Dispersive treatment of $K_S \to \gamma\gamma$ and $K_S \to \gamma\ell^+\ell^-$

Gilberto Colangelo, Ramon Stucki, and Lewis C. Tunstall

Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland

September 2016

Abstract. We analyse the rare kaon decays $K_S \to \gamma\gamma$ and $K_S \to \gamma\ell^+\ell^-$ ($\ell = e$ or $\mu$) in a dispersive framework in which the weak Hamiltonian carries momentum. Our analysis extends predictions from lowest-order $SU(3)_L \times SU(3)_R$ chiral perturbation theory ($\chi$PT) to fully account for effects from final-state interactions, and is free from ambiguities associated with extrapolating the kaon off-shell. Given input from $K_S \to \pi\pi$ and $\gamma\gamma^{(*)} \to \pi\pi$, we solve the once-subtracted dispersion relations numerically to predict the rates for $K_S \to \gamma\gamma$ and $K_S \to \gamma\ell^+\ell^-$. In the leptonic modes, we find sizeable corrections to the $\chi$PT predictions for the integrated rates.

PACS. 13.20.Eb Decays of $K$ mesons – 11.55.Fv Dispersion relations

1 Introduction

In the study of kaon decays, our ability to obtain precise predictions from the Standard Model (SM) depends on whether the underlying physics is predominantly of short- or long-distance nature. At one end of a broad spectrum of possible decay channels, there are “golden modes” like $K \to \pi\pi\nu\bar{\nu}$, where the amplitude factorises into a hadronic form factor and perturbative corrections — both of which are under excellent theoretical control [1]. In such cases, the resulting prediction can be at a level of precision that competes with (or even surpasses) current experimental measurements. This state of affairs can lead to powerful constraints on physics beyond the SM and drives much of the theoretical and experimental interest in these modes.

By contrast, non-leptonic decays such as $K \to \pi\pi$ and $K \to \pi\pi\pi$ are dominated by long-distance contributions involving hadronic matrix elements of four-quark operators. The evaluation of these matrix elements is a notoriously difficult non-perturbative problem, and this hinders the comparison of theory with experiment.

In between these extremes lies a range of decay modes in which a clean separation of the short- and long-distance physics can be achieved with varying degrees of success.

Since kaon decays occur at low energies, a systematic analysis can be undertaken within $SU(3)_L \times SU(3)_R$ chiral perturbation theory ($\chi$PT$_3$), where amplitudes are expanded as an asymptotic series in powers of $O(m_K)$ momentum and light quark masses $m_{u,d,s} = O(m_K^2)$. The application of $\chi$PT$_3$ to kaon decays is covered in a comprehensive review [2]; here we recall two important features that determine the quality of predictions arising from the 3-flavour expansion:

1. hadronic uncertainties are parametrised in terms of low-energy constants (LECs), whose values are not fixed by chiral symmetry alone. For several purely leptonic and semi-leptonic kaon decays, the corresponding LECs can be extracted from a combination of experimental data and input from lattice QCD. However, the situation for non-leptonic and weak radiative decays is far less certain, with many of the LECs essentially unconstrained at next-to-lowest-order (NLO) in the chiral expansion;

2. at energies above the $\pi\pi$ threshold, final-state interactions (FSI), especially in the $0^{-+}$ channel [3–6], can spoil the convergence of the $\chi$PT$_3$ expansion. These effects are related to the broad $f_0(500)$ resonance [7], whose $O(m_K)$ mass [8] implies a lack of scale separation between the Goldstone $\pi,K,\eta$ and non-Goldstone $f_0,\rho,\omega,\ldots$ sectors. In these cases, chiral-perturbative methods must be abandoned in favour of non-perturbative methods based on unitarity, analyticity, and crossing symmetry.\footnote{Scale separation can be restored in scenarios where $f_0$ belongs to the Goldstone sector, as in chiral-scale perturbation theory [9,10].}

Dispersion relations offer a means to address items 1 and 2 within a model-independent framework. These methods have been mostly applied in the context of pure strong processes such as pion form factors [11,12], $\pi\pi$-scattering [13–17], $\pi\pi$-scattering [18], $\gamma\gamma^{(*)} \to \pi\pi$ [19–21], $\pi N$ scattering [22–24], semi-leptonic kaon decays $K_{3\ell}$ [25–30] and $K_{4\ell}$ [31–33], or decays not involving kaons, e.g. $\eta \to \pi\pi\pi$ [34–39].

In view of current high-statistics kaon experiments such as NA62 [40], we believe it is timely to consider extending the scope of dispersive methods to $\Delta S = 1$ processes involving the effective weak Hamiltonian $\mathcal{H}_w$, and in particular to two-body decays. Such an extension was proposed sometime ago by B"uchler et al. [41,42], who treated the decay $K \to \pi\pi$ dispersively by allowing $\mathcal{H}_w$ to carry momentum, thereby overcoming the difficulty that the kinematics in two-body decays are completely fixed. The advantage of this approach over $\chi$PT$_3$ is that (a) only a few subtraction constants are required as input, and (b) $\pi\pi$ rescattering effects are fully accounted for in terms
of Omnès factors and calculable dispersive integrals in crossed channels. Moreover, by allowing \( \mathcal{H}_w \) to carry momentum, the ambiguities associated with taking the kaon off-shell [42, 43] are entirely avoided.

In this article, we extend the dispersive framework developed in [41] to the rare decays \( K_S \to \gamma \gamma \) and \( K_S \to \gamma \ell^+ \ell^- \) \( (\ell = e \text{ or } \mu) \). In lowest-order (LO) \( \chi \text{PT} \), the amplitudes for \( K_S \to \gamma \gamma^{(*)} \) possess the well known feature of ultraviolet finite \( \pi^\pm, K^0 \) one-loop diagrams coupled to the external photons. For the pure radiative decay, the chiral prediction [2,44,45] for the rate

\[
\text{BR}(K_S \to \gamma \gamma)_{\chi \text{PT}3} = 2.0 \times 10^{-6}
\]

is in reasonable agreement with the experimental average [46]

\[
\text{BR}(K_S \to \gamma \gamma) = (2.63 \pm 0.17) \times 10^{-6},
\]

while the predictions [47] for the leptonic modes are typically expressed in terms of the ratios

\[
\frac{\Gamma(K_S \to \gamma \ell^+ \ell^-)}{\Gamma(K_S \to \gamma \gamma)}_{\chi \text{PT}3} = \begin{cases} 1.6 \times 10^{-2} & (\ell = e) \\ 3.8 \times 10^{-4} & (\ell = \mu) \end{cases}.
\]

Although these decays have not yet been measured, they may lie within reach of the KLOE-2 experiment at DAΦNE [48], which is projected to be sensitive down to \( K_S \) branching ratios of \( O(10^{-7}) \). Given these projections, it is clearly of interest to determine what impact \( \pi \pi \) rescattering effects have on the \( \chi \text{PT}3 \) predictions (3).

The outline of this paper is as follows. In Section 2 we introduce the general formalism needed to analyse \( K_S \to \gamma \gamma' \) dispersively, and derive the decomposition of the decay amplitude into a basis of scalar functions that are free from kinematic zeros and singularities. In particular, we use this basis to extend the LO \( \chi \text{PT}3 \) calculation [47] to the case where \( \mathcal{H}_w \) carries non-zero momentum. Section 3 reviews the dispersive framework developed for \( K_S \to \pi \pi \) [41], which forms a key input in our analysis of \( K_S \to \gamma \gamma' \). In Section 4 we examine \( K_S \to \gamma \gamma \) and find that the inclusion of effects from FSI improves the agreement between theory and experiment. We also comment on how our results compare with previous work [49] based on extrapolating the kaon off-shell. Section 5 concerns \( K_S \to \gamma \ell^+ \ell^- \), where we observe that FSI and the pion vector form factor lead to sizeable corrections of the LO \( \chi \text{PT}3 \) predictions. Our summary is given in Section 6.

