Miniversal deformations of matrices of sesquilinear forms

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Abstract

V.I. Arnold [Russian Math. Surveys 26 (2) (1971) 29–43] constructed a miniversal deformation of matrices under similarity; that is, a simple normal form to which not only a given square matrix $A$ but all matrices $B$ close to it can be reduced by similarity transformations that smoothly depend on the entries of $B$. A miniversal deformation of matrices under congruence was constructed by V. Futorny and V.V. Sergeichuk [Miniversal deformations of matrices of bilinear forms, Preprint RT-MAT 2007-04, Universidade de São Paulo, 2007, 34 p. (arXiv:1004.3584v1)]. We similarly construct miniversal deformation of matrices under *congruence.

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1 Introduction

The reduction of a matrix to its Jordan form is an unstable operation: both the Jordan form and the reduction transformations depend discontinuously on the entries of the original matrix. Therefore, if the entries of a matrix are known only approximately, then it is unwise to reduce it to Jordan form. Furthermore, when investigating a family of matrices smoothly depending on parameters, then although each individual matrix can be reduced to a Jordan form, it is unwise to do so since in such an operation the smoothness relative to the parameters is lost.

For these reasons V. I. Arnold [1] (see also [2, 3]) constructed miniversal deformations of matrices under similarity; that is, a simple normal form to which not only a given square matrix \( A \) but all matrices \( B \) close to it can be reduced by similarity transformations that smoothly depend on the entries of \( B \). Miniversal deformations were also constructed for:

(a) real matrices with respect to similarity by Galin [11] (see also [2, 3]); his normal form was simplified in [13];

(b) complex matrix pencils by Edelman, Elmroth, and Kågström [8]; a simpler normal form of complex and real matrix pencils was constructed in [13];

(c) complex and real contragredient matrix pencils (i.e., matrices of pairs of counter linear operators \( U \rightleftarrows V \)) in [13];

(d) matrices of linear operators on a unitary space by Benedetti and Crag-nolini [4]; and

(e) matrices of selfadjoint operators on a complex or real vector space with scalar product given by a skew-symmetric, or symmetric, or Hermitian nonsingular form in [12, 7, 21, 23].

Futorny and Sergeichuk [9] constructed a miniversal deformation of matrices of complex bilinear forms; that is, of matrices under congruence transformations

\[ A \mapsto S^T A S, \quad S \text{ is nonsingular} \]

(and also miniversal deformations of pairs consisting of symmetric and skew-symmetric matrices since each square matrix is their sum).

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In this paper, we construct an analogous miniversal deformation of matrices of complex sesquilinear forms; that is, of matrices under *congruence transformations

\[ A \mapsto S^* AS, \quad S \text{ is nonsingular} \]

(and also miniversal deformations of pairs \((\mathcal{H}, \mathcal{K})\) of Hermitian matrices since each square matrix is uniquely represented in the form \(\mathcal{H} + i\mathcal{K}\); see Remark 3.1).

All matrices that we consider are complex matrices. In Sections 2 and 3 we give miniversal deformations of matrices of bilinear forms. In Sections 4–7 we prove that these deformations are miniversal.

2 The main theorem in terms of holomorphic functions

Define the \(n\)-by-\(n\) matrices:

\[
J_n(\lambda) := \begin{bmatrix} \lambda & 1 & 0 \\ 1 & \ddots & 1 \\ 0 & \ddots & \ddots & \ddots \\ \end{bmatrix}, \quad \Delta_n = \begin{bmatrix} 0 & \cdots & 1 \\ \cdots & \ddots & \ddots & \ddots \\ 1 & \cdots & i & 0 \\ \end{bmatrix}.
\]

The most important property of the symmetric matrices \(\Delta_n\) is that \(\Delta_n^* \Delta_n = \bar{\Delta}_n^{-1} \Delta_n\) is similar to \(J_n(1)\).

We use the following canonical form of complex matrices for *congruence.

**Theorem 2.1.** Each square complex matrix is *congruent to a direct sum, uniquely determined up to permutation of summands, of canonical matrices of the three types

\[
H_m(\lambda) := \begin{bmatrix} 0 & I_m \\ J_m(\lambda) & 0 \end{bmatrix} (|\lambda| > 1), \quad \mu \Delta_n (|\mu| = 1), \quad J_k(0) \quad (1)
\]

in which \(\lambda, \mu \in \mathbb{C}\).

This canonical form was obtained in [15] basing on [24, Theorem 3] and generalized to other fields in [18]; a direct proof that this form is canonical is given in [16, 17].
Let $A$ be a given $n$-by-$n$ matrix, and let

$$A_{\text{can}} = \bigoplus_i H_{\mu_i}(\lambda_i) \oplus \bigoplus_j \mu_j \Delta_{ij} \oplus \bigoplus_l J_{r_l}(0), \quad r_1 \geq r_2 \geq \ldots,$$

be its canonical form for congruence. All matrices that are close to $A$ are represented in the form $A + E$ in which $E \in \mathbb{C}^{n \times n}$ is close to $0$. Let $S(E)$ be a holomorphic $n \times n$ matrix function in some neighborhood of $0$ (this means that each of its entries is a power series in the entries of $E$ that is convergent in this neighborhood of $0$). Define $D(E)$ by

$$A_{\text{can}} + D(E) = S(E)^*(A + E)S(E), \quad S(0) = S.$$  

(3)

Then $D(E)$ is holomorphic at $0$ and $D(0) = 0$. In the next theorem we obtain $D(E)$ with the minimal number of nonzero entries that can be attained by using transformations (3). By a $(0,*,\circ,\bullet)$ matrix we mean a matrix whose entries are of the form $0$, $\ast$, $\circ$, and $\bullet$. The theorem involves the following $(0,*,\circ,\bullet)$ matrices:

- The $m \times n$ matrices

$$0^\leftarrow := \begin{bmatrix} * & \vdots & 0 \\ * & \ddots & \ast \\ \vdots & \ddots & \ast \end{bmatrix} \text{ if } m \leq n \text{ or } \begin{bmatrix} 0 \\ \ast \ldots \ast \end{bmatrix} \text{ if } m \geq n,$$

$$0^\rightarrow := \begin{bmatrix} \vdots \\ 0 & \ast & 0 \\ 0 & \ast \\ \ast \\ \ast & \ast & \ast \end{bmatrix} \text{ if } m \leq n \text{ or } \begin{bmatrix} 0 \\ \ast & \ast & \ast \ast \ast \end{bmatrix} \text{ if } m \geq n,$$

(choosing among the left and right matrices in these equalities, we take a matrix with the minimal number of stars; we can take any of them if $m = n$).

- The matrices

$$0^\leftarrow, \quad 0^\rightarrow, \quad 0^\leftrightarrow$$

that are obtained from $0^\leftarrow$ by the clockwise rotation through $90^\circ$, $180^\circ$, and $270^\circ$.

- The $n \times n$ matrices
\[
0^\searrow := \begin{cases} 
\text{diag}(\ast, \ldots, \ast, 0, \ldots, 0) & \text{if } n = 2k, \\
\text{diag}(\ast, \ldots, \ast, 0, \ldots, 0) & \text{if } n = 2k + 1, 
\end{cases}
\]

(4)

\[
0^\nw := \begin{cases} 
\text{diag}(\ast, \ldots, \ast, 0, \ldots, 0) & \text{if } n = 2k, \\
\text{diag}(\ast, \ldots, \ast, 0, \ldots, 0) & \text{if } n = 2k + 1. 
\end{cases}
\]

(5)

in which the number of \(*'s\) is equal to \(k\).

• The \(m \times n\) matrices

\[
0^i := \begin{bmatrix} \ast & \cdots & \ast \\ 0 & & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ \ast & \cdots & \ast \end{bmatrix}
\]

(0\(^i\) can be taken in any of these forms), and

\[
P_{mn} := \begin{bmatrix} 0 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 \ast \ldots \ast \end{bmatrix} \quad \text{with } m \leq n \quad \text{and } n - m - 1 \text{ stars.}
\]

(6)

Let \(A_{\text{can}} = A_1 \oplus A_2 \oplus \cdots \oplus A_t\) be the decomposition (2). Partition \(D\) in (3) conformably to the partition of \(A_{\text{can}}\):

\[
D = D(E) = \begin{bmatrix} D_{11} & \ldots & D_{1t} \\
\vdots & \ddots & \vdots \\
D_{t1} & \ldots & D_{tt} \end{bmatrix},
\]

(7)

and write

\[
D(A_i) := D_{ii}, \quad D(A_i, A_j) := (D_{ji}, D_{ij}) \quad \text{if } i < j.
\]

(8)

Our main result is the following theorem, which we reformulate in a more abstract form in Theorem 3.1.

