GORENSTEIN DIMENSION OF MODULES

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Notation.

$R$ ring (always commutative and Noetherian)
$(R, m, k)$ local ring with maximal ideal $m$ and $k = R/m$
$L, M, N, \ldots$ $R$-modules (always finitely generated)
$M^*$ $\text{Hom}_R(M, R)$, the dual of $M$
$D(M)$ the Auslander dual of $M$ (Definition 2)
$\sigma_M : M \longrightarrow M^{**}$ the natural evaluation map;
$K_M = \text{Ker}(\sigma_M), C_M = \text{Coker}(\sigma_M)$

G-dim$(R)$, G-dim$(M)$ Gorenstein dimension of $M$ (Definition 16)
G-dim$(M) <_{\text{loc}} \infty$ $M$ has locally finite Gorenstein dimension
(Remark #2 after Theorem 29)
pd$_R(M), \text{pd}(M)$ projective dimension of $M$
$\approx$ projective equivalence (Definition 3)

0. Introduction

In these expository notes I discuss several concepts and results in the theory of modules over commutative rings, revolving around the Gorenstein dimension of modules. Some of the related notions are the Auslander dual, $k$-torsionless modules, and $k^{th}$ syzygies.

Essentially everything in these notes can be found, in one form or another, in the memoir “Stable module theory” by M. Auslander and M. Bridger (Mem. A.M.S., 1991).
no. 94, 1969). The only difference is in presentation. In the Auslander–Bridger memoir many of the results are proved in the most general setting, e.g. over possibly non-commutative, non-Noetherian rings. The techniques used are quite abstract and unfamiliar to many commutative algebraists. Much space is devoted to the theory of satellites of functors which are exact only in the middle, etc. While such a degree of generality has many advantages, it does make the memoir difficult to read for the non-expert. My goal in writing these notes was to develop the theory in the context of commutative Noetherian rings, and to show that, in this important special case, the theory is fairly elementary and easy to build. As a practical matter, then, I wrote the notes using Matsumura’s “Commutative ring theory” as the only prerequisite; and indeed, my hope is that these notes can be read just like an extra chapter in Matsumura’s book. Still, some of the proofs given here are mere adaptations and simplification of those in [AB]. Incidentally, in §2 I fix a mistake in the proof of the analogue of the Auslander–Buchsbaum formula for G-dimension given in [AB].

Throughout these notes all rings are commutative and Noetherian and all modules are finitely generated.

Motivation. Here are some of the reasons why anyone would want to study the theory of Gorenstein dimension of modules:

- Modules of finite projective dimension also have finite Gorenstein dimension. On the other hand, all modules over a Gorenstein ring have locally finite Gorenstein dimension. There are many results in commutative algebra which begin with a hypothesis like, “assume that either \( \text{pd}(M) < \infty \) or that the ring \( R \) is Gorenstein.” And indeed, upon inspection, it turns out that the results hold for modules of (locally) finite Gorenstein dimension over any ring. The Gorenstein dimension provides a natural unifying language for such results.

- Gorenstein dimension has many of the good properties of projective dimension. (In particular, the Auslander–Buchsbaum formula holds with Gorenstein dimension instead of projective dimension, and a local ring \((R, m, k)\) is Gorenstein if and only if \( \text{G-dim}(k) < \infty \). Compare with: “\( R \) is regular if and only if \( \text{pd}(k) < \infty \).”) For this reason, many results proved initially over regular rings hold, in fact, over Gorenstein rings. One such example is the existence of Evans–Griffith presentations \([M, \delta]\).

- Some of the most useful characterizations of syzygies hold for modules of finite Gorenstein dimension (Theorem 40 in §3).

- In \([AF]\), Avramov and Foxby defined local ring homomorphisms of finite Gorenstein dimension and studied their properties. In particular, they defined a dualizing complex for such homomorphisms, very similar to the dualizing complex of a local ring.

Acknowledgement. In preparing and writing up these notes, I benefited from advice, encouragement and many useful discussions with L. Avramov, E. G. Evans, P. Griffith, N. Mohan Kumar, and D. Popescu. The treatment of several topics here owes much to their insights.
Definition 1. A module $M$ is torsionless, resp. reflexive, if the natural evaluation map $\sigma_M : M \rightarrow M^{**}$ is injective, resp. an isomorphism. That is, $M$ is torsionless if $K_M = 0$, and reflexive if $K_M = C_M = 0$, where $K_M = \text{Ker}(\sigma_M)$ and $C_M = \text{Coker}(\sigma_M)$. (See Remark after Definition 35 in §3 for a short comment on terminology.)

Auslander interpreted $K_M$ and $C_M$ cohomologically, in terms of the “Auslander dual”, $D(M)$, defined below. This interpretation explains the functorial properties of $K_M$ and $C_M$ and the good behavior of torsionless and reflexive modules, and can be used to define “$k$-torsionless” for all $k \geq 0$; see Definition 7 later in this section.

(1-torsionless is the same as torsionless, and 2-torsionless is the same as reflexive. All modules are 0-torsionless.)

Definition 2. Let $M$ be any module (finitely generated as always), and let

$$
\begin{array}{ccc}
P_1 & \xrightarrow{u} & P_0 \\
\downarrow \phi_1 & & \downarrow \phi_0 \\
M & \xrightarrow{f} & 0
\end{array}
$$

be a (finitely generated) projective presentation of $M$.

The Auslander dual, $D(M)$, of $M$, is defined as $D(M) = \text{Coker}(u^*: P_0^* \rightarrow P_1^*)$; in other words, dualizing $(\pi)$ we get an exact sequence

$$
\begin{array}{cccc}
0 & \rightarrow & M^* & \xrightarrow{f^*} & P_0^* & \xrightarrow{u^*} & P_1^* & \rightarrow & D(M) & \rightarrow & 0.
\end{array}
$$

$D(M)$ is sometimes called the Auslander transpose of $M$; see, for example, [Ei, p. 648].

Clearly, $D(M)$ depends on which projective presentation $(\pi)$ is used in the definition. Until the end of the proof of uniqueness of $D(M)$ (up to projective equivalence) in Proposition 4 below, we will denote $\text{Coker}(u^*)$ by $D_\pi(M)$.

Definition 3. Two modules $M$ and $N$ are projectively equivalent if $\exists P, Q$ projective with $M \oplus P \cong N \oplus Q$. Notation: $M \approx N$.

$\approx$ is an equivalence relation on the class of (finitely generated) $R$-modules.

Proposition 4. If $(\pi)$ as above and

$$
\begin{array}{ccc}
Q_1 & \xrightarrow{v} & Q_0 \\
\downarrow \phi_1 & & \downarrow \phi_0 \\
M & \xrightarrow{g} & 0
\end{array}
$$

are two projective presentations of $M$, then

$D_\pi(M) \approx D_\rho(M)$.

Proof. We say that $(\pi)$ strictly dominates $(\rho)$ if there are linear maps $\phi_i : P_i \rightarrow Q_i$ ($i = 0, 1$) such that

(a) $\phi_i$ is surjective, $i = 0, 1$;

(b) $\phi_0$ is a lifting of $\text{id}_M$ and $\phi_1$ is a lifting of $\phi_0$; i.e., the diagram

$$
\begin{array}{ccc}
P_1 & \xrightarrow{u} & P_0 \\
\downarrow \phi_1 & & \downarrow \phi_0 \\
Q_1 & \xrightarrow{v} & Q_0 \\
\downarrow & & \downarrow \\
M & \xrightarrow{g} & 0
\end{array}
$$

commutes;

(c) The map $\tilde{u} : K_1 \rightarrow K_0$ induced by $u$ is surjective, where $K_i = \text{Ker}(\phi_i)$ ($i = 0, 1$).
In other words, we should have a commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & 0 & \quad & \quad & \quad & \quad & 0 \\
\uparrow & \uparrow & \quad & \quad & \quad & \quad & \uparrow \\
K_1 & \overset{\bar{u}}{\longrightarrow} & K_0 & \longrightarrow & 0 \\
\uparrow & \uparrow & \quad & \quad & \quad & \quad & \uparrow \\
P_1 & \overset{u}{\longrightarrow} & P_0 & \overset{f}{\longrightarrow} & M & \longrightarrow & 0 \\
\uparrow & \uparrow & \quad & \quad & \quad & \quad & \uparrow \\
Q_1 & \overset{v}{\longrightarrow} & Q_0 & \overset{g}{\longrightarrow} & M & \longrightarrow & 0 \\
\uparrow & \uparrow & \quad & \quad & \quad & \quad & \uparrow \\
0 & 0 & \quad & \quad & \quad & \quad & 0 \\
\end{array}
\]

(\ast\ast)

We prove the Proposition in two steps. First we show that \( D_\pi(M) \approx D_\rho(M) \) under the extra hypothesis that \((\pi)\) strictly dominates \((\rho)\). Then we show that, given any two projective presentations \((\pi)\) and \((\rho)\), there is a third one, \((\sigma)\), which strictly dominates both \((\pi)\) and \((\rho)\).

For the time being, assume that \((\pi)\) strictly dominates \((\rho)\). Then we have the diagram \((\ast\ast)\). Since the \(Q_i\) are projective, the columns are split-exact and the \(K_i\) are projective. Therefore \(\bar{u}\) splits.

Dualizing \((\ast\ast)\) we get a commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
0 & 0 & \quad & \quad & \quad & \quad & 0 \\
\uparrow & \uparrow & \quad & \quad & \quad & \quad & \uparrow \\
0 & \longrightarrow & M^* & \longrightarrow & Q_0^* & \longrightarrow & Q_1^* \longrightarrow & D_\rho(M) & \longrightarrow & 0 \\
\uparrow & \uparrow & \quad & \quad & \quad & \quad & \uparrow \\
0 & \longrightarrow & M^* & \longrightarrow & P_0^* & \longrightarrow & P_1^* \longrightarrow & D_\pi(M) & \longrightarrow & 0 \\
\uparrow & \uparrow & \quad & \quad & \quad & \quad & \uparrow \\
0 & \longrightarrow & K_0^* & \overset{\bar{u}^*}{\longrightarrow} & K_1^* \longrightarrow & K & \longrightarrow & 0 \\
\uparrow & \uparrow & \quad & \quad & \quad & \quad & \uparrow \\
0 & 0 & \quad & \quad & \quad & \quad & 0 \\
\end{array}
\]

where \(K \overset{\text{def}}{=} \text{Coker}(\bar{u}^*)\). \(0 \rightarrow D_\rho(M) \rightarrow D_\pi(M) \rightarrow K \rightarrow 0\) is exact by the Snake Lemma, and \(K\) is projective, because \(\bar{u}^*\) is a split injective map of projective modules. Thus \(D_\pi(M) \cong D_\rho(M) \oplus K\), and \(D_\pi(M) \approx D_\rho(M)\).
Now let \((\pi), (\rho)\) be any two projective presentations of \(M\). We construct a new presentation, \((\sigma)\), which strictly dominates both \((\pi)\) and \((\rho)\).

