QUANTUM OSCILLATORS IN THE CANONICAL COHERENT STATES

R. de Lima Rodrigues\textsuperscript{(a)}*, A. F. de Lima\textsuperscript{(b)}, K. de Araújo Ferreira\textsuperscript{(b)} and A. N. Vaidya\textsuperscript{(c)}
\textsuperscript{(a)} Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud, 150
Rio de Janeiro-RJ-22290-180, Brazil
\textsuperscript{(b)}Departamento de Física, Universidade Federal de Campina Grande
Campina Grande, PB –58.109-970 – Brazil
\textsuperscript{(c)} Instituto de Física, Universidade Federal do Rio de Janeiro
Ilha do Fundão, Rio de Janeiro, RJ - 21.945-970 - Brazil

Abstract

The main characteristics of the quantum oscillator coherent states including the two-particle Calogero interaction are investigated. We show that these Calogero coherent states are the eigenstates of the second-order differential annihilation operator which is deduced via R-deformed Heisenberg algebra or Wigner-Heisenberg algebraic technique and correspond exactly to the pure uncharged-bosonic states. They possess the important properties of non-orthogonality and completeness. The minimum uncertainty relation for the Calogero interaction coherent states is investigated. New sets of Wigner oscillator even and odd coherent states are pointed out.

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*Permanent address: Departamento de Ciências Exatas e da Natureza, Universidade Federal de Campina Grande, Cajazeiras, PB – 58.900-000 – Brazil
I. INTRODUCTION

In the beginning of the sixties, the coherent states were investigated via three definitions, viz., states of minimal uncertainty, eigenstates of the annihilation operator and as being the states obtained by application of the displacement operator on the ground state [1]. In Ref. [1] it has been shown that these three definitions are equivalent for the simple harmonic oscillator. Due to the fact that the energy spectrum of a particle in a potential with centripetal barrier

\[ V(x) = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}gx^{-2}, \quad g = \lambda(\lambda + 1), \quad -\infty \leq x \leq \infty, \quad (1) \]

is equally spaced like that of the simple harmonic oscillator, this one-dimensional (1D) system is called an "isotonic oscillator" or two-particle Calogero interaction [2]. In 3-dimensional space this type of potential was first introduced by Davidson long ago [3]. Considered first by Weissman and Jortner [4] in the context of Gaussian wave functions, the dynamics and the energies of a coherent states for the 1D isotonic oscillator were studied. Elsewhere Nieto and Simmons Jr. found the minimum-uncertainty coherent states (MUCS) and discussed various properties of this system, and have also shown that the three definitions of coherent states are equivalent for the simple harmonic oscillator [5]. A year later it was shown by Gutshick, Nieto and Simmons Jr. [6] that the MUCS provide us with a better approximation to the classical motion than do the Gaussians. In another work Nieto [7] has shown the mathematical and physical connection of the charged-boson coherent states [8] with the MUCS. The canonical coherent states for the Wigner generalized oscillator in the Schrödinger representation were constructed by Sharma, Mehta and Sudarshan [9] and the representations and properties of para-bose oscillator operators were investigated in a Schrödinger description [10].

On the other hand, Leinaas and Myrheim [11] have investigated the relation between the fractal in 1+1 dimension and Calogero interaction, and Fernandez et al. have investigated the coherent states for SUSY partners of the oscillator [12].

In this work, we construct what we call canonical coherent states (CCS) [13], which are defined as the eigenstates of the annihilation operator \( B^- (\lambda) \) of the Calogero interaction Hamiltonian. Such annihilation operators are second-order differential ladder operators [14] and can be derived via the R-deformed Heisenberg algebra or Wigner-Heisenberg algebraic technique [15] which was recently super-realized for the SUSY isotonic oscillator [16,17]. The WH algebra has also been investigated for the three-dimensional non-canonical oscillator to generate a representation of the orthosympletic Lie superalgebra \( osp(3/2) \) [18]. The coherent states of \( SU(\ell, 1) \) groups have been explicitly constructed as orbits in some irreducible representations [19].

