Biquandle Arrow Weight Enhancements

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Abstract

We introduce a new infinite family of enhancements of the biquandle homset invariant called biquandle arrow weights. These invariants assign weights in an abelian group to intersections of arrows in a Gauss diagram representing a classical or virtual knot depending on the biquandle colors associated to the arrows. We provide examples to show that the enhancements are nontrivial and proper, i.e., not determined by the homset cardinality.

Keywords: Biquandles, homsets, enhancements, virtual knots, Gauss diagrams, biquandle arrow weights

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1 Introduction

A combinatorial approach to knot theory involves representing knots as equivalence classes of diagrams. The most common such scheme uses knot diagrams, 4-valent graphs with vertices enhanced with crossing information so the graph can be interpreted as a projection of the knot in space onto the plane of the paper with only double-point singularities represented by the vertices. A knot can then be identified as an equivalence class of knot diagrams under the equivalence relation generated by the Reidemeister moves and planar isotopy.

There are other combinatorial diagrammatic ways to represent knots, however, such as the triple-point diagrams and über-crossing diagrams in [1] and the Gauss diagrams used in [6], the latter of which we use in this paper. Equivalence classes of Gauss diagrams under the Gauss diagram Reidemeister moves are known as virtual knots, a larger category which includes classical knots as a subcategory. In papers such as [4,7], invariants of virtual knots such as the affine index polynomial are computed from Gauss diagrams using crossings of arrows, enigmatic features of Gauss diagrams which don’t correspond to anything obvious in the usual diagrams of knots. In this paper we adapt the idea from [3] of assigning weights to crossings in diagrams colored with an algebraic structure, in our case finite biquandles, such that the sum of the weights is preserved by Reidemeister moves. The resulting multiset of weight values over the set of colorings of a Gauss diagram is therefore an invariant of virtual (and hence classical) knots for each finite biquandle and biquandle arrow weight.

The paper is organized as follows. In Section 2 we review the basics of biquandles and Gauss diagrams. In Section 3 we define biquandle arrow weight systems and define the new family of invariants, including our main result showing that the resulting enhancement of the counting invariant is in fact an invariant of oriented classical and virtual knots. In Section 4 we collect some examples and in particular show that the enhancement is proper, i.e. not determined by the counting invariant. We conclude in Section 5 with some questions for future research. The first listed author thanks the second listed author and Shibaura Institute of Technology for their kind hospitality during the preparation of this paper.

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2 Biquandles and Gauss Diagrams

We recall the definition of biquandles; see [5] for more.

**Definition 1.** A biquandle is a set $X$ with operations $\sqsupseteq$, $\sqsubseteq : X \to X$ which satisfies for all $x, y, z \in X$

(i) $x \sqsupseteq x = x \sqsubseteq x$,

(ii) the maps $\alpha_x, \beta_x : X \to X$ and $S : X \times X \to X \times X$ defined by

$$\alpha_x(y) = y \sqsupseteq x, \ \beta_x(y) = y \sqsubseteq x \ \text{and} \ S(x, y) = (y \sqsubseteq x, x \sqsupseteq y)$$

are invertible, and

(iii) we have the exchange laws

$$(x \sqsupseteq y) \sqsupseteq (z \sqsupseteq y) = (x \sqsupseteq z) \sqsupseteq (y \sqsubseteq z),$$

$$(x \sqsupseteq y) \sqsubseteq (z \sqsubseteq y) = (x \sqsubseteq z) \sqsubseteq (y \sqsupseteq z), \ \text{and}$$

$$(x \sqsubseteq y) \sqsubseteq (z \sqsupseteq y) = (x \sqsubseteq z) \sqsupseteq (y \sqsubseteq z).$$

A biquandle coloring of an oriented knot or link diagram $D$ by a biquandle $X$, also known as an $X$-coloring, is an assignment of an element of $X$ to each semiarc in $D$ such that at every crossing we have the condition known as the biquandle coloring condition.

The biquandle axioms are chosen so that for every biquandle coloring of a diagram before a Reidemeister move, there is a unique coloring of the diagram after the move which coincides with the original coloring outside the neighborhood of the move. It then follows that the number of colorings of an oriented knot or link diagram by a finite biquandle $X$ is an integer-valued invariant, denoted $\Phi_X^K$ and called the biquandle counting invariant.

