A counterexample to the Hodge conjecture for Kähler varieties

Claire Voisin
Institut de mathématiques de Jussieu, CNRS, UMR 7586

1 Introduction

If $X$ is a smooth projective variety, it is in particular a Kähler variety, and its cohomology groups carry the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

A class $\alpha \in H^{2p}(X, \mathbb{Q})$ is said to be a rational Hodge class if its image in $H^{2p}(X, \mathbb{C})$ belongs to $H^{p,p}(X)$. As is well known, the classes which are Poincaré dual to irreducible algebraic subvarieties of codimension $p$ of $X$ are degree $2p$ Hodge classes. The Hodge conjecture asserts that any rational Hodge class is a combination with rational coefficients of such classes.

In the case of a general compact Kähler variety $X$, the conjecture above, where the algebraic subvarieties are replaced with closed analytic subsets, is known to be false (cf [13]). The simplest example is provided by a complex torus endowed with a holomorphic line bundle of indefinite curvature. If the torus is chosen general enough, it will not contain any analytic hypersurface, while the first Chern class of the line bundle will provide a Hodge class of degree 2.

In fact, another general method to construct Hodge classes is to consider Chern classes of holomorphic vector bundles. In the projective case, the set of classes generated this way is the same as the set generated by classes of subvarieties. To see this, one looks at a still more general set of classes, which is the set generated by the Chern classes of coherent sheaves on $X$. Since any coherent sheaf has a finite resolution by locally free sheaves, one does not get more classes than with locally free sheaves. On the other hand, this later set obviously contains the classes of subvarieties (one computes for this the top Chern class of $\mathcal{I}_Z$ for $Z$ irreducible of codimension $p$, and one shows that it is proportional to the class of $Z$). Finally, to see that the classes of coherent sheaves can be generated by classes of subvarieties, one puts a filtration on any coherent sheaf, whose associated graded consists of rank 1 sheaves supported on subvarieties, which makes the result easy.

In the general Kähler case, none of these equalities holds. The only obvious result is that the space generated by the Chern classes of analytic coherent sheaves contains both the classes which are Poincaré dual to irreducible closed analytic subspaces and the Chern classes of holomorphic vector bundles (or locally free analytic coherent sheaves). The example above shows that a Hodge class of degree 2 may be the Chern class of a holomorphic line bundle, even if $X$ does not contain any complex analytic subset. On the other hand, it may be the case that coherent sheaves do not admit a resolution by locally free sheaves, (although it is true in dimension 2 [9]), and that
more generally $X$ does not carry enough vector bundles to generate the Hodge classes of subvarieties or coherent sheaves (see the appendix for such examples). Hence the set generated by the Chern classes of analytic coherent sheaves is actually larger than the two others.

Notice that using the Grothendieck-Riemann-Roch formula (cf [3], extended in [7] to the complex analytic case), we can give the following alternative description of this set: it is generated by the classes

$$\phi_*c_i(E)$$

where $\phi: Y \rightarrow X$ is a morphism from another compact Kähler manifold, $E$ is a holomorphic vector bundle on $Y$, and $i$ is any integer. Another fact which follows from iterated applications of the Whitney formula, is that the set which is additively generated by the Chern classes of coherent sheaves is equal to the set which is generated as a subring of the cohomology ring by the Chern classes of coherent sheaves. Thus this set is as big and stable as possible.

Now, since we do not know other geometric ways of constructing Hodge classes, the following seems to be a natural extension of the Hodge conjecture to Kähler varieties.

*Are the rational Hodge classes of a compact Kähler variety $X$ generated over $\mathbb{Q}$ by Chern classes of analytic coherent sheaves on $X$?*

Our goal in this paper is to give a negative answer to this question. We show the following theorem

**Theorem 1** There exists a 4-dimensional complex torus $X$ which possesses a non-trivial Hodge class of degree 4, such that any analytic coherent sheaf $F$ on $X$ satisfies

$$c_2(F) = 0.$$ 

In the appendix, we also give a few geometric consequences of a result of Bando and Siu [1], extending Uhlenbeck-Yau’s theorem. We show in particular that for a general compact Kähler variety $X$, the analytic coherent sheaves on $X$ do not admit finite resolutions by locally free coherent sheaves. This answers a question asked to us by L. Illusie.

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## 2 A criterion for the vanishing of Chern classes of coherent sheaves

We consider in this section a compact Kähler variety $X$ of dimension $n \geq 3$ satisfying the following assumptions

1. The Néron-Severi group $NS(X)$ generated by the first Chern classes of holomorphic line bundles on $X$ is equal to 0.

2. $X$ does not contain any proper closed analytic subset of positive dimension.
3. For some Kähler class \( [\omega] \in H^2(X, \mathbb{R}) \cap H^{1,1}(X) \), the set of Hodge classes \( Hdg^4(X, \mathbb{Q}) \) is perpendicular to \( [\omega]^{n-2} \) for the intersection pairing
\[
H^4(X, \mathbb{R}) \otimes H^{2n-4}(X, \mathbb{R}) \to \mathbb{R}.
\]

Our aim is to show the following

**Proposition 1** If \( X \) is as above, any analytic coherent sheaf \( \mathcal{F} \) on \( X \) satisfies \( c_2(\mathcal{F}) = 0 \).

