ARGUMENT SHIFT METHOD AND GAUDIN MODEL

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Abstract. We construct a family of maximal commutative subalgebras in the tensor product of $n$ copies of the universal enveloping algebra $U(g)$ of a semisimple Lie algebra $g$. This family is parameterized by collections $\mu, z_1, \ldots, z_n$, where $\mu \in g^*$, and $z_1, \ldots, z_n$ are pairwise distinct complex numbers. The construction presented here generalizes the famous construction of the higher Gaudin hamiltonians due to Feigin, Frenkel, and Reshetikhin. For $n = 1$, our construction gives a quantization of the family of maximal Poisson-commutative subalgebras of $S(g)$ obtained by the argument shift method. Next, we describe natural representations of commutative algebras of our family in tensor products of finite-dimensional $g$-modules as certain degenerations of the Gaudin model. In the case of $g = sl_r$ we prove that our commutative subalgebras have simple spectrum in tensor products of finite-dimensional $g$-modules for generic $\mu$ and $z_i$. This implies simplicity of spectrum in the "generic" $sl_r$ Gaudin model.

1. Introduction

Let $g$ be a semisimple complex Lie algebra, and $U(g)$ its universal enveloping algebra. The algebra $U(g)$ bears a natural filtration by the degree with respect to the generators. The associated graded algebra is the symmetric algebra $S(g) = \mathbb{C}[g^*]$ by the Poincaré–Birkhoff–Witt theorem. The commutator on $U(g)$ defines the Poisson–Lie bracket on $S(g)$.

The argument shift method gives a way to construct subalgebras in $S(g)$ commutative with respect to the Poisson–Lie bracket. The method is as follows. Let $ZS(g) = S(g)^{g^*}$ be the center of $S(g)$ with respect to the Poisson bracket, and let $\mu \in g^*$ be a regular semisimple element. Then the algebra $A_\mu \subset S(g)$ generated by the elements $\partial_\mu^\alpha \Phi$, where $\Phi \in ZS(g)$, (or, equivalently, generated by central elements of $S(g) = \mathbb{C}[g^*]$ shifted by $t\mu$ for all $t \in \mathbb{C}$) is commutative with respect to the Poisson bracket, and has maximal possible transcendence degree equal to $\frac{1}{2}(\dim g + \text{rk } g)$ (see [MF]). Moreover, the subalgebras $A_\mu$ are maximal subalgebras in $S(g)$ commutative with respect to the Poisson–Lie bracket [Tar2]. In [Vin], the subalgebras $A_\mu \subset S(g)$ are named the Mischenko–Fomenko subalgebras.

In the present paper we lift the subalgebras $A_\mu \subset S(g)$ to commutative subalgebras in the universal enveloping algebra $U(g)$. More precisely, for any semisimple Lie algebra $g$, we construct a family of commutative subalgebras $A_\mu \subset U(g)$ parameterized by regular semisimple $\mu \in g^*$, so that $\text{gr } A_\mu = A_\mu$. For classical Lie algebras $g$, it was done (by other methods) by Olshanski and Nazarov (see [NO, Mol]), and also by Tarasov in the case $g = sl_r$ [Tar1].

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The construction presented here is a modification of the famous construction of the higher Gaudin Hamiltonians (see [FFR]). Gaudin model was introduced in [G1] as a spin model related to the Lie algebra $sl_2$, and generalized to the case of an arbitrary semisimple Lie algebra in [G], 13.2.2. The generalized Gaudin model has the following algebraic interpretation. Let $V_\lambda$ be an irreducible representation of $\mathfrak{g}$ with the highest weight $\lambda$. For any collection of integral dominant weights $(\lambda) = \lambda_1, \ldots, \lambda_n$, let $V_\lambda = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$. For any $x \in \mathfrak{g}$, consider the operator $x^{(i)} = 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1$ ($x$ stands at the $i$th place), acting on the space $V_\lambda$. Let $\{x_a\}$, $a = 1, \ldots, \dim \mathfrak{g}$, be an orthonormal basis of $\mathfrak{g}$ with respect to Killing form, and let $z_1, \ldots, z_n$ be pairwise distinct complex numbers. The Hamiltonians of Gaudin model are the following commuting operators acting in the space $V_\lambda$:

\begin{equation}
H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x^{(i)}_a x^{(k)}_a}{z_i - z_k}.
\end{equation}

We can regard the $H_i$ as elements of $U(\mathfrak{g})^\otimes n$. In [FFR], a large commutative subalgebra $A(z_1, \ldots, z_n) \subset U(\mathfrak{g})^\otimes n$ containing $H_i$ was constructed. For $\mathfrak{g} = sl_2$, the algebra $A(z_1, \ldots, z_n)$ is generated by $H_i$ and the central elements of $U(\mathfrak{g})^\otimes n$. In other cases, the algebra $A(z_1, \ldots, z_n)$ has also some new generators known as higher Gaudin Hamiltonians. The construction of $A(z_1, \ldots, z_n)$ uses the quite nontrivial fact [FF] that the completed universal enveloping algebra of the affine Kac–Moody algebra $\hat{\mathfrak{g}}$ at the critical level has a large center $Z(\hat{\mathfrak{g}})$. To any collection $z_1, \ldots, z_n$ of pairwise distinct complex numbers, one can naturally assign a homomorphism $Z(\hat{\mathfrak{g}}) \to U(\mathfrak{g})^\otimes n$. The image of this homomorphism is $A(z_1, \ldots, z_n)$.

In the present paper we construct a family of homomorphisms $Z(\hat{\mathfrak{g}}) \to U(\mathfrak{g})^\otimes n \otimes S(\mathfrak{g})$ parameterized by collections $z_1, \ldots, z_n$ of pairwise distinct complex numbers. For any collection $z_1, \ldots, z_n$, the image of such homomorphism is a certain commutative subalgebra $A(z_1, \ldots, z_n, \infty) \subset U(\mathfrak{g})^\otimes n \otimes S(\mathfrak{g})$. Taking value at any point $\mu \in \mathfrak{g}^* = \text{Spec } S(\mathfrak{g})$, we obtain a commutative subalgebra $A_\mu(z_1, \ldots, z_n) \subset U(\mathfrak{g})^\otimes n$ depending on $z_1, \ldots, z_n$ and $\mu \in \mathfrak{g}^*$. For $n = 1$, we obtain commutative subalgebras $A_\mu(z_1) = A_\mu \subset U(\mathfrak{g})$ which do not depend on $z_1$. We show that $\text{gr } A_\mu = A_\mu$ for regular semisimple $\mu$, i.e., the subalgebras $A_\mu \subset U(\mathfrak{g})$ are liftings of Mischenko–Fomenko subalgebras (in the case of $\mathfrak{g} = sl_r$ this can be deduced from Talalaev’s formula for higher Gaudin Hamiltonians, cf. [Tal, ChT]). For $\mu = 0$, we have $A_0(z_1, \ldots, z_n) = A(z_1, \ldots, z_n)$, i.e., the subalgebras $A_0(z_1, \ldots, z_n) \subset U(\mathfrak{g})^\otimes n$ are generated by (higher) Gaudin Hamiltonians. We show that the subalgebras $A_\mu(z_1, \ldots, z_n)$ for generic $z_1, \ldots, z_n$ and $\mu$ have maximal possible transcendence degree. These subalgebras contain the following ”non-homogeneous Gaudin Hamiltonians”:

\begin{equation}
H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x^{(i)}_a x^{(k)}_a}{z_i - z_k} + \sum_{a=1}^{\dim \mathfrak{g}} \mu(x_a) x^{(i)}_a.
\end{equation}

