Hierarchy of multipartite nonlocality in the nonsignaling scenario

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We propose a hierarchy of Bell-type inequalities for arbitrary n-partite systems that identify the different degrees of nonlocality ranging from standard to genuine multipartite nonlocality. After introducing the definition of nonsignaling m-locality, we show that the observed joint probabilities in any nonsignaling m-local realistic models should satisfy the (m − 1)-th Bell-type inequality. When m = 2 the corresponding inequality reduces to the one shown in [Phys. Rev. Lett. 112, 140404 (2014)] whose violation indicates genuine multipartite nonlocality, and when m = n the corresponding inequality is just Hardy’s inequality whose violation indicates standard multipartite nonlocality. Furthermore, several examples are provided to demonstrate their hierarchy of multipartite nonlocality.

I. INTRODUCTION

In 1964, Bell proved that the predictions of quantum theory for some bipartite quantum states are incompatible with those of deterministic local hidden-variable models by the violation of Bell’s inequality, and therefore physical theory of local hidden variables cannot reproduce all of the predictions of quantum mechanics [1, 2]. These quantum states which cannot be described by local hidden-variable models are nonlocal. Subsequently, Clause-Horne-Shimony-Holt (CHSH) inequality was introduced for bipartite systems with two different measurement settings for each observer and two possible outcomes for each measurement [3]. Furthermore, the CHSH inequality was generalized to the Collins-Gisin-Linden-Mass-Popescu (CGLMP) inequality with each measurement having more than two possible outcomes [4]. Quantum nonlocality is widely used in quantum information tasks [2], such as making the secure quantum communication [5, 6], decreasing the communication complexity [7], randomness generation [8], and measurement-based quantum computation [9].

The nonlocality issue for bipartite system is simple, it is either nonlocal or local. However, the situation is dramatically changed for multipartite case, since the structure of multipartite nonlocality is far from a simple extension of the bipartite one. For the multipartite case, quantum nonlocality has much richer and more complex structure. Consider an n-partite system, there exist n − 1 kinds of hierarchical multipartite nonlocality. The first one is the standard (or weakest) multipartite nonlocality which is a natural generalization from Bell’s bipartite nonlocality. Many Bell-type inequalities have been proposed for the standard multipartite nonlocality, such as the Mermin-Ardelhalli-Belinskii-Klyshko (MABK) inequalities [10], the Werner-Wolf-Zukowski-Brukner (WWZB) inequality [11], Hardy’s inequality [12, 13]. The last kind of (or strongest) multipartite nonlocality is genuine multipartite nonlocality, which shows that the nonlocality is truly established among all the parties of the system. The detection of genuine multipartite nonlocality has attracted much interest recently. In 1987, Svetlichny first introduced the notion of genuine multipartite nonlocality, and derived a Bell-type inequality for tripartite systems (i.e. Svetlichny inequality) to test the genuine tripartite nonlocality [14]. Moreover, Seevinck and Svetlichny and Collins et al. independently generalized the Svetlichny inequality from tripartite systems to arbitrary n-partite systems [15, 16]. Recently, the Svetlichny inequality has been generalized to arbitrary n-partite higher-dimensional systems [17]. However, Svetlichny’s notion of genuine multipartite nonlocality allows correlations capable of signaling among parties [18], which would be inconsistent with an operational viewpoint [19]. Fortunately, multipartite nonlocality in the nonsignaling scenario was proposed [18, 20], since allowing signaling is incongruous with a physical perspective.

As introduced above, the standard and genuine multipartite nonlocality have been studied in many papers. However, Bell-type inequalities for multipartite nonlocality between the standard and genuine one have never been proposed. In this paper, we propose a hierarchy of Bell-type inequalities for arbitrary n-partite systems, which can identify the different degrees of nonlocality ranging from standard to genuine multipartite nonlocal-
ity. After introducing the definition of nonsignaling m-locality, we show that the observed joint probabilities in any nonsignaling m-local realistic models should satisfy the \((m - 1)\)-th Bell-type inequality. When \(m = 2\) the corresponding inequality reduces to the one shown in Ref. 22 whose violation indicates genuine multipartite nonlocality, and when \(m = n\) the corresponding inequality is just Hardy’s inequality whose violation indicates standard multipartite nonlocality. Furthermore, several examples are provided to demonstrate the multipartite nonlocality hierarchy.

