APPROACH TO THE CONSISTENCY AND CONSENSUS OF PYTHAGOREAN FUZZY PREFERENCE RELATIONS BASED ON THEIR PARTIAL ORDERS IN GROUP DECISION MAKING

ZHENG MING MA*
School of Mathematics and Statistics, Linyi University
Linyi, 276005, China

ZE SHUI XU*
Business School, Sichuan University
Chengdu, 610064, China

WEI YANG
School of Mathematics and Statistics, Linyi University
Linyi, 276005, China

(Communicated by Yuanguo Zhu)

ABSTRACT. Although intuitionistic fuzzy preference relations have become powerful techniques to express the decision makers’ preference information over alternatives or criteria in group decision making, some limitations of them are pointed out in this paper, then they are overcome by developing the group decision making with Pythagorean fuzzy preference relations (PFPRs). Specially, we provide a partial order on the set of all the PFPRs, based on which, a deviation measure is defined. Then, we check and reach the acceptably multiplicative consistent and consensus of PFPRs associated with the partial order and mathematical programming. Concretely, acceptably multiplicative consistent PFPRs are defined by the deviation between a given PFPR and a multiplicative consistent PFPR constructed by a normal Pythagorean fuzzy priority vector. Then acceptable consensus of a collection of PFPRs is defined by the deviation of each PFPR and the aggregated result from symmetrical Pythagorean fuzzy aggregation operators. Based on which, a method which can simultaneously modify the unacceptable consistency and consensus of PFPRs in a stepwise way is provided. Particularly, we also prove that the collective PFPR obtained by aggregating several individual acceptably consistent PFPRs with various symmetric aggregation operators is still acceptably consistent. Then, a procedure is provided to solve group decision making with PFPRs and a numerical example is given to illustrate the effectiveness of our method.

1. Introduction. Group decision making involving to select the best alternative(s) by several decision makers from a given collection of finite feasible alternatives has been extensively used in politics, social psychology, engineering, management, business and economics, etc. In group decision making procedure, decision makers

2010 Mathematics Subject Classification. Primary: 68T35, 90B50, 62A86, 03E72.
Key words and phrases. Pythagorean fuzzy preference relations, partial order, acceptable consistency, acceptable consensus, group decision making.

The first author is supported by NSF of Shandong Province grant: ZR2017MG027.
* Corresponding author: Ze Shui Xu and Zhen Ming Ma.
are required to express their preferences on the collection of alternatives. It was pointed out in Ref. [32] that pairwise comparison methods are more accurate than non-pairwise comparison methods which generally conclude the following processes [17]:

1. Construct the preference relations.
2. Check and reach the consistency of individual preference relations.
3. Check and reach the consensus of a group.
4. The selection process.

For the first process, the classical preference relations are roughly divided into multiplicative preference relations (MPRs) [8, 28] and fuzzy preference relations (FPRs) [5, 7, 13, 24, 31], whose pairwise comparisons are expressed by single numerical values. However, due to the uncertainty of objects and the vagueness inherent in human thinking, it is more flexible to express pairwise comparisons as a certain information granule such as interval values [2, 15, 34], triangular fuzzy values [49] or linguistic values [23, 33, 35]. In spite of this, each element of them can only use a single value to describe the degree of one alternative preferred over the other and neglect the decision makers’ hesitations or indeterminacies. To circumvent this issue, intuitionistic fuzzy preference relations (IFPRs) [29, 30, 37] are presented. Therefore, IFPRs are more convenient to describe the uncertainties of pairwise comparisons between alternatives in decision making. In recent years, many scholars have paid much attention to group decision making with IFPRs [3, 17, 19, 20, 38, 39, 40, 41, 45].

The process of the consistency checking and reaching of individual preference relations could avoid misleading priority weights of alternatives. From the existing literature, MPRs and their consistency can be transformed into the FPRs and their corresponding consistency [13], and in FPRs there mainly are the multiplicative consistency and the additive consistency. Then, these consistencies were generalized to intuitionistic fuzzy environment. In [38], a multiplicative consistency of IFPRs was proposed by extending the multiplicative consistency of FPRs [31] which was redefined in [17]. In [49], another multiplicative consistency of IFPRs were defined by the multiplicative consistency of interval preference relations. Motivated by the additive consistency of FPRs, the additive and multiplicative consistencies of IFPRs were, respectively, proposed in [43] and [14]. As to checking and reaching the consistency of IFPRs, a consistency index of IFPRs for checking and reaching the consistency by modifying the whole IFPRs or a single element of IFPRs with maximum deviation was provided in [17, 50]. In [41], a mathematical programming approach to improve the consistency of IFPRs was made which can not only rapidly obtain the acceptably consistent IFPRs from the initial IFPRs, but also can make the obtained IFPRs preserve the preference information hidden in the initial IFPRs as much as possible. Recently, most definitions of consistency in intuitionistic fuzzy environment were reviewed in [16].

For the consensus checking and reaching process of a group which can be divided into hard consensus and soft consensus [6]. The latter is more pragmatic because it assesses the consensus degree in a more flexible way, according to which, various consensus-reaching methods were proposed in different cases, such as those of FPRs [5, 42], IFPRs [4, 18, 41] and linguistic preference relations [9, 10]. For the consensus of IFPRs, in [56], a consensus-reaching model by similarity measure was proposed for multiple criteria group decision making under intuitionistic fuzzy
environment. A group decision making model considering both the additive consistency and group consensus of IFPRs was investigated in [4]. A consensus checking and reaching procedure was developed in [18]. In [49], a feedback mechanism was designed to help decision makers modify their some pairwise comparisons to improve the consensus level. In [41], a novel method was developed for checking and improving the consistency of individual IFPRs and the consensus among experts, jointly.

For the last process, the aim of this process is to derive the best alternatives from the preference relations provided by decision makers which involves aggregation of individual preferences and exploitation of the collective preferences. In the first step, aggregating operators, such as multiplicative consistency induced order weighted averaging operator [49], consistency and confidence IOWA operator [32] and simple intuitionistic geometric operator [19] and so on are used. The second step is to exploit the collective preference relation to obtain the priority weights of alternatives by which the alternatives are ranked. In this regard, mathematical programming or approximate computation were often constructed to derive the priority weights of alternatives [3, 11, 17, 43, 44, 37].

Although the existing works have made significant contributions to solve group decision making problems with IFPRs, there are limitations as follows:

1. IFPRs are based on intuitionistic fuzzy sets (IFSs) [1] and the prominent characteristic of IFSs is that it assigns to each element a membership degree and a non-membership degree with their sum equal to or less than 1. However, it was pointed out in Refs. [51, 52] that, in some practical decision making process, the sum of the membership degree and the non-membership degree to which an alternative satisfying a criterion provided by a decision maker may be bigger than 1, but their square sum is equal to or less than 1. Therefore, Yager introduced the Pythagorean fuzzy sets (PFSs) to describe the above situation. At present, most works on PFSs are focused on Pythagorean fuzzy aggregation operators [12, 21, 25, 26] and multi-attribute group decision making [21, 27, 46, 47, 48], little attention has been paid to manage group decision making problem with preference relations in Pythagorean fuzzy environment.

2. When a given IFPR is not of acceptable consistency, iterative algorithms were proposed respectively by Liao et al. [17] and Wan et al. [50] to improve their consistency indices by a control parameter. To reach acceptable consistency, it may need to make a constant attempt for determining a suitable control parameter to repair this unacceptable consistent IFPR until it reaches acceptable consistency. It is time-consuming. It is also pointed by Xu in [41] that for the method of Liao’s, it is unknown how much the repaired IFPR preserves the preference information of the initial IFPRs, in fact, it is similar to Wan’s method [50]. Thus, Xu et al. [41] proposed a mathematical programming approach to improve the consistency and consensus simultaneously by minimizing the deviation between the initial IFPRs and the obtained IFPRs. Yet, some elements of the initial IFPRs may only be slightly modified, in most cases, the decision makers are reluctant to make such modification; in other words, Xu’s method can only preserve the initial preference information on the whole, but not do it on each element of the initial IFPRs.
In the last selection process, the existing methods of aggregating individual IFPRs into a collective one are all based on preserving acceptable consistency of the simple intuitionistic fuzzy weighted geometric (SIFWG) mean operator [19]. From theoretical aspect, the SIFWG operator cannot be reduced to fuzzy case; from application aspect, the single SIFWG operator cannot be suitable for various practical situations.

To overcome these limitations, this paper proposes a novel method for group decision-making problems with preference relations based on PFSs.

1. For expanding the scope of application of IFPRs, preference relations based on PFSs, that is, Pythagorean fuzzy preference relation (PFPRs) and their partial order are introduced.

2. For checking and reaching consistency and consensus of PFPRs, jointly, multiplicative consistency is introduced and then an acceptably multiplicative consistency is defined by the deviation between the given PFPRs and the constructed multiplicative consistent PFPRs by normal Pythagorean fuzzy priority weight vector. When PFPRs are unacceptably consistent, a mathematical programming is proposed to modify these given PFPRs as acceptably consistent PFPRs according to preserving the initial preference information as much as possible from the whole and local aspects where the consensus is reached at the same time.

3. An attractive property which keeps the collective PFPR obtained by aggregating several acceptably consistent individual PFPRs with various symmetric aggregation operators being acceptably consistent is proven.

The method proposed in this paper has the following significant features:

1. PFPRs are more capable than IFPRs to model the vagueness in the practical decision-making problems.

2. For several individual unacceptably consistent PFPRs, they are modified by a goal programming model which is constructed by the partial order of PFPRs to reduce the deviation between these individual PFPRs and the constructed consistent PFPRs step by step. These obtained individual PFPRs not only are all of acceptable consistency and consensus, but also embody the decision makers’ thought, that is, modification should be made step by step. Thus, the decision results derived from these obtained PFPRs may be more reasonable and easily be accepted by the decision makers.

3. An attractive property holds for various symmetric aggregation operators, not only for a single SIFWG operator.

The rest of this paper is structured as follows: In Section 2, some preliminaries regarding Pythagorean fuzzy sets (PFSs) are reviewed. Section 3 proposes PFPRs and their partial order. Particularly, multiplicatively consistent PFPRs are defined, and a formula for constructing the multiplicatively consistent PFPRs by normalized Pythagorean fuzzy priority weight vector is given; the concept of deviation measure between PFPRs is defined based on the proposed partial order. In Section 4, acceptably multiplicative consistent PFPR with respect to a multiplicatively consistent PFPR is defined, and a goal programming model is constructed by the partial order of PFPRs to reduce the deviation between the unacceptably multiplicative consistent PFPRs and its corresponding multiplicatively consistent PFPRs step by step. In Section 5, a novel method of improving the multiplicative consistency and reaching consensus of PFPRs in group decision-making and deriving the decision makers’
weights are provided. In Section 6, an example illustrates the validity of the proposed method and comparison analyses are conducted. The main conclusions are drawn in Section 7.

2. Preliminary. To make the presentation self-contained, in what follows, we review some basic concepts:

Recently, Yager [52] presented Pythagorean fuzzy sets (PFSs) whose membership grades are pairs, \((a, b)\) satisfying the requirement \(a^2 + b^2 \leq 1\). Obviously, this situation cannot be dealt with by using the IFS [1] in which the requirement is \(a + b \leq 1\). Here, we introduce the related concepts as follows:

**Definition 2.1.** [52] Let \(X\) be a universe of discourse. A Pythagorean fuzzy set \(A\) in \(X\) is defined as follows:

\[
A = \{x, \mu_A(x), \nu_A(x) | x \in X\}
\]

where \(\mu_A(x), \nu_A(x) \in [0, 1]\) define the membership and non-membership of \(x\) to \(A\), respectively and fulfill \(\mu_A(x)^2 + \nu_A(x)^2 \leq 1\) determinacy of \(x\).

