\]

\[
(-1)^{(p(a)+N-|J|)|N\setminus L'|} a(j|J) b_{(l+k-j-\#(J\setminus M)|L')},
\]

\[
L' := L \cup (J\setminus M) \cup (K\setminus M), \quad \nu := \tau \cdot \sigma(L, (J\setminus M) \cup (K\setminus M)) \sigma(L', N\setminus L')
\]

\[
= \tilde{\sigma}.
\]

(2.3.16)

Replacing \(k - j - |J\setminus M|\) by \(j\) and vice versa, we have \(\gamma + |N\setminus J| |N\setminus L'| \mapsto j + \alpha\), and (2.3.16) is equal to the first term in the right-hand side of the statement. By a similar calculation, we see the second term in (2.3.15) and the statement coincides. For the right-hand side of (2.3.15), we have by (2.3.6) that \(\text{res}_{Z-W}_K D^{(j|L)}_W \delta(Z, W) = \delta_{j,0} \delta_{j,0} \), which yields

\[
\text{res}_W \text{res}_{(Z-W)_K} \sum_{(j|J)} W^{l|L} (Z - W)_K^{k|L} (Z - W)_K^{1-j|l|L} \cdot a(j|L) b, W)
\]

\[
= (a(k|L))_{(l|L)}.
\]

Hence we have the consequence. \(\square\)

### 2.4 Commutative SUSY vertex algebras

Hereafter, the word “SUSY vertex algebra” means an \(N_W = N\) or \(N_K = N\) SUSY vertex algebra.
2.4.1 SUSY analogue of Borcherds’ equivalence

Recall that, in the even case, a commutative vertex algebra is equivalent to a unital commutative algebra with derivation [9, Sect. 4], [12, Sect. 1.4]. Lemmas 2.4.3 and 2.4.4 are analogues of this fact for \( N_W = N \) and \( N_K = N \) SUSY vertex algebras, respectively.

**Definition 2.4.1** A SUSY VA \( V \) is commutative if and only if \( Y(a, Z) \in (\text{End } V)[[Z]] \) for any \( a \in V \).

Similarly as in the even case, we have the following restatement:

**Lemma 2.4.2** A SUSY VA \( V \) is commutative if and only if \( Y(a, Z) \in (\text{End } V)[[Z]] \) for any \( a \in V \).

**Proof** The argument is quite similar to that for the even case [12, Sect. 1.4], but let us write down it for completeness. Assume that \( V \) is commutative. Then, for any \( a, b \in V \) of pure parity, we have

\[
Y(a, Z)b = Y(a, Z)Y(b, W)[0]_{W=0} = (-1)^{p(a)p(b)}Y(b, W)Y(a, Z)[0]_{W=0}.
\]

The right-hand side has no negative powers of \( z \) by the vacuum axiom, and hence, \( Y(a, Z)b \in V[[Z]] \). Thus, we have \( Y(a, Z) \in (\text{End } V)[[Z]]. \)

Conversely, assume \( Y(a, Z) \in (\text{End } V)[[Z]] \) for any \( a \in V \). Then, for any \( a, b \in V \) of pure parity, both of the series \( f_1(Z, W) : = Y(a, Z)Y(b, W) \) and \( f_2(Z, W) : = Y(b, W)Y(a, Z) \) are in \((\text{End } V)[[Z, W]]\). Now the locality axiom says that there exists \( n \in \mathbb{N} \) such that \( (z-w)^n f_1(Z, W) = (z-w)^n(-1)^{p(a)p(b)}f_2(Z, W) \). The last equality yields \( f_1(Z, W) = (-1)^{p(a)p(b)}f_2(Z, W) \), and thus \( V \) is commutative. \( \square \)

**Lemma 2.4.3** Let \((V, [0], T, S^i, Y)\) be a commutative \( N_W = N \) SUSY VA. Then, \( V \) is a commutative \( k \)-superalgebra with multiplication \( a \cdot b : = a_{(-1)^{|b|}}b, \text{ unit } [0], \) even derivation \( T \), and odd derivations \( S^i \) (see Definition 1.2.1). Conversely, given a commutative \( k \)-superalgebra \( V \) with multiplication \( \cdot \), unit \( 1 \), even and odd derivations \( T \) and \( S^i \), we have a commutative \( N_W = 1 \) SUSY VA \((V, 1, T, S^i, Y)\) with

\[
Y(a, Z) : = e^{Z\nabla}a = \left( \sum_{n \geq 0} \frac{1}{n!} \xi^n T^n \right) (1 + \xi^1 S^1) \cdots (1 + \xi^N S^N)(a \cdot ?) \in (\text{End } V)[[Z]],
\]

where the symbol \( a \cdot ? \) denotes the multiplication operator.

**Proof** This one is also a simple analogue of the even case [12, Sect. 1.4]. First, assume that \( V \) is a commutative \( N_W = 1 \) SUSY VA, and let \( a, b \in V \) be of pure parity. Then, (2.4.1) yields \( Y(a, Z)b = (-1)^{p(a)p(b)}b_{(-1)^{|a|}}a + O(Z, W) \), where \( O(Z, W) \) is an element vanishing at \( Z = W = 0 \). Taking the coefficient of \( Z^0W^0 \), we have
\[ a_{(-1|1)} b = (-1)^{p(a)p(b)} b_{(-1|1)} a. \] Then, denoting \( Y_a := a_{(-1|1)} \), we have \( Y_a Y_b = (-1)^{p(a)p(b)} Y_b Y_a \). By the argument in [12, Sect. 1.3.3], it shows the associativity and the commutativity of \( a \cdot b := Y_a Y_b \). The vacuum axiom yields that \(|0\rangle\) is the unit of this multiplication \( \cdot \), and Fact 2.2.7 means that \( T \) and \( S \) are an even and odd derivation, respectively. The converse statement is checked directly from the formula (2.4.2), and we omit the detail. \( \square \)

**Lemma 2.4.4** Let \( V \) be a \( k \)-linear superspace. Then, the structure \( (V, |0\rangle, S^i, Y) \) of commutative \( N_K = N \) SUSY VA is equivalent to the structure \( (V, \cdot, 1, S^i) \) of unital commutative \( k \)-superalgebra with odd derivations \( S^i (i \in [N]) \) which satisfy the commutation relation

\[ [S^i, S^j] = 2\delta_{i,j} T, \quad [T, S^j] = 0. \]

The correspondence is given by \( a_{(-1|N)} b = a \cdot b, |0\rangle = 1 \) and

\[ Y(a, Z) = e^{Z V} a = e^{\xi^T} (1 + \xi^1 S^1) \cdots (1 + \xi^N S^N) a. \] (2.4.3)

**Proof** The argument in Lemma 2.4.3 works, using Lemma 1.3.13 and Fact 2.3.10. \( \square \)

### 2.4.2 \( N_W = N \) SUSY VA structure of superjet algebras

Recall that in Sect. 1.3 we introduced the superjet algebra

\[ A^O = A^k[Z] = S^N(A_{\infty}) \]

for each \( A \in S\text{Com} \) (Proposition 1.3.5). Here \( O = k[Z] \) denotes the topological superalgebra \( O^{[1|N]} = k[z, \xi^1, \ldots, \xi^N] \). We argue that \( A^O \) has a natural structure of \( N_W = N \) SUSY VA (Proposition 2.4.5).

Let us recall some basic properties of \( A^O \). It is a commutative \( k \)-superalgebra equipped with the universal differentials \( d_{n|J} = d_{dR}^J d_n \) for \( n \in \mathbb{N} \) and \( J \subset [N] \). Since we are assuming \( k \) is of characteristic 0, we have \( d_n = \frac{1}{n!} d_1^n \) by Remark 1.3.2. Note that \( d_1 \) and \( d_{dR} \) is an even and odd derivation, respectively. The even derivation \( d_1 \) comes from the Hasse–Schmidt derivations \( A_{\infty} = HS_{A/k}^\infty \), and the odd derivation \( d_{dR} \) is the differential of the de Rham complex. Also, recall that \( A^O \) is generated by \( a^{[n|J]} := d_{n|J} a \) with \( a \in A, n \in \mathbb{N} \) and \( J \subset [N] \) (see Lemma 1.3.6).

By Lemma 1.3.7, the superjet algebra \( A^O \) has an even derivation \( T \) and odd derivations \( S^i \) for \( i \in [N] \) with

\[ T(a^{[n|J]}) := (n + 1)a^{[n+1|J]}, \quad S^i(a^{[n|J]}) := \sigma(e_i, J)a^{[n|J\setminus e_i]}. \] (2.4.4)

Then, Lemma 2.4.3 yields:

**Proposition 2.4.5** For every \( A \in S\text{Com} k \), the superjet algebra \( A^O \) has a structure of commutative \( N_W = N \) SUSY VA with \( T \) and \( S^i \) given by (2.4.4).
2.4.3 $N_K = N$ SUSY VA structure of superconformal jet algebras

Recall Definition 1.3.14, which says that for any $A \in S\text{Com}_k$ we have the superconformal jet algebra $A^{O_{sc}} = (A^O, S^K_i)$. It is a unital commutative superalgebra equipped with odd derivations

$$S^K_i(a^{[n|J]}) : = \begin{cases} 
\sigma(e_i, J)a^{[n|J \cup e_i]} & (i \notin J) \\
(n+1)\sigma(e_i, J \setminus e_i)a^{[n+1|J \setminus e_i]} & (i \in J) 
\end{cases} \quad (2.4.5)$$

$S^K_i$’s form the Lie superalgebra in Lemma 1.3.13. Hence, Lemma 2.4.4 yields:

**Proposition 2.4.6** For every $A \in S\text{Com}_k$, the superjet algebra $A^O$ has a structure of a commutative $N_K = N$ SUSY VA with $S^K_i$ being the odd derivation determined by (2.4.5).

3 Li filtration of SUSY vertex algebra

In this subsection, we introduce SUSY analogue of Li’s canonical filtration [22]. As in Sect. 2, we fix a positive integer $N$.

3.1 SUSY vertex Poisson algebras

Recall that, in the even case, a vertex Poisson algebra consists of data $(V, |0\rangle, T, Y_+, Y_-)$ combining a commutative VA structure $(|0\rangle, T, Y_+)$ and a vertex Lie algebra structure $(V, T, Y_-)$ with common derivation $T$, subject to the condition that $Y_-$ gives Poisson-like operations for the commutative multiplication associated $Y_+$. See [12, Chap. 16] and [22, Sect. 2] for the details. A SUSY analogue of this notion is introduced in [16], where the notion of Lie conformal algebra is used instead of vertex Lie algebras. Below we give a definition using vertex Lie algebras.

3.1.1 $NW = N$ SUSY vertex Lie algebras

We begin with the introduction of $NW = N$ SUSY vertex Lie algebras. Recall that the notion of an even vertex Lie algebra is obtained by taking singular part (called polar part in [12, 16.1.2. Definition]) of a vertex algebra. For a linear superspace $V$ and a series $f(Z) = \sum_{(j|J)} Z^{j|J} v_{j|J} \in V[Z^{\pm 1}]$ of a 1|$N$-supervariable $Z = (z, \zeta^1, \ldots, \zeta^N)$, we denote

$$\text{Sing}(f) : = \sum_{(j|J), j < 0} Z^{j|J} v_{j|J} \in Z^{-1} V[Z^{-1}] \quad (3.1.1)$$

and call it the singular part of $f$, where $Z^{-1} V[Z^{-1}] : = \{ \sum_{(j|J), j < 0} Z^{j|J} v_{j|J} | v_{j|J} \in V \}$. 

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Definition 3.1.1  An $N_W = N$ SUSY vertex Lie algebra is a data $(V, T, S^i, Y_-)$ consisting of

- a linear superspace $V$,
- an even operator $T \in \mathbb{End}(V)_T$,
- $N$ odd operators $S^i \in \mathbb{End}(V)_T$ for $i \in [N]$, and
- an even linear map $Y_-(\cdot, Z) : V \to \mathbb{Hom}(V, Z^{-1} V[Z^{-1}])$,

which satisfies the following axioms.

1. Translation invariance: $Y_-(T a, Z) = \partial_z Y_-(a, Z)$, $Y_-(S^i a, Z) = \partial_{\xi^i} Y_-(a, Z)$.
2. Skew-symmetry: $Y_-(a, Z)b = \text{Sing}((-1)^{p(a)p(b)} e^{Z \nabla} Y(b, -Z) a)$ with $Z \nabla := -zT + \sum_{i=1}^N \xi^i S^i$.
3. Supercommutator: $[a_{(m|M)}, Y_-(b, Z)] = \text{Sing}(Y(e^{-Z \nabla} a_{(m|M)} e^{Z \nabla} b, Z))$, where $a_{(m|M)}$ denotes the Fourier mode of the expansion $Y_-(a, Z) = \sum_{(m|M)} a_{(m|M)}$.

We often abbreviate the word “vertex Lie algebra” to VLA.

Lemma 3.1.2  Given an $N_W = N$ SUSY VA $(V, [0], T, S^i, Y)$, we have an $N_W = N$ SUSY VLA $(V, T, S^i, Y_-)$ by setting $Y_- (a, Z) := \text{Sing}(Y(a, Z))$.

Proof  The axioms in Definition 3.1.1 are consequences of the translation invariance in Definition 2.2.1, Facts 2.2.8 and 2.2.9.

We also have the notion of a module $M$ over an $N_W = N$ SUSY VLA $V$, similar to SUSY VA modules. For $a \in V$ and $m \in M$, we denote the SUSY vertex Lie action of $V$ on $W$ by

$$Y^W_-(a, Z)m = \sum_{(j|J), j \geq 0} Z^{1-j|N\setminus J} a_{(j|J)}m \in M((Z)).$$

(3.1.2)

Our Definition 3.1.1 is equivalent to the notion of an $N_W = N$ SUSY Lie conformal algebra in [16, Definition 3.2.2]. Using the latter, we can describe the “Lie algebra structure” more explicitly. Let us recall Notation 2.2.2 of the $\Lambda$-bracket for $N_W = N$ case. In particular, $\mathcal{L}_W = \mathbb{C}[\Lambda] = \mathbb{C}[\lambda, \chi^i]$ is the commutative superalgebra freely generated by even $\lambda$ and odd $\chi^i$’s. Note that the $\Lambda$-bracket $[a_{\Lambda} b]$ only involves $(j|J)$-operators for $j \geq 0$, so that it can be defined also for an $N_W = N$ SUSY VLA. Now, let $\mathcal{L}'_W$ be another commutative superalgebra freely generated by even $\gamma$ and odd $\eta^i$’s, and define $[a_{\Gamma} b]$ similarly using $\Gamma^m|M := \gamma^m \eta^M$.