**Theorem 2.2.** Let \(A\) be a square complex matrix and let (2) be its canonical matrix for congruence. All matrices \(A + E\) that are sufficiently close to \(A\) can be simultaneously reduced by transformations

\[
A + E \mapsto S(E)^*(A + E)S(E), \quad S(E) \text{ is nonsingular and holomorphic at zero,}
\]

(9)
to the form $A_{can} + \mathcal{D}$ in which $\mathcal{D}$ is a $(0, *, \circ, \bullet)$ matrix such that the number of zero entries in $\mathcal{D}$ is maximal that can be achieved by transformations (9), the symbols $\ast$, $\circ$, and $\bullet$ in $\mathcal{D}$ represent complex, real, and pure imaginary entries that depend holomorphically on the entries of $E$, and the blocks of $\mathcal{D}$ with respect to the partition (7) are defined in the notation (8) by the following equalities in which $|\lambda| > 1$, $|\lambda'| > 1$, and $|\mu| = |\mu'| = 1$:

(i) The diagonal blocks of $\mathcal{D}$ are defined by

\[
\mathcal{D}(H_m(\lambda)) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix};
\]

\[
\mathcal{D}(\mu \Delta_n) = \begin{cases} 0^{-} & \text{if } \mu \notin \mathbb{R}, \\ 0^{+} & \text{if } \mu \notin i\mathbb{R} \end{cases}
\]

(if $\mu \notin \mathbb{R} \cup i\mathbb{R}$ then we can use both $0^{-}$ and $0^{+}$);

\[
\mathcal{D}(J_n(0)) = 0^{+}.
\]

(ii) The off-diagonal blocks of $\mathcal{D}$ whose horizontal and vertical strips contain summands of $A_{can}$ of the same type are defined by

\[
\mathcal{D}(H_m(\lambda), H_n(\lambda')) = \begin{cases} (0, 0) & \text{if } \lambda \neq \lambda', \\ \begin{bmatrix} 0 & 0^{-} \\ 0^{+} & 0 \end{bmatrix}, 0 & \text{if } \lambda = \lambda' \end{cases}
\]

\[
\mathcal{D}(\mu \Delta_m, \mu' \Delta_n) = \begin{cases} (0, 0) & \text{if } \mu \neq \pm \mu', \\ (0^{-}, 0) & \text{if } \mu = \pm \mu' \end{cases}
\]

\[
\mathcal{D}(J_m(0), J_n(0)) = \begin{cases} (0^{+}, 0^{+}) & \text{if } m \geq n \text{ and } n \text{ is even}, \\ (0^{+} + \mathcal{P}_{nm}, 0^{+}) & \text{if } m \geq n \text{ and } n \text{ is odd}. \end{cases}
\]

(iii) The off-diagonal blocks of $\mathcal{D}$ whose horizontal and vertical strips contain summands of $A_{can}$ of different types are defined by

\[
\mathcal{D}(H_m(\lambda), \mu \Delta_n) = (0, 0);
\]

\[
\mathcal{D}(H_m(\lambda), J_n(0)) = \mathcal{D}(\mu \Delta_m, J_n(0)) = \begin{cases} (0, 0) & \text{if } n \text{ is even}, \\ (0^{+}, 0) & \text{if } n \text{ is odd}. \end{cases}
\]
For each \( A \in \mathbb{C}^{n \times n} \), the set
\[
T(A) := \{ C^*A + AC \mid C \in \mathbb{C}^{n \times n} \}
\] (18)
is a vector space over \( \mathbb{R} \), which is the tangent space to the congruence class of \( A \) at the point \( A \) since
\[
(I + \varepsilon C)^*A(I + \varepsilon C) = A + \varepsilon(C^*A + AC) + \varepsilon^2C^*AC
\] (19)
for all \( C \in \mathbb{C}^{n \times n} \) and \( \varepsilon \in \mathbb{R} \).

The matrix \( D \) from Theorem 2.2 was constructed such that
\[
\mathbb{C}^{n \times n} = T(A_{\text{can}}) \oplus D(\mathbb{C})
\] (20)
in which \( D(\mathbb{C}) \) is the vector space of all matrices obtained from \( D \) by replacing its entries \( *, \circ, \) and \( \bullet \) in \( D \) by complex, real, and pure imaginary numbers. Thus, the double number of stars plus the number of circles plus the number of bullets in \( D \) is the codimension over \( \mathbb{R} \) of the *congruence class of \( A_{\text{can}} \); it was independently calculated in \[6\]. Simplest miniversal deformations of matrix pencils and contagredient matrix pencils and of matrices under congruence were constructed in \[13, 9\] by an analogous method.

**Theorem 2.2** will be proved in Sections 4–7 as follows: we first prove in Lemma 4.2 that each \((0, *, \circ, \bullet)\) matrix that satisfies (20) can be taken as \( D \) in Theorem 2.2 and then verify that \( D \) from Theorem 2.2 satisfies (20).

### 3 The main theorem in terms of miniversal deformations

The notion of a miniversal deformation of a matrix with respect similarity was given by V. I. Arnold \[1\] (see also \[3, \S 30B\]) and can be extended to matrices with respect to *congruence as follows.

A deformation of a matrix \( A \in \mathbb{C}^{n \times n} \) is a holomorphic map \( \mathcal{A} : \Lambda \to \mathbb{C}^{n \times n} \) in which \( \Lambda \subset \mathbb{R}^k \) is a neighborhood of \( \tilde{0} = (0, \ldots, 0) \) and \( \mathcal{A}(\tilde{0}) = A \).

Let \( \mathcal{A} \) and \( \mathcal{B} \) be two deformations of \( A \) with the same parameter space \( \mathbb{R}^k \). \( \mathcal{A} \) and \( \mathcal{B} \) are considered as equal if they coincide on some neighborhood of \( \tilde{0} \) (this means that each deformation is a germ). We say that \( \mathcal{A} \) and \( \mathcal{B} \) are equivalent if the identity matrix \( I_a \) possesses a deformation \( \mathcal{I} \) such that
\[
\mathcal{B}(\tilde{\lambda}) = \mathcal{I}(\tilde{\lambda})^*A(\tilde{\lambda})\mathcal{I}(\tilde{\lambda})
\] (21)
for all \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_k) \) in some neighborhood of \( \tilde{0} \).

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Definition 3.1. A deformation \( A(\lambda_1, \ldots, \lambda_k) \) of a matrix \( A \) is called **versal** if every deformation \( B(\mu_1, \ldots, \mu_l) \) of \( A \) is equivalent to a deformation of the form \( A(\varphi_1(\mu_1), \ldots, \varphi_k(\mu_k)) \), in which all \( \varphi_i : \mathbb{R} \to \mathbb{R} \) are power series that are convergent in a neighborhood of \( 0 \) and \( \varphi_i(0) = 0 \). A versal deformation \( A(\lambda_1, \ldots, \lambda_k) \) of \( A \) is called **miniversal** if there is no versal deformation having less than \( k \) parameters.

For each \((0,*,o,\bullet)\) matrix \( D \), we denote by \( D(\mathbb{C}) \) the real space of all matrices obtained from \( D \) by replacing its entries \(*, o, \) and \( \bullet \) by complex, real, and pure imaginary numbers (as in (20)) and by \( D(\vec{\varepsilon}) \) the parameter matrix obtained from \( D \) by replacing each \((i,j)\) star with \( \varepsilon_{ij} + i\varepsilon'_{ij} \), each \((i,j)\) circle with \( \varepsilon_{ij} \), and each \((i,j)\) bullet with \( i\varepsilon'_{ij} \). This means that

\[
D(\mathbb{C}) := \bigoplus_{(i,j) \in \mathcal{I}_*(D)} \mathbb{C}E_{ij} \oplus \bigoplus_{(i,j) \in \mathcal{I}_o(D)} \mathbb{R}E_{ij} \oplus \bigoplus_{(i,j) \in \mathcal{I}_\bullet(D)} i\mathbb{R}E_{ij},
\]

\[
D(\vec{\varepsilon}) := \left( \sum_{(i,j) \in \mathcal{I}_*(D)} (\varepsilon_{ij} + i\varepsilon'_{ij})E_{ij} \right) + \left( \sum_{(i,j) \in \mathcal{I}_o(D)} \varepsilon_{ij}E_{ij} \right) + \left( \sum_{(i,j) \in \mathcal{I}_\bullet(D)} i\varepsilon'_{ij}E_{ij} \right),
\]

where

\[
\mathcal{I}_*(D), \mathcal{I}_o(D), \mathcal{I}_\bullet(D) \subseteq \{1, \ldots, n\} \times \{1, \ldots, n\}
\]

are the sets of indices of the stars of the circles, and of the bullets in \( D \), and \( E_{ij} \) is the elementary matrix whose \((i,j)\) entry is 1 and the others are 0.

We say that a miniversal deformation of \( A \) is **simplest** if it has the form \( A + D(\vec{\varepsilon}) \), where \( D \) is a \((0,*,o,\bullet)\) matrix. If all entries of \( D \) are stars, then it defines the deformation

\[
U(\vec{\varepsilon}) := A + \sum_{i,j=1}^{n} (\varepsilon_{ij} + i\varepsilon'_{ij})E_{ij}.
\]

Since each square matrix is *congruent to its canonical matrix, it suffices to construct miniversal deformations of canonical matrices (2). These deformations are given in the following theorem, which is a stronger form of Theorem 2.2.

