A canonical embedding of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$

FRANCISCO BRAUN$^*$ and FREDERICO XAVIER$^+$

Abstract

The group $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ of self-biholomorphisms of $\mathbb{C}^n$ consists of affine maps if $n = 1$, but in higher dimensions it is a large object that has not been described explicitly. Despite the intricacies involved when $n > 1$, surprisingly every $F \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ is uniquely determined inside the group by only two data, of infinitesimal and global nature: the 1-jet of $F$ at 0, and the complex Hessian of a certain plurisubharmonic function associated to $F$. If $n = 1$ this global datum is zero for all $F$, which is then determined solely by its 1-jet at 0, and one recovers $\text{Aut}_{\text{hol}}(\mathbb{C}) = \text{Aff}(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C}^*$. Our main result, formulated as the existence of a canonical embedding of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$, also singles out a natural candidate for moduli space of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$, for all $n > 1$.

1 Introduction

Let $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ be the group of all holomorphic automorphisms of $\mathbb{C}^n$, and $\text{Aff}(\mathbb{C}^n)$ the subgroup of invertible affine maps. It is elementary that $\text{Aut}_{\text{hol}}(\mathbb{C}) = \text{Aff}(\mathbb{C}) \cong \mathbb{C} \times \mathbb{C}^*$, but in higher dimensions $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ is so large and has such a rich structure $[1]$ that an explicit description of it is yet to be found.

In this paper we introduce an unified approach to the study of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ that puts this size dichotomy in context, and at the same time identifies in every dimension a natural candidate for moduli space of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$.

Our main result, Theorem 2.1, establishes the existence of a canonical embedding

$$\text{Aut}_{\text{hol}}(\mathbb{C}^n) \xrightarrow{X^n} \mathbb{C}^n \times GL(n, \mathbb{C}) \times \mathcal{L}^n(P\,SH),$$

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where $\mathfrak{L}^n(PSH)$ is the space of Levi matrices (complex Hessians) of real-analytic plurisubharmonic functions on $\mathbb{C}^n - \{0\}$.

The embedding $\mathcal{X}^n$ records the 1-jet of the automorphism $F$ at 0, together with a datum in $\mathfrak{L}^n(PSH)$. When $n = 1$ this global datum is zero for all $F$, and so all the information carried by $\mathcal{X}^n$ is already encoded in the 1-jet of $F$ at 0. From injectivity one then recovers the elementary description $\text{Aut}_{\text{hol}}(\mathbb{C}) = \text{Aff}(\mathbb{C})$, with the corresponding moduli space $\mathbb{C} \times \mathbb{C}^* \times \text{singleton} \cong \mathbb{C} \times \mathbb{C}^*$.

The range of $\mathcal{X}^n$ can be regarded as a “proto-moduli space” for the automorphism group, a concept that will be explained in the next section. At present, what stands between our results and a full-fledged moduli space for $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ is the lack of an intrinsic characterization of $\text{Ran}(\mathcal{X}^n)$ when $n > 1$.

Given the unusual nature of the map $\mathcal{X}^n$, it merits commenting on the process that led to its discovery. In the course of proving an embedding theorem for $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$, at some point of the argument one has to show that two holomorphic automorphisms $F, G$ are equal or, what is the same, that $F \circ G^{-1}$ is the identity map. In this regard, the work [4] is relevant, as it contains a characterization of the identity of $\mathbb{C}^n$ among injective local biholomorphisms. The statement of Theorem 2.1 was arrived at - in a manner akin to reverse engineering - by starting with a conjectural enhancement of the main result of [4], and then working backwards to figure out what the adequate hypotheses for Theorem 2.1 should be.

Although it may yet be possible to establish the desired technical improvement of [4], in which case Theorem 2.1 would be a corollary of such a result, it so happens that a sharper version of [4] is no longer needed to prove our embedding theorem. In an unexpected turn of events, the present authors realized that, since [4] has already led to the right hypotheses in Theorem 2.1 one can actually supply an independent, non-technical, proof of this result.