Fact 3.1.3  (c.f. [16, Sect. 3.2.1]) Let $V$ be an $N_W = N$ SUSY VLA. For $a, b, c \in V$ of pure parity, we have the following identity in $\mathcal{L}_W \otimes \mathcal{L}_W \otimes V$.

$$[a_{\Lambda} b_{\Gamma} c] = (-1)^{(p(a)+1)N} [a_{\Lambda} b_{\Gamma + \Lambda} c] + (-1)^{(p(a)+N)(p(b)+N)} [b_{\Gamma} [a_{\Lambda} c]].$$

We can regard it as a kind of Jacobi identity. In order to understand the signs, let us recall:
Definition 3.1.4 [16, Definition 3.2.5] A Lie superalgebra of parity \( q \in \mathbb{Z}_2 \) is a linear superspace \( V \) with a binary operation \([\cdot, \cdot]: V \otimes V \to V\) satisfying the following axioms for \( a, b, c \in V \) of pure parity.

1. Skew-symmetry: \([a, b] = -(−1)^{p(a)p(b)+q}[b, a]\).
2. Jacobi identity: \( [a, [b, c]] = -(−1)^{p(a)q+q}[a, [b, c]] + (−1)^{(p(a)+q)(p(b)+q)}[b, [a, c]]\).

We call \([\cdot, \cdot]\) the Lie bracket of parity \( q \). We also call \((V, [\cdot, \cdot])\) an even or odd Lie superalgebra according to \( q = 0 \) or \( 1 \).

Remark 3.1.5 Some comments are in order.

1. We slightly modify the terminology in [16] and use the phrase “of parity \( q \)” instead of “degree \( q \)” used therein.
2. An even Lie superalgebra is a Lie superalgebra in the standard sense.
3. We can find in Fact 3.1.3 that “the \( \Lambda = 0 \) part” of an NW = NSUSY VLA is a Lie superalgebra of parity \( N \mod 2 \). See [16, Lemma 3.2.7] for the precise statement.

3.1.2 \( N_W = N \) SUSY vertex Poisson algebras

Now we introduce \( N_W = N \) SUSY vertex Poisson structure.

Definition 3.1.6 (c.f. [16, Definition 3.3.16]) An \( N_W = N \) SUSY vertex Poisson algebra is a data \( (V, |0\rangle, T, S^i, Y_+, Y_-) \) consisting of

- a commutative \( N_W = N \) SUSY VA \((V, |0\rangle, T, S^i, Y_+)\),
- an \( N_W = N \) SUSY VLA \((V, T, S^i, Y_-)\)

such that the vertex Lie structure \( Y_- \) is a derivation for the commutative superalgebra structure \( ab = a_{(-1|N)}b \) coming from \( Y_+ \). More precisely, for \( a, b, c \in V \) of pure parity, we have

\[
Y_-(a, Z)(bc) = (Y_-(a, Z)b)c + (−1)^{(p(a)+N)p(b)}b(Y_-(a, Z)c). \tag{3.1.3}
\]

We often abbreviate the word “vertex Poisson algebra” to VPA.

Recalling Lemma 2.4.2 of commutative property, we denote the expansions of \( Y_+ \) and \( Y_- \) by

\[
Y_+(a, Z) = \sum_{(j|J), j < 0} Z^{-1-j|1-J}a_{(j|J)}, \quad Y_-(a, Z) = \sum_{(j|J), j \geq 0} Z^{-1-j|1-J}a_{(j|J)}. \tag{3.1.4}
\]

We also have the notion of a module over an \( N_W = N \) SUSY VPA \( V \), which is a combination of the module structure over the commutative superalgebra associated with \((V, |0\rangle, T, S^i, Y_+)\), and the module structure over the \( N_W = N \) SUSY VLA \((V, T, S^i, Y_-)\).

As for the derivation axiom (3.1.3), let us recall:
Definition 3.1.7 An Poisson superalgebra of parity $q \in \mathbb{Z}_2$ is a commutative superalgebra $P$ together with a Lie bracket $\{\cdot, \cdot\}: P \otimes P \rightarrow P$ of parity $q$ in the sense of Definition 3.1.4 such that
\[
\{a, bc\} = \{a, b\}c + (-1)^{(p(a)+q)p(b)}a\{b, c\}
\]
for $a, b, c \in P$ of pure parity. $\{\cdot, \cdot\}$ is called the Poisson bracket of parity $q$. We also call $(V, \{\cdot, \cdot\})$ an even or odd Poisson superalgebra according to $q = 0$ or $1$.

Remark 3.1.8 Some comments are in order.
(1) An even Poisson superalgebra is a Poisson superalgebra in the standard sense.
(2) The structure of an odd Poisson superalgebra is a part of the Gerstenhaber algebra structure.
(3) By Definition 3.1.7 and Remark 3.1.5 ((3)), we see that “the $\Lambda = 0$ part” of an $NW = N$ SUSY VPA is a Poisson superalgebra of parity $N$ mod 2.

3.1.3 $NW = N$ SUSY VPA structure on superjet algebra

Let us give a basic example of an $NW = N$ SUSY VPA. Recall the $1|N$-superjet algebra $A^O = Ak[[Z]] = S^N(A_\infty)$ of a commutative superalgebra $A$ (Proposition 1.3.5). It has a structure of commutative $NW = N$ SUSY VA (Proposition 2.4.5). If $A$ is moreover a Poisson superalgebra of parity $N$ mod 2 in the sense of Definition 3.1.7, then this structure has an enhancement to vertex Poisson structure.

Proposition 3.1.9 Let $(P, \{\cdot, \cdot\})$ be a Poisson superalgebra of parity $N$ mod 2, and $(P^O, T, S^i)$ be the commutative $NW = N$ SUSY VA. Then $P^O$ has an $NW = N$ SUSY VPA structure such that
\[
u_{(m|M)}u = \begin{cases} 
\{u, v\} & (m = 0, M = [N]) \\
0 & \text{(otherwise)}
\end{cases}
\]
for $u, v \in P \subset P^O$, where we used the expansion (3.1.4) of $Y_\theta$. We call is the level 0 SUSY VPA structure.

Remark 3.1.10 Our naming comes from the level 0 VPA structure [2, Proposition 2.3.1] of the jet algebra $P_\infty$ (see (1.3.1)) in the even case.

For the proof, we need the following $NW = N$ SUSY analogue of [21, Theorem 3.6, Proposition 3.10].

Lemma 3.1.11 Let $A$ be a unital commutative superalgebra equipped with an even derivation $T$ and odd derivations $S^i$ for $i \in [N]$. Also, let $B \subset A$ be a linear sub-superspace generating the differential algebra $(A, T, S^i)$ in the sense of Definition 1.3.8.
Assume that there is a morphism of linear superspaces \( Y_- (\cdot, Z) : A \to Z^{-1}(\text{Der} A)[Z] \) satisfying

(i) \( Y_- (a, Z) a' \in Z^{-1} A[Z^{-1}] \),
(ii) \( Y_- (1, Z) = 0 \),
(iii) \( Y_- (Ta, Z) = \partial_z Y_- (a, Z) \), \( Y_- (S^i a, Z) = \partial_{z^i} Y_- (a, Z) \), and
(iv) \( [T, Y_- (a, Z)] = \partial_z Y_- (a, Z) \), \([S^i, Y_- (a, Z)] = \partial_{z^i} Y_- (a, Z)\)

for \( a, a' \in A \) and \( i \in [N] \). Assume moreover that for \( a \in A \) and \( b \in B \) of pure parity, we have

\[
Y_- (a, Z)b = \text{Sing} \left( (-1)^{p(a)p(b)} e^{Z^\nabla} Y_- (b, -Z)a \right), \tag{3.1.6}
\]

where \( \text{Sing} \) is given in (3.1.1), and \( Z^\nabla := zT + \sum_{i=1}^N \xi^i S^i \). Then \( Y_- \) gives an \( N_W = N \) VPA structure on \( A \).

(2) Assume that there is a morphism of linear superspaces \( Y^0_- (\cdot, Z) : B \to Z^{-1}(\text{Hom}(B, A))[Z] \) satisfying

\[
Y^0_- (b, Z)b' = \text{Sing} \left( (-1)^{p(b)p(b')} e^{Z^\nabla} Y_- (b', -Z)b \right) \in Z^{-1} A[Z^{-1}] \tag{3.1.7}
\]

for \( b, b' \in B \) of pure parity. Then \( Y^0_- \) extends uniquely to \( Y_- (\cdot, Z) : A \to Z^{-1}(\text{Der} A)[Z] \) satisfying the conditions (i)–(iv) and (3.1.6) in ((1)).

**Proof** The arguments [21, Theorem 3.6, Proposition 3.10] work with little modification. We omit the details. \( \square \)

**Proof of Proposition 3.1.9** Apply Lemma 3.1.11 ((2)) to \( B = \mathcal{P}, A = P^O \) and \( Y^0_- (b, Z) : = b_{(0)}^{(0)} b' : = b_{(b)}^{(b')} \) for \( b, b' \in \mathcal{P} \), which obviously satisfies the condition (3.1.7). Then, we have an extension \( Y_- \) of \( Y^0_- \) on \( P^O \), which gives the desired VPA structure by Lemma 3.1.11 ((1)). \( \square \)

**Remark 3.1.12** The above proof lacks explicit formulas of \( a_{(m)}(M)b \) for \( a, b \in P^O \). For a comparison with the even case [2, Proposition 2.3.1] mentioned in Remark 3.1.10, we explain how to obtain concrete formulas.

Let us consider \( u_{(m)}(M)a \) with \( u \in \mathcal{P}, a \in P^O, m \geq 0 \) and \( M \subset [N] \). Note that every \( a \in P^O \) is written as a polynomial of \( S^L T^{(l)} v \) with \( L \subset [N], l \in \mathbb{N} \) and \( v \in \mathcal{P} \), where \( S^L := S^{l_1} S^{l_2} \cdots \) for \( L = \{ l_1, l_2, \ldots \} \) and \( T^{(l)} := \frac{1}{l!} T^l \). Hence, by the Leibniz rule (3.1.3), it is enough to determine \( u_{(m)}(M)(S^L T^{(l)} v) \) for \( u, v \in \mathcal{P} \) of pure parity. We determine them by the condition (iv) in Lemma 3.1.11, which is equivalent to (recall (2.2.6))

\[
[S^i, u_{(m)}(M)] = \begin{cases} 
\sigma(N \setminus M, e_m) a((m \setminus M) \varepsilon_i) & (i \in M) \\
0 & (i \notin M)
\end{cases}, \quad [T, u_{(m)}(M)] = -mu_{(m-1)}(M)
\]

with the convention \( u_{(-1)}(M) := 0 \). These are recursions for the desired \( u_{(m)}(M)(S^L T^{(l)} v) \), and using (3.1.5) as the initial condition, we can solve them to obtain:

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• In case \( l \geq m \), we set
\[
\begin{align*}
  u_{(m|N)}(S^l T^{(l)} u) &:= (-1)^{|L_p|} S^l T^{(l-m)} \{ u, v \}, \\
  u_{(m|N\setminus L)}(S^l T^{(l)} u) &:= (-1)^{|L_p|+|L|/2} \left( \prod_{i=1}^{|L|} \sigma(\{ l_i, l_i+2, \ldots \}, e_i) \right) T^{(l-m+|l|)} \{ u, v \},
\end{align*}
\] (3.1.8)
and otherwise \( u_{(m|M)}(T^{(l)} S^l v) := 0 \).

• In case \( l < m \), we set \( u_{(m|M)}(T^{(l)} S^l v) := 0 \).

We can check directly that these formulas satisfy the condition (iii) in Lemma 3.1.11. Let us also mention that the case \( L = \emptyset \) in (3.1.8) is analogous to the even case formula \( u_{(m)}(T^l v) = \frac{|l|}{(l-m)!} T^{l-m} \{ u, v \} \) for \( l \geq m \) in [2, Proposition 2.3.1, (10)].

**Remark 3.1.13** Let us sketch another description of the level 0 SUSY VPA structure, following [23, Lemma 3.1], where the level 0 VPA structure in the even case is treated in the language of Coisson algebra [7, Sect. 2.6]. Although it is not shown explicitly in [23] that the Coisson structure coincides with the level 0 VPA structure in [2], one can check it by writing down the Coisson product. The following argument has some overlap with [16, Remark 3.2.3], from which we borrow some symbols.

Let \( \mathcal{H}_W \) be the commutative superalgebra over the base field \( k \) generated by an even variable \( \partial \) and odd variables \( \delta^i \) for \( i \in \{ 1, \ldots, N \} \). We regard \( \mathcal{H}_W \) as a Hopf superalgebra with comultiplication \( \Delta(\partial) := \partial \otimes 1 + 1 \otimes \partial \), \( \Delta(\delta^i) := \delta^i \otimes 1 + 1 \otimes \delta^i \) and counit \( \epsilon(\partial) = \epsilon(\delta^i) = 0 \). On the category \( M \) of right \( \mathcal{H}_W \)-modules, we can define the \( \ast \)-pseudo-tensor structure in a similar way as the even case [7], [23, Sect. 3.1]. A Lie algebra object \( L \) in the corresponding pseudo-tensor category \( M^\ast \) is called a Lie\( ^\ast \) algebra. The Lie\( ^\ast \)-bracket \( [ \cdot, \cdot ] \) is an element of \( \text{Hom}_{\mathcal{H}_W}(L \otimes_k L, L \otimes_{\mathcal{H}_W} \mathcal{H}_W^2) \), where \( \mathcal{H}_W^2 \) denotes the tensor product algebra \( \mathcal{H}_W \otimes_k \mathcal{H}_W \), and we regard it as a right \( \mathcal{H}_W \)-module by \( \Delta \). Identifying \( L \otimes \mathcal{H}_W \mathcal{H}_W^2 \) with \( L[\partial_1, \delta^i_1] \), where \( \partial_1 := \partial \otimes 1 \) and \( \delta^i_1 := \delta^i \otimes 1 \), we can write \( [ \cdot, \cdot ] \) as \( [a, b] = \sum_{(j|j'), j \geq 0} \sigma(a_{(j|j')}b) \delta^i_{1(j|j')} \) for some \( a_{(j|j')}b \in L \). As in the even case, such a Lie\( ^\ast \) algebra is equivalent to our \( N_W = N \) SUSY VLA, and also to an \( N_W = N \) SUSY Lie conformal algebra in [16].

The category \( M \) has a tensor structure \( \otimes^\ast \) coming from the comultiplication \( \Delta \) on \( \mathcal{H}_W \), and the corresponding tensor category is denoted by \( M^\ast \). A commutative algebra object in \( M^\ast \) is called a commutative\( ^\ast \) algebra. A Coisson algebra \( C \) is a Lie\( ^\ast \) algebra and a commutative\( ^\ast \) algebra such that the multiplication \( C \otimes C \to C \) is a Lie\( ^\ast \) algebra morphism.

Now we can restate Proposition 3.1.9. Let \( \mathcal{H}_W \) be a Poisson superalgebra of parity \( N \) mod 2. Note that \( P^O \cong P \otimes_k \mathcal{H}_W \) with operators \( T \) and \( S^i \) corresponding to \( \partial \) and \( \delta^i \), respectively. The statement is that \( P \otimes_k \mathcal{H}_W \) has a Coisson structure whose Lie\( ^\ast \) bracket \( \{ \cdot, \cdot \} \) is given by \( \{ u \delta^i \delta^L, v \delta^m \delta^M \} = \{ u, v \} \delta^i_1 \delta^1_1 \delta^m_1 \delta^M_1 \) for \( u, v \in P, l, m \in \mathbb{N} \) and \( L, M \subseteq \{ 1, \ldots, N \} \), where \( \delta_2 := 1 \otimes \partial \) and \( \delta^i_2 := 1 \otimes \delta^i \).

