Perturbations to Generalized Kink-like Topological Defects in AdS

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Abstract

We explore perturbations to a kink-like (codimension 1) topological defect whose world brane is AdS\(_q\) embedded into AdS\(_{q+1}\). Previously, we found solutions in the limit the mass of the scalar field vanishes. In this article we extend a calculation previously done in AdS\(_2\) to higher-dimensional embedding spaces and find that all perturbations to the mass of the field are stable to first order as expected in a theory with topological defects. We find that the equation of motion to the correction strongly resembles a problem well-known in quantum mechanics.

1 Introduction

This is the third of a series of articles in which the authors explore the dynamics of topological defects in anti de Sitter space. These objects are interesting to study as a possible consequence of symmetry breaking in the early universe leading to cosmological-scale objects such as cosmic strings [1]. In more recent years, some study has gone into how these objects might influence the dynamics of nearby massive objects [2]. Our model concerns an SO(\(l\))-gauged Higgs-type field theory embedded into AdS\(_q\) with \(q, l \geq 1\). For convenience, we will often refer to these with the shorthand “\((q, l)\) defect” for the codimension-\(l\) topological defect with a worldbrane of AdS\(_q\) embedded in AdS\(_{q+l}\). In the first article [3], we laid out a framework for finding exact analytical spherically-symmetric topological defect solutions. The method followed in the spirit of Bogomolny, Prasad, and Sommerfield (BPS) [4, 5], who were able to study exact solutions to the ’tHooft-Polyakov Monopole and Julia-Zee dyon in flat spacetime. The solutions we found were affectionately dubbed the “double BPS” solutions for that reason. In the double BPS solution, the finite radius of curvature of AdS sets the length-scale for the physics, allowing one to take the limit in the equations of motion that the mass of the fields fall to zero. Lugo et al. [6, 7] and Ivanova et al. [8, 9] also did

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studies of topological defects in AdS, especially that of monopoles in the former and pure Yang Mills solutions in the latter. For a discussion on how these works relate to this model and more detail on the process, we refer the reader to the original article [3].

The second article [10] concerned an extension of this using perturbation theory. Given a double BPS solution, we allow a small perturbation in the potential to reintroduce the mass of the field. We took the case of the (1,1) kink-like defect in particular; that is, the codimension-1 topological defect with a worldbrane that is AdS$_1$ embedded in AdS$_2$. We found that these defects are stable under this perturbation and we were able to calculate the correction to the defect energy that results.

In this paper, we extend our discussion to a more general $(q,1)$ defect. We are able to show that these solutions are linearly stable in general as expected in a theory containing topological defects.

## 2 The metric and full EOM

The $(q,1)$ kink is a codimension-1 topological defect extending from a totally geodesic AdS$_q$ embedded in AdS$_{q+1}$. There is still only a scalar field $\Phi$, so we do not have a gauge field to worry about. We take the metric of AdS$_q$ to be the standard maximally invariant Lorentzian metric on AdS$_q$, $ds^2_{\text{AdS}_q} = \gamma_{ab}(\sigma) d\sigma^a \otimes d\sigma^b$ with $\sigma$ as coordinates along the submanifold. If $k < 0$ is the sectional curvature of AdS$_{q+1}$ (and by extension, the totally geodesic AdS$_q$), then the radius of curvature is defined by $\rho = |k|^{-1/2} > 0$. A point in AdS$_{q+1}$ will be given coordinates $(\sigma, \nu)$ where $\nu$ is the signed distance along a geodesic that is normal to AdS$_q$. Note that $\nu \in (-\infty, +\infty)$. The metric in AdS$_{q+1}$ is then

$$ds^2_{\text{AdS}_{q+1}} = \cosh^2 \left( \frac{\nu}{\rho} \right) ds^2_{\text{AdS}_q} + d\nu^2.$$  \hspace{1cm} (2.1)

From this point forward we will use dimensionless coordinates. These can be obtained by scaling everything by the radius of curvature: $\nu \to \rho \nu$, $\sigma \to \rho \sigma$, and $\Phi \to \Phi_0 \Phi$ where $\Phi_0$ the vacuum expectation value for the scalar field $\Phi(\sigma, \nu)$.

The action for a $(q,1)$-defect is

$$I = -\Phi_0^2 \int d^n x \sqrt{-g} \left( \frac{1}{2} g^{\mu \nu} \partial_\mu \Phi(x) \partial_\nu \Phi(x) + \epsilon U(\Phi(x)) \right).$$  \hspace{1cm} (2.2)

where $\epsilon$ is a bookkeeping parameter that will be used to keep track of the order of the perturbative expansion. We assume the potential $U(\Phi)$ is of the symmetry breaking type with minimum at $\pm 1$ after the field rescaling.

This gives rise to the equation of motion:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu \nu} \partial_\nu \Phi \right) - \epsilon U'(\Phi) = 0.$$  \hspace{1cm} (2.3)

\footnote{We use the mostly + convention for the signature.}
The first term is the negative of the d’Alembertian $\Box_{\text{AdS}_{q}}$ and which in our chosen coordinates may be expressed as:

$$\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-gg^{\mu\nu}}\partial_{\nu}) = \frac{1}{\cosh^{q}(\nu)\sqrt{-\gamma}}\left[\partial_{a}\left(\cosh^{q}(\nu)\sqrt{-\gamma}\frac{1}{\cosh^{q}(\nu)}\gamma^{ab}\partial_{b}\right)\right. $$

$$+ \partial_{\nu}\left(\cosh^{q}(\nu)\sqrt{-\gamma}\partial_{\nu}\right)$$

$$= \frac{1}{\cosh^{2}(\nu)}(-\Box_{\text{AdS}_{q}}) + \frac{1}{\cosh^{q}(\nu)}\partial_{\nu}(\cosh^{q}(\nu)\partial_{\nu})$$

$$\Box_{\text{AdS}_{n}} = \frac{1}{\cosh^{2}(\nu)}\Box_{\text{AdS}_{q}} - \frac{1}{\cosh^{q}(\nu)}\partial_{\nu}(\cosh^{q}(\nu)\partial_{\nu})$$

(2.4)

Using this operator, equation (2.3) becomes:

$$\Box_{\text{AdS}_{q+1}} \Phi = -\epsilon U'(\Phi).$$

(2.5)

This is the wave equation with a source. In the limit where $\epsilon \downarrow 0$, we obtain the wave equation $\Box_{\text{AdS}_{q+1}} \Phi = 0$. In addition, we retain the boundary conditions

$$\Phi \xrightarrow{\nu \to \pm \infty} \pm 1$$

(2.6)

that guarantee a kink-like solution. We seek solutions with localized energy, that is, energy density that falls off exponentially as $\nu \to \pm \infty$. It does not need to be localized in the $\sigma$ direction(s).

We found maximally symmetric solutions that depend only on the normal coordinate $\nu$, and we denoted them by $\phi(\nu)$. These are the so-called double BPS solutions and they satisfy the wave equation

$$\Box_{\text{AdS}_{q+1}} \phi(\nu) = 0,$$

(2.7)

with boundary conditions $\phi(\nu) \to \pm 1$ as $\nu \to \pm \infty$, and is discussed in [3]. We are interested in analyzing the corrections to our solutions by introducing a perturbation about the double BPS solution, and we write the field as the double BPS solution and a perturbation: $\Phi = \phi + \epsilon \psi$.

The first-order correction $\psi(\sigma, \nu)$ obeys a wave equation with a source term determined by the zeroth order double BPS solution:

$$\Box_{\text{AdS}_{q+1}} \psi = -U'(\phi).$$

(2.8)

The general solution will consist of a particular solution to the inhomogeneous equation and the general solution to the homogeneous equation: $\psi = \xi(\nu) + \eta(\sigma, \nu)$, where $\xi(\nu)$ solves the inhomogeneous equation with a source and $\eta(\sigma, \nu)$ solves the homogeneous equation. Our ansatz for the perturbative solution is therefore $\Phi = \phi + \epsilon \psi$.

