A Determinantal Formula for Catalan Tableaux and TASEP Probabilities

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Abstract

We present a determinantal formula for the steady state probability of each state of the TASEP (Totally Asymmetric Simple Exclusion Process) with open boundaries, a 1D particle model that has been studied extensively and displays rich combinatorial structure. These steady state probabilities are computed by the enumeration of Catalan tableaux, which are certain Young diagrams filled with $\alpha$’s and $\beta$’s that satisfy some conditions on the rows and columns. We construct a bijection from the Catalan tableaux to weighted lattice paths on a Young diagram, and from this we enumerate the paths with a determinantal formula, building upon a formula of Narayana that counts unweighted lattice paths on a Young diagram. Finally, we provide a formula for the enumeration of Catalan tableaux with a given number of rows, which corresponds to the steady state probability that in the TASEP on a lattice with $n$ sites, precisely $k$ of the sites are occupied by particles. This formula is an $\alpha/\beta$ generalization of the Narayana numbers.

1 Catalan Tableaux and the TASEP

The TASEP (Totally Asymmetric Simple Exclusion Process) is a model from statistical mechanics that describes particles hopping in one direction along a one-dimensional lattice. New particles can enter and exit the sides of the lattice, and particles can hop right as long as there is at most one particle per site.

In the one-dimensional TASEP with open boundaries, with parameters shown in Figure 1, the rules are as follows: particles hop right on a lattice of $n$ sites, such that

- There is at most one particle per site.
- A new particle can enter on the left at rate $\alpha$.
- A particle can exit on the right at rate $\beta$.
- Particles hop to the right at rate 1.

The TASEP is a special case of the ASEP (Asymmetric Simple Exclusion Process). The ASEP is one of the simplest and most investigated models for the dynamics of particle systems. The existence of exact solutions for this system is extremely useful as testing grounds for non-equilibrium problems in statistical mechanics (see Derrida and the references therein). The ASEP also displays rich algebraic and combinatorial structure; in particular, there are many nice combinatorial results for steady state probabilities in various levels of generality, due to Shapiro and Zeilberger, Duchi and Schaeffer, and Corteel and Williams. Corteel and Williams proved that the steady state probability for a given state of the ASEP can be computed by taking the sum of the weights of certain staircase tableaux that correspond to that state. For the TASEP, we define the following special case of a staircase tableau, an example of which is shown in Figure 2.

Definition 1.1. A Catalan tableau is a filling with $\alpha$’s and $\beta$’s of the Young diagram of shape $\{n, n-1, \ldots, 1\}$ according to the following rules:

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The parameters of the TASEP.}
\end{figure}
• Every diagonal entry must be $\alpha$ or $\beta$.
• All the boxes left of $\beta$ must be empty.
• All the boxes above $\alpha$ must be empty.
• Any box that sees $\alpha$ to the right and $\beta$ below must contain $\alpha$ or $\beta$.

The state that corresponds to a particular Catalan tableau is given by reading the diagonal from top to bottom, where $\alpha$ and $\beta$ are read as $\bullet$ and $\circ$ respectively.

**Definition 1.2.** The **weight** $wt(T)$ of a staircase tableau $T$ is the product of all the symbols it is filled with. For instance, the tableau in Figure 2 has weight $\alpha^6 \beta^5$.

In the case of the TASEP, the steady state probability of a given state is the sum of the weights of the Catalan tableaux that correspond to that state. Viennot [7] states a further characterization of the steady state probability that is given by the enumeration of certain weighted lattice paths, which we call Catalan paths and define in Section 2. A specialization of this result for the case $\alpha = \beta = 1$ is presented in [6].

To add a new column to the left of a tableau, we define a **free row** to be a row that is indexed by $\alpha$. This means that the leftmost symbol in this row is an $\alpha$, and hence it contains no $\beta$’s, so the box in the new column corresponding to that row is “available” to be filled with an $\alpha$ or $\beta$ unless it contains an $\alpha$ below. Hence every new column that we add must be, if starting from the bottom, a (possibly empty) sequence of $\beta$’s followed by an $\alpha$, or just a sequence of $\beta$’s, such that every free row is occupied by a $\beta$ until the $\alpha$ is reached. Figure 3 shows the two cases for the allowed additions of a new column to a Catalan tableau.

Now, we define a new object, the **condensed tableau**, that is in one-to-one correspondence with the Catalan tableaux. Note that condensed tableau were also studied in [7]. To construct a condensed tableau from a Catalan tableau, we start from the upper right and draw a lattice path by adding a step left for each $\beta$ on the diagonal of the Catalan tableau (starting from the upper right), and a step down for each $\alpha$ on the diagonal. The result will be a Young diagram of shape $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ with the number of rows $k$ equal to the number of $\alpha$’s on the diagonal, with a possible “empty” row above or below. Moreover, it is easy to verify that $\lambda_i$ is the number of holes to the right of particle $i$ in the corresponding TASEP configuration. The resulting Young tableau is then filled with $\alpha$’s and $\beta$’s such that the empty boxes are precisely those that lie above an $\alpha$ or left of a $\beta$.

It is easy to see that such a tableau corresponds to the Catalan tableau with all the shaded boxes removed, as in Figure 4. The weight of a condensed tableau is defined to be the product of all its symbols including the ones labeling the left- and down-steps (which correspond to the boxes on the diagonal of the Catalan tableau). From this point on, we will identify Catalan tableaux with their corresponding condensed tableaux. In particular, abusing notation, the shape of a Catalan tableau will mean the shape of its corresponding condensed tableau.

Our main results are new formulas for the enumeration of Catalan tableaux, which in turn provide...
new formulas for steady state probabilities in the TASEP. More specifically for $\lambda = \{\lambda_1, \lambda_2, \ldots\}$, let

$$P_\lambda(\alpha, \beta) = \sum_T \text{wt}(T),$$

where the sum is over staircase tableaux $T$ of shape $\lambda$, be the weight-generating function that enumerates the Catalan tableaux of shape $\lambda$. Our first main result is a determinantal formula for $P_\lambda(\alpha, \beta)$, which is given in Theorem 4.1. Our method of proving this result is to give a bijection between Catalan tableaux and certain weighted lattice paths, and then enumerate the weighted lattice paths, generalizing an argument of Narayana. As a corollary, we obtain a determinantal formula for the steady state probability of being in an arbitrary state of the TASEP. Our second main result, Theorem 5.1, is an explicit expression for the Catalan tableaux which fit exactly in an $m \times k$ box. This formula is in fact a 2-parameter generalization of the Narayana numbers (which are related to the Catalan numbers). As a corollary, we obtain an explicit formula for the steady state probability that in the TASEP on a lattice of $m + k$ sites, precisely $k$ of the states are occupied by particles.

In this paper, we first provide a proof of Narayana’s determinantal formula in Section 3, and then give an analogous proof for enumerating the weighted paths that represent the Catalan tableaux with
the $\alpha, \beta$ generalization $P_\lambda(\alpha, \beta)$. Section 2 defines the bijection from Catalan tableaux to weighted paths, providing the justification for the enumeration of the weighted paths in Section 4. Finally, Section 5 contains a formula for the number of Catalan tableaux with a given number of $\alpha$’s and $\beta$’s on the diagonal, and the related corollaries.