The motion of the peaks of the wavefunctions for the coherent states of the two-particle Calogero-Sutherland model were compared with the classical trajectory [20]. According to Calogero [2] the energy spectrum of the potential (1) and \( N \) bosons or fermions interacting are different by a energy shift proportional to \( \lambda = -\nu \). Using an operator formulation Brink et al. found all \( N \)-particle wave eigenfunctions and extended the approach to the supersymmetric Calogero model and Heisenberg algebra in the simple case of two particles [21]. The observable for the two-anyon problem, satisfy the same algebra as the observable in two-body Calogero problem.
Let us here point out the interesting connection between the mesoscopic effects and Calogero interaction for a Coulomb gas under a new universality in spectra of this chaotic system, which is described by a random matrix theory [22].

Another approach is the application of the time-dependent parameters in the potential (1) in many quantum-mechanical effects. For instance, Pedrosa et al. have used the Lewis-Riesenfeld invariant method and a unitary transformation to obtain the exact Schrödinger wave functions for a time-dependent harmonic oscillator with and without an inverse quadratic potential [23].

Recently, Witten’s supersymmetry formulation for Hamiltonian systems [24] has been extended to a system of annihilation operator eigenvalue equations associated with the supersymmetric unidimensional oscillator and supersymmetric isotonic oscillator (singular potential), which define supersymmetric canonical coherent states containing mixtures of both pure bosonic and pure fermionic counterparts [25]. The breaking of supersymmetry due to singular potentials in supersymmetric quantum mechanics given by Eq. (1) has been recently investigated [26]. In [17], we see that the main result was the observation of the intimate relationship between the generalized statistics and supersymmetry via R-deformed Heisenberg algebra: it was shown that the supersymmetry can be realized in purely parabosonic systems (it is realized there in linear or nonlinear form depending on the order of the paraboson). This application has been considered for the deformed Virasoro algebra so that a representation of the modified Virasoro algebra has been found [27].

Let us now point out that the R-deformed Heisenberg (or Wigner-Heisenberg) algebra is given by following (anti-)commutation relations ([A, B]+ ≡ AB + BA and [A, B]− ≡ AB − BA):

\[
H = \frac{1}{2}[a^-, a^+]_+, \quad [H, a^{\pm}]_+ = \pm a^{\pm}, \quad [a^-, a^+]_- = 1 + \nu R, \quad [R, a^{\pm}]_+ = 0, \quad R^2 = 1, \quad (2)
\]

where \(\nu\) is a real constant associated to the Wigner parameter [16]. Note that when \(\nu = 0\) we have the standard Heisenberg algebra. The generalized quantum condition given in Eq. (2) has been found relevant in the context of integrable models [28]. Furthermore, this algebra was used for solving the energy eigenvalue and eigenfunctions of the Calogero interaction, in the context of one-dimensional many-body integrable systems, in terms of a new set of phase space variables involving exchanged operators [29,30]. Recently it has been employed for bosonization of supersymmetry in quantum mechanics, and has also been demonstrated that finite-dimensional representations are representations of the deformed parafermionic algebra with internal \(Z_2\)–grading structure [31].

Recently, the coherent states à la Klauder-Perelomov for a particle moving in the Pöschl-Teller potential of the trigonometric type have been built up [32].

In this work, we display some graphs showing the behavior of the minimum uncertainty for a particular set of Wigner oscillator CCS. This present work is organized as follows. In Sec. II, we start by summarizing the R-deformed Heisenberg algebra or Heisenberg algebraic technique for the Wigner isotonic oscillator [16,17]. While Jayaraman and Rodrigues, in Ref. [16], adopt a super-realization of the R-deformed Heisenberg algebra as effective spectral resolution for the two-particle Calogero interaction, in Ref. [17], using the same super-realization, Plyushchay showed the various aspects of the R-deformed Heisenberg algebra. In [17], it was also shown that the nonlinear supersymmetry can be realized also at the
classical level via the appropriate simple modification of the model corresponding to the Witten supersymmetric quantum mechanics, and it was noted that the quantization results in a generic case in the quantum anomaly.

