Retrieving fields from proton radiography without source profiles

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Proton radiography is a technique in high energy density science to diagnose magnetic and/or electric fields in a plasma by firing a proton beam and detecting its modulated intensity profile on a screen. Current approaches to retrieve the integrated field from the modulated intensity profile require the unmodulated beam intensity profile before the interaction, which is rarely available experimentally due to shot-to-shot variability. In this paper, we present a statistical method to retrieve the integrated field without needing to know the exact source profile. We apply our method to experimental data, showing the robustness of our approach. Our proposed technique allows not only for the retrieval of the path-integrated fields, but also of the statistical properties of the fields.

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INTRODUCTION

Proton radiography is a popular method to diagnose magnetic and/or electric fields in laser-produced plasmas and inertial confinement fusion experiments 1–7. In proton radiography, a low density proton beam is fired into a plasma, gets deflected by its electric and magnetic fields, and the beam intensity profile is measured on a screen placed at some distance. The modulated intensity of the beam is related to the deflection of the proton beam and thus related to the integrated magnetic and electric field. Therefore, by comparing the unmodulated beam intensity and the modulated intensity one can retrieve the integrated magnetic and electric fields.

Earlier attempts to reconstruct fields from proton radiography utilize Poisson’s equation solver 8 and a diffusion model 9. However, those approaches use linear approximations of the proton radiography forward model, while most of the interesting interactions happen in the non-linear regime. Later, Graziani et al. 9 and Kasim et al. 10 independently realized that integrated field retrieval with proton radiography is a subclass of the optimal transport problem first posed in 1781 11. Establishing the connection between proton radiography and optimal transport opens up numerous algorithms for proton radiography field reconstruction even in the non-linear regime. This has been shown by Kasim et al. 10 and Bott et al. 12 by applying off-the-shelf optimal transport algorithms 13 14 to proton radiography, achieving remarkable results in some parameter regions.

Despite their successful implementations, current algorithms to solve the proton radiography inverse problem need the exact knowledge of the unmodulated beam intensity profile (source profile) as well as the modulated beam intensity profile. Unfortunately, there is no viable non-destructive method to simultaneously capture the proton beam intensity profiles before and after interacting with the plasma. Moreover, the source profiles are different from shot-to-shot by some considerable amount 15 making it harder to determine the source profile when the measurement is performed. Only the statistical properties of the source profile variability can be determined reliably, by measuring the beam intensity multiple times without any field interaction 15 16.

In this paper, we introduce a statistical method in retrieving the integrated field only by using the statistics of the source profiles, without the exact knowledge of the source profile. Besides retrieving the integrated field, the method also provides the probability distribution of the retrieved field allowing one to perform a statistical analysis of the results, such as checking if the measurement suffers from the inverse problem instability 17.

THEORY

A schematic of a typical proton radiography set-up can be seen in Figure 1. In a typical case, a proton beam is fired from a source through an object with magnetic and/or electric fields. The fields deflect the trajectory of the beam, forming an intensity modulation on the screen behind the object.

Without loss of generality, only magnetic fields are considered in this paper. It is also assumed that the deflection is small enough so that most of the deflected protons reach the screen. It is noted that the deflection can still be large enough to reach the non-linear regime.

If a proton beam is fired through a plasma with magnetic field, the beam at coordinate $(x_0, y_0)$ on the object plane will be deflected to an angle

$$\alpha(x_0, y_0) = -\nabla \Phi(x_0, y_0), \quad (1)$$

for $\alpha \ll 1$, where $\Phi$ is the integrated field given by 8

$$\Phi(x_0, y_0) = -\frac{e}{\sqrt{2mW}} \int A(x_0, y_0, z_0) \cdot dz_0. \quad (2)$$
FIG. 1: Schematics of a typical set up for proton radiography diagnostics.

The integrated field, $\Phi$, depends on the charge $e$ and mass $m$ of each proton, the kinetic energy of the beam $W$ and the magnetic vector potential in the object $\mathbf{A}$.

If the distance between the object and the screen is $L$, the beam at $(x_0, y_0)$ on the object plane is mapped to the coordinate $(x, y)$ on the screen where they are given by

$$
\begin{align*}
  x(x_0, y_0) &= x_0 + \alpha(x_0, y_0) \cdot \mathbf{x}L, \\
  y(x_0, y_0) &= y_0 + \alpha(x_0, y_0) \cdot \mathbf{y}L,
\end{align*}
$$

(3)

(4)

assuming the plasma size is much smaller than $L$ and the beam source is collimated. If the beam is diverging from a point source at distance $l$ from the object, then $x_0$ and $y_0$ are replaced by $x_0 \rightarrow x_0(1+L/l)$ and $y_0 \rightarrow y_0(1+L/l)$ in the equations above.