Let \(h : P_0 \oplus Q_0 \longrightarrow M\), \(h(p_0, q_0) = f(p_0) + g(q_0)\). Clearly, \(h\) is surjective. Let \(\alpha : E \longrightarrow P_0 \oplus Q_0\) be a linear map from a (finitely generated) projective module \(E\) onto \(\text{Ker}(h) \subseteq P_0 \oplus Q_0\); that is, consider an exact sequence

\[
E \xrightarrow{\alpha} P_0 \oplus Q_0 \xrightarrow{h} M \longrightarrow 0.
\]

Extend the projective presentations \((\pi)\) and \((\rho)\) of \(M\) to the left:

\[
P_2 \xrightarrow{u'} P_1 \xrightarrow{\alpha} P_0 \xrightarrow{f} M \longrightarrow 0
\]

and

\[
Q_2 \xrightarrow{v'} Q_1 \xrightarrow{v} Q_0 \xrightarrow{g} M \longrightarrow 0
\]

Define the projective presentation \((\sigma)\) of \(M\) as follows:

\[
(\sigma) \quad E \oplus P_2 \oplus Q_2 \xrightarrow{w} P_0 \oplus Q_0 \xrightarrow{h} M \longrightarrow 0,
\]

where \(w(e, p_2, q_2) \overset{\text{def}}{=} \alpha(e)\). \((\sigma)\) is clearly exact.

I claim that \((\sigma)\) strictly dominates both \((\pi)\) and \((\rho)\). Since the construction of \((\sigma)\) is symmetric with respect to \((\pi)\) and \((\rho)\), I will show only that \((\sigma)\) strictly dominates \((\rho)\).

Let \(\text{id}_M \circ \phi_0 : P_0 \rightarrow Q_0\); that is, \(g \phi_0 = f\). Define \(\chi_0 : P_0 \oplus Q_0 \rightarrow Q_0\), \(\chi_0(p_0, q_0) = \phi_0(p_0) + q_0\). \(\chi_0\) is clearly surjective, and \(g \chi_0 = h\).

Let \(\chi_0\) to \(\delta : E \rightarrow Q_1\); i.e., construct a commutative diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\alpha} & P_0 \oplus Q_0 \\
\downarrow{\delta} & & \downarrow{\chi_0} \\
Q_1 & \xrightarrow{v} & Q_0 \\
& & \downarrow{g} \\
& & M \\
& & 0
\end{array}
\]

Finally, define \(\chi_1 : E \oplus P_2 \oplus Q_2 \rightarrow Q_1, \chi_1(e, p_2, q_2) = \delta(e) + v'(q_2)\). Then \(v \chi_1(e, p_2, q_2) = v \delta(e) + v v'(q_2) = \chi_0 \alpha(e) = \chi_0 w(e, p_2, q_2)\); that is, \((\sigma)\) and \((\rho)\) sit in a commutative diagram:

\[
\begin{array}{ccc}
E \oplus P_2 \oplus Q_2 & \xrightarrow{u} & P_0 \oplus Q_0 \\
\downarrow{\chi_1} & & \downarrow{\chi_0} \\
Q_1 & \xrightarrow{v} & Q_0 \\
& & \downarrow{g} \\
& & M \\
& & 0
\end{array}
\]

\(\chi_1\) is surjective. Indeed, take any \(q_1 \in Q_1\). Then \(v(q_1) \in Q_0\), and \(h(0, v(q_1)) = f(0) + g v(q_1) = 0\); so \((0, v(q_1)) \in \text{Ker}(h) = \text{Im}(\alpha)\). Take \(e \in E\) with \(\alpha(e) = (0, v(q_1))\). Then \(v \delta(e) = \chi_0 \alpha(e) = \chi_0(0, v(q_1)) = v(q_1)\); thus \(q_1 - \delta(e) \in \text{Ker}(v) = \text{Im}(v')\), i.e. \(q_1 = \delta(e) + v'(q_2) = \chi_1(e, 0, q_2)\) for a suitable \(q_2 \in Q_2\), and we showed that \(q_1 \in \text{Im}(\chi_1)\).

Let \(K_i \overset{\text{def}}{=} \text{Ker}(\chi_i), i = 0, 1\); to finish the proof, we show that the map \(\tilde{w} : K_1 \rightarrow K_0\), induced by \(w\), is surjective.

Let \((p_0, q_0) \in K_0 = \text{Ker}(\chi_0) \subseteq \text{Ker}(h) = \text{Im}(\alpha)\). Take \(e \in E\) such that \(\alpha(e) = (p_0, q_0)\). \(v \delta(e) = \chi_0 \alpha(e) = 0 \implies \delta(e) \in \text{Ker}(v) = \text{Im}(v')\); take \(q_2 \in Q_2\) such that \(\delta(e) = v'(q_2)\). Then \((e, 0, -q_2) \in \text{Ker}(\chi_1) = K_1\), and \(w(e, 0, -q_2) = \alpha(e) = (p_0, q_0)\).

We showed that \(w\) takes \(K_1\) onto \(K_0\), as required. \(\square\)
Remarks. (1) When $M$ is given, $D(M)$ is defined only up to projective equivalence. However, we will work with $D(M)$ loosely, as though it were an $R$-module. We will be careful to specify, when necessary, that a particular representative is being used. In many instances, e.g. in definitions depending only on vanishing of various $\text{Ext}^i(D(M), R), i \geq 1$, the distinction is irrelevant.

(2) If $M \cong 0$ (that is, if $M$ is projective), then $D(M) \cong 0$. Also, $D(M_1 \oplus M_2) \cong D(M_1) \oplus D(M_2)$. Therefore $M \cong N \implies D(M) \cong D(N)$. 

(3) If we use $(\pi)$ to define $D(M)$ (or, rather, $D_\pi(M)$), then $(\pi^*)$ is a projective presentation of $D_\pi(M)$, and $D_{\pi^*}(D_\pi(M)) = \text{Coker}(u^{**}) \cong M$ (because $P_i \cong P_i^{**}$, and $u^{**}$ is identified canonically with $u$). Dropping the dependence on $(\pi)$ and $(\pi^*)$, we can write $D(D(M)) \cong M$.

(4) If $R$ is local, then we can give a more intrinsic definition of $D(M)$, requiring that $(\pi)$ be a minimal projective presentation of $M$. Then $D(M)$ is well-defined up to isomorphism (rather than projective equivalence). Note, however, that even then we will not have $D(D(M)) \cong M$ in general, since the dual of a minimal projective presentation is not necessarily minimal.

(5) The Auslander dual commutes with base change. That is, if $R \rightarrowtail R'$ is any homomorphism of (commutative Noetherian) rings and $M$ is a (finitely generated) $R$-module, then $D_{R'}(M \otimes_R R') \cong D_R(M) \otimes_R R'$ (we use the ring as a subscript to $D$ for emphasis). For example, if $S$ is a multiplicative system in $R$, then $D_{S^{-1}R}(S^{-1}M) \cong S^{-1}D_R(M)$. In particular, $D_{R, (M)} \cong D_R(M)_P, \forall P \in \text{Spec}(R)$.

Next we study the relationship between $D(M)$ and $K_M$ and $C_M$:

**Proposition 5.** Let $\sigma_M : M \longrightarrow M^{**}$ be the natural map, with kernel $K_M$ and cokernel $C_M$.

Then we have natural isomorphisms

$$K_M \cong \text{Ext}^1(D(M), R) \quad \text{and} \quad C_M \cong \text{Ext}^2(D(M), R).$$

Moreover, $\text{Ext}^i(D(M), R) \cong \text{Ext}^{i-2}(M^*, R), \forall i \geq 3$.

Note that the $\text{Ext}^i(D(M), R)$ do not depend on which particular $D(M)$ is being used.

**Proof.** Consider the projective presentation $(\pi)$ of $M$, as before. Dualizing $(\pi)$ we get $(\pi^*)$. Split $(\pi^*)$ into short exact sequences:

$$(\pi^*_0) \quad 0 \longrightarrow M^* \xrightarrow{f^*} P_0^* \xrightarrow{\beta_0} N \longrightarrow 0$$

and

$$(\pi^*_1) \quad 0 \longrightarrow N \xrightarrow{\beta_1} P_1^* \longrightarrow D(M) \longrightarrow 0,$$

where $N \overset{\text{def}}{=} \text{Coker}(f^*)$ and $\beta_1 \circ \beta_0 = u^*$.

Dualizing $(\pi^*_0)$ we get an exact sequence

$$(\pi^*_0^*) \quad 0 \longrightarrow N^* \xrightarrow{\beta_0^*} P_0^{**} \xrightarrow{f^{**}} M^{**} \longrightarrow \text{Ext}^1(N, R) \longrightarrow 0,$$

and dualizing $(\pi^*_1)$ we get an exact sequence

$$(\pi^*_1^*) \quad 0 \longrightarrow D(M)^* \xrightarrow{P_1^{**}} N^* \longrightarrow \text{Ext}^1(D(M), R) \longrightarrow 0.$$
Consider the commutative diagram with exact rows:

\[
\begin{array}{ccc}
P_1 & \xrightarrow{u} & P_0 \xrightarrow{f} M \xrightarrow{\sigma_M} 0 \\
\vert & & \vert \sigma_{P_0} \vert \\
0 & \xrightarrow{\beta_{P_1}^{-1}} & N^* \xrightarrow{\beta_{P_0}^{-1}} P_0^{**} \xrightarrow{f^{**}} M^{**}
\end{array}
\]

Since \(\sigma_{P_1}\) is an isomorphism, we have \(\text{Coker}(\beta_{P_1}^{-1} \circ \sigma_{P_1}) = \text{Coker}(\beta_{P_0}^{-1} \circ \sigma_{P_1}) \cong \text{Ext}^1(D(M), R)\), the isomorphism being given by \((\pi_1^{**})\). As \(\sigma_{P_0}\) is an isomorphism, the Snake Lemma gives

\[
K_M \overset{\text{def}}{=} \text{Ker}(\sigma_M) \cong \text{Coker}(\beta_{P_1}^{-1} \circ \sigma_{P_1}) \cong \text{Ext}^1(D(M), R).
\]

On the other hand, since \(f\) is surjective and \(\sigma_{P_0}\) is an isomorphism, we get

\[
\text{Im}(f^{**}) = \text{Im}(\sigma_M),
\]

and therefore \(C_M \overset{\text{def}}{=} \text{Coker}(\sigma_M) = \text{Coker}(f^{**}) \cong \text{Ext}^1(N, R) \cong \text{Ext}^2(D(M), R)\) (the last isomorphism comes from \((\pi_1^{**})\); the one before it from \((\pi_0^{**})\)).

