Covering homology

Morten Brun  Gunnar Carlsson *  Bjørn Ian Dundas †
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Abstract

We introduce the notion of covering homology of a commutative $S$-algebra with respect to certain families of coverings of topological spaces. The construction of covering homology is extracted from Bökstedt, Hsiang and Madsen’s topological cyclic homology. In fact covering homology with respect to the family of orientation preserving isogenies of the circle is equal to topological cyclic homology. Our basic tool for the analysis of covering homology is a cofibration sequence involving homotopy orbits and a restriction map similar to the restriction map used in Bökstedt, Hsiang and Madsen’s construction of topological cyclic homology.

Covering homology with respect to families of isogenies of a torus is constructed from iterated topological Hochschild homology. It receives a trace map from iterated algebraic K-theory and the hope is that the rich structure, and the calculability of covering homology will make covering homology useful in the exploration of J. Rognes’ “red shift conjecture”.

1 Introduction

Topological cyclic homology ($TC$), as defined by Bökstedt, Hsiang and Madsen in [1], is interesting for two reasons: firstly it is a good approximation to algebraic K-theory, secondly it is accessible through methods in stable homotopy theory. Along with motivic homotopy theory, topological cyclic homology is the main source for calculations of algebraic K-theory.

Topological cyclic homology is built from a diagram of categorical fixed point spectra of Bökstedt’s topological Hochschild homology. The main reason for the accessibility of $TC$ is the so-called “fundamental cofiber sequence” which inductively gives homotopical control of the categorical fixed points. There are many frameworks where people find conceptual reasons for the fundamental cofiber sequence – for instance it can be viewed as a concrete identification of the geometrical fixed points – but regardless of point of view it remains a marvellous fact at a crucial point of the theory.

Just as for other cyclic nerve constructions, if the input is commutative – in our case a connective commutative $S$-algebra $A$ – topological Hochschild homology extends to a functor of spaces $X \mapsto \Lambda_X A$, where the value at the circle $S^1$ recovers the usual definition $\Lambda_{S^1} A \simeq THH(A)$. If $X$ is a finite set, $\Lambda_X A$ is just a particular model for the $X$-fold smash product of $A$ with itself. Our preferred model $\Lambda_X A$ is extracted from Bökstedt’s construction of topological Hochschild homology and Street’s first construction [19] to enhance the functoriality of homotopy colimits. This functoriality has the side effect that the multiplicative structure of topological Hochschild homology can be realized on our concrete model, and using Bökstedt’s construction as our basis, we get that $\Lambda_X A$ is automatically a homotopy functor in both $X$ and $A$ without any cofibrant replacements.

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†Part of this work was done while the third author visited Stanford university, and we want to thank the institution for its hospitality. Part of this work was done while the second and third authors visited Institut Mittag-Leffler for the program in algebraic topology in the spring of 2006, and we want to thank the organizers for the invitation and the possibility to work there.
The main reason for our choice of model is that the fundamental cofiber sequence extends in a beautiful manner (see Lemma 5.1.2) giving full homotopy theoretic control over the categorical fixed points. Currently it is not clear how to extend it working with other models for the smash product of commutative ringspectra. The sequence becomes particularly transparent in the abelian case, which is the interesting part if one is mostly concerned with the case $X$ being a torus (which is the case for iterated topological Hochschild homology): if $G$ is a finite abelian group, $X$ a non-empty free $G$-space and $A$ a connective commutative $S$-algebra, then there is a cofiber sequence

$$[\Lambda_X(A)]_{hG} \to [\Lambda_X(A)]^G \to \holim_{0 \neq H \leq G} [\Lambda_{X/H}(A)]^{G/H},$$

where $[\Lambda_X(A)]_{hG}$ denotes the homotopy $G$-orbits, $[\Lambda_X(A)]^G$ the categorical fixed points and the homotopy limit is taken over all nontrivial subgroups of $G$. In the equivariant world, this could be viewed as an instance of the tom Dieck filtration gotten by taking fixed points of the sequence one gets by smashing $\Lambda_X(A)$ with the cofibration sequence $EG \to S^0 \to E\Sigma^n$, as discussed in [4], together with identifications, firstly of the geometric $H$-fixed-points of $\Lambda_X(A)$ and $\Lambda_{X/H}(A)$, and secondly of the categorical fixed points of $\Lambda_X(A)$ and the fixed points obtained by deloopings by representations in a universe. However, we also get that these deloopings are not necessary for developing the theory (with the exception of matters related to transfers, which will be important in a later paper), and we can stay with the concrete functorial model at hand and its associated categorical constructions.

More precisely, by induction on the order of the group, the fundamental cofibration sequence imply that we have full homotopical control over the categorical fixed points $(\Lambda_X A)^G$ if $G$ is a finite group acting freely on $X$.

As a result of this structure we get that if $X$ is connected, then there is a natural isomorphism

$$\pi_0[\Lambda_X A]^G \cong W_G(\pi_0 A)$$

where the right hand side is the Burnside-Witt ring of Dress and Siebeneicher [3]; and we recover Hesselholt and Madsen’s result $\pi_0[THH(A)]^{G_r} \cong W_{C_r}(\pi_0 A)$ [8] where $C_r$ is the cyclic group of order $r$.

Studying systems of coverings, the spectra $(\Lambda_X A)^G$ assemble into a diagram giving rise to the new notion of “covering homology”. In the particular case of finite orientation preserving self-coverings of the circle this is Bökstedt, Hsiang and Madsen’s topological cyclic homology. If we include reflections we get a definition of topological dihedral homology.

In the special case where $X$ is the $n$-torus $\mathbb{T}^n$, the spectrum $\Lambda_{\mathbb{T}^n} A$ is a model for the $n$-fold iterated topological Hochschild homology of $A$. This said, the covering homology is very different from iterated topological cyclic homology, having a vastly richer structure. We give examples at the very end of the paper where we see actions of various Galois groups, units in orders in division algebras (and so Morava stabilizer groups) and in the extreme case, all of $GL_n(\mathbb{Z})$. The study of this structure and concrete calculations will be followed up in a second paper. It is from this detailed analysis one should hope to glean insight into the chromatic behaviour of covering homology.

To give the reader an idea about the structure entering into the construction of covering homology, consider topological Hochschild homology $(THH)$ of the spherical group ring $S[G]$ of an abelian group $G$. It turns out to be the suspension spectrum $S[\text{Map}(\mathbb{T}, BG)]$ of the free loop space of the classifying space of the group in question, i.e. the space of unbased continuous maps of the circle into $BG$. The operators used to compute $TC$ arise from the evident circle action on the free loop space, as well as from the power maps of various degrees from the circle to itself. This free loop space interpretation of topological Hochschild homology shows that the $n$-fold iteration of $THH$ on $S[G]$ is equivalent to the suspension spectrum

$$\Lambda_{\mathbb{T}^n}(S[G]) \simeq S[\text{Map}(\mathbb{T}^n, B^n G)]$$

on the unbased mapping space of a higher dimensional torus into the iterated bar construction $B^n G$. This space supports many natural operations beyond the one-variable ones. For example, the group $GL_n(\mathbb{Z})$ acts on the $n$-torus, and hence on the mapping space from the torus into $B^n G$. In addition, generalizations of the power maps include all possible isogenies of the
torus to itself. This will be true in any sufficiently functorial model for $THH$; the important point is that the equivariant structure is “right” - a thing secured by the fundamental cofiber sequence.

This is a brief overview of the paper. In sections 2–4 we construct the Loday functor $X \mapsto \Lambda_X A$. Section 2 is a guide to the construction with references to related constructions. Section 3 contains combinatorial preliminaries, and in Section 4 we finally define $\Lambda_X A$. In Section 5 we present the fundamental cofibration sequence. In Section 6 this cofibration sequence is used to describe the zeroth homotopy group of fixed points of $\Lambda_X A$ in terms of the Burnside–Witt construction. Finally in the short Section 7 we define covering homology and show how it extends the definition of topological cyclic homology.

2 Guide to the construction

Let $X$ be a space. In the sections to follow we give a model, $\Lambda_X A$, for the $X$-fold smash-power of a connective commutative $S$-algebra $A$ (i.e., a symmetric monoid in $(\mathcal{T}S_+, \wedge, S)$). We have chosen to work with $\Gamma$-spaces, but our constructions work equally well on connective commutative symmetric ring-spectra. We shall call $\Lambda_X A$ the Loday functor of $A$ evaluated at $X$. It is important for the construction that $A$ is strictly commutative. The model $\Lambda_X A$ is functorial in both $X$ and $A$. Therefore, if a group $G$ acts on $X$, then we can consider the (categorically honest) $G$-fixed points of $\Lambda_X A$. In the particular situation where $G$ is finite and the action on $X$ is free we have good control on the fixed point spectrum $(\Lambda_X A)^G$. When $X$ is the underlying space of the simplicial circle group $T = \text{sin} U(1)$, given by the singular complex on the circle group $U(1) = \{x \in \mathbb{C} : |x| = 1\}$, the Loday functor evaluated at $X$ is a model for topological Hochschild homology.

Notation: $\Delta$ is the category of finite non-empty ordered sets, $\Delta^o$ its opposite, $\text{Fin}$ is the category of finite sets, $\Gamma^o$ is the category of finite pointed spaces and $\text{Ab}$ is the category of abelian groups.

2.1 Higher Hochschild homology

As a motivation, consider ordinary Hochschild homology $HH(A) : \Delta^o \to \text{Ab}$ of a flat ring $A$, given in each dimension by

$$HH_q(A) = A^q + 1.$$

If $A$ is commutative, then $HH(A)$ is a simplicial commutative ring, and Loday [12] observed that Hochschild homology factors through the category $\text{Fin}$ of finite sets:

$$\Delta^o \xrightarrow{HH(A)} \text{Ab} \xrightarrow{\text{Fin}} \text{Ab}.$$

(see e.g., [13]). Here $S^1 = \Delta[1]/\partial \Delta[1]$ is the standard simplicial circle. The diagonal functor is oftentimes called the Loday complex. Let us write simply

$$X \mapsto \Lambda^\mathbb{Z}_X A$$

for the diagonal functor. This is functorial in the finite set $X$, and so, if we extend to all sets by colimits and to all simplicial sets by applying the functor degreewise, we get a functor $X \mapsto \Lambda^\mathbb{Z}_X A$ from the category $\mathcal{S}$ of spaces (simplicial sets) to simplicial abelian groups, with classical Hochschild homology being

$$HH(A) = \Lambda^\mathbb{Z}_{S^1} A.$$

Pirashvili [18] uses the notation $HH^d(A, A)$ for the Locay complex for $A$ evaluated on the $d$-dimensional sphere and calls it “higher Hochschild homology of order $d$’.

There is no obstruction to apply the same construction to symmetric monoids in any symmetric monoidal category, Hochschild homology being the case when one considers $(\text{Ab}, \otimes, \mathbb{Z})$.  

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2.2 Higher topological Hochschild homology

Consider any of the popular symmetric monoidal categories of spectra. Then topological Hochschild homology of a cofibrant $S$-algebra $A$ is equivalent to the simplicial spectrum gotten by just replacing $\otimes$ with $\wedge$ in the definition of Hochschild homology, and in the commutative case we have a factorization $X \mapsto A \otimes X$ through $\text{Fin}$, where $\otimes$ is the categorical tensor in commutative $S$-algebras. Extending this functor to the category of spaces just as in the above subsection we obtain for every commutative $S$-algebra $A$ and every space $X$ the higher topological Hochschild homology $A \otimes X$. The above is another way of stating the result of McClure, Schwänzl and Vogt [17]: $THH(A) \simeq A \otimes S^1$. Here $THH$ is Bökstedt’s model for topological Hochschild homology. The above equivalence is an equivalence of cyclic spectra. However in the context of $\text{Fin}$-algebras in the sense of Elmendorf, Mandell, Kriz and May [6] the fixed point spectrum $(A \otimes sd_rS^1)^G$ of $A \otimes sd_rS^1$ with respect to the cyclic group $G$ with $r > 1$ elements does not have the same homotopy type as the $G$-fixed point spectrum of $sd_rTHH(A)$. In the language of McClure et al. it is easy to see what the iterated topological Hochschild homology is:

$$THH(THH(A)) \simeq (A \otimes S^1) \otimes S^1 \simeq A \otimes (S^1 \times S^1).$$

Hence taking the $n$th iterate of $THH$ is the same as tensoring with the $n$-torus.

In the situation where $X = \mathbb{T}$ is the circle group and $G$ is a finite subgroup of $\mathbb{T}$ there is a homotopy equivalence between the Loday functor $\Lambda \mathbb{T}A$ of $A$ evaluated at $\mathbb{T}$ and $THH(A)$, and this equivalence is $G$-equivariant in the sense that it induces an equivalence of $H$-fixed spectra for every subgroup $H$ of $G$. In this situation the control on the fixed point space derives from the “fundamental cofiber sequence”

$$THH(A)_hC_p \rightarrow THH(A)^{C_p^n} \rightarrow THH(A)^{C_p^{n-1}}.$$

In Lemma [17] we generalize the fundamental cofibration sequence to the situation where a finite group $G$ acts freely on a space $X$. In the toroidal case $X = \mathbb{T}^\times n$, the spectrum $\Lambda \mathbb{T}A$ is homotopy equivalent to iterated topological Hochschild homology, but the fact that the complexity of the subgroup lattice of torus increases with dimension makes the fundamental cofibration sequence more involved. This also gives rich and interesting symmetries on the collection of fixed point spectra $\Lambda \mathbb{T}^\times n(A)^{H}$ under varying finite $H \subseteq \mathbb{T}^\times n$. This structure gives actions by interesting groups possibly shedding light on the chromatic properties of the spectra, as in Rognes’ red shift conjecture.

Although iterated $TC$ involves iterations of fixed point spectra of $THH$, we consider the much simpler idea of taking the fixed point under the toroidal action on the iterated $THH$. This gives us in many ways a much cruder invariant, but also a much more computable one – essentially, the difference is that of $((A \wedge A)^{C_2} \wedge (A \wedge A)^{C_2})^{C_2}$ (the start of the iterated $TC$) and $((A \wedge A) \wedge (A \wedge A))^{C_2 \times C_2}$ (the start of our construction). Taken $A = S$ we see that these constructions give different results. Furthermore, the resulting theory displays a vastly richer symmetry, giving rise to a plethora of actions not visible if one only focuses on “diagonal” actions.

The underlying spectra of $X \mapsto \Lambda \mathbb{T}A$ and $A \otimes X$ are equivalent. The problem with the model $A \otimes X$ is that if a finite group $G$ acts on $X$ then we do not fully understand the $G$-fixed points of this model. Presumably this could be fixed along the lines of Brun and Lydakis’ version of $THH$ (unpublished) or Tore Kro’s thesis [10]. However, for now we choose a hands-on approach.

3 Encoding coherence with spans

In this section we shall use the category $V$ of spans of finite sets described below, to encode the coherence data for symmetric monoidal categories. More precisely, we shall encode the coherence data by a lax functor from $V$ to the category of categories. Since we are interested in group actions we will also investigate group actions on the morphism sets of $V$. The reader who is willing to believe in our coherence results may skip this quite technical section. However
these coherence results are essential for the construction of the Loday functor Λ in the next section.

3.1 The category of spans

The object class of the category $V$ of spans of finite set is the class of finite sets. Given finite sets $X$ and $Y$ the set of morphisms $V(Y, X)$ is the set of equivalence classes $[Y \leftarrow A \rightarrow X]$ of diagrams of finite sets of the form $Y \leftarrow A \rightarrow X$. Here $Y \leftarrow A \rightarrow X$ is equivalent to $Y \leftarrow A' \rightarrow X$ if there exists a bijection $A \rightarrow A'$ making the resulting triangles commute.

The composition of two morphisms $[Z \leftarrow B \rightarrow Y]$ and $[Y \leftarrow A \rightarrow X]$ is the morphism $[Z \leftarrow C \rightarrow X]$, where $C$ is the pull-back of the diagram $B \rightarrow Y \leftarrow A$.

If $G$ is a group acting on a finite set $X$, then $G$ acts on $X$ considered as an object of $V$ through the functor $g \mapsto g\cdot$. Consequently, $G$ acts on the morphism sets $V(Y, X)$. There is a function

$$\varphi: V(Y, X/G) \rightarrow V(Y, X)^G$$

defined by the formula

$$\varphi([Y \leftarrow A \rightarrow X/G]) = [Y \leftarrow A \times_{X/G} X \rightarrow X].$$

**Proposition 3.1.1** The function $\varphi$ is bijective.

**Proof:** The heart of the proof of this result is that the category $V$ has a categorical product given by the disjoint union of finite sets. This gives rise to a $G$-equivariant bijection

$$V(Y, X) \cong \prod_{x \in X} V(Y, \{x\}) \cong \text{Fin}(X, V(Y, *)),$$

where $G$ acts trivially on $V(Y, *)$ and on $\text{Fin}(X, V(Y, *))$ by conjugation. It is not hard to check that the composition

$$V(Y, X/G) \cong \text{Fin}(X, V(Y, *)) \cong \text{Fin}(X, V(Y, *))^G \cong V(Y, X)^G$$

is the map $\varphi$.

