Particle Spectrum
of
Non-Abelian Tensor Gauge Fields

George Savvidy

Institute of Nuclear Physics,
Demokritos National Research Center
Agia Paraskevi, GR-15310 Athens, Greece

Abstract

We present a brief review of the non-Abelian tensor gauge field theory and analyze its free field equations for lower rank gauge fields when the interaction coupling constant tends to zero. The free field equations are written in terms of the first order derivatives of extended field strength tensors similar to the electrodynamics and non-Abelian gauge theories. We determine the particle content of the free field equations and count the propagating modes which they describe. In four-dimensional space-time the rank-2 gauge field describes propagating modes of helicity two and zero. We show that the rank-3 gauge field describes propagating modes of helicity-three and a doublet of helicity-one gauge bosons. Only four-dimensional space-time is physically acceptable, because in five and higher-dimensional space-time the equation has solutions with negative norm states. We discuss the structure of the particle spectrum for higher rank gauge fields.

Dedicated to Professor Emmanuel Floratos at the occasion of his 60th birthday
1 Introduction

It is well understood that the concept of local gauge invariance allows to define non-Abelian gauge field [1, 2], to derive its dynamical field equations and to develop a universal point of view on matter interactions as resulting from the exchange of gauge quanta of spin one. It is appealing to extend the gauge principle so that it would define the interaction of matter fields which carry not only non-commutative internal charges, but also arbitrary spins [3, 4, 5].

In our recent approach the gauge fields are defined as rank-$(s + 1)$ tensors [3, 4, 5]

$$A^a_{\mu \lambda_1 \ldots \lambda_s},$$

and they are totally symmetric with respect to the indices $\lambda_1 \ldots \lambda_s$. The number of symmetric indices $s$ runs from zero to infinity \(^1\). The first member of this family of tensor gauge fields is the Yang-Mills vector boson $A^a_\mu$. The extended non-Abelian gauge transformation $\delta_\xi$ of tensor gauge fields is defined by the equation (2) and comprises a closed algebraic structure [3, 4, 5]. This allows to define generalized field strength tensors [3, 4, 5]

$$G^a_{\mu \nu \lambda_1 \ldots \lambda_s}$$

which are transforming homogeneously with respect to the extended gauge transformations $\delta_\xi$. Using these field strength tensors one can construct two infinite series of quadratic forms $L_s$ and $L'_s$ ($s = 2, 3, \ldots$) invariant with respect to the transformations $\delta_\xi$ [3, 4, 5]. These forms contain quadratic kinetic terms, as well as cubic and quartic terms describing nonlinear interaction of gauge fields with dimensionless coupling constant $g$. In order to make all tensor gauge fields dynamical one should add all these forms in the Lagrangian.

Thus the gauge invariant Lagrangian describing dynamical tensor gauge fields of all ranks has the form [3, 4, 5, 18, 19]

$$\mathcal{L} = \mathcal{L}_Y + g_2 \mathcal{L}_2 + g'_2 \mathcal{L}'_2 + g_3 \mathcal{L}_3 + g'_3 \mathcal{L}'_3 + \ldots$$

(1)

The coupling constants $g_s$ and $g'_s$ ($s = 2, 3, \ldots$) remain arbitrary because each term is separately invariant with respect to the extended gauge transformations $\delta_\xi$ leaves these

\(^1\)A priori the tensor fields have no symmetries with respect to the first index $\mu$. The $a$ is the adjoint index. The totally symmetric tensors were considered in [6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 20].
coupling constants undetermined. Our aim is to analyze the particle spectrum of the theory and to define coupling constants \( g_s \) and \( g'_s \) at which there are no negative norm states in the spectrum. Tensor gauge fields have many components, some of them define physical propagating modes of definite helicities, and some may correspond to unphysical negative norm states. As we shall see, only for specific values of the coupling constants \( g'_s \) and only in four-dimensional space-time the particle spectrum becomes completely physical.

Let us consider first a linear sum of two gauge invariant forms in (1)

\[
g_2 L_2 + g'_2 L'_2,
\]

which defines the kinetic operator and nonlinear interactions of the rank-2 tensor gauge field \( A^\mu_{\lambda 2} \). As we found in [3, 4, 5, 22], if one chooses the coupling constant \( g'_2 = g_2 \), then the sum exhibits invariance with respect to a bigger gauge group. In that case the free field equation (18)/(19) for the rank-2 tensor gauge field describes the propagation of helicity-two, \( \lambda = \pm 2 \), and helicity-zero, \( \lambda = 0 \), massless charged tensor gauge bosons and there are no propagating negative norm states. This result will be recapitulated in the third section.

Our aim here is to extend this analysis to the rank-3 tensor gauge field. Considering the linear sum

\[
g_3 L_3 + g'_3 L'_3
\]

we shall demonstrate that if one chooses the coupling constant \( g'_3 = \frac{4}{3}g_3 \) then the sum again exhibits invariance with respect to a bigger gauge group [19]. The explicit description of this symmetry together with the corresponding free field equation (42)/(46) for the rank-3 tensor gauge field \( A_{\mu \lambda_1 \lambda_2} \) will be given in the forth and fifth sections\(^3\). We shall demonstrate that in four-dimensional space-time the free equation (42)/(46) describes the propagation of helicity-three, \( \lambda = \pm 3 \), and a doublet of helicity-one, \( \lambda = \pm 1, \pm 1 \), massless charged gauge bosons and that there are no propagating negative norm states. The four-dimensional space-time is critical because in five- and higher-dimensional space-time the equation has solutions with negative norm states.

Summarizing our findings we can state that the Lagrangian \( \mathcal{L} \) describes the interacting system of gauge bosons of increasing helicities. The system has Yang-Mills gauge boson on the first level \( (s=0) \), the helicity-two and zero gauge bosons on the second level \( (s=1) \) and the helicity-three and a doublet of helicity-one gauge bosons on the third level \( (s=2) \).

The particle spectrum on higher levels is not yet known completely and to find it out remains a challenging problem. The problem consists in finding out the value of the coupling constant \( g'_{s+1} \) at which the corresponding free field equation for the rank-\( (s+1) \) gauge field is free from propagating negative norm states. As we have found for

\[
g'_{s+1} = \frac{2s}{s+1}g_{s+1}, \quad s = 0, 1, 2, \ldots
\]

\((g_1 = g_{YM})\) there are two solutions which describe the propagating positive norm states of helicities \( \lambda = \pm (s+1) \). But the difficulty in finding out all propagating modes for this value of the coupling constant \( g'_{s+1} \) lies in the fact that the number of field components

\(^2\)It has sixteen components in the four-dimensional space-time.

\(^3\)Its relation to the Schwinger equation for the symmetric rank-3 tensor gauge field is discussed in the fifth section. See also references [10, 19, 20].
dramatically increases with the rank of the tensor gauge field: in the case of rank-2 gauge field we had sixteen components and in the case considered in this article for the rank-3 gauge field we have to analyze an equation with forty components. The presented analysis shows that, most probably, the full system is unitary for all higher-rank non-Abelian tensor gauge fields and only in four-dimensional space-time.

First let us recapitulate the general form of the Lagrangian $\mathcal{L}$ in (1).

## 2 Non-Abelian Tensor Fields

The gauge fields are defined as rank-$(s + 1)$ tensors $[3, 4, 5]$

$$A^a_{\mu_1...\mu_s}(x), \quad s = 0, 1, 2, ...$$

and are totally symmetric with respect to the indices $\lambda_1...\lambda_s$. The index $a$ numerates the generators $L^a$ of a Lie algebra. The extended non-Abelian gauge transformations of the tensor gauge fields are defined by the following equations $[3, 4, 5]$

$$\delta A^a_{\mu} = (\delta^a_b \partial_\mu + gf^{acb} A^c_\mu) \xi^b, \quad (2)$$

$$\delta A^a_{\mu\nu} = (\delta^a_b \partial_\mu + gf^{acb} A^c_\mu) \xi^b_\nu + gf^{abc} A^c_{\mu\nu} \xi^b, \quad (3)$$

where $\xi^a_{\lambda_1...\lambda_s}(x)$ are totally symmetric gauge parameters. These extended gauge transformations generate a closed algebraic structure. The generalized field strengths are defined as $[3, 4, 5]$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu, \quad (3)$$

$$G^a_{\mu\nu,\lambda} = \partial_\mu A^a_{\nu\lambda} - \partial_\nu A^a_{\mu\lambda} + gf^{abc} (A^b_\mu A^c_{\nu\lambda} + A^b_{\mu\lambda} A^c_\nu), \quad (3)$$

and transform homogeneously with respect to the extended gauge transformations (2). The field strength tensors are antisymmetric in their first two indices and are totally symmetric with respect to the rest of the indices.

These field strength tensors allow to construct two series of gauge invariant quadratic forms. The first series is given by the formula $[3, 4, 5]$

$$\mathcal{L}_{s+1} = -\frac{1}{4} G^a_{\mu\nu,\lambda_1...\lambda_s} G^a_{\mu\nu,\lambda_1...\lambda_s} + \ldots$$

$$= -\frac{1}{4} \sum_{i=0}^{2s} a_i^s G^a_{\mu\nu,\lambda_1...\lambda_i} G^a_{\mu\nu,\lambda_{s+1}...\lambda_{s+i}} (\sum P \eta^{\lambda_1...\lambda_{2s}}), \quad (4)$$

where the sum $\sum P$ runs over all nonequal permutations of $i$'s in total $(2s - 1)!!$ terms and the numerical coefficients are $a_i^s = \frac{i!}{(2s - i)!}$.

The second series of gauge invariant quadratic forms is given by the formula $[3, 4, 5]$

$$\mathcal{L}'_{s+1} = \frac{1}{4} G^a_{\mu\rho,\lambda_3...\lambda_{s+1}} G^a_{\mu\rho,\lambda_3...\lambda_{s+1}} + \frac{1}{4} G^a_{\mu\nu,\lambda_3...\lambda_{s+1}} + \frac{1}{4} G^a_{\mu\rho,\lambda_3...\lambda_{s+1}} + \ldots$$

$$= \frac{1}{4} \sum_{i=1}^{2s+1} a_i^{s-1} G^a_{\mu\lambda_1...\lambda_i} G^a_{\mu\lambda_{2s+1}...\lambda_{2s+1}} (\sum P \eta^{\lambda_1...\lambda_{2s+1}}), \quad (5)$$


where the sum $\sum'_{\Lambda}$ runs over all nonequal permutations of $i'$s, with exclusion of the terms which contain $\eta^{i_1,i_2+i_2}$. In order to make all tensor gauge fields dynamical one should add the corresponding kinetic terms. Thus the invariant Lagrangian describing dynamical tensor gauge bosons of all ranks has the form

$$\mathcal{L} = \mathcal{L}_Y + \mathcal{L}_2 + \mathcal{L}'_2 + \mathcal{L}_3 + \mathcal{L}'_3 + \ldots,$$

(6)

where $\mathcal{L}_1 \equiv \mathcal{L}_Y$, and $g_s$ and $g'_{s}$ ($s = 2, 3, \ldots$) are arbitrary coupling constants.

