Decompositions of set-valued mappings

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On 100th anniversary of Professor V. S. Čarin*

Abstract. Let $X$ be a set, $B_X$ denotes the family of all subsets of $X$ and $F: X \to B_X$ be a set-valued mapping such that $x \in F(x)$, $\sup_{x \in X} |F(x)| < \kappa$, $\sup_{x \in X} |F^{-1}(x)| < \kappa$ for all $x \in X$ and some infinite cardinal $\kappa$. Then there exists a family $F$ of bijective selectors of $F$ such that $|F| < \kappa$ and $F(x) = \{f(x) : f \in F\}$ for each $x \in X$. We apply this result to $G$-space representations of balleans.

1. Decompositions

For a set $X$, $B_X$ denotes the family of all subsets of $X$. Given a set-valued mapping $F: X \to B_X$, any function $f: X \to X$ such that, for each $x \in X$, $f(x) \in F(x)$ is called a selector of $F$. We say that a selector $f$ is bijective if $f: X \to X$ is a bijection. For $x \in X$, we denote $F^{-1}(x) = \{y \in X : x \in F(y)\}$.

In section 1 we prove the main result and apply it to $G$-space representations of balleans in section 2.

Theorem 1. Let $F: X \to B_X$ be a set-valued mapping such that $x \in F(x)$, $\sup_{x \in X} |F(x)| < \kappa$, $\sup_{x \in X} |F^{-1}(x)| < \kappa$ for each $x \in X$ and some infinite cardinal $\kappa$. Then there exists a family $F$ of bijective selectors of $X$ such that $|F| < \kappa$ and $F(x) = \{f(x) : f \in F\}$ for each $x \in X$.

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Assume the contrary and choose if necessary) F of bijective selectors of F and y such that F(y) ∩ F(yi) ≠ ∅ for every i ∈ {1, ..., m^2}. Then yi ∈ F^{-1}(F(y)) but, by the choice of m, we have |F^{-1}(F(y))| < m^2.

We use the following simple fact [2]: if the local degree of each vertices of a graph Γ' does not exceed k then the chromatic number of Γ' does not exceed k + 1.

Hence the set P of vertices of Γ can be partition P_1, ..., P_{m^2} so that any two vertices from each P_i are not incident.

To construct the family F, we enumerate P_i = {F(y_α) : α < γ}. Let M = sup_{x ∈ X} |F(x)|. Then we enumerate each F(y_α) (with repetitions, if necessary) F(y_α) = {y_αj : j < M}, y_α0 = y_α. For each j < M, we define a bijective function f_j such that f_j acts as a transposition of y_α and y_αj at each F(y_α) and identically at all other elements of X. We put F_i = {f_j : j < M} and note that F = F_1 ∪ ... ∪ F_{m^2} is the desired family of selectors of F.

Case κ > ω. We take an infinite cardinal σ such that σ < κ and |F(x)| ≤ σ, |F^{-1}(x)| ≤ σ for each x ∈ X. Then we define a partition P of X such that each P ∈ P is the minimal by inclusion subset of X satisfying F(y) ∈ P, F^{-1}(y) ∈ P for each y ∈ P. Constructively, every P can be obtained applying to x ∈ P the sequence of operations F, F^{-1} : F(x), F^{-1}F(x), FF^{-1}F(x), .... Then P is the union of all numbers of this sequence.

By the choice of σ, we have |P| ≤ σ. We enumerate P = {P_α : α < γ}, P_α = {x_αj : j < γ}. For each j < σ, we choose a family F_j of bijective selectors of F such that |F_j| ≤ σ and F(x_αj) = {f(x_αj) : f ∈ F_j} for each α < γ, see the case κ = ω. Then ∪_{j<σ} F_j is the desired family F of bijective selectors of F.

2. Applications

Let X be a set. A family E of subsets of X × X is called a coarse structure if

• each E ∈ E contains the diagonal △_X, △_X = {(x, x) : x ∈ X};
• if E, E' ∈ E then E ∘ E' ∈ E and E^{-1} ∈ E, where E ∘ E' = {(x, y) : ∃z((x, z) ∈ E, (z, y) ∈ E')}, E^{-1} = {(y, x) : (x, y) ∈ E};
• if $E \in \mathcal{E}$ and $\triangle_X \subseteq E' \subseteq E$ then $E' \in \mathcal{E}$;
• for any $x, y \in X$, there exists $E \in \mathcal{E}$ such that $(x, y) \in E$.

A subset $\mathcal{E}' \subseteq \mathcal{E}$ is called a base for $\mathcal{E}$ if, for every $E \in \mathcal{E}$, there exists $E' \in \mathcal{E}'$ such that $E \subseteq E'$. For $x \in X$, $A \subseteq X$ we denote $E[x] = \{y \in X : (x, y) \in E\}$, $E[A] = \cup_{a \in A} E[a]$ and say $E(x)$ and $E(A)$ are balls of radius $E$ around $x$ and $A$.

The pair $(X, \mathcal{E})$ is called a coarse space [6] or a ballean [5].

