Inference for Multivariate Normal Mixtures

Jiahua Chen

Department of Statistics, University of British Columbia
Vancouver, BC, V6T 1Z2, Canada

Xianming Tan

LMPC and School of Mathematical Sciences, Nankai University
Tianjin, 300071, P.R. China

Abstract

Multivariate normal mixtures provide a flexible model for high-dimensional data. They are widely used in statistical genetics, statistical finance, and other disciplines. Due to the unboundedness of the likelihood function, classical likelihood-based methods, which may have nice practical properties, are inconsistent. In this paper, we recommend a penalized likelihood method for estimating the mixing distribution. We show that the maximum penalized likelihood estimator is strongly consistent when the number of components has a known upper bound. We also explore a convenient EM-algorithm for computing the maximum penalized likelihood estimator. Extensive simulations are conducted to explore the effectiveness and the practical limitations of both the new method and the ratified maximum likelihood estimators. Guidelines are provided based on the simulation results.

Key words: Multivariate normal mixture, Penalized maximum likelihood estimator, Strong consistency.

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1 Introduction

In the past few decades, there has been an exploding volume of literature on mixture models [22] [13] [15] [6]. Various mixture distributions including normal mixtures are used in a wide variety of situations. Schork et al. [19] reviewed...
the applications of mixture models in human genetics and Tadesse et al. [20] used a normal mixture model for clustering analysis. Application examples can be found in [5, 12, 16] and [1].

Finite mixtures of multivariate normals have also drawn substantial attention recently. Lindsay and Basak [14] devised a system of moment equations and a fast algorithm to estimate the parameters of multivariate normal mixture distributions under an equal-covariance-matrix assumption. However the equality assumption is crucial, and failing this condition leads to a substantial loss in the accuracy of the fit [15]. Unequal-variance normal mixture models have an ill effect on the likelihood function [3]. Placing a positive lower bound on the component variances helps, but the resulting statistical procedure can be awkward because it is not continuous in the data. Placing a positive lower bound on the ratio of the component variances is better. In the univariate case the resulting constrained maximum likelihood estimator is consistent for both constant and shrinking lower bounds [3, 21]. Though consistency is yet to be proved, Ingrassia [9] applied the constrained method to multivariate observations. Ray and Lindsay [17] found that in contrast to the univariate case, the multivariate normal mixture density can have more modes than the number of components. Inference on multivariate normal mixture models is hence more difficult.

In this paper, we investigate a penalized likelihood method for estimating the mixing distribution. The penalized likelihood estimations form a population class of methods, see [7, 4]. When the number of components has a known upper bound, the maximum penalized likelihood estimator (PMLE) is found to be strongly consistent. An EM-algorithm is developed and extensive simulations are conducted. Although after some ratification, the usual maximum likelihood estimators and the PMLE work similarly after the removal of degenerating local maxima in the univariate case [2], the PMLE is advantageous for multivariate normal mixture models.

The paper is organized as follows. In Section 2, the penalized likelihood method is introduced. Two theorems on strong consistency are presented with the proofs deferred to the Appendix. The EM-algorithm for solving the maximization problem for the penalized likelihood function is given. Section 3 contains the simulation results.
2 Penalized likelihood method

2.1 Consistency of the PMLE

Let $\varphi(x; \mu, \Sigma)$ be the multivariate normal density with $(d \times 1)$ mean vector $\mu$ and $d \times d$ covariance matrix $\Sigma$, i.e.,

$$
\varphi(x; \mu, \Sigma) = \left\{ 2\pi|\Sigma| \right\}^{-d/2} \exp \left\{ -\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu) \right\}.
$$

A $d$-dimensional random vector $X$ has a multivariate finite normal mixture distribution of order $p$ if its density function is given by

$$
f(x; G) = \pi_1 \varphi(x; \mu_1, \Sigma_1) + \pi_2 \varphi(x; \mu_2, \Sigma_2) + \cdots + \pi_p \varphi(x; \mu_p, \Sigma_p) \quad (1)
$$

where $G$ is the mixing distribution assigning probability $\pi_j$ to parameter set $(\mu_j, \Sigma_j)$ of the $j$th kernel density $\varphi(x; \mu_j, \Sigma_j)$.

Let $x_1, x_2, \ldots, x_n$ be a random sample from $f$. Then

$$
l_n(G) = \sum_{i=1}^{n} \log f(x_i, G)
$$

is the log-likelihood function. Even if $|\Sigma_j| > 0$ for all $j$, $l_n(G)$ is unbounded at $\mu_1 = x_1$ when $|\Sigma_1|$ gets arbitrarily small. The penalized log-likelihood function is of the form

$$
pl_n(G) = l_n(G) + p_n(G)
$$

where $p_n(G)$ is the penalty depending on the mixing distribution $G$ and the sample size $n$. Let $\hat{G}_n$ be the mixing distribution in the parameter space at which $pl_n(G)$ attains its maximum. We call $\hat{G}_n$ the penalized maximum likelihood estimator (PMLE).

We choose a penalty function such that:

C1. $p_n(G) = \sum_{j=1}^{p} \bar{p}_n(\Sigma_j)$,

C2. At any fixed $G$ such that $|\Sigma_j| > 0$ for all $j = 1, 2, \ldots, p$, we have $p_n(G) = o(n)$, and $\sup_G \max \{0, p_n(G)\} = o(n)$.

In addition, $p_n(G)$ is differentiable with respect to $G$ and as $n \to \infty$, $p_n'(G) = o(\sqrt{n})$ at any fixed $G$ such that $|\Sigma_j| > 0$ for all $j = 1, 2, \ldots, p$. Here we treat $G$ as a vector of parameters contained in the mixing distribution $G$.

C3. For large enough $n$, $\bar{p}_n(\Sigma) \leq 4(\log n)^2 \log |\Sigma|$, when $|\Sigma|$ is smaller than $cn^{-2d}$ for some $c > 0$.

These conditions are quite flexible and functions satisfying these conditions can be easily constructed. A class of such functions will be given in the simu-
lation section. Condition C1 simplifies the numerical computation. Condition C2 limits the effect of the penalty. The key condition is C3: it counters the damaging effect of a degenerate component covariance matrix. The order of the penalty size is well calibrated as will be seen in the proof, yet the exact value of the constant 4 is not important. The penalty function can also be viewed as a prior function via Bayesian analysis.

