$L^p$-Estimates for Singular Oscillatory Integral Operators

Per Sjölin

Abstract In this paper we study singular oscillatory integrals with a nonlinear phase function. We prove estimates of $L^2 \to L^2$ and $L^p \to L^p$ type.

Keywords Singular integral · Oscillatory integral · Nonlinear phase function

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1 Introduction

Let $K$ denote a singular kernel in $\mathbb{R}^n$. Singular integral operators $T$, defined by $Tf(x) = \int_{\mathbb{R}^n} K(x - y) f(y) dy$, $x \in \mathbb{R}^n$, $f \in C_0^\infty(\mathbb{R}^n)$, have been studied for a very long time. Since approximately 1970 there has also been a lot of interest in oscillatory integral operators. The following theorem describes a typical result.

Theorem 1.1 (see Stein [6], p. 377) Let $\psi_1 \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and $\lambda > 0$ and let $\Phi$ be real-valued and smooth. Set

$$U_\lambda f(x) = \int_{\mathbb{R}^n} e^{i\lambda \Phi(x, \xi)} \psi_1(x, \xi) f(x) dx, \quad \xi \in \mathbb{R}^n,$$

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and assume that \( \det \left( \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right) \neq 0 \) on \( \text{supp} \psi_1 \). Then one has

\[
\| \mathcal{U}_\lambda f \|_{L^2(\mathbb{R}^n)} \leq C \lambda^{-n/2} \| f \|_{L^2(\mathbb{R}^n)}.
\]

We shall here consider singular oscillatory integral operators, that is operators defined by integrals containing both a singular kernel and an oscillating factor. Operators of this type have been much studied in the theory of convergence of Fourier series and also in for instance Phong and Stein [4]. We shall continue this study.

Let \( \psi_0 \in C^\infty_0(\mathbb{R}^n \times \mathbb{R}^n - 1) \) and \( n \geq 2 \). For \( f \in L^2(\mathbb{R}^n - 1) \) set

\[
T_\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{i \lambda |x - (y', 0)|^\gamma} \psi_0(x, y') K(x - (y', 0)) f(y') dy'
\]

for \( x \in \mathbb{R}^n, \gamma > 0, \) and \( \lambda \geq 2 \). Here for \( \gamma > 1 \) we set

\[
K(z) = |z|^{-(n-m-1)}, \ z \in \mathbb{R}^n \setminus \{0\},
\]

and for \( 0 < \gamma \leq 1 \) we set

\[
K(z) = |z|^{-(n-m-1)} \omega(z), \ z \in \mathbb{R}^n \setminus \{0\},
\]

where \( \omega \in C^\infty(\mathbb{R}^n \setminus \{0\}) \), \( \omega \) is homogeneous of degree 0, and \( \omega(z) = 0 \) for all \( z \) with \( |z| = 1 \) and \( |z_n| \leq \varepsilon_0 \) for some given \( \varepsilon_0 > 0 \). We also assume that \( 0 < m < n - 1 \).

We shall study the norm of \( T_\lambda \) as an operator from \( L^p(\mathbb{R}^{n-1}) \) to \( L^p(\mathbb{R}^n) \) and denote this norm by \( \| T_\lambda \|_p \). In Aleksanyan et al. [1] the following theorem was proved.

**Theorem 1.2** Set \( \alpha = (n - 1)/2 \) and assume \( \gamma \geq 1 \). Then one has

\[
\| T_\lambda \|_2 \leq \begin{cases} 
C \lambda^{-(m+1)/2}/\gamma, & m < \gamma \alpha - 1/2, \\
C \lambda^{-\alpha} \log \lambda, & m = \gamma \alpha - 1/2, \\
C \lambda^{-\alpha}, & m > \gamma \alpha - 1/2.
\end{cases}
\]

The above choice of phase function is partially motivated by an application to an inhomogeneous Helmholtz equation where we give estimates for solutions. In this case we take \( \gamma = 1 \) (see [1], p. 544). It is also possible to use \( T_\lambda \) to give \( L^p \)-estimates for convolution operators. This will be studied in a forthcoming paper.

In [1] it is also proved that \( \| T_\lambda \|_2 \geq c \lambda^{-(m+1)/2}/\gamma \) for \( \gamma > 1 \), where \( c \) denotes a positive constant. We shall here prove that this also holds for \( \gamma = 1 \) and that \( \| T_\lambda \|_2 \geq c \lambda^{-\alpha} \) for \( \gamma \geq 1 \). It follows that the results in Theorem 1.2 are essentially sharp.

In this paper we shall first study the case \( n = 2 \) and \( 1 < p < \infty \). We have the following theorem.
Theorem 1.3 Assume \( n = 2 \) and \( 0 < \gamma \leq 1 \). Then \( \|T_\lambda\|_2 \leq C\lambda^{-1/2} \), and for \( 2 < p \leq 4 \) one has

\[
\|T_\lambda\|_p \leq \begin{cases} 
C\lambda^{-(1/p+m)/\gamma}, & 1/p + m < \gamma/2, \\
C\epsilon\lambda^{\gamma/2}, & 1/p + m \geq \gamma/2,
\end{cases}
\]

where \( \epsilon \) denotes an arbitrary positive number. Also set \( \beta(p) = 1 - 1/p \) for \( 1 < p < 2 \), and \( \beta(p) = 2/p \) for \( 4 < p < \infty \). For \( 1 < p < 2 \) and \( 4 < p < \infty \) one has

\[
\|T_\lambda\|_p \leq \begin{cases} 
C\lambda^{-(1/p+m)/\gamma}, & 1/p + m < \gamma\beta(p), \\
C\lambda^{-\beta(p)} \log \lambda, & 1/p + m = \gamma\beta(p), \\
C\lambda^{-\beta(p)}, & 1/p + m > \gamma\beta(p).
\end{cases}
\]

We shall also study the sharpness of the estimates in Theorem 1.3. We shall then estimate the operator \( S_\lambda \) given by

\[
S_\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{i\lambda|x-y|^\gamma} \psi_0(x, y) K(x - y) f(y) dy, \quad x \in \mathbb{R}^{n-1},
\]

where \( n \geq 2 \), \( \psi_0 \in C^\infty_0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}) \), and \( K(z) = |z|^{-(n-m-1)} \), \( z \in \mathbb{R}^{n-1} \setminus \{0\} \). We let \( \|S_\lambda\|_p \) denote the norm of \( S_\lambda \) as an operator from \( L^p(\mathbb{R}^{n-1}) \) to \( L^p(\mathbb{R}^{n-1}) \). We shall prove the following theorem.

