A 2k-Vertex Kernel for Maximum Internal Spanning Tree

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Abstract

We consider the parameterized version of the maximum internal spanning tree problem, which, given an n-vertex graph and a parameter k, asks for a spanning tree with at least k internal vertices. Fomin et al. [J. Comput. System Sci., 79:1–6] crafted a very ingenious reduction rule, and showed that a simple application of this rule is sufficient to yield a 3k-vertex kernel. Here we propose a novel way to use the same reduction rule, resulting in an improved 2k-vertex kernel. Our algorithm applies first a greedy procedure consisting of a sequence of local exchange operations, which ends with a locally-optimal spanning tree, and then uses this special tree to find a reducible structure. As a corollary of our kernel, we obtain a deterministic algorithm for the problem running in time $4^k \cdot n^{O(1)}$.

1 Introduction

A spanning tree of a connected graph G is a subgraph that includes all the vertices of G and is a tree. Spanning tree is a fundamental concept in graph theory, and finding a spanning tree of the input graph is a routine step of graph algorithms, though it usually induces no extra cost: most algorithms will start from exploring the input graph anyway, and both breadth- and depth-first-search procedures produce a spanning tree as a byproduct. However, a graph can have an exponential number of spanning trees, of which some might suit a specific application better than others. We are hence asked to find constrained spanning trees, i.e., spanning trees minimizing or maximizing certain objective functions. The most classic example is the minimum-weight spanning tree problem (in weighted graphs), which has an equivalent but less known formulation, i.e., maximum-weight spanning tree. Other constraints that have received wide attention include minimum diameter spanning tree [7], degree constrained spanning tree [10, 11], maximum leaf spanning tree [14], and maximum internal spanning tree [20]. Unlike the minimum-weight spanning tree problem [8], most of these constrained versions are NP-hard [16].

The optimization objective we consider here is to maximize the number of internal vertices (i.e., non-leaf vertices) of the spanning tree, or equivalently, to minimize the number of its leaves. More formally, the maximum internal spanning tree problem asks whether a given graph G has a spanning tree with at least k internal vertices. Containing the Hamiltonian path problem as a special case, it is clearly NP-hard. This paper approaches it by studying kernelization algorithms for its parameterized version; here the parameter is k, and hence we use the name k-internal spanning tree. Given an instance (G, k) of k-internal spanning tree, a kernelization algorithm produces in polynomial time an “equivalent” instance $(G', k')$ such that $k' \leq k$ and that the kernel size (i.e., the number of vertices in $G'$) is upper bounded by some function of $k'$. Prieto and Sloper [17] presented an $O(k^3)$-vertex kernel for the problem, and improved it to $O(k^2)$ in the journal version [18]. Fomin et al. [4] crafted a very ingenious reduction rule, and showed that a simple application of this rule is sufficient to yield a 3k-vertex kernel. Answering a question asked by Fomin et al. [4], we further improve the kernel size to 2k.

Theorem 1.1. The k-internal spanning tree problem has a 2k-vertex kernel.

We obtain this improved result by revisiting the reduction rule proposed by Fomin et al. [4]. A nonempty independent set X (i.e., a subset of vertices that are pairwise nonadjacent in G) as well as its neighborhood are called a reducible structure if $|X|$ is at least twice as the cardinality of its neighborhood. To apply the reduction rule one needs a reducible structure. Indeed, we are proving a stronger statement that implies Theorem 1.1 as a corollary.
**Theorem 1.2.** Given an $n$-vertex graph $G$, we can find in polynomial time either a spanning tree of $G$ with at least $n/2$ internal vertices, or a reducible structure.

The observation in [4] is that the leaves of a depth-first-search tree $T$ are independent. Therefore, if the graph has more than $3k - 3$ vertices, then either the problem has been solved (when $T$ has $k$ or more internal vertices), or the set of (at least $2k - 2$) leaves of $T$ will be the required independent set. It is, however, very nontrivial to find a reducible structure when $2k < n < 3k - 3$, and this will be the focus of this paper. We first preprocess the tree $T$ using a greedy procedure that applies a sequence of local-exchange operations to increase the number of its internal vertices. After a local optimal spanning tree is obtained, we show that if it has more leaves than internal vertices, then a subset of its leaves and its neighborhood make the reducible structure. We apply the reduction rule of [4] to reduce it and then repeat the process, which terminates on either a 2$k$-vertex kernel or a solution.