### 3.1.4 \( N_k = N \) SUSY vertex Poisson algebras

Here we give an \( N_k = N \) analogue of Sects. 3.1.1 and 3.1.2. We begin with:
Definition 3.1.14 An $N_K = N$ SUSY vertex Lie algebra is a data $(V, S^i, Y_-)$ consisting of

- a linear superspace $V$,
- $N$ odd operators $S^i \in \text{End}(V)_\Gamma$ for $i \in [N]$, and
- an even linear map $Y_-(\cdot, Z) : V \to \text{Hom}(V, Z^{-1}V[Z^{-1}])$, which satisfies the following axioms.

1. Translation invariance: $Y_-(S^i a, Z) = D^i_2 Y_-(a, Z)$.
2. Skew-symmetry: $Y_-(a, Z)b = \text{Sing}\left((-1)^{p(a)p(b)} e^{Z\nabla} Y(b, -Z^a)\right)$ with $Z\nabla := zT + \sum_{i=1}^{N} \xi^i S^i$, $T := (S^1)^2 = \cdots = (S^N)^2$.
3. Supercommutator: $[a_{(m|M)}, Y_-(b, Z)] = \text{Sing}\left(Y(e^{-Z\nabla} a_{(m|M)} e^{Z\nabla} b, Z)\right)$, where $a_{(m|M)}$ denotes the Fourier mode of the expansion $Y_-(a, Z) = \sum_{(m|M), m \geq 0} Z^{-m+1}\chi|_{M} a_{(m|M)}$.

We often abbreviate the word “vertex Lie algebra” to VLA.

As in Lemma 3.1.2, we have:

Lemma 3.1.15 Given an $N_K = N$ SUSY VA $(V, |0\rangle, S^i, Y)$, we have an $N_K = N$ SUSY VLA $(V, S^i, Y_-)$ by setting $Y_-(a, Z) := \text{Sing}(Y(a, Z))$.

Our Definition 3.1.14 is equivalent to the notion of an $N_K = N$ SUSY Lie conformal algebra in [16, Definition 4.10]. For later reference, we record the Jacobi identity. Recall Notation 2.3.3 of the $\Lambda$-bracket. In particular, $\mathcal{L}$ denotes the superalgebra generated by even $\lambda$ and odd $\chi$. Note that the $\Lambda$-bracket makes sense for an $N_K = N$ SUSY VLA $V = (V, S^i, Y_-)$, we have

$$[a_{\Lambda} b] = \sum_{(m|M), m \geq 0} \sigma(M, N\setminus M)(-1)^{|M|+1} \frac{1}{m!} \Lambda_{m|M} a_{(m|M)} b.$$ 

for $a, b \in V$. Also, let $\mathcal{L}'$ be the superalgebra generated by even $\gamma$ and odd $\eta^i$ subject to the relation $[\gamma, \eta^i] = 0$ and $[\eta^i, \eta^j] = -2\gamma \delta_{i,j}$, and define $[a_{\Gamma} b]$ similarly using $\Gamma_{m|M} := \text{sgn}(m) \eta^M$. 

Fact 3.1.16 (c.f. [16, Definition 4.10]) Let $V$ be an $N_K = N$ SUSY VLA. For $a, b, c \in V$ of pure parity, we have the following identity in $\mathcal{L} \otimes \mathcal{L}' \otimes V$.

$$[a_{\Lambda} [b_{\Gamma} c]] = (-1)^{(p(a)+1)N} [[a_{\Lambda} b]_{\Gamma+\Lambda} c] + (-1)^{(p(a)+N)(p(b)+N)} [b_{\Gamma} [a_{\Lambda} c]].$$

Definition 3.1.17 (c.f. [16, Sect. 4.10]) An $N_K = N$ SUSY vertex Poisson algebra $(N_K = N$ SUSY VPA for short) is a data $(V, |0\rangle, S^i_K, Y_+, Y_-)$ consisting of

- a commutative $N_K = N$ SUSY VA $(V, |0\rangle, S^i_K, Y_+)$,
- an $N_K = N$ SUSY VLA $(V, S^i_K, Y_-)$

such that the vertex Lie structure $Y_-$ is a derivation for the commutative superalgebra structure $ab = a_{(-1)^{|n|}b}$ coming from $Y_+$ (see (3.1.3)).

We also have the notion of a module $M$ over an $N_K = N$ SUSY VPA $V$. The VLA action of $V$ on $M$ is denoted as (3.1.2).
3.1.5 $N_K = N$ SUSY VPA structure on superconformal jet algebra

We have an $N_K = N$ analogue of Sect. 3.1.3. Recall the 1$|N$-superconformal jet algebra

$$A^{O_{\infty}} = (A^O, S^i_K) = (\Omega_{A_{\infty}}, S^i_K),$$

for a commutative superalgebra $A$ (Definition 1.3.14). It has a structure of a commutative $N_K = N$ SUSY VA (Proposition 2.4.6), and if $A$ is moreover a Poisson superalgebra, then this structure can be enhanced to a vertex Poisson structure.

**Proposition 3.1.18** Let $(P, \{\cdot, \cdot\})$ be a Poisson superalgebra of parity $N \mod 2$, and $P^{O_{\infty}} = (P^O, S^i_K)$ be the superconformal jet algebra of $P$. Then $P^{O_{\infty}}$ has an $N_K = N$ SUSY VPA structure such that

$$u_{(m|M)}v = \begin{cases} [u, v] & (m = 0, M = [N]) \\ 0 & \text{(otherwise)} \end{cases}$$

for $u, v \in P \subset P^O$, where we used the expansion (3.1.4) of $Y_{\cdot}$. We call is the level 0 SUSY VPA structure.

The proof is similar to Proposition 3.1.9, using the following lemma.

**Lemma 3.1.19** Let $A$ be a unital commutative superalgebra equipped with odd derivations $S^i_K$ for $i \in [N]$ subject to the commutation relation (c.f. Lemma 1.3.13)

$$[S^i_K, S^j_K] = 2T \delta_{i,j}, \quad [S^i_K, T] = 0.$$ 

Also, let $B \subset A$ be a sub-superspace generating the differential algebra $(A, S^i_K)$ (see Definition 1.3.8).

1. Assume that there is a morphism of linear superspaces $Y_{\cdot}(\cdot, Z): A \to Z^{-1}(\text{Der} A)[[Z]]$ satisfying
   (i) $Y_{\cdot}(a, Z)a' \in Z^{-1}A[Z^{-1}]$, 
   (ii) $Y_{\cdot}(1, Z) = 0$,
   (iii) $Y_{\cdot}(S^i_K a, Z) = D^i_Z Y_{\cdot}(a, Z)$,
   (iv) $[S^i_K, Y_{\cdot}(a, Z)] = D^i_Z Y_{\cdot}(a, Z)$

for $a, a' \in A$ and $i \in [N]$. Further assume that for $a \in A$ and $b \in B$ of pure parity, we have

$$Y_{\cdot}(a, Z)b = \text{Sing}\left((-1)^{p(a)p(b)}e^{Z\nabla}Y_{\cdot}(b, -Z) a\right),$$

where Sing is given in (3.1.1), and $Z\nabla := zT + \sum_{i=1}^N \xi^i S^i_K$ as in Definition 3.1.14. Then, $Y_{\cdot}$ gives an $N_K = N$ VPA structure on $A$. 

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Assume that there is a morphism of linear superspaces \( Y^- : B \rightarrow Z^{-1}(\text{Hom}(B, A))[Z] \) satisfying
\[
Y^- (b, Z)b' = \text{Sing}\left( (-1)^{p(b)p(b')} e^{Z\nabla} Y^-(b', -Z)b \right) \in Z^{-1}A[Z^{-1}] \quad (3.1.12)
\]
for \( b, b' \in B \) of pure parity. Then \( Y^- \) extends uniquely to \( Y^0 : A \rightarrow Z^{-1}(\text{Der } A)[Z] \) satisfying the conditions (i)–(iv) and (3.1.11) in ((1)).

**Remark 3.1.20** As in the \( N_W = N \) case (Remark 3.1.12), we have an explicit form of the VPA structure. We will describe \( u_{(m|M)}(S_K^L T^L v) \) for \( u, v \in P \) of pure parity. The condition (iv) in Lemma 3.1.19 is equivalent to (recall (2.3.2) and (2.3.3))
\[
[S_K^l, u_{(m|M)}] = \begin{cases} \sigma(N \setminus M, e_m) a_{(m|M\setminus e_i)} & (i \in M) \\ -\sigma(N \setminus (M \cup e_i), e_m) m a_{(m-1|M\cup e_i)} & (i \notin M) \end{cases}
\]
and using (3.1.10) as the initial condition, we can solve them to obtain:

- In case \( l \geq m \), we set
  \[
u_{(m|M)}(S_K^L T^L v) = (-1)^{|L|p(u)} S_K^L T^L \{u, v\},
  \]
  \[
u_{(m|M)}(S_K^L T^L v) = (-1)^{|L|p(u)+|L|/2} (\prod_{i=1}^{[L]} \sigma ([l_i+2, l_i+2, \ldots, e_i]) T^{[l-m+|L|/2]} \{u, v\},
  \]

- In case \( l < m \), we set \( u_{(m|M)}(T^L S_K^L v) = 0 \).

**Remark 3.1.21** We also have another description of the level 0 SUSY VPA structure via chiral algebra analogous to Remark 3.1.13. Instead of using the commutative superalgebra \( \mathcal{H}_W = k[\partial, \delta^l] \), we use the non-commutative superalgebra \( \mathcal{H}_K = k \langle \delta^l, \partial^l \rangle \) generated by odd variables \( \delta^l \) subject to the relation \( [\delta^l, \delta^j] = [\partial^l, \partial^j] \) and \( [\delta^l, \partial^j] = 0 \) for \( i \neq j \). Let us denote \( \partial = \frac{1}{2}[\delta^l, \delta^j] \). We regard \( \mathcal{H}_K \) as a Hopf superalgebra, similarly as \( \mathcal{H}_W \). The category \( \mathcal{M}_K \) of right \( \mathcal{H}_K \)-modules has the *-pseudo-tensor structure, and the Lie*-bracket \( [, \cdot] \) on an object \( L \in \mathcal{M}_K \) is an element of \( \text{Hom}_K(\mathcal{H}_K^2(L \otimes_k L, L \otimes_k \mathcal{H}_K^2)) \). Under the identification \( L \otimes_k \mathcal{H}_K^2 \cong L \langle \delta^l \rangle \) with \( \delta^l = \delta^l \otimes 1 \), we can write \( [, \cdot] \) as \( [a, b] = \sum_{(j|j), j \geq 0} (a_{(j|j)} b) \partial^l \langle \delta^l \rangle \) for some \( a_{(j|j)} b \in L \). A Lie*-algebra is equivalent to an \( N_K = N \) SUSY VLA, and also to an \( N_K = N \) SUSY Lie conformal algebra in [16]. The equivalence is given by \( a_{(2|j)} b = a_{(j|2)} b \).

The category \( \mathcal{M}_K \) has a tensor structure \( \otimes^l \) coming from the comultiplication \( \Delta \) on \( \mathcal{H}_K \) and has a compound tensor structure \( (\otimes^l, \otimes^1) \). As a result, we have the notion of a Coisson algebra on \( \mathcal{M}_K \).
Now we can restate Proposition 3.1.18. Let \((P, \{\cdot, \cdot\})\) be an odd Poisson superalgebra. Note that \(P \cong P \otimes_k \mathcal{H}_K\) with the odd operators \(S_K^i\) corresponding to \(\partial_K^i\). The statement is that \(P \otimes_k \mathcal{H}_K\) has a Coisson structure whose Lie* bracket \(\{\cdot, \cdot\}\) is given by
\[\{u \partial_l^i \partial^m, v \partial_m^l \partial^M\} = \{u, v\} \partial_l^i \partial^m \partial^K_{(l)} \partial^{(M)}_{(k)} \partial^m \partial^l \partial_K \partial^K_{(l)} \partial^{(M)}_{(k)} \] for \(u, v \in P, l, m \in \mathbb{N}\) and \(L, M \subset [N]\), where \(\partial_K^i := 1 \otimes \partial_K^i\).

### 3.2 Li filtration of \(NW = N\) SUSY vertex algebra

In this and the next subsections, we introduce a SUSY analogue of Li’s canonical filtration [22, Definition 2.7]. Let us give a brief recollection of the even case. Let \(V\) be an even vertex algebra, and \(M\) be a \(V\)-module. Then, the Li filtration of \(M\) is a decreasing sequence of subspaces
\[M = E_0(M) \supset E_1(M) \supset \cdots \supset E_n(M) \supset \cdots\]
which consists of
\[E_n(M) := \text{span} \left\{ a_1^{(i_1)} \cdots a_r^{(i_r)} | r \in \mathbb{Z}_{>0}, a_i^j \in V, m \in M, k_i \in \mathbb{N} \right\} \] satisfying \(k_1 + \cdots + k_r \geq n\).

(3.2.1)

Our strategy for SUSY case is to discard the odd index \(J\) of \((j|J)\)-operators and to apply Li’s arguments in [22]. Hence, some of the arguments in loc. cit. work as they are, but some do not. The points where Li’s argument need to be modified are the ones using Borcherds’ commutator formula (2.2.21) and iterate formula (2.2.23).

Hereafter until the end of this subsection, \(V = (V, |0\rangle, T, S^i, Y)\) denotes an \(NW = N\) SUSY VA.

The construction of Li filtration in the even case [22] starts with an argument [22, Proposition 2.6] on a general decreasing filtration of a vertex algebra whose associated graded space has a natural structure of a vertex Poisson algebra. The following statement is an \(NW = N\) SUSY analogue of [22, Proposition 2.6].

**Proposition 3.2.1** Let \(V = E_0 \supset E_1 \supset \cdots \supset E_n \supset \cdots\) be a decreasing filtration of linear sub-superspaces of \(V\) such that \(|0\rangle \in E_0\) and
\[a_{(j|J)} b \in E_{r+s-j-1} \quad (a \in E_r, b \in E_s, j \in \mathbb{Z}, J \subset [N]),\]
where we used the convention \(E_n := V\) for \(n \in \mathbb{Z}_{<0}\).

(1) The associated graded linear superspace
\[\text{gr}_E V := \bigoplus_{n \in \mathbb{N}} E_n / E_{n+1}\]
is an \(NW = N\) SUSY VA whose state-superfield correspondence is given by
\[(a + E_{r+1})_{(j|J)} (b + E_{s+1}) := a_{(j|J)} b + E_{r+s-j}\]
(3.2.3)
for \( a \in E_r, b \in E_s, j \in \mathbb{Z} \) and \( J \subset [N] \), whose vacuum is given by \( |0\rangle + E_1 \in E_0/E_1 \), and whose even operator \( \partial \) and odd operators \( \partial^i \) are given by

\[
\partial(a + E_{r+1}) = Ta + E_{r+2}, \quad \partial^i(a + E_{r+1}) = S^i a + E_{r+2}.
\]

(2) The \( N_W = N \text{ SUSY VA} \text{ gr}_E V \) in ((1)) is commutative if and only if

\[
a_{(m|M)} b \in E_{r+s-m}
\]

(3.2.4)

for all \( a \in E_r, b \in E_s, m \in \mathbb{N} \) and \( M \subset [N] \).

