A FIXED POINT THEOREM FOR TWIST MAPS

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Abstract. Poincaré’s last geometric theorem (Poincaré-Birkhoff Theorem [2]) states that any area-preserving twist map of annulus has at least two fixed points. We replace the area-preserving condition with a weaker intersection property, which states that any essential simple closed curve intersects its image under $f$ at least at one point. The conclusion is that any such map has at least one fixed point. Besides providing a new proof to Poincaré’s geometric theorem, our result also has some applications to reversible systems.

1. Introduction

In 1913, Birkhorff proved Poincaré’s Last Geometric Theorem [2] stating that an area-preserving homeomorphism of an annulus satisfying the twist condition has at least two fixed points. The proof has been subsequently improved by Barrar [1], Carter [5] and many others. In this note, we replace the area-preserving condition with a weaker intersection property and obtain a weaker, but sharp, result. We show that there is at least one fixed point under the intersection condition. Combining with Slaminka’s [14] result on removing isolated fixed point with zero index, our method provides another proof of Poincaré’s last geometric theorem for area-preserving case. It also provides a new proof to Carter’s theorem [5]. One of the interesting applications of our result is for the so-called reversible systems. It is interesting to note that both the KAM theory and Aubry-Mather theory are applicable to reversible systems (Siegel & Moser [13], Chow & Pei [6]). Here we also have a positive answer for Poincaré’s geometric theorem, albeit with just one fixed point.

Our proof is more of a simple standard dynamical systems approach, using techniques of Franks [9, 10], and Brouwer’s plane translation theorem (Brown [4]). We analyse the chain recurrent set, where Conley’s fundamental theorem for dynamical systems [7] provided a fine structure on the chain recurrent set. The key observation is very simple: the intersection property implies that two boundary components are in the same chain transitive component of the chain recurrent set. The techniques used in the proof are very much standard in dynamics.

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More precisely, let $\mathbb{A} = S^1 \times [0, 1]$ be an annulus with boundaries $A_0 = S^1 \times \{0\}$ and $A_1 = S^1 \times \{1\}$. Let $f : \mathbb{A} \to \mathbb{A}$ be a homeomorphism satisfying the twist condition and intersection property.

We say that $f$ satisfies the intersection property if any essential simple closed curve intersects its image under $f$ at least at one point. A simple closed curve is said to be essential if it is not contractible. As for the twist condition, we need to consider the covering space $\tilde{\mathbb{A}} = \mathbb{R} \times [0, 1]$ of $\mathbb{A}$, and the lift of $f$ denoted by $\tilde{f} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}}$. We say that $f$ satisfies the twist condition provided that $\tilde{f}$ moves the two boundary components of $\tilde{\mathbb{A}}$ in opposite directions. We may assume that $\tilde{f}(x, y) = (x + r_1(x), y)$ for $y = 1$ and $\tilde{f}(x, y) = (x - r_0(x), y)$ for $y = 0$, where $r_0(x), r_1(x) > 0$. Our main result is following theorem.

**Theorem 1.1.** If $f : \mathbb{A} \to \mathbb{A}$ is a homeomorphism satisfying twist condition and intersection property. Then $f$ has at least one fixed point.

Note that the intersection property is weaker than the area-preserving property, and Carter gives an example (Figure 1) in [5] that such $f$ may have exactly one fixed point. So, our weaker conclusion, $f$ has at least one fixed point instead of two, is actually sharp.

There are many different proofs and extensions of Poincaré’s geometric theorem. Some proofs are less clear and convincing than others. Our statement of theorem shares some common features with the works of Birkhoff [3] and Carter [5]. Carter obtained two fixed points, using a stronger intersection property.

An outline of this paper is as follows: In section 2, we recall some properties of chain recurrent set and introduce Conley’s fundamental theorem of
dynamical systems and the Brouwer planar translation theorem. In section 3, we will first assume \( f \) is rigid rotation on boundaries i.e. \( r_0(x) \) and \( r_1(x) \) are constant and prove our main theorem, then we will show for the general case, the theorem also holds true. In section 4, we show how our result applies to reversible systems.

