Complex Hyperbolic Geometry and Hilbert Spaces with the Complete Pick Property

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Abstract

Suppose $H$ is a finite dimensional reproducing kernel Hilbert space of functions on $X$. If $H$ has the complete Pick property then there is an isometric map, $\Phi$, from $X$, with the metric induced by $H$, into complex hyperbolic space, $CH^n$, with its pseudohyperbolic metric. We investigate the relationships between the geometry of $\Phi(X)$ and the function theory of $H$ and its multiplier algebra.

Key Words: complete Pick property; complex hyperbolic space

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1 Introduction and Summary

We begin with an informal overview; definitions and details are in the later sections.

The Hilbert spaces in this paper are finite dimensional.

Suppose $H$ is a reproducing kernel Hilbert space, RKHS, of functions on a set $X$. If $H$ has the complete Pick property, CPP, then there is a map, $\Phi$, of $X$ into the complex unit ball, $\mathbb{B}^n \subset \mathbb{C}^n$, so that $H$ is equivalent to $DA_n (\Phi(X))$, the subspace of the Drury Arveson space, $DA_n$, generated by the $DA_n$ kernel functions for points of $\Phi(X)$. The map $\Phi$ is an isometry, mapping $X$ with the
metric induced by $H$ to $\Phi(X)$ with the metric induced by $DA_n$. That latter metric is the restriction to $\Phi(X)$ of the pseudohyperbolic metric on $\mathbb{B}^n$. Thus, the passage from $H$ to $\Phi(X)$ establishes a correspondence between finite dimensional RKHS with the CPP and finite subsets of complex hyperbolic space, $\mathbb{CH}^n$. In this paper we study the relationship between the analytic structure of $H$ and its multiplier algebra, and the geometry of $\Phi(X)$.

Because we are working at the interface of different areas our expository material is richer than usual.

The next section contains notation and background about Hilbert spaces and about complex hyperbolic space. In the section after that we introduce and develop numerical invariants of RKHS; some are based on Gram matrix entries, others are defined using extremal problems in $H$ or its multiplier algebra. We also discuss related geometric invariants of sets in hyperbolic space. In Section 4 we consider rescalings of a RKHS $H$. For $H$ with the CPP we establish a close relation between rescalings and the action of the automorphism group of hyperbolic space on the associated set $\Phi(X)$. We also discuss other ways of modifying a RKHS to produce new spaces, and establish a relation between those modifications and conjugation operators on the Hilbert space.

Section 5 contains results about the existence and properties of the embedding $\Phi$. We begin with an analysis of a strengthened version of the triangle inequality which must hold if there is such a $\Phi$, but is not a sufficient condition. We then consider two dimensional $H$; there the embedding is always possible and is easy to describe. For a three dimensional $H$ many aspects of the general finite dimensional case appear, the most important being that the embedding may not be possible; for it to be possible $H$ must have the Pick property. To go beyond three dimensional spaces we use induction on dimension. The induction step resembles the three dimensional construction but it requires that $H$ have the more subtle complete Pick property.

If $H$ is three dimensional then describing $\Phi(X)$ up to automorphisms of $\mathbb{CH}^2$ is a version of the question studied systematically by Brehm [B] of describing congruence classes of triangles in complex hyperbolic space. He parameterizes those classes by the three distances between pairs of vertices and a fourth quantity, the ”shape invariant”, an invariant from hyperbolic geometry that has no Euclidean analog. In particular, even when $\Phi(X)$ only has three points, its geometric structure is richer than that encoded in its metric. One of our implicit goals is to understand this geometry which is more general than metric geometry.

In Section 6 we relate the Hilbert space invariants we introduced to the geometry of $\Phi(X)$. In particular we describe conditions on the numerical invariants which correspond to the having $\Phi(X)$ inside a single geodesic or inside a totally geodesically embedded copy of the Poincare disk, $\mathbb{CH}^1$, or of the real hyperbolic plane $\mathbb{RH}^2$. We also establish a relationship between $\Phi(X)$ being in a copy of the Poincare disk and $H$ having a conjugation operator which interchanges the basis of reproducing kernels with its dual basis.

In Section 7 we consider a class of RKHS of functions on trees and describe the associated maps $\Phi(X)$. That class of spaces includes the dyadic Dirich-
let space studied systematically in [ARS02] as well as more recent variants [ARSW18].

In Section 8 we briefly discuss the multiplier algebra of $H$. If $H$ has the CPP then one can recover $H$ using numerical data derived from its multiplier algebra. Less is known about general $H$.

A brief final section contains remarks and questions.

2 Background and Notation

General background on the material we discuss is in [Ru], [Go], [AM], [Sa], [Sh], and [ARSW18].

2.1 RKHS

An $n$ dimensional reproducing kernel Hilbert space, RKHS, is an $n$ dimension-

cal complex inner product space, $H$, together with a distinguished basis, $B = B(H) = \{k_i\}_{i=1}^n$, of vectors called reproducing kernels. Associated with $H$ is a set $X = X(H)$ with the same index set as $B$. Vectors $h \in H$ can be, and generally are, regarded as functions on $X$ by setting, for $x \in H$, $h(x) = \langle h, k_x \rangle$.

We suppose throughout that $H$ is irreducible; that is, for any $x, y \in X(H)$, $x \neq y$, the functions $k_x$ and $k_y$ are linearly independent and $\langle k_x, k_y \rangle \neq 0$.

Sometimes we will write $H$ for such a space without further comment, and, also without comment, write $X = \{x_i\}_{i=1}^n$ for $X(H)$. If $x = x_i$ and $y = x_j$ are in $X(H)$ we may write $k_x$ or $k_i$ for $k_{x_i}$, and write $k_{xy} = k_{x_j}$ for $\langle k_i, k_j \rangle = k_{x_i}(x_j)$.

We will denote the normalized kernel $k_i^{-1}\cdot k_i$ by $\hat{k}_i$, and write the inner product of two such as $\langle \hat{k}_x, \hat{k}_y \rangle$. The Gram matrix of $H$ is the positive $n \times n$ matrix $G(H) = (\langle k_i, k_j \rangle)$.

The function $\delta = \delta_H$ defined by; for $x_i, x_j \in X(H)$,

$$\delta^2_{i,j} = \delta^2_H(x_i, x_j) = 1 - \frac{|k_{ij}|^2}{k_{ii}k_{jj}} = 1 - |\hat{k}_{ij}|^2, \quad (1)$$

is a metric on $X(H)$. It is an elementary exercise that the same quantity is described by

$$\delta_H(x, y) = \delta(x, y) = \frac{1}{\|k_x\|} \sup \{ \text{Re} \ h(x) : h \in H, h(y) = 0, \|h\|_H \leq 1 \}. \quad (2)$$

Also, it is not hard to show that, with $P_x$ denoting the orthogonal projection onto the span of the kernel function $k_x$, $\delta_H$ can be described in terms of operator norms:

$$\delta_H(x, y) = \|P_x - P_y\|. \quad (3)$$

The metric $\delta$ is a generalization of the classic pseudohyperbolic metric, $\rho$, on the disk. If $H$ is the Hardy space $H^2$ then, on the unit disk $\delta_H = \rho$. For more about $\delta_H$ see [ARS07], [ARSW18].
2.2 Multiplier Algebras

Given a symbol function, $m$, defined on $X(H)$, the associated multiplier operator, $M_m$, is the linear operator on $H$ defined by, for $h \in H, x \in X, (M_m h)(x) = m(x)h(x)$. The collection of all multiplier operators on $H$ is the multiplier algebra of $H$, Mult($H$). With the operator norm Mult($H$) is a commutative Banach algebra generated by $n$ orthogonal idempotents. We denote its spectrum, its maximal ideal space, by Spec (Mult($H$)). The Gleason metric on the spectrum is defined by, for $x, y \in \text{Spec}(\text{Mult}(H))$,

$$
\delta_G(x, y) = \sup \{ \text{Re } m(x) : m \in \text{Mult}(H), m(y) = 0, ||m||_{\text{Mult}(H)} \leq 1 \}.
$$

(4)

It is an exercise in the use of von Neumann’s inequality that $\delta_G$ can also be described using the pseudohyperbolic metric $\text{ARSW15}$.

$$
\delta_G(x, y) = \sup \{ \rho(m(x), m(y)) : m \in \text{Mult}(H), ||m||_{\text{Mult}(H)} \leq 1 \}.
$$

(5)

Identifying $x_i \in X(H)$ with the maximal ideal of multipliers which vanish at $x_i$ gives a natural identification of $X(H)$ with Spec$(\text{Mult}(H))$. Using this identification we also regard $\delta_G$ as a metric on $X(H)$.

2.3 Rescaling and Invariance

We want to note when two RHKS are the same in a natural sense. We do this with the equivalence relation rescaling.

Suppose $H$ and $\tilde{H}$ are two RKHS of the same finite dimension. We say that $\tilde{H}$ is a rescaling of $H$, or is obtained from $H$ by rescaling, and write $H \sim \tilde{H}$, if there is a one to one map $\Xi : X(H) \rightarrow X(\tilde{H})$ and a nonvanishing complex valued function $\gamma$ defined on $X(H)$ so that, with $\{k_i\}$ and $\{\tilde{k}_i\}$ denoting the kernel functions for $H$ and $\tilde{H}$ respectively, we have for all $x \in X(H)$,

$$
\tilde{k}_{\Xi(x)}(\Xi(\cdot)) = \gamma(x)k_x(\cdot);
$$

(6)

or, equivalently, $\forall x, y \in X(H)$

$$
\tilde{k}_{\Xi(x)}\Xi(y) = \langle \tilde{k}_{\Xi(x)}, \tilde{k}_{\Xi(y)} \rangle = \gamma(x)\gamma(y)k_{xy}.
$$

(7)

Another equivalent formulation is that the linear map $A : H \rightarrow \tilde{H}$ defined by $A(k_i) = \tilde{k}_{\Xi(x)}$ and linearity has the property that $A^* A$ is diagonalized by the $\{k_i\}$ and has nonzero eigenvalues.

Rescaling is an equivalence relation, more details about it are [AM] Sec. 2.6. If $H \sim \tilde{H}$ we can use $\Xi$ to identify $X(\tilde{H})$ with $X(H)$, thus reducing to the case of $X(H) = X(\tilde{H})$ and $\Xi$ the identity map. We may do this without mention.

Associated with $\tilde{H}$ is the new Gram matrix, $G(\tilde{H})$. If $X(H) = X(\tilde{H})$ and if $\Xi$ is the identity map then $G(H)$ and $G(\tilde{H})$ are related by

$$
\Gamma(\gamma(x_1), \ldots, \gamma(x_n)) G(H) \Gamma(\gamma(x_1), \ldots, \gamma(x_n)) = G(\tilde{H}).
$$

(8)
Here $\Gamma(c_1, \ldots, c_n)$ is the $n \times n$ matrix with $c_1, \ldots, c_n$ on the diagonal and zeros elsewhere. If matrices $A$ and $B$ are related in this way then we will write $A \sim B$. Different choices, $\Gamma = \Gamma_1$ and $\Gamma = \Gamma_2$, produce different $G(\tilde{H})$ unless $\Gamma_1 = \alpha \Gamma_2$ for some unimodular $\alpha$.

One convenient rescaling is the basepoint rescaling. A point $x \in X(H)$ is selected as basepoint and $H$ is rescaled so that the rescaled kernel for $x$ is identically one. The Gram matrix of the rescaled space will have ones in the row and column corresponding to $y$. Two spaces are equivalent under rescaling if and only if they have the same Gram matrix after basepoint rescaling. Another useful rescaling is normalized kernels rescaling in which all the kernel functions are rescaled to be unit vectors. That rescaled space has a Gram matrix with all ones on the diagonal. That rescaling becomes unique after a further rescaling to insure, for instance, that the entries in the first row of the Gram matrix are real. We will encounter a different type of rescaling in the proof of Theorem 11.

We will call quantities built from $H$ invariant if they are unchanged under rescaling. For instance, neither the Gram matrix entries, $k_{ij}$, nor the normalized kernel functions $\hat{k}_i$ are invariant, but both $|\hat{k}_{xy}|$ and $\delta_H(x, y)$ are invariant. The Gram matrix of the basepoint normalized rescaling is invariant as is Gram matrix of the normalized kernel rescaling once it is further rescaled so that the first row is real. The multiplier algebra is invariant. That is, if $H \sim \tilde{H}$, then Mult $(H)$ and Mult $(\tilde{H})$ are the same sets of functions with the same algebraic structure and with the same norm.

Some statements which are not invariant under rescaling can be viewed as the specializations of invariant statements obtained by basepoint rescaling. For example, the statement $k_{xx} = k_{xy}$ is not invariant. However the statement

$$\frac{k_{xx}^2}{k_{xx}k_{ax}k_{aa}} = \frac{k_{xy}^2}{k_{xx}k_{ay}k_{ay}}$$

is invariant; and, after basepoint rescaling with $a$ as the basepoint, specializes to $k_{xx} = k_{xy}$.

There is an interesting discussion of this type of transformation in [Go, 7.2.3].

2.4 The Complete Pick Property and the Spaces $DA_n$

We are particularly interested in spaces $H$ with the CPP. There is a substantial literature on this class and we will take what we need from [AM], [Sa], [Sh], and [ARSW18].

The Pick property is an extension property for multipliers. Suppose an $n$–dimensional RKHS $H$ is given, $n \geq 2$, along with a subset $Y$ of $X(H)$. Let $H_Y$ be the RKHS that is the span of $\{k_y\}_{y \in Y}$. Given $M = M_m \in \text{Mult} (H)$, $\|M\| \leq 1$, we can define a multiplier $M_Y$ on $H_Y$ by restricting $m$, which is a function on $X(H)$, to a function, now called $m_Y$, on the subset $Y = X(H_Y) \subset X(H)$. We define $M_Y$ to be the multiplier on $H_Y$ with symbol function $m_Y$. The adjoint, $M_Y^*$, is the restriction of $M^*$ to the $M^*$ invariant subspace $H_Y$; hence $\|M_Y^*\| \leq 1$, and thus, also, $\|M_Y\| \leq 1$. The extension problem which defines
the Pick property is the converse question. Given $N_Y$, a multiplier on $H_Y$ of norm one, is there a multiplier $M$ on $H$, $\|M\| = 1$, so that $N_Y = M_Y$? If this question always has a positive answer then $H$ is said to have the Pick property (or the scalar Pick property). The stronger and more subtle CPP is defined by also having a positive answer to the matricial analog of that multiplier extension question.

In many cases below where we hypothesize that a space has the CPP, it would suffice to just assume the Pick property. We leave it to the interested reader to note those refinements as we go.

The Drury Arveson spaces, $DA_n$, are fundamental example of spaces with the CPP. Let $B_n \subseteq C^n$ be the ball in complex $n-$space, and denote the inner product on $C^n$ by $\langle \cdot, \cdot \rangle$. The space $DA_n$ is the RKHS of holomorphic functions on $B_n$ defined by the reproducing kernels $\{k_z(\cdot) = (1 - \langle \cdot, z \rangle)^{-1} : z \in B^n\}$. In particular $DA_1$ is the classical Hardy space $H^2$ on the unit disk. These spaces are discussed in detail in [Sh].

For any finite $Y \subseteq B_n = X(\text{DA}_n)$ let $DA_n(Y)$ be the subspace of $DA_n$ spanned by the subset $\{k_y\}_{y \in Y}$ of the $DA_n$ reproducing kernels. Each of these spaces inherits the CPP from its containing $DA_n$. Any $H$ with the CPP is a rescaling of a space $DA_n(X)$ and that fact is the starting point for our discussions.

**Theorem 1 ([AM, Thm. 8.2])** A finite dimensional RKHS $H$ has the complete Pick property if and only if there is a finite set $X$ in some $\mathbb{CH}^n$ such that $H \sim DA_n(X)$.

Thus, associated to any such $H$ is a map $\Phi$ of $X(H)$ into $\mathbb{CH}^n$ so that $H \sim DA_n(\Phi(X(H)))$. Our interest here is is the relation between the structural properties of $H$ and $\text{Mult}(H)$ and the geometry of $\Phi(X(H))$.

### 2.5 Complex Hyperbolic Space

We now discuss $\mathbb{CH}^n$, complex hyperbolic $n-$space. Our basic reference is [Ga].

We begin with $\mathbb{CH}^1$. The unit disk, $D = B^1 \subseteq C$, is a complex manifold which has a transitive group of holomorphic automorphism, $\text{Aut}(B^1)$, the Mobius maps of the disk to itself. $\mathbb{CH}^1$ carries the $\text{Aut}(B^1)$ invariant pseudohyperbolic metric, $\rho$,

$$\rho(z, w) = \frac{|z - w|}{|1 - \overline{w}z|},$$

which can also be defined by setting $\rho(0, z) = |z|$ and requiring that $\rho$ be $\text{Aut}(B^1)$ invariant. The complex manifold $B^1$, together with the metric $\rho$, and the isometry group $\text{Aut}(B^1)$ is the disk model of one dimensional complex hyperbolic space, $\mathbb{CH}^1$. The metric $\rho$ is not a length metric. The length metric which it induces, the Bergman-Poincare metric, is an $\text{Aut}(B^1)$ invariant Riemannian metric of constant curvature $-1/4$. (Care is needed here, the Bergman-Poincare metric is sometimes defined to be twice what we just offered,
in which case it has constant curvature $-1$. Our choice here ensures that $\beta$ is the length metric induced by $\rho$.) The full set of isometries of $\mathbb{CH}^1$ consists of the holomorphic isometries of $\text{Aut}(\mathbb{B}^1)$ and the complex conjugates of elements of $\text{Aut}(\mathbb{B}^1)$. For $X, Y \subset \mathbb{CH}^1$ we say $X$ and $Y$ are congruent, $X \sim Y$, if there is $\Lambda \in \text{Aut}(\mathbb{B}^1)$ with $X = \Lambda Y$. If $X$ and $Y$ are ordered sets we take the terminology and notation to include the requirement that $\Lambda$ respect the ordering.

Similar facts on the unit ball, $\mathbb{B}^n \subset \mathbb{C}^n$, give a model for complex hyperbolic $n$–space, $\mathbb{CH}^n$. Details about the ball are in [Ru], about the metric $\rho$ in [DW]. We just list some highlights.

The ball has a transitive group of holomorphic automorphisms, $\text{Aut}(\mathbb{B}^n)$. For each $a \in \mathbb{B}^n$ there is a $\phi_a \in \text{Aut}(\mathbb{B}^n)$, an involution of $\mathbb{B}^n$ which interchanges $0$ and $a$. Every unitary map of $\mathbb{C}^n$ is in $\text{Aut}(\mathbb{B}^n)$ and the unitary maps together with the involutions generate $\text{Aut}(\mathbb{B}^n)$. In particular, any automorphism which fixes the origin is given by a unitary map. As with $n = 1$, for $X, Y \subset \mathbb{CH}^n$ we will write $X \sim Y$ if there is an element of $\text{Aut}(\mathbb{B}^n)$ which takes $X$ to $Y$. Also, as with $n = 1$, there are $\rho$–isometries of $\mathbb{CH}^n$ which are not holomorphic but are complex conjugates of elements of $\text{Aut}(\mathbb{B}^n)$.

The pseudohyperbolic metric, $\rho$, on the ball can be defined by saying that for $z, w \in \mathbb{B}^n$ we have $\rho(z, w) = |\phi_z(w)| = |\phi_w(z)|$. Alternatively we can set $\rho(0, z) = |z|$ for $z \in \mathbb{B}^n$ and require that $\rho$ is $\text{Aut}(\mathbb{B}^n)$ invariant. The length metric generated by $\rho$ is $\beta$, the Bergman-Poincare metric; a Riemannian metric which is invariant under $\text{Aut}(\mathbb{B}^n)$ and agrees infinitesimally with the Euclidean metric at the origin. In contrast to one dimensional complex hyperbolic space, $\mathbb{CH}^1$, and to real hyperbolic $n$–space, $\mathbb{RH}^n$, the space $\mathbb{CH}^n$ with the metric $\beta$ does not have constant sectional curvature. This lack of isotropy is a fundamental feature in the metric geometry of $\mathbb{CH}^n$.

This same model of $\mathbb{CH}^n$ has an alternative description which is often used in geometric studies. In that model $\mathbb{CH}^n$ is defined as the set of "negative points in projective space". Begin with $\mathbb{C}^{n+1}$ and the Hermitian form $[\cdot, \cdot]$ of signature $(n, 1)$ given by

$$([x_1, x_2, ..., x_{n+1}], [y_1, y_2, ..., y_{n+1}]) = -x_{n+1}y_{n+1} + \sum_{i=1}^{n} x_i y_i,$$

Next, form the projective space $\mathbb{CP}^n$ from this $\mathbb{C}^{n+1}$. Although $[\cdot, \cdot]$ is not well defined on $\mathbb{CP}^n$, the quantity $[x, x]$ is always real and its sign is constant on lines in $\mathbb{C}^{n+1}$. Thus, this sign is well defined on $\mathbb{CP}^n$ and we define $\mathbb{CH}^n$ to be the subset of $\mathbb{CP}^n$ on which it is negative. That will never happen on a line which has $x_{n+1} = 0$, hence we can focus on the coordinate chart where $x_{n+1} \neq 0$. There we can use the inhomogenous coordinates on projective space obtained by representing points using $(n + 1)$–tuples with $x_{n+1} = 1$, and then abusing notation by writing $(x_1, x_2, ..., x_n)$ for $(x_1, x_2, ..., x_n, 1)$. In those coordinates the set of negative points, $\mathbb{CH}^n$, is \( \{(x_1, x_2, ..., x_n) : \sum |s_i|^2 < 1 \} = \mathbb{B}^n \). In those coordinates, we regard $[\cdot, \cdot]$ as being defined on $\mathbb{CH}^n$ by

$$[x, y] = ([x_1, x_2, ..., x_n], [y_1, y_2, ..., y_n]) = -1 + \sum_{i=1}^{n} x_i y_i, = -(1 - (x, y)), \]
In particular the $DA_n$ kernel functions can be written as $k_{xy} = -1/[y,x]$ and that relation allows translation between what we do here and the literature centered on the geometry of $\mathbb{CH}^n$.