(3) Under the condition (3.2.4), the commutative \( N_W = N \text{ SUSY VA} \text{ gr}_E V \) has an \( N_W = N \text{ SUSY VLA} \text{ for gr}_E V \) is then straightforward. It remains to check the derivation axiom (3.1.3), which is equivalent to

\[
a_{(l|L)}(b_{(−1|N)}c) + E_{r+s+t−l+1} = (a_{(l|L)}b_{(−1|N)}c) + (-1)^{(p(a)+|L|)}p(b) b_{(−1|N)}(a_{(l|L)}c) + E_{r+s+t−l+1}
\]

(3.2.5)

for \( l \in \mathbb{N}, L \subset [N], a \in E_r, b \in E_s \) and \( c \in E_{t} \). The commutator formula (2.2.22) yields

\[
a_{(l|L)}(b_{(−1|N)}c) − (-1)^{(p(a)+|L|)}p(b) b_{(−1|N)}(a_{(l|L)}c)
\]

\[= (a_{(l|L)}b_{(−1|N)}c) + \sum_{j=0}^{l-1} \pm \left(\begin{array}{c}l \\ j \end{array}\right) (a_{(j|N)}b_{(l−1−j|N)}c),
\]

where \( \pm \) denotes some sign. For each term of this equality, the condition (3.2.4) yields

\[a_{(l|L)}(b_{(−1|N)}c), b_{(−1|N)}(a_{(l|L)}c), (a_{(l|L)}b_{(−1|N)}c) \in E_{r+s+t−l},
\]

and since \( l − 1 − j \geq 0 \) in the summation, it also yields

\[(a_{(j|N)}b_{(l−1−j)}c) \in E_{(r+s−j)+t−(l−1−j)} = E_{r+s+t−l+1}.
\]

Hence, we have the desired equality (3.2.6).
Now, mimicking the even Li filtration (3.2.1) with the strategy of “discarding the odd index $J$ of the $(j|J)$-operator”, we introduce:

**Definition 3.2.2** For a $V$-module $M$ and $n \in \mathbb{Z}$, we define a linear sub-superspace $E_n(M) \subset M$ by

$$E_n(M) := \text{span} \left\{ a_1^{i_1} \cdots a_r^{i_r} m \mid r \in \mathbb{Z}_{>0}, a_i \in V, m \in M, i_j \in \mathbb{N}, K_i \subset [N] \text{ satisfying } k_1 + \cdots + k_r \geq n \right\}.$$

**Remark 3.2.3** (After the referee’s comments) This definition can also be obtained by viewing $V$ as an ordinary vertex superalgebra, taking expansions in ordinary fields instead of superfields, and applying the Li filtration of the vertex superalgebra. Although the Li filtration of a vertex superalgebra is not written down in [22], the definition is a straightforward translation of the even case (3.2.1).

Below we show that ${\{E_n(M) \mid n \in \mathbb{N}\}}$ satisfies the conditions in Proposition 3.2.1, so that for $M = V$, the associated graded space $\text{gr}_E V$ has a structure of $NW = N_{SUSY} V_{PA}$. Our argument follows that of [22, Lemma 2.8–Proposition 2.11].

The statements in the next lemma are analogue of [22, Lemmas 2.8, 2.9].

**Lemma 3.2.4** For a $V$-module $M$, we have the following.

1. $E_n(M) \supset E_{n+1}(M)$ for any $n \in \mathbb{Z}$.
2. $E_n(M) = M$ for any $n \in \mathbb{Z}_{\leq 0}$
3. $a_{(-1-k|K)} E_n(M) \subset E_{n+k}(M)$ for any $a \in V, k \in \mathbb{N}, K \subset [N]$ and $n \in \mathbb{Z}$.
4. For $n \geq 1$, $E_n(M)$ is equal to

$$E'_n(M) := \text{span} \left\{ a_{(-1-k|K)} m \mid a \in V, k \in \mathbb{Z}_{>0}, K \subset [N], m \in E_{n-k}(M) \right\}. \quad (3.2.7)$$

5. For $n \geq 1$, $E_n(M)$ is equal to

$$E''_n(M) := \text{span} \left\{ a_1^{i_1} \cdots a_r^{i_r} m \mid r \in \mathbb{Z}_{>0}, a_i \in V, m \in M, i_j \in \mathbb{N}, K_i \subset [N] \text{ satisfying } k_1 + \cdots + k_r \geq n \right\}. \quad (3.2.8)$$

Note that the difference with Definition 3.2.2 is the condition $k_i \geq 1$.

**Proof** The items ((1))–((3)) are immediate consequence of Definition 3.2.2, and we omit the proof. For the rest, we have by induction on $n$ that

$$E'_n(M) = E''_n(M). \quad (3.2.9)$$

So it is enough to prove ((4)) only. For that, we show that each element

$$u = a_1^{i_1} \cdots a_r^{i_r} m \in E_n(M)$$

satisfies

$$k_1 + \cdots + k_r \geq n.$$
belongs to $E'_n(W)$ by induction on the length $r$, using the $N_W = N$ SUSY commutator formula (2.2.22).

- If $r = 1$, then $u = a^{1}_{(-1-k_1|K_1)}m$ with $k_1 \geq n \geq 1$ and $m \in M = E_{n-k_1}(M)$. Thus, we have $u \in E'_n(M)$ by definition.
- Next, assume $r \geq 2$. We set

$$u = a^{1}_{(-1-k_1|K_1)}u', \quad u' : = a^{2}_{(-1-k_2|K_2)} \cdots a^{r}_{(-1-k_r|K_r)}m.$$ 

If $k_1 \geq 1$, then $u' \in E_{n-k_1}(M)$, and we have $u = a^{1}_{(-1-k_1|K_1)}u' \in E'_n(M)$ as desired. Hereafter we assume $k_1 = 0$. Then, $k_2 + \cdots + k_r \geq n$ and $u' \in E'_n(M)$, which yields $u' \in E'_n(M)$ by the induction hypothesis. Now the equality (3.2.9) yields $u' \in E''_n(M)$. Hence, we may assume $k_2, \ldots, k_r \geq 1$. Let us rewrite $a : = a^1$, $b : = a^2$ and $k : = k_2$, so that we have

$$u = a_{(-1|K_1)}b_{(-1-k_2|K_2)}u'' , \quad u'' : = a^3_{(-1-k_3|K_3)} \cdots a^{r}_{(-1-k_r|K_r)}m \in E_{n-k}(M).$$

The commutator formula (2.2.22) yields

$$u = \pm b_{(-1-k|K_2)}a_{(-1|K_1)}u'' + \sum_{(j|J), \ j \geq 0, J \supset K_1 \cap K_2} (\text{cst.}) \cdot (a_{(j|J)}b_{(-2-k-j|K)})u'' ,$$

where $\pm$ denotes a sign, $(\text{cst.})$ denotes a constant and $K = K_2 \cup (K_1 \ \backslash \ J)$. Since $u'' \in E_{n-k}(M)$, we have $a_{(-1|K_1)}u'' \in E_{n-k}(M)$ by (3), and the condition $k \geq 1$ yields

$$b_{(-1-k|K_2)}a_{(-1|K_1)}u'' \in E'_n(M).$$

Also, $u'' \in E_{n-k}(M) \subset E_{n-(k+j+1)}(M)$ for $j \geq 0$ by (1), and we have

$$(a_{(j|J)}b_{(-2-k-j|K)})u'' \in E'_n(M).$$

Hence we have $u \in E'_n(M)$.

\[ \square \]

The next lemma is an analogue of [22, Lemma 2.10].

**Lemma 3.2.5** Let $M$ be a $V$-module. For $a \in V$, $l \in \mathbb{Z}$ and $L \subset [N]$, we have

$$a_{(l|L)}E_n(M) \subset E_{n-l-1}(M). \quad (3.2.10)$$

If moreover $l \geq 0$, then we have

$$a_{(l|L)}E_n(M) \subset E_{n-l}(M). \quad (3.2.11)$$
Lemma 3.2.6

Let $M$ be a $V$-module. For $u \in M$, we have

$$a(\xi)A_{-1}b_{-1} \in E_{n-l-1}(M),$$

with $J' := K \cup (L \setminus J)$. As for the first term, by $n - k < n$, $l \geq 0$ and the induction hypothesis, we have $a(\xi)A_{-1} \in E_{n-k-l}(M)$. Then, since $-1 - k < 0$ and by the already-proved (3.2.10), we have

$$b_{-1}a(\xi)A_{-1} \in E_{n-k-l-1}(M) = E_{n-l}(M).$$

As for the terms in the summation, by (3.2.10) and $j \geq 0$, we have

$$(a(\xi)A_{-1}b_{-1})w \in E_{n-k-l-1}(M) = E_{n-l+j}(M) \subseteq E_{n-l}(M).$$

Hence, we have $a(\xi)A_{-1}b_{-1} \in E_{n-l}(M)$. \hfill $\square$

We also have an analogue of [22, Proposition 2.11]. The proof is again similar to loc. cit., but since the argument uses the iterate formula (Lemma 2.2.10), we write down it.

Lemma 3.2.6 Let $M$ be a $V$-module. For $u \in E_r(V), m \in E_s(M), l \in \mathbb{Z}$ and $L \subset [N]$, we have

$$u(\xi)L_m \in E_{r+s-l-1}(M).$$

(3.2.12)

If moreover $l \geq 0$, then we have

$$u(\xi)L_m \in E_{r+s-l}(M).$$

(3.2.13)

Proof We first consider the case $r \leq 0$. By Lemma 3.2.5, we have $u(\xi)L_m \in E_{s-l-1}(M) \subseteq E_{r+s-l-1}(M)$ for any $l \in \mathbb{Z}$, and $u(\xi)L_m \in E_{s-l}(M) \subseteq E_{r+s-l}(M)$ for $l \geq 0$. Thus, we have the conclusions.

Second, we show the case $r \geq 0$ by induction on $r$. Assume $u \in E_{r+1}(V)$. Then, by Lemma 3.2.4 ((4)), we can write $u = a(-2-i)A_{i}b$ with some $a \in V, 0 \leq i \leq r$, $I \subset [N]$ and $b \in E_{r+1}(V)$. We calculate $(a(-2-i)A_{i}b)(\xi)L_m$ using the iterate formula in Lemma 2.2.10. It yields

$$(a(-2-i)A_{i}b)(\xi)L_m = \sum_{j \geq 0, J \subset I} ((\text{est.}) \cdot a(-2-i-j)A_{j}b_{j+1} \cdot (\text{est.}) \cdot b_{j-i-j}A_{j}a(\xi)A_{j})m)$$

(3.2.14)
with $J' := L \cup (I \setminus J)$ and (cst.) being some constant. We distinguish two cases, according to the sign of $l$.

- Assume $l \geq 0$. For the first term in (3.2.14), we have

$$a_{(-2-i-j)}b_{(l+j)}m \in a_{(-2-i-j)}E_{(r-i)+s-(l+j)}(M) \subseteq E_{(r+i)+(s-j)-(l-2-i-j)-1}(M) = E_{(r+1)+s-l}(M),$$

where in the first line we used the induction hypothesis with $l+j \geq 0$ and $r-i \leq r$, and in the second line we used Lemma 3.3.4.

For the second term in (3.2.14), the induction hypothesis and Lemma 3.2.5 imply

$$b_{(l-2-i-j)}a_{(j)}m \in b_{(l-2-i-j)}E_{s-j}(M) \subseteq E_{(r-i)+(s-j)-(l-2-i-j)-1}(M) = E_{(r+1)+s-l}(M).$$

Hence we have $(a_{(-2-i-j)}b_{(l+j)})m \in E_{(r+1)+s-l}(M)$.

- Assume $l < 0$. By the induction hypothesis and Lemma 3.2.5, the first term in (3.2.14) is

$$a_{(-2-i-j)}b_{(l+j)}m \in a_{(-2-i-j)}E_{(r-i)+s-(l+j)-1}(M) \subseteq E_{(r+i)+(s-j)-(l-2-i-j)-1}(M) = E_{(r+1)+s-1}(M).$$

For the second term in (3.2.14), the argument (3.2.15) yields

$$b_{(l-2-i-j)}a_{(j)}m \in E_{(r+1)+s-l}(M) \subseteq E_{(r+1)+s-1}(M).$$

Hence we have $(a_{(-2-i-j)}b_{(l+j)})m \in E_{(r+1)+s-1}(M)$.

By these arguments, the induction step works, and we have the conclusion. \hfill \square

Now Proposition 3.2.1 and Lemma 3.2.6 yield the following analogue of [22, Theorem 2.12]:

**Theorem 3.2.7** Let $V = (V, [0], T, S', Y)$ be an $N_W = N$ SUSY VA, and $E_n = E_n(V)$ for $n \in \mathbb{Z}$ be the linear sub-superspaces of $V$ in Definition 3.2.2, which form a decreasing filtration

$$\cdots = E_{-1} = E_0 = V \supset E_1 \supset E_2 \supset \cdots \supset E_n \supset \cdots$$

by Lemma 3.2.4(1). Then, the associated graded space

$$\text{gr}_E V = \bigoplus_{n \in \mathbb{N}} E_n/E_{n+1}$$

has the following structure $(\cdot, 1, \partial, \delta^i, Y_\pm)$ of an $N_W = N$ SUSY VPA. Let $a \in E_r$ and $b \in E_s$. 

\[ \text{Springer} \]
• The commutative multiplication \( \cdot \) is
\[
(a + E_{r+1}) \cdot (b + E_{s+1}) = a_{(-1|N)} b + E_{r+s+1}.
\]

• The unit is \( 1 := |0\rangle + E_1 \).

• The even operator \( \partial \) is
\[
\partial(a + E_{r+1}) := T a + E_{r+2}.
\]

• The odd operator \( \delta^i \) for \( i \in [N] \) is
\[
\delta^i(a + E_{r+1}) := S^i a + E_{r+2}.
\]

• The SUSY VLA structure \( Y_\cdot \) is
\[
Y_\cdot(a + E_{r+1}, Z)(b + E_{s+1}) := \sum_{(j|J), j \geq 0} Z^{1-j|N\setminus J} (a_{(j|J)} b + E_{r+s-j+1}).
\]

Similarly as in [22, Proposition 2.13], we also have the module structure of the associated graded space of an \( NW = N \) SUSY VA module. The proof is straightforward, and we omit the details.

