LONG WAVE LIMIT FOR SCHröDINGER MAPS

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Abstract. We study long wave limits for general Schrödinger maps systems into Kähler manifolds with a constraining potential vanishing on a Lagrangian submanifold. We obtain KdV type systems set on the tangent space of the submanifold. Our general theory is applied to study the long wave limits of the Gross-Pitaevskii equation and of the Landau-Lifshitz systems for ferromagnetic and anti-ferromagnetic chains.

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1. Introduction

In this paper, we shall analyze a long wave limit problem for general Schrödinger map systems into Kähler manifolds. More precisely, we shall study the long wave limit in the presence of a nonlinear confining potential vanishing on a Lagrangian submanifold $L$. The "wave" regime, that is to say long time dynamics for data close to $L$ was studied in [31] where it was proven that the limit system is given by the wave map valued in the Lagrangian submanifold $L$. The aim of the present paper is to study the asymptotic behaviour on a longer time scale for data close to a point (that we denote by 0 without loss of generality) of $L$. We shall derive a system of KdV-type equations taking values in the tangent space $T_0L$.

Our motivation comes from the study of the Landau-Lifshitz systems and we shall apply our theory to these models. Our general equation is also linked to the study of long wave limits for Gross-Pitaevskii-type equations, that has received a lot of interest recently [7, 8, 10, 11] (the set up considered in these papers can be seen as a special case of the point of view adopted in the present article: namely choosing $M$ to be the Euclidean space $\mathbb{R}^2$, usually identified to $\mathbb{C}$, and the Lagrangian submanifold $L$ to be the unit circle). These authors derive the KdV equation, which was first proposed in [26]. Rigorous derivations of KdV-type limits have been established for various other equations, for example: general hyperbolic systems [4, 5], water-waves [2, 12], the Euler-Poisson [17, 27] or Vlasov-Poisson systems [18] for plasma, the Euler-Korteweg model for capillary fluids [6].

While all the results that were mentioned above concern scalar-valued KdV equation, our general framework leads to vector-valued KdV equations. Instances of vector-valued KdV equation are known: the Hirota-Satsuma system [19] was introduced based on its interesting integrability properties, see also [33] for its generalization; and the Gear-Grimshaw system [9, 15] to model the interaction of internal waves in a fluid. The model which we derive turns out to be related to the Gear-Grimshaw system, this point will be discussed in Section 6.

1.1. The geometric picture. We consider $M$ a $2d$ dimensional-Kähler manifold, and denote by $\nabla$ the connection compatible with the Riemannian metric (that we denote $\langle , \rangle$ or simply $\cdot$), and by $i$ the complex structure. The system that we shall study is of the form

$$\partial_s \Gamma = i \left( \frac{1}{2} \nabla y \partial_y \Gamma + B(\Gamma) \partial_y \Gamma - V'(\Gamma) \right)$$

(1.1)

where the unknown $\Gamma$ is a map, $\Gamma : \mathbb{R}_+ \times \mathbb{R} \to M$. We use the notation $\nabla_y$ for the riemannian connection on $\Gamma^{-1}TM$, the pull-back of $TM$ by $\Gamma$; it is uniquely defined by the condition that for a vector field $Y(y)$ along the curve $y \mapsto \Gamma(s, y)$, we have $\nabla_y Y(y) = (\nabla_{\partial_y} X) |_{\Gamma(s, y)}$ for any vector field $X$ on $M$ such that $X(\Gamma(s, y)) = Y(y), \forall y$. In the zero order term, we use the notation $V'(\Gamma)$ for the Riemannian gradient of the map $V : M \to \mathbb{R}$. For the first order term, $B(\Gamma) : T_{\Gamma}M \to T_{\Gamma}M$ is a smooth skew-symmetric tensor ($B^* = -B$). When there exists a vector field $W(\Gamma)$ such that

$$B(\Gamma) = \nabla W(\Gamma)^* - \nabla W(\Gamma)$$

(hence in particular when $B = 0$) the above system is formally the Hamiltonian flow of the energy functional

$$E(\Gamma) = \frac{1}{4} \int_{\mathbb{R}} |\partial_y \Gamma|^2 \, dy - \int_{\mathbb{R}} \langle W(\Gamma), \partial_y \Gamma \rangle \, dy + \int_{\mathbb{R}} V(\Gamma) \, dy.$$  

(1.2)
given the symplectic form on $L^2(\Gamma^{-1}TM)$

$$\omega(X, Y) = \int_{\mathbb{R}} \langle iX, Y \rangle \, dy.$$ 

As in [31], we shall assume that the potential $V$ is smooth and confining to a Lagrangian submanifold $L$:

$$V(p, N) \geq 0 \text{ on } M, \quad V(\Gamma) = 0 \text{ on } L \quad \text{and} \quad V''(p)|_{N_pL \times N_pL} = 2\lambda I, \text{ for some } \lambda > 0 \quad (H1)$$

where we use the notation $T_pL$ and $N_pL$ for the tangent and normal spaces to $L$ at $p$, which are subspaces of $T_pM$. We also use the notation $V''$ as a short hand for $\nabla V'$. Compared to [31], we make slightly stronger assumptions than in [31] in order to handle our asymptotic regime. Note that the case $B = 0$ is already interesting and will allow to handle many physical examples. Since $iB$ is skew symmetric, the first order term in (1.1), must be dominated by the two other terms in order to get a well-posed problem. This is the reason for the assumption $\mu < \lambda$ (see the energy estimates in the proof for more details).

It will be convenient to introduce a number $c > 0$ defined by

$$c^2 = \lambda - \mu \quad (1.3)$$

which will play the role of a sound speed.

To define the long wave regime that we will study, it is convenient to use the geodesic normal coordinate system in the vicinity of $L$ (we shall study precisely this coordinate system and its properties in section 5.1). Let us choose an arbitrary point of $\Gamma$.

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To define the long wave regime that we will study, it is convenient to use the geodesic normal coordinate system in the vicinity of $L$ (we shall study precisely this coordinate system and its properties in section 5.1). Let us choose an arbitrary point of $\Gamma$ that we denote by 0, we can then define a coordinate system for any $\Gamma$ in the vicinity of zero by $\Gamma = \Psi(p, N)$ with $p \in L$, $N \in N_pL$ and $\Psi(p, N) = \exp_M^p(N)$ with $\exp_M$ the exponential map on $M$. In this coordinate system, we obtain that $V$ is of the form

$$V(\Psi(p, N)) = \lambda|N|^2 + V_1(p)(N, N, N) + V_2(p, N), \quad (1.4)$$

where $V_1$ is smooth in $p$ and trilinear and symmetric in $N$, and $V_2$ is a smooth function of size $O(|N|^4)$, which will not contribute in the analysis.

1.2. Relevant scalings. To get the wave regime that was studied in [31], we look for solutions of (1.1) under the form

$$\Gamma(s, y) = u^\varepsilon(\varepsilon s, \varepsilon y), \quad u^\varepsilon(t, x) = \Psi(p^\varepsilon(t, x), \varepsilon n^\varepsilon(t, x)).$$

This yields the rescaled equation for $u^\varepsilon$

$$\partial_t u^\varepsilon = i\left(\frac{\varepsilon^2}{2} \nabla_x \partial_x u^\varepsilon + B \partial_x u^\varepsilon - \frac{1}{\varepsilon} V'(u^\varepsilon)\right), \quad u^\varepsilon = \Psi(p^\varepsilon, \varepsilon n^\varepsilon). \quad (1.5)$$

With our assumption on the shape of the nonlinear term $V(u)$, we get at leading order in $\varepsilon$ the system

$$\partial_t p - iB \partial_x p = -2i\lambda n, \quad \nabla^\perp_i n - iB \nabla^\perp_x n = \frac{1}{2} i(\nabla^\perp_x \partial_x p). \quad (1.6)$$

Here $\nabla^\perp$ and $\nabla^\perp$ stand for the covariant derivatives respectively on the tangent and normal bundles of $L$, and the two equalities above hold in $T_pL$ and $N_pL$ respectively; more details on these objects
and on the way to perform these computations will be given below. Note that since we assumed that \( L \) is Lagrangian, \( i \) maps \( T_pL \) on \( N_pL \) and vice-versa and that thanks to (H2), we have that on \( L, T_pL \) and \( N_pL \) are stable subspaces of \( iB \). By applying \( \nabla_t^{\top} - Bi\nabla_x^{\top} \) to the first equation of (1.6), we get (since \( \nabla i = 0 \)) that
\[
(\nabla_t^{\top} - Bi\nabla_x^{\top})(\partial_i - iB\partial_x)p = -2i(\nabla_t^{\top} - iB\nabla_x^{\top})n = \lambda \nabla_x^{\top}\partial_xp.
\]
Next,
\[
(\nabla_t^{\top} - Bi\nabla_x^{\top})(\partial_i - iB\partial_x)p = \nabla_t^{\top}\partial_tp - (Bi + iB)\nabla_x^{\top}\partial_xp - B^2\nabla_x^{\top}\partial_xp = \nabla_t^{\top}\partial_tp + \mu \nabla_x^{\top}\partial_xp
\]
thanks to (H2). Combining these two equalities, we obtain that \( p \in L \) solves
\[
\nabla_t^{\top}\partial_tp = c^2\nabla_x^{\top}\partial_xp
\]
with \( c \) defined by (1.3) which is a wave map system on \( L \).

The rigorous justification of this regime was performed in [31]. In particular, it is proven that in this regime smooth solutions of (1.5) exist on \([0,T]\) with \( T \) independent of \( \varepsilon \) and converge to the solutions of the wave map system.

Such a one-dimensional wave equation gives rise to one wave moving to the left, and one moving to the right; our aim here will be to study the dynamics of one of these waves on longer times in the case where the amplitude is smaller. We shall study the wave moving to the right close to some point of \( L \) that we denote by zero. We use the ansatz
\[
\Gamma(s, y) = u^\varepsilon(\varepsilon^3s, \varepsilon(y - cs)), \quad u^\varepsilon(t, x) = \Psi(p^\varepsilon, \varepsilon^2n^\varepsilon), \quad p = \Phi(\varepsilon\phi^\varepsilon)
\]
where \( \Phi \) is the exponential map at 0 on \( L \): \( \Phi = \exp_0^\varepsilon \). We are thus studying the wave moving to the right in a smaller amplitude regime (\( \varepsilon^2 \) instead of \( \varepsilon \) in the direction normal to \( L \)) and on a much longer time scale \( (1/\varepsilon^2) \) against \( 1/\varepsilon \). We get for \( u^\varepsilon \) the system
\[
\varepsilon^2\partial_t u^\varepsilon - c\partial_x u^\varepsilon - iB\partial_x u^\varepsilon = i\left(\frac{\varepsilon}{2}\nabla_x\partial_x u^\varepsilon - \frac{1}{\varepsilon}V'(u^\varepsilon)\right).
\]

### 1.3. Uniform well-posedness
We shall first prove that for appropriate initial data, smooth solutions of (1.8) exist on an interval of time \([0,T]\) with \( T \) independent of \( \varepsilon \) and satisfy uniform estimates. More precisely, we will also justify that on \([0,T]\), the representation
\[
u^\varepsilon(t, x) = \Psi(p^\varepsilon(t, x), \varepsilon^2n^\varepsilon(t, x)), \quad p^\varepsilon(t, x) = \Phi(\varepsilon\phi^\varepsilon(t, x))
\]
holds true and that \( \phi^\varepsilon \) and \( n^\varepsilon \) also satisfy uniform estimates. The following statement involves the energy functional \( E_s(\phi^\varepsilon, n^\varepsilon, t) \), which will be defined in Section 3 (see (1.3), (1.4)). By a slight abuse of notation, we shall often use the notation \( E_s(u^\varepsilon, t) \) in place of \( E_s(\phi^\varepsilon, n^\varepsilon, t) \). Without going into details for the moment, it satisfies
\[
||\partial_x\phi^\varepsilon(t)||_{H^s} + ||n^\varepsilon(t)||_{H^s} \leq E_s(\phi^\varepsilon, n^\varepsilon, t),
\]
and the reader can heuristically think of \( E_s \) as being equivalent to \( ||\partial_x\phi^\varepsilon(t)||_{H^s} + ||n^\varepsilon(t)||_{H^s} \).

We shall fix \( r > 0 \) such that \( \Phi = \exp_0^\varepsilon \) is well defined on \( B(0, r) \) (the ball in \( \mathbb{R}^d \)) and a diffeomorphism on its image.

**Theorem 1.1** (Uniform existence). Assume that (H1) and (H2) hold, and let \( s \geq 1, c_0 > 0 \) such that \( 2c_0 < r \). For every \( M > 0 \) such that
\[
E_s((\phi^\varepsilon, n^\varepsilon), t = 0) \leq M, \quad \varepsilon||\phi^\varepsilon||_{L^\infty} \leq c_0, \quad \forall \varepsilon \in (0, 1]
\]
there exists \( T > 0, E > 0 \) and \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0] \), we have a unique solution
\[
u^\varepsilon(t, x) = \Psi(p^\varepsilon, \varepsilon^2n^\varepsilon), \quad p^\varepsilon = \Phi(\varepsilon\phi^\varepsilon)
\]
of (1.8) on \([0,T]\) that satisfies the uniform estimates
\[
E_s((\phi^\varepsilon, n^\varepsilon), t) \leq E, \quad \varepsilon||\phi^\varepsilon(t)||_{L^\infty} \leq 2c_0 \quad \forall t \in [0,T].
\]
The estimate on the $L^\infty$ norm of $\varepsilon \phi^\varepsilon$ in the previous statement is needed in order to use the exponential coordinates in a vicinity of the reference point 0 on $L$.

The most difficult step in the proof of the above result is to provide a priori uniform estimates on $\phi^\varepsilon$ and $n^\varepsilon$. When dealing with the Gross-Pitaevskii equation, as in [11] for example, these estimates can be obtained by using the hydrodynamical form of the equation sometimes called quantum Euler equation which follows from the Madelung transform (or its modified version due to Grenier [16]).

In our more general framework, the representation $u^\varepsilon = \Psi(p^\varepsilon, \varepsilon^2 n^\varepsilon)$ can be thought of as a generalized Madelung transform. The first step in the analysis is thus to derive a suitable hydrodynamical system on $(p^\varepsilon, n^\varepsilon)$ (we can drop the subscript $\varepsilon$ for the sake of clarity for the moment) and to study its properties. It turns out there are several new difficulties that do not occur in the study of the standard hydrodynamical system derived from Gross-Pitaevskii, the main one being that our hydrodynamical system suffers from a lack of symmetry away from $L$. As a consequence, we are only able to prove an estimate on the hydrodynamical system with a loss of derivatives, but a gain in $\varepsilon$. In order to compensate for this loss of derivatives, we add to the natural energy for the hydrodynamical system, $\mathcal{E}_{s,1}$, a correction $\mathcal{E}_{s,2}$, and define

$$\mathcal{E}_{s}(u^\varepsilon, t) \overset{\text{def}}{=} \mathcal{E}_{s,1}(u^\varepsilon, t) + \mathcal{E}_{s,2}(u^\varepsilon, t).$$

Our plan is as follows:

- The control of the energy $\mathcal{E}_{s,1}$ is obtained by working directly on the Schrödinger map system in the spirit of [31]. This quantity controls higher order derivatives but has a singular behaviour in $\varepsilon$.
- Then we derive the hydrodynamical system and use it to estimate $\mathcal{E}_{s,2}$ that gives a control of $\|\partial_x \phi^\varepsilon\|_s + \|n^\varepsilon\|_s$. The terms with loss of derivatives that occur are controlled thanks to $\mathcal{E}_{s,1}$.

All these estimates are valid if $\|\varepsilon \phi^\varepsilon\|_{L^\infty} < r$ so that the exponential on $L$ is also well defined. In order to close the argument here, we thus have to estimate $\|\varepsilon \phi^\varepsilon\|_{L^\infty}$. Note that this estimate was not needed in the analysis for Gross-Pitaevskii: in this case, the hydrodynamical system does not involve $\phi^\varepsilon$ (only $\partial_x \phi^\varepsilon$ which plays the role of the fluid velocity and higher derivatives) and, since $L$ is the unit circle, the exponential is globally defined. The estimate does not follow from the control of $\mathcal{E}_s$ and Sobolev embeddings since $\mathcal{E}_s$ only controls $\|\partial_x \phi^\varepsilon\|_s$. Also, note that by direct time integration of the equation for $\phi$ in the hydrodynamical system, we only easily get an estimate of $\|\varepsilon^2 \phi^\varepsilon\|_{L^\infty}$. We will proceed as follows.

- To estimate $\|\varepsilon \phi^\varepsilon\|_{L^\infty}$, we use the hydrodynamical system to get that, up to well controlled terms, $\|\varepsilon \phi^\varepsilon\|_{L^\infty([0,T] \times \mathbb{R})}$ can be estimated by $\sup_x \int_0^T \frac{1}{\varepsilon} |W^\varepsilon(s, x)| ds$ where

$$W = (c + iB) D \Phi \partial_x \phi^\varepsilon - 2i \lambda n^\varepsilon.$$

We can control this quantity uniformly in $\varepsilon$ by observing that $|W^\varepsilon|^2$ solves a transport equation at speed $\varepsilon^{-2}$.

Let us now explain a little more the derivation of the hydrodynamical system, at least in a simplified framework. When deriving the hydrodynamical system and studying its structure, both the geometry of $L$ and the geometry of $\mathcal{M}$ play a role, together with the local structure of the potential $V$. In particular, in order to get the hydrodynamical system in the general case, we need to use a connection on $NL$ and to understand how the Schrödinger map system can be split into a tangential and a normal equation away from $L$, in such a way that $i$ still exchanges the tangential and the normal directions.

We begin with the study of the simpler case in which $L$ is a Lagrangian submanifold of $\mathbb{R}^{2d}$ (for instance with the complex structure of $\mathbb{C}^d$). In this case, the representation $u = \Psi(p^\varepsilon, \varepsilon^2 n^\varepsilon)$, $p^\varepsilon = \Phi(\varepsilon \phi^\varepsilon)$ is much simpler since it reads

$$u^\varepsilon = \Phi(\varepsilon \phi^\varepsilon) + \varepsilon^2 n^\varepsilon.$$
(indeed, the geodesics are straight lines in $\mathbb{R}^{2d}$ and thus only the exponential on $\mathcal{L}$ is nonlinear). It is then rather straightforward to get a hydrodynamical system by plugging this ansatz into (1.8) and by splitting the system into the tangential part and the normal part to $\mathcal{L}$. We obtain the system

$$
\begin{align*}
S_1 \left( \partial_t \phi^\varepsilon - \frac{c}{\varepsilon^2} \partial_x \phi^\varepsilon \right) - \frac{1}{\varepsilon^2} iB S_1 \partial_x \phi^\varepsilon &= i \left[ \frac{1}{2} \Pi^\perp (S_1 \partial_x \phi^\varepsilon, D\Phi \partial_x \phi^\varepsilon) + \frac{1}{2} (\nabla_x^\perp)^2 n^\varepsilon - 2 \lambda \frac{n^\varepsilon}{\varepsilon^2} \right. \\
\nabla_x^\perp n^\varepsilon - \frac{c}{\varepsilon^2} \nabla_x^\perp n^\varepsilon - \frac{1}{\varepsilon^2} iB \nabla_x^\perp n^\varepsilon &= i \left[ \frac{1}{2 \varepsilon^2} \nabla_x^\top (S_1 \partial_x \phi^\varepsilon) + \frac{1}{2} \Pi^\top (D\Phi \partial_x \phi^\varepsilon, \nabla_x^\perp n^\varepsilon) \right. \\
&\quad \left. - \frac{1}{\varepsilon^2} P^\top R^\perp (p^\varepsilon, \varepsilon^2 n^\varepsilon) \right]
\end{align*}
$$

where we denote

$$
S_1 \overset{\text{def}}{=} S_0 D\Phi \quad \text{with} \quad S_0 \overset{\text{def}}{=} \text{Id} + \varepsilon^2 \Pi^\top (\cdot, n^\varepsilon)
$$

and refer to the subsection 3.1.1 for the precise definition of the second fundamental form of the tangent bundle $\Pi^\perp$; the second fundamental form of the normal bundle $\Pi^\perp$; and the connection on the normal bundle $\nabla^\perp$. In this system these objects are always computed at the point $p = \Phi(\varepsilon \phi^\varepsilon) \in \mathcal{L}$. The terms involving $R^\perp$ are harmless terms that come from the higher order terms in the potential.

The case where $\mathbb{R}^{2d}$ is replaced by a general Kähler manifold $\mathcal{M}$ is more complicated since one needs to define appropriate generalizations of the tangential and normal projections away from $\mathcal{L}$.

### 1.4. Derivation of the vector KdV equation.

The uniform estimates of Theorem 1.1 are the key to the following result that justifies rigorously the KdV asymptotics. Before stating the result, recall that $\Pi^\perp_p$ is the second fundamental form of the tangent bundle of $\mathcal{L}$. Its definition and basic properties are recalled in section 3. In view of the expansion (1.4) of the potential, it is convenient to define $F_1(p)(N_1, N_2) \in N_p \mathcal{L}$ by the formula:

$$
F_1(p)(N_1, N_2) \cdot N_3 \overset{\text{def}}{=} 3V_1(p)(N_1, N_2, N_3), \quad \forall N_1, N_2, N_3 \in N_p \mathcal{L}. \tag{1.10}
$$

The second main result of this paper is the following.

**Theorem 1.2** (KdV limit). Assume (1.11), (1.12) and that, for some $s \geq 3$, and for some $M > 0$, we have the uniform estimate

$$
\mathcal{E}_s(u^\varepsilon, t = 0) \leq M, \quad \forall \varepsilon \in (0, 1].
$$

Assume furthermore that the initial data $u^\varepsilon_0 = \Psi(\Phi(\varepsilon \phi^\varepsilon_0), \varepsilon^2 n^\varepsilon_0)$ is such that $\varepsilon \phi^\varepsilon_0 \to 0$ in $L^\infty(\mathbb{R})$ and that there exists $A_0 \in L^2(\mathbb{R}, T_0 \mathcal{L})$ such that

$$
\varepsilon^2 \phi^\varepsilon_0 \to 0, \quad (c + iB)\Phi(\varepsilon \phi^\varepsilon_0)D\Phi \phi^\varepsilon_0 \to A_0, \quad 2i\lambda n^\varepsilon \to A_0
$$

in $L^2(\mathbb{R})$ when $\varepsilon$ tends to zero. Then $(c + iB)\Phi(\varepsilon \phi^\varepsilon)D\Phi \phi^\varepsilon$ and $2i\lambda n^\varepsilon$ converge to $A \in T_0 \mathcal{L}$ in $C([0, T], L^2)$ where $A$ is the unique solution of the KdV type system

$$
2c\partial_t A = \frac{1}{4} \partial_{xxx} A + \left( \frac{3}{2} - \frac{2\mu}{\lambda} - \frac{2c}{\lambda} i_0 B_0 \right) i_0 \Pi^\perp_0 (\partial_x A, A) - \frac{i_0}{2\lambda} F_{1,0}(i_0 \partial_x A, i_0 A), \tag{1.12}
$$

with initial data

$$
A(t = 0) = A_0.
$$
In the above system, the subscript 0 indicates that all the involved tensors are evaluated at 0, and we have set $F_{1,0} = F_{1}(0)$. The convergence in (1.11) should be understood in a natural way, either by embedding $\mathcal{M}$ (locally) in an Euclidean space, or by identifying tangent spaces through parallel transport.

Let us now explain how, at least formally, we expect that the limit evolves according to (1.12); we will focus on the simpler framework where the Kähler manifold is Euclidean, and the hydrodynamical system is simply given by (1.9). Due to the singular $\varepsilon^{-2}$ terms, we expect that in the limit, if $(\partial_x \phi^\varepsilon, n^\varepsilon)$ converges to $(\partial_x \phi, n)$, where

$$(c + iB)D\Phi \partial_x \phi = 2i\lambda n \overset{\text{def}}{=} A.$$  

In order to get the equation satisfied by $A$, we can apply $\nabla x^\top$ to the first equation of (1.9). The system then becomes

$$\begin{cases}
\nabla x^\top (D\Phi \partial_x \phi^\varepsilon) = \frac{1}{\varepsilon^2} \nabla x^\top W^\varepsilon + \frac{1}{2} (\nabla x^\top)^3 (in^\varepsilon) - 2iF_1(\nabla x^\top n^\varepsilon, n^\varepsilon) \\
+ i\Pi^\perp (D\Phi \partial_x \phi^\varepsilon, D\Phi \partial_x \phi^\varepsilon) - 4\lambda i\Pi^\perp (\nabla x^\top (in^\varepsilon), in^\varepsilon) + O(\varepsilon)
\end{cases}$$

$$\nabla x^\top (2\lambda in^\varepsilon) = -\frac{1}{\varepsilon^2} (c - iB) \nabla x^\top W^\varepsilon + 2i\lambda \Pi^\perp (D\Phi \partial_x \phi^\varepsilon, \nabla x^\top in^\varepsilon) + i\lambda \Pi^\perp (\nabla x^\top (D\Phi \partial_x \phi^\varepsilon), in^\varepsilon) + O(\varepsilon),$$

where $W^\varepsilon = (c + iB) D\Phi \partial_x \phi^\varepsilon - 2i\lambda n^\varepsilon$. Multiplying the first line by $(c - iB)$, and adding it to the second line, the singular $\frac{1}{\varepsilon^2}$ terms cancel, giving the equation

$$\nabla x^\top ((c - iB) D\Phi \partial_x \phi^\varepsilon + 2\lambda in^\varepsilon) = (c - iB) \left[ \frac{1}{2} (\nabla x^\top)^3 (in^\varepsilon) - 2iF_1(\nabla x^\top n^\varepsilon, n^\varepsilon) \right]$$

$$+ i\Pi^\perp (\nabla x^\top (D\Phi \partial_x \phi^\varepsilon), D\Phi \partial_x \phi^\varepsilon) - 4\lambda i\Pi^\perp (\nabla x^\top (in^\varepsilon), in^\varepsilon)$$

$$+ 2i\lambda \Pi^\perp (D\Phi \partial_x \phi^\varepsilon, \nabla x^\top in^\varepsilon) + i\lambda \Pi^\perp (\nabla x^\top (D\Phi \partial_x \phi^\varepsilon), in^\varepsilon) + O(\varepsilon).$$

If $\varepsilon \phi$ tends to zero and $(\partial_x \phi^\varepsilon, n^\varepsilon)$ converges to $(\partial_x \phi, n)$ sufficiently strongly with the constraint $(c + iB) D\Phi \partial_x \phi = 2i\lambda n = A$, by using some algebraic manipulations in order to express all the quantities with respect to $A$, we finally get (1.12).

In order to justify rigorously the above derivation, the main difficulty is to prove that $\varepsilon\|\phi^\varepsilon\|_{L^\infty}$ tends to zero and to get strong compactness for $n^\varepsilon$ and $\partial_x \phi^\varepsilon$ (the space compactness is a direct consequence of the uniform estimates of Theorem 1.1; the difficulty is the time compactness) in order to pass to the limit in (1.13). We point out again that in our geometric setting all the tensors involved in the hydrodynamical system are taken at $\varepsilon \phi$, hence we really need to prove a strong convergence of $\varepsilon \phi$ in order to pass to the limit.

To prove that $\varepsilon\|\phi^\varepsilon\|_{L^\infty}$ tends to zero, we use again the hydrodynamical system and the link with $W^\varepsilon = (c + iB) D\Phi \partial_x \phi^\varepsilon - 2i\lambda n^\varepsilon$. We first prove that it suffices to get that $\sup_{[0,T]} \|W^\varepsilon\|_{L^2}$ tends to zero. To get this we can use the conserved quantities of (1.1) (at least in the case when $B = \nabla W^\ast - \nabla W$). Note that $\|W^\varepsilon\|_{L^2}^2$ is of order $\sim \varepsilon^4$ and, to leading order in $\varepsilon$, conserved by the flow of (1.1). Indeed,

$$\frac{1}{\varepsilon^4} \|W^\varepsilon\|_{L^2}^2 = 4\lambda E + 4\lambda c P + O(\varepsilon),$$

where the energy $E$ and the momentum $P$

$$\begin{cases}
E = \frac{1}{4} \int [\varepsilon^2 |\partial_x u|^2 + V(u) + \varepsilon W(u) \cdot \partial_x u] \, dx \\
P = \int u \cdot \i \partial_x u \, dx
\end{cases}$$

are conserved quantities of (1.1). By assumption on the data $(\phi_0, n_0)$, we thus get that

$$\frac{1}{\varepsilon^4} \|W^\varepsilon(t)\|_{L^2}^2 = \frac{1}{\varepsilon^4} \|W^\varepsilon(0)\|_{L^2}^2 + O(\varepsilon) = o(1).$$
This provides the desired estimate on $W^\varepsilon$ which in turn implies that $\varepsilon \| \phi \|_{L^\infty}$ tends to zero uniformly on $[0,T]$.

Once we have obtained this estimate, we can use again the hydrodynamical system (1.9) to also obtain strong compactness for the quantity $U^\varepsilon = (c-iB)\partial_{\varepsilon} \partial_x \phi^\varepsilon + 2i\lambda n^\varepsilon$. This is a consequence of the fact that $\nabla^\perp_t U^\varepsilon = O(1)$. This in turn allows to justify the convergence towards a solution of (1.12) from some algebraic manipulations and standard (weak) convergence arguments.

1.5. **Organization of the article.**

- We show in Section 2 how our general theory can be used to obtain the long wave limit for the Gross-Pitaevskii equation as well as for the Landau-Lifshitz equations for ferromagnetic and anti-ferromagnetic chains.
- In Section 3, we recall some useful geometric facts and define the various norms and functionals that we will be using.
- In Section 4, we prove theorems 1.1 and 1.2 in the case where $\mathcal{M} = \mathbb{R}^{2d}$.
- Next, the proof of theorems 1.1 and 1.2 in the general case is presented in Section 5. The additional geometric notions that are needed for the proof are given in subsection 5.1.
- Finally, we say in Section 6 a few words about the properties of the KdV system that we derived.

2. **Examples**

In this section, we apply our general result to various physical situations

2.1. **The Gross-Pitaevskii equation.**

2.1.1. **Scalar case.** We consider first the Gross-Pitaevskii equation which is a classical model for nonlinear optics, superfluids and Bose-Einstein condensates (see [30] for a recent review) 

$$\partial_t \Gamma = i \left( \frac{1}{2} \partial_{xx} \Gamma + \Gamma (1 - |\Gamma|^2) \right), \quad t > 0, \quad x \in \mathbb{R}$$

(2.1)

where the unknown $\Gamma \in \mathbb{C}$.

We are thus in the case where $\mathcal{M}$ is the Euclidean space $\mathcal{M} = \mathbb{R}^2 \sim \mathbb{C}$, there is no first order terms so that the tensor $B$ is zero here and the Lagrangian submanifold is the unit circle, $\mathcal{L} = S^1$. The potential $V$ is given by 

$$V(\Gamma) = \frac{1}{4} (1 - |\Gamma|^2)^2.$$ 

Writing $\Gamma = p + n$ with $p \in S^1$ and $n \in N_p S^1$, we get

$$V(p + n) = (p \cdot n)^2 + p \cdot n |n|^2 + \frac{1}{4} |n|^4$$

and since in this simple situation $p = n/|n|$, this can also be written

$$V(p + n) = |n|^2 + p \cdot n |n|^2 + \frac{1}{4} |n|^4$$

which is under the form (1.4) with

$$\begin{cases} 
\Lambda = c = 1 \\
V_1(p)(n,n,n) = p \cdot n |n|^2 \\
F_1(p)(n_1,n_2) = (p \cdot n_1)n_2 + (p \cdot n_2)n_1 + (n_1 \cdot n_2)p.
\end{cases}$$

Moreover, the second fundamental form of $\mathcal{L} = S^1$ is given by

$$\Pi^\perp_p (X, Y) = -(X \cdot Y)p.$$
We use the KdV scaling (1.7) with base point 1 so that \( \Phi(z) = e^{i\phi} \). We can then write \( A \in T_1 S^1 \) as \( A = i\rho \), with \( \rho \in \mathbb{R} \). This gives
\[
\Pi_1^+(A, \partial_x A) = -\rho \partial_x \rho, \quad \text{and} \quad F_1(1) (iA, i\partial_x A) = 3\rho \partial_x \rho.
\]
By Theorem 1.2 we thus get that the long wave limit of the Gross-Pitaevskii equation (2.1) is described by the KdV equation
\[
2\partial_t \rho = \frac{1}{4} \partial_{xxx} \rho - 3\rho \partial_x \rho,
\]
recovering the result of [11], [7].

2.1.2. Vector case. We discuss here the case of two coupled Gross-Pitaevskii equations. It arises in Bose-Einstein condensates where two species are present [25] or in nonlinear optics [1]; the particular case of two coupled equations is sometimes called the Manakov equations. In general, it reads
\[
\partial_t \Gamma = i \left( \frac{1}{2} \Delta \Gamma - V'(\Gamma) \right),
\]
where \( \Gamma \) takes values in \( \mathcal{M} = \mathbb{C}^d \), and \( V(\Gamma) = G(\|\Gamma_1\|, \ldots, \|\Gamma_d\|) \), for \( G \) a function \([0, +\infty[^d \to \mathbb{R}\), hence
\[
V'(\Gamma) = \left( [\partial_1 G] \frac{\Gamma_1}{|\Gamma|}, \ldots, [\partial_d G] \frac{\Gamma_d}{|\Gamma|} \right).
\]
The associated Hamiltonian reads of course
\[
E(\Gamma) = \frac{1}{4} \int_\mathbb{R} |\partial_y \Gamma|^2 \, dx + \int_\mathbb{R} V(\Gamma) \, dx.
\]
Assume that \( G \) has a minimum at the point \( U^0 = (U^0_1, \ldots, U^0_d) \in (0, +\infty)^d \), with Hess \( G_{U^0} = 2\lambda \text{Id} \). Then \( V \) is minimal on the Lagrangian manifold \( \mathcal{L} = \{ [\Gamma_1] = U^0_1, \ldots, [\Gamma_d] = U^0_d \} \). Adopt now natural coordinates \((\phi_i, N_i)\) by decomposing \( \Gamma_i = U_i^0 e^{i\phi_i} + N_i \), where \( \phi_i \in \mathbb{R} \), and \( N_i \) is colinear to \( e^{i\phi_i} \). In these coordinates, \( V(\Gamma) \) admits the expansion
\[
V(\Gamma) = \lambda |N|^2 + F_1(N, N) \cdot N,
\]
for a function \( F_1 \) with the required symmetry.

It remains to describe the second fundamental form at the point \((U^0_1, \ldots, U^0_d) \in \mathcal{L}\). The tangent space is \( \sim i\mathbb{R} \times \cdots \times i\mathbb{R} \). For points \( r^1 = (ip_1, \ldots, ip_d) \) and \( r^2 = (ip_2, \ldots, ip_d) \) in the tangent space, we have as previously
\[
\Pi^{-1}(r^1, r^2) = (-\rho^1_1 \rho^2_1, \ldots, -\rho^1_d \rho^2_d).
\]
By using again the notation \( A = (ip_1, \ldots, ip_d)^t \), we obtain
\[
2c \partial_t \rho_k = \frac{1}{4} \partial_x^2 \rho_k - \frac{3}{2} \rho_k \partial_x \rho_k - \frac{1}{2\lambda} F_{1,0}(\rho, \partial_x \rho) , \quad k = 1, \ldots d
\]
with \( c = \sqrt{\lambda} \).

2.2. The Landau-Lifshitz equations for ferromagnetic chains.

2.2.1. General case. We quickly present the Landau-Lifshitz equation with only exchange and isotropy energies, referring to the textbook [20] for more. It describes, in the continuum approximation, the magnetic spin in a one-dimensional ferromagnetic chain and reads
\[
\partial_t \Gamma = \Gamma \times \left( \frac{1}{2} \partial_{xx} \Gamma - V'(\Gamma) \right)
\]
(2.2)
where \( \Gamma \in S^2 \) is the spin vector. We identify \( S^2 \) with the unit sphere, so that \( \Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) \in S^2 \) if and only if \( \Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 = 1 \). Again, we have \( B = 0 \) and the conserved energy for this equation is
\[
\int \left[ \frac{1}{2} |\partial_x \Gamma|^2 + V(\Gamma) \right] \, dx\]
where the term \( \frac{1}{2} |\partial_x \Gamma|^2 \) accounts for the exchange energy (molecular
magnetic fields tend to align) and \( V(\Gamma) \) for the anisotropy energy (in a crystal, not all directions of the molecular magnetic fields have the same energy).

This equation fits in our general framework by setting \( \mathcal{M} = S^2, \mathcal{L} = \{ V = 0 \} \) with the complex structure defined as \( i(u) = u \times \cdot \). For simplicity, we focus on the uniaxial case, in which the minimum of \( V \) is reached on sets of the form \( \{ \Gamma_3 = \gamma_0 \} \). We will distinguish two models: one with \( \gamma_0 = 0 \) ("easy plane anisotropy"), and one with \( \gamma_0 \neq 0 \) ("easy cone anisotropy").

### 2.2.2. Easy plane anisotropy.
We assume here that
\[
V(\Gamma) = \alpha \Gamma^2_3
\]
so that the Lagrangian submanifold is \( \mathcal{L} = S^1 \times \{ 0 \} \).

By using spherical coordinates, we have for \( p \in S^1 \) and \( n \in \mathbb{R} \)
\[
\Psi(p, n) = \cos(n) p + \sin(n) e_3
\]
and thus
\[
V(\Psi(p, n)) = \alpha^2 (\sin n)^2 = Kn^2 + O(n^4).
\]

This is under the form (1.4) with \( c = K \) and \( V_1 = 0 \). In the KdV regime (1.7), we can take \((1, 0, 0)^t\) as our reference point and set \( p = \Phi(\varepsilon \phi) = (\cos \varepsilon \phi, \sin \varepsilon \phi, 0)^t \). We note that the second fundamental form of the tangent bundle of \( S^1 \times \{ 0 \} \) as a submanifold of \( S^2 \) vanishes, \( \Pi^\perp = 0 \). Consequently, we get from Theorem 1.2 that the asymptotic regime is described by the linear KdV equation (Airy equation)
\[
2 \partial_t A = \frac{1}{4} \partial_x^3 A.
\]

A different scaling allowing larger data on shorter times has been studied in [10] in order to get nonlinear effects in the limit.

