Berry curvature and various thermal Hall effects

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Abstract
Applying the approach of semiclassical wave packet dynamics, we study various thermal Hall effects where carriers can be electron, phonon, magnon, etc. A general formula of thermal Hall conductivity is obtained to provide an essential physics for various thermal Hall effects, where the Berry phase effect manifests naturally. All the formulas of electron thermal Hall effect, phonon Hall effect, and magnon Hall effect can be directly reproduced from the general formula. It is also found that the Ströeda formula can not be directly applied to the thermal Hall effects, where only the edge magnetization contributes to the Hall effects. Furthermore, we obtain a combined formula for anomalous Hall conductivity, thermal Hall electronic conductivity and thermal Hall conductivity for electron systems, where the Berry curvature is weighted by a different function. Finally, we discuss particle magnetization and its relation to angular momentum of the carrier, change of which could induce a mechanical rotation; and possible experiments for thermal Hall effect associated with a mechanical rotation are also proposed.

1. Introduction
In analogy to an electronic Hall effect, a longitudinal temperature gradient can induce a transverse heat flow, which is named as a thermal Hall effect. For electron systems, the thermal Hall effect has been widely studied in various systems including high-Tc superconductor [1], impurity-doped iron [2], graphene [3, 4], etc. Recently thermal Hall effects of bosonic quasi-particles, such as phonons [5] and magnons [6, 7], have been observed, although the charge-free quasi-particles can not directly couple to an applied magnetic field. There have been many theoretical works in understanding the nontrivial physics for phonon and magnon Hall effects [8–11]. However, besides some theories were based on nonequilibrium Green’s function method, most of these theoretical studies adopted the Green–Kubo linear response formula. The Green–Kubo formula includes a term of nonobservable circulating thermal current, which should be subtracted since it does not contribute to the transport. Such correction was made in [12] and was also applied to phonon Hall effect [13]. But the derivation process is complicated and thus it cannot be easily applied to other systems. For magnon Hall effect studied in [14] the authors calculated the magnon Hall conductivity from the magnon edge current in analogy to electron edge current, and found an additional correction term to magnon Hall conductivity derived in [15]. Although some different theories are developed for different thermal Hall effects, so far there is no systematic theory based on which all the various thermal effects can be understood within a unified simple physical picture.

The semiclassical wave packet dynamics is a versatile tool which has been successfully applied to study various Berry phase effects on electronic properties with complete precision of the full quantum theory [16]. The Berry phase effect [16–18] is essential to understand the underlying mechanism of various quantum, spin, or anomalous Hall effects [19–21]. And there is a crucial role of the Berry phase and the Berry curvature in understanding the orbital angular momentum Hall effect and spin Hall effect associated with electron and optical vortex beams [22–25]. Recently Berry curvatures were also found in both of the phonon Hall effect [10, 13] and magnon Hall effect [14] through different methods. Therefore it is our interest to explore various thermal Hall effects under the semiclassical dynamics picture, through which the Berry phase effect of the carrier manifests itself naturally. In addition, for an insulator the Ströeda formula can be generalized and applied to the
thermal Hall effects [27]; however, for the thermal Hall effect in a general system people do not know whether it can be applied, which is also our interest in this work.

In this paper, in analogy to electronic Hall effect, using the wave packet dynamics we derive a theory for thermal Hall effect. The particle and thermal magnetization is derived, then the thermal current is obtained. From the transport thermal current, the general formula of thermal Hall conductivity is obtained, which reproduces all formulas for the thermal Hall effects of electrons, phonons and magnons. Thus the wave packet dynamics can be applied to various thermal Hall effects and it can give us a clear, simple and unified picture. We also find that the Strøeda formula can not be directly applied to the thermal Hall effects, where only the edge magnetization contributes to the Hall effects. A combined formula for anomalous Hall conductivity, thermal Hall electronic conductivity and thermal Hall conductivity for electron systems is derived, in which the Berry curvature is weighted by a function of $E_n (\varepsilon_n)$, which is dependent on the band energy. Based on the temperature-dependent particle magnetization and its relation to the angular momentum, the mechanical rotation could be observed with changing temperature. Before concluding, we discuss a possible experimental setup for the thermal Hall effect and associated mechanical rotation.

