Weak Continuity of Dynamical Systems for the KdV and mKdV Equations

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Abstract

In this paper we study weak continuity of the dynamical systems for the KdV equation in $H^{-3/4}(\mathbb{R})$ and the modified KdV equation in $H^{1/4}(\mathbb{R})$. This topic should have significant applications in the study of other properties of these equations such as finite time blow-up and asymptotic stability and instability of solitary waves. The spaces considered here are borderline Sobolev spaces for the corresponding equations from the viewpoint of the local well-posedness theory. We first use a variant of the method of [5] to prove weak continuity for the mKdV, and next use a similar result for a mKdV system and the generalized Miura transform to get weak continuity for the KdV equation.

1 Introduction

The purpose of this paper is to establish weak continuity of the dynamical system $S(t)$ in the Sobolev space $H^{-3/4}(\mathbb{R})$ of the KdV equation

$$\partial_t u + \partial_x^3 u - 6u \partial_x u = 0, \quad (1.1)$$

and the dynamical system $S_1(t)$ in $H^{1/4}(\mathbb{R})$ of the defocusing mKdV equation

$$\partial_t u + \partial_x^3 u - 6u^2 \partial_x u = 0. \quad (1.2)$$

Here, by the notion that $S(t)$ is a dynamical system of the KdV equation (1.1) in some Sobolev space $H^s(\mathbb{R})$ we mean that \{S(t) : t \in \mathbb{R}\} is a family of bounded and continuous (nonlinear) operators in $H^s(\mathbb{R})$ satisfying the following three conditions:
(a) \( \{S(t) : t \in \mathbb{R}\} \) is a strongly continuous group of bounded and continuous (nonlinear) operators in the space \( H^s(\mathbb{R}) \) (so that for any \( u_0 \in H^s(\mathbb{R}) \), we have that \( [t \mapsto S(t)u_0] \in C(\mathbb{R}, H^s(\mathbb{R})) \), \( S(0)u_0 = u_0 \), and \( S(t)S(t')u_0 = S(t + t')u_0 \) for all \( t, t' \in \mathbb{R} \). Note that since \( S(t)S(-t) = S(-t)S(t) = \text{id} \), we have \( S(t)^{-1} = S(-t) \).

(b) For every \( u_0 \in H^s(\mathbb{R}) \), the function \( u \) on \( \mathbb{R}^2 \) defined by \( u(x, t) = [S(t)u_0](x) \) (for \( (x, t) \in \mathbb{R}^2 \)) is a solution (in a weak sense) of the initial value problem

\[
\begin{aligned}
\partial_t u + \partial_x^3 u - 6u\partial_x u &= 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R}.
\end{aligned}
\tag{1.3}
\]

See Remark 1.1 below for more discussions on this condition.

(c) The mapping \( u_0 \mapsto [t \mapsto S(t)u_0] \) from \( H^s(\mathbb{R}) \) to \( C(\mathbb{R}, H^s(\mathbb{R})) \) is bounded and uniformly continuous on bounded sets.

For \( S_1(t) \) the definition is similar, except for replacing (1.3) with the problem

\[
\begin{aligned}
\partial_t u + \partial_x^2 u - 6u^2\partial_x u &= 0, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
u(x, 0) &= u_0(x), \quad x \in \mathbb{R}.
\end{aligned}
\tag{1.4}
\]

Remark 1.1 We note that the condition (b) implicitly implies that if \( s \) is not sufficiently large, say, \( s < 0 \), then the function \( u(x, t) = [S(t)u_0](x) \) must have certain additional regularity beyond what a general \( C(\mathbb{R}, H^s(\mathbb{R})) \) class function possesses so that the product \( 6u\partial_x u \) or its alternative form \( 3\partial_x (u^2) \) makes sense in the distribution category. In a concrete construction of a dynamical system \( S(t) \) this is usually achieved by building the solution \( u(x, t) \) of (1.3) in a subspace of \( C(\mathbb{R}, H^s(\mathbb{R})) \) with functions which have more regularity so that the product \( 6u\partial_x u \) is meaningful in the distribution sense. Choice of such a subspace is diversified and may be different for different situations, cf., e.g., \( [1], [3], [6], [12] \) and \( [14] \).

In general, for a given partial differential equation of the evolutionary type, if we can find a function space \( X \) in the space variable and a family of operators \( \{S(t) : t \in \mathbb{R}\} \) in \( X \) satisfying similar conditions as (a)–(c) above with \( H^{-3/4}(\mathbb{R}) \) replaced by \( X \) and (1.3) replaced by the corresponding initial value problem for that equation, then we say that \( S(t) \) is a dynamical system of that equation in \( X \), and we also say that the initial value problem for that equation is globally well-posed in the space \( X \). If \( S(t) \) is not defined for all \( t \in \mathbb{R} \), but for each \( u_0 \in X \) there exists a corresponding \( T > 0 \) such that \( S(t)u_0 \) is well-defined for \( t \in (-T, T) \) and conditions similar to (a)–(c) above with obvious modifications are satisfied, then we say that equation is locally well-posed in \( X \), and in this case we call \( S(t) \) a local dynamical system.
Existence of $S(t)$ for KdV and $S_1(t)$ for mKdV in Sobolev spaces has been a goal of study for many years, cf. [1], [3], [6]–[10], [12], [14] and references therein, culminating in the following result:

**Theorem 1.2** (i) For every $s \geq -3/4$, the KdV equation (1.1) is globally well-posed in $H^s(\mathbb{R})$. Moreover, for any $s < -3/4$, the KdV equation (1.1) is not locally well-posed in $H^s(\mathbb{R})$ in the sense that the solution operator, if it exists, is not uniformly continuous on bounded sets.

(ii) For every $s \geq 1/4$, the mKdV equation (1.2) is globally well-posed in $H^s(\mathbb{R})$. Moreover, for any $s < 1/4$, the mKdV equation (1.1) is not locally well-posed in $H^s(\mathbb{R})$ in the sense that the solution operator, if it exists, is not uniformly continuous on bounded sets.

Thus, for every $s \geq -3/4$ there exists a global dynamical system of the KdV equation in $H^s(\mathbb{R})$, and for every $s \geq 1/4$ there exists a global dynamical system of the mKdV equation in $H^s(\mathbb{R})$. These dynamical systems are unique. Indeed, we have:

**Lemma 1.3** For every $s \geq -3/4$, the dynamical system of the KdV equation in $H^s(\mathbb{R})$ is unique, and for every $s \geq 1/4$, the dynamical system of the mKdV equation in $H^s(\mathbb{R})$ is unique.

**Proof:** By a standard energy estimate argument, we can easily prove that the solution of the problem (1.3) in $C(\mathbb{R}, H^s(\mathbb{R}))$ is unique provided $u_0 \in H^s(\mathbb{R})$ and $s > 3/2$. In fact, by Zhou [26] we know that this unconditional uniqueness is actually true for all $s \geq 0$. This immediately implies that the dynamical system for the KdV equation in $H^s(\mathbb{R})$ is unique provided $s \geq 0$. Since $H^\infty(\mathbb{R}) = \cap_{s \geq 0} H^s(\mathbb{R})$ is dense in $H^s(\mathbb{R})$ for any $s \in \mathbb{R}$, by using the condition (c) in the definition of dynamical systems, we easily see that, for any $-3/4 \leq s < 0$, if $S^1(t)$ and $S^2(t)$ are two dynamical systems of the KdV equation in the space $H^s(\mathbb{R})$, then $S^1(t)u_0 = S^2(t)u_0$ for all $u_0 \in H^s(\mathbb{R})$ and $t \in \mathbb{R}$, which is exactly the desired assertion. The proof for the assertion (ii) is similar, because the energy method also ensures that the solution of the problem (1.4) is also unique in $C(\mathbb{R}, H^s(\mathbb{R}))$ when $s > 3/2$. □

Thus, for every $s > -3/4$, the dynamical system of the KdV equation in $H^s(\mathbb{R})$ is the restriction of $S(t)$ in $H^{-3/4}(\mathbb{R})$ to $H^s(\mathbb{R})$, and for every $s > 1/4$, the dynamical system of the mKdV equation in $H^s(\mathbb{R})$ is the restriction of $S_1(t)$ in $H^{1/4}(\mathbb{R})$ to $H^s(\mathbb{R})$.

