Three-Dimensional Euler Fluid Code for Fusion Fuel Ignition and Burning

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Abstract
The document describes a numerical algorithm to simulate plasmas and fluids in the 3 dimensional space by the Euler method, in which the spatial meshes are fixed to the space. The plasmas and fluids move through the spacial Euler mesh boundary. The Euler method can represent a large deformation of the plasmas and fluids. On the other hand, when the plasmas or fluids are compressed to a high density, the spatial resolution should be ensured to describe the density change precisely. The present 3D Euler code is developed to simulate a nuclear fusion fuel ignition and burning. Therefore, the 3D Euler code includes the DT fuel reactions, the alpha particle diffusion, the alpha particle deposition to heat the DT fuel and the DT fuel depletion by the DT reactions, as well as the thermal energy diffusion based on the three-temperature compressible fluid model.

1 Introduction
In inertial confinement fusion (ICF), the D (deuterium) and T (tritium) fuel should be imploded uniformly to reduce the input driver energy and to release a sufficient fusion energy output. The implosion non-uniformity should be less than a few per cent [1, 2]. Recent experimental results demonstrated that the DT fuel is compressed to one thousand to several thousand times the solid density [3]. The scientific issues include the DT fuel ignition and burning, as well as the implosion uniformity [4].

In order to investigate the DT fuel ignition, required are detail experimental [3], theoretical [2] and numerical studies [5, 6]. For the numerical studies, a 2-dimensional code for heavy ion ICF was developed to investigate the implosion non-uniformity smoothing control [2]. In this document, we present an Euler 3D fluid code algorithm toward a DT fuel ignition and burning. The three-temperature compressible fluid model is employed, together with the DT reaction, the alpha particle generation, diffusion and deposition to sustain the DT reaction in the burning phase [7].

2 Basic Equation
In this section the basic equations for the compressible plasma are listed below.

\[
\frac{\partial \rho}{\partial t} = -\rho(\nabla \cdot \textbf{u}) - (\textbf{u} \cdot \nabla)\rho
\]

\[
\frac{\partial \textbf{u}}{\partial t} = -\frac{1}{\rho} \nabla (p + q) - (\textbf{u} \cdot \nabla)\textbf{u}
\]

\[
\frac{\partial T_i}{\partial t} = -(\textbf{u} \cdot \nabla)T_i - \frac{k_B}{C_{V_i}} \left\{ \left( \rho B_{Ti} + \frac{p_i + q}{\rho} \right)(\nabla \cdot \textbf{u}) \right\}
\]

\[
\frac{\partial T_e}{\partial t} = -(\textbf{u} \cdot \nabla)T_e - \frac{k_B}{C_{V_e}} \left\{ \left( \rho B_{Te} + \frac{p_e}{\rho} \right)(\nabla \cdot \textbf{u}) \right\}
\]

\[
\frac{\partial T_r}{\partial t} = -(\textbf{u} \cdot \nabla)T_r - \frac{k_B}{C_{V_r}} \left\{ \left( \rho B_{Tr} + \frac{p_r}{\rho} \right)(\nabla \cdot \textbf{u}) \right\}
\]
Here $\rho$ is the mass density, $\mathbf{u} = (u, v, w)$ the velocity, $t$ the time, $p_{i,e,r}$ the pressure, $q$ the artificial viscosity, $T_{i,e,r}$ temperature for ion, electron and radiation, $k_B$ is the Boltzmann constant, $C_{V_{i,e,r}}$ is the specific heat for ion, electron and radiation, and $B_{T_{i,e,r}}$ the compressibility for the ion, the electron and the radiation.

The artificial viscosities are represented by the following equations.

\[ q = q_x + q_y + q_z \]  
\[ q_x = \begin{cases} 
\rho C_Q^2 \left( \frac{\partial u}{\partial t} \right)^2 + \rho C_L C_s | \frac{\partial u}{\partial t} | (\frac{\partial u}{\partial t} < 0) \\
0 \left( \frac{\partial u}{\partial t} \geq 0 \right)
\end{cases} \]  
\[ (7) \]
\[ q_y = \begin{cases} 
\rho C_Q^2 \left( \frac{\partial v}{\partial t} \right)^2 + \rho C_L C_s | \frac{\partial v}{\partial t} | (\frac{\partial v}{\partial t} < 0) \\
0 \left( \frac{\partial v}{\partial t} \geq 0 \right)
\end{cases} \]  
\[ (8) \]
\[ q_z = \begin{cases} 
\rho C_Q^2 \left( \frac{\partial w}{\partial t} \right)^2 + \rho C_L C_s | \frac{\partial w}{\partial t} | (\frac{\partial w}{\partial t} < 0) \\
0 \left( \frac{\partial w}{\partial t} \geq 0 \right)
\end{cases} \]  
\[ (9) \]

Here $C_Q = 2$, $C_L = 1.0$, and $C_s$ is the sound speed in our Euler code. The artificial viscosity $q$ would be a combination of $q_x$ in the $x$ direction, $q_y$ in the $y$ direction and $q_z$ in $z$ direction.

### 3 Normalization

The normalization factors of the time $t_0$, the length $L_0$ and the mass $M_0$ are determined as follows:

\[ t_0 = 1[\text{ns}] = 10^{-9}[\text{s}] \]  
\[ L_0 = 1[\text{mm}] = 10^{-3}[\text{m}] \]  
\[ M_0 = \frac{1}{2}(M_D + M_T) = 4.17638 \times 10^{-27}[\text{kg}] \]  
\[ (10) \]
\[ (11) \]
\[ (12) \]
$M_D$ and $M_T$ are the mass of the $D$ atom and the $T$ atom, respectively. Other physical quantities are normalized as follows:

\[
L = \tilde{L} L_0
\]

\[
t = \tilde{t} t_0
\]

\[
j = \tilde{j} j_0 \quad (j_0 = L_0^3)
\]

\[
u = \tilde{\nu} u_0 \quad (u_0 = \frac{L_0}{t_0})
\]

\[
M = \tilde{M} M_0
\]

\[
p = \tilde{p} p_0 \quad (p_0 = \frac{M_0}{L_0 t_0^4})
\]

\[
q = \tilde{q} q_0 \quad (q_0 = \frac{M_0}{L_0 t_0^4})
\]

\[
C_v = \tilde{C}_v C_{v0} \quad (C_{v0} = \frac{k_B}{M_0})
\]

\[
B_{T0} = \tilde{B}_T B_{T0} \quad (B_{T0} = \frac{T_0 L_0^3}{M_0^2})
\]

\[
\rho = \tilde{\rho} \rho_0 \quad (\rho_0 = \frac{M_0}{L_0 t_0^4})
\]

\[
T = \tilde{T} T_0 \quad (T_0 = \frac{M_0 L_0^2}{t_0^4})
\]

The normalization factor of the velocity $u = (u, v, w)$ is $u_0 = \frac{L_0}{t_0}$.