### 2.2.3. Easy cone anisotropy.
We assume here that
\[
V(\Gamma) = V(\Gamma_3)
\]
is nonnegative, equal to zero for \( \Gamma_3 = \gamma_0 \in [0, 1] \), and admits the following expansion
\[
V(\gamma_0 + s) = \alpha s^2 + \beta s^3 + O(s^4).
\]

Define \( \theta_0 \) by \( \cos \theta_0 = \gamma_0 \). For \( p \in \mathcal{L} = \{ \Gamma_3 = \gamma_0 \} \cap S^2 \), define \( N_0 \) to be the unit vector in \( N_p\mathcal{L} \) such that \( N_0 \cdot e_3 < 0 \). For \( s \in \mathbb{R} \), the map \( \Psi \) is then given by
\[
\Psi(p, sN_0) = \sin(\theta_0 + s) q + \cos(\theta_0 + s) e_3 \quad \text{with} \quad q = \frac{p - p \cdot e_3}{|p - p e_3|},
\]
and \( V \) can then be expanded as
\[
V(\Psi(p, sN_0)) = V(\cos(\theta_0 + s)) = \alpha (\sin \theta_0)^2 s^2 + (\alpha \sin \theta_0 \cos \theta_0 + \beta (\sin \theta_0)^3) s^3 + O(s^4).
\]
This means that
\[
F_1(N_1, N_2) = 3b(N_0 \cdot N_1)(N_0 \cdot N_2)N_0.
\]
On the other hand, a computation gives
\[
\Pi^\perp(X, Y) = - \cot \theta_0 (X \cdot Y)N_0.
\]
Therefore, Theorem 1.2 gives the following equation in the long-wave limit:
\[
2c \partial_t A = \frac{1}{4} \partial_x^3 A + \left( \frac{3}{2} \cot \theta_0 + \frac{3b}{2\lambda} \right) A \partial_x A.
\]
2.3. The Landau-Lifshitz equations for anti-ferromagnetic chains. The continuum limit for antiferromagnetic chains is described by the Landau-Lifshitz system

\[
\begin{align*}
\partial_t u &= u \times \left( -\frac{1}{2} \partial_{xx} u - \partial_x v + 2v \right), \\
\partial_t v &= v \times \left( -\frac{1}{2} \partial_{xx} v + \partial_x u + 2u \right)
\end{align*}
\]  

(2.3)

where the two unknowns \( u \) and \( v \) take values in \( S^2 \) (for the derivation of this equation, see \[29\], equation (3.4) taking into account that \( A = -B + O(\varepsilon) \)). This system also enters in our general framework with \( \Gamma = (u, v) \in \mathcal{M} = S^2 \times S^2 \). The complex structure is defined by \( i(\Gamma) \cdot (X, Y) = (u \times X, v \times Y)^t \), the tensor \( B \) is

\[
B(u, v)(X, Y) = (-P(u)Y, P(v)X), \quad \forall (X, Y) \in T_u S^2 \times T_v S^2
\]

where we denote by \( P(u) \) the orthogonal projection on the tangent space \( T_u S^2 \) (thus \( P(u)X = X - u \cdot X u, \quad \forall X \in \mathbb{R}^3 \)) and the potential \( V \) is given by

\[
V(\Gamma) = |u + v|^2
\]

so that \( \mathcal{L} \) is the anti diagonal \( \{ (u, v) \in \mathcal{M}, u + v = 0 \} \). Let us describe \( \Psi(p, n) \). We can write \( p \in \mathcal{L} \) under the form \( p = (\omega, -\omega), \omega \in S^2 \) and a normal vector \( n \) is under the form \( n = (X, X) \), \( X \in T_0 S^2 \). By choosing the axis of coordinates, we can always consider that \( \omega = (0, 0, 1)^t \) and by using spherical coordinates that \( X = \rho(\cos \phi, \sin \phi, 0)^t \). The geodesic on \( S^2 \) starting from \( \omega \) with initial speed \( X \) is thus given by

\[
\gamma(s) = (\sin(\rho s) \cos(\phi), \sin(\rho s) \sin(\phi), \cos(\rho s))^t.
\]

We thus get that

\[
\Psi(p, n) = (\psi(\omega, X), -\psi(\omega, -X))^t, \quad \psi(\omega, X) = (\sin(\rho) \cos(\phi), \sin(\rho) \sin(\phi), \cos(\rho))^t.
\]

Therefore, \( V \) can be expressed as

\[
V(\Psi(p, n)) = 4(\sin \rho)^2 = 4\rho^2 + O(\rho^4) = 2|n|^2 + O(|n|^4).
\]

Consequently, we have \( V_1 = 0 \) and \( \lambda = 2 \).

We note that on \( \mathcal{L} \), we have \( B(u, -u)(X, Y) = (-Y, X) \) and thus \( \mu = 1 \). This yields

\[
c^2 = \lambda - \mu = 1.
\]

In the KdV regime \[1.7\], we can take

\[
p_0 = ((1, 0, 0)^t, -(1, 0, 0)^t)
\]

as our reference point on \( \mathcal{L} \) for example and set \( p = (\omega, -\omega), \omega = \Phi(\varepsilon \phi), \Phi \) being the exponential map on \( S^2 \) at the point \( (1, 0, 0)^t \) (we do not need the precise expression). We also note that \( \mathcal{L} \) is a totally geodesic submanifold, therefore, \( \Pi^t = 0 \) on \( \mathcal{L} \). Consequently, we obtain from Theorem \[1.2\] that the long wave limit is described by the linear Airy system

\[
2\partial_t A = \frac{1}{4} \partial^3_{xx} A, \quad A \in \mathbb{R}^2.
\]

3. Preliminaries

3.1. Geometry. Consider \( \mathcal{M} \) a 2d-dimensional Kähler manifold. We denote its metric by \( (X, Y) \mapsto \langle X, Y \rangle \) or simply \( X \cdot Y \), its Levi-Civita connection by \( \nabla \), its Riemann curvature tensor by \( R \) and its complex structure by \( i \). The compatibility of \( i \) with the metric implies that

\[
\nabla i = 0 \quad \text{and} \quad \langle iX, iY \rangle = \langle X, Y \rangle \quad \text{for any} \ (X, Y) \in T\mathcal{M}.
\]
We also consider a Lagrangian submanifold \( L \) of \( M \). We denote \( T_pL \), respectively \( N_pL \), for the tangent, respectively normal, spaces of \( L \) as a submanifold of \( M \) at \( p \in L \). \( L \) being Lagrangian means that for any \( p \in L \),

\[ iT_pL = N_pL. \]

### 3.1. Covariant derivatives

We adopt the following notations:

- \( P^\top : T_pM \to T_pL \), respectively \( P^\perp : T_pM \to N_pL \), is the orthogonal projector on the tangent, respectively normal, space of \( L \).
- The covariant derivative on the tangent bundle of \( L \) reads \( \nabla^\top = P^\top \nabla \).
- The covariant derivative on the normal bundle of \( L \) reads \( \nabla^\perp = P^\perp \nabla \).
- We systematically abuse notations by not distinguishing between, say, the tangent space of \( L \) and its pull-back by a map. Assume for instance that \( f : t \mapsto f(t) \) is a map from \( \mathbb{R} \) to \( L \), and that \( X \) is a section of the pullback of \( TL \) by \( f \). In other words, \( X \) associates to each \( t \) in \( \mathbb{R} \) an element \( X(t) \) of \( T_{f(t)}L \). Denoting by \( \tilde{X} \) a vector field such that \( \tilde{X}(f(t)) = X(t) \), we will write

\[ \nabla_t X|_{t_0} = \nabla_{\partial_t f(t_0)} \tilde{X}|_{f(t_0)}. \]

#### 3.1.2. Differentiating tensors

Consider a tensor mapping, say, \((TL)^2 \) to \((NL) \). Then set, for \( U, V, W \) sections in \( TL \),

\[ \left[ \nabla^\perp_U A \right](V,W) \overset{def}{=} \nabla^\perp_U [A(V,W)] - A(\nabla^\perp_U V,W) - A(V,\nabla^\perp_U W). \tag{3.1} \]

This definition can be extended in an obvious way to general tensors. It will be also useful to view the covariant derivative of a tensor as a tensor with covariant index augmented by one. In the case of the above example, this yields

\[ (\nabla^\perp A)(U,V,W) \overset{def}{=} \left[ \nabla^\perp_U A \right](V,W). \]

Again this can be extended in an obvious way to general tensors.

#### 3.1.3. Second fundamental forms

The second fundamental form of \( TL \) is given by \((X \text{ and } Y \text{ being sections of } TL)\)

\[ \nabla_X Y = \nabla^\top_X Y + \Pi^\perp(X,Y) \quad \text{or} \quad \Pi^\perp(X,Y) = -\nabla P^\perp(X,Y). \tag{3.2} \]

The second fundamental form of \( NL \) is given by \((X \text{ and } N \text{ being sections of } TL \text{ and } NL \text{ respectively})\)

\[ \nabla_X N = \nabla^\top_X N + \Pi^\top(X,N) \quad \text{or} \quad \Pi^\top(X,N) = -\nabla P^\top(X,N). \tag{3.3} \]

**Proposition 3.1.** Let \( p \in L \), \( X, Y \in T_pL \) and \( N \in N_pL \). Denote simply \( \Pi^\top \) and \( \Pi^\perp \) for the second fundamental forms at \( p \). Then

1. \( \Pi^\perp(X,Y) = \Pi^\perp(Y,X) \).
2. \( i \Pi^\perp(X,N) = \Pi^\perp(X,iN) \).
3. \( \Pi^\top(\cdot, N) \) is symmetric on \( TL \) (for the metric scalar product).
4. \( i \Pi^\perp(\cdot, X) \) is symmetric on \( TL \) (for the metric scalar product).
5. \( \Pi^\top(i\cdot, X) \) is symmetric on \( NL \) (for the metric scalar product).
6. \( i \Pi^\top(X,\cdot) \) is symmetric on \( NL \) (for the metric scalar product).

As a corollary of these classical properties, we also obtain that

**Corollary 3.2.** Assuming \([H2]\), we have on \( L \)

1. \( iB \Pi^\top(X,N) = \Pi^\top(iBX,N) \), \( \forall X \in TL, \forall N \in NL \),
2. \( \Pi^\perp(X,iBY) = iB \Pi^\perp(X,Y) \), \( \forall X,Y \in TL \),
3. \( iB \Pi^\perp(X,\cdot) \) is symmetric on \( TL \) for the metric scalar product \( \forall X \in TL \).
Proof. Let us start with the first identity. We have thanks to the above properties and \[12\] that
\[iB \Pi^\top(X, N) = -Bi \Pi^\top(X, N) = B \Pi^\perp(X, iN) = B \Pi^\perp(iN, X) = BP^\perp \nabla_{iN}X\]
\[= -iP^\perp(iB) \nabla_{iN}X = -iP^\perp(iB) \nabla_{iN}(iBX) = -i \Pi^\perp(iN, iBX)\]
\[= -i \Pi^\perp(iBX, iN) = \Pi^\top(iBX, N).\]

For the second identity, it suffices to note that
\[\Pi^\perp(X, iBY) = P^\perp \nabla_X(iBY) = P^\perp(iB \nabla_X Y) = iBP^\perp(\nabla_X Y) = iB \Pi^\perp(X, Y).\]

For the last property, it suffices to combine (4) of Proposition \[3.1\] the previous identity and \[12\].

\[\square\]

Second fundamental forms can be differentiated as tensors. For instance, if \(U, V, W \in T\mathcal{L}\),
\[
\nabla_U \left[ \Pi^\perp(V, W) \right] = \left[ \nabla_U \Pi^\perp \right] (V, W) + \Pi^\perp \left( \nabla_U^\top V, W \right) + \Pi^\perp \left( V, \nabla_U^\top W \right)
\]
\[+ \Pi^\top \left( U, \Pi^\perp(V, W) \right). \tag{3.4}\]

where the first line gives the normal component, and the second the tangential one.

3.1.4. Normal coordinates on \(\mathcal{L}\). The coordinate system given by \(\Phi = \exp_0 : T_0 \mathcal{L} \to \mathcal{L}\) is normal at 0. It is well-known that in this coordinate system the Christoffel symbols vanish at 0. It can be expressed as
\[
\nabla^\top D\Phi|_0 = 0. \tag{3.5}\]

3.1.5. Commuting covariant derivatives with vector fields. For a coordinate system \((s, u)\), and \(F(s, u)\) a function valued on \(\mathcal{L}\), we get since the Levi-Civita connection is torsion free that
\[
\nabla^\top_s \partial_u F = \nabla_u^\top \partial_s F. \tag{3.6}\]

3.1.6. Commuting covariant derivatives. The tangent curvature tensor \(R^\top\) is defined by
\[
R^\top(X, Y)Z \overset{\text{def}}{=} \nabla_X \nabla^\top_Y Z - \nabla_Y \nabla^\top_X Z - \nabla^\top_{[X,Y]} Z \tag{3.7}\]
for \(X, Y, Z\) sections of the tangent bundle. It is given by the Gauss equation
\[
R^\top(X, Y)Z = P^\top R(X, Y)Z + \Pi^\top(Y, \Pi^\perp(X, Z)) - \Pi^\top(X, \Pi^\perp(Y, Z)).
\]

Similarly, the normal curvature tensor \(R^\perp\) is defined by
\[
R^\perp(X, Y)Z \overset{\text{def}}{=} \nabla_X \nabla^\perp_Y Z - \nabla_Y \nabla^\perp_X Z - \nabla^\perp_{[X,Y]} Z, \tag{3.8}\]
if \(X, Y\) are sections of the tangent bundle, and \(Z\) is a section of the normal bundle. It is given by
\[
R^\perp(X, Y)N = P^\perp R(X, Y)N + \Pi^\perp(Y, \Pi^\top(X, N)) - \Pi^\perp(X, \Pi^\top(Y, N)).
\]

3.2. Functional spaces. Recall first the classical Sobolev spaces. For a map \(F\) with values in \(\mathbb{R}^N\), for any \(s \in \mathbb{N}\), \(H^s\) is given by its norm
\[
\|F\|^2_{H^s} \overset{\text{def}}{=} \sum_{m \leq s} \|\partial_x^m F\|^2_{L^2}.
\]

For vector fields \(v \in u^{-1}T\mathcal{M}\), this definition is still valid since we can always assume that \(\mathcal{M}\) is embedded in \(\mathbb{R}^N\) as a Riemannian manifold. Nevertheless, it will be more convenient for us to use covariant derivatives in the definition:
\[
\|v\|^2_{H^s} \overset{\text{def}}{=} \sum_{m \leq s} \|
abla_x^m v\|^2_{L^2}.
\]
The two definitions coincide if $s > \frac{1}{2}$ and $\nabla u$ is at least as smooth as $v$, which will be the case for us below. A first variant which will be needed is $H^1_t$ whose norm reads

$$\| v \|_{H^1_t}^2 \overset{\text{def}}{=} \| v \|_{L^2}^2 + \varepsilon \| \nabla_x v \|_{L^2}^2.$$  

(3.9)

Next, we want to define Sobolev spaces which are anisotropic in space and time. First, let us set up our notation for multiindices: if $m = (m_0, m_1) \in \mathbb{N}^2$, define

$$\partial^m \overset{\text{def}}{=} (\varepsilon^2 \partial_t)^{m_0} (\partial_x)^{m_1}, \quad \nabla^m \overset{\text{def}}{=} (\varepsilon^2 \nabla_t)^{m_0} (\nabla_x)^{m_1}$$

and in a similar way

$$\left(\nabla^\top\right)^m \overset{\text{def}}{=} (\varepsilon^2 \nabla_t^\top)^{m_0} \left(\nabla^\top_x\right)^{m_1} \quad \text{and} \quad \left(\nabla^\perp\right)^m \overset{\text{def}}{=} (\varepsilon^2 \nabla_t^\perp)^{m_0} \left(\nabla^\perp_x\right)^{m_1}.$$  

The length of $m$ is denoted $|m| = m_0 + m_1$. When we do not want to keep track of the exact nature of the derivatives involved, but simply of their number, we shall abuse notations by writing $\partial^{\text{in}} m$ or $\nabla^{\text{in}} m$ instead of $\partial^m$ or $\nabla^m$. For instance,

$$\partial^2 f \quad \text{can denote} \quad \partial_x^2 f \quad \text{or} \quad \varepsilon^2 \partial_t \partial_x f.$$  

For maps $u(t, x) \in \mathcal{M}$, we define an anisotropic space-time semi norm $\| \cdot \|_{H^s}$, which we also abbreviate $\| \cdot \|_s$:

$$\| u(t) \|_{H^s}^2 = \| u(t) \|_{S}^2 \overset{\text{def}}{=} \sum_{|m| \leq s} \| \nabla^m u(t, \cdot) \|_{L^2(\mathbb{R})}^2.$$  

Note that this involves derivatives in time.

With the above definition of $\partial^m$, we record the following elementary product estimate in dimension 1, which we will use repeatedly:

$$\| \partial^m v(t) \partial^{m'} w(t) \|_{L^2(\mathbb{R})} \leq C \| v(t) \|_k \| w(t) \|_k, \quad |m| + |m'| \leq k, \quad k \geq 1$$  

(3.10)

with $C$ independent of $\varepsilon$ (again, recall that with our notation, $\partial^m$ depends on $\varepsilon$ when it involves time derivatives). The same estimate holds replacing $\partial^m$ by $\nabla^m$ for vector fields along $u^{-1}T \mathcal{M}$.

### 3.3. Notations

If $A$ and $B$ are two numbers, we denote

- $A \lesssim B$ or $A = O(B)$ if there exists $C > 0$ independent of $\varepsilon \in (0, 1]$ such that $A \leq C B$
- $A \sim B$ if $A \lesssim B$ and $B \lesssim A$

(of course, the value of $C$ can change between occurrences of $\lesssim$). If $f$ is a function, $X$ a Banach space, and $B$ a number, we use the notation

$$f = O_X(B) \quad \text{if} \quad \| f \|_X \lesssim B.$$  

For instance, $f = O_{L^2}(1)$ if, for some constant $C$ independent of $\varepsilon$, $\| f \|_{L^2} \leq C$.

### 4. The Euclidean case $\mathcal{M} = \mathbb{R}^d$

In this section we prove theorems 1.1 and 1.2 in the case where $\mathcal{M} = \mathbb{R}^d$ with the Euclidean metric.

In this simpler framework, the KdV scaling 1.7 takes the form

$$u = p + \varepsilon^2 n, \quad p = \Phi(\varepsilon \phi)$$  

(4.1)

where $\Phi$ is the exponential map at 0 on $\mathcal{L}$.

Since $\nabla i = 0$, the tensor $i$ is constant. To simplify the exposition, we shall furthermore also assume in this section that $B$ is a constant tensor. This implies that the properties stated in 1.2 hold for every $p \in \mathbb{R}^d$. In geometric terms, this means that $B$ and $i$ are two
anticommuting and parallel complex structures on $\mathbb{R}^{2d}$, which turns it into a hyperkähler manifold\footnote{To make things a little more concrete, consider the case $d = 2$, where $\mathbb{C}^2$ is viewed as $\mathbb{R}^4$ with the complex structure \[
abla = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \] Then $B$ is skew symmetric and satisfies $Bi = -iB$ et $B^2 = -\mu I$ if and only if $B = \begin{pmatrix} 0 & \alpha & 0 & -\beta \\ -\alpha & 0 & -\beta & 0 \\ 0 & \beta & 0 & -\alpha \\ -\beta & 0 & \alpha & 0 \end{pmatrix}$ with $\alpha^2 + \beta^2 = \mu$.}

The submanifold $\mathcal{L}$ is then assumed to be Lagrangian for both $i$ and $B$.

Next, recall that $V$ can be expanded as

$$V(p + n) = \lambda |n|^2 + V_1(p)(n, n, n) + V_2(p, n)$$

if $p \in \mathcal{L}$ and $n \in N_p \mathcal{L}$,

where $V_1$ is symmetrical and $V_2(p, n) = O(|n|^4)$. It is easy to see that

$$V'(p + n) = 2\lambda n + F_1(p)(n, n) + R^V(p, n),$$

where $F_1$ was defined in \[1.10\] and $R^V$ is at least cubic in $n$. In the scaling \[4.1\] above, this becomes

$$V'(p + \varepsilon^2 n) = 2\lambda \varepsilon^2 n + \varepsilon^4 F_1(p)(n, n) + R^V(p, \varepsilon^2 n). \quad (4.2)$$

### 4.1. Plan of the estimates.

As already explained, we shall proceed in two main steps. Set

\[
\begin{align*}
\mathcal{E}_{s,1}(u, t) & \overset{\text{def}}{=} \varepsilon \|\phi\|_{s+1} + \varepsilon^2 \|n\|_{s+1} + \varepsilon^2 \|\partial_x \phi\|_s + \varepsilon^3 \|\partial_x n\|_s \\
\mathcal{E}_{s,2}(u, t) & \overset{\text{def}}{=} \|\partial_x \phi\|_s + \|n\|_s + \varepsilon \|\partial_x n\|_s
\end{align*}
\]

(4.3)

and

$$\mathcal{E}_s(u, t) \overset{\text{def}}{=} \mathcal{E}_{s,1}(u, t) + \mathcal{E}_{s,2}(u, t). \quad (4.4)$$

Our plan is as follows:

- The a priori control of the energy $\mathcal{E}_{s,1}$ is obtained in Section 4.2 by working primarily on the Schrödinger equation.
- In section 4.3, we will commute high order derivatives with the hydrodynamical system as a preparation to controlling $\mathcal{E}_{s,2}$.
- The a priori control of the energy $\mathcal{E}_{s,2}$ is then obtained in Section 4.4 by working on the differentiated hydrodynamical system.
- The bootstrap argument is justified in section 4.5 where the estimate of $\varepsilon \|\phi\|_{L^\infty}$ is also performed.

For all the a priori estimates to come, we shall work on an interval of time $[0, T^c]$ on which we assume that a solution $u$ exists and satisfies the uniform estimates

$$\sup_{[0,T^c]} \varepsilon \|\phi\|_{L^\infty} + \varepsilon^2 \|n\|_{L^\infty} \leq r \quad \text{and} \quad \sup_{[0,T^c]} \mathcal{E}_s(u) \leq R \quad (4.5)$$

for two constants $r$ and $R$. We pick $r$ sufficiently small for the coordinate system \[4.1\] to be well defined (in view of the assumptions on the initial data in Theorem \[T.1\] this assumption is verified initially as soon as $\varepsilon$ is sufficiently small). As for $R$, it will be determined later.

Before going any further, we note the following lemma. Its proof uses the hydrodynamical system \[1.3\], which can be obtained by projecting the $u$ equation \[1.8\] on $T \mathcal{L}$ and $N \mathcal{L}$.

**Lemma 4.1.** On $[0, T^c]$, we have for $s \geq 2$, the estimates

$$\varepsilon^2 \|\partial_t \phi(t)\|_{s-1} = O(\mathcal{E}_s(u, t)), \quad (4.6)$$

$$\varepsilon \|\partial_x^2 \phi(t)\|_{s-1} = O(\mathcal{E}_s(u, t)). \quad (4.7)$$
Proof. Indeed, from the first equation in (1.9), we have
\[
\varepsilon^2 \partial_t \phi = c \partial_x \phi + S_1^{-1} \left( \frac{1}{2} \Pi^\perp (S_1 \varepsilon \partial_x \phi, D\Phi \varepsilon \partial_x \phi) + \frac{\varepsilon^2}{2} (\nabla_x^\perp)^2 n + BS_1 \partial_x \phi - 2\lambda n - \varepsilon^2 F_1 (p)(n, n) - \frac{1}{\varepsilon^2} P^\perp R^V (p, \varepsilon^2 n) \right).
\]
Consequently, by using the product law (3.10) and (3.11), we get that for \( s \geq 2 \)
\[
\| \varepsilon^2 \partial_t \phi(t) \|_{s-1} = O \left( \| n \|_{s-1} + \| \varepsilon \partial \phi \|_{s-2} + \| \partial_x (\phi, n) \|_{s-1} + \varepsilon^2 \| \partial_{xx} n \|_{s-1} \right).
\]
Note that the dependence in \( \| \varepsilon \partial \phi \|_{s-2} \) is due to the fact that \( D\Phi, \Pi^\perp, \) and \( F_1 \) depend on \( \varepsilon \phi \). In view of the definition of \( E_s \), this yields (4.6).

In a similar way, by writing the second equation of (1.9) under the form
\[
\frac{1}{2} \partial_{xx} \phi = S_1^{-1} \left[ -i (\varepsilon^2 \nabla_t n - (c + iB) \nabla_x^\perp n) - \frac{1}{2} \varepsilon^2 \Pi^\perp (D\Phi \partial_x \phi, \nabla_x^\perp n) - \frac{1}{2} \Pi^\perp (s^1 \partial_x \phi) + \frac{1}{\varepsilon^3} \Pi^\perp R^V (p, \varepsilon^2 n) \right],
\]
we also get
\[
\varepsilon \| \partial_{xx} \phi \|_{s-1} = O(\mathcal{E}_s(u, t)).
\]

\[\square\]

4.2. Estimates on the Schrödinger equation. We shall first prove.

Proposition 4.2. The following estimate holds if \( t \in [0, T^\varepsilon] \):
\[
\mathcal{E}_{s,1}^2 (u, t) \lesssim \mathcal{E}_s^2 (u, 0) + \varepsilon^2 O(\mathcal{E}_s^2 (u, t)) + \int_0^t O(\mathcal{E}_s^2 (u, \tau)) d\tau.
\]

To get this estimate, we shall use a cancellation in the singular terms that come from the underlying wave maps structure in the spirit of [31].

Proof. Step 1: initial decomposition. Taking into account the formula (4.2) for \( V' \), equation (1.8) reads
\[
(\varepsilon^2 \partial_t - (c + iB) \partial_x) u = i \left( \frac{1}{2} \varepsilon \partial_{xx} u - 2\lambda \varepsilon n - \varepsilon^3 F^1 (p)(n, n) - \frac{1}{\varepsilon} R^V (p, \varepsilon^2 n) \right).
\]
For a multiindex \( m \) such that \( |m| \leq s \), we shall apply \( (\varepsilon^2 \partial_t - (c + Bi) \partial_x)^m \) to the above equation. By using (1.2), we note that for the left hand side
\[
(\varepsilon^2 \partial_t - (c + Bi) \partial_x)(\varepsilon^2 \partial_t - (c + iB) \partial_x) = (\varepsilon^2 \partial_t - c \partial_x)^2 + (Bi + iB) \partial_x (\varepsilon^2 \partial_t - c \partial_x) + B^2 \partial_x^2
\]
while for the right hand side
\[
(\varepsilon^2 \partial_t - (c + Bi) \partial_x) i = i (\varepsilon^2 \partial_t - (c + iB) \partial_x).
\]
This yields
\[
(\varepsilon^2 \partial_t - c \partial_x)^2 \partial^m u + \mu \partial_x^2 \partial^m u = -\frac{1}{4} \varepsilon^2 \partial_{xx}^2 \partial^m u + \frac{1}{2} \varepsilon \partial_{xx} \partial^m \left[ 2\lambda \varepsilon n + \varepsilon^3 F^1 (p)(n, n) + \frac{1}{\varepsilon} R^V (p, \varepsilon^2 n) \right] - i (\varepsilon^2 \partial_t - (c + iB) \partial_x)^m \left[ 2\lambda \varepsilon n + \varepsilon^3 F^1 (p)(n, n) + \frac{1}{\varepsilon} R^V (p, \varepsilon^2 n) \right].
\]
Taking the scalar product in $L^2(\mathbb{R}^d)$ against $(\partial_t - \frac{c}{\varepsilon}x \partial_x) \partial^m u$ gives

$$
\frac{d}{dt} \int \left[ \frac{1}{2} \left( \varepsilon^2 \partial_t - c \partial_x \right) \partial^m u \right]^2 + \frac{1}{8} \varepsilon^2 \left| \partial_{xx} \partial^m u \right|^2 - \frac{\mu}{2} \left| \partial_x \partial^m u \right|^2 \right] \, dx
$$

$$
= \frac{1}{2} \int \partial_{xx} \partial^m \left[ 2\lambda + \varepsilon^2 F^1(p)(n,n) \right] \cdot \left( \varepsilon^2 \partial_t - c \partial_x \right) \partial^m u \, dx
$$

$$
- \frac{1}{\varepsilon} \int i(\varepsilon^2 \partial_t - (c + iB)\partial_x) \partial^m \left[ 2\lambda + \varepsilon^2 F^1(p)(n,n) \right] \cdot \left( \varepsilon^2 \partial_t - c \partial_x \right) \partial^m u \, dx + O(\varepsilon^2)
$$

(the $R^V$ terms above are easily seen to contribute $O(\varepsilon^2)$; also notice that the scalar product denoted by $\cdot$ above is simply that of $\mathbb{R}^d$). Next we decompose further $I$ and $II$ by splitting each scalar product into its tangential and normal parts:

$$
I = \int \left[ P^\top \partial_{xx} \partial^m \left[ 2\lambda + \varepsilon^2 F^1(p)(n,n) \right] \cdot P^\top \left( \varepsilon^2 \partial_t - c \partial_x \right) \partial^m u \, dx
$$

$$
+ \int P^\perp \partial_{xx} \partial^m \left[ 2\lambda + \varepsilon^2 F^1(p)(n,n) \right] \cdot P^\perp \left( \varepsilon^2 \partial_t - c \partial_x \right) \partial^m u \, dx
$$

and

$$
II = \frac{1}{\varepsilon} \int \left[ P^\top \left( i(\varepsilon^2 \partial_t - (c + iB)\partial_x) \partial^m \left[ 2\lambda + \varepsilon^2 F^1(p)(n,n) \right] \right) \cdot P^\top \left( \varepsilon^2 \partial_t - c \partial_x \right) \partial^m u \right]
$$

$$
+ \frac{1}{\varepsilon} \int P^\perp \left( i(\varepsilon^2 \partial_t - (c + iB)\partial_x) \partial^m \left[ 2\lambda + \varepsilon^2 F^1(p)(n,n) \right] \right) \cdot P^\perp \left( \varepsilon^2 \partial_t - c \partial_x \right) \partial^m u \, dx.
$$

Step 2: estimating $Ia$. Observe that

$$
\partial_{xx} \partial^m n - \nabla_x^{\perp} \nabla^m n = \Pi^\top \left( D\Phi \varepsilon \partial_x^2 \partial^m \phi, n \right) + O_{L^2}(\varepsilon).$$

Applying $P^\top$ gives

$$
P^\top \partial_{xx} \partial^m n = \Pi^\top \left( D\Phi \varepsilon \partial_x^2 \partial^m \phi, n \right) + O_{L^2}(\varepsilon).$$

Also notice that, since $F^1(p)(n,n)$ is valued in $N_p \mathcal{L}$,

$$
\varepsilon^2 P^\top \partial_{xx} \partial^m F^1(p)(n,n) = \varepsilon O_{L^2}(\varepsilon).$$

Therefore,

$$
Ia = \lambda \int \Pi^\top \left( D\Phi \varepsilon \partial_x^2 \partial^m \phi, n \right) \cdot \left( \varepsilon^2 \partial_t - c \partial_x \right) \partial^m u \, dx + O(\varepsilon^2).
$$

(4.9)

Notice that

$$
P^\top \left( \varepsilon^2 \partial_t - c \partial_x \right) \partial^m u = P^\top \left( \varepsilon^2 \partial_t - c \partial_x \right) D\Phi \varepsilon \partial^m \phi + \varepsilon O_{L^2}(\varepsilon).
$$

(4.10)

Therefore (4.9) can be written

$$
Ia = \lambda \int \Pi^\top \left( D\Phi \varepsilon \partial_x^2 \partial^m \phi, n \right) \cdot \left( \varepsilon^2 \partial_t - c \partial_x \right) D\Phi \varepsilon \partial^m \phi \, dx + O(\varepsilon^2).
$$
Finally, integrating by parts and relying on the symmetry of $\Pi^\top(\cdot, n)$ (see Proposition 3.1) gives

\[ I_a = -\frac{1}{2} \int \lambda \varepsilon^2 (\varepsilon^2 \partial_t - c \partial_x) \left[ \Pi^\top(D\Phi \partial_x \partial^m \phi, n) \cdot D\Phi \partial_x \partial^m \phi \right] dx + O(\varepsilon^2_s) \]

\[ = -\lambda \varepsilon^4 \frac{d}{dt} \int \Pi^\top(D\Phi \partial_x \partial^m \phi, n) \cdot D\Phi \partial_x \partial^m \phi dx + O(\varepsilon^2_s). \]

**Step 3: estimating $I_b$.** Start by noticing that

\[ P^\perp (\varepsilon^2 \partial_t - c \partial_x) \partial^m u = P^\perp (\varepsilon^2 \partial_t - c \partial_x) \partial^m \varepsilon^2 n + R, \]

where the remainder $R$ is such that $\partial_x R = \varepsilon O_L^2(\varepsilon_s)$. Therefore $I_b$ can be split into

\[ I_b = \frac{1}{2} \int P^\perp \partial_t \partial^m \n \cdot P^\perp (\varepsilon^2 \partial_t - c \partial_x) \partial^m \varepsilon^2 n dx \]

\[ + \frac{1}{2} \int P^\perp \partial_t \partial^m \varepsilon F_1(p)(n,n) \cdot P^\perp (\varepsilon^2 \partial_t - c \partial_x) \partial^m \varepsilon^2 n dx \]

\[ + \frac{1}{2} \int P^\perp \partial_t \partial^m (2\n \varepsilon^2 F_1(p)(n,n)) \cdot R dx. \]

In order to deal with (4.11a), it suffices to integrate by parts in $x$ as follows:

\[ \boxed{4.11a} \]

\[ -\lambda \varepsilon^2 \int P^\perp \partial_t \partial^m \n \cdot P^\perp (\varepsilon^2 \partial_t - c \partial_x) \partial^m \varepsilon^2 n dx + O(\varepsilon^2_s) \]

\[ = -\lambda \varepsilon^2 \int \n^{\perp} \partial^m \n \cdot (\varepsilon^2 \partial_t - c \partial_x) \n^{\perp} \partial^m \varepsilon^2 n dx + O(\varepsilon^2_s) \]

\[ = -\lambda \varepsilon^4 \frac{d}{dt} \int | \n^{\perp} \partial^m \varepsilon^2 n |^2 dx + O(\varepsilon^2_s). \]

We next estimate (4.11b), integrating by parts in $x$ and relying on the symmetry properties of $F_1$:

\[ \boxed{4.11b} \]

\[ -\varepsilon^6 \frac{d}{dt} \int F_1(p) \n^{\perp} \partial^m \varepsilon^2 n \cdot \n^{\perp} \partial^m \varepsilon^2 n dx + O(\varepsilon^2_s) \]

Finally, (4.11c) can be estimated directly (after an integration by parts in $x$)

\[ 4.11c \]

\[ | \varepsilon | \| \partial_x \partial^m \| (2\n \varepsilon^2 F_1(p)(n,n)) \| L^2 \| \frac{1}{\varepsilon} \partial_x R \| L^2 = O(\varepsilon^2_s). \]

**Step 4: estimating $II_a$.** First observe that

\[ P^\top (e^{-2} \partial_t - c \partial_x) \partial^m u = \left( e^{-2} \n^{\perp} \partial^m \right) \n^{\perp} \partial^m \Phi \varepsilon + \Pi^\top (D\Phi (e^{-2} \partial_t - c \partial_x) \varepsilon \partial^m \phi, \varepsilon^2 n) + \varepsilon^3 O_L^2(\varepsilon_s). \]

Therefore,

\[ II_a = \frac{2\lambda}{\varepsilon} \int P^\top (i (e^{-2} \partial_t - (c + iB) \partial_x) \partial^m n) \cdot \left( e^{-2} \n^{\perp} \partial^m \right) \n^{\perp} \partial^m \Phi \varepsilon dx \]

\[ + \varepsilon \int P^\top (i \partial_t \partial^m F_1(p)(n,n) \cdot \left( e^{-2} \n^{\perp} \partial^m \right) \n^{\perp} \partial^m \Phi \varepsilon dx \]

\[ + 2\lambda \varepsilon^2 \int P^\top (i \partial_t \partial^m F_1(p)(n,n) \cdot \Pi^\top (D\Phi (e^{-2} \partial_t - c \partial_x) \partial^m \phi, n) dx \]

\[ + O(\varepsilon^2_s). \]
Next, substituting first covariant to flat derivatives, and then using the differentiated hydrodynamical system \((1.3)\) in Proposition \((4.3)\) gives
\[
P^T i \left( \varepsilon^2 \partial_t - (c + iB) \partial_x \right) \partial^m n = i \left( \varepsilon^2 \nabla^\perp_t - (c + iB) \nabla^\perp_x \right) \nabla^\perp m n + \varepsilon O_{L^2}(\mathcal{E}_s)
\]
\[
= -\frac{1}{2} \nabla^T_x \nabla^T m \Phi^\varepsilon - \frac{1}{2} \varepsilon^2 \Pi^T \left( D\Phi (\partial^2_x \partial^m \phi) , n \right) + \varepsilon O_{L^2}(\mathcal{E}_s).
\]
(4.14)

Replacing \(P^T i \left( \varepsilon^2 \partial_t - (c + iB) \partial_x \right) \partial^m n\) by the above expression in \((4.13a)\), and then integrating by parts while keeping in mind that \(\Pi^T (\cdot , n)\) is symmetric, yields
\[
4.13a = - \int \lambda \nabla^T_x \nabla^T m \Phi^\varepsilon \cdot (\varepsilon^2 \partial_t - c \partial_x) \nabla^T m \Phi^\varepsilon \, dx 
- \int \lambda \varepsilon^2 \Pi^T \left( D\Phi \partial^2_x \partial^m \phi , n \right) \cdot (\varepsilon^2 \partial_t - c \partial_x) \nabla^T m \Phi^\varepsilon \, dx + O(\mathcal{E}_s^2)
\]
\[
= \frac{d}{dt} \left[ \frac{\lambda}{2} \varepsilon^2 \int |\nabla^T_x \nabla^T m \Phi^\varepsilon|^2 \, dx + \frac{\lambda}{2} \varepsilon^4 \int \Pi^T \left( D\Phi \partial_x \partial^m \phi , n \right) \cdot D\Phi \partial_x \partial^m \phi \, dx \right] + O(\mathcal{E}_s^2).
\]

To handle \((4.13b)\), we proceed in the same way, using in addition the symmetry properties of \(P^\perp\).