**Proof.** As in [2], the proof is by induction on the rank \( k \) of \( \mathcal{F} \). We note that because \( \dim X \geq 3 \), torsion sheaves supported on points on \( X \) have trivial \( c_1 \) and \( c_2 \). On the other hand assumption 2 implies that torsion sheaves are supported on points. This deals with the case where \( rk \mathcal{F} = 0 \). Furthermore this shows that we can restrict to torsion free coherent sheaves.

If \( \mathcal{F} \) contains a non trivial subsheaf \( \mathcal{G} \) of rank \( < k \), we have the exact sequence
\[
0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{F}/\mathcal{G} \to 0,
\]
with \( rk \mathcal{G} < k \) and \( rk \mathcal{F}/\mathcal{G} < k \). Assumption 2 and the induction hypothesis then give
\[
c_1(\mathcal{G}) = c_2(\mathcal{G}) = 0, \quad c_1(\mathcal{F}/\mathcal{G}) = c_2(\mathcal{F}/\mathcal{G}) = 0.
\]

Therefore Whitney formula implies \( c_2(\mathcal{F}) = 0 \).

We are now reduced to study the case where \( \mathcal{F} \) does not contain any non trivial proper subsheaf of smaller rank. By assumption 2, \( \mathcal{F} \) is locally free away from finitely many points \( \{x_1, \ldots, x_N\} \) of \( X \). One shows now that there exists a variety
\[
\tau : \tilde{X} \to X,
\]
which is obtained from \( X \) by finitely many successive blow-ups with smooth centers (and in particular is also Kähler), so that \( \tau \) restricts to an isomorphism
\[
\tilde{X} - \bigcup_i E_i \cong X - \{x_1, \ldots, x_N\},
\]
where \( E_i := \tau^{-1}(x_i) \) and such that there exists a locally free sheaf \( \tilde{\mathcal{F}} \) on \( \tilde{X} \) which is isomorphic to \( \mathcal{F} \) on the open set \( \tilde{X} - \bigcup_i E_i \) : indeed, the problem is local near each \( x_i \). Choosing a finite presentation
\[
\mathcal{O}_X^l \to \mathcal{O}_X^r \to \mathcal{F} \to 0
\]
of \( \mathcal{F} \) near \( x_i \), we get a morphism to the Grassmannian of \( l \)-dimensional subspaces of \( \mathbb{C}^r \), which is well defined away from \( x_i \), since \( \mathcal{F} \) is free away from \( x_i \). This morphism is easily seen to be meromorphic. Hence by Hironaka desingularization theorem [4], this morphism can be extended after finitely many blowups. Then the pull-back of the tautological quotient bundle on the Grassmannian will provide the desired extension.

Note that because \( \mathcal{F} \) does not contain any non zero subsheaf of smaller rank, the same is true of \( \tilde{\mathcal{F}} \), which implies that \( \tilde{\mathcal{F}} \) is \( h_\lambda \)-stable for any Kähler metric \( h_\lambda \) on \( \tilde{X} \). The theorem of existence of Hermitian-Yang-Mills connections ([10]) then provides \( \tilde{\mathcal{F}} \) with a Hermitian-Einstein metric \( k_\lambda \) for any Kähler metric \( h_\lambda \) on \( \tilde{X} \). This means
that the curvature $R_\lambda \in \Gamma(\text{Hom}(\tilde{\mathcal{F}}, \tilde{\mathcal{F}}) \otimes \Omega^2_{\tilde{\mathcal{X}}, \mathbb{R}})$ of the metric connection on $\tilde{\mathcal{F}}$ associated to $k_\lambda$ is the sum of a diagonal matrix with all coefficients equal to $\mu_\lambda \omega_\lambda$ and of a matrix $R^0_\lambda$ whose coefficients are $(1, 1)$-forms annihilated by the $\Lambda$ operator relative to the metric $h_\lambda$. (The connection is then said to be Hermitian-Yang-Mills.) Here $\omega_\lambda$ is the Kähler form of the metric $h_\lambda$ and $\mu_\lambda$ is a constant coefficient, equal to

$$2i\pi \frac{c_1(\tilde{\mathcal{F}})[\omega_\lambda]^{n-1}}{k[\omega_\lambda]^n},$$

(2.1)

where $[\omega_\lambda] \in H^2(\tilde{X}, \mathbb{R})$ denotes the de Rham class of the form $\omega_\lambda$.

