Mock theta functions and indefinite modular forms

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Abstract

In the explicit formula for the signed mock theta functions \( \Phi(\cdot)[m,s] \) obtained from the coroot lattice of \( D(2,1;a) \), functions with indefinite quadratic forms naturally take place. We compute their modular transformation properties by applying the Zwegers’ modification theory of mock theta functions and show that the \( \mathbb{C} \)-linear span of these functions is \( SL_2(\mathbb{Z}) \)-invariant.

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1 Introduction

For \( m \in \frac{1}{2} \mathbb{N} \) and \( s \in \frac{1}{2} \mathbb{Z} \), let \( \Phi_i^{(\pm)[m,s]} \) \((i = 1, 2)\) and \( \Phi^{(\pm)[m,s]} \) be the (signed) mock theta functions (cf. \([9]\)) defined by

\[
\Phi_1^{(\pm)[m,s]}(\tau, z_1, z_2, t) := e^{-2\pi i mt} \sum_{j \in \mathbb{Z}} (\pm 1)^j \frac{e^{2\pi i m(j + sj_j)} q^{m^2 + sj_j}}{1 - e^{2\pi i z_1 q^j}} \tag{1.1a}
\]

\[
\Phi_2^{(\pm)[m,s]}(\tau, z_1, z_2, t) := e^{-2\pi i mt} \sum_{j \in \mathbb{Z}} (\pm 1)^j \frac{e^{-2\pi i m(j + sj_j)} - 2\pi isz \Phi^{(\pm)[m,s]}(\tau, z_1, z_2, t)}{1 - e^{-2\pi i z_2 q^j}} \tag{1.1b}
\]

\[
\Phi^{(\pm)[m,s]}(\tau, z_1, z_2, t) := \Phi_1^{(\pm)[m,s]}(\tau, z_1, z_2, t) - \Phi_2^{(\pm)[m,s]}(\tau, z_1, z_2, t) \tag{1.1c}
\]

where \( q := e^{2\pi i \tau} \) \((\tau \in \mathbb{C}_+)\) and \( z_1, z_2, t \in \mathbb{C} \). We write simply \( \Phi_i^{[m,s]} \) and \( \Phi^{[m,s]} \) for \( \Phi_i^{(\pm)[m,s]} \) and \( \Phi^{(\pm)[m,s]} \) respectively. For \( m \in \frac{1}{2} \mathbb{N} \) and \( j \in \frac{1}{2} \mathbb{Z} \), we define the function \( R_{j,m}^{(\pm)}(\tau, z) \) by

\[
R_{j,m}^{(\pm)}(\tau, w) := \sum_{n \in \frac{1}{2} \mathbb{Z}} (\pm 1)^{n-j} \frac{\text{sgn}(n - \frac{1}{2} - j + 2m)}{\sqrt{\frac{\text{Im}(\tau)}{m}}} \sum_{E = 2mZ} E \left( n - 2m \frac{\text{Im}(w)}{\text{Im}(\tau)} \sqrt{\frac{\text{Im}(\tau)}{m}} \right) e^{-\frac{\pi i \tau n^2 + 2\pi i nw}} \tag{1.2}
\]

where \( (\tau, w) \in \mathbb{C}_+ \times \mathbb{C} \) and \( E(z) := \int_0^z e^{-\pi t^2} dt \). For \( m \in \frac{1}{2} \mathbb{N} \) and \( s \in \frac{1}{2} \mathbb{Z} \), we put

\[
\Phi_{\text{add}}^{(\pm)[m,s]}(\tau, z_1, z_2, t) := -\frac{1}{2} e^{-2\pi i mt} \sum_{k \in s + \mathbb{Z}} R_{k,m}^{(\pm)}(\tau, \frac{z_1 + z_2}{2}) [\theta_{k+m}^{(\pm)} - \theta_{-k,m}^{(\pm)}](\tau, z_1 + z_2) \tag{1.3a}
\]

\[
\Phi^{(\pm)[m,s]}(\tau, z_1, z_2, t) := \Phi^{(\pm)[m,s]}(\tau, z_1, z_2, t) + \Phi_{\text{add}}^{(\pm)[m,s]}(\tau, z_1, z_2, t) \tag{1.3b}
\]

where \( \theta_{k,m}^{(\pm)} \) is the Jacobi’s theta function:

\[
\theta_{k,m}^{(\pm)}(\tau, z) := \sum_{j \in \mathbb{Z}} (\pm 1)^j e^{2\pi i m(j + \frac{k}{2m})} q^{m(j + \frac{k}{2m})^2} \tag{1.4}
\]

We call \( \Phi^{(\pm)[m,s]} \) (resp. \( \Phi_{\text{add}}^{[m,s]} \)) the “modification” (resp. “correction term”) of \( \Phi^{(\pm)[m,s]} \).

It is known that the Jacobi’s theta functions satisfy the following properties:

**Lemma 1.1.** Let \( m \in \frac{1}{2} \mathbb{N}, j \in \frac{1}{2} \mathbb{Z} \) and \( a \in \mathbb{R} \) such that \( am \in \frac{1}{2} \mathbb{Z} \). Then

1) \( \theta_{j,m}^{(\pm)}(\tau, z + a\tau) = q^{-\frac{1}{4}} e^{-\pi i ma} \theta_{j+m,m}^{(\pm)}(\tau, z) \)

2) \( \theta_{j,m}^{(\pm)}(\tau, z + a) = \begin{cases} e^{\pi i ja} \theta_{j,m}^{(\pm)}(\tau, z) & \text{if } am \in \mathbb{Z} \\ e^{\pi i ja} \theta_{j,m}^{(\pm)}(\tau, z) & \text{if } am \in \frac{1}{2} \mathbb{Z}_{\text{odd}} \end{cases} \)
Lemma 1.2. \( \theta_{j+2am,m}^{(\pm)}(\tau, z) = (\pm 1)^a \theta_{j,m}^{(\pm)}(\tau, z) \) if \( m \in \frac{1}{2} \mathbb{N}, j \in \frac{1}{2} \mathbb{Z} \) and \( a \in \mathbb{Z} \).

Lemma 1.3. Let \( m \in \frac{1}{2} \mathbb{N}, j \in \frac{1}{2} \mathbb{Z} \). Then

\[
1) \quad \theta_{j,m}^{(+)} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m}} e^{\frac{\pi i m^2}{2\tau}} \times \left\{ \begin{array}{ll}
\sum_{k \in \mathbb{Z}} e^{\frac{\pi i k}{m}} \theta_{k,m}^{(+)}(\tau, z) & \text{if } j \in \mathbb{Z} \\
\sum_{0 \leq k < 2m} e^{\frac{\pi i k}{m}} \theta_{k,m}^{(+)}(\tau, z) & \text{if } j \in \frac{1}{2} \mathbb{Z}_{\text{odd}}
\end{array} \right.
\]

\[
2) \quad \theta_{j,m}^{(-)} \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) = \frac{1}{\sqrt{2m}} (-i\tau)^{\frac{1}{2}} e^{\frac{\pi i m^2}{2\tau}} \times \left\{ \begin{array}{ll}
\sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} e^{\frac{\pi i k}{m}} \theta_{k,m}^{(+)}(\tau, z) & \text{if } j \in \mathbb{Z} \\
\sum_{0 \leq k < 2m} e^{\frac{\pi i k}{m}} \theta_{k,m}^{(-)}(\tau, z) & \text{if } j \in \frac{1}{2} \mathbb{Z}_{\text{odd}}
\end{array} \right.
\]

Lemma 1.4. Let \( m \in \frac{1}{2} \mathbb{N}, j \in \frac{1}{2} \mathbb{Z} \). Then

\[
\theta_{j,m}^{(\pm)}(\tau + 1, z) = \begin{cases} 
eq e^{\frac{\pi i j^2}{2m}} \theta_{j,m}^{(\pm)}(\tau, z) & \text{if } m + j \in \mathbb{Z} \\
eq e^{\frac{\pi i j^2}{2m}} \theta_{j,m}^{(\mp)}(\tau, z) & \text{if } m + j \in \frac{1}{2} \mathbb{Z}_{\text{odd}}
\end{cases}
\]

We note that the Mumford’s theta functions \( \vartheta_{ab}(\tau, z) \) (cf. [12]) are related to the Jacobi’s theta functions by the following formulas:

Note 1.1.

\[
\begin{align*}
\vartheta_{00}(\tau, z) &= \vartheta_{0,\frac{1}{2}}^{(+)}(\tau, 2z) \\
\vartheta_{01}(\tau, z) &= \vartheta_{0,\frac{1}{2}}^{(-)}(\tau, 2z) \\
\vartheta_{10}(\tau, z) &= \vartheta_{1,\frac{1}{2}}^{(+)}(\tau, 2z) \\
\vartheta_{11}(\tau, z) &= i \vartheta_{1,\frac{1}{2}}^{(-)}(\tau, 2z)
\end{align*}
\]

In this paper, we write simply \( \vartheta_{j,m} \) for \( \vartheta_{j,m}^{(+)} \).

This paper is organized as follows. In section 2, we recall and prepare some of basic properties of mock theta functions \( \Phi_{1}^{(\pm)[m,s]} \) and \( \Phi_{1}^{(\pm)[m,s]} \), which are used in this paper.

In section 3, we deduce explicit formulas for \( \Phi_{1}^{(\pm)[m,s]} \) by using similar method as in [15] and, in sections 4 and 5, we compute Zwegers’ modification for these functions.

In sections 6 and 7, we study the functions \( \Xi_{1}^{[m,p]}(\tau, z) \) and \( \Upsilon_{1}^{[m,p]}(\tau, z) \) which have good modular properties. In section 8, we compute the modular transformation properties of indefinite modular forms \( g_{j}^{(i)[m,p]}(\tau) \) and show that their \( C \)-linear span is \( SL_{2}(\mathbb{Z}) \)-invariant.
2 Basic properties of $\Phi^{(\pm)[m,s]}$

Some of basic properties of $\Phi^{[m,s]}_i$ and $\Phi^{[m,s]}$ are shown in section 2 of [14] and [15]. In this section, we note that similar formulas hold for signed mock theta functions $\Phi^{(\pm)[m,s]}_i$ and $\Phi^{(\pm)[m,s]}$ and $\Phi^{(\pm)[m,s]}$.

**Note 2.1.** Let $m \in \frac{1}{2} \mathbb{N}$ and $s \in \frac{1}{2} \mathbb{Z}$. Then

1) $\Phi^{(\pm)[m,s]}_i(\tau, -z_2, -z_1, t) = \Phi^{(\pm)[m,s]}_j(\tau, z_1, z_2, t)$ for $i, j \in \{1, 2\}$ such that $i \neq j$.
2) $\Phi^{(\pm)[m,s]}(\tau, -z_2, -z_1, t) = -\Phi^{(\pm)[m,s]}(\tau, z_1, z_2, t)$
3) $\Phi^{(\pm)[m,s]}(\tau, z, -z, t) = 0$

**Lemma 2.1.** Let $m \in \frac{1}{2} \mathbb{N}$ and $s, s' \in \frac{1}{2} \mathbb{Z}$. Then

1) $\Phi^{(\pm)[m,s]}(\tau, z_1, z_2, t) = \Phi^{(\pm)[m,s']}(\tau, z_1, z_2, t)$ if $s - s' \in \mathbb{Z}$,
2) $\Phi^{(-)[m,s]}(\tau, z_1, z_2, t) = \tau e^{2\pi i m - z_1 z_2} \Phi^{(-)[m,s]}(\tau, z_1, z_2, t)$
3) $\Phi^{(\pm)[m,s]}(\tau + 1, z_1, z_2, t) = \Phi^{(\pm)[m,s]}(\tau, z_1, z_2, t)$ if $m + s \in \mathbb{Z}$.

**Lemma 2.2.** Let $m \in \frac{1}{2} \mathbb{N}$, $s \in \frac{1}{2} \mathbb{Z}$ and $p \in \mathbb{Z}$. Then

1) (i) $\Phi^{(\pm)[m,s]}(\tau, z_1, z_2 + pr, t) = e^{-2\pi im p} \Phi^{(\pm)[m,s+mp]}(\tau, z_1, z_2, t)$
    (ii) $\Phi^{(\pm)[m,s]}(\tau, z_1, z_2 + pr, t) = (\pm 1)^p e^{-2\pi im p} \Phi^{(\pm)[m,s+mp]}(\tau, z_1, z_2, t)$
2) $\Phi^{(\pm)[m,s]}(\tau, z_1 + z_2 + pr, t) = e^{-2\pi im p} \Phi^{(\pm)[m,s+mp]}(\tau, z_1, z_2, t)$ if $(\pm 1)^p = 1$.

**Proof.** The proof is similar with that of Lemma 2.4 in [15].

**Remark 2.1.** Lemmas 2.2 and 2.5 in [14] and Lemma 2.1 and formula (2.1) in [15] hold as well for $\Phi^{(\pm)[m,s]}$ if we replace $\Phi^{[m,s]}$ and $\theta_{j,m}$ with $\Phi^{(\pm)[m,s]}$ and $\theta^{(\pm)}_{j,m}$ respectively.

We note also that the following formulas hold.

**Lemma 2.3.** Let $m \in \frac{1}{2} \mathbb{N}$, $s \in \frac{1}{2} \mathbb{Z}$ and $a \in \mathbb{Z}$. Then

1) (i) $\Phi^{(\pm)[m,s]}(\tau, z_1 + a, z_2 + a, t) = (\pm 1)^a e^{2\pi im a} \Phi^{(\pm)[m,s-2a]}(\tau, z_1, z_2, t)$
    (ii) $\Phi^{(\pm)[m,s]}(\tau, z_1 + a, z_2 - a, t) = (\pm 1)^a e^{2\pi im a} \Phi^{(\pm)[m,s-2a]}(\tau, z_1, z_2, t)$
2) $\Phi^{(\pm)[m,s]}(\tau, z_1 + a, z_2 - a, t) = (\pm 1)^a e^{2\pi im a} \Phi^{(\pm)[m,s-2a]}(\tau, z_1, z_2, t)$

(2.1)
Lemma 2.4. Let \( \Phi(\pm) \) of \( \Phi_1^{(\pm)} \), we have

\[
\Phi_1^{(\pm)}(\tau, z_1 + a\tau, z_2 - a\tau, 0) = \sum_{j \in \mathbb{Z}} (-1)^j \frac{e^{2\pi imj(z_1 + z_2) + 2\pi is(z_1 + a\tau)} q^{mj^2 + sj}}{1 - e^{2\pi i(z_1 + a\tau)}q^j}
\]

Putting \( j = k - a \), this becomes

\[
= \sum_{k \in \mathbb{Z}} (-1)^{k-a} \frac{e^{2\pi im(k-a)(z_1 + z_2) + 2\pi isz_1} q^{m(k-a)^2 + sk}}{1 - e^{2\pi iz_1}q^k}
\]

\[
= (-1)^a e^{2\pi ima(z_1 - z_2)} q^{ma^2} \sum_{k \in \mathbb{Z}} (-1)^j \frac{e^{2\pi imk(z_1 + z_2) + 2\pi i(s-2am)z_1} q^{mk^2 + (s-2am)k}}{1 - e^{2\pi iz_1}q^k}
\]

proving (i). 1) (ii) is obtained from (i) by using Lemma 2.2 in [14] with Remark 2.1. 2) follows immediately from 1). 

Lemma 2.4. Let \( m \in \frac{1}{2} \mathbb{N} \), \( s \in \frac{1}{2} \mathbb{Z} \) and \( a \in \mathbb{Z}_{\geq 0} \). Then

\[
\Phi^{(\pm)}(\tau, z_1 + a\tau, z_2 - a\tau, 0) = (-1)^a e^{2\pi ima(z_1 - z_2)} q^{ma^2} \left\{ \Phi^{(\pm)}(\tau, z_1, z_2, 0) \right\}
\]

\[
- \sum_{k \in \mathbb{Z}} e^{-\pi i(k-s)(z_1 - z_2)} q^{\frac{(k-s)^2}{4m}} \left[ \theta^{(\pm)}_{k-s, m} - \theta^{(\pm)}_{-(k-s), m} \right] (\tau, z_1 + z_2)
\]

(2.2)

Proof. By Lemma 2.1 in [15] with Remark 2.1 we have

\[
\Phi^{(\pm)}(\tau, z_1, z_2, a, 0) = \Phi^{(\pm)}(\tau, z_1, z_2, 0)
\]

\[
= \sum_{k \in \mathbb{Z}} e^{\pi i(s+k)(z_1 - z_2)} q^{\frac{(s+k)^2}{4m}} \left[ \theta^{(\pm)}_{s+k, m} - \theta^{(\pm)}_{-(s+k), m} \right] (\tau, z_1 + z_2)
\]

Letting \( a \to 2am \), we have

\[
\Phi^{(\pm)}(\tau, z_1, z_2, 0) = \Phi^{(\pm)}(\tau, z_1, z_2, 0)
\]

\[
= \sum_{k \in \mathbb{Z}} e^{\pi i(s+k)(z_1 - z_2)} q^{\frac{(s+k)^2}{4m}} \left[ \theta^{(\pm)}_{s+k, m} - \theta^{(\pm)}_{-(s+k), m} \right] (\tau, z_1 + z_2)
\]

\[
- \sum_{l \in \mathbb{Z}} e^{\pi i(s-l)(z_1 - z_2)} q^{\frac{(s-l)^2}{4m}} \left[ \theta^{(\pm)}_{s-l, m} - \theta^{(\pm)}_{-(s-l), m} \right] (\tau, z_1 + z_2)
\]

(2.3)

Substituting this equation (2.3) into (2.1), we obtain (2.2), proving Lemma 2.4. 

\( \square \)
The following Lemma 2.5 is obtained immediately from Note 2.1 and Lemma 2.2.

**Lemma 2.5.** Let \( m \in \frac{1}{2} \mathbb{N} \), \( s \in \frac{1}{2} \mathbb{Z} \) and \( p \in \mathbb{Z} \). Then

1) (i) \( \Phi^{[m,s]}(\tau, z, z + pr, t) = 0 \)
   (ii) \( \Phi^{(-)[m,s]}(\tau, z, z + 2pr, t) = 0 \)

namely,

2) (i) \( \Phi^{[m,s]}(\tau, z, z + pr, t) = \Phi^{[m,s]}_2(\tau, z, z + pr, t) \)
   (ii) \( \Phi^{(-)[m,s]}_1(\tau, z, z + 2pr, t) = \Phi^{(-)[m,s]}_2(\tau, z, z + 2pr, t) \)

### 3 Explicit formula for \( \Phi^{(-)[m, \frac{1}{2}]} \)

Under the same setting and notations with Section 3 in [15], we consider \( \hat{D}(2, 1; a = \frac{-m}{m+1}) \) where \( m \in \frac{1}{2} \mathbb{N} \), and compute

\[
F^{(2)}_{a_{\Lambda_0}} := \sum_{j,k \in \mathbb{Z}} (-1)^k t_{j \alpha_j^\gamma + k \alpha_0^\gamma} \left( \frac{e^{a_{\Lambda_0}}}{1 - e^{-a_1}} \right)
\]

\[
A^{(2)}_{a_{\Lambda_0}} := \sum_{w \in \Pi} \varepsilon(w)w(F^{(2)}_{a_{\Lambda_0}})
\]

Then by just similar arguments with those in the proof of Lemma 3.1 and Propositions 3.1 and 3.2 and Corollary 3.1 in [15], we obtain the following Lemma 3.1 and Propositions 3.1 ~ 3.3.