2. Chain recurrent set and useful theorems

Chain recurrence is an important concept in dynamical systems. Given a compact metric space \( X \) and homeomorphism \( f : X \to X \). We have the following definitions.

Definition 2.1. Given two points \( p, q \in X \), an \( \epsilon \)-chain from \( p \) to \( q \) is a sequence \( x_1, x_2, ..., x_n \), where \( x_1 = p \), \( x_n = q \) and for all \( 1 \leq i \leq n - 1 \), \( d(f(x_i), x_{i+1}) < \epsilon \).

Definition 2.2. A point \( p \in X \) is chain recurrent if for all \( \epsilon > 0 \), there is an \( \epsilon \)-chain from \( p \) to itself. The set of all chain recurrent points in \( X \) is called the chain recurrent set, denoted by \( \mathcal{R}(f) \).

Now, we can define an equivalence relation on \( \mathcal{R}(f) \) as follows: For \( p, q \in \mathcal{R}(f) \), \( p \sim q \) if and only if for all \( \epsilon > 0 \), there are \( \epsilon \)-chains from \( p \) to \( q \) and from \( q \) to \( p \). And it’s easy to see that \( \sim \) is reflexive, symmetric and transitive.

Definition 2.3. The equivalence classes in \( \mathcal{R}(f) \) for \( \sim \) is called chain transitive components. A set is called chain transitive if any two points \( p, q \) in this set, we have \( p \sim q \).

The next lemma allows us to relate chain transitive components with connected components.

Lemma 2.1. The connected components of \( \mathcal{R}(f) \) are chain transitive.

Proof. Let \( K \) be a connected component of \( \mathcal{R}(f) \). Given any \( \epsilon > 0 \), we can cover \( K \) by disks with radius \( \frac{\epsilon}{4} \). Since \( K \) is closed and hence compact, there exists a finite subcover \( \{B_i\}_{i=1}^m \). Then for any \( x, y \in K \), we can find a sequence \( \{x = x_0, x_1, ..., x_{n-1}, x_n = y\} \) in \( \mathcal{R}(f) \) such that \( x_i \) and \( x_{i+1} \) lies in a same disk for \( i = 0, 1, ..., n - 1 \). Note each \( x_i \in \mathcal{R}(f) \), there is a \( \frac{\epsilon}{2} \)-chain from \( x_i \) to \( x_{i+1} \). We replace the last point by \( x_{i+1} \), then we get an \( \epsilon \)-chain form \( x_i \) to \( x_{i+1} \). And finally we get an \( \epsilon \)-chain form \( x \) to \( y \). Since \( x \) and \( y \) are arbitrary, \( K \) is chain transitive. \( \Box \)

In order to state the fundamental theorem of dynamical systems, we need to give this definition.

Definition 2.4. \( g : X \to \mathbb{R} \) is a complete Lyapunov function for \( f \) if:

1. \( \forall p \notin \mathcal{R}(f), g(f(p)) < g(p) \).
2. \( \forall p, q \in \mathcal{R}(f), g(p) = g(q) \) if and only if \( p \sim q \).
(3) \(g(\mathcal{R}(f))\) is compact and nowhere dense in \(\mathbb{R}\).

**Theorem 2.2** (Conley’s \([7]\) fundamental theorem of dynamical systems). Complete Lyapunov function exists for any homeomorphism on compact metric spaces.

This theorem states that we can get a complete Lyapunov function that stays constant only on the chain transitive components and strictly decreases along any orbit not in \(\mathcal{R}(f)\). More details of this theorem can be found in Franks \([11]\).

The next concept and theorem are introduced in \([9]\). It is very useful for showing the existence of a fixed point on plane. And it is easy to see from the definition that \(\epsilon\)-chain and disk chain have a closed relationship.