For a given \(x \in X\), the pair \((\mu_A(x), \nu_A(x))\) in Eq. (1) is called a Pythagorean fuzzy number (PFN), which is simply denoted as \(\alpha = (\mu_\alpha, \nu_\alpha)\), where \(\mu_\alpha, \nu_\alpha \in [0, 1]\), \(\mu_\alpha^2 + \nu_\alpha^2 \leq 1\). Let \(\mathcal{PFN}\) be the set of all PFNs. In order to compare the magnitudes of PFNs. For a PFN \(\alpha\), Zhang and Xu [55] provided a score function \(s_{\text{Zhang}}\), which is defined as the difference of membership and non-membership function, as \(s_{\text{Zhang}}(\alpha) = \mu_\alpha^2 - \nu_\alpha^2\), where \(s_{\text{Zhang}}(\alpha) \in [-1, 1]\). The larger the score \(s_{\text{Zhang}}(\alpha)\), the greater the PFN \(\alpha\). To make the comparison method more discriminatory, Peng and Yang [53] defined an accuracy function \(h_{\text{Peng}}\), which is defined as \(h_{\text{Peng}}(\alpha) = \mu_\alpha^2 + \nu_\alpha^2\), where \(h_{\text{Peng}}(\alpha) \in [0, 1]\). When the scores are the same, the larger the accuracy \(h_{\text{Peng}}(\alpha)\), the greater the PFN \(\alpha\). Evidently, it holds that \(h_{\text{Peng}}(\alpha) + (\pi_\alpha^P)^2 = 1\), where \(\pi_\alpha^P\) is usually called the hesitancy of \(\alpha\).

**Definition 2.2.** [53] Let \(\alpha, \beta\) be two PFNs. Then, we have the following:

1. If \(s_{\text{Zhang}}(\alpha) < s_{\text{Zhang}}(\beta)\), then \(\alpha\) is smaller than \(\beta\), i.e., \(\alpha \prec_{\text{Zhang}} \beta\).
2. If \(s_{\text{Zhang}}(\alpha) = s_{\text{Zhang}}(\beta)\),
   a. if \(h_{\text{Peng}}(\alpha) < h_{\text{Peng}}(\beta)\), then \(\alpha\) is smaller than \(\beta\), i.e., \(\alpha \prec_{\text{Zhang}} \beta\);
   b. if \(h_{\text{Peng}}(\alpha) = h_{\text{Peng}}(\beta)\), i.e., \(\alpha = \beta\).

Note that score function \(s_{\text{Zhang}}\) and accuracy function \(h_{\text{Peng}}\) were actually compressed after squaring each difference value. Here, a modified score function \(s\) and a modified accuracy function \(h\) [21] were defined as follows:

\[
s(\alpha) = \begin{cases} 
\sqrt{\mu_\alpha^2 - \nu_\alpha^2}, & \mu_\alpha \geq \nu_\alpha, \\
-\sqrt{\nu_\alpha^2 - \mu_\alpha^2}, & \mu_\alpha < \nu_\alpha.
\end{cases}
\]

where \(s(\alpha) \in [-1, 1]\), and \(h(\alpha) = \sqrt{\mu_\alpha^2 + \nu_\alpha^2}\), where \(h(\alpha) \in [0, 1]\). The larger the score \(s(\alpha)\), the greater the PFN \(\alpha\). When the scores are the same, the larger the accuracy \(h(\alpha)\), the greater the PFN \(\alpha\). Furthermore, we also have \(h^2(\alpha) + (\pi_\alpha^P)^2 = 1\).

**Definition 2.3.** Let \(\alpha, \beta\) be two PFNs. Then, we have

1. If \(s(\alpha) < s(\beta)\), then \(\alpha\) is smaller than \(\beta\), i.e., \(\alpha \prec \beta\).
2. If \(s(\alpha) = s(\beta)\),
   a. if \(h(\alpha) < h(\beta)\), then \(\alpha\) is smaller than \(\beta\), i.e., \(\alpha \prec \beta\);
   b. if \(h(\alpha) = h(\beta)\), i.e., \(\alpha = \beta\).
Similar to the case of IFNs, Yager [52] also defined the following partial order \( \leq \) on \( \mathcal{PFN} \), which is defined such that \( \alpha = (\mu_\alpha, \nu_\alpha) \) and \( \beta = (\mu_\beta, \nu_\beta) \), \( \beta \leq \alpha \) if and only if \( \mu_\beta \leq \mu_\alpha \) and \( \nu_\alpha \leq \nu_\beta \). We denote \( \beta < \alpha \) if and only if \( \beta \leq \alpha \) and \( \beta \neq \alpha \).

Additionally, to aggregate PFNs, some aggregation operators were introduced in [52, 21] as follows:

**Definition 2.4.** Let \( \alpha_j (j = 1, 2, \cdots, n) \) be a collection of PFNs, we denote \( \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \), \( \mathcal{PFN}^m \to \mathcal{PFN} \),

1. [52] if
\[
\text{PFWG}_\omega(\alpha) = \left( \prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j}, \prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j} \right)
\]
then PFWG\(_\omega\) is called a Pythagorean fuzzy weighted geometric (PFWG) operator of dimension \( n \),

2. [52] if
\[
\text{PFWA}_\omega(\alpha) = \left( \sum_{j=1}^{n} \omega_j \mu_{\alpha_j}, \sum_{j=1}^{n} \omega_j \nu_{\alpha_j} \right)
\]
then PFWA\(_\omega\) is called a Pythagorean fuzzy weighted averaging (PFWA) operator of dimension \( n \),

3. [21] if
\[
\text{SPFWG}_\omega(\alpha) = \left( \prod_{j=1}^{n} \mu_{\alpha_j}^{\omega_j}, \prod_{j=1}^{n} \nu_{\alpha_j}^{\omega_j} \right) \left( \prod_{j=1}^{n} (1 - \mu_{\alpha_j}^{2})^{\omega_j} + \prod_{j=1}^{n} \mu_{\alpha_j}^{2 \omega_j} \right)^{1/2}, \left( \prod_{j=1}^{n} (1 - \nu_{\alpha_j}^{2})^{\omega_j} + \prod_{j=1}^{n} \nu_{\alpha_j}^{2 \omega_j} \right)^{1/2}
\]
then SPFWG\(_\omega\) is called a symmetric Pythagorean fuzzy weighted averaging (SPFWA) operator of dimension \( n \),

where \( \omega = (\omega_1, \omega_2, \cdots, \omega_n)^\top \) is the weight vector of \( \alpha_j (j = 1, 2, \cdots, n) \) with \( \omega_j \in [0, 1] \) and \( \sum_{j=1}^{n} \omega_j = 1 \).

3. **Pythagorean fuzzy preference relations and their partial orders.** As mentioned above, PFNs can be suitable for some situations where IFNs are not, which leads that some preference information can not be provided by IFN, but can be done by PFN. Thus, we introduce the Pythagorean fuzzy preference relations for these situations.

**Definition 3.1.** A Pythagorean fuzzy preference relation (PFPR) \( P \) on \( X \) is characterized by a Pythagorean fuzzy judgment matrix \( P = (p_{ij})_{n \times n} \) with \( p_{ij} = (\mu_{ij}, \nu_{ij}) \), where \( p_{ij} \) is a PFN, and \( \mu_{ij} \) is the certainty degree to which the alternative \( x_i \) is preferred to \( x_j \), and \( \nu_{ij} \) is the certainty degree to which the alternative \( x_i \) is non-preferred to \( x_j \), and \( 0 \leq \mu_{ij}^2 + \nu_{ij}^2 \leq 1, \mu_{ij} = \nu_{ji}, \nu_{ij} = \mu_{ji}, \) and \( \mu_{ii} = \nu_{ii} = \sqrt{\frac{1}{2}} \), for \( i, j = 1, 2, \cdots, n \).

The set of all PFPRs is denoted as \( \mathcal{PFPR} \). Note that in practical decision making, due to the complexity of decision making problems and the ambiguity of the human thinking, decision makers rarely provide their preferences by (0, 1)
Proof. First, we prove that the PFPRs or (1)

Theorem 3.3. The matrix \( P = (p_{ij})_{n\times n} = (\mu_{pij}, \nu_{pij})_{n\times n} \) satisfying \( \mu_{pij}, \nu_{pij} \in (0, 1) \) for all \( i, j = 1, 2, \cdots, n \).

Consistency is a critical issue in decision making with preference relations owing to that the lack of consistency may lead to unreasonable conclusions. Then, we give the following multiplicative consistency for PFPRs.

**Definition 3.2.** A PFPR \( P = (p_{ij})_{n\times n} \) with \( p_{ij} = (\mu_{ij}, \nu_{ij}) \) is said to be multiplicatively consistent, if it satisfies the following multiplicative transitivity: \( \mu_{ij} \mu_{jk} \mu_{ki} = \nu_{ij} \nu_{jk} \nu_{ki}, i, j, k = 1, 2, \cdots, n \).

With the development of society and economics, the problems are becoming more complicated, uncertain and fuzzy than ever, crisp weights may not be able to rationally reflect the importance degrees of the alternatives in many group decision making problems. Thus, we introduce the Pythagorean fuzzy priority weight vector as follows:

Suppose that \( \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \) is a Pythagorean fuzzy priority weight vector of \( P = (p_{ij})_{n\times n} \), where \( \omega_i = (\mu_{\omega_i}, \nu_{\omega_i}) (i = 1, 2, \cdots, n) \) is a PNF, and \( \mu_{\omega_i}, \nu_{\omega_i} \in [0, 1], \mu_{\omega_i}^2 + \nu_{\omega_i}^2 \leq 1 \). \( \mu_{\omega_i} \) and \( \nu_{\omega_i} \) can be interpreted as the membership degree and non-membership degree of the importance of the alternative \( x_i \) (\( i = 1, 2, \cdots, n \)). A Pythagorean fuzzy priority weight vector \( \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \) is said to be normalized if it satisfies the following requirements: \( \sum_{i \neq j} \mu_{\omega_i}^2 \leq \nu_{\omega_i}^2 \) and \( \sum_{i \neq j} \nu_{\omega_i}^2 \leq \mu_{\omega_i}^2 \leq \mu_{\omega_i}^2 + \nu_{\omega_i}^2 \leq 1 \).

With the underlying normalized Pythagorean fuzzy priority weight vector \( \omega = (\omega_1, \omega_2, \cdots, \omega_n)^T \), a multiplicatively consistent PFPR \( \bar{P} = (\bar{p}_{ij})_{n\times n} \) can be established, where

\[
\bar{p}_{ij} = \begin{cases} \left( \frac{\sqrt{\mu_{\omega_i} \nu_{\omega_i}}}{\nu_{\omega_i} \mu_{\omega_i}}, \frac{\mu_{\omega_i} \nu_{\omega_i}}{\nu_{\omega_i} \mu_{\omega_i}} \right), & i = j, \\ \left( \frac{\sqrt{1 - \mu_{\omega_i}^2 \mu_{\omega_j}^2 - \nu_{\omega_i}^2 \nu_{\omega_j}^2}}{\nu_{\omega_i} \mu_{\omega_j}}, \frac{1 - \mu_{\omega_i}^2 \mu_{\omega_j}^2 - \nu_{\omega_i}^2 \nu_{\omega_j}^2}{\nu_{\omega_i} \mu_{\omega_j}} \right), & i \neq j, \end{cases} \tag{5}
\]

for all \( i, j = 1, 2, \cdots, n \).

**Theorem 3.3.** The matrix \( \bar{P} = (\bar{p}_{ij})_{n\times n} \) is a multiplicatively consistent PFPR, where \( \bar{p}_{ij} (i, j = 1, 2, \cdots, n) \) is defined in Eq. (5).