\[
\tilde{u} = \frac{\partial \tilde{x}}{\partial \tilde{t}}
\]

\[
\tilde{v} = \frac{\partial \tilde{y}}{\partial \tilde{t}}
\]

\[
\tilde{w} = \frac{\partial \tilde{z}}{\partial \tilde{t}}
\]

The equation of continuity is normalized as follows:

\[
\frac{\partial \rho}{\partial \tilde{t}} = -\rho(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\rho
\]

\[
\frac{\partial \rho}{\partial \tilde{t}} = \rho \left( \frac{\partial u}{\partial \tilde{x}} + \frac{\partial v}{\partial \tilde{y}} + \frac{\partial w}{\partial \tilde{z}} \right) - \left( u \frac{\partial \rho}{\partial \tilde{x}} + v \frac{\partial \rho}{\partial \tilde{y}} + w \frac{\partial \rho}{\partial \tilde{z}} \right)
\]

Therefore,

\[
\rho_0 \frac{\partial \tilde{\rho}}{t_0 \partial \tilde{t}} = -\frac{\rho_0 L_0}{L_0 t_0} \times \tilde{\rho} \left( \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} \right) - \frac{L_0 \rho_0}{t_0 L_0} \times \left( \frac{\partial \tilde{\rho}}{\partial \tilde{x}} + \frac{\partial \tilde{\rho}}{\partial \tilde{y}} + \frac{\partial \tilde{\rho}}{\partial \tilde{z}} \right).
\]

The normalized equation of continuity becomes as follows:

\[
\frac{\partial \tilde{\rho}}{\partial \tilde{t}} = -\tilde{\rho} \left( \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} \right) - \left( \tilde{u} \frac{\partial \tilde{\rho}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{\rho}}{\partial \tilde{y}} + \tilde{w} \frac{\partial \tilde{\rho}}{\partial \tilde{z}} \right)
\]
The equation of motion is represented by the following equation.
\[
\frac{\partial \mathbf{u}}{\partial t} = -(\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{1}{\rho} \nabla (p + q)
\]  
(2)

Each component of the equation of motion is listed here.
\[
\frac{\partial u}{\partial t} = - \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) - \frac{1}{\rho} \left( \frac{\partial p}{\partial x} + \frac{\partial q}{\partial x} \right)
\]  
(19)
\[
\frac{\partial v}{\partial t} = - \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) - \frac{1}{\rho} \left( \frac{\partial p}{\partial y} + \frac{\partial q}{\partial y} \right)
\]  
(20)
\[
\frac{\partial w}{\partial t} = - \left( u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) - \frac{1}{\rho} \left( \frac{\partial p}{\partial z} + \frac{\partial q}{\partial z} \right)
\]  
(21)

These equations are normalized:
\[
\frac{L_0}{t_0 \cdot t_0} \times \frac{\partial u}{\partial t} = -\frac{L_0}{t_0} \frac{1}{t_0} \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \right) - \frac{J_0}{M_0} \times \frac{1}{\mu_0} \left( \frac{\partial p}{L_0} + \frac{\partial q}{L_0} \right)
\]  
(22)
\[
\frac{L_0}{t_0 \cdot t_0} \times \frac{\partial v}{\partial t} = -\frac{L_0}{t_0} \frac{1}{t_0} \lambda \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \right) - \frac{J_0}{M_0} \times \frac{1}{\mu_0} \left( \frac{\partial p}{L_0} + \frac{\partial q}{L_0} \right)
\]  
(23)
\[
\frac{L_0}{t_0 \cdot t_0} \times \frac{\partial w}{\partial t} = -\frac{L_0}{t_0} \frac{1}{t_0} \lambda \left( \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} \right) - \frac{J_0}{M_0} \times \frac{1}{\mu_0} \left( \frac{\partial p}{L_0} + \frac{\partial q}{L_0} \right)
\]  
(24)

Finally, the equation of motion is normalized as follows:
\[
\frac{\partial \tilde{u}}{\partial t} = - \left( \tilde{u} \frac{\partial \tilde{u}}{\partial x} + \tilde{v} \frac{\partial \tilde{u}}{\partial y} + \tilde{w} \frac{\partial \tilde{u}}{\partial z} \right) - \frac{1}{\tilde{\rho}} \left( \frac{\partial \tilde{p}}{\partial x} + \frac{\partial \tilde{q}}{\partial x} \right)
\]  
(25)
\[
\frac{\partial \tilde{v}}{\partial t} = - \left( \tilde{u} \frac{\partial \tilde{v}}{\partial x} + \tilde{v} \frac{\partial \tilde{v}}{\partial y} + \tilde{w} \frac{\partial \tilde{v}}{\partial z} \right) - \frac{1}{\tilde{\rho}} \left( \frac{\partial \tilde{p}}{\partial y} + \frac{\partial \tilde{q}}{\partial y} \right)
\]  
(26)
\[
\frac{\partial \tilde{w}}{\partial t} = - \left( \tilde{u} \frac{\partial \tilde{w}}{\partial x} + \tilde{v} \frac{\partial \tilde{w}}{\partial y} + \tilde{w} \frac{\partial \tilde{w}}{\partial z} \right) - \frac{1}{\tilde{\rho}} \left( \frac{\partial \tilde{p}}{\partial z} + \frac{\partial \tilde{q}}{\partial z} \right)
\]  
(27)

The artificial viscosity in 3D is expressed by the following equations.
\[
q_x = \rho C_Q^2 \left( \frac{\partial \tilde{u}}{\partial \tilde{t}} \right)^2 + \rho C_L C_s \left| \frac{\partial \tilde{u}}{\partial \hat{i}} \right|
\]  
(28)
\[
q_y = \rho C_Q^2 \left( \frac{\partial \tilde{v}}{\partial \tilde{t}} \right)^2 + \rho C_L C_s \left| \frac{\partial \tilde{v}}{\partial \hat{j}} \right|
\]  
(29)
\[
q_z = \rho C_Q^2 \left( \frac{\partial \tilde{w}}{\partial \tilde{t}} \right)^2 + \rho C_L C_s \left| \frac{\partial \tilde{w}}{\partial \hat{k}} \right|
\]  
(30)

The artificial viscosity \(q_x\) is normalized.
\[
\tilde{q}_x = \frac{\rho_{00} u_0^2}{\eta_0} \times \left[ \hat{\rho}_{C_Q} \left( \frac{\partial \tilde{u}}{\partial \hat{j}} \right)^2 + \frac{\rho_{00} u_0^2}{\eta_0} \times \left[ \hat{\rho}_0 C_L \tilde{C}_s \left| \frac{\partial \tilde{u}}{\partial \hat{i}} \right| \right] \right]
\]  
(31)

