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Well-posedness of Linear Integro-Differential Equations with Operator-valued Kernels.

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Abstract. We study linear integro-differential equations in Hilbert spaces with operator-valued kernels and give sufficient conditions for the well-posedness. We show that several types of integro-differential equations are covered by the class of evolutionary equations introduced in [R. Picard. A structural observation for linear material laws in classical mathematical physics. Math. Methods Appl. Sci., 32(14):1768–1803, 2009]. We therefore give criteria for the well-posedness within this framework. As an example we apply our results to the equations of visco-elasticity.

Keywords and phrases: Integro-Differential Equations, Well-posedness and Causality, Evolutionary Equations, Linear Material Laws, Visco-Elasticity.

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1 Introduction

It appears that classical phenomena in mathematical physics, like heat conduction, wave propagation or elasticity, show some memory effects (see e.g. [11, 16]). One way to mathematically model these effects is to use integro-differential equations. In this work we give a unified approach to the well-posedness of linear integro-differential equations of hyperbolic and parabolic type, i.e. equations of the form

\[ \ddot{u}(t) + \int_{-\infty}^{t} g(t-s) \dot{u}(s) \, ds + A^* Au(t) - \int_{-\infty}^{t} A^* h(t-s)Au(s) \, ds = f(t) \quad (t \in \mathbb{R}), \] (1)

and of the form

\[ \dot{u}(t) + \int_{-\infty}^{t} g(t-s) \dot{u}(s) \, ds + A^* Au(t) - \int_{-\infty}^{t} A^* h(t-s)Au(s) \, ds = f(t) \quad (t \in \mathbb{R}), \] (2)

respectively. In both cases \( A \) denotes a closed, densely defined linear operator on some Hilbert space, which is in applications a differential operator with respect to the spatial variables. These type of equations were treated by several authors, mostly assuming that the kernels \( g \) and \( h \) are scalar-valued, while we allow \( g \) and \( h \) to be operator-valued. If the kernels are absolutely continuous, the well-posedness can easily be shown. However, we focus on kernels, which are just integrable in some sense without assuming any kind of differentiability.

The theory of integro-differential equations has a long history and there exists a large amount of works by several authors and we just mention the monographs [21, 12] and the references therein for possible approaches. Topics like well-posedness and the asymptotic behaviour of solutions were studied by several authors, even for semi-linear versions of (1) or (2) (e.g. [5, 2, 4] for the hyperbolic and [1, 6, 3] for the parabolic case).

Our approach to deal with integro-differential equations invokes the framework of evolutionary equations, introduced by Picard in [18, 19]. The main idea is to rewrite the equations as problems of the form

\[ (\partial_0 M(\partial_0^{-1}) + A) U = F. \] (3)

Here \( \partial_0 \) denotes the time-derivative established as a boundedly invertible operator in a suitable exponentially weighted \( L_2 \)-space. The operator \( M(\partial_0^{-1}) \), called the linear material law, is a bounded operator in time and space and is defined as an analytic, operator-value function of \( \partial_0^{-1} \). The operator \( A \) is assumed to be skew-selfadjoint (this can be relaxed to the assumption that \( A \) is maximal monotone, see [25, 26]). As it was already mentioned in [15], the operator \( M(\partial_0^{-1}) \) can be a convolution with an operator-valued function and we will point out, which kind of linear material laws yield integro-differential equations. By the solution theory for equations of the form (3) (see [18] Solution Theory or Theorem 2.9 in this article) it suffices to show the strict positive definiteness of \( \Re \partial_0 M(\partial_0^{-1}) \) in order to obtain well-posedness of the problem. Besides existence, uniqueness and continuous dependence we obtain the causality of the respective solution operators, which enables us to treat initial value problems.

In Section 2 we recall the notion of linear material laws, evolutionary equations and we state the solution theory for this class of differential equations. Section 3 is devoted to the well-posedness of hyperbolic- and parabolic-type integro-differential equations. We will show how
2 Evolutionary Equations

to reformulate the problem as an evolutionary equation and state conditions for the involved kernels, which imply the positive definiteness of $\Re \partial_0 M(\partial_0^{-1})$ and therefore yield the well-posedness of the problems. Furthermore, in Subsection 3.1 we will briefly discuss a way how to treat initial value problems (see Remark 3.10) as well as problems where the whole history of the unknown is given (Remark 3.11). Finally, we apply our findings in Section 4 to the equations of visco-elasticity. This problem was also treated by Dafermos [8, 7], even for operator-valued kernels but under the stronger assumption that the kernels are absolutely continuous.

Throughout, every Hilbert space is assumed to be complex and the inner product, denoted by $\langle \cdot | \cdot \rangle_H$ is linear in the second and anti-linear in the first argument. Norms are usually denoted by $| \cdot |$ except the operator-norm, which we denote by $\| \cdot \|$.

2 Evolutionary Equations

In this section we recall the notion of evolutionary equations due to [18, 19, 20]. We begin to introduce the exponentially weighted $L_2$-space and the time-derivative $\partial_0$, established as a normal, boundedly invertible operator on this space. Using the spectral representation of this time-derivative operator, we define so called linear material laws as operator-valued $H^\infty$-functions of $\partial_0^{-1}$. In the second subsection we recall the solution theory for evolutionary equations [18, Solution Theory] and the notion of causality.

2.1 The Time-derivative and Linear Material Laws

Throughout let $\nu \in \mathbb{R}$. As in [20, 18, 21] we begin to introduce the exponentially weighted $L_2$-space $H_{\nu,0}$. We define the Hilbert space

$$H_{\nu,0}(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ measurable, } \int_{\mathbb{R}} |f(t)|e^{-2\nu t} \, dt < \infty \right\}$$

edowed with the inner-product

$$\langle f | g \rangle_{H_{\nu,0}} := \int_{\mathbb{R}} f(t)^* g(t) e^{-2\nu t} \, dt \quad (f, g \in H_{\nu,0}(\mathbb{R})).$$

**Definition 2.1.**

**Remark 2.2.** Obviously the operator

$$e^{-im} : H_{\nu,0}(\mathbb{R}) \to L_2(\mathbb{R})$$

defined by $(e^{-im} f)(t) = e^{-\nu t} f(t)$ for $t \in \mathbb{R}$ is unitary.

\footnote{For convenience we always identify the equivalence classes with respect to the equality almost everywhere with their respective representers.}
We define the derivative \( \partial \) on \( L_2(\mathbb{R}) \) as the closure of the operator

\[
\partial|_{C_c^\infty(\mathbb{R})} : C_c^\infty(\mathbb{R}) \subseteq L_2(\mathbb{R}) \to L_2(\mathbb{R}) \\
\phi \mapsto \phi',
\]

where \( C_c^\infty(\mathbb{R}) \) denotes the space of infinitely differentiable functions on \( \mathbb{R} \) with compact support. This operator is known to be skew-selfadjoint (see [30, p. 198, Example 3]) and its spectral representation is given by the Fourier-Transform \( \mathcal{F} \), which is given by

\[
(\mathcal{F}\phi)(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} \phi(s) \, ds \quad (t \in \mathbb{R})
\]

for functions \( \phi \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \), i.e., we have

\[
\partial = \mathcal{F}^*(im)\mathcal{F},
\]

where \( m : D(m) \subseteq L_2(\mathbb{R}) \to L_2(\mathbb{R}) \) denotes the multiplication-by-the-argument operator \( ((mf)(t) = tf(t)) \) with maximal domain \( D(m) \).

**Definition 2.3.** We define the operator \( \partial_\nu \) on \( H_{\nu,0}(\mathbb{R}) \) by

\[
\partial_\nu := (e^{-\nu m})^{-1} \partial e^{-\nu m}
\]

and obtain again a skew-selfadjoint operator. From [41] we immediately get

\[
\partial_\nu = (e^{-\nu m})^{-1} \mathcal{F}^*im\mathcal{F}e^{-\nu m},
\]

which yields the spectral representation for \( \partial_\nu \) by the so-called Fourier-Laplace-Transform \( \mathcal{L}_\nu := \mathcal{F}e^{-\nu m} : H_{\nu,0}(\mathbb{R}) \to L_2(\mathbb{R}) \).

An easy computation shows, that for \( \phi \in C_c^\infty(\mathbb{R}) \) we get \( \phi' = \partial_\nu \phi + \nu \phi \), which leads to the following definition.

**Definition 2.4.** We define the operator \( \partial_{0,\nu} := \partial_\nu + \nu \), the *time-derivative* on \( H_{\nu,0}(\mathbb{R}) \). If the choice of \( \nu \in \mathbb{R} \) is clear from the context we will write \( \partial_0 \) instead of \( \partial_{0,\nu} \).

**Remark 2.5.** Another way to introduce \( \partial_{0,\nu} \) is by taking the closure of the usual derivative of test-functions with respect to the topology in \( H_{\nu,0}(\mathbb{R}) \), i.e.

\[
\partial_{0,\nu} = \overline{\partial|_{C_c^\infty(\mathbb{R})}}_{H_{\nu,0}(\mathbb{R})} \cap H_{\nu,0}(\mathbb{R}).
\]

We state some properties of the derivative \( \partial_{0,\nu} \) and refer to [15, 20] for the proofs.

