All-Sky Convolution for Polarimetry Experiments

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We discuss all-sky convolution of the instrument beam with the sky signal in polarimetry experiments, such as the Planck mission which will map the temperature anisotropy and polarization of the cosmic microwave background (CMB). To account properly for stray light (from, e.g., the galaxy, sun, and planets) in the far side-lobes of such an experiment, it is necessary to perform the beam convolution over the full sky. We discuss this process in multipole space for an arbitrary beam response, fully including the effects of beam asymmetry and cross-polarization. The form of the convolution in multipole space is such that the Wandelt-Górski fast technique for all-sky convolution of scalar signals (e.g. temperature) can be applied with little modification. We further show that for the special case of a pure co-polarized, axisymmetric beam the effect of the convolution can be described by spin-weighted window functions. In the limits of a small angle beam and large Legendre multipoles, the spin-weight 2 window function for the linear polarization reduces to the usual scalar window function used in previous analyses of beam effects in CMB polarimetry experiments. While we focus on the example of polarimetry experiments in the context of CMB studies, we emphasise that the formalism we develop is applicable to anisotropic filtering of arbitrary tensor fields on the sphere.

95.75.-Hi, 98.70.Vc, 98.80.-k

\section{I. INTRODUCTION}

Over the past decade a number of increasingly sophisticated experiments have reported detections of the temperature anisotropy in the cosmic microwave background (CMB). Following the detection of degree-scale anisotropy by the COBE satellite \cite{1}, ground-based and balloon-borne experiments have pushed back the limits on resolution and sensitivity to provide estimates of the anisotropy power spectrum $C_l^T$ up to Legendre multipoles of $l \approx 700$ (see e.g. Ref. \cite{2} for a review of the situation pre-BOOMERANG \cite{4} and MAXIMA-1 \cite{5}). The anisotropy power spectrum encodes a wealth of cosmological information (e.g. Ref. \cite{6}) in a highly compressed form, making it a very convenient data product from which to determine cosmological parameters (see Ref. \cite{6} and references therein).

The combination of Thomson scattering and the non-zero temperature quadrupole, as the radiation begins to free stream through recombination, leads to the robust prediction that the CMB should be linearly polarized, with an r.m.s. level of a few percent of the temperature anisotropies \cite{7}. Detection of polarization would provide complementary information to that obtained from temperature measurements, e.g. the unique polarization signature of gravitational waves (and vector modes) provides the best hope of detecting their presence at last scattering \cite{8,9}. Currently, only upper limits exist on the degree of linear polarization (e.g. Refs. \cite{10,11}).

see also Ref. \cite{12} and references therein for a recent review) but detections should be made with the MAP \cite{12} and Planck \cite{13} satellites, several experiments from the ground \cite{14–17} and the flights of the MAXIPOL \cite{15} and enhanced BOOMERANG \cite{17} balloon experiments.

In an ideal linear polarimetry experiment, a given detector is configured to respond only to a single component of the electric field of the incident radiation, along incident directions contained within a small solid angle $\Delta \Omega$ (the beam size). For such an ideal experiment, the detector measures the flux $(I + Q) \Delta \Omega / 2$ where $I$ (total intensity) and $Q$ are the Stokes parameters of the incident radiation along the beam direction, and the polarization basis vectors have been chosen with the $x$ direction aligned with the polarimeter. In practice, such ideals are never achieved, and for precision polarimetry experiments, e.g. Planck, it is essential to take full account of several beam effects on the measured signal. The polarization on the sky must be convolved with the response pattern of the detector, which will not be perfectly axisymmetric. Furthermore, the system will generally have some cross-polar contamination (non-zero response to more than one polarization component). Often the beam response pattern has non-negligible far side-lobes which are highly polarized due to reflection and diffraction effects in the instrument, making a full-sky convolution necessary if the effects of stray light from bright regions (such as the sun, moon and galaxy in CMB experiments) are to be properly accounted for. The implica-
tions of a subset of these effects for the analysis pipeline of total power (unpolarized) experiments have been studied recently in Ref. [24].

In this paper we present a formalism that allows all such non-ideal effects to be taken account of exactly and efficiently in multipole space. The form of the convolution for polarized data in multipole space is very similar to that for unpolarized radiation. This fact allows the recently suggested algorithm of Wandelt and Gorski [27] to be used to compute rapidly the detector output for an arbitrary pointing direction and orientation of the detector. The formalism presented here should prove useful in simulation and modelling of precision polarimetry experiments, as well as in the actual analysis of experimental data. Within the multipole formalism, it is simple to invoke approximations, such as axisymmetry of the beam, where appropriate, to reproduce the approximate results used in previous analyses of beam effects in CMB polarimetry experiments [28,29].