We first note that
\[
4.13b = 2 \int \varepsilon^2 i F_1(p) ((\varepsilon^2 \nabla^\perp_t - (c + iB) \nabla^\perp_x) \nabla^\perp m n, n) \cdot (\varepsilon^2 \partial_t - c \partial_x) D\Phi \partial^m \phi \, dx 
+ \varepsilon^2 \int \left( 2 i F_1(p) (iB \nabla^\perp_x \nabla^\perp m n, n) + 2 B F_1(p) (\nabla^\perp_x \nabla^\perp m n, n) \right) \cdot (\varepsilon^2 \partial_t - c \partial_x) D\Phi \partial^m \phi \, dx + O(\mathcal{E}_s^2)
\]
\[
= 2 \int \varepsilon^2 i F_1(p) ((\varepsilon^2 \nabla^\perp_t - (c + iB) \nabla^\perp_x) \nabla^\perp m n, n) \cdot (\varepsilon^2 \partial_t - c \partial_x) D\Phi \partial^m \phi \, dx + O(\mathcal{E}_s^2).
\]

Indeed, we have used that \(\varepsilon^2 \| \partial_x n \|_m \| \partial \phi \|_m\) is controlled by \(\mathcal{E}_s\). Then, by using again the hydrodynamical system, we find
\[
4.13b = 2 \int \varepsilon^2 i F_1(p) (i \frac{\partial^2}{2} D\Phi \partial^2_x \partial^m \phi , n) \cdot (\varepsilon^2 \partial_t - c \partial_x) D\Phi \partial^m \phi \, dx + O(\mathcal{E}_s^2)
\]
\[
= \int \varepsilon^2 F_1(p) (i D\Phi \partial_x \partial^m \phi , n) \cdot i (\varepsilon^2 \partial_t - c \partial_x) D\Phi \partial_x \partial^m \phi \, dx + O(\mathcal{E}_s^2)
\]
\[
= \frac{d}{dt} \frac{\varepsilon^4}{2} \int F_1(p) (i D\Phi \partial_x \partial^m \phi , n) \cdot i D\Phi \partial_x \partial^m \phi \, dx + O(\mathcal{E}_s^2).
\]

Once again replacing \(P^T i \left( \varepsilon^2 \partial_t - (c + iB) \partial_x \right) \partial^m n\) by the above expression \((4.14)\) in \((4.13a)\), integrating by parts and using the symmetry of \(\Pi^T (\cdot , n)\), we find
\[
4.13c = -\lambda \varepsilon^2 \int D\Phi \partial^2_x \partial^m \phi \cdot \Pi^T \left( D\Phi (\varepsilon^2 \partial_t - c \partial_x) \partial^m \phi , n \right) \, dx + O(\mathcal{E}_s^2)
\]
\[
= \frac{d}{dt} \frac{\varepsilon^4}{2} \int \Pi^T \left( D\Phi \partial_x \partial^m \phi , n \right) \cdot D\Phi \partial_x \partial^m \phi \, dx + O(\mathcal{E}_s^2).
\]

Step 5: Estimating IIb. Start by noticing that
\[
P^\perp i \left( \varepsilon^2 \partial_t - (c + iB) \partial_x \right) \partial^m n = i \Pi^T \left( D\Phi \varepsilon^2 \partial_t \partial^m \phi , n \right) + \varepsilon O_{L^2}(\mathcal{E}_s).
\]
while
\[
P^\perp i \left( \varepsilon^2 \partial_t - (c + iB) \partial_x \right) \partial^m \varepsilon^2 F_1(p)(n, n) = \varepsilon O_{L^2}(\mathcal{E}_s).
\]
Moreover, substituting first covariant to flat derivatives, and using the differentiated hydrodynamical equation derived in Proposition 4.3,

\[ P^\perp \left((\varepsilon^2 \partial_t - c \partial_x)\partial^m u\right) = \left(\varepsilon^2 \nabla_x^\perp - (c + iB)\nabla_x^\perp\right) \nabla^\perp m \varepsilon^2 n + iB \varepsilon^2 \nabla_x^\perp \nabla^\perp m n + \varepsilon O L^2(\varepsilon_s) \]

\[ = \frac{i}{2} \varepsilon^2 D\Phi \partial_x^2 \partial^m \phi + \varepsilon O L^2(\varepsilon_s). \]

Therefore, we find

\[ II_b = \lambda \varepsilon^2 \int \Pi^T (D\Phi(x^2 \partial_t)\partial^m \phi, n) \cdot D\Phi \partial_x^2 \partial^m \phi \, dx + O(\varepsilon_s^2) \]

and finally integrating by parts while using the symmetry of \( \Pi^T(\cdot, n) \) gives

\[ II_b = -\frac{\lambda \varepsilon^4}{2} \frac{d}{dt} \int \Pi^T (D\Phi \partial_x \partial^m \phi, n) \cdot D\Phi \partial_x \partial^m \phi \, dx + O(\varepsilon_s^2). \]

Step 6: Conclusion. Gathering the results of steps 1 to 5 gives

\[ \frac{d}{dt} E_m = O(\varepsilon_s^2) \quad (4.15) \]

with (notice the cancellation between \( Ia \) and \( IIb \) which, however, is not needed for the estimates to close)

\[ E_m \overset{\text{def}}{=} \int \left[ \left( \frac{1}{2} |(\varepsilon^2 \partial_t - c \partial_x)\partial^m u|^2 + \frac{1}{8} \varepsilon^2 |\partial_x \partial^m u|^2 - \frac{\mu}{2} |\partial_x \partial^m u|^2 + \frac{\lambda}{2} \varepsilon^2 \left| \nabla_x^\perp \nabla^\perp m \Phi \right|^2 + \frac{\lambda}{2} \varepsilon^4 \left| \nabla_x^\perp \nabla^\perp m n \right|^2 \right. \]

\[ + \lambda \varepsilon^4 \Pi^T (D\Phi \partial_x \partial^m \phi, n) \cdot D\Phi \partial_x \partial^m \phi + \frac{1}{2} \varepsilon^6 F_1(p)(\nabla_x^\perp \nabla^\perp m n, n) \cdot \nabla_x^\perp \nabla^\perp m n \]

\[ + \frac{1}{2} \varepsilon^4 F_1(p)(iD\Phi \partial_x \partial^m \phi, n) \cdot iD\Phi \partial_x \partial^m \phi \right] \, dx. \quad (4.16) \]

To conclude, it suffices to note that

\[ \sum_{|m| \leq s} E_m = \varepsilon_s^2 + \varepsilon^2 O(\varepsilon_s^2). \]

\[ \square \]

4.3. Differentiating the hydrodynamical system. In order to derive a priori bounds, we will need to differentiate the hydrodynamical form \((1.9)\) of the equation. Recall that this system is obtained by using the decomposition \((4.1)\) of \(\Phi\), the expression \((4.2)\) for \(V'\), and by noticing thanks to \((3.3)\) that

\( \partial u = \partial p + \varepsilon^2 \partial n = \partial p + \varepsilon^2 \Pi^T(\partial p, n) + \varepsilon^2 \nabla^\perp n = S_0 \partial p + \varepsilon^2 \nabla^\perp n \)

where the first term is tangent to \(\mathcal{L}\) and the second term is normal to \(\mathcal{L}\). The system \((1.9)\) results from projecting \((1.5)\) on \(N_p \mathcal{L}\) and \(T_p \mathcal{L}\) respectively.

The following proposition gives the system which is solved by \((\nabla^\perp m \Phi, \nabla^\perp m n)\) up to error terms which will not matter in the estimates. Note that we use the convention that when a covariant derivative hits a function it coincides with the standard derivative so that \(\frac{1}{\varepsilon} \nabla^\perp \Phi = D\Phi \partial \phi\). For notational convenience we shall set

\( \nabla^\perp m \Phi = \frac{1}{\varepsilon} \nabla^\perp m \Phi \)

which is the natural order one object since we roughly have that \(\nabla^\perp m \Phi = D\Phi \cdot \partial^m \phi + \varepsilon\) times lower order terms.
Proposition 4.3. For $1 \leq |m| \leq s$ and $s \geq 2$, we get the following system for $(\nabla^m \Phi^\varepsilon, \nabla^\perp m \eta)$ if $t \in [0, T^\varepsilon]$:

$$
\begin{cases}
\begin{aligned}
\left( S_0 \nabla_t^\varepsilon - \frac{1}{\varepsilon^2} (c + iB) S_0 \nabla_x^\varepsilon \right) \nabla^m \Phi^\varepsilon &= i \left[ \frac{1}{2} \Pi^\perp \left( S_0 \nabla_x^\varepsilon \nabla^m \Phi^\varepsilon, D\Phi \partial_x \Phi \right) + \frac{1}{2} \Pi^\perp \left( S_1 \partial_x \Phi, \nabla_x^\perp \nabla^m \Phi^\varepsilon \right) + \frac{1}{2} \left( \nabla_x^\perp \nabla^\perp m \eta - 2 F_1 (n, \nabla^\perp m \eta) - 2 \lambda \Pi^\perp (i n, \nabla^\perp m \eta) \right) \right] + O_H (\varepsilon_t)
\end{aligned}
\end{cases}
$$

(4.17)

(recall that $H^1$ was defined in [3.9]).

Proof. As a preliminary remark, we note that since $R^V (p, N)$ is $O(|N|^3)$, we get thanks to (3.10) that

$$
\frac{1}{\varepsilon^4} \partial^m \left( P^T R^V (p, \varepsilon^2 n) \right) = O_H (\varepsilon), \quad \frac{1}{\varepsilon^3} \partial^m \left( P^T R^V (p, \varepsilon^2 n) \right) = O_H (\varepsilon_t)
$$

so that the remainder from the potential does not contribute.

Step 1: the left-hand side of (1.9). We start by applying the operator $\nabla^m$ for $1 \leq |m| \leq s$ to the left-hand side of the first line of (1.9).

Observe that we can expand expand $\nabla^m \Phi^\varepsilon$

$$
\nabla^m \Phi^\varepsilon = D\Phi \partial^m \phi + \sum \varepsilon^{k+1} \ast_k \left( \nabla^k D\Phi \right) |_{\varepsilon \phi} \left( \partial^{a_1} \phi, \cdots, \partial^{a_{k+1}} \phi \right)
$$

(4.18)

where $\ast_k$ are harmless coefficients and the sum is for $k \geq 1$ and for $1 \leq a_1, \cdots, a_{k+1} < m$. Consequently, by using (4.6) and (3.10), we get that for $1 \leq |m| \leq s$, we have

$$
\| \nabla^m \Phi^\varepsilon \|_{L^2} = O (\varepsilon_t).
$$

(4.19)

In a similar way, by differentiating (1.18) in $x$, we also obtain that

$$
\| \nabla_x^\perp \nabla^m \Phi^\varepsilon - D\Phi \partial_x \partial^m \phi \|_{L^2} = \varepsilon O (\varepsilon_t)
$$

(4.20)

and hence that

$$
\| \nabla_x^\perp \nabla^m \Phi^\varepsilon \|_{L^2} = O (\varepsilon_t)
$$

(4.21)

We note that

$$
\nabla^m \left( S_1 (\partial_t \phi - \frac{c}{\varepsilon^2} \partial_x \phi) - \frac{1}{\varepsilon^2} i B S_1 \partial_x \phi \right) = \frac{1}{\varepsilon} \nabla^m \left( S_0 \left( \partial_t \phi - \frac{c}{\varepsilon^2} \partial_x \phi - \frac{1}{\varepsilon^2} S_0^{-1} i B S_0 \partial_x \phi \right) \right)
$$

and that thanks to Corollary 3.2, we have

$$
i B S_0 X = S_0 i B X, \quad \forall X \in \mathcal{T} \mathcal{L}.
$$

(4.22)

Therefore, we obtain

$$
\nabla^m \left( S_1 (\partial_t \phi - \frac{c}{\varepsilon^2} \partial_x \phi) - \frac{1}{\varepsilon^2} i B S_1 \partial_x \phi \right) = \frac{1}{\varepsilon} \nabla^m \left( S_0 \left( \partial_t \phi - \frac{1}{\varepsilon^2} (c + iB) \partial_x \phi \right) \right)
$$

(4.23)

$$
= \frac{1}{\varepsilon} S_0 \nabla^m \left( \partial_t \phi - \frac{1}{\varepsilon^2} (c + iB) \partial_x \phi \right) + \nabla^m \left( S_0 \frac{1}{\varepsilon} \right) \left( \partial_t \phi - \frac{1}{\varepsilon^2} (c + iB) \partial_x \phi \right)
$$

$$
\stackrel{\text{def}}{=} L_1^\phi + L_2^\phi.
$$
To express $L_1^\phi$, let us write following our conventions $\nabla^Tm = \nabla^T\tilde{m}\nabla^T$ with $|\tilde{m}| = |m| - 1$ and $\nabla^T = \varepsilon^2\nabla^T_t$ or $\nabla^T_x$. First note that since $\nabla_t = \nabla B = 0$, and $\nabla$ is torsion-free, we have

$$L_1^\phi = S_0\nabla^T\tilde{m}\left(\frac{\partial \Phi}{\varepsilon} + \frac{1}{\varepsilon^2}(c + iB)\nabla^T_x\right)\frac{\partial \Phi}{\varepsilon}.$$ 

Next, writing $\nabla^T\tilde{m} = (\varepsilon^2\nabla^T_t)^{\tilde{m}_0}(\nabla^T_x)^{\tilde{m}_1}$, we find by using the formula (3.7) that if $\tilde{m}_1 > 0$,

$$\nabla^T\tilde{m}\nabla^T_t\left(\frac{\partial \Phi}{\varepsilon}\right) = (\varepsilon^2\nabla^T_t)^{\tilde{m}_0}(\nabla^T_x)^{\tilde{m}_1-1}\nabla^T_t\nabla^T_x\left(\frac{\partial \Phi}{\varepsilon}\right) + \left((\varepsilon^2\nabla^T_t)^{\tilde{m}_0}(\nabla^T_x)^{\tilde{m}_1-1}\right)\left(R^T(\nabla^T_t\Phi, \nabla^T_x\Phi)\frac{\partial \Phi}{\varepsilon}\right).$$

Since the last term can be written under the form

$$(\varepsilon^2\nabla^T_t)^{\tilde{m}_0}(\nabla^T_x)^{\tilde{m}_1-1}\left(R^T(D\Phi\varepsilon^2\partial_t\phi, D\Phi\partial_x\phi)D\Phi\partial\phi\right),$$

we get by using (3.1) and (3.10) that for $s \geq 2$

$$\left\|\left((\varepsilon^2\nabla^T_t)^{\tilde{m}_0}(\nabla^T_x)^{\tilde{m}_1-1}\right)\left(R^T(D\Phi\varepsilon^2\partial_t\phi, D\Phi\partial_x\phi)D\Phi\partial\phi\right)\right\| H^1 = O\left(\|\partial_x\phi\|_s + \|\varepsilon^2\partial_t\phi\|_{s-1}\right)$$

and hence thanks to (4.6) that:

$$\left\|\left((\varepsilon^2\nabla^T_t)^{\tilde{m}_0}(\nabla^T_x)^{\tilde{m}_1-1}\right)\left(R^T(D\Phi\varepsilon^2\partial_t\phi, D\Phi\partial_x\phi)D\Phi\partial\phi\right)\right\| H^1 = O(\varepsilon^2).$$

By using again repeatedly (3.7), we can thus obtain from (4.24) that

$$\nabla^T\tilde{m}\nabla^T_t\left(\frac{\partial \Phi}{\varepsilon}\right) = \nabla^T_t\nabla^T\tilde{m}\left(\frac{\partial \Phi}{\varepsilon}\right) + O_{H^1}(\varepsilon).$$

which yields

$$S_0\nabla^T\tilde{m}\nabla^T_t\frac{\partial \Phi}{\varepsilon} = S_0\nabla^T_t\nabla^Tm\Phi + O_{H^1}(\varepsilon).$$

To study the second part of $L_1^\phi$, we can use the same arguments as above (note that $\frac{1}{\varepsilon^2}\partial_x$ and $\partial_t$ behaves in the same way, in our weighted spaces) and the fact that $\nabla(iB) = 0$ to also obtain

$$S_0\nabla^T\tilde{m}\frac{1}{\varepsilon^2}(c + iB)\nabla^T_x\frac{\partial \Phi}{\varepsilon} = \frac{1}{\varepsilon^2}S_0(c + iB)\nabla^T_x\nabla^Tm\Phi + O_{H^1}(\varepsilon).$$

Consequently, we have proven that

$$L_1^\phi = \left(S_0\nabla^T_t - \frac{1}{\varepsilon^2}(c + iB)S_0\nabla^T_x\right)\nabla^Tm\Phi + O_{H^1}(\varepsilon).$$

It remains to study $L_2^\phi$. By using the definition of $S_0$, we can reduce the estimate of this commutator to the estimate of terms of the form

$$(\nabla^\alpha\Pi^\top)\nabla^\gamma(D\Phi\varepsilon^2\partial_t\phi - (c + iB)D\Phi\partial_x\phi), \nabla^\beta n)$$

with $|\alpha| + |\beta| + |\gamma| \leq |m|$ and $|\gamma| < |m|$. By using again (3.10), we find

$$(\nabla^\alpha\Pi^\top)\nabla^\gamma(D\Phi\varepsilon^2\partial_t\phi - (c + iB)D\Phi\partial_x\phi), \nabla^\beta n) = O_{H^1}(\varepsilon)$$

except in the case $\beta = m$. Indeed, in this case the estimate would involve the norm $|\partial_x n|$ without the $\varepsilon$ weight which is in the definition of $E_{s,2}$. Consequently, it remains to estimate

$$\Pi^\top(D\Phi\varepsilon^2\partial_t\phi - (c + iB)D\Phi\partial_x\phi, \nabla^m n).$$

In order to get a simpler expression for the first term above, we can use the first equation of (1.9). This yields

$$\Pi^\top(D\Phi\varepsilon^2\partial_t\phi - (c + iB)D\Phi\partial_x\phi, \nabla^m n) = \Pi^\top(-i2\lambda n, \nabla^m n) + \Pi^\top(R, \nabla^m n)$$

(4.27)
where $R$ is such that
\[ \|R\|_{W^{1,\infty}} = \varepsilon O(\|\partial_x \phi\|_{W^{1,\infty}} + \|n\|_{W^{1,\infty}} + \varepsilon \|\partial_x^2 n\|_{W^{1,\infty}}) = \varepsilon O(\varepsilon) \]
for $s \geq 2$. Consequently, we find that
\[ \Pi^\top (D\Phi \varepsilon^2 \partial_t \phi - (c + iB)D\Phi \partial_x \phi, \nabla \perp^m n, ) = \Pi^\top ( - 2\lambda \im n, \nabla \perp^m n ) + O_{H^1}(\varepsilon). \]
In summary, we have proven that
\[ \varepsilon O(\varepsilon) \]
Consequently, thanks to (4.26), (4.28) and (4.23), we find
\[ \nabla \perp^m \left( S_0(\partial_t \phi - \frac{c}{\varepsilon^2} \partial_x \phi) - \frac{1}{\varepsilon^2} iBS_1 \partial_x \phi \right) \]
\[ = \left( S_0 \nabla \perp^T - \frac{1}{\varepsilon^2} (c + iB)S_0 \nabla \perp^T \right) \nabla \perp^m \Phi^\varepsilon + \Pi^\top ( - 2\lambda \im n, \nabla \perp^m n ) + O_{H^1}(\varepsilon). \]
Step 2: the right-hand side of (1.9).
We now differentiate the right-hand side of the first equation in (1.9). At first, we easily get that
\[ \nabla \perp^m \left( i\Pi^\perp (S_1 \partial_x \phi, D\Phi \partial_x \phi) \right) = \nabla \perp^m \left( i\Pi^\perp \left( S_0 \frac{\partial_x \Phi}{\varepsilon}, \frac{\partial_x \Phi}{\varepsilon} \right) \right) \]
\[ = i \left( \Pi^\perp \left( S_0 \nabla \perp^T \nabla \perp^m \Phi^\varepsilon, D\Phi \partial_x \phi \right) + \Pi^\perp \left( S_1 \partial_x \phi, \nabla \perp^T \nabla \perp^m \Phi^\varepsilon \right) \right) + O_{H^1}(\varepsilon) \]
for $s \geq 2$. Next, we need to expand
\[ \nabla \perp^m \left( \frac{i}{2} (\nabla \perp^T n) \right) = \frac{i}{2} \nabla \perp^m \left( (\nabla \perp^T n) \right). \]
We first note that if $\nabla \perp^m = \nabla \perp^T n$ there is no commutator. Consequently, denoting $m = (m_0, m_1)$, we only have to study the case $m_0 \geq 1$. We thus write
\[ \frac{i}{2} \nabla \perp^m \left( (\nabla \perp^T n) \right) = \frac{i}{2} \left( \varepsilon^2 \nabla \perp^T \right)^{m_0} (\nabla \perp^T n)^{m_1} \]
\[ = \frac{i}{2} \left( \varepsilon^2 \nabla \perp^T \right)^{m_0-1} \nabla \perp^T (\varepsilon^2 \nabla \perp^T )^{m_1-1} \left( \varepsilon^2 R \left( \varepsilon^2 D\Phi \partial_x \phi, D\Phi \partial_x \phi \right) \nabla \perp^T n \right) \]
thanks to (3.8). By using again (3.10) and (4.6), we see that the last term is $O_{H^1}(\varepsilon)$. By using repeatedly the commutator identity (3.8), we obtain
\[ \frac{i}{2} \nabla \perp^m \left( (\nabla \perp^T n) \right) = \frac{i}{2} (\nabla \perp^T n)^{m_1} + R_2, \]
where $R_2$ is a sum of terms of the type
\[ \frac{i}{2} \nabla \perp^\alpha \left( \varepsilon^2 R \left( \varepsilon^2 D\Phi \partial_x \phi, D\Phi \partial_x \phi \right) \nabla \perp^\beta n \right), \]
with $|\alpha| + |\beta| \leq s$ and $\alpha_1 + \beta_1 \geq 1$. This implies
\[ \frac{i}{2} \nabla \perp^m \left( (\nabla \perp^T n) \right) = \frac{i}{2} (\nabla \perp^T n)^{m_1} + O_{H^1}(\varepsilon). \]
It is clear that
\[ \nabla \perp^m F_1(p)(n, n) = 2F_1(p)(\nabla \perp^m n, n) + O_{H^1}(\varepsilon). \]
Consequently, by combining (3.29), (3.30), (4.31), (4.32) and the last line above, we obtain the first line of (4.17).
Step 3: the left-hand side of (1.9)\textsubscript{2}. By using (3.8), we get
\[ \nabla^{\perp m} \nabla^{\perp}_t n = \nabla^{\perp}_t \nabla^{\perp m} n + R_3 \]
where the commutator $R_3$ is a sum of terms of the form
\[ \nabla^{\perp} R_{\perp}^{\alpha} (\varepsilon^2 D\Phi \partial_t \phi, D\Phi \partial_x \phi) \nabla^{\perp \beta} n \]
with $|\alpha| + |\beta| \leq |m| - 1$. Consequently, by using (3.10) and (4.6) we find that
\[ R_3 = O_{H_1^s}(E_s) \]
(actually for this term, one could get a much better estimate). The estimate of the commutator
\[ \frac{1}{\varepsilon^2} \left[ \nabla^{\perp m}, \nabla^{\perp}_m \right] n \]
is very similar to what we already described. Since $\nabla(iB) = 0$, we thus get that
\[ \nabla^{\perp m} \left( \nabla^{\perp}_t - \frac{1}{\varepsilon^2} (c + iB) \nabla^{\perp}_x \right) n = \left( \nabla^{\perp}_t - \frac{1}{\varepsilon^2} (c + iB) \nabla^{\perp}_x \right) \nabla^{\perp m} n + O_{H_1^s}(E_s). \] (4.33)

Step 4: the right-hand side of (1.9)\textsubscript{2}. For the right hand side of the second line of (1.9), we first study the differentiation of
\[ L_1^n \overset{\text{def}}{=} \frac{i}{2\varepsilon^2} \nabla^{\top}_x (S_0 D\Phi \partial_x \phi). \]
As before, we note that $\nabla^{\top m}$ commutes with $\nabla^{\top}_x$ if $m_0 = 0$, hence we write
\[ \nabla^{\perp m} L_1^n = \frac{i}{2\varepsilon^2} (\varepsilon^2 \nabla^{\perp}_t) m_0 \nabla^{\top} \nabla^{\top m} (S_0 D\Phi \partial_x \phi) \]
and by using again (3.7), we have
\[ \nabla^{\perp m} L_1^n = \frac{i}{2\varepsilon^2} \nabla^{\top} \nabla^{\top m} (S_0 D\Phi \partial_x \phi) + O_{H_1^s}(E_s). \]
Indeed, the commutator involves a sum of terms of the form
\[ \frac{i}{2} \nabla^{\perp} R_{\perp}^{\alpha} \left( R_{\perp}^{\beta} (\varepsilon^2 D\Phi \partial_t \phi, D\Phi \partial_x \phi) \nabla^{\perp m} (S_0 D\Phi \partial_x \phi) \right) \]
with $|\alpha| + |\beta| \leq |m| - 1$ that can be again estimated by using (3.10) and (3.1). Commuting now $\nabla^{\top m}$ with $S_0$ gives
\[ \nabla^{\perp m} L_1^n = \frac{i}{2\varepsilon^2} \nabla^{\top}_x \left( S_0 \nabla^{\top m} (D\Phi \partial_x \phi) \right) + \frac{i}{2\varepsilon^2} \nabla^{\top}_x \left( \nabla^{\top m}, S_0 \right) \left( D\Phi \partial_x \phi \right) + O_{H_1^s}(E_s). \] (4.34)
To estimate the second term in the above right hand side, we can use the definition of $S_0$ and the derivation formula (3.4), to see that we have to estimate a sum of terms of the form
\[ \nabla^{\top}_x \left( (\nabla^{\top} R_{\perp}^{\alpha}) (D\Phi \partial_x \phi), \nabla^{\perp \gamma} n \right) \]
with $|\alpha| + |\beta| + |\gamma| = |m|$ and $\beta \neq m$. When $\gamma \neq m$, these terms are clearly estimated in $H_1^s$ by
\[ (1 + O(E_s^2)) \left( \| n \|_{H_1^s} + \| \partial_x \phi \|_s + \varepsilon^2 \| \nabla^{\top}_x n \|_s + \varepsilon^3 \| \nabla^{\top}_x^2 n \|_s + \varepsilon^2 \| \partial_{xx} \phi \|_s + \varepsilon \| \partial^2_x \phi \|_s - 1 \right) \]
By using (4.7) and the definition of $E_s$ (note that here we also use $E_{s,1}$), we thus get that these terms are $O_{H_1^s}(E_s)$. Consequently, since $n$ is a normal vector field, we get
\[ \frac{i}{2\varepsilon^2} \nabla^{\top}_x \left( \nabla^{\top m}, S_0 \right) \left( D\Phi \partial_x \phi \right) = \frac{i}{2} \nabla^{\top} \left( D\Phi \partial_x \phi, \nabla^{\perp m} n \right) + O_{H_1^s}(E_s) \] (4.35)

It remains to study the first term in the right-hand side of (4.34). Let us observe that
\[ \nabla^{\top m} (D\Phi \partial_x \phi) = \nabla^{\top m} \left( \frac{\partial_x \phi}{\varepsilon} \right) = \nabla^{\top} \bar{m} \left( \frac{\nabla^{\perp} \partial_x \phi}{\varepsilon} \right), \quad |\bar{m}| = s - 1. \]
This yields by using again (4.37) that
\[
\frac{i}{2\varepsilon^2} \nabla_x^\top \left( S_0 \nabla_x^\top (D\Phi_{\partial_x} \phi) \right) = \frac{i}{2\varepsilon^2} \nabla_x^\top \left( S_0 \nabla_x^\top \nabla^\top \Phi \phi \right) + O_{H^1} (\mathcal{E}_s). \tag{4.36}
\]
By combining (4.34), (4.35), and (4.36), we thus get
\[
\nabla^\perp m L^1_n = \frac{i}{2\varepsilon^2} \nabla_x^\top \left( S_0 \nabla_x^\top \nabla^\top m \phi \right) + \frac{i}{2} \Pi^\top (D\Phi_{\partial_x} \phi, \nabla^\perp m \phi) + O_{H^1} (\mathcal{E}_s). \tag{4.37}
\]
By using similar arguments, we also obtain
\[
\frac{1}{2} \nabla^\perp m \left( i \Pi^\top (D\Phi_{\partial_x} \phi, \nabla^\perp m \phi) \right) = \frac{1}{2} \iota \left( \Pi^\top (\nabla_x^\top \nabla^\top m \phi, \nabla^\perp m \phi) + \Pi^\top (D\Phi_{\partial_x} \phi, \nabla^\perp m \phi) \right) + O_{H^1} (\mathcal{E}_s).
\]
By combining the last identity, (4.34) and (4.33), we finally get the second line of (4.17). This ends the proof of Proposition 4.3.

\[\Box\]

4.4. Estimates on the hydrodynamical system. Our main result in this section will be the following:

**Proposition 4.4.** The following estimate holds if \( t \in [0, T^\varepsilon] \):
\[
\mathcal{E}_s^2(u, t) \lesssim \mathcal{E}_s^2(u, 0) + \varepsilon^2 O(\mathcal{E}_s^2(u, t)) + \int_0^t O(\mathcal{E}_s^2(u, \tau)) d\tau.
\]

**Proof.** Step 1: the multiplier method, and splitting of the resulting terms.

Observe first that by using again (4.22), the system (4.17) can be written equivalently
\[
\begin{align*}
\left( \nabla_x^\top - \frac{c}{\varepsilon^2} \nabla_x \right) \nabla^\top m \phi = & \ S_0^{-1} \left[ \frac{1}{2} \Pi^\top (S_0 \nabla_x^\top \nabla^\top m \phi, D\Phi_{\partial_x} \phi) + \frac{1}{2} \Pi^\top (S_1 \partial_x \phi, \nabla_x^\top \nabla^\top m \phi) \\
+ & \frac{1}{2} (\nabla_x^\perp)^2 \nabla^\perp m \phi - \frac{1}{2} B S_0 \nabla_x^\top \nabla^\top m \phi - 2 \lambda \nabla^\perp m \phi - 2 i \Pi^\top (i n, \nabla^\perp m \phi) \right] + O_{H^1} (\mathcal{E}_s) \\
\nabla_x^\top \nabla^\perp m \phi - & \frac{c}{\varepsilon^2} \nabla_x \nabla^\perp m \phi = \left[ \Pi^\top (D\Phi_{\partial_x} \phi, \nabla_x^\perp \nabla^\perp m \phi) + \frac{1}{2} \Pi^\top (\nabla_x^\top \nabla^\top m \phi, \nabla_x^\perp \nabla^\perp m \phi) \\
+ & \frac{1}{2} \nabla_x^\top (S_0 \nabla_x^\top \nabla^\top m \phi) + \frac{1}{2} B \nabla_x^\top \nabla^\perp m \phi \right] + O_{H^1} (\mathcal{E}_s) 
\end{align*}
\]

Now take the scalar product of the first and second lines of the above system with appropriate integrating factors:
\[
\text{(4.38)}_1 \cdot \left[ - \nabla_x^\top \left( S_0 S_0 \nabla_x^\top \nabla^\top m \phi \right) - 2 S_0 B \nabla_x^\top \nabla^\perp m \phi - 2 \varepsilon^2 \Pi^\top (D\Phi_{\partial_x} \phi, \nabla_x^\perp \nabla^\perp m \phi) \right] \\
+ \text{(4.38)}_2 \cdot \left[ - \varepsilon^2 (\nabla_x^\perp)^2 \nabla^\perp m \phi - 2 B S_0 \nabla_x^\top \nabla^\top m \phi - 2 \varepsilon^2 \Pi^\top (D\Phi_{\partial_x} \phi, \nabla_x^\top \nabla^\top m \phi) \\
+ 4 \lambda \nabla^\perp m \phi + 4 \varepsilon^2 F_1 (n, \nabla^\perp m \phi) + 4 \lambda \varepsilon^2 \Pi^\top (i n, \nabla^\perp m \phi) \right],
\]
then integrate over \( \mathbb{R} \). We rearrange the resulting terms by writing the above as
\[
I + II + III + IV = V + VI + O(\mathcal{E}_s^2) \tag{4.39}
\]
where: the term \( I \) corresponds to the contribution of the left-hand side of (4.38)_1 with the symmetric part of its multiplier
\[
I \overset{\text{def}}{=} \int_{\text{LHS}(4.38)_1} \left[ - \nabla_x^\top \left( S_0 S_0 \nabla_x^\top \nabla^\top m \phi \right) \right] dx
\]
\[
= \int \left[ \left( \nabla_x^\top - \frac{c}{\varepsilon^2} \nabla_x \right) \nabla^\top m \phi \right] \cdot \left[ - \nabla_x^\top \left( S_0 S_0 \nabla_x^\top \nabla^\top m \phi \right) \right] dx;
\]
the term $II$ corresponds to the contribution of the left-hand side of (4.38)$_2$ with the symmetric part of its multiplier

$$II \overset{\text{def}}{=} \int \text{LHS (4.38)$_2$} - \frac{\varepsilon^2}{2} (\nabla_x^2)^2 \nabla \perp m n + 4 \lambda \nabla \perp m n + 4 \varepsilon^2 F_1 (n, \nabla \perp m n) + 4 \lambda \varepsilon^2 \Pi (i n, \nabla \perp m n) dx$$

$$= \int \left[ \nabla \perp \nabla \perp m n - \frac{c}{\varepsilon^2} \nabla_x \nabla \perp m n \right]$$

$$\cdot \left[ -\frac{\varepsilon^2}{2} (\nabla_x^2)^2 \nabla \perp m n + 4 \lambda \nabla \perp m n + 4 \varepsilon^2 F_1 (n, \nabla \perp m n) + 4 \lambda \varepsilon^2 \Pi (i n, \nabla \perp m n) \right] dx;$$

the terms $III$ and $IV$ correspond to the contributions of the left-hand sides of (4.38)$_1$ and (4.38)$_2$ that involve the first order terms of the multipliers

$$III \overset{\text{def}}{=} \int \left[ \left( \nabla \perp - \frac{c}{\varepsilon^2} \nabla_x \right) \nabla \perp m \Phi \cdot \left( -2 S_0 B \nabla_x \nabla \perp m n \right) \right.$$ 

$$\left. + \left( \nabla \perp - \frac{c}{\varepsilon^2} \nabla_x \right) \nabla \perp m n \cdot \left( -2 S_0 B \nabla_x \nabla \perp m \Phi \right) \right];$$

$$IV \overset{\text{def}}{=} \int \left[ \left( \nabla \perp - \frac{c}{\varepsilon^2} \nabla_x \right) \nabla \perp m \Phi \cdot \left( -2 \varepsilon^2 \Pi \left( D \Phi \partial_x \phi, \nabla_x \nabla \perp m n \right) \right.$$ 

$$\left. + \left( \nabla \perp - \frac{c}{\varepsilon^2} \nabla_x \right) \nabla \perp m n \cdot \left( -2 \varepsilon^2 \Pi \left( D \Phi \partial_x \phi, \nabla_x \nabla \perp m \Phi \right) \right) \right];$$

the term $V$ contains the higher order terms coming from the right-hand sides of (4.38)$_1$ and (4.38)$_2$:

$$V \overset{\text{def}}{=} \frac{1}{2} \int S_0^{-1} i \left[ (\nabla \perp)^2 \nabla \perp m n + \frac{2}{\varepsilon^2} B S_0 \nabla_x \nabla \perp m \Phi + 2 \Pi \left( D \Phi \partial_x \phi, \nabla_x \nabla \perp m \Phi \right) \right.$$ 

$$- \frac{4 \lambda}{\varepsilon^2} \nabla \perp m n - 4 F_1 (n, \nabla \perp m n) - 4 \lambda \Pi (i n, \nabla \perp m n) \right]$$

$$\cdot \left[ -\nabla \perp \left(S_0 S_0 \nabla_x \nabla \perp m \Phi \right) - 2 S_0 B \nabla_x \nabla \perp m n - 2 \varepsilon^2 \Pi \left( D \Phi \partial_x \phi, \nabla_x \nabla \perp m n \right) \right] dx$$

$$+ \frac{1}{2 \varepsilon^2} \int i \left[ \nabla \perp (S_0 S_0 \nabla_x \nabla \perp m \Phi) + 2 B S_0 \nabla_x \nabla \perp m n + 2 \varepsilon^2 \Pi \left( D \Phi \partial_x \phi, \nabla_x \nabla \perp m n \right) \right.$$ 

$$\cdot \left[ -\frac{\varepsilon^2}{2} (\nabla_x^2)^2 \nabla \perp m n - 2 B S_0 \nabla_x \nabla \perp m \Phi - 2 \varepsilon^2 \Pi \left( D \Phi \partial_x \phi, \nabla_x \nabla \perp m \Phi \right) \right.$$ 

$$\left. + 4 \lambda \nabla \perp m n + 4 \varepsilon^2 F_1 (n, \nabla \perp m n) + 4 \lambda \varepsilon^2 \Pi (i n, \nabla \perp m n) \right] dx;$$

and finally the term $VI$ gathers the lower order terms from the right-hand sides of (4.38)$_1$ and (4.38)$_2$:

$$VI \overset{\text{def}}{=} \frac{1}{2} \int S_0^{-1} i \left[ \Pi \left( \Pi \nabla \perp m \Phi, n \right) D \Phi \partial_x \phi \right.$$ 

$$\left. + \Pi \left( \Pi \nabla \perp m \Phi, n \right) \nabla_x \nabla \perp m \Phi \right]$$

$$\cdot \left[ -\nabla \perp \left(S_0 S_0 \nabla_x \nabla \perp m \Phi \right) - 2 S_0 B \nabla_x \nabla \perp m n - 2 \varepsilon^2 \Pi \left( D \Phi \partial_x \phi, \nabla_x \nabla \perp m n \right) \right] dx$$

$$+ \int i \left[ \frac{1}{2} \Pi \left( \nabla \perp \nabla \perp m \Phi, \nabla \perp m \right) \right.$$ 

$$\cdot \left[ -\frac{\varepsilon^2}{2} (\nabla_x^2)^2 \nabla \perp m n - 2 B S_0 \nabla_x \nabla \perp m \Phi \right.$$ 

$$\left. - 2 \varepsilon^2 \Pi \left( D \Phi \partial_x \phi, \nabla_x \nabla \perp m \Phi \right) + 4 \lambda \nabla \perp m n + 4 \varepsilon^2 F_1 (n, \nabla \perp m n) + 4 \lambda \varepsilon^2 \Pi (i n, \nabla \perp m n) \right] dx$$

$$\overset{\text{def}}{=} VI_1 + VI_2.$$
Step 2: treating $I$. Since $S_0$ is self-adjoint, the first term gives

$$\int \nabla_t^\top \nabla^m \Phi^e \cdot \nabla_x \left( S_0 S_0 \nabla_x^\top \nabla^m \Phi^e \right) \, dx = \int S_0 \nabla_x^\top \nabla^m \Phi^e \cdot S_0 \nabla_x^\top \nabla^m \Phi^e \, dx$$

