CDF of non–central $\chi^2$ distribution revisited. 
Incomplete hypergeometric type functions approach

Dragana Jankov Maširević$^a$, Tibor K. Pogány$^{b,c,*}$

$^a$Department of Mathematics, University of Osijek, 31000 Osijek, Croatia  
$^b$Faculty of Maritime Studies, University of Rijeka, 51000 Rijeka, Croatia  
$^c$Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Hungary

Abstract

The cumulative distribution function of the non–central chi-square distribution $\chi^2(\lambda, \nu)$, $\nu \in \mathbb{R}^+$ possesses an integral representation in terms of a generalized Marcum $Q$–function. Regarding some already known results, here we derive a simpler form of the cumulative distribution function for $\nu = 2n \in \mathbb{N}$ degrees of freedom. Also, we express these representations in terms of an incomplete Fox–Wright function $\psi^{(\gamma)}_q$ and the generalized incomplete hypergeometric functions concerning the important special cases as $\Gamma_1$, $2\Gamma_1$ and $2\gamma_1$. New identities are established between $\Gamma_1$ and $2\Gamma_1$ as well.

Keywords: CDF of non–central $\chi^2$ distribution, Modified Bessel function of the first kind, Marcum $Q$–function, Incomplete gamma functions, (In)complete Fox–Wright function, Generalized (in)complete hypergeometric functions.

2010 MSC: Primary: 40H05, 60E05; Secondary: 33C10, 62E10.

1. Introduction and motivation

If $X_1, X_2, \ldots, X_\nu$ are independent homoscedastic normal $\mathcal{N}(\mu_j, \sigma^2)$, $\mu_j \in \mathbb{R}$, $j = 1, \nu$, $\sigma > 0$ random variables (rv) defined on a standard probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the rv $\xi = X_1^2 + \cdots + X_\nu^2$ has non–central $\chi^2$–distribution with $\nu \in \mathbb{N}$ degrees of freedom (the size number of the sum) and with the

*Corresponding author

Email addresses: djankov@mathos.hr (Dragana Jankov Maširević), poganj@pfri.hr (Tibor K. Pogány)
The non–centrality parameter $\lambda = \mu_1^2 + \cdots + \mu_n^2 \geq 0$. The distribution of the rv $\xi$ is usually denoted by $\chi^2_\nu(\lambda)$ [9, p. 433] and the appropriate probability density function (PDF) can be expressed as [6, 9]

$$f_{\nu,\lambda}(x) = \frac{1}{2} e^{-\frac{x}{\lambda}} \left( \frac{x}{\lambda} \right)^{\frac{\nu}{2}} I_{\frac{\nu}{2}-1}(\sqrt{\lambda x}), \quad x > 0,$$

(1.1)

where $I_\eta$ denotes the modified Bessel function of the first kind of the order $\eta$, which has the power series definition [13, p. 249, Eq. 10.25.2]

$$I_\eta(z) = \sum_{k \geq 0} \frac{1}{\Gamma(\eta + k + 1) k!} \left( \frac{z}{2} \right)^{2k+\eta}, \quad \Re(\eta) > -1, \ z \in \mathbb{C}. \quad (1.2)$$

The non–central $\chi^2$–distribution is, indeed, one of the most applied distributions, having application in some statistical tests [11], in finance, estimation and decision theory, time series analysis [2, 20], in mathematical physics [9, p. 435] and among others in communication theory in which case the appropriate cumulative distribution function (CDF) is given by [8, p. 66, Eq. (1.1)]

$$F_{\nu,\lambda}(x) = 1 - Q_{\frac{\nu}{2}}(\sqrt{\lambda}, \sqrt{x}), \quad x > 0,$$

(1.3)

where $Q_\mu$ stands for the generalized Marcum $Q$–function [3]

$$Q_\mu(a, b) = \frac{1}{a^{\mu-1}} \int_b^\infty t^\mu e^{-\frac{a^2+b^2}{2}} I_{\mu-1}(at) \, dt, \quad a, \mu > 0, \ b \geq 0; \quad (1.4)$$

in this case the non-centrality parameter $\lambda$ is interpreted as a signal–to–noise ratio [9, p. 435].

Although, in general, $\nu$ can be a nonnegative real number [18], considering, obviously, only the PDF (1.1), without bearing in mind the degrees of freedom ancestry and the structure of the rv $\xi \sim \chi^2_\nu(\lambda)$, most of the authors have dealt with PDF and CDF in the case when $\nu = n \in \mathbb{N}$ (see e.g. [9, 14, 15, 19, 25]). Quite recently, Brychkov derive a closed–form expression for the generalized Marcum $Q$–function [5, p. 178, Eq. (7)] in terms of the complementary error function $\text{erfc}(z)$ [13, p. 160, Eq. 7.2.2] which immediately implies a new formula for CDF (1.3) in the case when $n \in \mathbb{N}$ is odd; in the case of even number of the degrees of freedom Jankov Maširević derived [8] a novel expression for the appropriate CDF (1.3) in terms of the modified Bessel function of the second kind $K_\nu$ and its incomplete variant.
$K_\nu(z, w)$, see [1, p. 26, Eq. (1.30)], where, for $\Re(z) > 0$, $K_\nu(z, w) \rightarrow K_\nu(z)$, as $w \rightarrow \infty$, in pointwise sense. It is also worth to mention that Jankov Maširević established the computational efficiency (compare [8, Section 3]) of hers formulae versus the relations by Temme for even $n \in \mathbb{N}$, which ones, rewritten in our setting read [25, p. 58, Eq. (2.8)]

$$F_{n, \lambda}(x) = \begin{cases} 1 - \frac{1}{2} \left( \frac{x}{\lambda} \right)^{\frac{\nu}{2}} \left[ T_{\frac{n}{2}-1}(\sqrt{\lambda x}, \omega) - \sqrt{\frac{\lambda}{x}} T_{\frac{n}{2}}(\sqrt{\lambda x}, \omega) \right], & x > \lambda \\ \frac{1}{2} \left( \frac{x}{\lambda} \right)^{\frac{\nu}{2}} \left[ \sqrt{\frac{\lambda}{x}} T_{\frac{n}{2}}(\sqrt{\lambda x}, \omega) - T_{\frac{n}{2}-1}(\sqrt{\lambda x}, \omega) \right], & x < \lambda \end{cases}$$

