STRONG ADDITIVITY AND CONFORMAL NETS

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Dedicated to Masamichi Takesaki on the occasion of his 70th birthday

ABSTRACT. We show that the fixed point subnet of a strongly additive conformal net under the action of a compact group is strongly additive. Using the idea of the proof we define the notion of strong additivity for a pair of conformal nets and we show that a key result about the induction of the pair which we proved previously under the finite index assumption can be generalized to strongly additive pairs of conformal nets. These results are applied to classify conformal nets of central charge \( c = 1 \) which are not necessarily rational and satisfy a spectrum condition.

§1. INTRODUCTION

Let \( \mathcal{A} \) be a conformal net (or precosheaf) (cf. §2.1). Let \( I_1, I_2 \) be two disjoint intervals of the circle. In this paper an interval of the circle is defined to be an open connected proper subset of the circle. By [FJ] \( \mathcal{A} \) is additive in the sense that if \( I_n \) is a sequence of intervals which cover an interval \( J \), i.e., \( \cup_n I_n \supset J \), then \( \vee_n \mathcal{A}(I_n) \supset \mathcal{A}(J) \) where \( \vee_n \mathcal{A}(I_n) \) is the von Neumann algebra generated by \( \mathcal{A}(I_n) \), \( n \in \mathbb{N} \). \( \mathcal{A} \) is strongly additive if \( \mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I) \) where \( I_1 \cup I_2 \) is obtained by removing an interior point from an interval \( I \). A conformal net which violates strong additivity is given in [BS].

Strong additivity seems to be a rather technical condition, but it plays an important role in studying the representations of conformal nets (cf. [KLM], [Xo]).

In Prop. 2.8 of [Xo] we proved that the fixed point subnet of \( \mathcal{B} \subset \mathcal{A} \) (cf. §3) of a strongly additive net \( \mathcal{A} \) under the action of a finite group is strongly additive using Galois correspondence for the actions of finite groups on von Neumann algebras. This result is generalized in [Ls] to the case when \( \mathcal{B} \subset \mathcal{A} \) has finite Jones index using ideas of transplanting conformal subnets. Both proofs depend on the finite index condition and it is not immediately clear how to generalize them to the case when the finite group is replaced by a compact but infinite group. The first result Theorem 2.4 of this paper is to show that Prop. 2.8 of [Xo] is true when the finite

I’d like to thank Professors Chongying Dong, Jürgen Fuchs, Christoph Schweigert, Yasuyuki Kawahigashi for useful comments, and especially Professor Roberto Longo for pointing out reference [LRo] which helps to improve Prop. 3.4. This work is partially supported by a NSF grant. 1991 Mathematics Subject Classification. 46S99, 81R10.

Typeset by \texttt{AMS-\LaTeX}
group is replaced by a compact group using the results of [Ha]. As in [Xo], the idea is to show that

\[ \mathcal{A}(I_1) \vee \mathcal{B}(I_2) = \mathcal{A}(I) \]  

(1)

where \( I_1, I_2, I \) are as above, and we use a dilation argument similar to [Xo]. But the proof is different from [Xo] and [Ls], and the idea of the proof also gives a simpler proof of the strong additivity results of [Xo] and [Ls] (cf. Remark after Th. 2.4).

It turns out that (1) can be used to generalize a key result Th. 3.3 about inductions between \( \mathcal{B} \) and \( \mathcal{A} \) obtained in [Xb]. Since the results of [Xb] have many applications (cf. [BE1-3], [FRS], [PZ]), we define a pair of nets \( \mathcal{B} \subset \mathcal{A} \) to be strongly additive if (1) is satisfied (cf. §3.2) and we study such pairs in §3.2. In view of the applications in §4, we prove Theorem 3.8 in §3.2 which is a generalization of Th. 3.3 in [Xb], and the proof is different from the original proof in [Xb]. We note that a few other results in [Xb] also generalize to our current setting and we plan to return to them in a future publication.

In §4 we apply the results of §2 and §3 to classify conformal nets \( \mathcal{A} \) with central charge \( c = 1 \). The idea is very simple and we describe it here roughly as follows. The \( c = 1 \) Virasoro subnet \( \mathcal{B} \subset \mathcal{A} \) (cf. §4.1) has a representation (the Vector representation) with Jones index 4 (cf. Lemma 4.1), and since \( \mathcal{B} \) is strongly additive by Th. 2.4, the induced endomorphism to \( \mathcal{A} \) has Jones index 4 by Prop. 3.4 and Lemma 4.1. Hence the principal graph of this induced endomorphism is one of the “A-D-E” graphs listed in [GHJ], and in fact such subfactors are classified in [Po1]. Depending on the nature of the principal graph, we can obtain enough information on \( \mathcal{A} \) to identify \( \mathcal{A} \) as the “A-D-E” list given in §4.1 under a spectrum condition in §4.2. The spectrum condition states that a degenerate representation of the Virasoro net other than the vacuum representation must appear in \( \mathcal{A} \) if \( \mathcal{A} \neq \mathcal{B} \). This condition is true for all known examples and we conjecture that the spectrum condition is always true. We note that a “A-D-E” type classification for \( c < 1 \) has been given in [KL], and there are some similarities between our results and that of [KL]. But there are notable differences since \( c = 1 \) Virasoro net is not rational, and the results of §2 and §3 play a crucial role in our proofs. We also note that we have tried to give different proofs for each cases in §4.2 since we expect that the ideas of these proofs, as well as the general results of §2 and §3, will have applications beyond those described in §4.

The rest of the paper is organized as follows. In §2 after reviewing the concept of conformal nets, subnets and relevant notions we prove the strong additivity result Theorem 2.4 mentioned at the beginning of the introduction. In §3.1 we consider a pair of conformal nets and extend some of the results of [BE1-3], [LR] and [Xb] to this setting. Prop. 3.1 is essentially due to [LR], except under additivity assumptions we have stronger properties which was proved under finite index conditions in [Xb]. Lemma 3.2, 3.3, Prop. 3.4, 3.5 were also proved in [Xb] under finite index conditions, and are generalized with suitable modifications of the original proofs to our current case when the index is not necessarily finite. In §3.2 we define the notion of strongly additive pair. Lemma 3.6 and Prop. 3.7 give us many examples.
of such pairs. In Theorem 3.8 we generalize a key result Th. 3.3 of [Xb] to strongly additive pair. Cor. 3.9 is a direct consequence of Th. 3.8 and results of §3.1. We note that Theorem 3.8 and Cor. 3.9 give us powerful tools to determine the nature of induced endomorphisms as already shown in [Xb] under the finite index assumption.

In §4 we apply the results in §2 and §3 to pairs $B \subset A$ where $B$ is the Virasoro net with central charge $c = 1$. After reviewing the known “A-D-E” list of such pairs in §4.1, we first determine a distinguished class of irreducible representations of $B$ in Lemma 4.1, corresponding to the irreducible representations of $SU(2)$. The principal graph of the induced endomorphism of the vector representation is the affine A-D-E graph described in Lemma 4.2. In Lemma 4.3 we classify the covariant representations of a simple net associated with a Heisenberg group. In Lemma 4.4 we determine the principal graph of the induced endomorphism of the vector representation for the known list, showing that all possible graphs do appear for conformal nets. As a by-product of Lemma 4.4 we determine the fusion rules of $c = 1$ Virasoro net where one of the representation is degenerate in Prop. 4.5, which contains the result in [R1]. §4.2 is devoted to the proof of Th. 4.6. We treat the discrete cases §4.2.1-2 first where the reconstruction results of [DR1-2] are used. We then treat the general case according to the A-D-E type of the principal graph of the induced vector representation, under the spectrum assumption defined in §4.2. Finally we consider one application of Th. 4.6 to a concrete example appeared in our previous study of cosets.

After this paper appeared in the web, we have been informed by Professor Roberto Longo that the results in §4.2.1-2 have some overlap with unpublished results of S. Carpi (cf. [K]).

§2. Strong additivity of subnets

§2.1 Sectors. Let $M$ be a properly infinite factor and $\text{End}(M)$ the semigroup of unit preserving endomorphisms of $M$. In this paper $M$ will always be the unique hyperfinite $III_1$ factors. Let $\text{Sect}(M)$ denote the quotient of $\text{End}(M)$ modulo unitary equivalence in $M$. We denote by $[\rho]$ the image of $\rho \in \text{End}(M)$ in $\text{Sect}(M)$.

It follows from [L3] and [L4] that $\text{Sect}(M)$, with $M$ a properly infinite von Neumann algebra, is endowed with a natural involution $\theta \mapsto \bar{\theta}$; moreover, $\text{Sect}(M)$ is a semiring.

Let $\epsilon$ be a normal faithful conditional expectation from $M$ to $\rho(M)$. We define a number $d_{\epsilon}$ (possibly $\infty$) by:

$$d_{\epsilon}^{-2} := \text{Max}\{\lambda \in [0, +\infty) | \epsilon(m_+) \geq \lambda m_+, \forall m_+ \in M_+\}$$

(cf. [PP]).

We define

$$d_\rho = \text{Min}_{\epsilon}\{d_{\epsilon}\}.$$

$d_\rho$ is called the statistical dimension of $\rho$. $d_\rho^2$ will be called the (minimal) index of $\rho$. It is clear from the definition that the statistical dimension of $\rho$ depends only on
the unitary equivalence classes of $\rho$. The properties of the statistical dimension can be found in [L1], [L3] and [L4].

For $\lambda, \mu \in \text{Sect}(M)$, let $\text{Hom}(\lambda, \mu)$ denote the space of intertwiners from $\lambda$ to $\mu$, i.e. $a \in \text{Hom}(\lambda, \mu)$ iff $a\lambda(x) = \mu(x)a$ for any $x \in M$. When there are several von Neumann algebras, we will write $\text{Hom}(\lambda, \mu)_M$ to indicate the dependence on $M$.

$\text{Hom}(\lambda, \mu)$ is a vector space and we use $\langle \lambda, \mu \rangle$ to denote the dimension of this space. $\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover if $\nu, \lambda$ and $\mu$ has finite index, we have $\langle \nu\lambda, \mu \rangle = \langle \lambda, \nu\mu \rangle$, $\langle \nu\lambda, \mu \rangle = \langle \nu, \mu\lambda \rangle$ which follows from Frobenius duality (See [L2]). We will also use the following notation: if $\mu$ is a subsector of $\lambda$, we will write as $\mu \prec \lambda$ or $\lambda \succ \mu$. A sector is said to be irreducible if it has only one subsector. In this paper we will sometimes use 1 to denote the identity sector if there is no possible confusion.

If $\lambda$ is a sector with finite statistical dimension, the principal graph $\Gamma$ of $\lambda$ is a bi-parti graph defined as follows. The even vertices of $\Gamma$, denoted by $\Gamma_0$, are labeled by the irreducible sectors of $(\lambda\lambda)^n$, $n \in \mathbb{N}$, and the odd vertices of $\Gamma_1$ of $\Gamma$ are labeled by the irreducible sectors of $(\lambda\lambda)^n\lambda$, $n \in \mathbb{N}$. An even vertex $x$ is connected to an odd vertex $y$ by $\langle x\lambda, y \rangle$ edges. We say that $\lambda$ has finite depth if $\Gamma$ is a finite graph. Following [Po2] we say that $\lambda$ is amenable if $||\Gamma|| = d_\lambda$, where $\Gamma$ is considered a linear map from $l^2(\Gamma_0)$ to $l^2(\Gamma_1)$ as in [Po2].

§2.2 Conformal nets and subnets. In this section we recall the notion of irreducible conformal net (precosheaf) and its representation as described in [GL1].

By an interval we shall always mean an open connected subset $I$ of $S^1$ such that $I$ and the interior $I'$ of its complement are non-empty. We shall denote by $I$ the set of intervals in $S^1$. We shall denote by $\text{PSL}(2, \mathbb{R})$ the group of conformal transformations on the complex plane that preserve the orientation and leave the unit circle $S^1$ globally invariant. Denote by $G$ the universal covering group of $\text{PSL}(2, \mathbb{R})$. Notice that $G$ is a simple Lie group and has a natural action on the unit circle $S^1$.

Denote by $R(\vartheta)$ the (lifting to $G$ of the) rotation by an angle $\vartheta$. This one-parameter subgroup of $G$ will be referred to as rotation group (denoted by $\text{Rot}$) in the following. We may associate a one-parameter group with any interval $I$ in the following way. Let $I_1$ be the upper semi-circle, i.e. the interval $\{e^{i\vartheta}, \vartheta \in (0, \pi)\}$. By using the Cayley transform $C : S^1 \to \mathbb{R} \cup \{\infty\}$ given by $z \to -i(z - 1)(z + 1)^{-1}$, we may identify $I_1$ with the positive real line $\mathbb{R}_+$. Then we consider the one-parameter group $\Lambda_{I_1}(s)$ of diffeomorphisms of $S^1$ such that

$$C\Lambda_{I_1}(s)C^{-1}x = e^{sx}, \quad s, x \in \mathbb{R}. \quad C\Lambda_{I_1}(s)C^{-1}x = e^{sx}, \quad s, x \in \mathbb{R}.$$  

We also associate with $I_1$ the reflection $r_{I_1}$ given by $r_{I_1}z = \bar{z}$ where $\bar{z}$ is the complex conjugate of $z$. It follows from the definition that $\Lambda_{I_1}$ restricts to an orientation preserving diffeomorphisms of $I_1$, $r_{I_1}$ restricts to an orientation reversing diffeomorphism of $I_1$ onto $I_1'$.

Then, if $I$ is an interval and we choose $g \in G$ such that $I = gI_1$ we may set

$$\Lambda_I = g\Lambda_{I_1}g^{-1}, \quad r_I = gr_{I_1}g^{-1}.$$
Let $r$ be an orientation reversing isometry of $S^1$ with $r^2 = 1$ (e.g. $r_{I_1}$). The action of $r$ on $PSL(2, \mathbb{R})$ by conjugation lifts to an action $\sigma_r$ on $G$, therefore we may consider the semidirect product of $G \times_{\sigma_r} \mathbb{Z}_2$. Since $G \times_{\sigma_r} \mathbb{Z}_2$ acts on $S^1$. We call (anti-)unitary a representation $U$ of $G \times_{\sigma_r} \mathbb{Z}_2$ by operators on $\mathcal{H}$ such that $U(g)$ is unitary, resp. antiunitary, when $g$ is orientation preserving, resp. orientation reversing.

Now we are ready to define a conformal net (precosheaf).

An irreducible conformal net $\mathcal{A}$ of von Neumann algebras on the intervals of $S^1$ is a map

$$I \to \mathcal{A}(I)$$

from $\mathcal{I}$ to the von Neumann algebras on a separable Hilbert space $\mathcal{H}$ that verifies the following properties:

A. Isotony. If $I_1, I_2$ are intervals and $I_1 \subset I_2$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

B. Conformal invariance. There is a nontrivial unitary representation $U$ of $G$ on $\mathcal{H}$ such that

$$U(g)A(I)U(g)^* = A(gI), \quad g \in G, \quad I \in \mathcal{I}. $$

C. Positivity of the energy. The generator of the rotation subgroup $U(R(\vartheta))$ is positive.

D. Locality. If $I_0, I$ are disjoint intervals then $\mathcal{A}(I_0)$ and $\mathcal{A}(I)$ commute.

The lattice symbol $\lor$ will denote ‘the von Neumann algebra generated by’.

E. Existence of the vacuum. There exists a unit vector $\Omega$ (vacuum vector) which is $U(G)$-invariant and cyclic for $\lor_{I \in \mathcal{I}} \mathcal{A}(I)$.

F. Irreducibility. The only $U(G)$-invariant vectors are the scalar multiples of $\Omega$.

The term irreducibility is due to the fact (cf. Prop. 1.2 of [GL1]) that under the assumption of $\lor_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$.