In this description of hyperbolic space the automorphisms of $\mathbb{CH}^n$, which define the geometry of the model, are taken to be those natural automorphism of $\mathbb{CP}^n$ which preserve this set of negative points. Although it is not obvious, these are the same as the automorphism in $\text{Aut}(\mathbb{R}^n)$ which were discussed earlier, and so we have the same model.

We call properties of sets in $\mathbb{CH}^n$ invariant if they are preserved by automorphisms. Thus a set’s being in a geodesic is an invariant statement, that two geodesics cross at a right angle is not.

### 2.5.1 Invariant Submanifolds

We will be interested in some classes of submanifolds of $\mathbb{CH}^n$ which are preserved by automorphisms. Geodesic arcs are the totally geodesic submanifolds of $\mathbb{CH}^n$ of real dimension one. Because automorphisms are isometries they map geodesics to geodesics, and similarly for higher dimension totally geodesic submanifolds. In particular, the class of geodesic segments is invariant.

There are two classes of totally geodesic submanifolds of real dimension two and both are preserved by automorphisms. The first consists of totally real totally geodesic submanifolds. The slice $J = \{(x,y,0,...,0) : x,y \in \mathbb{R}, |x|^2 + |y|^2 < 1\}$ is a model case. The general elements of this class, which we will call real geodesic disks, are the images of $J$ under the action of $\text{Aut}(\mathbb{R}^n)$. $J$ is isometric to the real hyperbolic plane, $\mathbb{RH}^2$; however it is the Beltrami-Klein model of that plane, not the more familiar Poincare model. In the Beltrami-Klein model the geodesics are Euclidean straight line segments. The Poincare model is a conformal model of $\mathbb{RH}^2$, the Beltrami-Klein model is not. More discussion and useful figures are in [Go, Section 3.1.9].

The other class of totally geodesic submanifolds of real dimension two consists of complex geodesics. The horizontal slice $L = \{|z,0,...,0| : |z| < 1\}$, which is isometric to $\mathbb{CH}^1$, is a model case, the others are the images of $L$ under the automorphism group.

These classes also have higher dimensional analogs.

### 3 Numerical Parameters

Fix, for this section: a RKHS $H$ with associated set $\{x_i\}_{i=1}^n = X = X(H)$, kernel functions $\{k_i\}_{i=1}^n$, and multiplier algebra $A = \text{Mult}(H)$.

We are not supposing that $H = DA_n(X)$. However if $H$ is of that form then, recalling the relation $k_{xy} = -1/[y,x]$ discussed in Section 2.3 the parameters we describe can also be regarded as functionals of $X$. Furthermore, noting the discussion in Section 2.3 the values only depend on the congruence class of $X$. “
3.1 Invariant Parameters From the Gram Matrix

The Gram matrix of $H$ is $G(H) = (k_{ij})_{i,j=1}^n$. Those matrix entries change when $H$ is rescaled, but there are quantities built from those numbers which are invariant under rescaling and which will be useful. The first is the distance function $\delta = \delta_H$ which we introduced in (1). Here are several others.

3.1.1 The Angular Invariant

For $x, y, z \in X(H)$ we define the angular invariant $A(x, y, z)$, by

$$A(x, y, z) = \arg k_{xy}k_{yz}k_{zx} = \arg \widehat{k}_{xy}\widehat{k}_{yz}\widehat{k}_{zx}.$$ (9)

where we take $|\arg (\zeta)| \leq \pi$. (When working with classical function spaces, the ambiguity in $\arg (\cdot)$ is often removed by specifying that $\arg k_{xy}$ is a continuous function of both variables and vanishes when $x = y$. In that case, as shown by the family of spaces of holomorphic functions on the disk with $k_z(w) = (1 - zw)^{-\lambda}$ for $\lambda > 0$, there is no natural upper bound for $A$.) As with $\delta_{ij}$, we will write $A_{ijk}$. The interpretation of these invariants is subtle, we discuss it in Sections 5.3.2 and 5.3.3.

The $A_{ijk}$ are unchanged by rescaling of $H$, and unchanged by cyclic permutation of the indices, however they change sign when adjacent indices are interchanged. Also, it is straightforward from the definitions that $A$ satisfies a cocycle identity; for indices $i, j, k, l$

$$A_{i,j,k} - A_{i,j,l} + A_{i,k,l} - A_{j,k,l} = 0.$$ (10)

3.1.2 MQ Matrices

In Section 6.2 we will work with the matrices $MQ_r(H)$ used by McCullough and Quiggen in characterizing $H$ with the CPP [AM, Thm 7.6].

Suppose $H$ is $n$ dimensional. For $1 \leq r \leq n$ define the $(n-1) \times (n-1)$ matrices $MQ_r(H)$, by

$$MQ_r(H) = MQ_r(X(H)) = \left( 1 - \frac{k_{ir}k_{jr}}{k_{ij}k_{rr}} \right)_{1 \leq i,j \leq n \atop i \neq r, j \neq r}.$$ (11)

3.1.3 LF

Later we will also find the following invariants useful:

$$LF^2_{123} = LF^2(x_1, x_2, x_3) = \frac{1}{\delta_{12}^2} \left| 1 - \frac{k_{21}k_{13}}{k_{23}k_{11}} \right|^2 = \frac{1}{\delta_{12}^2} \left| 1 - \left| \frac{k_{21}k_{13}}{k_{23}k_{11}} e^{iA_{213}} \right|^2, \right.$$ (12)

with similar notation for other indices. We will describe the geometric interpretation of this quantity and the reason for its name in Section 5.3.1.
3.2 Describing Spaces and Counting Parameters

Describing $H$ requires that we specify the basis $\{k_i\}$ of $\mathbb{C}^n$ and that requires $n^2$ complex parameters, $2n^2$ real parameters. If we are only interested in the $\{k_{ij}\}$, the entries of $G(H)$, then the number is reduced; $G(H)$ is positive and hence determined by $n^2$ real parameters. Further, if we only consider equivalences classes modulo rescaling, then we have larger equivalence classes and fewer parameters. The rescaling is determined by the matrix $\Gamma$ in (8). That matrix is determined by $2n$ real parameters, but the comment there about $\alpha$ shows that the rescaling is actually described by $2n - 1$ parameters. Thus, our count of real parameter is $n^2$ for the Gram matrices, diminished by $2n - 1$ for possible rescalings, a total of $(n - 1)^2$. That is also the number of parameters required to describe a configuration of $n$ points in complex hyperbolic space modulo automorphisms.

We are particularly interested in describing $H$, up to rescaling, using geometric data about $X(H)$. The $\{\delta_{ij}\}$ are part of the answer, but, already in dimension $n = 3$, there are too few of them. The previous discussion suggests we need four parameters, and the distances only provide three. For a fourth we will use the angular invariant $A(x, y, z)$ given in (9).

For instance, a three dimensional space can be rescaled as a space $H$ with Gram matrix

$$
G(H) = \begin{pmatrix}
1 & 1 & 1 \\
1 & k_{22} & k_{23} \\
1 & k_{23} & k_{33}
\end{pmatrix}
$$

(13)

with $k_{22}, k_{33} > 0$. Thus the set $\kappa = \{k_{22}, k_{33}, \text{Re } k_{23}, \text{Im } k_{23}\}$ is a set of $(3 - 1)^2 = 4$ real numbers which determine the Gram matrix, and hence describes $H$ up to rescaling. The set $\delta = \{\delta_{12}, \delta_{13}, \delta_{23}, A_{123}\}$ carries the same information. We can write the elements of $\delta$ in terms of the elements of $\kappa$.

$$
\delta = \left\{1 - \frac{1}{k_{22}}, 1 - \frac{1}{k_{33}}, 1 - \frac{|k_{23}|^2}{k_{22}k_{33}}, \text{arg } k_{23}\right\};
$$

and the passage from $\delta$ to $\kappa$ is similarly straightforward. Our preference here is for the set of invariants $\delta$. Those numbers are invariant under rescaling and they also determine the Gram matrix of a rescaled version of $H$. Furthermore, if $H$ has the CPP and hence is of the form $H \sim DA_m(X)$ for some $X$, those numbers are geometric invariants of $X$ which determine $X$ up to congruence (Theorem 16 below).

A similar analysis holds if $H$ is $n$-dimensional. After basepoint rescaling $G(H)$ is determined by the $(n - 1)^2$ real parameters

$$
J(X) = \{\delta_{ij} : 1 \leq i < j \leq n\} \cup \{A_{1rs} : 1 < r < s \leq n\}.
$$

(14)

Again, these numbers are rescaling invariants and it is mechanical to pass between this set and the entries of $G(H)$. Taking note of Theorems 16 and 17 we see that if $H \sim DA_m(X)$ then these numbers also determine the congruence class of $X$. 

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3.3 Larger Spaces

Suppose we are given spaces \( H \subset H' \). Given \( x, y \in X(H) \subset X(H') \) we could measure the distance between \( x \) and \( y \) two ways; \( \delta_H(x, y) \) and \( \delta_{H'}(x, y) \). In fact, however, those two values are the same, and a similar comment holds for many of the invariants we consider. The invariant \( \delta_H \) defined by (1), as well as \( A_{ijk} \), and \( LF_{ijk} \), are defined using entries of the Gram matrix \( G(H) \) and those matrix entries do not change when \( G(H) \) is included in the natural way as a submatrix of \( G(H') \). Other invariants, such \( \delta_H \) defined using (2) or \( \Delta_H \) defined in (15) below, are defined using extremal problems which involve quantifying over all elements of \( H \). In those cases, the analogous extremal problem for \( H' \), involving quantifying over all of \( H' \), is not formally equivalent to the first problem. However in the problems we consider the two different extremal problems produce the same extremal value. That happens because in those problems if \( h' \in H' \) is a candidate to solve the extremal problem formulated in \( H' \), then \( h \), the orthogonal projection of \( h' \) onto \( H \), will give a superior candidate, one that meets the same conditions and has smaller norm. In those cases the larger set of candidates affects neither the value of the extremal, nor even the identity of the extremal function.

The situation with invariants such as \( \delta_G \) defined in (4) and \( \Delta_G \) defined in (16) is more subtle. It \( H \) is a subspace of \( H' \) then, algebraically, \( \text{Mult}(H) \) is the quotient of \( \text{Mult}(H') \) by the ideal of functions which vanish on \( X(H') \setminus X(H) \). However, in general there is no reason that the quotient norm should agree with the operator norm on \( \text{Mult}(H) \), which is what would insure that the values of \( \delta_G \) and \( \Delta_G \) were not influenced by bringing the larger space \( H' \) into consideration. In fact, it is exactly the statement that \( H' \) has the CPP which insures that the quotient norm for \( \text{Mult}(H) \) is the same as that operator norm. In all the cases where we consider a space \( H \) and there is a larger, containing, space \( H' \) lurking in the discussion, this will be the case.

3.4 Extremal Problems and Generalized Distances

We described distance \( \delta_G \) and \( \delta = \delta_H \) on \( X \) in terms of extremal problems (4) and (2). We now introduce generalizations of those quantities. For \( x, y, z \in X \) set

\[
\Delta_H(x; y, z) = \frac{1}{\| k_x \|} \sup \{ \text{Re} j(x) : j \in H, j(y) = j(z) = 0, \| j \|_H \leq 1 \}.
\]

\[
\Delta_G(x; y, z) = \sup \{ \text{Re} m(x) : m \in A, m(y) = m(z) = 0, \| m \|_A \leq 1 \}.
\]

Both of these are invariant.

Suppose \( m_\delta \in \text{Mult}(H) \) and \( h_\delta \in H \) are the functions which attain the extreme values in (4) and (2) respectively. It then follows from the definitions that \( m_\delta \tilde{k}_x \) is a competitor for the extremal problem which defines \( h_\delta \), and hence \( \delta_G \leq \delta_H \). A completely analogous argument, with \( M_\Delta \in \text{Mult}(H) \) and \( H_\Delta \in H \) the extremal functions for the problems (16) and (15), shows that \( \Delta_G \leq \Delta_H \).
The distinctive feature of RKHS with the CPP is that there is a particularly close relation between extremal problems in the multiplier algebra and in the space. In the particular case we just described the two inequalities are, in fact, equalities. The following is a special case of [AM, Theorem 9.33].

**Proposition 2** If $H$ has the CPP then the functions $m_{\delta}, h_{\delta}, M_{\Delta}$ and $H_{\Delta}$ are unique and satisfy

$$m_{\delta}k_{x} = h_{\delta} \quad \text{and} \quad M_{\Delta}k_{x} = H_{\Delta}. \quad (17)$$

In particular

$$\delta_{G} = \delta_{H} \quad \text{and} \quad \Delta_{G} = \Delta_{H}. \quad (18)$$

It is straightforward to solve the extremal problem (2) and obtain a formula for $h_{x}$. Using that and (17) then gives a formula for $m_{x}$. The two formulas are:

$$k_{\delta}(\cdot) = \frac{1}{\|k_{x}\| \delta(x, y)} \left( k_{x}(\cdot) - \frac{k_{xy}k_{y}(\cdot)}{k_{yy}} \right) \quad (19)$$

$$m_{xy}(\cdot) = m_{\delta}(\cdot) = \frac{1}{\delta(x, y)} \left( 1 - \frac{k_{xy}k_{y}(\cdot)}{k_{yy}k_{x}(\cdot)} \right). \quad (20)$$

There are also some simple relations between the $\delta$’s and the $\Delta$’s; for $x, y, z \in X$,

$$\delta_{G}(x, y)\delta_{G}(x, z) \leq \Delta_{G}(x; y, z) \leq \delta_{G}(x, y) \wedge \delta_{G}(x, z), \quad (21)$$

$$\Delta_{H}(x; y, z) \leq \delta_{H}(x, y) \wedge \delta_{H}(x, z).$$

The left inequality in the first line holds because the product of competitors in the extremal problems defining the $\delta_{G}$’s is a competitor for the extremal problem defining $\Delta_{G}$. The other estimates hold because of the monotonicity of the solution to a restricted maximum problem when the restrictions are loosened.

**3.4.1 Evaluating $\Delta_{G}$ and $\Delta_{H}$**

We now evaluate the quantities $\Delta_{G}$ and $\Delta_{H}$ for spaces $H$ with the CPP. Thus $H \sim DA_{n}(X)$, and by Proposition 2 $\delta_{G} = \delta_{H}, \Delta_{G} = \Delta_{H}$. We will generally drop the subscripts.

**Theorem 3** Suppose $H$ is a RKHS with the CPP and $X(H) = \{x_{i}\}_{i=1}^{n}$. Then

$$\Delta_{G}^{2} = \Delta_{H}^{2} = \frac{1}{\delta_{23}^{2}} \left( \delta_{23}^{2} + \delta_{12}^{2} + \delta_{13}^{2} - 2 + 2 \Re \widehat{k}_{12}\widehat{k}_{23}\widehat{k}_{31} \right) \quad (22)$$

$$= \delta_{12}^{2}\delta_{23}^{2} \left( \delta_{13}^{2} - LF_{123}^{2} (1 - \delta_{23}^{2}) \right) \quad (23)$$

$$= \delta_{12}^{2} \left( \delta_{23}^{2} (\delta_{13}^{2} - LF_{123}^{2}) + LF_{132}^{2} \right) \quad (24)$$

In the next section describe geometric conditions on $X$ which correspond to having $\Delta_{G} = \Delta_{H}$ simplify to $\delta_{12}\delta_{13}/\delta_{23}$, or to $\delta_{12}\delta_{13}$, or to $\delta_{12}$. 

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Proof. Taking note of the discussion in Section 3.3 we may assume \( H \) is three dimensional. Using the definitions and some algebra, including the fact that 
\[
2 \Re k_{23} = |1 - k_{23}|^2 - 1 - |k_{23}|^2,
\]
the formulas \( (22) \) and \( (23) \) are equivalent. Line \( (24) \) is an algebraic rewriting of \( (23) \) which will be convenient later.

We now compute \( \Delta_2^2 H \). Let \( v \in H \) be the function which takes the values 1, 0, 0 at \( x_1, x_2, \) and \( x_3 \). \( v \) spans the one dimensional subspace of functions in \( H \) which vanish at \( x_2 \) and \( x_3 \). Hence \( v/\|v\| \) is the extremal function in the problem defining \( \Delta_H \) and so \( \Delta_H = (|k_11||v|)^{-1} \). We now compute \( \|v\| \). The vector \( v \) can be written as 
\[
v = b_1k_1 + b_2k_2 + b_3k_3.
\]
By evaluating at the \( x_i \) and comparing with \( V = (1, 0, 0) \) we get a system of equations for the \( \{b_i\} \) which we write in matrix form. Let \( K = G(H) \) and set \( B = (b_1, b_2, b_3) \). Here and later we will use \( T^t \) to denote the transpose of the matrix \( T \).

We have
\[
B \bar{K} = V
\]
and hence, setting \( K^{-1} = (\gamma_{ij}) \), we have
\[
\|V\|^2 = B\bar{K}^t = (VK^{-1})^t V K^{-1} V^t = (\gamma_{11}),
\]
Thus our solution is
\[
\Delta_2^2 H = \frac{1}{k_{11}\gamma_{11}}.
\]
We now compute \( \gamma_{11} \) using Cramer’s rule.

Let \( K_{1 \to V^t} \) be the matrix obtained from \( K \) by replacing the first column of \( K \) with the column \( V^t \). Cramer’s rule tells us that \( \gamma_{11} = \det K_{1 \to V^t}/\det K \). Thus our solution is
\[
\Delta_2^2 H = \frac{\det K}{k_{11} \det K_{1 \to V^t}} \begin{vmatrix} k_{11}k_{22}k_{33} + 2\Re k_{12}k_{23}k_{31} - k_{22}\Re k_{13}^2 - k_{33}\Re k_{12}^2 \\ k_{11} \left( k_{22}k_{33} - |k_{23}|^2 \right) \end{vmatrix}
\]
Dividing top and bottom by \( k_{11}k_{22}k_{33} \) we get
\[
\Delta_2^2 H = \frac{1 + 2\Re \tilde{k}_{12} \tilde{k}_{23} \tilde{k}_{31} - |\tilde{k}_{23}|^2 - |\tilde{k}_{12}|^2 - |\tilde{k}_{13}|^2}{1 - |\tilde{k}_{23}|^2}
\]
Recalling that \( \delta_{ij}^2 = 1 - |\tilde{k}_{ij}|^2 \) we can rewrite that as
\[
\delta_{23}^2 \Delta_2^2 H = \delta_{23}^2 + \delta_{12}^2 + \delta_{13}^2 - 2 + 2\Re \tilde{k}_{12} \tilde{k}_{23} \tilde{k}_{31},
\]
which is what we wanted.

Finally, by Proposition 2 we also obtain the result for \( \Delta_G \).

An alternative proof, computing \( \Delta_2^2 G \), using the Pick matrix of the associated multiplier extremal problem, is of comparable length. ■
4 Modifying Spaces and Sets

4.1 Rescalings and Automorphisms; Normal Form

The involutive automorphism of the ball, $\varphi_a$, satisfy a number of useful identities [Rü]. For $a, z, w \in \mathbb{B}^n$, and $k$ the $DA_n$ kernel function,

$$|\varphi_a(z)|^2 = 1 - \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \sigma \cdot z|^2}, \quad (27)$$

$$\frac{1}{1 - \langle \varphi_a(w), \varphi_a(z) \rangle} = \frac{(1 - (w, a))(1 - (a, z))}{(1 - (a, a))^{1/2}} \frac{1}{(1 - (w, z))}, \quad (28)$$

$$k_z(w) = \frac{k_z(a)}{k_a(a)^{1/2}} k_w(a)^{1/2} k_{\varphi_a(z)}(\varphi_a(w)). \quad (29)$$

There is a natural identification of $X(DA_n)$ with $\mathbb{B}^n$. Using that identification the metric $\delta_{DA_n}$ can be regarded as a metric on $\mathbb{B}^n$ and that metric equals the pseudohyperbolic metric $\rho$ on $\mathbb{B}^n = \mathbb{C} \mathbb{H}^n$. This can be seen from (27) where the left side is $\rho(a, z)^2$, the square of the pseudohyperbolic distance between $a$ and $z$, and the right side is $\delta_{DA_n}(a, z)^2$.

