On the apparent horizon in fluid-gravity duality

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This article develops a computational framework for determining the location of boundary-covariant apparent horizons in the geometry of conformal fluid-gravity duality in arbitrary dimensions. In particular, it is shown up to second order and conjectured to hold to all orders in the gradient expansion that there is a unique apparent horizon which is covariantly expressible in terms of fluid velocity, temperature and boundary metric. This leads to the first explicit example of an entropy current defined by an apparent horizon and opens the possibility that in the near-equilibrium regime there is preferred foliation of apparent horizons for black holes in asymptotically-AdS spacetimes.

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I. INTRODUCTION

The AdS/CFT correspondence [1–3], or more generally gauge-gravity duality, demonstrates a deep and fascinating link between black hole physics and the plasma phase dynamics of certain (holographic) strongly coupled gauge theories. Over the last 10 years the gravitational side of the correspondence has also started to be used as a tractable theoretical model of strongly interacting non-Abelian media with properties similar to quark-gluon plasma studied first at RHIC and now also at the LHC (see [4] for the most recent review of these developments). Early efforts in the applications of gauge-gravity duality methods to hot QCD matter were motivated by hydrodynamic simulations of the expanding fireball created in heavy ion collisions and focused on obtaining transport properties of holographic plasmas by analyzing low-lying quasinormal modes and linear response theory. These results provided concrete numerical predictions for the simplest transport coefficients of strongly coupled non-Abelian media with \( \mathcal{N} = 4 \) super Yang-Mills plasma as the primary example \( ^1 \) and have eventually led to the formulation of fluid-gravity duality \( ^6 \). Fluid-gravity duality is a correspondence which maps solutions of relativistic Navier-Stokes equations describing holographic liquids to long-wavelength distortions of black branes in higher dimensional geometry. The direct connection between the dynamics of black objects in higher dimensional spacetimes and solutions of nonlinear hydrodynamics provided an opportunity to understand black brane geometries and their features in terms of dual fluids, as well as to gain insights about hydrodynamics from the properties of Einstein’s equations. These perspectives, as well as the possibility of applications, have generated significant interest in the nonlinear dynamics of black brane spacetimes.

Dynamical black holes and their characterization has also been an important research theme in mathematical relativity for the last couple of decades (see \( ^7–9 \) for useful reviews of this subject). The exact characterization of a dynamical black hole has proven to be a surprisingly thorny theoretical problem for general relativity. The standard textbook definition associates black hole interiors with regions of spacetime from which no signal can ever escape \( ^10 \). Thus, finding the exact extent of such a region is necessarily a teleological procedure: properly defining “ever” and “escape" means that one must examine the ultimate fate of all signals from a point before ruling whether or not that point is part of the black hole. Identifying the event horizon boundary of a causal black hole is similarly teleological. Thus, even though an event horizon is a congruence of null geodesics obeying the same rules as any other congruence, its evolution can appear to be acausal. For example the area increase of an event horizon is not directly driven by infalling matter or energy; instead the actual effect of an influx through the event horizon is a decrease in its rate of expansion.

These observations are not just mathematical curiosities. The non-local nature of the event horizon is acceptable as long as one treats it as a causal boundary removing the region containing a curvature singularity from the dynamics of the rest of spacetime \( ^2 \) and does not associate any physical characteristics with it. However, this is not the only role of the event horizon – for the last four decades, one of the most celebrated results of black hole physics has been the link between the area of the event horizon and entropy. Already in the 1960s it was established that event horizons necessarily increase in area \( ^10 \) and this has usually been interpreted as being equivalent to the second law of thermodynamics. Thus, the apparently acausal expansion of event horizons would seem to imply a similarly acausal evolution of entropy. This leads to problems since the origin of black hole entropy needs to be sought within microscopic theories underlying gravitational interactions in the sense of the holographic principle \( ^11, 12 \). For

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\[ ^1 \] For an excellent review of these early developments see \( ^5 \).

\[ ^2 \] Hence guaranteeing consistency of low energy description of black holes in terms of classical gravity.
asymptotically flat or asymptotically de Sitter spacetimes such theories are not known (in principle one could imagine any of these being non-local \[13\]), but in the anti-de Sitter context these are local quantum field theories in a suitably understood large \(N_c\) limit. At the superficial level it is hard to judge whether acausality of the event horizon is a real problem in the first two cases, but in the AdS/CFT context it definitely is.

The teleological nature of the event horizon is one of the main motivations for the ongoing research program to characterize black holes (quasi)locally, identifying their interiors from the presence of strong gravitational fields rather than on the basis of the causal structure of the entire spacetime. Quasilocal horizons go by such names as trapping \[14\], isolated or dynamical horizons (the last two are both reviewed in \[7\]). In all cases though, these horizons can be thought of as generalizations of the classical apparent horizons \[10\]. Recall that apparent horizons are associated with foliations of spacetimes. Areas of strong gravitational field are identified with the region on each surface that is covered by trapped surfaces. The boundary of that region is an apparent horizon and it is the outermost surface for which the outgoing light front does not expand in area. With a slight abuse of terminology the union of such surfaces over all time slices is often also referred to as an apparent horizon and it is in this sense that it will be used it here.

In the context of gauge-gravity duality characteristics of black holes with planar horizons in higher dimensional spacetimes are at the same time the quantities describing dual finite energy density or finite charge density states of local quantum field theories. In static situations, where the acausal nature of the event horizon plays no role, the entropy defined by the event horizon was identified with the thermodynamic entropy of a dual holographic field theory. Such entropy satisfies a very strong constraint: the first law of thermodynamics linking IR quantities (i.e. temperature and entropy density) with UV quantities (the energy density). Since the latter are well defined in the dual quantum field theories (energy density is the expectation value of one of the components of the energy-momentum tensor in thermal equilibrium), there are no controversies with associating thermodynamic entropy with the event horizon in time-independent situations. However in static situations (at least in the context of Kerr-Newman black holes/branes) the event horizon and one of apparent horizons coincide, so that by associating the entropy with the event horizon one actually associates it at the same time with an apparent horizon. The latter identification actually turns out to be more robust, as the example of conformal soliton flow \[15\] suggests \[16\].

Beyond equilibrium three problems arise. The first follows from the aforementioned nonlocality of the event horizon, the second comes from foliation dependence of apparent horizons, whereas the third one is related to the various ways in which one can associate points on any of horizons with points on the boundary (this is referred to as the bulk-boundary map \[17\]). The last of these is necessary to localize the entropy production in the dual field theory. To illustrate that the first issue is a serious problem quite disconnected from any ambiguities of the bulk-boundary map, one can consider the example of gravitational dynamics with a sharp distinction between a dual equilibrium regime without entropy production and a dynamical transition with dissipation. The relevant backgrounds, much in the spirit of the Vaidya solution, appeared in the context of the thermalization problem of strongly coupled non-Abelian media and describe gravitational processes in which the dual quantum field theory undergoes a transition between two equilibrium states in a finite time interval \[18\] \[19\]. In such a situation the event horizon evolves past the bulk lightcones spanned by the transition region on the boundary. The latter patch of bulk spacetime is dual to the boundary region where the holographic field theory is in equilibrium, so that its thermodynamic entropy stays constant. This

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\[3\] It is also required that ingoing light front shrinks in area and that inside this surface there are other surfaces, such that both ingoing and outgoing light fronts emitted from them shrink in area. For a more precise definition see Section \[III\] or the review articles \[3\] \[4\].
result strongly suggests that the causal boundary of a black hole is not the relevant entropy carrier regardless of any ambiguities of the bulk-boundary map \[16\]. In contrast with the area of the event horizon, the entropy defined by the unique apparent horizon respecting symmetries of 1-dimensional boundary dynamics considered in \[18, 19\] was constant before the transition process and eventually in the far future agreed with the one given by the event horizon.

In less symmetric situations, apart from the choice of bulk-boundary map, the foliation dependence of apparent horizons becomes a significant issue – different foliations of spacetime lead to different apparent horizons. The most trivial and at the same time pessimistic possibility is that the notion of local entropy does not extend beyond equilibrium situations and foliation dependence, as well as the freedom of bulk-boundary mapping, signal exactly this. It might also be that on the dual field theory side there are many relevant local notions of entropy and different apparent horizons correspond to such different notions. Yet another possibility is that the field theory notion of entropy does not suffer from ambiguities of kinds introduced by foliation dependence of apparent horizons, which might be used as a guiding principle for finding preferred apparent horizon.

Resolving these issues in the general case is very difficult if not impossible, so the only hope is to proceed example by example. In global equilibrium the foliation dependence essentially vanishes and the event and apparent horizons coincide. In the near-equilibrium regime one expects the horizons to be “close” (in a sense of \[20\] or \[21\]) so that a dynamical apparent horizon behaves almost like an isolated horizon. If one takes the near-equilibrium regime as point of departure for further studies, one is immediately led to considering apparent horizons in the geometry of fluid-gravity duality. This background captures the hydrodynamic regime on the field theory side starting from a locally boosted and dilated black brane supplemented with gradient corrections \[6\] and from this perspective hydrodynamics can be regarded as the simplest (because of its universality) type of collective dynamics that quantum field theory can undergo.