**Theorem 1** Assume that the true density function

\[ f(x; G_0) = \sum_{j=1}^{p_0} \pi_{0j} \phi(x; \mu_{0j}, \Sigma_{0j}) \]

satisfies \( \pi_{0j} > 0, |\Sigma_{0j}| > 0, \) and \( (\mu_{0j}, \Sigma_{0j}) \neq (\mu_{0k}, \Sigma_{0k}) \) for all \( j = 1, 2, \ldots, p_0 \) and \( j \neq k \).

Assume that the penalty function \( p_n(G) \) satisfies C1-C3 and \( \tilde{G}_n \) is a mixing distribution of order \( p_0 \) satisfying

\[ pl_n(\tilde{G}_n) - pl_n(G_0) \geq c > -\infty, \]

for all \( n \). Then, as \( n \to \infty \), \( \tilde{G}_n \to G_0 \), almost surely.

The proof is deferred to the Appendix.

Since \( pl_n(\tilde{G}_n) - pl_n(G_0) \geq 0 \), the PMLE \( \hat{G} \) is strongly consistent. Because \( \tilde{G}_n \) and \( G_0 \) have the same order, all elements in \( \tilde{G}_n \) converge to those of \( G_0 \) almost surely. Furthermore, let

\[ S_n(G) = \sum_{i=1}^{n} \frac{\partial \log f(x_i; G)}{\partial G} \]

be the vector score function at \( G \). Let

\[ S_n'(G) = \sum_{i=1}^{n} \frac{\partial S_n(G)}{\partial G} \]

be the matrix of the second derivative of the log-likelihood function. At \( G = G_0 \), the normal mixture model is regular and hence the Fisher information

\[ I_n(G_0) = nI(G_0) = -E\{S_n'(G_0)\} = E\{S_n(G_0)\}^T S_n(G_0) \]

is positive definite. Using classical asymptotic techniques as in [11], and under condition C2 such that \( p.ng\)'s \( = o_p(n^{1/2}) \), we have

\[ \hat{G}_n - G_0 = \{S_n'(G_0)\}^{-1} S_n(G_0) + o_p(n^{-1/2}). \]

Therefore, \( \hat{G}_n \) is an asymptotically normal and efficient estimator.
Theorem 2  Under the same conditions as in Theorem 1, as \( n \to \infty \),
\[
\sqrt{n}\{\hat{G}_n - G_0\} \to N(0, I(G_0))
\]
in distribution.

The proof is straightforward and omitted. In practice, we may know only an upper bound for \( p_0 \) rather than its exact value. The following theorem deals with this situation.

Theorem 3  Assume the same conditions as in Theorem 1, except that the order of the finite normal mixture model \( p_0 \) is known only to be smaller than or equal to \( p \). Let \( \tilde{G}_n \) be a mixing distribution of order \( p \) satisfying
\[
pl_n(\tilde{G}_n) - pl_n(G_0) \geq c > -\infty
\]
for all \( n \). Then, as \( n \to \infty \), \( G_n \xrightarrow{w} G_0 \) almost surely.

The proof is deferred to the Appendix.

2.2 The EM-algorithm

We recommend the EM-algorithm due to its simplicity in coding, and its guaranteed convergence to some local maximum under very general conditions \[24, 18, 7\]. In our simulations, we use a number of initial values to reduce the risk of poor local maxima. We also recommend some convenient and effective penalty functions for the EM-algorithm.

Let \( z_{ij} \) be the membership indicator variable such that it equals 1 when \( x_i \) is from the \( j \)th component of the normal mixture model, and equals 0 otherwise. The complete observation log-likelihood under a normal mixture model is then given by
\[
l_c(G) = \sum_{i=1}^{n} \sum_{k=1}^{p} z_{ik} \left\{ \log \pi_k - \frac{1}{2} \log |\Sigma_k| - \frac{1}{2}(x_i - \mu_k)^\top \Sigma_k^{-1}(x_i - \mu_k) \right\}.
\]

Given the current mixing distribution
\[
G^{(m)} = (\pi_1^{(m)}, \ldots, \pi_p^{(m)}, \mu_1^{(m)}, \ldots, \mu_p^{(m)}, \Sigma_1^{(m)}, \ldots, \Sigma_p^{(m)}),
\]
the EM-algorithm iterates as follows:

In the E-Step, we compute
\[
\pi_{ij}^{(m+1)} = E\{z_{ij}|x_1, \ldots, x_n, G^{(m)}\} = \frac{\pi_j^{(m)} \phi(x_i; \mu_j^{(m)}, \Sigma_j^{(m)})}{\sum_{j=1}^{p} \pi_j^{(m)} \phi(x_i; \mu_j^{(m)}, \Sigma_j^{(m)})}.
\]
Replacing $z_{ij}$ by $\pi_{ij}^{(m+1)}$ in $l_c(G)$, we get

$$Q(G; G^{(m)}) = E\{l_c(G) + p_n(G) | x_1, \ldots, x_n, G^{(m)}\}$$

$$= \sum_{j=1}^{p} (\log \pi_j) \sum_{i=1}^{n} \pi_{ij}^{(m+1)} - \frac{1}{2} \sum_{j=1}^{p} (\log |\Sigma_j|) \sum_{i=1}^{n} \pi_{ij}^{(m+1)}$$

$$- \frac{1}{2} \sum_{j=1}^{p} \sum_{i=1}^{n} \pi_{ij}^{(m+1)} (x_i - \mu_j)^\top \Sigma_j^{-1} (x_i - \mu_j) + p_n(G).$$

This completes the E-step.

In the M-step, we maximize $Q(G; G^{(m)})$ with respect to $G$ to obtain $G^{(m+1)}$.

We suggest the following penalty functions in practice:

$$p_n(G) = -a_n \sum_{j=1}^{p} \left\{ \text{tr}(S_x \Sigma_j^{-1}) + \log |\Sigma_j| \right\} \quad (2)$$

with $S_x$ being the sample covariance matrix, and $\text{tr}(\cdot)$ being the trace function. Using this penalty function, $Q(G; G^{(m)})$ is maximized at $G = G^{(m+1)}$ with

$$\begin{align*}
\pi_j^{(m+1)} &= \frac{1}{n} \sum_{i=1}^{n} \pi_{ij}^{(m+1)}, \\
\mu_j^{(m+1)} &= \frac{\sum_{i=1}^{n} \pi_{ij}^{(m+1)} x_i}{n \pi_j^{(m+1)}}, \\
\Sigma_j^{(m+1)} &= \frac{2a_n S_x + S_j^{(m+1)}}{2a_n + n \pi_j^{(m+1)}}
\end{align*}$$

where

$$S_j^{(m+1)} = \sum_{i=1}^{n} \pi_{ij}^{(m+1)} (x_i - \mu_j^{(m+1)})(x_i - \mu_j^{(m+1)})^\top.$$

From a Bayesian point of view, the penalty function (2) puts a Wishart distribution prior on $\Sigma_j$, and $S_x$ is the mode of the prior distribution. Increasing the value of $a_n$ implies a stronger conviction on $S_x$ as the possible value of $\Sigma_j$.