This was only to first order, but it’s not too hard to generalize this procedure to any order in perturbation theory. We can show that, in general:

$$\Box_{\text{AdS}_{q+1}} \Phi_{[j+1]} = -\epsilon U'(\Phi_{[j]}) + \mathcal{O}(\epsilon^{j+2}),$$

(2.9)
where we use $\Phi[j]$ to denote the $j$th order solution to the equations of motion. This gives an iterative scheme to solving the differential equation. At each order you will have to specify exactly how to choose the solution to the inhomogeneous equation because the d’Alembertian operator has a non-trivial kernel. To uniquely specify a perturbative solution we use the same method as in the $q = 1$ case.

3 Solving the EOM

3.1 Separating the EOM

The first order contribution $\psi(\sigma, \nu)$ must satisfy (2.8). Since the source term on the right is only a function of $\nu$, we can write $\psi(\sigma, \nu) = \xi(\nu) + \eta(\sigma, \nu)$, where $\xi(\nu)$ is a particular solution to the inhomogeneous equation and $\eta(\sigma, \nu)$ is a solution to the homogeneous equation below:

$$\frac{1}{\cosh^q(\nu)} \frac{d}{d\nu} \left( \cosh^q(\nu) \frac{d\xi}{d\nu} \right) = U'(\phi), \quad (3.1)$$

$$\frac{1}{\cosh^2(\nu)} □_{\text{AdS}} \eta - \frac{1}{\cosh^q(\nu)} \frac{\partial}{\partial \nu} \left( \cosh^q(\nu) \frac{\partial \eta}{\partial \nu} \right) = 0. \quad (3.2)$$

The boundary conditions are $\xi(\nu) \xrightarrow{|\nu| \to \infty} 0$ and $\eta(\sigma, \nu) \xrightarrow{|\nu| \to \infty} 0$. Equation (3.1) is an ordinary linear differential equation with source and we immediately discuss its solution. Afterwards, we will discuss how to solve PDE (3.2).

3.2 Solving for $\xi(\nu)$

The solution to (3.1) depends upon the choice of potential $U(\phi)$. The solution in the case of the $q = 1$ defect in the symmetry-breaking potential $U(\phi) = \frac{\Lambda}{8}(\phi^2 - \phi_0^2)^2$ was explored in our previous article [10]. We restate the results here, and extend the discussion of this toy model to a few other cases. The procedure for solving the equation is nearly identical for $q > 1$, though the difficulty in obtaining the solution varies significantly as $q$ increases, see Appendix C for details.

In the case of $q = 1$, the ODE becomes:

$$\xi''(\nu) + \tanh(\nu)\xi'(\nu) - \frac{\tilde{\mu}^2}{\pi^3} \arctan(\sinh(\nu))(4 \arctan(\sinh(\nu))^2 - \pi^2) = 0. \quad (3.3)$$

Here, $\tilde{\mu}^2 = \Lambda \phi_0^2 \rho^2$ is the squared flat-space mass of the scalar field $\phi$ scaled by the radius of curvature in order to keep dimensionless coordinates. The solution to this equation with the
Figure 1: Plot of the real valued functions $\xi_q(\nu)$ for $\tilde{\mu} = 1$ and $q = 1, 2, 3$.

given boundary conditions is

$$
\begin{align*}
\xi_1(\nu) &= \frac{\tilde{\mu}^2}{5\pi^3} \left( -20\operatorname{gd}^3(\nu) \log \left( 1 + e^{2\operatorname{gd}(\nu)} \right) - 5i\pi^2 \operatorname{gd}^2(\nu) \\
&\qquad - 5i \left( \pi^2 - 12\operatorname{gd}^2(\nu) \right) \operatorname{Li}_2 \left( e^{4i\tan^{-1}(e^{\nu})} \right) + 5\pi^2 \operatorname{gd}(\nu) \left( \log \left( \frac{\cosh(\nu)}{2} \right) + 2 \log \left( 1 + e^{2\operatorname{gd}(\nu)} \right) \right) \\
&\qquad - 90\operatorname{gd}(\nu) \operatorname{Li}_3 \left( e^{4i\tan^{-1}(e^{\nu})} \right) + 90\zeta(3)\operatorname{gd}(\nu) - 60i\operatorname{Li}_4 \left( e^{4i\tan^{-1}(e^{\nu})} \right) - i\pi^4 \right),
\end{align*}
$$

(3.4)

where $\operatorname{Li}_n(x)$ is the polylogarithm function of order $n$, and $\operatorname{gd}$ is the Gudermannian function.

When $q = 2$ the ODE is given in (C.1). The solution to this equation with the appropriate boundary conditions is

$$
\xi_2(\nu) = \frac{\tilde{\mu}^2}{4} \left( 2\nu - \tanh(\nu) \log \left( 4\cosh^2(\nu) \right) \right) = \frac{\tilde{\mu}^2}{2} \left( \nu - \tanh(\nu) \log(2\cosh(\nu)) \right).
$$

(3.5)

Though not immediately obvious, $\xi_1(\nu)$ is real-valued, see Appendix C. We also obtained an explicit expression for the case $q = 3$. This expression is lengthy and does not appear to yield novel information. It is included for completeness in the aforementioned appendix. These solutions are plotted in figure 1.

### 3.3 Solving the equation for $\eta(\sigma, \nu)$

We focus on solving the PDE (3.2) for $\eta$. We use the separation of variables method and try a separable solution $\eta(\sigma, \nu) = u(\sigma)w(\nu)$. Substituting into the equation, we obtain

$$
\begin{align*}
\Box_{\text{AdS}_q} \frac{u}{u} - \frac{1}{w \cosh^{q-2}(\nu)} \frac{d}{d\nu} \left( \cosh^q(\nu) \frac{dw}{d\nu} \right) &= 0.
\end{align*}
$$

(3.6)
The first term involves only $\sigma$-derivatives and the second only involves $\nu$-derivatives. Since these are independent, the only way they can cancel exactly is if they are equal to some constant $\lambda$. The first term gives:

$$\Box_{\text{AdS}_q} u = -\lambda u. \quad (3.7)$$

The second term gives:

$$-\frac{1}{w} \frac{1}{\cosh^{q-2}(\nu)} \frac{d}{d\nu} \left( \cosh^q(\nu) \frac{dw}{d\nu} \right) = \lambda, \quad (3.8)$$

which can be rewritten as

$$\frac{d^2 w}{d\nu^2} + q \tanh(\nu) \frac{dw}{d\nu} = -\lambda \sech^2(\nu) w \quad (3.9)$$

This is a one-dimensional problem. There is a gauge transformation we can do to eliminate the first derivative term and turn this equation into a Pöschl-Teller model in Schrödinger form.