We have the following (cf. Prop. 1.1 of [GL1]):

2.1 Proposition. Let $\mathcal{A}$ be an irreducible conformal net. The following hold:

(a) Reeh-Schlieder theorem: $\Omega$ is cyclic and separating for each von Neumann algebra $\mathcal{A}(I)$, $I \in \mathcal{I}$.

(b) Bisognano-Wichmann property: $U$ extends to an (anti-)unitary representation of $G \times_{\sigma_r} \mathbb{Z}_2$ such that, for any $I \in \mathcal{I}$,

$$U(\Lambda_I(2\pi t)) = \Delta^t_I$$

$$U(r_I) = J_I$$
where $\Delta_I$, $J_I$ are the modular operator and the modular conjugation associated with $(A(I), \Omega)$. For each $g \in G \times \sigma, Z_2$

$$U(g)A(I)U(g)^* = A(gI).$$

(c) Additivity: if a family of intervals $I_i$ covers the interval $I$, then

$$A(I) \subset \bigvee_i A(I_i).$$

(d) Haag duality: $A(I)' = A(I')$

A representation $\pi$ of $A$ is a family of representations $\pi_I$ of the von Neumann algebras $A(I)$, $I \in \mathcal{I}$, on a separable Hilbert space $\mathcal{H}_\pi$ such that

$$I \subset \overline{I} \Rightarrow \pi_{\overline{I}}|_{A(I)} = \pi_I \quad \text{(isotony)}$$

A covariant representation $\pi$ of $A$ is a family of representations $\pi_I$ of the von Neumann algebras $A(I)$, $I \in \mathcal{I}$, on a separable Hilbert space $\mathcal{H}_\pi$ and a unitary representation $U_\pi$ of the covering group $G$ of $PSL(2, \mathbb{R})$ with positive energy such that the following properties hold:

$$I \subset \overline{I} \Rightarrow \pi_{\overline{I}}|_{A(I)} = \pi_I \quad \text{(isotony)}$$

$$\text{ad} U_\pi(g) \cdot \pi_I = \pi_{gI} \cdot \text{ad} U(g) \quad \text{(covariance)}.$$  

A covariant representation $\pi$ is called irreducible if $\bigvee_{I \in \mathcal{I}} \pi(A(I)) = B(\mathcal{H}_\pi)$. By our definition the irreducible conformal net is in fact an irreducible representation of itself and we will call this representation the vacuum representation.

We note that by [GL2] if a representation has finite index, then it is covariant.

Let $H$ be a connected simply-laced compact Lie group. By Th. 3.2 of [FG], the vacuum positive energy representation of the loop group $LH$ (cf. [PS]) at level $k$ gives rise to an irreducible conformal net denoted by $A_{H_k}$. By Th. 3.3 of [FG], every irreducible positive energy representation of the loop group $LH$ at level $k$ gives rise to an irreducible covariant representation of $A_{H_k}$. We also note that the vacuum representation of the Virasoro algebra with central charge $c_0 > 0$ also give rise to a conformal net denoted by $A_{c=c_0}$ (cf. §3 of [FG]). We will see such examples in §4.

Next we recall some definitions from [KLM]. As in [GL1] by an interval of the circle we mean an open connected proper subset of the circle. If $I$ is such an interval then $I'$ will denote the interior of the complement of $I$ in the circle. We will denote by $\mathcal{I}$ the set of such intervals. Let $I_1, I_2 \in \mathcal{I}$. We say that $I_1, I_2$ are disjoint if $\overline{I_1} \cap \overline{I_2} = \emptyset$, where $\overline{I}$ is the closure of $I$ in $S^1$. When $I_1, I_2$ are disjoint, $I_1 \cup I_2$ is called a 1-disconnected interval in [Xj]. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $A$ be an irreducible conformal net as in §2.1. For $E = I_1 \cup I_2 \in \mathcal{I}_2$,
let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in $S^1$ where $I_3, I_4$ are disjoint intervals. Let

$$\mathcal{A}(E) := A(I_1) \vee A(I_2), \hat{\mathcal{A}}(E) := (A(I_3) \vee A(I_4))'.$$

Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is split if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is strongly additive if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from $I$.

**Definition 2.2 (Absolute rationality of [KLM]).** $\mathcal{A}$ is said to be absolute rational, or $\mu$-rational, if $\mathcal{A}$ is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ (it is independent of $E$ by Prop. 5 of [KLM]) is denoted by $\mu_\mathcal{A}$ and is called the $\mu$-index of $\mathcal{A}$.

Let $\mathcal{A}$ be a strongly additive conformal net. This net is not directed. So when we discuss covariant representations of $\mathcal{A}$, we have to specify the intervals. To simply our notations, we fix a point $\xi \in S^1$. Denote by $\mathcal{I}_\xi$ the set of intervals $I$ with $I \subset S^1 - \xi$. Note that $\mathcal{I}_\xi$ is a directed set under inclusion. Let $\mathcal{U}_\mathcal{A}$ be the associated quasi-local $C^*$-algebra $\mathcal{U}_\mathcal{A} = \bigcup_{J \in \mathcal{I}_\xi} \mathcal{A}(J)$ (norm closure). We note that any representation of $\lambda$ of $\mathcal{A}$ localized on $I$ restricts to a DHR endomorphism of $\mathcal{U}_\mathcal{A}$ localized on $I$ also denoted by $\lambda$ and vice versa (cf. Prop. 11 of [Ls]). We will use these two descriptions interchangeably without further specifications.

Fix an $I \subset \mathcal{I}_\xi$. Let $\lambda, \mu$ be representations of $\mathcal{A}$. Choose $I_-, I_+ \in \mathcal{I}_\xi$ so that $I_- \cap I = \emptyset = I_+ \cap I$, $I_-$ lies anti-clockwise to $I$, and $I_+$ lies clockwise to $I$. Choose $\lambda_+, \lambda_-$ covariant representations of $\mathcal{A}$ unitarily equivalent to $\lambda$ but localized on $I_+, I_-$ respectively and let $u_+, u_-$ be the unitary intertwiners. Note that we will not distinguish local and global intertwinners since they are the same when $\mathcal{A}$ is strongly additive. The *braiding operators* are defined by

$$\epsilon(\lambda, \mu) := \mu(u_+^*)u_+, \tilde{\epsilon}(\lambda, \mu) = \mu(u_-^*)u_-.$$

The properties of these operators are well known (cf. [Xb], [BE1]). We note that $\epsilon(\lambda, \mu), \tilde{\epsilon}(\lambda, \mu)$ are elements of $\mathcal{A}(I)$ and they are independent of the choices of $u_+, u_-, I_+, I_-$ as long as $I_- \cap I = \emptyset = I_+ \cap I$, $I_-$ lies anti-clockwise to $I$, and $I_+$ lies clockwise to $I$. These operators satisfy Yang-Baxter equation (YBE) and Braiding-Fusion equation (BFE), and we refer the reader to [Xb] and [BE1] for more details.

By a *conformal subnet* (cf. [Ls]) we shall mean a map

$$I \in \mathcal{I} \rightarrow \mathcal{B}(I) \subset \mathcal{A}(I)$$

which associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{B}(I)$ which is isotope

$$\mathcal{B}(I_1) \subset \mathcal{B}(I_2), I_1 \subset I_2,$$
and covariant with respect to the representation $U$, i.e.,

$$U(g)B(I)U(g)^* = B(gI), \quad g \in G, \quad I \in \mathcal{I}.$$  

Note that when restricting to the intervals from the set $\mathcal{I}_\xi$, the conformal nets $B \subset A$ is a standard net of inclusions as defined in §3.1 of [LR]. We denote by $\gamma_A$ be the canonical endomorphism from $A(J)$ to $B(J)$, $J \supset I, J \in \mathcal{I}_\xi$ which is an extension of canonical endomorphism from $A(I)$ to $B(I)$ as defined by Cor. 3.3 of [LR]. The restriction of $\gamma_A$ to $B$ will be simply denoted by $\gamma$, and when no confusion arises, we will also denote $\gamma_A$ by $\gamma$.

We have (cf. [KLM] or Prop. 2.4 of [Xo]):

**Proposition 2.2.** Suppose $B \subset A$ is a standard net of inclusions as defined in 3.1 of [LR]. Let $E \in \mathcal{I}_2$. If $A \subset B$ has finite index denoted by $[A : B]$ and $A$ and $B$ are split, then

$$[\hat{B}(E) : B(E)] = [A : B]^2[\hat{A}(E) : A(E)].$$

§2.3 Orbifolds. Let $A$ be an irreducible conformal net on a Hilbert space $\mathcal{H}$ and let $G$ be a compact group. Let $V : G \to U(\mathcal{H})$ be a faithful unitary representation of $G$ on $\mathcal{H}$.

**Definition 2.1.** We say that $G$ acts properly on $A$ if the following conditions are satisfied:

1. For each fixed interval $I$ and each $g \in G$, $\alpha_g(a) := V(g)aV(g^*) \in A(I), \forall a \in A(I)$;

2. For each $g \in G$, $V(g)\Omega = \Omega, \forall g \in G$.

Suppose that a finite group $G$ acts properly on $A$ as above. For each interval $I$, define $B(I) := \{a \in A(I) | V(g)aV(g^*) = a, \forall g \in G\}$. Let $\mathcal{H}_0 = \{x \in \mathcal{H} | V(g)x = x, \forall g \in G\}$ and $P_0$ the projection from $\mathcal{H}$ to $\mathcal{H}_0$. Notice that $P_0$ commutes with every element of $B(I)$ and $U(g), \forall g \in G$.

Define $A^G(I) := B(I)P_0$ on $\mathcal{H}_0$. The unitary representation $U$ of $G$ on $\mathcal{H}$ restricts to an unitary representation (still denoted by $U$) of $G$ on $\mathcal{H}_0$. By Prop. 2.2 of [Xo] the map $I \in \mathcal{I} \to A^G(I)$ on $\mathcal{H}_0$ together with the unitary representation (still denoted by $U$) of $G$ on $\mathcal{H}_0$ is an irreducible conformal net, denoted by $A^G$ and will be called the orbifold of $A$ with respect to $G$.

The net $B \subset A$ is a standard net of inclusions (cf. [LR]) when restricting to intervals in $\mathcal{I}_\xi$ with conditional expectation $E$ defined by

$$E(a) := \int_G \alpha_g(a)dg, \forall a \in A(I)$$

where $dg$ is the normalized Haar measure on $G$.

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1. If $V : G \to U(\mathcal{H})$ is not faithful, we can take $G' := G/\ker V$ and consider $G'$ instead.
Lemma 2.3. (1) For any interval \( I \), \( \mathcal{A}^G(I) \cap \mathcal{A}(I) = \mathbb{C} \);

(2) Let \( I \) be an interval, and \( I_1, I_2 \) are the connected components of a set obtained from \( I \) by removing an interior point of \( I \). Let \( g_n \in G \) be a sequence of elements such that \( g_nI_1 = I_1 \) and \( g_nI_2 \) is an increasing sequence intervals containing \( I_2 \), i.e., \( I_2 \subset g_nI_2 \subset g_{n+1}I_2 \), and \( \cup_n g_nI_2 = I_1' \) (One may take \( g_n \) to be a sequence of dilations). Let \( x \in B(I_1)' \cap \mathcal{A}(I_2) \), and suppose \( y \) is a weak limit of a subsequence of \( Ad_{g_k}(x) := g_kxg_k^{-1} \). Then \( y = \langle x\Omega, \Omega \rangle id \).

\[ \langle y\Omega, \Omega \rangle = \langle x\Omega, \Omega \rangle, \]

and so \( y = \langle x\Omega, \Omega \rangle id \).

\[ \square \]

Theorem 2.4. Let \( \mathcal{A} \) be an irreducible conformal net and let \( G \) be a compact group acting properly on \( \mathcal{A} \). Suppose that \( \mathcal{A} \) is strongly additive. Then \( \mathcal{A}^G \) is also strongly additive.

\[ \square \]

Proof: Let \( I \) be an interval, and \( I_1, I_2 \) are the connected components of a set obtained from \( I \) by removing an interior point of \( I \). To show \( \mathcal{A}^G \) is strongly additive, it is sufficient to show that \( B(I_1) \cap \mathcal{A}(I_2) = \mathcal{B}(I) \).

Let us show that \( B(I_1) \cap \mathcal{A}(I_2) = \mathcal{A}(I) \). Note that this will prove the theorem by simply applying \( E \) to both sides of the equality.

Since \( \mathcal{A} \) is strongly additive, it is sufficient to show that \( B(I_1) \cap \mathcal{A}(I_2) = \mathcal{A}(I_1) \cap \mathcal{A}(I_2) \), or, by taking commutants and using Haag duality

\[ N := \mathcal{B}(I_1)' \cap \mathcal{A}(I_2)' = M := \mathcal{A}(I_1)' \cap \mathcal{A}(I_2)' \]

Define

\[ N_0 := \mathcal{B}(I_1)' \supset M_0 := \mathcal{A}(I_1)' \]

Notice that by Remark 4.5 of [I], \( N_0 \) can be identified as the cross product of \( M_0 \) by \( G \). By (a) of Th. 3.1 of [H], for each continuous, positive definite function \( \phi \) on \( G \) there is a unique \( \sigma \)-weakly continuous linear map \( E_\phi \) on \( N_0 \) such that

\[ E_\phi(m_0xm_0'g) = m_0E_\phi(x)m_0', E_\phi(g) = \phi(g)g, \forall m_0, m_0' \in M_0, g \in G \]

Let \( g_n \in G \) be a sequence of elements such that \( g_nI_1 = I_1 \) and \( g_nI_2 \) is an increasing sequence intervals containing \( I_2 \), i.e., \( I_2 \subset g_nI_2 \subset g_{n+1}I_2 \), and \( \cup_n g_nI_2 = I_1' \) (One may take \( g_n \) to be a sequence of dilations). Consider

\[ F_\phi(n_0) := Ad_{g_k}E_\phi(Ad_{g_k}n_0), \forall n_0 \in N_0 \]
Note that
\[
F_\phi(m_0xm'_0) = m_0F_\phi(x)m'_0, \quad F_\phi(g) = \phi(g)g, \forall m_0, m'_0 \in M_0, g \in G
\]
Since \( M_0gM_0 \) is weakly closed in \( N_0 \), it follows that \( E_\phi = F_\phi \), i.e.,
\[
E_\phi(Ad_g n_0) = Ad_g(E_\phi(n_0)), \forall n_0 \in N_0
\]
Let \( x \in N \). Since \( A(I_2) \subset A(I'_1) \), \( E_\phi = \phi(1)id \) on \( A(I'_1) \), \( E_\phi(x) \in N \). Let \( a,b \in M \). Then
\[
\langle \frac{1}{\phi(1)}E_\phi(x)a\Omega, b\Omega \rangle = \langle \frac{1}{\phi(1)}E_\phi(b^*xa)\Omega, \Omega \rangle = \langle \frac{1}{\phi(1)}Ad_gE_\phi(b^*xa)\Omega, \Omega \rangle = \langle \frac{1}{\phi(1)}E_\phi(Ad_g(b^*xa))\Omega, \Omega \rangle
\]
Note that by Lemma 2.3, there is a subsequence of \( Ad_g b^*xa \) which converges weakly to \( \langle b^*xa\Omega, \Omega \rangle \). Since \( E_\phi \) is weakly continuous, we must have
\[
\langle \frac{1}{\phi(1)}E_\phi(x)a\Omega, b\Omega \rangle = \langle xa\Omega, b\Omega \rangle
\]
Since \( M\Omega \) is dense in \( H \), we have \( E_\phi(x) = \phi(1)x \). Let us choose \( x \) to be a projection in \( N \). By (b) of Th. 3.1 of [H],
\[
T(x) = \sup_{\phi \ll \delta} E_\phi(x)
\]
where \( T \) is the operator valued weights from \( N_0 \) to \( M_0 \), and we write \( \phi \ll \delta \) if \( \phi \) is less than the Dirac measure in the unit element of \( G \) with respect to the ordering of positive definite measures on \( G \). So when \( G \) is a finite group,
\[
T(x) = x
\]
implying that \( x \in M_0 \cap N = M \). When \( G \) is an infinite group
\[
T(x) = x \cdot \infty
\]
in the notation of extended positive part of \( M_0 \) on P. 150 of [SS]. And so \( x \in M_0 \cap N = M \). So we have shown that any projection of \( N \) is a projection of \( M \subset N \), and so \( M = N \).

\[\square\]
We note that the above proof is different from the one given in [X0]. The same idea also gives a different proof of the result in §3.5.2 of [Ls] under the assumption
that $B \subset A$ has finite index but without the assumption that $A$ is split if we modify the proof as follows. Instead of using $E_\phi$ we use $E$ the minimal conditional expectation from $N_0$ to $M_0$ which exists by finite index assumption. Let us check that just like $E_\phi$, we have

$$E(Ad_{g_k}n_0) = Ad_{g_k}(E(n_0)), \forall n_0 \in N_0$$

Note that $Ad_{g_k}E_\phi(Ad_{g_k} \cdot)$ is a conditional expectation from $N_0$ to $M_0$, and since $M_0 \subset N_0$ is irreducible by Lemma 14 of [LS], we must have $Ad_{g_k}E(Ad_{g_k} \cdot) = E(\cdot)$. Now using $E$ instead of $E_\phi$, the rest of the proof goes through, and we get $E(x) = x, \forall x \in N$, and so $x \in N \cap M_0 = M$, i.e., $N = M$.