The EM-algorithm iterates between the E-step and the M-step. The penalized likelihood increases after each iteration. At the same time, the penalized likelihood is bounded over the parameter space. Hence, the EM-algorithm converges to a non-degenerate local maximum. This is the dividing line between the penalized likelihood and the ordinary likelihood. In both cases, the EM-algorithm may converge to an undesired local maxima starting from a poor initial value. In the simulations, we use ten initial values including the true value for each data set to control this potential problem.
3 Simulation study.

When computing the MLE the local maxima located by the EM-algorithm with degenerate covariance matrices are first removed. The one that attains the largest likelihood value among those remaining is then identified as the MLE or the ratified MLE of the mixing distribution. Although this approach lacks solid theoretical support, it works well for univariate normal mixture models \cite{2}. The consistency result for the PMLE for multivariate normal mixture models does not guarantee its superiority in practice. Thus, we feel obliged to compare the performance of the PMLE with that of the ratified MLE. In addition, there is a general shortage of thorough simulation studies in the context of multivariate normal mixture models. This paper partially fills that knowledge gap.

We use bias and standard deviation to measure the accuracy of the ratified MLE and the PMLE. We also record the number of times that the EM-algorithm degenerates when the ratified MLE is attempted. For clarity, the simulation results are organized into two subsections.

3.1 Simulation models and settings

The size of the parameter space for the finite multivariate normal mixture model explodes with the dimension. It is difficult to use a few typical specific distributions to cover all aspects of this model. We struggled to come up with a few particularly important cases. We considered four categories of mixture models: two-component bivariate normal mixture models \((p = 2, \, d = 2)\); three-component bivariate normal mixture models \((p = 3, \, d = 2)\); two-component trivariate normal mixture models \((p = 2, \, d = 3)\); and three-component trivariate normal mixture models \((p = 3, \, d = 3)\).

In each category, we chose \(3 \times 6\) models formed by component mean vector and covariance matrix configurations. These combinations mimic practical situations and make the comparison of the performance of the ratified MLE and the PMLE meaningful.

The covariance matrices in the simulation models are designed to have the following general form when \(d = 2\):

\[
\Sigma = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta 
\end{bmatrix} \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2 
\end{bmatrix} \begin{bmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta 
\end{bmatrix}.
\]

By the choices of the eigenvalues \(\lambda_1, \lambda_2\), and the orientation angle \(\theta\), we obtain
various configurations of bivariate normal mixture models.

The covariance matrices in the simulation models are designed to have the following general form when \( d = 3 \):

\[
\Sigma = P(\alpha, \beta, \gamma) \text{diag}[\lambda_1, \lambda_2, \lambda_3] P^T(\alpha, \beta, \gamma)
\]

with

\[
P(\alpha, \beta, \gamma) = \\
\begin{bmatrix}
\cos \alpha \cos \gamma - \cos \beta \sin \alpha \sin \gamma & -\cos \beta \cos \gamma \sin \alpha - \cos \alpha \sin \gamma & \sin \alpha \sin \beta \\
\cos \gamma \sin \alpha + \cos \alpha \cos \beta \sin \gamma & \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \sin \beta \\
\sin \beta \sin \gamma & \cos \gamma \sin \beta & \cos \beta
\end{bmatrix},
\]

that is, a \( 3 \times 3 \) rotation matrix. For each multivariate normal mixture model, we specify the mixing proportion, covariance matrix, and mean vector for each component.

**Two-component bivariate normal mixture models.** We set the component proportions \((\pi_1, \pi_2) = (0.3, 0.7)\). No other cases are considered.

Due to the invariance property of the multivariate normal distribution, the distance between the two mean vectors is the only configuration that can make a difference. Thus, we simulated only three pairs of mean vectors representing the situation where two component mean vectors are in near, moderate, and distant locations as in the following table:

| Component 1 | near  | moderate | distant |
|-------------|-------|----------|---------|
| (0, -1)     | (0, -3) | (0, -5)  |
| Component 2 | (0, 1) | (0, 3)   | (0, 5)  |

There are many features in the pair of covariance matrices that may have an effect on the performance of the ratified MLE or PMLE. The sizes of the eigenvalues are most important in their ratio \( \lambda_2/\lambda_1 \). The angle \( \theta \) determines the relative orientation between two component densities. Our choices based
on these considerations are given in the following table:

| Component 1 | Component 2 |
|--------------|--------------|
| $\lambda_1$ $\lambda_2$ $\theta$ | $\lambda_1$ $\lambda_2$ $\theta$ |
| 1 1 1 0 | 1 1 0 |
| 2 1 5 0 | 1 1 0 |
| 3 1 5 $\pi/4$ | 1 1 0 |
| 4 1 5 $\pi/2$ | 1 1 0 |
| 5 1 5 $\pi/4$ | 1 5 0 |
| 6 1 5 $\pi/2$ | 1 5 0 |

Three-component bivariate normal mixture models. We set the component proportions $(\pi_1, \pi_2, \pi_3) = (0.15, 0.35, 0.50)$. The three mean vectors may form a straight line, an acute triangle, or an obtuse triangle. We select three representative ones as follows:

| Component | straight | acute | obtuse |
|-----------|----------|-------|--------|
| Component 1 | (0, -2) | (0, -2) | (0, -2) |
| Component 2 | (0, 0) | (3, 0) | (1, 0) |
| Component 3 | (0, 2) | (0, 2) | (0, 2) |

We select six triplets of covariance matrices as follows:

| Component 1 | Component 2 | Component 3 |
|--------------|--------------|--------------|
| $\lambda_1$ $\lambda_2$ $\theta$ | $\lambda_1$ $\lambda_2$ $\theta$ | $\lambda_1$ $\lambda_2$ $\theta$ |
| 1 1 1 0 | 1 1 0 | 1 1 0 |
| 2 1 1 0 | 1 1 0 | 1 5 0 |
| 3 1 1 0 | 1 5 0 | 1 5 $\pi/4$ |
| 4 1 1 0 | 1 5 0 | 1 5 $\pi/2$ |
| 5 1 5 0 | 1 5 $\pi/4$ | 1 5 $-\pi/4$ |
| 6 1 5 0 | 1 5 $\pi/4$ | 1 5 $-\pi/2$ |