Theorem 1.4 Assume \( n \geq 2 \), \( 0 < m < n - 1 \), \( \gamma > 0 \), and \( \gamma \neq 1 \). Then

\[
\|S_\lambda\|_2 \leq \begin{cases} 
C\lambda^{-m/\gamma}, & m < \gamma\alpha, \\
C\lambda^{-\alpha} \log \lambda, & m = \gamma\alpha, \\
C\lambda^{-\alpha}, & m > \gamma\alpha,
\end{cases}
\]

where \( \alpha = (n-1)/2 \). Here the constant \( C \) depends on \( n, m, \) and \( \gamma \).

We shall point out a relation between the operators \( T_\lambda \) and \( S_\lambda \). We choose \( \gamma > 1 \) and take \( K(z) = |z|^{-(n-m-1)} \), \( z \in \mathbb{R}^{n-1} \setminus \{0\} \), and let \( T_\lambda \) be defined as above. Then setting \( x = (x', x_n) \), where \( x' = (x_1, x_2, \ldots, x_{n-1}) \) we obtain

\[
T_\lambda f(x', 0) = \int_{\mathbb{R}^{n-1}} e^{i\lambda|x'-y'|^\gamma} \psi_0(x', 0, y') K(x' - y', 0) f(y') dy',
\]

that is we obtain an operator of type \( S_\lambda \). The reason for introducing the homogeneous function \( \omega \) in the above definition of \( T_\lambda \) for \( 0 < \gamma \leq 1 \) is that we want certain determinant conditions to be satisfied. This is discussed in [1, p. 539], and in this paper after the proof of Lemma 2.2.
We shall also make some remarks on an operator which is somewhat similar to $S_\lambda$.

Set

$$L(x) = \frac{e^{i|x|^a}}{|x|^\alpha}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

where $a > 0$, $a \neq 1$, and $\alpha < n$. Then $L$ belongs to the space $S'((\mathbb{R}^n)$ of tempered distributions and we set

$$Tf = L \ast f, \quad f \in C^\infty_0(\mathbb{R}^n).$$

We say that the operator $T$ is bounded on $L^p(\mathbb{R}^n)$ if

$$\|Tf\|_p \leq C_p \|f\|_p, \quad f \in C^\infty_0(\mathbb{R}^n).$$

In Sjölin [5] the following theorem is proved.

**Theorem 1.5** If $\alpha \geq n(1 - a/2)$ set $p_0 = na/(na - n + \alpha)$. Then $T$ is bounded on $L^p(\mathbb{R}^n)$ if and only if $p_0 \leq p \leq p'_0$. If $\alpha < n(1 - a/2)$ then $T$ is not bounded on any $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

We finally remark that Theorem 1.1 is due to Hörmander.

In Sect. 2 we shall give the proofs of Theorems 1.3 and 1.4. In Sect. 3 we shall discuss the sharpness of the results in these theorems.

## 2 Proofs of Theorems 1.3 and 1.4

We shall apply the following theorem.

**Theorem 2.1** (see Hörmander [3], p. 3) Let $\psi_1 \in C^\infty_0(\mathbb{R}^3)$, let $\varphi \in C^\infty(\mathbb{R}^3)$ be real-valued, and assume that the determinant

$$J = \begin{vmatrix} \varphi_{xt} & \varphi_{yt} \\ \varphi_{xtt} & \varphi_{ytt} \end{vmatrix} \neq 0$$

on supp$\psi_1$. Here $\varphi = \varphi(x, y, t)$ and $\varphi_{xt} = \frac{\partial^2 \varphi}{\partial x \partial t}$ etc. Set

$$U_N f(x, y) = \int_{\mathbb{R}} e^{iN\varphi(x, y, t)}\psi_1(x, y, t) f(t) dt, \quad N \geq 1,$$

for $f \in L^1(\mathbb{R})$ and $(x, y) \in \mathbb{R}^2$. It follows that

$$\|U_N f\|_{L^q(\mathbb{R}^2)} \leq CN^{-2/q(q/(q - 4))}^{1/4} \|f\|_{L^r(\mathbb{R})}$$

if $q > 4$ and $3/q + 1/r = 1$. 
We shall need an estimate of the norm of \( \mathcal{U}_N \) as an operator from \( L^p(\mathbb{R}) \) to \( L^p(\mathbb{R}^2) \). We denote this norm by \( ||\mathcal{U}_N||_p \). An application of Theorem 2.1 will give the inequalities in the following lemma.

**Lemma 2.2** Let \( \mathcal{U}_N \) be defined as in Theorem 2.1. Then one has

\[
||\mathcal{U}_N||_p \leq C N^{-\beta(p)}, \quad 1 < p < \infty,
\]

where

\[
\beta(p) = \begin{cases} 
1 - 1/p, & 1 < p \leq 2, \\
1/2 - \varepsilon, & 2 < p \leq 4, \\
2/p, & 4 < p < \infty.
\end{cases}
\]

Here \( \varepsilon \) is an arbitrary positive number and \( C \) depends on \( \varphi \) and \( p \), and in the case \( 2 < p \leq 4 \), also on \( \varepsilon \).

