Smooth Bosonization II: The Massive Case

P.H. Damgaard
CERN – Geneva

H.B. Nielsen and R. Sollacher
The Niels Bohr Institute
Blegdamsvej 17, DK-2100 Copenhagen, Denmark

August 10, 2018

Abstract

The (1+1)-dimensional bosonization relations for fermionic mass terms are derived by choosing a specific gauge in an enlarged gauge-invariant theory containing both fermionic and bosonic fields. The fermionic part of the generating functional subject to the gauge constraint can be cast into the form of a strongly coupled Schwinger model, which can be solved exactly. The resulting bosonic theory coupled to the scalar sources then exhibits directly the bosonic counterparts of the fermionic densities $\bar{\psi}\psi$ and $\bar{\psi}\gamma_5\psi$. 
In a previous paper [1], we have shown how the Abelian bosonization relations for the vector and axial vector currents in two dimensions [2, 3, 4] can be derived from a novel perspective, based on a new local “bosonization gauge symmetry” [5]. The idea is, briefly stated, that bosonic and fermionic formulations of the same theory should be understood as two different gauge fixings of a larger gauge-invariant action containing both bosonic and fermionic fields. Equivalence between certain purely bosonic and purely fermionic formulations should then amount to the usual gauge-fixing independence of S-matrix elements in gauge theories. Although this idea is appealing, it is a non-trivial matter to show in detail that it can be carried through. One needs two ingredients: (a) The local bosonization gauge symmetry and the “larger” gauge-invariant action, and, (b) a smooth interpolating gauge-fixing function which can bring one continuously from a “boson gauge” to a “fermion gauge” by tuning the gauge-fixing parameter.

In the usual path-integral approach to bosonization [6, 7, 8], it is clear that chiral fermion determinants play a fundamental rôle. The idea of [1] was therefore to start with a fermionic theory, and then promote chiral rotations to a local chiral gauge symmetry. Using the general scheme of [9], this can be done without changing the physical content of the theory through the introduction of a collective field $\theta(x)$. This collective field – in essence the chiral phase of the fermions – turns out to be the bosonized field in a suitable gauge. In [1] we showed how to introduce a particular gauge-fixing function $\Phi(x)$ depending on a gauge parameter $\Delta$ in such a way that one interpolates smoothly between purely fermionic formulations ($\Delta = 0$) and purely bosonic formulations ($\Delta = 1$). By comparing the couplings to external sources we recovered the usual bosonization relations for the currents. One of the most interesting aspects of this gauge-invariant approach to (1+1)-dimensional bosonization is that it demonstrates that these known equivalences are but two extreme cases of a continuum of equivalent theories that contain, in general, both fermions and bosons interactively. For this reason we dubbed our scheme smooth bosonization.

Although the current bosonization relations $ar{\psi}\gamma_\mu\psi \sim -\pi^{-1/2}\epsilon_{\mu\nu}\partial^\nu\theta$ and $\bar{\psi}\gamma_5\gamma_\mu\psi \sim \pi^{-1/2}\partial^\mu\theta$ can be established in this manner by a fairly direct route, the derivation of the much more subtle mass-term bosonization relations

$$
\bar{\psi}\psi \sim M \cos(2\sqrt{\pi}\theta) \quad \text{and} \quad \bar{\psi}\gamma_5\psi \sim M \sin(2\sqrt{\pi}\theta)
$$

(1)

was only outlined in [1]. The purpose of this letter is to fill in this gap.

Since we have already shown in [1] how to treat external vector and axial vector current sources, we shall only consider the generating functional

$$
\mathcal{Z}[M_\pm] = \int \mathcal{D}[\bar{\psi}, \psi] e^{i\int d^2x \mathcal{L}(x)}
$$

$$
\mathcal{L}(x) = \bar{\psi}(x)(i\partial + M_+(x)P_+ + M_-(x)P_-)\psi(x),
$$

(2)

which for constant sources $M_\pm$ just corresponds to having scalar and pseudoscalar mass terms. Here $P_\pm \equiv (1 \pm \gamma_5)$ are the usual chiral projectors.