(4.40)

after an integration by parts. Applying first (3.7), and keeping then in mind the definition of $S_0$, we can commute covariant derivatives to obtain

$$S_0 \nabla_x^\top \nabla^m \Phi^e = S_0 \nabla_x^\top \nabla^m \Phi^e + O_{H^1} (\mathcal{E}_s) = \nabla_t^\top \left( S_0 \nabla_x^\top \nabla^m \Phi^e \right) + O_{L_2} (\mathcal{E}_s).$$

Coming back to (4.40), this gives

$$\int \nabla_t^\top \nabla^m \Phi^e \cdot \nabla_x \left( S_0 S_0 \nabla_x^\top \nabla^m \Phi^e \right) \, dx = \frac{d}{dt} \frac{1}{2} \int \left| S_0 \nabla_x^\top \nabla^m \Phi^e \right|^2 \, dx + O (\mathcal{E}_s^2).$$

(4.41)

Proceeding similarly, we obtain

$$\int \frac{c}{\varepsilon^2} \nabla_x^\top \nabla^m \Phi^e \cdot \nabla_x \left( S_0 S_0 \nabla_x^\top \nabla^m \Phi^e \right) \, dx = - \frac{c}{\varepsilon^2} \int \nabla_t^\top \left( S_0 \nabla_x^\top \nabla^m \Phi^e \right) \cdot S_0 \nabla_x^\top \nabla^m \Phi^e \, dx + O (\mathcal{E}_s^2)$$

(4.42)

Putting together (4.41) and (4.42) gives

$$I = \frac{d}{dt} \frac{1}{2} \int \left| S_0 \nabla_x^\top \nabla^m \Phi^e \right|^2 \, dx + O (\mathcal{E}_s^2).$$

(4.43)

Step 3: treating $II$. We start with

$$\int \nabla_t^\perp \nabla^\perp n \cdot \left[ -\varepsilon^2 (\nabla_x^\perp)^2 \nabla^\perp m + 4\lambda \nabla^\perp m \right] \, dx = \frac{d}{dt} \left[ \varepsilon^2 |\nabla_x^\perp \nabla^\perp m|^2 + 4\lambda |\nabla^\perp m|^2 \right] + O (\mathcal{E}_s^2)$$

where we used (3.8) to estimate the term resulting from the commutation of $\nabla_x^\perp$ and $\nabla_t^\perp$. Next, by making use of the symmetry of $i \Pi^T (in, \cdot)$ (Proposition 3.1) and $F_1 (n, \cdot)$, we obtain

$$\int \nabla_t^\perp \nabla^\perp m \cdot 4\lambda \varepsilon^2 \Pi^T (in, \nabla^\perp m) \, dx = \varepsilon^2 \frac{d}{dt} \int 2\lambda \nabla^\perp m \cdot i \Pi^T (in, \nabla^\perp m) \, dx + O (\mathcal{E}_s^2),$$

and

$$\int \nabla_t^\perp \nabla^\perp m \cdot 4\varepsilon^2 F_1 (n, \nabla^\perp m) \, dx = \varepsilon^2 \frac{d}{dt} \int \nabla^\perp m \cdot 2F_1 (n, \nabla^\perp m) \, dx + O (\mathcal{E}_s^2).$$

Gathering the three previous equalities gives

$$\int \nabla_t^\perp \nabla^\perp m \cdot \left[ -\varepsilon^2 (\nabla_x^\perp)^2 \nabla^\perp m + 4\lambda \nabla^\perp m + 4\varepsilon^2 F_1 (n, \nabla^\perp m) + 4\lambda \varepsilon^2 \Pi^T (in, \nabla^\perp m) \right] \, dx$$

$$= \frac{d}{dt} \left[ \varepsilon^2 |\nabla_x^\perp \nabla^\perp m|^2 + 4\lambda |\nabla^\perp m|^2 + 4\varepsilon^2 F_1 (n, \nabla^\perp m) \cdot \nabla^\perp m + 4\lambda \varepsilon^2 \nabla^\perp m \cdot i \Pi^T (in, \nabla^\perp m) \right] \, dx$$

$$+ O (\mathcal{E}_s^2).$$

Similarly, one can show that

$$\int \frac{1}{\varepsilon^2} \nabla_x^\perp \nabla^\perp m \cdot \left[ -\varepsilon^2 (\nabla_x^\perp)^2 \nabla^\perp m + 4\lambda \nabla^\perp m + 4\varepsilon^2 F_1 (n, \nabla^\perp m) \cdot \nabla^\perp m$$

$$+ 4\lambda \varepsilon^2 \Pi^T (in, \nabla^\perp m) \right] \, dx = O (\mathcal{E}_s^2).$$
Gathering the two previous equalities gives

\[
II = \frac{d}{dt} \frac{1}{2} \int \left[ \varepsilon^2 |\nabla_x^{\perp} \nabla^{\perp m} n|^2 + 4\lambda |\nabla^{\perp m} n|^2 \\
+ 4\varepsilon^2 F_1(n, \nabla^{\perp m} n) \cdot \nabla^{\perp m} n + 4\lambda\varepsilon^2 \nabla^{\perp m} n \cdot i \Pi^\top (in, \nabla^{\perp m} n) \right] \, dx + O(\mathcal{E}_s^2). \quad (4.44)
\]

Step 4: treating \(III\). We shall first compute

\[
III_1 = -2 \int \left[ \nabla^{\perp m} n \cdot BS_0 \nabla^{\perp m} \Phi^\varepsilon + \nabla^{\perp m} n \cdot BS_0 \nabla^{\perp m} \Phi^\varepsilon \right] \, dx.
\]

We note that

\[
\int \nabla^{\perp m} n \cdot BS_0 \nabla^{\perp m} \Phi^\varepsilon = \frac{d}{dt} \left( \int \nabla^{\perp m} n \cdot BS_0 \nabla^{\perp m} \Phi^\varepsilon \, dx \right)
\]

\[
- \int \nabla^{\perp m} n \cdot \nabla^{\perp}_t (BS_0 \nabla^{\perp m} \Phi^\varepsilon) \, dx
\]

To compute the last term, we note that thanks to (112) and (3.7) we have

\[
\int \nabla^{\perp m} n \cdot \nabla^{\perp}_t (BS_0 \nabla^{\perp m} \Phi^\varepsilon) \, dx = - \int \nabla^{\perp m} n \cdot BS_0 \nabla^{\perp m} \Phi^\varepsilon \, dx + C(\psi^2)
\]

\[
= \int S_0 B \nabla^{\perp m} n \cdot \nabla^{\perp}_t \nabla^{\perp m} \Phi^\varepsilon \, dx + O(\mathcal{E}_s^2)
\]

since \(S_0\) is symmetric and \(B\) is skew symmetric. This yields

\[
III_1 = -2 \frac{d}{dt} \int \nabla^{\perp m} n \cdot BS_0 \nabla^{\perp m} \Phi^\varepsilon \, dx + O(\mathcal{E}_s^2).
\]

Since by using again the skew symmetry of \(B\) and the symmetry of \(S_0\), we have that

\[
III_2 = \frac{2\varepsilon}{\varepsilon^2} \int \left[ \nabla^{\perp m} n \cdot BS_0 \nabla^{\perp m} \Phi^\varepsilon + \nabla^{\perp m} n \cdot BS_0 \nabla^{\perp m} \Phi^\varepsilon \right] \, dx = 0
\]

we have thus proven that

\[
III = -2 \frac{d}{dt} \int \nabla^{\perp m} n \cdot BS_0 \nabla^{\perp m} \Phi^\varepsilon \, dx + O(\mathcal{E}_s^2). \quad (4.45)
\]

Step 5: treating \(IV\). Let us first remark that for any tangent vector field \(X \in T\mathcal{L}\) and normal vector field \(N \in N\mathcal{L}\), we have

\[
\Pi^\top (D\Phi \partial_x \phi, N) \cdot X + N \cdot \Pi^\top (D\Phi \partial_x \phi, X) = 0. \quad (4.46)
\]

Indeed, by using Proposition (3.1) we can write

\[
\Pi^\top (D\Phi \partial_x \phi, N) \cdot X = -i \Pi^\top (D\Phi \partial_x \phi, N) \cdot X = -i \Pi^\top (D\Phi \partial_x \phi, iN) \cdot X
\]

\[
= -iN \cdot i \Pi^\top (D\Phi \partial_x \phi, X) = -N \cdot \Pi^\top (D\Phi \partial_x \phi, X).
\]

This immediately yields that

\[
\int \left[ \nabla^{\perp m} n \cdot \Pi^\top (D\Phi \partial_x \phi, \nabla^{\perp m} n) + \nabla^{\perp m} n \cdot \Pi^\top (D\Phi \partial_x \phi, \nabla^{\perp m} \Phi^\varepsilon) \right] \, dx = 0.
\]
To handle the terms with time derivatives we can proceed as previously due to this skew symmetry property. We observe that

\[
\int \nabla_i \nabla^\perp m \cdot \left( -2\varepsilon^2 \Pi^\perp (D\Phi \partial_x \phi, \nabla^\perp m \Phi^\varepsilon) \right) \, dx \\
= -\frac{d}{dt} \int 2\varepsilon^2 \nabla^\perp m \cdot \Pi^\perp (D\Phi \partial_x \phi, \nabla^T \nabla^\perp m \Phi^\varepsilon) \, dx \\
+ 2\varepsilon^2 \int \nabla^\perp m \cdot \Pi^\perp (D\Phi \partial_x \phi, \nabla^T \nabla^\perp m \Phi^\varepsilon) \, dx + O(\varepsilon_s^2)
\]

Focusing on the last integral above, it can be transformed using successively (4.46), integrating by parts, and using the differentiated hydrodynamical equation (4.17) to replace \( \nabla_i \nabla^T \Phi^\varepsilon \), to give

\[
2\varepsilon^2 \int \nabla^\perp m \cdot \Pi^\perp (D\Phi \partial_x \phi, \nabla^T \nabla^\perp m \Phi^\varepsilon) \, dx = -2\varepsilon^2 \int \nabla^T \nabla^\perp m \Phi^\varepsilon \cdot \Pi^\perp (D\Phi \partial_x \phi, \nabla^\perp m) \, dx \\
= 2\varepsilon^2 \int \nabla^T \nabla^\perp m \Phi^\varepsilon \cdot \Pi^\perp (D\Phi \partial_x \phi, \nabla^\perp m) \, dx + 2\varepsilon^2 \int \nabla^T \nabla^\perp m \Phi^\varepsilon \cdot \Pi^\perp (D\Phi \partial_x \phi, \nabla^\perp m) \, dx + O(\varepsilon_s^2) \\
= 2\varepsilon^2 \int \nabla^T \nabla^\perp m \Phi^\varepsilon \cdot \Pi^\perp (D\Phi \partial_x \phi, \nabla^\perp m) \, dx + O(\varepsilon_s^2).
\]

Going back to the definition of \( IV \), and gathering the above equalities, it follows that

\[
IV = -2\frac{d}{dt} \int \varepsilon^2 \nabla^\perp m \cdot \Pi^\perp (D\Phi \partial_x \phi, \nabla^T \nabla^\perp m \Phi^\varepsilon) \, dx + O(\varepsilon_s^2).
\]

Step 6: treating \( V \). By using that \( S_0^{-1} \) is symmetric, that \( i \) is skew symmetric and that in the first integral

\[
\nabla^T_x (S_0 S_0 \nabla^T_x \nabla^\perp m \Phi^\varepsilon) = S_0 \nabla^T_x (S_0 \nabla^T_x \nabla^\perp m \Phi^\varepsilon) + [\nabla^T_x, S_0] S_0 \nabla^T_x \nabla^\perp m \Phi^\varepsilon,
\]

we observe that the two integrals cancel almost exactly, simply leaving the above commutator term, as well as another lower order commutator. In other words, \( V \) reduces to

\[
V = \frac{1}{2} \int S_0^{-1} i \left[ (\nabla^\perp_x)^2 \nabla^\perp m + \frac{2}{\varepsilon^2} B S_0 \nabla^T_x \nabla^\perp m \Phi^\varepsilon - \frac{4\lambda}{\varepsilon^2} \nabla^\perp m + 2 \Pi^\perp (D\Phi \partial_x \phi, \nabla^T_x \nabla^\perp m \Phi^\varepsilon) \right] \\
- 4F_1(n, \nabla^\perp m) - 4\lambda n \Pi^\perp (i n, \nabla^\perp m) \cdot \left[ (S_0, \nabla^T_x) \left( S_0 \nabla^T_x \nabla^\perp m \Phi^\varepsilon \right) \right] \, dx \\
+ \frac{1}{2} \int S_0^{-1} i \left[ (\nabla^\perp_x)^2 \nabla^\perp m + \frac{2}{\varepsilon^2} B S_0 \nabla^T_x \nabla^\perp m \Phi^\varepsilon - \frac{4\lambda}{\varepsilon^2} \nabla^\perp m + 2 \Pi^\perp (D\Phi \partial_x \phi, \nabla^T_x \nabla^\perp m \Phi^\varepsilon) \right] \\
- 4F_1(n, \nabla^\perp m) - 4\lambda n \Pi^\perp (i n, \nabla^\perp m) \cdot 2\varepsilon^2 (S_0 - Id) \Pi^\perp (D\Phi \partial_x \phi, \nabla^\perp_x \nabla^\perp m) \, dx \\
= V_1 + V_2.
\]

Since

\[
[S_0, \nabla^T_x] = [Id + \varepsilon^2 (\Pi^\perp)(\cdot, n), \nabla^T_x] = -\varepsilon^2 \Pi^\perp (\cdot, \nabla^T_x n) - \varepsilon^3 (\nabla D\Phi \partial_x \phi \Pi^\perp)(\cdot, n),
\]

it is easy to see that \( V_1 \) can be further reduced to

\[
V_1 = -\frac{\varepsilon^2}{2} \int S_0^{-1} i (\nabla^\perp_x)^2 \nabla^\perp m \cdot \Pi^\perp (S_0 \nabla^T_x \nabla^\perp m \Phi^\varepsilon, \nabla^\perp_x n) \, dx + O(\varepsilon_s^2).
\]

Finally, by using again that \( S_0 = Id + \varepsilon^2 \Pi^\perp (\cdot, n) \), this simplifies further to

\[
V_1 = -\frac{\varepsilon^2}{2} \int i (\nabla^\perp_x)^2 \nabla^\perp m \cdot \Pi^\perp (\nabla^T_x (\nabla^T \Phi^\varepsilon), \nabla^\perp_x n) \, dx + O(\varepsilon_s^2).
\]

The term \( V_2 \) is easy to handle. Again since \( S_0 - Id = O(\varepsilon^2) \), we obtain that

\[
V_2 = O(\varepsilon_s^2).
\]
Note that we use in a crucial way that the $E_{s,1}$ part of the energy allows to control $\varepsilon^3 \| \partial_{xx} n \|_s$. We have thus proven that

$$V = -\frac{\varepsilon^2}{2} \int i(\nabla_x^\perp)^2 \nabla^\perp m_n \cdot \Pi^\top \left( \nabla_x^\top \nabla^\top m \Phi^\varepsilon, \nabla_x^\perp n \right) \, dx + O(\varepsilon_s^2). \quad (4.48)$$

Step 7: treating $VI$. We start with the first integral in the definition of $VI$: using the definition of $S_0 = Id + \varepsilon^2 \Pi^\top (\cdot, n)$, and the symmetry of $i \Pi^\top (\cdot, D\Phi \partial_x \phi)$ (Proposition 3.1),

$$VI_1 = \frac{1}{2} \int \left[ S_0^{-1} i \Pi^\top (S_1 \partial_x \partial^m \phi, D\Phi \partial_x \phi) \cdot \left[ -\nabla_x^\top \left( S_0 S_0 \nabla_x^\top (D\Phi \partial^m \phi) \right) \right] \right] \, dx + O(\varepsilon_s^2)$$

$$= \frac{1}{2} \int \left[ i \Pi^\top (D\Phi \partial_x \partial^m \phi, D\Phi \partial_x \phi) \cdot \left[ -\nabla_x^\top (D\Phi \partial^m \phi) \right] \right] \, dx + O(\varepsilon_s^2)$$

$$= \frac{1}{2} \int \left[ i \Pi^\top (D\Phi \partial_x \partial^m \phi, D\Phi \partial_x \phi) \cdot \left[ -\nabla_x^\top (D\Phi \partial^m \phi) \right] \right] \, dx + O(\varepsilon_s^2)$$

$$= O(\varepsilon_s^2).$$

Finally, it is easy to see that

$$VI_2 = \frac{1}{2} \int \left[ i \Pi^\top \left( \nabla_x^\top \nabla^\top m \Phi^\varepsilon, \nabla_x^\perp n \right) \right] \cdot \left[ -\varepsilon^2 (\nabla_x^\perp)^2 \nabla^\perp m_n \right] \, dx + O(\varepsilon_s^2)$$

(notice that this last line and the expression we found for $III$ in (4.48) will cancel to leave only $O(\varepsilon_s^2)$). We have thus proven that

$$VI = \frac{1}{2} \int \left[ i \Pi^\top \left( \nabla_x^\top \nabla^\top m \Phi^\varepsilon, \nabla_x^\perp n \right) \right] \cdot \left[ -\varepsilon^2 (\nabla_x^\perp)^2 \nabla^\perp m_n \right] \, dx + O(\varepsilon_s^2), \quad (4.50)$$

so that

$$V + VI = O(\varepsilon_s^2).$$

Step 8: conclusion. From the identity (4.39) and (4.43), (4.44), (4.45), (4.47), (4.48) and (4.50) we deduce that

$$\frac{d}{dt} \int \left[ S_0 \nabla_x^\top \nabla^\top m \Phi^\varepsilon \right] \, dx \leq \varepsilon^2 |\nabla_x^\perp n^\perp|^2 + 4\lambda |\nabla^\perp m_n|^2 - 4\varepsilon^2 |\nabla^\perp m_n \cdot & B S_0 \nabla_x^\top \nabla^\top m \Phi^\varepsilon$$

$$-4\varepsilon^2 |\nabla^\perp m_n \cdot \Pi^\perp (D\Phi \partial_x \phi, \nabla_x^\top \nabla^\top m \Phi^\varepsilon) + 4\varepsilon^2 |\nabla^\perp m_n \cdot F_1(n, \nabla^\perp m_n) + 4\lambda \varepsilon^2 |\nabla^\perp m_n \cdot i \Pi^\top (in, \nabla^\perp m_n) \right] \, dx$$

$$= O(\varepsilon_s^2)$$

for $1 \leq |m| \leq s$. Let us call $E^m$, $|m| \geq 1$ the integral in the left hand side. In the case $m = 0$, we can get a similar estimate by using directly (1.9). Note that since the structure of (1.9) is slightly different from the structure of the system in Proposition 4.3 for $|m| \geq 1$, we can proceed as previously by using a slightly different multiplier. Let us set

$$E^0 = \frac{1}{2} \int \left[ 4\lambda |n^\varepsilon|^2 + \varepsilon^2 |\nabla_x^\perp n^\perp|^2 + \frac{4}{3} \varepsilon^2 F_1(n^\varepsilon, n^\varepsilon) \cdot n^\varepsilon + |S_0 D\Phi \partial_x \phi^\varepsilon|^2$$

$$-4iB \left( D\Phi \partial_x \phi^\varepsilon + \frac{1}{2} \varepsilon^2 \Pi^\top (D\phi \partial_x \phi^\varepsilon, n^\varepsilon) \right) \cdot in^\varepsilon \right].$$

Using the same arguments as above, we can also prove that

$$\frac{d}{dt} E^0 = O(\varepsilon_s^2).$$
(we shall perform a related more precise computation in the proof of Lemma 4.3 below). The conclusion follows if we prove that \( \tilde{E}_{s,2}^2 \sim E_{s,2}^2 \) with 
\[
\tilde{E}_{s,2}^2 = \sum_{|m| \leq s} E^m.
\]
By using (4.19), (4.20), we first easily get that 
\[
\tilde{E}_{s,2}(u, t)^2 = \sum_{|m| \leq s} \frac{1}{2} \int \left[ |S_0 \nabla^T_x \nabla^T m \Phi^e|^2 + \varepsilon^2 |\nabla^\perp_x \nabla^m n|^2 + 4\lambda |\nabla^m n|^2 - 4\nabla^\perp m \cdot B S_0 \nabla^T m \Phi^e \right] dx + \varepsilon^2 O(E_s(u, t)^2)
\]
\[= O(\varepsilon^2).\]
This also yields 
\[
\tilde{E}_{s,2}(u, t)^2 \geq \sum_{|m| \leq s} \frac{1}{2} \int \left[ |S_0 \nabla^T_x \nabla^T m \Phi^e|^2 + \varepsilon^2 |\nabla^\perp_x \nabla^m n|^2 + 4\lambda |\nabla^m n|^2 - 4|\nabla^\perp m| |B| |S_0 \nabla^T m \Phi^e| \right] dx - \varepsilon^2 O(E_s(u, t)^2).
\]
Note that, by using (H2) and (1.3), we have that 
\[|B|^2 = \mu < \lambda,\]
therefore the quadratic form 
\[Q(X_1, X_2) = X_1^2 + 4\lambda X_2^2 - 4|B|X_1X_2\]
is positive definite. This yields 
\[
\sum_{|m| \leq s} \frac{1}{2} \int \left[ |S_0 \nabla^T_x \nabla^T m \Phi^e|^2 + \varepsilon^2 |\nabla^\perp_x \nabla^m n|^2 + 4\lambda |\nabla^m n|^2 \right] dx \leq \tilde{E}_{s,2}(u, t)^2 + \varepsilon^2 O(E_s(u, t)^2)
\]
and by using again (4.19), (4.20), we finally obtain 
\[E_{s,2}(u, t)^2 \leq \tilde{E}_{s,2}(u, t)^2 + \varepsilon^2 O(E_s(u, t)^2).\]

\[\square\]

4.5. **Proof of Theorem 1.1 in the case \( \mathcal{M} = \mathbb{R}^{2d} \).** The local well-posedness of the Schrödinger system (1.8), which is semi linear, can be deduced from the a priori estimates which have been established in the previous subsections.

It gives a non-empty time interval where there exists a unique solution of (1.8) such that \( E_s(u, t) < +\infty \) and where the representation \( u = \Phi(\varepsilon \phi) + \varepsilon^2 n \) is well-defined. For constants \( R \) and \( r \), we can thus define a maximal existence time 
\[
T^e \overset{def}{=} \sup \left\{ T > 0, \sup_{[0, T]} E_s(u) \leq R \text{ and } \sup_{[0, T]} \varepsilon \| \phi \|_{L^\infty} + \varepsilon^2 \| n \|_{L^\infty} \leq r \right\}.
\]
Choose \( R \) to be a constant times the energy of the data, and \( r \) such that the representation \( (\varepsilon \phi, \varepsilon^2 n) \mapsto \Phi(\varepsilon \phi) + \varepsilon^2 n \) is a diffeomorphism if \( \varepsilon \| \phi \|_{L^\infty} + \varepsilon^2 \| n \|_{L^\infty} \leq r \) and \( r > 2c_0 \).

We shall prove that \( T^e \) is bounded from below by a positive time, uniformly in \( \varepsilon \). Consider \( T \leq T^e \). From the a priori estimates of Proposition 4.2 and Proposition 4.4 we get the existence of \( C_0 \) independent of \( \varepsilon \) and \( T \) that depends only on \( E_s(u, 0) \) such that 
\[
\sup_{[0, T]} E_s(u, t) \leq C_0 + (\varepsilon + T)O(R)
\]  
(4.51)
Next, simply by Sobolev embedding,
\[
\sup_{[0,T]} \varepsilon^2 \|n(t)\|_{L^\infty} \lesssim \varepsilon^2 \sup_{[0,T]} \mathcal{E}_s(u,t) \leq \varepsilon^2 R.
\] (4.52)

It remains to estimate \( \varepsilon \|\phi\|_{L^\infty} \). From the first line of (1.9), we first obtain by integrating in time and by using the uniform control of \( \mathcal{E}_s \) for \( s \geq 2 \) that
\[
|\varepsilon \phi(t,x)| \lesssim \sup_x \int_0^T \frac{1}{\varepsilon} |W(s,x)| \, ds + TO(r,R) + c_0, \quad \forall t \in [0,T].
\] (4.53)

where
\[
W \overset{\text{def}}{=} (c + iB)D\Phi \partial_x \phi - 2i\lambda n.
\] (4.54)

We shall thus estimate the quantity involving \( W \) in the right hand side of (4.53). From the first line of (4.17) with \( m = (0,1) \) i.e. \( \partial^m = \partial_x \), we have
\[
\nabla_t^T (D\Phi \partial_x \phi) = \frac{1}{\varepsilon^2} \nabla_x^T W + O_{L^2}(r,R)
\] (4.55)

In a similar way, by using that from (112), we have \( i(c + iB) = (c - iB)i \), we obtain from the second line of (1.9) that
\[
\nabla_t^T (in) = \frac{1}{\varepsilon^2} \nabla_x^T ((c - iB)in - \frac{1}{2} D\Phi \partial_x \phi) + O_{L^2}(r,R).
\] (4.56)

Now by using (112) again, we observe that \( (c-iB)(c+iB) = c^2 + \mu = \lambda \) and hence that
\[
(c \pm iB)^{-1} = \frac{1}{\lambda}(c \mp iB).
\]

This yields
\[
\nabla_t^T (in) = -\frac{1}{\varepsilon^2} \nabla_x^T \left( \frac{c-iB}{2\lambda} \right) W + O_{L^2}(r,R),
\] (4.57)

and therefore
\[
\nabla_t^T W = \frac{2c}{\varepsilon^2} \nabla_x^T W + O_{L^2}(r,R).
\]

Hence, \( |W|^2 \) solves the scalar transport equation
\[
\partial_t |W|^2 = \frac{2c}{\varepsilon^2} \partial_x |W|^2 + F(t,x), \quad F = O_{L^2}(r,R) \cdot W = O_{L^1}(R,r).
\]

Solving explicitly this equation, we find that
\[
|W(t,x)|^2 = \left| W_0(x + \frac{2c}{\varepsilon^2} t) \right|^2 + \int_0^t F(s,x + \frac{2c}{\varepsilon^2} (t-s)) \, ds.
\]

This yields by using the Fubini Theorem and a change of variable
\[
\int_0^T |W(t,x)|^2 \, dt \lesssim \varepsilon^2 \|W_0\|_{L^2}^2 + \left| \int_0^T \int_0^t F(s,x + \frac{2c}{\varepsilon^2} (t-s)) \, ds \, dt \right| \lesssim \varepsilon^2 \|W_0\|_{L^2}^2 + \varepsilon^2 \int_0^T \left| \int_x^{x+\frac{2c}{\varepsilon^2} (T-s)} F(s,y) \, dy \right| \, ds.
\]

Therefore, since \( F = O_{L^1}(R,r) \),
\[
\sup_x \int_0^T |W(t,x)|^2 \, dt \lesssim \varepsilon^2 (\|W_0\|_{L^2}^2 + TO(r,R)).
\]

This finally yields
\[
\sup_x \frac{1}{\varepsilon} \int_0^T |W(t,x)| \, dt \lesssim \sqrt{T} (\|W_0\|_{L^2} + \sqrt{T} O(r,R)).
\] (4.58)
By plugging this estimate in (4.53), we thus obtain
\[ \sup_{[0,T]} \| \varepsilon \phi \|_{L^\infty} \leq c_0 + (\sqrt{T} + T)O(r, R). \]  
(4.59)

By combining the last estimate and (4.51), (4.52), we get by a classical bootstrap argument that 
\( T^\varepsilon \) is bounded from below by \( T_0 > 0 \) that is uniform for \( \varepsilon \in [0, 1] \). This ends the proof of Theorem 1.1 in the case \( \mathcal{M} = \mathbb{R}^{2d} \).

4.6. Proof of Theorem 1.2 in the case \( \mathcal{M} = \mathbb{R}^{2d} \). In this section only, we add \( \varepsilon \) superscripts to \( u, p, n, \) etc... in order to make the dependence on \( \varepsilon \) more clear.

We first remark that \( u^\varepsilon = p^\varepsilon + \varepsilon^2 n^\varepsilon = \Phi(\varepsilon \phi^\varepsilon) + \varepsilon^2 n^\varepsilon \). Thanks to the uniform estimates of Theorem 1.1, we have by Sobolev embedding that \( n^\varepsilon \) is uniformly bounded in \( L^\infty \) and hence \( \varepsilon^2 n^\varepsilon \) converges strongly to zero in \( L^\infty([0, T] \times \mathbb{R}) \). The study of the convergence of \( \varepsilon \phi^\varepsilon \) is more delicate. This will be the first step in the proof of Theorem 1.2

Step 1: Convergence of \( \varepsilon \phi^\varepsilon \) to 0 in \( L^\infty \). Notice that thanks to Theorem 1.1, we already have the estimate
\[ \| \varepsilon \phi^\varepsilon \|_{L^\infty([0,T] \times \mathbb{R})} \leq 2c_0. \]
Moreover, by using again the first line of (1.11), we have
\[ \sup_x \int_0^T |\varepsilon \partial_t \phi^\varepsilon(t, x)| \, dt \lesssim \sup_x \int_0^T \frac{1}{\varepsilon} |((c + iB)D\phi^\varepsilon - 2i\lambda n^\varepsilon)(t, x)| \, dt + O(1), \quad \forall t \in [0, T]. \]
Consequently, by using (4.53) and (4.58), we get that
\[ \sup_x \int_0^T |\varepsilon \partial_t \phi^\varepsilon(t, x)| \, dt = O(1) \]  
(4.60)

We also have that \( \| \varepsilon \partial_x \phi^\varepsilon \|_{L^\infty([0,T] \times \mathbb{R})} = O(\varepsilon) \) thanks to the uniform estimates of Theorem 1.1. We thus get from the Arzela-Ascoli Theorem that a subsequence \( \varepsilon_n \phi_n^\varepsilon \) converges in \( C_{loc}([0, T] \times \mathbb{R}) \) towards some \( \gamma_\infty(t, x) \) and that since \( \varepsilon_n \partial_x \phi_n^\varepsilon \) converges to zero (in the distribution sense for example), we must have \( \partial_x \gamma_\infty(t, x) = 0 \). Hence \( \gamma_\infty \) is a function of time only that satisfies
\[ \sup_{[0,T]} |\gamma_\infty(t)| + \int_0^T |\partial_t \gamma_\infty(t)| \, dt \leq C \]  
(4.61)
for some \( C > 0 \). However, it is not possible to prove directly that \( \gamma_\infty = 0 \), and hence \( \varepsilon \phi^\varepsilon \to 0 \).

In order to prove that \( \varepsilon \phi^\varepsilon \to 0 \), we shall proceed differently and make a crucial use of the assumption (1.11) and the conserved (or almost conserved) quantities of the Schrödinger maps system.

From the one-dimensional Sobolev inequality, we first observe that
\[ (\varepsilon \| \phi^\varepsilon \|_{L^\infty})^2 \leq \frac{1}{2} \| \varepsilon \phi^\varepsilon \|_{L^2} \| \varepsilon \partial_x \phi^\varepsilon \|_{L^2} \leq C \varepsilon^2 \| \phi^\varepsilon \|_{L^2} \]
(4.62)
since \( \| \partial_x \phi^\varepsilon \|_{L^2} \) is uniformly bounded thanks to the estimates of Theorem 1.1. Next, by using again the first line of (1.9), we get that
\[ \| \varepsilon^2 \phi^\varepsilon(t) \|_{L^2} \lesssim \| \varepsilon^2 \phi^\varepsilon(0) \|_{L^2} + \int_0^t \| W^\varepsilon(s) \|_{L^2} \, ds + O(\varepsilon^2) \]  
(4.63)
with \( W^\varepsilon = (c + iB)D\phi^\varepsilon - 2i\lambda n^\varepsilon \).

Thus it suffices to prove that \( \lim_{\varepsilon \to 0} \sup_{[0,T]} \| W^\varepsilon(t) \|_{L^2} = 0 \) in order to deduce that \( \varepsilon \phi^\varepsilon \overset{L^\infty}{\to} 0 \). To this aim we now turn.
Observe that $|W|^2 = \lambda|D\Phi(\partial_x \phi^\varepsilon)|^2 + 4\lambda^2|n^\varepsilon|^2 - 4\lambda \text{Im} \cdot (c + iB) D\Phi \partial_x \phi^\varepsilon$. After integrating, the two first terms correspond, up to lower order terms, to the Hamiltonian, while the last term gives a quantity which can be thought of as momentum$^2$, which should be almost conserved.

Guided by this idea, we will derive directly an approximately conserved quantity by working directly on the hydrodynamical system (1.9). Keeping only the terms up to order 1 in $\varepsilon$ in (1.9), it reads

$$
\begin{aligned}
S_0 D\Phi \partial_t \phi &= \frac{1}{\varepsilon^2} i \left[ -2 \lambda n - i(c + iB) S_1 \partial_x \phi ight. \\
&
\left. + \varepsilon^2 \frac{1}{2} \Pi^\perp (D\Phi \partial_x \phi, D\Phi \partial_x \phi) + \frac{1}{2} \varepsilon^2 (\nabla_x^\perp)^2 n - \varepsilon^2 F_1(n, n) \right] + O_H(\varepsilon)
\end{aligned}
$$

$$
\nabla_x^\perp n = \frac{1}{\varepsilon^2} i \left[ \frac{1}{2} \nabla_x^\top (S_1 \partial_x \phi) - i(c + iB) \nabla_x^\perp n + \varepsilon^2 \frac{1}{2} \Pi^\perp \left( D\Phi \partial_x \phi, \nabla_x^\perp n \right) \right] + O_H(\varepsilon).
$$

In view of the above right hand side, we define

$$
H(u^\varepsilon) = \int \left[ \lambda |n^\varepsilon|^2 + \frac{1}{4} \varepsilon^2 (\nabla_x^\perp n^\varepsilon)^2 + \frac{1}{2} \varepsilon^2 F_1(n^\varepsilon, n^\varepsilon) \cdot n^\varepsilon + \frac{1}{4} |S_0 D\Phi \partial_x \phi^\varepsilon|^2 \\
- (c + iB) \left( D\Phi \partial_x \phi^\varepsilon + \varepsilon^2 \frac{1}{2} \Pi^\perp \left( D\Phi \partial_x \phi^\varepsilon, n^\varepsilon \right) \right) \cdot \nabla_x^\perp n \right].
$$

The above quantity is almost conserved in the sense that

\textbf{Lemma 4.5.} Under the assumptions of Theorem 1.1, we have

$$
\frac{d}{dt} H(u^\varepsilon)(t) = O(\varepsilon)
$$

uniformly on $[0, T]$.

We postpone the proof of this lemma until the end of this section.

We observe that for $t \in [0, T]$, we have

$$
H(u^\varepsilon(t)) = \frac{1}{2} \int \left[ 2\lambda |n^\varepsilon|^2 + \frac{1}{2} |D\Phi \partial_x \phi^\varepsilon|^2 - 2(c + iB) D\Phi \partial_x \phi^\varepsilon \cdot \nabla_x^\perp n \right] dx + O(\varepsilon^2)
$$

and hence that by using the definition (1.30) of $W^\varepsilon$ and (1.2), we have

$$
H(u^\varepsilon(t)) = \frac{1}{4\lambda} \int |W^\varepsilon|^2 dx + O(\varepsilon^2).
$$

By combining this observation with Lemma 4.3, we obtain that

$$
\sup_{t \in [0, T]} \|W^\varepsilon(t)\|_{L^2}^2 \lesssim \|W^\varepsilon(0)\|_{L^2}^2 + \varepsilon
$$

and hence since $\|W^\varepsilon(0)\|_{L^2} \to 0$ as $\varepsilon \to 0$, we finally obtain that

$$
\sup_{t \in [0, T]} \|W^\varepsilon(t)\|_{L^2} \to 0 \text{ as } \varepsilon \to 0.
$$

We have thus proven thanks to (4.62) and (4.63) that

$$
\lim_{\varepsilon \to 0} \sup_{[0, T]} \|\varepsilon \phi\|_{L^\infty} = 0.
$$

This ends the first step.