### 2 Preliminaries

We begin by considering the radiative decay

\[
K_S(k) \to \gamma(q_1) \gamma'(q_2),
\]

whose amplitude is given by

\[
M(K_S \to \gamma \gamma') = e_1^\mu e_2^\nu(q_1, \lambda_1)e_1^{\gamma(*)}(q_2, \lambda_2)A_{\mu \nu}(k, q_1, q_2),
\]

where \( e_1, e_2 \) are the polarization vectors of the photons. The tensor \( A_{\mu \nu} \) is defined in terms of the pure \( \Delta I = 1/2 \) matrix element\(^2\)

\[
A_{\mu \nu}(k, q_1, q_2) = -\int d^4x d^4y e^{i(q_1 \cdot x + q_2 \cdot y)} \langle \text{vac} | T \{J_\mu(x)J_\nu(y), \mathcal{H}_w^{1/2}(0) \} | K_S(k) \rangle,
\]

where \( J_\mu \) is the electromagnetic current of the light quarks \( u,d,s \), and we allow the weak Hamiltonian \( \mathcal{H}_w \) to carry non-zero momentum \( h_w \neq 0 \). Then the decay amplitude (5) becomes a function of the three Mandelstam variables

\[
s = (q_1 + q_2)^2, \quad t = (k - q_1)^2, \quad u = (k - q_2)^2,
\]

which satisfy

\[
s + t + u = m_K^2 + q_1^2 + q_2^2 + h^2.
\]

In what follows it is convenient to set \( h^2 = 0 \), while keeping \( h_w \neq 0 \) in general. Doing so does not result in a loss of generality, but does simplify several expressions derived in this paper. To recover the physical decay amplitude, one simply takes the limit \( h_w \to 0 \), in which case the kinematic variables become fixed at the values

\[
s = m_K^2, \quad t = q_2^2, \quad u = 0.
\]

### 2.1 Tensor decomposition

To set up a dispersive framework for \( K_S \to \gamma \gamma' \), the first step is to decompose \( A_{\mu \nu} \) in a basis of independent tensors, whose scalar coefficients are free from kinematic singularities and zeros. This can be achieved by applying the prescription of Bardeen and Tung [50], and Tarrach [51]; our approach resembles the tensor decomposition of \( \gamma' \gamma' \to \pi \pi \) discussed in [52–54].

Let \( q_i = \{ q_1, q_2, h - k \} \) label the three independent momenta and observe that Lorentz covariance and CP-invariance implies a decomposition in terms of ten tensors\(^3\)

\[
A_{\mu \nu} = g_{\mu \nu}A_1 + \sum_{i,j=1}^3 q_{i\mu}q_{j\nu}A_{ij}^2.
\]

The scalar functions \( \{ A_1, A_{ij}^2 \} \) are not all independent since \( A_{\mu \nu} \) is constrained by the electromagnetic Ward identities

\[
q_{i\nu}A_{\mu \nu} = q_{j\nu}A_{\mu \nu} = 0.
\]

A convenient way to impose the constraint (11) is to introduce the gauge projector

\[
P_{\mu \nu} = g_{\mu \nu} - q_{j\nu}q_{j\nu} \frac{q_{i\mu}}{q_{i1}q_{i2}}.
\]

\(^2\) In non-leptonic \( \Delta S = 1 \) processes, it is observed that amplitudes with \( \Delta I = 1/2 \) dominate over other isospin transitions. As in [41], we focus on this dominant contribution to \( K_S \to \gamma \gamma' \), noting that the dispersive framework can easily be adapted to a determination of the sub-dominant \( \Delta I = 3/2 \) amplitude.

\(^3\) The terms \( \sim \sum_{i,j}q_{i\nu}q_{j\nu}A_{ij}^2 \) are allowed by Lorentz covariance, but violate \( P \) and CP symmetry.
and let it act on both indices of $A_{\mu\nu}$:

$$A_{\mu\nu} = P_{\mu\alpha}P_{\nu\beta}A^{\alpha\beta} = \sum_{i=1}^{5} T_{\mu\nu}^{i} A_{i}.$$  \hfill (13)

By definition, this leaves the physical tensor $A_{\mu\nu}$ invariant and removes contributions that do not satisfy the Ward identities; with this procedure the set of scalar functions reduces to five. The new basis functions $A_{i}$ are free from kinematic singularities, but contain zeros because the tensors $T_{\mu\nu}^{i}$ contain single and double poles in $q_{1} \cdot q_{2}$. As shown in [52–54], the removal of these poles can be performed by adding suitable linear combinations of $T_{\mu\nu}^{i}$ with non-singular coefficients, followed by a rescaling in powers of $q_{1} \cdot q_{2}$. In our case, contraction with $\varepsilon_{i}$ and setting $q_{i}^{2} = 0$ imposes two additional constraints, so the final result is

$$A_{\mu\nu}(k, q_{1}, q_{2}) = \sum_{i=1}^{3} T_{\mu\nu}^{i} B_{i}(s, t, u, q_{i}^{2}) .$$  \hfill (14)

where the scalar functions $B_{i}$ are free from kinematic zeros and singularities, and the corresponding tensors are

$$T_{\mu\nu}^{1} = (q_{1} \cdot q_{2}) g_{\mu\nu} - q_{2\mu} q_{1\nu} ,$$  \hfill (15)

$$T_{\mu\nu}^{2} = (q_{1} \cdot q_{2}) g_{\mu\nu} - q_{2\mu} q_{1\nu} + \frac{1}{2} \left[ (t-u) - m_{K}^{2} \right] (q_{1} g_{\mu\nu} - q_{2\mu} q_{1\nu}) ,$$  \hfill (16)

$$T_{\mu\nu}^{3} = (q_{1} \cdot q_{2}) g_{\mu\nu} - \frac{1}{2} \left[ (t-u) - m_{K}^{2} \right] g_{\mu\nu} + \frac{1}{2} \left[ (t-u) + m_{K}^{2} \right] q_{2\mu} q_{1\nu} - \frac{1}{2} \left[ (t-u) - m_{K}^{2} \right] q_{2\nu} q_{1\mu} .$$  \hfill (17)

At the physical point (9) there are only two independent momenta, so $A_{\mu\nu}$ reduces to $T_{\mu\nu}^{i}$ times the coefficient

$$B_{1}(m_{K}^{2}, q_{2}^{2}) = B_{2}(m_{K}^{2}, q_{2}^{2}) + \frac{1}{2} (q_{2}^{2} + m_{K}^{2}) B_{3}(m_{K}^{2}, q_{2}^{2}) .$$  \hfill (18)

Evidently, the determination of the scalar functions $B_{i}$ completely fixes the prediction for the $K_{S} \to \gamma\gamma'$ amplitude (5).