Let us denote by $\eta^0_\lambda$ the closed 4-form

$$\eta^0_\lambda = \text{tr} \left( \frac{R^0_\lambda}{2i\pi} \right)^2.$$

We assume chosen small neighbourhoods $V_i$ of $x_i$ in $X$, and forms $\omega_i$ on $\tau^{-1}(V_i)$ which vanish near the boundary of $\tau^{-1}(V_i)$, and restrict to a Kähler form on $\tau^{-1}(x_i)$. Then we will choose

$$\omega_\lambda = \tau^* \omega + \lambda \left( \sum_i \omega_i \right)$$

(2.2)

which is easily seen to be Kähler for sufficiently small $\lambda$. We shall now prove that $R_\lambda$ tends to 0 with $\lambda$ in the $L^2$-sense away from the $V_i$’s. The argument is an extension of Lübke’s inequality \[6\] which proves that a Hermitian-Yang-Mills connection on a vector bundle $E$ with $c_1(E)[\omega]^{n-1} = c_1(E)^2[\omega]^{n-2} = c_2(E)[\omega]^{n-2} = 0$, where $[\omega]$ is the class of the Kähler form on the basis, is in fact flat. We first claim:

**Proposition 2** For any differential $2n - 4$-form $\alpha$ on $X$, the integral

$$\int_{X - \bigcup_i V_i} \eta^0_\lambda \wedge \alpha$$

tends to 0 with $\lambda$.

Before proving this proposition, we show how it implies that $c_2(\mathcal{F}) = 0$, thus completing the induction step.

Poincaré duality will provide an isomorphism

$$H^4(X - \bigcup_i V_i, \mathbb{R}) \cong H^{2n-4}(X, \bigcup_i V_i, \mathbb{R})^*,$$

which is realized by integrating closed 4-forms defined over $X - \bigcup_i V_i$ against $2n - 4$-forms vanishing on the $V_i$’s. Next because $\text{dim } X \geq 3$ we have isomorphisms

$$H^4(X, \mathbb{R}) \cong H^4(X - \bigcup_i V_i, \mathbb{R}),$$

$$H^{2n-4}(X, \bigcup_i V_i, \mathbb{R}) \cong H^{2n-4}(X, \mathbb{R})$$

which are compatible with Poincaré duality. Now the cohomology class of the closed 4-form $\eta^0_\lambda$ is easily computed to be

$$[\eta^0_\lambda] = -2c_2(\tilde{\mathcal{F}}) + c_1(\tilde{\mathcal{F}})^2 - \frac{\mu_\lambda}{2i\pi} [\omega_\lambda] \cup c_1(\tilde{\mathcal{F}}) + k \left( \frac{\mu_\lambda}{2i\pi} \right)^2 [\omega_\lambda]^2.$$

(2.3)
Hence its restriction to $\tilde{X} - \cup_i \tau^{-1}(V_i) = X - \cup_i V_i$ is equal to $-2c_2(\mathcal{F}) + k(\frac{\mu_\lambda}{2i\pi})^2[\omega]^2$, since $c_1(\mathcal{F}) = 0$ and $\omega_\lambda$ restricts to $\omega$ on $X - \cup_i V_i$.

In order to show that $c_2(\mathcal{F}) = 0$, it then suffices by the above to show that for any closed $2n - 4$-form $\alpha$ on $X$, vanishing on $\cup_i V_i$, we have

$$\int_X (\eta_\lambda^0 - k(\frac{\mu_\lambda}{2i\pi})^2[\omega]) \wedge \alpha = 0.$$  

But this integral is independent of $\lambda$ and so it suffices to show that

$$\lim_{\lambda \to 0} \int_{\tilde{X}} (\eta_\lambda^0 - k(\frac{\mu_\lambda}{2i\pi})^2[\omega]) \wedge \alpha = 0. \quad (2.4)$$

Now we claim that

$$\lim_{\lambda \to 0} \mu_\lambda = 0. \quad (2.5)$$

Indeed this follows from formula (2.1), and from the fact that the class $c_1(\tilde{\mathcal{F}})$ restricts to 0 on $\tilde{X} - \cup_i \tau^{-1}(x_i)$ because $\text{NS}(X) = 0$. Then the intersection pairing $< c_1(\tilde{\mathcal{F}}), \tau^*[\omega]^{n-1} >_{\tilde{X}}$ is equal to 0, and we conclude using the fact that $\lim_{\lambda \to 0} \omega_\lambda = \tau^* \omega$.

Then (2.4) follows from (2.5) and Proposition 2.