**Lemma 3.1.** For \( m \in \frac{1}{4} \mathbb{N} \) the following formula holds:

\[
\sum_{j \in \mathbb{Z}} (-1)^j q^{(m+1)j^2} e^{-2\pi ij(z_1 - z_2) + 2\pi ijm(z_1 - z_3)} \Phi^{(-)[m,0]}(\tau, z_1, -z_3 + 2j\tau, 0)
\]

\[
- \sum_{j \in \mathbb{Z}} (-1)^j q^{(m+1)j^2} e^{2\pi ij(z_1 - z_2) + 2\pi ijm(z_1 - z_3)} \Phi^{(-)[m,0]}(\tau, z_2, z_1 - z_2 - z_3 + 2j\tau, 0)
\]

\[
= \sum_{j \in \mathbb{Z}} (-1)^j q^{(m+1)j^2} e^{2\pi ij(z_1 - z_2) - 2\pi ijm(z_1 - z_3)} \Phi^{[1,0]}(\tau, z_1, -z_2 + 2j\tau, 0)
\]

\[
- \sum_{j \in \mathbb{Z}} (-1)^j q^{(m+1)j^2} e^{2\pi ij(z_1 - z_2) + 2\pi ijm(z_1 - z_3)} \Phi^{[1,0]}(\tau, z_3, z_1 - z_2 - z_3 + 2j\tau, 0)
\]

\[
(3.1)
\]

**Proposition 3.1.** For \( m \in \frac{1}{4} \mathbb{N} \) the following formula holds:

\[
\theta^{(-)}_{0, m+1}(\tau, \frac{(z_1 - z_2) + m(z_1 + z_3)}{m + 1}) \Phi^{(-)[m,0]}(\tau, z_1, -z_3, 0)
\]

\[
- \theta^{(-)}_{0, m+1}(\tau, \frac{(z_1 - z_2) + m(z_1 - 2z_2 - z_3)}{m + 1}) \Phi^{(-)[m,0]}(\tau, z_2, z_1 - z_2 - z_3, 0)
\]

\[
+ \sum_{j > 0} \sum_{k=1}^{2m} q^{(m+1)j^2} \omega^{[m]}_{j,k}(z_1, z_2, z_3) \left[ \theta^{(-)}_{k, m} - \theta^{(-)}_{-k, m} \right](\tau, z_1 - z_3)
\]
Proposition 3.2. For $m \in \mathbb{N}$, the following formula holds:

$$
\theta_{0,m+1}^{(-)}(\tau, \frac{-1}{2} + m(z_1 + z_3)) \left\{ \Phi^{(-)}(m,0)(\tau, z_1, z_3, 0) - \Phi^{(-)}(m,0)(\tau, z_1 + \frac{1}{2}, -z_3 - \frac{1}{2}, 0) \right\} = 2i\eta(\tau)^2\eta(2\tau)^2 \left\{ \frac{\theta_{0,m+1}^{(-)}(\tau, \frac{1}{2} + z_1 + z_3 + z_1 - z_3)}{\eta(\tau)} \frac{\theta_{0,m+1}^{(-)}(\tau, \frac{1}{2} + z_1 + z_3)}{\eta(\tau)} \right\}
$$

$$
+ 2 \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}_{odd}} \sum_{1 \leq k \leq m} (-1)^{j+r} q^{(m+1)j} \left\{ e^{\frac{\pi i(2m-j-k)(z_1+z_3)}{2m+1}} + e^{-\frac{\pi i(2m-j-k)(z_1+z_3)}{2m+1}} \right\} \times [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z_1 - z_3)
$$

$$
- 2 \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k \in \mathbb{Z}_{odd}} (-1)^{j+r} q^{(m+1)j} \left\{ e^{\frac{\pi i(2m+j-k)(z_1+z_3)}{2m+1}} + e^{-\frac{\pi i(2m+j-k)(z_1+z_3)}{2m+1}} \right\} \times [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z_1 - z_3)
$$

(3.3)

Proposition 3.3. For $m \in \frac{1}{2}\mathbb{N}$ the following formulas hold:

$$
\theta_{0,2m+1}^{(-)}(\tau, \frac{-1}{2} + 2m(z_1 - z_2)) \left\{ \Phi^{(-)}(m,\frac{1}{2})(2\tau, 2z_1, 2z_2, 0) \right\} = -i\eta(2\tau)^3 \left\{ \frac{\theta_{0,2m+1}^{(-)}(\tau, \frac{1}{2} + z_1 - z_2 + z_1 + z_2)}{\eta(2\tau, 2z_1)} \right\}
$$

$$
+ \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \mathbb{Z}_{odd}} \sum_{1 \leq k \leq 2m} (-1)^{j+r} q^{(2m+1)j} \left\{ e^{\frac{\pi i(4m-j-k)(z_1-z_2)}{4m+1}} + e^{-\frac{\pi i(4m-j-k)(z_1-z_2)}{4m+1}} \right\} \times [\theta_{k,2m}^{(-)} - \theta_{-k,2m}^{(-)}](\tau, z_1 + z_2)
$$

$$
- \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k \in \mathbb{Z}_{odd}} (-1)^{j+r} q^{(2m+1)j} \left\{ e^{\frac{\pi i(4m+j-k)(z_1-z_2)}{4m+1}} + e^{-\frac{\pi i(4m+j-k)(z_1-z_2)}{4m+1}} \right\} \times [\theta_{k,2m}^{(-)} - \theta_{-k,2m}^{(-)}](\tau, z_1 + z_2)
$$

(3.4)

Replacing $\tau$ and $z_i$ with $\frac{1}{\tau}$ and $\frac{1}{2}z_i$ respectively and using $\theta_{j,m}^{(+)}(\frac{\tau}{2}, \frac{z}{2}) = \theta_{j,m}^{(+) \frac{1}{2}m}(\tau, z)$, this
Lemma 3.2. Let formula (3.4) be rewritten as follows:

\[ \theta_{0,m+\frac{1}{2}}^{(-)}(\tau, \frac{1 + z_1 - z_2}{2m + 1} + z_1 + z_2) \]

\[ \sum_{j=1}^{\infty} \left\{ \theta_{0,m+\frac{1}{2}}^{(-)}(\tau, \frac{1 + z_1 - z_2}{2m + 1} + z_1 + z_2) \right\} \]

+ \frac{\tau(\pi)}{2} \sum_{j=m}^{\infty} \sum_{k=0}^{m-1} \sum_{\zeta \in \mathbb{Z}_{odd}} (-1)^j q^{2j} (2m+1) \left\{ e_{\frac{\pi}{2}}(4m-k)(z_1-z_2) + e_{\frac{-\pi}{2}}(4m-k)(z_1-z_2) \right\}

\times [\theta^{(-)}_k - \theta^{(-)}_{-k,m}](\tau, z_1 + z_2)

(3.5)

Applying this formula to \((z_1, z_2) = \left( \frac{\tau + \frac{\pi}{2} + p\tau, \frac{\pi}{2} + \frac{\pi}{2} - p\tau \right), \left( \frac{\tau + \frac{\pi}{2} + p\tau, \frac{\pi}{2} - \frac{\pi}{2} - p\tau \right)\)

and \((\frac{\tau}{2} + \frac{\pi}{2} + p\tau, \frac{\tau}{2} + \frac{\pi}{2} - p\tau)\), we obtain the following:

Lemma 3.2. Let \(m \in \frac{1}{2} \mathbb{N}_{odd} \) and \(p \in \mathbb{Z}_{0}. \) Then

1) \( q^{-\frac{m^2(2p+1)^2}{2(2m+1)}} \theta_{(2p+1)m,m+\frac{1}{2}}^{(-)}(\tau, 0) \Phi^{(-)}_{\{m, \frac{1}{2}\}}(\tau, z, 0) \]

\[ = -i \eta(\tau)^3 q^{m(2p+1)^2} \theta_{0,m+\frac{1}{2}}^{(-)}(\tau, z) \]

+ \sum_{j, r \in \mathbb{Z}_{odd}} \sum_{0 \leq r < j \leq \frac{m}{2}} \sum_{0 \leq k < m} (-1)^j e^{\pi ik} q^{(m+\frac{1}{2})^2} (2m+1) \left\{ e_{\frac{\pi}{2}}(2m-k)^2 - \frac{e^{-\pi i}}{2} (2m-k) \right\}

\times [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}](\tau, z)

2) \( q^{-\frac{m^2(2p+1)^2}{2(2m+1)}} \theta_{(2p+1)m,m+\frac{1}{2}}^{(-)}(\tau, 0) \cdot \Phi^{(-)}_{\{m, \frac{1}{2}\}}(\tau, z, 0) \]

\[ = (-1)^p \eta(\tau)^3 q^{m(2p+1)^2} \theta_{p+\frac{1}{2},m+\frac{1}{2}}^{(-)}(\tau, z) - \theta_{(p+\frac{1}{2}),m+\frac{1}{2}}^{(-)}(\tau, z) \]
Proof. 1) Letting $(\tau, z) = \left(\frac{\tau}{2} + \frac{z}{2} - \frac{1}{2} + p\tau, \frac{\tau}{2} + \frac{z}{2} + \frac{1}{2} - p\tau\right)$ in (3.5), we have

\[
\theta_{0, m + \frac{1}{2}}^{(-)}(\tau, \frac{-1 + 2m((2p + 1)\tau - 1)}{2m + 1}) \times \Phi^{(-)[m, \frac{1}{2}]}(\tau, \frac{z}{2} + \frac{\tau}{2} - \frac{1}{2} + p\tau, \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2} - p\tau, 0) = (I) + (II) \tag{3.6}
\]

where

\[
(I) := -i \eta(\tau)^3 \left\{ \frac{\theta_{0, m + \frac{1}{2}}^{(-)}(\tau, \frac{(2p + 1)\tau}{2m + 1} + z)}{\psi_{11}(\tau, \frac{\tau}{2} + \frac{z}{2} - \frac{1}{2} + p\tau)} + \frac{\theta_{0, m + \frac{1}{2}}^{(-)}(\tau, \frac{(2p + 1)\tau}{2m + 1} - z)}{\psi_{11}(\tau, \frac{\tau}{2} - \frac{z}{2} + \frac{1}{2} - p\tau)} \right\}
\]

\[
(II) := \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \mathbb{Z}_{\text{odd}}} (-1)^{j+r} q^{(m + \frac{1}{2})j^2 + 2pmj - \frac{1}{16m}((4m(j-r) + k)^2)}
\]

\[
\times \left[ \theta_{m, k}^{(-)}(\tau, z) - \theta_{-m, k}^{(-)}(\tau, z) \right]
\]
Using Lemma 1.1 and Note 1.1, the LHS of (3.6) and (I) become as follows:

\[ \text{LHS of (3.6)} = q^{-\frac{m^2(2p+1)^2}{2(2m+1)^2}} \theta^{(-)}_{(2p+1)m,m+\frac{1}{2}}(\tau, 0) \]

(3.7a)

\[ (I) = -i q^{-\frac{m(2p+1)^2}{2(2m+1)^2}} \eta(\tau)^3 \left[ \theta^{(-)}_{p+\frac{1}{2},m+\frac{1}{2}, \theta^{(-)}_{p+\frac{1}{2},m+\frac{1}{2}}} - \theta^{(-)}_{p+\frac{1}{2},m+\frac{1}{2}, \theta^{(-)}_{p+\frac{1}{2},m+\frac{1}{2}}} \right] (\tau, z) \]

(3.7b)

We compute (II) by putting \( k' = \frac{1}{2} k \):

\[ (II) = \sum_{j=1}^{\infty} \sum_{r=0}^{j} \sum_{k'=\frac{1}{2} k \in \mathbb{Z}_{\text{odd}}} \sum_{0<k' \leq m} (-1)^j + r q^{(m+\frac{1}{2})j^2 - \frac{1}{4m}(2m(j-r)+k')^2} \]

\[ \times \left\{ q^{\frac{1}{2}(2mr-k')(2p+1)} e^{-\pi i(2mr-k')} + q^{-\frac{1}{2}(2mr-k')(2p+1)} e^{\pi i(2mr-k')} \right\} \left[ \theta^{(-)}_{k',m} - \theta^{(-)}_{-k',m} \right] (\tau, z) \]

\[ - \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k'=\frac{1}{2} k \in \mathbb{Z}_{\text{odd}}} \sum_{0<k' \leq m} (-1)^j + r q^{(m+\frac{1}{2})j^2 - \frac{1}{4m}(2m(j-r)-k')^2} \]

\[ \times \left\{ q^{\frac{1}{2}(2mr+k')(2p+1)} e^{-\pi i(2mr+k')} + q^{-\frac{1}{2}(2mr+k')(2p+1)} e^{\pi i(2mr+k')} \right\} \left[ \theta^{(-)}_{k',m} - \theta^{(-)}_{-k',m} \right] (\tau, z) \]

\[ = \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k=\frac{1}{2} k \in \mathbb{Z}_{\text{odd}}} \sum_{0<k \leq m} (-1)^j \ e^{\pi ikq^{(m+\frac{1}{2})j^2} q^{-\frac{1}{4m}(2m(j-r)+k)^2}} \left\{ q^{\frac{2n+1}{2}(2mr-k)} - q^{-\frac{2n+1}{2}(2mr-k)} \right\} \]

\[ \times \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \]

\[ + \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k=\frac{1}{2} k \in \mathbb{Z}_{\text{odd}}} \sum_{0<k \leq m} (-1)^j \ e^{\pi ikq^{(m+\frac{1}{2})j^2} q^{-\frac{1}{4m}(2m(j-r)-k)^2}} \left\{ q^{\frac{2n+1}{2}(2mr+k)} - q^{-\frac{2n+1}{2}(2mr+k)} \right\} \]

\[ \times \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \]

Putting \( r = j - r' \), this equation is rewritten as follows:

\[ (II) = \sum_{j=1}^{\infty} \sum_{r'=0}^{j-1} \sum_{k=\frac{1}{2} k \in \mathbb{Z}_{\text{odd}}} \sum_{0<k \leq m} (-1)^j \ e^{\pi ikq^{(m+\frac{1}{2})j^2} q^{-\frac{1}{4m}(2mr'+k)^2}} \]
We decompose
\[ A := \sum_{j=1}^{\infty} \sum_{r'=1}^{\infty} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} \left[q^{(m+\frac{1}{2})j^2} q^{-\frac{1}{4m}(2mr')^2} - q^{-(2p+1)mj} q^{\frac{2p+1}{2}(2mr')^2} \right] \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \]
\[ + \sum_{j=1}^{\infty} \sum_{r'=1}^{\infty} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} \left[q^{(m+\frac{1}{2})j^2} q^{-\frac{1}{4m}(2mr')^2} - q^{-(2p+1)mj} q^{\frac{2p+1}{2}(2mr')^2} \right] \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \]
\[ = A_1 + A_2 + A_3 + A_4 \]
where \( A_1 \sim A_4 \) are as follows:

\[ \begin{align*}
A_1 &:= \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 + (2p+1)mj} q^{-\frac{1}{4m}(2mr+k)^2 - \frac{2p+1}{2}(2mr+k)} \\
&A \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \\
A_2 &:= -\sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 + (2p+1)mj} q^{-\frac{1}{4m}(2mr-k)^2 + \frac{2p+1}{2}(2mr-k)} \\
&A \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \\
A_3 &:= \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 + (2p+1)mj} q^{-\frac{1}{4m}(2mr-k)^2 - \frac{2p+1}{2}(2mr-k)} \\
&A \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \\
&= \sum_{j=0}^{\infty} \sum_{r=0}^{j} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 + (2p+1)mj} q^{-\frac{1}{4m}(2mr-k)^2 - \frac{2p+1}{2}(2mr-k)} \\
&A \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \\
A_4 &:= -\sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 - (2p+1)mj} q^{-\frac{1}{4m}(2mr-k)^2 + \frac{2p+1}{2}(2mr-k)} \\
&A \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \\
\end{align*} \]

We decompose \( A_2 \) and \( A_4 \) as \( A_2 = A_2' + A_2'' \) and \( A_4 = A_4' + A_4'' \), where

\[ \begin{align*}
A_2' &:= -\sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 - (2p+1)mj} q^{-\frac{1}{4m}(2mr+k)^2 + \frac{2p+1}{2}(2mr+k)} \\
&A \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \\
&= \sum_{j=0}^{\infty} \sum_{r=0}^{j} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 - (2p+1)mj} q^{-\frac{1}{4m}(2mr+k)^2 - \frac{2p+1}{2}(2mr-k)} \\
&A \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \\
A_4' &:= -\sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 - (2p+1)mj} q^{-\frac{1}{4m}(2mr-k)^2 + \frac{2p+1}{2}(2mr-k)} \\
&A \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \\
\end{align*} \]
\[
\begin{align*}
A_2'' &:= - \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} (-1)^r e^{r m j} q^{(m+\frac{1}{2})j^2 - (2p+1)mj} q^{-\frac{1}{4m}(2mr+m)^2 + \frac{2p+1}{2}(2mr+m)} \times \left[ \theta_{m,m}^{(-)} - \theta_{-m,m}^{(-)} \right](\tau, z) \\
A_4' &:= - \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} (-1)^r e^{r m j} q^{(m+\frac{1}{2})j^2 - (2p+1)mj} q^{-\frac{1}{4m}(2mr+m)^2 + \frac{2p+1}{2}(2mr+m)} \times \left[ \theta_{m,m}^{(-)} - \theta_{-m,m}^{(-)} \right](\tau, z) \\
A_3' &:= \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} (-1)^r e^{r m j} q^{(m+\frac{1}{2})j^2 - (2p+1)mj} q^{-\frac{1}{4m}(2mr+m)^2 + \frac{2p+1}{2}(2mr+m)} \times \left[ \theta_{m,m}^{(-)} - \theta_{-m,m}^{(-)} \right](\tau, z) \\
A_1 + A_4' &:= \left[ \sum_{j, r \in \mathbb{Z}, 0 \leq r < j} - \sum_{j, r \in \mathbb{Z}, j \leq r < 0} \right] \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^j e^{\pi i k q^{(m+\frac{1}{2})j^2 + (2p+1)mj} q^{-\frac{1}{4m}(2mr+k)^2 - \frac{2p+1}{2}(2mr+k)}} \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) \\
A_2' + A_3' &:= \left[ \sum_{j, r \in \mathbb{Z}, 0 \leq r < j} - \sum_{j, r \in \mathbb{Z}, j \leq r < 0} \right] \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^j e^{\pi i k q^{(m+\frac{1}{2})j^2 + (2p+1)mj} q^{-\frac{1}{4m}(2mr+k)^2 - \frac{2p+1}{2}(2mr+k)}} \times \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) \\
&- \sum_{j \in \mathbb{Z}, k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^j e^{\pi i k q^{(m+\frac{1}{2})j^2 + (2p+1)mj} q^{-\frac{1}{4m}k^2 + (p+\frac{1}{2})k} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right]((\tau, z),}
\end{align*}
\]
we have

\[ (II) = \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \sum_{0 \leq r < j} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 + (2p+1)m_j} q^{-\frac{1}{4m}(2mr+k)^2 - \frac{2p+1}{2}(2mr+k)} \times \left[ \theta_{-k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \]

\[ + \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \sum_{0 \leq r \leq j} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 + (2p+1)m_j} q^{-\frac{1}{4m}(2mr-k)^2 - \frac{2p+1}{2}(2mr-k)} \times \left[ \theta_{-k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \]

\[ - \sum_{j \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2 + (2p+1)m_j} q^{-\frac{1}{4m}k^2 + (p+\frac{1}{2})k} \left[ \theta_{-k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z) \quad (3.8) \]

Now the formulas (3.6), (3.7a), (3.7b) and (3.8) complete the proof of 1).