II. NONSIGNALING \(m\)-LOCALITY

Consider a system composed of \(n\) spacelike separated subsystems that are labeled with the index set \(I = \{1, 2, \ldots, n\}\). The measurement setting and outcome of the \(k\)-th subsystem \((k \in I)\) are denoted by \(M_k\) and \(r_k\), respectively. \(P(r_I|M_I)\) is the joint probability distribution with \(r_I = (r_1, \ldots, r_n)\) and \(M_I = (M_1, \ldots, M_n)\), when all \(n\) parties use the measurement settings \(M_1, \ldots, M_n\) and obtain the results \(r_1, \ldots, r_n\).

On the one hand, in a standard local hidden variable model, the joint probability distribution \(P(r_I|M_I)\) assumes the following form:

\[
P(r_I|M_I) = \int q(\lambda) \prod_{k=1}^{n} P_k(r_k|M_k, \lambda) d\lambda,
\]

where \(\lambda\) is a shared local hidden variable, \(q(\lambda) \geq 0\) with \(\int q(\lambda)d\lambda = 1\), and \(P_k(r_k|M_k, \lambda)\) is the probability of the \(k\)-th observer measuring observable \(M_k\) with outcome \(r_k\) for a given local hidden variable \(\lambda\) distributed according to \(q(\lambda)\). Violation of Eq. (1) indicates the standard (or weakest) multipartite nonlocality.

On the other hand, Svetlichny introduced the genuine multipartite nonlocality, which indicates that the joint probability distribution cannot be written as

\[
P(r_I|M_I) = \sum_{\alpha} \int q_\alpha(\lambda) P_\alpha(r_\alpha|M_\alpha, \lambda) P_{\bar{\alpha}}(r_{\bar{\alpha}}|M_{\bar{\alpha}}, \lambda) d\lambda,
\]

where \(\alpha \neq \emptyset\), \(\alpha \subset I\), \(\bar{\alpha} = I \setminus \alpha\), and \(|\alpha| \leq |\bar{\alpha}|\). Here, the set \(I\) has been divided into arbitrary two nonempty and disjoint subsets \(\alpha\) and \(\bar{\alpha}\), and \(P_\alpha(r_\alpha|M_\alpha, \lambda)\) and \(P_{\bar{\alpha}}(r_{\bar{\alpha}}|M_{\bar{\alpha}}, \lambda)\) denote the joint probability of all observers \(k \in \alpha\) measuring observable \(M_k\) with outcome \(r_k\) for a given local hidden variable \(\lambda\) distributed according to \(q_\alpha(\lambda)\). In the nonsignaling scenario, genuine multipartite nonlocality needs to obey the nonsignaling condition, i.e., \(P_\alpha(r_\alpha|M_\alpha, \lambda)\) and \(P_{\bar{\alpha}}(r_{\bar{\alpha}}|M_{\bar{\alpha}}, \lambda)\) in Eq. (2) satisfying

\[
\sum_{r_k} P_\beta(r_\beta|k, M_{\beta\setminus k}, M_k, \lambda) = \sum_{r_k} P_\beta(r_\beta|k, M_{\beta\setminus k}, M'_k, \lambda) := P_\beta(r_\beta|k, M_{\beta\setminus k}, \lambda)
\]

for all \(k \in \beta\), \(\beta = \alpha\) or \(\bar{\alpha}\), and \(|\beta| \geq 2\).

Violations of Eq. (1) and Eq. (2) indicate the standard (or weakest) and genuine (or strongest) multipartite nonlocality, respectively. Actually, there exist hierarchical kinds of multipartite nonlocality between the standard and the genuine one. We define the \(m\)-locality in the nonsignaling scenario as follows:

**Definition.** In a nonsignaling \(m\)-local hidden variable model \((2 \leq m \leq n)\), the joint probability distribution \(P(r_I|M_I)\) assumes the following form:

\[
P(r_I|M_I) = \sum_{(\alpha_i)} \int q(\alpha_i)(\lambda) \prod_{i=1}^{m} P_{\alpha_i}(r_{\alpha_i}|M_{\alpha_i}, \lambda)d\lambda,
\]

where \(\alpha_i \neq \emptyset\), \(\alpha_i \subset I\), \(\bigcup_{i=1}^{m} \alpha_i = I\), \(|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_m|\), and \(\alpha_i \cap \alpha_j = \emptyset\) for \(i, j = 1, \ldots, m\) and \(i \neq j\). If \(|\alpha_i| \geq 2\), \(P_{\alpha_i}(r_{\alpha_i}|M_{\alpha_i}, \lambda)\) should satisfy the nonsignaling condition Eq. (3) for all \(k \in \alpha_i\).