Theorem 3.1. Let \( A_{\text{can}} \) be a canonical matrix \((2)\) for congruence. A simplest miniversal deformation of \( A_{\text{can}} \) can be taken in the form \( A_{\text{can}} + D(\vec{\varepsilon}) \), where \( D \) is the \((0,*,o,\bullet)\) matrix partitioned into blocks \( D_{ij} \) (as in (7)) that are defined by (10) – (17) in the notation (8).
Remark 3.1. Theorem 3.1 also gives a miniversal deformation of a canonical pair for *congruence of Hermitian matrices \((H_{\text{can}}, K_{\text{can}})\) of the same size; that is, a normal form with minimal number of parameters to which all pairs of Hermitian matrices \((H, K)\) that are close to \((H_{\text{can}}, K_{\text{can}})\) can be reduced by *congruence transformations\
\[
(H, K) \mapsto (S^* H S, S^* K S), \quad S \text{ is nonsingular,}
\]
in which \(S\) smoothly depends on the entries of \(H\) and \(K\). All one has to do is to express \(A_{\text{can}} + D(\tilde{\epsilon})\) as the sum \(\mathcal{H}(\tilde{\epsilon}) + i \mathcal{K}(\tilde{\epsilon})\) in which \(\mathcal{H}(\tilde{\epsilon})\) and \(\mathcal{K}(\tilde{\epsilon})\) are Hermitian matrices. The canonical pair \((H_{\text{can}}, K_{\text{can}})\) such that \(H_{\text{can}} + i K_{\text{can}} = A_{\text{can}}\) was described in [17, Theorem 1.2(b)].

4 Beginning of the proof of Theorem 3.1

Let us give a method of constructing simplest miniversal deformations, which is used in the proof of Theorem 3.1.

The deformation (25) is universal in the sense that every deformation \(B(\mu_1, \ldots, \mu_l)\) of \(A\) has the form \(U(\varphi_1(\mu_1, \ldots, \mu_l), \ldots, \varphi_k(\mu_1, \ldots, \mu_l))\), where \(\varphi_{ij} : \mathbb{R} \to \mathbb{R}\) are power series that are convergent in a neighborhood of 0 and \(\varphi_{ij}(0) = 0\). Hence every deformation \(B(\mu_1, \ldots, \mu_l)\) in Definition 3.1 can be replaced by \(U(\tilde{\epsilon})\), which gives the following lemma.

Lemma 4.1. The following two conditions are equivalent for any deformation \(A(\lambda_1, \ldots, \lambda_k)\) of a matrix \(A\):

(i) The deformation \(A(\lambda_1, \ldots, \lambda_k)\) is versal.

(ii) The deformation (25) is equivalent to \(A(\varphi_1(\tilde{\epsilon}), \ldots, \varphi_k(\tilde{\epsilon}))\) for some power series \(\varphi_i : \mathbb{R} \to \mathbb{R}\) that are convergent in a neighborhood of 0 and such that \(\varphi_i(0) = 0\).

If \(U\) is a subspace of a vector space \(V\), then each set \(v + U\) with \(v \in V\) is called an affine subspace parallel to \(U\).

The proof of Theorem 3.1 is based on the following lemma, which gives a method of constructing miniversal deformations. A constructive proof of this lemma is given in Section ??.

Lemma 4.2. Let \(A \in \mathbb{C}^{n \times n}\) and let \(D\) be a \((0, *, o, \bullet)\) matrix of size \(n \times n\). The following three statements are equivalent:
(i) The deformation $A + D(\varepsilon)$ defined in (23) is miniversal.

(ii) The vector space $\mathbb{C}^{n \times n}$ decomposes into the direct sum

$$\mathbb{C}^{n \times n} = T(A) \oplus_{\mathbb{R}} D(\varepsilon)$$

in which $T(A)$ is the vector space over $\mathbb{R}$ defined in (18).

(iii) Each affine $\mathbb{R}$-subspace in $\mathbb{C}^{n \times n}$ parallel to $T(A)$ intersects $D(\varepsilon)$ at exactly one point.

Proof. Define the action of the group $GL_n(\mathbb{C})$ of nonsingular $n$-by-$n$ matrices on the space $\mathbb{C}^{n \times n}$ by

$$A^S := S^* AS, \quad A \in \mathbb{C}^{n \times n}, \quad S \in GL_n(\mathbb{C}).$$

The orbit $A^{GL_n}$ of $A$ under this action consists of all matrices that are *congruent to $A$.

By (19), the space $T(A)$ is the tangent space to the orbit $A^{GL_n}$ at the point $A$. Hence $D(\varepsilon)$ is transversal to the orbit $A^{GL_n}$ at the point $A$ if

$$\mathbb{C}^{n \times n} = T(A) + D(\varepsilon)$$

(see definitions in [3, §29E]; two subspaces of a vector space are called *transversal if their sum is equal to the whole space).

This proves the equivalence of (i) and (ii) since a transversal (of the minimal dimension) to the orbit is a (mini)versal deformation [2, Section 1.6]. The equivalence of (ii) and (iii) is obvious.

Recall that the orbits of canonical matrices [2] under the action (27) were also studied in [3].

Due to Lemma 4.2, a simplest miniversal deformation of $A \in \mathbb{C}^{n \times n}$ can be constructed as follows. Let $T_1, \ldots, T_r$ be a basis of the space $T(A)$, and let $E_1, \ldots, E_{n^2}, iE_1, \ldots, iE_{n^2}$, be the basis of $\mathbb{C}^{n \times n}$ over $\mathbb{R}$, in which $E_1, \ldots, E_{n^2}$ are all elementary matrices $E_{ij}$. Removing from the sequence $T_1, \ldots, T_r, E_1, \ldots, E_{n^2}, iE_1, \ldots, iE_{n^2}$ every matrix that is a linear combination of the preceding matrices, we obtain a new basis $T_1, \ldots, T_r, E_{i_1}, \ldots, E_{i_k}, E_{j_1}, \ldots, E_{j_\ell}$ of the space $\mathbb{C}^{n \times n}$ over $\mathbb{R}$. By Lemma 4.2, the deformation

$$A(\varepsilon_1, \ldots, \varepsilon_k, \varepsilon'_1, \ldots, \varepsilon'_\ell) = A + \varepsilon_1 E_{i_1} + \cdots + \varepsilon_k E_{i_k} + \varepsilon'_1 E_{j_1} + \cdots + \varepsilon'_\ell E_{j_\ell}$$

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is miniversal.

For each $M \in \mathbb{C}^{m \times m}$ and $N \in \mathbb{C}^{n \times n}$, define the real vector space

$$T(M, N) := \{(S^* M + NR, R^* N + MS) | S \in \mathbb{C}^{m \times n}, R \in \mathbb{C}^{n \times m}\}.$$  \hspace{1cm} (28)

**Lemma 4.3.** Let $A = A_1 \oplus \cdots \oplus A_t$ be a block-diagonal matrix in which every $A_i$ is $n_i \times n_i$. Let $[D_{ij}]$ be a $(0, *, \odot, \bullet)$ matrix of the same size and partitioned into blocks conformably to the partition of $A$. Then $A + D(\varepsilon)$ is a simplest miniversal deformation of $A$ for congruence if and only if

(i) each affine $\mathbb{R}$-subspace in $\mathbb{C}^{n_i \times n_i}$ parallel to $T(A_i)$ (defined in (18))

intersects $D_{ii}(\mathbb{C})$ at exactly one point, and

(ii) each affine $\mathbb{R}$-subspace in $\mathbb{C}^{n_j \times n_i} \oplus \mathbb{C}^{n_i \times n_j}$ parallel to $T(A_i, A_j)$

intersects $D_{ji}(\mathbb{C}) \oplus D_{ij}(\mathbb{C})$ at exactly one point.

**Proof.** By Lemma 4.2(iii), $A + D(\varepsilon)$ is a simplest miniversal deformation of $A$ if and only if for each $C \in \mathbb{C}^{n \times n}$ the affine $\mathbb{R}$-subspace $C + T(A)$ contains exactly one $D \in D(\mathbb{C})$; that is, exactly one

$$D = C + S^* A + A S \in D(\mathbb{C}) \quad \text{with} \quad S \in \mathbb{C}^{n \times n}.$$  \hspace{1cm} (29)

Partition $D$, $C$, and $S$ into blocks conformably to the partition of $A$. By (29), for each $i$ we have

$$D_{ii} = C_{ii} + S_{ii}^* A_i + A_i S_{ii},$$

and for all $i$ and $j$ such that $i < j$ we have

$$D_{ij} = C_{ij} + S_{ij}^* A_i + A_j S_{ij}, \quad D_{ji} = C_{ji} + S_{ji}^* A_j + A_i S_{ji}.$$  \hspace{1cm} (30)

Thus, (29) is equivalent to the conditions

$$D_{ii} = C_{ii} + S_{ii}^* A_i + A_i S_{ii} \in D_{ii}(\mathbb{C}) \quad \text{for} \quad 1 \leq i \leq t$$

and

$$(D_{ji}, D_{ij}) = (C_{ji}, C_{ij}) + (S_{ij}^* A_i + A_j S_{ji}, S_{ji}^* A_j + A_i S_{ij}) \in D_{ji}(\mathbb{C}) \oplus D_{ij}(\mathbb{C})$$  \hspace{1cm} (31)

for $1 \leq i < j \leq t$. Hence for each $C \in \mathbb{C}^{n \times n}$ there exists exactly one $D \in D$ of the form (29) if and only if

(i') for each $C_{ii} \in \mathbb{C}^{n_i \times n_i}$ there exists exactly one $D_{ii} \in D_{ii}$ of the form (30), and
(ii') for each $(C_{ji}, C_{ij}) \in \mathbb{C}^{n_i \times n_j} \oplus \mathbb{C}^{n_j \times n_i}$ there exists exactly one $(D_{ji}, D_{ij}) \in D_{ji}(\mathbb{C}) \oplus D_{ij}(\mathbb{C})$ of the form (31).

