CONVOLUTION OF A HARMONIC MAPPING WITH n-STARLIKE MAPPINGS AND ITS PARTIAL SUMS

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Abstract. We investigate the univalency and the directional convexity of the convolution \( \hat{\phi} \ast f = \hat{\phi} \ast h + \hat{\phi} \ast g \) of the harmonic mapping \( f = h + \bar{g} \) with a mapping \( \phi \) whose convolution with the mapping \( z + \sum_{k=2}^{\infty} k^n z^k \) is starlike (and such a mapping \( \phi \) is called \( n \)-starlike). In addition, we investigate the directional convexity of (i) the convolution of an analytic convex mapping with the slanted half-plane mapping, and (ii) the partial sums of the convolution of a 6-starlike mapping with the harmonic Koebe mapping and the harmonic half-plane mapping.

1. Introduction

Let \( \mathcal{H} \) consists of all complex-valued harmonic mappings \( f = h + \bar{g} \) in the unit disk \( \mathbb{D} := \{ z \in \mathbb{C}, |z| < 1 \} \), where \( h \) and \( g \) are analytic mappings. Let \( S_H^0 \) be the sub-class of \( \mathcal{H} \) consists of all mappings \( f \) in the class \( \mathcal{H} \) that are univalent, sense-preserving and normalized by the conditions \( f(0) = f_{\bar{z}}(0) = f_z(0) - 1 = 0 \). Let \( K_H^0 \) and \( S_H^0 \) denote the sub-classes of \( S_H^0 \) consisting of mappings which maps \( \mathbb{D} \) onto convex and starlike domains respectively. The sub-classes \( \mathcal{S}, K \) and \( S^* \) of analytic mappings consisting of univalent, convex and starlike mappings are respectively sub-classes of \( S_H^0, K_H^0 \) and \( S_H^0 \). Clunie and Sheil-Small [1] constructed two important mappings, the harmonic right-half plane mapping \( L \in K_H^0 \) and the harmonic Koebe mapping \( K \in S_H^0 \). These mappings are expected to play the role of extremal mappings respectively in classes \( S_H^0, K_H^0 \) as played by the analytic right-half plane mapping and analytic Koebe mapping respectively in the classes \( K \) and \( S^* \). These mappings \( K = H + \bar{G} \) and \( L = M + \bar{N} \) are defined in \( \mathbb{D} \) by

\[
H(z) = \frac{z - z^2/2 + z^3/6}{(1 - z)^3}, \quad G(z) = \frac{z^2/2 + z^3/6}{(1 - z)^3}
\]

and

\[
M(z) = \frac{z - z^2/2}{(1 - z)^2}, \quad N(z) = \frac{-z^2/2}{(1 - z)^2}.
\]

A domain \( D \) is said to be convex in direction \( \theta \) \( (0 \leq \theta < 2\pi) \), if every line parallel to the line joining 0 and \( e^{i\theta} \) lies completely inside or outside the domain \( D \). If \( \theta = 0 \) ( or \( \pi/2 \)), such a domain \( D \) is called convex in the direction of real (or imaginary) axis. In this paper we study the directional convexity of the convolution of these and some other mappings with \( n \)-starlike mapping introduced by Sălăgean [2] and their partial sums. Let \( A \) be the class of all analytic mappings \( f : \mathbb{D} \rightarrow \mathbb{C} \) with \( f(0) = 0 \), and \( f'(0) = 1 \). The function \( f \in A \) has the Taylor series expansion \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \). For the function

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By using this operator, Sălăgean introduced the class of \( n \)-starlike mappings is univalent and convex in a particular direction. In particular, for \( \pi/2 \)-half-plane mapping is univalent and convex in the direction of real-axis. Also, the convolution \( * \) of harmonic mapping \( f = h + \bar{g} \) and \( F = H + \bar{G} \) is defined by \( f * F := h * H + \bar{g} * \bar{G} \). It is well known that the convolution of two harmonic convex mappings is not necessarily convex/univalent. In [6], Dorff studied the directional convexity of harmonic mappings and proved that convolution of two right half-plane mappings is univalent and convex in the direction of real axis provided the convolution is locally univalent. Later, Dorff et al. [7] extended such results to slanted half-plane and strip mappings. Other recent related work in this direction can be found in [1,2,5,9,11,13,20,25].

In Section 2, we prove that the convolution of certain harmonic mappings with \( n \)-starlike mappings is univalent and convex in a particular direction. In particular, for \( 0 \leq \alpha < \pi \), we prove that the convolution of an analytic convex mapping with the slanted half-plane mapping is univalent and convex in the direction of \( \pi/2 - \alpha \). Lastly, in Section 3, we discuss the partial sums of \( n \)-starlike mappings and prove that all the partial sums of \( n \)-starlike mappings with \( n \geq 4 \) are \((n - 4)\)-starlike. By using this, we prove that all the partial sums of the convolution of \( 6 \)-starlike mappings with the mappings \( L \) and \( K \) are univalent and convex in the direction of real-axis.

2. CONVOLUTION OF SOME HARMONIC MAPPINGS WITH \( n \)-STARLIKE MAPPINGS

We first give some convolution properties of \( n \)-starlike mappings, which will be useful throughout the paper. From the definition of \( S_n(\alpha) \), one can easily see that

\[
(2.1) \quad f \in S_n(\alpha) \Leftrightarrow D^{n-m} f \in S_m(\alpha).
\]

Using this relation, we get the following result regarding the convolution of mappings in class \( S_n \).

**Lemma 2.1.** Let \( n + m \geq 1 \). If the function \( f \in S_n \) and the function \( g \in S_m \), then the convolution \( f * g \in S_{n+m-1} \).

**Proof.** Assume that \( n \geq 1 \). Since the function \( f \in S_n \) and the function \( g \in S_m \), by (2.1), the function \( D^{n-1} f \in K \) and the function \( D^m g \in S^* \). Therefore, from [22], we have

\[
D^{n+m-1}(f * g) = D^{n-1} f * D^m g \in S^*.
\]

Hence, by (2.1), it follows that the convolution \( f * g \in S_{n+m-1} \).
Theorem 2.2. \([24]\) If the function \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A\) satisfies the inequality \(\sum_{n=2}^{\infty} (n - \alpha) |a_n| \leq 1 - \alpha\), then the function \(f\) is starlike of order \(\alpha\).

By using \([21]\) and Theorem 2.2, we get the following result.