Generally speaking, equations which follow from this Lagrangian may contain propagating negative norm states. Our main task in this article is to analyze particle spectrum of the theory and to prove that there are no negative modes in rank-2 and rank-3 tensor gauge fields in four-dimensional space-time. The problem is that tensor gauge fields have many components, some of them define propagating physical modes of definite helicities, but some of them may correspond to unphysical negative norm states. As we shall see, only for specific values of the coupling constants $g'_{s}$ and only in four-dimensional space-time the particle spectrum becomes completely physical.

Indeed, analyzing a linear sum of gauge invariant forms [3, 4, 5]

$$g_2 \mathcal{L}_2 + g'_2 \mathcal{L}'_2,$$

which are describing the propagation of the rank-2 tensor gauge field $A_{\mu\lambda}^{a}$, we found that if one chooses the coupling constant $g'_2 = g_2$ then the sum exhibits invariance with respect to a bigger gauge group. In that case the free field equation (18)/(19) for the rank-2 tensor gauge field describes the propagation of helicity-two and helicity-zero massless charged tensor gauge bosons and there are no propagating negative norm states. Therefore the gauge invariant Lagrangian for the lower-rank tensor gauge fields has the form [3, 4, 5]:

$$\mathcal{L}_2 + \mathcal{L}'_2 = - \frac{1}{4} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} - \frac{1}{4} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} + \frac{1}{4} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} + \frac{1}{4} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} + \frac{1}{2} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda}.$$

(7)

Our aim here is to extend this analysis to the case of rank-3 tensor gauge field $A_{\mu\lambda_1\lambda_2}$. The explicit form of $\mathcal{L}_3$ and $\mathcal{L}'_3$ can be obtained from our general formulas (4), (5) and (6) by substituting $s = 2$. We shall consider the linear sum

$$g_3 \mathcal{L}_3 + g'_3 \mathcal{L}'_3$$

and demonstrate that for an appropriate choice of the coupling constant $g'_3 = \frac{4}{3} g_3$ the system exhibits invariance with respect to a bigger gauge group [19]. The explicit description of this symmetry together with the corresponding free field equation (42)/(46) for the rank-3 tensor gauge field will be given in the forth and fifth sections. We shall demonstrate that the free equation (42)/(46) describes the propagation of helicity-three and a doublet of helicity-one massless charged gauge bosons and that there are no propagating negative norm states. Thus the Lagrangian for rank-3 non-Abelian gauge field will take the form

$$\mathcal{L}_3 + \frac{4}{3} \mathcal{L}'_3 = - \frac{1}{4} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} - \frac{1}{8} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} - \frac{1}{2} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} - \frac{1}{2} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} + \frac{1}{3} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} + \frac{1}{3} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} + \frac{1}{3} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} + \frac{1}{3} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda} + \frac{1}{3} G_{\mu\nu,\lambda}^{a} G^{a}_{\mu\nu,\lambda}.$$

(8)
Propagating Modes of Rank-2 Gauge Field

Let us first consider the rank-2 gauge field $A^a_{\mu\lambda}$ [3, 4, 5]. There are two invariant forms for the rank-2 tensor gauge field $\mathcal{L}_2$ and $\mathcal{L}_2'$ and we have to consider their linear combination $g_2\mathcal{L}_2 + g'_2\mathcal{L}_2'$. A free field equation of motion is defined by the quadratic part of this invariant, its cubic and quartic parts define interaction. The quadratic part of the $\mathcal{L}_2$ is

$$\mathcal{L}_2^{\text{quadratic}} = \frac{1}{2} A^a_{\alpha\delta} H_{\alpha\delta\gamma\gamma} A^a_{\gamma\gamma},$$

where the kinetic operator $H$ in momentum representation has the form

$$H_{\alpha\gamma\gamma}(k) = (-k^2 \eta_{\alpha\gamma} + k_\alpha k_\gamma)\eta_{\delta\gamma}.$$

It is obviously invariant with respect to the gauge transformation $\delta A^a_{\mu\lambda} = \partial_\mu \xi^a_{\lambda}$, but it is not invariant with respect to the alternative gauge transformations $\delta A^a_{\mu\lambda} = \partial_\lambda \eta^a_\mu$. This can be seen, for example, from the following relations in momentum representation:

$$k_\alpha H_{\alpha\gamma\gamma}(k) = 0, \quad k_\delta H_{\alpha\gamma\gamma}(k) = -(k^2 \eta_{\alpha\gamma} - k_\alpha k_\gamma)k_\delta \neq 0. \quad (9)$$

The quadratic part of the $\mathcal{L}_2'$ is

$$\mathcal{L}_2'^{\text{quadratic}} = \frac{1}{2} A^a_{\alpha\delta} H'_{\alpha\delta\gamma\gamma} A^a_{\gamma\gamma}, \quad (10)$$

where the kinetic operator $H'$ has the form

$$H'_{\alpha\gamma\gamma}(k) = \frac{1}{2} (\eta_{\alpha\gamma} \eta_{\delta\gamma} + \eta_{\alpha\delta} \eta_{\gamma\gamma}) k^2 - \frac{1}{2} (\eta_{\alpha\gamma} k_\delta k_\gamma + \eta_{\delta\gamma} k_\alpha k_\gamma + \eta_{\alpha\delta} k_\gamma k_\delta + \eta_{\gamma\gamma} k_\alpha k_\delta - 2 \eta_{\alpha\gamma} k_\delta k_\gamma).$$

It is also invariant with respect to the gauge transformation $\delta A^a_{\mu\lambda} = \partial_\mu \xi^a_{\lambda}$, but it is not invariant with respect to the gauge transformations $\delta A^a_{\mu\lambda} = \partial_\lambda \eta^a_\mu$, as one can see from analogous relations

$$k_\alpha H'_{\alpha\gamma\gamma}(k) = 0, \quad k_\delta H'_{\alpha\gamma\gamma}(k) = (k^2 \eta_{\alpha\gamma} - k_\alpha k_\gamma)k_\delta \neq 0. \quad (11)$$

As it is obvious from (9) and (11), the sum $\mathcal{L}_2 + \mathcal{L}_2'$, when $g'_2 = g_2$, becomes invariant with respect to the alternative gauge transformations $\delta A^a_{\mu\lambda} = \partial_\lambda \eta^a_\mu$ and the kinetic operator now has both of the symmetries:

$$\delta A^a_{\mu\lambda} = \partial_\mu \xi^a_{\lambda} + \partial_\lambda \eta^a_\mu, \quad (12)$$

because

$$k_\alpha (H_{\alpha\gamma\gamma} + H'_{\alpha\gamma\gamma}) = 0, \quad k_\delta (H_{\alpha\gamma\gamma} + H'_{\alpha\gamma\gamma}) = 0. \quad (13)$$

Thus our kinetic operator is a sum

$$\mathcal{L}_2 + \mathcal{L}_2' \mid_{\text{quadratic}} = \frac{1}{2} A^a_{\alpha\delta} (H_{\alpha\gamma\gamma} + H'_{\alpha\gamma\gamma}) A^a_{\gamma\gamma} = \frac{1}{2} A^a_{\alpha\delta} H_{\alpha\gamma\gamma} A^a_{\gamma\gamma}, \quad (14)$$

\[4\] Longitudinal pieces in (9) and (11) cancel each other and the kinetic operator is fully transversal.
where
\[ H_{\alpha \dot{\alpha} \gamma \dot{\gamma}}(k) = (-\eta_{\alpha \gamma} \eta_{\dot{\alpha} \dot{\gamma}} + \frac{1}{2} \eta_{\alpha \dot{\gamma}} \eta_{\dot{\alpha} \gamma} + \frac{1}{2} \eta_{\alpha \dot{\alpha}} \eta_{\gamma \dot{\gamma}}) k^2 + \eta_{\alpha \gamma} k_{\dot{\alpha}} k_{\dot{\gamma}} + \eta_{\dot{\alpha} \dot{\gamma}} k_{\alpha} k_{\gamma} \]

\[ -\frac{1}{2} (\eta_{\alpha \gamma} k_{\dot{\alpha}} k_{\dot{\gamma}} + \eta_{\dot{\alpha} \dot{\gamma}} k_{\alpha} k_{\gamma} + \eta_{\alpha \dot{\alpha}} k_{\gamma} k_{\dot{\gamma}} + \eta_{\dot{\alpha} \dot{\gamma}} k_{\alpha} k_{\dot{\gamma}}). \]  

(15)

In terms of field strength tensor the quadratic part is

\[ L_2 + L_2' |_{\text{quadratic}} = - \frac{1}{4} F_{\mu \nu, \lambda} F^{a}_{\mu \nu, \lambda} + \frac{1}{4} F_{\mu \nu, \lambda} F^{a}_{\mu \lambda, \nu} + \frac{1}{4} F^{a}_{\mu \nu, \nu} F^{a}_{\mu \lambda, \lambda}, \]

(16)

where

\[ F_{\mu \nu, \lambda} = \partial_\mu A_{\nu \lambda} - \partial_\nu A_{\mu \lambda}, \]

(17)

and the equation of motion takes the form

\[ \partial^\mu F_{\mu \nu, \lambda} - \frac{1}{2} (\partial^\mu F^{a}_{\mu \lambda, \nu} + \partial^\mu F^{a}_{\lambda \nu, \mu} + \partial_\lambda F^{a}_{\mu \nu, \mu} + \eta_{\nu \lambda} \partial^\mu F^{a}_{\mu \rho, \rho}) = 0. \]

(18)