2) Letting \((z_1, z_2) = \left( \frac{\tau}{2} + \frac{p}{2}, \frac{\tau}{2} + \frac{p}{2} \right)\) in (3.5), we have

\[ \theta_{0,m+\frac{1}{2}}^{(-)} \left( \tau, \frac{1 + 2m(2p+1)\tau}{2m+1} \right) \phi(-)[m, \frac{1}{2}] \left( \tau, \frac{z}{2} + \frac{\tau}{2} + p\tau, \frac{z}{2} - \frac{\tau}{2} - p\tau, 0 \right) = (III) + (IV) \quad (3.9) \]

where

\[ (III) := - i \eta(\tau)^3 \left\{ \theta_{0,m+\frac{1}{2}}^{(-)} (\tau, \frac{1 + (2p+1)\tau}{2m+1} + z) + \theta_{0,m+\frac{1}{2}}^{(-)} (\tau, \frac{1 + (2p+1)\tau}{2m+1} - z) \right\} \]

\[ (IV) := \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \mathbb{Z}_{\text{odd}}} (-1)^{j+r} q^{(m+\frac{1}{2})j^2 - \frac{1}{16m}(4mj+r+k)^2} \times \left\{ e^{rac{2\pi i}{4m}(4mr-k)(2p+1)\tau} + e^{-\frac{2\pi i}{4m}(4mr-k)(2p+1)\tau} \right\} \left[ \theta_{-\frac{z}{2}, m}^{(-)} - \theta_{-\frac{z}{2}, m}^{(-)} \right] (\tau, z) \]

\[ - \sum_{j=1}^{\infty} \sum_{r=1}^{j-1} \sum_{k \in \mathbb{Z}_{\text{odd}}} (-1)^{j+r} q^{(m+\frac{1}{2})j^2 - \frac{1}{16m}(4mj-r-k)^2} \times \left\{ e^{rac{2\pi i}{4m}(4mr+k)(2p+1)\tau} + e^{-\frac{2\pi i}{4m}(4mr+k)(2p+1)\tau} \right\} \left[ \theta_{-\frac{z}{2}, m}^{(-)} - \theta_{-\frac{z}{2}, m}^{(-)} \right] (\tau, z) \]

Using Lemma 1.1 and Note 1.1 the LHS of (3.9) and (III) become as follows:

\[ \text{LHS of (3.9)} = q^{-\frac{m^2(2p+1)^2}{2(2m+1)}} \theta_{(2p+1)m,m+\frac{1}{4}}^{(-)} (\tau, 0) \quad (3.10a) \]

\[ (III) = (-1)^p q^{-\frac{m(2p+1)^2}{4(2m+1)}} \eta(\tau)^3 \frac{\theta_{p+\frac{1}{4}, m+\frac{1}{4}}^{(-)} - \theta_{-(p+\frac{1}{4}), m+\frac{1}{4}}^{(-)}}{\theta_{0, m+\frac{1}{4}}^{(-)} (\tau, z)} \quad (3.10b) \]
We compute (IV) by putting $k' = \frac{1}{2} k$:

\[
(IV) = \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k' \in \frac{1}{2} \mathbb{Z}} (-1)^{j+r} q^{(m+\frac{1}{2})j^2 - \frac{1}{4m}(2m(j-r)+k')^2} \\
\times \left\{ q^{\frac{1}{2}(2mr-k')(2p+1)} + q^{-\frac{1}{2}(2mr-k')(2p+1)} \right\} \left[ \theta_{k',m}^{(-)} - \theta_{-k',m}^{(-)} \right](\tau, z)
\]

\[
- \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k' \in \frac{1}{2} \mathbb{Z}} (-1)^{j+r} q^{(m+\frac{1}{2})j^2 - \frac{1}{4m}(2m(j-r)-k')^2} \\
\times \left\{ q^{\frac{1}{2}(2mr+k')(2p+1)} + q^{-\frac{1}{2}(2mr+k')(2p+1)} \right\} \left[ \theta_{k',m}^{(-)} - \theta_{-k',m}^{(-)} \right](\tau, z)
\]

Putting $r = j - r'$, this equation is rewritten as follows:

\[
(IV) = \sum_{j=1}^{\infty} \sum_{r'=0}^{j-1} \sum_{k \in \frac{1}{2} \mathbb{Z}} (-1)^{r'} q^{(m+\frac{1}{2})j^2} q^{-\frac{1}{4m}(2mr^2+k)^2} \\
\times \left\{ q^{(2p+1)mj} q^{-\frac{2p+1}{2}(2mr^2+k)} + q^{-(2p+1)mj} q^{\frac{2p+1}{2}(2mr^2+k)} \right\} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
- \sum_{j=1}^{\infty} \sum_{r'=1}^{j} \sum_{k \in \frac{1}{2} \mathbb{Z}} (-1)^{r'} q^{(m+\frac{1}{2})j^2} q^{-\frac{1}{4m}(2mr^2-k)^2} \\
\times \left\{ q^{(2p+1)mj} q^{-\frac{2p+1}{2}(2mr^2-k)} + q^{-(2p+1)mj} q^{\frac{2p+1}{2}(2mr^2-k)} \right\} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
= B_1 + B_2 + B_3 + B_4
\]

where $B_1 \sim B_4$ are as follows:

\[
B_1 := \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k \in \frac{1}{2} \mathbb{Z}} (-1)^{r} q^{(m+\frac{1}{2})j^2+(2p+1)mj} q^{-\frac{1}{4m}(2mr^2+k)^2 - \frac{2p+1}{2}(2mr+k)} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
B_2 := \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k \in \frac{1}{2} \mathbb{Z}} (-1)^{r} q^{(m+\frac{1}{2})j^2-(2p+1)mj} q^{-\frac{1}{4m}(2mr^2-k)^2 + \frac{2p+1}{2}(2mr+k)} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
B_3 := - \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \frac{1}{2} \mathbb{Z}} (-1)^{r} q^{(m+\frac{1}{2})j^2+(2p+1)mj} q^{-\frac{1}{4m}(2mr^2-k)^2 - \frac{2p+1}{2}(2mr-k)} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
B_4 := - \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \frac{1}{2} \mathbb{Z}} (-1)^{r} q^{(m+\frac{1}{2})j^2-(2p+1)mj} q^{-\frac{1}{4m}(2mr^2+k)^2 + \frac{2p+1}{2}(2mr+k)} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]
\(-\sum_{j=0}^{\infty} \sum_{r=0}^{j} \sum_{k \in \frac{1}{2}{\mathbb{Z}}} \frac{1}{4m} (2mr-k)^2 - \frac{2^{p+1}}{2} (2mr-k) \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z)\)

\[+ \sum_{j=0}^{\infty} \sum_{k \in \frac{1}{2}{\mathbb{Z}}} \frac{1}{4m} (2mr-k)^2 - \frac{2^{p+1}}{2} (2mr-k) \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z)\]

\[B_4 := -\sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \frac{1}{2}{\mathbb{Z}}} \frac{1}{4m} (2mr-k)^2 - \frac{2^{p+1}}{2} (2mr-k) \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z)\]

We decompose \(B_2\) and \(B_4\) as \(B_2 = B_2' + B_2''\) and \(B_4 = B_4' + B_4''\), where

\[B_2' := \sum_{j=0}^{\infty} \sum_{r=0}^{j} \sum_{k \in \frac{1}{2}{\mathbb{Z}}} \frac{1}{4m} (2mr-k)^2 - \frac{2^{p+1}}{2} (2mr-k) \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z)\]

\[B_2'' := \sum_{j=0}^{\infty} \sum_{r=0}^{j} \sum_{k \in \frac{1}{2}{\mathbb{Z}}} \frac{1}{4m} (2mr-k)^2 - \frac{2^{p+1}}{2} (2mr-k) \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z)\]

\[B_4' := \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \frac{1}{2}{\mathbb{Z}}} \frac{1}{4m} (2mr-k)^2 - \frac{2^{p+1}}{2} (2mr-k) \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z)\]

\[B_4'' := \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \frac{1}{2}{\mathbb{Z}}} \frac{1}{4m} (2mr-k)^2 - \frac{2^{p+1}}{2} (2mr-k) \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right] (\tau, z)\]
3) Letting \( (z_1, z_2) = (\frac{1}{2} + p\tau, \frac{1}{2} - p\tau) \) in (3.5), we have

\[
\theta_{0,m+\frac{1}{2}}^{(-)}(\tau, \frac{-1+2m(2p\tau-1)}{2m+1}) \Phi_{m,\frac{1}{2}}^{(-)}(\tau, \frac{z}{2} - \frac{1}{2} + p\tau, \frac{z}{2} + \frac{1}{2} - p\tau, 0) = (V) + (VI) \quad (3.12)
\]

where

\[
(V) := -i \eta(\tau)^3 \left\{ \frac{\theta_{0,m+\frac{1}{2}}^{(-)}(\tau, \frac{2p\tau}{2m+1} + z)}{\vartheta_{11}(\tau, \frac{z}{2} - \frac{1}{2} + p\tau)} + \frac{\theta_{0,m+\frac{1}{2}}^{(-)}(\tau, \frac{2p\tau}{2m+1} - z)}{\vartheta_{11}(\tau, \frac{z}{2} + \frac{1}{2} - p\tau)} \right\}
\]

Now the formulas (3.9), (3.10a), (3.10b) and (3.11) complete the proof of 2).

Then, noticing that

\[
B_2'' + B_4'' = 0
\]

\[
B_1 + B_4' = \left[ \sum_{j \in \mathbb{Z}} - \sum_{j \in \mathbb{Z}} \right] \sum_{k \in \mathbb{Z}_{\text{odd}}} \frac{1}{2} \quad k \leq m \quad 0 \leq k \leq m \quad \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}(\tau, z) \]

\[
B_2' + B_3 = \left[ \sum_{j \in \mathbb{Z}} - \sum_{j \in \mathbb{Z}} \right] \sum_{k \in \mathbb{Z}_{\text{odd}}} \frac{1}{2} \quad k \leq m \quad 0 < k < m \quad \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}(\tau, z)
\]

we have

\[
(IV) = \left[ \sum_{j \in \mathbb{Z}} - \sum_{j \in \mathbb{Z}} \right] \sum_{k \in \mathbb{Z}_{\text{odd}}} \frac{1}{2} \quad k \leq m \quad 0 < k < m \quad \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}(\tau, z)
\]

\[
- \left[ \sum_{j \in \mathbb{Z}} - \sum_{j \in \mathbb{Z}} \right] \sum_{k \in \mathbb{Z}_{\text{odd}}} \frac{1}{2} \quad k \leq m \quad 0 < k < m \quad \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}(\tau, z)
\]

\[
+ \left[ \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_{\text{odd}}} \frac{1}{2} \quad 0 < k < m \quad q^{(m+\frac{1}{2})^2 + \frac{2(2p+1)m^2}{2m+1}} k \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}(\tau, z)
\]

(3.11)
We compute (VI) by putting

\[
(VI) := \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \mathbb{Z}_{\text{odd}}} \sum_{1 \leq k \leq 2m} (-1)^{j+r} q^{(m+\frac{j}{2})j^2 - \frac{1}{4m}(4m(j-r)+k)^2} \\
\times \left\{ e^{\frac{-\pi i}{2}(4mr-k)(2p\tau -1)} + e^{-\frac{\pi i}{2}(4mr-k)(2p\tau -1)} \right\} \left[ \theta(-)_{\frac{k}{4m}} - \theta(-)_{\frac{m-k}{4m}} \right](\tau, z)
\]

Using Lemma 1.1 and Note 1.1, the LHS of (3.12) and (V) become as follows:

LHS of (3.12) = \[ q^{-\frac{m^2(2p)^2}{2(2m+1)}} \theta_{2pm,m+\frac{1}{2}}^\prime(\tau, 0) \] (3.13a)

(V) = \[ -i q^{\frac{mp^2}{2m+1}} \eta(\tau)^3 \frac{\left[ \theta_{\frac{k}{4m}}^\prime - \theta_{\frac{m-k}{4m}}^\prime \right](\tau, z)}{\theta_{\frac{k}{4m}}^\prime(\tau, z)} \] (3.13b)

We compute (VI) by putting \( k' = \frac{1}{2} k \):

\[
(VI) = \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k' \in \mathbb{Z}_{\text{odd}}} \sum_{0 < k' < m} (-1)^{j+r} q^{(m+\frac{j}{2})j^2 - \frac{1}{4m}(2m(j-r)+k')^2} \\
\times \left\{ q^{2m-k'}p \, e^{-\pi i(2m-k') \theta_{k',m}^\prime} + q^{-(2m-k')p} \, e^{\pi i(2m-k') \theta_{k',m}^\prime} \right\} \left[ \theta_{k',m}^\prime - \theta_{-k',m}^\prime \right](\tau, z)
\]

\[
- \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k' \in \mathbb{Z}_{\text{odd}}} \sum_{0 < k' < m} (-1)^{j+r} q^{(m+\frac{j}{2})j^2 - \frac{1}{4m}(2m(j-r)-k')^2} \\
\times \left\{ q^{2m+k'}p \, e^{-\pi i(2m+k') \theta_{k',m}^\prime} + q^{-(2m+k')p} \, e^{\pi i(2m+k') \theta_{k',m}^\prime} \right\} \left[ \theta_{k',m}^\prime - \theta_{-k',m}^\prime \right](\tau, z)
\]

\[
= \sum_{j=1}^{\infty} \sum_{r=1}^{j} \sum_{k \in \mathbb{Z}_{\text{odd}}} \sum_{0 < k \leq m} (-1)^{j} \, e^{\pi i k} q^{(m+\frac{j}{2})j^2} q^{\frac{1}{4m}(2m(j-r)+k)^2} \left\{ q^{p(2m-r-k)} - q^{-p(2m-r-k)} \right\} \\
\times \left[ \theta_{k,m}^\prime - \theta_{-k,m}^\prime \right](\tau, z)
\]

\[
+ \sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k \in \mathbb{Z}_{\text{odd}}} \sum_{0 < k \leq m} (-1)^{j} \, e^{\pi i k} q^{(m+\frac{j}{2})j^2} q^{\frac{1}{4m}(2m(j-r)-k)^2} \left\{ q^{p(2m+r+k)} - q^{-p(2m+r+k)} \right\}
\]
Putting $r = j - r'$, this equation is rewritten as follows:

\[
(VI) = \sum_{j=1}^{\infty} \sum_{r'=0}^{j-1} \sum_{k \in \{ \frac{1}{2} Z_{\text{odd}} \}} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2} q^{-\frac{1}{4m}(2mr'+k)^2} \\
\times \left\{ q^{2pmj} q^{-p(2mr'+k)} - q^{-2pmj} q^{p(2mr'+k)} \right\} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) \\
+ \sum_{j=1}^{\infty} \sum_{r'=1}^{j} \sum_{k \in \{ \frac{1}{2} Z_{\text{odd}} \}} \sum_{0 < k < m} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2} q^{-\frac{1}{4m}(2mr'-k)^2} \\
\times \left\{ q^{2pmj} q^{-p(2mr'-k)} - q^{-2pmj} q^{p(2mr'-k)} \right\} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) \\
= C_1 + C_2 + C_3 + C_4
\]

where $C_1 \sim C_4$ are as follows:

\[
C_1 := \sum_{j=1}^{\infty} \sum_{r'=0}^{j-1} \sum_{k \in \{ \frac{1}{2} Z_{\text{odd}} \}} \sum_{0 < k < m} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2+2pmj} q^{-\frac{1}{4m}(2mr+k)^2-p(2mr+k)} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
C_2 := -\sum_{j=1}^{\infty} \sum_{r'=0}^{j-1} \sum_{k \in \{ \frac{1}{2} Z_{\text{odd}} \}} \sum_{0 < k < m} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2+2pmj} q^{-\frac{1}{4m}(2mr+k)^2+p(2mr+k)} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
C_3 := \sum_{j=1}^{\infty} \sum_{r'=1}^{j} \sum_{k \in \{ \frac{1}{2} Z_{\text{odd}} \}} \sum_{0 < k < m} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2+2pmj} q^{-\frac{1}{4m}(2mr-k)^2-p(2mr-k)} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
= \sum_{j=0}^{\infty} \sum_{r'=0}^{j} \sum_{k \in \{ \frac{1}{2} Z_{\text{odd}} \}} \sum_{0 < k < m} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2+2pmj} q^{-\frac{1}{4m}(2mr-k)^2-p(2mr-k)} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
- \sum_{j=0}^{\infty} \sum_{k \in \{ \frac{1}{2} Z_{\text{odd}} \}} \sum_{0 < k < m} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2+2pmj} q^{-\frac{1}{4m}k^2+pk} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
C_4 := -\sum_{j=1}^{\infty} \sum_{r'=1}^{j} \sum_{k \in \{ \frac{1}{2} Z_{\text{odd}} \}} \sum_{0 < k < m} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})j^2-2pmj} q^{-\frac{1}{4m}(2mr-k)^2+p(2mr-k)} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]
We decompose $C_2$ and $C_4$ as $C_2 = C'_2 + C''_2$ and $C_4 = C'_4 + C''_4$, where

$$C'_2 := -\sum_{j=1}^{\infty} \sum_{r=0}^{j-1} \sum_{k \in \mathbb{Z}_{\text{odd}}} (-1)^j e^{\pi i k q^2} (m + \frac{1}{2}) j^2 - 2pmj q^{-\frac{1}{4m} (2mr + k)^2 + p(2mr + k)} [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)$$

$$= -\sum_{j=0}^{\infty} \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_{\text{odd}}} (-1)^j e^{\pi i k q^2} (m + \frac{1}{2}) j^2 + 2pmj q^{-\frac{1}{4m} (2mr + k)^2 - p(2mr - k)} \times [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)$$

$$C''_2 := -\sum_{j=0}^{\infty} \sum_{r = 0}^{j-1} (-1)^j e^{\pi i m q^2} (m + \frac{1}{2}) j^2 - 2pmj q^{-\frac{1}{4m} (2mr + m)^2 + p(2mr + m)} [\theta^{(-)}_{m,m} - \theta^{(-)}_{-m,m}] (\tau, z)$$

$$= -2\sum_{j=0}^{\infty} \sum_{r = 0}^{j-1} (-1)^j e^{\pi i m q^2} (m + \frac{1}{2}) j^2 - 2pmj q^{-\frac{1}{4m} (2r + 1)(2r - 4p + 1)} [\theta^{(-)}_{m,m} - \theta^{(-)}_{-m,m}] (\tau, z)$$

$$C'_4 := -\sum_{j=0}^{\infty} \sum_{r = 1}^{j} \sum_{k \in \mathbb{Z}_{\text{odd}}} (-1)^j e^{\pi i k q^2} (m + \frac{1}{2}) j^2 - 2pmj q^{-\frac{1}{4m} (2mr - k)^2 + p(2mr - k)} [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)$$

$$= -\sum_{j=0}^{\infty} \sum_{r \in \mathbb{Z}} \sum_{k \in \mathbb{Z}_{\text{odd}}} (-1)^j e^{\pi i k q^2} (m + \frac{1}{2}) j^2 + 2pmj q^{-\frac{1}{4m} (2mr + k)^2 - p(2mr + k)} \times [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)$$

$$C''_4 := \sum_{j=1}^{\infty} \sum_{r = 1}^{j} (-1)^j e^{\pi i m q^2} (m + \frac{1}{2}) j^2 - 2pmj q^{-\frac{1}{4m} (2mr - m)^2 + p(2mr - m)} [\theta^{(-)}_{m,m} - \theta^{(-)}_{-m,m}] (\tau, z)$$

$$= 2\sum_{j=1}^{\infty} \sum_{r' = 0}^{j-1} (-1)^j e^{\pi i m q^2} (m + \frac{1}{2}) j^2 - 2pmj q^{-\frac{1}{4m} (2r' + 1)(2r' - 4p + 1)} [\theta^{(-)}_{m,m} - \theta^{(-)}_{-m,m}] (\tau, z)$$

Then, noticing that

$$C'_2 + C''_2 = 0$$

$$C_1 + C'_4 = \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r' \in \mathbb{Z}} \right] \sum_{k \in \mathbb{Z}_{\text{odd}}} (-1)^j e^{\pi i k q^2} (m + \frac{1}{2}) j^2 + 2pmj q^{-\frac{1}{4m} (2mr + k)^2 - p(2mr + k)} \times [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)$$
Lemma 3.3. Let

Now the formulas (3.12), (3.13a), (3.13b) and (3.14) complete the proof of 3).