The generalization of entropy to hydrodynamics is provided by the notion of an entropy current. Such a current is constructed phenomenologically in the gradient expansion by requiring that in equilibrium it reproduces thermodynamic entropy and that its divergence evaluated on solutions of the equations of hydrodynamics is non-negative. A detailed analysis of the consequences of this generalized second law of thermodynamics on the form of the entropy current in \[22\] showed that even up to second order in gradients there is an ambiguity inherent in such a definition. From the point of view of fluid-gravity duality it was very natural to ask what is the gravity interpretation of the coefficients appearing in the boundary hydrodynamic entropy current and what might be the bulk counterpart of the ambiguity in its definition. In the pioneering work \[17\] a candidate entropy current was obtained by mapping the area theorem on the event horizon onto the boundary along ingoing null geodesics. In general, there are infinitely many directions in which such geodesics can propagate from the boundary, but hydrodynamic covariance requires that such geodesics close to the boundary move in the direction specified (at leading order of the gradient expansion) by the local fluid velocity \[17\]. Ambiguities appearing in such procedure appear at third and higher orders in the gradient expansion and were irrelevant in the second order construction of \[17, 22\]. This causal bulk-boundary map can be supplemented with suitably understood boundary diffeomorphisms and the latter turn out to capture precisely the ambiguity discussed in \[22\].

Furthermore, in a recent paper \[23\] it was shown that for a fixed bulk-boundary map the same freedom in entropy current can be understood as coming from different bulk hypersurfaces with a fixed foliation satisfying a generalized

\[4\] Part of the ambiguity is trivial and comes from a term whose divergence vanishes.
area theorem and asymptoting to the event horizon. Such surfaces were dubbed “generalized horizons” with the event horizon and the (asymptoting to it) apparent horizon being just two particular instances of the more general notion. From the perspective of the phenomenological definition of the hydrodynamic entropy current none of these hypersurfaces and none of the available bulk-boundary maps is favored over any other. However, causality of the boundary field theory seems to favor the entropy current dual to the apparent horizon – providing that it is free of the ambiguities related to foliation dependence and that the bulk-boundary map in use is causal. The task of this paper is to elaborate on the proposal [24] by presenting a derivation of the apparent horizon in the geometry of fluid-gravity duality, its features, as well as discussing the properties of the dual entropy current.

The organization of the paper is the following. Section II presents in a self-contained fashion the geometry dual to conformal fluid dynamics in arbitrary dimensions obtained in [25]. Section III is the main part of the paper and provides the detailed calculation of the relevant apparent horizon in the case of conformal fluid-gravity duality up to second order in gradients. Section IV focuses on the hydrodynamic side of fluid-gravity duality and analyzes the dual entropy current using the technology introduced in [24]. The general discussions of the results and possible future directions of research are provided in Section V. Appendix A provides some details on the Weyl-covariant derivative and Weyl-covariant hydrodynamic tensors, whereas Appendix B illustrates the methods developed in Section III by describing the construction of an apparent horizon in the Vaidya spacetime. Readers interested mostly in the general-relativistic aspects of these considerations can skip Section IV and regard the paper as an example of a perturbative calculation of an apparent horizon in a geometry governed by Einstein’s equations with negative cosmological constant.

II. THE GEOMETRY OF FLUID-GRAVITY DUALITY

The geometry of fluid-gravity duality in arbitrary dimensions [25] is a solution to Einstein gravity with negative cosmological constant

\[ I_{d+1} = \frac{1}{16\pi G_N} \int d^{d+1}x \sqrt{-G} \left\{ R + d(d - 1) \right\}, \]

where \( G_N \) is the \((d + 1)\)-dimensional Newton’s constant and the AdS radius is set to 1. The action (1) arises in the context of string theory (for \( d = 2, 3, 4 \) and 6, see [26] for details) and describes a sector of decoupled dynamics of the one-point function of the energy-momentum tensor operator in planar strongly coupled holographic conformal field theories [25]. The equations of motion derived from (1) support a \( d \)-parameter family of exact, static black hole solutions with planar horizons obtained by boosting and dilating the AdS-Schwarzschild black brane solution

\[ ds^2 = 2u_\mu dx^\mu dr - r^2 \left( 1 - \frac{1}{(rb)^d} \right) u_\mu u_\nu dx^\mu dx^\nu + r^2 (g_{\mu\nu} + u_\mu u_\nu) dx^\mu dx^\nu. \]

Here \( g_{\mu\nu} \) denotes components of the flat Minkowski metric on the conformal boundary of the asymptotically AdS spacetime (2). The boost parameter \( u^\mu \) is a \( d \)-component velocity in the \( x^\mu \) directions, normalized so that \( u_\mu u^\mu = -1 \) in the sense of the boundary metric \( g_{\mu\nu} \). The lines of constant \( x^\mu \) in (2) are ingoing null geodesic, for large \( r \) propagating in the direction set by \( u^\mu \), and the radial coordinate \( r \) parametrizes them in an affine way [17]. The geometry (2) may be regarded as a stack of constant-\( r \) \( d \)-dimensional planes, starting from the boundary at \( r = \infty \) (which is \( d \)-dimensional Minkowski spacetime) right down to the curvature singularity at \( r = 0 \). The latter is shielded by the event horizon at \( r = 1/b \), which is at the same time an (isolated) apparent horizon. The dilation parameter \( b \)
appearing in (2) is related to the Hawking temperature $T$ of the event horizon by

$$b = \frac{d}{4\pi T}. \quad (3)$$

Unlike black holes in asymptotically flat spacetime, the metric (2) supports perturbations varying much slower within the transverse planes than within the radial direction. The parameter controlling the scale of variations in the radial direction is $b$.

If $b$, $u^\mu$ and $g_{\mu\nu}$ are allowed to vary slowly compared to the scale set by $b$, the metric (2) should be an approximate solution of nonlinear Einstein’s equations with corrections organized in an expansion in the number of gradients in the $x^\mu$ directions. Direct calculations [6] have shown that proceeding in this way is a systematic way of solving Einstein’s equations, provided that the dual energy-momentum tensor [27, 28] depending on $b$ and $u^\mu$ is conserved. For the metric (2) the dual energy-momentum tensor is that of a relativistic perfect fluid with a conformal equation of state

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P^{\mu\nu}, \quad (4)$$

where

$$P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \quad (5)$$

is the projector operator onto the space transverse to $u^\mu$,

$$\epsilon = (d - 1) p = (d - 1) \frac{1}{16\pi G_N} b^{-d} \quad (6)$$

and $g_{\mu\nu}$ is some weakly curved metric in which the fluid lives.

Both the metric (2) and the energy-momentum tensor (4) receive gradient corrections. The separation of scales mentioned earlier implies that corrections to the metric (2) will be tensorial quantities made of $x^\mu$-derivatives of $b$, $u^\mu$ and $g_{\mu\nu}$ with scalar functions of $r$ and $b$ as coefficients. The relevance of these terms is suppressed by the number of gradients and for practical reasons the expansion is terminated at the 2-derivative level. The tensorial quantities in question are scalars $S$, transverse ($u_\mu V^\mu = 0$) vectors $V^\mu$ and transverse ($u^\mu T_{\mu\nu} = 0$) traceless symmetric rank 2 tensors $T_{\mu\nu}$. A priori one should consider all possible terms, as was done originally in [6]. This task can however be greatly simplified by utilizing the underlying conformal symmetry and seeking Weyl-invariant solutions of Einstein’s equations, i.e. solutions invariant under simultaneous rescalings of

$$g_{\mu\nu} \to e^{-2\phi} g_{\mu\nu}, \quad u^\mu \to e^\phi u^\mu, \quad b \to e^{-\phi} b \quad \text{and} \quad r \to e^\phi r \quad (7)$$

where $\phi$ depends on the coordinates $x^\mu$ [25]. The leading order metric (2) is Weyl-invariant, but due to the presence of $dr$ it does not retain its form at higher orders. It can however be written in a manifestly Weyl-invariant form upon introducing a vector field $A_\nu$ defined by [29]

$$A_\nu \equiv u^\lambda \nabla_\lambda u_\nu - \frac{\nabla_\lambda u^\lambda}{d - 1} u_\nu. \quad (8)$$

This quantity is of order one in the gradient expansion and transforms as a connection under Weyl-transformations

$$A_\nu \to A_\nu + \partial_\nu \phi. \quad (9)$$

It needs to be borne in mind, that all these quantities are also required to be independent when evaluated on lower order solutions of hydrodynamics.
The Weyl-invariant form of the metric (2) reads

\[ ds^2 = 2u_\mu dx^\mu (dr - A_\nu dx^\nu) - r^2 \left( 1 - \frac{1}{(rb)^d} \right) u_\mu u_\nu dx^\mu dx^\nu + r^2 (g_{\mu\nu} + u_\mu u_\nu) dx^\mu dx^\nu. \] (10)

This metric is a leading order approximation to a spacetime whose metric is of the form

\[ ds^2 = G_{ab} dx^a dx^b = -2u_\mu dx^\mu (dr + V_\alpha dx^\alpha) + G_{\mu\nu} dx^\mu dx^\nu \] (11)

with the condition \( u^\mu G_{\mu\nu} = 0 \) completely fixing the gauge freedom. The subleading corrections to (10) need to be Weyl-invariant and the simplest way to construct them is by summing individual Weyl-invariant contributions order by order in the gradient expansion. A single Weyl-invariant contribution to (11) can be represented as a scalar function of the Weyl-invariant combination \( rb \) multiplying a Weyl-covariant (i.e. transforming homogeneously under Weyl transformations of \( b, u^\mu \) and \( g_{\mu\nu} \), see Appendix A) tensor of a given weight \( w \) supplemented with a factor of \( b^w \).