Let us consider a large class of examples where Th. 2.4 can be applied. Let $A_{H_k}$ be the conformal net associated with representations of loop group $LH$ at level $k$ as in §2.2. Let $G \subset H$ be any closed subgroup. By [TL] $A_{H_k}$ is strongly additive, and it is easy to check that $G$ or $G'$ as defined in the footnote of Definition 2.1 acts properly on $A_{H_k}$. It follows that the fixed point net $A_{H_k}^{G'}$ is strongly additive by Th. 2.4. We will see a special case of such examples in §4.

**§3. Induction and strongly additive pairs**

**§3.1 Induction of a pair.** Let $B \subset A$ be a pair of conformal nets. In this section we assume that $B$ is strongly additive. Fix a point $\xi \in S^1$. Denote by $I_\xi$ the set of intervals $I$ with $I \subset S^1 - \xi$. Note that $I_\xi$ is a directed set under inclusion. Fix an $I \subset I_\xi$. All the intervals in this section will be in $I_\xi$ unless otherwise stated. Recall from §2.2 that $\gamma_A$ is the canonical endomorphism from $A(J)$ into $B(J), J \supset I, J \in I_\xi$ which is an extension of the canonical endomorphism from $A(I)$ into $B(I)$ as given by Cor. 3.3 of [LR]. The restriction of $\gamma_A$ to $B$ is denoted by $\gamma$. We note that $\gamma$ is a (DHR) representation of $B$ which is unitarily equivalent to the defining representation of $B$ on the vacuum Hilbert space of $A$ since $B$ is assumed to be strongly additive (cf. the proof of Prop. 17 in [Ls]). The following is essentially Prop. 3.9 of [LR], except that under our conditions we have further properties.

**Proposition 3.1.** Let $B \subset A$ be a pair of conformal nets with $B$ strongly additive. With every representation $\lambda$ of $B$ associate

$$\alpha_\lambda := \gamma_A^{-1}Ad_\epsilon \gamma_A, \tilde{\alpha}_\lambda := \gamma_A^{-1}Ad_\tilde{\epsilon} \gamma_A$$

(2)

where $\epsilon = \epsilon(\lambda, \gamma), \tilde{\epsilon} := \tilde{\epsilon}(\lambda, \gamma) \lambda \gamma \rightarrow \gamma \lambda$ are the braiding operators in $B(I)$. Then $\alpha_\lambda(A(I)) \subset A(I)$ and $\alpha_\lambda = \lambda$ on $B(I)$. Moreover, $\alpha_\lambda$ is localized on $I$ if and only if

$$\epsilon(\lambda, \gamma)\epsilon(\gamma, \lambda) = id.$$
Then for any $x \in \mathcal{A}(I)$ we have 

$$Ad_\epsilon \lambda \gamma (x) = Ad_\gamma (u^\star) \hat{\lambda} \gamma = \gamma (Ad_u^\star x)$$

Let $J$ be an interval in $I_\xi$ containing $I \cup I_+$. Then $\gamma (Ad_u^\star x) \in \gamma (\mathcal{A}(J))$, Note that the left hand side of the above depends on $J$ only through the braiding operator $\epsilon$, and by the invariance property of $\epsilon$ we can choose a decreasing sequence of intervals $J_n \supset \bar{I}$ such that $\cap_n J_n = I$. By (c) of Prop. 2.1 

$$\cap_n \mathcal{A}(J_n) = \mathcal{A}(I).$$

It follows that 

$$\cap_n \gamma (\mathcal{A}(J_n)) = \gamma (\mathcal{A}(I)).$$

So we have shown that 

$$Ad_\epsilon \lambda \gamma (\mathcal{A}(I)) \subset \gamma (\mathcal{A}(I)).$$

Hence $\alpha_\lambda (\mathcal{A}(I)) \subset \mathcal{A}(I)$ and 

$$\alpha_\lambda (x) = u^\star xu, \forall x \in \mathcal{A}(I) \quad (3)$$

Note that a similar formula hold for $\tilde{\alpha}_\lambda$ if we choose the unitary intertwinner accordingly.

The rest of the proof is the same as that of Prop. 3.9 of [LR].

□

The notation $\alpha_\lambda$ was introduced in [BE1]. In [Xb], a slightly different induction $\alpha_\lambda \in \text{End}(\mathcal{B}(I))$ was used motivated by certain questions in subfactors and the relations between these two are given in [Xi]. Let us point out one basic relation. Let $\rho \in \text{End}(\mathcal{B}(I))$ be such that 

$$\rho (\mathcal{B}(I)) = \gamma (\mathcal{A}(I)), \rho \bar{\rho} = \gamma,$$

then 

$$\alpha_\lambda (\gamma (a)) = \rho^{-1}(\gamma \alpha_\lambda (a)), \forall a \in \mathcal{A}(I)$$

We note that all the results of §3 can be written in terms of $a_\lambda$ using the above relation.

The following lemma is implicitly contained in [LR]:

**Lemma 3.2.** (1) If $x \in \text{Hom}(\lambda, \mu)_\mathcal{B}$, then $x \in \text{Hom}(\alpha_\lambda, \alpha_\mu)_\mathcal{A}$;

(2) $\alpha_{\lambda \mu} = \alpha_\lambda \alpha_\mu$;

(3) If $[\delta] = [\lambda] + [\mu]$, then $[\alpha_\delta] = [\alpha_\lambda] + [\alpha_\mu]$.

**Proof:** Let $u_\lambda$ and $u_\mu$ be the unitary intertwinners from $\lambda$ and $\mu$ localized on $I$ to $\hat{\lambda}$ and $\hat{\mu}$ localized on $I_+$ respectively. We can choose $I_+$ to share only one boundary point with $I$. Since $x \in \text{Hom}(\lambda, \mu)$, we have $u_\mu xu_\lambda^\star \in \mathcal{B}(I \cup I_+) \cap \mathcal{B}(I) = \mathcal{B}(I_+)$ by strong additivity of $\mathcal{B}$. It follows that $u_\mu xu_\lambda^\star$ commutes with every element of $\mathcal{A}(I)$, and (1) is proved. (2) and (3) follow from the definitions, YBE and BFE as in §3 of [BE1].

□
Lemma 3.3. (1) If $\lambda$ is a representation of $\mathcal{B}$ localized on $I$ and $\sigma \in \text{End}(\mathcal{A}(I))$, then

$$\langle \alpha_\lambda, \sigma \rangle_\mathcal{A} \leq \langle \lambda, \gamma \sigma \rangle_\mathcal{B}$$

(2) If $\alpha_\lambda, \alpha_\mu$ are localized on $I$, and denote by $\epsilon(\alpha_\lambda, \alpha_\mu)$ the braiding operator of $\alpha_\lambda, \alpha_\mu$ considered as (DHR) representations of $\mathcal{A}$, then

$$\epsilon(\alpha_\lambda, \alpha_\mu) = \epsilon(\lambda, \mu)$$

(3) If $\mathcal{B} \subset \mathcal{C} \subset \mathcal{A}$ such that $\mathcal{C} \subset \mathcal{A}$ and $\mathcal{B} \subset \mathcal{C}$ are conformal subnets, and $\lambda_1 := \alpha_{\lambda \mathcal{B} \rightarrow \mathcal{C}}^\mathcal{C} \mathcal{A}$ is localized on $I$. Then

$$\alpha_{\lambda_1 \mathcal{C} \rightarrow \mathcal{A}}(a) = \alpha_{\lambda \mathcal{B} \rightarrow \mathcal{A}}^\mathcal{C}(a), \forall a \in \mathcal{A}(I)$$

where $\mathcal{C} \rightarrow \mathcal{A}, \mathcal{B} \rightarrow \mathcal{A}$ means the induction as defined in Prop. 3.1 from $\mathcal{C}$ to $\mathcal{A}$ and from $\mathcal{B}$ to $\mathcal{A}$ respectively.

Proof. Ad (1): Let $E : \mathcal{A}(I) \rightarrow \mathcal{B}(I)$ be the faithful conditional expectation. Let $v \in \text{Hom}(id, \gamma)_\mathcal{B}$ be the isometry such that $E(\cdot) = v^* \gamma(\cdot) v$ (cf. [LR]). For any $x \in \text{Hom}(\alpha_\lambda, \sigma)_\mathcal{A}$, it is easy to check that $\gamma(x)v \in \text{Hom}(\lambda, \gamma \sigma)_\mathcal{B}$. To prove (1), we just have to show that the linear map

$$x \in \text{Hom}(\alpha_\lambda, \sigma)_\mathcal{A} \rightarrow \gamma(x)v \in \text{Hom}(\lambda, \gamma \sigma)_\mathcal{B}$$

is one-to-one. Assume that $\gamma(x)v = 0$, then

$$E(x^*x) = v^* \gamma(x^*)v \gamma(x)v = 0$$

It follows that $x = 0$ since $E$ is faithful.

Ad (2) and (3): Let $u_\lambda$ be a unitary intertwiner from $\lambda$ to $\hat{\lambda}$. By (1) of Lemma 3.2 $u_\lambda$ is also a unitary intertwiner from $\alpha_\lambda$ to $\alpha_{\hat{\lambda}}$. So

$$\epsilon(\alpha_\lambda, \alpha_\mu) = \alpha_\mu(u_\lambda)^* u_\lambda = \mu(u_\lambda)^* u_\lambda = \epsilon(\lambda, \mu)$$

and

$$\alpha_{\lambda_1 \mathcal{C} \rightarrow \mathcal{A}}(a) = u_\lambda^*(a) u_\lambda = \alpha_{\lambda \mathcal{B} \rightarrow \mathcal{A}}^\mathcal{C}(a), \forall a \in \mathcal{A}(I)$$

by formula (3) in Prop. 3.1.

$\Box$

Note that by Prop. 3.1, $\alpha_\lambda \in \text{End}(\mathcal{A}(I))$. We will use $d_{\alpha_\lambda}$ to denote the statistical dimension of $\alpha_\lambda$. 
Proposition 3.4. If $\lambda$ has finite index, then:

1) $d_\lambda = d_{\alpha\lambda}$;

2) (Commuting Squares) Let $E$ be the conditional expectation from $A(I)$ to $B(I)$, and $F_\lambda$ the minimal conditional expectation from $A(I) \to \alpha_\lambda(A(I))$. Then $EF_\lambda = F_\lambda E$.

Proof: Ad (1): The proof is completed in three steps.

In the first step we show that $d_\lambda \geq d_{\alpha\lambda}$. We give two different proofs of $d_\lambda \geq d_{\alpha\lambda}$.

Denote by $E_\lambda : B(I) \to \lambda_\epsilon(B(I))$ the unique minimal conditional expectation. Let us first show that $E_\lambda(\gamma(A(I))) \subset \gamma(A(I))$.

Note that by [L1], there exists an isometry $v \in Hom(id, \bar{\lambda}_\epsilon\lambda_\epsilon)$ such that $E_\lambda(\cdot) = \lambda_\epsilon(v^*)\lambda_\epsilon\bar{\lambda}_\epsilon(\cdot)\lambda_\epsilon(v)$ where $\epsilon := \epsilon(\lambda, \gamma) = \gamma(u^*)u, \bar{\epsilon} := \epsilon(\bar{\lambda}, \gamma) = \gamma(\bar{u}^*)\bar{u}, \lambda_\epsilon := Ad_\epsilon\lambda, \bar{\lambda}_\epsilon := Ad_{\bar{\epsilon}}\bar{\lambda}$ and $u, \bar{u} \in B(J), J \supset I$ are unitary intertwinners as in the definition of braiding operators (cf. §2.2). we have

$$\lambda_\epsilon(b) = \gamma(u^*)b\gamma(u), \lambda_\epsilon(b) = \gamma(\bar{u}^*)b\gamma(\bar{u})$$

and so

$$E_\lambda(\gamma(a)) = \gamma(u^*)v^*\gamma(\bar{u}^*a\bar{u})v\gamma(u)$$

Let us show that $v \in \gamma(A(I))$.

Since $v \in Hom(id, \bar{\lambda}_\epsilon\lambda_\epsilon)$ we have

$$vb = \gamma(\bar{u}^*u^*)b\gamma(u\bar{u})v, \forall b \in B(I)$$

and so

$$\gamma(u\bar{u})v \in B(I)^{'} \cap B(J) = B(I_+)$$

where the equality follows from the strong additivity of $B$. Hence

$$v = \gamma(\bar{u}^*u^*)\gamma(u\bar{u})v \in \gamma(B(J))$$
since $\gamma$ is localized on $I$. As in the proof of Prop. 3.1 we can choose a decreasing sequence of $J_n$ such that $\cap_n J_n = I$ and we have $v \in \cap_n \gamma(B(J_n)) = \gamma(B(I))$. From the expressions for $E_\lambda(\gamma(a))$ above we have proved that

$$E_\lambda(\gamma(A(I))) \in \cap_n \gamma(A(J_n)) = \gamma(A(I))$$

Let us show that

$$\gamma(A(I)) \cap \lambda_\epsilon(B(I)) = \lambda_\epsilon \gamma(A(I))$$

If $\lambda_\epsilon(b) = \gamma(a)$, then

$$\gamma(u^*b\gamma(u)) = \gamma(a)$$

and so

$$b = \gamma(uau^*) \in \gamma(A(J))$$

and by the same argument as above

$$b \in \gamma(A(I))$$

and this shows that

$$\gamma(A(I)) \cap \lambda_\epsilon(B(I)) = \lambda_\epsilon \gamma(A(I))$$

Hence

$$E_\lambda(\gamma(A(I))) \subset \gamma(A(I)) \cap \lambda_\epsilon(B(I)) = \lambda_\epsilon \gamma(A(I))$$

So the minimal index of $\lambda_\epsilon \gamma(A(I)) \subset \gamma(A(I))$ is less or equal to $d_\lambda^2$. Recall that $\lambda_\epsilon \gamma = \gamma(\alpha_\lambda)$ and so the minimal index of $\lambda_\epsilon \gamma(A(I)) \subset \gamma(A(I))$ is $d_{\alpha_\lambda}^2$. So we have shown that $d_{\alpha_\lambda} \leq d_\lambda$.