Although global chiral symmetry is broken explicitly by the sources, we still have what at first sight looks like a trivial discrete chiral symmetry,

$$
\psi(x) \rightarrow e^{i\pi n\gamma_5}\psi(x) = (-1)^n \psi(x)
$$

$$
\bar{\psi}(x) \rightarrow \bar{\psi}(x)e^{i\pi n\gamma_5} = (-1)^n \bar{\psi}(x),
$$

(3)

where $n$ is an integer. When we introduce a collective field $\theta(x)$ via a chiral rotation

$$
\psi(x) = e^{i\theta(x)\gamma_5}\chi(x),
$$

(4)
this degree of freedom is therefore only defined globally modulo $n\pi$, i.e., the gauge fixing $\delta$-function must be periodic. If, in the notation of [1], we take the case $\Delta = 0$, this gauge-fixing function is $\Phi(x) = \theta(x)/\pi$, and hence

$$
\delta(\Phi(x)) = \delta(\Phi(x) - n).
$$

We can give a convenient representation of such a globally periodic functional $\delta$-function by a Fourier transform:

$$
\delta(\Phi(x)) = \sum_{k=-\infty}^{\infty} \int D[b]_k \exp \left[ \int d^2x b(x) \Phi(x) \right],
$$

where the functional integral is performed over all $b$'s satisfying the constraint

$$
\frac{1}{\pi} \int d^2x b(x) = k,
$$

where $k$ is an integer.

The interpolating gauge-fixing function reads in detail, for general $\Delta$ [1]:

$$
\Phi(x) = \Delta \int_{-\infty}^{x} d\xi \bar{\chi}(\xi) \gamma_\nu \gamma_5 \chi(\xi) + \frac{1 - \Delta}{\pi} \theta(x),
$$

where, for convenience, we have shifted the arbitrary lower limit of the line integral all the way to $-\infty$. The periodicity of the $\delta$-function constraint is not immediately obvious if we choose $\Delta \neq 0$, since under the discrete chiral transformation (3), the contribution from the $\theta$-term above now becomes $-(1 - \Delta)n$. However, the line integral also transforms under the discrete chiral rotation,

$$
\Delta \int_{-\infty}^{x} d\xi \bar{\chi} \gamma_\nu \chi \rightarrow \Delta \int_{-\infty}^{x} d\xi \bar{\chi} \gamma_\nu \chi - n \Delta,
$$

as can be seen by choosing an extremely slowly varying function, $\alpha(x)$, tending to the limit of $n\pi$ everywhere (except at $-\infty$), and then performing the corresponding regularized discrete chiral rotation

$$
\bar{\chi} \rightarrow \bar{\chi} e^{i\alpha(x)\gamma_5}, \quad \chi \rightarrow e^{i\alpha(x)\gamma_5} \chi.
$$

Under this transformation, the line integral transforms on account of the axial anomaly:

$$
\Delta \int_{-\infty}^{x} d\xi \bar{\chi} \gamma_\nu \chi \rightarrow \Delta \int_{-\infty}^{x} d\xi \bar{\chi} \gamma_\nu \chi - \frac{\Delta}{\pi} \alpha(x) \rightarrow \Delta \int_{-\infty}^{x} d\xi \bar{\chi} \gamma_\nu \chi - n \Delta.
$$

The two pieces therefore add up, leaving us again with the result (5).

Using the Fourier representation (6), we can rewrite the exponent as an interaction term coupling the fermions and the boson $\theta$ to a vector potential $B_\mu(x)$ defined by

$$
B_\mu(x) = e_{\mu\nu} \int d^2y b(y) \int_{-\infty}^{x} d\xi \delta^{(2)}(\xi - x).
$$