Two-component trivariate normal mixture models. We again let $(\pi_1, \pi_2) =$
(0.3, 0.7). At the same time, only the distance between the two mean vectors matters. The two mean vectors are chosen to be:

| Component   | near   | moderate | distant |
|-------------|--------|----------|---------|
| Component 1 | (0, 0, -1) | (0, 0, -3) | (0, 0, -5) |
| Component 2 | (0, 0, 1)   | (0, 0, 3)   | (0, 0, 5)   |

The covariance matrix pairs are chosen as follows:

| Component 1 | Component 2 |
|-------------|-------------|
| \((\lambda_1, \lambda_2, \lambda_3)\) | \((\lambda_1, \lambda_2, \lambda_3)\) |
| \((\alpha, \beta, \gamma)\) | \((\alpha, \beta, \gamma)\) |

| Component 1 | Component 2 |
|-------------|-------------|
| \((1, 1, 1)\) | \((1, 1, 1)\) |
| \((0, 0, 0)\) | \((0, 0, 0)\) |

Three-component trivariate normal mixture models. We let the component proportions \((\pi_1, \pi_2, \pi_3)\) be (.15, .35, .50). Recall that any three points fall into one plane. Thus, the invariance property of the normal distribution allows us to set the first entry of the mean vector to 0:

| Component   | straight | acute   | obtuse  |
|-------------|----------|---------|---------|
| Component 1 | (0, 0, -2) | (0, 0, -2) | (0, 0, -2) |
| Component 2 | (0, 0, 0)   | (0, 3, 0)   | (0, 1, 0)   |
| Component 3 | (0, 0, 2)   | (0, 0, 2)   | (0, 0, 2)   |
The covariance matrix triplets are chosen as follows:

|   | Component 1 | Component 2 | Component 3 |
|---|-------------|-------------|-------------|
| 1 | (1, 1, 1) (0, 0, 0) | (1, 1, 1) (0, 0, 0) | (1, 1, 1) (0, 0, 0) |
| 2 | (1, 1, 1) (0, 0, 0) | (1, 1, 1) (0, 0, 0) | (1, 3, 10) (0, 0, 0) |
| 3 | (1, 1, 1) (0, 0, 0) | (1, 3, 10) (0, 0, 0) | (1, 3, 10) (-π, π, π)/3 |
| 4 | (1, 1, 1) (0, 0, 0) | (1, 3, 10) (0, 0, 0) | (1, 3, 10) (π, -π, π)/3 |
| 5 | (1, 3, 10) (0, 0, 0) | (1, 3, 10) (-π, π, π)/3 | (1, 3, 10) (π, -π, π)/3 |
| 6 | (1, 3, 10) (0, 0, 0) | (1, 3, 10) (π, -π, π)/3 | (1, 3, 10) (π, π, -π)/3 |

We let \( n = 200 \) for the two-component bivariate mixtures and \( n = 300 \) for the other mixtures to ensure a reasonable estimation of the mixing distribution. We generate 1000 data sets for each model.

We have presented four categories of finite normal mixture models. For ease of reference we use, for example, I.1.2 to refer to the model from Category I with mean vector configuration 1 and covariance matrix configuration 2. Even though there are many more mixing distribution configurations for which simulation studies are needed, there is a limit to how much one paper can achieve. We do not consider the case where \( p \) is unknown. All estimators in this case are expected to be poor although the consistency result for the PMLE remains true.

**Penalty term and initial values.** We compute the ratified MLE and two penalized MLEs corresponding to \( a_n = n^{-1} \) and \( a_n = n^{-1/2} \) in (2). We call these MLE, PMLE1, and PMLE2, respectively.

The ten initial values are chosen from two groups. The first group of initial values includes the true mixing distribution and four others obtained by perturbing the component mean vectors of the true mixing distribution. The second group of initial values was data-based. We first calculate the sample mean vector and the sample covariance matrix. Then we set the mixing proportions all equal to \( 1/p \) and the component covariance matrices all equal to the sample covariance matrix. We then apply similar perturbation to the sample mean vector to obtain another five sets of initial values.
3.2 Simulation results

**Number of Degeneracies.** When the EM-algorithm converges to a mixing distribution with singular component covariance matrices, we say that it degenerates. The EM-algorithm for the PMLE does not degenerate which is theoretically ensured. Regardless of the quality of the initial value, the corresponding EM-algorithm always converges to some non-degenerate local maximum. The PMLE is a good estimator if the largest local maximum is a good estimator.

When computing the ratified MLE, the EM-algorithm sometimes converges to a degenerate local maximum. We recorded the number of times that the EM-algorithm degenerated while computing the ratified MLE in our simulation. Since each data set had ten initial values, the number of degenerate outcomes is out of 10,000 for each entry.

For two-component bivariate normal mixture models, it is immediately clear that the number of degenerate outcomes increases when the mean vectors are more widely separated. The covariance structure is also important. For example, when the eigenvectors of one covariance matrix are rotated by an angle of $\pi/2$ (variance configurations 4 and 6), so that the two clusters of observations become more mixed, the number of degenerate outcomes declines. This observation is somewhat counter-intuitive but can be explained as follows. The success of the EM-algorithm is heavily dependent on sensible initial values. When the two mean vectors are close and the components are well mixed, different initial values do not matter as much. However, when the two mean vectors are distant, the location of the initial mean vectors is crucial. Thus the degenerate outcomes were mostly due to the second group of initial values.

In the other three categories, the above phenomenon persists. That is, the frequency of degeneracy increases when components are more widely separated. In addition, for these categories we observe a higher frequency of degeneracies on average. We believe this is because the EM-algorithm is more sensitive to the quality of the initial values when the mixture models are more complicated.

Degeneracy of the EM-algorithm should not be a serious problem for the ratified MLE, as long as the non-degenerate outcomes of the algorithm provide good estimates. We hence proceed to examine the bias and variance properties of the PMLE and the largest non-degenerate local maxima regarded as the ratified MLE.

**Bias and Standard Deviation.** We compute the element-wise mean bias and standard deviation based on 1000 simulated samples from each model. We present only a subset of representative outcomes from each category; the complete set is available upon request.
Two representative outcomes for models I.1.1 and I.2.4 in Category I are given in Table 2. There is about a 10% reduction in the standard deviation for PMLE2 compared to the ratified MLE or PMLE1 for the parameters in component 1 of Model I.1.1. The same is true for Models I.1.5 and I.1.6 (not presented). The PMLE2 also has a relatively lower bias in these models. The results for the remaining models are comparable to those for I.2.4: there is little appreciable difference between the three estimation methods.