(1.5)

here $\omega = (x + \lambda) (2\sqrt{\lambda x})^{-1} - 1$, while

$$T_\nu(\sqrt{\lambda x}, \omega) = \int_{\sqrt{\lambda x}}^{\infty} e^{-(\omega + 1)t} I_\nu(t) \, dt.$$  

Temme claimed that his formulae have certain computational advantages.\(^1\)

Introducing the function

$$S_\nu(\sqrt{\lambda x}, \omega) = \int_{0}^{\sqrt{\lambda x}} e^{-(\omega + 1)t} I_\nu(t) \, dt,$$  

(1.6)

with the help of the Laplace transform of the modified Bessel function [16, p. 313, Eq. 2.15.3.1] we conclude

$$S_\nu(\sqrt{\lambda x}, \omega) + T_\nu(\sqrt{\lambda x}, \omega) = \frac{(x + \lambda - |x - \lambda|)^\nu}{(2\sqrt{\lambda x})^{\nu-1}|x - \lambda|}, \quad \nu > -1, \min\{x, \lambda\} > 0,$$

it follows, by the corresponding considerations in [8], that (1.5) can be written in the modified symmetric form also for all $n \in \mathbb{N}$ and $\min\{x, \lambda\} > 0$:

$$F_{n, \lambda}(x) = \frac{1}{2} \left( \frac{x}{\lambda} \right)^{\frac{\nu}{2}} \left\{ S_{\frac{n}{2}-1}(\sqrt{\lambda x}, \omega) - \sqrt{\frac{\lambda}{x}} S_{\frac{n}{2}}(\sqrt{\lambda x}, \omega) \right\}$$

(1.7)

which is a more convenient representation for numerical calculations. In turn,

\(^{1}\)We mention that the formulae (1.5) are also listed in the book Johnson et al. [9, p. 441, Eq. (29.20)], but unfortunately in an erroneous form.
for $x = \lambda$ this expression reduces to a difference of two $S_\nu(\lambda, 0)$ integrals which are generalized hypergeometric $\text-_2F_2$ functions, viz.

$$F_{n,\lambda}(\lambda) = \frac{1}{\Gamma\left(\frac{n}{2} + 1\right)} \left(\frac{\lambda}{2}\right)^{\frac{n}{2}} \left\{ \text-_2F_2\left[ \frac{n-1}{2}, \frac{n}{2}, n-1 \left| -2\lambda \right. \right] - \frac{\lambda}{n + 2} \text-_2F_2\left[ \frac{n+1}{2}, \frac{n}{2} + 1 \left| -2\lambda \right. \right] \right\}.$$  \hspace{1cm} (1.8)

One of the main aims of this paper is to derive another elegant expression for $F_{2n,\lambda}$ related to (1.7). That result is presented in the next section. In the Section 3 we show that this new CDF formula can be explicitly expressed in terms of the incomplete Fox–Wright function $\text{}_p\Psi_q^{(\gamma)}$. The Section 4 consists from expressions for CDF which are established in terms of the incomplete confluent $1\Gamma_1$ and Gaussian hypergeometric function $\text{_2\Gamma_1}$. Some new identities between $1\Gamma_1$, $2\Gamma_1$ and $2\gamma_1$ end this part. The exposition closes the fifth section with a discussion and further related remarks.

2. On CDF of $\chi'^2_{2n}(\lambda)$–distribution regarding Temme’s result

In this section we will show that $F_{2n,\lambda}$ can be presented in simple form, containing only modified Bessel functions $I_n$, $n \in \mathbb{N}_0$ and the function $S_\nu$ of the order $\nu = 0$. The main tool we refer to is a formula by Jankov Maširević which consists Theorem 1 exposed in a slightly different, but more condensed way then in [8].

**Theorem 1.** [8, p. 4, Theorem 2.1] The CDF of the non–central chi-square distribution with an even number of the degrees of freedom can be represented in the form

$$F_{2n,\lambda}(x) = e^{-\frac{\lambda x}{2}} \sum_{k \geq n} \left(\frac{x}{\lambda}\right)^{\frac{k}{2}} I_k(\sqrt{\lambda x}), \hspace{1cm} (2.1)$$

where $n \in \mathbb{N}$ and $\min\{\lambda, x\} > 0$.

In 1971 Agrest and Maksimov [1, p. 139, Eq. (6.15)] concluded that

$$\int_0^z e^{\pm\alpha t} I_0(t) \, dt = \frac{1}{\sqrt{\alpha^2 - 1}} \left\{ 1 - e^{\mp\alpha z} \left[ I_0(z) + 2Y_2\left(\frac{z}{c}, z\right) \right] \right\},$$

where the parameters $\alpha, c$ are related as $2\alpha = c + c^{-1}$ and $Y_\nu(w, z)$ stands for the Lommel function of two variables of the order $\nu$, defined by the Neumann
series [1, p. 138, Eq. (6.5)]
\[ Y_\nu(w, z) = \sum_{k \geq 0} \left( \frac{w}{z} \right)^{\nu+2k} I_{\nu+2k}(z). \]

**Theorem 2.** For all \( n \in \mathbb{N} \) and \( \min\{\lambda, x\} > 0 \) there holds
\[ F_{2n,\lambda}(x) = \frac{1}{2} - e^{-\frac{\lambda+x}{2}} \left( \frac{1}{2} I_0(\sqrt{\lambda x}) + \sum_{k=1}^{n-1} \left( \frac{x}{\lambda} \right)^{k} I_k(\sqrt{\lambda x}) \right) - \frac{1}{4\sqrt{\lambda x}} S_0(\sqrt{\lambda x}, \omega), \]
where \( S_0 \) is defined in (1.6) and \( \omega = (x + \lambda) \left( 2\sqrt{\lambda x} \right)^{-1} - 1. \)