---

$^2$The same problem occurs for the usual nonlinear Schrödinger equation, where the conserved momentum is given by $P = \frac{i}{\varepsilon} \int i \partial_x u^\varepsilon \cdot u^\varepsilon dx$. 
Step 2: Derivation of the KdV limit. From the estimates of Theorem 1.1 that yield $\varepsilon^2 n^\varepsilon = O_L(\varepsilon^2)$, we get that

$$\|u^\varepsilon\|_{L^\infty([0,T]\times\mathbb{R})} \to 0.$$  \hfill (4.67)

This yields in particular that tensors such as $\Pi_p^\varepsilon$, $\Pi_p^\perp$ that implicitly depend on $p$ converge uniformly towards $\Pi_p^\varepsilon$, $\Pi_p^\perp$.

The uniform $H^s$ estimates of Theorem 1.1 provide local compactness in space, therefore we only need to get compactness in time in order to obtain strong convergence.

Define

$$A^\varepsilon = 2i\lambda n^\varepsilon \quad \text{and} \quad U^\varepsilon = (c - iB)D\Phi\partial_x\phi^\varepsilon + A^\varepsilon.$$  

Note that $W^\varepsilon = (c + iB)D\Phi\partial_x\phi^\varepsilon - A^\varepsilon$. By combining (4.55) and (4.57),

$$\nabla_t U^\varepsilon = O_{L^2}(1).$$

In particular, we obtain that $U^\varepsilon$ satisfies for every $t, s, 0 \leq s \leq t \leq T$

$$\|U^\varepsilon(t) - U^\varepsilon(s)\|^2_{L^2} = \int_0^t \int_s^t \nabla_t U(\tau, x) \cdot U(\tau, x) \, d\tau \, dx \lesssim |t - s|$$

by the Cauchy-Schwarz inequality.

From Theorem 1.1, we have that $U^\varepsilon$ is bounded in $L^\infty([0,T], H^s(u^{-1}T\mathcal{L}))$, where $H^s(u^{-1}T\mathcal{L})$ is given by the application of covariant derivatives. Since $\phi$ is bounded in $H^s(\mathbb{R}^d)$, this implies that $U^\varepsilon$ is bounded in $L^\infty([0,T], H^s(\mathbb{R}^{2d}))$, where this time we simply view $U^\varepsilon$ as a vector in $\mathbb{R}^{2d}$. We can now apply the Arzela-Ascoli Theorem, and get that there exists a sequence $\varepsilon_n$ such that $U^\varepsilon_n$ converges in $\mathcal{C}([0,T], H^s_{\text{loc}})$ to $U$ for every $\sigma, \sigma < s$ for some $U \in L^\infty([0,T], H^s)$. From (4.65), we already had that $W^\varepsilon$ converges strongly to 0 in $\mathcal{C}([0,T], L^2)$. Since $W^\varepsilon$ is also bounded in $L^\infty([0,T], H^s)$, this yields by interpolation that it converges strongly to 0 in $\mathcal{C}([0,T], H^\sigma)$ for every $\sigma, \sigma < s$. Next, since

$$D\Phi\partial_x\phi^\varepsilon = \frac{1}{2c}(U^\varepsilon + W^\varepsilon), \quad A^\varepsilon = 2\lambda n^\varepsilon = \frac{1}{2c}((c + iB)U^\varepsilon - (c - iB)W^\varepsilon)$$  \hfill (4.68)

we get that $D\Phi\partial_x\phi^\varepsilon$ and $in^\varepsilon$ also converge in $\mathcal{C}([0,T], H^\sigma_{\text{loc}})$. Moreover, by denoting $A \in L^\infty([0,T], H^s)$ the limit of $A^\varepsilon$, we get that

$$D\Phi\partial_x\phi^\varepsilon \to (c + iB)^{-1}A = \frac{1}{\lambda}(c - iB)A, \quad in^\varepsilon \to \frac{1}{2\lambda}A \quad \text{in} \mathcal{C}([0,T], H^\sigma_{\text{loc}})$$  \hfill (4.69)

To identify, the limit system, we need to make all the order one terms explicit. By applying $\nabla_x^T$ to the first line of (1.9) written as

$$D\Phi\partial_t\phi^\varepsilon - \frac{1}{\varepsilon^2}(c + iB)D\Phi\partial_x\phi^\varepsilon$$

$$= S_0^{-1}i \left[ \frac{1}{2} \Pi^\perp(S_1\partial_x\phi^\varepsilon, D\Phi\partial_x\phi^\varepsilon) + \frac{1}{2}(\nabla_x^T)^2 n^\varepsilon - 2\lambda n^\varepsilon \varepsilon_2 - F_1(p)(n^\varepsilon, \varepsilon^2) - \varepsilon^4 \Pi^\perp R^\perp(p^\varepsilon, \varepsilon^2n^\varepsilon) \right]$$

(with the help of Corollary 3.2), we find

$$\nabla_t^T(D\Phi\partial_x\phi^\varepsilon) = \frac{1}{\varepsilon^2}\nabla_x^T W^\varepsilon + \frac{1}{2}(\nabla_x^T)^3(in^\varepsilon) - 2iF_1(\nabla_x^T n^\varepsilon, \varepsilon^2) + i\Pi^\perp(D\Phi\partial_x\phi^\varepsilon), D\Phi\partial_x\phi^\varepsilon) - 4\lambda_2^\perp \Pi^\perp(\nabla_x^T(in^\varepsilon), in^\varepsilon) + O_{L^2}(\varepsilon)$$
Note that, for the last term, we have used that by the definition of $S_0$ and Proposition 3.1
\[
\nabla_x S_0^{-1} i \left( -2\frac{\lambda n}{\varepsilon^2} \right) = \nabla_x \left[ \text{Id} + \varepsilon^2 \Pi^\top (\cdot, n) \right]^{-1} i \left( -2\frac{\lambda n}{\varepsilon^2} \right)
\]
\[
= -\frac{2\lambda}{\varepsilon^2} \nabla_x (in^\varepsilon) - 2\lambda \nabla_x \Pi^\top (in^\varepsilon, in^\varepsilon) + O_L^2 (\varepsilon)
\]
\[
= -\frac{2\lambda}{\varepsilon^2} \nabla_x (in^\varepsilon) - 4i\lambda \Pi^\perp (\nabla_x (in^\varepsilon), in^\varepsilon) + O_L^2 (\varepsilon).
\]

Similarly, the second line of (1.9) can be written as a more precise version of (4.56) as follows
\[
\nabla^\top (2\lambda in^\varepsilon) = -\frac{1}{\varepsilon^2} (c - iB) \nabla^\top W^\varepsilon + 2i\lambda \Pi^\perp (D\Phi \partial_x \phi^\varepsilon, \nabla^\top in^\varepsilon) + i\lambda \Pi^\perp (\nabla^\top (D\Phi \partial_x \phi^\varepsilon), in^\varepsilon) + O_L^2 (\varepsilon).
\]
Note that we have again used Proposition 3.1. Consequently, we can combine the two equations to get
\[
\begin{align*}
\nabla^\top ((c - iB) D\Phi \partial_x \phi^\varepsilon + 2\lambda in^\varepsilon) &= (c - iB) \left[ \frac{1}{2} (\nabla^\top)^3 (in^\varepsilon) - 2iF_1 (\nabla^\top n^\varepsilon, n^\varepsilon) 
+ i\Pi^\perp (\nabla^\top (D\Phi \partial_x \phi^\varepsilon), D\Phi \partial_x \phi^\varepsilon) - 4\lambda \Pi^\perp (\nabla^\top (in^\varepsilon), in^\varepsilon) \right] 
+ 2i\lambda \Pi^\perp (D\Phi \partial_x \phi^\varepsilon, \nabla^\top in^\varepsilon) + i\lambda \Pi^\perp (\nabla^\top (D\Phi \partial_x \phi^\varepsilon), in^\varepsilon) + O_L^2 (\varepsilon).
\end{align*}
\]
(4.70)

Using (4.69), it is easy to pass to the limit weakly (in $S^' (\mathbb{R}^{2d})$) in all the terms above, except for the one involving a time derivative on the right hand side. Indeed, only poor estimates are available on $\varepsilon \partial_t \phi^\varepsilon$. Due to this term, we proceed differently: take $\psi \in C_\text{c}^\infty ([0, T] \times \mathbb{R}, \mathbb{R}^d)$ (identifying $T_0 \mathcal{L}$ and $\mathbb{R}^d$) and multiply the above by $D\Phi (\varepsilon \phi^\varepsilon) \psi$ before letting $\varepsilon \to 0$.

Thanks to (4.69), it is easy to see that $\varepsilon_n \to 0$,
\[
\int \int \text{RHS} [4.70] : D\Phi (\varepsilon \phi^\varepsilon) \psi \, dx \, dt \to \int \int \psi \cdot \left[ (c - iB) \left[ \frac{1}{4\lambda} (\nabla^\top)^3 A - \frac{1}{2\lambda^2} iF_{1,0} (i\nabla^\top A, iA) 
+ \frac{1}{\lambda^2} i\Pi^\perp ((c - iB) \nabla^\top A, (c - iB) A) - \frac{1}{\lambda} i\Pi^\perp (\nabla^\top A, A) \right] 
+ \frac{1}{\lambda} i\Pi^\perp ((c - iB) A, \nabla^\top A) + \frac{1}{2\lambda} i\Pi^\perp ((c - iB) \nabla^\top A, A) \right] \, dx \, dt.
\]
(4.71)

(where $F_{1,0} = F_1 (0)$ and $\Pi^\perp_0 = \Pi^\perp (0)$).

The left-hand side of (4.70) is more delicate, because of the time derivative which appears there, and of the poor estimates available on $\varepsilon \partial_t \phi^\varepsilon$. In order to pass to the limit on this term, we basically need to justify that if $X^\varepsilon$ is a vector field in $T_{\Phi (\varepsilon \phi^\varepsilon)} \mathcal{L}$ that converges strongly in $L^2_{\text{loc}} ([0, T] \times \mathbb{R})$ towards $X$ and which is bounded in $L^\infty ([0, T] \times \mathbb{R})$, then $\nabla^\top (X^\varepsilon) = P^\top (\varepsilon \phi^\varepsilon) \partial_t X^\varepsilon$ converges weakly towards $\nabla^\top X = P^\top (0) \partial_t X$. Taking as above $\psi \in C_\text{c}^\infty ([0, T] \times \mathbb{R}, \mathbb{R}^d)$, we have after integrating by parts
\[
\int_{\mathbb{R} \times \mathbb{R}} \nabla^\top X^\varepsilon \cdot D\Phi (\varepsilon \phi^\varepsilon) \psi \, dx \, dt = - \int_{\mathbb{R} \times \mathbb{R}} X^\varepsilon \cdot D\Phi (\varepsilon \phi^\varepsilon) \partial_t \psi \, dt \, dx 
- \int_{\mathbb{R} \times \mathbb{R}} X^\varepsilon \cdot \nabla^\top D\Phi (\varepsilon \phi^\varepsilon, (\varepsilon \partial_t \phi^\varepsilon, \psi)) \, dt \, dx.
\]
The first integral in the right-hand side above obviously converges towards
\[
- \int_{\mathbb{R} \times \mathbb{R}} X \cdot \partial_t \psi \, dt \, dx = \int_{\mathbb{R} \times \mathbb{R}} \nabla^\top X \cdot \psi \, dt \, dx,
\]
thus we just have to prove that the second integral tends to zero. By using (4.60), we obtain that
\[
\left| \int_{\mathbb{R} \times \mathbb{R}} X^\varepsilon \cdot \nabla^T D\Phi_{\varepsilon \phi^x}(\varepsilon \partial_t \phi^x, \psi) \, dt \, dx \right| 
\lesssim \| \nabla^T D\Phi_{\varepsilon \phi^x} \|_{L^\infty([0,T] \times \mathbb{R})} \left( \sup_x \int_0^T |\varepsilon \partial_t \phi^x| \right) \| \psi \|_{L^1(\mathbb{R})} \| X^\varepsilon \|_{L^\infty([0,T] \times \mathbb{R})}
\]
In the above estimate, all the terms are uniformly bounded, and since \( \varepsilon \phi^x \) converges uniformly to zero, we have thanks to (3.5) (which relies on the choice of normal coordinates on \( \mathcal{L} \)) that \( \| \nabla^T (D\Phi)(\varepsilon \phi^x) \|_{L^\infty([0,T] \times \mathbb{R})} \) tends to zero. We have thus proven that
\[
\left| \int \int LHS(4.70) \cdot D\Phi(\varepsilon \phi^x) \psi \, dx \, dt \right| \rightarrow \int \int \psi \cdot \nabla^T_t \left( \frac{(c - iB)^2}{\lambda} A + A \right) \, dx \, dt \quad (4.72)
\]
Combining (4.71) and (4.72) gives the following equality in \( \mathcal{S}'(\mathbb{R}^d) \) (identifying \( T_0 \mathcal{L} \) with \( \mathcal{R}^d \)):
\[
\nabla^T_t \left( \frac{(c - iB)^2}{\lambda} A + A \right) = (c - iB) \left[ \frac{1}{4\lambda} (\nabla^T_x)^3 A - \frac{1}{2\lambda^2} i F_{1,0} (i \nabla^T_x A, iA) 
\right. 
+ \frac{1}{\lambda^2} i \Pi_0^\perp ((c - iB) \nabla^T_x A, (c - iB)A) - \frac{1}{\lambda} i \Pi_0^\perp (\nabla^T_x A, A) 
\left. + \frac{1}{\lambda} i \Pi_0^\perp ((c - iB)A, \nabla^T_x A) + \frac{1}{2\lambda} i \Pi_0^\perp ((c - iB) \nabla^T_x A, A). \right) \quad (4.73)
\]
The above system (4.73) is the desired KdV type equation. We can simplify it a little bit, by noticing that thanks to (4.72), we have
\[
\frac{(c - iB)^2}{\lambda} A + A = 2c \frac{(c - iB)}{\lambda} A
\]
and by using Corollary 3.2 to obtain
\[
2c \nabla^T_t A = \frac{1}{4} \nabla^T_{xxx} A + \left( \frac{3}{2} - \frac{2\mu}{\lambda} - \frac{2c}{\lambda} i_0 B_0 \right) i_0 \Pi_0^\perp \left( \nabla^T_x A, A \right) - \frac{i_0}{2\lambda} F_{1,0} (i_0 \nabla^T_x A, i_0 A). \quad (4.74)
\]
Note that here, \( \nabla^T \) stands for \( P_0^T \nabla \) and therefore, since \( T_0 \mathcal{L} \) is a fixed vector space, we can also write it as
\[
2c \partial_t A = \frac{1}{4} \partial_{xxx} A + \left( \frac{3}{2} - \frac{2\mu}{\lambda} - \frac{2c}{\lambda} i_0 B_0 \right) i_0 \Pi_0^\perp \left( \partial_x A, A \right) - \frac{i_0}{2\lambda} F_{1,0} (i_0 \partial_x A, i_0 A). \quad (4.75)
\]
From the uniqueness for this KdV-type system (see section 5), we thus get that the whole family \( A^\varepsilon \), \( (c + iB) \Phi \partial_x \phi^x \) tends to \( A \) in \( C([0,T], H^s_{loc}) \).

Step 3: Global in space convergence. To obtain the convergence in \( C([0,T], L^2) \), we can proceed as follows. At first, we note that the convergence of \( U^\varepsilon \) also holds in \( C([0,T], L^2_{w}) \) (\( L^2 \) being equipped with the weak topology) and that \( U^\varepsilon \) tends to
\[
U = \frac{(c - iB)^2}{\lambda} A + A = 2c \frac{(c - iB)}{\lambda} A.
\]
We shall prove the global strong convergence in \( L^2(\mathbb{R}) \) of \( U^\varepsilon \). We note that by using that \( W^\varepsilon \) converges strongly to zero in \( C([0,T], H^s) \), for \( \sigma < s \), and the relations (4.68), we can rewrite (4.70) as
\[
\nabla^T_t U^\varepsilon = \frac{1}{8c} (\nabla^T_x)^3 \left( U^\varepsilon - \frac{(c - iB)^2}{\lambda} W^\varepsilon \right) - \frac{1}{8\lambda^2 c^2} (c - iB) iF_1 \left( (c - iB) i \nabla^T_x U^\varepsilon, (c - iB) i U^\varepsilon \right)
+ \frac{5}{8c^2} (c - iB) i \Pi^\perp \left( \nabla^T_x U^\varepsilon, U^\varepsilon \right) - \frac{1}{4\lambda c^2} (c - iB) i \Pi^\perp \left( (c + iB) i \nabla^T_x U^\varepsilon, (c + iB) U^\varepsilon \right) + o_{L^2}(1).
\]
Moreover, integrating by parts again, and also from the last property of Corollary 3.2 that strongly to zero in $C$.

Consequently, the $U$ converge in $C$.

This yields that $O(\varepsilon)$.

Moreover, integrating by parts again,

$$\left| \int (c - iB)^2 (\nabla_x^T U^\varepsilon \cdot U^\varepsilon) dx \right| \leq \|W^\varepsilon\|_{H^1} \|U^\varepsilon\|_{H^1} = o(1).$$

We have thus proven that

$$\frac{d}{dt} \frac{1}{2} \int_R |U^\varepsilon|^2 dx = o(1).$$

This yields for $t \in [0, T]$

$$\|U^\varepsilon(t)\|_{L^2}^2 = \|U^\varepsilon(0)\|_{L^2}^2 + o(1).$$

For the limit equation (see Section 3.1.73), we have that

$$\|A(t)\|_{L^2}^2 = \|A(0)\|_{L^2}^2$$

and we observe (again by (12)) that

$$\|U(t)\|_{L^2}^2 = \frac{4c^2}{\lambda^2} \|(c - iB)A\|_{L^2}^2 = \frac{4c^2}{\lambda} \|A(t)\|_{L^2}^2, \quad \forall t \in [0, T].$$

Consequently, the $L^2$ convergence at the initial time yields that $\|U^\varepsilon(t)\|_{L^2}^2 \to \frac{4c^2}{\lambda} \|A(t)\|_{L^2}^2$ uniformly in time. This yields that $U^\varepsilon$ converges in $C([0, T], L^2)$. Since we already had that $W^\varepsilon$ converges strongly to zero in $C([0, T], L^2)$, we finally obtain from (14.65) that $A^\varepsilon$ and $(c + iB)D\Phi \partial_x \phi^\varepsilon$ converge in $C([0, T], L^2(\mathbb{R}))$ to $A$. From the uniform $H^s$ estimates, this also yields convergence in $C([0, T], H^\sigma(\mathbb{R}))$, $\sigma < s$ and by Sobolev embedding in $L^\infty([0, T] \times \mathbb{R})$.

It remains to prove Lemma 4.5.
Proof of Lemma 4.5. Let us split $H$ into

$$H(u^\varepsilon) = H_1(u^\varepsilon) + H_2(u^\varepsilon) + H_3(u^\varepsilon)$$

with

$$H_1(u^\varepsilon) = \int 2\lambda n^\varepsilon - \varepsilon^2 (\nabla^\perp_x n^\varepsilon)^2 + \varepsilon^2 F_1(n^\varepsilon, n^\varepsilon) \cdot n^\varepsilon \, dx,$$

$$H_2(u^\varepsilon) = \int \frac{1}{4} |S_0 D\Phi \partial_x \phi^\varepsilon|^2 \, dx,$$

$$H_3(u^\varepsilon) = - \int (c + iB) \left( D\Phi \partial_x \phi^\varepsilon + \frac{1}{2} \varepsilon^2 \Pi^\top (D\Phi \partial_x \phi^\varepsilon, n^\varepsilon) \right) \cdot in^\varepsilon \, dx.$$

In the following computations, we shall make an extensive use of Proposition 3.1, Corollary 3.2, (3.5), (3.6), and the symmetry in its arguments of the trilinear application defined by (1.10). We first obtain easily that

$$\frac{d}{dt} H_1(u^\varepsilon) = \int \nabla^\perp_x n^\varepsilon \cdot \left( 2\lambda n^\varepsilon - \varepsilon^2 (\nabla^\perp_x n^\varepsilon)^2 + \varepsilon^2 F_1(n^\varepsilon, n^\varepsilon) \right) \, dx + O(\varepsilon). \quad (4.76)$$

Next,

$$\frac{d}{dt} H_2(u^\varepsilon) = \int \frac{1}{2} S_0 \nabla^\top_x (D\Phi \partial_t \phi^\varepsilon) \cdot S_0 D\Phi \partial_x \phi^\varepsilon + \frac{1}{2} \varepsilon^2 \Pi^\top (D\Phi \partial_x \phi^\varepsilon, n^\varepsilon) \cdot D\Phi \partial_x \phi^\varepsilon \, dx + O(\varepsilon)$$

Integrating by parts, we have

$$\int S_0 \nabla^\top_x (D\Phi \partial_t \phi^\varepsilon) \cdot S_0 D\Phi \partial_x \phi^\varepsilon \, dx = - \int S_0 D\Phi \partial_t \phi^\varepsilon \cdot \left( \nabla^\top_x (S_0 D\Phi \partial_x \phi^\varepsilon) + \varepsilon^2 \Pi^\top (D\Phi \partial_x \phi^\varepsilon, n^\varepsilon) \right) \, dx + O(\varepsilon)$$

and by using Proposition 3.1 we note that

$$Y \cdot \Pi^\top (X, N) = - \Pi^\top (X, Y) \cdot N, \quad \forall X, Y \in TL, \forall N \in NL \quad (4.77)$$

and hence

$$\int \Pi^\top (D\Phi \partial_x \phi^\varepsilon, \nabla^\perp n^\varepsilon) \cdot D\Phi \partial_x \phi^\varepsilon \, dx = - \int \nabla^\perp_x n^\varepsilon \cdot \Pi^\top (D\Phi \partial_x \phi^\varepsilon, D\Phi \partial_x \phi^\varepsilon) \, dx.$$

This yields

$$\frac{d}{dt} H_2(u^\varepsilon) = - \int \left[ S_0 D\Phi \partial_t \phi^\varepsilon \cdot \left( \frac{1}{2} \nabla^\top_x (S_0 D\Phi \partial_x \phi^\varepsilon) + \frac{1}{2} \varepsilon^2 \Pi^\top (D\Phi \partial_x \phi^\varepsilon, \nabla^\perp_x n^\varepsilon) \right) + \frac{1}{2} \nabla^\perp_x n^\varepsilon \cdot \varepsilon^2 \Pi^\top (D\Phi \partial_x \phi^\varepsilon, D\Phi \partial_x \phi^\varepsilon) \right] \, dx + O(\varepsilon). \quad (4.78)$$

Finally, let us study the evolution of $H_3(u^\varepsilon)$. Write first

$$\frac{d}{dt} H_3(u^\varepsilon) = \int \nabla^\perp_x n^\varepsilon \cdot i(c + iB) \left( D\Phi \partial_x \phi^\varepsilon + \frac{1}{2} \varepsilon^2 \Pi^\top (D\Phi \partial_x \phi^\varepsilon, n^\varepsilon) \right) + D\Phi \partial_t \phi^\varepsilon \cdot i(c + iB) \nabla^\perp_x n^\varepsilon$$

$$- \frac{1}{2} \varepsilon^2 (c + iB) \Pi^\top (\nabla^\top_x (D\Phi \partial_t \phi^\varepsilon), n^\varepsilon) \cdot in^\varepsilon - \frac{1}{2} \varepsilon^2 (c + iB) \Pi^\top (D\Phi \partial_x \phi^\varepsilon, \nabla^\perp_x n^\varepsilon) \cdot in^\varepsilon \, dx + O(\varepsilon)$$
Next, we observe that after integrating by parts,
\[-\frac{1}{2} \varepsilon^2 \int (c + iB) \Pi^\top (\nabla_x^\top (D\Phi \partial_t \phi_\varepsilon), n_\varepsilon) \cdot \text{in}_\varepsilon \, dx\]
\[= \frac{1}{2} \varepsilon^2 \int [(c + iB) \Pi^\top (D\Phi \partial_t \phi_\varepsilon, n_\varepsilon) \cdot i \nabla_x^\top n_\varepsilon + (c + iB) \Pi^\top (D\Phi \partial_t \phi_\varepsilon, \nabla_x^\top n_\varepsilon) \cdot \text{in}_\varepsilon] \, dx + O(\varepsilon)\]
\[= \varepsilon^2 \int \Pi^\top (D\Phi \partial_t \phi_\varepsilon, n_\varepsilon) \cdot i(c + iB) \nabla_x^\top n_\varepsilon \, dx + O(\varepsilon)\]
and that
\[-\frac{1}{2} \varepsilon^2 \int (c + iB) \Pi^\top (D\Phi \partial_x \phi_\varepsilon, \nabla_x^\top n_\varepsilon) \cdot \text{in}_\varepsilon \, dx = \frac{1}{2} \varepsilon^2 \int i(c + iB) \Pi^\top (D\Phi \partial_x \phi_\varepsilon, n_\varepsilon) \cdot \nabla_x^\top n_\varepsilon \, dx.\]
Consequently, we find
\[\frac{d}{dt} H_3(u_\varepsilon) = \int \left[ \nabla_x^\top n_\varepsilon \cdot i(c + iB) \left( D\Phi \partial_x \phi_\varepsilon + \varepsilon^2 \Pi^\top (D\Phi \partial_t \phi_\varepsilon, n_\varepsilon) \right) 
+ \left( D\Phi \partial_t \phi_\varepsilon + \varepsilon^2 \Pi^\top (D\Phi \partial_t \phi_\varepsilon, n_\varepsilon) \right) \cdot i(c + iB) \nabla_x^\top n_\varepsilon \right] \, dx\]
\[= \int \left[ \nabla_x^\top n_\varepsilon \cdot i(c + iB) S_0 D\Phi \partial_x \phi_\varepsilon + S_0 D\Phi \partial_t \phi_\varepsilon \cdot i(c + iB) \nabla_x^\top n_\varepsilon \right] \, dx.\]
(4.79)
By collecting (4.76), (4.78), (4.79), we thus find
\[\frac{d}{dt} H(u_\varepsilon) = -\int \left[ S_0 D\Phi \partial_x \phi_\varepsilon \cdot \left( \frac{1}{2} \nabla_x^\top (S_0 D\Phi \partial_x \phi_\varepsilon) + \frac{1}{2} \varepsilon^2 \Pi^\top (D\Phi \partial_x \phi_\varepsilon, \nabla_x^\top n_\varepsilon) - i(c + iB) \nabla_x^\top n_\varepsilon \right) \right] \, dx
- \int \left[ \nabla_x^\top n_\varepsilon \cdot \left( -2\lambda n_\varepsilon + \frac{1}{2} \varepsilon^2 (\nabla_x^\top n_\varepsilon)^2 n_\varepsilon - \varepsilon^2 F_1(n_\varepsilon, n_\varepsilon) - i(c + iB) S_0 D\Phi \partial_x \phi_\varepsilon 
+ \frac{1}{2} \varepsilon^2 \Pi^\top (D\Phi \partial_x \phi_\varepsilon, D\Phi \partial_t \phi_\varepsilon) \right) \right] \, dx + O(\varepsilon).\]
By using the hydrodynamical system (4.64) to express \(\nabla_x^\top n\) and \(S_0 D\Phi \partial_t \phi\) in each term, we obtain that the two integrals cancel up to the remainders \(O(\varepsilon)\) and hence that
\[\frac{d}{dt} H(u_\varepsilon) = O(\varepsilon).\]
This ends the proof of Lemma 4.5.

5. The case of a general Kähler manifold

With the KdV scaling (1.7), our Schrödinger map system reads
\[(\partial_t - \frac{c}{\varepsilon^2} \partial_x) u = i \left( \frac{1}{2e} \nabla_x \partial_x u + \frac{1}{\varepsilon^2} B \partial_x u - \frac{1}{\varepsilon^3} V'(u) \right).\]
(5.1)
Note that here, we deal with the general case where \(i = i(u)\) and \(B = B(u)\) depend on \(u\). To generalize the decomposition \(u = P + N\), with \(P = \Phi(\varepsilon \phi), N = \varepsilon^2 n\) that we have previously used, we define a parametrization of \(M\) in the vicinity of \(0 \in \mathcal{L}\) by
\[u = \Psi(p, N) \overset{\text{def}}{=} \exp^M_p(N), \quad p = \Phi(\varepsilon \phi) \overset{\text{def}}{=} \exp^\mathcal{L}_0(\varepsilon \phi), \quad N = \varepsilon^2 n,\]
(5.2)
where \(p \in \mathcal{L}, \phi \in T_0 \mathcal{L}, N, n \in N_p \mathcal{L},\) and \(\exp^M\) and \(\exp^\mathcal{L}\) are the Riemannian exponential maps on \(M\) and \(\mathcal{L}\) respectively. This yields a parametrization of \(M\) in the vicinity of zero by the normal bundle of \(\mathcal{L}\):
\[\Psi : N \mathcal{L} \to M.\]

We will assume in this section that
\[V(u) = \lambda \text{dist}(u, \mathcal{L})^2 \quad \text{or equivalently} \quad V(\Psi(p, N)) = \lambda |N|^2.\]
In other words, we assume that there are no cubic or higher order terms in the expansion (1.4) of \( V \), which will alleviate notations. Cubic terms and higher do not present any particular difficulty and can be treated by following the proof of the flat case \( M = \mathbb{R}^{2d} \).

5.1. Geometric preliminaries II.

5.1.1. Basic setup. We start with a Kähler manifold \( M \) with metric \((X, Y) \mapsto X \cdot Y = \langle X, Y \rangle\), Levi-Civita connection \( \nabla \), complex structure \( i \), and Riemannian curvature tensor \( R \); and a Lagrangian submanifold \( L \subset M \). We use the geometric notation defined in Section 3. If \( p \in L \), the tangent and normal bundles of \( L \) are denoted \( T_pL \) and \( N_pL \) respectively. Just like in the previous section, we use connections on these bundles denoted by \( \nabla^\top = P^\top \nabla \) and \( \nabla^\perp = P^\perp \nabla \), and second fundamental forms by \( \Pi^\top = -\nabla P^\top \) and \( \Pi^\perp = -\nabla P^\perp \).

As was already explained in the previous section, we do not distinguish in the notations between differentiation in a vector bundle and its pull-back by a map.

In order to write a hydrodynamical system, the first step is to understand the structure of \( T(NL) \). This was already done in [31]. We include it here for the sake of completeness. Then, we shall explain how we can extend in a natural way geometrical objects away from \( L \) and thus get a hydrodynamical system.

5.1.2. A connection on \( T(NL) \). We start by identifying \( T(NL) \) with \( TL \times NL \) in the following way: given a path \( \gamma(s) = (p(s), N(s)) \) in \( NL \), we identify

\[
\dot{\gamma}(0) \simeq (\partial_sp(0), \nabla^\perp_s N(0)) \in T_{p(0)}L \times N_{p(0)}L.
\]

(\( \nabla^\perp_s \) was defined in the previous section).

Having identified \( T(NL) \) with \( TL \times NL \), we define a scalar product by simply adopting the natural one on \( TL \times NL \):

\[
\langle \dot{\gamma}_1(0), \dot{\gamma}_2(0) \rangle \overset{\text{def}}{=} \langle \partial_sp_1(0), \partial_sp_2(0) \rangle + \langle \nabla^\perp_s N_1(0), \nabla^\perp_s N_2(0) \rangle.
\]

We can then define a connection on it in the following way: given a path \((X(s), N(s))\) in \( TL \times NL \), set

\[
\mathcal{D}_s(X, N) = (\nabla^\top_s X, \nabla^\perp_s N).
\]

This connection has the following properties:

- It is metric.
- It is torsion free on \( L \).

In other words: given a two-parameter function \((p(t, s), N(t, s))\) in \( NL \),

\[
\mathcal{D}_t(\partial_sp, \nabla^\perp_s N) = \mathcal{D}_s(\partial_tp, \nabla^\perp_t N) \quad \text{on} \quad L.
\]

This simply follows from the formula \( \mathcal{D}_t(\partial_sp, \nabla^\perp_s N) - \mathcal{D}_s(\partial_tp, \nabla^\perp_t N) = (0, R^\perp(\partial_tp, \partial_sp)N). \)

5.1.3. Exponential maps. We select from now on a point 0 in \( L \) and set

\[
\Phi(X) = \exp_0^L(X) \in L.
\]

We already saw that \( D\Phi|_0 = \text{Id} \) and \( \nabla D\Phi|_0 = 0 \). The next step is to define

\[
\Psi(p, N) = \exp_p^M N \in M.
\]

It is easy to see that

\[
\text{if } p \in L \text{ and } (X, N) \in T_pL \times N_pL, \quad D\Psi_{(p,0)}(X, N) = X + N. \tag{5.3}
\]

We will also need the second derivative of \( \Psi \): recall that it satisfies by definition

\[
\nabla^\perp_s D\Psi_{(p(s), n(s))}(X(s), N(s)) = \nabla D\Psi_{(p(s), n(s))}((\partial_sp(s), \nabla^\perp_s n(s)), (X(s), N(s)))
\]

\[
+ D\Psi_{(p(s), n(s))}(\nabla^\top_s X(s), \nabla^\perp_s N(s)).
\]

\footnote{We naturally identify \( L \) with \( L \times \{0\} \subset NL \).}
In order to prove this formula, first observe that
\[ \nabla D\Psi((X,N),(Y,M)) = \Pi^T(X,M) + \Pi^T(Y,N) + \Pi^\perp(X,Y). \] (5.4)

In order to prove this formula, first observe that \( \nabla D\Psi((X,N),(Y,M)) \) is symmetric in \( (X,N) \) and \( (Y,M) \); this follows from the connection \( D \) being torsion-free on \( L \). Thus, it suffices to compute \( \nabla D\Psi \) on \( ((X,0),(X,0)), ((0,N),(0,N)), \) and \( ((X,0),(0,N)) \). For the first of these, it suffices to differentiate \( \Psi \) twice along a path of the type \( (p(s),0) \), where \( p(s) \) is a geodesic. For the second, the same argument along a path of the type \( (p,sN) \), with \( N \in N_pL \) holds. The third one is a bit more delicate: consider \( D\Psi(p(s),0)(0,N(s)) = N(s) \), where \( N(s) \) is parallel in \( NL \), and differentiate in \( s \).

5.1.4. Extending the tangent and normal spaces and projectors. For any \( p \in L \), let \( \tau_1(p), \ldots \tau_d(p) \) be a basis of \( T_pL \); this implies that \( i\tau_1, \ldots i\tau_d \) is a basis of \( N_pL \) since \( L \) Lagrangian. At a point \( u = \Psi(p,N) \), we define \( \tau_1(u), \ldots \tau_d(u) \) as follows: these vectors are given by the parallel transport of \( \tau_1(p), \ldots \tau_d(p) \) along the geodesic from \( p \) to \( u \), which reads \( s \mapsto \Psi(p,sN) \). Define then
\[ \tilde{T}_uL = \text{span}\{\tau_1(u), \ldots \tau_d(u)\} \quad \text{and} \quad \tilde{N}_uL = \text{span}\{i\tau_1(u), \ldots i\tau_d(u)\}. \]

With this definition, \( \tilde{T}_uL \) and \( \tilde{N}_uL \) are orthogonal, and such that \( i\tilde{T}_uL = \tilde{N}_uL \). The latter property is clear, while the former follows from the fact that for all \( k \) and \( l \),
\[ i_u\tau_k(u) \cdot \tau_l(u) = i_p\tau_k(p) \cdot \tau_l(p) = 0 \]
since the \( \tau_j \) are parallel transported, \( \nabla i = 0 \) and \( L \) is a Lagrangian submanifold.

Define then \( P^T \) and \( P^\perp \) to be the orthogonal projectors from \( T_uM \) to \( \tilde{T}_uL \) and \( \tilde{N}_uL \) respectively. They satisfy
\[ P^T + P^\perp = \text{Id} \quad \text{and} \quad P^\perp i = iP^T. \]

Furthermore, we claim that
\[ \nabla_N P^T_p = 0 \quad \text{and} \quad \nabla_N P^\perp_p = 0. \] (5.5)

First, it suffices to prove the first of these two identities since \( P^T + P^\perp = \text{Id} \). Next, consider \( X \in T_pL \), which we extend to a parallel vector field \( X(s) \) along the geodesic \( s \mapsto \Psi(p,sN) \). It is then easy to see that \( P^T_{\Psi(p,sN)}X(s) = X(s) \). Since \( X(s) \) is parallel, the differentiation of this identity gives
\[ \nabla_N P^T_p X = 0. \]
Similarly, if \( N \in N_pL \) is extended to a parallel vector field \( N(s) \), the differentiation of the identity \( P^T_{\Psi(p,sN)}N(s) = N(s) \) yields
\[ \nabla_N P^T_p N = 0. \]

We finally extend \( \nabla^T \) and \( \nabla^\perp \) away from \( L \) by setting
\[ \nabla^T = P^T \nabla, \quad \nabla^\perp = P^\perp \nabla \] (5.6)
where \( \nabla \) is the Levi-Civita connection on \( M \).

