ALMOST EVERYWHERE CONVERGENCE OF BOCHNER-RIESZ MEANS FOR THE HERMITE OPERATORS

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Abstract. Let $H = -\Delta + |x|^2$ be the Hermite operator in $\mathbb{R}^n$. In this paper we study almost everywhere convergence of Bochner-Riesz means for the Hermite operator $H$. We prove that

$$\lim_{R \to \infty} S^\lambda_R(H)f(x) = f(x) \text{ a.e.}$$

for $f \in L^p(\mathbb{R}^n)$ provided that $p \geq 2$ and $\lambda > 2^{-1} \max\{n(1/2 - 1/p) - 1/2, 0\}$. Surprisingly, for the dimensions $n \geq 2$ our result reduces the borderline summability index $\lambda$ for a.e. convergence as small as only half of the critical index required for a.e. convergence of the classical Bochner-Riesz means for the Laplacian. For the dimension $n = 1$, we obtain that $\lim_{R \to \infty} S^\lambda_R(H)f(x) = f(x)$ a.e. for $f \in L^p(\mathbb{R})$ with $p \geq 2$ whenever $\lambda > 0$.

1. Introduction

Convergence of Bochner-Riesz means of the Fourier transform in the $L^p$ spaces is one of the most fundamental problems in classical harmonic analysis. For $\lambda \geq 0$ and $R > 0$, the Bochner-Riesz means of the Fourier transform on $\mathbb{R}^n$ are defined by

$$S^\lambda_R f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \left(1 - \frac{|\xi|^2}{R^2}\right)_+^{\lambda} \hat{f}(\xi) d\xi, \quad \forall \xi \in \mathbb{R}^n. \quad (1.1)$$

Here $t_+ = \max\{0, t\}$ for $t \in \mathbb{R}$ and $\hat{f}$ denotes the Fourier transform of $f$. The convergence of $S^\lambda_R f \to f$ in $L^p$-norm as $R \to \infty$ is equivalent to the boundedness of $S^\lambda := S^\lambda_1$ in $L^p(\mathbb{R}^n)$, and the longstanding open problem known as the Bochner-Riesz conjecture is that for $1 \leq p \leq \infty$ and $p \neq 2$, $S^\lambda$ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\lambda > \lambda(p) := \max\left\{n\left|\frac{1}{2} - \frac{1}{p}\right| - \frac{1}{2}, 0\right\}. \quad (1.2)$$

It was shown by Herz that the condition (1.2) on $\lambda$ is necessary for $L^p$ boundedness of $S^\lambda$, see [24]. Carleson and Sjölin [10] proved the conjecture when $n = 2$. Afterward, substantial progress has been made in higher dimensions, see [5, 23, 31, 40, 41] and references therein. However, the conjecture still remains open for $n \geq 3$. Concerning the pointwise convergence, Carbery, Rubio de
Francia and Vega [8] showed for any function \( f \in L^p(\mathbb{R}^n) \),
\[
\lim_{R \to \infty} S^1_R f(x) = f(x) \text{ a.e.}
\]
provided \( p \geq 2 \) and \( \lambda > \lambda(p) \). When \( n = 2 \) the results were previously obtained by Carbery [7] who proved the sharp \( L^p \) boundedness of the maximal Bochner-Riesz means. Also, see [14] for earlier partial result based on the maximal Bochner-Riesz estimate. Remarkably, the result by Carbery et al. [8] settled the a.e. problem up to the sharp index \( \lambda(p) \) for \( 2 \leq p \leq \infty \). There are also results at the critical exponent, i.e., \( \lambda = \lambda(p) \) (for example, see [1, 33]). Almost everywhere convergence of \( S^1_R f \) with \( f \in L^p \), \( 1 < p < 2 \), exhibits different nature and few results are known in this direction except \( n = 2 \) ([34, 38, 39]).

**Bochner-Riesz means for the Hermite operator.** In this paper we consider almost everywhere convergence of Bochner-Riesz means for the Hermite operator \( H \) on \( \mathbb{R}^n \), which is defined by
\[
(1.3) \quad H = -\Delta + |x|^2 = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + |x|^2.
\]
The operator \( H \) is non-negative and selfadjoint with respect to the Lebesgue measure on \( \mathbb{R}^n \). For each non-negative integer \( k \), the Hermite polynomials \( H_k(t) \) on \( \mathbb{R} \) are defined by \( H_k(t) = (-1)^k e^{t^2/2} \frac{d^k}{dt^k}(e^{-t^2}) \), and the Hermite functions \( h_k(t) := (2^k k! \sqrt{\pi})^{-1/2} H_k(t)e^{-t^2/2} \), \( k = 0, 1, 2, \ldots \) form an orthonormal basis of \( L^2(\mathbb{R}) \). For any multiindex \( \mu \in \mathbb{N}_0^n \), the \( n \)-dimensional Hermite functions are given by tensor product of the one dimensional Hermite functions:
\[
(1.4) \quad \Phi_{\mu}(x) = \prod_{i=1}^n h_{\mu_i}(x_i), \quad \mu = (\mu_1, \cdots, \mu_n).
\]
Then the functions \( \Phi_{\mu} \) are eigenfunctions for the Hermite operator with eigenvalue \( (2|\mu| + n) \) and \( \{\Phi_{\mu}\}_{\mu \in \mathbb{N}_0^n} \) form a complete orthonormal system in \( L^2(\mathbb{R}^n) \). Thus every \( f \in L^2(\mathbb{R}^n) \) has the Hermite expansion
\[
(1.5) \quad f(x) = \sum_{\mu} \langle f, \Phi_{\mu} \rangle \Phi_{\mu}(x) = \sum_{k=0}^{\infty} P_k f(x),
\]
where \( P_k \) denotes the Hermite projection operator given by
\[
(1.6) \quad P_k f(x) = \sum_{|\mu| = k} \langle f, \Phi_{\mu} \rangle \Phi_{\mu}(x).
\]
For \( R > 0 \) the Bochner-Riesz means for \( H \) of order \( \lambda \geq 0 \) is defined by
\[
(1.7) \quad S^\lambda_R(H)f(x) = \sum_{k=0}^{\infty} \left( 1 - \frac{2k + n}{R^2} \right)^{\lambda} P_k f(x).
\]
In one dimension, it was known from [2, 42] that if \( \lambda > 1/6 \), \( S^\lambda_R(H) \) is uniformly bounded on \( L^p \), \( 1 \leq p \leq \infty \) and, for \( 1/6 < \lambda \leq 0 \) and \( 1 \leq p \leq \infty \), \( S^\lambda_R(H) \) is uniformly bounded on \( L^p(\mathbb{R}) \) if and only if \( \lambda > (2/3)|1/p - 1/2| - 1/6 \). In higher dimensions \( (n \geq 2) \) the \( L^p \) boundedness of \( S^\lambda_R(H) \) is different and for \( \lambda > (n - 1)/2 \). Thangavelu [43] showed the uniform boundedness of \( S^\lambda_R(H) \)
on $L^p$, $1 \leq p \leq \infty$. In particular, $S^4_R(H)$ converges to $f$ in $L^1(\mathbb{R}^n)$ if and only if $\lambda > (n - 1)/2$. For $0 \leq \lambda \leq (n - 1)/2$ and $1 \leq p \leq \infty$, $p \neq 2$, it still seems natural to conjecture that $S^4_R(H)$ are uniformly bounded on $L^p(\mathbb{R}^n)$ if and only if $\lambda > \lambda(p)$ (see [46, p.259]). Thangavelu also showed $\|S^4_R(H)f\|_p \leq C\|f\|_p$ if and only if $\lambda > \lambda(p)$ under the assumption that $f$ is radial, thus the condition $\lambda > \lambda(p)$ is necessary for $L^p$ boundedness of $S^4_R(H)$. The necessity of the condition $\lambda > \lambda(p)$ for $L^p$ boundedness can also be shown by the transplantation result in [29] which deduces the $L^p$ boundedness of $S^4_R(H)$ from that of $S^4_R$. Karadzhov [27] verified the conjecture in the range $1 \leq p \leq 2n/(n + 2)$. The boundedness for $p \in [2n/(n - 2), \infty]$ follows from duality. However, it remains open to see if the conjecture is true in the range $2n/(n + 2) < p \leq 2n/(n + 1).

Almost everywhere convergence. Concerning a.e. convergence of $S^4_R(H)f$, it is known in [42, 43] (see also [45, Chapter 3]) that for every $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, $S^4_R(H)f(x)$ converges to $f(x)$ a.e. whenever $\lambda > (3n - 2)/6$. Recently, Chen, Lee, Sikora and Yan [12] studied $L^p$ boundedness of the maximal Bochner-Riesz means for the Hermite operator $H$ on $\mathbb{R}^n$ for $n \geq 2$, that is to say,

$$S^4_R(H)f(x) := \sup_{R>0} |S^4_R(H)f(x)|,$$

and it was shown that the operator $S^4_R(H)$ is bounded on $L^p(\mathbb{R}^n)$ whenever

$$p \geq \frac{2n}{n - 2} \text{ and } \lambda > \lambda(p).$$

(1.8)

As a consequence, we have

$$\lim_{R \to \infty} S^4_R(H)f(x) = f(x) \quad \text{a.e.}$$

(1.9)

for $f \in L^p(\mathbb{R}^n)$ and $p, \lambda$ satisfying (1.8). For more regarding the Hermite expansion (1.5) and the Bochner-Riesz means for the Hermite operator, we refer the reader to [13, 19, 28, 30, 43, 44, 46] and references therein.

The following is the main result of this paper which gives a new range of $p$ and $\lambda$ for the a.e convergence of the operator $S^4_R(H)$.

**Theorem 1.1.** Let $2 \leq p \leq \infty$. Then, for any function $f \in L^p(\mathbb{R}^n)$ we have (1.9) whenever $\lambda > \lambda(p)/2$. In particular, for $n = 1$, (1.9) holds for all $f \in L^p(\mathbb{R})$ whenever $\lambda > 0$.

As is already mentioned, $S^4_R(H)$ converges in $L^p$ only if $\lambda > \lambda(p)$. Surprisingly, our result tells that we only need half of the critical summability index in order to guarantee a.e. convergence of $S^4_R(H)f$. Unlike the classical Bochner-Riesz means the critical indices for $L^p$ convergence and a.e. convergence for Hermite operators do not match. Let us now recall from [9, pp.320-321] (also [33]) how the sharpness of the result in [8] can be justified. In order to study a.e. convergence of $S^4_Rf$ with $f \in L^p$, $S^4_Rf$ should be defined at least as a tempered distribution for $f \in L^p$. If so, by duality $S^4$ is defined from Schwartz class $\mathcal{S}$ to $L^p$. This implies the convolution kernel $K^4$ of $S^4$ is in $L^{p'}$, so it follows that $\lambda > \lambda(p)$ because $K^4 \in L^{p'}$ if and only if $\lambda > \lambda(p)$. However, such argument does not work for the Bochner-Riesz means for the Hermite operator since $S^4_R(H)f$ is well defined.
for any $f \in L^p$ and any $\lambda \geq 0$. For the present we do not have any evidence which supports the sharpness of the condition $\lambda > \lambda(p)/2$.

Our result also relies on the maximal estimate which is a typical device in the study of almost everywhere convergence. In order to prove Theorem 1.1 we consider the corresponding maximal operator $S^A_r(H)$ and prove the following, from which we deduce almost everywhere convergence of $S^A_r(H)f$.

**Theorem 1.2.** Let $0 \leq \alpha < n$. The operator $S^A_r(H)$ is bounded on $L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})$ if

$$\lambda > \max \left\{ \frac{\alpha - 1}{4}, 0 \right\}.$$ 

In the converse direction, if $S^A_r(H)$ is bounded on $L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})$, we necessarily have $\lambda \geq \max \{(\alpha - 1)/4, 0\}$. 

Once we have Theorem 1.2 it is easy to prove Theorem 1.1. Indeed, via a standard approximation argument (see, for example, [36] and [44, Theorem 2]) Theorem 1.2 establishes a.e. convergence of $S^A_r(H)f$ for all $f \in L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})$ provided that $\lambda > \max \{(\alpha - 1)/4, 0\}$. Now, for given $p \geq 2$ and $\lambda > \lambda(p)/2$ we choose an $\alpha$ such that $\alpha > n(1 - 2/p)$ and $\lambda > \max \{(\alpha - 1)/4, 0\}$. Our choice of $\alpha$ ensures that $f \in L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})$ if $f \in L^p$ as it follows by Hölder’s inequality. Therefore, this yields a.e. convergence of $S^A_r(H)f$ for $f \in L^p(\mathbb{R}^n)$ if $\lambda > \lambda(p)/2$.

The use of weighted $L^2$ estimate in the study of pointwise convergence for the Bochner-Riesz means goes back to Carbery et al. [8]. It turned out that the same strategy is also efficient for similar problems in different settings. For example, Bochner-Riesz means at the critical index $\lambda(p)$ for $p > 2n/(n - 1)$ (see [1, 33]), and for Bochner-Riesz means associated with sub-Laplacian on the Heisenberg group, see [21, 26].

**Square function estimate on weighted $L^2$-space.** The proof of the sufficiency part of Theorem 1.2 relies on a weighted $L^2$-estimate for the square function $\Xi_\delta$ which is defined by

$$\Xi_\delta f(x) = \left( \int_0^\infty \phi(\delta^{-1}(1 - \frac{H}{t^2}))[f(x)]^2 \frac{dt}{t} \right)^{1/2}, \quad 0 < \delta < 1/2,$$

where $\phi$ is a fixed $C^\infty$ function supported in $[2^{-3}, 2^{-1}]$ with $|\phi| \leq 1$. Here, for any bounded function $\mathfrak{M}$ the operator $\mathfrak{M}(H)$ is defined by $\mathfrak{M}(H) = \sum_{k=0}^\infty \mathfrak{M}(2n + k)P_k$. The following is our main estimate on which our results are based.