**Proposition 3.2.8** Let \( V \) be as in Theorem 3.2.7, \( M \) be a \( V \)-module, and \( \{ E_n(M) \mid n \in \mathbb{Z} \} \) be the decreasing filtration in Definition 3.2.2. Then, the associated graded \( \text{gr}_E M = \bigoplus_{n \in \mathbb{Z}} E_n(M)/E_{n+1}(M) \) is a module over the \( NW = N \) SUSY VPA \( \text{gr}_E V \) in Theorem 3.2.7 with the following structure for \( a \in E_r \) and \( m \in E_s(M) \):

• The module structure over commutative superalgebra is
\[
(a + E_{r+1}).(m + E_{s+1}(M)) := a_{(-1|N)} m + E_{r+s+1}(M).
\]

• The module structure over SUSY vertex Lie algebra (see (3.1.2)) is:
\[
Y_M(a + E_{r+1}, Z)(m + E_{s+1}(M)) := \sum_{(j|J), j \geq 0} Z^{1-j|N\setminus J} (a_{(-j|J)} m + E_{r+s-j+1}(M)).
\]

### 3.3 Li filtration of \( NK = N \) SUSY vertex algebra

Throughout this subsection, \( V = (V, |0\rangle, S'_K, Y) \) denotes an \( NK = N \) SUSY vertex algebra. We begin with an \( NK = N \) analogue of Proposition 3.2.1:

**Proposition 3.3.1** Let \( E = \{ E_n \mid n \in \mathbb{N} \} \) be a decreasing filtration of linear sub-superspaces of \( V \) such that \( |0\rangle \in E_0 \) and
\[
a_{(j|J)} b \in E_{r+s-j-1} \quad (a \in E_r, b \in E_s, j \in \mathbb{Z}, J \subset [N]),
\]
where we used the convention \( E_n := V \) for \( n \in \mathbb{Z}_{<0} \).

(1) The associated graded linear superspace
\[
\text{gr}_E V := \bigoplus_{n \in \mathbb{N}} E_n/E_{n+1}
\]
is an $N_K = N$ SUSY VA whose state-superfield correspondence is given by:

$$(a + E_{r+1})(j|J)(b + E_{s+1}) := a(j|J)b + E_{r+s-j} \quad (3.3.2)$$

for $a \in E_r$, $b \in E_s$, $j \in \mathbb{Z}$ and $J \subset [N]$, whose vacuum is given by $|0\rangle + E_1 \in E_0/E_1$, and whose odd operators $\mathcal{D}^i$ are given by

$$\mathcal{D}^i(a + E_{r+1}) := S_K^i a + E_{r+2}. \quad (3.3.2)$$

(2) The $N_K = N$ SUSY VA $\text{gr}_E V$ in (1) is commutative if and only if

$$a_{(m|M)}b \in E_{r+s-m} \quad (3.3.3)$$

for any $a \in E_r$, $b \in E_s$, $m \in \mathbb{N}$ and $M \subset [N]$.

(3) Under the condition (3.3.3), the commutative $N_K = N$ SUSY VA $\text{gr}_E V$ has an $N_K = N$ SUSY VPA structure

$$Y_-(a + E_{r+1}, Z)(b + E_{s+1}) := \sum_{(m|M), m \geq 0} Z^{-m-1|N}\mathcal{M}(a_{(m|M)}b + E_{r+s-m+1}) \quad (3.3.4)$$

for $a \in E_r$ and $b \in E_s$.

**Proof** Similarly as in Proposition 3.2.1, it is enough to check that (3) satisfies the derivation axiom (see (3.1.3)), which is equivalent to

$$a_{(l|L)}(b_{(-1|N)}c) + E_{r+s+t-m+1} = a_{(l|L)}b_{(-1|N)}c + (-1)^{(p(a)+|L|)p(b)} a_{(l|L)}b_{(-1|N)}(a_{(l|L)}c) + E_{r+s+t-m+1}$$

for $l \in \mathbb{N}$, $L \subset [N]$, $a \in E_r$, $b \in E_s$ and $c \in E_t$. The commutator formula (2.3.14) yields

$$a_{(l|L)}b_{(-1|N)}c - (-1)^{(p(a)+|L|)p(b)} b_{(-1|N)}(a_{(l|L)}c)$$

$$= \sum_{(j|J), j \geq 0} \pm \frac{1}{j!} (l)^{\downarrow}_{j+|J|L}(a_{(j|J)}b_{(-1|N)}(a_{(l|L)}c)$$

$$= (a_{(l|L)}b_{(-1|N)}c) + \sum_{(j|J), 0 \leq j \leq l-1, J \Delta L = \emptyset} \pm \frac{1}{j!} (l)^{\downarrow}_{j+|J|L}(a_{(j|J)}b_{(-1|N)}c,$$

where $\pm$ denotes some sign. For each term of this equality, the condition (3.3.3) yields

$$a_{(l|L)}b_{(-1|N)}c, b_{(-1|N)}(a_{(l|L)}c), (a_{(l|L)}b)_{(-1|N)}c \in E_{r+s+l-t}. \quad (3.3.4)$$
and since \( l - 1 - j \geq 0 \) in the summation, it also yields
\[
(a_{(j|j)b})(l-1-j)c \in E_{(r+s-j)+t-(l-1-j)} = E_{r+s+t-l+1}.
\]
Hence, we have the desired equality. \( \square \)

The definition of Li filtration for an \( N_K = N \) SUSY VA is the same as that for \( N_K = N \) case.

**Definition 3.3.2** For a \( V \)-module \( M \) and \( n \in \mathbb{Z} \), we define a linear sub-superspace \( E_n(M) \subset W \) by
\[
E_n(M) := \text{span} \left\{ a_{(r-1-k_1|K_1)} \cdots a_{(r-1-k_r|K_r)} m \middle| \begin{array}{l}
r \in \mathbb{Z}_{>0}, 
 a^i \in V, 
 m \in M, 
 k_i \in \mathbb{N}, 
 K_i \subset [N]
\end{array} \right. \text{satisfying } k_1 + \cdots + k_r \geq n \right\}.
\]

The following is an \( N_K = N \) analogue of Lemma 3.2.4.

**Lemma 3.3.3** For a \( V \)-module \( W \), we have the following.

1. \( E_n(M) \supset E_{n+1}(M) \) for any \( n \in \mathbb{Z} \).
2. \( E_n(M) = M \) for any \( n \in \mathbb{Z}_{\leq 0} \)
3. \( a_{(-1-k|K)} E_n(M) \subset E_{n+k}(M) \) for any \( a \in V, k \in \mathbb{N}, K \subset [N] \) and \( n \in \mathbb{Z} \).
4. For \( n \geq 1 \), \( E_n(M) \) is equal to
\[
E'_n(M) := \text{span} \left\{ a_{(r-1-k|K)} m \middle| a \in V, k \in \mathbb{Z}_{>0}, K \subset [N], m \in E_{n-k}(M) \right\}.
\]

(3.3.5)

5. For \( n \geq 1 \), \( E_n(M) \) is equal to
\[
E''_n(M) := \text{span} \left\{ a_{(r-1-k|K_1)} \cdots a_{(r-1-k_r|K_r)} m \middle| \begin{array}{l}
r \in \mathbb{Z}_{>0}, 
 a^i \in V, 
 m \in M, 
 k_i \in \mathbb{Z}_{>0}, 
 K_i \subset [N]
\end{array} \right. \text{satisfying } k_1 + \cdots + k_r \geq n \right\}.
\]

(3.3.6)

**Proof** Similarly as in Lemma 3.2.4, we only show (4) and (5). We can show by induction on \( n \) that
\[
E'_n(M) = E''_n(M),
\]
so that it is enough to prove ((4)) only. For that, we show that each element
\[
u = a_{(-1-k_1|K_1)} \cdots a_{(-1-k_r|K_r)} m \in E_n(M)
\]
Lemma 3.2.4. So let us assume condition $k$. Also, $u_{123}$ belongs to $E_{103}$. Hence we have $u_{k}$. Hereafter we assume $k$. Hence, we may assume $k$. The commutator formula (2.3.14) yields $u_{k}$. If moreover $l$, then we have $a_{(1)}$. Let $M$ be a $V$-module. For $a_{1}$, $b_{2}$: $=a_{2}$ and $k_{3}$. The commutator formula (2.3.14) yields

$$u = \sum_{(j|J), j \geq 0} (\text{cst.}) \cdot (a_{(j|J)} b_{(-2-k-j-\#K|K')}) u_{''},$$

where $\pm$ denotes a sign, (cst.) denotes a constant, $K = K_{1} \setminus J$ and $K' = K_{2} \cup (K_{1} \Delta J)$. Since $u_{''} \in E_{n-k}(M)$, we have $a_{(-1|K_{1})} u_{''} \in E_{n-k}(M)$ by (3), and the condition $k \geq 1$ yields

$$b_{(-1-k|K_{2})} a_{(-1|K_{1})} u_{''} \in E_{n}'(M).$$

Also, $u_{''} \in E_{n-k}(M) \subset E_{n-(k+j+1+\#K)}(M)$ for $j \geq 0$ by (1), and we have

$$(a_{(j|J)} b_{(-2-k-j-\#K|K')}) u_{''} \in E_{n}'(M).$$

Hence we have $u \in E_{n}'(W)$. □

The next lemma is an analogue of Lemma 3.2.5.

Lemma 3.3.4 Let $M$ be a $V$-module. For $a \in V$, $l$, $n \in \mathbb{Z}$ and $L \subset [N]$, we have

$$a_{(l|L)} E_{n}(M) \subset E_{n-l-1}(M).$$

(3.3.8)

If moreover $l \geq 0$, then we have

$$a_{(l|L)} E_{n}(M) \subset E_{n-l}(M).$$

(3.3.9)

Proof Similarly as Lemma 3.2.5, we show (3.3.9) by induction on $n$. The case $n \leq 0$ is done in the same way as Lemma 3.2.5. So assume $n \geq 1$. By Lemma 3.3.3 (4), $E_{n}(M)$ is spanned by elements $b_{(-1-k|K)} m$ with $b \in V$, $k \geq 1$, $K \subset [N]$ and
As in Lemma 3.2.6, the non-trivial case is
\[ a_{(l|L)}b_{(-1-k|K)}m \] belongs to \( E_{n-l}(M) \) using the commutator formula (2.3.14). It says
\[
a_{(l|L)}b_{(-1-k|K)}m = \pm b_{(-1-k|K)}a_{(l|L)}m + \sum_{(j|J), j \geq 0} (\text{cst.}) \cdot (a_{(j|J)}b_{(-1-k-j-\#J'|J')})m
\]
with \( J' := L \setminus J \) and \( J'' := K \cup (J \Delta L) \). As for the first term, by \( n - k < n, l \geq 0 \) and the induction hypothesis, we have \( a_{(l|L)}m \in E_{n-k-l}(M) \). Then, by \(-1-k < 0 \) and the already-proved (3.3.8), we have
\[
b_{(-1-k|K)}a_{(l|L)}m \in E_{(n-k-l)-(-1-k)-1}(M) = E_{n-l}(M).
\]
As for the terms in the summation, by (3.3.8) and \( j + \#J' \geq 0 \), we have
\[
(a_{(j|J)}b_{(-1-k-j-\#J'|J')})m \in E_{(n-k)-(l-1-k-j-\#J')-1}(M) = E_{n-l+j+\#J'}(M) \subset E_{n-l}(M).
\]
Hence we have \( a_{(l|L)}b_{(-1-k|K)}m \in E_{n-l}(M) \).

The next lemma is an \( N_K = N \) analogue of Lemma 3.2.6.

**Lemma 3.3.5** Let \( M \) be a \( V \)-module. For \( u \in E_r(V), m \in E_s(M), l \in \mathbb{Z} \) and \( L \subset [N] \), we have
\[
u_{(l|L)}m \in E_{r+x-l-1}(M).
\]
(3.3.10)

If moreover \( m \geq 0 \), then we have
\[
u_{(l|L)}m \in E_{r+x-l}(M).
\]
(3.3.11)

**Proof** As in Lemma 3.2.6, the non-trivial case is \( r \geq 0 \), which we show by induction on \( r \). Assume \( u \in E_{r+1}(M) \). By Lemma 3.3.3 ((4)), we can write \( u = a_{(-2-i|I)}b \) with some \( a \in V, 0 \leq i \leq r, I \subset [N] \) and \( b \in E_{r-i}(V) \). Thus, we want to calculate \( (a_{(-2-i|I)}b)_{(l|L)}m \), which is, by the iterate formula in Lemma 2.3.13, equal to
\[
(a_{(-2-i|I)}b)_{(l|L)}m = \sum_{(j|J), j \geq 0} \sum_{K \subset I \cap J} (\text{cst.}) \cdot a_{(-2-i-j-x|J)}b_{(l+j|J')}m
\]
+ (cst.) \cdot \( b_{(-2-i-j-x|J')}a_{(j|J)}m \)
(3.3.12)

with \( x := |J \setminus K|, J' := L \cup (I \setminus K) \cup (J \setminus K) \) and (cst.) being some constant.

- Assume \( l \geq 0 \). Then, the first term in (3.3.12) satisfies
\[
a_{(-2-i-j-x|J)}b_{(l+j|J')}m \in a_{(-2-i-j-x|J)}E_{(r-i)+s-(l+j)}(M)
\]
\[
\subset E_{(r+s-l-i-j)-(-2-i-j-x)-1}(M) = E_{(r+1)+s-l+x}(M) \subset E_{(r+1)+s-l}(M),
\]

\( \square \)
By these arguments, the induction step works, and we have the conclusion.

Let $V$ be as in Theorem 3.3.6, $M$ be a $V$-module, and Proposition 3.3.7

The commutative multiplication •

The odd operator $\mathcal{O}^i$ for $i \in [N]$ is $\mathcal{O}^i(a + E_{r+1}) := S_K^i a + E_{r+2}.$

The SUSY VLA structure $Y_-$ is

$$Y_-(a + E_{r+1}, Z)(b + E_{s+1}) := \sum_{(j | J), j \geq 0} Z^{-1-j|N\backslash J}(a(j|J)b + E_{r+s-j+1}).$$

By Proposition 3.3.1 and Lemma 3.3.5, we have the following $NK = N$ analogue of Theorem 3.2.7.

**Theorem 3.3.6** Let $V = (V, |0\rangle, S^i_K, Y)$ be an $NK = N$ SUSY VA, and \{$E_n = E_n(V) \mid n \in \mathbb{Z}$\} be the decreasing filtration in Definition 3.3.2. Then, the associated graded space $\text{gr}_E V = \bigoplus_{n \in \mathbb{N}} E_n / E_{n+1}$ has the following structure $(\cdot, 1, \mathcal{O}^i, Y_-)$ of an $NK = N$ SUSY VPA. Let $a \in E_r$ and $b \in E_s$.

- The commutative multiplication • is $(a + E_{r+1}) \cdot (b + E_{s+1}) := a(-1|N)b + E_{r+s+1}$.
- The unit is $1 := |0\rangle + E_1$.
- The odd operator $\mathcal{O}^i$ for $i \in [N]$ is $\mathcal{O}^i(a + E_{r+1}) := S_K^i a + E_{r+2}$.
- The SUSY VLA structure $Y_-$ is

$$Y_-(a + E_{r+1}, Z)(b + E_{s+1}) := \sum_{(j | J), j \geq 0} Z^{-1-j|N\backslash J}(a(j|J)b + E_{r+s-j+1}).$$

We also have an analogue of Proposition 3.2.8. The proof is omitted.

