Solving of an Inverse Boundary Value Problem for the Heat Conduction Equation by Using Lavrentiev Regularization Method

H K Al-Mahdawi

1 South Ural State University, Chelyabinsk, Russia

E-mail: hssnkid@gmail.com

Abstract. In this paper, the inverse problem for the boundary value of the heat equation is posed and solved. It is well known that this problem classified as an ill-posed problem. The boundary value problem can be represented as an integral equation of the first kind by using the separation of variables method. The discretization of the integral equation allowed us to reduce the integral equation to a system of linear algebraic equations or a linear operator equation of the first kind on Hilbert spaces. In order to find an approximation solution, we need to apply a regularization algorithm. In this type of equation and through the regularization step we faced a non-injective operator problem. The Lavrentiev regularization method was used to obtain the solution instead of the Tikhonov regularization method.

1. Introduction

Many applied problems formulated as inverse problems of mathematical physics belong to the class of ill-posed problems. The inverse boundary value problem for the heat equation is defined as an ill-posed problem in the sense that a “small” arbitrary change in data can lead to “large” errors in the solution [1]. The inverse problem under the study of the heat equation can be solved by many methods. For example, the method of regularization Tikhonov A.N. [2], the method of Lavrentiev M.M. [3], by the method of quasi-solutions Ivanova V.K. [4] and many others.

The inverse problems for the heat equation can be classified into two types depend on the type the unknown function for the initial condition or the boundary condition. This case can be modeled as a Cauchy problem for the heat equation. Various methods for solving this type of inverse problem have been proposed in many works. In [5] the mixed initial-boundary value problem for the heat conduction equation has solved after divided the problem into three intervals the first one is the heating, the second and third intervals are termination of the heating and free cooling. The Fourier transform method used to converts this problem into an ordinary differential equation. In other work, it used the sampling-type reconstruction method and showing the relation of its solution to the Green function for reconstructing the unknown inclusions inside a thermal conductor from boundary measurements, which is formulated as an inverse boundary value problem for the heat equation [6]. In [7] the inverse boundary-value problem of heat conduction was considered this problem is reduced to an integral equation of the first kind with a symmetrical kernel, by using the iterative method the approximation solution has been obtained. The inverse initial value problem for the heat equation has been solved in [8] [9] by using two different methods. In [8] the Tikhonov regularization method after reducing the
problem to an integral equation for the first kind and in [9] the author using the Ivanova method with Picard theorem for obtaining the approximation solution. There are many other methods for solving other ill-posed inverse problem described in [10][11].

The main idea in this work is solving the boundary value problem for the heat equation by using separation of the variable method the problem has reduced to an integral equation of first kind. Hence, the solution does not depend continuously on the data in conventional Banach spaces, which leads to the solution unstable so this problem classified as an ill-posed problem. In order to solve the ill-posed problems, an important role is played by the error estimates of the obtained solution. In this paper, we obtain estimates solution for the Lavrentiev method these results are new and practical interest.

All these steps are organized through the sections in this paper. Section 2 defines the linear partial differential equation and describes the solution as an integral equation of the first kind for the boundary value problem of the heat equation. Section 3 described the general discretization method for converting the integral equation to a linear system equation or linear operator equation. Section 4 implemented the discretization method and defined the inverse ill-posed problem for special conditions to the integral equation. After that in section 5, the Lavrentiev regularization method used to find an approximation solution for a linear operator equation with an injective operator. Then, in section 6 the example has been presented to verify the accuracy of our estimated solution. Finally, the explanation of the suggested method has been summarized in the conclusion section.