An generalization of the Heisenberg algebra which is written in terms of a functional of one generator of the algebra has been analyzed by Curado and Rego-Monteiro [33]. The creation and annihilation operators of correlated fermion pairs, in simple many body systems, satisfy a deformed Heisenberg algebra that can be approximated by $q$ oscillators [34]. In [35], a possibility of extending the $q$-deformed Heisenberg algebra to build a quantum field theory having fields that produce at any space-time point particles satisfying the same algebra. Also, Arik-Kiliç have extended the $SU_q(2)$ algebra and the coherent states have been investigated [36]. The $q$-coherent states have been investigated for $q$-algebra related with shape invariance condition by Fukui and Balantekin et al. [37].

Using the Gazeau-Klauder approach [38] for coherent states associated with quantum systems, Antoine et al. [39] have analyzed the spatial and temporal features of the coherent states associated to the infinite square-well and Pöschl-Teller potentials. Also, Daoud-Hussin have found new general sets of coherent states and the quantum optics Jaymes-Cummings model [40]. Moreover, Popov has constructed and investigated the pseudoharmonic oscillator in the Barut-Girardello coherent states and photon-added Barut-Girardello coherent states [41].

In Sec. III, we define and build without supersymmetry the two-particle Calogero interaction canonical coherent states as the eigenstates of the second-order differential annihilation operator ($B^-$). In Sec. IV, we discuss the MUCS from Wigner first-order differential ladder operators and Calogero interaction ladder operators. In this Section, new even and odd canonical coherent states are pointed out. Sec. V contains the conclusions.

II. THE SUPER WIGNER-HEISENBERG ALGEBRA

For convenience we choose units so that $\hbar = \omega = m = 1$. Thus, for a super-realization of the $R$-deformed Heisenberg algebra (2), the system governed by the potential (1) becomes identical to the potential of the bosonic sector of the Wigner Hamiltonian. The 1D Wigner oscillator Hamiltonian in terms of the Pauli’s matrices ($\sigma_i, i=1,2,3$) is given by

$$H(\lambda + 1) = \frac{1}{2} \left\{ -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}((\lambda + 1)(\lambda + 1) - 1)\sigma_3 \right\}$$

$$= \begin{pmatrix} H_-(\lambda) & 0 \\ 0 & H_+(\lambda) = H_-(\lambda + 1) \end{pmatrix},$$

(3)

where the even and odd sector Hamiltonians are respectively given by

$$H_-(\lambda) = \frac{1}{2} \left\{ -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}\lambda(\lambda + 1) \right\}$$

(4)

and

$$H_+(\lambda) = \frac{1}{2} \left\{ -\frac{d^2}{dx^2} + x^2 + \frac{1}{x^2}(\lambda + 1)(\lambda + 2) \right\} = H_-(\lambda + 1).$$

(5)
The even sector is the Hamiltonian of the oscillator with barrier.

Note that the Wigner oscillator ladder operators
\[ a^{\pm} = \frac{1}{\sqrt{2}}(\pm i\hat{p}_x - \hat{x}) \] of the R-deformed Heisenberg algebra may be written in terms of the super-realization of the position and momentum operators viz., \( \hat{x} = x\sigma_1 \) and \( \hat{p}_x = -i\sigma_1 \frac{d}{dx} + \frac{\nu}{2x}\sigma_2 \), satisfy the general quantum rule \([\hat{x}, \hat{p}_x]_\pm = i(1 + \nu R)\), where \( \nu = 2(\lambda + 1) \). Thus, in this representation the reflection operator becomes \( R = \sigma_3 \), where \( \sigma_3 \) is the diagonal Pauli matrix.

Thus, from the super-realized first order ladder operators given by [16,17]
\[ a^{\pm}(\lambda + 1) = \frac{1}{\sqrt{2}}\left\{ \pm \frac{d}{dx} \pm \frac{(\lambda + 1)}{x}\sigma_3 - x \right\} \sigma_1, \] the Wigner Hamiltonian becomes
\[ H(\lambda + 1) = \frac{1}{2}\left[a^+(\lambda + 1), a^-(\lambda + 1)\right]_+ \] and the Wigner-Heisenberg algebra ladder relations are readily obtained as
\[ \left[H(\lambda + 1), a^{\pm}(\lambda + 1)\right]_\pm = \pm a^{\pm}(\lambda + 1). \] Equations (8) and (9) together with the commutation relation
\[ \left[a^-(\lambda + 1), a^+(\lambda + 1)\right]_\pm = 1 + 2(\lambda + 1)\sigma_3 \] constitute the R-deformed Heisenberg algebra.