**Proof.** First, we prove that \( \bar{P} \) is a PFPR. It follows from the definition of PNF that \( 1 - \pi_{\omega_i}^2 = \mu_{\omega_i}^2 + \nu_{\omega_i}^2 \) and \( 1 - \pi_{\omega_j}^2 = \mu_{\omega_j}^2 + \nu_{\omega_j}^2 \), thus, we get \( 0 \leq (1 - \pi_{\omega_i}^2)(1 - \pi_{\omega_j}^2) = (\mu_{\omega_i}^2 + \nu_{\omega_i}^2)(\mu_{\omega_j}^2 + \nu_{\omega_j}^2) \leq 1 \), that is, \( \mu_{\omega_i} \mu_{\omega_j}^2 + \mu_{\omega_j} \mu_{\omega_i}^2 \leq 1 - \nu_{\omega_i} \nu_{\omega_j} - \mu_{\omega_i} \mu_{\omega_j} \).

Thus, it holds that

\[
\frac{\mu_{\omega_i} \nu_{\omega_j}}{\nu_{\omega_i} \mu_{\omega_j}} \left[ 1 - \mu_{\omega_j}^2 \nu_{\omega_j}^2 - \mu_{\omega_i}^2 \mu_{\omega_j}^2 \right] \leq 1 \leq \frac{\nu_{\omega_j} \mu_{\omega_i}^2}{\nu_{\omega_j} \mu_{\omega_i}^2} \left[ 1 - \mu_{\omega_j}^2 \nu_{\omega_j}^2 - \mu_{\omega_i}^2 \mu_{\omega_j}^2 \right] \leq 1. 
\]

It is obvious that \( \mu_{\bar{p}_{ij}} = \nu_{\bar{p}_{ij}} \). Thus, \( \bar{P} \) is a PFPR. Furthermore, we have

\[
\frac{\mu_{\bar{p}_{ij}}}{\nu_{\bar{p}_{ij}}} = \frac{\mu_{\omega_i} \nu_{\omega_j}}{\nu_{\omega_i} \mu_{\omega_j}}, \quad \frac{\nu_{\bar{p}_{ij}}}{\mu_{\bar{p}_{ij}}} = \frac{\nu_{\omega_j} \mu_{\omega_i}^2}{\nu_{\omega_j} \mu_{\omega_i}^2},
\]

and
it follows from Definition 3.2 that \( \tilde{P} \) is multiplicatively consistent which completes
the proof. \( \square \)

In order to obtain a reasonable result, the PFPR given by a decision maker should be
multiplicatively consistent, and then, by Theorem 3.3, it can be expressed as Eq. (5). However, in practical decision making, it is difficult for a decision maker to build such a multiplicatively consistent PFPR. Hence, it is expected that the deviation between the given PFPR and its corresponding multiplicatively consistent PFPR should be as small as possible or acceptable. As a result, we introduce the following partial order and deviation measure between PFPRs:

**Definition 3.4.** Let \( P, Q \) be two PFPRs. Then, a partial order, denoted as \( P \leq Q \), is defined between \( P \) and \( Q \) if it holds that \( p_{ij} \leq q_{ij} \) for all \( i < j \), \( i, j = 1, 2, \ldots, n \).

**Definition 3.5.** Let \( P, Q \) and \( R \) be three PFPRs. Then, a mapping \( D : \mathcal{PFPR} \times \mathcal{PFPR} \to [0, 1] \) is called a deviation measure, if it possesses the following properties:

1. \( D(P, P) = 0 \), for a PFPR \( P \);
2. \( D(P, Q) = D(Q, P) \), for two PFPRs \( P \) and \( Q \);
3. \( P \leq Q \leq R \) implies \( D(P, Q) \vee D(Q, R) \leq D(P, R) \), for three PFPRs \( P, Q \) and \( R \).

For given PFPRs \( P \) and \( Q \), a specific deviation measure between \( P \) and \( Q \) provided in Eq. (6) can be defined as follows:

\[
D(P, Q) = \left( \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} d^2(p_{ij}, q_{ij}) \right)^{\frac{1}{2}}, \tag{6}
\]

where

\[
d(p_{ij}, q_{ij}) = \left( \left\| \mu_{p_{ij}}^2 - \mu_{q_{ij}}^2 \right\| + \left\| \nu_{p_{ij}}^2 - \nu_{q_{ij}}^2 \right\| + \left\| \pi_{p_{ij}}^2 - \pi_{q_{ij}}^2 \right\| \right)^{\frac{1}{2}}. \tag{7}
\]

Note that although all kinds of deviation measures between PFNs have been proposed in Ref. [54], we observe that they were actually compressed after squaring each difference values which may have a negative effect to check the multiplicative consistency of PFPRs. Thus, Eq. (7) can be considered as a modified deviation measure between PFNs.

4. Improving the multiplicative consistency of a single PFPR. Based on Eq. (5), the partial order and the deviation measure (6) between PFPRs, we propose a novel method of improving the multiplicative consistency of a single PFPR through reducing the deviation between the given PFPR and its corresponding multiplicatively consistent PFPR according to the partial order step by step.

**Definition 4.1.** Let \( P \) be a PFPR provided by a decision maker and \( \tilde{P} \) be a multiplicatively consistent PFPR provided in Eq. (5). If \( D(P, \tilde{P}) < a \) for a given threshold \( a \in (0, 1] \), then \( P \) is said to be of the acceptably multiplicative consistency w. r. t. the multiplicatively consistent PFPR \( \tilde{P} \).

For a given PFPR \( P \), we hope that the absolute deviation \( D(P, \tilde{P}) \) between the PFPR \( P \) and the multiplicatively consistent PFPR \( \tilde{P} \) is as small as possible. In other word, if the deviation \( D(P, \tilde{P}) \) is not acceptable, the PFPR \( P \) should be modified.
The principle of modification is that the modified preference information should not only satisfy the acceptability requirement but also preserve the initial preference information as much as possible which can be interpreted from two aspects: one is that the whole deviation between the PFPR  \( P \) and the modified PFPR  \( \tilde{P} \) is small; the other is that it is easy to be acceptant for a decision maker to modify the elements of the PFPR  \( P \) one by one, otherwise, some elements may be slightly modified which leads that the decision maker refuses to do so. If we denote  \( Q \) as the modified PFPR by modifying one element of the PFPR  \( P \) and its corresponding multiplicatively consistent PFPR  \( \tilde{Q} \), then the issue can be mathematically described by the following multiobjective optimization model:

\[
\min(D(P, Q), D(Q, \tilde{Q})) \tag{8}
\]

By changing the weight coefficients transforming the multiobjective problem into a single objective one:

\[
\min \lambda D(P, Q) + (1 - \lambda) D(Q, \tilde{Q}) \tag{9}
\]

Associated with the proposed concepts above, a method to modify the elements of a given PFPR  \( P \) one by one to reach the multiplicative consistency can be established by the following steps:

**Algorithm 1.**

**Step 0** Let  \( P(0) = P \), \( \mu = 0 \) and \( a = 0.1 \).

**Step 1** Let  \( P(h) = P(0) \). Determine the multiplicatively consistent PFPR  \( \tilde{P}(0) \) such that  \( D(P(0), \tilde{P}(0)) \) is as small as possible and  \( P(0) \leq \tilde{P}(0) \) by the following Model 1:

**Model 1:**

\[
\text{min } D(P(0), \tilde{P}(0))
\]

\[
\text{s.t. } \begin{cases}
\mu^1_{ij} \leq \mu_{ij}, \\
\nu^1_{ij} \geq \nu_{ij}, \\
\mu^2_{ij}, \nu^2_{ij} \in [0, 1], \mu^2_{ij} + \nu^2_{ij} \leq 1,
\end{cases}
\]

\[
\sum_{j=1,j\neq i}^{n} \mu^2_{ij} + \sum_{j=1,j\neq i}^{n} \nu^2_{ij} \leq \mu^2_{ij} + \nu^2_{ij} + n - 2.
\]

for  \( i, j = 1, 2, \ldots, n \);

**Step 2** If  \( D(P(h), \tilde{P}(h)) < a \), then  \( P(h) \) is of the acceptably multiplicative consistency, go to Step 4; otherwise, there at least exists  \( i_0, j_0 \in \{1, 2, \ldots, n\} \) such that  \( d(p_{ij}, \tilde{p}_{ij}) = \max\{d(p_{ij}, \tilde{p}_{ij})|i, j = 1, 2, \ldots, n\} \); go to the next step.

**Step 3** Modify the PFN  \( p_{ij} \) in  \( P(h) \) as that in  \( P(h+1) \) by the following steps:

(a) Construct  \( P(h+1) \) with

\[
p_{ij}^{(h+1)} = \begin{cases}
p_{ij}^{(h)}, & (i, j) \notin \{(i_0^{(h)}, j_0^{(h)}), (j_0^{(h)}, i_0^{(h)})\}, \\
(\mu^{(h+1)}(i, j), \nu^{(h+1)}(i, j)), & (i, j) = (i_0^{(h)}, j_0^{(h)}), \\
(\nu^{(h+1)}(i, j), \mu^{(h+1)}(i, j)), & (i, j) = (j_0^{(h)}, i_0^{(h)}),
\end{cases}
\tag{10}
\]

where  \( p_{ij}^{(h)} \leq (\mu^{(h+1)}(i, j), \nu^{(h+1)}(i, j)) \) and  \( \mu^{(h+1)}, \nu^{(h+1)} \) are two variables to determine; that is, we construct a PFPR  \( P(h+1) \) such that  \( P(h) \leq P(h+1) \).
Example 1.
Suppose that a decision maker gives a PFPR proposed method, we provide an example as follows:

Let $P$ be a PFPR such that $P^h \leq P^{h+1} \leq P^{(h+1)} \leq \tilde{P}^h$. Return to Step 2.

Step 4 Output the modified PFPR $P^{(h)}$.

Model 2: $\min \lambda D(P^h, P^{(h+1)}) + (1 - \lambda)D(P^{(h+1)}, \tilde{P}^{(h+1)})
\leq \min \lambda D(P^{(h)}, P^{(h+1)}) + (1 - \lambda)D(P^{(h+1)}, \tilde{P}^{(h+1)})$

for $i, j = 1, 2, \ldots, n$.

The prominent characteristics of the above algorithm are listed as follows:

(1) when the multiplicative consistency of the given PFPR is not acceptable, the proposed model could tell the decision maker which element of the PFPR should be repaired and how to repair it, which accords with the decision maker’s thinking, that is, the modification should be performed in a stepwise manner.

(2) The proposed algorithm can reflect the principle of modification, that is, reaching the acceptability requirement and preserving the initial preference information as much as possible.

Theorem 4.2. Let $P$ be a PFPR, $\{P^h\}$ and $\{\tilde{P}^h\}$ be the sequences of PFPRs generated from Algorithm I. Then, it holds that

$$\lim_{h \to \infty} D(P^h, \tilde{P}^h) = 0.$$  \hspace{1cm} (11)

Proof. It follows from Definition 3.5 that $D(P^h, \tilde{P}^h) \geq 0$, which shows that the sequence $D(P^h, \tilde{P}^h)$ has a lower bound. By Step 3 in Algorithm I, we have $P^h \leq P^{h+1} \leq \tilde{P}^{(h+1)} \leq \tilde{P}^h$. Thus, we get $D(P^{h+1}, \tilde{P}^{(h+1)}) \leq D(P^h, \tilde{P}^h)$, that is, the sequence $D(P^h, \tilde{P}^h)$ is monotone decreasing for any $h$. Thus, it holds that $\lim_{h \to \infty} D(P^h, \tilde{P}^h) = 0$ which completes the proof.

The above theorem indicates that Algorithm I is convergent. To illustrate the proposed method, we provide an example as follows:

Example 1. Suppose that a decision maker gives a PFPR $P$ on an object set $X = \{x_1, x_2, x_3, x_4\}$ as follows:

$$P = \begin{pmatrix}
0.7071, 0.7071 & 0.469, 0.8216 & 0.7969, 0.3674 & 0.9301, 0.2646 \\
0.8216, 0.469 & 0.7071, 0.7071 & 0.8689, 0.2121 & 0.9618, 0.1449 \\
0.3674, 0.7969 & 0.2121, 0.8689 & 0.7071, 0.7071 & 0.7583, 0.3873 \\
0.2646, 0.9301 & 0.1449, 0.9618 & 0.3873, 0.7583 & 0.7071, 0.7071
\end{pmatrix}.$$

We use the following steps to check and improve its multiplicative consistency:

Step 1 Let $P^{(0)} = P$, $h = 0$, $\lambda = 0.2$ and $\alpha = 0.1$.