**Proposition 1.3.** Let $0 < \delta \leq 1/2, 0 < \epsilon \leq 1/2$, and let $0 \leq \alpha < n$. Then, there exists a constant $C > 0$, independent of $\delta$ and $f$, such that

$$\int_{\mathbb{R}^n} |\Xi_\delta f(x)|^2(1 + |x|)^{-\alpha} dx \leq C\delta A_{\alpha,\epsilon}(\delta) \int_{\mathbb{R}^n} |f(x)|^2(1 + |x|)^{-\alpha} dx,$$
where

\( A_{\alpha,n}(\delta) := \left\{ \begin{array}{ll} \delta^{-\alpha}, & 0 \leq \alpha \leq 1, \quad \text{if } n = 1, \\ \delta^{-\frac{\alpha}{n}}, & 1 < \alpha < n, \quad \text{if } n \geq 2. \end{array} \right. \)  

(1.12)

A similar weighted \( L^2 \) estimate with the homogeneous weight \(|x|^{-\alpha}\) was obtained in [8] for the square function associated to the Laplacian \( \Delta \):

\[ S_\delta f(x) := \left( \int_0^\infty \left| \phi\left( \delta^{-1}\left(1 + t^{-2}\Delta\right)\right)f(x) \right|^2 \frac{dt}{t} \right)^{1/2}. \]

Though we make use of the weighted \( L^2 \) estimate as in [8] there are notable differences which are due to special properties of the Hermite operator and they eventually lead to improvement of the summability indices. Let \( P_k \) be the Littlewood-Paley projection operator which is given by \( \hat{P}_k \hat{f}(\xi) = \phi(2^{-k}|\xi|)\hat{f}(\xi) \) for \( \phi \in C_c^\infty(2^{-1}, 2) \). Because of the scaling property of the Laplacian, the estimate for \( S_\delta(P_k f) \) can be reduced to the equivalent estimate for \( S_\delta(P_0 f) \). This tells that contributions from different dyadic frequency pieces are essentially identical. However, this is not the case for \( \varepsilon_\delta f \). As for the Hermite case estimate (1.11), the high and low frequency parts exhibit considerably different natures. Unlike the classical Bochner-Riesz operator, we need to handle them separately.

Indeed, as is to be seen in Section 4 below, the proof of Proposition 1.3 depends heavily on the following two facts. The first is the estimate

\[ \|(1 + |x|)^{2\alpha} f\|_2 \leq C\|(1 + H)^\alpha f\|_2 \]  

(1.13)

which holds for all \( \alpha \geq 0 \) and \( f \in L^2(\mathbb{R}^n) \) (see Proposition 2.1 below). Clearly, this can not be true if \( H \) is replaced by the Laplacian. It should be noted that the estimate (1.13) is more efficient when we deal with the low frequency part of the function. The second is a type of trace lemma (Lemma 3.1) for the Hermite operator. In fact, we obtain the estimate

\[ \|\chi_{[k,k+1]}(H)\|_{L^2(\mathbb{R}^{n}) \to L^2(\mathbb{R}^{n}, (1+|x|)^{-\alpha})} \leq Ck^{-\frac{n}{2}} \]  

(1.14)

for every \( k \in \mathbb{N} \) and all \( \alpha > 1 \). In our proof of (1.11) this inequality (1.14) takes the place of the classical trace lemma (see (3.1)), which played an important role in establishing the weighted \( L^2 \) estimate [8]. In contrast with the case of Laplacian where the corresponding trace inequality should take a scaling-invariant form, that is to say, the weight should be homogeneous (cf. (3.1)), we have the inhomogeneous weight \((1 + |x|)^{-\alpha}\) in both of the estimates (1.13) and (1.14). As to be seen later, this is due to the fact that the spectrum of the Hermite operator \( H \) is bounded away from the origin.

We show Proposition 1.3 by making use of both of the estimates (1.13) and (1.14). The proof of Proposition 1.3 divides into two parts depending on size of frequency in the spectral decomposition (1.6). For the high frequency part \( (k \geq \delta^{-1} \text{ in (1.6)}) \) the key tool is the estimate (1.14), which we combine with spatial localization argument based on the finite speed of propagation of the wave operator \( \cos(t\sqrt{H}) \). The estimate (1.14) can be compared with the restriction-type estimate due to
Karadzhov [27]:

\[ \| \chi_{[k,k+1]}(H) \|_{2 \to p} \leq C k^{(1/p - 1/2) - 1/2}, \quad \forall k \geq 1. \] (1.15)

The bound in (1.14) is much smaller than that in (1.15) when \( k \) is large. So, the estimate (1.14) becomes more efficient in the high frequency regime. In fact, the estimate (1.15) was used to show the sharp \( L^p \)-bounds on \( \mathcal{E}_\delta \) for \( 2n/(n-2) \leq p \leq \infty, n \geq 2, \) [12, Proposition 5.6]. In the low frequency part \( (k \lesssim \delta^{-1} \text{ in (1.6)}) \), inspired by [12, Lemma 5.7] (see also [19]), we directly obtain the estimate using the estimate (1.13). The estimate (1.13) does not seem to be so efficient since the bound gets worse as the frequency increases, but it is remarkable that this bound is good enough to yield the sharp result in Theorem 1.2 via balancing the estimates for low and high frequencies (see Remark 4.2).

The remainder of the paper is organized as follows. In Section 2 we obtain some weighted \( L^2 \)-estimates for the operators \((1 + H)\alpha, \alpha \geq 0\), and the Littlewood-Paley inequality for the Hermite operator. In Section 3 we prove a trace lemma for the Hermite operator \( H \) and its generalization, which play a crucial role in the proof of Theorem 1.2. The proof of Theorem 1.2 will be given in Section 4 (sufficiency) and Section 5 (necessity).

2. \( L^2 \)-estimates for the Hermite operator

In this section, we prove the estimate (1.13) and a Littlewood-Paley inequality for the Hermite operator in \( \mathbb{R}^n \), which is to be used in the proof of Theorem 1.2 in Section 4. In the following, \( \mathcal{S}(\mathbb{R}^n) \) stands for the class of Schwartz functions in \( \mathbb{R}^n \).

Proposition 2.1. For all \( \alpha \geq 0 \) the estimate (1.13) holds for any \( f \in \mathcal{S}(\mathbb{R}^n) \).

In order to show Proposition 2.1, we use the following Lemmata 2.2, 2.3 and 2.4.

Lemma 2.2. For all \( \phi \in \mathcal{S}(\mathbb{R}^n) \), \( \| \phi \|_2 \leq \| H\phi \|_2 \) and we also have \( \| H^k \phi \|_2 \leq \| H^{k+m} \phi \|_2 \) for any \( k, m \in \mathbb{N} \).

Proof. This follows from the fact that the first eigenvalue of \( H \) is bigger than or equal to 1. \( \square \)

Lemma 2.3. Let \( H \) be the Hermite operator in \( \mathbb{R} \). Then, for all \( \phi \in \mathcal{S}(\mathbb{R}) \), we have

\[ \| x^2 \phi \|_2^2 + \| -\Delta \phi \|_2^2 + \left\| \frac{d}{dx} \phi \right\|_2^2 \leq 3 \| H\phi \|_2^2. \]

Proof. Since \( \| H\phi \|_2^2 = \langle (-\Delta + x^2)\phi, (-\Delta + x^2)\phi \rangle \), a simple calculation shows

\[ \| H\phi \|_2^2 = \| -\Delta \phi \|_2^2 + 2 \text{Re} \left( \frac{d}{dx} \phi, 2x \phi \right) + 2 \left\| \frac{d}{dx} \phi \right\|_2^2 + \| x^2 \phi \|_2^2. \]
We now observe $2\text{Re}\left(\frac{d}{dx}\phi, 2x\phi\right) = \left(\frac{d}{dx}\phi, 2x\phi\right) + \langle 2x\phi, \frac{d}{dx}\phi \rangle = -2\langle \phi, \phi \rangle$. This and the above give
\[
\|x^2\phi\|^2 + \| - \Delta \phi\|^2 + 2 \left\| \frac{d}{dx} \phi \right\|^2 = \|H\phi\|^2 + 2\|\phi\|^2 \leq 3\|H\phi\|^2
\]
as desired. For the last inequality we use Lemma 2.2.

\section*{Lemma 2.4} Let $n = 1$. Then, for $\phi \in \mathcal{S}(\mathbb{R})$ and for $k \in \mathbb{N}$, we have
\[
\|x^k\|_2 \leq C_k\|H^k\phi\|_2 \quad \text{and} \quad \|Hx^{2(k-1)}\phi\|_2 \leq D_k\|H^k\phi\|_2.
\]

\section*{Proof} We begin with noting that, if $k = 1$, the first estimate in (2.1) holds with $C_1 = \sqrt{3}$ by Lemma 2.3, and the second with $D_1 = 1$. We now proceed to prove (2.1) for $k \geq 2$ by induction. Assume that (2.1) holds for $k - 1$ with some constants $C_{k-1}$ and $D_{k-1}$. A computation gives
\[
H(x^{2(k-1)}\phi) = -[(2k-2)(2k-3) - 2(2k-2)(2k-4)]x^{2(k-2)}\phi
\]
\[
- 2(2k-2)x\frac{d}{dx}(x^{2(k-2)}\phi) + x^{2k-2}H\phi.
\]
By (2.2), Lemma 2.3, and our induction assumption we see that
\[
\|H(x^{2(k-1)}\phi)\|_2 \leq (2k-2)(3k-5)\|x^{2(k-2)}\phi\|_2 + 2(2k-2)\left\| \frac{d}{dx} (x^{2(k-2)}\phi) \right\|_2 + \|x^{2k-2}H\phi\|_2
\]
\[
\leq (2k-2)(3k-5)C_{k-2}\|H^{k-2}\phi\|_2 + 2(2k-2)C_1\|H(x^{2(k-2)}\phi)\|_2 + C_{k-1}\|H^k\phi\|_2
\]
\[
\leq (2k-2)(3k-5)C_{k-2}\|H^{k-2}\phi\|_2 + 2(2k-2)C_1D_{k-1}\|H^{k-1}\phi\|_2 + C_{k-1}\|H^k\phi\|_2.
\]
Hence, we get the estimate
\[
\|H(x^{2(k-1)}\phi)\|_2 \leq D_k\|H^k\phi\|_2
\]
with $D_k = (2k-2)(3k-5)C_{k-2} + 4(2k-2)C_1D_{k-1} + C_{k-1}$. On other hand, we also have
\[
\|x^k\|_2 \leq C_1\|H(x^{2(k-1)}\phi)\|_2 \leq C_k\|H^k\phi\|_2
\]
with $C_k = C_1D_k$, $k \geq 2$. So, we readily get the estimates in (2.1). This completes the proof.

Now we are ready to prove Proposition 2.1.

\section*{Proof of Proposition 2.1} Define $H_i = -\frac{d^2}{dx_i^2} + x_i^2$, $i = 1, 2, \ldots, n$. By Lemma 2.4
\[
\|x_i|^{2k}f\|_2 \leq C_k\|H_i^k\|_2
\]
for all positive natural numbers $k \in \mathbb{N}$. Hence, we have
\[
\| (1 + |x|)^{2k}f \|_2 \leq C_k(\|f\|_2^2 + \sum_{i=1}^n \| |x_i|^{2k}f \|_2^2) \leq C_k(\|f\|_2^2 + \sum_{i=1}^n \|H_i^k\|_2^2).
\]
Since all $H_i$ are non-negative self-adjoint operators and commute strongly (that is, their spectral resolutions commute), the operators $\prod_{i=1}^n H_i^{\ell_i}$ are non-negative self-adjoint for all $\ell_i \in \mathbb{Z}_+$. Hence
\[
1 + \sum_{i=1}^n H_i^{2k} \leq (1 + \sum_{i=1}^n H_i)^{2k} = (1 + H)^{2k}
\]
for all $k \in \mathbb{N}$. Combining this with the above inequality we get

$$
\|(1 + |x|)^{2k}f\|_2^2 \leq C_k \left( \left(1 + \sum_{i=1}^n H_i \right)^{2k} f, f \right) = C_k \|(1 + H)^k f\|_2^2.
$$

This proves estimates (1.13) for all $\alpha \in \mathbb{N}$. Now, by virtue of Löwner-Heinz inequality (see, e.g., [15, Section I.5]) we can extend this estimate to all $\alpha \in [0, \infty)$. This completes the proof of Proposition 2.1.

We now recall a few standard results in the theory of spectral multipliers of non-negative selfadjoint operators (see for example, [18, 19]). The important fact is that the Feynman-Kac formula implies the Gaussian upper bound on the semigroup kernels $p_t(x, y)$ associated to $e^{-tH}$:

$$(2.3) \quad 0 \leq p_t(x, y) \leq (4\pi t)^{-n/2} \exp \left(-\frac{|x-y|^2}{4t}\right)$$

for all $t > 0$, and $x, y \in \mathbb{R}^n$.