**Remark 1.4** We note that uniqueness of the dynamical system for an evolution equation in a Banach space $X$ does not mean uniqueness of the solution for the initial value problem of that equation in the space $C(\mathbb{R}, X)$, because it does not exclude the possibility of existence of solutions which cannot be expressed in forms of orbits of the dynamical system.
By the condition (c) we see that if $S(t)$ is a dynamical system of an evolution equation in some Sobolev space $H^s(\mathbb{R})$ then the mapping $u_0 \mapsto [t \mapsto S(t)u_0]$ from $H^s(\mathbb{R})$ to $C(\mathbb{R}, H^s(\mathbb{R}))$ is continuous. A natural question is: Is the dynamical system also weakly continuous? If the evolution equation under consideration is linear, then the answer to this question is trivially positive, because we know that every continuous linear operator in Banach spaces is also weakly continuous. Since, however, we are considering nonlinear equations, this question by no means has an obvious answer. Our motivation of asking this question is inspired by the important series of works of Martel and Merle [16]–[19], where the authors studied finite time blow-up and asymptotic stability and instability of solitary waves for the generalized KdV equations. One key step in their strategy in these works is a reduction to a nonlinear Liouville type theorem, which was further reduced into a corresponding linear one, involving the linearized operator around the solitary wave. It is in both these steps that the weak continuity of the flow map for generalized KdV in suitable Sobolev spaces plays a central role.

Recently, Kenig and Martel [11] studied the asymptotic stability of solitons for the Benjamin-Ono equation in $H^{1/2}(\mathbb{R})$, following the program initiated by Martel and Merle for the generalized KdV. Thus, a key step in [11] is to establish weak continuity of the dynamical system of the BO in $H^{1/2}(\mathbb{R})$. The proof is very simple and reduces matters to the uniform continuity of the dynamical system in spaces of strictly smaller indices. This reduction relies on the fact that BO is well-posed in $L^2(\mathbb{R})$ (cf. [7]). The same method shows this weak continuity for KdV in $H^s(\mathbb{R})$ with $s > -3/4$ and mKdV in $H^s(\mathbb{R})$ with $s > 1/4$. We can then ask if the dynamical systems $S(t)$ of KdV in $H^{-3/4}(\mathbb{R})$ and $S_1(t)$ of the mKdV in $H^{1/4}(\mathbb{R})$ are weakly continuous? Note that since the KdV and mKdV equations do not have a uniformly continuous flow map when restricted to bounded sets in the spaces $H^s(\mathbb{R})$ with $s < -3/4$ and $s < 1/4$, respectively, the approach used in [11] does not work in these critical cases.

Weak continuity of dynamical systems in critical Sobolev spaces which are critical from the viewpoint of local well-posedness was first studied by Goubet and Molinet in the reference [5], where the cubic nonlinear Schrödinger equation on the line was studied. For this equation the global well-posedness in $L^2(\mathbb{R})$ was established in [25], while in [15] (focusing case) and [2] (defocusing case) it was shown that the flow map is not uniformly continuous in any Sobolev space of negative index. Thus, the weak continuity in $L^2(\mathbb{R})$ of the flow map cannot be treated by the approach reviewed in the above paragraph. Goubet and Molinet [5] affirmatively settled this problem by taking advantage of the “local smoothing” effect estimates together with a suitable uniqueness result.

We would also like mention two recent interesting preprints by L. Molinet [20, 21], which disprove the weak continuity of the flow maps in $L^2(\mathbb{T})$ for both the cubic Nonlinear Schrödinger equation and the Benjamin-Ono equation, though we know that the initial
value problems of these equations are globally well-posed in $L^2(\mathbb{T})$.

In this paper we give a positive answer to the weak continuity question for KdV and mKdV. More precisely, the main purpose of this paper is to prove the following results:

**Theorem 1.5** The dynamical system $S(t)$ of the KdV equation (1.1) in $H^{-3/4}(\mathbb{R})$ is weakly continuous for any fixed $t \in \mathbb{R}$. In fact, we have the following stronger assertion: Assume that $u_{0n} \in H^{-3/4}(\mathbb{R})$ $(n = 1, 2, \cdots)$ and $u_{0n} \to u_0$ weakly in $H^{-3/4}(\mathbb{R})$ as $n \to \infty$. Let $u_n(x, t) = [S(t)u_{0n}](x)$ $(n = 1, 2, \cdots)$ and $u(x, t) = [S(t)u_0](x)$. Then for any $T > 0$ and any $\varphi \in H^{-3/4}(\mathbb{R})$ we have

$$\lim_{n \to \infty} \sup_{|t| \leq T} |(u_n(\cdot, t) - u(\cdot, t), \varphi)_{H^{-3/4}(\mathbb{R})}| = 0. \quad (1.5)$$

**Theorem 1.6** The dynamical system $S_1(t)$ of the mKdV equation (1.2) in $H^{1/4}(\mathbb{R})$ is weakly continuous for any fixed $t \in \mathbb{R}$. In fact, we have also the following stronger assertion: Assume that $u_{0n} \in H^{1/4}(\mathbb{R})$ $(n = 1, 2, \cdots)$ and $u_{0n} \to u_0$ weakly in $H^{1/4}(\mathbb{R})$ as $n \to \infty$. Let $u_n(x, t) = [S_1(t)u_{0n}](x)$ $(n = 1, 2, \cdots)$ and $u(x, t) = [S_1(t)u_0](x)$. Then for any $T > 0$ and any $\varphi \in H^{1/4}(\mathbb{R})$ we have

$$\lim_{n \to \infty} \sup_{|t| \leq T} |(u_n(\cdot, t) - u(\cdot, t), \varphi)_{H^{1/4}(\mathbb{R})}| = 0. \quad (1.6)$$

As in [5] and [4], we shall use some compactness arguments together with suitable uniqueness results to prove the above results. The proof of Theorem 1.6 is easier than that of Theorem 1.5. The idea of the proof of Theorem 1.6 (following [5] in a simplified situation) is as follows: If a sequence of solutions $\{u_n\}$ of the equation (1.2) is bounded in $C([-T, T], H^{1/4}(\mathbb{R}))$, then $\{\partial_t u_n\}$ is bounded in $C([-T, T], H^{-11/4}(\mathbb{R}))$, so that $\{u_n\}$ has a subsequence which is strongly convergent in $L^2([-R, R] \times [-T, T])$. By this fact and a certain uniqueness result, the desired conclusion follows. See Section 3 for details of the proof. This argument clearly does not apply to the equation (1.1) (because here we deal with Sobolev spaces of negative index). Thus, to prove Theorem 1.5 we shall appeal to the generalized Miura transform introduced by Christ, Colliander and Tao in [2] to reduce the problem into the corresponding problem for a mKdV system, for which the above argument applies. See Section 4 for details.

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2 Review of proofs of Theorem 1.2 and the Miura transform

In order to prove Theorems 1.5 and 1.6, we need to have a basic knowledge about the proofs of Theorem 1.2 and the Miura transform. In this section we recall these materials.