**Corollary 3.1.2** There is an isomorphism of categories $\psi: V/(X/G) \rightarrow (V/X)^G$.

**Proof:** The functor $\psi$ takes an object $\alpha \in V(Y, X/G)$ to $\varphi(\alpha) \in V(Y, X)^G$. By the definition of $\varphi$ it is clear that it is functorial in the sense that the diagram

$$\begin{array}{ccc}
V(Z, Y) \times V(Y, X/G) & \rightarrow & V(Z, X/G) \\
\downarrow \text{id} \times \varphi & & \downarrow \varphi \\
V(Z, Y) \times V(Y, X)^G & \rightarrow & V(Z, X)^G
\end{array}$$

commutes for all finite sets $Y$ and $Z$. Therefore, on morphisms, we can define the functor $\psi$ to be given by the identity.

To get the multiplicative structure on our construction we will need the following lemma

**Lemma 3.1.3** Disjoint union of sets is the coproduct on $V$.

**Proof:** To be precise, the disjoint union of two maps $[X \leftarrow A \rightarrow Y]$ and $[X' \leftarrow A' \rightarrow B']$ is $[X \sqcup X' \leftarrow A \sqcup A' \rightarrow Y \sqcup Y']$. The required isomorphism $V(X, Y) \times V(X', Y) \cong V(X \sqcup X', Y)$ is given by

$$\left(\begin{array}{c}
[ X \leftarrow f \rightarrow A \rightarrow g' \rightarrow Y ] , [ X' \leftarrow f' \rightarrow A' \rightarrow g' \rightarrow Y ] \\
\end{array}\right) \mapsto [ X \sqcup X' \leftarrow f \sqcup f' \rightarrow A \sqcup A' \rightarrow g + g' \rightarrow Y ]$$
with inverse given by the restrictions to the appropriate inverse images

\[
\begin{align*}
\left[ X \sqcup X' \xleftarrow{\bar{f}} A \xrightarrow{g} Y \right] \\
\leftrightarrow \left( \left[ X \xleftarrow{\bar{f}} f^{-1}(X) \xrightarrow{g} Y \right], \left[ X' \xleftarrow{\bar{f}} f^{-1}(X') \xrightarrow{g} Y \right] \right).
\end{align*}
\]

### 3.2 A lax functor

In this section we shall work heavily with the concepts around bicategories. Unfortunately the relevant terminology is not universal, and since the book [21] of Leinster is freely available on the arXiv we shall use the notation he uses throughout.

Given a bicategory \( \mathcal{B} \) and 0-cells \( A \) and \( B \) there is a category \( \mathcal{B}(A,B) \). The objects of \( \mathcal{B}(A,B) \) are the 1-cells from \( A \) to \( B \), and the morphisms in \( \mathcal{B}(A,B) \) are 2-cells in \( \mathcal{B} \).

Let \( W \) denote the following bicategory of spans (compare [15] pp. 283–285). The bicategory \( W \) has the class of finite sets as class of zero-cells, and \( W(X,Y) \) is the category of spans \( f = (X \xleftarrow{\bar{f}} A \xrightarrow{g} Y) \). A morphism \( \alpha : f \to g \) from \( f \) to \( g \) is \( \alpha : (X \xleftarrow{\bar{f}} A \xrightarrow{g} Y) \to (X' \xleftarrow{\bar{f}} A' \xrightarrow{g} Y) \) in \( W(X,Y) \), that is, a 2-cell \( \alpha \) in \( W \), consists of a map from \( \alpha : A \to A' \) making the diagram

\[
\begin{align*}
\begin{array}{c}
\xymatrix{X \ar@{..>}[r]_{f_1} & A \
\downarrow^{g_1} \ar@{..>}[r]_{\alpha} \ar@{..>}[u] & A' \ar@{..>}[l]_{f_2} \ar@{..>}[u]^{g_2}
}\end{array}
\end{align*}
\]

commute. The composition functor

\[
W(Y,Z) \times W(X,Y) \to W(X,Z)
\]

takes a pair \((g,f)\) of spans \( Y \xleftarrow{\bar{g_1}} B \xrightarrow{g_2} Z \) and \( X \xleftarrow{\bar{f_1}} A \xrightarrow{f_2} Y \) to the span \( g \circ f \) represented by the diagram

\[
X \xleftarrow{(g \circ f)_1} A \times Y \xrightarrow{(g \circ f)_2} Z
\]

where

\[
A \times Y B = \{ (a,b) \in A \times B : f_2(a) = g_1(b) \}
\]

is a functorial choice of fiber product. Here \((g \circ f)_1(a,b) = f_1(a)\) and \((g \circ f)_2(a,b) = g_2(b)\).

On morphisms the composition functor is defined by the same formula. The identity functor \( 1 \to W(X,X) \) is given by the one-cell \( X \xleftarrow{1_X} X \xrightarrow{1_X} X \). Using the universal property of pull-backs it is easy to verify that \( W \) is a bicategory.

The (strict) bicategory \( \text{Cat} \) has small categories as 0-cells, functors as 1-cells and natural transformations as 2-cells.

In the notation we have adopted a weak functor \( F : \mathcal{B} \to \mathcal{C} \) is a lax functor with the property that the 2-cells \( \phi_{g,f} : Fg \circ Ff \to F(g \circ f) \) and \( \phi_\alpha : \text{id}_{FA} \to F(\text{id}_A) \) are isomorphisms. At many places weak functors are called pseudo-functors. A strict functor \( F : \mathcal{B} \to \mathcal{C} \) is a lax functor with the property that the 2-cells \( \phi_{g,f} : Fg \circ Ff \to F(g \circ f) \) and \( \phi_\alpha : \text{id}_{FA} \to F(\text{id}_A) \) are identity morphisms.

**Definition 3.2.1** We define a lax functor \( L : W \to W \). On 0-cells \( L \) is the identity. Let \( f = (X \xleftarrow{\bar{f}} A \xrightarrow{g} Y) \) be a 1-cell in \( W \). We define

\[
L(f) = X \xleftarrow{L f_1} LA \xrightarrow{L f_2} Y,
\]

where \( LA \) is the image of the map \((f_1, f_2) : A \to X \times Y\), that is,

\[
LA = \{ (x,y) \in X \times Y : \exists a \in A \text{ such that } f_1(a) = x, \text{ and } f_2(a) = y \}.
\]
The maps $Lf_1$ and $Lf_2$ are the projection maps to the first and the second factor. Given a 2-cell $\alpha$ of the form (3.2.0) in $W$, the set $LA$ is a subset of $LA'$, and we define $L\alpha$ to be the 2-cell 
\[
\begin{array}{c}
X & \overset{Lf_1}{\longrightarrow} & LA \\
\downarrow & & \downarrow \text{incl.} \\
LA' & \overset{Lf_2}{\longrightarrow} & Y
\end{array}
\]
given by the inclusion $LA \subseteq LA'$. Let $g$ be a 1-cell in $W$ of the form $g = (Y \overset{g_1}{\longrightarrow} B \overset{g_2}{\longrightarrow} Z)$. There is a natural invertible 2-cell
\[
L(g) \circ L(f) \cong (X \xleftarrow{(x,y,z) \in X \times Y \times Z} (x,y) \in LA \text{ and } (y,z) \in LB) \to Z,
\]
where the maps are the projections, and precomposing with (the inverse of) this 2-cell, the structure map
\[
L(g,f): L(g) \circ L(f) \to L(g \circ f)
\]
corresponds to the projection taking $(x,y,z)$ to $(x,z)$.

If $f_1$ is the identity, then
\[
LA = \{(x, f_2(x)) \in X \times Y : x \in X\}
\]
is the graph of $f_2$ and $A \cong LA$. Note that if also $g_1$ is the identity, then the structure map $L(g,f): L(g) \circ L(f) \to L(g \circ f)$ is an identity morphism. Thus considering $\text{Fin}$ as a bicategory with only identity 2-cells, there is a strict functor $\text{Fin} \to W$ which is the identity on 0-cells and which takes a map $f: X \to Y$ to the 1-cell $(X \overset{f}{\to} X \overset{f}{\to} Y)$.

The following lemma is straightforward

**Lemma 3.2.2** Considered as an endofunctor in $V$, $L$ preserves coproducts.

We can use the lax functor $L$ to define a bicategory $\text{Rel}$ of relations. The 0-cells of $\text{Rel}$ is the class of finite sets. Given finite sets $X$ and $Y$ the category $\text{Rel}(X,Y)$ is the full subcategory of $W(X,Y)$ consisting of 1-cells in $W$ of the form $f = (X \overset{f_1}{\longrightarrow} A \overset{f_2}{\longrightarrow} Y)$, where $A$ is a subset of $X \times Y$. The composition $g \circ_{\text{Rel}} f$ of two 1-cells in $\text{Rel}$ is the 1-cell $L(g \circ f)$, where $g \circ f$ is the composition of $f$ and $g$ considered as 1-cells in $W$. It is easily verified that this defines a bicategory $\text{Rel}$.

**Remark 3.2.3** The lax functor $L: W \to \text{Rel}$ is left adjoint to the inclusion $\text{Rel} \subseteq W$ in the following sense: A lax functor $F: B \to C$ is left adjoint to the lax functor $G: C \to B$ if there exist a transformation $\beta: 1 \to G \circ F$ and a transformation $\gamma: F \circ G \to 1$ such that the composites
\[
\begin{array}{c}
B(A, GX) \xrightarrow{F} C(FA, FGX) \xrightarrow{C(\text{id}_{FA}, \gamma_X)} C(FA, X)
\end{array}
\]
and
\[
\begin{array}{c}
C(FA, X) \xrightarrow{G} B(GFA, GX) \xrightarrow{B(\beta_A, \text{id}_{GX})} B(A, GX)
\end{array}
\]
are adjoint functors for all 0-cells $A$ in $B$ and $X$ in $C$.

Note that $L: W \to \text{Rel}$ is a strict functor and that if $\alpha$ is an invertible 2-cell in $W$, then $L\alpha$ is an identity 2-cell in $\text{Rel}$. This implies that there is a strict functor $L_0: V \to \text{Rel}$ taking a morphims, that is, 1-cell $f$ to $Lf$. On the other hand the inclusion $\text{Rel} \subseteq W$ is only a lax functor.

**Definition 3.2.4** Consider the category $V$ as a bicategory with only identity 2-cells. The lax functor $T: V \to W$ is defined to be the composition
\[
V \xrightarrow{L_0} \text{Rel} \subseteq W
\]
of the strict functor $L_0$ and the inclusion of $\text{Rel}$ into $W$. 

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The functors $\text{Fin}^o \to V$, $(f: X \to Y) \mapsto f^* = [Y \xrightarrow{\delta} X \xrightarrow{\gamma} X]$ and $\text{Fin} \to V$, $(g: Y \to X) \mapsto g_* = [Y \xrightarrow{\alpha} Y \xrightarrow{\beta} X]$ allow us to consider both $\text{Fin}$ and $\text{Fin}^o$ as subcategories of $V$. Note that the composite lax functors $f \mapsto T(f^*)$ and $f \mapsto T(f_*)$ are both strict functors. By abuse of notation we shall write $T: \text{Fin}^o \to W$ and $T: \text{Fin} \to W$ for these strict functors.

**Definition 3.2.5** Let $\mathcal{C}$ be a category with finite coproducts. We define a weak functor $\mathcal{C}: W \to \text{Cat}$. Given a finite set $X$ we let $\mathcal{C}(X) = \mathcal{C}^X$ be the $X$-fold product of $\mathcal{C}$. Given a one-cell $f = (X \xrightarrow{\delta} A \xrightarrow{\beta} Y)$ the functor $\mathcal{C}(f): \mathcal{C}(X) \to \mathcal{C}(Y)$ is given by the formula

$$(\mathcal{C}(f)(c))(y) = \coprod_{a \in f^{-1}_2(y)} c(f_1(a)).$$

Given a two-cell of the form $\alpha: f \to f'$ we define $\mathcal{C}(\alpha): \mathcal{C}(f) \to \mathcal{C}(f')$ to be the natural transformation with $(\mathcal{C}(\alpha)(c))(y)$ equal to the composition

$$\coprod_{a \in f^{-1}_2(y)} c(f_1(a)) \cong \coprod_{a' \in (f_2)^{-1}(y)} \coprod_{a \in (f_1)^{-1}(a')} c(f_1(a')) \to \coprod_{a' \in (f_2)^{-1}(y)} c(f_1(a')),$$

where the last map is the sum map.

Composing the weak functor $\mathcal{C}: W \to \text{Cat}$ with the lax functor $T: V \to W$ we obtain a lax functor $\mathcal{C}T: V \to \text{Cat}$, which we by abuse of notation refer to simply as $\mathcal{C}: V \to \text{Cat}$.

Composing with the inclusion $\text{Fin} \subseteq V$ we obtain a weak functor

$$\mathcal{C}: \text{Fin} \to \text{Cat}, \quad (X \xrightarrow{f} Y) \mapsto \mathcal{C}^X \xrightarrow{T(f)} \mathcal{C}^Y,$$

$$c \mapsto y \mapsto \coprod_{a \in f^{-1}(y)} c(x).$$

On the other hand, composing with the inclusion $\text{Fin}^o \subseteq V$ we obtain a strict functor

$$\mathcal{C}: \text{Fin}^o \to \text{Cat}, \quad (X \xleftarrow{f} Y) \mapsto \mathcal{C}^X \xrightarrow{T(f)} \mathcal{C}^Y,$$

$$c \mapsto c \circ f.$$

### 4 The Loday functor

In this section we shall construct the Loday functor $\Lambda_X A$. It is a functor in the unpointed space $X$ and the commutative $S$-algebra $A$. When $X$ is the circle group, this a version of topological Hochschild homology, which we think of as the cyclic nerve of $A$ in the category of $S$-modules.

The construction of $\Lambda_X A$ is quite technical. It can be summarized in the following steps. The crucial ingredient of Bökstedt’s definition of topological Hochschild homology is the stabilizations indexed by the skeleton category $I$ of the category of finite sets and injective functions. Firstly, we note that $I$ is a category with finite coproducts, and we apply the results of the previous section to obtain an op-lax functor $II: V \to \text{Cat}$ from the category of spans $X \to A \to Y$ of finite sets to the category of categories. Secondly, we construct the transformation $G^A$ associated to a commutative $S$-algebra $A$. This is a transformation from the lax functor given by the composition $\text{Fin} \xrightarrow{\delta} W \xrightarrow{\beta} \text{Cat}$ to the constant functor from $\text{Fin}$ to $\text{Cat}$ with value $\mathbb{I}_S$. Thirdly, using Street’s first construction we rectify $I$ to a strict functor $\tilde{I}: V \to \text{Cat}$. Finally, applying the homotopy colimit construction, we obtain a functor $X \mapsto \Lambda_X A$ from $\text{Fin}$ to $\mathbb{I}_S$.

#### 4.1 On the functoriality of homotopy colimits

Bökstedt’s construction of topological Hochschild homology involves in each degree a stabilization by means of a homotopy colimit over a certain category. When the degrees vary, so
do the categories, and since book-keeping becomes rather involved we offer an overview of the functoriality properties of homotopy colimits.

First some notation. Given bicategories \( \mathcal{B} \) and \( \mathcal{C} \), the bicategory \( \text{Lax}(\mathcal{B}, \mathcal{C}) \) has the class of lax functors \( F: \mathcal{B} \to \mathcal{C} \) as 0-cells. The 1-cells of \( \text{Lax}(\mathcal{B}, \mathcal{C}) \) are transformations and the 2-cells are modifications. The bicategory \( [\mathcal{B}, \mathcal{C}] \subset \text{Lax}(\mathcal{B}, \mathcal{C}) \) has the class of strict functors \( F: \mathcal{B} \to \mathcal{C} \) as 0-cells. The 1-cells of \([\mathcal{B}, \mathcal{C}]\) are strict transformations and the 2-cells are modifications.