Now we give a second proof of $d_{\alpha_\lambda} \leq d_\lambda$.

Let $F_\lambda : B(I) \to \lambda(B(I))$ be the minimal conditional expectation. Since $\lambda$ has finite index, there are two isometries

$$R_{\lambda\bar{\lambda}} \in (id, \lambda\bar{\lambda}), \; R_{\bar{\lambda}\lambda} \in (id, \bar{\lambda}\lambda)$$

in $B(I)$ such that

$$R_{\lambda\bar{\lambda}}^* R_{\lambda\bar{\lambda}} = R_{\bar{\lambda}\lambda}^* \bar{\lambda} R_{\bar{\lambda}\lambda} = \frac{1}{d(\lambda)}$$

By [L1] (also cf. [LR]) $F_\lambda$ is induced by an isometry $R_{\lambda\bar{\lambda}} \in Hom(id, \lambda\bar{\lambda})$ such that

$$F_\lambda(\cdot) = \lambda(R_{\lambda\bar{\lambda}}(\cdot)\bar{\lambda})\lambda(R_{\lambda\bar{\lambda}}(\cdot))$$

By (1) of Lemma 3.2 we have

$$R_{\lambda\bar{\lambda}} \in Hom(id, \alpha_\lambda\alpha_{\bar{\lambda}}) R_{\lambda\bar{\lambda}} \in Hom(id, \alpha_{\lambda\bar{\lambda}})$$
and it follows from the properties of $R_{\lambda\lambda}, R_{\lambda\lambda}^*$ that (cf. [LR])

$$F_\lambda(\cdot) = \lambda(R_{\lambda\lambda})\alpha_\lambda \alpha_\lambda^\ast(\cdot)\lambda(R_{\lambda\lambda}^*)$$

is a conditional expectation from $\mathcal{A}(I) \to \alpha_\lambda(\mathcal{A}(I))$ with index $d_\lambda^2$. It follows that $d_{\alpha_\lambda} \leq d_\lambda$.

In the second step we show that if $\lambda$ is amenable (cf. §2.1), then $d_\lambda = d_{\alpha_\lambda}$.

Let $\Gamma$ be the principal graph of $\lambda$, and $\Gamma_0$ the set of even vertices. By Lemma 3.2 and properties of statistical dimensions $V := (d_{\alpha_x})_{x \in \Gamma_0}$ is a vector which verifies $\Gamma^t V = d_{\alpha_\lambda} d_{\alpha_\lambda}^\ast V$.

By Prop. 1.3.5 of [Po2],

$$||\Gamma||^2 = \lim_{n \to \infty} ((\Gamma^t)^n \delta, \delta)^{\frac{1}{n}}$$

where $\delta \in l^2(\Gamma_0)$ is a vector which is 1 at the identity sector and 0 elsewhere. Hence

$$||\Gamma||^2 = \lim_{n \to \infty} ((\Gamma^t)^n \delta, \delta)^{\frac{1}{n}}$$

$$\leq \lim_{n \to \infty} ((\Gamma^t)^n \delta, V)^{\frac{1}{n}}$$

$$= \lim_{n \to \infty} (\delta, (\Gamma^t)^n V)^{\frac{1}{n}}$$

$$= \lim_{n \to \infty} (\delta, (d_{\alpha_\lambda} d_{\alpha_\lambda}^\ast)^n V)^{\frac{1}{n}}$$

$$= d_{\alpha_\lambda} d_{\alpha_\lambda}^\ast$$

Since $\lambda$ is amenable, we have $d_\lambda^2 = ||\Gamma||^2$, and it follows from above $d_\lambda^2 \leq d_{\alpha_\lambda} d_{\alpha_\lambda}^\ast$. By the first step we must have $d_\lambda = d_{\alpha_\lambda}$.

Finally, by Theorem 5.31 and remarks on Page 122 of [LRo], $\lambda$ is always amenable. Combining this with above (1) is proved.

Ad(2): Since by (1) $d_\lambda = d_{\alpha_\lambda}$ it follows that $F_\lambda$ defined at the the proof of (1) above is the minimal conditional expectation.

For any $a \in \mathcal{A}(I)$, we first note that

$$E(\alpha_{\lambda^\ast}(a)) = E(u^* au) = u^* E(a) u = \alpha_{\lambda^\ast}(E(a))$$

where $u$ is the unitary intertwiner transporting $\lambda^\ast$ to $\lambda^\ast$ which is localized on $I_+$. We have:

$$E(F_\lambda(a)) = E(\lambda(R_{\lambda\lambda})\alpha_\lambda \alpha_\lambda^\ast(a)\lambda(R_{\lambda\lambda}^*))$$

$$= \lambda(R_{\lambda\lambda}) E(\alpha_\lambda \alpha_\lambda^\ast(a)) \lambda(R_{\lambda\lambda}^*)$$

$$= F_\lambda(E(a))$$

□
Proposition 3.5. Suppose $\mu, \lambda$ are representations of $\mathcal{B}$ localized on $I$. Then:

(1) Let $x$ be any subsector of $\alpha_\mu$, then

$$[\alpha_\lambda][x] = [x][\alpha_\lambda];$$

(2) Let $z$ and $y$ be subsectors of $\alpha_\lambda$ and $\tilde{\alpha}_\mu$ respectively, then $[z][y] = [y][z].$

Proof: Ad (1): Let $\epsilon(\lambda, \mu)$ be the braiding operator. Then

$$\alpha_\mu \alpha_\lambda = \alpha_{Ad(\epsilon(\lambda, \mu))\lambda \mu}$$

$$= \alpha_{Ad(\epsilon(\lambda, \mu))\alpha_\lambda \alpha_\mu}$$

$$= Ad(\epsilon(\lambda, \mu))\alpha_\lambda \alpha_\mu$$

Now let $v_x \in \text{Hom}(x, \alpha_\mu)$ be the isometry such that $x(\cdot) = v_x^* \alpha_\mu(\cdot)v_x$. As in the proof of Th. 3.6 on Page 377 of [Xb], it is sufficient to show that

$$\alpha_\lambda(v_x v_x^*) = \epsilon(\lambda, \mu)v_x v_x^* \epsilon(\lambda, \mu)^*$$

Applying $\gamma$ to the above equality, it is sufficient to show that

$$\gamma \alpha_\lambda(v_x v_x^*) = \gamma(\epsilon(\lambda, \mu)v_x v_x^* \epsilon(\lambda, \mu)^*)$$

This follows from YBE and BFE as on Page 377 of [Xb].

Ad (2): We first prove that

$$Ad(\epsilon(\lambda, \mu))\alpha_\lambda \tilde{\alpha}_\mu = \tilde{\alpha}_\mu \alpha_\lambda$$

Let $u_{\lambda+}$ and $u_{\mu-}$ be the intertwiners as in the definition of $\alpha_\lambda$ and $\tilde{\alpha}_\mu$ in Prop. 3.1. Then the above equality is equivalent to

$$\epsilon(\lambda, \mu)u_{\lambda+}^* u_{\mu-}^* = u_{\mu-}^* u_{\lambda+}^*$$

Since $\epsilon(\lambda, \mu) = \mu(u_{\lambda+}^*)u_{\lambda+}$, we need to show that

$$u_{\mu-}(u_{\lambda+}^*)u_{\mu-}^* = u_{\lambda+}^*$$

which follows from the fact that $u_{\lambda+}^* \in \mathcal{B}(I \cup I_+)$ and $\tilde{\mu}$ is localized on $I_-$. The rest of the proof follows by YBE and BFE as on Page 385 of [Xb].

We note that many properties of relative braidings implicitly used in [Xb] and further studied in [BE3] can also be proved in our current setting, but we will not use them in this paper.

§3.2 Strongly additive pairs. Let $\mathcal{A}$ be a conformal net and let $\mathcal{B} \subset \mathcal{A}$ be a conformal subnet as defined in §2.1. Motivated by the proof of Theorem 2.4, we give the following definition:
Definition 3.2. The pair $B \subset A$ is said to be strongly additive if

$$B(I_1) \vee A(I_2) = A(I)$$

for any intervals $I, I_1, I_2$ such that $I_1, I_2$ are the connected components of a set obtained from $I$ by removing an interior point of $I$.

Note that the above definition can be generalized to nets of algebras without conformal invariance, but conformal nets give most interesting examples of strongly additive pairs, and so we shall consider only conformal nets in this paper. Also note that by conformal invariance, it is sufficient to check the condition in the above definition for a particular $I, I_1, I_2$.

Lemma 3.6. (1) If the pair $B \subset A$ is strongly additive, then $B$ and $A$ are strongly additive, and

$$B(I_1)' \cap A(I_1) = \mathbb{C}, \forall I_1$$

(2) If $G$ is a compact group acting properly on $A$ and $B$ is the fixed point subnet under the action of $G$, then the pair $B \subset A$ is a strongly additive pair if and only if either $A$ is strongly additive or $B$ is strongly additive.

(3) Let $B \subset C \subset A$ be conformal subnets. Then the pair $B \subset A$ is strongly additive iff the pairs $B \subset C$ and $C \subset A$ are strongly additive.

Proof: Ad(1): The first follows trivially from the definition and applying conditional expectation from $A$ to $B$. For the second part, choose $I_2$ which share only one boundary point with $I_1$ and let $I$ be the smallest interval containing $I_1 \cup I_2$. Then by the assumption we have

$$B(I_1) \vee A(I') = A(I_2')$$

Taking the commutants and applying the Haag Duailty, we get

$$B(I_1)' \cap A(I) = A(I_2),$$

and so

$$B(I_1)' \cap A(I_1) \subset A(I_2)$$

for all $I_2$ sharing only one boundary point with $I_1$. Choose a sequence of $I_2^{(n)}$ so that $\cap_n I_2^{(n)}$ is one point, by (c) of Prop. 2.1 we get

$$B(I_1)' \cap A(I_1) \subset \cap_n A(I_2^{(n)}) = \mathbb{C}$$

Ad(2): By Theorem 2.4 and (1), it is sufficient to show that if $B$ is strongly additive, then the pair $B \subset A$ is strongly additive. Let $I$ be an arbitrary interval, and $I_1, I_2$ are the connected components of a set obtained from $I$ by removing an interior point of $I$. Then we have the following inclusions:

$$B(I) \subset B(I_1) \vee A(I_2) \subset A(I)$$
where the first inclusion follows from the strong additivity of $\mathcal{B}$. By [I], there exists a closed subgroup $G_1$ of $G$ such that $\mathcal{B}(I_1) \vee \mathcal{A}(I_2)$ is the fixed point subalgebra of $\mathcal{A}(I)$ under the action of $G_1$. It follows that there is normal faithful conditional expectation form $\mathcal{A}(I)$ to $\mathcal{B}(I_1) \vee \mathcal{A}(I_2)$ preserving the vector state $(\cdot \Omega, \Omega)$. Since $\mathcal{A}(I_2)\Omega$ is dense in $H$ by Reeh-Schlieder theorem in Prop. 2.1 it follows that

$$\mathcal{B}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$$

by Takesaki’s theorem (cf. §9 of [SS])..

(3) follows directly from the definitions and applying suitable conditional expectations.

□

Note that orbifolds as discussed in §2 are examples of strongly additive pairs. The following proposition will be used in §4 to give more strongly additive pairs (cf. the list in §4.1).

**Proposition 3.7.** Let $\mathcal{A}$ be a conformal net and $\mathcal{B} \subset \mathcal{A}$ is a conformal subnet. Assume that $\mathcal{B}$ is strongly additive and $U(g) \in \vee_1 \mathcal{B}(I), \forall g \in G$ where $U$ is the representation of conformal group $G$ as defined in $\mathcal{B}$ of §2.1. Then the pair $\mathcal{B} \subset \mathcal{A}$ is strongly additive.

**Proof.** Let $I$ be an interval and $I_1, I_2$ the intervals obtained by removing an interior point of $I$. Let $E$ be the unique conditional expectation from $\mathcal{A}(I)$ to $\mathcal{B}(I)$ such that $\psi(E(\cdot)) = \psi(\cdot)$ where $\psi(\cdot) = (\cdot \Omega, \Omega)$ is the normal faithful state on $\mathcal{A}(I)$ and $\Omega$ is the vacuum vector. Denote by $\Delta_\psi$ and $\Delta$ the modular operator of $\mathcal{A}(I)$ and $\mathcal{B}(I) \vee \mathcal{A}(I_1)$ with respect to $\Omega$. Notice that $Ad_{\Delta_\psi}$ and $Ad_{\Delta_{\psi^t}}$, $t \in \mathbb{R}$ induce the same automorphism on $\mathcal{B}(I)$ and $\mathcal{B}(I')$, and so $\Delta_{\psi^t} \Delta^{-it} \in \mathcal{B}(I') \cap \mathcal{B}(I')$. By the geometric nature of $\Delta_{\psi^t}$ (cf. Prop. 2.1) and our assumption, $\Delta_{\psi^t} \in \vee_1 \mathcal{B}(I)$, and by strong additivity of $\mathcal{B}$ we have $\Delta_{\psi^t} \in \mathcal{B}(I) \vee \mathcal{B}(I')$, $\forall t \in \mathbb{R}$. So $\Delta_{\psi^t} \Delta^{-it}$ commute with $\Delta_{\psi^{t'}}$ for all $t, t' \in \mathbb{R}$, hence $\Delta_{\psi^t}$ commutes with $\Delta_{\psi^{t'}}$. It follows that for any $t'$, $Ad_{\Delta_{\psi^{t'}}}$ is a one parameter automorphism of $Ad_{\Delta_{\psi^{t'}}}(\mathcal{B}(I) \vee \mathcal{A}(I_1))$ preserving the vector state $(\cdot \Omega, \Omega)$. By KMS condition (cf. Page 28 pf [SS]) $Ad_{\Delta_{\psi^{t'}}}$ is the modular automorphism of $Ad_{\Delta_{\psi^{t'}}}(\mathcal{B}(I) \vee \mathcal{A}(I_1))$ with respect to $\Omega$. Notice that $\overline{\mathcal{A}(I)\Omega} = \overline{\mathcal{A}(I)\Omega}$ for any interval $I$ by Reeh-Schlieder’s Theorem, it follows by a Theorem of Takesaki (cf. §9 of [SS]) that

$$Ad_{\Delta_{\psi^{t'}}}(\mathcal{B}(I) \vee \mathcal{A}(I_1)) = \mathcal{B}(I) \vee \mathcal{A}(I_1), \forall t' \in \mathbb{R}$$

Since $\cup_{t'} \Lambda_I(2\pi t'I_1 = I$, it follows that

$$\mathcal{B}(I) \vee \mathcal{A}(I_1) = \mathcal{A}(I)$$

which proves the proposition since $\mathcal{B}$ is strongly additive.

□

The following is a generalization of Th. 3.3 of [Xb] and is the key result of §3:
Theorem 3.8. Let $\mathcal{B} \subset \mathcal{A}$ be a strongly additive pair of conformal nets, and suppose $\mu, \lambda$ are representations of $\mathcal{B}$ localized on $I$.

(1) If $x \in \mathcal{A}(I)$ satisfies $x\lambda(b) = \mu(b)x$, $\forall b \in \mathcal{B}(I)$, then $x\alpha\lambda(a) = \alpha\mu(a)x$, $\forall a \in \mathcal{A}(I)$.

(2) If $\mu, \lambda$ have finite index, then

$$\langle \alpha\mu, \alpha\lambda \rangle = \langle \mu\bar{\lambda}, \gamma \rangle$$

where $\gamma$ is a representation of $\mathcal{B}$ unitarily equivalent to the defining representation of $\mathcal{B}$ on the vacuum Hilbert space of $\mathcal{A}$.

