GALOIS MODULE STRUCTURE OF ORIENTED ARAKELOV CLASS GROUPS

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Abstract. We show that Chinburg’s Ω(3) conjecture implies tight restrictions on the Galois module structure of oriented Arakelov class groups of number fields. We apply our findings to formulating a probabilistic model for Arakelov class groups in families, offering a correction of the Cohen–Lenstra–Martinet heuristics on ideal class groups.

1. Introduction

The present paper is concerned with the Galois module structure of oriented Arakelov class groups of number fields, and with the consequences for the Cohen–Lenstra–Martinet heuristics on class groups of “random” number fields.

The oriented Arakelov class group $\tilde{\text{Pic}}_0^0 F$ of a number field $F$ was first defined by Schoof in [Sch08], and we will recall the definition in Section 3.

It is a compact abelian group, which “knows” the unit group $O_F^\times$ and the class group $\text{Cl}_F$ of the ring of integers $O_F$ of $F$. Its Pontryagin dual $\tilde{\text{Ar}}_F$ is a finitely generated $\mathbb{Z}[^{\text{Aut}} F]$-module.

Henceforth all modules will be assumed to be left modules unless stated otherwise. Recall that for a ring $T$, the Grothendieck group $G_0(T)$ of the category of finitely generated $T$-modules is the additive group generated by expressions $[M]$, one for each isomorphism class of finitely generated $T$-modules $M$, with a relation $[L] + [N] = [M]$ whenever there exists a short exact sequence $0 \to L \to M \to N \to 0$ of finitely generated $T$-modules. In Section 5 we will show the following.

Theorem 1.1. Let $F/K$ be a finite Galois extension of number fields, let $d$ be the degree of $K$ over $\mathbb{Q}$, and let $G$ be the Galois group of $F/K$. Suppose that “Chinburg’s $\Omega(3)$ conjecture modulo the kernel group”, Conjecture 5.6, holds for $F/K$. Then the equality

$$[\tilde{\text{Ar}}_F] = d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}]$$

holds in $G_0(\mathbb{Z}[G])$.

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In particular, Theorem 1.1 gives a conditional answer to Question 5.5 in [BLJ20].

For a set $P$ of prime numbers, let

$$\mathbb{Z}(P) = \{a/b : a, b \in \mathbb{Z}, b \not\in \bigcup_{p \in P \cup \{0\}} p\mathbb{Z} \}.$$  

If $P$ is such a set, then the conclusion of Theorem 1.1 implies an analogous statement for the $\mathbb{Z}(P)[G]$-module $\mathbb{Z}(P) \otimes \tilde{\text{Ar}}_F$ in the Grothendieck group $G_0(\mathbb{Z}(P)[G])$.

Of particular interest to us will be the special case when $P$ does not contain any prime divisors of $2 \cdot \# G$, and in this case that analogous statement can sometimes be proven unconditionally. For example in [BLJ20, Theorem 5.4], this was proven in the special case that $G$ is abelian and $K = \mathbb{Q}$. Another example is given by the following special case of Proposition 5.1, which will be proven in Section 5.

**Proposition 1.2.** Let $F/\mathbb{Q}$ be a Galois extension of degree less than 112, let $G$ be the Galois group, and let $P$ be a set of prime numbers not dividing $2 \cdot \# G$. Then the equality

$$[\mathbb{Z}(P) \otimes \mathbb{Z} \tilde{\text{Ar}}_F] = [\mathbb{Z}(P)[G]] - [\mathbb{Z}(P)]$$

holds in $G_0(\mathbb{Z}(P)[G])$.

Our main motivation for this work comes from the Cohen–Lenstra–Martinet heuristics. These were originally formulated in [CLJ84] to explain the behaviour of ideal class groups in families of quadratic number fields, and were extended in [CM90] to much more general families of number fields. It was shown in [BLJ20] that these heuristics can be conceptually understood via a certain subgroup of $\tilde{\text{Ar}}_F$, the Pontryagin dual $\text{Ar}_F$ of the (unoriented) Arakelov class group. Namely, the heuristics can be interpreted as saying that locally at a finite set of “good” primes, $\text{Ar}_F$ looks like a random Galois module that is isomorphic to a given module $X$ with a probability that is inversely proportional to $\# \text{Aut} X$. However, for abelian number fields Theorem 5.4 in [BLJ20], mentioned earlier, poses an obstruction to this being true globally. Theorem 1.1 allows us to understand such global obstructions for general Galois extensions, and thus to formulate a global Cohen–Lenstra–Martinet heuristic. This will be carried out in Section 4. Roughly speaking, our heuristic postulates that the dual Arakelov class group of a “random” number field is equidistributed among all Galois modules that are compatible with Theorem 1.1 with respect to the measure that gives an isomorphism class of such modules $X$ a probability weight that is proportional to $1/\# \text{Aut} X$ – see Heuristic 4.3 for the precise formulation.

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In this section we review some standard facts about Grothendieck groups of orders, and examine the effect of some duality operations upon these Grothendieck groups.

Let $R$ be a Dedekind domain and let $k$ be the field of fractions of $R$. An $R$-order is an $R$-algebra that is finitely generated and projective as an $R$-module. For example if $G$ is a finite group, then the group ring $\Lambda = R[G]$ is an $R$-order.

Let $\Lambda$ be an $R$-order. A finitely generated $\Lambda$-module that is projective over $R$ will be referred to as a $\Lambda$-lattice. Let $G_0^R(\Lambda)$ denote the Grothendieck group of the category of $\Lambda$-lattices. By definition, $G_0^R(\Lambda)$ is the additive group generated by expressions $[M]$, one for each isomorphism class of $\Lambda$-lattices $M$, with a relation $[L] + [N] = [M]$ whenever there exists a short exact sequence $0 \to L \to M \to N \to 0$ of $\Lambda$-lattices. By [CR87, (38.42)], the inclusion of the category of $\Lambda$-lattices into the category of all finitely generated $\Lambda$-modules induces an isomorphism $G_0^R(\Lambda) \cong G_0(\Lambda)$.

Let $\Lambda^{op}$ denote the opposite ring of $\Lambda$. If $M$ is a $\Lambda$-lattice, then $M^* = \text{Hom}_R(M, R)$ is a $\Lambda^{op}$-lattice. This defines a contravariant functor from the category of $\Lambda$-lattices to the category of $\Lambda^{op}$-lattices, given on objects by $M \mapsto M^*$ for every $\Lambda$-lattice $M$, and on morphisms by $f \mapsto (\nu \mapsto \nu \circ f) \in M^*$ for every morphism $f : M \to N$ of $\Lambda$-lattices and every $\nu \in N^*$. This functor is easily seen to be exact, and to induce a group isomorphism $G_0^R(\Lambda) \to G_0^R(\Lambda^{op})$, and hence an isomorphism $G_0(\Lambda) \to G_0(\Lambda^{op})$, which we will denote by $\sigma$.