**Remark 1.** More precisely, to every oriented knot or link $K$ there is an associated biquandle called the fundamental biquandle or the knot biquandle, denoted $B(K)$, which can be described from any diagram of $K$ with a presentation consisting of generators associated to semiarcs and relations at the crossings. Any $X$-coloring then determines and is determined by a unique biquandle homomorphism $f \in \text{Hom}(B(K), X)$. This set of biquandle homomorphisms, known as a homset, is an invariant of $K$. Choosing a different diagram for $K$ changes the representations of each homset element, with colored diagrams representing the same homset element if they are related by $X$-colored Reidemeister moves. We can think of choosing a diagram for $K$ as analogous to choosing input and output bases for vector spaces; then $X$-colored diagrams represent homset elements analogously to the way matrices represent linear transformations, and Reidemeister moves play the role of change of basis matrices.

Next, we recall the definition of Gauss diagrams. Let $K$ be a virtual knot and $D$ a virtual knot diagram of $K$. Then, $D$ is regarded as the image $f(S^1)$ of a generic immersion $f : S^1 \to \mathbb{R}^2$. A Gauss diagram for $D$ is the preimage of $D$ with chords, each of which connects the preimages of each real crossing. We suppose that virtual knot diagrams are oriented. We specify over/under information of each real crossing on the
corresponding chord by directing the chord toward the under path and assigning each chord with the sign of the crossing as shown:

\[
\begin{array}{c}
\text{+ 1} \\
\text{-.}
\end{array}
\]

It is well-known that there exists a bijection from the set of virtual knots to the set of equivalence classes of their Gauss diagrams by the generalized Reidemeister moves of Gauss diagrams:

We identify a virtual knot with an equivalence class of Gauss diagrams under these moves.

A biquandle coloring of a Gauss diagram by a biquandle \( X \) is an assignment of an element of \( X \) to each segment of the circle between the arrowheads and tails such that at every arrow we have

\[
\begin{array}{c}
(x, y) + \\
(x, y) -
\end{array}
\]

as shown. In particular, to each arrow in a biquandle-colored Gauss diagram we associate an ordered pair \((x, y)\) of biquandle elements together with a sign.

As in Remark 1, the set of biquandle colorings of a Gauss diagram \( D \) representing a virtual knot \( K \) by a biquandle \( X \) can be identified with the homset \( \text{Hom}(B(K), X) \), with each coloring representing a homset element and with our choice of diagram \( D \) analogous to a choice of basis for a vector space.

### 3 Biquandle Arrow Weights

We begin with a definition.

**Definition 2.** Let \( X \) be a biquandle and \( A \) an abelian group. A **biquandle arrow weight** is a function \( \phi : X^4 \to A \) satisfying the conditions

(i) For all \( x, y, u, v \in X \) we have

\[
\phi((x, y), (u, v)) = \phi((u, v), (x, y)),
\]
(ii) For all \( x, y \in X \), \( \phi((x, y), (x, y)) = 0 \),

(iii) For all \( x, y, z \in X \) we have

\[
\phi((x, y), (y, z)) = \phi((x, z), (y, z)) + \phi((x, z), (y, z))
\]

and

(iv) For all \( x, y, z \in X \) we have

\[
\phi((x \uparrow z, y \uparrow z), (x \downarrow z, y \downarrow z)) = \phi((x, y), (x \downarrow z, y \downarrow z)) + \phi((y, z), (x \downarrow z, y \downarrow z))
\]

(v) For all \( u, v, x, y, z \in X \) we have

\[
\begin{align*}
\phi((u, v), (x, y)) + \phi((u, v), (y, z)) &= \phi((u, v), (x \uparrow z, y \uparrow z)) + \phi((u, v), (y \downarrow z, z \downarrow z)) \\
\phi((u, v), (x, z)) + \phi((u, v), (y \downarrow z, z \downarrow z)) &= \phi((u, v), (x, z)) + \phi((u, v), (x \downarrow z, y \downarrow z)) \\
\phi((u, v), (x, y)) + \phi((u, v), (x \downarrow z, y \downarrow z)) &= \phi((u, v), (x, y)) + \phi((u, v), (x \downarrow z, y \downarrow z)).
\end{align*}
\]

The motivation for this definition is the following: let \( D \) be a Gauss diagram with coloring by a biquandle \( X \). Whenever two arrows with biquandle colors \((x, y)\) and \((u, v)\) and signs \(\epsilon\) and \(\epsilon'\) cross, we want to assign a weight of \(\epsilon\epsilon'\phi((x, y), (u, v))\) to the crossing point. For example, in the case

we have a weight of \((+1)(-1)\phi((x, y), (u, v)) = -\phi((x, y), (u, v))\). The conditions in Definition 2 are then the conditions needed for invariance of the sum of weights in a diagram under the Gauss diagram Reidemeister moves. More precisely, we have:

**Proposition 1.** Let \( X \) be a biquandle, \( Y \) a biquandle arrow weight and \( D \) a Gauss diagram with a choice of \( X \)-coloring. Then the sum of \( \phi \)-values over all arrow crossings in \( D \), denoted \( \Sigma_D \), is not changed by \( X \)-colored Reidemeister moves.