**Proposition 2.6.** Let \( \nu > 0 \). Then the following statements hold:

(a) The operator \( \partial_{0,\nu} \) is normal and \( 0 \in \sigma(\partial_{0,\nu}) \) with \( \|\partial_{0,\nu}^{-1}\| = \frac{1}{\nu} \).

(b) \( \partial_{0,\nu} = \mathcal{L}_\nu^*(im + \nu)\mathcal{L}_\nu \).
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(c) For \( u \in H_{\nu,0}(\mathbb{R}) \) we have \( (\partial_{0,\nu}^{-1}u)(t) = \int_{-\infty}^{t} u(s) \, ds \) for almost every \( t \in \mathbb{R} \).\(^2\)

Of course, the operator \( \partial_{0,\nu} \) can be lifted in the canonical way to Hilbert space-valued functions and for convenience we will use the same notation for the derivative on scalar-valued and on Hilbert space-valued functions. The space of Hilbert space-valued functions, which are square-integrable with respect to the exponentially weighted Lebesgue measure will be denoted by \( H_{\nu,0}(\mathbb{R}; H) \) for \( \nu \in \mathbb{R} \). Using the spectral representation for the inverse time-derivative \( \partial_{0,\nu}^{-1} \) for \( \nu > 0 \), we introduce linear material laws as follows.

**Definition 2.7.** For \( r > 0 \) let \( M : B_{C}(r,r) \rightarrow L(H) \) be a bounded, analytic function. Then we define the bounded linear operator

\[
M \left( \frac{1}{im + \nu} \right) : L_{2}(\mathbb{R}; H) \rightarrow L_{2}(\mathbb{R}; H)
\]

for \( \nu > \frac{1}{2r} \) by \( M \left( \frac{1}{im + \nu} \right) f \) \( f \) \( t \). linear material law \( M(\partial_{0,\nu}^{-1}) \) by

\[
M(\partial_{0,\nu}^{-1}) := L_{\nu} \left( \frac{1}{im + \nu} \right) \quad \text{for } \nu \geq 0\]

Note that the operator \( M(\partial_{0,\nu}^{-1}) \), as a function of \( \partial_{0,\nu}^{-1} \), commutes with the derivative \( \partial_{0,\nu} \), in the sense that \( \partial_{0,\nu} M(\partial_{0,\nu}^{-1}) \geq M(\partial_{0,\nu}^{-1}) \partial_{0,\nu} \).

**Remark 2.8.** The assumed analyticity of the mapping \( M \) is needed to ensure the *causality* (see Theorem 2.9) of the operator \( M(\partial_{0,\nu}^{-1}) \) using a Paley-Wiener-type result (cf. [23]).

### 2.2 Well-posedness and Causality of Evolutionary Equations

In [18] the following type of a differential equation was considered:

\[
\left( \partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A \right) u = f,
\]

where \( A : D(A) \subseteq H \rightarrow H \) is a skew-selfadjoint operator, \( f \in H_{\nu,0}(\mathbb{R}; H) \) is an arbitrary source term and \( u \in H_{\nu,0}(\mathbb{R}; H) \) is the unknown. For this class of problems the following solution theory was established.

**Theorem 2.9 ([18] Solution Theory).** Let \( A : D(A) \subseteq H \rightarrow H \) be a skew-selfadjoint operator and let \( M : B_{C}(r,r) \rightarrow L(H) \) be analytic, bounded and such that there exists \( c > 0 \) such that for all \( z \in B_{C}(r,r) \) the following holds

\[
\Re z^{-1} M(z) \geq c.
\]

Then for each \( \nu > \frac{1}{2r} \) the operator \( \left( \partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A \right) \) is boundedly invertible as an operator on \( H_{\nu,0}(\mathbb{R}; H) \) and the inverse is causal, i.e.

\[
\chi_{\mathbb{R}_{\leq a}}(m) \left( \partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A \right)^{-1} \chi_{\mathbb{R}_{\leq a}}(m) = \chi_{\mathbb{R}_{\leq a}}(m) \left( \partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A \right)^{-1}
\]

\(^2\)This shows, that for \( \nu > 0 \) the operator \( \partial_{0,\nu}^{-1} \) is causal, while for \( \nu < 0 \) we get the anti-causal operator given by \( \partial_{0,\nu}^{-1} u = - \int_{-\infty}^{t} u(s) \, ds \) (see [18]).
for each $a \in \mathbb{R}$.\footnote{Here we denote by $\chi_{\mathbb{R} \geq m}$ the cut-off operator given by $(\chi_{\mathbb{R} \geq m} f)(t) = \chi_{\mathbb{R} \geq m}(t) f(t)$.}

This means that under the hypotheses of Theorem 2.9 Problem (5) is well-posed, i.e. the uniqueness, existence and continuous dependence on the data $f$ of a solution $u$ is guaranteed. However, (5) just holds in the sense of

$$(\partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A) u = f,$$

where the closure of the operator is taken with respect to the topology on $H_{\nu,0}(\mathbb{R}; H)$. To avoid the closure, one can use the concept of extrapolation spaces, so-called Sobolev-chains with respect to the operator $A + 1$ and $\partial_{0,\nu}$ (see \cite{17, 20 Chapter 2}). In this context Equation (5) holds in the space $H_{\nu,-1}(\mathbb{R}; H_{-1}(A + 1))$, where we denote by $(H_{\nu,k}(\mathbb{R}))_{k \in \mathbb{Z}}$ the Sobolev-chain associated to $\partial_{0,\nu}$. Using that $M(\partial_{0,\nu}^{-1})$ and $A$ commute with $\partial_{0,\nu}$, one derives the following corollary from Theorem 2.9.

**Corollary 2.10.** Under the conditions of Theorem 2.9, the solution operator $\left(\partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A\right)^{-1}$ extends to a bounded linear operator on $H_{\nu,k}(\mathbb{R}; H)$ for each $k \in \mathbb{Z}$.

**Remark 2.11** (\cite{20 Chapter 6], \cite{28 Theorem 1.4.2}). The solution theory is independent of the particular choice of $\nu > \frac{1}{2r}$ in the sense that for right-hand sides $f \in H_{\nu,k}(\mathbb{R}; H) \cap H_{\mu,k}(\mathbb{R}; H)$ for $\mu, \nu > \frac{1}{2r}$, $k \in \mathbb{Z}$ we have

$$\left(\partial_{0,\nu} M(\partial_{0,\nu}^{-1}) + A\right)^{-1} f = \left(\partial_{0,\mu} M(\partial_{0,\mu}^{-1}) + A\right)^{-1} f.$$

## 3 Integro-Differential Equations

In this section we introduce an abstract type of integro-differential equations with operator-valued kernels, which covers hyperbolic- and parabolic-type equations. This abstract type allows to treat convolutions with the unknown as well as with the derivatives (with respect to time and space) of the unknown. We introduce the space $L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H))$ for $\mu \in \mathbb{R}$ as the space of weakly measurable functions $B : \mathbb{R}_{\geq 0} \rightarrow L(H)$ (i.e. for every $x, y \in H$ the function $t \mapsto (B(t)x|y)$ is measurable) such that the function $t \mapsto \|B(t)\|$ is measurable\footnote{If $H$ is separable, then the weak measurability implies the measurability of $t \mapsto \|B(t)\|$.} and

$$|B|_{L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H))} := \int_{\mathbb{R}_{\geq 0}} e^{-\mu t} \|B(t)\| \, dt < \infty.$$ 

Note that $L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H)) \hookrightarrow L_{1,\nu}(\mathbb{R}_{\geq 0}; L(H))$ for $\mu \leq \nu$. For a function $B \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H))$ we can establish the Fourier-transform of $B$ as a function on the lower half-plane $[\mathbb{R}]_{-1}[\mathbb{R}_{\geq \mu}] := \{t - i\nu \mid t \in \mathbb{R}, \nu \geq \mu\}$ by defining

$$\hat{B}(t - i\nu)x|y) := \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-is} e^{-\nu s} (B(s)x|y) \, ds \quad (t \in \mathbb{R}, \nu \geq \mu)$$
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for \(x, y \in H\). Obviously the function \(t - i\nu \mapsto \hat{B}(t - i\nu)\) is bounded on \([\mathbb{R}] - i[\mathbb{R} \geq \mu]\) with values in the bounded operators on \(H\) and satisfies

\[
|\hat{B}|_{L_{\infty}([\mathbb{R}] - i[\mathbb{R} \geq \mu]; L(H))} \leq \frac{1}{\sqrt{2\pi}} |B|_{L_{1, \mu}(\mathbb{R} \geq 0; L(H))}.
\]

Moreover it is analytic on the open half plane \([\mathbb{R}] - i[\mathbb{R} \geq \mu]\). For \(B \in L_{1, \mu}(\mathbb{R} \geq 0; L(H))\) we define the convolution operator as follows.