In addition,
\[
\frac{\rho_{00} u_0^2}{\eta_0} = \frac{M_0 T_0^2}{L_0 W_0^2} = 1
\]  
(32)
Then the normalized artificial viscosities are following:

\[
\begin{align*}
\hat{\eta}_x &= \hat{\rho} C_Q^2 \left( \frac{\partial \hat{u}}{\partial \hat{t}} \right)^2 + \hat{\rho} C_L \hat{C}_v \left| \frac{\partial \hat{u}}{\partial \hat{t}} \right| \\
\hat{\eta}_y &= \hat{\rho} C_Q^2 \left( \frac{\partial \hat{v}}{\partial \hat{t}} \right)^2 + \hat{\rho} C_L \hat{C}_v \left| \frac{\partial \hat{v}}{\partial \hat{t}} \right| \\
\hat{\eta}_z &= \hat{\rho} C_Q^2 \left( \frac{\partial \hat{w}}{\partial \hat{t}} \right)^2 + \hat{\rho} C_L \hat{C}_v \left| \frac{\partial \hat{w}}{\partial \hat{t}} \right|
\end{align*}
\]

(33)  

(34)  

(35)

The energy equation is expressed as follows:

\[
\begin{align*}
\frac{\partial T_i}{\partial \hat{t}} &= - (\mathbf{u} \cdot \nabla) T_i - \frac{k_B}{C_{V_i}} \left\{ \left( \rho B_{T_i} + \frac{p_i + q}{\rho} \right) (\nabla \cdot \mathbf{u}) \right\} \\
\frac{\partial T_e}{\partial \hat{t}} &= - (\mathbf{u} \cdot \nabla) T_e - \frac{k_B}{C_{V_e}} \left\{ \left( \rho B_{T_e} + \frac{p_e}{\rho} \right) (\nabla \cdot \mathbf{u}) \right\} \\
\frac{\partial T_r}{\partial \hat{t}} &= - (\mathbf{u} \cdot \nabla) T_r - \frac{k_B}{C_{V_r}} \left\{ \left( \rho B_{T_r} + \frac{p_r}{\rho} \right) (\nabla \cdot \mathbf{u}) \right\}
\end{align*}
\]

(36)  

(37)  

(38)

The ion temperature equation is normalized by the following process, except the term including the compressibility \( B_T \):

\[
\frac{T_o}{t_0} \frac{\partial \hat{T}_i}{\partial \hat{t}} = -\frac{L_o}{t_o} \frac{T_o}{L_o} \left( \frac{\partial \hat{T}_i}{\partial \hat{x}} + \frac{\partial \hat{T}_i}{\partial \hat{y}} + \frac{\partial \hat{T}_i}{\partial \hat{z}} \right) - M_0 \frac{k_B}{k_B} \frac{T_o}{L_o} \left[ \left( \frac{M_o T_o L_o^2}{M_o L_o} \rho B_{T_i} + \frac{M_0}{L_o t_0^3} \frac{L_o}{M_0} \frac{\rho}{\rho} \rho \right) \right]
\]

(39)

The temperature itself is normalized as follows:

\[
T_o = \frac{M_0 L_o^2}{t_0}
\]

(40)

Finally the ion temperature equation is normalized as follows:

\[
\frac{\partial \hat{T}_e}{\partial \hat{t}} = - \left( \hat{\nu} \frac{\partial \hat{T}_e}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{T}_e}{\partial \hat{y}} + \hat{w} \frac{\partial \hat{T}_e}{\partial \hat{z}} \right) - \frac{1}{C_{V_e}} \left[ \left( \rho B_{T_e} + \frac{\rho}{\rho} \right) \left( \frac{\partial \hat{T}_e}{\partial \hat{x}} + \frac{\partial \hat{T}_e}{\partial \hat{y}} + \frac{\partial \hat{T}_e}{\partial \hat{z}} \right) \right]
\]

(41)

Similarly, the equations for the electron temperature and the radiation temperature are normalized as follows:

\[
\begin{align*}
\frac{\partial \hat{T}_e}{\partial \hat{t}} &= - \left( \hat{\nu} \frac{\partial \hat{T}_e}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{T}_e}{\partial \hat{y}} + \hat{w} \frac{\partial \hat{T}_e}{\partial \hat{z}} \right) - \frac{1}{C_{V_e}} \left[ \left( \rho B_{T_e} + \frac{\rho}{\rho} \right) \left( \frac{\partial \hat{T}_e}{\partial \hat{x}} + \frac{\partial \hat{T}_e}{\partial \hat{y}} + \frac{\partial \hat{T}_e}{\partial \hat{z}} \right) \right] \\
\frac{\partial \hat{T}_r}{\partial \hat{t}} &= - \left( \hat{\nu} \frac{\partial \hat{T}_r}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{T}_r}{\partial \hat{y}} + \hat{w} \frac{\partial \hat{T}_r}{\partial \hat{z}} \right) - \frac{1}{C_{V_r}} \left[ \left( \rho B_{T_r} + \frac{\rho}{\rho} \right) \left( \frac{\partial \hat{T}_r}{\partial \hat{x}} + \frac{\partial \hat{T}_r}{\partial \hat{y}} + \frac{\partial \hat{T}_r}{\partial \hat{z}} \right) \right]
\end{align*}
\]

(42)  

(43)
4 Discretization

4.1 Discretization of equation of continuity

The equation of continuity Eq. (1) is discretized.

\[
\frac{\partial \rho}{\partial t} = -\rho (\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla) \rho
\]  

(1)

When the equation of continuity is normalized, the following equation is obtained.

\[
\frac{\partial \tilde{\rho}}{\partial \tilde{t}} = -\tilde{\rho} \left( \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} \right) - \left( \tilde{u} \frac{\partial \tilde{\rho}}{\partial \tilde{x}} + \tilde{v} \frac{\partial \tilde{\rho}}{\partial \tilde{y}} + \tilde{w} \frac{\partial \tilde{\rho}}{\partial \tilde{z}} \right)
\]  

(18)

The left side of this equation is discretized.

\[
\left( \frac{\partial \tilde{\rho}}{\partial \tilde{t}} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} = \frac{\rho_{i+1,j+1,k+1}^{n+1} - \rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n}}{\Delta t_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1}}
\]  

(44)

Therefore, the discretized equation of continuity is expressed as follows:

\[
\rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} = \rho_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} - \Delta t_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} \left[ \frac{n}{n+1} \left( \frac{\partial \tilde{u}}{\partial \tilde{x}} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} + \left( \frac{\partial \tilde{v}}{\partial \tilde{y}} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} + \left( \frac{\partial \tilde{w}}{\partial \tilde{z}} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n} \right] + \left\{ \frac{\partial \rho}{\partial \tilde{x}} \right\}_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1}
\]  

(45)
Each term at the right side is shown below.