### 2. Semiclassical wave packet theory for thermal Hall effects

We know that to study the electron dynamics driven by perturbations varying in space, the semiclassical theory provides a very useful way by describing the motion of a narrow wave packet, which is obtained through superposing the Bloch states of an electron band. For phonon or magnon transport in crystals, we also can describe them by a wave packet which is the superposition of Bloch states within a phonon or magnon band. In this paper by using the wave packet dynamics we study various thermal Hall effects, the carriers of which could be electron, phonon or magnon. Assuming the perturbations in crystal are sufficiently weak such that transitions between different bands can be neglected, that is, the carrier dynamics can be confined within a single band [16]. Thus we construct a wave packet: $|W_0\rangle = \int dq w(q, t) |\psi(q)\rangle$ using the Bloch functions $|\psi(q)\rangle = e^{i q \cdot \mathbf{r}} |u_n(q)\rangle$ from the nth band. The envelope function of $w(q, t)$ have a sharp distribution around the wave vector of $\mathbf{q}_{c}$.

\[
\int [w(q, t)]^2 dq = q_{c}
\]

In the Brillouin zone, and the wave packet must be narrowly localized around its center of mass $\mathbf{r}_c$ in the real space, with $\langle W_0 | \mathbf{r} | W_0 \rangle = r_c$. The wave packet possess a self-rotation around its center $r_c$, which will in turn give rise to orbital magnetic moment $m_{Q}^{N}(\mathbf{q}) = -\frac{e}{2} \langle W_0 | (\mathbf{r} - r_c) \times \mathbf{v} | W_0 \rangle$ [16]. We can define $m_{Q}^{N}(\mathbf{q}) = -\mathbf{c} m_{Q}^{N}(\mathbf{q})$, where $m_{Q}^{N}(\mathbf{q})$ is called particle magnetic moment and reads as

\[
m_{Q}^{N}(\mathbf{q}) = \frac{1}{2} \langle W_0 | (\mathbf{r} - r_c) \times \mathbf{v} | W_0 \rangle.
\]

The energy and thermal magnetic moment [12] are written as

\[
m_{Q}^{E}(\mathbf{q}) = \varepsilon_n(\mathbf{q}) m_{Q}^{N}(\mathbf{q}), \quad m_{Q}^{N}(\mathbf{q}) = m_{Q}^{E}(\mathbf{q}) = \mu m_{Q}^{N}(\mathbf{q}),
\]

where $\varepsilon_n$ is the energy of nth Bloch band, and $\mu$ is the chemical potential. By calculating the angular momentum of a wave packet directly, one finds [26] $\langle W_0 | H | W_0 \rangle = -\frac{e}{2} \langle \nabla \cdot u_n | \mathbf{H} | \nabla \cdot u_n \rangle$ where $H(\mathbf{q}) = e^{-i \mathbf{q} \cdot \mathbf{r}} H(\mathbf{r})$ is the q-dependent Hamiltonian.

In analogy to electron orbital magnetization, thermal orbital magnetization $M_Q$ has two parts, one is the thermodynamic average of the thermal orbital moments, which comes from the self rotation of the wave packet, $M_{Q}^{self} = \frac{1}{V} \sum_{n} f(\mathbf{q}) m_{Q}^{N}(\mathbf{q})$, where $f(\mathbf{q})$ is the Fermi–Dirac (or Bose–Einstein) distribution function for the fermionic (or bosonic) carriers; another is the contribution from the motion of the center of wave packet we denote it as $M_{Q}^{edge}$ since it vanishes in the bulk for a uniform system. We consider a finite system with a confining potential $U(\mathbf{r})$ [16], as illustrated in figure 1(a), which vanishes in the bulk and varies slowly at atomic length scale such that the wave packet description of the electron is still valid on the edge. Near the edge, the carrier’s energy is tilted up due to the increase of $U(\mathbf{r})$. We can derive the Lagrangian by using $L = \langle W_0 | i \hbar (\partial / \partial t - H) | W_0 \rangle$, where $\langle W_0 | H | W_0 \rangle = \varepsilon_n - U(\mathbf{r}_c)$ and $\langle W_0 | i \hbar (\partial / \partial t) | W_0 \rangle = -\hbar (\partial / \partial t) \arg \mathbf{w}(\mathbf{q}_C, t)$. By using $\mathbf{r}_c = -\partial (\partial / \partial \mathbf{q}_C) \arg \mathbf{w} + \mathbf{A}_n(\mathbf{q}_C)$ (A$_n$ = i$\partial / \partial \mathbf{q}$ $\mathbf{u}_n$ is the Berry connection of the nth band) [16], we find that the Lagrangian is given by, up to some unimportant total time-derivative terms (dropping the subscript Con $\mathbf{q}_C$),