Let \( \text{Cat} \) be the category of small categories, and consider \( J \in \text{Cat} \) as a bicategory with only identity 2-cells. Given strict functors \( E, F: J \to \text{Cat} \) a strict transformation \( G: F \to E \) consists of the following data: for every \( i \in J \) we have a functor \( G_i: F(i) \to E(i) \) and for every \( f: i \to j \in J \) we have a natural transformation \( G_f: E(f)G_i \Rightarrow G_jF(f) \) such that if \( g: j \to k \in J \) we have that \( G_{gf} = G_gG_f \) and \( G_{id_j} = id_{G_j} \). The category \([J, \text{Cat}]\) has as objects pairs \((F, G)\) of a functor \( F: J \to \text{Cat} \) and a transformation \( G: F \to E \). A morphism \((F, G) \to (F', G')\) in \([J, \text{Cat}]/E\) consists of a natural transformation \( \epsilon: F \Rightarrow F' \) and a modification \( \eta: G \Rightarrow G' \epsilon \). That \( \eta \) is a modification means that for each \( i \in J \) we have a natural transformation \( \eta_i: G_i \Rightarrow G'_i \epsilon_i \) such that for each \( f: i \to j \in J \) we have \( \eta_jG_f = G'_j\eta_i \circ G_i \epsilon_j F(f) = G'_iF'(f)\epsilon_i \).

Let \( K \) be a category with all coproducts. Considering \( K \) as a constant functor from \( J \) to (big) categories, we can extend the above construction to give a category \([J, \text{Cat}]/K\). The homotopy colimit is the functor

\[
\text{holim}: [J, \text{Cat}]/K \to [J \times \Delta^\omega, K]
\]

sending \((F, G)\) to \( \text{holim}^K G \), which is the functor taking \((j, [\eta]) \in J \times \Delta^\omega\) to

\[
\prod_{x_0, \ldots, x_q \in F(j)} G_j(x_\eta).
\]

If \((\epsilon, \eta): (F, G) \to (F', G')\) is a morphism, we write \( \text{holim}_\epsilon^K \eta \) for the corresponding natural transformation \( \text{holim}^K G \Rightarrow \text{holim}^K G' \) given as

\[
\prod_{x_0, \ldots, x_q \in F(j)} G_j(x_\eta) \xrightarrow{\prod \eta} \prod_{x_0, \ldots, x_q \in F(j)} G'_j\epsilon(x_\eta) \xrightarrow{\prod \epsilon} \prod_{y_0, \ldots, y_q \in F'(j)} G'_j(y_\eta)
\]

where the last map is the obvious one.

### 4.2 The transformation \( G^A \)

Let \( \Sigma \) be the category of finite sets and bijections and choose a strong symmetric monoidal functor \( S: \Sigma \to S_* \) with \( S(i) = S^i \). We may consider \( S \) as a transformation

\[
\begin{array}{ccc}
\text{Fin} & \xrightarrow{\delta} & S_* \\
\Sigma \downarrow & & \downarrow \\
\text{Cat} & \xrightarrow{S} & \text{Cat}.
\end{array}
\]

Likewise, given a commutative \( S \)-algebra \( A \), we can use the multiplication in \( A \) to construct a transformation

\[
\begin{array}{ccc}
\text{Fin} & \xrightarrow{\delta} & S_* \\
\Sigma \downarrow & & \downarrow \\
\text{Cat} & \xrightarrow{S} & \text{Cat}.
\end{array}
\]

with \( A(j) \cong \bigwedge_{x \in X} A(S^{i(x)}) \) for \( j \in \Sigma^X \). We shall use the transformations \( S \) and \( A \) to construct a transformation \( G^A \) from the weak functor \( \mathcal{I}: \text{Fin} \to \text{Cat} \) with \( \mathcal{I}(S) = \mathcal{I}S \) to the constant functor \( \text{Fin} \to \text{Cat} \) sending everything to \( IS_* \).

If \( X \) and \( Y \) are pointed simplicial sets, \( \text{Map}_\ast(X, Y) \) denotes the \( \Gamma \)-space given by sending \( Z \in \Gamma^S \) to the space of pointed maps from \( X \) to the singular complex of the geometric realization of \( Z \wedge Y \).
For a finite set $T$, let \( G^A_T : T^T \to \mathbb{I}S \)
be the functor which to the object \( i \in T^T \) assigns the \( \Gamma \)-space
\[
\text{Map}_*(S(i), A(i)) \cong \text{Map}_* \left( \bigwedge_{t \in T} S^{i(t)}, \bigwedge_{t \in T} A(S^{i(t)}) \right),
\]
and to a morphism \( \alpha : i \to j \) assigns the map
\[
\text{Map}_* \left( \bigwedge_{t \in T} S^{j(t)}, \bigwedge_{t \in T} A(S^{j(t)}) \right) \xrightarrow{\alpha} \text{Map}_* \left( \bigwedge_{t \in T} S^{i(t)} - \alpha(t) \bigwedge_{t \in T} S^{i(t)}, \bigwedge_{t \in T} S^{j(t)} - \alpha(t) \bigwedge_{t \in T} A(S^{j(t)}) \right) \xrightarrow{\alpha}
\]
\[
\text{Map}_* \left( \bigwedge_{t \in T} S^{j(t)}, \bigwedge_{t \in T} A(S^{j(t)}) \right),
\]
where the first map is suspension and the second map is given by the isomorphism
\[
\bigwedge_{t \in T} S^{j(t)} - \alpha(t) \bigwedge_{t \in T} S^{i(t)} \cong \bigwedge_{t \in T} S^{i(t)}
\]
and the map
\[
\bigwedge_{t \in T} S^{j(t)} - \alpha(t) \bigwedge_{t \in T} A(S^{j(t)}) \cong \bigwedge_{t \in T} \left( S^{i(t)} - \alpha(t) \bigwedge_{t \in T} A(S^{i(t)}) \right)
\]
\[
\bigwedge_{t \in T} A(S^{i(t)} - \alpha(t) \bigwedge_{t \in T} S^{i(t)}) \cong \bigwedge_{t \in T} A(S^{i(t)}).
\]

If \( \phi : S \to T \) is a function of finite sets, there is a natural transformation
\[
G^A_\phi : G^A_T \to G^A_T \circ T(\phi)
\]
given on \( i \in T^S \) by sending a map \( f : S(i) \to A(i) \) to the composite
\[
\left. \bigwedge_{t \in T} S^{\pi_{i,t}} \right|_{t \in T} \xrightarrow{\cong} \bigwedge_{t \in T} A(S^{\pi_{i,t}}) \cong \left. \bigwedge_{a \in S} S^{\pi_{a,t}} \right|_{a \in S} \xrightarrow{\phi} \left. \bigwedge_{a \in S} A(S^{\pi_{a,t}}) \right|_{a \in S}
\]
where the top vertical maps are structure maps for \( S \), being a transformation, and the un-marked arrow is multiplication in \( A \).

This structure assembles into the fact that \( G^A \) is a transformation from the weak functor \( S \to T^S \) to the constant functor \( \text{Fin} \to \text{Cat} \) sending everything to \( \mathbb{I}S \). (one has to check that
for every $\psi: R \to S \in \text{Fin}$ the natural transformations

\[
\begin{tikzcd}
& T^R & \ar[dl]\text{G}^A & \ar[dl] \ar[d] \ar[d] & & & \ar[d] \ar[d] & \ar[dl]\text{G}^A & \ar[dl]

T^S & \ar[dl] & \ar[dl] & \ar[d] & \ar[d] & \ar[dl] & \ar[dl] & \ar[d] \ar[d] & \ar[dl] & \ar[dl] & \ar[d] \ar[d] & \ar[dl] & \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl]

\end{tikzcd}
\]

and

\[
\begin{tikzcd}
& T^R & \ar[dl]\text{G}^A & \ar[dl] \ar[d] \ar[d] & & & \ar[d] \ar[d] & \ar[dl]\text{G}^A & \ar[dl]

T^S & \ar[dl] & \ar[dl] & \ar[d] & \ar[d] & \ar[dl] & \ar[dl] & \ar[d] \ar[d] & \ar[dl] & \ar[dl] & \ar[d] \ar[d] & \ar[dl] & \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl] & \ar[d] \ar[d] & \ar[dl] \ar[dl]

\end{tikzcd}
\]

are equal - the unitality degenerates as $G^A_{i\alpha_S} = \text{id}_{G^A_S}$.

**Note 4.2.1** The reader may wonder why $G^A$ is only defined on $\text{Fin}$ (rather than $V$). The reason is that there exists no extension of $G^A$ to $V$, for if one did, a factorization $p \circ i: \{1\} \subseteq \{1, 2\} \to \{1\} \in \text{Fin}$ of the identity would then induce a splitting

\[
G^A_{i\alpha_S} = (p^*J) \circ G^A_{i\alpha_S}(j): \text{Map}_*(S^2, S^2) \to \text{Map}_*(S^4, S^4) \to \text{Map}_*(S^2, S^2)
\]

(here $j \in T^{(1)}$ is represented by $\{1, 2\} \in T$), which obviously does not exist since $\mathbb{Z}/2\mathbb{Z} = \pi_1 \Omega^4 S^4$ does not contain $\mathbb{Z} = \pi_1 \Omega^2 S^2$ as a retract (evaluate the $\Gamma$-spaces on $S^0$).

However, in section 5 and in subsequent applications, it is crucial for the equivariant structure of the Loday functor that $T$ is defined on $V$.

In section 5.2 we will give a very weak extension of $\pi_0 G^A(S^0)$ to cover also the maps $[T \leftarrow T \times K = T \times K] \in V$ induced by projection onto the first factor.

### 4.3 Rectifying $G^A$ and the definition of $\Lambda(A)$

We make a tiny recollection on Street’s first construction [9] p. 226 and make a remark about right cofinality, and then we construct the restriction of the Loday functor of $A$ to finite sets.

Let $F: J \to \text{Cat}$ be a lax functor. Street’s first construction is a functor $\tilde{F}: J \to \text{Cat}$ defined as follows: Given an object $j$ of $J$, the category $\tilde{F}(j)$ has as objects pairs $(\varphi, x_1)$ where $\varphi: j_1 \to j$ is a morphism in $J$ and $x_1$ is an object of $F(j_1)$. A morphism from $(\varphi_1, x_1)$ to $(\varphi_0, x_0)$ consists of a morphism $\psi: j_0 \to j_1$ with $\varphi_0 = \varphi_1 \psi$ and a morphism $\alpha: x_1 \to F(\varphi)(x_0)$ in $F(j_1)$. The composition of two morphisms $(\psi_2, \alpha_2): (\varphi_2, x_2) \to (\varphi_1, x_1)$ and $(\psi_1, \alpha_1): (\varphi_1, x_1) \to (\varphi_0, x_0)$ is the pair $(\psi_2 \psi_1, \beta)$ where $\beta$ is the composition

\[
F(\psi_2 \psi_1)x_0 \xleftarrow{F(\psi_2) \alpha_1} F(\psi_2)F(\psi_1)x_0 \xrightarrow{F(\psi_2) \alpha_2} F(\psi_2)x_1 \xrightarrow{\alpha_2} x_2.
\]

If $\psi: j \to j'$ is a morphism in $J$, then $\tilde{F}(\psi): \tilde{F}(j) \to \tilde{F}(j')$ is given by the formula $\tilde{F}(\psi)(\varphi, x) = (\psi \varphi, x)$.

For every object $j$ of $J$ there is a functor $r = r_j: \tilde{F}(j) \to F(j)$ with $r_j(\varphi, x_1) = F(\varphi_1)(x_1)$ and with $r_j(\psi, \alpha)$ equal to the composite

\[
F(\varphi_1)x_1 \xleftarrow{F(\psi) \alpha_1} F(\varphi_1)F(\psi)x_0 \xrightarrow{F(\psi) \alpha_2} F(\varphi_1)x_0 = F(\varphi_0)x_0.
\]

These functors assemble to a transformation $r: \tilde{F} \to F$. Note that for every object $x \in F(j)$ the structure morphism $x \to r_j(id_j, x) = F(id_j)(x)$ is an object of the category $x/r_j$. Given another object $\alpha: x \to r_j(\varphi_0, x_0) = F(\varphi_0)(x_0)$ of the category $x/r_j$, the pair $(\varphi_0, \alpha)$ is the only morphism in $x/r_j$ from $x \to F(id_j)(x)$ to $\alpha$. Thus the category $x/r_j$ has an initial element so the functor $r_j$ is right cofinal.

Now we use Street’s first construction to build the Loday functor. Let $V$ denote the category of spans of finite sets. In the notation introduced below Definition 4.24 we have a lax functor $I: V \to \text{Cat}$ such that $I(f^*) = ([V \overset{f}{\rightarrow} X \overset{\pi}{\rightarrow} X])$: $I^V \to I^X$ is the functor
given by precomposing with \( f: X \to Y \) and \( \mathcal{I}(f_*) = \mathcal{I}([X \xrightarrow{\sim} X \xrightarrow{f} Y]): \mathcal{I}^X \to \mathcal{I}^Y \) is a functor with
\[
\mathcal{I}(f_*)(j)(y) \cong \prod_{x \in f^{-1}(y)} j(x),
\]
for \( j \in \mathcal{I}^X \).

**Definition 4.3.1** The functor \( \tilde{\mathcal{I}}: V \to \text{Cat} \) and the transformations \( r: \tilde{\mathcal{I}} \to \mathcal{I} \) are defined by applying Street’s first construction with \( J = V \) and \( F = \mathcal{I} \).

Composing with the inclusion \( \text{Fin} \to V \) (precomposition with functors takes lax functors to lax functors) we get a transformation of weak functors
\[
G^A \circ r: \tilde{\mathcal{I}} \to \mathcal{I}^\text{Fin}.
\]

**Definition 4.3.2** Given a finite set \( S \) and a commutative \( S \)-algebra \( A \) the Loday functor of \( A \) evaluated at \( S \) is the \( \Gamma \)-space \( \Lambda^A_S \) given by the homotopy colimit
\[
\text{holim}_{\tilde{\mathcal{I}}(S)} G^A \circ r_S.
\]

By the functoriality of the homotopy colimit explained in Section 3.1 this construction is functorial in \( S \in \text{Fin} \), so that we have a functor \( \Lambda(A): \text{Fin} \to \mathcal{I}^\text{Fin} \).

Note that we do not use the ring structure on \( A \) in the construction of \( \Lambda^A_S \) for a fixed finite set \( S \), and also note that the action of the automorphism group of \( S \) on \( \Lambda^A_S \) does not depend on the ring structure.

We have introduced the category \( V \) in order to have an isomorphism \( \tilde{\mathcal{I}}(S)^G \cong \tilde{\mathcal{I}}(S/G) \) whenever \( G \) is a finite group acting on a finite set \( S \). This gives us the following crucial lemma.

**Lemma 4.3.3** Let \( S \) be a finite set and let \( G \) be a group acting on \( S \). For every functor \( Z: \tilde{\mathcal{I}}^S \to \text{Spaces} \) the map
\[
\text{holim}_{\tilde{\mathcal{I}}(S)} Z \circ r_S \to \text{holim}_{\tilde{\mathcal{I}}^S} Z
\]
induces a weak equivalence on \( G \)-fixed points.

**Proof:** First we treat the case where \( G \) is the trivial group. We have seen that the functor \( r_S: \tilde{\mathcal{I}}(S) \to \tilde{\mathcal{I}}^S \) is cofinal. Therefore the map
\[
\text{holim}_{\tilde{\mathcal{I}}(S)} Z \circ r_S \to \text{holim}_{\tilde{\mathcal{I}}^S} Z
\]
is an equivalence (right cofinality).

In the case where \( G \) is nontrivial we note that there is an isomorphism
\[
(holim_{\tilde{\mathcal{I}}(S)} Z)^G \cong holim_{\tilde{\mathcal{I}}^S} Z \circ G^S.
\]

Further, it is an immediate consequence of Corollary 3.1.2 that the categories \( \tilde{\mathcal{I}}(S)^G \) and \( \tilde{\mathcal{I}}(S/G) \) are isomorphic. We can now easily reduce to the situation where \( G \) is the trivial group.

Given an \( S \)-module \( A \) and a totally ordered finite set \( T = \{1, \ldots, t\} \) we define the \( S \)-module \( \Lambda_T A \) inductively by letting \( \Lambda_T A = S \) for \( t = 0 \) and \( \Lambda_T A = (\Lambda_{T\setminus\{t\}} A) \wedge A \) for \( t \geq 1 \).
Lemma 4.3.4 For every finite set $S$ the functor $A \mapsto \Lambda_S A$ preserves connectivity of maps of commutative $S$-algebras, and sends stable equivalences to pointwise equivalences. If $A$ is cofibrant and $T$ is a finite totally ordered set then there is a chain of stable equivalences between $\Lambda_T A$ and $\wedge_T A$.