The Wigner eigenfunctions that generate the eigenspace associated with even(odd) \( \sigma_3 \)-parity for even(odd) quanta \( n = 2m(n = 2m + 1) \) are given by
\[ | n = 2m, \lambda + 1 > = \begin{pmatrix} m, \lambda \\ 0 \end{pmatrix}, \quad | n = 2m + 1, \lambda > = \begin{pmatrix} 0 \\ m, \lambda \end{pmatrix} \] and satisfy the following eigenvalue equation
\[ H(\lambda + 1) | n, \lambda + 1 > = E^{(n)} | n, \lambda + 1 >, \] where the non-degenerate energy eigenvalues are obtained by the application of the raising operator on the ground eigenstate and are given by
\[ E^{(n)} = \lambda + \frac{3}{2} + n, \quad n = 0, 1, 2, \ldots. \] For the oscillator with barrier the energy eigenvectors satisfy the following equations
\[ H_-(\lambda) | m, \lambda > = E^{(m)}_\lambda | m, \lambda >, \] where the eigenvalues are exactly constructed via R-deformed Heisenberg algebra ladder relations and are given by [16].
\[ E^{(m)}_m = \lambda + \frac{3}{2} + 2m, \quad m = 0, 1, 2, \ldots \]  

(15)

Also from the Wigner-Heisenberg algebra we obtain the second-order differential raising and lowering operators for the energy spectrum of the 1D oscillator with barrier, viz., on \( \frac{1}{2}(1 + \sigma_3) \) projection, the R-deformed Heisenberg algebra representations decouple, \( [H(\lambda + 1), a^{\pm 2}(\lambda + 1)]_\pm = \pm 2a^{\pm 2}(\lambda + 1). \) Indeed, the left hand side leads us

\[ \frac{1}{2}(1 + \sigma_3)[H(\lambda + 1), a^{\pm 2}(\lambda + 1)]_\pm = \begin{pmatrix} [H_-(\lambda), B^\pm(\lambda)]_\pm & 0 \\ 0 & 0 \end{pmatrix} \]

and the right hand side becomes

\[ \frac{1}{2}(1 + \sigma_3)a^{\pm 2}(\lambda + 1) = \begin{pmatrix} B^\pm(\lambda) & 0 \\ 0 & 0 \end{pmatrix}, \]

where

\[ B^-(\lambda) = \frac{1}{2} \left\{ \frac{d^2}{dx^2} + 2x \frac{d}{dx} + x^2 - \frac{\lambda(\lambda + 1)}{x^2} + 1 \right\} \]

(16)

and

\[ B^+(\lambda) = \frac{1}{2} \left\{ \frac{d^2}{dx^2} - 2x \frac{d}{dx} + x^2 - \frac{\lambda(\lambda + 1)}{x^2} - 1 \right\}. \]

(17)

Thus, these ladder operators obey the following commutation relations:

\[ [B^-(\lambda), B^+(\lambda)]_\pm = 4H_-(\lambda) \]

\[ [H_-, B^\pm(\lambda)]_\pm = \pm 2B^\pm(\lambda). \]

(18)

Hence, the quadratic operators \( B^\pm(\lambda) \) acting on the orthonormal basis of eigenstates of \( H_-(\lambda), \) \( \{ |m, \lambda \rangle \} \) where \( m = 0, 1, 2, \ldots \) have the effect of raising or lowering the quanta by two units so that we can write

\[ B^-(\lambda) |m, \lambda \rangle = \sqrt{2m(2m + 2\lambda + 1)} |m - 1, \lambda \rangle \]

(19)

and

\[ B^+(\lambda) |m, \lambda \rangle = \sqrt{2(m + 1)(2m + 2\lambda + 3)} |m + 1, \lambda \rangle \]