\[
L = \hbar \mathbf{q} \cdot \dot{\mathbf{r}}_c + \hbar \mathbf{q} \mathbf{A}_n(\mathbf{q}) \cdot \dot{\varepsilon}_n(\mathbf{q}) - \varepsilon_n(\mathbf{q}) - U(\mathbf{r}_c).
\]

From $\partial L / \partial \mathbf{r}_c = (d/dt)(\partial L / \partial (\partial / \partial \mathbf{r}_c))$, we obtain $-\nabla U = \hbar \mathbf{q}$. From $\partial L / \partial \mathbf{q} = (d/dt)(\partial L / \partial (\partial / \partial \mathbf{q}))$, we obtain $\hbar \dot{\mathbf{r}}_c = \hbar \dot{\varepsilon}_n(\mathbf{q}) + \hbar (\partial / \partial \mathbf{q})(\mathbf{q} \mathbf{A}_n) + \hbar (d/dt)(\mathbf{A}_n)$, and we have $(\partial / \partial \mathbf{q})(\mathbf{q} \mathbf{A}_n) = (d/dt)(\mathbf{A}_n) = \mathbf{q} \times (\nabla \mathbf{q} \times \mathbf{A}_n) = \dot{\mathbf{q}} \times \nabla \mathbf{A}_n$ (A$_n$ is the Berry curvature of nth band: $\mathbf{A}_n = \nabla \mathbf{q} \times \mathbf{A}_n = i \left( \frac{\partial \mathbf{u}_n}{\partial \mathbf{q}} \right) \times \left( \frac{\partial \mathbf{u}_n}{\partial \mathbf{q}} \right)$), thus the equation of motion is given by
which can be verified by the calculation of $\hat{\tau}_C = \langle \hat{\tau}_C \rangle$ with the perturbed band wave function $|\underline{u}'_n\rangle = |\underline{u}_n\rangle + \sum_{n' \neq n} |\underline{u}_n\rangle \langle \underline{u}_n | \nabla U | \underline{u}_n\rangle$. As illustrated in figures 1 (a) and (b), we can derive the particle orbital magnetization per unit volume at the edge as

$$M^\text{edge}_N = \frac{1}{\pi R^2 d} \sum_n \int \int \int \frac{dq}{(2\pi)^3} R d\theta dr$$

$$\times f (\epsilon (q) + U (r)) \frac{1}{2} \left[ -R \cdot \frac{1}{\hbar} \frac{\partial U}{\partial r} \Omega^\text{N}_n (q) \right].$$

Using the relation of $\frac{1}{V} \sum_q = \int \frac{dq}{(2\pi)^3}$ we can obtain the particle magnetization per unit volume at the edge as

$$M^\text{edge}_N = -\frac{1}{V \hbar} \sum_{n, q} \int_{\epsilon (q)}^\infty f (\epsilon) d\epsilon \Omega^\text{N}_n (q).$$

Then the energy magnetization and thermal magnetization at the edge can also be obtained as

$$M^\text{edge}_E = -\frac{1}{V \hbar} \sum_{n, q} \int_{\epsilon (q)}^\infty f (\epsilon) d\epsilon \Omega^\text{N}_n (q)$$

and

$$M^\text{edge}_Q = -\frac{1}{V \hbar} \sum_{n, q} \int_{\epsilon (q)}^\infty (\epsilon - \mu) f (\epsilon) d\epsilon \Omega^\text{N}_n (q),$$

respectively. Although in the beginning we assume the existence of a confining potential $U$, the thermal orbital magnetization does not depend on the form of $U$. The $M^\text{edge}_N$, $M^\text{edge}_E$, $M^\text{edge}_Q$ do not depend on the boundary conditions but only depend on the bulk Bloch functions. In a uniform system, they will vanish in the bulk and only be nonzero along the edge. Here the boundary confining potential is only a tool to investigate the contribution from the global movement of the wave-packet center since it always vanishes in the bulk in a uniform system.

