WELL-POSEDNESS AND EXPONENTIAL DECAY OF SOLUTIONS FOR THE BLACKSTOCK–CRIGHTON–KUZNETSOV EQUATION

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Abstract. The present work provides well-posedness and exponential decay results for the Blackstock–Crighton–Kuznetsov equation arising in the modeling of nonlinear acoustic wave propagation in thermally relaxing viscous fluids.

We prove local well-posedness by combining regularity results for the classical heat equation and the linearized Westervelt equation and applying the Banach fixed-point theorem. Global well-posedness is obtained by performing energy estimates and a classical barrier argument which finally leads to exponential decay of the energy with respect to appropriate norms.

1. Introduction

1.1. Nonlinear acoustics. An acoustic wave travels through a medium as a local pressure change. Nonlinear phenomena in acoustic wave propagation occur at high acoustic pressures which are used for several medical and industrial purposes such as lithotripsy, thermotherapy, ultrasound cleaning and sonochemistry. Driven by this broad range of applications, nonlinear acoustics is currently a highly active field of research (see, e.g., [3], [4], [9], [10], [11], [12], [13], [14], [15], [19] and [20]).

The classical models in nonlinear acoustics are partial differential equations of second order in time which are characterized by the presence of a viscoelastic damping. The most general classical model is Kuznetsov’s equation

\[ \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \left( \frac{1}{c^2} \frac{B}{2A} (\psi_t)^2 + |\nabla \psi|^2 \right)_t \]

where \( \psi \) denotes the acoustic velocity potential, \( c > 0 \) is the speed of sound, \( b \geq 0 \) is the diffusivity of sound and \( B/A \) is the parameter of nonlinearity. Neglecting local nonlinear effects (in the sense that the expression \( c^2 |\nabla \psi|^2 - (\psi_t)^2 \) is sufficiently small) one arrives at the Westervelt equation

\[ \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \left( \frac{1}{c^2} \left( 1 + \frac{B}{2A} \right) (\psi_t)^2 \right)_t. \]

Both, the Kuznetsov and the Westervelt equation, can alternatively be formulated in terms of the acoustic pressure \( p \) via the relation \( \rho \psi_t = p \) where \( \rho \) denotes the mass density. The quantities \( A \) and \( B \) occurring in the parameter of nonlinearity are the coefficients of the first and second order terms of the Taylor series expansion

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of the pressure-density relation. For more information on the physical concepts and mathematical models in nonlinear acoustics the reader is referred to [8].

The Kuznetsov equation (1.1) can in some sense be regarded as a simplification (for a small ratio $\nu \Pr^{-1}$ between the kinematic viscosity $\nu$ and the Prandtl number $\Pr$) of the fourth order in space equation

\[(1.3) \quad \left( \frac{\nu}{\Pr} \Delta - \partial_t \right) \left( \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t \right) = \left( \frac{1}{c^2} \frac{B}{2A} (\psi_t)^2 + |\nabla \psi|^2 \right)_{tt}.
\]

This equation results from two higher order models from the original paper [5] (see equations (11) and (13) there), namely

\[(1.4) \quad -c^2 \frac{\nu}{\Pr} \Delta^2 \psi + \left( \frac{\nu}{\Pr} + b \right) \Delta \psi_{tt} + \left( c^2 \Delta \psi - \psi_{ttt} \right) = \left( |\nabla \psi|^2 + \frac{B}{2A} \psi_t \Delta \psi \right)_{\psi}.
\]

Equations (1.4) and (1.5) are approximate equations derived from the basic equations (mass conservation, momentum conservation, entropy balance and thermodynamic state) describing the general motion of a thermally relaxing, viscous fluid. For the derivation of (1.4) and (1.5), the reader is referred to [2].

We replace of $\Delta \psi$ in the last term of (1.4), (1.5) by $\frac{1}{c^2} c^2 \psi_{tt}$, which can be justified by the main part of the differential operator which corresponds to the wave equation $\psi_{tt} - c^2 \Delta \psi = 0$. Moreover, we consider potential diffusivity as appearing in (1.4). Therewith, equation (1.5) becomes (1.3). We call (1.3) Blackstock–Crighton–Kuznetsov equation.

A simplified version of (1.3), namely

\[(1.6) \quad \left( \frac{\nu}{\Pr} \Delta - \partial_t \right) \left( \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t \right) = \left( \frac{1}{c^2} \left( 1 + \frac{B}{2A} \right) (\psi_t)^2 \right)_{\psi_{tt},}
\]

is obtained by neglecting local nonlinear effects which are taken into account by the gradient on the right-hand side as it is done when reducing the Kuznetsov to the Westervelt equation. For this equation, in [3] the name Blackstock-Crighton-Westervelt equation has been introduced.

The Westervelt and the Kuznetsov equation as well as the Khoklov-Zabolotskaya-Kuznetsov equation which is another standard model in nonlinear acoustics have recently been quite extensively investigated (see [4], [9], [10], [11], [12], [13], [15] and [19]). Research on higher order models governing nonlinear acoustic wave propagation such as (1.3) and (1.6) is still in an early stage. The starting point was [3] where well-posedness and exponential decay of solutions for (1.3) together with homogeneous Dirichlet boundary conditions was shown.

The goal of the present paper is to provide results on well-posedness and exponential decay for the more general Blackstock-Crighton-Kuznetsov equation (1.3) which is one more step towards closing the gap of missing results on higher order nonlinear acoustic wave equations. We abbreviate $a = \nu \Pr^{-1}$ and $\sigma = \frac{1}{c^2} \frac{B}{2A}$.

Our object of investigation is the initial boundary value problem

\[(1.7) \quad \begin{cases}
(a \Delta - \partial_t) \left( \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t \right) = (\sigma (\psi_t)^2 + |\nabla \psi|^2)_{tt} & \text{in } \Omega \times (0, T], \\
(\psi, \psi_t, \psi_{tt}) = (\psi_0, \psi_1, \psi_2) & \text{on } \Omega \times \{ t = 0 \}, \\
\psi = 0, \Delta \psi = 0 & \text{on } \partial \Omega \times [0, T],
\end{cases}
\]
on an open and bounded subset $\Omega \subset \mathbb{R}^d, d \in \{1, 2, 3\}$ with boundary $\partial \Omega$, where $\psi_0, \psi_1, \psi_2 : \Omega \to \mathbb{R}$ are given and $\psi : \overline{\Omega} \times [0, T] \to \mathbb{R}$ is the unknown, $\psi = \psi(x,t)$.

The restriction on the dimension of the spatial domain $\Omega$ is imposed in order to be able to use various embeddings. Increasing the space dimension to $d \geq 4$ would not be of relevance in applications anyway.

1.2. Notation. Throughout the paper we will assume that $\Omega \subset \mathbb{R}^d, d \in \{1, 2, 3\}$ is an open and bounded domain with smooth boundary $\partial \Omega$.

We denote by $L^2(\Omega)$ the space of (classes of) Lebesgue square integrable functions $\Omega \to \mathbb{R}$ equipped with the inner product $\langle u, v \rangle_{L^2(\Omega)} = \int_\Omega uv$ and the induced norm $\|u\|_{L^2(\Omega)}$.

More generally, we will always write $\| \cdot \|_X$ for the $X$-norm of a function in a Banach space $X$ and $C^k_X \hookrightarrow Y$ for the embedding constant of the continuous embedding $X \hookrightarrow Y$ of $X$ into another Banach space $Y$ where we emphasized that the constant depends on $K$.

The space $C^k(0, T; X)$ consists of all $k$-times continuously differentiable functions $u : [0, T] \to X$ where $k \in \mathbb{N}_0$. We equip it with the norm $\|u\|_{C^k(0, T; X)} = \sum_{i=0}^{k} \|\partial_t^i u\|_X$.

By $H^s(\Omega) := W^{s,2}(\Omega)$ we denote the Sobolev space of order $s \in \mathbb{N}$ and exponent $p = 2$. The space $H^1_0(\Omega)$ contains all $H^1(\Omega)$-functions with zero trace. The norm of a function $u \in H^s(\Omega) \cap H^1_0(\Omega), s \geq 1$ is given by $\|u\|_{H^s(\Omega)} := \|(-\Delta)^{s/2} u\|_{L^2(\Omega)}$ where $\Delta$ stands for the negative Laplacian on $L^2(\Omega)$ with domain of definition $\mathcal{D}(-\Delta) = H^2(\Omega) \cap H^1_0(\Omega)$. Recall that $\Delta$ is closed, densely defined, self-adjoint and that its spectrum $\sigma(-\Delta)$ is contained in the positive half-line. For additional information on Sobolev spaces, in particular embedding theorems, we refer to [1].