**Definition 2.5.** Let \(f : M \to M\) be a homeomorphism of a surface. A disk chain for \(f\) is a finite set \(U_1, U_2, \ldots, U_n\) of embedded open disks in \(M\) satisfying

1. \(f(U_i) \cap U_i = \emptyset\) for \(1 \leq i \leq n\).
2. If \(i \neq j\), then either \(U_i = U_j\) or \(U_i \cap U_j = \emptyset\).
3. For \(1 \leq i \leq n\), there exists \(m_i > 0\) with \(f^{m_i}(U_i) \cap U_{i+1} \neq \emptyset\).

If \(U_1 = U_n\), we will say that \(U_1, U_2, \ldots, U_n\) is a periodic disk chain.

**Theorem 2.3** (Brouwer planar translation theorem). Let \(f : \mathbb{R}^2 \to \mathbb{R}^2\) be an orientation preserving homeomorphism which possesses a periodic disk chain. Then there is a simple closed curve \(\gamma\) in \(\mathbb{R}^2\) such that \(I(\gamma, f) = 1\). If \(f\) has only isolated fixed points, then \(f\) has a fixed point of positive index.

The original version of Brouwer’s planar translation theorem states that a fixed point free homeomorphism of \(\mathbb{R}^2\) can be viewed locally as plane translations.

### 3. Proof of the Main Theorem

With the above preparation, we can prove our main theorem.

The rough idea of the proof is as follows: we first consider the case that \(f\) is rigid rotation on the boundaries of the annulus i.e. \(r_0(x)\) and \(r_1(x)\) are both constant. The general case can be embedded in this case and we will explain this in more details in the end. In the simple case, note that \(A_0\) and \(A_1\) are both in \(\mathcal{R}(f)\), we will first show, in lemma \([3.1]\) that we can find an \(\epsilon\)-chain from \(A_0\) to \(A_1\) and an \(\epsilon\)-chain from \(A_1\) to \(A_0\). Suppose, on the contrary, that \(A_0\) and \(A_1\) are not in the same chain transitive component, by Conley’s fundamental theorem of dynamical systems, there is a complete Lyapunov function \(g\) defined on \(A\), such that \(g(A_0) \neq g(A_1)\). And this allows us to find an essential closed curve, close to some level curves of \(g\), that doesn’t satisfies the intersection property.

With Lemma \([3.1]\) and the twist condition on the boundary, there is a periodic \(\epsilon\)-chain in the covering space. Then, similar to Franks \([9]\), we can
use this ϵ-chain to construct a periodic disk chain. And the existence of a
fixed point follows from the Brouwer plane translation theorem.

We now proceed with details.

**Lemma 3.1.** If \( f \) is rigid rotation on the boundaries of the annulus, there
is a chain transitive component \( D \) of \( \mathcal{R}(f) \) such that \( D \cap A_0 \neq \emptyset \) and
\( D \cap A_1 \neq \emptyset \)

**Proof.** We prove by contradiction. First, by Conley’s fundamental theorem
of dynamical system, there is a complete Lyapunov function \( g \) on \( \mathcal{A} \) with the
following three properties:

1. \( \forall p \notin \mathcal{R}(f), g(f(p)) < g(p) \).
2. \( \forall p, q \in \mathcal{R}(f), g(p) = g(q) \) if and only if \( p \sim q \).
3. \( g(\mathcal{R}(f)) \) is compact and nowhere dense in \( \mathbb{R} \).

Suppose that, contrary to the conclusion of the lemma, \( A_0 \) and \( A_1 \) are not
in the same chain transitive component, then, by property (2), the function
\( g \) takes different values on \( A_0 \) and \( A_1 \). Assume \( g|_{A_0} = a < g|_{A_1} = b \), where
\( a, b \) are real numbers.

By property (3), there is a number \( c \in [a, b] \) such that the set
\( C = g^{-1}(c) \subseteq \mathcal{A} \) does not intersect the chain recurrent set of \( f \), therefore, by
property (1), \( g(f(x)) < g(x) = c \), therefore, \( f(C) \cap C = \emptyset \), and \( C \) separates
\( A_0 \) and \( A_1 \).

Let
\[
\epsilon = \inf_{x \in C} (g(x) - g(f(x))) = \inf_{x \in C} (c - g(f(x))),
\]
then \( \epsilon > 0 \), by the compactness of \( C \).