It is interesting to point out that our kernelization algorithm will never end with a NO situation, which is common in kernelization algorithms in literature. It either returns a trivial YES instance, or continuously reduces the graph until it has a spanning tree with at least half internal vertices. This also means that our kernelization algorithm does not rely on the parameter $k$. One should be noted that to further improve a 2$k$-vertex kernel for a graph problem seems to be a very challenging, if possible, undertaking: more and more such kernels have appeared in literature, which have stubbornly withstood all subsequent attacks, however hard they were.

Priesto and Sloper [18] also initiated the study of parameterized algorithms (i.e., algorithms running in time $O(f(k) \cdot n^{O(1)})$ for some function $f$ independent of $n$) for k-internal spanning tree, which have undergone a sequence of improvement. Closely related here is to the $k$-internal out-branching problem, which, given a directed graph $G$ and a parameter $k$, asks if $G$ has an out-branching (i.e., a spanning tree having exactly one vertex of in-degree 0) with at least $k$ vertices of positive out-degrees. It is known that any $O^*(f(k))$-time algorithm for $k$-internal out-branching can solve $k$-internal spanning tree in the same time, but not necessarily the other way. After a successive sequence of studies [6, 2, 5, 23, 3], the current best deterministic and randomized parameterized algorithms for $k$-internal out-branching run in time $O^*(6.86^k)$ and $O^*(4^k)$ respectively, which are also the best known for $k$-internal spanning tree. Table 1 summarizes the history of this line of research.

**Table 1:** Known parameterized algorithms for problems $k$-internal out-branching and $k$-internal spanning tree (note that an algorithm for the former applies to the latter as well).

| Problem                  | Running time | Reference               | Remark          |
|--------------------------|--------------|-------------------------|-----------------|
| $k$-internal out-branching | $O^*(k^{O(1)})$ | Gurin et al. [6]       |                 |
|                          | $O^*(55.8^k)$ | Cohen et al. [2]       |                 |
|                          | $O^*(16^k-o(k))$ | Fomin et al. [5]      |                 |
|                          | $O^*(4^k)$   | Daligault and Kim [3]  | randomized      |
|                          | $O^*(6.86^k)$ | Shachnai and Zehavi [23]|                 |
| $k$-internal spanning tree | $O^*(k^{2.5k})$ | Priesto and Sloper [18]|                 |
|                          | $O^*(2.14^k)$ | Binkele-Raible et al. [1]| cubic graphs |
|                          | $O^*(8^k)$   | Fomin et al. [4]       |                 |
|                          | $O^*(4^k)$   | This paper             |                 |

The $O^*(4^k)$-time randomized algorithm for $k$-internal out-branching [3, Theorem 180] was obtained using a famous algebraic technique developed by Kouits and Williams [12], which, however, is very unlikely to be derandomized. As a corollary of Theorem 1.1, we obtain an $O^*(4^k)$-time deterministic algorithm for $k$-internal spanning tree,—it suffices to apply the $O^*(2^n)$-time algorithm of Nederlof [15] to the $2k$-vertex kernel produced by Theorem 1.1,—matching the running time of the best randomized algorithm.

**Theorem 1.3.** The $k$-internal spanning tree problem can be solved in time $O^*(4^k)$.

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*Following convention, we use the $O^*(f(k))$ notation to suppress the polynomial factor $n^{O(1)}$ in the running time.*
It remains an open problem to develop a deterministic $O^*(4^k)$-time algorithm for $k$-internal outbranching. Note that the minimum spanning tree problem has been long known to be solvable in randomized linear time [8], while a deterministic linear-time algorithm is still elusive. As a final remark, there is also a line of research devoted to developing approximation algorithms for maximum internal spanning tree [17, 9, 19, 21]. In a companion paper [13], we have used a similar local-exchange procedure to improve the approximation ratio to 1.5.