### 2.2 $K_{S} \to \gamma\gamma'$ in lowest order $\chi PT_{3}$

Before discussing our dispersive treatment of the scalar functions $B_{i}$, it is instructive to extend the LO $\chi PT_{3}$ calculation of $K_{S} \to \gamma\gamma'$ [47] to the case where $\mathcal{H}_{u}$ carries momentum. In the conventions of [2], the graphs shown in Figure 1 yield

$$A_{\mu\nu}|_{\chi PT_{3}} = -i G_{8} F_{8}(3s + m_{K}^{2} - 4m_{K}^{2}) I_{\mu\nu} - \left\{ m_{\pi}^{2} \to m_{K}^{2} \right\} ,$$  \hfill (19)}

where $G_{8} = 9.1 \times 10^{-6}$ GeV$^{-2}$ is the octet coupling at $O(p^{2})$, $F_{8} = 92.2$ MeV is the pion decay constant [46], and the loop integral is

$$I_{\mu\nu} = \int \frac{d^{4}q}{(2\pi)^{4}} \frac{g_{\mu\nu}((q^{2})^{2} - m_{\pi}^{2}) - (q^{2} + q_{1}\mu)(q^{2} - q_{1}\nu)}{((q^{2} + q_{1}\mu)(q^{2} - q_{1}\nu))^{2} - m_{\pi}^{2}((q^{2})^{2} - m_{\pi}^{2})} ,$$  \hfill (20)

where $\phi = \pi^{\pm}$ or $K^{\pm}$. The integral is ultraviolet finite and can be evaluated in terms of Feynman parameters:

$$I_{\mu\nu} = \int \frac{16\pi^{2}}{s} du \int_{0}^{1-u} dv \frac{4\pi T_{\mu\nu}^{3} - 2v(1-2v)T_{\mu\nu}^{2}}{D(u,v,m_{K}^{2})} ,$$  \hfill (21)

where the denominator is given by

$$D(u,v,m_{K}^{2}) = m_{K}^{2} - su - v(1-u-v)q_{K}^{2} - i\epsilon .$$  \hfill (22)

In (19), the second tensor

$$T_{\mu\nu}^{4} = (q_{1} \cdot q_{2}) q_{1\mu} q_{2\nu} - q_{2} q_{1\mu} q_{1\nu}$$  \hfill (23)

vanishes upon contraction with $\varepsilon_{i}$, so we find that only $B_{1}$ contributes to $M(K_{S} \to \gamma\gamma')$ at LO, with

$$B_{1}(s, q_{2}^{2})|_{\chi PT_{3}} = \frac{G_{8} F_{8}}{4\pi^{2}} \left\{ 3 s + m_{K}^{2} - 4 m_{K}^{2} \right\} H(s, m_{\pi}^{2}, q_{2}^{2}) - \left\{ m_{\pi}^{2} \to m_{K}^{2} \right\} .$$  \hfill (24)

Here, the quantity

$$H(s, m_{\pi}^{2}, q_{2}^{2}) = \frac{s^{2}}{2(s - q_{2}^{2})^{2}}$$  \hfill (25)

is defined [2] in terms of the one-loop functions

$$F(a) = \begin{cases} 1 - \frac{4}{a} \arcsin^{2} \left( \sqrt{a}/2 \right) & a \leq 4 , \\ 1 + \frac{1}{a} \left( \ln 1 - \sqrt{1 - 4/a} + i\pi \right) & a > 4 , \end{cases}$$  \hfill (26)

$$G(a) = \begin{cases} 1 - \frac{2}{\sqrt{4-a}} \arcsin \left( \sqrt{a}/2 \right) & a \leq 4 , \\ \frac{2}{\sqrt{4-a}} \left( \ln 1 + \sqrt{1 - 4/a} - i\pi \right) & a > 4 . \end{cases}$$  \hfill (27)

At the physical point (9), the expression in (22) agrees with the original $\chi PT_{3}$ result [47], as it should.

As emphasised in [55], tadpole cancellation completely eliminates the weak mass operator at $O(p^{2})$ in the $\chi PT_{3}$ expansion. The argument can be extended to $O(p^{4})$ [56] and remains valid when $\mathcal{H}_{u}$ carries momentum.

### 2.3 Unitarity and $\pi\pi$ intermediate states

Let us now analyse the unitarity relation due to the intermediate $\pi\pi$ state. In the s-channel, this contribution reads (Figure 2)}

$$\text{disc}_{\mu\nu} A_{\mu\nu} = \frac{1}{2} \int \frac{d^{4}p_{1}}{(2\pi)^{4}E_{1}} \frac{d^{4}p_{2}}{(2\pi)^{4}E_{2}} (2\pi)^{4}$$  \hfill (28)

$$\times \delta^{4}(q_{1} + q_{2} - p_{1} - p_{2}) A_{\pi\pi}(s, t', u') W_{\mu\nu}(q_{1}, q_{2}, p_{1}) ,$$  \hfill (29)
where $A_{\pi\pi}$ and $W_{\mu\nu}$ are the amplitudes for the subprocesses $K_S \to \pi\pi$ and $\gamma\gamma^* \to \pi\pi$ respectively. On the left-hand side of the cut, the Mandelstam variables are
\[
\hat{s} = (k - p_1)^2, \quad \hat{t} = (k - p_2)^2,
\]
while on the right-hand side, $W_{\mu\nu}$ can be decomposed into a basis of three independent tensors $[21, 52-54]$:
\[
W_{\mu\nu}(q_1, q_2, p_1) = \sum_{i=1}^{3} t_{i\mu\nu}(s, t', u', q_2^2),
\]
where
\[
t' = (q_1 - p_1)^2, \quad u' = (q_1 - p_2)^2,
\]
and
\[
t_{1\mu\nu} = (q_1 \cdot q_2) \delta_{\mu\nu} - q_2 \mu q_1 \nu,
\]
\[
t_{2\mu\nu} = (q_1 \cdot q_2) \Delta_{\mu}(q_2 - q_2^0 q_1) + \frac{1}{2} (t' - u') (q_2^2 \delta_{\mu\nu} - q_2 \mu q_2 \nu),
\]
\[
t_{3\mu\nu} = (q_1 \cdot q_2) \Delta_{\mu} \Delta_{\nu} - \frac{1}{2} (t' - u')^2 \delta_{\mu\nu},
\]
\[
+ \frac{1}{2} (t' - u') (\Delta_{\mu} q_1 - q_2 \Delta_{\nu}), \quad \Delta = p_2 - p_1.
\]

The phase space integration (25) must project each of the tensors $t_{i\mu\nu}$ onto linear combinations of $T_{\mu\nu}$. However, the integration is trivial if contributions from $D$ waves and higher are neglected. This is because in this approximation, $A_{\pi\pi}$ is independent of $t'$ and $u'$, while the scalar functions $W_i$ can be expressed in terms of a single helicity partial wave $[53, 54]$,
\[
W_i = -\frac{2}{s - q_2^2} h_i^0(s, q_2^2), \quad W_2 = W_3 = 0.
\]
Since $W_1$ is independent of the pion momenta, the tensor $t_{1\mu\nu}$ can be pulled under the phase space integral (25). Equating the scalar coefficients then gives the analytical result
\[
\text{disc.} B_1(s, q_2^2) = \frac{1}{32\pi^2} \int d\Omega' A_{\pi\pi}(s) W_1(s, q_2^2)
\]
\[
= -\frac{\sigma_0(s)}{8\pi} A_{\pi\pi}(s) |h_0^0(s, q_2^2)|^2,
\]
where we have introduced the kinematic factor
\[
\sigma_0(s) = \sqrt{1 - 4m_\pi^2/s}.
\]

At higher energies, other intermediate states like $4\pi, K\bar{K}$ etc. will contribute to the $s$-discontinuity of $A_{\mu\nu}$. Moreover, for a complete dispersive treatment one should also consider discontinuities in the $t$- and $u$-channels. We will not consider any of these contributions to the dispersion relation for $A_{\mu\nu}$, and we explain below on what grounds these approximations can be justified.

### 3 Dispersive framework for $K \to \pi\pi$

The construction of a dispersion relation for $K_S \to \gamma\gamma$ requires input from $K_S \to \pi\pi$ and $\gamma\gamma^* \to \pi\pi$. It is well known that one-loop chiral corrections to the $K_S \to \pi\pi$ amplitude are substantial, and largely due to significant rescattering effects of pions in the final state $[57-59]$. An understanding of FSI is thus essential in order to make sense of puzzles such as the $\Delta I = 1/2$ rule or the SM prediction for $\epsilon'/\epsilon$. As noted in Section 1, dispersive techniques are well suited to addressing FSI; here we review the dispersive framework $[41, 42]$ developed for $K \to \pi\pi$.