For a $\Lambda$-module $N$, we define $N^\vee = \text{Hom}_R(N, k/R)$, which is also a $\Lambda^{op}$-module. If $N$ is a finitely generated $\Lambda$-module that is $R$-torsion, then $N^\vee$ is finitely generated over $\Lambda^{op}$ and $R$-torsion.

**Proposition 2.1.** Let $N$ be a finitely generated $\Lambda$-module that is $R$-torsion. Then the equality $[N^\vee] = -\sigma[N]$ holds in $G_0(\Lambda^{op})$.

**Proof.** Since $R$ is a Dedekind domain, every $\Lambda$-submodule of a $\Lambda$-lattice is itself a $\Lambda$-lattice. Hence there exists a presentation $0 \to M_1 \to M_2 \to N \to 0$ for $N$ by $\Lambda$-lattices, so that $[N] = [M_2] - [M_1]$ in $G_0(\Lambda)$. We claim that $N^\vee$ is canonically isomorphic as a $\Lambda^{op}$-module to $M_1^*/M_2^*$.

Since $M_1$ and $M_2$ are projective over $R$, applying the functors $\text{Hom}_R(M_i, \bullet)$ for $i = 1, 2$ to the short exact sequence $0 \to R \to k \to k/R \to 0$ yields the commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \to & \text{Hom}_R(M_2, R) & \to & \text{Hom}_R(M_2, k) & \to & \text{Hom}_R(M_2, k/R) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Hom}_R(M_1, R) & \to & \text{Hom}_R(M_1, k) & \to & \text{Hom}_R(M_1, k/R) & \to & 0,
\end{array}
$$

where the vertical maps are induced by the injection $M_1 \to M_2$. Of these, the middle map $\text{Hom}_R(M_2, k) \to \text{Hom}_R(M_1, k)$ is an isomorphism. Indeed,
it is the $k$-linear dual of the map $k \otimes_R M_1 \rightarrow k \otimes_R M_2$, which is clearly an isomorphism, since the cokernel $N$ of $M_1 \rightarrow M_2$ is $R$-torsion. The snake lemma therefore gives an isomorphism of right $A$-modules from the kernel of $\text{Hom}_R(M_2, k/R) \rightarrow \text{Hom}_R(M_1, k/R)$ to the cokernel of $\text{Hom}_R(M_2, R) \rightarrow \text{Hom}_R(M_1, R)$. Since $\text{Hom}_R(k, k/R)$ is left exact, that kernel is exactly $N^\vee$, while the cokernel is precisely $M_1^\vee/M_2^\vee$, as claimed. The proposition immediately follows.

\[ \square \]

3. Oriented Arakelov class group

In this section we recall from [Sch08] the definitions of the Arakelov class group and the oriented Arakelov class group of a number field, prove some properties of oriented Arakelov class groups as Galois modules, and formulate the main working hypothesis that we will assume for the purposes of statistical heuristics in Section 4. We refer to [Sch08] for further details on Arakelov class groups.

For the remainder of this section, let $F$ be a number field, let $\text{Id}_F$ be the group of fractional ideals of $O_F$, let $S_{\infty}$ denote the set of Archimedean places of $F$, and let $F_\mathbb{R}$ denote the étale $\mathbb{R}$-algebra $F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{w \in S_{\infty}} F_w$, where $F_w$ denotes the completion of $F$ at $w$. We have canonical maps $\text{Id}_F \rightarrow \mathbb{R}_{>0}$ and $|\text{Nm}|: F_\mathbb{R}^\times \rightarrow \mathbb{R}_{>0}$, the first given by the ideal norm, and the second given by the absolute value of the $\mathbb{R}$-algebra norm. Let $\text{Id}_F \times_{\mathbb{R}_{>0}} F_\mathbb{R}^\times$ denote the fibre product with respect to these maps. The oriented Arakelov class group $\widetilde{\text{Pic}}_F^0$ of $F$ is defined as the cokernel of the map $F_\mathbb{R}^\times \rightarrow \text{Id}_F \times_{\mathbb{R}_{>0}} F_\mathbb{R}^\times$ that sends $\alpha \in F_\mathbb{R}^\times$ to $(\alpha O_F, \alpha)$. It follows from Dirichlet’s unit theorem and from the finiteness of the class group of $O_F$, that this is a compact abelian group.

For every $w \in S_{\infty}$ we have a direct product decomposition $F_\mathbb{R}^\times \cong \mathbb{R}_{>0} \times c(F_w^\times)$, where $c(F_w^\times)$ is the maximal compact subgroup of $F_w^\times$, which is equal to $\{\pm 1\}$ if $w$ is real, and to the circle group in $F_w$ if $w$ is complex. The maximal compact subgroup $c(F_w^\times) = \prod_{w \in S_{\infty}} c(F_w^\times)$ of $F_\mathbb{R}^\times$ is contained in the kernel of the map $|\text{Nm}|$. Define the Arakelov class group $\text{Pic}_F^0$ of $F$ to be the quotient of $\widetilde{\text{Pic}}_F^0$ by the image of $\{1\} \times c(F_\mathbb{R}^\times) \subset \text{Id}_F \times_{\mathbb{R}_{>0}} F_\mathbb{R}^\times$ in $\widetilde{\text{Pic}}_F^0$.

We briefly recall some facts on Pontryagin duality and refer the reader to [NSW08] Chapter 1, §1 for a more detailed overview. If $A$ and $B$ are topological groups, then $\text{Hom}_{cts}(A, B)$ denotes the group of continuous group homomorphisms from $A$ to $B$. Let $\mathcal{C}$ be the category of Hausdorff, abelian and locally compact topological groups. If $A$ is an object of $\mathcal{C}$, then its Pontryagin dual is defined to be $\text{Hom}_{cts}(A, \mathbb{R}/\mathbb{Z})$. This defines an involutory contravariant automorphism on $\mathcal{C}$ and induces an equivalence between the full subcategories of compact abelian groups and of discrete abelian groups. In particular, since $\text{Hom}_{cts}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$, we have $\text{Hom}_{cts}(\mathbb{R}/\mathbb{Z}, \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}$.

Suppose that $F/K$ is a finite Galois extension of number fields, and let $G$ be the Galois group. The ring $\mathbb{Z}[G]$ is equipped with an involution $\iota$ induced by $g \mapsto g^{-1}$ for all $g \in G$. If $M$ is a finitely generated $\mathbb{Z}[G]$-module, then we view the $(\mathbb{Z}[G]^{op})$-module $M^* = \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ as a finitely generated $\mathbb{Z}[G]$-module via $\iota$, and we view the map $\sigma$ introduced in Section 2 with $\Lambda = \mathbb{Z}[G]$, as an automorphism of $G_0(\mathbb{Z}[G])$. Similarly, if $A$ is an object of $\mathcal{C}$
on which $G$ acts by continuous automorphisms, then we view its Pontryagin dual $\text{Hom}_{cts}(A, \mathbb{R}/\mathbb{Z})$ as a $\mathbb{Z}[G]$-module via $\iota$. If $N$ is a $\mathbb{Z}[G]$-module of finite cardinality, then $\text{Hom}_{cts}(N, \mathbb{R}/\mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Q}/\mathbb{Z}) = N^\vee$ is a finitely generated $\mathbb{Z}[G]$-module, and if $M$ is a $\mathbb{Z}[G]$-lattice, then $\text{Hom}_{cts}(M \otimes_{\mathbb{Z}} (\mathbb{R}/\mathbb{Z}), \mathbb{R}/\mathbb{Z})$ is isomorphic to $M^\ast$.

