NERON-SEVERI LIE ALGEBRA, AUTOEQUIVALENCES OF THE DERIVED CATEGORY, AND MONODROMY

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Abstract. Let $X$ be a smooth complex projective variety. The group of autoequivalences of the derived category of $X$ acts naturally on its singular cohomology $H^\bullet(X, \mathbb{Q})$ and we denote by $G^{eq}(X) \subset GL(H^\bullet(X, \mathbb{Q}))$ its image. Let $\overline{G^{eq}(X)} \subset GL(H^\bullet(X, \mathbb{Q}))$ be its Zariski closure. We study the relation of the Lie algebra $\text{Lie} \overline{G^{eq}(X)}$ and the Neron-Severi Lie algebra $\mathfrak{g}_{NS}(X) \subset \text{End}(H(X, \mathbb{Q}))$ in case $X$ has trivial canonical line bundle.

At the same time for mirror symmetric families of (weakly) Calabi-Yau varieties we consider a conjecture of Kontsevich on the relation between the monodromy of one family and the group $G^{eq}(X)$ for a very general member $X$ of the other family.

1. Introduction

1.1. Lie algebra $\mathfrak{g}_{NS}(X)$ and the group $G^{eq}(X)$. Let $X$ be a smooth complex projective variety of dimension $n$. Consider the semi-simple operator $h \in \text{End}(H^\bullet(X, \mathbb{Q}))$ which acts as multiplication by $i - n$ on the space $H^i(X)$. Every ample class $\kappa \in H^2(X, \mathbb{Q})$ defines a Lefschetz operator $e_\kappa := \cup \kappa : H^\bullet(X) \to H^{\bullet+2}(X)$ i.e. $e_\kappa^i : H^{n-i}(X) \to H^{n+i}(X)$ is an isomorphism. In classical Hodge theory one also considers the (unique) operator $f_\kappa : H^\bullet(X) \to H^{\bullet-2}(X)$ such that $(e_\kappa, h, f_\kappa) \subset \text{End}(H^\bullet(X))$ is an $sl_2$-triple. Let $\mathfrak{g}_{NS}(X) \subset \text{End}(H^\bullet(X))$ be the Lie algebra generated by such $sl_2$-triples $(e_\kappa, h, f_\kappa)$ for all ample classes $\kappa \in H^2(X, \mathbb{Q})$. This Lie algebra is graded by the adjoint action of $h$. It is called the Neron-Severi Lie algebra of $X$ [LL]. This Lie algebra is semi-simple [LL Prop.1.6].

On the other hand, one has the group of autoequivalences of the derived category $D^b(\text{coh}X)$. This group acts naturally on the cohomology $H(X, \mathbb{Q})$ and we denote by $G^{eq}(X)$ its image in $GL(H(X, \mathbb{Q}))$. Let $\overline{G^{eq}(X)} \subset GL(H(X, \mathbb{Q}))$ be the algebraic $\mathbb{Q}$-subgroup which is the Zariski closure of $G^{eq}(X)$, and let $L^{eq}(X) := \text{Lie} \overline{G^{eq}(X)} \subset \text{End}(H(X, \mathbb{Q}))$ be its Lie algebra.

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The following theorem was proved in [GLO].

**Theorem 1.1.** Let $A$ be an abelian variety. Then there is an equality of Lie subalgebras of $\text{End}(H(A, \mathbb{Q}))$:

$$L^{eq}(A) = \mathfrak{g}_{NS}(A)$$

In [Pol] the groups $\overline{G^{eq}}(A)$ were studied and classified according to the type of the abelian variety $A$. In [LL] a similar classification of Lie algebras $\mathfrak{g}_{NS}(A)$ was obtained. Theorem 1.1 follows from the comparison of the two lists.

We expect a similar phenomenon for hyperkahler manifolds.

**Conjecture 1.2.** Let $X$ be a projective hyperkahler manifold. Then we have an equality of Lie subalgebras of $\text{End}(H^\bullet(X, \mathbb{Q}))$:

$$L^{eq}(X) = \mathfrak{g}_{NS}(X)$$

The conjecture is easily verified for K3 surfaces (Corollary 3.4). By a recent result of Taelman [Tael] it also holds for $\text{Hilb}^2(X)$ of a K3 surface $X$.

**Remark 1.3.** Consider the ”biggest” $\mathfrak{g}_{NS}(X)$-submodule of $H^\bullet(X, \mathbb{Q})$, which is generated by $1 \in H^0(X, \mathbb{Q})$. Denote it by $\mathfrak{g}_{NS}(X) \cdot 1$. A weaker version of Conjecture 1.2 would say that the Lie algebra $L^{eq}(X)$ preserves this subspace and we have the equality of subalgebras of $\text{End}(\mathfrak{g}_{NS}(X) \cdot 1)$:

$$L^{eq}(X)|_{\mathfrak{g}_{NS}(X) \cdot 1} = \mathfrak{g}_{NS}(X)$$

Using results of [Tael] we prove (Theorem 3.5) one inclusion in this weak version of the conjecture:

$$L^{eq}(X)|_{\mathfrak{g}_{NS} \cdot 1} \subseteq \mathfrak{g}_{NS}(X)$$

For a general smooth projective variety $X$ a priori it is not clear that either side of (1.2) is contained in the other. Conjecture 1.2 is false for any positive dimensional smooth projective variety $X$ which is Fano or of general type. Note however that for any smooth projective variety $X$ the Lie algebra of the subgroup of $\overline{G^{eq}}(X)$ corresponding to tensoring with line bundles is by definition contained in $\mathfrak{g}_{NS}(X)$ (this is the only immediately visible relation between the group $G^{eq}(X)$ and the Lie algebra $\mathfrak{g}_{NS}(X)$).

It is natural to ask if Conjecture 1.2 holds for Calabi-Yau varieties. An easy counterexample is given by a smooth hypersurface $X \subset \mathbb{P}^n$ of degree $n+1$, assuming that $n = 2k \geq 4$. Both $G^{eq}(X)$ and $\mathfrak{g}_{NS}(X)$ preserve the space $H^{even}(X, \mathbb{Q})$ with its skew-symmetric Mukai
pairing. Clearly $\mathfrak{g}_{NS}(X) = sl_2$, but $L^{eq}(X) = sp(H^{even}(X, \mathbb{Q}))$ (Theorem 6.2). So we have the strict inclusion

$$L^{eq}(X)_{\mathfrak{g}_{NS}} \supseteq \mathfrak{g}_{NS}(X)$$

Nevertheless we expect that Conjecture 1.2 hold for "most" CY varieties as well. Let us explain why. The reason that Conjecture 1.2 fails in the above counter example is that the Picard rank of $X$ is 1 and hence $\mathfrak{g}_{NS}(X)$ is too small. The classification Theorem 6.8 in [LL] suggests that one should expect $\mathfrak{g}_{NS}(X)$ to be either small, that is to be nonzero only in degrees $-2, 0, 2$ (which is rare), or else to be maximal, i.e. to be the full Lie algebra preserving a nondegenerate form. For example, Conjecture 1.2 should hold if in the above example one replaces $\mathbb{P}^n$ with a smooth toric Fano variety (which is not a product) with Picard rank $\geq 2$, and take $X$ to be a smooth anticanonical divisor.

1.2. Kontsevich’s conjecture for mirror symmetric families of (weakly) CY varieties. The following sentence appears in the introduction section of [BorHor]: "Kontsevich conjectured that the action on cohomology of the group of self-equivalences of the bounded derived category of coherent sheaves on a smooth projective Calabi–Yau variety matches the monodromy action on the cohomology of the mirror Calabi–Yau variety associated to the variations of complex structures."

Below we state our version of Kontsevich’s conjecture (Conjecture 1.3). First let us make a few definitions and reminders.

Let $\mathcal{X}/S$ be a family of smooth complex projective varieties over a connected base $S$. Fixing a point $s \in S$ we get the fundamental group $\pi_1(S, s)$ and its monodromy representation

$$\mu : \pi_1(S, s) \to GL(H^\bullet(X_s, \mathbb{Q}))$$

in the cohomology $H^\bullet(X_s, \mathbb{Q})$ of the fiber $X_s$. We denote by $G^{mon}(\mathcal{X})$ the image of $\mu$. This is a discrete group whose isomorphism class does not depend on the choice of a point $s \in S$. It is called the monodromy group of $\mathcal{X}$. Also denote by $G^{eq}(\mathcal{X})$ the group $G^{eq}(X_s)$ for a very general fiber $X_s$ of $\mathcal{X}$. (A fiber is very general if it lies outside of a countable union of analytic subvarieties of the base.)

**Definition 1.4.** The equivalence relation $\sim$ on the collection of discrete groups is generated by allowing to replace a group $G$ by a subgroup of finite index or by the quotient of $G$ by a finite normal subgroup. If $G \sim G'$ we say that $G$ and $G'$ are isomorphic up to finite groups.

In the mathematical literature there exist at least two series of ”mirror symmetric” (MS) families of CY varieties. Namely, one has

(I) Mirror symmetric (MS) families of lattice polarized K3 surfaces [Dolg, Pink].
(II) MS families of anticanonical divisors in dual Fano toric varieties [Bat].

Since the author is not aware of a mathematical definition of MS, we use this term in quotation marks. However, the definition of the above families is indeed based on symmetry relations: of lattices in case (I) and of polytopes in case (II). In Section 5 below we introduce a third series:

(III) MS families of abelian varieties.

This is defined using a simple symmetry relation of $\mathbb{Q}$-algebraic groups.

In our understanding the point of MS is that a simple minded duality (as in (I), (II), (III)) implies a duality relation involving highly sophisticated objects, like the derived category. Let us formulate the following principle (Kontsevich’s conjecture), which we call ”a conjecture” for simplicity of statement and of reference.

**Conjecture 1.5.** Let $X$ and $X^\vee$ be mirror symmetric families of complex smooth projective varieties with trivial canonical line bundle. Then the groups $G^{\text{mon}}(X)$ and $G^{\text{eq}}(X^\vee)$ are isomorphic up to finite groups.

We prove Conjecture 1.5 for families (I), (III) and present some evidence for it in case (II). Let us briefly summarize our results on Conjecture 1.5 in the three examples.

1.2.1. Lattice polarized K3 surfaces. Let $L$ be the lattice of a K3 surface. Recall [Dolg] that primitive sublattices $M, M^\vee \subset L$ of signatures $(1, s)$ and $(1, 18 - s)$ respectively, are called mirror symmetric if

$$M^\perp = M^\vee \oplus U$$

Following the works [Dolg], [Pink] we consider the ample $M$- and $M^\vee$-polarized families, $U_M^\circ$ and $U_{M^\vee}^\circ$ respectively, of K3 surfaces. We check Conjecture 1.5 for these families. This is just a pleasant exercise, since one knows everything about the group $G^{\text{eq}}$ and the moduli of K3 surfaces.

1.2.2. Abelian varieties. We extend the work [GLO] by defining the notion of mirror symmetric families of abelian varieties. In loc.cit. we considered algebraic pairs $(A, \omega_A)$ where $A$ is an abelian variety and $\omega_A$ is an element of the complexified ample cone $C_A$ of $A$. Then we defined the mirror symmetry relation between algebraic pairs $(A, \omega_A)$ and $(B, \omega_B)$. One feature of this relation is the natural inclusion of algebraic $\mathbb{Q}$-groups

$$Hdg_B \subseteq G^{\text{eq}}(A), \quad Hdg_A \subseteq G^{\text{eq}}(B)$$

Now we say that the pairs $(A, \omega_A)$ and $(B, \omega_B)$ and perfectly mirror symmetric (PMS) if the inclusions (1.5) are equalities. Such PMS algebraic pairs naturally give rise to families of abelian varieties

$$A := \{A_{\eta_B} | \eta_B \in C_B\} \quad \text{and} \quad B := \{B_{\eta_A} | \eta_A \in C_A\}$$
with bases $C_B$ and $C_A$ respectively. We call these families mirror symmetric (Definition 5.24). A proof of Conjecture 1.5 for such families is in Section 5.14.

1.2.3. Anticanonical hypersurfaces in Fano toric varieties. Batyrev [Bat] constructs mirror symmetric families of CY varieties in the following way: he starts with two dual lattices $M \simeq \mathbb{Z}^{n+1}$ and $N = M^*$ and a pair of dual reflexive polytopes $\Delta \subset M_{\mathbb{Q}}$, $\Delta^\vee \subset N_{\mathbb{Q}}$. These dual polytopes define a pair of projective Gorenstein toric Fano varieties $\mathbb{P}_\Delta$ and $\mathbb{P}_{\Delta^\vee}$. The induced families $\mathcal{X}$ and $\mathcal{X}^\vee$ of anticanonical divisors consist of Gorenstein CY varieties. These are the MS families (see Section 6 for details).

Assume that the toric variety $\mathbb{P}_\Delta$ is smooth. Then the family $\mathcal{X}$ will consist of smooth CY varieties $X$ and the group $G^{\text{eq}}(\mathcal{X})$ is defined. Assume in addition that $n$ is odd. Then for any member $X$ of the family $\mathcal{X}$ we have $G^{\text{eq}}(X) \subset Sp(H^{\text{even}}(X, \mathbb{Q}))$ and also $G^{\text{mon}}(\mathcal{X}^\vee) \subset Sp(H^n(X^\vee, \mathbb{Q}))$. The spaces $H^{\text{even}}(X, \mathbb{Q})$ and $H^n(X^\vee, \mathbb{Q})$ are isomorphic and we expect that $G^{\text{eq}}(\mathcal{X})$ and $G^{\text{mon}}(\mathcal{X}^\vee)$ are arithmetic subgroups in the corresponding isomorphic symplectic groups. We prove a little weaker statement for the group $G^{\text{eq}}(X)$.

Namely we show that the Zariski closure $\overline{G^{\text{eq}}(X)}$ is equal to $Sp(H^{\text{even}}(X, \mathbb{Q}))$ for any $X$ in the family $\mathcal{X}$. Some evidence is also provided for the group $G^{\text{mon}}(\mathcal{X}^\vee)$.

1.2.4. Strictly speaking, among the 3 families mentioned above, Conjecture 1.5 can be tested only in case (II), where the actual universal family exists. In cases (I) and (III) one typically has only the coarse moduli space $\overline{S}$, which is the quotient by a discrete group of an analytic space $S$ that is a base of an actual family $\mathcal{X} \to S$. (For example, $\overline{S}$ can the quotient of the Lobachevsky upper half plane $S$ by the group $SL(2, \mathbb{Z})$ and $\mathcal{X} \to S$ the natural family of elliptic curves). So in order to make Conjecture 1.5 applicable to cases (I) and (III) we need to extend appropriately the notion of the monodromy. This is done in 2.3 below.

1.2.5. Some aspects of Kontsevich’s conjecture were already studied in [Hor] and in [Szen]. However, the intersection of results in loc. cit. with ours appears to be minimal.

1.3. Organization of the paper. Section 2 collects some general results on Fourier-Mukai transforms and gives the definition of the monodromy in a somewhat nonstandard situation. In section 3 we discuss Conjecture 1.2 for hyperkahler varieties. Sections 4 and 5 deal with mirror symmetry for families of K3 surfaces and abelian varieties respectively. In both cases the Conjecture 1.5 is proved. In the last Section 6 we discuss the case of Calabi-Yau hypersurfaces in toric varieties and prove some partial results in the direction of the conjecture.
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2. Some generalities and extension of the notion of monodromy

2.1. Notation. We consider smooth complex projective varieties. For such a variety $X$, $H(X, \mathbb{Q})$ denotes the singular cohomology of the corresponding analytic manifold. The bounded derived category of coherent sheaves on $X$ is denoted by $D^b(X)$. Its group of autoequivalences is $\text{Aut} D^b(X)$. There exists the Chern character map $ch : D^b(X) \to H(X, \mathbb{Q})$

Let $\sqrt{td_X} \in H(X, \mathbb{Q})$ be the square root of the Todd class of $X$. For $F \in D^b(X)$ one defines its Mukai vector $v(F)$ as $v(F) := ch(F) \cup \sqrt{td_X} \in H(X, \mathbb{Q})$

2.2. Action of the group $\text{Aut} D^b(X)$ on the cohomology $H(X, \mathbb{Q})$. There is a natural homomorphism of groups $\rho_X : \text{Aut} D^b(X) \to \text{Gl}(H(X, \mathbb{Q}))$. Let us recall it.

Consider the two projections $X \xleftarrow{p} X \times X \xrightarrow{q} X$. It is known that any autoequivalence $\Phi \in \text{Aut} D^b(X)$ is given by a Fourier-Mukai functor $\Phi_E$ for a unique kernel $E \in D^b(X \times X)$. That is $\Phi(-) = \Phi_E(-) := Rq_*(p^*(-) \otimes E)$

This operation is compatible with the Mukai vector in the following sense. Any $e \in H(X \times X)$ defines the corresponding cohomological transform $\Phi^H_e$

$\Phi^H_e(-) = q_*(p^*(-) \cup e) : H(X, \mathbb{Q}) \to H(X, \mathbb{Q})$

Then for any kernel $E \in D^b(X \times X)$, and $F \in D^b(X)$ we have $\Phi^H_{v(E)}(v(F)) = v(\Phi_E(F))$

and the correspondence $\Phi_E \mapsto \Phi^H_{v(E)}$ is the group homomorphism $\rho_X : \text{Aut} D^b(X) \to \text{Gl}(H(X, \mathbb{Q}))$.