We begin with the standard isospin decomposition for the $K^0 \to \pi\pi$ amplitude $[2]$:
\[
\frac{A_{\pi\pi}}{\sqrt{2}} = A_{1/2},
\]
where $A_{1/2}$ is generated by the $\Delta I = 1/2$ component of $\mathcal{H}_u$, and we have omitted a term involving $\Delta I = 3/2$. As in Section 2, we allow the effective weak Hamiltonian $\mathcal{H}_u$ to carry momentum $h_\mu \neq 0$, so the amplitude reads
\[
A_{1/2}(s, t', u') = \langle (\pi(p_1)\pi(p_2))|\mathcal{H}_u|K^0(k)\rangle,
\]
where the corresponding Mandelstam variables are given in (26), and satisfy
\[
s + t' + u' = 2m_\pi^2 + m_K^2.
\]
The physical $K^0 \to \pi\pi$ decay amplitude is then obtained by taking the limit $h_\mu \to 0$, at which point we have
\[
s = m_K^2 \quad \text{and} \quad t' = u' = m_\pi^2.
\]
If contributions from the imaginary parts of $D$ waves and higher are neglected, it is possible to decompose $A_{1/2}$ in terms of single-variable functions
\[
A_{1/2}(s, t', u') = M_0(s) + C(s, t', u'),
\]
where the angular dependence is contained in
\[
C(s, t', u') = \frac{1}{3} N_0(t') + 2R_0(t')
\]
\[
+ \frac{1}{2} \left[ s - u' - \frac{m_\pi^2(m_K^2 - m_\pi^2)}{t'} \right] N_1(t') + \{t' \leftrightarrow u' \},
\]
and the explicit expressions for $N_i$ and $R_i$ can be found in [41].

As a result of this simplification, the dispersive treatment of the full amplitude $A_{1/2}$ is reduced to solving a coupled set of dispersion relations of the single-variable functions appearing in the right-hand side of (37). As shown in [41], these relations can be solved numerically, with a minimum of two subtraction constants$^4$ needed to ensure convergence of the dispersive integrals. One of these constants $a_{\pi\pi}$ can be determined at the soft-pion point

$$ s = u' = m_{\pi}^2 \quad \text{and} \quad t' = m_K^2, $$

where $A_{1/2}$ is related to the on-shell $K \to \pi$ amplitude $A_{\pi}$:

$$ \frac{A_{\pi}}{2F_{\pi}} = \frac{A_{1/2}(m_{\pi}^2, m_K^2, m_{\pi}^2)}{m_K^2 - m_{\pi}^2(4 + 3X)} + O(m_{\pi}^2), $$

Note that with both $K$ and $\pi$ on-shell, the weak operator $\mathcal{H}_w$ in $A_{\pi}$ necessarily carries momentum. The relevance of lattice calculations of $A_{\pi}$ in connection with the $\Delta I = 1/2$ rule has recently been discussed in [62].

On the other hand, the second constant $b_{\pi\pi}$ can be obtained by considering e.g. the derivative $\partial A_{1/2}/\partial s$ at the soft-pion point (39). Ideally, lattice techniques would be used to determine $a_{\pi\pi}$ and $b_{\pi\pi}$, although such calculations remain to be undertaken. Thus the approach taken in [41] was essentially pragmatic: to illustrate the role of FSI, the value of $b_{\pi\pi}$ was fixed by applying $\chi$PT3, so that

$$ b_{\pi\pi} = \frac{3a_{\pi\pi}(1 + X)}{m_K^2 - m_{\pi}^2(4 + 3X)} + O(m_{\pi}^2), $$

where the dimensionless parameter $X$ controls the size of the expected NLO corrections; on the basis of the 3-flavour expansion it can be varied between $X = \pm 0.3$. We note that the relation (41) is not affected by the weak mass term in $\mathcal{H}_w$; see Section 2.2.

From the solutions to the dispersion relations, it is a straightforward matter to reconstruct the $K \to \pi\pi$ amplitude. For $u'$ fixed near the physical value $m_{\pi}^2$, it has been shown [63] that the contribution due to $C(s, t', u')$ is negligible relative to $M_0$ in the low-energy region $s \gtrsim 1.5$ GeV$^2$. Thus to a good approximation, we can write

$$ A_{1/2}(s, m_K^2 + m_{\pi}^2 - s, m_{\pi}^2) \simeq a_{\pi\pi} \left[1 + E(X)s/m_{\pi}^2\right] \Omega_0^0(s), $$

where the quantity

$$ E(X) = \frac{3m_{\pi}^2(1 + X)}{m_K^2 - m_{\pi}^2(4 + 3X)} $$

parametrises the NLO corrections, and

$$ \Omega_0^0(s) = \exp \left(\frac{s}{\pi} \int_{m_{\pi}^2}^{\infty} dz \frac{\delta_0^0(z)}{z(z - s - i\epsilon)}\right) $$

is the Omnès function [64] subtracted at $s = 0$, with $\delta_0^0$ the $\pi\pi$ scattering phase shift in the $I = 0$ channel.

The $K \to \pi\pi$ amplitude in (42) can be determined up to the unknown subtraction constant $a_{\pi\pi}$, modulo chiral corrections parametrised by $X$. As a result, a first principles prediction for $K \to \pi\pi$ is not currently possible within this framework. Fortunately, this does not pose a problem for $K_S \to \gamma\gamma$ since we can eliminate the dependence on $a_{\pi\pi}$ by matching to $A_{1/2} = A_0 e^{i\theta}$ at the physical point (36):

$$ \left|a_{\pi\pi}\right| = \frac{A_0}{\Omega_0^0(m_{\pi}^2)\left[1 + E(X)\right]}, $$

where $A_0 = (2.704 \pm 0.001) \times 10^{-7}$ GeV

4 Dispersion relations for $K_S \to \gamma\gamma$

As a first application of our dispersive framework, here we consider the case where both photons are on-shell. A complete dispersive treatment of $K_S \to \gamma\gamma$ (with $\mathcal{H}_w$ carrying momentum) would require an analysis of all possible intermediate states in all three channels $s, t, u$ — clearly a daunting task. A simplification which has proven to be particularly effective for other scattering processes at low energies is to neglect the contributions to discontinuities coming from $D$ waves and higher. This leads to a dispersive representation of the scattering amplitude in terms of single-variable functions, much like in the case of the $K \to \pi\pi$ amplitude discussed in Section 3. As in that case, we expect that at the physical point (9), the contributions to the $S$ wave coming from discontinuities in the $t$ and $u$ channels are negligible, and so will not consider them. Effectively this means that we construct a dispersion relation of the form-factor type (i.e. with a right-hand cut only), and only for the $S$ wave. Moreover, we will explicitly consider only the effect of $\pi\pi$ rescattering, which at low energies should be far the most important one. Indeed, this expectation is borne out by the LO $\chi$PT3 result discussed in Section 2.2.