Theorem 2.3. \([24]\) Let \(0 \leq \alpha < 1\). If the function \(f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in D\) satisfies the inequality \(\sum_{n=2}^{\infty} n^{m-1} (n - \alpha) |a_n| \leq 1 - \alpha\), then the function \(f\) is \((m-1)\)-starlike of order \(\alpha\).

The harmonic mappings \(f\) considered in this paper are assumed to be normalized by \(f(0) = f_1(0) - 1 = f_2(0) = 0\), unless otherwise specified. The following result due to Clunie and Sheil-Small is used for constructing univalent harmonic mappings convex in a given direction.

Lemma 2.4. \([3]\) A locally univalent harmonic mapping \(f = h + \overline{g}\) on \(D\) is univalent and maps \(D\) onto a domain convex in the direction of \(\phi\) if and only if the analytic mapping \(h - e^{2i\beta}g\) is univalent and maps \(D\) onto a domain convex in the direction of \(\phi\).

Lemma 2.5. Let the function \(f = h + \overline{g}\) be harmonic and the function \(\phi\) be analytic in \(D\). If the function \((h - e^{-2i\beta}g) \ast \phi\) is convex and, for some real number \(\gamma\),

\[
(2.2) \quad \text{Re} \left( \frac{(\phi \ast h)'(z)}{(\phi \ast h)' - e^{-2i\gamma}(\phi \ast g)'}(z) \right) > \frac{1}{2} \quad \text{for } z \in D,
\]

then the convolution \(f \ast \phi \in S^0_H\) and is convex in the direction of \(-\beta\).

Proof. Since the function \((h - e^{-2i\beta}g) \ast \phi\) is convex and hence convex in the direction of \(-\beta\), in view of Lemma 2.4, it is enough to prove that the mapping \(f \ast \phi\) is locally univalent. Clearly \((2.2)\) shows that \((h \ast \phi)'(z) \neq 0\) for \(z \in D\). Therefore, using \((2.2)\), we see that the dilatation \(w_{e^{ri} f \ast \phi} = (e^{-r} g \ast \phi)' / (e^{r} h \ast \phi)'\) of \(e^{ri} f \ast \phi\) satisfies

\[
(2.3) \quad \text{Re} \left( \frac{(\phi \ast h)'(z)}{(\phi \ast h)' - e^{-2i\gamma}(\phi \ast g)'}(z) \right) - 1 > \frac{1}{2}, \quad z \in D.
\]

This shows that \(|w_{e^{ri} f \ast \phi}(z)| < 1\) for \(z \in D\), or equivalently \(|w_{f \ast \phi}(z)| < 1\) for \(z \in D\), where \(w_{f \ast \phi} = (g \ast \phi)' / (h \ast \phi)'\) is dilatation of the function \(f \ast \phi\). The result now follows by Lewis theorem.

Theorem 2.6. Let the harmonic function \(f = h + \overline{g}\) in \(D\) satisfies \(h(z) - e^{-2i\gamma}g(z) = z\) for some real number \(\gamma\). If the function \(h \ast \phi \in S_2\) and the function \((h - e^{-2i\beta}g) \ast \phi \in K\) for some analytic function \(\phi\), then the convolution \(f \ast \phi \in S^0_H\) and is convex in the direction of \(-\beta\).

Proof. Since the function \(h \ast \phi \in S_2\), we have \(z(h \ast \phi)' \in K\). Hence, from \([23]\) Corollary 1, p.251], we get

\[
\text{Re} \left( \frac{(\phi \ast h)'(z)}{(\phi \ast h)' - e^{-2i\gamma}(\phi \ast g)'}(z) \right) = \text{Re}(h \ast \phi)'(z) > \frac{1}{2} \quad \text{for } z \in D.
\]

Also, the function \((h - e^{-2i\beta}g) \ast \phi\) is convex. The result now follows from Lemma 2.4.

Corollary 2.7. If the function \(\phi(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S_2\) satisfies the inequality \(\sum_{n=2}^{\infty} n^2 |a_n| \leq 1 / \sqrt{2(1 - \cos 2\theta)}\) for some \(\theta \in [0, \pi/2]\), then, for real number \(\gamma\), the function \(\phi + e^{2i\gamma}(\phi - z) \in S^0_H\) and is convex in the direction \(-\beta\) for all \(\beta\) satisfying \(|\beta + \gamma| \leq \theta\).
Proof. The harmonic mapping \( f = h + \bar{g} \), with \( h(z) = z/(1 - z) \) and \( g(z) = e^{2\gamma}(z/(1 - z) - z) \), satisfy \( h(z) - e^{-2\gamma}g(z) = z \). Also, the function \( h * \phi = \phi \in S_2 \). Furthermore, we see that

\[
(h(z) - e^{-2i\beta}g(z)) * \phi(z) = \phi(z) - e^{-2i(\beta + \gamma)}(\phi(z) - z)
\]

\[
= z + \sum_{n=2}^{\infty} \left(1 - e^{-2i(\beta + \gamma)}\right) a_n z^n
\]

Now, for \(|\beta + \gamma| \leq \theta\), we have

\[
\sum_{n=2}^{\infty} n^2 |\left(1 - e^{-2i(\beta + \gamma)}\right) a_n| = |1 - e^{-2i(\beta + \gamma)}| \sum_{n=2}^{\infty} n^2 |a_n|
\]

\[
= \sqrt{2(1 - \cos 2(\beta + \gamma))} \sum_{n=2}^{\infty} n^2 |a_n|
\]

\[
\leq \sqrt{2(1 - \cos 2(\beta + \gamma))} \frac{1}{\sqrt{2(1 - \cos 2\theta)}} \leq 1.
\]

Therefore, by Lemma 2.3, the function \((h - e^{-2i\beta}g) * \phi\) is convex for every \(\beta\) such that \(|\beta + \gamma| \leq \theta\). Hence, by Theorem 2.6, the function \(f * \phi = \phi + e^{2\gamma}(\phi - z) \in S_H^0\), and is convex in every direction \(-\beta\) satisfying \(|\beta + \gamma| \leq \theta\).

Remark 2.8. For \(\theta = \pi/2\) in Corollary 2.7, we see the function \(\phi + e^{2\gamma}(\phi - z) \in K_H^0\), if the function \(\phi = z + \sum_{n=2}^{\infty} a_n z^n \in S_2\) and satisfies the inequality \(\sum_{n=2}^{\infty} n^3 |a_n| \leq 1/2\).