We now go to the proof of proposition 2. We first claim that

$$\lim_{\lambda \to 0} \int_{\tilde{X}} \eta_\lambda^0 \wedge [\omega_\lambda]^{n-2} = 0. \quad (2.6)$$

Indeed, we know that the space $Hdg^4(X)$ is perpendicular to $[\omega]^{n-2}$ for the intersection pairing. Hence we have

$$< c_2(\tilde{\mathcal{F}}), \tau^*[\omega]^{n-2} >_{\tilde{X}} = 0.$$ 

On the other hand this is equal to

$$\lim_{\lambda \to 0} < c_2(\tilde{\mathcal{F}}), [\omega_\lambda]^{n-2} >_{\tilde{X}}$$

since $\lim_{\lambda \to 0} \omega_\lambda = \tau^* \omega$. Exactly by the same argument, we show that

$$\lim_{\lambda \to 0} < c_1^2(\tilde{\mathcal{F}}), [\omega_\lambda]^{n-2} >_{\tilde{X}} = 0.$$ 

Then the result follows from formula (2.3) and from (2.5).

Next we recall that the endomorphism $R_0^0$ of $\tilde{\mathcal{F}}$, with forms coefficients is anti-self-adjoint with respect to the metric $k_\lambda$. This follows from the fact that $R_\lambda$ is the curvature of the metric connection with respect to $k_\lambda$. In a local orthonormal basis of $\tilde{\mathcal{F}}$, this will be translated into the fact that $R_0^0$ is represented by a matrix, whose coefficients are differential 2-forms, which satisfies

$$^t R_0^0 = - R_0^0.$$ 

The second crucial property of $R_0^0$ is the fact that its coefficients are primitive differential $(1,1)$-forms on $\tilde{X}$, with respect to the metric $h_\lambda$. It is well known that this implies the following equality

$$_* \lambda g = - g \wedge \frac{\omega_\lambda^{n-2}}{(n-2)!}.$$
where \( \ast_{\lambda} \) is the Hodge \( \ast \)-operator for the metric \( h_{\lambda} \). Since \( h_{\lambda} \) restricts to \( h \) on \( X - \cup_i V_i \), these forms satisfy as well
\[
\ast  \gamma = -\gamma \wedge \frac{\omega^{n-2}}{(n-2)!}
\]
(2.7) on \( X - \cup_i V_i \).

Now let \( \alpha \) be a differential \( 2n-4 \)-form on \( X \). Then it follows from (2.7) that there exists a positive constant \( c_\alpha \) such that for any primitive \( (1,1) \)-form \( \gamma \) on \( X \), we have the following pointwise inequality of pseudo-volume forms on \( X \):
\[
| \gamma \wedge \eta \wedge \alpha | \leq c_\alpha \gamma \wedge \ast \gamma \wedge \frac{\omega^{n-2}}{(n-2)!}.
\]
(2.8)

Working locally in a orthonormal basis of \( \tilde{\mathcal{F}} \) and using the fact that the matrix \( R_{\lambda}^0 \) is anti-self-adjoint and with primitive coefficients of \( (1,1) \)-type, we now get the pointwise inequality of pseudo-volume forms on \( X - \cup_i V_i \)
\[
| \text{tr} (R_{\lambda}^0)^2 \wedge \alpha | \leq c_\alpha \text{tr} (R_{\lambda}^0)^2 \wedge \frac{\omega^{n-2}}{(n-2)!}.
\]

Therefore we get the inequality
\[
| \int_{X - \cup_i V_i} \text{tr} (R_{\lambda}^0)^2 \wedge \alpha | \leq c_\alpha \int_{X - \cup_i V_i} \text{tr} (R_{\lambda}^0)^2 \wedge \frac{\omega^{n-2}}{(n-2)!}.
\]

But by (2.6), and because \( \eta_{\lambda}^0 = \text{Tr} (\frac{R_{\lambda}^0}{2\pi i})^2 \), we know that
\[
\lim_{\lambda \to 0} \int_X \text{tr} (R_{\lambda}^0)^2 \wedge \omega^2 = 0.
\]
Because the integrand is positive and \( \omega = \omega_{\lambda} \) on \( X - \cup_i V_i \), this implies that
\[
\lim_{\lambda \to 0} \int_{X - \cup_i V_i} \text{tr} (R_{\lambda}^0)^2 \wedge \omega^2 = 0.
\]
Hence
\[
\lim_{\lambda \to 0} \int_{X - \cup_i V_i} \text{tr} (R_{\lambda}^0)^2 \wedge \alpha = 0 = \lim_{\lambda \to 0} \int_{X - \cup_i V_i} \eta_{\lambda}^0 \wedge \alpha.
\]
Proposition 2 is proven.

Remark 1 A. Teleman mentioned to me the possibility of using the result of [1] (see the appendix) to give a shorter proof of the equality \( c_2(\mathcal{F}) = 0 \) for stable \( \mathcal{F} \). In this paper, the results of Uhlenbeck and Yau are extended to reflexive coherent stable sheaves, and Lübke’s inequality, together with the fact that equality implies projective flatness, is proven.