**Lemma 3.1.** Let \(B \in L_{1, \mu}(\mathbb{R} \geq 0; L(H))\) for some \(\mu \in \mathbb{R}\). We denote by \(S(\mathbb{R}; H)\) the space of simple functions on \(\mathbb{R}\) with values in \(H\). Then for each \(\nu \geq \mu\) the convolution operator

\[
B^* : S(\mathbb{R}; H) \subseteq H_{\nu,0}(\mathbb{R}; H) \to H_{\nu,0}(\mathbb{R}; H)
\]

is bounded with \(\|B^*\|_{L(H_{\nu,0}(\mathbb{R}; H))} \leq |B|_{L_{1, \nu}(\mathbb{R} \geq 0; L(H))}\). Hence, it can be extended to a bounded linear operator on \(H_{\nu,0}(\mathbb{R}; H)\).

**Proof.** Let \(\nu \geq \mu\). Then we estimate for \(u \in S(\mathbb{R}; H)\) using Young’s inequality

\[
\int_{\mathbb{R}} e^{-2\nu t} |(B^* u)(t)|^2 \, dt \leq \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-\nu (t-s)} \|B(t-s)\| e^{-\nu s} |u(s)| \, ds \right)^2 \, dt
\]

\[
\leq \left( \int_{\mathbb{R}} e^{-\nu t} \|B(t)\| \, dt \right)^2 \int_{\mathbb{R}} |u(t)|^2 e^{-2\nu t} \, dt,
\]

which yields \(B^* u \in H_{\nu,0}(\mathbb{R}; H)\) and

\[
|B^* u|_{H_{\nu,0}(\mathbb{R}; H)} \leq \int_0^\infty e^{-\nu t} \|B(t)\| \, dt |u|_{H_{\nu,0}(\mathbb{R}; H)}.
\]

This completes the proof.

**Remark 3.2.** Note that since \(B^*\) commutes with \(\partial_{0, \nu}\), we can extend \(B^*\) to a bounded linear operator on \(H_{\nu, k}(\mathbb{R}; H)\) for each \(k \in \mathbb{Z}\).

**Corollary 3.3.** Let \(B \in L_{1, \mu}(\mathbb{R} \geq 0; L(H))\). Then \(\lim_{\nu \to \infty} |B|_{L_{1, \nu}(\mathbb{R} \geq 0; L(H))} = 0\) and thus, in particular \(\lim_{\nu \to \infty} \|B^*\|_{L(H_{\nu,0}(\mathbb{R}; H))} = 0\).

**Proof.** This is an immediate consequence of Lemma 3.1 and the theorem of monotone convergence.

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Note that scalar analyticity on a norming set and local boundedness is equivalent to analyticity (see [13, Theorem 3.10.1]).

The integral is defined in the weak sense.
Lemma 3.4. Let $B \in L_{1,\mu}(\mathbb{R}; L(H))$ for some $\mu \geq 0$ and $u \in H_{\nu,0}(\mathbb{R}; H)$ for $\nu \geq \mu$. Then
\[(L_\nu(B \ast u))(t) = \sqrt{2\pi}B(t - i\nu) (L_\nu u(t))\]
for almost every $t \in \mathbb{R}$.

Proof. The proof is a classical computation using Fubini’s Theorem and we omit it. \qed

From now on let $H_0, H_1$ be complex Hilbert spaces, $A : D(A) \subseteq H_0 \to H_1$ be a densely defined closed linear operator, $B_1, B_2, C_1, C_2 \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H_0))$ and $B_3, B_4, C_3, C_4 \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H_1))$ for some $\mu \geq 0$. We consider linear material laws of the form
\[M(z) = \begin{pmatrix} Q_1 \left( \hat{C}_1(-iz^{-1}) \right)^{-1} & P_1 \left( \hat{B}_1(-iz^{-1}) \right) \\ 0 & Q_2 \left( \hat{C}_2(-iz^{-1}) \right)^{-1} P_2 \left( \hat{B}_2(-iz^{-1}) \right) \\ 0 & 0 & Q_3 \left( \hat{C}_3(-iz^{-1}) \right)^{-1} P_3 \left( \hat{B}_3(-iz^{-1}) \right) \end{pmatrix} + z \begin{pmatrix} z \end{pmatrix} \]
for $z \in \mathbb{C} \left( \frac{1}{2\pi}, \frac{1}{2\pi} \right)$, where $P_i, Q_i$ are complex affine linear functions, such that $Q_i \left( \hat{C}_i(-iz^{-1}) \right)$ is boundedly invertible for $z \in \mathbb{C} \left( \frac{1}{2\pi}, \frac{1}{2\pi} \right)$ and $z \mapsto Q_i \left( \hat{C}_i(-iz^{-1}) \right)^{-1} P_i \left( \hat{B}_i(-iz^{-1}) \right)$ is bounded for every $i \in \{1, \ldots, 4\}$. Under these assumptions $M$ defines a linear material law and we consider evolutionary problems of the form
\[\left( \partial_{t,\nu} M \left( \partial_{t,\nu}^{-1} \right) + \begin{pmatrix} 0 \\ -A \end{pmatrix} \right) U = F.\]  
(7)

According to Theorem 2.5 it suffices to check the solvability condition (9), which will be done in concrete cases in the forthcoming subsections. Throughout we set $H := H_0 \oplus H_1$.

### 3.1 Hyperbolic-type equations

In this subsection we consider the case $P_3, P_4 = 0$, $P_1(x) = 1 + \sqrt{2\pi}x$, $Q_1, Q_2 = 1$ and $Q_2(x) = 1 - \sqrt{2\pi}x$. Thus the material law reads as follows
\[M(z) = \begin{pmatrix} 1 + \sqrt{2\pi}\hat{C}(-iz^{-1}) & 0 \\ 0 & (1 - \sqrt{2\pi}\hat{B}(-iz^{-1}))^{-1} \end{pmatrix} \quad (z \in \mathbb{C}(r, r)) \]
(8)
for some $C \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H_0)), B \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H_1))$, where $\mu \geq 0$. According to Corollary 3.3 there exists $\mu_0 > \mu$, such that $\|B \ast \| L_{\nu}(H_{\nu,0}(\mathbb{R}; H_1)) < 1$ for each $\nu \geq \mu_0$. To ensure that the function $M$ defines a linear material law, we choose $r := \frac{1}{2\mu_0}$. Note that the system (7) with $M$ given by (8) and $F = \begin{pmatrix} f \\ g \end{pmatrix} \in H_{\nu,0}(\mathbb{R}; H)$ reads as
\[\left( \begin{pmatrix} 1 + C \ast & 0 \\ 0 & (1 - B \ast)^{-1} \end{pmatrix} \right) \left( \begin{pmatrix} v \\ q \end{pmatrix} \right) = \begin{pmatrix} f \\ g \end{pmatrix}.\]  
(9)
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In the special case that \( g = 0 \) we obtain
\[
\partial_0(1 - B^*)^{-1}q = Av
\]
or, equivalently,
\[
q = \partial_0^{-1}(1 - B^*)Av.
\]
If we plug this representation of \( q \) into the first line of Equation (9) we get
\[
\partial_0(1 + C^*)v + A^* \partial_0^{-1}(1 - B^*)Av = f
\]
which gives, by defining \( u := \partial_0^{-1}v \)
\[
\partial_0^2(1 + C^*)u + A^*(1 - B^*)Au = f.
\]
A semi-linear version of this equation was treated in [4] for scalar-valued kernels, where criteria for the well-posedness and the exponential stability were given. Also in [22] this type of equation was treated for scalar-valued kernels and besides well-posedness, the polynomial stability was addressed. In both works the well-posedness (the existence and uniqueness of mild solutions) was shown under certain conditions on the kernel by techniques developed for evolutionary integral equations (see [21]). We will show that the assumptions on the kernels made in both articles can be weakened such that the well-posedness of the problem can still be shown, even for operator-valued kernels.

To ensure the well-posedness of (9), we have to guarantee that there exist \( r_1, c > 0 \) with \( r_1 \leq r \) such that for all \( z \in B_C(r_1, r_1) \):
\[
\Re z^{-1}(1 - \sqrt{2\pi \hat{B}(-iz^{-1})})^{-1} \geq c \quad (10)
\]
and
\[
\Re z^{-1}(1 + \sqrt{2\pi \hat{C}(-iz^{-1})}) \geq c. \quad (11)
\]

Remark 3.5. One standard assumption for scalar-valued kernels is absolute continuity. In our case this means that there exists a function \( G \in L^1_{\mu}(\mathbb{R} \geq 0; L(H_1)) \) for some \( \mu \geq 0 \) such that
\[
B(t) = \int_0^t G(s) \, ds + B(0) \quad (t \in \mathbb{R} \geq 0)
\]
for the kernel \( B \).