Step 2 Utilize Model 1 to compute the multiplicatively consistent PFPR $\tilde{P}^{(0)}$ listed in Table 1 such that $P^{(0)} \leq \tilde{P}^{(0)}$. 

Step 3 Check the multiplicative consistency. Compute the deviation \(D(P^{(0)}, \tilde{P}^{(0)}) = 0.1987 > 0.1\) in Eq. (6) between \(P^{(0)}\) and \(\tilde{P}^{(0)}\), that is, \(P^{(0)}\) is not of acceptably multiplicative consistency. Thus, \(P^{(0)}\) needs to be modified. By Eq. (7), we compute the distance measures \(d(p_{ij}^{0}, \tilde{p}_{ij}^{0})\) \((i < j)\) between the elements \(p_{ij}^{0}\) in \(P_{ij}^{0}\) and \(\tilde{p}_{ij}^{0}\) in \(\tilde{P}_{ij}^{0}\), respectively. We find that \(i_0 = 1, j_0 = 3\) such that \(d(p_{13}^{0}, \tilde{p}_{13}^{0}) = 0.3198 = \max\{d(p_{ij}^{0}, \tilde{p}_{ij}^{0}) | i < j\}\).

Step 4 Modify the PFPR \(P^{(0)}\) as \(P^{(1)}\) according to Eq. (10) such that \(P_{13}^{(1)} \leq P_{13}^{(0)}\); that is, \(P^{(0)} \leq P^{(1)}\) with \(P_{13}^{(0)}\) as an unknown variable. Then, determine the PFN \(p_{13}^{(1)}\) by Model 2, we have \(p_{13}^{(1)} = (0.8559, 0.3462)\) such that \(P^{(0)} \leq P^{(1)} \leq P^{(0)}\) which are listed in Table 1. And then return the Step 3, we have \(D(P^{(1)}, \tilde{P}^{(1)}) = 0.1467 > 0.1; \) that is, \(P^{(1)}\) is still not of acceptably multiplicative consistency, and needs to be modified.

Step 5 By Eq. (7), we compute the distance measures \(d(p_{ij}^{(1)}, \tilde{p}_{ij}^{(1)})\) \((i < j)\) between the elements \(p_{ij}^{(1)}\) in \(P_{ij}^{(1)}\) and \(\tilde{p}_{ij}^{(1)}\) in \(\tilde{P}_{ij}^{(1)}\), respectively. We find that \(i_0 = 2, j_0 = 3\) such that \(d(p_{23}^{(1)}, \tilde{p}_{23}^{(1)}) = 0.2957 = \max\{d(p_{ij}^{(1)}, \tilde{p}_{ij}^{(1)}) | i < j\}\).

Step 6 Modify the PFPR \(P^{(1)}\) as \(P^{(2)}\) according to Eq. (10) such that \(P_{23}^{(2)} \leq P_{23}^{(1)}\); that is, \(P^{(1)} \leq P^{(2)}\) with \(P_{23}^{(1)}\) as a unknown variable. Then, determine the PFN \(p_{23}^{(2)}\) by Model 2, we have \(p_{23}^{(2)} = (0.9178, 0.2121)\) such that \(P^{(0)} \leq P^{(1)} \leq P^{(2)} \leq \tilde{P}^{(1)} \leq P^{(0)}\) which are listed in Table 1. And then return the Step 3, we have \(D(P^{(2)}, \tilde{P}^{(2)}) = 0.0833 < 0.1; \) that is, \(P^{(2)}\) is of acceptably multiplicative consistency with respect to the multiplicatively consistent PFPR \(\tilde{P}^{(2)}\).

Step 7 Output the modified PFPR \(P^{(2)}\).

5. A novel method for group decision making with PFPRs. In this section, we propose a method to synthesize the individual Pythagorean fuzzy information into the collective preference information in group decision making problem which can be described as follows:

Let \(X = \{x_1, x_2, \cdots, x_n\}\) and \(E = \{e_1, e_2, \cdots, e_s\}\) be the set of alternatives under consideration and the set of \(s\) decision makers who are invited to evaluate the alternatives, respectively. In many cases, since the problem is very complicated or the decision makers are not familiar with the problem and thus they cannot give explicit preferences over the alternatives, it is suitable to use the PFNs, which express the preference information from three aspects: “preferred”, “not preferred”, and “indeterminate”, to represent their opinions.

Suppose that the decision maker \(e_j\) provides his/her preference values for the alternative \(x_i\) against the alternative \(x_j\) as \(p_{ij}^{(l)} = (\mu_{p_{ij}^{(l)}}, \nu_{p_{ij}^{(l)}})\) in which \(\mu_{p_{ij}^{(l)}}\) denotes the degree to which the object \(x_i\) is preferred to the object \(x_j\), \(\nu_{p_{ij}^{(l)}}\) indicates the degree to which the object \(x_i\) is not preferred to the object \(x_j\), and \(\pi_{ij}^{(l)} = \sqrt{1 - \mu_{p_{ij}^{(l)}}^2 - \nu_{p_{ij}^{(l)}}^2}\) is interpreted as an indeterminacy degree or a hesitancy degree, subject to and 0 \(\leq \mu_{p_{ij}^{(l)}}^2 + \nu_{p_{ij}^{(l)}}^2 \leq 1\), \(\mu_{p_{ij}^{(1)}} = \nu_{p_{ij}^{(1)}} = \mu_{p_{ij}^{(2)}} = \nu_{p_{ij}^{(2)}} = 0\), and \(\mu_{p_{ij}^{(l)}} = \nu_{p_{ij}^{(l)}} = 1\) for \(i, j = 1, 2, \cdots, n\). The PFPR for the \(l\)th decision maker can be written as \(P^{(l)} = \left(p_{ij}^{(l)}\right)_{n \times n}\). (12)
In order to check and reach consensus of the individual PFPRs, it is needed to aggregate all these individual PFPRs with the form in Eq. (12). In order to find the final result of the problem, it is needed to aggregate all these individual PFPRs into a collective one by an aggregation operator. Before doing this, as presented in the introduction, consistency and consensus of PFPRs should be checked; otherwise the unreasonable results may be produced.

5.1. Checking and reaching consensus of PFPRs in group decision making. In order to check and reach consensus of the individual PFPRs $P^{(l)}$, where $l = 1, 2, \cdots, s$, we develop the following results:

**Proposition 1.** Let $P^{(1)}, P^{(2)}, \cdots, P^{(s)}$ be a collection of PFPRs. Then,

$$P_{Agg} = (a_{ij}) = (Agg_w(\mu_{p_i^{(1)}}, \mu_{p_i^{(2)}}, \cdots, \mu_{p_i^{(s)}}), Agg_w(\nu_{p_i^{(1)}}, \nu_{p_i^{(2)}}, \cdots, \nu_{p_i^{(s)}})),$$

(13)

is a PFPR, where Agg is a symmetric Pythagorean fuzzy aggregation operator with the weight vector $\omega$.

**Proof.** It is obvious that $a_{ij}$ for all $i, j = 1, 2, \cdots, n$ is a PFN. Since $\mu_{p_i^{(l)}} = \nu_{p_i^{(l)}} = \sqrt{T}$ for all $l = 1, 2, \cdots, s$, we have $A_{i}g_{w}(\mu_{p_i^{(1)}}, \mu_{p_i^{(2)}}, \cdots, \mu_{p_i^{(s)}}) = A_{i}g_{w}(\nu_{p_i^{(1)}}, \nu_{p_i^{(2)}}, \cdots, \nu_{p_i^{(s)}})$.
The original PFPR is provided by the following algorithm:

\[ (a_{ij})^C = (\text{Agg}_\omega(\nu_{p_{ij}}^{(1)}, \nu_{p_{ij}}^{(2)}, \ldots, \nu_{p_{ij}}^{(s)}), \text{Agg}_\omega(\mu_{p_{ij}}^{(1)}, \mu_{p_{ij}}^{(2)}, \ldots, \mu_{p_{ij}}^{(s)})) \]

\[ = (\text{Agg}_\omega(\mu_{p_{ij}}^{(1)}, \mu_{p_{ij}}^{(2)}, \ldots, \mu_{p_{ij}}^{(s)}), \text{Agg}_\omega(\nu_{p_{ij}}^{(1)}, \nu_{p_{ij}}^{(2)}, \ldots, \nu_{p_{ij}}^{(s)})) \]

Thus, \( P_{\text{Agg}} \) is a PFPR.

**Proposition 2.** Let \( P^{(1)}, P^{(2)}, \ldots, P^{(s)} \) be a collection of PFPRs. Then,

\[ N = (n_{ij}) = \begin{cases} \left( \min_{l=1}^{s} \mu_{p_{ij}}^{(l)}, \max_{l=1}^{s} \nu_{p_{ij}}^{(l)} \right), & i < j, \\ \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), & i = j, \\ \left( \max_{l=1}^{s} \nu_{p_{ij}}^{(l)}, \min_{l=1}^{s} \mu_{p_{ij}}^{(l)} \right), & i > j, \end{cases} \]  

\[ M = (m_{ij}) = \begin{cases} \left( \max_{l=1}^{s} \mu_{p_{ij}}^{(l)}, \min_{l=1}^{s} \nu_{p_{ij}}^{(l)} \right), & i < j, \\ \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), & i = j, \\ \left( \min_{l=1}^{s} \nu_{p_{ij}}^{(l)}, \max_{l=1}^{s} \mu_{p_{ij}}^{(l)} \right), & i > j, \end{cases} \]

are PFPRs.

**Proof.** We only prove that \( N \) is a PFPR, and \( M \) can be proven in a similar way. It is obvious that \( \min_{l=1}^{s} \mu_{p_{ij}}^{(l)}, \max_{l=1}^{s} \nu_{p_{ij}}^{(l)} \in [0,1] \). Without loss of generality, we assume that \( i < j, \min_{l=1}^{s} \mu_{p_{ij}}^{(l)} = \mu_{p_{ij}}^{(i)}, \max_{l=1}^{s} \nu_{p_{ij}}^{(l)} = \nu_{p_{ij}}^{(j)} \). Then, \( \mu_{p_{ij}}^{(i)} + \nu_{p_{ij}}^{(j)} \leq 1 \) and \( \nu_{p_{ij}}^{(i)} \geq \nu_{p_{ij}}^{(j)} \), thus, \( \mu_{p_{ij}}^{(i)} + \nu_{p_{ij}}^{(j)} \leq 1 \); that is, \( n_{ij} \) is a PFN. It follows immediately from Definition 3.1 that \( N \) is a PFPR.

**Proposition 3.** Let \( P^{(1)}, P^{(2)}, \ldots, P^{(s)} \) be a collection of PFPRs and \( P_{\text{Agg}}, N, M \) be PFPRs defined in Eqs. (13), (14) and (15), respectively. Then, \( N \leq P_{\text{Agg}} \leq M \).

**Proof.** Since \( \text{Agg} \) is a Pythagorean fuzzy aggregation operator, we get

\[ \mu_{n_{ij}} = \min_{l=1}^{s} \mu_{p_{ij}}^{(l)} \leq \text{Agg}_\omega(\mu_{p_{ij}}^{(1)}, \mu_{p_{ij}}^{(2)}, \ldots, \mu_{p_{ij}}^{(s)}) \leq \max_{l=1}^{s} \mu_{p_{ij}}^{(l)} = \mu_{m_{ij}} \]

\[ \nu_{n_{ij}} = \max_{l=1}^{s} \nu_{p_{ij}}^{(l)} \geq \text{Agg}_\omega(\nu_{p_{ij}}^{(1)}, \nu_{p_{ij}}^{(2)}, \ldots, \nu_{p_{ij}}^{(s)}) \geq \min_{l=1}^{s} \nu_{p_{ij}}^{(l)} = \nu_{m_{ij}} \]

Thus, \( N \leq P_{\text{Agg}} \leq M \).