The main problem in Gaudin model is the problem of simultaneous diagonalization of (higher) Gaudin Hamiltonians. The bibliography on this problem is enormous (cf. [Fr1, Fr2, FFR, MV]). It follows from the [FFR] construction that
all elements of $A(z_1, \ldots, z_n) \subset U(\mathfrak{g})^{\otimes n}$ are invariant with respect to the diagonal action of $\mathfrak{g}$, and therefore it is sufficient to diagonalize the algebra $A(z_1, \ldots, z_n)$ in the subspace $V_{(\lambda)}^{\text{sing}} \subset V(\lambda)$ of singular vectors with respect to $\text{diag}_n(\mathfrak{g})$ (i.e., with respect to the diagonal action of $\mathfrak{g}$). The standard conjecture says that generic $z_i$ the algebra $A(z_1, \ldots, z_n)$ has simple spectrum in $V_{(\lambda)}^{\text{sing}}$. This conjecture is proved in [MV] for $\mathfrak{g} = \mathfrak{sl}_r$ and $\lambda_i$ equal to $\omega_1$ or $\omega_{r-1}$ (i.e., for the case when every $V_{\lambda_i}$ is the standard representation of $\mathfrak{sl}_r$ or its dual) and in [SV] for $\mathfrak{g} = \mathfrak{sl}_2$ and arbitrary $\lambda_i$.

It is also natural to set up a problem of diagonalization of $A_{\mu}(z_1, \ldots, z_n)$ in the space $V(\lambda)$. We show that the representation of the algebra $A_{\mu}(z_1, \ldots, z_n)$ in the space $V(\lambda)$ is a limit of the representations of $A(z_1, \ldots, z_{n+1})$ in $[V(\lambda) \otimes M^*_{z_{n+1}}]^\text{sing}$ as $z_{n+1} \to \infty$. Here $M^*_{z_{n+1}}$ is the contragredient module of the Verma module with highest weight $z_{n+1}\mu$, and the space $[V(\lambda) \otimes M^*_{z_{n+1}}]^\text{sing}$ consists of all singular vectors in $V(\lambda) \otimes M^*_{z_{n+1}}$ with respect to $\text{diag}_{n+1}(\mathfrak{g})$. This means that the representation of $A_{\mu}(z_1, \ldots, z_n)$ in $V(\lambda)$ is in some sense a limit case of Gaudin model.

We prove the conjecture on the simplicity of the spectrum for the representation of $A_{\mu}(z_1, \ldots, z_n)$ in the space $V(\lambda)$ for $\mathfrak{g} = \mathfrak{sl}_r$. The point of our proof is the fact that the closure of the family $A_{\mu}$ contains the Gelfand–Tsetlin subalgebra (on the level of Poisson algebras, this fact was proved by Vinberg [Vin]). Hence, for $\mathfrak{g} = \mathfrak{sl}_r$, we conclude that the algebra $A_{\mu}(z_1, \ldots, z_n)$ for generic $\mu$ and $z_1, \ldots, z_n$ has simple spectrum in $V(\lambda)$ for any $V(\lambda)$. As a consequence, we obtain that the spectrum of the algebra $A_{0}(z_1, \ldots, z_n)$ in $V(\lambda)^{\text{sing}}$ is simple for generic $z_i$ and $\lambda$.

The paper is organized as follows. In sections 2 and 3 we collect some well-known facts on Mischenko–Fomenko subalgebras and the center $Z(\hat{\mathfrak{g}})$ at the critical level, respectively. In section 4 we describe the construction of the subalgebras $A_{\mu}$ and prove that $\text{gr}A_{\mu} = A_{\mu}$. In section 5 we describe the general construction of the subalgebras $A_{\mu}(z_1, \ldots, z_n) \subset U(\mathfrak{g})^{\otimes n}$ and prove that these subalgebras have the maximal possible transcendence degree. In section 6 we describe the representation of $A_{\mu}(z_1, \ldots, z_n)$ in $V(\lambda)$ as a “limit” Gaudin model. And, finally, in section 7 we prove the assertions on simplicity of spectrum for $\mathfrak{g} = \mathfrak{sl}_r$.

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2. ARGUMENT SHIFT METHOD

Argument shift method is a particular case of the famous Magri–Lenart construction [Ma]. Let $R$ be a commutative algebra equipped with two compatible Poisson brackets, $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$, (i.e., any linear combination of $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ is a Poisson bracket). Let $Z_t$ be the Poisson center of $R$ with respect to $\{\cdot, \cdot\}_1 + t\{\cdot, \cdot\}_2$. Let $A$ be the subalgebra of $R$ generated by all $Z_t$ for generic $t$.

**Fact 1.** (cf. [BB], Proposition 4) The subalgebra $A \subset R$ is commutative with respect to any Poisson bracket $\{\cdot, \cdot\}_1 + t\{\cdot, \cdot\}_2$.

**Proof.** Suppose $a \in Z_{t_1}$, $b \in Z_{t_2}$ with $t_1 \neq t_2$. The expression $\{a, b\}_1 + t\{a, b\}_2$ is linear in $t$, and, on the other hand, it vanishes at two distinct points, $t_1$ and $t_2$. This means that $\{a, b\}_1 + t\{a, b\}_2 = 0$ for all $t$. 

Now suppose \( a, b \in \mathbb{Z}_{t_0} \). Since \( t_0 \) is generic, there exists a continuous function \( a(s) \) such that \( a(t_0) = a \), and for \( s \) in a certain neighborhood of \( t_0 \) we have \( a(s) \in \mathbb{Z} \). For any \( s \) in a punctured neighborhood of \( t_0 \) we have \( \{a(s), b\}_{1} + t\{a(s), b\}_{2} = 0 \), and therefore \( \{a, b\}_{1} + t\{a, b\}_{2} = 0 \).

**Corollary 1.** Suppose that \( ZS(\mathfrak{g}) = S(\mathfrak{g})^0 \) is the center of \( S(\mathfrak{g}) \) with respect to the Poisson bracket, and let \( \mu \in \mathfrak{g}^* \). Then the algebra \( A_\mu \subset S(\mathfrak{g}) \) generated by the elements \( \partial_\mu^q \Phi \), where \( \Phi \in ZS(\mathfrak{g}) \), (or, equivalently, generated by central elements of \( S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*] \) shifted by \( t\mu \) for all \( t \in \mathbb{C} \)) is commutative with respect to the Poisson bracket.

**Proof.** Take the Poisson–Lie bracket as \( \{\cdot, \cdot\}_1 \), and the "frozen argument" bracket as \( \{\cdot, \cdot\}_2 \); this means that for the generators we have

\[
\{x, y\}_2 = \mu([x, y]) \quad x, y \in \mathfrak{g}.
\]

Then the algebra \( Z_t \) is generated by central elements of \( \mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g}) \) shifted by \( t\mu \).

Since the Lie algebra \( \mathfrak{g} \) is semisimple we can identify \( \mathfrak{g} \) with \( \mathfrak{g}^* \) and write \( \mu \in \mathfrak{g} \).