Remark 2.9. If the function \(\phi = z + \sum_{n=2}^{\infty} a_n z^n\) satisfies the inequality \(\sum_{n=2}^{\infty} n^2 |a_n| \leq 1/2\) and, by Lemma 2.3, the function \(\phi \in S_2\). Therefore, Remark 2.8 shows that \(\phi + e^{2\gamma}(\phi - z) \in K_H^0\).

Theorem 2.10. Let the function \(\phi \in S_2\) and the function \(f = h + \bar{g}\) be a harmonic mapping in \(D\) satisfying \(h(z) - e^{-2\gamma}g(z) = h(z) * \log(1/(1 - z))\) for some real number \(\gamma\) with the function \(h \in S^*\). Then the convolution \(f * \phi \in S_H^0\) and is convex in the direction \(-\gamma\). Furthermore, if for any \(\beta\) real, the function \(h - e^{-2i\beta}g \in S^*\), then the convolution \(f * \phi\) is convex in the direction \(-\beta\).

Proof. Since the function \(h \in S^*\) and the function \(\log(1/(1 - z)) \in S_2\), by Lemma 2.1, we have the function \(h - e^{-2\gamma}g = h * \log 1/(1 - z) \in K\). Hence, from [23, Corollary 1, p.251], we have

\[
\text{Re} \left( \frac{h(z)}{h(z) * \log \frac{1}{1 - z}} \right) > \frac{1}{2}, \quad z \in D.
\]

Note that, we can write

\[
\text{Re} \left( \frac{\phi * h'}{((\phi * h)' - e^{-2\gamma}(\phi * g)')}(z) \right) = \text{Re} \left( \frac{z\phi'(z) * (h(z) * \log \frac{1}{1 - z})}{z\phi'(z) * (h(z) * \log \frac{1}{1 - z})} \right),
\]

where the function \(z\phi' \in K\) and the function \(h * \log 1/(1 - z) \in S^*\). Therefore, in view of (2.4), (2.5) and [8, Theorem 2.4, p.54], it follows that

\[
\text{Re} \left( \frac{\phi * h'}{((\phi * h)' - e^{-2\gamma}(\phi * g)')}(z) \right) > \frac{1}{2}, \quad z \in D.
\]

As the function \(h \in S^*\) and the function \(\phi \in S_2\), Lemma 2.1 gives that the function \((h - e^{-2\gamma}g) * \phi = h * \log(1/(1 - z)) * \phi \in K\). Similarly, the function \((h - e^{-2i\beta}g) * \phi \in K\). Therefore, in view of (2.6), the result follows by Lemma 2.5.
Corollary 2.11. Let the function $\phi \in \mathcal{S}_2$ and the function $f = h + \bar{g}$ be a harmonic mapping in $\mathbb{D}$ satisfying $h(z) - g(z) = h(z) \ast \log(1/(1 - z))$. Then, we have the following.

1. If the function $h \in \mathcal{S}^*$, then the convolution $f \ast \phi \in \mathcal{S}_H^*$ and is convex in the direction of real axis.

2. If, for some $\theta$ ($0 \leq \theta < \pi$), the function $h(z) = z + \sum_{n=2}^\infty |a_n|\sqrt{2n(n-1)}(1 - \cos 2\theta) + 1 \leq 1$, then the convolution $f \ast \phi \in \mathcal{S}_H^*$ and is convex in every direction $-\beta$ such that $|\beta| \leq \theta$.

3. If, for some $\theta$ ($0 \leq \theta < \pi$) such that $\cos 2\theta \leq 1/4$, the function $h(z) = z + \sum_{n=2}^\infty a_nz^n$ satisfies the inequality $\sum_{n=2}^\infty |a_n|\sqrt{2n(n-1)}(1 - \cos 2\theta) + 1 \leq 1$, then the convolution $f \ast \phi \in \mathcal{S}_H^*$ and is convex in every direction $-\beta$ such that $|\beta| \leq \theta$.

Proof. (1) is obvious from Theorem 2.10. For (2) and (3), in view of Theorem 2.10, it is enough to prove that the function $h - e^{-2i\beta}g \in \mathcal{S}^*$ for every $\beta$ such that $|\beta| \leq \theta$ and $h \in \mathcal{S}^*$. If $\cos 2\theta \leq 1/4$, then the inequality $\sum_{n=2}^\infty |a_n|\sqrt{2n(n-1)}(1 - \cos 2\theta) + 1 \leq 1$ implies the inequality $\sum_{n=2}^\infty |a_n|n \leq 1$. Hence, by Theorem 2.2, the function $h \in \mathcal{S}^*$.

Also, for $\beta$ real, we have

$$h(z) - e^{-2i\beta}g(z) = h(z) - e^{-2i\beta}(h(z) - h(z) \ast \log(1/(1 - z))) = z + \sum_{n=2}^\infty a_n(1 - e^{-2i\beta}(n - 1)/n).$$

Therefore, in view of (2) and (3), we see, for $|\beta| \leq \theta$, that the function $h - e^{-2i\beta}g$ satisfies the inequality

$$\sum_{n=2}^\infty n|a_n(1 - e^{-2i\beta}(n - 1)/n)| = \sum_{n=2}^\infty |a_n|\sqrt{2n(n-1)}(1 - \cos 2\beta) + 1 \leq \sum_{n=2}^\infty |a_n|\sqrt{2n(n-1)}(1 - \cos 2\theta) + 1 \leq 1.$$

Hence, by Theorem 2.2, the function $h - e^{-2i\beta}g \in \mathcal{S}^*$.

Remark 2.12. Taking $\theta = \pi/2$ in Corollary 2.11, we see that, if the function $\phi \in \mathcal{S}_2$ and the function $h(z) = z + \sum_{n=2}^\infty a_nz^n$ satisfies the inequality $\sum_{n=2}^\infty |a_n| \leq 1$, then the convolution $f \ast \phi \in \mathcal{K}_H^*$, where the function $f = h + g$ and the function $g(z) = h(z) - h(z) \ast \log(1/(1 - z))$.