Since in our case we have a much stronger assumption than stability, namely stability of any desingularization of \( \mathcal{F} \) with respect to any Kähler metric, it seemed however appropriate to avoid the reference to the technically hard result of [1] and to content ourselves with an argument which appeals only to [10] and elementary computations.
3 Constructing an example

Our example will be of Weil type \[12\]. The Hodge classes described below have been constructed by Weil in the case of algebraic tori, as a potential counterexample to the Hodge conjecture for algebraic varieties. In the case of a general complex torus, the construction is still simpler. These complex tori have been also considered in \[13\] by Zucker, who proves there some of the results stated below. (I thank P. Deligne and C. Peters for pointing out this reference to me.)

We start with a $\mathbb{Z}[I]$-action on $\Gamma := \mathbb{Z}^8$, where $I^2 = -1$, which makes $\Gamma \otimes \mathbb{Q}$ into a $K$-vector space, where $K$ is the quadratic field $\mathbb{Q}[I]$.

Let

$$\Gamma_C = \Gamma \otimes \mathbb{C} = C^4_i \oplus C^4_{-i}$$

be the associated decomposition into eigenspaces for $I$. A four dimensional complex torus $X$ with underlying lattice $\Gamma$ and inheriting the $I$-action is determined by a 4 dimensional complex subspace $W$ of $\Gamma_C$, which has to be the direct sum

$$W = W_i \oplus W_{-i}$$

of its intersections with $C^4_i$ and $C^4_{-i}$. It has furthermore to satisfy the condition that

$$W \cap \Gamma_R = \{0\}. \tag{3.9}$$

Given $W$, $X$ is given by the formula

$$X = \Gamma_C/\left( W \oplus \Gamma \right).$$

We will choose $W$ so that

$$\dim W_i = \dim W_{-i} = 2.$$ 

Then $W$, hence $X$ is determined by the choice of the 2-dimensional subspaces

$$W_i \subset C^4_i, \ W_{-i} \subset C^4_{-i},$$

which have to be general enough so that the condition \[3.9\] is satisfied.

We have isomorphisms

$$H^4(X, \mathbb{Q}) \cong H_4(X, \mathbb{Q}) \cong \bigwedge^4 \Gamma_Q. \tag{3.10}$$

Consider the subspace

$$\bigwedge^K \Gamma_Q \subset \bigwedge \Gamma_Q.$$

Since $\Gamma_Q$ is a 4-dimensional $K$-vector space, $\bigwedge^K \Gamma_Q$ is a one dimensional $K$-vector space, and its image is a 2 dimensional $\mathbb{Q}$-vector space. The claim is that $\bigwedge^K \Gamma_Q$ is made of Hodge classes, that is, is contained in the subspace $H^{2,2}(X)$ for the
Hodge decomposition. Notice that under the isomorphisms \((3.10)\), tensorized by \(\mathbb{C}\), 
\[ H^2,2(X) \] 
identifies to the image of 
\[ 2 \bigwedge W \otimes 2 \bigwedge W \]
in \(\bigwedge^4 \Gamma_\mathbb{C}\).

To prove this claim, note that we have the decomposition 
\[ \Gamma_K := \Gamma_\mathbb{Q} \otimes K = \Gamma_{K,i} \oplus \Gamma_{K,-i} \]
into eigenspaces for the \(I\) action. Then \(\bigwedge^4 \Gamma_\mathbb{Q} \subset \bigwedge^4 \Gamma_\mathbb{C}\) is defined as the image of 
\[ \bigwedge^4 \Gamma_{K,i} \subset \bigwedge^4 \Gamma_K \]
via the trace map 
\[ \bigwedge^4 \Gamma_K = \bigwedge^4 \Gamma_\mathbb{Q} \otimes K \to \bigwedge^4 \Gamma_\mathbb{Q}. \]

Now we have the inclusion 
\[ \Gamma_K \subset \Gamma_\mathbb{C} \]
and the equality 
\[ \Gamma_{K,i} = \Gamma_K \cap \mathbb{C}_i^4. \]
The space \(\Gamma_{K,i}\) is a 4 dimensional \(K\)-vector space which generates over \(\mathbb{R}\) the space \(\mathbb{C}_i^4\). It follows that the image of \(\bigwedge^4 \Gamma_{K,i}\) in \(\bigwedge^4 \Gamma_\mathbb{C}\) generates over \(\mathbb{C}\) the line \(\bigwedge^4 \mathbb{C}_i^4\).

But we know that \(\mathbb{C}_i^4\) is the direct sum of the two spaces \(W_i\) and \(\overline{W_{-i}}\), which are 2-dimensional. Hence 
\[ \bigwedge^4 \mathbb{C}_i^4 = \bigwedge^2 W_i \otimes \bigwedge^2 \overline{W_{-i}} \]
is contained in \(\bigwedge^2 W \otimes \bigwedge^2 \overline{W}\), that is in \(H^{2,2}(X)\).