### 3.4 The Pöschl-Teller model and the equation for $w(\nu)$

Consider the first order differential operator

$$D = \frac{d}{d\nu} + \frac{q}{2} \tanh(\nu). \quad (3.10)$$

Squaring this operator leads to

$$D^2 = \frac{d^2}{d\nu^2} + q \tanh(\nu) \frac{d}{d\nu} + \left( \frac{q}{2} \sech^2(\nu) + \frac{q^2}{4} \tanh^2(\nu) \right). \quad (3.11)$$

We can see that the first two terms are exactly the operator we have in (3.9), and with a little bit of algebra and hyperbolic function identities we can rewrite (3.9) as

$$-D^2 w - \left( \lambda + \frac{q^2}{4} - \frac{q}{2} \right) \sech^2(\nu) w = -\frac{q^2}{4} w. \quad (3.12)$$

The covariant derivative like operator $D$ can be turned into an ordinary derivative via a gauge transformation. We note that

$$D = \cosh^{-q/2}(\nu) \frac{d}{d\nu} \cosh^{q/2}(\nu), \quad (3.13)$$

and

$$D^2 w = \cosh^{-q/2}(\nu) \frac{d^2}{d\nu^2} \left( \cosh^{q/2}(\nu) w \right). \quad (3.14)$$
If we define the auxiliary function $\tilde{w}(\nu) = \cosh^{q/2}(\nu)w(\nu)$ then we have

$$D^2w = \operatorname{sech}^{q/2}(\nu) \frac{d^2\tilde{w}}{d\nu^2}. \quad (3.15)$$

Substituting (3.15) into (3.12) gives

$$- \frac{d^2\tilde{w}}{d\nu^2} - \left( \lambda + \frac{q^2 - q}{2} \right) \operatorname{sech}^2(\nu)\tilde{w} = -\frac{q^2}{4} \tilde{w}. \quad (3.16)$$

This is a Schrödinger-type differential equation with a Pöschl-Teller-style potential

$$V(\nu) = -\left( \lambda + \frac{q(q-2)}{4} \right) \operatorname{sech}^2(\nu), \quad (3.17)$$

with eigenvalue of $-q^2/4$. The strategy is to determine all values of the parameter $\lambda + \frac{q(q-2)}{4}$ that lead to an eigenvalue $-q^2/4$. We work out the solution to this problem in Appendix A.1.

For brevity, the normalized solution found there is:

$$w_m(\nu) = \frac{(-1)^m}{\mathcal{N}_{q/2}} \sqrt{(m+1)! (q+2)^m} \cosh^{m+1}(\nu) \left( \operatorname{sech}(\nu) \frac{d}{d\nu} \right)^m \operatorname{sech}^{q+1}(\nu), \quad (3.18)$$

$$\mathcal{N}_{q/2}^2 = \int_{-\infty}^{\infty} d\nu \operatorname{sech}^{q}(\nu),$$

where $m = 0, 1, 2, 3, 4, \ldots$, and $x^m = x(x+1)(x+2)\cdots(x+m-1)$ is the rising factorial of $x$. The values of $\lambda$ that lead to an eigenvalue of $-q^2/4$ for (3.16) are the infinite sequence

$$\lambda_m = m(m + q - 1) \text{ with } m = 0, 1, 2, 3, \ldots. \quad (3.19)$$

We note that a consequence of $q \geq 2$ and $m \geq 0$ is that $\lambda_m \geq 0$.

### 3.5 The Eigenvalue Problem for $u(\sigma)$

Next we study equation (3.7) incorporating the correct expression for the eigenvalues (3.19). The equation for $u(\sigma)$ to be solved is

$$\left( \square_{\text{AdS}_q} + \lambda_m \right) u = 0, \quad \lambda_m = m(m + q - 1). \quad (3.20)$$

This is the massive Klein-Gordon equation on the submanifold $\text{AdS}_q$ with mass $\sqrt{\lambda_m}$ that depends on the dimension $q$ and the index $m$. We can work out the explicit form of $\square_{\text{AdS}_q}$. First, write the metric for $\text{AdS}_q$ as

$$ds_{\text{AdS}_q}^2 = -\cosh^2(r) \, dt^2 + dr^2 + \sinh^2(r) \, ds_{S^{q-2}}^2. \quad (3.21)$$
In these coordinates, we have split $\sigma$ into the coordinates $t$, $r$ and $(q - 2)$ angles $\theta$ on $S^{q-2}$.

The metric determinant is therefore

$$\sqrt{-\det \gamma} = \cosh(r) \sinh^{q-2}(r),$$

and (3.20) becomes

$$
\left[-\frac{1}{\cosh^2(r)} \frac{\partial^2}{\partial t^2} + (\tanh(r) + (q - 2) \coth(r)) \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{\sinh^2(r)} \nabla^2_{S^{q-2}} - \lambda_m \right] u = 0.\tag{3.22}
$$

The metric is diagonal, so we try separation of variables with ansatz

$$u(\sigma) = R_{mJ\omega}(r)Y^J_M(\theta)e^{-i\omega t},$$

where $R_{mJ\omega}(r)$ is the radial wave function, and $Y^J_M$ are orthonormal real spherical harmonics on $S^{q-2}$. Here $J = 0, 1, 2, 3, \cdots$ is the angular momentum that labels the irreducible representation of $\text{SO}(q-1)$. The index $M$ takes

$$(2J + q - 3) \frac{(J + q - 4)!}{J!(q-3)!}.$$ different values. Note that for $q = 3$, the formula above gives 2, which is the dimensionality of the real irreducible representations of $\text{SO}(2)$.

For now, we will suppress the indices. If we plug this back in and divide the whole equation by $RYe^{-i\omega t}$ per the usual procedure in separation of variables, we wind up with the equation:

$$- \frac{d^2 R}{dr^2} - \left[ \tanh(r) + (q - 2) \coth(r) \right] \frac{dR}{dr} - \frac{\omega^2 R}{\sinh^2(r)} + \frac{\ell R}{\cosh^2(r)} + \lambda R = 0,$$

where $\ell = J(J + q - 3)$ is an eigenvalue of the Laplacian on $S^{q-2}$. Note that the ODE is parametrized by $q, J, m, \omega$. We ignore $q$ because it is fixed, and this is why we had previously written $R_{mJ\omega}$.

We have managed to reduce our problem to an ODE in $r$. Since this is a 1D problem, let’s try a gauge transformation (as we did prior) to see if we can eliminate the first derivative terms:

$$D \equiv \frac{d}{dr} + \frac{1}{2} \left[ \tanh(r) + (q - 2) \coth(r) \right],$$

$$D^2 = \frac{d^2}{dr^2} + \left( \tanh(r) + (q - 2) \coth(r) \right) \frac{d}{dr} + \left( \frac{1}{2} \text{sech}^2(r) - \frac{q-2}{2} \text{csch}^2(r) + \frac{1}{4} \tanh^2(r) \right)$$

$$+ \left( \frac{q - 2}{4} \coth^2(r) + \frac{q - 2}{2} \right).$$

The term inside the large parentheses is

$$\frac{1}{2} \text{sech}^2(r) - \frac{q-2}{2} \text{csch}^2(r) + \frac{1}{4} \tanh^2(r) + \frac{(q - 2)^2}{4} \coth^2(r) + \frac{q - 2}{2}$$

$$= \frac{1}{4} \text{sech}^2(r) + \frac{1}{4} [(q - 2)^2 - 2(q - 2)] \text{csch}^2(r) + \frac{1}{4} (q - 1)^2$$

(3.28)
Putting all this together, we have

\[
\frac{d^2}{dr^2} + (\tanh(r) + (q - 2) \coth(r)) \frac{d}{dr} = D^2 - \left[ \frac{1}{4} \operatorname{sech}^2(r) + \frac{(q - 2)(q - 4)}{4} \operatorname{csch}^2(r) + \frac{(q - 1)^2}{4} \right] \quad (3.29)
\]

We can use this to rewrite (3.25) as

\[
-D^2 R + \left( \frac{1}{4} - \omega^2 \right) \operatorname{sech}^2(r) R
+ \left( \frac{1}{4}(q - 2)(q - 4) + \ell \right) \operatorname{csch}^2(r) R + \left( \frac{1}{4}(q - 1)^2 + \lambda \right) R = 0 \quad (3.30)
\]

To eliminate the covariant derivative, let \( \Upsilon = e^{\frac{i}{2} \ln(\cosh(r))} e^{\frac{q - 2}{2} \ln(\sinh(r))} \), then \( D = \Upsilon^{-1} \frac{d}{dr} \Upsilon \). If we also set \( \tilde{R}(r) = \Upsilon R(r) \) then we can rewrite (3.30) as

\[
-\frac{d^2 \tilde{R}}{dr^2} - \left( \omega^2 - \frac{1}{4} \right) \operatorname{sech}^2(r) \tilde{R}
+ \left( \ell + \frac{1}{4}(q - 2)(q - 4) \right) \operatorname{csch}^2(r) \tilde{R} + \left( \lambda + \frac{1}{4}(q - 1)^2 \right) \tilde{R} = 0 \quad (3.31)
\]