Here, the set \(I\) has been divided into arbitrary \(m\) nonempty and disjoint subsets \(\alpha_i\) for \(i = 1, \ldots, m\), and \(P_{\alpha_i}(r_{\alpha_i}|M_{\alpha_i}, \lambda)\) denotes the joint probability of all observers \(k \in \alpha_i\) measuring observable \(M_k\) with outcome \(r_k\) for a given local hidden variable \(\lambda\) distributed according to \(q(\alpha_i)(\lambda)\). The sum in Eq. (4) is taken over all possible partitions of \(I\) into \(m\) nonempty subsets. When \(m = 2\), Eq. (4) reduces to Eq. (2) whose violation indicates genuine multipartite nonlocality. When \(m = n\), Eq. (4) reduces to Eq. (1) whose violation indicates standard multipartite nonlocality.

III. HIERARCHY OF BELL-TYPE INEQUALITIES

As introduced in Sec. I, a Hardy-like Bell-type inequality has been proposed in Ref. 22:

\[
P(0|a_I) - \sum_{k \in I} P(0|b_k a_k) - \sum_{k \in I \setminus \{k'\}} P(1|k'0_k b_k a_k) \leq 0,
\]

where the \(k\)-th local observer measures two alternative observables \((a_k, b_k)\) with two outcomes labeled by \((0, 1)\), \(k = I \setminus \{k\}\), \(k'k = I \setminus \{k, k'\}\), and \(k' \in I\) is fixed. All nonsignaling \(2\)-local hidden variable models \((m = 2)\) satisfy this inequality, and the violation indicates genuine multipartite nonlocality. Moreover, Hardy’s inequality has been proposed for \(n\)-local hidden variable models \((m = n)\) 12 14:

\[
P(0|a_I) - \sum_{k \in I} P(0|b_k a_k) - P(1|b_I) \leq 0,
\]

and the violation indicates standard multipartite nonlocality. However, when \(2 < m < n\) Bell-type inequalities for nonsignaling \(m\)-local hidden variable models are still missing. In the following, we will present those Bell-type inequalities for nonsignaling \(m\)-local hidden variable models with \(2 \leq m \leq n\).
Theorem. In any nonsignaling \( m \)-local hidden variable model \((2 \leq m \leq n)\), the joint outcome probabilities should satisfy the following \((m - 1)\)-th Bell-type inequality:

\[
P(0_j | a_j) - \sum_{k \in I} P(0_j | b_k a_k) - \sum_{k_1 \cdots k_m \in I \setminus I(k')} P(1_{k'k_1 \cdots k_m} | 0_{k_1 k_2 \cdots k_m}) \leq 0, \tag{7}
\]

where the \( k \)-th local observer measures two alternative observables \( \{a_k, b_k\} \) with two outcomes labeled by \( \{0, 1\} \), \( k = I \setminus \{j, k'k_1 \cdots k_{m-1} = \{k', k_1, \ldots, k_m\}, k' k_1 \cdots k_{m-1} = I \setminus \{k', k_1, \ldots, k_m\}\}, \) and \( k' \in I \) is fixed.

Proof. By linearity of Eq. (4), we only have to prove that the correlation \( \prod_{i=1}^n P(\alpha_i | M_{\alpha_i}, \lambda) \) satisfies the inequality Eq. (7) for any given \( \{\alpha_i\} \) and \( \lambda \). It is worth noticing that \( k' \in I \) is fixed but can be an arbitrary number with \( 1 \leq k' \leq n \). Since every \( \alpha_i \) is a nonempty subset of \( I \), without loss of generality, we suppose \( k' \in \alpha_i \) and each of the rest \( \alpha_i \) \( (2 \leq i \leq m) \) contains at least one element denoted as \( j_i \) \( (2 \leq i \leq m) \). The mathematical induction method will be used in the following proof.