2. Statement of the boundary heat transfer problem

We consider an inverse boundary value problem given in [5] with time interval \([0, T]\).

\[
\frac{\partial u(x, t)}{\partial t} = \frac{\partial^2 u(x, t)}{\partial x^2}; \quad 0 < x < 1, 0 < t \leq T,
\]

(1)

\[u(x, 0) = 0; \quad 0 \leq x \leq 1,
\]

(2)

\[
\frac{\partial u(0, t)}{\partial x} = 0; \quad 0 < t \leq T,
\]

(3)

\[u(1, t) = h(t); 0 < t \leq T,
\]

(4)

suppose that the \(h(t)\) is a function such that

\[h(t) \in H^4[0, T], h(0) = h(0) = h''(0) = h''(T) = h'(T) = 0,
\]

(5)

and \(\int_0^T |h''(t)|^2 \, dt \leq r^2, r \text{ – known number.}

By using the separation of variables method we obtained the following solution

\[u(x_0, t) = \sum_{n=0}^{\infty} C_n(t) \cos \left( n + \frac{1}{2} \right) \pi x_0 + h(t),
\]

(6)

where the \(x_0 \in (0, 1), t \in [0, T]\)

\[C_n(t) = \frac{2e^{-\left( n + \frac{1}{2} \right)^2 \pi^2 t}}{\left( n + \frac{1}{2} \right) \pi} \int_0^t h'(\tau) e^{\left( n + \frac{1}{2} \right)^2 \pi^2 \tau} \, d\tau,
\]

(7)

integrating by parts the right-hand side of formula (7) twice, we obtain
\[
C_n(t) = \frac{2}{(n + \frac{1}{2}) \pi} \int_0^t h'''(\tau)e^{-\left(\frac{n+1}{2}\right)^2\pi^2(t-\tau)} d\tau + \frac{2}{(n + \frac{1}{2}) \pi^3} h'(t)C_n(t) \\
- \left[ \frac{2}{(n + \frac{1}{2}) \pi^3} h'(0) + \frac{2}{(n + \frac{1}{2}) \pi^5} h''(t) + \frac{2}{(n + \frac{1}{2}) \pi^5} h''(0) \right],
\]

(8)

\[
C_n(t) = \frac{2}{(n + \frac{1}{2}) \pi^5} \int_0^t h'''(\tau)e^{-\left(\frac{n+1}{2}\right)^2\pi^2(t-\tau)} d\tau \\
- \left[ \frac{2}{(n + \frac{1}{2}) \pi^3} h'(0) + \frac{2}{(n + \frac{1}{2}) \pi^5} h''(0) \right],
\]

(9)

where \( h'(0) = h''(0) = 0 \) by (5)

\[
C_n(t) = \frac{2}{(n + \frac{1}{2}) \pi^5} \int_0^t h'''(\tau)e^{-\left(\frac{n+1}{2}\right)^2\pi^2(t-\tau)} d\tau.
\]

(10)

Next step it is important to know the solution \( u(x, t) \) belongs to the space \( H^4[0,1] \), this proofed in lemma 2. in [5].

Lemma 2. The function \( u(x, t) \) defined by formulas (6) and (8) belongs to the space \( H^4[0,1] \).

From (6) and (10) we have the following integral equation of first kind

\[
f(t) = u(x_0, t) = h(t) + \int_0^t \sum_{n=0}^\infty \frac{2 \cos \left(\frac{n+1}{2}\right)\pi x_0}{(n + \frac{1}{2}) \pi^5} e^{-\left(\frac{n+1}{2}\right)^2\pi^2(t-\tau)} h'''(\tau) d\tau,
\]

(11)

3. Discretization of the Integral Equation of the First Kind

The discretization algorithm described in [12], the algorithm implemented on inverse initial value problem for the heat equation [8]. The integral equation is considered in the following form to simplify the discretization algorithm with many derivative times for kernel.

\[
Au(s) = \int_a^b P(s, t)u(s) ds = f(t), \quad c \leq t \leq d,
\]

(12)

where \( u(s) \in L_2[a, b] \), \( f(t) \in L_2[c, d] \), the kernel \( P(s, t) \) represent the closed operator \( A \). \( P(s, t) \)

limited to interval rectangle \([a, b] \times [c, d], \forall t \in [c, d], P(s, t) \) piecewise continuously on \([a, b]\), and 

\( P(s, t) \in ([a, b] \times [c, d]). \)