The last statement of the Proposition follows from \((\pi_0^{**})\) and \((\pi_1^{**})\). \(\square\)

**Remark.** Essentially the same proof shows the existence of an exact sequence of functors:

\[
0 \to \text{Ext}^1(D(M), \star) \to M \otimes \star \xrightarrow{\sigma_M^M} \text{Hom}(M^*, \star) \to \text{Ext}^2(D(M), \star) \to 0;
\]

that is, for every \(R\)-module \(N\) there is a natural exact sequence

\[
0 \to \text{Ext}^1(D(M), N) \to M \otimes N \xrightarrow{\sigma_M^N} \text{Hom}(M^*, N) \to \text{Ext}^2(D(M), N) \to 0.
\]

(The evaluation map \(\sigma_M^N\) is defined by \(\sigma_M^N(m \otimes n) = \phi_{m,n}, \phi_{m,n}(u) = u(m) \cdot n, \forall u \in M^*.\) The naturality is with respect to homomorphisms \(N \longrightarrow N'.\) We leave the proof as an exercise.

**Lemma 6.** Let

\[
0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0
\]

be an exact sequence. Then, for a suitable choice of Auslander duals, we have a long exact sequence:

\[
0 \longrightarrow M''^* \longrightarrow M^* \longrightarrow M'^* \longrightarrow D(M'') \longrightarrow D(M) \longrightarrow D(M') \longrightarrow 0.
\]

**Proof.** Let

\[
P'_1 \longrightarrow P'_0 \longrightarrow M' \longrightarrow 0
\]

and

\[
P''_1 \longrightarrow P''_0 \longrightarrow M'' \longrightarrow 0
\]
be any projective presentations of $M'$ and $M''$, respectively. Fit these in a commutative diagram with exact rows and columns:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
& & \\
P_1' & P_0' & M' \\
& & \\
P_1' \oplus P_1'' & P_0' \oplus P_0'' & M \\
& & \\
P_1'' & P_0'' & M'' \\
& & \\
0 & 0 & 0 \\
\end{array}
\]

The first two columns are split exact, so they remain exact after dualizing. Dualize the whole diagram, writing the cokernels of the dual rows as $D(M''), D(M',)$, and $D(M')$; the conclusion follows from the Snake Lemma.

Remark. Let $C = \text{Coker}(M^* \to M^{**})$. $0 \to M' \to M \to M'' \to 0$ is dual exact if $C = 0$, i.e. if the dual sequence $0 \to M''^* \to M^* \to M'^* \to 0$ is exact. Lemma 6 shows that the sequence is dual-exact if and only if $0 \to D(M'') \to D(M') \to D(M') \to 0$ is exact for a suitable choice of Auslander duals.

Note also that $C$ is (isomorphic to) a submodule of $\text{Ext}^1(M'', R)$.

**k-torsionless modules.** In view of Definition 6 and Proposition 3, we define $k$-torsionless modules as follows:

**Definition 7.** A module $M$ is $k$-torsionless if $\text{Ext}^i(D(M), R) = 0$, $\forall i = 1, \ldots, k$. Equivalently, $M$ is $k$-torsionless if it is torsionless (for $k = 1$), if it is reflexive (for $k = 2$), resp. if it is reflexive and $\text{Ext}^i(M^*, R) = 0$, $\forall i = 1, \ldots, k - 2$ (for $k \geq 3$). By definition, every module is 0-torsionless.

Note that $M$ is $k$-torsionless if and only if $M_P$ is $k$-torsionless over $R_P$, $\forall P \in \text{Spec}(R)$ (because both the Ext functors and the Auslander dual localize).

**Proposition 8.** Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence, and put $C = \text{Coker}(M^* \to M^{**})$ (cf. Remark after Lemma 6).

(a) If $M'$ and $M''$ are $k$-torsionless and $\text{grade}(C) \geq k$, then $M$ is $k$-torsionless.

(b) If $M$ is $k$-torsionless, $M''$ is $(k - 1)$-torsionless and $\text{grade}(C) \geq k - 1$, then $M'$ is $k$-torsionless.

(c) If $M'$ is $(k + 1)$-torsionless, $M$ is $k$-torsionless and $\text{grade}(C) \geq k + 1$, then $M''$ is $k$-torsionless.

Recall the definition of grade: $\text{grade}(C) = \inf\{i \geq 0 \mid \text{Ext}^i(C, R) \neq 0\}$. Since $C$ is finitely generated, we have $\text{grade}(C) = \text{depth}_I(R) = \inf\{\text{depth}(R_P) \mid P \in \text{Supp}(C)\}$, where $I = \text{Ann}_R(C)$. 

Proof. Using Lemma 8, we get an exact sequence
\[ 0 \to C \to D(M'') \to D(M) \to D(M') \to 0, \]
which we can split into short exact sequences
\[ 0 \to C \to D(M'') \to L \to 0 \]
and
\[ 0 \to L \to D(M) \to D(M') \to 0 \]
for suitable L. Now the assertions of the Proposition can be proved by looking at the corresponding long exact sequences of Ext’s. \( \square \)

Remark. As C is (isomorphic to) a submodule of \( \text{Ext}^1(M'', R) \), the grade condition on C holds automatically if it holds for \( \text{Ext}^1(M'', R) \). For example (see Theorem 10 in §3), in statements (a) and (b) of the Proposition, if \( \text{G-dim}(M'') < \text{loc} \times \infty \) then the grade condition on C follows from the torsionless condition on \( M'' \).

On the other hand, in part (c) the condition on grade(C) cannot be omitted or weakened, as the following example illustrates:

Example. Let \( R \) be any Cohen–Macaulay local ring of dimension \( n \geq 1 \). Let \( \{x_1, \ldots, x_n\} \) be a system of parameters. Fix an integer \( k, 0 \leq k \leq n - 1 \). Let \( F = R^{k+1}, x = (x_1, \ldots, x_{k+1}) \in F \), and \( M = F/Rx \). We have an exact sequence:
\[ 0 \to R \xrightarrow{R^{k+1}} M \to 0. \]

We have \( C = \text{Ext}^1(M, R) \cong R/I \), where \( I = Rx_1 + \cdots + Rx_{k+1} \). Thus \( \text{grade}(C) = \text{depth}_I(R) = k + 1 \); by Proposition 8(c) we see that \( M \) is \( k \)-torsionless. (The Proposition applies because \( R^{k+1} \) and \( R \) are free modules.) However, \( M \) is not \( (k+1) \)-torsionless, or else we would have \( \text{grade}(\text{Ext}^1(M, R)) \geq k + 2 \) by Theorem 10 in §3 (which applies because \( \text{pd}(M) = 1 \), and in particular \( \text{G-dim}(M) < \infty \)). \( \square \)

Universal pushforward. Let \( M \) be any module (finitely generated with \( R \) Noetherian, as always). Let \( f_1, \ldots, f_n \in M^* \) generate \( M^* \). Let \( f : M \to R^n \) be the map \( (f_1, \ldots, f_n) \). Then \( f \) is injective if and only if \( M \) is torsionless; in fact, \( \ker(f) = K_M \).

Thus if \( M \) is torsionless we get an exact sequence
\( \text{(u.p.f.)} \)
\[ 0 \to M \xrightarrow{f} R^n \xrightarrow{N} 0 \]
with \( N = \text{Coker}(f) \), and this sequence is also dual-exact (\( f^* \) takes the canonical basis of \( (R^n)^* \) onto \( (f_1, \ldots, f_n) \); thus \( f^* \) is surjective). Therefore \( \text{Ext}^1(N, R) = 0 \).

Such an exact and dual-exact sequence, obtained from a system of generators of \( M^* \) – with \( M \) torsionless – is called a universal pushforward of \( M \).

By (c) of Proposition 8, if \( M \) is \( k \)-torsionless \( (k \geq 1) \) and \( \text{(u.p.f.)} \) as above is a universal pushforward, then \( N \) is \( (k - 1) \)-torsionless. Indeed, \( C = \text{Ext}^1(N, R) = 0 \) in this case.

We conclude this section by showing that \( k \)-torsionless \( \implies k^\text{th syzygy} \implies \text{property } S_k \).

Definition 9. A module \( M \) is a \( k^\text{th syzygy} \) \( (k \geq 1) \) if there exists an exact sequence
\[ 0 \to M \to P_0 \to \cdots \to P_{k-1} \]
with \( P_j \text{ projective}, j = 0, \ldots, k - 1 \).

For \( k = 0 \), every module is a \( 0^\text{th syzygy} \) (this is part of the definition).

Proof. Using Lemma 8, we get an exact sequence
\[ 0 \to C \to D(M'') \to D(M) \to D(M') \to 0, \]
which we can split into short exact sequences
\[ 0 \to C \to D(M'') \to L \to 0 \]
and
\[ 0 \to L \to D(M) \to D(M') \to 0 \]
for suitable L. Now the assertions of the Proposition can be proved by looking at the corresponding long exact sequences of Ext’s. \( \square \)

Remark. As C is (isomorphic to) a submodule of \( \text{Ext}^1(M'', R) \), the grade condition on C holds automatically if it holds for \( \text{Ext}^1(M'', R) \). For example (see Theorem 10 in §3), in statements (a) and (b) of the Proposition, if \( \text{G-dim}(M'') < \text{loc} \times \infty \) then the grade condition on C follows from the torsionless condition on \( M'' \).

On the other hand, in part (c) the condition on grade(C) cannot be omitted or weakened, as the following example illustrates:

Example. Let \( R \) be any Cohen–Macaulay local ring of dimension \( n \geq 1 \). Let \( \{x_1, \ldots, x_n\} \) be a system of parameters. Fix an integer \( k, 0 \leq k \leq n - 1 \). Let \( F = R^{k+1}, x = (x_1, \ldots, x_{k+1}) \in F \), and \( M = F/Rx \). We have an exact sequence:
\[ 0 \to R \xrightarrow{R^{k+1}} M \to 0. \]

We have \( C = \text{Ext}^1(M, R) \cong R/I \), where \( I = Rx_1 + \cdots + Rx_{k+1} \). Thus \( \text{grade}(C) = \text{depth}_I(R) = k + 1 \); by Proposition 8(c) we see that \( M \) is \( k \)-torsionless. (The Proposition applies because \( R^{k+1} \) and \( R \) are free modules.) However, \( M \) is not \( (k+1) \)-torsionless, or else we would have \( \text{grade}(\text{Ext}^1(M, R)) \geq k + 2 \) by Theorem 10 in §3 (which applies because \( \text{pd}(M) = 1 \), and in particular \( \text{G-dim}(M) < \infty \)). \( \square \)

Universal pushforward. Let \( M \) be any module (finitely generated with \( R \) Noetherian, as always). Let \( f_1, \ldots, f_n \in M^* \) generate \( M^* \). Let \( f : M \to R^n \) be the map \( (f_1, \ldots, f_n) \). Then \( f \) is injective if and only if \( M \) is torsionless; in fact, \( \ker(f) = K_M \).

