A Theoretical Analysis of Learning with Noisily Labeled Data

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Abstract

Noisy labels are very common in deep supervised learning. Although many studies tend to improve the robustness of deep training for noisy labels, rare works focus on theoretically explaining the training behaviors of learning with noisily labeled data, which is a fundamental principle in understanding its generalization. In this draft, we study its two phenomena, clean data first and phase transition, by explaining them from a theoretical viewpoint. Specifically, we first show that in the first epoch training, the examples with clean labels will be learned first. We then show that after the learning from clean data stage, continuously training model can achieve further improvement in testing error when the rate of corrupted class labels is smaller than a certain threshold; otherwise, extensively training could lead to an increasing testing error.

1 Introduction

Noisy labels are very common in the real world [Xiao et al., 2015, Li et al., 2017, Lee et al., 2018, Song et al., 2019] due to some reasons such as data labeling by human mistake and the complexity of label. In many tasks, even if wrongly labeled examples could be recognized by a human expert, it is impossible to manually review all training examples [Koh and Liang, 2017]. Learning with noisily labeled data is probably inevitable and becomes a common challenge in supervised learning, especially for large-scale deep learning problems.

It has been shown that noisy labels would lead to performance degradation when training deep neural networks (DNNs) because of over-fitting to the noise [Arpit et al., 2017]. However, it is a fundamental principle for understanding the generalization of deep learning models [Zhang et al., 2016]. Empirically, learning with noisily labeled data has been examined by many recent studies such as [Song et al., 2019, Lyu and Tsang, 2019, Wang et al., 2019a, Li et al., 2020a, Liu et al., 2020, Nguyen et al., 2020]. Most of them are designed to improve learning performance by make a supervised learning process more robust to label noise [Song et al., 2020, Algan and Ulusoy, 2021]. Rare studies focus on explaining the training behavior of noisily labeled data from the view of theory. Empirical results have shown that even with a significant portion of the training data assigned to random labels, we are still able to learn an appropriate prediction model that can make accurate prediction [Li et al., 2020a, Liu et al., 2020]. Two interesting phenomena have been observed from the task of learning from noisily labeled data. The first observation is clean data first, i.e. data points with clean labels will be learned first. Although it was shown “intuitively” through a linear
binary classification case [Liu et al., 2020], the full story behind this belief is unclear. In the first
epoch, we can consider the standard optimization analysis due to the statistical independence
property of current model and sampled example(s)\(^1\). The convergence results shows that stochastic
gradient descent (SGD) essentially converges to the objective function of examples with clean labels
in the order of \(\tilde{O}(1/m)\) \(^2\), where \(m\) is the number of training examples with clean labels. The second
observation is phase transition, i.e. when \(\gamma\), the percentage of data with corrupted class labels,
is smaller than a certain threshold, continuously training model can lead to a further reduction in
testing error after the stage of learning from clean data. However, when \(\gamma\) exceeds the certain
threshold, extensively training could result in an increase in testing error. In this draft, we aim to
explain both phenomenon from a theoretical viewpoint, which could be considered as a theoretical
understanding of learning with noisy labels.

\section{Preliminaries and Notations}

We introduce preliminaries in this section. We first give some notations and the problem defini-
tion, then we present some assumptions that will be used in the convergence analysis.

Let \(\nabla_{\theta} f(\theta)\) denote the gradient of a function \(f(\theta)\). When the variable to be taken a gradient is
obvious, we use \(\nabla f(\theta)\) for simplicity. We use \(|| \cdot ||\) to denote the Euclidean norm. Let \(\langle \cdot, \cdot \rangle\) be the inner product.

The classification problem aims to seek a classifier to map an instance \(x \in \mathbb{R}^d\) onto one of labels
\(y \in \{+1, -1\}\). Suppose the instances are draw from a distribution \(\mathcal{P}\), i.e., \(x \sim \mathcal{P}\), where \(\mathcal{P}\)
is usually unknown. For simplicity, we assume \(y = y(x)\), i.e. the class label of instance \(x\) is decided by
a deterministic function \(y(x)\). Our goal is to learn a prediction function \(f(x; \theta) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}\)
that is as close as possible to \(y\), where \(\theta \in \mathbb{R}^d\) is the model parameter. To achieve this goal,
traditional method is to minimize the following expected loss
\(\min_{\theta \in \mathbb{R}^d} \mathbb{E}_{x \sim \mathcal{P}} [1/2 \| f(\theta; x) - y(x) \|^2]\), which is also known as expected risk minimization. Since the distribution \(\mathcal{P}\) is unknown, solving
the risk minimization is difficulty. However, one can sample \(n\) examples \(\{(x_1, y_1), \ldots, (x_n, y_n)\}\) from
\(\mathcal{P}\) independently, then we aim to minimize an empirical loss
\(\min_{\theta \in \mathbb{R}^d} \frac{1}{2n} \sum_{i=1}^n \| f(\theta; x_i) - y_i(x_i) \|^2\),
which is also known as empirical risk minimization. In this study, the training data set contains
the full story behind this belief is unclear. In the first
clean labels and noisy labels, we denote by \(\gamma\), the percentage of data points with noisy labels. We
divide the training data into two parts:

- **clean data**: \(D_a = \{(x_i^a, y_i^a), i = 1, \ldots, m\}\), where \(x_i^a \in \mathbb{R}^d\) is sampled independently from
  a given distribution \(\mathcal{P}\) and \(y_i^a = y(x_i^a)\) is a clean class label decided by function \(y : \mathbb{R}^d \rightarrow \{-1, +1\}\), and

- **corrupted data**: \(D_b = \{(x_i^b, y_i^b), i = 1, \ldots, n\}\), where \(x_i^b\) is sampled independently from \(\mathcal{P}\) and
  \(y_i^b\) is assigned randomly with equal chance to be either \(-1\) or \(+1\).