It should be mentioned that the Hilbert space that \( \tilde{R} \) belongs to is plain old square integrable functions on \((0, \infty)\). The reason is that \( \Upsilon(r)^2 \) is just the volume element factor (3.22) that enters the inner product required for the function \( R \). We next substitute the basic parameters into the above and algebraically simplify to obtain

\[
-\frac{d^2 \tilde{R}}{dr^2} - \left( \omega^2 - \frac{1}{4} \right) \operatorname{sech}^2(r) \tilde{R}
+ \left( \left( J + \frac{q - 3}{2} \right)^2 - \frac{1}{4} \right) \operatorname{csch}^2(r) \tilde{R} = -\left( m + \frac{q - 1}{2} \right)^2 \tilde{R} \quad (3.32)
\]

This is a Schrödinger-type equation with a generalized Pöschl-Teller type potential. We examine how to analyze the spectrum for this type of problem for \( q \geq 3 \) in Appendix B. The question we address may be phrased as follows.

**Remark 1.** We assume that the immutable parameter \( q \) is *a priori* fixed by the dimensionality of space-time. There are two parameters at our disposal in (3.32): the parameter \( m \) is fixed by the solution of a previous eigenvalue problem (3.19), and the parameter \( J \) is fixed by the rotational properties of the solution. How do you find \( \omega \) and \( \tilde{R} \) to make equation (3.32) valid?

You can recast the question in more familiar language by observing that if you multiply both sides of eq. (3.32) by \( \cosh^2 r \), then the above is an eigenvalue problem for a complicated second order ODE with regular singular points depending on parameters \( m \) and \( J \) with eigenvalue \( \omega^2 - 1/4 \) and eigenfunction \( \tilde{R} \) to be determined.
The case $q = 2$ is special and does not require new tools because $J = 0$ automatically, consequently the “centrifugal barrier potential term” $1/\sinh^2 r$ is absent. The Hamiltonian for the radial wave function $\tilde{R}$ is just a Pöschl-Teller Hamiltonian of the type discussed in Appendix A.1. You can analyze this problem in two equivalent ways: (1) Take the range of $\nu$ to be $(-\infty, +\infty)$. The Hamiltonian is parity invariant so the energy eigenfunctions may be taken to also be eigenfunctions of parity. (2) Mimic the discussion for $q \geq 3$, by taking the range of $\nu$ to be $[0, \infty)$ and thinking of the symmetry group of $S^{q-2} = \{+1, -1\}$ as being the group with two elements $\{+1, -1\}$. The role of the spherical harmonics can be replaced studying two classes of functions. Those that vanish at $r = 0$ and functions whose derivative vanishes at $r = 0$. In viewpoint (1), there are respectively the even functions and the odd functions.

4 Conclusion

We have found the first order correction to the double BPS solution $\phi(\nu)$ of a generalized $(q, 1)$ kink-like topological defect under a small perturbation in the mass of the scalar field. Putting everything together, the solution to the equation of motion (2.5) is a linear superposition

$$
\Phi(t, r, \theta, \nu) = \phi(\nu) + \epsilon \left( \xi(\nu) + \sum_{m=0}^{\infty} \sum_{J, \omega} C_{m,J,\omega} w_m(\nu) R_m(J,\omega)(r) Y_J^M(\theta) e^{\pm i\omega t} \right) + O(\epsilon^2),
$$

where $\sigma = (t, r, \theta)$ are the coordinates of $\text{AdS}_q$, and where the functions $\xi(\nu), R_m(J,\omega)$ and $w_m(\nu)$ in the expression above take specific forms once $q$ is specified. The summation set $A_m$ is described in Figure 6. The takeaway here is that the boundary conditions, which serve as a check on topological constraints, ensure that the function $\xi(\nu)$ die off as $\nu \to \infty$, the stability of these solutions depends on what values the parameter $\omega$ can have. By determining that this frequency is bounded from below and is always a positive integer, we have shown that all solutions are stable to first order. Furthermore, once these functions are determined, the correction to the energy can be calculated. The correction to the energy was explicitly computed in the case of the $(1, 1)$ defect in our previous article [10].

We mention an interesting topic that we have not explored. The case with $m = 0$ corresponds to $\lambda_0 = 0$ in the Klein-Gordon eq. (3.20). Looking at the full solution (4.1), we see that this corresponds to a linearized perturbation of the form $w_0(\nu) R_0(J,\omega) Y_J^M(\theta) e^{\pm i\omega t}$. This excitation is bound to $\text{AdS}_q$ because the factor $w_0(\nu)$ decays exponentially in the direction normal to $\text{AdS}_q$ with the length scale set by the radius of curvature. This excitation has a factor $R_0(J,\omega) Y_J^M(\theta) e^{\pm i\omega t}$ which corresponds to a massless particle on $\text{AdS}_q$. From the viewpoint of $\text{AdS}_q$ this looks like a massless excitation, but from the bulk view of $\text{AdS}_{q+1}$ these are massive excitations.
A Solving the Pöschl-Teller equation

A.1 1D problem

Begin by defining ladder-like operators for this problem:

\[ a_\beta = \frac{d}{dx} + \beta \tanh(x), \quad a_\beta^\dagger = -\frac{d}{dx} + \beta \tanh(x) \]  \hspace{1cm} (A.1)

We then compute:

\[ a_\beta^\dagger a_\beta = -\frac{d^2}{dx^2} - \beta(\beta + 1) \text{sech}^2(x) + \beta^2 \]  \hspace{1cm} (A.2)

The Pöschl-Teller hamiltonian operator is

\[ h_\beta = -\frac{d^2}{dx^2} - \beta(\beta + 1) \text{sech}^2(x). \]  

In our problem we need eigenfunctions with a negative eigenvalue, thus we require \(-\beta(\beta + 1) < 0\). The hamiltonian is invariant under \(\beta \to -\beta + 1\), so we restrict to the parameter range to \(\beta > 0\).

We are interested in the eigenvalue problem

\[ h_\beta |\beta, \epsilon\rangle = \epsilon |\beta, \epsilon\rangle. \]

We note that \(a_\beta a_\beta = h_\beta + \beta^2\)  \hspace{1cm} (A.3)

Two immediate consequences of these equations is that

\[ a_{\beta+1} a_{\beta+1}^\dagger = h_\beta + (\beta + 1)^2 \]  \hspace{1cm} (A.4)

Two immediate consequences of these equations are that \(\langle h_\beta \rangle \geq -\beta^2\) and \(\langle h_\beta \rangle \geq -(\beta + 1)^2\). Remember that we are in the parameter range \(\beta > 0\). If \(\epsilon\) is an eigenvalue of \(h_\beta\), then eigenvalues are forbidden in the region \(\epsilon < -\beta^2\) and in the region \(\epsilon < -(\beta + 1)^2\). Since the latter region is a subset of the first, we conclude that if \(\epsilon\) is an eigenvalue of \(h_\beta\), then we have the rigorous Rayleigh eigenvalue lower bound \(\epsilon \geq -\beta^2\).

We can verify two identities

\[ h_{\beta-1}(a_\beta |\beta, \epsilon\rangle) = \epsilon(a_\beta |\beta, \epsilon\rangle) \]  \hspace{1cm} (A.5)

\[ h_{\beta+1}(a_{\beta+1}^\dagger |\beta, \epsilon\rangle) = \epsilon(a_{\beta+1}^\dagger |\beta, \epsilon\rangle) \]  \hspace{1cm} (A.6)

We see that \(a_\beta |\beta, \epsilon\rangle\) is an eigenstate of \(h_{\beta-1}\) with the same energy eigenvalue \(\epsilon\). Also \(a_{\beta+1}^\dagger |\beta, \epsilon\rangle\) is an eigenstate of \(h_{\beta+1}\) with the same eigenvalue \(\epsilon\).