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2 Bijection from Catalan tableaux to weighted lattice paths

Definition 2.1. A Catalan path is a path that starts from the top right corner of a Young diagram and ends at the bottom left corner, only taking steps down and left, and never crossing the boundary of the Young diagram. To each such path, we associate a unique labelling of its steps by $\alpha$, $\beta$, and 1 as follows:

- A down-step that does not lie on the left-most boundary of the Young diagram receives a $\beta$.
- A down-step that lies on the left-most boundary of the Young diagram receives a 1.
- A step left that lies strictly above the lower boundary of the Young diagram receives an $\alpha$.
- A step left that lies on the lower boundary of the Young diagram receives a 1.

Such a path is called a weighted Catalan path, and its total weight is the product of all the weights of its edges. In Figure 6 (c) we see an example of a weighted Catalan path.

![Figure 5: An example of a path on a tableau of shape \{6, 4, 4, 2\} with weight $\alpha^4\beta^3$.](image)

In this section we prove the following.

Proposition 2.1. There is a weight-preserving bijection between Catalan tableaux of a fixed shape $\lambda$ and weighted Catalan paths contained in a Young diagram of the same shape $\lambda$.

For the purpose of the proof, it will be useful to make the following definition.

Definition 2.2. A modified Catalan tableau is a filling of a Young tableau of shape $\lambda$ with $\alpha$’s and $\beta$’s with the following properties:

1. There is at most one $\beta$ in each row.

2. The $\beta$’s must all be in consecutive rows starting from the top one. (If there is a $\beta$ in some row, there must also be a $\beta$ in the row above it.)
3. There cannot be a $\beta$ to the right and below another $\beta$.

4. There cannot be an $\alpha$ to the left and in the same row as a $\beta$.

5. Every box on the lower boundary of the Young tableau contains an $\alpha$ if permissible according to Property \[4\].

Notice that Properties \[1,3\] imply that if the $\beta$’s are associated with down-steps, they will form a Catalan path.

Proof. (Prop \[2.1\]) We obtain the bijection from Catalan tableaux to Catalan paths by first constructing a modified tableau that is in bijection with both the Catalan tableaux and the Catalan paths. The direct correspondence of modified Catalan tableaux with Catalan tableaux of the same shape can be observed in the example given in Figure \[6\](b). The modified tableau is constructed as follows from a given Catalan tableau.

- Take a Catalan filling and drop each $\alpha$ to the bottom of the column that contains it.
- Reading the $\beta$’s from right to left, place them at the highest row possible (within the same column) such that there is at most one $\beta$ per row, so that in the end we have some set of $\alpha$’s lining the lower boundary of the tableau, and a set of $\beta$’s such that if read right to left, they will be decreasing in height.

For each column, the number of boxes below the lowest $\beta$ is the number of free rows (i.e. $\alpha$-indexed rows) remaining in that column. Hence the modified tableau keeps invariant the number of free rows in each column of the tableau. This property uniquely determines the Catalan tableau.

A weighted Catalan path can now be constructed from the modified tableau by reading the columns right to left:

- If the column contains some $\beta$’s, draw a down-step that is labelled by $\beta$ directly to the right of each $\beta$. This labeling is Catalan-path consistent, since these down-steps are all off the left-most edge of the Young diagram by construction.
- If that column also contains an $\alpha$, continue the path with a horizontal step to the left that is labelled by $\alpha$. Note that this horizontal step will be strictly above the lower boundary of the Young diagram, since the $\beta$’s (and the corresponding down-steps) in the modified tableau are strictly above the lower boundary as they all lie above an $\alpha$ in that column - so the labeling of the new step by an $\alpha$ is Catalan-path consistent.
- If that column contains no $\alpha$, continue the path with a horizontal step labeled by 1. A column contains an $\alpha$ if and only if the number of $\alpha$-indexed rows after this column is at least 1. Hence, since the column does not contain $\alpha$, there are no free (or $\alpha$-indexed) rows, and thus the height of the lowest $\beta$ in that column is zero. And so, this horizontal step lies on the lower boundary of the Young tableau, and hence its labeling with a 1 is Catalan-path consistent.

- Once all the columns have been read, the path is at the left-most edge of the Young diagram, so complete the path by drawing some down-edges labeled by 1 directly down to the bottom left corner of the tableau. This segment is by construction Catalan-path consistent.

In the other direction, given a weighted Catalan path, for each vertical segment that is not on the left-most edge of the Young diagram, draw a $\beta$ in the box directly to the left. Then, reading from right to left, for each column, drop the $\beta$’s to the lowest possible locations in that same column (i.e. to the bottom-most free row) so that in the end there is at most one $\beta$ per row. Then, fill in $\alpha$’s
in the first available spots reading from right to left, so that in the end there is at most one $\alpha$ per column. Figure 6 shows an example of a Catalan tableau and its corresponding weighted Catalan path.

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\begin{tikzpicture}
\draw[blue, line width=1pt] (0,0) -- (4,0) -- (4,3) -- (0,3) -- cycle;
\node at (2,1) {$\alpha\beta$};
\node at (2,2) {$\alpha\alpha$};
\node at (2,3) {$\beta\alpha$};
\node at (1,2) {$\alpha\beta$};
\node at (1,1) {$\alpha\alpha$};
\node at (1,0) {$\beta\alpha$};
\node at (0,1) {$\alpha\beta$};
\node at (0,2) {$\alpha\alpha$};
\node at (0,3) {$\beta\alpha$};
\end{tikzpicture}
\caption{A (condensed) Catalan tableau}
\end{subfigure}
\hfill
\begin{subfigure}{0.4\textwidth}
\centering
\begin{tikzpicture}
\draw[blue, line width=1pt] (0,0) -- (4,0) -- (4,3) -- (0,3) -- cycle;
\node at (2,1) {$\alpha\beta$};
\node at (2,2) {$\alpha\alpha$};
\node at (2,3) {$\beta\alpha$};
\node at (1,2) {$\alpha\beta$};
\node at (1,1) {$\alpha\alpha$};
\node at (1,0) {$\beta\alpha$};
\node at (0,1) {$\alpha\beta$};
\node at (0,2) {$\alpha\alpha$};
\node at (0,3) {$\beta\alpha$};
\end{tikzpicture}
\caption{A modified Catalan tableau}
\end{subfigure}
\hfill
\begin{subfigure}{0.4\textwidth}
\centering
\begin{tikzpicture}
\draw[blue, line width=1pt] (0,0) -- (4,0) -- (4,3) -- (0,3) -- cycle;
\node at (2,1) {$\alpha\beta$};
\node at (2,2) {$\alpha\alpha$};
\node at (2,3) {$\beta\alpha$};
\node at (1,2) {$\alpha\beta$};
\node at (1,1) {$\alpha\alpha$};
\node at (1,0) {$\beta\alpha$};
\node at (0,1) {$\alpha\beta$};
\node at (0,2) {$\alpha\alpha$};
\node at (0,3) {$\beta\alpha$};
\end{tikzpicture}
\caption{A weighted Catalan path}
\end{subfigure}
\caption{A Catalan filling of shape \{11, 8, 6, 6, 6, 2\} that corresponds to a weighted Catalan path on this shape.}
\end{figure}

\section{Narayana’s path-counting formula}

Narayana \cite{Narayana} provided the following formula for counting the number of Catalan paths on a Young tableau of shape $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ (i.e. of at most $k$ rows). An example of such a path on a tableau of shape $\{6, 4, 4, 2\}$ is shown in Figure 5.

\begin{theorem} \textbf{(Narayana)} The number of Catalan paths on a Young tableau of shape $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$
is
\[ \det(A) = \det\left( \left( \frac{\lambda_j + 1}{j - i + 1} \right) \right)_{(i,j)}. \]

For clarity, we write out this matrix \( A_{\{\lambda_1, \ldots, \lambda_k\}} \) of which we are taking the determinant:
\[
A = \begin{pmatrix}
\lambda_1 + 1 & \lambda_2 + 1 & \ldots & \lambda_k + 1 \\
1 & \lambda_2 + 1 & \ldots & \lambda_k + 1 \\
0 & 1 & \ldots & \lambda_k + 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_k + 1
\end{pmatrix}
\]

We include the proof of Narayana’s Theorem as a warmup for the proof of our main result, Theorem 4.1.