**Proof** Assume that \( \text{supp} \psi_1 \subset B_2 \times B_1 \), where \( B_1 \) is a ball in \( \mathbb{R} \) and \( B_2 \) a ball in \( \mathbb{R}^2 \). We then have \( \mathcal{U}_N f = \mathcal{U}_N(\mu f) \) if \( \mu \in C_0^\infty(\mathbb{R}) \) and \( \mu(t) = 1 \) for \( t \in B_1 \). Now take \( q > 4 \) and assume that \( 3/q + 1/r = 1 \). It follows that \( 1 < r < 4 \) and using Hölder’s inequality twice and Theorem 2.1 we obtain

\[
||\mathcal{U}_N f||_4 \leq C||\mathcal{U}_N f||_q = C||\mathcal{U}_N(\mu f)||_q \leq \]

\[
CN^{-2/q}||\mu f||_r \leq CN^{-2/q}||\mu f||_4 \leq CN^{-2/q}||f||_4.
\]

Hence

\[
||\mathcal{U}_N f||_4 \leq C N^{\varepsilon - 1/2}||f||_4 \quad (2.1)
\]

for every \( \varepsilon > 0 \), where the constant depends on \( \varepsilon \). Then we shall obtain an \( L^2 \)-estimate for the operator \( \mathcal{U}_N \). From the condition on \( J \) in Theorem 2.1 it follows that there exists a number \( \delta_0 > 0 \) such that

\[
\delta_0 \leq |J| \leq C_0(\varphi_{xt} + \varphi_{yt})
\]

on \( \text{supp} \psi_1 \), where \( C_0 \) depends on \( \varphi \).

Choose \( \mu_j \in C_0^\infty(\mathbb{R}^3), j = 2, 3, \ldots, M \), such that \( \sum_j \mu_j(x, y, t) = 1 \) for \( (x, y, t) \in Q \) and each \( \mu_j \) has support in a small cube. Here \( Q \) is a cube in \( \mathbb{R}^3 \) with center at the origin and \( \text{supp} \psi_1 \subset Q \). It follows that

\[
\psi_1 = \sum_{2}^{M} \psi_1 \mu_j = \sum_{2}^{M} \psi_j,
\]
where $\psi_j = \psi_1 \mu_j$. Setting

$$U_N^{(j)} f(x, y) = \int_{\mathbb{R}} e^{iN\varphi(x, y, t)} \psi_j(x, y, t) f(t) dt$$

we have

$$U_N = \sum_{j=2}^{M} U_N^{(j)}$$

and shall estimate each $U_N^{(j)}$.

If $(x_0, y_0, t_0) \in \text{supp} \psi_j$ then $(x_0, y_0, t_0) \in \text{supp} \psi_1$ and $|\varphi_{xt}| \geq \delta/2$ or $|\varphi_{yt}| \geq \delta/2$ at $(x_0, y_0, t_0)$, where $\delta = \delta_0/C_0$. Say that $|\varphi_{xt}| \geq \delta/2$. Then $|\varphi_{xt}| \geq \delta/4$ on $\text{supp} \psi_j$ since $\text{supp} \psi_j$ is contained in a small cube.

Invoking Theorem 1.1 we get

$$\left( \int |U_N^{(j)} f(x, y)|^2 dx \right)^{1/2} \leq CN^{-1/2} \left( \int |f(t)|^2 dt \right)^{1/2}$$

for every $y$. Integrating in $y$ and summing over $j$ we then obtain

$$||U_N f||_{L^2(\mathbb{R}^2)} \leq CN^{-1/2} ||f||_{L^2(\mathbb{R})}.$$ (2.2)

Interpolating between the inequalities (2.1) and (2.2) one has

$$||U_N f||_{L^p(\mathbb{R}^2)} \leq CN^{q-1/2} ||f||_{L^p(\mathbb{R})}, \quad 2 < p \leq 4$$ (2.3)

for every $\varepsilon > 0$.

We then assume $q > 4$. Choosing $\mu$ as above we have $U_N(f) = U_N(\mu f)$ and it follows that

$$||U_N f||_q \leq CN^{-2/q} ||\mu f||_r \leq CN^{-2/q} ||\mu f||_q \leq CN^{-2/q} ||f||_q,$$ (2.4)

where we have used Hölder’s inequality. It remains to study the case $1 < p < 2$. Interpolating between (2.2) and the trivial estimate $||U_N f||_1 \leq C||f||_1$ one obtains

$$||U_N f||_p \leq CN^{-(1-1/p)} ||f||_p, \quad 1 < p < 2,$$ (2.5)

and Lemma 2.2 follows from (2.2), (2.3), (2.4), and (2.5).

Now let $\varphi(x, y, t) = d^\gamma$, where $d = ((x-t)^2 + y^2)^{1/2}$ and $0 < \gamma \leq 1$. A computation shows that

$$J = \gamma^2(\gamma - 2)y((\gamma - 1)(x-t)^2 - y^2)$$
for $d = 1$. Since $\mathcal{J}$ is a homogeneous function of degree $2\gamma - 5$ of $(x_0, y)$ where $x_0 = x - t$, we conclude that if $1/2 \leq d \leq 2$ and $|y| \geq c > 0$ on $\text{supp}\psi_1$, then $|\mathcal{J}| \geq c_1 > 0$ on $\text{supp}\psi_1$. Hence (2.2)–(2.5) hold in this case.

We remark that in the case $\gamma = 1$ $\mathcal{J}$ was computed in Carleson and Sjölin [2], and that in the case $\gamma = 1$ (2.2) and (2.3) are proved in [2] in the case $\psi_1(x, y, t) = \chi_1(t)\chi_2(x, y)$, where $\chi_1$ is the characteristic function for the interval $[0, 1]$ and $\chi_2$ is the characteristic function for the square $[0, 1] \times [2, 3]$. We shall now prove Theorem 1.3.