The paper is arranged as follows. In Sec. II we discuss the derivation of the detector response from its far-field radiation pattern, and introduce the beam response tensor. The multipole expansion of the beam in spherical scalar and tensor harmonics is presented in Sec. III. The scanning of the detector on the sky is described by a (time-dependent) rotation of the beam from some standard configuration. This rotation is most conveniently handled in multipole space, and is described in Sec. IV, where an efficient algorithm for performing the rotation and convolution is also described. Section V deals with the case of an axisymmetric beam. It is shown that for a certain geometry of the polarized beam response, it is possible to describe the convolution of the beam with the incident linear polarization in terms of spin-weight 2 window functions, first introduced in Ref. [29]. We give the window function for Gaussian beams of arbitrary angular size, and show that in the small-angle limit we reproduce the results of Ng and Liu [29]. We summarise our discussion in Sec. VI. An appendix provides some details on the rotation properties of the tensor harmonics which are needed for Sec. IV.

II. FAR-FIELD RADIATION PATTERN AND THE BEAM RESPONSE

It is convenient to characterise the response of the detector and feed system by the far-field radiation pattern it would emit if used as a transmitter, rather than a receiver [28]. Assuming a (quasi-)monochromatic system, the electric field in the far field is of the form

\[ E \propto \frac{1}{r} \Re \{ \tilde{E} \exp[i(kr - \omega t)] \}, \]

where \( \tilde{E} \) is a complex, transverse vector function on the sphere. (We refer to geometric objects having no components outside the surface of the sphere as being transverse.) Here \( r \) is radial distance and \( \omega = ck \) is the (mean) angular frequency of the radiation, where \( k \gg 1/r \) is the wavenumber. The most general detector and feed system will produce a partially polarized signal in transmission, so that \( \tilde{E} \) will generally be a slowly varying function of time (compared to \( \omega \)). However, only the stationary statistical properties of \( \tilde{E} \) are important for determining the response of the system to incident radiation. With the system in some specified orientation, the power received \( dW_{\text{tot}} \) when illuminated by the sky along some direction \( \mathbf{e} \) is proportional to the intensity in that polarization mode which is the time reverse of Eq. (1). It follows that

\[ dW_{\text{tot}} \propto \langle |\mathbf{E} \cdot \tilde{\mathbf{E}}|^2 \rangle d\Omega, \]

where \( \mathbf{E} \) is the complex representative (or analytic signal) of the incident electric field propagating along \( -\mathbf{e} \), \( d\Omega \) is the element of solid angle, and angle brackets denote time averaging over the slow variations in \( \mathbf{E} \) and \( \tilde{\mathbf{E}} \). Writing the components of the fields \( \mathbf{E} \) and \( \tilde{\mathbf{E}} \) on the orthonormal basis vectors \( \mathbf{e}_\sigma \) and \( \mathbf{e}_\phi \) of a spherical polar coordinate system as e.g. \( \mathbf{E}_\theta \) and \( \mathbf{E}_\phi \), the contribution to \( W_{\text{tot}} \) can be written as

\[ \frac{dW_{\text{tot}}}{d\Omega} \propto \frac{1}{2} (I + Q \dot{Q} + U \dot{U} - V \dot{V}), \]

where \( \{ I, Q, U, V \} \) are the Stokes parameters of the incoming radiation on the \( \{ \mathbf{e}_\sigma, -\mathbf{e}_\phi \} \) basis:

\[ I = \langle |\mathbf{E}_\theta|^2 + |\mathbf{E}_\phi|^2 \rangle, \]
\[ Q = \langle |\mathbf{E}_\theta|^2 - |\mathbf{E}_\phi|^2 \rangle, \]
\[ U = -2\Re \langle \mathbf{E}_\theta \mathbf{E}_\phi^* \rangle, \]
\[ V = 2\Re \langle i \mathbf{E}_\theta \mathbf{E}_\phi^* \rangle, \]

and \( \{ \tilde{I}, \tilde{Q}, \tilde{U}, \tilde{V} \} \) are effective Stokes parameters for the beam:

\[ \tilde{I} = \langle |\tilde{\mathbf{E}}_\theta|^2 + |\tilde{\mathbf{E}}_\phi|^2 \rangle, \]
\[ \tilde{Q} = \langle |\tilde{\mathbf{E}}_\theta|^2 - |\tilde{\mathbf{E}}_\phi|^2 \rangle, \]
\[ \tilde{U} = -2\Re \langle \tilde{\mathbf{E}}_\theta \tilde{\mathbf{E}}_\phi^* \rangle, \]
\[ \tilde{V} = 2\Re \langle i \tilde{\mathbf{E}}_\theta \tilde{\mathbf{E}}_\phi^* \rangle. \]

Note that the effective Stokes parameters for the beam are defined on the same basis as the incoming radiation, which is responsible for the minus sign in front of the \( V \dot{V} \) term in Eq. (3).