Proof.: Ad (1): Let $u_1$ (resp. $u_2$) be unitary intertwiner in $\mathcal{B}(J), J \supset I$ transporting $\lambda$ (resp. $\mu$) to $\bar{\lambda}$ (resp. $\bar{\mu}$) localized on $I_+$ as in the definition of braiding operator (cf. §2.2). Since $x\lambda(a) = \mu(a)x$, $\forall a \in \mathcal{B}(I)$ and $x \in \mathcal{A}(I)$, it follows that by formula (3) of Prop. 3.1 $u_2xu_1^* \in \mathcal{A}(J) \cap \mathcal{B}(J)'$. Let us choose $J$ so that $I, I_+$ are the intervals obtained by removing an interior point of $J$. By the strong additive pair assumption, we have

$$\mathcal{A}(J') \cup \mathcal{B}(I) = \mathcal{A}(I_+)$$

and so

$$\mathcal{A}(J) \cap \mathcal{B}(I)' = \mathcal{A}(I_+)$$

It follows that

$$u_2xu_1^* \in \mathcal{A}(I_+)$$

and (1) follows from formula (3) in Prop. 3.1 and and locality of the net $\mathcal{A}$.

Ad (2): Since $\lambda$ has finite index, there are two isometries

$$R_{\lambda\bar{\lambda}} \in (id, \lambda\bar{\lambda}), R_{\bar{\lambda}\lambda} \in (id, \bar{\lambda}\lambda)$$

in $\mathcal{B}(I)$ such that

$$R_{\lambda\bar{\lambda}}^*\lambda(R_{\bar{\lambda}\lambda}) = R_{\bar{\lambda}\lambda}^*\bar{\lambda}(R_{\lambda\bar{\lambda}}) = \frac{1}{d(\lambda)}$$

Let $x \in Hom(\alpha\lambda, \alpha\mu)$. Then

$$xR_{\lambda\bar{\lambda}}b = \mu\bar{\lambda}(b)xR_{\lambda\bar{\lambda}}, \forall b \in \mathcal{B}(I)$$

Note that $xR_{\lambda\bar{\lambda}} \in \mathcal{A}(I)$. Since $\mathcal{B}(I) \subset \mathcal{A}(I)$ is irreducible by Lemma 3.6, the vector space $H_{\mu\bar{\lambda}} := \{ y \in \mathcal{A}(I) | yb = \mu\bar{\lambda}(b)y, \forall b \in \mathcal{B}(I) \}$ is a finite dimensional vector space with dimension

$$\langle \mu\bar{\lambda}, \gamma \rangle$$

by Th. 3.3 (i) of [I]. Note that the map

$$x \in Hom(\alpha\lambda, \alpha\mu)_A \rightarrow xR_{\lambda\bar{\lambda}} \in H_{\mu\bar{\lambda}}$$
is one-to-one by the relations satisfied by $R_{\lambda\bar{\lambda}}, R_{\bar{\lambda}\lambda}$. To prove (2) it is sufficient to show that the following one-to-one map

$$y \to \mu(R_{\lambda\lambda}^*)y$$

where $y \in H_{\mu\bar{\lambda}}$ is a map from $H_{\mu\bar{\lambda}}$ to $Hom(\alpha_{\lambda}, \alpha_{\mu})_\mathcal{A}$. Note that

$$\mu(R_{\lambda\lambda}^*)y(\lambda(b)) = \mu(b)\mu(R_{\lambda\lambda}^*)y$$

and $\mu(R_{\lambda\lambda}^*)y \in \mathcal{A}(I)$. By (1)

$$\mu(R_{\lambda\lambda}^*)y \in Hom(\alpha_{\lambda}, \alpha_{\mu})_\mathcal{A}$$

□

**Corollary 3.9.** With the notations as in Theorem 3.8, and assume that $\lambda$ has finite index. Then (1) $[\alpha_{\lambda}] = [\bar{\alpha}_{\lambda}]$;

(2) Let $H_\Lambda := \{x \in \mathcal{A}(I) | xb = \lambda(b)x, \forall b \in \mathcal{B}(I)\}$. $H_\Lambda$ is called the space of charged intertwinners associated with $\lambda$ as in [LR]. Then $H_\Lambda = Hom(id, \alpha_{\lambda})$ and $\dim H_\Lambda = \langle \gamma, \lambda \rangle \leq d_{\lambda}$.

(3) Let $\mathcal{A}_f(I) \subset \mathcal{A}$ be the subalgebra generated by $\mathcal{B}(I)$ and $H_\Lambda, \lambda \in \mathcal{S}$, where $\mathcal{S}$ is a set of (DHR) irreducible representations of $\mathcal{B}$ with finite statistical dimensions, and is closed under fusion and conjugation. Then $\mathcal{A}_f(I)$ is invariant as a set under the modular automorphism $AdU(\Lambda_I(t))$ (cf. Prop. 2.1), and there exists a unique conformal subnet $\mathcal{L} \subset \mathcal{A}$ such that

$$\mathcal{B}(J) \subset \mathcal{L}_f(J) \subset \mathcal{A}(J), \forall J \in \mathcal{I}$$

and $\mathcal{A}_f(I) = \mathcal{L}(I)$. Moreover, the vacuum representation $H_\mathcal{L}$ of the conformal net $\mathcal{L}$ as a representation of $\mathcal{B}$ decomposes as $H_\mathcal{L} \simeq \oplus_{\lambda \in \mathcal{S}} \dim H_\lambda \lambda$.

**Proof.** Ad (1): The proof is the same as that of [BE1] by Lemma 3.2 and (2) of Theorem 3.8 as follows:

$$\langle \alpha_{\lambda}, \alpha_{\lambda} \rangle = \langle \lambda\bar{\lambda}, \gamma \rangle$$

$$= \langle \alpha_{\lambda\lambda}, id \rangle$$

$$= \langle \alpha_{\lambda\lambda}, id \rangle$$

$$= \langle \alpha_{\lambda}, \bar{\alpha}_{\lambda} \rangle$$

Replace $\lambda$ by $\bar{\lambda}$ we get

$$\langle \alpha_{\lambda\lambda}, \alpha_{\lambda} \rangle = \langle \alpha_{\lambda\lambda}, \bar{\alpha}_{\lambda} \rangle$$

Thus we have

$$\langle \alpha_{\lambda\lambda}, \alpha_{\lambda} \rangle = \langle \alpha_{\lambda\lambda}, \bar{\alpha}_{\lambda} \rangle = \langle \alpha_{\lambda\lambda}, \bar{\alpha}_{\lambda} \rangle$$
Since $d_{\alpha_\lambda} < \infty, d_{\alpha_{\lambda^*}} < \infty$, by Prop. 3.4, the above identities imply (1).

(2) follows directly from (2) of Theorem 3.8 and (1) of Prop. 3.4.

Ad (3): Let $F_\lambda$ be the minimal conditional expectation as in (3) of Prop. 3.4. By [L1] we can choose a set of isometries $v_i \in Hom(id, \alpha_\lambda), 1 \leq i \leq dim H_\lambda$ such that $F_\lambda(v_i v_i^*) = \frac{1}{d_\lambda}\delta_{ij}$. Note that $E(v_i v_i^*) \in B(I) \cap A(I) = \mathbb{C}$, and by (3) of Prop. 3.4 we have

$$E(v_i v_i^*) = F_\lambda(E(v_i v_i^*)) = E(F_\lambda(v_i v_i^*)) = \frac{1}{d_\lambda}\delta_{ij}$$

It follows that the operator $\alpha_\lambda$ as defined on Page 39 of [I] is the identity operator (our $\lambda$ corresponds to $\xi$ in [I]). Then the argument on Page 41 of [I] shows that $\sigma^\psi_E(v_i) = v_i$ where $\psi$ is a dominant weight on $B(I)$, and $\sigma^\psi_E$ is the modular automorphisms associated with the weight $\psi E$. By Haagerup’s Theorem (cf. Page 156 of [SS]),

$$AdU(\Lambda_I(t))((\cdot)) = Ad_{u_t}\sigma^\psi_E((\cdot))$$

where $u_t \in B(I)$. It follows that $A_f(I)$ is invariant as a set under the modular automorphism $AdU(\Lambda_I(t))$. Since $H_\lambda$ is finite dimensional, the rest of the proof is the same as the proof on Page 18 of [Ls] as follows. Note that $\Lambda_I(\mathbb{R})$ is exactly the subgroup of $PSL(2, \mathbb{R})$ which leaves $I$ globally fixed. For each $J \in \mathcal{L}$, set $\mathcal{L}(J) = Ad_{U(g)}(A_f(I))$, where $g \in PSL(2, \mathbb{R}), gI = J$. It is easy to check that $\mathcal{L}(J)$ is independent of the choice of $g$ as long as $gI = J$. Note that $\mathcal{L}(J)$ verifies locality since $\mathcal{L}(J) \subset A(J)$. To show that $\mathcal{L}$ is a conformal net we just have to check the isotony property, namely $\mathcal{L}(J_1) \subset \mathcal{L}(J_2)$ if $J_1 \subset J_2$. By conformal invariance we may assume that $J_1 = I$ and that $J_2 = gI$ for some $g \in PSL(2, \mathbb{R})$, and it is sufficient to show that $Ad_{U(g)}(A_f(I)) \supset A_f(I)$. Since $H_\lambda$ is finite dimensional, By the second part of Cor. 19 in [Ls] $Ad_{U(g)}H_\lambda = z_\lambda(g)^*H_\lambda$, where $z_\lambda(g) \in B(J_2)$ is a unitary defined by formula (14) of [Ls]. Hence

$$Ad_{U(g)}(A_f(I)) = \{Ad_{U(g)}(B(I)), Ad_{U(g)}H_\lambda, \lambda \in S\}'' = \{B(J_2), z_\lambda(g)^*H_\lambda, \lambda \in S\}'' = \{B(J_2), H_\lambda, \lambda \in S\}'' \supset \{B(I), H_\lambda, \lambda \in S\}'' = A_f(I)$$

Now let $\Omega$ be the vacuum vector for $A$, since $\mathcal{L}$ is a conformal net, the vacuum representation space of $\mathcal{L}$ can be identified as $\overline{A_f(I)\Omega}$ by Reeh-Schiller theorem in Prop. 2.1. For each $\lambda \in S$, we choose isometries $v_{\lambda,i}, 1 \leq i \leq dim H_\lambda$ as in the beginning of the proof of (3) (we add a subscript $\lambda$ to emphasize its dependence on $\lambda$). Then the set consisting of $\sum_{\lambda \in S} v_{\lambda,i} x_{\lambda,i}, x_{\lambda,i} \in B(I)$ where the sum is a finite sum is a dense subalgebra of $A_f(I)$. Note that the space $X_{\lambda,i} := v_{\lambda,i}^* B(I)\Omega$ is invariant under the action of $\mathcal{B}$, and the restriction of $\mathcal{B}$ to this space is a representation of $\mathcal{B}$ unitarily equivalent to $\lambda$. Also note that $X_{\lambda,i} \perp X_{\lambda',i'}$ if $(\lambda, i) \neq (\lambda', i')$ since $E(v_{\lambda,i} v_{\lambda',i'}^*) = \frac{1}{d_\lambda^2}\delta_{\lambda\lambda'}\delta_{ii'}$. Hence $\overline{A_f(I)\Omega}$ is a direct sum of $X_{\lambda,i}$, and this proves the last part of (3).
We will call the conformal net $L$ constructed in (3) of Cor. 3.9 the conformal subnet of $A$ generated by $B$ and charged intertwinners associated with the set $S$.

§4. Applications

§4.1 Conformal nets with central charge 1. The irreducible representations of Virasoro algebra with central charge 1 are classified as follows. For each $n \geq 0$, there is an irreducible representation with lowest weight $n$, and such representation is denoted by $L(1, n)$ as in [D]. Here 1 is the central charge. When $n = m^2$, $2m \in \mathbb{Z}$, $L(1, n)$ will be called the degenerate representations due to the degenerate nature of certain Verma modules. The vacuum representation is $L(1, 0)$. All $L(1, n)$ can be “exponentiated” to give irreducible projective representations of $Diff(S^1)$ (cf. [GW]), where $Diff(S^1)$ is the group of smooth differmorphisms of $S^1$. On the vacuum representation $L(1, 0)$, one can define a conformal net $A_{c=1}$ as in §3 of [FG], called Virasoro net with central charge 1.

Let $A_{SU(2)}$ be the conformal net associated with loop group $LSU(2)$ at level 1. The adjoint action of group $SO(3)$ acts properly on $A_{SU(2)}$, in the sense of §2.3, and the fixed point is identified as $A_{c=1}$, the Virasoro net with central charge $c = 1$, in [R3].

It follows from Theorem 2.4 that $A_{c=1}$ is strongly additive. It is already pointed out in [R3] that the statistical dimension $d_{L(1, m^2)}$ of $L(1, m^2)$ is $2m + 1$ when $m$ is a non-negative integer, and the fusion rules among $L(1, m^2)$ are the same as representation rings of $SO(3)$. The following lemma generalize this to the case when $2m$ is a non-negative integer:

**4.1 Lemma.** Assume that $2m$ are non-negative integers. Then $d_{L(1, m^2)} = 2m + 1$ and the fusion ring generated by $L(1, m^2)$ is isomorphic to the representation ring of $SU(2)$.

**Proof.** Consider the following conformal inclusion (cf. §3.1 of [Xj])

$$LSU(2)_1 \times LU(1)_2 \subset LU(2)_1$$

The group $SU(2)$ acts properly on the net $A_{U(2)}$ with fixed point net $A_{c=1} \times A_{U(1)}$. We note that the net $A_{U(2)}$ is not local, but satisfies the twisted duality (cf. §15 of [Wa]). So Th. 3.6 in [DR2] apply in this case, and the fusion of those irreducible representations of $A_{c=1} \times A_{U(1)}$ appearing in $A_{U(2)}$ are given by the fusion ring of finite dimensional representations of $SU(2)$. Since irreducible covariant representations of $A_{U(1)}$ have statistical dimension equal to 1 and generate an abelian group $\mathbb{Z}_2$ under the fusion, the lemma follows.

Let $A$ be a conformal net. Following [KL], we say that $A$ is a diffeomorphism covariant net if there exists a unitary projective representation $U$ of $Diff(S^1)$ on $H$ extending the unitary representation of $PSL(2, \mathbb{R})$ such that

$$U(g)A(I)U(g^*) = A(g.I), g \in Diff(S^1), I \in I$$
We say that $A$ is a \textit{conformal net with central charge 1} if $A$ is a diffeomorphism covariant net containing $A_{c=1}$ as a conformal subnet such that $U(Diff(I))'' = (A_{c=1})(I), \forall I \in \mathcal{I}$, where $Diff(I)$ denotes the group of smooth diffeomorphisms $g$ of $S^1$ satisfying $g(t) = t, t \in I$.

Let us describe the known list of such nets. Let $G$ be a closed group of $SO(3)$. Such groups are well known to be of $A-D-E$ groups corresponding to affine $A-D-E$ graphs. Let $\hat{G}$ be two-fold covering group of $G$ in $SU(2)$. We note the Perron-Frobenius eigenvectors given on Page 14 of [GHJ] are the dimensions of the irreducible representations of the two-fold covering group $\hat{G}$. Since $A_{c=1}$ can be identified with $A_{SU(2),1}, A_{SU(2),1}$ is a conformal net with central charge $c = 1$.

The remaining two cases are $A_{U(1),2n}$ and its $\mathbb{Z}_2$ orbifold $A_{U(1),2n}^{\mathbb{Z}_2}$ as studied in [Xo], where $n$ is not the square of an integer. So the known list of conformal nets with central charge 1 is:

$$A_{SU(2),1}^G, A_{U(1),2n}, A_{U(1),2n}^{\mathbb{Z}_2} \quad (*)$$

where $G$ is a closed subgroup of $SO(3)$ and $n$ is not the square of an integer.