\[\text{In a manner somewhat reminiscent of the prescription used by Mandelstam [4] in relating boson and fermion operators. Here, the point } -\infty \text{ should simply be viewed as any point "outside" the space-time region we are considering. A more rigorous treatment could either keep } x_0, \text{ the lower limit of the line integral finite, and then modify the analysis by the inclusion of zero modes, } - \text{ or choose appropriate boundary conditions in a finite volume.}\]
The Pontryagin index density of this vector potential is
\[ \frac{1}{\pi} \epsilon^{\mu \nu} \partial_\mu B_\nu(x) = -\frac{1}{\pi} b(x); \] (13)
by the requirement (7), this implies
\[ -\frac{1}{\pi} \int d^2 x \epsilon^{\mu \nu} \partial_\mu B_\nu(x) = k \] (14)
where the integer \( k \) is the instanton number of the gauge potential \( B_\mu \). In general, this gauge potential can be decomposed as (see, e.g., ref. [10])
\[ B_\mu(x) = k C_\mu(x) + \epsilon_{\mu \nu} \partial_\nu \tilde{b}(x) + \partial_\mu \varphi(x), \] (15)
but we can always remove the third term by a local phase rotation of the fermion fields. This simply corresponds to choosing a gauge for \( B_\mu \). The field \( C_\mu(x) \) is a background which is only constrained to have the volume integral of \( \pi \epsilon^{\mu \nu} \partial_\nu C_\mu \) fixed, i.e. it carries topological number 1. We can choose \( C_\mu \) such that \( \partial_\mu C_\nu - \partial_\nu C_\mu \) is constant [11].

The relation of the measure defined by (6) to those in terms of \( k, \tilde{b}, \) respectively \( B_\mu \) is given by
\[ +\infty \sum_{k=-\infty}^{+\infty} \int \mathcal{D}[b] \left| \frac{\partial^2}{\partial^2} \right| \left| \det \left[ \frac{\partial^2}{\partial^2} \right] \right| = +\infty \sum_{k=-\infty}^{+\infty} \int \mathcal{D}'[\tilde{b}] \mathcal{D}'[\varphi] \left| \delta(\varphi) \right| \left| \det \left[ \frac{\partial}{\partial^2} \right] \right|, \] (16)
where the prime denotes that the zero-mode sector (with respect to \( \partial^2 \)) is excluded. The last expression is nothing but the measure for the vector field \( B_\mu \) in the decomposition (15) for the gauge \( \varphi = 0 \).

Let us now add a term
\[ \frac{1}{2g^2} b(x)^2 = -\frac{1}{4g^2} B_\mu(x) B^{\mu \nu}(x) \] (17)
to the action. The field strength is \( B^{\mu \nu}(x) = \partial_\mu B_\nu - \partial_\nu B_\mu \). Surprisingly, the dimensionful coupling constant \( g \) will turn out to play the rôle of an ultraviolet cut-off. The effect of adding such a term is to smear the \( \delta \)-function of the gauge-fixing into a Gaussian. The advantage is that, before integrating over \( \theta \), our action now contains one part which is the Schwinger model coupled to an external source. We can then directly make use of results which have been established for that model [10]. Later, we of course send the cut-off \( g \) to infinity, thereby reducing the gauge-fixing function to the original \( \delta \)-function constraint.

**The case \( \Delta = 1 \): Bosonization**

Putting everything together, we can now represent our gauge-fixed Lagrangian as
\[ \mathcal{L} = \bar{\chi} \left( i \partial \gamma_5 + B + M_+ e^{2i\theta} P_+ + M_- e^{-2i\theta} P_- \right) \chi \]
\[ -\frac{1}{4g^2} B_\mu B^{\mu \nu} + \frac{1}{2\pi} \partial_\mu \theta \partial^\nu \theta + M_+ (e^{2i\theta} - 1) \frac{\kappa_1(L)}{4\pi} \]
\[ + M_- (e^{-2i\theta} - 1) \frac{\kappa_1(L)}{4\pi} - \frac{1}{8\pi} \left( M_+^2 (e^{4i\theta} - 1) + M_-^2 (e^{-4i\theta} - 1) \right). \] (18)
The additional terms depending on $M_\pm$ result from the introduction of the field $\theta$ via a chiral transformation. The corresponding Jacobian of such a transformation, calculated with a Pauli-Villars regularization, contains also mass-dependent terms (see [12, 1]). With the constants $c_i, k_i$ obeying the relations
\begin{equation}
  c_1 k_1 + c_2 k_2 = 0, \quad c_1 + c_2 = 1,
\end{equation}
the cut-off–dependent constant $\kappa_1(\Lambda)$ is defined as
\begin{equation}
  \kappa_1(\Lambda) = \Lambda \sum_i c_i k_i \log k_i^2.
\end{equation}

For convenience, and in order to compare our results with the work of Dorn [13], we use a scheme where $c_1 = c_2 = \frac{1}{2}, k_1 = -k_2 = 1$. With this choice $\kappa_1(\Lambda) = 0$.