For simplicity let us assume \( C = 0 \). Note that due to the absolute continuity, \( B \) is an element of \( L^1_{\nu}(\mathbb{R} \geq 0; L(H_1)) \) for each \( \nu > \mu \) and we choose \( \nu \) large enough, such that \(|B|_{L^1_{\nu}(\mathbb{R} \geq 0; L(H_1))} < 1 \). In this case (10) can be easily verified. Using the Neumann-series we obtain
\[
z^{-1}(1 - \sqrt{2\pi \hat{B}(-iz^{-1})})^{-1} = z^{-1} \sum_{k=0}^{\infty} \left( \sqrt{2\pi \hat{B}(-iz^{-1})} \right)^k
\]
\[
= z^{-1} + z^{-1} \sqrt{2\pi \hat{B}(-iz^{-1})} \sum_{k=0}^{\infty} \left( \sqrt{2\pi \hat{B}(-iz^{-1})} \right)^k
\]
for $z \in B_C(\frac{1}{2\nu}, \frac{1}{2\nu})$. The Fourier-transform of $B$ can be computed by

$$\hat{B}(-iz^{-1}) = z \left( \hat{G}(-iz^{-1}) + \frac{1}{\sqrt{2\pi}} B(0) \right)$$

and hence we can estimate

$$\Re z^{-1}(1 - \sqrt{2\pi} \hat{B}(-iz^{-1}))^{-1} = \Re z^{-1} + \Re \left( \sqrt{2\pi} \hat{G}(-iz^{-1}) + B(0) \right) \sum_{k=0}^{\infty} (\sqrt{2\pi} \hat{B}(-iz^{-1}))^k$$

$$\geq \nu \frac{|\sqrt{2\pi} \hat{G}(-i(-1))|_{L^\infty(B_C(\frac{1}{2\nu}, \frac{1}{2\nu}); L(H_1))} + \|B(0)\|_{L(H_1)}}{1 - |\sqrt{2\pi} \hat{B}(-i(-1))|_{L^\infty(B_C(\frac{1}{2\nu}, \frac{1}{2\nu}); L(H_1))}}$$

$$\geq \nu - \frac{|G|_{L^1,\nu(R_{\geq 0}; L(H_1))} + \|B(0)\|_{L(H_1)}}{1 - |B|_{L^1,\nu(R_{\geq 0}; L(H_1))}}$$

for every $z \in B_C(\frac{1}{2\nu}, \frac{1}{2\nu})$. Since $\frac{|G|_{L^1,\nu(R_{\geq 0}; L(H_1))} + \|B(0)\|_{L(H_1)}}{1 - |B|_{L^1,\nu(R_{\geq 0}; L(H_1))}} \to \|B(0)\|_{L(H_1)}$ as $\nu \to \infty$, this yields the assertion.

In the case, when $C$ and $B$ are not assumed to be differentiable in a suitable sense, the conditions (10) and (11) are hard to verify. We now state some hypotheses for $B$ and $C$ and show in the remaining part of this subsection, that these conditions imply (10) and (11).

**Hypotheses.** Let $T \in L_{1,\mu}(R_{\geq 0}; L(G))$, where $G$ is an arbitrary Hilbert space and $\mu \geq 0$. Then $T$ satisfies the hypotheses (10), (11) and (13), respectively, if

(i) for all $t \in R_{\geq 0}$ the operator $T(t)$ is selfadjoint,

(ii) for all $s, t \in R_{\geq 0}$ the operators $T(t)$ and $T(s)$ commute,

(iii) there exists $\nu_0 \geq \mu$ such that for all $t \in R$

$$t \Im \hat{T}(t - i\nu_0) \leq 0.$$

**Remark 3.6.**

(a) If $T$ satisfies the hypothesis (10), then

$$\Im \hat{T}(t - i\nu_0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin(-ts)e^{-i\nu_0 s} G(s) \, ds = -\Im \hat{T}(t - i\nu_0) \quad (t \in R)$$

and thus (13) holds if and only if

$$\Im \hat{T}(t - i\nu_0) \leq 0 \quad (t \in R_{\geq 0}).$$

(b) Note that in [22] and [4] the kernel is assumed to be real-valued. Thus, (10) and (13) are trivially satisfied. In [22] we find the assumption, that the kernel should be non-increasing and non-negative, i.e., $T(s) \geq 0$ and $T(t) - T(s) \leq 0$ for each $t \geq s \geq 0$. Note that these
Lemma 3.7. Assume that
\[ (e^{it}T(t) - e^{-\nu t}T(s)) x | x \rangle = e^{-it} \langle (T(t) - T(s)) x | x \rangle + (e^{-it} - e^{-\nu t}) \langle T(s) x | x \rangle \leq 0 \]
for every \( t \geq s \geq 0, \nu \geq 0 \) and \( x \in G \). Hence, we estimate for \( t > 0 \) and \( x \in G \)
\[
(\Im \hat{T}(t - i\nu_0) x | x) \\
= \frac{1}{\sqrt{2\pi}} \int_0^\infty \sin(-ts) e^{-\nu_0 s} \langle T(s) x | x \rangle \ ds \\
= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \left( \int_{2k\pi}^{(2k+1)\pi} \sin(-ts) e^{-\nu_0 s} \langle T(s) x | x \rangle \ ds + \int_{(2k+1)\pi}^{2(k+1)\pi} \sin(-ts) e^{-\nu_0 s} \langle T(s) x | x \rangle \ ds \right) \\
= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \left( \int_{2k\pi}^{(2k+1)\pi} \sin(-ts) e^{-\nu_0 s} \langle T(s) x | x \rangle \ ds + \int_{2k\pi}^{2(k+1)\pi} \sin(-ts) e^{-\nu_0 s} \langle T(s) x | x \rangle \ ds \right) \\
= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \int_{2k\pi}^{2(k+1)\pi} \sin(-ts) \langle T(s) x | x \rangle \ ds \\
= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \int_{2k\pi}^{2(k+1)\pi} \sin(ts) \langle T(s) x | x \rangle \ ds \\
= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^\infty \int_{2k\pi}^{2(k+1)\pi} \sin(ts) | T(s) x |^2 \ ds \\
\leq 0,
\]
which yields \( (11) \) according to (a). The authors of \([4]\) assume that the integrated kernel defines a positive definite convolution operator on \( L_2(\mathbb{R}_{\geq 0}) \). However, according to \([4]\) Proposition 2.2 (a)], this condition also implies \( (11) \).

**Lemma 3.7.** Assume that \( T \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(G)) \) satisfies the hypotheses \([1]\) and \( (11) \). Then we have for all \( \nu \geq \nu_0 \) and \( t \in \mathbb{R} \)
\[
t \Im \hat{T}(t - i\nu) \leq 0.
\]

**Proof.** Let \( x \in G \) and \( \nu \geq \mu \). We define the function
\[
f(t) := \langle T(t) x | x \rangle \quad (t \in \mathbb{R})
\]
which is real-valued, due to the selfadjointness of \( T(t) \) and we estimate
\[
\int_{\mathbb{R}} |f(t)| e^{-\mu t} \ dt \leq \int_{\mathbb{R}} ||T(t)|| e^{-\mu t} \ dt |x|^2
\]
which shows \( f \in L_{1,\mu}(\mathbb{R}_{>0}) \). We observe that
\[
\langle \hat{T}(t - iv)x|x \rangle = \left\langle \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{-its} e^{-\nu s} T(s) \, ds \, x \right\rangle x \\
= \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{its} e^{-\nu s} \langle T(s)x|x \rangle \, ds \\
= \hat{f}(-t - iv)
\]
for each \( t \in \mathbb{R}, \nu \geq \mu \). Hence, by
\[
\langle \mathfrak{Im} \hat{T}(t - iv)x|x \rangle = \mathfrak{Im} \langle \hat{T}(t - iv)x|x \rangle = \mathfrak{Im} \hat{f}(-t - iv)
\]
it suffices to prove \( t \mathfrak{Im} \hat{f}(-t - iv) \leq 0 \) for \( \nu \geq \nu_0 \), \( t \in \mathbb{R} \) under the condition that \( t \mathfrak{Im} \hat{f}(-t - iv_0) \leq 0 \) for each \( t \in \mathbb{R} \). For this purpose we follow the strategy in [3 Lemma 3.4] and employ the Poisson formula for the half plain (see [24 p. 149]) in order to compute the values of the harmonic function \( \mathfrak{Im} \hat{f} : [\mathbb{R}] - i [\mathbb{R}_{\geq \mu}] \rightarrow \mathbb{R} \). This gives, using \( \mathfrak{Im} \hat{f}(-s - iv) = -\mathfrak{Im} \hat{f}(s - iv) \)
\[
\mathfrak{Im} \hat{f}(-t - iv) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\nu - \nu_0}{(t - s)^2 + (\nu - \nu_0)^2} \mathfrak{Im} \hat{f}(s - iv_0) \, ds \\
= \frac{\nu - \nu_0}{\pi} \left( \int_{0}^{\infty} \frac{1}{(t + s)^2 + (\nu - \nu_0)^2} \mathfrak{Im} \hat{f}(s - iv_0) \, ds + \right. \\
+ \int_{0}^{\infty} \frac{1}{(t - s)^2 + (\nu - \nu_0)^2} \mathfrak{Im} \hat{f}(s - iv_0) \, ds \Bigg) \\
= \frac{\nu - \nu_0}{\pi} \int_{0}^{\infty} \left( \frac{1}{(t - s)^2 + (\nu - \nu_0)^2} - \frac{1}{(t + s)^2 + (\nu - \nu_0)^2} \right) \mathfrak{Im} \hat{f}(s - iv_0) \, ds \\
= 4t \frac{\nu - \nu_0}{\pi} \int_{0}^{\infty} \left( \frac{s}{((t - s)^2 + (\nu - \nu_0)^2)((t + s)^2 + (\nu - \nu_0)^2)} \right) \mathfrak{Im} \hat{f}(s - iv_0) \, ds,
\]
which implies
\[
t \mathfrak{Im} \hat{f}(-t - iv) \\
= 4t^2 \frac{\nu - \nu_0}{\pi} \int_{0}^{\infty} \left( \frac{s}{((t - s)^2 + (\nu - \nu_0)^2)((t + s)^2 + (\nu - \nu_0)^2)} \right) \mathfrak{Im} \hat{f}(s - iv_0) \, ds \\
\leq 0.
\]

\[\square\]