**Lemma 3.1.** Let $F/K$ be a finite Galois extension of number fields, let $G$ be the Galois group, let $S^\infty$ be the set of Archimedean places of $F$, and let $d$ be the degree of $K$ over $\mathbb{Q}$. Then the equality

$$[\text{Hom}_{cts}(c(F^\times_K), \mathbb{R}/\mathbb{Z})] = d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}[S^\infty]]$$

holds in $G_0(\mathbb{Z}[G])$.

**Proof.** If $v$ is an Archimedean place of $K$, let $I_v \subset G$ denote an inertia subgroup at $v$, and let $\tau_v$ be a $\mathbb{Z}[I_v]$-module defined as follows: if $v$ is real and $I_v$ is the trivial group, then let $\tau_v = \mathbb{F}_2$; if $v$ is real and $I_v$ has order 2, then let $\tau_v$ be free over $\mathbb{Z}$ of rank 1, and with the generator of $I_v$ acting by $-1$; and if $v$ is complex, so that $I_v$ is necessarily trivial, let $\tau_v = \mathbb{Z}$. Then it is easy to see that we have an isomorphism

$$\text{Hom}_{cts}(c(F^\times_K), \mathbb{R}/\mathbb{Z}) \cong \bigoplus_v \text{Ind}_{G/I_v} \tau_v$$

of $\mathbb{Z}[G]$-modules, where the direct sum runs over the Archimedean places of $K$, and $\text{Ind}_{G/I_v}$ denotes induction from $I_v$ to $G$.

If $v$ is a real place of $K$ such that $I_v$ is trivial, then the exact sequence

$$0 \rightarrow \text{Ind}_{G/I_v} \mathbb{Z} \rightarrow \text{Ind}_{G/I_v} \mathbb{Z} \rightarrow \text{Ind}_{G/I_v} \mathbb{F}_2 \rightarrow 0$$

shows that $[\text{Ind}_{G/I_v} \tau_v] = 0$ in $G_0(\mathbb{Z}[G])$. We deduce that for all Archimedean places $v$ of $K$ one has

$$[\text{Ind}_{G/I_v} \tau_v] = \delta_v[\mathbb{Z}[G]] - [\mathbb{Z}[G/I_v]],$$

where $\delta_v = 1$ if $v$ is real, and $\delta_v = 2$ if $v$ is complex. The result follows by summing (3.2) over all Archimedean places $v$ of $K$. \hfill $\square$

Every permutation $\mathbb{Z}[G]$-module $M$ is isomorphic to $M^\ast$, so one has $\sigma[M] = [M]$ in $G_0(\mathbb{Z}[G])$. We will repeatedly use this observation. If $S$ is a set of places of $F$ containing all Archimedean places, let $\mathcal{O}_{F,S}$ denote the ring of $S$-integers in $F$, let $\mathcal{O}_{F,S}^\times$ denote its unit group, and let $\text{Cl}_{F,S}$ denote its class group. If $S$ is $G$-stable, then $\mathcal{O}_{F,S}^\times$ and $\text{Cl}_{F,S}$ are $\mathbb{Z}[G]$-modules. Write $\text{Ar}_F$ and $\text{Pic}_F$ for the Pontryagin duals of $\mathcal{O}^0_F$ and $\text{Pic}^0_F$, respectively. These are finitely generated $\mathbb{Z}[G]$-modules.

**Proposition 3.3.** Let $F/K$ be a finite Galois extension of number fields, let $G$ be the Galois group, let $d$ be the degree of $K$ over $\mathbb{Q}$, and let $S$ be a finite $G$-stable set of places of $F$ containing all Archimedean places. Then the equality

$$[\text{Ar}_F] = d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}[S]] + \sigma[\mathcal{O}_{F,S}^\times] - \sigma[\text{Cl}_{F,S}]$$

holds in $G_0(\mathbb{Z}[G])$. 

\hfill $\square$
Proof. Let $S_f = S \setminus S_{\infty}$ denote the set of non-Archimedean places in $S$. The subgroup of $\text{Id}_F$ generated by the prime ideals corresponding to the places in $S_f$ is free abelian on the set $S_f$. Below, when we write $Z^{S_f}$, we will mean that subgroup. Let $\text{Id}_{F,S}$ be the quotient of $\text{Id}_F$ by the subgroup $Z^{S_f}$. It is naturally isomorphic to the group of fractional ideals of $O_{F,S}$. The preimage of $Z^{S_f} \times F_R^{\times} \subset \text{Id}_F \times F_R^{\times}$ in $F^{\times}$ is $O_{F,S}^{\times}$. There is thus a commutative diagram of $\mathbb{Z}[G]$-modules with exact rows and columns

$\begin{array}{cccccc}
0 & \to & 0 & \to & 0 & \to \\
0 & \to & O_{F,S}^{\times} & \to & Z^{S_f} \times F_R^{\times} & \to & T^0_{F,S} & \to & 0 \\
0 & \to & F^{\times} & \to & \text{Id}_F \times F_R^{\times} & \to & \widetilde{\text{Pic}}_F & \to & 0 \\
0 & \to & F^{\times}/O_{F,S}^{\times} & \to & \text{Id}_{F,S} & \to & \text{Cl}_{F,S} & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0,
\end{array}$

where $T^0_{F,S}$ is defined by the exactness of the first row, and the exactness of the last column follows from the snake lemma.

Taking the Pontryagin dual $\text{Hom}_{\text{cts}}(\bullet, \mathbb{R}/\mathbb{Z})$ of the right column, we deduce that

$$[\widetilde{\text{Ar}}_F] = [\text{Cl}^\vee_{F,S}] + [\text{Hom}_{\text{cts}}(T^0_{F,S}, \mathbb{R}/\mathbb{Z})]$$

in $G_0(\mathbb{Z}[G])$.