It should be noted that the terms $M_\pm^2 (e^{\pm 4i\theta} - 1)$ in eq. (18) arise as a consequence of this particular regularization scheme. As we shall see later, they correspond to contact terms of the $\delta$-function kind in certain Green functions. Such $\delta$-function contributions are not naively expected to occur in a regularized theory. However, Pauli-Villars regularization is defined here as a subtraction scheme of connected n-point functions (see [12]), not as a regularization of the free propagator itself. This leads to $\delta$-function contributions when calculating quantities involving more than one loop, like the two-loop contribution $M_\pm^2 (e^{\pm 4i\theta} - 1)$ to the Jacobian mentioned above. It will turn out that these contact terms proportional to $M_\pm^2$ appear in the bosonized version of the generating functional as well. One could try to avoid the occurrence of these terms by using a different regularization method; however, some very pleasant features would then have to be given up. In particular, this Pauli-Villars regularization guarantees that independent (commuting) functional derivatives of $Z[V_\mu, A_\mu, M_\pm]$ yield the same result, irrespective of the order in which they are taken, as was shown in ref. [12].

The Lagrangian (18) looks almost like a Schwinger model coupled to sources $M_\pm(x) e^{\pm 2i\theta(x)}$, except for the derivative coupling of $\theta$ to the fermions\(^2\). This coupling can be removed by the following set of transformations:

\begin{align}
  \chi &\rightarrow e^{i a \theta \gamma_5} \chi \\
  B_\mu &\rightarrow B_\mu + (1 + a) \epsilon_{\mu\nu} \partial^\nu \theta \\
  \theta &\rightarrow (1 + a)^{-1} \theta.
\end{align}

These transformations have to be carried out in the order shown. Here, $a$ is a non-local operator:
\begin{equation}
  a = -\partial^2 \left( \partial^2 + \frac{g^2}{\pi} \right)^{-1}.
\end{equation}

As a consequence the kinetic term for $\theta$ acquires an additional term, which leads to a Pauli-Villars regularized propagator with regulator mass $\frac{\sqrt{g}}{\sqrt{\pi}}$. The Lagrangian reads as follows:
\begin{align}
  \mathcal{L} &= \mathcal{L}_{Sch} + \mathcal{L}_{int} + \mathcal{L}_\theta \\
  \mathcal{L}_{Sch} &= \bar{\chi} (i \partial - B) \chi - \frac{1}{4g^2} B_{\mu\nu} B^{\mu\nu}.
\end{align}

\(^2\)And of course the additional part of the Lagrangian depending only on $\theta$. Since $\theta$ is a dynamical field, the full theory is not completely decoupled into separate sectors. But as we shall henceforth perform a perturbative expansion in the terms $\bar{\chi} M_\pm(x) \exp[\pm 2i\theta(x)]$, this has no consequences for the following analysis.
\[ \mathcal{L}_{\text{int}} = \bar{\chi}(M_+ e^{2i\theta} P_+ + M_- e^{-2i\theta} P_-)\chi \]
\[ \mathcal{L}_\theta = \frac{1}{2 g^2} \partial \mu \theta \left( \partial^2 + \frac{g^2}{\pi} \right) \partial \nu \theta - \frac{1}{8 \pi} \left( M_+^2 (e^{4i\theta} - 1) + M_-^2 (e^{-4i\theta} - 1) \right). \] (23)

Now we are in a position to derive an effective bosonic theory in terms of the field \( \theta \). Let us therefore treat \( \mathcal{L}_{\text{int}} \) as a perturbation of the Schwinger model part, \( i.e. \)
\[ \langle \exp \left( i \int d^2 x \mathcal{L}_{\text{int}}(x) \right) \rangle = \exp \left( i \int d^2 x \mathcal{L}(x) \right) + \frac{i^2}{2} \int d^2 x \int d^2 y \langle \mathcal{L}_{\text{int}}(x) \mathcal{L}_{\text{int}}(y) \rangle_c + \ldots \] (24)