**Proof of Theorem 1.3.** We shall estimate the norm of $T_\lambda$ where

$$T_\lambda f(x) = \int_{\mathbb{R}} e^{i\lambda|x-(y',0)|^\gamma} \psi_0(x, y')K(x - (y', 0)) f(y')dy',$$

where $x \in \mathbb{R}^2$. Here $\lambda \geq 2$, $0 < \gamma \leq 1$, and $\psi_0 \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R})$. Also $K(z) = |z|^{m-1}\omega(z)$, $z \in \mathbb{R}^2 \setminus \{0\}$, where $0 < m < 1$ and $\omega$ is described in the introduction.

We first observe that there exists $\psi \in C_0^\infty(\mathbb{R}^2)$, with support in $\{x \in \mathbb{R}^2 : 1/2 \leq |x| \leq 2\}$ such that $K(z) = \sum_{k=-\infty}^{\infty} 2^{k(1-m)}\psi(2^kz)\omega(z)$ (see Stein [6, p. 393]). Since $\text{supp}\psi_0$ is bounded it follows that there exists an integer $k_0$ such that $K(z) = \sum_{k=k_0}^{\infty} 2^{k(1-m)}\psi(2^kz)\omega(z)$ for all $z = x - (y', 0)$ with $(x, y') \in \text{supp}\psi_0$.

We shall assume that $k_0 = 0$. The proof in the general case is the same as for $k_0 = 0$. Also choose $\chi \in C_0^\infty(\mathbb{R})$ such that $\text{supp}\chi \subset [-1/2 - 1/10, 1/2 + 1/10]$ and $\sum_{j=-\infty}^{\infty} \chi(t - j) = 1$.

We have $T_\lambda f = \sum_{k=0}^{\infty} T_{\lambda, k} f$ where

$$T_{\lambda, k} f(x) = \int_{\mathbb{R}} e^{i\lambda|x-(y',0)|^\gamma} \psi_0(x, y')2^{k(1-m)}\psi(2^k(x - (y', 0)))\omega(x - (y', 0)) f(y')dy',$$

Also $T_{\lambda, k} f = \sum_{j} T_{\lambda, k} f_j$ where $f_j(t) = f(t)\chi(2^k(t-2^{-k}j))$. Assuming $1 < p < \infty$ and invoking Hölder’s inequality we obtain

$$|T_{\lambda, k} f(x)|^p \leq C \sum_{j} |T_{\lambda, k} f_j(x)|^p,$$

since the number of terms in the above sum is bounded.

Setting $y' = 2^{-k}z'$ we get

$$T_{\lambda, k} f_j(x) = \int_{\mathbb{R}} e^{i\lambda|x-(y',0)|^\gamma} 2^{k(1-m)}\psi_0(x, y')\psi(2^k(x - (y', 0)))\omega(x - (y', 0)) f_j(y')dy'.$$
\[= 2^{-mk} \int_{\mathbb{R}} e^{i\lambda x - 2^{-k}(\xi', 0)^{y'}} \psi_0(x, 2^{-k} z') \psi(2^k x - (\xi', 0)) \omega(x - 2^{-k}(\xi', 0)) f_j(2^{-k} z') dz' \]

\[= 2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-k} y [2^k x - (\xi, 0)^{y'}]} \psi_0(x, 2^{-k} z') \psi(2^k x - (\xi', 0)) \omega(2^k x - (\xi', 0)) f(2^{-k} z') \chi(z' - j) dz' \]

\[= [\text{with } y' = z' - j] 2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-ky}[2^k x - (y' + j, 0)^{y'}]} \psi_0(x, 2^{-k} (y' + j)) \psi(2^k x - (y' + j, 0)) \times \omega(2^k x - (y' + j, 0)) f(2^{-k} (y' + j)) \chi(y') dy' \]

\[= 2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-ky}[2^k x - (y' - k j, 0)^{y'}]} \psi_0(x, 2^{-k} (y' - k j, 0)) - (y', 0) \omega(2^k x - (2^{-k} j, 0)) - (y', 0) \times f(2^{-k} j + 2^{-k} y') \chi(y') dy'. \]

We also have
\[
\int_{\mathbb{R}^2} |T_{\lambda,k} f_j(x)|^p dx = [\text{with } x = u + (2^{-k} j, 0)]
\int_{\mathbb{R}^2} |T_{\lambda,k} f_j(u + (2^{-k} j, 0))|^p du = [\text{with } \xi = 2^k u]
2^{-2k} \int_{\mathbb{R}^2} |T_{\lambda,k} f_j(2^{-k} \xi + (2^{-k} j, 0))|^p d\xi. \tag{2.6}
\]

Now let \( \bar{\chi} \in C_0^\infty(\mathbb{R}) \) be so that \( \bar{\chi} = 1 \) on \( \text{supp} \chi \) and \( \text{supp} \bar{\chi} \subset [-1, 1] \). We then have
\[
T_{\lambda,k} f_j(2^{-k} \xi + (2^{-k} j, 0))) = 2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-ky}[\xi - (y', 0)^{y'}]} \psi_0(2^{-k} \xi)
+ (2^{-k} j, 0), 2^{-k} j + 2^{-k} y') \psi(\xi - (y', 0)) \times \omega(\xi - (y', 0)) f(2^{-k} j + 2^{-k} y') \chi(y') dy' \]
\[= 2^{-mk} \int_{\mathbb{R}} e^{i\lambda 2^{-ky} \Phi(y', \xi)} \psi_1(y', \xi) g(y') dy'
= 2^{-mk} U_{\lambda 2^{-k} y} g(\xi), \]

where
\[\Phi(y', \xi) = |\xi - (y', 0)|^y = (|\xi' - y'|^2 + \xi_2^2)^{\gamma/2},\]
\[\psi_1(y', \xi) = \psi(\xi - (y', 0)) \omega(\xi - (y', 0)) \psi_0(2^{-k} \xi + (2^{-k} j, 0), 2^{-k} j + 2^{-k} y') \bar{\chi}(y'),\]
and
\[g(y') = f(2^{-k} j + 2^{-k} y') \chi(y').\]

Here \( \xi = (\xi_1, \xi_2) = (\xi', \xi_2). \)
It is clear that $\psi_1$ has a support which is uniformly bounded in $j$ and $k$, and the derivatives of $\psi_1$ can be bounded uniformly in $j$ and $k$. Here we use the fact that $k \geq 0$.