2 A greedy local search procedure

All graphs discussed in this paper shall always be undirected and simple, and the input graph is assumed to be connected. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$ respectively. For a vertex $v \in V(G)$, let $N_G(v)$ denote the neighborhood of $v$ in $G$, and let $d_G(v) := |N_G(v)|$ be its degree in $G$. The neighborhood of a subset $U \subseteq V(G)$ of vertices is defined to be $N_G(U) = \bigcup_{v \in U} N_G(v) \setminus U$. A tree $T$ is a spanning tree of a graph $G$ if $V(T) = V(G)$ and $E(T) \subseteq E(G)$; edges not in $T$, i.e., $E(G) \setminus E(T)$, are cotree edges of $T$. A vertex $u \in V(T)$ is a leaf of $T$ if $d_T(u) = 1$, and an internal vertex of $T$ otherwise; let $L(T)$ and $I(T)$ denoted the set of leaves and the set of internal vertices of $T$ respectively. An internal vertex $u$ of $T$ is a branchpoint if $d_T(u) \geq 3$. Let $I_3(T)$ denote the set of branchpoints of $T$, and let $I_2(T)$ denote other internal vertices (having degree 2 in $T$); the three vertex sets $L(T)$, $I_2(T)$, and $I_3(T)$ partition $V(T)$.

Since $|L(T)| = |V(T)| - |I(T)|$, to maximize it is equivalent to minimizing the number of leaves. Also connecting leaves and internal vertices, especially branchpoints, of a tree $T$ is the following elementary fact:

$$|L(T)| - 2 = \sum_{v \in I(T)} (d_T(v) - 2) = \sum_{v \in I_3(T)} (d_T(v) - 2).$$

Therefore, informally speaking, we need to decrease the number and degrees of branchpoints. We start from an arbitrary spanning tree $T$ of $G$. We may assume that $T$ is not a path, (as otherwise the problem has been solved,) to which we apply some local exchanges to increase the number of internal vertices of $T$. We start with a leaf exchange: we take any leaf $l$ of $T$, and then replace a tree edge in $T$ by a cotree edge incident to $l$.

**Exchange Rule 1 ([17]).** If there is a cotree edge connecting two leaves $l_1$ and $l_2$ of $T$, then find an edge $uv$ from $P_T(l_1, l_2)$ such that $u$ is a branchpoint, and substitute $l_1l_2$ for $uv$ in $T$.

After an exhaustive application of Rule 1, all leaves in the resulting tree are pairwise nonadjacent. Henceforth we may assume that each cotree edge is incident to at least one internal vertex. We remark that Fomin et al. [4] achieved this by using depth-first-search tree at first place. We can surely use the same way to get the initial spanning tree, but we still need Rule 1, as later operations of the other exchange rule to follow may introduce cotree edges connecting leaves that are originally not.

**Definition 1.** A cotree edge of $T$ is good if it connects a leaf $l$ and an internal vertex $w$ of $T$. We say that $lw$ crosses every edge in the path $P_T(l,w)$.

For notational convenience, when referring to a good cotree edge $lw$, we always put the leaf $l$ first, and when referring to an edge $uv$ crossed by it, we always put the vertex closer to $l$ first; hence, $P_T(l,w)$ can be written as $l \cdots uv \cdots w$. We would like to point out that the same edge $uv$ can be crossed by two different cotree edges and they may be referred to by different orders.

Let us consider the impact of a substitution on the involved vertices. We will avoid the tree edge incident to $l$ but do allow $v = w$, and hence there are either three of four vertices involved. The two vertices of the cotree edge are clear: $l$ always becomes an internal vertex, and $w$ always remains internal (independent of $w = v$ or not). On the other hand, $u$ and $v$ will remain internal if they are
in $I_3(T)$, but one or both of them may become leaves if they are in $I_2(T)$ (with the only exception $v = w$). Although some operation does not increase, or even decreases, the number of internal vertices, by switching vertices in $I(T)$ and $I(T)$, it may introduce cotree edge(s) between leaves in the new tree, which enable us to subsequently apply Rule 1 and serve our purpose. This is never the case for the first edge in $P_T(u, v)$ and hence we avoid it. This observation is formalized in the next reduction rule, for which we need a technical definition that characterize those vertices in $I_2(T)$ that can participate in the aforementioned successive exchanges.