The biases of all three estimators for estimating $\mu_2$ are high under I.1.1 and I.1.5 in which the two mean vectors are lined up in the $\mu_1$ direction. Due to the orientation of the two component covariance matrices, it is hard to tell the two mean vectors apart. The biases and standard deviations for estimating $\sigma_{22}$ under I.1.1, I.1.2, ..., I.1.6 are also high or relatively high.

Table 2 about here.

We present outcomes for two models (II.1.1, II.2.4) in Category II in Tables 3 and 4. For both models, for the parameters in component 1, there is a 10% to 20% reduction in the standard deviation for PMLE2 compared to the other two estimators. The bias of PMLE2 is also lower. Some reductions in components 2 and 3 are also noticed but to varying degrees. In the other models, the performance of PMLE2 does not dominate that of the ratified MLE or PMLE1.

Under a straight-line configuration of the component mean vectors, the bias for estimating $\mu_2$ is relatively high. For a triangle configuration, the roles of $\mu_1$ and $\mu_2$ are no longer different. This bias problem is not estimator dependent although PMLE2 helps slightly.

The estimation of $\sigma_{22}$ again comes with both higher bias and higher standard deviation in general. For this category of models, the problem spreads into other parts of the covariance matrix.

Tables 3 4 about here.

We report simulation results for three models (III.1.1, III.2.4, III.3.6) in Category III in Tables 5 6 and 7. We again observe that PMLE2 has smaller bias and standard deviation for estimating the parameters in the first component where the mixing proportion is small, and in model III.1.1 where the two mean vectors are close. The gain is as much as 30% for $\sigma_{33}$.

The gains seem to disappear when the two component mean vectors are far from each other. Nevertheless, PMLE2 still appears to be the best estimator in terms of both bias and standard deviation.
We report simulation results for three models (IV.1.1, IV.2.4, IV.3.6) in Category IV in Tables 8, 9, and 10. Again, PMLE2 has the lowest standard deviations for estimating the parameters in the first component where the mixing proportion is small. The comparison is the sharpest in model IV.2.4 for $\sigma_{13}$. In contrast to the models for the other categories, here the superiority of PMLE2 is widespread. In fact, PMLE2 is superior for parameters in component 2, and mixed for parameters in component 3.

We caution that even the best estimator is not necessarily a good estimator for trivariate mixture models. Overall, none of the three estimators does a great job at estimating mixing distributions, possibly due to their fundamental nature, e.g., small Fisher Information for high-dimension multivariate normal mixture models. This problem is expected to disappear with increased sample size.

Summary of the simulation results. To conclude, the penalized likelihood estimators, both PMLE1 and PMLE2, are completely free from degeneracy problems. Moreover, PMLE2 has the best general performance in terms of bias and standard deviation. This is most obvious when the components are not well separated. In applications, it is unnecessary to first judge whether it is safe to use the ratified MLE, when a superior PMLE2 is available. Although we do not completely dismiss the use of the ratified MLE, it is clearly advantageous to use PMLE2 outright. We further caution against the use of high-dimension multivariate normal mixture models in practice when the sample size is not large. In these situations, even the best performing estimator may not be a good estimator.

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Appendix

The ordinary likelihood function is unbounded because when the covariance matrix of a kernel density becomes close to singular, the likelihood contribution of the observations near its mean vector goes to infinity. Thus, a key step in our proof is to assess the number of such observations. In the univariate case, Chen et al. [2] obtained the following result:

**Lemma 1:** Assume that \( x_1, x_2, \ldots, x_n \) is a random sample from a finite normal mixture distribution with density \( f(x), x \in \mathbb{R} \). Let \( F_n \) be the empirical distribution function and define \( M = \max \{ \sup_x f(x), 8 \} \), and \( \delta_n(\sigma) = -M\sigma \log(\sigma) + n^{-1} \). Except for a zero-probability event not depending on \( \sigma \), we have for all large enough \( n \),

(a) for \( \sigma \) between \( \exp(-2) \) and \( 8/(nM) \),

\[
\sup_{\mu} [F_n(\mu - \sigma \log(\sigma)) - F_n(\mu)] \leq 4\delta_n(\sigma);
\]

(b) for \( \sigma \) between 0 and \( 8/(nM) \),

\[
\sup_{\mu} [F_n(\mu - \sigma \log(\sigma)) - F_n(\mu)] \leq 2n^{-1}(\log n)^2.
\]

The consistency result for the multivariate normal mixture model is built on a generalized result. More specifically, the following lemma gives a bound for the multivariate normal mixture model:

**Lemma 2:** Let \( x_1, x_2, \ldots, x_n \) be a random sample from a \( d \)-dimensional multivariate normal mixture model with \( p \) components such that its density function is given by

\[
f(x, G_0) = \sum_{j=1}^{p} \pi_{j0} \varphi(x; \mu_{j0}, \Sigma_{j0}).
\]

Assume that all \( \Sigma_{j0} \) are positive definite. For any mean and covariance matrix pair \( (\mu, \Sigma) \) such that \( |\Sigma| < \exp(-4d) \), except for a zero probability event not depending on \( (\mu, \Sigma) \), we have, for \( n \) large enough, that

\[
H_n(\mu, \Sigma) = \sum_{i=1}^{n} I\{ (x_i - \mu)^T \Sigma^{-1} (x_i - \mu) \leq -(\log |\Sigma|)^2 \}
\]

\[
\leq 4(\log^2 n) I(|\Sigma| \leq \alpha_n) + 8n\delta_n(|\Sigma|) I(\alpha_n \leq |\Sigma|),
\]

where

\[
\left\{ \begin{array}{l}
\alpha_n = (4/Md)^{2d} n^{-2d}, \\
\delta_n(|\Sigma|) = -M|\Sigma|^{1/2d} \log |\Sigma| + n^{-1},
\end{array} \right.
\]

and \( M = \max \{ 8, \lambda_0^{-1/2} \} \) with \( \lambda_0 \) being the smallest eigenvalue among those of \( \Sigma_{j0}, (j = 1, 2, \ldots, p) \).
Proof of Lemma 2: Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_d$ and $(a_1, \ldots, a_d)$ be the eigenvalues and corresponding eigenvectors of unit length of $\Sigma$. We have that

$$\{x : (x - \mu)^T \Sigma^{-1} (x - \mu) \leq -(\log |\Sigma|)^2 \}$$

$$= \{x : \sum_{j=1}^{d} \lambda_j^{-1} |a_j^T (x - \mu)|^2 \leq -(\log |\Sigma|)^2 \}$$

$$\subseteq \{x : |a_j^T (x - \mu)| \leq -\sqrt{\lambda_j} \log |\Sigma|, j = 1, \ldots, d \}$$

$$\subseteq \{x : |a_1^T (x - \mu)| \leq -\sqrt{\lambda_1} \log |\Sigma| \}.$$

Furthermore, let

$$Q = \{b_i : i = 1, 2, \ldots \}$$

be a sequence of unit vectors so that $Q$ forms a dense subset of unit vectors in $R^d$. Hence, for any given $a_1$ and any bounded subset $B \in R^d$, we can find a vector $b$ in $Q$ such that they are arbitrarily close so that

$$\{x \in B : |a_1^T (x - \mu)| \leq -\sqrt{\lambda_1} \log |\Sigma| \} \subseteq \{x \in B : |b^T (x - \mu)| \leq -\sqrt{2\lambda_1} \log |\Sigma| \}.$$