Example 2.13. Let the harmonic function $f = h + \bar{g}$ be given by the function $h(z) = z/(1 - z)^2$ and the function $g(z) = z/(1 - z)^2 - z/(1 - z) = z/(1 - z)^2 - z/(1 - z)^2 \ast \log(1/(1 - z))$. Now, taking the function $\phi(z) = \log(1/(1 - z))$ in Theorem 2.10, we get the convolution $f(z) \ast \log(1/(1 - z)) = z/(1 - z) + z/(1 - z) - \log(1/(1 - z)) \in \mathcal{S}_H^*$ and is convex in the direction of real axis.

Example 2.14. Let the harmonic function $f = h + \bar{g}$ be given by the function $h(z) = z + z^2/3$ and the function $g(z) = z^2/6$ and let the function $\phi(z) = \log(1/(1 - z))$. Then, the functions $f$ and $\phi$ satisfy the conditions in Remark 2.12 and hence the convolution $(f \ast \phi)(z) = z + z^2/6 + z^2/12 \in \mathcal{K}_H^*$.

We denote the convolution $f \ast f \ast \cdots \ast f$ ($n$-times) by $(f)_n$. A simple calculation shows that, for $|\alpha| = 1$ and $|z| < 1$,

$$\left(\frac{z}{(1 - z\alpha)^2}\right)_n \ast \left(\alpha \log \frac{1}{1 - z/\alpha}\right)_n = \frac{z}{1 - z} \quad \text{for all } n \in \mathbb{N}.$$
Since the function \(z/(1-z)\) is convolution identity, the inverse under convolution of the function \(\left(\frac{z}{(1-z\alpha)^2}\right)_n\) is the function \(\left(\alpha \log \frac{1}{1-z/\alpha}\right)_n\). For the function \(f\), we write the inverse of \((f)_n\) by \((f)^{-n}_n\).

**Theorem 2.15.** Let \(n \in \mathbb{N}\), \(|\alpha| = |\gamma| = 1\) and \(0 \leq \beta < 2\pi\). Let the function \(f = h + \bar{g}\) be a harmonic mapping in \(\mathbb{D}\) satisfying

\[
h(z) - e^{-2i\beta}g(z) = \frac{z}{(1-\alpha z)^2} * \left(\frac{z}{(1-\gamma z)^2}\right)_n, \quad z \in \mathbb{D},
\]

and let the function \(\phi \in S_n\). If the function \((h - e^{-2i\delta}g)\ast \phi \in \mathcal{K}\) for some real number \(\delta\) and

\[
\text{Re} \left\{ \frac{(1-\alpha z)^2}{z} \left(h(z) * \left(\gamma \log \frac{1}{1-z/\gamma}\right)_n\right) \right\} > \frac{1}{2}, \quad z \in \mathbb{D},
\]

then the convolution \(f \ast \phi \in S^n_H\) and is convex in the direction of \(-\delta\).

**Proof.** We have

\[
(h(z) - e^{-2i\beta}g(z)) \ast \phi = \frac{z}{(1-\alpha z)^2} * \left(\frac{z}{(1-\gamma z)^2}\right)_n \ast \phi,
\]

where the functions \(z/(1-\alpha z)^2\), \(z/(1-\gamma z)^2\) \(\in S^*\) and the function \(\phi \in S_n\). Therefore, by repeated application of Theorem 2.1, we see that the function \((h - e^{-2i\delta}g) \ast \phi \in \mathcal{K}\). Also, we have

\[
\frac{(\phi \ast h)'(z)}{((\phi \ast h)' - e^{-2i\beta}(\phi \ast g)')(z)} = \frac{z\phi'(z) \ast \left\{ h(z) * \left(\frac{z}{(1-\gamma z)^2}\right)_n \ast \left(\gamma \log \frac{1}{1-z/\gamma}\right)_n\right\}}{z\phi'(z) \ast (h - e^{-2i\beta}g)(z)}
\]

\[
= \frac{z\phi'(z) \ast \left(\frac{z}{(1-\gamma z)^2}\right)_n \ast \left(\frac{z}{1-\alpha z}\right) \ast \left(h(z) * \left(\gamma \log \frac{1}{1-z/\gamma}\right)_n\right)}{z\phi'(z) \ast \left(\frac{z}{(1-\gamma z)^2}\right)_n \ast \left(\frac{z}{1-\alpha z}\right)}
\]

\[
(2.8)
\]

Since the function \(\phi \in S_n\) and the function \(z/(1-\gamma z)^2 \in S^*\), from Lemma 2.1 we have

\[
(2.9) \quad z\phi'(z) \ast \left(\frac{z}{(1-\gamma z)^2}\right)_n \in \mathcal{K}.
\]

Since the function \(z/(1-\alpha z)^2 \in S^*\), in view of (2.7), (2.8) and (2.9), it follows, by Theorem 2.4, p.54, that

\[
\text{Re} \left\{ \frac{(\phi \ast h)'(z)}{((\phi \ast h)' - e^{-2i\beta}(\phi \ast g)')(z)} \right\} > 1/2, \quad z \in \mathbb{D}.
\]

The result now follows from Lemma 2.5.

**Remark 2.16.** Take \(n = 2\) in Theorem 2.15. Let \(|\alpha| = 1\), \(0 \leq \beta < 2\pi\). Let the harmonic mapping \(f = h + \bar{g}\) satisfy

\[
(2.10) \quad \text{Re} \left\{ \frac{(1-\alpha z)^2}{z}h(z) > \frac{1}{2} \right\}, \quad h(z) - e^{-2i\beta}g(z) = \frac{z}{(1-\alpha z)^2}, \quad z \in \mathbb{D},
\]

and the function \((h - e^{-2i\delta}g) \in S^*\) for some real number \(\delta\). If the function \(\phi \in S_2\), then the convolution \(f \ast \phi \in S^0_H\) and is convex in the direction \(-\delta\), and in particular in the direction \(-\beta\).

**Remark 2.17.** Take \(n = \gamma = 1\) in Theorem 2.15 and notice that

\[
\frac{z}{(1 - \alpha z)^2} \ast \left( \frac{z}{(1 - z)^2} \right)^{-1} = \frac{z}{(1 - \alpha z)^2} \ast \log \frac{1}{1 - z} = \frac{z}{(1 - \alpha z)},
\]

and

\[
\frac{(1 - \alpha z)^2}{z} \left( h(z) \ast \left( \log \frac{1}{1 - z} \right)^{-1} \right) = \frac{(1 - \alpha z)^2}{z} \left( h(z) \ast \frac{z}{(1 - z)^2} \right) = \frac{(1 - \alpha z)^2}{z} D h(z).
\]

Let \(|\alpha| = 1, 0 \leq \beta < 2\pi\). Let the harmonic mapping \(f = h + \bar{g}\) satisfy

\[
(2.11) \quad h(z) - e^{-2i\beta}g(z) = \frac{z}{(1 - \alpha z)}, \quad \text{Re} \left\{ \frac{(1 - \alpha z)^2}{z} D h(z) \right\} > \frac{1}{2}, \quad z \in \mathbb{D},
\]

and the function \(h - e^{-2i\delta}g \in K\) for some real number \(\delta\). If the function \(\phi \in K\), then the convolution \(f \ast \phi \in S^0_H\) and is convex in the direction \(-\delta\), and in particular in the direction \(-\beta\).

