A product integration rule on equispaced nodes for highly oscillating integrals

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Abstract

This paper provides a product integration rule for highly oscillating integrands, based on equally spaced nodes. The stability and the error estimate are proven in the space of continuous functions, and some numerical tests which confirm such estimates are provided.

Keywords: Approximation by polynomials, Boolean iterated sums of Bernstein operators, quadrature rules

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1 Introduction

From a numerical point of view, the main difficulty to treat integrals of the type

\[
\int_{-a}^{a} e^{-i\omega(x-y)} f(x) dx, \quad a > 0, \quad i = \sqrt{-1}, \quad y \in [-a, a]
\]  

depends on the presence of the kernel \( e^{-i\omega(x-y)} \), which highly oscillates for frequencies \( |\omega| >> 1 \). For the evaluation of such integrals, many accurate formulas exist in the literature, mainly based on the zeros of orthogonal polynomials; see e.g. \[1, 2, 6, 7, 9, 15\]. On the other hand, in many practical applications, the function \( f \) is known only at equispaced nodes.

In order to adopt the values of \( f \) at equidistant nodes, a widely used technique is based on composite quadrature rules of “lower” degree, such as Filon type rules, whose degree of approximation cannot be improved over the saturation class of the approximation process.

Recently, in [3] starting from the knowledge of \( f \) at a finite set of equally spaced nodes, an approach has been introduced to efficiently compute weighted integrals, but not advisable in the case of oscillating integrands. For a short review on the main techniques the interested reader can consult [1].

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Finding accurate quadrature rules based on equidistant points is still an open problem. Here, we introduce a product integration rule based on the approximation of the function \( f \) through the generalized Bernstein (shortly GB) polynomials \( \bar{B}_{m,\ell}(f) \) of degree \( m \) and parameter \( \ell \) defined in the interval \([-a, a]\). Such polynomials represent an adequate tool for our aims, since they make use of the samples of \( f \) at \( m+1 \) equally spaced nodes; see e.g. [4, 11, 12]. Moreover, unlike the classical Bernstein polynomials \( B_m(f) \), the speed of the uniform convergence to \( f \) accelerates as the smoothness of \( f \) increases [3].

The outline of the paper is as follows. In Section 2 we provide definition and main properties of GB polynomials. Section 3 contains the main results regarding the product integration rules and its error estimates. In Section 4 two numerical examples are given to corroborate the theoretical estimate. The proofs are given in the last Section 5.

2 Notation and Preliminary Results

In the sequel \( \mathcal{C} \) denotes any positive constant having different meanings at different occurrences and the writing \( \mathcal{C} \neq \mathcal{C}(a,b,..) \) has to be understood as \( \mathcal{C} \) not depending on \( a,b,... \)

As usual, \( \mathbb{P}_m \) denotes the space of the algebraic polynomials of degree less than or equal to \( m \), and \( \mathcal{C}^0 := \mathcal{C}^0([-a,a]) \) is the space of the continuous functions on \([-a,a]\) equipped with the uniform norm \( \| f \| := \max_{x \in [-a,a]} |f(x)| \). Moreover, for any bivariate function \( g(x,y) \), by \( g_x(y) \) we refer to \( g \) as function of the only variable \( y \). Finally, for a given integer \( m \), we set \( N_0^m := \{0,1,2,\ldots,m\} \).

2.1 Iterated Boolean sums of Bernstein Operators in \([-a,a]\)

For a fixed positive \( a \in \mathbb{R} \), consider the \( m \)-th Bernstein polynomial \( \bar{B}_m f \) of a given \( f \in \mathcal{C}^0 \),

\[
\bar{B}_m(f,x) := \sum_{k=0}^{m} f(t_k)\bar{p}_{m,k}(x), \quad t_k := -a + k \frac{2a}{m}, \quad x \in [-a,a] \tag{2.1}
\]

where

\[
\bar{p}_{m,k}(x) := \binom{m}{k} \left( \frac{a+x}{2a} \right)^k \left( \frac{a-x}{2a} \right)^{m-k}. \tag{2.2}
\]

Based on \( \bar{B}_m(f) \), the \( \ell \)-iterated boolean sum \( \bar{B}_{m,\ell}(f) \in \mathbb{P}_m \), \( \ell \in \mathbb{N} \), is defined for any \( f \in \mathcal{C}^0 \) as

\[
\bar{B}_{m,\ell}(f) = f - (f - \bar{B}_m(f))^\ell, \quad \bar{B}_{m,1}f = \bar{B}_m f.
\]