Proof: The first statement follows from Bökstedt’s Lemma (see e.g. [2, Lemma 2.5.1]) since, for given $n$, there is an $i \in \mathcal{I}^S$ for which

$$\text{Map}_*(\bigwedge_{s \in S} S^{\delta(s)}, K \wedge \bigwedge_{s \in S} A(S^{\delta(s)})) \cong G^A_S(K)(i) \to \Lambda_S(A)(K)$$

is $n$-connected. But as $n$ increases and $K$ varies, the term to the left is just a concrete model for the (derived) $S$-fold smash of $A$ with itself.

Let $\tilde{G}^A_T : \mathcal{I}^T \to \mathcal{I} S$, be the functor which to the object $i \in \mathcal{I}^T$ assigns the $\Gamma$-space

$$K \mapsto \text{Map}_*(\bigwedge_{t \in T} S^{\delta(t)}, (\bigwedge_{t \in T} A)(K \wedge \bigwedge_{t \in T} S^{\delta(t)})),$$

and with structure maps similar to those of $G^A_T$. There are natural maps

$$\Lambda_T A(K) \leftarrow \text{holim} \tilde{G}^A_T(i)(K) \to \tilde{\Lambda}_T A(K) := \text{holim} \tilde{G}^A_T(i)(K) \mapsto (\bigwedge_{t \in T} A)(K).$$

The first map is a weak equivalence by Lemma 4.3.3, the third map is always a stable equivalence, and the second map is a stable equivalence if $A$ is cofibrant [14, Proposition 5.22].

The smash product of $S$-modules is the coproduct in the category of commutative $S$-algebras, and so the category of commutative $S$-algebras is “tensored” over totally ordered finite sets through the formula

$$T \otimes A = \bigwedge_{T} A.$$

Let us choose a total order on every finite set. Using the universal property of the coproduct we see that $S \mapsto S \otimes A$ is a functor from the category of finite sets to the category of commutative $S$-algebras. The following corollary of the proof of Lemma 4.3.4 implies that up to homotopy $\Lambda_S(A)$ is a commutative $S$-algebra.

Corollary 4.3.5 If $A$ is a commutative $S$-algebra which is cofibrant as an $S$-module, then $\Lambda_S(A)$ is stably equivalent as an $S$-module to the commutative $S$-algebra $S \otimes A$ via maps that are natural in the finite set $S$.

Corollary 4.3.6 Given an $S$-module $A$ and finite sets $R$ and $S$ there is an equivalence

$$\Lambda_{R \times S}(A) \simeq \Lambda_R(\Lambda_S(A)).$$

4.4 Multiplicative structure

One disadvantage with Bökstedt’s formulation of $THH(A)$ for a commutative $S$-algebra $A$ is that the multiplicative structure needs some elaboration. In our model, $\Lambda_X A$ will automatically be an $S$-algebra without any amendments, and it is appropriately functorial in $X$ and $A$. In particular, our fattening up of Bökstedt’s model could be viewed as a way of getting the multiplicative structure in one sweep (though, if one did not care about the equivariant structure to come, one would drop the complicating spans, and work with functors from $\text{Fin}$ only). That said, our model is not strictly commutative, so there is still some advantage to the categorical smash product constructions in this regard.

Lemma 4.4.1 The coproduct in $\mathcal{I}$ gives a transformation with 2-cells consisting of isomorphisms $\mu : (\mathcal{I} \times \mathcal{I}) \circ \Delta \Rightarrow \mathcal{I}$ of lax functors $V \to \text{Cat}$, where $\Delta : V \to V \times V$ is the diagonal.
Proof: We build the 2-cells consisting of isomorphism from the natural isomorphisms we get in

\[ \mathcal{I}^X \times \mathcal{I}^X \rightarrow \mathcal{I}^X \]
\[ \downarrow \]
\[ \mathcal{I}^Y \times \mathcal{I}^Y \rightarrow \mathcal{I}^Y \]

where the vertical maps are induced by \([X \xrightarrow{f} A \xrightarrow{g} Y]\) \(\in V\) and the horizontal maps by the coproduct in \(\mathcal{I}\), simply (when evaluated on \((i, j) \in \mathcal{I}^X \times \mathcal{I}^X\) and \(y \in Y\)) by the coherence isomorphism that permutes \(\coprod_{a \in (Lg)^{-1}(y)} i(Lf(a))\) to \(\coprod_{a \in (Lg)^{-1}(y)} j(Lf(a))\).

Even more simply, we obtain a natural transformation \(\tilde{\mu}: (\tilde{I} \times \tilde{I}) \circ \Delta \Rightarrow \tilde{I}\): if \(X \in \text{ob}V\) then \(\tilde{\mu}_X: \tilde{I}(X) \times \tilde{I}(X) \rightarrow \tilde{I}(X)\) is given by

\[
(f_1: T_1 \rightarrow X, i_1 \in \tilde{I}(T_1), f_2: T_2 \rightarrow X, i_2 \in \tilde{I}(T_2)) \mapsto (f_1 + f_2: T_1 \sqcup T_2 \rightarrow X, (i_1, i_2) \in \tilde{I}(T_1 \sqcup T_2))
\]

(here \(f_1\) and \(f_2\) are maps in \(V\), and we have identified \(\tilde{I}(T_1 \sqcup T_2)\) with \(\tilde{I}(T_1) \times \tilde{I}(T_2)\) for the purpose of naming elements). Since we have chosen our category \(\mathcal{I}\) of finite sets and injective functions so that the monoidal structure is strictly associative and unital, we see that so is \(\tilde{\mu}\).

Note that \(X \mapsto \tilde{I}(X) \times \tilde{I}(X)\) is not the rectification of \(X \mapsto \tilde{I}(X)\), so the comparison between these is not the formal one.

However, there is an invertible modification \(M: r \circ \tilde{\mu} \Rightarrow \mu \circ (r \times r)\) defined as follows. If \(X \in V\) we let \(M_X: r_X \circ \tilde{\mu}_X \Rightarrow \mu_X \circ (r_X \times r_X)\) be the transformation which, when applied to \(((g_1^1 f_1^1), i_1), (g_2^2 f_2^2), i_2)\) \(\in \tilde{I}(X) \times \tilde{I}(X)\), is provided by the canonical isomorphism between \(\mathcal{L}([g_1^1 f_1^1] + [g_2^2 f_2^2])(i_1, i_2)\) and \(x \mapsto \left(\coprod_{(s, x) \in \mathcal{L}(A_1 \sqcup A_2)} (t_1, i_2)(s)\right)\) and \(x \mapsto \left(\coprod_{(t_1, x) \in \mathcal{L}(A_1)} i_1(t_1)\right)\).}

Lemma 4.4.2 Let \(G^A \wedge G^A\) be the transformation from \((\tilde{I} \times \tilde{I}) \circ \Delta\) to \(\text{Fin}_*\). which evaluated on \((i, j) \in \mathcal{I}^X \times \mathcal{I}^X\) is given by \(G^A_S(i) \wedge G^A_S(j)\). Multiplication in \(A\) defines a modification \(\mu^A\) from \(G^A \wedge G^A\) to the composite

\[
\begin{array}{ccc}
\text{Fin} & \xrightarrow{\text{Fin}} & \text{Fin} \\
\downarrow & & \downarrow \\
V & \Rightarrow & \text{v} \Rightarrow \text{JS}_* \\
\text{Cat} & \Rightarrow & \text{Cat} \\
\end{array}
\]

where the transformation in the lower left square comes from Lemma 4.4.1.

Proof: Explicitly, on objects \(i, j \in \mathcal{I}^S\), the natural transformation \(\mu_S^A(i, j): G^A_S(i) \wedge G^A_S(j) \rightarrow G^A_S(i \sqcup j)\) is given by the composite

\[
\begin{align*}
\text{Map}_*(\bigwedge_{s \in S} A(S^{(i)}(s))) \wedge \text{Map}_*(\bigwedge_{s \in S} A(S^{(s)}(s))) \\
\downarrow \\
\text{Map}_*(\left(\bigwedge_{s \in S} S^{(i)}(s)\right) \wedge \left(\bigwedge_{s \in S} A(S^{(s)}(s))\right)) \wedge \left(\bigwedge_{s \in S} A(S^{(s)}(s))\right)) \\
\downarrow \\
\text{Map}_*(\left(\bigwedge_{s \in S} S^{(i)}(s) \wedge S^{(j)}(s))\right) \wedge \left(\bigwedge_{s \in S} A(S^{(s)}(s))\right)) \\
\downarrow \\
\text{Map}_*(\left(\bigwedge_{s \in S} (S^{(i)}(s) \wedge A(S^{(s)}(s)))\right) \\
\end{align*}
\]
by using the functoriality of homotopy colimits

\[ \begin{array}{ccc}
G^A_S(i) \wedge G^A_S(j) & \xrightarrow{\mu^A_S(i,j)} & G^A_S(i \sqcup j) \\
\downarrow & & \downarrow \\
G^A_S(i') \wedge G^A_S(j') & \xrightarrow{\mu^A_S(i',j')} & G^A_S(i' \sqcup j')
\end{array} \]

commutes.

Checking that this defines a modification involves proving naturality in maps \( f : S \to T \in Fin \), and basically boils down to the fact that diagrams like

\[
\begin{array}{ccc}
(\bigwedge_{s \in S} A(S_{(s)})) \wedge (\bigwedge_{s \in S} A(S_{(s)})) & \xrightarrow{\cong} & \bigwedge_{s \in S} (A(S_{(s)}) \wedge A(S_{(s)})) \\
\downarrow & & \downarrow \\
(\bigwedge_{t \in T} \bigwedge_{s \in f^{-1}(t)} A(S_{(s)})) \wedge (\bigwedge_{t \in T} \bigwedge_{s \in f^{-1}(t)} A(S_{(s)})) & \xrightarrow{\cong} & \bigwedge_{s \in S} (A(S_{(s)}) \wedge A(S_{(s)})) \\
\downarrow & & \downarrow \\
\bigwedge_{t \in T} A(S_{(s)} \preceq f^{-1}(t)) & \xrightarrow{\cong} & \bigwedge_{t \in T} A(S_{(s)} \preceq f^{-1}(t)) \\
\downarrow & & \downarrow \\
\bigwedge_{t \in T} A(S_{(s)} \preceq f^{-1}(t)) \wedge A(S_{(s)} \preceq f^{-1}(t)) & \xrightarrow{\cong} & \bigwedge_{t \in T} A(S_{(s)} \preceq f^{-1}(t))
\end{array}
\]

commute, where the marked isomorphisms are the obvious rearrangements, and the nonmarked arrows are multiplication.

**Corollary 4.4.3** Multiplication in \( A \) and coherence in \( \mathcal{I} \) gives a modification

\[
(\mathcal{I} \times \mathcal{I}) \circ \Delta \xrightarrow{(r \times r)} (\mathcal{I} \times \mathcal{I}) \circ \Delta \xrightarrow{G^A \wedge G^A} \mathcal{I} \mathcal{S}_A
\]

of transformations of lax functors \( Fin \to \text{Cat} \). Consequently we get a natural multiplication map

\[ \mu^A_S : \Lambda_S A \wedge \Lambda_S A \to \Lambda_S A \]

by using the functoriality of homotopy colimits

\[
(\tilde{\mu}_S, M_S \mu^A_S) : \text{holim} \xrightarrow{\cong} \text{lim} \xrightarrow{\cong} \text{holim} \xrightarrow{\cong} \text{lim}
\]

plus the obvious natural transformation

\[
\left( \text{holim} \xrightarrow{\cong} \text{lim} \right) \wedge \left( \text{holim} \xrightarrow{\cong} \text{lim} \right) \to \left( \text{holim} \xrightarrow{\cong} \text{lim} \right)
\]

**Theorem 4.4.4** Let \( A \) be a connective commutative \( S \)-algebra. The multiplication \( \mu^A_S : \Lambda_S A \wedge \Lambda_S A \to \Lambda_S A \) is associative and unital.
Proof: Proving associativity and unitality is done by direct checking using the explicit formula for \( \mu^A \) given in the proof of Lemma 4.4.2 and is left as an exercise.

Remark 4.4.5 Note that this multiplication is also \( E_\infty \). This may be shown, for instance, by adapting Hesselholt’s and Madsen’s “spherewise” model for the multiplication, as in [3, 1.7], to the current setup. This approach was pointed out to us by C. Schlichtkrull, and details will appear in a future publication of Schlichtkrull. An alternative approach, which works if \( A \) is cofibrant as an \( S \)-module is to shorten the comparison map in 4.3.4 by only using \( \tilde{I} \)'s and observe that it is multiplicative, and the right hand side is commutative.

4.5 The Loday functor as a functor of unbased spaces
If \( X \) is a finite (unbased) space, then we define
\[
\Lambda_X(A) = \text{diag}^* \{ [q] \mapsto \Lambda_{X_q}(A) \}.
\]
If \( X \) is any (unbased) space, then we define
\[
\Lambda_X(A) = \lim_{\to \to} \Lambda_S A
\]
where \( S \) varies over the finite subspaces of \( X \).

More generally, let \( F \) be a functor from the category \( \text{Fin} \) of finite (unbased) sets to the category of \( S \)-modules (\( \Gamma \)-spaces). If \( X \) is a finite (unbased) space, then we define \( F(X) = \text{diag}^* \{ [q] \mapsto F(X_q) \} \). If \( X \) is any (unbased) space, then we define \( F(X) = \lim_{\to \to} F(S) \) where \( S \) varies over the finite subspaces of \( X \). Note that the following lemma in particular applies to the Loday functor. A map of \( \Gamma \)-spaces \( X \to Y \) is a pointwise equivalence (resp. pointwise \( n \)-connected) if for all finite pointed sets \( S \) the map \( X(S) \to Y(S) \) is a weak equivalence (resp. \( n \)-connected). This is stronger than claiming that the map of associated spectra is a stable equivalence (resp. stably \( n \)-connected).

Lemma 4.5.1 Let \( f: X \to Y \) be a map of simplicial sets and let \( f_+ : X_+ \to Y_+ \) denote the map obtained by adding a disjoint base point.

1. If \( f \) is a weak equivalence (resp. \( n \)-connected), then the induced map
\[
F(f): F(X) \to F(Y)
\]
is a pointwise equivalence (resp. \( n \)-connected).

2. If \( f \) is a weak equivalence (resp. \( n \)-connected) after \( p \)-completion, then \( F(f)^p \) is a pointwise equivalence (resp. \( n \)-connected).

3. If \( E \) is a spectrum and \( E \wedge f_+ \) is a stable equivalence, then \( E \wedge F(f) \) is a stable equivalence.

Proof: Let \( LF \) denote the functor from simplicial sets to pointed simplicial \( \Gamma \)-spaces with
\[
(LF)(X)_k = \bigvee_{S_0, \ldots, S_k} F(S_k) \wedge \text{Fin}(S_k, S_{k-1})_+ \wedge \cdots \wedge \text{Fin}(S_1, S_0)_+ \wedge S(S_0, X)_+
\]
The functor \( X \mapsto (LF)(X) \) clearly has the stated properties (for spaces \( B \) and \( C \) we have equivalences \( (C \wedge B)_+^p \simeq (C_{p,+} \wedge B_{p,+})^p \) and \( S(C, B)_+^p \simeq S(C, B^p_0) \)). The result now is a consequence of the fact that there is a natural pointwise weak equivalence \( (LF)(X) \to F(X) \).

The above proof also gives the following result.

Lemma 4.5.2 For every space \( X \) the map \( F(X)^p_+ \to F(X^p_+)^p \) is a weak equivalence.
4.5.3 Adams operations

Any endomorphism \( X \to X \) gives by functoriality rise to an “operation” \( \Lambda_X(A) \to \Lambda_X(A) \).

For instance, if \( X = T^1 \cong \sin |S^1| \), the \( r \)-th power map \( T^1 \to T^1 \) gives an operation on ordinary topological Hochschild homology \( THH(A) \cong \Lambda_{T^1}(A) \) corresponding to the one described by McCarthy [16]. There is a difference in description in that he uses the usual simplicial circle \( S^1 \), and so has to subdivide to express the operation, whereas in our situation this can be viewed as a reflection of the fact that (the image of the acyclic cofibration) \( sd_1 S^1 \cong \sin |sd_1 S^1| \cong T^1 \) is one of the legitimate finite simplicial subsets over which we perform our colimit.