**Remark 2.18.** Take \(n = 3, \gamma = 1\) in Theorem 2.15. First notice that

\[
\frac{z}{(1 - \alpha z)^2} \ast \frac{z}{(1 - z)^2} = \frac{z + \alpha z^2}{(1 - \alpha z)^3}.
\]

Let \(|\alpha| = 1, 0 \leq \beta < 2\pi\). Let the harmonic mapping \(f = h + \bar{g}\) satisfy

\[
h(z) - e^{-i\beta}g(z) = \frac{z + \alpha z^2}{(1 - \alpha z)^3}, \quad \text{Re} \left\{ \frac{(1 - \alpha z)^2}{z} \left( h(z) \ast \log \frac{1}{1 - z} \right) \right\} > 1/2, \quad z \in \mathbb{D}.
\]

If the function \(\phi \in S_3\), then the convolution \(f \ast \phi \in S^0_H\) and is convex in the direction \(-\beta\).

Taking \(\alpha = 1, \beta = 0\) in Remark 2.16, we get the following result.

**Corollary 2.19.** Let the function \(\phi \in S_2\) and the function \(f = h + \bar{g}\) be a harmonic mapping in \(\mathbb{D}\) satisfying \(h(z) - g(z) = z/(1 - z)^2\) for all \(z \in \mathbb{D}\). If

\[
\text{Re} \left( \frac{1 - z}{z} h(z) \right) > 1/2, \quad \text{for} \quad z \in \mathbb{D},
\]

then the convolution \(f \ast \phi \in S^0_H\) and is convex in the the direction of real axis.

Next, we give two examples of non-univalent convolution products.

**Example 2.20.** For \(a \geq -1 \ (a \neq 0)\), consider the harmonic mapping \(f_a = h + \bar{g}\) given by \(h(z) = (1 + z/a)l(z)\) and \(g(z) = zl(z)/a\), where \(l(z) = z/(1 - z)\) is analytic right-half plane mapping. Then, for the function \(\phi(z) = z + z^2/2 \in S^*\), we have

\[
(f_a \ast \phi)(z) = z + \frac{1 + a}{2a}z^2 + \frac{1}{2a}z^2.
\]

Its Jacobian \(J_{f_a \ast \phi}\), given by

\[
J_{f_a \ast \phi}(z) = |(h \ast \phi)'(z)|^2 - |(g \ast \phi)'(z)|^2 = 1 + \frac{2 + a}{a} |z|^2 + \frac{1 + a}{a} \text{Re} z,
\]

vanishes at \(z = -a/(a + 2) \in \mathbb{D}\).
**Example 2.21.** For $0 < b < 1/2$, consider the harmonic mapping $F_b = h + \bar{g}$ given by

$$h(z) = \frac{z + (1 + 2b)z^2}{(1 - z)^3} \quad \text{and} \quad g(z) = \frac{2bz^2}{(1 - z)^3}.$$ 

Then, for the function $\phi(z) = z + z^2/8 \in S_2$, the Jacobian $J_{F_b,\bar{\phi}}$ of the convolution $F_b \ast \phi$, given by

$$J_{F_b,\bar{\phi}}(z) = 1 + (1 + b)|z|^2 + (2 + b)\text{Re} z,$$

vanishes at $z = 1/(1 + b) \in \mathbb{D}$.

Examples 2.20\[2.21\] show that if the function $\phi \in S^*$ and $S_2$, then respectively the convolutions $f_a \ast \phi$ and $F_b \ast \phi$ need not be univalent, where the functions $f_a$ and $F_b$ are respectively given in the Examples 2.20\[2.21\]. However, the results are true respectively for the function $\phi \in K$ and $S_3$. In fact we have the following the results.

**Corollary 2.22.** For $a \geq 6$, let the function $f_a = h + \bar{g}$ be the harmonic mapping given in Example 2.20. If the function $\phi \in K$, then the convolution $f_a \ast \phi \in S_H$ and is convex in the direction of real axis.

**Proof.** We have $h(z) - g(z) = l(z) = z/(1 - z)$. Also, for $a \geq 6$, we see that

$$\text{Re} \left\{ \frac{(1 - z)^2}{z} \mathcal{D} h(z) \right\} = \text{Re} \left\{ (1 - z)^2 h'(z) \right\} = \text{Re} \left\{ 1 + \frac{z}{a} (2 - z) \right\} > 1/2 \quad \text{for } z \in \mathbb{D}.$$ 

By Remark 2.17, the result follows.

**Corollary 2.23.** Let the function $F_b = h + \bar{g}$ be the harmonic mapping given in Example 2.21. If $\phi \in S_3$, then $F_b \ast \phi \in S^0_H$ and is convex in the direction of real axis.

**Proof.** From the definition of $F_b$, we have

$$h(z) - g(z) = \frac{z + z^2}{(1 - z)^3},$$

and

$$\mathcal{D} \left( \frac{z + bz^2}{(1 - z)^2} \right) = \frac{z + (1 + 2b)z^2}{(1 - z)^3} = h(z).$$

Therefore, for $|b| \leq 1/2$,

$$\text{Re} \left\{ \frac{(1 - z)^2}{z} \left( h(z) \ast \log \frac{1}{1 - z} \right) \right\} = \text{Re} \left\{ \frac{(1 - z)^2}{z} \left( \mathcal{D} \frac{z + bz^2}{(1 - z)^2} \ast \log \frac{1}{1 - z} \right) \right\} = \text{Re} \left\{ \frac{(1 - z)^2}{z} \frac{z + bz^2}{(1 - z)^2} \right\} = \text{Re}(1 + bz) > 1/2.$$ 

The result now follows from Remark 2.18.