**Definition 5.1.** Let \( P^{(1)}, P^{(2)}, \ldots, P^{(s)} \) be a collection of PFPRs. If \( D(P^{(l)}, P_{\text{Agg}}) < b \) for a given threshold \( b \in [0,1] \), then \( P^{(l)} (l = 1, \ldots, s) \) is said to be of acceptable consensus.

If \( P^{(1)} = P^{(2)} = \ldots = P^{(s)} \), then \( D(P^{(l)}, P_{\text{Agg}}) = 0 \), which implies a full consensus is reached. Otherwise, the bigger \( D(P^{(l)}, P_{\text{Agg}}) \), the lower the consensus among these decision makers, at this time, the decision makers are requested to modify their PFPRs to reach consensus. Here, a method of how to modify these PFPRs is provided by the following algorithm:

**Algorithm II.**

Step 0 The original PFPR \( P^{(l)} \) for \( l = 1, 2, \ldots, s, h = 0 \) and the threshold \( b = 0.1 \).
Step 1 Compute the collective PFPR \( P_{\text{Agg}}^{(h)} \) and the deviation \( D(P^{(l)}, P_{\text{Agg}}^{(h)}) \) between \( P^{(h)} \) and \( P_{\text{Agg}}^{(h)} \). If \( D(P^{(l)}, P_{\text{Agg}}^{(h)}) < b \), then \( P^{(h)} \) for \( l = 1, 2, \ldots, s \) are of acceptable consensus, go to Step 5; otherwise, there exists \( \{l_h, l_2, \ldots, l_t\} \subseteq \{1, 2, \ldots, s\} \) such that \( D(P^{(l_h)}, P_{\text{Agg}}^{(h)}) \geq b \) for \( k = 1, 2, \ldots, t \).

Step 2 Compute the distance measures \( d(p_{ij}^{(l_h)}, p_{ij}^{(h)}) \) \( (i < j \) and \( k = 1, \ldots, t) \) between the elements \( p_{ij}^{(l_h)} \) in \( P^{(l_h)} \) and \( p_{ij}^{(h)} \) in \( P_{\text{Agg}}^{(h)} \) respectively. We take \( (i, j, l_h) \in \{1, 2, \cdots, n\} \) and \( l_h \in \{l_1, l_2, \cdots, l_t\} \) such that \( \max\{d(p_{ij}^{(l_h)}, p_{ij}^{(h)}) | i < j, k = 1, \ldots, t\} \).

Step 3 Modify the PFPR \( P^{(l_h)} \) as \( P^{(l_h+1)} \) where

\[
\begin{align*}
p_{ij}^{(l_h+1)} = \begin{cases} p_{ij}^{(l_h)}, & (i, j) \notin \{(i_0^{(h)}, j_0^{(h)}), (j_0^{(h)}, i_0^{(h)}), (i_0^{(h)}, j_0^{(h)})\}, \
\mu_{l_h+1}^{(i, j)} p_{ij}^{(l_h+1)}, & (i, j) = (i_0^{(h)}, j_0^{(h)}), \
\nu_{l_h+1}^{(i, j)} p_{ij}^{(l_h+1)}, & (i, j) = (j_0^{(h)}, i_0^{(h)}),
\end{cases}
\end{align*}
\]

such that \( N^{(h)} \leq N^{(h+1)} \) and \( M^{(h)} \leq M^{(h+1)} \) with \( N^{(h+1)}, M^{(h+1)} \) defined by Eqs. (14) and (15).

Step 4 Determine the PFN \( \mu_{l_h+1}^{(i, j)}, \nu_{l_h+1}^{(i, j)} \) such that the deviation \( D(N^{(h+1)}, M^{(h+1)}) \) is as small as possible by Model 4, and then return Step 1.

Step 5 Output all the PFPRs \( P^{(l)} \) for \( l = 1, 2, \ldots, s \).

Model 4: min \( D(N^{(h+1)}, M^{(h+1)}) \)

s.t. \[
\begin{align*}
\mu_{N^{(h+1)}}, \nu_{N^{(h+1)}} & \leq \mu_{N^{(h+1)}}, \nu_{N^{(h+1)}} \leq \nu_{N^{(h+1)}}, \\
\mu_{M^{(h+1)}}, \nu_{M^{(h+1)}} & \leq \mu_{M^{(h+1)}}, \nu_{M^{(h+1)}} \leq \nu_{M^{(h+1)}};
\end{align*}
\]

Theorem 5.2. Let \( P^{(1)}, P^{(2)}, \cdots, P^{(s)} \) be a collection of PFPRs and \( \{N^{(h)}\} \) and \( \{M^{(h)}\} \) be the sequences of PFPRs generated from Algorithm II. Then, it holds that

\[
\lim_{h \to \infty} D(N^{(h)}, M^{(h)}) = 0.
\]

Proof. It is similar to that of Theorem 4.2, so we omit it.

Example 2. Suppose that three decision makers give the PFPRs \( P^{(1)}, P^{(2)} \) and \( P^{(3)} \) on an object set \( X = \{x_1, x_2, x_3\} \) as in Table 2 and We aggregate these PFPRs by Eq. (3) with the weight vector \( w = (0.3, 0.3, 0.4) \). Iterative process computed by

| \( P^{(1)} \) | \( P^{(2)} \) | \( P^{(3)} \) |
|---|---|---|
| \( (0.7071, 0.7071) \) | \( (0.9487, 0.1) \) | \( (0.8367, 0.3162) \) |
| \( (0.1, 0.9487) \) | \( (0.7071, 0.7071) \) | \( (0.7746, 0.4472) \) |
| \( (0.3162, 0.8367) \) | \( (0.4472, 0.7746) \) | \( (0.7071, 0.7071) \) |

| \( P^{(2)} \) | \( P^{(3)} \) |
|---|---|
| \( (0.7071, 0.7071) \) | \( (0.7746, 0.3162) \) | \( (0.8367, 0.4472) \) |
| \( (0.3162, 0.7746) \) | \( (0.7071, 0.7071) \) | \( (0.7071, 0.3162) \) |
| \( (0.4472, 0.8367) \) | \( (0.3162, 0.7071) \) | \( (0.7071, 0.7071) \) |

| \( P^{(3)} \) |
|---|
| \( (0.7071, 0.7071) \) |
| \( (0.4472, 0.7746) \) | \( (0.7071, 0.7071) \) |
| \( (0.3162, 0.8944) \) | \( (0.7071, 0.7071) \) |

Algorithm II with mathematical software Sagemath is provided in Table 3, where we find that when \( h = 9 \), three modified PFPRs are of acceptable consensus.
Table 3. Iterative process of the proposed method for the acceptable consensus

| $h$ | $(D(P^{(1h)}, P^{(1h)}_{\text{Agg}}), D(P^{(2h)}, P^{(2h)}_{\text{Agg}}), D(P^{(3h)}, P^{(3h)}_{\text{Agg}}))$ | $l_0$ | $i_0, j_0$ |
|-----|-------------------------------------------------|------|-----------|
| 0   | $(0.3589, 0.2902, 0.2578)$                      | 1    | $(1, 2)$  |
| 1   | $(0.271, 0.2774, 0.2244)$                       | 1    | $(2, 3)$  |
| 2   | $(0.2205, 0.2606, 0.2291)$                      | 2    | $(1, 3)$  |
| 3   | $(0.2175, 0.2288, 0.2203)$                      | 2    | $(2, 3)$  |
| 4   | $(0.2037, 0.1762, 0.1986)$                      | 2    | $(1, 2)$  |
| 5   | $(0.1943, 0.1377, 0.1875)$                      | 1    | $(1, 3)$  |
| 6   | $(0.1493, 0.1540, 0.1598)$                      | 3    | $(1, 2)$  |
| 7   | $(0.1188, 0.1355, 0.1076)$                      | 2    | $(1, 3)$  |
| 8   | $(0.1017, 0.0787, 0.0814)$                      | 1    | $(1, 2)$  |
| 9   | $(0.0780, 0.0628, 0.0757)$                      |      |           |

5.2. Checking and reaching the multiplicative consistency of PFPRs in group decision making. In group decision making problem, there are $s$ PFPRs, in order to assure that the aggregated individual PFPRs is of acceptable consistency, we develop the following results:

**Definition 5.3.** Let $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ be a collection of PFPRs, and $\tilde{P}$ be a multiplicatively consistent PFPR. If $D(P^{(l)}, \tilde{P}) < a$ for a given threshold $a \in [0, 1]$ and $l = 1, 2, \ldots, s$, then $P^{(l)}$ is said to be of acceptably multiplicative consistency with respect to the multiplicatively consistent PFPR $\tilde{P}$.

Note that the above definition about acceptably multiplicative consistency is different from that in Ref. [19]. Here, we emphasize that with respect to different PFPRs $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$, a single multiplicatively consistent PFPR $\tilde{P}$ is provided; in Ref. [19], different multiplicatively consistent IFPRs are determined by a formula and the given PFPRs.

**Theorem 5.4.** Let $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$ be a collection of PFPRs which are of acceptably multiplicative consistency with respect to a multiplicatively consistent PFPR $K$ satisfying $P^{(l)} \leq K$ for all $l = 1, 2, \ldots, s$. If $N$ defined in Eq. (14) is of acceptably multiplicative consistency with respect to $K$, then $P_{\text{Agg}}$ is of acceptably multiplicative consistency with respect to $K$.

**Proof.** Since $P^{(l)}$ is of acceptably multiplicative consistency with respect to $K$ satisfying $P^{(l)} \leq K$; i.e., $D(P^{(l)}, K) < a$ and $P^{(l)} \leq K$. Thus, $N \leq P_{\text{Agg}} \leq M \leq K$. Due to that $N$ defined in Eq. (14) is of acceptably multiplicative consistency with respect to $K$; i.e., $D(N, K) < a$, we have $D(P_{\text{Agg}}, K) \leq D(N, K) < a$; i.e. $P_{\text{Agg}}$ is of acceptably multiplicative consistency with respect to $K$.

Here, we take $K = \tilde{N}$ to provide a concrete multiplicatively consistent PFPR for PFPRs $P^{(1)}, P^{(2)}, \ldots, P^{(s)}$, which can be determined by the following algorithm: Algorithm III.

**Algorithm III.**

**Step 0** The original PFPR $P^{(ih)}$ for $l = 1, 2, \ldots, s$, $h = 0$ and the threshold $a = 0.1$.

**Step 1** Construct PFPRs $N^{(h)}$ and $M^{(h)}$ by Eqs. (14) and (15) for the individual PFPRs $P^{(lh)}$ for $l = 1, 2, \ldots, s$.

**Step 2** Compute the multiplicatively consistent PFPR $\tilde{N}^{(h)}$ such that the deviation $D(N^{(h)}, \tilde{N}^{(h)})$ is as small as possible and $M^{(h)} \leq \tilde{N}^{(h)}$ by the following
model:

**Model 3:** \( \min D(N^{(h)}, \tilde{N}^{(h)}) \)

\[
\begin{aligned}
\mu_{m_{ij}^{(h)}} & \leq \mu_{n_{ij}^{(h)}}, \quad \nu_{m_{ij}^{(h)}} \geq \nu_{n_{ij}^{(h)}}, & i < j; \\
\mu_{\omega_i^{(h)}}, \nu_{\omega_i^{(h)}} & \in [0, 1], \quad \mu_{\omega_i^{(h)}}^2 + \nu_{\omega_i^{(h)}}^2 \leq 1, \\
\sum_{j=1, i \neq j} \mu_{\omega_i^{(h)}}^2 & \leq \nu_{\omega_i^{(h)}}, \quad \sum_{j=1, i \neq j} \nu_{\omega_i^{(h)}}^2 \leq \mu_{\omega_i^{(h)}}^2 + n - 2.
\end{aligned}
\]

Step 3 Compute the deviation measures \( D(P^{(l)}, \tilde{N}^{(h)}) \) between the PFPRs \( P^{(l)} \) and \( \tilde{N}^{(h)} \) by Eq. (6) for \( l = 1, 2, \ldots, s \). If \( D(P^{(l)}, \tilde{N}^{(h)}) < a \) for all \( l = 1, 2, \ldots, s \), go to step 7; otherwise, there always exists \( \{l_i, h \} \subseteq \{1, 2, \ldots, s\} \) such that \( D(P^{(l_i)}, \tilde{N}^{(h)}) \geq a \) for \( k = 1, 2, \ldots, t \).