1.3. Outline. In Section 2 we recall some results for the linearized version of (1.7) which has been studied in [3]. In particular, the underlying semigroup is analytic which, together with a negative spectral bound of its generator, yields exponential decay results for the homogeneous equation. Moreover, we provide regularity results for the homogeneous and linearized inhomogeneous version of the Blackstock–Crighton–Kuznetsov equation.

In Section 3 we prove local well-posedness for the nonlinear equation (1.7), i.e. we show existence, uniqueness and continuous dependence on the initial conditions of solutions for (1.7) on a sufficiently small time interval. The key ingredient is the Banach fixed-point theorem. In order to motivate the space used in the fixed-point argument we combine certain regularity results for the classical heat equation and the linearized Westervelt equation.

Finally, Section 4 is devoted to the proof of global well-posedness. As a preparation we perform energy estimates which we use in a classical barrier argument in order to show that there exists a well-posed solution for all times. Moreover, as a consequence, we get exponential decay for solutions of (1.7).

2. Semigroup framework

The linearized version of (1.7) has been investigated in [3] in an abstract form. It was considered on a general Hilbert space $\mathcal{H}$ and $\Delta$ was replaced by a self-adjoint, strictly positive, closed and densely defined operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$. Here we always take $\mathcal{H} = L^2(\Omega)$, $\mathcal{A} = -\Delta$ and $\mathcal{D}(\mathcal{A}) = H^2(\Omega) \cap H^1_0(\Omega)$ and only recall the main facts. For more details and the proofs of the results presented below the
We consider the linear partial differential equation

\begin{equation}
(a\Delta - \partial_t) (\psi_{tt}(t) - c^2\Delta \psi(t) - b\Delta \psi_t(t)) = f(t),
\end{equation}

where \( f \) is the inhomogeneity, defined on \( L^2(\Omega) \) with the initial conditions

\begin{equation}
\psi(0) = \psi_0, \quad \psi_0(0) = \psi_1, \quad \psi_t(0) = \psi_2,
\end{equation}

which can be represented as an abstract ordinary differential equation

\begin{equation}
\Psi(t) = \Psi(0) = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 - c^2\Delta \psi_0 - b\Delta \psi_1 \end{pmatrix},
\end{equation}

with initial the conditions

\begin{equation}
\Psi_0 = \Psi(0) = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 - c^2\Delta \psi_0 - b\Delta \psi_1 \end{pmatrix}.
\end{equation}

If we choose

\begin{equation}
A = \begin{pmatrix} 0 & I & 0 \\ c^2\Delta & b\Delta & I \\ 0 & 0 & a\Delta \end{pmatrix}, \quad D(A) = (H^2(\Omega) \cap H^1_0(\Omega))^3,
\end{equation}

and

\begin{equation}
\Psi(t) = \begin{pmatrix} \psi(t) \\ \psi_0(t) \\ \psi_t(t) - c^2\Delta \psi(t) - b\Delta \psi_t(t) \end{pmatrix}.
\end{equation}

**Theorem 2.1** ([3] Theorems 3.7 and 3.10). The operator \( A \) given by (2.5) generates an analytic semigroup on the spaces

\[
H_1 := H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega),
\]

\[
H_2 := H^2(\Omega) \cap H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega).
\]

**Remark 2.2.** In [3], the presence of the damping parameter \( b > 0 \) was essential in order to obtain analyticity of the semigroup generated by the operator \( A \) in (2.5).

The spectral bound \( s(A) := \{\text{Re}(\lambda) : \lambda \in \sigma(A)\} \) of \( A \) is given by

\[
s(A) = -\min \left\{ a\mu, \frac{b\mu}{2}, \frac{c^2}{\mu} \right\},
\]

where \( \mu = \min \sigma(-\Delta) \) (cf. Lemma 3.12 in [3]). As a consequence, we obtain an exponential decay result for the homogeneous equation.

**Theorem 2.3** ([3] Theorem 3.14). Suppose \( f \equiv 0 \). Then there exist positive constants \( M_1, M_2, \omega_1, \omega_2 \) such that

\[
E_1[\psi](t) \leq M_1 e^{-\omega_1 t} E_1[\psi](0) \quad \text{and} \quad E_2[\psi](t) \leq M_2 e^{-\omega_2 t} E_2[\psi](0),
\]

where

\[
E_1[\psi](t) := \|\psi(t)\|^2_{H^1(\Omega)} + \|\psi_t(t)\|^2_{L^2(\Omega)} + \|\psi_{tt}(t) - c^2\Delta \psi(t) - b\Delta \psi_t(t)\|^2_{L^2(\Omega)},
\]

\[
E_2[\psi](t) := \|\psi(t)\|^2_{H^2(\Omega)} + \|\psi_t(t)\|^2_{L^2(\Omega)} + \|\psi_{tt}(t) - c^2\Delta \psi(t) - b\Delta \psi_t(t)\|^2_{L^2(\Omega)}.
\]
Corollary 2.4 ([3 Corollary 3.16]). The homogeneous initial boundary value problem

\[
\begin{align*}
(a\Delta - \partial_t) (\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t) &= 0 & \text{in } \Omega \times (0, T], \\
(\psi, \psi_t, \psi_{tt}) &= (\psi_0, \psi_1, \psi_2) & \text{on } \Omega \times \{t = 0\}, \\
\psi = 0, \Delta \psi &= 0 & \text{on } \partial \Omega \times [0, T],
\end{align*}
\] (2.7)

has a unique solution

\[\psi \in C^1(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^2(0, T; L^2(\Omega))\]

for all \(T > 0\) provided \(\psi_0 \in H^1_0(\Omega), \psi_1 \in L^2(\Omega)\) and \(\psi_2 - b \Delta \psi_1 - c^2 \Delta \psi \in L^2(\Omega)\). In particular, it has a unique solution of this regularity, if \(\psi_0 \in H^2(\Omega) \cap H^1_0(\Omega), \psi_1 \in H^2(\Omega) \cap H^1_0(\Omega)\) and \(\psi_2 \in L^2(\Omega)\).

Corollary 2.5 ([3 Corollary 3.18]). Let \(\psi_0 \in H^1_0(\Omega), \psi_1 \in L^2(\Omega)\) and \(\psi_{tt} - b \Delta \psi_1 - c^2 \Delta \psi \in L^2(\Omega)\). Furthermore, suppose \(f \in L^1(0, T; L^2(\Omega))\) is locally Hölder-continuous on \((0, T]\). Then the initial boundary value problem

\[
\begin{align*}
(a\Delta - \partial_t) (\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t) &= f & \text{in } \Omega \times (0, T], \\
(\psi, \psi_t, \psi_{tt}) &= (\psi_0, \psi_1, \psi_2) & \text{on } \Omega \times \{t = 0\}, \\
\psi = 0, \Delta \psi &= 0 & \text{on } \partial \Omega \times [0, T],
\end{align*}
\] (2.8)

has a unique solution

\[\psi \in C^1(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap C^2(0, T; L^2(\Omega)).\]

In particular, it has a unique solution of this regularity, if \(\psi_0 \in H^2(\Omega) \cap H^1_0(\Omega), \psi_1 \in H^2(\Omega) \cap H^1_0(\Omega)\) and \(\psi_2 \in L^2(\Omega)\).

3. Local well-posedness

In this section we show local in time well-posedness of the nonlinear initial boundary value problem (2.4), i.e. existence and uniqueness of a solution \(\psi(x, t), (x, t) \in \overline{\Omega} \times [0, \bar{T}]\) with \(\bar{T}\) sufficiently small, as well as its continuous dependence on the initial conditions.