\[
\begin{align*}
\left( \frac{\partial u}{\partial x} \right)_n^{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} &= \frac{u^n_{i+1,j+\frac{1}{2},k+\frac{1}{2}} - u^n_{i,j+\frac{1}{2},k+\frac{1}{2}}}{Dx_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \\
\left( \frac{\partial v}{\partial y} \right)_n^{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} &= \frac{v^n_{i+\frac{1}{2},j+1,k+\frac{1}{2}} - u^n_{i+\frac{1}{2},j,k+\frac{1}{2}}}{Dy_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \\
\left( \frac{\partial w}{\partial z} \right)_n^{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} &= \frac{v^n_{i+\frac{1}{2},j+\frac{1}{2},k+1} - u^n_{i+\frac{1}{2},j+\frac{1}{2},k}}{Dz_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} \\
\left( \frac{\partial \rho}{\partial x} \right)_n^{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} &= \begin{cases} u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - \rho^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} & (u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \geq 0) \\
\rho^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} & (u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} < 0) \end{cases} \\
\left( \frac{\partial \rho}{\partial y} \right)_n^{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} &= \begin{cases} u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - \rho^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} & (u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \geq 0) \\
\rho^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} & (u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} < 0) \end{cases} \\
\left( \frac{\partial \rho}{\partial z} \right)_n^{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} &= \begin{cases} u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - \rho^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} & (u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} \geq 0) \\
\rho^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} - u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} & (u^n_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}} < 0) \end{cases}
\end{align*}
\]
4.2 Discretization of equation of motion

The basic equation of the equation of motion is expressed by the following equation.

\[
\frac{\partial u}{\partial t} = -\frac{1}{\rho} \nabla (p + q) - (u \cdot \nabla) u \tag{2}
\]

When this equation is normalized, the following equation is obtained:

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial \tilde{t}} &= -\left( \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} \right) - \frac{1}{\tilde{\rho}} \frac{\partial (\tilde{p} + \tilde{q})}{\partial \tilde{x}} \\
\frac{\partial \tilde{v}}{\partial \tilde{t}} &= -\left( \frac{\partial \tilde{v}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} \right) - \frac{1}{\tilde{\rho}} \frac{\partial (\tilde{p} + \tilde{q})}{\partial \tilde{y}} \\
\frac{\partial \tilde{w}}{\partial \tilde{t}} &= -\left( \frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} \right) - \frac{1}{\tilde{\rho}} \frac{\partial (\tilde{p} + \tilde{q})}{\partial \tilde{z}}
\end{align*}
\]

The time derivative of \( u \), that is, the \( x \) component of the velocity becomes as follows:

\[
\left( \frac{\partial u}{\partial t} \right)^n_{i + \frac{1}{2}, j + \frac{1}{2}, k + \frac{1}{2}} = \frac{u^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} - u^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}}{Dt^{n+\frac{1}{2}}}
\]
Therefore, the discretized equation of motion is as follows:

\[
\begin{align*}
\frac{u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n+\frac{1}{2}}}{\partial t} &= u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} - Dr^n \left\{ \left( \frac{\partial u}{\partial x} \right)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} + \left( \frac{\partial u}{\partial y} \right)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} + \left( \frac{1}{\rho} \frac{\partial (p + q)}{\partial x} \right)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} \right\} \\
\left( \frac{\partial u}{\partial x} \right)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} &= \begin{cases} u_{i+1,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} - u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \left( u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \geq 0 \right) \\ u_{i+1,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} - u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \left( u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} < 0 \right) \end{cases} \\
\left( \frac{\partial u}{\partial y} \right)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} &= \begin{cases} u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} - u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \left( v_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \geq 0 \right) \\ u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} - u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \left( v_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} < 0 \right) \end{cases} \\
\left( \frac{\partial u}{\partial z} \right)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} &= \begin{cases} u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} - u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \left( w_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \geq 0 \right) \\ u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} - u_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} \left( w_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n-\frac{1}{2}} < 0 \right) \end{cases} \\
\left( \frac{1}{\rho} \frac{\partial (p + q)}{\partial x} \right)_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} &= \frac{2}{\rho_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} + \rho_{i+1,j+\frac{1}{2},k+\frac{1}{2}}^{n}} \left\{ p_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} + q_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n} \frac{1}{2} - \frac{\left( p_{i+1,j+\frac{1}{2},k+\frac{1}{2}}^{n} + q_{i+1,j+\frac{1}{2},k+\frac{1}{2}}^{n} \right)}{Dr_{i,j+\frac{1}{2},k+\frac{1}{2}}^{n}} \right\}
\end{align*}
\]

The y direction and the z direction are similarly represented.
4.3 Discretization of artificial viscosity

The artificial viscosity is represented by the following equation.

\[ q = q_x + q_y + q_z \]  

\[ q_x = \begin{cases} 
\rho C_Q^2 \left( \frac{\partial u}{\partial i} \right)^2 + \rho C_L C_s \left| \frac{\partial u}{\partial i} \right| & \left( \frac{\partial u}{\partial i} < 0 \right) \\
0 & \left( \frac{\partial u}{\partial i} \geq 0 \right) 
\end{cases} \]  

(52)

\[ q_y = \begin{cases} 
\rho C_Q^2 \left( \frac{\partial v}{\partial j} \right)^2 + \rho C_L C_s \left| \frac{\partial v}{\partial j} \right| & \left( \frac{\partial v}{\partial j} < 0 \right) \\
0 & \left( \frac{\partial v}{\partial j} \geq 0 \right) 
\end{cases} \]  

(53)

\[ q_z = \begin{cases} 
\rho C_Q^2 \left( \frac{\partial w}{\partial k} \right)^2 + \rho C_L C_s \left| \frac{\partial w}{\partial k} \right| & \left( \frac{\partial w}{\partial k} < 0 \right) \\
0 & \left( \frac{\partial w}{\partial k} \geq 0 \right) 
\end{cases} \]  

(54)

Here, \( C_Q = 2 \), \( C_L = 1.0 \), \( C_s \) is the sound speed. When these equations are standardized, the following equations are obtained.

\[ \tilde{q}_x = \tilde{\rho} C_Q^2 \left( \frac{\partial \tilde{u}}{\partial i} \right)^2 + \tilde{\rho} C_L \tilde{C}_s \left| \frac{\partial \tilde{u}}{\partial i} \right| \]  

(55)

\[ \tilde{q}_y = \tilde{\rho} C_Q^2 \left( \frac{\partial \tilde{v}}{\partial j} \right)^2 + \tilde{\rho} C_L \tilde{C}_s \left| \frac{\partial \tilde{v}}{\partial j} \right| \]  

(56)

\[ \tilde{q}_z = \tilde{\rho} C_Q^2 \left( \frac{\partial \tilde{w}}{\partial k} \right)^2 + \tilde{\rho} C_L \tilde{C}_s \left| \frac{\partial \tilde{w}}{\partial k} \right| \]  

(57)
When these equations are discretized, they are expressed by the following equations.