The results in this paper are part of a larger program whose aim is to use tools from differential geometry, analysis, topology, and dynamical systems in order to understand the various mechanisms behind the phenomenon of global injectivity of maps (for more on this, see [5] and the references therein).
2 A proto-moduli space for $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$

In order to present our findings about $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ in a systematic way, and with a view towards future developments, it is convenient to adopt a more formal approach to the notion of “moduli problem” for a given “space” $X$. This problem is generally understood to be the search for a parametrization of $X$ by a set whose elements have special, often geometric, properties. The following set-theoretic definitions make the idea of having “special properties” precise. Given an injective map $f : X \rightarrow Y$ with range $Z$, the bijection $Z \overset{f^{-1}}{\rightarrow} X$ can be thought of as a proto-parametrization of $X$, and $Z$ can be regarded as a proto-moduli space of $X$.

If, in addition, the membership relation “$y \in Z$” can be described alternatively using only properties that do not involve the set $X$ (a fortiori, invoking neither $f$ nor $Z$), then we say that $Z$ admits an intrinsic description, $Z \overset{f^{-1}}{\rightarrow} X$ is a parametrization of $X$, and $Z$ is a moduli space of $X$.

To summarize, the moduli spaces of $X$ are precisely those ranges of embeddings of $X$ that admit an intrinsic description. The desideratum that the moduli of $X$ should have “special properties” is reflected in the condition that membership in $Z$ can be described by properties that do not refer to $X$. A concrete example illustrating these simple concepts will be given later in this section, in connection with our main theorem.

The advantage of this recasting of the moduli problem lies in the fact that the problem is purposely split into two sub-problems, which can then be examined separately: one first needs to find a “natural” proto-parametrization $f : X \rightarrow Y$ and then, in a second stage, argue that the associated proto-moduli space can be upgraded to a full-fledged moduli space.

In the present context, the set $X$ is $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$. Theorem 2.1 below provides a solution of the first half of the moduli problem in all dimensions that, when specialized to $n = 1$, also gives the expected solution of the second half of the problem. We conjecture that these proto-moduli spaces for $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ are actual moduli spaces when $n > 1$ as well.

Before stating our results, we review some basic concepts from the theory of several complex variables (a suitable reference is [2, chapter 4]). Given a $C^2$ real-valued function $g$ defined on an open set of $\mathbb{C}^n$, the $n \times n$ Hermitean matrix $L(g) = \left( \frac{\partial^2 g}{\partial z_i \partial \bar{z}_j} \right)$ is called the Levi matrix, or complex Hessian, of $g$. When $L(g) \geq 0$ (resp., $L(g) = 0$) everywhere, $g$ is said to be plurisubharmonic (resp., pluriharmonic).

Alternatively, the plurisubharmonicity (pluriharmonicity) of $g$ can be characterized by the property that the restriction of $g$ to every complex line intersecting the domain of $g$ is
subharmonic (resp., harmonic). Both notions are invariant under holomorphic changes of coordinates. A prototypical example of a smooth plurisubharmonic function is $\log \|G\|$, where $G : \mathbb{C}^m \to \mathbb{C}^n - \{0\}$ is holomorphic. A pluriharmonic function defined on a simply connected set is the real part of a holomorphic function on this set.

We denote by $\mathcal{L}^n(PSH)$ the subset of $C^\omega(\mathbb{C}^n - \{0\}, \mathbb{C}^{n \times n})$ consisting of the Levi matrices of real-analytic plurisubharmonic functions defined on $\mathbb{C}^n - \{0\}$.

Theorem 2.1. The map $\mathcal{X}^n : \text{Aut}_{\text{hol}}(\mathbb{C}^n) \to \mathbb{C}^n \times GL(n, \mathbb{C}) \times \mathcal{L}^n(PSH)$,

$$\mathcal{X}^n(F) = (F(0), DF(0), L(\log \|DF(0)^{-1}(F - F(0))\|)),$$  \hspace{1cm} (2.1)

is injective for all $n \geq 1$.