**Proof.** This is a matter of checking the statement for a generating set of Reidemeister moves. We note that one such set consists of all four RI moves, all four RII moves and the RIII move with all positive crossings; see [8].

First, we observe that there’s no particular ordering on the set of arrows, so we set

\[
\phi((x, y), (u, v)) = \phi((u, v), (x, y))
\]

as axiom (i).

The RI moves involve introduction or removal of single arrows which do not cross other arrows, so the contribution to weight sum the on both sides of the move is the same, namely zero, and the RI move
conditions are satisfied.

The RII moves involve introduction or removal of pairs of arrows with opposite signs and the same pair of biquandle colors, either crossing or not. These moves impose two axioms on the weights.

For the RII moves where the arrows cross, we need the crossing weight to be zero; since these two arrows both have biquandle colors \((x,y)\), we set \(\phi((x,y),(x,y)) = 0\) and obtain axiom (ii). For the RII moves where the arrows don’t cross, the weight contributions from the two arrows of the move on both sides are zero.

More generally, the two arrows of either type of RII move will both cross the same set of arrows but with
opposite signs

so we arrange for the weights from the positive and negative signed arrows of the RII move to cancel by our weighting rule.

Our chosen generating set of Reidemeister moves contains only one RIII move, but the two cyclic orderings of the three strands of the move impose different conditions on biquandle arrow weights. Labeling the strands as shown

we have one case
which gives us axiom (iii) and the other

which yields axiom (iv).

Finally, we must consider the crossings of other arrows with the arrows involved in the Reidemeister III moves. We note that such arrows cross a pair of arrows on each side of the move and that it suffices to consider positive arrows since negative arrows yield equivalent equations. We further note that the two cyclic orderings of the move produce the same requirements, namely the conditions in axiom (v). We illustrate the case of the first equation in axiom (v); the others are similar.

Since this set of moves is a generating set, it follows that that the list of axioms obtained from it suffices to guarantee invariance.

We can represent a biquandle arrow weight as a 4-tensor or matrix of matrices with values in the coefficient ring such that \( \phi((i, j), (k, l)) \) is the row \( k \) column \( l \) entry in the matrix in row \( i \) column \( j \).

**Example 1.** Our Python computations indicate that the 4-tensor

\[
\begin{bmatrix}
0 & 3 & 3 & 0 \\
3 & 0 & 6 & 9 \\
3 & 6 & 0 & 9 \\
0 & 9 & 9 & 0 \\
\end{bmatrix}
\]
defines a biquandle arrow weight over the biquandle structure on the set \( X = \{1, 2\} \) specified by the operation tables
\[
\begin{array}{c|cc}
\circ & 1 & 2 \\
\hline
1 & 2 & 2 \\
2 & 1 & 1
\end{array}
\quad
\begin{array}{c|cc}
\circ & 1 & 2 \\
\hline
1 & 2 & 2 \\
2 & 1 & 1
\end{array}
\]
with coefficient group \( A = \mathbb{Z}_{12} \). Then for instance we have \( \phi((1, 2), (1, 1)) = 3 \) and \( \phi((1, 1), (2, 2)) = 0 \).

**Definition 3.** Let \( X \) be a finite biquandle and \( W \) a biquandle arrow weight with values in an abelian group \( A \). We define the **biquandle arrow weight multiset** for Gauss diagram \( G \) to be the multiset of \( \Sigma_D \) values over the set of biquandle colorings \( D \) of \( G \),
\[
\Phi_{X, M}^{W}(G) = \{ \Sigma_D \mid D \in \text{Hom}(B(G), X) \}.
\]
We define the **biquandle arrow weight polynomial** of \( G \) to be the expression
\[
\Phi_{X}^{W}(G) = \sum_{D \in \text{Hom}(B(G), X)} u^{\Sigma_D}
\]
for a formal variable \( u \).

**Corollary 2.** \( \Phi_{X, M}^{W}(G) \) and \( \Phi_{X}^{W}(G) \) are invariants of oriented virtual (and hence classical) knots and links.

**Proof.** This follows immediately from Proposition [1]. \( \square \)

4 Examples

In this section we collect a few examples of the new invariants.