**Proof.** Taking \( z = \sqrt{\lambda x}, c = \sqrt{\lambda/x} \) in (2.2) we rewrite the expression in brackets into
\[ I_0(\sqrt{\lambda x}) + 2Y_2(x, \sqrt{\lambda x}) + 2Y_1(x, \sqrt{\lambda x}) \]
\[ = I_0(\sqrt{\lambda x}) + 2 \sum_{k \geq 0} \left( \frac{x}{\lambda} \right)^{k+1} I_{2k+2}(\sqrt{\lambda x}) + 2 \sum_{k \geq 0} \left( \frac{x}{\lambda} \right)^{k+\frac{1}{2}} I_{2k+1}(\sqrt{\lambda x}) \]
\[ = -I_0(\sqrt{\lambda x}) + 2 \sum_{k \geq 0} \left( \frac{x}{\lambda} \right)^{k} I_{2k}(\sqrt{\lambda x}) + 2 \sum_{k \geq 0} \left( \frac{x}{\lambda} \right)^{k+\frac{1}{2}} I_{2k+1}(\sqrt{\lambda x}) \]
\[ = -I_0(\sqrt{\lambda x}) + 2 \sum_{k \geq 0} \left( \frac{x}{\lambda} \right)^{k} I_k(\sqrt{\lambda x}), \]
where in the last equality we employed the elementary identity
\[ \sum_{k \geq 0} a_k = \sum_{k \geq 0} a_{2k} + \sum_{k \geq 0} a_{2k+1}. \]

By simple considerations it follows from the relations (2.2) and (2.4) that
\[ \sum_{k \geq 0} \left( \frac{x}{\lambda} \right)^{k} I_k(\sqrt{\lambda x}) = \frac{1}{2} \left( e^{\frac{\lambda+x}{2}} + I_0(\sqrt{\lambda x}) \right) - \frac{e^{\frac{\lambda+x}{2}} |\lambda - x|}{4\sqrt{\lambda x}} \int_0^{\sqrt{\lambda x}} e^{-\frac{1}{2\sqrt{\lambda x}} t} I_0(t) \, dt. \]
We deduce the assertion of the theorem combining the above inferred formula with the identity (2.1) given in Theorem 1. \( \square \)
3. CDF in terms of incomplete Fox–Wright function

In Theorem 2 we derive a new representation for the CDF of \( \xi \sim \chi_{2n}^2(\lambda) \) in terms of the integral \( S_0 \). In this section, we derive new expression for \( S_0 \) which yields, in combination with Theorem 2, a new form for the CDF.

The Fox–Wright generalized hypergeometric function with \( p \) numerator parameters \( a_1, \cdots, a_p \) and \( q \) denominator parameters \( b_1, \cdots, b_q \) is defined by the series [26, pp. 286–287]

\[
p\Psi_q \left[ \begin{array}{c} (a_1, A_1), \cdots, (a_p, A_p) \\ (b_1, B_1), \cdots, (b_q, B_q) \end{array} \right] | z \] = \sum_{n \geq 0} \frac{\prod_{j=1}^{p} \Gamma(a_j + nA_j)}{\prod_{j=1}^{q} \Gamma(b_j + nB_j)} z^n n!,
\]

(3.1)

where \( A_i, B_j \geq 0, i = 1, \cdots, p, j = 1, \cdots, q \). The defining series converges in the whole complex \( z \)-plane when

\[
\Delta := 1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i > 0;
\]

when \( \Delta = 0 \) the series in (3.1) converges for \( |z| < \nabla \), and \( |z| = \nabla \) under the condition \( \Re(\mu) > 1/2 \) where

\[
\nabla := \left( \prod_{i=1}^{p} A_i^{-A_i} \right) \left( \prod_{j=1}^{q} B_j^{B_j} \right), \quad \mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}.
\]

If in (3.1) we set \( A_1 = \cdots = A_p = 1 \) and \( B_1 = \cdots = B_q = 1 \) we get the generalized hypergeometric function \( pF_q \), up to the multiplicative constant:

\[
p\Psi_q \left[ \begin{array}{c} (a_p, 1) \\ (b_q, 1) \end{array} \right] | z \] = \frac{\Gamma(a_1) \cdots \Gamma(a_p)}{\Gamma(b_1) \cdots \Gamma(b_q)} pF_q \left[ \begin{array}{c} a_p \\ b_q \end{array} \right] | z \].

In what follows, the symbol \( p\Psi_q^{(\gamma)} \) stands for the incomplete Fox–Wright function as a generalization of the complete Fox-Wright function \( p\Psi_q \) [24].
The series definition reads [24, p. 131, Eq. (6.1)]

\[
p^q \Psi^{(\gamma)} \left[ (\mu, M, x), (a_{p-1}, A_{p-1}); (b_q, B_q) \mid z \right] = \sum_{n \geq 0} \gamma(\mu + nM, x) \frac{\prod_{j=1}^{p-1} \Gamma(a_j + nA_j)}{\prod_{j=1}^{q} \Gamma(b_j + nB_j)} \frac{z^n}{n!},
\]

where \( \gamma(a, x) \) denotes the lower incomplete gamma function

\[
\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt, \quad \Re(a) > 0.
\]

Parameters \( M, A_j, B_j > 0 \) should satisfy the constraint

\[
\Delta^{(\gamma)} = 1 + \sum_{j=1}^{q} B_j - M - \sum_{j=1}^{p-1} A_j \geq 0,
\]

while the convergence conditions coincide with the ones regarding the complete Fox–Wright (3.1). The case \( p = q = 1 \) leads to the confluent incomplete Fox–Wright hypergeometric function.

**Lemma 1.** For all positive real numbers \( \min \{\lambda, x\} > 0 \) there holds

\[
S_\nu \left( \sqrt{\lambda x}, \frac{x + \lambda}{2\sqrt{\lambda x}} - 1 \right) = \frac{2(\lambda x)^{\nu+1}}{(x + \lambda)^{\nu+1}} \Psi_1^{(\gamma)} \left[ (\nu + 1, 2, \frac{x+\lambda}{2}); (\nu + 1, 1) \mid \frac{\lambda x}{(x + \lambda)^2} \right].
\]

**Proof.** By expanding the Bessel function in (1.6), we obtain an infinite series in terms of the lower incomplete gamma functions (3.3):

\[
S_\nu(\sqrt{\lambda x}, \omega) = \sum_{k \geq 0} \frac{2^{-2k-\nu}}{\Gamma(\nu + k + 1) k!} \int_0^{\sqrt{\lambda x}} e^{-u(\omega+1)\nu} u^{2k-\nu+1} du
\]

\[
= \frac{1}{2^\nu(\omega+1)^{\nu+1}} \sum_{k \geq 0} \frac{(2(\omega + 1))^{-2k}}{\Gamma(\nu + k + 1) k!} \int_0^{\frac{x+\lambda}{2}} e^{-u\nu} u^{2k-\nu} du
\]

\[
= \frac{1}{2^\nu(\omega+1)^{\nu+1}} \sum_{k \geq 0} \frac{\gamma(\nu + 1 + 2k, (x + \lambda)/2)}{\Gamma(\nu + k + 1) k! (2(\omega + 1))^{2k}}
\]

which is equivalent to the statement, being \( \omega = (x + \lambda)(2\sqrt{\lambda x})^{-1} - 1 \).
The previous expression is very convenient for computing CDF $F_{n,\lambda}$ for not too large values of the variables.