To this end, we employ the Banach fixed-point theorem. In order to motivate the space we use in our fixed point argument, we recall and thereafter combine regularity results for the heat equation and the linearized Westervelt equation. Using the differential operators

\[D_h := a\Delta - \partial_t \quad \text{and} \quad D_w := \partial^2_t - c^2 \Delta - b \Delta \partial_t,\]

the linearized version of (2.4) reads

\[D_h D_w \psi = f.\]

Explicitly, we have

\[
\begin{align*}
D_w \psi(t) &= \psi_{tt}(t) - c^2 \Delta \psi(t) - b \Delta \psi_t(t) = \tilde{f}(t), \\
D_h \tilde{f}(t) &= a \Delta \tilde{f}(t) - \tilde{f}_t(t) = f(t).
\end{align*}
\] (3.1) (3.2)

Here, (3.1) is the linearized Westervelt equation and (3.2) is the heat equation.
Proposition 3.1. Consider the heat equation
\[
\begin{aligned}
\begin{cases}
  a \Delta \tilde f - \tilde f_t = f & \text{in } \Omega \times (0, T], \\
  \tilde f = 0 & \text{in } \partial \Omega \times [0, T], \\
  \tilde f = \tilde f_0 & \text{in } \Omega \times \{t = 0\}.
\end{cases}
\end{aligned}
\]
Suppose \( f \in L^2(0, T; H^1_0(\Omega)) \) and \( \tilde f_0 \in H^2(\Omega) \cap H^1_0(\Omega) \). Then
\[
\tilde f \in \tilde X := C(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap H^1(0, T; H^1_0(\Omega)).
\]

Proof. Taking inner products of \( a \Delta \tilde f - \tilde f_t = f \) with \( \Delta \tilde f \) in \( L^2(\Omega) \) and integrating with respect to time yields
\[
a \int_0^t \langle \Delta \tilde f, \Delta \tilde f_t \rangle_{L^2(\Omega)} d\tau - \int_0^t \langle \tilde f_t, \Delta \tilde f \rangle_{L^2(\Omega)} d\tau = \int_0^t \langle f, \Delta \tilde f \rangle_{L^2(\Omega)} d\tau
\]
which after integration by parts with respect to space and time and estimating
\[
\langle f, \Delta \tilde f \rangle_{L^2(\Omega)} = \langle -\nabla f, \nabla \tilde f \rangle_{L^2(\Omega)} \leq \frac{1}{2} \| \nabla f \|^2_{L^2(\Omega)} + \frac{1}{2} \| \nabla \tilde f \|^2_{L^2(\Omega)}
\]
reads
\[
\frac{1}{2} \int_0^t \| \nabla \tilde f \|^2_{L^2(\Omega)} d\tau + \frac{a}{2} \| \Delta \tilde f \|^2_{L^2(\Omega)} \big|_0^t \leq \frac{1}{2} \int_0^t \| \nabla f \|^2_{L^2(\Omega)} d\tau.
\]
Thus, under the assumptions \( f \in L^2(0, T; H^1_0(\Omega)) \) and \( \tilde f_0 \in H^2(\Omega) \cap H^1_0(\Omega) \), we get
\( \tilde f \in C(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap H^1(0, T; H^1_0(\Omega)) \). \( \square \)

Proposition 3.2. Consider the linearized Westervelt equation
\[
\begin{aligned}
\begin{cases}
  \psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t = \tilde f & \text{in } \Omega \times (0, T], \\
  (\psi, \psi_t) = (\psi_0, \psi_1) & \text{in } \Omega \times \{t = 0\}, \\
  \psi = 0 & \text{on } \partial \Omega \times [0, T].
\end{cases}
\end{aligned}
\]

(i) Suppose \( \psi \in C(0, T; H^1_0(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)), \psi_0 \in H^4(\Omega) \cap H^1_0(\Omega), \psi_1 \in H^3(\Omega) \cap H^1_0(\Omega) \) and \( \psi_2 \in H^3(\Omega) \), then we also have
\( \psi \in C(0, T; H^4(\Omega) \cap H^1_0(\Omega)) \cap C^1(0, T; H^3(\Omega) \cap H^1_0(\Omega)) \).

(ii) If, in addition to (i), \( \tilde f \in H^1(0, T; H^1_0(\Omega)), \psi_2 \in H^2(\Omega) \cap H^1_0(\Omega), \) then we also have
\( \psi \in C^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap H^3(0, T; H^1_0(\Omega)) \).

Proof. The result in (i) follows from Proposition 7.2(ii) in [13].

In order to prove (ii), we differentiate the linearized Westervelt equation \( D_w \psi = \tilde f \) with respect to time, take inner products in \( L^2(\Omega) \) with \( -\Delta \psi_{ttt} \) and integrate with respect to time to obtain
\[
\int_0^t \langle \psi_{ttt} - c^2 \Delta \psi_t - b \Delta \psi_{tt}, -\Delta \psi_{ttt} \rangle_{L^2(\Omega)} d\tau = \int_0^t \langle \tilde f_t, -\Delta \psi_{ttt} \rangle_{L^2(\Omega)} d\tau,
\]
which, after integration by parts with respect to space and time, reads
\[
\int_0^t \| \nabla \psi_{ttt} \|^2_{L^2(\Omega)} d\tau + \frac{b}{2} \| \Delta \psi_{tt} \|^2_{L^2(\Omega)} \big|_0^t = \int_0^t \langle \nabla \tilde f_t + c^2 \Delta \psi_t, \nabla \psi_{ttt} \rangle_{L^2(\Omega)} d\tau.
\]
Estimating the right-hand side yields
\[ \int_0^t \langle \nabla (\tilde{f}_t + c^2 \Delta \psi_t), \nabla \psi_{tt} \rangle_{L^2(\Omega)} \, dt \]
\[ \leq \int_0^t \| \nabla (\tilde{f}_t + c^2 \Delta \psi_t) \|_{L^2(\Omega)} \| \nabla \psi_{tt} \|_{L^2(\Omega)} \, dt \]
\[ \leq \frac{1}{2} \int_0^t \| \psi_{tt} \|_{H^1(\Omega)}^2 \, dt + \frac{1}{2} \int_0^t \| \tilde{f}_t + c^2 \Delta \psi_t \|_{H^1(\Omega)}^2 \, dt \]
\[ \leq \frac{1}{2} \int_0^t \| \psi_{tt} \|_{H^1(\Omega)}^2 \, dt + \int_0^t \| \tilde{f}_t \|_{H^1(\Omega)}^2 + c^2 \| \psi_t \|_{H^3(\Omega)}^2 \, dt \]
which implies
\[ \frac{1}{2} \int_0^t \| \psi_{tt} \|_{H^1(\Omega)}^2 \, dt + \frac{b}{2} \int_0^t \| \psi_t \|_{H^2(\Omega)}^2 \, dt \leq \int_0^t \| \tilde{f}_t \|_{H^1(\Omega)}^2 \, dt + c^2 \int_0^t \| \psi_t \|_{H^3(\Omega)}^2 \, dt. \]
Invoking the assumptions and noting that for any finite time horizon \( T < \infty \) we have \( C^1(0, T; H^4(\Omega) \cap H^1_0(\Omega)) \subset H^1(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \) finally yields the claim. \( \square \)

Demanding \( \psi_1 \in H^1(\Omega) \cap H^1_0(\Omega) \) together with assumptions \( \psi_0 \in H^4(\Omega) \cap H^1_0(\Omega), \psi_2 \in H^2(\Omega) \cap H^1_0(\Omega) \) ensures \( \tilde{f}_0 \in H^2(\Omega) \cap H^1_0(\Omega) \) which allows us to combine Propositions \( 3.2 \) and \( 3.3 \) and begs us to introduce the space
\[ \mathcal{V} := C(0, T; H^4(\Omega) \cap H^1_0(\Omega)) \cap C^1(0, T; H^3(\Omega) \cap H^1_0(\Omega)) \]
\[ \cap C^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \cap H^3(0, T; H^1_0(\Omega)) \]
which we equip with the norm
\[ \| \cdot \| := \| \cdot \|_{C(0,T;H^4(\Omega))} + \| \cdot \|_{C^1(0,T;H^3(\Omega))} + \| \cdot \|_{C^2(0,T;H^2(\Omega))} + \| \cdot \|_{H^3(0,T;H^1(\Omega))}. \]

Our strategy in order to show local existence and uniqueness of solutions of \( \text{(1.7)} \) is to apply the Banach fixed-point theorem to the map
\[ \mathcal{T} : \mathcal{W} \to \mathcal{V}, \]
\[ \varphi \mapsto \psi \]
where
\[ \mathcal{W} := B_{\overline{m}}^V(0) = \{ v \in \mathcal{V} : \| v \|_{\mathcal{V}} \leq \overline{m} \} \]
with \( \overline{m} \) sufficiently small and where \( v \) is a solution of
\[ (a \Delta - \partial_t) (\psi_{tt} - c^2 \Delta \psi - b \Delta \psi_t) = \left( \sigma (\varphi_t)^2 + |\nabla \varphi|^2 \right)_t. \]

**Step 1:** \( \mathcal{T} \) is a self-mapping on \( \mathcal{W} \).

**Lemma 3.3.** Suppose \( \varphi \in \mathcal{V} \). Then \( f = (\sigma (\varphi_t)^2 + |\nabla \varphi|^2)_{tt} \in L^2(0, T; H^1_0(\Omega)) \) and we have the estimate
\[ \| f \|_{L^2(0, T; H^1(\Omega))} \leq C \| \varphi \|_{\mathcal{V}}^2 \]
for some constant \( C > 0 \).

