BKP and CKP revisited: the odd KP system

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Received 6 October 2008, in final form 8 January 2009
Published 3 February 2009
Online at stacks.iop.org/IP/25/045001

Abstract

By restricting a linear system for the KP hierarchy to those independent variables $t_\ell$ with odd $\ell$, its compatibility (Zakharov-Shabat conditions) leads to the ‘odd KP hierarchy’. The latter consists of pairs of equations for two dependent variables, taking values in an (typically noncommutative) associative algebra. If the algebra is commutative, the odd KP hierarchy is known to admit reductions to the BKP and the CKP hierarchy. We approach the odd KP hierarchy and its relation to BKP and CKP in different ways, and address the question of whether noncommutative versions of the BKP and the CKP equation (and some of their reductions) exist. In particular, we derive a functional representation of a linear system for the odd KP hierarchy, which in the commutative case produces functional representations of the BKP and CKP hierarchies in terms of a tau function. Furthermore, we consider a functional representation of the KP hierarchy that involves a second (auxiliary) dependent variable and features the odd KP hierarchy directly as a subhierarchy. A method to generate large classes of exact solutions to the KP hierarchy from solutions to a linear matrix ODE system, via a hierarchy of matrix Riccati equations, then also applies to the odd KP hierarchy, and this in turn can be exploited, in particular, to obtain solutions to the BKP and CKP hierarchies.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Many integrable partial differential and difference equations (PDEs) admit generalizations to versions where the dependent variable takes values in an arbitrary associative and typically noncommutative algebra (see e.g. [1–4]). This fact can be exploited to generate large classes of exact solutions to a scalar integrable PDE via simple solutions to the corresponding matrix PDE (see also [5, 6]). In particular, the existence of families of solutions like multi-solitons
is then a consequence of the existence of certain solutions to the matrix PDE universally for arbitrary matrix size.

There are, however, integrable equations that do not admit a direct noncommutative generalization in the above sense. The Sawada–Kotera equation [7] belongs to these exceptions [3]. This equation is a reduction of the BKP equation, the first member of the BKP hierarchy [8–14] (see also [15–56]), which also lacks a noncommutative version (the latter should not be confused with the multi-component version of BKP). The BKP hierarchy and also the CKP hierarchy [9, 10] (see also [15, 31, 41, 43–45, 50, 55, 57–60]) originate from the ‘commutative’ KP hierarchy in the Gelfand–Dickey–Sato (GDS) formalism (see section 2.6) by first restricting the Lax equations to only odd-numbered variables $t_1, t_3, t_5, \ldots$, and then imposing additional reduction conditions. The first step clearly also works in the noncommutative case. It leads to the (noncommutative) ‘odd KP hierarchy’.

The GDS formulation of the KP hierarchy involves an infinite number of dependent variables. All except one can be eliminated, resulting in PDEs for a single dependent variable. In the same way, the odd KP hierarchy (in the GDS formalism) leaves us with PDEs for two dependent variables. These PDEs admit (generalized) symmetries by means of which the full KP hierarchy can be restored (and the two dependent variables reduced to a single one). This shows that the odd KP hierarchy is a part (subhierarchy) of the KP hierarchy, something that is obvious in its GDS form. So why should we deal with a subhierarchy if we could treat the full hierarchy? The crucial point is that the BKP and CKP reductions of the odd KP hierarchy are not compatible with the abovementioned KP-restoring symmetries. The general message is that a subhierarchy can admit a reduction that does not extend to a reduction of the full hierarchy. And this is the reason why BKP and CKP retain their individuality, despite their KP origin.

In section 2 we derive the first member of the odd KP hierarchy in an elementary way. This ‘odd KP system’ is a system of two PDEs for the KP variable and one additional dependent variable. Within this system we can then look for noncommutative versions of reductions known in the commutative case, and this is done in some subsections of section 2. BKP and CKP possess a certain noncommutative extension with a single dependent variable, but severely constrained. It turns out, in particular, that these extensions are solved by any solution to the first two equations of the ‘noncommutative’ (potential) KdV hierarchy, and this result remains true in the commutative case (where the constraints disappear). Furthermore, there is a natural noncommutative generalization of the CKP equation, though as a system with two dependent variables. Nothing similar is found in the BKP case.

In section 3 we derive a linear system, in a functional form, for the whole odd KP hierarchy and deduce the corresponding results for the BKP and CKP hierarchies. Section 4 takes a different route, starting from a functional representation of the KP hierarchy that involves an auxiliary dependent variable [61]. In this formulation, the odd KP hierarchy appears as the subhierarchy that consists of equations containing only partial derivatives with respect to the odd-numbered variables, $t_1, t_3, t_5, \ldots$. The auxiliary dependent variable then takes the role of the second dependent variable of the odd KP system. A certain symmetry reduction for the (odd) KP hierarchy is then introduced, which plays a crucial role in the step from odd KP to BKP and CKP.

Several classes of solutions to the matrix KP hierarchy and, if a rank 1 condition holds (see e.g. [62]), then also the scalar KP hierarchy, can be obtained from solutions to a system of linear matrix ordinary differential equations, via a system of matrix Riccati equations

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3 Throughout we will work with a potential $\phi$ related to the KP variable $u$ by $u = \phi_1$, hence this system may rather be called ‘potential odd KP system’.
This is a finite-dimensional version of the famous Sato theory for the KP hierarchy. Using the abovementioned formulation of the KP hierarchy that exhibits the odd KP hierarchy directly as a subhierarchy, this immediately also generates solutions to the odd KP hierarchy. This is elaborated in section 5. Furthermore, we show how solutions to the BKP and CKP hierarchies can be obtained from solutions to the matrix odd KP hierarchy. Some final remarks are collected in section 6.

2. The odd KP system

The Kadomtsev–Petviashvili (KP) hierarchy (see e.g. [66]) is given by the integrability (or zero curvature) conditions

\[ B_{m,n} = B_{n,m} + [B_m, B_n] = 0 \]  

of the linear system [67]

\[ \psi_{t_n} = B_n \psi, \quad n = 1, 2, 3, \ldots, \]  

where

\[ B_n = 3 \theta_{t_1} + 3 \phi_{t_1} \text{ and } \psi_{t_n} = 3 \phi_{t_1}. \]  

By exploiting the integrability condition and introducing potentials \( \phi \) and \( \theta \) via

\[ b_{n,0} = 3 \theta_{t_1} + 3 \phi_{t_1}, \quad b_{n,1} = 3 \phi_{t_1}. \]  

the coefficients of the linear system are fixed in terms of \( \phi \) and \( \theta \),

\[ \psi_{t_n} = 3 \phi_{t_1} \text{ and } \psi_{t_n} = 3 \phi_{t_1}. \]  

4 This requires some additional structure that we need not specify here. If \( A \) is an algebra of real or complex matrices, the usual differential structure will be assumed.

5 The shift by \( \frac{1}{2} \phi_{t_1} \) leads to a more ‘symmetric’ form of the resulting equations (2.10) and (2.11).
\[
\psi_t = (\partial^3 + 5\phi_t \partial + 5(\phi_t + \frac{3}{4} \phi_{ttt}) \partial^2 + 5(\phi_{ttt} + \frac{1}{2} \phi_t + \frac{1}{2} \phi_{ttt}) \partial + b_{5,0}) \psi, \tag{2.8}
\]

where
\[
b_{5,0} = \frac{5}{3} \theta_t + \frac{10}{3} (\partial_{ttt} + 5(\phi_t + \frac{3}{4} \phi_{ttt}) \partial^2 + \frac{5}{3} (\phi_{ttt} + \frac{1}{2} \phi_t + \frac{1}{2} \phi_{ttt}) \partial + b_{5,0}) \psi_t + \frac{5}{3} \left[ \phi_t, \phi_{ttt} \right] + \frac{5}{3} \int \left[ \phi_t, \phi_t \right] \mathrm{d}t_1.
\tag{2.9}
\]

Here \([,]\) and \([,]\) mean commutator and anti-commutator, respectively. The remaining integrability conditions then result in the following pair of equations,
\[
\left( 9\phi_t - 5\phi_{ttt} + \phi_{tttt} - \frac{15}{2} \left[ \phi_t, \phi_t - \phi_{ttt} - \phi_t^2 \right] + \frac{45}{4} (\phi_{ttt}^2 - 4\phi_t^2) \right)_{t_t} - 5\phi_{ttt} + 15 \left[ [\phi_t, \theta_t] - \theta_{ttt} + \theta_t \phi_t + \frac{1}{2} \phi_{ttt} \right] + \frac{3}{2} \left[ \theta_{ttt}, \phi_{ttt} \right] + \left[ \phi_t, \int [\phi_t, \phi_t] \mathrm{d}t_1 \right] = 0.
\tag{2.10}
\]

and
\[
\left[ \theta_t - 5\theta_{ttt} + \theta_{tttt} + \frac{15}{2} \left[ \phi_t, \theta_t \right] + \left[ \theta_{ttt}, \phi_t \right] + \frac{1}{5} \left[ \phi_t, \theta_{ttt} \right] \right]_{t_t} + 6\phi_t \theta_t \phi_t + \frac{1}{6} \left[ \phi_t, \phi_t \right]_{t_t} + \frac{1}{6} \left[ \phi_t, \phi_t \right]_{t_{ttt}} - \frac{1}{4} \left[ \phi_t, \phi_{ttt} \right]_{t_t} + 15 \left[ \theta_t, \theta_t + \frac{1}{2} \theta_{ttt} + \int [\phi_t, \phi_t] \mathrm{d}t_1 \right] - 5\theta_{ttt} - \frac{15}{2} \left[ \theta_t, \phi_t \right]_{t_t} + 15 \left[ \theta_t, \theta_t \right] + \frac{15}{2} \left[ \theta_{ttt}, \theta_t \right] + \frac{25}{4} \left[ \theta_{ttt}, \phi_t \right] - 5 \left( \int [\phi_t, \phi_t] \mathrm{d}t_1 \right) + \frac{15}{2} \left[ \phi_t, - \phi_{ttt} - (\phi_t)^2 \right] + \frac{45}{4} (\phi_t, (\phi_t)^2) = 0.
\tag{2.11}
\]

In the following we refer to (2.10) and (2.11) as the 'odd KP system'. We note that by introducing
\[
\tilde{\theta} := \theta + \frac{1}{5} \int [\phi_t, \phi_t] \mathrm{d}t_1,
\tag{2.12}
\]

which implies \(\theta_t = \tilde{\theta}_t - \frac{1}{5} [\phi_t, \phi_t] - \int [\phi_t, \phi_t] \mathrm{d}t_1\), the resulting equations no longer involve integrals, see also section 4.

**Remark 2.1.** Switching on 'even flows', we have in particular \(\psi_{\tilde{t}} = (\partial^2 + b_{2,0})\psi\). Compatibility with (2.4) (using (2.6)) then leads to \(b_{2,0} = 2\phi_t, \tilde{\theta}_t = \frac{1}{2} \phi_t\), and the (potential) KP equation for \(\phi\).

### 2.2. Recovering BKP and CKP in the commutative case

If \(\mathcal{A}\) is commutative, then the above pair of equations reduces to
\[
(9\phi_t - 5\phi_{ttt} + \phi_{tttt} - 15\phi_t \phi_t + 15\phi_t \phi_{ttt} + 15\phi_t^3 + \frac{45}{4} \phi_t^2 - 45\phi_t^2)_{t_t} - 5\phi_{ttt} = 0,
\tag{2.13}
\]

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\[(9\theta_t - 5\theta_{t\Phi_t} + 3\theta_{t\Phi} + 15\phi_t \theta_t + 15\phi_t \phi_{t\phi} + \frac{15}{2} (\phi_{t\Phi_t} \theta_t)_{\phi_t} + 45\phi_t^2 \theta_t)_{\phi_t} - 5(\theta_t + 3\phi_t \theta_t) \phi_t = 0.\]  
\[(2.14)\]

Setting
\[\theta = k\phi_t,\]  
\[(2.15)\]

it turns out that the second equation is a consequence of the first if
\[k = 0, \pm \frac{1}{2}.\]  
\[(2.16)\]

If \(k = \pm 1/2, (2.13)\) becomes the BKP equation
\[(9\phi_t - 5\phi_{t\phi_t} + 3\phi_{t\phi} + 15\phi_t \phi_t + 15\phi_t \phi_{t\phi_t} + 15\phi_t^3)_{\phi_t} - 5\phi_t \phi_t = 0.\]  
\[(2.17)\]

Setting \(\phi_t = 0\) reduces \((2.17)\) to the (potential) Sawada–Kotera equation [7, 9, 47]
\[9\phi_t + \phi_{t\phi_t} + 15(\phi_t \phi_{t\phi_t} + \phi_t^3) = 0,\]  
\[(2.18)\]

which is known not to possess a noncommutative (e.g. matrix) version [3]. Setting \(\phi_t = 0\) in \((2.17)\) yields the Ramani equation [9, 68] (also called (potential) bidirectional Sawada–Kotera (bSK) equation [50, 69–71]),
\[(-5\phi_{t\phi_t} + \phi_{t\phi_t\phi_t} - 15\phi_t \phi_t + 15\phi_t \phi_{t\phi_t} + 15\phi_t^3)_{\phi_t} - 5\phi_t \phi_t = 0.\]  
\[(2.19)\]

If \(k = -1/2\) we have \(b_{3,0} = b_{5,0} = 0\) and thus the familiar linear system for the BKP equation [9, 10],
\[\psi_t = (\partial^3 + 3\phi_t \partial) \psi,\]  
\[(2.20)\]

\[\psi_t = (\partial^3 + 5\partial \phi_t \partial^2 + \frac{5}{2} (\phi_t + 2\phi_{t\phi_t} + 3\phi_t^2) \partial) \psi.\]  
\[(2.21)\]

If \(k = 1/2\), we obtain another linear system for the BKP equation:
\[\psi_t = 3\phi_t \partial \psi + 3\phi_t \psi_t + \psi_{\Phi_t \phi_t} = (\partial^3 + 3\partial \phi_t) \psi,\]  
\[(2.22)\]

\[\psi_t = \frac{5}{2} (\partial^3 + 5\partial \phi_t \partial^2 + \frac{5}{2} (\phi_t + 2\phi_{t\phi_t} + 3\phi_t^2) \partial) \psi_t,\]
\[\psi_t + 10\phi_t \psi_{\phi_t} + 5\phi_t \psi_{\Phi_t \phi_t} + \psi_{\Phi_t \phi_t \phi_t} = (\partial^3 + 5\partial^2 \phi_t \partial + \frac{5}{2} \partial (\phi_t + 2\phi_{t\phi_t} + 3\phi_t^2) \partial) \psi,\]  
\[(2.23)\]

which is thus simply an adjoint of the first linear system.