The group $Z^{S_f} \times F_R^{\times}$ can be explicitly described as follows: one has

$$Z^{S_f} \times F_R^{\times} = \left\{ ((a_p), (b_w)) \in Z^{S_f} \times (\mathbb{R}_{>0})^{S_{\infty}} : \prod_{p \in S_f} \#(O_{F,p})^{a_p} = \prod_{w \in S_{\infty}} b_{\delta_w}^{\delta_w w} \right\} \rtimes c(F_R^{\times}),$$

where $\delta_w = 1$ if $w$ is real, and $\delta_w = 2$ if $w$ is complex. That group naturally embeds into $\mathbb{R}^{S_f} \times F_R^{\times}$, where the fibre product is taken with respect to the map

$$\mathbb{R}^{S_f} \to \mathbb{R}_{>0},
(a_p)_{p \in S_f} \mapsto \prod_{p \in S_f} \#(O_{F,p})^{a_p},$$

and to the same map $|\text{Nm}| : F_R^{\times} \to \mathbb{R}_{>0}$ as before, so that we have an exact sequence

$$0 \to T^0_{F,S} \to \frac{\mathbb{R}^{S_f} \times \mathbb{R}_{>0}}{O_{F,S}^{\times}} \to (\mathbb{R}/\mathbb{Z})^{S_f} \to 0. \quad (3.5)$$
Lemma 3.1, we see that there is an equality in $G_0$ of the two equations lies in the torsion subgroup of $G_0$ of the two equations lies in the torsion subgroup of $G_0$. By \cite{Sch08, Proposition 2.2} there is an exact sequence

$$0 \to c(F_F^\times)_{/\mu_F} \to \mathbb{R}^{S\setminus\infty}_{/\mathbb{Z}_{\mathbb{R}_{>0}}} F_F^\times \to (\mathcal{O}_{F,S}^\times/\mu_F) \otimes_{\mathbb{Z}} \mathbb{R} \to 0. \quad (3.6)$$

Combining \cite{S21} with the Pontryagin duals of \cite{S21} and \cite{3.6} and applying Lemma \ref{S3.1} we see that there is an equality

$$[\tilde{\mathcal{A}}_{\mathcal{F}}] = [\mathcal{F}_{F,S}] + d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}[S_{\infty}]] - [\mathbb{Z}[S]] + \mathbb{Z}[S]] = [\mathbb{Z}[S]]$$

holds in $G_0(Z[G])$. The result now follows by applying Proposition \ref{S1} and noting that there are $\mathbb{Z}[G]$-module isomorphisms $\mathbb{Z}[S]] = \mathbb{Z}[S]$ and $\mathbb{Z}[S] = \mathbb{Z}[S_{\infty} \oplus Z[S]]$.

\begin{corollary}
Let $F/K$ be a finite Galois extension of number fields, let $G$ be the Galois group, let $d$ be the degree of $K$ over $\mathbb{Q}$, and let $S$ be a finite $G$-stable set of places of $F$ containing all Archimedean places, and large enough such that $\mathcal{C}_{F,S}$ is trivial. Then the equality

$$[\tilde{\mathcal{A}}_{\mathcal{F}}] = d \cdot [\mathbb{Z}[G]] - \sigma([\mathbb{Z}[S]] - [\mathbb{Z}[F,S]])$$

holds in $G_0(Z[G])$.
\end{corollary}

\begin{proof}
The result follows by combining Proposition \ref{S3.3} with the observation that $\sigma[Z[S]] = [Z[S]]$.
\end{proof}

\begin{corollary}
Let $F/K$ be a finite Galois extension of number fields, let $G$ be the Galois group, and let $S$ and $S'$ be two finite $G$-stable sets of places of $F$, both containing all Archimedean places. Then the equality

$$[\mathbb{Z}[S']] - [\mathcal{O}_{F,S'}] + [\mathcal{C}_{F,S}] = [\mathbb{Z}[S]] - [\mathcal{O}_{F,S}] + [\mathcal{C}_{F,S}]$$

holds in $G_0(Z[G])$.
\end{corollary}

\begin{proof}
The result follows by combining Proposition \ref{S3.3} with the observation that $\sigma[Z[S]] = [Z[S]]$ and $\sigma[Z[S']] = [Z[S']]$.
\end{proof}

\begin{conjecture}
Let $F/K$ be a finite Galois extension of number fields, let $G$ be the Galois group, let $d$ be the degree of $K$ over $\mathbb{Q}$, let $S_{\infty}$ be the set of Archimedean places of $F$, let $\mu_F$ be the group of roots of unity in $F$, and let $\sigma$ be the automorphism of $G_0(Z[G])$ induced by the involution $g \mapsto g^{-1}$ on $Z[G]$, as defined in Section 2. Then the following equalities hold in $G_0(Z[G])$:

(a) $[\tilde{\mathcal{A}}_{\mathcal{F}}] = d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}]$;
(b) $[\tilde{\mathcal{A}}_{\mathcal{F}}] = [\mathbb{Z}[S_{\infty}]] - [\mathbb{Z}] - \sigma[\mu_F]$.
\end{conjecture}

\begin{lemma}
The two assertions of Conjecture \ref{S3.9} are equivalent. Moreover, the difference between the left hand side and the right hand side of each of the two equations lies in the torsion subgroup $G_0(Z[G])_{\text{tors}}$ of $G_0(Z[G])$.
\end{lemma}

\begin{proof}
By \cite{Sch08, Proposition 2.2} there is an exact sequence

$$0 \to (\mathcal{O}_F^\times/\mu_F) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \to \text{Pic}_F^0 \to \mathcal{C}_F \to 0.$$
Taking the Pontryagin dual of this sequence and applying Proposition 2.11 gives the equality
\[ \text{Ar}_F = \sigma(O_F^\times / \mu_F) - \sigma[\text{Cl}_F] \] (3.11)
in \( G_0(\mathbb{Z}[G]) \). By combining this equality with Proposition 3.3 with \( S = S_\infty \), we deduce that
\[ \text{Ar}_F - \text{Ar}_F = d \cdot [\mathbb{Z}[G]] - [\mathbb{Z}[S_\infty]] + \sigma[\mu_F], \] (3.12)
which is also the difference between the right hand sides of the equalities in 3.10(a) and 3.10b. This proves the equivalence of these equalities.

We now prove the last assertion. By Dirichlet’s unit theorem, there is an isomorphism \( (\mathbb{R} \otimes \mathbb{Z} O_F^\times) \otimes \mathbb{R} \cong \mathbb{R}[S_\infty] \) of \( \mathbb{R}[G] \)-modules and by the Noether–Deuring theorem [CR81, Exercise 6.6], there is an analogous isomorphism of \( \mathbb{Q}[G] \)-modules. Let \( \theta : G_0(\mathbb{Z}[G]) \rightarrow G_0(\mathbb{Q}[G]) \) be the map induced by the flat ring homomorphism \( \mathbb{Z}[G] \rightarrow \mathbb{Q}[G] \). For every finitely generated \( \mathbb{Q}[G] \)-module \( M \) we have \( \theta(\sigma[M]) = \theta([M]) \). Therefore applying \( \theta \) to (3.11), we have
\[ [\mathbb{Q} \otimes \mathbb{Z} \text{Ar}_F] - [\mathbb{Q}[S_\infty]] + [\mathbb{Q}] = [\mathbb{Q} \otimes \mathbb{Z} O_F^\times] - [\mathbb{Q}[S_\infty]] + [\mathbb{Q}] = 0 \]
in \( G_0(\mathbb{Q}[G]) \). This shows that the difference between the left hand side and the right hand side of Conjecture 3.13 lies in \( \ker(\theta) \), which is equal to \( G_0(\mathbb{Z}[G])_{\text{tors}} \) by [CR81] (39.14). The conclusion for Conjecture 3.13 follows from this and from (3.12).