**Proof.** We prove Narayana’s formula using induction on the number of rows.

First, if the Young diagram has a single row of length \( \lambda_1 \), all paths in that tableau have exactly one down-step, for which there are \( \lambda_1 + 1 \) possible locations including the left-most and right-most edges of the tableau. Thus there are \( \lambda_1 + 1 \) such paths, which equals the single entry and hence the determinant of the matrix \( A_{\{\lambda_1\}} \).

Now, let us assume the formula above holds for counting the paths in all tableaux containing up to \( k - 1 \) rows. We first observe that the bottom right \( m \times m \) minor of \( A_{\{\lambda_1, \ldots, \lambda_k\}} \) equals the matrix \( A_{\{\lambda_{k-m+1}, \lambda_{k-m+2}, \ldots, \lambda_k\}} \), and hence its determinant counts the number of paths on the shape \( \{\lambda_{k-m+1}, \lambda_{k-m+2}, \ldots, \lambda_k\} \).

We also note that any path can be uniquely represented by its set of down-steps: to reconstruct the path from the set of down-steps, we simply have to draw horizontal lines between them. **Rule 1** below gives us an if-and-only-if condition to check whether some set of down-steps that lies within the \( \{\lambda_1, \ldots, \lambda_k\} \) shape gives rise to a valid path.

**Rule 1:** In a valid path, a down-step in row \( \lambda_j \) cannot be to the left of a down-step in row \( \lambda_{j+1} \).
A path is valid if and only if there is a unique down-step in each row and Rule 1 holds for each \( 1 \leq j \leq k - 1 \).

We expand the determinant by its top row. Since the matrix is upper triangular with ones below the main diagonal, it expands as:
\[
\det A_{\{\lambda_1, \ldots, \lambda_k\}} = \left( \frac{\lambda_1 + 1}{1} \right) \det A_{\{\lambda_2, \ldots, \lambda_k\}} - \left( \frac{\lambda_2 + 1}{2} \right) \det A_{\{\lambda_3, \ldots, \lambda_k\}} + \ldots \pm \left( \frac{\lambda_k + 1}{k} \right) \det A_{\emptyset},
\]
where by convention, \( \det A_{\emptyset} = 1 \).

**Step 1:** A path on \( \{\lambda_1, \ldots, \lambda_k\} \) contains a down-step somewhere in the top row, and there are \( \lambda_1 + 1 \) choices for that down-step, as we see in Figure 7. Combinatorially, \( \left\lfloor \lambda_1 + 1 \right\rfloor \) \( \det A_{\{\lambda_2, \ldots, \lambda_k\}} \) is the combination of all choices for the down-step in the top row with all possibilities for paths that start in the upper-right corner of the shape \( \{\lambda_2, \ldots, \lambda_k\} \) (i.e. the paths in all rows below the first one).
All such combinations will certainly be counting the collections of down-steps that represent all the possible paths, but they will also be counting some illegal collections of down-steps that violate Rule 1 at rows 1 and 2, such as in Figure 8 (a).

**Step 2:** Specifically, Rule 1 is violated at rows 1 and 2 when the down-step in the top row lies to the left of the first down-step of the path starting in the second row, such as in Figure 8 (b). Let us subtract out those combinations. In particular, all such illegal combinations will be counted by the set of an illegal pair of down-steps in the top two rows, combined with all possible paths starting from the third row. The ways of selecting this illegal pair of down-steps in rows 1 and 2 is simply a choice of two disjoint columns \( C_1 \) and \( C_2 \) with \( C_1 < C_2 \) such that the top row gets a down-step in column \( C_1 \) and the row below it gets a down-step in column \( C_2 \). The number of such choices is \( \binom{\lambda_2+1}{2} \), and the number of all paths starting at the third row is \( \det A_{\lambda_3,\ldots,\lambda_k} \). Their product is the second term in the expansion of the determinantal formula.

**Step 3:** Now we have subtracted all collections of down-steps that violate Rule 1 in rows 1 and 2, but some of the terms that we subtracted were not actually counted in Step 1, so we have to add those back in. Those paths are those where Rule 1 is violated not only in rows 1 and 2, but also in rows 2 and 3. This is because Step 1 presumes that the collection of down-steps starting from row 2 is legal, in particular that the pair of down-steps in rows 2 and 3 is legal. Thus, we must add back in all combinations of the form shown in Figure 8 (c). The possibilities for the top 3 rows are counted by \( \binom{\lambda_3+1}{3} \), and the possibilities for all paths starting from row 4 are given by \( \det A_{\lambda_4,\ldots,\lambda_k} \), so their product is the third term of the determinantal expansion.

**Steps 4 through \( k \):** This subtraction and re-addition of terms is repeated for \( k \) steps, where at step \( j \) we subtract/add the terms that violated Rule 1 in rows 1 and 2, 2 and 3, \ldots, and \( (j-1) \) and \( j \), since the steps 1 through \( (j-1) \) only accounted for terms that had a valid collection of down-steps starting from row \( (j-1) \). Hence at step \( j \), we subtract/add the product \( \binom{\lambda_{j+1}}{j} \det A_{\lambda_{j+1},\ldots,\lambda_k} \), the \( j \)'th term in the determinantal expansion - thus we have accounted for all the terms in Narayana’s formula.

**4 Enumeration of the weighted paths corresponding to Catalan fillings of shape \( \lambda \)**

From the bijection from Catalan tableaux to paths, and the determinantal formula that counts the number of paths, we construct the formula for enumerating tableaux by assigning a weight to each
path which equals the weight of the Catalan tableau that path corresponds to.