If \(k = 0\) (i.e. \(\theta = 0\), (2.13) becomes the CKP equation [9]
\[(9\phi_t - 5\phi_{t\phi_t} + \phi_{t\phi_t\phi_t} - 15\phi_t \phi_t + 15\phi_t \phi_{t\phi_t} + 15\phi_t^3 + \frac{45}{4} \phi_t \phi_t^2)_{\phi_t} - 5\phi_t \phi_t = 0.\]  
\[(2.24)\]

The linear system in this case turns out to be given by half the sum of the respective equations of the above two BKP linear systems.

Setting \(\phi_t = 0\) reduces \((2.24)\) to the (potential) Kaup–Kuperschmidt equation [72]
\[9\phi_t + \phi_{t\phi_t\phi_t} + 15(\phi_t \phi_{t\phi_t} + \frac{1}{2} \phi_t \phi_t^2 + \phi_t^3) = 0.\]  
\[(2.25)\]

Setting \(\phi_t = 0\) in \((2.24)\), yields the bidirectional Kaup–Kuperschmidt (bKK) equation [50, 69–71, 73].
2.3. BKP and the noncommutative KdV hierarchy

Imposing the BKP condition

\[ \theta = -\frac{1}{2} \phi_t, \]  

(i.e., \( k = -\frac{1}{2} \) in (2.15)) in the noncommutative case, we have \( b_{3,0} = 0 \) and

\[ b_{5,0} = \frac{5}{3} \int \left[ \phi_t - \phi_t \phi_t, \phi_t \right] dt. \]  

(2.27)

Then (2.10) reduces to

\[ (9\phi_{t} - 5\phi_{t} \phi_{t} + \phi_{t} \phi_{t} \phi_{t} - 15(\phi_{t} (\phi_{t} - \phi_{t} \phi_{t}) - \phi_{t}^{3})), - 5\phi_{t} \phi_{t} \]  

\[ + 15\left[ \phi_{t}, \int \left[ \phi_{t}, \phi_{t} \right] dt \right] = 0. \]  

(2.28)

and (2.11), after use of (2.28), becomes

\[ [\phi_{t}, - \phi_{t} \phi_{t} \phi_{t}], \phi_{t}]_{t_1} - \left( \int \left[ \phi_{t}, - \phi_{t} \phi_{t} \phi_{t} \phi_{t}, \phi_{t} \right] dt \right)_{t_1} + 3\phi_{t} \left[ \phi_{t}, - \phi_{t} \phi_{t} \phi_{t}, \phi_{t} \right] = 0. \]  

(2.29)

The latter equation represents a constraint which, however, is not in general preserved under the flow with evolution variable \( t_1 \), given by (2.28).\(^6\) Now we observe that (2.29) is obviously solved if

\[ \phi_{t} = \phi_{t} \phi_{t} + f(\phi_{t}), \]  

(2.30)

where \( f \) is an arbitrary polynomial in \( \phi_{t} \) with coefficients in the center of \( \mathcal{A} \). But only for a special choice of \( f \), equation (2.30) has a chance to be compatible with (2.28). Addressing integrability, evolution equations like (2.30), with the restriction that the right-hand side is a homogeneous differential polynomial, appear to be distinguished. This reduces (2.30) to the potential KdV equation, where \( f(\phi_{t}) = a\phi_{t}^{2} \) with a constant \( a \), or the mKdV equation, where \( f(\phi_{t}) = a\phi_{t}^{3} \). But only in the KdV case the weighting of terms is compatible with (2.28). Using the KdV equation in (2.28), yields

\[ (9\phi_{t} - 9\phi_{t} \phi_{t} \phi_{t} \phi_{t} - 15a(\phi_{t}^{2}), - 5(\phi_{t} - 3a\phi_{t}^{2})), \phi_{t}^{3} + 5(9 - a^{3})\phi_{t} \phi_{t} \phi_{t} \phi_{t} \phi_{t} = 0. \]  

(2.31)

Choosing \( a = 3 \), this can be integrated to

\[ \phi_{t} - \phi_{t} \phi_{t} \phi_{t} \phi_{t} = 5(\phi_{t}^{2} - 3a\phi_{t}^{2} + 5\phi_{t}^{3}), \quad 5\phi_{t} \phi_{t}^{2} - 10 \phi_{t}^{3} = 0, \]  

(2.32)

and (2.30) reads

\[ \phi_{t} = \phi_{t} \phi_{t} + 3\phi_{t}^{2}. \]  

(2.33)

(2.33) and (2.32) are the first two equations of the noncommutative potential KdV (ncpKdV) hierarchy\(^7\). Hence, any solution to the first two ncpKdV hierarchy equations (2.33) and (2.32) also solves the above noncommutative extension of the BKP equation\(^8\). This relation then also holds for the ‘commutative’ scalar equations, of course. But to find this result the step into the noncommutative framework was extremely helpful.

\(^6\) Taking a look at this problem in the Sawada–Kotera case, where \( \phi_{t} = 0 \) simplifies the equations a lot, we have to compute the derivative of (2.29) with respect to \( t_1 \) and then eliminate \( \phi_{t_1} \) by use of (2.28). Already the resulting terms quadratic in (derivatives of) \( \phi \) do not cancel as a consequence of (2.29) and its derivatives with respect to \( t_1 \).

\(^7\) With \( x = -\phi_{t} \), where \( x = t_1 \) we obtain from (2.33) and (2.32), respectively, the potential versions of (3.46) and (3.47) in [74].

\(^8\) An analogous relation exists between the first two equations of the (noncommutative) Burgers hierarchy and the KP equation [61, 75].
Remark 2.2. If we impose the conditions $b_{1,0} = b_{3,0} = 0$ on the noncommutative odd KP system, we obtain (2.26) and $[\phi_n - \phi_{i+1}, \phi_n] = 0$, which leads more directly to (2.30).

Another noncommutative extension of the BKP equation is obtained for $\theta = \frac{1}{2} \phi_1$ (i.e. $k = \frac{1}{2}$ in (2.15)), and one finds corresponding results.

2.4. CKP and the noncommutative KdV hierarchy

Imposing the CKP condition $\theta = 0$, (2.10) reduces to

$$
\left(9\phi_{i+1} - 5\phi_{i+1,i+1} + \phi_{i+1,i+1,i+1} - \frac{15}{2} \left[\phi_{i+1}, \phi_{i+1} - \phi_{i+1,2}\right] + \frac{45}{4} \left(\phi_{i+1,2}\right)^2\right)_{t_i} - 5\phi_{i+1} + 15 \left[\phi_{i+1}, \int \left[\phi_{i+1}, \phi_{i+1}\right] d_{t_i}\right] = 0
$$

(2.34)

and (2.11) yields a constraint, involving only commutators, which is not in general preserved under the flow of (2.34). The constraint turns out to be satisfied as a consequence of the ncpKdV equation in the form

$$
\phi_i = \frac{1}{2} \phi_{i+1,i+1} + \frac{1}{2} \left(\phi_i\right)^2,
$$

(2.35)

and (2.34) then integrates to

$$
\phi_i = \frac{1}{16} \phi_{i+1,i+1} + \frac{5}{8} \left(\phi_i\right)^2_{t_i} - \frac{5}{8} \left(\phi_{i+1,2}\right)^2 + \frac{5}{2} \left(\phi_i\right)^3,
$$

(2.36)

which is the second equation of the ncpKdV hierarchy. As a consequence, any solution to the first two equations of the ncpKdV hierarchy (with coefficients as given above) is also a solution to the constrained noncommutative extension of the CKP equation. In the commutative case, the corresponding statement then also holds, of course. i.e. any solution to the first two equations of the noncommutative KdV hierarchy (with coefficients as given above) is also a solution to the CKP equation.

2.5. Further reductions of the odd KP system in the noncommutative case

Imposing $\phi_i = \theta_i = 0$, we obtain from (2.10) and (2.11) the following noncommutative generalization of the (potential) Sawada–Kotera (2.18) and Kaup–Kupershmidt equation (2.25),

$$
9\phi_{i+1} + \phi_{i+1,i+1} + \frac{15}{2} \left[\phi_{i+1}, \phi_{i+1}\right] + \frac{45}{4} \left(\phi_{i+1}\right)^2 + 15\theta_{i+1,3} + 15 \left[\theta_{i+1}, \phi_{i+1}\right] + \frac{15}{2} \left[\phi_i, \phi_{i+1}\right] - 45\theta_{i+1}^2 = 0
$$

(2.37)

and

$$
9\theta_{i+1} + \theta_{i+1,i+1} + \frac{15}{2} \left[\theta_{i+1}, \phi_{i+1}\right] + \frac{1}{2} \left[\theta_i, \phi_{i+1}\right]_{ni} + 6\theta_i \theta_{i+1} + 6 \left[\phi_i, \phi_{i+1}\right]_{ni} + 45 \left[\theta_i\right]_2^2 - \frac{1}{2} \left(\phi_{i+1,2}\right)^2, \phi_i
$$

$$
+ 15 \left[\phi_{i+1}, \left[\phi_i, \theta_i\right]\right] + \frac{15}{2} \left[\phi_i, \left[\phi_{i+1}, \theta_i\right]\right] - \frac{15}{2} \left[\phi_{i+1}, \left(\phi_i\right)^2\right] = 0.
$$

(2.38)

In the commutative case, the last equation can be integrated with respect to $t_1$, and we recover an integrable system that appeared in [76, 77] (see also [78]),

$$
9u_i + u_{i+1,i+1} + 10uu_{i+1,i+1} + 25u_i u_{i+1,i+1} + 20u^2 u_{i+1,i+1} - 135u_i \theta_i u_{i+1,i+1} = 0,
$$

(2.39)

$$
9\theta_i + \theta_{i+1,i+1} + 10u\theta_{i+1,i+1} + 5 \left(u_i \theta_i\right)_{t_i} + 20u^2 \theta_i = 0,
$$

(2.40)

We note that (2.35) and (2.36) can be obtained from (2.33) and (2.32) via $\theta_i \mapsto 2\theta_i$. 


where \( u := \frac{1}{2} \phi_t.\) In [79] an attempt was made to find a noncommutative version of ‘coupled systems of Kaup–Kupershmidt and Sawada–Kotera type’, but without success. The above equations (2.37) and (2.38) constitute a solution to this problem.

Setting \( \phi_t = \theta_t = 0 \) in (2.10) and (2.11), we obtain a system that may be regarded as a noncommutative generalization of the Ramani (or bSK) equation (2.19) and the bidirectional Kaup–Kupershmidt (bKK) equation.

**Remark 2.3.** System (2.37) and (2.38) possesses the symmetry \( \phi_t = 2 \theta_t \) (see also remark 2.1), by use of which we obtain from it the first and the third member of the (noncommutative) Boussinesq hierarchy. The latter is the 3-reduction of the (noncommutative) KP hierarchy (also called third Gelfand–Dickey hierarchy [66]). This means that the system (2.37) and (2.38) can also be obtained as a reduction of the KP hierarchy, and not just as a reduction of the odd KP hierarchy. The crucial point is that the reduction condition is compatible with the equations (like \( \phi_t = 2 \theta_t \)) that are needed to complete the odd KP hierarchy (cf section 4). This is not so for the reductions of odd KP to BKP or CKP. In the same way, the noncommutative generalization of the bSK and bKK equations is related to the 5-reduction of the (noncommutative) KP hierarchy (fifth Gelfand–Dickey hierarchy).

### 2.6. The Gelfand–Dickey–Sato formulation of the odd KP hierarchy

The odd KP system can be extended to a hierarchy by restricting the GDS formulation (see e.g. [66]) of the KP hierarchy,

\[
L_{\theta_T} = [B_{\theta_T}, L],
\]

where

\[
B_{\theta_T} = (L^\theta)_{\geq 0}, \quad L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \cdots,
\]

to odd-numbered variables \( \theta_T. \) Here \( \partial^{-1} \) is the formal inverse of \( \partial \) and \( (L^\theta)_{\geq 0} \) means the projection of a pseudodifferential operator to its differential operator part (see e.g. [66]). We have in particular

\[
B_3 = (L^\theta)_{\geq 0} = \partial^3 + 3u_2 \partial + 3(u_3 + u_{2,2}),
\]

\[
B_5 = (L^\theta)_{\geq 0} = \partial^5 + 5u_2 \partial^3 + 5(u_3 + 2u_{2,1}) \partial^2 + 5(u_4 + 2u_{3,1} + 2u_{2,1,1} + 2u_2^2) \partial
\]
\[
+ 5(u_5 + 2u_{4,1} + 2u_{3,2,1} + u_{2,2,1,1} + 2(u_2 + u_3)) + 2(u_2^2),
\]

Equation (2.41) is known to be equivalent to the zero curvature conditions (2.1), with \( B_{\theta_T} \) defined in (2.42). By comparison with \( B_3 \) and \( B_5 \) computed in section 2.1, we find

\[
u_2 = \phi_t, \quad u_3 = \theta_t - \frac{1}{2} \phi_{t,1}, \quad u_4 = -\theta_{t,1} + \frac{1}{2} \phi_{t,1} + \frac{1}{6} \phi_{t,1,1,1} = \left( \phi_t \right)^2,
\]

\[
u_5 = \frac{1}{3} \theta_t + \frac{2}{3} \phi_{t,1,1} - \frac{1}{6} \phi_{t,1,1} = \left[ \phi_t, \phi_t \right] + \frac{1}{3} \int \left[ \phi_t, \phi_t \right] \partial t_1.
\]

If \( A \) is commutative, the CKP reduction of the KP hierarchy is determined by \( L + L^* = 0, \) and the BKP reductions by \( \partial L + L^* \partial = 0, \) respectively \( L \partial + \partial L^* = 0 \) [10]. Here \( L^* \) denotes the adjoint of the pseudodifferential operator \( L \) (see e.g. [66]). We summarize these well-known relations together with those found in section 2.2 in the following table.

|       | \( \partial L + L^* \partial = 0 \) | \( \theta = -\frac{1}{2} \phi_t \) |
|-------|-----------------------------------|----------------------------------|
| BKP   |                                   |                                  |
| CKP   |                                   |                                  |

|       | \( L \partial + \partial L^* = 0 \) | \( \theta = \frac{1}{2} \phi_t \) |
|-------|-----------------------------------|----------------------------------|
| BKP   |                                   |                                  |
| CKP   |                                   |                                  |

|       | \( L + L^* = 0 \) | \( \theta = 0 \) |
|-------|-----------------|-----------------|
| BKP   |                 |                 |
| CKP   |                 |                 |
If $\mathcal{A}$ is matrix algebra over $\mathbb{R}$ or $\mathbb{C}$, we can generalize the adjoint by setting $(A\partial)^* := -\partial A^T$, where $A \in \mathcal{A}$ with transpose $A^T$.\(^{10}\) The CKP condition then generalizes to

$$
\text{matrix CKP} \quad L + L^* = 0 \quad \phi^T = \phi, \quad \theta^T = -\theta
$$

The conditions for $\phi$ and $\theta$ indeed yield a consistent reduction of the odd KP system, which may thus be regarded as a noncommutative version of the CKP equation. For $m > 1$, it is a pair of equations for two dependent (matrix) variables, however. The corresponding hierarchy will be called matrix CKP hierarchy. In the following, ‘CKP equation’ or ‘CKP hierarchy’ throughout refers to the familiar scalar (commutative) case, i.e. $m = 1$, and we will add ‘matrix’ whenever we mean the matrix generalization. In contrast to the CKP case, the above BKP reduction condition for $L$ does not consistently generalize to the noncommutative case.