**Proposition 2.5.** Fix a non-zero $C^\infty$ bump function $\varphi$ on $\mathbb{R}$ such that $\text{supp} \varphi \subseteq (1, 3)$. Let $\varphi_k(t) = \varphi(2^{-k}t)$, $k \in \mathbb{Z}$, for $t > 0$. Then, for any $-n < \alpha < n$,

$$(2.4) \quad \left\| \left( \sum_{k=-\infty}^{\infty} |\varphi_k(\sqrt{t}H)f|^2 \right)^{1/2} \right\|_{L^2(\mathbb{R}^n, (1+|x|)^\alpha)} \leq C_p \|f\|_{L^2(\mathbb{R}^n, (1+|x|)^\alpha)}.$$ 

This can be proved by following the standard argument, for example, see [35, Chapter IV]. We include a brief proof for convenience of the reader.

**Proof.** Let us denote by $\{r_k\}_{k \in \mathbb{Z}}$ the Rademacher functions, which is defined as follows: i) The function $r_0(t)$ is defined by $r_0(t) = 1$ on $[0, 1/2]$ and $r_0(t) = -1$ on $(1/2, 1)$, and then extended to $\mathbb{R}$ by periodicity; ii) For $k \in \mathbb{Z}\setminus\{0\}$, $r_k(t) = r_0(2^k t)$. Set

$$F(t, \lambda) := \sum_{k=-\infty}^{\infty} r_k(t)\varphi_k(\lambda).$$

A straightforward computation shows that $\sup_{R>0} |\eta F(t, R\lambda)|_{C^\beta} \leq C_\beta$ uniformly in $t \in [0, 1]$ for every integer $\beta > n/2 + 1$. On another hand, we note that the function $|x|^\alpha$ belongs to the weight $A_2$ if and only if $-n < \alpha < n$ (see [22, Example 7.1.7]). Thus, $1 + |x|^\alpha \in A_2$ and so is $(1 + |x|)^\alpha$ for $-n < \alpha < n$. Then we apply [18, Theorem 3.1] to obtain

$$\left\| \sum_{k=-\infty}^{\infty} r_k(t)\varphi_k(H)f \right\|_{L^2(\mathbb{R}^n, (1+|x|)^\alpha)}^2 = \left\| F(t, H)f \right\|_{L^2(\mathbb{R}^n, (1+|x|)^\alpha)}^2 \leq C \left\| f \right\|_{L^2(\mathbb{R}^n, (1+|x|)^\alpha)}^2$$

with $C > 0$ uniformly in $t \in [0, 1]$. Since $\sum_{k=-\infty}^{\infty} |\varphi_k(H)f|^2 \approx \int_0^1 \sum_{k=-\infty}^{\infty} |r_k(t)\varphi_k(H)f|^2$ by the property of the Rademacher functions, taking integration in $t$ on both sides of the above inequality yields (2.4). This proves Proposition 2.5. \qed
3. A trace lemma for the Hermite operator

In the work of Carbery, Rubio de Francia and Vega [8], the main tool was the trace lemma, which states a function in the Sobolev space $\dot{W}^{\alpha,2}({\mathbb R}^n)$ can be restricted to $S^{n-1}$ as an $L^2$ function on $S^{n-1}$. An alternative formulation is that, for all $0 < \varepsilon < 1/2$, $$\int_{|x| - \varepsilon \leq \varepsilon} |f(x)|^2 dx \leq C\varepsilon \|f\|_{\dot{W}^{\alpha,2}}^2, \quad 1 < \alpha < n,$$ which in turn, by taking Fourier transform and Plancherel’s theorem, is equivalent to

$$\int_{\mathbb{R}^n} |\chi_{[1-\varepsilon,1]}(\sqrt{-\Delta})f(x)|^2 dx \leq C\varepsilon \int_{\mathbb{R}^n} |f(x)|^2 \alpha^\alpha dx.$$  

In the following we establish the estimate (1.14) which is the counterpart of the above estimate (3.1) in the setting of the Hermite operator.

**Lemma 3.1.** For $\alpha > 1$, there exists a constant $C > 0$ such that the estimate (1.14) holds for every $k \in \mathbb{N}$.

**Proof.** The proof of (1.14) is inspired by the result of Bongioanni-Rogers [4, Theorem 3.3]. To show (1.14), it is sufficient to show

$$\int_{|x| \leq M} |\chi_{[k,k+1]}(H)f(x)|^2 dx \leq CMk^{-\frac{1}{\alpha}}\|f\|_2^2$$  

for every $M \geq 1$. Indeed, the estimate (1.14) immediately follows by decomposing $\mathbb{R}^n$ into dyadic shells and applying the condition $\alpha > 1$ and (3.2) to each of them.

Let us prove (3.2). For every $f \in \mathcal{S}(\mathbb{R}^n)$, we may write its Hermite expansion $f(x) = \sum_\mu \langle f, \Phi_\mu \rangle \Phi_\mu(x)$ as in (1.5). Considering this spectral decomposition, we set

$$f_i = \sum_\mu c(\mu, i) \Phi_\mu, \quad i = 1, 2, \ldots, n,$$

where $\mu = (\mu_1, \ldots, \mu_n)$ and

$$c(\mu, i) := \begin{cases} \langle f, \Phi_\mu \rangle, & \mu_i \geq \mu_j \text{ for all } j \neq i, \text{ and } \mu_i \neq \mu_\ell \text{ for all } \ell < i; \\ 0, & \text{otherwise.} \end{cases}$$

Then we decompose $f = \sum_{i=1}^n f_i$ such that $f_1, \ldots, f_n$ are orthogonal and $\mu_i \geq |\mu|/n$ whenever $\langle f_i, \Phi_\mu \rangle \neq 0$ (see for example, [4]). Recalling that the Hermite functions $\Phi_\mu$ are eigenfunctions for the Hermite operator $H$, it is clear that $\chi_{[k,k+1]}(H)f_i(x) = \sum_{2\mu + n = k} \langle f_i, \Phi_\mu \rangle \Phi_\mu(x)$. Note that $\mu_i \sim |\mu|$ if $\langle f_i, \Phi_\mu \rangle \neq 0$.

So for (3.2), it is enough to show

$$\int_{|x| \leq M} |\chi_{[k,k+1]}(H)f_i(x)|^2 dx \leq CM \sum_{2\mu + n = k} \mu_i^{-\frac{1}{\alpha}}|\langle f_i, \Phi_\mu \rangle|^2$$

for each $i = 1, \ldots, n$. By symmetry we have only to show (3.5) with $i = 1$. For the purpose we do not need the particular structure of $f_1$, so let us set $g := f_1$ for a simpler notation.
Let us write \( g(x) = \sum_{\mu} c(\mu) \Phi_\mu(x) \) with \( c(\mu) = \langle g, \Phi_\mu \rangle \). Hence, we have

\[
|\chi_{(k,k+1)}(H)g(x)|^2 = \sum_{2|\mu|+n=k} \sum_{2|v|+n=k} c(\mu)c(v)\Phi_\mu(x)\overline{\Phi_v(x)}.
\]

Thus, by Fubini’s theorem it follows that, for \( M > 0 \),

\[
\int_{[-M,M]^n} |\chi_{(k,k+1)}(H)g(x)|^2 \, dx \leq \int_{[-M,M]} \int_{\mathbb{R}^n} |\chi_{(k,k+1)}(H)g(x)|^2 \, d\bar{x} \, dx_1
\]

\[
\leq \sum_{2|\mu|+n=k} \sum_{2|v|+n=k} c(\mu)c(v) \int_{-M}^{M} h_{\mu_1}(x_1) \overline{h_{v_1}(x_1)} \, dx_1 \prod_{i=2}^{n} \langle h_{\mu_i}, h_{v_i} \rangle.
\]

From the orthogonality of \( h_{\mu_i} \), we have \( \mu_i = v_i \) for \( i = 2, \ldots, n \) whenever \( \langle h_{\mu_i}, h_{v_i} \rangle \neq 0 \) and we also have \( \mu_1 = v_1 \) since \( 2|\mu| + n = k = 2|v| + n \). Thus,

\[
\int_{[-M,M]^n} |\chi_{(k,k+1)}(H)g(x)|^2 \, dx \leq \sum_{2|\mu|+n=k} |c(\mu)|^2 \int_{-M}^{M} h_{\mu_1}^2(x_1) \, dx_1.
\]

Therefore, to complete the proof we need only to show that \( \int_{-M}^{M} h_{\mu_1}^2(t) \, dt \leq CM \mu_1^{-1/2} \). If \( \mu_1 \leq M^2 \), the estimate is trivial because \( \|h_{\mu_1}\|_2 = 1 \). Hence, we may assume \( \mu_1 > M^2 \). By the property of the Hermite functions (see [45, Lemma 1.5.1]) there exists a constant \( C \) such that \( |h_{\mu_1}(t)| \leq C \mu_1^{-1/4} \) provided that \( t \in [-M, M] \) and \( \mu_1 > M^2 \). Thus, we get the desired estimate, which completes the proof of Lemma 3.1.

Our next aim is to find a suitable trace lemma in our setting of the Hermite operator \( H \) on \( \mathbb{R}^n \). For any function \( F \) with support in \([0, 1]\) and \( 2 \leq q < \infty \), we define

\[
\|F\|_{N^2,q} := \left( \frac{1}{N^2} \sum_{\ell=1}^{N^2} \sup_{x \in \left( \frac{-\ell}{N}, \frac{\ell}{N} \right)} |F(\lambda)|^q \right)^{1/q}, \quad N \in \mathbb{N}.
\]

For \( q = \infty \), we put \( \|F\|_{N^2,\infty} = \|F\|_{\infty} \) (see [13, 17, 19]). Then we have the following result which is a consequence of Lemma 3.1.

**Lemma 3.2.** For \( \alpha > 1 \) we have

\[
\int_{\mathbb{R}^n} |F(\sqrt{H})f(x)|^2(1+|x|)^{-\alpha} \, dx \leq CN\|\delta_N F\|_{N^2,2}^2 \int_{\mathbb{R}^n} |f(x)|^2 \, dx
\]

for any function \( F \) with support in \([N/2, N]\) and \( N \in \mathbb{N} \), where \( \delta_N F(\lambda) \) is defined by \( F(N\lambda) \).

**Proof.** Since the operator \( F(\sqrt{H}) \) is selfadjoint, it is sufficient to show the dual estimate

\[
\int_{\mathbb{R}^n} |F(\sqrt{H})f(x)|^2 \, dx \leq CN\|\delta_N F\|_{N^2,2}^2 \int_{\mathbb{R}^n} |f(x)|^2(1+|x|)^\alpha \, dx.
\]

By orthogonality it is clear that

\[
\int_{\mathbb{R}^n} |F(\sqrt{H})f(x)|^2 \, dx \leq \sum_{\ell=N/16}^{N^2} \|\chi_{[\ell, \ell+1)}(\sqrt{H})F(\sqrt{H})f\|_2^2
\]
because \( F \) is supported in \([N/4, N]\). Note \( \|\chi_{(\ell-\frac{1}{N})^2}^{(\ell-\frac{1}{N})^2} (H^\alpha f)\|_2^2 \leq \|\chi_{(\ell-\frac{1}{N})^2}^{(\ell-\frac{1}{N})^2} (H f)\|_2^2 \). Hence, it follows that

\[
\int_{\mathbb{R}^n} |F(\sqrt{H}) f(x)|^2 \, dx \leq \sum_{\ell=N^2/16}^{N^2} \sup_{\lambda \in \mathfrak{A}_{\ell, 2}} |F(\lambda)|^2 \|\chi_{(\ell-\frac{1}{N})^2}^{(\ell-\frac{1}{N})^2} (H f)\|_2^2.
\]

Since \( \ell \sim N \), applying (1.14) we obtain

\[
\int_{\mathbb{R}^n} |F(\sqrt{H}) f(x)|^2 \, dx \leq C N^{-1} \sum_{\ell=N^2/16}^{N^2} \sup_{\lambda \in \mathfrak{A}_{\ell, 2}} |F(\lambda)|^2 \int_{\mathbb{R}^n} |f(x)|^2 (1 + |x|)^\nu \, dx.
\]

Thus, the estimate (3.8) follows from (3.7). \( \square \)

In Lemma 3.1, the estimate (1.14) is established for \( \alpha > 1 \) (see also [4, Theorem 3.3]). In what follows, we use a bilinear interpolation theorem to extend the range of \( \alpha \) to \( 0 < \alpha \leq 1 \). We recall that \([\cdot, \cdot]_\theta\) stands for the complex interpolation bracket ([3, 6]).

**Lemma 3.3.** Let \((A_i, B_i), i = 1, 2\) and \((A, B)\) be interpolation pairs. Suppose \( T(x_1, x_2), x_i \in A_i \cap B_i \) is a bilinear operation defined on \( \bigoplus_{i=1}^2 (A_i \cap B_i) \) with values in \( A \cap B \) such that

\[
\|T(x_1, x_2)\|_A \leq M_0 \prod_{i=1}^2 \|x_i\|_{A_i}, \quad \|T(x_1, x_2)\|_B \leq M_1 \prod_{i=1}^2 \|x_i\|_{B_i}.
\]

Then, for \( \theta \in [0, 1] \), we have

\[
\|T(x_1, x_2)\|_{[A, B]_\theta} \leq M_0^{1-\theta} M_1^{\theta} \prod_{i=1}^2 \|x_i\|_{[A_i, B_i]_\theta}.
\]

Thus \( T \) can be extended continuously to a bilinear mapping of \( \bigoplus_{i=1}^2 [A_i, B_i]_\theta \) into \([A, B]_\theta\) for any \( \theta \in [0, 1] \).