The following conjecture is implied by Conjecture 3.9 and will suffice for the applications to the Cohen–Lenstra–Martinet heuristics in Section 4.

**Conjecture 3.13.** Let \( F/K \) be a finite Galois extension of number fields, let \( G \) be the Galois group, and let \( P \) a set of prime numbers not dividing \( 2 \cdot \#G \). Then the equalities
\[ [\mathbb{Z}(P) \otimes \mathbb{Z} \text{Ar}_F] = d \cdot [\mathbb{Z}(P)[G]] - [\mathbb{Z}(P)], \]
\[ [\mathbb{Z}(P) \otimes \mathbb{Z} \text{Ar}_F] = [\mathbb{Z}(P)[S_\infty]] - [\mathbb{Z}(P)] - \sigma[\mathbb{Z}(P) \otimes \mathbb{Z} \mu_F] \]
hold in \( G_0(\mathbb{Z}(P)[G]) \).

**Remark 3.14.** Conjecture 3.13 is equivalent to an affirmative answer to [BLJ20, Question 5.5]. Thus in the special case that \( G \) is abelian and \( K = \mathbb{Q} \), Conjecture 3.13 was proven unconditionally in [BLJ20, Theorem 5.4].

In Section 5 we will give further evidence for Conjectures 3.9 and 3.13 and in particular show that they are implied by Chinburg’s \( \Omega(3) \) conjecture.

### 4. Cohen–Lenstra–Martinet heuristics

In this section we propose a correction of the Cohen–Lenstra–Martinet heuristics [CLJ84, CM90], taking into account the implications of Conjecture 3.13. The notation introduced in the next two paragraphs will remain in force throughout this section.

Let \( G \) be a finite group, let \( P \) be a set of prime numbers not dividing \( 2 \cdot \#G \), let the \( \mathbb{Q} \)-algebra \( A \) be a quotient of \( \mathbb{Q}[G] \) by a 2-sided ideal containing \( \sum_{g \in G} g \), let \( \Lambda \) be the image of \( \mathbb{Z}(P)[G] \) in \( A \), and let \( V \) be a finitely generated \( \mathbb{Q}[G] \)-module. For brevity, set \( V_A = A \otimes_{\mathbb{Q}[G]} V \). Let \( \mathcal{M}_V \) be a set of finitely generated \( \Lambda \)-modules \( M \) that satisfy \( A \otimes_{\Lambda} M \cong_A V \), and with the property that for every finitely generated \( \Lambda \)-module \( M' \) satisfying \( A \otimes_{\Lambda} M' \cong_A V \),
there exists a unique $M \in \mathcal{M}_V$ such that $M' \cong M$. Note that the set $\mathcal{M}_V$ is countable. In [BLJ17] it was shown that there is a unique "automorphism index" function $\text{ia}: \mathcal{M}_V \times \mathcal{M}_V \to \mathbb{Q}_{>0}$ that behaves, in a precise sense explained in [BLJ17. Theorem 1.1], like $(L, M) \mapsto \# \text{Aut}_M$, even when the automorphism groups of $M$ and of $L$ are infinite. Fix $M \in \mathcal{M}_V$. If $\mathcal{N}$ is a subset of $\mathcal{M}_V$ and $X$ is a positive real number, let $\mathcal{N}_X$ be the finite set of all $L \in \mathcal{N}$ whose torsion subgroup has order less than $X$. For $\mathcal{N} \subset \mathcal{M}_V$ and for a function $f: \mathcal{N} \to \mathbb{C}$, define the expected value of $f$ on $\mathcal{N}$ by

$$E_\mathcal{N}(f) = \lim_{X \to \infty} \left( \sum_{L \in \mathcal{N}_X} \text{ia}(L, M)f(L) / \sum_{L \in \mathcal{N}_X} \text{ia}(L, M) \right)$$

when the limit exists. One of the defining properties of the function $\text{ia}$ is that for all $L$, $M$, and $N \in \mathcal{M}_V$ one has $\text{ia}(L, M)\text{ia}(M, N) = \text{ia}(L, N)$, whence it follows that when $E_\mathcal{N}(f)$ is defined, it is independent of the choice of $M$.

Let $K$ be a number field, and let $\bar{K}$ be an algebraic closure of $K$. Given a pair $(F, i)$, where $F \subset \bar{K}$ is a Galois extension of $K$ and $i$ is an isomorphism between the Galois group of $F/K$ and $G$, we view $\text{Gal}(F/K)$-modules as $G$-modules via $i$. Let $\mathcal{F}$ be the set of all pairs $(F, i)$, where $F \subset \bar{K}$ is a Galois extension of $K$ that contains no primitive $p$-th root of unity for any prime number $p \in P$, and $i$ is an isomorphism between the Galois group of $F/K$ and $G$ such that there is an isomorphism $\mathbb{Q}[G] \otimes_{\mathbb{Z}[G]} \mathcal{O}_F^\times \cong V$ of $\mathbb{Q}[G]$-modules. Assume that $\mathcal{F}$ is infinite.

For $(F, i) \in \mathcal{F}$, let $c_{F/K}$ be the ideal norm of the product of the prime ideals of $\mathcal{O}_K$ that ramify in $F/K$. For a positive real number $B$, let $\mathcal{F}_{c \leq B} = \{(F, i) \in \mathcal{F} : c_{F/K} \leq B\}$. If $M$ is a finitely generated $\Lambda$-module satisfying $A \otimes_{\Lambda} M \cong V_A$, and $f$ is a function defined on $\mathcal{M}_V$, then we write $f(M)$ for the value of $f$ on the unique element of $\mathcal{M}_V$ that is isomorphic to $M$. In [BLJ20] the following version of the Cohen–Lenstra–Martinet heuristic was proposed (with slightly different notation).

**Conjecture 4.1.** Assume that the set $P$ is finite, and let $f$ be a “reasonable” $\mathbb{C}$-valued function on $\mathcal{M}_V$. Then the limit

$$\lim_{B \to \infty} \frac{\sum_{(F, i) \in \mathcal{F}_{c \leq B}} f(A \otimes_{\mathbb{Z}[G]} \text{Ar}_F)}{\# \mathcal{F}_{c \leq B}}$$

exists, and is equal to $E_{\mathcal{M}_V}(f)$.

If the assumption of finiteness on $P$ is dropped, however, then Conjecture 4.1 can be an obstruction to the conclusions of Conjecture 4.1. For example it was shown in [BLJ20 §4], as a consequence of a proven special case of Conjecture 3.13 that without the finiteness assumption the conclusion of Conjecture 4.1 does not hold, in general, for functions of the form $M \mapsto \chi([M])$, where $\chi: G_0(\Lambda) \to \mathbb{C}^\times$ is a homomorphism of finite order. In the remainder of the section, we formulate a Cohen–Lenstra–Martinet heuristic without the hypothesis that $P$ be finite.