This proves the lemma.

\[ \Box \]

Corollary 4.1. In the notation of Lemma 4.3 $A + D(\varepsilon)$ is a miniversal deformation of $A$ if and only if each submatrix of the form

\[
\begin{bmatrix}
A_i + D_{ii}(\varepsilon) & D_{ij}(\varepsilon) \\
D_{ji}(\varepsilon) & A_j + D_{jj}(\varepsilon)
\end{bmatrix}
\]

with $i < j$

is a miniversal deformation of $A_i \oplus A_j$. A similar reduction to the case of canonical forms for congruence with two direct summands was used in [6] for the solution of the equation $XA + AX^* = 0$.

Let us start to prove Theorem 2.2. Let $A_{\text{can}} = A_1 \oplus A_2 \oplus \cdots \oplus A_t$ be the canonical matrix (2), and let $D = [D_{ij}]_{i,j=1}^t$ be the $(0,*,\circ,\bullet)$ matrix that has been constructed in Theorem 3.1. Each $A_i$ has the form $H_n(\lambda)$, or $\mu \Delta_n$, or $J_n(0)$, and so there are 9 types of diagonal blocks $D(A_i) = D_{ii}$ and pairs of off-diagonal blocks $D(A_i, A_j) = (D_{ji}, D_{ij})$, $i < j$; they have been given in Theorem 2.2. It suffices to prove that (10)–(17) satisfy the conditions (i) and (ii) from Lemma 4.3.

5 Diagonal blocks of $D$

First we verify that the diagonal blocks of $D$ defined in part (i) of Theorem 2.2 satisfy the condition (i) of Lemma 4.3.

5.1 Diagonal blocks $D(H_n(\lambda))$ with $|\lambda| > 1$

Due to Lemma 4.3(i), it suffices to prove that each 2n-by-2n matrix $A = [A_{ij}]_{i,j=1}^2$ can be reduced to exactly one matrix of the form (10) by adding

\[
\begin{bmatrix}
S_{11}^* & S_{21}^* \\
S_{12}^* & S_{22}^*
\end{bmatrix}
\begin{bmatrix}
0 & I_n \\
J_n(\lambda) & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & I_n \\
J_n(\lambda) & 0
\end{bmatrix}
\begin{bmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{bmatrix}
= \begin{bmatrix}
S_{21}^*J_n(\lambda) + S_{21} & S_{11}^* + S_{22} \\
S_{22}^*J_n(\lambda) + J_n(\lambda)S_{11} & S_{12}^* + J_n(\lambda)S_{12}
\end{bmatrix}
\]

(32)
in which $S = [S_{ij}]_{i,j=1}^2$ is an arbitrary $2n$-by-$2n$ matrix. Taking $S_{22} = -A_{12}$ and the other $S_{ij} = 0$, we obtain a new matrix $A$ with $A_{12} = 0$. To preserve $A_{12}$, we hereafter must take $S$ with $S_{11} + S_{22} = 0$. Therefore, we can add $S_{21}J_n(\lambda) + S_{21}$ to (the new) $A_{11}$, $S_{12} + J_n(\lambda)S_{12}$ to $A_{22}$, and $S_{11}J_n(\lambda) + J_n(\lambda)S_{11}$ to $A_{21}$. Using these additions, we can reduce $A$ to the form (10) due to the following 3 lemmas.

**Lemma 5.1.** Adding $SJ_n(\lambda) + S^*$, in which $\lambda$ is a fixed complex number, $|\lambda| \neq 1$, and $S$ is arbitrary, we can reduce each $n$-by-$n$ matrix to the zero matrix.

**Proof.** Let $A = [a_{ij}]$ be an arbitrary $n$-by-$n$ matrix. We will reduce it along its skew diagonals starting from the upper left hand corner:

$$a_{11}, \ (a_{21}, a_{12}), \ (a_{31}, a_{22}, a_{13}), \ \ldots, \ a_{nn}, \quad (33)$$

by adding $\Delta A := SJ_n(\lambda) + S^*$ in which $S = [s_{ij}]$ is any $n$-by-$n$ matrix. For instance, if $n = 4$ then $\Delta A$ is

$$\begin{bmatrix}
\lambda s_{11} + 0 + \bar{s}_{11} & \lambda s_{12} + s_{11} + \bar{s}_{21} & \lambda s_{13} + s_{12} + \bar{s}_{31} & \lambda s_{14} + s_{13} + \bar{s}_{41} \\
\lambda s_{21} + 0 + \bar{s}_{12} & \lambda s_{22} + s_{21} + \bar{s}_{22} & \lambda s_{23} + s_{22} + \bar{s}_{32} & \lambda s_{24} + s_{23} + \bar{s}_{42} \\
\lambda s_{31} + 0 + \bar{s}_{13} & \lambda s_{32} + s_{31} + \bar{s}_{23} & \lambda s_{33} + s_{32} + \bar{s}_{33} & \lambda s_{34} + s_{33} + \bar{s}_{43} \\
\lambda s_{41} + 0 + \bar{s}_{14} & \lambda s_{42} + s_{41} + \bar{s}_{24} & \lambda s_{43} + s_{42} + \bar{s}_{34} & \lambda s_{44} + s_{43} + \bar{s}_{44}
\end{bmatrix}$$

We reduce $A$ to 0 by induction: Assume that the first $t - 1$ skew diagonals of $A$ are zero. To preserve them, we take the first $t - 1$ skew diagonals of $S$ equalling zero. If the $t^{th}$ skew diagonal of $S$ is $(x_1, \ldots, x_r)$, then we can add

$$(\lambda x_1 + \bar{x}_r, \ \lambda x_2 + \bar{x}_{r-1}, \ \lambda x_3 + \bar{x}_{r-2}, \ \ldots, \ \lambda x_r + \bar{x}_1) \quad (34)$$

to the $t^{th}$ skew diagonal of $A$. Let us show that each vector $(c_1, \ldots, c_r) \in \mathbb{C}^r$ is represented in the form (34); that is, the corresponding system of linear equations

$$\lambda x_1 + \bar{x}_r = c_1, \ \ldots, \ \lambda x_j + \bar{x}_{r-j+1} = c_j, \ \ldots, \ \lambda x_r + \bar{x}_1 = c_r \quad (35)$$

has a solution. This is clear if $\lambda = 0$. Suppose that $\lambda \neq 0$.

Let $r = 2k + 1$. By (35), $x_j = \lambda^{-1}(c_j - \bar{x}_{r-j+1})$. Replace $j$ by $r - j + 1$:

$$x_{r-j+1} = \lambda^{-1}(c_{r-j+1} - \bar{x}_j). \quad (36)$$
Substituting (36) into the first \( k + 1 \) equations of (35), we obtain
\[
\lambda x_j + \tilde{\lambda}^{-1}(\tilde{c}_{r-j+1} - x_j) = (\lambda - \tilde{\lambda}^{-1})x_j + \tilde{\lambda}^{-1}\tilde{c}_{r-j+1} = c_j, \quad j = 1, \ldots, k + 1.
\]
Since \(|\lambda| \neq 1\), \( \lambda - \tilde{\lambda}^{-1} \neq 0 \) and we have
\[
x_j = \frac{c_j - \tilde{\lambda}^{-1}\tilde{c}_{r-j+1}}{\lambda - \tilde{\lambda}^{-1}} = \frac{\tilde{\lambda}c_j - \tilde{c}_{r-j+1}}{\lambda\lambda - 1}, \quad j = 1, \ldots, k + 1. \tag{37}
\]
The equalities (36) and (37) give a solution of (35).

If \( r = 2k \), then (35) is solved in the same way, but we take \( j = 1, \ldots, k \) in (37).

**Lemma 5.2.** Adding \( J_n(\lambda)R + R^* \), in which \( \lambda \) is a fixed complex number, \(|\lambda| \neq 1\), and \( R \) is arbitrary, we can reduce each \( n \times n \) matrix to the zero matrix.

**Proof.** By Lemma 5.1 for each \( n \times n \) matrix \( B \) there exists \( S \) such that
\[
B^* + J_n(\lambda)^*S^* + S = 0.
\]
Write
\[
Z := \begin{bmatrix} 0 & \cdots & 1 \\ 1 & \cdots & 0 \end{bmatrix}.
\]
Because \( Z J_n(\lambda)^*Z = J_n(\tilde{\lambda}) \), we have
\[
ZB^*Z + J_n(\tilde{\lambda})(ZSZ)^* + ZSZ = 0.
\]
This ensures Lemma 5.2 since \( ZB^*Z \) is arbitrary. \( \square \)

**Lemma 5.3.** Adding \( SJ_n(\lambda) + J_n(\lambda)S \), we can reduce each \( n \times n \) matrix to exactly one matrix of the form \( 0^* \).