To conclude the construction of an example satisfying the conclusion of proposition \([1]\), and hence the proof of theorem \([1]\), it remains now only to prove that a general \(X\) as above satisfies the assumptions stated at the beginning of section \([3]\). Since \(X\) is a complex torus, assumption \([3]\) will be a consequence of assumption \([1]\) and of the fact that \(X\) is simple. Indeed it is known that if \(Y \subset X\) is a proper positive dimensional subvariety of a simple complex torus, then \(Y\) has positive canonical bundle. But \(X\) being simple, \(Y\) must generate \(X\) as a group, and then \(X\) must be algebraic, contradicting the fact that \(NS(X) = 0\).

Next we show that the Hodge classes in \(\bigwedge^4 \Gamma_\mathbb{Q}\) constructed above are perpendicular to \([\omega]^2\) for a Kähler class \([\omega] \in H^{1,1}(X)\). To see this, note that with the notations as above, these classes lie in \(\bigwedge^2 W_i \otimes \bigwedge^2 \overline{W_{-i}}\), with 
\[ W_i \subset W, \overline{W_{-i}} \subset W. \]
The space \(H^{1,0}(X)\) identifies to \(\overline{W^*}\) and accordingly the space \(H^{1,1}(X)\) identifies to \(\overline{W^*} \otimes W^*\). For \([\omega] \in \overline{W^*} \otimes W^*\), the pairing \(<[\omega]^2, H^{2,2}(X)\rangle\), restricted to \(\bigwedge^2 W_i \otimes \bigwedge^2 \overline{W_{-i}}\), is obtained by squaring \([\omega]\) to get an element of 
\[ \bigwedge^2 W^* \otimes \bigwedge^2 W^* \cong \bigwedge^2 W^* \otimes \bigwedge^2 W^*. \]
and by projecting to $\bigwedge^2 W_i^* \otimes \bigwedge^2 W_{-i}^*$.

Now choose $[\omega] \in W_i^* \otimes W_i^* \oplus W_{-i}^* \otimes W_{-i}^*$. Since $W^* = W_i^* \oplus W_{-i}^*$, we can find a Kähler class $[\omega]$ in this space. On the other hand, we see that $[\omega]^2$ belongs to the space $\bigwedge^2 W_i^* \otimes \bigwedge^2 W_{-i}^*$.

Hence its projection (after switching the factors in the tensor product) to $\bigwedge^2 W_i^* \otimes \bigwedge^2 W_{-i}^*$ is equal to 0.

In conclusion, the assumptions at the beginning of section 2 will be a consequence of the following facts:

**Proposition 3** For a general $X$ as above, we have:

1. $NS(X) = 0$.
2. $X$ is simple.
3. The space $H^{d^4}(X)$ is equal to the space $\bigwedge^4 \Gamma_Q \subset \bigwedge^4 \Gamma_Q = H^4(X, \mathbb{Q}) \cong H^4(X, \mathbb{Q})$.

**Proof.** The analogues of these statements have been proven in the algebraic case in [12] (see also [11]). The result is that for a general abelian 4-fold of Weil type, the Néron-Severi group is of rank 1, generated by a class $\omega$, and the space $H^{d^4}(X)$ is of rank 3, generated over $\mathbb{Q}$ by the space $\bigwedge^4 \Gamma_Q$ and by the class $\omega^2$. Furthermore property 2 is true for the generic abelian variety $X$ of Weil type.

Property 2 for the general complex torus of Weil type follows immediately, since this is a property satisfied away from the countable union of closed analytic subsets of the moduli space of complex tori of Weil type.

As for properties 1 and 3, we prove them by an infinitesimal argument, starting from an abelian 4-fold of Weil type $X$ satisfying the properties stated above. Assume we can show that for some first order deformation $u \in H^1(T_X)$, tangent to the moduli space of complex tori of Weil type (which is smooth), we have

$$\text{int}(u)(\omega) \neq 0 \text{ in } H^2(O_X),$$

where the interior product here is composed of the cup-product

$$H^1(T_X) \otimes H^1(\Omega_X) \to H^2(T_X \otimes \Omega_X)$$

and of the map induced by the contraction

$$H^2(T_X \otimes \Omega_X) \to H^2(O_X).$$

Then from the general theory of Hodge loci ([4]), it will follow that for a general complex torus of Weil type, we have $NS(X) = 0$. Furthermore, it will also follow that

$$\text{int}(u)(\omega^2) \neq 0 \text{ in } H^3(\Omega_X),$$

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because it is equal to $2\omega \cup \text{int}(u)(\omega)$ and the cup product with $\omega$ from $H^2(O_X)$ to $H^3(\Omega_X)$ is injective because $\omega$ is a Kähler class on $X$. But as before this will imply by the theory of Hodge loci that for a general complex torus of Weil type we have $\text{rk} Hdg^k(X) = 2$ so that $Hdg^4(X) = \frac{4}{K} \Gamma_Q$.