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\(^1\)The notion of a “reasonable” function was made precise in [BLJ20], but will not concern us further here.
Let Proposition 5.1. Conjecture 3.1]. We will then prove the following concrete result. Let $\Lambda$ be a commutative ring, and let $\Lambda^\times$ be the group of units of $\Lambda$. By definition, $K_0(\Lambda)$ is the Grothendieck group of the category of finitely generated $\Lambda$-modules.

The purpose of this section is to give some evidence for Conjectures 3.9 and 3.13. We will first explain that Conjecture 3.9 follows from Conjecture 3.11. By Dirichlet’s unit theorem, there is an isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}^\times_F \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{O}^\times_{F'}$, where $\mathcal{O}^\times_F$ is the group of units of $\mathcal{O}_F$. By the Noether–Deuring theorem [CR81, Exercise 6.6], there are analogous isomorphisms of $\mathbb{Q}[G]$-modules. Since all point stabilisers for $S_\infty$ and $S'_\infty$ are inertia groups at Archimedean places, they are all cyclic. It follows from Artin’s induction theorem (e.g. by combining [Ser77] Corollary 13.1, §38, and comparing dimensions) that if $S$ and $S'$ are finite G-sets with cyclic point stabilisers such that there is an isomorphism $\mathbb{Q}[S] \cong \mathbb{Q}[S']$ of $\mathbb{Q}[G]$-modules, then the G-sets $S$ and $S'$ are isomorphic. In particular, there is then an isomorphism $\mathbb{Z}[S] \cong \mathbb{Z}[S']$ of $\mathbb{Z}[G]$-modules. The result follows by applying this observation to the G-sets $S_\infty$ and $S'_\infty$. \hfill \Box

We define $C(\mathcal{F})$ to be the common class of $\Lambda \otimes_{\mathbb{Z}[G]} \mathbb{Z}[S_\infty]$ in $G_0(\Lambda)$ for all $(F,i) \in \mathcal{F}$, where $S_\infty$ is the G-set of Archimedean places of $F$.

**Heuristic 4.3.** Let $\mathcal{N} = \{M \in \mathcal{M}_V : |M| = C(\mathcal{F}) \text{ in } G_0(\Lambda)\}$, and let $f$ be a “reasonable” $C$-valued function on $\mathcal{M}_V$. Then the limit

$$\lim_{B \to \infty} \frac{\sum_{(F,i) \in \mathcal{F} \leq B} f(\Lambda \otimes_{\mathbb{Z}[G]} \text{Art}_F)}{\# \mathcal{F} \leq B}$$

exists, and is equal to $E_\mathcal{N}(f)$.\hfill \□

5. THE RELATION TO CHINBURG’S $\Omega(3)$ CONJECTURE

The purpose of this section is to give some evidence for Conjectures 3.9 and 3.13. We will first explain that Conjecture 3.9 follows from Conjecture 5.1 below, which is a weaker variant of Chinburg’s $\Omega(3)$ conjecture [Chi85, Conjecture 3.1]. We will then prove the following concrete result.

**Proposition 5.1.** Let $F/K$ be a finite Galois extension of number fields, let $G$ be the Galois group, and let $G'$ be its commutator subgroup.

(a) If $F/Q$ is abelian, then Conjecture 3.9 holds for $F/K$.

(b) If $F[G]/Q$ is abelian and $\#G < 122$, then Conjecture 5.1 holds for $F/K$.

We first briefly review some material from algebraic K-theory. We refer the reader to [CR87] §38, §39, §49 for further details.

Let $R$ be a Dedekind domain and let $\Lambda$ be an $R$-order. Let $K_0(\Lambda)$ denote the Grothendieck group of the category of finitely generated projective $\Lambda$-modules. By definition, $K_0(\Lambda)$ is the additive group generated by expressions $[P]$, one for each isomorphism class of finitely generated projective $\Lambda$-modules $P$, with relations $[P_1 \oplus P_2] = [P_1] + [P_2]$ for all such modules $P_1, P_2$. By [CR87] (38.50), the group $K_0(\Lambda)$ can be identified with the Grothendieck group of the category of finitely generated $\Lambda$-modules of finite projective dimension.
For each maximal ideal $p$ of $R$, let $R_p$ denote the localisation of $R$ at $p$, and define $\Lambda_p = R_p \otimes_R \Lambda$. A $\Lambda$-lattice $M$ is said to be locally free if $\Lambda_p \otimes_{\Lambda} M$ is free over $\Lambda_p$ for every such $p$. Let $K_0(\Lambda) \to K_0(\Lambda_p)$ be the map induced by the ring homomorphism $\Lambda \to \Lambda_p$. Then the locally free class group $Cl(\Lambda)$ is defined to be the kernel of the homomorphism $K_0(\Lambda) \to \prod_p K_0(\Lambda_p)$, where the product runs over all maximal ideals $p$ of $R$ (see [CR87] (39.12)). By [CR87] (39.13) we have

$$Cl(\Lambda) = \{[\Lambda] - [L] \in K_0(\Lambda) : L \neq 0\}$$

Note that there are several equivalent definitions of $Cl(\Lambda)$ (see [CR87] §49A, particularly [CR87] p. 223).

We now recall the statement of Chinburg’s $\Omega(3)$ conjecture. For the rest of the section, let $F/K$ be a finite Galois extension of number fields and let $G$ be the Galois group. For any finite $G$-stable set $S$ of places of $F$, let $X_S$ be the kernel of the augmentation map $\mathbb{Z}[S] \to \mathbb{Z}$. Henceforth let $S$ be a finite $G$-stable set of places of $F$ such that

1. $S$ contains the Archimedean places $S_\infty$ of $F$,
2. $S$ contains the ramified places of $F/K$, and
3. for every subfield $N$ of $F$ containing $K$, the ideal class group of $N$ is generated by the classes $\{p \cap \mathcal{O}_N : p \in S \setminus S_\infty\}$.

Tate [Tat66] p. 711 defined a canonical class $\alpha = \alpha_S \in \text{Ext}^2_{\mathbb{Z}[G]}(X_S, \mathcal{O}_F^\times)$, and showed the existence of so-called Tate sequences [Tat84] II, Théorème 5.1, that is, four term exact sequences of finitely generated $\mathbb{Z}[G]$-modules

$$0 \to \mathcal{O}_F^\times \to A \to B \to X_S \to 0$$

representing $\alpha$, where $A$ and $B$ are of finite projective dimension. In [Ch85], Chinburg defined $\Omega(F/K, 3) = [A] - [B] \in K_0(\mathbb{Z}[G])$. Moreover, he showed that $\Omega(F/K, 3)$ lies in the locally free class group $Cl(\mathbb{Z}[G])$, and depends only on the extension $F/K$; in particular, it does not depend on the choice of $S$ or on the choice of exact sequence (5.3).