The expectation values have to be taken with respect to \( \chi, \bar{\chi} \) and \( B_\mu \). The expression \( \langle \ldots \rangle_c \) means the connected part of the correlation function. The resulting effective bosonic action reads

\[ \mathcal{L}_{\text{eff}} = \mathcal{L}_\theta + \mathcal{L}_{M \pm} \]
\[ \mathcal{L}_{M \pm} = M_+(x)e^{2i\theta(x)} \langle \bar{\chi}(x) P_+ \chi(x) \rangle + (+ \leftrightarrow -) \]
\[ + \frac{i}{2} \int d^2 y M_+(x)e^{2i\theta(x)} \langle \bar{\chi}(x) P_+ \chi(x) \bar{\chi}(y) P_+ \chi(y) \rangle_c M_+(y)e^{2i\theta(y)} \]
\[ + \frac{i}{2} \int d^2 y M_+(x)e^{2i\theta(x)} \langle \bar{\chi}(x) P_+ \chi(x) \bar{\chi}(y) P_- \chi(y) \rangle_c M_-(y)e^{-2i\theta(y)} \]
\[ + ( + \leftrightarrow - ) + \ldots \] (25)

All constants independent of \( \theta \) or \( M_\pm \) have been dropped. They are unimportant for our discussion because they only contribute to the normalization of the functional integral.

We can now use known results for the Schwinger model, in particular the cluster decomposition property \cite{14, 10}. First of all, for the chiral condensates we obtain the well-known expression
\[ \langle \bar{\chi} P_+ \chi \rangle = \frac{g e^\gamma}{\sqrt{\pi} 4\pi}. \] (26)
where \( \gamma \) is Euler’s constant. For the 2-point functions of fermion bilinears we get (in Euclidean space, and including the contributions from the Pauli-Villars regulators for the fermions \cite{10, 13}):
\[ \langle \bar{\chi}(x) P_\pm \chi(x) \bar{\chi}(y) P_\pm \chi(y) \rangle_c = \frac{1}{4\pi} \delta(x - y) + \left( \frac{g e^\gamma}{\sqrt{\pi} 4\pi} \right)^2 \left( e^{-2K_0(\frac{\sqrt{\pi} |x - y|}{\sqrt{\pi}})} - 1 \right), \] (27)
and
\[ \langle \bar{\chi}(x) P_+ \chi(x) \bar{\chi}(y) P_\pm \chi(y) \rangle_c = \left( \frac{g e^\gamma}{\sqrt{\pi} 4\pi} \right)^2 \left( e^{2K_0(\frac{\sqrt{\pi} |x - y|}{\sqrt{\pi}})} \Theta(e^\gamma \Lambda |x - y| - 1) - 1 \right), \] (28)
valid up to terms of \( \mathcal{O}(\Lambda^{-1}) \). Here \( K_0 \) is a modified Bessel function of the second kind. Of course we take the limit where \( \Lambda \) is sent to infinity, and as a consequence the step-function \( \Theta(e^\gamma \Lambda |x - y| - 1) \) above simply equals unity for physical distances.\footnote{That is, distances larger than \( 1/\Lambda \).} The scheme-dependent contribution from \( \langle \bar{\chi}(x) P_\pm \chi(x) \bar{\chi}(y) P_\pm \chi(y) \rangle_c \) to the effective action of eq. (25) just cancels the \( (1/8\pi)M_\pm(x)^2 e^{\pm 4i\theta(x)} \)-terms of \( \mathcal{L}_\theta \). This cancellation is already one highly non-trivial step towards identifying the bosonization relations.
The cluster decomposition property (shown in [10] to hold even in this massless Schwinger model in which we are computing averages) states that the connected part of the correlation functions vanishes for distances $|x - y| \gg \frac{\sqrt{\pi}}{g}$. This can be seen quite easily from the above expressions, taking into account the properties of the Bessel function $K_0(z)$ at large argument. If we take the limit $g \to \infty$ in order to recover the old $\delta$-function gauge, all these connected correlation functions vanish for all non-vanishing distances. It may appear surprising that the gauge parameter $g$ plays such a crucial role in simplifying the analysis; after all, any physical answer is independent of this parameter. However, although we may recover the same final results also for finite $g$, the corresponding equivalent bosonic theory will in general be highly non-local, and not very illuminating. As we shall see below, it is only in the limit $g \to \infty$ that we recover the standard local bosonized action. So the most convenient bosonization gauge remains the $\delta$-function choice of [1], which corresponds precisely to $g \to \infty$.