The formulation (2.41), with $n \in \mathbb{N}$, of the KP hierarchy depends on an infinite number of dependent variables. Elimination of $u_3, u_4, \ldots$ leads to PDEs that only involve the variable $u_2 (= \phi_t)$. Omitting some of the equations (2.41), it will no longer be possible to eliminate all the auxiliary variables $u_3, u_4, \ldots$. In the step to the odd KP hierarchy, where all equations (2.41) involving derivatives with respect to even-numbered variables are dropped, one of the remaining variables is retained, namely $u_3$, which leads to the appearance of $\theta$. It would be desirable to find a way to explicitly eliminate all the remaining auxiliary variables $u_4, u_5, \ldots$ from the sequence of equations (2.41) with odd $n$. In section 3 we solve this problem on the level of the corresponding linear system. The odd KP hierarchy expressed in terms of $\phi$ and $\theta$ (without auxiliary variables) then arises from the integrability conditions.

Also in the case of the full KP system, (2.41) with $n \in \mathbb{N}$, we may think of eliminating only $u_4, u_5, \ldots$. The resulting equations then depend on $u_2$ and $u_3$, and further elimination of $u_3$ would lead to the KP equation and its companions. The more interesting aspect, however, is that in such a formulation of the KP hierarchy, we should expect the odd KP system (and its hierarchy companions) to form a subhierarchy. In fact, in section 4, we start from a functional form of the KP hierarchy that involves one additional (auxiliary) variable that, by now not surprisingly, turns out to be related to $\theta$. In this representation of the KP hierarchy, the odd KP hierarchy is indeed nicely described as a subhierarchy. We note that in this picture a solution to the odd KP hierarchy in general still depends on the even-numbered variables $t_{2n}$, which are constants with respect to the odd KP hierarchy.

3. A linear system for the odd KP hierarchy in functional form

In this section we present a linear system for the whole noncommutative odd KP hierarchy in functional form. This extends the linear system for the odd KP system obtained in section 2.1. The bilinear identity for the KP hierarchy (see e.g. [66]), restricted to odd-numbered variables, is

$$
\text{res}[(s_0, z) \psi(t_0, z)] = 0, \quad \text{(3.1)}
$$

where $t_0 = (t_1, t_3, t_5, \ldots)$,

$$
\psi(t_0, z) = w(t_0, z) e^{\xi(t_0, z)}, \quad \bar{\psi}(t_0, z) = \bar{w}(t_0, z) e^{-\xi(t_0, z)}, \quad \text{(3.2)}
$$

with $\bar{\xi} = \sum_{n \geq 1} t_{2n-1} z^{2n-1}$ and

$$
w(t_0, z) = I + \sum_{n \geq 1} w_n(t_0) z^{-n}, \quad \bar{w}(t_0, z) = I + \sum_{n \geq 1} \bar{w}_n(t_0) z^{-n}. \quad \text{(3.3)}
$$

\(^{10}\) More generally, we may consider an algebra $\mathcal{A}$ with an involution $\ast$, and define $(A\partial)^* := -\partial A^\ast$.\(^{10}\)
We will often omit the argument $t_n$, for simplicity. Inserting (3.2) in (3.1), the bilinear identity reads
\[ \text{res}(w(s_n, z) \bar{w}(t_0, z) e^{\hat{\theta}t_0 - t_n z}) = 0. \] (3.4)

The residue $\text{res} f(z)$ of a formal series $f(z) = \sum_{n=-\infty}^{+\infty} f_n z^{-n}$ is the coefficient $f_1$. In particular, setting $s_n = t_0$, (3.4) implies
\[ \bar{w}_1 = -w_1 =: \phi. \] (3.5)

We write
\[ w_2 = -\hat{\theta} + \frac{1}{2}(\phi_1 + \phi^2), \] (3.6)

with a variable $\hat{\theta}$. We shall see that $\phi$ can be identified with the variable of the same name introduced in section 2.1, and that $\hat{\theta}$ coincides with the variable defined in (2.12). Below we use the Miwa shift notation $\phi_{2j}(t_0) = \phi(t_0 + [\lambda j])$, $[\lambda] = (\lambda, \lambda^2/3, \lambda^3/5, \ldots)$. The proof of the following theorem is presented in appendix A.

**Theorem 3.1.** The bilinear identity implies
\[ \frac{1}{\lambda} F(\lambda)(\psi_{2[\lambda]} - \psi) - (\psi_{2[\lambda]} + \psi)_{\hat{\theta}} \]
\[ = \frac{\lambda}{2} \left( \hat{\theta}_{2[\lambda]} - \hat{\theta} + \frac{1}{2}(\phi_{2[\lambda]} - \phi)_{\hat{\theta}} - \frac{1}{2}[\phi, \phi_{2[\lambda]}] \right) F(\lambda)^{-1}(\psi_{2[\lambda]} + \psi), \] (3.7)

where
\[ F(\lambda) := I - \frac{\lambda}{2}(\phi_{2[\lambda]} - \phi). \] (3.8)

Equation (3.7) is a functional representation of the linear system for the odd KP hierarchy. By expansion in powers of the indeterminate $\lambda$, we recover from the lowest orders the linear system of the odd KP system derived in section 2.1. Indeed, at order $\lambda^2$ we obtain
\[ \psi_{t_1} = (\hat{\theta}^3 + 3\phi_{t_1} \hat{\theta} + \frac{3}{2}(2\hat{\theta}_{t_1} + \phi_{t_1} + [\phi_{t_1}, \phi])) \psi, \] (3.9)

which is (2.7) by use of (2.12). At order $\lambda^3$ we obtain the derivative of the above equation with respect to $t_1$. At order $\lambda^4$ we get an equation that contains $\psi_{t_2}$, which can be replaced with the help of (3.9). This results in
\[ \psi_{t_2} = (\hat{\theta}^5 + 5\phi_{t_1} \hat{\theta}^3 + \frac{5}{2}(2\hat{\theta}_{t_1} + \phi_{t_1} + [\phi_{t_1}, \phi])) \hat{\theta}^2 + \frac{5}{6}(6\hat{\theta}_{1t_1} + 7\phi_{1t_1} + 2\phi_{t_2}) \]
\[ + 6\phi_{t_1} \hat{\theta}^2 + \frac{5}{2}[\phi_{t_1}, \phi] \hat{\theta} + \frac{5}{2}(\phi_{t_1} + \phi_{1t_1} + \phi_{11t_1} + \frac{1}{2}\phi_{t_2}) \hat{\theta} + 5[\phi_{t_1}, \phi] \]
\[ + \frac{5}{2}(\phi_{t_1} \hat{\theta})_{1t_1} + \frac{5}{6}[\phi_{t_1}, \phi] + \frac{5}{2}[\phi_{t_1}, \phi] \hat{\theta} + \frac{5}{2}[\phi_{t_1}, \phi] \psi, \] (3.10)

which by use of (2.12) becomes (2.8).

### 3.1. The commutative case

If $\mathcal{A}$ is commutative, imposing the reduction condition (2.15), i.e. $\hat{\theta} = \theta = k\phi_{t_1}$ with a constant $k$, the linear system (3.7) takes the form
\[ \frac{1}{\lambda}(\psi_{2[\lambda]} - \psi) = F(\lambda)^{k-\frac{1}{2}}(F(\lambda)^{-k-\frac{1}{2}}(\psi_{2[\lambda]} + \psi))_{\hat{\theta}}. \] (3.11)

Hence
\[ \frac{1}{\lambda}(\psi_{2[\lambda]} - \psi) = \begin{cases} F(\lambda)^{-1}(\psi_{2[\lambda]} + \psi)_{\hat{\theta}}, & k = -\frac{1}{2} \text{ (BKP)} \\ (F(\lambda)^{-1}(\psi_{2[\lambda]} + \psi))_{\hat{\theta}}, & k = \frac{1}{2} \text{ (BKP)} \\ F(\lambda)^{-\frac{1}{2}}(F(\lambda)^{-\frac{1}{2}}(\psi_{2[\lambda]} + \psi))_{\hat{\theta}}, & k = 0 \text{ (CKP)}. \end{cases} \] (3.12)
The CKP functional linear equation is half of the sum of the two BKP functional linear equations. In the remainder of this section we consider the case where \( \phi \) is a \( \mathbb{C} \)-valued function and write

\[
\phi = (\ln t^2)_n, \quad (3.13)
\]

with a function \( t \). (In sections 5.1 and 5.2 we use a different function \( t \) given by \( \phi = (\ln t)_n, \).

**Lemma 3.1.** The bilinear identity (3.1) with the reduction \( \theta = k\phi_t \) implies

\[
w(\lambda^{-1}) = \frac{\tau_{2[2]} - \lambda^{k+\frac{1}{2}} F(-\lambda)}{\tau}, \quad \tilde{w}(\lambda^{-1}) = \frac{\tau_{2[2]} - \lambda^{k+\frac{1}{2}} F(-\lambda)}{\tau}, \quad (3.14)
\]

where \( F(\lambda) \) now takes the form

\[
F(\lambda) = 1 - \frac{\lambda}{\tau} \left( \ln \frac{\tau_{2[2]}}{\tau} \right)_t, \quad (3.15)
\]

**Proof.** We refer to some consequences of (3.1) derived in appendix A. (A.3) can be written as

\[
\tilde{w}(\lambda^{-1}) = \frac{k \tau}{\tau_{2[2]}} F(\lambda). \quad (3.13)
\]

From (A.4) we obtain

\[
w(\lambda^{-1})' = \frac{1}{\lambda} (F(\lambda) - 1) F(\lambda) + \frac{1}{2} F'(\lambda) - \frac{\lambda}{2} (\theta_{2[2]} - \theta),
\]

where a prime indicates a partial derivative with respect to \( t_1 \), and thus

\[
(\ln w(\lambda^{-1}))' = -\frac{1}{2} (\theta_{2[2]} - \phi) + \frac{1}{2} (\ln F(\lambda))' - \frac{\lambda}{2} (\theta_{2[2]} - \theta).
\]

Using (3.13), the preceding equation can be integrated to

\[
w(\lambda^{-1}) = \frac{\tau_{2[2]}}{\tau_{2[2]}} F(\lambda) + \frac{1}{2},
\]

which is equivalent to the first equation in (3.14). With its help, the equation we started with becomes the second of (3.14).

By use of the lemma, and setting \( z = \lambda^{-1} \), we find

\[
\tilde{w}(z) = \begin{cases} 
  w(-z) - z^{-1} w(-z)_n & k = -\frac{1}{2} \quad \text{(BKP)} \\
  \tilde{w}(-z) + z^{-1} \tilde{w}(-z)_n & k = \frac{1}{2} \quad \text{(BKP)} \\
  w(-z) & k = 0 \quad \text{(CKP)}
\end{cases} \quad (3.16)
\]

and thus the following relations for the Baker–Akhiezer function \( \psi \) and its adjoint \( \tilde{\psi} \),

\[
\begin{align*}
  \tilde{\psi}(z) &= -z^{-1} \psi(-z)_n & k = -\frac{1}{2} \quad \text{(BKP)} \\
  \psi(z) &= z^{-1} \tilde{\psi}(-z)_n & k = \frac{1}{2} \quad \text{(BKP)} \\
  \tilde{\psi}(z) &= \psi(-z) & k = 0 \quad \text{(CKP)}.
\end{align*} \quad (3.17)
\]

**Proposition 3.1.** The bilinear identity (3.1) with the reduction \( \theta = k\phi_t \) implies the ‘differential Fay identity’

\[
\begin{align*}
  \lambda + \mu & \tau_{2[2]} (\lambda F_2(\lambda)(\mu)^{k+\frac{1}{2}} F(\mu)^{-k+\frac{1}{2}} - \mu F_2(\mu)(\lambda)^{k+\frac{1}{2}} F(\lambda)^{-k+\frac{1}{2}}) \\
  &= (\lambda + \mu) \tau_{2[2]} - \lambda \mu ((\tau_{2[2]} + \tau)_{n_1} \tau - \tau_{2[2]} + \tau_{2[2]}). \quad (3.18)
\end{align*}
\]

**Proof.** This is obtained from (A.5) using (3.14) and (A.7).
In the BKP case \((k = \pm 1/2)\), the differential Fay identity (3.18) is bilinear,
\[
\begin{align*}
(\lambda^{-1} + \mu^{-1})((\tau_{2[\lambda + 1]} - \tau_{2[\lambda]})(\tau_{2[\lambda]} - \tau_{2[\lambda + 1]})) \\
= (\tau_{2[\lambda + 1]})(\tau_{2[\lambda + 2]} - \tau_{2[\lambda]})(\tau_{2[\lambda]} - \tau_{2[\lambda + 1]} - \tau_{2[\lambda + 1]}(\tau_{2[\lambda]}),
\end{align*}
\]
whereas in the CKP case \((k = 0)\) it is not\(^{11}\),
\[
\begin{align*}
\frac{\lambda + \mu}{\lambda - \mu} \tau_{2[\lambda]}(\lambda F_{2[\lambda]}(\mu))^{2} F(\mu)^{2} - \mu F_{2[\lambda]}(\lambda) \tau_{2[\lambda]}(\tau_{2[\lambda + 1]}),
\end{align*}
\]
Expansion in powers of the indeterminates \(\lambda\) and \(\mu\) generates the BKP, respectively CKP, hierarchy equations.