The root number class $W_{F/K} \in Cl(\mathbb{Z}[G])$ was defined by Ph. Cassou-Noguès in the case that $F/K$ is at most tamely ramified, and was generalised to wildly ramified extensions $F/K$ by Fröhlich [Frö78]. It is an element of order at most 2, and is defined in terms of the Artin root numbers of the irreducible symplectic characters of $G$. Moreover, if $G$ has no irreducible symplectic characters (for example, if $G$ is abelian or of odd order), then $W_{F/K}$ is trivial by definition.

**Conjecture 5.4** (Chinburg’s $\Omega(3)$ conjecture). *There is an equality $\Omega(F/K, 3) = W_{F/K}$.*

**Theorem 5.5.** *If $F/\mathbb{Q}$ is abelian, then $\Omega(F/K, 3) = W_{F/K} = 0$.*

**Proof.** This is a special case of [BF06] Corollary 1.4. \qed

Fix, for the rest of the section, a maximal order $\mathcal{M}$ in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$, let $\rho : Cl(\mathbb{Z}[G]) \to Cl(\mathcal{M})$ be the map induced by the ring homomorphism $\mathbb{Z}[G] \to \mathcal{M}$, and define the kernel subgroup $D(\mathbb{Z}[G])$ of $Cl(\mathbb{Z}[G])$ to be the kernel of $\rho$. If $\mathcal{M}'$ is any other maximal order in $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$, and
\( \rho' : \text{Cl}(\mathbb{Z}[G]) \to \text{Cl}(\mathcal{M}') \) is the analogous map, then by \( \text{[CR87, (49.34)]} \) the kernel of \( \rho \) is equal to that of \( \rho' \).

**Conjecture 5.6** (Chinburg’s \( \Omega(3) \) conjecture modulo the kernel group).

We have \( \Omega(F/K, 3) \equiv W_{F/K} \mod D(\mathbb{Z}[G]) \).

**Proposition 5.7.** Conjecture 5.6 for \( F/K \) implies Conjecture 3.9 for \( F/K \).

In order to prove this result, we will make use of the following lemma. Let \( \Psi(F/K) = [O_F^\times] - [X_{S_{\infty}}] - [\text{Cl}_F] \in G_0(\mathbb{Z}[G]) \).

**Lemma 5.8.** Conjecture 3.9 for \( F/K \) is equivalent to the vanishing of \( \Psi(F/K) \). Moreover, we always have \( \Psi(F/K) \in G_0(\mathbb{Z}[G])_{\text{tors}} \).

**Proof.** By Corollary 3.7, Conjecture 3.9(a) is equivalent to the assertion that

\[ [Z] = \sigma([Z][S]) - [O_{F,S}^\times] \]

in \( G_0(\mathbb{Z}[G]) \). Since \( \sigma[Z] = [Z] \), this is equivalent to

\[ [O_{F,S}^\times] = [Z][S] - [Z] = [X_S] \]

By Corollary 3.8 with \( S' = S_{\infty} \), this in turn is equivalent to \( \Psi(F/K) = 0 \).

The last claim follows from Lemma 3.10. \( \square \)

It is well known to experts in Galois module theory that Conjecture 5.6 for \( F/K \) implies the vanishing of \( \Psi(F/K) \) (see \text{[Chi83, III], [CNCFT91, §4, Proposition 6] or [CKPS98, §1]}). We will recall this argument in the following proof and give some additional references.

**Proof of Proposition 5.7.** We begin with some observations that are unconditional, i.e. do not depend on Conjecture 5.6. Let \( \mu : K_0(\mathbb{Z}[G]) \to G_0(\mathbb{Z}[G]) \) denote the Cartan map, which is induced by letting \( \mu([A]) = [A] \) if \( A \) is a finitely generated projective \( \mathbb{Z}[G] \)-module. Then it follows from the definition of \( \Omega(F/K, 3) \), from the exact sequence (5.3), and from Corollary 3.8 with \( S' = S_{\infty} \), this in turn is equivalent to \( \Psi(F/K) = 0 \).

Moreover, a special case of a result of Queyrut \text{[Que85, Proposition 2.3]} shows that \( \mu(W_{F/K}) = 0 \).

Let \( \xi \in \text{Cl}(\mathbb{Z}[G]) \). Write \( \xi = [Z[G]] - [L] \), where \( L \) is a locally free left ideal of \( \mathbb{Z}[G] \), which we may do by \text{[5.7.2]}. By \text{[CR81, Exercise 31.10]} we have \( \xi = [Z[G]] - [L] = [\mathcal{M}] - [\mathcal{M} \otimes_{\mathbb{Z}[G]} L] \) in \( G_0(\mathbb{Z}[G]) \).

Hence we have the following commutative diagram of abelian groups:

\[
\begin{array}{ccc}
\text{Cl}(\mathbb{Z}[G]) & \xrightarrow{\mu} & G_0(\mathbb{Z}[G]) \\
\rho \downarrow & & \alpha \uparrow \\
\text{Cl}(\mathcal{M}) & \xrightarrow{\mu'} & G_0(\mathcal{M}),
\end{array}
\]

where \( \alpha \) is induced by restriction, and \( \mu' \) is defined analogously to \( \mu \).

Now suppose that Conjecture 5.6 holds for \( F/K \). By definition of \( D(\mathbb{Z}[G]) \), this is equivalent to \( \rho(\Omega(F/K, 3)) = \rho(W_{F/K}) \). Since \( \mu \) factors via \( \rho \), we have

\[ \Psi(F/K) = \mu(\Omega(F/K, 3)) = \mu(W_{F/K}) = 0. \]

The result now follows from Lemma 5.8. □

Proof of Proposition 5.7. If $F/Q$ is abelian, then Conjecture 3.9 for $F/K$ follows from Theorem 5.5 and Proposition 5.7. This proves part (a) of the proposition.

We now prove part (b). Suppose that $F^G/Q$ is abelian and $\# G < 112$. For an element $C$ of $G_0(\mathbb{Z}(G))$, let $\mathbb{Z}(G) \otimes C$ denote the image of $C$ in $G_0(\mathbb{Z}(G))$. Then by Lemma 5.8 we have $\mathbb{Z}(G) \otimes \Psi(F/K) \in G_0(\mathbb{Z}(G))_{\text{tors}}$, and the claim is equivalent to the assertion that $\mathbb{Z}(G) \otimes \Psi(F/K) = 0$.