**Lemma 3.8.** Let \( B \) satisfy the hypotheses \([\text{H}_1], [\text{H}_2],[\text{H}_3]\). Then there exists \( 0 < r_1 \leq r \) such that for all \( z \in B \subset (r_1, r_1) \) the condition \([\text{H}_4]\) is satisfied.
Proof. Let \(x \in H_1\) and set \(r_1 := \min \left\{ \frac{1}{2\nu_0}, r_1 \right\}\). Let \(z \in B_C(r_1, r_1)\) and note that \(z^{-1} = it + \nu\) for some \(t \in \mathbb{R}, \nu > \nu_0\). Since the operator \(1 - \sqrt{2\pi} \hat{B}(t - i\nu)\) is bounded and boundedly invertible, so is its adjoint, which is given by \(1 - \sqrt{2\pi} \hat{B}(-t - i\nu)\) since \(B(s)\) is selfadjoint for each \(s \in \mathbb{R}\). We compute

\[
\Re((it+\nu)(1-\sqrt{2\pi} \hat{B}(t-i\nu))^{-1}x|x) = \Re((it+\nu)(1-\sqrt{2\pi} \hat{B}(t-i\nu))^2(1-\sqrt{2\pi} \hat{B}(-t-i\nu))x|x).
\]

We define the operator \(C := |1 - \sqrt{2\pi} \hat{B}(t - i\nu)|^{-1}\). Furthermore, note that due to the assumption that the operators pairwise commute, we have that the operators \(\hat{B}(\cdot)\) commute, too. This especially implies, that \(\hat{B}(t - i\nu)\) is normal and hence \(C\) and \(1 - \sqrt{2\pi} \hat{B}(-t - i\nu)\) commute. Thus, we can estimate the real part by

\[
\Re((it+\nu)C^2(1 - \sqrt{2\pi} \hat{B}(-t - i\nu))x|x) = \Re((it+\nu)(1 - \sqrt{2\pi} \hat{B}(t-i\nu))C|x|Cx) = \nu(1 - \sqrt{2\pi} \Re \hat{B}(-t - i\nu)) C|x|Cx + t(\sqrt{2\pi} \Im \hat{B}(-t - i\nu)C|x|Cx \\
\geq \nu(1 - \|\sqrt{2\pi} \hat{B}(-t - i\nu)\|)^2 - \sqrt{2\pi} t \Im \hat{B}(t - i\nu)C|x|Cx) \\
\geq \nu(1 - |B|_{L_{1,\nu}}(\mathbb{R}_{\geq 0}; L(H_1)))|Cx|^2,
\]

where we have used Lemma 3.7. Using now the inequality

\[
|x| = |C^{-1}Cx| = \left| (1 - \sqrt{2\pi} \hat{B}(t - i\nu)) Cx \right| \leq (1 + |B|_{L_{1,\nu}}(\mathbb{R}_{\geq 0}; L(H_1)))|Cx|
\]

we arrive at

\[
\Re((it+\nu)C^2(1 - \sqrt{2\pi} \hat{B}(-t - i\nu))x|x) \geq \nu \frac{1 - |B|_{L_{1,\nu}}(\mathbb{R}_{\geq 0}; L(H_1))}{1 + |B|_{L_{1,\nu}}(\mathbb{R}_{\geq 0}; L(H_1))} |x|^2 \\
\geq \nu_0 \frac{1 - |B|_{L_{1,\nu}}(\mathbb{R}_{\geq 0}; L(H_1))}{1 + |B|_{L_{1,\nu}}(\mathbb{R}_{\geq 0}; L(H_1))} |x|^2,
\]

which shows the assertion.

After these preparations we can state our main theorem.

**Theorem 3.9** (Solution theory for hyperbolic-type integro-differential equations). Let \(A : D(A) \subseteq H_0 \rightarrow H_1\) be a densely defined closed linear operator and \(C \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H_0))\), \(B \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H_1))\) for some \(\mu \geq 0\). Assume that \(C\) and \(B\) are absolutely continuous or that \(B\) satisfies the hypotheses (ii) (iii) and \(C\) satisfies (i) and (ii). Then the problem (3) is well-posed in the sense of Theorem 2.9.

Proof. In case of absolute continuity of \(B\) or \(C\) the positive definiteness condition is satisfied according to Remark 3.5. If \(B\) satisfies (ii) (iii) then Lemma 3.8 yields (ii). If \(C\) satisfies (i)
and \(\text{then we estimate}\)
\[
\Re (it + \nu) \left( 1 + \sqrt{2\pi} \hat{C}(t - iv) \right) = \nu \left( 1 + \sqrt{2\pi} \Re \hat{C}(t - iv) \right) - \sqrt{2\pi} \Re \hat{C}(t - iv) \geq \nu \left( 1 - |C|_{L^1,\nu(\mathbb{R}_0; L(H))} \right)
\]
for each \(t \in \mathbb{R}\) and \(\nu \geq \nu_0\), where we have used Lemma 3.7. Using Corollary 3.3 this yields the assertion. 

In applications it turns out that Equation (11) is just assumed to hold for positive times, i.e. on \(\mathbb{R}_{>0}\) and the equation is completed by initial conditions. So for instance, we can require that the unknowns \(v\) and \(q\) are supported on the positive real line and attain some given initial values at time \(0\). Then we arrive at a usual initial value problem. Since, due to the convolution with \(B\) and \(C\) the history of \(v\) and \(q\) has an influence on the equation for positive times, we can, instead of requiring an initial value at \(0\), prescribe the values of \(v\) and \(q\) on the whole negative real-line. This is a standard problem in delay-equations and it is usually treated by introducing so-called history-spaces (see e.g. [13, 9]). However, following the idea of [15] we can treat this kind of equations as a problem of the form (9) with a modified right-hand side. Let us treat the case of classical initial value problems first.

**Remark 3.10 (Initial value problem).** For \((f, g) \in H_{\nu,0}(\mathbb{R}; H)\) with supp \(f\), supp \(g \subseteq \mathbb{R}_{\geq 0}\) we consider the differential equation
\[
\left( \partial_0 \left( 1 + C^* \begin{array}{cc} 0 & 0 \\ 0 & (1-B^*)^{-1} \end{array} \right) + \left( \begin{array}{cc} 0 & A^* \\ -A & 0 \end{array} \right) \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}
\]
on \(\mathbb{R}_{>0}\) completed by initial conditions of the form
\[
v(0^+) = v(0) \in D(A)\) and \(q(0^+) = q(0) \in D(A^*)\).

We assume that the solvability conditions (10) and (11) are fulfilled. Assume that a pair \((v, q) \in \chi_{\mathbb{R}_{>0}}(m_0)[H_{\nu,1}(\mathbb{R}; H)]\) solves this problem. Then we get
\[
\left( \partial_0 \left( 1 + C^* \begin{array}{cc} 0 & 0 \\ 0 & (1-B^*)^{-1} \end{array} \right) + \left( \begin{array}{cc} 0 & A^* \\ -A & 0 \end{array} \right) \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}
\]
on \(\mathbb{R}\), which is equivalent to
\[
\left( \partial_0 \left( 1 + C^* \begin{array}{cc} 0 & 0 \\ 0 & (1-B^*)^{-1} \end{array} \right) + \left( \begin{array}{cc} 0 & A^* \\ -A & 0 \end{array} \right) \right) \begin{pmatrix} v \\ q \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} + \left( 1 + C^* \begin{array}{cc} 0 & 0 \\ 0 & (1-B^*)^{-1} \end{array} \right) \delta \otimes \begin{pmatrix} v(0) \\ q(0) \end{pmatrix}.
\]
(12)