An analytical equation to determine the intensity on the screen is given by

$$
I(x, y) = I_0(x, y) \left| \frac{\partial(x, y)}{\partial(x_0, y_0)} \right|^{-1}.
$$

(5)

The term $I_0(x, y)$ denotes the intensity profile without any deflections from the object and the term $|\partial(x, y)/\partial(x_0, y_0)|$ is the absolute determinant of the Jacobian matrix of $(x, y)$ with respect to $(x_0, y_0)$. If the deflection is large enough, the determinant of the Jacobian can be close to zero at some points and the intensity at the corresponding positions can reach a very high value. These are known as caustics.

One way to describe the deflection strength in proton radiography is using a dimensionless variable, as introduced by Kugland et al. [8], which is defined as

$$
\mu = L\alpha/a,
$$

(6)

with $L$ the distance from the system to the screen, $\alpha$ the magnitude of the deflection angle, and $a$ the size of the field perturbation.

If $\mu \ll 1$, equation [8] can be linearized and the integrated field can be obtained by solving Poisson’s equation

$$
\nabla^2 \Phi \approx \frac{2W}{e} \left( \frac{I}{I_0} - 1 \right). 
$$

(7)

The two integrals in Poisson’s equation solver amplify the lower frequency components of $I/I_0$ by a factor proportional to $1/k^2$ with the wavenumber $k$.

When $\mu$ exceeds a certain value, $\mu \geq \mu_c$, caustics form and parts of the beam cross each other. In this region, the relation between $\Phi$ and $I$ is no longer injective [8][12]. This means that there are multiple profiles of $\Phi$ that correspond to the same intensity profile $I$.

If $\mu$ is on the order of unity and less than $\mu_c$, it gets into the so-called non-linear injective regime. In this regime, Poisson’s equation is no longer accurate, but the relation between $\Phi$ and $I$ is still injective. The integrated field, $\Phi$, can be obtained by solving the optimal transport problem: find a way in “transporting” the protons from the source profile to resemble the modulated profile with total squared displacement as small as possible. Numerous algorithms are available in solving this type of problem, for example [13][14].

For the rest of the paper, it is assumed that the proton deflection is always in the injective domain where there is no beam crossing. Field reconstruction in the non-injective domain is beyond the scope of this paper.

METHOD

Optimal transport based algorithms to retrieve the integrated field from a source profile and a modulated intensity profile are deterministic. Therefore, the probability of getting $\Phi$ from $I$ and $I_0$ can be written as:

$$
\mathbb{P}(\Phi|I, I_0) = \delta(\Phi - \Phi'(I, I_0)),
$$

(8)

where $\delta(\cdot)$ is the Dirac delta and $\Phi'(I, I_0)$ is the integrated field profile retrieved using the retrieval algorithms given the modulated intensity profile $I$, and the source profile $I_0$.

When we have no exact knowledge of the source profile $I_0$, one thing that we can do is to marginalize it by integrating the term $I_0$ over its probability distribution,

$$
\mathbb{P}(\Phi|I) = \int \mathbb{P}(\Phi|I, I_0)\mathbb{P}(I_0)\ dI_0,
$$

(9)

with $\mathbb{P}(\Phi|I, I_0)$ given by equation [8]. From the equation above, we can get the probability distribution of the integrated field $\Phi$ by generating multiple source profile samples according to $\mathbb{P}(I_0)$ and apply the retrieval algorithm for every generated profile. In this paper we use the algorithm from [14] to retrieve the integrated field from a modulated intensity profile and a source profile.

Now the problem is shifted to determining the probability distribution of the source profile, $\mathbb{P}(I_0)$. Based on
where \( \theta \) parameters that represents the correlation of two points with hypersimilarity for the source profile, and \( \kappa \) profile with hyperparameters where \( \theta \) is the expected deviation, and \( \theta \) is larger for TNSA-generated proton sources compared to DD and D\(^3\)He fusion source deviations. One way to capture correlated deviations is by representing them using a Gaussian Process (GP) where each element in the covariance matrix \( K \) is given by

\[
K_{ij} = \kappa(|x_i - x_j|, \theta) = \sigma^2 \exp\left(-\frac{|x_i - x_j|^2}{2d^2}\right),
\]

where \( \kappa \) is the kernel, \( \theta \) is the correlated distance. In this case, the variable \( y \) is the vectorized source profile \( I_0 \), and \( n \) is the number of pixels. The inputs to the kernel are the position of each pixel in \( I_0 \).

One of the most commonly used kernels is the squared exponential kernel,

\[
\kappa_{\text{SE}}(x_1, x_2|\theta) = \sigma^2 \exp\left(-\frac{|x_1 - x_2|^2}{2d^2}\right),
\]

where \( \theta = \{\sigma, d\} \) are the hyperparameters of the kernel, \( \sigma \) is the expected deviation, and \( d \) is the correlated distance. Increasing \( \sigma \) will increase the standard deviation from the mean of the profiles. Increasing \( d \) will make the deviation less oscillatory. To illustrate the effect of \( \sigma \) and \( d \), figure 2 shows some samples taken from the Gaussian Process distribution with various values of \( \sigma \) and \( d \).

FIG. 2: Some samples taken from the one-dimensional Gaussian Process with various values of \( \sigma \) and \( d \).