\[
q_{x i+\frac{1}{2} j+\frac{1}{2}, k+\frac{1}{2}} = \rho_i^n \frac{\partial u}{\partial x} + \frac{u_i^n}{\rho_i^n} + \frac{\partial v_i^n}{\partial y} + \frac{\partial w_i^n}{\partial z} \left[ \left( \rho B_{T_e} + \frac{u_i^n}{\rho_i^n} \right) \right] (58)
\]

\[
q_{y i+\frac{1}{2} j+\frac{1}{2}, k+\frac{1}{2}} = \rho_i^n \frac{\partial v}{\partial y} + \frac{v_i^n}{\rho_i^n} + \frac{\partial w_i^n}{\partial z} \left[ \left( \rho B_{T_e} + \frac{v_i^n}{\rho_i^n} \right) \right] (59)
\]

\[
q_{z i+\frac{1}{2} j+\frac{1}{2}, k+\frac{1}{2}} = \rho_i^n \frac{\partial w}{\partial z} + \frac{w_i^n}{\rho_i^n} \left[ \left( \rho B_{T_e} + \frac{w_i^n}{\rho_i^n} \right) \right] (60)
\]

4.4 Discretization of energy equation

The basic equation of the energy equation is represented by the following equation.

\[
\frac{\partial T_i}{\partial t} = -(u \cdot \nabla) T_i - \frac{k_B}{C_V} \left[ \left( \rho B_{T_e} + \frac{u_i^n}{\rho_i^n} \right) (\nabla \cdot u) \right] (62)
\]

\[
\frac{\partial T_e}{\partial t} = -(u \cdot \nabla) T_e - \frac{k_B}{C_{V_e}} \left[ \left( \rho B_{T_e} + \frac{v_i^n}{\rho_i^n} \right) (\nabla \cdot u) \right] (63)
\]

\[
\frac{\partial T_r}{\partial t} = -(u \cdot \nabla) T_r - \frac{k_B}{C_{V_r}} \left[ \left( \rho B_{T_r} + \frac{r_i^n}{\rho_i^n} \right) (\nabla \cdot u) \right] (64)
\]

When these equations are normalized, they are expressed by the following equations.

\[
\frac{\partial \tilde{T}_i}{\partial t} = - \left( \tilde{u} \frac{\partial \tilde{T}_i}{\partial x} + \tilde{v} \frac{\partial \tilde{T}_i}{\partial y} + \tilde{w} \frac{\partial \tilde{T}_i}{\partial z} \right) - \frac{1}{C_{V_i}} \left[ \left( \tilde{p}_i + \tilde{q}_i \right) \left( \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} \right) \right] (65)
\]

\[
\frac{\partial \tilde{T}_e}{\partial t} = - \left( \tilde{u} \frac{\partial \tilde{T}_e}{\partial x} + \tilde{v} \frac{\partial \tilde{T}_e}{\partial y} + \tilde{w} \frac{\partial \tilde{T}_e}{\partial z} \right) - \frac{1}{C_{V_e}} \left[ \left( \tilde{p}_e + \tilde{q}_e \right) \left( \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} \right) \right] (66)
\]

\[
\frac{\partial \tilde{T}_r}{\partial t} = - \left( \tilde{u} \frac{\partial \tilde{T}_r}{\partial x} + \tilde{v} \frac{\partial \tilde{T}_r}{\partial y} + \tilde{w} \frac{\partial \tilde{T}_r}{\partial z} \right) - \frac{1}{C_{V_r}} \left[ \left( \tilde{p}_r + \tilde{q}_r \right) \left( \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + \frac{\partial \tilde{w}}{\partial z} \right) \right] (67)
\]

Here, \(B_{T_e} = 0\). The ion temperature equation is expressed by the following equation when the left side is discretized.

\[
\left( \frac{\partial T_i}{\partial x} \right)_{i+\frac{1}{2} j+\frac{1}{2}, k+\frac{1}{2}} = \frac{T_{i+\frac{1}{2} j+\frac{1}{2}, k+\frac{1}{2}} - T_{i+\frac{1}{2} j+\frac{1}{2}, k+\frac{1}{2}}}{D_T \frac{T^{n+1}}{T^{n+1}}} (68)
\]
Therefore, the energy equation for the discretized ion temperature is expressed by the following equation.

\[
T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} = T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n - D_{T_i}^{n+\frac{1}{2}} \left\{ \left( \frac{\partial T_i}{\partial x} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n + \left( \frac{\partial T_i}{\partial y} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n \right\} + \frac{1}{C V_i} \left[ \left( \frac{\partial u}{\partial x} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n + \left( \frac{\partial v}{\partial y} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n + \left( \frac{\partial w}{\partial z} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n \right] \right\}
\]

(69)

\[
\left( \frac{\partial T_i}{\partial x} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \begin{cases} 
T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n & \frac{T_{i-\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n - T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n}{D_{x,i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} (u_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n \geq 0) \\
T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n & \frac{T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n - T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^-}{D_{x,i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} (u_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n < 0)
\end{cases}
\]

\[
\left( \frac{\partial T_i}{\partial y} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \begin{cases} 
v_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n & \frac{T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n - T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^-}{D_{y,i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} (v_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n \geq 0) \\
v_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n & \frac{T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^- - T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n}{D_{y,i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} (v_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n < 0)
\end{cases}
\]

\[
\left( \frac{\partial T_i}{\partial z} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \begin{cases} 
w_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n & \frac{T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n - T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^-}{D_{z,i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} (w_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n \geq 0) \\
w_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n & \frac{T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^- - T_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n}{D_{z,i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}} (w_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n < 0)
\end{cases}
\]

\[
\left( \frac{\partial u}{\partial x} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \frac{u_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} - u_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n}{D_{x,i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}
\]

\[
\left( \frac{\partial v}{\partial y} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \frac{v_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} - v_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n}{D_{y,i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}
\]

\[
\left( \frac{\partial w}{\partial z} \right)_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n = \frac{w_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^{n+1} - w_{i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}^n}{D_{z,i+\frac{1}{2},j+\frac{1}{2},k+\frac{1}{2}}}
\]

The electron temperature and radiation temperature are also expressed in the same way.
5 Three temperature relaxation

In this study, we employ the three-temperature model for the ion, the electron and the radiation [7]. In this model, it assumed the radiation is in its equilibrium. The assumption means that the radiation becomes the Planck distribution. For example, the heavy ion beams (HIBs) deposit their energy inside a material of the energy absorber [2, 4, 5]. The temperature in the energy absorber becomes around 300eV during the HIBs pulse length of ∼10 ns, and all the three temperatures are almost equilibrated during the fusion fuel target implosion. However, at the fuel ignition and burning phases induced by the energy deposition of the alpha particles created by the DT fusion reactions, the three temperatures may be different among them. We need to compute the energy transfer between the three temperatures.