Based on this observation, we get

$$\sup_{\mu} H_n(\mu, \Sigma) = \sup_{\mu} \sum_{i=1}^{n} I\{(x_i - \mu)^T \Sigma^{-1} (x_i - \mu) \leq -(\log |\Sigma|)^2 \}$$

$$\leq \sup_{b \in Q} \sup_{\mu} \sum_{i=1}^{n} I\{|b^T (x_i - \mu)| \leq \sqrt{2\lambda_1} \log |\Sigma| \}.$$

On the other hand, given any non-random unit vector $b$, $b^T x_i, i = 1, 2, \ldots, n$ is a random sample from the univariate normal mixture model with density

$$f^b(x) = \sum_{j=1}^{p} \pi_{j0} \phi(x; b^T \mu_{j0}, b^T \Sigma_{j0} b).$$

We remark that since some pairs of $(b^T \mu_{j0}, b^T \Sigma_{j0} b)$ can be equal, this univariate mixture distribution can have fewer than $p$ components. This does not affect the following derivation. Recall that $\lambda_0$ is the smallest eigenvalue among those of $\Sigma_{j0}, j = 1, \ldots, p$. Then

$$\sup_{b \in Q} \sup_{x} f^b(x) \leq \sup_{b \in Q} \max_{j=1, \ldots, p} \{(b^T \Sigma_{j0} b)^{-1/2}, j = 1, \ldots, p \} = \lambda_0^{-1/2}.$$

Applying Lemma 1 to the univariate data $b^T x_i, i = 1, \ldots, n$, except for a zero-event not depending on $\Sigma$, as $n \to \infty$, we have
\[
\sup_{\mu} \sum_{i=1}^{n} I\{|b^\top (x_i - \mu)| \leq \sqrt{\lambda_1} \log |\Sigma|\} \\
\leq 4(\log^2 n) I(\{|\Sigma| \leq \alpha_n\}) + 8n\delta_n(\{|\Sigma|\} I(\alpha_n \leq |\Sigma|).
\]

The conclusion of the lemma simply claims that the above inequality is true over all \(b \in Q\) with only a zero-probability-event exception. The zero-probability claim remains true because \(Q\) is countable.

**Proof of Theorem 1**: We give a proof for the case \(p = 2\); the proof for the general case is similar. Let \(\Gamma\) be the parameter space for \(G\) and define

\[
\begin{align*}
\Gamma_1 &= \{G \in \Gamma : |\Sigma_1| \leq |\Sigma_2| \leq \varepsilon_0\} \\
\Gamma_2 &= \{G \in \Gamma : |\Sigma_1| \leq \tau_0, |\Sigma_2| \geq \varepsilon_0\} \\
\Gamma_3 &= \Gamma - (\Gamma_1 \cup \Gamma_2)
\end{align*}
\]

where \(\varepsilon_0 > \tau_0 > 0\) are two small positive constants to be specified soon. The first subspace represents the case where the two components have nearly singular covariance matrices. Hence the observations inside the small ellipse centered at the mean parameter make a large contribution to the log likelihood function.

Let \(K_0 = E\{\log f(X; G_0)\}\). The constants \(\varepsilon_0, \tau_0\) must satisfy the following four conditions:

1: \(0 < \varepsilon_0 < \exp\{-4d\}\);
2: \(-\log \varepsilon_0 - (\log \varepsilon_0)^2 \leq 4(K_0 - 2)\);
3: \(16M\varepsilon_0^{1/2d}(\log \varepsilon_0)^2 \leq 1\);
4: \(16M d\tau_0 (\log \tau_0)^2 \leq \frac{2}{5}\delta_0\);

for some \(\delta_0 > 0\) to be specified. The existence of \(\varepsilon_0, \tau_0\) is obvious.

We proceed with the proof in three steps.

**Step 1.** For any \(G \in \Gamma_1\), we show that almost surely,

\[
\sup_{\Gamma_1} pl_n(G) - pl_n(G_0) \rightarrow -\infty.
\]

Define two index sets

\[
\begin{align*}
A &= \{i : (x_i - \mu_1)^\top \Sigma_1^{-1} (x_i - \mu_1) \leq (\log |\Sigma_1|)^2\}, \\
B &= \{i : (x_i - \mu_2)^\top \Sigma_2^{-1} (x_i - \mu_2) \leq (\log |\Sigma_2|)^2\},
\end{align*}
\]
and for any index set \( S \subseteq \{1, 2, \ldots, n\} \), denote

\[
l_n(G; S) = \sum_{i \in S} \log f(X_i, G).
\]

We can write \( l_n(G) = l_n(G; A) + l_n(G; A^c B) + l_n(G; A^c B^c) \), where \( A^c \) and \( B^c \) are the complement sets of \( A \) and \( B \) respectively. For any index set \( S \), denote \( n(S) \) as its cardinality. It is easy to see that

\[
l_n(G; A) \leq n(A) \log |\Sigma_1|^{-\frac{1}{2}},
\]

\[
l_n(G; B) \leq n(B) \log |\Sigma_2|^{-\frac{1}{2}}.
\]

Applying Lemma 2 to \( n(A) \) and \( n(B) \), noting that \( |\Sigma_1| \leq \epsilon_0 \) for \( G \) in \( \Gamma_1 \), and C3 on the penalty function, we find that

\[
l_n(G; A) + \tilde{p}_n(\Sigma_1) \leq 16d \log n + 8M \epsilon_0 \frac{d}{2} \left( \log \epsilon_0 \right)^2 n
\]

\[
l_n(G; A^c B) + \tilde{p}_n(\Sigma_2) \leq 16d \log n + 8M \epsilon_0 \frac{d}{2} \left( \log \epsilon_0 \right)^2 n.
\]