In terms of tensor gauge field the free equation of motion (18) is

\[ \partial^2 (A_{\nu \lambda} - \frac{1}{2} A^{a}_{\lambda \nu}) - \partial_\nu \partial_\mu (A^{a}_{\mu \lambda} - \frac{1}{2} A^{a}_{\lambda \mu}) - \partial_\lambda \partial_\mu (A^{a}_{\mu \nu} - \frac{1}{2} A^{a}_{\nu \mu}) + \partial_\nu \partial_\lambda (A^{a}_{\mu \nu} - \frac{1}{2} A^{a}_{\nu \mu}) + \frac{1}{2} \eta_{\nu \lambda} (\partial_\mu \partial_\rho A^{a}_{\mu \rho} - \partial^2 A^{a}_{\mu \mu}) = 0. \]

(19)

and it describes the propagation of massless charged gauge bosons of helicity two and zero. Indeed, this can be seen by decomposition of the rank-2 gauge field into symmetric and antisymmetric parts. For the symmetric tensor gauge fields \( A_{\nu \lambda} = A^{a}_{\lambda \nu} \) our equation reduces to the Einstein and Fierz-Pauli equation

\[ \partial^2 A_{\nu \lambda} - \partial_\nu \partial_\mu A_{\mu \lambda} - \partial_\lambda \partial_\mu A_{\mu \nu} + \partial_\nu \partial_\lambda A_{\mu \mu} + \eta_{\nu \lambda} (\partial_\mu \partial_\rho A^{a}_{\mu \rho} - \partial^2 A^{a}_{\mu \mu}) = 0, \]

which describes the propagation of massless gauge boson of helicity two. For the antisymmetric fields it reduces to the Kalb-Ramond equation

\[ \partial^2 A_{\nu \lambda} - \partial_\nu \partial_\mu A_{\mu \lambda} + \partial_\lambda \partial_\mu A_{\mu \nu} = 0 \]

and describes the propagation of helicity-zero state.

A more direct way to solve the free equation of motion (18)/(19) is to consider it in the momentum representation [22]:

\[ H_{\alpha \dot{\alpha} \gamma \dot{\gamma}}(k) A^a_{\gamma \dot{\gamma}} = 0. \]

(20)

The vector space of independent solutions \( A_{\gamma \dot{\gamma}} \) crucially depends on the rank of the matrix \( H_{\alpha \dot{\alpha} \gamma \dot{\gamma}}(k) \). If the matrix operator \( H \) has dimension \( d \times d \) and its rank is \( rank H = r \), then the vector space of solutions has the dimension

\[ \mathcal{N} = d - r. \]

Because the matrix operator \( H_{\alpha \dot{\alpha} \gamma \dot{\gamma}}(k) \) explicitly depends on the momentum \( k_\mu \), its \( rank H = r \) also depends on momenta and therefore the number of independent solutions \( \mathcal{N} \) depends on momenta

\[ \mathcal{N}(k) = d - r(k). \]  

(21)
Analyzing the \( \text{rank}H \) of the matrix operator \( H \) one can observe that it depends on the value of momentum square \( k^2_\mu \). When \( k^2_\mu \neq 0 \) - off mass-shell momenta - the vector space consists of pure gauge fields. When \( k^2_\mu = 0 \) - on mass-shell momenta - the vector space consists of pure gauge fields and propagating modes. Therefore the number of propagating modes can be calculated from the following relation:

\[
\sharp \text{ of propagating modes} = \mathcal{N}(k)|_{k^2=0} - \mathcal{N}(k)|_{k^2\neq0} = \text{rank}H|_{k^2\neq0} - \text{rank}H|_{k^2=0}. \tag{22}
\]

Our field equation (20) for the tensor gauge field \( A_{\mu\lambda} \) is defined by the matrix operator (15), which in the four-dimensional space-time is a \( 16 \times 16 \) matrix\(^5\). In the reference frame, where \( k^\gamma = (\omega, 0, 0, k) \), it has a particularly simple form. If \( \omega^2 - k^2 \neq 0 \), the rank of the 16-dimensional matrix \( H_{a\alpha\gamma\gamma}(k) \) is equal to \( \text{rank}H|_{\omega^2-k^2\neq0} = 9 \) and the number of linearly independent solutions is \( 16 - 9 = 7 \). These seven solutions are

\[
\gamma = \begin{pmatrix}
-\omega^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & k^2 \\
0 & 0 & k & 0
\end{pmatrix}, \quad \begin{pmatrix}
\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
k & 0 & 0 & 0 \\
0 & k & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & \omega & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & \omega & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \tag{23}
\]

and they are pure gauge fields

\[
e_{\gamma\gamma} = k_\gamma \xi + k_\gamma \eta. \tag{24}\]

When \( \omega^2 - k^2 = 0 \), then the rank of the matrix \( H_{a\alpha\gamma\gamma}(k) \) drops and is equal to \( \text{rank}H|_{\omega^2-k^2=0} = 6 \). This leaves us with \( 16 - 6 = 10 \) solutions. These are 7 solutions, the pure gauge potentials (23), (24), and three new solutions representing the propagating modes:

\[
e^{(1)}_{\gamma\gamma} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad e^{(2)}_{\gamma\gamma} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad e^{A}_{\gamma\gamma} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}. \tag{25}
\]

Thus the general solution of the equation on the mass-shell is

\[
e_{\gamma\gamma} = \xi_\gamma k_\gamma + \eta_\gamma k_\gamma + c_1 e^{(1)}_{\gamma\gamma} + c_2 e^{(2)}_{\gamma\gamma} + c_3 e^{A}_{\gamma\gamma}, \tag{26}\]

where \( c_1, c_2, c_3 \) are arbitrary constants. We see that the number of the propagating modes is three:

\[
\text{rank}H|_{\omega^2-k^2\neq0} - \text{rank}H|_{\omega^2-k^2=0} = 9 - 6 = 3.
\]

These are the propagating modes of helicity-two and helicity-zero \( \lambda = \pm 2, 0 \) charged gauge bosons [3, 4, 5].

The above consideration brings the final form of the gauge invariant Lagrangian for the rank-2 tensor gauge field to the form (7) with its free equation of motion (18)/(19). And, as we have seen, has a well defined physical spectrum (26).

\(^5\)The multi-index \( A \equiv (\mu, \lambda) \) takes sixteen values.
4 Rank-3 Tensor Gauge Field

Let us turn now to the rank-3 gauge field. There are two invariant forms $L_3$ and $L'_3$ for the rank-3 tensor gauge field $A^a_{\mu\nu\lambda}$ and we have to consider their linear combination $g_3 L_3 + g'_3 L'_3$. The Lagrangian $L_3$ has the form (4)\(^6\) [3, 4, 5]

$$L_3 = -\frac{1}{4} G^a_{\mu\nu,\lambda\rho} G^a_{\mu\nu,\lambda\rho} - \frac{1}{8} G^a_{\mu\nu,\lambda\lambda} G^a_{\mu\nu,\rho\rho} - \frac{1}{2} G^a_{\mu\nu,\lambda\lambda} G^a_{\mu\nu,\rho\rho} - \frac{1}{8} G^a_{\mu\nu} G^a_{\mu\nu,\lambda\lambda\rho\rho},$$

(27)

where higher rank field strength tensors are:

$$G^a_{\mu\nu,\lambda\rho\sigma} = \partial_\mu A^a_{\nu\lambda\rho\sigma} - \partial_\nu A^a_{\mu\lambda\rho\sigma} + g f^{abc} \{ A^b_{\mu} A^c_{\nu\lambda\rho\sigma} + A^b_{\mu\lambda} A^c_{\nu\rho\sigma} + A^b_{\mu\rho\sigma} A^c_{\nu\lambda\sigma} + A^b_{\mu\lambda\rho\sigma} A^c_{\nu} \} +$$

and

$$G^a_{\mu\nu,\lambda\rho\sigma\delta} = \partial_\mu A^a_{\nu\lambda\rho\sigma\delta} - \partial_\nu A^a_{\mu\lambda\rho\sigma\delta} + g f^{abc} \{ A^b_{\mu} A^c_{\nu\lambda\rho\sigma\delta} + \sum_{\lambda+\rho, \sigma, \delta} A^b_{\mu\lambda} A^c_{\nu\rho\sigma\delta} +$$

$$+ \sum_{\lambda, \rho+\sigma, \delta} A^b_{\mu\lambda\rho} A^c_{\nu\sigma\delta} + \sum_{\lambda, \rho, \sigma+\delta} A^b_{\mu\lambda\rho\sigma} A^c_{\nu\delta} + A^b_{\mu\lambda\rho\sigma\delta} A^c_{\nu} \}.$$  

The terms in parentheses are symmetric over $\lambda \rho \sigma$ and $\lambda \rho \sigma \delta$ respectively. The Lagrangian $L_3$ is invariant with respect to the extended gauge transformations (2) of the low-rank gauge fields $A_\mu, A_{\mu\nu}, A_{\mu\nu\lambda}$ together with the fourth- and fifth-rank gauge fields

$$\delta_\xi A_{\mu\nu\lambda} = \partial_\xi A_{\mu\nu\lambda} = -\partial_\nu A_{\mu\lambda\rho\sigma} + g f^{abc} \{ A^b_{\mu} A^c_{\nu\lambda\rho\sigma} + A^b_{\mu\lambda} A^c_{\nu\rho\sigma} + A^b_{\mu\rho\sigma} A^c_{\nu\lambda\sigma} + A^b_{\mu\lambda\rho\sigma} A^c_{\nu} \} +$$

$$+ g f^{abc} \{ A^b_{\mu} A^c_{\nu\lambda\rho\sigma\delta} + \sum_{\lambda+\rho, \sigma, \delta} A^b_{\mu\lambda} A^c_{\nu\rho\sigma\delta} + \sum_{\lambda, \rho+\sigma, \delta} A^b_{\mu\lambda\rho} A^c_{\nu\sigma\delta} + \sum_{\lambda, \rho, \sigma+\delta} A^b_{\mu\lambda\rho\sigma} A^c_{\nu\delta} + A^b_{\mu\lambda\rho\sigma\delta} A^c_{\nu} \}.$$  

The gauge parameters $\xi_{\nu\lambda\rho}$ and $\xi_{\nu\lambda\rho\sigma}$ are totally symmetric tensors. The second Lagrangian $L'_3$ has the form (5)\(^7\) [3, 4, 5]

$$L'_3 = \frac{1}{4} G^a_{\mu\nu,\lambda\rho} G^a_{\mu\nu,\lambda\rho} + \frac{1}{4} G^a_{\mu\nu,\lambda\lambda} G^a_{\mu\nu,\rho\rho} + \frac{1}{4} G^a_{\mu\nu,\lambda\lambda} G^a_{\mu\nu,\rho\rho} +$$

$$+ \frac{1}{4} G^a_{\mu\nu,\lambda\rho\rho} + \frac{1}{2} G^a_{\mu\nu,\lambda\rho\rho} + \frac{1}{4} G^a_{\mu\nu,\lambda\rho\rho} + \frac{1}{4} G^a_{\mu\nu,\lambda\rho\rho}.$$  

(28)

We wish to know if there exists a special value of the constant $g'_3$ at which the system will have higher symmetry, as it happens in the case of the rank-2 gauge field. We shall see that this indeed takes place.