**Theorem 4.1.** The formula to enumerate the Catalan tableaux for the fixed shape \( \lambda = \{ \lambda_1, \lambda_2, \ldots, \lambda_k \} \) (here we assume \( \lambda_{k+1} = \lambda_{k+2} = \ldots = 0 \) for a Young tableau of \( k \) rows) is

\[
P_{\{\lambda_1, \ldots, \lambda_k\}}(\alpha, \beta) = \det A^\alpha, \beta
\]

where \( A^\alpha, \beta = (A_{ij})_{i,j=1}^k \) is given by

\[
A_{ij} = \begin{cases} 
0 & \text{for } j < i - 1 \\
1 & \text{for } j = i - 1 \\
\beta^{j-i} \alpha^{\lambda_i - \lambda_{j+1}} \left( \binom{\lambda_{j+1}}{j-i} + \beta \binom{\lambda_{j+1}}{j-i+1} \right) \\
+ \beta^{j-i} \alpha^{\lambda_i - \lambda_j} \sum_{\ell=0}^{\lambda_j - \lambda_{j+1} - 1} \alpha^\ell \left( \binom{\lambda_{j+1}}{j-i-1} + \beta \binom{\lambda_j - \ell - 1}{j-i} \right) & \text{for } j \geq i
\end{cases}
\]

For example, for \( k = 2 \), we have:

\[
A^\alpha, \beta = \begin{pmatrix}
\alpha^{\lambda_1 - \lambda_2} (1 + \lambda_2 \beta) + \beta \sum_{\ell=0}^{\lambda_1 - \lambda_2 - 1} \alpha^\ell & \alpha^{\lambda_1 - \lambda_2} \sum_{\ell=0}^{\lambda_2 - 1} \alpha^\ell (1 + \beta(\lambda_2 - \ell - 1)) \\
1 & \alpha^{\lambda_2} + \beta \sum_{\ell=0}^{\lambda_2 - 1} \alpha^\ell
\end{pmatrix}
\]

And for \( k = 3 \), we have:

\[
A^\alpha, \beta = \begin{pmatrix}
\alpha^{\lambda_1 - \lambda_2} (1 + \lambda_2 \beta) + \beta \sum_{\ell=0}^{\lambda_1 - \lambda_2 - 1} \alpha^\ell & X & Z \\
1 & \alpha^{\lambda_2 - \lambda_3} (1 + \lambda_3 \beta) + \beta \sum_{\ell=0}^{\lambda_2 - \lambda_3 - 1} \alpha^\ell & Y \\
0 & 1 & \alpha^{\lambda_3} + \beta \sum_{\ell=0}^{\lambda_3 - 1} \alpha^\ell
\end{pmatrix}
\]

where

\[
X = \beta \alpha^{\lambda_1 - \lambda_2} \sum_{\ell=0}^{\lambda_2 - \lambda_3 - 1} \alpha^\ell (1 + \beta(\lambda_2 - \ell - 1)) + \alpha^{\lambda_1 - \lambda_3} \left( \lambda_3 + \beta \binom{\lambda_3}{2} \right),
\]

\[
Y = \beta \alpha^{\lambda_2 - \lambda_3} \sum_{\ell=0}^{\lambda_3 - 1} \alpha^\ell (1 + \beta(\lambda_3 - \ell - 1)),
\]

\[
Z = \beta^2 \alpha^{\lambda_1 - \lambda_3} \sum_{\ell=0}^{\lambda_3 - 1} \alpha^\ell \left( \lambda_3 - \ell - 1 + \beta \binom{\lambda_3 - \ell - 1}{2} \right).
\]
Observe that similarly to Narayana’s formula, each $m \times m$ lower-left minor of $A^{\alpha,\beta}_{\{\lambda_1,\ldots,\lambda_k\}}$ equals $A^{\alpha,\beta}_{\{\lambda_{k-m+1},\lambda_{k-m+2},\ldots,\lambda_k\}}$, whose determinant enumerates the weighted Catalan paths on the shape $\{\lambda_{k-m+1},\lambda_{k-m+2}, \ldots, \lambda_k, 0, \ldots\}$.

We prove this formula analogously to the proof of Narayana’s path-counting formula by an inductive argument, except that now instead of letting each path have weight 1, we assign the weights to the Catalan paths according to the bijection given in Section 2. In particular, in Narayana’s formula, since the weight of each new component is 1, we enumerated the paths by adding up all possible contributions row by row. Similarly, to prove Theorem 4.1 we enumerate the possible Catalan paths by, for each successive row of the path, taking the sum of the weight contributions of all the possibilities for that row. The subtlety here is that the contribution from the segment of the path corresponding to row $j$ is given by the weight of row $j$ in the modified filling. This rule is well-defined since the modified fillings are in a weight-preserving bijection with the Catalan paths. In Figure 9 we see an example of the weight contributions that arise from each row segment of a Catalan path.

Observe that in the modified filling, the $\alpha$’s can only lie on the "shelves" of the tableau, i.e. in row $j$ they can only lie in locations $\{\lambda_{j+1}+1, \lambda_{j+1}+2, \ldots, \lambda_j\}$, since to create the modified fillings, we had all the $\alpha$’s drop the bottom of each column containing them.

From here on we will no longer be referring to the modified fillings, but in the proof that follows, it is assumed that all weight contributions are taken from the rows of the modified fillings.

![Figure 9: The weight contributions from each row of a Catalan path are given by the weight of the corresponding row in the modified filling.](image)

In the following, we reproduce the proof of Narayana’s path counting formula, except now for the weighted Catalan paths.

**Proof.** First, when $k = 1$, a Catalan filling of one row of length $\lambda_1$ can either have some number of $\alpha$’s capped by a $\beta$ when read right to left, or the entire row can be filled with $\alpha$’s. Hence the sum of the weights of Catalan tableau of shape $\{\lambda_1\}$ is

$$\alpha^{\lambda_1} + \beta \sum_{j=0}^{\lambda_1-1} \alpha^j,$$

which equals $\det A^{\alpha,\beta}_{\lambda_1}$.

For $k > 1$, we expand the determinant of $A^{\alpha,\beta}_{\lambda_1,\ldots,\lambda_k}$ by its top row. Since the matrix is upper triangular with ones below the main diagonal, it expands as:

$$\det A^{\alpha,\beta}_{\{\lambda_1,\ldots,\lambda_k\}} = W_1 \times \det A^{\alpha,\beta}_{\{\lambda_2,\ldots,\lambda_k\}} - W_2 \times \det A^{\alpha,\beta}_{\{\lambda_3,\ldots,\lambda_k\}} + W_3 \times \det A^{\alpha,\beta}_{\{\lambda_4,\ldots,\lambda_k\}} - \ldots \pm W_k$$

where we denote by $W_k$ the entry in column $k$ of the first row of $A^{\alpha,\beta}_{\lambda_1,\ldots,\lambda_k}$. 

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Figure 10: The four cases of choosing a set of down-steps at locations $C_1 < \ldots < C_j$ in corresponding rows $1, \ldots, j$ and the weights associated with them.
Step 1: A Catalan path on \( \{ \lambda_1, \ldots, \lambda_k \} \) contains a down-step somewhere in the top row including the left-most and right-most edges. We label the position of the possible down-step from left to right by \( C_1 \) for \( 0 \leq C_1 \leq \lambda_1 \), where \( C_1 = 0 \) corresponds to that down-step being on the left-most edge of the tableau and hence has weight 1, and otherwise, that down-step is on the edge to the right of column \( C_1 \) and hence carries weight \( \beta \). When \( C_1 = 0 \), the Catalan path contains no \( \beta' \)'s in the top row, otherwise it contains a \( \beta \) in position \( C_1 \). The weight contribution of this segment of the path is thus

\[
\begin{align*}
(a) & \quad \alpha^{\lambda_1 - \lambda_2} \quad C_1 = 0 \\
(b) & \quad \beta \alpha^{\lambda_1 - \lambda_2} \quad 1 \leq C_1 \leq \lambda_2 \\
(c) & \quad \beta \alpha^{\lambda_1 - C_1} \quad \lambda_2 + 1 \leq C_1 \leq \lambda_1
\end{align*}
\]

Hence the sum of weights of all possible modified fillings of Row 1, from taking the sum over all choices of \( 0 \leq C_1 \leq \lambda_1 \), is

\[
W_1 = \alpha^{\lambda_1 - \lambda_2} (1 + \lambda_2 \beta) + \beta \sum_{\ell=0}^{\lambda_1 - \lambda_2 - 1} \alpha^\ell.
\]

\( W_1 \times \det A^{\alpha,\beta}_{\lambda_2,\ldots,\lambda_k} \) is the sum of the weights of the combination of all choices for the down-step in the top row, with all possibilities for Catalan paths that start in the upper-right corner of the shape \( \{ \lambda_2, \ldots, \lambda_k \} \). Here all possible Catalan paths have been accounted for, but we have also included some illegal collections of down-steps that violate Rule 1 at rows 1 and 2.