A powerful tool in generating Weyl-covariant gradient terms is the Weyl-covariant derivative \( D_\mu \), which uses the connection \( A_\mu \) to compensate for derivatives of the Weyl factor coming from derivatives of Weyl-covariant tensors. It has the property that a Weyl-covariant derivative of a Weyl-covariant expression is itself Weyl-covariant with the same weight (see Appendix A or the original publications [17, 25, 29] for details).

At first order in gradients there is only a single Weyl-covariant term available, which is the shear tensor of the fluid \( \sigma_{\mu\nu} \). It reads

\[ \sigma_{\mu\nu} = \frac{1}{2} (D_\mu u_\nu + D_\nu u_\mu) \] (12)

and transforms with Weyl-weight 3. At second order in gradients, there are in total 10 Weyl-covariant terms: 3 scalars, 2 transverse vectors and 5 transverse traceless symmetric rank 2 tensors. For convenience these objects can be defined with appropriate powers of \( b \) to render them Weyl-invariant. The scalar contributions read

\[ S_1 = b^2 \sigma_{\mu\nu} \sigma^{\mu\nu}, \quad S_2 = b^2 \omega_{\mu\nu} \omega^{\mu\nu} \quad \text{and} \quad S_3 = b^2 R, \] (13)

where \( \omega \) is the vorticity of the flow and \( R \) is the Weyl-covariant curvature tensor and curvature scalar (see Appendix A for details). The Weyl-invariant transverse vectors are

\[ V_1_\mu = b P_{\mu\nu} D_\rho \omega^{\nu\rho} \quad \text{and} \quad V_2_\mu = b P_{\mu\nu} D_\rho \omega^{\nu\rho} \] (14)

Finally, the Weyl-invariant tensors read

\[ T_{1\mu\nu} = \omega^{\rho} D_\rho \sigma_{\mu\nu}, \quad T_{2\mu\nu} = C_{\mu\rho\sigma\nu} u^\rho u^\sigma, \quad T_{3\mu\nu} = \omega^\rho \sigma_{\rho\mu} + \omega_\rho \sigma_{\rho\nu}, \quad T_{4\mu\nu} = \sigma^\rho \sigma_{\rho\mu} - \frac{1}{d-1} P_{\rho\sigma\alpha\beta} \sigma^{\alpha\beta} \quad \text{and} \quad T_{5\mu\nu} = \omega^\rho \omega_{\rho\mu} + \frac{1}{d-1} P_{\rho\sigma\alpha\beta} \omega^{\alpha\beta}, \] (15)

where \( C_{\mu\rho\sigma\nu} \) is a Weyl-covariantized curvature tensor (consult Appendix A for its detailed form).

The metric (11) up to second order in gradients can be expressed in terms of (12), (13), (14) and (15) and takes the
where

\[ \mathcal{V}_\mu = r^2 B u_\mu + r A_\mu \]

+ \frac{1}{d-2} (-b^{-1}(V_{2\mu} + V_{1\mu}) - b^{-2}u_\mu(S_2 - S_1 + \frac{1}{2(d-1)}S_3)) - \frac{2r}{(br)^d} LV_{1\mu} + 

\frac{1}{4} \frac{r^2}{(d-1)(br)^d} K_2 S_1 \]

\[ \mathcal{G}_{\mu
u} = r^2 F_{\mu
u} + 2br^2 F\sigma_{\mu
u} + \]

\(- (T_{5\mu
u} - \frac{1}{d-1} P_{\mu
u}b^{-2}S_2) + 2b^2r^2 F^2(T_{4\mu\nu} + \frac{1}{d-1} P_{\mu\nu}b^{-2}S_1) - \frac{2r^2}{d-1} K_1 S_1 P_{\mu\nu} + 

2b^2r^2 H_1 (T_{1\mu\nu} + T_{4\mu\nu} + T_{2\mu\nu}) + 2b^2r^2 H_2 (T_{1\mu\nu} + T_{3\mu\nu}) \]

The quantities \( F, H_1, H_2, K_1, K_2, L \) are functions of \( br \) introduced in \( \text{[25]} \) and read

\[ F(br) \equiv \int_{br}^{\infty} \frac{y^{d-1} - 1}{y(y^d - 1)} dy, \]

\[ H_1(br) \equiv \int_{br}^{\infty} \frac{y^{d-2} - 1}{y(y^d - 1)} dy, \]

\[ H_2(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi(\xi^d - 1)} \int_{1}^{\xi} y^{d-3} dy \left[ 1 + (d-1)yF(y) + 2y^2 F'(y) \right] \]

\[ = \frac{1}{2} F(br)^2 - \int_{br}^{\infty} \frac{d\xi}{\xi(\xi^d - 1)} \int_{1}^{\xi} \frac{y^{d-2} - 1}{y(y^d - 1)} dy, \]

\[ K_1(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi^2} \int_{1}^{\xi} dy y^2 F'(y)^2, \]

\[ K_2(br) \equiv \int_{br}^{\infty} \frac{d\xi}{\xi^2} \left[ 1 - \xi(\xi - 1)F'(\xi) - 2(d-1)\xi^{d-1} 

+ (2(d-1)\xi^d - (d-2)) \int_{\xi}^{\infty} dy y^2 F'(y)^2 \right] , \]

\[ L(br) \equiv \int_{br}^{\infty} \xi^{d-1} \frac{d\xi}{\xi^2} \int_{1}^{\xi} dy \frac{y - 1}{y^3(y^d - 1)}. \]

The metric given above is a solution of Einstein equations with negative cosmological constant up to second order in gradients, provided that \( b \) and \( u^\mu \) satisfy the equations of dual hydrodynamics, i.e. the equations of covariant conservation of the energy-momentum tensor obtained from \( \text{[110]} \) by holographic renormalization

\[ T_{\mu\nu} = p(g_{\mu\nu} + du_\mu u_\nu) - 2\eta\sigma_{\mu\nu} - 2\eta\tau_{\omega}(T_{1\mu\nu} + T_{3\mu\nu}) + 2\eta b(T_{1\mu\nu} + T_{2\mu\nu} + T_{4\mu\nu}) , \]

where

\[ \eta = \frac{s}{4\pi} = \frac{1}{16\pi G_N b^{d-1}}, \quad \tau_{\omega} = b \int_{1}^{\infty} \frac{y^{d-2} - 1}{y(y^d - 1)} dy, \quad p = \frac{1}{16\pi G_N} b^d. \]
The geometric picture emerging is that of spacetime locally approximated by tubes of uniform black branes spanned along ingoing null geodesics given by lines of constant $x^\mu$. The dilation and boost parameters $b$ and $u^\mu$, as well as the boundary metric $g_{\mu\nu}$ vary from tube to tube, but, as anticipated, the scales of these variations are small compared to variations of the bulk metric along the radial null direction $x^\mu$. Due to this tubewise approximation, the leading order geometry of fluid-gravity duality inherits the causal structure of static black brane, i.e. the event horizon located at $r = 1/b$, now with $b$ depending on $x^\mu$. It is to be expected, and is confirmed by direct calculation further in the text, that the event horizon of (2) with slowly varying $b$, $u^\mu$ and $g_{\mu\nu}$ is at the same time an apparent horizon. Such an apparent horizon is called isolated and does not lead to entropy production. The isolated apparent horizon at $r = 1/b$ is expected to become dynamical once corrections to (2) are included and its position will also be modified. The dynamics of this almost isolated apparent horizon can be described in a gradient expansion much in the spirit of slowly evolving horizons [30–32].

III. LOCATING THE APPARENT HORIZON IN THE GEOMETRY OF FLUID-GRAVITY DUALITY

This section is devoted to identifying an apparent horizon for the spacetimes defined by the metric (11). This search will be based on two criteria: 1) from the isolated horizon contained in the unperturbed geometry (2) it is natural to expect the apparent horizon to be a perturbation of the hypersurface $r = 1/b(x)$ and 2) to ensure compatibility with the dual conformal fluid solution those perturbations are required to be manifestly Weyl-invariant.