The intensity \( I \) and circular polarization \( V \) are invariant under rotations of the polarization basis vectors, whilst \( Q \) and \( U \) transform like the components of a second-rank tensor. Introducing the linear polarization tensor for the incident radiation

\[ \mathcal{P}^{ab}(\mathbf{e}) = \frac{1}{2} [ Q (\mathbf{e} \otimes \mathbf{e}) - \mathbf{e} \otimes \mathbf{e}], \]

we can write the total power received in the basis-independent form
\[
W_{\text{tot}} \propto \frac{1}{2} \int (I \tilde{\mathcal{V}} - \mathcal{V} \tilde{\mathcal{V}} + 2 \mathcal{P}_{ab} \mathcal{B}^{ab}) \, d\Omega. \quad (13)
\]

Here, \( \mathcal{B}_{ab} \) is the (linear) beam response tensor:
\[
\mathcal{B}^{ab}(e) = \frac{1}{2} \left( \tilde{Q}(\sigma_{\theta} \otimes \sigma_{\theta} - \sigma_{\phi} \otimes \sigma_{\phi}) - \tilde{U}(\sigma_{\theta} \otimes \sigma_{\phi} + \sigma_{\phi} \otimes \sigma_{\theta}) \right) = \Re[(\tilde{\mathcal{E}} \otimes \tilde{\mathcal{E}}^{*})]^{TT}, \quad (14)
\]
where \( ^{TT} \) denotes the transverse, trace-free part. Our main task now is to derive the dependence of the total power received on the pointing direction and orientation of the detector.

### A. Co- and cross-polarized basis

The polar basis \( \{\sigma_{\theta}, \sigma_{\phi}\} \) is fixed relative to the sky and is singular at the north and south poles. For describing the beam, it is standard practice to use an alternative basis which is fixed relative to the detector, and has its only singularity in the opposite direction to the main beam. We define a set of Cartesian basis vectors \( \{\sigma'_{\theta}, \sigma'_{\phi}\} \) which are fixed relative to detector. It is convenient to take \( \sigma'_{\phi} \) to be along the (nominal) main beam, and \( \sigma'_{\theta} \) along the polarization direction on axis. Using this Cartesian frame we derive a set of polar basis vectors \( \{\sigma'_{\phi}, \sigma'_{\theta}\} \) on the sphere in the standard manner. The co- and cross-polar basis vectors are then derived by parallel-transporting \( \sigma'_{y} \) and \( \sigma'_{z} \) respectively from the north pole along great circles through the poles:
\[
\sigma'_{co} = \sin \phi' \sigma'_{\theta} + \cos \phi' \sigma'_{\phi}, \quad (16)
\]
\[
\sigma'_{cross} = \cos \phi' \sigma'_{\theta} - \sin \phi' \sigma'_{\phi}, \quad (17)
\]
where \( \theta' \) and \( \phi' \) are spherical polar coordinates. A well-defined linearly polarized receiver has \( |\tilde{\mathcal{E}}_{\text{cross}}| \ll |\tilde{\mathcal{E}}_{\text{co}}| \) along the main beam, where e.g. \( \tilde{\mathcal{E}}_{\text{co}} \) is the component of \( \tilde{\mathcal{E}} \) along \( \sigma'_{co} \). Cross-polar contamination arises from a non-zero \( |\tilde{\mathcal{E}}_{\text{cross}}| \). To rotate from the co- and cross-polarized basis to the spherical polar basis we have to rotate through \( \pi/2 - \phi' \) in a right-handed sense about the inward normal to the sphere. Transforming the Stokes parameters for the beam from the co- and cross-polar basis to the spherical polar basis, we have:
\[
\tilde{I} = |\tilde{\mathcal{E}}_{\text{co}}|^2 + |\tilde{\mathcal{E}}_{\text{cross}}|^2, \quad (18)
\]
\[
\tilde{Q} = -2 \mathcal{R} \langle \tilde{\mathcal{E}}_{\text{co}} \tilde{\mathcal{E}}_{\text{cross}}^{*} \rangle \cos 2\phi', \quad (19)
\]
\[
\tilde{U} = -2 \mathcal{R} \langle \tilde{\mathcal{E}}_{\text{co}} \tilde{\mathcal{E}}_{\text{cross}}^{*} \rangle \sin 2\phi', \quad (20)
\]
\[
\tilde{V} = -2 \mathcal{R} \langle \tilde{\mathcal{E}}_{\text{cross}} \tilde{\mathcal{E}}_{\text{cross}}^{*} \rangle, \quad (21)
\]
which reflect the spin-2 nature of the linear polarization. For simulation purposes, \( \tilde{\mathcal{E}} \) is usually determined with physical optics codes, the results being reported on the co- and cross-polar basis. For real experiments the Stokes parameters for the beam must be calibrated using sources with known surface brightness and polarization. Note that we do not assume that the beam response is fully polarized, so the formalism developed here can also be applied to total power experiments (only \( I \) non-zero). In practice, the optics and feeds will introduce some beam polarization in the side-lobes even for a nominal total power experiment. Although small, the role of such effects in total power experiments could be quantified with our formalism.

### III. MULTIPOLE EXPANSIONS

The dependence of the total power received on the direction and orientation of the telescope is most easily formulated in multipole space. We say that the detector is in its reference orientation when it is oriented so that the basis \( \{\sigma'_{x}, \sigma'_{y}, \sigma'_{z}\} \) coincides with the \( \{\sigma_{x}, \sigma_{y}, \sigma_{z}\} \) basis, which is fixed relative to the sky. We describe the beam via a set of constant multipole coefficients which are extracted on the sky basis when the detector is in its reference orientation. To describe an arbitrary orientation of the detector at some time along the scan, we can rotate the beam (which is most easily performed in multipole space) to obtain the rotated beam response which is convolved with the sky, as in Eq. \([13]\).