**Proof:** We prove that, provided \( \varphi \in \mathcal{V} \), we have \( \nabla f \in L^2(0, T; L^2(\Omega)) \) and an estimate of the form \( \| \nabla f \|_{L^2(0, T; L^2(\Omega))} \leq C \| \varphi \|_{\mathcal{V}}^2 \) for some constant \( C > 0 \). Explicitly we have
\[ f = 2\sigma (\varphi_{tt})^2 + 2\varphi_t \varphi_{tt} + 2|\nabla \varphi_t|^2 + 2\nabla \varphi \nabla \varphi_{tt} \]
where \( L^2(0,T;L^2(\Omega)) \) and can be estimated in terms of \( \| \varphi \|_V^2 \).

(1) Note that

\[
\varphi \in V \implies \varphi \in C^2(0,T;H^2(\Omega) \cap H^1_0(\Omega)) \\
\implies \varphi_{tt} \in C(0,T;H^2(\Omega) \cap H^1_0(\Omega)) \rightarrow C(0,T;L^\infty(\Omega)),
\]

\[
\varphi \in V \implies \varphi \in H^3(0,T;H^3_0(\Omega)) \implies \nabla \varphi_{tt} \in H^1(0,T;L^2(\Omega)) \rightarrow L^2(0,T;L^2(\Omega))
\]
due to the embeddings \( H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^\infty(\Omega) \) and \( H^1(0,T) \hookrightarrow L^2(0,T) \). Since \( C(0,T;L^\infty(\Omega)) \) is an ideal in \( L^2(0,T;L^2(\Omega)) \), we finally conclude \( \varphi_{tt} \nabla \varphi_{tt} \in L^2(0,T;L^2(\Omega)) \) and estimate

\[
\| \varphi_{tt} \nabla \varphi_{tt} \|_{L^2(0,T;L^2(\Omega))} \leq \| \varphi_t \|_{C(0,T;L^\infty(\Omega))} \| \nabla \varphi_t \|_{L^2(0,T;L^2(\Omega))}
\]

\[
\leq C^G_{H^2,L^\infty} C^T_{H^1,L^2} \| \varphi_t \|_{C(0,T;H^2(\Omega))} \| \varphi_{tt} \|_{H^1(0,T;H^1(\Omega))}
\]

\[
\leq C^G_{H^2,L^\infty} C^T_{H^1,L^2} \| \varphi \|_{C^2(0,T;H^2(\Omega))} \| \varphi_t \|_{H^3(0,T;H^1(\Omega))}
\]

\[
\leq C^G_{H^2,L^\infty} C^T_{H^1,L^2} \| \varphi \|_V^2,
\]

where \( C^G_{H^2,L^\infty} \) and \( C^T_{H^1,L^2} \) denote the embedding constants for the embeddings \( H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^\infty(\Omega) \) and \( H^1(0,T) \hookrightarrow L^2(0,T) \), respectively.

(2) For the second term we have

\[
\varphi \in V \implies \varphi \in C^1(0,T;H^3(\Omega) \cap H^1_0(\Omega))
\]

\[
\implies \nabla \varphi_t \in C(0,T;H^2(\Omega) \cap H^1_0(\Omega)) \hookrightarrow C(0,T;L^\infty(\Omega)),
\]

\[
\varphi \in V \implies \varphi \in H^3(0,T;H^3_0(\Omega)) \implies \varphi_{ttt} \in L^2(0,T;H^1_0(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega)),
\]

where we have used \( H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^\infty(\Omega) \) and \( H^3_0(\Omega) \hookrightarrow L^2(\Omega) \). Hence \( \varphi_{ttt} \nabla \varphi_t \in L^2(0,T;L^2(\Omega)) \) and

\[
\| \varphi_{ttt} \nabla \varphi_t \|_{L^2(0,T;L^2(\Omega))} \leq \| \nabla \varphi_t \|_{C(0,T;L^\infty(\Omega))} \| \varphi_{ttt} \|_{L^2(0,T;L^2(\Omega))}
\]

\[
\leq C^G_{H^3,L^\infty} C^T_{H^1,L^2} \| \nabla \varphi_t \|_{C(0,T;H^2(\Omega))} \| \varphi_{ttt} \|_{L^2(0,T;H^3(\Omega))}
\]

\[
\leq C^G_{H^3,L^\infty} C^T_{H^1,L^2} \| \varphi \|_{C^1(0,T;H^3(\Omega))} \| \varphi \|_{H^3(0,T;H^1(\Omega))}
\]

\[
\leq C^G_{H^3,L^\infty} C^T_{H^1,L^2} \| \varphi \|_V^2,
\]

with \( C^G_{H^3,L^\infty} \) the norm of the embedding \( H^3_0(\Omega) \hookrightarrow L^2(\Omega) \).

(3) Using the embedding \( H^3(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^\infty(\Omega) \), we obtain

\[
\varphi \in V \implies \varphi \in C^1(0,T;H^3(\Omega) \cap H^1_0(\Omega))
\]

\[
\implies \varphi_t \in C(0,T;H^3(\Omega) \cap H^1_0(\Omega)) \hookrightarrow C(0,T;L^\infty(\Omega)),
\]

\[
\varphi \in V \implies \varphi \in H^3(0,T;H^3_0(\Omega)) \implies \nabla \varphi_{ttt} \in L^2(0,T;L^2(\Omega)),
\]
which yields \( \varphi_t \nabla \varphi_{tt} \in L^2(0, T; L^2(\Omega)) \). Furthermore,

\[
\| \varphi_t \nabla \varphi_{tt} \|_{L^2(0, T; L^2(\Omega))} \leq \| \nabla \varphi_t \|_{C(0, T; L^\infty(\Omega))} \| \nabla \varphi_{tt} \|_{L^2(0, T; L^2(\Omega))} \\
\leq C_{H^3, L^\infty}^\Omega \| \nabla \varphi_t \|_{C(0, T; H^2(\Omega))} \| \varphi_{tt} \|_{L^2(0, T; H^1(\Omega))} \\
\leq C_{H^3, L^\infty}^\Omega \| \varphi_t \|^2_V,
\]

where \( C_{H^3, L^\infty}^\Omega \) denotes the constant for the embedding \( H^3(\Omega) \cap H^1_0(\Omega) \to L^\infty(\Omega) \).

(4) We use \( H^2(\Omega) \cap H^1_0(\Omega) \to L^\infty(\Omega) \), \( H^1_0(\Omega) \to L^2(\Omega) \) and \( C(0, T) \to L^2(0, T) \) to obtain

\[
\varphi \in V \implies \varphi \in C^1(0, T; H^3(\Omega) \cap H^1_0(\Omega)) \\
\implies \nabla \varphi_t \in C(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \to C(0, T; L^\infty(\Omega)), \\
\varphi \in V \implies \varphi \in C^1(0, T; H^3(\Omega) \cap H^1_0(\Omega)) \\
\implies \Delta \varphi \in C(0, T; H^1_0(\Omega)) \to L^2(0, T; L^3(\Omega)).
\]

Thus we have \( \Delta \varphi_t \nabla \varphi_t \in L^2(0, T; L^2(\Omega)) \) and estimate

\[
\| \nabla \varphi_t \Delta \varphi_t \|_{L^2(0, T; L^2(\Omega))} \leq \| \nabla \varphi_t \|_{C(0, T; L^\infty(\Omega))} \| \Delta \varphi_t \|_{L^2(0, T; L^2(\Omega))} \\
\leq C_{H^2, L^\infty}^\Omega C_{C, L^2}^T C_{H^1, L^2}^\Omega \| \nabla \varphi_t \|_{C(0, T; H^2(\Omega))} \| \Delta \varphi_t \|_{C(0, T; H^1(\Omega))} \\
\leq C_{H^2, L^\infty}^\Omega C_{C, L^2}^T C_{H^1, L^2}^\Omega \| \varphi_t \|_{C(0, T; H^3(\Omega))} \| \varphi \|_{C(0, T; H^1(\Omega))} \\
\leq C_{H^2, L^\infty}^\Omega C_{C, L^2}^T C_{H^1, L^2}^\Omega \| \varphi \|^2_V,
\]

where \( C_{C, L^2}^T \) is the embedding constant of \( C(0, T) \to L^2(0, T) \).