Step 4 Compute the distance measures \( d(p_{ij}^{(l)}, \tilde{N}_{ij}^{(h)}) \) for \( i < j \) and \( k = 1, 2, \ldots, t \) between the elements \( p_{ij}^{(l)} \) in \( P^{(l)} \) and \( \tilde{N}_{ij}^{(h)} \) in \( \tilde{N}^{(h)} \), respectively. We take \( l_0^{(h)}, j_0^{(h)} \in \{1, 2, \ldots, n\} \) and \( h \in \{l_1, l_2, \ldots, l_t\} \) such that

\[
d(p_{ij}^{(l)}, \tilde{N}_{ij}^{(h)}) = \max\{d(p_{ij}^{(l)}, \tilde{N}_{ij}^{(h)}) | i < j, k = 1, \ldots, t \}.
\]

Step 5 Modify the PFPR \( P^{(l_h)} \) as \( P^{(l_h+1)} \) where

\[
p_{ij}^{(l_h+1)} = \begin{cases} p_{ij}^{(l_h)}, & (i, j) \notin \{(l_0^{(h)}, l_0^{(h)}), (j_0^{(h)}, j_0^{(h)})\}, \\ \mu_{(l_h+1)}, & (i, j) = (l_0^{(h)}, l_0^{(h)}), \\ \nu_{(l_h+1)}, & (i, j) = (j_0^{(h)}, j_0^{(h)}), \end{cases}
\]

such that \( N^{(h)} \leq N^{(h+1)} \) and \( M^{(h)} \leq M^{(h+1)} \) with \( M^{(h+1)}, N^{(h+1)} \) defined by Eqs. (15) and (14).

Step 6 Determine the PFN \( (\mu^{(l_h+1)}, \nu^{(l_h+1)}) \) such that the deviation \( D(N^{(h+1)}), \tilde{N}^{(h+1)} \) is as small as possible and \( M^{(h+1)} \leq \tilde{N}^{(h+1)} \leq \tilde{N}^{(h)} \) by Model 4, and then return Step 3.

Step 7 Output all the PFPRs \( P^{(l_h)} \) for \( l = 1, 2, \ldots, s \).

**Model 4:** \( \min D(N^{(h+1)}, \tilde{N}^{(h+1)}) \)

\[
\begin{aligned}
\mu_{\tilde{N}^{(h+1)}} & \leq \mu_{N^{(h)}}, \quad \nu_{\tilde{N}^{(h+1)}} \geq \nu_{N^{(h)}}, \\
\mu_{M^{(h+1)}} & \leq \mu_{N^{(h+1)}}, \quad \nu_{M^{(h+1)}} \geq \nu_{N^{(h+1)}}, \\
\mu_{N^{(h)}} & \leq \mu_{N^{(h+1)}}, \quad \nu_{N^{(h)}} \geq \nu_{N^{(h+1)}}, \\
\mu_{\omega_j^{(h+1)}}, \nu_{\omega_j^{(h+1)}} & \in [0, 1], \quad \mu_{\omega_j^{(h+1)}}^2 + \nu_{\omega_j^{(h+1)}}^2 \leq 1, \\
\sum_{j=1, i \neq j} \mu_{\omega_j^{(h+1)}}^2 & \leq \nu_{\omega_j^{(h+1)}}, \quad \sum_{j=1, i \neq j} \nu_{\omega_j^{(h+1)}}^2 \leq \mu_{\omega_j^{(h+1)}}^2 + n - 2.
\end{aligned}
\]

where \( k = 1, 2, \ldots, t \) and \( i = 1, 2, \ldots, n \).

**Theorem 5.5.** Let \( P^{(l)} \) for \( l = 1, 2, \ldots, s \) be a collection of PFPRs and \( \{N^{(h)}\} \) and \( \{\tilde{N}^{(h)}\} \) be the sequences of PFPRs generated from Algorithm III. Then, it holds that

\[
\lim_{h \to \infty} D(N^{(h)}, \tilde{N}^{(h)}) = 0.
\]

**Proof.** It is similar to that of Theorem 4.2, so we omit it. \(\square\)

**Theorem 5.5** shows that Algorithm III is convergent. For the consensus process, although several consensus indices were developed [18, 49, 56], there is no study on improving the consistency degrees of several individual IFPRs and consensus among
the decision makers simultaneously. Since \( N^{(h)} \leq M^{(h)} \leq \tilde{N}^{(h)} \) in Algorithm III, then it holds that
\[
\lim_{h \to \infty} D(N^{(h)}, M^{(h)}) = 0,
\]
from Definition 3.5, that is, the collective PFPR and all the individual PFPRs become identical with the increasing \( h \), in other word, all the decision makers can ultimately reach a consensus.

5.3. Deriving the decision makers’ weights and reaching acceptable consistency and consensus simultaneously. The weight vector of the decision makers can usually be determined subjectively or objectively according to the decision makers’ decision making, experience, judgment quality and related knowledge.

Let \( P^{(1)}, \ldots, P^{(s)} \) be a collection of PFPRs provided by the decision makers. Then, we derive the objective weights by the following formula:
\[
w_l = \frac{1 + T(P^{(l)})}{\sum_{k=1}^{s} (1 + T(P^{(k)}))},
\]
for \( l = 1, \ldots, s \), where \( T(P^{(k)}) = \sum_{r \neq k}^{s} \text{Sup}(P^{(k)}, P^{(r)}) \) and \( \text{Sup}(P^{(k)}, P^{(r)}) \) are the supports for the \( k \)th decision maker’s PFPR \( P^{(k)} \) from the \( r \)th decision maker’s PFPR \( P^{(r)} \), with the conditions:

1. \( \text{Sup}(P^{(k)}, P^{(r)}) \in [0, 1] \);
2. \( \text{Sup}(P^{(k)}, P^{(r)}) = \text{Sup}(P^{(r)}, P^{(k)}) \);
3. \( \text{Sup}(P^{(k)}, P^{(r)}) \geq \text{Sup}(P^{(u)}, P^{(v)}) \), if \( D(P^{(k)}, P^{(r)}) < D(P^{(u)}, P^{(v)}) \), where \( D \) is the above distance measure in Eq. (6).

For convenience, we always assume that
\[
\text{Sup}(P^{(k)}, P^{(r)}) = \sqrt{1 - D^2(P^{(k)}, P^{(r)})}.
\]
Suppose that the subjective weights of decision makers are \( \lambda_l \) for \( l = 1, \ldots, s \), which are assigned according to the knowledge of the decision makers, then a comprehensive weight for the \( l \)th decision maker can be given by a combination of his/her objective and subjective weights, that is,
\[
\omega_l = \theta w_l + (1 - \theta) \lambda_l
\]
where \( \theta \) ( \( 0 \leq \theta \leq 1 \) ) is a control parameter and can tradeoff the \( l \)th decision maker’s objective and subjective weights. Particularly, if \( \theta = 1 \), then the weight \( w_l \) is reduced to \( l \)th decision maker’s objective weights; If \( \theta = 0 \), then the weight \( \lambda_l \) is reduced to \( l \)th decision maker’s subjective weights. In practical applications, we usually consider both the objective weights and the subjective weights and take \( \theta \in (0, 1) \).

Algorithm IV.

Step 0 The original individual PFPRs \( P^{(100)} \) for \( l = 1, 2, \ldots, s, h = 0, k = 0 \), the consistency thresholds \( a \), the consensus threshold \( b \) and \( \delta = 0.9 \).

Step 1 Compute Algorithm III with \( a = a \times \delta^k \).
Step 2 Aggregate these individual PFPRs $P^{(hk)}$ into a collective PFPR $P^{(hk)}$ by symmetric Pythagorean fuzzy aggregation operators provided in Definition 2.4, where

$$p_{ij}^{(hk)} = \left(\text{Agg}_w(\mu_{p_{ij}}^{(hk)}, \cdots, \mu_{p_{ij}}^{(shk)}), \text{Agg}_w(\nu_{p_{ij}}^{(hk)}, \cdots, \nu_{p_{ij}}^{(shk)})\right),$$

and compute the deviation measures $D(P^{(lhk)}, P^{(hk)})$ between the individual PFPRs $P^{(lhk)}$ and the collective PFPR $P^{(hk)}$ by Eq. (6). If $D(P^{(lhk)}, P^{(hk)}) < b$ for $l = 1, 2, \cdots, s$, go to next step; otherwise, taking $k = k + 1$, and go to step 1.

Step 3 Employ the Pythagorean fuzzy aggregation operator to aggregate all PFNs $p_{ij}^{(hk)}$ corresponding to the object $x_i$ into the collective PFN $p_i^{(hk)}$; Step 4 Calculate the score value $s(p_i^{(hk)})$ and accuracy values $h(p_i^{(hk)})$ and then rank all the objects by Definition 2.3.

6. A practical example of a virtual enterprise partner selection and comparative analyses. In this section, a practical example of a virtual enterprise partner selection is provided to illuminate the application of the proposed method. Comparative analyses are carried out to demonstrate the desirable advantages of the proposed method over other methods.

6.1. A practical example of a virtual enterprise partner selection. Here, we use the example in Ref. [45]. Due to the constant change of today’s market environment, a single real enterprise is unable to adjust itself rapidly to satisfy market demand. In this scenario, virtual enterprise, a dynamic alliance composed of many members (real enterprises), emerges. In a virtual enterprise, members can share cost, risk, technology and key competitiveness, by which win-win among members can be achieved. As an example, in order to develop a new product which cannot be finished by a single enterprise, a real enterprise (also called a key enterprise) will seek other enterprises (also called virtual enterprise partners) based on its demand. Thus, the key enterprise and other ones compose a virtual enterprise. While developing, designing, producing and selling this product, each enterprise gives full play to its advantages and obtains corresponding benefit. Once this process ends, this virtual enterprise disintegrates and each enterprise can seek new members again. In this process, selecting suitable partners is very important for this key enterprise.

AHEAD Information Technology Co., LTD (AHEAD for short), a famous software enterprise of China, has been identified to focus on medical information integrating and service since its inception in 2003. AHEAD desires to develop a new-type rural cooperative medical care management information system for facilitating health reimbursement management. This system consists of some software systems and a hardware device integrating chips. AHEAD can develop software systems and only needs to seek a partner to produce the hardware device. After primary screening, four different partners \{A_1, A_2, A_3, A_4\} chosen from the virtual industry cluster remain for further evaluation. To select the best partner, AHEAD asks three decision makers $e_1, e_2, e_3$ to evaluate these four partners. By conducting pairwise comparisons on four partners, decision makers furnish their PFPRs listed in Table 4 as follows:

In what follows, the proposed method in this paper is employed to solve this example:
Step 2 Aggregate all the individual PFPRs $P^{(1.0.0)}$, $P^{(2.0.0)}$ and $P^{(3.0.0)}$ by Algorithm II, computing with mathematical software Sagemath, we have $h = 44$, $k = 0$ and the modified individual PFPRs and the multiplicatively consistent PFPR $\tilde{N}^{(44)}$ listed in Table 5 and then we have $D(P^{(1.44.0)}, \tilde{N}^{(44)}) = 0.097 < 0.1$.