In the reference orientation, the total intensity and circular polarization parts of the beam response can be expanded in scalar spherical harmonics, e.g.
\[
\tilde{I}(e) = \sum_{lm} b_{lm}^{I} Y_{lm}(e), \quad (22)
\]
where the sum is over \( l \geq 0 \) and \( |m| \leq l \). The multipoles \( b_{lm}^{I} \) for the circular polarization are defined analogously. For the beam response tensor \( \mathcal{B}_{ab} \) we must expand in the transverse, trace-free tensor harmonics. Here we follow the coordinate-dependent approach of Ref. \([13]\) (although for some applications the coordinate-free approach of Ref. \([22]\) is more convenient):
\[
\mathcal{B}_{ab}(e) = \sum_{Plm} b_{lm}^{P} Y_{lm(ab)}, \quad (23)
\]
where the sum is over \( l \geq 2 \), \( |m| \leq l \), and the two types of transverse trace-free harmonics \( P = G \) (for Gradient, often called electric) or \( C \) (Curl, often called magnetic). All multipoles satisfy \( b_{lm}^{P} = (-1)^{m} b_{l(-m)}^{P} \) since the fields are real and we have adopted the Condon-Shortley phase for the spherical harmonics. The Stokes parameters \( I \) and \( V \) for the sky can be similarly expanded in multipoles \( a_{lm}^{I} \) and \( a_{lm}^{V} \), and the linear polarization in multipoles \( a_{lm}^{G} \) and \( a_{lm}^{C} \).

The tensor harmonics are derived from the scalar harmonics by covariant differentiation over the sphere \([14]\).
(see also the Appendix). Performing the differentiation gives

\[ Y^{G\,ab}_{(lm)} = \frac{1}{\sqrt{2}} \left( -2Y_{(lm)}\mathbf{m} \otimes \mathbf{m} + 2Y_{(lm)}\mathbf{m}^* \otimes \mathbf{m}^* \right), \]

\[ Y^{C\,ab}_{(lm)} = \frac{1}{i\sqrt{2}} \left( -2Y_{(lm)}\mathbf{m} \otimes \mathbf{m} - 2Y_{(lm)}\mathbf{m}^* \otimes \mathbf{m}^* \right), \]

where \( \mathbf{m} \equiv (\sigma_\phi + i\sigma_y)/\sqrt{2} \). The \( \pm 2 \) harmonics are the spin-weight 2 harmonics:

\[ \pm 2Y_{(lm)} = \frac{N_l}{\sqrt{2}} W_{(lm)}(\mp iX_{(lm)}), \]

where \( N_l \equiv |2(l-2)!/(l+2)!|^{1/2}, \) and

\[ W_{(lm)}(\theta, \phi) = \frac{\partial^2}{\partial \theta^2} Y_{(lm)}(\theta, \phi) - \cot \theta \frac{\partial}{\partial \theta} Y_{(lm)}(\theta, \phi) - \csc^2 \theta \frac{\partial^2}{\partial \phi^2} Y_{(lm)}(\theta, \phi), \]

\[ X_{(lm)}(\theta, \phi) = 2 \csc \theta \left( \frac{\partial^2}{\partial \theta \partial \phi} Y_{(lm)}(\theta, \phi) - \cot \theta \frac{\partial}{\partial \phi} Y_{(lm)}(\theta, \phi) \right). \]

Explicit expressions for \( W_{(lm)} \) and \( X_{(lm)} \) with the derivatives eliminated are given in Ref. [24]. Note that our convention for the spin-weight functions follows Ref. [29], which differs from Goldberg et al. [32] by the inclusion of the factor \((-1)^m\) in the definition of the spherical harmonics.

**A. Extracting the beam multipoles**

With the detector in the reference orientation, the spherical polar bases fixed relative to the sky and detector coincide. We can extract the beam multipoles from the effective Stokes parameters on this polar basis using the orthonormality of the scalar and tensor harmonics (e.g. Ref. [13]). For the linear polarization, we have

\[ b_{(lm)}^G(\pm iU) = \frac{1}{\sqrt{2}} \int \left( \tilde{Q} \mp i\tilde{U} \right) \pm 2Y_{(lm)}^* \ d\Omega, \]

which is the inverse of the expansion of \( \tilde{Q} \) and \( \tilde{U} \) in spin-weight 2 harmonics:

\[ \frac{1}{\sqrt{2}} \left( \tilde{Q} \pm i\tilde{U} \right) = \sum_{lm} (b_{(lm)}^G \pm ib_{(lm)}^C) \pm 2Y_{(lm)}. \]

(The sum is over \( l \geq 2 \) and \( |m| \leq l \).) Note that with our conventions for the polarization basis vectors, \( \tilde{Q} \pm i\tilde{U} \) is a spin-weight \( \mp 2 \) quantity, which differs from some authors (notably Ref. [14]).

![FIG. 1. The orientation of the detector is specified by the three Euler angles \{\phi, \theta, \psi\} which takes the \{\sigma_i\} frame \(i = x, y, z\), which is fixed relative to the sky, onto the \{\sigma_i'\} frame which is fixed relative to the detector. The nominal main beam of the detector is along \( \sigma_x'\), and the co-polar direction along the main beam is \( \sigma_y'\).]