The polynomial \( \bar{B}_{m,\ell}(f) \) can be expressed \( \forall x \in [-a,a] \), as

\[
\bar{B}_{m,\ell}(f,x) = \sum_{j=0}^{m} \bar{p}_{m,j}^{(\ell)}(x)f(t_j), \quad \text{with} \quad \bar{p}_{m,j}^{(\ell)}(x) = \sum_{i=0}^{m} \bar{p}_{m,i}(x)c_{i,j}^{(m,\ell)}, \tag{2.3}
\]

where \( c_{i,j}^{(m,\ell)} \) are the entries of the matrix \( C_{m,\ell} \in \mathbb{R}^{(m+1)\times(m+1)} \),

\[
C_{m,\ell} = I + (I - A) + \ldots + (I - A)^{\ell-1}, \quad C_{m,1} = I,
\]

being \( I \) the identity matrix of order \( m+1 \) and \( A \in \mathbb{R}^{(m+1)\times(m+1)} \) the matrix

\[
(A)_{i,j} = p_{m,j}(t_i), \quad i,j \in N_0^m.
\]
By induction on \( \ell = 2^p, \ p \in \mathbb{N} \), the following recurrence relation holds

\[
C_{m,2^p} = C_{m,2^{p-1}} + (I - A)^{2^{p-1}} C_{m,2^{p-1}},
\]

which allows to a fast construction of the subsequence \( \{B_{m,2^p}\}_{p=1,2,\ldots} \), since

\[
B_{m,2^p}(f,x) = 2B_{m,2^{p-1}}(f,x) - B_{m,2^{p-1}}^2(f,x).
\]

A survey containing properties and applications of Generalized Bernstein polynomials is given in [14]. Moreover, specific employments of such polynomials to Fredholm and Volterra integral equations can be found in [5, 13].

In order to evaluate numerically the above integral, we propose to approximate \( f \) by \( B_{m,\ell}(f) \), getting

\[
\|f - B_{m,\ell}(f)\| = O(\sqrt{m^{-\ell}}).
\]

In other words, the rate of convergence behaves like the square root of the best approximation error for this space of functions.

### 3 A product integration rule

By separating the kernel \( e^{-i\omega(x-y)} \) into the real and imaginary parts, we are dealing with integrals of the type

\[
\mathcal{I}(f,y) := \int_{-a}^{a} \kappa(\omega(y-x)) f(x) \, dx,
\]

where \( \kappa(x) = \sqrt{a^2 - x^2} \), and \( \mathcal{A}C \) denotes the space of all locally absolutely continuous functions on \([-a,a]\), then for each \( f \in \mathcal{W}_r = \mathcal{W}_r, \) with \( 0 < r < 2\ell \), we have

\[
\|f - B_{m,\ell}(f)\| = O(\sqrt{m^{-r}}).
\]

Let us note that, in the above formula, the pathology of the integrand, that is the presence of the oscillating kernel, has been isolated in the coefficients \( q_i \) that we approximate by using a suitable technique explained in the next subsection.

Next theorem establishes conditions assuring that the rule \( \text{[2.3]} \) is stable and convergent for any \( f \in C^0 \). Moreover, it also provides an error estimate in suitable subspaces of \( C^0 \).
Theorem 3.1. For any $f \in C^0$, and for any $k$ defined as in (3.1), and for each fixed $\ell$, the rule (3.3) is stable, that is
\[\sup_m ||I_{m,\ell}(f)|| \leq C ||f||, \quad C \neq C(m, f).\] (3.4)
Moreover, for any $f \in W_r$, $0 < r \leq 2\ell$, we have
\[||R_{m,\ell}(f)|| \leq C \left( \frac{a^{r+1} ||f||_{W_r}}{(\sqrt{m})^r} \right), \quad C \neq C(m, f).\] (3.5)

3.1 The approximation of the coefficients

Now we approach to the computation of the coefficients $q_i(y)$, by proposing a suitable technique that treats the pathology of the kernel.

Let $N = \lceil \frac{\omega a}{\pi} \rceil + 1$, introduce the partition
\[-a, a] = \bigcup_{h=1}^N [x_{h-1}, x_h], \quad x_0 = -a, \quad x_h = -a + h\eta, \quad \eta = \frac{2a}{N}, \quad h = 1, 2, \ldots, N,
and consider the following decomposition in sum
\[q_i(y) = \sum_{h=1}^{N} \int_{x_{h-1}}^{x_h} \kappa(\omega(y - x)) \bar{p}_{m,i}(x) dx, \quad i = 1, \ldots, m.
\]