For the proof of Lemma 3.3, we refer the reader to [6, 10.1, page 118]. Making use of Lemma 3.3, we obtain the following result.

**Lemma 3.4.** Let \( 0 < \alpha \leq 1, N \in \mathbb{N}, \) and let \( F \) be a function supported in \([N/4, N]\). Then, for any \( \varepsilon > 0 \), we have

\[
\int_{\mathbb{R}^n} |F(\sqrt{H}) f(x)|^2 (1 + |x|)^{-\alpha} \, dx \leq C_\varepsilon N^{\frac{\alpha}{1+\varepsilon}} \|\delta_N F\|_{\mathcal{M}_q}^2 \int_{\mathbb{R}^n} |f(x)|^2 \, dx
\]

with \( q = 2\alpha^{-1}(1 + \varepsilon) \) for some constant \( C_\varepsilon > 0 \) independent of \( f \) and \( F \).

**Proof.** Fixing \( N \in \mathbb{N} \), we consider a set \( \mathcal{A} \) of functions defined on \( \mathbb{R} \) with supports contained in \([N/4, N]\) as follows:

\[
\mathcal{A} := \left\{ G \in L^1_{\text{loc}} : G(x) = \sum_{i=(N^2+1)/4}^{N^2} a_i \chi_{(\ell-\frac{1}{N})^2}^{(\ell-\frac{1}{N})^2}(x), \; a_i \in \mathbb{C} \right\}.
\]
Then we define a normalized counting measure $\nu$ on $\mathbb{R}$ by setting, for any Borel set $Q$,

$$
\nu(Q) := \frac{4}{3N^2} \# \left\{ i : \frac{2i-1}{2N} \in Q, i = (N^2 + 1)/4, \ldots, N^2 \right\}.
$$

We also define an $L^q$ norm on $\mathcal{A}$ by

$$
\|G\|_{L^q(\nu)} := \left( \int_{N/4}^N |G(x)|^q d\nu(x) \right)^{1/q}.
$$

Hence, $\|G\|_{L^q(\nu)} \sim \|\delta_N G\|_{N^2,q}$, and the space $\mathcal{A}$ equipped with this norm becomes a Banach space which is denoted by $\mathcal{A}_q$. It also follows (for example, see [3]) that

$$
[A_2, A_\infty]_s = A_q, \quad \frac{1}{q} = \frac{1-s}{2}, \quad s \in [0, 1].
$$

Let us denote by $B_a$ the space $B_a := \{ f \in L^1_{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n} |f(x)|^2 (1 + |x|)^a dx < \infty \}$ equipped with the norm $\|f\|_{B_a} := \|f\|_{L^2(\mathbb{R}^n, (1+|x|)^a)}$. Thus we have $[B_a, B_0]_{s} = B_s, s = (1- \theta) \nu$.

We consider the bilinear mapping $T$ given by

$$
T(G, f) := G(\sqrt{H}) f(x).
$$

From Lemma 3.2 and duality, for any $\epsilon > 0$ we have

$$
\|T(G, f)\|_{L^2(\mathbb{R}^n)} \leq CN^{1/2} \|G\|_{A_2} \|f\|_{B_{1+\epsilon}}.
$$

Since $\int_{\mathbb{R}^n} |G(\sqrt{H}) f(x)|^2 dx \leq \|\delta_N G\|_\infty \int_{\mathbb{R}^n} |f(x)|^2 dx$, we also have

$$
\|T(G, f)\|_{L^2(\mathbb{R}^n)} \leq \|G\|_{A_\infty} \|f\|_{B_0}.
$$

Now taking $A_1 = A_2, A_2 = B_{1+\epsilon}, B_1 = A_\infty, B_2 = B_0$, and $A = B = L^2(\mathbb{R}^n)$, we apply Lemma 3.3 to get

$$
\|T(G, f)\|_{L^2(\mathbb{R}^n)} \leq CN^{1/2} \|G\|_{A_2(1-\theta)} \|f\|_{B_{1-\theta(1+\epsilon)}}
$$

for $\theta \in [0, 1]$. Let $(1- \theta)(1 + \epsilon) = \alpha$. Then, equivalently, for $G \in A$ we have

$$
(3.9) \quad \int_{\mathbb{R}^n} |G(\sqrt{H}) f(x)|^2 dx \leq C_N N^\frac{\alpha}{2} \|\delta_N G\|_{N^2, 2(1+\epsilon)/\alpha}^2 \int_{\mathbb{R}^n} |f(x)|^2 (1 + |x|)^\alpha dx.
$$

We now extend the estimate to general functions supported in $[N/4, N]$. For a function $F$ supported in $[N/4, N]$, set $a_i = \sup_{k \in [N/4, N]} |F(\lambda_i)|, i = (N^2 + 1)/4, \ldots, N^2$ and define $G \in A$ by

$$
G = \sum_{i=(N^2+1)/4}^{N^2} a_i \chi_{[\frac{i}{N^2}, \frac{i+1}{N^2}]}(x).
$$

Then, it follows that $\|\delta_N F\|_{N^2,q} = \|\delta_N G\|_{N^2,q}$ and $|F(x)| \leq |G(x)|$ for $x \in \mathbb{R}$. Since $\langle |F|^2(\sqrt{H}) f, f \rangle = \sum_{k \in 2N+q} |F|^2(\sqrt{N}) \sum_{n+2\mu = k} |f, \Phi_\mu|^2$, we have

$$
\int_{\mathbb{R}^n} |F(\sqrt{H}) f(x)|^2 dx \leq \sum_{k \in 2N+q} |G|^2(\sqrt{N}) \sum_{n+2\mu = k} |f, \Phi_\mu|^2 = \int_{\mathbb{R}^n} |G(\sqrt{H}) f(x)|^2 dx.
$$
Since \( \| \delta_N F \|_{N^2,q} = \| \delta_N G \|_{N^2,q} \), using (3.9) we get
\[
\int_{\mathbb{R}^n} |F(\sqrt{H}) f(x)|^2 \, dx \leq C_n N^{2+\varepsilon} \| \delta_N F \|^2_{N^2,2/(1+\varepsilon)/\alpha} \int_{\mathbb{R}^n} |f(x)|^2 (1 + |x|)^{\alpha} \, dx.
\]
By duality we get the desired estimate, which completes the proof of Lemma 3.4.

\[\square\]

4. Proof of Theorem 1.2: Sufficiency

In this section we prove the sufficiency part of Theorem 1.2, that is to say, the operator \( S_\ast^1(H) \) is bounded on \( L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha}) \) whenever \( 0 \leq \alpha < n \) and \( \lambda > \max \left\{ \frac{\alpha}{n}, 0 \right\} \). To do so, we make use of the square function to control the maximal operator (see [7, 8, 12, 14, 31]).

4.1. Reduction to square function estimate. We begin with recalling the well known identity, for \( \lambda > 0 \) and \( \lambda > \rho \),
\[
(1 - |m|^2)^{\lambda} = C_{\lambda, \rho} R^{-2\lambda} \int_{|m|}^R (R^2 - r^2)^{1-\rho} r^{2\rho+1} \left( 1 - \frac{|m|^2}{r^2} \right) \, dt
\]
with \( C_{\lambda, \rho} = 2\Gamma(\lambda + 1)/(\Gamma(\rho + 1)\Gamma(\lambda - \rho)) \). By the spectral theory, using the argument in [36, p.278–279], we obtain
\[
S_\ast^1(H) f(x) \leq C_{\lambda, \rho} \sup_{0 < R < \infty} \left( \frac{1}{R} \right) \int_0^R \left| S_H^1 f(x) \right|^2 \, dt \right)^{1/2}
\]
provided that \( \rho > -1/2 \) and \( \lambda > \rho + 1/2 \). By a dyadic decomposition, we write \( x_n^\rho = \sum_{k \in \mathbb{Z}} 2^{-k\rho} \phi^\rho(2^k x) \) for some \( \phi^\rho \in C_0^\infty(2^{-3}, 2^{-1}) \). Thus
\[
(1 - |\xi|^2)^{\rho} =: \phi_0^\rho(\xi) + \sum_{k=1}^{\infty} 2^{-k\rho} \phi_k^\rho(\xi),
\]
where \( \phi_k^\rho = \phi(2^k (1 - |\xi|^2)) \), \( k \geq 1 \). We also note that \( \text{supp} \phi_0^\rho \subset \{ \xi : |\xi| \leq 7 \times 2^{-3} \} \) and \( \text{supp} \phi_k^\rho \subset \{ \xi : 1 - 2^{-1-k} \leq |\xi| \leq 1 - 2^{-k-3} \} \). Using (4.1), for \( \lambda > \rho + 1/2 \) we have
\[
S_\ast^1(H) f(x) \leq C \left( \sup_{0 < R < \infty} \left( \frac{1}{R} \right) \int_0^R \left| \phi_0^\rho(2^{-1} \sqrt{H}) f \right|^2 \, dt \right)^{1/2} + C \sum_{k=1}^{\infty} 2^{-k\rho} \left( \int_0^\infty \left| \phi_k^\rho(2^{-1} \sqrt{H}) f \right|^2 \, dt \right)^{1/2}.
\]

Proof of Theorem 1.2: Sufficiency. Since \( \lambda > 0 \), choosing \( \eta > 0 \) which is to be taken arbitrarily small later such that \( \lambda - \eta > 0 \), we set \( \rho = \lambda - \frac{1}{2} - \eta \). With our choice of \( \rho \) we can use (4.2). It is easy to obtain estimate for the first term in the right hand side of (4.2). Since \( (1 + |x|)^{-\alpha} \) is an \( A_2 \) Muckenhoupt weight, by virtue of (2.3) a standard argument (see for example [12, Lemma 3.1]) yields that
\[
\left\| \sup_{0 < R < \infty} \left| \phi_0^\rho(2^{-1} \sqrt{H}) f \right| \right\|_{L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})} \leq C \| \mathcal{M} f \|_{L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})},
\]
where \( \mathcal{M} \) is the Hardy-Littlewood maximal operator. By the Hardy-Littlewood maximal estimate the right hand side is bounded by \( C \| f \|_{L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})} \). Hence, in order to prove Theorem 1.2, it is sufficient to handle the remaining terms in the right hand side of (4.2).
We first consider \( n = 1 \). Using Minkowski’s inequality and the estimate (1.11) with \( \delta = 2^{-k} \), with our choice of \( \rho \) we obtain

\[
\left\| \sum_{k=1}^{\infty} 2^{-k\rho} \left( \int_{0}^{\infty} \left| \phi_k' \left( t^{-1} \sqrt{H} \right) f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^2(\mathbb{R}, (1+|x|)^{-\alpha})} \leq C \sum_{k=1}^{\infty} 2^{-k(\lambda - \epsilon/2)} \| f \|_{L^2(\mathbb{R}, (1+|x|)^{-\alpha})}.
\]

Taking \( \eta \) and \( \epsilon \) small enough, the right hand side is bounded by \( C \| f \|_{L^2(\mathbb{R}, (1+|x|)^{-\alpha})} \) for any \( \lambda > 0 \). Hence, this gives the desired boundedness of \( S_\lambda(H) \) on \( L^2(\mathbb{R}, (1+|x|)^{-\alpha}) \) for \( n = 1 \) and \( 0 \leq \alpha < 1 \).

To handle the case \( n \geq 2 \), we assume \( \alpha > 1 \) for the moment and the range of \( \alpha \) is later to be extended by interpolation. Similarly as before, we use (1.11) with \( \delta = 2^{-k} \) back into (4.2) to get

\[
\left\| \sum_{k=1}^{\infty} 2^{-k\rho} \left( \int_{0}^{\infty} \left| \phi_k' \left( t^{-1} \sqrt{H} \right) f \right|^2 \frac{dt}{t} \right)^{1/2} \right\|_{L^2(\mathbb{R}^n, (1+|x|)^{-\alpha})} \leq C \sum_{k=0}^{\infty} 2^{-k(\lambda - \eta/2)} \| f \|_{L^2(\mathbb{R}^n, (1+|x|)^{-\alpha})}.
\]

Thus, taking small enough \( \eta \) we see that \( S_\lambda(H) \) is bounded on \( L^2(\mathbb{R}^n, (1+|x|)^{-\alpha}) \) for \( n \geq 2 \) and \( 1 < \alpha < n \) provided that \( \lambda > \frac{\alpha-1}{4} \). On the other hand, we note that \( S_\lambda(H) \) is bounded on \( L^2(\mathbb{R}^n) \) for any \( \lambda > 0 \), see for example, \([12, \text{Corollary 3.3}]\). Interpolation between these two estimates (\([37, \text{Theorem (2.9)}]\)) gives \( S_\lambda(H) \) is bounded on \( L^2(\mathbb{R}^n, (1+|x|)^{-\alpha}) \) for any \( 0 < \alpha \leq 1 \) as long as \( \lambda > 0 \). This proves the sufficient part of Theorem 1.2.

To complete the proof of the sufficiency part of Theorem 1.2, it remains to prove Proposition 1.3.