To go further, we note also the following:

\[
\text{we have}
\]

\[(VI) = \left[ \sum_{j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}} 0 < k < m} (-1)^j e^{\pi i k q (m + \frac{1}{2})^2 + 2pmj q - \frac{1}{4m} (2mr-k)^2 - p(2mr-k)} \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} (\tau, z)
\]

\[- \sum_{j \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \cdot \sum_{0 < k < m} (-1)^j e^{\pi i k q (m + \frac{1}{2})^2 + 2pmj q - \frac{1}{4m} k^2 + pk \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} (\tau, z),}
\]

Now the formulas (3.12), (3.13a), (3.13b) and (3.14) complete the proof of 3).

To go further, we note also the following:

**Lemma 3.3.** Let \( m \in \frac{1}{2} \mathbb{N}_{\text{odd}} \) and \( p \in \mathbb{Z}_{\geq 0} \). Then

1) For \((z_1, z_2) = (\frac{s}{2} + \frac{r}{2} - \frac{1}{2}, \frac{t}{2} - \frac{1}{2} + \frac{1}{2}),\)

\[
\Phi^{(-)[m, \frac{1}{2}]} (\tau, z_1 + pr, z_2 - pr, 0) = q^{mp(p+1)} \left\{ \Phi^{(-)[m, \frac{1}{2}]} (\tau, z_1, z_2, 0) + \right.
\]

\[\left. - \sum_{-p < r \leq p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} e^{\pi i k q \frac{(2mr+k)(2m-r+1)k}{4m}} \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} (\tau, z) - q^{-mp(p+1)} \right. \]

\[\left. \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \sum_{0 < k < m} e^{\pi i k q \frac{1}{4m} k^2 + (2p+1)\frac{m}{4} \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} (\tau, z) + \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \sum_{0 < k < m} e^{\pi i k q \frac{k(2m-k)}{4m} \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} (\tau, z) - \sum_{-p \leq r \leq p} q^{\frac{1}{m}(2r-1)(2r+1)} \theta_{m,m}^{(-)} (\tau, z) + e^{\pi i m} q^{\mp \theta_{m,m}^{(-)} (\tau, z)} \right\} \quad (3.15a)
\]
2) For \((z_1, z_2) = \left(\frac{\tau}{2} + \frac{p}{2}, \frac{\tau}{2} - \frac{p}{2}\right)\),
\[
\Phi^{(-)}_{m, \frac{1}{2}}(\tau, z_1 + p\tau, z_2 - p\tau, 0) = (-1)^p q^{mp(p+1)} \left\{ \Phi^{(-)}_{m, \frac{1}{2}}(\tau, z_1, z_2, 0) + \sum_{-p < r < p} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} (\tau, z) \right. \\
\left. \sum_{0 < k < m} \frac{(-1)^r q^{(2m(r+k)(2m(r-1)+k))}}{4m} \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \\
\sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0 < k < m} q^{k(2m-k)(2m-1)+k} \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \right. \\
\left. \sum_{0 < k < m} \frac{q^{k(2m-k)}}{4m} \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \\
\sum_{-p < r < p} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} (\tau, z) \right. \\
\left. \sum_{0 < k < m} q^{k(2m-k)} \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \right\} 
(3.15b)
\]

3) For \((z_1, z_2) = \left(\frac{\tau}{2} - \frac{1}{2}, \frac{\tau}{2} + \frac{1}{2}\right)\),
\[
\Phi^{(-)}_{m, \frac{1}{2}}(\tau, z_1 + p\tau, z_2 - p\tau, 0) = q^{mp(p+1)} \left\{ \Phi^{(-)}_{m, \frac{1}{2}}(\tau, z_1, z_2, 0) - \sum_{-p < r < p} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} (\tau, z) \right. \\
\left. \sum_{0 < k < m} e^{\pi i k} q^{(2m(r+k)))} \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \\
\sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0 < k < m} e^{\pi i k} q^{k(2m-k)} \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \right. \\
\left. \sum_{-p < r < p} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} (\tau, z) \right. \\
\left. \sum_{0 < k < m} q^{k(2m-k)} \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \right\} 
(3.15c)
\]

Proof. By Lemma 2.4, we have
\[
\Phi^{(-)}_{m, \frac{1}{2}}(\tau, z_1 + p\tau, z_2 - p\tau, 0) = \left\{ \Phi^{(-)}_{m, \frac{1}{2}}(\tau, z_1, z_2, 0) - U(z_1, z_2) \right\} \\
\times \left\{ q^{mp(p+1)} \right. \text{if } (z_1, z_2) = \left(\frac{\tau}{2} + \frac{p}{2}, \frac{\tau}{2} - \frac{p}{2} \right) \text{ or } \left(\frac{\tau}{2} - \frac{1}{2}, \frac{\tau}{2} + \frac{1}{2} \right) \\
\left. (-1)^p q^{mp(p+1)} \right. \text{if } (z_1, z_2) = \left(\frac{\tau}{2} + \frac{p}{2}, \frac{\tau}{2} - \frac{p}{2} \right) \\
\right\} 
(3.16a)
\]
where
\[
U(\frac{\tau}{2} + \frac{p}{2}, \frac{\tau}{2} - \frac{p}{2} + \frac{1}{2}) = \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} e^{\pi i k} q^{k \frac{2}{4m}} \frac{k}{2} \left[ \theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m} \right] (\tau, z) \Rightarrow (I)
\]
\[
U(\frac{\tau}{2} + \frac{\pi}{2}, \frac{\tau}{2} - \frac{\pi}{2}) = \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0<k<2pm} q^{- \frac{k^2}{4m} - \frac{1}{2} \left( \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right)}(\tau, z) =: (\text{II}) \quad (3.16b)
\]
\[
U(\frac{\tau}{2} - \frac{1}{2}, \frac{\tau}{2} + \frac{1}{2}) = \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0<k<2pm} e^{\pi ik} q^{- \frac{k^2}{4m} - \frac{1}{2} \left( \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right)}(\tau, z) =: (\text{III})
\]

Since
\[
\left\{ k \in \frac{1}{2}\mathbb{Z}_{\text{odd}} ; \ 0 < k < 2pm \right\} = \bigcup_{r=0}^{p-1} \left\{ k \in \frac{1}{2}\mathbb{Z}_{\text{odd}} ; \ 2rm < k < 2(r+1)m \right\}
\]

the formulas for (I) \sim (III) are rewritten as follows by using Lemma 1.2

\[
(I) = \sum_{r=0}^{p-1} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0<k<2m} e^{\pi i(2mr+k)} q^{- \frac{(2mr+k)^2}{4m} - \frac{2mr+k}{2} \left( \theta_{2mr+k,m}^{(-)} - \theta_{-(2mr+k),m}^{(-)} \right)}(\tau, z)
\]
\[
= \sum_{r=1}^{p} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0<k<2m} e^{\pi ik} q^{- \frac{(2mr+k)^2}{4m} - \frac{2mr+k}{2} \left( \theta_{2mr+k,m}^{(-)} - \theta_{-(2mr+k),m}^{(-)} \right)}(\tau, z)
\]
\[
= \sum_{r=0}^{p-1} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0<k<2m} (-1)^r q^{- \frac{(2mr+k)^2}{4m} - \frac{2mr+k}{2} \left( \theta_{2mr+k,m}^{(-)} - \theta_{-(2mr+k),m}^{(-)} \right)}(\tau, z)
\]
\[
(II) = \sum_{r=0}^{p-1} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0<k<2m} q^{- \frac{(2mr+k)^2}{4m} - \frac{2mr+k}{2} \left( \theta_{2mr+k,m}^{(-)} - \theta_{-(2mr+k),m}^{(-)} \right)}(\tau, z)
\]
\[
= \sum_{r=1}^{p} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0<k<2m} e^{\pi ik} q^{- \frac{(2mr+k)^2}{4m} - \frac{2mr+k}{2} \left( \theta_{2mr+k,m}^{(-)} - \theta_{-(2mr+k),m}^{(-)} \right)}(\tau, z)
\]
\[
(III) = \sum_{r=0}^{p-1} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0<k<2m} e^{\pi ik} q^{- \frac{(2mr+k)^2}{4m} - \frac{2mr+k}{2} \left( \theta_{2mr+k,m}^{(-)} - \theta_{-(2mr+k),m}^{(-)} \right)}(\tau, z)
\]

Due to the decomposition \[
\sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} = \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} + \sum_{k=m} + \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} , \text{ each of (I) \sim (III) decomposes into the sum of 3 parts:}
\]

\[
(I) = (I)_A + (I)_B + (I)_C, \quad (II) = (II)_A + (II)_B + (II)_C, \quad (III) = (III)_A + (III)_B + (III)_C
\]

where
\[
(I)_A := \sum_{r=1}^{p} \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{0<k<m} e^{\pi ik} q^{- \frac{(2mr+k^2)(2mr+k)}{4m} - \frac{2mr+k}{2} \left( \theta_{2mr+k,m}^{(-)} - \theta_{-(2mr+k),m}^{(-)} \right)}(\tau, z)
\]
\( (I)_B := \sum_{r=1}^{p} e^{\pi im} q^{-\frac{(2m(r-1)+m)(2mr+m)}{4m}} \left[ \theta_{m,m}^{(-)} - \theta_{-m,m}^{(-)} \right](\tau, z) \)

\( = 2 e^{\pi im} \sum_{r=1}^{p} q^{-\frac{m}{4}(2r-1)(2r+1)} \theta_{m,m}^{(-)}(\tau, z) \)

\( (I)_C := \sum_{r=1}^{p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\(\sum_{r=1}^{p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\( (I)_B := -\sum_{r=1}^{p} (-1)^r q^{-\frac{m}{4}(2r-1)(2r+1)} \theta_{m,m}^{(-)}(\tau, z) \)

\( = -2 \sum_{r=1}^{p} (-1)^r q^{-\frac{m}{4}(2r-1)(2r+1)} \theta_{m,m}^{(-)}(\tau, z) \)

\( (I)_C := -\sum_{r=1}^{p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{m}{4}(2r-1)(2r+1)} \left[ \theta_{k,m}^{(-)} \right] \left[ \theta_{-k,m}^{(-)} \right](\tau, z) \)

\( = -2 \sum_{r=1}^{p} (-1)^r q^{-\frac{m}{4}(2r-1)(2r+1)} \theta_{m,m}^{(-)}(\tau, z) \)

\( (II)_A := -\sum_{r=0}^{p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{m}{4}(2r-1)(2r+1)} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\(\sum_{r=0}^{p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{m}{4}(2r-1)(2r+1)} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\( (II)_B := \sum_{r=0}^{p} e^{\pi im} q^{-\frac{(2mr+m)^2}{4m}} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\( = 2 e^{\pi im} \sum_{r=1}^{p} q^{-\frac{m}{4}(2r-1)^2} \theta_{m,m}^{(-)}(\tau, z) \)

\( (II)_C := \sum_{r=0}^{p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{m}{4}(2r-1)^2} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\( = 2 e^{\pi im} \sum_{r=1}^{p} q^{-\frac{m}{4}(2r-1)^2} \theta_{m,m}^{(-)}(\tau, z) \)

\( (III)_A := \sum_{r=0}^{p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{m}{4}(2r-1)^2} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\(\sum_{r=0}^{p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{m}{4}(2r-1)^2} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\( (III)_B := \sum_{r=0}^{p} e^{\pi im} q^{-\frac{(2mr+m)^2}{4m}} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\( = 2 e^{\pi im} \sum_{r=1}^{p} q^{-\frac{m}{4}(2r-1)^2} \theta_{m,m}^{(-)}(\tau, z) \)

\( (III)_C := \sum_{r=0}^{p} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{m}{4}(2r-1)^2} \left[ \theta_{m,m}^{(-)} \right] \left[ \theta_{-m,m}^{(-)} \right](\tau, z) \)

\( = 2 e^{\pi im} \sum_{r=1}^{p} q^{-\frac{m}{4}(2r-1)^2} \theta_{m,m}^{(-)}(\tau, z) \)

Computing \((I)_C\), \((II)_C\) and \((III)_C\) by putting \(k = 2m - k'\), we have

\( (I)_C = \sum_{r=1}^{p} \sum_{k' \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{m}{4}(2m(r-1)+k')^2} \left[ \theta_{k',m}^{(-)} \right] \left[ \theta_{-k',m}^{(-)} \right](\tau, z) \)
Then we have

\[(I) = \{ (I)_A + (I)_C \} + (I)_B\]

\[
= \sum_{r=1}^{p} \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} e^{\pi i k'} q^{-\frac{(2mr'+k')(2mr'-k')}{4m}} \left[ \theta_k^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[(II)_C = - \sum_{r=1}^{p} \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{(2mr-k')(2mr+1-k')}{4m}} \left[ \theta_k^{(-)} - \theta_{-k',m}^{(-)} \right](\tau, z)
\]

\[
= - \sum_{r=1}^{p} \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^{r'} q^{-\frac{(2mr'+k')(2mr'+1-k')}{4m}} \left[ \theta_k^{(-)} - \theta_{-k',m}^{(-)} \right](\tau, z)
\]

\[(III)_C = \sum_{r=0}^{p-1} \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} e^{\pi i k'} q^{-\frac{(2mr+1-k')^2}{4m}} \left[ \theta_k^{(-)} - \theta_{-k',m}^{(-)} \right](\tau, z)
\]

\[
= \sum_{r=0}^{p-1} \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} e^{\pi i k'} q^{-\frac{(2mr+1-k')^2}{4m}} \left[ \theta_k^{(-)} - \theta_{-k',m}^{(-)} \right](\tau, z)
\]

Then we have

\[(I) = \{ (I)_A + (I)_C \} + (I)_B\]

\[
= \left[ \sum_{r=1}^{p} \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} e^{\pi i k} q^{-\frac{(2mr-k')(2mr+k)}{4m}} \left[ \theta_k^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) + (I)_B \right)
\]

\[
= \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} e^{\pi i k} q^{-\frac{(2mr-k)(2mr+k)}{4m}} \left[ \theta_k^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
- \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} k^{-\frac{(2mr-k)^2}{4m}} \left[ \theta_k^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
+ 2 e^{\pi im} \sum_{r=1}^{p} q^{-\frac{m}{4}(2r-1)(2r+1)} \theta_{m,m}^{(-)}(\tau, z), \quad \text{proving } 1.
\]

\[(II) = \{ (II)_A + (II)_C \} + (II)_B\]

\[
= \left[ \sum_{r=1}^{p} \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{(2mr-k)(2mr+k)}{4m}} \left[ \theta_k^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) + (II)_B \right)
\]

\[
= - \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-1)^r q^{-\frac{(2mr-k)(2mr+k)}{4m}} \left[ \theta_k^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]
Proposition 3.4. Thus the proof of Lemma 3.3 is completed.

\[ (III) = \{ (III)_A + (III)_C \} + (III)_B \]

\[
\begin{align*}
+ \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} q^{\frac{k(2m-k)}{4m}} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) \\
- 2 \sum_{r=1}^{p} (-1)^r q^{\frac{r(2r-1)(2r+1)}{4m}} \theta_{m,m}^{(-)}(\tau, z), \quad \text{proving 2).}
\end{align*}
\]

(III) = \{ (III)_A + (III)_C \} + (III)_B

\[
\begin{align*}
\sum_{r=0}^{p-1} + \sum_{r \in \mathbb{Z}} \left[ \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}} \atop -p \leq r < 0} e^{\pi i k} q^{\frac{2mr+k}{4m}} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) + (III)_B \\
\sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}} \atop -p \leq r < p} e^{\pi i k} q^{\frac{2mr+k}{4m}} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) \\
+ 2 e^{\pi i m} \sum_{r=1}^{p} q^{\frac{r(2r-1)}{4m}} \theta_{m,m}^{(-)}(\tau, z), \quad \text{proving 3).}
\end{align*}
\]

Thus the proof of Lemma 3.3 is completed. 

