Precipitation storm property distributions with heavy tails follow tempered stable density relationships

Peng Jiang\textsuperscript{1,2}, Shawn Dawley\textsuperscript{3}, Binqing Lu\textsuperscript{3}, Yong Zhang\textsuperscript{3,6}, Geoffrey R Tick\textsuperscript{3}, HongGuang Sun\textsuperscript{4} and Chunmiao Zheng\textsuperscript{5}

\textsuperscript{1}State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, Hohai University, Nanjing, 210098, China
\textsuperscript{2}Division of Hydrologic Sciences, Desert Research Institute, Las Vegas, NV 89119, USA
\textsuperscript{3}Department of Geological Sciences, University of Alabama, Tuscaloosa, AL 35487, USA
\textsuperscript{4}Institute of Soft Matter Mechanics, Department of Engineering Mechanics, Hohai University, 1 Xikang Road, Nanjing, Jiangsu 210098, China
\textsuperscript{5}School of Environmental Science & Engineering, Southern University of Science and Technology, Shenzhen, Guangzhou, China
\textsuperscript{6}Email: yzhang264@ua.edu

Abstract. Precipitation storm properties, especially high impact and low probability extremes in storm intensity, duration, and frequency, are important for hydrologic engineering design/control and hydrologic response to changing climate. This study reveals that the property distributions of precipitation storms in the southwestern United States exhibit heavy tails, which follow a tempered stable density function, one possible universal density for hydrologic variables. The precipitation time series from four regions in different states are decomposed as sequences of storm intensity, duration, and frequency, where the underlying anomaly or heavy tailed distribution for each characteristic is explored. Analysis shows that the average storm intensity, storm duration, and interstorm period distribute as a rescaled and tempered stable density function with a variable index. The probability distribution is also space dependent, likely due to climatological variation, which can be represented by a space-dependent index and/or truncation parameter in the tempered stable density function. These sequences, or the cumulative rainfall, therefore might be treated as a realization of a three-stage random walk process, where each stage contains unique tempered stable random variables. This finding leads to a distributed-order, fractional-derivative model with exponentially truncated power-law densities to quantify precipitation storm properties, while the practical application, of which, remains to be shown.

1. Introduction
Major storm characteristics including intensity, duration, and frequency of precipitation storm events, are critical for appropriate hydrologic engineering design and control. In particular, safety, risk, and economic analyses of engineering constructions such as storm sewer, street and urban drainage, and channel design are sensitive to precipitation properties and characteristics. In addition, recent extreme rainfall events in the United States, including severe drought in California, enhanced Southwest monsoons due to increasing yearly temperatures (i.e., global warming induces a large increase in
atmospheric water vapor content which accelerates hydrologic cycle [1,2]), and recent destructive storms in the southeastern US trigged by tropical cyclones (such as hurricane Irma and Harvey), also motivate us to explore the properties of precipitation storms, especially the low probability and high impact tailing behavior in storm property distributions.

This study will focus on the following two questions. First, what is the distribution and spatial variation of precipitation storm properties in the southwestern US? Second, is there a universal density function that can capture the whole property distribution, especially for extreme rainfall and drought events? Answers to these questions help us interpret the possible nonstationary evolution of future extreme precipitation scenarios, which is critical for improved hydrologic engineering design and control measures under climate change [3]. The calculation of storm properties is based on the identification of independent storm events, which had been done by the arbitrary separation method [4] and the autocorrelation method [5], among others. A general method and a universal density function are still needed for theoretical analysis and practical applications, a primary motivation of this study.

To explore these questions, historical time series data for precipitation in various states within southwestern US are analyzed, and state-of-the-art statistics based on the tempered stable density function are applied in this study. A generalized fractional-calculus based physical model charactering the identified rainfall properties will eventually be proposed for future applications, especially involving extreme precipitation events (both storms and drought).

2. Study sites and storm time series
Four study sites (figure 1) representing the large climatological diversity in the southwestern US were selected to explore the potential heavy tail anomalies associated with precipitation storm time series.