We assume there is a solution \( u_0(s) \) for (12) in the set \( M_r \) this solution for function \( f(t)=f_0(t) \), the set \( M_r \) defined as the following

\[
M_r = \left\{ u(s): \quad u''(s) \in L_2[a, b], \right\}
\]

\[
\left\{ u(a) = u'(a) = u''(a) = 0, \quad \int_a^b |u''(s)|^2 ds \leq r^2 \right\}.
\]

(13)
The function \( f_0(t) \) is unknown but instead of the \( f_0(t) \in L_2[c,d] \), and number \( \delta > 0 \) are given such that
\[
\|f_0(t) - f_0(t)\|_{L_2}^2 \leq \delta^2, \tag{14}
\]
we can estimate the approximation solution \( u_\delta(s) \) of equation (12) from given \( f_0(t), \delta \) and \( M_r \).

We need define an operator \( B: L_2[a,b] \rightarrow L_2[a,b] \) by the following formula
\[
u(s) = Bv(s) = \int_a^s \frac{(s - \zeta)^2}{2} v(\zeta) d\zeta, \quad v(s), Bv(s) \in L_2[a,b], \tag{15}
\]
the operator \( C \) can be defined as following:
\[
Cv(s) = ABv(s), \quad v(s) \in L_2[a,b], \quad Cv(s) \in L_2[c,d], \tag{16}
\]
\[
K(t,s) = \int_s^b \frac{(\zeta - s)^2}{2} P(\zeta, t) d\zeta, \tag{17}
\]
from (15), (16) and (17)
\[
Cv(s) = \int_a^s K(s,t) v(s) ds, \quad t \in [c,d], \tag{18}
\]
In order to find a numerical solution for equation (18), we find finite-dimensional operator \( C_{n,m} \), where \( C_{n,m} \rightarrow C \) and \( C_{n,m}^* \rightarrow C^* \), \( \forall v(s) \in L_2[a,b] \Rightarrow C_{n,m}v(s) \rightarrow Cv(s) \) and \( \forall g(s) \in L_2[a,b], \Rightarrow C_{n,m}^*g(s) \rightarrow C^*g(s) \).

We need define the value for \( \eta_{n,m} \) which satisfy the following:
\[
\|C_{n,m} - C\| \leq \eta_{n,m}, \tag{19}
\]
for finding the \( \eta_{n,m} \) we need define
\[
N(t) = \max_{a \leq s \leq b, c \leq t \leq d} |K'_s(s,t)|, \quad t \in [c,d], \tag{20}
\]
and
\[
N_1 = \max_{a \leq s \leq b} \lbrace |K'_s(s,t)|: a \leq s \leq b, c \leq t \leq d \}, \tag{21}
\]
where \( N(t) \in [c,d] \) because the \( P(s, t) \) and \( P'(s, t) \in C([a,b] \times [c,d]) \).

We need divide the intervals \( [a, b] \) and \( [c, d] \) into \( n \) and \( m \) equal parts respectively. Where interval \( [a, b] \) divided by points \( s_i = a + \frac{(b-a)}{n}i, i = 0, 1, ..., n - 1 \), and interval \( [c, d] \) divided by points \( t_j = c + \frac{(d-c)}{m}j, j = 0, 1, ..., m - 1 \). From [12] we define \( \eta_{n,m} \) by following
\[
\eta_{n,m} = \sqrt{(b-a)(d-c)N_1 \frac{d-c}{m} + \sqrt{b-a}\|N(t)\|_{L_2}} \frac{b-a}{n}. \tag{22}
\]
Now we define the kernel function:
\[
K_i(t) = K(s_i, t), \tag{23}
\]
\[
K_i(s, t) = K_i(t); s_i \leq s \leq s_{i+1}, \quad t \in [c, d], i = 0, 1, ..., n - 1 \tag{24}
\]
\[
K_{nm}(s, t) = K_i(t_j); s_{i} \leq s \leq s_{i+1}, t_{j} \leq t \leq t_{j+1}, \quad i = 0, 1, ..., n - 1, j = 0, 1, ..., m - 1. \tag{25}
\]
\[ C_{n,m} v(s) = \int_a^b \overline{K}_{nm}(s,t)v(s)ds = f_\delta(t), \ t \in [c,d] \]  

(26)