Please note that the labels \(y_i^b\) from corrupted data is independent of instance \(x_i^a\), whose expectation
is 0, i.e. \(\mathbb{E}[y_i^b] = 0\). The percentage of noisily labeled data points is \(\gamma = n/(m + n)\). We consider
the following expected risk to be optimized

\[ \mathcal{L}(\theta) = (1 - \gamma)\mathcal{L}_a(\theta) + \gamma \mathcal{L}_b(\theta), \]  

\(^1\)We could use the uniform sampling strategy without replacement when implementing stochastic gradient descent
during the training process.

\(^2\)\(\tilde{O}(\cdot)\) suppresses a logarithmic factor
where $L_a(\theta)$ is the loss function for clean labeled data and $L_b(\theta)$ is the loss function related to noisily labeled data, whose definitions are given by

$$
\begin{align*}
L_a(\theta) &= \mathbb{E}_{x \sim \mathcal{P}} \left[ \frac{1}{2} f(x; \theta) - y(x) \right]^2, \\
L_b(\theta) &= \mathbb{E}_{x \sim \mathcal{P}} \left[ \frac{1}{2} |f(x; \theta)|^2 \right].
\end{align*}
$$

In practice, we want to solve the following empirical risk minimization problem

$$
\min_{\theta \in \mathbb{R}^d} \hat{\mathcal{L}}(\theta) := (1 - \gamma)\hat{L}_a(\theta) + \gamma\hat{L}_b(\theta),
$$

where $\hat{L}_a(\theta)$ is the loss function over training data with clean labels and $\hat{L}_b(\theta)$ is the loss function over training data with noisy labels, whose definitions are given by

$$
\begin{align*}
\hat{L}_a(\theta) &= \frac{1}{2n} \sum_{i=1}^m |f(x_i^a; \theta) - y_i^a|^2, \\
\hat{L}_b(\theta) &= \frac{1}{2n} \sum_{i=1}^n |f(x_i^b; \theta) - y_i^b|^2.
\end{align*}
$$

Since $\mathbb{E}[\hat{L}_b(\theta)] = L_b(\theta) + \frac{1}{2}$, we may consider $\mathcal{L}(\theta)$ as the final objective function. As a widely used method, SGD can be employed to solve problem (3) and it updates solutions iteratively by

$$
\theta_{t+1} = \theta_t - \eta g_t
$$

where $t$ is the iteration number, $\eta > 0$ is a learning rate, and $g_t$ is the stochastic gradient given by

$$
g_t = (f(x_i; \theta_t) - y_i) \nabla f(x_i; \theta_t)
$$

with $(x_i, y_i)$ is uniformly sampled from $D_a \cup D_b$.

To establish the theoretical analysis, we introduce some commonly used assumptions in the literature of optimization [Ghadimi and Lan, 2013, Xu et al., 2019].

**Assumption 1.** Assume the following conditions hold:

(i) There exists two constants $F > 0$ and $G > 0$, such that

$$
\|f(x; \theta)\| \leq F, \quad \|f(x; \theta) - y(x)\| \leq F,
$$

and

$$
\|(f(x; \theta) - y(x))\nabla f(x; \theta)\| \leq G, \quad \|f(x; \theta)\nabla f(x; \theta)\| + \|\nabla f(x; \theta)\| \leq G.
$$

(ii) The loss function $\mathcal{L}$ is smooth with an $L$-Lipchitz continuous gradient, i.e., it is differentiable and there exists a constant $L > 0$ such that

$$
\mathcal{L}(\theta') \leq \mathcal{L}(\theta'') + \langle \nabla \mathcal{L}(\theta''), \theta' - \theta'' \rangle + \frac{L}{2} \|\theta' - \theta''\|^2, \forall \theta', \theta'' \in \mathbb{R}^d.
$$

We now introduce an important functional property, which is called Polyak-Lojasiewicz (PL) condition [Polyak, 1963]. PL condition has been observed in training deep neural networks [Allen-Zhu et al., 2019, Yuan et al., 2019], and it is widely used to establish convergence in the literature of optimization [Yuan et al., 2019, Wang et al., 2019b, Karimi et al., 2016, Li and Li, 2018, Charles and Papailiopoulos, 2018, Li et al., 2020b].

**Definition 1.** A function $\mathcal{L}$ satisfies $\tau$-PL condition if there exists a constant $\tau > 0$ such that

$$
2\tau (\mathcal{L}(\theta) - \mathcal{L}^*) \leq \|\nabla \mathcal{L}(\theta)\|^2, \forall \theta \in \mathbb{R}^d, \text{where } \mathcal{L}^* = \min_{\theta \in \mathbb{R}^d} \mathcal{L}(\theta) \text{ is the optimal value}.
$$

For the simplicity, we set $\mathcal{L}_a^{\min} := \min_{\theta \in \mathbb{R}^d} L_a(\theta) = 0$ and $\mathcal{L}_b^{\min} := \min_{\theta \in \mathbb{R}^d} L_b(\theta) = 0$ in our proofs without loss of generality, which is a common property observed in training deep neural networks [Zhang et al., 2016, Allen-Zhu et al., 2019, Du et al., 2018, 2019, Arora et al., 2019, Chizat et al., 2019, Hastie et al., 2019, Yun et al., 2019, Zou et al., 2020].
3 Main Results

We first study the observation of learning from data with clean labels first, and then the observation of phase transition in terms of $\gamma$.

3.1 Learning from data with clean labels first

To this end, we will focus on the optimization of the first epoch. Note that in the first epoch, it satisfies the condition of statistical independence and therefore we can apply the standard analysis of stochastic gradient descent.