The open set \( B = g^{-1}((c - \epsilon, c)) \) separates \( A_0 \) and \( A_1 \). Moreover, \( f(B) \cap B = \emptyset \). One can easily choose an essential simple closed curve \( \gamma \) inside \( B \).
We have \( f(\gamma) \cap \gamma = \emptyset \). This contradicts the intersection property. \( \square \)

One can prove the above lemma directly by using the so-called attractor-
repeller pair. However, using Lyapunov function is more conceptual.

To prove our main theorem, all we need to do is to construct a periodic
disk chain in the covering space.

**Proof of theorem 1.1.** First, we consider a simple case that \( f \) is rigid rotation
on the boundaries of the annulus. Let’s extend \( \tilde{f} : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}} \) to \( h : \mathbb{R}^2 \to \mathbb{R}^2 \)
as follows:

\[
h(x, y) = \begin{cases} 
(x + r_1, y), & \text{if } y \geq 1 \\
\tilde{f}(x, y) & \text{if } 0 < y < 1 \\
(x - r_0, y), & \text{if } y \leq 0 
\end{cases}
\]

By lemma 3.1, there exists \( \tilde{p}_0 \in \tilde{A}_0, \tilde{p}_1 \in \tilde{A}_1 \) s.t. for any \( \epsilon > 0 \), we have
\( \epsilon \)-chains with respect to \( h \) from \( \tilde{p}_0 \) to \( \tilde{p}_1 \) and from \( \tilde{p}_1 + (l_1, 0) \) to \( \tilde{p}_0 + (l_0, 0) \).
for some \( l_0, l_1 \in \mathbb{Z} \). Because of our boundary assumption, we can choose a suitable \( l \in \mathbb{Z} \) such that there exist \( \epsilon \)-chains from \( \tilde{p}_0 \) to \( \tilde{p}_1 + (l_1 + l, 0) \) and from \( \tilde{p}_0 + (l_0 + l, 0) \) to \( \tilde{p}_0 \). Note that \( h(p + (1, 0)) = h(p) + (1, 0) \). We obtain an \( \epsilon \)-chain from \( \tilde{p}_0 \) to \( \tilde{p}_0 \) passing through \( \tilde{p}_1, \tilde{p}_1 + (l_1 + l, 0) \) and \( \tilde{p}_0 + (l_0 + l, 0) \).

Prove by contradiction. Suppose \( \tilde{f} : \tilde{A} \to \tilde{A} \) doesn’t have fixed points, since \( \tilde{A} \) is compact, it follows that \( \exists \delta > 0 \) s.t. \( \|\tilde{f}(p) - p\| \geq \delta \) for all \( p \in \tilde{A} \). Hence, \( \|h(p) - q\| \geq \delta \) for all \( p \in \tilde{A} \).

Take \( \epsilon < \frac{\delta}{4} \), and assume \( \{z_0 = \tilde{p}_0, z_1, ..., z_{n-1}, z_n = z_0\} \) is the above \( \epsilon \)-chain from \( \tilde{p}_0 \) to \( \tilde{p}_0 \). We will construct a periodic disk chain and apply the Brouwer planar translation theorem to get a contradiction.

Let \( U_0 = B_\epsilon(z_0), U_1 = B_\epsilon(z_1), ..., U_{n-1} = B_\epsilon(z_{n-1}), U_n = B_\epsilon(z_n) = U_0 \), where \( B_\epsilon(z) = \{q : \|q - z\| < \epsilon\} \). Then we have the following properties.

1. \( \forall 0 \leq i \leq n, h(U_i) \cap U_{i+1} \neq \emptyset \).
2. \( \forall 0 \leq i \leq n, h(U_i) \cap U_i = \emptyset \).

Now, let’s consider two cases.