Thus if \( M \) is torsionless we get an exact sequence
\( \text{(u.p.f.)} \)
\[ 0 \to M \xrightarrow{f} R^n \xrightarrow{N} 0 \]
with \( N = \text{Coker}(f) \), and this sequence is also dual-exact (\( f^* \) takes the canonical basis of \( (R^n)^* \) onto \( (f_1, \ldots, f_n) \); thus \( f^* \) is surjective). Therefore \( \text{Ext}^1(N, R) = 0 \).

Such an exact and dual-exact sequence, obtained from a system of generators of \( M^* \) – with \( M \) torsionless – is called a universal pushforward of \( M \).

By (c) of Proposition 8, if \( M \) is \( k \)-torsionless \( (k \geq 1) \) and \( \text{(u.p.f.)} \) as above is a universal pushforward, then \( N \) is \( (k - 1) \)-torsionless. Indeed, \( C = \text{Ext}^1(N, R) = 0 \) in this case.

We conclude this section by showing that \( k \)-torsionless \( \implies k^\text{th syzygy} \implies \text{property } S_k \).

Definition 9. A module \( M \) is a \( k^\text{th syzygy} \) \( (k \geq 1) \) if there exists an exact sequence
\[ 0 \to M \to P_0 \to \cdots \to P_{k-1} \]
with \( P_j \text{ projective}, j = 0, \ldots, k - 1 \).

For \( k = 0 \), every module is a \( 0^\text{th syzygy} \) (this is part of the definition).
Example. Every dual is a second syzygy. Indeed, let $M$ be any module; then dualize any projective presentation of $M$ to see that $M^*$ is (at least) a second syzygy.

**Definition 10.** A module $M$ satisfies property $\tilde{S}_k$ if

$$\text{depth}(M_P) \geq \min\{k, \text{depth}(R_P)\}, \quad \forall P \in \text{Spec}(R).$$

$\tilde{S}_k$ is weaker than the better-known Serre condition $S_k$. A projective module over any ring $R$ satisfies $\tilde{S}_k$ for every $k$; a projective module over $R$ satisfies $S_k$ only if $R$ itself satisfies $S_k$. From this point of view, property $\tilde{S}_k$ is more like $k$-torsionlessness and being a $k$th syzygy; a projective module is always $k$-torsionless and a $k$th syzygy for every $k$, no matter what the ring $R$ is.

**Proposition 11.** Let $M$ be any module over a ring $R$. Let $k \geq 0$. Consider the following three conditions on $M$:

(a) $M$ is $k$-torsionless;
(b) $M$ is a $k$th syzygy; and
(c) $M$ satisfies $\tilde{S}_k$.

Then (a) $\implies$ (b) $\implies$ (c).

Later in §3 we will show that (a), (b) and (c) are equivalent if $M$ has locally finite Gorenstein dimension, and we will give several other equivalent conditions.

**Proof.** For $k = 0$ all three conditions are automatically true (by definition) for every $M$. Let $k \geq 1$.

(a) $\implies$ (b). $M$ is at least 1-torsionless, and therefore there is a universal pushforward (u.p.f.). Thus $M$ is at least a first syzygy. If $k \geq 2$, then $N$ is $(k-1)$-torsionless; by induction, $N$ is a $(k-1)^{st}$ syzygy, and therefore $M$ is a $k$th syzygy.

(b) $\implies$ (c) follows directly from the (otherwise trivial) Depth Lemma:

**Depth Lemma.** Let $(R, m, k)$ be a local ring. Let $0 \to K \to L \to M \to 0$ be an exact sequence of finitely generated modules. Then

$$\text{depth}(K) \geq \min\{\text{depth}(L), \text{depth}(M) + 1\}.$$ 

If moreover $\text{depth}(L) > \text{depth}(M)$, then $\text{depth}(K) = \text{depth}(M) + 1$.

The Depth Lemma follows at once from the cohomological characterization of depth [Theorem 16.7], by looking at the long exact sequence of Ext.$^i(k, \ast)$. \qed

2. **Gorenstein Dimension of Modules**

**Definition 12.** A module $M$ has Gorenstein dimension 0 if $M$ is reflexive and Ext.$^i(M, R) = \text{Ext}^i(M^*, R) = 0$, $\forall i \geq 1$. Equivalently, G-dim$(M) = 0$ if Ext.$^i(M, R) = \text{Ext}^i(D(M), R) = 0$, $\forall i \geq 1$. Notation: G-dim$(M) = 0$, or G-dim$_R(M) = 0$.

Clearly, if G-dim$(M) = 0$, then G-dim$(M^*) = 0$ as well, and [G-dim$(M) = 0] \iff [\text{G-dim}(D(M)) = 0]$ (recall that $D(D(M)) \approx M$).

**Lemma 13.** (a) $M$ projective $\implies$ G-dim$(M) = 0$.

(b) If $0 \to K \to L \to M \to 0$ is exact, with G-dim$(M) = 0$, then:

$$[\text{G-dim}(K) = 0] \iff [\text{G-dim}(L) = 0].$$

(c) G-dim$_R(M) = 0 \iff$ G-dim$_{R_P}(M_P) = 0$, $\forall P \in \text{Spec}(R)$.

These are all easy to prove.
Lemma 14. If $\text{pd}(M) < \infty$ and $\text{G-dim}(M) = 0$, then $\text{pd}(M) = 0$.

Proof. If $\text{pd}(M) \leq 1$, we have an exact sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

with $P_j$ projective, $j = 0, 1$. As $\text{Ext}^1(M, R) = 0$, we have $0 \to M^* \to P_0^* \to P_1^* \to 0$ exact. Therefore $D(M) \approx 0$, and then $M \approx D(D(M)) \approx D(0) \approx 0$, i.e. $M$ is projective.

If $\text{pd}(M) \geq 2$, consider $0 \to K \to P_0 \to M \to 0$ exact, with $P_0$ projective; then $\text{pd}(K) < \text{pd}(M)$, and $\text{G-dim}(K) = 0$ by Lemma 13, (a) and (b). By induction on $\text{pd}(M)$ we see that $K$ is projective. But then $\text{pd}(M) \leq 1$, contradicting $\text{pd}(M) \geq 2$.

Remark. This proof shows that, if $\text{pd}(M) < \infty$ and $\text{Ext}^i(M, R) = 0$, $\forall i \geq 1$, then already $\text{pd}(M) = 0$. More generally, if $\text{pd}(M) < \infty$ and $\text{Ext}^i(M, R) = 0$, $\forall i \geq k+1$, then $\text{pd}(M) \leq k$. The analogue for Gorenstein dimension is given in Lemma 23.

Lemma 15. Let $0 \to M_1 \to M_0 \to M \to 0$ and $0 \to E_1 \to E_0 \to M \to 0$ be exact sequences with $M_1$, $M_0$ and $E_0$ having Gorenstein dimension 0. Then $\text{G-dim}(E_1) = 0$ as well.

Proof. Consider the fiber product diagram of $M_0$ and $E_0$ over $M$:

$$
\begin{array}{ccc}
0 & 0 \\
\uparrow & \downarrow \\
E_1 & E_1 \\
\uparrow & \downarrow \\
0 & M_1 & M_0 \times_M E_0 & E_0 & 0 \\
\uparrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & M_1 & M_0 & M & 0 \\
\uparrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 & 0
\end{array}
$$

Lemma 13(b) applies twice: first to the top row to give $\text{G-dim}(M_0 \times_M E_0) = 0$, and then to the left column to give $\text{G-dim}(E_1) = 0$.

Definition 16. A module $M$ has Gorenstein dimension at most $k$ for some integer $k \geq 0$ (notation: $\text{G-dim}(M) \leq k$) if there exists an exact sequence

$$
0 \to \cdots \to M_{k-1} \to M_{k-2} \to \cdots \to M_0 \to M \to 0
$$

with $\text{G-dim}(M_j) = 0$, $j = 0, \ldots, k$.

If $\text{G-dim}(M) \leq k$ for some $k \geq 0$, then we write $\text{G-dim}(M) < \infty$. Otherwise $\text{G-dim}(M) = \infty$. If $\text{G-dim}(M) < \infty$, we define $\text{G-dim}(M)$ as the smallest $k$ such that $\text{G-dim}(M) \leq k$.

Note the perfect similarity with the definition of projective dimension.
Characterization of Gorenstein rings in terms of G-dimension.

**Theorem 17.** Assume that \((R, m, k)\) is local. Let \(\dim(R) = n\). Then the following conditions are equivalent:

(a) \(R\) is Gorenstein;
(b) \(\text{G-dim}(M) \leq n, \forall M\) finitely generated \(R\)-module;
(c) \(\text{G-dim}(M) < \infty, \forall M\) finitely generated \(R\)-module;
(d) \(\text{G-dim}(k) < \infty\);
(e) \(\text{G-dim}(k) = n\).

Compare with the similar characterization of regular local rings in terms of projective dimension.

**Proof.** We show that \((a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e)\). Then we show that \((a) + (b) + (c) + (d) \Rightarrow (e); (c) \Rightarrow (d)\) is obvious.

Since \((b) \Rightarrow (c) \Rightarrow (d)\) are clear, we will show that \((a) \Rightarrow (b), (d) \Rightarrow (a), \) and \((a) + \cdots + (d) \Rightarrow (e)\).

Recall the various equivalent definitions of Gorenstein local rings [M, Theorem 18.1]:

\(\hat{R}\) is Gorenstein \iff \(\text{inj. dim}(R) < \infty \iff \text{inj. dim}(R) = n\)

\(\iff \text{Ext}^i(k, R) = 0\) for \(i \neq n\) and \(\text{Ext}^n(k, R) \cong k\)

\(\iff \text{Ext}^i(k, R) = 0\) for some \(i > n\)

\(\iff \text{Ext}^i(k, R) = 0\) for \(i < n\) and \(\text{Ext}^n(k, R) \cong k\).

\((a) \Rightarrow (b).\) Assume that \(R\) is Gorenstein. If \(n = 0\) then \(R\) is self-injective, and therefore \(\text{Ext}^i(M, R) = \text{Ext}^i(D(M), R) = 0, \forall i \geq 1\) and \(\forall M\); i.e., all (finitely generated) modules \(M\) have Gorenstein dimension 0.

Now assume that \(n \geq 1\). Let \(M\) be a finitely generated \(R\)-module. Let

\[0 \to K \to F_{n-1} \to \cdots \to F_0 \to M \to 0\]

be the beginning of a free resolution of \(M\) \((P_j \text{ free, } j = 0, \ldots, n - 1)\). Then \(\text{Ext}^i(K, R) = \text{Ext}^{i+n}(M, R) = 0, \forall i \geq 1\), for \(\text{inj. dim}(R) = n\).