**Definition 2.** All vertices in $I_3(T)$ are detachable. A vertex $w \in I_2(T)$ is detachable if there exists a good cotree edge $lw$ of $T$ satisfying at least one of the following:

1. $P_T(l, w)$ visits a branchpoint, or
2. some vertex $v$ in $P_T(l, w)$ is incident a good cotree edge $l'v$ of $T$ with $l' \neq l$.

Let $D(T)$ denote the set of detachable vertices of $T$, and let $D_2(T)$ denote those detachable vertices in $I_2(T)$. Then $D(T) = D_2(T) \cup I_3(T) \subseteq I(T)$. Note that the vertex $v$ in item (2) of the definition of $D_2(T)$ necessarily has degree two, and possibly $v = w$.

**Exchange Rule 2.** Let $uv$ be an edge crossed by a good cotree edge $lw$ of $T$. Substitute $lw$ for $uv$ if any of the following is true.

- $u \in I_3(T)$, and
  - $(a)$ $v = w$ or $v \in I_3(T)$; or
  - $(b)$ $v \in D_2(T)$ and there is a good cotree edge $l_vv$ with $l_v \neq l$.

- $u \in D_2(T)$, there is a good cotree edge $l_uu$ with $l_u \neq l$, and
  - $(c)$ $v = w$ or $v \in I_3(T)$; or
  - $(d)$ $v \in D_2(T)$ and there is a good cotree edges $l_vv$ with $l_v \not\in \{l, l_u\}$.

Moreover, after (b) and (c), apply Rule 1 to $l_vv$ and $l_uu$ respectively; after (d), apply Rule 1 to $l_uu$ and then to $l_vv$ (if still applicable).

One can check in polynomial time whether Rule 2 (and which case of it) is applicable. To show that the whole procedure can be finished in polynomial time, we need to argue that each invocation increases the number of internal vertices by at least one. Note that Rule 2 is only applied after Rule 1 is no longer applicable.

**Lemma 2.1.** Applying Rule 1 or Rule 2 to a spanning tree $T$ of $G$ results in a new spanning tree $T'$ of $G$ satisfying $|I(T')| \geq |I(T)| + 1$.

**Proof.** In Rule 1 and Rule 2, the replaced edge $uv$ is in $P_T(l_1, l_2)$ and $P_T(l, w)$ respectively, so the resulting subgraph $T'$ must be a spanning tree of $G$. To compare the number of internal vertices, it suffices to consider these vertices incident to the added/deleted edges.

After the application of Rule 1, $u$ remains an internal vertex as $d_{T'}(u) = d_T(u) - 1 \geq 2$. Note that possibly $v \in \{l_1, l_2\}$, and in this case $v$ remains a leaf but the other leaf becomes an internal vertex. Otherwise, both $l_1$ and $l_2$ become internal vertices, while $v$ might become a leaf. In either case, the number of internal vertices increases by at least 1.

We now consider Rule 2. Case (a) is straightforward: after its application, all vertices in $\{l, w, u, v\}$ are internal vertices of the resulting tree $T'$; hence $|I(T')| = |I(T)| + 1$. In case (b), after the substitution, $w$ and $u$ remain internal vertices, while $l$ becomes an internal vertex and $v$ becomes a leaf of the new tree. Albeit the number of leaves does not change, the cotree edge of the new tree between $l_v$ and the new leaf $v$ enables us to apply Rule 1, which increases the number of internal vertices by 1. Case (c) is similar as case (b): the first substitution does not affect the number of internal vertices, but the subsequent application of Rule 1 increases it by 1. In case (d), after the first substitution, $w$ remains an internal vertex, $l$ becomes an internal vertex, while both $u$ and $v$ become leaves; hence the number of
leaves decreases by 1. Since \( l_u, u, l_v, v \) are four distinct leaves in the resulting tree, applying Rule 1 to \( l_u u \) results in a spanning tree of \( G \) that has at least \( I(T) \) internal vertices. Either \( l_v \) and \( v \) (which remain leaves) are the only leaves and we are done, or we can apply Rule 1 to \( l_v v \) to increase the number of internal vertices. This obtained tree \( T' \) has at least one more internal vertex than \( T \).