The other case is that \( \exists i_0 \neq j_0 \in \{0, 1, ..., n-1\} \) s.t. \( U_{i_0} \cap U_{j_0} \neq \emptyset \). Here, we can take \( i_0 < j_0 \) and consider the sequence \( \{U_{i_0}, U_{i_0+1}, ..., U_{j_0-1}, U_{j_0}\} \). We may assume that this sequence has the property: for any \( i \neq j \in \{i_0, i_0 + 1, ..., j_0 - 1, j_0\} \) except for \( i = i_0 \) and \( j = j_0 \) simultaneously, we have \( U_i \cap U_j = \emptyset \). Because if not, \( \exists i_1 \neq j_1 \in \{i_0, i_0 + 1, ..., j_0 - 1, j_0\} \) s.t. \( U_{i_1} \cap U_{j_1} \neq \emptyset \) and \( \{i_1, i_1 + 1, ..., j_1 - 1, j_1\} \) is a shorter sequence than \( \{i_0, i_0 + 1, ..., j_0 - 1, j_0\} \). Then we can let the new \( i_0 = i_1 \) and new \( j_0 = j_1 \). Moreover, since \( U_i \cap U_{i+1} = \emptyset \), \( \forall i = 0, 1, ..., n-1 \), we can always find the shortest sequence satisfying the above property and this guarantees the existence of such \( i_0 \) and \( j_0 \).

Let \( V_0 := U_{i_0} \cup U_{j_0}, V_1 := U_{i_0+1}, ..., V_{k-1} := U_{j_0-1}, \) and \( V_k := V_0 \). We will show that \( \{V_0, V_1, ..., V_k\} \) is a periodic disk chain.

Firstly, since \( U_{i_0} \cup U_{j_0} \neq \emptyset \), we have

\[
\begin{align*}
\text{d}(h(U_{i_0}), U_{j_0}) := & \inf_{p \in U_{i_0}, q \in U_{j_0}} \|h(p) - q\| \\
& \geq \inf_{p \in U_{i_0}, q \in U_{j_0}} (\|h(p) - q\| - \|q - p\|) \\
& \geq \delta - \sup_{p \in U_{i_0}, q \in U_{j_0}} \|p - q\| \\
& \geq \delta - 4\epsilon > 0.
\end{align*}
\]

and similarly, \( \text{d}(h(U_{j_0}), U_{i_0}) > 0 \). Thus,

\[
h(V_0) \cap V_0 = h(U_{i_0} \cup U_{j_0}) \cap (U_{i_0} \cup U_{j_0}) = (h(U_{j_0}) \cap U_{i_0}) \cup (h(U_{i_0}) \cap U_{j_0}) = \emptyset
\]

Along with the property that \( \forall 0 \leq i \leq n, h(U_i) \cap U_i = \emptyset \), we have \( h(V_i) \cap V_i = \emptyset \) for \( 0 \leq i \leq k \).
Secondly, according to the property that for any $i \neq j \in \{i_0, i_0 + 1, \ldots, j_0 - 1, j_0\}$ except for $i = i_0$ and $j = j_0$ simultaneously we have $U_i \cap U_j = \emptyset$, it is easy to see that if $i \neq j \in \{0, 1, \ldots, k - 1\}$, then $V_i \cap V_j = \emptyset$.

Thirdly, since $\forall 0 \leq i \leq n, h(U_i) \cap U_{i+1} \neq \emptyset$,

$h(V_0) \cap V_1 = (h(U_{i_0}) \cup h(U_{j_0})) \cap U_{i_0+1} \supset h(U_{i_0}) \cap (U_{i_0+1}) \neq \emptyset$

and similarly, $h(V_{k-1}) \cap V_k \neq \emptyset$. Thus, $\forall 0 \leq i \leq k, h(V_i) \cap V_{i+1} \neq \emptyset$.

Therefore, $h : \mathbb{R}^2 \to \mathbb{R}^2$ possesses a periodic disk chain, and hence $h$ has at least one fixed point. By the definition of $h$, the fixed point must lie in the interior of $\tilde{A}$, so this point is also fixed by $\tilde{f}$, a contradiction to our assumption. We have thus proved the theorem for the case that $f$ is rigid rotation on the boundaries.