Assume we have found an eigenvector \(|\beta^*, \epsilon_*\rangle\) of the hamiltonian \(h_{\beta^*}\). The state

\[ a_{\beta^*-k} \cdots a_{\beta^*-2} a_{\beta^*-1} a_{\beta^*} |\beta^*, \epsilon_*\rangle \]

is in principle an eigenvector of \(h_{\beta^*-k}\) with the same eigenvalue \(\epsilon_*\). The big issue is that for \(k\) sufficiently large you will cross into the forbidden pink region of figure 2 and the state construction process has to stop. In other words, there exists an eigenvector \(|\beta^*, \epsilon_*\rangle \neq 0\) with the point \((\beta^*, \epsilon_*)\) in the allowed region, but \(a_{\beta^*} |\beta^*, \epsilon_*\rangle = 0\) because the point \((\beta^* - 1, \epsilon_*)\) is in the forbidden region. Note that as a consequence of \(A.3\), we have that \(h_{\beta^*} |\beta^*, \epsilon_*\rangle = \epsilon |\beta^*, \epsilon_*\rangle = \epsilon_\ast |\beta^*, \epsilon_*\rangle \neq 0\).
Figure 2: In the coupling-eigenvalue $(\beta, \epsilon)$-plane for the Pöschl-Teller model, the pink region $\epsilon < -\beta^2$ is forbidden by the rigorous Rayleigh eigenvalue lower bound. At the point $(\beta, \epsilon)$, the lowering ladder operator $a_\beta$ moves you horizontally to the left while the raising ladder operator $a^\dagger_{\beta+1}$ moves you horizontally to the right according to equations (A.5) and (A.6). It is a consequence of the analysis discussed in the text that $\beta_*$ is on the boundary curve.

$-\beta_*^2 |\beta_*, \epsilon_*\rangle$ and we conclude that $\epsilon_* = -\beta_*^2$. In other words, the point $(\beta_*, \epsilon_*)$ must be on the boundary of the Rayleigh bound given by the red curve $\epsilon = -\beta^2$. To find all values of $\beta$ with the same energy $\epsilon_*$, the procedure is to start at the Rayleigh bound by choosing $\beta_* = \sqrt{-\epsilon_*}$ and operate on the state $|\beta_*, \epsilon_*\rangle$ with a product of appropriately indexed $a^\dagger_\beta$ that move you horizontally to the right, and obtain the infinite sequence $\beta_*, \beta_* + 1, \beta_* + 2, \ldots$.

The problem of finding all the eigenvectors and eigenvalues of $h_{\beta_*}$ is a different problem, and the solution is indicated without explanation by asking at how you would construct the green circles in the figure. For the illustrated value of $\beta_* = 2$, there are only two eigenvectors of $h_2$ with eigenvalues $-1$ and $-4$.

We now determine an explicit formula for the state $|\beta_*, \epsilon_*\rangle$. Let $\varphi_{\beta_*}(x) = \langle x |\beta_*, \epsilon_*\rangle$ then $a^\dagger_{\beta_*} \varphi_{\beta_*}(x) = 0$ implies

$$\left(\frac{d}{dx} + \beta_* \tanh(x)\right) \varphi_{\beta_*} = 0$$

The normalized solution is easily found to be

$$\varphi_{\beta_*}(x) = N^{-1}_{\beta_*} \text{sech}^{\beta_*}(x), \text{ where } N_{\beta_*}^2 = \int_{-\infty}^{\infty} \text{d}x \text{sech}^{2\beta_*}(x) \quad \text{(A.7)}$$

Now we will construct all the other choices of $\beta$ that have the same $\epsilon = \epsilon_* = -\beta_*^2$. Consider a normalized state $|\beta, \epsilon_*\rangle$. We can apply $a^\dagger_{\beta+1}$ to this state and get the relation $a^\dagger_{\beta+1} |\beta, \epsilon_*\rangle = $
\( \mathcal{N}_{\beta+1} |\beta + 1, \epsilon_\star \rangle \), where the normalization constant is determined via

\[
|\mathcal{N}_{\beta+1}|^2 \langle \beta + 1, \epsilon_\star | \beta + 1, \epsilon_\star \rangle = \langle \beta + 1, \epsilon_\star | a_{\beta+1} \dagger a_{\beta+1} | \beta + 1, \epsilon_\star \rangle
\]

\[
|\mathcal{N}_{\beta+1}|^2 = \langle \beta + 1, \epsilon_\star | (\hbar_\beta + (\beta + 1)^2) | \beta + 1, \epsilon_\star \rangle
\]

\[
= \langle \beta + 1, \epsilon_\star | h_\beta | \beta + 1, \epsilon_\star \rangle + (\beta + 1)^2
\]

\[
= \epsilon_\star + (\beta + 1)^2
\]

\[
= -\beta_\star^2 + (\beta + 1)^2
\]

\[
|\mathcal{N}_{\beta+1}|^2 = (\beta + 1 - \beta_\star)(\beta + 1 + \beta_\star)
\]

(A.8)

We can generalize equation (A.8) to any \( \beta = \beta_\star + m \), with \( m = 0, 1, 2, 3, \ldots \):

\[
|\mathcal{N}_{\beta_\star + m}|^2 = (m + 1)(2\beta_\star + m + 1)
\]

(A.9)

With this factor, we can now write down an “excited” state as

\[
|\beta_\star + m, \epsilon\rangle = \frac{1}{\mathcal{N}_{\beta_\star + m}} a_{\beta_\star + m} \dagger | \beta_\star + m - 1, \epsilon \rangle
\]

\[
= \frac{1}{\mathcal{N}_{\beta_\star + m} \mathcal{N}_{\beta_\star + m - 1}} a_{\beta_\star + m} \dagger a_{\beta_\star + m - 1} \dagger | \beta_\star + m - 2, \epsilon \rangle
\]

\[
|\beta_\star + m, \epsilon\rangle = \frac{1}{\mathcal{N}_{\beta_\star + m} \mathcal{N}_{\beta_\star + m - 1} \cdots \mathcal{N}_{\beta_\star + 1}} a_{\beta_\star + m} \dagger a_{\beta_\star + m - 1} \dagger \cdots a_{\beta_\star + 1} \dagger | \beta_\star, \epsilon \rangle
\]

(A.10)

We can obtain explicit expressions for the eigenfunctions. We need to work out \( \langle x | \beta_\star + m, \epsilon \rangle \).