Let us define $A_{\gamma\gamma}(s) = e^2B_1(s)$, whose imaginary part coincides with the $s$-discontinuity in (31) once we set $q_1^2: 0$.

$$ \text{Im}_1 A_{\gamma\gamma}(s) = -\alpha \sigma_{\pi\pi}(s) \Omega_0^0(s) A_0 \left[1 + E(X)s/m_{\pi}^2\right] \left\{H_{0,+++}^0(s)\right\}^*. $$

Here $\alpha = e^2/4\pi$ is the fine-structure constant, and $H_{0,+++}^0$ is the projection of $H_{1,+++}$ onto the $I = 0$ channel. The real part then
follows from a once-subtracted dispersion relation at \( s = s_0 \):

\[
A_{\gamma\gamma}(s) = a_{\gamma\gamma} + \frac{s - s_0}{\pi} \int_{4m_\pi^2}^{\infty} dz \frac{\text{Im} A_{\gamma\gamma}(z)}{(z - s_0)(z - s - i\varepsilon)},
\]

where \( a_{\gamma\gamma} \) is the constant term. The subtraction is necessary because the \( \pi^+ K^0 \) loop contribution to the \( \chi \)PT amplitude vanishes at the point

\[
s_0 = -0.098 \text{ GeV}^2,
\]

and moreover to ensure convergence of the dispersive integral. This feature can be deduced from the explicit form of the \( \chi \)PT amplitude in (22), with \( q_2^2 = 0 \). It follows that matching \( A_{\gamma\gamma}(s_0) \) onto LO \( \chi \)PT fixes \( a_{\gamma\gamma} = 0 \), although in general, \( a_{\gamma\gamma} \) will receive \( SU(3) \) corrections due to terms at \( O(p^6) \) in the chiral expansion. It is important to note that by matching below the \( \pi\pi \) threshold, we make use of the single-variable functions in (37) is only valid in the elastic scattering region \( 4m_\pi^2 < s < 16m_\pi^2 \), even though the first significant inelastic correction is due to the \( KK \) intermediate state when \( s > 4m_K^2 \). Taking this into account would require a coupled-channel analysis of \( K_S \to \pi\pi \) and \( K_S \to KK \), which is beyond the scope of this work. Moreover, it is unclear whether this would lead to better precision, because there are no sources of experimental information on \( K_S \to KK \), and we would have to rely completely on \( \chi \)PT to determine the subtraction constants, with correspondingly large uncertainties.

To compute the integral in (48), we require input for \( h_{0,++}^0 \) and the \( S \) wave of the \( K_S \to \pi\pi \) amplitude which, in our representation, is given by \( \Omega_{0}^{S} \). Concerning the latter, the dispersive representation of the single-variable functions in (37) is only valid in the elastic scattering region \( 4m_\pi^2 < s < 16m_\pi^2 \), even though the first significant inelastic correction is due to the \( KK \) intermediate state when \( s > 4m_K^2 \). Taking this into account would require a coupled-channel analysis of \( K_S \to \pi\pi \) and \( K_S \to KK \), which is beyond the scope of this work. Moreover, it is unclear whether this would lead to better precision, because there are no sources of experimental information on \( K_S \to KK \), and we would have to rely completely on \( \chi \)PT to determine the subtraction constants, with correspondingly large uncertainties.

We thus conclude that our preferred phase is the one given in Fig. 3. Energy dependence of phase shift inputs (top) and magnitude of the corresponding Omnès functions (bottom).
it is necessary to impose a cutoff $\Lambda$ in our dispersion integral (48). At the physical point $s = m_K^2$, a comparison of the cutoff dependence is shown in Figure 4, where $|\text{Re} \ A_{\gamma\gamma}|$ is seen to exhibit a very mild sensitivity to variations in $\Lambda$.

Taking $\Lambda = 1.2$ as a benchmark value, the energy dependence of the real and imaginary parts of $A_{\gamma\gamma}$ is shown in Figure 5. As expected, the dispersive representation agrees with LO $\chi$PT below the $\pi\pi$ threshold. However, for $s > 4m_K^2$, the effects from FSI distort the amplitude, producing a significant enhancement (suppression) of the real (imaginary) part. These effects lead to an enhanced prediction for the branching ratio

$$\text{BR}(K_S \to \gamma\gamma) = \frac{m_K^3 |A_{\gamma\gamma}(m_K^2)|^2}{64\pi} \frac{1}{T(K_S)_{\text{tot}}} = (2.34 \pm 0.31) \times 10^{-6}$$

which brings the SM and experiment (2) into much better agreement. The uncertainty has been determined by considering the variation $X = \pm 0.3$, shifting the value of $s_0$ by 30%, the comparison of the two Omnès inputs (Figure 3), and an estimate of contributions from the high-energy region $\Lambda > 1.2$ GeV, where the phase of $\Omega_3^0$ is guided to $\pi$ and the helicity partial wave is fixed to a constant value $|h_{0,++}| \approx 4$. Combined in quadrature, the final uncertainty has turned out to be remarkably modest.

### 4.1 Comparison to the literature

As shown in Figure 5, the real part of $A_{\gamma\gamma}$ receives a significant enhancement in absolute value at $s = m_K^2$ due to FSI. A similar observation has been made by Kambor and Holstein (KH) [49], who estimated the effects of $\pi\pi$ rescattering in $K_S \to \gamma\gamma$ and $K_L \to \pi^0\gamma\gamma$ by extrapolating the kaon mass off-shell. Focusing on the former process, we can adapt their notation to ours by defining

$$A_{\gamma\gamma}^{\text{KH}}(s) = -2a_F F_2 B(s)/s,$$

where $B(s)$ is a scalar function whose definition is given in [49]. In our comparison, we have updated the input used in [49] to account for improved determinations [19] of the Omnès factor and helicity partial wave. The resulting predictions at the benchmark value of $\Lambda = 1.2$ GeV are shown in Table 1, where we also list the pure octet predictions from $\chi$PT3. We note that although the KH formalism produces a branching ratio consistent with experiment, it relies on the assumption that one can extrapolate the kaon mass off the mass shell. As discussed in [42,43], this procedure suffers from an inherent ambiguity as there is no unique way in which to perform the off-shell extrapolation. By contrast, our framework always involves on-shell states, and is free from such ambiguities.

### 5 Dispersion relations for $K_S \to \gamma\ell^+\ell^-$

We now consider the case where the photon momentum in $K_S \to \gamma\ell^+\ell^-$ can remain off-shell $q_\ell^2 \neq 0$. As in Section 4, we focus on contributions from $S$ waves and define $A_{\gamma\ell^+\ell^-}(s, q_\ell^2) = e^2 B_1(s, q_\ell^2)$.
The charge basis to the isospin one \([21]\).

The corresponding dispersion integral is given by Table 1. The numbers in the row labelled “This work” have been obtained with input from GMM.

| Input | Re\(A_{\gamma\gamma}^0\) [10^{-9} GeV^{-1}] | Im\(A_{\gamma\gamma}^0\) [10^{-9} GeV^{-1}] | \(|A_{\gamma\gamma}^0|\) [10^{-9} GeV^{-1}] | BR\(K_S \to \gamma\gamma\) [10^{-6}] |
|-------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| \(\chi PT_3\) | -2.38 | 4.19 | 4.82 | 1.9 |
| KH | -4.28 | 3.47 | 5.51 | 2.54 |
| This work | -4.00 ± 0.47 | 3.47 | 5.30 ± 0.35 | 2.34 ± 0.31 |
| PDG | - | - | 5.62 ± 0.18 | 2.63 ± 0.17 |

In the presence of \(\pi\pi\) rescattering in the \(I = 0\) channel, the \(s\)-discontinuity reads

\[
\text{disc}_s A_{\gamma\gamma}^0(s,q^2) = -\frac{\alpha}{\sqrt{2}} \frac{\sigma_\gamma(s) \Omega_\gamma^0(s) A_0 \left[ 1 + E(X)s/m_K^2 \right]}{\left[ 1 + E(X) \right]} \left[ \frac{b_{0,++}^0(s,q^2)}{s-q^2} \right],
\]

so the corresponding dispersion integral is given by

\[
A_{\gamma\gamma}^0(s,q^2) = a_{\gamma\gamma}^0(q^2) + \frac{s}{\pi} \int_{4m^2}^{\infty} dz \frac{\text{disc}_s A_{\gamma\gamma}^0(z,q^2)}{z(z-s-ie^i)},
\]

where we have subtracted at \(s_0 = 0\) to ensure convergence of the dispersive integral, and fixed the subtraction constant by matching to the \(\chi PT_3\) amplitude (22):

\[
a_{\gamma\gamma}^0(q^2) = e^2 B_1(0,q^2) \big|_{\chi PT_3} \approx A_{\gamma\gamma}^0(0,q^2) \big|_{\chi PT_3}.
\]

To evaluate (55), we begin by decomposing the helicity partial wave

\[
h_{0,++}^0(s,q^2) = h_{0,++}^{0,\text{Born}}(s,q^2) + h_{0,++}^{0,\text{scatt}}(s,q^2),
\]

noting that Low’s theorem [67] implies the Born-subtracted partial wave \(h_{0,++}^{0,\text{scatt}}\) has a zero at \(s = q^2\) (i.e. when the on-shell photon becomes soft \(q_1 \to 0\)).