(20)

giving

\[ |m, \lambda \rangle = 2^{-m} \left\{ \frac{\Gamma(\lambda + 3/2)}{m!\Gamma(\lambda + m + 3/2)} \right\}^{1/2} \{B^+(\lambda)\}^m |0, \lambda \rangle, \]

(21)

where \( \Gamma(x) \) is the ordinary Gamma Function. Note that \( B^\pm(\lambda) |m, \lambda \rangle \) are associated with the energy eigenvalues \( E^{(m \pm 1)}_\pm = \lambda + \frac{3}{2} + 2(m \pm 1), \quad m = 0, 1, 2, \ldots. \)

Let us conclude this section presenting a very simple question: what is the structure generated by the new operators pointed out in this section from quantum oscillator? Note
that the operators $\pm i (a^\pm (\lambda + 1))^2$ and $\frac{1}{2}H(\lambda + 1)$ can be chosen as a basis for a realization of the $SO(2, 1) \sim SU(1, 1) \sim SL(2, \mathbb{R})$ Lie algebra. When projected the $-\frac{1}{2} (a^\pm)^2$ operators in the even sector we obtain that $-\frac{1}{2}B^\pm$ and together with $\frac{1}{2}H_-$ generate once again the Lie algebra $SU(1, 1)$.

Consequently, for Calogero interaction the resultanting Lie algebra is $SU(1, 1)$:

$$[K_0, K_1]_+ = iK_2, \quad [K_1, K_2]_+ = -iK_0, \quad [K_2, K_0]_+ = -iK_1. \quad (22)$$

Indeed, from (4), (16) and (17) we obtain $K_0 = \frac{H}{2}, K_1 = -\frac{1}{4}(B^- + B^+)$ and $K_2 = -\frac{i}{4}(B^- - B^+)$. Therefore one can generate the generalized coherent states according to Perelomov [42,43].

### III. CALOGERO INTERACTION CANONICAL COHERENT STATES

Now, we define the Calogero interaction canonical coherent states, $| \alpha, \lambda >$, as the eigenkets of the annihilation operator $B^- (\lambda)$,

$$B^- (\lambda) | \alpha, \lambda > = \alpha | \alpha, \lambda >, \quad (23)$$

where the eigenvalue $\alpha$ can be any complex number. Writing

$$| \alpha, \lambda > = \sum_{m=0}^{\infty} b_m | m, \lambda > \quad (24)$$

we obtain a recursion relation for the coefficients $b_m$

$$b_m = \frac{\alpha}{2} \left\{ m(m + \lambda + \frac{1}{2}) \right\}^{-\frac{1}{2}} b_{m-1} = \frac{\left( \frac{\alpha}{2} \right)^m}{m! \Gamma(m + \lambda + \frac{3}{2})} b_0 \quad (25)$$

which provides us with the normalized canonical coherent states ($< \alpha, \lambda | \alpha, \lambda >$) for the Calogero interaction in the form

$$| \alpha, \lambda > = \{ g(| \alpha |) \}^{-\frac{1}{2}} \sum_{m=0}^{\infty} \frac{\left( \frac{\alpha}{2} \right)^m}{m! \Gamma(m + \lambda + \frac{3}{2})} | m, \lambda >, \quad (26)$$

where the normalization constant $b_0$ is given by

$$b_0^2 = g(| \alpha |) = \left\{ \frac{2}{| \alpha |} \right\}^{(\lambda + \frac{1}{2})} I_{\lambda + \frac{1}{2}}(| \alpha |) \quad (27)$$

and $I_\nu(| \alpha |)$ is the modified Bessel function of the first kind,

$$I_\nu(| \alpha |) = \sum_{m=0}^{\infty} \frac{\left( \frac{| \alpha |}{2} \right)^{2m+\nu}}{m! \Gamma(m + \nu + 1)}. \quad (28)$$