Invoking (2.6) we conclude that
\[
\left( \int_{\mathbb{R}^2} |T_{\lambda,k} f_j(x)|^p dx \right)^{1/p} = 2^{-2k/p} 2^{-mk} \left( \int_{\mathbb{R}^2} |\mathcal{U}_{\lambda^2-ky} g(\xi)|^p d\xi \right)^{1/p}.
\]

We set $d = (|\xi'| - y'|^2 + \xi_2^2)^{1/2}$. It follows from the definitions of $\psi$ and $\omega$ that $1/2 \leq d \leq 2$ and $|\xi_2| \geq c > 0$ on supp $\psi_1$. Hence the determinant $J$ for the phase function $\Phi_1$ satisfies $|J| \geq c > 0$ on supp $\psi_1$, as we remarked after the proof of Lemma 2.2. We can therefore apply Lemma 2.2 and one obtains
\[
\left( \int_{\mathbb{R}^2} |\mathcal{U}_{\lambda^2-ky} g(\xi)|^p d\xi \right)^{1/p} \leq C(\lambda 2^{-ky})^{-\beta(p)} ||g||_{L^p(\mathbb{R})}.
\]

We have $g = g_{j,k}$ and
\[
\int_{\mathbb{R}} |g_{j,k}|^p dy' \leq \int_{-1}^{1} |f(2^{-k} j + 2^{-k} y')|^p dy' = 2^k \int_{|z'| \leq 2^{-k}} |f(2^{-k} j + z')|^p dz'
\]
and it follows that
\[
\sum_{j=-\infty}^{\infty} \int_{\mathbb{R}} |g_{j,k}|^p dy' \leq C 2^k ||f||_p^p.
\]

Hence
\[
\int_{\mathbb{R}^2} |T_{\lambda,k} f_j|^p dx \leq C \sum_j \int_{\mathbb{R}^2} |T_{\lambda,k} f_j|^p dx \leq C 2^{-2k} 2^{-mkp} (\lambda 2^{-ky})^{-\beta(p)}
\]
\[
\sum_j \int_{\mathbb{R}} |g_{j,k}|^p dy' \leq C 2^{-2k} 2^{-mkp} (\lambda 2^{-ky})^{-p\beta(p)} ||f||_p^p
\]
and we obtain the inequality
\[
||T_{\lambda,k}||_p \leq C 2^{-k/p} 2^{-mk} (\lambda 2^{-ky})^{-\beta(p)}.
\]

Making a trivial estimate we also have
\[
||T_{\lambda,k}||_p \leq C 2^{-k/p} 2^{-mk}.
\]
Invoking the inequality $\|T_\lambda\|_p \leq \sum_{0}^{\infty} \|T_{\lambda,k}\|_p$ we obtain

$$\|T_\lambda\|_p \leq C \lambda^{-\beta(p)} \sum_{2^k \leq \lambda^{1/\gamma}} 2^{k(-1/p-m+\gamma\beta(p))} + C \sum_{2^k > \lambda^{1/\gamma}} 2^{-k(1/p+m)} = A + B.$$ 

It is clear that $B \leq C \lambda^{-\gamma\beta(p)}$ and in the case $1/p + m < \gamma\beta(p)$ we get $A \leq C \lambda^{-\beta(p)} \lambda^{(-1/p-m+\gamma\beta(p))/\gamma} = C \lambda^{-(1/p+m)/\gamma}$

and

$$\|T_\lambda\|_p \leq C \lambda^{-(1/p+m)/\gamma}.$$ 

In the case $1/p + m = \gamma\beta(p)$ we get $A \leq C \lambda^{-\beta(p)} \log \lambda$ and $\|T_\lambda\|_p \leq C \lambda^{-\beta(p)} \log \lambda$.

Finally, in the case $1/p + m > \gamma\beta(p)$ we have $A \leq C \lambda^{-\beta(p)}$ and $\|T_\lambda\|_p \leq C \lambda^{-\beta(p)}$.

We remark that in the case $p = 2$ only the case $1/p + m > \gamma\beta(p)$ can occur. The proof of Theorem 1.3 is complete. □

Before proving Theorem 1.4 we shall make a preliminary observation. Set $\xi = (\xi', \xi_n)$ where $\xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1})$ and $n \geq 2$. Also set $x' = (x_1, x_2, \ldots, x_{n-1})$ and $\Phi(x', \xi) = d^\gamma$ where $\gamma > 0$ and $d = (|\xi' - x'|^2 + \xi_n^2)^{1/2}$. In [1, Section 4.1], we studied the determinant

$$P(x', \xi', \xi_n) = \det \left( \frac{\partial^2 \Phi}{\partial x_i \partial \xi_j} \right)_{i,j=1}^{n-1}$$

for $1/2 \leq d \leq 2$. In [1] it is proved that

$$P(x', \xi', \xi_n) = (-\gamma d^{\gamma-2})^{n-1} (\gamma - 1)|\xi' - x'|^2 + \xi_n^2 \over d^2. \quad (2.7)$$

Now let $\Phi_1(x', \xi') = |\xi' - x'|^\gamma = d_1^\gamma$ where $d_1 = |\xi' - x'|$. We shall need the determinant

$$P_1(x', \xi') = \det \left( \frac{\partial^2 \Phi_1}{\partial x_i \partial \xi_j} \right)_{i,j=1}^{n-1}.$$ 