5.1.5. Extending the second fundamental forms. It is natural to extend the second fundamental forms by setting for \( u \in M \)
\[ \Pi^T(X,N) = -(\nabla_X P^T)N \quad \text{if} \ (X,N) \in T_uM \times \tilde{N}_uL \]
\[ \Pi^\perp(X,Y) = -(\nabla_X P^\perp)Y \quad \text{if} \ (X,Y) \in T_uM \times \tilde{T}_uL \]
(notice that the second argument, \( N \) or \( Y \) above, is still required to be normal, respectively tangent, but not the first one). With these definitions,\[
\nabla_X N = \nabla^\perp_X N + \Pi^\perp(X, N) \quad \text{if } (X, N) \in T_u M \times \tilde{N}_u L
\]
\[
\nabla_X Y = \nabla^\perp_X Y + \Pi^\perp(X, Y) \quad \text{if } (X, Y) \in T_u M \times \tilde{T}_u L.
\]

Notice that (5.5) implies that
\[
\Pi^\perp_p (N, \cdot) = 0 \quad \text{and} \quad \Pi^\perp_p (N, \cdot) = 0.
\]
If \( N \in \tilde{N}_u L \), with \( u \notin L \), \( \Pi^\perp(\cdot, N) \) is not exactly symmetrical anymore on \( \tilde{T}_u L \). Namely, the following holds if \( (N, X, Y) \in \tilde{N}_u L \times \tilde{T}_u L \times \tilde{T}_u L \):
\[
\langle \Pi^\perp(X, N), Y \rangle = \langle \Pi^\perp(Y, N), X \rangle - \langle N, [X, Y] \rangle.
\]

It follows from the elementary computation:
\[
\langle \Pi^\perp(X, N), Y \rangle = \langle P^\perp \nabla_X N, Y \rangle = \langle \nabla_X N, Y \rangle = -\langle N, \nabla_X Y \rangle = -\langle N, \nabla_Y X \rangle - \langle N, [X, Y] \rangle
\]
\[
= \langle \nabla_Y N, X \rangle - \langle N, [X, Y] \rangle = \langle \Pi^\perp(Y, N), X \rangle - \langle N, [X, Y] \rangle.
\]

Note that since \( \tilde{N}_u \) is still orthogonal to \( \tilde{T}_u \), \( \langle N, [X, Y] \rangle \) is indeed a tensor for \( N \in \tilde{N}_u \) and \( X, Y \in \tilde{T}_u \). It implies in particular that, at the point \( u = \Psi(p, \epsilon^2 n) \), if \( (N, X, Y) \in \tilde{N}_u L \times \tilde{T}_u L \times \tilde{T}_u L \):
\[
|\langle \Pi^\perp(X, N), Y \rangle - \langle \Pi^\perp(Y, N), X \rangle| \lesssim \epsilon^2 |X||Y|.
\]

5.1.6. Extending the tangent curvature tensor. Commuting tangential derivatives \( \nabla^\perp \) away from \( L \) also gives rise to a curvature tensor: if \( X, Y \in T M \) and \( Z \in \tilde{T} L \),
\[
\nabla^\perp_X \nabla^\perp_Y Z - \nabla^\perp_Y \nabla^\perp_X Z - \nabla^\perp_{[X,Y]} Z = R^\perp(X, Y) Z
\]
with
\[
R^\perp(X, Y) Z = P^\perp R(X, Y) Z + \Pi^\perp(Y, \Pi^\perp(X, Z)) - \Pi^\perp(X, \Pi^\perp(Y, Z))
\]
(recall that \( R \) is the Riemannian curvature tensor of \( M \)). This identity follows from the computation:
\[
\nabla^\perp_X \nabla^\perp_Y Z - \nabla^\perp_Y \nabla^\perp_X Z - \nabla^\perp_{[X,Y]} Z = P^\perp \nabla_X P^\perp \nabla_Y Z - P^\perp \nabla_Y P^\perp \nabla_X Z - \nabla^\perp_{[X,Y]} Z
\]
\[
= P^\perp (\nabla_X P^\perp) \nabla_Y Z - P^\perp (\nabla_Y P^\perp) \nabla_X Z + P^\perp R(X, Y) Z
\]
\[
= P^\perp \nabla_Y \left[ (\nabla_X P^\perp) Z \right] - P^\perp \nabla_X \left[ (\nabla_Y P^\perp) Z \right] + P^\perp \left( \left| \nabla_X \nabla_Y - \nabla_Y \nabla_X \right| P^\perp \right) Z + P^\perp R(X, Y) Z.
\]

Starting from the identity \( P^\perp (\nabla_X \nabla_Y - \nabla_Y \nabla_X) Z = P^\perp (\nabla_X \nabla_Y - \nabla_X \nabla_Y) P^\perp Z \), it is easy to see that \( P^\perp \left( \left| \nabla_X \nabla_Y - \nabla_Y \nabla_X \right| P^\perp \right) Z = 0 \). Therefore, by definition of the second fundamental forms,
\[
\nabla^\perp_X \nabla^\perp_Y Z - \nabla^\perp_Y \nabla^\perp_X Z - \nabla^\perp_{[X,Y]} Z = P^\perp \nabla_Y \Pi^\perp(X, Z) - P^\perp \nabla_X \Pi^\perp(Y, Z) + P^\perp R(X, Y) Z
\]
\[
= \Pi^\perp(Y, \Pi^\perp(X, Z)) - \Pi^\perp(X, \Pi^\perp(Y, Z)) + P^\perp R(X, Y) Z.
\]

5.1.7. Extending the normal curvature tensor. A computation similar to the preceding one gives:
if \( X, Y \in T M \) and \( N \in \tilde{N} L \),
\[
\nabla^\perp_X \nabla^\perp_Y N - \nabla^\perp_Y \nabla^\perp_X N - \nabla^\perp_{[X,Y]} N = R^\perp(X, Y) N
\]
with
\[
R^\perp(X, Y) N = P^\perp R(X, Y) Z + \Pi^\perp(Y, \Pi^\perp(X, N)) - \Pi^\perp(X, \Pi^\perp(Y, N)).
\]
5.1.8. New coordinates for \( u \). We shall describe \( u \) by coordinates \((\varepsilon \phi, \varepsilon^2 n) \in T_0 \mathcal{L} \times N_{\Phi(\varepsilon \phi)} \mathcal{L}:\)

\[
u = \Psi(p, \varepsilon^2 n) \quad \text{with} \quad p = \Phi(\varepsilon \phi) \quad \text{or equivalently} \quad u = \Psi(\Phi(\varepsilon \phi), \varepsilon^2 n).
\]

It will be convenient to denote

\[ D\Psi_{(p, N)} = \Sigma_{(p, N)}. \]

Viewing \( \Sigma \) as a map \( N_{\mathcal{L}} \cong T_p \mathcal{L} \times N_p \mathcal{L} \to T_u \mathcal{L} \times \tilde{N}_u \mathcal{L} \), it is natural to adopt a block matrix notation, where the first coordinate is the tangential, and the second the normal one. Then we claim that \( \Sigma \) can be written as

\[
\Sigma_{(p, \varepsilon^2 n)} = \begin{pmatrix} S_{TT}(\Phi, \varepsilon^2 n) & \varepsilon^4 S_{T\perp}(\Phi, n) \\ \varepsilon^4 S_{\perp T}(\Phi, n) & S_{\perp\perp}(\Phi, \varepsilon^2 n) \end{pmatrix}
\]

(5.11)

where \( S_{TT}, S_{T\perp}, S_{\perp T}, S_{\perp\perp} \) depend smoothly on their arguments, with bounds uniform in \( \varepsilon \). To check that \( P^\top \Sigma P^\perp \) is indeed \( O(\varepsilon^4) \), set

\[
G(s) = P_{\Psi(p, sN)}^\top \Sigma_{(p, sN)} \begin{pmatrix} 0 \\ N \end{pmatrix}
\]

It follows from (5.3) that \( G(0) = 0 \). Furthermore, by (5.4),

\[
\nabla_s G(s)_{|s=0} = \nabla_N P^\top_p \Sigma_p \begin{pmatrix} 0 \\ N \end{pmatrix} + P^\top_p \nabla_N \Sigma \begin{pmatrix} 0 \\ N \end{pmatrix} = 0.
\]

Therefore, \( G(s) = O(s^2) \), giving the desired result. A similar argument gives that \( P^\perp \Sigma P^\top \) is \( O(\varepsilon^4) \).

Further properties of \( S_{TT} \) and \( S_{\perp\perp} \) that will be useful are that for every \( p \in \mathcal{L} \) and for every \( X \in T_p \mathcal{L}, N \in N_p \mathcal{L}, \) we have

\[
(S_{TT})_p = Id, \quad (S_{T\perp})_p = Id, \quad (\nabla^\perp X S_{\perp\perp})_p = 0, \quad (\nabla^\top X S_{TT})_p = 0, \quad (\nabla_N^\top S_{TT})_p = \Pi^\top (\cdot, N).
\]

(5.12)

Let us prove the above properties. We start with the properties of \( S_{\perp\perp} \). Let us recall that by definition \( S_{\perp\perp}(p, sN)Z = P^\perp(\Phi(p, sN))D\Psi_{(p, sN)}(0, Z) \) for \( s \) sufficiently small and \( N, Z \in \mathcal{N}_p \mathcal{L} \). For \( s = 0 \), we get from (5.3) that \( S_{\perp\perp} Z = Z \). This also implies that \( \nabla^\perp_s S_{\perp\perp} = 0 \). By applying \( \nabla^\perp_s \) and taking the value at \( s = 0 \), we also obtain that

\[
(\nabla^\perp_N S_{\perp\perp})_p Z = (\nabla_N^\top P^\perp)Z + P^\perp \nabla^\top \nabla^\top \Psi_{(p, 0)}(0, Z), (0, N)) = 0
\]

by using (5.4) and (5.5). In a similar way, since \( S_{TT}(p, sN)Y = P^\top(\Phi(p, sN))D\Psi_{(p, sN)}(Y, 0) \) for \( s \) sufficiently small and \( N \in \mathcal{N}_p \mathcal{L}, Y \in T_p \mathcal{L}, \) we also find that \( (S_{TT})_p Y = Y \) and hence that \( \nabla^\top_X S_{TT} = 0 \). By taking \( \nabla^\top_X \) at \( s = 0 \), we obtain that

\[
(\nabla^\top_X S_{TT})_p Y = P^\top \nabla^\top \nabla^\top \Psi_{(p, 0)}((Y, 0), (0, N)) = \Pi^\top (Y, N).
\]

5.1.9. Computing \( V'(u) \). Recall that in this section, we assume that \( V(u) = \lambda \text{dist}(u, \mathcal{L})^2 \), or in other words

\[
V(\Psi(p, N)) = \lambda |N|^2.
\]

To compute \( V'(u) \), consider a path \((p(s), N(s))\) such that \( p(s) \in \mathcal{L}, N(s) \in \mathcal{N}_{p(s)} \mathcal{L}, \) and let \( u(s) = \Psi(p(s), N(s)) \). Differentiate then

\[
\partial_s V(u(s)) = \lambda \partial_s |N(s)|^2 = 2\lambda \left< \nabla^\perp_s N(s), N(s) \right>.
\]

Recall that \( \partial_s u = D\Psi \left( \begin{pmatrix} \partial_s p \\ \nabla^\perp_s N(s) \end{pmatrix} \right) \), therefore

\[
\left( \begin{pmatrix} \partial_s p \\ \nabla^\perp_s N(s) \end{pmatrix} \right) = (D\Psi)^{-1} \partial_s u \quad \text{and, coming back to the above,}
\]

\[
\partial_s V(u(s)) = 2\lambda \left< (D\Psi)^{-1} \partial_s u, \begin{pmatrix} 0 \\ N(s) \end{pmatrix} \right> = 2\lambda \left< \partial_s u, (D\Psi)^{-\ast} \begin{pmatrix} 0 \\ N(s) \end{pmatrix} \right>.
\]
(using the notation \((D\Psi)^{-*}\) as a shorthand for \(((D\Psi)^{-1})^*\) which means that
\[
V'(\Psi(p, N)) = 2\lambda (D\Psi)^{-*}_{(p, N)} \begin{pmatrix} 0 \\ N \end{pmatrix} \quad \text{and} \quad V'(\Psi(p, \varepsilon^2 n)) = 2\lambda \varepsilon^2 (D\Psi)^{-*}_{(p, \varepsilon^2 n)} \begin{pmatrix} 0 \\ n \end{pmatrix}.
\] (5.13)
We claim that
\[
(D\Psi)_{(p, \varepsilon^2 n)} \begin{pmatrix} 0 \\ n \end{pmatrix} = (D\Psi)^{-*}_{(p, \varepsilon^2 n)} \begin{pmatrix} 0 \\ n \end{pmatrix} + O(\varepsilon^4). \tag{5.14}
\]
To prove this, let for \(p \in \mathcal{L}, \) and \(n, N \in \mathcal{N}_p \mathcal{L}\)
\[
G(s) = D\Psi_{(p, sn)} \begin{pmatrix} 0 \\ n \end{pmatrix} - (D\Psi)^{-*}_{(p, sn)} \begin{pmatrix} 0 \\ n \end{pmatrix}.
\]
It satisfies \(G(0) = 0\) and
\[
G'(0) = \nabla_N D\Psi \begin{pmatrix} 0 \\ n \end{pmatrix} - (D\Psi)^{-*}(\nabla_N D\Psi)^* (D\Psi)^{-*} \begin{pmatrix} 0 \\ n \end{pmatrix} = 0 - 0 = 0.
\]
Furthermore, it is easy to see that \((D\Psi)_{(p, sn)} \begin{pmatrix} 0 \\ n \end{pmatrix}, \tau_k(s)\) = 0; this quantity is indeed zero for \(s = 0\), and has a zero derivative (in \(s\)). Coming back to \(V'(u)\), we obtain
\[
V'(\Psi(p, \varepsilon^2 n)) = 2\lambda \varepsilon^2 \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} + \varepsilon^6 R_V(p, n)
\]
\[
P^T V'(\Psi(p, \varepsilon^2 n)) = \varepsilon^6 R_V^T(p, n)
\]
\[
P^\perp V'(\Psi(p, \varepsilon^2 n)) = 2\lambda \varepsilon^2 S_{\perp\perp}(p, \varepsilon^2 n) + \varepsilon^6 R_V^\perp(p, n)
\]
where \(R_V\) is a smooth function of \(p\) and \(n\) and we have set
\[
R_V^T(p, n) = P^T R_V(p, n) + 2\lambda S_{\perp\perp}(p, n), \quad R_V^\perp(p, n) = P^\perp R_V(p, n)
\]
by using (5.11).

5.1.10. Action of \(iB\). We shall also need to describe the action of \(iB \Sigma_{(p, \varepsilon^2 n)}\). We can write it in block matrix form
\[
iB \Sigma_{(p, \varepsilon^2 n)} = \begin{pmatrix} (iB)_{\perp\perp} & \varepsilon^4 (iB)_{\perp\perp} \\ \varepsilon^4 (iB)_{\perp\perp} & (iB)_{\perp\perp} \end{pmatrix},
\] (5.16)
where we have set
\[
(iB)_{\perp\perp} \begin{pmatrix} X \\ 0 \end{pmatrix} = P^T (iB) \Sigma_{(p, \varepsilon^2 n)}(X, 0), \quad (iB)_{\perp\perp} \begin{pmatrix} 0 \\ N \end{pmatrix} = P^\perp (iB) \Sigma_{(p, \varepsilon^2 n)}(0, N),
\]
\[
(iB)_{\perp\perp} \begin{pmatrix} n \end{pmatrix} = \frac{1}{\varepsilon^4} P^T (iB) \Sigma_{(p, \varepsilon^2 n)}(0, N), \quad (iB)_{\perp\perp} \begin{pmatrix} n \end{pmatrix} = \frac{1}{\varepsilon^4} P^\perp (iB) \Sigma_{(p, \varepsilon^2 n)}(0, N),
\]
for every \((X, N) \in T_\Psi \mathcal{L} \times N_\Psi \mathcal{L}\).

Again, the tensors \((iB)_{\perp\perp}, (iB)_{\perp\perp},...\) are smooth with derivatives uniformly bounded in \(\varepsilon\). This is due to the fact that
\[
P^T (iB) \Sigma \begin{pmatrix} 0 \\ N \end{pmatrix} = O(\varepsilon^4)N, \quad P^\perp (iB) \Sigma \begin{pmatrix} X \\ 0 \end{pmatrix} = O(\varepsilon^4)X.
\]
Indeed, for the first identity, this follows again by setting
\[
G(s) = \left( P^T (iB) \Sigma \right)_{(p, sn)} \begin{pmatrix} 0 \\ N \end{pmatrix}
\]
and by noticing that \(G(0) = 0\) and \(\nabla_s G(0) = 0\) since \(\nabla i = 0, \nabla B = 0\) on \(\mathcal{L}, \nabla_n P^T = 0\) (thanks to (5.5), and by using (5.4). The second one can be obtained following the same lines.
Finally, we observe that on \( \mathcal{L} \), that is to say, for every \( p \in \mathcal{L} \) and for every \( Y \in T_p \mathcal{L} \), \( N \in N_p \mathcal{L} \), we have that
\[
\nabla_Y^\top (iB)_{\top\top} = 0, \quad \nabla_N^\top (iB)_{\top\top} = \Pi^\top (iB \cdot N), \quad (5.17)
\]
\[
\nabla^\perp (iB)_{\perp\perp} = 0. \quad (5.18)
\]
Indeed, by using the definitions of these tensors, we obtain that on \( \mathcal{L} \), and for every \( X, Y \in T_p \mathcal{L} \) and \( N \in N_p \mathcal{L} \), prolonging \( X \) to be a parallel vector field,
\[
(\nabla_Y^\top (iB)_{\top\top}) X = (\nabla_Y^\top \Pi^\top)(iBX) = 0
\]
and
\[
(\nabla_N^\top (iB)_{\top\top}) X = (\nabla_N^\top \Pi^\top)(iBX) = \Pi^\top (iBY, N)
\]
thanks to (5.14), (5.15) and Corollary 3.2.

The proof of the second identity follows from the same arguments.

5.2. **The hydrodynamical system.** First of all, we will work under the Bootstrap hypothesis. As in Section 4, we work on an interval of time \([0, T^*] \) such that the estimate (4.3) is satisfied for some \( r \) sufficiently small. Note that this ensures that our exponential coordinate system provides a nice parametrization of the solution.

Written in the \((\Phi, n)\) coordinates, (5.1) reads
\[
\Sigma(\Phi, \varepsilon^2 n) \left( \partial_t - \frac{c}{\varepsilon^2} \partial_x \Phi, \varepsilon^2 (\nabla_i^\perp n - \frac{c}{\varepsilon^2} \nabla_x^\perp n) \right) - \frac{1}{\varepsilon^2} iB \Sigma(\Phi, \varepsilon^2 n) (\partial_x \Phi, \varepsilon^2 \nabla_x^\perp n)
\]
\[
= \frac{1}{2\varepsilon} \nabla_x^\perp (\Sigma(\Phi, \varepsilon^2 n) (\partial_x \Phi, \varepsilon^2 \nabla_x^\perp n)) - \frac{1}{\varepsilon^3} \nabla' (u). \quad (5.19)
\]
In order to write a system, we rely on the decomposition of \( \Sigma \) obtained in (5.11).

First take the tangential component of the above to obtain
\[
S^\top\top(\Phi, \varepsilon^2 n) \left( \partial_t - \frac{c}{\varepsilon^2} \partial_x \Phi, \varepsilon^5 \nabla_i^\perp n - \frac{c}{\varepsilon^2} \nabla_x^\perp n \right) - \frac{1}{\varepsilon^2} (iB)_{\top\top} \partial_x \Phi - \varepsilon^3 (iB)_{\top\perp} \nabla_x^\perp n
\]
\[
= \frac{1}{2\varepsilon} \nabla_x^\perp \left( S_{\top\perp}(\Phi, \varepsilon^2 n) \nabla_x^\perp n \right) + \frac{1}{2\varepsilon^2} \nabla_x^\perp \left( P^\top \Sigma(\Phi, \varepsilon^2 n) \cdot (\partial_x \Phi, \varepsilon^2 \nabla_x^\perp n) \right)
\]
\[
+ \frac{\varepsilon^3}{2} \nabla_x^\perp \left( S_{\top\top}(\Phi, \varepsilon^2 n) \partial_x \Phi \right) - \frac{1}{\varepsilon^4} P^\perp V' (u). \quad (5.19)
\]
Next, observe that
\[
\frac{1}{2\varepsilon^2} \nabla_x^\perp \left( P^\top \Sigma(\Phi, \varepsilon^2 n) (\partial_x \Phi, \varepsilon^2 \nabla_x^\perp n) \right)
\]
\[
= \frac{1}{2} \Pi^\perp \left( \Sigma(\Phi, \varepsilon^2 n) \cdot \left( \frac{\partial_x \Phi}{\varepsilon}, \varepsilon \nabla_x^\perp n \right), P^\top \Sigma(\Phi, \varepsilon^2 n) \cdot \left( \frac{\partial_x \Phi}{\varepsilon}, \varepsilon \nabla_x^\perp n \right) \right),
\]
while formula (5.13) gives the expression of \( P^\perp V'(u) \). We thus get the equation for the tangential component:
\[
S^\top\top(\partial_t - \frac{c}{\varepsilon^2} \partial_x \Phi) - \frac{1}{\varepsilon^2} (iB)_{\top\top} \partial_x \Phi + \varepsilon^5 S_{\top\perp} (\nabla_i^\perp n - \frac{c}{\varepsilon^2} \nabla_x^\perp n) - \varepsilon^3 (iB)_{\top\perp} \nabla_x^\perp n
\]
\[
= i \left( \frac{1}{2} \nabla_x^\perp \left( S_{\top\perp}(\Phi, \varepsilon^2 n) \nabla_x^\perp n \right) - \frac{2\lambda}{\varepsilon^2} S_{\top\perp} n + \frac{1}{2} \Pi^\perp \left( \Sigma \left( \frac{\partial_x \Phi}{\varepsilon}, \varepsilon \nabla_x^\perp n \right), P^\top \Sigma \left( \frac{\partial_x \Phi}{\varepsilon}, \varepsilon \nabla_x^\perp n \right) \right) \right)
\]
\[
+ \frac{\varepsilon^3}{2} \nabla_x^\perp \left( S_{\top\top}(\Phi, \varepsilon^2 n) \partial_x \Phi \right) + \varepsilon^2 P^\perp V', \quad (5.19)
\]
In a similar way, we can get the equation for the normal component. Combining it with the above, we find the hydrodynamic form of (5.19):

\[
S_{\Sigma}(\partial_t - \frac{c}{\varepsilon^2}\partial_x)\Phi - \frac{1}{\varepsilon^2}(iB)_{\Sigma} \partial_x \Phi + \varepsilon^5 S_{\Sigma}^{-1}(\nabla^T_t n - \frac{c}{\varepsilon^2} \nabla^x n) - \varepsilon^3 (iB)_T \nabla^x n
\]

\[
= i \left( \frac{1}{2} \nabla^T_x (S_{\Sigma}^{-1} n) - \frac{2\lambda}{\varepsilon^2} S_{\Sigma}^{-1} n + \frac{1}{2} \Pi^\perp \left( \Sigma \left( \frac{\partial_x \Phi}{\varepsilon \nabla^x n} \right), P^\Sigma \left( \frac{\partial_x \Phi}{\varepsilon \nabla^x n} \right) \right) + \frac{3}{2} \nabla^T_x (S_{\Sigma}^{-1} \partial_x \Phi) + \varepsilon^3 R_V \right).
\]

As a first consequence of this hydrodynamic formulation and (4.5), we easily get the following

\[
\begin{align*}
\varepsilon^2 \| \partial_t \phi(t) \|_{s-1} &= O(\varepsilon s) \\
\varepsilon \| \partial_{xx} \phi \|_{s-1} &= O(\varepsilon s)
\end{align*}
\]

5.3. Estimates on \( u \).

**Proposition 5.2.** The following a priori estimate holds on \([0,T^*]\):

\[
\varepsilon_s(u,t) \lesssim \varepsilon_s(u,0) + \int_0^t O(\varepsilon_s(u,\tau)) \, d\tau.
\]

**Proof.** Step 1: first decomposition. Start with the equation satisfied by \( u \)

\[
(\varepsilon^2 \partial_t - (c + iB) \partial_x)u = i \left( \frac{1}{2} \varepsilon \nabla_x \partial_x u - \frac{1}{\varepsilon} V'(u) \right)
\]

which we write

\[
LHS = RHS1 + RHS2.
\]

The plan is the following: apply \((\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m\) to the above, take the scalar product with \(\frac{1}{\varepsilon^2}(\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} u\), and then estimate the resulting terms. The first such term is the following; we will use integration by parts and commutation of derivatives to estimate it.

\[
\frac{1}{\varepsilon^2} \int (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m LHS \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} u \, dx
\]

\[
= \frac{1}{\varepsilon^2} \int (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m (\varepsilon^2 \partial_t - (c + iB) \partial_x) u \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} u \, dx \quad (5.23)
\]

\[
= \frac{d}{dt} \left[ \frac{1}{2} \int |(\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} u|^2 \, dx - \frac{\mu}{2} \int |\nabla_x \nabla^{m-1} u|^2 \, dx \right] + O(\varepsilon^2).
\]

Here, the term \( O(\varepsilon_s^2) \) results from commutators arising when commuting \( \nabla_t \) and \( \nabla_x \) derivatives. Typical instances are

\[
\frac{1}{\varepsilon^2} \int (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^{m-2} R(\varepsilon^2 \partial_t u, \partial_x u) \partial_x u \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} u \, dx
\]

\[
\frac{1}{\varepsilon^2} \int (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^{m-1} i \nabla B(\nabla u, \partial_x u) \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} u \, dx;
\]
simply counting derivatives and using (5.3), it becomes clear that it can be estimated by $O(\mathcal{E}_s^2)$. Next, we turn to the term involving RHS1 and transform it by first commuting derivatives, and then using the equation satisfied by $u$:

\[
(\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m \text{RHS}1 = (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m \frac{i}{2} \varepsilon \nabla_x \partial_x u \\
= \frac{i \varepsilon}{2} \nabla^2_x \nabla^m (\varepsilon^2 \partial_t - (c + iB) \partial_x) u + \varepsilon^3 O_L(\mathcal{E}_s) \\
= \frac{i \varepsilon}{2} \nabla^2_x \nabla^m i \left( \frac{1}{2} \varepsilon \nabla_x \partial_x u - \frac{1}{\varepsilon} V'(u) \right) + \varepsilon^3 O_L(\mathcal{E}_s).
\]

(once again, commutators are easily dealt with). We now take the scalar product with $\frac{1}{\varepsilon^2}(\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u$ and once again commute derivatives and use the equation satisfied by $u$ to obtain

\[
\frac{1}{\varepsilon^2} \int (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m \text{RHS1} \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx \\
= -\frac{1}{4} \int \nabla^2_x \nabla^m \nabla^2_x u \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx \\
+ \frac{1}{2\varepsilon^2} \int \nabla^{m} \nabla^2_x \nabla'(t) \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx + O(\mathcal{E}_s^2) \\
= -\frac{\varepsilon^2}{8} \frac{d}{dt} \int \nabla^{m} \nabla^2_x \nabla'(u) \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx + O(\mathcal{E}_s^2).
\]

(5.24)

Finally, the contribution of RHS2 is simply

\[
\frac{1}{\varepsilon^2} \int (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m \text{RHS2} \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx \\
= -\frac{1}{\varepsilon^3} \int (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m i V'(u) \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx.
\]

Putting everything together gives

\[
\frac{d}{dt} \left[ \frac{1}{2} \int |(\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u|^2 \, dx - \frac{\mu}{2} \int |\nabla^m \partial_x u|^2 \, dx + \frac{\varepsilon^2}{8} \int |\nabla^m \nabla_x \partial_x u|^2 \, dx \right] \\
= \frac{1}{2\varepsilon^2} \int \nabla^m \nabla^2_x \nabla'(t) \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx \\
- \frac{1}{\varepsilon^3} \int (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m i V'(u) \cdot (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx + O(\mathcal{E}_s^2)
\]

We decompose further $I$ and $II$ by distinguishing for each scalar product between the tangent and normal part:

\[
I = \frac{1}{2\varepsilon^2} \int P^\top \nabla^m \nabla^2_x \nabla'(u) \cdot P^\top (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx + \frac{1}{2\varepsilon^2} \int P^\perp \ldots \cdot P^\perp \ldots \, dx \\
II = \frac{1}{\varepsilon^3} \int P^\perp (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m i V'(u) \cdot P^\perp (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^{m-1} \partial u \, dx + \frac{1}{\varepsilon^3} \int P^\perp \ldots \cdot P^\perp \ldots \, dx.
\]

We will in the following examine separately $Ia$, $Ib$, $IIa$ and $IIb$. 
Step 2: estimating $Ia$. First, (5.15) gives

$$
\frac{1}{\varepsilon^2} \nabla^m \nabla^2_x V'(u) = 2\lambda \nabla^m \nabla^2 \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} + O_{L^2}(\mathcal{E}_s).
$$

Due to the vanishing of the second fundamental form on the normal bundle (5.7),

$$
P^T \nabla^m \nabla_x \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} = \Pi^T \left( \nabla^m \nabla_x \partial_x u, \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} \right) + O_{L^2}(\mathcal{E}_s)
$$

$$
= \Pi^T \left( \nabla^T m \nabla_x P^T \partial_x u, \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} \right) + O_{L^2}(\mathcal{E}_s).
$$

Combining the two previous equalities, we obtain

$$
P^T \frac{1}{\varepsilon^2} \nabla^m \nabla^2_x V'(u) = 2\lambda \Pi^T \left( \nabla^T m \nabla_x P^T \partial_x u, \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} \right) + O_{L^2}(\mathcal{E}_s).
$$

On the other hand,

$$
P^T (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^m u = (\varepsilon^2 \nabla_t^T - c \nabla_x^T) \nabla^T m^{-1} P^T \partial u + \varepsilon O_{H^1}(\mathcal{E}_s).
$$

Taking the scalar product of the two last equalities,

$$
Ia = \lambda \int \Pi^T \left( \nabla^T m \nabla_x \partial_x u, \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} \right) \cdot (\varepsilon^2 \nabla_t^T - c \nabla_x^T) \nabla^T m^{-1} P^T \partial u \, dx + O(\mathcal{E}_s^2)
$$

$$
= -\frac{\varepsilon^2 \lambda}{2} \frac{d}{dt} \int \Pi^T \left( \nabla^T m P^T \partial_x u, \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} \right) \cdot \nabla^T m P^T \partial_x u \, dx + O(\mathcal{E}_s^2)
$$

where we used for the last equality the almost symmetry property (5.8).

Step 3: estimating $Ib$. First observe that, on the one hand, (5.15) gives

$$
P^\perp \nabla^m \nabla^2_x V'(u) = 2\lambda \varepsilon^2 P^\perp \nabla^m \nabla^2 \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} + O_{L^2}(\mathcal{E}_s)
$$

$$
= 2\lambda \varepsilon^2 \nabla^\perp m \nabla^2_x \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} + O_{L^2}(\mathcal{E}_s),
$$

while on the other hand, by (5.4) and (5.11),

$$
P^\perp (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^m u = (\varepsilon^2 \nabla_t^\perp - c \nabla_x^\perp) \nabla^\perp m^{-1} P^\perp \Sigma \begin{pmatrix} \varepsilon \partial \phi \\ \varepsilon^2 \nabla^\perp_n \end{pmatrix} + \varepsilon O_{H^1}(\mathcal{E}_s)
$$

$$
= (\varepsilon^2 \nabla_t^\perp - c \nabla_x^\perp) \nabla^\perp m^{-1} \Sigma \begin{pmatrix} 0 \\ \varepsilon^2 \nabla^\perp_n \end{pmatrix} + \varepsilon O_{H^1}(\mathcal{E}_s) + \varepsilon^3 O_{L^2}(\mathcal{E}_s)
$$

$$
= (\varepsilon^2 \nabla_t^\perp - c \nabla_x^\perp) \nabla^\perp m \Sigma \begin{pmatrix} 0 \\ \varepsilon^2 \nabla^\perp_n \end{pmatrix} + \varepsilon O_{H^1}(\mathcal{E}_s) + \varepsilon^3 O_{L^2}(\mathcal{E}_s).
$$

Taking the scalar product of the two gives, after integrating by parts in $x$, and commuting derivatives,

$$
Ib = \lambda \int \nabla^\perp m \nabla^2_x \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} \cdot (\varepsilon^2 \nabla_t^\perp - c \nabla_x^\perp) \nabla^\perp m \Sigma \begin{pmatrix} 0 \\ \varepsilon^2 \nabla^\perp_n \end{pmatrix} \, dx + O(\mathcal{E}_s^2)
$$

$$
= -\frac{\lambda \varepsilon^4}{2} \frac{d}{dt} \int \left| \nabla^\perp m \nabla^\perp_x \Sigma \begin{pmatrix} 0 \\ n \end{pmatrix} \right|^2 \, dx + O(\mathcal{E}_s^2).
$$

Step 4: estimating $IIa$. On the one hand, it is easy to see from formula (5.15) that

$$
P^T (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m iV'(u) = 2\lambda i (\varepsilon^2 \nabla_t^T - (c + iB) \nabla_x^T) \nabla^\perp m \Sigma \begin{pmatrix} 0 \\ \varepsilon^2 \nabla^\perp_n \end{pmatrix} + \varepsilon^3 O_{L^2}(\mathcal{E}_s) \quad (5.25)
$$
On the other hand, it follows from the equation (5.1) satisfied by $u$ that
\[
\nabla \lambda^m \Sigma \left( (\varepsilon^2 \nabla_t^\top - (c + iB) \nabla_x^\top) e^2 \right) = \frac{i \varepsilon}{2} \nabla^T m \nabla_x^\top P^\top \partial_x u + \varepsilon^3 O_L^2 (\mathcal{E}_s).
\]

If $p \in \mathcal{L}$, and $N, N' \in N \mathcal{L}$, denote $G(s) = \Sigma_{(p,sN)} \left( \begin{array}{c} 0 \\ iB N' \end{array} \right) - iB \Sigma_{(p,sN)} \left( \begin{array}{c} 0 \\ N' \end{array} \right)$, one checks thanks to (5.3) and (5.4) that $G(0) = 0$ and $\nabla_s G(s)_{s=0} = 0$. Therefore, $\Sigma_{(p,s^2n)} \left( \begin{array}{c} 0 \\ iBN \end{array} \right) = iB \Sigma_{(p,s^2n)} \left( \begin{array}{c} 0 \\ N \end{array} \right) + O(\varepsilon^4)$. Combined with the last equality, this leads to
\[
(\varepsilon^2 \nabla_t^\top - (c + iB) \nabla_x^\top) \nabla \lambda^m \Sigma \left( \begin{array}{c} \varepsilon^2 \lambda^m \\ n \end{array} \right) = \frac{i \varepsilon}{2} \nabla^T m \nabla_x^\top P^\top \partial_x u + \varepsilon^3 O_L^2 (\mathcal{E}_s).
\] (5.26)

The equalities (5.25) and (5.26) lead to
\[
P^\top (\varepsilon^2 \nabla_t - (c + Bi) \nabla_x) \nabla^m V'(u) = -\lambda \nabla^T m \nabla_x^\top P^\top \partial_x u + \varepsilon^3 O_L^2 (\mathcal{E}_s).
\]

Using this last equality together with
\[
P^\top (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^m u = (\varepsilon^2 \nabla_t^\top - c \nabla_x^\top) \nabla^T (m-1) P^\top \partial_x u + \varepsilon^3 O_L^2 (\mathcal{E}_s).
\]
leads to
\[
IIa = -\frac{\lambda}{\varepsilon^2} \int (\varepsilon^2 \nabla_t^\top - c \nabla_x^\top) \nabla^T (m-1) P^\top \partial_x u \cdot \nabla^T m \nabla_x^\top P^\top \partial_x u dx + O_L^2 (\mathcal{E}_s^2)
\]
\[
= \frac{\lambda}{2} \frac{d}{dt} \int |\nabla^T m P^\top \partial_x u|^2 dx + O_L^2 (\mathcal{E}_s^2).
\]

Step 5: estimating $IIb$. Proceeding as in the previous steps,
\[
\frac{i \varepsilon}{2} P^\top (\varepsilon^2 \nabla_t - (c + iB) \nabla_x) \nabla^m \Sigma (\varepsilon^2 \nabla_t^\top - (c + iB) \nabla_x^\top) e^2 = \frac{2\lambda}{\varepsilon} \int P^\top (\varepsilon^2 \nabla_t - (c + iB) \nabla_x^\top) \nabla^T m \nabla_x^\top P^\top \partial_x u + O_L^2 (\mathcal{E}_s)
\]
\[
= \frac{2\lambda}{\varepsilon} \int \Pi^\top \left( \nabla^T m (\varepsilon^2 \nabla_t^\top - c \nabla_x^\top) u, \Sigma \right) + O_L^2 (\mathcal{E}_s).
\]

Pairing this identity with
\[
P^\perp (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^m \Sigma \partial_x u = \frac{i \varepsilon}{2} \nabla^T m \nabla_x^\top P^\top \partial_x u + \varepsilon O_L^2 (\mathcal{E}_s)
\]
(where we used the equation satisfied by $u$) yields the desired estimate for $IIb$:
\[
IIb = \lambda \int \Pi^\top \left( \nabla^T m (\varepsilon^2 \nabla_t^\top - c \nabla_x^\top) u, \Sigma \right) \cdot \nabla^T m \nabla_x^\top P^\top \partial_x u dx + O(\mathcal{E}_s^2)
\]
\[
= -\frac{\lambda \varepsilon^2}{2} \frac{d}{dt} \int \Pi^\top \left( \nabla^T m P^\top \partial_x u, \Sigma \right) \cdot \nabla^T m P^\top \partial_x u dx + O(\mathcal{E}_s^2),
\]
where we used in the last line the almost symmetry of $\Pi^\top$ (5.3).