For $0 \leq \alpha < 2\pi$, let $S^0(H_\alpha) \subset S^0_H$ denote the class of all harmonic mappings that maps $\mathbb{D}$ onto $H_\alpha$, where

$$H_\alpha := \left\{ z \in \mathbb{C} : \text{Re}(e^{i\alpha} z) > -\frac{1}{2} \right\}.$$

In [7], it is shown that if $f = h + \bar{g} \in S^0(H_\alpha)$, then

$$h(z) + e^{-2\alpha} g(z) = \frac{z}{1 - e^{i\alpha} z}. \quad (2.12)$$
Using this result, we check the convexity of the convolution \( f \ast \phi \), where the function \( f \in S^0(H) \) and the function \( \phi \in K \). Before this, we give an example of a mapping in class \( H_\alpha \).

**Example 2.24.** In [3], it is shown that the harmonic right half-plane mapping \( L = M + N \), where
\[
M(z) = \frac{z - z^2/2}{(1 - z)^2} \quad \text{and} \quad N(z) = -\frac{z^2/2}{(1 - z)^2},
\]
maps \( \mathbb{D} \) onto the right-half plane \( \{ w : \Re w > -1/2 \} \). Consider the mapping \( f_\alpha = h_\alpha + \bar{g}_\alpha \), where
\[
h_\alpha(z) = \frac{z - e^{2\alpha} z^2/2}{(1 - e^{2\alpha} z)^2} \quad \text{and} \quad g_\alpha(z) = -\frac{e^{3\alpha} z^2/2}{(1 - e^{2\alpha} z)^2}.
\]
Now, for \( w = e^{i\alpha} z \), we have
\[
e^{i\alpha} f_\alpha(z) = e^{i\alpha} h_\alpha(z) + e^{-i\alpha} g_\alpha(z) = \frac{ze^{i\alpha} - e^{2i\alpha} z^2/2}{(1 - e^{2\alpha} z)^2} + \frac{-e^{2i\alpha} z^2/2}{(1 - e^{2\alpha} z)^2}
= \frac{w - w^2/2}{(1 - w)^2} + \frac{-w^2/2}{(1 - w)^2} = L(w).
\]
Therefore the function \( e^{i\alpha} f_\alpha \) maps \( \mathbb{D} \) onto the right-half plane \( \{ w' : \Re w' > -1/2 \} \). Hence the function \( f_\alpha \in S^0(H) \).

**Example 2.25.** Consider the function \( \phi = z + z^2/2 \in S^* \). Then, for the harmonic mapping \( f_\alpha \) given in Example 2.24, the convolution \( f_\alpha \ast \phi \) is given by
\[
f_\alpha \ast \phi(z) = (h_\alpha \ast \phi)(z) + (g_\alpha \ast \phi)(z) = z + \frac{3}{4} e^{i\alpha} z^2 - \frac{1}{2} e^{3i\alpha} z^2.
\]
Its Jacobian \( J_{f_\alpha \ast \phi} \), given by
\[
J_{f_\alpha \ast \phi}(z) = |(h_\alpha \ast \phi)'(z)|^2 - |(g_\alpha \ast \phi)'(z)|^2 = 1 + 2|z|^2 + 3 \Re(e^{i\alpha} z),
\]
vanesishes at \( z = -e^{-i\alpha}/2 \in \mathbb{D} \).

Example 2.25 shows that for the function \( \phi \in S^* \), the convolution \( f_\alpha \ast \phi \) is not univalent, where \( f_\alpha \) is given in Example 2.24. However, the result is true for the function \( \phi \in K \). In fact we have the following strong result.

**Theorem 2.26.** Let the harmonic mapping \( f = h + \bar{g} \in S^0(H_\alpha) \). If the function \( \phi \in K_\alpha \), then the convolution \( f \ast \phi \in S^0(H_\alpha) \) and is convex in the direction \( \pi/2 - \alpha \).

**Proof.** Since the harmonic mapping \( f = h + \bar{g} \in S^0(H_\alpha) \), from equation (2.12), we have
\[
h(z) + e^{-2i\alpha} g(z) = \frac{z}{(1 - e^{i\alpha} z)} \quad \text{or} \quad h(z) - e^{-2i(\alpha - \pi/2)} g(z) = \frac{z}{(1 - e^{i\alpha} z)}.
\]
Upon differentiating (2.13) and writing the dilatation \( g'/h' \) of \( f \) by \( \omega \), we get
\[
h'(z) = \frac{1}{(1 - e^{i\alpha} z)^2(1 + e^{-2i\alpha} \omega(z))}.
\]
Therefore
\[
\Re \frac{(1 - e^{i\alpha} z)^2}{z} \mathcal{D} h(z) = \Re \frac{1}{1 + e^{-2i\alpha} \omega(z)} > \frac{1}{2} \quad \text{for} \ z \in \mathbb{D}.
\]
Using (2.14) and (2.15) in Remark 2.17, we see that the convolution \( f \ast \phi \in S^0_\alpha \) and is convex in the direction \( \pi/2 - \alpha \).
In the next result, we show that the convolution of the harmonic mapping \( f_\alpha = h_\alpha + \overline{\alpha} \), given in Example 2.24 with the mappings in class \( S_2 \) is convex in two perpendicular directions.

**Theorem 2.27.** Let the function \( f_\alpha = h_\alpha + \overline{\alpha} \) be the harmonic mapping defined in Example 2.24. If the function \( \phi \in S_2 \), then the convolution \( f_\alpha \ast \phi \in S^0_\alpha \) and is convex in the directions \( -\alpha \) and \( \pi/2 - \alpha \).

**Proof.** From Example 2.24, we have

\[
(1) \quad (1) \quad (2) \quad (3)
\]

For \( f_\alpha \) and \( \phi \), follows from Theorem 2.2. For \( f_\alpha \ast \phi \in S^0_\alpha \) and is convex in the directions \( -\alpha \) and \( \pi/2 - \alpha \).

The partial sums satisfies:

1. \( f_{1,2}(z) = z + z^2/2 \in S^* \);
2. \( f_{2,3}(z) = z + z^2/2 + z^3/3 \in S^* \);
3. \( f_{3,n}(z) = \sum_{l=1}^{n} z^l/l^3 \in S^* \), for all \( n \in \mathbb{N} \).