Let \(F \to K \to 0\) be a free resolution of \(K\). As \(\text{Ext}^i(K, R) = 0, \forall i \geq 1\), we have \(0 \to K^* \to F^*\) exact. In particular, \(K^*\) is an \(n\)th syzygy, and then \(\text{Ext}^i(K^*, R) = 0, \forall i \geq 1\), for the same reason as for \(K\).

Finally, we show that \(K\) is reflexive. As \(K\) is at least a first syzygy, it is torsionless. Now take \(0 \to K' \to F \xrightarrow{f} K \to 0\) exact, with \(F\) free. Then \(K'\) is an \((n+1)\)st syzygy; with the same proof as for \(K\), we get \(\text{Ext}^i(K', R) = \text{Ext}^i(K'^*, R) = 0, \forall i \geq 1\). In particular, \(0 \to K' \to F \to K \to 0\) is dual-exact and double-dual-exact. Therefore \(f^{**} : F^{**} \to K^{**}\) is surjective. As \(\sigma_K \circ f = f^{**} \circ \sigma_F\) and \(\sigma_F\) is an isomorphism, we see that \(\sigma_K\) is surjective, and therefore an isomorphism, as required.

We have shown that \(\text{G-dim}(K) = 0\), and therefore that \(\text{G-dim}(M) \leq n\).

\((d) \Rightarrow (a).\) If \(0 \to M_k \to \cdots \to M_0 \to M \to 0\) is an exact sequence with \(\text{G-dim}(M_j) = 0, j = 0, \ldots, k, \) then \(\text{Ext}^{i+k}(M, R) = \text{Ext}^i(M_k, R) = 0, \forall i \geq 1\). In particular, if \(\text{G-dim}(k) < \infty\), then \(\text{Ext}^i(k, R) = 0\) for \(i > \text{G-dim}(k)\), so that \(R\) is Gorenstein in this case.

\((a) + \cdots + (d) \Rightarrow (e).\) By \((b)\), \(\text{G-dim}(k) \leq n\). On the other hand, \(\text{Ext}^n(k, R) \cong k \neq 0\), and the proof of \((d) \Rightarrow (a)\) above shows that \(\text{G-dim}(k) \geq n\). \(\square\)
Gorenstein dimension in exact sequences. In this subsection we prove the following theorem:

**Theorem 18.** If $0 \to K \to L \to M \to 0$ is an exact sequence, then:

(a) $\text{G-dim}(K) \leq \max\{\text{G-dim}(L), \text{G-dim}(M) - 1\}$;
(b) $\text{G-dim}(L) \leq \max\{\text{G-dim}(K), \text{G-dim}(M)\}$;
(c) $\text{G-dim}(M) \leq 1 + \max\{\text{G-dim}(K), \text{G-dim}(L)\}$.

In particular, if two of the three modules $K$, $L$ and $M$ have finite Gorenstein dimension, then so does the third one.

First we prove several preliminary results. Some of them, in particular Theorem 20 and Corollary 22, are of independent interest.

**Lemma 19.** Let $0 \to K \to L \to M \to 0$ be an exact sequence with $\text{G-dim}(M) = 0$. Then $\text{G-dim}(K) = \text{G-dim}(L)$.

*Proof.* 1. Assume first that $\text{G-dim}(L) \leq k < \infty$. We show that $\text{G-dim}(K) \leq k$. The case $k = 0$ is covered by Lemma 13(b). Assume that $k \geq 1$.

Let $0 \to F \to T_0 \to L \to 0$ be an exact sequence with $\text{G-dim}(T_0) = 0$ and $\text{G-dim}(F) \leq k - 1$; such an exact sequence exists by the definition of $\text{G-dim}(L) \leq k$. Consider the commutative diagram

$$
\begin{array}{cccccc}
0 & 0 \\
& & & & & \\
F & F \\
& & & & & \\
0 & S & T_0 & M & 0 \\
& & & & & \\
0 & K & L & M & 0 \\
& & & & & \\
0 & 0 & & & & \\
\end{array}
$$

with exact rows and columns, where $T_0 \to M$ in the top row is the composite of $T_0 \to L$ and $L \to M$, and $S$ is the kernel of $T_0 \to M$. (Then the two columns have the same kernel by the Snake Lemma.)

$\text{G-dim}(S) = 0$ by Lemma 13(b) applied to the top row; therefore $\text{G-dim}(K) \leq k$, because $\text{G-dim}(F) \leq k - 1$.

2. Conversely, assume that $\text{G-dim}(K) \leq k < \infty$. We show that $\text{G-dim}(L) \leq k$, by induction on $k$. The case $k = 0$ is covered by Lemma 13(b). Assume that $k \geq 1$.

Let $0 \to F \to S \to K \to 0$ be an exact sequence with $\text{G-dim}(S) = 0$ and $\text{G-dim}(F) \leq k - 1$. Let $0 \to D \to P \to M \to 0$ be exact, with $P$ projective; then
G-dim(D) = 0. Complete the diagram:

By the inductive hypothesis applied to the top row, G-dim(H) ≤ k − 1; therefore G-dim(L) ≤ k, as required.

**Theorem 20.** Let k ≥ 1. Assume that

\[ 0 \rightarrow K \rightarrow M_{k-1} \rightarrow \cdots \rightarrow M_0 \rightarrow N \rightarrow 0 \]

is exact, with G-dim(N) ≤ k and G-dim(M_j) = 0, j = 0, . . . , k − 1.

Then G-dim(K) = 0.

(Compare with the similar statement for projective dimension.)

**Proof.** It suffices to show that \([0 \rightarrow K_1 \rightarrow M_0 \rightarrow N \rightarrow 0]\) exact, with G-dim(N) ≤ k and G-dim(M_0) = 0 \(\Rightarrow [G\text{-dim}(K_1) \leq k - 1]\) - if k ≥ 1.

The proof is essentially the same as that of Lemma 19. Since G-dim(N) ≤ k, there is an exact sequence \(0 \rightarrow F \rightarrow T_0 \rightarrow N \rightarrow 0\) with G-dim(T_0) = 0 and G-dim(F) ≤ k − 1. Now consider the fiber product diagram:

Lemma 19 applies twice: first to the left column, giving G-dim(M_0 \times_N T_0) ≤ k − 1, and then to the top row, giving G-dim(K_1) ≤ k − 1.
Corollary 21. If \( \text{pd}(M) < \infty \), then \( \text{G-dim}(M) = \text{pd}(M) \).

Proof. If \( \text{pd}(M) < \infty \), we clearly have \( \text{G-dim}(M) \leq \text{pd}(M) \).

Now assume that \( \text{G-dim}(M) = k \) (and \( \text{pd}(M) < \infty \)). Let

\[
0 \to K \to P_{k-1} \to \cdots \to P_0 \to M \to 0
\]

be an exact sequence with \( P_j \) projective, \( j = 0, \ldots, k-1 \). Then \( \text{G-dim}(K) = 0 \) by Theorem 20. As \( \text{pd}(K) < \infty \), Lemma 14 shows that \( K \) is, in fact, projective – and therefore \( \text{pd}(M) \leq k \).

Alternatively, we could use the Remark after Lemma 14: we showed in the proof of (d) \( \implies \) (a) in Theorem 17 that \( [\text{G-dim}(M) = k < \infty] \implies [\text{Ext}^i(M, R) = 0 \text{ for all } i > k] \), and this together with \( \text{pd}(M) < \infty \) implies \( \text{pd}(M) \leq k \).

Corollary 22. \( \text{G-dim}(M) \leq k \iff \text{G-dim}_{R_P}(M_P) \leq k, \forall P \in \text{Spec}(R) \).

In other words, \( \text{G-dim}(M) = \sup\{\text{G-dim}_{R_P}(M_P) \mid P \in \text{Spec}(R)\} \).

Proof. \( \implies \) is clear, as “\( \text{G-dim}(M_j) = 0 \)” is a local property. Conversely, assume that \( \text{G-dim}_{R_P}(M_P) \leq k, \forall P \in \text{Spec}(R) \). Let

\[
0 \to K \to M_{k-1} \to \cdots \to M_0 \to M \to 0
\]

be exact, with \( \text{G-dim}(M_j) = 0, j = 0, \ldots, k-1 \). Then localizing at any \( P \) and using Theorem 20 we get \( \text{G-dim}_{R_P}(K_P) = 0 \); thus \( \text{G-dim}(K) = 0 \), and therefore \( \text{G-dim}(M) \leq k \).

Proof of Theorem 18. Take a truncated free resolution of \( 0 \to K \to L \to M \to 0 \).

For each \( K \) we get an exact sequence of \( k \)th syzygies, \( 0 \to K_k \to L_k \to M_k \to 0 \).

Take, for example, assertion (c). Assume that \( \text{G-dim}(K) \leq k \) and \( \text{G-dim}(L) \leq k \).

Then Theorem 20 gives \( \text{G-dim}(K_k) = \text{G-dim}(L_k) = 0 \). Thus \( \text{G-dim}(M_k) \leq 1 \), and \( \text{G-dim}(M) \leq 1 + k \).

(a) and (b) are similar.

Gorenstein dimension and depth. In this subsection and the next one, we prove the analogue of the Auslander–Buchsbaum formula:

\([R \text{ local}, M \neq 0, \text{ and } \text{G-dim}(M) < \infty] \implies [\text{G-dim}(M) + \text{depth}(M) = \text{depth}(R)]\).

Lemma 23. Assume that \( \text{G-dim}(M) \leq k < \infty \). Then:

(a) \( \text{Ext}^i(M, R) = 0, \forall i > k \);

(b) \( \text{If } \text{Ext}^k(M, R) = 0 \text{ (and } k \geq 1 \text{), then } \text{G-dim}(M) \leq k-1 \);

(c) \( \text{If } M \neq 0, \text{ then } \text{G-dim}(M) = \max\{t \geq 0 \mid \text{Ext}^t(M, R) \neq 0\} \).

Remark. Say \( M \neq 0 \) and \( \text{G-dim}(M) < \infty \). Compare: \( \text{G-dim}(M) = \max\{t \geq 0 \mid \text{Ext}^t(M, R) \neq 0\} \); \( \text{grade}(M) = \min\{t \geq 0 \mid \text{Ext}^t(M, R) \neq 0\} \). In particular, \( \text{grade}(M) \leq \text{G-dim}(M) \).

Proof of Lemma 23. We have already seen (a) in the proof of (d) \( \implies \) (a) in Theorem 17, and (c) clearly follows from (a) and (b).