The following equations are used for the basic equation[7].

\[
\begin{align*}
C_V \frac{dT}{dt} & = -K_{ie} \\
C_V \frac{dT}{dt} & = K_{ie} - K_{re} \\
C_V \frac{dT}{dt} & = K_{re}
\end{align*}
\]  

(70)

Here \(C_V\) is ion constant volume specific heat [J/K·kg]. \(C_V\) is electron constant volume specific heat [J/K·kg]. \(C_V\) is radiation constant volume specific heat [J/K·kg]. \(T_i\) is ion temperature[K], \(T_e\) is electron temperature[K] and \(T_r\) is radiation temperature[K]. \(K_{ie}\) is energy exchange rate between the ions and the electrons, and \(K_{re}\) is energy exchange rate between the radiation and the electrons.

The energy exchange rate is expressed by the following equation.

\[
\begin{align*}
K_{ie} & = C_V \omega_{ie}(T_i - T_e) \\
K_{re} & = C_V \omega_{re}(T_e - T_r)
\end{align*}
\]  

(71)

Here \(\omega_{ie}\) and \(\omega_{re}\) are the collision frequencies between the ion-electron and the radiation-electron, respectively. They are calculated by the following equations. The inverse Compton scattering is also included in the collision frequency between the radiation and the electrons.

\[
\omega_{ie} = \frac{Z^2 e^4 n \log \Lambda \sqrt{m_e}}{2 \sqrt{2 \pi} e^2 M m_p (kT)^{3/2}} = 6.5758 \times 10^{-10} \times \frac{n_i \log \Lambda Z^2}{MT_e^{3/2}} [1/s]
\]  

(72)

\[
\omega_{re} = \omega_{re}' + \omega_{re''}
\]

\[
\omega_{re}' = 8.5 \times 10^{-14} \frac{(Z^2)(Z)n_i I_g}{M T_e^{3/2} T_e} [1/s]
\]  

(73)

\[
I_g = \int_0^\infty \frac{\xi(e^\xi - 1) - (e^\xi - 1)(e^{\xi u} - 1)}{(\xi - 1)(e^{\xi u} - 1)(e^u - 1)} du
\]

\[
\omega_{re''} = \frac{128 \pi e^4 \sigma}{3 (m_e e^2)^3} T_r^4 = 7.362 \times 10^{-22} T_r^4 [1/s]
\]  

(74)

Here, \(u = \frac{h \nu}{kT_e}\), \(\xi = \frac{T_e}{T_i}\), \(h\) is Planck’s constant, and \(\nu\) is the radiation frequency.

When the basic equations are discretized, they are expresses as follows:

\[
\begin{align*}
C_{V_i} \frac{T_i^{n+1} - T_i^n}{\Delta t^{n+\frac{1}{2}}} & = -K_{ie}^{n+\frac{1}{2}} \\
C_{V_e} \frac{T_e^{n+1} - T_e^n}{\Delta t^{n+\frac{1}{2}}} & = K_{ie}^{n+\frac{1}{2}} - K_{re}^{n+\frac{1}{2}} \\
C_{V_r} \frac{T_r^{n+1} - T_r^n}{\Delta t^{n+\frac{1}{2}}} & = -K_{re}^{n+\frac{1}{2}}
\end{align*}
\]  

(75)
Here $T^*$ indicates the temperatures after the calculation of the energy equations in Subsection 4.4.

By introducing the expressions of $\xi_{ie} = T_i - T_e$, $\xi_{re} = T_e - T_r$, the energy exchange rates are expressed as follows:

\[
\begin{align*}
K_{ie}^{n+\frac{1}{2}} &= C_V i \omega_{ie}^{n+\frac{1}{2}} \xi_{ie}^{n+\frac{1}{2}} \\
K_{re}^{n+\frac{1}{2}} &= C_V e \omega_{re}^{n+\frac{1}{2}} \xi_{re}^{n+\frac{1}{2}}
\end{align*}
\]

(76)

Here $\xi_{ie}^{n+\frac{1}{2}}$ and $\xi_{re}^{n+\frac{1}{2}}$ are expressed as follows:

\[
\begin{align*}
\xi_{ie}^{n+\frac{1}{2}} &= C_i A + \left[ \xi_{ie}^{n} - \left( \frac{\alpha_i}{\gamma} \right)^{n+\frac{1}{2}} \right] B + \left( \frac{\alpha_i}{\gamma} \right)^{n+\frac{1}{2}} \\
\xi_{re}^{n+\frac{1}{2}} &= C_r A + \left[ \xi_{re}^{n} - \left( \frac{\alpha_r}{\gamma} \right)^{n+\frac{1}{2}} \right] B + \left( \frac{\alpha_r}{\gamma} \right)^{n+\frac{1}{2}}
\end{align*}
\]

(77)

(78)

Here each symbol definition is displayed below:

\[
\begin{align*}
\alpha_i &= (\phi_i + \beta_i \phi_r) \omega_{re} \\
\alpha_r &= (\phi_r G + \beta_i \phi_r) \omega_{ie} \\
\beta_i &= 1 + \frac{C_i}{C_{Vi}} \\
\beta_r &= 1 + \frac{C_r}{C_{Ve}} \\
G &= \frac{C_V e}{C_{Ve}} \\
\gamma &= (\beta_i \beta_r - G) \omega_{ie} \omega_{re} \\
A &= \left[ \exp \left( X \Delta t^{n+\frac{1}{2}} \right) - 1 \right] - \left[ \exp \left( Y \Delta t^{n+\frac{1}{2}} \right) - 1 \right] \\
B &= \left[ \exp \left( Y \Delta t^{n+\frac{1}{2}} \right) - 1 \right] \\
X &= -\frac{1}{2} \lambda + \frac{1}{2} (\lambda^2 - 4 \gamma)^{n+\frac{1}{2}} \\
Y &= -\frac{1}{2} \lambda - \frac{1}{2} (\lambda^2 - 4 \gamma)^{n+\frac{1}{2}} \\
\lambda &= \beta_i \omega_{ie} + \beta_r \omega_{re} \\
C_i &= \frac{1}{(\lambda^2 - 4 \gamma)^{n+\frac{1}{2}}} \left[ \phi_i - \beta_i \omega_{ie} \xi_{i0} + \omega_{ir} \xi_{r0} + \frac{1}{2} \lambda \left( \xi_{i0} - \frac{\alpha_i}{\gamma} \right) \right] + \frac{1}{2} \lambda \left( \xi_{i0} - \frac{\alpha_i}{\gamma} \right) \\
C_r &= \frac{1}{(\lambda^2 - 4 \gamma)^{n+\frac{1}{2}}} \left[ \phi_r - \beta_r \omega_{re} \xi_{r0} + G \omega_{ie} \xi_{i0} + \frac{1}{2} \lambda \left( \xi_{r0} - \frac{\alpha_r}{\gamma} \right) \right] + \frac{1}{2} \lambda \left( \xi_{r0} - \frac{\alpha_r}{\gamma} \right) \\
\phi_i &= \frac{W_i}{C_{Vi}} - \frac{W_e}{C_{Ve}} \\
\phi_r &= \frac{W_r}{C_{Ve}} - \frac{W_r}{C_{Ve}} \\
\end{align*}
\]
6 Heat conduction