4. From a functional representation of the KP hierarchy to odd KP

A functional representation of the \(m \times m\) matrix KP hierarchy is determined by \([61]\)
\[
\begin{align*}
\lambda^{-1}(\phi - \phi(-\lambda)) - \phi_{t} - (\phi - \phi(-\lambda))\phi = \tilde{\theta} - \tilde{\theta}(-\lambda)
\end{align*}
\]
with an additional dependent variable \(\tilde{\theta}\), and \(\phi_{\lambda}(t) = \phi(t + [\lambda])\) where \(t = (t_{1}, t_{2}, t_{3}, \ldots)\) and \([\lambda] = (\lambda, \lambda^{2}/2, \lambda^{3}/3, \ldots)\). By expansion in powers of the indeterminate \(\lambda\) and elimination of \(\tilde{\theta}\) one recovers the equations of the KP hierarchy. We note that although (4.1) contains a ‘bare’ \(\phi\) besides derivatives of it with respect to \(t_{\lambda}\), after elimination of \(\tilde{\theta}\) the resulting equations do not. Writing
\[
\begin{align*}
\tilde{\theta} = \theta - \frac{1}{2}(\phi_{t} + \phi^{2})
\end{align*}
\]
(4.1) takes the following form, after a Miwa shift,
\[
\begin{align*}
\lambda^{-1}(\phi_{\lambda} - \phi) - \frac{1}{2}(\phi_{\lambda} + \phi)^{2} - \frac{1}{2}(\phi_{\lambda} - \phi)^{2} - [\phi_{\lambda}, [\phi_{\lambda}, \phi]] = \tilde{\theta}_{\lambda} - \tilde{\theta}.
\end{align*}
\]
Clearly, now one recovers the equations of the matrix KP hierarchy by expansion in powers of \(\lambda\) and elimination of \(\tilde{\theta}\). The first four equations from expansion of (4.3) can be written as\(^{12}\)
\[
\begin{align*}
\phi_{t_{1}} = 2\tilde{\theta}_{1} - [\phi, \phi_{1}].
\end{align*}
\]
\[
\begin{align*}
\tilde{\theta}_{1} = \frac{2}{3}\phi_{t_{1}} - \frac{1}{6}\phi_{t_{1}t_{1}} - (\phi_{t_{1}})^{2} - \frac{1}{2}[\phi_{1}, [\phi_{1}, \phi_{1}]] + [\phi, \theta_{1}].
\end{align*}
\]
\[
\begin{align*}
\phi_{t_{2}} = \frac{4}{3}\phi_{t_{1}} + \frac{4}{3}\theta_{t_{1}} + 2[\phi_{1}, \tilde{\theta}_{1}] - \frac{1}{2}[\phi, 2\phi_{1} + \phi_{t_{1}t_{1}}] - [\phi_{1}, (\phi_{1})^{2}].
\end{align*}
\]
\[
\begin{align*}
\tilde{\theta}_{1} = \frac{2}{3}\phi_{t_{1}} - \frac{1}{6}\phi_{t_{1}t_{1}t_{1}} - [\phi_{1}, 4\phi_{1} - \phi_{t_{1}t_{1}}] + [\phi, (\phi_{t_{1}})^{2}]
\end{align*}
\]
\[
\begin{align*}
&- 2(\tilde{\theta}_{1})^{2} - [\phi_{1}, \tilde{\theta}_{1}t_{1}] + \frac{1}{2}[\phi, 2\tilde{\theta}_{1} + \tilde{\theta}_{t_{1}}] - \frac{1}{6}[\phi_{1}, [\phi, 2\phi_{1} + \phi_{t_{1}}]] \\
&+ [\phi, [\phi_{1}, \tilde{\theta}_{1}]] + [\tilde{\theta}_{1}, [\phi, \phi_{1}]] + \frac{1}{2}[\phi_{1}, [\phi, \phi_{t_{1}}]] - \frac{1}{6}[\phi, (\phi_{t_{1}})^{2}]
\end{align*}
\]
Solving the first equation for \(\tilde{\theta}_{1}\) and using the resulting expression to eliminate \(\tilde{\theta}\) from the second, results in the (potential) KP equation
\[
\begin{align*}
4\phi_{t_{1}} - \phi_{t_{1}t_{1}} - 6(\phi_{t_{1}})^{2} - 3\int \phi_{t_{1}} \, dt_{1} + 6\int [\phi_{1}, \phi_{t_{1}}] \, dt_{1} = 0.
\end{align*}
\]
\(^{11}\)This is in agreement with the fact that the CKP hierarchy cannot be expressed in Hirota bilinear form with a single \(\tau\)-function \([10]\).

\(^{12}\)Here we used e.g. the first equation to eliminate \(\phi_{t_{2}}\) from the second. By use of (2.12), (4.4) simplifies to \(\phi_{t_{1}} = 2\tilde{\theta}_{1}\) (see also remark 2.1).
Instead of eliminating \( \tilde{\theta} \) from (4.3), which yields the matrix KP hierarchy, we can eliminate the derivatives of \( \phi \) and \( \tilde{\theta} \) with respect to the even-numbered variables, \( t_{2n} \). This means we solve the equations resulting from (4.3) for the derivatives of \( \phi \) and \( \tilde{\theta} \) with respect to \( t_{2n} \), as in (4.4), (4.5), etc., compute their integrability conditions, and further use them to eliminate in the latter all derivatives with respect to even-numbered variables. In particular, \( \phi_{t_{2n}} = \phi_{t_{2n}} \) yields, after elimination of ‘even’ derivatives,

\[
\begin{align*}
(9\phi_n - 5\phi_{t_{3n}} + \phi_{t_{3n}t_{3n}} - \frac{15}{2} \{ \phi_n, \phi_n - \phi_{t_{3n}t_{3n}} \} + 15(\phi_n)^3 + \frac{45}{2} ((\phi_n)^2 - 4(\theta_n)^2) \\
&+ \frac{45}{2} \left[ \tilde{\theta}_n, \left[ \phi, \phi_n \right] \right] - \frac{15}{4} \left[ \left[ \phi, \phi_n \right], \phi_n \right] - \frac{15}{4} \left[ \left[ \phi, \phi_{t_{3n}} \right], \phi_n \right] \\
&- \frac{45}{2} \left[ \phi, \phi_n \right] \frac{1}{2} \left[ \{ \phi_n, \tilde{\theta}_n - \tilde{\theta}_{t_{3n}} \} + \{ \tilde{\theta}_n, \phi_n + \frac{1}{2} \phi_{t_{3n}} \} \right] \\
&+ \frac{1}{2} \left[ \tilde{\theta}_{t_{3n}}, \phi_{t_{3n}} \right] - \frac{1}{2} \left[ \phi, \left[ \phi_n, \phi_n \right] \right] \right) = 0, \tag{4.9}
\end{align*}
\]

and from \( \tilde{\theta}_{t_{2n}} = \tilde{\theta}_{t_{2n}} \) one obtains another quite lengthy equation for the two dependent variables \( \phi \) and \( \tilde{\theta} \), involving only derivatives with respect to \( t_1, t_3, t_5 \). We verified independently with FORM [80] and Mathematica [81] that via (2.12) these two equations are equivalent to (2.10) and (2.11), which is our odd KP system.

The structure displayed in (4.4)–(4.7) in fact extends to the whole hierarchy, since the expansion of (4.3) in powers of \( \lambda \) has the following leading derivatives (which do not appear in the remaining terms, represented by dots),

\[
\lambda^{2n-1} \cdot \frac{1}{2n} \phi_{t_{2n}} = \frac{1}{2n-1} \tilde{\theta}_{t_{2n-1}} + \cdots, \quad \lambda^{2n-2} \cdot \frac{1}{2n} \tilde{\theta}_{t_{2n}} = \frac{1}{2n+1} \phi_{t_{2n-1}} + \cdots, \tag{4.10}
\]

where \( n = 1, 2, \ldots \). Hence the method of computing the integrability conditions \( \phi_{t_{2n}t_{2n}} = \tilde{\theta}_{t_{2n}t_{2n}} = \phi_{t_{2n}t_{2n}} = \tilde{\theta}_{t_{2n}t_{2n}} \) and then eliminating all derivatives of \( \phi \) and \( \tilde{\theta} \) with respect to even-numbered variables, extends to the whole KP hierarchy. This yields a hierarchy of equations involving only derivatives with respect to odd-numbered variables and we have shown that its first member is our odd KP system. Because of the hierarchy property, it should then coincide with the odd KP hierarchy as formulated in section 2.6, or generated by the linear system derived in section 3.

Above we started with a formulation of the KP hierarchy in terms of two dependent variables, \( \phi \) and \( \tilde{\theta} \) (or equivalently \( \theta \)). \( \tilde{\theta} \) entered the stage as an auxiliary variable and its elimination leads to an expression for the KP hierarchy in terms of a single dependent variable, which is \( \phi \). In this formulation of the KP hierarchy, the odd KP hierarchy is directly described as a subhierarchy (without further auxiliary variables as in the GDS formulation of section 2.6). A particular consequence is that any method to construct exact solutions to the KP hierarchy in the formulation using the auxiliary dependent variable \( \theta \) (or \( \tilde{\theta} \)) automatically yields solutions to the odd KP hierarchy. This fact will be used in section 5.

We note that (4.4), (4.5), etc., are symmetries of the odd KP hierarchy equations, with the help of which one recovers the whole KP hierarchy.

The next result will turn out to be crucial for establishing a relation between solutions to the (noncommutative) odd KP hierarchy and solutions to the BKP and CKP hierarchies. From now on we consider matrices over \( \mathbb{R} \) or \( \mathbb{C} \).

**Proposition 4.1.** The functional representation (4.3) of the \( m \times m \) matrix KP hierarchy is invariant under

\[
\phi \mapsto \phi^T \circ \varepsilon, \quad \tilde{\theta} \mapsto -\tilde{\theta}^T \circ \varepsilon, \tag{4.11}
\]

where \( \varepsilon(t_1, t_2, t_3, t_4, \ldots) := (t_1, -t_2, t_3, -t_4, \ldots) \), and \( \phi^T \) is the transpose of \( \phi \).
Proof. We consider (4.1) with \( \phi \) and \( \hat{\theta} \) replaced by \( \phi^T \circ \varepsilon \) and \( -\hat{\theta}^T \circ \varepsilon \), respectively. Taking the transpose of the resulting equation, noting that \( (f \circ \varepsilon)_{[\lambda]} = (f_{-[-\lambda]}) \circ \varepsilon \), and composing with \( \varepsilon \) (which has the property \( \varepsilon \circ \varepsilon = \text{id} \)), leads to

\[
\lambda^{-1}(\phi_{-[-\lambda]} - \phi) - \frac{1}{2}(\phi_{-[-\lambda]} + \phi)_{t_1} - \frac{1}{2}(\phi_{-[-\lambda]} - \phi)^2 - \frac{1}{2}[\phi, \phi_{-[-\lambda]}] = -\hat{\theta}_{-[-\lambda]} + \hat{\theta}.
\]

With the substitution \( \lambda \rightarrow -\lambda \) and a Miura shift with \([\lambda]\), this becomes (4.1). \( \square \)

As a consequence, the (matrix) KP hierarchy admits the symmetry reduction

\[
\phi = \phi^T \circ \varepsilon, \quad \hat{\theta} = -\hat{\theta}^T \circ \varepsilon.
\] (4.12)

Restricting to the odd KP hierarchy, and setting \( t_{2n} = 0, n = 1, 2, \ldots \), we have \( \phi \circ \varepsilon = \phi \) and \( \hat{\theta} \circ \varepsilon = \hat{\theta} \), hence the last conditions simplify to

\[
\phi = \phi^T, \quad \hat{\theta} = -\hat{\theta}^T.
\] (4.13)

In particular, for \( m = 1 \) we obtain \( \hat{\theta} = 0 \), hence \( \theta = 0 \) by (2.12), and thus the CKP hierarchy. Conditions (4.13) are equivalent to those that determine the matrix CKP hierarchy, see section 2.6.

Obviously the reduction (4.13) is not compatible with the symmetries (the flows associated with \( t_{2n} \)) that extend the odd KP to the KP hierarchy. This example shows that a subhierarchy can admit a (symmetry) reduction that is not a reduction of the complete hierarchy.

Remark 4.1. A functional representation of the (noncommutative) discrete KP hierarchy is given by [64]

\[
\lambda^{-1}(\phi - \phi_{-[-\lambda]}) - (\phi^T - \phi_{-[-\lambda]])\phi = \hat{\theta}^+ - \hat{\theta}_{-[-\lambda]}.
\] (4.14)

where \( n \in \mathbb{Z} \) and \( (\phi^*)_n = \phi_{n+1} \). To order \( \lambda^0 \), we obtain

\[
\phi_{t_1} - (\phi^+ - \phi)\phi = \hat{\theta}^+ - \hat{\theta}.
\] (4.15)

Subtracting this from (4.14) yields (4.1), hence each \( \phi_n, n \in \mathbb{Z} \), has to satisfy the KP hierarchy, thus also \( \phi^+ \).13 The transformation (4.2) converts the discrete KP hierarchy into

\[
\lambda^{-1}(\phi_{[\lambda]} - \phi) - \frac{1}{2}(\phi_{[\lambda]} + \phi)_{t_1} - \frac{1}{2}(\phi_{[\lambda]} - \phi)^2 + \frac{1}{2}[\phi, \phi_{[\lambda]}] = \hat{\theta}_{[\lambda]} - \hat{\theta}.
\] (4.16)

\[
\frac{1}{2}(\phi^+ + \phi)_{t_1} + \frac{1}{2}(\phi^+ - \phi)^2 + \frac{1}{2}[\phi, \phi^+] = \hat{\theta}^+ - \hat{\theta}.
\] (4.17)

According to proposition 4.1, \( \phi^+ = \phi^T \circ \varepsilon \) and \( \hat{\theta}^+ = -\hat{\theta}^T \circ \varepsilon \) solve (4.16) if \( \phi \) and \( \hat{\theta} \) do. Restricting the KP hierarchy (in the form presented in this section) to the odd KP hierarchy, in the scalar case \( (m = 1) \) these conditions read

\[
\phi^+ = \phi, \quad \hat{\theta}^+ = -\hat{\theta},
\] (4.18)

and (4.17) becomes \( \tilde{\theta} = \hat{\theta} = -\frac{1}{2}\phi_{t_1} \), which is the BKP reduction! We also refer to [82] (p. 969) for a related result.