It has been conjectured (cf [DVVV]) that the list (*) exhausts all conformal theories with central charge 1.

When $G$ is a finite group, $A_{SU(2),1}^G$ is absolutely rational by Prop. 2.2.

$A_{U(1),2n}, A_{U(1),2n}^{\mathbb{Z}_2}$ are also absolutely rational and all irreducible representations are obtained in [Xo]. The irreducible representations of $A_{U(1),2n}$ will be denoted by $\pi_i, i \in \mathbb{Z}_{2n}$. They generate a fusion ring isomorphic to $\mathbb{Z}_{2n}$.

When $G = U(1)$, $A_{SU(2),1}^G$ is the net corresponding to the Heisenberg group, denoted by $H(1)$. $H(1)$ is the set $C^\infty(S^1, \mathbb{R}) \times S^1$ with multiplication defined by (cf. §9.5 of [PS])

$$(f_1, x_1) \cdot (f_2, x_2) = (f_1 + f_2, e^{f_1 f_2} x_1 x_2)$$

Note that $(x, 1)$ where $x \in \mathbb{R}$ is considered as a constant map is in the center of $H(1)$. For each real number $q$, there is an irreducible representation of $H(1)$ denoted by $F_q$ where $(x, 1)$ acts on $F_q$ as $(x, 1) \rightarrow q x$, and these are all the irreducible representations of $H(1)$ (cf. Prop. 9.5.10 of [PS]). The net $A_{SU(2),1}^{U(1)}$ is related to $H(1)$ as follows. $F_0$ is the vacuum representation of $A_{SU(2),1}^{U(1)}$, and $A_{SU(2),1}^{U(1)}(I) = \pi_{F_0}(C^\infty_0(I, \mathbb{R}))''$ where $C^\infty_0(I, \mathbb{R})$ is the set of smooth maps from $I$ to $\mathbb{R}$ which vanishes on the boundary, and is considered as a subspace of $C^\infty(S^1, \mathbb{R})$. Each $F_q$ is also an irreducible representation of $A_{SU(2),1}^{U(1)}$. The net $A_{SU(2),1}^{U(1)}$ was studied in [BMT].

Some decompositions of the vacuum representation of the nets in the list above when restricting to $A_{c=1}$ are also known (cf. Prop. 2.2, Th. 2.7 and Th. 2.9 of [D]) where our $2n$ corresponds to $n$ in [D]) as follows:
If \( n \) is not the square of an integer, then
\[
H_{A_U(1)_{2n}} = (\oplus_{p \geq 0} L(1, p^2)) \oplus (\oplus_{m > 0} 2L(1, m^2 n))
\]
\[
H_{\tilde{A}_{SU(2)}^{U(1)}_{2n}} = (\oplus_{p \geq 0} L(1, 4p^2)) \oplus (\oplus_{m > 0} L(1, m^2 n))
\]

If \( n = k^2 \) where \( k \) is a nonnegative integer then
\[
H_{A_{U(1)_2}} = \oplus_{m \geq 0} \oplus_{0 \leq p \leq k-1} (2m + 1)L(1, (mk + p)^2)
\]

When \( G = U(1) \) or \( G = D_\infty \) (infinite dihedral group), we have:
\[
H_{A_{U(1)}^{SU(2)_1}} = \oplus_{p \geq 0} L(1, p^2)
\]
\[
H_{A_{D_\infty}^{SU(2)_1}} = \oplus_{p \geq 0} L(1, 4p^2)
\]

Recall that \( F_q \) is the irreducible representation of \( A_{SU(2)_1}^{U(1)} \) corresponding to an irreducible representation of \( H(1) \) labeled by a real number \( q \). The decompositions of \( F_q \) with respect to \( A_{c=1} \) is also well known to be (cf. [D]) the following:

If \( q = \frac{p^2}{4} \) for some non-negative integer \( p \), then
\[
F_q = \oplus_{-\frac{1}{2}p \leq m \leq \frac{1}{2}p, m + \frac{1}{2} p \in \mathbb{Z}} L(1, m^2)
\]

If \( 4q \) is not the square of an integer, then \( F_q = L(1, \sqrt{q}) \).

We note that by definition if \( A \) is a conformal net with central charge 1, then the pair \( A_{c=1} \subset A \) satisfies the condition of Prop. 3.7, and hence is a strongly additive pair in the sense of definition 3.2. By Lemma 4.1, the principal graph of \( L(1, \frac{1}{4}) \) is \( A_\infty \), and \( d_{L(1, \frac{1}{4})} = 2 \). It follows from Prop. 3.4 that \( \alpha_{L(1, \frac{1}{4})} = d_{L(1, \frac{1}{4})} = 2p + 1 \) for all non-negative integer \( 2p \). Since \( \alpha_f := \alpha_{L(1, \frac{1}{4})} \) has minimal index 4, its principal graph are determined in [GHJ] and [Po]. In the following Lemma we list properties of \( \alpha_f \):

**Lemma 4.2.** The possible principal graphs of \( \alpha_f \) are given by the A-D-E graphs on Page 19 of [GHJ]. More precisely:

1. \([\alpha_f] = [\bar{\alpha}_f];
2. If \( \alpha_f \) is irreducible, then its principal graph is given by D, E graph on Page 19 of [GHJ] or \( A_\infty \) and \( D_\infty \) on Page 217 of [GHJ].
3. If \( \langle \alpha_f, \alpha_f \rangle = 4 \), then its principal graph is \( A_1^{(1)} \).
4. If \( \langle \alpha_f, \alpha_f \rangle = 2 \), then the principal graph is given by either \( A_{\infty, \infty} \) or \( A_n^{(1)} \).

**Proof:** (1) follows from Cor. 3.9. (2), (3) and (4) follows from §4 of [Po1].

As a warm up exercise, let us work out \( \alpha_f \) for the list (*). We first prove the following Lemma which will also be used in §4.2:
Lemma 4.3. Every covariant representation of $A_{SU(2)_1}^{U(1)}$ is a direct sum of irreducible representations, and every irreducible covariant representation of $A_{SU(2)_1}^{U(1)}$ is isomorphic to some $F_q$.

Proof: Let $\pi$ be a covariant representation of $A_{SU(2)_1}^{U(1)}$. Recall the product rule in $H(1)$: for any $f_i : S^1 \to \mathbb{R}$, $i = 1, 2$,

$$(f_1(x), f_2(x)) = (f_1 + f_2, \frac{f_1}{2}x f_2)$$

We will write $(f_i, 1)$ simply as $f_i$ in the following.

Let $I_i \in \mathcal{I}$, $1 \leq i \leq n$ be an open covering of $S^1$ and $\phi_i$, $1 \leq i \leq n$ a partition of unity such that $\text{supp}(\phi_i) \in I_i$, $1 \leq i \leq n$. If $f : S^1 \to \mathbb{R}$, we assume that in $H(1)$:

$$f = \prod_k \pi_{I_k}(\pi_{F_0}(f_\phi_k))C(f, \phi)$$

where $C(f, \phi) \in \mathbb{C}$ is a phase coming from the product rules. we have $\pi_{F_0}(f_\phi_k) \in A_{SU(2)_1}^{U(1)}(I_k)$, and we define

$$\pi(f) := \prod_k \pi_{I_k}(\pi_{F_0}(f_\phi_k))C(f, \phi)$$

It is routine to check that $\pi(f)$ is independent of the choices of open coverings and partition of unity by using the product rules in $H(1)$ and isotony, and it gives a representation of $H(1)$ with positive energy. The Lemma now follows from Prop. 9.5.10 of [PS] and its proof.

□

Lemma 4.4. (1) If $A = A_{c=1}$, then the principal graph of $\alpha_f$ is $A_\infty$;

(2) If $A = A_{U(1)_{2k^2}}$, then the principal graph of $\alpha_f$ is $A_{2k-1}^{(1)}$;

(3) If $A = A_{U(1)_{2n}}$, where $n$ is not the square of an integer, or $A = A_{SU(2)_1}^{U(1)}$, then the principal graph of $\alpha_f$ is $A_{\infty, -\infty}$;

(4) If $A = A_{U(1)_{2k^2}}$, then the principal graph of $\alpha_f$ is $D_k^{(1)}$;

(5) If $A = A_{U(1)_{2n}}$, where $n$ is not the square of an integer, or $A = A_{SU(2)_1}^{D_{\infty}}$, then the principal graph of $\alpha_f$ is $D_{\infty, -\infty}$;

(6) If $A = A_{SU(2)_1}^{E_i}, i = 6, 7, 8$, then the principal graph of $\alpha_f$ is $E_i$.

Proof: (1) follows from Lemma 4.1. If $A = A_{U(1)_{2k^2}}$, it follows from the branching rules, Lemma 4.1 and Prop. 3.1 that $\alpha_f$ is localized on $I$, and by (2) of Th 3.8 $\langle \alpha_f, \alpha_f \rangle = 4$ when $k = 1$ and $\langle \alpha_f, \alpha_f \rangle = 2$ when $k > 1$. So (2) is proved for $k = 1$ by Lemma 4.2. We note that when $k = 1$, $A_{U(1)}$ can be identified with $A_{SU(2)_1}$, and it is easy to check that $[\alpha_f] = 2[\tau]$, where $\tau$ is the irreducible representation of $A_{SU(2)_1}$, which is not the vacuum representation and $[\tau]^2 = [1]$. 


When \( k > 1 \), \( [\alpha_f] = [\sigma_1] + [\sigma_2] \) where \( \sigma_1, \sigma_2 \) are representations of \( \mathcal{A}_{U(1)_{2k^2}} \) which are classified, and by (1) of Lemma 3.3 we must have \( [\alpha_f] = [\pi_k] + [\pi_{-k}] \) and (2) follows. If \( \mathcal{A} = \mathcal{A}_{U(1)_{2n}} \) where \( n \) is not the square of an integer, or \( \mathcal{A} = \mathcal{A}_{SU(2)_1}^{U(1)} \) it follows from the branching rules, Lemma 4.1 and (2) of Th 3.8 that \( \langle \alpha_{L(1,m^2)}, id \rangle = 1, \langle \alpha_{L(1,m^2)}, \alpha_{L(1,m^2)} \rangle = 2m + 1, \forall m \in \mathbb{N}, \langle \alpha_f, \alpha_f \rangle = 2 \). So \( [\alpha_f] = [\sigma_1] + [\sigma_2] \) and \( \sigma_1 = d_{\sigma_2} = 1 \). By lemma 4.2 \( [\alpha_f] = [\sigma_f] \), so either \( [\sigma_f] = [\sigma_2] \) or \( [\sigma_f] = [\sigma_1], i = 1, 2 \). If \( [\sigma_f] = [\sigma_i], i = 1, 2 \), then \( [\sigma_i^2] = [1], i = 1, 2 \) and

\[
[\alpha_{L(1,1)}] = [\alpha_f^2] - [1] = [\sigma_1\sigma_2] + [\sigma_1\sigma_2] + [1]
\]

Note that by (1) of Prop. 3.5 we have

\[
[\sigma_1\alpha_f] = [\sigma_1\sigma_2] + [1] = [\alpha_f\sigma_1] = [\sigma_2\sigma_1] + [1]
\]

Hence \( [\sigma_1\sigma_2] = [\sigma_2\sigma_1] \) implying that \( \langle \alpha_{L(1,1)}, \alpha_{L(1,1)} \rangle \) is 5, which contradicts

\[
\langle \alpha_{L(1,1)}, \alpha_{L(1,1)} \rangle = 3
\]

So we have \( [\alpha_f] = [\sigma_1] + [\sigma_2], [\sigma_f] = [\sigma_2], \) and it follows that

\[
[\alpha_{L(1,m^2)}] = \sum_{-m \leq k \leq m} [\sigma_1^{2k}]
\]

Since \( \langle \alpha_{L(1,m^2)}, 1 \rangle = 1 \), it follows that \( [\sigma_k^m] \neq [1], \forall k \), and the principal graph of \( \alpha_f \) is \( A_{\infty, -\infty} \).

When \( \mathcal{A} = \mathcal{A}_{U(1)_{2k^2}} \) or \( \mathcal{A} = \mathcal{A}_{SU(2)_1}^{E_i}, i = 6, 7, 8, \alpha_f \) is irreducible and a representation of \( \mathcal{A}_c \) denoted by \( f_1 \). We will call this representation the vector representation. Since \( \mathcal{A} \) is completely rational, it follows that the graph of \( \alpha_f \) must be finite and hence must be finite \( D, E \) type. Consider the following inclusions of conformal nets:

\[
\mathcal{A}_{c=1} \subset \mathcal{A} \subset \mathcal{A}_{SU(2)_1}
\]

consider the induction of \( f_1 \) (as a local representation of \( \mathcal{A} \)) from \( \mathcal{A} \) to \( \mathcal{A}_{SU(2)_1} \), denoted by \( \alpha_{f_1} \) which by (3) of Lemma 3.3, is the same as \( \alpha_{L(1,\frac{1}{4})} \). But from the proof above \( [\alpha_{L(1,\frac{1}{4})}] = 2[\tau] \), where \( \tau \) is the irreducible representation of \( \mathcal{A}_{SU(2)_1} \), which is not the vacuum representation and \( [\tau]^2 = [1] \). For any irreducible representations \( \lambda < f_1^{2n} \) for some \( n \in \mathbb{N} \), let \( m_\lambda := \langle \alpha_\lambda, 1 \rangle \), it follows that \( m_\lambda = d_\lambda \), and by (2) of Th. 3.8, \( m_\lambda \) is the multiplicity of \( \lambda \) which appears in the vacuum representation of \( \mathcal{A}_{SU(2)_1} \), and is equal to the dimension of the representation of the corresponding \( D - E \) groups since \( \mathcal{A} \) is the fixed point net of \( \mathcal{A}_{SU(2)_1} \) under the action of such group. Note that the Perron-Frobenius eigenvectors listed on Page 14 of [GHJ] are the statistical dimensions associated with the corresponding representations. Hence the graph is uniquely determined by the corresponding groups to be those of (4) and (6).
(5) follows by inspecting the branching rules and using Th. 3.8 as we have done in proving (1)-(3).