Global well-posedness of the KdV initial value problem (1.3) in the Sobolev space $H^{-3/4}(\mathbb{R})$ was established recently by Guo [6] in the framework of the function space $\tilde{F}^s$ ($s \geq -3/4$), which is a dyadic Bourgain-type space with modifications in the low frequency part of functions by considering the smoothing effect estimate of the Airy equation. Similar spaces of this type have previously been used by some other authors, cf. [7], [23], [24] and references therein. In [6] the author first used a contraction mapping argument in the space $\tilde{F}^{-3/4}$ to get local well-posedness of (1.3) in $H^{-3/4}(\mathbb{R})$, and next he used the $I$-operator introduced by Colliander et al in [3] to establish almost conservation of a modified energy quantity which ensures that the local solution can be extended into a global one. The function space $\tilde{F}^s$ ($s \geq -3/4$) is defined as follows. Let $\eta_0 : \mathbb{R} \rightarrow [0,1]$ denote an even function supported in $[-8/5,8/5]$ and equal to 1 in $[-5/4,5/4]$. For $k \in \mathbb{Z}$, $k \geq 1$, let $\eta_k(\xi) = \eta_0(2^{-k}\xi) - \eta_0(2^{-(k+1)}\xi)$. We also denote, for all $k \in \mathbb{Z}$, $\chi_k(\xi) = \eta_0(2^{-k}\xi) - \eta_0(2^{-k+1}\xi)$. It follows that

$$\sum_{k=0}^{\infty} \eta_k(\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R},$$

and

$$\sum_{k=-\infty}^{\infty} \chi_k(\xi) = 1 \quad \text{for} \quad \xi \in \mathbb{R}\backslash\{0\}.$$ 

Note that $\text{supp} \chi_k \subseteq [-((8/5)2^k, -((5/8)2^k)] \cup [(5/8)2^k, (8/5)2^k]$ for all $k \in \mathbb{Z}$, and $\text{supp} \eta_k \subseteq [-(8/5)2^k, -(5/8)2^k] \cup [(5/8)2^k, (8/5)2^k]$ for $k \geq 1$. For $k \in \mathbb{N}$ we denote

$$I_k = [-2^{k+1}, -2^{k-1}] \cup [2^{k-1}, 2^{k+1}],$$

and let $X_k$ be the function space

$$X_k = \{ f \in L^2(\mathbb{R} \times \mathbb{R}) : f \text{ supported in } I_k \times \mathbb{R} \text{ and } \| f \|_{X_k} := \sum_{j=0}^{\infty} 2^{j/2} \| \eta_j(\tau - \xi^3) f(\xi, \tau) \|_{L^2_{\xi, \tau}} < \infty \}.$$
The function space $\bar{F}^s$ is defined as follows:

$$\bar{F}^s = \{ u \in S'(\mathbb{R} \times \mathbb{R}) : \| u \|_{F^s}^2 := \sum_{k=1}^{\infty} 2^{2sk} \| \eta_k(\xi) \tilde{u}(\xi, \tau) \|_{L^2_{\tau}}^2 + \| \mathcal{F}[\eta_0(\xi) \tilde{u}(\xi, \tau)] \|_{L^2_{\tau}L^\infty_{\tau}}^2 < \infty \},$$

where $\tilde{u} = \mathcal{F}(u)$ represents Fourier transform of $u$ (in two variables). It can be easily shown that

$$\bar{F}^{-3/4} \subseteq C(\mathbb{R}, H^{-3/4}(\mathbb{R})) \cap L^\infty_x(\mathbb{R}, L^2_{t,\text{loc}}(\mathbb{R})) \subseteq C(\mathbb{R}, H^{-3/4}(\mathbb{R})) \cap L^2_{\text{loc}}(\mathbb{R}^2),$$

so that for any $u \in \bar{F}^{-3/4}$, $u^2$ makes sense. We let $\bar{F}^s_T$ be the restriction of $\bar{F}^s$ on $\mathbb{R} \times [-T, T]$, i.e., $u \in \bar{F}^s_T$ if and only if there exists $w \in \bar{F}^s$ such that $w|_{\mathbb{R} \times [-T, T]} = u$, and the norm

$$\| u \|_{F^s_T} := \inf\{ \| w \|_{F^s} : w \in \bar{F}^s, w|_{\mathbb{R} \times [-T, T]} = u \}.$$

By using the method of first establishing a bilinear estimate in the space $\bar{F}^s$ to get a local solution and next using the $I$-operator to prove that the norm $\| u(\cdot, t) \|_{H^{-3/4}(\mathbb{R})}$ grows only polynomially fast so that it cannot blow-up in finite time, Guo [6] proved the following result:

**Theorem 2.1** There exists a bounded and locally Lipschitz continuous mapping $\Psi : H^{-3/4}(\mathbb{R}) \to C(\mathbb{R}, H^{-3/4}(\mathbb{R}))$ such that (i) for any $u_0 \in H^{-3/4}(\mathbb{R})$, $t_0 \in \mathbb{R}$ and $T > 0$, the function $u = \Psi(u_0)$ belongs to $\bar{F}^{-3/4}_T$ when restricted to $\mathbb{R} \times [t_0 - T, t_0 + T]$, and (ii) $u$ is a solution (in distribution sense) of the initial value problem (1.3), and it is the unique solution of (1.3) satisfying the property ensured by (i). Moreover, there exists a constant $C > 0$ such that

$$\| u(\cdot, t) \|_{H^{-3/4}(\mathbb{R})} \leq C(1 + |t|) \| u_0 \|_{H^{-3/4}(\mathbb{R})} \text{ for all } t \in \mathbb{R}. \quad (2.1)$$

Combining this result with the global well-posedness of (1.3) in $H^s(\mathbb{R})$ for $s > -3/4$ (cf [3] and [14]) and the result of [2] which states that the solution operator (if it exists) of (1.3) is not locally uniformly continuous in $H^s(\mathbb{R})$ for $s < -3/4$, we see that the assertion (i) of Theorem 1.2 follows.

However, we are unable to directly use Theorem 2.1 to prove Theorem 1.5 by following the approach of [4] and [5]. The reason is that, though we are able to establish estimates of the form

$$\| D^\theta_x u \|_{L^\infty_x L^2_{\tau}} \leq C \| u \|_{\bar{F}^{-3/4}_T}$$

for $0 \leq \theta < 1/4$, unfortunately we are unable to get an integral estimate for $D^\theta_x u$ by $\| u \|_{\bar{F}^{-3/4}_T}$ even locally in both $x$ and $t$, no matter how small that $\theta > 0$ is, and even worse, the equation (1.1) does not seem to help for a such estimate (these are crucial...
techniques used in [4] and [5]). To prove Theorem 1.5 we shall have to appeal to the so-called \textit{generalized Miura transform}, which we shall recall later in this section.

We now turn our attention to the mKdV equation (1.2). Local well-posedness of the problem (1.4) in $H^s(\mathbb{R})$ with $s \geq 1/4$ was established by Kenig, Ponce and Vega in [12]. For our purpose we recall this result in the critical case $s = 1/4$. For any time interval $I = [t_0, t_0 + T]$, let $X = X(\mathbb{R} \times I)$ denote the function space defined as follows: We first introduce a norm $\| \cdot \|_{X(\mathbb{R} \times I)}$ for measurable functions $u$ on $\mathbb{R} \times I$:

$$\|u\|_{X(\mathbb{R} \times I)} := \|u\|_{L^\infty_t L^{1/4}_x} + \|u\|_{L^1_t L^\infty_x} + \|\partial_x u\|_{L^1_t L_2^5_x} + \|\partial_x J^{1/4}_x u\|_{L^{\infty}_t L^2_x}.$$

Next we define

$$X(\mathbb{R} \times I) = \{ u \in C(I, H^{1/4}(\mathbb{R})) : \|u\|_{X(\mathbb{R} \times I)} < \infty \}.$$

In [12], it was proved that for any $u_0 \in H^{1/4}(\mathbb{R})$ there exists corresponding $T = T(\|u_0\|_{H^{1/4}(\mathbb{R})}) > 0$ such that the problem (1.4) has a unique solution in the space $X(\mathbb{R} \times I)$. Thus, the dynamical system $S_1(t)$ of the mKdV equation (1.2) in $H^{1/4}(\mathbb{R})$ is well-defined at least locally, and $S_1(t_0)u_0 \in X(\mathbb{R} \times [0, T])$ for any $u_0 \in H^{1/4}(\mathbb{R})$, where $T = T(\|u_0\|_{H^{1/4}(\mathbb{R})})$. To show that $S_1(t)$ is actually defined for all $t \in \mathbb{R}$ we need the \textit{Miura transform} $v = M(u)$, which is defined by

$$v = \partial_x u + u^2. \quad (2.2)$$

Indeed, by an argument of Colliander, Keel, Staffilani, Takaoka and Tao [3], the Miura transform can be used to prove global well-posedness of the mKdV equation in $H^{1/4}(\mathbb{R})$ from that of the KdV equation in $H^{-3/4}(\mathbb{R})$ ensured by Theorem 2.1. For our purpose we review a few more details of this argument in the following paragraphs.