We note that the action of $\text{Aut} D^b(X)$ on $H(X, \mathbb{Q})$ preserves the Mukai pairing, which is nondegenerate and is a modification of the Poincare pairing.
Definition 2.1. Denote by $G^{eq}(X)$ the image of the homomorphism $\rho_X$.

The group $G^{eq}(X)$ rarely preserves the integral cohomology $H(X, \mathbb{Z}) \subset H(X, \mathbb{Q})$ but it preserves a different lattice. Consider the topological $K$-group $K_{top}(X) = K^0_{top}(X) \oplus K^1_{top}(X)$ and the map

$$v_{top} : K_{top}(X) \to H(X, \mathbb{Q}), \ F \mapsto \sqrt{td_X} \cdot ch(F)$$

The image $\text{im}(v_{top})$ is a lattice in $H(X, \mathbb{Q})$ of full rank. This lattice is preserved by the group $G^{eq}(X)$ \footnote{AdTho}. In particular, $G^{eq}(X)$ is a discrete subgroup of $GL(H(X, \mathbb{Q}))$.

2.3. Monodromy group. Besides considering mirror dual universal families of CY varieties we also want to study the case when only coarse moduli spaces exist. Let us make a rather ad hoc definition of the monodromy group in that case. The definition seems reasonable and suffices for our purposes.

Let $f : \mathcal{X} \to S$ be a continuous map of topological spaces which is a locally trivial fibration and whose fibers are compact complex manifolds (with certain additional structure, for example, an embedding in a projective space (a polarization) or a multi-polarization, or a fixed sublattice in the Neron-Severi group). Assume that $S$ is connected. Assume also that the (graded) local system $R^*f_*\mathbb{Q}_\mathcal{X}$ is trivial. Let $G$ be a discrete group that acts on $S$ and this action lifts to an action on the local system $R^*f_*\mathbb{Q}_\mathcal{X}$. Let $K \subset G$ denote the kernel of the $G$-action on $S$. So elements of $K$ act by fiberwise automorphisms of the local system $R^*f_*\mathbb{Q}_\mathcal{X}$. Suppose that the following holds:

1. The $G$-action on the space of global sections $H^0(S, R^*f_*\mathbb{Q}_\mathcal{X})$ is effective (i.e. every $1 \neq g \in G$ acts nontrivially).
2. The $G/K$-action on $S$ is generically free. More precisely there exists a countable union $Z \subset S$ of closed subsets such that the complement $S^0 := S \setminus Z$ is everywhere dense and $G/K$-action on $S^0 := S \setminus Z$ is free.
3. The quotient space $\overline{S^0} := S^0/G = S^0/(G/K)$ is the coarse moduli space of complex manifolds (with the given additional structure) appearing as fibers in the family $f$ over $S^0$ (that is, points of $\overline{S^0}$ are in bijection with isomorphism classes of fibers in the family $\mathcal{X}|_{S^0}$.)

Definition 2.2. In the above situation we call $G/K$ the monodromy group of the family $f : \mathcal{X} \to S$. We denote this group $G^{mon}(\mathcal{X})$. (In case the $G/K$-action on $S^0$ is free only modulo a finite kernel, we say the $G/K$ is the monodromy group up to finite groups).

If the $G$-action of $S$ is free and $\overline{S} = S/G$ is a fine moduli space, i.e. the family $f : \mathcal{X} \to S$ descends to a universal family $\overline{f} : \mathcal{X} \to \overline{S}$, the group $G = G^{mon}(\mathcal{X})$ coincides with the monodromy group of the local system $R^*\overline{f}_*\mathbb{Q}_{\overline{\mathcal{X}}}$. 

For a nontrivial example one can take $f : \mathcal{X} \to S$ to be the natural family of elliptic curves over the Lobachevsky upper half plane $S$. Then the discrete group $G = SL(2, \mathbb{Z})$ acts on $S$ and the quotient $\overline{S} = S/G$ is the coarse moduli space of elliptic curves. This $G$ action is not free, so there is no universal family $f : \mathcal{X} \to S$ and $G$ is not the topological fundamental group of $\overline{S}$. However, according to Definition 2.2, $G$ is the monodromy group of the family $f : \mathcal{X} \to S$. This fits well with Conjecture 1.5. Indeed, the family of elliptic curves $f : \mathcal{X} \to S$ is mirror symmetric to itself (Definition 5.24) and the group $G^\text{eq}(E)$ of a general elliptic curve is the group $G$.

Remark 2.3. We will show that in case of mirror symmetric families of abelian varieties or lattice polarized families of K3 surfaces there exists the monodromy group in the sense of Definition 2.2.

3. Conjecture 1.2 for hyperkahler manifolds

3.1. Construction of the Lie algebras $\mathfrak{g}_{NS}$ and $\mathfrak{g}_{\text{tot}}$ for a hyperkahler manifold.

For a smooth projective variety $X$ one defines the total Lie algebra (sometimes called the LLV Lie algebra) $\mathfrak{g}_{\text{tot}}(X) \subset \text{End}(H(X, \mathbb{Q}))$ in the same way as $\mathfrak{g}_{NS}(X)$ but using all Lefschetz elements in $H^2(X, \mathbb{Q})$ and not just in $NS(X)$ [LL, Verb]. We recall the description of these Lie algebras for a hyperkahler manifold.

Let $V$ be a finite dimensional $\mathbb{Q}$-vector space with a nondegenerate symmetric bilinear form $q$. Consider the graded vector space

$$\tilde{V} := \mathbb{Q}e \oplus V \oplus \mathbb{Q}\eta$$

where $\deg(e) = 0$, $\deg(V) = 2$, $\deg(\eta) = 4$. Extend the form $q$ to a form $\tilde{q}$ on $\tilde{V}$ by putting $\tilde{q}(e, \eta) = 1$, $\tilde{q}(e, V) = \tilde{q}(\eta, V) = 0$.

We make $\tilde{V}$ into a graded commutative algebra by defining multiplication

$$xe := x, \quad \eta e := \eta, \quad xy := q(x, y)\eta$$

for $x, y \in V$. Every nonisotropic $x \in V$ defines a Lefschetz operator on $\tilde{V}$, hence gives rise to an $sl_2$-triple. All such triples generate a graded Lie subalgebra $\mathfrak{g}(V) \subset \text{End}(\tilde{V})$. This is a graded Lie algebra

$$\mathfrak{g}(V) = \mathfrak{g}(V)_{-2} \oplus \mathfrak{g}(V)_0 \oplus \mathfrak{g}(V)_2$$

and $\mathfrak{g}(V) = \mathfrak{so}(\tilde{V}, \tilde{q})$ [Verb, Sect.9]. Moreover, $\mathfrak{g}(V)_0 = \mathfrak{so}(V, q) \oplus \mathbb{Q}$.

If $V' \subset V$ is a subspace such that the form $q' := q|_{V'}$ is nondegenerate, consider a similar extension $(\tilde{V}', \tilde{q}') \subset (\tilde{V}, \tilde{q})$. One can generate a Lie subalgebra $\mathfrak{g}(V') \subset \text{End}(\tilde{V})$ by using only the Lefschetz operators from $V'$. Then again $\mathfrak{g}(V') = \mathfrak{so}(\tilde{V}', \tilde{q}')$ and $\mathfrak{g}(V')_0 = \mathfrak{so}(V', q') \oplus \mathbb{Q}$. 
The above construction is applicable to any smooth projective surface $Y$. Namely, by taking $V = H^2(Y, \mathbb{Q})$ and $V' = NS(Y)_\mathbb{Q}$ we get

$$g(V) = g_{tot}(Y), \quad g(V') = g_{NS}(Y)$$

More interestingly, if $X$ is a projective hyperkahler manifold, $V = H^2(X, \mathbb{Q})$ with the Bogomolov-Beauville (BB) form $q_X$ and $V' = NS(X)_\mathbb{Q}$, we again obtain $g(V) = g_{tot}(X), \quad g(V') = g_{NS}(X)$

In this case the extended lattice $(\tilde{H}^2(X, \mathbb{Q}), \tilde{q}_X)$ is called the rational Mukai lattice of $X$. It has the obvious integral structure

$$\Lambda = \mathbb{Z}e \oplus H^2(X, \mathbb{Z}) \oplus \mathbb{Z} \eta \subset \tilde{H}^2(X, \mathbb{Q})$$

and we equip it with the Hodge structure of weight zero:

$$\tilde{H}^2(X, \mathbb{Q}) = \mathbb{Q}e \oplus H^2(X, \mathbb{Q}(1)) \oplus \mathbb{Q} \eta$$

Denote by $O_{hdg}(\Lambda)$ the discrete group of Hodge isometries of $\Lambda$ and let $O_{hdg}(\Lambda) \subset O(\tilde{H}^2(X, \mathbb{Q}), \tilde{q}_X)$ be its Zariski closure.

**Lemma 3.1.** Let $X$ be a projective hyperkahler manifold. Then we have the equality of Lie subalgebras of $\text{End}(\tilde{H}^2(X, \mathbb{Q}))$

$$\text{Lie}(O_{hdg}(\Lambda)) = g_{NS}(X)$$

**Proof.** Recall that the signature of the BB form $q_X$ on $H^2(X, \mathbb{Z})$ is $(3, b_2 - 3)$ and the signature of its restriction to $NS(X)$ is $(1, s)$ [Huyb2]. Denote by $T(X) \subset H^2(X, \mathbb{Z})$ the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$. Then

$$H^2(X, \mathbb{Q}) = NS(X)_\mathbb{Q} \oplus T(X)_\mathbb{Q}$$

and the signature of the restriction of $q_X$ to $T(X)$ is $(2, b_2 - 3 - s)$. Moreover the Hodge structure on $H^2(X, \mathbb{Z})$ restricts to one on $T(X)$. Let $O_{hdg}(T(X))$ be the corresponding group of Hodge isometries.

First we claim that the group $O_{hdg}(T(X))$ is finite. We copy the argument from [Huyb3 3.3.4]: Consider the real space $T(X)_\mathbb{R}$ and its orthogonal decomposition $T(X)_\mathbb{R} = W \oplus W^\perp$ where $W = H^2(X, \mathbb{R}) \cap (H^{2,0} \oplus H^{0,2})$. The form $q$ is positive definite on $W$ and hence is negative definite on $W^\perp$. The group $O_{hdg}(T(X))$ preserves $W$ and $W^\perp$ so it is contained in a finite subgroup of $O(W) \times O(W^\perp)$.

Define the sublattice

$$\Lambda' := NS(X) \oplus \mathbb{Z}e \oplus \mathbb{Z} \eta \subset \Lambda$$
Then the sublattices $\Lambda'$ and $T(X)$ are preserved by the group $O_{hhdg}(\Lambda)$ and $\Lambda' \oplus T(X) \subset \Lambda$ is a sublattice of full rank. Consider the restriction homomorphism

$$r : O_{hhdg}(\Lambda) \to O(\Lambda')$$

As explained above the kernel of $r$ is finite. It is also clear that the image of $r$ is a subgroup of finite index in $O(\Lambda')$. Therefore

$$\text{Lie}(O_{hhdg}(\Lambda)) = \text{Lie}(O(\Lambda')) = \mathfrak{so}(\Lambda'_Q) = \mathfrak{g}_{\text{NS}}(X)$$

□

Remark 3.2. In Lemma 3.1 the hyperkahler manifold $X$ was used only through the associated Hodge structure on $H^2(X)$.

So essentially it is a statement about $K3$-type Hodge structures.

Later we will need the following fact.

Lemma 3.3. Let $\Gamma \subset O(\tilde{H}^2(X, \mathbb{Q}), \tilde{q}_X)$ be a discrete subgroup of Hodge isometries, and let $\overline{\Gamma} \subset O(\tilde{H}^2(X, \mathbb{Q}), \tilde{q}_X)$ be its Zariski closure. Then

$$\text{Lie}\overline{\Gamma} \subset \mathfrak{g}_{\text{NS}}(X)$$

Proof. Similar to the proof of Lemma 3.1. Namely, in the above notation consider the orthogonal decomposition of rational Hodge structures

(3.3) $\tilde{H}^2(X, \mathbb{Q}) = \Lambda'_Q \oplus T(X)_Q$

This decomposition is preserved by the group $O_{hhdg}(\tilde{H}^2(X, \mathbb{Q}), \tilde{q}_X)$, and so

$$O_{hhdg}(\tilde{H}^2(X, \mathbb{Q}), \tilde{q}_X) = O(\Lambda'_Q) \times O_{hhdg}(T(X)_Q)$$

As in the proof of Lemma 3.1 we conclude that the group $\Gamma \cap O_{hhdg}(T(X)_Q)$ is finite.

Therefore

$$\text{Lie}\overline{\Gamma} \subset \text{Lie}(O(\Lambda')) = \mathfrak{so}(\Lambda'_Q) = \mathfrak{g}_{\text{NS}}(X)$$

□

Corollary 3.4. Conjecture 1.2 holds for projective $K3$ surfaces.

Proof. Let $X$ be a projective $K3$ surface. Then $G^{eq}$ is a discrete subgroup of $GL(\tilde{H}^2(X, \mathbb{Z}), \tilde{q}_X)$ [Huyb, Ch.10]. Moreover $G^{eq}$ is a subgroup of $O_{hhdg}(\tilde{H}^2(X, \mathbb{Z}))$ and its index is at most 2 [Huyb, Ch.10]. It remains to apply Lemma 3.1. □
3.2. Towards a proof of Conjecture [1.2] for hyperkahler manifolds. The following theorem establishes one inclusion in the weak form of Conjecture [1.2] (see Remark [1.3]).

**Theorem 3.5.** Let $X$ be a hyperkahler manifold of dimension $2d$. The action of the Lie algebra $L^q(X)$ on $H(X,\mathbb{Q})$ preserves the subspace $\mathfrak{g}_{NS}(X)\cdot 1$. Moreover we have the inclusion of rational Lie subalgebras of $\text{End}(\mathfrak{g}_{NS}(X)\cdot 1)$:

$$L^q(X)|_{\mathfrak{g}_{NS}(X)}\cdot 1 \subset \mathfrak{g}_{NS}(X).$$

**Proof.** We will use some results of [Tael].

**Theorem 3.6.** [Tael, Thm.A,B] Let $\Phi : D(X) \to D(X)$ be an autoequivalence and $\Phi^H \in \text{GL}(H(X,\mathbb{Q}))$ the corresponding operator on the cohomology. Then the following holds.

1. The operator $\text{Ad}_{\Phi^H}$ preserves the Lie subalgebra $\mathfrak{g}_{\text{tot}}(X) \subset \text{End}(H(X,\mathbb{Q}))$.
2. $\Phi^H$ preserves the irreducible $\mathfrak{g}_{\text{tot}}(X)$-submodule $\mathfrak{g}_{\text{tot}}(X)\cdot 1 \subset H(X,\mathbb{Q})$.

It follows from part (2) of Theorem 3.6 that there is a group homomorphism $G^q(X) \to \text{GL}(\mathfrak{g}_{\text{tot}}(X)\cdot 1)$. Denote by $G^q(X)|_{\mathfrak{g}_{\text{tot}}(X)}\cdot 1$ its image.

The subspace $\mathfrak{g}_{\text{tot}}(X)\cdot 1 \subset H^{even}(X,\mathbb{Q})$ inherits the Hodge structure of weight zero, given by

$$H^{even}(X,\mathbb{Q}) = \bigoplus_{s} H^{2s}(X,\mathbb{Q}(s))$$

The group $G^q(X)|_{\mathfrak{g}_{\text{tot}}(X)}\cdot 1$ is a discrete group of Hodge isometries of $\mathfrak{g}_{\text{tot}}(X)\cdot 1$.

Theorem 3.6 implies that the conjugation action of the group $G^q(X)|_{\mathfrak{g}_{\text{tot}}(X)}\cdot 1$ gives the group homomorphism

$$\alpha : G^q(X)|_{\mathfrak{g}_{\text{tot}}(X)}\cdot 1 \to \text{Aut}(\mathfrak{g}_{\text{tot}}(X))$$

and an element $g \in \ker(\alpha)$ is an automorphism of the simple $\mathfrak{g}_{\text{tot}}(X)$-module $\mathfrak{g}_{\text{tot}}(X)\cdot 1$. Hence $g$ is a scalar operator on $\mathfrak{g}_{\text{tot}}(X)\cdot 1$. But $g$ is an isometry, so $g = \pm 1$.

The Lie algebra $\mathfrak{g}_{\text{tot}}(X)$ is simple, so $G^q(X)|_{\mathfrak{g}_{\text{tot}}(X)}\cdot 1$ has a subgroup $P$ of finite index whose image under $\alpha$ is contained in the adjoint group $\text{Ad}(\mathfrak{g}_{\text{tot}}(X))$ of the Lie algebra $\mathfrak{g}_{\text{tot}}(X)$.