We prove (b). First assume that \( \text{G-dim}(M) \leq 1 \) and \( \text{Ext}^1(M, R) = 0 \); we show that \( \text{G-dim}(M) = 0 \). Let \( 0 \to M_1 \to M_0 \to M \to 0 \) be an exact sequence with \( \text{G-dim}(M_j) = 0, j = 0, 1 \). As \( \text{Ext}^1(M, R) = 0 \), the sequence is dual-exact, and therefore we have an exact sequence \( 0 \to D(M) \to D(M_0) \to D(M_1) \to 0 \) for suitable choices of Auslander duals (Remark after Lemma 6). \( \text{G-dim}(M_j) = \)
Corollary 24. If $\text{G-dim}(M) = 0$ and $0 \to M \to R^n \to N \to 0$ is a universal pushforward, then $\text{G-dim}(N) = 0$.

Proof. $\text{Ext}^1(N, R) = 0$, and on the other hand $\text{G-dim}(N) \leq 1$; part (b) of the Lemma gives $\text{G-dim}(N) = 0$.

Lemma 25. (a) Let $0 \to M_1 \to M_0 \to M \to 0$ be an exact sequence with $M_1$ and $M_0$ reflexive and $\text{Ext}^1(M_0, R) = 0$. Then $\text{Ext}^1(M, R)^* = 0$.

(b) Let $(R, m, k)$ be local, with depth$(R) = 0$. If $F$ is a finitely generated $R$-module with $F^* = 0$, then $F = 0$.

Corollary 26. If $\text{G-dim}(M) < \infty$ and $(R, m, k)$ is local with depth$(R) = 0$, then $\text{G-dim}(M) = 0$.

Proof. It suffices to consider the case $\text{G-dim}(M) \leq 1$. Let $0 \to M_1 \to M_0 \to M \to 0$ be exact, with $\text{G-dim}(M_j) = 0$, $j = 0, 1$. Then part (a) of Lemma 25 gives $\text{Ext}^1(M, R)^* = 0$, and part (b) gives $\text{Ext}^1(M, R) = 0$. Therefore $\text{G-dim}(M) = 0$ by Lemma 25(b).

Proof of Lemma 25. (a) Dualizing the given sequence we get $0 \to M^* \to M_0^* \to M_1^* \to \text{Ext}^1(M, R) \to 0$ exact. Dualizing again, we get $0 \to \text{Ext}^1(M, R)^* \to M_1^* \to M_0^*$ exact. But $M_1^{**} \to M_0^{**}$ is the same as the injective map $M_1 \to M_0$ (up to canonical isomorphisms $\sigma_{M_1}$ and $\sigma_{M_0}$), and therefore $\text{Ext}^1(M, R)^* = 0$.

(b) If $F \neq 0$, then $F/mF$ is a non-zero $k$-vector space. Let $\pi : F \to F/mF$ be the canonical surjection, and let $\sigma : F/mF \to k$ be any non-zero (and therefore surjective) $k$-linear map. Finally, depth$(R) = 0 \implies \exists \alpha : k \to R$, a non-zero (and therefore injective) $R$-linear map. Then $u = \alpha \circ \sigma \circ p \in F^*$ is non-zero, and therefore $F^* \neq 0$.

Lemma 27. Let $(R, m, k)$ be local, with depth$(R) = 0$. Let $E$ be any nonzero, finitely generated $R$-module. Then depth$(E^*) = 0$.

In particular, if $M \neq 0$ is any reflexive module (for example, if $\text{G-dim}(M) = 0$), then depth$(M) = 0$.

Proof. As in the proof of Lemma 25(b), we can find a surjective homomorphism $E \to k$. Dualizing we get an injective homomorphism $k^* \to E^*$. $k^*$ is nonzero (because depth$(R) = 0$) and of finite length, and therefore depth$(E^*) = 0$.

Lemma 28. Let $(R, m, k)$ be local, of arbitrary depth, and let $\text{G-dim}(M) = 0$ with $M \neq 0$. Then depth$(M) = \text{depth}(R)$.

Proof. Put depth$(R) = d$.

$M$ is $k$-torsionless for every $k \geq 0$. In particular, $M$ is $d$-torsionless. By Proposition 1, $M$ satisfies $\bar{S}_d$, and in particular depth$(M) \geq d$. 

Now we show that depth$(M) = d$. The case $d = 0$ is covered by Lemma 27, therefore we may assume that $d \geq 1$. We will show that Ext$^d(k, M) \neq 0$ (which means that depth$(M) \leq d$).

We have G-dim$(M^*) = 0$. Consider the beginning of a minimal free resolution of $M^*$:

$$F_1 \xrightarrow{u^*} F_0 \xrightarrow{f} M^* \rightarrow 0.$$ 

Put $K_1 = \text{Ker}(f)$, $K_2 = \text{Ker}(u)$; thus we have exact sequences:

$$0 \rightarrow K_2 \rightarrow F_1 \rightarrow K_1 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K_1 \rightarrow F_0 \rightarrow M^* \rightarrow 0,$$

and G-dim$(K_j) = 0$, $j = 1, 2$. Dualizing, we get exact sequences:

$$0 \rightarrow M \rightarrow F^*_0 \rightarrow K^*_1 \rightarrow 0 \quad \text{and} \quad 0 \rightarrow K^*_1 \rightarrow F^*_1 \rightarrow K^*_2 \rightarrow 0$$

(using $M^{**} \cong M$).

G-dim$(K^*_1) = 0$ $\implies$ depth$(K^*_1) \geq d$ $\implies$ Ext$^{d-1}(k, K^*_1) = 0$; therefore we have an exact sequence

$$0 \rightarrow \text{Ext}^d(k, M) \rightarrow \text{Ext}^d(k, F^*_0) \rightarrow \text{Ext}^d(k, K^*_1).$$

We show that $\alpha = 0$; then Ext$^d(k, M) = \text{Ext}^d(k, F^*_0) \neq 0$ (because depth$(F^*_0) = \text{depth}(R) = d$), as required.

To show that $\alpha = 0$, consider the commutative diagram

$$\begin{array}{ccc}
\text{Ext}^d(k, F^*_0) & \xrightarrow{\gamma} & \text{Ext}^d(k, F^*_1) \\
\alpha \downarrow & & \beta \downarrow \\
\text{Ext}^d(k, K^*_1) & & \\
\end{array}$$

$\text{Ker}(\beta) = \text{Ext}^{d-1}(k, K^*_2) = 0$, so that $\beta$ is injective. On the other hand, $\gamma$ is induced by $u^* : F^*_0 \rightarrow F^*_1$, and therefore $\gamma = 0$. Consequently $\alpha = 0$, as advertised. \hfill \square

**Theorem 29** (The Auslander–Bridger Formula). *Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring, and let $M \neq 0$ be a finitely generated $R$-module. If G-dim$(M) < \infty$, then

$$\text{G-dim}(M) + \text{depth}(M) = \text{depth}(R).$$

Proof. Induction on $k = \text{G-dim}(M)$. If $k = 0$, the result is contained in Lemma 28.

If $k \geq 1$, then depth$(R) \geq 1$ by Corollary 24. Put depth$(R) = d$.

Assume first that $k = 1$. Then we have an exact sequence

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$$

with $F$ free and G-dim$(K) = 0$. Then depth$(K) = \text{depth}(F) = d$. By the Depth Lemma, depth$(M) \geq d - 1$. The crux of the whole proof is to show that depth$(M) = d - 1$ in this case. We will prove this later, in the next subsection, after we will have studied the behavior of Gorenstein dimension under reduction modulo a regular element.

If $k \geq 2$, we have an exact sequence

$$0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$$

with $F$ free and G-dim$(K) = k - 1$. By induction, depth$(K) = d - (k - 1)$, and then depth$(M) = \text{depth}(K) - 1 = d - k$ by the Depth Lemma, for depth$(F) =$
\[ d > \text{depth}(K) \text{ (so that } \text{depth}(K) \geq \text{depth}(M) + 1, \text{ and in particular } \text{depth}(F) > \text{depth}(M)). \]

**Remarks.** (1) The proof in [AB, pp. 118–119] contains a serious mistake. Namely, when \( \text{G-dim}(M) \geq 1 \), the authors consider an exact sequence

\[ 0 \to K \to P \to M \to 0 \]

with \( P \) projective, and then claim that “It is well-known and easily proved that \( \text{depth}(K) = \text{depth}(M) + 1 \).” Unfortunately, this formula is not necessarily true when \( \text{depth}(K) = \text{depth}(P) \); this is exactly why the case \( \text{G-dim}(M) = 1 \) is the most difficult.

At that point in the memoir, Auslander and Bridger had already included reduction modulo a regular element in the theory. They used it for the case \( \text{G-dim}(M) = 0 \). As we have seen, that case is, in fact, elementary (i.e. does not require reduction modulo a regular element). I was unable to find a similar elementary proof for the case \( \text{G-dim}(M) = 1 \). Compare with the much easier proof of the usual Auslander–Buchsbaum formula for projective dimension in [M, p. 155].

(2) Let \( R \) be a “global” (i.e. not necessarily local) Noetherian ring. Assume that \( \text{G-dim}(M) <_{\text{loc}} \infty \), i.e. \( \text{G-dim}_R(M_P) < \infty \), \( \forall P \in \text{Spec}(R) \). If \( \text{dim}(R) < \infty \) (\( \text{dim}(R) \) is the Krull dimension of \( R \)), then \( \text{G-dim}(M) \leq \text{dim}(R) \); this follows from the Auslander–Bridger formula. However, if \( \text{dim}(R) = \infty \) it is possible to have \( \text{G-dim}(M) <_{\text{loc}} \infty \) but \( \text{G-dim}(M) = \infty \). For example, let \( k \) be a field, and let \( A = k[x_1, \ldots, x_n, \ldots] \) be the polynomial ring over \( k \) in countably many variables. Let \( m_1 < m_2 < \cdots \) be positive integers such that \( m_2 - m_1 < m_3 - m_2 < \cdots \).

Let \( m_i \) be the prime ideal of \( A \) generated by \( \{X_j \mid m_i \leq j < m_{i+1}\} \), let \( S \) be the complement of \( \bigcup m_i \) in \( A \), and let \( R = S^{-1}A \). Then \( R \) is a Noetherian ring of infinite Krull dimension [K, p. 203]. In this example, \( k \) is naturally a finitely generated \( R \)-module, and \( \text{G-dim}(k) <_{\text{loc}} \infty \), but \( \text{G-dim}(k) = \infty \).

**Corollary 30.** If \( \text{G-dim}(M) <_{\text{loc}} \infty \), then

\[ \text{grade}(\text{Ext}^i(M, R)) \geq i, \forall i \geq 1. \]

**Proof.** \( \text{grade}(\text{Ext}^i(M, R)) = \min\{\text{depth}(RP) \mid P \in \text{Supp}(\text{Ext}^i(M, R))\} \).

If \( \text{depth}(R_P) < i \), then \( \text{G-dim}_R(M_P) \leq \text{depth}(R_P) < i \) by the Auslander–Bridger formula, so that \( \text{Ext}^i_R(M_P, R_P) = 0 \) by Lemma 23(a). Therefore \( \text{Ext}^i(M, R)_P = 0 \), i.e. \( P \notin \text{Supp}(\text{Ext}^i(M, R)) \).