We first write:

**Lemma 2.2** For any $s \geq 0$, the Miura transform $M$ is a bounded and continuous mapping from $H^s(\mathbb{R})$ to $H^{s-1}(\mathbb{R})$, and it is Lipschitz continuous when restricted to any bounded set in $H^s(\mathbb{R})$. Moreover, if $s \geq 1/4$ then it is injective, and for any subset $S$ of $H^s(\mathbb{R})$, if $S$ is bounded in $L^2(\mathbb{R})$ and $M(S)$ is bounded in $H^{s-1}(\mathbb{R})$, then $S$ is bounded in $H^s(\mathbb{R})$, or more precisely, there exists constant $C > 0$ such that

$$\|u\|_{H^s} \leq C(\|M(u)\|_{H^{s-1}} + \|u\|_2^2) \quad (2.3)$$

for all $u \in H^s(\mathbb{R})$.

**Proof**: From the proofs of Lemma 9.1 in [3] and Lemma 9.1 in [2], we easily see that for any $0 \leq s < 1$ and $u_1, u_2 \in H^s(\mathbb{R})$, the following inequality holds:

$$\|M(u_1) - M(u_2)\|_{H^{s-1}(\mathbb{R})} \leq C(\|u_1\|_{H^s(\mathbb{R})} + \|u_2\|_{H^s(\mathbb{R})}) \|u_1 - u_2\|_{H^s(\mathbb{R})}.$$
Since $H^s(\mathbb{R})$ is an algebra when $s > 1/2$, the above inequality is trivially true for any $s \geq 1$. Thus, for any $s \geq 0$, $M$ is a bounded and continuous mapping from $H^s(\mathbb{R})$ to $H^{s-1}(\mathbb{R})$, and it is Lipschitz continuous when restricted to any bounded set in $H^s(\mathbb{R})$. The assertion that $M$ is injective when $s \geq 1/4$ follows from Lemma 2.3 below, and the last assertion follows from the proof of Lemma 9.2 in [3]. □

**Lemma 2.3** Let $s \geq 1/4$, $a \in L^4(\mathbb{R})$ and $w \in H^s(\mathbb{R})$. Assume that

$$w'(x) + a(x)w(x) = 0 \text{ for } x \in \mathbb{R}$$

(in distribution sense). Then $w = 0$.

**Proof:** We fist note that $w \in H^s(\mathbb{R})$ and $s \geq 1/4$ implies that $w \in L^4(\mathbb{R})$. Thus, $aw \in L^2(\mathbb{R})$. To prove $w = 0$ we only need to show that for any $\varphi \in C_0^\infty(\mathbb{R})$,

$$\int_{-\infty}^{\infty} w(x)\varphi(x)dx = 0.$$ 

Let $\varphi_0 \in C_0^\infty(\mathbb{R})$ be such that $\varphi_0(x) = 1$ for $|x| \leq 1$, $\varphi_0(x) = 0$ for $|x| \geq 2$, and $0 \leq \varphi_0 \leq 1$. Set $\varphi_n(x) = \varphi_0(x/n)$, $n = 1, 2, \cdots$. Then $\|\varphi_n\|_\infty = 1$, $n = 1, 2, \cdots$, and $\|\varphi'_n\|_2 \leq Cn^{-1/2} \to 0$ as $n \to \infty$. Given $\varphi \in C_0^\infty(\mathbb{R})$, let

$$\psi(x) = \int_{-\infty}^{x} \varphi(y)e^{\int_y^{x}a(t)dt}dy \quad \text{and} \quad \psi_n(x) = \varphi_n(x)\psi(x), \quad n = 1, 2, \cdots.$$ 

We have that $\psi \in L^\infty(\mathbb{R}) \cap H^1_{loc}(\mathbb{R})$, $\psi_n \in H^1_c(\mathbb{R})$ (i.e., $\psi_n \in H^1(\mathbb{R})$ and has compact support), $n = 1, 2, \cdots$, and

$$\psi'_n - a\psi_n = \varphi_n(\psi' - a\psi) + \varphi'_n\varphi = \varphi_n\varphi + \varphi'_n\psi \to \varphi \in L^2(\mathbb{R})$$

(as $n \to \infty$). Hence

$$\int_{-\infty}^{\infty} w(x)\varphi(x)dx = \lim_{n \to \infty} \int_{-\infty}^{\infty} w(x)[\psi'_n(x) - a(x)\psi_n(x)]dx
=- \lim_{n \to \infty} \int_{-\infty}^{\infty} [w'(x) + a(x)w(x)]\psi_n(x)dx = 0.$$

□

As well-known, if $u$ is a solution of (1.2) then its Miura transform $v = M(u)$ is a solution of the KdV equation (1.1). Thus, by using a similar argument as in the proof of Lemma 1.3 we conclude that

$$M(S_1(t)u_0) = S(t)(M(u_0))$$

for any $u_0 \in H^{1/4}(\mathbb{R})$ and any $t \in \mathbb{R}$ such that $S_1(t)u_0$ is well-defined (note that $S(t)u_0$ is well-defined for all $t \in \mathbb{R}$ whereas so far we only know that $S_1(t)u_0$ is well-defined for
small $|t|$. Using this relation, the $L^2$ conservation law for the mKdV equation, the growth estimate (2.1) for solutions of the KdV equation, and Lemma 2.2, we can easily infer that for any $u_0 \in H^{1/4}(\mathbb{R})$, the solution $u = S_1(t)u_0$ of the initial value problem (1.4) satisfies a similar growth estimate as (2.1) in its existence time interval, so that it cannot blow-up in finite time. Thus, the problem (1.4) is globally well-posed and the dynamical system $S_1(t)$ is well-defined for all $t \in \mathbb{R}$. We thus have the following result which is implicitly stated in [6]:

**Theorem 2.4** There exists a bounded and locally Lipschitz continuous mapping $\Psi_1 : H^{1/4}(\mathbb{R}) \to C(\mathbb{R}, H^{1/4}(\mathbb{R}))$ such that for any $u_0 \in H^{1/4}(\mathbb{R})$, the function $u = \Psi_1(u_0)$ is a solution (in distribution sense) of the initial value problem (1.4), and it defines a dynamical system $S_1(t)$ in $H^{1/4}(\mathbb{R})$ of the mKdV equation (1.2). Moreover, there exists a constant $C > 0$ such that

$$
\|u(\cdot,t)\|_{H^{1/4}(\mathbb{R})} \leq C(1 + |t|)\|u_0\|_{H^{1/4}(\mathbb{R})} \quad \text{for all } t \in \mathbb{R}. 
$$

□

In Section 4 we shall use this theorem to prove Theorem 1.6.