By comparing (7) and (29) we see that automorphisms of the ball induce rescalings; if $X = \{x_i\}$ is a finite subset of $\mathbb{B}^n$ and $\Phi \in \text{Aut}(\mathbb{B}^n)$ then $DA_n(X) \sim DA_n(\Phi(X))$. We now introduce a notion of normal form for a set in $\mathbb{B}^n$ and use it to prove a converse statement; if $Y \subset \mathbb{B}^n$ and $DA_n(X) \sim DA_n(Y)$ then $X$ and $Y$ are congruent, $X \sim Y$.

We say a finite ordered set $X = \{x_i\}_{i=1}^M \subset \mathbb{C} \mathbb{H}^n = \mathbb{B}^n$ is in normal form, $X \in \mathcal{N}$, if the coordinate description of $X$ with respect to the standard orthonormal basis, $\{e_i\}_{i=1}^n$, of $\mathbb{C}^n$ takes the following roughly triangular form. The first point, $x_1$, is at the origin, and the coordinates of the remaining points have the form

$$x_j = (a_{j1}, a_{j2}, ..., a_{jN(j)}, 0, ..., 0)$$

with $\{N(k)\}$ a nondecreasing, sequence with differences, $N(k+1) - N(k)$, always 0 or 1. We further require the positivity conditions that if $N(k + 1) > N(k)$ then $a_{(k+1),N(k+1)} > 0$.

Let $\mathcal{N}$ be collection of sets in normal form.

**Proposition 4** Suppose $X$ is a finite ordered set, $X = \{x_i\}_{i=1}^M$, contained in $\mathbb{B}^n = \mathbb{C} \mathbb{H}^n$. There is a unique $\Psi_X \in \text{Aut}(\mathbb{B}^n)$ such that $Y = \Psi_X(X) \in \mathcal{N}$. In particular there is exactly one $Y \in \mathcal{N}$ with $Y \sim X$.

**Proof.** First apply the involution $\varphi_{x_1}$ to $X$. That produces a congruent set with (the new) $x_1$ at the origin. Now split $X$ as a disjoint union $X = \{y_i\}_{i=1}^r \cup \{z_j\}_{j=1}^s = Y \cup Z$. The set $Y$ is constructed by setting $y_1 = x_1 = 0$ and then going through the remaining $x_i$’s in the order of their indices and designating each $x_r$ to be the next $y_i$ if that $x_r$ is not in the linear span of the $y_i$ already selected. Otherwise put $x_r$ in $Z$. Thus, for instance, $y_2 = x_2$. Now set $v_1 = 0$ and apply
the Gram-Schmidt process to the vectors \( y_2, \ldots, y_r \) to produce an orthonormal sequence \( v_2, \ldots, v_j \) with \( j - 1 \leq n \). If \( j - 1 < n \) then complete the sequence in an arbitrary way to an orthonormal basis of \( \mathbb{C}^n \). The structure of the Gram-Schmidt process insures that the coordinate representation of the \( \{x_i\} \) with respect to the basis \( \{v_j\} \) have nonzero entries in the pattern required for a set in \( \mathcal{N} \). Next, replace the basis \( \{v_j\} \) with an orthonormal basis \( \{\alpha_j v_j\} \) where the \( \{\alpha_i\} \) are unimodular constants selected so that the coordinate entries in the positions where positivity is required are, in fact, positive. This is possible because the positivity rule requires that each \( v_j \) be modified at most once.

If the basis \( \{\alpha_j v_j\} \) which we constructed happened to be the canonical basis \( \{e_i\} \) we would be done. Otherwise we now move \( X \) using the unitary map \( U \) which takes the elements \( \{\alpha_j v_j\} \) to the elements \( \{e_i\} \). This is possible because any two orthonormal bases of \( \mathbb{C}^n \) are connected by a unitary map. Because the \( \{x_i\} \) are linear combinations of the \( \{\alpha_j v_j\} \) with coefficients having the desired pattern, the points \( \{Ux_i\} \) have coordinate representations in the desired pattern with respect to the basis \( \{U\alpha_j v_j\} = \{e_i\} \). Finally, recall that any unitary map is in \( \text{Aut}(\mathbb{B}^n) \). Combining \( U, \varphi_{z_1} \), and the rotations used to generate the \( \alpha_j \) produces the required \( \Psi_X \).

Suppose now there were another automorphism \( \tilde{\Psi}_X \) with \( \tilde{\Psi}_X(X) \in \mathcal{N} \). Consider the automorphism \( \Lambda = \tilde{\Psi}_X \Psi_X^{-1} \). Tracing through the definitions shows \( \Lambda(0) = 0 \) hence \( \Lambda \) is a unitary map. Tracing the definitions again shows that \( \Lambda e_1 \) must be a positive multiple of \( e_1 \); but \( \Lambda \) is unitary and hence \( \Lambda e_1 = e_1 \). This pattern continues through the \( e_i \)’s and that is enough to conclude that \( \Lambda \) is the identity on the span of \( \Psi_X(X) \). That establishes the uniqueness of \( \Psi_X \) and hence of the normal form. \( \blacksquare \)

In the proof we possibly did not use all of the dimensions of \( \mathbb{B}^n \).

**Corollary 5** If \( X \subset \mathbb{B}^n \), \( |X| = k \) then \( X \sim Y \) for some \( Y \) in the \( \mathbb{B}^{k-1} \) in \( \mathbb{B}^n \) consisting of all points with their last \( n - k + 1 \) coordinates zero.

**Corollary 6** If \( H \sim DA_n(X) \) for some \( X \) and \( \dim(H) = k \) then \( H \sim DA_{k-1}(Y) \) for some \( Y \).

**Theorem 7** Suppose \( X = \{x_i\} \) and \( Y = \{y_i\} \) are ordered finite sets in \( \mathbb{CH}^n \),

The following are equivalent:

1. \( X \) is congruent to \( Y \) : \( X \sim Y \).
2. \( X \) and \( Y \) have the same normal forms: \( \Psi_X X = \Psi_Y Y \).
3. The spaces \( DA_n(X) \) and \( DA_n(Y) \) are rescalings of each other: \( DA_n(X) \sim DA_n(Y) \).
4. The Gram matrices of the associated spaces are equivalent: \( G(DA_n(X)) \sim G(DA_n(Y)) \).
5. The triangles of \( X \) are congruent to the triangles of \( Y \) : For any triple \( i, j, k \) there is a \( \Gamma_{ijk} \in \text{Aut}(\mathbb{B}^n) \) taking \( \{x_i, x_j, x_k\} \) to \( \{y_i, y_j, y_k\} \).
With this result as background, the discussion going forward is in the spirit of Klein’s Erlangen Program. The geometry of $X \subset \mathbb{CH}^n$ is described by numerical data that is invariant under the automorphism group of $\mathbb{CH}^n$. The structure of a RKHS $H$ is described by numerical data invariant under the rescaling group. Much of the work here focuses on identifying useful invariants and establishing a dictionary between analytic and geometric invariants.

**Proof.** If (1) holds, and thus $Y = \Lambda X$ for $\Lambda \in \text{Aut}(\mathbb{E}^n)$, then $\Psi_Y Y = \Psi_Y \Lambda X$ is in $\mathcal{N}$ and is congruent to $X$. Hence by the uniqueness statement in the previous proposition, $\Psi_Y \Lambda X = \Psi_X X$. Hence (1) implies (2). If (2) holds then $DA_n(\Psi_X X) = DA_n(\Psi_Y Y)$. Also, as we noted earlier, $X \sim \Psi_X X$ implies that $DA_n(\Psi_X X) \sim DA_n(\Psi_X X)$, with a similar statement for $Y$. Combining those equivalences we see that (3) holds. If (3) holds then, by formula (5) and the discussion surrounding it, (4) holds.

We now show that (4) implies (1). We know from the previous paragraph that $G(\Lambda A_n(X)) \sim G(\Lambda A_n(\Psi_X X))$ and similarly for $Y$. Hence we can replace (4) with $G(\Lambda A_n(\Psi_X X)) \sim G(\Lambda A_n(\Psi_X X))$. Consider now $G(\Lambda A_n(\Psi_X X))$. The set $\Psi_X X$ has its point $\Psi_{x_1}$ at the origin and hence $DA_n(\Psi_X X)$ is basepoint normalized with $\Psi_{x_1}$ as the basepoint, similarly with $Y$. We noted earlier that Gram matrices of basepoint normalized spaces are equivalent if and only if they are equal. Hence we are reduced to the case of equal Gram matrices. Thus we will be finished if we can show that if $Z \in \mathcal{N}$ then $G = DA_n(Z)$ determines $Z$. The matrix $G$ has entries $k_{ij} = (1 - \langle z_j, z_i \rangle)^{-1}$ and hence knowing $G$ insures that we know the matrix $((z_j, z_i))_{i,j=2}^m$. This matrix is the Gram matrix of a set of $m - 1$ points in $\mathbb{C}^n$ and hence determines that set of points up to unitary equivalence. In the case of interest to us, $Z = \Psi_X X$, the set is assumed to be in normal form and that removes the ambiguity associated with the unitary equivalence.

Certainly (1) implies (5). To finish we show that (5) implies (4). To do this it suffices to show that if $X$ and $Y$ are both in normal form then they have the same Gram matrix. The first row and first column of those matrices agree by construction. Select $i, j > 1$, $i \neq j$ and consider the triple $X_{ij} = \{x_1 = 0, x_i, x_j\}$ and similarly for $Y_{ij}$. By assumption the two are congruent. Hence the invariant data set $\delta = \delta(X_{ij})$ defined as in Section 3.2 is equal to the corresponding set $\delta(Y_{ij})$. As noted there, this implies the corresponding data sets $\kappa(X_{ij})$ and $\kappa(Y_{ij})$ also agree, hence, also, the associated three by three Gram matrices agree. Further, the elements in those small matrices are determined by the position of the points, independently of any containing superset. Hence the corresponding entries on the Gram matrices for $X$ and $Y$ agree.

The previous result is specific to finite dimensional spaces. If $X$ is infinite then $DA_n(X)$ gives more complicated information about $X$. For instance, if $X \subset \mathbb{CH}^1 = \mathbb{D}$ satisfies $DA_1(X) = DA_1 = H^2$ then we can only conclude that $X$ contains a sequence which fails the Blaschke condition. More information about the general, infinite dimensional, situation is in [SH].

In [HS], [BE], and [G] the authors study congruence classes of finite point sets in $\mathbb{CH}^n$ and obtain results that are similar to the equivalence of conditions (1), (4), and (5) in the previous theorem. Their proofs follow the same general line
as the previous proof; they move from the point set to an associated matrix, develop an appropriate notion of normal form for the matrix, and show that equality of the normal forms is equivalent to the congruence of the sets. However the details of their analysis differ. They view \( \mathbb{C}P^n \) as the negative points of \( \mathbb{C}P^n \) and study the matrix \((x_i, x_j)\) using tools from projective geometry. We view \( \mathbb{C}H^n \) as the ball in \( \mathbb{C}^n \) and use Euclidean coordinate geometry to study the matrix \((k_{ij}) = (1/[x_i, x_j])\).

Using this theorem we see two sets of data which can be used to describe \( X \) up to congruence. The first, \( E(X) \), is the set of \((n - 1)^2\) real numbers which specify the Euclidean coordinates of the points of \( X \) in normal form. This is an inductive description of the set, adding points to the set one at a time and describing each new point by its relation to the previous points. It is similar in spirit to an inductive description which was suggested by Hakin and Sandler in [HS]. A second set of data which describes \( X \) is \( J(X) \) introduced in (14). Taking into account the cocycle identity for angular invariants \( J(X) \) is described by \((n - 1)^2\) real numbers. That data is rescaling invariant and determines the Gram matrix of a rescaled version of \( H \). Those numbers are also invariant under automorphisms and hence should be viewed as geometric descriptors of \( X \). In particular, considering the previous theorem and the discussion in Section 3.2 we see that this data determines the congruence class of triangles with vertices in \( X \) and that data determines \( X \).

The Euclidean parameters \( E(X) \) do not clearly capture the hyperbolic geometry of \( X \), but they do allow a very simple description of which parameter sets are attainable. In contrast, the set \( J(X) \), which contains explicit information about the hyperbolic geometry, does not give a clear vision of the allowable parameter set. The description for three point sets is given in (44) of Theorem 16, but the situation for \( n > 3 \) is unclear.

4.2 The Conjugate Space, \( \overline{H} \)

A RKHS, \( H \), consists of a vector space, a Hermitian inner product, and a distinguished basis, called reproducing kernels. In this section and the next we describe two ways of constructing a new RKHS from \( H \); one by modifying the inner product, the other by changing to a new set of reproducing kernels. If \( K \) is the Gram matrix for \( H \) then the new spaces will have Gram matrices \( \overline{K} \) and \( K^{-1} \) respectively. We then discuss the particularly interesting case when the two constructions give identical spaces. That happens when the matrix \( K \) is orthogonal, \( \overline{K} = K^t = K^{-1} \).

Given \( H \), we define \( \overline{H} \), the conjugate space of \( H \), to be the RKHS formed using the same vector space, the same set of vectors as reproducing kernels, but a different Hermitian inner product, \([\cdot, \cdot]\), defined by

\[
[k_i, k_j] = \langle k_i, k_j \rangle. \quad (30)
\]

It is immediate that \( G(\overline{H}) = \overline{G(H)} = G(H)^t \). It is also immediate that the
conjugate linear map $\Lambda$ defined by
\[
\Lambda \left( \sum \alpha_i k_i \right) = \sum \bar{\alpha}_i k_i.
\] (31)
is an isometry from $H$ to $\overline{H}$; that is
\[
\left\| \sum \alpha_i k_i \right\|_H^2 = \sum \alpha_i \bar{\alpha}_j \langle k_i, k_j \rangle = \sum \bar{\alpha}_i a_j [k_i, k_j] = \left\| \sum \bar{\alpha}_i k_i \right\|_{\overline{H}}^2.
\]
If $H$ has the CPP and thus satisfies $H \sim DA_n(X)$ for some $X \subset \mathbb{C}^n$ then $\overline{H}$ also has the CPP and satisfies $\overline{H} \sim DA_n(\overline{X})$. Here $\overline{X}$ is the set of points obtained by expressing the points of $X$ in terms of coordinates with respect to standard basis and then conjugating those coordinates. In fact, if we knew from the start that $H$ had the CPP and thus $H \sim DA_n(X)$ then we could have based the construction of $\overline{H}$ on the conjugate linear isometry of $\mathbb{C}^n$ given by conjugating the coordinates.

If $X = \overline{X}$ then $H \sim \overline{H}$. This holds, for instance, for the Hilbert spaces of functions on trees which we discuss in Section 7 and for the RKHS obtained as subspaces of the diameter spaces of $\mathcal{A}$.

4.3 The Dualized Space, $H^#$

A RKHS $H$ is a Hilbert space together with the distinguished basis $B = B(H) = \{k_i\}$. Associated with $B$ is the dual basis $B^\# = \{f_j\}$ defined by the requirement that $\langle k_i, f_j \rangle = \delta_{ij}$. We define the dualized space $H^#$ to be the RKHS obtained by using the same Hilbert space, $H$, but selecting $B^\#$ as the distinguished basis rather than $B$.

Let $K$ be the Gram matrix of $H$ and $K^\#$ the Gram matrix of $H^\#$, $K^\# = \langle f_i, f_j \rangle = (f_{ij})$. Let $\Theta = (\theta_{ij})$ be the matrix which takes $B$ to $B^\#$; for all $i$
\[
f_i = \sum_j \theta_{ij} k_j
\] (32)
The transformation in the other direction is then given by $\Theta^{-1} = (\gamma_{ij})$, $k_i = \sum_j \gamma_{ij} f_j$.

**Proposition 8**
\[
\Theta K = I \\
K^\# = \Theta K \Theta^*
\]

*Hence the matrices $K$, $K^\#$ and $\Theta$ are self adjoint and*
\[
\Theta = K^\# = K^{-1}.
\] (34)

**Proof.** The calculation
\[
\delta_{ij} = \langle f_i, k_j \rangle = \left\langle \sum_s \theta_{is} k_s, k_j \right\rangle = \sum_s \theta_{is} k_{sj},
\]
gives the first equation. The second follows from

$$\langle f_i, f_j \rangle = \langle \sum_s \theta_{is} k_s, \sum_t \theta_{jt} k_t \rangle = \sum_{s,t} \theta_{is} \overline{\theta_{jt}}.$$ 

4.4 Orthogonal Spaces and Pick Spaces

Associated with the construction of $H^\#$ is a mapping $\Omega$, the conjugate linear map from $H$ to itself that takes the reproducing kernel basis $\{k_i\}$ to the dual basis $\{f_i\}$:

$$\Omega(\sum \alpha_i k_i) = \sum \overline{\alpha_i} \Omega(k_i) = \sum \overline{\alpha_i} f_i.$$ 

(35)

Using this operator and $\Lambda$ defined by (31) we define the operator $S = \Omega \Lambda$. Thus $S : \overline{H} \to H^\#$,

$$S(\sum \alpha_i k_i) = \sum \alpha_i f_i.$$ 

A conjugate linear map $\Gamma$ from a Hilbert space to itself which is an involutive, i.e. $\Gamma^2 = 1$, and an isometry, i.e. $\forall h \parallel \Gamma h \parallel = \parallel h \parallel$; is called a conjugation. We will be particularly interested in cases where the operator $\Omega$ we just defined is a conjugation. Because $\Omega$ has the additional structural property of taking the kernel basis to the dual basis the conditions for $\Omega$ to be a conjugation simplify.

**Theorem 9** Suppose $\Omega$ is defined by (35). The following are equivalent:

1. The matrix $K$ is orthogonal: $K^t K = KK = I$,

2. Let $\sigma = \sum^n_{i=1} k_i$. The matrix $(k_i k_j, \sigma)$ is the identity.

3. $\Omega$ is an isometry of $H$: $K^t = \Theta K \Theta^*$.

4. $\Omega$ is an involution of $H$: $\Theta^* \Theta = \Theta \Theta^* = I$.

5. $\Omega$ is a conjugation of $H$.

6. $S$ is an an isometry between $\overline{H}$ and $H^\#$.

**Proof.** The second statement is a rewriting of the first. The definition of $\Omega$ together with the previous proposition shows that the equations in statements one, three, and four are equivalent, we must show why the verbal statements correspond to the equations. The matrix $K$ is selfadjoint and hence the equations in the first statement follow from the definition. For the third, suppose $\Omega$ is an isometry. In that case we must have $\parallel \sum_i \alpha_i k_i \parallel = \parallel \sum_i \overline{\alpha_i} f_i \parallel$. Squaring and expanding gives

$$\sum_i \alpha_i \overline{\alpha_j} \langle k_i, k_j \rangle = \sum_i \overline{\alpha_i} \alpha_j \langle f_i, f_j \rangle.$$ 

The right hand side is real and hence we can replace it with its complex conjugate. This produces an equality which will hold for all $\{a_i\}$ if and only if $\forall i, j$

$$\langle k_i, k_j \rangle = \langle f_i, f_j \rangle.$$ 

(36)
We now use (32) in that equality to obtain $\bar{K} = \Theta K \Theta^t$, which is equivalent to the equation in the third statement. Similarly straightforward calculations show that the fourth statement, requiring that $\Omega \Omega^* h = h$ for a general $h \in H$, is equivalent to the equation $\Theta \Theta^t = I$. The fifth statement is, by definition, the union of two before it.

Given the definition of $S$ the final statement is equivalent to the equality of inner products

$$[k_i, k_j] = \langle f_i, f_j \rangle.$$ 

Given the definition (30) this is equivalent to (36) and hence to the fourth statement. \hfill \blacksquare

We will say that a RKHS $H$ is orthogonal if any, and hence all, of the conditions in the previous theorem hold. Thus the orthogonal $H$ are those which have a conjugation operator taking reproducing kernel basis to the dual basis. Using (36) we see that the orthogonal $H$ are also those for which the linear map $S$ of $\overline{H}$ to $H^*$ given by $S(\sum \alpha_i k_i) = \sum \alpha_i f_i$ is an isometry of RKHS. We say that a RKHS is $r$-orthogonal if it is a rescaling of an orthogonal $H$.

**Corollary 10** Given a RKHS $H$, either all or none of the spaces $\{H, H^*, \overline{H} \}$ are $r$-orthogonal.

We now show that every space of the form $DA_1(X)$, is an $r$-orthogonal RKHS. In fact we do not know of any other examples. Later, in Theorem 29 we will show that there are no other three dimensional examples.

The spaces $DA_1(X)$ are exactly the generic finite dimensional model spaces; that is, model spaces corresponding to finite Blaschke products with only simple zeros. Model spaces are discussed systematically in [GMR]. Here we collect some facts about them and about conjugation operators acting on them.

Recall that $DA_1$ is the classical Hardy space, $H^2$. Given a finite Blaschke product, $\Theta \in H^2$, the associated finite dimensional model space is the subspace $J_\Theta \subset H^2$ which is the orthogonal complement of $\Theta H^2$, $J_\Theta = H^2 \ominus \Theta H^2$. If $\Theta$ has only simple zeros then $J_\Theta$ can be regarded as a space of functions on $X = X_\Theta = \{x_i\}$, the zero set for $\Theta$. This space inherits from $H^2$ the structure of a RKHS, and the reproducing kernel functions for $J_\Theta$ are the restrictions to $X$ of the Hardy space kernels. Thus $J_\Theta = DA_1(X_\Theta)$. We will call such a space, $DA_1(X)$ for a finite $X$, a Pick space, both in recognition of the fact that the classical Pick interpolation theorem can be cast as a theorem about the multiplier algebra of such a space, and in parallel with the usage in [CLW] where algebras isomorphic to multiplier algebras of such a space are called Pick algebras. We will call a RKHS which is a rescaling of a Pick space an $r$-Pick space.