**Theorem 1.** Under Assumption 1, suppose $\mathcal{L}_a$ and $\mathcal{L}_b$ satisfies $\mu_a$-PL condition and $\mu_b$-PL condition, respectively. Let $\theta_0$ be our initial solution, and $\theta_1$ be the solution after running through the first epoch of data, and $\tau := \min(2\mu_a, 2\mu_b)$. By selecting $\eta \leq 1/L$ and $m \geq \frac{4G^2\log m}{\tau^2L(\theta_0)}$. We have

$$E[\mathcal{L}_a(\theta_1)] \leq O\left(\frac{4G^2\log m}{(1 - \gamma)\tau^2m}\right).$$

**Remark.** Through the first epoch of optimization, since we do not reuse any data point, either clean or corrupted one, SGD essentially optimizes the objective function $\mathcal{L}(\theta)$, instead of $\mathcal{L}_a(\theta)$. As indicated by the above analysis, despite of the presence of noisy labels, the resulting solution is still able to reach a low value of $\mathcal{L}_a(\theta)$, which is on the order of $O(\log(m)/m)$. Of course, the presence of noisy labels indeed hinders the progress of optimization, through parameter $\gamma$ (i.e. the percentage of noisily labeled data), $G^2$ and $\tau$. This analysis explains why for learning from noisily labeled data, it appears that the model could learn clean data first because for early epoch, noisily labeled data behave like regularizer to our solution.

3.2 Phase Transition for $\gamma$

Now, let’s examine the late stage of the optimization beyond the first epoch, in which we do not have the statistical independence between the model to be updated and the training data. As a result, our objective function “becomes” $\hat{\mathcal{L}}(\theta) = (1 - \gamma)\hat{\mathcal{L}}_a(\theta) + \gamma\hat{\mathcal{L}}_b(\theta)$. Let $\theta_1$ be the solution obtained from the first epoch. Define

$$\Omega(\tilde{r}, s) = \left\{ \theta : \frac{1}{m}\sum_{i=1}^{m} \| f(x_i^a; \theta) - y(x_i^a) \|^2 \leq \tilde{r}^2, \frac{1}{m}\sum_{i=1}^{m} \| f(x_i^a; \theta) - y(x_i^a) \| \leq \sqrt{s} \right\}$$

where

$$\tilde{r}^2 := \frac{1}{m}\sum_{i=1}^{m} \| f(x_i^a; \theta_1) - y(x_i^a) \|^2, \quad \sqrt{s} := \frac{1}{m}\sum_{i=1}^{m} \| f(x_i^a; \theta_1) - y(x_i^a) \|$$

We assume that $s \ll m$, implying that vector $(f(x_1; \theta), \ldots, f(x_m; \theta))$ is a sparse vector. We first give the following results of bounding the variances of the stochastic gradients, which will be used in our analysis of the main result.
Theorem 2. Under the assumption \( m \geq 64F^2 \log(2/\delta) \), we have, with a probability \( 1 - 3\delta \),

\[
\max_{\theta \in \Omega(\hat{r}, s)} \| \nabla \mathcal{L}_a(\theta) - \nabla \hat{\mathcal{L}}_a(\theta) \| \leq C \left( \frac{GF \log(1/\delta)}{n} + GF \left[ \frac{\log(1/\delta)}{n} + \frac{s}{n} \log \frac{2n}{s} (1 + \log n) \right] \right),
\]

\[
(7)
\]

\[
\max_{\theta \in \Omega(\hat{r}, s)} \| \nabla \mathcal{L}_b(\theta) - \nabla \hat{\mathcal{L}}_b(\theta) \| \leq C \left( \frac{GF \log(1/\delta)}{n} + GF \left[ \frac{\log(1/\delta)}{n} + \frac{s}{n} \log \frac{2n}{s} (1 + \log n) \right] \right),
\]

\[
(8)
\]

where \( C \) is a universal constant.

To understand the phenomenon of phase transition (i.e. the optimizer may continuously reduce the testing error when \( \gamma \) is smaller than a threshold, and will lead to an increase in testing error after extensive training if \( \gamma \) is above the threshold), we will examine the condition for all the solutions in the later iterations after the first epoch that will stay within \( \Omega(\hat{r}, s) \). If solution \( \theta \) continuously stays in \( \Omega(\hat{r}, s) \), we will not see the degradation in performance. Otherwise, we may find a large error after the training of the first epoch. To this end, we will assume that is the case, and see when the condition will be violated. In this case, the stochastic gradient \( g_t \) is not a unbiased estimator of \( \nabla \mathcal{L}(\theta'_t) \), i.e. \( E[g_t] \neq \nabla \mathcal{L}(\theta'_t) \)

Theorem 3. Under Assumption 1, suppose \( \mathcal{L}_a \) and \( \mathcal{L}_b \) satisfies \( \mu_a \)-PL condition and \( \mu_b \)-PL condition, respectively. Let \( \theta_1 \) be the solution after running through the first epoch of data, and \( \tau := \min(2\mu_a, 2\mu_b) \). By selecting \( \eta = \frac{2}{\tau} \log \left( \frac{m\tau^2 \mathcal{L}(\theta_0)}{8G^2L} \right) \) and \( m \geq \frac{2L}{\eta} \log \left( \frac{m\tau^2 \mathcal{L}(\theta_0)}{8G^2L} \right) \). If we have

\[
1 - \gamma \geq \max \left\{ C'' G^2 \Delta (\hat{r}^2 + F^2) \frac{16G^2L \log \left( \frac{\mathcal{L}(\theta_1)}{m(1-\gamma)} \right)}{\tau^2 \hat{r}^2} \right\}
\]

\[
(9)
\]

where \( C'' \) is a universal constant, \( \Delta := \left( \log \left( \frac{m}{n} \right) + s \log \left( \frac{2\max(m,n)}{s} \log(\max(m,n)) \right) \right) \), then

\[
E \left[ \mathcal{L}_a(\theta'_t) \right] \leq \frac{\hat{r}^2}{2}.
\]

Remark. The theorem shows that when (9) holds, the solution \( \theta \) continuously stays in \( \Omega(\hat{r}, s) \) and we will not see the degradation in performance. The condition (9) essentially leads to the so called phase transition phenomenon, i.e. the performance over the testing data would degrade with more training epochs. In addition, if the condition (9) does not hold, then we may have a worse performance over the testing data after the first epoch.