A spanning tree \( T \) of a graph \( G \) is called maximal if neither exchange rule is applicable to it. By Lemma 2.1, each application of an exchange rule to a spanning tree increases its number of internal vertices at least 1. Since a spanning tree of \( G \) has at most \( |V(G)| - 2 \) internal vertices, we have the following theorem.

**Theorem 2.2.** A maximal spanning tree of a graph can be constructed in polynomial time.

### 3 The kernelization algorithm

Although neither exchange rule reduces the graph size, their exhaustive application provides a maximal spanning tree whose structural properties will be crucial for our kernelization algorithm. In this section we are only concerned with maximal spanning trees. We will use the reduction rule of Fomin et al. [4], which is recalled below. Let \( \text{opt}(G) \) denote the maximum number of internal vertices a spanning tree of \( G \) can have.

**Lemma 3.1 ([4]).** Let \( L' \) be an independent set of \( G \) such that \( |L'| \geq 2|N_G(L')| \). We can find in polynomial time nonempty subsets \( S \subseteq N_G(L') \) and \( L \subseteq L' \) such that:

1. \( N_G(L) = S \), and
2. the graph \( (S \cup L, E(G) \cap (S \times L)) \) has a spanning tree such that all vertices of \( S \) and \( |S| - 1 \) vertices of \( L \) are internal.

Moreover, let \( G' \) be obtained from \( G \) by adding a vertex \( v_s \) adjacent to every vertex in \( N_G(S) \setminus L \), adding a vertex \( v_t \) adjacent to \( v_s \), and removing all vertices of \( S \cup L \), then \( \text{opt}(G') = \text{opt}(G) - 2|S| + 2 \).

Note that \( |L| \geq 2 \) (otherwise the graph in Lemma 3.1(2) cannot have internal vertices), and hence each application of the reduction rule decreases the number of vertices by at least 1. The safeness of the following reduction rule is ensured by Lemma 3.1.

**Reduction Rule** ([4]): Find nonempty subsets \( S \) and \( L \) of vertices as in Lemma 3.1. Return \( (G', k') \) where \( G' \) is defined in Lemma 3.1 and \( k' = k - 2|S| + 2 \).

The main technical obstacle is then to identify a vertex set \( L' \) with \( |L'| \geq 2|N_G(L')| \). The is trivial when \( |V(G)| \geq 3k - 3 \). In any spanning tree \( T \) of \( G \) with \( |L(T)| < k \) it holds that \( |L(T)| \geq 2k - 2 \geq 2|I(T)| = 2|N_G(L(T))| \); hence we can use \( L(T) \) as \( L' \) and a \( 3k \)-vertex kernel follows. However, it becomes very nontrivial to find such a set when \( 2k < |V(G)| < 3k - 3 \). Our approach here is to separate a maximal spanning tree \( T \) into several subtrees and bound the number of \( L(T) \) by the number of \( I(T) \) residing in each subtree individually. It is worth mentioning that a leaf of a subtree may not be a leaf of \( T \).

Recall that the removal of any edge \( uv \in E(T) \) from a tree \( T \) breaks it into two components, one containing \( u \) and the other containing \( v \). In general, the removal of all edges of an edge subset \( E' \subseteq E(T) \) from \( T \) breaks it into \( |E'| + 1 \) components, each being a subtree of \( T \). We would like to divide \( T \) in a way that the two ends of any good cotree edge always reside in the same subtree, hence the following definition.

**Definition 3.** An edge \( uv \in E(T) \) connecting two internal vertices \( u, v \) of \( T \) is critical if there is no good cotree edge connecting the two components of \( T - uv \).

Let \( C(T) \) denote the (possibly empty) set of all critical edges in \( T \). Note that for each non-critical edge \( uv \) with both \( u, v \in D(T) \), there must be a good cotree edge connecting the two components of \( T - uv \).
Lemma 3.2. Let $u$ and $v$ be two detachable vertices of a maximal spanning tree $T$ and $uv \in E(T)$. If there is a good cotree edge connecting the two components of $T - uv$, then there exists a good cotree edge $lw$ such that

1. $u \in D_2(T)$, the only good cotree edge incident to $u$ is $lu$, and $lw$ crosses $uv$; or
2. $v \in D_2(T)$, the only good cotree edge incident to $v$ is $lv$, and $lw$ crosses $vu$.