**Theorem 11** Any finite dimensional $r$-Pick space $H$ is $r$-orthogonal.

**Proof.** It is a basic fact about Pick spaces that each space carries a conjugation operator taking the basis of reproducing kernels to a rescaled version of its dual basis [GMR]. Specifically, if we denote the basis of $J_\Theta$ consisting of reproducing
kernels by \( \{j_i\} \) and its dual basis by \( \{g_i\} \); \( \langle j_r, g_s \rangle = \delta_{rs} \), then the conjugate linear map \( \Omega \) which satisfies

\[
\Omega(j_i) = \Theta'(x_i)g_i
\]  

is a conjugation. Hence if we rescale \( J_\Theta \) we obtain an orthogonal space. Specifically, let \( H \) be the rescaling of \( \tilde{H} \) which is the same Hilbert space, but with the new distinguished basis of kernel functions \( \tilde{B} = \{r_j\} = \{\Theta'(x_i)^{-1/2}j_i\} \). Direct computation shows that the dual basis of \( \tilde{B} \), \( \tilde{B}^\# = \{s_i\} \), is given by setting 

\[
s_i = \Theta'(x_i)^{1/2}g_i, \quad i = 1, \ldots, n.
\]  

Using (37) we check that \( \Omega \) takes the basis \( \tilde{B} \) to its dual basis \( \tilde{B}^\# \):

\[
\Omega(r_i) = \Omega(\Theta'(x_i)^{-1/2}j_i) = \Theta'(x_i)^{-1/2}\Omega(j_i) = \Theta'(x_i)^{-1/2}\Theta'(x_i)g_i = s_i
\]

The rescaled space \( \tilde{H} \) has the same norm as \( H \) and hence \( \Omega \) is also isometric on \( \tilde{H} \). Thus we have shown that the previous theorem applies to \( \Omega \) and that \( \Omega \) satisfies condition (3) of that theorem. Hence, by that theorem, \( \Omega \) is a conjugation operator on \( \tilde{H} \). Thus \( \tilde{H} \) is orthogonal and hence our original space, \( H = J_\Theta \), is \( r \)-orthogonal. ■

The previous result together with Theorem 9 shows that for \( X \subset \mathbb{D} \) there is a very close relation between the Gram matrix of \( DA_1(X) \) and the Gram matrix of \( DA_1(X)^\# \). That relationship has been used very effectively in analysis of interpolating sequences for the Hardy space; see [AM, 9.5, 9.6] or [Sa, Ch 5, Remark 26]. The explicit analyses there as well as the facts used here about model spaces make crucial use of the theory of Blaschke products. It is not clear what, if any, analogous results hold for spaces \( DA_n(X) \), \( n > 1 \).

5 Embedding \( X \) in \( \mathbb{CH}^n \)

5.1 The Strong Triangle Inequality

The metric \( \rho \) is not a length metric and so there is no reason to believe equality could happen in the triangle inequality for \( \rho \). In fact it never does, and points in \( \mathbb{B}^n \) satisfy a strengthened triangle inequality, STI. For any \( a, b, c \in \mathbb{B}^n \)

\[
\frac{|\rho(a, b) - \rho(b, c)|}{1 - \rho(a, b)\rho(b, c)} \leq \rho(a, c) \leq \frac{\rho(a, b) + \rho(b, c)}{1 + \rho(a, b)\rho(b, c)}.
\]  

(STI)

One way to verify this is to note that the Poincare-Bergman metric, \( \beta \), on the disk is a length metric and so satisfies the standard triangle inequality, including the possibility of equality. Further \( \rho = \tanh c\beta \); Here \( c \) is a constant which we set to one. (The choice \( c = 1/2 \) is also common.) Combining the addition theorem for \( \tanh \) and the triangle inequality for the metric \( c\beta \) produces (STI).

As this suggests, the same configuration which produce equality in the triangle inequality for \( \beta \), namely three points on the same hyperbolic geodesic, will also produce equality in (STI). More discussion of \( \rho \), including a free-standing proof of (STI), is in [DW].
We are interested in understanding conditions on \( H \) related to the possibility that \( H \sim DA_n(X) \), as in Theorem \( \text{[1]} \). If there is such an \( X \) then the metric space \((X, \delta_H)\) must satisfy the STI, so we begin by examining that.

**Proposition 12** Suppose for \( i, j = 1, 2, 3 \) we have \( \delta_{ij} > 0, k_{ij}, \) and \( \hat{k}_{ij} \), and they are related by

\[
\hat{k}_{ij} = k_{ii}^{-1/2} k_{jj}^{-1/2} k_{ij}, \quad \delta_{ij}^2 = 1 - |\hat{k}_{ij}|^2,
\]

then the following are equivalent:

1. \[
\frac{|\delta_{12} - \delta_{13}|}{1 - \delta_{12} \delta_{13}} \leq \delta_{23} \leq \frac{\delta_{12} + \delta_{13}}{1 + \delta_{12} \delta_{13}},
\]

2. \[
1 - \left| \frac{k_{21} k_{13}}{k_{23} k_{11}} \right| = 1 - \left| \frac{k_{21} k_{13}}{k_{23}} \right| \leq \delta_{12} \delta_{13},
\]

3. \[
\frac{1}{|k_{12}|^2} + \frac{1}{|k_{23}|^2} + \frac{1}{|k_{13}|^2} - 1 \leq \frac{2}{|k_{12}| |k_{23}| |k_{13}|}.
\]

**Proof.** We square all three expressions in \((38)\), replace \( \delta_{23}^2 \) by \( 1 - |\hat{k}_{23}|^2 \) and rearrange to obtain

\[
1 - \left( \frac{\delta_{12} - \delta_{13}}{1 - \delta_{12} \delta_{13}} \right)^2 \geq |\hat{k}_{23}|^2 \geq 1 - \left( \frac{\delta_{12} + \delta_{13}}{1 + \delta_{12} \delta_{13}} \right)^2.
\]

Now note that

\[
1 - \left( \frac{\delta_{12} + \delta_{13}}{1 + \delta_{12} \delta_{13}} \right)^2 = \frac{(1 - \delta_{12}^2)}{(1 + \delta_{12} \delta_{13})^2} = \left| \frac{k_{12}}{1 + \delta_{12} \delta_{13}} \right|^2 \left| \frac{k_{13}}{1 + \delta_{12} \delta_{13}} \right|^2
\]

and there is a similar formula for the left side of \((41)\). Hence from \((41)\) we move to

\[
\frac{|\hat{k}_{12}|^2 |\hat{k}_{13}|^2}{(1 - \delta_{12} \delta_{13})^2} \geq |\hat{k}_{23}|^2 \geq \frac{|\hat{k}_{12}|^2 |\hat{k}_{13}|^2}{(1 + \delta_{12} \delta_{13})^2}.
\]

We now extract square roots, divide by \( |\hat{k}_{12}| |\hat{k}_{13}| \), take reciprocals, and rearrange to obtain

\[
1 - \delta_{12} \delta_{13} \leq \frac{|\hat{k}_{12}| |\hat{k}_{13}|}{|k_{23}|} \leq 1 + \delta_{12} \delta_{13},
\]

\[23\]
or, equivalently
\[
\left| \frac{\hat{k}_{12}}{\hat{k}_{23}} \right| - 1 \leq \delta_{12} \delta_{13}, \tag{43}
\]
which gives (39). To obtain (40) we square both sides of (43) and replace the \(\delta\)'s with their definition in terms of the \(k\)'s and obtain
\[
\left( \frac{\hat{k}_{12}}{\hat{k}_{23}} \right)^2 - 2 \left( \frac{\hat{k}_{12}}{\hat{k}_{23}} \right) + 1 \leq \left( 1 - \left| \hat{k}_{12} \right|^2 \right) \left( 1 - \left| \hat{k}_{13} \right|^2 \right).
\]
Dividing by \(\left| \hat{k}_{12} \right|^2 \left| \hat{k}_{13} \right|^2\) and rearranging then produces (40). ■

This result is just a statement that several numerical inequalities are equivalent. However, if the \(k_{ij}\) are the Gram matrix entries for some RKHS \(H\) and the \(\delta\)'s are the \(\delta_H\) distances between points in \(X(H)\), then the proposition shows how an inequality about the distances can be reformulated using Gram matrix entries. In particular, if \(H = DA_n(X)\) then the strong triangle inequality for \(DA_n\) insures that the first statement holds, and the proposition then insures that the other two also hold. Furthermore, if \(H\) has a complete Pick kernel then there is an \(X\) so that \(H \sim DA_n(X)\). In that case \(\delta_H = \delta_{DA_n(X)}\) and the STI, which is automatic for \(\delta_{DA_n(X)}\), also holds for \(\delta_H\). Hence, also in that case all three statements hold for \(\delta_H\) and the kernels from \(H\).

**Example 13** Here is an example of a space \(H\) for which the points of \((X(H), \delta_H)\) fail to satisfy (STI). Suppose \(0 < r < 1\) and let \(K\) be the \(3 \times 3\) matrix with entries
\[
\begin{align*}
k_{12} &= k_{22} = k_{32} = k_{21} = k_{23} = 1 \\
k_{11} &= k_{33} = (1 - r^2)^{-2} \\
k_{13} &= k_{31} = (1 + r^2)^{-2}.
\end{align*}
\]
The matrix \(K\) is positive definite and hence is the Gram matrix of a RKHS \(H\). We write \(X(H) = \{x_1, x_2, x_3\}\) and \(\delta = \delta_H\). For small values of \(r\) we have
\[
\begin{align*}
\delta_{13} &= 2\sqrt{2}r - 4\sqrt{2}r^3 + O(r^5) \\
\frac{\delta_{12} + \delta_{23}}{1 + \delta_{12} \delta_{23}} &= 2\sqrt{2}r - \frac{9}{2}\sqrt{2}r^3 + O(r^5).
\end{align*}
\]
For small \(r\) the second line is smaller than the first and the STI fails.

To see this example in a larger context, recall that the Bergman space, \(A^2 = A^2(\mathbb{D})\), has kernel functions \(k_z(w) = (1 - \bar{z}w)^{-2}\). The \(A^2\) kernel functions for the points \(\{-r, 0, r\}\) have Gram matrix \(K\) and hence their span is (a rescaling of) \(H\). Either because the points \(\{-r, 0, r\}\) lie on a hyperbolic geodesic, or by direct computation, the pseudohyperbolic distances, \(\rho\), of the three points satisfy the STI with equality:
\[
\rho_{13} = \frac{2\rho_{12}}{1 + \rho_{12}^2}.
\]
The Hardy space, $H^2$, has kernel functions $k_z(w) = (1 - \bar{z}w)^{-1}$ and $\delta_{H^2} = \rho$. Using this fact, the formulas for the kernel functions, and the definition of $\delta$, we find that $\delta_{A^2}^2 = \rho^2 (2 - \rho^2)$. In particular, for small distances

$$\delta_{A^2} \sim \sqrt{2}\rho.$$ 

These last two displays are not compatible with what the STI calls for in $H$, which is

$$\delta_{A^2}^2(1,3) \leq \frac{2\delta_{A^2}(1,2)}{1 + \delta_{A^2}(1,2)^2}.$$ 

In this example the failure of (STI) insures that we do not have $H \sim DA_n(X)$. However we will see in Example 19 below that having (STI) is not enough to insure that $H \sim DA_n(X)$. On the other hand, if we are only interested in the metric structure of a three point set, and not any additional structure, then (STI) is a complete condition for isometric embedding in hyperbolic space.

**Proposition 14** A three point metric space $(Z, \delta)$ with $\delta < 1$ can be mapped isometrically into $(CH^n, \rho)$ if and only if it satisfies (STI). If that holds then the map $\Phi$ can be chosen to map into $D = CH^1$, in which case the image is uniquely determined up to the action of (a possibly antiholomorphic) isometry of $CH^1$.

**Proof.** We noted when we introduced (STI) that the inequality is always satisfied by points of $(CH^n, \rho)$. Hence, if we have the mapping of $Z$ then (STI) follows.

Now suppose we have $(Z, \delta)$ which satisfies (STI) and write $Z = \{z_i\}_{i=1}^3$. We want to find $\Phi$ mapping $Z$ into $CH^1$. By considering composition with Mobius transformations we see that if we can find a map $\Phi$ with the right mapping property, then we can find a $\Phi$ with $\Phi(z_1) = 0$ and $\Phi(z_2) = \delta(z_1, z_2) = s$. Further, this normalization determines $\Phi$ uniquely up to possible complex conjugation. Thus we are reduced to showing that if we set $\Phi(z_1) = 0$ and $\Phi(z_2) = \delta(z_1, z_2)$ then we can find a $\Phi(z_3) = w$, unique up to complex conjugation, so $\rho(0, w) = \delta(z_1, z_3)$ and $\rho(s, w) = \delta(z_2, z_3)$.

Those conditions state that $w$ must lie on the intersection of two pseudo-hyperbolic circles, one centered at 0, the other centered at $s$, with radii given by the $\delta$’s. However those pseudo-hyperbolic circles are also Euclidean circles with centers on the real axis. From this we see that the intersection is either empty, or one point on the real axis, or two points, conjugate to each other. The condition that the intersection be nonempty is exactly that the triangle inequality for the hyperbolic metric be satisfied. However that is equivalent to the pseudohyperbolic metric satisfying the STI. If the intersection is nonempty, then selecting $w$ to be an intersection point completes the proof. ■

In short, the isometric congruence class of a three point set in $CH^1$ is uniquely determined by its distances. We are not claiming, and it is not true, that the same holds for three point sets in $CH^n, n > 1$.

The fact that there are isometries of $CH^1$ that are not holomorphic persists in higher dimensions and is part of the discussion of congruence in $CH^n$, see,
for instance, [BE]. Going forward when we refer to isometries we will mean the holomorphic ones.

5.2 Two Dimensional Spaces

We now look in more detail at the possibility, given $H$, of finding $\Phi$ such that $H \sim DA_n(\Phi(X(H)))$.

- If $\dim(H) = 1$, there is nothing to say.
- If $\dim(H) = 2$, then $H$ can be rescaled so that the Gram matrix is

$$G(H) = \begin{pmatrix} 1 & 1 \\ 1 & g \end{pmatrix},$$

and because $G(H)$ is positive we must have $g > 1$. Set $\gamma = \sqrt{1 - 1/g}$. The Gram matrix of $J = DA_1(\{0, \gamma\})$ is identical to $G(H)$. Hence $H \sim J$, and thus any two dimensional RKHS $H$ is a rescaling of a space $DA_1(X)$.

We can also describe the multiplier algebra, $\text{Mult}(H)$. The multipliers are diagonal operators on a two dimensional space, and hence can be analyzed without recourse to general theory. However, it is convenient to take advantage of von Neumann’s inequality which insures us that if $M_m$ is the operator of multiplication by $m$ and it satisfies $\|M_m\| = 1$, and if $\varphi$ is a conformal automorphism of the disk, then $\varphi(M_m) = M_{\varphi(m)}$ is also a multiplier of norm one. We also want the following elementary computational fact about $\rho$.

**Lemma 15** Given $\alpha, \beta \in \mathbb{C}$ and $0 < \gamma < 1$, there is a unique $\lambda > 0$ such that $\rho(\lambda \alpha, \lambda \beta) = \gamma$.

Given a nonzero $M_m \in \text{Mult}(J)$ the lemma produces a unique $\lambda > 0$ such that $\rho(\lambda m(0), \lambda m(\gamma)) = \gamma = \rho(0, \gamma)$. Given that equality of distances, there is a unique $\sigma \in \text{Aut}(\mathbb{B}^1)$ with $\sigma(0) = \lambda m(0)$ and $\sigma(\gamma) = \lambda m(\gamma)$. The coordinate multiplier, $M_z$, has norm one. That can be checked quickly by computing the norm of the adjoint, $M_z^*$, using the basis of kernel functions. Hence, by von Neumann’s inequality the multiplier $N = \sigma(M_z) = M_{\sigma(z)}$ also has norm one. By comparing values we see that $N = \lambda M_m$ and hence $\|M_m\| = 1/\lambda$. Furthermore, $\lambda$ could be written explicitly in terms of the values taken by $m$ and the parameter $\gamma$ which is determined by the space $J$.

5.3 Three Dimensional Spaces

We now look at the case $\dim(H) = 3$ in some detail. The situation is more complicated than $\dim(H) = 2$ because the realization of $H$ as $DA_n(X)$ is not automatically possible. On the other hand, in three dimensions the Pick property is equivalent to the CPP and hence some complications which appear in higher dimensions are avoided.

**Theorem 16** Suppose $H$ is a three dimensional RKHS, $X = X(H) = \{x_i\}_{i=1}^3$. The following are equivalent:
1. $H$ has the complete Pick property.

2. $H$ has the Pick property.

3. $\exists i, j, i \neq j$ with $\delta_H(x_i, x_j) = \delta_G(x_i, x_j)$.

4. $LF_{123}^2 \leq \delta_{13}^2$.

5. \[ \frac{1}{|k_{12}|^2} + \frac{1}{|k_{23}|^2} + \frac{1}{|k_{13}|^2} - 1 \leq \frac{2 \cos A_{123}}{|k_{12}| |k_{23}| |k_{13}|} \quad (44) \]

6. There are $w \in \mathbb{C}$, $s, t > 0$ such that with

$$\Phi(X) = \{(0, 0), (s, 0), (w, t)\} \subset \mathbb{B}^2 = \mathbb{C}H^2, \quad (45)$$

we have $H \sim DA_2(\Phi(X))$, 

Furthermore, the location of the points of $\Phi(X)$, the rescaling equivalence class of $H$, and the congruence class of the triangle with vertices $\Phi(X)$ are uniquely determined by the rescaling invariant parameters $\delta = \{\delta_{12}, \delta_{13}, \delta_{23}, A_{123}\}$.

**Corollary 17** If $H$ is a three dimensional RKHS with the CPP then $\cos A_{123} > 0$, $|A_{123}| \leq \pi/2$.

**Proof of the Corollary.** An application of the Cauchy-Schwartz inequality shows $|\hat{k}_{ij}| < 1$. Hence the left hand side of (44) is positive, which shows $\cos A_{123}$ must be positive. 

The first two statements in the theorem are general properties of $H$ and $\text{Mult}(H)$, the next three concern numerical invariants derived from those function spaces. Statement (5) is Brehm’s classical description of parameters which determine the congruence class of triangles in $\mathbb{C}H^2$, as given in [BE, Pg. 92] and translated into our notation. The final statement describes a set $\Phi(X)$ in $\mathbb{C}H^n$ whose existence is required by Theorem [1].

Even if $n = \text{dim}(H) > 3$ it is true that (1) implies (2) implies (3) implies (4), and that (4) and (5) are equivalent. However in that range (3) is weaker than (2) which is weaker than (1). Also, in that range a simple statement in the style of (4) is not enough to get a representation such as (6). Our work for $n > 3$ centers on understanding how to replace (4). The path to proving (6) implies (1) depends on how the CPP is defined. We will avoid any work at that spot by accepting Theorem which states that for finite dimensional spaces the existence of a representation as in (6) is implies the CPP.

**Proof of the Theorem.** If (1) holds then so does (2) which is just a restricted version of (1). Condition (2) is enough to appeal to Proposition whose proof only uses the Pick property, not the CPP) and obtain the equality of $\delta_G$ and $\delta_H$, i.e. (18), for each pair of indices. (3) is the weaker statement that the equality holds for a single pair of indices. However any one equality $\delta_{Gij} = \delta_{Hij}$ is enough to give the formula (20) for the extremal multiplier for that particular pair of
indices; and that is what we need to go forward. By renumbering, and without loss of generality, we suppose we have the particular case that \( \delta_{G12} = \delta_{H12} \). In that case, we know from (20) that

\[
M_{x_2,x_1}(\zeta) = \frac{1}{\delta_{12}} \left( 1 - \frac{k_{21}k_1}{k_{11}k_2} \right)
\]

is a multiplier of norm one. Because of that and the fact that \( M_{x_2,x_1}(x_2) = 0 \) we must have

\[
|M_{x_2,x_1}(x_3)| \leq \delta_{G23} \leq \delta_{H23};
\]

the first inequality by the definition of \( \delta_{G23} \), the second because, as we mentioned in Section 3.4, the \( \delta_{G}^i \)'s are always dominated the \( \delta_{H}^i \)'s. Rearranging that inequality gives statement (4).

Statements (4) and (5) are equivalent by an algebraic rewriting, similar to that connecting (39) and (40) in the proof of Proposition 12. However instead of starting with

\[
\left| 1 - \frac{k_{21}k_{13}}{k_{23}} \right| \leq \delta_{12}\delta_{13},
\]

we start with the stronger statement (4), which, written out using (12), is

\[
\left| 1 - \frac{k_{21}k_{13}}{k_{23}} \right| \leq \delta_{12}\delta_{13}.
\]  

We now follow the proof of Proposition 12. We square both sides of (46) and replace the \( \delta \)'s with their definition in terms of the \( k \)'s and obtain

\[
\left| k_{12}^2 \right|^2 \left| k_{13}^2 \right|^2 - 2 \left| k_{12} \right|^2 \left| k_{13} \right|^2 \left| k_{23} \right|^2 \Re \cos \arg \left( \frac{k_{21}k_{13}}{k_{23}} \right) + 1 \leq \left( 1 - \left| k_{12} \right|^2 \right) \left( 1 - \left| k_{13} \right|^2 \right).
\]

Dividing by \( \left| k_{12} \right|^2 \left| k_{13} \right|^2 \), using the definition of \( A_{123} \), and rearranging then produces (5).