Step 2: We subtract out illegal combinations of down-steps where \( C_1 < C_2 \) for \( C_1 \) the location of the down-step in Row 1, and \( C_2 \) that of Row 2. The ways of selecting this illegal pair of down-steps in rows 1 and 2 is simply a choice of two disjoint columns \( C_1 \) and \( C_2 \) with \( C_1 < C_2 \) such that the top row gets a down-step in column \( C_1 \) and the row below it gets a down-step in column \( C_2 \). Figure \[10\] with \( j = 2 \) gives the different cases for the possibilities for \( C_1 \) and \( C_2 \), with the following weight contributions:

\[
\begin{align*}
(a) & \quad \beta \alpha^{\lambda_1 - \lambda_3} \quad C_1 = 0 \text{ and } 1 \leq C_2 \leq \lambda_3 \\
(b) & \quad \beta \alpha^{\lambda_1 - C_2} \quad C_1 = 0 \text{ and } \lambda_3 + 1 \leq C_2 \leq \lambda_2 \\
(c) & \quad \beta^2 \alpha^{\lambda_1 - \lambda_3} \quad C_1 > 0 \text{ and } C_1 < C_2 \leq \lambda_3 \\
(d) & \quad \beta^2 \alpha^{\lambda_1 - C_2} \quad C_1 > 0 \text{ and } \lambda_3 + 1 < C_2 \leq \lambda_2 \text{ and } C_1 < C_2
\end{align*}
\]

Taking the sum over all choices of \( C_1 \) and \( C_2 \) with \( 0 \leq C_1 < C_2 \leq \lambda_2 \), we obtain

\[
W_2 = \beta \alpha^{\lambda_1 - \lambda_3} \left( \left( \beta \left( \frac{\lambda_3}{2} \right) \right) + \beta \alpha^{\lambda_1 - \lambda_2} \sum_{\ell=0}^{\lambda_2 - \lambda_1 - 1} \alpha^\ell \left( 1 + \left( \frac{\lambda_2 - \ell - 1}{1} \right) \right) \right)
\]

\( W_2 \times \det A^{\alpha,\beta}_{\lambda_3,\ldots,\lambda_k} \) is the sum of the weights of all combinations of violations of Rule 1 at rows 1 and 2 with all possible Catalan paths starting from row 3. This is the second term in the expansion of the determinantal formula.

Step 3: We now add back in those combinations of down-steps where Rule 1 is violated not only in rows 1 and 2, but also in rows 2 and 3 (i.e. of the form shown in Figure ??). We enumerate these combinations by selecting \( 0 \leq C_1 < C_2 < C_3 \leq \lambda_3 \) where \( C_1 \) is the location of the down-step in row
i. Figure 10 for $j = 3$ shows the different cases, with the following weight contributions:

\[
\begin{align*}
\text{(a)} \quad & \beta^2 \alpha \lambda_1 - \lambda_1 \quad C_1 = 0 \text{ and } 1 \leq C_2 < \ldots < C_j \leq \lambda_i + 1 \\
\text{(b)} \quad & \beta^2 \alpha \lambda_1 - C_j \quad C_1 = 0 \text{ and } \lambda_i + 1 \leq C_2 < \ldots < C_j \leq \lambda_i + 1 \quad \text{and } 1 \leq C_2 < \ldots < C_j \\
\text{(c)} \quad & \beta^2 \alpha \lambda_1 - \lambda_i \quad C_1 > 0 \text{ and } 1 \leq C_1 < C_2 < \ldots < C_j \leq \lambda_i \\
\text{(d)} \quad & \beta^2 \alpha \lambda_1 - C_3 \quad C_1 > 0 \text{ and } \lambda_i + 1 \leq C_2 < \ldots < C_j \leq \lambda_i \quad \text{and } 1 \leq C_2 < \ldots < C_j < C_3
\end{align*}
\]

The sum over all such $C_1, C_2, C_3$ is:

\[
W_3 = \beta^2 \alpha \lambda_1 - \lambda_4 \left( \frac{\lambda_4}{2} + \beta \left( \frac{\lambda_4}{3} \right) \right) + \beta^2 \alpha \lambda_1 - \lambda_3 \sum_{\ell=0}^{\lambda_3 - \ell - 1} \alpha^\ell \left( \frac{\lambda_3 - \ell - 1}{1} \right) + \beta \left( \frac{\lambda_3 - \ell - 1}{2} \right).
\]

The possibilities for all paths starting from row 4 are given by $\det A^\alpha_{\lambda_1, \ldots, \lambda_k}$, so the product with $W_3$ is the third term of the determinantal expansion.

**Step 4 through k:** We repeat the above for $k$ steps, where at step $j$ we subtract/add the combinations of down-steps that violated Rule 1 in rows 1 and 2, 2 and 3, ..., and $(j-1)$ and $j$. We enumerate these combinations by selecting $0 \leq C_1 < \ldots < C_j \leq \lambda_i$ where $C_i$ is the location of the down-step in row $i$. Figure 10 shows the different cases, with the following weight contributions:

\[
\begin{align*}
\text{(a)} \quad & \beta^{j-1} \alpha \lambda_1 - \lambda_{j+1} \quad C_1 = 0 \text{ and } 1 \leq C_2 < \ldots < C_j \leq \lambda_j + 1 \\
\text{(b)} \quad & \beta^{j-1} \alpha \lambda_1 - C_j \quad C_1 = 0 \text{ and } \lambda_{j+1} + 1 \leq C_2 < \ldots < C_j \leq \lambda_j \quad \text{and } 1 \leq C_2 < \ldots < C_j \\
\text{(c)} \quad & \beta \alpha \lambda_1 - \lambda_{j+1} \quad C_1 > 0 \text{ and } C_1 < C_2 < \ldots < C_j \leq \lambda_j + 1 \\
\text{(d)} \quad & \beta \alpha \lambda_1 - C_j \quad C_1 > 0 \text{ and } \lambda_{j+1} + 1 \leq C_2 < \ldots < C_j \leq \lambda_j \quad \text{and } C_1 < C_2 < \ldots < C_j
\end{align*}
\]

The sum over all such $C_1, \ldots, C_j$ is:

\[
W_j = \beta^{j-1} \alpha \lambda_1 - \lambda_{j+1} \left( \frac{\lambda_{j+1}}{j-1} + \beta \left( \frac{\lambda_{j+1}}{j} \right) \right) + \beta^{j-1} \alpha \lambda_1 - \lambda_j \sum_{\ell=0}^{\lambda_j - \ell - 1} \alpha^\ell \left( \frac{\lambda_j - \ell - 1}{j-2} \right) + \beta \left( \frac{\lambda_j - \ell - 1}{j-1} \right).
\]

This is because, once we have chosen $C_j$, there remain $\binom{C_j - 1}{j-2}$ choices for $\{C_2, \ldots, C_{j-1}\}$ such that $1 \leq C_2 < \ldots < C_{j-1} < C_j$ when $C_1 = 0$, and $\binom{C_j - 1}{j-1}$ choices for $\{C_1, \ldots, C_{j-1}\}$ such that $1 \leq C_1 < C_2 < \ldots < C_{j-1} < C_j$ when $C_1 > 0$, and the formula above is obtained by taking the sum over all $C_j$.