A. Preliminaries

Apparent horizons are defined in terms of trapped and marginally trapped surfaces. In both cases the term “surface” means a codimension-two hypersurface $\Omega$ embedded in a larger spacetime. The normal space to such a surface is spanned at any point by a pair of null vectors $\ell$ and $n$. The following considerations apply to spacetimes where it makes sense to specify that both of these are future-oriented and respectively outwards and inwards pointing. It is convenient to cross-normalize them so that $\ell \cdot n = -1$ (this leaves a degree of scaling freedom).

The induced metric on $\Omega$ can be written as

$$\tilde{q}_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a,$$

while the outward and inward null expansions of $\Omega$ are

$$\theta(\ell) = \tilde{q}^{ab} \nabla_a \ell_b = \mathcal{L}_\ell \log \sqrt{\tilde{q}} \quad \text{and} \quad \theta(n) = \tilde{q}^{ab} \nabla_a n_b = \mathcal{L}_n \log \sqrt{\tilde{q}}$$

or, more generally, for an arbitrary normal vector $X = A\ell + Bn$

$$\theta(X) = A\theta(\ell) + B\theta(n).$$

Now $\Omega$ is said to be outer trapped if $\theta(\ell) < 0$, trapped if $\theta(\ell) < 0$ and $\theta(n) < 0$ and untrapped if $\theta(\ell) = 0$ and $\theta(n) < 0$. It is outer marginally trapped if $\theta(\ell) = 0$ and marginally trapped if $\theta(\ell) = 0$ and $\theta(n) < 0$. Trapped surfaces are indicative of black hole regions, with well-known theorems linking them to both singularities and the existence of event horizons [10]. As recalled in the introduction they are also used to define apparent horizons [10]. Given a foliation of spacetime
into spacelike hypersurfaces $\Sigma_t$ (“instants” of time) one can define the total trapped region on each $\Sigma_t$ as the union of all the outer trapped surfaces. Then (up to some technicalities which will be ignored here) the boundary of each of those regions $\Omega_t$ is outer marginally trapped and known as the apparent horizon. A common abuse of terminology (adopted in the following) also uses the term apparent horizon to refer to the hypersurface defined by the evolving $\Omega_t$ (that is the union of the $\Omega_t$).

In practical calculations, this definition of an apparent horizon is not very usable and instead one just searches directly for hypersurfaces foliated by outer marginally trapped surfaces. This is common practice in numerical relativity (see, for example, [9] and references therein). More generally, the teleological nature of classical black holes and their event horizons has lead many to search for a (quasi)local and properly causal replacement. Horizons foliated by (outer) marginally trapped surfaces which (hopefully) bound regions of trapped surfaces are the most obvious and mathematically tractable candidates.

For example, the boundaries of stationary black holes (or branes) are taken to be weakly isolated horizons: codimension-one hypersurfaces that are foliated by outer marginally trapped surfaces or isolated horizons if their extrinsic geometry is also invariant (see for example the discussions in [33, 34] or review articles such as [7–9]). These are closely related (though more general than) Killing horizons and under many circumstances do a good job of characterizing a stationary black hole boundary without reference to causal structure or infinities. This is particularly so if one adds extra conditions to ensure that there are fully trapped surfaces “just inside” the horizon. For Hayward’s future outer trapping horizons (FOTHs) one assumes that

$$\theta(n) < 0 \text{ and } L_n \theta(t) < 0.$$  

That is, the inward expansion is negative and under a small inwards deformation the outward expansion also becomes negative. The black branes considered in this paper are examples of FOTHs.

For the classical definition, it is clear that time-evolved apparent horizons are foliation dependent: different foliations will sample a different set of trapped surfaces and so give rise to a different “time-evolved” horizon. Alternatively, focusing on the time-evolved horizon itself, it can be shown (see, for example [31, 35]) that a hypersurface foliated by outer marginally trapped surfaces is not rigid and may be deformed while maintaining its properties. The non-uniqueness of apparent horizons has been explicitly demonstrated in several papers [36, 37].

### B. Finding the horizon: strategy

Problems with uniqueness are somewhat alleviated in the present calculation by the requirement that perturbations of the horizon be manifestly Weyl covariant. Then, the time-evolved apparent horizon $\Delta$ should be specified as the level set of a scalar function

$$S(r, x) = b(x)r - g(x),$$

where $g(x)$ is a Weyl-invariant scalar defined by

$$g(x) = g_1(x) + g_2(x) + \ldots$$
where \( g_k \) denotes a linear combination of all Weyl-invariant scalars at order \( k \) in the gradient expansion. There are no Weyl-invariant scalars at order 1, and 3 at order 2, so one expects to find

\[
\begin{align*}
g_1(x) &= 0 \\
g_2(x) &= h_1 S_1(x) + h_2 S_2(x) + h_3 S_3(x),
\end{align*}
\]

where the \( S_i \) are the 3 independent Weyl-invariant scalars \( \text{(31)} \) and the constants \( h_i \) will be determined by solving \( \theta(\ell) = 0 \) and the conditions \( \text{(30)} \). Once this is done, the expression for the position of the apparent horizon will take the form

\[
r = r_H(x) \equiv \frac{1}{b} (1 + h_1 S_1(x) + h_2 S_2(x) + h_3 S_3(x)).
\]

(34)

This is a strong constraint, but a reasonable one to impose in a perturbative regime where physical considerations suggest that the horizon should be given by a Weyl covariant structure. Testing these surfaces as potential horizons means that one must consider their possible foliations and find out whether any of them satisfy \( \theta(\ell) = 0 \) and the conditions \( \text{(30)} \). Again however, one can lean on the Weyl covariance to simplify the calculation. Specifically, the outer marginally trapped surfaces of \( \Delta \) will have their own (in \( \Delta \)) normal \( v \). This vector is required to be expressible as a sum of Weyl invariant terms and further that it be surface forming

\[
v \wedge dv = 0.
\]

(35)

Though this only really needs to apply on the horizon itself, it turns out to be computationally much easier to check this condition for \( v \) specified not only on the putative horizon but also in some neighbourhood. Thus, in practice one should look for Weyl-covariant one-form fields that are surface forming in some neighbourhood of \( \Delta \).

Thus the search domain will not be arbitrarily large, but rather be restricted to potential horizons and foliations that are essentially Weyl-covariant perturbations of the unperturbed boosted black brane solution \( \text{(2)} \). Marginally outer trapped surfaces are to be sought among intersections of these classes. It will be shown in the following that for the geometry of fluid-gravity duality, up to second order in the gradient expansion the conditions \( \theta(\ell) = 0 \) and \( \text{(35)} \) determine \( v \) (as well as the \( h_i \) in \( \text{(31)} \)) uniquely.

### C. Finding the horizon: hypersurfaces and intersections

The program outlined above can be implemented as follows. The normal covector to a surface of the form \( \text{(31)} \) is

\[
m = dS,
\]

(36)

which up to second order in the gradient expansion is

\[
m = r \partial_\mu b \, dx^\mu + b \, dr.
\]

(37)

The function \( g \) does not contribute above, since the leading term involves \( \partial_\mu g_2 \), which is of third order in gradients. It is convenient to write the normal in terms of the Weyl-covariant derivative, which acting on \( b \) (Weyl weight \(-1\)) is

\[
D_\mu b = \partial_\mu b - A_\mu b.
\]

(38)
One then has

\[ m = r \left( \mathcal{D}_\mu b + b A_\mu \right) dx^\mu + b \, dr. \quad (39) \]

Raising the index using the metric \((11)\) one gets (using the formula for the inverse given in \([25]\)), up to second order

\[ m^\mu = b u^\mu + b^2 \left( V_1^\mu \left( -\frac{2}{d} \frac{1}{dr} + \frac{1}{(d-2)(br)^2} + \frac{2}{(br)^d} L \right) + V_2^\mu \frac{1}{(d-2)(br)^2} \right) \]

\[ m^r = 2 Br^2 b - A_\mu b u^\mu + \frac{1}{b} \left( \left( \frac{2}{d-2} + \frac{1}{2(br)^2} \right) S_2 + \left( \frac{2}{d(d-1)b} - \frac{2}{d-2} + \frac{1}{(d-1)(br)^d - 2K_2} S_1 \right) + \frac{1}{(d-2)(d-1)} S_3 \right) \]. \quad (40)

Next one must consider potential foliations of \(\Delta\). As noted earlier, the foliation of the apparent horizon can be specified by a vector field \(v\) which is tangent to \(\Delta\) and but otherwise normal to the leaves. Given a parametrization \(y^\mu\) of the horizon so that \(S(r(y), x(y)) \equiv \text{const}\), this means that

\[ \frac{\partial S}{\partial r} \frac{\partial}{\partial r} + \frac{\partial S}{\partial x^\beta} \frac{\partial}{\partial y^\alpha} = 0. \quad (42) \]

In terms of this coordinate system the tangent vectors are, of course, \(\partial/\partial y^\alpha\) which push-forward into the full space time as

\[ \tilde{e}_\mu = \frac{\partial x^\alpha}{\partial y^\mu} \frac{\partial}{\partial x^\alpha} \quad (43) \]

and applying \([22]\) one finds

\[ \tilde{e}_\mu = \frac{\partial x^\alpha}{\partial y^\mu} \left\{ \frac{\partial}{\partial x^\beta} - \left( \frac{\partial S}{\partial r} \right)^{-1} \left( \frac{\partial S}{\partial x^\beta} \right) \frac{\partial}{\partial r} \right\}. \quad (44) \]

By construction these vectors all satisfy \(v \cdot m = 0\) (as they should).