**Proof.** Let \( A = [a_{ij}] \) be an arbitrary \( n \times n \) matrix. Adding
\[
SJ_n(\lambda) - J_n(\lambda)S = SJ_n(0) - J_n(0)S
\]

\[
= \begin{bmatrix}
  s_{21} - 0 & s_{22} - s_{11} & s_{23} - s_{12} & \cdots & s_{2n} - s_{1,n-1} \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  s_{n-1,1} - 0 & s_{n-1,2} - s_{n-2,1} & s_{n-1,3} - s_{n-2,2} & \cdots & s_{n-1,n} - s_{n-2,n-1} \\
  s_{n1} - 0 & s_{n2} - s_{n-1,1} & s_{n3} - s_{n-1,2} & \cdots & s_{nn} - s_{n-1,n-1} \\
  0 - 0 & 0 - s_{n1} & 0 - s_{n2} & \cdots & 0 - s_{n,n-1}
\end{bmatrix},
\]

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we reduce $A$ along the diagonals
\[
a_{n1}, \ (a_{n-1,1}, a_{n2}), \ (a_{n-2,1}, a_{n-1,2}, a_{n3}), \ldots, \ a_{1n}
\]
to the form $0^\omega$.

\[\square\]

5.2 Diagonal blocks $\mathcal{D}(\mu \Delta_n)$ with $|\mu| = 1$

Due to Lemma 4.3(i), it suffices to prove that each $n \times n$ matrix $A$ can be reduced to exactly one matrix of the form $0^\omega$ if $\mu \not\in \mathbb{R}$ or $0^\omega$ if $\mu \not\in i\mathbb{R}$ by adding $\Delta A := \mu(S^* \Delta_n + \Delta_n S)$ in which $S = [s_{ij}]$ is any $n$-by-$n$ matrix.

For example, if $n = 4$ then $\Delta A$ is
\[
\mu \begin{bmatrix}
\bar{s}_{41} + s_{41} + i(0 + 0) & \bar{s}_{31} + s_{32} + i(\bar{s}_{41} + 0) & \ldots & \bar{s}_{11} + s_{44} + i(\bar{s}_{21} + 0) \\
\bar{s}_{42} + s_{31} + i(0 + s_{11}) & \bar{s}_{32} + s_{32} + i(\bar{s}_{42} + s_{22}) & \ldots & \bar{s}_{12} + s_{44} + i(\bar{s}_{22} + s_{44}) \\
\bar{s}_{43} + s_{31} + i(0 + s_{31}) & \bar{s}_{33} + s_{32} + i(\bar{s}_{43} + s_{33}) & \ldots & \bar{s}_{13} + s_{44} + i(\bar{s}_{23} + s_{34}) \\
\bar{s}_{44} + s_{31} + i(0 + s_{21}) & \bar{s}_{34} + s_{32} + i(\bar{s}_{44} + s_{22}) & \ldots & \bar{s}_{14} + s_{44} + i(\bar{s}_{24} + s_{44})
\end{bmatrix}.
\]

Let $\Delta A = \mu[\delta_{ij}]$. Write
\[
s_{n+1,i} := 0 \quad \text{for} \ j = 1, \ldots, n. \quad (38)
\]

Then
\[
\delta_{ij} = \bar{s}_{n+1-i,j} + s_{n+1-i,j} + i(\bar{s}_{n+2-j,i} + s_{n+2-i,j}). \quad (39)
\]

Step 1: Let us prove that
\[
\exists S : \ A + \Delta A \text{ is a diagonal matrix.} \quad (40)
\]

Let $A = \mu[a_{ij}]$. We need to prove that the system of equations
\[
\delta_{ij} = -a_{ij}, \quad i, j = 1, \ldots, n, \quad i \neq j \quad (41)
\]

with unknowns $s_{ij}$ is consistent for all $a_{ij}$.

Since
\[
\bar{s}_{ij} = \bar{s}_{n+2-j,i} + s_{n+2-i,j} - i(\bar{s}_{n+2-j,i} + s_{n+2-i,j}) = -\bar{a}_{ij}
\]
we have
\[
\bar{s}_{n+1-i,j} + s_{n+1-i,j} = (\delta_{ij} + \bar{s}_{ij})/2 = (-a_{ij} - \bar{a}_{ij})/2
\]
\[
\bar{s}_{n+2-j,i} + s_{n+2-i,j} = (\delta_{ij} - \bar{s}_{ij})/(2i) = (-a_{ij} + \bar{a}_{ij})/(2i)
\]

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Thus, the system of equations (41) is equivalent to the system

\begin{align*}
\bar{s}_{n+1-j,i} + s_{n+1-i,j} &= b_{ij} \quad i, j = 1, \ldots, n, \quad i < j. \\
\bar{s}_{n+2-j,i} + s_{n+2-i,j} &= c_{ij}
\end{align*}

(42)

in which

\[ b_{ij} := (-a_{ij} - \bar{a}_{ji})/2, \quad c_{ij} := (-a_{ij} + \bar{a}_{ji})/(2i). \]

For \( k, l = 1, \ldots, n \), write

\[ u_{kl} := \begin{cases} 
-s_{kl} & \text{if } k + l \geq n + 2, \\
\bar{s}_{kl} & \text{if } k + l \leq n + 1.
\end{cases} \]

(43)

Then the system (42) takes the form

\begin{align*}
 u_{n+1-j,i} - u_{n+1-i,j} &= b_{ij} \\
u_{n+2-j,i} - u_{n+2-i,j} &= c_{ij} \quad i, j = 1, \ldots, n, \quad i < j.
\end{align*}

(44)

Rewrite it as follows

\begin{align*}
u_{kl} - u_{pq} &= b'_{kl}, \quad &k + q = l + p = n + 1, & k < p, \\
u_{kl} - u_{p'q'} &= c'_{kl}, \quad &k + q' = l + p' = n + 2, & k < p'.
\end{align*}

(45)

Since \( k-l = p-q = p'-q' \), the system (45) is partitioned into subsystems with unknowns \( u_{ij}, i-j = \text{const} \). Each of these subsystems has the form

\[ \ldots, u_{kl} - u_{p+1,q+1} = c'_{kl}, \quad u_{kl} - u_{pq} = b'_{kl}, \quad u_{k+1,l+1} - u_{pq} = c'_{k+1,l+1}, \quad \ldots \]

(46)

and is consistent. This proves (40).

**Step 2:** Let us prove that for each diagonal matrix \( A \)

\[ \exists S: \quad A + \Delta A \text{ has the form } 0 \text{ if } \mu \notin \mathbb{R} \text{ or } 0 \text{ if } \mu \notin i\mathbb{R}. \]

(47)

Since \( A \), \( 0 \), and \( 0 \) are diagonal, the matrix \( \Delta A \) must be diagonal too. Thus, the entries of \( S \) must satisfy the system (41) with \( a_{ij} = 0 \). Reasoning as in Step 1, we obtain the system (45) with \( b'_{kl} = c'_{kl} = 0 \), which is partitioned into subsystems (46). Each of these subsystems is represented in the form

\[ u_{1,r+1} = u_{2,r+2} = \cdots = u_{n-r,n} \]

(48)

in which \( r \geq 0 \), or

\[ u_{r+1,1} = u_{r+2,2} = \cdots = u_{n,n-r} = u_{n+1,n-r+1} = 0 \quad (\text{see (38)}) \]

(49)
in which \( r \geq 1 \). By (13), \( S \) is upper triangular and

\[
s_{1,r+1} = \cdots = s_{z,r+z} = -s_{z+1,r+z+1} = \cdots = -s_{n-r,n}
\]

in which \( z \) is the integer part of \((n + 1 - r)/2\) and \( r = 0, 1, \ldots, n - 2\).

Let \( n = 2m \) or \( 2m + 1 \). By (39), the first \( m \) entries of the main diagonal of \( \mu^{-1}\Delta A \) are

\[
\tilde{s}_{n1} + s_{n1} \\
s_{n-1,2} + s_{n-1,2} + i(\tilde{s}_{n2} + s_{n2}) \\
\hspace{1cm} \cdots \\
\tilde{s}_{n1-m,m} + s_{n1-m,m} + i(\tilde{s}_{n2-m,m} + s_{n2-m,m}).
\]

They are zero and so we cannot change the first \( m \) diagonal entries of \( A \).

The last \( m \) entries of the main diagonal of \( \mu^{-1}\Delta A \) are

\[
\tilde{s}_{m,n-m+1} + s_{m,n-m+1} + i(\tilde{s}_{m1,n-m+1} + s_{m1,n-m+1}) \\
\hspace{1cm} \cdots \\
\tilde{s}_{2,n-1} + s_{2,n-1} + i(\tilde{s}_{3,n-1} + s_{3,n-1}) \\
\tilde{s}_{1n} + s_{1n} + i(\tilde{s}_{2n} + s_{2n}).
\]

They are arbitrary and we make zero the last \( m \) entries of the main diagonal of \( A \). This proves (47) for \( n = 2m \).

Let \( n = 2m + 1 \). Since \( s_{m+2,m+1} = 0 \), the \((m + 1)\)st entry of \( \mu^{-1}\Delta A \) is

\[
\tilde{\delta}_{m+1,m+1} = \tilde{s}_{m+1,m+1} + s_{m+1,m+1},
\]

which is an arbitrary real number. Thus, we can add \( \mu r \) with an arbitrary \( r \in \mathbb{R} \) to the \((m + 1)\)st entry of \( A \). This proves (47) for \( n = 2m + 1 \).