Then, by Lemma 3.2 and Lemma 3.3, we obtain the following:

Proposition 3.4. Let \( m \in \frac{1}{2} \mathbb{N}_{\text{odd}} \) and \( p \in \mathbb{Z} \geq 0 \). Then

1) For \((z_1, z_2) = (\frac{\tau}{2} + \frac{\tau}{2} - \frac{1}{2}, \frac{\tau}{2} - \frac{\tau}{2} + \frac{1}{2})\),

\[
q^{\frac{m}{4}} \theta_{(2p+1)m,m+\frac{1}{2}}^{(-)}(\tau, 0) \left\{ \Phi_{[m, \frac{3}{2}]}(\tau, z_1, z_2, 0) \right. \\
+ \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}} \atop 0 < k < m} e^{\pi i k} q^{\frac{k(2m-k)}{4m}} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) \\
+ 2 e^{\pi i m} \sum_{r=1}^{p} q^{\frac{r(2r-1)}{4m}} \theta_{m,m}^{(-)}(\tau, z) \left. \right\}
\]

\[
= - i \eta(\tau)^3 \frac{\theta_{p+\frac{1}{2}, m+\frac{1}{2}}^{(-)}(\tau, z) - \theta_{-(p+\frac{1}{2}), m+\frac{1}{2}}^{(-)}(\tau, z)}{\theta_{0, \frac{1}{2}}^{(-)}(\tau, z)}
\]

\[
+ \left[ \sum_{r \in \mathbb{Z} \atop 0 \leq r < j} - \sum_{r \in \mathbb{Z} \atop j \leq r < 0} \right] \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}} \atop 0 < k < m} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})(j+m(2p+1)+2)-m(r+p+m+1)^2} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]

\[
+ \left[ \sum_{r \in \mathbb{Z} \atop 0 \leq r \leq j} - \sum_{r \in \mathbb{Z} \atop j < r < 0} \right] \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}} \atop 0 < k < m} (-1)^j e^{\pi i k} q^{(m+\frac{1}{2})(j+m(2p+1)+2)-m(r+p+m+1)^2} \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)
\]
\[ + \theta_{(2p+1)m,m+\frac{1}{2}}^{(-)}(\tau,0) \sum_{r \in \mathbb{Z}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \frac{1}{\mathbb{Z} \leq 0} e^{i\pi k q^{-m}(r+\frac{k-m}{2m})^2} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z) \]

\[ + e^{i\pi m} \theta_{(2p+1)m,m+\frac{1}{2}}^{(-)}(\tau,0) \sum_{r \in \mathbb{Z}} \sum_{-p \leq r \leq p} q^{-mr^2} \theta_{m,m}^{(-)}(\tau, z) \] (3.17)

2) For \((z_1, z_2) = (\frac{\tau}{2} + \frac{\pi}{2}, \frac{\tau}{2} - \frac{\pi}{2})\),

\[ q^{-\frac{m}{4}} \theta_{(2p+1)m,m+\frac{1}{2}}^{(-)}(\tau,0) \left\{ \Phi_{m}^{(-)}[m,\frac{1}{2}](\tau, z_1, z_2, 0) \right\} - \frac{\pi \theta_{m,m}^{(-)}(\tau, z)}{0, \frac{1}{2}} \]

\[ \theta_{p,m+\frac{1}{4}}^{(-)}(\tau, z) - \theta_{-(p+\frac{1}{4}),m+\frac{1}{4}}^{(-)}(\tau, z) \]

\[ + (-1)^p \left[ \sum_{r \in \mathbb{Z}} \sum_{-\frac{1}{2} \leq r \leq \frac{1}{2}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \frac{1}{\mathbb{Z} \leq 0} \right] \sum_{-\frac{1}{2} \leq r \leq \frac{1}{2}} \sum_{-\frac{1}{2} \leq r \leq \frac{1}{2}} (\tau)^r q^{(m+\frac{1}{2})(j+\frac{m(2p+1)}{2m+1})^2 - m(r+p+\frac{m+k}{2m})^2} \times [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z) \]

\[ - \frac{\pi \theta_{m,m}^{(-)}(\tau, z)}{0, \frac{1}{2}} \]

3) For \((z_1, z_2) = (\frac{\tau}{2} - \frac{1}{4}, \frac{\tau}{2} + \frac{1}{4})\),

\[ \theta_{2pm,m+\frac{1}{2}}^{(-)}(\tau,0) \Phi_{m}^{(-)}[m,\frac{1}{2}](\tau, z_1, z_2, 0) \]

\[ = -i \eta(\tau)^3 \frac{\theta_{p,m+\frac{1}{4}}^{(-)}(\tau, z) - \theta_{-p,m+\frac{1}{4}}^{(-)}(\tau, z)}{\theta_{p+\frac{1}{2},m+\frac{1}{2}}(\tau, z)} \]

\[ + \left[ \sum_{r \in \mathbb{Z}} \sum_{-\frac{1}{2} \leq r \leq \frac{1}{2}} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \frac{1}{\mathbb{Z} \leq 0} \right] \sum_{-\frac{1}{2} \leq r \leq \frac{1}{2}} \sum_{-\frac{1}{2} \leq r \leq \frac{1}{2}} (\tau)^r e^{i\pi k q^{(m+\frac{1}{2})(j+\frac{m(p+1)}{2m+1})^2 - m(r+p+\frac{m+k}{2m})^2} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z)
In order to compute \( R_{j,m}(\tau, a\tau + b) \), we consider functions \( P_{j,m}(\tau, w) \) and \( Q_{j,m}(\tau, w) \) defined by

\[
P_{j,m}(\tau, w) := \sum_{k \in \mathbb{Z}} (-1)^k E \left( (j + 2mk - 2m) \frac{\text{Im}(w)}{\text{Im}(\tau)} \right) \sqrt{\frac{\text{Im}(\tau)}{m}} e^{-\frac{\pi in}{2m}(j+2mk)^2\tau + 2\pi i(j+2mk)w} \tag{4.1a}
\]

\[
Q_{j,m}(\tau, w) := \sum_{n \equiv j \mod 2m} (-1)^{n-2j} \frac{1}{2m} \text{sgn}(n - \frac{1}{2} - j + 2m) e^{-\frac{\pi in^2}{2m} \tau + 2\pi imw} \tag{4.1b}
\]

so that

\[
R_{j,m}(\tau, w) = P_{j,m}(\tau, w) + Q_{j,m}(\tau, w) \tag{4.1c}
\]

Then it is easy to see the following:

\[
P_{j+2m,m}(\tau, w) = -P_{j,m}(\tau, w) \tag{4.2a}
\]

\[
Q_{j+2m,m}(\tau, w) = -Q_{j,m}(\tau, w) + 2e^{-\frac{\pi in^2}{2m} \tau + 2\pi ijw} \tag{4.2b}
\]

It is also easy to see the following formulas hold for \( a, b \in \mathbb{R} \):

\[
P_{j,m}(\tau, a\tau + b) = e^{2\pi ijb} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi iimbk} e^{(j+2mk)(k-a)} E \left( (j + 2m(k-a)) \frac{\text{Im}(\tau)}{m} \right) e^{-\frac{\pi in^2}{2m}(j+2mk)(j-2m(2a-k))} \tag{4.3a}
\]

\[
Q_{j,m}(\tau, a\tau + b) = e^{2\pi ijb} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi iimbk} \text{sgn}(2m(k+1) - \frac{1}{2}) e^{-\frac{\pi in^2}{2m}(j+2mk)(j-2m(2a-k))} \tag{4.3b}
\]

We claim the following:
Lemma 4.1. Let $m \in \frac{1}{2}\mathbb{N}$, $j \in \frac{1}{2}\mathbb{Z}$, $a \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ and $b \in \mathbb{R}$ such that $4mb \in \mathbb{Z}$. Then

1) \[ P_{j,m}^{(-)}(\tau, a\tau + b) - (-1)^{2a}e^{4\pi ijb}e^{8\pi i m a b}P_{2m-j,m}^{(-)}(\tau, a\tau + b) = 0 \]

2) \[ Q_{j,m}^{(-)}(\tau, a\tau + b) - (-1)^{2a}e^{4\pi ijb}e^{8\pi i m a b}Q_{2m-j,m}^{(-)}(\tau, a\tau + b) \]

\[ = 2(-1)^{2a}e^{2\pi ijb}e^{8\pi i m a b} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} \]

Proof. 1) Letting $j \to -j$ and $k = 2a - k'$ in (4.3a), we have

\[ P_{-j,m}^{(-)}(\tau, a\tau + b) = e^{-2\pi ijb}e^{8\pi i m a b} \]

\[ \times \sum_{k' \in \mathbb{Z}} (-1)^{2a-k'}e^{-4\pi i m b k'}E \left( (-j + 2m(a-k')) \sqrt{\frac{\text{Im}(\tau)}{m}} \right) q^{-\frac{1}{4m}(-j+2m(2a-k'))(-j-2mk')} \]

\[ = -(-1)^{2a}e^{-2\pi ijb}e^{8\pi i m a b} \]

\[ \times \sum_{k' \in \mathbb{Z}} (-1)^{k'}e^{4\pi i m b k'}E \left( (j + 2m(k'-a)) \sqrt{\frac{\text{Im}(\tau)}{m}} \right) q^{-\frac{1}{4m}(j-2m(2a-k'))(j+2mk')} \]

\[ = -(-1)^{2a}e^{-4\pi ijb}e^{8\pi i m a b}P_{j,m}^{(-)}(\tau, a\tau + b) \]

Then we have

\[ P_{j,m}^{(-)}(\tau, a\tau + b) = -(-1)^{2a}e^{4\pi ijb}e^{8\pi i m a b}P_{-j,m}^{(-)}(\tau, a\tau + b) \]

\[ = (-1)^{2a}e^{4\pi ijb}e^{8\pi i m a b}P_{2m-j,m}^{(-)}(\tau, a\tau + b) \]

by (4.2a), proving 1).

2) Decomposing the sum $\sum_{k \in \mathbb{Z}}$ into two parts in the RHS of (4.3b), we have

\[ Q_{j,m}^{(-)}(\tau, a\tau + b) = (I)_{j} + (II)_{j} \]

where

\[ (I)_{j} := e^{2\pi ijb} \sum_{k \in \mathbb{Z}_{\geq 0}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2m(2a-k))} \]

\[ (II)_{j} := e^{2\pi ijb} \sum_{k \in \mathbb{Z}_{<0}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2m(2a-k))} \]
First we compute \((II)_j\) by putting \(k = 2a - k'\):

\[
(II)_j = - e^{2\pi i j b} \sum_{k' \in \mathbb{Z} \atop k' > 2a} (-1)^{k' + 2a} \left( e^{4\pi i m b (2a - k')} q^{- \frac{1}{4m} (j + 2m (2a - k')) (j - 2mk')} e^{8\pi i m b k'} \right)
\]

\[
= - (-1)^{2a} e^{2\pi i j b} e^{8\pi i m a b} \sum_{k' \in \mathbb{Z} \atop 0 \leq k' \leq 2a} (-1)^{k'} e^{4\pi i m b k'} q^{- \frac{1}{4m} (j - 2m (2a - k')) (j - 2mk')} (I)_{-j}
\]

Then, by \((I)_{-j}\), we have

\[
(II)_{-j} = \text{letting } j \to -j \text{ in } (4.4a)
\]

\[
(II)_{-j} = - (-1)^{2a} e^{-4\pi i j b} e^{8\pi i m a b} \times (I)_{j}
\]

\[
+ (-1)^{2a} e^{-2\pi i j b} e^{8\pi i m a b} \sum_{k \in \mathbb{Z} \atop 0 \leq k \leq 2a} (-1)^k e^{4\pi i m b k} q^{- \frac{1}{4m} (j + 2m (2a - k)) (j - 2mk)}
\]

We note that the formula \((4.4a)\) is rewritten as follows:

\[
(I)_{-j} = - (-1)^{2a} e^{-4\pi i j b} e^{-8\pi i m a b} \times (II)_j
\]

\[
+ e^{-2\pi i j b} \sum_{k \in \mathbb{Z} \atop 0 \leq k \leq 2a} (-1)^k e^{4\pi i m b k} q^{- \frac{1}{4m} (j + 2m (2a - k)) (j - 2mk)}
\]

Then, by \((4.4b)\) and \((4.4c)\), we have

\[
Q_{-j,m}^{(-)}(\tau, a\tau + b) = (I)_{-j} + (II)_{-j}
\]

\[
= - (-1)^{2a} e^{-4\pi i j b} e^{-8\pi i m a b} \times \left\{ (I)_j + (II)_j \right\}
\]

\[
Q_{j,m}^{(-)}(\tau, a\tau + b)
\]
\[ + e^{-2\pi ij} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} \]
\[ + (-1)^{2a} e^{-2\pi ij} e^{8\pi i m a b} \sum_{0 \leq k \leq 2a} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j+2mk)} \]

\[ (4.5\text{a}) \]

\[ \text{namically,} \]
\[ Q_{j,m}^{(-)}(\tau, a\tau + b) = -(-1)^{2a} e^{4\pi i j} e^{8\pi i m a b} Q_{j,m}^{(-)}(\tau, a\tau + b) \]
\[ + (-1)^{2a} e^{2\pi i j} e^{8\pi i m a b} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} \]
\[ + e^{2\pi i j} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j+2mk)} \]
\[ (4.5\text{b}) \]
\[ = -(-1)^{2a} e^{4\pi i j} e^{8\pi i m a b} Q_{j,m}^{(-)}(\tau, a\tau + b) \]
\[ + (-1)^{2a} e^{2\pi i j} e^{8\pi i m a b} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} \]
\[ + e^{2\pi i j} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j+2mk)} \]
\[ + 2(-1)^{2a} e^{2\pi i j} e^{8\pi i m a b} q^{-\frac{1}{4m}j(4ma)} \]

\[ (4.5\text{c}) \]

Note that, by (4.2b), we have
\[ Q_{j,m}^{(-)}(\tau, a\tau + b) = -Q_{2m-j,m}^{(-)}(\tau, a\tau + b) + 2e^{-\frac{\pi i}{2m}j^2 - 2\pi ij(a\tau + b)} \]
\[ = -Q_{2m-j,m}^{(-)}(\tau, a\tau + b) + 2e^{-2\pi ij} q^{-\frac{1}{4m}j(4ma)} \]

Substituting this equation into (4.5c), we have
\[ Q_{j,m}^{(-)}(\tau, a\tau + b) = (-1)^{2a} e^{4\pi i j} e^{8\pi i m a b} Q_{2m-j,m}^{(-)}(\tau, a\tau + b) \]
\[ + (-1)^{2a} e^{2\pi i j} e^{8\pi i m a b} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} \]
\[ (\text{III}) \]
\[ + e^{2\pi i j} \sum_{k \in \mathbb{Z}} (-1)^k e^{4\pi i m b k} q^{-\frac{1}{4m}(j+2m(2a-k))(j+2mk)} \]
\[ (\text{IV}) \]
We compute (IV) by putting \( k = 2a - k' \) as follows:

\[
(IV) = e^{2\pi i jb} \sum_{k' \in \mathbb{Z}} (-1)^{2a-k'} e^{4\pi imb(2a-k')} q^{-\frac{1}{4m}(j+2m(2a-k'))(j-2mk')}
\]

\[
1 \leq k' \leq 2a
\]

\[
= (-1)^{2a} e^{2\pi i jb} e^{8\pi imab} \sum_{k' \in \mathbb{Z}} (-1)^{k'} e^{-4\pi imbk'} q^{-\frac{1}{4m}(j+2m(2a-k'))(j-2mk')}
\]

\[
= (III)
\]

Then we have

\[
Q^{(-)}_{j,m}(\tau, a\tau + b) = (-1)^{2a} e^{4\pi isb} e^{8\pi imab} Q^{(-)}_{2m-j,m}(\tau, a\tau + b)
\]

\[
+ 2 (-1)^{2a} e^{2\pi i jb} e^{8\pi imab} \sum_{k \in \mathbb{Z}} (-1)^{k} e^{4\pi imbk} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)}
\]

Thus the proof of 2) is completed. \(\square\)

From this lemma, we obtain the following:

**Proposition 4.1.** Let \( m \in \frac{1}{2} \mathbb{N}, s, b \in \frac{1}{2} \mathbb{Z}, a \in \frac{1}{2} \mathbb{Z}_{\geq 0} \) and \( j \in s + \mathbb{Z} \) satisfying the condition

\[
(-1)^{2a} e^{4\pi isb} e^{8\pi imab} = -1 \quad (4.6)
\]

Then

\[
R^{(-)}_{j,m}(\tau, a\tau + b) + R^{(-)}_{2m-j,m}(\tau, a\tau + b)
\]

\[
= -2 e^{2\pi i jb} e^{4\pi isb} \sum_{k \in \mathbb{Z}} (-1)^{k} e^{4\pi imbk} q^{-\frac{1}{4m}(j+2m(2a-k))(j-2mk)} \quad (4.7)
\]

**Proof.** Under this setting, we have \( e^{4\pi isb} = e^{4\pi jb} \in \{ \pm 1 \} \). So the condition (4.6) means that

\[
(-1)^{2a} e^{8\pi imab} = - e^{4\pi isb}
\]

Then we have

\[
(-1)^{2a} e^{2\pi i jb} e^{8\pi imab} = - e^{2\pi i jb} e^{4\pi isb} \quad (4.8)
\]

and the formula (4.7) follows from (4.1c) and Lemma 4.1 and (4.8). \(\square\)

We note that this Proposition 4.1 gives the following:

**Note 4.1.** In the case when \( m \in \frac{1}{2} \mathbb{N}_{\text{odd}} \) and \( s = \frac{1}{2} \),

1) the condition (4.6) is satisfied if \( (a, b) \) is \((\frac{1}{2}, -\frac{1}{2})\) or \((\frac{1}{2}, 0)\) or \((0, -\frac{1}{2})\).

2) \( R^{(-)}_{j,m}(\tau, a\tau + b) + R^{(-)}_{2m-j,m}(\tau, a\tau + b) \) is as follows for \( j \in \frac{1}{2} + \mathbb{Z} \):

   (i) if \( (a, b) = (\frac{1}{2}, -\frac{1}{2}) \), \( R^{(-)}_{j,m}(\tau, a\tau + b) + R^{(-)}_{2m-j,m}(\tau, a\tau + b) = -2 e^{\pi i j} q^{-\frac{1}{4} j(j-2m)} \)

   (ii) if \( (a, b) = (\frac{1}{2}, 0) \), \( R^{(-)}_{j,m}(\tau, a\tau + b) + R^{(-)}_{2m-j,m}(\tau, a\tau + b) = 2 q^{-\frac{1}{4} j(j-2m)} \)

   (iii) if \( (a, b) = (0, -\frac{1}{2}) \), \( R^{(-)}_{j,m}(\tau, a\tau + b) + R^{(-)}_{2m-j,m}(\tau, a\tau + b) = 0 \)
5 Modified function $\widetilde{\Phi}^{(\pm)}[m, \frac{1}{2}]$

In this section we compute $\Phi^{(\pm)}[m, \frac{1}{2}] (\tau, \frac{\pi}{2} + a \tau + b, \frac{\pi}{2} - a \tau - b, 0)$ in the case when $m \in \frac{1}{2} \mathbb{N}_{\text{odd}}$ and $(a, b) = (\frac{1}{2}, -\frac{1}{2}), (\frac{1}{2}, 0)$ or $(0, -\frac{1}{2})$.