Table 4. Individual preference information from three decision makers

| $P^{(1)}$   | (0.707, 0.707) | (0.707, 0.447) | (0.837, 0.316) | (0.707, 0.548) |
|            | (0.447, 0.707) | (0.707, 0.707) | (0.775, 0.447) | (0.548, 0.775) |
|            | (0.316, 0.837) | (0.447, 0.775) | (0.707, 0.707) | (0.548, 0.775) |
|            | (0.548, 0.707) | (0.775, 0.548) | (0.775, 0.548) | (0.707, 0.707) |
| $P^{(2)}$   | (0.707, 0.707) | (0.775, 0.316) | (0.894, 0.447) | (0.775, 0.548) |
|            | (0.316, 0.775) | (0.707, 0.707) | (0.707, 0.316) | (0.548, 0.837) |
|            | (0.447, 0.894) | (0.316, 0.707) | (0.707, 0.707) | (0.632, 0.775) |
|            | (0.548, 0.775) | (0.837, 0.548) | (0.775, 0.632) | (0.707, 0.707) |
| $P^{(3)}$   | (0.707, 0.707) | (0.775, 0.447) | (0.894, 0.316) | (0.837, 0.447) |
|            | (0.447, 0.775) | (0.707, 0.707) | (0.775, 0.316) | (0.447, 0.837) |
|            | (0.316, 0.894) | (0.316, 0.775) | (0.707, 0.707) | (0.447, 0.548) |
|            | (0.447, 0.837) | (0.837, 0.447) | (0.548, 0.447) | (0.707, 0.707) |

Step 1 For the origin PFPRs $P^{(1.0.0)}$, $P^{(2.0.0)}$ and $P^{(3.0.0)}$ by Algorithm II, computing with mathematical software Sagemath, we have $h = 44$, $k = 0$ and the modified individual PFPRs and the multiplicatively consistent PFPR $\tilde{N}^{(44)}$ listed in Table 5 and then we have $D(P^{(1.44.0)}, \tilde{N}^{(44)}) = 0.097 < 0.1$.

Table 5. Modified individual preference information from three decision makers

| $P^{(1.44.0)}$ | (0.707, 0.707) | (0.767, 0.341) | (0.89, 0.149) | (0.88, 0.136) |
|               | (0.316, 0.767) | (0.707, 0.707) | (0.769, 0.316) | (0.748, 0.271) |
|               | (0.149, 0.39)  | (0.316, 0.769) | (0.707, 0.707) | (0.626, 0.549) |
|               | (0.136, 0.88)  | (0.271, 0.748) | (0.549, 0.626) | (0.707, 0.707) |
| $P^{(2.44.0)}$ | (0.707, 0.707) | (0.775, 0.316) | (0.894, 0.154) | (0.883, 0.128) |
|               | (0.316, 0.775) | (0.707, 0.707) | (0.772, 0.322) | (0.76, 0.271)  |
|               | (0.154, 0.894) | (0.322, 0.772) | (0.707, 0.707) | (0.624, 0.549) |
|               | (0.128, 0.883) | (0.271, 0.76)  | (0.549, 0.624) | (0.707, 0.707) |
| $P^{(3.44.0)}$ | (0.707, 0.707) | (0.764, 0.327) | (0.894, 0.155) | (0.874, 0.13)  |
|               | (0.327, 0.764) | (0.707, 0.707) | (0.775, 0.316) | (0.756, 0.274) |
|               | (0.155, 0.894) | (0.316, 0.775) | (0.707, 0.707) | (0.628, 0.548) |
|               | (0.13, 0.874)  | (0.274, 0.756) | (0.548, 0.628) | (0.707, 0.707) |
| $\tilde{N}^{(44)}$ | (0.707, 0.707) | (0.75, 0.316)  | (0.894, 0.149) | (0.883, 0.128) |
|               | (0.316, 0.775) | (0.707, 0.707) | (0.775, 0.316) | (0.76, 0.271)  |
|               | (0.149, 0.894) | (0.316, 0.775) | (0.707, 0.707) | (0.628, 0.548) |
|               | (0.128, 0.883) | (0.271, 0.76)  | (0.548, 0.628) | (0.707, 0.707) |
| $P^{(44.0)}$   | (0.707, 0.707) | (0.769, 0.328) | (0.893, 0.153) | (0.879, 0.131) |
|               | (0.328, 0.769) | (0.707, 0.707) | (0.772, 0.318) | (0.755, 0.272) |
|               | (0.153, 0.893) | (0.318, 0.772) | (0.707, 0.707) | (0.626, 0.549) |
|               | (0.131, 0.879) | (0.272, 0.755) | (0.549, 0.626) | (0.707, 0.707) |

$D(P^{(2.44.0)}, \tilde{N}^{(44)}) = 0.045 < 0.1$, $D(P^{(3.44.0)}, \tilde{N}^{(44)}) = 0.083 < 0.1$, that is, all the PFPRs are of acceptably multiplicative consistency.

Step 2 Aggregate all the individual PFPRs $P^{(1.44.0)}$, $P^{(2.44.0)}$ and $P^{(3.44.0)}$ into a collective PFPR $P^{(44.0)}$ listed in Table 5 by Eq. (2). Compute with
Step 4 Calculate the scores $s$.

Step 3 Aggregate the rows of the collective PFPR $P^{(44,0)}$ by Eq. (2), and we have $p_1 = (0.808, 0.261)$, $p_2 = (0.606, 0.466)$, $p_3 = (0.383, 0.719)$ and $p_4 = (0.343, 0.736)$.

Step 4 Calculate the scores $s(p_i)$ for $i = 1, 2, 3, 4$, we get $s(p_1) = 0.547 > s(p_2) = 0.140 > s(p_3) = -0.336 > s(p_4) = -0.393$, thus $y_1 > y_2 > y_3 > y_4$, that is, $y_1$ is the best one.

6.2. Comparative analyses. First, we use Xu’s method [41] to solve the example in Subsection 6.1 as follows:

Step 1 Solve the model

**Model 5:**

\[
\begin{align*}
\min & \quad D(P^{(1)}, \tilde{P}^{(1)}) + D(P^{(2)}, \tilde{P}^{(2)}) + D(P^{(3)}, \tilde{P}^{(3)})/3 \\
\text{s.t.} & \quad CI(\tilde{P}^{(1)}) < 0.1; \\
& \quad CI(\tilde{P}^{(2)}) < 0.1; \\
& \quad CI(\tilde{P}^{(3)}) < 0.1; \\
& \quad CM(\tilde{P}^{(1)}, \tilde{P}^{(2)}, \tilde{P}^{(3)}) < 0.1. \\
\end{align*}
\]

where $CI(P) = \frac{1}{n} \sum_{i<j<k} |\mu_{ij} \mu_{jk} \mu_{ki} - \nu_{ij} \nu_{jk} \nu_{ki}|$ and

\[
CM(\tilde{P}^{(1)}, \tilde{P}^{(2)}, \tilde{P}^{(3)}) = \frac{D(\tilde{P}^{(1)}, \tilde{P}^{(2)}) + D(\tilde{P}^{(2)}, \tilde{P}^{(3)}) + D(\tilde{P}^{(1)}, \tilde{P}^{(3)})}{3},
\]

we have the modified PFPRs $\tilde{P}^{(1)}$, $\tilde{P}^{(2)}$ and $\tilde{P}^{(3)}$ as listed in Table 6, at this time, $CI(\tilde{P}^{(1)}) = 0.0544$, $CI(\tilde{P}^{(2)}) = 0.0543$, $CI(\tilde{P}^{(3)}) = 0.0542$ and $CM(\tilde{P}^{(1)}, \tilde{P}^{(2)}, \tilde{P}^{(3)}) = 0.99999$, that is, these modified PFPRs are of acceptable consistency and consensus.

| $\tilde{P}^{(1)}$ | (0.707, 0.707) | (0.71, 0.445) | (0.01, 0.056) | (0.011, 0.553) |
|------------------|----------------|----------------|---------------|---------------|
|                  | (0.445, 0.71) | (0.707, 0.707) | (0.047, 0.013) | (0.512, 0.802) |
|                  | (0.056, 0.01) | (0.013, 0.047) | (0.707, 0.707) | (0.529, 0.619) |
|                  | (0.553, 0.011) | (0.802, 0.512) | (0.619, 0.529) | (0.707, 0.707) |
| $\tilde{P}^{(2)}$ | (0.707, 0.707) | (0.71, 0.445) | (0.01, 0.056) | (0.011, 0.553) |
|                  | (0.445, 0.71) | (0.707, 0.707) | (0.047, 0.014) | (0.512, 0.802) |
|                  | (0.056, 0.011) | (0.014, 0.047) | (0.707, 0.707) | (0.529, 0.619) |
|                  | (0.553, 0.011) | (0.802, 0.512) | (0.619, 0.529) | (0.707, 0.707) |
| $\tilde{P}^{(3)}$ | (0.707, 0.707) | (0.728, 0.447) | (0.011, 0.132) | (0.011, 0.553) |
|                  | (0.447, 0.728) | (0.707, 0.707) | (0.046, 0.03)  | (0.509, 0.804) |
|                  | (0.132, 0.011) | (0.03, 0.046)  | (0.707, 0.707) | (0.5, 0.573)   |
|                  | (0.553, 0.011) | (0.804, 0.509) | (0.573, 0.5)   | (0.707, 0.707) |
| $\tilde{P}$      | (0.707, 0.707) | (0.716, 0.446) | (0.01, 0.074)  | (0.011, 0.553) |
|                  | (0.446, 0.716) | (0.707, 0.707) | (0.047, 0.018) | (0.511, 0.803) |
|                  | (0.074, 0.011) | (0.018, 0.047) | (0.707, 0.707) | (0.519, 0.603) |
|                  | (0.553, 0.011) | (0.803, 0.511) | (0.603, 0.519) | (0.707, 0.707) |
Step 2 Aggregate all the individual PFPRs $\tilde{P}^{(1)}$, $\tilde{P}^{(2)}$ and $\tilde{P}^{(3)}$ into a collective PFPR $\tilde{P}$ listed in Table 6 by Eqs. (2) and (20).

Step 3 Aggregate the rows of the collective PFPR $\tilde{P}$ by Eq. (2), and we have $p_1 = (0.086, 0.337)$, $p_2 = (0.295, 0.292)$, $p_3 = (0.149, 0.119)$ and $p_4 = (0.66, 0.213)$.

Step 4 Calculate the scores $s(p_i)$ of $p_i$ for $i = 1, 2, 3, 4$, we get $s(p_1) = 0.293 > s(p_4) = 0.061 > s(p_2) = -0.047 > s(p_3) = -0.294$, thus $y_1 > y_4 > y_2 > y_3$, that is, $y_1$ is the best one, which is in accordance with that obtained by the proposed method in this paper.

Compared with Xu’s method [41], although the optimal alternative is identical, the proposed method in this paper has the following advantages:

(1) For several individual unacceptably consistent PFPRs, a goal programming model is constructed by the partial order of PFPRs to stepwisely reduce the deviation between these individual PFPRs and the consistent PFPRs by which the derived decision results may be more reasonable and be easily accepted by the decision makers.

(2) In the progress of aggregation, various symmetrical aggregation operators can be used in the proposed method, but only the SIFWG operator can be used in Xu’s method [41].

7. Concluding remarks. We have summarized the systematic comparison in the following aspects:

(1) Although IFSs are convenient to simultaneously express the support and against aspects of the preference information, their use is limited by the condition that the sum of the membership degree and non-membership degree is less than or equal to 1; PFSs relax the constraint condition by replacing that of IFSs with their square sum being less than or equal to 1. Thus, PFPRs are more capable than IFPRs to model the vagueness in the practical decision making problems.

(2) Checking and reaching the consistency and consensus are necessary to guarantee the transitivity and rationality of the decision makers’ preferences which could avoid misleading priority weights of alternatives and the decision result could be accepted by the group. Although the existing methods [17, 50, 41] can achieve the goal that check and reach the consistency and consensus, they have their own shortcomings listed in the introduction. In this paper, the above shortcomings are overcome by the partial order of PFPRs and a goal programming model to stepwisely improve the consistency and consensus.