Proof. Let $l'w'$ be a good cotree edge connecting the two components of $T - uv$. Without loss of generality, we may assume that $l'w'$ cross $uv$—then $l'$ and $w'$ are in the components of $T - uv$ containing $u$ and $v$ respectively—the other case follows by symmetry.

We argue first that $u \in D_2(T)$. Suppose for contradiction, $u \in I_3(T)$. Since Rule 2(a) is not applicable to $l'w'$ and $uv$, we must have $v \notin I_2(T)$. Then $v \in D_2(T)$, and by definition, there is a good cotree edge $l_vv$ of $T$. Again, since Rule 2(a) is not applicable to $l_vv$ and $uv$, we must have $l_v \neq l'$. However, $l_v \neq l'$ and $v \neq w$ would imply that Rule 2(b) is applicable to $l'w'$, $uv$, and $l_vv$, a contradiction. Now that $u \in D_2(T)$, by definition, there is a good cotree edge incident to $u$. The proof is now completed if $l'u$ is a good cotree edge of $T$ and the only one incident to $u$: we are in case (1) and $l'w'$ is the claimed edge. In the rest of the proof we may assume otherwise—that is, there is a good cotree edge $lu$ of $T$ with $l \neq l'$.

Since Rule 2(c) cannot be applicable to $l'w'$, $uv$, and $lu$, the vertex $v$ is also in $D_2(T)$ and $v \neq w'$. There is also a good cotree edge $l_vv$ of $T$. Similarly it can be inferred that $l_v \neq l'$: otherwise Rule 2(c) is applicable to $l'v$ (i.e., $l_vv$, $uv$, and $lu$. Now that $l'$ is different from both $l$ and $l_v$, but Rule 2(d) is not applicable to $l'w'$, $uv$, $lu$ and $l_vv$, we must have $l = l_v$, i.e., both $lu$ and $lv$ are good cotree edges of $T$. We consider now which component of $T - uv$ the leaf $l$ belongs to. If it is with $u$, then there cannot be another good cotree edge $l_1u$ with $l' \neq l$: otherwise, Rule 2(c) would be applicable on $lv$, $uv$, and $l_1u$. In other words, $lu$ is the only good cotree edge incident to $u$, and $lv$ is the claimed good cotree edge crossing $uv$. We are in case (1) and $lv$ is the claimed edge. A symmetric argument implies that we are in case (2) and $lu$ is the claimed edge if $l$ is in the component of $T - uv$ with $v$. This concludes the proof. 

The vertices $u$ or $v$ stipulated in Lemma 3.2 turns out to be our main trouble in analyzing the size of reduced instance. We use $D_1(T)$ to denote this set of vertices, which need our special attention. For the pair of vertices $u$, $v$ as in Lemma 3.2, $u \in D_B(T)$ if case (1) holds true, and $v \in D_B(T)$ otherwise. Note that $D_B(T) \subseteq D_2(T) \subseteq I_2(T)$. This vertex has degree 2 in $T$, and its other neighbor (different from $v$ or $u$) cannot be a leaf of $T$: suppose that it is $u'$; the component of $T - uv$ containing $u$ must contain another $l \in L(T)$ that is nonadjacent to $u$ in $T$. Therefore, the only good cotree edge mentioned in Lemma 3.2 is actually the only edge between it and $L(T)$ in $G$. The following corollary follows easily.

Corollary 3.3. For each $u \in D_B(T)$ for a maximal spanning tree $T$, we have $|L(T) \cap N_{G}(u)| = 1$.

Note that $I_3(T) \subseteq D(T) \setminus D_B(T)$ and $I(T) \setminus D(T) \subseteq I_2(T)$.