### 5.3 Diagonal blocks \( \mathcal{D}(J_n(0)) \)

Due to Lemma 4.3(i), it suffices to prove that each \( n \)-by-\( n \) matrix \( A \) can be reduced to exactly one matrix of the form \( 0^\cdot \) by adding

\[
\Delta A := S^*J_n(0) + J_n(0)S
\]

where

\[
\begin{bmatrix}
0 + s_{21} & \tilde{s}_{11} + s_{22} & \tilde{s}_{21} + s_{23} & \ldots & \tilde{s}_{n-1,1} + s_{2n} \\
0 + s_{31} & \tilde{s}_{12} + s_{32} & \tilde{s}_{22} + s_{33} & \ldots & \tilde{s}_{n-1,2} + s_{3n} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 + s_{1n} & \tilde{s}_{1,n-1} + s_{2n} & \tilde{s}_{2,n-1} + s_{3n} & \ldots & \tilde{s}_{n-1,n-1} + s_{nn} \\
0 + 0 & \tilde{s}_{1n} + 0 & \tilde{s}_{2n} + 0 & \ldots & \tilde{s}_{n-1,n} + 0
\end{bmatrix}, \quad (50)
\]
We call the entries (52) and (53) the main entries; the pairs of indices in (52) and in (53) are equivalent: all entries of $\Delta A$ have the form $s_{kl} + s_{l+1,k+1}$. The transitive closure of $(k, l) \sim (l+1, k+1)$ is an equivalence relation on the set $\{1, \ldots, n\} \times \{1, \ldots, n\}$. Decompose $\Delta A$ into the sum of matrices

$$\Delta A = B_{n1} + B_{n-1,1} + \cdots + B_{i1} + B_{12} + \cdots + B_{1,n-1}$$

that correspond to the equivalence classes and are defined as follows. Each $B_{ij}$ ($j = 1, 2, \ldots, n$) is obtained from $\Delta A$ by replacing with 0 all of its entries except for

$$\bar{s}_{ij} + s_{j+1,2}, \bar{s}_{j+1,2} + s_{3,j+2}, \bar{s}_{3,j+2} + s_{j+3,4}, \ldots \quad (52)$$

and each $B_{ij}$ ($i = 2, 3, \ldots, n$) is obtained from $\Delta A$ by replacing with 0 all of its entries except for

$$0 + s_{i1}, \bar{s}_{i1} + s_{2,i+1}, \bar{s}_{2,i+1} + s_{i+2,3}, \bar{s}_{i+2,3} + s_{i+3,4}, \bar{s}_{i+3,4} + s_{i+4,5}, \ldots \quad (53)$$

the pairs of indices in (52) and in (53) are equivalent:

$$(1, j) \sim (j + 1, 2) \sim (3, j + 2) \sim (j + 3, 4) \sim \ldots$$

and

$$(i, 1) \sim (2, i + 1) \sim (i + 2, 3) \sim (4, i + 3) \sim (i + 4, 5) \sim \ldots$$

We call the entries (52) and (53) the main entries of $B_{ij}$ and $B_{ij}$ ($i > 1$). The matrices $B_{n1}, \ldots, B_{11}, B_{12}, \ldots, B_{1n}$ have no common $s_{ij}$, and so we can add to $A$ each of these matrices separately.

The entries of the sequence (52) are independent: an arbitrary sequence of complex numbers can be represented in the form (52). The entries (53) are dependent only if the last entry in this sequence has the form $s_{kn} + 0$ (see (50)); then $(k, n) = (2p, i - 1 + 2p)$ for some $p$, and so $i = n + 1 - 2p$. Thus the following sequences (53) are dependent:

$$0 + s_{n-1,1}, \bar{s}_{n-1,1} + s_{2n}, \bar{s}_{2n} + 0;$$

$$0 + s_{n-3,1}, \bar{s}_{n-3,1} + s_{2n-2}, \bar{s}_{2n-2} + s_{n-1,3}, \bar{s}_{n-1,3} + s_{4n}, \bar{s}_{4n} + 0; \ldots$$

One of the main entries of each of the matrices $B_{n-1,1}, B_{n-3,1}, B_{n-5,1}, \ldots$ is expressed through the other main entries of this matrix, which are arbitrary. The main entries of the other matrices $B_{i1}$ and $B_{1j}$ are arbitrary. Adding $B_{i1}$ and $B_{1j}$, we reduce $A$ to the form $0^\ast$. 

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6 Off-diagonal blocks of $\mathcal{D}$ that correspond to summands of $A_{\text{can}}$ of the same type

Now we verify the condition (ii) of Lemma 4.3 for those off-diagonal blocks of $\mathcal{D}$ (defined in Theorem 2.2(ii)) whose horizontal and vertical strips contain summands of $A_{\text{can}}$ of the same type.

6.1 Pairs of blocks $\mathcal{D}(H_m(\lambda), H_n(\mu))$ with $|\lambda|, |\mu| > 1$

Due to Lemma 4.3(ii), it suffices to prove that each pair $(B, A)$ of $2n \times 2m$ and $2m \times 2n$ matrices can be reduced to exactly one pair of the form (13) by adding

$$(S^* H_m(\lambda) + H_n(\mu) R, R^* H_n(\mu) + H_m(\lambda) S), \quad S \in \mathbb{C}^{m \times n}, \quad R \in \mathbb{C}^{n \times m}.$$  

Putting $R = 0$ and $S = -H_m(\lambda)^{-1} A$, we reduce $A$ to 0. To preserve $A = 0$ we hereafter must take $S$ and $R$ such that $R^* H_n(\mu) + H_m(\lambda) S = 0$; that is,

$$S = -H_m(\lambda)^{-1} R^* H_n(\mu),$$

and so we can add

$$\Delta B := -H_n(\mu)^* R H_m(\lambda)^{-*} H_m(\lambda) + H_n(\mu) R$$

to $B$.

Write $P := -H_n(\mu)^* R$, then $R = -H_n(\mu)^{-*} P$ and

$$\Delta B = P \begin{bmatrix} J_m(\lambda) & 0 \\ 0 & J_m(\lambda)^{-T} \end{bmatrix} - \begin{bmatrix} J_n(\mu)^{-T} & 0 \\ 0 & J_n(\mu) \end{bmatrix} P$$

(54)

Partition $B$, $\Delta B$, and $P$ into $n \times m$ blocks:

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad \Delta B = \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix}, \quad P = \begin{bmatrix} X & Y \\ Z & T \end{bmatrix}.$$  

By (54),

$$\Delta B_{11} = X J_m(\lambda) - J_n(\mu)^{-T} X, \quad \Delta B_{12} = Y J_m(\lambda)^{-T} - J_n(\mu)^{-T} Y, \quad \Delta B_{21} = Z J_m(\lambda) - J_n(\mu) Z, \quad \Delta B_{22} = T J_m(\lambda)^{-T} - J_n(\mu) T.$$
These equalities ensure that we can reduce each block $B_{ij}$ separately by adding $\Delta B_{ij}$.

(i) First we reduce $B_{11}$ by adding $\Delta B_{11} = X J_m(\lambda) - J_n(\bar{\mu})^{-T}X$.
Since $|\lambda| > 1$ and $|\mu| > 1$, we have that $J_m(\lambda)$ and $J_n(\bar{\mu})^{-T}$ have no common eigenvalues and so $\Delta B_{11}$ is an arbitrary matrix. We make $B_{11} = 0$.

(ii) Let us reduce $B_{12}$ by adding $\Delta B_{12} = Y J_m(\bar{\lambda})^{-T} - J_n(\bar{\mu})^{-T}Y$.
If $\lambda \neq \mu$, then $\Delta B_{12}$ is arbitrary; we make $B_{12} = 0$.
Let $\lambda = \mu$. Write $F := J_n(0)$. Since

$$
J_n(\bar{\lambda})^{-1} = (\lambda I_n + F)^{-1} = \bar{\lambda}^{-1}I_n - \bar{\lambda}^{-2}F + \bar{\lambda}^{-3}F^2 - \ldots,
$$

we have

$$
\Delta B_{12} = Y (J_m(\bar{\lambda})^{-T} - \bar{\lambda}^{-1}I_m) - (J_n(\bar{\lambda})^{-T} - \bar{\lambda}^{-1}I_n)Y
$$

$$
= -\bar{\lambda}^{-2} \begin{bmatrix}
y_{12} & \cdots & y_{1m} & 0 \\
y_{22} & \cdots & y_{2m} & 0 \\
y_{32} & \cdots & y_{3m} & 0 \\
\cdots & \cdots & \cdots & \cdots
\end{bmatrix} + \bar{\lambda}^{-2} \begin{bmatrix}
y_{11} & \cdots & y_{1m} & 0 \\
y_{21} & \cdots & y_{2m} & 0 \\
\cdots & \cdots & \cdots & \cdots
\end{bmatrix} + \ldots
$$

We reduce $B_{12}$ to the form $0^\ast$ along its diagonals starting from the upper right hand corner.

(iii) Let us reduce $B_{21}$ by adding $\Delta B_{21} = Z J_m(\lambda) - J_n(\mu)Z$.
If $\lambda \neq \mu$, then $\Delta B_{21}$ is arbitrary; we make $B_{21} = 0$.
If $\lambda = \mu$, then

$$
\Delta B_{21} = Z (J_m(\lambda) - \lambda I_m) - (J_n(\lambda) - \lambda I_n)Z
$$

$$
= \begin{bmatrix}
0 & z_{11} & \cdots & z_{1,m-1} \\
& \cdots & \cdots & \cdots \\
0 & z_{n-1,1} & \cdots & z_{n-1,m-1} \\
0 & z_{n1} & \cdots & z_{nm}
\end{bmatrix} - \begin{bmatrix}
z_{21} & \cdots & z_{2m} \\
& \cdots & \cdots & \cdots \\
z_{n1} & \cdots & z_{nm} \\
0 & \cdots & 0
\end{bmatrix}.
$$

we reduce $B_{12}$ to the form $0^\ast$ along its diagonals starting from the lower left hand corner.