The Born contribution to the helicity partial wave

\[
h_{0,++}^{0,\text{Born}}(s,q^2) = \frac{4 F_K^2(q^2)}{3} \frac{m^2}{s-q^2} \left[ \frac{4 m^2}{\sigma_\gamma(s)} \right] \left( 1 - \frac{1}{\sigma_\gamma(s)} - 2 q_2^2 \right)
\]

produces a double pole \(\sim (s-q^2)^2\) in \(\text{disc}_s A_{\gamma\gamma}\), so a decomposition of the integrand

\[
\frac{1}{(z-s)(z-q^2)^2} = \frac{1}{(s-q^2)^2} \left[ \frac{1}{z-s} - \frac{1}{z-q^2} \right] = \frac{1}{s-q^2} \frac{1}{z-q^2},
\]

is required in order to evaluate the dispersive integral numerically. In the above, \(F_K^2\) denotes the vector form factor of the

\[7\] The absence of anomalous thresholds in \(K_S \to \gamma\gamma\) follows from the same arguments used for \(\gamma\gamma^* \to \pi\pi\) [21]; see also [66] for a general treatment.

\[8\] The Clebsch-Gordan factor of \(\sqrt{4/3}\) is due to the rotation from the charge basis to the isospin one [21].

pion, and is set to unity in LO \(\chi PT_3\). Using the identity in (59), we get the Born part of the \(K_S \to \gamma\gamma^*\) amplitude

\[
A_{\gamma\gamma}^{\text{Born}}(s,q^2) = \frac{s}{\pi} \left\{ \frac{Q(s,q^2) - Q(q^2,q^2)}{(s-q^2)^2} - \frac{1}{s-q^2} \left[ \frac{\partial}{\partial q^2} Q(\lambda,q^2) \right] \right\},
\]

where we have defined

\[
Q(s,q^2) = \int_{4m^2}^{\infty} dz \frac{(z-q^2)^2 \text{disc}_s A_{\gamma\gamma}^{\text{Born}}(z,q^2)}{z(z-s-ie^i)}.
\]

Similarly, for the rescattering contribution, we use the identity

\[
\frac{1}{(z-s)(z-q^2)} = \frac{1}{s-q^2} \left[ \frac{1}{z-s} - \frac{1}{z-q^2} \right]
\]

so that

\[
A_{\gamma\gamma}^{\text{scatt}}(s,q^2) = \frac{s}{\pi} \left\{ \frac{R(s,q^2) - R(q^2,q^2)}{s-q^2} \right\},
\]

where

\[
R(s,q^2) = \int_{4m^2}^{\infty} dz \frac{(z-q^2)^2 \text{disc}_s A_{\gamma\gamma}^{\text{scatt}}(z,q^2)}{z(z-s-ie^i)}.
\]

In the evaluation of (60) and (63), we use the two Omnès inputs discussed in Section 4, as well as the pion form factor and helicity partial waves \(h_{0,0,++}\) obtained from Moussallam’s single-channel analysis of \(\gamma\gamma^* \to \pi\pi\) [21]. The range of validity of \(h_{0,0,++}\) can be inferred by comparing the result from the single-channel analysis at \(q^2 = 0\) with that from GMM’s coupled-channel analysis of \(\gamma\gamma \to \pi\pi\) [19]. As shown in Figure 6, the real parts begin to differ for \(\sqrt{s} \gtrsim 0.8\) GeV, while the imaginary parts differ for \(\sqrt{s} \gtrsim 0.5\) GeV. The reason\(^7\) why the imaginary part differs at relatively small energies is because it is related to the real part via Watson’s theorem

\[
\text{Im} h_{0,0,++}^0(s) = \pm \text{Re} h_{0,0,++}^0(s) \times \tan h_{0,0,++}^0(s).
\]

Near \(\sqrt{s} = 0.8\), the phase is close to \(\pi/2\), so small variations in the zero of \(\text{Re} h_{0,0,++}^0\) can lead to a large variation in \(\text{Im} h_{0,0,++}^0\). From a conservative viewpoint, this suggests that the cutoff be fixed to \(\Lambda \approx 0.8\) GeV. However, we have checked that increasing the cutoff to \(\Lambda = 1.2\) GeV does not lead to a difference of more than \(\approx 7\%\) in the resulting predictions for \(A_{\gamma\gamma}^0\). Note that

\[^9\] B. Moussallam, private communication.
this 7% is the effect of a 100% uncertainty on our input between 0.8 and 1.2 GeV. Since this small change is covered by our estimate of the systematic uncertainty, we take the larger cutoff as a benchmark value in our numerics and stress that only a coupled-channel analysis for this process would allow one to better assess this source of uncertainty and push the cutoff to yet higher energies. As noted in Section 4, however, there are non-trivial difficulties in performing a coupled-channel analysis for two-body K decays.

By combining (60) and (63), we obtain the desired result for the total $K_S \rightarrow \gamma \gamma$ amplitude:

$$A_{\gamma\gamma}(s, q_2^2) = A_{\gamma\gamma}^\text{Born}(s, q_2^2) + A_{\gamma\gamma}^\text{scat}(s, q_2^2).$$

(66)

For fixed values of $q_2^2$, we first compare the predictions arising from (66) against those of $\chi$PT. In Figure 7, we show the energy dependence of the amplitude for three values of $q_2^2$. As shown in the Figure, when $q_2^2 < 4m_K^2$, the effect of FSI resembles that previously seen in $K_L \rightarrow \gamma \gamma$ (Figure 5), with the real (imaginary) parts enhanced (suppressed) relative to $\chi$PT. However, as $q_2^2$ increases above the $\pi \pi$ threshold, the pion form factor $F_\pi^V$ becomes progressively more important, and both real and imaginary parts in the dispersive amplitude are enhanced relative to LO $\chi$PT. This feature can be clearly seen in Figure 8, where we keep $s = m_K^2$ fixed and vary $q_2^2$ within the physical region

$$4m_K^2 \leq q_2^2 \leq m_K^2$$

(67)

of the three-body decay. The effect of including the pion form factor in the $\chi$PT amplitude shows a moderate enhancement at large $q_2^2$, especially for the real part. We also note that even for small values of $q_2^2$, the dispersive amplitude differs from $\chi$PT due to the effects of FSI.

We now consider the predictions for the $K_S \rightarrow \gamma \ell^+ \ell^-$ decay rates. Here the differential decay rate is [47]

$$\frac{d\Gamma_{\ell\ell}}{dq_2^2} = \frac{m_K}{32\pi q_2^2} \left(1 - \frac{q_2^2}{m_K^2}\right)^3 \left|A_{\gamma\gamma}(m_K^2, q_2^2)\right|^2 \frac{1}{\pi} \Pi(q_2^2),$$

(68)

where the electromagnetic spectral function is given by

$$\frac{1}{\pi} \Pi(q_2^2) = \frac{\alpha}{3\pi} \left(1 + \frac{m_\ell^2}{q_2^2}\right) \sqrt{1 - 4m_\ell^2/q_2^2} \theta(q_2^2 - 4m_\ell^2).$$

(69)

In Figure 9, we compare the $\chi$PT prediction [47] for the differential decay rate involving muons against our dispersive result. Evidently, the corrections are large for $q_2^2 \gtrsim 0.05$: again, this can be inferred from the $q_2^2$ behaviour shown in Figure 8. We also see that, for this mode, the dominant source of the enhancement is due to the pion form factor.