□

As a by-product of the proof above, we have the following proposition which contains the main result of [R1]:

**Proposition 4.5.** Let \(2m\) be a non-negative integer, and \(n \geq 0\). Then

\[
[L(1, m^2)][L(1, n)] = \sum_{-m \leq k \leq m, k + m \in \mathbb{Z}} [L(1, (k + \sqrt{n})^2)]
\]

**Proof.** By Lemma 4.1 it is sufficient to consider the case when \(L(1, n)\) is generic, i.e., when \(4n\) is not the square of an integer. Consider the inclusion \(A_{c=1} \subset A^{U(1)}_{SU(2)}\). It is a strongly additive pair. As in the proof of (3) in Lemma 4.4, by the branching rules, Lemma 4.1 and Prop. 3.1, \(\alpha_f\) is a (DHR) representation of \(A^{U(1)}_{SU(2)}\), and \([\alpha_f] = [\sigma_1] + [\sigma_2], [\bar{\sigma}_1] = [\sigma_2]\). By Lemma 4.3, \([\sigma_1] = [F_q]\) for some real \(q\), and since by (1) of Lemma 3.3

\[
\langle \alpha_f, F_q \rangle \leq \langle L(1, \frac{1}{4}), \gamma F_q \rangle,
\]

inspecting the branching rules we must have \([\alpha_{L(1, \frac{1}{4})}] = [F_{\frac{1}{2}}] + [F_{-\frac{1}{2}}]\). It follows by Lemma 4.1

\[
[\alpha_{L(1, m^2)}] = \sum_{-m \leq k \leq m, k + m \in \mathbb{Z}} [F_k]
\]

Note that \([\gamma(F_{\sqrt{n}})] = [L(1, n)]\) by the branching rules and Prop 3.1 of [LR]. We have:

\[
[L(1, m^2)][L(1, n)] = [L(1, m^2)][\gamma F_{\sqrt{n}}]
= [\gamma \alpha_{L(1, m^2)}] F_{\sqrt{n}}
= [\gamma] \sum_{-m \leq k \leq m, k + m \in \mathbb{Z}} [F_k F_{\sqrt{n}}]
= [\gamma] \sum_{-m \leq k \leq m, k + m \in \mathbb{Z}} [F_k + \sqrt{n}]
= \sum_{-m \leq k \leq m, k + m \in \mathbb{Z}} [L((k + \sqrt{n})^2)]
\]

□

§4.2 **Classifications.** As in [KL], two conformal nets \(\mathcal{A}\) and \(\mathcal{B}\) are said to be isomorphic, denoted by \(\mathcal{A} \simeq \mathcal{B}\), if there is a uniatry operator \(U : \mathcal{H}_\mathcal{A} \to \mathcal{H}_\mathcal{B}\) such that \(U^*\mathcal{B}(J)U = \mathcal{A}(J), \forall J \in \mathcal{T}, U\OmegaA = \OmegaB\), where \(\OmegaA\) and \(\OmegaB\) are the vacuum vectors of \(\mathcal{A}\) and \(\mathcal{B}\) respectively.
Notice by the proof of Lemma 4.4, we can infer the type of principal graphs for $\alpha_f$ from branching rules and $\gamma$ from Th. 3.8. On the other hand, for the same reason due to Th. 3.8, we can deduce information about $\gamma$ from the type of principal graph of $\alpha_f$. This is the basic strategy which we will follow to classify conformal nets with central charge $c = 1$. This will work out under the following spectrum condition:

**Spectrum condition.** We say that a conformal net with central charge $c = 1$ verifies the spectrum condition if a degenerate representation of the Virasoro net other than the vacuum representation must appear in the vacuum representation of $\mathcal{A}$ if $\mathcal{A} \neq \mathcal{A}_{c=1}$.

**Theorem 4.6.** If a conformal net $\mathcal{A}$ with central charge $c = 1$ verifies the spectrum condition, then $\mathcal{A}$ is isomorphic to one of the nets in the list (*).

The proof is divided into the following steps:

### 4.2.1 Discrete case: Full spectrum.
In this section we assume that
\[
\gamma = \bigoplus_{0 \leq m} (2m + 1) L(1, m^2).
\]
We’d like to show that $\mathcal{A} \simeq \mathcal{A}_{SU(2)_1}$. This is essentially an application of reconstruction theorem of Doplicher-Roberts (cf. [DR1-2]). We give a sketch of the proof and refer to [DR1-2] for details. First by our assumption we can assume that $\mathcal{A}$ and $\mathcal{A}_{SU(2)_1}$ acting on the same Hilbert space, and $\mathcal{A}_{c=1}$ is a common conformal subnet. Let $\Delta := \{ L(1, p^2), p \in \mathbb{Z} \}$. We note that $L(1, 1)$ has permutation symmetry and satisfies special conjugate property, and $\Delta$ is generated by $L(1, 1)$, and is specially directed as defined on Page 98 of [DR1].

Let us fix an interval $I$. Choose charged intertwinners $\psi^\Delta \in \mathcal{A}(I), \psi \in \mathcal{A}_{SU(2)_1}(I)$ for the set $\Delta$ (it is enough to choose charged intertwinners for $L(1, 1)$). Denote by $U_{\Delta}$ (resp. $U$) the $C^*$ algebra generated by $U_{\Delta_{c=1}}$ and $\psi^\Delta$ (resp. $\psi$). Note that $U_{\Delta} \cap U'_{\Delta_{c=1}} = \mathbb{C}$. By Page 93-4 of [DR1] there exists an epimorphism $\phi : U \to U_{\Delta}$ such that $\phi = id$ on $U_{\Delta_{c=1}}$. Using $\phi$ one can define an action of $SO(3)$ which commutes with $U_{\Delta_{c=1}}$. One checks that $\phi$ commutes with the adjoint action of $SO(3)$, and so $\ker \phi$ is $SO(3)$ invariant. Then the argument of Page 95 of [DR1] shows that $\ker \phi = \{0\}$, and so $\phi$ is an isomorphism. Now define a unitary operator $U$ on $H$ by $U m \Omega = \phi(m) \Omega$ where $\Omega$ is the vacuum vector. Then we have
\[
UU^* = U_{\Delta}
\]
and $U$ commutes with $U_{\Delta_{c=1}}$. Passing to the von Neumann algebra generated by $U$ and $U_{\Delta}$, we have $UA(I)U^* = A_{SU(2)_1}(I), U\Omega = \Omega$. Since $U$ commutes with $U_{\Delta_{c=1}}$, $U$ commutes with $PSL(2, \mathbb{R})$ by the strong additivity of $A_{c=1}$, and it follows that
\[
UA(J)U^* = A_{SU(2)_1}(J), \forall J \in \mathcal{I}, U\Omega = \Omega.
\]

### §4.2.2 General discrete case.
In this section we assume that
\[
\gamma = \bigoplus_{p \geq 0} m_p L(1, p^2)
\]
Notice that by (2) of Cor. 3.9 $m_p \leq 2p + 1$. By Lemma 4.1 and Prop. 3.1, $\alpha_{L(1,p^2)}$ is localized. Consider the following set of representations $\Delta : \{ \lambda < \alpha_{L(1,1)}^p, n \in \mathbb{N} \}$
of $\mathcal{A}$. Since $\epsilon(\alpha_{L(1,1)}, \alpha_{L(1,1)}) = \epsilon(L(1,1), L(1,1))$ by (2) of Lemma 3.3, the set $\Delta$ has permutation symmetry, and $\mathcal{A}$ is strongly additive, we can apply Doplicher-Roberts reconstruction as in Prop. 3.9 of [Mu] to obtain conformal net $\mathcal{C}$ such that $\mathcal{A}_{c=1} \subset \mathcal{A} \subset \mathcal{C}$. Consider the conformal nets $\mathcal{A}_{c=1} \subset \mathcal{C}$. We claim that $m_p^c = 2p + 1$ in this case, where $m_p^c$ is the multiplicity of $L(1,p^2)$ which appears in the vacuum representation of $\mathcal{C}$. We have $m_p^c \geq \sum_{\lambda} \langle \alpha_{L(1,p^2)}, \lambda \rangle$, where the sum is over those irreducible representations $\lambda$ of $\mathcal{A}$ which appears in the vacuum representation of $\mathcal{C}$. We note that $\sum_{\lambda} \langle \alpha_{L(1,p^2)}, \lambda \rangle$ is completely determined by the principal graph of $\alpha_f$, which corresponds to A-D-E groups. So the number $\sum_{\lambda} \langle \alpha_{L(1,p^2)}, \lambda \rangle$ depends only the type of A-D-E graph associated with $\alpha_f$. Since all types of such graphs have appeared in Lemma 4.4, and in each case, it is easy to check the number $\sum_{\lambda} \langle \alpha_{L(1,p^2)}, \lambda \rangle$ is $2p + 1$, since for $\mathcal{A}_{c=1} \subset \mathcal{A}_{SU(2)}$, where $G$ is a closed subgroup of $SO(3)$, the above reconstruction give us $\mathcal{A}_{SU(2)}$. It follows that $m_p^c \geq 2p + 1$, and by (2) of Cor. 3.9 $m_p^c = 2p + 1$. Now considered the conformal subnet $\mathcal{D} \subset \mathcal{C}$ generated by $\mathcal{A}_{c=1}$ and charged intertwiners for $\{L(1, p^2) : \forall p \geq 0\}$ whose existence is shown by Cor. 3.9. $\mathcal{D}$ now has full spectrum as in §4.2.1, and so $\mathcal{A}_{c=1} \subset \mathcal{A} \subset \mathcal{D} \simeq \mathcal{A}_{SU(2)}$. Since $\mathcal{A}_{c=1}$ is the fixed point net of $\mathcal{A}_{SU(2)}$ under the action of $SO(3)$, by [I] there exists a closed subgroup $G_1$ of $SO(3)$ such that $\mathcal{A}(I)$ is the fixed point subalgebra of $\mathcal{A}_{SU(2)}(I)$ under the action of $G_1$. Since $G_1$ commutes with $PSL(2, \mathbb{R})$, $\mathcal{A}$ is the fixed point net of $\mathcal{A}_{SU(2)}$, under the action of $G_1$. Since $G_1$ is classified as one of the $A-D-E$ groups, it follows that $\mathcal{A}$ is in the list $(\ast)$. 

4.2.3 $\mathcal{A}_{c=1} \subset \mathcal{A}_{U(1)_{2n}} \subset \mathcal{A}$ or $\mathcal{A}_{c=1} \subset \mathcal{A}^{U(1)}_{SU(2)} \subset \mathcal{A}$ case. Let us first consider the case when $\mathcal{A}_{c=1} \subset \mathcal{A}_{U(1)_{2n}} \subset \mathcal{A}$. 

Recall that $\mathcal{A}_{U(1)_{2n}}$ is completely rational, and its representations are labeled by $\pi_i, 0 \leq i \leq 2n - 1$, with conformal dimensions (the eigenvalues of the action of rotations) $\frac{i^2}{4n^2} + N$ and statistical dimension 1. The fusion ring generated by $\pi_i$ is $\mathbb{Z}_{2n}$. Since $\mathcal{A}_{U(1)_{2n}} \subset \mathcal{A}$ is a strongly additive pair by (3) of Lemma 3.6, the vacuum representation $H_\mathcal{A}$ decomposes into representation of $\mathcal{A}_{U(1)_{2n}}$ as $H_\mathcal{A} = \oplus_i m_i \pi_i$ with $m_i \leq 1$ by (2) of Cor. 3.9. Also not that if $m_i = 1$, then again by (2) of Cor. 3.9 $m_i = 1 = \langle 1, \alpha_i \rangle$, hence $[\alpha_i] = [1]$ since $d_{\alpha_i} = d_i = 1$. So the set of $H_i$ which appears in the decompositions of $H_\mathcal{A}$ form an abelian subgroup of $\mathbb{Z}_{2n}$. Let $i_0 \geq 0$ be the generator of this abelian subgroup. Then there is a positive integer $k$ such that $ki_0 = 2n$. On the other hand, we must have $\frac{k^2}{4n^2} \in \mathbb{Z}$, and we assume that $k_0$ is a positive integer such that $i_0^2 = 4nk_0$. It follows that $n = k^2k_0$. Let us compare the inclusions $\mathcal{A}_{U(1)_{2k^2k_0}} \subset \mathcal{A}$ and $\mathcal{A}_{U(1)_{2k^2k_0}} \subset \mathcal{A}_{U(1)_{2k^2k_0}}$. Since the vacuum representations $H_\mathcal{A}$ and $H_{\mathcal{A}_{U(1)_{2k^2k_0}}}$ have the same decompositions with respect to $\mathcal{A}_{U(1)_{2k^2k_0}}$, we can identify $H_\mathcal{A}$ and $H_{\mathcal{A}_{U(1)_{2k^2k_0}}}$ so that we can assume that $\mathcal{A}_{U(1)_{2k^2k_0}}$ is a common conformal subnet of $\mathcal{A}$ and $\mathcal{A}_{U(1)_{2k^2k_0}}$ on the same Hilbert space. Now choose unitary charged intertwiners $\phi_{i_0} \in \mathcal{A}(I)$ and $\psi_{i_0} \in \mathcal{A}_{U(1)_{2k^2k_0}}(I)$ such that $Ad_{\phi_{i_0}}$ and $Ad_{\psi_{i_0}}$ induce the same representation $\pi_{i_0}$ of $\mathcal{A}_{U(1)_{2k^2k_0}}$. Define a unitary operator $U$ commuting with $\mathcal{A}$ such that $U\phi_{i_0}^m\Omega = \psi_{i_0}^m\Omega, \forall m \in \mathbb{Z}$, where
Ω is the vacuum vector. One checks easily that $UA(I)U^* = A_{U(1)2k_0}(I), U\Omega = \Omega$. Since $U$ commutes with $A$, and so it commutes with the action of $PSL(2, \mathbb{R})$, we have $UA(J)U^* = A_{U(1)2k_0}(J), \forall J \in \mathcal{I}, U\Omega = \Omega$, thus proving that $A \cong A_{U(1)2k_0}$.

The second case when $\mathcal{A}_{c=1} \subset \mathcal{A}_{SU(2),1}^{U(1)} \subset \mathcal{A}$ is similar as above. $\mathcal{A}_{SU(2),1}^{U(1)}$ has a continuous series of irreducible representations labeled by a real number $q$ which generates a fusion ring that is isomorphic to $\mathbb{R}$. By Lemma 4.3 and Cor. 3.9 we have $H_A = \oplus_{q \in \mathbb{R}} H_q$ where $S \subset \mathbb{R}$ is an abelian subgroup, and $q^2 \in 2\mathbb{Z}, \forall q \in S$. If $S = \{0\}$, then $\mathcal{A} = \mathcal{A}_{SU(2),1}^{U(1)}$. Assume that $S \neq \{0\}$, and let $q_0 > 0$ be the least positive number in the discrete set $S$. It follows that $S = \{\mathbb{Z}q_0\}$. Let $n$ be the positive integer such that $q_0 = \sqrt{2n}$. So we have the decompositions $H_A = \oplus_{k \in \mathbb{Z}} F(\sqrt{2n})$. Compare this with $\mathcal{A}_{SU(2),1}^{U(1)} \subset \mathcal{A}_{U(1)2n}$, and a similar argument using unitary charged intertwiners as above shows that $\mathcal{A} \cong \mathcal{A}_{U(1)2n}$.

4.2.4 $\mathcal{A}_{c=1} \subset \mathcal{A}_{SU(2),1}^{D_{\infty}} \subset \mathcal{A}$ case. Let $\mathcal{O}$ be the the nontrivial one-dimensional representation of $D_{\infty}$ (the infinite Dihedral group). We will also use $\mathcal{O}$ to denote the corresponding irreducible representation of $\mathcal{A}_{SU(2),1}^{D_{\infty}}$. We note that the conformal dimensions of $\mathcal{O}$ are integers, and by [R2], we can choose a representative of $[\mathcal{O}]$ such that $\mathcal{O}^2 = id$. Also the braiding operator $\epsilon(\mathcal{O}, \mathcal{O})$ is a scalar with property $\epsilon(\mathcal{O}, \mathcal{O})^2 = 1$.

Let us first consider the case when $\alpha_{\mathcal{O}}$ is localized on $I$, i.e., $\alpha_{\mathcal{O}}$ is a DHR representation of $\mathcal{A}$. Apply Doplicher-Roberts reconstruction to $\mathcal{A}$ and $\alpha_{\mathcal{O}}$ as in Prop. 3.8 of [Mu], we get a conformal net $\mathcal{A}_1$ such that $\mathcal{A} \subset \mathcal{A}_1$ and $\mathcal{A}_{SU(2),1}^{U(1)} \subset \mathcal{A}_1$.