Some care is needed in extracting the beam multipoles at the north and south poles since the Stokes parameters on the polar basis are ill-defined there. However, using Eqs. [19] and [20], it is simple to show that any well-defined polarization field must have \( \phi \) dependence going like \( Q \pm iU \propto \exp(\pm 2i\phi) \) on the polar basis at the north pole (\( \theta = 0 \)). This is consistent with Eq. [30] since the spin-weight \( \pm 2 \) harmonics satisfy

\[ \pm 2Y_{(lm)} = \delta_m \mp 2 \sqrt{\frac{2l+1}{4\pi}} e^{\mp 2i\phi} \]

at \( \theta = 0 \). With finitely sampled, simulated data for \( \tilde{E}_{\phi\phi} \) and \( \tilde{E}_{x\phi} \), the contribution to \( b_{(lm)}^P \) from samples on (or very near to) the north pole can be treated by approximating \( \pm 2Y_{(lm)} \) with Eq. [30], and absorbing the \( \exp(\pm 2i\phi) \) into \( \left( \tilde{Q} \mp i\tilde{U} \right) \exp(\pm 2i\phi) \) which is then well-defined by the data. Similar problems occur at the south pole, but since the beam has virtually no power there this is not problematic.

**IV. BEAM ROTATION AND CONVOLUTION**

The kinematics of the experiment can be specified by a scan strategy which describes the rotation necessary to take each detector from the reference orientation to its orientation at the specified time in the scan. For simplicity we consider only a single detector, but our approach
could easily be generalised to experiments with multiple detectors. The rotation is specified by its Euler angles \( \{ \phi, \theta, \psi \} \), such that first we rotate in a right-handed sense by \( \psi \) about \( \sigma_z \), then by \( \theta \) about \( \sigma_y \), and finally by \( \phi \) about \( \sigma_x \) again. We denote the rotation by \( D(\phi, \theta, \psi) \) so that the image of \( \mathbf{r} \) is \( D(\phi, \theta, \psi) \mathbf{r} = \mathbf{r}' \) for \( i = x, y, z \). For \( \psi = 0 \),

\[
\sigma_z' = \sin \theta (\cos \phi \sigma_x + \sin \phi \sigma_y) + \cos \theta \sigma_z \\
\sigma_y' = -\sin \phi \sigma_x + \cos \phi \sigma_y,
\]

(32)

For non-zero \( \psi, \sigma_z' \) and \( \sigma_y' \) are additionally rotated by \( \psi \) about \( \sigma_z' \) (see Fig. 1).

Under the rotation \( D(\phi, \theta, \psi) \) the beam Stokes parameters \( I \) and \( V \) rotate as scalar fields so that e.g.

\[
\tilde{I}(e) \to \tilde{I}[D(-\psi, -\theta, -\phi) e],
\]

(33)

where \( D(-\psi, -\theta, -\phi) \) is the inverse rotation. The beam response tensor \( B_{ab}(e) \) rotates as a rank-two tensor, so that

\[
B_{ab}(e) \to \Lambda_a^{c1} \Lambda_b^{c2} B_{c1c2}[D(-\psi, -\theta, -\phi) e],
\]

(34)

where \( \Lambda_a^{c} \) is the SO(3) rotation matrix representing \( D(\phi, \theta, \psi) \) (see Appendix). Since we are describing the beams in multipole space, we must consider the transformation properties of the scalar and tensor harmonics under the rotations given in Eqs. (33) and (34) respectively. The scalar spherical harmonics transform irreducibly under rotations as \( \tilde{Y}_{lm}(e) \to \tilde{Y}_{lm} \).

\[
Y_{lm}(e) \to \sum_{|m'| \leq l} D_{lm'}^l(\phi, \theta, \psi) Y_{lm'}(e),
\]

(35)

and, as we show in the Appendix, the same is true of the tensor harmonics:

\[
Y_{lmab}(e) \to \sum_{|m'| \leq l} D_{lm'm'}^l(\phi, \theta, \psi) Y_{lm'm'}(e).
\]

(36)

Here, the \( D_{lm'm'}^l(\phi, \theta, \psi) \) are Wigner’s D-matrices. With our conventions for the Euler angles, we have

\[
D_{lm'm'}^l(\phi, \theta, \psi) = e^{-im'\phi} d_{m'm'}^l(\theta) e^{-im'\psi},
\]

(37)

where

\[
d_{m'm}^l(\theta) = \sum_t (-1)^t \frac{[(l+m)![(l+m)![(l+n)![(l+n)!]]^{1/2}}{[(l-m)![(l-m)![(l-n)![(l-n)!]]!} \times [\cos(\theta/2)]^{2l+m-n-2t}[\sin(\theta/2)]^{n-m+2t}.
\]

(38)

The sum is over integers \( t \) such that the arguments of the factorials are non-negative.