13 By eliminating \( \hat{\theta} \) and \( \phi^+ \), one obtains the modified KP (mKP) hierarchy for \( v \), where \( v_{t_1} = \phi^+ - \phi \), and the Miura transformation.
5. Solutions to the odd KP system and some of its reductions via a matrix Riccati system

We consider the matrix linear system

$$Z_{tn} = H_n Z_n = 1, \quad 2, \ldots, \quad H = \begin{pmatrix} R & Q \\ S & L \end{pmatrix}, \quad Z = \begin{pmatrix} X \\ Y \end{pmatrix},$$

(5.1)

where $L, Q, R, S$ are, respectively, constant $M \times M, N \times M, N \times N$ and $M \times N$ matrices over $\mathbb{C}$, $X$ is an $N \times N$ and $Y$ an $M \times N$ matrix. With suitable technical assumptions, the size of the matrices may also be infinite. The solution to the above linear system is given by

$$Z = \exp(\xi(t, H)) Z_0$$

where \(\xi(t, H) = \sum_{k=1}^{\infty} t_k H^k\).

(5.2)

For the new variable \(\Phi_1 = YX^{-1}\), assuming that $X$ possesses an inverse, (5.1) implies the following hierarchy of matrix Riccati equations

$$\Phi_{tn} = S_n + L_n \Phi - \Phi R_n - \Phi Q_n \Phi \quad n = 1, 2, \ldots,$$

(5.4)

where

$$ \begin{pmatrix} R_n \\ S_n \\ L_n \end{pmatrix} := H^n $$

(5.5)

(see [61, 63–65]). Using its functional representation

$$\lambda^{-1}(\Phi - \Phi_{-[\lambda]}) = S + L \Phi - \Phi R_{-[\lambda]} R - \Phi Q_{-[\lambda]} Q \Phi,$$

(5.6)

it turns out (see [61] for details) that \(\Phi\) together with

$$\hat{\Theta} = \Phi R$$

(5.7)

solves the $M \times N$ matrix KP$_Q$ hierarchy, which is determined by

$$\lambda^{-1}(\Phi - \Phi_{-[\lambda]}) - \Phi_{\hat{\Theta} - \lambda} - (\Phi - \Phi_{-[\lambda]}) Q \Phi = \hat{\Theta} - \hat{\Theta}_{-[\lambda]}.$$

(5.8)

If \(\text{rank}(Q) = m\), hence

$$Q = VU^\top$$

(5.9)

with an $M \times m$ matrix $U$ (with transpose $U^\top$) and an $N \times m$ matrix $V$, then \(\Phi := U^\top \Phi V\)

(5.10)

solves the $m \times m$ matrix KP hierarchy (4.1). By use of the first Riccati equation

$$\Phi_{\hat{\Theta}} = S + L \Phi - \Phi R - \Phi Q \Phi,$$

(5.11)

in \(\hat{\Theta} = \hat{\Theta} - \frac{1}{2} (\Phi_{\hat{\Theta}} + \Phi Q \Phi)\) (cf (4.2)), and using (5.7), we obtain

$$\hat{\Theta} = \frac{1}{2}(S + L \Phi + \Phi R).$$

(5.12)

Here we shall drop $S$ since it cancels out in $\hat{\Theta}_{[\lambda]} - \hat{\Theta}$. It follows that the $Q$-modified version of (4.3) is satisfied as a consequence of the Riccati system. Recalling (2.12), which now takes the form

$$\hat{\Theta} = \Theta + \frac{1}{2} \int (\Phi Q \Phi_{\hat{\Theta}} - \Phi_{\hat{\Theta}} Q \Phi) \, dt,$$

(5.13)

we arrive at the following conclusion.
Proposition 5.1. Any solution $\Phi$ to the odd Riccati hierarchy, i.e. the Riccati hierarchy (5.4) restricted to odd $n$, together with\(^{14}\)
\[ \Theta = \frac{1}{2} \left( L \Phi + \Phi R - \int \left( \Phi Q \Phi_t - \Phi_t Q \Phi \right) dt \right), \tag{5.14} \]
solves the odd KP$_Q$ hierarchy.\(^{15}\) Furthermore, if (5.9) holds, then
\[ \phi = U^T \Phi V \quad \text{and} \quad \theta = U^T \Theta V \tag{5.15} \]
solve the $m \times m$ matrix odd KP hierarchy (hence in particular the odd KP system (2.10) and (2.11)). If $m = 1$, then
\[ \phi = U^T \Phi V \quad \text{and} \quad \theta = \frac{1}{2} U^T (L \Phi + \Phi R) V \tag{5.16} \]
solve the scalar odd KP hierarchy (thus in particular (2.13) and (2.14)).

Remark 5.1. For some fixed $r \in \mathbb{N}$, $r > 1$, let us impose the condition
\[ H' Z_0 = Z_0 P, \tag{5.17} \]
with an $N \times N$ matrix $P$, on the solution (5.2) of the linear matrix system (5.1). This implies $H'' Z_0 = Z_0 P^n$ and thus $H'' Z = Z P^n$ for $n \in \mathbb{N}$. Hence $R_{mn} X + Q_{mn} Y = X P^n$ and $S_{mn} X + L_{mn} Y = Y P^n$, which leads to the algebraic Riccati equations
\[ S_{mn} + L_{mn} \Phi = Y P^n X^{-1} = \Phi X P^n X^{-1} = \Phi (R_{mn} + Q_{mn} \Phi) \quad n \in \mathbb{N}. \tag{5.18} \]
The corresponding equations of the Riccati hierarchy then imply $\Phi_{mn} = 0$, for all $n \in \mathbb{N}$. Condition (5.17) thus ensures that $\phi$ solves the $r$-reduction of the KP hierarchy (rth Gelfand–Dickey hierarchy). If $r$ is odd, this also yields a reduction of the odd KP hierarchy. Hence, adding condition (5.17) to the assumptions of proposition 5.1, (5.15) constitutes a solution to the $r$-reduction of the $m \times m$ matrix odd KP hierarchy. For $r = 3$ this is the hierarchy with the pair (3.27), (3.38) as its first member, for $r = 5$ it starts with the noncommutative generalization of the bSK and bKK equations, see section 2.5.

In proposition 5.1 ‘odd KP hierarchy’ more directly refers to the form in section 4, where it has been described as a subhierarchy of the KP hierarchy, in the formulation of the latter involving the auxiliary variable $\theta$. In the scalar case, this hierarchy then admits reductions to the CKP and BKP hierarchy by imposing $\theta = 0$, respectively $\theta = -\frac{1}{2} \phi_0$ (see section 2). In the following we show how the preceding proposition generates solutions to the BKP and the (matrix) CKP hierarchy.

Lemma 5.1. Let $M = N$. The transformation given by
\[ L \mapsto -R^T, \quad R \mapsto -L^T, \quad Q \mapsto \pm Q^T, \quad S \mapsto \pm S^T, \quad \Phi \mapsto \pm \Phi^T \circ \epsilon, \tag{5.19} \]
with $\epsilon$ defined in proposition 4.1, leaves the Riccati hierarchy (5.4) invariant.

Proof. The first four replacement rules in (5.19) can be combined into
\[ H \mapsto -TH^T T^{-1} \quad \text{with} \quad T = \begin{pmatrix} 0 & \mp I_N \\ I_N & 0 \end{pmatrix}. \tag{5.20} \]
\(^{14}\)By use of the Riccati system (5.4), this can also be written as $\Theta = \int (S_2 \Phi - \Phi R_2 - \Phi Q_2 \Phi) dt_1$. The integrand is the right-hand side of the Riccati equation for the variable $t_2$ (which, however, is prohibited in proposition 5.1), so that $\Theta_1 = -\frac{1}{2} \phi_2$, a symmetry of the odd KP (here odd KP$_Q$) hierarchy which we already met in remark 2.1.

\(^{15}\)Hence it solves in particular (2.10) and (2.11) with $\phi$ and $\theta$ replaced by matrices $\Phi$ and $\Theta$, and with the product modified by the constant matrix $Q$. 

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This implies
\[ H^n \mapsto (-1)^n T (H^n)^T T^{-1}, \]
and thus
\[ L_n \mapsto (-1)^n L_n^T, \quad R_n \mapsto (-1)^n R_n^T, \quad Q_n \mapsto \mp (-1)^n Q_n^T, \quad S_n \mapsto \mp (-1)^n S_n. \]
Applying the map to the Riccati hierarchy (5.4), taking the transpose and using \((\Phi \circ \varepsilon)_n = (-1)^n \Phi_n \circ \varepsilon\), reproduces (5.4).

As a consequence of the preceding lemma, we have the following symmetry reduction of the Riccati hierarchy (5.4),
\[ R = -L^T, \quad Q^T = \pm Q, \quad S^T = \pm S, \quad (5.21) \]
and thus
\[ \Phi = \pm \Phi^T \circ \varepsilon. \quad (5.22) \]
Restricting to the odd Riccati hierarchy, we are allowed to set \(t_{2n} = 0, n = 1, 2, \ldots\) Then \(\Phi\) given by (5.3) solves the odd Riccati hierarchy and has the property \(\Phi \circ \varepsilon = \Phi\). Furthermore, (5.21) and (5.22), which now reads \(\Phi = \pm \Phi^T\), constitute a symmetry reduction of the odd Riccati hierarchy.

**Proposition 5.2.** Let \(M = N\) and \(\Phi\) a solution to the odd Riccati hierarchy with
\[ R = -L^T, \quad S = S^T, \quad Q = VV^T, \quad (5.23) \]
where \(V\) is a constant \(N \times m\) matrix. If
\[ \Phi^T = \Phi, \quad (5.24) \]
then
\[ \phi = V^T \Phi V \quad \text{and} \quad \theta = V^T \Theta V \quad (5.25) \]
with \(\Theta\) given in (5.14) solve the \(m \times m\) matrix CKP hierarchy (see section 2.6).

**Proof.** Conditions (5.23) and (5.24) correspond to the upper signs in (5.21). According to proposition 5.1, \(\phi \) and \(\theta\) solve the \(m \times m\) matrix odd KP hierarchy. Using (5.14), (5.24) and \(Q^T = Q\), one easily verifies that \(\theta^T = -\theta\) holds, which is the reduction to the matrix CKP hierarchy.

**Corollary 5.1.** Let \(M = N\) and \(\Phi\) a solution to the odd Riccati hierarchy with (5.23), where \(V\) is a constant \(N\)-component vector. If \(\Phi^T = \Phi\), then \(\phi = V^T \Phi V\) solves the CKP hierarchy.

**Proof.** The assertion follows from the last proposition, with \(\Theta \) defined in (5.14) and \(m = 1\), in which case the CKP reduction condition \(\theta = 0\) holds.

To obtain BKP solutions via proposition 5.1 is a bit less direct.

**Proposition 5.3.** Let \(M = N\) and \(\Phi\) a solution to the odd Riccati hierarchy subject to the conditions (5.23) with a constant \(N\)-component vector \(V\). If \(\Phi\) satisfies
\[ S + L\Phi + \Phi^T L^T - \Phi^T Q \Phi = 0, \quad (5.26) \]
then \(\phi = V^T \Phi V\) solves the BKP hierarchy.

**Proof.** First we note that the fractional linear transformation \(\Phi \mapsto \Phi^* := (S+L\Phi)(R+Q\Phi)^{-1}\) (provided the inverse exists) leaves the Riccati hierarchy (5.4) invariant. This is so because this
transformation is induced by $Z \mapsto HZ$, which leaves the linear matrix system (5.1) invariant. We may then impose the symmetry reduction $\Phi^T = \Phi^*$, i.e.

$$\Phi^T = (S + L\Phi)(R + Q\Phi)^{-1},$$

which is (5.26). Using the definitions (5.16) with $U = V$, the first Riccati equation (5.11), and then the last equation, we show that the BKP reduction condition is satisfied,

$$2(\theta + \varphi_1) = V^T(L \Phi - \Phi^T + \phi_1)V$$

$$= V^T(S + L\Phi - L^T \Phi^T - \Phi^T \Phi)V$$

$$= V^T(L \Phi - \Phi^T L^T + (\Phi^T - \Phi)V V^T \Phi)V$$

$$= V^T L \Phi V - V^T \Phi^T L^T V = 0.$$

(One also finds $\phi^* = \phi$ and $\theta^* = -\theta$, cf (4.18).) \(\Box\)

**Remark 5.2.** The discrete KP hierarchy is solved by a sequence $\Phi = (\Phi_n)_{n \in \mathbb{Z}}$ of solutions to the Riccati hierarchy (5.4) if $L \Phi - \Phi^T R - \Phi^T Q \Phi = 0$, where $\Phi_1 = \Phi_{n+1}$. This is the fractional linear transformation appearing in the proof of proposition 5.3, with $S = 0$. It follows from (4.15) by use of (5.7) and (5.11).

**Remark 5.3.** The case with the lower signs in lemma 5.1 might be expected to be related to BKP. But it requires a skew-symmetric $Q$ and thus does not quite fit together with proposition 5.1. However, writing $Q = Q L - L^T Q$ with a rank 1 matrix $\tilde{Q} = V V^T$, it turns out that $\phi = V^T (L \Phi - \Phi^T L^T) V = 2V^T L \Phi V$ solves the BKP equation (and its hierarchy), if $\Phi$ satisfies the conditions of lemma 5.1 with the lower signs. We shall elaborate on the underlying structure elsewhere.

**Remark 5.4.** As a consequence of (5.1) and (5.21), which implies $H = -TH^T T^{-1}$ with $T$ defined in (5.20), we have

$$(Z^T T H^k Z)_n = 0 \quad \text{for all odd } n$$

(5.27)

and $k = 0, 1, \ldots$. Choosing $T$ with the minus sign, our CKP condition $\Phi^T = \Phi$ originates from $Z^T T Z = 0$, and the BKP condition (5.26) corresponds to $Z^T T H Z = 0$. These conditions are the first two in a sequence that offers additional possibilities,

$$Z^T T H^k Z = 0 \quad k = 0, 1, 2, \ldots$$

(5.28)

We note that $(TH^k)^T = (-1)^{k+1} TH^k$, so that the left-hand side of (5.28) is a symmetric bilinear form if $k$ is odd, and skew-symmetric if $k$ is even. Invariance under a transformation $Z \mapsto GZ$, with a constant invertible matrix $G$, requires $G^T T H^k G = TH^k$. If the bilinear form is non-degenerate\(^{16}\), this means that $G$ has to be (complex) orthogonal if $k$ is odd, and symplectic if $k$ is even. This connects with original work like [10]. It should be noticed, however, that the above method to construct solutions to the BKP hierarchy also works if the bilinear form is degenerate.