The heat conduction is also solved to include the energy transport inside the target materials. [6].

\[ C_V \frac{DT}{Dt} = \frac{1}{\rho} \nabla \cdot (\kappa \nabla T_k) \quad (k = i, e, r) \]  
(79)

\[ \kappa_i = 4.3 \times 10^{-12} T_i^{5/2} (\log \Lambda) m^{-1/2} Z^{-4} \quad [\text{W/mK}] \]  
(80)

\[ \kappa_e = 1.83 \times 10^{-10} T_e^{5/2} (\log \Lambda)^{-1} Z^{-1} \quad [\text{W/mK}] \]  
(81)

\[ \kappa_r = \frac{16}{3} \sigma L_R T_r^3 \quad [\text{W/mK}] \]

The variables are defined as follows: \( \kappa_k \) is the thermal conductivity, \( T_k \) represents one of the temperatures for the ions, electrons and the radiation[K], \( \log \Lambda \) the Coulomb logarithm, \( m \) the atomic weight, \( Z \) the ionization degree. \( \sigma \) the Stefan-Boltzmann constant, and \( L_R \) is the Rosseland mean free path [8].

The thermal conductivity \( k_r \) of the radiation is expressed together with a flux limit approximation. The energy flux should be limited to prevent an excess energy transport by a steep temperature gradient in ICF.

\[ k_r = k_r(1 + \frac{4}{5} \frac{L_R}{T_r} \delta T_r)^{-1} \]  
(82)

The basic equation is shown again below.

\[ C_V \frac{DT}{Dt} = \frac{1}{\rho} \nabla \cdot (\kappa \nabla T) \]  
(83)

When Eq. (82) is discretized, the following equation is obtained.

\[
\begin{aligned}
\frac{T^{n+1}_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} - T^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}}{dt^{n+\frac{1}{2}}} &= \frac{1}{M_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} C_V_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}} \\
&\times \left\{ \left( \kappa^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} \frac{T^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} - T^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}}{Dx^2} \right) \\
&- \left( \kappa^n_{i, j+\frac{1}{2}, k+\frac{1}{2}} \frac{T^n_{i+\frac{1}{2}, j, k+\frac{1}{2}} - T^n_{i+\frac{1}{2}, j, k+\frac{1}{2}}}{Dy^2} \right) \\
&+ \left( \kappa^n_{i+\frac{1}{2}, j+1, k+\frac{1}{2}} \frac{T^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}} - T^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}}{Dz^2} \right) \right\} \\
&+ \left\{ \left( \kappa^n_{i, j+\frac{1}{2}, k+\frac{1}{2}} \frac{T^n_{i+\frac{1}{2}, j, k+\frac{1}{2}} - T^n_{i+\frac{1}{2}, j, k+\frac{1}{2}}}{Dy^2} \right) \\
&- \left( \kappa^n_{i+\frac{1}{2}, j, k+\frac{1}{2}} \frac{T^n_{i+\frac{1}{2}, j+1, k+\frac{1}{2}} - T^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}}}{Dz^2} \right) \right\} \\
&+ \left\{ \left( \kappa^n_{i, j+\frac{1}{2}, k+1} \frac{T^n_{i+\frac{1}{2}, j, k+\frac{1}{2}} - T^n_{i+\frac{1}{2}, j, k+\frac{1}{2}}}{Dz^2} \right) \\
&- \left( \kappa^n_{i+\frac{1}{2}, j, k+1} \frac{T^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+1} - T^n_{i+\frac{1}{2}, j+\frac{1}{2}, k+1}}{Dz^2} \right) \right\} \\
\end{aligned}
\]

The equations of the heat conduction is solved by the ADI (Alternating Directional Implicit) method [9].
7  Fusion reaction

7.1  Fusion reaction

In the document we focus mainly on the reaction of the deuterium (D) and tritium (T). Additionally, the DD reaction is considered. The reaction equations are shown below.

\[
\begin{align*}
D + T & \rightarrow \text{He}^4(3.5\text{MeV}) + n(14.1\text{MeV}) \\
D + D & \xrightarrow{50\%} T(1.01\text{MeV}) + p(3.02\text{MeV}) \\
& \xrightarrow{50\%} \text{He}^3(0.82\text{MeV}) + n(2.45\text{MeV})
\end{align*}
\]  

(84)

The number of reactions \( N_{DT} \) per unit time in the D-T reaction is represented by the following equation from equation.

\[
N_{DT} = \langle \sigma v \rangle_{DT} n_D n_T
\]  

(85)

Similarly, the number of reactions \( N_{DD} \) per unit time in the D-D reaction is expressed by the following equation.

\[
N_{DD} = \frac{1}{2} \langle \sigma v \rangle_{DD} n_D n_D
\]  

(86)

According to the formula (84), D is reduced by the D-D reaction and the D-T reaction. When Eqs. (85) and (86) are used, the amount of change in the number density \( n_D \) of D per minute time is expressed as follows.

\[
\frac{\partial n_D}{\partial t} = -N_{DD} - N_{DT}
\]  

\[
= -\frac{1}{2} \langle \sigma v \rangle_{DD} n_D n_D - \langle \sigma v \rangle_{DT} n_D n_T
\]  

(87)

Further, according to Eq. (84), T is created by the DD reaction and consumed by the DT reaction. Therefore, the number density \( n_T \) of T is expressed as follows:

\[
\frac{\partial n_T}{\partial t} = +\frac{1}{2} N_{DD} - N_{DT}
\]  

\[
= +\frac{1}{4} \langle \sigma v \rangle_{DD} n_D n_D - \langle \sigma v \rangle_{DT} n_D n_T
\]  

(88)

From Eq. (84) \( \text{He}^4 \), that is, the \( \alpha \) particle is generated by the DT reaction.

\[
\frac{\partial n_\alpha}{\partial t} = +N_{DT}
\]  

\[
= +\langle \sigma v \rangle_{DT} n_D n_T
\]  

(89)

The \( \alpha \) particles collide with the target ions and electrons during the diffusion process. Then, the diffusion of the \( \alpha \) particles and the \( \alpha \) particle energy deposition are expressed by the following equation:

\[
\frac{\partial n_\alpha}{\partial t} = +\langle \sigma v \rangle_{DT} n_D n_T - \nabla \cdot F - \omega_\alpha n_\alpha
\]  

(90)
7.2 Reaction rate

Here, the reaction rates of the DD reaction and the DT reaction are described. In this study, the fusion reaction is calculated using the analytical curves corresponding to each reaction rate[5, 10]. The formulae fitted are shown below:

\[
\langle \sigma v \rangle_{\text{DD}} = \exp \left( x_1 - \frac{x_2}{x_5 T_i} + \frac{x_3 T_i}{(T_i + x_4)^2} \right) 
\]

(91)

\[
\langle \sigma v \rangle_{\text{DT}} = \exp \left( x_1 - \frac{x_2}{x_5 T_i} + \frac{x_3}{1 + x_4} \right) 
\]

(92)

Here \( T_i \) is the ion temperature.