Let $\mathcal{Q}^n$ denote the quotient space of the equivalence relation that identifies two real-analytic plurisubharmonic functions defined on $\mathbb{C}^n - \{0\}$ if their difference is pluriharmonic. There is a natural bijection between $\mathcal{L}^n(PSH)$ and $\mathcal{Q}^n$, given by $L(g) \to [g]$. Accordingly, Theorem 2.1 can be stated with a more intrinsic flavor:

Theorem 2.2. Any $F \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ is uniquely determined inside this group by only two data: the 1-jet of $F$ at 0 and the equivalence class of $\log \|DF(0)^{-1}(F - F(0))\|$ in $\mathcal{Q}^n$.

The case $n = 1$ is included in the statement of Theorem 2.1 for the sake of uniformity, but it can be checked directly. Notice that Ran($\mathcal{X}^1$) $\cong \mathbb{C} \times \mathbb{C}^* \times \text{singleton}$, a description that does not involve $\text{Aut}_{\text{hol}}(\mathbb{C})$. In particular, Ran($\mathcal{X}^n$) is a proto-moduli space of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ for all $n > 1$, and a moduli space if $n = 1$ (see the discussion below for further details.)

Conjecture. Ran($\mathcal{X}^n$) is a moduli space for all $n \geq 1$.

The central issue here is whether Ran($\mathcal{X}^n$) admits an intrinsic description when $n > 1$, in which case we would have a complete solution of the moduli problem for $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$, via the bijection

$$\text{Ran}(\mathcal{X}^n) \xrightarrow{(\mathcal{X}^n)^{-1}} \text{Aut}_{\text{hol}}(\mathbb{C}^n).$$

For illustration purposes, let us assume that $\mathcal{X}^1$ is injective, as stated in Theorem 2.1, and proceed to show that Ran($\mathcal{X}^1$) is a moduli space. Along the way, we will recover $\text{Aut}_{\text{hol}}(\mathbb{C}) = \text{Aff}(\mathbb{C})$ and provide an informal reasoning that sheds light into the conceptual question of why $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ is so much bigger in higher dimensions. As we shall see, ultimately, this dichotomy is linked to a simple property of plurisubharmonic/pluriharmonic functions (Corollary 2.3).
Since $\Delta \log |h| = 0$ for any non-vanishing holomorphic function $h$ of a single variable, the Levi matrix in (2.1) vanishes identically when $n = 1$. It then follows that $\mathcal{X}^1(F) = \mathcal{X}^1(F(0) + DF(0))$. As $\mathcal{X}^1$ was assumed to be injective, $F = F(0) + DF(0)$, and so $\text{Aut}_{\text{hol}}(\mathbb{C}) = \text{Aff}(\mathbb{C})$. Manifestly, $\text{Ran}(\mathcal{X}^1) \cong \mathbb{C} \times \mathbb{C}^* \times \{0\}$, a description independent of $\text{Aut}_{\text{hol}}(\mathbb{C})$, and so $\mathbb{C} \times \mathbb{C}^* \times \{0\}$ qualifies as a moduli space for $\text{Aut}_{\text{hol}}(\mathbb{C})$, as per our definition of moduli space. The corresponding parametrization of $\text{Aut}_{\text{hol}}(\mathbb{C})$ is

$$\text{Ran}(\mathcal{X}^1) \cong \mathbb{C} \times \mathbb{C}^* \times \{0\} \xrightarrow{(\mathcal{X}^1)^{-1}} \text{Aut}_{\text{hol}}(\mathbb{C}), \quad (b, a, 0) \rightarrow \Phi_{(b,a,0)}, \quad \Phi_{(b,a,0)}(z) = az + b.$$  

A map $F \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ lies in $\text{Aff}(\mathbb{C}^n)$ if and only if $F = F(0) + DF(0)$. From Theorem 2.1 one has

**Corollary 2.3.** An automorphism $F \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ is affine if and only if

$$\log \left( \frac{\|DF(0)^{-1}(F - F(0))\|}{\|I\|} \right)$$

is pluriharmonic away from zero.