A note should also be made here concerning the distance scales involved at this point. We started out with an ultraviolet cut-off $\Lambda$ from the Pauli-Villars regularization of the original fermion theory. Through the modified gauge-fixing function [the addition of a kinetic energy term for the gauge potential (17)], we introduced what turned out to be a Pauli-Villars regulator for the boson field $\theta(x)$. Finally, we also need an infrared cut-off $\mu$ for the computation of certain Green functions (see below). Although both ultraviolet cut-offs $\Lambda$ and $g$ are eventually taken to infinity, we are always performing the analysis in a certain distance regime set by these two scales. First of all, from the beginning we clearly restrict the whole analysis to distance scales $|x - y| \gg 1/\Lambda$. The cluster decomposition [10] sets in for distances $|x - y| > 1/g$. Taking the limit $g \to \infty$ guarantees that all higher connected correlation functions vanish at all physical distances. However, if we wish to retain a finite cut-off $g$, then all terms in the expansion contribute to the effective bosonic theory, suppressed only by powers of $g^{-1}$.

Finally, adding all the above ingredients and rescaling $\theta$ by $\sqrt{\pi}$, we arrive at a bosonized theory described by a Lagrangian

$$L_{\text{bos}} = \frac{\pi}{2 g^2} \partial_\mu \theta \left( \partial^2 + \frac{g^2}{\pi} \right) \partial^\mu \theta + M_+ \frac{g e^\gamma}{\sqrt{\pi} 4 \pi} e^{2i \sqrt{\pi} \theta} + M_- \frac{g e^\gamma}{\sqrt{\pi} 4 \pi} e^{-2i \sqrt{\pi} \theta} - \frac{1}{8 \pi} (M_+^2 + M_-^2),$$

(29)

where the limit $g \to \infty$ is to be taken. This coincides with the result of Dorn [13] if we identify $g/\sqrt{\pi}$ with his Pauli-Villars regulator mass $M_S$. Indeed, the kinetic term for $\theta$ is precisely the corresponding Pauli-Villars regularized kinetic energy. Taking the sources $M_\pm = m$ to be constant, this is the Sine-Gordon action with a Pauli-Villars mass $g/\sqrt{\pi}$ and one particular normal-ordering prescription in the operator formalism [3]. We hope that the present derivation of the bosonization relations (1) for fermionic mass terms has underlined some of the subtleties behind the statement that the massive Thirring model is to be viewed as equivalent to the 2D Sine-Gordon theory. But the path-integral manipulations of the present paper can also be viewed as a much simplified derivation of the classic result of Coleman [3].

---

4 It should again be emphasized that there are corrections to this simple effective action, down by powers of $g^{-1}$. The shown terms should really be viewed as the result of the limit $g \to \infty$. Everything is here expressed in terms of bare quantities.
Conclusion

We have shown how to regain the usual mass-term bosonization identities by means of the gauge $\Phi$ with $\Delta = 1$. The case $\Delta = 0$ of course trivially yields the purely fermionic formulation. As in previous work [3], the equivalent bosonic Lagrangian has been derived only through its perturbative expansion. However, in contrast with [3] and other similar approaches using path integral methods [6], we do not explicitly need an order-by-order comparison in the perturbative expansion. Using the established cluster decomposition property of the Schwinger model [which a priori would seem to be unrelated to the present treatment of the ungauged action (2)], we need only the first two orders of the expansion (24). All higher orders vanish. Thus, although we have not yet demonstrated a truly non-perturbative boson-fermion equivalence for these mass terms, we feel nevertheless that the present derivation represents a substantial improvement.

As for the restriction at intermediate steps to very definite distance scales, we see here an effect which is bound to occur if the same collective field technique is used to derive effective Lagrangians (partly bosonized, or not) in higher dimensions. The analysis of [1] was in that sense rather particular in that for fermionic couplings to just vector and axial vector currents in two dimensions, no true field theoretic regularization was required beyond the calculation of functional Jacobians. Here we have seen the more general machinery at work. If we insist on a finite cut-off $\Lambda$, the fermionic theory (2) can only be partly be bosonized in a local fashion, with rather unpleasant non-local corrections of $O(\Lambda^{-1})$. In addition, only in the limit $g \to \infty$ do we immediately achieve a local action in the bosonic representation.