4 Proofs

In this section, we provide the proofs of theorems in previous section.
4.1 Proof of Theorem 1

Proof. For each iteration $t$, we have

$$
E[\mathcal{L}(\theta_{t+1}) - \mathcal{L}(\theta_t)]
\leq E[\langle \nabla \mathcal{L}(\theta_t), \theta_{t+1} - \theta_t \rangle] + \frac{L}{2}E[\|\theta_{t+1} - \theta_t\|^2]
$$

(a)

$$
\leq -\eta \left(1 - \frac{\eta L}{2}\right) E\|\nabla \mathcal{L}(\theta_t)\|^2 + 2\eta^2 G^2 L
$$

(b)

$$
\leq -\frac{\eta}{2} E[(1 - \gamma)\nabla \mathcal{L}_a(\theta_t) + \gamma \nabla \mathcal{L}_b(\theta_t)]^2 + 2\eta^2 G^2 L
$$

(c)

$$
= -\frac{\eta}{2} E[(1 - \gamma)^2 \|\nabla \mathcal{L}_a(\theta_t)\|^2 + \gamma^2 \|\nabla \mathcal{L}_b(\theta_t)\|^2 + 2(1 - \gamma)\gamma \langle \nabla \mathcal{L}_a(\theta_t), \nabla \mathcal{L}_b(\theta_t) \rangle] + 2\eta^2 G^2 L,
$$

(11)

where (a) uses the smoothness of $\mathcal{L}$ in Assumption 1 (ii); (b) uses $\theta_{t+1} = \theta_t - \eta g_t$ where $E[g_t] = \nabla \mathcal{L}(\theta_t)$, and $E[\|\mathcal{L}(\theta_t) - g_t\|^2] \leq 4G^2$ due to Assumption 1 (i) with Jensen’s inequality; (c) uses $\eta L \leq 1$ and the definition of $\mathcal{L}(\theta)$. We consider two cases. Case I. when $1 > \gamma > \frac{1}{2}$, define $\alpha = \frac{1 - \gamma}{2\gamma - 1}$, by Young’s inequality we get

$$
(1 - \gamma)^2 \|\nabla \mathcal{L}_a(\theta_t)\|^2 + \gamma^2 \|\nabla \mathcal{L}_b(\theta_t)\|^2 - 2(1 - \gamma)\nabla \mathcal{L}_a(\theta_t), \nabla \mathcal{L}_b(\theta_t))
\leq (1 - \gamma)^2 \|\nabla \mathcal{L}_a(\theta_t)\|^2 + \gamma^2 \|\nabla \mathcal{L}_b(\theta_t)\|^2 + \frac{1 + \alpha}{\alpha} (1 - \gamma)^2 \|\nabla \mathcal{L}_a(\theta_t)\|^2 + \frac{\alpha + 1}{\alpha} \gamma^2 \|\nabla \mathcal{L}_b(\theta_t)\|^2
= (1 - \gamma)\|\nabla \mathcal{L}_a(\theta_t)\|^2 + \gamma \|\nabla \mathcal{L}_b(\theta_t)\|^2.
$$

Case II. when $0 < \gamma \leq \frac{1}{2}$, define $\alpha = \frac{\gamma}{1 - 2\gamma}$, by Young’s inequality we get

$$
(1 - \gamma)^2 \|\nabla \mathcal{L}_a(\theta_t)\|^2 + \gamma^2 \|\nabla \mathcal{L}_b(\theta_t)\|^2 - 2(1 - \gamma)\nabla \mathcal{L}_a(\theta_t), \nabla \mathcal{L}_b(\theta_t))
\leq (1 - \gamma)^2 \|\nabla \mathcal{L}_a(\theta_t)\|^2 + \gamma^2 \|\nabla \mathcal{L}_b(\theta_t)\|^2 + \frac{\alpha}{\alpha + 1} (1 - \gamma)^2 \|\nabla \mathcal{L}_a(\theta_t)\|^2 + \frac{\alpha + 1}{\alpha} \gamma^2 \|\nabla \mathcal{L}_b(\theta_t)\|^2
= (1 - \gamma)\|\nabla \mathcal{L}_a(\theta_t)\|^2 + \gamma \|\nabla \mathcal{L}_b(\theta_t)\|^2.
$$

Therefore, we get

$$
E[\mathcal{L}(\theta_{t+1}) - \mathcal{L}(\theta_t)] \leq -\frac{\eta}{2} E[(1 - \gamma)\|\nabla \mathcal{L}_a(\theta_t)\|^2 + \gamma \|\nabla \mathcal{L}_b(\theta_t)\|^2] + 2\eta^2 G^2 L
\leq -\frac{\eta}{2} E[2\mu_a(1 - \gamma)(\mathcal{L}_a(\theta_t) - \mathcal{L}_a^{\min}) + 2\mu_b(\mathcal{L}_b(\theta_t) - \mathcal{L}_b^{\min})] + 2\eta^2 G^2 L,
$$

(12)

where the last inequality is due to PL-condition of $\mathcal{L}_a(\theta)$ and $\mathcal{L}_b(\theta)$. Let define $\tau = \min(2\mu_a, 2\mu_b)$, then

$$
E[(1 - \gamma)\mathcal{L}_a(\theta_{t+1}) + \gamma \mathcal{L}_b(\theta_{t+1}) - (1 - \gamma)\mathcal{L}_a(\theta_t) - \gamma \mathcal{L}_b(\theta_t)]
\leq -\frac{\eta \tau}{2} E[(1 - \gamma)(\mathcal{L}_a(\theta_t) - \mathcal{L}_a^{\min}) + \gamma(\mathcal{L}_b(\theta_t) - \mathcal{L}_b^{\min})] + 2\eta^2 G^2 L,
$$