Similar to the orbital magnetization the total thermal current has two contributions: one is from the motion of the wave-packet center and the other is from the self-rotation of the wave-packet center, which can be written as

$$J^\text{total}_Q = \frac{1}{V} \sum_{n, q} [g (r, q) (\epsilon - \mu) \hat{\tau}_C + \nabla \times g (r, q) m^\text{N}_n (q)].$$

For a first-order calculation, the distribution function $g (r, q)$ in the second term can be replaced by Fermi–Dirac (for electron) or Bose–Einstein (for phonon or magnon) distribution function $f (r, q)$.

In the total current $J^\text{total}_Q$, there is one part of magnetization current $J^\text{mag}_Q = \nabla \times M_Q$ which is divergence-free. Subsequently the magnetization current does not contribute to the net current measured by conventional transport experiments [28]. Therefore, the transport current is given by

$$J^\text{tr}_Q = J^\text{total}_Q - \nabla \times M_Q, M_Q = M^\text{self}_Q + M^\text{edge}_Q.$$
thus we obtain
\[ J^r_Q = \frac{1}{V} \sum_{n,q} g(r, q)(\epsilon - \mu) \hat{\mathbf{c}} - \nabla \times \mathbf{M}^\text{edge}_Q. \]  
(11)

In a system with temperature \( T(r) \), the second term can be written as
\[ -\nabla T \times \frac{\partial \mathbf{M}^\text{edge}_Q}{\partial T}. \]

The thermal Hall conductivity \( \kappa_{th} \) can be written as
\[ \kappa_{th} = \frac{\partial \mathbf{M}^\text{edge}_Q}{\partial T} = \frac{1}{V V T} \sum_{n,q} \int_{\epsilon_n(q)}^{\infty} (\epsilon - \mu) \frac{\partial f}{\partial \epsilon} d\epsilon \left[ \mathbf{\Omega}_n(q) \right]. \]
(12)

3. Thermal Hall effects for electron, phonon and magnon

The equation (12) is the general formula for thermal Hall conductivity. For electron, we could rewrite equation (12) as
\[ \kappa_{th} = -\frac{1}{e^2 V} \int (\epsilon - \mu) \frac{\partial f}{\partial \epsilon} \sigma(\epsilon) d\epsilon, \]
where \( f(\epsilon) = 1/(e^{(\epsilon-\mu)/k_B T} + 1) \) and \( \sigma(\epsilon) \) is the zero-temperature electron Hall conductivity for a system with Fermi energy \( \epsilon \). \( \sigma(\epsilon) = -\frac{e^2}{h} \sum_{n,q \epsilon_n(q) < \epsilon} \mathbf{\Omega}_n(q) \), which is the formula equation (25) derived in the electron thermal Hall effect [12] from a perturbation theory. For phonon Hall effect, due to the spin-phonon interaction, the eigenvalue problem is not a standard one, people need consider both the coordinates and momenta to solve an eigenvalue problem with 2nd branches [9, 10, 13], where \( n \) is the number of particles in one unit cell and \( d \) is the dimension of the motion. And the effective Hamiltonian is not Hermitian, one need calculate the left and right eigenvectors, using which the Berry connection and Berry curvature can be calculated. In phonon Hall system, 2r phonon modes are considered, in which both positive and negative eigen frequencies are included, thus we need replace \( V \) with \( 2V \) outside of the summation in equation (12). Therefore the phonon Hall conductivity can be written as
\[ \kappa_{th} = \frac{1}{2 e^2 V} \sum_{n,q \epsilon_n(q) < \epsilon} \int_{\epsilon_n(q)}^{\infty} (\epsilon - \mu) \frac{\partial f}{\partial \epsilon} d\epsilon \left[ \mathbf{\Omega}_n(q) \right]. \]