We now go to the basic construction, showing that (4) insures that we can select the required points in hyperbolic space. We know from our analysis of normal forms that if we can find some \( X \subset \mathbb{C}^3 \) so that \( H \sim DA_k(X) \) then we can find a \( X = (x_1, x_2, x_3) \subset \mathbb{C}^3 \) in normal form, i.e. as described in (5), and having \( H \sim DA_2(X) \). Hence the question is if we can find \( s, w, t \) so that the following system is satisfied. Here the \( \delta \)'s and \( k \)'s are data from \( H \); \( s, w, t \)
are the unknowns:

\[
\delta_{12}^2 = 1 - \frac{|k_{12}|^2}{k_{22}k_{11}} = 1 - |k_{12}|^2 = s^2 \\
\delta_{13}^2 = 1 - \frac{|k_{13}|^2}{k_{33}k_{11}} = 1 - |k_{13}|^2 = |w|^2 + t^2 \\
\delta_{23}^2 = 1 - \frac{|k_{23}|^2}{k_{22}k_{33}} = 1 - |k_{23}|^2 = 1 - \frac{(1 - \delta_{12}^2)(1 - \delta_{13}^2)}{|1 - sw|^2} \\
A_{123} = \arg k_{12}k_{23}k_{31} = -\arg (1 - sw) 
\]

We start by setting \(x_1 = (0, 0)\) and \(s = \delta_{12}\) so that \(x_2 = (s, 0)\). Once that is done, then (47) and (50) force the value of \(1 - sw\), and hence of \(w\). If we can show that \(|w| \leq \delta_{13}\) then we can select a unique nonnegative \(t\) such that (48) holds. At that point we will have that \((w, t)\) is in the ball and all the required equations are satisfied, and we will be finished. To obtain the required estimate for \(w\) note that, using \(x_1 = (0, 0)\) and \(x_2 = (s, 0)\) and \(x_3 = (w, t)\) and the formula for the kernel function, the definition of \(LF_{123}\) in (12) gives \(LF_{123} = |w|\). Thus statement 4 simplifies to the required \(|w| \leq \delta_{13}\).

Combined with the earlier comments this completes the proof. 

**Corollary 18** In the situation of the previous theorem the following are equivalent:

4. \(LF_{123}^2 = \delta_{13}^2\)

6. For some \(w \in \mathbb{C}\) and \(s > 0\), and with \(\Phi(X) = \{0, s, w\} \subset \mathbb{B}^1 \subset \mathbb{CH}^1\), we have \(H \sim DA_1(\Phi(X))\).

**Proof.** Using the fact \(LF_{123} = |w|\) from the previous proof we see that 4’ is equivalent to \(t = 0\). 

5.3.1 About \(LF_{123}^2\)

Condition (4) on \(LF_{123}^2\) is related to the positivity of one of the matrices \(MQ\) introduced in (11). It is a basic fact from the theory of spaces with the CPP that a necessary and sufficient condition for a finite dimensional \(H\) to have the CPP is that the matrices (11) be positive semidefinite, \[AM\] Thm. 7.6 and Theorem 22. If \(\dim(H) = n\) then the general theorem requires consideration of \(n\) matrices of size \((n-1) \times (n-1)\). However in three dimensions the situation simplifies and we only need consider the positivity of a single \(2 \times 2\) matrix from (11):

\[
MQ = \begin{pmatrix}
1 - 1/k_{22} & 1 - 1/k_{23} \\
1 - 1/k_{32} & 1 - 1/k_{33}
\end{pmatrix}.
\]

That matrix has positive diagonal elements and hence its positivity reduces to the positivity of \(\det(MQ)\), which is equivalent to Condition (4).
The statement $LF_{123}^2 \leq \delta_{13}^2$ is also an inequality between two Euclidean distances in $\mathbb{B}^n$; it compares the length of the hypotenuse of a right triangle to the length of one of the other sides. After having placed the points $\Phi(x_1)$ and $\Phi(x_2)$ at $(0, 0)$ and $(s, 0)$ we want to find $w$ and $t$ so that if we place $\Phi(x_3)$ at $(w, t)$ then the required equalities hold. We do that in two steps. First we locate the point $(w, 0)$, the projection of the not-yet-located final point $\Phi(x_3)$ onto $(0, 0)$, $(s, 0)$, and $(w, 0)$. The inequality is the statement that $|w| \leq \delta_{13}$ which can be reformulated as $LF_{123}^2 \leq \delta_{13}^2$. Thus we split finding the final vector into computing $(w, 0)$, its footprint in the span of the vectors already selected, and the length of its footprint, $LF$. If the length of the footprint is not longer than the length of the final vector then there is no obstruction to locating the final point by specifying a nonnegative height for that vector above the footprint.

This scheme for placing $\Phi(x_3)$ is similar to one we used in adjoining points to sets in the construction of normal forms in Theorem 7 and to the methods we use later in Theorems 21 and 35. The general situation is that we have placed points $\{x_j = 0, x_2, ..., x_k\}$ in the ball and need to place a new point, $x_{k+1}$. We let $S_k$ be the span of $\{x_j\}_{j=k}$. We place $x_{k+1}$ by first identifying an auxiliary point $P_k(x_{k+1})$, the point that, if we knew the location of $x_{k+1}$, would be the orthogonal projection of $x_{k+1}$ onto $S_k$; the nearest point projection in terms of both the Euclidean and hyperbolic distances. The first coordinates of $x_{k+1}$ will be the first coordinates of $P_k(x_{k+1})$. The remaining data needed to describe the location of $x_{k+1}$ is its distance from the point $P_k(x_{k+1})$, and that Euclidean distance $d$ becomes the $k^{th}$ coordinate of $x_{k+1}$, the height of $x_{k+1}$ above $S_k$. In this language the estimate $LF_{123}^2 \leq \delta_{13}^2$ in the previous proof is essentially the requirement that $d$ not be negative. A similar comment applies to the estimate $(58)$ at the end of the proof of Theorem 21.

We discuss the geometric interpretation of the special values $LF_{123}^2 = 0$ and $LF_{123}^2 = \delta_{13}^2$ in Section 6.

5.3.2 About $A(x, y, z)$ and Area

By comparing (44) to (40) we see that the conditions in the previous theorem imply STI. However STI itself is not sufficient for the statements in the theorem. Here are examples of spaces with the STI which fail the conclusion of Corollary 17 about the size of $A_{123}$ and hence do not have the CPP.

Example 19 Pick $r, \lambda$ with $0 < r < 1$, $\lambda > 0$. Let $\omega$ be a primitive cube root of unity. Set $y_j = r\omega^j$, $j = 1, 2, 3$. Let $K(r, \lambda)$ be the $3 \times 3$ matrix with entries

$$k_{ij} = (1 - y_i \overline{y_j})^{-\lambda}$$
It can be verified by hand that $K(r, \lambda)$ is a positive matrix and hence determines a three dimensional RKHS, $H = H(r, \lambda)$. Alternatively, $k(y, w) = (1 - yw)^{-\lambda}$ is the reproducing kernel for a space $D_\lambda$ of holomorphic functions on the disk, and $K(r, \lambda)$ is the Gram matrix of the $D_\lambda$ kernel functions for the points $y_1, y_2, y_3$.

The space $D_\lambda$ has the CPP if, but only if, $\lambda \leq 1$. In those cases $H(r, \lambda)$ inherits the CPP from the containing $D_\lambda$ and hence there is a map $\Phi$ of $X(H(r, \lambda))$ into $\mathbb{C}H^n$ so that $H(r, \lambda) \sim DA_n(\Phi(X(H(r, \lambda)))$.

However for some $\lambda$ there is no embedding. The inequality (STI) is not the problem. The symmetry of the configuration under rotations of $2\pi/3$ insures that all the $\delta_{ij}$ are the same, in which case (STI) is automatic. However for some parameter values the space $H(r, \lambda)$ fails to satisfy the conclusion of Corollary 17 which requires $\cos A_{123} > 0$. For $H(r, \lambda)$ we have

$$A_{123} = \arg k_{12}k_{23}k_{31}$$

$$= \arg \left( \frac{1}{(1 - r^2\omega^{1-2}) (1 - r^2\omega^{2-3}) (1 - r^2\omega^{3-1})} \right)^{\lambda}$$

$$= 3\lambda \arg (1 - r^2\omega)$$

Thus $\cos A_{123} < 0$ for some $r, \lambda$.

Suppose $H$ is three dimensional. The previous example shows that the values of $A$, which are determined by the kernel functions in $H$, can indicate an obstruction to having $H \sim DA_n(X)$. On the other hand, recalling the comments in Section 3 if there is such a representation then $A$ is also a geometric invariant of $X$. In that case it makes sense to ask for its geometric interpretation.

We regard a triple of points $X = \{x, y, z\} \subset \mathbb{C}H^n$ as the vertices of a geodesic triangle, $\Delta \subset \mathbb{C}H^n$, a triangle with vertices $X$ and sides which are geodesic segments connecting the vertices. There are natural ways to measure the size of the sides of $\Delta$, either with $\rho$ or with $\beta$, but there is not a simple notion of the area of $\Delta$. In this section and the next we discuss the relation between the values of $A$ and two substitutes for the area for $\Delta$. Another theme that runs through both discussions, although we will not give it a quantitative formulation, is that $A$ measures how well $\Delta$ fits into a single complex geodesic.

The congruence class of a Euclidean triangle is determined by its three side lengths, but the analogous statement fails in complex hyperbolic space. As suggested by our parameter count, as shown by Brehm in his classic analysis of triangles in both projective and hyperbolic space [2], and as can be seen from (44), the side length data is not enough to determine the triangle. An additional parameter is needed. Various quantities are used for a fourth parameter; here we are using the angular invariant $A$, a version of the invariente angulaire introduced by E. Cartan in [C]. Related invariants are discussed in [Go, Ch. 7].

If $n = 1$ we are in the unit disk where there is a natural notion of the surface of the triangle. In that case we can use the classical Poincare-Bergman area element to define/compute the area of $\Delta$, Area($\Delta$). Furthermore, in that case Area($\Delta$) = $2A(x, y, z)$. That can be proved by taking advantage of the classical formula relating Area($\Delta$) to the angles of $\Delta$, a detailed discussion is in [C] Sec
However if $\Delta$ is in general position in $\mathbb{C}H^n$ then there is no natural notion of the surface of $\Delta$ on which to base a notion of "surface area". Nevertheless it is still possible to define the symplectic area of $\Delta$, $\mathcal{S}A(\cdot)$. Complex hyperbolic space carries a natural symplectic two form, $\omega$, a type of area form. Given the sides of $\Delta$, select a smooth real two manifold $\Sigma(\Delta)$ connecting the three sides of $\Delta$. Define the symplectic area of the triangle $\Delta$ by $\mathcal{S}A(\cdot) = \int_{\Sigma(\Delta)} \omega$. Because $\omega$ is a closed form Stokes' theorem allows us to evaluate this as a boundary integral over the sides of $\Delta$, in particular the value does not depend on the choice of $\Sigma(\Delta)$. Because that is the only use we make of $\Sigma(\Delta)$, we need not be explicit about the details of its construction. On the disk $D = \mathbb{C}H^1 \subset \mathbb{C}H^n$ the symplectic form $\omega$ is the same as the hyperbolic area element and so, in that case $\mathcal{S}A(\cdot) = \text{Area}(\Delta) = 2A(x, y, z)$. However much more is true. For general $\Delta \subset \mathbb{C}H^n$, $2A(x, y, z) = \mathcal{S}A(\cdot)$. This general fact requires more work. It was proved by Hangen and Masalla [HM] by explicit evaluation of the double integral. It can also be proved, both in this context and much more general ones, using Stokes' theorem, see the discussion in [C]. Of course once $\Delta$ is in general position $\mathcal{S}A(\Delta)$ is only an "area" in a metaphorical sense. Note for instance that for the triangle $\Delta$ with vertices \{$(0, 0)$, $(s, 0)$, $(0, t)$\} $\subset \mathbb{C}H^2$, $s, t \in \mathbb{R}$ we have $k_{23} = 1$ and hence $A(x, y, z) = 0$. Alternatively, note that the two-form $\omega$ vanishes on the real two-plane spanned by the vertices of $\Delta$. Still, it is satisfying to phrase Brehm's theorem as saying the congruence class of a triangle is determined by its side lengths and its area.

In fact, $A$ and variations on it have a much richer life than we have discussed. One suggestion of this is that in complex projective space there is a similar formula relating the argument of a product of kernel functions to the invariant area of a triangle [Go, Sec 1.3.6], Another similar formula, using kernels of the Fock space, gives the area of Euclidean triangles in the plane. Also, an invariant similar to $A$ can be defined using the Bergman kernel function and hence has a natural definition on general symmetric domains, and even more widely. The cocycle identity persists and, in many cases, so does the fact that $A$ can be evaluated by integrating a natural symplectic form over a triangle. All this suggests that $A$ might be a valuable cohomological tool in studying symmetric domains and, more generally, complex and symplectic manifolds. This is true, but we will not even begin discussing details of these relations. More information as well as further references are in [BS, Thm 4.8], [C, Section 5], [BIW, Introduction], and [BH, Sections 1,2,3].

5.3.3 More About $A(x, y, z)$ and Area

In the previous section we introduced $\mathcal{S}A(\Delta)$, a functional related to area which gave a geometric interpretation to the invariant $A_{123}$. We now introduce another geometric functional, also related to area, which turns out to equal $|\mathcal{S}A(\Delta)|$ and hence gives a slightly different geometric interpretation of $A_{123}$.

Suppose, again, $X = \{x_1, x_2, x_3\} \subset \mathbb{C}H^n$ and let $\Delta$ is the associated geodesic triangle. Set $H = DA_n(X)$.

If $X$ is contained in a complex geodesic, $G$, then there is a natural way to
define the area of $\Delta$. Because $G$ is a complex geodesic there is a hyperbolically isometric map of $\mathbb{CH}^1$ onto $G$. By using the geometry from $\mathbb{CH}^1$ there is then a natural interpretation of the region of $G$ inside $\Delta$. That map also can be used to carry the Poincare Bergman area element to $G$ where it can be used to compute the area of $\Delta$, $\text{Area}(\Delta)$.

If $X$ is not in a complex geodesic then we can push $X$ into a nearby complex geodesic using a map $\Pi$, and then use the functional $\text{Area}()$ compute the area of the triangle with vertices $\Pi X$. More precisely, the suppose $S$ is a side of $\Delta$. It is a geodesic segment and hence is contained in a unique complex geodesic $G$. Let $\Pi$ be the hyperbolic nearest point projection of $\mathbb{CH}^n$ to $G$, and let $\Delta_{\Pi X}$ be the triangle in $G$ with geodesic sides and with vertices $\Pi X$. We define the projected area of $X$ to be $\text{Area}(\Delta_{\Pi X})$. It is a consequence of the next theorem that the value of $\text{Area}(\Delta_{\Pi X})$ would be the same if we did the similar construction using a one of the other sides of $\Delta$.

If $X$ is in a complex geodesic $J$ then $G = J$ and $\Pi$ is the identity on $\Delta$. In that case, combining this with the discussion in the previous section we have the following chain of equalities: $\text{Area}(\Delta_{\Pi X}) = \text{Area}(\Delta_X) = |\mathcal{S}A(\Delta_X)/2| = |A_{123}|$. Although $\text{Area}(\Delta_X)$ is not defined for $X$ in general position, the other three quantities are defined and, in fact, are equal. Thus, although $\text{Area}(\Delta_{\Pi X})$ is not new numerical data, it does give an alternative geometric interpretation of $A_{123}$.

**Theorem 20** For $X$ a three point set in $\mathbb{CH}^n$ and $\Delta_X$ the triangle with $X$ as its vertex set, we have

$$\text{Area}(\Delta_{\Pi X}) = |\mathcal{S}A(\Delta_X)/2| = |A_{123}|.$$ 

**Proof.** The theorem is a consequence of the following:

1. The construction of $\Delta_{\Pi X}$, and hence also the final statement, are invariant under automorphisms of $\mathbb{CH}^n$.

2. If the complex geodesic $G$ equals $\mathbb{D}$, the intersection of $\mathbb{B}^n$ with the $z_1$ axis; then the nearest point projection of $\mathbb{B}^n$ onto $G$ is the same as the Euclidean orthogonal projection of $\mathbb{B}^n$ onto $\mathbb{D}$.

3. If $X \in \mathcal{N}$ then the hyperbolic area of the triangle in $\mathbb{D}$ with vertices given by the Euclidean projection of the set $X$ into $\mathbb{D}$ is $|\mathcal{S}A(\Delta)|$.

4. For any $\Delta$, $\mathcal{S}A(\Delta) = 2A_{123}$.

The first statement holds by inspection of the definitions. The second is an elementary exercise after expressing the pseudohyperbolic distance between points in terms of Euclidean coordinates in $\mathbb{B}^n$. The third statement is proved in [Go] as part of the proof of Theorem 7.1.11. As mentioned in the previous section, the final statement is a result of Hangan and Masala [HM] and also has alternative proofs as described in [C].
5.4 \( \dim(H) > 3 \) and the CPP

Theorem 4 stated that the finite dimensional reproducing kernel Hilbert spaces with the CPP are exactly the rescalings of spaces \( DA_n(X) \). We now prove part of that theorem, namely:

**Theorem 21 ([AM, Thm. 7.28])** If \( H^+ \) is a finite dimensional RKHS with the CPP then there is a finite set \( X^+ \) in some \( \mathbb{C}^n \) such that \( H^+ \sim DA_n(X^+) \).

This is a well established result. Our goal here is to showcase a geometric argument similar to what we just used for \( \dim(H) = 3 \). With that in mind, we will be less than fully detailed.

**Proof.** Theorem 6 proves the result in the case \( \dim(H^+) = 3 \). With that as a starting point we prove the theorem by induction on the dimension of \( H \). Thus we need to know that we can extend the definition of the function \( \Phi \) from a set \( X \) to a larger set \( X^+ \) so that certain conditions are met. Here is the precise formulation.

Suppose \( H^+ \) is an \( n+1 \) dimensional RKHS with the CPP, with \( X^+ = X(H^+) = \{x_1\}_{i=1}^{n+1} \), with kernel functions \( \{k_i\}_{i=1}^{n+1} \) and with Gram matrix \( K^+ = (k_{ij}) \). Let \( H \) be the subspace spanned by \( \{k_i\}_{i=1}^{n} \), and hence \( X = X(H) = \{x_i\}_{i=1}^{n} \). The subspace \( H \) inherits the CPP and hence, by our induction hypothesis, and taking note of Corollary 2 there is a map \( \Phi : X \rightarrow \mathbb{C}^n \) so that \( H \sim DA_n(\Phi(X)) \). We can suppose \( \Phi(X) \) is in normal form in which case \( \Phi(x_1) = z_1 = 0 \) and \( \Phi(X) \) is contained in the subspace of \( \mathbb{C}^{n-1} \) of \( \mathbb{C}^n \) characterized by having the last coordinate equal to zero. We write \( \Phi(x_1) = z_i, \, i = 1, \ldots, n \). Hence we have the following formula for some of the entries of the \((n+1) \times (n+1)\) matrix \( K^+ \); for \( 1 \leq i, j \leq n \)

\[
k_{ij} = (1 - \overline{z}_{ij})^{-1}.
\]

We want to find \( z_{n+1} \in \mathbb{C}^n \) with the property that if we extend \( \Phi \) to a map \( \Phi^+ \) defined on all of \( X^+ \) by setting \( \Phi^+(x_{n+1}) = z_{n+1} \) then we will have \( H^+ \sim DA_n(\Phi^+(X^+)) \).

We write the candidate for \( z_{n+1} \) as

\[
z_{n+1} = w + ce_n = (c_1, \ldots, c_{n-1}, 0) + c(0, \ldots, 0, 1)
\]

We will be finished if we construct \( z_{n+1} \) so that \( 51 \) holds for the full Gram matrix \( K^+ \). This involves conditions on the inner products, \( \langle z_j, z_{n+1} \rangle \) for \( 1 \leq j \leq n + 1 \), but our construction insures that for \( j < n + 1 \) the last coordinate of \( z_j \) is 0 and hence \( \langle z_j, z_{n+1} \rangle = \langle z_j, w \rangle \). Thus we can write all of those conditions as requirements for \( w \):

\[
\langle w, z_j \rangle = 1 - 1/k_{n+1,j} \quad j = 1, \ldots, n
\]

We now suppose temporarily that \( H \) is generic, that is, the set \( Z = \{z_i\}_{i=2}^{n} \) is linearly independent. We consider the other case later. The set \( Z \) is a basis of \( \mathbb{C}^{n-1} \). Let \( Z^* = \{z_i^*\}_{i=2}^{n} \) be the dual basis. We set

\[
w = \sum_{i=2}^{n} \langle w, z_i \rangle z_i^* = \sum_{i=2}^{n} (1 - 1/k_{ii}) z_i^*,
\]

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The first equality holds for any \( w \in \mathbb{B}^{n-1} \), the second insures the that holds.