**Fact 2.** [MF] For regular semisimple \( \mu \in \mathfrak{g} \) the algebra \( A_\mu \) is a free commutative subalgebra in \( S(\mathfrak{g}) \) with \( \frac{1}{2}(\dim \mathfrak{g} + \text{rk} \mathfrak{g}) \) generators (this means that \( A_\mu \) is a commutative subalgebra of maximal possible transcendence degree). One can take the elements \( \partial_\mu^n \Phi_k \), \( k = 1, \ldots, \text{rk} \mathfrak{g}, n = 0, 1, \ldots, \deg \Phi_k \), where \( \Phi_k \) are basic \( \mathfrak{g} \)-invariants in \( S(\mathfrak{g}) \), as free generators of \( A_\mu \).

In [Sh] Shuvalov described the closure of the family of subalgebras \( A_\mu \subset S(\mathfrak{g}) \) under the condition \( \mu \in \mathfrak{h}^{\text{reg}} \) (i.e., for regular \( \mu \) in the fixed Cartan subalgebra). In particular, the following assertion is proved in [Sh].

**Fact 3.** Suppose that \( \mu(t) = \mu_0 + t\mu_1 + t^2\mu_2 + \cdots \in \mathfrak{h}^{\text{reg}} \) for generic \( t \). Set

\[
\mathfrak{z}_k = \bigcap_{i=0}^{k} \mathfrak{z}_i(\mu_i) \quad \text{(where} \quad \mathfrak{z}_i(\mu_i) \quad \text{is the centralizer of} \quad \mu_i \quad \text{in} \quad \mathfrak{g}, \quad \mathfrak{z}_{-1} = \mathfrak{g}. \quad \text{Then we have}
\]

1. the subalgebra \( \lim_{t \to 0} A_{\mu(t)} \subset S(\mathfrak{g}) \) is generated by all elements of \( S(\mathfrak{z}_k) \) and their derivatives (of any order) along \( \mu_{k+1} \) for all \( k \).
2. \( \lim_{t \to 0} A_{\mu(t)} \) is a free commutative algebra.

This means, in particular, that the closure of the family \( A_\mu \) for \( \mathfrak{g} = sl_r \) contains the Gelfand–Tsetlin algebra (see [Vin], 6.1–6.4). We shall discuss this case in section 7.

The following results were obtained by Tarasov.

**Fact 4.** [Tar2] The subalgebras \( A_\mu \) and the limit subalgebras of the type \( \lim_{t \to 0} A_{\mu(t)} \) are maximal commutative subalgebras, i.e., they coincide with their Poisson centralizers in \( S(\mathfrak{g}) \).

The symmetrization map \( \sigma : S(\mathfrak{g}) \to U(\mathfrak{g}) \) is defined by the following property:

\[
\sigma(x^k) = x^k \quad \forall x \in \mathfrak{g}, \quad k = 0, 1, 2, \ldots
\]
Fact 5. [Tar1] [Tar3] For \( g = sl_r \), a certain system of generators of \( A_\mu \) and of the limit subalgebras of the type \( \lim_{t \to 0} A_\mu(t) \) can be lifted to commuting elements of \( U(g) \) by the symmetrization map. This gives rise to a unique lifting of \( A_\mu \) to the universal enveloping algebra.

Remark. The system of generators of \( A_\mu \) of the limit subalgebras of the type \( \lim_{t \to 0} A_\mu(t) \) to be lifted by the symmetrization is chosen explicitly in [Tar1]. It is, up to proportionality, the system of the elements \( \partial^n \Phi_k \), \( k = 1, \ldots, r - 1 \), \( n = 0, 1, \ldots, \deg \Phi_k \), (where \( \Phi_k \in S(sl_r)^{sl_r} \) are the coefficients of the characteristic polynomial as functions on \( sl_r \)) and their limits, respectively. We shall only use that this system of generators up to proportionality is continuous in the parameter \( \mu \).

3. Center at the critical level

Let \( \hat{g} \) be the affine Kac–Moody algebra corresponding to \( g \). The Lie algebra \( \hat{g} \) is a central extension of the formal loop algebra \( g((t)) \) by an element \( K \). The commutator relations are defined as follows:

\[
[g_1 \otimes x(t), g_2 \otimes y(t)] = [g_1, g_2] \otimes x(t)y(t) + \kappa_c(g_1, g_2) \text{Res}_{t=0} x(t)dy(t) \cdot K,
\]

where \( \kappa_c \) is the invariant scalar product on \( g \) defined by the formula

\[
\kappa_c(g_1, g_2) = -\frac{1}{2} \text{Tr}_g \text{ad}(g_1) \text{ad}(g_2).
\]

Set \( \hat{g}_+ = g[[t]] \subset \hat{g} \) and \( \hat{g}_- = t^{-1}g[[t^{-1}]] \subset \hat{g} \).

Define the completion \( \hat{U}(\hat{g}) \) of \( U(\hat{g}) \) as the inverse limit of \( U(\hat{g})/U(\hat{g})(t^n g[[t]]) \), \( n > 0 \). The action of \( \hat{U}(\hat{g}) \) is well-defined on \( \hat{g} \)-modules from the category \( O^0 \) (i.e., \( \hat{g} \)-modules on which the Lie subalgebra \( \hat{g}_+ \) acts locally finitely). We set \( \hat{U}(\hat{g})_c = \hat{U}(\hat{g})/(K - 1) \). This algebra acts on \( \hat{g} \)-modules of the critical level (i.e., \( \hat{g} \)-modules on which the element \( K \) acts as unity). The name "critical" is explained by the fact that the representation theory at this level is most complicated. In particular, the algebra \( \hat{U}(\hat{g})_c \) has a non-trivial center \( Z(\hat{g}) \). The following fact shows that this center is rather large.

Fact 6. [FF] [Fr2]

1. The natural homomorphism \( Z(\hat{g}) \to (U(\hat{g})/U(\hat{g})(\hat{g}_+ + \mathbb{C}(K - 1)))^{\hat{g}_+} \) is surjective.

2. The Poincaré–Birkhoff–Witt filtration on the enveloping algebra yields a filtration on the \( \hat{g}_+ \)-module \( U(\hat{g})/U(\hat{g})(\hat{g}_+ + \mathbb{C}(K - 1)) \). We have \( \text{gr}(U(\hat{g})/U(\hat{g})(\hat{g}_+ + \mathbb{C}(K - 1)))^{\hat{g}_+} = (S(\hat{g})/S(\hat{g})(\hat{g}_+ + \mathbb{C}K))^{\hat{g}_+} \) with respect to this filtration.

Now let us give an explicit description of the algebra \( (S(\hat{g})/S(\hat{g})(\hat{g}_+ + \mathbb{C}K))^{\hat{g}_+} \). Since \( \hat{g} = \hat{g}_+ \oplus \hat{g}_- \oplus \mathbb{C}K \) as vector spaces, every element of \( U(\hat{g})/U(\hat{g})(\hat{g}_+ + \mathbb{C}(K - 1)) \) (respectively, \( S(\hat{g})/S(\hat{g})(\hat{g}_+ + \mathbb{C}K) \)) has a unique representative in \( U(\hat{g}_-) \) (respectively, in \( S(\hat{g}_-) \)). Thus we obtain the following natural embeddings

\[
(U(\hat{g})/U(\hat{g})(\hat{g}_+ + \mathbb{C}(K - 1)))^{\hat{g}_+} \hookrightarrow U(\hat{g}_-)
\]
and
\( (S(\hat{g})/S(\hat{g})(\hat{g}_+ + \mathbb{C}K))^{\hat{g}_+} \hookrightarrow S(\hat{g}_-) ) \).