**Proposition 5.1.** Let $m \in \frac{1}{2} \mathbb{N}_{\text{odd}}$. Then

1) $\Phi^{(\pm)}[m, \frac{1}{2}] (\tau, \frac{\pi}{2} + \frac{\pi}{2} - \frac{1}{2}, \frac{\pi}{2} - \frac{1}{2}, 0)$

\[
\sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} e^{\pi ik} q^{\frac{1}{4m} k (k-2m)} [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z) + e^{\pi im} q^{\frac{1}{4m} k (k-2m)} [\theta^{(2)}_{k,m} - \theta^{(2)}_{-k,m}] (\tau, z)
\]

2) $\Phi^{(\pm)}[m, \frac{1}{2}] (\tau, \frac{\pi}{2} + \frac{\pi}{2} - \frac{1}{2}, \frac{\pi}{2} - \frac{1}{2}, 0)$

\[
- \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} q^{\frac{1}{4m} k (k-2m)} [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z) - q^{\frac{1}{4m}} \theta^{(-)}_{m,m} (\tau, z)
\]

3) $\Phi^{(\pm)}[m, \frac{1}{2}] (\tau, \frac{\pi}{2} - \frac{1}{2}, \frac{\pi}{2} + \frac{1}{2}, 0) = 0$

**Proof.** When $z_1 - z_2 = 2(a \tau + b)$ and $z_1 + z_2 = z$, the formula (1.3a) gives

\[
\Phi^{(-)}[m, \frac{1}{2}] (\tau, z_1, z_2, 0) = - \frac{1}{2} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} R^{(-)}_{k,m} (\tau, a \tau + b) [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)
\]

\[
= - \frac{1}{2} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} R^{(-)}_{2m-k,m} (\tau, a \tau + b) [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)
\]

\[
= - \frac{1}{4} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} \left[ R^{(-)}_{k,m} (\tau, a \tau + b) + R^{(-)}_{2m-k,m} (\tau, a \tau + b) \right] [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)
\]

We compute this equation (5.1c) in each case by using Note 4.1 as follows:

**In the case** $(z_1, z_2) = (\frac{\pi}{2} + \frac{\pi}{2} - \frac{1}{2}, \frac{\pi}{2} - \frac{1}{2} + \frac{1}{2})$ i.e., $(a, b) = (\frac{1}{2}, -\frac{1}{2})$,

\[
\Phi^{(\pm)}[m, \frac{1}{2}] (\tau, z_1, z_2, 0) = - \frac{1}{4} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} (-2) e^{\pi ik} q^{\frac{k(2m-k)}{4m}} [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)
\]

\[
= \frac{1}{2} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}} e^{\pi ik} q^{\frac{k(2m-k)}{4m}} [\theta^{(-)}_{k,m} - \theta^{(-)}_{-k,m}] (\tau, z)
\]
In the case \((z_1, z_2) = (\frac{\pi}{2} + \frac{\pi}{2}, \frac{\pi}{2} - \frac{\pi}{2})\) i.e., \((a, b) = (\frac{1}{2}, 0)\),
\[
\Phi_{add}^{(\pm)[m, \frac{1}{2}]}(\tau, z_1, z_2, 0) = -\frac{1}{4} \sum_{k \in \frac{1}{2} \mathbb{Z}_{odd}}^{0 < k < 2m} q^{k(2m-k)\frac{1}{4m}} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}] (\tau, z)
\]

\[
= -\frac{1}{2} \sum_{k \in \frac{1}{2} \mathbb{Z}_{odd}}^{0 < k < 2m} q^{k(2m-k)\frac{1}{4m}} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}] (\tau, z)
\]

In the case \((z_1, z_2) = (\frac{\pi}{2} - \frac{1}{2}, \frac{\pi}{2} + \frac{1}{2})\) i.e., \((a, b) = (0, -\frac{1}{2})\),
\[
\Phi_{add}^{(\pm)[m, \frac{1}{2}]}(\tau, z_1, z_2, 0) = -\frac{1}{4} \sum_{k \in \frac{1}{2} \mathbb{Z}_{odd}}^{0 < k < 2m} \sum_{m < k < 2m} \left\{ R_{k,m}^{(-)}(\tau, -\frac{1}{2}) + R_{2m-k,m}^{(-)}(\tau, -\frac{1}{2}) \right\} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}] (\tau, z) = 0
\]

Thus the proof of Proposition 5.1 is completed. \(\square\)

Then, by Proposition 3.4 and Proposition 5.1 we obtain the expression for the modified function \(\tilde{\Phi}^{(-)[m, \frac{1}{2}]}\) as follows:

**Theorem 5.1.** Let \(m \in \frac{1}{2} \mathbb{N}_{odd}\) and \(p \in \mathbb{Z}_{\geq 0}\). Then

1) \(q^{-\frac{\pi}{4m}} \theta_{(2p+1)m+m+1}^{(-)}(\tau, 0) \tilde{\Phi}^{(-)[m, \frac{1}{2}]}(\tau, \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{2}, \frac{\pi}{2} - \frac{\pi}{2} + \frac{\pi}{2}, 0)\)

\[
= - \eta(\tau)^3 . \frac{\theta_{p+1, m+1}^{(-)}(\tau, z) - \theta_{-p+1, m+1}^{(-)}(\tau, z)}{\theta_{0, \frac{1}{2}}^{(-)}(\tau, z)}
\]
\[
\sum_{\substack{r \in \mathbb{Z} \\
0 \leq r < j}} + \sum_{\substack{q \in \mathbb{Z} \\
0 < k < m}} (-1)^j e^{\pi i k q (m + \frac{j}{2}) (j + \frac{m(2p+1)}{2m+1})^2 - m(r + p + \frac{m+k}{2m})^2} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z)
\]

\[
\sum_{\substack{r \in \mathbb{Z} \\
0 \leq j < r}} + \sum_{\substack{q \in \mathbb{Z} \\
0 < k < m}} (-1)^j e^{\pi i k q (m + \frac{j}{2}) (j + \frac{m(2p+1)}{2m+1})^2 - m(r + p + \frac{m+k}{2m})^2} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z)
\]

\[
\theta^{(-)}_{(2p+1)m,m+\frac{1}{2}}(\tau, 0) \sum_{\substack{r \in \mathbb{Z} \\
-p \leq r \leq p}} q^{-mr^2} \theta_{m,m}^{(-)}(\tau, z)
\]

2) \[
q^{-\frac{m}{2}} \theta_{(2p+1)m,m+\frac{1}{2}}^{(-)}(\tau, 0) \bar{\Phi}^{(-)}(m, \frac{1}{2})(\tau, \frac{z}{2} + \frac{r}{2}, \frac{z}{2} - \frac{r}{2}, 0)
\]

\[= \eta(\tau)^3 \cdot \frac{\theta_{p+m+\frac{1}{2}}^{(-)}(\tau, z) - \theta_{-(p+\frac{1}{2}),m+\frac{1}{2}}^{(-)}(\tau, z)}{\theta_{0,\frac{1}{2}}^{(-)}(\tau, z)}
\]

\[-(1)^p \left[ \sum_{\substack{r \in \mathbb{Z} \\
0 \leq r < j}} - \sum_{\substack{q \in \mathbb{Z} \\
0 < k < m}} \sum_{\substack{k \in \mathbb{Z}_{\text{odd}}}} (-1)^r q^{m \frac{1}{2} (j + \frac{m(2p+1)}{2m+1})^2 - m(r + p + \frac{m+k}{2m})^2} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z)
\]

\[-(1)^p \left[ \sum_{\substack{r \in \mathbb{Z} \\
0 \leq j < r}} - \sum_{\substack{q \in \mathbb{Z} \\
0 < k < m}} \sum_{\substack{k \in \mathbb{Z}_{\text{odd}}}} (-1)^r q^{m \frac{1}{2} (j + \frac{m(2p+1)}{2m+1})^2 - m(r + p + \frac{m+k}{2m})^2} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z)
\]

\[= - \theta_{(2p+1)m,m+\frac{1}{2}}^{(-)}(\tau, 0) \sum_{\substack{r \in \mathbb{Z} \\
-p \leq r \leq p}} q^{-mr^2} \theta_{m,m}^{(-)}(\tau, z)
\]

3) \[
\theta_{2pm,m+\frac{1}{2}}^{(-)}(\tau, 0) \bar{\Phi}^{(-)}(m, \frac{1}{2})(\tau, \frac{z}{2} - \frac{1}{2}, \frac{z}{2} + \frac{1}{2}, 0)
\]

\[= - i \eta(\tau)^3 \cdot \frac{\theta_{p+m+\frac{1}{2}}^{(-)}(\tau, z) - \theta_{-(p+m+\frac{1}{2}),\frac{1}{2}}^{(-)}(\tau, z)}{\theta_{\frac{1}{2},\frac{1}{2}}^{(-)}(\tau, z)}
\]

\[+ \left[ \sum_{\substack{r \in \mathbb{Z} \\
0 \leq r < j}} - \sum_{\substack{q \in \mathbb{Z} \\
0 < k < m}} \sum_{\substack{k \in \mathbb{Z}_{\text{odd}}}} (-1)^j e^{\pi i k q (m + \frac{j}{2}) (j + \frac{m(2p+1)}{2m+1})^2 - m(r + p + \frac{m+k}{2m})^2} [\theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)}](\tau, z)
\]
6 Functions $\tilde{\psi}^{(i)[m]}(\tau, z)$

For $m \in \frac{1}{2} \mathbb{N}_{\text{odd}}$ and $i \in \{1, 2, 3\}$, we consider functions $\tilde{\psi}^{(i)[m]}(\tau, z)$ defined by

$$\tilde{\psi}^{(i)[m]}(\tau, z) := \Phi^{(-)}[m, \frac{1}{2}]\left(\tau, \frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2}, 0\right)$$

$$= -\Phi^{(-)}[m, \frac{1}{2}]\left(\tau, \frac{z}{2} + \frac{\tau}{2} + \frac{1}{2}, \frac{z}{2} - \frac{\tau}{2} - \frac{1}{2}, 0\right)$$

$$\tilde{\psi}^{(2)[m]}(\tau, z) := \Phi^{(-)}[m, \frac{1}{2}]\left(\tau, \frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, 0\right)$$

$$\tilde{\psi}^{(3)[m]}(\tau, z) := \Phi^{(-)}[m, \frac{1}{2}]\left(\tau, \frac{z}{2} - \frac{\tau}{2} + \frac{1}{2}, \frac{z}{2} + \frac{\tau}{2}, 0\right)$$

$$= -\Phi^{(-)}[m, \frac{1}{2}]\left(\tau, \frac{z}{2} + \frac{1}{2}, \frac{z}{2} - \frac{1}{2}, 0\right)$$

These functions $\tilde{\psi}^{(i)[m]}(\tau, z)$ satisfy the following modular transformation properties:

**Lemma 6.1.** Let $m \in \frac{1}{2} \mathbb{N}_{\text{odd}}$, then

1) (i) $\tilde{\psi}^{(1)[m]}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -\tau e^{\pi i m} e^{\frac{\pi i m}{2\tau}} e^{-\frac{\pi i m}{2\tau}} q^{-\frac{m}{4} - \frac{m}{4}} \tilde{\psi}^{(1)[m]}(\tau, z)$

(ii) $\tilde{\psi}^{(2)[m]}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \tau e^{\frac{\pi i m}{2\tau}} e^{-\frac{\pi i m}{2\tau}} \tilde{\psi}^{(3)[m]}(\tau, z)$

(iii) $\tilde{\psi}^{(3)[m]}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -\tau e^\frac{\pi i m}{2\tau} q^{-\frac{m}{4}} \tilde{\psi}^{(2)[m]}(\tau, z)$

2) (i) $\tilde{\psi}^{(1)[m]}(\tau + 1, z) = \tilde{\psi}^{(2)[m]}(\tau, z)$

(ii) $\tilde{\psi}^{(2)[m]}(\tau + 1, z) = -\tilde{\psi}^{(1)[m]}(\tau, z)$

(iii) $\tilde{\psi}^{(3)[m]}(\tau + 1, z) = \tilde{\psi}^{(3)[m]}(\tau, z)$

**Proof.** These are obtained easily from Lemma [2.1] as follows.

1) (i) $\tilde{\psi}^{(1)[m]}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = -\Phi^{(-)}[m, \frac{1}{2}]\left(-\frac{1}{\tau}, \frac{z}{2\tau} - \frac{1}{2}, \frac{z}{2\tau} + \frac{1}{2}, 0\right)$

$$= -\Phi^{(-)}[m, \frac{1}{2}]\left(-\frac{1}{\tau}, \frac{z}{2} - \frac{1}{2}, \frac{z}{2} + \frac{1}{2}, 0\right)$$
Thus the proof of Lemma 6.1 is completed.

\[ (ii) \quad \overline{\psi}^{(2)}[m]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \Phi(-)[\frac{m}{2}](\tau + 1, \frac{z}{2} + \frac{\tau + 1}{2} - \frac{1}{2}, \frac{z}{2} - \frac{\tau + 1}{2} + \frac{1}{2}, 0) \]

\[ = \Phi(-)[\frac{m}{2}](\tau + 1, \frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, 0) \]

\[ = \Phi(-)[\frac{m}{2}](\tau, \frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, 0) = \overline{\psi}^{(2)}[m](\tau, z), \quad \text{proving (i).} \]

\[ (iii) \quad \overline{\psi}^{(3)}[m]\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = \Phi(-)[\frac{m}{2}](\tau + 1, \frac{z}{2} + \frac{\tau}{2} + \frac{1}{2}, \frac{z}{2} - \frac{\tau}{2} - \frac{1}{2}, 0) \]

\[ = \Phi(-)[\frac{m}{2}](\tau + 1, \frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, 0) \]

\[ = \Phi(-)[\frac{m}{2}](\tau, \frac{z}{2} + \frac{\tau}{2}, \frac{z}{2} - \frac{\tau}{2}, 0) = \overline{\psi}^{(3)}[m](\tau, z), \quad \text{proving (ii).} \]

Thus the proof of Lemma 6.1 is completed.
7 Functions $\Xi^{(i)[m,p]}(\tau, z)$ and $\Upsilon^{(i)[m,p]}(\tau, z)$

For $m \in \frac{1}{2}N_{\text{odd}}$ and $p \in \mathbb{Z}$ such that $0 \leq p \leq 2m$ and $i \in \{1, 2, 3\}$, we define functions $\Xi^{(i)[m,p]}(\tau, z)$ and $\Upsilon^{(i)[m,p]}(\tau, z)$ as follows:

\[
\begin{align*}
\Xi^{(1)[m,p]}(\tau, z) & := q^{-\frac{m}{2}} \theta_{(2p+1)m,m+\frac{1}{2}}(\tau, 0) \cdot \bar{\psi}^{(1)[m]}(\tau, z) \\
\Xi^{(2)[m,p]}(\tau, z) & := q^{-\frac{m}{2}} \theta_{(2p+1)m,m+\frac{1}{2}}(\tau, 0) \cdot \bar{\psi}^{(2)[m]}(\tau, z) \\
\Xi^{(3)[m,p]}(\tau, z) & := \theta_{2pm,m+\frac{1}{2}}(\tau, 0) \cdot \bar{\psi}^{(3)[m]}(\tau, z)
\end{align*}
\]

and

\[
\begin{align*}
\Upsilon^{(1)[m,p]}(\tau, z) & := -i \eta(\tau)^3 \cdot \frac{\theta_{p+\frac{1}{2},m+\frac{1}{2}}(\tau, z) - \theta_{-p+\frac{1}{2},m+\frac{1}{2}}(\tau, z)}{\theta_{0,\frac{1}{2}}(\tau, z)} \\
\Upsilon^{(2)[m,p]}(\tau, z) & := \eta(\tau)^3 \cdot \frac{\theta_{p+\frac{1}{2},m+\frac{1}{2}}(\tau, z) - \theta_{-p+\frac{1}{2},m+\frac{1}{2}}(\tau, z)}{\theta_{0,\frac{1}{2}}(\tau, z)} \\
\Upsilon^{(3)[m,p]}(\tau, z) & := -i \eta(\tau)^3 \cdot \frac{\theta_{p,m+\frac{1}{2}}(\tau, z) - \theta_{-p,m+\frac{1}{2}}(\tau, z)}{\theta_{0,\frac{1}{2}}(\tau, z)}
\end{align*}
\]

To compute the $S$-transformation of $\Xi^{(i)[m,p]}(\tau, z)$, we use the following formulas which are obtained easily from Lemma 13

**Note 7.1.** Let $m \in \frac{1}{2}N_{\text{odd}}$ and $p \in \mathbb{Z}_{\geq 0}$. Then

1. $\theta_{(2p+1)m,m+\frac{1}{2}}(-\frac{1}{\tau}, 0) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{4\pi i m^{2}}{2m+1}\ell(2p+1)} \theta_{2\ell m,m+\frac{1}{2}}(\tau, 0)$

2. $\theta_{(2p+1)m,m+\frac{1}{2}}(-\frac{1}{\tau}, 0) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{4\pi i m^{2}}{2m+1}\ell(2\ell+1)} \theta_{(2\ell+1)m,m+\frac{1}{2}}(\tau, 0)$

3. $\theta_{2pm,m+\frac{1}{2}}(-\frac{1}{\tau}, 0) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{4\pi i m^{2}}{2m+1}\ell(2\ell+1)} \theta_{(2\ell+1)m,m+\frac{1}{2}}(\tau, 0)$

The $S$-transformation properties of these functions $\Xi^{(i)[m,p]}$ and $\Upsilon^{(i)[m,p]}$ are given by the following formulas:

**Lemma 7.1.** Let $m \in \frac{1}{2}N_{\text{odd}}$ and $p \in \mathbb{Z}_{\geq 0}$. Then

1. (i) $\Xi^{(1)[m,p]}(-\frac{1}{\tau}, \frac{z}{\tau}) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}} e^{\frac{i \pi m^{2}}{2p+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p+1)(2\ell+1)} \Xi^{(1)[m,\ell]}(\tau, z)$

(ii) $\Xi^{(2)[m,p]}(-\frac{1}{\tau}, \frac{z}{\tau}) = \frac{(-i\tau)^{\frac{1}{2}}}{\sqrt{2m+1}} e^{\frac{i \pi m^{2}}{2p+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p+1)(2\ell+1)} \Xi^{(1)[m,\ell]}(\tau, z)$
2) (i) $\Xi^{(2)}[m,p](- \frac{1}{\tau}, \frac{z}{\tau}) = i \left( -i\tau \right)^{\frac{3}{2}} \frac{\xi_{m}^{2} e^{2m}}{\sqrt{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i}{2m+1} (2\ell+1) \ell} \Xi^{(3)}[m,\ell] (\tau, z)$