We also want \( z_{n+1} \) given by (52) to satisfy \(|z_{n+1}|^2 = 1 - 1/k_{n+1,n+1}\). We have specified \( w \) and from (52) we know \(|w|^2 + |c|^2 = |z_{n+1}|^2\). Hence, to insure we can find the required \( c \) we need to show that \( w \) satisfies

\[
|w|^2 \leq 1 - 1/k_{n+1,n+1}, \tag{54}
\]

This requirement; that the length of the projection of the as yet undiscovered target vector \( z_{n+1} \) onto the linear span of the points already identified is, in fact, less than that the desired length of \( z_{n+1} \), is the higher dimensional analog of the statement \( LF_{123}^2 \leq \delta_{13}^2 \) in Theorem 16.

We now use the CPP hypothesis to obtain the length estimate. There is no loss in passing to a rescaling of \( H^+ \), and hence we suppose \( k_1 \) is identically one. In that case the McCullough-Quiggen matrix of (11), with subscript 1, is

\[
MQ^+ = MQ_1(H^+) = \left( 1 - \frac{1}{k_{rs}} \right)_{2 \leq r,s \leq n+1}. \tag{55}
\]

The fundamental fact about this matrix is the following.

**Theorem 22** (McCullough-Quiggen [AM, Thm. 7.6]) If \( H^+ \) has the CPP then the matrix \( MQ^+ \) is positive semidefinite.

**Proof.** We assumed \( H^+ \) is generic so we would know that the vectors of \( Z \) are linearly independent, which is equivalent to \( MQ \) being strictly positive definite. We write \( MQ^+ \) in block form as

\[
MQ^+ = \begin{pmatrix}
MQ & v^* \\
v & c
\end{pmatrix}.
\tag{56}
\]

Here \( MQ \) is the sub-matrix of \( MQ^+ \) obtained by deleting the last row and last column,

\[
MQ = \left( 1 - \frac{1}{k_{rs}} \right)_{2 \leq i,s \leq n}
\]

The other terms are what are needed to complete \( MQ^+ \); \( v = \left( 1 - \frac{1}{k_{ns}} \right)_{2 \leq s \leq n} \), \( v^* \) is the conjugate transpose of \( v \), and \( c \) is the scalar \( 1 - k_{n+1,n+1}^{-1} \). A computational lemma now lets us recast the hypothesis on \( MQ^+ \) in the form we will use.

**Lemma 23** \( MQ^+ \) is positive definite if and only if \( MQ \) is positive definite and

\[
vMQ^{-1}v^* < c \tag{57}
\]
Proof of Lemma. This is the characterization of the positivity of $MQ^+$ in terms of the Schur complement of $MQ$ [MA pg. 86].

Our desired estimate for $|w|$ is given by (54). We want

$$|w|^2 = \sum_{i,j=1}^n (1 - 1/k_{in}) \langle z_i^*, z_j^* \rangle (1 - 1/k_{jn}) \leq 1 - 1/k_{n+1,n+1}. \quad (58)$$

Defining the matrix $J$ by $J = (\langle z_i^*, z_j^* \rangle)_{i,j=2}^n$ and recalling the notation used in (56) we can rewrite the desired inequality as $vJv^* < c$. If we show $J = MQ^{-1}$ then, appealing to Lemma 23 we will be finished. That matrix equality is the standard relation between the Gram matrix of a basis and the Gram matrix of its dual basis. We saw it earlier in (34) where $K^*$ had the role of $J$.

Suppose now that $H^+$ is not generic. Renumber the points in $Z = \{z_i\}_{i=2}^{n+1}$ so that $W = \{z_i\}_{i=2}^s$ is a maximal linearly independent set. Form a basis $V$ of $\mathbb{C}^{n-1}$ by adding $n-s$ vectors to $W$, each orthogonal to the vectors in $Z$. We temporary abuse notation and let $Z^* = \{z_i^*\}_{i=1}^n$ be the basis dual to $V$. Now we set

$$w = \sum_{i=2}^s \langle w, z_i \rangle z_i^* = \sum_{i=2}^s (1 - 1/k_{in}) z_i^*,$$

and note that the upper index of summation is $s < n$. It is immediate from the definitions that (55) holds for $1 \leq j \leq s$. For the remaining values of $j$ note that $z_j$ is a linear combination of $z_i$ with $i \leq s$ and that $(1 - 1/k_{jn})$ is the same linear combination of $(1 - 1/k_{in})$. Combining these two facts insures that (55) holds for the full range of $j$.

One reason we are considering the non-generic situation separately is to avoid having to work with the more complicated analog of Lemma 23 that holds when $MQ^+$ is semidefinite. In this case we want the estimate

$$|w|^2 = \sum_{i,j=2}^s (1 - 1/k_{in}) \langle z_i^*, z_j^* \rangle (1 - 1/k_{jn}) \leq 1 - 1/k_{n+1,n+1}. \quad (59)$$

where now we have a different upper limit of summation. In this case we start from the matrix $MQ^+$, which is positive semidefinite, and remove rows and columns $s + 1, ..., n$. The resulting matrix $MQ^-$ is still positive semidefinite and furthermore its upper left block obtained by deleting the last row and last column is strictly positive definite. This last property by the fact that the vectors in $W$ are linearly independent. Given the strict positivity of that block we have the analog of $vK^{-1}v^* < c$ and can finish the argument as before.

This result includes a statement about the Gram matrix of a four dimensional $H$ which insures there is an embedding $\Phi(X)$, but the geometric content of the result is elusive. Consider the following specific question. Given a four point set $X$ in $\mathbb{C}H^3$, we can think of $X$ as the vertices of a tetrahedron. That configuration is described by nine parameters; the six distances between pairs of points together with the angular invariants associated with any three of the
triangular faces, the fourth angular invariant being determined by the cocycle identity \( (10) \). The inverse question is this; given four triangles in \( \mathbb{CH}^3 \), can they be assembled as the faces of a tetrahedron? That is, is there a tetrahedron in \( \mathbb{CH}^3 \) whose four triangular faces are congruent to the four given triangles? The triangles must satisfy the obvious necessary conditions; side lengths must match and the cocycle identity for the faces must hold. A configuration which meets these conditions is described by the same nine parameters used to describe a tetrahedron. However that is not the full story. An example due to Quiggen [AM, Pg. 94] shows that there must be additional conditions. He gives a four dimensional RKHS, \( H \) such that each of the four natural three dimensional RKHS subspaces, \( \{ H_i \}_{i=1}^4 \) has the CPP, but \( H \) does not. The fact that each \( H_i \) has the CPP insures that for each \( i \) we can find a triple \( X_i \) in \( \mathbb{C}^n \) with \( H_i \sim DA_n(X_i) \). The kernel functions for \( H_i \) are kernel functions from \( H \) and hence the side lengths of the triangle \( X_i \) and its angular invariant are the same as would be computed from the Gram matrix of \( H \). This insures that these four triangles satisfy the matching side length conditions and also that the \( A \)'s satisfy the cocycle condition. However if the triangles could be assembled into a tetrahedron with vertices \( X \) then we would have \( H \sim DA_n(X) \) and hence \( H \) would have the CPP; but it does not. The obstruction to there being such an \( X \) must be that the inequality \( (57) \) fails; but the geometry associated with that failure is not clear.

In the proof of Proposition 33 we will see an example of how, under restrictive assumptions, locally coherent information about three dimensional subspaces can be spliced together to completely describe a larger space.

6 The Geometry of Sets

6.1 Three Point Sets

We now look at three dimensional spaces \( H \sim DA_n(X) \) and the relation between the analytic and algebraic properties of \( H \) and the geometric properties of \( X \). We focus on the complex hyperbolic analogs of the Euclidean statements that a set of points is colinear or coplanar. Taking note of the comments in Section 3.3, some of the results apply \textit{mutatis mutandis} to subspaces of larger spaces and subsets of larger sets. We consider additional results for larger spaces and sets in the next section.

We are interested in properties of \( H \) that are unchanged by rescaling, and properties of \( X \) that are unchanged by automorphisms. With that in mind we focus on \( H = DA_2(X) \) with \( X \) a three point set in normal form. We denote the collection of all such sets by \( \mathcal{N} \).

\[
\mathcal{N} = \{ X = \{ x_1, x_2, x_3 \} = \{ (0,0), (s,0), (w,t), s, t > 0, w \in \mathbb{C} \} \subset \mathbb{CH}^2 \} \quad (60)
\]

We will be particularly interested the certain subsets of \( \mathcal{N} \);
\[ A = \{ X \in \mathcal{N} : s > 0, w \in \mathbb{R}, t = 0 \} \quad \text{in a geodesic curve} \]
\[ B = \{ X \in \mathcal{N} : s > 0, w \in \mathbb{R}, t > 0 \} \quad \text{in a real geodesic disk} \]
\[ C = \{ X \in \mathcal{N} : s > 0, w = 0, t > 0 \} \quad \text{.....with a right angle at } x_1 = 0 \]
\[ D = \{ X \in \mathcal{N} : s > 0, w = s, t > 0 \} \quad \text{.....with a right angle at } x_2 \]
\[ E = \{ X \in \mathcal{N} : s > 0, w \in \mathbb{C}, t = 0 \} \quad \text{in a complex geodesic} \]

6.1.1 Points on a Geodesic

For \( X \) in \( \mathcal{N}, x_1 = 0 \). Hence any hyperbolic geodesic segment containing \( x_1 \) sits in a Euclidean line through the origin. That observation will sometimes let us use Euclidean coordinate geometry rather than hyperbolic incidence geometry.

Any two points in \( \mathbb{C}H^2 \) determine a unique geodesic segment. If a three point set \( Y \) which lies on a geodesic is put in normal form then one point will be at the origin and hence the geodesic will lie in a real line through the origin. That is, if \( Y \sim X \in \mathcal{N} \) then the points of \( Y \) are on a single geodesic exactly if \( X \in A \). One invariant characterization of that configuration is that the points produce equality in the triangle inequality for the hyperbolic metric (and hence the "triangle" is degenerate); equivalently, equality holds in the strong triangle inequality for the pseudohyperbolic metric. In either case one can tell which point is between the others by noting which of the two possible equalities holds.

Alternative characterizations can also be given; for instance a geodesic segment lies on the intersection of a complex geodesic and a real geodesic disk. Combining that with the results below characterizing those geometric conditions we have the following.

**Proposition 24** Suppose \( X \) is a three point set in \( \mathbb{C}H^n \) and \( H = DA_n(X) \). The points of \( X \) lie on geodesic if and only if \( LF_{123} = \delta_{13} \) and \( A_{123} = 0 \).

6.1.2 Points in a Real Geodesic Disk

The totally geodesic submanifolds of \( \mathbb{C}H^n \) of real dimension one are the geodesics, we just considered those. We now look at the two types of totally geodesic submanifolds of real dimension two introduced in Section 2.5. We begin with the real geodesic disks.

Any real geodesic disk is equivalent under an automorphism to the intersection of the ball with the real linear span of \((1, 0, ..., 0)\) and \((0, 1, 0, ..., 0)\). Hence a general three point set \( Y \) is in a real geodesic disk if and only if it is congruent to a set in \( B \). If \( X \in \mathcal{N} \) this is equivalent to having \( k_{23} \in \mathbb{R} \). An equivalent invariant statement is the following.

**Proposition 25** The set \( Y \) lies in a real geodesic disk if and only if \( Y \sim X \) for some \( X \in B \), if and only if \( A_{123} = 0 \).

This proposition describes the geometry associated with the minimal value of \( A_{123} \). The maximal value of \( A_{123} \) for \( H = DA_2(Y), |A_{123}| = \pi/2, \) is not
attained for any \( Y \subset \mathbb{C}H^2 \). However, if we extend the definition of \( A_{123} \) by continuity to distinct triples in \( \partial \mathbb{H}^2 = \mathbb{S}^3 \), then that value can be attained. It was shown by E. Cartan that the value \( \pi/2 \) is attained exactly if \( Y \) lies on the intersection of the boundary with the closure of a geodesic disk (i.e. the three points lie in a chain). For a full discussion see [Go, Cor 7.1.3].

There are two subsets of \( B \) that we want to look at more closely; \( C \) and \( D \). The points of any \( X \in B \) sit inside the real disk \( \{ (s, t) : s, t \in \mathbb{R}, s^2 + t^2 < 1 \} \subset \mathbb{B}^2 \). As we mentioned, that disk, with the metric \( \delta_{D,\mathbb{A}_2} = \rho \), is the Beltrami-Klein model of \( \mathbb{R}H^2 \), and in that model the Euclidean lines segments are the hyperbolic geodesics. Hence the hyperbolic triangle with vertices at the points of \( X \in B \) is the same as the Euclidean triangle. For sets \( X \) in \( C \) or \( D \) that triangle is a (Euclidean and hyperbolic) right triangle. For \( X \in C \) the right angle is at \( x_1 = 0, x_2 \perp x_3 \); for \( X \in D \) the right angle is at \( x_2, x_2 \perp x_2 - x_3 \). These two configurations are actually not very different. The negative of the ball involution interchanging \( x_1 \) and \( x_2 \) interchanges sets in \( C \) with those in \( D \). These types of Euclidean orthogonality are pervasive in subsets of the sets \( X \) we construct in Section 7 where we study spaces \( DA_n(X) \) associated with spaces of functions on trees.

Given the previous results, including Theorem 3, the following equivalences are straightforward.

**Proposition 26** Given a three point set \( Y \) in \( \mathbb{C}H^n \), let \( X \) be the normal form of \( Y \), and \( H = DA_2(X) \). The following are equivalent:

1. \( X \in C \).
2. \( x_2 \perp x_3 \).
3. \( k_{23} = 1 \).
4. \( LF_{123} = 0 \).
5. \( \Delta = \delta_{12}\delta_{13}/\delta_{23} \).

Note that the first two statements are unchanged if \( Y \) is replaced by \( \Theta Y \) with \( \Theta \in \text{Aut}(\mathbb{C}H^n) \), and hence they are statements about the hyperbolic geometry of \( Y \). The last two are algebraic/analytic statements about \( H = DA_n(Y) \) that are invariant under rescaling of \( H \).

The analogous result for \( D \) is

**Proposition 27** Given a three point set \( Y \) in \( \mathbb{C}H^n \), let \( X \) be the normal form of \( Y \), and \( H = DA_2(X) \). The following are equivalent:

1. \( X \in D \).
2. \( x_2 \perp x_2 - x_3 \).
3. \( k_{22} = k_{23} \).
4. $LF_{213} = 0$.

5. $\Delta = \delta_{12}$.

Note that the fourth statements in the previous two propositions have different strings of indices.

**Proof.** The equivalence of the first four statements is straightforward. We then look at the last condition. Recall that $X$ is in normal form. In that case Condition (3), $k_{22} = k_{23}$, implies $|k_{23}|^2 = |k_{13}|^2$. Using this with $\delta_{ij}^2 = 1 - |\hat{k}_{ij}|^2$ leads to $\delta_{23}^2 = \delta_{13}^2 - \delta_{12}^2 (1 - \delta_{23}^2)$. Noting again that $k_{22} = k_{23}$ and using (12) this can be rewritten as $\delta_{23}^2 = \delta_{13}^2 (1 - LF_{123}^2 (1 - \delta_{23}^2))$. That equality combined with (25) gives Condition (5). On the other hand if we have Condition (5) then we have equality between solutions to two extremal problems, one for a function required to vanish at $x_2$ and one for a function required to vanish at both $x_2$ and $x_3$. Both of the problems have unique solutions, hence the solutions agree. In particular the solution to the first problem must vanish at $x_3$. Using that fact and the explicit formula for that extremal function given by (19) with $x = x_1$ and $y = x_2$ we see that we must have $k_{22} = k_{23}$ which is Condition (3).

**6.1.3 Points on a Complex Geodesic, Pick Spaces**

If $X \in \mathcal{N}$ and $X$ is contained in a complex geodesic then, recalling that $x_1$ is at the origin, the complex geodesic must be the intersection of the ball with the complex line $\mathbb{C} x_2$. In short, $X \in \mathcal{E}$. Direct substitution in (12) then yields $LF_{123}^2 = \delta_{13}^2$. Furthermore note from (24) that $LF_{123}^2 = \delta_{13}^2$ is equivalent to

$$\Delta = \delta_{12} \delta_{13}. \quad (62)$$

There are relations between multipliers that are a consequence of this. For $i = 2, 3$ let $m_i$ be the multiplier of unit norm which vanishes at $x_i$ and maximizes $\text{Re} m_i(x_1)$; and let $m_{23}$ be the multiplier of unit norm which vanishes at both $x_2$ and $x_3$ and maximizes $\text{Re} m_{23}(x_1)$. In this situation $m_{23} = m_2 m_3$. The second function is certainly a competitor for the extremal problem defining the first function. Further, that space of competitors is one dimensional. Hence $m_{23}$ must equal $m_2 m_3/ \|m_2 m_3\|$. Hence,

$$\delta_{G12} \delta_{G13} = \Delta G = m_{23}(x_1) = \frac{m_2(x_1)m_3(x_1)}{\|m_2 m_3\|} = \frac{\delta_{G12} \delta_{G13}}{\|m_2 m_3\|}.$$ 

The first equality is (12), the second is from the definition of $m_{23}$, the next is from the analysis we just did of $m_{23}$, and the last is from the definitions of $m_1$ and $m_2$.

This string of equalities is equivalent to (62) and it is apparent that it holds if and only if $\|m_2 m_3\| = 1$. Hence those conditions are equivalent. Further, given our analysis of $m_{23}$, those conditions are also equivalent to $m_{23} = m_2 m_3$.

Let $M_2$ be the multiplier of norm one which vanishes at $x_1$ and maximizes $\text{Re} M(x_2)$, and similarly for $M_3$. The statement $LF_{123}^2 = \delta_{13}^2$ is equivalent to

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the statement that $M_2$ and $M_3$ are unimodular multipliers of each other. This holds because comparing (12) and (4) shows that $LF_{123}^2 = \delta_{13}^2$ is equivalent to $|M_2(x_3)| = \delta_{13}$. Hence a unimodular multiple of $M_2$ is a solution to the extremal problem defining $M_3$; but that solution is unique.

Continuing the notation of the previous paragraphs, we have proved the following result.

**Proposition 28** Given a three point set $Y$ in $\mathbb{C}^n$, let $X$ be the normal form of $Y$ and $H = DA_2(X)$. The following are equivalent:

1. The points of $Y$ lie in a single complex geodesic.
2. $X \in \mathcal{E}$.
3. $LF_{123}^2 = \delta_{13}^2$.
4. $M_2 = \alpha M_3$ for some $\alpha$, $|\alpha| = 1$
5. $m_{23} = m_2 m_3$
6. $\|m_2 m_3\| = 1$.
7. $\Delta = \delta_{12} \delta_{13}$.

Statements (3) through (7) are all invariant under rescaling or automorphism. Hence we could have started our analysis by using Corollary 18 and Condition (3) to reduce consideration of (4) – (7) to statements about multipliers on $H \subset H^2$. By the CPP for $H^2$ the questions could then be reduced to statements about Hardy space extremal problems. Those particular problems are elementary ones which are solved by Blaschke products of degree one and two.

That Statements (5) or (6) imply Statement (1) also follows from results by Cole, Lewis, and Wermer [CLW] in their work characterizing multiplier algebras of Pick spaces.

In the next result we do not assume $H \sim DA_2(X)$; the existence of $X$ and the fact that it can be selected in $\mathbb{C}H^1$ are the main conclusions.

Recall the language of Section 4.4; if $H \sim DA_1(X)$ for some $X \in \mathcal{E}$, then $H$ is called an $r$-Pick space, and $H$ is called $r$-orthogonal if it is the rescaling of a space whose Gram matrix is an orthogonal.

**Theorem 29** A three dimensional RKHS $H$ is $r$-orthogonal if and only if it is an $r$-Pick space.

**Proof.** That such a space is $r$-orthogonal is the three dimensional case of Theorem 11.

In the other direction, we start with an $H$ which is $r$-orthogonal and, with no loss of generality, do a preliminary basepoint rescaling with $x_1$ as basepoint. Let $K$ be the Gram matrix of $H$. Let $\tilde{H}$, with Gram matrix $\tilde{K}$, be the orthogonal RKHS which is a rescaled version of $H$. That is, $\tilde{K} \tilde{K}^t = I$. Because $H$ is a...
rescaling of $H$ there is a diagonal matrix, $\Gamma$, with nonzero diagonal entries $\{\gamma_i\}$, such that

$$\tilde{K} = \Gamma K \Gamma = (\gamma_i k_{ij} \gamma_j)_{i,j=1}$$

is an orthogonal matrix.