The coefficients \( x_n (n = 1 \sim 5) \) in Eq. (91) are listed below for the 50% of the DD reaction:

\[
x_1 = -49.1789720673151 \\
x_2 = 15.3267580380585 \\
x_3 = -4168271.58512757 \\
x_4 = 36677.9694366768 \\
x_5 = 0.365303247159742 
\]

For another 50% of the DD reactions, the coefficients \( x_n (n = 1 \sim 5) \) in Eq. (91) are listed below:

\[
x_1 = -48.9931165228571 \\
x_2 = 15.6125104498645 \\
x_3 = -4168271.58512757 \\
x_4 = 36677.9694366768 \\
x_5 = 0.363023326564475 
\]

The corresponding coefficients for the D-T reaction in Eq. (92) are shown as follows:

\[
x_1 = -48.9580509680824 \\
x_2 = 18.1155080330636 \\
x_3 = 895.149425658926 \\
x_4 = 135.88636700177 \\
x_5 = 0.366290140024939 
\]
8 $\alpha$ particle heating

8.1 $\alpha$ particle diffusion

The flux of the $\alpha$ particle is shown below $F$.

$$ F = -D_\alpha \nabla n_\alpha \quad (93) $$

Here $D_\alpha$ is the diffusion coefficient and is expressed by the following equation.

$$ D_\alpha = \frac{\frac{1}{3} v_\alpha \lambda_\alpha}{1 + \frac{4}{3} \frac{\lambda_\alpha}{\lambda_\alpha} \left| \nabla n_\alpha \right|} \quad (94) $$

Here $v_\alpha$ is the speed of $\alpha$ particle and $\lambda_\alpha$ the mean free path of $\alpha$. The second term of the denominator in Eq. (94) expresses the flux limiting effect, which limits the excess flux by the steep gradient of the $\alpha$ density. The flux $F$ of the $\alpha$ particles in the $xy$ and $z$ directions are expressed by the following equations:

$$ F_x = -\frac{\frac{1}{3} n_\alpha v_\alpha \lambda_\alpha}{n_\alpha + \frac{4}{3} \lambda_\alpha} \frac{\partial n_\alpha}{\partial x} \quad (95) $$

$$ F_y = -\frac{\frac{1}{3} n_\alpha v_\alpha \lambda_\alpha}{n_\alpha + \frac{4}{3} \lambda_\alpha} \frac{\partial n_\alpha}{\partial y} \quad (96) $$

$$ F_z = -\frac{\frac{1}{3} n_\alpha v_\alpha \lambda_\alpha}{n_\alpha + \frac{4}{3} \lambda_\alpha} \frac{\partial n_\alpha}{\partial z} \quad (97) $$

8.2 $\alpha$ particle deposition

In the fusion target plasma, the $\alpha$ particles collide with the ions and the electrons. When only the collision term is considered, it is expressed by the following equation:

$$ \frac{\partial n_\alpha}{\partial t} = -n_\alpha \omega_\alpha \quad (98) $$

During the short time interval of $dt$, we can assume that $\omega_\alpha$ is constant. Then an analytical solution is obtained.

$$ n_\alpha \propto e^{-\omega_\alpha t} \quad (99) $$

The energy deposited to the electrons and the ions are expressed by the following equation:

$$ \rho C_v \Delta T = +E_\alpha n_\alpha f \quad (100) $$

In Eq. (100), $f$ represents the partition ratio between the ions and the electrons. Here $f_i$ is the $\alpha$-particle energy deposition ratio to the ions, and $f_e$ the deposition ratio to the electrons [11].

$$ f_i = \frac{1}{1 + \frac{32}{T_e}} \quad (101) $$

$$ f_e = 1 - f_i \quad (102) $$
The energy increase by the $\alpha$ particle energy deposition is expressed by the following equation:

$$\Delta T_i = \frac{E_\alpha n_\alpha v_i}{\rho C_{v_i}}$$  \hspace{1cm} (103)

$$\Delta T_e = \frac{E_\alpha n_\alpha v_e}{\rho C_{v_e}}$$  \hspace{1cm} (104)

9 Algorithm review

Here we summarize the computation cycle.

1. Initial setup and computation preparations.

2. The time step $dt$ is controlled to avoid the numerical instability. $dt = C_{FL} \min\{dl/(V + C_s)\}$. The time step of $dt$ is evaluated at each mesh direction. The minimum $dt$ is employed. Here $V$ shows the absolute value of the plasma velocity and $C_s$ the sound speed. In plasmas and fluids shock waves may appear, and the shock speed would be larger than the sound speed. The coefficient of $C_{FL}$ should be less than 1.0. Normally we use $C_{FL} < 0.1$ to ensure the numerical stability and the numerical accuracy.

3. The artificial viscosity $q_{i+1/2,j+1/2,k+1/2}$ is obtained.

4. The velocity $u_{i+1/2,j+1/2,k+1/2}^{n+1}$, $v_{i+1/2,j+1/2,k+1/2}^{n+1}$, and $w_{i+1/2,j+1/2,k+1/2}^{n+1}$ are obtained by equation of motion.

5. The mass density $\rho_{i+1/2,j+1/2,k+1/2}^{n+1}$ is obtained by equation of continuity.

6. The temperature $T_{i+1/2,j+1/2,k+1/2}^{n+1}$ is obtained by energy equation.

7. The pressure $p_{i+1/2,j+1/2,k+1/2}^{n+1}$, the specific heat $C_{V,i+1/2,j+1/2,k+1/2}^{n+1}$, and the compressibility $B_{i+1/2,j+1/2,k+1/2}^{n+1}$ are obtained by equation of state.

8. The $\alpha$ particles are created by the DT fusion reactions, and the $\alpha$ particles are diffused. The new temperature $T_{i+1/2,j+1/2,k+1/2}^{n+1}$ is obtained by the $\alpha$ particle energy deposition.

9. The new temperature $T_{i+1/2,j+1/2,k+1/2}^{n+1}$ is obtained by the temperature relaxation among the three temperatures. In addition, the electron energy and radiation energy are conducted.

10 Summary

In this document we described a numerical algorithm for a 3D Euler fluid code with a uniform spatial mesh to simulate the nuclear fusion fuel ignition and burning. At the fusion fuel stagnation, ignition and burning phases, the fusion fuel spatial deformation would be serious. In order to avoid the spatial mesh crush, we use the Euler method in these phases, though the implosion phase can be simulated by the Lagrange method [5, 12] or the ALE (Arbitrary Lagrangian Eulerian) method [13].
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