The Miura transform is not a surjection from $H^{1/4}(\mathbb{R})$ to $H^{-3/4}(\mathbb{R})$, cf. [2]. Thus, we cannot use the relation (2.4) and the weak continuity of $S_1(t)$ to get weak continuity of $S(t)$. In order to prove Theorem 1.4 we shall use a generalized version of the Miura transform — the generalized Miura transform introduced by Christ, Colliander and Tao [2], which is the mapping $(v, w) \mapsto u = M_g(v, w)$ defined by

$$
u = \partial_x v + v^2 + w. \quad (2.6)
$$

It can be easily verified that if $(v, w)$ is a solution of the initial value problem

$$
\begin{align*}
\partial_t v + \partial_x^2 v &= 6(v^2 + w)\partial_x v, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
\partial_t w + \partial_x^2 w &= 6(v^2 + w)\partial_x w, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}, \\
v(x, 0) &= v_0(x), \quad x \in \mathbb{R}, \\
w(x, 0) &= w_0(x), \quad x \in \mathbb{R},
\end{align*}
$$

(2.7)

and $u_0 = v'_0 + v_0^2 + w_0$, then $u = M_g(v, w)$ is a solution of the problem (1.3). Using Lemma 2.2, we see immediately that $M_g$ maps $H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$ into $H^{-3/4}(\mathbb{R})$, and it is a bounded and locally Lipschitz continuous mapping. An important feature of $M_g$ is that it is a surjection from $H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$ to $H^{-3/4}(\mathbb{R})$. More precisely, we have:

**Lemma 2.5** For any $A > 0$ there exists a Lipschitz continuous mapping $W_A : H^{-3/4}(\mathbb{R}) \to H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$ such that $M \circ W_A = \text{id}$ when restricted to the ball $B_A = \{ u \in H^{-3/4}(\mathbb{R}) : \|u\|_{H^{-3/4}} \leq A \}$.

**Proof:** See Lemma 10.1 of [2]. □
In [2] it was proved that the initial value problem (2.6) is locally well-posed in $H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$. Since we shall use this result later on, in the sequel we review some details of its proof.

Let $\chi_k (k = 0, \pm 1, \pm 2, \cdots)$ be the functions introduced in the beginning of this section, and let 

$$P_k u = F^{-1}(\chi_k(\xi) \hat{u}(\xi)) \quad \text{for} \quad u \in S'(\mathbb{R})$$

$(k = 0, \pm 1, \pm 2, \cdots)$. Given a time interval $I = [t_0, t_0 + T]$, let $X = X(\mathbb{R} \times I)$ be as before, and define the space $X^* = X^*(\mathbb{R} \times I)$ by setting the norm 

$$||u||_{X^*} = ||u||_X + \left( \sum_{k=-\infty}^{\infty} ||P_k u||^2_X \right)^{1/2}. \quad (2.8)$$

Next we define the space $X^{**} = X^{**}(\mathbb{R} \times I)$ for vector functions $(v, w)$ by setting the norm 

$$||(v, w)||_{X^{**}} = ||v||_{X^*} + ||w||_{X^*}. \quad (2.9)$$

Since $X \subseteq C(I, H^{1/4}(\mathbb{R}))$ and the embedding mapping is continuous, we see easily that $X^{**} \subseteq C(I, H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R}))$, and the embedding mapping is continuous. The local well-posedness result for the problem (2.7) is as follows:

**Proposition 2.6** Let $t_0 = 0$. For any $(v_0, w_0) \in H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$ there exists $T = T(\|v_0\|_{H^{1/4}(\mathbb{R})}, \|w\|_{H^1(\mathbb{R})}) > 0$, such that the problem (2.7) has a unique solution $(v, w)$ in the space $X^{**}$, and the mapping $(v_0, w_0) \mapsto (v, w)$ from $H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$ to $X^{**}$ is locally Lipschitz continuous.

**Proof:** See Proposition 1 in [2] and its proof. \qed

By Proposition 2.6, it follows that there exists a local dynamical system $S^{**}(t)$ in $H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$ for the system of equations (2.7). Since the generalized Miura transform $M_g$ maps a solution of (2.7) into a solution of (1.3) with $u_0 = v'_0 + v_0^2 + w_0$, by a similar argument as in the proof of Lemma 2.2 it follows that 

$$M_g[S^{**}(t)(v_0, w_0)] = S(t)M_g(v_0, w_0) \quad (2.10)$$

for any $(v_0, w_0) \in H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$ and any $t \in \mathbb{R}$ such that $S^{**}(t)(v_0, w_0)$ makes sense. In Section 5 we shall use this relation and Proposition 2.6 to prove Theorem 1.5.

### 3 Proof of Theorem 1.6

In this section we give the proof of Theorem 1.6.
We denote $I = [-T, T]$. Let $u_{0n} \in H^{1/4}(\mathbb{R})$, $n = 1, 2, \ldots$, and $u_0 \in H^{1/4}(\mathbb{R})$ be such that $u_{0n} \to u_0$ weakly in $H^{1/4}(\mathbb{R})$ as $n \to \infty$. Then there exists constant $M > 0$ such that

$$\|u_{0n}\|_{H^{1/4}} \leq M, \quad n = 1, 2, \ldots, \quad \text{and} \quad \|u_0\|_{H^{1/4}} \leq M. \quad (3.1)$$

Let $u_n(x, t) = [S_1(t)u_{0n}](x), n = 1, 2, \ldots$, and $u(x, t) = [S_1(t)u_0](x)$. Let $T > 0$ be given, and set $M_1 = C(1 + T)M$, where $C$ is the constant appearing in (2.5). Then we have

$$\|u_n(\cdot, t)\|_{H^{1/4}} \leq M_1, \quad n = 1, 2, \ldots, \quad \text{and} \quad \|u(\cdot, t)\|_{H^{1/4}} \leq M_1 \quad (3.2)$$

for all $t \in I$. Using the equation (1.4), we further obtain

$$\|\partial_t u_n(\cdot, t)\|_{H^{-11/4}} \leq M_2, \quad n = 1, 2, \ldots, \quad \text{and} \quad \|\partial_t u(\cdot, t)\|_{H^{-11/4}} \leq M_2 \quad (3.3)$$

for all $t \in I$. Indeed, by (1.4) we have

$$\|\partial_t u_n(\cdot, t)\|_{H^{-11/4}} \leq \|\partial_t^3 u_n(\cdot, t)\|_{H^{-11/4}} + 2\|\partial_x u_n^3(\cdot, t)\|_{H^{-11/4}} \leq \|u_n(\cdot, t)\|_{H^{1/4}} + C\|u_n^3(\cdot, t)\|_{H^{-5/6}}. \quad (3.4)$$

Since $H^{1/4}(\mathbb{R}) \subseteq L^4(\mathbb{R}), L^{4/3}(\mathbb{R}) \subseteq H^{-5/6}(\mathbb{R})$, and the embeddings are continuous, we have

$$\|u_n^3(\cdot, t)\|_{H^{-5/6}} \leq C\|u_n^3(\cdot, t)\|_{L^{4/3}} = C\|u_n(\cdot, t)\|_{L^4}^3 \leq C\|u_n(\cdot, t)\|_{H^{1/4}}^3. \quad (3.5)$$

Hence

$$\|\partial_t u_n(\cdot, t)\|_{H^{-11/4}} \leq \|u_n(\cdot, t)\|_{H^{1/4}} + C\|u_n(\cdot, t)\|_{H^{1/4}}^3 \leq M_2$$

for all $t \in I$. The proof of the last inequality in (3.3) is similar. We note that (3.2) also implies that for any $2 \leq p \leq 4$,

$$\|u_n\|_{L^p(\mathbb{R} \times I)} \leq M_p, \quad n = 1, 2, \ldots \quad (3.6)$$

In addition, by the local well-posedness result for the problem (1.4) that we reviewed in Section 2, from (3.2) we can also get the following estimate

$$\|u_n\|_{X(\mathbb{R} \times I)} \leq C(T, M_1), \quad n = 1, 2, \ldots \quad (3.7)$$

We note that to get this estimate we need to divide the interval $I$ into small subintervals, with the number of them depending only on $T$ and $M_1$.