### 4.2 Weighted inequality for the square function

In this subsection, we establish Proposition 1.3. For the purpose we decompose \( \mathcal{E}_\delta \) into high and low frequency parts. Let us set

\[
\mathcal{E}_\delta f(x) = \left( \int_{1/2}^{1} \left| \phi(\delta^{-1}(1 - \frac{H}{t^2}))f(x) \right|^2 \frac{dt}{t} \right)^{1/2},
\]

\[
\mathcal{E}_\delta^h f(x) = \left( \int_{\delta^{-1/2}}^{1} \left| \phi(\delta^{-1}(1 - \frac{H}{t^2}))f(x) \right|^2 \frac{dt}{t} \right)^{1/2}.
\]

Since the first eigenvalue of the Hermite operator is larger than or equal to 1, \( \phi(\delta^{-1}(1 - \frac{H}{t^2})) = 0 \) if \( t \leq 1 \) because \( \text{supp} \phi \subset (2^{-3}, 2^{-1}) \) and \( \delta \leq 1/2 \). Thus, it is clear that

\[
\mathcal{E}_\delta f(x) \leq \mathcal{E}_\delta^h f(x) + \mathcal{E}_\delta^h f(x).
\]

In order to prove Proposition 1.3 it is sufficient to show the following.

**Lemma 4.1.** Let \( A_{\alpha,n}(\delta) \) be given by (1.12). Then, for all \( 0 < \delta \leq 1/2 \) and \( 0 < \epsilon \leq 1/2 \), we have the following estimates:

\[
\int_{\mathbb{R}^n} |\mathcal{E}_\delta^h f(x)|^2 (1 + |x|)^{-\alpha} dx \leq C \delta A_{\alpha,n}(\delta) \int_{\mathbb{R}^n} |f(x)|^2 (1 + |x|)^{-\alpha} dx,
\]

\[
\int_{\mathbb{R}^n} |\mathcal{E}_\delta^h f(x)|^2 (1 + |x|)^{-\alpha} dx \leq C \delta A_{\alpha,n}(\delta) \int_{\mathbb{R}^n} |f(x)|^2 (1 + |x|)^{-\alpha} dx.
\]
Both of the proofs of the estimates (4.4) and (4.5) heavily rely on the generalized trace lemmata, Lemma 3.2 and Lemma 3.4. Though, there are distinct differences in their proofs. As for (4.4) we additionally use the estimate (1.13) which is efficient for the low frequency part. Regarding the estimate (4.5) we use the spatial localization argument which is based on the finite speed of propagation of the Hermite wave operator \( \cos(t\sqrt{H}) \). Similar strategy has been used to related problems, see for example [12]. In this regards, our proof of the estimate (4.5) is similar to that in [8]. In high frequency regime the localization strategy becomes more advantageous since the associated kernels enjoy tighter localization. This allows us to handle the weight \((1 + |x|)^{-\alpha}\) in an easier way. The choice of \(\delta^{-\frac{1}{4}}\) in the definitions of \(\Xi_{\delta}^{'}, \Xi_{\delta}^{k}\) is made by the optimizing the estimates which result from two different approaches, see Remark 4.2.

4.3. Proof of (4.4): low frequency part. We start with the Littlewood-Paley decomposition associated with the operator \(H\). Fix a function \(\varphi \in C^\infty_0\) supported in \([1 \leq |s| \leq 3\) such that \(\sum_{\infty}^\infty \varphi(2^ks) = 1\) on \(\mathbb{R}\setminus\{0\}\). By the spectral theory we have that, for any \(f \in L^2(\mathbb{R}^n)\),

\[
(4.6) \quad \sum_k \varphi_k(\sqrt{H})f := \sum_k \varphi(2^{-k}\sqrt{H}) = f.
\]

Using (4.6), we have

\[
(4.7) \quad |\Xi_{\delta}^{'}(f(x))|^2 \leq C \sum_{0 \leq k \leq 1 - \log_2 \sqrt{\delta}} \int_{2^{i-1}}^{2^{i+2}} |\phi(\delta^{-1}(1 - \frac{H}{t^2}))\varphi_k(\sqrt{H})f(x)|^2 \frac{dt}{t}
\]

for \(f \in L^2(\mathbb{R}^n) \cap L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})\). To exploit disjointness of the spectral support \(\phi(\delta^{-1}(1 - \frac{H}{t^2}))\) we make additional decomposition in \(t\). For \(k \in \mathbb{Z}\) and \(i = 0, 1, \cdots, i_0 = [8/\delta] + 1\) we set

\[
(4.8) \quad I_i = \left[2^{i-1} + i2^{i-1}\delta, 2^{i-1} + (i + 1)2^{i-1}\delta\right]
\]

so that \([2^{k-1}, 2^{k+2}] \subseteq \bigcup_{i=0}^{i_0} I_i\). Define a smooth cutoff function \(\eta_i\) adapted to the interval \(I_i\) by setting

\[
(4.9) \quad \eta_i(s) = \eta\left(i + \frac{2^{i-1} - s}{2^{i-1}\delta}\right),
\]

where \(\eta \in C^\infty_0(-1, 1)\) and \(\sum_{i \in \mathbb{Z}} \eta(\cdot - i) = 1\). For simplicity we also set

\[
\phi_\delta(s) := \phi(\delta^{-1}(1 - s^2)).
\]

We observe that, for \(t \in I_i\), \(\phi_\delta(s/t) \eta_i'(s) \neq 0\) only if \(i - i\delta - 3 \leq i' \leq i + i\delta + 3\). Hence, for \(t \in I_i\) we have

\[
\phi_\delta(t^{-1}\sqrt{H})\varphi_k(\sqrt{H}) = \sum_{i' = i - 10}^{i + 10} \phi_\delta(t^{-1}\sqrt{H})\varphi_k(\sqrt{H})\eta_i'(\sqrt{H}),
\]

and thus

\[
\int_{2^{i-1}}^{2^{i+2}} \left|\phi_\delta(t^{-1}\sqrt{H})\varphi_k(\sqrt{H})f\right|^2 \frac{dt}{t} \leq C \sum_{i} \sum_{i' = i - 10}^{i + 10} \int_{I_i} \left|\phi_\delta(t^{-1}\sqrt{H})\varphi_k(\sqrt{H})\eta_i'(\sqrt{H})f\right|^2 \frac{dt}{t}.
\]
Substituting this into (4.7), we have that

\[
|\mathcal{Z}_\delta f(x)|^2 \leq C \sum_{0 \leq k \leq 1 - \log_2 \sqrt{t}} \sum_{i} \sum_{i' = i - 10}^{i + 10} \int_{I_i} |\phi_\delta \left( t^{-1/\sqrt{t}} H \right) \eta_{i'}(\sqrt{t} H) \varphi_k(\sqrt{t} H) f(x) |^2 \frac{dt}{t}.
\]

Now we claim that, for \( 1 \leq t \leq \delta^{-1/2} \),

\[
\int_{\mathbb{R}^n} |\phi_\delta \left( t^{-1/\sqrt{t}} H \right) g(x)|^2 (1 + |x|)^{-\alpha} dx \leq C A_{\alpha,n}^\varepsilon(\delta) \int_{\mathbb{R}^n} |(1 + H)^{-\alpha/4} g(x)|^2 dx.
\]

Before we begin to prove it, we show that this concludes the proof of estimate (4.4). Combining (4.11) with (4.10), we see that \( \int_{\mathbb{R}^n} |\mathcal{Z}_\delta f(x)|^2 (1 + |x|)^{-\alpha} dx \) is bounded by

\[
CA_{\alpha,n}^\varepsilon(\delta) \sum_{0 \leq k \leq 1 - \log_2 \sqrt{t}} \sum_{i} \sum_{i' = i - 10}^{i + 10} \int_{I_i} \int_{\mathbb{R}^n} \left| \eta_{i'}(\sqrt{t} H) \varphi_k(\sqrt{t} H) (1 + H)^{-\alpha/4} f(x) \right|^2 dx \frac{dt}{t}.
\]

Since the length of interval \( I_i \) is comparable to \( 2^{k-1} \delta \), taking integration in \( t \) and using disjointness of the spectral supports, we get

\[
\int_{\mathbb{R}^n} |\mathcal{Z}_\delta f(x)|^2 (1 + |x|)^{-\alpha} dx \leq C \delta A_{\alpha,n}^\varepsilon(\delta) \int_{\mathbb{R}^n} |(1 + H)^{-\alpha/4} f(x)|^2 dx.
\]

This, being combined with Proposition 2.1, yields the desired estimate (4.4).

We now show the estimate (4.11). Let us consider the equivalent estimate

\[
\int_{\mathbb{R}^n} |\phi_\delta \left( t^{-1/\sqrt{t}} H \right) (1 + H)^{\alpha/4} g(x)|^2 (1 + |x|)^{-\alpha} dx \leq C A_{\alpha,n}^\varepsilon(\delta) \int_{\mathbb{R}^n} |g(x)|^2 dx.
\]

We first show the estimate for the case \( n \geq 2 \). Let \( N = 8|t| + 1 \). Note that \( \text{supp} \phi_\delta(\cdot/t) \subset [N/4, N] \). By Lemma 3.2,

\[
\int_{\mathbb{R}^n} |\phi_\delta \left( t^{-1/\sqrt{t}} H \right) (1 + H)^{\alpha/4} g(x)|^2 (1 + |x|)^{-\alpha} dx \leq C \left\| \phi_\delta(t^{-1} Nu) (1 + N^2 u^2)^{\alpha/4} \right\|_{N^{2,2}} \int_{\mathbb{R}^n} |g(x)|^2 dx.
\]

We now estimate \( \left\| \phi_\delta(t^{-1} Nu) (1 + N^2 u^2)^{\alpha/4} \right\|_{N^{2,2}} \). Note that \( \text{supp} \phi_\delta(t^{-1} Nu) \subset \left[ \frac{N^{1/2} - \delta}{N}, \frac{N^{1/2} + \delta}{N} \right] \). Since the length of the interval \( \left[ \frac{N^{1/2} - \delta}{N}, \frac{N^{1/2} + \delta}{N} \right] \sim \delta \) and \( N \sim t \leq \delta^{-1/2} \), we get

\[
\left\| \phi_\delta(t^{-1} Nu) (1 + N^2 u^2)^{\alpha/4} \right\|_{N^{2,2}} \leq C N^{\alpha/2} \left\| \phi_\delta(t^{-1} Nu) (1 + N^2 u^2)^{\alpha/4} \right\|_{N^{2,2}} \leq C N^{\alpha/2}.
\]

Thus, noting \( 1/2 \leq t \leq \delta^{-1/2} \) and \( \alpha > 1 \), we obtain

\[
\int_{\mathbb{R}^n} |\phi_\delta \left( t^{-1/\sqrt{t}} H \right) (1 + H)^{\alpha/4} g(x)|^2 (1 + |x|)^{-\alpha} dx \leq C N^{\alpha/2 - 1} \int_{\mathbb{R}^n} |g(x)|^2 dx \leq C \delta^{1/2 - \alpha/2} \int_{\mathbb{R}^n} |g(x)|^2 dx,
\]

which gives (4.11) in the dimensional case \( n \geq 2 \).

Next we prove (4.11) with \( n = 1 \). Let \( 0 \leq \alpha < 1 \) and \( N = 8|t| + 1 \). Note that \( \text{supp} \phi_\delta(\cdot/t) \subset [N/4, N] \). By Lemma 3.4, for any \( \varepsilon > 0 \) we have

\[
\text{LHS of (4.11) } \leq C_{\varepsilon} N^{1/2} \left\| \phi_\delta(t^{-1} Nu) (1 + N^2 u^2)^{\alpha/4} \right\|_{N^{2,2/1+(\varepsilon)}} \int_{\mathbb{R}^n} |g(x)|^2 dx.
\]
As before, in the same manner as in (4.13) we have $\|\varphi_\delta(t^{-1}Nu)(1 + N^2u^2)^{\eta/4}\|_{L^2(\nu^{(n+1)/n})}^2 \leq C N^{\alpha(n-1)/(n+1)}$, so it follows that

$$\int_{\mathbb{R}^n} |\varphi_\delta(t^{-1}\sqrt{H})(1 + H)^{\eta/4}g(x)|^2(1 + |x|)^{-\eta}dx \leq C\delta^{-\frac{\eta}{2(n+1)}}\int_{\mathbb{R}^n} |g(x)|^2dx$$

because $1 \leq t \leq \delta^{-1/2}$. This gives (4.11) with $n = 1$ and the proof of estimate (4.4) is complete. □

4.4. Proof of (4.5): high frequency part. We now make use of the finite speed of propagation of the wave operator $\cos(t\sqrt{H})$. From (2.3), it is known (see for example [16]) that the kernel of the operator $\cos(t\sqrt{H})$ satisfies

$$\text{supp} K_{\cos(t\sqrt{H})} \subseteq D(t) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t \}, \quad \forall t > 0.$$  

For any even function $F$ with $\hat{F} \in L^1(\mathbb{R})$ we have $F(t^{-1}\sqrt{H}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(\tau) \cos(\tau t^{-1}\sqrt{H}) d\tau$. Thus from the above we have

$$\text{supp} K_{F(t^{-1}\sqrt{H})} \subseteq D(t^{-1}r)$$

whenever $\text{supp} \hat{F} \subseteq [-r, r]$. This will be used in the sequel.