For phonon \( \mu = 0 \) and \( f(\epsilon) = 1/(e^{\epsilon/k_B T} + 1) \). The formula of the phonon Hall effect is the same with that derived in [13], where the Berry curvature should be calculated by using left and right eigenvector. For magnon Hall effect, using the relation
\[ \int_{\epsilon_n(q)}^{\infty} (\epsilon - \mu) \frac{\partial f}{\partial \epsilon} d\epsilon = -k_B^2 T \int_{\epsilon_n(q)}^{\infty} (\log(1 + \rho^{-1})^2 d\rho, \]
one can immediately obtain the formula:
\[ \kappa_{th} = -\frac{k_B^2 T}{e} \sum_{n,q} \mathbf{\Omega}_n(q) \int_{0}^{\epsilon_n(q)} (\log(1 + \rho^{-1})^2 d\rho \text{ with } f(\epsilon_n(q)) = 1/(e^{\epsilon/k_B T} + 1), \]
which is the equation (5) in [14].

Therefore, the thermal Hall conductivity written as equation (12) is a general formula for various systems, such as electron, phonon, magnon, etc. Due to orbital magnetization along the edge, if the eigen wave functions have nontrivial Berry curvature, we can observe the thermal Hall effect for a system with a temperature gradient. For an electron system, the applied magnetic field can directly couple to electron which induce the normal thermal Hall effect; while the system with nontrivial Berry curvature, we could observe the anomalous thermal Hall effect. For phonon and magnon systems, we only could observe the anomalous thermal Hall effect due to the Berry curvature since the magnetic field cannot couple to them by Lorentz forces. In the thermal Hall effect, the nontrivial Berry curvature of Bloch bands introduces anomalous velocity for the wave packet, which will give rise to a transverse thermal Hall current.

4. Str\'ed\'a formula

For insulators, we could consider an adiabatic process such that a time-dependent magnetic flux generates an emf along the boundary then a Hall current is induced perpendicular to the boundary, that is, a electron current flows in or out across the boundary thus the electron density inside changes. So the Hall conductivity can be written as
\[ \sigma_{xy} = -e\left( \frac{\partial n}{\partial B} \right)_\mu = -e\left( \frac{\partial M}{\partial B} \right)_\mu. \]
However, for a semiconductor or metal, such argument is invalid due to the breakdown of the adiabaticity, while the Maxwell relation \( \left( \frac{\partial n}{\partial B} \right)_\mu = \left( \frac{\partial M}{\partial B} \right)_\mu \) always holds. For a general system, electron density can be written as
\[ \frac{1}{V} \sum_{n,q} f(\epsilon_n(q) \left( 1 + \frac{e}{h} \mathbf{B} \cdot \mathbf{\Omega}(q) \right), \]
which means that the time-dependent magnetic field not only induces a Hall current to change the electron density inside (through the term \( 1 + \frac{e}{h} \mathbf{B} \cdot \mathbf{\Omega}(q) \)), but also the inbuilt thermodynamic process changes the distribution function \( f(\epsilon_n(q)) \). For the derivation of the Hall conductivity, one need to deduct such thermodynamic part. Therefore the formula
should be written as: \( \sigma_{\text{II}} = -e \left( \frac{\partial M_q}{\partial \mu} \right)_B - e \sum_{n,q} m^{*}_n(q) \frac{\partial f}{\partial \mu} \). The second term is \(-e \left( \frac{\partial M_{\text{sd}}}{\partial \mu} \right)_B \), which is zero for insulators. Thus, for the electron Hall conductivity, 
\[
\sigma_{\text{H}} = -e \left( \frac{\partial M_{\text{edge}}}{\partial \mu} \right)_B,
\]
where \( M_{\text{edge}} = -e M^*_n \).