The key point underlying the above two inequalities is that they are bounded by an arbitrarily small fraction of \( n \). Further, for observations away from \( \mu_1 \) and \( \mu_2 \), we have

\[
l_n(G; A^c B^c)
\]

\[
\leq \sum_{i \in A^c B^c} \log \pi_1 \exp\{\log |\Sigma_1|^{-\frac{1}{2}} - \frac{1}{2}(\log |\Sigma_1|)^2\} + \pi_2 \exp\{\log |\Sigma_2|^{-\frac{1}{2}} - \frac{1}{2}(\log |\Sigma_2|)^2\}
\]

\[
\leq \sum_{i \in A^c B^c} \left\{ -\frac{1}{2} \log \epsilon_0 - \frac{1}{2}(\log \epsilon_0)^2 \right\}
\]

\[
\leq n(K_0 - 2)
\]

The last line in the above derivation is obtained by choosing a small enough \( \epsilon_0 \) as specified earlier. Combining these inequalities, we get \( pl_n(G) \leq n(K_0 - 1) \), and hence almost surely

\[
\sup_{G_1} p l_n(G) - p l_n(G_0) \leq -n + 16d \log n.
\]

That is,

\[
\sup_{G_1} p l_n(G) - p l_n(G_0) \to -\infty
\]

almost surely which completes the first step.

**Step 2.** For \( G \in \Gamma_2 \), we also show that almost surely

\[
\sup_{G_2} p l_n(G) - p l_n(G_0) \to -\infty.
\]
Recall that for each $i \in A$, $(x_i - \mu_1)^T \Sigma_1^{-1} (x_i - \mu_1)$ is bounded by $(\log \Sigma_1)^2$.

Hence, it is easy to verify that for $i \in A$,

$$\varphi(x_i; \mu_1, \Sigma_1) \leq |\Sigma_1|^{-1/2} \exp \{ -\frac{1}{4} (x_i - \mu_1)^T \Sigma_1^{-1} (x_i - \mu_1) \}.$$  

For $i \not\in A$,

$$\varphi(x_i; \mu_1, \Sigma_1) \leq \exp \{ -\frac{1}{4} (x_i - \mu_1)^T \Sigma_1^{-1} (x_i - \mu_1) \}.$$  

Therefore, letting (not a density itself)

$$g(x; G) = \pi_1 \exp \{ -\frac{1}{4} (x - \mu_1)^T \Sigma_1^{-1} (x - \mu_1) \} + \pi_2 \varphi(x; \mu_2, \Sigma_2),$$

we have

$$\log f(x; G) \leq \log g(x; G) + I(i \in A) \log |\Sigma_1|^{-1/2}.$$  

Hence, we get

$$l_n(G; A) \leq n(A) \log |\Sigma_1|^{-\frac{1}{2}} + \sum_{i=1}^{n} g(x_i; G).$$

It is obvious that for any $G \in \Gamma_2$, (a) $E_0 \{ \log g(X; G)/f(X; G_0) \} < 0$ by Jensen’s inequality and the fact that the integration of $g(x, G)$ is less than 1; (b) $g(x; G) \leq \epsilon_0^{-1}$ by the definition of $\Gamma_2$. Hence for each given $G \in \Gamma_2$, by the law of large numbers,

$$\frac{1}{n} \sum_{i=1}^{n} \log \{ g(X_i; G)/f(X_i; G_0) \} \rightarrow E \{ g(X; G)/f(X; G_0) \} < 0.$$

For each fixed $x$, we can extend the definition of $g(x; G)$ in $G$ onto the compacted $\Gamma_2$ while maintaining properties (a) and (b) and its continuity in $G$. Thus, a classical technique as in [23] can be readily employed to show that as $n \to \infty$,

$$\sup_{G \in \Gamma_2} \left\{ \frac{1}{n} \sum_{i=1}^{n} \log \left( \frac{g(X_i; G)}{f(X_i; G_0)} \right) \right\} \rightarrow -\delta(\tau_0) < 0$$

for some decreasing function $\delta(\tau_0)$. Hence, it is possible to choose a small enough $\tau_0 \leq \epsilon_0$, such that

$$\sup_{\Gamma_2} pl_n(G) - pl_n(G_0)$$

$$\leq \sup_{\Gamma_2} \{ n(A) \log |\Sigma_1|^{-\frac{1}{2}} + p_n(G) \} + \sup_{\Gamma_2} \sum_{i=1}^{n} \log \left\{ \frac{g(X_i; G)}{f(X_i; G_0)} \right\}$$

$$\leq 8M \tau_0 (\log \tau_0)^2 n - \frac{9}{10} \delta(\epsilon_0) n$$

$$\leq -\frac{1}{2} \delta(\epsilon_0) n.$$
The first term of the third line above is from the assessment of \( n(A) \), C3 on \( p_n(G) \). Note also that \( p_n(G_0) = o(n) \). Therefore, almost surely,

\[
\sup_{\Gamma_2} p l_n(G) - pl_n(G_0) \to -\infty.
\]

**Step 3.** From the above two steps, we know that \( \tilde{G}_n \in \Gamma_3 \) with probability 1. At the same time, when \( G \in \Gamma_3 \), we have \( p_n(G) = o(1) \). By the definition of the maximum penalized likelihood estimator, we have

\[
l_n(\tilde{G}_n) - l_n(G_0) \geq p_n(G) - p_n(G_0) = o(1).
\]

Since the parameter space \( \Gamma_3 \) is now completely regular, an estimator with property (4) is easily shown to be consistent by the classical technique \[23\] even with a penalty of size \( o(n) \).

**Proof of Theorem 3:** When \( p_0 < p < \infty \), we cannot expect that every part of \( G \) converges to that of \( G_0 \). Instead, we measure their difference as two distributions. Let

\[
H(G, G_0) = \int_{\mathcal{R}^d \times \mathcal{A}} |G(\lambda) - G_0(\lambda)| \exp\{-|\lambda|\} d\lambda
\]

where

\[
\lambda = (\mu_1, \mu_2, \ldots, \mu_d, \sigma_{11}, \sigma_{12}, \sigma_{22}, \ldots, \sigma_{dd}) \in \mathcal{R}^d \times \mathcal{A},
\]

\[
|\lambda| = \sum_{j=1}^d |\mu_j| + \sum_{i=1}^d \sum_{j=1}^i |\sigma_{ij}|,
\]

and \( \mathcal{A} \) is a subset of \( \mathcal{R}^{d \times (d+1)/2} \) containing all eligible combinations of \( d \times (d + 1)/2 \) real numbers which form a symmetric positive definite matrix. It is well known that \( \mathcal{A} \) is an open connected subset of \( \mathcal{R}^{d \times (d+1)/2} \) and is regular enough although it may not be easy to visualize its shape.