This is general: for any functor \( F \) from \( Fin \) to, say, spaces, the \( r \)th Adams operation on \( \pi_q(F(S^1)) \) is given through McCarthy’s interpretation as the composite

\[
\pi_q(F(S^1)) \cong \pi_q(F(sd_1(S^1))) \to \pi_q(F(S^1))
\]

where the last map is induced by a certain map \( sd_1 S^1 \to S^1 \). That this corresponds to our definition is seen through the commutativity of

\[
\begin{array}{ccc}
\sin |sd_1 S^1| & \cong & \sin |S^1| \\
\cong & & \cong \\
sd_1 S^1 & \to & S^1
\end{array}
\]

For higher dimensional tori one gets operations for every integral matrix \( \alpha \) with nonzero determinant. In this case Lemma 4.5.2 gives \( \Lambda_{\mathbb{T}^n}(A) \cong \Lambda_{\mathbb{T}^n}(A)_p \) and so we have an action by \( GL_n(\mathbb{Z}_p) \) which in the one-dimensional case corresponds to the action by the \( p \)-adic units.

5 The fundamental cofibration sequence

In this section we investigate fixed points of the Loday functor \( \Lambda_X \) with respect to group actions on \( X \). This leads to a generalized version of the fundamental cofibration sequence for Bökstedt’s topological Hochschild homology. On the way we describe the norm map from homotopy orbits to homotopy fixed points.

5.1 The norm cofibration sequence

Let \( G \) be a finite group. For convenience, we introduce the following shorthand notation: if \( S \) is a finite set and \( A \) is a commutative \( S \)-algebra and \( j \in T^S \) we write \( A(j) \) for the space

\[
\bigwedge_{s \in S} A(S^{(s)}),
\]

so that (c.f. [12]) \( G^A_S(j) = Map_*(S(j), A(j)) \).

We say that a set \( \mathcal{F} \) of subgroups of \( G \) is a closed family if it has the property that if \( H \in \mathcal{F} \) and \( H \leq gKg^{-1} \), for \( g \in G \) and a subgroup \( K \) of \( G \), then \( K \in \mathcal{F} \). If \( G \) acts on \( S \) and \( \mathcal{F} \) is a closed family of subgroups in \( G \), we define

\[
G^A_S(\mathcal{F})(j) = Map_*(\bigcup_{H \in \mathcal{F}} S(j)^H, A(j)).
\]

This is a transformation from the weak functor \( S \mapsto T^S \) to the constant functor from finite \( G \)-sets to categories sending everything to \( IS_\ast \). Just like when we constructed \( \Lambda(A) \), we can define a functor \( \Lambda^\mathcal{F}(\mathcal{F}) \) from the category of \( G \)-spaces to the category of spectra, with

\[
\Lambda^\mathcal{F}_*(\mathcal{F}) = \limcolim_{\mathcal{F}(S)} G^A_S(\mathcal{F}) \circ r_S
\]

when \( S \) is a finite \( G \)-set. In particular, if \( \mathcal{F} \) is the empty family of subgroups of \( G \), then \( \Lambda^\mathcal{F}_*(\mathcal{F}) = * \).
Lemma 5.1.1 Let $N$ be a normal subgroup in $G$ and let $\mathcal{F}$ be the closed family of subgroups of $G$ consisting of the subgroups containing $N$. There is, for every $j \in \mathcal{T}$, a natural isomorphism of $G/N$-spaces of the form

$$G^S_{\mathcal{F}}(j)^N = \text{Map}_*(S(j)^N, A(j)^N).$$

In particular, if $\mathcal{F}$ is the family of all subgroups of $G$, then $\Lambda^n_{\mathcal{F}}(G) = \Lambda_*\Lambda^n_{\mathcal{F}}(A)$. 

Note that if $\mathcal{G} \subseteq \mathcal{F}$ is an inclusion of closed families of subgroups of $G$, then the inclusion

$$\bigcup_{H \in \mathcal{G}} S(j)^H \subseteq \bigcup_{H \in \mathcal{F}} S(j)^H$$

induces a $G$-map

$$\text{res} : \Lambda^n_{\mathcal{F}}(G) \to \Lambda^n_{\mathcal{G}}(G).$$

We refer to these maps as restriction maps. However they should not be confused with the restriction maps $R$ in Section 5.2 generalizing the restriction maps of Hesselholt and Madsen [8]. If $\mathcal{F}$ is the family of all subgroups in $G$, then by Lemma 5.1.1 the restriction map takes the form

$$\text{res} : \Lambda_*\Lambda^n_{\mathcal{F}}(A) \to \Lambda_*\Lambda^n_{\mathcal{G}}(A).$$

If the complement of $\mathcal{G}$ in $\mathcal{F}$ is equal to the conjugacy class of a subgroup $K$ in $G$ we shall say that $\mathcal{F}$ and $\mathcal{G}$ are $K$-adjacent. Moreover we let $W_G K = N_G K / K$ denote the Weyl group of the subgroup $K$ in $G$. Here $N_G K$ is the normalizer of $K$ in $G$ consisting of those $g$ in $G$ with $gKg^{-1} = K$.

In the next lemma we shall describe the homotopy fibre of the restriction map when $\mathcal{G} \subseteq \mathcal{F}$ are $K$-adjacent. In the proof we will need a norm map $Z_{hG} \to Z^{hG}$ for $Z$ a $\Gamma$-space with action of $G$. Later we shall relate the norm map to a transfer map, and therefore we need to choose a specific representative of it. The discussion below is modeled on Weiss and William’s paper [20 Section 2]. Given $n \geq 0$ we let $S^{nG}$ denote the $G$-fold smash product of the $n$-sphere. This is our model for the one-point compactification of the regular representation of $G$. The map $\alpha : G_+ \sma S^{nG} \to \text{Map}_*(G_+, S^{nG})$, with $\alpha(g, x)(g) = x$ and with $\alpha(g, x)(h)$ equal to the base point in $S^{nG}$ if $h \neq g$, and the diagonal inclusion $S^{nG} \to \text{Map}_*(G_+, S^{nG})$ induce $G$-maps

$$\text{Map}_*(S^{nG}, Z(S^{nG})) \cong \text{Map}_*(G_+ \sma S^{nG}, Z(S^{nG}))^G \leftarrow \text{Map}_*(\text{Map}_*(G_+, S^{nG}), Z(S^{nG}))^G \to \text{Map}_*(S^{nG}, Z(S^{nG}))^G.$$ 

Here $G$ acts by conjugation on the first space, through the left action of $G$ on itself on the second space and through the right action of $G$ on itself on the third space. On the last space $G$ acts trivially. Passing to homotopy colimits we obtain a weak $G$-map

$$\lim_{\prod} \text{Map}_*(S^{nG}, Z(S^{nG})) \cong \lim_{\prod} \text{Map}_*(\text{Map}_*(G_+, S^{nG}), Z(S^{nG}))^G \to \lim_{\prod} \text{Map}_*(S^{nG}, Z(S^{nG}))^G$$

of $\Gamma$-spaces. This is an additive transfer associated to $G$. Observing that there is a chain of stable equivalences between $Z$ and $\text{holim}_G \text{Map}_*(S^{nG}, Z(S^{nG}))$ we denote the resulting map in the homotopy category of (naive) $G$-$\Gamma$-spaces by $V^G : Z \to \text{holim}_G \text{Map}_*(S^{nG}, Z(S^{nG}))^G$. The homotopy category we have in mind is the one where we invert the $G$-maps whose underlying non-equivariant maps are stable equivalences. Since $G$ acts trivially on the target, there is an induced map $V^G : Z_{hG} \to \text{holim}_G \text{Map}_*(S^{nG}, Z(S^{nG}))^G$. The norm map is the weak map

$$Z_{hG} \to (\text{Map}_*(EG_+, Z))_{hG}$$

$$\prod \to \text{holim}_G \text{Map}_*(S^{nG}, \text{Map}_*(EG_+, Z(S^{nG})))^G \cong Z^{hG}$$

of $\Gamma$-spaces. This is an additive transfer associated to $G$. Observing that there is a chain of stable equivalences between $Z$ and $\text{holim}_G \text{Map}_*(S^{nG}, Z(S^{nG}))$ we denote the resulting map in the homotopy category of (naive) $G$-$\Gamma$-spaces by $V^G : Z \to \text{holim}_G \text{Map}_*(S^{nG}, Z(S^{nG}))^G$. The norm map is the weak map

$$Z_{hG} \to (\text{Map}_*(EG_+, Z))_{hG}$$

$$\prod \to \text{holim}_G \text{Map}_*(S^{nG}, \text{Map}_*(EG_+, Z(S^{nG})))^G \cong Z^{hG}$$

It is an easy matter to check that the composite homomorphism

$$\pi_*Z \to \pi_*Z_{hG} \xrightarrow{N} \pi_*Z^{hG} \to \pi_*Z$$

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Lemma 5.1.2 (the norm cofiber sequence) Let \( G \) be a finite group, let \( \mathcal{G} \subseteq \mathcal{F} \) be \( K \)-adjacent families of subgroups of \( G \) and let \( X \) be a non-empty free \( G \)-space. For every commutative \( S \)-algebra \( A \) the homotopy fiber of the restriction map

\[
\text{res}: [A^G_S(\mathcal{F})]^G \to [A^G_S(\mathcal{G})]^G
\]

is equivalent to the homotopy orbit spectrum

\[
[A_{\mathcal{X}_K}^G, A]_{hW_GK}.
\]

Proof: Since we work stably it suffices to prove the result for every discrete set \( X = S \). Moreover, since fixed points, finite homotopy limits and homotopy orbits commute with filtered colimits, it suffices to consider the case where \( S \) is a finite set. Let \( (\phi, i) \in [\mathcal{I}(S)]^G \), write \( j = \mathcal{I}(\phi)i \in T^S \) and consider the restriction map

\[
\text{Map}_* \left( \bigcup_{H \in \mathcal{F}} (S(j))^H, A(j) \right)^G \to \text{Map}_* \left( \bigcup_{H \in \mathcal{G}} (S(j))^H, A(j) \right)^G.
\]

The fiber of this fibration is

\[
\text{Map}_* \left( Z(j), A(j) \right)^G
\]

where

\[
Z(j) = \left( \bigcup_{H \in \mathcal{F}} (S(j))^H \bigg/ \bigcup_{H \in \mathcal{G}} (S(j))^H \right).
\]

Apart from the base point, the \( G \)-orbits of \( Z(j) \) are all isomorphic to \( G/K \). Therefore we have isomorphisms

\[
\text{Map}_* \left( Z(j), A(j) \right)^G \cong \text{Map}_* \left( Z(j)^K, A(j) \right)^{N_GK}
\]

\[
\cong \text{Map}_* \left( Z(j)^K, A(j)^K \right)^{W_GK}.
\]

Since \( Z(j)^K \) is a free \( W_GK \)-space, the map

\[
\text{Map}_* \left( Z(j)^K, A(j)^K \right)^{hW_GK} \to \text{Map}_* \left( Z(j)^K, A(j)^K \right)^{W_GK}
\]

is an equivalence. Since \( W_GK \) is a finite group and \( Z(j)^K \) is a finite \( W_GK \)-space, the norm map

\[
N: \text{Map}_* \left( Z(j)^K, A(j)^K \right)^{hW_GK} \to \text{Map}_* \left( Z(j)^K, A(j)^K \right)^{hW_GK}
\]

is a stable equivalence, so we can deduce that

\[
\text{holim}_{(\phi, i) \in [\mathcal{I}(S)]^G} \text{Map}_* \left( Z(\mathcal{I}(\phi)i), A(\mathcal{I}(\phi)i) \right)^G
\]
is equivalent to
\[
\holim_{(\phi,i) \in [\mathcal{I}(S)]^G} \Map_*(Z(\mathcal{I}(\phi))i^K, \mathcal{B}(\mathcal{I}(\phi))i^K)_{hW_GK}.
\]

Note that \( H \in \mathcal{G} \) implies that \( K \) is a proper subgroup of the subgroup \( H - K \) of \( G \) generated by \( H \) and \( K \). Since \( K \) acts freely on \( S \), the space
\[
\left[ \bigcup_{H \in \mathcal{G}} S(j)^H \right]^K = \bigcup_{H \in \mathcal{G}} S(j)^{H - K}
\]
is at most \( \sum_{s \in S} j(s)/2|K| \)-dimensional and \( S^K \wedge A(j)^K \) is \( k - 1 + \sum_{s \in S} j(s)/|K| \)-connected. Using that \( K \)-fixed points commute with quotients by sub-\( K \)-spaces we can conclude that the map
\[
\Map_*(Z(j)^K, S^K \wedge A(j)^K)_{hW_GK} \to \Map_*(S(j)^K, S^K \wedge A(j)^K)_{hW_GK}
\]
is \( k - 1 + \sum_{s \in S} j(s)/2|K| \)-connected.

These considerations are functorial in \((\phi, i)\), and since \( X \) is non-empty, together with Lemma 4.3.3 and Bökstedt’s Lemma (see e.g. [2, Lemma 2.5.1]) they give that the homotopy fiber of the map \([Z_S^\mathcal{A}(\mathcal{F})]^G \to [Z_S^\mathcal{A}(\mathcal{G})]^G\) is equivalent to
\[
\holim_{(\phi,i) \in [\mathcal{I}(S)]^G} \Map_*(Z(\mathcal{I}(\phi))i^K, \mathcal{A}(\mathcal{I}(\phi))i^K)^G
\]
\[
\simeq \holim_{(\phi,i) \in [\mathcal{I}(S)]^G} \Map_*(S(\mathcal{I}(\phi))i^K, \mathcal{A}(\mathcal{I}(\phi))i^K)_{hW_GK}.
\]

The \( G \)-maps
\[
\holim_{(\phi,i) \in [\mathcal{I}(S)]^G} \Map_*(S(\mathcal{I}(\phi))i^K, \mathcal{A}(\mathcal{I}(\phi))i^K)
\]
\[
\to \holim_{(\phi,i) \in [\mathcal{I}(S)]^G} \Map_*(S(\mathcal{I}(\phi))i^K, \mathcal{A}(\mathcal{I}(\phi))i^K)
\]
\[
\to \holim_{(\phi,i) \in [\mathcal{I}(S/K)]^G} \Map_*(S(\mathcal{I}(\phi))i^K, \mathcal{A}(\mathcal{I}(\phi))i^K)
\]
induced by the inclusion \([\mathcal{I}(S)]^G \subseteq [\mathcal{I}(S)]^K\) and the isomorphism \([\mathcal{I}(S/K)]^K \cong \mathcal{I}(S/K)\) are by Lemma 4.3.3 equivalences, and hence we get that the homotopy fiber is equivalent to
\[
[Z_S/KA]_{hW_GK}.
\]

**Corollary 5.1.3** Let \( G \) be a finite group and let \( X \) be a non-empty free \( G \)-space. For every closed family \( \mathcal{F} \) of subgroups of \( G \) the functor \( A \mapsto [\Lambda^\mathcal{A}_X(\mathcal{F})]^G \) preserves connectivity of maps of commutative \( S \)-algebras, and has values in very special \( \Gamma \)-spaces. Furthermore we have a natural equivalence \([\Lambda^\mathcal{A}_X(\mathcal{F})]^G \simeq [\Lambda^\mathcal{A}_X(\mathcal{G})]^G\).

**Proof:** We make induction on the partially ordered set of closed families ordered by inclusion. If \( \mathcal{F} \) is empty, then \([\Lambda_X(\mathcal{F})]^G\) is contractible, and thus the result holds. Otherwise we may choose a minimal subgroup \( K \) in \( \mathcal{F} \) and a closed family \( \mathcal{G} \subseteq \mathcal{F} \) of subgroups of \( G \) so that \( \mathcal{G} \) is \( K \)-adjacent to \( \mathcal{F} \). By Lemma 5.1.2 there is a cofibration sequence
\[
[\Lambda_X/K(A)]_{hW_GK} \to [\Lambda^\mathcal{A}_X(\mathcal{F})]^G \to [\Lambda^\mathcal{A}_X(\mathcal{G})]^G.
\]

By Lemma 13.3.1 the functor \( A \mapsto [\Lambda_X/K(A)]_{hW_GK} \) preserves connectivity, and thus also the functor \( A \mapsto [\Lambda^\mathcal{A}_X(\mathcal{F})]^G \) preserves connectivity. Together with the inductive assumption and the five lemma, this implies that the functor \( A \mapsto [\Lambda^\mathcal{A}_X(\mathcal{G})]^G \) preserves connectivity. Since completion commutes with homotopy orbits the second statement is proved in the same way.
**Corollary 5.1.4** Let $\mathcal{G} \subseteq \mathcal{F}$ be closed families of subgroups of a finite group $G$. For every commutative $S$-algebra $A$ and non-empty free $G$-space $X$ the restriction map $[\Lambda^A_X(F)]^G \to [\Lambda^A_X(G)]^G$ is 0-connected.