Concerning (1.7) and Lemma 1 we deduce the following result.

**Theorem 3.** For all positive real numbers $\min\{\lambda, x\} > 0$ there holds

$$F_{n,\lambda}(x) = \frac{x^n}{(x+\lambda)^{n/2}} \left\{ \psi_1^{(\gamma)} \left[ \left( \frac{n}{2}, 2, \frac{x+\lambda}{2} \right) \left| \frac{x\lambda}{(x+\lambda)^2} \right] \right. \\
- \frac{\lambda}{x+\lambda} \psi_1^{(\gamma)} \left[ \left( \frac{n}{2} + 1, 2, \frac{x+\lambda}{2} \right) \left| \frac{x\lambda}{(x+\lambda)^2} \right] \right\}.$$  

We express now $S_0$ in terms of the incomplete Fox–Wright function (3.2).

**Lemma 2.** For all positive real numbers $p, b > 0$ there holds

$$S_0(p, b) = \frac{1}{p} \psi_1^{(\gamma)} \left[ \left( 1, 2, pb \right) \left| \frac{1}{4p^2} \right] \right.$$  

$$= \frac{1}{2p^3} \psi_1^{(\gamma)} \left[ \left( 2, 2, pb \right) \left| \frac{1}{4p^2} \right] \right] - \frac{e^{-pb}}{p} I_0(b).$$

**Proof.** Using the definition of the modified Bessel function (1.2) and the special case of the confluent hypergeometric (Kummer) function [13, p. 327, Eq. 13.6.2]

$$1 \psi_1^{(\gamma)} \left[ \left( \frac{1}{2} \right) \left| \frac{1}{z} \right] = e^z - \frac{1}{z},$$

we obtain

$$S_0(p, b) = \int_0^b e^{-px} I_0(x) \, dx = \sum_{k \geq 0} \frac{4^{-k}}{\Gamma(k+1) k!} \int_0^b e^{-px} x^{2k} \, dx$$  

$$= \sum_{k \geq 0} \frac{4^{-k}}{\Gamma(k+1) k!} \left( \frac{\partial}{\partial p} \right)^{2k} e^{-px} \, dx$$  

$$= \sum_{k \geq 0} \frac{4^{-k}}{\Gamma(k+1) k!} \left( \frac{\partial}{\partial p} \right)^{2k} \frac{1 - e^{-pb}}{p}$$  

$$= b \sum_{k \geq 0} \frac{4^{-k}}{\Gamma(k+1) k!} \left( \frac{\partial}{\partial p} \right)^{2k} 1 \psi_1^{(\gamma)} \left[ \left( \frac{1}{2} \right) \left| \frac{1}{pb} \right] \right].$$
As the derivative of the hypergeometric function equals

\[
\left( \frac{\partial}{\partial p} \right)^{2k} {}_{1}F_{1} \left[ \frac{1}{2} \left| -pb \right. \right] = \frac{(1)_{2k} b^{2k}}{(2)_{2k}} {}_{1}F_{1} \left[ \frac{2k + 1}{2k + 2} \left| -pb \right. \right] = \frac{b^{2k}}{2k + 1} {}_{1}F_{1} \left[ \frac{2k + 1}{2k + 2} \left| -pb \right. \right] =: H_1,
\]

moreover

\[
H_1 = \frac{b^{2k}}{2k + 1} \cdot \frac{2k + 1}{(pb)^{2k+1}} \gamma(2k + 1, pb) = \frac{\gamma(2k + 1, pb)}{b p^{2k+1}}.
\]

We have

\[
S_0(p, b) = \frac{1}{p} \sum_{k \geq 0} \frac{\gamma(2k + 1, pb)}{\Gamma(k + 1) k!} \frac{1}{(4p^2)^k},
\]

which is equivalent with the first equality in (3.4). Finally, having in mind the contiguous recurrence formula [13, p. 178, Eq. 8.8.1]

\[
\gamma(a + 1, x) = a \gamma(a, x) - x^a e^{-x},
\]

we transform (3.6) into an elegant (seemingly new) formula

\[
S_0(p, b) = \frac{2}{p} \sum_{k \geq 1} \frac{\gamma(2k, pb)}{(4p^2)^k} \frac{1}{k!} \frac{1}{\Gamma(k)} - e^{-pb} I_0(b)
\]

which completes the proof of (3.5).

Remark 1. The recursion formula (3.7) should be used with care in computations, because the numerical recursion is rather unstable, consult for instance [7, p. 114].

Now, combining Theorem 2 and Lemma 2 we infer the following result.

Theorem 4. For all \( n \in \mathbb{N} \) and \( \min\{\lambda, x\} > 0 \) there holds

\[
F_{2n, \lambda}(x) = \frac{1}{2} - e^{-\frac{\lambda x}{2}} \left( \frac{1}{2} I_0 \left( \sqrt{\lambda x} \right) + \sum_{k=1}^{n-1} \left( \frac{x}{\lambda} \right)^k I_k(\sqrt{\lambda x}) \right).
\]
Moreover

\[ F_{2n,\lambda}(x) = \frac{1}{2} - e^{-\frac{x+\lambda}{2}} \left( \frac{1}{2} I_0(\sqrt{\lambda}x) + \frac{1}{2\lambda x} \Psi_1^{(\gamma)} \left[ \left( \frac{1}{2}, 2, \frac{(\sqrt{x} - \sqrt{\lambda})^2}{2} \right) \left( \frac{1}{2}, 1 \right) \right] \left( \frac{1}{4\lambda x} \right) \right) \]

\[ - \frac{\lambda - x}{4\lambda x} \left( \frac{1}{2\lambda x} \Psi_1^{(\gamma)} \left[ \left( 2, 2, \frac{(\sqrt{x} - \sqrt{\lambda})^2}{2} \right) \left( 2, 1 \right) \right] \left( \frac{1}{4\lambda x} \right) \right) \]

\[ - e^{-\frac{(\sqrt{x} - \sqrt{\lambda})^2}{2}} I_0 \left( \frac{x + \lambda}{2\sqrt{\lambda}x} - 1 \right) . \]