We now consider the general case where $f$ on the boundaries are not rigid rotations. Recall that $\tilde{f}(x, 1) = (x + r_1(x), 1)$ and $\tilde{f}(x, 0) = (x - r_0(x), 0)$ with $r_0(x), r_1(x) > 0$. Suppose the rotation numbers for $\tilde{f}|_{\mathbb{R} \times \{0\}}$ and $\tilde{f}|_{\mathbb{R} \times \{1\}}$ are $\alpha$ and $\beta$ respectively. Take a small $\delta > 0$, we will extend $\tilde{f}$ to $\tilde{f}_\delta : \mathbb{R} \times [-\delta, 1 + \delta] \to \mathbb{R} \times [-\delta, 1 + \delta]$ in the following way:

$$
\tilde{f}_\delta(x, y) = \begin{cases} 
\frac{1+\delta-y}{\delta} \tilde{f}(x, 1) + \frac{y-1}{\delta}(x + \beta, 1 + \delta), & \text{if } 1 \leq y \leq 1 + \delta \\
\tilde{f}(x, y) & \text{if } 0 < y < 1 \\
\frac{y+\delta}{\delta} \tilde{f}(x, 0) - \frac{y}{\delta}(x + \alpha, -\delta), & \text{if } -\delta \leq y \leq 0
\end{cases}
$$

We can project $\tilde{f}_\delta$ through the covering map to the extended annulus, we have a map $f_\delta$ that is rigid rotation on both boundaries of the extended annulus. Let $A_{\delta} := S^1 \times [-\delta, 1 + \delta]$ with two boundaries $A_{0,\delta}$ and $A_{1,\delta}$; the two extended regions are $D_0 := S^1 \times [-\delta, 0]$ and $D_1 := S^1 \times [1, 1 + \delta]$; the covering space of them are $\tilde{A}_{\delta}, A_{0,\delta}, A_{1,\delta}, D_0$, and $D_1$ respectively.

Now, We should check that $f_\delta$ still satisfies the intersection property on $A_{\delta}$. For any essential simple closed curve $\gamma$, there are three cases.

The first one is the curve $\gamma$ is fully contained in $\tilde{A}$. Then it must satisfy the intersection property.

The second case is the curve $\gamma$ is fully contained in $D_0$ or $D_1$. We may assume $\gamma$ is in $D_0$. Consider the covering space $\tilde{A}_{\delta} = \mathbb{R} \times [-\delta, 1 + \delta]$, let $\pi_y : \mathbb{R} \times [-\delta, 1 + \delta] \to [-\delta, 1 + \delta]$ and $\pi_x : \mathbb{R} \times [-\delta, 1 + \delta] \to \mathbb{R}$ be the projections. And we can find points $p, q \in \tilde{\gamma}$ such that $\pi_y(p) = \min \pi_y(\tilde{\gamma})$ and $\pi_y(q) = \max \pi_y(\tilde{\gamma})$. Let $\tilde{\gamma}_{p, q}$ be the part of $\tilde{\gamma}$ connecting $p$ and $q$, then $\tilde{\gamma}(\tilde{f}(p)) \cap \tilde{\gamma} \neq \emptyset$, since $\pi_y(\tilde{f}(p)) \leq \pi_y(\tilde{\gamma}|_{\pi_x(\tilde{f}(p))})$ and $\pi_y(\tilde{f}(q)) \geq \pi_y(\tilde{\gamma}|_{\pi_x(\tilde{f}(q))})$. Therefore, the curve $\gamma$ satisfies the intersection property.

The third case is that part of the curve $\gamma$ is contained in $\tilde{A}$ denoted by $\gamma_0$ and part of $\gamma$ is contained in $D_0$ or $D_1$ denoted by $\gamma_1$. We first construct $\gamma'$ in the way that we fix $\gamma_0$ in $\tilde{A}$ and replace $\gamma_1$ with parts of $A_0$ or $A_1$ fixing the endpoints s.t. $\gamma'$ is an essential simple closed curve. Then $f(\gamma') \cap \gamma' \neq \emptyset$. If there is an intersection point is in int($\tilde{A}$), then it is also the intersection
point of $f(\gamma) \cap \gamma$. Otherwise, parts of $\gamma'$ in $A_0$ or $A_1$ must intersect its image under $f$. Similar argument as the second case implies $f(\gamma_1) \cap \gamma_1 \neq \emptyset$. So, the curve $\gamma$ still satisfies the intersection property.