Let \( \mathcal{A} \subset U(\hat{g}_-) \) and \( A \subset S(\hat{g}_-) \) be the images of these embeddings, respectively. Consider the following derivations of the Lie algebra \( \hat{g}_- \):

\[ \partial_t (g \otimes t^m) = mg \otimes t^{m-1} \quad \forall g \in \hat{g}, m = -1, -2, \ldots \]

\[ t \partial_t (g \otimes t^m) = mg \otimes t^m \quad \forall g \in \hat{g}, m = -1, -2, \ldots \]

The derivations (6), (7) extend to the derivations of the associative algebras \( S(\hat{g}_-) \) and \( U(\hat{g}_-) \). The derivation (7) induce a grading of these algebras.

Let \( i_{-1} : S(\hat{g}) \hookrightarrow S(\hat{g}_-) \) be the embedding, which maps \( g \in \hat{g} \) to \( g \otimes t^{-1} \). Let \( \Phi_k, k = 1, \ldots, \text{rk} \hat{g} \) be the generators of the algebra of invariants \( S(\hat{g})^g \).

**Fact 7.** [BD, Fr2, Mu] The subalgebra \( A \subset S(\hat{g}_-) \) is freely generated by the elements \( \partial_t^n \overline{S}_k, k = 1, \ldots, \text{rk} \hat{g}, n = 0, 1, 2, \ldots \), where \( \overline{S}_k = i_{-1}(\Phi_k) \).

It follows from the Fact 6 that the generators \( \overline{S}_k \) can be lifted to the (commuting) generators of \( A \). This means that we have

**Corollary 2.**

1. There exist the homogeneous with respect to \( t \partial_t \) elements \( S_k \in \mathcal{A} \) such that \( \text{gr} S_k = \overline{S}_k \).
2. \( \mathcal{A} \) is a free commutative algebra generated by \( \partial_t^n S_k, k = 1, \ldots, \text{rk} \hat{g}, n = 0, 1, 2, \ldots \).

In the further consideration we use only the existence of the commutative subalgebra \( A \subset U(\hat{g}_-) \) and its description from the Corollary 2.

**Remark.** No general explicit formulas for the elements \( S_k \) are known at the moment. For the quadratic Casimir element \( \Phi_1 \), the corresponding element \( S_1 \in \mathcal{A} \) is obtained from \( \overline{S}_1 = i_{-1}(\Phi_1) \) by the symmetrization map. For \( \hat{g} = sl_r \) explicit formulas for \( S_k \) were obtained by Talalaev in [ChT, Tal].

**Remark.** The construction of the higher Gaudin hamiltonians is as follows. The commutative subalgebra \( \mathcal{A}(z_1, \ldots, z_n) \subset U(\hat{g})^{\otimes n} \) is the image of the subalgebra \( A \subset U(\hat{g}_-) \) under the homomorphism \( U(\hat{g}_-) \rightarrow U(\hat{g})^{\otimes n} \) of specialization at the points \( z_1, \ldots, z_n \) (see [FPR, ER]). We discuss this in Section 5.

4. **Maximal commutative subalgebras in \( U(\hat{g}) \)**

For any \( z \neq 0 \), we have an evaluation homomorphism

\[ \varphi_z : U(\hat{g}_-) \rightarrow U(\hat{g}), \quad g \otimes t^m \mapsto z^m g. \]

Furthermore, there is a homomorphism

\[ \varphi_\infty : U(\hat{g}_-) \rightarrow S(\hat{g}), \quad g \otimes t^{-1} \mapsto g, \quad g \otimes t^m \mapsto 0, \quad m = -2, -3, \ldots \]

Let \( \Delta : U(\hat{g}_-) \rightarrow U(\hat{g}_-) \otimes U(\hat{g}_-) \) be the comultiplication. For any \( z \neq 0 \), we have the following homomorphism:

\[ \varphi_{z, \infty} = (\varphi_z \otimes \varphi_\infty) \circ \Delta : U(\hat{g}_-) \rightarrow U(\hat{g}) \otimes S(\hat{g}). \]

More explicitly,

\[ \varphi_{z, \infty}(g \otimes t^m) = z^m g \otimes 1 + \delta_{-1,m} \otimes g. \]
We set
\[ A(z, \infty) = \varphi_{z, \infty}(A) \subset U(\mathfrak{g}) \otimes S(\mathfrak{g}) \]

**Proposition 1.** The subalgebra \( A(z, \infty) \) is generated by the coefficients of the principal part of the Laurent series for the functions \( S_k(w) = \varphi_{w-z, \infty}(S_k) \) about \( z \) and by the values of these functions at \( \infty \).

**Proof.** Indeed, \( A(z, \infty) \) is generated by the elements \( \varphi_{z, \infty}(\partial_t^n S_k) \). These elements are Taylor coefficients of \( S_k(w) = \varphi_{w-z, \infty}(S_k) \) about \( w = 0 \). Since \( S_k(w) \) has a unique pole at \( z \), the Taylor coefficients of \( S_k(w) \) about \( w = 0 \) are linear expressions in the coefficients of the principal part of the Laurent series for the same function about \( z \) and its value at \( \infty \), and vice versa. \( \square \)

**Corollary 3.** The subalgebra \( A(z, \infty) \subset U(\mathfrak{g}) \otimes S(\mathfrak{g}) \) does not depend on \( z \).

**Proof.** Indeed, the Laurent coefficients of the functions \( S_k(w) = \varphi_{w-z, \infty}(S_k) \) about the point \( z \) and the values of these functions at \( \infty \) do not depend on \( z \). \( \square \)

Every \( \mu \in \mathfrak{g}^* \) defines the homomorphism of "specialization at the point \( \mu \)" \( S(\mathfrak{g}) \to \mathbb{C} \). We denote this homomorphism also by \( \mu \). Consider the following family of commutative subalgebras of \( U(\mathfrak{g}) \), which is parameterized by \( \mu \in \mathfrak{g}^* \):

\[ A_\mu := (id \otimes \mu)(A(z, \infty)) \subset U(\mathfrak{g}). \]

**Proposition 2.** All elements of the subalgebra \( A_\mu \subset U(\mathfrak{g}) \) are \( \mathfrak{z}_\mathfrak{g}(\mu) \)-invariant (where \( \mathfrak{z}_\mathfrak{g}(\mu) \) is the centralizer of \( \mu \) in \( \mathfrak{g} \)).

**Proof.** Indeed, we have \( A(z, \infty) \subset [U(\mathfrak{g}) \otimes S(\mathfrak{g})]^\Delta(\mathfrak{g}) \), and the homomorphism \( \mu \) is \( \mathfrak{z}_\mathfrak{g}(\mu) \)-equivariant. Therefore \( A_\mu \subset U(\mathfrak{g})^\mathfrak{z}_\mathfrak{g}(\mu) \). \( \square \)

Now let us prove that the subalgebras \( A_\mu \subset U(\mathfrak{g}) \) give a quantization of the Mischenko–Fomenko subalgebras in \( S(\mathfrak{g}) \) obtained by the argument shift method.