Let $G_{\text{tot}}(X) \subset \text{Gl}(\mathfrak{g}_{\text{tot}}(X)\cdot 1)$ be the connected Lie subgroup with the Lie algebra $\mathfrak{g}_{\text{tot}}(X)$. The adjoint surjective homomorphism $\beta : G_{\text{tot}}(X) \to \text{Ad}(\mathfrak{g}_{\text{tot}}(X))$ has a finite kernel. Put $A := \beta^{-1}(\alpha(P)) \subset G_{\text{tot}}(X)$. So we have the diagram

$$\begin{array}{ccc}
P & \xrightarrow{\alpha} & \text{Ad}(\mathfrak{g}_{\text{tot}}(X)) \\
& \uparrow^{\beta} & \\
A = \beta^{-1}(\alpha(P)) & \hookrightarrow & G_{\text{tot}}(X)
\end{array}$$

The group $P$ is a discrete group of Hodge isometries of $\mathfrak{g}_{\text{tot}}\cdot 1$. If $p \in P, a \in A$ are such that $\alpha(p) = \beta(a)$, then $pa^{-1}$ acts as a scalar on $\mathfrak{g}_{\text{tot}}\cdot 1$. It follows that $A$ is a discrete
subgroup of $G_{\text{tot}}(X)$ which acts by Hodge isometries on $g_{\text{tot}} \cdot 1$ and we have the equality of Lie subalgebras of $\text{End}(g_{\text{tot}} \cdot 1)$:

$$\text{Lie} \overline{A} = L^{eq}(X)|_{g_{\text{tot}} \cdot 1}$$

For the proof of the theorem it suffices to establish the inclusion of Lie subalgebras of $\text{End}(g_{\text{tot}} \cdot 1)$:

(3.5) $$\text{Lie} \overline{A} \subset g_{NS}(X)$$

Recall a lemma from [Tael].

**Lemma 3.7.** Let $\dim X = 2d$. Then there exists a unique map

$$\Psi : g_{\text{tot}} \cdot 1 \to \text{Sym}^d \tilde{H}^2(X, \mathbb{Q})$$

with the following properties.

1. $\Psi(1) = e^d/d!$
2. $\Psi$ is a morphism of $g_{\text{tot}}$-modules.

This map is an injective isometry and a morphism of Hodge structures (3.1).

**Proof.** See [Tael, Prop. 3.5, 3.7, Lemma 4.6].

Denote by $\tilde{G}_{\text{tot}}(X) \subset GL(\tilde{H}^2(X, \mathbb{Q}))$ the connected algebraic subgroup with the Lie algebra $g_{\text{tot}}(X) \subset \text{End}(\tilde{H}^2(X, \mathbb{Q}))$. The group $\tilde{G}_{\text{tot}}(X)$ acts naturally on the space $\text{Sym}^d \tilde{H}^2(X, \mathbb{Q})$, and the restriction to the subspace $\Psi(g_{\text{tot}} \cdot 1)$ gives (by Lemma 3.7) a surjective group homomorphism with finite kernel

$$\theta : \tilde{G}_{\text{tot}}(X) \to G_{\text{tot}}(X)$$

Put $B = \theta^{-1}(A) \subset \tilde{G}_{\text{tot}}(X)$. This is a discrete subgroup of isometries of $\tilde{H}^2(X, \mathbb{Q})$. We claim that it also preserves the Hodge structure. Indeed, the Hodge structure on $\tilde{H}^2(X, \mathbb{Q})$ is given by a group homomorphism $h : S^1 \to SO(\tilde{H}^2(X, \mathbb{R})) \subset \tilde{G}_{\text{tot}}(X)(\mathbb{R})$. The induced Hodge structure on the subspace $\Psi(g_{\text{tot}} \cdot 1) \subset \text{Sym}^d \tilde{H}^2(X, \mathbb{Q})$ is given as the composition

$$S^1 \xrightarrow{h} \tilde{G}_{\text{tot}}(X)(\mathbb{R}) \xrightarrow{\theta} G_{\text{tot}}(X)(\mathbb{R}) \subset GL(\Psi(g_{\text{tot}} \cdot 1), \mathbb{R})$$

For every $b \in B$, the element $\theta(b) \in A$ commutes with the Hodge structure on $\Psi(g_{\text{tot}} \cdot 1)$ (since $\Psi$ is a morphism of Hodge structures). It follows that $b$ commutes with the Hodge structure on $\tilde{H}^2(X, \mathbb{Q})$ up to an element in the kernel of $\theta$, which is a finite group. But the group $S^1$ is connected, hence $b$ and the image of $S^1$ commute.

We conclude that $B$ is a discrete group of Hodge isometries of $\tilde{H}^2(X, \mathbb{Q})$. Lemma 3.3 implies that

$$\text{Lie} \overline{B} \subset g_{NS}(X)$$
But then
\[ \text{Lie} \overline{A} = \text{Lie} \overline{B} \subset \mathfrak{g}_{NS}(X) \]
which proves Theorem 3.5. □

4. Conjecture [1.5] for dual families of K3 surfaces

We define the notion of mirror symmetric families of lattice polarized families of K3 surfaces following the work of Dolgachev-Nikulin [Dolg] and Pinkham [Pink]. Then we prove Conjecture [1.5] for such families (Theorem 4.13).

Let us recall the notion of a lattice polarized K3 surface and their moduli spaces following [Dolg]. First we review the classical theory of moduli space of K3 surfaces [K3]. Let \( L \) be the even unimodular lattice of signature \((3,19)\) which is the direct sum
\[ L = (-E_8)^{62} \oplus (U)^{83} \]
Recall that for any K3 surface \( X \) the lattice \( H^2(X, \mathbb{Z}) \) is isomorphic to \( L \). Unless stated otherwise we consider K3 surfaces which are not necessarily algebraic.

**Definition 4.1.** A marked K3 surface \((X, u)\) is a K3 surface \( X \) with an isomorphism of lattices \( u : H^2(X, \mathbb{Z}) \to L \). Marked surfaces \((X, u)\) and \((X', u')\) are isomorphic if there exists an isomorphism \( f : X \to X' \) such that \( u' = u \cdot f^* \).

The following theorem is proved in [K3, Exp. XIII].

**Theorem 4.2.** There exists a fine moduli space \( \mathcal{M} \) of marked K3 surfaces.

The moduli space \( \mathcal{M} \) is a non-separated analytic space. By definition it comes with the universal family \( f : \mathcal{U} \to \mathcal{M} \) of marked K3 surfaces. The orthogonal group \( \Gamma = O(L) \) acts naturally on \( \mathcal{M} \) by changing the marking \( \gamma \cdot (X, u) = (X, \gamma \cdot u) \) and the quotient \( \mathcal{M}/\Gamma \) is the set of isomorphism classes of K3 surfaces, i.e. \( \mathcal{M}/\Gamma \) is the coarse moduli space of K3 surfaces. However, the action of \( \Gamma \) on \( \mathcal{M} \) is not proper (because the stabilizer of a point \((X, u)\) is isomorphic to the automorphism group of \( X \), which may be infinite) and there is no reasonable analytic structure on the set \( \mathcal{M}/\Gamma \).

The space \( \mathcal{M} \) has two connected components which are interchanges by the involution \((X, u) \leftrightarrow (X, -u)\). Choose one of these components \( \mathcal{M}^0 \) and let \( \Gamma^0 \subset \Gamma \) be its stabilizer (a subgroup of index 2). Clearly \( \mathcal{M}^0/\Gamma^0 = \mathcal{M}/\Gamma \). We denote by \( f^0 : \mathcal{U}^0 \to \mathcal{M}^0 \) the restriction of the universal family \( f : \mathcal{U} \to \mathcal{M} \).

Given a marked K3 surface \((X, u)\), the image of the line \( H^{2,0}(X) \) under the map \( u_C : H^2(X, \mathbb{C}) \to L_C = \mathbb{C}^{22} \) defines a point in the corresponding projective space \( \mathbb{P}(L_C) = \mathbb{P}^{21} \).
This point lies in the period domain $\Omega \subset \mathbb{P}^{21}$ consisting of points $\{ \omega \in \mathbb{P}^{21} \mid (\omega, \omega) = 0, (\omega, \overline{\omega}) > 0 \}$ and so one gets the period map

\[(4.1) \quad P : \mathcal{M} \to \Omega\]

It is known that the map $P$ is holomorphic, etale and surjective \([K3, \text{Ex. XIII}]\). Its restriction to $\mathcal{M}^0$ is also surjective.

**Lemma 4.3.** Let $g \in O(L), g \neq \pm 1$. The collection of marked K3 surfaces

$$E_g := \{(X, u) \mid u \in \mathcal{C}(H^{2,0}) \text{ is an eigenvector of } g \}$$

is contained in a proper analytic subspace of $\mathcal{M}$.

**Proof.** It suffices to prove that the image $P(E_g)$ is contained in a proper analytic subspace of the period domain $\Omega$. Our assumption on $g$ means that it is not a scalar operator. Thus eigenvectors of $g_\mathcal{C}$ are contained in a union of proper linear subspaces of $L_\mathcal{C}$. But the period domain $\Omega$, being an open subset of a nondegenerate quadric, is not contained in any hyperplane in $\mathbb{P}^{21}$, which proves the lemma. □

**Corollary 4.4.** Consider $f^0 : \mathcal{U}^0 \to \mathcal{M}^0$ as the family of unmarked K3 surfaces. Then the group $\Gamma^0$ is its monodromy group (Definition 2.2).

**Proof.** In terms of Definition 2.2 we have $S = \mathcal{M}^0, X = \mathcal{U}^0, G = \Gamma^0, K = \{1\}$. The marking defines a canonical trivialization of the local system $R^\bullet f^0_* \mathbb{Q}_{\mathcal{U}^0}$. Clearly the $\Gamma^0$-action on $\mathcal{Q} \oplus L_\mathcal{Q} \oplus \mathbb{Q} = H^0(\mathcal{M}, R^\bullet f^0_* \mathbb{Q}_{\mathcal{U}^0})$ is effective. Since $\mathcal{M}^0/\Gamma^0$ is a coarse moduli space of K3 surfaces, it remains to show that away from a countable number of analytic subsets the $\Gamma^0$-action on $\mathcal{M}^0$ is free.

Let $1 \neq g \in \Gamma^0$ and assume that $(X, u) \in (\mathcal{M}^0)^g$. For simplicity of notation let us identify $H^2(X, \mathbb{Z})$ with $L$ by means of $u$. Then there exists an automorphism $\phi : X \to X$ such that $\phi^* : H^2(X, \mathbb{C}) \to H^2(X, \mathbb{C})$ equals $g_\mathcal{C}$. In particular the line $H^{2,0}(X)$ is contained in an eigenspace of $\phi^*$. Lemma 4.3 implies that $(X, u)$ belongs to a proper analytic subspace $E_g$ of $\mathcal{M}^0$, unless $g = \pm 1$. We excluded the case $g = 1$, and $g = -1$ does not belong to $\Gamma^0$.

Since $\Gamma^0$ has countably many elements, the subset of $\mathcal{M}^0$ on which the $\Gamma^0$-action is free is the complement of countably many proper analytic subsets, hence in particular it is everywhere dense. □

4.1. **Lattice polarized K3 surfaces and their moduli spaces.** Let $X$ be a projective K3 surface. It is known that the first Chern class map

$$c : Pic(X) \to H^2(X, \mathbb{Z})$$
is injective. By Hodge index theorem $\text{Pic}(X)$ is a lattice of signature $(1, t)$.

Let $M$ be an even non-degenerate lattice of signature $(1, s)$. Let

$$\Delta(M) = \{ \delta \in M \mid (\delta, \delta) = -2 \}$$

Fix a subset $\Delta(M)^+ \subset \Delta(M)$ such that

(i) $\Delta(M) = \Delta^+ \coprod (-\Delta(M)^+)$;

(ii) if $\delta_1, ..., \delta_k \in \Delta(M)^+$ and $\delta = \sum n_i \delta_i \in \Delta(M)$ with $n_i \geq 0$, then $\delta \in \Delta(M)^+$.

The choice of a subset $\Delta(M)^+ \subset \Delta(M)$ defines the subset

$$C(M)^+ = \{ h \in M \mid (h, h) > 0 \text{ and } (h, \delta) > 0 \text{ for all } \delta \in \Delta(M)^+ \}$$

**Definition 4.5.** An $M$-polarized K3 surface is a pair $(X, j)$, where $X$ is K3 surface and $j : M \to \text{Pic}(X)$ is a primitive lattice embedding. We say that $(X, j)$ is ample polarized if in addition $j(M)$ contains the class of an ample divisor on $X$. Two $M$-polarized K3 (resp. ample polarized) surfaces $(X, j)$ and $(X', j')$ are called isomorphic if there exists an isomorphism $f : X \to X'$ such that $j = f^* \cdot j'$.

**Remark 4.6.** Notice that any $M$-polarized K3 surface $X$ is projective. Indeed, by the signature assumption there exists $q \in M$ such that $(q, q) > 0$. So there exists a line bundle $\mathcal{L} \in \text{Pic}(X)$ with $c_1(\mathcal{L})^2 > 0$. This implies that $X$ is projective [Kod, Thm. 8].

Now assume that we are given a primitive embedding of lattices $a : M \to L$.

**Definition 4.7.** A marked $M$-polarized K3 surface is a triple $(X, j, u)$ such that $(X, u)$ is a marked K3 surface, $(X, j)$ is an $M$-polarized K3 surface and in addition

$$a = u \cdot j : M \to L$$

We say that $(X, j, u)$ is marked ample $M$-polarized if $(X, j)$ is ample $M$-polarized. Two marked $M$-polarized K3 surfaces are isomorphic if they are isomorphic as marked K3 surfaces (and hence also as $M$-polarized K3 surfaces).

Clearly, a marked $M$-polarized K3 surface surface $(X, j, u)$ is uniquely determined by the corresponding marked K3 surface $(X, u)$.

Let $N := M_L^\perp$ be the orthogonal complement of $M$ in $L$. We have the inclusion of projective spaces $\mathbb{P}(N_C) \subset \mathbb{P}(L_C)$ and put $\Omega_M := \Omega \cap \mathbb{P}(N_C)$. This is the period domain for $M$-polarized K3 surfaces. It has 2 connected components.

For any $\delta \in \Delta(N) := \{ a \in N \mid (a, a) = -2 \}$ set

$$H_\delta := \{ z \in N_C \mid (z, \delta) = 0 \}$$
and define
\[ \Omega^0_M := \Omega_M \setminus \left( \bigcup_{\delta \in \Delta(N)} H_\delta \cap \Omega_M \right) \]

Since \( \Omega_M \) has two connected components, so does \( \Omega^0_M \).

Similarly to Theorem 4.2 one can prove the following [Dolg, Cor. 3.2]

**Theorem 4.8.** (1) There exists a fine moduli space \( \mathcal{M}_M \) of marked \( M \)-polarized K3 surfaces. It is a non-separated analytic space, which is an analytic subspace of \( \mathcal{M} \). The universal family \( f_M : \mathcal{U}_M \to \mathcal{M}_M \) is the restriction of the universal family \( f : \mathcal{U} \to \mathcal{M} \).

(2) The obvious period map \( P_M : \mathcal{M}_M \to \Omega_M \) is analytic, étale and surjective.

(3) The diagram of analytic maps
\[
\begin{array}{ccc}
\mathcal{M}_M & \hookrightarrow & \mathcal{M} \\
P_M \downarrow & & P \downarrow \\
\Omega_M & \hookrightarrow & \Omega
\end{array}
\]

commutes.

(4) Let \( \mathcal{M}^a_M \subset \mathcal{M}_M \) denote the subspace parametrizing marked ample \( M \)-polarized K3 surfaces. Then the restriction to \( \mathcal{M}^a_M \) of the family \( f_M : \mathcal{U}_M \to \mathcal{M}_M \) is the universal family \( f^a_M : \mathcal{U}^a_M \to \mathcal{M}^a_M \) of marked ample \( M \)-polarized K3 surfaces. The subset \( \mathcal{M}^a_M \subset \mathcal{M}_M \) is open. The restriction of the period map \( P_M \) is an isomorphism
\[ P^a_M : \mathcal{M}^a_M \sim \to \Omega^0_M \]

In particular, the space \( \mathcal{M}^a_M \) has 2 connected components.

Consider the group
\[ \Gamma_M = \{ \sigma \in O(L) \mid \sigma(m) = m \text{ for all } m \in a(M) \} \]

This group acts on the space \( \mathcal{M}_M \) in the obvious way: \( \sigma(X, j, u) = (X, j, \sigma \cdot u) \). It preserves the subspace \( \mathcal{M}^a_M \).

Notice that the above concepts of a marked \( M \)-polarized K3 surface, the moduli space \( \mathcal{M}_M \), and the group \( \Gamma_M \) only make sense after we have made a choice of a primitive lattice embedding \( a : M \hookrightarrow L \). As in [Dolg] we consider the following condition on the lattice \( M \):

(U) For any two primitive embeddings \( a_1, a_2 : M \hookrightarrow L \), there exists an isometry \( \sigma : L \to L \) such that \( \sigma \cdot a_1 = a_2 \).

**Lemma 4.9.** Assuming condition (U), the quotient space \( \mathcal{M}_M / \Gamma_M \) is the coarse moduli space of \( M \)-polarized K3 surfaces. Hence also \( \mathcal{M}^a_M / \Gamma_M \) is the coarse moduli space of ample \( M \)-polarized K3 surfaces.
Proof. The assumption (U) means that any $M$-polarized K3 surface $(X, j)$ can be complemented to a marked $M$-polarized K3 surface. Indeed, choose any lattice isomorphism $u : H^2(X, \mathbb{Z}) \rightarrow L$. Then by condition (U) there exists an automorphism $\sigma : L \rightarrow L$, such that $(X, j, \sigma \cdot u)$ is a marked $M$-polarized K3 surface. In particular, the forgetful map $(X, j, u) \mapsto (X, j)$ from isomorphism classes of marked $M$-polarized K3 surfaces to isomorphism classes of $M$-polarized K3 surfaces is surjective. Obviously, the group $\Gamma_M$ acts on the fibers of this map.