**Proof.** (1) and (2) follows from Theorem 2.2. For \( l \in \mathbb{N} \), let \( a_l = 1/l^3 \) for \( 1 \leq l \leq n \) and \( a_l = 0 \) for all \( l > n \). Then, we have

\[
\sum_{l=2}^{\infty} l|a_l| < -1 + \sum_{n=1}^{\infty} \frac{1}{n^2} = -1 + \frac{\pi^2}{6} < 1.
\]

Therefore, (3) follows from Theorem 2.2.

Using \( D^m f_{m+l,n} = f_{l,n} \), Lemma 3.1 and equation (2.1) gives the following:

**Lemma 3.2.** The partial sums satisfies:

1. \( f_{m,2}(z) = z + z^2/2^m \in S_{m-1} \), if \( m \geq 1 \);
2. \( f_{m,3}(z) = z + z^2/2^m + z^3/3^m \in S_{m-2} \), if \( m \geq 2 \);
3. \( f_{m,n}(z) = z + z^2/2^m + z^3/3^m + \cdots + z^n/m^m \in S_{m-3} \), for all \( n \in \mathbb{N} \), if \( m \geq 3 \).

**Lemma 3.3.** Let \( \phi \) denotes thepth partial sum of the function \( \phi \in S_m \). Then, we have the following:

1. \( \phi_2 \in S_{m-2} \), if \( m \geq 2 \);
2. \( \phi_3 \in S_{m-3} \), if \( m \geq 3 \);
3. \( \phi_p \in S_{m-4} \), for all \( p \in \mathbb{N} \), if \( m \geq 4 \).
Proof. Let \( \phi(z) = z + a_2 z^2 + a_3 z^3 + \ldots, z \in \mathbb{D} \). Then we can write \( \phi_p \) as
\[
\phi_p(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_p z^p
\]
\[
= \left( z + 2^n a_2 z^2 + 3^n a_3 z^3 + \ldots \right) \ast \left( z + \frac{z^2}{2^n} + \frac{z^3}{3^n} + \cdots + \frac{z^p}{p^n} \right)
\]
\[
= \mathcal{D}^p \phi(z) \ast f_{n,p}(z).
\]
Since the function \( \phi \in S_m \), (2.1) shows that the function \( \mathcal{D}^p \phi \in S_{m-n} \). Therefore, by using Lemma 3.2 and Lemma 2.1, we get the result.

Let the function \( f = h + \tilde{g} \) be a harmonic mapping, where
\[
h(z) = \sum_{k=1}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k.
\]
We define the \( n \)th partial sum of \( f \) by
\[
f_n(z) := \sum_{k=1}^{n} a_k z^k + \sum_{k=1}^{n} b_k z^k.
\]
Therefore, we can write \( f_n = f \ast l_n \), where \( l_n(z) = \sum_{k=1}^{n} z^k \) is the \( n \)th partial sum of right-half plane mapping \( l(z) = z/(1-z) \).

Theorem 3.4. For \( n \in \mathbb{N} \), let the function \( f = h + \tilde{g} \) be a harmonic mapping in \( \mathbb{D} \) with
\[
h(z) - g(z) = \left( \frac{z}{(1-z)^2} \right)^{n-1} \text{ for } z \in \mathbb{D},
\]
and
\[
\text{Re} \left( \frac{(1-z)^2}{z} \left\{ h(z) \ast \left( \log \frac{1}{1-z} \right)^{n-2} \right\} \right) > 1/2 \text{ for } z \in \mathbb{D}.
\]
Also, let the function \( \phi \in S_m \). Then, for the partial sum \( (f \ast \phi)_p \) of the convolution \( f \ast \phi \), we have the following:

1. If \( m \geq n + 2 \), then \( (f \ast \phi)_2 \in S_{H}^0 \) and is convex in the the direction of real axis;
2. If \( m \geq n + 3 \), then \( (f \ast \phi)_2, (f \ast \phi)_3 \in S_{H}^0 \) and are convex in the the direction of real axis;
3. If \( m \geq n + 4 \), then \( (f \ast \phi)_p \in S_{H}^0 \) and is convex in the the direction of real axis for all \( p \in \mathbb{N} \).

Proof. We know that \( (f \ast \phi)_p = (f \ast \phi) \ast l_p = f \ast (\phi \ast l_p) = f \ast \phi_p \), where \( \phi_p \) is the \( p \)th partial sum of the function \( \phi \). Therefore, in order to apply Theorem 2.15 we need \( \phi_p \) to be in the class \( S_n \), which follows from Lemma 3.3. The result now follows by Theorem 2.15.

Corollary 3.5. For \( a \geq 6 \), let the function \( f_n = h + \tilde{g} \) be the harmonic mapping given in Example 2.20. Then, we have the following:

1. If the function \( \phi \in S_3 \), then \( (f_6 \ast \phi)_2 \in S_{H}^0 \) and is convex in the direction of real axis;
2. If the function \( \phi \in S_4 \), then \( (f_6 \ast \phi)_2, (f_6 \ast \phi)_3 \in S_{H}^0 \) and are convex in the direction of real axis;
3. If the function \( \phi \in S_5 \), then \( (f_6 \ast \phi)_p \in S_{H}^0 \) and is convex in the direction of real axis for all \( p \in \mathbb{N} \).
Proof. We have $h(z) - g(z) = l(z) = z/(1 - z)$. Also, for $a \geq 6$ we have
\[
\text{Re} \left( \frac{(1-z)^2}{z} \left( h(z) * \left( \frac{z}{(1-z)^2} \right) \right) \right) = \text{Re}(1-z)^2 h'(z)
= \text{Re} \left( 1 + \frac{z}{a}(2-z) \right) > \frac{1}{2}.
\]
Therefore, the mapping $f_a$ satisfies (3.1) and (3.2) with $n = 1$. Hence, the result follows from the Theorem 3.4.

**Corollary 3.6.** For the harmonic Koebe mapping $K = H + G$ and the harmonic half-plane mapping $L = M + \overline{N}$, we have the following:

1. If the function $\phi \in S_4$, then $(K\tilde{\phi})_2, (L\tilde{\phi})_2 \in S^0_H$ and are convex in the direction of real axis;
2. If the function $\phi \in S_5$, then $(K\tilde{\phi})_2, (K\tilde{\phi})_3, (L\tilde{\phi})_2, (L\tilde{\phi})_3 \in S^0_H$ and are convex in the direction of real axis;
3. If the function $\phi \in S_6$, then $(K\tilde{\phi})_p, (L\tilde{\phi})_p \in S^0_H$ and are convex in the direction of real axis for all $p \in \mathbb{N}$.