**Example 2.** Let us start with a basic illustration of how to compute the invariant. Let \( X \) be the biquandle structure on the set \( X = \{1, 2\} \) defined by the operation tables
\[
\begin{array}{c|cc}
\circ & 1 & 2 \\
\hline
1 & 2 & 2 \\
2 & 1 & 1
\end{array}
\quad
\begin{array}{c|cc}
\circ & 1 & 2 \\
\hline
1 & 2 & 2 \\
2 & 1 & 1
\end{array}
\]
and let \( W \) be the biquandle arrow weight with coefficients in \( A = \mathbb{Z}_{8} \) given by the 4-tensor
\[
\begin{bmatrix}
0 & 2 \\
6 & 4
\end{bmatrix}
\begin{bmatrix}
2 & 0 \\
0 & 2
\end{bmatrix}
\begin{bmatrix}
6 & 0 \\
0 & 6
\end{bmatrix}
\begin{bmatrix}
4 & 2 \\
6 & 0
\end{bmatrix}
\]
with coefficients in \( \mathbb{Z}_{8} \). Let us consider the virtual knot 4.72, given by the Gauss diagram

![Gauss Diagram](image-url)
There are two $X$-colorings of this diagram,

![Diagram of two knots]

For the coloring on the left we have biquandle arrow weight

$$\phi((1,1,+),(2,1,-)) + \phi((2,2,+),(1,2,-)) + \phi((2,2,+),(1,1,+)) = -6 - 2 + 4 = 4$$

and on the right we have

$$\phi((2,2,+),(1,2,-)) + \phi((1,1,+),(2,1,-)) + \phi((1,1,+),(2,2,+)) = -2 - 6 + 4 = 4.$$ 

Then the multiset version of the invariant is

$$\Phi^{W, M}_X(4.72) = \{4, 4\}$$

and the polynomial version is

$$\Phi^W_X(4.72) = 2u^4.$$ 

We note that since the unknot $0_1$ has values respectively $\Phi^{W, M}_X(0_1) = \{0, 0\}$ and $\Phi^W_X(0_1) = 2$, this example shows that the invariant is nontrivial since it distinguishes 4.72 from the unknot. Moreover, this example shows that the invariant is a proper enhancement, i.e., it determines but is not determined by the biquandle counting invariant.

**Example 3.** Let $X$ be the biquandle in Example 2. We randomly selected two biquandle arrow weights with coefficients in $\mathbb{Z}_4$

$$W_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}, \quad W_2 = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 3 & 2 & 1 & 0 \\ 3 & 0 & 2 & 1 \\ 0 & 3 & 3 & 0 \end{bmatrix}$$

and computed the invariant values for the virtual knots with up to four classical crossings in Jeremy Green’s table at the knot atlas as shown in the table.

| $\Phi^{W, 1}_X(K)$ | $K$ |
|---------------------|-----|
| 2                   | 3.1, 3.5, 3.6, 3.7, 4.1, 4.2, 4.3, 4.6, 4.7, 4.8, 4.10, 4.12, 4.13, 4.16, 4.17, 4.19, 4.21, 4.23, 4.24, 4.25, 4.26, 4.29, 4.30, 4.33, 4.34, 4.37, 4.38, 4.39, 4.40, 4.43, 4.46, 4.47, 4.50, 4.51, 4.53, 4.55, 4.56, 4.60, 4.61, 4.62, 4.63, 4.64, 4.69, 4.75, 4.76, 4.77, 4.79, 4.80, 4.85, 4.86, 4.89, 4.90, 4.91, 4.93, 4.96, 4.97, 4.98, 4.99, 4.100, 4.102, 4.103, 4.105, 4.106, 4.107, 4.108 |
| $2u^2$              | 2.1, 3.2, 3.2, 3.4, 4.4, 4.5, 4.9, 4.11, 4.14, 4.15, 4.18, 4.20, 4.22, 4.27, 4.28, 4.31, 4.32, 4.35, 4.36, 4.41, 4.42, 4.44, 4.45, 4.48, 4.49, 4.52, 4.54, 4.57, 4.58, 4.59, 4.65, 4.66, 4.67, 4.68, 4.70, 4.71, 4.72, 4.73, 4.74, 4.78, 4.81, 4.82, 4.83, 4.84, 4.87, 4.88, 4.92, 4.94, 4.95, 4.101, 4.104 |
Let $\Phi^w_X(K)$. 