4. CDF in terms of incomplete hypergeometric function

Besides the already mentioned lower incomplete gamma function (3.3), there is also its complementary function, i.e. upper incomplete gamma function, defined by

\[ \Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} \, dt, \quad \Re(a) > 0 \quad (4.1) \]

and those two functions satisfy the following decomposition formula [13, p. 174, Eq. 8.2.3]

\[ \gamma(a, x) + \Gamma(a, x) = \Gamma(a), \quad \Re(a) > 0. \]

By means of the incomplete gamma functions, in order to generalize the Pochhammer symbol [17, p. 22]

\[ (\lambda)_\mu := \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } \mu = 0; \lambda \in \mathbb{C} \setminus \{0\} \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1), & \text{if } \mu = n \in \mathbb{N}; \lambda \in \mathbb{C} \end{cases} \]

in 2012 Srivastava et al. [22] introduced the incomplete Pochhammer symbols which lead to a natural generalization and decomposition of a class of hypergeometric functions. Precisely, the incomplete Pochhammer symbols
are defined as

\[(a; x)_\nu := \frac{\gamma(a + \nu, x)}{\Gamma(a)}, \quad [a; x]_\nu := \frac{\Gamma(a + \nu, x)}{\Gamma(a)}, \quad a, \nu \in \mathbb{C}, x \geq 0,\]

satisfying the decomposition

\[(a; x)_\nu + [a; x]_\nu = (a)_\nu, \quad a, \nu \in \mathbb{C}, x \geq 0.\]

Accordingly, the generalized incomplete hypergeometric functions have the definitions

\[p \gamma_q \left[ \begin{array}{c} (a_1, x), a_2, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right] (z) = p \gamma_q \left[ \begin{array}{c} (a_1, x), a_{p-1} \\ b_q \end{array} \right] = \sum_{n \geq 0} \frac{(a_1; x)_n}{\prod_{j=2}^{p} (b_j)_n} \frac{z^n}{n!},\]

and

\[p \Gamma_q \left[ \begin{array}{c} [a_1, x], a_2, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \right] (z) = p \Gamma_q \left[ \begin{array}{c} [a_1, x], a_{p-1} \\ b_q \end{array} \right] = \sum_{n \geq 0} \frac{[a_1; x]_n}{\prod_{j=2}^{p} (b_j)_n} \frac{z^n}{n!},\]

provided that the infinite series in each case absolutely converges where the appropriate convergence constraints coincide with the ones for the generalized hypergeometric function \(p F_q\), compare [22, p. 675, Remark 7].

One of the most important and widely used cases of generalized incomplete hypergeometric functions are those when \(p = q = 1\), the so-called incomplete confluent hypergeometric (or Kummer) function while for \(p = 2, q = 1\) one have the incomplete Gauss hypergeometric function. Regarding to our results, Srivastava et al. proved that [22, p. 680]

\[Q_M(\sqrt{2a}, \sqrt{2x}) = e^{-a} 1 \Gamma_1 \left[ \begin{array}{c} [M, x] \\ M \end{array} \right] a.\]

Also, it is not difficult to show the connection between the generalized Marcum \(Q\)–function and the Gaussian hypergeometric function \(2 \Gamma_1\), which we realize by the familiar confluence principle.
Lemma 3. For all \( \min\{a, x\} > 0 \) there holds

\[
Q_M(\sqrt{2a}, \sqrt{2x}) = e^{-a} \lim_{b \to \infty} 2 \Gamma_1 \left[ \begin{array}{c} [M, x], b \\ M \end{array} \right] \frac{a}{b}.
\]

Proof. Considering the limit representation formula of the Kummer function

\[
I_{\nu}(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu + 1)} \lim_{b \to \infty} \frac{1}{b^{\nu + 1}} \sum_{n \geq 0} \frac{(b)_n}{(M)_n n!} \left( \frac{a}{2b} \right)^n \int^\infty_{\sqrt{2x}} t^{2n+2\nu+1} e^{-t^2/2} dt
\]

the integral representation (1.4) yields

\[
Q_M(\sqrt{2a}, \sqrt{2x}) = e^{-a} \lim_{b \to \infty} \frac{1}{2^{M-1} \Gamma(M)} \sum_{n \geq 0} \frac{(b)_n}{(M)_n n!} \left( \frac{a}{2b} \right)^n \int^\infty_{\sqrt{2x}} t^{2M+2n-1} e^{-t^2/2} dt
\]

\[
= e^{-a} \lim_{b \to \infty} \frac{1}{\Gamma(M)} \sum_{n \geq 0} \frac{(b)_n}{(M)_n n!} \left( \frac{a}{2b} \right)^n \int^\infty_{x} t^{M+n-1} e^{-t} dt
\]

\[
= e^{-a} \lim_{b \to \infty} \frac{1}{\Gamma(M)} \sum_{n \geq 0} \frac{\Gamma(M+n,x)}{(M)_n n!} \frac{(b)_n}{(a)_n \left( \frac{a}{b} \right)^n}.
\]

By the definition (4.2) this proves the assertion of the lemma. \( \square \)

The result (4.3) and the formula proven in the Lemma 3 imply the next limit representation formula, which follows by the confluence principle for hypergeometric functions.