(13)

which implies

$$
E[(1 - \gamma)(\mathcal{L}_a(\theta_{t+1}) - \mathcal{L}_a^{\min}) + \gamma \mathcal{L}_b(\theta_{t+1})] 
\leq (1 - \frac{\eta \tau}{2}) E[(1 - \gamma)(\mathcal{L}_a(\theta_t) - \mathcal{L}_a^{\min}) + \gamma(\mathcal{L}_b(\theta_t) - \mathcal{L}_b^{\min})] + 2\eta^2 G^2 L.
$$

(14)
We thus have
\[
E[(1 - \gamma)(L_a(\theta_t) - \ell_a^0) + \gamma L_0(\theta_t)] \leq \left(1 - \frac{\eta \gamma}{2}\right) \mathcal{L}(\theta_0) + \frac{A G^2 L}{\tau}. \tag{15}
\]
Since \(\eta = \frac{2}{\tau m} \log \left(\frac{m \tau^2 L(\theta_0)}{8G^2 L}\right)\) and \(\ell_a^0 = 0\), we have
\[
E[\mathcal{L}_a(\theta_m)] \leq \left(1 - \frac{\eta \gamma}{2}\right) \left(\frac{m \mathcal{L}(\theta_0)}{1 - \gamma} + \frac{A G^2 L}{\tau (1 - \gamma)}\right) \\
\leq \exp \left(-\frac{\eta \gamma m}{2}\right) \left(\frac{m \mathcal{L}(\theta_0)}{1 - \gamma} + \frac{A G^2 L}{\tau (1 - \gamma)}\right) \\
= \frac{8G^2 L}{m \tau^2 (1 - \gamma)} \left(1 + \log \left(\frac{m \tau^2 L(\theta_0)}{8G^2 L}\right)\right) \\
\leq O \left(\frac{4G^2 L \log(m)}{m \tau^2 (1 - \gamma)}\right). \tag{16}
\]

### 4.2 Proof of Theorem 2

**Proof.** Since we have
\[
\max_{\theta \in \Omega(\hat{f}, \hat{s})} \left\| \nabla \mathcal{L}_a(\theta) - \nabla \mathcal{L}_a(\hat{\theta}) \right\| = \max_{\theta \in \Omega(\hat{f}, \hat{s})} \left\| \frac{1}{m} \sum_{i=1}^{m} (f(x_i^0; \theta) - y_i^0) \nabla f(x_i^0; \theta) - \nabla \mathcal{L}_a(\theta) \right\|,
\]
using the Bousquet inequality [Koltchinskii, 2011], with appropriate definition of the function space, we have, with a probability \(1 - \delta\)
\[
\max_{\theta \in \Omega(\hat{f}, \hat{s})} \left\| \frac{1}{m} \sum_{i=1}^{m} (f(x_i^0; \theta) - y_i^0) \nabla f(x_i^0; \theta) - \nabla \mathcal{L}_a(\theta) \right\| \\
\leq 2E \left[ \max_{\theta \in \Omega(\hat{f}, \hat{s})} \left\| \frac{1}{m} \sum_{i=1}^{m} (f(x_i^0; \theta) - y_i^0) \nabla k f(x_i^0; \theta) - \nabla k \mathcal{L}_a(\theta) \right\| + \frac{4F \log(1/\delta)}{3m} + \sqrt{\frac{2\sigma^2}{m} \log \frac{1}{\delta}} \right]
\]
where \(\sigma^2 = \frac{1}{\tau^2} \max_{\theta \in \Omega(\hat{f}, \hat{s})} E \left[ (f(x_i^0; \theta) - y_i^0)^2 | \nabla f(x_i^0; \theta)|^2 \right] \leq \max_{\theta \in \Omega(\hat{f}, \hat{s})} E \left[ (f(x^0; \theta) - y^0)^2 \right] \leq 2 \mathcal{L}_a(\theta_1) := r^2\) with the Assumption 1 that \(\| \nabla f(x; \theta) \| \leq G\). Using the Berstein inequality, we have, with a probability \(1 - \delta\),
\[
r^2 \geq r^2 - \frac{2F^2}{3m} \log \frac{2}{\delta} - 2F \sqrt{\frac{r^2}{m} \log \frac{2}{\delta}}.
\]
Using the assumption that \(m \geq 64F^2 \log(2/\delta)\), we have, with a probability \(1 - \delta\)
\[
r^2 \leq 2r^2 + \frac{4F^2}{3m} \log \frac{2}{\delta},
\]
and therefore
\[
\sigma^2 \leq 2r^2 + \frac{4F^2}{3m} \log \frac{2}{\delta}.
\]
As a result, with a probability $1 - 2\delta$, we have

$$
\max_{\theta \in \Omega(\vec{r}, s)} \left| \frac{1}{mG} \sum_{i=1}^{m} (f(x_i^0; \theta) - y_i^0) \nabla f(x_i^0; \theta) - \nabla L_a(\theta) \right|
$$

$$
\leq 2E \left[ \max_{\theta \in \Omega(\vec{r}, s)} \left| \frac{1}{mG} \sum_{i=1}^{m} (f(x_i^0; \theta) - y_i^0) \nabla_k f(x_i^0; \theta) - \nabla_k L_a(\theta) \right| \right] + \frac{3Fm}{m} \log \frac{1}{\delta} + 2\bar{r} \sqrt{\frac{2\log(1/\delta)}{m}}
$$

$$
\leq 4E_{x,\sigma} \left[ \max_{\theta \in \Omega(\vec{r}, s)} \left| \frac{1}{mG} \sum_{i=1}^{m} \sigma_i(f(x_i^0; \theta) - y(x_i^0)) \nabla f(x_i^0; \theta) \right| \right] + \frac{4Fm \log(1/\delta)}{m} + 2\bar{r} \sqrt{\frac{2\log(1/\delta)}{m}}.
$$