**Regular elements and Gorenstein dimension.** In this subsection we complete the proof of the Auslander–Bridger formula. Before we can do that, however, we must bring regular elements into the picture. In particular, we need to study the behavior of Gorenstein dimension under reduction modulo a regular element.

**Notation.** Throughout this subsection, fix a ring \( R \) and a non-unit \( x \in R \). Write \( \bar{R} \) for \( R/xR \), \( \bar{M} \) for \( M/xM \), etc. Note that \( \bar{M} = \text{Hom}_R(\bar{M}, \bar{R}) \), while \( \bar{M}^* = \text{Hom}_R(M, \bar{R}) \otimes_R \bar{R} \).

Similarly with \( \bar{D}(\bar{M}) \) and \( \bar{D}(\bar{M}) \).

Consider the functors \( \star \otimes_R \bar{R} \) and \( \text{Hom}_R(\star, \bar{R}) \) and their composite, \( \text{Hom}_R(\star, \bar{R}) \); note that we have an obvious functorial isomorphism

\[ \text{Hom}_R(\bar{M}, \bar{R}) \cong \text{Hom}_R(M, \bar{R}) \].
In this case the abstract Leray spectral sequence becomes
\[ \text{Ext}^1_R(\text{Tor}^R_j(M, \bar{R}), \bar{R}) \Rightarrow \text{Ext}^{i+j}_R(M, \bar{R}). \]

In particular, if \( x \) is both \( R \)-regular and \( M \)-regular, then \( \text{Tor}^R_j(M, \bar{R}) = 0, \forall j \geq 1 \), and we get canonical isomorphisms
\[ \text{Ext}^i_R(M, \bar{R}) \cong \text{Ext}^i_R(M, \bar{R}), \forall i \geq 0. \]

Of course, these isomorphisms can be proved easily (when \( x \) is \( R \)-regular and \( M \)-regular) without using spectral sequences. Here is a quick argument for the benefit of the graduate student:

First, \( \text{Hom}_R(\bar{M}, \bar{R}) \cong \text{Hom}_R(M, \bar{R}) \) is clear.
Next, let \( 0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0 \) be an exact sequence with \( F \) free of finite rank.
Then \( x \) is also \( K \)-regular, and the sequence \( 0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0 \) is exact (by the Snake Lemma). We have a commutative diagram with exact rows:

\[
\begin{array}{cccc}
\text{Hom}_R(F, \bar{R}) & \longrightarrow & \text{Hom}_R(K, \bar{R}) & \longrightarrow & \text{Ext}^1_R(M, \bar{R}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Hom}_R(\bar{F}, \bar{R}) & \longrightarrow & \text{Hom}_R(\bar{K}, \bar{R}) & \longrightarrow & \text{Ext}^1_R(M, \bar{R}) & \longrightarrow & 0
\end{array}
\]

and \( \text{Ext}^i_R(\bar{M}, \bar{R}) \cong \text{Ext}^i_R(M, \bar{R}) \) follows.
Finally, for \( i \geq 2 \) we have by induction on \( i \):
\[ \text{Ext}^i_R(\bar{M}, \bar{R}) \cong \text{Ext}^{i-1}_R(\bar{K}, \bar{R}) \cong \text{Ext}^{i-1}_R(K, \bar{R}) \cong \text{Ext}^i_R(M, \bar{R}). \]

Since the Auslander dual commutes with base change (Remark \#5 after Proposition \[1\]), we have:

**Proposition 31.** If \( x \) is any element of \( R \) and \( M \) is any \( R \)-module, then
\[ D_R(\bar{M}) \approx D_R(M). \]

\[ \square \]

**Corollary 32.** If \( x \) is \( R \)-regular and \( \text{G-dim}(M) = 0 \), then \( x \) is \( M \)-regular and \( \text{G-dim}_R(\bar{M}) = 0 \).

**Proof.** First we show that \( x \) is \( M \)-regular. Indeed, \( M \) is 1-torsionless, and therefore a first syzygy (Proposition \[1\]). Since \( x \) is \( R \)-regular and \( M \) is isomorphic to a submodule of a free module, \( x \) is \( M \)-regular.

Therefore \( \text{Ext}^i_R(\bar{M}, \bar{R}) \cong \text{Ext}^i_R(M, \bar{R}), \forall i \geq 0 \). From \( 0 \rightarrow R \overset{x}{\rightarrow} R \rightarrow \bar{R} \rightarrow 0 \) exact we get
\[ \text{Ext}^i_R(M, R) \longrightarrow \text{Ext}^i_R(M, \bar{R}) \longrightarrow \text{Ext}^{i+1}_R(M, \bar{R}) \]
exact. Since \( \text{G-dim}(M) = 0 \), we get \( \text{Ext}^i_R(\bar{M}, \bar{R}) = \text{Ext}^i_R(M, \bar{R}) = 0, \forall i \geq 1 \).

On the other hand, \( \text{G-dim}_R(M) = 0 \;\Rightarrow\; \text{G-dim}_R(D(M)) = 0 \); by what we have just proved, \( \text{Ext}^i_R(D_R(\bar{M}), \bar{R}) = 0, \forall i \geq 1 \). But \( D_R(\bar{M}) \approx D_R(M) \). Thus
\[ \text{Ext}^i_R(\bar{D}_R(\bar{M}), \bar{R}) = 0, \forall i \geq 1 \]
– that is, \( \text{G-dim}_R(\bar{M}) = 0 \).

\[ \square \]
Corollary 33. If $x$ is $R$-regular and $M$-regular and $\text{G-dim}(M) < \infty$, then $\text{G-dim}_R(M) \leq \text{G-dim}_R(M)$. If moreover $x \in J(R)$ (the Jacobson radical of $R$), then $\text{G-dim}_R(M) = \text{G-dim}_R(M)$.

Proof. Induction on $k = \text{G-dim}_R(M)$; the case $k = 0$ is covered by the previous Corollary. Assume that $k \geq 1$.

Consider an exact sequence

$$0 \to K \to F \to M \to 0$$

with $F$ free and $\text{G-dim}(K) = k - 1$. Since $x$ is $M$-regular and $F$-regular, it is also $K$-regular, and we have an exact sequence of $R$-modules

$$0 \to \bar{K} \to F \to \bar{M} \to 0.$$

By induction, $\text{G-dim}_R(\bar{K}) \leq k - 1$, and therefore $\text{G-dim}_R(\bar{M}) \leq k$.

To show equality when $x \in J(R)$, it suffices to show that $\text{Ext}^k_R(\bar{M}, \bar{R}) \neq 0$ in that case (see Lemma 23(a)).

Equivalently, we will show that $\text{Ext}^k_R(M, \bar{R}) \neq 0$. We have an exact sequence:

$$\text{Ext}^k_R(M, R) \xrightarrow{x} \text{Ext}^k_R(M, R) \to \text{Ext}^k_R(M, \bar{R}).$$

As $x \in J(R)$, Nakayama’s Lemma shows that $[\text{Ext}^k_R(M, \bar{R}) = 0] \implies [\text{Ext}^k_R(M, R) = 0]$; but if that happens, then $\text{G-dim}_R(M) < k$, contradiction.

Now we are ready to finish the proof of the Auslander–Bridger formula.

Proof of Theorem 22: Conclusion.

The only thing left to prove is: If $(R, \mathfrak{m})$ is local, $d = \text{depth}(R)$, and $\text{G-dim}_R(M) = 1$, then $\text{depth}(M) = d - 1$.

We prove this by induction on $d$. The case $d = 0$ is vacuously true (in that case there are no modules $M$ with $\text{G-dim}(M) = 1$). Let $d \geq 1$.

We have already proved that $\text{depth}(M) \geq d - 1$. By way of contradiction, assume that $\text{depth}(M) > d$. Then there exists an element $x \in \mathfrak{m}$ which is both $R$-regular and $M$-regular. We have: $\text{depth}(\bar{R}) = d - 1$ and $\text{G-dim}_R(\bar{M}) = \text{G-dim}_R(M) = 1$.

By induction, $\text{depth}_R(\bar{M}) = d - 2$; equivalently, $\text{depth}_R(M) = d - 2$.

But now $\text{depth}(M) = d$ and $\text{depth}(\bar{M}) = d - 2$ taken together contradict the Depth Lemma applied to the exact sequence $0 \to M \to \bar{M} \to 0$.

Exercise. (cf. [M, Theorem 16.9] for the case of projective dimension.)

Let $R$ be a Noetherian ring, and let $M, N$ be finitely generated $R$-modules with $M \neq 0$, grade$(M) = k > 0$, and $\text{G-dim}(N) = l < k$.

Show that $\text{Ext}^i(M, N) = 0$, $\forall i < k - l$.

I include this exercise to further illustrate the theme of results which remain true if one replaces projective dimension with Gorenstein dimension (sometimes at the price of more delicate proofs).

Hint: Induction on $l = \text{G-dim}(N)$. The case $l = 0$ is the most serious. (The case with $\text{pd}(N) = 0$ is trivial by comparison.)

Assume that $\text{G-dim}(N) = 0$. First show that $\text{Hom}(M, N) = 0$: as grade$(M) = k > 0$, $\exists x \in \text{Ann}(M)$ with $x$ $R$-regular. Then $x$ is also $N$-regular, and $\text{Hom}(M, N) = 0$ follows easily. Then take a universal pushforward $0 \to N \to F \to N' \to 0$ (with $F$ free and $\text{G-dim}(N') = 0$; see Corollary 24). As $\text{Ext}^i(M, F) = 0$, $\forall i < k$, we get $\text{Ext}^i(M, N) = 0$, $\forall i < k$, by bootstrapping.
3. $k$th syzygies of finite Gorenstein dimension

In this section we study the relationship between $k$-torsionless, being a $k$th syzygy and condition $\tilde{S}_k$ for modules of locally finite Gorenstein dimension. Proposition 11 in §1 gives implications which hold for an arbitrary module $M$. (We will refine that result in Proposition 36 below.)

First we introduce $k$-torsionfreeness:

**Definition 34.** Let $(R, m)$ be a local ring, and let $M$ be an $R$-module. Fix an integer $k \geq 0$. $M$ is $k$-torsionfree if every $R$-regular sequence of length at most $k$ is also $M$-regular. (Note that “1-torsionfree” is usually called “torsionfree”.)

**Definition 35.** Let $R$ be any ring (not necessarily local). A module $M$ is locally $k$-torsionfree if $M_P$ is $k$-torsionfree over $R_P$, $\forall P \in \text{Spec}(R)$.

As usual, every module is (locally) 0-torsionfree.