Fixing an even function $\vartheta \in C_0^\infty$ which is identically one on $\{|s| \leq 1\}$ and supported on $\{|s| \leq 2\}$, let us set $j_0 = \lfloor -\log_2 \delta \rfloor - 1$ and $\zeta_{j_0}(s) := \vartheta(2^{-j_0}s)$ and $\zeta_j(s) := \vartheta(2^{-j}s) - \vartheta(2^{-j+1}s)$ for $j > j_0$. Then, we clearly have

$$1 = \sum_{j \geq j_0} \zeta_j(s), \quad \forall s > 0.$$  

Recalling that $\phi_\delta(s) = \phi(\delta^{-1}(1 - s^2))$, for $j \geq j_0$ we set

$$\phi_{\delta, j}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \zeta_j(u) \hat{\phi}_\delta(u) \cos(\nu u)du.$$  

By a routine computation it can be verified that

$$|\phi_{\delta, j}(s)| \leq \begin{cases} C_N 2^{(j_0-j)N}, & |s| \in [1 - 2\delta, 1 + 2\delta], \\ C_N 2^{j-j_0} (1 + 2|s - 1|)^{-N}, & \text{otherwise}, \end{cases}$$

for any $N$ and all $j \geq j_0$ (see also [14, page 18]). By the Fourier inversion formula, we have

$$\phi\left(\delta^{-1}(1 - s^2)\right) = \sum_{j \geq j_0} \phi_{\delta, j}(s), \quad s > 0.$$  

By the finite speed propagation property (4.15), we particularly have

$$\text{supp} K_{\phi_{\delta, j}(\sqrt{\nu}t)} \subseteq D(t^{-1}2^{j+1}) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq 2^{j+1}/t\}.$$  

Now from (4.6), it follows that

$$|\tilde{\zeta}_{j}^2f(x)|^2 \leq 5 \sum_{k \geq j_0 - \log_2 \sqrt{\delta}} \int_{2^{j+1}}^{2^{k+2}} \left|\phi\left(\delta^{-1}\left(1 - \frac{H}{t^2}\right)\right)\varphi_k(\sqrt{H})f(x)\right|^2dt.$$
For $k \geq 1 - \log_2 \sqrt{\delta}$ and $j \geq j_0$, let us set
\[ E^{k,j}(t) := \left| \langle \phi_{\delta,j} (t^{-1}H) \varphi_k(\sqrt{H}) f(x) \rangle \right|^2, \quad (1 + |x|)^{-\alpha}. \]
Using the above inequality (4.20), (4.18), and Minkowski’s inequality, we have
\[ \int_{\mathbb{R}^n} |\mathbb{E}^k_t f(x)|^2 (1 + |x|)^{-\alpha} \, dx \leq C \sum_{k \geq 1 - \log_2 \sqrt{\delta}} \left( \sum_{j \geq j_0} \left( \int_{2^{k-1}}^{2^{k+2}} E^{k,j}(t) \, dt \right)^{1/2} \right)^2. \]

In order to make use of the localization property (4.19) of the kernel, we need to decompose $\mathbb{R}^n$ into disjoint cubes of side length $2^{j-k+2}$. For a given $k \in \mathbb{Z}$, $j \geq j_0$, and $m = (m_1, \cdots, m_n) \in \mathbb{Z}^n$, let us set
\[ Q_m = \left[ 2^{j-k+2}(m_1 - \frac{1}{2}), 2^{j-k+2}(m_1 + \frac{1}{2}) \right] \times \cdots \times \left[ 2^{j-k+2}(m_n - \frac{1}{2}), 2^{j-k+2}(m_n + \frac{1}{2}) \right], \]
which are disjoint dyadic cubes centered at $2^{j-k+2}m$ with side length $2^{j-k+2}$. Clearly, $\mathbb{R}^n = \cup_{m \in \mathbb{Z}^n} Q_m$. For each $m$, we define $\overline{Q_m}$ by setting
\[ \overline{Q_m} := \bigcup_{m' \in \mathbb{Z}^n : \text{dist}(Q_{m'}, Q_m) \leq \sqrt{\delta} 2^{j-k+3}} Q_m', \]
and denote
\[ M_0 := \{ m \in \mathbb{Z}^n : Q_0 \cap \overline{Q_m} \neq \emptyset \}. \]
For $t \in [2^{k-1}, 2^{k+2}]$ it follows by (4.19) that $\chi_{Q_m} \phi_{\delta,j} (t^{-1}H) \chi_{Q_m} = 0$ if $\overline{Q_m} \cap Q_m' = \emptyset$ for every $j, k$.
Hence, it is clear that
\[ \phi_{\delta,j}(t^{-1}H) \varphi_k(\sqrt{H}) f = \sum_{m, m' : \text{dist}(Q_m, Q_m') \leq \sqrt{\delta} 2^{j-k+3}} \chi_{Q_m} \phi_{\delta,j} (t^{-1}H) \chi_{Q_m} \varphi_k(\sqrt{H}) f, \]
which gives
\[ E^{k,j}(t) \leq C \sum_{m} \left( \left| \chi_{Q_m} \phi_{\delta,j} (t^{-1}H) \chi_{Q_m} \varphi_k(\sqrt{H}) f(x) \right|^2, (1 + |x|)^{-\alpha} \right). \]

To exploit orthogonality generated by the disjointness of spectral support, we further decompose $\phi_{\delta,j}$ which is not compactly supported. We choose an even function $\theta \in C_c^\infty(-4, 4)$ such that $\theta(s) = 1$ for $s \in (-2, 2)$. Set
\[ \psi_{0,\delta}(s) := \theta(\delta^{-1}(1 - s)), \quad \psi_{\ell,\delta}(s) := \theta(2^{-\ell} \delta^{-1}(1 - s)) - \theta(2^{-\ell+1} \delta^{-1}(1 - s)) \]
for all $\ell \geq 1$ such that $1 = \sum_{\ell=0}^{\infty} \psi_{\ell,\delta}(s)$ and $\phi_{\delta,j}(s) = \sum_{\ell=0}^{\infty} (\psi_{\ell,\delta} \phi_{\delta,j})(s)$ for all $s > 0$. We put it into (4.22) to write
\[ \left( \int_{2^{k-1}}^{2^{k+2}} E^{k,j}(t) \, dt \right)^{1/2} \leq \sum_{\ell=0}^{\infty} \left( \sum_{m} \int_{2^{k-1}}^{2^{k+2}} E_{m}^{k,\ell}(t) \, dt \right)^{1/2}, \]
where
\[ E_{m}^{k,\ell}(t) = \left| \chi_{Q_m} \left( \psi_{\ell,\delta} \phi_{\delta,j} \right) (t^{-1}H) \chi_{Q_m} \varphi_k(\sqrt{H}) f(x) \right|^2, (1 + |x|)^{-\alpha}. \]
Recalling (4.8) and (4.9), we observe that, for every \( t \in I_i \), it is possible that \( \psi_{\ell,\delta} (s/t) \eta_i(s) \neq 0 \) only when \( i - 2^{i+6} \leq i' \leq i + 2^{i+6} \). Hence,

\[
(\psi_{\ell,\delta} \phi_{\delta,i})(t^{-1} \sqrt{H}) = \sum_{i' = i - 2^{i+6}}^{i + 2^{i+6}} (\psi_{\ell,\delta} \phi_{\delta,i})(t^{-1} \sqrt{H}) \eta_i \sqrt{H}, \quad t \in I_i.
\]

From this and Cauchy-Schwarz’s inequality we have

\[
E_{m,i}^{k,j,\ell}(t) \leq C 2^\ell \sum_{i' = i - 2^{i+6}}^{i + 2^{i+6}} E_{m,i'}^{k,j,\ell}(t)
\]

for \( t \in I_i \) where

\[
E_{m,i'}^{k,j,\ell}(t) := \left( \left| \chi_{Q_m} (\psi_{\ell,\delta} \phi_{\delta,i})(t^{-1} \sqrt{H}) \eta_i \sqrt{H} \chi_{Q_m} (\sqrt{H} f)(x) \right|^2, (1 + |x|)^{-\alpha} \right).
\]

Combining this with (4.24), we get

\[
\left( \int_{2^{k-1}}^{2^{k+2}} E^{k,j}(t) \frac{dt}{t} \right)^{1/2} \leq C \sum_{\ell = 0}^\infty 2^{\ell/2} \left( \sum_{m} \sum_{i} \int_{I_i} \sum_{i' = i - 2^{i+6}}^{i + 2^{i+6}} E_{m,i'}^{k,j,\ell}(t) \frac{dt}{t} \right)^{1/2}.
\]  

To continue, we distinguish two cases: \( j > k \); and \( j \leq k \). In the latter case the associated cubes have side length \( \leq 4 \) so that the weight \((1 + |x|)^\alpha \) behaves like a constant on each cube \( Q_m \), so the desired estimate is easier to obtain. The first case is more involved and we need to distinguish several cases which we need to deal with separately.

### 4.4.1 Case \( j > k \)

From the above inequality (4.25) we now have

\[
\left( \int_{2^{k-1}}^{2^{k+2}} E^{k,j}(t) \frac{dt}{t} \right)^{1/2} \leq I_1(j,k) + I_2(j,k) + I_3(j,k),
\]

where

\[
I_1(j,k) := \sum_{\ell = 0}^{[-\log_2 6] - 3} 2^{\ell/2} \left( \sum_{m \in M} \sum_{i} \int_{I_i} \sum_{i' = i - 2^{i+6}}^{i + 2^{i+6}} E_{m,i'}^{k,j,\ell}(t) \frac{dt}{t} \right)^{1/2},
\]

\[
I_2(j,k) := \sum_{\ell = [-\log_2 6] - 2}^\infty 2^{\ell/2} \left( \sum_{m \in M} \sum_{i} \int_{I_i} \sum_{i' = i - 2^{i+6}}^{i + 2^{i+6}} E_{m,i'}^{k,j,\ell}(t) \frac{dt}{t} \right)^{1/2},
\]

\[
I_3(j,k) := \sum_{\ell = 0}^\infty 2^{\ell/2} \left( \sum_{m \in M} \sum_{i} \int_{I_i} \sum_{i' = i - 2^{i+6}}^{i + 2^{i+6}} E_{m,i'}^{k,j,\ell}(t) \frac{dt}{t} \right)^{1/2}.
\]

We first consider the estimate for \( I_1(j,k) \) which is the major one. The estimates for \( I_2(j,k), I_3(j,k) \) are to be obtained similarly but easier. In fact, concerning \( I_3(j,k) \), the weight \((1 + |x|)^{-\alpha} \) behave as if it were a constant, and the bound on \( I_2(j,k) \) is much smaller because of rapid decay of the associated multipliers.
Estimate of the term $I_1(j,k)$. We claim that, for any $N > 0$,

$$
I_1(j,k) \leq C_N 2^{j_0-jN} \left( \delta A_{\alpha,\nu}(\delta) \right)^{1/2} \left( \int_{\mathbb{R}^n} |\varphi_k(\sqrt{H}) f(x)|^2 (1 + |x|)^{-\alpha} dx \right)^{1/2},
$$

where $A_{\alpha,\nu}(\delta)$ is defined in (1.12).

Let us first consider the case $n \geq 2$. For (4.30), it suffices to show

$$
E^{k,i,\ell}_{m,r}(t) \leq C_N 2^{-\ell N} 2^{j_0-jN} 2^k \delta \int_{\mathbb{R}^n} |\eta_r(\sqrt{H}) [\chi_{Q_m^c} \varphi_k(\sqrt{H}) f](x)|^2 dx
$$

for any $N > 0$ while $t \in I_i$ being fixed and $i - 2^{\ell+6} \leq i' \leq i + 2^{\ell+6}$. Indeed, since the supports of $\eta_i$ are boundedly overlapping, (4.31) gives

$$
\sum_i \sum_{i',\ell} E^{k,i,\ell}_{m,r}(t) \frac{dt}{t} \leq C_N 2^{-\ell(N-1)} 2^{(j_0-jN)2^k} \delta \|\chi_{Q_m^c} \varphi_k(\sqrt{H}) f \|_2.
$$

Recalling (4.27), we take summation over $\ell$ and $m \in M_0$ to get

$$
I_1(j,k) \leq C_N 2^{j_0/2} 2^{j_0-j(N-\alpha)/2} 2^{k(1-\alpha)/2} \delta \left( 2^{(k-j)\alpha} \sum_{m \in M_0} \|\chi_{Q_m^c} \varphi_k(\sqrt{H}) f \|_2 \right)^{1/2}.
$$

Since $j > k$ and $m \in M_0$, we note that $(1 + |x|)^\alpha \leq C 2^{(j-k)\alpha}$ if $x \in Q_m$. It follows that

$$
2^{(k-j)\alpha} \sum_{m \in M_0} \|\chi_{Q_m^c} \varphi_k(\sqrt{H}) f \|_2 \leq C \int_{\mathbb{R}^n} |\varphi_k(\sqrt{H}) f(x)| (1 + |x|)^{-\alpha} dx.
$$

Noting that $j_0 = [-\log_2 \delta] - 1$ and $k \geq [-\frac{1}{2} \log_2 \delta]$, we obtain

$$
I_1(j,k) \leq C_N 2^{j_0-j(N-\alpha)/2} 2^{k} \delta^{3/4-\alpha/4} \left( \int_{\mathbb{R}^n} |\varphi_k(\sqrt{H}) f(x)|^2 (1 + |x|)^{-\alpha} dx \right)^{1/2},
$$

which clearly gives (4.30) since $N > 0$ is arbitrary.