We note that

\[
a_{\beta} \dagger = -\cosh^\beta(x) \frac{d}{dx} \cosh^{-\beta}(x)
\]

(A.11)

We need to figure out what the product of all the operators amounts to. For example,

\[
a_{\beta_\star + 3} \dagger a_{\beta_\star + 2} \dagger a_{\beta_\star + 1} \dagger = (-1)^3 \cosh^{\beta_\star + 3}(x) \frac{d}{dx} \cosh^{-\beta_\star - 3}(x) \cosh^{\beta_\star + 2}(x)
\]

\[
\times \frac{d}{dx} \cosh^{-\beta_\star - 2}(x) \cosh^{\beta_\star + 1}(x) \frac{d}{dx} \cosh^{-\beta_\star - 1}(x)
\]

\[
= (-1)^3 \cosh^{\beta_\star + 3}(x) \frac{d}{dx} \text{sech}(x) \frac{d}{dx} \text{sech}(x) \frac{d}{dx} \text{sech}^{\beta_\star + 1}(x)
\]

Generalizing the observation above, we can write the function:

\[
\langle x | \beta_\star + m, \epsilon \rangle = \left( \prod_{l=0}^{m} \frac{1}{\mathcal{N}_{\beta_\star + l}} \right) (-1)^m \cosh^{\beta_\star + m + 1}(x) \left( \text{sech}(x) \frac{d}{dx} \right)^m \text{sech}^{\beta_\star + 1}(x) \mathcal{N}_{\beta_\star}^{-1} \text{sech}^{\beta_\star}(x)
\]

\[
\varphi_{\beta_\star + m}(x) = \left( \prod_{l=0}^{m} \frac{1}{\mathcal{N}_{\beta_\star + l}} \right) (-1)^m \cosh^{\beta_\star + m + 1}(x) \left( \text{sech}(x) \frac{d}{dx} \right)^m \text{sech}^{2\beta_\star + 1}(x)
\]

(A.12)

We have to make a note that \( \mathcal{N}_{\beta_\star} \) is computed differently than the other normalization constants. Next we observe that in the product

\[
\mathcal{N}_{\beta_\star + 1}^2 \mathcal{N}_{\beta_\star + 2}^2 \cdots \mathcal{N}_{\beta_\star + m}^2 = [2(2\beta_\star + 2)][3(2\beta_\star + 3)] \cdots [(m + 1)(2\beta_\star + m + 1)]
\]

(A.13)
there are factors that combine

\[(m + 1)! = (m + 1)(m)(m - 1) \cdots 2 \]
\[(2\beta_s + 2)^m = (2\beta_s + m + 1)(2\beta_s + m) \cdots (2\beta_s + 2)\]

Thus,

\[N_{\beta_s+1}^2 N_{\beta_s+2}^2 \cdots N_{\beta_s+m}^2 = (m + 1)!(2\beta_s + 2)^m \quad (A.14)\]

So if we put this back in, our normalized eigenfunctions are

\[\varphi_{\beta_s+m}(x) = \frac{(-1)^m}{N_{\beta_s}} \sqrt{(m + 1)! (2\beta_s + 2)^m} \cosh^{\beta_s+m+1}(x) \left( \frac{\mathrm{sech}(x)}{\frac{d}{dx}} \right)^m \frac{\mathrm{sech}^{2\beta_s+1}(x)}{m} \quad (A.15)\]

In summary, for a fixed eigenvalue \(\epsilon_* = -\beta_s^2\), we have found all possible values of \(\beta\) where the associated Pöschl-Teller model has an eigenfunction with the same eigenvalue

\[\left(-\frac{d^2}{dx^2} - \beta(\beta - 1) \mathrm{sech}^2(x)\right)\varphi_{\beta}(x) = -\epsilon_* \varphi_{\beta}(x). \quad (A.16)\]

Our solution was that \(\beta = \beta_s + m\), where \(m = 0, 1, 2, 3, \ldots\). Thus, the equation takes the form

\[\left(-\frac{d^2}{dx^2} - (\beta_s + m)(\beta_s + m - 1) \mathrm{sech}^2(x)\right)\varphi_{\beta_s+m}(x) = -\beta_s^2 \varphi_{\beta_s+m}(x) \quad (A.17)\]

Remember that \(\varphi_{\beta_s+m}(x) = \langle x | \beta_s + m, \epsilon_* \rangle\).

### A.2 Making contact with equation (3.16)

We can make contact with our original equation (3.16) by noting that

\[(\beta_s + m)(\beta_s + m - 1) = \lambda_m + \frac{q^2}{4} - \frac{q}{2},\]

therefore,

\[\left(\beta_s + m - \frac{q}{2}\right)\left(\beta_s + m + \frac{q}{2} - 1\right) = \lambda_m \quad (A.18)\]

In our problem, the energy eigenvalue \(\epsilon_* = -\beta_s^2 = -\frac{q^2}{4}\) is fixed by the dimension of AdS\(_q\). This reduces the expression above to

\[\lambda_m = m(m + q - 1), \quad m = 0, 1, 2, 3, \ldots \quad (A.19)\]
Since $\beta_*$ always has the fixed value of $q/2$, we index the solutions differently to simplify the notation. The solution to (3.16) previously written as $\varphi_{\beta_*,m}(x)$ will be simplified to

$$\tilde{w}_m(\nu) = \frac{(-1)^m}{\mathcal{N}_{q/2}} \cosh^{2m+1}(\nu) \left( \text{sech}(\nu) \frac{d}{d\nu} \right)^m \text{sech}^{q+1}(\nu),$$  \hspace{1cm} (A.20)

$$|\mathcal{N}_{q/2}|^2 = \int_{-\infty}^{\infty} d\nu \text{sech}^{q}(\nu)$$

Note that as $|\nu| \to \infty$, we have that $\tilde{w}_m(\nu) \sim e^{-(q/2)|\nu|}$ as expected for a state with $\epsilon_* = -(q/2)^2$. In terms of the original function $w$ we have

$$w_m(\nu) = \frac{(-1)^m}{\mathcal{N}_{q/2}} \cosh^{m+1}(\nu) \left( \text{sech}(\nu) \frac{d}{d\nu} \right)^m \text{sech}^{q+1}(\nu),$$  \hspace{1cm} (A.21)

$$|\mathcal{N}_{q/2}|^2 = \int_{-\infty}^{\infty} d\nu \text{sech}^{q}(\nu)$$

Note that the parity of $\tilde{w}_m$ or $w_m$ is $(-1)^m$ as expected.

### B The two parameter Pöschl-Teller Model

The eigenvalue problem for the two parameter Pöschl-Teller model is to find solutions of the ODE

$$-\frac{d^2\psi}{dr^2} - \beta(\beta + 1) \cosh^2(r) \psi + \frac{\gamma(\gamma + 1)}{\sinh^2(r)} \psi = \epsilon \psi$$  \hspace{1cm} (B.1)

In our problem of interest we have $q \geq 3$ and therefore the eigenvalue in the right hand side of equation (3.32) satisfies $\epsilon \leq -1$. Consequently, we require the function $\psi(r) \sim e^{-\sqrt{-\epsilon}r} \sim e^{-(m+(q-1)/2)r}$ as $r \to \infty$. The Hilbert space consists of the square integrable function in the domain $(0, \infty)$. Define the two parameter Pöschl-Teller hamiltonian by

$$h_{\beta,\gamma} = -\frac{d^2}{dr^2} - \beta(\beta + 1) \cosh^2(r) + \gamma(\gamma + 1) \text{csch}^2(r)$$  \hspace{1cm} (B.2)

The parameter space for the hamiltonian is the $(\beta, \gamma)$-plane. Note that the hamiltonian is invariant under the two distinct transformation of parameters: (1) $\beta \to -(1 + \beta)$, and (2) $\gamma \to -(1 + \gamma)$. Respectively, these transformation correspond to reflection about the line $\beta = -1/2$, and reflection about the line $\gamma = -1/2$. Because of this symmetry we can restrict the parameter region to semi-infinite rectangular region $\beta \geq -1/2$ and $\gamma \geq -1/2$.