**Remark.** Auslander and Bridger use the term “$k$-torsionfree” for what I call “$k$-torsionless” (and they do not use any name for what I call “$k$-torsionfree”). It seems to me that the meaning of “torsionfree” as given in Definition 34 above is standard in algebra – and therefore I use “torsionless” for what [AB] calls “torsionfree”.

Local $k$-torsionfreeness lies between being a $k$th syzygy and property $\tilde{S}_k$. In fact, we have the following refinement of Proposition 11:

**Proposition 36.** Let $M$ be any module over a ring $R$. Let $k \geq 0$. Consider the following conditions on $M$:

(a) $M$ is $k$-torsionless;
(b) $M$ is a $k$th syzygy;
(c) $\exists 0 \rightarrow M \rightarrow M_0 \rightarrow \cdots \rightarrow M_{k-1}$ exact, with $G\text{-dim}(M_j) = 0$, $j = 0, \ldots, k-1$;
(d) $M$ is locally $k$-torsionfree;
(e) $M$ satisfies $\tilde{S}_k$.

Then (a) $\implies$ (b) $\implies$ (c) $\implies$ (d) $\implies$ (e).

**Proof.** We have seen that (a) $\implies$ (b) in Proposition 11, and (b) $\implies$ (c) and (d) $\implies$ (e) are clear.

We prove (c) $\implies$ (d). By localizing at any prime $P$, it suffices to show that (c) $\implies [M$ is $k$-torsionfree$]$ over a local ring $R$. We do induction on $k$, the case $k = 0$ being trivial. If $k \geq 1$, let $N = \text{Coker}(M \rightarrow M_0)$; then we have a short exact sequence

$$0 \rightarrow M \rightarrow M_0 \rightarrow N \rightarrow 0$$

where $N$ satisfies (c) for $k-1$, and therefore $N$ is $(k-1)$-torsionfree by induction. $M_0$ is $l$-torsionfree for every $l \geq 0$; this follows easily from Corollary 32. Therefore the conclusion follows from the following Lemma:

**Lemma 37.** If $0 \rightarrow M \rightarrow B \rightarrow N \rightarrow 0$ is a short exact sequence with $B$ $k$-torsionfree and $N$ $(k-1)$-torsionfree, then $M$ is $k$-torsionfree.

Compare this with Proposition 8(b).

**Proof.** If $k \geq 1$ and $x_1$ is $R$-regular, then $x_1$ is $B$-regular and therefore $M$-regular as well. Thus $M$ is at least 1-torsionfree in this case.
If \( k \geq 2 \) and \( x_1, \ldots, x_s \) is an \( R \)-regular sequence with \( 2 \leq s \leq k \), then \( x_1 \) is \( N \)-regular, \( B \)-regular and \( M \)-regular, and we have an exact sequence of \( \bar{R} \)-modules,

\[
0 \to \bar{M} \to \bar{B} \to \bar{N} \to 0,
\]

where \( \bar{R} = R/xR, \bar{M} = M/xM, \) etc.

As \( B \) is \( k \)-torsionfree over \( R \) and \( x_1 \) is \( R \)-regular, we see that \( \bar{B} \) is \((k - 1)\)-torsionfree over \( \bar{R} \). Similarly, \( \bar{N} \) is \((k - 2)\)-torsionfree over \( \bar{R} \). By induction, \( \bar{M} \) is \((k - 1)\)-torsionfree over \( \bar{R} \), and in particular \( x_2, \ldots, x_s \) is \( M \)-regular. But then \( x_1, x_2, \ldots, x_s \) is \( M \)-regular, as required.

When \( \text{G-dim}(M) <_{\text{loc}} \infty \), all the conditions in Proposition 36 are equivalent. The cycle of implications is closed via yet another equivalent condition, the prototype of which we have already seen in Corollary 30.

**Proposition 38.** Fix an integer \( k \geq 0 \). Assume that \( \text{G-dim}(M) <_{\text{loc}} \infty \) and that \( M \) satisfies \( \bar{S}_k \).

Then \( \text{grade}(\text{Ext}^i(M, R)) \geq i + k, \forall i \geq 1 \).

The proof is essentially the same as that of Corollary 30.

**Proof.** Fix \( i \geq 1 \), and take \( P \in \text{Spec}(R) \) with \( \text{depth}(R_P) < i + k \). Then we must show that \( \text{Ext}^i_{R_P}(M_P, R_P) = 0 \), i.e. that \( P \notin \text{Supp}(\text{Ext}^i(M, R)) \). We have \( \text{G-dim}_{R_P}(M_P) = \text{depth}(R_P) - \text{depth}(M_P) \leq \text{depth}(R_P) - \text{min}\{k; \text{depth}(R_P)\} = \max\{0; \text{depth}(R_P) - k\} \). Since \( \text{depth}(R_P) < i + k \) and \( i \geq 1 \), we have in any case \( \text{G-dim}_{R_P}(M_P) < i \), and therefore \( \text{Ext}^i_{R_P}(M_P, R_P) = 0 \) by Lemma 23(a).

**Proposition 39.** Fix an integer \( k \geq 0 \). Assume that \( \text{G-dim}(M) = g < \infty \), and that \( \text{grade}(\text{Ext}^i(M, R)) \geq i + k \) for \( 1 \leq i \leq g \). Then \( M \) is \( k \)-torsionless.

**Proof.** Induction on \( g \geq 0 \). For \( g = 0 \) the hypothesis is just that \( \text{G-dim}(M) = 0 \) – but in that case \( M \) is \( k \)-torsionless for every \( k \geq 0 \).

Assume that \( g \geq 1 \). Let

\[
0 \to M' \to P \to M \to 0
\]

be an exact sequence with \( P \) projective and \( \text{G-dim}(M') = g - 1 \). For \( 1 \leq i \leq g - 1 \), we have

\[
\text{grade}(\text{Ext}^i(M', R)) = \text{grade}(\text{Ext}^{i+1}(M, R)) \geq i + k + 1;
\]

by the inductive hypothesis, \( M' \) is \((k + 1)\)-torsionless. \( P \) is projective, and in particular \((k + 1)\)-torsionless; therefore \( M \) is \( k \)-torsionless by Proposition 3(c). Note that the hypothesis \( \text{grade}(\text{Ext}^1(M, R)) \geq 1 + k \) is used here, while the other grade conditions are used to show (by induction) that \( M' \) is \((k + 1)\)-torsionless.

Putting all these results together, we get:

**Theorem 40.** Fix \( k \geq 0 \), and assume that \( \text{G-dim}(M) <_{\text{loc}} \infty \) (where \( M \) is a module over any ring \( R \), not necessarily local).

Then the following conditions on \( M \) are equivalent:

(a) \( M \) is \( k \)-torsionless;
(b) \( M \) is a \( k \)-th syzygy;
(c) \( \exists 0 \to M \to M_0 \to \cdots \to M_{k-1} \) exact, with \( \text{G-dim}(M_j) = 0, j = 0, \ldots, k - 1 \);
(d) \( M \) is locally \( k \)-torsionfree.
(e) $M$ satisfies $\tilde{S}_k$;
(f) $\text{grade}(\text{Ext}^i(M, R)) \geq i + k$, $\forall i \geq 1$.

In particular, all these conditions are equivalent if $M$ is any module over a Gorenstein ring $R$, or if $\text{pd}(M) < \text{loc} \infty$.

The equivalence of (a) and (b) holds in more general conditions. A typical situation where the result below can be used is that of modules which have (locally) finite projective (or Gorenstein) dimension on the punctured spectrum of a local ring of sufficiently large depth.

**Theorem 41.** Fix $k \geq 0$. Let $R$ be a ring and $M$ an $R$-module. Assume that

$$[P \in \text{Spec}(R) \text{ and } \text{depth}(R_P) \leq k - 2] \implies [\text{G-dim}_{R_P}(M_P) < \infty].$$

Then the following conditions on $M$ are equivalent:

- (a) $M$ is $k$-torsionless;
- (b) $M$ is a $k^{th}$ syzygy.

**Proof.** We already know that (a) $\implies$ (b), with no conditions on G-dimension. Conversely, assume that $M$ is a $k^{th}$ syzygy and that $\text{G-dim}_{R_P}(M_P) < \infty$ whenever $\text{depth}(R_P) \leq k - 2$. We show that $M$ is $k$-torsionless by induction on $k$. For $k = 0$ there is nothing to prove, and the case $k = 1$ is easy. Now let $k \geq 2$. Since $M$ is a $k^{th}$ syzygy, there is an exact sequence

$$0 \to M \to F \to N \to 0$$

with $F$ free and $N$ a $(k - 1)^{st}$ syzygy. By induction, $N$ is $(k - 1)$-torsionless; if we can show that $\text{grade}(\text{Ext}^1(N, R)) \geq k - 1$, then Proposition 41(a) gives that $M$ is $k$-torsionless, as required.

To finish the proof, we show that $\text{grade}(\text{Ext}^1(N, R)) \geq k - 1$. Let $P \in \text{Spec}(R)$ with $\text{depth}(R_P) < k - 1$. We must show that $P \notin \text{Supp}(\text{Ext}^1(N, R))$. But this is clear, since

$$\text{depth}(R_P) < k - 1 \implies \text{G-dim}_{R_P}(M_P) < \infty \implies \text{G-dim}_{R_P}(N_P) < \infty \implies \text{grade}(\text{Ext}^1_{R_P}(N_P, R_P)) \geq 1 + (k - 1),$$

by Proposition 41(b) ($N_P$ is a $(k - 1)^{st}$ syzygy, and therefore satisfies $\tilde{S}_{k-1}$); then, in fact, $\text{Ext}^1_{R_P}(N_P, R_P) = 0$, because the grade of a non-zero module over $R_P$ cannot exceed $\text{depth}(R_P)$.  

**Definition 42.** A ring $R$ is $q$-Gorenstein ($q \geq 0$ an integer) if it satisfies Serre’s condition $S_q$ and is Gorenstein in codimension $q - 1$; equivalently, $R$ is $q$-Gorenstein if $R_P$ is Gorenstein for every prime ideal $P$ of $R$ with $\text{depth}(R_P) \leq q - 1$.

In particular, all rings are 0-Gorenstein, and a ring is 1-Gorenstein if and only if it satisfies $S_1$ and is Gorenstein in codimension 0 (this includes, in particular, all reduced rings).

Notice the similarity with the definition of $q$-regular rings: a ring is $q$-regular if $R_P$ is regular whenever $\text{depth}(R_P) \leq q - 1$, or equivalently, if $R$ satisfies $S_q$ and $R_{q-1}$ (thus 1-regular is the same as reduced, and 2-regular is the same as normal). Of course, $q$-regular implies $q$-Gorenstein.

**Corollary 43.** If $R$ is $(k - 1)$-Gorenstein and $M$ is an $R$-module, then $M$ is $k$-torsionless if and only if it is a $k^{th}$ syzygy.
For example, over a reduced ring $R$, every second syzygy (and in particular every
dual) is reflexive.

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