It is very convenient to choose the coordinates on the horizon \(y^\mu = x^\mu\). This should be reasonable as long as the horizon does not “fold over”; this should be the case in this perturbative, gradient expansion limit. This choice also has the advantage of making the bulk-boundary map trivial (as discussed in Section \([11]\)). Then the tangent vectors to \(\Delta\) can be written as

\[ \tilde{e}_\beta = \frac{\partial}{\partial x^\beta} - \left( \frac{\partial S}{\partial r} \right)^{-1} \left( \frac{\partial S}{\partial x^\beta} \right) \frac{\partial}{\partial r} \quad (45) \]

and a general vector field tangent to the horizon is given by

\[ v = u^\mu \tilde{e}_\mu. \quad (46) \]

In terms of the coordinate basis in the bulk one then has

\[ v = u^\beta \left\{ \frac{\partial}{\partial x^\beta} - \left( \frac{\partial S}{\partial r} \right)^{-1} \left( \frac{\partial S}{\partial x^\beta} \right) \frac{\partial}{\partial r} \right\}. \quad (47) \]

---

6 Recall now that the geometry found in \([25]\) satisfies Einstein equations provided that the equations of hydrodynamics are satisfied by the quantities \(b, u^\mu\) (in terms of which also \(A_\mu\) is expressed). These equations imply \([25]\) that \(\mathcal{D}b\) is of the second order in gradients.
Requiring that the vector \( v \) be Weyl covariant fixes (up to second order)

\[
v^\mu = bu^\mu + \{V_1^\mu b^2 c_1 + V_2^\mu b^2 c_2 + u^\mu (S_1 b e_1 + S_2 b e_2 + S_3 b e_3)\}
\]

\[
v^r = -r A_\mu b u^\mu + S_1 r \frac{2}{d(d-1)},
\]

where \( c_1, c_2, e_1, e_2 \) and \( e_3 \) are some constants. It is computationally convenient to normalize \( v \) so that

\[
m^2 + v^2 = 0,
\]

in which case the coefficients of the longitudinal terms in (48) vanish

\[
e_1 = e_2 = e_3 = 0.
\]

The remaining coefficients \((c_1, c_2)\) appearing in \( v \) are also not arbitrary. As discussed earlier, to ensure that the vector \( v \) defines a foliation one has to impose the Frobenius condition (35). There are two types of terms, which turn out to be given by

\[
v_{\mu} \partial^\nu v^\rho = 0
\]

\[
v_{\mu} \partial^\nu v^\nu = d \left( c_1 + \frac{1}{d} - \frac{1}{d-2} \right) V_1[^\nu u^\mu] + d \left( c_2 - \frac{1}{d-2} \right) V_2[^\nu u^\mu]
\]

up to terms of higher order in the gradient expansion. This determines the coefficients \( c_1 \) and \( c_2 \)

\[
c_1 = \frac{2}{d(d-2)}
\]

\[
c_2 = \frac{1}{d-2}.
\]

This way one finds that the foliation vector \( v \) is completely determined once \( \Delta \) is fixed

\[
v^\mu = bu^\mu + \frac{1}{d-2} b^2 \left( \frac{2}{d} V_1^\mu + V_2^\mu \right).
\]

Since the Frobenius condition was imposed for the full spacetime (rather than just on the horizon), this vector \( v \) actually gives rise to foliation of the full spacetime, at least in a neighborhood of the horizon.

It is interesting that one gets a unique result. It seems plausible that this will also be the case at higher orders in the gradient expansion. To see this, note that at a given order \( k \), \( v \) is entirely specified in terms of its \( v^\mu \) components, and its \( v^r \) component does not depend on the \( k \)-th order contribution to \( v^\mu \). In complete analogy with the second order, \( v^\mu \) at order \( k \) will be a linear combination of all available transverse and longitudinal vectors. The hypersurface \( \Delta \) is specified as the level set of a scalar function \( S(r, x) \) (see (31)), which at order \( k \) contains all the available hydrodynamic scalars of order \( k \). The vector \( m \) normal to \( \Delta \) is defined by \( m = dS \), so the construction does not introduce any further coefficients to be determined. Now consider the normalization condition (50) and expand the \( v^2 \) contribution

\[
G_{\mu\nu} v^\mu v^\nu - 2u_\mu V_\nu v^\mu v^\nu - 2u_\mu v^\mu v^r + m^2 = 0.
\]

In order to evaluate the \( k \)-th order contribution to (55) from \( v^\mu \) it is sufficient to take the zeroth order metric. Note however that since at leading order \( v^\mu \) is proportional to \( u^\mu \) and \( G_{\mu\nu} \) is transverse, the first term on the left hand side of (55) vanishes for all \( r \). Since \( v^r \) does not receive corrections from the \( k \)-th order \( v^\mu \), the only term which depends
on this is actually \( u_\mu V_\nu v^\nu \). But since \( V_\mu \) is also proportional to \( u_\mu \) at leading order, the whole left hand side of (55) at order \( k \) depends only on the longitudinal contributions to \( v^\mu \) at this order. If so, formula (55) fixes them uniquely and does not constrain the transverse contributions. What is left at order \( k \) are scalar contributions to \( S(r, x) \) and transverse contributions to \( v^\mu \). But at a given order, transverse and longitudinal quantities are independent, so the scalar condition \( \theta_l \) at order \( k \) fixes all the contributions to \( S(r, x) \). The transverse components of \( v^\mu \), relevant for the foliation of \( \Delta \), are likely to be fixed by the Frobenius condition (35) in analogy with what happens at second order. It can be checked that the contributions in question will appear in the Frobenius conditions, but it seems difficult to show that by choosing the transverse parts appropriately one can satisfy Frobenius conditions at any order. One argument that this is indeed the case is that such a condition must be satisfied for \( v^\mu \) on the event horizon at arbitrary order and in this case \( v^\mu \) is fixed and given by \( m^\mu \). It would certainly be interesting to make these statements more precise.

### D. Horizons

Dynamical quasilocal horizons are spacelike and so \( m \) should be timelike and \( v \) spacelike. Without loss of generality one can assume that \( m \) is future pointing and \( v \) is outward pointing. Then the null normals to the surfaces of constant \( S \) and \( v \) are

\[
v = \ell - Cn \]

\[
m = \ell + Cn,
\]

where the scalar \( C \) is called the evolution parameter \([30, 31]\). In this case

\[
C = \frac{1}{2}v^2.
\]

The sign of the evolution parameter indicates whether \( \Delta \) is spacelike or timelike (or null if \( C = 0 \)). The signs of the coefficients in (56) have been chosen to ensure that both \( \ell \) and \( n \) are future-pointing, and \( \ell \) is outward-pointing while \( n \) is inward-pointing.

The null normals are then

\[
\ell^\mu = bu^\mu + \frac{1}{2}b^2 \left( V_2^\mu \left( \frac{1}{d-2} \frac{1}{(br)^2} + c_2 \right) + V_1^\mu \left( -2 \frac{1}{d} \frac{1}{rb} + \frac{1}{(d-2)} \frac{1}{(rb)^2} + 2L \frac{1}{(rb)^2} + c_1 \right) \right)
\]

\[
\ell^r = -A_{\mu} br v^\mu + Br^2 b + b^{-1} \left( S_2 \left( \frac{1}{d-2} + \frac{1}{4} \frac{1}{(br)^d} \right) + S_1 \left( -\frac{1}{d-2} + \frac{1}{2} \frac{1}{d} \frac{1}{(br)^d} + \frac{1}{2(2d-2)} \frac{1}{(br)^d} \frac{1}{d-2} K_2 \right) + \frac{1}{2(d-2)} \frac{1}{(d-1)} S_3 \right)
\]

and

\[
n^\mu = \frac{1}{2Br^2} \left( V_1^\mu \left( \frac{2}{d} \frac{1}{rb} - \frac{1}{(d-2)} \frac{1}{(rb)^2} - 2 \frac{1}{(rb)^d} L + c_1 \right) + V_2^\mu \left( -\frac{1}{(d-2)} \frac{1}{(rb)^2} + c_2 \right) \right)
\]

\[
n^r = -\frac{1}{b}.
\]
As a check on the results obtained so far one can determine the location of the event horizon and compare with [25].