For the thermal Hall effect, we have the same argument: the thermodynamic part \( \frac{\partial M_{\text{sd}}}{\partial T} \) needs to be subtracted from the total change \( \frac{\partial M_{\text{sd}}}{\partial T} \) to calculate thermal conductivity

\[
\kappa_{\text{II}} = \frac{\partial M_{\text{sd}}}{\partial T} = \frac{\partial M_{\text{sd}}}{\partial T} - \frac{\partial M_{\text{sd}}}{\partial T},
\]
which justifies the formula equation (12). Thermal Hall conductivity \( \kappa_{\text{II}} \) generally is not equal to \( \frac{\partial M_{\text{sd}}}{\partial T} \) and only does for insulators where \( \frac{\partial f}{\partial T} = 0 \).

5. Various Hall effects for electrons

In the thermal Hall effect, we could obtain the thermal Hall electronic conductivity \( \sigma_T \) through \( \sigma_T = \frac{\partial M_{\text{edge}}}{\partial T} \) in analogy to the thermal Hall conductivity \( \kappa_{\text{II}} \). We obtain

\[
\sigma_T = -e \frac{1}{VT} \sum_{n,q} \int_{\varepsilon_n(q)}^{\infty} (\varepsilon - \mu) \frac{\partial f}{\partial \varepsilon} d\varepsilon \Omega_n(q).
\]
Therefore we obtain a combined formula for the three conductivities

\[
\sigma, \kappa = \frac{(-e)^2}{VT} \sum_{n,q} \int_{\varepsilon_n(q)}^{\infty} (\varepsilon - \mu)^m \frac{\partial f}{\partial \varepsilon} d\varepsilon \Omega_n(q)
\]
\[
= -\frac{e^2}{VT} \left[ k_B \right]^m \sum_{\varepsilon \varepsilon} F_m(\varepsilon_n(q)) \Omega_n(q).
\]

With a weighting function \( F_m(\varepsilon_n(q)) = \frac{1}{\omega_m} \left[ \log (\rho^{-1} - 1) \right]^m d\rho \). Here \( \alpha = 0, 1 \) for anomalous Hall effect \( \sigma_{\text{II}} \) and thermal Hall effect \( \sigma_T \) and \( \kappa_{\text{II}} \) respectively, \( m = 0, 1, 2 \) corresponds to anomalous Hall conductivity \( \sigma_{\text{II}} \), thermal Hall electronic conductivity \( \sigma_T \), thermal Hall conductivity \( \kappa_{\text{II}} \), respectively.

Equation (16) shows how the Berry curvature is weighted in the contribution to the various Hall effects for electron systems by the function of \( F_m(\varepsilon_n) \) which is dependent on the band energy \( \varepsilon_n \). For the anomalous Hall conductivity, \( F_0 = f(\varepsilon_n) \) is the standard Fermi function as expected. We plot the function \( F_m(\varepsilon_n) \), as shown in figure 2. The anomalous Hall conductivity looks like a step function due to the Fermi distribution function, which is as expected. However, we find that the thermal Hall electronic conductivity is peaked at \( \varepsilon = \mu \), but for both \( (\varepsilon - \mu)/(k_B T) \ll 0 \) and \( (\varepsilon - \mu)/(k_B T) \gg 0 \), it approaches zero. Therefore in the thermal Hall effect, only the modes with energy around Fermi level contribute to the thermal Hall electronic conductivity. And for thermal Hall conductivity, besides it has the similar effect as anomalous Hall conductivity that it approaches constant for \( (\varepsilon - \mu)/(k_B T) \ll 0 \) and tends to zero for \( (\varepsilon - \mu)/(k_B T) \gg 0 \), it has another turning point at \( \varepsilon = \mu \), where it equals to half of the maximum value. As temperature increases, all the steps and peak are smoothed by the temperature.