Performing the integral over the sphere in Eq. (13) is now straightforward in multipole space using the orthogonality of the harmonics. The final result for the total power as a function of orientation of the detector is

\[
W_{tot}(\phi, \theta, \psi) \propto \sum_{lm'm'} \left[ \frac{1}{2} \left( a_{lm}^l b_{lm'}^l - a_{lm'}^l b_{lm}^l \right) + \sum_p a_{lm}^p b_{lm'}^p \right] D_{mm'}^l(\phi, \theta, \psi).
\]

(39)

The sum is over \( l \geq 0 \) with \( |m| \) and \( |m'| \leq l \), where we defined the linear polarization multipoles to be zero for \( l = 0 \) and 1. Our result for the dependence of the total power on orientation is quite general; we have made no assumptions about the beam profile and level of cross-polar contamination. Equation (39) is one of the main results of this paper. Note that the function \( W_{tot}(\phi, \theta, \psi) \) is expressed as a linear combination of the \( D \)-matrices, which form a complete set for expanding single-valued (square-integrable) functions on the three-sphere.

### A. Fast convolution algorithms

The right-hand side of Eq. (39) can be evaluated rapidly by making only minor modifications to the algorithm developed recently by Wandelt and Górski [27] for the case of an unpolarized detector. The key to the algorithm is to factor the rotation \( D(\phi, \theta, \psi) \) as follows:

\[
D(\phi, \theta, \psi) = D(\phi - \pi/2, -\pi/2, \theta) D(0, \pi/2, \psi + \pi/2),
\]

(40)

so we may write

\[
D_{m'm}^l(\phi, \theta, \psi) = \sum_{|M| \leq l} [D_{m'M}^l(\phi - \pi/2, -\pi/2, \theta) \\
\times D_{m'm'}^l(0, \pi/2, \psi + \pi/2)].
\]

(41)

The advantage of factoring the rotation in this way is that now the Euler angles only occur in complex exponentials, and we only need evaluate \( d_{m'm}(\theta) \) at \( \theta = \pi/2 \) (since \( d_{m'm}^l(-\theta) = d_{m'm}^l(\theta) \)). The full three-sphere of rotations can now be calculated with a three-dimensional Fast Fourier Transform. The \( d_{m'm}^l(\theta) \) can be computed with the accurate recursive method in Ref. [27] (which can be further enhanced by making use of the symmetries of the \( d_{m'm}^l \)). To perform the convolution to a resolution corresponding to multipoles \( \ell_{\text{max}} \), for all possible orientations of the system, requires \( O(\ell_{\text{max}}^4) \) operations. This should be compared with the \( O(\ell_{\text{max}}^3) \) operations required for a brute force computation in pixel space. For experiments such as the Planck mission, where \( \ell_{\text{max}} \) is of the order of a few thousand, the saving is considerable. For many experiments, the approximate azimuthal symmetry of the beam limits the sum over \( m' \) in Eq. (39) to \( |m'| \ll \ell_{\text{max}} \). Since \( W_{tot} \) is then a slowly varying function of \( \psi \), it is possible to sample the \( \psi \) variation much more sparsely than for \( \theta \) and \( \phi \), which effectively reduces the operations count to \( O(\ell_{\text{max}}^3) \).
V. AXISYMMETRIC BEAMS

In this section we consider the limiting form of the general result, Eq. (38), when the beam is approximated as being pure co-polarized (i.e. having no cross-polar contamination), and axisymmetric. Typically, these approximations hold well across the main beam for a well-defined linear polarimetry system. However, the approximations are not valid for describing the response of the system in the far side-lobes, where the complete expression, Eq. (39), should be used. To quantify the errors introduced in a given experiment by assuming an axisymmetric beam requires detailed simulation with the apparatus developed in the earlier sections of this paper. Techniques for propagating these errors to the band power estimates of the power spectrum in a total power experiment have been developed recently [26], but the detailed development of the full analysis pipeline for polarized tension to polarized experiments must await the detailed development of the full analysis pipeline for polarized data.

For a pure co-polar beam, $\tilde{E}_{\text{cross}}$ vanishes in Eqs. (33)–(34). If we further assume axisymmetry, then $\tilde{E}_{\text{co}}$ is a function of $\theta'$ alone. Writing $|\tilde{E}_{\text{co}}|^2 = B(\theta')$, we have $\tilde{I} = B(\theta')$, $\tilde{V} = 0$, and

$$\tilde{Q} \pm i\tilde{U} = -B(\theta') e^{\pm 2i\phi'}.$$  \hspace{1cm} (42)

It follows that $b^l_{(i0)}(\theta') = 0$ unless $|m| = 2$. Furthermore, using Eq. (23) with $m = 2$, we see that $b^C_{(i2)} = ib^G_{(i2)}$ and hence $b^C_{(-i2)} = -ib^G_{(-i2)}$. The beam response tensor for linear polarization, $E_{ab}$, is now fully specified by $b^G_{(i2)}$.