**Theorem 1.** \( \text{gr } A_\mu = A_\mu \) for regular semisimple \( \mu \in \mathfrak{g}^* \).

**Proof.** Let \( E \) be a \( \mathfrak{g} \)-invariant derivation of \( U(\hat{\mathfrak{g}}_-) \otimes S(\mathfrak{g}) \) acting on the generators as follows:

\[ (12) \quad E((g \otimes x(t)) \otimes 1) = 1 \otimes g \text{Res}_{t=0} x(t) dt, \quad E(1 \otimes g) = 0 \quad \forall \ g \in \mathfrak{g}. \]

In other words, we have

\[ (g \otimes t^{-m}) \otimes 1 \mapsto \delta_{-1,m} \otimes g, \quad 1 \otimes g \mapsto 0 \quad \forall \ g \in \mathfrak{g}. \]

**Lemma 1.** The subalgebra \( A(z, \infty) \subset U(\mathfrak{g}) \otimes S(\mathfrak{g}) \) is generated by the elements

\[ (\varphi_z \otimes \text{id})(E^j(S_k \otimes 1)) \in U(\mathfrak{g}) \otimes S^j(\mathfrak{g}). \]

**Proof.** Note that

\[ (\text{id} \otimes \varphi_\infty \circ \Delta)(\partial_t^n S_k) = (\exp E)(\partial_t^n S_k) \in U(\hat{\mathfrak{g}}_-) \otimes S(\mathfrak{g}). \]

Since the elements \( S_k \) are homogeneous with respect to \( t \partial_t \), the mentioned elements \( (\varphi_z \otimes \text{id})(E^j(S_k \otimes 1)) \) are Laurent coefficients of the function \( S_k(w) = \varphi_{w-z, \infty}(S_k) = \varphi_{w-z}(\exp E)(S_k \otimes 1) \) about the point \( w = z \). Now it remains to take advantage of Proposition \( \square \)
Now let $e$ be a $\mathfrak{g}$-invariant derivation of $S(\mathfrak{g}) \otimes S(\mathfrak{g})$ acting on the generators as follows:

$$
ed(g \otimes 1) = 1 \otimes g, \quad e(1 \otimes g) = 0.$$  

Clearly, for any $f \in S(\mathfrak{g})$ we have $(\text{id} \otimes \mu) \circ e^j(f \otimes 1) = \partial^j_i f$.

Now let us note that

$$\text{gr}(\varphi_z \otimes \text{id})(E^j(S_k \otimes 1)) = z^{(-\deg \Phi_k + j)}e^j(\Phi_k \otimes 1) \in S(\mathfrak{g}) \otimes S^j(\mathfrak{g}),$$

since $\text{gr} S_k = i_{-1}(\Phi_k)$. Hence we have

$$\text{gr}(\text{id} \otimes \mu) \circ (\varphi_z \otimes \text{id})(E^j(S_k \otimes 1)) = z^{(-\deg \Phi_k + j)} \partial^j_i(\Phi_k).$$

Since the elements $\partial^j_i(\Phi_k)$ generate $A_\mu$, we have $\text{gr} A_\mu \supset A_\mu$. The elements $\partial^j_i(\Phi_k)$ are algebraically independent by Fact 2 and the Lemma says that the elements $(\text{id} \otimes \mu) \circ (\varphi_z \otimes \text{id})(E^j(S_k \otimes 1))$ generate $A_\mu$. Thus $\text{gr} A_\mu = A_\mu$. \hfill \Box

5. Commutative subalgebras in $U(\mathfrak{g})^\otimes^n$

Now let us generalize our construction. Let $U(\mathfrak{g})^\otimes^n$ be the tensor product of $n$ copies of $U(\mathfrak{g})$. We denote the subspace $1 \otimes \cdots \otimes 1 \otimes \mathfrak{g} \otimes 1 \otimes \cdots \otimes 1 \subset U(\mathfrak{g})^\otimes^n$, where $\mathfrak{g}$ stands at the $r$th place, by $\mathfrak{g}^{(i)}$. Respectively, for any $u \in U(\mathfrak{g})$ we set

$$u^{(i)} = 1 \otimes \cdots \otimes 1 \otimes u \otimes 1 \otimes \cdots \otimes 1 \in U(\mathfrak{g})^\otimes^n.$$  

Let $\text{diag}_n : U(\mathfrak{g}^\cdots) \hookrightarrow U(\mathfrak{g}^\cdots)^\otimes^n$ be the diagonal embedding. For any collection of pairwise distinct complex numbers $z_i, i = 1, \ldots, n$, we have the following homomorphism:

$$\varphi_{z_1, \ldots, z_n, \infty} = (\varphi_{z_1} \otimes \cdots \otimes \varphi_{z_n} \otimes \varphi_{\infty}) \circ \text{diag}_{n+1} : U(\mathfrak{g}^\cdots) \rightarrow U(\mathfrak{g}^\otimes^n) \otimes S(\mathfrak{g}).$$

More explicitly, we have

$$\varphi_{z_1, \ldots, z_n, \infty}(g \otimes t^m) = \sum_{i=1}^n z_i^m g^{(i)} \otimes 1 + \delta_{-1,m} \otimes g.$$  

Set

$$A(z_1, \ldots, z_n, \infty) = \varphi_{z_1, \ldots, z_n, \infty}(A) \subset U(\mathfrak{g})^\otimes^n \otimes S(\mathfrak{g}).$$

The following assertion is proved in the same way as Proposition 1 and Corollary 2.

**Proposition 3.**  
(1) The subalgebras $A(z_1, \ldots, z_n, \infty)$ are generated by the coefficients of the principal parts of the Laurent series of the functions

$$S_k(w; z_1, \ldots, z_n) = \varphi_{w-z_1, \ldots, w-z_n, \infty}(S_k)$$

at the points $z_1, \ldots, z_n$, and by their values at $\infty$.

(2) The subalgebras $A(z_1, \ldots, z_n, \infty)$ are stable under simultaneous affine transformations of the parameters $z_i \mapsto a_z + b$.

(3) All the elements of $A(z_1, \ldots, z_n, \infty)$ are invariant with respect to the diagonal action of $\mathfrak{g}$. 

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Let us consider the following family of commutative subalgebras in $U(\mathfrak{g})^\otimes n$, which is parameterized by $z_1, \ldots, z_n \in \mathbb{C}$ and $\mu \in \mathfrak{g}^*$:

$$A_\mu(z_1, \ldots, z_n) := (\text{id} \otimes \mu)(\mathcal{A}(z_1, \ldots, z_n, \infty)) \subset U(\mathfrak{g})^\otimes n.\quad(16)$$

Directly from Proposition 3, we obtain

**Proposition 4.** (1) The subalgebras $A_\mu(z_1, \ldots, z_n)$ are stable under simultaneous translations of the parameters $z_i \mapsto z_i + b$.

(2) All the elements of $A_\mu(z_1, \ldots, z_n)$ are invariant with respect to the diagonal action of $\mathfrak{h}(\mu)$.

**Remark.** The subalgebra $A_0(z_1, \ldots, z_n) \subset U(\mathfrak{g})^\otimes n$ can be obtained as the image of the subalgebra $A \subset U(\hat{\mathfrak{g}}_-)$ under the homomorphism

$$\varphi_{z_1, \ldots, z_n} = (\varphi_{z_1} \otimes \cdots \otimes \varphi_{z_n}) \circ \text{diag}_n : U(\hat{\mathfrak{g}}_-) \to U(\mathfrak{g})^\otimes n.$$ 

These subalgebras are just the subalgebras of the higher Gaudin hamiltonians $\mathcal{A}(z_1, \ldots, z_n) \subset U(\mathfrak{g})^\otimes n$ introduced in [FPR] (see also [ER]). The quadratic Gaudin hamiltonians $(\mathbb{I})$ are linear combinations of the elements $\varphi_{z_1, \ldots, z_n}(\partial_t^k S_1)$, $n = 0, 1, 2, \ldots$.