By (3.2), (3.3) and a standard compactness result, it follows that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ and a function $u' \in L^2_{\text{loc}}(\mathbb{R} \times I)$, such that for any $R > 0$,

$$u_{n_k} \to u' \quad \text{strongly in} \quad L^2([-R, R] \times I).$$

This further implies, after passing to a subsequence when necessary, that

$$u_{n_k} \to u' \quad \text{almost everywhere in} \quad \mathbb{R} \times I. \quad (3.8)$$
By (3.4), we have \( u' \in L^p(\mathbb{R} \times I) \) for any \( 2 \leq p \leq 4 \), and by (3.5), we also have \( u' \in \tilde{X}(\mathbb{R} \times I) \), where \( \tilde{X}(\mathbb{R} \times I) \) denotes the function space of all measurable functions \( u \) on \( \mathbb{R} \times I \) such that \( \| u \|_{\tilde{X}(\mathbb{R} \times I)} < \infty \). The last assertion is ensured by the fact that \( \tilde{X}(\mathbb{R} \times I) \) is a \( L^\infty \)-type space, i.e., it is the dual of a separable Banach space.

From (3.4) we see that \( \{ u_n \} \) is bounded in \( L^p(\mathbb{R} \times I) \), \( 2 \leq p \leq 4 \), which further implies that

\[
\{ u_n^3 \} \text{ is bounded in } L^p(\mathbb{R} \times I), \quad 1 \leq p \leq \frac{4}{3}.
\]

Hence, by using the Vitali convergence theorem (see Corollary A.2 of [1]), we infer from (3.6)–(3.8) that for any finite \( R > 0 \),

\[
u_{nk} \to u' \text{ strongly in } L^p([-R,R] \times I), \quad 1 \leq p < 4,
\]

\[
u_{nk}^3 \to (u')^3 \text{ strongly in } L^p([-R,R] \times I), \quad 1 \leq p < \frac{4}{3}.
\]

From (3.7)–(3.10) and the density of \( C_0^\infty(\mathbb{R} \times (-T,T)) \) in \( (L^p(\mathbb{R} \times I))' = L^{p'}(\mathbb{R} \times I) \) for \( 1 < p < \infty \), we deduce that

\[
u_{nk} \to u' \text{ weakly in } L^p(\mathbb{R} \times I), \quad 1 < p < 4,
\]

\[
u_{nk}^3 \to (u')^3 \text{ weakly in } L^p(\mathbb{R} \times I), \quad 1 < p < \frac{4}{3}.
\]

Thus, by letting \( k \to \infty \) in the equation

\[
u_{nk}(\cdot,t) = W(t)u_{0nk} + 2\partial_x \int_0^t W(t-t')u_{nk}^3(\cdot,t')dt', \quad k = 1, 2, \ldots,
\]

we see that \( u' \) satisfies the equation

\[
u'(\cdot,t) = W(t)u_0 + 2\partial_x \int_0^t W(t-t')(u')^3(\cdot,t')dt'.
\]

Thus, since \( u' \in \tilde{X}(\mathbb{R} \times I) \) and, from the proofs of Theorems 2.3 and 2.4 of [12] we see that the solution of this equation in \( \tilde{X}(\mathbb{R} \times I) \) is unique, we conclude that \( u' = u \). Thus, we have proved that \( \{ u_n \} \) has a subsequence \( \{ u_{nk} \} \) converging to \( u \) almost everywhere in \( \mathbb{R} \times I \). Since the above argument works when the sequence \( \{ u_n \} \) is replaced by any of its subsequences, it follows that the following assertion holds:

\[
u_n \to u \quad \text{almost everywhere in } \mathbb{R} \times I.
\]
As a consequence of this assertion, (4.10) (arbitrarily take $8/7 < p \leq 4/3$) and the Vitali convergence theorem, we see that also the following assertion holds:

\[ u_n^3 \to u^3 \quad \text{strongly in } L^{8/7}([-R, R] \times I) \quad \text{for any } R > 0. \]  

(3.12)

Next, by (3.1), (3.2) and the density of $S(\mathbb{R})$ in $H^{1/4}(\mathbb{R})$, we see that in order to prove Theorem 1.6 it suffices to prove that (1.6) holds for any $\varphi \in S(\mathbb{R})$. Since

\[ (u, \varphi)_{H^{1/4}} = (u, \psi)_{L^2}, \quad \forall \varphi \in S(\mathbb{R}), \]

where $\psi = \mathcal{F}^{-1}[(1 + |\xi|^2)^{1/4}\mathcal{F}_1(\varphi)] \in S(\mathbb{R})$, it follows that in order to prove that (1.6) holds for any $\varphi \in S(\mathbb{R})$, it suffices to prove that the following holds for any $\varphi \in S(\mathbb{R})$:

\[ \lim_{n \to \infty} \sup_{|t| \leq T} |(u_n(\cdot, t) - u(\cdot, t), \varphi)_{L^2}| = 0. \]  

(3.13)

Let

\[ v_n(\cdot, t) = W(t)u_{n0}, \quad w_n(\cdot, t) = 2\partial_x \int_0^t W(t - t')u_n^3(\cdot, t')dt', \quad n = 1, 2, \ldots, \]

\[ v(\cdot, t) = W(t)u_0, \quad w(\cdot, t) = 2\partial_x \int_0^t W(t - t')u^3(\cdot, t')dt'. \]

Then $u_n(\cdot, t) - u(\cdot, t) = [v_n(\cdot, t) - v(\cdot, t)] + [w_n(\cdot, t) - w(\cdot, t)]$, $n = 1, 2, \ldots$. It can be easily shown that (cf. the proof of Assertion 2 in Section 2.3 of [H])

\[ \lim_{n \to \infty} \sup_{|t| \leq T} |(v_n(\cdot, t) - v(\cdot, t), \varphi)_{L^2}| = 0 \]

for any $\varphi \in S(\mathbb{R})$. Thus (3.13) follows if we prove that

\[ \lim_{n \to \infty} \sup_{|t| \leq T} |(w_n(\cdot, t) - w(\cdot, t), \varphi)_{L^2}| = 0 \]  

(3.14)

for any $\varphi \in S(\mathbb{R})$. Let

\[ z(\cdot, t) = \int_0^t W(t - t')u^3(\cdot, t')dt' \quad \text{and} \quad z_n(\cdot, t) = \int_0^t W(t - t')u_n^3(\cdot, t')dt' \]

\[ (n = 1, 2, \ldots). \]  

Since $w_n(\cdot, t) - w(\cdot, t) = 2\partial_x[z_n(\cdot, t) - z(\cdot, t)]$, $n = 1, 2, \ldots$, we see that (3.14) follows if we prove that

\[ \lim_{n \to \infty} \sup_{|t| \leq T} |(z_n(\cdot, t) - z(\cdot, t), \varphi)_{L^2}| = 0. \]  

(3.15)

The proof of this relation follows from a similar argument as in the proof of Assertion 3 in Section 2.3 of [H]. Indeed, let $\chi_R$ be the characteristic function of the interval $[-R, R]$ (for $x$ variable), and denote

\[ z_n^{(1)}(\cdot, t) = \int_0^t W(t - t')(\chi_Ru_n^3(\cdot, t'))dt', \quad z_n^{(2)}(\cdot, t) = \int_0^t W(t - t')((1 - \chi_R)u_n^3(\cdot, t'))dt', \]

14
\[
    z^{(1)}(\cdot, t) = \int_0^t W(t - t')(\chi_R u^3(\cdot, t'))dt', \quad z^{(2)}(\cdot, t) = \int_0^t W(t - t')(1 - \chi_R)u^3(\cdot, t'))dt'.
\]

Then we have
\[
    (z_n(\cdot, t) - z(\cdot, t), \varphi)_{L^2} = \int_0^\infty [z_n(x, t) - z(x, t)] \varphi(x) dx
\]
\[
    = \int_{-\infty}^\infty [z_n^{(1)}(x, t) - z^{(1)}(x, t)] \varphi(x) dx + \int_{-\infty}^\infty [z_n^{(2)}(x, t) - z^{(2)}(x, t)] \varphi(x) dx
\]
\[
    \equiv I_n^R(t) + J_n^R(t).
\]

Let
\[
    f_n^R(x, t) = \chi_R(x)u^3_n(x, t), \quad f^R(x, t) = \chi_R(x)u^3(x, t).
\]