If we use Eqs. (23) and (28) to express the spin-weight harmonics in terms of Legendre functions, we find the following expression for $b^G_{(i2)}$:

$$b^G_{(i2)} = \frac{\pi N^2}{\sqrt{2}} \sqrt{\frac{2l+1}{4\pi}} \int_{-1}^{1} dB(x) \left\{ (l+2)(x-2)P''_l(x) + 2(l-1)x - \frac{1}{2}l(l-1)(1-x^2) - (l-4) \right\} P^0_m(x),$$ \hspace{1cm} (43)

where primes denote differentiation with respect to $x \equiv \cos \theta'$, and $P_l(x)$ is the $l$th Legendre polynomial. Note that $b^G_{(i2)}$ is real, so $b^G_{(-i2)}$ is imaginary. For the intensity multipoles we have $b^l_{(i0)}(\theta') = 0$ unless $m = 0$, with

$$b^l_{(i0)} = 2\pi \sqrt{\frac{2l+1}{4\pi}} \int_{-1}^{1} P_l(x) B(x) \, dx.$$ \hspace{1cm} (44)

Given the restrictions on the beam multipoles for an axisymmetric, co-polar beam, it is possible to simplify Eq. (39) for the total power received. Using the relation [33]

$$D^l_{m-s}(\phi, \theta, \psi) = (-1)^s \sqrt{\frac{4\pi}{2l+1}} Y^*_{(im)}(\theta, \phi)e^{is\psi},$$ \hspace{1cm} (45)

between the $D$-matrices and the spin-weight $s$ harmonics for integer $l$, $m$, and $s$, and the reality of $b^l_{(i0)}$ and $b^G_{(i2)}$, Eq. (39) can be written in the form

$$W_{\text{tot}}(\theta, \phi, \psi) \sim \frac{1}{2} [I_{\text{eff}} - Q_{\text{eff}} \cos 2\psi + U_{\text{eff}} \sin 2\psi].$$ \hspace{1cm} (46)

Here, $I_{\text{eff}}$ is the usual beam smoothed intensity:

$$I_{\text{eff}}(\theta, \phi) = \sum_{lm} W_{l} a^l_{(lm)} Y^*_{(lm)}(\theta, \phi),$$ \hspace{1cm} (47)

where the window function

$$W_{l} = \sqrt{\frac{4\pi}{2l+1}} b^l_{(i0)},$$ \hspace{1cm} (48)

Similarly, $Q_{\text{eff}}$ and $U_{\text{eff}}$ are beam smoothed Stokes parameters on the spherical-polar basis, given by

$$I_{\text{eff}}(\theta, \phi) = \sum_{lm} 2W_{l} a^G_{(lm)} \mp ia^{C}_{(lm)} \mp 2Y_{(lm)}.$$ \hspace{1cm} (49)

where, following Ref. [34], we have introduced the spin-weight 2 window function

$$2W_{l} = -2\sqrt{\frac{\pi}{2l+1}} b^G_{(i2)}.$$ \hspace{1cm} (50)

Note that Eq. (44) shows that the signal obtained by convolving the pure co-polar, axisymmetric beam with the sky is equivalent to the response of an idealised co-polar detector, with vanishing beam width, on the smoothed sky. This result does not depend on any assumptions about the angular size of the beam response; for the polarized contribution it is a consequence of the definition of the co-polar vector field, Eq. (16), which is the obvious generalization of a constant vector field to the surface of sphere. Equations (43)–(50) provide a complete description of the power received in polarimetry experiments with axisymmetric, co-polar beams.

A. Gaussian beams

It is often the case that the axisymmetric beam profile is approximately Gaussian:

$$B(\theta') = B \exp[-(1 - \cos \theta')/\sigma^2],$$ \hspace{1cm} (51)

where $\sigma$ is a measure of the beam width. Note that we follow Bond and Efstathiou [38] in taking the beam to be Gaussian in $2\sin(\theta'/2)$ rather than $\theta'$. The former allows us to derive simple analytic results valid for any $\sigma$. However, in most cases a Gaussian profile is only appropriate close to the beam axis, in which case the two definitions are almost indistinguishable. For the beam profile in Eq. (51), the window functions $W_{l}$ and $2W_{l}$ can be evaluated analytically:
and

\[ W_l = 4\pi Be^{-\alpha}i_l(\alpha), \]  \quad (52)

and

\[ 2W_l = 2\pi BN_l^2e^{-\alpha}\{2(-1)^l(l+2)(l-1)+6\alpha e^{-\alpha} \\
+[(l^2-4\alpha)(l-1)^2+12\alpha^2]i_l(\alpha) \\
+4\alpha(l^2+l+1-3\alpha)i_{l-1}(\alpha)\}, \quad (53)\]

where \(i_l(x)\) is a modified spherical Bessel function, and \(\alpha \equiv 1/\sigma^2\). For \(\sigma^2 \ll 1\), which is always the case for high resolution experiments, the window functions are very well approximated by their asymptotic expansions, which give

\[ W_l \approx 2\pi B\sigma^2 \exp[-l(l+1)\sigma^2/2], \]  \quad (54)

\[ 2W_l \approx 2\pi B\sigma^2 \exp[-l(l+1)-4\sigma^2/2], \]  \quad (55)

in full agreement with Ng and Liu [29]. For \(l \gg 1\), the polarized and unpolarized window functions in the small-scale limit are approximately equal, which can easily be verified by making a flat sky approximation.