In conclusion, we show that $f$ still satisfies the intersection property on $A$. Follow the proof of rigid rotation case, we can show $\bar{f}_\delta : \bar{A}_\delta \to \bar{A}_\delta$ has a fixed point. And it’s easy to see $\bar{f}_\delta$ has no fixed point on $D_0$ and $D_1$ since $\bar{f}_\delta$ in these regions just moves all points in $D_0$ or $D_1$ in one direction. Thus, the fixed point we get must lie in $A$. Hence, it is fixed by $f$.

This completes the proof of our main theorem. □

4. AN APPLICATION TO REVERSIBLE SYSTEMS

We first introduce the definition and some properties of reversible systems.

Definition 4.1. A map $R : \mathbb{R}^n \to \mathbb{R}^n$ is called an involution if it satisfies $R^2 = \text{Id}$.

An easy example of involution is $R : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $R(x, y) = (-x, y)$.

Next, we define general reversible system for both continuous and discrete dynamical systems on $\mathbb{R}^n$.

Definition 4.2. A vector field in $\mathbb{R}^n$ 

\[ \dot{x} = F(t, x) \]

is said to be reversible if there exists an $C^1$ involution $R : \mathbb{R}^n \to \mathbb{R}^n$ such that 

\[ DR \circ F(-t, R(x)) = -F(t, x) \]

where $DR$ is the derivative of $R$.

A homeomorphism $f$ is said to be reversible if there is a continuous involution $R : \mathbb{R}^n \to \mathbb{R}^n$ such that 

\[ f^{-1} = R \circ f \circ R \]

Intuitively, a system is reversible if under some involution it is transformed to a system which is the same as the original one except that the time direction is reversed. Reversible system naturally arises in mechanics (Devaney [8]). A simple non-Hamiltonian example is the system derived following second order equation 

\[ \ddot{x} + f(t, x)\dot{x} + g(t, x) = 0 \]

where 

\[ f(-t, -x) = -f(t, x), \quad g(-t, -x) = -g(t, x). \]

On the annulus, we take the standard involution for reversible systems. Let $R : \mathbb{A} \to \mathbb{A}$ be the involution that takes $(x, y)$, where $x \in \mathbb{R}/\mathbb{Z}$, $y \in [0, 1]$ to $(-x, y)$. A map on the annulus $f : \mathbb{A} \to \mathbb{A}$ is said to be reversible if $f^{-1} = R \circ f \circ R$.

Reversible homeomorphisms may not necessarily have the general intersection property (cf. Sevryuk [12]). Given an essential closed curve $\gamma$, $f(\gamma)$
may not intersect with $\gamma$. So our theorem does not apply directly. However, the proof of our theorem is based on analysis of the chain recurrent set $R(f)$. It is easy to see that the chain recurrent set is symmetric with respect $R$ for reversible systems, i.e., $R(R(f)) = R(f)$, furthermore, the intersection property holds for any symmetric simple closed curve $\gamma$. More precisely, if $R(\gamma) = \gamma$, then by reversibility,

$$R(f^{-1}(\gamma)) = f(R(\gamma)) = f(\gamma).$$

Suppose, without loss of generality, $f(\gamma)$ is inside the annulus bounded by $A_0$ and $\gamma$, then $f^{-1}(\gamma)$ must be outside of the annulus bounded by $A_0$ and $\gamma$ which contains $f(\gamma)$, a contradiction.

Our proof works under the intersection property for symmetric simple closed curves.

We conclude that, by the proof of our theorem, any reversible twist map on the annulus has at least one fixed point. It turns out that there must be two fixed points in this case. One can remove index zero fixed point (Slaminka [14]) by some local modifications without breaking the intersection property. Carter’s [5] result does not apply in this case.

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