(ii) $\Upsilon^{(2)}[m,p](- \frac{1}{\tau}, \frac{z}{\tau}) = i \left( -i\tau \right)^{\frac{3}{2}} \frac{\xi_{m}^{2} e^{2m}}{\sqrt{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i}{2m+1} (2\ell+1) \ell} \Upsilon^{(3)}[m,\ell] (\tau, z)$

3) (i) $\Xi^{(3)}[m,p](- \frac{1}{\tau}, \frac{z}{\tau}) = -i \frac{\xi_{m}^{2} e^{2m}}{\sqrt{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i}{2m+1} (2\ell+1) \ell} \Xi^{(2)}[m,\ell] (\tau, z)$

(ii) $\Upsilon^{(3)}[m,p](- \frac{1}{\tau}, \frac{z}{\tau}) = -i \frac{\xi_{m}^{2} e^{2m}}{\sqrt{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i}{2m+1} (2\ell+1) \ell} \Upsilon^{(2)}[m,\ell] (\tau, z)$

Proof. 1) (i) $\Xi^{(1)}[m,p](- \frac{1}{\tau}, \frac{z}{\tau}) = e^{-2\pi i \frac{m}{2} \left( \frac{1}{\tau} - \frac{1}{\tau} z \right)} \theta(-)_{2(p+1),m+\frac{1}{2}} \left( \frac{1}{\tau}, 0 \right) \Xi^{(1)}[m,\frac{1}{2}](\tau, z)$

(ii) $\Upsilon^{(1)}[m,p](- \frac{1}{\tau}, \frac{z}{\tau}) = -i \eta \left( -\frac{1}{\tau} \right)^{3} \frac{\xi_{m}^{2} e^{2m}}{\sqrt{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i}{2m+1} (2\ell+1) \ell} \Upsilon^{(1)}[m,\ell] (\tau, z)$
\[ \Xi^{(2)[m,p]}(\frac{-1}{\tau}, \frac{z}{\tau}) = e^{-2\pi i m (\frac{-1}{\tau})} \frac{\psi(2p+1) m, m + \frac{1}{2} \left( -\frac{1}{\tau}, 0 \right)}{\tau} \Xi^{(2)[m,\frac{1}{2}]}(\frac{-1}{\tau}, \frac{z}{\tau}) \]

\[ = e^{\frac{\pi i m^2}{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i m^2}{2m+1} (2(p+1) - 2\ell)} \left( \frac{-1}{\tau}, 0 \right) \Xi^{(3)[m,\frac{1}{2}]}(\frac{-1}{\tau}, \frac{z}{\tau}) \]

\[ = e^{\frac{\pi i m^2}{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i m^2}{2m+1} (2(p+1) - 2\ell)} \left( \frac{-1}{\tau}, 0 \right) \Xi^{(3)[m,\frac{1}{2}]}(\frac{-1}{\tau}, \frac{z}{\tau}) \]

\[ = e^{\frac{\pi i m^2}{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i m^2}{2m+1} (2(p+1) - 2\ell)} \left( \frac{-1}{\tau}, 0 \right) \Xi^{(3)[m,\frac{1}{2}]}(\frac{-1}{\tau}, \frac{z}{\tau}) \]

(ii) \( \Upsilon^{(2)[m,p]}(\frac{-1}{\tau}, \frac{z}{\tau}) = \eta(\frac{-1}{\tau})^3 \left[ \frac{\theta_{\ell, m + \frac{1}{2}}}{\psi(r, \frac{1}{2})} \psi_{\Xi(3)[m,\ell]}(\tau, \frac{z}{\tau}) \right] \]

\[ = \eta(\frac{-1}{\tau})^3 \sum_{\ell=0}^{2m} e^{\frac{\pi i m^2}{2m+1} (2(p+1) - 2\ell)} \eta(\tau) \Xi^{(3)[m,\ell]}(\tau, \frac{z}{\tau}) \]

3) (i) \( \Xi^{(3)[m,p]}(\frac{-1}{\tau}, \frac{z}{\tau}) = \theta_{p, m, m + \frac{1}{2}}(\frac{-1}{\tau}, 0) \Xi^{(3)[m,\frac{1}{2}]}(\frac{-1}{\tau}, \frac{z}{\tau}) \]

\[ = e^{\frac{\pi i m^2}{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i m^2}{2m+1} (2(p+1)) \theta_{(2\ell+1) m, m + \frac{1}{2}}(\tau, 0) \Xi^{(2)[m,\frac{1}{2}]}(\tau, \frac{z}{\tau})} \]

\[ = e^{\frac{\pi i m^2}{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i m^2}{2m+1} (2(p+1)) \theta_{(2\ell+1) m, m + \frac{1}{2}}(\tau, 0) \Xi^{(2)[m,\frac{1}{2}]}(\tau, \frac{z}{\tau})} \]

\[ = e^{\frac{\pi i m^2}{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i m^2}{2m+1} (2(p+1)) \Xi^{(2)[m,\frac{1}{2}]}(\tau, \frac{z}{\tau})} \]

(ii) \( \Upsilon^{(3)[m,p]}(\frac{-1}{\tau}, \frac{z}{\tau}) = \eta(\frac{-1}{\tau})^3 \left[ \frac{\theta_{\ell, m + \frac{1}{2}}}{\psi_{r, \frac{1}{2}}} \right] \Xi^{(3)[m,\ell]}(\tau, \frac{z}{\tau}) \]

\[ = e^{\frac{\pi i m^2}{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i m^2}{2m+1} (2(p+1)) \Xi^{(3)[m,\ell]}(\tau, \frac{z}{\tau})} \]

\[ = e^{\frac{\pi i m^2}{2m+1}} \sum_{\ell=0}^{2m} e^{-\frac{\pi i m^2}{2m+1} (2(p+1)) \Xi^{(3)[m,\ell]}(\tau, \frac{z}{\tau})} \]
Lemma 7.2. Let $m \in \frac{1}{2} \mathbb{N}_{\text{odd}}$ and $p \in \mathbb{Z}_{\geq 0}$. Then

\[ \Xi^{(i)[m,p]}(\tau + 1, z) = e^{-\frac{\pi i m}{4}(\tau + 1)} \theta^{(-)}_{(2p+1)m,m+\frac{1}{2}}(\tau, 0) \tilde{\psi}^{(1)[m]}(\tau + 1, z) \]
\[ = q^{-\frac{m}{4}} e^{-\frac{\pi m}{2}} \cdot e^{\frac{\pi i}{2m+1}(2p+1)^2m^2} \theta^{(-)}_{(2p+1)m,m+\frac{1}{2}}(\tau, 0) \cdot \tilde{\psi}^{(2)[m]}(\tau, z) \]
\[ = e^{\frac{\pi i (2p+1)^2m^2}{4(2m+1)^2}} q^{-\frac{m}{4}} \theta^{(-)}_{(2p+1)m,m+\frac{1}{2}}(\tau, 0) \tilde{\psi}^{(2)[m]}(\tau, z) \]
\[ \Xi^{(2)[m,p]}(\tau + 1, z) = e^{-\frac{\pi i m}{4}(\tau + 1)} \theta^{(-)}_{(2p+1)m,m+\frac{1}{2}}(\tau + 1, 0) \tilde{\psi}^{(1)[m]}(\tau + 1, z) \]
\[ = q^{-\frac{m}{4}} e^{-\frac{\pi m}{2}} \cdot e^{\frac{\pi i}{2m+1}(2p+1)^2m^2} \theta^{(-)}_{(2p+1)m,m+\frac{1}{2}}(\tau + 1, 0) \cdot (-1)\tilde{\psi}^{(1)[m]}(\tau, z) \]
\[ = -e^{\frac{\pi i (2p+1)^2m^2}{4(2m+1)^2}} q^{-\frac{m}{4}} \theta^{(-)}_{(2p+1)m,m+\frac{1}{2}}(\tau, 0) \tilde{\psi}^{(1)[m]}(\tau, z) \]
\[ \Xi^{(1)[m,p]}(\tau + 1, z) = e^{-\frac{\pi i m}{4}(\tau + 1)} \theta^{(-)}_{2pm,m+\frac{1}{2}}(\tau + 1, 0) \tilde{\psi}^{(3)[m]}(\tau + 1, z) \]
\[ = e^{\frac{\pi i (2pm)^2}{4(2m+1)^2}} \theta^{(-)}_{2pm,m+\frac{1}{2}}(\tau, 0) \cdot \tilde{\psi}^{(3)[m]}(\tau, z) \]

Thus the proof of Lemma 7.1 is completed. 

The $T$-transformation properties of $\Xi^{(i)[m,p]}$ and $\Upsilon^{(i)[m,p]}$ are given by the following formulas:

Lemma 7.2. Let $m \in \frac{1}{2} \mathbb{N}_{\text{odd}}$ and $p \in \mathbb{Z}_{\geq 0}$. Then
2) (i) \( \Upsilon^{(1)[m,p]}(\tau+1,z) = -i \eta(\tau+1)^3 \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau+1,z)} \)

\[= -i \left[ e^{\frac{2\pi i}{4}} \eta(\tau) \right]^3 e^{-\frac{2\pi i}{4} p^{(p+\frac{m}{2})^3} \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau+1,z)} \]

\[= e^{\frac{2\pi i}{4} (2p+1)^3} \eta(\tau)^3 \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau,z)} \]

\[\Upsilon^{(2)[m,p]}(\tau,z) \]

(ii) \( \Upsilon^{(2)[m,p]}(\tau+1,z) = \eta(\tau+1)^3 \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau+1,z)} \)

\[= [e^{\frac{2\pi i}{4}} \eta(\tau)]^3 e^{\frac{2\pi i}{4} (p+\frac{m}{2})^2} \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau,z)} \]

\[= e^{\frac{2\pi i}{4} (p+\frac{m}{2})^2} \eta(\tau)^3 \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau,z)} \]

\[= -e^{\frac{2\pi i}{4} (p+\frac{m}{2})^2} \Upsilon^{(1)[m,p]}(\tau,z) \]

(iii) \( \Upsilon^{(3)[m,p]}(\tau+1,z) = -i \eta(\tau+1)^3 \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau+1,z)} \)

\[= -i \left[ e^{\frac{2\pi i}{4}} \eta(\tau) \right]^3 e^{-\frac{2\pi i}{4} p^{(p+\frac{m}{2})^3} \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau,z)} \]

\[= -i \left[ e^{\frac{2\pi i}{4}} \eta(\tau) \right]^3 e^{-\frac{2\pi i}{4} p^{(p+\frac{m}{2})^3} \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau,z)} \]

\[= -i \left[ e^{\frac{2\pi i}{4}} \eta(\tau) \right]^3 e^{-\frac{2\pi i}{4} p^{(p+\frac{m}{2})^3} \frac{[\theta_{p+\frac{m}{2},m+\frac{1}{2}}^{(-)} - \theta_{-(p+\frac{m}{2}),m+\frac{1}{2}}^{(-)}]}{\theta_{0,\frac{1}{2}}^{(-)}(\tau,z)} \]

\[= e^{\frac{2\pi i}{4} (p+\frac{m}{2})^2} \Upsilon^{(3)[m,p]}(\tau,z) \]

Thus the proof of Lemma 7.2 is completed.
8 Indefinite modular forms $g_k^{(i)[m,p]}(\tau)$

For $m \in \frac{1}{2}\mathbb{N}_{\text{odd}}$ and $p \in \mathbb{Z}$ such that $0 \leq p \leq 2m$ and $i \in \{1, 2, 3\}$, we put

$$G^{(i)[m,p]}(\tau, z) := \Xi^{(i)[m,p]}(\tau, z) - \Upsilon^{(i)[m,p]}(\tau, z) \quad (8.1a)$$

Then, by Theorem 5.1, $G^{(i)[m,p]}(\tau, z)$ can be written in the following form:

$$G^{(i)[m,p]}(\tau, z) = \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} g^{(i)[m,p]}_k(\tau) \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z)$$

$$= \sum_{k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} g^{(i)[m,p]}_k(\tau) \left[ \theta_{k,m}^{(-)} - \theta_{-k,m}^{(-)} \right](\tau, z) + 2g^{(i)[m,p]}_m(\tau) \theta_{m,m}^{(-)}(\tau, z) \quad (8.1b)$$

The modular transformation properties of $G^{(i)[m,p]}(\tau, z)$ are obtained immediately from (8.1a) and Lemma 7.1 and Lemma 7.2 as follows:

**Proposition 8.1.** Let $m \in \frac{1}{2}\mathbb{N}_{\text{odd}}$ and $p \in \mathbb{Z}_{\geq 0}$ such that $0 \leq p \leq 2m$. Then

1) (i) $G^{(1)[m,p]}(\tau + 1, z) = \frac{-i}{\sqrt{2m+1}} e^{\frac{\pi m y^2}{2m+1}} \sum_{p'=0}^{2m} e^{-\frac{\pi i}{2m+1} (2p+1)(2p'+1)} G^{(1)[m,p']}(\tau, z) \quad (8.2a)$

   (ii) $G^{(2)[m,p]}(\tau + 1, z) = \frac{i}{\sqrt{2m+1}} e^{\frac{\pi m y^2}{2m+1}} \sum_{p'=0}^{2m} e^{-\frac{\pi i}{2m+1} (2p+1)(2p'+1)} G^{(2)[m,p']}(\tau, z) \quad (8.2b)$

   (iii) $G^{(3)[m,p]}(\tau + 1, z) = -\frac{i}{\sqrt{2m+1}} e^{\frac{\pi m y^2}{2m+1}} \sum_{p'=0}^{2m} e^{-\frac{\pi i}{2m+1} (2p+1)(2p'+1)} G^{(3)[m,p']}(\tau, z) \quad (8.2c)$

2) (i) $G^{(1)[m,p]}(\tau + 1, z) = e^{\frac{\pi i (2p+1)^2}{2m+1}} \Xi^{(1)[m,p]}(\tau, z)$

   (ii) $G^{(2)[m,p]}(\tau + 1, z) = -e^{\frac{\pi i (2p+1)^2}{2m+1}} \Xi^{(1)[m,p]}(\tau, z)$

   (iii) $G^{(3)[m,p]}(\tau + 1, z) = e^{\frac{\pi i p^2}{2m+1}} \Xi^{(3)[m,p]}(\tau, z)$

Also one can see easily, from (8.1b) and Theorem 5.1, that $g_k^{(i)[m,p]}(\tau)$’s are written explicitly as follows:

$$g_k^{(1)[m,p]}(\tau) = \begin{cases} \sum_{r \in \mathbb{Z}} - \sum_{r \in \mathbb{Z}} \\ 0 \leq r < j \quad j \leq r < 0 \end{cases} (-1)^j e^{\frac{i \pi k}{2} \left(\frac{m}{2} + \frac{r}{2m+1}\right)^2} q^{\frac{m}{2} + \frac{r}{2m+1}} - m (r+p+\frac{m+k}{2m})^2$$

$$+ \begin{cases} \sum_{r \in \mathbb{Z}} - \sum_{r \in \mathbb{Z}} \\ 0 \leq r < j \quad j < r < 0 \end{cases} (-1)^j e^{\frac{i \pi k}{2} \left(\frac{m}{2} + \frac{r}{2m+1}\right)^2} q^{\frac{m}{2} + \frac{r}{2m+1}} - m (r+p+\frac{m-k}{2m})^2$$
\[
\begin{align*}
&+ \theta^{(-)}_{m, m + \frac{1}{2}}(\tau, 0) \sum_{-p < r < p} e^{\pi i k q - m (r + \frac{k + m}{2m})^2} \\
g^{(1)}_{m}(\tau, 0) &= \left[ \sum_{r \in \mathbb{Z}} - \sum_{r \in \mathbb{Z}} \right] (-1)^j e^{\pi i m q - m (r + \frac{1}{2})^2} - m (r + p)^2 \\
&+ \frac{1}{2} e^{\pi i m} \theta^{(-)}_{m, m + \frac{1}{2}}(\tau, 0) \sum_{-p < r < p} q^{-m r^2} \\
g^{(2)}_{m}(\tau, 0) &:= -(-1)^p \left[ \sum_{r \in \mathbb{Z}} - \sum_{r \in \mathbb{Z}} \right] (-1)^r q - m (r + \frac{k + m}{2m})^2 \\
&- \theta^{(-)}_{m, m + \frac{1}{2}}(\tau, 0) \sum_{-p < r < p} q^{-m r^2} \\
g^{(3)}_{m}(\tau, 0) &:= \left[ \sum_{r \in \mathbb{Z}} - \sum_{r \in \mathbb{Z}} \right] (-1)^j e^{\pi i k q - m (r + \frac{1}{2})^2} - m (r + p + \frac{k}{2m})^2 \\
&+ \left[ \sum_{r \in \mathbb{Z}} - \sum_{r \in \mathbb{Z}} \right] (-1)^j e^{\pi i k q - m (r + \frac{1}{2})^2} - m (r + p - \frac{k}{2m})^2 \\
&+ \theta^{(-)}_{m, m + \frac{1}{2}}(\tau, 0) \sum_{-p < r < p} e^{\pi i k q - (2mr + k)^2} \\
g^{(3)}_{m}(\tau, 0) &:= e^{\pi i m} \left[ \sum_{r \in \mathbb{Z}} - \sum_{r \in \mathbb{Z}} \right] (-1)^j q - m (r + \frac{1}{2})^2 - m (r - \frac{1}{2} + p)^2 \\
&+ e^{\pi i m} \theta^{(-)}_{m, m + \frac{1}{2}}(\tau, 0) \sum_{r=1}^{p} q^{-m (2r - 1)^2} 
\end{align*}
\]
First we compute Lemma 8.1.