The integrated rates (normalised to the total $K_S$ decay width) are shown in Table 2, where the uncertainties are determined as in Sec. 4, except for the subtraction constant: here we keep the subtraction point fixed and vary the $\chi$PT amplitude by 30%. In both cases, the corrections are sizeable: for the electron mode we see a shift of $\approx 50\%$ compared to the $\pi \pi$ matrix element, whereas for the muon mode we have a shift of $\approx 10\%$. The origin of these shifts is different in each case. For the electron mode, the phase space is peaked near the origin $q_2^2 = 0$, so the role of $F_\pi^V$ is suppressed and the dominant effect is due to FSI. On the other hand, the enhancement in the muon mode is predominantly due to the form factor (Figure 9).

6 Summary

Current and near-future searches for rare kaon decays are reaching sensitivities where a better control over the long-distance contribution to the relevant amplitudes is needed. Chi- ral perturbation theory and lattice QCD are two of the main
tools which allow a systematic calculation of these contributions, but getting FSI under good control in either of these approaches is challenging. Dispersion relations offer a different, complementary methodology to the previous two, which addresses specifically the treatment of FSI. If one can match the dispersive and the chiral representation, and solve the dispersion relation, one can usually obtain much better control over FSI effects. In this paper, we have taken a first step in this direction by introducing a dispersive framework for $K_S \rightarrow \gamma \gamma$ and $K_S \rightarrow \gamma \ell^+ \ell^-$. A key feature of our analysis is that by allowing the weak Hamiltonian to carry momentum, there is no need to extrapolate the kaon mass off-shell. Moreover, the input for the subamplitudes $K_S \rightarrow \pi \pi$ and $\gamma \gamma^{(*)} \rightarrow \pi \pi$ provide a strong constraint on the dispersive amplitude, and when expressed in terms of measurable quantities we find relatively small uncertainties in our final predictions. In particular, the Born contribution to $\gamma \gamma^{(*)} \rightarrow \pi \pi$ has a negligible uncertainty because the pion vector form factor is known to high precision: for this particular contribution, going off-shell in the photon momentum does not lead to larger uncertainties.

In general, we find that the effects due to FSI provide sizeable corrections to the predictions from LO $\chi$PT. For $K_S \rightarrow \gamma \gamma$, these effects distort the amplitude such that the relative size of the real and imaginary parts are interchanged. That LO $\chi$PT predicts too large an imaginary part can be concluded on the basis of unitarity alone and by taking as input the experimental measurements of $K_S \rightarrow \pi \pi$ and $\gamma \gamma \rightarrow \pi \pi$ at $s = m_K^2$: LO $\chi$PT overshoots the correct value by 21%. As for the real part, we need to rely on analyticity and on a dispersive treatment of both $K_S \rightarrow \pi \pi$ as well as $\gamma \gamma \rightarrow \pi \pi$, where the latter is also well constrained by data. The uncertainties involved here are larger, but still allow us to firmly conclude that the prediction of LO $\chi$PT has the correct sign (negative), but substantially underestimates the absolute value: we obtain an enhancement of about 70%. This feature has been observed earlier by Kambar and Holstein [49], who noted that the reasonable agreement between the rates from LO $\chi$PT and experiment should not be viewed as a success of the effective theory, since unitarization methods produce nearly identical results. Our results confirm this observation and places it on a stronger footing since we do not rely on off-shell extrapolations.

For $K_S \rightarrow \gamma \ell^+ \ell^-$, we found that the pion vector form factor produces an additional source of enhancement over LO $\chi$PT. Since the form factor is well known experimentally in both the timelike and the spacelike region, we can evaluate this particular correction very reliably, which is an important outcome of this analysis. Although less pronounced in the electron mode due to phase space suppression, we observed a particularly large increase in the rate for the muon mode. In view of this result, we believe the muon mode has good prospects of being observed at the projected sensitivities of KLOE-2.

In our analysis, we have restricted ourselves to the case where at most one photon is off-shell. It would be interesting to extend our dispersive framework to the doubly off-shell amplitude $K_S \rightarrow \gamma \gamma^{(*)}$, which provides the dominant contribution to the rare decay $K_S \rightarrow \ell^+ \ell^-$. For the muon mode, LHCb [68] has recently placed an upper bound on the rate $\text{BR}(K_S \rightarrow \mu^+ \mu^-) < 2 \times 10^{-6}$.

| Input          | $\text{BR}(K_S \rightarrow \gamma e^+ e^-)$ | $\text{BR}(K_S \rightarrow \gamma \mu^+ \mu^-)$ |
|----------------|---------------------------------------------|-------------------------------------------------|
| $\chi$PT$_3$  | $3.09 \times 10^{-8}$                       | $7.25 \times 10^{-10}$                          |
| $\chi$PT$_3$ ($F_\pi^V \neq 1$) | $3.17 \times 10^{-8}$                       | $9.97 \times 10^{-10}$                          |
| This work     | $(4.38 \pm 0.57) \times 10^{-8}$            | $(1.45 \pm 0.27) \times 10^{-9}$                |

Table 2. Predictions for the branching ratio of $K_S \rightarrow \gamma \ell^+ \ell^-$. The second row indicates the effect of including the pion vector form factor $F_\pi^V$ in the $\chi$PT$_3$ amplitude.

Fig. 6. Energy dependence of helicity partial waves obtained from dispersive analyses of $\gamma \gamma^{(*)} \rightarrow \pi \pi$ [19, 21].

Fig. 8. Dependence of the $K_S \rightarrow \gamma \gamma^{(*)}$ amplitude on the photon momentum $q_\gamma^2$ for fixed $s = m_K^2$. The real parts are denoted by the solid curves, while the imaginary parts are dashed. The bands on the dispersive results correspond to the systematic uncertainty.
This work was partially funded by the Swiss National Science Foundation and partial support during the completion of this work by the Mainz Institute for Theoretical Physics (MITP) for the hospitality and partial support during the completion of this work.

Acknowledgements

We are indebted to Bachir Moussallam for providing us with numerical data and code from his analyses of $\gamma\gamma \rightarrow \pi\pi$ and $\gamma\gamma \rightarrow \pi\pi$. We also thank him for extensive correspondence and helpful suggestions on the work presented in this paper. We thank Martin Hoferichter and Peter Stoffer for useful discussions and correspondence; we also thank them and Gerhard Ecker and Toni Pich for providing comments on the manuscript. We thank Bastian Kubis for insightful remarks on the role of Born terms in $K_S \rightarrow \gamma\gamma^*$. The authors are grateful to the Mainz Institute for Theoretical Physics (MITP) for the hospitality and partial support during the completion of this work. This work was partially funded by the Swiss National Science Foundation.