We shall write $A(z_1, \ldots, z_n)$ instead of $A_0(z_1, \ldots, z_n)$.

**Proposition 5.** The subalgebras $A_\mu(z_1, \ldots, z_n)$ contain the following “non-homogeneous Gaudin hamiltonians”:

$$H_i = \sum_{k \neq i} \frac{\text{dim } \mathfrak{g}}{z_i - z_k} \sum_{a=1}^{\dim \mathfrak{g}} x_a^{(i)} (x_a)^{(k)} + \sum_{a=1}^{\dim \mathfrak{g}} x_a x_a^{(i)}.$$ 

**Proof.** Since the element $S_1 \in A$ is the symmetrization of $\overline{S_1} = i_-(\Phi_1)$, the element $H_i$ is the coefficient of $\frac{1}{z_i - z}$ in the expansion of $S_1(w; z_1, \ldots, z_n) = \varphi_{w - z_1, \ldots, w - z_n, \infty}(S_1)$ at the point $w = z_i$. Now it remains to apply Proposition 3. □

The algebra $U(\mathfrak{g})^\otimes n \otimes S(\mathfrak{g})^\otimes m$ has an increasing filtration by finite-dimensional spaces, $U(\mathfrak{g})^\otimes n \otimes S(\mathfrak{g})^\otimes m = \bigcup_{k=0}^{\infty} (U(\mathfrak{g})^\otimes n \otimes S(\mathfrak{g})^\otimes m)_{(k)}$ (by degree with respect to the generators). We define the limit $\lim_{s \to \infty} B(s)$ for any one-parameter family of subalgebras $B(s) \subset U(\mathfrak{g})^\otimes n \otimes S(\mathfrak{g})^\otimes m$ as $\bigcup_{k=0}^{\infty} \lim_{s \to \infty} B(s) \cap (U(\mathfrak{g})^\otimes n \otimes S(\mathfrak{g})^\otimes m)_{(k)}$.

It is clear that the limit of a family of commutative subalgebras is a commutative subalgebra. It is also clear that passage to the limit commutes with homomorphisms of filtered algebras (in particular, with the projection onto any factor and with finite-dimensional representations).

**Theorem 2.** $\lim_{s \to \infty} A_\mu(s z_1, \ldots, s z_n) = A^{(1)}_\mu \otimes \cdots \otimes A^{(n)}_\mu \subset U(\mathfrak{g})^\otimes n$ for regular semisimple $\mu \in \mathfrak{g}^*$.

**Proof.**

**Lemma 2.** $\lim_{z \to \infty} \varphi_z = \varepsilon$, where $\varepsilon : U(\hat{\mathfrak{g}}_-) \to \mathbb{C} \cdot 1 \subset U(\mathfrak{g})$ is the co-unit.
Proof. It is sufficient to check this on the generators. We have
\[
\lim_{z \to \infty} \varphi_z(g \otimes t^m) = \lim_{z \to \infty} z^m g = 0 \quad \forall \ g \in \mathfrak{g}, \ m = -1, -2, \ldots
\]
\[\square\]

Now let us choose the generators of \( \mathcal{A}(s z_1, \ldots, s z_n, \infty) \) as in Proposition 3. The coefficients of the Laurent expansion of \( S_k(w; s z_1, \ldots, s z_n) \) at any point \( s z_i \) are equal to the Laurent coefficients of \( S_k(w + s z_i; s z_1, \ldots, s z_n) \) at the point 0. On the other hand, by Lemma 2 we have
\[
\lim_{s \to \infty} S_k(w + s z_i; s z_1, \ldots, s z_n) = \lim_{s \to \infty} \varphi_{w-s(s z_1),\ldots,w-s(s z_n),\infty}(S_k) =\]
\[
= (\varepsilon \otimes \cdots \otimes \varphi_w \otimes \varepsilon \otimes \cdots \otimes \varepsilon \otimes \varphi_\infty) \circ \text{diag}_{s+1}(S_k) = S_k^{(i)}(w; 0).
\]
This means that the generators of \( \mathcal{A}(s z_1, \ldots, s z_n, \infty) \) give the generators of \( \mathcal{A}(z_1, \infty)^{(1)} \cdot \cdots \cdot \mathcal{A}(z_n, \infty)^{(n)} \) as the limit. Hence we conclude
\[
\lim_{s \to \infty} \mathcal{A}(s z_1, \ldots, s z_n, \infty) \supset \mathcal{A}(z_1, \infty)^{(1)} \cdot \cdots \cdot \mathcal{A}(z_n, \infty)^{(n)},
\]
and therefore
\[
\lim_{s \to \infty} \mathcal{A}_\mu(s z_1, \ldots, s z_n) \supset \mathcal{A}_\mu^{(1)} \cdot \cdots \cdot \mathcal{A}_\mu^{(n)}.
\]
By Fact 2, the subalgebra \( \mathcal{A}_\mu^{(1)} \cdot \cdots \cdot \mathcal{A}_\mu^{(n)} \subset U(\mathfrak{g})^\otimes n \) coincides with its own centralizer. Thus we have
\[
\lim_{s \to \infty} \mathcal{A}_\mu(s z_1, \ldots, s z_n) = \mathcal{A}_\mu^{(1)} \cdot \cdots \cdot \mathcal{A}_\mu^{(n)}.
\]
\[\square\]

**Corollary 4.** For generic values of the parameters, the commutative subalgebra \( \mathcal{A}_\mu(z_1, \ldots, z_n) \subset U(\mathfrak{g})^\otimes n \) has the maximal possible transcendence degree (which is equal to \( \frac{1}{2}(\dim \mathfrak{g} + \text{rk} \mathfrak{g}) \)).

**Proof.** Indeed, for generic \( \mu \) the subalgebra \( \mathcal{A}_\mu^{(1)} \cdot \cdots \cdot \mathcal{A}_\mu^{(n)} \subset U(\mathfrak{g})^\otimes n \) has the maximal possible transcendence degree due to the Fact 2. Since those subalgebras are contained in the closure of the family \( \mathcal{A}_\mu(z_1, \ldots, z_n) \), the subalgebra \( \mathcal{A}_\mu(z_1, \ldots, z_n) \) for generic values of the parameters has the maximal possible transcendence degree as well. \[\square\]

Consider the one-parameter family \( U(\mathfrak{g})_t \) of associative algebras whose space of generators is \( \mathfrak{g} \) and the defining relations are as follows:
\[
xy - yx = t[x, y] \quad \forall \ x, y \in \mathfrak{g}.
\]
(17)
For any \( t \neq 0 \), the map \( \mathfrak{g} \to \mathfrak{g}, \ x \mapsto t^{-1}x \) induces the associative algebra homomorphism
\[
\psi_t : U(\mathfrak{g}) \to U(\mathfrak{g})_t.
\]
(18)
For \( t = 0 \), we have \( U(\mathfrak{g})_0 = S(\mathfrak{g}) \).