**Proof.**

1) Substituting (8.1b) into (8.2a), one has

\[ \sum_{0<k\leq m} \sin \frac{\pi k\ell}{m} \cdot g_k^{(1)[m,p]} \left( -\frac{1}{\tau} \right) = \frac{\tau \sqrt{2m}}{2\sqrt{2m+1}} \sum_{\ell'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p'+1)} g^{(1)[m,p']}_{\ell'}(\tau) \]

(8.3a)

2) Substituting (8.2b) into (8.3a), one has

\[ \sum_{0<k\leq m} \sin \frac{\pi k\ell}{m} \cdot g_k^{(2)[m,p]} \left( -\frac{1}{\tau} \right) = \frac{i \tau \sqrt{2m}}{2\sqrt{2m+1}} \sum_{\ell'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p'+1)} g^{(1)[m,p']}_{\ell'}(\tau) \]

(8.3b)

3) Substituting (8.3a) into (8.4c), one has

\[ \sum_{0<k\leq m} \sin \frac{\pi k\ell}{m} \cdot g_k^{(3)[m,p]} \left( -\frac{1}{\tau} \right) = \frac{-i \tau \sqrt{2m}}{2\sqrt{2m+1}} \sum_{\ell'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p'+1)} g^{(3)[m,p']}_{\ell'}(\tau) \]

(8.3c)

2) Substituting (8.1b) into (8.2a), one has

\[ \frac{1}{2} \sum_{0<k\leq m} \sin \pi k \cdot g_k^{(1)[m,p]} \left( -\frac{1}{\tau} \right) = \frac{\tau \sqrt{2m}}{2\sqrt{2m+1}} \sum_{\ell'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p'+1)} g^{(1)[m,p']}_{\ell'}(\tau) \]

(8.4a)

2) Substituting (8.2b) into (8.3a), one has

\[ \frac{1}{2} \sum_{0<k\leq m} \sin \pi k \cdot g_k^{(2)[m,p]} \left( -\frac{1}{\tau} \right) = \frac{i \tau \sqrt{2m}}{2\sqrt{2m+1}} \sum_{\ell'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p'+1)} g^{(2)[m,p']}_{\ell'}(\tau) \]

(8.4b)

2) Substituting (8.3a) into (8.4c), one has

\[ \frac{1}{2} \sum_{0<k\leq m} \sin \pi k \cdot g_k^{(3)[m,p]} \left( -\frac{1}{\tau} \right) = \frac{-i \tau \sqrt{2m}}{2\sqrt{2m+1}} \sum_{\ell'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p'+1)} g^{(3)[m,p']}_{\ell'}(\tau) \]

(8.4c)

Proof. 1) Substituting (8.1b) into (8.2a), one has

LHS of (8.2a) = \[ G^{(1)[m,p]} \left( -\frac{1}{\tau} \right) - \frac{1}{\tau} \frac{\sin z}{\tau} \]

\[ \sum_{k\in \frac{1}{Z_{odd}} \cap 0<k\leq m} g_k^{(1)[m,p]} \left( -\frac{1}{\tau} \right) \left[ \theta_k^{(-)} - \theta_{-k,m}^{(-)} \right] \left( -\frac{1}{\tau}, \frac{z}{\tau} \right) \]

\[ \times \left\{ \sum_{\ell\in \frac{1}{Z_{odd}} \cap 0<\ell<m} \sin \frac{\pi k\ell}{m} \cdot \left[ \theta_{\ell,m}^{(-)} - \theta^{(-)}_{-\ell,m} \right] (\tau, z) + \sin \pi k \cdot \theta_{m,m}^{(-)} (\tau, z) \right\} \]

(8.5a)

RHS of (8.2a) = \[ \frac{(-i\tau)^{\frac{3}{2}}}{\sqrt{2m+1}} \frac{\sin z}{\tau} \sum_{\ell'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p'+1)} G^{(1)[m,p']} (\tau, z) \]
Note 8.1. Let $m \in \frac{1}{2}\mathbb{N}_{\text{odd}}$ and $j, k \in \frac{1}{2}\mathbb{Z}_{\text{odd}}$ such that $0 < j, k \leq m$. Then

\[
\sum_{\ell \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sin \frac{\pi j \ell}{m} \sin \frac{\pi k \ell}{m} + \frac{1}{2} \sin \pi j \cdot \sin \pi k = \frac{m}{2} c_j \delta_{j,k}
\]  

(8.6a)

where

\[
c_j := \begin{cases} 
  1 & \text{if } j < m \\
  2 & \text{if } j = m 
\end{cases}
\]  

(8.6b)

Summing up the above, we arrive at the following theorem:

Theorem 8.1. Let $m, j \in \frac{1}{2}\mathbb{N}_{\text{odd}}$ and $p \in \mathbb{Z}$ such that $0 \leq p \leq 2m$ and $0 < j \leq m$. Then $S$- and $T$-transformation of $g_j^{(i)[m,p]}(\tau)$ are given by the following formulas:

1) $S$-transformation:

\begin{enumerate}
  \item[(i)] $g_j^{(1)[m,p]}(-\frac{1}{\tau}) = \frac{\tau}{c_j \sqrt{m(m+\frac{1}{2})}} \sum_{\ell \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{p'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p+1)(2p'+1)} \sin \frac{\pi j \ell}{m} g_{\ell}^{(1)[m,p']}(\tau)$
  
  \item[(ii)] $g_j^{(2)[m,p]}(-\frac{1}{\tau}) = \frac{i \tau}{c_j \sqrt{m(m+\frac{1}{2})}} \sum_{\ell \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{p'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)}(2p+1)p'} \sin \frac{\pi j \ell}{m} g_{\ell}^{(3)[m,p']}(\tau)$
  
  \item[(iii)] $g_j^{(3)[m,p]}(-\frac{1}{\tau}) = \frac{-i \tau}{c_j \sqrt{m(m+\frac{1}{2})}} \sum_{\ell \in \frac{1}{2}\mathbb{Z}_{\text{odd}}} \sum_{p'=0}^{2m} e^{-\frac{\pi i}{2(2m+1)p(2p'+1)}} \sin \frac{\pi j \ell}{m} g_{\ell}^{(2)[m,p']}(\tau)$
\end{enumerate}

where $c_j$ is defined by (8.6b).

2) $T$-transformation:

\begin{enumerate}
  \item[(i)] $g_j^{(1)[m,p]}(\tau + 1) = e^{\frac{\pi i(2p+1)^2}{4m} - \frac{\pi i^2}{2m} - \frac{\pi i}{4}} g_j^{(2)[m,p]}(\tau)$
  
  \item[(ii)] $g_j^{(2)[m,p]}(\tau + 1) = -e^{\frac{\pi i(2p+1)^2}{4m} - \frac{\pi i^2}{2m} - \frac{\pi i}{4}} g_j^{(1)[m,p]}(\tau)$
  
  \item[(iii)] $g_j^{(3)[m,p]}(\tau + 1) = e^{\frac{\pi i^2}{2m+1} - \frac{\pi i^2}{2m}} g_j^{(3)[m,p]}(\tau)$
\end{enumerate}
Proof. 1) (i) For \( j \in \frac{1}{2} \mathbb{N}_{\text{odd}} \) such that \( \frac{1}{2} \leq j \leq m \), we compute
\[
\sum_{\ell \in \frac{1}{2} \mathbb{Z}_{\text{odd}}, \atop 0 < \ell < m} \sin \frac{\pi j \ell}{m} \times (8.3a) + \sin \pi j \times (8.4a).
\]
Then
\[
\sum_{\ell \in \frac{1}{2} \mathbb{Z}_{\text{odd}}, \atop 0 < \ell < m} \sin \frac{\pi j \ell}{m} \times \text{LHS of } (8.3a) + \sin \pi j \times \text{LHS of } (8.4a)
\]
\[
= \sum_{\ell \in \frac{1}{2} \mathbb{Z}_{\text{odd}}, \atop 0 < \ell < m} \sin \frac{\pi j \ell}{m} \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}, \atop 0 < k \leq m} \sin \frac{\pi k \ell}{m} \cdot g_k^{(1)[m,p]} \left( -\frac{1}{\tau} \right) + \frac{1}{2} \sin \pi j \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}, \atop 0 < k \leq m} \sin \pi k \cdot g_k^{(1)[m,p]} \left( -\frac{1}{\tau} \right)
\]
\[
= \sum_{k \in \frac{1}{2} \mathbb{Z}_{\text{odd}}, \atop 0 < k \leq m} \left\{ \sum_{\ell \in \frac{1}{2} \mathbb{Z}_{\text{odd}}, \atop 0 < \ell < m} \sin \frac{\pi j \ell}{m} \cdot \sin \frac{\pi k \ell}{m} + \frac{1}{2} \sin \pi j \cdot \sin \pi k \right\} g_k^{(1)[m,p]} \left( -\frac{1}{\tau} \right)
\]
\[
= \frac{m}{2} c_j g_j^{(1)[m,p]} \left( -\frac{1}{\tau} \right) - \frac{1}{\tau} \delta_{j,k}.
\]
and
\[
\sum_{\ell \in \frac{1}{2} \mathbb{Z}_{\text{odd}}, \atop 0 < \ell < m} \sin \frac{\pi j \ell}{m} \times \text{RHS of } (8.3a) + \sin \pi j \times \text{RHS of } (8.4a)
\]
\[
= \frac{\tau \sqrt{2m}}{2 \sqrt{2m} + 1} \sum_{\ell \in \frac{1}{2} \mathbb{Z}_{\text{odd}}, \atop 0 < \ell < m} \sin \frac{\pi j \ell}{m} \sum_{p' = 0}^{2m} e^{-\pi^2 \frac{\ell^2}{m} (2p+1)(2p'+1)} g_{\ell}^{(1)[m,p']} \left( \tau \right).
\]
Then, by (8.7a) = (8.7b), we obtain the formula 1) (i). The formulas 1) (ii) and (iii) can be proved in similar way. And 2) follows immediately from Proposition 8.1 and Lemma 1.4.

Remark 8.1. We note that \( g_m^{(3)[m,0]}(\tau) = 0 \) and this is, of course, in consistency with S- and T-transformation formulas.

9 An example ~ the case \( m = \frac{1}{2} \)

In this section, we compute the functions \( g_j^{(i)[m,p]}(\tau) \) and their modular transformation in the case \( m = \frac{1}{2} \) to show the following:

Proposition 9.1.
Proof. In the case $m = \frac{1}{2}$, the functions $g(1,2,3)_{}\frac{1}{2}(i,\tau) \quad (i \in \{1, 2, 3\}$ and $p \in \{0, 1\})$ are as follows:

$$
\begin{align*}
1) \quad \left[ \sum_{0 < r \leq j, j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] (-1)^j q^{(1)\frac{j^2}{2} - \frac{1}{2}r^2} &= \frac{1}{2} \left\{ \frac{\eta(\tau)}{\eta(2\tau)} \right\} \\
2) \quad \left[ \sum_{0 < r \leq j, j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] (-1)^r q^{(1)\frac{j^2}{2} - \frac{1}{2}r^2} &= \frac{1}{2} \left\{ \frac{\eta(\tau)}{\eta(2\tau)} \right\} \\
3) \quad \left[ \sum_{0 < r \leq j, j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] (-1)^j q^{(1)\frac{j^2}{2} - \frac{1}{2}(r + \frac{1}{2})^2} &= \eta(\tau)\eta(2\tau)
\end{align*}
$$

Modular transformation of these functions are obtained from Theorem 8.1 as follows:

$$
\begin{align*}
\begin{cases}
\begin{aligned}
g(1,2,3)_{\frac{1}{2}}(\tau - \frac{1}{\tau}) &= -i\tau g(1,2,3)_{\frac{1}{2}}(\tau) \\
g(2,1,2)_{\frac{1}{2}}(\tau - \frac{1}{\tau}) &= \frac{-\tau}{\sqrt{2}} \cdot g(2,1,2)_{\frac{1}{2}}(\tau) \\
g(3,2)_{\frac{1}{2}}(\tau - \frac{1}{\tau}) &= \frac{1}{\sqrt{2}} \cdot g(3,2)_{\frac{1}{2}}(\tau)
\end{aligned}
\end{cases}
\begin{cases}
\begin{aligned}
g(1,2,3)_{\frac{1}{2}}(\tau + 1) &= i e^{i\pi} g(2,1,2)_{\frac{1}{2}}(\tau) \\
g(2,1,2)_{\frac{1}{2}}(\tau + 1) &= i e^{i\pi} g(1,2,3)_{\frac{1}{2}}(\tau) \\
g(3,2)_{\frac{1}{2}}(\tau + 1) &= e^{i\pi} g(3,2)_{\frac{1}{2}}(\tau)
\end{aligned}
\end{cases}
\end{align*}
$$

Putting $\bar{g}_1(\tau) := -i g(1,2,3)_{\frac{1}{2}}(\tau)$, $\bar{g}_2(\tau) := g(2,1,2)_{\frac{1}{2}}(\tau)$ and $\bar{g}_3(\tau) := \frac{-i}{\sqrt{2}} g(3,2)_{\frac{1}{2}}(\tau)$, these formulas are rewritten in terms of $\bar{g}_i(\tau)$ as follows:

$$
\bar{g}_1(\tau) = \left[ \sum_{0 < r \leq j, j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] (-1)^j q^{(1)\frac{j^2}{2} - \frac{1}{2}r^2} + \frac{1}{2} \theta_{\frac{1}{2}, 1}(\tau, 0) = \frac{1}{2} q^{\frac{1}{16}} + \cdots
$$

$$
\bar{g}_2(\tau) = \left[ \sum_{0 < r \leq j, j, r \in \mathbb{Z}} - \sum_{j, r \in \mathbb{Z}} \right] (-1)^r q^{(j)\frac{j^2}{2} - \frac{1}{2}r^2} + \frac{1}{2} \theta_{\frac{1}{2}, 1}(\tau, 0) = \frac{1}{2} q^{\frac{1}{16}} + \cdots \quad (9.1)
$$
\[ \bar{g}_3(\tau) = \frac{1}{\sqrt{2}} \left[ \sum_{j,r \in \mathbb{Z}, 0 < r \leq j} \sum_{j,r \in \mathbb{Z}, j < r \leq 0} (-1)^j q^{(j+\frac{1}{2})^2 - \frac{1}{2}(r+\frac{1}{2})^2} = \frac{1}{\sqrt{2}} q^{\frac{\tau}{2}} + \cdots \right] \]

and

\[
\begin{cases}
\bar{g}_1 \left( -\frac{1}{\tau} \right) = (-i\tau) \bar{g}_1(\tau), \\
\bar{g}_2 \left( -\frac{1}{\tau} \right) = (-i\tau) \bar{g}_3(\tau), \\
\bar{g}_3 \left( -\frac{1}{\tau} \right) = (-i\tau) \bar{g}_2(\tau) \\
\bar{g}_1(\tau + 1) = e^{\frac{\pi i}{\tau}} \bar{g}_2(\tau), \\
\bar{g}_2(\tau + 1) = e^{\frac{\pi i}{\tau}} \bar{g}_1(\tau), \\
\bar{g}_3(\tau + 1) = e^{\frac{\pi i}{\tau}} \bar{g}_3(\tau)
\end{cases}
\]

Now we put \( f_i(\tau) := \frac{\bar{g}_i(\tau)}{\eta(\tau)\eta(2\tau)} \) for \( i \in \{1, 2, 3\} \). Since \( \eta(-\frac{1}{\tau}) = (-i\tau)^\frac{\tau}{2} \eta(\tau) \) and \( \eta(\tau+1) = e^{\frac{\pi i}{\tau}} \eta(\tau) \), these functions \( f_i(\tau) \) satisfy the following equations:

\[
\begin{cases}
f_1 \left( -\frac{1}{\tau} \right) = f_1(\tau), \\
f_2 \left( -\frac{1}{\tau} \right) = f_3(\tau), \\
f_3 \left( -\frac{1}{\tau} \right) = f_2(\tau) \\
f_1(\tau + 1) = e^{-\frac{\pi i}{\tau}} f_2(\tau), \\
f_2(\tau + 1) = e^{-\frac{\pi i}{\tau}} f_1(\tau), \\
f_3(\tau + 1) = e^{-\frac{\pi i}{\tau}} f_3(\tau) \\
f_1(\tau) = \frac{1}{2} q^{\frac{\tau}{2^3} + \cdots}, \\
f_2(\tau) = \frac{1}{2} q^{\frac{\tau}{2^3} + \cdots}, \\
f_3(\tau) = \frac{1}{\sqrt{2}} q^{\frac{\tau}{2^3} + \cdots}
\end{cases}
\]

Then, by Lemma 4.8 in [3] (or Lemma 4.9 in [13]), we have

\[ f_1(\tau) = \frac{1}{2} \cdot \frac{\eta(\tau)^2}{\eta(\tau)\eta(2\tau)}, \quad f_2(\tau) = \frac{1}{2} \cdot \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}, \quad f_3(\tau) = \frac{1}{2} \cdot \sqrt{2} \frac{\eta(2\tau)}{\eta(\tau)} \]

so

\[ \bar{g}_1(\tau) = \frac{1}{2} \cdot \frac{\eta(\tau)^4}{\eta(\frac{\tau}{2})\eta(2\tau)}, \quad \bar{g}_2(\tau) = \frac{1}{2} \cdot \eta(\tau)\eta\left(\frac{\tau}{2}\right), \quad \bar{g}_3(\tau) = \frac{1}{\sqrt{2}} \cdot \eta(\tau)\eta(2\tau) \]

proving Proposition 9.1 since

\[ \theta_{\frac{1}{2},1}(\tau, 0) = \frac{\eta(\tau)^2}{\eta\left(\frac{\tau}{2}\right)} \quad \text{and} \quad \theta^{(-)}_{\frac{1}{2},1}(\tau, 0) = \frac{\eta\left(\frac{\tau}{2}\right)\eta(2\tau)}{\eta(\tau)}. \]
References

[1] V. G. Kac : Infinite-Dimensional Lie Algebras, 3rd edition, Cambridge University Press, 1990.

[2] V. G. Kac, S.-S. Roan and M. Wakimoto : Quantum reduction for affine superalgebras, Commun. Math. Phys. 241 (2003), 307-342.

[3] V. G. Kac and M. Wakimoto : Modular and conformal invariance constraints in representation theory of affine algebras, Advances in Math. 70 (1988), 156-236.

[4] V. G. Kac and M. Wakimoto : Integrable highest weight modules over affine superalgebras and Appell’s function, Commun. Math. Phys. 215 (2001), 631-682.

[5] V. G. Kac and M. Wakimoto : Quantum reduction and representation theory of superconformal algebras, Advances in Math. 185 (2004), 400-458.

[6] V. G. Kac and M. Wakimoto : Quantum reduction in the twisted case, Progress in Math. 237 Birkhäuser (2005), 85-126. [math-ph/0404049]

[7] V. G. Kac and M. Wakimoto : Representations of affine superalgebras and mock theta functions, Transformation Groups 19 (2014), 387-455. [arXiv:1308.1261]

[8] V. G. Kac and M. Wakimoto : Representations of affine superalgebras and mock theta functions II, Advances in Math. 300 (2016), 17-70. [arXiv:1402.0727]

[9] V. G. Kac and M. Wakimoto : Representations of affine superalgebras and mock theta functions III, Izv. Math. 80 (2016), 693-750. [arXiv:1505.01047]

[10] V. G. Kac and M. Wakimoto : A characterization of modified mock theta functions, Transformation Groups 22 (2017). [arXiv:1510.05683]

[11] V. G. Kac and M. Wakimoto : Representation of superconformal algebras and mock theta functions, Trudy Moskow Math. Soc. 78 (2017), 64-88. [arXiv:1701.03344]

[12] D. Mumford : Tata Lectures on Theta I, Progress in Math. 28, Birkhäuser Boston, 1983.

[13] M. Wakimoto : Infinite-Dimensional Lie Algebras, Translation of Mathematical Monographs Vol.195, American Mathematical Society 2001.

[14] M. Wakimoto : Mock theta functions and characters of N=3 superconformal modules, [arXiv:2202.03098]

[15] M. Wakimoto : Mock theta functions and characters of N=3 superconformal modules II, [arXiv:2204.01473]

[16] S. Zwegers : Mock theta functions, PhD Thesis, Universiteit Utrecht, 2002, arXiv:0807.483.