A free field equation of motion is defined by the quadratic part of invariant $g_3 L_3 + g'_3 L'_3$, the interaction - by cubic and quartic. The quadratic part of the Lagrangian $L_3$ comes from the terms

$$-\frac{1}{4} G^a_{\mu\nu,\lambda\rho} G^a_{\mu\nu,\lambda\rho} - \frac{1}{8} G^a_{\mu\nu,\lambda\lambda} G^a_{\mu\nu,\rho\rho}$$

(29)

\(^6\)In (4) one should take $s=2$.

\(^7\)In (5) one should take $s=2$.  

9
and has the form

\[ \mathcal{L}_{3}^{\text{quadratic}} = \frac{1}{2} A_{\alpha\alpha'}^a \partial_{\gamma}^a H_{\alpha\alpha' \gamma'} H_{\alpha\alpha' \gamma'} \]

where the kinetic operator \( H \) in the momentum representation is

\[ H_{\alpha\alpha' \gamma' \gamma'} (k) = -\frac{1}{2} H_{\alpha\gamma} (\eta_{\alpha' \gamma'} \eta_{\alpha' \gamma'} + \eta_{\gamma' \gamma'} \eta_{\gamma' \gamma'} + \eta_{\alpha' \alpha''} \eta_{\gamma' \gamma'}) \]

and \( H_{\alpha\gamma} = k^2 \eta_{\alpha\gamma} - k_{\alpha} k_{\gamma} \). It is invariant with respect to the gauge transformation \( \delta A_{a\mu} = \partial_{\mu} \xi_{a\mu} \), which can be seen from the relation \( k_{\alpha} H_{\alpha\alpha' \gamma' \gamma'} (k) = 0 \). But it is not invariant with respect to the alternative gauge transformations \( \delta A_{a\mu} = \partial_{\mu} \zeta_{a\mu} + \partial_{\lambda} \zeta_{a\lambda} \), where the gauge parameter \( \zeta_{a\mu} \) is a totally symmetric tensor. This can be seen from the following relation in momentum representation

\[ k_{\alpha} H_{\alpha\alpha' \gamma' \gamma'} (k) = -\frac{1}{2} H_{\alpha\gamma} (k_{\gamma'} \eta_{\alpha' \gamma'} + k_{\gamma' \gamma'} \eta_{\alpha' \gamma'} + k_{\alpha''} \eta_{\gamma' \gamma'}) \neq 0. \]

The quadratic part of the Lagrangian \( \mathcal{L}_{3} \) comes from the terms

\[ \frac{1}{4} G_{\mu\nu,\lambda\rho}^a G_{\alpha\lambda,\nu\rho} + \frac{1}{4} G_{\mu\rho,\nu\lambda}^a G_{\mu\nu,\rho\lambda} + \frac{1}{4} G_{\mu\rho,\nu\lambda}^a G_{\mu\rho,\nu\lambda} \]

and has the form

\[ \mathcal{L}_{3}^{\prime \text{quadratic}} = \frac{1}{2} A_{\alpha\alpha' \gamma'} H_{\alpha\alpha' \gamma'} H_{\alpha\alpha' \gamma'} \]

The kinetic operator \( H' \) is (see Appendix A for derivation)

\[ H_{\alpha\alpha' \gamma' \gamma'} (k) = \frac{1}{8} \left( + k_{\alpha} k_{\gamma'} (\eta_{\alpha\gamma'} \eta_{\alpha' \gamma'} + \eta_{\gamma' \gamma'} \eta_{\alpha' \gamma'} + \eta_{\alpha' \alpha''} \eta_{\gamma' \gamma'}) 
+ k_{\gamma'} k_{\alpha'} (\eta_{\gamma\gamma'} \eta_{\alpha' \gamma'} + \eta_{\alpha' \gamma'} \eta_{\gamma' \gamma'} + \eta_{\alpha' \alpha''} \eta_{\gamma' \gamma'}) 
+ k_{\alpha'} k_{\gamma'} (\eta_{\alpha\alpha'} \eta_{\gamma' \gamma'} + \eta_{\alpha\gamma} \eta_{\alpha' \gamma'} + \eta_{\gamma' \gamma'} \eta_{\alpha' \gamma'}) 
+ k_{\gamma} k_{\gamma'} (\eta_{\alpha\alpha'} \eta_{\gamma' \gamma'} + \eta_{\alpha\gamma} \eta_{\alpha' \gamma'} + \eta_{\gamma' \gamma'} \eta_{\alpha' \gamma'}) \right) 
- \frac{1}{8} \left( + k_{\gamma} k_{\alpha} (\eta_{\alpha\gamma'} \eta_{\alpha' \gamma'} + \eta_{\alpha\gamma} \eta_{\alpha' \gamma'} + \eta_{\alpha\alpha'} \eta_{\gamma' \gamma'}) 
+ k_{\gamma} k_{\alpha'} (\eta_{\gamma\gamma'} \eta_{\alpha' \gamma'} + \eta_{\alpha' \gamma'} \eta_{\gamma' \gamma'} + \eta_{\alpha' \alpha''} \eta_{\gamma' \gamma'}) 
+ k_{\gamma} k_{\gamma'} (\eta_{\alpha\alpha'} \eta_{\gamma' \gamma'} + \eta_{\alpha\gamma} \eta_{\alpha' \gamma'} + \eta_{\gamma' \gamma'} \eta_{\alpha' \gamma'}) \right) 
+ \frac{1}{4} \left( + \eta_{\alpha\gamma'} (k_{\gamma'} k_{\alpha'} \eta_{\alpha' \gamma'} + k_{\gamma'} k_{\alpha'} \eta_{\gamma' \gamma'} + k_{\alpha'} k_{\gamma} \eta_{\alpha' \gamma'} 
+ k_{\alpha'} k_{\gamma'} \eta_{\alpha' \gamma'} + k_{\gamma'} k_{\alpha} \eta_{\gamma' \gamma'} + k_{\gamma'} k_{\gamma} \eta_{\alpha' \gamma'}) \right). \]

It is again invariant with respect to the transformation \( \delta A_{a\mu\lambda} = \partial_{\mu} \xi_{a\mu\lambda} \) which is translated into the relation \( k_{\alpha} H_{\alpha\alpha' \gamma' \gamma'} (k) = 0 \), but it is not invariant with respect to the transformation \( \delta A_{a\mu\lambda} = \partial_{\mu} \zeta_{a\mu\lambda} + \partial_{\lambda} \zeta_{a\mu\mu} \), as one can see from the relation (see also Appendix A for derivation)

\[ k_{\alpha} H_{\alpha\alpha' \gamma' \gamma'} (k) = \frac{1}{8} \left( + H_{\alpha\alpha'} (k_{\gamma'} \eta_{\gamma' \gamma'} + k_{\gamma'} \eta_{\gamma' \gamma'}) 
+ H_{\alpha\gamma'} (k_{\gamma' \gamma'} + k_{\gamma'} \eta_{\gamma' \gamma'}) \right) \]
We have to see now whether the total longitudinal part of the kinetic operator
\[
k_\alpha'(g_3 H'_{\alpha\alpha''\alpha''} + g_3 H'_{\alpha\alpha''\alpha''})
\]
can be made equal to zero by an appropriate choice of the coupling constant $g_3'$. For that let us compare the expressions (31) and (34) for longitudinal terms. As one can see, only the last term in (34) $H_{\alpha\gamma}(k_{\gamma}\eta_{\alpha\gamma''} + k_{\gamma}k_{\gamma''}\eta_{\alpha\gamma''} - 3\eta_{\alpha\gamma''}k_{\alpha''}k_{\gamma''})$ and the whole term (31) can cancel each other if we choose $g_3' = 2g_3$, but this will leave the terms of the first case untouched, thus $H_\alpha'(g_3 H_{\alpha\alpha''\alpha''} + g_3 H'_{\alpha\alpha''\alpha''}) \neq 0$ for all values of $g_3'$. This situation differs from the case of the rank-2 gauge field. In the last case we were able to choose coupling constant $g_2'$ so that longitudinal pieces (9) and (11) cancel each other.

In order to understand the reason, why in the case of the rank-3 gauge field it is impossible to fully cancel longitudinal pieces, we have to remind a beautiful result obtained long ago by Schwinger [10]. It has been proven by Schwinger [10, 19, 20] that it is impossible to derive free field equation for the totally symmetric rank-3 tensor which is invariant with respect to the gauge group of transformations $\delta A_{\mu\nu\lambda} = \partial_\alpha \xi^\alpha_{\mu\nu\lambda} + \partial_\beta \xi^\beta_{\mu\nu\lambda} + \partial_\gamma \xi^\gamma_{\mu\nu\lambda}$ without imposing some restriction on the gauge parameters $\xi_{\mu\nu\lambda}$. As Schwing demonstrated, the gauge parameter should be traceless: $\xi_{\mu\nu\lambda} = 0$. We shall see that similar phenomena take place also in our case, that is, the gauge parameter $\xi^\alpha_{\mu\nu\lambda}$ should fulfill the restriction (36).