**Proof.** The harmonic Koebe mapping $K(z) = H(z) + G(z)$ is given by
\[
H(z) = \frac{z - z^2/2 + z^3/6}{(1-z)^3} \quad \text{and} \quad G(z) = \frac{z^2/2 + z^3/6}{(1-z)^3}.
\]
Therefore,
\[
H(z) - G(z) = \frac{z}{(1-z)^2}
\]
and
\[
\text{Re} \left( \frac{(1-z)^2}{z} H(z) \right) = \text{Re} \left( \frac{1-z/2 + z^2/6}{1-z} \right) = \text{Re} \left( \frac{2/3}{1-z} + \frac{1}{3} + \frac{z}{6} \right) > \frac{1}{2}.
\]
Also, the half plane mapping $L(z) = M(z) + \overline{N(z)}$ is given by
\[
M(z) = \frac{z - z^2/2}{(1-z)^2} \quad \text{and} \quad N(z) = \frac{-z^2/2}{(1-z)^3}.
\]
Then,
\[
M(z) - N(z) = \frac{z}{(1-z)^2} \quad \text{and} \quad \text{Re} \left( \frac{(1-z)^2}{z} M(z) \right) = \text{Re} \left( 1 - \frac{z}{2} \right) > \frac{1}{2}.
\]
Therefore, the functions, $K$ and $L$ satisfies (3.1) and (3.2) with $n = 2$. Hence, the result follows from the Theorem 3.4.

**Example 3.7.** For $m \in \mathbb{N}$, we can easily see that the function
\[
f_m(z) := z + \frac{z^2}{2^m} + \frac{z^3}{3^m} + \cdots \in S_{m+1}.
\]
Therefore, by Corollary 3.6 the following functions
\[
(1) \ (K\tilde{f}_3)_2(z) = z + \frac{5}{16}z^2 + \frac{1}{16}z^2;
(2) \ (K\tilde{f}_4)_3(z) = z + \frac{5}{16}z^2 + \frac{14}{243}z^3 + \frac{14}{243}z^3 + \frac{5}{243}z^3;
(3) \ (K\tilde{f}_5)_4(z) = z + \frac{5}{64}z^2 + \frac{1}{729}z^3 + \frac{15}{2048}z^4 + \frac{1}{64}z^2 + \frac{5}{729}z^3 + \frac{7}{2048}z^4.
\]
begins belong to $S^0_H$ and are convex in the direction of real axis.

**Corollary 3.8.** For $|b| \leq 1/2$, let the function $F_b = h + \tilde{g}$ be the harmonic mapping given in Example 2.21. Then, we have the following:
(1) If the function \( \phi \in S_5 \), then \( (F_b \ast \phi)_2 \in S^0_H \) and is convex in the direction of real axis.

(2) If the function \( \phi \in S_6 \), then \( (F_b \ast \phi)_2, (F_b \ast \phi)_3 \in S^0_H \) and are convex in the direction of real axis.

(3) If the function \( \phi \in S_7 \), then \( (F_b \ast \phi)_p \in S^0_H \) and are convex in the direction of real axis for all \( p \in \mathbb{N} \).

Proof. We have

\[
h(z) - g(z) = \frac{z + z^2}{(1 - z)^3} = \left( \frac{z}{1 - z} \right)^2,
\]

and

\[
D \left( \frac{z + b z^2}{(1 - z)^2} \right) = \frac{z + (1 + 2b) z^2}{(1 - z)^3} = h(z).
\]

Using above equation, we get, for \( |b| \leq 1/2 \),

\[
\text{Re} \left( \frac{1 - z^2}{z} \right) \left( h(z) \ast \log \frac{1}{1 - z} \right) = \text{Re} \left( \frac{1 - z^2}{z} \right) \left( D \frac{z + b z^2}{(1 - z)^2} \ast \log \frac{1}{1 - z} \right) = \text{Re} \left( \frac{1 - z^2}{z} \right) (1 + b z) > 1/2.
\]

Therefore, the function \( F_b \) satisfies \((3.1)\) and \((3.2)\) with \( n = 3 \). Hence, the result follows from the Theorem 3.4.

Theorem 3.9. For the mapping \( f = h + \bar{g} \in S^0(H_\alpha) \), we have the following:

(1) If the function \( \phi \in S_3 \), then \( (f \ast \phi)_2 \in S^0_H \) and is convex in the direction \((\pi/2 - \alpha)\).

(2) If the function \( \phi \in S_4 \), then \( (f \ast \phi)_2, (f \ast \phi)_3 \in S^0_H \) and are convex in the direction \((\pi/2 - \alpha)\).

(3) If the function \( \phi \in S_5 \), then \( (f \ast \phi)_p \in S^0_H \) and are convex in the direction \((\pi/2 - \alpha)\) for all \( p \in \mathbb{N} \).

Proof. Since \( (f \ast \phi)_p = f_a \ast \phi_p \), where \( \phi_p \) is the \( p \)th partial sum of the function \( \phi \), therefore, in order to apply Theorem 2.26, we need the function \( \phi_p \) to be in the class \( K \), which follows from Lemma 3.3. The result now follows from Theorem 2.26.

Theorem 3.10. For the harmonic mapping \( f_\alpha \) defined in the Example 2.24, we have the following:

(1) If the function \( \phi \in S_4 \), then \( (f_\alpha \ast \phi)_2 \in S^0_H \) and is convex in the directions \(-\alpha \) and \((\pi/2 - \alpha)\).

(2) If the function \( \phi \in S_5 \), then \( (f_\alpha \ast \phi)_2, (f_\alpha \ast \phi)_3 \in S^0_H \) and are convex in the directions \(-\alpha \) and \((\pi/2 - \alpha)\).

(3) If the function \( \phi \in S_6 \), then \( (f_\alpha \ast \phi)_p \in S^0_H \) and is convex in the directions \(-\alpha \) and \((\pi/2 - \alpha)\) for all \( p \in \mathbb{N} \).

Proof. Since \( (f_\alpha \ast \phi)_p = f_\alpha \ast \phi_p \), where \( \phi_p \) is the \( p \)th partial sum of the function \( \phi \), therefore, in order to apply Theorem 2.27, we need the function \( \phi_p \) to be in the class \( S_2 \), which follows from Lemma 3.3. The result now follows from Theorem 2.27.

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