**Proposition 3.3.7** Let $V$ be as in Theorem 3.3.6, $M$ be a $V$-module, and \{$E_n(M) \mid n \in \mathbb{Z}$\} be the decreasing filtration in Definition 3.3.2. Then, the associated graded $\text{gr}_E M = \bigoplus_{n \in \mathbb{N}} E_n(M) / E_{n+1}(M)$ is a module over the $NK = N$ SUSY VPA $\text{gr}_E V$ in Theorem 3.3.6 with the following structure for $a \in E_r$ and $m \in E_s(M)$:

- The module structure over the commutative superalgebra is

$$(a + E_{r+1})(m + E_{s+1}(M)) := a(-1|N)m + E_{r+s+1}(M).$$
Li filtrations of SUSY vertex algebras

• The module structure over SUSY vertex Lie algebra (see (3.1.2)) is

\[ Y^M_-(a + E_{r+1}, Z)(m + E_{s+1}(M)) : = \sum_{(j,j): j \geq 0} Z^{-1-j[N\backslash J]}(a_{(j|J)}m + E_{r+s-j+1}(M)). \]

4 Associated superschemes and singular supports for SUSY vertex algebras

In this section, \( V \) denotes an \( NW = N \) or \( NK = N \) SUSY vertex algebra over the field \( k \) of characteristic 0.

4.1 \( C_2 \)-Poisson superalgebra and associated superscheme

For the even case, Li showed in [22, Proposition 3.8] that the degree zero subspace \( V/E_1(V) \subset \text{gr}_E V \) of the Li filtration is a Poisson algebra which coincides with the one introduced by Zhu [28, Sect. 4.4]. Following [3, Sect. 2.3.1, p. 11614], we call this Poisson algebra Zhu's \( C_2 \)-Poisson algebra of \( V \).

There are several notions of finiteness condition on vertex algebras. One of them is the \( C_2 \)-cofiniteness, which guarantees the modular property for a vertex operator algebra [25, 28]. Though the \( C_2 \)-cofiniteness looks like a technical condition, Arakawa illustrated in [2] that it has a clear geometric meaning, by introducing the notion of associated variety.

In this subsection, we give SUSY analogue of \( C_2 \)-Poisson algebras, and of the theory of associated varieties.

4.1.1 \( C_2 \)-Poisson superalgebra of SUSY vertex algebra

Let \( V \) be an \( NW = N \) or \( NK = N \) SUSY VA. For a \( V \)-module \( M \), we denote by \( \{ E_n(M) \mid n \in \mathbb{Z} \} \) the Li filtration of \( M \), given in Definitions 3.2.2 and 3.3.2. We also abbreviate \( E_n := E_n(V) \) as before.

**Definition 4.1.1** For a \( V \)-module \( M \), we denote

\[ C_2(M) : = \text{span}\{a_{(-j|J)}m \mid a \in V, \ m \in M, \ j \geq 2, \ J \subset [N]\}, \]

and call \( M \) \( C_2 \)-cofinite if \( \dim_k M/C_2(M) < \infty \). We also say \( V \) is \( C_2 \)-cofinite if it is \( C_2 \)-cofinite as a \( V \)-module.

By Lemmas 3.2.4(4) and 3.3.3(4), we have

\[ C_2(M) = E_1(M), \]

which yields the first half of Proposition 4.1.2.
Proposition 4.1.2 For an $N_W = N$ or $N_K = N$ SUSY VA $V$, the quotient space

$$R_V := V / C_2(V)$$

is equal to the degree zero subspace $E_0(V)/E_1(V) = V/E_1(V) \subset \text{gr}_E V$ of the Li filtration. Moreover, it is a Poisson superalgebra of parity $N \mod 2$ in the sense of Definition 3.1.7 whose commutative multiplication $\cdot$ and Poisson bracket $\{\cdot, \cdot\}$ are given by

$$\overline{a} \cdot \overline{b} := \overline{a_{(1-N)}} b, \quad \{\overline{a}, \overline{b}\} := \overline{a_{(0-N)}} b$$

for $\overline{a} := a + C_2(V), \overline{b} := b + C_2(V)$ with $a, b \in V$. We call $R_V$ the $C_2$-Poisson superalgebra of $V$.

Proof The first half is already explained. The second half can be shown directly using Definitions 3.1.6 and 3.1.17 of SUSY VPAs. $\square$

 Remark 4.1.3 (After the referees’ comments) Let $V$ be an $N_W = N$ SUSY VA. Then, by (2.2.20), we see that the even derivation $T$ is 0 on the $C_2$-Poisson superalgebra $R_V$, but the odd derivation $S^i$ may not be. The induced linear map $S^i$ on $R_V$ is nilpotent and, by Fact 2.2.7, satisfies

$$\overline{S^i (\overline{a} \cdot \overline{b})} = (\overline{S^i a}) \cdot \overline{b} + (-1)^{p(a)} (\overline{S^i b}), \quad \{\overline{a}, \overline{b}\} = \{\overline{S^i a}, \overline{b}\} + (-1)^{p(a)} \{\overline{a}, \overline{S^i b}\}$$

for $a, b \in V$ of pure parity. Similarly, we can show using (2.3.12) and Fact 2.2.7 that for an $N_K = N$ SUSY VA $V$, the odd derivation $S^i_K$ induces a nilpotent linear map $\overline{S^i_K}$ on $R_V$ satisfying the same properties. Hence, the $C_2$-Poisson superalgebra of a SUSY VA has an extra structure compared to the non-SUSY case. This structure corresponds to the degree zero part of the graded SUSY VPA $\text{gr}_E V$ in Theorems 3.2.7 and 3.3.6.

Example 4.1.4 Let us study the $C_2$-Poisson superalgebra of the Neveu–Schwarz SUSY vertex algebra $V$ (Example 2.3.5). Recall that it is an $N_K = 1$ SUSY VA over $k = \mathbb{C}$, and as a linear superspace we have

$$V = \mathbb{C}[S^m_K \tau \mid n \in \mathbb{N}]$$

with $\tau = G_{-\frac{1}{2}} |0\rangle$ being the Neveu–Schwarz element. Then, $E_1(V) = C_2(V) = \text{span}\{a_{(-m|M)} b \mid a, b \in V, \ m \in \mathbb{Z}_{\geq 2}, \ M \in \{0, 1\}\}$, and by (2.3.12), we have $S^m_K \tau = (S^m_K \tau)_{(-1[1])} |0\rangle = (S^{m-2}_K \tau)_{(-2[1])} |0\rangle \in E_1(V)$ for $m \geq 2$. Also, by (3.3.10), we have $E_1(V)_{(-1[1])}E_1(V) \subset E_2(V) \subset E_1(V)$. These imply $E_1(V) = \mathbb{C}[S^m_K \tau \mid m \in \mathbb{Z}_{\geq 2}]$, and we have

$$R_V \cong \mathbb{C}[\tau, \overline{S_K \tau}] = \mathbb{C}[\tau, \overline{\nu}]$$
as commutative superalgebras, where $\nu := \frac{1}{2} S^K \tau$ is the Virasoro element of $V$. The Poisson structure is given by

$$\{\tau, \tau\} = 2\nu, \quad \{\tau, \nu\} = \{\nu, \nu\} = 0.$$  

Here is the calculation: Using (2.3.10), (2.3.9) and (2.3.11), we have

$$\tau(0|0)\tau = G_{-\frac{1}{2}} G_{-\frac{3}{2}}|0\rangle = 2 L_{-2}|0\rangle = 2\nu,$$

which gives the first equality. Similarly, we have

$$\tau(0|1)\nu = G_{-\frac{1}{2}} L_{-2}|0\rangle = \frac{1}{2} G_{-\frac{5}{2}}|0\rangle = \frac{1}{2} \tau(-2|1)|0\rangle,$$

which belongs to $C_2(V) = E_1(V)$. Hence $\{\tau, \nu\} = 0$. Finally, we have

$$\nu(0|1)\nu = L_{-1} L_{-2}|0\rangle = L_{-3}|0\rangle = \frac{1}{2} \tau(-2|0)|0\rangle,$$

which belongs to $E_1(V)$, implying $\{\nu, \nu\} = 0$.

In the even case, Arakawa showed in [2, Proposition 2.5.1] that the embedding $R_V = E_0/E_1 \hookrightarrow \text{gr}_E V = \bigoplus_{n \geq 0} E_n/E_{n+1}$ can be extended to a surjection $(R_V)_\infty \twoheadrightarrow \text{gr}_E V$ of vertex Poisson algebras, where $(R_V)_\infty$ is the jet algebra of $R_V$ equipped with the level 0 vertex Poisson structure (Remark 3.1.10). We have the following SUSY analogue of this fact.

**Proposition 4.1.5** Let $V$ be a SUSY VA, and $\phi: R_V \hookrightarrow \text{gr}_E V$ be the embedding in Proposition 4.1.2.

1. If $V$ is an $N_W = N$ SUSY VA, then $\phi$ extends to a surjective morphism
   
   $$\Phi: (R_V)^O \twoheadrightarrow \text{gr}_E V$$

   of $N_W = N$ SUSY VPAs, where $(R_V)^O$ is the $1|N$-superjet algebra of $R_V$ equipped with the level 0 SUSY VPA structure (Proposition 3.1.9).

2. If $V$ is an $N_K = N$ SUSY VA, then $\phi$ extends to a surjective morphism
   
   $$\Phi: (R_V)^{O_{sc}} \twoheadrightarrow \text{gr}_E V$$

   of $N_K = N$ SUSY VPAs, where $(R_V)^{O_{sc}}$ is the $1|N$-superconformal jet algebra of $R_V$ with the level 0 SUSY VPA structure (Proposition 3.1.18).

For the proof, we need some preliminaries. The following is an analogue of [22, Lemma 4.2, (4.7)].

**Lemma 4.1.6** Let $V$ be a SUSY VA. We regard the SUSY VPA $\text{gr}_E V$ in Theorems 3.2.7 and 3.3.6 as a commutative superalgebra, and denote it by $A$. Hence, for a $V$-module $M$, the $\text{gr}_E V$-module $\text{gr}_E M$ in Propositions 3.2.8 and 3.3.7 can be regarded as an $A$-module. Then the $A$-module $\text{gr}_E M$ is generated by $E_0(M)/E_1(M)$.

**Proof** Considering the decomposition $\text{gr}_E M = \bigoplus_{n \in \mathbb{N}} E_n(M)/E_{n+1}(M)$, we show by induction on $n$ that every element of $E_n(M)/E_{n+1}(M)$ can be written in the form $u.m$ with $u \in A$ and $m \in E_0(M)/E_1(M)$. We may assume $n \geq 1$. The rest of the argument is divided into $N_W = N$ case and $N_K = N$ case.
Assume \( V = (V, [0], T, S', Y) \) is an \( N_W = N \) SUSY VA. Then, by Lemma 3.2.4 ((4)), \( E_n(M) \) is linearly spanned by the subspaces \( a_{(-2-j|J)} E_{n-1-j}(M) \) with \( a \in V, 0 \leq j \leq n - 1 \) and \( J \subset [N] \). For \( m \in E_{n-1-j}(M) \), using (2.2.20) and \( T(j) := \frac{1}{j!} T^j \), we have

\[
a_{(-2-j|J)m} + E_{n+1}(M) = \pm (S^{N\setminus J} a)_{(-2-j|N)} m + E_{n+1}(M) \\
= \pm (T^{(j+1)} S^{N\setminus J} a)_{(-1|N)} m + E_{n+1}(M) \\
= \pm (T^{(j+1)} S^{N\setminus J} a + E_{j+2}(V))_{(-1|N)} (m + E_{n-j}(M)) + E_{n+1}(M).
\]

Denoting \( \overline{m} := m + E_n(M) \) for \( m \in E_n(M) \), we have

\[
\overline{a}_{(-2-j|J)m} = \pm (\partial^{j+1} \delta^{N\setminus J} \overline{a}) \overline{m},
\]

where \( \partial \) and \( \delta^i \) are the operators of the \( N_W = N \) SUSY VPA \( \text{gr}_E V \) (see Theorem 3.2.7). It implies that \( \overline{a}_{(-2-j|J)m} \overline{m} \) belongs to the subspace generated by \( E_0(M)/E_1(M) \). Thus, the induction step works, and we have the result.

A similar argument works in the case \( V = (V, [0], S^i_K, Y) \) is an \( N_W = N \) SUSY VA. By Lemma 3.3.3 ((4)), \( E_n(M) \) is linearly spanned by the subspaces \( a_{(-2-j|J)} E_{n-1-j}(M) \) with \( a \in V, 0 \leq j \leq n - 1 \) and \( J \subset [N] \). For \( m \in E_{n-1-j}(M) \), using (2.3.12), we have

\[
a_{(-2-j|J)m} + E_{n+1}(M) = \pm (T^{(j+1)} S_K^{N\setminus J} a + E_{j+2}(V))_{(-1|N)} (m + E_{n-j}(M)) + E_{n+1}(M).
\]

Using the odd operator \( \delta^i \) of the \( N_K = N \) SUSY VPA \( \text{gr}_E V \) (Theorem 3.3.6), we have

\[
\overline{a}_{(-2-j|J)m} = \pm ((\delta^i)^{(2j+2)} \delta^{N\setminus J} \overline{a}) \overline{m},
\]

The rest part is the same as the \( N_W = N \) case.

\[
\square
\]

**Proof of Proposition 4.1.5** First we consider the \( N_W = N \) case. By Lemmas 1.3.10 and 4.1.6, we find that the embedding \( R_V \hookrightarrow \text{gr}_E V \) extends to a surjection \( \Phi: (R_V)^O \twoheadrightarrow \text{gr}_E V \) of differential superalgebras. By Theorem 3.2.7, the \( N_W = N \) SUSY VPA structure on \( \text{gr}_E V \) is restricted to the sub-superspace \( R_V = E_0/E_1 \) as

\[
\overline{a}_{(m|M)} \overline{b} = \begin{cases} 
\overline{a}_{(0|M)} \overline{b} & (m = 0, M = [N]) \\
0 & \text{(otherwise)}
\end{cases}
\]

for \( m \in \mathbb{N}, M \subset [N] \) and \( \overline{a} = a + E_1, \overline{b} = b + E_1 \) with \( a, b \in E_0 = V \). It coincides with the level 0 SUSY VPA structure on \( (R_V)^O \) (Proposition 4.1.2). Then, an obvious
SUSY analogue of [21, Lemma 3.3] shows that the surjection $\Phi$ is a morphism of $N_K = N$ SUSY VPAs.

The $N_K = N$ case can be treated similarly, using Lemma 1.3.16 and Theorem 3.3.6 instead.

4.1.2 Associated superscheme of SUSY vertex algebra

Following the argument of the even case [2, Sects. 3.1 and 3.2], we give an analogue of the theory of associated varieties.

**Notation 4.1.7** (After the referees’ comments) Below we will use the language of superschemes. Let us give a brief recollection.