Let $(X, \mathcal{E})$, $(X', \mathcal{E}')$ be coarse spaces. A mapping $f : X \to X'$ is called macro-uniform if, for every $E \in \mathcal{E}$ there exists $E' \in \mathcal{E}'$ such that $E[x] \subseteq E'[f(x)]$. If $f$ is a bijection such that $f, f^{-1}$ are macro-uniform then $f$ is called an asymorphism.

Now we describe some general way of constructing balleans. Let $G$ be a group. A family $\mathcal{I}$ of subsets of $G$ is called an ideal if, for every $A, B \in \mathcal{I}$ and $A' \subseteq A$, we have $A \cup B \in \mathcal{I}$ and $A' \in \mathcal{I}$. An ideal $\mathcal{I}$ is called a group ideal if $F \in \mathcal{I}$ for every finite subset of $G$ and $A, B \in \mathcal{I}$ imply $AB^{-1} \in \mathcal{I}$.

Let a group $G$ acts transitively on a set $X$ by the rule $(g, x) \mapsto gx$, $g \in G$, $x \in X$. Every group ideal $\mathcal{I}$ on $G$ defines the ballean $(X, G, \mathcal{I})$ on $X$ with the base of entourages $\{(x, y) : y \in Ax : A \in \mathcal{I}\}$. By Theorem 1 from [3], for every ballean $(X, \mathcal{E})$, there exist a group $G$ of permutations of $X$ and a group ideal $\mathcal{I}$ on $G$ such that $(X, \mathcal{E})$ is asymorphic to $(X, G, \mathcal{I})$.

**Theorem 2.** Let $(X, \mathcal{E})$ be a ballean and let $\kappa$ be an infinite cardinal such that, for each $E \in \mathcal{E}$, $\sup_{x \in E} |E[x]| < \kappa$. Then there exist a group $G$ of permutations of $X$ and a group ideal $\mathcal{I}$ on $G$ such that $(X, \mathcal{E})$ is asymorphic to $(X, \mathcal{E}, \mathcal{I})$ and $|A| < \kappa$ for each $A \in \mathcal{I}$.

**Proof.** For each $E \in \mathcal{E}$, we define a mapping $F_E : X \to B_X$ by $F_E(x) = E[x]$. By Theorem 1, there exists a family $F_E$ of permutations of $X$ such that $|F_E| < \kappa$ and $F_E(x) = \{f(x) : f \in F_E\}$ for each $x \in X$. We denote by $\mathcal{I}$ the minimal by inclusion group ideal of $G$ such that $F_E \in \mathcal{I}$ for each $E \in \mathcal{E}$. Then $(X, \mathcal{E})$ is asymorphic to $(X, G, \mathcal{I})$. \hfill $\square$

In the case $\kappa = \omega$, Theorem 2 was proved in [4]. For its applications see Remark 3.5 in [1].

A ballean $(X, \mathcal{E})$ is called cellular if $\mathcal{E}$ has a base consisting of equivalence relations. By Theorem 3 from [3], every cellular ballean is asymorphic to some ballean $(X, G, \mathcal{I})$ such that $\mathcal{I}$ has a base consisting of subgroups of $G$.

A ballean $(X, \mathcal{E})$ is called finitary if, for every $E \in \mathcal{E}$ there exists a natural number $m$ such $|E[x]| < m$ for each $x \in X$. The finitary ballean
of a $G$ space $X$ is the ballean $(X, G, \mathcal{I})$, where $\mathcal{I}$ is the ideal of all finite subsets of $G$.

**Theorem 3.** For every finitary cellular ballean $(X, \mathcal{E})$ there exists a locally finite group of permutations of $X$ such that $(X, \mathcal{E})$ is asymorphic to the finitary ballean of $G$-space $X$.

**Proof.** We take a base $\mathcal{E}'$ of consisting of partitions of $X$. For every $\mathcal{P} \in \mathcal{E}$ we pick a natural number $n_{\mathcal{P}}$ such that $|P| \leq n_{\mathcal{P}}$ for each $P \in \mathcal{P}$. We denote by $G_{\mathcal{P}}$ the direct product of the family of symmetric groups $\{S_m : m \leq n_{\mathcal{P}}\}$ and note that $G_{\mathcal{P}}$ acts on each $P \in \mathcal{P}$ so that $G_{\mathcal{P}}x = P$ for each $x \in P$. Then the group $G$ generated by the family $\{G_{\mathcal{P}} : \mathcal{P} \in \mathcal{E}'\}$ satisfies the conclusion of Theorem 3.

**References**

[1] Cornulier Y. *On the space of ends of infinitely generated groups*, arXiv: 1901.11073.
[2] A. Harary, *Graph Theory*, Addison-Wesley, 1994.
[3] O. V. Petrenko, I.V. Protasov, *Balleans and $G$-spaces*, Ukr. Mat. Zh. **64** (2012), 344-350.
[4] I.V. Protasov, *Balleans of bounded geometry and $G$-space*, Algebra Discrete Math. 2008, no 2, 101-108.
[5] I. Protasov, M. Zarichnyi, *General Asymptology*, Mat. Stud. Monogr. Ser, vol. 12, VNTL, Lviv, 2007.
[6] J. Roe, *Lectures on Coarse Geometry*, Univ. Lecture Ser., vol. 31, American Mathematical Society, Providence RI, 2003.

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