Then by the Cauchy inequality and the inhomogeneous Strichartz estimate we have (note that 8/7 is the dual of 8 and (2, \infty), (8, 8) are admissible pairs)
\[
    \sup_{0 \leq t \leq T} |I_n^R(t)| \leq \sup_{0 \leq t \leq T} \|z_n^{(1)}(t) - z^{(1)}(t)\|_2 \cdot \|\varphi\|_2 \leq C\|f_n^R - f^R\|_{L^6_x(\mathbb{R} \times I)} \|\varphi\|_2
\]
\[
    (n = 1, 2, \cdots). \text{ Hence, by (4.14) we have}
\]
\[
    \lim_{n \to \infty} \sup_{0 \leq t \leq T} |I_n^R(t)| = 0 \quad \text{for any fixed } R > 0. \quad (3.16)
\]

Next, we compute
\[
    J_n^R(t) = \int_{-\infty}^\infty [z_n^{(2)}(x, t) - z^{(2)}(x, t)] \varphi(x) dx
\]
\[
    = \int_{-\infty}^\infty \int_0^t W(t - t')\{(1 - \chi_R)[u^3_n(\cdot, t') - u^3(\cdot, t')]\} \varphi(x) dt' dx
\]
\[
    = \int_0^t \int_{-\infty}^\infty [1 - \chi_R(x)][u^3_n(x, t') - u^3(x, t')] \cdot W(t - t') \varphi dx dt'
\]
\[
    = \int_0^t \int_{-\infty}^\infty [u^3_n(x, t') - u^3(x, t')] \cdot [1 - \chi_R(x)]W(t - t') \varphi dx dt'.
\]

From this expression and the Hölder inequality we have
\[
    \sup_{0 \leq t \leq T} |J_n^R(t)| \leq \left( \int_0^T \int_{-\infty}^\infty \left| u^3_n(x, t') - u^3(x, t') \right|^{8/7} dx dt' \right)^{7/8} \left( \int_0^T \int_{|x| \geq R} |W(t)\varphi|^8 dx dt \right)^{1/8}.
\]

By (4.10) (take \( p = 8/7 \)), the first term on the right-hand side is bounded by a constant independent of \( n \). Besides, since (8, 8) is an admissible pair, we have \( \|W(t)\varphi\|_{L^8_x(\mathbb{R} \times I)} \leq C \|\varphi\|_2 \), so that
\[
    \lim_{R \to \infty} \int_0^T \int_{|x| \geq R} |W(t)\varphi|^8 dx dt = \lim_{R \to \infty} \int_{|x| \geq R} \int_0^T |W(t)\varphi|^8 dx dt = 0.
\]

Hence
\[
    \lim_{R \to \infty} \sup_{0 \leq t \leq T} |J_n^R(t)| = 0 \quad \text{uniformly for } n \in \mathbb{N}. \quad (3.17)
\]

By (3.16) and (3.17), we obtain (3.15). This completes the proof of Theorem 1.6. \( \square \)
4 Proof of Theorem 1.5

In this section we give the proof of Theorem 1.5.

We first prove a preliminary lemma. Let $M_g$ be the generalized Miura transform given by (2.6) and $W_A$ be the Lipschitz continuous mapping from the ball $B_A = \{ \phi \in H^{-3/4}(\mathbb{R}) : \| \phi \|_{H^{-3/4}} \leq A \}$ of $H^{-3/4}(\mathbb{R})$ to $H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$ ensured by Lemma 2.5 such that

$$M_g \circ W_A = id.$$

**Lemma 4.1** (i) Let $(v_n, w_n) \in H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$, $n = 1, 2, \cdots$, and $(v, w) \in H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$. Let $u_n = M_g(v_n, w_n)$, $n = 1, 2, \cdots$, and $u = M_g(v, w)$. Assume that $(v_n, w_n) \rightharpoonup (v, w)$ weakly in $H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$. Then $u_n \to u$ weakly in $H^{-3/4}(\mathbb{R})$.

(ii) Conversely, let $u_n \in H^{-3/4}(\mathbb{R})$, $n = 1, 2, \cdots$, and $u \in H^{-3/4}(\mathbb{R})$. Let $(v_n, w_n) = W_A(u_n)$, $n = 1, 2, \cdots$, and $(v, w) = W_A(u)$. Assume that $u_n \to u$ weakly in $H^{-3/4}(\mathbb{R})$. Then $(v_n, w_n) \to (v, w)$ weakly in $H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$.

**Proof:** (i) We write

$$u_n = \partial_x v_n + v_n^2 + w_n \quad \text{and} \quad u = \partial_x v + v^2 + w.$$

Clearly, $\partial_x v_n \to \partial_x v$ weakly in $H^{-3/4}(\mathbb{R})$ and $w_n \to w$ weakly in $H^{-3/4}(\mathbb{R})$. Thus, we only need to prove that $v_n^2 \to v^2$ weakly in $H^{-3/4}(\mathbb{R})$, which is almost obvious. Indeed, by the compact embedding $H^{1/4}(\mathbb{R}) \hookrightarrow L^2[a, b]$ for any $-\infty < a < b < \infty$, we easily see that $v_n^2 \to v^2$ strongly in $L^1[a, b]$ for any $-\infty < a < b < \infty$, so that for any $\varphi \in C_0^\infty(\mathbb{R})$ we have

$$|(v_n^2 - v^2, \varphi)_{L^1}| \leq \| v_n^2 - v^2 \|_{L^1[a, b]} \| \varphi \|_\infty \to 0 \quad \text{as} \ n \to \infty,$$

where $a, b$ are real numbers such that supp$\varphi \subseteq [a, b]$. This implies that for any $\psi \in J_x^{3/2}(C_0^\infty(\mathbb{R})) \subseteq H^{-3/4}(\mathbb{R})$ we have

$$\langle v_n^2 - v^2, \psi \rangle_{H^{-3/4}} \to 0 \quad \text{as} \ n \to \infty.$$

Since $C_0^\infty(\mathbb{R})$ is dense in $H^{3/4}(\mathbb{R})$ and $J_x^{3/2}$ is an isomorphism of $H^{3/4}(\mathbb{R})$ onto $H^{-3/4}(\mathbb{R})$, we see that $J_x^{3/2}(C_0^\infty(\mathbb{R}))$ is dense in $H^{-3/4}(\mathbb{R})$. Thus, by the boundedness of the sequence $\{ v_n^2 - v^2 \}$ in $H^{-3/4}(\mathbb{R})$, we conclude that the above relation holds for all $\psi \in H^{-3/4}(\mathbb{R})$. This proves the assertion (i).

(ii) We first recall the construction of the mapping $W_A : H^{-3/4}(\mathbb{R}) \to H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R})$ (cf. the proof of Lemma 10.1 of [2]). We know that there are sufficiently large constants $C_A, C'_A > 0$ such that, denoting by $P$ the Fourier projection to the frequency region $|\xi| \geq C_A$, for any given $\phi \in H^{-3/4}(\mathbb{R})$ such that $\| \phi \|_{H^{-3/4}} \leq A$, the mapping $\varphi \mapsto \partial_x^{-1} P(\phi - \varphi^2)$
is a contraction on the ball $\|\varphi\|_{H^{1/4}} \leq C_A' A$ in $H^{1/4}(\mathbb{R})$. Let $\varphi = L(\phi)$ be the unique fixed point of this mapping and $\psi = (I - P)(\phi - \varphi^2)$. Then we have $W_A(\phi) = (\varphi, \psi)$.