- A superspace means a locally ringed space $(X, O_X)$, where the structure sheaf $O_X$ is equipped with a $\mathbb{Z}_2$-grading $O_X = O_{X, 0} \oplus O_{X, 1}$ making it into a sheaf of commutative superrings in the sense of Notation 1.1.3.
- A superscheme means a superspace $(X, O_X)$ such that $(X, O_{X, 0})$ is a scheme and $O_{X, 1}$ is a quasi-coherent sheaf of $O_{X, 0}$-modules.
- For a commutative superring $R$, we have a superscheme $\text{Spec } R$, called the affine superscheme of $R$. The underlying topological space is $\text{Spec } R = \text{Spec } R_{\overline{0}}$, the underlying space of the affine scheme $\text{Spec } R_{\overline{0}}$. The structure sheaf $O_{\text{Spec } R} = O_{\text{Spec } R, 0} \oplus O_{\text{Spec } R, 1}$ consists of the structure sheaf $O_{\text{Spec } R, 0}$ of $\text{Spec } R_{\overline{0}}$ and the quasi-coherent sheaf $O_{\text{Spec } R, 1}$ of $O_{\text{Spec } R, 0}$-modules corresponding to the $R_{\overline{0}}$-module $R_{\overline{1}}$.

We refer to [20, Sect. 1.1] for the basics of the theory of superschemes.

Hereafter, $V$ denotes an $N_W = N$ or $N_K = N$ SUSY VA.

**Definition 4.1.8** The associated superscheme of $V$ is defined to be

$$X_V := \text{Spec } R_V.$$

By Proposition 4.1.2, it is a Poisson superscheme of parity $N \mod 2$ (see Definition 3.1.7).

Let us study a condition when $R_V$ is finitely generated as a superalgebra (so that $X_V$ is of finite type).

**Definition 4.1.9** $V$ is called finitely strongly generated if there exists a finite subset $G \subset V$ such that $V$ is linearly spanned by the elements of the form

$$a^1_{(-p_1|p_1)} \cdots a^r_{(-p_r|p_r)}[0] \quad (r \in \mathbb{N}, \; a^i \in G, \; p_i \in \mathbb{Z}_{>0}, \; P_i \subset [N]).$$

We call $G$ a set of strong generators of $V$.

Below is an analogue of a half of [2, Corollary 2.6.2].
Lemma 4.1.10 If $V$ is finitely strongly generated, then the $C_2$-superalgebra $R_V$ is finitely generated. More precisely, if $\overline{G} = \{a^1, \ldots, a^r\}$ is a set of strong generators of $V$, then $\overline{G} := \{a^1, \ldots, a^r\}$ with $a^i := a^i + C_2(V)$ generates the superalgebra $R_V = V/C_2(V)$.

Proof By Definitions 3.2.2 and 3.3.2 of the Li filtration, $C_2(V) = E_1$ is linearly spanned by elements of the form $a^1_{(-p_1|P_1)} \cdots a^1_{(-p_r|P_r)} b$ with $r \geq 1$, $a^i, b \in V$, $p_i \in \mathbb{Z}_{\geq 0}$ and $P_i \subset [N]$ satisfying $p_1 + \cdots + p_r \geq r + 1$. Hence, both $V$ and $E_1$ are linearly spanned by the elements of the form (4.1.1), and $R_V = V/E_1$ is generated as a superalgebra by $\overline{G}$. □

Next, we turn to the associated scheme of a SUSY VA module. Let $M$ be a $V$-module. Then the $gr_F V$-module structure on $gr_E M$ (Propositions 3.2.8 and 3.3.7) induces a Poisson $R_V$-module structure on the quotient superspace

$$M/C_2(M) = E_0(M)/E_1(M).$$

In other words, it has a superalgebra $R_V$-action

$$\overline{a}.\overline{m} := \overline{a}(-1|N)\overline{m} \quad (a \in V, \ m \in M)$$

(4.1.2)

with $\overline{a} := a + C_2(V)$ and $\overline{m} := m + C_2(M)$, and has a Lie superalgebra $R_V$-action (of parity $N \mod 2$)

$$\{\overline{a}, \overline{m}\} := \overline{a}(0|N)\overline{m} \quad (a \in V, \ m \in M),$$

which are compatible in the sense that

$$\{\overline{a}, \overline{b}.\overline{m}\} = \{\overline{a}, \overline{b}\} \overline{m} + (-1)^{(p(a)+N)p(b)}\overline{b}.\{\overline{a}, \overline{m}\}.\ (4.1.3)$$

(c.f. Definition 3.1.7 of Poisson superalgebras.) The last formula (4.1.3) shows the following SUSY analogue of [2, Lemma 3.2.1 (1)]:

Lemma 4.1.11 Let $M$ be a $V$-module, and $\text{Ann}_{R_V} (M/C_2(M))$ be the annihilator of $M/C_2(M)$ in the superalgebra $R_V$ (see (4.1.2)). Then $\text{Ann}_{R_V} (M/C_2(M))$ is a Poisson ideal of the Poisson superalgebra $R_V$ of parity $N \mod 2$.

Now, extending Definition 4.1.8, we introduce:

Definition 4.1.12 For a $V$-module $M$, we define the associated superscheme of $M$ to be

$$X_M := \text{Supp}_{R_V} (M/C_2(M)) = \{p \in \text{Spec } R_V \mid (M/C_2(M))_p \neq 0\}.$$
Definition 4.1.13 A $V$-module $M$ is finitely strongly generated over $V$ if $M/C_2(M)$ is finitely generated over $R_V$ as a superalgebra.

If $M$ is finitely strongly generated over $V$, then

$$X_M = \{ p \in \text{Spec } R_V \mid p \supset \text{Ann}_{R_V}(M/C_2(M)) \},$$

and by Lemma 4.1.11, $X_M$ is a closed Poisson subscheme of $X_V$. We have an immediate consequence:

Lemma 4.1.14 [2, Lemma 3.2.2] Assume $V$ is finitely strongly generated, so that $R_V$ is a finitely generated superalgebra by Lemma 4.1.10. Then, for a finitely strongly generated $V$-module $M$, we have

$$M$$

is $C_2$-cofinite (Definition 4.1.1) $\iff$ $\dim X_M = 0$.

4.2 Singular support

In this subsection, we introduce the notion of singular supports for SUSY VAs and study the relation to the lisse condition. The contents are more or less straightforward analogues of the even case [2, Sect. 3.3]. Throughout of this subsection, we take the base field $k$ to be $\mathbb{C}$. Also, a SUSY VA means an $NW = N$ or $NK = N$ SUSY VA.

First, we need to restrict the class of SUSY VAs and modules.

Definition 4.2.1 Let $V$ be a SUSY VA.

(1) $V$ is graded if it is equipped with an even semisimple operator $H$, called a Hamiltonian, such that

$$[H, a_{(j|J)}] = -(j + 1)a_{(j|J)} + (Ha)_{(j|J)}$$

for any $a \in V$, $j \in \mathbb{Z}$ and $J \subset [N]$. We denote the eigenspace decomposition by $V = \bigoplus_{\Delta \in \mathbb{C}} V_{\Delta}, V_{\Delta} : = \{ a \in V \mid Ha = \Delta a \}$. An element of $V_{\Delta}$ for some $\Delta$ will be called homogeneous of weight $\Delta$. For $a \in V_{\Delta_a}$ and $b \in V_{\Delta_b}$, we have $a_{(j|J)}b \in V_{\Delta_a + \Delta_b - j - 1}$.

(2) For a subset $I \subset \mathbb{C}$, $V$ is $I$-graded if it is graded and $V_{\Delta} = 0$ for $\Delta \notin I$.

(3) Assume $V$ is graded with Hamiltonian $H$. Then a $V$-module $M$ is graded if

$$[H, a^M_{(j|J)}] = -(j + 1)a^M_{(j|J)} + (Ha)^M_{(j|J)}$$

for any $a \in V$, $j \in \mathbb{Z}$ and $J \subset [N]$, and moreover if there is a decomposition

$$M = \bigoplus_{d \in \mathbb{C}} M_d, \quad M_d : = \{ m \in M \mid Hm = dm \}. \quad (4.2.1)$$

An element of $M_d$ for some $d$ is called homogeneous of weight $d$. For $a \in V_{\Delta_a}$ and $m \in M_d$, we have $a^M_{(j|J)}m \in M_{d + \Delta_a - j - 1}$.
Remark 4.2.2 An $NW = N$ conformal SUSY VA (Definition 2.2.4) is graded with Hamiltonian $\omega_{(1|0)}$. An $NK = N$ conformal SUSY VA (Definition 2.3.6) is graded with Hamiltonian $\tau_{(1|0)}$.

We consider the following class of SUSY VAs and their modules. The conditions are simple analogue of those in [2].

Assumption 4.2.3 Let $V$ be a SUSY VA which is
- $\frac{1}{r_0}\mathbb{N}$-graded with some $r_0 \in \mathbb{Z}_{>0}$, and
- finitely strongly generated (Definition 4.1.9).

Also, let $M$ be a $V$-module which is
- graded and lower truncated, i.e., there exists a finite subset $\{d_1, \ldots, d_s\} \subset \mathbb{C}$ such that $M_d = 0$ unless $d \in d_i + \frac{1}{r_0}\mathbb{N}$ in the decomposition (4.2.1), and
- finitely strongly generated (Definition 4.1.13).

Remark 4.2.4 Strongly conformal SUSY VAs (Definitions 2.2.5 and 2.3.7) satisfy Assumption 4.2.3.

By Lemma 4.1.10, the $C_2$-superalgebra $RV$ is finitely generated. We also have the following lemma, the even version of which is stated in [2, Sect. 3.1, p. 570].

Lemma 4.2.5 Let $V$ be a SUSY VA and $M$ be a $V$-module, both satisfying Assumption 4.2.3. Then the Li filtration $\{E_n(M) \mid n \in \mathbb{Z}\}$ is separated, i.e., $\bigcap_{n\in\mathbb{Z}} E_n(M) = 0$.

Proof It is essentially shown in [22, Lemma 2.14], but let us write down a proof for completeness. Let $E_n(M) = \bigoplus_d E_n(M)_d$, $E_n(M)_d := E_n(M) \cap M_d$ be the induced decomposition. It is enough to show $E_n(M)_d = 0$ unless $d \geq n + d_i$ for some $i \in \{1, \ldots, s\}$.

The case $n \leq 0$ holds by the condition on $M$. For $n \geq 1$, recall Definition 3.3.2, and consider a basis element of $E_n(M)$:

$$v := a^1_{(-1-k_1|K_1)} \cdots a^r_{(-1-k_r|K_r)} m$$

with $r \geq 1$, $a^j \in V$, $m \in M$, $k_j \in \mathbb{Z}_{>0}$ and $K_j \subset [N]$ satisfying $k_1 + \cdots + k_r \geq n$. We may assume that $a^j$ and $m$ are homogeneous, say $a^j \in V_{\Delta_j}$ and $m \in M_d$. By the assumption on $M$, we have $d \geq d_i$ for some $i \in \{1, \ldots, s\}$. Then $v$ is also homogeneous of weight

$$(\Delta_1 + k_1) + \cdots + (\Delta_1 + k_1) + d \geq k_1 + \cdots + k_r + d \geq n + d \geq n + d_i.$$ 

Thus, we have $E_n(M)_d = 0$ unless $d \geq n + d_i$. 

The following lemma corresponds to a half of [2, Lemma 3.1.5].

Lemma 4.2.6 Let $V$ be a SUSY VA and $M$ be a $V$-module, both satisfying Assumption 4.2.3. Then $\text{gr}_E M$ is a finitely generated module over the superalgebra $\text{gr}_E V$. 

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**Proof** By Lemma 4.1.6, the \( \gr_E V \)-module \( \gr_E M \) is generated by \( E_0(M)/E_1(M) = M/C_2(M) \). By Assumption 4.2.3, \( M/C_2(M) \) is finitely generated over the superalgebra \( R_V \subseteq \gr_E V \). So we have the consequence. \( \square \)

Let us recall Proposition 4.1.5, by which we have a surjection of commutative superalgebras

\[
\Phi : (R_V)^O \longrightarrow \gr_E V
\]  

(4.2.2)

from the superjet algebra. On the other hand, for a \( V \)-module \( M, \gr_E M \) is an \( N_K = N \) SUSY VPA module of \( \gr_E V \) by Proposition 3.3.7. Hence, \( \gr_E M \) can also be regarded as a module over the commutative superalgebra \( (R_V)^O \). Under Assumption 4.2.3, \( \gr_E M \) is finitely generated over \( \gr_E V \), so that it is also finitely generated over \( (R_V)^O \).

Now, following \([2, \text{Sect. } 3.3]\), we introduce:

**Definition 4.2.7** Let \( V \) be a SUSY VA \( V \) and a \( V \)-module \( M \). We define the singular support of \( M \) to be

\[
\SS(M) := \text{Supp}_{(R_V)^O}(\gr_E M).
\]

Here we regard \( (R_V)^O \) as a commutative superalgebra. Under Assumption 4.2.3, it can be written as

\[
\SS(M) = \left\{ p \in \text{Spec}(R_V)^O \mid p \supset \text{Ann}_{(R_V)^O}(\gr_E M) \right\}.
\]

For a superscheme \( X = (X, \mathcal{O}) \), its reduced part \([20, (1.1.5)]\) is a scheme defined by

\[
X_{\text{red}} := (X, \mathcal{O}_\text{n}/\sqrt{\mathcal{O}_\text{n}}).
\]

Mimicking the even case \([2, \text{Sect. } 3.3]\), let us also introduce:

**Definition 4.2.8** Let \( V \) be a SUSY VA.

1. A \( V \)-module \( M \) is called lisse if it is finitely strongly generated and \( \SS(M)_{\text{red}} \) is 0-dimensional.
2. \( V \) itself is called lisse if it is lisse as a \( V \)-module.

We have the following analogue of \([2, \text{Theorem } 3.3.3]\).

**Theorem 4.2.9** Let \( V \) be a SUSY VA satisfying Assumption 4.2.3. We have

\[
V \text{ is } C_2\text{-cofinite (Definition 4.1.1)} \iff V \text{ is lisse.}
\]

For the proof, we need some preliminaries. First, recall the projection \( (R_V)^O_m \rightarrow (R_V)^O \) of superjet algebras in (1.3.4). We rewrite it as a morphism of superschemes

\[
p_m : (X_V)^O : = \text{Spec}(R_V)^O \longrightarrow (X_V)^O_m : = \text{Spec}(R_V)^O_m.
\]  

(4.2.3)
Then we have the following analogue of [2, Lemma 3.3.1].

**Lemma 4.2.10** Let V be a SUSY VA and M be a V-module, both satisfying Assumption 4.2.3. Then we have \( X_M = p_0(\text{SS}(M)) \) (scheme-theoretic image).