(5) We observe

\[
\varphi \in V \implies \varphi \in C(0, T; H^4(\Omega) \cap H^1_0(\Omega)) \\
\implies \Delta \varphi \in C(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \to C(0, T; L^\infty(\Omega)), \\
\varphi \in V \implies \varphi \in H^3(0, T; H^1_0(\Omega)) \implies \nabla \varphi_{tt} \in H^1(0, T; L^2(\Omega)) \to L^2(0, T; L^2(\Omega)),
\]

where we have used \( H^2(\Omega) \cap H^1_0(\Omega) \to L^\infty(\Omega) \) and \( H^1(0, T) \to L^2(0, T) \). Hence \( \Delta \varphi \nabla \varphi_{tt} \in L^2(0, T; L^2(\Omega)) \) and

\[
\| \Delta \varphi \nabla \varphi_{tt} \|_{L^2(0, T; L^2(\Omega))} \leq \| \Delta \varphi \|_{C(0, T; L^\infty(\Omega))} \| \nabla \varphi_{tt} \|_{L^2(0, T; L^2(\Omega))} \\
\leq C_{H^3, L^\infty}^\Omega C_{H^1, L^2}^\Omega \| \Delta \varphi \|_{C(0, T; H^2(\Omega))} \| \nabla \varphi_{tt} \|_{H^1(0, T; L^2(\Omega))} \\
\leq C_{H^3, L^\infty}^\Omega C_{H^1, L^2}^\Omega \| \varphi \|_{C(0, T; H^3(\Omega))} \| \varphi \|_{H^1(0, T; H^1(\Omega))} \\
\leq C_{H^3, L^\infty}^\Omega C_{H^1, L^2}^\Omega \| \varphi \|^2_V,
\]

(6) Finally, invoking \( H^3(\Omega) \cap H^1_0(\Omega) \to L^\infty(\Omega) \) and \( C(0, T) \to L^2(0, T) \), we have

\[
\varphi \in V \implies \varphi \in C(0, T; H^4(\Omega) \cap H^1_0(\Omega)) \\
\implies \nabla \varphi \in C(0, T; H^3(\Omega) \cap H^1_0(\Omega)) \to C(0, T; L^\infty(\Omega)), \\
\varphi \in V \implies \varphi \in C^2(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \\
\implies \Delta \varphi_{tt} \in C(0, T; L^2(\Omega)) \to L^2(0, T; L^2(\Omega)).
\]
which yields $\Delta \varphi_t \nabla \varphi \in L^2(0, T; L^2(\Omega))$ and allows us to estimate
\[
\|\Delta \varphi_t \nabla \varphi\|_{L^2(0, T; L^2(\Omega))} \leq \|\nabla \varphi\|_{C(0, T; L^\infty(\Omega))} \|\Delta \varphi_t\|_{L^2(0, T; L^2(\Omega))}
\leq C_{H^3, L^\infty}^0 C_{C, L^2}^T \|\nabla \varphi\|_{C(0, T; H^1(\Omega))} \|\Delta \varphi_t\|_{C(0, T; L^2(\Omega))}
\leq C_{H^3, L^\infty}^0 C_{C, L^2}^T \|\varphi\|_{C(0, T; H^1(\Omega))} \|\varphi\|_{C^2(0, T; L^2(\Omega))}
\leq C_{H^3, L^\infty}^0 C_{C, L^2}^T \|\varphi\|_V^2.
\]
Combining steps (1)–(6), we obtain $\nabla f \in L^2(0, T; L^2(\Omega))$, i.e. $f \in L^2(0, T; H^1_0(\Omega))$, and, using Minkowski’s inequality, we finally have (3.7) with
\[
C = 2C_{H^3, L^\infty}^0 (\sigma + C_{C, L^2}^T) + 2C_{H^2, L^\infty}^0 (\sigma C_{H^1, L^2}^0 + C_{H^1, L^2}^T (2C_{C, L^2}^T + 2\sigma + 1))
\]
and hence we are done. \qed

Now, Proposition 3.1 and Lemma 3.3 provide us with the estimate
\[
\|\tilde{f}\|_\chi \leq C_h \left( \|f\|_{L^2(0, T; H^1(\Omega))} + \|f_0\|_{H^2(\Omega)} \right)
\leq C_h \left( C \|\varphi\|_V^2 + \|f_0\|_{H^2(\Omega)} \right)
\]
for some constant $C_h$. Furthermore, from Proposition 3.2 we obtain
\[
\|\psi\|_V \leq C_w \left( \|\tilde{f}\|_\chi + \|\psi_0\|_{H^1(\Omega)} + \|\psi_1\|_{H^2(\Omega)} + \|\psi_2\|_{H^2(\Omega)} \right)
\leq C_w \left( C_h \left( C \|\varphi\|_V^2 + \|f_0\|_{H^2(\Omega)} \right) + \|\psi_0\|_{H^1(\Omega)} + \|\psi_1\|_{H^2(\Omega)} + \|\psi_2\|_{H^2(\Omega)} \right)
\]
for some other constant $C_w$. Let now $\varphi \in W$, i.e. $\|\varphi\|_V \leq \overline{m}$ with $\overline{m} \leq \frac{1}{2 C_C C_w}$.

Provided initial data are sufficiently small,
\[
(3.9) \quad C_h \|f_0\|_{H^2(\Omega)} + \|\psi_0\|_{H^1(\Omega)} + \|\psi_1\|_{H^2(\Omega)} + \|\psi_2\|_{H^2(\Omega)} \leq \lambda
\]
with $\lambda = \frac{\overline{m}}{2 C_w}$, we conclude $\|\psi\|_V \leq \overline{m}$. Hence $\mathcal{T} W \subseteq W$ under the above assumptions.

\textit{Step 2: $W$ is a closed subset of $V$.}
This is trivial as $W \subseteq V$ is a closed ball of radius $\overline{m}$.

\textit{Step 3: $\mathcal{T} : W \to W$ is a contraction.}
In order to show contractibility of $\mathcal{T} : W \to W$, suppose $\psi_i, i = 1, 2$ are two solutions of (3.6), $\psi_i = \mathcal{T} \varphi_i$. Then $\hat{\varphi} = \varphi_1 - \varphi_2$ and $\hat{\psi} = \psi_1 - \psi_2$ solve the equation
\[
(a \Delta - \partial_t)(\hat{\psi}_{tt} - c^2 A \Delta \hat{\psi} - A \hat{\psi}) = \hat{f},
\]
i.e. $D_w \hat{\psi} = \hat{f}, D_w \hat{\varphi} = \hat{f}$ where
\[
\hat{f} = \left( \sigma (\varphi_1)^2 - \sigma (\varphi_2)^2 + |\nabla \varphi_1|^2 - |\nabla \varphi_2|^2 \right)_{tt}.
\]

\textbf{Lemma 3.4.} Suppose $\varphi_1, \varphi_2 \in V$. Then $\hat{f} \in L^2(0, T; H^1_0(\Omega))$ and we have
\[
(3.10) \quad \|\hat{f}\|_{L^2(0, T; H^1(\Omega))} \leq \overline{C} \|\hat{\varphi}\|_V (\|\varphi_1\|_V + \|\varphi_2\|_V)
\]
for some constant $\overline{C} > 0$.