We claim that this equation is the proper formulation of the initial value problem in our framework. According to Corollary 2.10 this equation admits a unique solution \((v, q) \in H_{\nu,-1}(\mathbb{R}; H)\) and due to the causality of the solution operator we get supp \(v\), supp \(q \subseteq \mathbb{R}_{\geq 0}\). We derive from (12) that
\[
\left( \partial_0 \left( 1 + C^* \begin{array}{cc} 0 & 0 \\ 0 & (1-B^*)^{-1} \end{array} \right) + \left( \begin{array}{cc} 0 & A^* \\ -A & 0 \end{array} \right) \right) \begin{pmatrix} v - \chi_{\mathbb{R}_{>0}} \otimes v(0) \\ q - \chi_{\mathbb{R}_{>0}} \otimes q(0) \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} - \chi_{\mathbb{R}_{>0}} \otimes \begin{pmatrix} A^* q(0) \\ -A v(0) \end{pmatrix}
\]
\(\text{This means that we find a pair} (w, p) \in H_{\nu,1}(\mathbb{R}; H)\) that coincides with \((v, q)\) for positive times.
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which gives \( \left( \frac{v - \chi_{\mathbb{R}_{>0}} \otimes v^{(0)}}{q - \chi_{\mathbb{R}_{>0}} \otimes q^{(0)}} \right) \in H_{v,0}(\mathbb{R}; H) \). However, we also get that

\[
\partial_{b} \left( \begin{array}{cc}
1 + C^{*} & 0 \\
0 & (1 - B^{*})^{-1}
\end{array} \right) \left( \begin{array}{c}
v - \chi_{\mathbb{R}_{>0}} \otimes v^{(0)} \\
q - \chi_{\mathbb{R}_{>0}} \otimes q^{(0)}
\end{array} \right) = \left( \begin{array}{c}
f \\
g
\end{array} \right) - \left( \begin{array}{cc}
0 & A^{*} \\
-A & 0
\end{array} \right) \left( \begin{array}{c}
v \\
q
\end{array} \right) \in H_{v,0}(\mathbb{R}; H_{-1}(|A^{*}| + i) \oplus H_{-1}(|A| + i)),
\]

and hence,

\[
\left( \begin{array}{c}
v - \chi_{\mathbb{R}_{>0}} \otimes v^{(0)} \\
q - \chi_{\mathbb{R}_{>0}} \otimes q^{(0)}
\end{array} \right) \in H_{v,1}(\mathbb{R}; H_{-1}(|A^{*}| + i) \oplus H_{-1}(|A| + i)).
\]

Using the Sobolev-embedding Theorem (see [20, Lemma 3.1.59] or [15, Lemma 5.2]) we obtain that \( \left( \begin{array}{c}
v - \chi_{\mathbb{R}_{>0}} \otimes v^{(0)} \\
q - \chi_{\mathbb{R}_{>0}} \otimes q^{(0)}
\end{array} \right) \) is continuous with values in \( H_{-1}(|A^{*}| + i) \oplus H_{-1}(|A| + i) \) and hence

\[
0 = \left( v - \chi_{\mathbb{R}_{>0}} \otimes v^{(0)} \right)(0-) = \left( v - \chi_{\mathbb{R}_{>0}} \otimes v^{(0)} \right)(0+)
\]
in \( H_{-1}(|A^{*}| + i) \) and thus

\[
v(0+) = v^{(0)} \text{ in } H_{-1}(|A^{*}| + i).
\]

Analogously we get

\[
q(0+) = q^{(0)} \text{ in } H_{-1}(|A| + i).
\]

**Remark 3.11** (Problems with prescribed history). For \( (f, g) \in H_{v,0}(\mathbb{R}; H) \) with supp \( f, \text{ supp } g \subseteq \mathbb{R}_{\geq 0} \) we again consider the equation

\[
\partial_{b} \left( \begin{array}{cc}
1 + C^{*} & 0 \\
0 & (1 - B^{*})^{-1}
\end{array} \right) + \left( \begin{array}{cc}
0 & A^{*} \\
-A & 0
\end{array} \right) \left( \begin{array}{c}
v \\
q
\end{array} \right) = \left( \begin{array}{c}
f \\
g
\end{array} \right) \quad (13)
\]

on \( \mathbb{R}_{>0} \) and the initial conditions

\[
v|_{\mathbb{R}_{<0}} = v(-\infty), \quad v(0+) = v(-\infty)(0-) \quad \text{and} \quad q|_{\mathbb{R}_{<0}} = q(-\infty), \quad q(0+) = q(-\infty)(0-).
\]

We assume that \( v(-\infty) \in H_{v,0}(\mathbb{R}; H_{0}) \) with supp \( v(-\infty) \subseteq \mathbb{R}_{\leq 0} \) and such that \( v(-\infty)(0-) \in D(A) \) and \( (1 + C^{*}) v(-\infty) \in \chi_{\mathbb{R}_{>0}}(m_{0})[H_{v,1}(\mathbb{R}; H_{0})] \) as well as \( q(-\infty) \in H_{v,0}(\mathbb{R}; H_{1}) \) with supp \( q(-\infty) \subseteq \mathbb{R}_{\leq 0}, \quad q(-\infty)(0-) \in D(A^{*}) \) and \( (1 - B^{*})^{-1} q(-\infty) \in \chi_{\mathbb{R}_{>0}}(m_{0})[H_{v,1}(\mathbb{R}; H_{1})] \). We want to determine an evolutionary equation for \( w := \chi_{\mathbb{R}_{>0}} v \) and \( p := \chi_{\mathbb{R}_{>0}} q \). We have that

\[
\left( \begin{array}{c}
f \\
g
\end{array} \right) = \chi_{\mathbb{R}_{>0}} \left( \begin{array}{cc}
1 + C^{*} & 0 \\
0 & (1 - B^{*})^{-1}
\end{array} \right) + \left( \begin{array}{cc}
0 & A^{*} \\
-A & 0
\end{array} \right) \left( \begin{array}{c}
v \\
q
\end{array} \right) = \chi_{\mathbb{R}_{>0}} \left( \begin{array}{cc}
1 + C^{*} & 0 \\
0 & (1 - B^{*})^{-1}
\end{array} \right) + \left( \begin{array}{cc}
0 & A^{*} \\
-A & 0
\end{array} \right) \left( \begin{array}{c}
w + v(-\infty) \\
p + q(-\infty)
\end{array} \right) = \chi_{\mathbb{R}_{>0}} \left( \begin{array}{cc}
1 + C^{*} & 0 \\
0 & (1 - B^{*})^{-1}
\end{array} \right) + \left( \begin{array}{cc}
0 & A^{*} \\
-A & 0
\end{array} \right) \left( \begin{array}{c}
w \\
p
\end{array} \right)
\]

\[\text{Note that } A \text{ and } A^{*} \text{ can be extended to bounded operators } A : H_{0} \rightarrow H_{-1}(|A^{*}| + i) \text{ and } A^{*} : H_{1} \rightarrow H_{-1}(|A| + i) \text{ respectively (cf. [20, Lemma 2.1.16]).}\]
3.2 Parabolic-type Equations

\[ + \chi_{\mathbb{R} > 0} \partial_0 \left( \frac{(1 + C_*) v_{(-\infty)}}{(1 - B_*)^{-1}} \right) . \]  

(14)

Hence, we arrive at the following equation for \((w, p)\):

\[ \chi_{\mathbb{R} > 0} \left( \partial_0 \left( \frac{1 + C_*}{0} \right) \left[ (1 - B_*)^{-1} \right] \right) \left( \begin{array}{c} w \\ p \end{array} \right) = \left( \begin{array}{c} f \\ g \end{array} \right) - \chi_{\mathbb{R} > 0} \partial_0 \left( \frac{(1 + C_*) v_{(-\infty)}}{(1 - B_*)^{-1} q_{(-\infty)}} \right) . \]

Note that we can omit the cut-off function on the left hand side due to the causality of the operators. The conditions \(v(0^+) = v_{(-\infty)}(0^-)\) and \(q(0^+) = q_{(-\infty)}(0^-)\) are now classical initial conditions for the unknowns \(w\) and \(p\). Hence, following Remark \(3.10\) we end up with the following evolutionary equation for \((w, p)\):

\[ \left( \begin{array}{c} \partial_0 \left( \frac{1 + C_*}{0} \right) \left[ (1 - B_*)^{-1} \right] \end{array} \right) \left( \begin{array}{c} w \\ p \end{array} \right) = \left( \begin{array}{c} f \\ g \end{array} \right) - \chi_{\mathbb{R} > 0} \partial_0 \left( \frac{(1 + C_*) v_{(-\infty)}}{(1 - B_*)^{-1} q_{(-\infty)}} \right) + \left( \begin{array}{c} 1 + C_* \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ (1 - B_*)^{-1} \end{array} \right) \delta \otimes \left( \begin{array}{c} v_{(-\infty)}(0-) \\ q_{(-\infty)}(0-) \end{array} \right) . \]

(15)

This equation possesses a unique solution in \(H_{\nu,-1}(\mathbb{R}; H)\) with \(\text{supp} w, \text{supp} p \subseteq \mathbb{R} > 0\) due to the causality of the solution operator. Like in Remark \(3.10\) we get that

\[ \left( \begin{array}{c} w - \chi_{\mathbb{R} > 0} \otimes v_{(-\infty)}(0-). \\ p - \chi_{\mathbb{R} > 0} \otimes q_{(-\infty)}(0-) \end{array} \right) \in H_{\nu,0}(\mathbb{R}; H) \]

from which we derive, using Equation \(15\), that

\[ w(0+) = v_{(-\infty)}(0-) \quad \text{and} \quad p(0+) = q_{(-\infty)}(0-) \]

in \(H_{-1}(|A| + i)\) and \(H_{-1}(|A^*| + i)\) respectively. We are now able to define the original solution by setting

\[ v := w + v_{(-\infty)} \in H_{\nu,0}(\mathbb{R}; H_0) \quad \text{and} \quad q = p + q_{(-\infty)} \in H_{\nu,0}(\mathbb{R}; H_1) . \]

Indeed, \(v\) and \(q\) satisfy the initial conditions by definition and on \(\mathbb{R} > 0\) the solution \((v, q)\) satisfies the differential equation \(15\) according to the computation done in \(14\).