Proposition 1. The following limit representation holds true

\[
\lim_{b \to \infty} 2 \Gamma_1 \left[ \begin{array}{c} [a, x], b \\ c \end{array} \right] \frac{z}{b} = \Gamma_1 \left[ \begin{array}{c} [a, x] \\ c \end{array} \right] \frac{z}{b}.
\]

Proof. Having in mind the definitions of the generalized incomplete hypergeometric functions (4.2) and the upper incomplete gamma function (4.1), respectively, we get

\[
2 \Gamma_1 \left[ \begin{array}{c} [a, x], b \\ c \end{array} \right] \frac{z}{b} = \sum_{n \geq 0} \frac{\Gamma(a+n,x)}{(a)_n n!} \left( \frac{z}{b} \right)^n
\]

\[
= \int^\infty_x e^{-t} \frac{t^{a-1}}{\Gamma(a)} \sum_{n \geq 0} \frac{(b)_n}{(c)_n n!} \left( \frac{zt}{b} \right)^n dt = \int^\infty_x e^{-t} \frac{t^{a-1}}{\Gamma(a)} \Gamma_1 \left[ \begin{array}{c} b \\ c \end{array} \right] \frac{zt}{b} dt.
\]
Now, employing the limit representation \([27]\) for the confluent hypergeometric function \(_0F_1\) and the Kummer function \(_1F_1\), we have
\[
\lim_{b \to \infty} _1F_1 \left[ \frac{b}{c} \left| \frac{z}{b} \right. \right] = _0F_1 \left[ \frac{-c}{b} \right].
\]
Therefore,
\[
\lim_{b \to \infty} _2\Gamma_1 \left[ \frac{a, x}{c} \left| \frac{z}{b} \right. \right] = \int_x^\infty e^{-t} \frac{t^{a-1}}{\Gamma(a)} \lim_{b \to \infty} _1F_1 \left[ \frac{b}{c} \left| \frac{zt}{b} \right. \right] \, dt
\]
\[
= \int_x^\infty e^{-t} \frac{t^{a-1}}{\Gamma(a)} _0F_1 \left[ \frac{-c}{b} \right] \, dt = \frac{1}{\Gamma(a)} \sum_{n \geq 0} \frac{z^n}{(c)_n n!} \int_x^\infty e^{-t} t^{n+a-1} \, dt
\]
\[
= \sum_{n \geq 0} \frac{\Gamma(a+n, x)}{\Gamma(a) (c)_n \, n!} = \sum_{n \geq 0} \frac{[a; x]_n \, z^n (c)_n \, n!}{n!} = \frac{1}{\Gamma(a)} _1F_1 \left[ \frac{a}{c} \left| \frac{z}{b} \right. \right],
\]
which completes the proof.

**Remark 2.** Upon setting \(x = 0\) in Proposition 1 we arrive at the familiar limit inter-connection representation for the Gaussian and Kummer hypergeometric functions \([4, \text{p. 189}]\)
\[
\lim_{b \to \infty} _2F_1 \left[ \frac{a, b}{c} \left| \frac{z}{b} \right. \right] = _1F_1 \left[ \frac{-c}{b} \right].
\]

**Corollary 1.1.** The CDF of the rv \(\xi \sim \chi^2_\nu(\lambda)\) having \(\nu\) degrees of freedom possesses the representations
\[
F_{\nu, \lambda}(x) = 1 - e^{-\frac{\lambda}{2}} \frac{\nu}{\lambda} \Gamma_1 \left[ \frac{\nu}{\nu} \left| \frac{x}{\nu} \right. \right],
\]
and
\[
F_{\nu, \lambda}(x) = 1 - e^{-\frac{\lambda}{2}} \lim_{b \to \infty} _2\Gamma_1 \left[ \frac{\nu}{\nu} \left| \frac{x}{\nu} \right. \right], b \left| \frac{\lambda}{2b} \right. .
\]
Both formulae are valid for all \(\min\{\nu, \lambda, x\} > 0\).

Finally, it is important to observe that using the expression (2.3) derived in Section 2 the CDF of the rv \(\xi \sim \chi^2_{2n}(\lambda)\), \(n \in \mathbb{N}\), can be presented explicitly in terms of the lower incomplete Gauss hypergeometric function \(_2\gamma_1\).
Theorem 5. For all \( n \in \mathbb{N} \) and \( \min\{\lambda, x\} > 0 \) there holds
\[
F_{2n,\lambda}(x) = \frac{1}{2} - e^{-\frac{x}{\lambda}} \left( \frac{1}{2} I_0(\sqrt{\lambda x}) + \sum_{m=1}^{n-1} \left( \frac{x}{\lambda} \right)^m I_m(\sqrt{\lambda x}) \right)
- \frac{|x-\lambda|}{2(\sqrt{x}+\sqrt{\lambda})^2} 2\gamma_1 \left[ (1, (\sqrt{x}+\sqrt{\lambda})^2/2), \frac{1}{2} \left| \frac{4\sqrt{\lambda x}}{(\sqrt{\lambda}+\sqrt{x})^2} \right. \right].
\]

Proof. The modified Bessel function of the first kind can be expressed in terms of the Kummer confluent hypergeometric function [13, p. 328, Eq. 13.6.9]
\[
I_\nu(z) = \frac{e^{-z}}{\Gamma(\nu+1)} \left( \frac{z}{2} \right)^\nu \frac{1}{\nu} \frac{1}{2} F_1 \left[ \nu + \frac{1}{2} \left| \frac{1}{2} \right| 2z \right],
\]
and the definition (1.6) of \( S_\nu \) yields
\[
S_0(\sqrt{\lambda x}, \omega) = \int_0^{\sqrt{\lambda x}} e^{-(\omega+2)t} \frac{1}{\Gamma(\nu+1)} \left( \frac{t}{2} \right)^\nu \frac{1}{\nu} \frac{1}{2} F_1 \left[ \nu + \frac{1}{2} \left| \frac{1}{2} \right| 2t \right] dt = \sum_{n\geq0} \frac{\left(\frac{1}{1}\right)_n}{(1)_n n!} \int_0^{\sqrt{\lambda x}} t^n e^{-(\omega+2)t} dt
= \frac{1}{\omega+2} \sum_{n\geq0} \frac{\left(\frac{1}{1}\right)_n}{(1)_n n!} \left( \frac{2}{\omega+2} \right)^n \int_0^{(\omega+2)\sqrt{\lambda x}} u^n e^{-u} du
= \frac{1}{\omega+2} \sum_{n\geq0} \gamma \left( n + 1, \left( \frac{\sqrt{x}+\sqrt{\lambda}}{2} \right)^2 \right) \frac{(\frac{1}{1})_n}{(1)_n n!} \left( \frac{2}{\omega+2} \right)^n
= \frac{1}{\omega+2} \sum_{n\geq0} \frac{\left(\frac{1}{1}\right)_n}{(1)_n n!} \left( \frac{2}{\omega+2} \right)^n
= \frac{1}{\omega+2} \left[ (1, (\sqrt{x}+\sqrt{\lambda})^2/2), \frac{1}{2} \left| \frac{2}{\omega+2} \right. \right].
\]

Now, by virtue of (2.3) the desired representation follows. \( \square \)

5. Discussion. Related results. Further remarks

In this section we discuss some results stated in the introduction, mostly in order to precisely describe some remaining cases.