By §4.2.3 we can identify $\mathcal{A}_1$ as $\mathcal{A}_{SU(2),1}^{U(1)}$ or $\mathcal{A}_{U(1)2n}$, $n \in \mathbb{N}$. If $\mathcal{A}_1 = \mathcal{A}_{SU(2),1}^{U(1)}$, we have $\mathcal{A}_{SU(2),1}^{D_{\infty}} \subset \mathcal{A} \subset \mathcal{A}_{SU(2),1}^{U(1)}$, and it follows that $\mathcal{A} \cong \mathcal{A}_{SU(2),1}^{D_{\infty}}$ or $\mathcal{A} \cong \mathcal{A}_{SU(2),1}^{U(1)}$ since $\mathcal{A}_{SU(2),1}^{D_{\infty}}$ is the fixed point subnet under the action of $\mathbb{Z}_2$. If $\mathcal{A}_1 = \mathcal{A}_{U(1)2n}$, note that $\mathcal{A}_{SU(2),1}^{D_{\infty}} \subset \mathcal{A}_{SU(2),1}^{U(1)}$ under the action of $\mathbb{Z}_2$, and so $\mathcal{A}$ is the fixed point net under the action of a closed subgroup of $D_{\infty}$ as in §4.2.2. It follows that $\mathcal{A}$ is isomorphic to either $\mathcal{A}_{SU(2),1}^{D_{\infty}} \subset \mathcal{A}_{SU(2),1}^{U(1)}$ or $\mathcal{A}_{SU(2),1}^{U(1)}$ for some $m \in \mathbb{N}$.

Let us now return to the general case. Consider the following inclusions

$$\mathcal{A}_{SU(2),1}^{D_{\infty}}(I) \subset \mathcal{A}(I) \subset \mathcal{A}_{SU(2),1}^{D_{\infty}}(I')'$$

on $H_A$. Let $\gamma_1 : \mathcal{A}_{SU(2),1}^{D_{\infty}}(I')' \to \mathcal{A}(I)$ be the canonical endomorphism such that its restriction to $\mathcal{A}(I)$ is the canonical endomorphism from $\mathcal{A}(I)$ to $\mathcal{A}_{SU(2),1}^{D_{\infty}}(I)$ (cf. [LR]). Let $v_1 \in Hom(id_i, \mathcal{O}_i \mathcal{O}_e)\mathcal{A}_{SU(2),1}^{D_{\infty}}(I)$ be a unitary operator, and let $v_\mathcal{O} \in \mathcal{A}_{SU(2),1}^{D_{\infty}}(I')'$ be the unique unitary operator such that $\gamma_1(v_\mathcal{O}) = v_1$. Notice that $\gamma(\alpha_{\mathcal{O}}) = \mathcal{O}_e \mathcal{O}_e \gamma$, and it follows that $v_\mathcal{O} a = \tilde{\alpha}_{\mathcal{O}} a v_\mathcal{O}, \forall a \in \mathcal{A}(I)$. Restrict to $\mathcal{A}_{SU(2),1}(I)$ and recall that $O^2 = id$ we have that $v_\mathcal{O} \in \mathcal{A}_{SU(2),1}(I)' \cap \mathcal{A}_{SU(2),1}(I')'$. So $v_\mathcal{O}$ commutes with $PSL(2, \mathbb{R})$ by the strong additivity of $\mathcal{A}_{SU(2),1}^{D_{\infty}}$. On the other hand we note that by (2) of Prop. 3.5 $\tilde{\alpha}_{\mathcal{O}} = \alpha_{\mathcal{O}} \tilde{\alpha}_{\mathcal{O}}$ since $\epsilon(\mathcal{O}, \mathcal{O})$ is a scalar,
and so $v_O^2 \in \mathcal{A}^{D_\infty}_{SU(2)_1} \cap \mathcal{A}(I) = \mathbb{C}$. Multiplying by a scalar if necessary, we can assume that $v_O^2 = id, v_O = v_O$. Hence the subnet defined by $\hat{\mathcal{A}}(J) := \mathcal{A}(J) \cap \{v_O\}'$ is a conformal subnet of $\mathcal{A}$ and $\mathcal{A}^{U(1)}_{SU(2)_1} \subset \hat{\mathcal{A}}$ is a conformal subnet. Now consider

$$\beta_O := \hat{\alpha}^{U(1)}_{SU(2)_1} \rightarrow \hat{\mathcal{A}}, \quad \alpha^{U(1)}_{SU(2)_1} \rightarrow \hat{\mathcal{A}}.$$ By the definition we have $v_O a = \beta_O(a)v_O, \forall a \in \hat{\mathcal{A}}(I)$, and by the definition of $\hat{\mathcal{A}}$ we get

$$v_O a = av_O = \beta_O(a)v_O, \forall a \in \hat{\mathcal{A}}(I)$$

and so $\beta_O = id$. It follows that

$$[\hat{\alpha}^{U(1)}_{SU(2)_1} \rightarrow \hat{\mathcal{A}}] = [\alpha^{U(1)}_{SU(2)_1} \rightarrow \hat{\mathcal{A}}]$$

and so $\alpha^{U(1)}_{SU(2)_1} \rightarrow \hat{\mathcal{A}}$ is localized on $I$. So we can apply the first part of this section to identify $\mathcal{A}$ as either $\mathcal{A}^{D_\infty}_{SU(2)_1}, \mathcal{A}^{U(1)}_{SU(2)_1}$ or $2^{\mathcal{A}_{U(1)}_{2m}}$ or $U_{(1)}_{2m}$ for some positive integer $m$. Since $\mathcal{A} \subset \mathcal{A}$, to identify $\mathcal{A}$ it is enough to consider the case when $\mathcal{A} \simeq \mathcal{A}^{D_\infty}_{SU(2)_1}$ or $\mathcal{A} \simeq \mathcal{A}^{U(1)}_{SU(2)_1}$ since the other cases have been treated in §4.2.3. Let us show that in this case $\mathcal{A} = \mathcal{A}$. Note that by definition the index of $\mathcal{A}(I) \subset \mathcal{A}$ is at most 2. So we just have to show that the index is not 2. Consider the inclusion $\mathcal{A}(I) \subset \mathcal{A}$. If the index is 2, one checks easily that $\alpha_O$ is not localized on $I$ and $[\gamma, \mathcal{A}] = [1 + \hat{\alpha}_O, \alpha_O]$.

Let $f$ be the vector representation of $\mathcal{A}(I)$. Note that $[Of] = [f]$. It is now easy to check that

$$\langle \alpha_f, \alpha_f \rangle = 1, \langle \hat{\alpha}_f \alpha_f, 1 \rangle \geq 1$$

It follows that $\alpha_f$ is localized on $I$, and so $\alpha_O \prec \alpha^2_f$ is also localized on $I$, a contradiction.

**4.2.5** $\mathcal{A}_{c=1} \subset \mathcal{A}^{G}_{SU(2)_1} \subset \mathcal{A}, G = E_i, i = 6, 7, 8$. Note that since $\mathcal{A}^{G}_{SU(2)_1}$ is absolutely rational, it follows that $\mathcal{A}$ by is also absolutely rational by Prop. 2.3 of [KL]. If $\mathcal{A} \subset \mathcal{A}_1$ is a conformal subnet of $\mathcal{A}_1$ which is irreducible, i.e., $\mathcal{A}(I)' \cap \mathcal{A}_1(I) = \mathbb{C}$, by Prop. 2.3 of [KL] $\mathcal{A}_1$ is also absolutely rational, and $\mu_{\mathcal{A}_1} \leq \mu_{\mathcal{A}}$ with equality iff $\mathcal{A} = \mathcal{A}_1$ by Prop. 2.2. Also note that if $\mathcal{A} \subset \mathcal{A}_1 \subset \mathcal{A}_2$ are conformal subnets with $\mathcal{A} \subset \mathcal{A}_1$ and $\mathcal{A} \subset \mathcal{A}_2$ irreducible, by Prop. 2.3 of [KL] again we have that $\mathcal{A} \subset \mathcal{A}_2$ has finite index and so $\mathcal{A} \subset \mathcal{A}_2$ is also irreducible by Lemma 14 of [Ls]. We claim that among the irreducible conformal extensions of $\mathcal{A}$ there must be a conformal net $\mathcal{A}_{\text{max}}$ such that if $\mathcal{A}_{\text{max}} \subset \mathcal{B}$ is an irreducible conformal subnet, then $\mathcal{A}_{\text{max}} = \mathcal{B}$. If not, we have $\mathcal{A} \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_n \subset \ldots$ where $\mathcal{A}_i \subset \mathcal{A}_{i+1}$ is irreducible and $\mathcal{A}_i \neq \mathcal{A}_{i+1}, \forall i$. By Jones’s theorem (cf. [J]) the index $[\mathcal{A}_{i+1}, \mathcal{A}_i] \geq 2$, and it follows that from Prop. 2.2 $1 \leq \mu_{\mathcal{A}_i} \leq \frac{1}{2^{\mu_{\mathcal{A}_i}}} \forall i$, contracting the fact that $\mu_{\mathcal{A}} < \infty$.

Let $\mathcal{A}_{\text{max}}$ be a maximal conformal extension of $\mathcal{A}$ in the sense above such that $\mathcal{A}^{G}_{SU(2)_1} \subset \mathcal{A} \subset \mathcal{A}_{\text{max}}$. Let $\mathcal{B}$ be the conformal subnet of $\mathcal{A}_{\text{max}}$ generated by the $\mathcal{A}_{c=1}$ and charged intertwinners associated with the set $\{L(1, p^2), \forall p \in \mathbb{Z}\}$. Note
that $\mathcal{A}_{SU(2)}^G \subset \mathcal{B}$. By §4.2.2, we can identify $\mathcal{B}$ with $\mathcal{A}_{SU(2)}^G$, where $G'$ a closed A-D-E subgroup of $SO(3)$. If $G'$ is of type $D$, using §4.2.4 to identify all possible $\mathcal{A}_{\text{max}}$, we see that $\mathcal{A}_{\text{max}}$ can be further extended, contradicting the maximality of $\mathcal{A}_{\text{max}}$.

If $G'$ is of type $E$, let $\hat{G}'$ be the two-fold covering group of $G'$ in $SU(2)$. For any irreducible representation $\lambda$ of $\hat{G}'$, we use the same $\lambda$ to label the covariant representation of $\mathcal{A}_{SU(2)}^G$. Note that the statistical dimensions of $\lambda$ are given by the Perron-Frobenius eigenvectors labeled in [GHJ]. Consider the induction for the pair $\mathcal{B} = \mathcal{A}_{SU(2)}^G \subset \mathcal{A}_{\text{max}}$. By the definition of $\mathcal{B}$ and (3) of Lemma 3.3 each $\alpha_\lambda$ is irreducible. Consider the index set $S := \{\lambda|\lambda \in \text{Irrep}G', [\alpha_\lambda] = [\hat{\alpha}_\lambda]\}$ and $\hat{S} := \{\lambda|\lambda \in \text{Irrep}\hat{G}', [\alpha_\lambda] = [\hat{\alpha}_\lambda]\}$. Note that the set $S$ consists of representations of $\mathcal{A}_{\text{max}}$ which have permutation symmetry. By Doplicher-Roberts reconstruction as in Prop. 3.9 of [Mu], we conclude there is an irreducible conformal extension of $\mathcal{A}_{\text{max}}$ and it follows that $S = 1$ where 1 denotes the trivial representation by the maximality of $\mathcal{A}_{\text{max}}$. Since $\hat{S}^2 \subset S$, it follows that there is at most one $\lambda \in \hat{S} - \text{irrep}G'$ such that $\lambda^2 = 1$. By inspecting the $E$ graph on Page 14 of [GHJ], we conclude that $\hat{S} = \{1\}$, and so

$$\langle \alpha_\lambda \hat{\alpha}_\lambda, \alpha_\mu \hat{\alpha}_\mu \rangle = \langle \alpha_\lambda \alpha_\mu \hat{\alpha}_\mu \rangle$$

$$= \langle \sum_{\delta \in \text{irrep}G'} N_{\mu \lambda}^{\delta} \alpha_\delta, \sum_{\delta' \in \text{irrep}G'} N_{\mu \lambda}^{\delta'} \hat{\alpha}_{\delta'} \rangle$$

$$= \delta_{\lambda \mu}$$

On the other hand $\langle \gamma_A, \alpha_\lambda \hat{\alpha}_\lambda \rangle \geq 1$ by Frobenius reciprocity and definitions, and we have

$$\gamma_A \succ \sum_{\lambda \in \text{irrep}G'} [\alpha_\lambda \hat{\alpha}_\lambda]$$

This implies

$$[\mathcal{A}_{\text{max}} : \mathcal{A}_{SU(2)}^G]^2 \geq |\hat{G}'|^2 = 4|G|^2$$

and by Prop. 2.2

$$1 \leq \mu_{\mathcal{A}_{\text{max}}} = \frac{2|G'|^2}{[\mathcal{A}_{\text{max}} : \mathcal{A}_{SU(2)}^G]^2} \leq \frac{1}{2}$$

a contradiction.

It follows that $G'$ must be of type $A$, and so $\mathcal{A}_{\text{max}}$ can be identified with $\mathcal{A}_{U(1)_{2m}}$ for some positive integer by §4.2.3. By the maximality of $\mathcal{A}_{\text{max}}$, either $m = 1$ or $m$ must be square free, i.e., $m$ is not divisible by $k^2, \forall k > 1$. In the later case the principal graph of $\alpha_{L(1, 1/2)}^A \rightarrow \mathcal{A}_{\text{max}}$ is $\mathcal{A}_{\infty, -\infty}$ by (3) of Lemma 4.4. On the other hand by (3) of Lemma 3.3, $\alpha_{L(1, 1/2)}^A \rightarrow \mathcal{A}_{\text{max}}$ is also the induced endomorphism of the vector representation $f$ from $\mathcal{A}_{SU(2)}^G$ to $\mathcal{A}_{\text{max}}$, and it has finite depth since $\mathcal{A}_{SU(2)}^G$,
is absolutely rational. This contraction shows that $A_{max}$ must be identified with $A_{U(1)_2} = A_{SU(2)_1}$. Now we have inclusions

$$A_{c=1} \subset A \subset A_{SU(2)_1}$$

Since $A_{c=1}$ is the fixed point subnet of $A_{SU(2)_1}$ under the action of $SO(3)$, it follows that $A$ is the fixed point subnet of $A_{SU(2)_1}$ under the action of of a closed subgroup of $SO(3)$ as in the end of §4.2.2.

**Proof of Theorem 4.6.**

If $A_{c=1} = A$ there is nothing to prove. Let us assume that $A_{c=1} \neq A$. Let $A_f \subset A$ be the conformal subnet generated by charged intertwiners associated with the set $\{L(1,p^2), \forall p \in \mathbb{Z}\}$ as in (3) of Cor. 3.9. By the spectrum condition the conformal subnet $A_f$ is larger than $A_{c=1}$, i.e., $A_{c=1} \subset A_f$ but $A_{c=1} \neq A_f$. By (3) of Cor. 3.9 $A_{c=1} \subset A_f$ is discrete in the sense of §4.2.2, and by §4.2.2 $A_f$ can be identified as conformal net containing $A_{SU(2)_1}^{U(1)}$, $A_{SU(2)_1}^{D_{\infty}}$, or $A_{SU(2)_1}^G$, $G = E_6, E_7, E_8$, and the theorem follows from §4.2.3-5. □

We note that if the spectrum condition is violated, then the canonical endomorphism $\gamma$ satisfies strange properties, and we have been able to rule out some cases in [Xt]. In general it is still an open question to show that the spectrum condition is always satisfied.

We point out one application of Th. 4.6. By a general result of [Xc] the net associated with the coset $SU(2)_4 \subset SU(2)_2 \times SU(2)_2$ has 13 irreducible representations, and this net has central charge $c = 1$. Using the known character formulas in this case one can verify that the spectrum condition is satisfied in this case, so this net must be identified with an element in the list (*) by Th. 4.6. From the fusion rules of the coset in [Xc] one immediately identifies the coset with $A_{SU(2)_1}^{Z_2}$. This observation was first pointed out to me by C. Dong during our discussions on the “fixed point resolution” problem about the cosets, of which $SU(2)_4 \subset SU(2)_2 \times SU(2)_2$ is the first nontrivial example (cf. [Xc]).

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