\textit{Proof.} Explicitly,
\[
\hat{f} = 2 \left( \sigma \hat{\varphi}_{tt} (\varphi_{1, tt} + \varphi_{2, tt}) + \sigma \varphi_{1, tt} \hat{\varphi}_t + \sigma \varphi_{2, tt} \hat{\varphi}_t 
\right.
\]
\[
+ \nabla \hat{\varphi}_t (\nabla \varphi_{1, t} + \nabla \varphi_{2, t}) + \nabla \hat{\varphi}_t \nabla \varphi_1 + \nabla \hat{\varphi}_t \nabla \varphi_2)
\]
and
\[
\nabla \hat{f} = 2 \left( \sigma (\varphi_{1,t} + \varphi_{2,t}) \nabla \hat{\phi}_{tt} + \sigma \hat{\phi}_{tt} (\nabla \varphi_{1,t} + \nabla \varphi_{2,t}) + \sigma \hat{\phi}_{tt} \nabla \varphi_{2,t} + \sigma \hat{\phi}_{tt} \nabla \varphi_{t} + \sigma \hat{\phi}_{tt} \nabla \varphi_{1,t} + \sigma \hat{\phi}_{tt} \nabla \varphi_{t}ight)
\]
\[
+ \Delta \hat{\phi}_{t} (\nabla \varphi_{1,t} + \nabla \varphi_{2,t}) + (\Delta \varphi_{1,t} + \Delta \varphi_{2,t}) \nabla \hat{\phi}_{t}
\]
\[
+ \Delta \hat{\phi}_{tt} \nabla \varphi_{1} + \Delta \varphi_{2,t} \nabla \hat{\phi}_{t} + \Delta \varphi_{1} \nabla \hat{\phi}_{tt} + \Delta \hat{\phi} \nabla \varphi_{2,t}^{(T)}.
\]

The terms can be treated analogously as in the proof of Lemma 3.3 which shows that \( \varphi_{1}, \varphi_{2} \in V \) implies \( \hat{f} \in L^{2}(0, T; H_{0}^{1}(\Omega)) \) and that \( \Delta \varphi_{1,t} + \Delta \varphi_{2,t} \nabla \hat{\phi}_{t} \) holds with
\[
C = 2\sigma (C^{T}_{H_{1,2}^{2}}H_{1,2}^{2}C^{T}_{H_{1,2}^{2}}H_{1,2}^{2} + C^{0}_{H_{1,2}^{2}}H_{1,2}^{2}C^{0}_{H_{1,2}^{2}}H_{1,2}^{2})
\]
\[
+ 4C^{T}_{H_{1,2}^{2}}H_{1,2}^{2}C^{0}_{H_{1,2}^{2}}H_{1,2}^{2} + 2C^{T}_{H_{1,2}^{2}}H_{1,2}^{2}C^{0}_{H_{1,2}^{2}}H_{1,2}^{2} + 2C^{T}_{H_{1,2}^{2}}H_{1,2}^{2}C^{0}_{H_{1,2}^{2}}H_{1,2}^{2}
\]
which completes the proof. □

As \( \hat{\psi}(0) = 0, \partial_{t} \hat{\psi}(0) = 0 \) and \( \partial_{t}^{2} \hat{\psi}(0) = 0 \) we have \( \hat{\psi}(0) = D_{w} \hat{\psi}(0) = 0 \) and therefore, by Proposition 3.1 and Lemma 3.4, we obtain
\[
\| \hat{\psi} \|_{X} \leq C_{h} \| \hat{\psi} \|_{L^{2}(0, T; H^{1}(\Omega))}
\]
\[
\leq C_{h} \| \hat{\psi} \|_{V} (\| \varphi_{1} \|_{V} + \| \varphi_{2} \|_{V})
\]
\[
\leq 2C_{h} \| \hat{\psi} \|_{V}.
\]
Now, using Proposition 3.2 leads to \( \| \hat{\psi} \|_{V} \leq 2C_{h}C_{w}m \| \hat{\psi} \|_{V} \) and a choice \( m < \frac{1}{2C_{h}C_{w}} \) finally implies contractivity.

**Theorem 3.5** (Local well-posedness). Suppose \( \psi_{0} \in H^{4}(\Omega) \cap H_{0}^{1}(\Omega), \psi_{1} \in H^{4}(\Omega) \cap H_{0}^{1}(\Omega), \psi_{2} \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega) \) and \( \psi_{2} - b \Delta \psi_{1} - c^{2} \Delta \psi_{0} \in H^{2}(\Omega) \cap H_{0}^{2}(\Omega) \). For any finite \( T > 0 \) there exists a sufficiently small \( \kappa_{T} > 0 \) such that if
\[
\| \psi_{0} \|_{H^{4}(\Omega)} + \| \psi_{1} \|_{H^{4}(\Omega)} + \| \psi_{2} \|_{H^{2}(\Omega)} \leq \kappa_{T},
\]
there exists a unique weak solution
\[
\psi \in W := \{ \phi \in V : \| \phi \|_{V} \leq m \}
\]
where \( m \) is sufficiently small and the space \( V \) is given by (3.3).

Proof. On the strength of the Banach fixed point theorem, the map \( T : W \to W \) has a unique fixed point, i.e. there exists a unique \( \psi \in W \) such that \( T(\psi) = \psi \). Condition (3.11) is needed due to (3.3). □

4. Global well-posedness and exponential decay

4.1. Energy estimates. In order to show global in time well-posedness of (1.7) by means of barrier’s method, we need to perform energy estimates first.

We consider the equation
\[
(\partial_{t} - a \Delta)(\psi_{tt} - c^{2} \Delta \psi - b \Delta \psi) = -(\sigma(\psi_{t}^{2}) + |\nabla \psi|^{2})_{tt}
\]
or, equivalently,
\[
(\partial_{tt} - c^{2} \Delta - b \Delta \partial_{t})(\psi_{t} - a \Delta \psi) = -(\sigma(\psi_{t}^{2}) + |\nabla \psi|^{2})_{tt},
\]
i.e. \( D_{w}w = f \) and \( D_{h}w = w \), where \( D_{h} = \partial_{t} - a \Delta, D_{w} = \partial_{tt} - c^{2} \Delta - b \Delta \partial_{t}, \)
\[
f = (\sigma(\psi_{t}^{2}) + |\nabla \psi|^{2})_{tt}.
\]
In order to interchange the order of differentiation, we assume here that $\psi$ is sufficiently smooth, a restriction which can finally be removed by density arguments.

**Notation 4.1.** We introduce the energy functionals

$$(4.1) \quad e_1[\psi](t) := \|D_w \psi(t)\|_{H^1(\Omega)}^2 + \|D_\psi \psi(t)\|_{H^1(\Omega)}^2 + \|D_h \psi(t)\|_{H^2(\Omega)}^2,$$

$$(4.2) \quad e_2[\psi](t) := \|\psi(t)\|_{H^1(\Omega)}^2 + \|\psi(t)\|_{H^2(\Omega)}^2 + \|\psi(t)\|_{H^3(\Omega)}^2,$$

$$(4.3) \quad k[\psi](t) := \|\psi(t)\|_{H^2(\Omega)}^2 + \|\psi(t)\|_{H^3(\Omega)}^2 + \|\psi(t)\|_{H^4(\Omega)}^2,$$

$$(4.4) \quad e[\psi](t) := e_1[\psi](t) + e_2[\psi](t).$$

**Lemma 4.2 (§ Lemma 4.3).** For any $f \in H^1(0,T;L^2(\Omega))$ any solution $\psi$ to $D_wD_h\psi = f$ satisfies

$$e_1[\psi](t) + b \int_0^t \left\{ \|D_h \psi(t)\|_{L^2(\Omega)}^2 + \|D_\psi \psi(t)\|_{H^1(\Omega)}^2 + \|D_h \psi(t)\|_{H^2(\Omega)}^2 \\
+ \|D_h \psi(t)\|_{H^2(\Omega)}^2 \right\} d\tau \leq e \left( e_1[\psi](0) + \int_0^t \left\{ \|f(t)\|_{L^2(\Omega)}^2 + \|f(t)\|_{H^2(\Omega)}^2 \right\} d\tau \right)$$

with $b > 0$ sufficiently small and $e > 0$ sufficiently large.