It remains to show that given marked $M$-polarized K3 surfaces $(X, j, u)$ and $(X', j', u')$ such that $(X, j) \simeq (X', j')$, there exists a $\tau \in \Gamma_M$, such that $(X, j, u) \simeq (X', j', \tau \cdot u')$. So assume that there exists an isomorphism $\phi : X \rightarrow X'$ such that

$$j = \phi^* \cdot j' : M \rightarrow \text{Pic}(X) \subset H^2(X, \mathbb{Z})$$

Then the automorphism $\tau := u \cdot \phi^* \cdot (u')^{-1} : L \rightarrow L$ is the identity on $a(M)$, i.e. $\tau \in \Gamma_M$, which means that $\phi$ induces an isomorphism $(X, j, u) \simeq (X', j', \tau \cdot u')$. □

Let $\mathcal{M}_M^{a,0} \subset \mathcal{M}_M^a$ be one of the connected components (Theorem 18) and let $f^{a,0}_M : \mathcal{U}^{a,0}_M \rightarrow \mathcal{M}_M^{a,0}$ be the restriction of the universal family $f^{a}_M : \mathcal{U}_M^a \rightarrow \mathcal{M}_M^a$. Let $\Gamma_0^M \subset \Gamma_M$ be the stabilizer of the component $\mathcal{M}_M^{a,0}$. So the index of $\Gamma_0^M$ in $\Gamma_M$ is at most 2.

Remark 4.10. We note for future reference that $\Gamma_M$ (and hence also $\Gamma_0^M$) is a subgroup of finite index in the orthogonal group $O(N)$ ($N = M_L^\perp$) [Dolg Prop. 3.3].

Proposition 4.11. Assume that condition (U) holds. Consider $f^{a,0}_M : \mathcal{U}^{a,0}_M \rightarrow \mathcal{M}_M^{a,0}$ as a family of unmarked ample $M$-polarized K3 surfaces. Then its monodromy group is isomorphic to $\Gamma_0^M$ up to finite groups (Definition 2.2).

Proof. The marking defines a canonical trivialization of the local system $R^\bullet f^{a,0}_M_* \Omega_{\mathcal{U}^{a,0}_M}$ on $\mathcal{M}_M^{a,0}$ and clearly the $\Gamma_0^M$-action on $\mathbb{Q} \oplus L \otimes \mathbb{Q} = H^0(\mathcal{M}_M^{a,0}, R^\bullet f^{a,0}_M_* \Omega_{\mathcal{U}^{a,0}_M})$ is effective. Since $\mathcal{M}_M^{a,0}/\Gamma_0^M$ is the coarse moduli space of ample $M$-polarized K3 surfaces appearing in the family $\mathcal{U}^{a,0}_M$ (Lemma 14), it remains to show that the $\Gamma_0^M$-action on $\mathcal{M}_M^{a,0}$ is generically free modulo a finite kernel.

As in the proof of Corollary 14 it is enough to show that $\Gamma_M$ acts generically free modulo a finite kernel on the period domain $\Omega_M = \Omega \cap \mathbb{P}(N_C)$ (Theorem 18). Since $\Gamma_M \subset O(N)$ it suffices to analyze the $O(N)$-action on $\Omega_M$. Applying a version of Lemma 13 with $O(N)$ and $\Omega_M$ instead of $O(L)$ and $\Omega$, we find that $O(N)$ acts generically free on $\Omega_M$ modulo its center $\{\pm 1\}_O(N)$. It follows that $\Gamma_M$ acts generically free on $\Omega_M$ modulo its central subgroup $\Gamma_M \cap \{\pm 1\}_O(N)$. Hence $\Gamma_M$ acts on $\mathcal{M}_M$ either generically free or generically free modulo $\Gamma_M \cap \{\pm 1\}_O(N)$.
4.2. Mirror symmetric families of lattice polarized K3 surfaces. Let $M$ be a lattice as above with a fixed primitive embedding of lattices $M \hookrightarrow L$. We will identify $M$ with its image in $L$. We call a primitive sublattice $M^\vee \subset L$ a mirror dual of $M$ if there is a direct sum decomposition

$$L = M \oplus U \oplus M^\vee$$

The signature of $M^\vee$ is $(1, 18 - s)$ (if the signature of $M$ is $(1, s)$). It is clear that $M = M^{\vee \vee}$.

(Our definition of a mirror dual sublattice is a somewhat simplified version of [Dolg]).

Definition 4.12. In the above notation we consider the universal families $f_{M^0}^0 : \mathcal{U}^{a,0}_M \to \mathcal{M}^{a,0}_M$ and $f_{M^\vee}^0 : \mathcal{U}^{a,0}_{M^\vee} \to \mathcal{M}^{a,0}_{M^\vee}$ as mirror symmetric families of ample lattice polarized K3 surfaces.

Our main result is the following.

Theorem 4.13. In the above notation assume that the lattices $M$ and $M^\vee$ satisfy condition $(U)$. Then the groups $G_{\text{mon}}(\mathcal{U}^{a,0}_{M^\vee})$ and $G_{\text{eq}}(\mathcal{U}^{a,0}_M)$ are isomorphic up to finite groups. That is, Conjecture 1.5 holds for mirror symmetric families of ample lattice polarized K3 surfaces.

4.3. Proof of Theorem 4.13. The proof will take several steps.

By assumption we have sublattices $M, M^\vee \subset L$ of signatures $(1, s)$ and $(1, 18 - s)$ respectively that satisty

$$(M^\vee)^\perp_L = M \oplus U$$

By Proposition 4.11 the monodromy group of the family $\mathcal{U}^{a,0}_{M^\vee}$ is isomorphic up to finite groups to

$$\Gamma_{M^\vee} = \{ \sigma \in O(L) \mid \sigma(m) = m \text{ for all } m \in M^\vee \}$$

We have the natural injective homomorphism $\Gamma_{M^\vee} \hookrightarrow O((M^\vee)^\perp_L) = O(M \oplus U)$ and by Remark 4.10 the image is a subgroup of finite index. Therefore $G_{\text{mon}}(\mathcal{U}^{a,0}_{M^\vee})$ is isomorphic up to finite groups to the group $O(M \oplus U)$, and it suffices to prove the following proposition.

Proposition 4.14. For a general $M$-polarized K3 surface $X$ the group $G_{\text{eq}}(X)$ is isomorphic up to finite groups to $O(M \oplus U)$.

Proof. By Remark 4.6 any $M$-polarized K3 surface is projective. For any projective K3 surface $Y$ the group $G_{\text{eq}}(Y)$ is well known: it is a subgroup of the group $O_{\text{hdlg}}(\tilde{H}^2(Y, \mathbb{Z}))$ (Section 3.1) of index at most two [Huyb Ch. 10]. So for the proof of Proposition 4.14 it remains to show that for a general $M$-polarized K3 surface $X$ the groups $O(M \oplus U)$ and $O_{\text{hdlg}}(\tilde{H}^2(X, \mathbb{Z}))$ are isomorphic up to finite groups.
Lemma 4.15. For a general \( M \)-polarized K3 surface \( X \) we have the equality \( M = \text{Pic}(X) \).

Proof. It follows from the assumption (U) on the lattice \( M \) that any \( M \)-polarized K3 surface can be complemented to a marked \( M \)-polarized K3 surface (see proof of Lemma 4.9). So it suffices to prove the equality \( M = \text{Pic}(X) \) for a general marked \( M \)-polarized K3 surface \( X \). Given such a surface \( X \), we consider the corresponding point \([X] \in \mathcal{M}_M\) and its image \( P_M([X]) \in \Omega_M \cap l_L^1\). If \( l \in \text{Pic}(X) \), then \( P_M([X]) \in \Omega_M \cap l_L^1\) and unless \( l \in M \), this intersection \( \Omega_M \cap l_L^1\) is a proper analytic subset of \( \Omega_M \). So \( P_M^{-1}(\Omega_M \cap l_L^1) \) is a proper analytic subset of \( M \). Because \( \mathcal{M}_M \) is a Baire space it is not a countable union of nowhere dense subsets. This proves the lemma. \( \square \)

For an \( M \)-polarized K3 surface \((X,j)\) we will identify \( M \) with its image \( j(M) \subset \text{Pic}(X)\). Consider the extension of the sublattice \( M \subset H^2(X,\mathbb{Z}) \) to the primitive sublattice \( \tilde{M} := M \oplus H^0(X,\mathbb{Z}) \oplus H^4(X,\mathbb{Z}) \) of \( \tilde{H}^2(X,\mathbb{Z}) \). Then abstractly \( \tilde{M} \simeq M \oplus U \). In particular \( O(\tilde{M}) = O(M \oplus U) \). Assuming that \( X \) is general, by Lemma 4.15 we may assume that \( M = \text{Pic}(X) \). Then the group \( O(\tilde{M}) \) and \( O_{\text{hdg}}(\tilde{H}^2(X,\mathbb{Z})) \) are isomorphic up to finite groups as is shown in the proof of Lemma 3.1. This proves Proposition 4.14 and Theorem 4.13. \( \square \)

5. Conjecture 1.5 for dual families of abelian varieties

In [GLO] there was defined a notion of mirror symmetry for algebraic pairs (see Definition 5.16 below). An algebraic pair \((A,\omega)\) consists of an abelian variety \( A \) and an element \( \omega \) of the complexified ample cone of \( A \) (Definition 5.13). Building on this work we define the notion of mirror symmetric families of abelian varieties (Definition 5.24). Then we prove Conjecture 1.5 for such families. We start by recalling some relevant facts about abelian varieties.

5.1. Complex tori and abelian varieties. [Mu1], [BirLa], [GLO].

5.1.1. Let \( \Gamma \simeq \mathbb{Z}^{2n} \) be a lattice, \( V = \Gamma \otimes \mathbb{R} \simeq \mathbb{R}^{2n} \) and \( J \in \text{End}_\mathbb{R}(V) \), s.t. \( J^2 = -1 \). (Here a lattice means a discrete subgroup of finite covolume). That is \( J \) is a complex structure on \( V \). This way we obtain an \( n \)-dimensional complex torus \( A = (V/\Gamma, J) \).

Note the canonical isomorphisms

\[ \Gamma = H_1(A,\mathbb{Z}), \quad V = H_1(A,\mathbb{R}). \]

Sometimes we will write \( \Gamma_A, V_A, J_A \).
Given another complex torus \( B = (V_B/\Gamma_B, J_B) \), the group \( \text{Hom}(A, B) \) consists of homomorphisms \( f : \Gamma_A \to \Gamma_B \) such that
\[
J_B \cdot f_R = f_R \cdot J_A : V_A \to V_B
\]
Thus the abelian group \( \text{Hom}(A, B) \) can be considered as a subgroup of \( \text{Hom}(\Gamma_A, \Gamma_B) \).

5.1.2. One has the dual torus \( \hat{A} \) defined as follows. Put \( \Gamma^*_A = \text{Hom}_\mathbb{Z}(\Gamma_A, \mathbb{Z}) \), \( V^*_A = \Gamma^*_A \otimes \mathbb{R} = \text{Hom}(V, \mathbb{R}) \) and \( \hat{J} : V^* \to V^* \), s.t. \( (\hat{J}w)(v) = w(-Jv) \) for \( v \in V, w \in V^* \). Then by definition
\[
\hat{A} = (V^*/\Gamma^*, \hat{J}).
\]

5.1.3. Denote by \( \text{Pic}_A \) the Picard group of \( A \). Let \( \text{Pic}_A^0 \subset \text{Pic}_A \) be the subgroup of line bundles with the trivial Chern class. It has a natural structure of a complex torus. Moreover, there exists a natural isomorphism of complex tori
\[
\hat{A} \simeq \text{Pic}_A^0.
\]
Every line bundle \( L \) on \( A \) defines a morphism \( \phi_L : A \to \hat{A} \) by the formula
\[
\phi_L(a) = T^*_a L \otimes L^{-1}.
\]
(Here \( T_a : A \to A \) is the translation by \( a \)). We have \( \phi_L = 0 \) iff \( L \in \text{Pic}_A^0 \), and \( \phi_L \) is an isogeny if \( L \) is ample. Thus the correspondence \( L \mapsto \phi_L \) identifies the Néron-Severi group \( NS_A := \text{Pic}_A / \text{Pic}_A^0 \) as a subgroup in \( \text{Hom}(A, \hat{A}) \). Also \( NS_A \) is naturally a subgroup of \( H^2(A, \mathbb{Z}) \): to a line bundle \( L \) there corresponds its first Chern class, which can be considered as a skew-symmetric bilinear form on \( \Gamma \). Put \( c_1(L) = c \). Then the morphism \( \phi_L \) is given by the map
\[
V_A \to V_A, \quad v \mapsto c(v, \cdot).
\]
We will identify \( NS_A \) either as a subgroup of \( \text{Hom}(A, \hat{A}) \) or \( \text{Hom}(\Gamma_A, \Gamma_{\hat{A}}) \) or as a set of (integral) skew-symmetric forms \( c \) on \( \Gamma_A \) such that the extension \( c_R \) on \( V_A \) is \( J \)-invariant.

5.1.4. Given a morphism of complex tori \( f : A \to B \), the dual morphism \( \hat{f} : \hat{B} \to \hat{A} \) is defined.

The double dual torus \( \hat{A} \) is naturally identified with \( A \) by means of the Poincaré line bundle on \( A \times \hat{A} \) and \( \hat{A} \times \hat{A} \). So given a morphism \( f : A \to \hat{A} \) we will consider \( \hat{f} : \hat{A} \to \hat{A} \) as a morphism from \( A \) to \( \hat{A} \) again. Then for \( L \in NS_A \) we have \( \hat{\phi}_L = \phi_L \).
5.1.5. Consider the lattice $\Lambda = \Lambda_A := \Gamma_A \oplus \Gamma_A^*$ with the canonical symmetric bilinear form $Q: \Lambda \times \Lambda \rightarrow \mathbb{Z}$ defined as follows

$$Q((a_1, b_1), (a_2, b_2)) = b_1(a_2) + b_2(a_1).$$

Let $O(\Lambda, Q) \subset GL(\Lambda)$ be the corresponding orthogonal group. It is equal to

$$O(\Lambda, Q) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \left( \begin{array}{cc} \text{Hom}(\Gamma, \Gamma) & \text{Hom}(\Gamma^*, \Gamma) \\ \text{Hom}(\Gamma, \Gamma^*) & \text{Hom}(\Gamma^*, \Gamma^*) \end{array} \right) \mid g^{-1} = \begin{pmatrix} \hat{d} & -\hat{b} \\ -\hat{c} & \hat{a} \end{pmatrix} \right\}$$

where $\Gamma = \Gamma_A$.

Notice that if $A = (V/\Gamma_A, J_A)$ is a complex torus, then the complex structure $J_{A \times \hat{A}}$ of the product $A \times \hat{A} = (\Lambda_R/\Lambda, J_{A \times \hat{A}})$ belongs to the special orthogonal group $SO(\Lambda_R, Q_R)$.

5.1.6. A complex torus $A = (V/\Gamma, J)$ is algebraic, i.e. an abelian variety, if there exists $c \in NS_A$ such that the symmetric bilinear form $c_\mathbb{R}(\cdot, \cdot)$ on $V$ is positive definite. If a line bundle $L \in \text{Pic}_A$ is ample then the induced map

$$\phi_L: \Gamma_A, Q \rightarrow \Gamma_{\hat{A}}, Q$$

is an isomorphism.

We will only be interested in complex tori which are abelian varieties.

5.2. Hodge group of an abelian variety. Let $W$ be a finite dimensional $\mathbb{Q}$-vector space and $J \in \text{End}(W_\mathbb{R})$ a complex structure, i.e. $J^2 = -1$. This defines an embedding of $\mathbb{R}$-algebras $\mathbb{C} \subset \text{End}(W_\mathbb{R})$ and in particular an inclusion of groups $h: S^1 \hookrightarrow \text{Aut}(W_\mathbb{R})$ such that $h(\sqrt{-1}) = J$.

**Definition 5.1.** The Hodge group of the complex structure $J$ is the smallest algebraic $\mathbb{Q}$-subgroup $H \subset \text{Aut}(W)$ such that $h(S^1) \subset H(\mathbb{R})$. We denote it by $\langle J \rangle_\mathbb{Q}$. If $A = \langle V_A/\Gamma_A, J_A \rangle$ is an abelian variety, the Hodge group $\langle J_A \rangle_\mathbb{Q} \subset Gl(\Gamma_{A, \mathbb{Q}})$ is also denoted by $Hdg_A$.

**Remark 5.2.** Since the Lie group $S^1$ is connected, so is the algebraic $\mathbb{Q}$-group $\langle J \rangle_\mathbb{Q}$.