Now for reducing the equation (26) to the system of linear algebraic equations, we used the discretization algorithm that defines in [12] and it has implemented in [8].

\[ C_{n,m}(v_i) = f_j^\delta, \]  

(27)

where \( v(s) \rightarrow (v_i), \ i = 0, 1, \ldots n - 1 \). \( v_i = v(s_i) \) and \( f_\delta(t) \rightarrow (f_j^\delta), j = 0, 1, \ldots m - 1 \), \( f_j^\delta = f_\delta(t_j) \).

\[
\sqrt{\frac{b-a}{n}} \begin{bmatrix}
\overline{K}_{nm}(s_0,t_0) & \overline{K}_{nm}(s_1,t_0) & \ldots & \overline{K}_{nm}(s_{n-1},t_0) \\
\overline{K}_{nm}(s_0,t_1) & \overline{K}_{nm}(s_1,t_1) & \ldots & \overline{K}_{nm}(s_{n-1},t_1) \\
\vdots & \vdots & \ddots & \vdots \\
\overline{K}_{nm}(s_0,t_{m-1}) & \overline{K}_{nm}(s_1,t_{m-1}) & \ldots & \overline{K}_{nm}(s_{n-1},t_{m-1})
\end{bmatrix} 
\begin{bmatrix}
v(s_0) \\
v(s_1) \\
\vdots \\
v(s_{n-1})
\end{bmatrix} 
= \sqrt{\frac{b-a}{n}} \begin{bmatrix}
f_\delta(t_0) \\
f_\delta(t_1) \\
\vdots \\
f_\delta(t_{m-1})
\end{bmatrix}. \]  

(28)

The problem (27) is ill-posed in the sense that the inverse operator \( C_{n,m}^{-1} \) of \( C_{n,m} \) not exist. That is meaning the numerical attempting to find a direct solution for (27) will be fail.

4. Discretization of equations (11)

We used the algorithm in section 2 with a specified derivative term for the kernel and some other conductions as explained below. We assumed \( h'''(t) = g(t) \)

\[ A[h(t)] = f(t) = h(t) + \int_0^t \sum_{n=0}^{\infty} \frac{2 \cos \left( n + \frac{1}{2} \right) \pi x_0}{(n + \frac{1}{2})^5 \pi^5} e^{-\left( n + \frac{1}{2} \right)^2 \pi^2(t-\tau)} g(\tau) \, d\tau, \]  

(29)

from (29) the kernel \( K(t, \tau) \) corresponding integral equation, and its derivative with respect of time \( K'_i(t, \tau) \) belong and continues in \( C([0,1] \times [0, T]) \) equation (29) refers to the type of equations (12). In particular, after implementing the discretization steps (12–28), we reduce the integral equation (29) to a system of linear algebraic equations.

\[ A[Bg(\tau)] = \int_0^t K(t, \tau) g(\tau) \, d\tau = f(t), \ t \in [0, T], \]  

(30)

where

\[ h(t) = Bg(\tau) = \int_0^t \frac{(t-\tau)^2}{2} g(\tau) \, d\tau, \ g(\tau), Bg(\tau) \in L_2[0, T], \]  

(31)

the operator \( C \) can be defined as following:

\[ C g(\tau) = ABg(\tau), \quad g(\tau) \in L_2[0, T], \quad C g(\tau) \in L_2[0, T], \]  

(32)

\[ K(t, \tau) = \frac{(t-\tau)^2}{2} + \sum_{n=0}^{\infty} \frac{2 \cos \left( n + \frac{1}{2} \right) \pi x_0}{(n + \frac{1}{2})^5 \pi^5} e^{-\left( n + \frac{1}{2} \right)^2 \pi^2(t-\tau)}, \]  

(33)

\[ \overline{K}_{i}(t) = K(\tau_i, t), \]  

(34)
\[ K_i(t_i); \tau_i \leq t \leq t_{i+1}, \ t \in [0, T], i = 0, 1, ..., n - 1 \]  
(35)

\[ K_{ij}(t_i, t_j); \tau_i \leq t \leq t_{i+1}, \tau_j \leq t \leq t_{j+1}, i = 0, 1, ..., n-1, j = 0, 1, ..., m - 1. \]  
(36)

\[ K_{nm}(t, \tau) = \bar{K}_i(t_i). \]  
(37)