Hence at step $j$, we subtract/add the product $W_j \times \det A^\alpha_{\lambda_{j+1}, \ldots, \lambda_k}$, the $j$’th term in the determinantal expansion - thus we have accounted for all the terms in the determinantal formula.

**Corollary 4.2.** The un-normalized steady state probability that the TASEP with $n$ sites has particles in precisely the locations $1 \leq x_1 < \ldots < x_k \leq n$ for $1 \leq x_1 < x_2 < \ldots < x_k \leq n$ is:

\[
P \left[ \{x_1, \ldots, x_k\} \right] = \det A^\alpha_{\lambda_j}
\]

where $A^\alpha_{\lambda_j}$ is given by

\[
A_{ij} = \begin{cases} 
0 & \text{for } j < i - 1 \\
1 & \text{for } j = i - 1 \\
\beta^{j-i} \alpha^{i-(j+1)+x_{j+1}-x_i} \left( \frac{n-k+j+1-x_{j+1}}{j-1} + \beta \left( \frac{n-k+j+1-x_{j+1}}{j} \right) \right) + \beta^{j-i} \alpha^{i-j+x_j-x_i} \sum_{\ell=0}^{x_j+x_i-x_j-1} \alpha^\ell \left( \frac{n-k+j-x_j-\ell-1}{j-1} \right) + \beta \left( \frac{n-k+j-x_j-\ell-1}{j-1} \right) & \text{for } j \geq i
\end{cases}
\]
Proof: Applying the construction of a condensed tableau, the TASEP configuration of \{x_1, \ldots, x_k\} corresponds to a condensed tableau of shape \(\lambda = \{n-k+1-x_1, n-k+2-x_2, \ldots, n-k+k-x_k\}\) (equivalently, \(\lambda = \{\lambda_1, \ldots, \lambda_k\}\) where \(\lambda_j\) is the number of holes to the right of particle \(j\), meaning \(\lambda_j = n-k+j-x_j\)). Hence Theorem 4.1 gives the desired formula.

5 Formula for the Catalan Tableaux inside an \(m \times k\) box

Let \(N_{m,k}(\alpha, \beta)\) be the sum of the weights of the \(\alpha - \beta\) staircase tableaux with diagonal of size \(m + k\) with \(k\) \(\alpha\)'s on the diagonal. Such tableaux correspond to the Catalan tableaux that fit inside an \(m \times k\) box, meaning that there are at most \(k\) rows, and the largest row of such a tableau has length at most \(m\). Hence if \(\lambda = \{\lambda_1, \ldots, \lambda_k\}\) is the shape of such a tableau, then \(0 \leq \lambda_k \leq \ldots \leq \lambda_1 \leq m\).

Let \(N'_{m',k'}(\alpha, \beta)\) be the sum of the weights of the \(\alpha - \beta\) staircase tableaux of size \(m' + k'\) with \(k'\) \(\alpha\)'s on the diagonal, and with \(\beta\) in the bottom-most diagonal entry and \(\alpha\) in the top-most diagonal entry, up to a factor of \(\alpha^{k'}\beta^m\) (the weight of the diagonal). Such tableaux correspond to the Catalan tableaux that lie in a \(m' \times k'\) box and have exactly \(k'\) rows with largest row of length exactly \(m'\). Hence if \(\lambda' = \{\lambda'_1, \ldots, \lambda'_{k'}\}\) is the shape of such a tableau, then \(1 \leq \lambda'_{k'} \leq \ldots \leq \lambda'_1 = m'\). The following gives the relation between \(N_{m,k}(\alpha, \beta)\) and \(N'_{m',k'}(\alpha, \beta):\)

\[
N_{m,k}(\alpha, \beta) = \sum_{m' = 0}^{m} \sum_{k' = 0}^{k} \alpha^k \beta^m N'_{m',k'}(\alpha, \beta).
\]

Here we multiplied by a factor of \(\alpha^k\beta^m\) to account for the weight of the diagonal itself.

Enumerating all the Catalan tableaux that fit exactly in a box of width \(m\) and height \(k\) is equivalent to taking the sum

\[
N'_{m,k}(\alpha, \beta) = \sum_{1 \leq \lambda_k \leq \ldots \leq \lambda_1 \leq m} \det A_{\{m, \lambda_2, \lambda_3, \ldots, \lambda_k\}}.
\]

The above gives rise to the following Theorem.

Theorem 5.1. The formula for the Catalan tableaux that fit exactly in an \(m \times k\) box is:

\[
N'_{m,k}(\alpha, \beta) = \alpha^k \beta^m \left( \sum_{\ell=0}^{k-1} \sum_{j=0}^{m-1} \alpha^j \beta^{m-\ell} \left( \binom{m+\ell-2}{m-1} \binom{k+j-2}{k-1} - \binom{m+\ell-2}{m} \binom{k+j-2}{k} \right) + \alpha^m \sum_{\ell=0}^{k-1} \beta^{m-\ell} \left( \binom{m+k-2}{m-1} \binom{m+\ell-1}{m-1} - \binom{m+k-2}{m} \binom{m+\ell-1}{m} \right) + \beta^k \sum_{j=0}^{m-1} \alpha^j \left( \binom{m+k-2}{k-1} \binom{k+j-1}{k-1} - \binom{m+k-2}{k} \binom{k+j-1}{k} \right) \right).
\]

Summation of (2) according to (1) yields the following:
$$N_{m,k}(\alpha, \beta) = \alpha^k \beta^m \sum_{j=0}^{m} \sum_{\ell=0}^{k} \alpha^j \beta^\ell \left( \binom{k+j-1}{j} \binom{m+\ell-1}{\ell} - \binom{k+j-1}{j-1} \binom{m+\ell-1}{\ell-1} \right)$$ \hspace{1cm} (3)$$

And so the generating function for Catalan tableaux parametrized by \(x\) and \(y\) to give the number of \(\alpha\)'s and \(\beta\)'s on the diagonal respectively is:

$$F_{\alpha, \beta}(x, y) = \sum_{m \geq 0} \sum_{k \geq 0} x^k y^m \alpha^k \beta^m \sum_{j=0}^{k} \sum_{\ell=0}^{m} \alpha^j \beta^\ell \left( \binom{k+j-1}{j} \binom{m+\ell-1}{\ell} - \binom{k+j-1}{j-1} \binom{m+\ell-1}{\ell-1} \right)$$

**Proof.** We prove Formula (2) by induction on \(m\) and \(k\). As seen in Figure 11, the tableaux with exactly \(k\) rows and largest row of length \(m\) can be formed by the addition of a \(k - m\) hook with a row of length \(m\) and column of length \(k\) to the top and left edges of a Catalan tableau that fits inside a \(k - 1 \times m - 1\) box.