What we would like to prove is that our equation has enhanced invariance with respect to the gauge group of transformations
\[
\tilde{\delta} A^a_{\mu\nu\lambda} = \partial_\rho \xi^a_{\rho\mu\nu\lambda} + \partial_\lambda \xi^a_{\rho\mu\nu\lambda},
\]
only if the gauge parameter $\xi^a_{\mu\nu\lambda}$ fulfills the following restriction:
\[
\partial_\rho \xi^a_{\rho\mu\nu\lambda} - \partial_\lambda \xi^a_{\rho\mu\nu\lambda} = 0.
\]
This takes place when we choose $g_3' = \frac{4}{3}g_3$. Indeed, let us consider the equation of motion. From $\mathcal{L}_{3}^{\text{quadratic}}$ we have:
\[
H_{\alpha\alpha'\alpha''\gamma\gamma''\gamma''} = \partial^2 A_{\alpha\alpha'\alpha''} - \partial_\gamma \partial_\rho A_{\rho\alpha'\alpha''} + \frac{1}{2} \eta_{\alpha\alpha'\alpha''} (\partial^2 A_{\alpha\rho\rho} - \partial_\gamma \partial_\rho A_{\rho\gamma\gamma''}),
\]
and from $\mathcal{L}_{3}^{\text{quadratic}}$
\[
H'_{\alpha\alpha'\alpha''\gamma\gamma''\gamma''} = -\frac{1}{8} \{ \partial^2 (A_{\alpha'\alpha''} + A_{\alpha'\alpha''} + A_{\alpha''\alpha'}) - 
- \partial_\alpha \partial_\rho (A_{\alpha'\alpha''} + A_{\alpha'\alpha''} + A_{\alpha''\alpha'})
- \partial_\alpha \partial_\rho (A_{\alpha'\alpha''} + A_{\alpha'\alpha''} + A_{\alpha''\alpha'})
- \partial_\alpha \partial_\rho (A_{\alpha'\alpha''} + A_{\alpha'\alpha''} + A_{\alpha''\alpha'})
- \partial_\alpha \partial_\rho (A_{\alpha'\alpha''} + A_{\alpha'\alpha''} + A_{\alpha''\alpha'})\} - 
\]

\[ -\partial_a \partial_a \left( A_{\alpha' \alpha'' \rho} + A_{\alpha' \rho} + A_{\rho \alpha''} \right) + \partial_a \partial_a \left( A_{\alpha' \alpha'' \rho} + A_{\rho \alpha'} + A_{\rho \alpha''} \right) + 2\partial_a \partial_a \left( A_{\alpha' \rho} \right) - \\
\frac{1}{8} \left\{ \eta_{\alpha' \alpha''} \left[ \partial^2 (A_{\alpha' \alpha'' \rho} + A_{\rho \alpha''}) - \partial_{\alpha'} \partial_{\alpha''} A_{\rho \alpha''} - \partial_{\alpha} \partial_{\rho} \left( A_{\rho \alpha' \alpha''} + A_{\rho \alpha''} \right) \right] \right. \\
+ \eta_{\alpha' \alpha''} \left[ \partial^2 (A_{\alpha' \rho} + A_{\rho \alpha'}) - \partial_{\alpha'} \partial_{\rho} A_{\rho \alpha'} - \partial_{\alpha} \partial_{\rho} \left( A_{\rho \alpha'} + A_{\rho \alpha''} \right) \right] \right. \\
+ \left. \eta_{\alpha' \alpha''} \left[ \partial^2 (A_{\rho \alpha'} + A_{\rho \alpha''}) - \partial_{\rho} \partial_{\alpha'} A_{\rho \alpha'} - \partial_{\rho} \partial_{\alpha''} A_{\rho \alpha''} \right] \right\}. \tag{38} \]

Summing these two pieces together we shall get the following free field equation of motion for the rank-3 tensor gauge field:

\[
\left( H_{\alpha' \alpha''} + \frac{4}{3} H_{\alpha' \alpha''} \right) \delta A_{\alpha' \alpha''} = \partial^2 A_{\alpha' \alpha''} - \frac{c}{4} A_{\alpha' \alpha''} - \frac{c}{4} A_{\alpha' \alpha''} - \\
- \frac{c}{4} \partial_{\alpha'} \partial_{\alpha''} A_{\rho \alpha''} - \frac{c}{4} \partial_{\alpha'} \partial_{\rho} \left( A_{\rho \alpha''} + A_{\rho \alpha''} \right) - \\
- \frac{c}{4} \partial_{\alpha'} \partial_{\rho} \left( A_{\rho \alpha'} + A_{\rho \alpha'} \right) + \frac{c}{8} \partial_{\alpha} \partial_{\rho} \left( A_{\rho \alpha' \alpha''} + A_{\rho \alpha''} \right) + \\
+ \frac{c}{8} \partial_{\alpha} \partial_{\rho} \left( A_{\rho \rho \alpha' \alpha''} + A_{\rho \rho \alpha''} \right) - \frac{c}{4} \partial_{\alpha} \partial_{\rho} A_{\rho \alpha' \alpha''} - \\
- \frac{c}{8} \partial_{\alpha} \partial_{\rho} A_{\rho \rho \alpha' \alpha''} - \partial_{\rho} \partial_{\alpha} A_{\rho \rho \alpha' \alpha''} + \\
+ \frac{1}{2} \partial_{\alpha} \partial_{\rho} A_{\rho \rho \alpha' \alpha''} - \partial_{\alpha} \partial_{\rho} A_{\rho \rho \alpha' \alpha''} = 0, \\
\]  
where \( c = g_3 \). Performing the gauge transformation (35) of the gauge field one can see that the terms which originate from differential operators \( \partial^2 \), \( \partial_{\alpha} \partial_{\rho} \), \( \partial_{\alpha} \partial_{\rho} \), and \( \partial_{\rho} \partial_{\alpha} \rho \) in the above equation cancel each other if we choose \( g_3 = \frac{4}{3} g_3 \). The rest of the terms have the following form:

\[
\left( H_{\alpha' \alpha''} + \frac{4}{3} H_{\alpha' \alpha''} \right) \delta A_{\alpha' \alpha''} = \\
+ \frac{1}{3} \partial_{\alpha} \partial_{\alpha'} \partial_{\rho} A_{\rho \alpha''} + \frac{1}{3} \partial_{\alpha} \partial_{\alpha''} \partial_{\rho} A_{\rho \alpha'} - \frac{4}{3} \partial_{\alpha} \partial_{\alpha'} \partial_{\rho} A_{\rho \alpha''} + \frac{2}{3} \partial_{\alpha} \partial_{\rho} \partial_{\alpha} A_{\rho \alpha''} - \\
- \frac{1}{6} \eta_{\alpha' \alpha''} \left[ \partial_{\rho} \partial_{\rho} A_{\rho \alpha''} - 4 \partial_{\alpha'} \partial_{\rho} A_{\rho \alpha''} + 2 \partial_{\alpha'} \partial_{\rho} A_{\rho \alpha''} \right] \right. \\
- \frac{1}{6} \eta_{\alpha' \alpha''} \left[ \partial_{\rho} \partial_{\rho} A_{\rho \alpha''} - 4 \partial_{\alpha'} \partial_{\rho} A_{\rho \alpha''} + 2 \partial_{\alpha'} \partial_{\rho} A_{\rho \alpha''} \right] + \\
+ \frac{1}{3} \eta_{\alpha' \alpha''} \left( \partial_{\rho} \partial_{\rho} A_{\rho \alpha''} - \partial_{\alpha} \partial_{\rho} A_{\rho \alpha''} \right) \right). \tag{39} \]

and can be rewritten in the form which makes the desired invariance explicit:

\[
\left( H_{\alpha' \alpha''} + \frac{4}{3} H_{\alpha' \alpha''} \right) \delta A_{\alpha' \alpha''} = \\
+ \frac{1}{3} \partial_{\alpha} \partial_{\alpha'} \left( \partial_{\rho} A_{\rho \alpha''} - \partial_{\alpha} A_{\rho \alpha''} \right) + \frac{1}{3} \partial_{\alpha} \partial_{\alpha'} \left( \partial_{\rho} A_{\rho \alpha''} - \partial_{\alpha} A_{\rho \alpha''} \right) - \\
- \frac{1}{3} \eta_{\alpha' \alpha''} \left[ \partial_{\rho} \partial_{\rho} A_{\rho \alpha''} - \partial_{\alpha} \partial_{\rho} A_{\rho \alpha''} + 2 \partial_{\alpha'} \partial_{\rho} A_{\rho \alpha''} \right] + \\
- \frac{1}{3} \eta_{\alpha' \alpha''} \left[ \partial_{\rho} \partial_{\rho} A_{\rho \alpha''} - \partial_{\alpha} \partial_{\rho} A_{\rho \alpha''} + 2 \partial_{\alpha'} \partial_{\rho} A_{\rho \alpha''} \right] + \\
+ \frac{1}{3} \eta_{\alpha' \alpha''} \left[ \partial_{\rho} A_{\rho \alpha''} - \partial_{\alpha} A_{\rho \alpha''} + \partial_{\alpha} \partial_{\rho} A_{\rho \alpha''} - \partial_{\rho} A_{\rho \alpha''} \right]. \tag{40} \]
From that we see that if the gauge parameter satisfies the restriction (36) the equation is indeed invariant with respect to a larger group of gauge transformations \[ \delta A_{\mu\nu\lambda}^a = \partial_\mu \zeta^a_{\nu\lambda} + \partial_\nu \zeta^a_{\mu\lambda} + \partial_\lambda \zeta^a_{\mu\nu}, \]

because

\[
(H_{\alpha\alpha'}a'' \gamma'' \gamma'' + \frac{4}{3} H_{\alpha\alpha'}a'' \gamma'' \gamma'' \gamma''(\kappa)) \delta A_{\gamma\gamma''} = H_{\alpha\alpha'}a'' \gamma'' \gamma''(k) \delta A_{\gamma\gamma''} = 0. \tag{41}
\]

The final form of the equation is

\[
\partial^2 (A_{\alpha\alpha'}^a - \frac{1}{3} A_{\alpha\alpha' a'' a''}^a - \frac{1}{3} A_{a'' a'' a''}^a) - \partial_\alpha \partial_\rho (A_{\alpha\rho a''}^a - \frac{1}{3} A_{\alpha\rho a'' a''}^a - \frac{1}{3} A_{a'' a'' \rho a''}^a) - \frac{1}{3} \partial_\alpha \partial_\rho (A_{\alpha \rho a''}^a + A_{\alpha a'' \rho}^a - A_{\rho a'' a''}^a) + \frac{1}{6} \partial_\alpha \partial_\rho (A_{\alpha a'' \rho}^a + A_{a'' \rho a''}^a + A_{\rho a'' a''}^a) + \frac{1}{6} \partial_\alpha \partial_\rho (A_{\alpha a'' \rho}^a + A_{\alpha a'' \rho}^a) + \frac{1}{6} \partial_\alpha \partial_\rho (A_{\alpha a'' \rho}^a + A_{\alpha a'' \rho}^a) + \frac{1}{6} \partial_\alpha \partial_\rho (A_{\alpha a'' \rho}^a + A_{\alpha a'' \rho}^a) + \frac{1}{6} \partial_\alpha \partial_\rho (A_{\alpha a'' \rho}^a + A_{\alpha a'' \rho}^a) + \frac{1}{6} \partial_\alpha \partial_\rho (A_{\alpha a'' \rho}^a + A_{\alpha a'' \rho}^a) + \frac{1}{6} \partial_\alpha \partial_\rho (A_{\alpha a'' \rho}^a + A_{\alpha a'' \rho}^a) = 0.
\tag{42}
\]

and, it is invariant with respect to the group of gauge transformations

\[
\delta A_{\mu\nu\lambda}^a = \partial_\mu \zeta_{\nu\lambda}^a, \quad \tilde{\delta} A_{\mu\nu\lambda}^a = \partial_\nu \zeta_{\mu\lambda}^a + \partial_\lambda \zeta_{\mu\nu}^a, \quad \partial_\mu \zeta_{\nu\lambda}^a - \partial_\nu \zeta_{\mu\lambda}^a = 0. \tag{43}
\]

The above invariance of the equation (42) with respect to the transformations (43) can be checked now directly without referring to the previous analysis.