The Calogero interaction CCS are normalized however they are non-orthogonal since
\[ <\xi,\lambda|\alpha,\lambda>=\{g(|\alpha|)g(|\xi|)\}^{-\frac{1}{2}}g\left((\xi^*\alpha)^{\frac{1}{2}}\right), \] (29)

which means that the CCS is an over-complete. The resolution of unity is given by

\[ \int |\alpha,\lambda><\alpha,\lambda|\frac{1}{2\pi}K_{\lambda+\frac{1}{2}}(|\alpha|)I_{\lambda+\frac{1}{2}}(|\alpha|)d^2\alpha = \sum_{m=0}^{\infty} |m,\lambda><m,\lambda|=1, \] (30)

where \(x=|\alpha|, z=1\) and \(t=\sinh u\) with

\[ K_\nu(|\alpha|) = 2^\nu \frac{\Gamma(\nu+\frac{1}{2})}{|\alpha|^{\nu} \sqrt{\pi}} \int_0^{\infty} \frac{\cos(|\alpha|t)}{(t^2+1)^{\nu+\frac{1}{2}}} dt \]
\[ = \frac{2\Gamma(\nu+\frac{1}{2})}{|\alpha|^{\nu} \sqrt{\pi}} \int_0^{\infty} \cos(\alpha u)^{-2\nu} \cosh(|\alpha|\sinh u) du, \] (31)

which is a particular form of \(K_\nu(\alpha)z\) is the modified Bessel function of the third kind [44].

The completeness property here deduced is formally analogous to the resolution of the identity for the isotonic oscillator minimum-uncertainty coherent states [7]. However, one obtain this properties from our operators deduced via the super-realization of the R-deformed Heisenberg.

**IV. THE MINIMUM UNCERTAINTY COHERENT STATES**

Let us begin by making some remarks about the CCS and the minimum uncertainty coherent states (MUCS) of Wigner oscillator. The CCS of Wigner annihilation operator which satisfies the R-deformed Heisenberg algebra are defined by

\[ a^-|\zeta>_W = \zeta|\zeta>_W, \] (32)

and can be written in terms of the Wigner oscillator eigenstates

\[ |\zeta>_W = \sum_{n=0}^{\infty} c_n |n,\lambda+1>, \] (33)

where the eigenvalue \(\zeta\) can be any complex number.

From the definition of the position \(\hat{x}\) and momentum \(\hat{p}\), quantum operators given by Eqs. (6) and (10) we have the following commutation relation

\[ [\hat{x},\hat{p}_x]_- = i[1+2(\lambda+1)\sigma_3]. \] (34)

Thus, there is a generalised uncertainty relation for \(\hat{x}\) and \(\hat{p}\), given by

\[ \Delta\hat{x}\Delta\hat{p} \geq \frac{1}{2} |1+2(\lambda+1)<\sigma_3>|. \] (35)

We show that the product of uncertainties of the Wigner oscillator position \(\hat{x}\) and momentum \(\hat{p}\), in the CCS becomes minimal among those values which are permissible by quantum mechanics, viz., \(\Delta\hat{x}\Delta\hat{p} = |\frac{1}{2} + (\lambda+1)<\sigma_3>|\), where \(<\sigma_3>\) is the average value in the Wigner
oscillator CCS. For instance note that when \( \lambda = -1 \) the unidimensional oscillator MUCS is re-obtained. A detailed analysis of Wigner oscillator coherent state s is in preparation.

To complete our analysis on coherent states for the Calogero interaction, we trace below the construction of minimum uncertainty coherent states. Now let us consider new definitions of the position \( \hat{X} \) and momentum \( \hat{P} \) so that

\[
B^\pm = \frac{1}{\sqrt{2}}(\mp i \hat{P} - \hat{X})
\]

which leads us to the following commutation relation

\[
[\hat{X}, \hat{P}] = 4iH_-
\]

In this case the minimum uncertainty states \( |\alpha \rangle_M \) with equal dispersions for \( \hat{X} \) and \( \hat{P} \) are given by

\[
B^- |\alpha \rangle_M = \alpha |\alpha \rangle_M, \quad \alpha = -\frac{1}{\sqrt{2}}(<\hat{X}> + i <\hat{P}>)
\]

where \( |\alpha, \lambda \rangle_M \) is an eigenstate particular set of \( B^- \).