Let $R_i$, $i = 1, 2, 3$ be the rows of $\tilde{K}$, $R_i = (\gamma_i k_{i1} \gamma_1, \gamma_i k_{i2} \gamma_2, \gamma_i k_{i3} \gamma_3)$; and let $R^*_i$ be the column vectors that are their adjoints. Noting that $K^t = \bar{K}$ we see that the $R^*_i$ are also the column vectors of $\tilde{K}$. Hence the fact that $\tilde{K}$ is orthogonal implies that, for $i \neq j$, $R_i R^*_j = 0$;

$$0 = R_i R^*_j = \sum_s \gamma_i k_{is} \gamma_j k_{js} = \gamma_i \gamma_j \sum_s |\gamma_s|^2 k_{is} k_{js}.$$ 

We now consider the cases $(i, j) = (1, 2), (3, 1), (2, 3)$. By the basepoint rescaling we have $k_{ij} = 1$ if $i$ or $j$ is 1. Also, we can cancel the initial factor $\gamma_i \gamma_j$ in each equation. The resulting equations are

$$(1, 2) : |\gamma_1|^2 + |\gamma_2|^2 k_{22} + |\gamma_3|^2 k_{23} = 0.$$ 

$$(3, 1) : |\gamma_1|^2 + |\gamma_2|^2 k_{32} + |\gamma_3|^2 k_{33} = 0.$$ 

$$(2, 3) : |\gamma_1|^2 + |\gamma_2|^2 k_{22} k_{32} + |\gamma_3|^2 k_{23} k_{33} = 0.$$ 

We have assumed that there is a nontrivial set $\{ |\gamma_i|^2 \}$ for which the equations hold. For that to happen the matrix of coefficients of the $\{ |\gamma_i|^2 \}$ must be singular, and hence has determinant 0;

$$0 = \det \begin{pmatrix} 1 & k_{22} & k_{23} \\
1 & k_{32} & k_{33} \\
1 & k_{22} k_{32} & k_{23} k_{33} \end{pmatrix},$$

Expanding this, recalling that $k_{23} = \bar{k}_{32}$, and rearranging gives

$$|k_{23}|^2 (k_{33} + k_{22} - 1) + k_{22} k_{33} - 2k_{22} k_{33} \text{ Re } k_{23} = 0 \quad (63)$$

After dividing by $|k_{23}|^2 k_{33} k_{22}$ this yields

$$1 - \frac{1}{k_{22}} - \frac{1}{k_{33}} + \frac{1}{k_{22} k_{33}} = 1 + \frac{1}{|k_{23}|^2} - 2 \text{ Re } \frac{1}{k_{23}}.$$ 

This equation is $L F^2_{123} = \delta^2_{13}$ written in terms of the Gram matrix entries. Once we have that, by Theorem 10 we conclude that $H \sim DA_3(X)$ for some $X$, and then the previous proposition insures that we can select $X$ in $E$. ■

As we mentioned earlier, we know of no counterexample to a higher dimensional version of the previous result.

### 6.2 Larger Sets

In the previous section we considered three dimensional Hilbert spaces $H = DA_n(X)$ and related the structure of $H$ to the geometry of the three point set $X$. Now we consider $H = DA_n(X)$ with larger $X$. Several times we will use the argument used to show that a set in Euclidean space lies in a line if every three points in it are colinear.
Suppose \( H = DA_n(X) \) and we have applied a preliminary automorphism to \( X \) so that \( x_1 \) is at the origin; and hence, also, \( H \) is basepoint normalized. In that case the matrix \( MQ_1(X) \) defined in (11) is the Gram matrix of the set of vectors \( \{ x_i \}_{i=2}^n \subset \mathbb{C}^n \). Hence that matrix and variations on it can be used to study linear independence among the \( x_i \). Also, because \( H \) is basepoint normalized at \( x_1 \), the formulas for \( MQ_1(H) \) are quite simple. For instance, if \( n = 4 \) then

\[
MQ_1(H) = MQ_1(X) = \left( 1 - \frac{1}{k_{ij}} \right)_{2 \leq i,j \leq 4}.
\]

(64)

**Lemma 30** Given \( X = \{ x_i \}_{i=1}^4 \subset \mathbb{C}^4 = \mathbb{H}^4 \), with \( x_1 = 0 \), The set \( X \) lies on a complex line through the origin if and only if

\[
\det MQ_1(X \setminus \{ x_3 \}) = \det MQ_1(X \setminus \{ x_2 \}) = 0.
\]

The set \( X \) sits in a complex subspace of dimension two if and only if \( \det MQ_1(X) = 0 \).

**Proof.** We just look at the second case. From (64) we find that \( MQ_1(X) \) is the Gram matrix \( \langle x_i, x_j \rangle_{i,j=1,2,3,4} \). That matrix is nonsingular exactly if the three \( x \)'s are linearly independent. \( \blacksquare \)

The second statement in the next result is included because of the analogy with Proposition 32 below.

**Proposition 31** \( X \) lies in a geodesic

1. if and only if for any \( 1 \leq i < j < k \leq r \) we have \( A_{ijk} = 0 \) and \( LF_{ijk} = \delta_{ik} \).

2. if and only if for any \( 1 \leq i < j < k \leq r \) we have \( A_{ijk} = 0 \) and \( \det MQ_1(\{ x_i, x_j, x_k \}) = 0 \).

**Proof.** If three of the points are on a geodesic then Proposition 24 insures that we have the two equalities in the first statement. In the other direction, note that both those equations and the fact of lying on a geodesic are invariant under automorphisms of \( \mathbb{C}^n \). Hence we can suppose that \( x_1 \) is at the origin. In that case, by Proposition 24 we see that, for any index \( j \), the three points \( \{ x_1, x_2, x_j \} \) lie on a geodesic. Because \( x_1 \) is at the origin that geodesic must be a Euclidean line. Thus \( x_j \) is on \( L \), the Euclidean line through \( x_2 \) and the origin. Now note that \( j \) was arbitrary; hence all of the points lie on \( L \). That completes the proof for the first statement. That the second statement is equivalent to the first can be seen by writing the two differing expressions in terms of kernel functions. \( \blacksquare \)
6.2.2 Sets in Real Geodesic Disks or Totally Real Subspaces

We suppose $X = \{x_i\}_{i=1}^r \subset \mathbb{C}H^n$, $r > 3$, $H = DA_n(X)$. From Section 6.1.2 we know that if, for instance, $A_{123} = 0$ then $\{x_1, x_2, x_3\}$ lies in real geodesic disk; similarly if $A_{124} = 0$. However we cannot concatenate those results. Knowing both is not sufficient to insure that all four points lie in a single real geodesic disk. Consider, for instance, the origin and real vectors $a, b,$ and $c$ which are mutually orthogonal. However if we control the dimension of the real span of $\{x_i\}$ then we can go forward; and we can control that dimension using the matrices $MQ$. We will give a result with that dimension bounded by two but the general pattern will be clear.

The first statement in the next proposition is a variation on Lemma 2.1 of work by Burger and Iozzi, [BI], in which they consider sets that sit inside totally real subspaces. We will follow their language. More information about the geometry and properties of totally real subspaces is their paper and in [Go].

We will say that a subspace $S$ of $\mathbb{C}H^n$ is a totally real subspace of dimension $k$ if it is a totally geodesic submanifold isometric to $\mathbb{R}H^k$. In particular, if $k = 1$ then $S$ is an ordinary geodesics and for $k = 2$ it is a real geodesic disk. As before, the description is clearer if we use a preliminary automorphism to reduce to the case of $S$ containing the origin of $\mathbb{C}H^n = \mathbb{B}^n$. The geodesic connecting the origin to any other point is a radial line segment. Hence $S$ is the intersection of $\mathbb{B}^n$ with a totally real vector subspace of $\mathbb{C}^n$ of dimension $k$; that is, a real vector subspace of $\mathbb{C}^n$ spanned by vectors $\{v_i\}_{i=1}^k$ with all $\langle v_i, v_j \rangle$ real.

**Proposition 32** Suppose $X = \{x_i\}_{i=1}^N \subset \mathbb{C}H^n$ and $H = DA_n(X)$.

1. If $A_{ijk} = 0$ for every $i, j, k$ then $X$ is inside a totally real subspace of $\mathbb{C}H^n$.

2. If $A_{ijk} = 0$ for every $i, j, k$ and, furthermore, for every $i, j, k, l$ we have, in the notation of (17),

$$\det MQ_1(\{x_i, x_j, x_k, x_l\}) = 0;$$

then $X$ is contained in a real geodesic disk.

**Proof.** As before, we first use an automorphism to reduce to the case of $x_1$ at the origin. Having done that, $A_{ijk} = 0$ implies $k_{jk}$ is real which, in turn, implies $\langle x_i, x_j \rangle$ is real. Having that for all $j, k$ gives the first conclusion. For the second statement note that if all of $X$ sits in a single geodesic containing the origin then we are done. Otherwise we can find $x_i$ and $x_j$ which are linearly independent. Select any $x_k$ and consider the matrix $M = MQ_1(\{x_1, x_i, x_j, x_k\})$. By the hypothesis on the $A$'s the entries of $M$ are real and by the second part of the hypothesis $\det M = 0$. Hence $x_k$ is in the real linear span of $x_i$ and $x_j$. Because $x_k$ was arbitrary we have our conclusion. \[\blacksquare\]

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6.2.3 Sets in a Complex Geodesic, Pick Spaces

Some of the results in this paper have been for general $H$, others for $H$ which have the CPP. The next result considers an intermediate case, the assumptions of the result make it automatic that every subspace spanned by three reproducing kernels has the CPP. We do not know if that is enough to reach the conclusion of the proposition. On the other hand, if we assume that $H$ itself has the CPP then the desired conclusions follow easily. In the actual proposition we make the intermediate assumption that each subspace spanned by four kernel functions has the CPP.

Suppose $H$ is a RKHS with kernel functions $\{k_i\}_{i=1}^m$, $m \geq 4$, and $X = X(H) = \{x_i\}_{i=1}^m$. For any set $\Lambda$ of indices let $H(\Lambda)$ be the subspace spanned by $\{k_i\}_{i \in \Lambda}$.

**Proposition 33** Suppose that for every four element set of indices, $\Lambda$, the space $H(\Lambda)$ has the CPP. Then $H \sim DA_1(Y)$ for some $Y \subset \mathbb{C}H^1$, that is, $H$ is an $r$-Pick space, if and only if for any $1 \leq i < j < k < m$, $LF_{ijk} = \delta_{ik}$.

**Proof of a simpler result.** If $H$ is an $r$-Pick space then we can apply Proposition 28 to all the three dimensional subspaces of $H$ and obtain the condition on the $LF$’s. In the other direction, if we had the stronger assumption that $H$ has the CPP then we could start with $H \sim DA_4(Y)$ for some $Y = \{y_i\}$ in $\mathbb{C}H^n$. Then, again by Proposition 28 we would see that every three element subset of $Y$ lies in a complex geodesic. However if two complex geodesics share a pair of points then they are the same. Hence $Y$ actually sits in a single geodesic. The rescaling induced by the automorphism placing $Y$ on the $z_1$ axis produces the required Pick space.

**Proof.** Without loss of generality $H$ is basepoint normalized with $x_1$ as basepoint. From our analysis of multipliers on two dimensional spaces in Section 5.2 we know that there is a unique multiplier $M$ on $H(\{1, 2\})$ which is of norm one and has $M(x_1) = 0$ and $M(x_2) = \delta_{(1, 2)}(x_1, x_2) = \delta_H(x_1, x_2)$. Suppose $2 < r \neq s \leq m$. By the hypothesis $H(\{1, 2, r, s\})$ has the CPP. Hence there is a norm one extension of $M$ to a multiplier $M^{rs}$ on $H(\{1, 2, r, s\})$. Furthermore $M^{rs}$ must be given by (23) which shows that $M^{rs}$ is unique and also shows that $M^{rs}(x_s)$ does not depend on the index $s$. Define the set $Y = \{y_i\}_{i=1}^m \subset \mathbb{D} = \mathbb{C}H^1$ by $y_1 = M(x_1) = 0$, $y_2 = M(x_2) = \delta_{12}$ and, for $r > 2$, $y_r = M^{rs}(x_r)$ (which we just noted does not depend on $s$).

To complete the proof we will show that $H \sim DA_1(Y)$. We will establish that by showing that the Gram matrix $K(H) = (k_h)$ equals the Gram matrix $G(DA_1(Y)) = J = (j_{rs})$. By construction both matrices have only 1’s in their first row and first column. Now select $r, s$ with $2 \leq r, s \leq m$; we want to show $k_{rs} = j_{rs}$. Gram matrix elements are stable under passage to subspaces spanned by reproducing kernels, so we can study the gram matrix element $k_{rs}$ in the context of the Hilbert space $H(\{1, 2, r, s\})$ (with the obvious modifications in interpretation if $r = s$). We assumed that the four dimensional space $H(\{1, 2, r, s\})$ has the CPP, and hence we have $H(\{1, 2, r, s\}) \sim DA_3(\{a, b, c, d\})$ for $\{a, b, c, d\} \subset \mathbb{C}H^3$. The argument we gave in the "Proof of a simpler result"
shows that, in fact, \( \{a, b, c, d\} \) lies in a complex geodesic. Hence, using an automorphism we can reduce to the case \( \{a, b, c, d\} \subset \mathbb{D} = \mathbb{C}^1 \), \( a = 0 \), \( b = \delta_{12} \).

The Gram matrix entries of \( H(\{1, 2, r, s\}) \), computed in the space \( H \), match the Gram matrix entries of \( DA_1(\{a, b, c, d\}) \) computed by regarding that space as a subspace of \( DA_1 \). In particular \( k_{rs} = (1 - \cd) \). If we can establish that \( y_r = c \) and \( y_s = d \) we will have the desired match. The same argument is used for both equalities and we will just look at the first.

The spaces \( H_r = H(\{1, 2, r\}) \) and \( DA_1(\{a, b, c\}) = DA_1(\{0, \delta_{12}, c\}) \) are corresponding subspaces of the two four dimensional spaces we just looked at, and hence \( H_r \sim DA_1(\{0, \delta_{12}, c\}) \). We know from Theorem 16 that this uniquely determines \( c \). Consider now any three dimensional space \( J \) with \( X(J) = \{j_1, j_2, j_3\} \) and for which we know \( J \sim DA_1(\{\alpha, \beta, \gamma\}) \) for some \( \{\alpha, \beta, \gamma\} \subset \mathbb{D} \). Let \( N \) be the unique multiplier on \( J \) of norm one with \( N(j_1) = 0 \) and \( N(j_2) = \delta_{12} \). By noting formula (20) and also looking at the proof of Theorem 16 we see that that

\[
J \sim DA_1(\{N(j_1), N(j_2), N(j_3)\}) = DA_1(\{0, \delta_{12}, N(j_3)\}).
\]

We now compare these facts. If \( J = H_r \) then \( j_3 = x_r \) and \( N \) is \( M^\ast \) restricted to \( H_r \). Hence \( N(j_3) = M^\ast(x_r) = y_r \). Thus \( H_r \sim DA_1(\{0, \delta_{12}, y_r\}) \). Comparing this with the earlier unique representation in that form we conclude \( y_r = c \), which is what we needed to finish.

The argument in the proof gives a type of description of \( \text{r-Pick spaces} \).

**Corollary 34** Suppose \( H \) is an \( \text{r-Pick space} \) with \( X(H) = \{x_i\}_{i=1}^m \), and \( M \) is a multiplier on \( H \) of norm one which, and for some \( i, j \), has \( \rho(M(x_i), M(x_j)) = \delta_H(x_i, x_j) \). Then \( H \simeq DA_1(\{M(x_i)\}) \).

## 7 Function Spaces on Trees

### 7.1 Defining the Spaces

In this section we study a class of Hilbert spaces \( H \) of functions on trees \( T \). Many natural examples of this type of space are infinite dimensional and certainly some of what we do extends to that setting, but we continue to assume our \( H \) are finite dimensional.

We start with a rooted tree \( T \), a connected loopless graph with a root vertex \( o \). For vertices \( x, y \in T \), we let \( [x, y] \) denote the non-overlapping path connecting \( x \) and \( y \). We will be informal about whether that path consists of vertices, edges, or both. If \( w, x, y, z \) are vertices we will write \( w < x \) if \( w \in [o, x) \), write \( x \setminus y = z \) if \( z = \sup\{[o, x] \cap [o, y]\} \), and denote the immediate predecessor of \( y \), \( \sup\{[o, y]\} \), by \( y^- \).

One way to form a RKHS \( H \) of functions on \( T \) with properties related to the structure of \( T \) is suggested by the metaphor that " < " reflects a flow of time or a flow of influence. With that in mind, we define kernel functions \( \{k_x\} \) with the value \( k_{xy}, x, y \in T \), determined by the "shared past" of \( x \) and \( y \). Explicitly,
we select a function \( \Omega \) defined on \( T \) which satisfies, for \( x, y \in T \),

\[
\Omega(o) = 1, \quad \Omega(y) < \Omega(x) \text{ if } y < x.
\]

(65)

and define \( k = k_\Omega \) by

\[
k_{xy} = k_{x \land y} x \land y = \Omega(x \land y).
\]

(66)

These conditions insure that \( k_{xy} > 0 \) and that \( k \) satisfies the Cauchy Schwarz inequality, \( k_{xy}^2 \leq k_{xx} k_{yy} \).

In fact this definition insures that \( k_{xy} \) is the reproducing kernel for a space \( H \) and that \( H \) has the CPP. We will establish both facts by explicitly constructing a map \( \Phi \) of \( T \) into \( \mathbb{C}^n \).

The entries of the Gram matrix are real and hence, in the language of Section 4.2, the space is equal to its conjugate, \( H = \overline{H} \); and, also, \( \Phi(T) = \Phi(T) \subset \mathbb{R}^n \subset \mathbb{C}^n \). Furthermore, if the kernel function satisfies (65) then it also satisfies the weaker condition

\[
\text{if } x < y \text{ then } k_{xy} = k_{xx}.
\]

(67)

That condition is reflected in the shape of \( \Phi(T) \); triples of points in \( \Phi(T) \) have the type of orthogonality described in Proposition 27.

In addition, independently of the construction of \( \Phi \), we will use the algebraic structure of \( k_{xy} \) to show that it is a reproducing kernel and has the CPP.

**Theorem 35** Let \( K \) be the kernel function for \( DA_n \). If \( T \) and \( \Omega \) are as described above, then:

1. The function \( k_{xy} \) in (66) is the reproducing kernel for a RKHS, \( H = H(T, \Omega) \), of functions on \( T \).
2. The space \( H \) has the CPP.
3. There is a map \( \Phi_\Omega : T \rightarrow \mathbb{R}^n \) with \( \Phi_\Omega(o) = 0 \), and for all \( x, y \in T \),

\[
k_{xy} = K_{\Phi_\Omega(x) \Phi_\Omega(y)} = \Omega(x \land y). \text{ Thus } H = H(T, \Omega) = DA_n(\Phi_\Omega(T)).
\]

**Proof.** First we will construct the map \( \Phi_\Omega \) required for (3). Once we have that then statements (1) and (2) follow from general facts about spaces \( DA_n(X) \). We will then give an alternate proof of (1) and (2) using a summation by parts formula for kernel functions of the form described by (65) and (66).

To construct \( \Phi \) we first construct the **spine of \( T \)\**, \( \text{Sp}(T) \), a set of strings of orthonormal vectors in \( \mathbb{R}^n \) which is indexed by elements of \( T \). Let \( E = \{e_x\}_{x \in T} \) be a set of orthonormal vectors in \( \mathbb{R}^n \). For each \( y \in T \) let \( \{o, y_1, y_2, ..., y_l\} \) be the ordered string of vertices in the interval \( [o, y] \). Let \( st(y) \) be the corresponding ordered string of elements of \( E \), \( st(y) = \{e_o, e_{y_1}, ..., e_{y_l}\} \). Set \( \text{Sp}(T) = \{st(y) : y \in T\} \).

Using \( \text{Sp}(T) \) we construct \( \Phi \) by selecting appropriate positive scalars \( \{c_x\}_{x \in T} \) and setting

\[
\Phi(y) = \sum_{w \in st(y)} c(w)e_w.
\]

(68)
We define the coefficients \( \{c(w)\}_{w \in T} \) by induction on the parameter \( n(w) \), the number of edges in the path \([o, w]\). The only \( w \) with \( n(w) = 0 \) is \( w = o \) and we begin by setting \( c(o) = 0 \); that is, we map the root vertex to the origin. Suppose now we have defined the \( \{c(w)\} \) for all \( w \) with \( n(w) \leq N \). Select \( z \) with \( n(z) = N + 1 \). We have \( n(z^-) = N \), hence by our induction hypotheses and the definition of \( \Phi \), \( \Phi(z^-) \) is already defined. Set \( \Phi(z) = \Phi(z^-) + c(z)e_z \) with \( c(z) \) the positive number which we now define. In order to have

\[
k_{zz} = K_{\Phi(z)\Phi(z)} = \Omega(z)
\]  
(69)

we need

\[
\langle \Phi(z), \Phi(z) \rangle = 1 - \frac{1}{\Omega(z)}.
\]  
(70)

By our construction of the string \( st(z) \), the \( e_w \) corresponding to \( w \in [o, z] \) are orthogonal to \( e_z \). Hence we want

\[
1 - \frac{1}{\Omega(z)} = ||\Phi(z)||^2 = ||\Phi(z^-)||^2 + ||c(z)e_z||^2 = 1 - \frac{1}{\Omega(z^-)} + c(z)^2.
\]

Thus we want \( c(z)^2 = \Omega(z^-)^{-1} - \Omega(z)^{-1} \). Because \( \Omega \) is increasing that quantity is positive. Hence we can select \( c(z) > 0 \) and complete the definition of \( \Phi(z) \). There is no obstacle in repeating this process through the set of \( z \) with \( n(z) = N + 1 \) to complete the inductive step in the definition. Thus we have \( \Phi(z) \) defined for all \( z \), Note that the construction insures that

\[
1 - \frac{1}{\Omega(z)} = ||\Phi(z)||^2
\]  
(71)

holds for every \( z \).