We now proceed to prove (4.31). Note that $\text{supp}\psi_{\ell,\delta} \subseteq (1 - 2^{\ell+2}\delta, 1 + 2^{\ell+2}\delta)$, and so $\text{supp}\left(\psi_{\ell,\delta} \phi_{j_0,i} \right) (\cdot/t) \subset [t(1 - 2^{\ell+2}\delta), t(1 + 2^{\ell+2}\delta)]$. Thus, setting $R = [t(1 + 2^{\ell+2}\delta)]$, by Lemma 3.2 we get

$$
E^{k,i,\ell}_{m,r}(t) \leq R \|\psi_{\ell,\delta} \phi_{j_0,i}(R \cdot /t)\|_{L^2,R^2} \int_{\mathbb{R}^n} |\eta_r(\sqrt{H}) [\chi_{Q_m^c} \varphi_k(\sqrt{H}) f](x)|^2 dx
$$

for $0 \leq \ell \leq [-\log_2 \delta] - 3$. Since $\text{supp}(\psi_{\ell,\delta} \phi_{j_0,i}(R \cdot /t)) \subset [R^{-1}t(1 - 2^{\ell+2}\delta), R^{-1}t(1 + 2^{\ell+2}\delta)]$ and $R^2 \delta \geq 1$, we get

$$
\|\psi_{\ell,\delta} \phi_{j_0,i}(1 + 2^{\ell+2}\delta \cdot)\|_{L^2,R^2} \leq C \|\psi_{\ell,\delta} \phi_{j_0,i}(1 + 2^{\ell+2}\delta \cdot)\|_{L_\infty} \left( 2^{\ell+3} \right)^{1/2}.
$$

On the other hand, if $\ell \geq 1$, then $\psi_{\ell,\delta}(s) = 0$ for $s \in (1 - 2^{\ell}\delta, 1 + 2^{\ell}\delta)$, which together with (4.17) and $j_0 = [-\log_2 \delta] - 1$ shows that

$$
\|\psi_{\ell,\delta} \phi_{j_0,i}(1 + 2^{\ell+2}\delta \cdot)\|_{L^\infty} \leq C_N 2^{(j_0-j)N} 2^{-\ell N}, \quad \ell \geq 0
$$

for any $N < \infty$. Since $R \sim 2^k$, combining these two estimates with (4.34) we get the desired (4.31).
Now we prove (4.30) with \( n = 1 \). This case can be handled in the same manner as before, so we shall be brief. The only difference is that we use Lemma 3.4 instead of Lemma 3.2. Indeed, by following the same argument in the above and using Lemma 3.4, we get

\[
E_{m,j,k}^{l}(t) \leq C_N 2^{-\ell N} 2^{(j_0 - j)N} (\delta 2^{\ell+k})^N \int_{\mathbb{R}^N} |\eta_r(\sqrt{H})[\chi_{Q_m} \varphi_k(\sqrt{H})]f(x)|^2 \, dx
\]

for any \( N > 0 \). Once we have the above estimate, one can deduce without difficulty the estimate (4.30).

**Estimate of the term \( I_2(j,k) \).** As is clear in the decomposition of \( E^{k,j} \), the term \( I_2(j,k) \) is a tail part and we can obtain an estimate which is stronger than we need to have. In fact, we show

\[
I_2(j,k) \leq C_N 2^{(j_0 - j)N} \delta^N \left( \int_{\mathbb{R}^N} |\varphi_k(\sqrt{H})f(x)|^2 (1 + |x|)^{-\alpha} \, dx \right)^{1/2}
\]

for any \( N > 0 \). Indeed, we clearly have

\[
E_{m,j,k}^{l}(t) \leq \left\| (\psi_{t,\delta} \phi_{\delta,i}) \left( t^{-1} \sqrt{H} \right) \eta_r(\sqrt{H})[\chi_{Q_m} \varphi_k(\sqrt{H})f] \right\|_2^2
\]

\[
\leq \left\| (\psi_{t,\delta} \phi_{\delta,i})(\cdot/t) \right\|_2^2 \int_{\mathbb{R}^N} |\eta_r(\sqrt{H})[\chi_{Q_m} \varphi_k(\sqrt{H})f](x)|^2 \, dx.
\]

From the definition of \( \psi_{t,\delta} \) and (4.17) we have \( \| (\psi_{t,\delta} \phi_{\delta,i})(\cdot/t) \|_2 \leq C_N 2^{j_0 - j} (2^{l+\delta})^{-N}, \ l \geq \lfloor -\log_2 \delta \rfloor - 2. \) Thus,

\[
E_{m,j,k}^{l}(t) \leq C_N 2^{2^{j_0 - j} (2^{l+\delta})^{-N}} \int_{\mathbb{R}^N} |\eta_r(\sqrt{H})[\chi_{Q_m} \varphi_k(\sqrt{H})f](x)|^2 \, dx.
\]

After putting this in (4.28) we take summation over \( m \in M_0 \) to obtain

\[
E_2(j,k) \leq C_N \delta^{1/2 - N} 2^{(j_0 - j)N/2} \sum_{\ell = \lfloor -\log_2 \delta \rfloor - 2} 2^{-\ell(N-2)} \left( 2^{(k-j)\alpha} \sum_{m \in M_0} \left\| \chi_{Q_m} \varphi_k(\sqrt{H})f \right\|_2^2 \right)^{1/2}.
\]

As before we may use (4.33) since \( j > k \). Since \( j_0 = \lfloor -\log_2 \delta \rfloor - 1 \) and \( k \geq \lfloor -\frac{1}{2} \log_2 \delta \rfloor \), taking sum over \( \ell \) we obtain (4.36).

**Estimate of the term \( I_3(j,k) \).** We now prove the estimate

\[
I_3(j,k) \leq C_N 2^{(j_0 - j)N} \delta^{1/2} \left( \int_{\mathbb{R}^N} |\varphi_k(\sqrt{H})f(x)|^2 (1 + |x|)^{-\alpha} \, dx \right)^{1/2}.
\]

We begin with making an observation that

\[
C^{-1} (1 + |2^{j-k+2} m|) \leq 1 + |x| \leq C (1 + |2^{j-k+2} m|), \quad x \in Q_m
\]

provided that \( m \notin M_0 \). Thanks to this observation the estimates for \( E_{m,j,k}^{l}(t) \) is much simpler. By (4.38) it is clear that

\[
E_{m,j,k}^{l}(t) \leq (1 + |x_m|)^{-\alpha} \left\| (\psi_{t,\delta} \phi_{\delta,i}) \left( t^{-1} \sqrt{H} \right) \right\|_2^2 \left\| \eta_r(\sqrt{H})[\chi_{Q_m} \varphi_k(\sqrt{H})f] \right\|_2^2.
\]

Since \( \| (\psi_{t,\delta} \phi_{\delta,i})(t^{-1}\sqrt{H}) \|_2 \leq \|(\psi_{t,\delta} \phi_{\delta,i}) \|_2 \), it follows from (4.35) that we have

\[
E_{m,j,k}^{l}(t) \leq C_N (1 + |x_m|)^{-\alpha} 2^{(j_0 - j)N} 2^{-2\ell N} \left\| \eta_r(\sqrt{H})[\chi_{Q_m} \varphi_k(\sqrt{H})f] \right\|_2^2.
\]
Using this and disjointness of the spectral supports, successively, we get
\[
\sum_{m \in M_n} \sum_{i} \int_{l_i} \sum_{j,l=2^{4i+6}} E_{m,j}^{k,j}(t) \frac{dt}{t} \leq C 2^{(j_0-j)N} 2^{-2N} \delta \sum_{m \in M_n} (1 + |x_m|)^{-\alpha} \|x \varphi_k(\sqrt{H})f\|_2^2
\leq C 2^{(j_0-j)N} 2^{-2N} \delta \sum_{m \in M_n} |\varphi_k(\sqrt{H})f(x)|^2 (1 + |x|)^{-\alpha} dx
\leq C 2^{(j_0-j)N} 2^{-2N} \delta \int_{\mathbb{R}^n} |\varphi_k(\sqrt{H})f(x)|^2 (1 + |x|)^{-\alpha} dx.
\]
Finally, recalling (4.29) and taking sum over \( \ell \) yields the estimate (4.37).

Therefore, recalling \( \delta \leq \delta A_{\alpha,n}(\delta) \), we combine the estimates (4.30), (4.36), and (4.37) with (4.26) to obtain
\[
\left( \int_{2^{k-1}}^{2^{k+2}} E^{k,j}(t) \frac{dt}{t} \right)^{1 \over 2} \leq C_n 2^{(j_0-j)N} \left( \delta A_{\alpha,n}(\delta) \right)^{1 \over 2} \left( \int_{\mathbb{R}^n} |\varphi_k(\sqrt{H})f(x)|^2 (1 + |x|)^{-\alpha} dx \right)^{1 \over 2}
\]
for any \( N > 0 \) if \( j > k \).

4.4.2. **Case 2**: \( j \leq k \). In this case, the side length of each \( Q_m \) is less than 4. Thus, (4.38) holds for any \( m \in \mathbb{Z}^n \). Thus, the same argument in the proof of (4.37) works without modification. Similarly as before, we get
\[
\left( \int_{2^{k-1}}^{2^{k+2}} E^{k,j}(t) \frac{dt}{t} \right)^{1 \over 2} \leq C_n 2^{(j_0-j)N} \delta^{1 \over 2} \left( \int_{\mathbb{R}^n} |\varphi_k(\sqrt{H})f(x)|^2 (1 + |x|)^{-\alpha} dx \right)^{1 \over 2}
\]
for any \( N > 0 \), which is stronger than (4.39).

4.4.3. **Completion of the proof of (4.5)**. Finally, we are in position to complete the proof of (4.5). By the estimates (4.39) and (4.40) we now have the estimate (4.39) for any \( j \geq j_0 \) and \( k \). Putting (4.39) in the right hand side of (4.21) and then taking sum over \( j \), we obtain
\[
\int_{\mathbb{R}^n} |\varphi_k(\sqrt{H})f(x)|^2 (1 + |x|)^{-\alpha} dx \leq C \delta A_{\alpha,n}(\delta) \int_{\mathbb{R}^n} \sum_k |\varphi_k(\sqrt{H})f(x)|^2 (1 + |x|)^{-\alpha} dx.
\]
Using Proposition 2.5 we get the estimate (4.5) and this completes the proof of Lemma 4.1.

**Remark 4.2.** Let us generalize the square functions by setting
\[
\varphi_{\tau}(f(x)) = \left( \int_{1/2}^{\tau} \left| \phi(\delta^{-1}(1 - \frac{H}{\tau}))f(x) \right|^2 \frac{dt}{t} \right)^{1 \over 2}, \quad \varphi^{\tau}(f(x)) = \left( \int_{1/2}^{\infty} \left| \phi(\delta^{-1}(1 - \frac{H}{\tau}))f(x) \right|^2 \frac{dt}{t} \right)^{1 \over 2}
\]
for \( \tau \gg 1 \). By examining the proofs in the above one can obtain the bounds on \( \varphi_{\tau} \) and \( \varphi^{\tau} \) in the space \( L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha}) \). In fact, it is not difficult to see that, for \( n \geq 2 \) and \( \alpha > 1 \),
\[
\|\varphi_{\tau}\|_{L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})} \leq C \delta (\alpha - 1)^{n \over 2}, \quad \|\varphi^{\tau}\|_{L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})} \leq C \delta^{1 - \tau} (\alpha - 1)^{n \over 2}.
\]
Optimization between these two estimates gives the choice \( \tau = \delta^{-1} \).
5. Proof of Theorem 1.2: Necessity

In this section we will discuss the necessary condition for boundedness of the Bochner-Riesz means on $L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})$ and show that Theorem 1.2 is sharp up to the endpoint.

**Theorem 5.1.** Let $0 \leq \alpha < n$. Suppose that

$$\sup_{R>0} \left\| S^\lambda_R (H) \right\|_{L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha}) \rightarrow L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})} \leq C < \infty.$$  

Then, we have $\lambda \geq \max\{0, \frac{\alpha - 1}{n}\}$.

This clearly implies the necessary part of Theorem 1.2 because $\sup_{R>0} \left\| S^\lambda_R (H) \right\|_{L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha}) \rightarrow L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})} \leq \sup_{R>0} \| S^\lambda_R (f) \|_{L^2(\mathbb{R}^n, (1 + |x|)^{-\alpha})}$. The proof of Theorem 5.1 is based on the following lemma on the weighted estimates of the normalized Hermite functions.

**Lemma 5.2.** Let $\alpha \geq 0$. Then, if $k \in \mathbb{N}$ is large enough, we have

$$\int_{-\infty}^{\infty} h_k^2(x) (1 + |x|)^\alpha dx \geq C k^{\alpha/2}, \tag{5.2}$$

$$\int_{-\infty}^{\infty} h_k^2(x) (1 + |x|)^{-\alpha} dx \geq C \max\{k^{-\alpha/2}, k^{-1/2}\}. \tag{5.3}$$

To prove the lower bounds (5.2) and (5.3), we make use of the following asymptotic property of the Hermite function (see [45, 1.5.1, p. 26]):

$$h_k(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \left(N - x^2\right)^{-\frac{1}{4}} \cos \left(\frac{N(\theta - \sin \theta) - \pi}{4}\right) + O\left(N^{\frac{1}{4}}(N - x^2)^{-\frac{3}{4}}\right), \tag{5.4}$$

where $N = 2k + 1$, $0 \leq x \leq N^{1/2} - N^{-1/6}$ and $\theta = \arccos(xN^{-1/2})$.