(3) In the selection process of the group decision making problems, the existing methods can only use the SIFWG operator [19] because of an attractive property which obviously can not used in all decision making situations. The proposed method keeps the attractive property holding for various symmetric aggregation operators, naturally, can satisfy the need of selecting special aggregation operators in different decision making situations.

Furthermore, the decision makers may adopt other types of preference values, such as intuitionsitic multiplicative preference relations [22], linguistic preference relations [36], probabilistic linguistic preference relations [57]. How to extend the proposed method to solve these preference relations, especially in group decision making, is the future work.
Acknowledgments. The authors would like to thank the editors and the anonymous reviewers for their insightful and constructive comments and suggestions that have led to this improved version of the paper. This research was supported by the NSF of Shandong Province (No. ZR2017MG027).

REFERENCES

[1] K. Atanassov, Intuitionistic fuzzy set, Fuzzy Sets and Systems, 20 (1986), 87–96.
[2] E. Barrenechea, J. Fernandez, M. Pagola, F. Chiclana and H. Bustince, Construction of interval-valued fuzzy preference relations from ignorance functions and fuzzy preference relations, Application to Decision Making, Knowledge Based Systems, 58 (2014), 33–44.
[3] H. Behret, Group decision making with intuitionistic fuzzy preference relations, Knowledge-Based Systems, 70 (2014), 33–43.
[4] J. Chu, X. Liu, Y. Wang and K. Chin, A group decision making model considering both the additive consistency and group consensus of intuitionistic fuzzy preference relations, Computers and Industrial Engineering, 101 (2016), 227–242.
[5] F. J. Cabrerizo, R. Urena, W. Pedrycz and E. Herrera-Viedma, Building consensus in group decision making with an allocation of information granularity, Fuzzy Sets and Systems, 255 (2014), 115–127.
[6] F. J. Cabrerizo, J. M. Moreno, I. J. Perez and E. Herrera-Viedma, Analyzing consensus approaches in fuzzy group decision making: Advantages and drawbacks, Soft Computing, 14 (2010), 451–463.
[7] F. Chiclana, F. Herrera and E. Herrera-Viedma, Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations, Fuzzy Sets and Systems, 97 (1998), 33–48.
[8] F. Chiclana, E. Herrera-Viedma, F. Herrera and S. Alonso, Induced ordered weighted geometric operators and their use in the aggregation of multiplicative preference relations, International Journal of Intelligent Systems, 19 (2004), 233–255.
[9] Y. Dong, X. Chen and F. Herrera, Minimizing adjusted simple terms in the consensus reaching process with hesitant linguistic assessments in group decision making, Information Sciences, 297 (2015), 95–117.
[10] Y. Dong, J. Xiao, H. Zhang and T. Wang, Managing consensus and weights in iterative multiple-attribute group decision making, Applied Soft Computing, 48 (2016), 80–90.
[11] Z. W. Gong, L. S. Li, J. Forrest and Y. Zhao, The optimal priority models of the intuitionistic fuzzy preference relation and their application in selecting industries with higher meteorological sensitivity, Expert Systems with Applications, 38 (2011), 4394–4402.
[12] H. Garg, New Logarithmic operational laws and their aggregation operators for Pythagorean fuzzy set and their applications, International Journal of Intelligent Systems, 34 (2019), 82–106.
[13] E. Herrera-Viedma, F. Herrera, F. Chiclana and M. Luque, Some issues on consistency of fuzzy preference relations, European Journal of Operational Research, 154 (2004), 98–109.
[14] F. Jin, Z. Ni, H. Chen and Y. Li, Approaches to group decision making with intuitionistic fuzzy preference relations based on multiplicative consistency, Knowledge-Based Systems, 97 (2016), 48–59.
[15] J. B. Lan, M. M. Hu, X. M. Ye and S. Q. Sun, Deriving interval weights from an interval multiplicative consistent fuzzy preference relation, Knowledge-Based Systems, 26 (2012), 128–134.
[16] H. C. Liao and Z. S. Xu, Consistency and consensus of intuitionistic fuzzy preference relations in group decision making, imprecision and uncertainty in information representation and processing, Studies in Fuzziness and Soft Computing, 332 (2016), 189–206.
[17] H. C. Liao and Z. S. Xu, Priorities of intuitionistic fuzzy preference relation based on multiplicative consistency, IEEE Transactions on Fuzzy Systems, 22 (2014), 1669–1681.
[18] H. C. Liao, Z. S. Xu, X. J. Zeng and J. M. Merigo, Framework of group decision making with intuitionistic fuzzy preference information, IEEE Transactions on Fuzzy Systems, 23 (2014), 1211–1227.
[19] H. C. Liao and Z. S. Xu, Consistency of the fused intuitionistic fuzzy preference relation in group intuitionistic fuzzy analytic hierarchy process, Applied Soft Computing, 35 (2015), 812–826.
[20] H. C. Liao, Z. S. Xu and X. J. Zeng, An enhanced consensus reaching process in group decision making with intuitionistic fuzzy preference relations, *Information Sciences*, 329 (2016), 274–286.

[21] Z. M. Ma and Z. S. Xu, Symmetric Pythagorean fuzzy weighted geometric/averaging operators and their application in multicriteria decision-making problems, *International Journal of Intelligent Systems*, 31 (2016), 1198–1219.

[22] Z. M. Ma and Z. S. Xu, Hyperbolic scales involving appetites-based intuitionistic multiplicative preference relations for group decision making, *Information Sciences*, 451/452 (2018), 310–325.

[23] F. Mata, L. G. Perez, S. M. Zhou and F. Chiclana, Type-1 OWA methodology to consensus reaching processes in multi-granular linguistic contexts, *Knowledge-Based Systems*, 58 (2014) 11–22.

[24] S. A. Orlovsky, Decision-making with a fuzzy preference relation, *Fuzzy Sets and Systems*, 1 (1978), 155–167.

[25] X. Peng and Y. Yang, Fundamental properties of interval-valued Pythagorean fuzzy aggregation operators, *International Journal of Intelligent Systems*, 31 (2016), 444–487.

[26] X. Peng and G. Selvachandran, Pythagorean fuzzy set: State of the art and future directions, *Artificial Intelligence Review*, 52 (2019), 1873–1927.

[27] X. Peng, New operations for interval-valued Pythagorean fuzzy set, *Scientia Iranica*, 26 (2019), 1049–1076.

[28] T. L. Saaty, *The Analytic Hierarchy Process*, McGraw-Hill, New York, 1980.

[29] E. Szmidt and J. Kacprzyk, A consensus-reaching process under intuitionistic fuzzy preference relations, *International Journal of Intelligent Systems*, 18 (2003), 837–852.

[30] E. Szmidt and J. Kacprzyk, A new concept of a similarity measure for intuitionistic fuzzy sets and its use in group decision making, *Lecture Notes in Computer Science*, 3558 (2005), 272–282.

[31] T. Tanino, Fuzzy preference orderings in group decision making, *Fuzzy Sets and Systems*, 12 (1984), 117–131.

[32] R. Urena, F. Chiclana, J. A. Morente-Molinera and E. Herrera-Viedma, Managing incomplete preference relations in decision making: A review and future trends, *Information Sciences*, 302 (2015), 14–32.

[33] Z. S. Xu, A method based on linguistic aggregation operators for group decision making with linguistic preference relations, *Information Sciences*, 166 (2004), 19–30.

[34] Z. S. Xu, On compatibility of interval fuzzy preference relations, *Fuzzy Optimization and Decision Making*, 3 (2004), 217–225.

[35] Z. S. Xu, Deviation measures of linguistic preference relations in group decision making, *Omega*, 33 (2005), 249–254.

[36] Z. S. Xu, Incomplete linguistic preference relations and their fusion, *Information Fusion*, 7 (2006), 331–337.

[37] Z. S. Xu, Intuitionistic preference relations and their application in group decision making, *Information Sciences*, 177 (2007), 2363–2379.

[38] Z. S. Xu, X. Q. Cai and E. Szmidt, Algorithms for estimating missing elements of incomplete intuitionistic preference relations, *International Journal of Intelligent Systems*, 26 (2011), 787–813.

[39] Z. S. Xu and H. Liao, Intuitionistic fuzzy analytic hierarchy process, *IEEE Transactions on Fuzzy Systems*, 22 (2014), 749–761.

[40] Z. S. Xu and H. Liao, A survey of approaches to decision making with intuitionistic fuzzy preference relations, *Knowledge-Based Systems*, 80 (2015), 131–142.

[41] G. Xu, S. Wan, F. Wang, J. Dong and Y. Zeng, Mathematical programming methods for consistency and consensus in group decision making with intuitionistic fuzzy preference relations, *Knowledge-Based Systems*, 98 (2016), 30–43.

[42] Y. J. Xu, K. W. Li and H. M. Wang, Distance-based consensus models for fuzzy and multiplicative preference relations, *Information Sciences*, 253 (2013), 56–73.

[43] Z. J. Wang, Derivation of intuitionistic fuzzy weights based on intuitionistic fuzzy preference relations, *Applied Mathematical Modelling*, 37 (2013), 6377–6388.

[44] S. P. Wan, F. Wang, L. L. Lin and J. Y. Dong, An intuitionistic fuzzy linear programming method for logistics outsourcing provider selection, *Knowledge-Based Systems*, 82 (2015), 80–94.
[45] S. P. Wan, F. Wang and J. Dong, A preference degree for intuitionistic fuzzy values and application to multi-attribute group decision making, *Information Sciences*, 370/371 (2016), 127–146.
[46] S. P. Wan, Z. Jin and J. Y. Dong, Pythagorean fuzzy mathematical programming method for multi-attribute group decision making with Pythagorean fuzzy truth degrees, *Knowledge and Information Systems*, 55 (2018), 437–466.
[47] S. P. Wan, Z. Jin and J. Y. Dong, A three-phase method for Pythagorean fuzzy multi-attribute group decision making and application to haze management, *Computers & Industrial Engineering*, 123 (2018), 348–363.
[48] S. P. Wan, Z. Jin and J. Y. Dong, A new order relation for Pythagorean fuzzy numbers and application to multi-attribute group decision making, *Knowledge and Information Systems*, 62 (2020), 751–785.
[49] J. Wu and F. Chiclana, Multiplicative consistency of intuitionistic reciprocal preference relations and its application to missing values estimation and consensus building, *Knowledge-Based Systems*, 71 (2014), 187–200.
[50] S. P. Wan, G. Xu and J. Dong, A novel method for group decision making with interval-valued Atanassov intuitionistic fuzzy preference relations, *Information Sciences*, 372 (2016), 53–71.
[51] R. R. Yager and A. M. Abbasov, Pythagorean membership grades, complex numbers, and decision making, *International Journal of Intelligent Systems*, 28 (2013), 436–452.
[52] R. R. Yager, Pythagorean membership grades in multicriteria decision making, *IEEE Transactions on Fuzzy Systems*, 22 (2014), 958–965.
[53] X. Peng and Y. Yang, Some results for Pythagorean fuzzy sets, *International Journal of Intelligent Systems*, 30 (2015), 1133–1160.
[54] X. Peng, H. Yuan and Y. Yang, Pythagorean fuzzy information measures and their applications, *International Journal of Intelligent Systems*, 32 (2017), 991–1029.
[55] X. L. Zhang and Z. S. Xu, Extension of TOPSIS to multiple criteria decision making with Pythagorean fuzzy sets, *International Journal of Intelligent Systems*, 29 (2014), 1061–1078.
[56] L. Y. Zhang, T. Li and X. H. Xu, Consensus model for multiple criteria group decision making under intuitionistic fuzzy environment, *Knowledge-Based Systems*, 57 (2014), 127–135.
[57] Y. Zhang, Z. S. Xu and H. Liao, A consensus process for group decision making with probabilistic linguistic preference relations, *Information Sciences*, 414 (2017), 260–275.

Received October 2019; revised January 2020.

E-mail address: dmyuto@126.com
E-mail address: xuzeshui@263.net
E-mail address: wyoeng@126.com