![Figure 11: Constructing a Catalan tableau that fits exactly in a \(m \times k\) box by adding a \(k - m\) hook.](image)

Let \(H_{m,k}^{m,k}\) be the sum of the weights of the possible fillings of the \(k - m\) hook, when the inside tableau has \(p\) rows that are \(\alpha\)-indexed and \(q\) columns that are \(\beta\)-indexed. If the inside tableau has weight \(\alpha^j \beta^\ell\), then it must contain \(\ell\) \(\beta\)'s, and so there are \(k - 1 - \ell\) rows that are \(\alpha\)-indexed since there is at most one \(\beta\) per row. Similarly, the inside tableau contains \(j\) \(\alpha\)'s, and hence then there must be \(m - 1 - j\) columns that are \(\beta\)-indexed, since there is at most one \(\alpha\) per column. Figure 12 shows the cases that result in the following expression:

$$H_{k-1-\ell, m-1-j}^{m,k} = \alpha^{m-j} \sum_{s=0}^{k-\ell-1} \beta^s + \beta^{k-\ell} \sum_{t=0}^{m-j-1} \alpha^t + \sum_{t=1}^{m-j-1} \sum_{s=1}^{k-\ell-1} \alpha^t \beta^s. \hspace{1cm} (4)$$

Hence for \(m, k \geq 2\) we obtain the following recursion:

$$N'_{m,k}(\alpha, \beta) = \sum_{j=0}^{m-1} \sum_{\ell=0}^{k-1} H_{k-1-\ell, m-1-j}^{m,k} \alpha^j \beta^\ell \left[ \alpha^j \beta^\ell \right] N_{m-1,k-1}(\alpha, \beta). \hspace{1cm} (5)$$
The expression $[\alpha^j \beta^k] N_{m-1,k-1}(\alpha, \beta)$ means the coefficient of $\alpha^j \beta^k$ in $N_{m-1,k-1}(\alpha, \beta)$, which by the induction hypothesis and from (3) we know to be
\[ \left( \binom{k+j-2}{j} \binom{m+\ell-2}{\ell} - \binom{k+j-2}{j-1} \binom{m+\ell-1}{\ell-1} \right). \]

The recursion is now straightforward to verify. On the RHS of (4), we have:

\[
\begin{align*}
\sum_{s=0}^{k-1} \alpha^m \beta^s \sum_{u=0}^m \sum_{j=0}^{m-1} & \left( \binom{m+u-2}{m-2} \binom{k+j-2}{k-2} - \binom{m+u-2}{m-1} \binom{k+j-2}{k-1} \right) \\
+ \sum_{t=0}^{m-1} \alpha^t \beta^s & \sum_{v=0}^t \sum_{\ell=0}^{k-1} \left( \binom{m+\ell-2}{m-2} \binom{k+v-2}{k-2} - \binom{m+\ell-2}{m-1} \binom{k+v-2}{k-1} \right) \\
+ \sum_{s=1}^{k-1} \sum_{t=1}^{m-1} \alpha^t \beta^s & \sum_{u=0}^{m-1} \sum_{v=0}^{s-1} \sum_{\ell=0}^{t-1} \left( \binom{m+u-2}{m-2} \binom{k+v-2}{k-2} - \binom{m+u-2}{m-1} \binom{k+v-2}{k-1} \right).
\end{align*}
\]

Using the fact that $\sum_{i=0}^{a} \binom{b+i}{c} = \binom{b+a+1}{c+1}$, the above simplifies to obtain:

\[
\begin{align*}
\alpha^m & \sum_{s=0}^{k-1} \beta^s \left( \binom{m+s-1}{m-1} \binom{k+m-2}{k-1} - \binom{m+s-1}{m} \binom{k+m-2}{k} \right) \\
+ \beta^k & \sum_{t=0}^{m-1} \alpha^t \left( \binom{m+k-2}{m-1} \binom{k+t-1}{k-1} - \binom{m+k-2}{m} \binom{k+t-1}{k} \right) \\
+ \sum_{s=1}^{k-1} \sum_{t=1}^{m-1} \alpha^t \beta^s & \left( \binom{m+s-2}{m-1} \binom{k+t-2}{k-1} - \binom{m+s-2}{m} \binom{k+t-2}{k} \right).
\end{align*}
\]

This formula equals (2) for $s = \ell$ and $t = j$, which is the LHS that we desire.
It remains to check the base cases for $N_{m,k}^{'}(\alpha, \beta)$ when $m = 1$ or $k = 1$. If we plug $m = 1$ into (2), we obtain

$$N_{1,k}^{'}(\alpha, \beta) = \beta^k + \alpha \sum_{\ell = 0}^{k-1} \beta^{\ell},$$

which is the sum of the weights of Catalan fillings of the shape $\lambda = \{1, \ldots, 1\}$ of $k$ rows. Similarly, plugging $k = 1$ into (2) yields

$$N_{m,1}^{'}(\alpha, \beta) = \alpha^m + \beta \sum_{i = 0}^{m-1} \alpha^i,$$

which is the sum of the weights of Catalan fillings of the shape $\lambda = \{m\}$, and so the proof is complete.

Remark. For $k + m = n$, Derrida provides a formula in [3] for $\sum_{k=0}^{n} N_{n-k,k}(\alpha, \beta)$, which enumerates all staircase tableaux of size $n$.

$$Z_n = \alpha^n \beta^n \sum_{p=1}^{n} \frac{p}{2n-p} \left( \frac{2n-p}{n} \right) \frac{\alpha^{-p-1} - \beta^{-p-1}}{\alpha^{-1} - \beta^{-1}}.$$  \hspace{1cm} (6)

This expression normalizes the stationary probabilities of the TASEP that were derived earlier, as we see in the following Corollary. We now show that (3) implies the above after some manipulations of the sums.

$$\sum_{k=0}^{n} N_{n-k,k} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{\ell=0}^{k} \alpha^{j+k} \beta^{\ell+n-k} \left( \binom{k+j-1}{k-1} \binom{n-k-1}{n-k-1} \right) - \left( \binom{k+j-1}{k-1} \binom{n-k-1}{n-k-1} \right)$$

We extract the coefficient of $\alpha^{n-t} \beta^{n-t-s}$ for $s, t \leq n$:

$$[\alpha^{n-t} \beta^{n-t-s}] \sum_{k=0}^{n} N_{n-k,k} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{\ell=0}^{k} \alpha^{j+k} \beta^{\ell+n-k} \left( \binom{k+j-1}{k-1} \binom{n-k-1}{n-k-1} \right)$$

Using the Vandermonde convolution identity, we get the following:

$$[\alpha^{n-t} \beta^{n-t-s}] \sum_{k=0}^{n} N_{n-k,k} = \left( \frac{2n-s-2}{n-2} \right) - \left( \frac{2n-s-2}{n} \right) = \frac{s}{2n-s} \left( \frac{2n-s}{n} \right).$$  \hspace{1cm} (7)
Corollary 5.2. The stationary probability of a TASEP of length $k + m$ and containing $k$ particles, normalized by $Z$ from [6], is:

$$N_{m,k}(\alpha, \beta) = \sum_{j=0}^{m} \sum_{\ell=0}^{k} \alpha^{k+j} \beta^{m+\ell} \left(\binom{k+j-1}{k} \binom{m+\ell-1}{m-1} - \binom{k+j-1}{k} \binom{m+\ell-1}{m} \right).$$