| $\Phi^w_X(K)$ | $K$ |
|---------------|-----|
| 2             | 2.1, 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.7, 4.8, 4.9, 4.10, 4.11, 4.12, 4.13, 4.14, 4.15, 4.16, 4.17, 4.18, 4.19, 4.20, 4.21, 4.22, 4.23, 4.24, 4.25, 4.26, 4.27, 4.28, 4.43, 4.44, 4.45, 4.46, 4.47, 4.48, 4.49, 4.50, 4.51, 4.52, 4.53, 4.54, 4.55, 4.56, 4.73, 4.74, 4.75, 4.76, 4.77, 4.78, 4.80, 4.81, 4.82, 4.83, 4.84, 4.85, 4.86, 4.87, 4.88, 4.89, 4.90, 4.91, 4.92, 4.93, 4.94, 4.95, 4.96, 4.97, 4.98, 4.99, 4.100, 4.101, 4.102, 4.103, 4.104, 4.105, 4.106, 4.107, 4.108 |
| $2u^2$        | 4.29, 4.30, 4.31, 4.32, 4.33, 4.34, 4.35, 4.36, 4.37, 4.38, 4.39, 4.40, 4.41, 4.42, 4.57, 4.58, 4.59, 4.60, 4.61, 4.62, 4.63, 4.64, 4.65, 4.66, 4.67, 4.68, 4.69, 4.70, 4.71, 4.72 |

**Example 4.** Let $X$ be the biquandle structure on $\{1, 2, 3\}$ given by the operation tables

| $\varphi$ | 1 | 2 | 3 |
|-----------|---|---|---|
| 1         | 2 | 2 | 2 |
| 2         | 3 | 3 | 3 |
| 3         | 1 | 1 | 1 |

| $\varphi$ | 1 | 2 | 3 |
|-----------|---|---|---|
| 1         | 2 | 2 | 2 |
| 2         | 3 | 3 | 3 |
| 3         | 1 | 1 | 1 |

Our python computations indicate that the 4-tensors

$$w_3 = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

and

$$w_4 = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \\ 2 & 0 & 3 \\ 3 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ 0 & 3 & 2 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 0 & 1 \\ 3 & 2 & 0 \\ 2 & 0 & 3 \\ 3 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

define biquandle arrow weight on $X$ with coefficients in $\mathbb{Z}_3$ and $\mathbb{Z}_6$ respectively, giving us the following table of nontrivial invariant values for the prime virtual knots with up to 4 classical crossings in the table at the Knot Atlas [2]:

| $\Phi^w_X(K)$ | $K$ |
|---------------|-----|
| $3u$          | 4.10, 4.13, 4.15, 4.18, 4.19, 4.20, 4.24, 4.29, 4.33, 4.34, 4.35, 4.38, 4.39, 4.40, 4.41, 4.42, 4.49, 4.50, 4.51, 4.57, 4.58, 4.70, 4.72, 4.78 |
| $3u^2$        | 4.11, 4.17, 4.22, 4.23, 4.32, 4.61, 4.62, 4.63, 4.66, 4.67, 4.68, 4.79 |

and

$\Phi^w_X(K)$ | $K$ |
|---------------|-----|
| $3u$          | 4.17, 4.22, 4.23, 4.32, 4.61, 4.62, 4.67 |
| $3u^2$        | 4.13, 4.15, 4.20, 4.24, 4.29, 4.34, 4.38, 4.41, 4.42, 4.49, 4.51, 4.58, 4.72 |
| $3u^3$        | 2.1, 3.1, 3.2, 4.4, 4.5, 4.9, 4.14, 4.26, 4.27, 4.30, 4.37, 4.44, 4.47, 4.48, 4.52, 4.54, 4.60, 4.64, 4.69, 4.74, 4.80, 4.82, 4.84, 4.91, 4.93, 4.94, 4.102 |
| $3u^4$        | 4.11, 4.63, 4.66, 4.68, 4.79 |
| $3u^5$        | 4.10, 4.18, 4.19, 4.33, 4.35, 4.39, 4.40, 4.50, 4.57, 4.70, 4.78 |
Virtual knots not listed have the trivial invariant value of $\Phi^W_X(K) = 3$.

5 Questions

We conclude with some questions for future research.

The big question seems to be: what is the relationship between these invariants and the (bi)quandle 2-cocycle and 3-cocycle invariants from [3] and later generalizations? Despite taking four biquandle colors as inputs, these biquandle arrow weights do not seem to be identifiable as 4-cocycles in an obvious way.

What conditions analogous to cohomology make two biquandle arrow weights define the same invariant? What is the geometric meaning of these invariants?

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