We have canonical identifications

$$Hdg_A = Hdg_{\hat{A}} = Hdg_{A \times \hat{A}}$$

Depending on the context we may view $Hdg_A$ as a subgroup of $Gl(\Gamma_{A, \mathbb{Q}})$ or $GL(\Gamma_{A, \mathbb{Q}})$ or $SO(\Lambda_\mathbb{Q}, Q_\mathbb{Q})$. (Indeed, by construction $J_{A \times \hat{A}} \in O(\Lambda_R, Q_R)$ and so $Hdg_{A \times \hat{A}} \subset O(\Lambda_\mathbb{Q}, Q_\mathbb{Q})$; hence $Hdg_{A \times \hat{A}} \subset SO(\Lambda_\mathbb{Q}, Q_\mathbb{Q})$ because $Hdg_{A \times \hat{A}}$ is connected.)
The following facts about the group $Hdg_A$ are known ([Mu2], [Del1]):

**Theorem 5.3.** Assume that $A$ is an abelian variety.

(a) $Hdg_A$ is a connected reductive algebraic $\mathbb{Q}$-group without simple factors of exceptional type.

(b) The adjoint action of $J_A$ on the Lie group $Hdg_A(\mathbb{R})^0$ is a Cartan involution, i.e. it is an involution whose fixed subgroup is a maximal compact subgroup $K$.

(c) The symmetric space $Hdg_A(\mathbb{R})^0/K$ is of Hermitian type.

5.3. **Derived category of an abelian variety.** Let $A$ be an abelian variety. In this case the action of the group $AutD^b(A)$ preserves the integral cohomology of $A$, i.e. we have the homomorphism

$$
\rho_A : AutD^b(A) \to Gl(H^\bullet(A, \mathbb{Z}))
$$

(In [GLO] the image of $\rho_A$ is denoted $Spin(A)$, but here we denote it $G_{eq}(A)$.) This group tends to be big and there exists a precise description of this group in terms of the Mukai-Polishchuk group $U(A)$. Let us recall it.

**Definition 5.4.** For an abelian variety $A$ put

$$
U(A) = \left\{ g = \begin{pmatrix} \ast & b \\ c & \ast \end{pmatrix} \in \left( \begin{array}{cc} \text{End}(A) & \text{Hom}(\hat{A}, A) \\
\text{Hom}(A, \hat{A}) & \text{End}(\hat{A}) \end{array} \right) \left| g^{-1} = \begin{pmatrix} \hat{d} & -\hat{b} \\ -\hat{c} & \hat{a} \end{pmatrix} \right. \right\}
$$

So by definition we have $U(A) = Aut(A \times \hat{A}) \cap O(\Lambda, \mathbb{Q})$, which also equals $Aut(A \times \hat{A}) \cap SO(\Lambda, \mathbb{Q})$, because elements of $Aut(A \times \hat{A})$ have positive determinant (as they preserve the complex structure on $V_A \oplus \hat{V}_A$).

5.3.1. For us the group $U(A)$ is important because of the following facts.

**Proposition 5.5.** There exists a natural exact sequence of groups

$$
0 \to \mathbb{Z} \times A \times \hat{A} \to Auteq(D^b(A)) \to U(A) \to 1
$$

The homomorphism $\rho_A : Aut(D^b(A)) \to Gl(H^\bullet(A, \mathbb{Z}))$ almost factors through the group $U(A)$. Namely, we have the exact sequence of groups

$$
0 \to \mathbb{Z}/2\mathbb{Z} \to G_{eq}(A) \to U(A) \to 1
$$

**Remark 5.6.** It follows that the groups $G_{eq}(A)$ and $U(A)$ are isomorphic up to finite groups.

As explained in [GLO] the group $SO(\Lambda, \mathbb{Q})$ does not act on the space $H^\bullet(A, \mathbb{Z})$, but its double cover does. Namely, there is a discrete group $Spin(\Lambda, \mathbb{Q})$ and an exact sequence

$$
0 \to \mathbb{Z}/2\mathbb{Z} \to Spin(\Lambda, \mathbb{Q}) \to SO(\Lambda, \mathbb{Q})
$$
The group $Spin(\Lambda, Q)$ acts on $H^\bullet(A, \mathbb{Z}) = \Lambda^\bullet \Gamma_A^*$ via the spinorial representation. Moreover we have the commutative diagram

$$
\begin{array}{ccc}
G^eq(A) & \hookrightarrow & Spin(\Lambda, Q) \\
\downarrow & & \downarrow \\
U(A) & \hookrightarrow & SO(\Lambda, Q)
\end{array}
$$

and the action of $G^eq(A)$ on $H^\bullet(A, \mathbb{Z})$ is the restriction of the spinorial representation of $Spin(\Lambda, Q)$.

5.4. **The algebraic $\mathbb{Q}$-group $U_{A,Q}$**. Let $A$ be an abelian variety and let $Aut_{\mathbb{Q}}(A \times \hat{A})$ be the group of invertible elements in $\text{End}^0(A \times \hat{A}) := \text{End}(A \times \hat{A}) \otimes \mathbb{Q}$. Define the algebraic $\mathbb{Q}$-group $U_{A,Q}$ as follows

$$
U_{A,Q} = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Aut_{\mathbb{Q}}(A \times \hat{A}) \left| g^{-1} = \begin{pmatrix} \hat{d} & -\hat{b} \\ -\hat{c} & \hat{a} \end{pmatrix} \right. \right\}
$$

So $U_{A,Q} = Aut_{\mathbb{Q}}(A \times \hat{A}) \cap O(\Lambda_Q, \mathbb{Q}) = Aut_{\mathbb{Q}}(A \times \hat{A}) \cap SO(\Lambda_Q, \mathbb{Q})$ and $U(A)$ is the arithmetic subgroup of $U_{A,Q}$ consisting of elements that preserve the lattice $\Lambda$.

**Remark 5.7.** Note that the algebraic $\mathbb{Q}$-group $U_{A,Q}$ is the centralizer in $SO(\Lambda_Q, \mathbb{Q})$ of the group $Hdg_A$.

The group $U_{A,Q}$ was introduced and studied in [Pol].

**Theorem 5.8.** [Pol] For an abelian variety $A$ the group $U_{A,Q}$ is reductive.

5.5. **The algebraic $\mathbb{Q}$-group $U(A)_Q$**. It will be convenient for us to consider a slightly smaller algebraic $\mathbb{Q}$-group. Namely, first consider the $\mathbb{Q}$-Zariski closure in $U_{A,Q}$ of its arithmetic subgroup $U(A)$. (In [GLO] this group was denoted $U(\bar{A})$.) Let $U(A)_Q := U(\bar{A})^0$ be its connected component. This is the algebraic $\mathbb{Q}$-group, that we will be interested in.

The main properties of this group are summarized in the following proposition.

**Proposition 5.9.** Let $A$ be an abelian variety.

1. The group $U(A)_Q$ is semisimple.
2. The semi-simple Lie group $U(A)_Q(\mathbb{R})^0$ consists of all the non-compact factors of the reductive Lie group $U_{A,Q}(\mathbb{R})^0$.
3. The arithmetic subgroup $U(A)^0 := U(A) \cap U(A)_Q(\mathbb{R})^0$ of $U(A)_Q(\mathbb{R})^0$ is Zariski dense.

**Proof.** (1) [GLO] 5.3.5], (2) [GLO] 7.2.1, (3) [Bor Thm.1].
Remark 5.10. The subgroup $U(A)^0 \subset U(A)$ has finite index. Since the groups $U(A)$ and $G^{eq}(A)$ are isomorphic up to finite groups, so are $U(A)^0$ and $G^{eq}(A)$ (Remark 5.6).

Remark 5.11. Let $A$ and $B$ be abelian varieties with an identification of lattices $\Lambda_A = \Lambda_B = \Lambda$ which is compatible with the form $Q$. Assume that under this identification we have the equality of algebraic groups $U_{A,Q} = U_{B,Q}$. Then clearly $U(A)_Q = U(B)_Q$, $U(A)^0 = U(B)^0$, etc. (The equality $U_{A,Q} = U_{B,Q}$ holds for example when $Hdg_A = Hdg_B$ as subgroups in $SO(\Lambda_Q, Q)$.)

5.6. Action of the Lie group $U_{A,Q}(\mathbb{R})$ on a Siegel domain. Let $A$ be an abelian variety. Let us define a rational (i.e. not everywhere defined) action of the Lie group $U_{A,Q}(\mathbb{R})$ on the complex space $NS_{A,C} \subset \text{Hom}(A, \hat{A}) \otimes \mathbb{C}$:

$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \omega := (c + d\omega)(a + b\omega)^{-1},$

$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in U_{A,Q}(\mathbb{R}), \quad \omega \in NS_{A,C} \subset \text{Hom}(A, \hat{A}) \otimes \mathbb{C}.$

Here the multiplication is understood as composition of maps.

$NS_{A,C}$ contains a Siegel domain of the first kind [Piat] on which this action is well defined. Namely, let $C^a_A \subset NS_{A,R}$ be the ample cone of $A$, which is defined as the set of $\mathbb{R}^+$-linear combinations of ample classes in $NS_A$. It is an open subset in $NS_{A,R}$. Consider the complexified ample cone

$C_A := NS_{A,R} + iC^a_A \subset NS_{A,C}$

(Note that in [GLO] $C_A$ denotes the bigger set $NS_{A,R} \pm iC^a_A$ which has two connected components.)

Theorem 5.12. Let $A$ be an abelian variety.

1. The action of $U_{A,Q}(\mathbb{R})$ on $C_A$ is well defined and is transitive.

2. The stabilizer of a point in $C_A$ is a maximal compact subgroup in $U_{A,Q}(\mathbb{R})$.

3. The action of the subgroup $U(A)_Q(\mathbb{R})^0 \subset U_{A,Q}(\mathbb{R})$ on $C_A$ is also transitive and the stabilizer of a point is a maximal compact subgroup of $U(A)_Q(\mathbb{R})^0$. Hence $C_A$ is the Hermitian symmetric space for the semi-simple Lie group $U(A)_Q(\mathbb{R})^0$.

Proof. [GLO] 8.2, 8.3. □

5.7. Mirror symmetry for algebraic pairs $(A, \omega)$ following [GLO].

Definition 5.13. An algebraic pair is a pair $(A, \omega)$, where $A$ is an abelian variety and $\omega \in C_A$. 
Let us recall the notion of mirror symmetry for algebraic pairs from [GLO]. Consider the $U_{A,Q}(\mathbb{R})$-action on $C_A$ defined above. Given $\omega = \phi_1 + i\phi_2 \in C_A$, define

$$I_\omega := \begin{pmatrix} \phi_2^{-1}\phi_1 & -\phi_2^{-1} \\ \phi_2 + \phi_1\phi_2^{-1}\phi_1 & -\phi_1\phi_2^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ \phi_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\phi_2^{-1} \\ \phi_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\phi_1 & 1 \end{pmatrix} \in U_{A,Q}(\mathbb{R})$$

The following proposition together with Theorem 5.12 should be compared with Theorem 5.3.

**Proposition 5.14.** Consider the $U(A)_{Q}(\mathbb{R})^0$-action on $C_A$ (Theorem 5.12). Let $\omega \in C_A$ and let $K_\omega \subset U(A)_{Q}(\mathbb{R})^0$ be its stabilizer. Then the following holds.

1. The operator $I_\omega$ belongs to the center of $K_\omega$ (in particular $I_\omega \in U(A)_{Q}(\mathbb{R})^0$) and the adjoint action of $I_\omega$ on $U(A)_{Q}(\mathbb{R})$ is the Cartan involution corresponding to $K_\omega$.
2. We have $I_\omega^2 = -1$, hence $I_\omega$ defines a complex structure on $\Lambda_\mathbb{R}$.
3. The correspondence $\omega \mapsto I_\omega$ is injective.

**Proof.** [GLO] 8.4.1.

**Remark 5.15.** Let $\omega \in C_A$. It follows from Proposition 5.14 that there is an inclusion of algebraic $Q$-groups $(I_\omega)_Q \subset U(A)_Q$ (Definition 5.1).

Consider the real vector space $V_A \oplus \hat{V}_A$. It has the complex structure $J_{A \times \hat{A}}$. Since the group $U(A)_{Q}(\mathbb{R})^0$ acts on $V_A \oplus \hat{V}_A$, for each $\omega \in C_A$ the operator $I_\omega$ defines another complex structure on $V_A \oplus \hat{V}_A$. These complex structures commute.

**Definition 5.16.** [GLO] 9.2] Algebraic pairs $(A, \omega_A)$ and $(B, \omega_B)$ are **mirror symmetric** if there is an isomorphism of lattices

$$\alpha : \Lambda_A \xrightarrow{\sim} \Lambda_B$$

which identifies the bilinear forms $Q_A$ and $Q_B$ and satisfies the following conditions

$$\alpha_R \cdot J_{A \times \hat{A}} = I_{\omega_B} \cdot \alpha_R,$$

$$\alpha_R \cdot I_{\omega_A} = J_{B \times \hat{B}} \cdot \alpha_R$$

Let algebraic pairs $(A, \omega_A)$ and $(B, \omega_B)$ be mirror symmetric. We may assume that $\Lambda = \Lambda_A = \Lambda_B$ and $\alpha = \text{id}$. Then we obtain the inclusions of algebraic $Q$-groups

$$Hdg_A \subset U(B)_Q, \quad \text{and} \quad Hdg_B \subset U(A)_Q$$

as subgroups of $SO(\Lambda_Q, Q_Q)$ (Remark 5.15).
5.8. **How to find a mirror symmetric pair.** Let \((A, \omega_A)\) be an algebraic pair. As explained above we obtain two commuting complex structures on the real vector space \(V_A \oplus V_A^*: J_{A\times A}^*\text{ and } I_{\omega_A}\). To find a mirror pair \((B, \omega_B)\) we need the following:

1. Find a \(Q_A\)-isotropic decomposition \(\Lambda_A = \Gamma_1 \oplus \Gamma_2\) such that the vector subspaces \(\Gamma_{1R}, \Gamma_{2R} \subseteq V_A \oplus V_A^*\) are \(I_{\omega_A}\)-invariant. This will give a complex torus \(B = (\Gamma_{1R}/\Gamma_1, I_{\omega_A}|_{\Gamma_{1R}})\) and the dual torus \(\hat{B} = (\Gamma_{2R}/\Gamma_2, I_{\omega_A}|_{\Gamma_{2R}})\) and an isomorphism of complex tori

\[
\left(\frac{V_A \oplus V_A^*}{(\Gamma_A \oplus \Gamma_A^*)}, I_{\omega_A}\right) \simeq B \times \hat{B}
\]

2. Show that \(C_B \neq \emptyset\) (i.e. \(B\) is an abelian variety) and there exists \(\omega_B \in C_B\) such that the operator \(I_{\omega_B}\) on \(V_B \oplus V^*_B = V_A \oplus V_A^*\) coincides with \(J_{A\times A}^*\). This will show that the algebraic pairs \((A, \omega_A)\) and \((B, \omega_B)\) are mirror symmetric (take \(\alpha = \text{id}\)).

Actually if (1) is achieved, then (2) is automatic [GLO] 9.4.6].

**Remark 5.17.** (1) It is not true that for every algebraic pair \((A, \omega_A)\) there exists a mirror symmetric pair. The problem may occur if the group \(U(A)_\mathbb{Q}\) is too big and \(\omega_A \in C_A\) is chosen too general [GLO] 9.5.1. But for every abelian variety \(A\) there exists an element \(\omega \in C_A\) such that the pair \((A, \omega)\) has a mirror symmetric pair [GLO] 10.4.3].

(2) The mirror pair, if exists, may not be unique. However for a given pair \((A, \omega_A)\) the collection of isomorphism classes of abelian varieties \(B\) for which there exists \(\omega_B \in C_B\) such that the pairs \((A, \omega_A)\) and \((B, \omega_B)\) are mirror symmetric is finite [GLO] 9.2.3].

5.9. **A useful lemma.** We recall a result from [GLO] that will be useful later. Let \(A\) be an abelian variety, \(I \in U(A)_\mathbb{Q}(\mathbb{R})^0\) such that \(I^2 = -1\). Then \(I\) defines a complex structure on \(\Lambda_\mathbb{R}\). The complex structures \(I\) and \(J_{A\times A}\) commute and preserve the bilinear form \(Q_\mathbb{R}\). The operator \(c := I \cdot J_{A\times A}\) also preserves \(Q_\mathbb{R}\) and the bilinear form \(Q_\mathbb{R}(c(-), -)\) is symmetric. Denote by \(E_I\) the corresponding quadratic form on \(\Lambda_\mathbb{R}\).

**Lemma 5.18.** The quadratic form \(E_I\) is positive definite if and only if \(I = I_\omega\) for some \(\omega \in C_A\).

**Proof.** This is a special case of [GLO] 9.4.2].

5.10. **Perfect algebraic pairs.**

**Definition 5.19.** An algebraic pair \((A, \omega)\) is **perfect** if \(U(A)_\mathbb{Q} = \langle I_\omega \rangle_{\mathbb{Q}}\).