3.2 Parabolic-type Equations

In this subsection we treat the case \(P_1(x) = 1 + \sqrt{2\pi x}, Q_1 = 1, P_2 = 0, P_3 = 0, P_4 = 1\) and \(Q_4(x) = 1 - \sqrt{2\pi x}\). Hence, we end up with a linear material law of the form

\[ M(z) = \left( \begin{array}{cccc} 1 + \sqrt{2\pi} \hat{C}(-iz^{-1}) & 0 \\ 0 & 1 - \sqrt{2\pi} \hat{B}(-iz^{-1}) \end{array} \right) \right) \quad (z \in \mathbb{C}^\ast) . \]

where \(B \in L_{1,\mu}(\mathbb{R} > 0; L(H_1))\) and \(C \in L_{1,\mu}(\mathbb{R} > 0; L(H_0))\) for some \(\mu > 0\) and \(r > 0\) is chosen suitably. The corresponding integro-differential equation is given by

\[ \left( \begin{array}{cccc} 1 + C_* & 0 \\ 0 & 1 - B_* \end{array} \right) \right) \left( \begin{array}{c} u \\ q \end{array} \right) = \left( \begin{array}{c} f \\ g \end{array} \right) , \]

(16)
which, in the case \( g = 0 \), yields the parabolic equation

\[
\partial_0 u + C \ast \partial_0 u + A^\ast Au - A^\ast (B \ast Au) = f.
\] (17)

This problem was already considered in [3] for scalar-valued kernels. As it turns out the solution theory for this kind of problem is quite easy in comparison to the solution theory for the hyperbolic case.

**Theorem 3.12** (Solution theory for parabolic-type integro-differential equations). Let \( A : D(A) \subseteq H_0 \to H_1 \) be a densely defined closed linear operator and \( B \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H_1)), C \in L_{1,\mu}(\mathbb{R}; L(H_0)) \) for some \( \mu \geq 0 \). Assume that \( C \) satisfies the hypotheses \( \text{(iii)} \) and \( \text{(iii)} \) (see Subsection 3.1). Then problem (16) is well-posed in the sense of Theorem 2.9.

**Proof.** To verify the solvability condition \( \text{(iii)} \), we have to show that the operators \( \Re \sqrt{2\pi B} (-iz^{-1}) \) and \( \Re (1 - \sqrt{2\pi B} (-iz^{-1}))^{-1} \) are uniformly strictly positive definite on some ball \( B_{\mathbb{C}}(r, r) \). The first term can be estimated as in Theorem 3.9. For the second term we use the Neumann-series and estimate

\[
\Re \left( 1 - \sqrt{2\pi B} (-iz^{-1}) \right)^{-1} = 1 + \Re \sqrt{2\pi B} (-iz^{-1}) \sum_{k=0}^{\infty} \left( \sqrt{2\pi B} (-iz^{-1}) \right)^k
\]

\[
\geq 1 - \frac{\sup_{z \in B_{\mathbb{C}}(r, r)} \| \sqrt{2\pi B} (-iz^{-1}) \|_{L(H_1)}}{1 - \sup_{z \in B_{\mathbb{C}}(r, r)} \| \sqrt{2\pi B} (-iz^{-1}) \|_{L(H_1)}} \rightarrow 1 \quad (r \to 0+),
\]

which yields the strict positive definiteness of the second term.

\[\square\]

**Remark 3.13.** In [27] the following kind of a parabolic-type integro-differential equation was considered:

\[
\partial_0 u + C \ast \partial_0 u + Au - B \ast Au = f,
\] (18)

where \( A : D(A) \subseteq H_0 \to H_0 \) is a selfadjoint strictly positive definite operator and \( B, C \in L_{1,\mu}(\mathbb{R}_{\geq 0}; L(H_0)) \). Equivalently we can consider the equation

\[
\partial_0 u + (1 + C \ast)^{-1} (1 - B \ast) Au = (1 + C \ast)^{-1} f.
\]

In this case the well-posedness can be shown, without imposing additional hypotheses on \( C \). First we write the problem in the form given in [17]. For doing so, consider the operator \( A : D(A) \subseteq H_1(\sqrt{A}) \to H_0 \). We compute the adjoint of this operator. First observe that for \( g \in H_1(\sqrt{A}) \) we get

\[
\langle g | Af \rangle_{H_0} = \langle \sqrt{A}g | \sqrt{A}f \rangle_{H_0} = \langle g | f \rangle_{H_1(\sqrt{A})}
\]

for each \( f \in D(A) \) and thus \( g \in D(A^\ast) \). Furthermore, if \( g \in D(A^\ast) \) there exists \( h \in H_1(\sqrt{A}) \) such that for all \( f \in D(A) \)

\[
\langle g | Af \rangle_{H_0} = \langle h | f \rangle_{H_1(\sqrt{A})} = \langle \sqrt{A}h | \sqrt{A}f \rangle_{H_0} = \langle h | Af \rangle_{H_0}.
\]
Since $A$ has dense range we conclude that $g = h \in H_1(\sqrt{A})$. Thus, the adjoint is given by the identity $1 : H_1(\sqrt{A}) \subseteq H_0 \rightarrow H_1(\sqrt{A})$. We rewrite Equation \text{18} in the following way

$$\left( \partial_0 \begin{pmatrix} \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \end{pmatrix} + \begin{pmatrix} \begin{array}{cc} 0 & 0 \\ 0 & (1-B^*)^{-1}(1+C^*) \end{array} \end{pmatrix} + \begin{pmatrix} \begin{array}{cc} 0 & 1 \\ -A & 0 \end{array} \end{pmatrix} \right) \begin{pmatrix} u \\ q \end{pmatrix} = \begin{pmatrix} (1 + C^*)^{-1}f \\ 0 \end{pmatrix}.$$  

Note that this is now an equation in the space $H_{\nu,0}(\mathbb{R}; H_1(\sqrt{A}) \oplus H_0)$. The strict positive definiteness of $\Re(1-B^*)^{-1}(1+C^*)$ follows from the strict positive definiteness of $\Re(1-B^*)^{-1}$ (compare the proof of Theorem 3.12) and the fact that $\| (1 - B^*)^{-1} C^* \|_{L(H_{\nu,0}(\mathbb{R}; H_0))} \rightarrow 0$ as $\nu \rightarrow \infty$.

4 Application to Visco-Elasticity

In this section we apply our findings of Subsection 3.1 to the equations of visco-elasticity. For doing so we first need to introduce the involved differential operators. Throughout let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain.

**Definition 4.1.** We consider the space $$L_2(\Omega)^{n \times n} := \{ \Psi = (\Psi_{ij})_{i,j \in \{1,...,n\}} \mid \forall i,j \in \{1,...,n\} : \Psi_{ij} \in L_2(\Omega) \}$$ equipped with the inner product $$\langle \Psi | \Phi \rangle := \int_{\Omega} \text{trace}(\Psi(x)^*\Phi(x)) \, dx \quad (\Psi, \Phi \in L_2(\Omega)^{n \times n}).$$

It is obvious that $L_2(\Omega)^{n \times n}$ becomes a Hilbert space and that $$H_{\text{sym}}(\Omega) := \{ \Psi \in L_2(\Omega)^{n \times n} \mid \Psi(x)^T = \Psi(x) \ (x \in \Omega \ a.e.) \}$$ defines a closed subspace of $L_2(\Omega)^{n \times n}$ and therefore $H_{\text{sym}}(\Omega)$ is also a Hilbert space. We introduce the operator

$$\text{Grad} |_{C_c^{\infty}(\Omega)^n} : C_c^{\infty}(\Omega)^n \subseteq L_2(\Omega)^n \rightarrow H_{\text{sym}}(\Omega)$$

$$(\phi_i)_{i \in \{1,...,n\}} \mapsto \left( \frac{1}{2}(\partial_i \phi_j + \partial_j \phi_i) \right)_{i,j \in \{1,...,n\}},$$

which turns out to be closable and we denote its closure by $\text{Grad}_c$. Moreover we define $\text{Div} := - \text{Grad}_c^*.$

For $\Phi \in C_c^1(\Omega)^{n \times n} \cap H_{\text{sym}}(\Omega)$ one can compute $\text{Div} \Phi$ by

$$(\text{Div} \Phi)i \in \{1,...,n\} = \left( \sum_{j=1}^n \partial_j \Phi_{ij} \right)_{i \in \{1,...,n\}}.$$
4 Application to Visco-Elasticity

The equations of linear elasticity in a domain $\Omega \subseteq \mathbb{R}^n$ read as follows (see e.g. [10] p. 102 ff.)

\[
\begin{align*}
\partial_0 (g \partial_0 u) - \text{Div } T &= f \quad (19) \\
T &= C \text{Grad}_c u, \quad (20)
\end{align*}
\]

where $u \in H_{\nu,0}(\mathbb{R}; L_2(\Omega)^n)$ denotes the displacement field and $T \in H_{\nu,0}(\mathbb{R}; H_{\text{sym}}(\Omega))$ denotes the stress tensor. Note that due to the domain of the operator Grad$_c$ we have assumed an implicit boundary condition, which can be written as

\[u = 0 \text{ on } \partial \Omega\]

in case of a smooth boundary. The function $g \in L_\infty(\Omega)$ describes the pressure and is assumed to be real-valued and strictly positive. The operator $C \in L(H_{\text{sym}}(\Omega))$, linking the stress and the strain tensor Grad$_c$ $u$ is assumed to be selfadjoint and strictly positive definite. In viscous media it turns out that the stress $T$ does not only depend on the present state of the strain tensor, but also on its past. One way to model this relation is to add a convolution term in $\partial_0 u$, i.e.,

\[T(t) = C \text{Grad}_c u(t) - \int_{-\infty}^t B(t-s) \text{Grad}_c u(s) \, ds, \quad (t \in \mathbb{R}), \quad (21)\]

where $B \in L_{1,\nu}(\mathbb{R}_{\geq 0}; L(H_{\text{sym}}(\Omega)))$. If we plug (21) into (19) we end up with the equation, which was considered in [8] under the assumption, that $B$ is absolutely continuous.