Let $\text{Irr}_{\text{na}}(G)$ denote the set of complex irreducible characters of $G$ of degree greater than 1 (“na” stands for “non-abelian”), and for $\chi, \chi' \in \text{Irr}_{\text{na}}(G)$ write $\chi \sim \chi'$ if there exists $\tau \in \text{Gal}(\overline{Q}/Q)$ such that $\chi = \tau \circ \chi'$. Then there is a direct product decomposition of $\mathbb{Q}$-algebras

$$\mathbb{Q}[G] \cong \mathbb{Q}[G/G'] \times \bigoplus_{\chi \in \text{Irr}_{\text{na}}/\sim} A_{\chi},$$

where the product is taken over all the quotient of representatives of Galois orbits of non-abelian characters of $G$, and each $A_{\chi}$ is a simple $\mathbb{Q}$-algebra. By [CR81 (27.1)], the $\mathbb{Z}(G)$-order $\mathbb{Z}(G)$ is maximal in $\mathbb{Q}[G]$, so there is a corresponding direct product decomposition $\mathbb{Z}(G)[G] \cong \mathbb{Z}(G)[G'] \times \bigoplus_{\chi \in \text{Irr}_{\text{na}}/\sim} R_{\chi}$, where each $R_{\chi}$ is a maximal $\mathbb{Z}(G)$-order in $A_{\chi}$, and accordingly a direct sum decomposition of abelian groups

$$G_0(\mathbb{Z}(G))_{\text{tors}} \cong G_0(\mathbb{Z}(G)[G/G'])_{\text{tors}} \bigoplus \bigoplus_{\chi \in \text{Irr}_{\text{na}}/\sim} G_0(R_{\chi})_{\text{tors}}.$$

The claim that $\mathbb{Z}(G) \otimes \Psi(F/K) = 0$ in $G_0(\mathbb{Z}(G))_{\text{tors}}$ is therefore equivalent to the claim that the image of $\Psi(F/K)$ in each of the above summands is 0.

Since $P$ does not contain any prime divisors of $\#G'$, it is easily seen that the image of $\Psi(F/K)$ in $G_0(\mathbb{Z}(G)[G/G'])_{\text{tors}}$ is equal to the image of $\Psi(F'G'/K)$. Since $F^G/Q$ is abelian, that image is 0 by part (a) of the proposition.

For every $\chi \in \text{Irr}_{\text{na}}(G)$, let $Z_{\chi}$ denote the centre of $A_{\chi}$. Then by [CR81 (38.67)], the group $G_0(R_{\chi})_{\text{tors}}$ is the quotient of the narrow class group of $Z_{\chi}$ if $\chi$ is symplectic, and of the usual ideal class group of $Z_{\chi}$ otherwise, by the subgroup generated by the non-zero integral prime ideals of the ring of integers of $Z_{\chi}$ not dividing any element of $P$. A direct computation, e.g. using the computational algebra system MAGMA [BCP97], shows that since $\#G < 112$, all these quotients are trivial, which completes the proof. □

Remark 5.9. There are exactly two groups of order 112 that have an irreducible character $\chi$ of degree greater than 1 such that, in the notation of the proof of Proposition 5.1, the group $G_0(R_{\chi})_{\text{tors}}$ is non-trivial. Each has exactly one such character, in both cases of degree 2. The two groups are both semidirect products of a cyclic group $C$ of order 56 and a group $H$ of order 2. Let $x \in C$ be an element of order 7, and let $y \in C$ be an element of order 8. In one semidirect product, the non-trivial element of $H$ acts on $C$ by $x \mapsto x^{-1}$ and $y \mapsto y^5$; and in the other it acts on $C$ by $x \mapsto x^{-1}$ and $y \mapsto y^3$. These are the two smallest Galois groups $G$ for which we do not currently know Conjecture 3.13 with $K = \mathbb{Q}$. 

References

[BCP97] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993).

[BF06] D. Burns and M. Flach, On the equivariant Tamagawa number conjecture for Tate motives. II, Doc. Math. (2006), no. Extra Vol., 133–163. MR 2290586

[BLJ17] A. Bartel and H. W. Lenstra Jr., Commensurability of automorphism groups, Compos. Math. 153 (2017), no. 2, 323–346. MR 3705226

[BLJ20] ———, On class groups of random number fields, Proc. Lond. Math. Soc., to appear, arXiv:1803.06903v3 (2020).

[Chi83] T. Chinburg, On the Galois structure of algebraic integers and $S$-units, Invent. Math. 74 (1983), no. 3, 321–349. MR 724009

[Chi85] ———, Exact sequences and Galois module structure, Ann. of Math. (2) 121 (1985), no. 2, 351–376. MR 786352

[CKPS98] T. Chinburg, M. Kolster, G. Pappas, and V. Snaith, Galois structure of $K$-groups of rings of integers, $K$-Theory 14 (1998), no. 4, 319–369. MR 1641555

[CLJ84] H. Cohen and H. W. Lenstra Jr., Heuristics on class groups of number fields, Number theory, Noordwijkerhout 1983 (Noordwijkerhout, 1983), Lecture Notes in Math., vol. 1068, Springer, Berlin, 1984, pp. 33–62. MR 756082

[CM90] H. Cohen and J. Martinet, Étude heuristique des groupes de classes des corps de nombres, J. Reine Angew. Math. 404 (1990), 39–76. MR 1037430

[CNCFT91] Ph. Cassou-Noguès, T. Chinburg, A. Fröhlich, and M. J. Taylor, $L$-functions and Galois modules, $L$-functions and arithmetic (Durham, 1989), London Math. Soc. Lecture Note Ser., vol. 153, Cambridge Univ. Press, Cambridge, 1991, Based on notes by D. Burns and N. P. Byott, pp. 75–139. MR 1110391

[CR81] C. W. Curtis and I. Reiner, Methods of representation theory. Vol. I, John Wiley & Sons, Inc., New York, 1981, With applications to finite groups and orders, Pure and Applied Mathematics, A Wiley-Interscience Publication. MR 632548

[CR87] ———, Methods of representation theory. Vol. II, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1987. With applications to finite groups and orders, A Wiley-Interscience Publication. MR 892316

[Frö78] A. Fröhlich, Some problems of Galois module structure for wild extensions, Proc. London Math. Soc. (3) 37 (1978), no. 2, 193–212. MR 0507063

[Lan94] S. Lang, Algebraic number theory, second ed., Graduate Texts in Mathematics, vol. 110, Springer-Verlag, New York, 1994. MR 1282723

[NSW08] J. Neukirch, A. Schmidt, and K. Wingberg, Cohomology of number fields, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR 2392026

[Que85] J. Queyrut, Anneaux d’entiers dans le même genre, Illinois J. Math. 29 (1985), no. 1, 157–179. MR 769765

[Sch08] R. Schoof, Computing Arakelov class groups, Algorithmic number theory: lattices, number fields, curves and cryptography, Math. Sci. Res. Inst. Publ., vol. 44, Cambridge Univ. Press, Cambridge, 2008, pp. 447–495. MR 2467554

[Ser77] J.-P. Serre, Linear representations of finite groups, Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. MR 0450380

[Tat66] J. Tate, The cohomology groups of tori in finite Galois extensions of number fields, Nagoya Math. J. 27 (1966), 709–719. MR 0207680

[Tat84] ———, Les conjectures de Stark sur les fonctions $L$ d’Artin en $s = 0$, Progress in Mathematics, vol. 47, Birkhäuser Boston, Inc., Boston, MA, 1984, Lecture notes edited by Dominique Bernardi and Norbert Schappacher. MR 782485