Lemma 3.4. Let $T$ be a maximal spanning tree. For every pair of vertices $u, w \in D(T) \setminus D_B(T)$ that are in the same component of $T - C(T)$, the path $P_T(u, w)$ visits at least one vertex $v \in I(T) \setminus D(T)$ such that for any $l \in L(T)$, the path $P_T(l, v)$ visits $D(T)$.

Proof. No generality will be lost by assuming that $P_T(u, w)$ is minimal (in the sense that it visits no other vertex in $D(T) \setminus D_B(T)$). By assumption, $P_T(u, w)$ is retained in $T - C(T)$. We argue first $uw \notin E(T)$. Suppose for contradiction, $uw \in E(T)$. Since $uv \notin C(T)$, there must be some good cotree edge crossing it; however, by Lemma 3.2 and the definition of $D_B(T)$, at least one of $u$ and $w$ is then in $D_B(T)$, a contradiction. Let $P_T(u, w) = uv_1 \cdots v_pw$, where $p \geq 1$; note that $d_T(v_i) = 2$ and $v_i \notin D(T) \setminus D_B(T)$ for each $1 \leq i \leq p$.

We now find an internal vertex of $P_T(u, w)$ that is not in $D(T)$ as follows. If $v_1 \notin D(T)$, then we are done. Otherwise, $v_1 \in D_B(T)$, and there is a unique good cotree edge $lv_1$. We prove by contradiction that $l$ is in the same component of $T - uv_1$ with $v_1$. Suppose the contrary, then

- if $u \in I_3(T)$, then Rule 2(a) is applicable to $lv_1$ and $uv_1$; or
if \( u \in D_2(T) \setminus D_B(T) \), then there is a good cotree edge \( l'u \) with \( l' \neq l \), and hence Rule 2(c) is applicable to \( lv_1, uv_1 \), and \( l'u \).

Noting that every internal vertex of \( P \) has degree 2, this actually implies that \( l \) must be in the same component of \( T - v_w \) with \( w \). The first \( v_i \) such that \( lv_i \) is not the only good cotree edge incident to \( v_i \) is the vertex we need. Its existence can be argued by contradiction as follows. Suppose for contradiction, there is no such a vertex \( v_i \), then \( lv_i \) is the only good cotree edge incident to \( v_p \), and

- if \( w \in I_2(T) \), then Rule 2(a) is applicable to \( lv_p \) and \( wv_p \); or
- if \( w \in D_2(T) \setminus D_B(T) \), then there is a good cotree edge \( l'w \) with \( l' \neq l_1 \), and hence Rule 2(c) is applicable to \( lv_p, wv_p \), and \( l'w \).

These contradictions imply that \( p \geq 2 \) and there must be \( 2 \leq i \leq p \) such that \( v_i \) is incident to a good cotree edge \( l'v_i \) with \( l' \neq l_i \). Since Rule 2(c) is not applicable to \( lv_{i-1}, v_{i-1}, \) and \( l'v_i \), we can conclude that \( v_i \notin D_2(T) \). Noting that \( v_i \in I_2(T) \), we have verified that \( v_i \) is an internal vertex of \( P(T, u, w) \) not in \( D(T) \).

Noting that every internal vertex of \( P(T, u, w) \) has degree 2, for any \( l \in L(T) \), the path \( P(T, l, v) \) necessarily visits either \( u \) or \( w \), which is in \( D(T) \). This concludes the proof.

We are now ready for proving the main result of this section.

**Lemma 3.5.** Let \( T \) be a maximal spanning tree of \( G \) with \( |V(G)| \geq 4 \). If \( |L(T)| > |I(T)| \), then we can find in polynomial time an independent set \( l' \) of \( G \) such that \( |l'| \geq 2|N_G(l')| \).