We now show that (19) and (21) can be written as a system of the form (9). For this purpose we define $v := \partial_0 u$. Note that the operator $C - B* = C^{\frac{1}{2}} \left(1 - C^{-\frac{1}{2}} (B*) C^{-\frac{1}{2}}\right) C^{\frac{1}{2}}$ is boundedly invertible, since $C$ is boundedly invertible and $t \mapsto C^{-\frac{1}{2}} B(t) C^{-\frac{1}{2}} \in L_{1,\nu}(\mathbb{R}_{\geq 0}; L(H_{\text{sym}}(\Omega)))$ which gives that $\left(1 - C^{-\frac{1}{2}} (B*) C^{-\frac{1}{2}}\right)$ is boundedly invertible on $H_{\nu,0}(\mathbb{R}; H_{\text{sym}}(\Omega))$ for large $\nu$ (Corollary 3.3). Therefore we can write Equation (21) as

\[
\left(1 - C^{-\frac{1}{2}} (B*) C^{-\frac{1}{2}}\right)^{-1} C^{-\frac{1}{2}} T = C^{\frac{1}{2}} \text{Grad}_c u
\]

and by differentiating the last equality we obtain

\[
\partial_0 \left(1 - C^{-\frac{1}{2}} (B*) C^{-\frac{1}{2}}\right)^{-1} C^{-\frac{1}{2}} T = C^{\frac{1}{2}} \text{Grad}_c v.
\]

Thus, we get formally

\[
\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left(1 - C^{-\frac{1}{2}} (B*) C^{-\frac{1}{2}}\right)^{-1} \right) + \begin{pmatrix} 0 & -g^{-1} \text{Div } C^{\frac{1}{2}} \\ -C^{\frac{1}{2}} \text{Grad}_c & 0 \end{pmatrix} \begin{pmatrix} v \\ C^{-\frac{1}{2}} T \end{pmatrix} = \begin{pmatrix} g^{-1} f \\ 0 \end{pmatrix}.
\]

Following the strategy used in [20] Section 5.2.1 we define a new inner-product on $H$ in the following way.

\[\text{By suitable realizations of the operators Grad and Div one can model also more complicated boundary conditions (see [25] p. 98 ff.).} \]
Definition 4.2. Let $H$ be a Hilbert space and $M \in L(H)$ be selfadjoint and strictly positive definite. Then we define an inner product on $H$ by\footnote{Note that $M^{-1} \in L(H)$ is also selfadjoint and strictly positive definite.}

$$\langle x|y \rangle_{M[H]} := \langle M^{-1} x|y \rangle_H, \quad (x, y \in H)$$

which yields an equivalent norm on $H$. We denote the Hilbert space $(H, \langle \cdot | \cdot \rangle_{M[H]})$ by $M[H]$.

Lemma 4.3. We define the operator

$$A := C^{\frac{1}{2}} \text{Grad}_c : D(\text{Grad}_c) \subseteq \varrho^{-1}[L_2(\Omega)^n] \rightarrow H_{\text{sym}}(\Omega).$$

Then $A^* = -\varrho^{-1} \text{Div} C^{\frac{3}{2}}$.

Proof. Let $\Phi \in D(\text{Div} C^{\frac{3}{2}})$. Then we compute for $\psi \in D(\text{Grad}_c)$

$$\langle A\psi|\Phi \rangle_{H_{\text{sym}}(\Omega)} = \langle \psi| \text{Div} C^{\frac{1}{2}} \Phi \rangle_{L_2(\Omega)^n}$$

$$= \langle \psi| -\varrho^{-1} \text{Div} C^{\frac{1}{2}} \Phi \rangle_{\varrho^{-1}[L_2(\Omega)^n]},$$

showing that $-\varrho^{-1} \text{Div} C^{\frac{3}{2}} \subseteq A^*$. If $\Phi \in D(A^*)$, then there exists $\eta \in L_2(\Omega)^n$ such that

$$\langle C^{\frac{1}{2}} \text{Grad} \psi|\Phi \rangle_{H_{\text{sym}}(\Omega)} = \langle A\psi|\Phi \rangle_{H_{\text{sym}}(\Omega)} = \langle \psi|\eta \rangle_{\varrho^{-1}[L_2(\Omega)^n]} = \langle \varrho \psi|\eta \rangle_{L_2(\Omega)^n}.$$

The latter yields that $C^{\frac{3}{2}} \Phi \in D(\text{Div})$ and $-\text{Div} C^{\frac{3}{2}} \Phi = \varrho \eta$, which gives the remaining operator inclusion. $\square$

The previous lemma shows that (22) as an equation in $H_{\nu,0}(\mathbb{R}; \varrho^{-1}[L_2(\Omega)^n] \oplus H_{\text{sym}}(\Omega))$ is of the form given in (9). Thus our solution theory (Theorem 3.9) applies. So, in the case that $B$ is absolutely continuous (as it was assumed in [8]) we get the well-posedness. If we do not assume any smoothness for $B$ we end up with the following result.

Theorem 4.4. Assume that $B$ satisfies the hypotheses (1)-(3) and that $C$ and $B(t)$ commute for each $t \in \mathbb{R}$. Then (22) is well-posed as an equation in $H_{\nu,0}(\mathbb{R}; \varrho^{-1}[L_2(\Omega)^n] \oplus H_{\text{sym}}(\Omega))$ for $\nu$ large enough.

Proof. Note that $C^{-\frac{1}{2}}(B \ast)C^{-\frac{1}{2}} \ast = \left(C^{-\frac{1}{2}}B(\cdot)C^{-\frac{1}{2}} \ast \right)^\ast$, so according to Theorem 3.9 it suffices to verify the hypotheses (1)-(3) for the kernel $C^{-\frac{1}{2}}B(\cdot)C^{-\frac{1}{2}}$. The conditions (1) and (3) are obvious. Furthermore

$$\left(C^{-\frac{1}{2}}B(\cdot)C^{-\frac{1}{2}} \right)(t-i\nu) = C^{-\frac{1}{2}} \hat{B}(t-i\nu)C^{-\frac{1}{2}}$$

for each $t \in \mathbb{R}, \nu > \mu$ and hence by the selfadjointness of $C$

$$\Im \left(C^{-\frac{1}{2}}B(\cdot)C^{-\frac{1}{2}} \right)(t-i\nu) = C^{-\frac{1}{2}} \Im \hat{B}(t-i\nu)C^{-\frac{1}{2}}.$$
This gives
\[
\left\langle t \Im \left( C^{-\frac{1}{2}} \hat{B}(\cdot) C^{-\frac{1}{2}} \right) (t - i \nu_0) \Phi \right| \Phi \right\rangle_{H_{\text{sym}}(\Omega)} = \left\langle t \Im \hat{B}(t - i \nu_0) C^{-\frac{1}{2}} \Phi \left| C^{-\frac{1}{2}} \Phi \right\rangle_{H_{\text{sym}}(\Omega)} \leq 0
\]
for each $\Phi \in H_{\text{sym}}(\Omega)$ and $t \in \mathbb{R}$. 

\section{5 Conclusion}

We have shown that linear integro-differential equations with operator-valued kernels of hyperbolic and parabolic type are covered by the class of evolutionary equations and we gave sufficient conditions for the well-posedness in both cases. Moreover, using the causality of the solution operators and its continuous extensions to the Sobolev-chain associated to the time-derivative, we have proposed a way to treat initial value problems without introducing history spaces (see Remark \[5.10\] and Remark \[5.11\]). Note that most of the results can also be formulated for differential inclusions, where we replace the operator $A^*A$ by a maximal monotone relation, using the techniques developed in \[26, 24\].

So far, we have restricted ourselves to the case of affine linear functions $P_i, Q_i$ in (8). However, one could also treat polynomials instead, which would yield integro-differential equations with compositions of convolution operators. Also one could allow non-vanishing off-diagonal entries in the block operator matrices in (8). For these cases it is left to state some sufficient conditions for the well-posedness, which are easier to verify than the abstract solvability condition (6).

Another possible generalization would be integro-differential equations with convolutions with unbounded operators, for example equations of the form
\[
(\partial_0^2 + A^*(1 - B^*)A) u = f,
\]
where the operators $B(t)$ are unbounded, but continuous as operators on the Sobolev-chain associated with the operator $|A^*| + i$. Also in this case the solution theory for evolutionary equations is applicable, if we assume that the resolvent $(1 - B^*)^{-1}$ is bounded. However, the main problem remains to give sufficient conditions for the solvability condition (6) in this case.

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