We find finite-dimensional operator \( C_\delta \), where \( C_n \to C \)

\[ C_n g(\tau) = \int_0^t K_{nm}(t, \tau) g(\tau) \, d\tau = f(t), t \in [0, T], \]  
(38)

by implementing the algorithm in [12] we can define \( g(\tau_i) \) and \( f_0(t_j) \)

\[ C_n g(\tau_i) = f_0(t_j), \]  
(39)

where (39) equivalent the following system of linear equations

\[
\begin{pmatrix}
K(\tau_0, t_0) & 0 & \cdots & 0 \\
K(\tau_0, t_1) & K(t_1, t_1) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
K(\tau_0, t_{m-1}) & K(\tau_1, t_{m-1}) & \cdots & K(\tau_{m-1}, t_{m-1})
\end{pmatrix}
\begin{pmatrix}
g(\tau_0) \\
g(\tau_1) \\
\vdots \\
g(\tau_{m-1})
\end{pmatrix} =
\begin{pmatrix}
f_0(t_0) \\
f_0(t_2) \\
\vdots \\
f_0(t_{m-1})
\end{pmatrix},
\]  
(40)

the value of kernel depended on following condition

\[ \bar{K}_i(t_j) = \begin{cases} K(\tau_i, t_j) & \text{при } i \leq j \\ 0 & \text{при } i > j \end{cases} \]

In (1–4) we consider the inverse boundary value problem for the heat equation, we need to find \( h_0(t) \), where the \( h_0(t) \in H^4 [0, T] \)

\[ h_0(t) = B g_0(\tau), \]  
(41)

it is necessary defined the \( g_0(\tau) \in M_r \),

\[ M_r = \{ g(\tau): g(\tau) \in L^2 [0, T], \| g \| \leq r \}, \]  
(42)

Suppose we know a function \( f_0(t) \in L^2 [0, T] \), which is a solution to the direct problem.

\[ f_0(t) = u_0(x_0, t), \]  
(43)

where \( x_0 \in [0, 1], t \in [0, T] \), additionally assume we do not know the exact value of the function \( f_0(t) \) instated of the \( f_0(t) \in L^2 [0, T] \) and \( \delta > 0 \) are given such that

\[ \| f_0(t) - \tilde{f}_0(t) \|^2 \leq \delta^2. \]  
(44)

Using the given data of the problem \( f_0(t), \delta \) and \( M_r \) it is required to determine the approximate value \( h_\delta(t) \), and also get an error estimate \( \| h_\delta(t) - h_0(t) \|_{H^4} \).

5. The solution of the inverse problem by using Lavrentieva method

The operator \( C_\delta \) in (39) is non-injective operator, because the matrix of this operator is triangular matrix and \( N(C_\delta) \neq \{0\} \). In this case the Tikhonov’s regularization method is not suitable because \( N(C_\delta) \neq \{0\} \) and \( f_\delta \perp N(C_\delta) \Rightarrow C_\delta f_\delta = 0 \). Therefore, the Lavrentieva method is suggested for solving the problem (39).

This method is described in [13] and borrowed from [14]. It is based on the substitution of operator equation (39) with a family of operator equation of the second kind depending on the parameter \( \alpha > 0 \) as shown in (45).

\[ Ah + \alpha Qh = f. \]  
(45)
The equation (45) ill-posed in sense of the operator $A^{-1}$ exists and $\|A^{-1}\| = \infty$. By using the polar decomposition $\tilde{A} = \sqrt{A^*A}$, $\tilde{B} = \sqrt{BB^*}$ and $A = QA$, where $Q$ is a unitary operator. We can find $Q$ by using singular value decomposition (SVD) [15].

$$A = U\Sigma V^*, \quad Q = UV^*$$

(46)

(47)