(i) When \( m = 2 \), we substitute the probability distribution \( P_{a_1}(r_{a_1} | M_{a_1}, \lambda)P_{a_2}(r_{a_2} | M_{a_2}, \lambda) \) into the left hand side (LHS) of Eq. (7). Thus, we have

\[
\text{LHS} = P_{a_1}(0_1 | a_1, \lambda)P_{a_2}(0_2 | a_2, \lambda) - \sum_{k \in a_1} P_{a_1}(0_1 | a_1 \setminus k | b_k a_{a_1 \setminus k}, \lambda)P_{a_2}(0_2 | a_2, \lambda)
\]

\[
= \sum_{k \in a_2} P_{a_1}(0_1 | a_1, \lambda)P_{a_2}(0_2 | 0_{a_2} \setminus k | b_k a_{a_2 \setminus k}, \lambda) - \sum_{k \in a_1} P_{a_1}(1_k 0_1 | \{k', k\} | b_k b_{a_1 \setminus \{k', k\}}, \lambda)P_{a_2}(0_2 | a_2, \lambda)
\]

\[
= \sum_{k_1 \cdots k_m \in I \setminus I(k')} P(1_{k'k_1 \cdots k_m} | 0_{k_1 k_2 \cdots k_m}) \leq 0.
\]

The above proof was given in the appendix of Ref. [22].

(ii) Suppose the inequality holds for \( m - 1 \) \((3 \leq t \leq n)\),

\[
\text{LHS} \leq \prod_{i=1}^{t-1} P_{a_1}(0_1 | a_1, \lambda) - \sum_{i=2}^{t-1} P_{a_1}(0_j | a_{1 \setminus j}, b_j a_{a_{1 \setminus j}}, \lambda) \prod_{i'=1, i' \neq i}^{t-1} P_{a_{i'}}(0_{a_{i'}}, \lambda)
\]

\[
- P_{a_1}(0_k 0_{a_1 \setminus k} | b_k a_{a_1 \setminus k}, \lambda) \prod_{i=2}^{t-1} P_{a_1}(0_{a_1 \setminus a_i} | a_i, \lambda) - P_{a_1}(1_k 0_{a_1 \setminus k} | b_k a_{a_1 \setminus k}, \lambda) \prod_{i=2}^{t-1} P_{a_1}(1_{a_1 | a_i} | b_j a_{a_1 \setminus j}, \lambda) \leq 0. \tag{9}
\]

It is worth noticing that when \( t = 3 \) \((i.e. m = 2)\), this inequality reduces to case (i).

(iii) We now prove the inequality holds for \( m = t \) \((3 \leq t \leq n)\),

\[
\text{LHS} \leq \prod_{i=1}^{t} P_{a_1}(0_1 | a_1, \lambda) - \sum_{i=2}^{t} P_{a_1}(0_j | a_{1 \setminus j}, b_j a_{a_{1 \setminus j}}, \lambda) \prod_{i'=1, i' \neq i}^{t} P_{a_{i'}}(0_{a_{i'}}, \lambda)
\]

\[
- P_{a_1}(0_k 0_{a_1 \setminus k} | b_k a_{a_1 \setminus k}, \lambda) \prod_{i=2}^{t} P_{a_1}(0_{a_1 \setminus a_i} | a_i, \lambda) - P_{a_1}(1_k 0_{a_1 \setminus k} | b_k a_{a_1 \setminus k}, \lambda) \prod_{i=2}^{t} P_{a_1}(1_{a_1 | a_i} | b_j a_{a_1 \setminus j}, \lambda) \leq 0. \tag{10}
\]
If $\prod_{i=1}^{n-1} P_{a_i}(0_{a_i}, |a_{\alpha_i}, \lambda) \leq P_{a_i}(1_{a_i}, 0_{a_i}, |b_{\alpha_i'}, \lambda') P_{a_i}(0_{a_i}, |a_{\alpha_i}, \lambda) \prod_{i=1}^{n-2} P_{a_i}(1_{a_i}, 0_{a_i}, |b_{\alpha_i}, \lambda') + P_{a_i}(0_{a_i}, 1_{a_i}, |b_{\alpha_i}, \lambda') \prod_{i=2}^{n-1} P_{a_i}(0_{a_i}, |a_{\alpha_i}, \lambda) \times P_{a_i}(0_{a_i}, |a_{\alpha_i}, \lambda) \leq 0$. If $\prod_{i=1}^{n-1} P_{a_i}(0_{a_i}, |a_{\alpha_i}, \lambda) \leq P_{a_i}(1_{a_i}, 0_{a_i}, |b_{\alpha_i'}, \lambda') P_{a_i}(0_{a_i}, |a_{\alpha_i}, \lambda) \prod_{i=1}^{n-2} P_{a_i}(1_{a_i}, 0_{a_i}, |b_{\alpha_i}, \lambda') + P_{a_i}(0_{a_i}, 1_{a_i}, |b_{\alpha_i}, \lambda') \prod_{i=2}^{n-1} P_{a_i}(0_{a_i}, |a_{\alpha_i}, \lambda) \times P_{a_i}(0_{a_i}, |a_{\alpha_i}, \lambda) \leq 0$. Therefore, Eq. (7) holds in any nonsignaling $m$-local hidden variable model ($2 \leq m \leq n$).