Site 1: Los Angeles, California: Subtropical-Mediterranean climate. Climate in Los Angeles is characterized by nearly no rain in summer and a winter/early spring strong rainy season (with an average precipitation of 15.1 inches per year). Although the overall precipitation is quite low in Los Angeles, the rain sometimes comes in brief heavy storms which often cause landslides and debris flows. The warm-phase of an El Nino-Southern Oscillation (ENSO) regime typically coincides with above average precipitation across southern California, including Los Angeles.

Site 2: Las Vegas, Nevada: Subtropical hot desert climate. The scarce rainfall (5.0 inches per year) disperses between ~26 rainy days per year. Las Vegas is among the driest and least humid locations in North America.

Figure 1. Study locations with precipitation regions and categories from cluster analysis. Figure 2. Decomposed precipitation series as a sequence of storm properties.
Site 3: Tucson, Arizona: Western desert climate transitioning to eastern semi-arid steppe climate. The annual precipitation is 11.8 inches, and this desert climate includes the North American Monsoon.

Site 4: Cimarron, New Mexico: Semi-arid steppe climate. It receives an average of 18.0 inches of rain and 46 inches of snowfall per year. Frequent thunderstorms occur in the summer, and occasional heavy snowfall occurs in the cold winter season.

Hourly precipitation data are obtained from the National Oceanic and Atmospheric Administration Nation Climatic Data Center [6]. Data are available between 1949 and 2010 with less than 1% of missing values for the selected four stations (regions).

3. Storm property analysis

3.1. Precipitation time series data decomposition

A precipitation storm can be defined as a precipitation period separated from preceding and succeeding precipitation by a minimum dry period, such as six hours [7]. The storm event can be characterized by its duration, average storm intensity, and inter-storm events [8] (figure 2). After storm events are identified, the monthly mean storm duration, inter-storm period, and storm intensity can be calculated from the precipitation time series using the following equations:

\[
SD_{\text{mean}} = \frac{\sum_{i=1}^{n} SD_i}{n}
\]

(1)

\[
IP_{\text{mean}} = \frac{\sum_{i=1}^{n} IP_i}{n}
\]

(2)

\[
SI_{\text{mean}} = \frac{\sum_{i=1}^{n} SD_i \times SI_i}{\sum_{i=1}^{n} SD_i}
\]

(3)

where \(SD_i\), \(IP_i\), and \(SI_i\) are storm duration, inter-storm period, and average storm intensity, respectively, for the \(i\)-th (total \(n\)) storm events in the selected month. \(SD_{\text{mean}}\), \(IP_{\text{mean}}\), and \(SI_{\text{mean}}\) are monthly mean of storm duration, inter-storm period, and average storm intensity, respectively. Note that a single storm event can contain multiple rainfalls (figure 2). The above formulas are used for the four locations shown in figure 1, and the calculated precipitation storm properties are shown by symbols in figures 3 and 4.

3.2. Density function

The tempered stable density analysis selected here is motivated by the pioneering work of Benson et al. [9], who found that extreme events, such as precipitation storms, can have power-law interarrival times. A heavy-tailed distribution of the continuous time random maximum (CTRM) model for interstorm periods was therefore proposed by Benson et al. [9]. Natural processes however are usually bounded. Hence, we extend the pure power-law density function (such as the classical stable density) to the bounded power-law density function, which is the tempered stable density and represents an exponentially truncated power-law function. We analyze the major properties of storm events in addition to their interarrival times.

We use the following exponentially truncated power-law function to represent the memory function, which can define the distribution of waiting times for geophysical processes, such as the interarrival times between two subsequent precipitation events [10]:

\[
g(t) = \int_{T'}^{\infty} e^{-\lambda r} \frac{r^{-\gamma-1}}{\Gamma(1-\gamma)} dr ,
\]

(4)

where \(t\) is time, \(\lambda [T']\) is a truncation parameter, \(\gamma\) [dimensionless] denotes the stable index (where \(0 < \gamma < 1\) in this study), and the symbol \(\Gamma(\cdot)\) denotes the Gamma function. When the truncation parameter \(\lambda \to 0\) (i.e., without truncation), the memory function (4) reduces to the power-law memory function:
which was used to describe sub-diffusion due to multi-rate mass exchange in geological media [10].