VI. CONCLUSION

In this paper we have presented a multipole method for describing the response of an arbitrary detector and feed system. We fully include the effects of non-zero beam size, asymmetric beam patterns, and cross-polar contamination. Such inclusions are essential for the accurate modelling and interpretation of precision polarimetry data, such as that expected from the Planck mission. Working in multipole space, we derived a simple expression for the response of the system, when convolved with the sky, as a function of the three Euler angles needed to describe a general orientation of the system. Given the form of this expression, it is straightforward to modify the fast algorithm of Wandelt and Górski [27] to compute the system response for the entire three-sphere of orientations in \(O(l_{\text{max}})\) operations. Finally, we showed how, for the case of a pure co-polar, axisymmetric beam, the response can be described by spin-weighted window functions. This extended the results of Ref. [28] to arbitrary size beams, and we gave the exact form of the window functions for a beam with a Gaussian profile. Although our discussion has been in the context of CMB polarimetry experiments, the formalism introduced here should be useful in any applications involving anisotropic filtering of tensor fields on the sphere.

The techniques described in this paper have now been implemented in the simulation pipeline for polarized channels of the Planck mission.

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APPENDIX A: ROTATING THE TENSOR HARMONICS

In this appendix we establish the transformation properties of the transverse, trace-free tensor harmonics under active rotations. The action of the rotation \(D(\phi, \theta, \psi)\) can be represented by a rotation matrix \(\Lambda^a_b\) such that an arbitrary vector \(v^a\) rotates to \(v'^a = \Lambda^a_b v^b\). Orthogonality of the rotation implies \(\Lambda^a_b \Lambda_d^c = \delta^c_d\), so the inverse rotation is \(v^a = v'^b \Lambda_b^a\). Under an active rotation of the beam, the beam response tensor rotates to \(B'_{ab}(\epsilon^c)\) which is obtained by forward rotating the original tensor evaluated at the back rotated position:

\[ B'_{ab}(\epsilon^c) = \Lambda_a^{c_1} \Lambda_b^{c_2} B_{c_1c_2} (\epsilon^d \Lambda_d^c). \]  \quad (A1)

Under this transformation, the transverse nature of the tensor field \(B_{ab}(\epsilon^c)\) is preserved. We shall demand the same transformation properties for the tensor harmonics \(Y_{(lm)ab}\).

It is convenient to write the tensor harmonics in terms of covariant derivatives on the sphere of the scalar harmonics (e.g. Ref. [34]):

\[ Y_{(lm)ab}^G = N_l \left( \hat{\nabla}_a \hat{\nabla}_b Y_{(lm)} - \frac{1}{2} \hat{g}_{ab} \hat{\nabla}^2 Y_{(lm)} \right) \]  \quad (A2)

\[ Y_{(lm)ab}^C = \frac{N_l}{2} \left( \hat{\epsilon}^{c_1} \hat{\nabla}_a \hat{\nabla}_c Y_{(lm)} + \hat{\epsilon}^{c_2} \hat{\nabla}_b \hat{\nabla}_c Y_{(lm)} \right), \]  \quad (A3)

where \(\hat{\nabla}_a\) is the covariant derivative on the unit sphere, \(\hat{g}_{ab} = g_{ab} - \epsilon_a \epsilon_b\) is the (induced) metric (with \(g_{ab}\) the metric of Euclidean 3-space) and \(\hat{\epsilon}_a = \epsilon_{abc} \epsilon^c\) is the projected alternating tensor. The covariant derivative \(\hat{\nabla}_a\) is obtained from the 3-dimensional (flat) covariant derivative \(\nabla_a\) by total projection:

\[ \hat{\nabla}_a T_{b\ldots c} = \hat{g}_a^{d} \hat{g}_b^{e} \ldots \hat{g}_d^{f} \nabla_d T_{b\ldots f}, \]  \quad (A4)

for an arbitrary tensor \(T_{b\ldots c}\). Making use of the results

\[ (\nabla_a Y_{(lm)})(\epsilon^c) = \Lambda_a^b (\hat{\nabla}_b Y_{(lm)})(\epsilon^c \Lambda_d^c) \]  \quad (A5)

\[ (\nabla_a \hat{\nabla}_b Y'_{(lm)})(\epsilon^c) = \Lambda_a^d \Lambda_b^e (\nabla_d \hat{\nabla}_e Y'_{(lm)})(\epsilon^c \Lambda_d^c), \]  \quad (A6)

it is straightforward to prove that the rotated tensor harmonics are obtained by replacing the scalar harmonics by their rotated counterparts \(Y'_{(lm)}(\epsilon^c) = Y_{(lm)}(\epsilon^d \Lambda_d^c)\) in Eqs. (A2) and (A3). Since the \(l\)th scalar harmonics transform irreducibly under rotations as (e.g. Ref. [33])

\[ Y'_{(lm)}(\epsilon) = \sum_{|m'| \leq l} D_{m'm}^{m}(\phi, \theta, \psi) Y_{(lm')}(\epsilon), \]  \quad (A7)
the tensor harmonics inherit the same transformation law:

$$Y^P_{(lm)}(e) = \sum_{|m'| \leq l} D^P_{l'lm}(\phi, \theta, \psi) Y^P_{(lm')}(e). \quad (\text{A8})$$

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