Hence it remains only to find such $u$, which is equivalent to prove that if for any $u$ tangent to the moduli space of complex tori of Weil type, $\text{int}(u)(\omega) = 0$ in $H^2(O_X)$, then $\omega = 0$ in $H^1(\Omega_X)$. Here the notations are as in the beginning of this section. The tangent space to the deformations of the complex torus $X$ identifies to

$$Hom(W, \Gamma_C/W) = Hom(W, \overline{\Omega}^*) = W^* \otimes \overline{\Omega}.$$  \hspace{1cm} (3.11)

The tangent space to the deformations of $X$ as a complex torus of Weil type is then the subspace

$$Hom(W_i, C^4_i/W_i) \oplus Hom(W_{-i}, C^4_{-i}/W_{-i})$$

$$= W_i^* \otimes \overline{W}_{-i} \oplus W_{-i}^* \otimes \overline{W}_i.$$  

Via the identification (3.11), the interior product

$$H^1(X, T_X) \otimes H^1(X, \Omega_X) \rightarrow H^2(X, O_X) = \bigwedge^2 W^*$$

identifies to the contraction followed by the wedge product

$$W^* \otimes \overline{\Omega} \otimes \overline{W} \otimes W^* \rightarrow \bigwedge W^*.$$  

We now write

$$\omega = \omega_1 + \omega_2 + \omega_3 + \omega_4,$$

where

$$\omega_1 \in \overline{W}_i^* \otimes W_i^*, \quad \omega_2 \in \overline{W}_i^* \otimes W_{-i}^*,$$

$$\omega_3 \in \overline{W}_{-i}^* \otimes W_i^*, \quad \omega_4 \in \overline{W}_{-i}^* \otimes W_{-i}^*.$$  

Then clearly for $u_1 \in W_i^* \otimes \overline{W}_{-i}$ we have

$$\text{int}(u_1)(\omega_1) = \text{int}(u_1)(\omega_2) = 0,$$

$$\text{int}(u_1)(\omega_3) \in \bigwedge W_i^*, \quad \text{int}(u_1)(\omega_4) \in W_i^* \otimes W_{-i}^*.$$  

Similarly, for $u_2 \in W_{-i}^* \otimes \overline{W}_i$ we have

$$\text{int}(u_2)(\omega_3) = \text{int}(u_1)(\omega_4) = 0,$$

$$\text{int}(u_2)(\omega_2) \in \bigwedge W_{-i}^*, \quad \text{int}(u_2)(\omega_1) \in W_{-i}^* \otimes W_{-i}^*.$$  

The condition

$$\text{int}(u_1)(\omega) = 0 = \text{int}(u_2)(\omega)$$

for any $u_1, u_2$ then implies that

$$\text{int}(u_1)(\omega_3) = 0 \in \bigwedge W_i^*, \quad \text{int}(u_1)(\omega_4) = 0 \in W_i^* \otimes W_{-i}^*,$$

$$\text{int}(u_2)(\omega_1) = 0 \in W_i^* \otimes W_{-i}^*, \quad \text{int}(u_2)(\omega_2) = 0 \in \bigwedge W_{-i}^*$$

for any $u_1, u_2$. But it is obvious that it implies $\omega_1 = \omega_2 = \omega_3 = \omega_4 = 0$.  

Hence Proposition 3 is proven, which together with Proposition 1 completes the proof of Theorem 1.

4 Appendix

Our goal in this appendix is to give a few geometric consequences of the following theorem due to Bando and Siu [1]:

Theorem 2 Let $X$ be a compact Kähler variety, endowed with a Kähler metric $h$ and let $\mathcal{F}$ be a reflexive $h$-stable sheaf on $X$. Then there exists a Hermite-Einstein metric on $\mathcal{F}$ relative to $h$. Furthermore, if we have $< c_2(\mathcal{F}), [\omega]^{n-2} > _X = 0$ and $c_1(\mathcal{F}) = 0$, $\mathcal{F}$ is locally free and the associated metric connection is flat.

Where $[\omega]$ is the Kähler class of the metric $h$.

Remark 2 Once we know that the metric connection is flat away from the singular locus $Z$ of $\mathcal{F}$, the fact that $\mathcal{F}$ is locally free is immediate. Indeed the flat connection is associated to a local system on $X - Z$. But since codim $Z \geq 2$, this local system extends to $X$. Hence there exists a holomorphic vector bundle $E$ on $X$, which admits a unitary flat connection and is isomorphic to $\mathcal{F}$ away from $Z$. But since $\mathcal{F}$ is reflexive, the isomorphism $E \cong \mathcal{F}$ on $X - Z$ extends to $X$.