Corollary 5.3. Formula [3] can be rewritten in terms of power series as:

$$N_{m,k}(\alpha, \beta) = \frac{1}{(1-\alpha)^k(1-\beta)^m} \left[ 1 - \beta^{m+k} \sum_{j=0}^{m-1} \left( \frac{\beta}{1-\beta} \right)^j \left( \binom{m+k}{j} \right) \left[ 1 - \alpha^{m+k} \sum_{i=0}^{m-1} \left( \frac{\alpha}{1-\alpha} \right)^i \left( \binom{m+k}{i} \right) \right] \right] - \frac{1}{(1-\alpha)^k(1-\beta)^m} \left[ 1 - \beta^{m+k-2} \sum_{j=0}^{m-2} \left( \frac{\beta}{1-\beta} \right)^j \left( \binom{m+k-2}{j} \right) \left[ 1 - \alpha^{m+k-2} \sum_{i=0}^{m-2} \left( \frac{\alpha}{1-\alpha} \right)^i \left( \binom{m+k-2}{i} \right) \right] \right].$$

This corollary was obtained by applying the following power series equivalence:

$$\sum_{i=0}^{a} x^i \binom{b-1+i}{b-1} = \frac{1}{(1-x)^b} - x^a \sum_{i=0}^{b-1} \left( \frac{x}{1-x} \right)^{b-i} \binom{a+b}{i}.$$  

The properties presented in the following two corollaries are mentioned in [4].

Corollary 5.4. $N_{n-k,k}$ is an $\alpha / \beta$ generalization of the Narayana number $N(n+1,k+1)$. That is, $N_{n-k,k}(\alpha = 1, \beta = 1) = \frac{1}{n+1} \binom{n+1}{k+1}$. 

Proof: A summation of the binomial coefficients yields

$$N_{n-k,k}(\alpha = 1, \beta = 1) = \sum_{j=0}^{n-k} \sum_{\ell=0}^{k} \left( \binom{k+j-1}{k-1} \binom{n-k+l-1}{n-k-1} - \binom{k+j-1}{k} \binom{n-k+l-1}{n-k} \right)$$

$$= \binom{n}{k} \binom{n}{n-k} - \binom{n}{k+1} \binom{n}{n-k+1}$$

$$= \frac{1}{n+1} \binom{n+1}{k+1} \binom{n}{k+1}.$$ 

Corollary 5.5. The total number of Catalan tableaux of size $n$ is the $n+1$st Catalan number, that is, $\sum_{k=0}^{n} N_{n-k,k}(1,1) = \frac{1}{n+2} \binom{2n+2}{n+1}$. 

Proof: Using Vandermonde’s convolution, we obtain from the above corollary:

$$\sum_{k=0}^{n} N_{n-k,k}(1,1) = \binom{2n}{n} - \binom{2n}{n+2} = \frac{1}{n+2} \binom{2n+2}{n+1}.$$ 

Remark. Corollary [5.4] can be extended to provide a $q$-refinement of the Narayana numbers by setting $\alpha = \beta = q$ or $\alpha = 1$ and $\beta = q$ in [3]. Table 1 shows some of the resulting $q$-polynomials.
| \(n\) | \(k\) | \(q^{-n}N_{n-k,k}(q,q)\) | \(q^{-n}N_{n-k,k}(q,1) = q^{-n}N_{k,n-k}(1,q)\) |
|---|---|---|---|
| 6 | 1 | \(2q^5 + 3q^4 + 4q^3 + 5q^2 + 6q + 1\) | \(q^5 + 2q^4 + 3q^3 + 4q^2 + 5q + 6\) |
| 6 | 2 | \(20q^5 + 30q^4 + 28q^3 + 20q^2 + 6q + 1\) | \(15q^4 + 24q^3 + 27q^2 + 24q + 15\) |
| 6 | 3 | \(40q^5 + 60q^4 + 48q^3 + 20q^2 + 6q + 1\) | \(50q^3 + 60q^2 + 45q + 20\) |
| 6 | 4 | \(20q^5 + 30q^4 + 28q^3 + 20q^2 + 6q + 1\) | \(50q^2 + 40q + 15\) |
| 6 | 5 | \(2q^5 + 3q^4 + 4q^3 + 5q^2 + 6q + 1\) | \(15q + 6\) |
| 7 | 1 | \(2q^6 + 3q^5 + 4q^4 + 5q^3 + 6q^2 + 7q + 1\) | \(q^6 + 2q^5 + 3q^4 + 4q^3 + 5q^2 + 6q + 7\) |
| 7 | 2 | \(30q^6 + 45q^5 + 46q^4 + 40q^3 + 27q^2 + 7q + 1\) | \(21q^5 + 35q^4 + 42q^3 + 42q^2 + 35q + 21\) |
| 7 | 3 | \(100q^6 + 150q^5 + 130q^4 + 75q^3 + 27q^2 + 7q + 1\) | \(105q^4 + 140q^3 + 126q^2 + 84q + 35\) |
| 7 | 4 | \(100q^6 + 150q^5 + 130q^4 + 75q^3 + 27q^2 + 7q + 1\) | \(175q^3 + 175q^2 + 105q + 35\) |
| 7 | 5 | \(30q^6 + 45q^5 + 46q^4 + 40q^3 + 27q^2 + 7q + 1\) | \(105q^2 + 70q + 21\) |
| 7 | 6 | \(2q^6 + 3q^5 + 4q^4 + 5q^3 + 6q^2 + 7q + 1\) | \(21q + 7\) |
| 8 | 1 | \(2q^7 + 3q^6 + 4q^5 + 5q^4 + 6q^3 + 7q^2 + 8q + 1\) | \(q^7 + 2q^6 + 3q^5 + 4q^4 + 5q^3 + 6q^2 + 7q + 8\) |
| 8 | 2 | \(42q^7 + 63q^6 + 68q^5 + 65q^4 + 54q^3 + 35q^2 + 8q + 1\) | \(28q^6 + 48q^5 + 60q^4 + 64q^3 + 60q^2 + 48q + 28\) |
| 8 | 3 | \(210q^7 + 315q^6 + 292q^5 + 205q^4 + 110q^3 + 35q^2 + 8q + 1\) | \(196q^5 + 280q^4 + 280q^3 + 224q^2 + 140q + 56\) |
| 8 | 4 | \(350q^7 + 525q^6 + 460q^5 + 275q^4 + 110q^3 + 35q^2 + 8q + 1\) | \(490q^4 + 560q^3 + 420q^2 + 224q + 70\) |
| 8 | 5 | \(210q^7 + 315q^6 + 292q^5 + 205q^4 + 110q^3 + 35q^2 + 8q + 1\) | \(490q^3 + 420q^2 + 210q + 56\) |
| 8 | 6 | \(42q^7 + 63q^6 + 68q^5 + 65q^4 + 54q^3 + 35q^2 + 8q + 1\) | \(196q^2 + 112q + 28\) |
| 8 | 7 | \(2q^7 + 3q^6 + 4q^5 + 5q^4 + 6q^3 + 7q^2 + 8q + 1\) | \(28q + 8\) |

Table 1: Specialization of the \(\alpha/\beta\)-Narayana numbers \(N_{n-k,k}(\alpha, \beta)\).
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