**Proof:** Choose adjacent closed families $\mathcal{G}_1 \subseteq \cdots \subseteq \mathcal{G}_n$ of subgroups of $G$ with $\mathcal{G}_1 = \mathcal{G}$ and $\mathcal{G}_n = \mathcal{F}$ and apply Lemma 5.2.2 ■

### 5.2 The restriction map

Given a normal subgroup $N$ of $G$ we let $p_N^S : S \to S/N$ be the projection and we let $\mathcal{F}$ denote the closed family consisting of all subgroups of $G$ containing $N$. Notice that Corollary 5.1.4 implies that the induced functor $(p_N^S)_{\mathcal{F}} : \tilde{I}(S/N) \to \tilde{I}(S)$ induces an isomorphism $(p_N^S)^* : \tilde{I}(S/N) \cong \tilde{I}(S)^N.$

**Lemma 5.2.1** Let $N$ be a normal subgroup of a finite group $G$ and let $S$ be a finite $G$-set. For every commutative $S$-algebra $A$ the $G/N$-spaces $[\Lambda^A_S(F)]^N$ and

\[
\text{holim}_{\tilde{I}(S/N)} \left[ G^A_S(F) \circ r_S \circ (p_N^S)^* \right]^N
\]

are isomorphic.

**Proof:** The $G/N$-space $[\Lambda^A_S(F)]^N$ is defined to be the $N$-fix points of the homotopy colimit $\text{holim}_{\tilde{I}(S)} G^A_S(F) \circ r_S$. Homotopy colimits and fixed points commute in the sense that this $N$-fix point space is isomorphic to the homotopy colimit $\text{holim}_{\tilde{I}(S)^N} \left[ G^A_S(F) \circ r_S \right]^N.$ The result now follows from the fact that $(p_N^S)^* : \tilde{I}(S/N) \cong \tilde{I}(S)^N$ is an isomorphism. ■

Let $(\phi,i) \in \tilde{F}(S/N)$ and let $j = \tilde{I}(p_N^S)^* \phi(i)$. The isomorphisms $S(j)^N \cong S(I(\phi)(i))$ and $A(j)^N \cong A(I(\phi)(i))$ induce a natural isomorphism of the form

\[
[(G^A_S(F) \circ r_S \circ (p_N^S)^*)(\phi,i)]^G = Map_*(\cup_{N \leq H} S(j)^H, A(j))^G \\
= Map_*(\cup_{N \leq H} S(j)^H, S(j)^N)^G \\
= Map_*(S(j)^N, A(j)^N)^{G/N} \\
\cong [(G^A_S(S/N) \circ r_{S/N})(\phi,i)]^{G/N}. \\
\]

Thus the inclusion of $\mathcal{F}$ in the family of all subgroups of $G$ induces a natural transformation of functors of the form

\[
\left[ G^A_S \circ r_S \circ (p_N^S)^* \right]^G \Rightarrow \left[ G^A_S(F) \circ r_S \circ (p_N^S)^* \right]^G \cong \left[ G^A_S(S/N) \circ r_{S/N} \right]^{G/N}
\]

of functors $\tilde{I}(S/N) \to I_S$, given on $(\phi,i)$ by restricting to $N$-fixed points. This induces a modification, and hence a natural map

\[
R^G_{N} : \left[ \text{holim}_{\tilde{I}(S)} [G^A_S \circ r_S] \right]^G \to \left[ \text{holim}_{\tilde{I}(S/N)} G^A_{S/N} \circ r_{S/N} \right]^{G/N},
\]

that is, a map

\[
R^G_{N} : [\Lambda_S(A)]^G \to [\Lambda_{S/N}(A)]^{G/N}. \\
\]

If $G$ is a finite group and $X$ is a simplicial $G$-set, let $\mathcal{F}_X^G$ be the filtered category of finite $G$-subspaces of $X$ and inclusions. Notice that $\mathcal{F}_X^G \subseteq \mathcal{F}_X^{(1)}$ is right cofinal, and so colimits over $\mathcal{F}_X^G$ and $\mathcal{F}_X^{(1)}$ are isomorphic, and the isomorphism is $G$-equivariant.

**Definition 5.2.2** Let $A$ be a commutative $S$-algebra, $G$ a finite group acting on a simplicial set $X$ and $N$ a normal subgroup of $G$. The restriction map

\[
R^G_{N} : [\Lambda_X(A)]^G \to [\Lambda_{X/N}(A)]^{G/N}
\]
is the composite

\[ [\Lambda X(A)]^G \cong \lim_{S \in F_N^G} [\Lambda S(A)]^G \]

\[ \to \lim_{S \in F_N^G} [\Lambda S/N(A)]^{G/N} \cong \lim_{U \in F_{G/N}^S} [\Lambda U(A)]^{G/N} \cong [\Lambda X/N(A)]^{G/N}. \]

The restriction map \( R_N^G \) is natural in the commutative \( S \)-algebra \( A \) and in the \( G \)-space \( X \). Moreover Corollary 4.3.1 implies that \( R_N^G \) is 0-connected. The following lemma is direct from the definition, but is important for future reference:

**Lemma 5.2.3** Let \( A \) be a commutative \( S \)-algebra, \( X \) a \( G \)-space with \( G \) a finite group, and let \( H \subseteq N \subseteq G \) be normal subgroups. Then

\[ R_N^G = R_{N/H}^G R_H^G. \]

If \( F_N^G : [\Lambda X(A)]^G \subseteq [\Lambda X(A)]^N \) denotes the inclusion of fixed points, then we also have that

\[ [\Lambda X(A)]^G \xrightarrow{R_N^G} [\Lambda X/H(A)]^{G/H} \]

\[ F_N^G \]

\[ [\Lambda X(A)]^N \xrightarrow{R_N^G} [\Lambda X/H(A)]^{N/H} \]

commutes.

Most importantly for our applications, we have the fundamental cofiber sequence:

**Lemma 5.2.4** Let \( G \) be a finite abelian group, \( X \) a non-empty free \( G \)-space and \( A \) a commutative \( S \)-algebra. The homotopy fibre of the map

\[ [\Lambda X(A)]^G \to \holim_{\phi \neq H \leq G} [\Lambda X/H(A)]^{G/H} \]

induced by the restriction maps is equivalent to the homotopy orbit spectrum \([\Lambda S(A)]_{hG}\).

**Proof:** Let \( \mathcal{F} = \{ H : 0 \neq H \leq G \} \). In order to apply Lemma 5.1.2 it suffices to show that the space

\[ [\Lambda X(\mathcal{F})]^G \cong \holim_{(\phi,i) \in \overline{G}(\mathcal{F})} \text{Map}_* \left( \bigcup_{\phi \neq H \leq G} (S(I(\phi)i))^H, A(I(\phi)i) \right)^G \]

is equivalent to

\[ \holim_{\phi \neq H \leq G} [\Lambda S/H(A)]^G = \holim_{\phi \neq H \leq G} [\Lambda X/H(A)]^{G/H}. \]

First we notice that the natural map

\[ \holim_{(\phi,i) \in \overline{G}(\mathcal{F})} \text{Map}_* \left( \bigcup_{\phi \neq H \leq G} (S(I(\phi)i))^H, A(I(\phi)i) \right)^G \]

\[ \to \left[ \holim_{\phi \neq H \leq G} \holim_{(\phi,i) \in \overline{G}(\mathcal{F})} \text{Map}_* \left( (S(I(\phi)i))^H, A(I(\phi)i) \right)^H \right]^G \]

\[ \cong \left[ \holim_{\phi \neq H \leq G} \holim_{(\phi,i) \in \overline{G}(\mathcal{F})} \text{Map}_* \left( (S(I(\phi)i))^H, A(I(\phi)i) \right)^H \right]^G \]

is an equivalence (again by Lemma 4.3.3 and since \( G \) is finite). The last term is isomorphic to

\[ \left[ \holim_{\phi \neq H \leq G} G_{S/H}^G \circ \tau_{S/H} \circ (p_H^G)^* \right]^G \]
which again is isomorphic to
\[ \left( \varprojlim_{\substack{H \leq G \nmid \vartriangleleft \leq H}} A_{S/H}(A) \right)^G. \]
That the induced maps are the restriction maps is now clear.

6 The Burnside-Witt construction

Hesselholt and Madsen prove in \cite{8} that if \( A \) is a discrete commutative ring and \( TR(A) \) is the homotopy inverse limit over the restriction maps between the fixed points of the (one dimensional) topological Hochschild homology under finite subgroups of the circle, then \( \pi_0 TR(A) \) is isomorphic to the ring of Witt vectors over \( A \).

In this section we prove an analogous result for arbitrary finite groups. Let \( G \) be a finite group, let \( X \) be a connected free \( G \)-space and let \( A \) be a commutative \( S \)-algebra \( A \). There is a canonical isomorphism of the form
\[ \pi_0 \Lambda_X(A)^G \cong W_G(\pi_0 A), \]
where the latter ring is Dress and Siebeneicher’s “Burnside-Witt”-ring \cite{5} of the commutative ring \( \pi_0 A \) over the group \( G \).

6.1 The Burnside-Witt ring

We shall review some elementary facts about the \( G \)-typical Burnside-Witt ring \( W_G(B) \) of a commutative ring \( B \). For more details on the Burnside-Witt construction, the reader may consult for instance \cite{7} and \cite{3} in addition to \cite{5}. The underlying set of \( W_G(B) \) is the set
\[ \left[ \prod_{H \leq G} B \right]^G \]
where \( G \) acts on the product by taking the factor of \( B \) corresponding to a subgroup \( H \) of \( G \) identically to the factor corresponding to \( gHg^{-1} \). For every subgroup \( K \leq G \) there is a ring homomorphism \( \phi_K : W_G(B) \to B \) taking \( x = (x_H)_{H \leq G} \) to
\[ \phi_K(x) = \sum_{[H]} \|(G/H)^K\| x_H^{|H|/|K|}, \]
where the sum runs over the conjugacy classes of subgroups of \( G \). The endofunctor \( B \mapsto W_G(B) \) on the category of commutative rings is uniquely determined by the following facts:
Firstly, the underlying set of \( W_G(B) = \left[ \prod B \right]^G \) is functorial in \( B \) and secondly, \( \phi_K \) is a natural ring homomorphism for every subgroup \( K \) of \( G \). Dress and Siebeneicher establish the existence of the ring structure on \( W_G(B) \) by making a detailed study of the combinatorics of finite \( G \)-sets. Below we use the functor \( \Lambda \) to give a different proof of the existence of the ring structure on \( W_G(B) \).

Note that when the underlying additive group of \( B \) is torsion free the map
\[ \phi : W_G(B) \to \left[ \prod_{H \leq G} B \right]^G \]
with \( \phi(x)_K = \phi_K(x) \) is injective. For every closed family \( \mathcal{F} \) of subgroups of \( G \) we define \( W_{\mathcal{F}}(B) \) to be the set
\[ W_{\mathcal{F}}(B) = \left[ \prod_{H \in \mathcal{F}} B \right]^G. \]
6.2 The Teichmüller map

We shall show by induction on the size of $\mathcal{F}$, that for every connected free $G$-space $X$ there is a bijection $\mathcal{W}_F(\pi_0 A) \to \pi_0 \Lambda^G_X(F)^G$. In order to do this we need to recollect some structure on $\Lambda^G_X(F)^G$.

Suppose that $S$ is a finite free $G$-set, let $K$ be a subgroup of $G$, let $j \in [I^S]^G$ and let $L$ be a pointed space. Taking homotopy colimits over the maps

$$\text{Map}_*(\bigcup_{H \in \mathcal{F}} S(j)^H, A(j) \wedge L)^K \cong \text{Map}_*(G_+ \wedge K \bigcup_{H \in \mathcal{F}} S(j)^H, A(j) \wedge L)^G$$

we obtain a weak map $V^G_j$ of the form $\Lambda^G_X(\mathcal{F})^H \leftarrow \Lambda^G_X(F)^H \to \Lambda^G_\mathcal{A}(A)^G$, where $\Lambda^G_X(F)^H$ is the homotopy colimit of the middle term in the diagram displayed above. Extending from finite sets $S$ to $G$-spaces $X$ we denote the induced homomorphism on $\pi_0$ by

$$V^G_K : \pi_0 \Lambda^G_X(\mathcal{F})^K \to \pi_0 \Lambda^G_X(\mathcal{F})^G,$$

and call it the additive transfer. Clearly, if $G \subseteq \mathcal{F}$ is an inclusion of closed families of subgroups of $G$, then the restriction maps defined in (1.1) commute with additive transfers:

$$\pi_0 \Lambda^G_X(\mathcal{F})^K \xrightarrow{V^G_K} \pi_0 \Lambda^G_X(\mathcal{F})^G$$

$$\text{res} \downarrow \quad \text{res} \downarrow$$

$$\pi_0 \Lambda^G_X(\mathcal{F})^K \xrightarrow{V^G_K} \pi_0 \Lambda^G_X(\mathcal{F})^G$$

The homomorphism $V^G_K$ is the additive transfer associated to the inclusion $K \leq G$. We also need a kind of multiplicative transfer. Given a pointed space $L$, we can consider the $K$-fold smash power $L^\wedge K$, where $K$ acts by permuting the smash-factors. Let $S$ be a finite $G$-set, let $j \in I^S$ and let $p^* j \in [I^K \times S]^K$ correspond to $j$ under the isomorphism $I^S \cong [I^K \times S]^K$ induced by the projection $p : K \times S \to S$. The map

$$\text{Map}_*(S(j), A(j) \wedge L)^\wedge K \to \text{Map}_*(S(p^* j), A(p^* j) \wedge L)^\wedge K)$$

taking a map $\alpha$ to its $K$-th smash power $\alpha^\wedge K$ induces a map

$$[\Lambda^G_X(A)(L)^\wedge K]^K \to [\Lambda^G_{K \times X}(A)(L)^\wedge K]^K.$$

Thus there is a map

$$\Lambda^G_X(A)(L) \cong [\Lambda^G_X(A)(L)^\wedge K]^K \to [\Lambda^G_{K \times X}(A)(L)^\wedge K]^K \xrightarrow{\Delta^G(p)} [\Lambda^G_X(A)(L)^\wedge K]^K.$$

When $L = S^0$ we denote the induced map on $\pi_0$ by

$$\Delta_K : \pi_0 \Lambda^G_X(A) \to \pi_0 \Lambda^G_X(A)^K,$$

and call it the multiplicative transfer. The following lemma summarizes the properties of the additive and multiplicative transfers that we shall need. Recall that $F^G_H : \Lambda^G_X(A) \to \Lambda^G_X(A)^H$ is the inclusion of fixed points.

**Lemma 6.2.1** Let $H$ and $K$ be subgroups of $G$ and let $X$ be a $G$-space. Let $\phi_{K,H} : \pi_0 \Lambda^G_X(A) \to \pi_0 \Lambda^G_X(A)$ be the map defined by $\phi_{K,H}(x) = [(G/H)^K, x^{[H/H]}]$ and let $q$ be the quotient map $q : X \to X/H$. The following diagram commutes:

$$\begin{array}{ccc}
\pi_0 \Lambda^G_X(A) & \xrightarrow{\phi_{K,H}} & \pi_0 \Lambda^G_X(A) \\
\downarrow \Delta_K & & \downarrow \mu^H_H \\
\pi_0 \Lambda^G_X(A)^K & \xrightarrow{V^G_K} & \pi_0 \Lambda^G_X(A)^G \\
\end{array}$$

$$\begin{array}{ccc}
\pi_0 \Lambda^G_X(A)^K & \xrightarrow{V^G_K} & \pi_0 \Lambda^G_X(A)^G \\
\downarrow \pi_0 \Lambda^G_X(A)^K & & \downarrow \pi_0 \Lambda^G_X(A)^G \\
\pi_0 \Lambda^G_X(A)^H & \xrightarrow{F^G_H} & \pi_0 \Lambda^G_X(A)^H. \\
\end{array}$$
Lemma 6.2.2 Let $\mathcal{F}$ be a closed family of subgroups of $G$ and let $K$ be a minimal element of $\mathcal{F}$. For every connected free $G$-space $X$ the following diagram commutes:

\[
\begin{array}{cccccc}
\pi_0 \Lambda_X(A) & \xrightarrow{\Delta_K} & \pi_0 \Lambda_X(A)^K & \xrightarrow{V^G_K} & \pi_0 \Lambda_X(A)^G \\
\pi_0 \Lambda_X(A) & \cong & R^K_{\mathcal{F}^{X/K}} & \cong & \pi_0 \Lambda_X^A(F)^K & \cong & \pi_0 \Lambda_X^A(F)^G \\
\end{array}
\]

Here $q: X \to X/K$ is the quotient map, the map $\rho$ is induced by the natural isomorphism in Lemma 5.2.1, and the unlabeled map is given by the fundamental coﬁbrieration sequence in Lemma 5.2.4.