**Remark 5.5.** Adding the $r$-reduction condition (5.17) to the assumptions of corollary 5.1, respectively proposition 5.3, they generate solutions to the $r$-reduction of the CKP, respectively BKP, hierarchy. For $r = 3$, this yields solutions to the Kaup–Kupershmidt, respectively the Sawada–Kotera equation. For $r = 5$, we obtain solutions to the bKK, respectively the bSK equation (see section 2.2). We will not elaborate this further in this work, but a comparison with the results in [50, 60, 69–71] would certainly be of interest.

\(^{16}\)In the CKP case ($k = 0$) this is fulfilled. In the BKP case (and more generally for $k > 0$), and if $S = 0$ and $R = -L^T$, which is the case we address in more detail below, the bilinear form is non-degenerate iff $\det(L) \neq 0$. 18
In the following subsections we elaborate some classes of solutions more explicitly. We consider the odd Riccati hierarchy with \( M = N \), impose the conditions (5.23) with \( S = 0 \), and treat the rank 1 case where \( Q = V V^\top \) with a vector \( V \). The choices (5.29) and (5.53) below have their origin in certain normal forms of the matrix \( H \), see [64].

### 5.1. A class of BKP and CKP solutions

Setting

\[
Q = RK - KL = -(L^\top K + KL)
\]

with a symmetric matrix \( K \) (i.e. \( K^\top = K \)), (5.2) can be computed explicitly (cf [64]) and we find the following solution to the odd Riccati hierarchy,

\[
\Phi = e^{\tilde{\xi}(t_0,L)}\Phi_0(e^{-\tilde{\xi}(t_0,L^\top)}(I_N + K\Phi_0) - K e^{\tilde{\xi}(t_0,L)}\Phi_0)^{-1},
\]

where \( \Phi_0 = Y_0X_0^{-1} \) and

\[
\tilde{\xi}(t_0, L) = \sum_{k=0}^{\infty} t_{2k+1} L^{2k+1}.
\]

Assuming \( \Phi_0 \) invertible, this simplifies to

\[
\Phi = (e^{-\tilde{\xi}(t_0,L)}(\Phi_0^{-1} + K) e^{-\tilde{\xi}(t_0,L)} - K)^{-1}.
\]

Using \( Q = V V^\top \), the cyclicity of the trace, and \( \text{tr} \ln = \ln \det \), we obtain

\[
\phi = V^\top \Phi V = \text{tr}(Q\Phi) = -\text{tr}((L^\top K + KL)\Phi) = (\ln \tau)_t, \quad \text{with} \quad \tau = \det \left( \Phi_0^{-1} + K - e^{\tilde{\xi}(t_0,L^\top)}K e^{\tilde{\xi}(t_0,L)} \right).
\]

Here \( K, L, V \) have to solve the rank 1 condition

\[
L^\top K + KL = -V V^\top.
\]

In order that (5.33) solves the CKP or the BKP hierarchy, (5.24), respectively (5.26), still has to be satisfied.

**CKP** If \( \Phi_0 \) is symmetric, i.e. \( \Phi_0^\top = \Phi_0 \), then also \( \Phi \) given in (5.32). We can thus express \( \tau \) in (5.33) as

\[
\tau = \det \left( C - e^{\tilde{\xi}(t_0,L^\top)}K e^{\tilde{\xi}(t_0,L)} \right)
\]

with an arbitrary constant symmetric \( N \times N \) matrix \( C \), i.e. \( C^\top = C \). According to corollary 5.1, this determines a solution \( \phi = (\ln \tau)_t \) to the CKP hierarchy, provided that \( K \) and \( L \) satisfy (5.34).

**BKP** We have to elaborate the BKP condition (5.26) (with \( S = 0 \)). Using (5.29), it can be expressed as

\[
L^\top (\Phi^{-1} + K) = -(\Phi^{-1} + K)^\top L.
\]

Inserting (5.32), written in the form

\[
\Phi^{-1} + K = e^{-\tilde{\xi}(t_0,L^\top)}(\Phi_0^{-1} + K) e^{-\tilde{\xi}(t_0,L)}
\]

this reduces to

\[
\frac{1}{2} C := L^\top (\Phi_0^{-1} + K) = -(\Phi_0^{-1} + K)^\top L = -\frac{1}{2} C^\top,
\]

i.e. \( C \) has to be a skew-symmetric matrix.
It is known that BKP $\tau$-functions can be expressed as the square of a Pfaffian. In the following we translate (5.33) into such a form, assuming that $L$ is invertible. We may replace $\tau$ given in (5.33) by

$$\tau = \det \left( C - 2 \mathcal{V} (t, L^T L) L^T K \mathcal{V} (t, L) \right),$$

(5.39)

since the two expressions differ only by a constant factor that drops out in $\phi = (\ln \tau)_t$. Using $L^T K = -KL - VV^T$, this becomes

$$\tau = \det (\mathcal{V} (t, L^T L))^2 \det (A + VV^T) \quad \text{where} \quad A := \mathcal{V} (t, L^T) C \mathcal{V} (t, L) - L^T K + KL.$$

(5.40)

If the size $N$ of the matrices is even, then $\det (A + VV^T) = \det (A)$ (for skew-symmetric $A$, see e.g. (2.92) in [47]) leads to

$$\tau = \det \left( C - \mathcal{V} (t, L^T) (L^T K - KL) \mathcal{V} (t, L) \right).$$

(5.41)

This is the determinant of a skew-symmetric matrix; hence $\tau$ can be expressed as the square of the Pfaffian of this matrix. If $N$ is odd, then $\det (A) = 0$, but (5.40) with a suitable choice of $V$ can still lead to non-trivial solutions. In this case we can use the identity

$$\det (A + VV^T) = \det \begin{pmatrix} 0 & V^T \\ -V & A \end{pmatrix} = \left( \text{Pf} \begin{pmatrix} 0 & V^T \\ -V & A \end{pmatrix} \right)^2$$

(5.42)

(see appendix B) to express $\tau$ as the square of a Pfaffian.$^{17}$

A subclass of solutions is obtained by choosing

$$L = \text{diag}(p_1, \ldots, p_N)$$

(5.43)

with constants $p_i$, $i = 1, \ldots, N$. The solution to (5.34) is then given by

$$K_{ij} = -\frac{v_i v_j}{p_i + p_j} \quad i, j = 1, \ldots, N,$$

(5.44)

assuming $p_i + p_j \neq 0$ for all $i, j$ and writing $V^T = (v_1, \ldots, v_N)$. From this one recovers in particular BKP and CKP multi-soliton solutions (see also [8, 10, 82] for different approaches).

5.1.1. Examples.

Example 5.1. We consider the CKP case with (5.43). For $N = 1$, (5.35) becomes $\tau = 1 + b \mathcal{V} (t, p)$ with $b = \frac{v_i^2}{2}$, dropping an irrelevant factor $c$. This yields a regular solution if $b > 0$, and $u = \phi_1$ then describes a single line soliton. For $N = 2$ and $C = \text{diag}(c_1, c_2)$ we obtain, dropping an irrelevant factor $c_1 c_2$,

$$\tau = 1 + b_1 \mathcal{V} (t, p_1) + b_2 \mathcal{V} (t, p_2) + b_1 b_2 \left( \frac{p_1 - p_2}{p_1 + p_2} \right)^2 \mathcal{V} (t, p_1)^2 \mathcal{V} (t, p_2), \quad b_i := \frac{v_i^2}{2c_i p_i},$$

(5.45)

assuming $p_1, p_2, c_1, c_2 \neq 0$ and $p_2 \neq -p_1$. If the parameters are real and such that $b_1, b_2 > 0$, this yields a regular CKP solution $\phi$, and $u = \phi_1$ generically describes two oblique line solitons. In this case we can simplify the above expression by writing $b_i = \exp(2a_i)$ with constants $a_i, i = 1, 2$.

Example 5.2. In the BKP case with (5.43), we consider $N = 2$, hence

$$L = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}, \quad V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

(5.46)

$^{17}$ The factor $\det (\mathcal{V} (t, L^T L))^2$ in (5.40) can be dropped since it does not influence $\phi_0$. 


Equation (5.41) leads to \( \tau = p^2 \) with
\[
p = c + v_1 v_2 \frac{p_1 - p_2}{p_1 + p_2} e^{i(t, p_1) + i(t, p_2)},
\]
if \( p_1 + p_2 \neq 0 \). Without restriction of generality we can set \( v_1 = v_2 = 1 \). For real \( c, p_1, p_2 \), the function \( \phi \) is then regular (for all \( t_1, t_2, \ldots \)) if \( c \left( p_1^2 - p_2^2 \right) > 0 \), and \( u = \phi \) describes a single line soliton.

Solutions can be superposed as follows. If \((L_i, V_i, K_i, C_i), i = 1, 2\), are two sets of matrix data that determine (BKP or CKP) solutions, then
\[
L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 & K_{12} \\ K_{12}^T & K_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}
\]
(5.48)
determine a new solution, provided that a solution \( K_{12} \) exists to
\[
L_1^T K_{12} + K_{12} L_2 = -V_1 V_2^T.
\]
(5.49)

Example 5.3. We consider the BKP case. By superposition of two solutions of the form given in example 5.2, setting \( V_1 = V_2 = (1, 1)^T \), one obtains \( \tau = p^2 \) with
\[
p = b (\tilde{c}_1 \tilde{c}_2 + \tilde{c}_1 e^{i(t, p_1) + i(t, p_3)} + \tilde{c}_2 e^{i(t, p_1) + i(t, p_2)} + a e^{i(t, p_2) + i(t, p_3) + i(t, p_1)}),
\]
(5.50)
where
\[
a = \frac{(p_1 - p_3)(p_2 - p_3)(p_1 - p_4)(p_2 - p_4)}{(p_1 + p_3)(p_2 + p_3)(p_1 + p_4)(p_2 + p_4)}, \quad b = \frac{(p_1 - p_2)(p_1 - p_4)}{(p_1 + p_2)(p_1 + p_4)}
\]
(5.51)
and \( \tilde{c}_1 = c_1 (p_1 + p_3)/(p_1 - p_3), \tilde{c}_2 = c_2 (p_3 + p_4)/(p_3 - p_4) \). If \( p_1 \neq p_2 \) and \( p_3 \neq p_4 \), we may drop the factor \( b \). With real parameters and \( \tilde{c}_1, \tilde{c}_2 > 0 \), one recovers a well-known expression for the 2-soliton solution \( a > 0 \) to the BKP hierarchy [16, 47], see also figure 1. Allowing the parameters to be complex, we can superpose the solution data (5.46) and the complex conjugate data, so that
\[
p = \left| c + \frac{p_1 - p_2}{p_1 + p_2} e^{i(t, p_1) + i(t, p_2)} \left| + \frac{\text{Im}(p_1) \text{Im}(p_2)}{\text{Re}(p_1) \text{Re}(p_2) - \left| \frac{p_1 - p_2}{p_1 + p_2} \right|^2} \left| e^{2 \text{Re}(i(t, p_1) + i(t, p_2))} \right| \right|^2 \right| \frac{p_1 - p_2}{p_1 + p_2} \right|^2 \left| e^{2 \text{Re}(i(t, p_1) + i(t, p_2))} \right| \right|
\]
(5.52)
A regular solution from this family is plotted in figure 2. See also appendix C for a general recipe to obtain real solutions from complex matrix data.
5.2. Another class of BKP and CKP solutions

Now we set $R = L$, so that $L$ is skew-symmetric, i.e. $L^T = -L$, and

$$Q = I_N + [L, K]$$

with a symmetric matrix $K$. Assuming $\Phi_0$ invertible, computation of (5.2) (cf [64]) leads to the following solution to the odd Riccati hierarchy,

$$\Phi = \left(e^{\tilde{\xi}(t_0, L)}(\Phi_0^{-1} + K) e^{-\tilde{\xi}(t_0, L)} + \tilde{\xi}'(t_0, L) - K\right)^{-1}, \quad (5.54)$$

where

$$\tilde{\xi}'(t_0, L) := \sum_{n=0}^{\infty} (2n + 1)_{2n+1} L^{2n}. \quad (5.55)$$

If also $Q = VV^T$ with a vector $V$, hence $K, L, V$ have to satisfy the rank 1 condition

$$I_N + [L, K] = VV^T, \quad (5.56)$$

then we obtain

$$\phi = V^T \Phi V = \text{tr}((I_N + [L, K]) \Phi) = (\ln \tau)_{t_0} \quad \text{with} \quad \tau = \det\left(e^{\tilde{\xi}(t_0, L)}(\Phi_0^{-1} + K) e^{-\tilde{\xi}(t_0, L)} + \tilde{\xi}'(t_0, L) - K\right). \quad (5.57)$$

In order that (5.57) solves the CKP or the BKP hierarchy, the condition (5.24), respectively (5.26), still has to be elaborated.

**CKP.** If $\Phi_0$ is symmetric, then also $\Phi$. We can then replace the above function $\tau$ by

$$\tau = \det(\text{e}^{\tilde{\xi}(t_0, L)} C e^{-\tilde{\xi}(t_0, L)} + \tilde{\xi}'(t_0, L) - K), \quad (5.58)$$

with an arbitrary constant symmetric $N \times N$ matrix $C$. According to corollary 5.1, this determines a solution $\phi$ to the CKP hierarchy, if $K$ and $L$ satisfy (5.56).

**BKP.** The condition (5.26) (with $S = 0$) can be written in the form

$$L(\Phi^{-1} + K) - (\Phi^{-1} + K)^T L = -I_N. \quad (5.59)$$

Inserting (5.54), rewritten as

$$\Phi^{-1} + K = e^{\tilde{\xi}(t_0, L)}(\Phi_0^{-1} + K) e^{-\tilde{\xi}(t_0, L)} + \tilde{\xi}'(t_0, L), \quad (5.60)$$
leads to
\[ L(\Phi_0^{-1} + K) - (\Phi_0^{-1} + K)^T L = -I_N, \tag{5.61} \]
which is
\[ C^T = -C \quad \text{where} \quad C := 2L(\Phi_0^{-1} + K) + I_N, \tag{5.62} \]
i.e. \( C \) has to be skew-symmetric.