We then bound $E_{x,\sigma}[\cdot]$. Using Klei-Rio bound [Koltchinskii, 2011], with a probability $1 - 2\delta$, we have

$$
E_{x,\sigma} \left[ \max_{\theta \in \Omega(\vec{r}, s)} \left| \frac{1}{m} \sum_{i=1}^{m} \sigma_i(f(x_i^0; \theta) - y(x_i^0)) \nabla f(x_i^0; \theta) \right| \right]
\leq 2E_{\sigma} \left[ \max_{\theta \in \Omega(\vec{r}, s)} \left| \frac{1}{m} \sum_{i=1}^{m} \sigma_i(f(x_i^0; \theta) - y(x_i^0)) \nabla f(x_i^0; \theta) \right| \right] + 2\bar{r} \sqrt{\frac{\log(1/\delta)}{m}} + \frac{9Fm}{m} \log \frac{1}{\delta}
$$

We can bound the expectation term using Dudley entropy bound (Lemma A.5 of [Bartlett et al., 2017]), i.e.

$$
E_{\sigma} \left[ \max_{\theta \in \Omega(\vec{r}, s)} \left| \frac{1}{mG} \sum_{i=1}^{m} \sigma_i(f(x_i^0; \theta) - y(x_i^0)) \nabla f(x_i^0; \theta) \right| \right]
\leq \frac{4\alpha}{\sqrt{m}} + \frac{12}{m} \int_{\alpha}^{\sqrt{m}} \sqrt{\log \mathcal{N}(\mathcal{G}(\vec{r}, s); \epsilon) d\epsilon}
$$

(17)

where $\mathcal{N}(\mathcal{G}(r, s); \epsilon)$ is the covering number of proper $\epsilon$-net over $\mathcal{G}(r, s)$. Here $\mathcal{G}(r, s)$ is defined as

$$
\mathcal{G}(\vec{r}, s) = \left\{ v \in \mathbb{R}^m : |v| \leq \frac{\sqrt{m\bar{r}}}{F}, |v|_1 \leq \frac{\sqrt{ms\bar{r}}}{F} \right\}
$$

Using the covering number result for sparse vectors (Lemma 3.4 of [Plan and Vershynin, 2013]), we have

$$
\log \mathcal{N} \left( \mathcal{G}(\vec{r}, s); \frac{\sqrt{m\bar{r}}}{F} \epsilon \right) \leq \frac{Cs}{\epsilon^2} \log \frac{2m}{s}
$$

and therefore

$$
\log \mathcal{N} \left( \mathcal{G}(\vec{r}, s); \epsilon \right) \leq \frac{Cm\bar{r}^2s}{F^2\epsilon^2} \log \frac{2m}{s}
$$

Plugging into the Dudley entropy bound, we have

$$
E_{\sigma} \left[ \max_{\theta \in \Omega(\vec{r}, s)} \left| \frac{1}{mF} \sum_{i=1}^{m} \sigma_i(f(x_i^0; \theta) - y(x_i^0)) \right| \right] \leq \inf_{\alpha} \left( \frac{4\alpha}{\sqrt{m}} + \frac{12\bar{r}}{F} \sqrt{\frac{Cs}{m} \log \frac{2m}{s} \log \frac{\sqrt{m}}{\alpha}} \right)
$$
By choosing $\alpha$ as 
\[ \alpha = \frac{3\bar{r}}{F} \sqrt{Cs \log \frac{2m}{s}}, \]
we have 
\[ E_{\sigma} \left[ \max_{\theta \in \Omega(\bar{r}, s)} \left| \frac{1}{m} \sum_{i=1}^{m} \sigma_i (f(x_i^0; \theta) - y(x_i^0)) \right| \right] \leq 12\bar{r} \sqrt{\frac{Cs}{m} \log \frac{2m}{s}} \left( 1 + \frac{1}{2} \log \left( \frac{m}{9\sqrt{Cs \log (2m/s)}} \right) \right) \]

Combining the above results together, with a probability $1 - 3\delta$, we prove the inequality (7). Similarly, we can bound $\max_{\theta \in \Omega(\bar{r}, s)} |\nabla L_b(\theta) - \nabla \tilde{L}_b(\theta)|$.

### 4.3 Proof of Theorem 3

**Proof.** Using the standard analysis of optimization, we have
\[
E[\mathcal{L}(\theta'_{t+1}) - \mathcal{L}(\theta'_t)] \\
\overset{(a)}{=} -\eta E[\langle \nabla \mathcal{L}(\theta'_t), \hat{g}_t \rangle] + \frac{\eta^2 L}{2} E[\|\hat{g}_t\|^2] \\
\overset{(b)}{=} \frac{\eta}{2} E[\|\nabla \mathcal{L}(\theta'_t) - \nabla \tilde{L}(\theta'_t)\|^2] - \frac{\eta(1 - \eta L)}{2} E[\|\nabla \tilde{L}(\theta'_t)\|^2] - \frac{\eta^2 L}{2} E[\|\hat{g}_t - \nabla \tilde{L}(\theta'_t)\|^2] \\
\overset{(c)}{=} \frac{\eta}{2} E[\|\nabla \mathcal{L}(\theta'_t) - \nabla \tilde{L}(\theta'_t)\|^2] - \frac{\eta}{2} E[\|\nabla \tilde{L}(\theta'_t)\|^2] + \frac{\eta^2 \sigma^2 L}{2} \\
\overset{(d)}{=} \frac{\eta}{2} E[(1 - \gamma)\|\nabla \mathcal{L}_a(\theta'_t) - \nabla \tilde{L}_a(\theta'_t)\|^2 + \gamma \|\nabla \mathcal{L}_b(\theta'_t) - \nabla \tilde{L}_b(\theta'_t)\|^2] - \frac{\eta}{2} E[\|\nabla \tilde{L}(\theta'_t)\|^2] + \frac{\eta^2 \sigma^2 L}{2}. 
\] (18)