The Lavrentieva method defined in [2 p. 14] using the regularizing family of operators $\{R_\alpha: 0 < \alpha \leq \alpha_0\}$, $R_\alpha: H \rightarrow H$ defined by the following formula

$$R_\alpha = \tilde{B}(\tilde{C} + \alpha E)^{-1}Q^*, \quad 0 < \alpha \leq \alpha_0,$$

(48)

where $\tilde{C} = \tilde{A}\tilde{B}$, but we have a finite-dimensional operator $\tilde{C}_n$ instead of $\tilde{C}$

$$\|\tilde{C}_n - \tilde{C}\| \leq \eta_n.$$ (49)

The regularizing family of finite-dimensional operators $\{R_{na} : 0 < \alpha \leq \alpha_0\}$, $R_{na} : H \rightarrow H$ defined by the formula

$$R_{na} = \tilde{B}_n(\tilde{C}_n + \alpha E)^{-1}Q_n, \quad 0 < \alpha \leq \alpha_0,$$

(50)

the regularized solution $h^a_\delta$ of the inverse problem is defined by the formula

$$h^a_\delta = R_{na}f_\delta,$$

(51)

we consider the variational problem in order to give the definition of finite-dimensional approximation in the method of M.M. Lavrentieva.

$$\inf\{\|\tilde{C}_n g + \alpha Q_n \tilde{B}_n g - f_\delta\| : g \in M_r\},$$

(52)

now we estimate the deviation $\|h^a_\delta - h_0\|$ of the approximate solution $h^a_\delta$ from the exact $h_0$

$$\|h^a_\delta - h_0\| \leq \sup\{\|h^a_\delta - h^R_\delta\| : h_0 \in L_2[0, T], \|f_\delta - A_n h_0\| \leq \delta\} + \sup\{\|h^a_\delta - h_0\| : h_0 \in L_2[0, T]\},$$

(53)

where $h^R_\delta = R_{na}f_\delta$, from (52)

$$\|h^a_\delta - h_0\| \leq \|R_{na}\|\delta + \sup\{\|R_{na}\tilde{C}_n g_0 - \tilde{B}_n g_0\|\}.$$

(54)

Next step we need to define the value of regularization parameter $\alpha(\delta)$ in (51) by the method of V.N. Strakhov [13] from the condition equation.

$$\inf\left\{\|R_{na}\|\delta + \sup\_{\|g_0\|_{L^2}}\|R_{na}\tilde{C}_n g_0 - \tilde{B}_n g_0\|\right\}$$

(55)

where $\|R_{na}\|\delta = \|h^a_\delta - h^R_\delta\|$, let define $\Delta_1(\alpha(\delta))$ and $\Delta_2(\alpha)$

$$\Delta_1(\alpha(\delta)) = \|R_{na}\|\delta,$$

(56)

$$\Delta_2(\alpha) = \sup\_{\|g_0\|_{L^2}}\|R_{na}\tilde{C}_n g_0 - \tilde{B}_n g_0\|,$$

(57)

where the $\Delta_1(\alpha(\delta)) = \Delta_2(\alpha)$ for $\alpha = \alpha(\delta)$ and $\alpha(\delta) \rightarrow 0$ for $\delta \rightarrow 0$ from that the equation (54) becomes

$$\|h^a_\delta - h_0\| \leq 2\Delta_1(\alpha(\delta)),$$

(58)

from (56)

$$2\Delta_1(\alpha(\delta)) = 2\|R_{na(\delta)}\|\delta,$$

(59)

from (58) and (59)
\[ \| h_0^g - h_0 \| \leq 2 \| R_n(\alpha) \| \delta, \]  

(60)

by using (60) we can select the best value of regularization parameter \( \alpha(\delta) \) in (51).

6. Numerical example

Considering the inverse boundary value problem (1–5) for the heat equation, we need to find the \( h_0^g(t) \in H^4[0,T] \) by using Lavrenteva method, the exact solution \( h(t) \) showed in ‘figure 1’.

![Figure 1. The exact solution for inverse problem \( h_0(t) \).](image1)

The input function for inverse problem \( u(x_0,t) = f_0(t) \) where the \( x_0 \in (0,1) \), \( t \in [0,T] \), \( x_0 = 0.5 \) and \( T = 5 \), as shown in ‘figure 2’.