It is clear that

$$P_1(x', \xi') = P(x', \xi', 0) = (-\gamma d_1^{\gamma-2})^{n-1} (\gamma - 1)$$

and for $\gamma > 0$, $\gamma \neq 1$, it follows that

$$|P_1(x', \xi')| \geq c > 0 \text{ for } 1/2 \leq d_1 \leq 2. \quad (2.8)$$
Proof of Theorem 1.4. We shall use the method in the proof of Theorem 1.3 and omit some details. We assume that

$$K(z) = \sum_{k=0}^{\infty} 2^{k(n-1-m)} \psi(2^k z),$$

where $\text{supp} \psi \subset \{ x \in \mathbb{R}^{n-1}, 1/2 \leq |x| \leq 2 \}$. One obtains

$$S^\lambda f = \sum_{k=0}^{\infty} S^\lambda f_k,$$

where

$$S^\lambda f_k(x) = \int_{\mathbb{R}^{n-1}} e^{i \lambda |x-y|} \psi_0(x,y) 2^{k(n-1-m)} \psi(2^k (x-y)) f(y) dy.$$

We also have

$$f = \sum_{j \in \mathbb{Z}^{n-1}} f_j,$$

where

$$f_j(t) = f(t) \chi(2^k(t - 2^{-k}j)), \ j \in \mathbb{Z}^{n-1}, \ t \in \mathbb{R}^{n-1},$$

and $\chi \in C_0^\infty(\mathbb{R}^{n-1})$ is like $\chi$ in the proof of Theorem 1.3.

The Schwarz inequality gives the estimate

$$|S^\lambda f_k(x)|^2 \leq C \sum_j |S^\lambda f_j(x)|^2$$

and arguing as in the proof of Theorem 1.3 we get

$$S^\lambda f_j(x) = 2^{-mk} \int_{\mathbb{R}^{n-1}} e^{i \lambda 2^{-k}y} |2^k(x - 2^{-k}j) - y| \psi_0(x, 2^{-k}j + 2^{-k}y) \psi(2^k(x - 2^{-k}j) - y) \times f(2^{-k}j + 2^{-k}y) \chi(y) dy$$

and

$$\int_{\mathbb{R}^{n-1}} |S^\lambda f_j(x)|^2 dx = 2^{-k(n-1)} \int_{\mathbb{R}^{n-1}} |S^\lambda f_j(2^{-k} \xi + 2^{-k}j)|^2 d\xi.$$
It follows that

\[ S_{\lambda, k} f_j (2^{-k} \xi + 2^{-k} j) = 2^{-mk} \int_{\mathbb{R}^{n-1}} e^{i \lambda 2^{-k} y' |y'|} \psi_0 (2^{-k} \xi + 2^{-k} j, 2^{-k} j + 2^{-k} y) \times \psi (\xi - y) f (2^{-k} j + 2^{-k} y) \chi (y) dy \]

\[ = 2^{-mk} \int_{\mathbb{R}^{n-1}} e^{i \lambda 2^{-k} y' \Phi_1 (y, \xi)} \psi_1 (y, \xi) g (y) dy \]

where \( \Phi_1 (y, \xi) = |\xi - y'|, \psi_1 (y, \xi) = \psi (\xi - y) \psi_0 (2^{-k} \xi + 2^{-k} j, 2^{-k} j + 2^{-k} y) \chi (y), \)

and \( g (y) = f (2^{-k} j + 2^{-k} y) \chi (y). \)

Invoking the determinant condition (2.8) and Theorem 1.1 we conclude that

\[ ||U_{\lambda, 2^{-k} y} g||_{L^2 (\mathbb{R}^{n-1})} \leq C (\lambda 2^{-k} y)^{-\alpha} ||g||_{L^2 (\mathbb{R}^{n-1})} \]

where \( \alpha = (n - 1)/2. \) Arguing as in the proof of Theorem 1.3 we then obtain

\[ ||S_{\lambda, k}||_2 \leq C 2^{-mk} \lambda^{-\alpha} 2^{k y' \alpha} \]

and \( ||S_{\lambda, k}||_2 \leq C 2^{-mk}. \)

Hence

\[ ||S_{\lambda}||_2 \leq C \lambda^{-\alpha} \sum_{2^k \leq \lambda^{1/\gamma}} 2^{k (\gamma \alpha - m) k} + \sum_{2^k \geq \lambda^{1/\gamma}} 2^{-mk} \]

and Theorem 1.4 follows easily from this inequality. \( \square \)

3 Counter-examples

Assume \( \gamma > 0, 1 < p < \infty, \) and

\[ T_{\lambda} f (x) = \int_{\mathbb{R}^{n-1}} e^{i \lambda |x - (y', 0)|} \psi_0 (x, y', K (x - (y', 0)) f (y') dy', \]

where \( x \in \mathbb{R}^n, n \geq 2, \) and \( K (z) = |z|^{m-n+1} \) with \( 0 < m < n - 1. \) We shall estimate the norm \( ||T_{\lambda}||_p = ||T_{\lambda}||_{L^p (\mathbb{R}^{n-1}) \rightarrow L^p (\mathbb{R}^n)} \) from below. We take \( y'_0 \in \mathbb{R}^{n-1} \) and set \( E = B (y'_0; c_0 \lambda^{-\rho}) \) where \( B (x; R) \) denotes a ball with center \( x \) and radius \( R. \) Also let \( F \) denote a cube in \( \mathbb{R}^n \) with center \( (y'_0, 100 c_0 \lambda^{-\rho}) \) and side length \( c_0 \lambda^{-\rho}. \) We assume that \( \psi_0 (x, y') = 1 \) for \( x \in F \) and \( y' \in E. \)
Setting $f = \chi_E$ and taking $x \in F$ we obtain

$$T_\lambda f(x) = \int_E K(x - (y', 0)) dy' + \int_E (e^{i\lambda|x-(y',0)|^\gamma} - 1) K(x - (y', 0)) dy'$$

$$= P(x) + R(x).$$

Setting $\rho = 1/\gamma$ we have

$$|e^{i\lambda|x-(y',0)|^\gamma} - 1| \leq \lambda|x-(y',0)|^\gamma \leq Cc_0 \lambda^{\rho\gamma} = Cc_0, \ y' \in E,$$

and

$$|R(x)| \leq Cc_0 \int_E K(x - (y', 0)) dy'.$$

Now taking $c_0$ small we obtain

$$|T_\lambda f(x)| \geq c \int_E K(x - (y', 0)) dy' \geq c \int_E \lambda^{-\rho(m-n+1)} dy' = C\lambda^{-\rho m}$$

and

$$\int_F |T_\lambda f(x)|^p dx \geq c\lambda^{-\rho m}(\lambda^{-\rho n})^{1/p} = c\lambda^{-m/\gamma}\lambda^{-n/p}.$$