We recall the following identity for the heat equation (cf. (4.11) in §)

$$\int_0^t \|D_h v(\tau)\|_{L^2}^2 d\tau = \int_0^t \|v_0(\tau) - a\Delta v\|_{L^2(\Omega)}^2 d\tau$$

$$= a\|v(t)\|_{H^1(\Omega)}^2 - a\|v(0)\|_{H^1(\Omega)}^2 + \int_0^t \left\{ \|v_0(\tau)\|_{H^1(\Omega)}^2 + a^2\|v(\tau)\|_{H^2(\Omega)}^2 \right\} d\tau.$$

Applying (4.5) to $v = \psi_{tt}$, $v = \nabla \psi_t$, $v = \Delta \psi_t$, and $v = \Delta \psi$, we obtain that the left-hand side terms under the time integrals in the estimate in Lemma 4.2 provide us with estimates on $\psi$ via the identities

$$\int_0^t \|D_h \psi_{tt}(\tau)\|_{L^2(\Omega)}^2 d\tau = a\|\psi_{tt}(t)\|_{H^1(\Omega)}^2 - a\|\psi_{tt}(0)\|_{H^1(\Omega)}^2$$

$$+ \int_0^t \left\{ \|\psi_{ttt}(\tau)\|_{L^2(\Omega)}^2 + a^2\|\psi_{ttt}(\tau)\|_{H^2(\Omega)}^2 \right\} d\tau,$$

$$\int_0^t \|D_h \psi_{tt}(\tau)\|_{H^1(\Omega)}^2 d\tau = a\|\psi_{tt}(t)\|_{H^2(\Omega)}^2 - a\|\psi_{tt}(0)\|_{H^2(\Omega)}^2$$

$$+ \int_0^t \left\{ \|\psi_{ttt}(\tau)\|_{H^1(\Omega)}^2 + a^2\|\psi_{ttt}(\tau)\|_{H^2(\Omega)}^2 \right\} d\tau,$$

$$\int_0^t \|D_h \psi_t(\tau)\|_{H^2(\Omega)}^2 d\tau = a\|\psi_t(t)\|_{H^3(\Omega)}^2 - a\|\psi_t(0)\|_{H^3(\Omega)}^2$$

$$+ \int_0^t \left\{ \|\psi_{ttt}(\tau)\|_{H^2(\Omega)}^2 + a^2\|\psi_{ttt}(\tau)\|_{H^3(\Omega)}^2 \right\} d\tau,$$

$$\int_0^t \|D_h \psi(\tau)\|_{H^2(\Omega)}^2 d\tau = a\|\psi(t)\|_{H^3(\Omega)}^2 - a\|\psi(0)\|_{H^3(\Omega)}^2$$

$$+ \int_0^t \left\{ \|\psi_{ttt}(\tau)\|_{H^2(\Omega)}^2 + a^2\|\psi_{ttt}(\tau)\|_{H^3(\Omega)}^2 \right\} d\tau.$$
Therewith, by splitting \( \|D_h \psi_t(t)\|_{H^1(\Omega)}^2 \), \( \|D_h \psi_t(\tau)\|_{H^1(\Omega)}^2 \) and \( \|D_h \psi(\tau)\|_{H^2(\Omega)}^2 \) on the left-hand side of the estimate in Lemma 4.2, we arrive at the estimate

\[
e_1[\psi](t) + e_2[\psi](t) + b_1 \int_0^t \left\{ \|D_h \psi_{tt}(\tau)\|_{L^2(\Omega)}^2 + \|D_h \psi_t(\tau)\|_{H^1(\Omega)}^2 \right. \\
+ \|D_h \psi(\tau)\|_{H^1(\Omega)}^2 + \|D_h \psi(\tau)\|_{H^2(\Omega)}^2 + \|\psi_{ttt}(\tau)\|_{L^2(\Omega)}^2 \\
+ \|\psi_{ttt}(\tau)\|_{H^1(\Omega)}^2 + \|\psi_{tt}(\tau)\|_{H^1(\Omega)}^2 + \|\psi_{tt}(\tau)\|_{H^2(\Omega)}^2 + \|\psi(\tau)\|_{H^1(\Omega)}^2 \left\} d\tau \\
\leq c_1 \left( e_1[\psi](0) + e_2[\psi](0) + \int_0^t \left\{ \|f(\tau)\|_{L^2(\Omega)}^2 \right. \\
+ \left. \|f_t(\tau)\|_{L^2(\Omega)}^2 \right\} d\tau \right)
\]

where \( b_1 > 0 \) is sufficiently small and \( c_1 > 0 \) is sufficiently large. This, by definition of \( k[\psi] \) and \( e_2[\psi] \leq (C_{H^1,L^2})^2 k[\psi] \) yields our next intermediate result.

**Lemma 4.3.** Provided \( f \in H^1(0,T;L^2(\Omega)) \), any solution \( \psi \) of \( D_wD_h \psi = f \) satisfies

\[
e[\psi](t) + \int_0^t \{ e[\psi](\tau) + k[\psi](\tau) \} d\tau \\
\leq c_2 \left( e[\psi](0) + \int_0^t \left\{ \|f(\tau)\|_{L^2(\Omega)}^2 + \|f_t(\tau)\|_{L^2(\Omega)}^2 \right\} d\tau \right),
\]

for some \( b_2 > 0 \) sufficiently small and some \( c_2 > 0 \) sufficiently large.

It remains to estimate the terms \( \|f_t(\tau)\|_{L^2(\Omega)}^2 \) and \( \|f(\tau)\|_{L^2(\Omega)}^2 \) under the integral sign on the right-hand side of (4.6). Explicitly we have

\[
f = 2\sigma \psi(t)^2 + 2\sigma \psi_t \psi_{ttt} + 2|\nabla \psi(t)|^2 + 2\nabla \psi \nabla \psi_{ttt},
\]

\[
f_t = 6\sigma \psi_t \psi_{ttt} + 2\sigma \psi_t \psi_{tttt} + 6\nabla \psi_t \nabla \psi_{ttt} + \nabla \psi \nabla \psi_{tttt}.
\]

For all \( \tau \in (0,t) \), we have

\[
\|f(\tau)\|_{L^2(\Omega)}^2 \leq (2\sigma \|\psi_t(\tau)\|_{L^2(\Omega)}^2 + 2\sigma \|\psi_t(\tau)\|_{H^1(\Omega)}^2) \\
+ 2|\nabla \psi(\tau)|^2 \psi_{ttt}(\tau) + 2|\nabla \psi_{tt}(\tau)|^2 \psi_{ttt}(\tau)
\]

\[
\leq 16\sigma^2 \|\psi_t(\tau)\|_{L^2(\Omega)}^2 + 16\sigma^2 \|\psi_t(\tau)\|_{H^1(\Omega)}^2 \\
+ 16\|\nabla \psi(\tau)|^2 \psi_{ttt}(\tau) + 16\|\nabla \psi(\tau)|^2 \psi_{ttt}(\tau)
\]

\[
\leq 16\sigma^2 \left( C_{H^2,L^2}^2 \right)^2 \left( C_{H^1,L^2}^2 \right)^2 \sup_{s \in (0,t)} \|\psi_t(s)\|_{H^2(\Omega)}^2 \|\psi_t(\tau)\|_{H^1(\Omega)}^2 \\
+ 16\sigma^2 \left( C_{H^3,L^2}^2 \right)^2 \left( C_{H^2,L^2}^2 \right)^2 \sup_{s \in (0,t)} \|\psi_t(s)\|_{H^3(\Omega)}^2 \|\psi_t(\tau)\|_{H^2(\Omega)}^2 \\
+ 16 \left( C_{H^2,L^\infty}^2 \right)^2 \left( C_{H^3,L^2}^2 \right)^2 \sup_{s \in (0,t)} \|\psi_t(s)\|_{H^3(\Omega)}^2 \|\psi_t(\tau)\|_{H^3(\Omega)}^2 \\
+ 16 \left( C_{H^2,L^\infty}^2 \right)^2 \left( C_{H^3,L^2}^2 \right)^2 \sup_{s \in (0,t)} \|\psi(s)\|_{H^3(\Omega)}^2 \|\psi_t(\tau)\|_{H^3(\Omega)}^2
\]
and
\[ \|f_\tau\|_{L^2(\Omega)}^2 \leq (6\sigma\|\psi_{tt}(\tau)\|_{L^2(\Omega)} + 2\sigma\|\psi_t(\tau)\|_{L^2(\Omega)} + 2\|\nabla\psi_{tt}(\tau)\|_{L^2(\Omega)})^2 \\
+ 6\|\nabla\psi_{tt}(\tau)\|_{L^2(\Omega)} + 2\|\nabla\psi_t(\tau)\|_{L^2(\Omega)})^2 \\
\leq 144\sigma^2 \left( C_{H^3_L}^2 \right)^2 \sup_{s \in (0,t)} \|\psi_t(s)\|_{H^3(\Omega)} \|\psi_{tt}(\tau)\|_{L^2(\Omega)} \]
\[ + 16\sigma^2 \left( C_{H^3_L}^2 \right)^2 \sup_{s \in (0,t)} \|\psi_t(s)\|_{H^3(\Omega)} \|\psi_{tt}(\tau)\|_{L^2(\Omega)} \]
\[ + 144 \left( C_{H^3_L}^2 \right)^2 \sup_{s \in (0,t)} \|\psi_t(s)\|_{H^3(\Omega)} \|\psi_{tt}(\tau)\|_{L^2(\Omega)} \]
\[ + 16 \left( C_{H^3_L}^2 \right)^2 \sup_{s \in (0,t)} \|\psi(s)\|_{H^3(\Omega)} \|\psi_{tt}(\tau)\|_{L^2(\Omega)}. \]

Therewith, we obtain
\[ \int_0^t \|f(\tau)\|_{L^2(\Omega)}^2 + \|f_\tau(\tau)\|_{L^2(\Omega)}^2 \, d\tau \leq c_3 \sup_{s \in (0,t)} e_2[\psi](s) \int_0^t k[\psi](\tau) \, d\tau \]
for some constant \( c_3 > 0 \) and arrive at our final energy estimate.