**Remark.** When $m = 2$ and $m = n$, Eq. (7) reduces to Eq. (5) and Eq. (9), respectively. In Eq. (7), there are $n$ terms in the first sum and $(n-1)$ terms in the second sum. All nonsignaling $m$-local models will satisfy Eq. (7). For an arbitrary $n$-particle quantum state $\rho$, to violate Eq. (7) one must find two measurement settings $\{\{a_i\}, \{b_i\}\}$ for each particle $i$,

$$\langle a_i | b_i \rangle = \sum_{k \in I} \langle b_k a_k | b_k a_k \rangle - \sum_{2 \leq k_1 \leq \ldots \leq k_{m-1} \leq n} \langle \hat{b}_1 \hat{b}_2 \ldots \hat{b}_{k_{m-1}} a_{1k_1 \ldots k_{m-1}} | b_i \rangle b_i \hat{b}_1 \hat{b}_2 \ldots \hat{b}_{k_{m-1}} a_{1k_1 \ldots k_{m-1}} \rangle > 0, \quad (11)$$

where $|a_{\alpha_i} \rangle = \otimes_{k \in \alpha_i} |a_k \rangle$ and $|b_k \rangle$ is orthogonal to $|b_k \rangle$. For simplicity we just set $k' = 1$.

### IV. EXAMPLES

In this section, we will present several examples to demonstrate their multipartite nonlocality hierarchy. For example, when $n = 4$, there are 3 Bell-type inequalities in Eq. (7). To violate them, one must find two measurement settings $\{\{a_i\}, \{b_i\}\}$ for each particle $i$ such that,

$$\langle a_1 a_2 a_3 a_4 \rangle - \langle \hat{b}_1 \hat{b}_2 a_3 a_4 \rangle - \langle \hat{a}_1 \hat{b}_2 a_3 a_4 \rangle = \langle \hat{a}_1 \hat{a}_2 a_3 a_4 \rangle - \langle \hat{a}_1 \hat{b}_2 a_3 a_4 \rangle = \langle \hat{a}_1 a_2 a_3 a_4 \rangle - \langle \hat{a}_1 b_2 a_3 a_4 \rangle - \langle \hat{a}_1 \hat{b}_2 a_3 a_4 \rangle - \langle \hat{a}_1 a_2 a_3 b_4 \rangle - \langle \hat{a}_1 \hat{b}_2 b_3 a_4 \rangle > 0, \quad (12)$$

$$\langle \hat{a}_1 \hat{a}_2 a_3 a_4 \rangle - \langle \hat{a}_1 \hat{b}_2 a_3 a_4 \rangle - \langle \hat{a}_1 \hat{b}_2 \hat{b}_3 a_4 \rangle - \langle \hat{a}_1 a_2 \hat{b}_3 a_4 \rangle - \langle \hat{a}_1 \hat{b}_2 b_3 a_4 \rangle - \langle \hat{a}_1 a_2 \hat{b}_3 a_4 \rangle - \langle \hat{a}_1 \hat{b}_2 a_3 b_4 \rangle - \langle \hat{a}_1 a_2 a_3 b_4 \rangle - \langle \hat{a}_1 \hat{b}_2 b_3 b_4 \rangle - \langle \hat{a}_1 a_2 \hat{b}_3 b_4 \rangle > 0, \quad (13)$$

where $\hat{a}_k = |a_k \rangle \langle a_k |$, $\hat{b}_k = |b_k \rangle \langle b_k |$, $\hat{b}_k = \mathbb{1} - |b_k \rangle \langle b_k |$. Eqs. (12) and (13) indicate the genuine and standard multipartite nonlocality, respectively. Our new Bell-type inequality is Eq. (13) which indicates the nonlocality between the genuine and the standard one.