If source profiles are available experimentally, then one can fit the hyperparameters \( \theta \) to the experimental source profiles by finding the maximum log-likelihood,

\[
\theta_{ML} = \arg \max_{\theta} \sum \log P(I_{0j}|\theta),
\]

where \( I_{0j} \) is the \( j \)-th source profile obtained experimentally. However, if there is no experimental data available, one should make a reasonable assumption over the prior probability of the hyperparameters, \( P(\theta) \). This changes Eq. (9) to

\[
P(I|\theta) = \int P(I|I_0)P(I_0|\theta)P(\theta) \, d\theta \, dI_0.
\]

One way to choose the prior distribution of the hyperparameters when there is no experimental data available is to set it based on known references, for example [13, 16]. Another way is to choose a weak prior of the hyperparameters such as Jeffreys prior [21], a uniform, or a log uniform prior for certain range.

We present a summary of our method in algorithm 1.

**Algorithm 1 Retrieving \( \Phi \) without explicit \( I_0 \)**

**Input:** the modulated intensity profile \( I \)

**Output:** pool of \( \Phi \) samples

1. if experimental source profiles available then
2. find \( \theta_{ML} \) according to equation [13]
3. else
4. set a prior distribution on \( \theta, P(\theta) \)
5. end if
6. while not enough samples do
7. use \( \theta_s = \theta_{ML} \) or draw a sample for \( \theta_s \sim P(\theta) \)
8. draw a sample of source profile, \( I_{0s} \sim P(I_0|\theta_s) \)
9. retrieve \( \Phi_s \) using \( I \) and \( I_{0s} \)
10. add \( \Phi_s \) to the pool of samples
11. end while

FIG. 3: (a) The proton radiography modulated intensity profile captured on an RCF (false color) and (b) the region-of-interest where the proton beam has no obstruction. The data was taken from [2] with a permission from the corresponding author.
RESULTS

We apply our method in retrieving the integrated field as well as the uncertainty to an experiment of dynamo amplification of magnetic fields [2]. In the experiment, the authors used proton radiography to retrieve the integrated magnetic field using a proton beam generated from a D<sub>3</sub>He fusion capsule. One of the measured beam intensity profiles is shown in figure 3.

Because there is no information on the source profile statistics from this experiment, we take $\sigma$ to be $\sim$10%, close to the $\sim$13% standard deviation for proton beams generated from a D<sub>3</sub>He fusion capsule given in [16]. For the correlated length $d$, we use a prior distribution which is log-uniform from 0.7 cm to 7 cm. The range was chosen to capture the variation of source profile that could give considerably different integrated field profiles. It is to obtain an upper estimate of the variance and to avoid over-confidence. The bounding box that we are analyzing has the maximum size of 7 cm, therefore setting the correlated length to be greater than 7 cm would produce an almost uniform source profile and provide little variation of the integrated fields. On the other hand, when $d$ is smaller than 0.7 cm, the result is similar to white noise and gives a similar integrated field to a uniform source profile.

In each iteration, we choose a value of $d$ from the prior distribution above, generate a source profile from the Gaussian Process distribution using the chosen hyperparameters, retrieve the integrated field profile using the generated source, and put the integrated field profile into the pool of samples. Figure 4(a,b) shows the mean and standard deviation of the integrated field from 15,000 samples. From the figure, we can see that with $\sigma = 10\%$, the integrated magnetic field has standard deviation of about $\sim$20%. The error magnification factor of only 2 is similar to the error propagated by a quadratic equation.

Besides calculating the mean and standard deviation of the integrated field, we also calculated the power spectra from the integrated field samples. This is shown in figure 4(c). From the figure, we can see that the power spectrum uncertainty is high at small and large wavenumbers. High uncertainty at small wavenumbers is due to the nature of the retrieval algorithm that amplifies low wavenumber elements (i.e. amplification $\sim 1/k^2$). High uncertainty at large wavenumbers is due to limited precision in retrieving features with small amplitudes which happen to be at large wavenumber.

Figure 4(d) shows the power of the wavenumber in relation to the spectrum. Here we can see that the samples distribution agrees with Kolmogorov’s power law ($S(k) \propto k^{-5/3}$) better than with the $k^{-1}$ power law, although most samples show that the gradient should be shallower than suggested by Kolmogorov’s power law. Note that this result is just from the uncertainty of the source profile. One needs to include more uncertainty factors to reach a meaningful conclusion, such as non-injectivity of crossing beams and uncertainty in proton energy, which are not the focus of this paper.

CONCLUSIONS

We have presented a statistical method to retrieve integrated fields from proton radiography without knowing the exact source profile. The probability distribution of the integrated field is obtained by marginalizing out the probability distribution of the source profile. The distribution of source profiles can be obtained by collecting a number of proton radiography samples without any interaction with electric and magnetic fields, or by making a weak prior assumption. The method has been applied to an experiment to retrieve the integrated magnetic field and its statistics, showing the robustness of the proposed approach.
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