Proof: Apart from the lower part of the diagram everything follows directly from the definitions. We need to jump back to the proof of Lemma 5.1.2 where we in particular considered the spaces $Z(j)$. The commutativity of the lower part of the diagram follows from the commutativity of the diagram

\[
\begin{array}{cccccc}
\text{Map}_*(Z(j), A(j))^K & \xrightarrow{V^G_K} & \text{Map}_*(Z(j), A(j))^G \\
\cong & & \cong & & \cong \\
\text{Map}(Z(j)^K, A(j)^K) & \xrightarrow{V^G_{W_G K}} & \text{Map}_*(Z(j)^K, A(j)^K)^{W_G K} \\
\text{can} & & \text{can} & & \text{can} \\
\text{Map}_*(Z(j)^K, A(j)^K)^{W_G K} & \xrightarrow{N} & \text{Map}_*(Z(j)^K, A(j)^K)^{W_G K} \\
\end{array}
\]

in the homotopy category of the category of $\Gamma$-spaces. Here $N$ is the norm map from Section 5.1 and the commutativity of this diagram is a direct consequence of the definitions.

We define the extended Teichmüller map $\tau_\mathcal{F}^F: \mathcal{W}_F(\pi_0 \Lambda_X(A)) \to \pi_0 \Lambda_X^A(F)^G$ to be the composition

\[
\mathcal{W}_F(\pi_0 \Lambda_X(A)) \to \pi_0 \Lambda_X(A)^G \xrightarrow{\text{res}} \pi_0 \Lambda_X^A(F)^G,
\]

where the first map takes $x = (x_h)_{h \leq G}$ to $\sum_{|K|} V^G_K \Delta_K(x_K)$, the sum runs over the conjugacy classes of subgroups in $\mathcal{F}$ and the second map is the restriction map from Section 5.1.

Taking $\mathcal{F}$ to be the family of all subgroups of $G$ we obtain the extended Teichmüller map $\tau_\mathcal{F}^F: \mathcal{W}_G(\pi_0 \Lambda_X(A)) \to \pi_0 \Lambda_X(A)^G$.

Lemma 6.2.3 Let $\mathcal{F}$ be a closed family of subgroups of $G$ and let $K$ be a minimal element of $\mathcal{F}$. If the underlying abelian group of $\pi_0 A$ is torsion free, then the composite

\[
\pi_0 \Lambda_X(A) \to \mathcal{W}_F(\pi_0 \Lambda_X(A)) \xrightarrow{\tau_\mathcal{F}^F} \pi_0 \Lambda_X^A(F)^G
\]

is injective. Here the left map is the diagonal inclusion in the factor corresponding to $K$.

Proof: By Lemma 6.2.1 the composite

\[
\pi_0 \Lambda_X(A) \xrightarrow{\Delta_K \pi_0 \Lambda_X(A)^K} \pi_0 \Lambda_X(A)^G \xrightarrow{\text{res}} \pi_0 \Lambda_X^A(F)^G \xrightarrow{\rho^G_K} \pi_0 \Lambda_X^A(F)^K = \pi_0 \Lambda_X(A)^K \xrightarrow{K^G} \pi_0 \Lambda_X/K(A) \cong \pi_0 \Lambda_X(A)
\]

is equal to $\phi_{K,K}$, that is, it is multiplication by the cardinality of $W_GK$.

The existence of the endofunctor $B \mapsto \mathcal{W}_G(B)$ is a corollary of the proof of the following proposition.
Proposition 6.2.4 Let $G$ be a finite group, $X$ a connected free $G$-space and $A$ a commutative $S$-algebra. The extended Teichmüller map $\tau^G_A: \mathcal{W}_G(\pi_0 A) \to \pi_0 [\Lambda_X(A)]^G$ is an isomorphism of rings.

Proof: We first consider the case where $\pi_0 A$ is torsion free. If $F = \{G\}$, then it is a consequence of Lemma 6.2.2 and Lemma 6.2.4 that $\tau^G_A$ is bijective since $\Lambda^X_A(\emptyset) = \ast$. If $G \subseteq F$ are $K$-adjacent closed families of subgroups of $G$, then by Lemma 6.2.2 there is a commutative diagram of the form

\[
\begin{array}{ccc}
\pi_0 \Lambda_X(A) & \longrightarrow & \mathcal{W}_F(A) \\ \cong & \vcenter{\xymatrix{ \tau^F_A \ar[d] & \mathcal{W}_G(A) \ar[d] } } & \tau^G_A \\
\pi_0 \Lambda_X/K(A)_hW_K & \longrightarrow & \pi_0 \Lambda^X_A(F)^G \longrightarrow \Lambda_X(G)^G
\end{array}
\]

Here the upper left map is the diagonal inclusion in the factor of the product corresponding to the conjugacy class of $K$ and the upper right map is the projection away from that factor. If $\tau^F_A$ is a bijection, then by Lemma 6.2.4 and the 5-lemma it follows readily that also $\tau^G_A$ is a bijection. Thus by induction $\tau^G_A$ is a bijection for every $F$, and in particular $\tau^G_A$ is a bijection.

When $\pi_0 A$ has torsion one proceeds as follows.

First notice that $\tau^G_q$ is natural in $A$. Let $P,q \to \pi_0 A$ be a simplicial resolution of the discrete ring $\pi_0 A$ such that $P_q$ is torsion free for all $q$. Then $HP \to H\pi_0 A$ is a fibration, and therefore $Q = HP \times_{H\pi_0 A} A$ is a simplicial resolution of $A$. Note that $\pi_0 (Q_q) \cong P_q$ for every $q$. It is easy to see that $\pi_0 ([q] \mapsto \mathcal{W}_G(\pi_0 (Q_q))) \cong \mathcal{W}_G(\pi_0 A)$, and likewise $\pi_0 ([q] \mapsto \pi_0 [\Lambda_X(Q_q)]^G) \cong \pi_0 [\Lambda_X(A)]^G$ (for the last one must show that $A \to [\Lambda_X(A)]^G$ may be “calculated degreewise”, a fact which is readily proved by induction using the fundamental cofibration sequence). Thus, for each $q \pi_0 \tau^G_q$ is an isomorphism, and so $\tau^G_q = \pi_0 \tau^G_q$ is an isomorphism.

Now it follows from Lemma 6.2.4 that the composition

\[
\mathcal{W}_G(\pi_0 A) \cong \mathcal{W}_G(\pi_0 \Lambda_X(A)) \xrightarrow{\tau^G_A} \pi_0 \Lambda_X(A)^G \xrightarrow{\phi_H} \pi_0 \Lambda_X(A)^H \xrightarrow{R^H_G} \pi_0 \Lambda_X/H(A) \cong \pi_0 (A)
\]

is equal to $\phi_H$. Taking $A$ to be the Eilenberg MacLane spectrum on a commutative ring $B$ we see that the ring structure on $\mathcal{W}_G(\pi_0 A)$ obtained from the bijection to $\pi_0 \Lambda_X(A)^G$ satisfies the two criteria mentioned in 6.1 determining the ring structure on the ring $\mathcal{W}_G(B)$ of Witt vectors on $B$ uniquely. Thus we may conclude the existence of the ring $\mathcal{W}_G(B)$ of Witt vectors and moreover that $\pi_0 \Lambda_X(A)^G$ is isomorphic to $\mathcal{W}_G(\pi_0 A)$.

Dress and Siebeneicher also define Burnsides-Witt rings over profinite groups. In order to recall their construction we note that, if $N$ is a normal subgroup of a finite group $G$ and if $f: G \to G' = G/N$ is the quotient homomorphism, then there is a natural (surjective) ring homomorphism $R^G_{N'}: \mathcal{W}_G(B) \to \mathcal{W}_{G'}(B)$ taking $x = (x_H)_{H \subseteq G}$ to $y = (y_H')_{H' \subseteq G'}$ where $y_H' = x_{f^{-1}H}$.

Lemma 6.2.5 Let $A$ be a commutative $S$-algebra, let $N$ be a normal subgroup of a finite group $G$, let $X$ be a $G$-space and let $q: X \to X/N$ be the quotient map. For every subgroup $K$ of $G$ containing $N$ the diagram

\[
\begin{array}{ccc}
\pi_0 \Lambda_X A & \xrightarrow{\pi_0 \Delta X A} & \pi_0 \Lambda_{X/N} A \\ \Delta_K \downarrow & & \Delta_{K/N} \downarrow \\
\pi_0 \Lambda_X A^K & \xrightarrow{\pi_0 R^K} & \pi_0 \Lambda_{X/N} A^{K/N} \\ \varphi^G_K \downarrow & & \varphi^{G/N}_K \downarrow \\
\pi_0 \Lambda_X A^G & \xrightarrow{\pi_0 R^G} & \pi_0 \Lambda_{X/N} A^{G/N}
\end{array}
\]

commutes.
Given a profinite group $G$, we let $\mathcal{W}_G(B)$ be the projective limit of the rings $\mathcal{W}_{G/U}(B)$ with respect to the homomorphisms $R^{G/U}_V$ for $U \leq V$ open (i.e. finite index) normal subgroups of $G$. Let $X$ be a $G$-space with the property that $X/U$ is a free connected $G/U$-space for every open normal subgroup $U$ in $G$. (The guiding example we have in mind is when $G = (\mathbb{Z}^n)^{p}$ is the profinite completion of $\mathbb{Z}^n$ or $G = (\mathbb{Z}_p)^{\infty}$ is the $p$-completion of $\mathbb{Z}^n$ and $X = G \times_{\mathbb{Z}^n} \mathbb{R}^n$.) Then it follows from Lemma 6.2.5 that the diagram

$$
\begin{array}{ccc}
\mathcal{W}_{G/U}(\pi_0\Lambda_X/U/A) & \xrightarrow{\pi_0\Lambda_X/U/A} & \pi_0\Lambda_X/U/A^{G/U} \\
R^{G/U}_V \mathcal{W}_{G/U}(\pi_0\Lambda_X/U) & \xrightarrow{R^{G/U}_V} & R^{G/U}_V \\
\mathcal{W}_{G/V}(\pi_0\Lambda_X/V/A) & \xrightarrow{\pi_0\Lambda_X/V/A} & \pi_0\Lambda_X/V/A^{G/V}
\end{array}
$$

commutes for $U \leq V$ open normal subgroups of $G$. Since the Mittag Leffler condition is satisfied we conclude that there are isomorphisms

$$
\mathcal{W}_G(\pi_0A) \cong \lim_U \mathcal{W}_{G/U}(\pi_0A) \cong \lim_U \pi_0\Lambda_X/U/A^{G/U} \cong \pi_0 \text{holim}_U \Lambda_X/U/A^{G/U}
$$

where the limits are taken over restriction maps. We summarize the above discussion with a very special case.

Given a prime $p$, consider the set $\mathcal{O}_p$ consisting of subgroups $U \subseteq \mathbb{Z}^n$ with index a power of $p$. Choose a free contractible $\mathbb{Z}^n$-space $E$ (e.g., $\sin \mathbb{R}^n$), and consider the diagram of spaces $E/U$ where $U$ varies over $\mathcal{O}_p$. Notice that this diagram is equivalent to a diagram of isogenies of the $n$-torus.

**Corollary 6.2.6** Let $A$ be a connective commutative $S$-algebra. Then there is a natural ring isomorphism “preserving $F$ and $V$” between Dress and Siebeneicher’s Burnside Witt ring $\mathcal{W}_{\mathbb{Z}^n}(\pi_0A)$ and $\pi_0 \text{holim}_{E \in \mathcal{O}_p} (\Lambda_{E/U}A)^{\mathbb{Z}^n/U}$.

Note that we have a cofinal subsystem given by the powers of $p$, so that both groups in the corollary are isomorphic to $\pi_0 \text{holim}_E (\Lambda_{E/p^n\mathbb{Z}^n}A)^{\mathbb{Z}^n/p^n\mathbb{Z}^n}$.

## 7 Covering Homology

In this section we define covering homology through appropriate homotopy limits of the restriction and Frobenius maps. The name derives from the fact that in our main examples the limit is indexed by systems of self-coverings.

### 7.1 Spaces with finite actions

We shall use a category $\mathcal{E}$ to index the homotopy limit defining covering homology. The objects of $\mathcal{E}$ consist of triples $(G, H, X)$ where $G$ is a simplicial group, $H$ is a discrete subgroup of $G$ and $X$ is a non-empty $G$-space with the property that the image of $H$ in $\text{Aut}(X)$ is finite. A morphism in $\mathcal{E}$ from $(G, H, X)$ to $(G', H', X')$ consists of a pair $(\varphi, f)$ where $\varphi: G \rightarrow G'$ is a group homomorphism such that $H' \subseteq \varphi(H)$ and $f: X \rightarrow \varphi^*X'$ is a $G$-map. The composition in $\mathcal{E}$ is given by composing in each of the two entries.

Covering homology is built from functors into $\mathcal{E}$. Before we give examples of functors into $\mathcal{E}$ we recall the definition of the twisted arrow category.

**Definition 7.1.1** If $\mathcal{C}$ is a category, then the twisted arrow category $A(\mathcal{C})$ of $\mathcal{C}$ is the category whose objects are arrows in $\mathcal{C}$. A morphism from $d: x \rightarrow y$ to $b: z \rightarrow w$ in $A(\mathcal{C})$ is a commutative diagram

$$
\begin{array}{ccc}
x & \xrightarrow{c} & y \\
\downarrow_d & & \downarrow_b \\
z & \xleftarrow{a} & w
\end{array}
$$
in \( C \), i.e., every equation \( abc = d \) represents an arrow \((a,c)\) from \( d \) to \( b \), and composition is horizontal composition of squares: \((a_1, c_1)(a_0, c_0) = (a_0a_1, c_1c_0)\).

The shorthand notation \( a^* = (a, id) \) and \( e_* = (id, c) \) is usual, giving formulae like \( a^* e_* = c_* a^* \), \((ab)^* = b^* a^* \) and \( b_* c_* = (bc)_* \).

Given a simplicial group \( G \) we let \( I(G) \) be the monoid of isogenies of \( G \), that is, group-endomorphisms of \( G \) with finite discrete kernel and cokernel.

**Definition 7.1.2** Let \( G \) be a simplicial group and consider \( I(G) \) as a category with one object. Let \( \mathcal{A}_G \) be the subcategory of \( \mathcal{A}(I(G)) \) containing all objects and with morphism set \( \mathcal{A}_G(\delta, \beta) \) equal to the set of pairs \((\gamma, \alpha)\) of isogenies of \( G \) with the property that \( \text{Ker} \beta \subseteq \gamma(\text{Ker} \delta) \). The functor \( S_G : \mathcal{A}_G \rightarrow E \) takes an object \( \alpha : G \rightarrow G \) of \( \mathcal{A}_G \) to the triple \( S_G(\alpha) = (G, \text{Ker}(\alpha), X) \), where \( X = G \) is the \( G \)-space where \( G \) acts on itself by translation. A morphism

\[
\begin{array}{ccc}
G & \xrightarrow{\gamma} & G \\
\delta \downarrow & & \downarrow \beta \\
G & \xleftarrow{\alpha} & G
\end{array}
\]

in \( \mathcal{A}_G \) is taken to the morphism

\[
S_G(\alpha, \gamma) = (\gamma, f) : (G, \text{Ker}(\delta), X) \rightarrow (G, \text{Ker}(\beta), X)
\]

in \( E \), where \( f : X \rightarrow \gamma^* X \) is equal to \( \gamma \) considered as a map of \( G \)-spaces.

Given a set \( X \) we let \( EX \) denote the contractible simplicial set defined by the formula \( EX_k = \text{Map}([k], X) \), where \([k] \in \Delta \) is considered as a set with \( k + 1 \) elements. Note that there is an inclusion \( X \cong EX_0 \subseteq EX \). If \( X = K \) is a discrete group, then \( EK = EX \) is a simplicial group containing \( K \). In particular, \( K \) acts freely on \( EK \). Note that if \( K \) is abelian, then \( EK \) is abelian.