![Figure 2. \( u(x_0,t) = f_0(t) \) where \( t \in [0,T] \), \( x_0 = 0.5 \) and \( T = 5 \).](image2)

We divided the intervals for kernel in (36) into \( m = 100 \) and \( n = 100 \). We considered the \( \delta \) and by implanting the Lavrenteva method we obtained some solutions as shown in the ‘figure 3’ and ‘figure 4’.
7. Conclusion
This work deals with the algorithm for solving the boundary value problem for the heat equation and some results have been collected. This problem is an ill-posed problem and a special method needs to solve such a problem. The separation of variables method used to represent the partial differential equation as an integral equation of the first kind. By using the discretization method we converted the integral equation to a system of linear equations or linear operator equation first kind. Operator $C$ of this problem is a non-injective operator this led to the null-space self-adjoint operator $N(C^*) \neq \emptyset$. Tikhonov’s regularization method is not suitable. Therefore, the Lavrenteva method is suggested for solving this problem we convert the linear operator equation first kind to the second kind by adding unitary operator $Q$ by using the polar decomposition multiplied by regurgitation parameter $\alpha$.

References
[1] Kabanikhin S I 2012 Inverse and Ill-Posed Problems: Theory and Applications (Inverse and Ill-Posed Problems Ser. 55) (De Gruyter) p 24
[2] Tikhonov A N 1963 On the solution of incorrectly posed problem and the method of regularization Sov. Math. 4 1035–1038
[3] Menikhes L D and Tanana V P 1998 The finite-dimensional approximation for the Lavrent'ev method Sib. Zh. Vychisl. Mat 1 59–66
[4] Ivanov V K 1968 About Application of Picard Method to the Solution of Integral Equations for the First Kind Bui. Inst. Politehn. Iasi. 4 (34) 71–78
[5] Sidikova A I 2019 A Study of an Inverse Boundary Value Problem for the Heat Conduction Equation Sibirskii Zhurnal Vychislitel'noi Matematiki 22 (1) 79–95
[6] Wang H and Li Y 2018 Numerical solution of an inverse boundary value problem for the heat equation with unknown inclusions Journal of Computational Physics 369 1–15
[7] Dmitriev V I and Stolyarov L V 2017 Numerical Method for the Inverse Boundary-Value Problem of the Heat Equation *Comput Math Model* **28** 141–147

[8] Al-Mahdawi H K 2019 Development of a Numerical Method for Solving the Inverse Cauchy Problem for the Heat Equation *Bulletin of the South Ural State University. Series: Computational Mathematics and Software Engineering* **8** (2) 22–31

[9] Al-Mahdawi H K 2019 Studying the Picard’s Method for Solving the Inverse Cauchy Problem for Heat Conductivity Equations *Bulletin of the South Ural State University. Series: Computational Mathematics and Software Engineering* **8** (4) 5–14

[10] Tanana V P 1975 On the optimality of methods of solving nonlinear unstable problems *Dokl. Akad. (Nauk SSSR)* **220** (5) 1035–1037

[11] Tanana V P 1979 On an iterative projection algorithm for solving ill posed problems equations with perturbed operator *Dokl. (AN SSSR)* **224** (15) 1025–1029

[12] Tanana V P and Sidikova A I 2017 On Estimating the Error of an Approximate Solution Caused by the Discretization of an Integral Equation of the First Kind *Steklov Institute of Mathematics* **299** (1) 217–224

[13] Tanana V P and Sidikova A I 2018 *Optimal Methods for Solving Ill-Posed Heat Conduction Problems* (Inverse and Ill-Posed Problems Series vol 62) (De Gruyter)

[14] Lavrent’ev M M 1963 *On Some Ill-Posed Problems of Mathematical Physics* (Novosibirsk: Siberian Branch of the Academy of Sciences of the USSR)

[15] Gallier J 2011 *Singular Value Decomposition (SVD) and Polar Form BT - Geometric Methods and Applications: For Computer Science and Engineering* (New York: Springer) pp 367–385