(iv) Finally, reduce $B_{22}$ by adding $\Delta B_{22} = T J_m(\bar{\lambda})^{-T} - J_n(\mu)T$.
Since $|\lambda| > 1$ and $|\mu| > 1$, $\Delta B_{22}$ is arbitrary; we make $B_{22} = 0$. 

20
6.2 Pairs of blocks $\mathcal{D}(\mu \Delta_m, \nu \Delta_n)$ with $|\mu| = |\nu| = 1$

Due to Lemma 4.3(ii), it suffices to prove that each pair $(B, A)$ of $n \times m$ and $m \times n$ matrices can be reduced to exactly one pair of the form $(0, 0)$ if $\mu \neq \pm \nu$ and $(0^\top, 0)$ if $\mu = \pm \nu$ by adding

$$(\mu S^* \Delta_m + \nu \Delta_n R, \nu R^* \Delta_n + \mu \Delta_m S), \quad S \in \mathbb{C}^{m \times n}, \quad R \in \mathbb{C}^{n \times m}.$$  

We put $R = 0$ and $S = -\mu \Delta_m^{-1} A$, which reduces $A$ to 0. To preserve $A = 0$ we hereafter must take $S$ and $R$ such that $\nu R^* \Delta_n + \mu \Delta_m S = 0$; that is, $S = -\mu \nu \Delta_m^{-1} R^* \Delta_n$, and so we can add

$$\Delta B := \nu \Delta_n R - \mu^2 \bar{\nu} \Delta_n^* R \Delta_m^{-1} \Delta_m$$

to $B$.

Write $P := \Delta_n^* R$, then

$$\Delta B = \bar{\nu}[\nu^2 (\Delta_n \Delta_n^*) P - \mu^2 P (\Delta_m^{-1} \Delta_m)]. \quad (55)$$

Since

$$\Delta_n^* = \begin{bmatrix} * & i & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ i & 1 & \cdots & 0 \end{bmatrix},$$

we have

$$\Delta_n \Delta_n^* = \begin{bmatrix} 1 & 0 \\ 2i & 1 & \cdots & \cdots \\ \cdots & \ddots & \ddots & \ddots \\ * & 2i & 1 \end{bmatrix} \quad (56)$$

and

$$\Delta_m^{-1} \Delta_m = (\Delta_n \Delta_n^*)^T = \begin{bmatrix} 1 & 2i & \cdots & * \\ \vdots & 1 & \ddots & \ddots \\ \cdots & \ddots & 2i & \cdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix}. \quad (57)$$

If $\mu \neq \pm \nu$, then $\mu^2 \neq \nu^2$, the matrices $\nu^2 (\Delta_n \Delta_n^*)$ and $\mu^2 (\Delta_m^{-1} \Delta_m)$ have distinct eigenvalues, and so $\Delta B$ can be made arbitrary. We make $B = 0$.

If $\mu = \pm \nu$, then

$$\frac{1}{2i \nu} \Delta B = \begin{bmatrix} 0 & 0 & \cdots & \cdots \\ 1 & 0 & \ddots & \cdots \\ \cdots & \ddots & \ddots & \ddots \\ * & \cdots & 1 & 0 \end{bmatrix} P - \begin{bmatrix} 0 & 1 & \cdots & * \\ \vdots & \ddots & \ddots & \ddots \\ \cdots & \ddots & 1 & \cdots \\ 0 & \cdots & \cdots & 1 \end{bmatrix},$$
and we reduce \( B \) to the form \( 0^\ast \) along its skew diagonals starting from the upper left hand corner.

### 6.3 Pairs of blocks \( \mathcal{D}(J_m(0), J_n(0)) \) with \( m \geq n \)

Due to Lemma \[13\] (ii), it suffices to prove that each pair \((B, A)\) of \( n \times m \) and \( m \times n \) matrices with \( m \geq n \) can be reduced to exactly one pair of the form \((0^\ast, 0^\ast)\) if \( n \) is even and of the form \((0^\ast + \mathcal{P}_{nm}, 0^\ast)\) if \( n \) is odd by adding the matrices

\[
\Delta A = R^* J_n(0) + J_m(0)S, \quad \Delta B^* = J_m(0)^T S + R^* J_n(0)^T
\]

(58)

to \( A \) and \( B^* \) (we prefer to reduce \( B^* \) instead of \( B \)).

Write \( S = [s_{ij}] \) and \( R^* = [-r_{ij}] \) (they are \( m \)-by-\( n \)). Then

\[
\Delta A = \begin{bmatrix}
s_{21} - 0 & s_{22} - r_{11} & s_{23} - r_{12} & \cdots & s_{2n} - r_{1,n-1} \\
s_{m-1,1} - 0 & s_{m-1,2} - r_{m-2,1} & s_{m-1,3} - r_{m-2,2} & \cdots & s_{m-1,n} - r_{m-2,n-1} \\
s_{m1} - 0 & s_{m2} - r_{m-1,1} & s_{m3} - r_{m-1,2} & \cdots & s_{mn} - r_{m-1,n-1} \\
0 - 0 & 0 - r_{m1} & 0 - r_{m2} & \cdots & 0 - r_{m,n-1}
\end{bmatrix}
\]

and

\[
\Delta B^* = \begin{bmatrix}
0 - r_{12} & 0 - r_{13} & \cdots & 0 - r_{1n} & 0 - 0 \\
s_{11} - r_{22} & s_{12} - r_{23} & \cdots & s_{1,n-1} - r_{2n} & s_{1n} - 0 \\
s_{m-1,1} - r_{m-2,2} & s_{m-1,2} - r_{m-2,3} & \cdots & s_{m-2,n-1} - r_{m-2,n} & s_{m-2,n} - 0 \\
s_{m-1,1} - r_{m2} & s_{m-1,2} - r_{m3} & \cdots & s_{m-1,n-1} - r_{mn} & s_{m-1,n} - 0
\end{bmatrix}
\]

Adding \( \Delta A \), we reduce \( A \) to the form

\[
0^\dagger := \begin{bmatrix} 0_{m-1,n} \end{bmatrix}.
\]

(59)

To preserve this form, we hereafter must take

\[
s_{21} = \cdots = s_{m1} = 0, \quad s_{ij} = r_{i-1,j-1} \quad (2 \leq i \leq m, \ 2 \leq j \leq n).
\]

(60)

Write

\[
(r_{00}, r_{01}, \ldots, r_{0,n-1}) := (s_{11}, s_{12}, \ldots, s_{1n}),
\]

Write
then
\[
\Delta B^* = \begin{bmatrix}
0 - r_{12} & 0 - r_{13} & \ldots & 0 - r_{1n} & 0 - 0 \\
r_{00} - r_{22} & r_{01} - r_{23} & \ldots & r_{0,n-2} - r_{2n} & r_{0,n-1} - 0 \\
0 - r_{32} & r_{11} - r_{33} & \ldots & r_{1,n-2} - r_{3n} & r_{1,n-1} - 0 \\
0 - r_{42} & r_{21} - r_{43} & \ldots & r_{2,n-2} - r_{4n} & r_{2,n-1} - 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 - r_{m2} & r_{m-2,1} - r_{m3} & \ldots & r_{m-2,n-2} - r_{mn} & r_{m-2,n-1} - 0
\end{bmatrix}
\] . (61)

If \(r_{ij}\) and \(r_{i'j'}\) are parameters of the same diagonal of \(\Delta B^*\), then \(i - j = i' - j'\). Hence, the diagonals of \(\Delta B^*\) have no common parameters, and so we can reduce the diagonals of \(B^*\) independently.

The first \(n\) diagonals of \(\Delta B^*\) starting from the upper right hand corner are

\[
0, \; (-r_{1n}, r_{0,n-1}), \; (-r_{1,n-1}, r_{0,n-2} - r_{2n}, r_{1,n-1}),
\]

\[
(-r_{1,n-2}, r_{0,n-3} - r_{2,n-1}, r_{1,n-2} - r_{3n}, r_{2,n-1}),
\]

\[
(-r_{1,n-3}, r_{0,n-4} - r_{2,n-2}, r_{1,n-3} - r_{3,n-1} r_{2,n-2} - r_{4n}, r_{3,n-1}), \ldots
\]

(we underline linearly dependent entries), adding them we reduce the first \(n\) diagonals of \(B^*\) to the form \(0^\times\).

The \((n + 1)^{st}\) diagonal of \(\Delta B^*\) is

\[
\begin{cases}
(r_{00} - r_{22}, r_{11} - r_{33}, \ldots, r_{n-2,n-2} - r_{nn}) & \text{if } m = n,
(r_{00} - r_{22}, r_{11} - r_{33}, \ldots, r_{n-2,n-2} - r_{nn}, r_{n-1,n-1}) & \text{if } m > n.
\end{cases}
\]

Adding it, we make the \((n + 1)^{st}\) diagonal of \(B^*\) zero.