**Lemma 5.20.** Let \(A\) be an abelian variety.

1. The set \(\{I_\omega \mid \omega \in C_A\}\) is a conjugacy class in \(U(A)_\mathbb{Q}(\mathbb{R})^0\).

2. Let \(\sigma\) be an automorphism of the Lie group \(U(A)_\mathbb{Q}(\mathbb{R})^0\) such that \(\sigma(I_{\omega_1}) = I_{\omega_2}\) for some \(\omega_1, \omega_2 \in C_A\). Then \(\sigma\) preserves the conjugacy class \(\{I_\omega \mid \omega \in C_A\}\).
Proof. (1) Notice that the center $Z(U(A)_Q(\mathbb{R})^0)$ is discrete (as the group $U(A)_Q(\mathbb{R})^0$ is semi-simple) and the adjoint action of $I_\omega$ on the Lie algebra $\text{Lie}(U(A)_Q(\mathbb{R})^0)$ is the Cartan involution corresponding to the maximal compact subgroup $K_\omega \subset U(A)_Q(\mathbb{R})^0$ (Proposition 5.14). Therefore for each $\omega \in C_A$ the set of elements in $U(A)_Q(\mathbb{R})^0$ whose adjoint action on the Lie algebra is the corresponding Cartan involution is the discrete set $I_\omega Z(U(A)_Q(\mathbb{R})^0)$. All maximal compact subgroups of $U(A)_Q(\mathbb{R})^0$ are conjugate [Helg, Theorem 2.2]. If $gK_\omega g^{-1} = K_{\omega'}$ for some $g \in U(A)_Q(\mathbb{R})^0$, then $gI_\omega Z(U(A)_Q(\mathbb{R})^0)g^{-1} = I_{\omega'} Z(U(A)_Q(\mathbb{R})^0).$ As $g$ belongs to the connected group $U(A)_Q(\mathbb{R})^0$ and elements $I_\omega$ depend continuously on $\omega$, we must have $gI_\omega g^{-1} = I_{\omega'}$.

(2) This follows from (1). \qed

Lemma 5.21. Let $A$ be an abelian variety. Then there exists a subset $Z \subset C_A$ which is a countable union of proper analytic subsets such that for every $\omega \in C_A \setminus Z$ the pair $(A, \omega)$ is perfect.

Proof. We claim that for some $\tau \in C_A$ the Lie group $\langle I_\tau \rangle_Q(\mathbb{R})$ contains the whole conjugacy class $\{I_\omega \mid \omega \in C_A\}$. Indeed, there exists a countable number of algebraic $Q$-subgroups in $U(A)_Q$. Let $B$ be one such subgroup. If $\{I_\omega \mid \omega \in C_A\} \notin B(\mathbb{R})$, then

$$B(\mathbb{R}) \cap \{I_\omega \mid \omega \in C_A\}$$

is a proper algebraic subset, so it is nowhere dense in $\{I_\omega \mid \omega \in C_A\}$. Since $\{I_\omega \mid \omega \in C_A\}$ is a Baire space, it is not a countable union of nowhere dense subsets. Therefore there exists $\tau \in C_A$ such that $\{I_\omega \mid \omega \in C_A\} \subset \langle I_\tau \rangle_Q(\mathbb{R})$. Fix one such $\tau \in C_A$. We claim that $\langle I_\tau \rangle_Q = U(A)_Q$. Both these groups are connected, so it is enough to prove the equality of dimensions. For this it suffices to show the equality of the groups of $\mathbb{R}$-points $\langle I_\tau \rangle_Q(\mathbb{R})^0 = U(A)_Q(\mathbb{R})^0$.

First notice that $\langle I_\tau \rangle_Q(\mathbb{R})^0$ is a normal subgroup in $U(A)_Q(\mathbb{R})^0$. Indeed, for any $g \in U(A)_Q(\mathbb{Q})$, we have

$$g(\langle I_\tau \rangle_Q)g^{-1} = \langle I_{g(\tau)} \rangle_Q \subset \langle I_\tau \rangle_Q$$

Since the semisimple Lie group $U(A)_Q(\mathbb{R})^0$ has no compact factors, the group $U(A)_Q(\mathbb{Q})$ is Zariski dense in $U(A)_Q(\mathbb{R})^0$ [Bor, Thm.1]. So $\langle I_\tau \rangle_Q(\mathbb{R})^0$ is a normal (closed) subgroup in $U(A)_Q(\mathbb{R})^0$. Thus the Lie algebra of $\langle I_\tau \rangle_Q(\mathbb{R})^0$ consists of a number of simple factors of the Lie algebra of $U(A)_Q(\mathbb{R})^0$. But the adjoint action of $\langle I_\tau \rangle_Q(\mathbb{R})^0$ on $\text{Lie}(U(A)_Q(\mathbb{R})^0)$ contains a Cartan involution. Hence $\langle I_\tau \rangle_Q(\mathbb{R})^0 = U(A)_Q(\mathbb{R})^0$. \qed

Definition 5.22. For an abelian variety $A$ put $C_A^0 = C_A \setminus Z$ (in the notation of Lemma 5.21), i.e. $C_A^0$ consists of elements $\omega$, such that $(A, \omega)$ is a perfect pair. By Lemma 5.21 $C_A^0$ is a dense subset of $C_A$. 
5.11. Mirror symmetric families of abelian varieties.

**Definition 5.23.** Algebraic pairs \((A,\omega_A)\) and \((B,\omega_B)\) which are mirror symmetric (Definition 5.10) are called **perfectly mirror symmetric** if in addition these pairs are perfect (Definition 5.19).

5.12. Construction of mirror symmetric families of abelian varieties. Assume that the algebraic pairs \((A,\omega_A)\) and \((B,\omega_B)\) are perfectly mirror symmetric. Identify the lattices \(\Lambda = \Lambda_A = \Gamma_A \oplus \Gamma_A^*\) and \(\Lambda_B = \Gamma_B \oplus \Gamma_B^*\) via the isomorphism \(\alpha\) (Definition 5.16), and consider the algebraic \(\mathbb{Q}\)-groups \(U(A)_\mathbb{Q}, U(B)_\mathbb{Q}, Hdg_A, Hdg_B\) as subgroups of the special orthogonal group \(SO(\Lambda_\mathbb{Q},\mathbb{Q})\). By assumption the group \(Hdg_A\) (resp. \(Hdg_B\)) is the Zariski \(\mathbb{Q}\)-closure of the complex structure \(I_{\omega_B}\) (resp. \(I_{\omega_A}\)). Hence we have the equalities

\[(5.4) \quad U(A)_\mathbb{Q} = Hdg_B, \quad U(B)_\mathbb{Q} = Hdg_A\]

(Vice versa: if equalities (5.4) hold for mirror symmetric pairs \((A,\omega_A)\) and \((B,\omega_B)\) then these pairs are perfect). Since the group \(Hdg_A\) preserves the subspace \(\Lambda_A^0 \subset \Lambda_\mathbb{R}\), so does the group \(U(B)_\mathbb{Q}\). It follows that for any \(\eta_B \in C_B\) the complex structure \(I_{\eta_B}\) on \(\Lambda_{\mathbb{R}}\) restricts to a complex structure on \(\Gamma_{A,\mathbb{R}} \subset \Lambda_{\mathbb{R}}\), i.e. we get the abelian variety \(A_{\eta_B} = (V_A/\Gamma_A, I_{\eta_B})\) (5.8). Symmetrically, for any \(\eta_A \in C_A\) we have the abelian variety \(B_{\eta_A} = (V_B/\Gamma_B, I_{\eta_A})\).

In sum we obtain two families of abelian varieties

\[(5.5) \quad A := \{A_{\eta_B} \mid \eta_B \in C_B\} \quad \text{and} \quad B := \{B_{\eta_A} \mid \eta_A \in C_A\}\]

with bases \(C_B\) and \(C_A\) respectively.

**Definition 5.24.** We call the families \(A\) and \(B\) as in (5.5) the **mirror symmetric families of abelian varieties**.

5.13. Properties of mirror symmetric families. If in the above notation in addition \(\eta_B \in C_B^0\), then

\[(5.6) \quad Hdg_{A_{\eta_B}} = U(B)_\mathbb{Q} = Hdg_A\]

Therefore \(U_{A_{\eta_B},\mathbb{Q}} = U_{A,\mathbb{Q}}\) (Remark 5.7) and \(U(A_{\eta_B})_\mathbb{Q} = U(A)_\mathbb{Q}\) (Remark 5.11). So \(I_{\omega_A} \in U(A_{\eta_B})_\mathbb{Q}(\mathbb{R})^0\). We claim that \((A_{\eta_B},\omega_A)\) is an algebraic pair, i.e. that \(\omega_A \in C_{A_{\eta_B}}\). To see this we consider the operator

\[c := I_{\omega_A} \cdot J_{A_{\eta_B} \times A_{\eta_B}} = J_{B \times B} \cdot I_{\eta_B}\]
By Lemma 5.18 $I = I_\omega$ equals $I_{\omega_{A \eta B}}$ for some $\omega_{A \eta B} \in C_{A \eta B}$ (i.e. $\omega_A \in C_{A \eta B}$ ) if and only if the quadratic form associated with the bilinear form $Q(c(-),-)$ is positive definite on $A_R$. But the same Lemma 5.18 implies that this form is positive definite, as $\eta_B \in C_B$.

So we obtain perfectly symmetric pairs $(A_{\eta_B},\omega_A)$ and $(B,\eta_B)$.

Symmetrically, for any $\eta_A \in C_A^0$ we have the abelian variety $B_{\eta_A} = (V_B/\Gamma_B,I_{\eta_A})$ with

$$Hdg_{B_{\eta_A}} = Hdg_B \quad \text{and} \quad U(B_{\eta_A})_Q = U(B)_Q$$

and perfectly symmetric pairs $(A,\eta_A)$ and $(B_{\eta_A},\omega_B)$.

In particular we obtain the following corollary.

**Corollary 5.25.** (1) For any parameter $\omega \in C_A$ there is an inclusion $Hdg_{B_\omega} \subset Hdg_B$ and therefore the inclusions $U_{B,\omega} \subset U_{B,\omega} \subset U_\omega \subset U(B_\omega),U(B) \subset U(B_\omega), U(B)_Q \subset U(B_\omega)_Q \subset U(B_\omega)^0 \subset U(B_\omega)_Q^0$. Also $End(B) \subset End(B_\omega), \text{Aut}(B) \subset \text{Aut}(B_\omega)$. (And similarly for the abelian varieties $A_{\eta}, \eta \in C_B$.)

(2) For any $\omega \in C_A^0$ the groups $Hdg_{B_\omega}, U_{B_\omega}, U(B_\omega), U(B_\omega)_Q, U(B_\omega)^0, \text{End}(B_\omega), \text{Aut}(B_\omega)$ coincide with the corresponding groups for $B$. (And similarly for the abelian varieties $A_\eta, \eta \in C_B^0$.)

Actually also the ample cones of the abelian varieties $\{B_\omega \mid \omega \in C_A^0\}$ are the same. Hence the complexified cone $C_{B_\omega}$ is independent of $\omega \in C_A^0$ (Corollary 5.27).

**Lemma 5.26.** Let $A = (V/\Gamma,J)$ be an abelian variety, $Hdg_A = \langle J \rangle_Q$. Let $g \in Hdg_A(\mathbb{R})$, and consider $J' := gJg^{-1}$ as another complex structure on $V$, so that we have the complex torus $A' := (V/\Gamma,J')$.

(1) Then $A'$ is also an abelian variety and we have the inclusions of Neron-SEveri groups $NS_A \subset NS_{A'}$ and of ample cones $C_A^a \subset C_{A'}^a$.

(2) Assume in addition that $\langle J' \rangle_Q = Hdg_A$, i.e. $Hdg_A = Hdg_{A'}$. Then $C_A^a = C_{A'}^a$.

**Proof.** (1) By definition an integral skew symmetric form $s$ on $V$ belongs to $NS_A$ if $s$ is $J$-invariant. This happens if and only if $s$ is invariant under all elements of $Hdg_A(\mathbb{R})$. We have

$$Hdg_A = \langle J \rangle_Q \subset Hdg_A$$

Hence $NS_A \subset NS_{A'}$.

A skew-symmetric form $s \in NS_A$ represents an ample class if and only if the quadratic form $s(J(-),-)$ is positive definite on $V$. In this case for $0 \neq x \in V$ we have

$$s(J'x,x) = s(gJg^{-1}x,x) = s(Jg^{-1}x,g^{-1}x) > 0$$

Thus $s$ represents an ample class in $A'$, so $A'$ is an abelian variety and also $C_A^a \subset C_{A'}^a$.

(2) follows from (1), because we can interchange $A$ and $A'$. □
**Corollary 5.27.** (1) For any parameter \( \omega \in C_A \) we have \( C_B \subset C_{B,\omega} \).

(2) For all \( \omega \in C_A^0 \) there is the equality \( C_B = C_{B,\omega} \).

**Proof.** (1) Let \( \omega \in C_A \). By Lemma 5.20 the operators \( I_\omega \) and \( I_{\omega A} \) are conjugate in the Lie group \( U(A)_\mathbb{Q}(\mathbb{R})^0 = Hdg_B \). So by Lemma 5.26 \( NS_B = NS_{B,\omega} \subset NS_{B,\omega} \) and similarly for the ample cones. Therefore also \( C_B \subset C_{B,\omega} \).

(2) If in addition \( \omega \in C_A^0 \), then by definition \( \langle I_\omega \rangle_\mathbb{Q} = \langle I_{\omega A} \rangle_\mathbb{Q} \) and hence \( Hdg_{B,\omega} = Hdg_B \). So it remains to apply Lemma 5.20 and Lemma 5.26. \( \square \)

5.14. **Proof of Conjecture 1.5** for mirror families of abelian varieties. By Corollary 5.25 we have \( U(A,\omega) = U(A) \) for \( \omega \in C_A^0 \) and by Remark 5.6, for any abelian variety \( C \), the groups \( G_{eq}(C) \) and \( U(C) \) are isomorphic up to finite groups. We conclude that for the family \( A \) the group \( G_{eq}(A) \) (1.5) is isomorphic up to finite groups to \( U(A) \) and also to \( U(A)^0 \) (Remark 5.10).

Now consider the mirror dual family

\[
f : B \to C_A
\]

We need to determine the monodromy group (Definition 2.2) of this family and to prove that it is isomorphic up to finite groups to \( U(A)^0 \). This will require a few steps.

Recall that by our assumption we have the equality of algebraic \( \mathbb{Q} \)-subgroups of \( SO(\Lambda_\mathbb{Q},Q_\mathbb{Q}) : U(A)_\mathbb{Q} = Hdg_B \) (5.4). Hence in particular the group \( U(A)_\mathbb{Q} \) preserves the subspace \( \Gamma_{B,\mathbb{Q}} \subset \Lambda_\mathbb{Q} \) and the groups \( U(A)_\mathbb{Q} \) and \( Hdg_B \) can (and will) be considered as subgroups of \( Gl(\Gamma_{B,\mathbb{Q}}) \).

**Definition 5.28.** Let \( G \subset Gl(\Gamma_B) \) be the set of elements \( g \) for which there exist \( \omega_1,\omega_2 \in C_A^0 \) (that may depend on \( g \) ) such that

\[
gI_{\omega_1}g^{-1} = I_{\omega_2}
\]

(5.7)

**Proposition 5.29.** The following holds for the set \( G \):

(1) For each \( g \in G \) we have \( gU(A)_\mathbb{Q}(\mathbb{R})^0g^{-1} = U(A)_\mathbb{Q}(\mathbb{R})^0 \) and \( gU(A)^0g^{-1} = U(A)^0 \).

(2) The conjugation action of \( g \) on \( U(A)_\mathbb{Q}(\mathbb{R})^0 \) preserves the conjugacy class \( \{ I_\omega \mid \omega \in C_A \} \). In particular \( G \) is a subgroup of \( Gl(\Gamma_{B,\mathbb{Q}}) \).

(3) We have the inclusion of groups \( U(A)^0 \subset G \).

(4) \( G \) acts on the space \( C_A \) and this action lifts to an action on the family of abelian varieties \( f : B \to C_A \).

(5) The quotient \( C_A^0/G \) is the coarse moduli space of abelian varieties which appear in the family \( B|_{C_A^0} \).
Proof. (1) Let \( g \in G \), and let \( \omega_1, \omega_2 \in C_A^0 \) such that \( gI_{\omega_1}g^{-1} = I_{\omega_2} \). Since \( U(A)_Q = \langle I_{\omega_1} \rangle_Q = \langle I_{\omega_2} \rangle_Q \) and \( g \) is an integral matrix it follows that \( gU(A)_Qg^{-1} = U(A)_Q \). Hence also \( gU(A)_Q(R)^0g^{-1} = U(A)_Q(R)^0 \). The group \( U(A)_Q \) is the subgroup of integral points in \( U(A)_Q(R)^0 \). Hence also \( gU(A)_Qg^{-1} = U(A)_Q \).

(2) The first assertion follows from Lemma 5.20. For the second one notice that \( g \) consists of integral matrices, hence the action of \( gU \) on the family \( B \) preserves the subset \( \{ I_\omega \mid \omega \in C_A \} \). So \( G \) is a subgroup of \( GL(\Gamma_B) \).

(3) Since \( U(A)_Q \subset GL(\Gamma_B) \) consists of integral matrices, its adjoint action on the conjugacy class \( \{ I_\omega \mid \omega \in C_A \} \) preserves the subset \( \{ I_\omega \mid \omega \in C_A^0 \} \). Therefore \( U(A)_Q \subset G \).

(4) Recall that the correspondence \( \omega \mapsto I_\omega \) is injective, which means that \( G \) acts on \( C_A \). If for \( g \in G \) and \( \omega_1, \omega_2 \in C_A \) we have \( gI_{\omega_1}g^{-1} = I_{\omega_2} \) then \( g : \Gamma_B \to \Gamma_B \) defines an isomorphism of abelian varieties \( g : B_{\omega_1} \sim B_{\omega_2} \). Therefore the \( G \) action on \( C_A \) lifts to an action on the family \( B \).