6. Angular momentum

From the above semiclassical theory we also can obtain the total particle magnetic moment per volume as

\[
M_N = \frac{1}{V} \sum_{n,q} \left[ f(q) m^N_n(q) \right] \int_{\varepsilon_n(q)}^{\infty} f(\varepsilon) d\varepsilon \Omega_n(q).
\]

For an electron system, we will immediately get the total angular momentum of the electron as

\[
L = 2m_e M_N.
\]

For quasi-particle systems, we replace \( m_e \) with an effective one \( m^* = \hbar^2 / (\alpha q^2) \) to measure the angular momentum, where \( m^* \) is a \( q \)-dependent and must be moved into the summation. While only the motion of the
The center of the wave packet contributes the thermal Hall current, both the self rotation of the wave packet and the motion of its center contribute to the total angular momentum. In Einstein–de Haas effect \cite{29, 30}, the angular momentum of electrons changes due to the magnetization changes. Due to the conservation of angular momentum, the change of angular momentum of electrons is taken to be equal in magnitude but opposite in sign to the change of lattice angular momentum (for the system with spin-phonon interaction, the phonon can have a nonzero angular momentum \cite{31, 32}, we can ignore it at room temperature or higher temperatures), which corresponds to mechanical rotation. Therefore if the temperature changes, the total orbital angular momentum of electrons will change, similar to the Einstein–de Haas effect we could observe the mechanical rotation of the system.

### 7. Possible experiment of mechanical rotation associated with thermal hall effect

In the Corbino effect, if a magnetic field is perpendicularly applied to the plane of a circular disk with radial current, one could observe a circular current through the disk \cite{33}. Similar to the Corbino effect, we propose an experimental setup to measure the mechanical rotation accompanying with thermal Hall effect as illustrated in figure 3(a). When we heat the center of the disk which has nontrivial Berry curvature, then we will observe circular thermal Hall current when heat current flows to the edge. For figure 3(b), when heat flows upwards, we will also observe circular thermal Hall current along the surface of the cylinder. Due to increase of the temperature in the center of the disk in figure 3(a), the magnetization will gradually increase along the plane of the disk, such inhomogeneous distribution make the total magnetization current non-vanishing thus contribute to the emergent thermal Hall current. In figure 3(b), if we heat the cylinder from the bottom, the magnetization will gradually increase along the surface of the cylinder, thus the total non-vanishing magnetization gives a nonzero circular thermal Hall current along the surface and concentric with the axis of the cylinder. Besides the thermal Hall effect perpendicular to the heat current, with increasing of magnetization the total electron angular momentum changes. Similar to the mechanical rotation induced by magnetization change in the Einstein–de Haas effect, by heating the system we could observe the mechanical rotation through the laser and mirror setup.

From equations (17) and (18), with the temperature changing, the change of angular momentum for each unit cell is proposed to be changed as \( a \hbar \), \( a \) is a constant. Then the angular velocity of the rotation would be \( \omega \approx \frac{N_a a \hbar}{J} \), where the moment of inertia could be estimated as \( J \approx \frac{N_A m R^2}{2} \), here \( N_A \) is the Avogadro's number, \( m \) is the mass of each unit cell, \( R \) is the radius of the disk or cylinder. If we assume \( m \sim 10^{-25} \text{ kg}, R \sim 10^{-2} \text{ cm}, a \sim 10^{-1}, \) then \( \omega \sim 10^{-6} \text{ rad s}^{-1} \). Due to the torsion of the fine wire connecting the cylinder or disk, there will be a negative angular acceleration which makes \( \omega \) decrease. The rotation angle will be about \( 10^{-6} \text{ rad} \) after 1 s, which can be observed by a displacement around 1000 nm of the laser spot on the screen if the distance between the mirror and the screen is 1 m.
8. Conclusion

Based on the semiclassical wave packet dynamics, we present a unified theory for various thermal Hall effects. A general formula of thermal Hall conductivity is derived, which can be applied to any kind of carriers, such as electrons, phonons and magnons. For electron systems, a combined formula for anomalous Hall conductivity, thermal Hall electronic conductivity and thermal Hall conductivity is derived, from which one can analyze their properties near Fermi surface. It is also found that, the Středa formula can not directly be applied to the thermal Hall effects, where the magnetization from the self rotation of the wave packet must be subtracted. Finally we propose a possible experiment for the nonequilibrium thermal Hall effect associated with a mechanical rotation.

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Figure 3. Experimental setup for measurement of mechanical rotation with the thermal Hall effect. The mechanical rotation is measured by the laser-mirror setup. (a) A thin rounded disk is heated in the center. (b) A cylinder is heated from the bottom.
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