Therefore, using the two identities

\[
\hat{X}^2 = \frac{1}{2} \left( (B^-)^2 + (B^+)^2 + 2B^+B^- + 4H_- \right),
\]

\[
\hat{P}^2 = -\frac{1}{2} \left( (B^-)^2 + (B^+)^2 - 2B^+B^- - 4H_- \right)
\]

we obtain the following expectation values in the CCS:

\[
<\hat{X}> = -\frac{1}{\sqrt{2}}(\alpha^* + \alpha) = -\sqrt{2}Re(\alpha),
\]

\[
<\hat{P}> = \frac{i}{\sqrt{2}}(\alpha^* - \alpha) = \sqrt{2}Im(\alpha)
\]

\[
<\hat{X}^2> = 2[Re(\alpha)]^2 + 2 <H_->, \quad <\hat{P}^2> = 2[Im(\alpha)]^2 + 2 <H_->,
\]

where

\[
<H_-> = <\alpha | H_- | \alpha \rangle_M = |\alpha | \frac{I_{\lambda-\frac{1}{2}}(|\alpha \rangle)}{I_{\lambda+\frac{1}{2}}(|\alpha \rangle)} + \frac{1}{2} - \lambda.
\]

The variances of position operator \( (\Delta \hat{X})^2 = <\hat{X}^2> - <\hat{X}>^2 \) and momentum operator \( (\Delta \hat{P})^2 = <\hat{P}^2> - <\hat{P}>^2 \) on the coherent states \( |\alpha \rangle_M \) are identical. Next, \( I_{\lambda-\frac{1}{2}}(|\alpha \rangle) \) is the modified Bessel function of the first kind given by Eq. (28), for \( x >> 1 \) is given by

\[
I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}} \left( 1 - \frac{4n^2 - 1}{8x} + \frac{(4n^2 - 1)(4n^2 - 3)^2}{2!(8x)^2} + \cdots \right).
\]
Thus, from Eq. (40) the minimum uncertainty relation for \( | \alpha | \gg 1 \), \( \lambda = \frac{1}{2} \) and \( \lambda = 0 \), respectively, becomes

\[
\begin{align*}
\Delta \hat{X} \Delta \hat{P} &= | \alpha | \\
\Delta \hat{X} \Delta \hat{P} &= | \alpha | + \frac{1}{2}.
\end{align*}
\]  
(43)

Indeed, we find that the variances in new position and momentum satisfy

\[
< (\Delta \hat{X})^2 > = 2 < H > = < (\Delta \hat{P})^2 >
\]

\[
\Delta \hat{X} \Delta \hat{P} = 2 | \alpha | \frac{I_{\lambda - \frac{1}{2}}(| \alpha |)}{I_{\lambda + \frac{1}{2}}(| \alpha |)} + 1 - 2\lambda.
\]  
(44)

Therefore, we show that a Calogero interaction in CCS leads us to minimum uncertainty relation. In figures I, II and III we plot \( < \hat{X}^2 > \) and \( (\Delta \hat{X})(\Delta \hat{P}) \), for two particular values of \( \lambda \).

V. CONCLUDING REMARKS

We have presented the canonical coherent states (CCS) associated with the unidimensional harmonic oscillator plus a centripetal barrier (a Calogero interaction [2,42] with two particles for the relative coordinate \( x = x_1 - x_2 \) or isotonic oscillator [14]), which preserve the property of non-orthogonality. These CCS were deduced via R-deformed Heisenberg (or Wigner-Heisenberg) algebra in non-relativistic quantum mechanics. Although we have mainly treated the Calogero interaction CCS, similar results can be adequately extracted for any physical D-dimensional radial oscillator system by the Hermitian replacement of

\[-i \frac{d}{dx} \rightarrow -i \left( \frac{d}{dx} + \frac{D-1}{2} \right)\]

and the Wigner deformation parameter \( \lambda + 1 \rightarrow \ell_D + \frac{1}{2}(D - 1) \)

where \( \ell_D (\ell_D = 0, 1, 2, ...) \) is the D-dimensional oscillator angular momentum. In tridimensional space this Hermitian replacement left us exactly to the potential first investigated by Davidson in the beginning of thirties [3].