Define the operators

$$a_{\beta,\gamma} = \frac{d}{dr} + \beta \tanh(r) - (\gamma + 1) \coth(r)$$  \hspace{1cm} (B.3)

$$a_{\beta,\gamma}^\dagger = -\frac{d}{dr} + \beta \tanh(r) - (\gamma + 1) \coth(r)$$  \hspace{1cm} (B.4)
Figure 3: Parameter space region for the hamiltonian $h_{\beta,\gamma}$. The qualitative behavior of the effective potential $V_{\text{eff}}(r) = -\beta(\beta+1) \text{sech}^2(r) + \gamma(\gamma+1) \text{csch}^2(r)$ that enters into hamiltonian $h_{\beta,\gamma}$ is plotted. The energy eigenvalue $\epsilon < -1$, therefore the region in pink where $V_{\text{eff}}(r) > 0$ for all $r > 0$ is ruled out because the expectation value of each summand in the hamiltonian $h_{\beta,\gamma}$ is non-negative. We expect that the values of $(\beta, \gamma)$ that admit bound states will be contained in the parameter regions with a green background. We will pin down the region later on.
Just as in the ordinary Pöschl-Teller model we have relationships

\[ a^\dagger_{\beta,\gamma} a_{\beta,\gamma} = h_{\beta,\gamma} + (\beta - \gamma - 1)^2 \]  
\[ a_{\beta+1,\gamma-1} a^\dagger_{\beta+1,\gamma-1} = h_{\beta,\gamma} + (\beta - \gamma + 1)^2 \]

These equations lead to Rayleigh bounds that constrain the eigenvalues:

\[ \langle h_{\beta,\gamma} \rangle \geq - (\beta - \gamma - 1)^2 \]  
\[ \langle h_{\beta,\gamma} \rangle \geq - (\beta - \gamma + 1)^2 \]

### B.1 Relationship to our problem

The original equation we are interested in solving is (3.32). Comparing to (B.1) we see that the parameters are related by:

\[ \epsilon = - \left( m + \frac{q - 1}{2} \right)^2, \]  
\[ \beta (\beta + 1) = \omega^2 - \frac{1}{4}, \]  
\[ \gamma (\gamma + 1) = \left( J + \frac{q - 3}{2} \right)^2 - \frac{1}{4}. \]

It is worthwhile recollecting that the three parameters that appear in (3.32) are all integers, and are constrained by \( m \geq 0, J \geq 0, \) and \( q \geq 3 \). In addition, we chose the parameter range of \( \beta \) and \( \gamma \) to be \( \beta + 1/2 \geq 0 \) and \( \gamma + 1/2 \geq 0 \). The first observation we make is that we are interested in studying generalized Pöschl-Teller equation when the eigenvalue

\[ \epsilon \leq -1 \]

Next we observe that eqs. (B.10) and (B.11) may be rewritten as

\[ \left( \beta + \frac{1}{2} \right)^2 = \omega^2, \]  
\[ \left( \gamma + \frac{1}{2} \right)^2 = \left( J + \frac{q - 3}{2} \right)^2. \]

The advantage of writing the equations in this form is that they automatically incorporate the parameter range of \( \beta \) and \( \gamma \), and the reflection symmetry about \( \beta = -1/2 \) and \( \gamma = -1/2 \). When we take the square root of the two equations above, the parameter range tells us that we have to take the positive branch. Time reversal allows us to restrict to \( \omega \geq 0 \). Note that \( J + (q - 3)/2 \geq 0 \) so we have

\[ \beta + \frac{1}{2} = \omega, \]  
\[ \gamma + \frac{1}{2} = J + \frac{q - 3}{2}. \]
Figure 4: The wedge shaped region $\mathcal{W}$ in $(\beta, \gamma, \epsilon)$ parameter space is between the green shaded plane $\epsilon = -1$, and the yellow parabolic cylinder $\epsilon = -(\beta - \gamma - 1)^2$. It is defined by $\mathcal{W} = \{(\beta, \gamma, \epsilon) : (\beta > 3/2) \land (\gamma > -1/2) \land (\epsilon \leq -1) \land (\epsilon \geq -(\beta - \gamma - 1)^2)\}$. The symbol $\land$ is the "logical and" operator. The straight line that appears in the plot is the intersection of the horizontal plane $z = 1$ with the vertical plane $\beta - \gamma = 2$.

In summary we have that

$$\epsilon = -\left(m + \frac{q - 1}{2}\right)^2,$$  \hfill (B.13)

$$\gamma = J + \frac{q - 4}{2} \geq \frac{q - 4}{2},$$  \hfill (B.14)

$$\beta = \omega - \frac{1}{2} \geq -\frac{1}{2}.$$  \hfill (B.15)

Note that for even integer $q \geq 4$, we have that $\gamma \geq 0$ is an integer. For an odd integer $q \geq 3$ we have that $\gamma \geq -1/2$ and $\gamma$ is a half-integer, i.e., $\gamma \in \mathbb{Z} + \frac{1}{2}$. Figure 3 tells that that the values of $(\beta, \gamma)$ that give bound states will be found in the parameter region that is complimentary to the shaded pink region where $V(r) > 0$ for $r > 0$. A study of equations (B.13), (B.14), (B.15), and bounds (B.7) and (B.8) produces a wedge shaped region $\mathcal{W}$ in $(\beta, \gamma, \epsilon)$ parameter space, see Figure 4. Solutions to the question posed in Remark 1 must belong to $\mathcal{W}$.

B.2 The ladder

If $|\beta, \gamma, \epsilon\rangle$ is an eigenvector of $h_{\beta, \gamma}$ with eigenvalue $\epsilon$ then you can verify the “ladder” relations:

$$h_{\beta-1,\gamma+1}(a_{\beta,\gamma}|\beta, \gamma, \epsilon\rangle) = \epsilon (a_{\beta,\gamma}|\beta, \gamma, \epsilon\rangle)$$  \hfill (B.16)

$$h_{\beta+1,\gamma-1}(a_{\beta,\gamma}^\dagger|\beta, \gamma, \epsilon\rangle) = \epsilon (a_{\beta,\gamma}^\dagger|\beta, \gamma, \epsilon\rangle)$$  \hfill (B.17)

In summary, the state $a_{\beta,\gamma}|\beta, \gamma, \epsilon\rangle$ is an eigenvector of $h_{\beta-1,\gamma+1}$ with the same eigenvalue $\epsilon$, and the state $a_{\beta+1,\gamma-1}^\dagger|\beta, \gamma, \epsilon\rangle$ is an eigenvector of $h_{\beta+1,\gamma-1}$ with the same eigenvalue $\epsilon$. We
Figure 5: Once you choose a value of \( m \), the energy eigenvalue \( \epsilon \) is determined by eq. (B.13). Let \( \epsilon_* \) be one such value. The horizontal reddish region is the intersection of the horizontal plane \( \epsilon = \epsilon_* \) with the wedge \( \mathcal{W} \). The appropriate ladder operators move you diagonally in the \( \epsilon = \epsilon_* \) plane.

also have the rigorous Rayleigh bound on the eigenvalues of \( h_{\beta, \gamma} \):

\[
\epsilon \geq - (\beta - \gamma - 1)^2. \tag{B.18}
\]

First we observe that according to eq. (B.13), choosing \( m \) determines energy eigenvalue \( \epsilon_* \). Our task is to determine all \( (\beta, \gamma) \) that give eigenvectors with eigenvalue \( \epsilon_* \). This means that we are interested in admissible choices of \( \beta \) and \( \gamma \) that belong to the red plane in Figure 5 and we restrict our selves to that plane. We also observe that the different admissible \( \gamma \) are determined by (B.14) with \( J = 0, 1, 2, \ldots \). All we need is to determine the admissible \( \beta \). Assume there is a state \( |\beta_\sharp, \gamma_\sharp, \epsilon_*\rangle \) that is an eigenvector of \( h_{\beta_\sharp, \gamma_\sharp} \) with eigenvalue \( \epsilon_* \leq -1 \). Operating on this state with a product of appropriate \( a_{\beta, \gamma} \) moves us diagonally to the NW. In Figure 6 the direction of the arrows indicate how the operators \( a_{\beta, \gamma} \) change the value of \( (\beta, \gamma) \) according to eq. (B.17). Here we are interested in the action of \( a_{\beta, \gamma} \) which according to (B.16) would correspond to arrows in Figure 6 with the opposite orientation. Under the action of the various \( a_{\beta, \gamma} \) we are moving towards the Rayleigh bound. Eventually we will cross the line \( -(\beta - \gamma - 1)^2 = \epsilon_* \) and violate the bound. The same argument that was made in the Pöschl-Teller model tells us that there is a state \( |\beta_*, \gamma_*, \epsilon_*\rangle \) such that

\[
a_{\beta_*, \gamma_*} |\beta_*, \gamma_*, \epsilon_*\rangle = 0. \tag{B.19}
\]