**Proposition 4.4.** The estimate

\[ e[\psi](t) + \hat{b} \int_0^t \{e[\psi](\tau) + k[\psi](\tau)\} \, d\tau \]
\[ \leq \hat{c} \left( e[\psi](0) + \sup_{s \in (0,t)} e_2[\psi](s) \int_0^t k[\psi](\tau) \, d\tau \right) \]

holds with \( \hat{b} > 0 \) sufficiently small and \( \hat{c} > 0 \) sufficiently large.

**4.2. The global well-posedness result.** Relying on Proposition 4.4, we now prove global in time well-posedness. To this end, we suppose that \( \psi \) is a local solution according to Theorem 3.3 and let \( T > 0 \) be the maximal time horizon for which this solution exists. The results on global well-posedness and exponential decay of the energy \( e[\psi] \) follow analogously to [3]. For the sake of self-completeness we include their proofs below.

**Theorem 4.5** (Global well-posedness). Suppose \( \psi_0 \in H^4(\Omega) \cap H^1_0(\Omega) \), \( \psi_1 \in H^4(\Omega) \cap H^1_0(\Omega) \) with \( \psi_1 \neq \frac{1}{2\sigma} \), \( \psi_2 \in H^3(\Omega) \cap H^1_0(\Omega) \) and \( \psi_{ttt}(0) \in H^3_0(\Omega) \), where
\[ \psi_{ttt}(0) = \left(1 - 2\sigma\psi_1\right)^{-1}[(a + b)\Delta\psi_2 - c^2a\Delta^2\psi_0 + c^2\Delta\psi_1 - ab\Delta^2\psi_1 + 2\sigma(\psi_2)^2 + 2|\nabla\psi_1|^2 + 2\nabla\psi_2 \nabla\psi_0]. \]

For all initial values satisfying
\[ e[\psi](0) \leq \kappa \]
with \( \kappa \) sufficiently small,
\[ \kappa \leq \frac{\hat{b}}{4\hat{c}\max\{1, \hat{c}\}}, \]
we get that for all \( t > 0 \)
\[ e[\psi](t) \leq 2\max\{1, \hat{c}\}\kappa. \]
In particular, if there exists a sufficiently small constant $\rho$ such that
\begin{equation}
\|\psi_t(0)\|_{H^1(\Omega)}^2 + \|\psi_2\|_{H^1(\Omega)}^2 + \|\psi_1\|_{H^1(\Omega)}^2 + \|\psi_0\|_{H^1(\Omega)}^2 \leq \rho,
\end{equation}
then for all times $T > 0$ there exists a unique weak solution
\begin{align*}
\psi &\in C(0, T; H^1(\Omega) \cap H^1_0(\Omega)) \cap C^1(0, T; H^2(\Omega) \cap H^1_0(\Omega)) \\
&\cap C^2(0, T; H^3(\Omega) \cap H^1_0(\Omega)) \cap C^3(0, T; H^4(\Omega) \cap H^1_0(\Omega)) \\
&\cap H^2(0, T; H^3(\Omega) \cap H^1_0(\Omega)) \cap H^3(0, T; H^4(\Omega) \cap H^1_0(\Omega)) \\
&\cap H^4(0, T; L^2(\Omega)).
\end{align*}

**Proof.** We prove the claim by contradiction. To this end, we assume that there exists a finite time such that (4.10) is violated. We denote by $T_0$ the minimal such time (observe that $T_0 > 0$ since $e(0) < 2\max\{1, C\}$) and have
\begin{equation}
e(\psi)(T_0) \geq 2\max\{1, \hat{c}\} \kappa.
\end{equation}
Moreover, (4.10) holds for all $t \in (0, T_0)$. From Proposition 4.4 we get
\begin{equation}
e(\psi)(t) + \hat{b} \int_0^t \{e(\psi)(\tau) + k(\psi)(\tau)\} d\tau \\
\leq \hat{c} \left(e(\psi)(0) + 2\max\{1, \hat{c}\} \kappa \int_0^t k(\psi)(\tau)d\tau\right)
\end{equation}
for all $t \in (0, T_0)$, which, by $e(\psi)(0) \leq \kappa$ and $2\hat{c}\max\{1, \hat{c}\} \rho \leq \frac{\hat{b}}{2}$, gives
\begin{equation}
e(\psi)(t) + \hat{b} \int_0^t \{e(\psi)(\tau) + k(\psi)(\tau)\} d\tau \leq \hat{c} e(\psi)(0).
\end{equation}
Thus we have $e(\psi)(t) \leq \hat{c}\kappa$ for all $t \in (0, T_0)$ and hence by continuity $e(\psi)(T_0) \leq \hat{c}\kappa$ which is a contradiction to (4.12). This proves that the bound (4.10) holds for all $t > 0$ provided (4.8) holds with (4.9). For the condition of the initial values (4.11) note that
\begin{equation}
e(\psi)(t) \leq c_4 \left\{\|\psi_{tt}(t)\|_{H^1(\Omega)}^2 + \|\psi_{tt}(t)\|_{H^1(\Omega)}^2 + \|\psi(t)\|_{H^1(\Omega)}^2 + \|\psi(t)\|_{H^1(\Omega)}^2\right\}.
\end{equation}
Hence, (4.11) with $\rho = \frac{\hat{c}}{c_4} e(\psi)$ ensures (4.8).

The regularity of the solution is obtained by (4.14) together with the definitions of $e(\psi)$ and $k(\psi)$ and noting that $\psi_t \in C(0, T; H^3(\Omega) \cap H^1_0(\Omega))$ together with $\psi_t - a\Delta\psi \in C(0, T; H^2(\Omega) \cap H^1_0(\Omega))$ and $\psi_t - a\Delta\psi \in C(0, T; H^4(\Omega) \cap H^1_0(\Omega))$ and $\psi_t - a\Delta\psi \in C(0, T; H^4(\Omega) \cap H^1_0(\Omega))$.

\[\Box\]

**Remark 4.6.** (i) Note that $\psi_t \neq \frac{\hat{b}}{c_4} e(\psi)$ will be satisfied by $e(\psi)(0) \leq \kappa$ for $\kappa$ sufficiently small and the embedding $H^2(\Omega) \cap H^1_0(\Omega) \hookrightarrow L^\infty(\Omega)$.

(ii) The term $k(\psi)$ in (4.14) is actually not needed for the proof of global well-posedness, but it provides us with higher regularity.

(iii) Under the additional assumptions imposed in Theorem 4.3 the regularity of the local solution according to Theorem 3.5 is preserved for all times $t > 0$.

**Theorem 4.7 (Exponential decay).** Provided (4.8) holds with (4.9) we have
\begin{equation}
e(\psi)(t) \leq e^{-\hat{b}t} e(\psi)(0)
\end{equation}
where $\hat{b}$ and $\hat{c}$ are the same constants as in Proposition 4.4.
Proof. From the proof of Theorem 4.5 we see that, provided (4.8) holds with (4.9), we have (4.14). Hence,

$$\int_0^t e^{\psi(\tau)} \, d\tau \leq \frac{\dot{c}}{b} e^{\psi(0)}$$

which, by a standard argument, implies (4.15). □

5. Conclusions and Outlook

Based on a decomposition of (1.3) into the classical heat equation and the linearized Westervelt equation, we have shown local and global well-posedness of the initial boundary value problem (1.7) as well exponential decay of solutions. The result on global well-posedness needs stronger assumptions than the result on local well-posedness. However, under these stronger assumptions, the regularity of the local solution is preserved for all times.

Considering equation (1.3) together with application relevant boundary conditions (e.g., inhomogeneous Neumann boundary conditions for modeling excitation or absorbing boundary conditions for modeling boundary dissipation) will be subject of further research.

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