Step 6: conclusion. The above estimates lead to the differential inequality
\[
\frac{d}{dt} E_m = O(\mathcal{E}_s^2)
\]
where

\[
E_m = \int \left[ \frac{1}{2} (\varepsilon^2 \nabla_t - c \nabla_x) \nabla^m u^2 + \frac{e^2}{8} \nabla^m \nabla_x u^2 - \mu \nabla^m \partial_x u^2 + \frac{\lambda}{2} \nabla^m P^T \partial_x u^2 + \frac{\varepsilon^4 \lambda}{2} \nabla^m \nabla_x^{\perp} \left( \begin{array}{c} 0 \\ n \end{array} \right) \right]^2 dx.
\]

This gives the desired result since

\[
\sum_{|m| \leq s} E_m \gtrsim \mathcal{E}_{s,1}.
\]

\[5.4. \quad \text{Differentiating the hydrodynamical system.} \quad \text{We shall again use the notation}
\]

\[
\nabla^T \nabla^m \Phi^\varepsilon = \frac{1}{\varepsilon} \nabla^T \nabla^m \Phi.
\]

**Proposition 5.3.** For \(1 \leq |m| \leq s\) and \(s \geq 2\), we obtain on \([0, T^\varepsilon]\) the following system for \((\nabla^\perp \nabla^m \Phi^\varepsilon, \nabla^\perp n, n)\),

\[
\begin{cases}
S_{\perp \perp} (\nabla^T t - \frac{c}{\varepsilon^2} \nabla^T x) \nabla^T \nabla^m \Phi^\varepsilon - \frac{1}{\varepsilon^2} (iB)_{\perp \perp} \nabla^T \nabla^T \nabla^m \Phi^\varepsilon + \varepsilon^5 S_{\perp \perp} (\nabla^T t - \frac{c}{\varepsilon^2} \nabla^T x) \nabla^\perp m n \\
= i \left[ \frac{1}{2} \nabla^T \left( S_{\perp \perp} \nabla^T x \nabla^T \nabla^m \Phi^\varepsilon + \varepsilon^5 S_{\perp \perp} (\nabla^T t - \frac{c}{\varepsilon^2} \nabla^T x) \nabla^\perp m n \right) - \frac{\varepsilon^2}{2} S_{\perp \perp} \nabla^\perp m n - 2 \lambda \Pi T (iS_{\perp \perp} n, S_{\perp \perp} \nabla^\perp m n) \right. \\
\left. + \frac{1}{2} \Pi T (S_{\perp \perp} \nabla^T x \nabla^T \nabla^m \Phi^\varepsilon, S_{\perp \perp} D \partial_x \phi) + \frac{1}{2} \Pi T (S_{\perp \perp} \nabla^T \nabla^T \nabla^m \Phi^\varepsilon + O_{H^1}(\mathcal{E}) \right) \\
S_{\perp \perp} (\nabla^T t - \frac{c}{\varepsilon^2} \nabla^T x) \nabla^T \nabla^m \Phi^\varepsilon - \frac{1}{\varepsilon^2} (iB)_{\perp \perp} \nabla^T x \nabla^T \nabla^m \Phi^\varepsilon + \varepsilon^5 S_{\perp \perp} (\nabla^T t - \frac{c}{\varepsilon^2} \nabla^T x) \nabla^\perp m n \\
= i \left[ \frac{1}{2} \nabla^T \left( S_{\perp \perp} \nabla^T x \nabla^T \nabla^m \Phi^\varepsilon + \varepsilon^5 S_{\perp \perp} (\nabla^T t - \frac{c}{\varepsilon^2} \nabla^T x) \nabla^\perp m n \right) \right. \\
\left. + \Pi T (S_{\perp \perp} \nabla^T x \nabla^T \nabla^m \Phi^\varepsilon, S_{\perp \perp} \nabla^T \nabla^T \nabla^m \Phi^\varepsilon) + O_{H^1}(\mathcal{E}) \right]
\end{cases}
\]

(5.27)

**Proof.** We shall apply \((\nabla^T)^m\) to the first equation of (5.20), and \((\nabla^\perp)^m\) to the second one.

We first note that (4.19), (4.20), (1.21) still hold. We shall use these estimates intensively again. We already noticed note that the equivalents of (4.6), (4.7), namely (5.21) and (5.22), are still true.

**Step 1:** Left hand side of (5.20). Let us apply \(\nabla^T \nabla^m\) to the left hand side of (5.20) and let us expand the resulting line \(L^\phi\) as

\[
L_1^\phi = S_{\perp \perp} \nabla^T \nabla^m (\partial_t - \frac{c}{\varepsilon^2} \partial_x) \frac{\Phi}{\varepsilon} - \frac{1}{\varepsilon^2} (iB)_{\perp \perp} \nabla^T \nabla^m \frac{\partial_x \Phi}{\varepsilon},
\]

\[
L_2^\phi = \left[ \nabla^T \nabla^m, S_{\perp \perp} \right] (\partial_t - \frac{c}{\varepsilon^2} \partial_x) \frac{\Phi}{\varepsilon} - \left[ \nabla^T \nabla^m, (iB)_{\perp \perp} \right] \frac{\partial_x \Phi}{\varepsilon} = C_1 - C_2
\]

\[
L_3^\phi = \nabla^T \nabla^m \left( \varepsilon^5 S_{\perp \perp} \left( \nabla^T t - \frac{c}{\varepsilon^2} \nabla^T x \right) \nabla^\perp m n \right) - \nabla^T \nabla^m \left( \varepsilon^3 (iB)_{\perp \perp} \nabla^T \nabla^T \nabla^m \Phi^\varepsilon \right).
\]

Note that for \(L_2^\phi\), we still use commutator notations, but that in the definition of

\[
\left[ \nabla^T \nabla^m, S_{\perp \perp} \right] X := \nabla^T \nabla^m (S_{\perp \perp} X) - S_{\perp \perp} \nabla^T \nabla^m X,
\]

the operator \(\nabla^T\) is not the same in the first and in the second term. Indeed, in the first term \(\nabla^T = P^T \nabla\) with \(P^T\) the orthogonal projector on \(T_u \mathcal{L}\) while in the second term \(\nabla^T = P^T \nabla\) with
$P^\top$ the orthogonal projection on $T_\Phi \mathcal{L}$. We shall keep on using this notation in the following. To estimate $L_1^\Phi$, we can proceed as in (4.24) because of the identity (3.7), we thus get as in (4.26)

$$L_1^\Phi = S_{\top\top_T} \left( \nabla^\top_T \left( \nabla^\top_T - \frac{c}{\varepsilon^2} \nabla^\top_T \right) \nabla^\top_T \Phi - \frac{1}{\varepsilon^2} \left( iB \right)_{\top\top_T} \nabla^\top_T \nabla^\top_T \Phi + O_{H'}(\mathcal{E}_s) \right).$$

(5.28)

To estimate $L_2^\Phi$, let us start with the commutator $C_1$. We can expand it as a sum of terms under the form

$$\nabla^{\top\alpha} \nabla^{\top\beta} \left( \nabla^\top_T, S_{\top\top_T} \right) \nabla^\top_T \left( \partial_t - \frac{c}{\varepsilon^2} \partial_x \right) \frac{\Phi}{\varepsilon}, \quad |\alpha| + |\beta| = |m| - 1.

Note that the $\nabla^\top_T$ in the middle commutator can be either $\nabla^\top_T$ or $\varepsilon^2 \nabla^\top_T$. We thus obtain that the commutator can be expanded into a sum of terms of the type

$$\nabla^{\top\alpha} \left( (\nabla^\top_T S_{\top\top_T})(\nabla^\top_T \Phi, \varepsilon^2 \nabla^\top_T), \nabla^{\top\beta} \left( \partial_t - \frac{c}{\varepsilon^2} \partial_x \right) \frac{\Phi}{\varepsilon} \right)$$

with $|\alpha| + |\beta| = |m| - 1$.

Thanks to (5.12), we first observe that for every $X \in T_\Phi \mathcal{L}$, the tensor $(\nabla^\top_T S_{\top\top_T}) = \nabla^\top T S_{\top\top_T}(X, \cdot)$ vanishes on $\mathcal{L}$, thus we can write that

$$(\nabla^\top T S_{\top\top_T})(X, Y) = \varepsilon^2 T_{(\Phi, n)}(X, Y)$$

(5.29)

for some smooth tensor $T$. This yields

$$\nabla^{\top\alpha} \left( (\nabla^\top_T S_{\top\top_T})(\Phi, \varepsilon^2 n) \left( (\nabla^\top T \Phi, 0), \nabla^{\top\beta} \left( \partial_t - \frac{c}{\varepsilon^2} \partial_x \right) \frac{\Phi}{\varepsilon} \right) \right)$$

$$= \nabla^{\top\alpha} \left( T_{(\Phi, n)} \left( \nabla^\top T \Phi, \nabla^{\top\beta} \left( \varepsilon^2 \partial_t - c \partial_x \right) \frac{\Phi}{\varepsilon} \right) \right) = O_{H'}(\mathcal{E}_s).$$

Next, we also observe thanks to (5.12) that for every $Z \in N_\Phi \mathcal{L}, Y \in T_\Phi \mathcal{L}$, the tensor $(\nabla^\top T S_{\top\top_T})(0, Z, Y) - \Pi_{\pi}^\top(S_{\top\top_T}, S_{\perp\perp}, Z)$ vanishes on $\mathcal{L}$. Thus, we can write

$$(\nabla^\top T S_{\top\top_T})(0, Z, Y) = \Pi_{\pi}^\top \left( S_{\top\top_T}, S_{\perp\perp}, Z \right) + \varepsilon^2 T_{(\Phi, n)}(Z, Y)$$

(5.30)

for some smooth tensor $T$ (in this proof, we shall always use the notation $T(\Phi, n)$ for a harmless smooth tensor). We thus easily obtain that

$$\nabla^{\top\alpha} \left( (\nabla^\top T S_{\top\top_T})(\Phi, \varepsilon^2 n) \left( (0, \varepsilon^2 \nabla^\top_T), \nabla^{\top\beta} \left( \partial_t - \frac{c}{\varepsilon^2} \partial_x \right) \frac{\Phi}{\varepsilon} \right) \right)$$

$$= \nabla^{\top\alpha} \left( \Pi_{\pi}^\top \left( S_{\top\top_T}, \nabla^{\top\beta} \left( \varepsilon^2 \partial_t - c \partial_x \right) \frac{\Phi}{\varepsilon}, S_{\perp\perp} \nabla^\top_T \right) \right) + O_{H'}(\mathcal{E}_s)$$

(5.31)

Observe then that all the above terms are $O_{H'}(\mathcal{E}_s)$ as long as $\beta \neq 0$. We now focus on the term for which $\beta = 0$. It can be written as

$$\nabla^{\top m-1} \left( \Pi_{\pi}^\top \left( S_{\top\top_T} \left( \varepsilon^2 \partial_t - c \partial_x \right) \frac{\Phi}{\varepsilon}, S_{\perp\perp} \nabla^\top_T \right) \right)$$

$$= \Pi_{\pi}^\top \left( S_{\top\top_T} \left( \varepsilon^2 \partial_t - c \partial_x \right) \frac{\Phi}{\varepsilon}, S_{\perp\perp} \nabla^{\top m-1} \right) + O_{H'}(\mathcal{E}_s).$$

We have thus proven that

$$C_1 = \Pi_{\pi}^\top \left( S_{\top\top_T} \left( \varepsilon^2 \partial_t - c \partial_x \right) \frac{\Phi}{\varepsilon}, S_{\perp\perp} \nabla^{\top m-1} \right) + O_{H'}(\mathcal{E}_s).$$

(5.32)

We can proceed in a similar way for the commutator $C_2$. We thus have to estimate

$$\frac{1}{\varepsilon^2} \nabla^{\top\alpha \beta \gamma} \left( (\nabla^\top_T (iB)_{\top\top_T}) \left( (\nabla^\top T \Phi, \varepsilon^2 \nabla^\top_T), \nabla^{\top\beta} \partial_x \frac{\Phi}{\varepsilon} \right) \right)$$

(5.33)
with \((\alpha, \beta)\) as above. Thanks to (5.17), we first observe that
\[
\nabla^\top \alpha \left( \frac{1}{\varepsilon^2} (\nabla^\top (iB)_{\top\top}) \left( (\nabla^\top \Phi, 0), \nabla^\top \beta \partial_x \frac{\Phi}{\varepsilon} \right) \right) = \left( \nabla^\top \alpha T_{(\Phi, n)} \left( \nabla^\top \Phi, \nabla^\top \beta \partial_x \frac{\Phi}{\varepsilon} \right) \right) = O_{H^1}(\mathcal{E}_s).
\]

Thus, it remains to estimate
\[
\nabla^\top \alpha \left( (\nabla^\top (iB)_{\top\top}) \left( (0, \nabla^\perp n), \nabla^\top \beta \partial_x \frac{\Phi}{\varepsilon} \right) \right).
\]

We note that for every \(X \in \mathcal{T}_\Phi \mathcal{L}, Z \in N_\Phi \mathcal{L}\), we have that
\[
\left( \nabla^\perp (iB)_{\top\top} \right) (\Phi, \varepsilon^2 n) X = \Pi_u^\top \left( (iB)_{\top\top} X, S_{\perp\perp} Z \right) + \varepsilon^2 T(\Phi, n)(Z, X).
\]

Indeed, this follows from the fact that the tensor \((\nabla^\perp (iB)_{\top\top}) - \Pi_u^\top \left( (iB)_{\top\top}, S_{\perp\perp} Z \right)\) vanishes on \(\mathcal{L}\) thanks to (5.17). By plugging this identity in the terms (5.33), we finally obtain that
\[
C_2 = \Pi_u^\top \left( (iB)_{\top\top} \partial_x \frac{\Phi}{\varepsilon}, S_{\perp\perp} \nabla^\perp m \right) + O_{H^1}(\mathcal{E}_s).
\]

By collecting (5.32) and (5.34), we get
\[
L_2^\alpha = \Pi_u^\top \left( \left( S_{\top\top} (\varepsilon^2 \partial_t - c \partial_x) - (iB)_{\top\top} \partial_x \right) \frac{\Phi}{\varepsilon}, S_{\perp\perp} \nabla^\perp m \right) + O_{H^1}(\mathcal{E}_s)
\]

and thanks to the first line of (5.20), we can simplify its expression as
\[
L_2^\alpha = -2\lambda \Pi_u^\top \left( iS_{\perp\perp} n, S_{\perp\perp} \nabla^\perp m \right) + O_{H^1}(\mathcal{E}_s).
\]

It remains to deal with \(L_3^\alpha\) which is easy to handle. Just by counting derivatives, we get
\[
L_3^\alpha = \varepsilon^5 S_{\perp\perp} (\Phi, \varepsilon^2 n) \left( \nabla^\perp \partial_t - \frac{c}{\varepsilon^2} \nabla^\perp \partial_x \right) \nabla^\perp m + O_{H^1}(\mathcal{E}_s).
\]

Note that we cannot consider the term involving \(\nabla^\perp \nabla^\perp m\) as a remainder term since it is not controlled in \(H^1\) by \(\mathcal{E}_{s,1}\) (at least when \(\nabla^m\) involves only time derivatives). We also keep the term involving \(\nabla^\perp \nabla^\perp m\) for symmetry reasons.

Looking at (5.36), (5.35), (5.25), we have thus proven that
\[
\nabla^\top m \text{LHS } (5.20) \downarrow_1 = S_{\top\top} \left( \nabla^\top \partial_t - \frac{c}{\varepsilon^2} \nabla^\top \partial_x \right) \nabla^\top m \Phi^\varepsilon - \frac{1}{\varepsilon^2} (iB)_{\top\top} \nabla^\top \nabla^\top m \Phi^\varepsilon
\]
\[
+ \varepsilon^5 S_{\perp\perp} \left( \nabla^\perp \partial_t - \frac{1}{\varepsilon^2} \nabla^\perp \partial_x \right) \nabla^\perp m + O_{H^1}(\mathcal{E}_s).
\]

We shall again omit to specify that \(\Pi^\top\) is evaluated at \(u\).

\underline{Step 2: Right-hand side of (5.20) \downarrow_1.} Simply counting derivatives,
\[
\nabla^\perp m \left( \Pi^\perp \left( \sum \left( \frac{\partial_x \Phi}{\varepsilon \nabla^\perp n}, P^\top \Sigma \left( \frac{\partial_x \Phi}{\varepsilon \nabla^\perp n} \right) \right) \right) \right)
\]
\[
= \Pi^\perp \left( \sum \left( \begin{array}{c}
\nabla^\perp \nabla^\perp m \Phi^\varepsilon \\
\varepsilon \nabla^\perp \nabla^\perp m \Phi^\varepsilon
\end{array} \right), P^\top \Sigma \left( \begin{array}{c}
D \Phi \partial_x \Phi \\
\varepsilon \nabla^\perp \nabla^\perp m \Phi^\varepsilon
\end{array} \right) \right) + \Pi^\perp \left( \sum \left( \begin{array}{c}
D \Phi \partial_x \Phi \\
\varepsilon \nabla^\perp \nabla^\perp m \Phi^\varepsilon
\end{array} \right), P^\top \Sigma \left( \begin{array}{c}
\nabla^\perp \nabla^\perp m \Phi^\varepsilon \\
\varepsilon \nabla^\perp \nabla^\perp m \Phi^\varepsilon
\end{array} \right) \right) + O_{H^1}(\mathcal{E}_s)
\]
\[
= T_1 + T_2 + O_{H^1}(\mathcal{E}_s)
\]

as in the proof of (4.30). Using the properties of \(\Sigma\) and \(\Pi^\perp\), we can further simplify the expression of these terms: by (5.11),
\[
T_1 = \Pi^\perp \left( S_{\top\top} \nabla^\top \nabla^\top m \Phi^\varepsilon, S_{\top\top} D \Phi \partial_x \Phi \right) + O_{H^1}(\mathcal{E}_s)
\]
\[
T_2 = \Pi^\perp \left( S_{\top\top} D \Phi \partial_x \Phi, S_{\top\top} \nabla^\perp \nabla^\top m \Phi^\varepsilon \right) + O_{H^1}(\mathcal{E}_s)
\]
Therefore,

\[
\nabla^m (\Pi^\perp \left( \Sigma \left( \frac{\partial \Phi}{\varepsilon \nabla^\perp_2 n} \right), P^\perp \Sigma \left( \frac{\partial \Phi}{\varepsilon \nabla^\perp_2 n} \right) \right)) = \Pi^\perp \left( S_{\perp \perp} \nabla_x^T \nabla^T m \Phi^\varepsilon, S_{\perp \perp} D \Phi \partial_x \phi, S_{\perp \perp} \nabla_x^T \nabla^T m \Phi^\varepsilon \right) + O_{H^1}(\mathcal{E}_s). \quad (5.38)
\]

To treat

\[
\frac{1}{\varepsilon^2} \nabla^m (S_{\perp \perp} n),
\]

we use (5.12) which gives that

\[
\left( \nabla^\perp S_{\perp \perp} \right)_{(\Phi, \varepsilon^2 n)} = \varepsilon^2 \mathcal{T}(\Phi, n). \quad (5.39)
\]

This equality enables to write the commutation of one derivative with \( S_{\perp \perp} \) as follows

\[
\frac{1}{\varepsilon} \nabla^m (S_{\perp \perp} n) = \frac{1}{\varepsilon} \nabla^m-1 (S_{\perp \perp} \nabla^\perp n) + \frac{1}{\varepsilon} \nabla^m-1 (\nabla^\perp \nabla^\perp S_{\perp \perp} ((\nabla^\perp \Phi, \varepsilon^2 \nabla^\perp n), n))
\]

\[
= \frac{1}{\varepsilon} \nabla^m-1 (S_{\perp \perp} \nabla^\perp n) + O_{H^1}(\mathcal{E}_s)
\]

Repeating this operation gives the desired identity:

\[
\frac{1}{\varepsilon^2} \nabla^m (S_{\perp \perp} n) = \frac{1}{\varepsilon^2} S_{\perp \perp} \nabla^m n + O_{H^1}(\mathcal{E}_s). \quad (5.40)
\]

It remains to consider

\[
\nabla^m (\nabla^\perp (S_{\perp \perp} \nabla^\perp n)).
\]

The first step is to commute \( \nabla^m \) with the first \( \nabla^\perp \) using (5.10). Observe that if \( \nabla^m \) does not contain time derivatives, there is no commutator, we can thus write \( \nabla^m = \nabla^m \varepsilon^2 \nabla^\perp \) and get

\[
\nabla^m (\nabla^\perp (S_{\perp \perp} \nabla^\perp n)) = \nabla^m (\nabla^\perp \varepsilon^2 \nabla^\perp (S_{\perp \perp} \nabla^\perp n)) + \nabla^m (\varepsilon^2 R^\perp \left( \frac{\partial u}{\varepsilon}, \frac{\partial u}{\varepsilon} \right) S_{\perp \perp} \nabla^\perp n)
\]

\[
= \nabla^m (\nabla^\perp \varepsilon^2 \nabla^\perp (S_{\perp \perp} \nabla^\perp n)) + O_{H^1}(\mathcal{E}_s).
\]

Iterating this manipulation yields

\[
\nabla^m (\nabla^\perp (S_{\perp \perp} \nabla^\perp n)) = \nabla^\perp \left( \nabla^m (S_{\perp \perp} \nabla^\perp n) \right) + O_{H^1}(\mathcal{E}_s).
\]

Commuting one derivative with \( S_{\perp \perp} \) gives

\[
\nabla_x^\perp (\nabla^m (S_{\perp \perp} \nabla^\perp n)) = \nabla_x^\perp \left( \nabla^{m-1} \nabla_x^\perp (S_{\perp \perp} \nabla^\perp n) \right) + \nabla_x^\perp \left( \nabla^{m-1} \nabla_x^\perp \left( (\nabla^\perp \Phi, \varepsilon^2 \nabla^\perp n), \nabla^\perp n \right) \right)
\]

\[
= \nabla_x^\perp \left( \nabla^{m-1} \nabla_x^\perp (S_{\perp \perp} \nabla^\perp n) \right) + O_{H^1}(\mathcal{E}_s);
\]

here, the last equality follows from (5.39). Repeating this argument leads to

\[
\nabla^m (\nabla^\perp (S_{\perp \perp} \nabla^\perp n)) = \nabla^\perp \left( S_{\perp \perp} \nabla^m \nabla^\perp n \right) + O_{H^1}(\mathcal{E}_s).
\]

Finally, using once again the commutation relation (5.10) enables us to write

\[
\nabla^m (\nabla^\perp (S_{\perp \perp} \nabla^\perp n)) = \nabla^\perp \left( S_{\perp \perp} \nabla_x^\perp \nabla^m n \right) + O_{H^1}(\mathcal{E}_s). \quad (5.41)
\]

The term \( \varepsilon^3 \nabla^m \nabla^\perp (S_{\perp \perp} \frac{\partial \Phi}{\varepsilon}) \) can be easily expanded as

\[
\varepsilon^3 \nabla^m \nabla^\perp (S_{\perp \perp} \frac{\partial \Phi}{\varepsilon}) = \varepsilon^3 \nabla^\perp (S_{\perp \perp} \nabla_x^\perp \nabla^m \Phi^\varepsilon) + O_{H^1}(\mathcal{E}_s)
\]
just by counting the derivatives of the commutator. Gathering \((5.38), (5.40), (5.41)\) and the last identity, we obtain

\[
\nabla^m \text{RHS}_{(5.20)} = i \left[ \frac{1}{2} \nabla_x^\perp \left( S_{\perp \perp} \nabla^m \nabla_x^\perp n \right) + \frac{1}{2} \epsilon^3 \nabla_x^\perp \left( S_{\perp \perp} \nabla_x^\top \nabla^m \Phi^\epsilon \right) - \frac{2}{\epsilon^2} S_{\perp \perp} \nabla^m n \right] (5.42)
\]

\[
+ \frac{1}{2} \Pi^\perp \left( S_{\perp \perp} \nabla_x^\top \nabla^m \Phi^\epsilon, S_{\perp \perp} D \Phi \partial_x \phi \right) + \frac{1}{2} \Pi^\perp \left( S_{\perp \perp} D \Phi \partial_x \phi, S_{\perp \perp} \nabla_x^\top \nabla^m \Phi^\epsilon \right) + O_{H^1}(\mathcal{E}_s).
\]

(5.43)

**Step 3:** Left-hand side of \((5.20)\)_2. By using similar arguments as above, we can first write

\[
\nabla^m \text{LHS}_{(5.20)} = S_{\perp \perp} \left( \nabla^\perp_t - \frac{c}{\epsilon^2} \nabla^\perp_x \right) \nabla^m n - \frac{1}{\epsilon^2} \nabla^m \left( (iB)_{\perp \perp} \nabla_x^\perp n \right)
\]

\[
+ \epsilon^3 S_{\perp \perp} \left( \nabla^\top_t - \frac{c}{\epsilon^2} \nabla^\top_x \right) \nabla^m \Phi^\epsilon + O_{H^1}(\mathcal{E}_s).
\]

To handle \(\frac{1}{\epsilon^2} \nabla^m \left( (iB)_{\perp \perp} \nabla_x^\perp n \right)\), we can expand this term as previously and use \((5.18)\) to handle the commutator. This yields

\[
\frac{1}{\epsilon^2} \nabla^m \left( (iB)_{\perp \perp} \nabla_x^\perp n \right) = \frac{1}{\epsilon^2} (iB)_{\perp \perp} \nabla_x^\perp \nabla^m n + O_{H^1}(\mathcal{E}_s).
\]

We have thus proven that

\[
\nabla^m \text{LHS}_{(5.20)} = S_{\perp \perp} \left( \nabla^\perp_t - \frac{c}{\epsilon^2} \nabla^\perp_x \right) \nabla^m n - \frac{1}{\epsilon^2} (iB)_{\perp \perp} \nabla_x^\perp \nabla^m n
\]

\[
+ \epsilon^3 S_{\perp \perp} \left( \nabla^\top_t - \frac{c}{\epsilon^2} \nabla^\top_x \right) \nabla^m \Phi^\epsilon + O_{H^1}(\mathcal{E}_s). \tag{5.44}
\]

**Step 4:** Right-hand side of \((5.20)\)_2. Again simply using commutator estimates, we obtain

\[
\nabla^m \nabla_x^\perp \left( i \epsilon^3 S_{\perp \perp} \nabla_x^\perp n \right) = i \nabla_x^\top \left( \epsilon^3 S_{\perp \perp} \nabla_x^\perp \nabla^m n \right) + O_{H^1}(\mathcal{E}_s). \tag{5.45}
\]

In a similar way, there is no new difficulty in commuting \(\nabla^m \) and the term involving \(\Pi^\top\) in order to get

\[
\nabla^m \left( \Pi^\top \left( \sum \left( \frac{D \Phi \partial_x \phi}{\epsilon \nabla_x^\perp n} \right), P^\perp \sum \left( \frac{D \Phi \partial_x \phi}{\epsilon \nabla_x^\perp n} \right) \right) \right)
\]

\[
= \Pi^\top \left( \sum \left( \frac{\nabla_x^\top \nabla^m \Phi^\epsilon}{\epsilon \nabla_x^\perp \nabla^m n} \right), P^\perp \sum \left( \frac{D \Phi \partial_x \phi}{\epsilon \nabla_x^\perp n} \right) \right) + O_{H^1}(\mathcal{E}_s).
\]

Next, proceeding as in the proof of \((5.38)\), we obtain

\[
\nabla^m \left( \Pi^\top \left( \sum \left( \frac{D \Phi \partial_x \phi}{\epsilon \nabla_x^\perp n} \right), S_{\perp \perp} \nabla_x^\perp n \right) \right)
\]

\[
= \Pi^\top \left( S_{\perp \perp} \nabla_x^\top \nabla^m \Phi^\epsilon, S_{\perp \perp} \nabla_x^\perp n \right) + \Pi^\top \left( S_{\perp \perp} D \Phi \partial_x \phi, S_{\perp \perp} \nabla_x^\perp \nabla^m n \right). \tag{5.46}
\]

We shall now study more precisely the term \(\frac{1}{\epsilon^2} \nabla^m \nabla_x^\top \left( S_{\perp \perp} D \Phi \partial_x \phi \right)\). By using \((5.9)\), we first get

\[
\frac{1}{\epsilon^2} \nabla^m \nabla_x^\top \left( S_{\perp \perp} D \Phi \partial_x \phi \right) = \frac{1}{\epsilon^2} \nabla_x^\top \left( \nabla^m \left( S_{\perp \perp} D \Phi \partial_x \phi \right) \right).
\]
We now commute $\nabla^T m$ with $S_{TT}$. It yields a sum of terms of the type
\[
\frac{1}{\varepsilon^2} \nabla^T \nabla^T \alpha \left[ \nabla^T S_{TT} \right] \left( (\nabla^T \Phi, \varepsilon^2 \nabla^T n), \nabla^T \gamma D \Phi \partial_x \phi \right)
= \frac{1}{\varepsilon^2} \nabla^T \nabla^T \alpha \left[ \nabla^T S_{TT} \right] \left( (0, \varepsilon^2 \nabla^T n), \nabla^T \gamma D \Phi \partial_x \phi \right) + \frac{1}{\varepsilon^2} \nabla^T \nabla^T \alpha \left[ \nabla^T S_{TT} \right] \left( (\nabla^T \Phi, 0), \nabla^T \gamma D \Phi \partial_x \phi \right),
\]
where $|\alpha| + |\gamma| = m - 1$. The second term on the above right-hand side is always $O_{H_1}(E_s)$ by (5.12); as for the first term on the above-right hand side, due in particular to (5.22), it is also $O_{H_1}(E_s)$ unless $\gamma = 0$. In other words,
\[
\frac{1}{\varepsilon^2} \nabla^T m \nabla^T \left( S_{TT} D \Phi \partial_x \phi \right)
= \frac{1}{\varepsilon^2} \nabla^T \left( S_{TT} \nabla^T m D \Phi \partial_x \phi \right) + \frac{1}{\varepsilon^2} \left[ \nabla^T S_{TT} \right] \left( (0, \varepsilon^2 \nabla^T \nabla^T m n), D \Phi \partial_x \phi \right) + O_{H_1}(E_s).
\]
Using once again (5.12), the second term on the above right-hand side can be replaced as follows
\[
\frac{1}{\varepsilon^2} \nabla^T m \nabla^T \left( S_{TT} D \Phi \partial_x \phi \right)
= \frac{1}{\varepsilon^2} \nabla^T \left( S_{TT} \nabla^T m D \Phi \partial_x \phi \right) + \Pi^T \left( S_{TT} D \Phi \partial_x \phi, S_{\perp \perp} \nabla^T \nabla^T m n \right) + O_{H_1}(E_s)
= \frac{1}{\varepsilon^2} \nabla^T \left( S_{TT} \nabla^T \Phi \nabla^T m \Phi \right) + \Pi^T \left( S_{TT} D \Phi \partial_x \phi, S_{\perp \perp} \nabla^T \nabla^T m n \right) + O_{H_1}(E_s). \tag{5.47}
\]
Combining (5.45), (5.46), (5.47), we have thus proven that
\[
\nabla^T m RHS \tag{5.20}_2 = i \left[ \frac{1}{2\varepsilon^2} \nabla^T \left( S_{TT} \nabla^T \nabla^T m \Phi \varepsilon + \varepsilon^5 S_{\perp \perp} \nabla^T \nabla^T m n \right) + \Pi^T \left( S_{TT} \nabla^T \nabla^T m \Phi \varepsilon, S_{\perp \perp} \nabla^T \nabla^T m n \right) + O_{H_1}(E_s) \right] \tag{5.48}
\]
The proof of Proposition 5.3 follows by combining (5.37), (5.42), (5.44), (5.48). □

5.5. Estimates on the hydrodynamical system. The aim of this section is to use the hydrodynamical system derived in Proposition 5.3 to prove the following

**Proposition 5.4.** The following a priori estimate holds
\[
E^2_{s,2}(u, t) \lesssim E^2_s(u, 0) + \varepsilon O(E_s(u, t)) + \int_0^t O(E_s(u, \tau)) d\tau, \quad \forall t \in [0, T^\varepsilon].
\]

We shall use the same idea as in the proof of Proposition 1.14. The additional difficulty is that the coupling between the two equations of the system (5.27) is slightly stronger than before. It is thus more convenient to deal directly with the whole system and use vectors and block matrix notations.

We introduce some notations before stating the proposition: let $U^m \overset{\text{def}}{=} (\nabla^T m \Phi \varepsilon, \nabla^T m n)^T = (U^m_1, U^m_2)^T$ $\in T_0 \mathcal{L} \times N_0 \mathcal{L}$.

For any vector $U \in \tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L}$ along $u \in \mathcal{M}$, it will be convenient to use the notation
\[
D_i U \overset{\text{def}}{=} (\nabla^T_i U_1, \nabla^T_i U_2), \quad i = t, x.
\]
Note that in this notation in the definition of the connection, $P^T$ and $P^\perp$ are to be taken at $u$. Define the matrix $J_u \in \mathcal{L}(\tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L}, \tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L})$ by
\[
J_u \overset{\text{def}}{=} \begin{pmatrix} 0 & i_u \\ i_u & 0 \end{pmatrix}.
\]
Note that $J$ is skew symmetric for the scalar product on $\tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L}$ defined by
\[
\langle U, V \rangle_u = \langle U_1, V_1 \rangle_u + \langle U_2, V_2 \rangle_u. \tag{5.49}
\]
In a similar way, we define $B_u \in \mathcal{L}(\tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L}, \tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L})$ defined by
\[
B_u = \begin{pmatrix}
0 & P_u \top B(u) P_u \top \\
P_u \top B(u) P_u \top & 0
\end{pmatrix}.
\]
Again $B$ is skew-symmetric for the above scalar product. We shall also use $\mathcal{M}(\Phi, \varepsilon^2 n) \in \mathcal{L}(T_\Phi \mathcal{L} \times N_\Phi \mathcal{L}, \tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L})$ defined as
\[
\mathcal{M} \overset{\text{def}}{=} \begin{pmatrix}
S_{\top \top} & \varepsilon S_{\top \perp} \\
\varepsilon^3 S_{\top \top} & S_{\perp \perp}
\end{pmatrix}.
\]
Note that the only difference between $\mathcal{M}$ and $\Sigma$ defined in ($5.11$) is in the power of $\varepsilon$ in front of the off-diagonal terms that come from the anisotropic scaling of the KdV limit.

We now rewrite in a more concise form the system ($5.27$): first the left hand-side of ($5.27$), which, thanks to ($5.16$), we can write
\[
\text{LHS \ (5.27)} = \mathcal{M}
\left(D_t U^m - \frac{c}{\varepsilon^2} D_x \right) U^m - \frac{1}{\varepsilon^2} J B M D_x U^m + \left( \frac{O_{H^1}(\mathcal{E}_s)}{O_{H^1}(\mathcal{E}_s)} \right),
\]
Next, the right hand-side of ($5.27$), which can be symmetrized a little more by observing that
\[
\frac{1}{2} \Pi_u^\perp \left( S_{\top \top} \nabla^m_x \nabla^\top \Phi^\varepsilon, S_{\top \top} D \Phi \partial_x \phi \right) + \frac{1}{2} \Pi_u^\perp \left( S_{\top \top} D \Phi \partial_x \phi, S_{\top \top} \nabla^m_x \nabla^\top \Phi^\varepsilon \right)
= \Pi_u^\perp \left( S_{\top \top} D \Phi \partial_x \phi, S_{\top \top} \nabla^m_x \nabla^\top \Phi^\varepsilon \right) + O_{H^1}(\mathcal{E}_s).
\]
Indeed, our generalized second fundamental form $\Pi^\perp$ is not symmetric in its arguments, but if we define the tensor
\[
T(u)(D \Phi \partial_x \phi, X) = \frac{1}{2} \Pi_u^\perp (S_{\top \top} u(X), S_{\top \top} D \Phi \partial_x \phi) + \frac{1}{2} \Pi_u^\perp (S_{\top \top} u(D \Phi \partial_x \phi, S_{\top \top} X)
- \Pi_u^\perp (S_{\top \top} u(D \Phi \partial_x \phi, S_{\top \top} X)
acting on $T_\Phi \mathcal{L} \times T_\Phi \mathcal{L}$, it vanishes if $u \in \mathcal{L}$ by the symmetry in its arguments of the second fundamental form of the tangent bundle of $\mathcal{L}$. We thus obtain that
\[
T(u)(D \Phi \partial_x \phi, X) = \varepsilon^2 T(\Phi, n)(D \Phi \partial_x \phi, X)
\]
and hence we get ($5.51$).

Using this simplification, define the tensor $B_\Pi(u, \partial_x \phi) \in \mathcal{L}(\tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L}, \tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L})$ as
\[
B_\Pi(u, \partial_x \phi) = \begin{pmatrix}
0 & \Pi_u^\top (S_{\top \top} D \Phi \partial_x \phi, \cdot) \\
\Pi_u^\top (S_{\top \top} D \Phi \partial_x \phi, \cdot) & 0
\end{pmatrix}
\]
so that
\[
\begin{pmatrix}
i \Pi_u^\top (S_{\top \top} D \Phi \partial_x \phi, S_{\top \top} \nabla^m_x \nabla^\top \Phi^\varepsilon) \\
i \Pi_u^\top (S_{\top \top} D \Phi \partial_x \phi, S_{\perp \perp} \nabla^m_x \nabla^\top \Phi^\varepsilon)
\end{pmatrix}
= J B_\Pi \mathcal{M} D_x U^m + \left( \frac{O_{H^1}(\mathcal{E}_s)}{O_{H^1}(\mathcal{E}_s)} \right).
\]
By definition of $\Pi^\top$ and $\Pi^\perp$, we have that
\[
\langle \Pi_u^\top (X, N), Y \rangle + \langle N, \Pi_u^\perp (X, Y) \rangle = 0
\]
for every vector fields along $u$ such that $X \in \tilde{T}_u \mathcal{L}$, $Y \in \tilde{T}_u \mathcal{L}$, $N \in \tilde{N}_u \mathcal{L}$. This yields that $B_\Pi$ is skew symmetric on $\tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L}$ for the scalar product defined by ($5.49$).

To include the zero order terms in the right hand-side of ($5.27$), we introduce the tensor $\mathcal{V}(u, n) \in \mathcal{L}(\tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L}, \tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L})$ defined by
\[
\mathcal{V}(u, n) = \begin{pmatrix}
0 & 0 \\
2 \lambda + 2 \varepsilon^2 \lambda \Pi^\top (i S_{\perp \perp} n) & 0
\end{pmatrix}.
to obtain that
\[ \frac{1}{\varepsilon^2} \left( i (2\lambda S_{\perp \perp} \nabla_{\perp \perp}^m n + 2\varepsilon^2 \lambda i \Pi^\top (iS_{\perp \perp} n, S_{\perp \perp} \nabla_{\perp \perp}^m n)) \right) = \frac{1}{\varepsilon^2} \mathcal{J} \mathcal{V} \mathcal{M} U^m + \left( O_{H^1}(\mathcal{E}_s) \right). \]

Moreover, let us notice that \( i \Pi^\top \mathcal{U} (X, \cdot) \) remains symmetric on \( \tilde{N}_u \mathcal{L} \). Consequently, we obtain that \( \mathcal{V} \) is symmetric on \( \tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L} \) for the scalar product defined by (5.49).