Moreover, this state saturates the Rayleigh bound \( \epsilon_* = -(\beta_* - \gamma_* - 1)^2 \), i.e., \( (\beta_*, \gamma_*) \) is on the boundary line. The normalized solution to (B.19) is

\[
\langle x |\beta_*, \gamma_*, \epsilon_*\rangle = \frac{1}{\mathcal{N}_{\beta_*, \gamma_*}^2} \frac{\sinh^{\gamma_*+1}(r)}{\cosh^{\beta_*}(r)} \quad \text{where} \quad \mathcal{N}_{\beta_*, \gamma_*}^2 = \int_0^\infty \frac{\sinh^{2\gamma_*+2}(r)}{\cosh^{2\beta_*}(r)} \, dr. \tag{B.20}
\]

\(^2\)This is a straight line because \( \beta - \gamma - 1 = \sqrt{-\epsilon_*} \).
Rayleigh bound:
$$\beta - \gamma - 1 = m + \frac{q - 1}{2} = \sqrt{-\epsilon}$$

$$\gamma = \gamma_{\text{min}} = \frac{q - 4}{2}$$

$$J = 0$$
$$J = 1$$
$$J = 2$$
$$J = 3$$
$$J = 4$$
$$J = 5$$
$$J = \ldots$$

All allowed \((\beta, \gamma)\) that give hamiltonians \(h_{\beta, \gamma}\) with \(\epsilon = -(m + (q - 1)/2)^2\) where \(q \geq 3\) and \(q \neq 4\). The direction of the arrows represent the change in \((\beta, \gamma)\) under the action of the appropriate \(a_{\beta, \gamma}^\dagger\). The grid squares are \(\frac{1}{2} \times \frac{1}{2}\).

Figure 6: The angle and its interior determine the allowed values of \(\omega\) that provide normalizable solutions to eq. (3.32). The required value of \(\omega\) at each point is simply \(\omega = \beta + 1/2\). The \((\beta, \gamma)\) allowed parameter region is \(\{(\beta, \gamma) : \gamma \geq (q - 4)/4 \text{ and } \beta - \gamma - 1 \geq m + (q - 1)/2\}\). The vertex of the angle is at the point \((m + q - 3/2, (q - 4)/2)\). Every circle on the Rayleigh bound gives a solution of type (B.20). The set of lattice points denoted by the black circles will be denoted by \(A_m\).

The asymptotic behavior of the function above as \(r \to \infty\) is \(e^{-(\beta_{\ast} - \gamma_{\ast} - 1)} r = e^{-\sqrt{-\epsilon_{\ast}} r}\) since the Rayleigh bound is saturated. Also note that it behaves as \(r^{\gamma_{\ast} + 1}\) as \(r \downarrow 0\) as expected. Beginning with this state we can operate sequentially with \(a_{\beta, \gamma}^\dagger\) to generated other states with the same energy eigenvalues. We are typically interested in states

$$a_{\beta_{\ast} + l, \gamma_{\ast} - l}^\dagger a_{\beta_{\ast} + l - 1, \gamma_{\ast} - l + 1}^\dagger \cdots a_{\beta_{\ast} + 2, \gamma_{\ast} - 2}^\dagger a_{\beta_{\ast} + 1, \gamma_{\ast} - 1}^\dagger |\beta_{\ast}, \gamma_{\ast}, \epsilon_{\ast}\rangle.$$  

This process is described in Figure 6.

We can now make some observation about the three parameters \(m, J\) and \(\omega\) in the original
equation \([3.32]\), and the allowed saturated bound values \(\epsilon_*, \beta_*\) and \(\gamma_*\).

\[
\sqrt{-\epsilon_*} = m + \frac{q - 1}{2}, \quad (B.21)
\]

\[
\gamma_* = J + \frac{q - 4}{2}, \quad (B.22)
\]

\[
\beta_* = \omega_* - \frac{1}{2} = m + J + q - \frac{3}{2}, \quad (B.23)
\]

where

\[
\omega_* = m + J + (q - 1). \quad (B.24)
\]

Equation \([B.24]\) is a rewriting of the saturated bound. In particular note that for \(q \geq 3\), the frequency \(\omega_* \geq m + J + 2 \geq 2\) and it is an integer. Therefore we have that \(\beta_* \geq 3/2\) and \(\beta_* \in \mathbb{Z} + \frac{1}{2}\). The first remark is that once \(m\) is specified, \(\epsilon_*\) is determined and we have the Rayleigh bound line in Figure 6. The various points on the bound line are associated with the different values of \(J = 0, 1, 2, \ldots\) according to eq. \([B.22]\). Note that action of \(a_{\beta+1, \gamma-1}^{\dagger}\) takes \((\beta, \gamma)\) to \((\beta + 1, \gamma - 1)\) this has the consequences that \(\omega \rightarrow \omega + 1\) and \(J \rightarrow J - 1\). We know that \(J \geq 0\) so the process stops by at \(\gamma_{\text{min}} = (q - 4)/2\) because smaller \(\gamma\) are outside the parameter range specified by the original Hamiltonian \([3.32]\).

C More detail on solving the inhomogeneous equation for the first-order correction

The general solution to \([3.1]\) depends on the type of potential chosen in the action. Here we discuss the procedure to solve this equation in our symmetry-breaking toy model, where \(U(\phi) = \frac{A}{8}(\phi^2 - \phi_0^2)^2\). When \(q = 1\), the ODE for the correction is \([3.3]\):

\[
\xi''(\nu) + \tanh(\nu)\xi'(\nu) + \frac{\bar{\mu}}{\pi^3} \arctan(\sinh(\nu)) (\pi^2 - 4 \arctan(\sinh(\nu))^2) = 0.
\]

We solved the equation by making use of a coordinate transformation. Using \(x = \arctan(\sinh(\nu))\), the equation becomes

\[
\xi''(x) + \frac{\bar{\mu}}{\pi^3} x \sec^2(x) (\pi^2 - 4x^2) = 0.
\]

The boundary conditions for this equation are \(\xi(x) \xrightarrow{|x| \to \pi/2} 0\). Using Mathematica, a solution can be found to this equation. When translated back into the original coordinates, the expression is \([3.4]\).

In the case of \(q = 2\), the substitution is carried out with \(x = \tanh(\nu)\) with the boundary conditions \(\xi_2(x) \xrightarrow{|x| \to 1} 0\). In this form the ODE is:

\[
\bar{\mu} \tanh(\nu) \text{sech}^2(\nu) + 2\xi''(\nu) + 4\tanh(\nu)\xi'(\nu) = 0. \quad (C.1)
\]

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Figure 7: The solution that Mathematica returns for the case $q = 3$. This function is real, but we did not find it to be illuminating.

This is in a form that Mathematica can solve. It returns:

$$\xi_2(x) = \frac{1}{4} \mu^2 \left( x \log \left( x^2 - 1 \right) - x \log(4) - \log(1 - x) + \log(x + 1) + i\pi \right).$$

(C.2)

Transforming back to the original coordinates gives the solution in the text, (3.5).

For completeness, note that this same procedure can be carried out in other dimensions. We were able to find a solution to the case of $q = 3$ by using the substitution $x = \sinh(\nu)$. The solution is lengthy; see figure 7.

We did not find this solution illuminating, but we did verify that it was real, see figure 1.

Note that in some cases, Mathematica and other computer solvers can return a solution with an imaginary part. However, in all cases, this is merely a complex superposition of solutions of the homogeneous equation and so it can be subtracted off without loss of generality.

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