Using the explicit form of the evolution parameter $C$ (obtained from (57))

$$ C = -B b^2 r^2 + \left( \frac{1}{(d-2)} - \frac{1}{2} \frac{1}{(d-1)} (br)^{d-2} K_2 - \frac{2}{d-1} (br) S_1 \right) - \left( \frac{1}{(d-2)} + \frac{1}{4(rb)^d} \right) S_2 - \frac{1}{2} \frac{1}{(d-2)} \frac{1}{(d-1)} S_3 \tag{62} $$

it is easy to calculate where the event horizon is located. On general grounds one expects a result of the form (34).

Solving $C(r_{EH}) = 0$ one finds (34) with

$$ h^{(EH)}_1 = \frac{2(d^2 + d - 4)}{d^2(d-1)(d-2)} - \frac{1}{d(d-1)} K_2(1) $$

$$ h^{(EH)}_2 = -\frac{d + 2}{2d(d-2)} $$

$$ h^{(EH)}_3 = -\frac{d - 1}{d(d-1)(d-2)} \tag{63} $$

which matches the results of [25].

Note that because the event horizon is null it must be the case that $v$ is proportional to $m$. Using (63) and the explicit form of $m$ and $v$, it may be checked directly that this is indeed the case.

To determine the position of the apparent horizon one needs to calculate the null expansions from the forms (28)

$$ \theta^{(\ell)} = \tilde{q}^{ab} \nabla_a \ell_b \quad \text{and} \quad \theta^{(n)} = \tilde{q}^{ab} \nabla_a n_b \tag{64} $$

where the metric induced on the foliation slices is calculated from (27). Using the results of the previous section one finds (up to second order)

$$ \theta^{(\ell)} = (d - 1) \left( B b r + \frac{1}{br} \left( S_1 \left( \frac{1}{(d-2)} \frac{1}{2(d-1)} (br)^{d-2} K_2 - \frac{1}{d-1} B b^2 r^2 K'_2(1) \right) \right) \right) + \frac{1}{br} \left( S_2 \left( B b r + \frac{1}{br} \left( S_1 \frac{1}{2(d-1)} (br)^2 S_2 + S_3 S_1 \right) \right) \right) \tag{65} $$

$$ \theta^{(n)} = -\frac{d - 1}{br} + \frac{1}{br} \left( \frac{1}{br} S_2 + r K'_2 S_1 \right) \tag{66} $$

Note that the results are manifestly Weyl-invariant. In particular, there is no correction at first order (as required by Weyl invariance). With these results in hand, it is straightforward to determine the location of the apparent horizon by solving $\theta^{(\ell)}(r_{AH}) = 0$. One again finds [34] with

$$ h^{(AH)}_1 = \frac{2}{(d-2)d} - \frac{1}{d(d-1)} K_2(1), $$

$$ h^{(AH)}_2 = -\frac{d + 2}{2d(d-2)}, $$

$$ h^{(AH)}_3 = -\frac{1}{d(d-1)(d-2)} \tag{67} $$

Only $h_1$ differs from the result for the event horizon [25].

---

7 This computation is fairly lengthy.
The expression

$$r_{EH} - r_{AH} = \frac{4S_1}{bd^2(d-1)} \geq 0$$  \hfill (68)

explicitly shows that the apparent horizon lies within (or coincides with) the event horizon in the sense that an ingoing radial null geodesic will cross first the event horizon and only then the apparent horizon, since $r$ is an affine parameter on such geodesics. It is also easy to check that the apparent horizon is spacelike or null

$$C(r_{AH}) = \frac{2S_1}{d(d-1)} \geq 0,$$  \hfill (69)

as required.

**IV. THE HYDRODYNAMIC ENTROPY CURRENT DEFINED BY THE APPARENT HORIZON**

In hydrodynamics the entropy current is a phenomenological notion constructed order-by-order in the gradient expansion starting from the term describing the flow of thermodynamic entropy. Subleading contributions are given as a sum of all available hydrodynamic vectors (not necessarily transverse) chosen in such a way that the divergence of the current is non-negative when evaluated on the solutions of equations of hydrodynamics. In the conformal case, up to second order in gradients, there are in total 5 available contributions consisting of the 3 hydrodynamic Weyl-invariant scalars \cite{13} multiplied by the velocity $u^\mu$ and 2 Weyl-invariant transverse vectors \cite{14}

$$J^\mu = \frac{1}{4G_N} b^{1-d} \left\{ u^\mu + b \left( j_{1}^\parallel V_1^\mu + j_{2}^\parallel V_2^\mu \right) + \left( j_{1}^\parallel S_1 + j_{2}^\parallel S_2 + j_{3}^\parallel S_3 \right) u^\mu \right\}.$$ \hfill (70)

The overall factor of $1/4G_N$ in (70) comes from the holographic expression for thermodynamic entropy. The on-shell divergence\(^8\) of the current (70) was evaluated in reference \cite{25} and reads, up to third order in gradients,

$$4G_N b^{d-1} D_\mu J_S^\mu = \frac{2b}{d} \sigma^{\mu\nu} \left[ \sigma_{\mu\nu} - bd(d-2) \left( j_3^\parallel - \frac{2(j_2^\parallel + j_3^\parallel)}{d(d-2)} \right) \omega_{\mu\lambda} \omega^\lambda_\nu \right. $$

$$- bd(d-2) \left( j_3^\parallel + \frac{1}{d(d-2)} \right) \left( \sigma_{\mu\nu} - bd(d-2) \left( j_3^\parallel - \frac{2(j_2^\parallel + j_3^\parallel)}{d(d-2)} \right) \omega_{\mu\lambda} \omega^\lambda_\nu \right)$$

$$+ \left( (j_1^\parallel - j_3^\parallel) bd + \tau_\omega \right) u^{\lambda} D_\lambda \sigma_{\mu\nu} \right]$$

$$+ b^2 (j_1^\parallel + 2j_3^\parallel) D_\mu D_\nu \sigma^{\mu\nu} + \ldots$$  \hfill (71)

As understood in \cite{38} for $d = 4$, this expression makes it possible to constrain some of the coefficients appearing in (70). These arguments are based on the observation that local non-negativity should hold both when the shear tensor vanishes at a given point, as well as when it is arbitrary small (if it is large enough, then $\sigma_{\mu\nu} \sigma^{\mu\nu}$ dominates over other contributions and there are no further constraints). The first condition automatically implies that

$$j_1^\parallel = -2j_3^\parallel,$$ \hfill (72)

whereas the second sets to zero all contributions which spoil non-negativity for very small $\sigma_{\mu\nu}$, i.e.

$$j_2^\parallel + j_2^\parallel = \frac{1}{2} (d-2) j_3^\parallel \quad \text{and} \quad j_3^\parallel = -\frac{1}{d(d-2)}.$$ \hfill (73)

\(^8\) In the sense of conservation of the energy-momentum tensor given by \cite{26}
Note that $j^\perp_x$ appears in the divergence only in the combination $j^\perp_x + j^\parallel_x$, so that shifting $j^\perp_x$ and $j^\parallel_x$ keeping the sum constant does not change the divergence. This ambiguity comes from the freedom of modifying the entropy current by adding a multiple of the divergence-free term $b^{\mu\nu}D_\lambda \omega^{\lambda \mu} = b^{1-b} (-b V^\mu_2 + S_2 u^\mu)$ \cite{17, 24, 25} and does not affect the local rate of entropy production. At third order there are no further constraints available so that $j^\parallel_x$ remains the only unspecified parameter affecting the divergence \cite{71}. Results of \cite{17, 24, 25} make it clear that $j^\parallel_x$ is not fixed by some higher order argument – dual gravitational constructions, which all guarantee non-negativity of the divergence, lead to different values of $j^\parallel_x$. Thus, if the notion of local entropy production in the near-equilibrium regime makes sense, there must be some further constraints on the form of the hydrodynamic entropy current. This paper argues that one such constraint might be causality, which leads to considering the holographic entropy current based on the apparent horizon in the dual gravity description.

The problem of constructing a candidate hydrodynamic entropy current on the gravity side of the correspondence was first solved in \cite{17} and then generalized to weakly curved boundary \cite{39} and to arbitrary dimensions \cite{25}. These articles relied on using the bulk-boundary map defined by ingoing null geodesics supplemented with boundary diffeomorphisms \cite{17} to map the area form of the black brane event horizon satisfying the area theorem onto a dual current of non-negative divergence. The main motivation for mapping bulk data along ingoing null geodesics was causality. Note however that such a constraint on the bulk-boundary map is self-consistent only when the bulk entropy carrier is causal \cite{17}. Such a notion is provided by an apparent horizon, which along with the event horizon provides an example of a “generalized horizon” introduced in \cite{24}.