Also, related novel results which concern the complete and incomplete Fox–Wright functions connection are obtained.
A. In the Section 2 we have listed the Theorem 1 in condensed and more elegant form than it is exposed in the original version, namely [8, p. 4, Theorem 2.1]

\[
F_{2n,\lambda}(x) = e^{-\frac{\lambda}{x}} \left( \sum_{n\geq 0} \left( \frac{x}{\lambda} \right)^{\frac{n}{2}} I_n(\sqrt{x\lambda}) - \sqrt{\frac{\lambda}{x}} \sum_{m=1}^{n} \left( \frac{x}{\lambda} \right)^{\frac{m}{2}} I_{m-1}(\sqrt{x\lambda}) \right),
\]

where \( n \in \mathbb{N} \) and \( \min\{\lambda, x\} > 0 \).

B. The lower an upper incomplete Fox–Wright functions \( _p\Psi_q[\cdot|\gamma(x, w)] \), and \( _p\Psi_q[\cdot|\Gamma(x, w)] \), say, were introduced by Srivastava and Pogany [23, pp. 196–197, Eqs. (6) and (7)] in a study about the generalized Voigt–functions. We mention that Srivastava et al. [24] considered not only the power series definitions, but also the Mellin–Barnes integral forms of the incomplete Fox–Wright \( \Psi \) functions as well. Similar definition to [24, p. 131, Eq. (6.1)] with the detailed discussion of the lower incomplete Fox–Wright function is exploited in [12].

C. One can notice that Temme obtained the formulae (1.5) using the representations of the generalized Marcum \( Q \)–function [25, pp. 57–58, Eqs. (2.6) and (2.8)], being a survival function (consult the relation (1.3)) which codomain is the unit interval \([0, 1]\) and the fact that his results exclude the case \( x = \lambda \). Repeating the derivation procedure applied in [25, p. 57], in order to present the Marcum \( Q \)–function in terms of the integral \( T_{\nu} \), but in our settings for \( x = \lambda \) we conclude that \textit{mutatis mutandis}

\[
Q_{\mu}(a, a) = \frac{1}{2} \int_{2a}^{\infty} e^{-t} I_{\mu-1}(t) \, dt - \frac{1}{2} \int_{2a}^{\infty} e^{-t} I_\mu(t) \, dt,
\]

which yields in combination with the formula (1.3) the extended CDF formula (1.5), in the upper case expanded into \( x \geq \lambda \). Consequently, the CDF formula in terms of the finite \( S \)–integrals can be written in the modified symmetric
form which includes both (1.7) and (1.8), reads as follows

\[
F_{n,\lambda}(x) = \begin{cases} 
\frac{1}{2} \left( \frac{x}{\lambda} \right)^{\frac{n}{2}} \left\{ S_{\frac{n}{2}-1}(\sqrt{\lambda x}, \omega) - \sqrt{\frac{\lambda}{x}} S_{\frac{n}{2}}(\sqrt{\lambda x}, \omega) \right\} & x \neq \lambda \\
\frac{1}{\Gamma\left(\frac{n}{2} + 1\right)} \left( \frac{\lambda}{2} \right)^{\frac{n}{2}} \left\{ \binom{n/2 + 1/2}{n/2} \right\} & x = \lambda 
\end{cases}
\]

(5.1)

Also, it is worth to mention that Brychkov has proved the identities [5, p. 178, Eq. (5)]

\[
Q_{n+1}(a, a) = \frac{1}{2} \left( 1 + e^{-a^2} I_0(a^2) \right) + e^{-a^2} \sum_{k=1}^{n} I_k(a^2),
\]

(5.2)

and [5, p. 178, Eq. (7)]

\[
Q_{n+\frac{1}{2}}(a, a) = \frac{1}{2} \left( 1 + \text{erfc}(\sqrt{2a}) \right) + e^{-a^2} \sum_{k=1}^{n} I_{k-\frac{1}{2}}(a^2),
\]

(5.3)

valid for any non-negative integer \( n \); these formulae also cower the remaining case \( x = \lambda \) in Temme’s article. Both Brychkov’s displays are important since \( F_{n,\lambda}(x) \) is expressed either for even indices \( \mu = n + 1 \) associated with (5.2), or are related to the half-integer case (5.3).

D. The question about non–negativity of \( F_{n,\lambda}(x) \) in (1.7) was not discussed in detail in the earlier article [8]. The non–negativity of expressions in (1.5) are evident, being survival functions. In turn, skipping this approach we can prove the non–negativity of (1.7) by analytical tools for any \( x > 0 \), using into account certain bounding inequalities for the ratio of modified Bessel \( I \) functions. To do this, start with the input set of \( n \) homosedastic normal rvs \( X_j \sim \mathcal{N}(\mu_j, \sigma^2) \), which build the quadratic sum rv \( \xi \) (see the introduction) having non–centrality parameter \( \lambda > 0 \), and denote \( \mu = \max_{1 \leq j \leq n} |\mu_j| \).

Then, consider the ‘normalized’ set of rvs \( X_j' = X_j \mu_j^{-1}, j = 1, n \) having term–wise \( \mathcal{N}(\mu_j \mu^{-1}, \sigma^2 \mu^{-2}) \) distributions, respectively. The ‘normalized’ non–centrality parameter \( \lambda' = \lambda \mu^{-2} \leq n \). However, multiplication with an absolute constant provides the identical structure of the set of input random variables which defines \( \xi \).
Now, assuming \( x > \lambda' \) and rewriting the difference of two \( S \)-terms in (1.7) in their integral form, then applying Soni’s bound [21, p. 406, Eq. (A)]

\[
I_{\nu+1}(x) < I_{\nu}(x), \quad \nu > -\frac{1}{2}, x > 0,
\]

by setting \( \nu = \frac{n}{2} - 1 \), we deduce the estimate

\[
J = \int_0^{\sqrt[\lambda']x} e^{-(\omega + 1)t} I_{\frac{\nu}{2}-1}(t) \left( 1 - \sqrt{\frac{\lambda'}{x}} \frac{I_{\frac{n}{2}}(t)}{I_{\frac{\nu}{2}-1}(t)} \right) dt \\
\geq \int_0^{\sqrt[\lambda']x} e^{-(\omega + 1)t} I_{\frac{\nu}{2}-1}(t) \left( 1 - \sqrt{\frac{\lambda'}{x}} \right) dt.
\]

Hence, the integral \( J \) is obviously positive.