On the other hand

$$||f||_p = \left(\int_E dy'\right)^{1/p} = C\lambda^{-\rho(n-1)/p} = C\lambda^{-(n-1)/\gamma p}$$

and we have

$$||T_\lambda||_p \geq c\frac{\lambda^{-m/\gamma}\lambda^{-n/p}}{\lambda^{-(n-1)/\gamma p}} = c\lambda^{-m/\gamma}\lambda^{-1/\gamma p} = c\lambda^{-(1/p+m)/\gamma}.$$

The same proof works also in the case $K(z) = |z|^{m-n+1}\omega(z)$.

In Theorems 1.2 and 1.3 we proved estimates of the type

$$||T_\lambda||_p \leq C\lambda^{-(1/p+m)/\gamma}$$

and the inequality (3.1) shows that these estimates are sharp.

In Theorem 1.4 we proved the estimate

$$||S_\lambda||_2 \leq C\lambda^{-m/\gamma}.$$ (3.2)
We shall now prove that also this estimate is sharp. We shall use the same method as in the above counter-example.

We take \( x_0 \) and \( y_0 \) in \( \mathbb{R}^{n-1} \) with \( |x_0 - y_0| = 100 c_0 \lambda^{-\rho} \) and set \( E = B(y_0; c_0 \lambda^{-\rho}) \) and \( F = B(x_0; c_0 \lambda^{-\rho}) \). Here \( E \) and \( F \) are balls in \( \mathbb{R}^{n-1} \). Setting \( f = \chi_E \) and arguing as above one obtains

\[
|S_\lambda f(x)| \geq c \lambda^{-\rho m} \quad \text{for} \quad x \in F.
\]

It follows that

\[
||S_\lambda f||_2 \geq c \lambda^{-m/\gamma} \lambda^{-(n-1)/2}.
\]

and

\[
||f||_2 = C \lambda^{-(n-1)/2}.
\]

We conclude that

\[
||S_\lambda||_2 \geq c \lambda^{-m/\gamma}
\]

and it follows that (3.2) is sharp.

In Theorems 1.2 and 1.3 we have

\[
T_\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{i \lambda \varphi(x, y')} \psi_0(x, y') K(x - (y', 0)) f(y') dy'
\]

where \( x = (x', x_n) \) and \( \varphi(x, y') = (|x' - y'|^2 + x_n^2)^{\gamma/2} \).

We let \( a \) denote the point \((0, 1) = (0, 0, \ldots, 0, 1)\) in \( \mathbb{R}^n \). We assume that \( \psi_0(x, y') = 1 \) in a neighbourhood of \((a, 0)\) and let \( f = \chi_B \) where \( B = B(0; c_0 \lambda^{-1}) \) is a ball in \( \mathbb{R}^{n-1} \). For \( x \) in a neighbourhood of \( a \) one obtains

\[
T_\lambda f(x) = \int_B e^{i \lambda \varphi(x, y')} K(x - (y', 0)) dy'.
\]

It follows from the mean value theorem that

\[
|\varphi(x, y') - \varphi(x, 0)| \leq C c_0 \lambda^{-1} \quad \text{for} \quad y' \in B
\]

and choosing \( c_0 \) small we obtain

\[
|\lambda \varphi(x, y') - \lambda \varphi(x, 0)| \leq c_1 \quad \text{for} \quad y' \in B,
\]

where \( c_1 \) is small. It follows that there is no cancellation in the above integral and we get

\[
|T_\lambda f(x)| \geq c_2 \lambda^{-(n-1)}
\]
in a neighbourhood of $a$. Hence

$$\|T_\lambda f\|_2 \geq c_3 \lambda^{-(n-1)}.$$  

We have $\|f\|_2 = c_4 \lambda^{-(n-1)/2}$ and we obtain

$$\frac{\|T_\lambda f\|_2}{\|f\|_2} \geq \frac{c_3 \lambda^{-(n-1)}}{c_4 \lambda^{-(n-1)/2}} = c_5 \lambda^{-(n-1)/2}.$$  

Hence

$$\|T_\lambda f\|_2 \geq c_5 \lambda^{-(n-1)/2} \tag{3.3}$$  

and thus the estimates $\|T_\lambda f\|_2 \leq C \lambda^{-(n-1)/2}$ in Theorems 1.2 and 1.3 are sharp.

We shall then construct a similar counter-example for the operator $S_\lambda$ in Theorem 1.4. Here we have

$$S_\lambda f(x) = \int_{\mathbb{R}^{n-1}} e^{i\lambda \varphi(x,y)} \psi_0(x,y) K(x-t,y) f(y) dy, \ x \in \mathbb{R}^{n-1},$$  

where $\varphi(x,y) = |x-y|^\gamma$. Take $a = (0,0,\ldots,0,1)$ and assume that $\psi_0(x,y) = 1$ in a neighbourhood of $(a,0)$. Also let $f = \chi_B$ where $B$ is as in the previous counter-example. The same argument as above then gives the estimate $\|S_\lambda f\|_2 \geq c \lambda^{-(n-1)/2}$ and it follows that the estimate $\|S_\lambda f\|_2 \leq C \lambda^{-(n-1)/2}$ in Theorem 1.4 is sharp.