References

1. A. J. Buras, D. Buttazzzo, J. Girrbach-Noe and R. Kneijens, JHEP 1511 (2015) 033 [arXiv:1503.02693].
2. V. Cirigliano, G. Ecker, H. Neufeld, A. Pich and J. Portoles, Rev. Mod. Phys. 84 (2012) 399 [arXiv:1107.6001].
3. A. Neveu and J. Scherk, Annals Phys. 57 (1970) 39.
4. T. N. Truong, Acta Phys. Polon. B 15 (1984) 633.
5. T. N. Truong, Phys. Lett. B 207 (1988) 495.
6. A. Dobado, M. J. Herrero and T. N. Truong, Phys. Lett. B 235 (1990) 134.
7. J. R. Peláez, arXiv:1510.00653.
8. I. Caprini, G. Colangelo and H. Leutwyler, Phys. Rev. Lett. 96 (2006) 132001 [arXiv:hep-ph/0512364].
9. R. J. Crewther and L. C. Tunstall, arXiv:1203.1321.
10. R. J. Crewther and L. C. Tunstall, Phys. Rev. D 91 (2015) 034016 [arXiv:1312.3319].
11. J. F. Donoghue, J. Gasser and H. Leutwyler, Nucl. Phys. B 343 (1990) 341.
12. B. Ananthanarayan, I. Caprini, G. Colangelo, J. Gasser and H. Leutwyler, Phys. Lett. B 602 (2004) 218 [arXiv:hep-ph/0402222].
13. S. M. Roy, Phys. Lett. B 36 (1971) 353.
14. B. Ananthanarayan, G. Colangelo, J. Gasser and H. Leutwyler, Phys. Rept. 353 (2001) 207 [arXiv:hep-ph/0005297].
15. G. Colangelo, J. Gasser and H. Leutwyler, Nucl. Phys. B 603 (2001) 125 [arXiv:hep-ph/0103088].
16. S. Descotes-Genon, N. H. Fuchs, L. Girlanda and J. Stern, Eur. Phys. J. C 24 (2002) 469 [arXiv:hep-ph/0112088].
17. R. Kamifinski, J. R. Peláez and F. J. Ynduráin, Phys. Rev. D 77 (2008) 054015 [arXiv:0710.1150].
18. P. Büttiker, S. Descotes-Genon and B. Moussallam, Eur. Phys. J. C 33 (2004) 409 [arXiv:hep-ph/0310283].
19. R. García-Martín and B. Moussallam, Eur. Phys. J. C 70 (2010) 155 [arXiv:1006.5373].
20. M. Hoferichter, D. R. Phillips and C. Schat, Eur. Phys. J. C 71 (2011) 1743 [arXiv:1106.4147].
21. B. Moussallam, Eur. Phys. J. C 73 (2013) 2539 [arXiv:1305.3143].
22. C. Ditsche, M. Hoferichter, B. Kubis and U.-G. Meißner, JHEP 1206 (2012) 043 [arXiv:1203.4758].
23. M. Hoferichter, C. Ditsche, B. Kubis and U. G. Meißner, JHEP 1206 (2012) 063 [arXiv:1204.6251].
24. M. Hoferichter, J. Ruiz de Elvira, B. Kubis and U. G. Meißner, Phys. Rept. 625 (2016) 1 [arXiv:1510.06039].
25. M. Jamin, J. A. Oller and A. Pich, Nucl. Phys. B 622 (2002) 279 [arXiv:hep-ph/0110193].
26. M. Jamin, J. A. Oller and A. Pich, JHEP 0402 (2004) 047 [arXiv:hep-ph/0404108].
27. M. Jamin, J. A. Oller and A. Pich, Phys. Rev. D 74 (2006) 074009 [arXiv:hep-ph/0605095].
28. V. Bernard, M. Oertel, E. Passemar and J. Stern, Phys. Lett. B 638 (2006) 480 [arXiv:hep-ph/0603202].
29. V. Bernard, M. Oertel, E. Passemar and J. Stern, Phys. Rev. D 80 (2009) 034034 [arXiv:0903.1654].
30. G. Abbas, B. Ananthanarayan, I. Caprini and I. Sentieys Immsong, Phys. Rev. D 82 (2010) 094018 [arXiv:1008.0925].
31. T. N. Truong, Phys. Lett. B 99 (1981) 154.
32. J. Bijnens, G. Colangelo and J. Gasser, Nucl. Phys. B 427 (1994) 427 [arXiv:hep-ph/9403390].
33. G. Colangelo, E. Passemar and P. Stoffer, Eur. Phys. J. C 75 (2015) 172 [arXiv:1501.05627].
34. C. Roiesnel and T. N. Truong, Nucl. Phys. B 187 (1981) 293.
35. J. Kambor, C. Wiesendanger and D. Wyler, Nucl. Phys. B 465 (1996) 215 [arXiv:hep-ph/9509374].
36. A. V. Anisovich and H. Leutwyler, Phys. Lett. B 375 (1996) 335 [arXiv:hep-ph/9601237].
37. G. Colangelo, S. Lanz, H. Leutwyler and E. Passemar, PoS EPS -HEP2011 (2011) 304.
38. K. Kamp, M. Knecht, J. Novotný and M. Zdralhá, Phys. Rev. D 84 (2011) 114015 [arXiv:1103.0928].
39. P. Guo, I. V. Danilkin, D. Schott, C. Fernández-Ramírez, V. Mathieu and A. P. Szczepaniak, Phys. Rev. D 92 (2015) 054016 [arXiv:1505.01715].
40. G. Collazuol [NA62 Collaboration], PoS EPS -HEP2009 (2009) 260.
41. M. Büchler, G. Colangelo, J. Kambor and F. Orellana, Phys. Lett. B 521 (2001) 22 [arXiv:hep-ph/0102287].
42. G. Colangelo, Nucl. Phys. Proc. Suppl. 106 (2002) 53 [arXiv:hep-lat/0110003].
43. M. Büchler, G. Colangelo, J. Kambor and F. Orellana, Phys. Lett. B 521 (2001) 29 [arXiv:hep-ph/0102289].
44. G. D’Ambrosio and D. Espriu, Phys. Lett. B 175 (1986) 237.
45. J. L. Goity, Z. Phys. C 34 (1987) 341.
46. K. A. Olive et al. [Particle Data Group Collaboration], Chin. Phys. C 38 (2014) 090001.
47. G. Ecker, A. Pich and E. de Rafael, Nucl. Phys. B 303 (1988) 665.
48. G. Amelino-Camelia et al., Eur. Phys. J. C 68 (2010) 619 [arXiv:1003.3868].
49. J. Kambor and B. R. Holstein, Phys. Rev. D 49 (1994) 2346 [arXiv:hep-ph/9310324].
50. W. A. Bardeen and W. K. Tung, Phys. Rev. 173 (1968) 1423 Erratum: [Phys. Rev. D 4 (1971) 3229].
51. R. Tarrach, Nuovo Cim. A 28 (1975) 409.
52. G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, JHEP 1409 (2014) 091 [arXiv:1402.7081].
53. P. Stoffer, arXiv:1412.5171.
54. G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, JHEP 1509 (2015) 074 [arXiv:1506.01386].
55. R. J. Crewther, Nucl. Phys. B 264 (1986) 277.
56. J. Kambor, J. H. Missimer and D. Wyler, Nucl. Phys. B 346 (1990) 17.
57. J. Kambor, J. H. Missimer and D. Wyler, Phys. Lett. B 261 (1991) 496.
58. S. Bertolini, J. O. Eeg, M. Fabbrichesi and E. I. Lashin, Nucl. Phys. B 514 (1998) 63 [arXiv:hep-ph/9705244].
59. E. Pallante, A. Pich and I. Scimemi, Nucl. Phys. B 617 (2001) 441 [arXiv:hep-ph/0010501].
60. M. Froissart, Phys. Rev. 123 (1961) 1053.
61. A. Martin, Nuovo Cim. A 42 (1966) 930.
62. R. J. Crewther and L. C. Tunstall, PoS CD 15 (2015) 132 [arXiv:1510.01322].
63. L. Mercolli, Ph.D. thesis, University of Bern, 2012.
64. R. Omnès, Nuovo Cim. 8 (1958) 316.
65. D. Morgan and M. R. Pennington, Phys. Lett. B 137 (1984) 411.
66. M. Hoferichter, G. Colangelo, M. Procura and P. Stoffer, Int. J. Mod. Phys. Conf. Ser. 35 (2014) 1460400 [arXiv:1309.6877].
67. F. E. Low, Phys. Rev. 110 (1958) 974.
68. R. Aaij et al. [LHCb Collaboration], JHEP 1301 (2013) 090 [arXiv:1209.4029].
69. T. Yamanaka, arXiv:1412.5919.
70. G. Isidori and R. Unterdorfer, JHEP 0401 (2004) 009 [arXiv:hep-ph/0311084].