**Proof** By Lemma 4.1.6, the \( \text{gr}_E \) V-module \( \text{gr}_E M \) is generated by \( E_0(M)/E_1(M) = M/C_2(M) \), which implies

\[
RV \cap \text{Ann}_{\text{gr}_E V}(\text{gr}_E M) = \text{Ann}_{RV}(M/C_2(M)). \tag{4.2.4}
\]

It yields the consequence. \( \square \)

Now we start:

**Proof of Theorem 4.2.9** For the direction \( \Rightarrow \), \( C_2 \)-cofiniteness of V implies that the reduced part of \( (X_V)^O = \text{Spec}(RV)^O \) is 0-dimensional. Since \( \text{SS}(V) \subset (X_V)^O \) by Definition 4.2.7, we find that \( \text{SS}(V)_{\text{red}} \) is 0-dimensional. For the converse \( \Leftarrow \), using (4.2.3), we find that \( p_0(\text{SS}(V)_{\text{red}}) \) is 0-dimensional. Then Lemma 4.2.10 yields that \( (X_V)_{\text{red}} \) is 0-dimensional. \( \square \)

We have a similar statement for V-modules as Theorem 4.2.9 if we assume V to be strongly conformal (Definitions 2.2.5 and 2.3.7). Recall that such a strongly conformal V is graded with Hamiltonian \( H = \nu(1|0) \) or \( \tau(1|0) \) (Remark 4.2.2).

**Theorem 4.2.11** Let V be a strongly conformal SUSY VA whose grading for the Hamiltonian H satisfies Assumption 4.2.3. Also, let M be a V-module satisfying Assumption 4.2.3. Then, we have

\[
M \text{ is } C_2 \text{-cofinite } \iff \ M \text{ is lisse.}
\]

For the proof, we need a few lemmas.

**Lemma 4.2.12** (c.f. [2, Lemma 3.2.1 (ii)]) Let V and M be as in Theorem 4.2.11.

1. In the \( NW = N \) case, the annihilator \( \text{Ann}_{(RV)^O}(\text{gr}_E M) \) is a vertex Poisson ideal of the level 0 \( NW = N \text{ SUSY VPA} \ ((RV)^O, T, S^i) \) (Propositions 3.1.9 and 4.1.5).
2. In the \( NK = N \) case, the annihilator \( \text{Ann}_{(RV)^O}(\text{gr}_E M) \) is a vertex Poisson ideal of the level 0 \( NK = N \text{ SUSY VPA} \ ((RV)^O, S^i_K) \) (Propositions 3.1.18 and 4.1.5).

**Proof** We give an argument only for the \( NK = N \) case, since the \( NW = N \) case is quite similar. Recall the surjection \( \Phi: (RV)^O_{\text{sc}} \to \text{gr}_E V \) of \( NK = N \text{ SUSY VPAs} \) (Proposition 4.1.5) and the argument around (4.2.2). In particular, \( \text{gr}_E M \) is a module over the commutative superalgebra \( (RV)^O \). We also have

\[
I := \text{Ann}_{(RV)^O}(\text{gr}_E M) = \Phi^{-1}(\text{Ann}_{\text{gr}_E V}(\text{gr}_E M)),
\]

and it is an ideal of \( (RV)^O \). Since \( \Phi \) is a VLA morphism, it is also a VLA ideal. Thus, it is enough to show that \( I \) is a differential ideal over the differential algebra \( (RV)^O_{\text{sc}} = \quad \square \)
\((R_V)^{O}, S^i_K\). Now, recall that \(gr_E V\) is a differential algebra with derivations \(S^i_K\) (Definition 1.3.8). We have that \(gr_E M\) is a differential module over \(gr_E V\). Indeed, using the conformal element \(\tau \in V\), we have the action of \(S^i_K\) by \(S^i_K \cdot m = \overline{\tau(0, e_i)}m\) for \(i \in [N]\). Hence, \(I' := \text{Ann}_{gr_E V}(gr_E M)\) is a differential ideal, and so is \(I = \Phi^{-1}(I')\).

\[\square\]

**Lemma 4.2.13** (c.f. [2, Lemma 3.3.1 (ii)]) Let \(V\) and \(M\) be as in Theorem 4.2.11. Then, we have \(SS(M) \subset (X_M)^O\).

**Proof** Again, we only give a proof for \(N_K = N\) case. By Lemma 4.2.12, \(\text{Ann}_{gr_E V}(gr_E M)\) is a differential ideal of the differential algebra \(gr_E V\) with derivations \(S^i_K\). Then (4.2.4) implies that \(\text{Ann}_{gr_E V}(gr_E M)\) contains the ideal \(I \subset \mathbb{C}(V)^O\) which is minimal among the \(S^i_K\)-stable ideals containing \(\text{Ann}_{V}(M/C_2(M))\). Since \(I\) is the defining ideal of \((X_M)^O\), we have the conclusion. \(\square\)

**Proof of Theorem 4.2.11** The direction \(\Leftarrow\) can be shown in the same way as Theorem 4.2.9 using Lemma 4.2.10. For the direction \(\Rightarrow\), the reduced part of \((X_M)^O\) is 0-dimensional by the assumption. Then, from Lemma 4.2.10, we find that \(SS(M)_{\text{red}}\) is 0-dimensional. \(\square\)

### 4.2.1 Example: Neveu–Schwarz SUSY vertex algebra

We continue Examples 2.3.5 and 4.1.4. Let us denote by \(g\) the Neveu–Schwarz Lie superalgebra with central charge \(c \in \mathbb{C}\). We use the basis \(\{L_n \mid n \in \mathbb{Z}\} \cup \{G_r \mid r \in \frac{1}{2} + \mathbb{Z}\}\) satisfying (2.3.9). Let us also denote by \(V^c\) the Neveu–Schwarz SUSY VA with central charge \(c \in \mathbb{C}\). Recall that \(V^c\) is an \(N_K = 1\) SUSY VA.

As a \(g\)-module, we have \(V^c = M_{0,c}/M^c_{\frac{1}{2},c}\) with \(M_{h,c}\) the Verma module of \(g\) generated by the highest weight vector \(|h, c\rangle\) satisfying \(L_0. |h, c\rangle = h\) and \(1.|h, c\rangle = |h, c\rangle\). \(V^c\) has a unique maximal sub-\(g\)-module \(N_c\). Now let us cite:

**Fact 4.2.14** [13, 5.2.1. Theorem (ii), (iv)] Let \(Y := \{(p, q) \in \mathbb{Z}_{\geq 1}^2 \mid p - q \in 2\mathbb{Z}, (\frac{p-q}{2}, q) = 1\}\), and for \((p, q) \in Y\) we set \(c^S_{p,q} := \frac{3}{2}(1 - \frac{2(p-q)^2}{pq})\). Then the following conditions are equivalent.

- \(V^c\) is not simple as a \(g\)-module.
- \(c \neq c^S_{p,q}\) for \((p, q) \in Y\) and \(p > q \geq 2\).
- \(\dim \mathbb{C}V^c/C^G_{2K}\) is finite, where \(C^G_{2K} \subset V^c\) is the linear sub-superspace given by
  \[
  C^G_{2K} := \text{span}\{L_{-n}v, G_{-r}v \mid n, r - \frac{1}{2} > 2, v \in V^c\}.
  \]

We can restate this fact as follows, which is a Neveu–Schwarz analogue of the statement [2, Proposition 3.4.1] for the universal Virasoro vertex algebra.

**Proposition 4.2.15** The following conditions are equivalent.

- \(V^c\) is \(C_2\)-cofinite as a SUSY VA module over itself (Definition 4.1.1).
• $V^c$ is not simple as a $\mathfrak{g}$-module.
• $c \neq c_{p,q}^S$ for $(p, q) \in Y$ and $p > q \geq 2$.

**Proof** It is enough to show $C_{2GK}^C = C_2(V^c)$. Recall the Neveu–Schwarz element $\tau$ and $\nu := \frac{1}{2} S_K \tau$ of $V^c$. By $V^c = \mathbb{C}[S_K^c \tau | n \in \mathbb{Z}]|0\rangle$ and (2.3.12), we have $C_2(V^c) = \text{span}\{\tau_{(-n|J)} v | n \in \mathbb{Z}_{\geq 2}, J \in \{0, 1\}, v \in V^c\}$. On the other hand, by (2.3.10) and (2.3.12), the Fourier modes of $\tau$ satisfy $G_r = \tau_{(r+\frac{1}{2}|1)}$ and $L_n = \frac{1}{2} \tau_{(n+1|0)}$. Thus, we have

$$C_{2GK}^C = \text{span}\{\tau_{(-n|0)} v, \tau_{(-n|1)} v | n \in \mathbb{Z}_{\geq 2}, v \in V^c\} = C_2(V^c).$$

We also have a Neveu–Schwarz analogue of the statement [2, Proposition 3.4.2] on the relation between $C_2$-cofiniteness of Virasoro modules and the zero singular support condition in the sense of Beilinson et al. [8].

Let us denote by $\{U_p(\mathfrak{g}) | p \in \mathbb{N}\}$ the PBW filtration of the universal enveloping algebra $U(\mathfrak{g})$. The associated graded $\text{gr} U(\mathfrak{g})$ is a commutative superalgebra isomorphic to the symmetric algebra $S(\mathfrak{g})$. Let $M$ be a highest weight representation of $\mathfrak{g}$ with central charge $c$ and the highest weight vector $v$. Then $\{U_p(\mathfrak{g}).v | \mathbb{N}\}$ gives a filtered module structure on $M$ over the PBW-filtered algebra $U(\mathfrak{g})$. The associated graded space, denoted by $\text{gr}_{\text{PBW}} M$, is an $S(\mathfrak{g})$-module generated by the image of $v$.

Now, following [8, Sect. 7.1.1] and borrowing the symbol in [2, Sect. 3.4], we define

$$SS_{\text{BFM}} M := \text{Supp}_{S(\mathfrak{g})}(\text{gr}_{\text{PBW}}(M)).$$

On a highest weight representation $M$ over $\mathfrak{g}$, we have a standard module structure over the Neveu–Schwarz vertex superalgebra, which is equivalent to a module structure over the $N_K = 1$ SUSY VA $V^c$. Thus we can discuss the $C_2$-cofiniteness of $M$.

**Proposition 4.2.16** For a highest weight representation $M$ over the Neveu–Schwarz Lie superalgebra $\mathfrak{g}$ of central charge $c$, the following conditions are equivalent.

• $M$ is $C_2$-cofinite as a $V^c$-module in the sense of Definition 4.1.1.
• $SS_{\text{BFM}} M = \{0\}$.

**Proof** The argument for the Virasoro algebra in [2, Proposition 3.4.2] works by replacing $L_{-1}, L_{-2}$ therein by $G_{-\frac{1}{2}}, G_{-\frac{3}{2}}$. We omit the details. \qed

In [8, 7.3.10 Proposition], it is shown that the condition $SS_{\text{BFM}} M = \{0\}$ for a highest weight module $M$ over the Virasoro Lie algebra of central charge $c$ is equivalent to the condition that $M$ belongs to the minimal series representations (or the Belavin–Polyakov–Zamolodchikov representations). This equivalence and [2, Proposition 3.4.2] yield that a $C_2$-cofinite module over the universal Virasoro vertex algebra is nothing but a minimal series representation of Virasoro Lie algebra [2, Theorem 3.4.3].

The Neveu–Schwarz Lie superalgebra $\mathfrak{g}$ also has the family of minimal series representations $M_{c,h}$ with $(c, h) = (c_{p,q}^S, h_{p,q}^S)$, where $c_{p,q}^S$ is given in Fact 4.2.14,
and $h_{r,s}^{p,q} = \frac{(sp-rq)^2-(p-q)^2}{8pq}$ for $(r, s) \in \mathbb{Z}^2$ satisfying $0 < r < p$, $0 < s < q$ and $r-s \in 2\mathbb{Z}$. See [19, Sect. 3] and [1, Sect. 1] for the detail. At this moment, the author is not sure whether the argument in [8] works for Neveu–Schwarz minimal representations, but believes that it should do. Let us state it as:

**Conjecture 4.2.17** Let $M$ be a simple module over the Neveu–Schwarz Lie superalgebra $\mathfrak{g}$ with central charge $c$. Then the following conditions are equivalent.

- $M$ is $C_2$-cofinite as a module over the Neveu–Schwarz SUSY vertex algebra $V^c$.
- $M$ is a minimal series representation of $\mathfrak{g}$.

5 Concluding comments

As mentioned in Sect. 0, this note is written as a first step toward a semi-infinite Poisson-geometric study of SUSY vertex algebras. There are already several geometric studies of SUSY vertex algebras, and our next step would be to relate this note and those studies. Here is a list of possible directions.

1. The $C_2$-cofiniteness guarantees the modular property of conformal blocks in the even case (see, e.g., [25, 28]), as mentioned in Sect. 4.1. Mathematical study of conformal blocks of a SUSY vertex algebra seems to be still in infancy. One remarkable study in this direction is, as mentioned in Sect. 0, the work of Heluani and Van Ekeren [17], which shows that the normalized character of a positively graded module over a charge cofinite $N = 2$ topological vertex algebra can be regarded as a solution of a flat connection over the moduli space of elliptic super-curves, and actually are Jacobi forms. An $N = 2$ topological vertex algebra has a structure of a strongly conformal $N_W = 1$ SUSY vertex algebra by the result of Heluani and Kac [16], and by the work of Heluani [14], we have the vector bundle with flat connection of conformal blocks on the moduli space. It is shown in [17, A.2. Lemma] that the charge cofiniteness is equivalent to the $C_2$-cofiniteness for the underlying vertex superalgebra. We can show that it is also equivalent to the $C_2$-cofiniteness of the associated $N_W = 1$ SUSY vertex algebra in the sense of Definition 4.1.1. Thus, the result of [17] implies that modules over a $C_2$-cofinite $N_W = 1$ SUSY vertex algebra satisfying Assumption 4.2.3 would enjoy super-modularity.

2. We also expect an analogous result of the item ((1)) for the $N_W = N > 1$ case. The argument of [17] is based on a SUSY analogue [14, Theorem 3.4] of Huang’s coordinate change formula, applied to a natural $N_W = 1$ SUSY analogue $(z, \zeta) \mapsto (e^{2\pi i z} - 1, e^{2\pi i \zeta})$ of Zhu’s coordinate change $z \mapsto e^{2\pi i z} - 1$, and based on $N = 1$ super analogue of Weierstrass elliptic functions. Both have straightforward $N > 1$ analogue, but the analysis gets much more complicated, and at this moment the author has no clear picture.

3. Along the line of (1) and (2), the $N_K = N$ case is mysterious for the author. The $N_K = N$ SUSY analogue of Huang’s formula is given in [14, Theorem 3.6]. Mimicking the argument in [16], the author tried to find a clean coordinate change which preserves the superconformal structure (also called SUSY structure or super Riemann surface structure), but has failed until this moment.
(4) Also related to super-modularity, let us recall the notion of quasi-lisse vertex algebras introduced in the work of Arakawa and Kawasetsu [5]. It is a finitely strongly generated vertex algebra whose associated variety (the reduced part of the associated scheme in our terminology) has only finitely many symplectic leaves. In [5, Theorem 5.1], it is shown that the normalized character of a module over a quasi-lisse vertex operator algebra enjoys modularity. An \( N_W = 1 \) SUSY analogue of this result might be obtained by borrowing some arguments in [17], but one must take care to establish a super (non-reduced) analogue of the results of Poisson geometry used in [5]. The other \( N_W = N > 1 \) and \( N_K = N \) cases are much more unclear at this moment.

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