The geometric setup described in Section \textbf{III} contains a distinguished vector field $v$ tangent to the horizon $\Delta$. As anticipated in \cite{24} in the context of “generalized horizons” one motivation for introducing $v$ is that the change of the area form on the horizon sections can be written in terms of the expansion $\theta$ along $v$

$$\theta(v) = \frac{1}{\sqrt{h}} C_v \sqrt{h},$$

(74)

where $h$ is the determinant of the induced metric on the section. The generalized second law of thermodynamics is then the statement that the area of the leaves is non-decreasing under the above flow

$$\theta(v) \geq 0.$$  

(75)

On the apparent horizon this area law is guaranteed by $\theta(\ell) = 0$, $\theta(n) < 0$ and $C \geq 0$. The boundary entropy current is obtained from $v$ by means of rewriting the left hand side of (73) within a chosen bulk-boundary map as a divergence of boundary current. This current is interpreted as a candidate boundary current and is given by \cite{24}.

$$J^\mu = \frac{1}{4G_N} \frac{1}{b} \sqrt{\frac{G}{g}} v^\mu.$$  

(76)

where the prefactor involving $G_N$ has been introduced to reproduce thermodynamic entropy at leading order and the AdS radius has been set to 1, as in \cite{11}. The technical assumptions used to derive (76) match those in Section \textbf{III} In

\footnote{The bulk-boundary map along ingoing null geodesics associates points on the apparent horizon, the event horizon or any other “generalized horizon” with boundary points lying on the same null geodesics moving close to the boundary in a direction specified by a given vector field. This vector field is taken to be proportional to $u^\mu$ in the leading order with subleading corrections modifying dual entropy current at orders higher than 2. As anticipated in section \textbf{III} in the gauge \cite{11} this bulk-boundary map acts trivially and maps points of the same $x^\mu$ position. Any such bulk-boundary map may be supplemented with boundary diffeomorphisms, which are generated by another vector field specified on the boundary. Such a vector field, if non-zero at leading order of the gradient expansion, must be also proportional to $u^\mu$, which modifies the dual entropy current at second and higher orders. The only parameter in \cite{10} shifted by boundary diffeomorphisms of such form is $j^\parallel_x$. For a detailed discussion of bulk-boundary maps see \cite{13}.}

\footnote{Relaxing the assumption of causality of bulk-boundary maps has so far not been explored. Note at this point that although the mapping along ingoing null geodesics seems (at least superficially) to be causal, boundary diffeomorphisms composed with a given bulk-boundary map might lead to causality violations (see Section \textbf{V} for a discussion of this point).}
particular, the formula (76) is valid for a trivial bulk-boundary map, i.e. along null geodesics, which in the vicinity of the boundary move in the direction defined by $u^\mu$. In the conformal case this direction can be modified only by second and higher order terms which change the entropy current at third order, and thus are beyond the scope of this article. The bulk-boundary map used here is not supplemented with boundary diffeomorphisms partly due to causality reasons (see Section V for more details). Because of this, the formula (76) leads to the unique causal second order entropy current.

To apply (76) to the gravitational setup of Section (III) one needs the form of $v$ and a computation of the determinant of the bulk metric $G$. One finds, up to second order in gradients

$$G = r^{2(d-1)} g \left( 1 - K_1 S_1 + \frac{1}{2} \frac{1}{(br)^2} S_2 \right).$$  \hspace{1cm} (77)

As discussed earlier, the vector $v$ is completely fixed by the self-consistency of the bulk construction; the second order result (54) reads

$$v^\mu = bu^\mu + b^2 \left( \frac{2}{d(d-2)} V_1^\mu + \frac{1}{d-2} V_2^\mu \right).$$  \hspace{1cm} (78)

The right hand side of (76) evaluated on the apparent horizon (67) leads to (70) where $j^\perp_1$ and $j^\perp_2$ are fixed by Frobenius condition and equal

$$j^\perp_1 = \frac{2}{d(d-2)}, \quad j^\perp_2 = \frac{1}{d-2}$$  \hspace{1cm} (79)

while the $j^\parallel_i$'s depend on the radial position of the apparent horizon and read

$$j^\parallel_1 = -K_1(1) - \frac{1}{d} K_2(1) + \frac{2(d-1)}{d(d-2)}$$
$$j^\parallel_2 = \frac{(2-3d)}{2d(d-2)},$$
$$j^\parallel_3 = -\frac{1}{d(d-2)}.$$  \hspace{1cm} (80)

As a crosscheck one can easily see that the coefficients (80) satisfy conditions (72) and (73). Comparing with the event horizon result [24, 25] one can see that the only difference is in the choice of $j^\parallel_1$

$$j^\parallel_{1,EH} = j^\parallel_{1,AH} - \frac{4}{d^2}.$$  \hspace{1cm} (81)

Calculating the area theorem on the apparent horizon up to the second order in gradients one obtains

$$\theta(\ell) - C\theta(n)|_{r=r_{AH}} = \frac{2S_1}{d},$$  \hspace{1cm} (82)

which indeed matches the hydrodynamic result up to second order in gradients (71). Calculating the third order contribution in the bulk requires third order geometry, which has so far not been obtained. Nevertheless the match is guaranteed by the formula (76) relating divergence of the entropy current to the area theorem on the apparent horizon modulo modifications of bulk-boundary map.

V. SUMMARY

This paper discusses the construction of apparent horizons in the geometry of conformal fluid-gravity duality. The motivation for this work are interrelated questions of local definition of entropy beyond equilibrium and foliation
dependence of apparent horizons of black holes. The reason for focusing on apparent horizons is that they are causal objects: they evolve in response to a flux of gravitational radiation or infall of matter. This makes them a preferable carrier for the notion of entropy beyond equilibrium in dual holographic field theories. There are however two caveats, which need to be taken into account: foliation dependence of apparent horizons \cite{16} and locality of entropy production being directly related to the mapping of horizon information onto the boundary \cite{17}. This paper focuses only on the first issue since the latter appears most severely at higher orders of the gradient expansion than are available for the geometry under consideration.

The key idea behind this paper is that the apparent horizons of interest are only those which are covariant in the sense of the dual hydrodynamic description, i.e. can be covariantly specified (in the boundary sense) in terms of $b$, $u^\mu$, $g_{\mu\nu}$ and their gradients. This requirement chooses only those apparent horizons which have covariant dual hydrodynamic entropy currents. Apparent horizons which do not satisfy this condition evade a clear physical interpretation in terms of the dual field theory and are not studied in this paper. Because of the requirement of hydrodynamic covariance, this paper adopts the somewhat unusual strategy of first finding suitable null normals and only afterwards confirming that they are foliation-forming, rather than starting with a foliation and then proceeding to normals. The approach adopted here relies on the near-equilibrium regime, where one expects that at least one of the apparent horizons will “closely” follow the dynamics of the event horizon \cite{16}. In the case of fluid-gravity duality this requirement is indeed satisfied – using even the results of \cite{15} alone one can easily check that the leading order event horizon is at the same time an (isolated) apparent horizon. It would certainly be interesting to try to apply similar methods to find apparent horizons in other black hole spacetimes, perhaps making contact with the framework of slowly evolving horizons \cite{30,32}.

The main result of this paper is that, up to second order in gradients in conformal fluid-gravity duality, there exists a unique apparent horizon covariant in the hydrodynamic sense. It is very plausible that the uniqueness of this apparent horizon holds to all orders of the gradient expansion, as arguments in Section \ref{sec:main_result} suggest. The apparent horizon in the geometry of fluid-gravity duality is isolated at leading and first subleading orders of the gradient expansion and becomes spatial once second order gradient contributions are included. As expected \cite{17} and confirmed by an explicit calculation in \cite{24}, the apparent horizon gives rise to a notion of hydrodynamic entropy current when the area form on the apparent horizon is mapped to the boundary in an appropriate way. Reference \cite{17} introduced the map spanned along ingoing null geodesics, the main motivation for it being the causal structure of bulk spacetime. Such geodesics are specified by their tangent vector at the boundary and hydrodynamic covariance forces this to be proportional to the fluid velocity at leading order, but starting at second order additional contributions appear. These terms will modify the form of the entropy current at third and higher orders of the gradient expansion and are beyond the scope of this paper. The only freedom, which affects the divergence of an entropy current at second order comes from combining the bulk-boundary map specified by ingoing bulk geodesics with boundary diffeomorphisms \cite{17}. It is not clear however, whether or not this leads to causality violations. One argument suggesting that it does was presented in \cite{24}. There, using the results from both \cite{17,25} and the present paper, it was shown that up to the second order in gradients the entropy current on the event horizon is equivalent to the entropy current on the apparent horizon when the bulk-boundary map in the latter case is supplemented with a particular boundary diffeomorphism. It would be very interesting to understand better the constraints on the form of the bulk-boundary map which follow from causality.