Let us now map each interval $[x_{h-1}, x_h]$ into $[-1, 1]$ through the linear transformations $z = \gamma_h(x) := \frac{2x - x_{h-1}}{\eta} - 1$. In this way we get
\[q_i(y) = \frac{\eta}{2} \sum_{h=1}^{N} \int_{-1}^{1} \bar{p}_{m,i}(\gamma_h^{-1}(z)) \kappa(\omega(y - \gamma_h^{-1}(z))) dz.
\]
Now, by approximating each integral by a $m$-point Gauss-Legendre, we have
\[q_{m,i}(y) = \frac{a}{N} \sum_{h=1}^{N} \left( \sum_{k=1}^{m} \lambda_k \bar{p}_{m,i}(\gamma_h^{-1}(z_k)) \kappa(\omega(y - \gamma_h^{-1}(z_k))) + \varepsilon_i^{m,h}(y) \right).\] (3.6)
where $\{z_k\}_{k=1}^{m}$ are the zeros of the $m$th Legendre polynomial and $\{\lambda_k\}_{k=1}^{m}$ are the Christoffel numbers.

Theorem 3.2. Fixed $y \in [-a, a]$, for any $1 \leq h \leq N$ and $0 \leq i \leq m$, we get
\[|\varepsilon_i^{m,h}(y)| \leq \frac{C}{\sqrt{2\pi i}} \left( \frac{a}{N} \right)^m \left( m \frac{1}{m} \cdot \frac{m + \omega}{2m - 1} \right)^m,
\]
$C \neq C(m, f)$.

Hence, by replacing the approximation (3.6) into (3.3), we have the following product integration rule
\[\tilde{I}_{m,\ell}^\omega(f, y) = \frac{a}{N} \sum_{j=0}^{m} f(t_j) \sum_{i=0}^{m} c_{i,j}^{(m,\ell)} \left( \sum_{h=1}^{N} \sum_{k=1}^{m} \lambda_k \bar{p}_{m,i}(\gamma_h^{-1}(z_k)) \kappa(\omega(\gamma_h^{-1}(y - z_k))) \right).\] (3.7)
Combining estimates by Ths. 3.1-3.2 assuming \( f \in W_r \), with \( 1 \leq r \leq 2\ell \), for any \( y \in [-a,a] \) and for \( m \) sufficiently large (say \( m > m_0 \), \( m_0 \) fixed), the following error estimate holds:

\[
|\mathcal{I}_\omega(f, y) - \tilde{\mathcal{I}}_{m,\ell}(f, y)| \leq C\|f\|_{W_r} \left[ \frac{a^{r+1}}{(\sqrt{m})^r} + m^{\frac{3}{2}}\|C_{m,\ell}\|_\infty \left( \frac{a}{N} \cdot \frac{m + \omega}{2m - 1} \right)^m \right],
\]

where \( C \neq C(m, f) \).

4 Numerical tests

In this section, we show the accuracy of our product rule as well as the reliability of our theoretical estimates, to the following two integrals

\[
\mathcal{I}_\omega(f_1, y) = \int_{-1}^{1} \sin \omega(y - x) f_1(x) \, dx, \quad \mathcal{I}_\omega(f_2, y) = \int_{-2}^{2} \cos \omega(y - x) f_2(x) \, dx,
\]

with \( f_1(x) = \tanh(x + 1) \), and \( f_2(x) = |x + 1|^{9/2} \). In both cases, the exact value of the integrals is not known and then we consider as exact the value provided by our quadrature rule with \( m = 512 \). Table 1 contains, for increasing values of \( m \), the absolute errors \( e_{m,\ell}(f_j, y) = |\tilde{\mathcal{I}}_{512,\ell}(f_j, y) - \tilde{\mathcal{I}}_{m,\ell}(f_j, y)| \), \( j = 1, 2 \) for the first and second integral, respectively. In both text, we fix \( \ell = 2^8 \) and we show the results in two different points and for several values of \( \omega \).

As we can observe, in the first integral the convergence is very fast due to the analyticity of the function \( f_1 \). On the contrary, in the second integral the convergence is slower since \( f_2 \in W_{9/2} \).

5 The proofs

**Proof of Theorem 3.1** First, let us prove the stability, i.e. (3.4). By the first identity of (3.3), for any \( y \in [-a,a] \) we have

\[
|\mathcal{I}_{m,\ell}(f, y)| \leq \|f\| \left| \int_{-a}^{a} \sum_{j=0}^{m} \sum_{i=0}^{m} c_{i,j}^{(m,\ell)} \tilde{p}_{m,i}(x) \right| dx \leq \|f\| \left| \int_{-a}^{a} \sum_{i=0}^{m} \bar{p}_{m,i}(x) \sum_{j=0}^{m} c_{i,j}^{(m,\ell)} \right| dx
\]

\[
\leq 2a \|f\| \|C_{m,\ell}\|_{\infty} \leq C \|f\|,
\]

since \( \|C_{m,\ell}\|_{\infty} \leq 2^\ell - 1 \).