It is only when $n = 1$ that condition (2.2) is trivially satisfied, as the norm becomes absolute value, and

$$\Delta \log \left| \frac{DF(0)^{-1}(F(z) - F(0))}{z} \right| = 0, \quad z \neq 0.$$  

Once again, one recovers $\text{Aut}_{\text{hol}}(\mathbb{C}) = \text{Aff}(\mathbb{C})$. One can think of Corollary 2.3 as providing an informal “explanation” to why $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ is much larger than $\text{Aff}(\mathbb{C}^n)$ when $n > 1$.

### 3 Proof of the embedding theorem

We write $\text{Aut}^0_{\text{hol}}(\mathbb{C}^n)$ for the subgroup of $\text{Aut}_{\text{hol}}(\mathbb{C}^n)$ consisting of those $F$ that are normalized at 0 by $F(0) = 0$ and $DF(0) = I$. The map $\Theta^n : \text{Aut}_{\text{hol}}(\mathbb{C}^n) \rightarrow \text{Aut}^0_{\text{hol}}(\mathbb{C}^n),$

$$\Theta^n(F) = DF(0)^{-1}(F - F(0)),$$

is a retraction, in the sense that the restriction of $\Theta^n$ to $\text{Aut}^0_{\text{hol}}(\mathbb{C}^n)$ is the identity map.

Also, $\Theta^n(F) = \Theta^n(G)$ if and only if $F = H \circ G$ for some $H \in \text{Aff}(\mathbb{C}^n)$. In particular, $\Theta^n$ induces a bijection between a coset space and the subgroup of normalized holomorphic
automorphisms, namely,
\[ \text{Aut}_{\text{hol}}(\mathbb{C}^n)/\text{Aff}(\mathbb{C}^n) \cong \text{Aut}_{\text{hol}}^0(\mathbb{C}^n). \]

Thus, modulo affine maps, the study of Aut_{hol}(\mathbb{C}^n) is reduced to that of Aut_{hol}^0(\mathbb{C}^n).

**Theorem 3.1.** The map \( \mathcal{X}_0^n : \text{Aut}_{\text{hol}}^0(\mathbb{C}^n) \rightarrow \mathcal{L}^n(PSH), \)
\[ \mathcal{X}_0^n(F) = L(\log \|F\|), \]
is injective for all \( n \geq 1. \)

In words, Theorem 3.1 states that two normalized self-biholomorphisms \( F \) and \( G \) of \( \mathbb{C}^n \) are equal if and only if the difference between the plurisubharmonic functions \( \log \|F\| \) and \( \log \|G\| \) is pluriharmonic away from zero.

As in Theorem 2.1, we include the case \( n = 1 \) in the statement of Theorem 3.1 for the sake of uniformity. In fact, one can check directly that both \( \text{Aut}_{\text{hol}}^0(\mathbb{C}) \) and \( \text{Ran}(\mathcal{X}_0^1) \) are singletons (i.e. the identity map of \( \mathbb{C} \) versus the zero \( 1 \times 1 \) matrix), and so the theorem holds trivially in this case.

Theorem 2.1 follows directly from Theorem 3.1. Indeed, from
\[ \mathcal{X}_0^n(F) = \left( F(0), DF(0), \mathcal{X}_0^n(\Theta^n(F)) \right) \] (3.1)
one sees that if \( \mathcal{X}_0^n \) is injective so is \( \mathcal{X}^n \) (and conversely).