If \(m > n + 1\), then the \((n + 2)^{nd}, \ldots, m^{th}\) diagonals of \(\Delta B^*\) are

\[
(-r_{32}, r_{21} - r_{43}, r_{32} - r_{54}, \ldots, r_{n,n-1}),
\]

\[
(-r_{m-n+1,2}, r_{m-n,1} - r_{m-n+2,3}, r_{m-n+1,2} - r_{m-n+3,4}, \ldots, r_{m-2,n-1}).
\]

Each of these diagonals contains \(n\) elements. If \(n\) is even, then the length of each diagonal is even and its elements are linearly independent; we make the corresponding diagonals of \(B^*\) equal to zero. If \(n\) is odd, then the length of each diagonal is odd and the set of its odd-numbered elements is linearly dependent; we make all elements of the corresponding diagonals of \(B^*\) equal
to zero except for their last elements (they correspond to the stars of $P_{nm}$, which is defined in (6)).

It remains to reduce the last $n-1$ diagonals of $B^*$ (the last $n-2$ diagonals if $m = n$). The corresponding diagonals of $\Delta B^*$ are

$$-r_{m2},$$

$$(-r_{m-1,2}, r_{m-2,1} - r_{m3}),$$

$$(-r_{m-2,2}, r_{m-3,1} - r_{m-1,3}, r_{m-2,2} - r_{m4}),$$

$$(-r_{m-3,2}, r_{m-4,1} - r_{m-2,3}, r_{m-3,2} - r_{m-1,4}, r_{m-2,3} - r_{m5}),$$

$$\cdots$$

$$(-r_{m-n+3,2}, r_{m-n+2,1} - r_{m-n+4,3}, \cdots, r_{m-2,n-3} - r_{m,n-1}),$$

and, only if $m > n$,

$$(-r_{m-n+2,2}, r_{m-n+1,1} - r_{m-n+3,3}, \cdots, r_{m-2,n-2} - r_{mn}).$$

Adding these diagonals, we make the corresponding diagonals of $B^*$ zero. To preserve the zero diagonals, we hereafter must take $r_{m2} = r_{m4} = r_{m6} = \cdots = 0$ and arbitrary $r_{m1}, r_{m3}, r_{m5}, \cdots$.

Recall that $A$ has the form $0^\dagger$ (see (59)). Since $r_{m1}, r_{m3}, r_{m5}, \cdots$ are arbitrary, we can reduce $A$ to the form

$$\begin{bmatrix}
0_{m-1,n} \\
* & 0 & \ast & 0 & \cdots
\end{bmatrix}$$

by adding $\Delta A$; these additions preserve $B$.

If $m = n$, then we may alternatively reduce $A$ to the form

$$\begin{bmatrix}
\cdots \\
0 & 0 & \cdots & 0 \\
* & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
* & 0 & \cdots & 0
\end{bmatrix}$$

preserving the form $0^{\ast}$ of $B$.

7 Off-diagonal blocks of $\mathcal{D}$ that correspond to summands of $A_{\text{can}}$ of distinct types

Finally, we verify the condition (ii) of Lemma 4.3 [1] for those off-diagonal blocks of $\mathcal{D}$ (defined in Theorem 2.2(iii)) whose horizontal and vertical strips contain summands of $A_{\text{can}}$ of different types.
7.1 Pairs of blocks \( D(H_m(\lambda), \mu \Delta_n) \) with \( |\lambda| > 1 \) and \( |\mu| = 1 \)

Due to Lemma 4.3(ii), it suffices to prove that each pair \((B, A)\) of \( n \times 2m \)
and \(2m \times n\) matrices can be reduced to the pair \((0, 0)\) by adding

\[
(S^* H_m(\lambda) + \mu \Delta_n R, R^* \mu \Delta_n + H_m(\lambda) S), \quad S \in \mathbb{C}^{2m \times n}, \quad R \in \mathbb{C}^{n \times 2m}.
\]

Reduce \( A \) to 0 by putting \( R = 0 \) and \( S = -H_m(\lambda)^{-1} A \). To preserve \( A = 0 \),
we hereafter must take \( S \) and \( R \) such that \( R^* \mu \Delta_n + H_m(\lambda) S = 0 \); that is,

\[ S = -H_m(\lambda)^{-1} R^* \mu \Delta_n. \]

Hence, we can add

\[
\Delta B := \mu \Delta_n R - \bar{\mu} \Delta_n^* R H_m(\lambda)^{-*} H_m(\lambda)
\]

to \( B \). Write \( P = \bar{\mu} \Delta_n^* R \), then

\[
\Delta B = \mu \bar{\mu}^{-1} \Delta_n \Delta_n^* P - P \left( J_m(\lambda) \oplus J_m(\bar{\lambda})^{-T} \right).
\]

By \( (56) \), \( \mu \bar{\mu}^{-1} \Delta_n \Delta_n^* \) has the single eigenvalue \( \mu \bar{\mu}^{-1} \), which is of modulus 1.
Since \( |\lambda| > 1 \), \( \mu \bar{\mu}^{-1} \Delta_n \Delta_n^* \) and \( J_m(\lambda) \oplus J_m(\bar{\lambda})^{-T} \) have no common eigenvalues.
Thus, \( \Delta B \) is an arbitrary matrix and we make \( B = 0 \).

7.2 Pairs of blocks \( D(H_m(\lambda), J_n(0)) \) with \( |\lambda| > 1 \)

Due to Lemma 4.3(ii), it suffices to prove that each pair \((B, A)\) of \( n \times 2m \)
and \(2m \times n\) matrices can be reduced to exactly one pair of the form \((0, 0)\) if \( n \) is even and to
the form \((0^T, 0)\) if \( n \) is odd by adding

\[
(S^* H_m(\lambda) + J_n(0) R, R^* J_n(0) + H_m(\lambda) S), \quad S \in \mathbb{C}^{2m \times n}, \quad R \in \mathbb{C}^{n \times 2m}.
\]

Putting \( R = 0 \) and \( S = -H_m(\lambda)^{-1} A \), we reduce \( A \) to 0. To preserve \( A = 0 \)
we hereafter must take \( S \) and \( R \) such that \( R^* J_n(0) + H_m(\lambda) S = 0 \); that is,

\[ S = -H_m(\lambda)^{-1} R^* J_n(0). \]

Hence we can add

\[
\Delta B := J_n(0) R - J_n(0)^T R H_m(\lambda)^{-*} H_m(\lambda)
\]

\[
= J_n(0) R - J_n(0)^T R \left( J_m(\lambda) \oplus J_m(\bar{\lambda})^{-T} \right)
\]
to \( B \).

Divide \( B \) and \( R \) into two blocks of size \( n \times m \):

\[
B = [M \; N], \quad R = [U \; V].
\]

We can add to \( M \) and \( N \) the matrices

\[
\Delta M := J_n(0) U - J_n(0)^T U J_m(\lambda), \quad \Delta N := J_n(0) V - J_n(0)^T V J_m(\bar{\lambda})^{-T}.
\]

We reduce \( M \) as follows. Let \((u_1, u_2, \ldots, u_n)^T\) be the first column of \( U \). Then we can add to the first column \( \vec{b}_1 \) of \( M \) the vector

\[
\Delta \vec{b}_1 := (u_2, \ldots, u_n, 0)^T - \lambda(0, u_1, \ldots, u_{n-1})^T = \begin{cases} 0 & \text{if } n = 1, \\ (u_2, u_3 - \lambda u_1, u_4 - \lambda u_2, \ldots, u_n - \lambda u_{n-2}, -\lambda u_{n-1})^T & \text{if } n > 1. \end{cases}
\]

The elements of this vector are linearly independent if \( n \) is even, and they are linearly dependent if \( n \) is odd. We reduce \( \vec{b}_1 \) to zero if \( n \) is even, and to the form \( (*, 0, \ldots, 0)^T \) or \( (0, \ldots, 0, *)^T \) if \( n \) is odd. Then we successively reduce the other columns transforming \( M \) to 0 if \( n \) is even, and to the form \( 0^\top_{n,m} \) if \( n \) is odd.

We reduce \( N \) in the same way starting from the last column.

### 7.3 Pairs of blocks \( \mathcal{D}(\lambda \Delta_m, J_n(0)) \) with \( |\lambda| = 1 \)

Due to Lemma 4.3(ii), it suffices to prove that each pair \((B, A)\) of \( n \times m \) and \( m \times n \) matrices can be reduced to exactly one pair of the form \((0, 0)\) if \( n \) is even and to the form \((0^4, 0)\) if \( n \) is odd by adding

\[
(S^* \lambda \Delta_m + J_n(0) R, R^* J_n(0) + \lambda \Delta_m S), \quad S \in \mathbb{C}^{m \times n}, \quad R \in \mathbb{C}^{n \times m}.
\]

Putting \( R = 0 \) and \( S = -\lambda \Delta_m^{-1} A \), we reduce \( A \) to 0. To preserve \( A = 0 \) we hereafter must take \( S \) and \( R \) such that \( R^* J_n(0) + \lambda \Delta_m S = 0 \); that is, \( S = -\lambda \Delta_m^{-1} R^* J_n(0) \). By (57), we can add

\[
\Delta B := J_n(0) R - \lambda^2 J_n(0)^T R \Delta_m^{+} \Delta_m
\]

to \( B \). We reduce \( B \) to 0 if \( n \) is even and to \( 0^4 \) if \( n \) is odd along its columns starting from the first column.
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