Next we translate (5.57) in the BKP case into a form, where \( \tau \) is the determinant of a skew-symmetric matrix, under the assumption that \( \det(L) \neq 0 \). According to remark 5.4, the latter condition corresponds to the genuine BKP case. A function equivalent to \( \tau \) given in (5.57) is then
\[ \tau = \det(e^{\xi(t_0, L)} C - e^{-\xi(t_0, L)} + 2L(\xi'(t_0, L) - K)) \]
\[ = \det(A - VV^T) \quad \text{where} \quad A := e^{\xi(t_0, L)} C e^{-\xi(t_0, L)} - (KL + LK) + 2L\xi'(t_0, L). \tag{5.63} \]
This is the determinant of the sum of the skew-symmetric matrix \( A \) and a rank 1 matrix. If \( N \) is even, then \( \det(A - VV^T) = \det(A) \) and thus
\[ \tau = \det(e^{\xi(t_0, L)} C e^{-\xi(t_0, L)} - (KL + LK) + 2L\xi'(t_0, L)), \tag{5.64} \]
which is then the square of the Pfaffian of \( A \).

**Remark 5.6.** (5.53) implies \( \text{tr}(L^k(Q - I_N)) = 0 \), \( k = 0, 1, \ldots, N - 1 \). These constraints are obstructions to solving (5.53) for \( k \). In particular, we have \( \text{tr}(Q) = N \). Hence \( V \) lies on a sphere in \( N \) dimensions. Since the (complex) orthogonal group acts transitively on a (complexified) sphere (see e.g. lemma 4.1 in [84]), \( V \) can be transformed to \( V = (1, \ldots, 1)^T \). Since a similarity transformation of the matrices leaves (5.57) invariant, this means that without loss of generality we can set \( V = (1, \ldots, 1)^T \), as long as no restrictions are placed on the antisymmetric matrix \( L \).

Choosing \( \Phi_0 \) such that
\[ [\Phi_0^{-1} + K, L] = 0, \tag{5.65} \]
the above solutions become rational functions of \( t_1, t_3, t_5, \ldots \). We confine ourselves to this case in the following examples. For the matrix \( C \) (which has to be symmetric in the CKP and skew-symmetric in the BKP case), (5.65) implies \([C, L] = 0\).

**5.2.1. Examples.**

**Example 5.4.** Let \( N = 2 \) and
\[ L = \begin{pmatrix} 0 & p \\ -p & 0 \end{pmatrix} \tag{5.66} \]
with a constant \( p \). According to the last remark we can set \( V^T = (1, 1) \) without restriction of generality. The solution to (5.56) is then given by
\[ K = cI_2 + \frac{1}{2p} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5.67} \]
with an arbitrary constant \( c \). Condition (5.65) leads to \( C = aI_2 + bL \) with constants \( a, b \).

---

\[18\] There are no independent equations for \( k > N - 1 \) because of the Cayley–Hamilton theorem.

\[19\] For other approaches to rational solutions see [9, 21, 48] in the BKP and [10, 43, 44] in the CKP case.
In the CKP case, $b = 0$ and the resulting term in (5.54) involving $a$ can be absorbed by redefinition of $c$. We obtain
\[
\tau = \eta^2 - \frac{1}{4p^2} \quad \text{where} \quad \eta(t_0, p, c) = \sum_{n=0}^{\infty} (-1)^n (2n + 1) t_{2n+1} p^{2n} - c.
\] 
(5.68)

If $p$ is imaginary, the corresponding CKP solution is real and regular. Treating $t_5$ as a ‘time’ variable (and freezing the higher variables), $u = \phi_t = (\ln \tau) t_1 t_3$ describes a line soliton (with rational decay) moving in $t_1 t_3$-space.

In the BKP case, (5.62) requires $b = \frac{1}{2} p^{-2}$, hence $C = 2aL$. In (5.64), $a$ can be absorbed by redefinition of $c$. Hence we can set $C = 0$ without loss of generality. We obtain $P(A) = 2p\eta(t_0, p, c)$, which cannot provide us with a real and regular BKP solution.

Given two sets of matrix data $(L_i, V_i, K_i), i = 1, 2$, that determine (BKP or CKP) solutions, we can superpose them as follows,
\[
L = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}, \quad V = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 & K_{12} \\ K_{12}^T & K_2 \end{pmatrix}.
\]
(5.69)

Then (5.56) is satisfied if $K_{12}$ solves
\[
L_1 K_{12} - K_{12} L_2 = V_1 V_2^T.
\]
(5.70)

Example 5.5. We superpose two solutions of the form given in example 5.4. The solution to (5.70) is then
\[
K_{12} = \begin{pmatrix} \frac{1}{p_1 + p_2} & -\frac{1}{p_1 p_2} \\ -\frac{1}{p_1 p_2} & \frac{1}{p_1 - p_2} \end{pmatrix}.
\]
(5.71)

In the CKP case, (5.57) with $C = 0$ yields
\[
\tau = \left( \eta_1 \eta_2 - \frac{1}{(p_1 - p_2)^2} \right) - \frac{1}{(p_1 + p_2)^2} - \frac{1}{4p_1 p_2} \left( \frac{\eta_1}{p_2} - \frac{\eta_2}{p_1} \right)^2 - \frac{1}{p_1 p_2 (p_1 - p_2)^2},
\]
(5.72)

where $\eta_i = \eta(t_0, p_i, c_i), i = 1, 2$ (see (5.68)). If $p_2 = p_1^*$ (the complex conjugate of $p_1$), $c_2 = c_1^*$, and $\text{Re}(p_1) \text{Im}(p_1) \neq 0$, this expression is real (see also appendix C) and strictly positive, and thus determines a regular solution to the CKP hierarchy. See also figure 3.
In the BKP case, we obtain
\[ Pf(A) = 4p_1p_2\left(\eta_1\eta_2 - 2 \frac{p_1^2 + p_2^2}{(p_1^2 - p_2^2)^2}\right). \] (5.73)
where again \( \eta_i = \eta(t_n, p_i, c_i), i = 1, 2. \) Choosing \( p_1^* = p_1 =: p \) and \( c_1^* = c_1 =: c \), this takes the form
\[ Pf(A) = 4|p|^2\left(\frac{\text{Re}(p^2)}{\text{Im}(p^2)^2} + |\eta(t_n, p, c)|^2\right). \] (5.74)
which is strictly positive if \( \text{Re}(p^2) > 0 \), hence the solution is regular. Writing \( p = \alpha + i\beta \), the last condition means \( |\alpha| > |\beta| \). This solution appeared in [21] (with the opposite inequality \( |\alpha| < |\beta| \), since our \( p \) corresponds to \( ip \) in that work). Figure 4 shows a plot. The factor 4\(|p|^2\) in (5.74) drops out in the passage to \( \phi \) and can thus be omitted. Setting \( t_{2n+1} = 0 \) for \( n > 2 \), the maximum value of \( u = \phi(t) \) for the above solution is given by \( u_{\text{max}} = 4 \text{Im}(p^2)^2/\text{Re}(p^2) \) and the maximum moves, in ‘time’ \( t_5 \), according to
\[ t_1 = 5|p|^4t_5 + \text{Re}(c) - \frac{\text{Re}(p^2)}{\text{Im}(p^2)}\text{Im}(c), \quad t_3 = \frac{10}{3} \text{Re}(p^2)t_5 - \frac{\text{Im}(c)}{3\text{Im}(p^2)}. \] (5.75)
The solution has two minima with \( u_{\text{min}} = -\text{Im}(p^2)/(2\text{Re}(p^2)) \), located symmetrically with respect to the maximum. See also figure 4.

**Example 5.6.** Let
\[ L_i = \begin{pmatrix} 0 & p_i & 0 & 0 \\ -p_i & 0 & 0 & 0 \\ 0 & 0 & 0 & p_i^* \\ 0 & 0 & -p_i^* & 0 \end{pmatrix}, \quad V_i = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad i = 1, 2. \] (5.76)
In the preceding example we have seen that these data determine single BKP lumps, and the corresponding \( K_i \) are obtained from (5.67) and (5.71). The superposition condition (5.70) is then solved by
\[ K_{12} = \begin{pmatrix} \frac{1}{p_1^*+p_2^*} & \frac{1}{p_2^*+p_1^*} & \frac{1}{p_1-p_1^*} & \frac{1}{p_2-p_2^*} \\ \frac{1}{p_1^*+p_2^*} & \frac{1}{p_2^*+p_1^*} & \frac{1}{p_1-p_1^*} & \frac{1}{p_2-p_2^*} \\ \frac{1}{p_1^*+p_2^*} & \frac{1}{p_2^*+p_1^*} & \frac{1}{p_1-p_1^*} & \frac{1}{p_2-p_2^*} \\ \frac{1}{p_1^*+p_2^*} & \frac{1}{p_2^*+p_1^*} & \frac{1}{p_1-p_1^*} & \frac{1}{p_2-p_2^*} \end{pmatrix}. \] (5.77)
Figure 5. A 2-lump solution to the BKP equation. Plot of $u = \varphi_t$ at $t_5 = -50, 0, 50$ (and
$t_7, t_9, \ldots = 0$) for the solution in example 5.6 with $C = 0$ and $p_1 = \frac{1}{2} + \frac{i}{4}, p_2 = \frac{1}{3} + \frac{i}{4}$. The two
lumps never merge but seem to exchange their identities at a certain minimal separation.

All this determines BKP 2-lump solutions via (5.64) and figure 5 displays an example.

Example 5.7. Let $N = 3$. The general skew-symmetric $3 \times 3$ matrix $L$ is

$$L = \begin{pmatrix}
0 & p_1 & p_3 \\
-p_1 & 0 & p_2 \\
-p_3 & -p_2 & 0
\end{pmatrix}$$

with constants $p_1, p_2, p_3$. Without loss of generality we may set $V^T = (1, 1, 1)$. From
$	ext{tr}(L^2(Q - I_3)) = 0$ we obtain the constraint $p_1p_2 - p_1p_3 - p_2p_3 = 0$, which we solve for
$p_3 = p_1p_2/(p_1 + p_2)$, assuming $p_1 + p_2 \neq 0$. The solution to (5.56) is then given by

$$K = k_1I_3 + \begin{pmatrix}
k_2(1 + \frac{p_2}{p_1}) & \frac{k_2}{p_1} & \frac{k_2}{p_1} \\
\frac{k_2}{p_1} & k_2(\frac{p_2}{p_1} + \frac{p_2}{p_1 + p_3}) - \frac{k_2}{p_1} & k_2 \\
-\frac{k_2}{p_1} & k_2 & 0
\end{pmatrix}$$

with arbitrary constants $k_1, k_2$. In the CKP case, the resulting function $\tau$ cannot be real and
regular (since e.g. at $t_3 = t_5 = \cdots = 0$ it is a third order polynomial in $t_1$). In the BKP case, it
is not really justified to use (5.63), since it has been derived under the condition $\det(L) \neq 0$, but
here $N$ is odd and thus $\det(L) = 0$ (because $L$ is skew-symmetric). Nevertheless, (5.63)
yields a solution, though an uninteresting one, since $\tau = -p^2$ with $p$ linear in $t_1, t_3, \ldots$. We
should rather go back to (5.62) and (5.65), but it turns out that these equations cannot both be
satisfied non-trivially in the case under consideration.

6. Conclusions

The odd KP system studied in this work is a system of two PDEs for two dependent variables,
$\phi$ and $\theta$, taking values in any associative (and typically noncommutative) algebra $A$. We
have shown how this is embedded in the KP hierarchy, if the latter is expressed with the
help of an auxiliary dependent variable (related to $\theta$). In particular, this allowed one to
adapt a construction of exact solutions for the KP hierarchy to the odd KP system (and the
Corresponding hierarchy). We further demonstrated how this can be exploited to generate
solutions to the BKP and the CKP equation (and their hierarchies). In the latter cases we
worked out only comparatively simple examples of solutions explicitly. The general formu-}
lae, however, involve constant matrices of arbitrary size, with little restrictions, and with certain
choices they may lead to further interesting solutions.

If $A$ is commutative, the odd KP system admits reductions to the BKP and the CKP
equations. In the noncommutative case, these reductions lead to severely constrained
extensions of these equations. Nevertheless, they turned out to be helpful since they allowed
one to uncover some properties of the commutative equations (see the relations with the
KdV hierarchy in sections 2.3 and 2.4) that are hardly recognizable without the step into the noncommutative realm. Whereas the CKP equation possesses a natural noncommutative generalization, though as a system with two dependent variables, nothing comparable has been found for BKP. We also considered some other reductions of the odd KP system with noncommutative $\mathcal{A}$ and obtained in particular a noncommutative version of a coupled system of Kaup–Kupershmidt and Sawada–Kotera type. The odd KP system, (2.10) and (2.11) with noncommutative $\mathcal{A}$, and its reductions, have not been studied previously according to our knowledge.

Furthermore, we presented different formulations of the odd KP hierarchy (with noncommutative $\mathcal{A}$), and derived in particular a functional representation of a linear system for the whole hierarchy. We verified that all these hierarchy formulations possess the odd KP system as their simplest member. Because of the KP hierarchy origin and the hierarchy property one then expects the equivalence of all these hierarchy formulations, but a formal proof would nicely complement this work.

The relation between KP and BKP (CKP) via odd KP shows that a subhierarchy can admit a symmetry reduction that does not extend to a symmetry reduction of the whole hierarchy. This suggests to take a corresponding look at other subhierarchies of KP, and moreover subhierarchies of other hierarchies. Besides the odd KP there is evidently also an ‘even KP’ subhierarchy of the KP hierarchy. In the GDS formulation, this means restricting (2.41) to even-numbered variables. We shall report on this elsewhere.

Appendix A. Proof of theorem 3.1

For the evaluation of the bilinear identity (3.1), we will use the residue formula (which is Lemma 6.3.2 in [66])

$$\text{res}_{z = \lambda} \frac{f(z)}{1 - \lambda z} = \lambda^{-1} f_{<0}(\lambda^{-1}),$$  \hspace{1cm} (A.1)

where $f_{<0}(z) = \sum_{n=1}^{\infty} f_n z^{-n}$. In the following, a prime denotes a partial derivative with respect to $t_1$, hence e.g. $\phi' := \phi_1$.