**Proof.** We find all critical edges \( C(T) \), and take the forest \( T - C(T) \). By assumption, there must be some component \( T_0 \) of \( T - C(T) \) of which more than half vertices are from \( L(T) \). Let \( X \) and \( Y \) denote \( L(T) \setminus V(T_0) \) and \( I(T) \cap V(T_0) \) respectively; then \( |X| \geq |Y| + 1 \). Since \( T \) is maximal (Rule 1 is not applicable), \( X \) is an independent set. We divide \( X \) into the following three subsets:

\[
X_1 = X \cap N_G(D_B(T)); \quad X_2 = X \cap N_G(D(T)) \setminus X_1; \quad \text{and} \quad X_3 = X \setminus (X_1 \cup X_2).
\]

We will show that \( |X_2| \geq 2|N_G(X_2)| \), and hence \( X_2 \) satisfies the claimed condition and can be used as \( l' \). By the definition of critical edges, there is no good cotree edge of \( T \) connecting two different components of \( T - C(T) \); hence \( N_G(X) \subseteq Y \). We accordingly divide \( Y \) into subsets. The detachable vertices are either in \( Y_1 := D_B(T) \cap V(T_0) \) or \( Y_2 := (D(T) \setminus D_B(T)) \cap V(T_0) \), while a vertex \( y \in Y \setminus D(T) \) is in \( Y_3 \) if there exists \( l \in L(T) \) such that the path \( P(T, l, y) \) does not visit \( D(T) \), or in \( Y_4 \) otherwise. Note that \( |X| = |X_1| + |X_2| + |X_3| \) and \( |Y| = |Y_1| + |Y_2| + |X_3| + |Y_4| \).

We argue first that \( N_G(X_2) \subseteq Y_2 \). It suffices to show \( N_G(X_2) \subseteq D(T) \) (the definition of \( X_2 \) requires that a vertex in it is nonadjacent to \( D_B(T) \) in \( G \), which further boils down to showing \( N_G(X_2) \cap I_2(T) \subseteq D(T) \): since \( T \) is maximal (Rule 1 is not applicable), \( X_2 \) has no neighbor in \( L(T) \); on the other hand, \( I_3(T) \subseteq D(T) \). Consider a vertex \( x \in X_2 \), and let \( y \) be the unique neighbor of \( x \) in \( T \). By assumption, \( y \in D_B(T) \setminus D_B(T) \), and hence there is a good cotree edge \( ly \) of \( T \) with \( l \neq x \). For each \( y' \in N_G(x) \cap I_2(T) \) different from \( y \), the path \( P_T(x, y') \) visits \( y \), using the definition of \( D(B) \) we can conclude that \( y' \in D(T) \).

Each \( x \in X_1 \) has a neighbor \( y \in Y_1 \). By Corollary 3.3, \( x \) is the only vertex in \( N_G(y) \cap L(T) \). Thus, \( |X_1| \leq |Y_1| \). The unique neighbor \( y \) of a vertex \( x \in X_3 \) in \( T \) must be in \( I_2(T) \setminus D_2(T) \). Since the trivial path \( P_T(x, y) \) (consisting of a single edge \( xy \)) does not visit \( D(T) \), we have \( y \notin Y_3 \). The other neighbor of \( y \) in \( T \) cannot be a leaf of \( T \) (\( G \) has at least four vertices). Thus, \( |X_3| \leq |Y_3| \). By Lemma 3.4, for any two different vertices \( u \) and \( w \) of \( Y_2 \), the path \( P_T(u, w) \) visits at least one vertex in \( Y_4 \). Since \( T_0 \) is a tree, using induction it is easy to show \( |Y_4| \geq |Y_2| - 1 \).

Summarizing above, we have

\[
|X_2| = |X| - |X_1| - |X_3| = |X| - |X_1| - |X_3| \\
\quad \geq |Y| + 1 - |Y_1| - |Y_3| = |Y_2| + |Y_4| + 1 \\
\quad \geq 2|Y_2| = 2|N_G(X_2)|.
\]

(because \( |X| = |X_1| + |X_2| + |X_3| \))

(because \( |X| \geq |Y| + 1; |X_1| \leq |Y_1|; |X_3| \leq |Y_3| \))

(because \( |Y_4| \geq |Y_2| - 1 \))

(because \( |Y_4| \geq |Y_2| - 1 \))

(because \( |N_G(X_2)| \leq |Y_2| \)).

Therefore, \( |X_2| \geq 2|N_G(X_2)| \), which is the claim.

\[\square\]
Hence $X_2$ can be used as the independent set $L'$. This concludes the proof.

Lemmas 3.5 and 3.1, together with Theorem 2.2, imply Theorem 1.2.

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