On the other hand, the case \( x < \lambda' \) we handle in a similar way, but now with the aid of the simple functional bound by Joshi and Bissu [10, p. 255]

\[
\frac{I_{\nu+1}(x)}{I_{\nu}(x)} < \frac{x}{2(\nu + 1)}, \quad \nu > -1, x > 0.
\]

This results in

\[
J \geq \int_0^{\sqrt[\lambda']x} e^{-(\omega + 1)t} I_{\frac{\nu}{2}-1}(t) \left( 1 - \sqrt{\frac{\lambda'}{x}} \frac{t}{n} \right) dt \geq \left( 1 - \frac{\lambda'}{n} \right) S_{\frac{\nu}{2}-1}(\sqrt[\lambda']x, \omega);
\]

the integral \( J \) is non-negative as \( \lambda' \leq n \). Finally, we point out that the case \( x = \lambda \) is self-explanatory, compare (5.1).

### Acknowledgments

The authors are immensely grateful to Professor Nico Temme for his careful reading, constructive suggestions and helpful comments on an earlier version of the manuscript. His remarks significantly contribute to encompass the final version. The research of TKP was partially supported by the University of Rijeka, Croatia; project codes uniri-pr-prirod-19-16 and uniri-tehnic-18-66.
References

[1] M. M. Agrest, M. S. Maksimov. *Theory of Incomplete Cylindrical Functions and their Applications*. Springer–Verlag, New York, 1971.

[2] S. András, Á. Baricz. Properties of the probability density function of the non–central chi–squared distribution. J. Math. Anal. Appl. 346 (2008), 395–402.

[3] S. András, Á. Baricz, Y. Sun. The generalized Marcum $Q$–function: an orthogonal polynomial approach. Acta Univ. Sapientiae Mathematica. 3 (2011), No. 1, 60–76.

[4] G. E. Andrews, R. Askey, R. Roy. *Special Functions*. Encyclopedia of Mathematics and it Applications 71. Cambridge University Press, Cambridge, 1999.

[5] Yu. A. Brychkov. On some properties of the Marcum $Q$ function. Integral Transforms Spec. Funct. 23 (2012), No. 3, 177–182.

[6] R. A. Fisher. The general sampling distribution of the multiple correlation coefficient. Proc. Royal Soc. London (A) 121 (1928), 654–673.

[7] A. Gil, J. Segura, N. M. Temme. *Numerical Methods for Special Functions*. SIAM, Philadelphia, 2007.

[8] D. Jankov Maširević. On new formulas for the cumulative distribution function of the non-central chi-square distribution. Mediterr. J. Math. 14 (2017), No. 2, Paper No. 66, 13 pp.

[9] N. L. Johnson, S. Kotz, N. Balakrishnan. *Continuous Univariate Distributions*. Volume 2. John Wiley & Sons, Inc., New York, 1995.

[10] C. M. Joshi, S. K. Bissu. Inequalities for some special functions. J. Comput. Appl. Math. 69 (1996), 251–259.

[11] A. S. Kamel, A. I. Abdel–Samad. On the computation of non-central chi-square distribution function. Comm. Statist. Simul. Comput. 19 (1990), No. 4, 1279–1291.

[12] K. Mehrez, T. K. Pogány, Integrals of ratios of Fox–Wright and incomplete Fox–Wright functions with applications. (submitted manuscript)
[13] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark (Eds.) *NIST Handbook of Mathematical Functions*. NIST and Cambridge University Press, Cambridge, 2010. Online: https://dlmf.nist.gov

[14] P. B. Patnaik. The non-central $\chi^2$– and the $F$–distributions and their applications. Biometrika. 36 (1949), 202–234.

[15] E. S. Pearson. Note on an approximation to the distribution of non-central $\chi^2$. Biometrika 46 (1959), 202–232.

[16] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev. *Integrals and series*. Volume 4. *Direct Laplace Transforms*. Gordon and Breach Science Publishers, New York, 1992.

[17] E. D. Rainville, Special Functions, Macmillan Company, New York, 1960; Reprinted by Chelsea publishing Company, Bronx, New York, 1971.

[18] C. Robert. Modified Bessel functions and their applications in probability and statistics. Statist. Probab. Lett. 9 (1990), 155–161.

[19] M. Sankaran. Approximations to the non-central chi-square distribution. Biometrika 50 (1963), 199–204.

[20] L. L. Scharf. *Statistical Signal Processing: Detection, Estimation, and Time Series Analysis*. Addison–Wesley Publishing Co. 1990.

[21] R. P. Soni. On an inequality for modified Bessel functions. J. Math. and Phys. 44 (1965), 406–407.

[22] H. M. Srivastava, M. A. Chaudhry, R. P. Agarwal. The incomplete Pochhammer symbols and their applications to hypergeometric and related functions. Integral Transforms Spec. Funct. 23 (2012), No. 9, 659–683.

[23] H. M. Srivastava, T. K. Pogány, Inequalities for a unified family of Voigt functions in several variables. Russ. J. Math. Phys. 14 (2007), No. 2, 194–200.

[24] H. M. Srivastava, R. K. Saxena, R. K. Parmar, Some families of the incomplete $\Gamma$–functions and the incomplete $H$–functions and associated integral transforms and operators of fractional calculus with applications, Russ. J. Math. Phys. 25 (2018), No. 1, 116–138.
[25] N. M. Temme. Asymptotic and numerical aspects of the non-central chi-square distribution. Comput. Math. Appl. 25 (1993), No. 5, 55–63.

[26] E. M. Wright, The asymptotic expansion of the generalized hypergeometric function. J. London. Math. Soc. 10 (1935), No. 4, 286–293.

[27] https://functions.wolfram.com/HypergeometricFunctions/Hypergeometric0F1/09/0003/