Let us now estimate the error. We have

\[
|R_{m,\ell}(f, y)| \leq \int_{-a}^{a} |f(x) - \bar{B}_{m,\ell}(f, x)| k(\omega(y - x)) \, dx \leq C \|f - \bar{B}_{m,\ell}(f)\|.
\]

The thesis follows taking into account [8, Th. 2.1]. 

\[\square\]
Table 1: Numerical results for integral \( I^\omega(f_1, y) \) to the left and for integral \( I^\omega(f_2, y) \) to the right

| \( m \) | \( \omega \) | \( e_{m,25}(f_1, -0.7) \) | \( e_{m,25}(f_1, 0.5) \) | \( m \) | \( \omega \) | \( e_{m,25}(f_2, -1.5) \) | \( e_{m,25}(f_2, 1) \) |
|---|---|---|---|---|---|---|---|
| 4 | 10 | 7.20e-04 | 1.30e-04 | 4 | 10 | 1.98e-02 | 6.34e-03 |
| 8 | 1.63e-05 | 2.52e-05 | 8 | 3.78e-03 | 3.47e-03 |
| 16 | 5.91e-09 | 4.15e-09 | 16 | 1.88e-04 | 1.79e-04 |
| 32 | 2.82e-12 | 8.81e-12 | 32 | 2.41e-05 | 2.75e-05 |
| 64 | 1.32e-15 | 7.42e-15 | 64 | 3.64e-08 | 1.30e-07 |
| 128 | 8.26e-15 | 3.23e-16 | 128 | 7.76e-10 | 3.92e-10 |
| 256 | 3.27e-15 | 7.28e-15 | 256 | 1.01e-11 | 4.75e-12 |

\[ |I^\omega(f_1, y)| \leq C \left| \frac{\varphi(x)}{1-x^2} \right| \]

where here and in the next \( \|g\| = \sup_{z \in [-1,1]} |g(z)| \). Hence, we can write

\[ |e_{m,h}(y)| \leq \frac{C}{(2m-1)^m} \left\{ \left\| \tilde{p}_{m,i} \left( \gamma_h^{-1} \right) \kappa_g \left( \omega \left( \gamma_h^{-1} \right) \right) \right\| + \left\| \left[ \tilde{p}_{m,i} \left( \gamma_h^{-1} \right) \kappa_g \left( \omega \left( \gamma_h^{-1} \right) \right) \right]^{(m)} \varphi \right\| \right\} . \]

Now, taking into account that for \( 1 \leq i \leq m-1 \)

\[ \left\| \tilde{p}_{m,i} \right\| = \left( \frac{m}{i} \right) \left( \frac{i}{m} \right)^i \left( \frac{m-i}{m} \right)^{m-i} \]

by Stirling’s formula we have

\[ \left\| \tilde{p}_{m,i} \right\| \sim \frac{\sqrt{m}}{\sqrt{2\pi i}} \quad \text{as} \quad m \to \infty, \]

and by Bernstein inequality for polynomials (see e.g. [10]) we deduce for \( 0 \leq r \leq m \), \( r \in \mathbb{N} \)

\[ \left\| \tilde{p}_{m,i}^{(m-r)} \left( \gamma_h^{-1} \right) \varphi^{m-r} \right\| \leq C \left( \frac{a}{N} \right)^{m-r} m^{m-r} \left\| \tilde{p}_{m,i} \right\| \leq \frac{C}{\sqrt{2\pi i}} \sqrt{m} \left( \frac{a}{N} \right)^{m-r} m^{m-r} . \]
Moreover, by the definition of the kernel $k_y$ in (3.1) we have
\[
\left\| k_y^{(r)}(\omega (\gamma_h^{-1})) \right\| \leq \left( \frac{\omega a}{N} \right)^r.
\]
Consequently, by applying Leibniz formula, we get
\[
\left\| (\bar{p}_{m,i}(\gamma_h^{-1}) \; k_y (\omega \gamma_h^{-1}))^{(m)} \right\| \leq C \frac{\sqrt{m}}{\sqrt{2} \pi i} \left( \frac{a}{N} \right)^m \sum_{r=0}^{m} \binom{m}{r} m^{m-r} \omega^r
\]
\[
= C \frac{\sqrt{m}}{\sqrt{2} \pi i} \left( \frac{a}{N} \right)^m (m+\omega)^m,
\]
and then
\[
| \varepsilon_i^{h,m}(y) | \leq C \frac{\sqrt{m}}{\sqrt{2} \pi i} \left( \frac{a}{N} \right)^m \left( \frac{m+\omega}{2m-1} \right)^{m}.
\]

\[\square\]

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