Let us now estimate, how many independent gauge parameters are at our disposal. Because there are no restrictions on the symmetric gauge parameter \( \zeta^a_{\mu\nu} \), we have ten independent gauge parameters in the four-dimensional space-time. To estimate the amount of independent gauge parameters in \( \zeta^a_{\mu\nu} \) one should solve the restriction (36)

\[
\omega \zeta_{03} + \kappa \zeta_{33} + \kappa(\zeta_{00} - \zeta_{11} - \zeta_{22} - \zeta_{33}) = 0, \\
\omega \zeta_{01} + \kappa \zeta_{31} = 0, \\
\omega \zeta_{02} + \kappa \zeta_{32} = 0, \\
\omega \zeta_{00} + \kappa \zeta_{30} - \omega(\zeta_{00} - \zeta_{11} - \zeta_{22} - \zeta_{33}) = 0,
\]

where \( k^\mu = (\omega, 0, 0, \kappa) \), therefore

\[
\zeta_{00} = \zeta_{11} + \zeta_{22} - \frac{\omega}{\kappa} \zeta_{03}, \quad \zeta_{31} = -\frac{\omega}{\kappa} \zeta_{01}, \\
\zeta_{33} = -\zeta_{11} - \zeta_{22} - \frac{\omega}{\kappa} \zeta_{03}, \quad \zeta_{32} = -\frac{\omega}{\kappa} \zeta_{02}, 
\tag{44}
\]

and we have six independent gauge parameters\(^8\) \( \zeta_{01}, \zeta_{02}, \zeta_{03}, \zeta_{11}, \zeta_{22}, \zeta_{12} \). We shall present the free equation of motion (42) also in terms of field strength tensors. The quadratic

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\(^8\)One can also use a different set of independent parameters, in particular, \( \zeta_{11}, \zeta_{12}, \zeta_{13}, \zeta_{22}, \zeta_{23}, \zeta_{33} \).
The aim of this section is to analyze the free field equation (42) written in terms of field strength tensor $F_{\mu\nu,\lambda\rho}$.

The corresponding free field equation of motion (42) invariant with respect to the gauge transformations (43).

5 Propagating Modes of Rank-3 Gauge Field

The aim of this section is to analyze the free field equation (42)/(46) for the rank-3 gauge field. It is convenient to decompose the rank-3 gauge field into irreducible pieces. The gauge field $A_{\gamma'\gamma''}$ is symmetric over the last two indices $\gamma' \leftrightarrow \gamma''$ and has no symmetries with respect to the index $\gamma$. Let us consider the transformation $T$ of the form [21]

\[
A^T_{\mu\lambda_1} = A_{\lambda_1\mu},
A^T_{\mu\lambda_1\lambda_2} = \frac{2}{3}(A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}) - \frac{1}{3}A_{\mu\lambda_1\lambda_2},
\]

It has the property of the standard transposition $(A^T)^T = A$ and allows to define symmetric $A^S$ and anti-symmetric $A^A$ tensors as

\[
A^S = \frac{1}{2}(A + A^T), \quad A^A = \frac{1}{2}(A - A^T).
\]

In the case of the rank-3 gauge field they are

\[
A^S_{\mu\lambda_1\lambda_2} = \frac{1}{3}(A_{\mu\lambda_1\lambda_2} + A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}), \quad A^A_{\mu\lambda_1\lambda_2} = \frac{2}{3}A_{\mu\lambda_1\lambda_2} - \frac{1}{3}(A_{\lambda_1\mu\lambda_2} + A_{\lambda_2\mu\lambda_1}).
\]

One should also define vector fields associated with rank-3 tensor field:

\[
B_\mu = A_{\mu\lambda\lambda}, \quad C_\mu = A_{\lambda\lambda\mu}, \quad D_\mu = \partial_\lambda \partial_\rho A_{\lambda\mu\rho}, \quad E_\mu = \partial_\lambda \partial_\rho A_{\lambda\rho\mu}.
\]
Equation for these fields follow from our main equation (42)/ (46), if one takes its trace

\( (\eta_{\mu\nu}\partial^2 - \partial_{\mu}\partial_{\nu})(\frac{7}{8}B_{\nu} - C_{\nu}) = (D_\mu - E_\mu). \) (50)

These equations show that these vector fields (49) fulfill Maxwell equation. Our aim is to find explicit solutions for all these fields. This will allow to clarify the physical content of the equation (42)/ (46) and the propagating modes which it describes.

A convenient way to solve the free equation of motion is to consider it in momentum representation

\[
(H_{\alpha\alpha'}\gamma' \gamma'' + \frac{4}{3}H'_{\alpha\alpha'}\gamma' \gamma'')A^\alpha_{\gamma' \gamma''}(k) A^\alpha_{\gamma' \gamma''} = 0, \quad (51)
\]

as we did in the case of the rank-2 gauge field in section three. The matrix operator \( H_{\alpha\alpha'}\gamma' \gamma''(k) \) is the sum of \( H \) and \( H' \) given by (30) and (33) and in the four-dimensional space-time it is a square matrix \( 40 \times 40 \). Indeed, the gauge field \( A^\alpha_{\gamma' \gamma''} \) is symmetric over the last two indices \( \gamma' \leftrightarrow \gamma'' \) and has no symmetries with respect to the index \( \gamma \), thus the multi-index \( N \equiv (\gamma, \gamma', \gamma'') \) runs \( 4 \times 10 = 40 \) values and the matrix \( H_{NM} \) is \( 40 \times 40 \). It is convenient to represent the gauge field in the form of four symmetric matrices \( A_{\gamma' \gamma''} = (A_{0 \gamma' \gamma''}, ..., A_{3 \gamma' \gamma''}) = (e_{0 \gamma' \gamma''}, ..., e_{3 \gamma' \gamma''}) \exp\{i k x}\)

\[
e_{\gamma' \gamma''} = \begin{pmatrix}
e_{000} & e_{001} & e_{002} & e_{003} \\
e_{010} & e_{011} & e_{012} & e_{013} \\
e_{020} & e_{021} & e_{022} & e_{023} \\
e_{030} & e_{031} & e_{032} & e_{033}
\end{pmatrix}, ..., \begin{pmatrix}
e_{300} & e_{301} & e_{302} & e_{303} \\
e_{310} & e_{311} & e_{312} & e_{313} \\
e_{320} & e_{321} & e_{322} & e_{323} \\
e_{330} & e_{331} & e_{332} & e_{333}
\end{pmatrix}
\] (52)

each of which has ten independent components. In the reference frame, where \( k^\gamma = (\omega, 0, 0, k) \), the matrix \( H_{NM} \) has a particularly simple form. If \( \omega^2 - k^2 \neq 0 \), the rank of the 40-dimensional matrix \( H_{NM}(k) \) is equal to rank \( H_{|\omega^2-k^2=0} = 25 \) and the number of linearly independent solutions is \( 40 - 25 = 15 \). These are pure gauge fields (43)

\[e_{\gamma' \gamma''} = k_{\gamma} \xi_{\gamma' \gamma''} + k_{\gamma'} \xi_{\gamma \gamma''} + k_{\gamma''} \xi_{\gamma \gamma'}\] (53)

with ten independent gauge parameters \( \xi_{\gamma' \gamma''} \) and five independent gauge parameters \( \xi_{\gamma \gamma'} \).

When \( \omega^2 - k^2 = 0 \), then the rank of the matrix \( H_{NM}(k) \) drops and is equal to \( rank H_{|\omega^2-k^2=0} = 18 \). This leaves us with \( 40 - 18 = 22 \) solutions. These are 15+1=16 solutions, the pure gauge fields (43), (53) and six solutions representing propagating modes. On the mass-shell the number of pure gauge fields increases by one unit: instead of the pure gauge field

\[e_{\gamma' \gamma''} = k_{\gamma'}(e_{\gamma'}^{(1)}e_{\gamma''}^{(2)} + e_{\gamma''}^{(1)}e_{\gamma'}^{(2)}) + k_{\gamma''}(e_{\gamma'}^{(1)}e_{\gamma'}^{(2)} + e_{\gamma'}^{(2)}e_{\gamma''}^{(1)})\] (54)

two new linearly independent solutions appear

\[e_{\gamma' \gamma''} = k_{\gamma'}e_{\gamma'}^{(1)}e_{\gamma''}^{(2)} + k_{\gamma''}e_{\gamma'}^{(2)}e_{\gamma''}^{(1)}\]

\[e_{\gamma' \gamma''} = k_{\gamma'}e_{\gamma'}^{(2)}e_{\gamma''}^{(1)} + k_{\gamma''}e_{\gamma'}^{(1)}e_{\gamma''}^{(2)}\] (55)
where \( e_\mu^{(1)} = (0, 1, 0, 0), \quad e_\mu^{(2)} = (0, 0, 1, 0) \). Therefore on the mass-shell we have sixteen pure gauge fields and six propagating modes \( 22-16=6 \). The first two solutions are:

\[
e^{(1)}_{\gamma\gamma',\gamma''} = \left( \begin{array}{cccc} 0, & (0, 0, 0, 0), & (0, 0, 0, 0), & 0 \end{array} \right),\]

\[
e^{(2)}_{\gamma\gamma',\gamma''} = \left( \begin{array}{cccc} 0, & (0, 0, 0, 0), & (0, 0, 0, 0), & 0 \end{array} \right).
\]