Consider the commutative subalgebra
\[
(id^\otimes n \otimes \psi_{z^{-1}})(\mathcal{A}(z_1, \ldots, z_n, z)) \subset U(\mathfrak{g})^\otimes n \otimes U(\mathfrak{g})_z^{-1}.
\]
Passing to the limit as \( z \to \infty \), we obtain a certain commutative subalgebra in \( U(\mathfrak{g})^\otimes n \otimes S(\mathfrak{g}) \).
Theorem 3.
\[
\lim_{z \to \infty} (\text{id} \otimes \psi_{z-1})(A(z_1, \ldots, z_n, z)) = A(z_1, \ldots, z_n, \infty) \subset U(g)^{\otimes n} \otimes S(g).
\]

Proof.

Lemma 3. \[
\lim_{z \to \infty} \psi_{z-1} \circ \varphi_z = \varphi_\infty.
\]

Proof. It suffices to check this on the generators. We have
\[
\psi_{z-1} \circ \varphi_z(g \otimes t^m) = z \cdot z^m g \in U(g)_{z-1} \quad \forall \ g \in g, \ m = -1, -2, \ldots.
\]

Hence,
\[
\lim_{z \to \infty} \psi_{z-1} \circ \varphi_z(g \otimes t^m) = \delta_{-1,m} g = \psi_\infty(g \otimes t^m) \in S(g).
\]

Using Lemma 3, we obtain
\[
\lim_{z \to \infty} (\text{id} \otimes \psi_{z-1})(A(z_1, \ldots, z_n, z)) = \lim_{z \to \infty} (\varphi_{z_1} \otimes \cdots \otimes \varphi_{z_n} \otimes (\psi_{z-1} \circ \varphi_z)) \circ \text{diag}_{n+1}(A) = \]
\[
= (\varphi_{z_1} \otimes \cdots \otimes \varphi_{z_n} \otimes \varphi_\infty) \circ \text{diag}_{n+1}(A) = A(z_1, \ldots, z_n, \infty) \subset U(g)^{\otimes n} \otimes S(g).
\]

□

6. "Limit" Gaudin Model

Let \( V_\lambda \) be a finite-dimensional irreducible \( g \)-module of the highest weight \( \lambda \).

We consider the following \( U(g)^{\otimes n} \)-module:

\[
(19) \quad V(\lambda) := V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}.
\]

The subalgebra \( A(z_1, \ldots, z_n) \subset U(g)^{\otimes n} \) consists of \( \text{diag}_n(g) \)-invariant elements, and therefore acts on the space \( V(\lambda)^{\text{sing}} \subset V(\lambda) \) of singular vectors with respect to \( \text{diag}_n(g) \). This representation of \( A(z_1, \ldots, z_n) \) is known as the \((n\text{-point})\) Gaudin model.

We will show that the representation of the subalgebra \( A_\mu(z_1, \ldots, z_n) \subset U(g)^{\otimes n} \) in the space \( V(\lambda) \) for semisimple \( \mu \in g^* \) is a limit case of the \((n+1)\text{-point}\) Gaudin model.

Let \( M_\chi^* \) be the contragredient module of the Verma module with highest weight \( \chi \). This module can be constructed as follows. Suppose that \( \Delta_+ \) is the set of positive roots of \( g \). Then we have \( M_\chi^* = \mathbb{C}[x_\alpha]_{\alpha \in \Delta_+} \) (the generators \( x_\alpha \) have (multi-) degree \( \alpha \)), and the elements of \( g \) act by the following formulas:

1. The elements \( e_\alpha, \alpha \in \Delta_+ \) of the subalgebra \( n_+ \) act as
\[
\frac{\partial}{\partial x_\alpha} + \sum_{\beta > \alpha} P^\alpha_\beta \frac{\partial}{\partial x_\beta},
\]
where \( P^\alpha_\beta \) is a certain polynomial of degree \( \beta - \alpha \).

2. The elements \( h \in h \) act as
\[
h = \chi(h) - \sum_{\beta \in \Delta_+} \beta(h) x_\beta \frac{\partial}{\partial x_\beta}.
\]
(3) The generators $e_{-\alpha_i}$ (where $\alpha_i$ are the simple roots) of the subalgebra $n_-$ act as
\[
e_{-\alpha_i} = \chi(h_{\alpha_i}) x_{\alpha_i} + \sum_{\beta \in \Delta_+} Q_{\beta}^{\alpha_i} \frac{\partial}{\partial x_{\beta}},
\]
where $Q_{\beta}^{\alpha_i}$ is a certain polynomial of degree $\beta + \alpha_i$.

Consider the $U(\mathfrak{g})^{\otimes n} \otimes U(\mathfrak{g})$-module $V(\lambda) \otimes M^*_z$. We identify the vector space $M^*_z$ with $\mathbb{C}[x_{\alpha}]$ and re-scale the generators setting $y_\alpha = z^{ht(\alpha)} x_\alpha$, where $ht(\alpha)$ stands for the height of a root $\alpha$. The formulas for the action of the Lie algebra $\mathfrak{g}$ on $M^*_z = \mathbb{C}[y_\alpha]$ now look as follows:
\[
e_\alpha = z^{ht(\alpha)} \frac{\partial}{\partial y_\alpha} + \sum_{\beta > \alpha} z^{ht(\alpha)} P_{\beta}^\alpha \frac{\partial}{\partial y_\beta},
\]
\[h = z \mu(h) - \sum_{\beta \in \Delta_+} \beta(h) y_\beta \frac{\partial}{\partial y_\beta},
\]
\[e_{-\alpha_i} = \mu(h_{\alpha_i}) y_{\alpha_i} + z^{-1} \sum_{\beta \in \Delta_+} Q_{\beta}^{\alpha_i} \frac{\partial}{\partial y_\beta},
\]

Thus we can assume that the basis of $V(\lambda) \otimes M^*_z = V(\lambda) \otimes \mathbb{C}[y_\alpha]$ does not depend on $z$, and the operators from $U(\mathfrak{g})^{\otimes n} \otimes U(\mathfrak{g})$ do depend on $z$. The subspace of singular vectors $[V(\lambda) \otimes M^*_z]^{\text{sing}} \subset V(\lambda) \otimes \mathbb{C}[y_\alpha]$ now becomes depending on $z$ as well. Furthermore, the space $V(\lambda) \otimes M^*_z = V(\lambda) \otimes \mathbb{C}[y_\alpha]$ is graded by weights of the diagonal action of $\mathfrak{g}$, where the homogeneous components do not depend on $z$ and have finite dimensions. The subspace $[V(\lambda) \otimes M^*_z]^{\text{sing}} \subset V(\lambda) \otimes \mathbb{C}[y_\alpha]$ is contained in a finite sum of homogeneous components, and hence the following limit is well-defined: $\lim_{z \to \infty} [V(\lambda) \otimes M^*_z]^{\text{sing}} \subset V(\lambda) \otimes \mathbb{C}[y_\alpha]$. Moreover, the limit of the image of $A(z_1, \ldots, z_n, z)$ in $\text{End}([V(\lambda) \otimes M^*_z]^{\text{sing}})$ as $z \to \infty$ is a commutative subalgebra in $\text{End}([V(\lambda) \otimes M^*_z]^{\text{sing}})$.

**Theorem 4.** For $z \to \infty$ we have

1. the limit of $[V(\lambda) \otimes M^*_z]^{\text{sing}} \subset V(\lambda) \otimes \mathbb{C}[y_\alpha]$ is $V(\lambda) \otimes 1$;
2. the limit of the image of $A(z_1, \ldots, z_n, z)$ in $\text{End}([V(\lambda) \otimes M^*_z]^{\text{sing}})$ contains the image of $A_{\mu}(z_1, \ldots, z_n)$ in $\text{End}(V(\lambda) \otimes 1) = \text{End}(V(\lambda))$.