We now proceed to prove the assertion (ii). Since $u_n \to u$ weakly in $H^{-3/4}(\mathbb{R})$, $\{u_n\}$ is bounded in $H^{-3/4}(\mathbb{R})$, which implies that $\{v_n\}$ is bounded in $H^{1/4}(\mathbb{R})$. It follows that there exists subsequence of $\{v_n\}$ which weakly converges in $H^{1/4}(\mathbb{R})$. Thus, to prove that $v_n \to v$ weakly in $H^{1/4}(\mathbb{R})$ we only need to prove that if a subsequence of $\{v_n\}$ weakly converges to an element $v' \in H^{1/4}(\mathbb{R})$, then $v' = v$. For simplicity of notation we assume that the whole sequence $v_n \to v'$ weakly in $H^{1/4}(\mathbb{R})$. From the proof of (i) we see that this implies that $v_n^2 \to (v')^2$ weakly in $H^{-3/4}(\mathbb{R})$. For every $n \in \mathbb{N}$ we have

$$v_n = \partial_x^{-1} P(u_n - v_n^2),$$

or equivalently,

$$\partial_x v_n = P(u_n) - P(v_n^2).$$

Letting $n \to \infty$ and considering the weak limits, we get

$$\partial_x v' = P(u) - P((v')^2).$$

This shows that $v'$ is a fixed point of the mapping $\varphi \mapsto \partial_x^{-1} P(u - \varphi^2)$. Since $W_A(u) = (v, w)$, we see that $v$ is also a fixed point of this mapping. By uniqueness of the fixed point, we obtain $v' = v$. Hence, the desired assertion follows. From this assertion and the relations $w_n = (I - P)(v_n - v_n^2)$ $(n = 1, 2, \cdots)$ and $w = (I - P)(u - u^2)$ it follows immediately that $w_n \to w$ weakly in $H^1(\mathbb{R})$. This completes the proof. □

We are now ready to prove Theorem 1.5.

Let $u_{0n} \in H^{-3/4}(\mathbb{R})$, $n = 1, 2, \cdots$, and $u_0 \in H^{-3/4}(\mathbb{R})$ be such that $u_{0n} \to u_0$ weakly in $H^{-3/4}(\mathbb{R})$ as $n \to \infty$. Then there exists constant $M > 0$ such that

$$\|u_{0n}\|_{H^{-3/4}} \leq M, \quad n = 1, 2, \cdots, \quad \text{and} \quad \|u_0\|_{H^{-3/4}} \leq M. \quad (4.1)$$

Let $u_n(x, t) = [S(t)u_0_n](x)$, $n = 1, 2, \cdots$, and $u(x, t) = [S(t)u_0](x)$. Given $T > 0$ be given, we set $A = C(1 + T)M$, where $C$ is the constant appearing in (2.1). Then we have

$$\|u_n(\cdot, t)\|_{H^{-3/4}} \leq A, \quad n = 1, 2, \cdots, \quad \text{and} \quad \|u(\cdot, t)\|_{H^{-3/4}} \leq A \quad (4.2)$$

for all $t \in I$. Let $(v_{0n}, w_{0n}) = W_A(u_{0n})$, $n = 1, 2, \cdots$, and $(v_0, w_0) = W_A(u_0)$. Let

$$M = M_A := \sup_{(\varphi, \psi) \in W_A(B_A)} \max\{\|\varphi\|_{H^{1/4}}, \|\psi\|_{H^{1/4}}\}.$$ 

By Proposition 2.6, for this constant $M$ there is a corresponding constant $\delta > 0$ such that for any $t_0 \in \mathbb{R}$ and any $(\varphi, \psi) \in H^{1/4}(\mathbb{R}) \times H^{1}(\mathbb{R})$ satisfying $\|\varphi\|_{H^{1/4}} \leq M$ and
we have the estimate
\[ \| \psi \|_{H^1} \leq M, \]
the initial value problem
\[
\begin{cases}
\partial_t v + \partial_x^2 v = 6(v^2 + w)\partial_x v, & x \in \mathbb{R}, \quad t_0 - \delta \leq t \leq t_0 + \delta, \\
\partial_t w + \partial_x^2 w = 6(v^2 + w)\partial_x w, & x \in \mathbb{R}, \quad t_0 - \delta \leq t \leq t_0 + \delta,
\end{cases}
\]
has a unique solution \((v, w)\) in the space \(X^{**} = X^{**}(\mathbb{R} \times [t_0 - \delta, t_0 + \delta])\) (see (2.9) for the definition of this space), which we also denote as \((v(\cdot, t), w(\cdot, t)) = S^{**}_{t_0}(\varphi, \psi)\). Moreover, we have the estimate
\[
\|(v, w)\|_{X^{**}} \leq C(\max\{\|\varphi\|_{H^{1/4}}, \|\psi\|_{H^1}\}) \leq C(M),
\]
where \(C: [0, \infty) \rightarrow [0, \infty)\) is a nondecreasing function. Let
\[
(v_n(\cdot, t), w_n(\cdot, t)) = S^{**}_{t_0}(v_0, w_0), \quad n = 1, 2, \ldots,
\]
and \((v(\cdot, t), w(\cdot, t)) = S^{**}(v_0, w_0)\). Since \((v_0, w_0) \in W_A(B_A)\), we have \(\|v_0\|_{H^{1/4}} \leq M\) and \(\|w_0\|_{H^1} \leq M, n = 1, 2, \ldots\). Thus
\[
\|(v_n, w_n)\|_{X^{**}} \leq C(M), \quad n = 1, 2, \ldots
\]
Moreover, since \(u_0 \rightarrow u_0\) weakly in \(H^{-3/4}(\mathbb{R})\) as \(n \rightarrow \infty\), by Lemma 4.1 (ii) we see that
\[
(v_0, w_0) \rightarrow (v_0, w_0) \text{ weakly in } H^{1/4}(\mathbb{R}) \times H^1(\mathbb{R}) \text{ as } n \rightarrow \infty.
\]
From (4.5) and the definition of the space \(X^{**}\) it follows immediately that
\[
\|v_n\|_{L^\infty_T H^{1/4}_x} \leq C(T, M)
\]
and
\[
\|w_n\|_{L^\infty_T H^1_x} \leq C(T, M)
\]
for all \(n = 1, 2, \ldots\).

Using the estimates (4.8) and a similar argument as in the proof of Theorem 1.6, we conclude that for any \(\varphi \in H^{1/4}(\mathbb{R})\) and \(\psi \in H^1(\mathbb{R})\) we have
\[
\lim_{n \rightarrow \infty} \left\{ \sup_{|t| \leq \delta} |(v_n(\cdot, t) - v(\cdot, t), \varphi)_{H^{1/4}}| + \sup_{|t| \leq \delta} |(w_n(\cdot, t) - w(\cdot, t), \psi)_{H^1}| \right\} = 0
\]
Using this relation, the relation (2.10) and a similar argument as in the proof of Lemma 4.1 (i), we obtain that for any \(\phi \in H^{-3/4}(\mathbb{R})\),
\[
\lim_{n \rightarrow \infty} \sup_{|t| \leq \delta} |(u_n(\cdot, t) - u(\cdot, t), \phi)_{H^{-3/4}}| = 0.
\]
Now let \( m = T/\delta \) if \( T/\delta \) is an integer, and \( m = \lfloor T/\delta \rfloor + 1 \) otherwise. We divide the time interval \( I \) into \( 2m \) subintervals \( I_{\pm 1}, I_{\pm 2}, \cdots, I_{\pm m} \), where

\[
I_j = [(j-1)\delta, j\delta], \quad j = 1, 2, \cdots, m-1, \quad I_m = [(m-1)\delta, T],
\]

and \( I_{-j} = -I_j, \ j = 1, 2, \cdots, m \). By inductively using the result we have just proved to every pair of intervals \( I_{\pm j}, j = 1, 2, \cdots, m \), we see that for any \( \phi \in H^{-3/4}(\mathbb{R}) \),

\[
\lim_{n \to \infty} \sup_{t \in I_j} |(u_n(\cdot, t) - u(\cdot, t), \phi)_{H^{-3/4}}| = 0, \quad j = \pm 1, \pm 2, \cdots, \pm m.
\]

Hence, for any \( \phi \in H^{-3/4}(\mathbb{R}) \),

\[
\lim_{n \to \infty} \sup_{|t| \leq T} |(u_n(\cdot, t) - u(\cdot, t), \phi)_{H^{-3/4}}| = 0.
\]

This completes the proof of Theorem 1.5. □

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