where (a) uses the smoothness of $\mathcal{L}$ and $\theta'_{t+1} = \theta'_t - \eta g_t$; (b) uses $E[g_t] = \nabla \tilde{L}(\theta_t)$; (c) uses $E[\|\mathcal{L}(\theta_t) - g_t\|^2] \leq \sigma^2$ and $\eta \leq \frac{1}{L}$; (d) uses the convexity of norm operation $\|\cdot\|^2$. Hence, we need to bound both $\|\nabla \mathcal{L}_a(\theta'_t) - \nabla \tilde{L}_a(\theta'_t)\|$ and $\|\nabla \mathcal{L}_b(\theta'_t) - \nabla \tilde{L}_b(\theta'_t)\|$. Since we assume that $\theta'_t \in \Omega(\bar{r}, s)$, it is thus sufficient to bound $\max_{\theta \in \Omega(\bar{r}, s)} \|\nabla \mathcal{L}_a(\theta) - \nabla \tilde{L}_a(\theta)\|$ and $\max_{\theta \in \Omega(\bar{r}, s)} \|\nabla \mathcal{L}_b(\theta) - \nabla \tilde{L}_b(\theta)\|$.

Plugging the bounds for $\|\nabla \mathcal{L}_a(\theta'_t) - \nabla \tilde{L}_a(\theta'_t)\|$ and $\|\nabla \mathcal{L}_b(\theta'_t) - \nabla \tilde{L}_b(\theta'_t)\|$ in Theorem 2, we have, with a probability $1 - \delta$,
\[
E[\mathcal{L}(\theta'_{t+1}) - \mathcal{L}(\theta'_t)] \\
\leq \frac{C'' \eta G^2}{n + m} \left( \log \left( \frac{6}{\delta} \right) + s \log \left( \frac{2\max(m, n)}{s} \right) \log(\max(m, n)) \right) \left( \bar{r}^2 + F^2 \right) - \frac{\eta}{2} E[\|\nabla \mathcal{L}(\theta'_t)\|^2] + \frac{\eta^2 \sigma^2 L}{2}. 
\] (19)

Using the similar analysis as the proof of Theorem 1, after $t$ iterations, we have
\[
E[(1 - \gamma)(\mathcal{L}_a(\theta'_t) - \mathcal{L}_a^{\min})] \leq \exp(-\eta rt) \mathcal{L}(\theta_1) + \frac{C'' \eta G^2 \Delta}{\tau (n + m)} \left( \bar{r}^2 + F^2 \right) + \frac{4\eta G^2 L}{2\tau}. 
\] (20)
To ensure that the solution in the later iteration will fall into the range of $\Omega(\tilde{r}, s)$, we need the following conditions:

$$n + m \geq \frac{C''G^2 \Delta (\tilde{r}^2 + F^2)}{8\tau\tilde{r}^2 (1 - \gamma)},$$

$$\eta = \frac{\tau\tilde{r}^2 (1 - \gamma)}{16G^2 L},$$

$$t \geq \frac{\log \left( \frac{8\mathcal{L}(\theta_1)}{\tilde{r}^2 (1 - \gamma)} \right)}{\eta \tau}.$$

That is

$$1 - \gamma \geq \max \left\{ \frac{C''G^2 \Delta (\tilde{r}^2 + F^2)}{8\tau\tilde{r}^2 (n + m)}, \frac{16G^2 L \log \left( \frac{\mathcal{L}(\theta_1)}{\tilde{r}^2 (1 - \gamma)} \right)}{\tau^2 \tilde{r}^2 t} \right\}.$$  

\[\square\]

5 Conclusions

Learning with noisily labeled data has been studied in many deep supervised learning tasks, and its two interesting phenomena called clean data first and phase transition have been empirically observed. We provide a theoretical analysis for rethinking about these two empirical phenomena from the view of learning theory and non-convex optimization. The result reveals that the models first learn clean data and then after that the testing performance would not be degraded when the percentage of data with corrupted class labels is not too large.

References

Görkem Algan and Ilkay Ulusoy. Image classification with deep learning in the presence of noisy labels: A survey. Knowledge-Based Systems, 215:106771, 2021.

Zeyuan Allen-Zhu, Yuanzhi Li, and Zhao Song. A convergence theory for deep learning via over-parameterization. In International Conference on Machine Learning, pages 242–252, 2019.

Sanjeev Arora, Simon S Du, Wei Hu, Zhiyuan Li, and Ruosong Wang. Fine-grained analysis of optimization and generalization for overparameterized two-layer neural networks. arXiv preprint arXiv:1901.08584, 2019.

Devansh Arpit, Stanislaw Jastrzebski, Nicolas Ballas, David Krueger, Emmanuel Bengio, Maninder S Kanwal, Tegan Maharaj, Asja Fischer, Aaron Courville, Yoshua Bengio, et al. A closer look at memorization in deep networks. In International Conference on Machine Learning, pages 233–242, 2017.

Peter Bartlett, Dylan J Foster, and Matus Telgarsky. Spectrally-normalized margin bounds for neural networks. arXiv preprint arXiv:1706.08498, 2017.
Zachary Charles and Dimitris Papailiopoulos. Stability and generalization of learning algorithms that converge to global optima. In *International Conference on Machine Learning*, pages 745–754, 2018.