Finally, we will denote
\[ A_\delta \overset{\text{def}}{=} \left( \begin{array}{cc} \text{Id} & 0 \\ 0 & \delta \text{Id} \end{array} \right). \]

Then the right hand side of (5.27) reads
\[ \text{RHS} (5.27) = \frac{1}{\varepsilon^2} J \left( \frac{1}{2} \mathcal{D}_x A_\varepsilon \mathcal{M} \mathcal{D}_x U^m + \varepsilon^2 \mathcal{B}_{\Pi \perp} \mathcal{M} \mathcal{D}_x U^m - \mathcal{V} \mathcal{M} U^m \right) + \left( 0 \right) + \left( \frac{1}{2} \Pi^\top \left( S_{\top \top} \nabla_{\top \perp}^m \Phi, S_{\perp \perp} \nabla_{\perp \perp}^m n \right) \right) + \left( O_{H^1}(\mathcal{E}_s) \right). \]

Gathering the expressions for the left and right hand sides of (5.27), we proved the following proposition

**Proposition 5.5.** On \([0, T^\varepsilon]\), we have that \( U^m = (\nabla_{\top \perp}^m \Phi, \nabla_{\perp \perp}^m n)^t \) solves for \( 1 \leq |m| \leq s \) the system
\[
(D_t - \frac{c}{\varepsilon^2} \mathcal{D}_x) U^m = \frac{1}{\varepsilon^2} \mathcal{M}^{-1} J \left( \frac{1}{2} \mathcal{D}_x (A_\varepsilon \mathcal{M} \mathcal{D}_x U^m) + (\mathcal{B} + \varepsilon^2 \mathcal{B}_{\Pi \perp}) \mathcal{M} \mathcal{D}_x U^m - \mathcal{V} \mathcal{M} U^m \right) + \mathcal{M}^{-1} \left( \frac{1}{2} \Pi^\top \left( S_{\top \top} \nabla_{\top \perp}^m \Phi, S_{\perp \perp} \nabla_{\perp \perp}^m n \right) \right) + \left( O_{H^1}(\mathcal{E}_s) \right) \quad (5.52)
\]

where \( \mathcal{B}, \mathcal{B}_{\Pi \perp} \) are skew symmetric and \( \mathcal{V} \) is symmetric on \( \tilde{T}_u \mathcal{L} \times \tilde{N}_u \mathcal{L} \) for the scalar product defined by (5.49).

As in the proof of Proposition 4.4, we shall prove Proposition 5.4 by taking the scalar product of (5.52) with \( \mathcal{L} U^m \) for a well chosen operator \( \mathcal{L} \). Set
\[ \mathcal{L} U^m = -\mathcal{D}_x (\mathcal{M}^* A_\varepsilon \mathcal{M} \mathcal{D}_x U^m) + 2 \mathcal{M}^* \mathcal{V} \mathcal{M} U^m - 2 \mathcal{M}^* (\mathcal{B} + \varepsilon^2 \mathcal{B}_{\Pi \perp}) \mathcal{M} \mathcal{D}_x U^m \]

and the \( L^2 \) scalar product
\[ (U, V) \overset{\text{def}}{=} \int_R \langle U_1, V_1 \rangle_u + \langle U_2, V_2 \rangle_u \ dx. \]

We shall prove the following crucial result for the proof of Proposition 5.4

**Proposition 5.6.** For any \( m \) such that \( 1 \leq |m| \leq s \), we have that on \([0, T^\varepsilon]\),
\[
\frac{d}{dt} \left( \frac{1}{2} (\mathcal{M} \mathcal{D}_x U^m, A_\varepsilon \mathcal{M} \mathcal{D}_x U^m) + (\mathcal{V} \mathcal{M} U^m, \mathcal{M} U^m) \right)
- 2 \left( U_2^m, S_{\perp \perp} P^\perp B P^\top S_{\top \top} \nabla_{\top \perp}^m U_1^m \right) - 2 \varepsilon^2 \left( U_2^m, S_{\perp \perp} P^\perp \left( S_{\top \top} D \Phi \partial_x \phi, S_{\top \top} \nabla_{\top \perp}^m U_1^m \right) \right)
= O(\varepsilon^2). \]

**Proof.** Taking the scalar product of (5.52) with \( \mathcal{L} U^m \), we get
\[ (\text{LHS} (5.52), \mathcal{L} U^m) = (\text{RHS} (5.52), \mathcal{L} U^m). \quad (5.53) \]
Split the left hand side into
\begin{align*}
I &= \left( \left( \mathcal{D}_t - \frac{c}{\varepsilon^2} \mathcal{D}_x \right) U^m, -\mathcal{D}_x (\mathcal{M}^* A_{e2} \mathcal{M} \mathcal{D}_x U^m) \right), \\
II &= 2 \left( \left( \mathcal{D}_t - \frac{c}{\varepsilon^2} \mathcal{D}_x \right) U^m, \mathcal{M}^* \nabla \mathcal{M} U^m \right), \\
II_B &= -2 \left( \left( \mathcal{D}_t - \frac{c}{\varepsilon^2} \mathcal{D}_x \right) U^m, \mathcal{M}^* \mathcal{B} \mathcal{M} \mathcal{D}_x U^m \right), \\
II \mathcal{I} &= -2 \varepsilon^2 \left( \left( \mathcal{D}_t - \frac{c}{\varepsilon^2} \mathcal{D}_x \right) U^m, \mathcal{M}^* \mathcal{B}_\mathcal{I} \mathcal{M} \mathcal{D}_x U^m \right),
\end{align*}
and the right hand side into
\begin{align*}
III &= \frac{1}{2 \varepsilon^2} (\mathcal{M}^{-1} J \left( \mathcal{D}_x (A_{e2} \mathcal{M} \mathcal{D}_x U^m) + 2 (\mathcal{B} + \varepsilon^2 \mathcal{B}_1) \mathcal{M} \mathcal{D}_x U^m - 2 \nabla \mathcal{M} U^m \right), LU^m), \\
IV &= \left( \lambda^{-1} \left( 0 \right. \mathcal{I}^\top \left( S_{\mathcal{I}} \nabla^\top \nabla^\top \Phi^\varepsilon, S_{\mathcal{I}} \nabla^\top \nabla^\top \right), LU^m \right),
\end{align*}
so that (5.53) becomes
\begin{equation}
I + II + II_B + II \mathcal{I} = III + IV. \tag{5.54}
\end{equation}
Note that the above terms have very similar properties to the ones that were defined in the proof of Proposition 4.3. To handle I, we can rely on (5.9), (5.10) and (5.12) which yield \( \mathcal{D}_x \mathcal{M} = O(\varepsilon^2), \mathcal{D}_t \mathcal{M} = O(1) \) to obtain that
\begin{equation}
2I = \frac{d}{dt} (\mathcal{M} \mathcal{D}_x U^m, A_{e2} \mathcal{M} \mathcal{D}_x U^m) + O(\varepsilon^2). \tag{5.55}
\end{equation}
To handle II, we use that \( \mathcal{D}_t (\mathcal{M}^* \nabla \mathcal{M}) = O(1), \mathcal{D}_x (\mathcal{M}^* \nabla \mathcal{M}) = O(\varepsilon^2) \) and that \( \mathcal{M}^* \nabla \mathcal{M} \) is symmetric. This yields
\begin{equation}
II = \frac{d}{dt} (\nabla \mathcal{M} U^m, \mathcal{M} U^m). \tag{5.56}
\end{equation}
The term II_B requires some more care. We first note that since \( \mathcal{M}^* \mathcal{B} \mathcal{M} \) is skew-symmetric, we have that
\begin{equation*}
-II_B = 2 (\mathcal{D}_t U^m, \mathcal{M}^* \mathcal{B} \mathcal{M} \mathcal{D}_x U^m).
\end{equation*}
Next, by using the definition of \( \mathcal{M} \) and \( \mathcal{B} \), we get that
\begin{equation*}
-II_B = 2 \left( (\nabla_t U_1^m, S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t U_2^m) + (\nabla_t U_2^m, S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t U_1^m) \right) + O(\varepsilon^2)
\end{equation*}
and we shall manipulate the second term. We write
\begin{equation*}
\left( \nabla_t U_2^m, S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t U_1^m \right) = \frac{d}{dt} \left( U_2^m, S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t U_1^m \right) - \left( U_2^m, \nabla_t \left( S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t U_1^m \right) \right)
\end{equation*}
and observing that \( \nabla_t (S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}}) = O(\varepsilon^2) \mathcal{X} \) (using again (5.12) and the fact that \( \nabla B = 0 \) on \( \mathcal{L} \)), we obtain
\begin{equation*}
\left( U_2^m, \nabla_t \left( S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t U_1^m \right) \right) = \left( U_2^m, S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t \nabla_t U_1^m \right) + O(\varepsilon^2).
\end{equation*}
Note that to get this estimate, it is crucial that the \( x \) derivative is applied to \( U_1^m \) and not \( U_2^m \). Next, using (5.9), the skew symmetry of \( \mathcal{B} \) and similar arguments as above, we obtain
\begin{equation*}
\left( \nabla_t U_2^m, S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t U_1^m \right) = \frac{d}{dt} \left( U_2^m, S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t U_1^m \right) - \left( \nabla_t U_1^m, S_{\mathcal{I}} \nabla_t P^\perp B \mathcal{P}^\perp S_{\mathcal{I}} \nabla_t U_2^m \right) + O(\varepsilon^2).
This finally yields
\[
- II_B = 2 \frac{d}{dt} \left( U_2^m, S_{\perp \perp}^* D^1 BP^T S_{\perp \perp} \nabla_x U_1^m \right) + O(\mathcal{E}_s^2) + O(\mathcal{E}_s^2). \tag{5.57}
\]
The term \( II_B \) can be handled by similar arguments (note in particular that \( M^* B_{II} M \) is also skew symmetric) and is actually slightly easier due to the factor \( \varepsilon^2 \) in front. We obtain
\[
- II_B = 2 \varepsilon^2 \frac{d}{dt} \left( U_2^m, S_{\perp \perp}^* \Pi_1^1 \left( S_{\perp \perp} D \Phi \partial_x \phi, S_{\perp \perp} \nabla_x U_1^m \right) \right). \tag{5.58}
\]
Let us now turn to the terms from the right hand-side. By the skew symmetry of \( V \) and the choice of \( L \),
\[
III = \frac{1}{2 \varepsilon^2} \left( M^{-1} J \left( \mathcal{D}_x (A_{\varepsilon} M \mathcal{D}_x U^m) + 2(B + \varepsilon^2 B_{II} \cdot M \mathcal{D}_x U^m - 2V M U^m), -[\mathcal{D}_x, M^*] \cdot M A_{\varepsilon} \mathcal{D}_x U^m \right) \right). \tag{5.59}
\]
For the commutator in the right hand side, by using the rough estimate \( [\nabla_x^\perp, S_{\perp \perp}^*] = O(\varepsilon) \), \( [\nabla_x^\perp, S_{\perp \perp}] = O(\varepsilon^2) \), that comes from (5.12), we obtain that
\[
III = - \frac{1}{2 \varepsilon^2} \left( S_{\perp \perp}^* i S_{\perp \perp} \nabla_x^\perp U_2^m, [\nabla_x^\perp, S_{\perp \perp}] S_{\perp \perp} \nabla_x U_1^m \right) + O(\mathcal{E}_s^2).
\]
Since \( [\nabla_x^\perp, S_{\perp \perp}] = [\nabla_x^\perp, S_{\perp \perp}]^* \), we can use (5.29), (5.30) and (5.8) to get that
\[
III = - \frac{1}{2} \left( i S_{\perp \perp} \nabla_x^\perp U_2^m, \Pi_{\perp} \left( S_{\perp \perp} \nabla_x U_1^m, S_{\perp \perp} \nabla_x^\perp n \right) \right) + O(\mathcal{E}_s^2). \tag{5.60}
\]
Finally it remains to handle \( IV \). By counting powers of \( \varepsilon \), we can easily simplify it into
\[
IV = \frac{1}{2} \left( S_{\perp \perp}^* i \Pi_1^1 \left( S_{\perp \perp} \nabla_x^\perp U_1^m, -S_{\perp \perp} \nabla_x^\perp n \right), S_{\perp \perp}^* S_{\perp \perp} \nabla_x^\perp U_2^m \right) + O(\mathcal{E}_s^2)
\]
\[
= - \frac{1}{2} \left( i \Pi_{\perp} \left( S_{\perp \perp} \nabla_x U_1^m, S_{\perp \perp} \nabla_x^\perp n \right), S_{\perp \perp} \nabla_x^\perp U_2^m \right) + O(\mathcal{E}_s^2).
\]
Note that again there is a crucial cancellation when we sum up \( III \) and \( IV \) thanks to the skew symmetry of \( i \).

To conclude the proof of Proposition 5.6 it suffices to collect (5.55), (5.58), (5.57), (5.58), (5.59), (5.60).

**Proof of Proposition 5.4.** It suffices to integrate in time the estimate of Proposition 5.6 and to prove that the left hand side gives a control of \( \mathcal{E}_s^2 \). Using rough expansions, we can first write that
\[
\left( \frac{1}{2} \left( M \mathcal{D}_x U^m, A_{\varepsilon} M \mathcal{D}_x U^m \right) + (\mathcal{V} M U^m, M U^m) \right)
\]
\[
- 2 \left( U_2^m, S_{\perp \perp}^* D^1 BP^T S_{\perp \perp} \nabla_x U_1^m \right) - 2 \varepsilon^2 \left( U_2^m, S_{\perp \perp}^* \Pi_1^1 \left( S_{\perp \perp} D \Phi \partial_x \phi, S_{\perp \perp} \nabla_x U_1^m \right) \right)
\]
\[
= \int_{\mathbb{R}} \left( \frac{1}{2} |\nabla_x^\perp \nabla_x^\perp \Phi \varepsilon|^2 + \frac{1}{2} \varepsilon^2 |\nabla_x^\perp \nabla_x^\perp m|^2 + 2\lambda |\nabla_x^\perp m|^2 - 2 \nabla_x^\perp m \cdot B_\Phi \nabla_x^\perp \Phi \varepsilon \right) dx + \varepsilon O(\mathcal{E}_s(u, t)).
\]
Moreover, we can also easily find a slightly modified energy for the case \( m = 0 \). We can thus conclude as in the end of the proof of Proposition 4.4. \( \square \)

### 5.6. Reduction of (5.20).

The aim is to prove that the control of \( \mathcal{E}_s, s \geq 2 \) provided on \( [0, T^*] \) by Proposition 5.2 and Proposition 5.4 allows to reduce (5.20) to a hydrodynamical system set on \( T_{\Phi} \mathcal{L} \times N_{\Phi} \mathcal{L} \) which is very similar at leading order to (1.9). As a preliminary, we shall establish that
Lemma 5.7. The following expansions hold on $[0,T^c]$ for the following tensors acting from $T_\Phi L$ or $N_\Phi L$ to $T_\Phi L$ or $N_\Phi L$, 
\begin{align*}
(S^{-1}_{TT}(iB)_{TT})_{(\Phi,\varepsilon^2n)} &= (iB)_\Phi + O(\varepsilon^4), \\
(S^{-1}_{LL}(iB)_{LL})_{(\Phi,\varepsilon^2n)} &= (iB)_\Phi + O(\varepsilon^4), \\
(S^{-1}_{TT}iS_{LL})_{(\Phi,\varepsilon^2n)} &= (S_0^{-1}i)_\Phi + O(\varepsilon^4), \\
(S^{-1}_{LL}iS_{TT})_{(\Phi,\varepsilon^2n)} &= (iS_0)_\Phi + O(\varepsilon^4), \\
(S^{-1}_{LL}(\nabla^T_{(Y,N)}S_{TT})_{(\Phi,\varepsilon^2n)}) &= i\Pi_{\Phi}^T(\cdot, N) + O(\varepsilon^2), \quad \forall \ Y \in T_\Phi L, \ N \in N_\Phi L, \\
(S^{*}_{LL}(S_{LL})^{-1})_{(\Phi,\varepsilon^2n)} &= (S_0^{-1}i)_\Phi + O(\varepsilon^4)
\end{align*}

(recall that $S_0$ is defined by $S_0 = Id + \varepsilon^2 \Pi^T(\cdot, n)$ on $TL$).

Proof. Let us prove the first expansion. Fix $X \in T_\Phi L, \ n \in N_\Phi L$ and define 

\[ f(s) = (S^{-1}_{TT}(iB)_{TT})_{(\Phi,sn)} X = \left( S^{-1}_{TT}P^T iBS_{TT} \right)_{(\Phi,sn)} (X, 0) + O(s^2)X \]

by definition of $(iB)_{TT}$. Then, we have $f(0) = (iB)_\Phi X$. Moreover, by using that $\nabla i = 0, \nabla B = 0$ on $L$, and that $(\nabla_n S_{TT})_{\Phi} = \Pi^T(\cdot, n)$ thanks to (5.12) and (5.5), we obtain that 

\[ f'(0) = -\Pi^T_{\Phi}(iBX, n) + iB \Pi^T_{\Phi}(X, n) = 0 \]

where the final cancellation comes from Corollary 3.2. Thus by Taylor expansion $f(s) = f(0) + O(s^2)$.

The other expansions follow from the same arguments and (5.12).

\[ \square \]

We shall now simplify the expression of the system (5.20). Start with the first line: multiplying it by $S^{-1}_{TT}$ and dropping higher order terms in $\varepsilon$ gives 

\[ \frac{\partial \Phi}{\varepsilon} - \frac{1}{\varepsilon^2} (c + S^{-1}_{TT}(iB)_{TT}) \frac{\partial_x \Phi}{\varepsilon} = S^{-1}_{TT} i \left[ \frac{1}{2} \nabla_x^T (S_{LL} \nabla_x^T n) - \frac{2\lambda}{\varepsilon^2} (S_{LL})^{-1} n + \frac{1}{2} \Pi^{\perp} \left( \sum \left( \frac{\partial_x \Phi}{\varepsilon} \right), \sum \left( \frac{\partial_x \Phi}{\varepsilon} \right) \right) \right] + \varepsilon O_H^2(\varepsilon_s). \]

Next, note that 

\[ S^{-1}_{TT} i \nabla_x^T (S_{LL} \nabla_x^T n) = S^{-1}_{TT} i S_{LL} (\nabla_x^T)^2 n + (\nabla_x^{\perp} (\epsilon D\Phi \partial_x \phi, \epsilon^2 \nabla_x^T n) S_{LL}) \nabla_x^T n = S^{-1}_{TT} i S_{LL} (\nabla_x^T)^2 n + \varepsilon O_H^2(\varepsilon_s) \]

and hence by using Lemma 5.7 we get 

\[ S^{-1}_{TT} i \nabla_x^T (S_{LL} \nabla_x^T n) = i\phi (\nabla_x^T)^2 n + \varepsilon O_H^2(\varepsilon_s). \]

We can also use Lemma 5.7 to expand $\frac{1}{\varepsilon^2} S^{-1}_{TT}(iB)_{TT}$ and $\frac{2\lambda}{\varepsilon^2} S^{-1}_{TT} i S_{LL}^{-1} n$. Moreover, an immediate expansion gives 

\[ S^{-1}_{TT} i \Pi^{\perp} \left( \sum \left( \frac{\partial_x \Phi}{\varepsilon} \right), \sum \left( \frac{\partial_x \Phi}{\varepsilon} \right) \right) = i\phi \Pi_{\Phi}^T (D\Phi \partial_x \phi, D\Phi \partial_x \phi) + \varepsilon O_H^2(\varepsilon_s). \]

This yields the following equation on $T_\Phi L$ where all the involved tensors are evaluated on $L$ at $\Phi$: 

\[ D\Phi \partial_t \phi - \frac{1}{\varepsilon^2} (c + iB) D\Phi \partial_x \phi = \frac{1}{2} i (\nabla_x^T)^2 n + \frac{2}{\varepsilon^2} \lambda S_0^{-1} n + \frac{1}{2} \Pi^{\perp} (D\Phi \partial_x \phi, D\Phi \partial_x \phi) + \varepsilon O_H^2(\varepsilon_s). \]

By Corollary 3.2 we can also write it as 

\[ S_1 \partial_t \phi - \frac{1}{\varepsilon^2} (c + iB) S_1 \partial_x \phi = i \left[ \frac{1}{2} \left( (\nabla_x^T)^2 n + \frac{2}{\varepsilon^2} \lambda n + \frac{1}{2} \Pi^{\perp} (D\Phi \partial_x \phi, D\Phi \partial_x \phi) \right) + \varepsilon O_H^2(\varepsilon_s). \]
Note that up to $O(\varepsilon)$ term, we get back the same expression as in the first line of (1.9) (here we have simplified a little bit by assuming that $F_1 = R' = 0$).

We can proceed in the same way for the second line of (5.20), we multiply it by $S_{-1}^{-1}$ and we use Lemma 5.7 again. Let us give some details for the most involved term in the right hand side:

$$\frac{1}{\varepsilon^2} S_{-1}^{-1} i \nabla^T_x \left( S_{TT} \frac{\partial_x \Phi}{\varepsilon} \right) = \frac{1}{\varepsilon^2} S_{-1}^{-1} i S_{TT} \nabla^T_x \frac{\partial_x \Phi}{\varepsilon} + \frac{1}{\varepsilon^2} S_{-1}^{-1} i \left( \nabla^T_x (\varepsilon D \Phi \partial_x \varepsilon^2 \nabla^T_x n) S_{TT} \right) \frac{\partial_x \Phi}{\varepsilon},$$

and hence by using Lemma 5.7 again, we obtain

$$\frac{1}{\varepsilon^2} S_{-1}^{-1} i \nabla^T_x \left( S_{TT} \frac{\partial_x \Phi}{\varepsilon} \right) = \frac{1}{\varepsilon^2} i \left( S_{0} \frac{\partial_x \Phi}{\varepsilon} \right) + i \Pi^T \left( \nabla^T_x n, \frac{\partial_x \Phi}{\varepsilon} \right) + O_{H^2}(\mathcal{E}_s),$$

which can also be written, using the definition of $S_0$ and Proposition 3.1, under the form

$$\frac{1}{\varepsilon^2} S_{-1}^{-1} i \nabla^T_x \left( S_{TT} \frac{\partial_x \Phi}{\varepsilon} \right) = \frac{1}{\varepsilon^2} i \left( S_{0} \frac{\partial_x \Phi}{\varepsilon} \right) + i \Pi^T \left( \nabla^T_x n, \frac{\partial_x \Phi}{\varepsilon} \right) + O_{H^2}(\mathcal{E}_s).$$

We thus obtain for the second line of (5.20)

$$\nabla^T_x n = \frac{1}{\varepsilon^2} \left( c + iB \right) \nabla^T_x n = i \left( \frac{1}{\varepsilon^2} \nabla^T_x \left( S_{0} \frac{\partial_x \Phi}{\varepsilon} \right) + \frac{1}{2} \Pi^T \left( D \Phi \partial_x \varepsilon^2 \nabla^T_x n \right) \right) + O_{H^2}(\mathcal{E}_s) \quad (5.62)$$

where again all the tensors are evaluated at $\Phi(\varepsilon \phi) \in \mathcal{L}$.

Looking at (5.61), (5.62), we can use again (12) and Corollary 3.2 to obtain that

**Proposition 5.8.** On $[0, T^*]$, we have that the solution $u = \Psi(\Phi(\varepsilon \phi), \varepsilon^2 n)$ of (5.1) verifies the system

$$\begin{cases}
S_1 \partial_t \phi = \frac{1}{\varepsilon^2} \left[ -2 \lambda n - i(c + iB) S_1 \partial_x \phi \\
\quad + \varepsilon^2 \left( \frac{1}{2} \Pi^T \left( D \Phi \partial_x \phi, D \Phi \partial_x \phi \right) + \frac{1}{2} \varepsilon^2 (\nabla^T_x \phi)^2 n \right) \right] + O_{H^2}(\mathcal{E}_s) \\
\nabla^T_x n = \frac{1}{\varepsilon^2} \left[ \frac{1}{2} \nabla^T_x \left( S_1 \partial_x \phi \right) - i(c + iB) \nabla^T_x n + \varepsilon^2 \left( \frac{1}{2} \Pi^T \left( D \Phi \partial_x \phi, \nabla^T_x n \right) \right) \right] + O_{H^2}(\mathcal{E}_s)
\end{cases} \quad (5.63)$$

where all the tensors are evaluated at $\Phi \in \mathcal{L}$.

Note that up to the $O(\varepsilon)$ remainders, this is a system on $T_0 \mathcal{L} \times N_\Phi \mathcal{L}$!

5.7. **Proof of Theorem 1.1 in the case of a general Kähler manifold $\mathcal{M}$.** The local well-posedness of smooth solutions for (5.1) is classical (see for example [22], [13]) and we can proceed as in section 4.5. We define $T^*$ in the same way and use Propositions 5.2, 5.4 for a bootstrap argument. The only difficulty as before is to estimate $\|\varepsilon \phi\|_{L^\infty}$, but this can be done as previously by using the system (5.63). Note that here $i$ and $B$ are not necessarily constant tensors but since $\nabla \phi = \nabla B = 0$ on $\mathcal{L}$, we can indeed proceed in the same way when we apply $\nabla^T$ to the first equation of (5.63).

5.8. **Proof of Theorem 1.2 in the case of a general Kähler manifold $\mathcal{M}$.** Again, by using the system (5.63), we can proceed exactly as in section 4.6.

6. **Remarks on the limit KdV system**

The Cauchy problem for our KdV system on $T_0 \mathcal{L}$ reads

$$\begin{cases}
2c \partial_t A = \frac{1}{2} \partial_x A + \left( \frac{3}{2} - \frac{2c}{x} \right) i B(iB) \partial_x A, A(t=0) = A_0.
\end{cases} \quad (6.1)$$
Rescaling coordinates appropriately, and taking into account of the symmetry properties of \( i_0 \Pi^L_0 \) and \( F_1(0) \) (see [11]), this can be written under the general form
\[
\begin{align*}
\partial_t u - \partial_{xxx} u + \partial_x Q(u, u) &= 0 \\
u(t = 0) &= u_0,
\end{align*}
\] (6.2)
where the unknown \( u \) is valued in \( \mathbb{R}^d \), and \( Q \) is a bilinear tensor \( \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) such that
\[
(u, v, w) \mapsto Q(u, v) \cdot w \text{ is symmetric in } (u, v, w).
\] (6.3)
Indeed, \( (u, v, w) \mapsto i_0 B_0 i_0 \Pi^L_0 (u, v) \cdot w = B_0 \Pi^L_0 (u, v) \cdot w = -i_0 \Pi^L_0 (u, v) \cdot i_0 B_0 w \) is symmetric in all its arguments thanks to Proposition [3.1] and Corollary [3.2].

6.1. Hyperbolic structure. Since the matrix \( Q(u, \cdot) \) is symmetric for any \( u \),
\[
\partial_t u + \partial_x Q(u, u) = 0
\] (6.4)
is a symmetric hyperbolic system. From the classical theory, we therefore get

**Theorem 6.1.** For \( s > 3/2 \), \( A_0 \in H^s(\mathbb{R}) \), there exists \( T > 0 \) such that for every \( \delta \in \mathbb{R} \) there is a unique solution of
\[
\partial_t u - \delta \partial_{xxx} u + \partial_x Q(u, u) = 0,
\] in \( C([0, T], H^s) \cap C^1([0, T], H^{s-1}) \).

Notice that this result does not use any dispersive properties.

The system [6.4] is a system of conservation laws with flux \( f \) given by
\[
f(u) = Q(u, u).
\]
Let us restrict to the region where it is strictly hyperbolic and consider \( \lambda(u) \) an eigenvalue of \( Df(u) \), and \( r(u) \) an associated unitary eigenvector. We have
\[
Df(u) \cdot r(u) = \lambda(u) r(u), \quad \forall u \in \mathcal{U}.
\]
Let us differentiate this identity in the direction \( r(u) \). We find
\[
D^2 f(u)(r, r) + Df(u) \cdot (Dr(u) \cdot r(u)) = D\lambda(u) \cdot r(u) r(u) + \lambda(u) Dr(u) \cdot r(u).
\]
Taking the scalar product of this identity with \( r(u) \), and using that \( (Dr(u) \cdot r(u)) \cdot r(u) = 0 \) (since \( r(u) \) is unitary) and that \( D^2 f = 2Q \), we obtain
\[
2Q(r, r) \cdot r + (D f \cdot (Dr \cdot r)) \cdot r = D\lambda \cdot r.
\]
By symmetry of \( Df \),
\[
(D f \cdot (Dr \cdot r)) \cdot r = (Dr \cdot r) \cdot (D f \cdot r) = \lambda(Dr \cdot r) \cdot r = 0,
\]
therefore, we finally obtain that
\[
D\lambda \cdot r = 2Q(r, r) \cdot r.
\]
Consequently the characteristic field \( \lambda \) is genuinely nonlinear in an open set \( \mathcal{U} \) if and only if
\[
Q(r(u), r(u)) \cdot r(u) \neq 0, \quad \forall u \in \mathcal{U}.
\]
In particular, assuming that in an open set \( \mathcal{U} \), the system is strictly hyperbolic and such that for any eigenvector \( r \), \( Q(r, r) \cdot r \neq 0 \), the result of John [21], for example, gives that singularities occur in finite time.
6.2. Hamiltonian structure. The equation \(6.2\) derives formally from the Hamiltonian
\[
H(u) = \int_{\mathbb{R}} \left[ \frac{1}{2} |\partial_x u|^2 + \frac{1}{3} Q(u, u) \cdot u \right] \, dx
\]
given the symplectic form
\[
\omega(f, g) = \int_{\mathbb{R}} \partial_x^{-1} fg \, dx.
\]
Other conserved quantities are the mass (which is related via the Noether theorem to the invariance by translation of \(6.2\))
\[
M(u) = \int_{\mathbb{R}} |u|^2 \, dx.
\]
and the “momentum”
\[
P(u) = \int_{\mathbb{R}} u \, dx.
\]
The class of Hamiltonians given by (6.5) is equal to the class of Hamiltonians of the type
\[
\int \left[ \frac{1}{2} |\partial_x u|^2 + P(u) \right] \, dx,
\]
where \(P\) is a trilinear form.

A more general class consists of Hamiltonians of the type \(\int [Q(\partial_x u) + P(u)] \, dx\), with \(Q\) a bilinear form, and \(P\) a trilinear form. In the case \(d = 2\), this class of Hamiltonians gives the Gear-Grimshaw equations, which were derived in the context of water waves [15]. A mathematical investigation of their properties was conducted in [9].

6.3. Local and global well-posedness. Let us first mention the central result in [9]. As already mentioned, this paper focuses on Hamiltonians of the type \(\int [Q(\partial_x u) + P(u)] \, dx\), with \(Q\) a bilinear form, and \(P\) a trilinear form. Relying on methods introduced in [23], global well-posedness is established for data in \(H^1\) when the bilinear form \(Q\) is coercive (which is automatically the case for \(H\) as in (6.5)).

Focusing now on the equation \((6.2)\), we observe that the linearized problem is simply \(\partial_t u - \partial_{xxx} u = 0\), therefore only one dispersion relation is present in the problem, namely \(\xi^3\). It is then clear that the argument in [24] applies, giving the following theorem (we refer to [24] for a definition of the \(X^{s,b}\) space).

**Theorem 6.2.** Let \(s \in (-\frac{3}{4}, 0]\) and \(b \in (\frac{1}{2}, 1)\). If \(u_0 \in H^s\), there exists \(T > 0\) and a unique solution of \((6.2)\) belonging to \(X^{s,b}(0, T)\) and \(C([0, T], H^s)\).

Combining this theorem with the conservation of the \(L^2\) norm, one obtains global well-posedness for \(L^2\) data.

6.4. Solitary waves. In the case of scalar KdV \((d = 1, Q(u, u) = u^2)\), solitary waves are of the form \(cq(\sqrt{c}(x + ct))\), where \(c > 0\), and \(q\) is real-valued and solves the ODE
\[
q' - q''' + (q^2)' = 0.
\]
It is known since the original paper of Korteweg and De Vries [26] that periodic wave solutions are given by Jacobi elliptic functions, while a finite energy solitary wave on \(\mathbb{R}\) is given by
\[
q = -3 \text{sech}^2.
\]
For the equation \((6.2)\) in the general case, solitary waves are also of the type \(cQ(\sqrt{c}(x + ct))\), where \(c > 0\), and \(Q\) is valued in \(\mathbb{R}^d\) and solves the ODE
\[
Q' - Q''' + Q(Q, Q)' = 0.
\]
The following lemma allows to reduce partially \((6.7)\) to \((6.6)\).
Lemma 6.3. For $Q$ non zero satisfying (6.3), there exists $z \in \mathbb{R}^d$, $z \neq 0$, such that $Q(z, z) = z$.

Proof. A solution of $Q(z, z) = z$ is a critical point of the functional $K(z) \overset{\text{def}}{=} \frac{1}{3} Q(z, z) \cdot z - \frac{1}{2} |z|^2$. The existence of a nonzero critical point can be deduced from the (finite dimensional) mountain pass lemma: it suffices to observe first that $K(0) = 0$; that there exists $\delta, \varepsilon > 0$ such that $K(z) < -\delta$ if $|z| = \varepsilon$; and that $K(\lambda y) \xrightarrow{\lambda \to \infty} \infty$ if $y$ is such that $B(y, y) \cdot y > 0$.

It remains to check the (finite dimensional) Palais-Smale condition: suppose that $z^n$ is a sequence such that $K(z^n)$ is bounded and $K'(z^n) = Q(z^n, z^n) - z^n \to 0$. But then

$$|3K(z^n) - K'(z^n) \cdot z^n| \lesssim 1 + o(z^n) \quad \text{and} \quad 3K(z^n) - K'(z^n) \cdot z^n = -\frac{1}{2} |z^n|^2.$$

This implies that $(z^n)$ is bounded, hence the desired conclusion. \qed

This lemma implies that if $Q = qz$, with $z$ given by the previous lemma, and $q \in \mathbb{R}$, $Q$ solves (6.7) if and only if $q$ solves (6.8). This results in the following proposition.

Proposition 6.4. If $z$ is given by the previous lemma, and $q = -3 \text{sech}^2$, then $cq(\sqrt{c}(x + ct))z$ is a solution of (6.2) for any $c > 0$.

An interesting question is whether there are other stationary waves than the above; one might even think that the set of traveling waves of finite energy is of dimension $d + 1$, since the stable and unstable manifolds of (6.7) at $(Q, Q') = (0, 0)$ have dimension $d$.  

6.5. Miura transform and integrability. The classical Miura transform $u = \partial_x v + \frac{1}{6} v^2$ turns a solution of the mKdV equation $\partial_t v - \partial_x^3 v + \frac{1}{6} v^2 \partial_x v = 0$ into a solution of the KdV equation $\partial_t u - \partial_x^3 u + u \partial_x u = 0$.

We wish to generalize this transformation to the vector case (6.2). We therefore look for a symmetric quadratic form $B$ such that the transformation

$$u = \partial_x v + B(v, v)$$

maps solutions of the vector mKdV

$$\partial_t v - \partial_x^3 v + \mathcal{T}(v, v, \partial_x v)$$

to solutions of (6.2). We assume here that $\mathcal{T}$ is symmetrical in its arguments, but lifting this restriction does not lead to an improvement of the conditions on $Q$.

A short computation gives the equalities

$$\partial_t u - \partial_x^3 u = -2\mathcal{T}(u, \partial_x v, \partial_x v) - \mathcal{T}(v, v, \partial_x v) - 4B(v, \mathcal{T}(v, v, \partial_x v)) - 6B(\partial_x v, \partial_x v)$$

$$- \partial_x Q(u, u) = -2Q(\partial_x v, \partial_x v) - 4Q(\partial_x v, B(v, \partial_x v)) - 2Q(\partial_x v, B(v, v)) - 4Q(B(v, v), B(v, \partial_x v)).$$

Identifying similar terms on the right-hand sides leads to the necessary condition: for any $X, Y, Z \in \mathbb{R}^d$, $Q(X, Q(Y, Z)) = Q(Y, Q(X, Z))$, and further

$$B = \frac{1}{3} Q \quad \text{and} \quad \mathcal{T}(X, Y, Z) = \frac{2}{3} Q(X, Q(Y, Z)).$$

Summarizing, we proved the following lemma.

Lemma 6.5. If $Q$ satisfies $Q(X, Q(Y, Z)) = Q(Y, Q(X, Z))$ for all $X, Y, Z \in \mathbb{R}^d$, then the Miura transform

$$u = \partial_x v + \frac{1}{3} Q(v, v)$$

maps solutions of the equation

$$\partial_t v - \partial_x^3 v + \frac{2}{3} Q(v, Q(v, \partial_x v)) = 0$$

to solutions of (6.2).
Observe that the $v$ equation above derives from the Hamiltonian
\[ \int_{\mathbb{R}} \left[ \frac{1}{2} |\partial_x v|^2 + \frac{1}{6} \mathcal{Q}(v, \mathcal{Q}(v, v)) \cdot v \right] \, dx. \]

When the above lemma applies, this bi-Hamiltonian structure results in an infinite number of conserved quantities [28].

6.6. The case $d = 2$. To illustrate the above, we consider the first non-trivial case: $d = 2$. It is then convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$. The condition (6.3) is satisfied for
\[ \mathcal{Q}(x, y) = \alpha xy + \beta \overline{xy} + \overline{\alpha}(xy + x\overline{y}) \quad \text{with } \alpha, \beta \in \mathbb{C}. \]

The condition for the existence of a Miura transform in Lemma 6.5 becomes then $|\alpha| = |\beta|$.

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