**Proof.** We begin with showing that there exists a constant $C > 0$ such that, for any large $N$,

$$|E| \geq C \sqrt{N}, \tag{5.5}$$

where

$$E = \left\{ x \in \left[\frac{\sqrt{N}}{2}, \frac{\sqrt{N}}{\sqrt{2}}\right] : \cos \left(\frac{N(\theta - \sin \theta) - \pi}{4}\right) \geq \frac{\sqrt{2}}{2}\right\}, \quad \theta = \arccos(xN^{-1/2}).$$

For (5.5), it is enough to show

$$\left\| \left\{ t \in \left[1, \frac{1}{\sqrt{2}}\right] : \cos \left(\frac{N(\tilde{\theta} - \sin \tilde{\theta}) - \pi}{4}\right) \geq \frac{\sqrt{2}}{2}\right\} \right\| \geq C, \quad \tilde{\theta} = \arccos(t) \tag{5.6}$$

with $C$ independent of $N$, which is equivalent to (5.5) as is easy to see by change of variables. In order to see (5.6), we make change of variables $y = 2\tilde{\theta} - \sin \tilde{\theta}$. The condition $t \in [1/2, 1/\sqrt{2}]$ implies that $\tilde{\theta} \in [\pi/4, \pi/3]$ and $y \in [\pi/2 - \sqrt{2}/2, 2\pi/3 - \sqrt{3}/2]$. We note that $-\frac{\sqrt{2}}{2} < \frac{dy}{dt} = -\frac{2 - y}{\sqrt{1 - y^2}} < -\frac{2 - \sqrt{2}}{2}$.
for $t \in [1/2, 1/\sqrt{2}]$. So, (5.6) follows if we show that there exists a constant $C > 0$ independent on $N$ such that

$$\left| \left\{ y \in \left[ \frac{\pi}{2} - \frac{\sqrt{2}}{2}, \frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right] : \cos \left( \frac{Ny - \pi}{4} \right) \geq \frac{\sqrt{2}}{2} \right\} \right| \geq C,$$

but this is clear from an elementary computation.

Once we have (5.5), the desired estimate (5.2) follows because $h_k(x) \geq CN^{-1/4}, x \in E$ by (5.4).

Clearly, we also have the following estimate $\int_{-\infty}^{\infty} h_k^2(x)(1 + |x|)^{-\alpha} dx \geq Ck^{-\alpha/2}$. To complete the proof it remains to show

$$\int_{-\infty}^{\infty} h_k^2(x)(1 + |x|)^{-\alpha} dx \geq Ck^{-1/2}.$$

In the similar manner as before it is easy to show (see also [4, Lemma 3.4]) that

$$\left| \left\{ x \in [0, 1] : \cos \left( \frac{N(2\theta - \sin \theta - \pi)}{4} \right) \geq \frac{\sqrt{2}}{2} \right\} \right| \geq C,$$

with $C$ independent of $N$. Combining this with (5.4) we get $\int_0^1 h_k^2(x)dx \geq CN^{-1/2} \geq Ck^{-1/2}$ and hence the desired estimate.

**Lemma 5.3.** Let $\alpha > 0$. Then, for all $f \in L^2$, we have the estimate

$$k^{\alpha/4} \| \chi_{[k,k+1)}(H)f \|_2 \leq C \| \chi_{[k,k+1)}(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)^\alpha)}. \tag{5.7}$$

**Proof.** We write $f(x) = \sum_{\mu} c(\mu)\Phi_\mu(x)$ with $c(\mu) = \langle f, \Phi_\mu \rangle$. Following Lemma 3.1, we decompose $f = \sum_{j=1}^n f_j$ in which $f_j = \sum_{\mu} c(\mu, j)\Phi_\mu$ is defined in (3.3). Then we have $\| \chi_{[k,k+1)}(H)f \|_2 = \sum_{j=1}^n \| \chi_{[k,k+1)}(H)f_j \|_2^2$ since the functions $f_1, \cdots, f_n$ are orthogonal. Hence there is a $j \in \{1, \cdots, n\}$ such that $\| \chi_{[k,k+1)}(H)f_j \|_2^2 \geq n^{-1} \| \chi_{[k,k+1)}(H)f \|_2^2$. So, it is sufficient for (5.7) to show that

$$k^{\alpha/2} \| \chi_{[k,k+1)}(H)f_j \|_2 \leq C \| \chi_{[k,k+1)}(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)^\alpha)}. \tag{5.8}$$

Without loss of generality, we may assume $j = 1$.

We now proceed to show (5.8) for $j = 1$. Since $(1 + |x|)^\alpha \geq (1 + |x_j|)^\alpha$, we have

$$\| \chi_{[k,k+1)}(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)^\alpha)}^2 \geq \int_{\mathbb{R}^n} \sum_{2|\mu|+n=k} \sum_{2|\nu|+n=k} c(\mu)c(\nu)\Phi_\mu(x)\Phi_\nu(x)(1 + |x_1|)^\alpha dx,$$

where $c(\mu) = \langle f, \Phi_\mu \rangle$. This gives

$$\| \chi_{[k,k+1)}(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)^\alpha)}^2 \geq \sum_{2|\mu|+n=k} \sum_{2|\nu|+n=k} c(\mu)c(\nu) \int_{-\infty}^{\infty} h_{\mu_1}(x_1)h_{\nu_1}(x_1)(1 + |x_1|)^\alpha dx_1 \prod_{i=2}^n \langle h_{\mu_i}, h_{\nu_i} \rangle.$$

By orthonormality of the Hermite functions and the relation $2|\mu| + n = k = 2|\nu| + n$ we get

$$\| \chi_{[k,k+1)}(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)^\alpha)}^2 \geq \sum_{2|\mu|+n=k} |c(\mu)|^2 \int_{-\infty}^{\infty} h_{\mu_1}^2(x_1)(1 + |x_1|)^\alpha dx_1. \tag{5.9}$$
Using (5.2) in Lemma 5.2, we have
\[ \| \chi_{[k,k+1]}(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)\nu)}^2 \geq \sum_{2^j\mu+n=k} (c(\mu))^2 \mu_1^{\alpha/2}. \]
This yields the desired estimate (5.8) for \( j = 1 \) because \( \mu_1 \geq |\mu|/n \) whenever \( c(\mu,1) \neq 0 \) and so
\[ \sum_{2^j\mu+n=k} (c(\mu))^2 \mu_1^{\alpha/2} \geq \sum_{2^j\mu+n=k} (c(\mu,1))^2 \mu_1^{\alpha/2} \sim k^{\alpha/2} \sum_{2^j\mu+n=k} |c(\mu,1)|^2 = k^{\alpha/2} \| \chi_{[k,k+1]}(H)f \|_2^2. \]
The proof of Lemma 5.3 is complete. \( \square \)

In order to prove Theorem 5.1, we use the distributions \( \chi_\nu \) (see [25]) which is defined by
\[ \chi_\nu = \frac{x_\nu}{\Gamma(\nu+1)}, \quad \Re \nu > -1, \]
where \( \Gamma \) is the Gamma function and \( x_\nu = |x| \) if \( x \leq 0 \) and \( x_\nu = 0 \) if \( x > 0 \). For \( \Re \nu > -1 \), the distribution \( \chi_\nu \) is clearly well defined.

For \( \Re \nu \leq -1 \), straightforward observation \( \frac{d}{dx} x_\nu = -\nu x_\nu^{-1} \), it follows that \( \frac{d}{dx} \chi_\nu = -\chi_\nu^{-1} \) for all \( \Re \nu > 0 \). One can use the above relation to extend the family of functions \( \chi_\nu \) to a family of distributions on \( \mathbb{R} \) defined for all \( \nu \in \mathbb{C} \), see [25, Ch III, Section 3.2] for details. Since \( 1 - \chi_0^0(x) \) is the Heaviside function, it follows that \( \chi_{-\nu} = (-1)^k \delta_0^{(k-1)}, k = 1,2,\ldots, \) where \( \delta_0 \) is the \( \delta \)-Dirac measure. For compactly supported function \( F \) such that \( \text{supp} \, F \subset [0, \infty) \), we then define the Weyl fractional derivative of \( F \) of order \( \nu \) by the formula
\[ F^{(\nu)} = F * \chi_{-\nu}^{-1}, \quad \nu \in \mathbb{C}, \]
and we note that for every \( \nu \in \mathbb{C} \), \( F^{(\nu)} * \chi_{-\nu}^{-1} = F * \chi_{-\nu}^{-1} * \chi_{-\nu}^{-1} = F, \) see [20, p. 308] or [11, 19]. It follows from the above equality and Fubini’s theorem that for every \( \nu \geq 0 \),
\[ F(H) = \frac{1}{\Gamma(\nu)} \int_0^\infty F^{(\nu)}(t)(t-H)_+^{\nu-1} \, dt = \frac{1}{\Gamma(\nu)} \int_0^\infty F^{(\nu)}(R)^{\nu-1} \, S_R^{\nu-1}(H) \, dR \]
for all \( F \) compactly supported in \([0, \infty)\). Relation (5.12) plays an important role in the proof of Theorem 5.1 below.

**Proof of Theorem 5.1.** Since we are assuming that (5.1) holds, we have the equivalent dual form
\[ \sup_{R > 0} \| S_R^{\nu}(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)\nu)} \leq C < \infty, \]
Hence, for compactly supported function \( F \) such that \( \text{supp} \, F \subset [0, \infty) \), we apply (5.12) with \( \nu = \lambda + 1 \) to obtain
\[ \| F(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)\nu)} \leq C \sup_{R > 0} \| S_R^{\nu}(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)\nu)} \int_0^\infty |F^{(\lambda+1)}(s)|s^\lambda ds. \]
Let \( \eta \) be a non-negative smooth function such that \( \eta(0) = 1 \) and \( \text{supp} \, \eta \subset [-1,1] \). Taking \( F(t) = \eta(t-k) \) in the estimate (5.13), we get
\[ \| \chi_{[k,k+1]}(H)f \|_{L^2(\mathbb{R}^n,(1+|x|)\nu)} \leq CK|k| \| f \|_{L^2(\mathbb{R}^n,(1+|x|)\nu)} \int_0^\infty |F^{(\lambda+1)}(s)|s^\lambda ds. \]
because \( \int_0^\infty |F^{(\lambda+1)}(s)|s^\lambda ds \sim k^\lambda, \) and \( \eta(H-k)f = \chi_{[k,k+1]}(H)f. \)
We now consider a specific functions \( g_k, G_k \) which are given by

\[
g_k(x) = h_k(x_1)h_0(x_2) \cdots h_0(x_n), \quad G_k(x) = g_k(x)(1 + |x|)^{-\alpha},
\]

where \( 2k + n = k \). Then we have the estimate

\[
\|\chi_{\{k,k+1\}}(H)G_k\|_{L^2(\mathbb{R}^n, (1+|x|)^\alpha)} \leq Ck^4 \min \{k^{\alpha/4}, k^{1/4}\} \|\chi_{\{k,k+1\}}(H)G_k\|_2
\]

with \( C \) independent of \( k \). Indeed, since \( \|G_k\|_2 = 1 \), we have \( \|\chi_{\{k,k+1\}}(H)G_k\|_2 \geq \langle \chi_{\{k,k+1\}}(H)G_k, g_k \rangle = \langle G_k, \chi_{\{k,k+1\}}(H)g_k \rangle \). Thus, noting that \( \chi_{\{k,k+1\}}(H)g_k = g_k \) from our choice of \( g_k \) and \( g_k(x)(1 + |x|)^{-\alpha/2} = G_k(x)(1 + |x|)^{\alpha/2} \), we also have

\[
\|\chi_{\{k,k+1\}}(H)G_k\|_2 \geq |\langle G_k, g_k \rangle| = \|(1 + |x|)^{\alpha/2}G_k\|_2(1 + |x|)^{-\alpha/2}g_k\|_2.
\]

Since \( \|g_k\|_{L^2(\mathbb{R}^n, (1+|x|)^{\alpha}))} \geq \int_0^\infty |h_k(x_1)|^2(1 + |x_1|)^{-\alpha}dx_1(\int_0^1 |h_0(t)|^2dt)^{n-1} \), by the estimate (5.3) it follows that \( \|g_k\|_{L^2(\mathbb{R}^n, (1+|x|)^{\alpha}))} \geq C \max(k^{-\alpha/2}, k^{-1/2}) \). Now, combining this with the above inequality, we get

\[
\|\chi_{\{k,k+1\}}(H)G_k\|_2 \geq C \max(k^{-\alpha/4}, k^{-1/4})\|G_k\|_{L^2(\mathbb{R}^n, (1+|x|)^{\alpha}))}.
\]

On the other hand, by the estimate (5.14) we have \( k^{\alpha/4}\|G_k\|_{L^2(\mathbb{R}^n, (1+|x|)^{\alpha}))} \geq C\|\chi_{\{k,k+1\}}(H)G_k\|_{L^2(\mathbb{R}^n, (1+|x|)^{\alpha}))} \). Thus we have the estimate (5.15).

We apply (5.7) to the function \( G_k \) and combine the consequent estimate with (5.15) to get

\[
k^{\alpha/4}\|\chi_{\{k,k+1\}}(H)G_k\|_2 \leq Ck^4 \min \{k^{\alpha/4}, k^{1/4}\} \|\chi_{\{k,k+1\}}(H)G_k\|_2
\]

with \( C \) independent of \( k \). Since \( \langle G_k, \Phi_{\mu_0} \rangle = \int_{\mathbb{R}^n} |h_k(x_1)|h_0(x_2) \cdots h_0(x_n)|^2(1 + |x|)^{-\alpha}dx \neq 0 \) for \( \mu_0 = (k, 0, \ldots, 0) \), it follows that \( \|\chi_{\{k,k+1\}}(H)G_k\|_2 \neq 0 \). Thus, (5.16) implies \( k^{\alpha/4} \leq Ck^4 \min \{k^{\alpha/4}, k^{1/4}\} \) with \( C \) independent of \( k \). Letting \( k \) tend to infinity we have \( \lambda \geq \max \{\frac{\alpha}{4}, 0\} \) as desired. \( \square \)

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