We now check that for any \( z, w \in T \) we have \( k_{wz} = K_{\Phi(w)\Phi(z)} = \Omega(w \wedge z) \)

Taking note of the formula for \( K \) it suffices to show that

\[
\langle \Phi(w), \Phi(z) \rangle = \langle \Phi(w \wedge z), \Phi(w \wedge z) \rangle = 1 - \frac{1}{\Omega(w \wedge z)}.
\]

The structure of the tree insures that \( w \wedge z \) is a point on the geodesic \([o, w]\) and on the geodesic \([o, z]\). Taking note of the orthogonality relations in \( st(w) \) and \( st(z) \) and the formula (69) we see that \( \Phi(w) = \Phi(w \wedge z) + r(w, z) \) with \( r(w, z) \perp \Phi(w) \) and also \( \Phi(z) = \Phi(w \wedge z) + t(z, w) \) with \( t(z, w) \perp \Phi(z) \). Furthermore, taking note of the definition of \( w \wedge z \), the substrings \( st(w) \setminus st(w \wedge z) \) and \( st(z) \setminus st(w \wedge z) \) are disjoint. That implies \( r(w, z) \perp t(z, w) \). Combining these facts gives the first equality in the previous display follows. To obtain the second equality follows from (71).

That completes the proof of (3) which, as we noted, implies (1) and (2). We now give an independent proof of (1) and (2) using a summation by parts formula for bilinear forms with kernel functions such as \( \{k_{xy}\} \) which are functions of \( x \wedge y \).
It is convenient to introduce several operators on functions defined on \( T \). For \( g \) a function on \( T \), \( a, b, c \in T \) we set
\[
I g(\alpha) = \sum_{\tau \leq \alpha} g(\tau),
\]
\[
I^* g(b) = \sum_{\tau \geq \beta} g(\tau)
\]
\[
D g(c) = g(c) - g(c^-) \text{ if } c \neq o, \quad D g(o) = 0.
\]
The notation follows the usage in \([ARS02]\) where \( I \) as a discrete model for integration and the operator \( I^* \) is the adjoint of \( I \) with respect to the pairing of \( \ell^2(T) \). The operator \( D \), the difference operator, is the one sided inverse to \( I \).

The following summation by parts formula is Lemma 3 of \([ARS10]\). It is proved there by several lines of straightforward computation.

**Lemma 36 (Summation by parts.)** For any functions \( h \) and \( f \) defined on \( T \) we have
\[
\sum_{x,y \in T} h(x \wedge y) f(x) \overline{f(y)} = h(o) |I^* f(o)|^2 + \sum_{z \in T} (h(z) - h(z^-)) |I^* f(z)|^2.
\]

To establish (1) we need to know that for any function \( f \) defined on \( T \) we have
\[
\sum_{x,y \in T} k_{xy} f(x) \overline{f(y)} \geq 0
\]
with equality only if \( f \) is the zero function. We apply the lemma with \( h = \Omega \). Because \( \Omega \) is increasing the term \( (h(z) - h(z^-)) \) is positive and hence the resulting bilinear form is positive definite. Because \( k \) is defined in terms of \( \Omega \) by (66), this shows that \( k \) defines a positive definite form, and hence is the kernel function of some Hilbert space \( H \). Furthermore, by Theorem 7.28 of \([AM]\), to show \( H \) has the CPP it suffices to show that, in addition, \( 1 - 1/k_{xy} \) generates a positive bilinear form. That follows from the same lemma, applied this time to the function \( h = 1 - 1/\Omega \) which is increasing because \( \Omega \) is.

**Corollary 37 (Infinite divisibility of kernel functions)** If \( k_{xy} \) and \( \Omega \) are as in the previous theorem, and if \( \Lambda \) is any strictly increasing function with \( \Lambda(1) = 1 \) then \( k^\Lambda_{xy} = \Lambda(k_{xy}) \), the kernel function associated with \( \Omega \lambda = \Lambda(\Omega) \) through (66), is the kernel function of the RKHS with CPP. In particular, for \( 0 < \lambda < \infty \), \( k^\Lambda_{xy} \) is such a kernel.

This is a consequence of the theorem and the observation that if \( \Omega \) satisfies (65) then so does \( \Lambda(\Omega) \). The spaces with kernel function \( k^\Lambda_{xy} \) arise naturally in the study of Hankel forms on \( H \) and have an independent intrinsic description, there is some discussion of this and further references in \([FR]\).
space with the CPP [AM, Remark 8.10]. However that range of \( \lambda \) is sharp. That is shown by the family of spaces \( D_\lambda \) of Example 19, a family which includes the Hardy space for \( \lambda = 1 \) and the Bergman space at \( \lambda = 2 \).

It is possible to reverse the construction in the theorem and recover the tree from the Hilbert space. For instance, suppose we have a RKHS \( H \) with its set of reproducing kernels \( \{k_\lambda\}_{\lambda \in \Lambda} \) and that all \( k_{\lambda\mu} = \langle k_\lambda, k_\mu \rangle \) are real. In analogy with (67), define a partial order \( \preceq \) on \( \Lambda \) by

\[
\sigma \preceq \tau \text{ if } k_{\sigma\tau} = k_{\sigma\sigma}.
\]

Suppose there is an element \( \alpha \) so that for all \( \lambda \in \Lambda \) we have \( k_{\alpha\lambda} = 1 \), or, equivalently, \( k_{\alpha\alpha} = 1 \) and for all \( \lambda, \alpha \preceq \lambda \). Suppose further that for each \( \lambda \) the segment \([\alpha, \lambda]\) = \( \{\mu \in \Lambda : \mu \preceq \lambda\} \) is totally ordered by \( \preceq \). This is enough data to form \( T \), a rooted tree with \( \Lambda \) as its vertex set and \( \alpha \) as the root. If we define \( \Omega \) on \( T \) by requiring (69) hold then our space \( H \) is the space \( H(T, \Omega) \) produced by the earlier construction. In fact, if we do not start with a Hilbert space, but just start with a real valued function \( k \) on \( \Lambda \times \Lambda \) which induces a partial order of the type described then the previous discussion produces a tree and the Hilbert space of functions on that tree having a kernel function with the CPP.

Special cases of the previous theorem are proved in [Haa] and [N]. Although those proofs are formulated very differently, they center on constructing strings of orthonormal vectors similar to our \( \text{Sp}(T) \).

It is not hard to see that such strings of orthonormal vectors must provide the framework of any mapping such as \( \Phi \).

7.2 Formulas for the Norm

We can think of \( \Omega(t) \) as defining the length of the path \([0, t]\) and let \( \omega \) as the length of the individual segments. We then have \( \Omega(t) = \sum_{o<s \leq t} \omega(s) \), or, equivalently, \( \omega = D\Omega \).

When the kernel function of \( H \) is of the form \( k_{xy} = \Omega(x \wedge y) \) we can write the distance function \( \delta_H \) using \( \Omega \). For \( y \in T \) and \( y^- \) the predecessor of \( y \) we have \( k_{y^-y^-} = k_{yy^-} = \Omega(y^-) \) and \( k_{yy} = \Omega(y) \). Thus

\[
\delta_H^2(y, y^-) = 1 - \frac{\Omega(y^-)^2}{\Omega(y^-)\Omega(y)} = \frac{\Omega(y) - \Omega(y^-)}{\Omega(y)} = \frac{D\Omega(y)}{\Omega(y)} = \frac{\omega(y)}{\Omega(y)}.
\]

The final expressions suggest an analogy with the expression \( \partial_y \log \|k_y\| \) for a continuous variable \( y \).

Using the definition of \( \omega \) and the summation by parts formula we can write the norm of

\[
f(y) = \sum c_x k_x(y).
\]

in two ways, one involving the values of \( f(y) \), the other involving the coefficients \( c_x \). The sets of data \( \{f(y)\} \) and \( \{c_x\} \) are dual to each other; the reproducing kernels generate the evaluation functionals and the vectors in the basis which is dual to the basis of reproducing kernels generate the coefficient functionals.
Corollary 38 Given $f$ as in (72) we have

$$
\|f\|^2 = |I^* f(o)|^2 + \sum_{z > o} \omega(z) |I^*(c_y)(z)|^2, \quad \text{and}
$$

$$
\|f\|^2 = |I^* f(o)|^2 + \sum_{z > o} \omega(z)^{-1} |Df(z)|^2.
$$

(73)

Proof. The first statement follows directly from the summation by parts formula. The second follows from the first as soon as we show that $Df(z) = \omega(z) I^*(c_y)(z)$. Both sides are linear functions of $f$ and hence it suffices to do the verification for $f = k_x$ Select $x$ and $z$. If $x > z$ or $x$ is not comparable to $z$ then $k_x(z) = k_x(z^-)$ and hence $Dk_x(z) = 0 = I^*(c_y)(z)$. The other possibility is that $x \leq z$ in which case, taking note of the definitions of $k, \Omega,$ and $\omega,$ we have

$$
Dk_x(z) = -k_x(z^-) + k_x(z) = -\Omega(z^-) + \Omega(z) = \omega(z) I^*(c_y)(z).
$$

7.3 Examples

7.3.1 Dirichlet-Sobolev Spaces

Classical Dirichlet type spaces and Sobolev spaces are characterized by integrability conditions on derivatives. Analogous spaces on trees are obtained putting summability conditions on differences.

The dyadic Dirichlet space is a basic example. Let $T_2$ be a rooted dyadic tree and let $I, I^*$, and $D$ be as above, and select $\omega$ to be identically one. Define $D(T_2)$, the dyadic Dirichlet space to be the Hilbert space $H(T_2, \Omega)$ produced in the previous theorem, the space of functions $f$ on $T_2$ for which

$$
\|f\|^2 = |I^* f(o)|^2 + \sum_{z \in T} |Df(z)|^2 < \infty.
$$

That space models the classical Dirichlet space, the space of functions $f$ holomorphic on the disk for which

$$
\|f\|^2 = |f(0)|^2 + \frac{1}{\pi} \int \int_{|z| < 1} \left| \frac{d}{dz} f(z) \right|^2 dA(z) < \infty.
$$

The space $D(T_2)$, and and related spaces have been studied by the author and collaborators, both for their intrinsic interest and as a tool in the study of spaces of smooth functions: [ARS02], [ARS06], [ARSW11], [ARSW14], [ARSW18].

7.3.2 Exponentials of Distances

Suppose that a rooted tree $T$ carries a geodesic distance function $d$; a non-negative function such that, for $x, y, z \in T$ with $y \in [x, z]$ we have $d(x, z) = \ldots$
$d(x, y) + d(y, z)$. Any such function is obtained by assigning a nonnegative length to each edge and letting $d(x, z)$ be the length of the geodesic path connecting $x$ and $y$. Such distance functions automatically satisfy the following useful relationship: for $x, y \in T$

$$d(x, y) = d(o, y) - 2d(o, x \wedge y). \quad (74)$$

Interestingly, this can be rewritten as

$$d(o, x \wedge y) = (d(o, y) + d(o, y) - d(x, y))/2.$$ 

Hence, by definition, $d(o, x \wedge y)$ equals the Gromov product $(x|y)_o$. For more about that quantity see, for instance, [V].

Select $\Lambda > 1$. Given $T$ and $d$ we consider the space $H = H(T, d, o) = H(T, d, \Lambda, o)$ with kernel functions

$$k_{xy} = k_{x \wedge y, x \wedge y} = \Lambda^{d(o, x \wedge y)} = \Lambda^{(x|y)_o}.$$ 

This is an instance of our earlier construction with $\Omega(s) = \Lambda^{d(o, s)}$ and it has several attractive computational properties.

If we change the choice of root vertex on the tree then we can build a new Hilbert space using the same distance function. If $\tilde{o}$ is the new root then there is also a new order structure $\tilde{\prec}$ and hence, also a new meet operation $\tilde{\wedge}$. We can then form the Hilbert space $\tilde{H} = H(T, d, \tilde{o})$ with kernel function

$$\tilde{k}_{xy} = \tilde{k}_{x \wedge y, x \wedge y} = \Lambda^{d(\tilde{o}, x \wedge y)}.$$ 

Although we have changed the root, we have not changed the tree or the distance function.

**Proposition 39** Changing the root of $T$ produces a rescaling of $H$;

$$H(T, d, o) \sim H(T, d, \tilde{o}).$$

**Proof.** This is an immediate consequence of the definitions, the computational properties of the function $\Lambda^x$, and the following equation which relates the new geometry to the old;

$$d(\tilde{o}, x \wedge y) = d(o, x \wedge y) - d(o, \tilde{o}) + d(o, x \wedge \tilde{o}) + d(o, \tilde{o} \wedge y). \quad (75)$$

That equation is Lemma 4 of [ARS10], where it is described as “clear after making sketches for the various cases”. 

Another interesting rescaling of $H(T, d, \Lambda, o)$ is the normalized kernel rescaling. That is, we pass to the space defined by the new kernel functions

$$j_{xy} = \frac{k_{xy}}{k_{xx}^{1/2} k_{yy}^{1/2}}.$$ 

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In this rescaling all the kernel functions are unit vectors, we always have $|j_{xy}| \leq 1$, and the point $o$ does not play a distinguished role. With $\Gamma = \Lambda^{-1/2}$, and using (74), we have

$$j_{xy} = \frac{\Lambda^d(o,x \Lambda y)}{\Lambda^d(o,x) / 2 \Lambda^d(o,y) / 2} = \Gamma^d(x,y).$$

Thus the new kernel function only depends on the distance between the points and, in particular does not depend on the root. We denote the space with these kernel functions by $J(T,\Gamma,d)$.

**Proposition 40** Given $T$, $d$, and $\Gamma$, $0 < \Gamma < 1$. The space $J(T,\Gamma,d)$ with reproducing kernels $j_{x,y} = \Gamma^d(x,y)$ is a RKHS which has the CPP. For $\Lambda = \Gamma^{-2}$ and any choice of basepoint $o$ in $T$, $J(T,\Gamma,d) \sim H(T,d,\Lambda,o)$.

**Proof.** The space $J$ was constructed as a rescaling of $H$ and thus it is a RKHS with the CPP. The second statement then follows from the previous proposition.

The spaces $H(T,d,\Lambda,o)$ are instances of tree Dirichlet-Sobolev spaces characterized by (73). They also show up in other places and for other reasons, [Haa], [N], [ARS10]. One practical fact about the spaces is that they are well suited for making explicit computations and estimates, [Haa], [N]. The spaces are also useful models for spaces holomorphic functions on $B^n$ and, more generally, Hilbert spaces that arise in the harmonic analysis of $\text{Aut}(B^n)$ and related groups, [N], [ARS10].

There is an additional use of these spaces which goes beyond our discussion here but we would like to at least mention. Questions involving function spaces on the disk often lead to questions in the function theory of the boundary circle. Those questions can be quite delicate, with subtle issues in capacity theory replacing more familiar analysis of smooth functions. A similar thing happens with function theory on trees, analysis in the Hilbert space of functions on the tree leads to questions about functions on the ideal boundary of the tree. In some cases that analysis on the ideal boundary is much more transparent and tractable than its continuous analog, and it gives both a tool and a guide for the more classical case. For instance, this is a basic theme in [N] and is explored [ARSW14].

8 The Multiplier Algebra

If $A$ is the multiplier algebra of a finite dimensional RKHS $H$ with the CPP, $A = \text{Mult}(H)$, $H \sim DA_n(X)$, $X \subset \mathbb{C}H^n$, then many of the results in Section 6 can be used to pass analytic and geometric information between $A$, $H$ and $X$. In fact much more is true. It is a theorem of Hartz [HA, Sec. 3] that $H$ and $X$ are determined (up to the natural equivalence relations) by the structure of $A$. Here is his theorem formulated to emphasize the geometry of the unit ball of $A$.

We are assuming $A = \text{Mult}(DA_n(X))$, and without loss of generality we assume $X$ is in normal form. An $m \in A$ is determined by the vector $a(m) =$
(α₁, ..., αₙ) where m(xᵢ) = aᵢ. Using those vectors as coordinates we identify A with the space ℂⁿ, with coordinatewise multiplication, and with the norm ||·|| induced by A. Let (A)₁ be the closed unit ball of A viewed as a subset of ℂⁿ. For 1 ≤ i ≤ n let Sᵢ be the hyperplane on which the i-th coordinate vanishes, Sᵢ = {(α₁, ..., αₙ) ∈ ℂⁿ : aᵢ = 0}. For 1 < j ≤ n let eᵢ be the point of Sᵢ ∩ (A)₁ that gives the maximum value of the functional Re aᵢ. Thus the coordinates of eᵢ are the values taken by the multiplier m_j which satisfies ||m_j|| ≤ 1, m_j(x₁) = 0, and Re m_j(x_j) is maximal. Because we are assuming that A = Mult(H) for an H with the CPP, we know that that m_j is unique and is given by (20). Because X is in normal form that formula simplifies. We have

\[ e_j = (e_{j1}, e_{j2}, ..., e_{jn}) = (m_{1j}(x_1), ..., m_{nj}(x_n)) \]

\[ m_{1j}(x_r) = \frac{1}{δ_j(x_1, x_j)} \left( 1 - \frac{k_{jl}k_{lr}}{k_{1r}} \right) = (1 - k_{jl}^{-1})^{-1/2} \left( 1 - \frac{1}{k_{lr}} \right) \]

For all j, e_{j1} = 0. It is clear from this formula that the (n - 1)² remaining e_{jk} are sufficient to reconstruct the Gram matrix of DAₙ(X). Thus

**Theorem 41 ([Ha, Sec. 3])** If

\[ A = \text{Mult}(DAₙ(X)), \]  \hfill (76)

then the Hilbert space H = DAₙ(X) is determined up to rescaling, equivalently, the set X ⊂ ℂℐⁿ is determined up to automorphism, by the (n - 1)² complex numbers \( F = \{e_{jk}\}_{j,k=2}^{n} \).

Here is a slightly weaker variation on the theorem using parameters that are more algebraic. Given A = Mult(DAₙ(X)) we extend the notation of (16) to

\[ \Delta(i; j, k) = \sup \{ \text{Re } m(x_i) : m ∈ A, m(x_j) = m(x_k) = 0, ||m||_A ≤ 1 \}. \]

A geometric description of these numbers is that \( \Delta(i; j, k) \) is the maximal value of \( \text{Re } a_i \) in \( S_j ∩ S_k ∩ (A)₁ \).

Set

\[ \mathcal{D}(X) = \{ δ_{ij} : 1 ≤ i < j ≤ n \} \cup \{ Δ(1; j, k) : 1 < j < k ≤ n \} \]

The set of invariants \( \mathcal{D} \) determines the congruence class of X up to a finite set of ambiguity.

**Theorem 42** Given A = Mult(DAₙ(X)), there are at most \( 2^{(n^2 - 3n)/2} \) distinct congruence classes \( \mathcal{Y} \) of sets in ℂℐⁿ for which Y ∈ \( \mathcal{Y} \) implies \( \mathcal{D}(Y) = \mathcal{D}(X) \).

**Proof.** We see from Theorem 3 that once we have \( \mathcal{D} \) then we know cos A₁jk and hence we know the A₁jk up to sign. By Theorem 16 and the comments which follow it, we then know the congruence class of the triangle with vertices \( \{ x_1, x_j, x_k \} \) up to a possible anticonformal conjugation. Thus \( (n^2 - 3n)/2 \) binary
choices determine the set of congruence classes of those triangles. From Theorem 7 we see that each set of choices corresponds to at most one class \( Y_i \).

In fact that bound is attained, see [BE].

The previous two theorems, as well as many of the previous results were specifically about algebras of the form \( A = \text{Mult}(DA_n(X)) \). There are closely related classes of algebras, for instance commutative finite dimensional algebras of operators on Hilbert space, and one can ask about their properties or ask how to recognize algebras of the type \( \text{Mult}(DA_n(X)) \) among them. There is interesting literature on these questions, including in particular the question of how to identify Pick algebras, algebras of the type \( \text{Mult}(DA_1(X)) \). Here are references to some of that work that seems related in spirit to what we do here: [CW], [CLW], [L], [MP], [P], [P2], [PS].

9 Beyond Spaces with Complete Pick Kernels

Geometers who study moduli for finite subsets of \( \mathbb{C}P^n \) frequently also consider similar questions for finite subsets of complex projective space, \( \mathbb{C}P^n \), and there are very strong analogies between those results and the results for \( \mathbb{C}H^n \), [BE], [BE], [HS]. It would be interesting to know how questions about point sets in \( \mathbb{C}P^n \) are related to Hilbert space questions. With that in mind we mention that there are RKHS, \( H \), on the Riemann sphere for which the associated \( \delta_H \) is the natural metric for \( \mathbb{C}P^1 \), see, for instance, the discussion of spin coherent states in, for instance, [P].

Finally, finite sets \( X \) in \( \mathbb{C}H^n \) are finite metric spaces with additional structure inherited from \( \mathbb{C}H^n \). It would be interesting to have an intrinsic, geometric, description of that type of structure on a set \( X \), one not dependent on its realization inside hyperbolic space and perhaps without references to Hilbert spaces or multiplier algebras.

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