**Proof.** Let us prove the first assertion. The subspace $[V(\lambda) \otimes M^*_z]^{\text{sing}} \subset V(\lambda) \otimes M^*_z$ is the intersection of kernels of the operators $\text{diag}_{n+1}(e_\alpha) = \sum_{i=1}^{n+1} e^{(i)}_\alpha$, $\alpha \in \Delta_+$. Clearly,
\[
\lim_{z \to \infty} z^{-ht(\alpha)} \text{diag}_{n+1}(e_\alpha) = 1^{\otimes n} \otimes \left( \frac{\partial}{\partial y_\alpha} + \sum_{\beta > \alpha} P_{\beta}^\alpha \frac{\partial}{\partial y_\beta} \right).
\]
Hence,
\[
\lim_{z \to \infty} [V(\lambda) \otimes M^*_z]^{\text{sing}} \subset \bigcap_{\alpha \in \Delta_+} \text{Ker} 1^{\otimes n} \otimes \left( \frac{\partial}{\partial y_\alpha} + \sum_{\beta > \alpha} P_{\beta}^\alpha \frac{\partial}{\partial y_\beta} \right) = V(\lambda) \otimes 1.
\]
Lemma 4. From Shuvalov’s results (Fact 3) it follows that the associated graded algebra $\lim_{t \to 0} [V(\lambda) \otimes M^*_{\mu}]^{sing}$ can be regarded as $U(g)^{\otimes n} \otimes U(g)_{\mu^{-1}}$-module with highest weight $(\lambda_1, \ldots, \lambda_n, \mu)$. Using the formulas for the action of the Lie algebra $g$ on $\mathbb{C}[y_\alpha]$, we see that

$$
\lim_{z \to \infty} 1 \otimes \cdots \otimes 1 \otimes e_{-\alpha_i} = \lim_{z \to \infty} z^{-1} 1 \otimes \cdots \otimes 1 \otimes \psi_{z^{-1}}(e_{-\alpha_i}) = 0
$$

for any simple root $\alpha_i \in \Delta_+$.

Therefore, the subspace $V(\lambda) \otimes 1 \subset V(\lambda) \otimes \mathbb{C}[y_\alpha]$ is stable with respect to the action of $\lim_{z \to \infty} U(g)^{\otimes n} \otimes U(g)_{\mu^{-1}} = U(g)^{\otimes n} \otimes S(g)$. Moreover, the algebra $1 \otimes \cdots \otimes 1 \otimes S(g)$ acts on this space through the character $\mu$. By Theorem 3, we have

$$
\lim_{z \to \infty} (\text{id}^{\otimes n} \otimes \psi_{z^{-1}})(A(z_1, \ldots, z_n, z)) = A(z_1, \ldots, z_n, \infty) \subset U(g)^{\otimes n} \otimes S(g).
$$

This means that the limit of the image of $A(z_1, \ldots, z_n, z)$ in $\text{End}(V(\lambda) \otimes M^*_{\mu})^{sing}$ contains the image of the algebra $(\text{id} \otimes \mu)(A(z_1, \ldots, z_n, \infty)) = A_{\mu}(z_1, \ldots, z_n)$ in $\text{End}(V(\lambda) \otimes 1)$.

7. The case of $sl_r$

In this section we set $g = sl_r$.

Lemma 4. For $g = sl_r$ and $\mu(t) = E_{11} + tE_{22} + \cdots + t^{n-1}E_{nn}$, the limit subalgebra $\lim_{t \to 0} A_{\mu(t)}$ is the Gelfand–Tsetlin subalgebra in $U(sl_r)$.

Proof. From Shuvalov’s results (Fact 3) it follows that the associated graded algebra $\lim_{t \to 0} A_{\mu(t)} \subset S(g)$ is the Gelfand–Tsetlin subalgebra in $S(g)$. Indeed, in this case $\mathfrak{z}_k$ is the Lie algebra $sl_{r-k-1} \oplus \mathbb{C}^{k+1}$ consisting of all matrices $A \in sl_r$ satisfying

$$A_{ij} = A_{ji} = 0, \quad i = 1, \ldots, k + 1, \quad j = 1, \ldots, r, \quad i \neq j.
$$

The subalgebra of $S(sl_r)$ that is generated by $S(\mathfrak{z}_k)^{\otimes k}$ for all $k$ is the Gelfand–Tsetlin subalgebra.

For any $\mu$, the generators of $A_{\mu}$ are the images of the generators of $A_{\mu}$ under the symmetrization map (Fact 5). Therefore, the generators of $\lim_{t \to 0} A_{\mu(t)} \subset U(g)$ are the images of the generators of $\lim_{t \to 0} A_{\mu(t)} \subset S(g)$ under the symmetrization map as well.

The uniqueness of the lifting (Fact 5) implies that $\lim_{t \to 0} A_{\mu(t)}$ is the subalgebra in $U(sl_r)$ generated by all elements of $ZU(\mathfrak{z}_k)$ for all $k$, i.e., it is the Gelfand–Tsetlin subalgebra in $U(sl_r)$.

Theorem 5. The algebra $A_{\mu}(z_1, \ldots, z_n)$ has simple spectrum in $V(\lambda)$ for generic values of the parameters $\mu$ and $z_1, \ldots, z_n$.

Proof. (1) The Gelfand–Tsetlin subalgebra in $U(sl_r)$ has simple spectrum in $V_\lambda$ for any $\lambda$ – it is a well-known classical result.

(2) Since the Gelfand–Tsetlin subalgebra is a limit of $A_{\mu}$, the algebra $A_{\mu}$ for generic $\mu$ has simple spectrum in $V_\lambda$ as well.
Corollary 5. There exists a subset \( W \subset \Lambda_+ \times \cdots \times \Lambda_+ \), which is Zariski dense in \( h^* \) (where \( \Lambda_+ \) is the set of integral dominant weights), such that for any \( (\lambda) = (\lambda_1, \ldots, \lambda_n) \in W \) the Gaudin subalgebra \( A(z_1, \ldots, z_n) \) has simple spectrum in \( V_{(\lambda)}^{\text{sing}} \) for generic values of the parameters \( z_1, \ldots, z_n \).

Proof. For fixed \( \lambda_1, \ldots, \lambda_{n-1} \), the condition of non-simplicity of the spectrum of \( A(z_1, \ldots, z_n) \) in the space \( \left[V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n-1}} \otimes M_{\lambda_n}^*\right]^{\text{sing}} \) is an algebraic condition on \( \lambda_n \in h^* \) for any \( z_1, \ldots, z_n \). By Theorems 4 and 5 this condition is not always satisfied. This means that the set of \( \lambda_n \in \Lambda_+ \) such that the spectrum of the algebra \( A(z_1, \ldots, z_n) \) in \( \left[V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{n-1}} \otimes M_{\lambda_n}^*\right]^{\text{sing}} \) is simple for generic \( z_1, \ldots, z_n \), is Zariski dense in \( h^* \) for any collection \( \lambda_1, \ldots, \lambda_{n-1} \). Since \( V_{\lambda_n} \subset M_{\lambda_n}^* \), the spectrum of the algebra \( A(z_1, \ldots, z_n) \) in the space \( V_{(\lambda)}^{\text{sing}} \) is simple for any of these collections \( \lambda_1, \ldots, \lambda_n \). □

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