Given an inclusion \( H \subseteq K \) of discrete abelian groups, let \( \mathcal{C}(H, K) \) be the monoid of group automorphisms \( a : K \rightarrow K \) with the property that \( a(H) \) is a subgroup of finite index in \( H \). There is a monoid homomorphism \( \varphi : \mathcal{C}(H, K) \rightarrow I(\mathbb{E}K/H) \), taking \( a \in \mathcal{C}(H, K) \) to the surjection

\[
\varphi(a) : EK/H \rightarrow EK/H
\]

induced by \( a \).

In the situation of Definition 7.1.2 let \( G = EK/H \). Since the functor \( \mathcal{A}(\varphi) : \mathcal{A}(\mathcal{C}) \rightarrow \mathcal{A}(I(G)) \) factors through the subcategory \( \mathcal{A}_G \) we can make the following definition.

**Definition 7.1.3** Let \( H \subseteq K \) be an inclusion of discrete abelian groups. Given a submonoid \( \mathcal{C} \) of \( \mathcal{C}(H, K) \) we let \( S(\mathcal{C}) : \mathcal{A}(\mathcal{C}) \rightarrow E \) be the functor taking \( c \in \mathcal{C} \) to the triple \( S(\mathcal{C})(c) = (EK/H, \text{Ker}(\varphi(c)), EK/H) \).

### 7.2 Covering homology

To every commutative \( S \)-algebra \( A \) there is an associated functor \( \Lambda A \) from the category \( E \) of \( \Delta \)-sets to the category of spectra, taking an object \((G, H, X)\) to \( [A_{X/H}]^H \). For a morphism \((\varphi, f) : (G, H, X) \rightarrow (G', H', X')\) in \( E \) we write \( K = \text{Ker}(\varphi) \cap H \), and by abuse of notation we write \( \Lambda f \) for the map

\[
[A_{X/K}]^H \rightarrow [A^*(X')^A]^H = [A_{X'}^A]^\varphi(H)
\]

induced by \( f \). We define the map \( \Lambda A(f, \varphi) \) to be the composition

\[
[A_X^A]^H \xrightarrow{\Lambda f} [A_{X/K}^A]^H \xrightarrow{\Lambda_\varphi} [A_{X}^A]^\varphi(H) \xrightarrow{E \varphi} [A_{X'}^A]^H
\]

In order to check that \( \Lambda A \) is a functor, let

\[
(\varphi, f) : (G, H, X) \rightarrow (G', H', X')
\]
and

$$(\psi, g): (G', H', X') \to (G'', H'', X'')$$

be morphisms in $\mathcal{E}$. Let $K = \text{Ker}(\varphi) \cap H$, $K' = \text{Ker}(\psi) \cap H'$ and $K'' = \text{Ker}(\varphi \circ \psi) \cap H$. By the naturality and transitivity properties of the maps $R$ and $F$ stated in Lemma 5.2.3 the diagram

\[
\begin{array}{ccc}
[A_X A]^H & \xrightarrow{R} & [A_{X/K} A]^H/K \\
R & \downarrow & \downarrow \\
[A_{X/K} A]^H/K' & \xrightarrow{R} & [A_{X'/K'} A]^H/K'
\end{array}
\]

commutes.

**Definition 7.2.1** Let $A$ be a commutative $S$-algebra and let $S: \mathcal{A} \to \mathcal{E}$ be a functor from an arbitrary category $\mathcal{A}$. The covering homology of $A$ with respect to the functor $S$ is

$$TC_S(A) := \text{holim} A A S$$

Let $G$ be a simplicial group and let $I(G)$ be the monoid of isogenies of $G$, that is, group-endomorphisms $\alpha$ of $G$ with finite and discrete kernel and cokernel. In Definition 7.1.2 we constructed a functor $S_G: A_G \to \mathcal{E}$. Thus there is a covering homology $A \mapsto TC_{S_G}(A)$ associated to every simplicial group. The fact that surjective isogenies are simplicial versions of finite covering maps is the reason for our choice of the name “covering homology”. Note that given $\alpha$ and $\beta$ in $I(G)$, the map $\Lambda_\alpha$ is the composite

$$[\Lambda_{G/\text{Ker} \alpha} A]^{\text{Ker}(\beta \alpha)/\text{Ker} \alpha} \to [\Lambda_{\alpha \circ G} A]^{\text{Ker}(\beta \alpha)/\text{Ker} \alpha} = [\Lambda G A]^{\text{Ker}(\beta \alpha)}.$$

If $\alpha$ is surjective, the map $G/\text{Ker} \alpha \to G$ induced by $\alpha$ is an isomorphism, and thus $\Lambda_\alpha$ is an isomorphism in this case.

We shall occasionally use the notations $R_\alpha = \Lambda A (\gamma_*)$ and $F^\alpha = \Lambda (\alpha^*)$ for $\Lambda A$ applied to morphisms of the form

\[
\begin{array}{ccc}
G & \xrightarrow{\gamma} & G \\
\delta & & \beta \\
G & \xleftarrow{\alpha} & G
\end{array}
\]

in $A_G$.

**Example 7.2.2** The situation where $G$ is the simplicial $n$-torus $T^n = \sin(\mathbb{R}^n/\mathbb{Z}^n)$ is particularly interesting. There is an isomorphism

$$\text{Hom}(\mathbb{Z}^n, \mathbb{Z}^n) \to \text{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathbb{R}^n/\mathbb{Z}^n)$$

taking $\alpha$ to the map induced by $\mathbb{R} \otimes_{\mathbb{Z}} \alpha: \mathbb{R}^n \to \mathbb{R}^n$, and the induced homomorphism

$$\text{Hom}(\mathbb{Z}^n, \mathbb{Z}^n) \subseteq \text{Hom}(T^n, T^n), \quad \alpha \mapsto \sin(\mathbb{R} \otimes_{\mathbb{Z}} \alpha),$$

is injective. This allows us to consider the monoid $M_\alpha$ of injective linear endomorphisms of $\mathbb{Z}^n$ as a submonoid of $I(G)$. Thus, given a submonoid $C$ of $M_\alpha$, we can consider the covering homology $A \mapsto TC_{S(C)}(A)$.  

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In the special case where \( n = 1 \) and where \( \mathcal{C} = \mathcal{M}_1 = (\mathbb{Z} \setminus \{0\}, \cdot) \), the covering homology \( TC_{S(\mathcal{M}_1)}(A) \) gives us what might be called topological dihedral homology. If \( \mathcal{C} \) is the submonoid \( (\mathbb{N}, \cdot) \), giving just orientation-preserving coverings of the circle, the covering homology \( TC_{S(\mathcal{C})}(A) \) is weakly equivalent to Bökstedt, Hsiang and Madsen’s topological cyclic homology.

Let \( G \) be a simplicial group, let \( \mathcal{C} \) be a submonoid of \( I(G) \) and let \( \mathcal{A}_G := \mathcal{A}^G \cap \mathcal{A}_G \). If \( \varphi: G \to G \) is a group automorphism with the property that \( \varphi x \varphi^{-1} \in \mathcal{C} \) for every \( x \in \mathcal{C} \), then we define the functor
\[
c_{\varphi}: \mathcal{A}_G \to \mathcal{A}_G, \quad x \mapsto \varphi x \varphi^{-1},
\]
and the commutative diagram
\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G \\
\downarrow & & \downarrow \\
G & \xrightarrow{\varphi^{-1}} & G,
\end{array}
\]
specifies a natural transformation \( \eta_{\varphi} \) from the inclusion \( j: \mathcal{A}_G \subseteq \mathcal{A}_G \) to the functor \( j \circ c_{\varphi} \).

Since \( (\varphi, \varphi^{-1}) = \varphi^* \circ \varphi^{-1} \) we have
\[
\Lambda A(\varphi, \varphi^{-1}) = \Lambda A(\varphi^*) \circ \Lambda A(\varphi^{-1}) = R_{\varphi} \circ F_{\varphi^{-1}}: \Lambda G A \to \Lambda G A.
\]
Let \( S_C \) be the composite
\[
\mathcal{A}_G \xrightarrow{j} \mathcal{A}_G \xrightarrow{S_G} \mathcal{E}.
\]

**Definition 7.2.3** Let \( G \) be a simplicial group, let \( \mathcal{C} \) be a submonoid of \( I(G) \) and let \( I \) be a group contained in \( I(G) \) with the property that \( \varphi x \varphi^{-1} \in \mathcal{C} \) for every \( x \in \mathcal{C} \) and every \( \varphi \in I \). We define an action of \( I \) on \( TC_{S_C}(A) \) by letting the action of \( \varphi \in I \) by given by the endomorphism
\[
\Lambda A \circ S_C \xrightarrow{\text{holim } \Lambda A \circ S_C \circ c_{\varphi}} \text{holim } \Lambda A \circ S_C \xrightarrow{\text{holim } \Lambda A \circ S_C} \text{holim } \Lambda A \circ S_C.
\]
Note that The group \( I \) acts through maps.

**Example 7.2.4** Let \( G = T^x \), let \( p \) be a prime and let \( \mathcal{C} \) denote the submonoid of \( I(G) \) consisting of isogenies of the form
\[
p^r: T^x \to T^x, \quad (x_1, \ldots, x_n) \mapsto p^r(x_1, \ldots, x_n) = (p^r x_1, \ldots, p^r x_n).
\]
for \( r \geq 0 \) corresponding to the submonoid of \( \mathcal{M}_n \) consisting of endomorphisms of \( \mathbb{Z}^n \) of the form \( p^r: \mathbb{Z}^n \to \mathbb{Z}^n \). The group of automorphisms of \( \mathbb{Z}^n \) is contained in \( \mathcal{M}_n \). Since the endomorphisms in \( C \) correspond to multiplication by a number, they are fixed under conjugation by elements in the group \( GL_n \mathbb{Z} \) of group automorphisms of \( \mathbb{Z}^n \). Thus Definition 7.2.3 specifies an action of \( I \) on \( TC_{S_C} A \). In this particular situation the category
\[
\cdots \xrightarrow{p^r} \xrightarrow{p^r} \xrightarrow{p^r} \xrightarrow{p^r} (1),
\]
with \((p^n)\) equal to the endomorphism of \( G \) by multiplication by \( p^n \), is cofinal in the category \( \mathcal{A}_C \). Thus \( TC_{S_C} \) is homotopy equivalent to the homotopy limit of the diagram
\[
\cdots \xrightarrow{F_p} (\Lambda T^x A) \xrightarrow{p^r} (\Lambda T^x A) \xrightarrow{F_p} (\Lambda T^x A) \xrightarrow{p^r} \Lambda T^x A.
\]

If \( n = 1 \) we may consider the action of \( \{\pm 1\} = GL_1(\mathbb{Z}) \), and the homotopy fixed point spectrum picks up the “part relevant to \( p^r \)” of the topological dihedral homology in Example 7.2.4.
Working with the $p$-complete torus instead, we get operations by all of $GL_n(\mathbb{Z}_p)$. Note that if $n = 1$ the map from $TC(A)^p \cong TC_{S_1}(A)^p$ to $THH(A)^p \cong \Lambda_{(T_1)^p}(A)^p$ then sends the operation of a $p$-adic unit on $TC(A)^p$ to the corresponding Adams operation on $THH(A)^p$, as discussed in \[E.3\].

**Example 7.2.5** Let $R \subseteq B$ be an inclusion of (discrete) commutative rings and let $M$ be a flat $R$-module. In the context of Definition 7.2.4 let $H = M = R \otimes_R M$ and let $\mathcal{C} \subseteq \mathcal{C}$ be the inclusion of discrete abelian groups induced by the $R$-module homomorphism $R \to B$. Applying Definition 7.2.3 to a submonoid $\mathcal{C} \subseteq \mathcal{C}$ of $\mathcal{C}(M, B \otimes_R M)$ we obtain a functor $S(\mathcal{C}) : \mathcal{C}(\mathcal{C}) \to \mathcal{C}$, and we may form the covering homology $A \mapsto TC_{S(\mathcal{C})}(A)$. If $I$ is a group contained in $\mathcal{C}(M, B \otimes_R M)$ with the property, that $\varphi x \varphi^{-1} \in \mathcal{C}$ for every $x \in \mathcal{C}$ and every $\varphi \in I$, then Definition 7.2.3 specifies an action of $I$ on $TC_{S(\mathcal{C})}(A)$.

Let us emphasize that if $R \subseteq B$ is the inclusion $\mathbb{Z} \subseteq \mathbb{Q}$ and if $M = \mathbb{Z}^n$, then $G = E(\mathbb{Q} \otimes \mathbb{Z})/(\mathbb{Z} \otimes \mathbb{Z})$ is a model for the classifying space $B\mathbb{Z}^n$. In fact, the homomorphisms $R^n/\mathbb{Z}^n \hookrightarrow (R^n \times |E(\mathbb{Q} \otimes \mathbb{Z})^n|)/\mathbb{Z}^n \to |E(\mathbb{Q} \otimes \mathbb{Z})^n|/\mathbb{Z}^n \cong |G|$ of topological abelian groups are homotopy equivalences. In Example 7.2.2 we have seen that under these equivalences $M_n = \mathbb{C}(\mathbb{Z}^n, \mathbb{Q}^n)$ corresponds to the monoid of isogenies of the $n$-torus $\mathbb{R}^n/\mathbb{Z}^n$. The spectrum $TC_{S(\mathcal{C})(\mathbb{Z}^n, \mathbb{Q}^n)}(A)$ is related to iterated topological cyclic homology. In fact, in Example 7.2.2 we have seen that when $n = 1$ and $\mathcal{C}$ is the submonoid $(\mathbb{N}_{\geq 0}, +)$ of $\mathbb{C}(\mathbb{Z}, \mathbb{Q}) = (\mathbb{Z} \setminus \{0\}, +)$, then $TC_{S(\mathcal{C})}(A)$ is weakly equivalent to Bökstedt, Hsiang and Madsen’s topological cyclic homology. Note that Definition 7.2.3 gives an action of $I = \{−1, +1\}$ on $TC_{S(\mathcal{C})}(A)$ whose homotopy fixed point spectrum is the topological dihedral homology of Example 7.2.2.

Consider the situation where $B$ is the quotient field of an integral domain $R$ and $M = \mathcal{O}$ is a possibly non-commutative $R$-algebra. In this situation we can choose the monoid $\mathcal{C}$ to be the intersection of $\mathcal{C}(\mathcal{O}, B \otimes_R \mathcal{O})$ and image of the homomorphism $\psi : \mathcal{O} \to \text{End}_R(B \otimes_R \mathcal{O})$ from $\mathcal{O}$ to the monoid $\text{End}_R(B \otimes_R \mathcal{O})$ of group-endomorphisms of $B \otimes_R \mathcal{O}$ with $\psi(x)(b \otimes y) = b \otimes xy$. If $f : \mathcal{O} \to \mathcal{O}$ is an $R$-algebra automorphism, then the diagram

$$
\begin{array}{ccc}
B \otimes_R \mathcal{O} & \xrightarrow{B \otimes_R f} & B \otimes_R \mathcal{O} \\
\psi(x) & & \psi(f(x)) \\
\downarrow & & \downarrow \\
B \otimes_R \mathcal{O} & \xrightarrow{B \otimes_R f} & B \otimes_R \mathcal{O}
\end{array}
$$

commutes. This implies that if we let $I$ be the group of $R$-algebra automorphisms of $B \otimes_R \mathcal{O}$, of the form $\varphi = B \otimes_R f$, then $\varphi \psi(x) \varphi^{-1} \in \mathcal{O}$ for every $\varphi \in I$ and every $\psi(x) \in \mathcal{C}$. Thus the group of $R$-algebra automorphisms of $\mathcal{O}$ acts on $TC_{S(\mathcal{C})}(A)$.

Explicit examples are listed in the figure below, where $G$ is a finite group, $K \subseteq L$ is a finite Galois extension of (local) number fields and $\mathcal{O}(K) \subseteq \mathcal{O}(L)$ is the induced inclusion of rings of integers.

| $R$ | $B$ | $\mathcal{O}$ | $\mathcal{C}$ | $\text{Aut}_R(\mathcal{O})$ |
|-----|-----|------|------|-----------------|
| $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{Z}^n$ | $(\mathbb{Z} \setminus \{0\})^n$ | $\Sigma_n$ |
| $\mathbb{Z}_p$ | $\mathbb{Q}_p$ | $(\mathbb{Z}_p)^n$ | $(\mathbb{Z}_p \setminus \{0\})^n$ | $\Sigma_n$ |
| $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{Z}[G]$ | $\mathbb{Z}[G] \cap \mathbb{Q}[G]^*$ | $\text{Aut}(G)$ |
| $\mathcal{O}(K)$ | $K$ | $\mathcal{O}(L)$ | $\mathbb{Z}[L] \cap L^*$ | $\text{Gal}(L/K)$ |

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