We shall then again consider the operator $T_\lambda$ in Theorem 1.3. Here we have $n = 2$ and the above counter-example also gives

$$\|T_\lambda f\|_p \geq \frac{\|T_\lambda f\|_p}{\|f\|_p} \geq c \frac{\lambda^{n-1}}{\lambda^{1/p}} = c \lambda^{-(1-1/p)}$$  

for $1 \leq p < 2$. It follows that the estimate

$$\|T_\lambda f\|_p \leq C \lambda^{-\beta(p)}$$  

for $1 < p < 2$ in Theorem 1.3 is sharp (since $\beta(p) = 1 - 1/p$).

In Theorem 1.3 we have

$$T_\lambda f(x,y) = \int_{\mathbb{R}} e^{i\lambda \varphi(x,y,t)} \psi_0(x,y,t) K(x-t,y) f(t) dt, \ (x,y) \in \mathbb{R}^2,$$  

where $\varphi(x,y,t) = ((x-t)^2 + y^2)^{\gamma/2}$ and $K(z) = |z|^{m-1} \omega(z)$.

Setting

$$T_\lambda^* g(t) = \int_{\mathbb{R}^2} e^{-i\lambda \varphi(x,y,t)} \psi_0(x,y,t) K(x-t,y) g(x,y) dx dy, \ t \in \mathbb{R},$$
we get

\[(T_\lambda f, g)_2 = (f, T_\lambda^* g)_1, \ f \in C_0^\infty(\mathbb{R}), \ g \in C_0^\infty(\mathbb{R}^2),\]

where \((,)_2\) and \((,)_1\) denote the inner products in \(L^2(\mathbb{R}^2)\) and \(L^2(\mathbb{R})\). It follows that

\[
||T_\lambda||_p = ||T_\lambda||_{L^p(\mathbb{R}) \to L^p(\mathbb{R}^2)} \geq ||T_\lambda^*||_{L^r(\mathbb{R}^2) \to L^r(\mathbb{R})}
\]

where \(1/p + 1/r = 1\). We shall use this inequality for \(4 \leq p < \infty\).

Let \(B\) denote a disc in \(\mathbb{R}^2\) with center \((0, 1)\) and radius \(c_0\lambda^{-1}\). Take \(g \in C_0^\infty(\mathbb{R}^2)\) with support in \(B\), \(0 \leq g \leq 1\), and \(g = 1\) in \(\frac{1}{2}B\). Then

\[
||g||_r \leq \left( \iint_B dx dy \right)^{1/r} = c\lambda^{-2/r}
\]

and choosing \(\psi_0\) such that \(\psi_0(x, y, t) = 1\) in a neighbourhood of \((0, 1, 0)\) we get

\[
|T_\lambda^* g(t)| \geq c\lambda^{-2}
\]

in a neighbourhood of 0. Hence

\[
||T_\lambda^* g||_r \geq c\lambda^{-2}
\]

and

\[
||T_\lambda^*||_r \geq \frac{||T_\lambda^* g||_r}{||g||_r} \geq c \frac{\lambda^{-2}}{\lambda^{-2/r}} = c\lambda^{-2(1-1/r)}.
\]

Since \(1 - 1/r = 1/p\) we conclude that

\[
||T_\lambda||_p \geq c\lambda^{-2/p}, \ 4 \leq p < \infty
\]

and it follows that the estimate

\[
||T_\lambda||_p \leq C\lambda^{-\beta(p)}, \ 4 < p < \infty,
\]

in Theorem 1.3 is sharp (since \(\beta(p) = 2/p\)).

In Theorem 1.3 we also have an estimate of the type

\[
||T_\lambda||_p \leq C\lambda^{-1/2+\varepsilon}
\]

for \(2 < p < 4\). We shall finally discuss the sharpness of this estimate in the case \(\gamma = 1\). We shall study the statement

\[
||T_\lambda||_p \leq C\lambda^{-1/2-\delta} \text{ for some } p \text{ with } 2 < p < 4 \text{ and some } \delta > 0. \tag{3.4}
\]
Omitting details we shall describe how (3.4) leads to a contradiction. Following Stein [6], p. 393, we have

\[ \frac{1}{|x|^{3/2}} = u(x) + \sum_{k=1}^{\infty} 2^{-3k/2} \psi \left( \frac{x}{2^k} \right), \quad x \in \mathbb{R}^2 \setminus \{0\}, \]

where \( u \in L^1(\mathbb{R}^2) \), \( \psi \) is smooth, and \( \text{supp}\psi \subset \{x \in \mathbb{R}^2; \ 1/2 \leq |x| \leq 2\} \). We set

\[ K_0(x) = \frac{e^{i|x|}}{|x|^{3/2}} = e^{i|x|}u(x) + \sum_{k=1}^{\infty} 2^{-3k/2} e^{i|x|} \psi \left( \frac{x}{2^k} \right), \quad x \in \mathbb{R}^2 \setminus \{0\}, \]

and \( S_0 f = K_0 \ast f \). We define the operator \( V_k \) by setting

\[ V_k f = 2^{-3k/2} 2^{2k} (e^{i2^k|x|} \psi) \ast f = 2^{k/2} (e^{i2^k|x|} \psi) \ast f = \lambda^{1/2} (e^{i\lambda|x|} \psi) \ast f, \]

where \( \lambda = 2^k \). Using (3.4) we can prove that

\[ ||V_k||_p = ||V_k||_{L^p(\mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)} \leq C \lambda^{-\delta} = C 2^{-k\delta}, \]

and the inequality

\[ \sum_{k=1}^{\infty} ||V_k||_p < \infty \]

implies that \( S_0 \) is a bounded operator on \( L^p(\mathbb{R}^2) \). It follows that the characteristic function of the unit disc is a Fourier multiplier for \( L^p(\mathbb{R}^2) \). This contradicts Fefferman’s multiplier theorem.

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