Lemma A.1. The following are consequences of the bilinear identity (3.1). We have

$$\tilde{w}_2 = \bar{\theta} + \frac{1}{2}(\phi' + \phi^2),$$  \hspace{1cm} (A.2)

and

$$w(\lambda^{-1})\tilde{w}(\lambda^{-1}) = F(\lambda),$$  \hspace{1cm} (A.3)

with $F(\lambda)$ defined in (3.8). Furthermore,

$$w'(\lambda^{-1}) + \lambda^{-1} w(\lambda^{-1}) \tilde{w}(\lambda^{-1}) = \lambda^{-1} F(\lambda)^2 - \frac{\lambda}{2} (\tilde{\theta}_{2[1]} - \tilde{\theta}) + \frac{\lambda}{4} [\phi, \phi_{2[1]}],$$  \hspace{1cm} (A.4)

$$\mu^{-1} w_{2[1]}(\mu^{-1}) \tilde{w}(\mu^{-1}) - \lambda^{-1} w_{2[1]}(\lambda^{-1}) \tilde{w}(\lambda^{-1}) = (\mu^{-1} - \lambda^{-1}) F(\lambda, \mu),$$  \hspace{1cm} (A.5)

$$\mu^{-1} (w'(\mu^{-1}) + \mu^{-1} w(\mu^{-1}) \tilde{w}(\mu^{-1}) - \lambda^{-1} (w'(\lambda^{-1}) + \lambda^{-1} w(\lambda^{-1}) \tilde{w}(\lambda^{-1})$$

$$= (\mu^{-2} - \lambda^{-2}) F(\lambda, \mu)^2 - \frac{1}{2} \frac{\lambda - \mu}{\lambda + \mu} (\tilde{\theta}_{2[1]} + 2[\mu] - \tilde{\theta}) + \frac{1}{4} \frac{\lambda - \mu}{\lambda + \mu} [\phi, \phi_{2[1]} + 2[\mu]],$$  \hspace{1cm} (A.6)

where $\tilde{\theta} := \bar{\theta} + \frac{1}{2} \phi'$, and

$$F(\lambda, \mu) := 1 - \frac{1}{2} \frac{\lambda \mu}{\lambda + \mu} (\phi_{2[1]} + 2[\mu] - \phi) = \frac{1}{\lambda + \mu} (\mu F_{2[1]}(\lambda) + \lambda F(\mu)).$$  \hspace{1cm} (A.7)
Proof. Taking the derivative of (3.4) with respect to $s_1$ and then setting $s_o = t_o$, leads to
\[
0 = \text{res}(w'(z)\tilde{w}(z) + zw'(z)\hat{w}(z)) = w'_t + w_2 + w_1\tilde{w}_1 + \hat{w}_2.
\]
Using (3.5) and (3.6), this becomes (A.2). With the help of the identities
\[
\exp\left(\pm \frac{(\lambda z)^n}{n}\right) = (1 - \lambda z)^{-n}, \quad \text{hence} \quad \exp\left(2 \sum_{n \geq 1} \frac{(\lambda z)^{2n-1}}{2n - 1}\right) = \frac{1 + \lambda z}{1 - \lambda z},
\]
(3.4) for $s_o = t_o + 2[\lambda]$ becomes
\[
0 = \text{res}\left(\frac{1 + \lambda z}{1 - \lambda z}w'_{2[\lambda]}(z)\tilde{w}(z)\right) = 2\lambda^{-1}w_{2[\lambda]}(\hat{w}(\lambda^{-1})) - 2\lambda^{-1} - (w_{1}[2[\lambda]} - \tilde{w}_1),
\]
which is (A.3). Next we differentiate (3.4) with respect to $s_1$ and then set $s_o = t_o + 2[\lambda]$ to obtain
\[
\text{res}\left(\frac{1 + z\lambda}{1 - z\lambda} w'_{2[\lambda]+2[\mu]}(z)\tilde{w}(z)\right) = 0.
\]
With the partial fraction decomposition
\[
\frac{1 + \lambda z}{1 - \lambda z} = 1 + \frac{\lambda + \mu}{\lambda - \mu} \frac{1}{1 - \lambda z},
\]
this results in (A.5). Finally, we differentiate (3.4) with respect to $s_1$, and then set $s_o = t_o + 2[\lambda] + 2[\mu]$ to obtain
\[
\text{res}\left(\frac{1 + z\lambda}{1 - z\lambda} w'_{2[\lambda]+2[\mu]}(z)\tilde{w}(z)\right) = 0,
\]
which evaluates to (A.6). \hfill \Box

Proof of the theorem. With the help of (A.3), we can write (A.4) in the form
\[
(w'(\lambda^{-1}) + \lambda^{-1}w(\lambda^{-1}))_{2[\lambda]} = \left(\lambda^{-1}F(\lambda) = \frac{\lambda}{2} \left(\tilde{\theta}_{2[\lambda]} - \hat{\theta} - \frac{1}{2}[\phi, \phi_{2[\lambda]}]\right)F(\lambda)^{-1}\right)w_{2[\lambda]}(\lambda^{-1}).
\]
Now we apply a Miwa shift with $2[\mu]$ and then multiply by $\tilde{w}(\lambda^{-1})$ from the right to obtain
\[
(w'(\lambda^{-1}) + \lambda^{-1}w(\lambda^{-1}))_{2[\lambda]+2[\mu]}\tilde{w}(\lambda^{-1}) = \left(\lambda^{-1}F(\lambda) = \frac{\lambda}{2} \left(\tilde{\theta}_{2[\lambda]} - \hat{\theta} - \frac{1}{2}[\phi, \phi_{2[\lambda]}]\right)F(\lambda)^{-1}\right)w_{2[\lambda]+2[\mu]}(\lambda^{-1})\tilde{w}(\lambda^{-1}).
\]
Inserting this in (A.6) leads to
\[
\mu^{-1}(w'(\mu^{-1}) + \mu^{-1}w(\mu^{-1}))_{2[\lambda]+2[\mu]}\tilde{w}(\mu^{-1}) = -\left(\lambda^{-1}F(\lambda)^{-2} = \frac{\lambda}{2} \left(\tilde{\theta}_{2[\lambda]} - \hat{\theta} + \frac{\lambda}{4}[\phi, \phi_{2[\lambda]}]\right)\right)_{2[\mu]}F_{2[\mu]}(\lambda^{-1})\lambda^{-1}w_{2[\lambda]+2[\mu]}(\lambda^{-1})\tilde{w}(\lambda^{-1}) = \frac{\lambda^2 - \mu^2}{\lambda^2 \mu^2} F(\lambda, \mu)^2 - \frac{1}{\lambda + \mu} \left(\frac{1}{2}[\phi, \phi_{2[\lambda]+2[\mu]}]\right).
Next we use (A.5) to eliminate the factor $\lambda^{-1} w_{2[\lambda]}(\lambda^{-1}) \tilde{w}(\lambda^{-1})$,
\[
\begin{align*}
w'_{2[\lambda]}(\mu^{-1}) \tilde{w}(\mu^{-1}) + \left( \mu^{-1} F(\lambda) - \lambda^{-1} F(\lambda)^2 + \frac{\lambda}{2} (\hat{\beta}_{2[\lambda]} - \tilde{\beta}) - \frac{\lambda}{4} [\phi, \phi_{2[\lambda]}] \right)_{2[\mu]} \times F_{2[\mu]}(\lambda^{-1}) w_{2[\lambda]}(\mu^{-1}) \tilde{w}(\mu^{-1}) \\
= \frac{\lambda - \mu}{2} \left( \hat{\beta}_{2[\lambda]} - \tilde{\beta} \right)_{2[\mu]} F_{2[\mu]}(\lambda^{-1}) F(\lambda, \mu) - \frac{\mu - \mu}{\lambda + \mu} (\hat{\beta}_{2[\lambda]} - \tilde{\beta})_{2[\mu]} F_{2[\mu]}(\lambda^{-1}) F(\lambda, \mu) \\
+ \frac{\mu}{4} \frac{\lambda - \mu}{\lambda + \mu} [\phi, \phi_{2[\lambda]}]_{2[\mu]} [\phi, \phi_{2[\lambda]}]_{2[\mu]} F_{2[\mu]}(\lambda^{-1}) F(\lambda, \mu) \\
= \frac{\lambda - \mu}{\lambda + \mu} \left( F(\mu) F(\lambda) + \frac{1}{2} \frac{\lambda - \mu}{\lambda + \mu} (\hat{\beta}_{2[\lambda]} - \tilde{\beta})_{2[\mu]} F_{2[\mu]}(\lambda^{-1}) F(\lambda, \mu) \\
+ \frac{\mu}{2} [\phi, \phi_{2[\lambda]}]_{2[\mu]} - \frac{\mu}{2} [\phi, \phi_{2[\lambda]}]_{2[\mu]} F_{2[\mu]}(\lambda^{-1}) F(\lambda, \mu) \\
+ \frac{\lambda - \mu}{\lambda + \mu} (w(\mu^{-1}) + \mu^{-1} w(\mu^{-1}))_{2[\lambda]} \tilde{w}(\mu^{-1}) \right),
\end{align*}
\]
taking into account of (A.7), (A.4), $(\lambda^{-1} + \mu^{-1}) F(\lambda, \mu) - \mu^{-1} F(\lambda) = \lambda^{-1} F_{2[\mu]}(\lambda)$, and
\[
[F(\mu), F_{2[\mu]}(\lambda)] = \frac{\lambda}{4} [\phi_{2[\lambda]} - \phi, \phi_{2[\lambda]}]_{2[\mu]} - \phi_{2[\mu]}].
\]
Now we use (A.3) to replace the factor $F(\mu)$, divided by $\tilde{w}(\mu^{-1})$, and then apply a Miwa shift with $-2[\mu]$ to obtain
\[
\begin{align*}
\frac{\lambda + \mu}{\lambda - \mu} \left[ w'_{2[\lambda]}(\mu^{-1}) + \left( \mu^{-1} F(\lambda) - \lambda^{-1} F(\lambda)^2 + \frac{\lambda}{2} (\hat{\beta}_{2[\lambda]} - \tilde{\beta}) - \frac{\lambda}{4} [\phi, \phi_{2[\lambda]}] \right) \right] \\
\times F(\lambda^{-1}) w_{2[\lambda]}(\mu^{-1}) \\
= w'_{2[\lambda]}(\mu^{-1}) + \mu^{-1} w(\mu^{-1}) \\
+ \left( \frac{1}{\lambda} F(\lambda) + \frac{\lambda}{2} (\hat{\beta}_{2[\lambda]} - \tilde{\beta}) F(\lambda^{-1}) - \frac{\lambda}{4} [\phi, \phi_{2[\lambda]}] F(\lambda^{-1}) \right) \tilde{w}(\mu^{-1}).
\end{align*}
\]
Setting $\mu = z^{-1}$, after some rearrangements this takes the form
\[
\begin{align*}
\frac{1 + \lambda z}{1 - \lambda z} \left( w'_{2[\lambda]}(z) + z w_{2[\lambda]}(z) \right) + w'(z) + z w(z) - \frac{1}{\lambda} F(\lambda) \left( \frac{1 + \lambda z}{1 - \lambda z} w_{2[\lambda]}(z) - w(z) \right) \\
+ \frac{\lambda}{2} \left( \hat{\beta}_{2[\lambda]} - \tilde{\beta} - \frac{1}{2} [\phi, \phi_{2[\lambda]}] \right) F(\lambda^{-1}) \left( \frac{1 + \lambda z}{1 - \lambda z} w_{2[\lambda]}(z) + w(z) \right) = 0.
\end{align*}
\]
Multiplying by $e^{\phi(z,z)}$ and using $\psi_{2[\lambda]} = w_{2[\lambda]}(z) \frac{1 + \lambda z}{1 - \lambda z} e^{\phi(z,z)}$, we arrive at (3.7). □
Appendix B. A determinant identity

According to (2.90) in [47], we have

\[ \det \begin{pmatrix} zV & V^T \\ -V & A \end{pmatrix} = \det(A)z + \sum_{i,j=1}^{N} \Delta_{i,j} v_i v_j, \tag{B.1} \]

where \( A \) is an \( N \times N \) matrix, \( \Delta_{i,j} \) is the cofactor with respect to the component \( A_{i,j} \) of \( A \), \( z \) a parameter, and \( v_i, i = 1, \ldots, N \), are the components of a vector \( V \). If \( N \) is odd and \( A \) skew-symmetric, then \( \det(A) = 0 \) and thus

\[ \det \begin{pmatrix} z & V^T \\ -V & A \end{pmatrix} = \sum_{i,j=1}^{N} \Delta_{i,j} v_i v_j, \tag{B.2} \]

which is thus independent of \( z \). Since

\[ \det \begin{pmatrix} 1 & V^T \\ -V & A \end{pmatrix} = \det \begin{pmatrix} 1 & V^T \\ 0 & A + VV^T \end{pmatrix} = \det(A + VV^T), \tag{B.3} \]

we obtain

\[ \det(A + VV^T) = \det \begin{pmatrix} 0 & V^T \\ -V & A \end{pmatrix}, \tag{B.4} \]

which is the determinant of a skew-symmetric matrix, and thus the square of the Pfaffian of this matrix.

Appendix C. Reality conditions

In order to obtain real solutions to the BKP or CKP hierarchy from the matrix linear system in section 5 with complex matrices, a reality condition is needed.

**Proposition C.1.** Let \( T \) be a constant invertible \( N \times N \) matrix with the properties

\[ T^* = T^T = T^{-1} \]  \( \tag{C.1} \)

(where \( T^* \) denotes the complex conjugate of \( T \)). Let \( C, K, L \) be constant complex \( N \times N \) matrices and \( V \) an \( N \)-vector satisfying

\[ C^* = TCT^{-1}, \quad K^* = TKT^{-1}, \quad L^* = TLT^{-1}, \quad V^* = TV. \]  \( \tag{C.2} \)

The function \( \tau \) given by \((5.35), (5.41), (5.58)\) or \((5.64)\) in terms of \((C, K, L, V)\) (subject to the corresponding rank 1 condition \((5.34)\) or \((5.56)\), and \(C^T = C\), respectively \(C^T = -C\), is then real.

**Proof.** The assertion is easily verified. \((C.1)\) ensures the compatibility of \((C.2)\) with \(C^T = \pm C\), \((5.34)\) and \((5.56)\).

If \( N = 2n \), then

\[ T = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}, \]  \( \tag{C.3} \)

where \( I_n \) is the \( n \times n \) unit matrix, satisfies the conditions \((C.1)\). Decomposing the matrix \( L \) into \( n \times n \) blocks, \((C.2)\) leads to

\[ L = \begin{pmatrix} L_1 & L_{12} \\ L_{12}^T & L_1 \end{pmatrix}. \]  \( \tag{C.4} \)

In section 5 we presented examples with such conjugate diagonal blocks (and \(L_{12} = 0\)).
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