We consider $n$-qubit mixed states, i.e., the noisy GHZ state and W state,

$$\rho_{\text{GHZ}} = p(\text{GHZ}) \langle \text{GHZ} | + (1 - p) \frac{\mathbb{1}_n}{2^n}, \quad (15)$$

$$\rho_{W} = p(W) \langle W | + (1 - p) \frac{\mathbb{1}_n}{2^n}, \quad (16)$$

where $\mathbb{1}_n$ is a $2^n \times 2^n$ identity matrix, $|\text{GHZ} \rangle = (|000 \cdots 0 \rangle + |111 \cdots 1 \rangle) / \sqrt{2}$, $|W \rangle = \sum_{\text{perm}(0 \cdots 01)} / \sqrt{n}$ with the sum taking over all possible permutation cases of 1 one and $n - 1$ zeros. For simplicity, we choose all the measurement settings in the X-Z plane of the Bloch Sphere and assume $\hat{a}_k = \hat{a}$, $\hat{b}_k = \hat{b}$ for all $2 \leq k \leq n$ due to the symmetry of the inequalities. After numerical search for $n = 4, 5, 6$, we find that when $p > p_i (1 \leq i \leq n - 1)$ the corresponding state $\rho_{\text{GHZ}}$ or $\rho_W$ will violate the $i$-th Bell-type inequality in Eq. (7) derived from nonsignaling $(i+1)$-local hidden variable models, which demonstrates hierarchical multipartite nonlocality for different $p_i$. All the $p_i$ are listed in Table II and Table III for $\rho_{\text{GHZ}}$ and $\rho_W$ with $n = 4, 5, 6$, respectively.

### V. DISCUSSIONS AND CONCLUSION

It is worth noticing that the nonsignaling $m$-local hidden variable models are related with $k$-separable states. An $n$-particle pure quantum state $|\Psi_{k \text{-sep}} \rangle$ is called $k$-separable if only if there is a $k$-partition $\alpha_1 |\alpha_2 \cdots |\alpha_3 \rangle$ such that $|\Psi_{k \text{-sep}} \rangle$ can be written as a product...
uct of \( k \) substates:

\[
|\Psi_{k-\text{sep}}\rangle = |\psi_1\rangle_{\alpha_1} \otimes |\psi_2\rangle_{\alpha_2} \otimes \cdots \otimes |\psi_k\rangle_{\alpha_k},
\]

where the set \( I = \{1, 2, \ldots, n\} \) has been split into arbitrary \( k \) nonempty and disjoint subsets \( \alpha_i \), i.e., \( \alpha_i \neq \emptyset \), \( \alpha_i \subset I \), \( \bigcup_{i=1}^{k} \alpha_i = I \), and \( \alpha_i \cap \alpha_j = \emptyset \) for \( i \neq j \). A \( n \)-partite mixed state \( \varrho_{k-\text{sep}} \) is called \( k \)-separable, if and only if it can be written as a convex combination of \( k \)-separable pure states:

\[
\varrho_{k-\text{sep}} = \sum_i p_i |\Psi_{k-\text{sep}}^i\rangle \langle \Psi_{k-\text{sep}}^i|,
\]

where \( |\Psi_{k-\text{sep}}\rangle \) might be \( k \)-separable under different \( \alpha \) partitions. Comparing Eq. (17) with Eq. (18), one can conclude that all \( k \)-separable states are nonsignaling \( k \)-local, since we can always find a nonsignaling \( k \)-local hidden variable model to describe the joint probability distribution from any \( k \)-separable states. This is similar to that all bipartite separable states are bipartite local. Therefore, violation of Eq. (17) guarantees its \( m \)-nonseparable, i.e., it is a sufficient condition for detecting \( m \)-nonseparable states.

In conclusion, we have proposed a hierarchy of Bell-type inequalities for arbitrary \( n \)-partite systems, which can identify the different degrees of nonlocality ranging from standard to genuine multipartite nonlocality. After introducing the definition of nonsignaling \( m \)-locality, we have shown that the observed joint probabilities in any nonsignaling \( m \)-local realistic models should satisfy the \((m - 1)\)-th Bell-type inequality. When \( m = 2 \) the corresponding inequality reduces to the one shown in Ref. 22 whose violation indicates genuine multipartite nonlocality, and when \( m = n \) the corresponding inequality is just Hardy’s inequality whose violation indicates standard multipartite nonlocality. Furthermore, several examples have been provided to demonstrate the multipartite nonlocality hierarchy, and the relations between \( m \)-locality and \( k \)-separable states have been discussed.

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