Lenaic Chizat, Edouard Oyallon, and Francis Bach. On lazy training in differentiable programming. In *Advances in Neural Information Processing Systems*, pages 2937–2947, 2019.

Simon Du, Jason Lee, Haochuan Li, Liwei Wang, and Xiyu Zhai. Gradient descent finds global minima of deep neural networks. In *International Conference on Machine Learning*, pages 1675–1685, 2019.

Simon S Du, Xiyu Zhai, Barnabas Poczos, and Aarti Singh. Gradient descent provably optimizes over-parameterized neural networks. *arXiv preprint arXiv:1810.02054*, 2018.

Saeed Ghadimi and Guanghui Lan. Stochastic first-and zeroth-order methods for nonconvex stochastic programming. *SIAM Journal on Optimization*, 23(4):2341–2368, 2013.

Trevor Hastie, Andrea Montanari, Saharon Rosset, and Ryan J Tibshirani. Surprises in high-dimensional ridgeless least squares interpolation. *arXiv preprint arXiv:1903.08560*, 2019.

Hamed Karimi, Julie Nutini, and Mark Schmidt. Linear convergence of gradient and proximal-gradient methods under the polyak-lojasiewicz condition. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pages 795–811. Springer, 2016.

Pang Wei Koh and Percy Liang. Understanding black-box predictions via influence functions. In *International Conference on Machine Learning*, pages 1885–1894. PMLR, 2017.

Vladimir Koltchinskii. *Oracle Inequalities in Empirical Risk Minimization and Sparse Recovery Problems: Ecole d'Eté de Probabilités de Saint-Flour XXXVIII-2008*, volume 2033. Springer Science & Business Media, 2011.

Kuang-Huei Lee, Xiaodong He, Lei Zhang, and Linjun Yang. Cleannet: Transfer learning for scalable image classifier training with label noise. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 5447–5456, 2018.

Junnan Li, Richard Socher, and Steven CH Hoi. Dividemix: Learning with noisy labels as semi-supervised learning. *arXiv preprint arXiv:2002.07394*, 2020a.

Wen Li, Limin Wang, Wei Li, Eirikur Agustsson, and Luc Van Gool. Webvision database: Visual learning and understanding from web data. *arXiv preprint arXiv:1708.02862*, 2017.

Xiaoyu Li, Zhenxun Zhuang, and Francesco Orabona. Exponential step sizes for non-convex optimization. *arXiv preprint arXiv:2002.05273*, 2020b.

Zhize Li and Jian Li. A simple proximal stochastic gradient method for nonsmooth nonconvex optimization. In *Advances in Neural Information Processing Systems*, pages 5564–5574, 2018.

Sheng Liu, Jonathan Niles-Weed, Narges Razavian, and Carlos Fernandez-Granda. Early-learning regularization prevents memorization of noisy labels. *arXiv preprint arXiv:2007.00151*, 2020.
Yueming Lyu and Ivor W Tsang. Curriculum loss: Robust learning and generalization against label corruption. In *International Conference on Learning Representations*, 2019.

Tam Nguyen, C Mummadi, T Ngo, L Beggel, and Thomas Brox. Self: learning to filter noisy labels with self-ensembling. In *International Conference on Learning Representations (ICLR)*, 2020.

Yaniv Plan and Roman Vershynin. One-bit compressed sensing by linear programming. *Communications on Pure and Applied Mathematics*, 66(8):1275–1297, 2013.

Boris Teodorovich Polyak. Gradient methods for minimizing functionals. *Zhurnal Vychislitel’noi Matematiki i Matematicheskoi Fiziki*, 3(4):643–653, 1963.

Hwanjun Song, Minseok Kim, and Jae-Gil Lee. Selfie: Refurbishing unclean samples for robust deep learning. In *International Conference on Machine Learning*, pages 5907–5915. PMLR, 2019.

Hwanjun Song, Minseok Kim, Dongmin Park, and Jae-Gil Lee. Learning from noisy labels with deep neural networks: A survey. *arXiv preprint arXiv:2007.08199*, 2020.

Yisen Wang, Xingjun Ma, Zaiyi Chen, Yuan Luo, Jinfeng Yi, and James Bailey. Symmetric cross entropy for robust learning with noisy labels. In *Proceedings of the IEEE/CVF International Conference on Computer Vision*, pages 322–330, 2019a.

Zhe Wang, Kaiyi Ji, Yi Zhou, Yingbin Liang, and Vahid Tarokh. Spiderboost and momentum: Faster variance reduction algorithms. In *Advances in Neural Information Processing Systems*, pages 2403–2413, 2019b.

Tong Xiao, Tian Xia, Yi Yang, Chang Huang, and Xiaogang Wang. Learning from massive noisy labeled data for image classification. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pages 2691–2699, 2015.

Yi Xu, Rong Jin, and Tianbao Yang. Non-asymptotic analysis of stochastic methods for non-smooth non-convex regularized problems. *arXiv preprint arXiv:1902.07672*, 2019.

Zhuoning Yuan, Yan Yan, Rong Jin, and Tianbao Yang. Stagewise training accelerates convergence of testing error over sgd. In *Advances in Neural Information Processing Systems*, pages 2604–2614, 2019.

Chulhee Yun, Suvrit Sra, and Ali Jadbabaie. Small relu networks are powerful memorizers: a tight analysis of memorization capacity. In *Advances in Neural Information Processing Systems*, pages 15558–15569, 2019.

Chiyuan Zhang, Samy Bengio, Moritz Hardt, Benjamin Recht, and Oriol Vinyals. Understanding deep learning requires rethinking generalization. *arXiv preprint arXiv:1611.03530*, 2016.

Difan Zou, Yuan Cao, Dongruo Zhou, and Quanquan Gu. Gradient descent optimizes over-parameterized deep relu networks. *Machine Learning*, 109(3):467–492, 2020.