Therefore we can construct new spherical coherent states for diatomic molecules with Davidson interaction [45], so that complete diatomic molecule energy spectra and eigenfunctions can be deduced algebraically via R-deformed Heisenberg algebra or Wigner-Heisenberg factorization method [16,17].

We also consider a succinct anlysis of the construction of minimum uncertainty coherent states (MUCS) for the Wigner oscillator position \( \hat{x} \) and momentum \( \hat{p} \), and a detailed anlysis of minimum uncertainty coherent states for the Calogero interaction.

Let us point out that a CCS \( |z >_W \) is an eigenstate of the Wigner annihilation operator according to Eq. (32), it is possible to show that the analogous of so called even and odd CCS \( |z, \pm > \), which appear in the coherent states for the usual harmonic oscillator and Quantum Optics for uncharged quanta [47] and charge quanta [48], are eigenstates of the operator \( (a^-(\lambda + 1))^2 \) (but not of \( a^-(\lambda + 1) \)). A detailed analysis of the generation of even and odd canonical coherent states via R-deformed Heisenberg algebra will be published elsewhere.

Recently Alexanian et al. have built a star product associated with an arbitrary two-dimensional Poisson structure using generalized coherent states on the complex plane [46].
The coherent states for the isotonic oscillator has been considered in the coordinate representation by Bagchi and Bhaumik [49]. The correspondence between eigenvalue Eqs. (26) in Ref. [49] and our Eq. (23) provided us $z = \sqrt{2}$, where $2z^2$ is the eigenvalue in Ref. [49].

Following [21,22,42] it is of interest to note that defining

$$A_j^\pm = \frac{1}{\sqrt{2}} \left\{ p_j \pm i \left( \frac{\partial}{\partial x_j} W(x_j) \right) \right\}, \quad p_j = -i \frac{d}{dx_j}$$

we obtain in one dimension the factorized Hamiltonian of the Calogero interaction for a many particle system with Davidson interactions

$$H_- = \sum_j A_j^+ A_j^- = \frac{1}{2} \sum_j \left\{ p_j^2 + \left( \frac{\partial}{\partial x_j} W(x_j) \right)^2 \right\} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 W(x_j)}{\partial x_i \partial x_j}.$$ 

Thus, following [22], under the context of mesoscopic physics, in the case of $N$-boson or fermion Hamiltonian we can obtain the relation between the Brownian motion and Calogero model.

For instance in our case with two particles the choice $W(x) = \frac{1}{2} x^2 + (\lambda + 1) \ell n(x)$ gives us $H_-$ belonging to the even sector of $H(\lambda + 1)$. These aspects will be considered elsewhere in construction of the supercoherent states [25] for diatomic molecules [45]. However, note that in this work the $A_j^\pm$ operators become

$$A_j^\pm \to A^\pm = \frac{1}{\sqrt{2}} \left( -i \frac{d}{dx} \pm i \frac{(\lambda + 1)}{x} \pm ix \right).$$

Therefore, the present work opens a new route for future investigations on the Calogero interaction coherent states, for instance let us point out that the R-deformed Heisenberg (or Wigner-Heisenberg) algebra can be applied for a complete spectral resolution of the complex Calogero model with real energies [50], too. Finally, let us point out that one can consider an analysis of the Calogero interaction coherent states as reported in the works of references [51–53].

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(a) E-mail: rafaelr@cbpf.br or rafael@fisica.ufpb.br
(b) E-mail: aerlima@df.ufpb.br
(c) E-mail: vaidya@if.ufrj.br

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FIG. 1. The minimum uncertainty relation for the Calogero system coherent states given by Eqs. (41) and (44), for $\lambda = \frac{1}{2}$.

FIG. 2. The minimum uncertainty relation for the Calogero system coherent states for $\lambda = \frac{1}{2}$ and $|\alpha| >> 1$ is given by Eq. (43).
FIG. 3. The minimum uncertainty relation for the Calogero system coherent states given by Eqs. (41) and (44), for $\lambda = 10.0$