(5) Assume that for \( \omega_1, \omega_2 \in C_A^0 \) the abelian varieties \( B_{\omega_1} \) and \( B_{\omega_2} \) are isomorphic. Then there exists an element \( g \in GL(\Gamma_B) \) such that \( gI_{\omega_1}g^{-1} = I_{\omega_2} \). By definition we have \( g \in G \), i.e. \( \omega_1, \omega_2 \) lie in the same orbit of the group \( G \). \( \square \)

Corollary 5.30. (1) The group \( G \) acts by automorphisms of the Lie group \( U(A)_Q(R)^0 \) and we have the exact sequence of groups

\[
1 \to \text{Aut}(B) \to G \to \text{Aut}(U(A)_Q(R))
\]

(2) The monodromy group \( G^{\text{mon}}(\mathcal{B}) \) of the family \( \mathcal{B} \) is \( G/\text{Aut}(B) \) (Definition 2.3).

Proof. (1). By Proposition 5.29 the group \( H \) acts on the Lie group \( U(A)_Q(R)^0 = Hgd_B(R)^0 \) by conjugation. The kernel of this action consists precisely of operators \( g \in GL(\Gamma_B) \) such that \( g \) commutes with the complex structure \( J_B \) on \( \Gamma_{B,R} \), i.e. \( g \) is an automorphism of the abelian variety \( B \).

(2) Obviously \( R\text{^*}f_\ast Q_B \) is the trivial local system on \( C_A \) with fiber \( \wedge^\bullet \Gamma_{B,Q} \). The discrete group \( G \) acts on this family and its action on the space of global sections of this local system is clearly effective. By (1) the kernel of the \( G \)-action on \( C_A \) is \( \text{Aut}(B) \). Moreover by Proposition 5.29 the quotient space \( C_A^0/G \) is the coarse moduli space of abelian varieties in the family \( \mathcal{B}_{|C_A^0} \). Recall that \( C_A^0 \) is the complement of a countably many analytic subsets in \( C_A \). It remains to show that the \( G/\text{Aut}(B) \)-action on \( C_A \) is generically free, i.e. it is free outside a countable number of analytic subsets.

The group \( G/\text{Aut}(B) \) acts effectively on the space \( C_A \) which is a conjugacy class in \( U(A)_Q(R)^0 \), hence an algebraic variety. The fixed subset \( C_A^0 \subset C_A \) for any \( 1 \neq g \in \)}
$G/\text{Aut}(B)$ is a proper algebraic subvariety. Hence the set

$$Z := \bigcup_{1 \neq g \in G/\text{Aut}(B)} C^g_A$$

is a union of countably many proper analytic subsets and the $G/\text{Aut}(B)$-action on its complement $C_A \setminus Z$ is free. (Note that $C_A \neq Z$, since $C_A$ is a Baire space.) So the action on $C^0_A \setminus (Z \cap C^0_A)$ is also free. Thus according to Definition 2.2 the group $G/\text{Aut}(B)$ is the monodromy group of the family $B$. □

It remains to prove the following

**Proposition 5.31.** The group $G/\text{Aut}(B)$ is isomorphic up to finite groups to the group $U(A)^0$.

**Proof.** Denote the group $G/\text{Aut}(B)$ by $H$. By Corollary 5.30 $H \subset \text{Aut}(U(A)_\mathbb{Q}(\mathbb{R})^0)$. Since the Lie group $U(A)_\mathbb{Q}(\mathbb{R})^0$ is semi-simple, the group of inner automorphisms $\text{Inn}(U(A)_\mathbb{Q}(\mathbb{R})^0)$ has finite index in $\text{Aut}(U(A)_\mathbb{Q}(\mathbb{R})^0)$. Put $H' := H \cap \text{Inn}(U(A)_\mathbb{Q}(\mathbb{R})^0)$. Again semi-simplicity of $U(A)_\mathbb{Q}(\mathbb{R})^0$ implies that its center is finite. Denote by $H''$ the preimage of $H'$ in $U(A)_\mathbb{Q}(\mathbb{R})^0$ under the conjugation action homomorphism

$$\text{Ad} : U(A)_\mathbb{Q}(\mathbb{R})^0 \to \text{Inn}(U(A)_\mathbb{Q}(\mathbb{R})^0)$$

The groups $H''$ and $H$ are isomorphic up to finite groups, so it suffices to show that $H''$ and $U(A)^0$ are isomorphic up to finite groups.

Note that the inclusion $U(A)^0 \subset G$ (Proposition 5.29) induces the inclusion $U(A)^0 \subset H''$ and it suffices to prove that $U(A)^0$ is a subgroup of finite index in $H''$. The conjugation action of $H''$ on $U(A)_\mathbb{Q}(\mathbb{R})^0$ preserves its arithmetic subgroup $U(A)^0$.

The assertion now follows from the general lemma.

**Lemma 5.32.** Let $G$ be an algebraic $\mathbb{Q}$-group such that $G(\mathbb{R})$ is a semisimple Lie group without compact factors and let $L \subset G(\mathbb{R})$ be an arithmetic subgroup. Then $L$ has finite index in its normalizer $N := N_{G(\mathbb{R})}(L)$.

**Proof.** First notice that by Borel’s theorem [Bor Thm.2] the normalizer $N$ is contained in the group $G(\mathbb{Q})$ of rational points of $G$. But $N$ is a closed subgroup of $G(\mathbb{R})$, so $N$ is discrete.

By a theorem of Borel and Harish-Chandra [BorHC Thm.1] the arithmetic subgroup $L \subset G(\mathbb{R})$ is a lattice, i.e. the homogeneous space $G(\mathbb{R})/L$ has finite volume.

Let $1 \in U \subset G(\mathbb{R})$ be a neighborhood of identity with $U \cap N = 1$, and let $V$ be a symmetric neighborhood of $1$ with $V^2 \subset U$. The subsets $\{nV\}_{n \in \mathbb{N}}$ are disjoint sets of the same positive Haar measure and there are $[N : L]$-many of them that project injectively
into \( G(\mathbb{R})/L \). The latter has finite volume, so \( N/L \) is finite. This proves the lemma and completes the proof of Proposition 5.31. □

Summarizing the discussion we can formulate the final result.

**Theorem 5.33.** Conjecture 1.5 holds for mirror families of abelian varieties.

### 5.15. An example of mirror symmetric families of abelian varieties.

We will construct an example of perfectly mirror symmetric algebraic pairs (which then by construction in 5.12 gives rise to mirror symmetric families of abelian varieties).

Start with a lattice \( \Gamma = \oplus_{i=1}^{2n} \mathbb{Z} e_i \) and a skew symmetric form \( s \) on \( \Gamma \) defined by

\[
s(e_i, e_j) = \delta_{ij} \quad \text{for} \quad 1 \leq i, j \leq n\]

Consider the operator \( J_0 \in \text{Gl}(\Gamma) \)

\[
J_0(e_i) = -e_{i+n}, \quad J_0(e_{i+n}) = e_i \quad \text{for} \quad 1 \leq i \leq n
\]

Then \( J_0^2 = -1 \). Moreover, \( J_0 \) preserves the bilinear form and for each \( 0 \neq \gamma \in \Gamma \) we have \( s(J_0 \gamma, \gamma) > 0 \). That is the symmetric bilinear form \( b(v, w) := s_\mathbb{R}(J_0 v, w) \) is positive definite on \( V := \Gamma_\mathbb{R} \). Therefore \( J_0 \) is a complex structure on the \( \mathbb{R} \)-vector space \( V \) and \( A_0 = (V/\Gamma, J_0) \) is an abelian variety with polarization \( s \).

Consider the symplectic Lie group \( Sp(V, s_\mathbb{R}) \). We have \( J_0 \in Sp(V, s_\mathbb{R}) \) and the centralizer of \( J_0 \) is the maximal compact subgroup

\[
K_{J_0} := \{ g \in Sp(V, s_\mathbb{R}) \mid b(gv, gw) = b(v, w) \quad \text{for all} \quad v, w \in V \}
\]

The operator \( AdJ_0 \) is a Cartan involution on the Lie algebra \( \text{Lie}(Sp(V, s_\mathbb{R})) \) corresponding to the subgroup \( K_{J_0} \). Let \( C(J_0) \subset Sp(V, s_\mathbb{R}) \) be the conjugacy class of \( J_0 \). We get a family of abelian varieties \( \{ A_J := (V/\Gamma, J) \mid J \in C(J_0) \} \) with polarization \( s \). As in the proof of Proposition 5.21 one can show that for a general \( J \in C(J_0) \)

\[
\langle J \rangle_\mathbb{Q} = Sp(\Gamma_\mathbb{Q}, s_\mathbb{Q}) = Sp_{2n, \mathbb{Q}}
\]

Fix \( J \in C(J_0) \) such that (5.9) holds. Then the corresponding abelian variety \( A = (V/\Gamma, J) \) has \( \text{NS}_A = \mathbb{Z}s \). Moreover, \( \text{End}(A) = \text{End}(\hat{A}) = \mathbb{Z} \) and \( \text{Hom}(A, \hat{A}) = \mathbb{Z}s \), \( \text{Hom}(\hat{A}, A) = \mathbb{Z}s^{-1} \). Therefore the group \( U(A) \) is isomorphic to \( SL_2(\mathbb{Z}) \), and

\[
U_{A, \mathbb{Q}} = U(A)_\mathbb{Q} = SL_{2, \mathbb{Q}}
\]

As usual we may consider the group \( Hdg_A = Sp_{2n, \mathbb{Q}} \) as a subgroup of the special orthogonal group \( SO(\Lambda_{A, \mathbb{Q}}) \). Then it is easy to see that \( Hdg_A \) is the centralizer of \( U(A)_\mathbb{Q} \) in \( SO(\Lambda_{A, \mathbb{Q}}) \). Hence \( Hdg_A \) and \( U(A)_\mathbb{Q} \) are mutual centralizers in \( SO(\Lambda_{A, \mathbb{Q}}) \).
Since $NS_A = Z$, for any $\omega_A \in C_A$ the algebraic pair $(A, \omega_A)$ has a mirror symmetric pair $(B, \omega_B)$ (moreover in our case $B$ is a power of an elliptic curve) [GLO, 9.6.3]. As usual we may assume that $\Lambda = \Lambda_B = \Lambda$. Let us choose $\omega_A$ so that the pair $(A, \omega_A)$ is perfect, i.e. $\langle I_{\omega_A} \rangle_Q = U(A)_Q$ (Proposition 5.21). Then we claim that the corresponding pair $(B, \omega_B)$ is also perfect. Indeed, by definition

$$\langle I_{\omega_B} \rangle_Q = \langle J \rangle_Q = Hdg_A = Sp_{2n,Q}$$

which is the centralizer of

$$U(A)_Q = \langle I_{\omega_A} \rangle_Q = Hdg_B$$

in $SO(\Lambda_Q)$. But $U(B)_Q$ is contained in the centralizer of $Hdg_B$, hence $\langle I_{\omega_B} \rangle_Q = U(B)_Q$, i.e. the pair $(B, \omega_B)$ is perfect.

We have constructed perfectly mirror symmetric pairs of abelian varieties $(A, \omega_A)$ and $(B, \omega_B)$, which give rise to a mirror symmetric pairs of abelian varieties as in 5.12.

6. Towards Conjecture [1.5] for Calabi-Yau hypersurfaces in dual toric varieties

6.1. Many examples of mirror symmetric families of Calabi-Yau varieties were constructed by Batyrev [Bat]. He starts with two dual lattices $M \simeq \mathbb{Z}^{n+1}$ and $N = M^*$ and a pair of dual reflexive polytopes $\Delta \subset M_Q$, $\Delta^* \subset N_Q$. These polytopes define a pair of dual projective Gorenstein toric Fano varieties $P_\Delta$ and $P_{\Delta^*}$. The corresponding families $Y$ and $Y^*$ of anticanonical Gorenstein Calabi-Yau divisors in $P_\Delta$ and $P_{\Delta^*}$ are expected to be mirror symmetric [Bat] (one takes the family of all anticanonical divisors and removes the ones that are not Gorenstein).

From our perspective the problem here is that both families $Y$ and $Y^*$ may consist of singular Calabi-Yau varieties, in which case we do not want to consider their derived categories. We can only test our Conjecture [1.5] in case the general member of the family is smooth.

It is proved in [Bat] that there always exist maximal projective crepant partial (MPCP) toric resolutions $\hat{P}_\Delta \to P_\Delta$ and $\hat{P}_{\Delta^*} \to P_{\Delta^*}$. The projective toric varieties $\hat{P}_\Delta$ and $\hat{P}_{\Delta^*}$ correspond to simplicial fans (in $N_Q$ and $M_Q$ respectively), hence they have only quotient singularities. Moreover the pullbacks $X$ and $X^*$ of families $Y$ and $Y^*$ will consist of projective $n$-dimensional Gorenstein Calabi-Yau varieties which are quasi-smooth, i.e. have only quotient singularities. In particular, for members $X$ and $X^*$ of these families the rational cohomology spaces $H^*(X, Q)$ and $H^*(X^*, Q)$ satisfy Poincare duality and possess pure Hodge structure.
The main result of [BatBor] implies the mirror symmetry for the corresponding Hodge numbers

$$h^{p,q}(X) = h^{n-p,q}(X^*), \quad 0 \leq p, q \leq n$$

It might happen that varieties $X$ and $X^*$ are smooth (this is always so if $n = 3$ [Bat]) in which case one can test Conjecture [5] for the families $\mathcal{X}$ and $\mathcal{X}^*$ and we believe that it holds. Unfortunately we are unable to prove this in full generality. Instead we have some partial results in the direction of the conjecture. Let us make two technical assumptions.

**Assumption A.** We assume that the polytope $\Delta$ is integral, i.e. $\Delta \subset M$, and that the projective Gorenstein toric Fano variety $\mathbb{P}_\Delta$ is smooth.

This implies that $\hat{\mathbb{P}}_\Delta = \mathbb{P}_\Delta$, $\mathcal{X}$ is the family of (very ample) smooth anticanonical divisors in $\mathbb{P}_\Delta$. So all members $X$ of the family $\mathcal{X}$ are smooth projective Calabi-Yau varieties.

**Assumption B.** We assume that $n = \dim X$ is odd.

The two assumptions imply that the Hodge diamond of $X$ is a cross:

$$H^\bullet(X, \mathbb{Q}) = H^n(X, \mathbb{Q}) \oplus (\bigoplus_p H^{p,p}(X, \mathbb{Q}))$$

Indeed, $X$ is a hyperplane section of a smooth projective toric variety $\mathbb{P}_\Delta$ whose cohomology consists of algebraic cycles. It remains to apply the weak Lefschetz theorem to the pair $X \subset \mathbb{P}_\Delta$.

The group $G^{eq}(X)$ preserves the Mukai pairing on the even cohomology $H^{even}(X, \mathbb{Q}) = \bigoplus_p H^{p,p}(X, \mathbb{Q})$. Because $c_1(X) = 0$ and $\dim X$ is odd, this Mukai pairing is skew-symmetric [Huyb, Exercise 5.43]. Therefore $G^{eq}(X) \subset Sp(H^{even}(X, \mathbb{Q}))$.

On the mirror side we have no reason to believe that the general member $X^*$ of the dual family $\mathcal{X}^*$ is smooth. However the relation (6.1) implies that

$$\dim H^{even}(X, \mathbb{Q}) = \dim H^n(X^*, \mathbb{Q}) \quad \text{and} \quad \dim H^{even}(X^*, \mathbb{Q}) = \dim H^n(X, \mathbb{Q})$$

The monodromy group of $G^{mon}(\mathcal{X}^*)$ acts trivially on the even cohomology $H^{even}(X^*, \mathbb{Q})$ and preserves the Poincare pairing on the middle cohomology $H^n(X^*, \mathbb{Q})$. Since $n$ is odd this pairing is skew-symmetric and therefore $G^{mon}(\mathcal{X}^*) \subset Sp(H^n(X^*, \mathbb{Q}))$.

We find that the discrete groups in question, $G^{eq}(X)$ and $G^{mon}(\mathcal{X}^*)$, are contained in isomorphic symplectic groups.

**Remark 6.1.** (1) Notice that in fact $G^{mon}(\mathcal{X}^*)$ is contained in $Sp(H^n(X^*, \mathbb{Z}))$ and we expect that it is a subgroup of finite (small) index. For the family of smooth hypersurfaces in a projective space this is a theorem of Beauville [Beau].
(2) On the other hand the action of $G_{eq}(X) = G_{eq}(\mathcal{X})$ on $H^{even}(X, \mathbb{Q})$ does not in general preserve the lattice $H^{even}(X, \mathbb{Z})$. However it preserves a different lattice, which is the image of the topological K-theory $K^0_{top}(X)$ under the Mukai vector isomorphism $v : K^0_{top}(X) \otimes \mathbb{Q} \to H^{even}(X, \mathbb{Q})$ [AdTho].

We expect the groups $G^{mon}(\mathcal{X}^*)$ and $G_{eq}(\mathcal{X})$ to be arithmetic subgroups in the corresponding isomorphic symplectic groups $Sp(H^n(X^*, \mathbb{Q}))$ and $Sp(H^{even}(X, \mathbb{Q}))$ (which would prove Conjecture 1.5 for families $X$ and $X^*$).

6.2. The next theorem is an indication that $G_{eq}(X)$ may indeed be an arithmetic subgroup of $Sp(H^{even}(X, \mathbb{Q}))$.