To begin the proof of Theorem 3.1, assume
\[ \mathcal{X}_0^n(F) = L(\log \|F\|) = \mathcal{X}_0^n(G) = L(\log \|G\|). \]

Hence
\[ L\left( \log \frac{\|F\|}{\|G\|} \right)(z) = 0, \; z \neq 0, \]
and so \( \log \frac{\|F\|}{\|G\|} \) is pluriharmonic for \( z \neq 0. \) Because of the normalizations, \( \frac{\|F\|}{\|G\|} \) extends continuously to \( z = 0, \) with value 1. If \( n = 1 \) the harmonic function \( \log \frac{\|F\|}{\|G\|} \) is bounded on a punctured neighborhood of 0, and so the singularity is removable. In particular, \( \log \frac{\|F\|}{\|G\|} \) is the real part of an entire function \( f. \)

If \( n > 1, \) the pluriharmonic function \( \log \frac{\|F\|}{\|G\|} \) is defined on the simply-connected set \( \mathbb{C}^n - \{0\}, \) and so it is the real part of a holomorphic function on \( \mathbb{C}^n - \{0\}. \) In higher
dimensions holomorphic functions have no isolated singularities, and so one also obtains an entire function \( f \) as before.

Therefore, regardless of the dimension, one can write,

\[
\log \frac{\|F\|}{\|G\|}(z) = \text{Re} f(z), \quad \|F(z)\| = |e^{f(z)}| \|G(z)\|,
\]

with \( f : \mathbb{C}^n \to \mathbb{C} \) holomorphic. Clearly, \( \text{Re} f(0) = 0 \) and by adding a suitable purely imaginary constant, if necessary, we may assume that \( f(0) = 0 \). Composing with \( F^{-1} \) in the last equation, we have

\[
\|z\| = |e^{(f \circ F^{-1})(z)}| \|(G \circ F^{-1})(z)\|.
\]

Setting \( H = e^{(f \circ F^{-1})} (G \circ F^{-1}) \), one has \( \|H(z)\| = \|z\| \). Hence, the holomorphic map \( H \) applies the unit ball into itself, \( H(0) = 0 \) and from \( f(0) = 0 \) one has \( DH(0) = I \). It now follows from a result of Cartan \([3\), p.66\] that \( H = I \). Unwinding this condition one has

\[
G \circ F^{-1} = hI, \quad h = e^{-(f \circ F^{-1})}, \quad h(0) = 1.
\]

In order to show that \( F = G \), it suffices to show that \( h = 1 \) everywhere. It follows from \( G \circ F^{-1} = hI \) that the map \( hI \) is injective. Since the restriction of \( hI \) to the line determined by 0 and any \( z \in \mathbb{C}^n \setminus \{0\} \) is injective, so is the map \( \mathbb{C} \ni t \to h(tz)(tz) \in \mathbb{C}^n \). If \( z_i \) is a non-zero coordinate of \( z \), a short argument shows that the function \( \mathbb{C} \ni t \to h(tz)(tz_i) \in \mathbb{C} \) is injective as well. Since injective entire maps of \( \mathbb{C} \) are affine linear, it follows that \( h(tz) \) is constant as a function of \( t \), and setting \( t = 0,1 \) one has \( h(z) = h(0) = 1 \). As \( z \in \mathbb{C}^n \setminus \{0\} \) was arbitrary, \( h = 1 \) everywhere, thus concluding the proof of Theorem 3.1.

In view of (3.1), the key to solving the conjecture stated in section 2, hence the moduli problem for \( \text{Aut}^0_{\text{hol}}(\mathbb{C}^n) \), is to find an intrinsic description of \( \text{Ran}(\mathcal{X}_n) \) for \( n > 1 \).

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Departamento de Matemática, Universidade Federal de São Carlos, São Carlos, SP, Brazil.
franciscobraun@dm.ufscar.br

Department of Mathematics, Texas Christian University, Fort Worth, TX, USA.
f.j.xavier@tcu.edu