It may be worthwhile to add a few comments on the computation of chiral condensates, and in general fermionic correlation functions, of the original fermionic theory (3). Even for finite $g$ the physical quantities should not depend on this parameter. Thus, if one calculates expectation values (or Green functions) in the effective $\theta$-theory, occurrences of $g$ should cancel with contributions from the $\theta$-integration. Indeed, this is what happens. Take as an example the chiral condensate $\langle \bar{\psi} P_\pm \psi \rangle$ in a free theory, given by a functional derivative with respect to $M_\pm$, which is subsequently set equal to zero. The integration over $\theta$ has to be done with respect to the kinetic term defined in $L_\theta$, even for finite $g$. This yields

$$\langle \bar{\psi} P_\pm \psi \rangle = \frac{g e^\gamma}{\sqrt{\pi} 4\pi} \left( e^{2i\theta} \right) = \lim_{x \to 0} \frac{g e^\gamma}{\sqrt{\pi} 4\pi} e^{-K_0(\mu|x|) + K_0\left(\frac{g}{\sqrt{\pi}}|x|\right)} = \frac{\mu e^\gamma}{4\pi},$$

(30)

where $\mu$ is an infrared cut-off mass. This means that in going from the formulation in terms of rotated fermions $\tilde{\chi}, \chi$ to the physical fermion fields $\tilde{\psi}$ and $\psi$, $g/\sqrt{\pi}$ is replaced by the infrared cut-off $\mu$, which is finally taken to zero. One then recovers the correct result for the chiral condensate of a free massless theory.

Acknowledgements

We thank A. Wirzba for discussions. H.B. Nielsen and, in parts, R. Sollacher acknowledge support by EEC grant CS1-D430-C. The work of R. Sollacher has been supported mainly by the Deutsche Forschungsgemeinschaft.

References

5Which, however, should be related through a suitable non-local field redefinition to the form (29).
[1] P.H. Damgaard, H.B. Nielsen and R. Sollacher, CERN preprint CERN–TH-6460/92 (1992).

[2] A. Casher, J. Kogut and L. Susskind, Phys. Rev. D10 (1974) 732.

[3] S. Coleman, Phys. Rev. D11 (1975) 2088.
   A. Luther and I. Peschel, Phys. Rev. B9 (1974) 2911.
   A. Luther and V. Emery, Phys. Rev. Lett. 33 (1974) 589.

[4] S. Mandelstam, Phys. Rev. D11 (1975) 3026.

[5] P.H. Damgaard, H.B. Nielsen and R. Sollacher, CERN preprint CERN–TH-6486/92 (1992), to appear in Proc. 1st German-Polish Symposium on Particles and Fields, (World Scientific Publ. Co.).

[6] K. Furuya, R.E. Gamboa Saravi and F.A. Schaposnik, Nucl. Phys. B208 (1982) 159.
   R.E. Gamboa Saravi, M.A. Muscietti, F.A. Schaposnik and J.E. Solomin, Ann. Phys. (NY) 157 (1984) 360.
   C.M. Naon, Phys. Rev. D31 (1985) 2035.
   L.C.L. Botelho, Phys. Rev. D33 (1986) 1195.

[7] A.V. Kulikov, Teor. Mat. Fiz. 54 (1983) 157.

[8] R. Sollacher and H. Hofmann, Z. Phys. A339 (1991) 1.

[9] J. Alfaro and P.H. Damgaard, Ann. Phys. (NY) 202 (1990) 398.
   For an earlier discussion, see also D. Förster, H.B. Nielsen and M. Ninomiya, Phys. Lett. B94 (1980) 135.

[10] C. Jayewardena, Helv. Phys. Acta 61 (1988) 636.

[11] I. Sachs and A. Wipf, ETH Zürich preprint ETH-TH/91-15.

[12] R.D. Ball, Phys. Rep. 182 (1989) 1.

[13] H. Dorn, Phys. Lett. B167 (1986) 86.

[14] N.K. Nielsen and B. Schroer, Nucl. Phys. B120 (1977) 62.
    K.D. Rothe and J. Swiega, Ann. Phys. (NY) 117 (1979) 382.