**Theorem 6.2.** For every member $X$ of the family $\mathcal{X}$ the discrete group $G_{eq}(X)$ is Zariski dense in $Sp(H^{even}(X, \mathbb{Q}))$.

**Proof.** As explained above, the Mukai pairing on $H^{even}(X, \mathbb{Q})$ is skew-symmetric and $G_{eq}(X) \subset Sp(H^{even}(X, \mathbb{Q}))$. Let $\overline{G_{eq}(X)} \subset Sp(H^{even}(X, \mathbb{Q}))$ be the algebraic $\mathbb{Q}$-subgroup which is the Zariski closure of $G_{eq}(X)$. To prove the equality $\overline{G_{eq}(X)}(\mathbb{C}) = Sp(H^{even}(X, \mathbb{C}))$ it suffices to show the equality of the Lie groups $\overline{G_{eq}(X)}(\mathbb{C}) = Sp(H^{even}(X, \mathbb{C}))$.

Since the smooth projective variety $X$ is Calabi-Yau, every line bundle $L$ on $X$ is a spherical object in $D^b(X)$ and as such it defines the corresponding spherical twist functor $T_L$ [Huyb, Def. 8.3] which is an autoequivalence of the derived category $D^b(X)$. For any spherical object $E \in D^b(X)$ the action of the corresponding twist functor on the cohomology $H^\bullet(X, \mathbb{C})$ is the reflection with respect to the Mukai vector $v(E)$:

$$r_{v(E)}(x) := x - \langle v(E), x \rangle v(E)$$

where $\langle -, - \rangle$ is the Mukai pairing on $H^\bullet(X, \mathbb{C})$ [Huyb, 8.12]. Let

$$Q = \{ \delta \in H^{even}(X, \mathbb{C}) | r_\delta \in \overline{G_{eq}(X)}(\mathbb{C}) \}$$

Note that $Q$ is a closed subset in $H^{even}(X, \mathbb{C})$ and $v(M) \in Q$ for all line bundles $M \in Pic(X)$. If $\delta \in Q$ and $g \in \overline{G_{eq}(X)}(\mathbb{C})$, then also

$$r_{g(\delta)} = g \cdot r_\delta \cdot g^{-1} \in \overline{G_{eq}(X)}(\mathbb{C})$$

So $Q$ is $\overline{G_{eq}(X)}(\mathbb{C})$-invariant.

Moreover, if $\delta \in Q$, then for $k \in \mathbb{Z}$

$$r_\delta^k(x) = x - k\langle \delta, x \rangle \delta \quad \text{for all} \quad x \in H^{even}(X, \mathbb{C})$$

(6.4)

It follows that for any $\delta \in Q$ the 1-parameter subgroup

$$U_\delta = \{ x \mapsto x + \lambda \langle x, \delta \rangle \delta \mid \lambda \in \mathbb{C} \}$$
This is Lemma 4.4.2

Proof. Then $G$ to show that the Chern characters of line bundles span $H_{\text{even}}(X, \mathbb{C})$.

At this point we recall the following lemma of Deligne.

**Proposition 6.3.** Let $(V, \psi)$ be a finite dimensional symplectic $\mathbb{C}$-vector space, $G \subset \text{Sp}(V, \psi)$ an algebraic subgroup. Let $R \subset V$ be an $G$-orbit, which spans $V$. Assume that for every $\delta \in R$, $G$ contains the 1-parameter subgroup $U_\delta = \{x \mapsto x + \lambda(x, \delta) \mid \lambda \in \mathbb{C}\}$. Then $G = \text{Sp}(V, \psi)$.

**Proof.** This is [De3, Lemma 4.4.2]

To apply this proposition to our case $V = H_{\text{even}}(X, \mathbb{C})$, $G = \overline{G_{\text{eq}}(X)}(\mathbb{C})$ it suffices to find an element of $Q$, whose $\overline{G_{\text{eq}}(X)}(\mathbb{C})$-orbit spans $H_{\text{even}}(X, \mathbb{C})$. We will show that the fundamental class $\eta \in H^{2n}(X, \mathbb{C})$ is such an element. For this we will analyze Mukai vectors $v(L)$ of line bundles and their Mukai pairing $\langle v(L_1), v(L_2) \rangle$.

For $a \in H_{\text{even}}(X, \mathbb{Q})$ we denote its $i$-th homogeneous component by $a_i$.

**Lemma 6.4.** The Mukai vectors $v(L)$ of line bundles $L \in \text{Pic}(X)$ span the vector space $H_{\text{even}}(X, \mathbb{Q})$.

**Proof.** First notice that the ring $H_{\text{even}}(X, \mathbb{Q})$ is generated by $H^2(X, \mathbb{Q}) = \text{NS}(X)\mathbb{Q}$. Indeed, it is well known that the cohomology ring $H^\bullet(\mathbb{P}_\Delta, \mathbb{Q})$ of the nonsingular projective toric variety $\mathbb{P}_\Delta$ is generated by $H^2(\mathbb{P}_\Delta, \mathbb{Q}) = \text{NS}(\mathbb{P}_\Delta)\mathbb{Q}$. The smooth subvariety $X \subset \mathbb{P}_\Delta$ is a hyperplane section, so by the weak Lefschetz theorem the part $H^{< n}(X, \mathbb{Q})$ is generated by $H^2(X, \mathbb{Q}) = \text{NS}(X)\mathbb{Q}$. Now the hard Lefschetz theorem for $X$ implies that the whole ring $H^\bullet(X, \mathbb{Q})$ is generated by $H^2(X, \mathbb{Q})$.

To prove the lemma recall that

$$v(F) = ch(F) \cup \sqrt{td_X}$$

where $(\sqrt{td_X})_0 = 1$ and hence $\sqrt{td_X}$ is invertible in the ring $H_{\text{even}}(X, \mathbb{Q})$. So it suffices to show that the Chern characters of line bundles span $H_{\text{even}}(X, \mathbb{Q})$. Let $L_1, \ldots, L_p$ be line bundles such that $c_1(L_1), \ldots, c_1(L_p)$ form a basis of $H^2(X, \mathbb{Q})$. Put $x_i := c_1(L_i)$. Then monomials $x_1^{m_1} \cdots x_p^{m_p}$ span $H_{\text{even}}(X, \mathbb{Q})$ and hence the Chern characters

$$\{ch(L_1^{k_1} \otimes \cdots \otimes L_p^{k_p}) = \sum_{m_1, \ldots, m_p \geq 0} k_1^{m_1} \cdots k_p^{m_p} x_1^{m_1} \cdots x_p^{m_p} \mid k_1, \ldots, k_p \in \mathbb{Z}\}$$

span $H_{\text{even}}(X, \mathbb{Q})$ as well. \hfill \Box

Let $\eta \in H^{2n}(X, \mathbb{Q})$ be the fundamental class. Fix an ample line bundle $L$. Then the top component $(v(L^m))_{2n}$ of the Mukai vector $v(L^m)$ as a function of $m$ is

$$am^n \eta + \text{lower terms}, \quad \text{with } a > 0$$
whereas the components \( (v(L^m))_{2d} \) with \( d < n \) grow no faster than \( m^d \). So as \( m \to \infty \) the lines spanned by the Mukai vectors \( v(L^m) \) will tend to the line \( H^{2n}(X, \mathbb{C}) \). Since \( Q \) is a closed subset of \( H^{2n}(X, \mathbb{C}) \) which is a cone over the origin, we find that the line \( H^{2n}(X, \mathbb{C}) \subset Q \). Therefore \( \eta \in Q \).

Note that for any line bundle \( M \in \text{Pic}(X) \)

\[
 r_{v(M)}(\eta) = \eta - \langle v(M), \eta \rangle v(M) = \eta - v(M)
\]

[Huyb, 5.42]. Let \( M_1, ..., M_t \) be line bundles whose Mukai vectors span \( H^{even}(X, \mathbb{C}) \). Then the vectors \( \eta \) and \( r_{v(M_i)}(\eta) \) span \( H^{even}(X, \mathbb{C}) \) and belong to an orbit of \( \text{G}_{\text{eq}}(X)(\mathbb{C}) \), which completes the proof of Theorem 6.2.

6.3. On the mirror side we have no definite results. However let us recall the situation with the universal family of \( d \)-dimensional hypersurfaces \( Y \to S \) in a projective space \( \mathbb{P}^{d+1} \). Assume that \( d \) is odd. Let \( S^0 \subset S \) be the subset parametrizing smooth hypersurfaces, \( s \in S^0 \) and \( Y_s \) the corresponding smooth hypersurface. We are interested in the monodromy representation of \( \pi_1(S^0, s) \) in the middle cohomology \( H^d(Y_s) \). A theorem of Beauville [Beau] asserts that the monodromy group is a subgroup of finite index in the corresponding arithmetic group \( Sp(H^d(Y_s, \mathbb{Z})) \). The proof of this theorem uses a trick and relies on earlier results of [Jan]. However it is relatively easy to prove that the monodromy group is Zariski dense in \( Sp(H^d(X, \mathbb{Q})) \). Recall the relevant well known facts [Voi].

(1) For most projective lines \( \Delta \simeq \mathbb{P}^1 \subset S \) the restriction of the universal family \( Y \to S \) to \( \Delta \) is a Lefschetz pencil. That is there exist a finite number of critical values \( t_1, ..., t_n \in \Delta \) and the corresponding singular fibers \( Y_t \) have a unique nondegenerate singular point.

(2) The map of the fundamental groups \( \pi_1(\Delta \cap S^0, s) \to \pi_1(S^0, s) \) is surjective.

(3) For each \( t_i \) there exists a unique vanishing cycle \( \delta_i \in H^d(Y_s, \mathbb{Q}) \) with the following properties:

- The cycles \( \delta_i, i = 1, ..., n \) span \( H^d(Y_s, \mathbb{Q}) \).
- The local monodromy around \( t_i \) is the reflection about \( \delta_i \), i.e. it is \( x \mapsto x \pm (x, \delta_i)\delta_i \), where \( (\cdot, \cdot) \) is the skew-symmetric Poincare pairing on \( H^d(Y_s, \mathbb{Q}) \).
- The monodromy representation of \( \pi_1(\Delta \cap S^0, s) \) in \( H^d(Y_s, \mathbb{Q}) \) is irreducible.

It is not difficult to deduce from (3) that the monodromy group is Zariski dense in \( Sp(H^d(Y_s, \mathbb{Q})) \) [Del2, 5.11].

We recalled the case of hypersurfaces in the projective space to stress the analogy between the monodromy group \( G^{\text{mon}} \) and the group \( G^{\text{eq}} \) as in Theorem 6.2. Indeed, both groups contain "many reflections".
Coming back to our family $\mathcal{X}^*$ of quasi-smooth Calabi-Yau varieties in $\mathbb{P}_{\Delta^*}$, we don't know if Lefschetz pencils with the properties (1), (2), (3) exist, and so we do not know how to analyze the monodromy group $G^{\text{mon}}(\mathcal{X}^*)$.

**References**

[AdTho] N. Addington, R. Thomas, "Hodge theory and derived categories of cubic fourfolds", Duke Math. J. 163 (2014), no. 10, 1885-1927.

[Bat] V. Batyrev, "Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties", J. Algebraic Geom. 3 (1994), no. 3, 493-535.

[BatBor] V. Batyrev, L. Borisov, "Mirror duality and string-theoretic Hodge numbers", Invent. Math. 126, 183-203 (1996).

[Beau] A. Beauville, "Le group de monodromie des familles universelles d’hypersurfaces et d’intersections completes”, LNM 1194, (1986) pp. 8-18.

[Beau2] A. Beauville, "Variétés Kähleriennes dont la première classe de Chern est nulle", (French) [Kähler manifolds whose first Chern class is zero] J. Differential Geom. 18 (1983), no. 4, 755–782 (1984)

[BirLa] Ch. Birkenhake, H. Lange, "Complex abelian varieties", Springer, 2004.

[BKR] T. Bridgeland, A. King, M. Reid, "The McKay correspondence as an equivalence of derived categories", J. Amer. Math. Soc. 14 (2001), no. 3, 535-554.

[Bor] A. Borel, "Density and maximality of arithmetic subgroups", J. Reine Angew. Math. 224 (1966), 78-89

[BorHC] A. Borel, Harish-Chandra, "Arithmetic subgroups of algebraic groups", Bull. Amer. Math. Soc. 67 (1961), 579-583.

[BorHor] L. Borisov, P. Horja, "Mellin–Barnes integrals as Fourier–Mukai transforms", Advances in Mathematics 207 (2006) 876–927.

[Bou] N. Bourbaki, "Groupes et Algèbres de Lie", Hermann, 1971.

[Del1] P. Deligne, "Variétés de Shimura: interpretation modulaire, et techniques de construction de modes canoniques”, Proc. of Symp. in Pure Math., vol.33(1979), part 2, pp.247-290.

[Del2] P. Deligne, "La conjecture de Weil : I", Publ. Math. IHES, tome 43, (1974), p.273-307.

[Del3] P. Deligne, "La conjecture de Weil : II", Publ. Math. IHES, tome 43, (1980), p.137-252.

[Dolg] I. Dolgachev, "Mirror symmetry for lattice polarized K3 surfaces”, J. Math. Sci. 81 (1996), No. 3, 2599-2630.

[FT] B. L. Feigin, B. L. Tsygan, "Additive K-theory and crystalline cohomology", Functional. Anal. i Prilozh., 1985, Vol. 19, no. 2, 52-62.

[GLO] V. Golyshev, V. Lunts, D. Orlov, "Mirror symmetry for abelian varieties”, J. Algebraic Geometry 10 (2001) 433-496.

[Ha] M. Haiman, Hilbert schemes, polygraphs, and the MacDonald positivity conjecture”, J. Amer. Math. Soc. 14 (2001), no. 4, 941–1006.

[Hart] R. Hartshorne, "Algebraic de Rham cohomology”, Manuscripta Math. 7, 125 - 140 (1972)

[Hart2] R. Hartshorne, "Residues and duality”, LNM 20.

[Helg] S. Helgason, "Differential geometry, Lie groups, and Symmetric spaces”, Academic press, 1978.

[Hor] P. Horja, "Hypergeometric functions and mirror symmetry in toric varieties", preprint, math.AG/9912109
[Huyb] D. Huybrechts, "Fourier-Mukai transforms in algebraic geometry", Claredon press, Oxford, 2006.

[Huyb2] D. Huybrechts, Compact hyper-Kähler manifolds: basic results. Invent. Math. 135 (1999), no. 1, 63–113.

[Huyb3] D. Huybrechts, "Lectures on K3 surfaces", Cambridge univ. press (2016).

[Jan] W.A.M. Janssen "Skew-symmetric vanishing lattices and their monodromy group", Math. Ann. 266 (1983), 115-133; 272 (1985), 17-22.

[Kel1] B. Keller, "On cyclic homology of ringed spaces and schemes", Doc. Math. 3 (1998) 231-259.

[Kod] K. Kodaira, "On the structure of Compact Analytic Surfaces, I", Amer. J. Math. Vol. 86, (1964) pp.751-798.

[Kon] M. Kontsevich, Lecture at Rutgers University, November 11, 1996 (unpublished).

[K3] "Géométrie des surfaces K3: modules et périodes", Astérisque, 126 (1985).

[LL] E. Looijenga, V.A. Lunts, "A Lie algebra attached to a projective variety", Invent. math. 129, 361-412 (1997).

[Mu1] D. Mumford, "Abelian varieties", Tata lectures, Bombay, 1968.

[Mu2] D. Mumford, "Families of abelian varieties", 1966 Algebraic Groups and Discontinuous subgroups (Proc. Sympos. Pure Math., Boulder, Colo, 1965) pp.347-351, Amer. Math. Soc., Providence, R.I.

[Or1] D. Orlov, "On equivalences of derived categories and K3 surfaces", J. Math. Sci. (New York) 84 (1997) 1361-1381.

[Or2] D. Orlov, "Derived categories of coherent sheaves on abelian varieties and equivalences between them", Izv. Math. 66 (2002), 569-594.

[Pink] H. Pinkham, "Singularités exceptionnelles, la dualité étrange d’Arnold et les surfaces K3,” C. R. Acad. Sci. Paris, Set. A-B, 284, 615-618 (1977).

[Plo] D. Ploog, "Equivariant autoequivalences for finite group actions", Advances of Mathematics 216 (2007) 62-74.

[Pol] A. Polishchuk, Thesis, Harvard (1996).

[Pjat] I.I. Pjateckij-Shapiro, "Geometry of classical domains and theory of automorphic functions", (in Russian) Moscow, 1961.

[Szen] B. Szendroi, "Diffeomorphisms and families of Fourier-Mukai transforms in mirror symmetry", in: Applications of Algebraic Geometry to Coding Theory, Physics and Computation, NATO Science Series, pp. 317-337, Kluver 2001. MR1866907 (2003a:14059)

[Tael] L. Taelman, "Derived equivalence of hyperkahler varieties", [arXiv:1906.08081]

[Verb] M. Verbitsky, "Cohomology of compact hyperkahler manifolds. [arXiv:alg-geom/9501001]

[Voi] C. Voisin, "Hodge theory and complex algebraic geometry II", Cambridge univ. press, 2003.

[Wei] C. Weibel, "The Hodge filtration and cyclic homology", K-Theory 12 (1997), no. 2, 145–164

[Wei-Gel] C. Weibel, S. Geller, "Etale descent for Hochshild and cyclic homology", Comment. Math. Helv. 66 (1991), 368-388.

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