These are traceless tensors \( (B_\mu = C_\mu = D_\mu = E_\mu = 0) \). Their linear combinations describe positive norm states with helicities \( \lambda = \pm 3 \), because one can represent these solutions as a direct product of helicity-one and helicity-two tensors \( e_{\gamma\gamma',\gamma''} = e_{\gamma}^{\pm 1} \otimes e_{\gamma'}^{\pm 2} \). The next two solutions are:

\[
e^{(5)}_{\gamma\gamma',\gamma''} = \left( \begin{array}{cccc} 0, & (0, 0, 0, 0), & (0, 0, 0, 0), & 0 \end{array} \right) - \frac{1}{3} (\eta_{\gamma\gamma'} e_{\gamma}^{(1)} + \eta_{\gamma\gamma''} e_{\gamma}^{(1)}) \quad (56)
\]

\[
e^{(6)}_{\gamma\gamma',\gamma''} = \left( \begin{array}{cccc} 0, & (0, 0, 0, 0), & (0, 0, 0, 0), & 0 \end{array} \right) - \frac{1}{3} (\eta_{\gamma\gamma'} e_{\gamma}^{(2)} + \eta_{\gamma\gamma''} e_{\gamma}^{(2)}).
\]

In accordance with \( (56) \) we have

\[
B^{(5,6)}_\mu = -\frac{2}{3} e^{(1,2)}_\mu, \quad C^{(5,6)}_\mu = -\frac{2}{3} e^{(1,2)}_\mu, \quad D_\mu = E_\mu = 0
\]

and they fulfill the free Maxwell equation \((46)\). Their linear combinations describe positive norm states of helicities \( \lambda = \pm 1 \). The last two solutions are:

\[
e^{(3)}_{\gamma\gamma',\gamma''} = \left( \begin{array}{cccc} 0, & (0, 0, 0, 0), & (0, 0, 0, 0), & 0 \end{array} \right) - \frac{1}{8} e_{\gamma} e_{\gamma'} e_{\gamma''} \quad (57)
\]

\[
e^{(4)}_{\gamma\gamma',\gamma''} = \left( \begin{array}{cccc} 0, & (0, 0, 0, 0), & (0, 0, 0, 0), & 0 \end{array} \right) - \frac{1}{8} e_{\gamma} e_{\gamma'} e_{\gamma''}.
\]

In accordance with solutions \((57)\) we have

\[
B^{(3,4)}_\mu = \frac{1}{2} e^{(1,2)}_\mu, \quad C^{(3,4)}_\mu = -\frac{1}{8} e^{(1,2)}_\mu, \quad D_\mu = E_\mu = 0
\]

and they also fulfill the free Maxwell equation \((46)\). Their linear combinations describe positive norm states of helicities \( \lambda = \pm 1 \). As one can check, the last solutions can not
be decomposed into symmetric and antisymmetric pieces (48). The reason is that the kinetic operator $\mathcal{H}_{\alpha\alpha',\gamma'\gamma''}$ in (51) cannot be represented as a sum of symmetric and anti-symmetric operators. It has non-diagonal matrix elements and these solutions are a mixture of the both symmetries. This is a new phenomenon which appears in the case of rank-3 gauge field. In the case of rank-2 gauge field the kinetic operator $\mathcal{H}_{\alpha\alpha',\gamma'\gamma''}$ in (15) can be decomposed into symmetric and anti-symmetric pieces, as we have seen in section three.

Thus the general solution of the equation on the mass-shell is:

$$e_{\gamma',\gamma''} = k_\gamma \xi_{\gamma',\gamma''} + k_{\gamma'} \zeta_{\gamma,\gamma''} + k_{\gamma''} \zeta_{\gamma',\gamma''} + \sum_{i=1}^{6} c_i e_{\gamma',\gamma''}^{(i)},$$

(58)

where $c_i$ are arbitrary constants. Thus we see that there are six propagating modes of helicity-three and a doublet of helicity-one charged gauge bosons: $\lambda = \pm 3, \pm 1, \pm 1$.

It is also interesting to see what happens if we consider free field equation (42)/(46) in $D$-dimensional space-time. As one can see the number of potentially negative norm states increases as $(3D^3 - 5D + 4)/2$ while the number of gauge parameters grows as $D^2$. Only in 3+1 dimensional space-time there is a chance for full cancelation of negative norm states, and, indeed, as we have seen, the particle spectrum is physical in 3+1 dimensions. In five dimensions the matrix $\mathcal{H}_{NM}$ has dimension $75 \times 75$. In the reference frame, where $k^\gamma = (\omega, 0, 0, k)$ and $\omega^2 - k^2 \neq 0$, the rank of the matrix $\mathcal{H}_{NM}(k)$ is equal to $\text{rank} \mathcal{H}_{|\omega^2-k^2\neq 0} = 50$ and the number of linearly independent solutions is 25. These are pure gauge fields (43), (53) with fifteen independent gauge parameters $\xi_{\gamma',\gamma''}$ and ten independent gauge parameters $\zeta_{\gamma',\gamma''}$. When $\omega^2 - k^2 = 0$, then $\text{rank} \mathcal{H}_{|\omega^2-k^2=0} = 30$. This leaves us with 45 solutions. These are 25 pure gauge solutions and 20 new solutions representing propagating modes. Only 18 modes can be positive definite.

Let us also consider equations for the higher-rank tensor gauge fields. The Lagrangian form for the rank-4 gauge field is a sum of the following two terms (s=3 in (4), (5)) [18]:

$$\mathcal{L}_4 = - \frac{1}{4} G_{\mu\nu,\rho\sigma} \lambda G_{\mu\nu,\rho\sigma} - \frac{3}{8} G_{\mu\nu,\rho\rho} G_{\mu\nu,\sigma\lambda\lambda} - \frac{3}{4} G_{\mu\nu,\rho\sigma} G_{\mu\nu,\rho\sigma\lambda\lambda}$$

$$- \frac{3}{16} G_{\mu\nu,\rho\rho} G_{\mu\nu,\sigma\sigma\lambda\lambda} - \frac{3}{8} G_{\mu\nu,\rho} G_{\mu\nu,\rho\sigma\sigma\lambda\lambda} - \frac{1}{16} G_{\mu\nu} G_{\mu\nu,\rho\rho\sigma\sigma\lambda\lambda}$$

(59)

and

$$\mathcal{L}_4' = \frac{1}{4} G_{\mu\nu,\rho\sigma} \lambda G_{\mu\nu,\rho\sigma} + \frac{1}{4} G_{\mu\nu,\rho\sigma} G_{\mu\lambda,\rho\sigma} + \frac{1}{8} G_{\mu\nu,\rho\sigma} G_{\mu\lambda,\nu\lambda\lambda} + \frac{1}{2} G_{\mu\nu,\rho\sigma} G_{\mu\lambda,\nu\rho\rho}$$

$$+ \frac{1}{8} G_{\mu\nu,\rho\rho} G_{\mu\sigma,\sigma\lambda\lambda} + \frac{1}{2} G_{\mu\nu,\rho\sigma} G_{\mu\lambda,\nu\sigma\lambda\lambda} + \frac{1}{2} G_{\mu\nu,\rho\sigma} G_{\mu\lambda,\nu\rho\sigma\lambda\lambda}$$

$$+ \frac{1}{2} G_{\mu\nu,\rho\sigma} G_{\mu\lambda,\rho\sigma\sigma\lambda\sigma} + \frac{1}{2} G_{\mu\nu,\rho\sigma} G_{\mu\lambda,\rho\lambda\lambda\sigma\sigma} + \frac{1}{2} G_{\mu\nu,\rho\sigma} G_{\mu\lambda,\nu\rho\lambda\lambda\sigma\sigma}$$

$$+ \frac{1}{8} G_{\mu\nu,\rho\sigma} G_{\mu\rho,\rho\sigma\sigma\lambda\lambda}. (60)$$

Deriving field equations from this Lagrangian one can see that the free equation of motion has two solutions which describe the propagating $\lambda = \pm 4$ helicity states only if

$$g_4' = \frac{3}{2} g_4.$$

(61)
In the case of rank-(s+1) gauge field, as it follows from (4) and (5), the free equation of motion has two solutions describing the propagating $\lambda = \pm(s + 1)$ helicity states only if

$$g'_{s+1} = \frac{2s}{s + 1} g_{s+1},$$  \hspace{1cm} (62)

where $s = 0, 1, 2, ...$ and $g_1 = g_{YM}$. Therefore the Lagrangian (1) has the following form:

$$\mathcal{L} = \mathcal{L}_{YM} + g_2(\mathcal{L}_2 + \mathcal{L}'_2) + g_3(\mathcal{L}_3 + \frac{4}{3} \mathcal{L}'_3) + ... + g_{s+1}(\mathcal{L}_{s+1} + \frac{2s}{s + 1} \mathcal{L}'_{s+1}) + ...,$$  \hspace{1cm} (63)

where the coupling constants $g_{s+1}$ still remain undefined. Therefore, let us consider the dependence of the Lagrangian on the coupling constant $g_2$. As we shall see the coupling constant $g_2$ can be eliminated from the Lagrangian by redefinition of fields and other coupling constants. Indeed, let us define the transformation of tensor gauge fields as follows:

$$A_a^{\mu_1...\lambda_s} \rightarrow \frac{1}{g_2^{s/2}} A_a^{\mu_1...\lambda_s}.$$  \hspace{1cm} (64)

This transformation should be complemented by the transformation of gauge parameters

$$\xi^{a}_{\lambda_1...\lambda_s} \rightarrow \frac{1}{g_2^{s/2}} \xi^{a}_{\lambda_1...\lambda_s}.$$  \hspace{1cm} (65)

in order to protect the extended gauge transformations (2). In that case the extended field strength tensors will also transform homogeneously:

$$\mathcal{G}_{\mu_\nu,\lambda_1...\lambda_s}^{a} \rightarrow \frac{1}{g_2^{s/2}} \mathcal{G}_{\mu_\nu,\lambda_1...\lambda_s}^{a}.$$  \hspace{1cm} (66)

and the invariant forms will transform as follows:

$$\mathcal{L}_s \rightarrow \frac{1}{g_2^{s-1}} \mathcal{L}_s.$$  \hspace{1cm} (67)

Therefore the Lagrangian will take the form

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_2 + \mathcal{L}'_2 + \frac{g_3}{g_2^2}(\mathcal{L}_3 + \frac{4}{3} \mathcal{L}'_3) + \frac{g_4}{g_2^3}(\mathcal{L}_4 + \frac{3}{2} \mathcal{L}'_4) + ... \rightarrow$$

$$\rightarrow \mathcal{L}_{YM} + \mathcal{L}_2 + \mathcal{L}'_2 + g_3(\mathcal{L}_3 + \frac{4}{3} \mathcal{L}'_3) + g_4(\mathcal{L}_4 + \frac{3}{2} \mathcal{L}'_4) + ...$$  \hspace{1cm} (68)

and the coupling constant $g_2$ is fully eliminated from the theory. This cannot be done with the coupling constant $g_3$.