It seems unlikely that explicit causality violations due to the choice of the bulk-boundary map can be visible at low orders of the gradient expansion in the same way as in the near-equilibrium regime it is hard to tell whether the
event or one of apparent horizons is a better entropy carrier. Instead, one probably needs to look for some
general principles or at concrete examples based on numerical solutions in asymptotically AdS spacetimes. One such
a simple example is the gravity dual to boost-invariant flow. In the boost-invariant case, the boundary
dynamics depend on a single variable, the proper time $\tau$, and its large proper time limit is governed by boost-invariant
hydrodynamics. Consider now the setup introduced in where the initial state given by the vacuum AdS space
is excited in a boost-invariant way in the vicinity of $\tau = 0$ by a time-dependent boundary metric. Such a quench
leads to the emission of gravitational waves, and these propagate into the bulk and collapse forming a black hole.
The black hole equilibrates and in the end is well-described in terms of fluid-gravity duality. In this example, the
causal behavior of the apparent horizon is clearly visible only in the vacuum (before the quench) and in the far-
from-equilibrium regime. However, close to equilibrium the apparent and event horizon follow each other closely as
expected, and from that perspective there seems to be no reason to choose one over the other. Consider now a
slight modification of the setup. As initial state at some late time $\tau_i \gg 0$ one can choose a boost-invariant black
brane solution dual to perfect fluid hydrodynamics. Such a solution might have an arbitrary temperature and does not
produce any entropy as required by perfect fluid hydrodynamics. Consider now the same kind of quench as considered
in, but now in the vicinity of $\tau_i$. Such a quench will excite both far-from-equilibrium and hydrodynamics modes.
The former equilibrate over a time scale set by the inverse of temperature, so by taking temperature to be large, one
can effectively decouple them from analysis. Thus such setup serves as a causally clear example of entropy production,
governed entirely by hydrodynamics. It would be very interesting to see what are the constraints on the part of the
bulk-boundary map which corresponds to boundary diffeomorphisms following from this and similar examples.

As for more obvious further projects, it would be very interesting to calculate the location of apparent horizons in
the cases of charged, non-conformal, and superfluid fluid-gravity dualities. In those examples there are more gradient terms available so that the relevant backgrounds may serve as further testing grounds for the
claims of this paper. It would be also interesting to analyze the interplay between the position of the event and
apparent horizons in the context of the AdS/CFT correspondence making use of the technology introduced in.

In conclusion, the gravity dual to second order hydrodynamics of conformal media in arbitrary dimensions has a
unique apparent horizon, which is covariant in the hydrodynamic sense. Possible ambiguities, which appear in the
gravity construction, should not affect the amount of entropy produced between two equilibrium states (represented
on the gravity side by two isolated horizons). Furthermore if the foliation of the apparent horizon is fixed to all orders
in the gradient expansion and there are very stringent constraints on the bulk-boundary map, then it is plausible that
the local rate of entropy production in the near-equilibrium regime is a meaningful observable. This would be a very
interesting result from the point of view of non-equilibrium statistical mechanics.

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Appendix A: Weyl-covariance and available gradient terms

The Weyl-covariant derivative $\mathcal{D}$ is defined so that for an arbitrary tensor $Q^{\mu_1...\mu_d}_{\nu_1...\nu_d}$ of weight $w$ (i.e. one that obeys $Q^{\mu_1...\mu_d}_{\nu_1...\nu_d} \to e^{w\phi}Q^{\mu_1...\mu_d}_{\nu_1...\nu_d}$ under Weyl transformations (1)), $\mathcal{D}_\lambda Q^{\mu_1...\mu_d}_{\nu_1...\nu_d}$ transforms homogeneously with weight $w$

\begin{align}
\mathcal{D}_\lambda Q^{\mu_1...\mu_d}_{\nu_1...\nu_d} &\equiv \nabla_\lambda Q^{\mu_1...\mu_d}_{\nu_1...\nu_d} + w A_\lambda Q^{\mu_1...\mu_d}_{\nu_1...\nu_d} \\
&= \left[ g_{\lambda\alpha} A^\alpha - \delta^\alpha_\lambda A_\alpha - \delta_\alpha^\lambda A_\alpha \right] Q^{\mu_1...\mu_d}_{\nu_1...\nu_d} + \ldots \\
&- \left[ g_{\lambda\nu} A^\nu - \delta^\nu_\lambda A_\nu - \delta_\nu^\lambda A_\nu \right] Q^{\mu_1...\mu_d}_{\nu_1...\nu_d} + \ldots
\end{align}

(A1)

The field $A_\mu$ is given by (8) and transforms as a connection under Weyl transformations (8).

At first order in gradients there are two available contributions: $\mathcal{D}_\mu u_\nu$ and $\mathcal{D}_\mu b$. The latter quantity vanishes at this order of the gradient expansion when evaluated on solutions of the equations of perfect fluid hydrodynamics [29] and thus at the first order the only nontrivial contribution comes from $\mathcal{D}_\mu u_\nu$. Taking its symmetric part one obtains the shear tensor $\sigma_{\mu\nu}$

$$\sigma_{\mu\nu} = \frac{1}{2} \mathcal{D}_{(\mu} u_{\nu)},$$

(A2)

whereas its antisymmetric part is the vorticity of the flow

$$\omega_{\mu\nu} = \frac{1}{2} \mathcal{D}_{[\mu} u_{\nu]}.$$  

(A3)

Here $X_{(\mu\nu)} = X_{\mu\nu} + X_{\nu\mu}$ denotes symmetrization, whereas $X_{[\mu\nu]} = X_{\mu\nu} - X_{\nu\mu}$ antisymmetrization. One can show that both tensors (A2) and (A3) are transverse, i.e. $u^\mu \sigma_{\mu\nu} = u^\mu \omega_{\mu\nu} = 0$.

Following [25] one defines the Weyl-covariant Riemann tensor $\mathcal{R}_{\mu\nu\lambda\sigma}$

$$\mathcal{R}_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} + \nabla_{[\mu A_\nu]} g_{\lambda\sigma} - \sum_{\alpha} \hat{g}^\alpha_{\mu\nu} \delta^\beta_{\lambda\sigma} \left( \nabla_\alpha A_\beta + A_\alpha A_\beta - \frac{A^2}{2} g_{\alpha\beta} \right)$$

(A4)

Note that the Weyl-covariantized curvature tensors do not vanish even if the fluid lives in a flat spacetime. With this, the Weyl-covariant Ricci tensor $\mathcal{R}_{\mu\nu}$ and Weyl-covariant Ricci scalar $\mathcal{R}$ appearing in (13) can be defined:

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\lambda\nu}^\lambda \quad \text{and} \quad \mathcal{R} = \mathcal{R}^\lambda_\lambda$$

(A5)

Finally, the Weyl curvature tensor entering (15) is given by

$$C_{\mu\nu\lambda\sigma} = R_{\mu\nu\lambda\sigma} + \frac{1}{d-2} \sum_{\alpha} \hat{g}^\alpha_{\mu\nu} \delta^\beta_{\lambda\sigma} \left( \mathcal{R}_{\mu\nu} - \frac{\mathcal{R} g_{\mu\nu}}{2(d-1)} \right).$$

(A6)
Appendix B: Example: the Vaidya metric

To see how the method for finding apparent horizon introduced in the context of “generalized horizons” in [24] and reviewed in Section III works, consider the example of the Vaidya metric:

\[ ds^2 = 2dwdr - f(r, w)dw^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (B1)

with

\[ f(r, w) = 1 - \frac{2m(w)}{r}. \] (B2)

In this case symmetry suggests taking \( S = r - g(w) \) (i.e. spherically symmetric apparent horizon). Then the covector \( m \) is

\[ m_a = (dr - g'(w)dw)_a \] (B3)

and raising the index leads to

\[ m^a = (\partial_w + (f(r, w) - g'(w))\partial_r)^a. \] (B4)

Here latin indices are used for \((r, w, \theta, \phi)\) and greek ones (below) for \((w, \theta, \phi)\) – this is sort of analogous to the conventions used earlier for the AdS case. Choosing slicing on the horizon given by \( v^\mu = (\partial_w)^\mu \) leads to

\[ v^a = (\partial_w + g'(w)\partial_r)^a \] (B5)

Using (B6) gives to

\[ l = \partial_w + \frac{1}{2}f(r, w) \partial_r \quad \text{and} \quad n = \lambda (f(r, w) - 2g'(w)) \partial_r. \] (B6)

Imposing the normalization condition (50) to fix \( \lambda \) gives the expected result [32]

\[ l = \partial_w + \frac{1}{2}f(r, w) \partial_r \quad \text{and} \quad n = -\partial_r. \] (B7)

Note that in spherical symmetry vectors \( l \) and \( n \) are fixed up to an overall scaling by conditions

\[ l_al^a = n_an^a = 0 \quad \text{and} \quad l_an^a = -1. \] (B8)

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11 The letter \( w \) denotes the ingoing Eddington-Finkelstein time coordinate to avoid clashing with the choice of \( v \) for the vector which defines the slicing