We assume now that $X$ is compact Kähler and satisfies the condition that the group $Hdg^2(X)$ of rational Hodge classes of degree 2 vanishes, and that the group $Hdg^4(X)$ is perpendicular for the intersection pairing to $[\omega]^{n-2}$ for some Kähler class $\omega$ on $X$. Under these assumptions, $X$ does not contain any proper analytic subset of codimension less or equal to 2, and any coherent sheaf $\mathcal{F}$ satisfies the conditions

$$c_1(\mathcal{F}) = 0, < c_2(\mathcal{F}), [\omega]^{n-2} > _X = 0.$$

We now prove:

Proposition 4 If $X$ is as above, for any torsion free coherent sheaf $\mathcal{F}$ on $X$, there exists a holomorphic vector bundle $E$ on $X$, whose all rational Chern classes $c_i(E)$, $i > 0$ vanish, and an exact sequence

$$0 \rightarrow \mathcal{F} \rightarrow E \rightarrow T \rightarrow 0$$

where $T$ is a torsion sheaf on $X$.

Before we prove the proposition, we state the following corollaries.

Corollary 1 If $E$ is a holomorphic vector bundle on $X$, then all rational Chern classes $c_i(E)$, $i > 0$ vanish.

Indeed, we know that there exists an inclusion

$$E \hookrightarrow E',$$

where $E'$ is a vector bundle of the same rank as $E$ and satisfies the property that all rational Chern classes $c_i(E')$, $i > 0$ vanish. Now since $Hdg^2(X) = 0$, $X$ does not contain any hypersurface, and it follows that the inclusion above is an isomorphism.
**Corollary 2** If $X$ is as above and $Z \subset X$ is a non-empty proper analytic subset, the ideal sheaf $I_Z$ does not admit a finite free resolution. Indeed, if such a resolution

$$0 \to E^n \to \ldots \to E^i \to E^{i-1} \to E^0 \to I_Z \to 0$$

would exist, then we would get the equality

$$c(I_Z) = \prod_i c(E_i)^{e_i}$$

with $e_i = (-1)^i$. But the left hand side is non zero in positive degrees since its term of degree $r = \text{codim } Z$ is a non zero multiple of the class of $Z$. On the other hand the right hand side vanishes in positive degrees by corollary [1].

Note that the assumptions are satisfied by a general complex torus of dimension at least 3. Taking for $Z$ a point, we get an explicit example of a coherent sheaf which does not admit a finite locally free resolution.

**Proof of proposition 4.** We use again induction on the rank. Let $\mathcal{F}$ be a torsion free coherent sheaf of rank $k$ on $X$, and assume first that $\mathcal{F}$ does not contain any non zero subsheaf of smaller rank. There is an inclusion

$$\mathcal{F} \hookrightarrow \mathcal{F}^{**}$$

whose cokernel is a torsion sheaf, where the bidual $\mathcal{F}^{**}$ of $\mathcal{F}$ is reflexive. Then $\mathcal{F}^{**}$ does not contain any non zero subsheaf of smaller rank and hence is stable with respect to the given Kähler metric $h$ on $X$. The theorem of Bando and Siu together with the fact that

$$c_1(\mathcal{F}^{**}) = 0, \quad <c_2(\mathcal{F}^{**}), [\omega]^{n-2}>_X = 0$$

implies that $\mathcal{F}^{**}$ is a holomorphic vector bundle which is endowed with a flat connection, hence has trivial Chern classes and the result is proved in this case.

Assume otherwise that there is an exact sequence

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0,$$

where the ranks of $\mathcal{G}$ and $\mathcal{H}$ are smaller than the rank of $\mathcal{F}$, and $\mathcal{G}$ and $\mathcal{H}$ are without torsion. This exact sequence determines (and is determined by) an extension class

$$e \in \text{Ext}^1(\mathcal{H}, \mathcal{G}).$$

Now, by induction on the rank we may assume that we have inclusions

$$\mathcal{G} \hookrightarrow E_1, \mathcal{H} \hookrightarrow E_2,$$

whose cokernels $\mathcal{T}_i$ are of torsion and where the $E_i$’s are holomorphic vector bundles with vanishing Chern classes. The extension class $e$ gives first an extension class $f \in \text{Ext}^1(\mathcal{H}, E_1)$, which provides a sheaf $E'$ containing $\mathcal{F}$ in such way that $E'/\mathcal{F}$ is of torsion, and fitting into an exact sequence

$$0 \to E_1 \to E' \to \mathcal{H} \to 0.$$
Next, because the torsion sheaf $\mathcal{T}_2$ is supported in codimension $\geq 3$, the restriction map provides an isomorphism

$$\text{Ext}^1(E_2, E_1) \cong \text{Ext}^1(\mathcal{H}, E_1)$$

as one sees by applying Serre’s duality to these Ext groups. Hence it follows that there is a holomorphic vector bundle $E$ on $X$, which is an extension of $E_2$ by $E_1$, and which contains $E'$ as a subsheaf, such that the quotient $E/E'$ is of torsion. $E$ has vanishing Chern classes, because $E_i$ satisfy this property for $i = 1, 2$, and contains $\mathcal{F}$ as a subsheaf such that the quotient $E/\mathcal{F}$ is of torsion. This completes the proof by induction.

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