Following the procedure in Zhang et al. [11], we build the following time-fractional derivative equation for precipitation storm dynamics with the memory function (4):

\[
\frac{\partial P(t, s)}{\partial t} = -\beta e^{-\lambda s} \frac{\partial^\gamma}{\partial s^\gamma} \left[ e^{\lambda t} P(t, s) \right] + \beta \lambda^\gamma P(t, s),
\]

where \( s \) denotes the property of the target storm, \( P \) is the transition probability of \( s \), \( \beta \) is the scale factor that controls the expansion (or width) of the distribution, and \( \partial^\gamma / \partial s^\gamma \) is the Riemann-Liouville fractional derivative. The solution of (6) in Laplace space \( (s \rightarrow u) \) is:

\[
P(t, u) = e^{-\lambda(t-u) + u \beta \lambda^\gamma} f_\gamma \left( \frac{t-u}{(\beta u)^{1/\gamma}} \right) (\beta u)^{-1/\gamma},
\]

where \( f_\gamma \) is a stable density function with index \( 0 < \gamma < 1 \). The Laplace inverse transform of \( P(t, u) \) at time \( t=1 \) is the scaled, tempered stable density, or the stable density with an exponential truncation. Equation (6) can be solved by the inverse transform of (7), or using the Lagrangian solver [11].

Another reason that we select (7) to quantify storm properties is due to the suggestion of Cvetkovic [12], who found that the tempered one-sided stable density distribution, expressed by equation (7), recovers virtually all random time distributions considered in the literature for hydrological dynamics. In the next section, we test whether Cvetkovic’s [12] conclusion is valid for various storm data.

3.3. Application and results

We apply (7) to build the density function for the storm properties obtained above. The best-fit results are shown in figures 3 and 4. The resultant tempered stable density parameters are listed in table 1.

| Location     | Climate                    | Storm properties | Index \( \gamma \) | Truncation parameter \( \lambda \) | Scale factor \( \beta \) |
|--------------|-----------------------------|------------------|---------------------|-------------------------------------|---------------------------|
| Los Angeles  | Subtropical-Mediterranean   | Intensity        | 0.60                | 0.18                                | 1.2                       |
|              |                             | Duration         | 0.40                | 0.03                                | 2.8                       |
|              |                             | Interstorm Period| 0.20                | 0.0012                              | 5.1                       |
| Las Vegas    | Subtropical desert          | Intensity        | 0.70                | 0.02                                | 0.7                       |
|              |                             | Duration         | 0.45                | 0.05                                | 2.1                       |
|              |                             | Interstorm Period| 0.25                | 0.0004                              | 5.5                       |
| Tucson       | Western desert to eastern   | Intensity        | 0.70                | 0.02                                | 0.85                      |
|              | semi-arid steppe climate    | Duration         | 0.45                | 0.05                                | 2.1                       |
|              |                             | Interstorm Period| 0.25                | 0.0018                              | 5.5                       |
| Cimarron     | Semi-arid steppe climate    | Intensity        | 0.70                | 0.02                                | 1.0                       |
|              |                             | Duration         | 0.45                | 0.05                                | 2.1                       |
|              |                             | Interstorm Period| 0.25                | 0.0018                              | 5.5                       |

Results show that the distribution of precipitation storm properties in the southwestern US can be captured overall by a universal law – the tempered stable density function. The extreme values for all storm properties, as shown by the right tail of each curve in figures 3 and 4, transition gradually from power-law to exponential, likely due to the natural upper limit of extreme events.

There is also subtle spatial variation of the tailing behavior in storm property distributions. For example, the storm intensity distribution at Los Angeles exhibits a relatively shorter power law regime than that for Las Vegas (figures 3a, d), implying that the shorter rainy season in Las Vegas may,
however, exhibit higher variation in rainfall rates. This discrepancy may also distinguish subtropical-Mediterranean climates from subtropical desert climates.

Figure 3. Los Angeles vs. Las Vegas: the measured storm properties (symbols) versus the best-fit tempered stable density function (lines).

4. Discussion and conclusions
Precipitation storm properties can affect engineering construction design, and have significant implications for related environmental, social and economic consequences. Understanding and defining the major characteristics of precipitation storms and how they can be quantified efficiently are of great practical importance. Statistical analysis of historical storm data in the southwestern US in this study lead to the following five results (i.e., the research contribution of this work) and brief discussion.