Summarizing our findings we can state that the Lagrangian $\mathcal{L}$ describes the interacting system of gauge bosons of increasing helicities. The system has Yang-Mills gauge boson on the first level ($s=0$), the helicity-two and -zero gauge bosons on the second level ($s=1$) and the helicity-three and a doublet of helicity-one gauge bosons on the third level ($s=2$).

The particle spectrum on higher levels is not yet known completely and to find it out remains a challenging problem. The problem consists in finding out the value of the coupling constant $g'_{s+1}$ at which the corresponding free field equation for the rank-(s+1) gauge field is free from propagating negative norm states. As we have found for

$$g'_{s+1} = \frac{2s}{s + 1} g_{s+1}, \hspace{1cm} s = 0, 1, 2, ...$$
(g_1 = g_{YM}) there are two solutions which describe the propagating positive norm states of helicities \( \lambda = \pm (s + 1) \). But the difficulty in finding out all propagating modes for this value of the coupling constant \( g'_{s+1} \) lies in the fact that the number of field components dramatically increases with the rank of the tensor gauge field: in the case of rank-2 gauge field we had sixteen components and in the case considered in this article for the rank-3 gauge field we had to analyze an equation with forty components. The presented analysis shows that, most probably, the full system is unitary for all higher-rank non-Abelian tensor gauge fields.

In conclusion let us discuss the relation between the present field theoretical model and the Coleman-Mandula theorem which imposes strict restrictions on the possible theories consistent with the fundamental principles of quantum field theory [24]. The results of the Coleman-Mandula paper were generally accepted as most powerful in a series of “no-go” theorems, destroying the hope for a fusion between internal symmetries and the Poincaré group. It is applicable if five conditions formulated in the Coleman-Mandula article are hold. One of these conditions - Particle-finiteness condition - states that: “(2) For any finite \( M > 0 \), there is only a finite number of one-particle states with mass less than \( M \).” The equivalent formulation can also be found in the book of Wess and Bagger [27] and in an important discussion in the article [25].

The particle-finiteness condition is not applicable to the field-theoretical model studied in the present article because there are \( \alpha \) massless particles in the spectrum and \( \beta \) the number of massless particles is infinite, in which case the theorem does not apply.

There are well known cases when the Coleman-Mandula theorem is not applicable. First of all it is the case already mentioned in Coleman-Mandula article, the so called ”infinite-supermultiplet theories” and the second case is the supersymmetric extension of the Poincaré algebra [26, 27]. Our model belongs to the first exceptional case.

In this article we study the spectrum of the non-Abelian tensor gauge fields and describe in details the helicity content of these tensor fields. These studies comprise a necessary step in any serious investigation, without which it is impossible to accept or reject any theory. The article does not contain claims that the suggested model is a fully consistent field theoretical model of interacting non-Abelian tensor gauge fields, but takes the necessary steps in order to get an answer to the above question.

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6 Appendix A

The quadratic form \( H'_{\alpha'\alpha''\gamma'\gamma''} \) can be extracted from (32) and should be symmetrized over \( \alpha' \leftrightarrow \alpha'' \), \( \gamma' \leftrightarrow \gamma'' \) and over the exchange of two sets of indices \( \alpha' \alpha'' \leftrightarrow \gamma' \gamma'' \) so that in the momentum representation it has the form

\[
H'_{\alpha'\alpha''\gamma'\gamma''}(k) = \frac{k^2}{8} \left\{ \eta_{\alpha'\alpha''} (\eta_{\gamma'\gamma''} + \eta_{\gamma'\gamma''} + \eta_{\gamma'\gamma''} + \eta_{\gamma'\gamma''}) + \eta_{\alpha'\alpha''} (\eta_{\gamma'\gamma''} + \eta_{\gamma'\gamma''} + \eta_{\gamma'\gamma''} + \eta_{\gamma'\gamma''}) \right\}
\]
This expression can be used to calculate divergences. Indeed,

\[ k_\alpha' H'_{\alpha'\alpha''\gamma\gamma'}(k) = \frac{1}{8} \left\{ \begin{array}{l}
+ (k^2 \eta_{\alpha''} - k_\alpha k_\alpha')(\eta_{\alpha''}\eta_{\gamma'} + \eta_{\alpha''}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
+ k_\gamma k_\alpha'(\eta_{\gamma'}\eta_{\gamma'} + \eta_{\alpha''}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
+ k_\gamma k_\alpha''(\eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
+ k_\gamma k_\alpha''(\eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
+ k_\gamma k_\alpha''(\eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
\end{array} \right\} \]

This expression can be used to calculate divergences. Indeed,

\[ k_\alpha H_{\alpha'\alpha''\gamma\gamma'}(k) = \frac{1}{8} \left\{ \begin{array}{l}
+ (k^2 \eta_{\alpha''} - k_\alpha k_\alpha'')(\eta_{\alpha''}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
+ (k^2 \eta_{\alpha''} - k_\alpha k_\alpha')(\eta_{\alpha''}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
+ k_\gamma k_\alpha''(\eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
+ k_\gamma k_\alpha''(\eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
+ k_\gamma k_\alpha''(\eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
\end{array} \right\} \]

or using the operator \( H_{\alpha\gamma} = k^2 \eta_{\alpha\gamma} - k_\alpha k_\gamma \) one can get

\[ k_\alpha H'_{\alpha'\alpha''\gamma\gamma'}(k) = \frac{1}{8} \left\{ \begin{array}{l}
+ H_{\alpha\alpha''}(\eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'} + \eta_{\gamma'}\eta_{\gamma'}) \\
\end{array} \right\} \]
and canceling the identical terms we shall get

\[
\begin{align*}
&+ \ H_{\alpha\gamma'} \left( k_\gamma \eta^{\alpha''\gamma''} + k_\gamma \eta^{\alpha''\gamma} + k_\alpha'' \eta^{\gamma''\gamma} \right) \\
&+ \ H_{\alpha''\gamma} \left( k_\gamma \eta^{\alpha\gamma'} + k_\gamma \eta^{\alpha'\gamma} + k_\alpha'' \eta^{\gamma'\gamma} \right) \\
&- \frac{1}{8} \left\{ \ + \ k_\gamma k_\gamma \left( 2k_\gamma \eta^{\alpha\gamma'} + 2k_\gamma \eta^{\alpha'\gamma} + 2k_\alpha'' \eta^{\gamma'\gamma} \right) \\
&+ \ k_\gamma k_\gamma \left( 2k_\gamma \eta^{\alpha''\gamma''} + k_\alpha'' \eta^{\gamma''\gamma} \right) + k_\gamma k_\gamma k_\alpha'' \eta^{\gamma''\gamma'} \right\} \\
&+ \frac{1}{4} \left\{ \ + \ k_\alpha'' k_\gamma^{\alpha'} \eta^{\gamma''\gamma'} + k_\alpha'' k_\alpha'' \eta^{\gamma''\gamma'} + k_\alpha'' k_\alpha'' \eta^{\gamma''\gamma'} + 3k_\alpha'' k_\gamma^{\alpha'} k_\gamma^{\alpha'' \gamma'} \right\} 
\end{align*}
\]

and canceling the identical terms we shall get

\[
k'_{\alpha'} H_{\alpha''\alpha'}^{\alpha''\gamma'}(k) = \frac{1}{8} \left\{ \ + \ H_{\alpha''\alpha'}(k_\gamma \eta^{\alpha''\gamma''} + k_\alpha'' \eta^{\gamma''\gamma}) \\
&+ \ H_{\alpha''\gamma'}(k_\gamma \eta^{\alpha''\gamma'} + k_\alpha'' \eta^{\gamma''\gamma}) \\
&+ \ H_{\alpha''\gamma'}(k_\gamma \eta^{\alpha''\gamma'} + k_\alpha'' \eta^{\gamma''\gamma'}) \right\} \\
- \frac{1}{4} \left\{ \ + \ k_\gamma k_\gamma \left( k_\gamma \eta^{\alpha''\gamma''} + k_\gamma \eta^{\alpha''\gamma'} + k_\gamma k_\gamma k_\gamma \eta^{\gamma''\gamma'} \right) \right\} \\
+ \frac{1}{4} \left\{ \ + \ H_{\alpha\gamma'} k_\gamma \eta^{\alpha''\gamma''} + H_{\alpha\gamma'} k_\gamma \eta^{\alpha''\gamma'} + H_{\alpha\gamma'} k_\gamma \eta^{\alpha''\gamma'} \\
&+ \ 3k_\alpha k_\gamma^{\alpha'} k_\gamma^{\alpha'' \gamma'} \right\} \tag{71} 
\]

Again collecting terms we shall get the final expression:

\[
k'_{\alpha'} H_{\alpha''\alpha'}^{\alpha''\gamma'}(k) = \frac{1}{8} \left\{ \ + \ H_{\alpha''\alpha'}(k_\gamma \eta^{\alpha''\gamma''} + k_\alpha'' \eta^{\gamma''\gamma}) \\
&+ \ H_{\alpha''\gamma'}(k_\gamma \eta^{\alpha''\gamma'} + k_\alpha'' \eta^{\gamma''\gamma'}) \\
&+ \ H_{\alpha''\gamma'}(k_\gamma \eta^{\alpha''\gamma'} + k_\alpha'' \eta^{\gamma''\gamma'}) \right\} \\
- \frac{1}{4} \left\{ \ + \ k_\gamma k_\gamma \left( k_\gamma \eta^{\alpha''\gamma''} + k_\gamma \eta^{\alpha''\gamma'} + k_\gamma k_\gamma k_\gamma \eta^{\gamma''\gamma'} \right) \right\} \\
+ \frac{1}{4} \left\{ \ + \ H_{\alpha\gamma'} k_\gamma \eta^{\alpha''\gamma''} + k_\gamma \eta^{\alpha''\gamma'} + 3k_\alpha k_\gamma^{\alpha'} k_\gamma^{\alpha'' \gamma'} \right\} 
\]

which has been used in the main text.

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