First, the PDFs of average storm intensity, storm duration and interstorm period for the four study sites follow the same functional form – represented by the tempered stable density function. Hence, a universal law may exist for storm properties across climatologic zones. This finding extends the applicability of Cvetkovic’s [12] single-side stable density distribution to precipitation data, and changes the unlimited interarrival times proposed by Benson et al. [9] to a bounded domain.

Second, it is observed that Los Angeles receives larger probabilities for extreme rainfall (which also lasts longer) than the other three sites. This discrepancy (which is likely related to the stronger average rainfall at Los Angeles) can be characterized by a smaller index $\gamma$ and a larger scale factor $\beta$ in
the tempered stable density function than those for the other sites. Conversely, the interstorm periods at Los Angeles exhibit a relatively narrower distribution represented by a relatively smaller factor $\beta$ than those of the other sites, indicating that Los Angeles has the lowest probability for extremely long dry periods. This larger probability for long duration extreme rainfall compared to the other sites is likely due to the winter and spring frontal systems originating in the North Pacific Ocean, which provides the largest source of moisture to the southwestern US and tends to generate long-duration extreme winter rainfall.

Figure 4. Tucson vs. Cimarron: the measured storm properties (symbols) versus the best-fit tempered stable density function (lines).

Third, Las Vegas has similar PDFs to Tucson for rainfall intensity and duration. This similarity is due to the fact that the moisture at both sites mainly comes from Pacific storms passing over the Sierra Nevada and Spring Mountains and summer monsoon. Despite the similar PDFs in storm intensity and duration, the interstorm period distribution at Las Vegas (characterized by the truncation parameter $\lambda=0.0004$) is slightly heavier (i.e. more tailing toward the end of the distribution) than that at Tucson ($\lambda=0.0018$), showing that extreme drought events can last longer in Las Vegas than Tucson. This discrepancy is most likely a result of the fact that Las Vegas is dominated by a subtropical hot desert climate while Tucson is characterized by a western desert climate transitioning to eastern semi-arid steppe climate which is wetter than Las Vegas.

Fourth, Tucson and Cimarron share similar density functions for storm properties. This similarity, however, contains high uncertainty. First, the noise in observational data, especially for extreme
values, leads to high uncertainty for the actual density. Second, here we consider the annual average, which may not be fine enough to reveal seasonal storm variations. Tucson contains two precipitation seasons: the monsoon season and the winter precipitation season. Precipitation at Cimarron, however, has no apparent seasonal fluctuations.

Fifth, we propose a distributed-order, fractional continuous-time random walk (F-CTRW) model with exponential truncations to quantify the precipitation storm processes, or the cumulative rainfall:

\[
\beta_D \frac{\partial^{\gamma_D, \lambda_D} F}{\partial t^{\gamma_D, \lambda_D}} + \beta_P \frac{\partial^{\gamma_P, \lambda_P} F}{\partial t^{\gamma_P, \lambda_P}} = \beta_I \frac{\partial^{\gamma_I, \lambda_I} F}{\partial s^{\gamma_I, \lambda_I}},
\]

where \( \frac{\partial^{\gamma, \lambda}}{\partial t^{\gamma, \lambda}} \) denotes the tempered fractional derivative, and the suffix “\( D, P, I \)” denote storm Duration, interstorm Period, and storm Intensity, respectively. The first, second, and third terms in (8) represent random immobile waiting times (representing interstorm period), mobile operational times (representing storm duration), and discrete jump sizes (representing storm intensity), which all distribute as tempered stable random variables. Hence, equation (8) is a three-stage random walk process. End members of (8) include the CTRM model proposed by Benson et al. [9] and the time-fractional model proposed by Golder et al. [13] to simulate the heavy-tailed rainfall process. Properties of precipitation from other regions may follow the same law defined above (and the same for (8)), since the tempered stable density (7) captures a broad range of transition from the light tailed, normal law density. Equation (8) accounts for the evolution of extreme rainfalls with random interarrival (immobile) and active (mobile) periods, whose validation remains to be shown. We will explore the applicability of (8) in the next study.

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