It can be shown that \( H(G_n, G_0) \to 0 \) implies \( G_n \to G_0 \) in distribution. An estimator \( \tilde{G}_n \) is strongly consistent if \( H(G_n, G_0) \to 0 \) almost surely.

Again, for the sake of clarity, we consider only the special case with \( p = 2, p_0 = 1 \), that is, to fit a non-mixture multivariate normal model with a two-component multivariate normal mixture model. The extension of our proof to general situations is straightforward and the major hurdle is merely a complicated presentation. Most intermediate conclusions in the proof of consistency of the PMLE when \( p = p_0 = 2 \) are still applicable; some need minor changes. We use many of these results and notations to establish a brief proof.

For an arbitrarily small positive number \( \delta \), define \( \mathcal{H}(\delta) = \{ G : G \in \Gamma, H(G, G_0) \geq \delta \} \),
\( \delta \). That is, \( \mathcal{H}(\delta) \) contains all mixing distributions with up to \( p \) components that are at least \( \delta > 0 \) distance from the true mixing distribution \( G_0 \).

Since \( G_0 \not\in \mathcal{H}(\delta) \), we have \( E[\log\{g(X; G)/f(X; G_0)\}] < 0 \) for any \( G \in \mathcal{H}(\delta) \). Thus, (3) remains valid after being slightly revised as follows:

\[
\sup_{G \in \mathcal{H}(\delta) \cap \Gamma_2} n^{-1} \sum_{i=1}^{n} \log\left\{g(X_i; G)/f(X_i; G_0)\right\} \to -\eta(\tau)
\]

for some positive \( \eta(\tau) \) depending on \( \Gamma_2 \). Because of this, the derivations in the proof of Theorem 1 still apply after \( \Gamma_k \) is replaced by \( \mathcal{H}(\delta) \cap \Gamma_k \ (k = 1, 2) \).

That is, with proper choice of \( \epsilon_0 \) and \( \tau_0 \), we similarly get \( \sup_{G \in \mathcal{H}(\delta) \cap \Gamma_k} pl_n(G) - pl_n(G_0) \to -\infty \) for \( k = 1, 2 \).

With what we have proved, it is seen that the penalized maximum likelihood estimator of \( G \), \( \tilde{G}_n \), must almost surely belong to \( \mathcal{H}^c(\delta) \cup \Gamma_3 \), where \( \mathcal{H}^c(\delta) \) is the complement of \( \mathcal{H}(\delta) \). Since \( \delta \) is arbitrarily small, \( \tilde{G}_n \in \mathcal{H}^c(\delta) \) implies \( H(\tilde{G}_n, G_0) \to 0 \). On the other hand, \( \tilde{G}_n \in \Gamma_3 \) is equivalent to putting a positive lower bound on the component variances, which also implies \( H(\tilde{G}_n, G_0) \to 0 \) by [10]. That is, consistency of the PMLE is also true when \( p = 2 \) but \( p_0 = 1 \).

A generalization of the above derivation leads to the conclusion of Theorem 3.
Table 1
Number of Degeneracies

| Mean,Var,Config               | 1      | 2     | 3     | 4     | 5     | 6     |
|-------------------------------|--------|-------|-------|-------|-------|-------|
| 2-component bivariate normal mixture |        |       |       |       |       |       |
| near                          | 0      | 11    | 19    | 5     | 40    | 8     |
| moderate                      | 1911   | 3256  | 441   | 6     | 2523  | 157   |
| distant                       | 4997   | 4998  | 4966  | 4782  | 4998  | 4943  |
| 3-component bivariate normal mixture |       |       |       |       |       |       |
| straight                      | 3049   | 5058  | 4947  | 1998  | 2306  | 2491  |
| acute                         | 2888   | 4505  | 4812  | 4052  | 4057  | 4561  |
| obtuse                        | 3253   | 4980  | 4983  | 2885  | 3022  | 3511  |
| 2-component trivariate normal mixture |       |       |       |       |       |       |
| near                          | 1      | 4872  | 5003  | 4866  | 4961  | 1466  |
| moderate                      | 4011   | 5000  | 5001  | 5000  | 5000  | 4900  |
| distant                       | 5000   | 5000  | 5000  | 5000  | 5000  | 5000  |
| 3-component trivariate normal mixture |       |       |       |       |       |       |
| straight                      | 5009   | 5010  | 5002  | 5002  | 5000  | 5000  |
| acute                         | 5006   | 5034  | 5000  | 5002  | 5000  | 5000  |
| obtuse                        | 5009   | 5038  | 5002  | 5004  | 5000  | 5001  |
Table 2
Bias (std) under 2-component bivariate normal mixture models.

| Parameter | Model I.1.1, component 1 | Model I.1.1, component 2 | Model I.2.4, component 1 | Model I.2.4, component 2 |
|-----------|---------------------------|--------------------------|--------------------------|--------------------------|
| $\pi = 0.3$ | -0.03 (0.11) | -0.02 (0.11) | -0.01 (0.10) | -0.01 (0.03) |
| $\mu_1 = 0$ | -0.16 (0.53) | -0.16 (0.53) | -0.13 (0.50) | -0.00 (0.09) |
| $\mu_2 = -1$ | 0.72 (1.17) | 0.72 (1.17) | 0.71 (1.14) | 0.00 (0.08) |
| $\sigma_{11} = 1$ | -0.14 (0.41) | -0.14 (0.40) | -0.13 (0.37) | -0.01 (0.12) |
| $\sigma_{12} = 0$ | -0.01 (0.39) | 0.00 (0.38) | 0.00 (0.34) | 0.00 (0.08) |
| $\sigma_{22} = 1$ | -0.03 (0.71) | -0.03 (0.70) | -0.01 (0.64) | -0.01 (0.12) |
| $\pi_2 = 0.7$ | 0.03 (0.11) | 0.02 (0.11) | 0.01 (0.10) | 0.00 (0.03) |
| $\mu_1 = 0$ | 0.04 (0.19) | 0.04 (0.19) | 0.04 (0.19) | 0.00 (0.09) |
| $\mu_2 = 1$ | -0.39 (0.47) | -0.39 (0.47) | -0.37 (0.48) | -0.01 (0.12) |
| $\sigma_{11} = 1$ | -0.07 (0.18) | -0.07 (0.18) | -0.07 (0.18) | -0.01 (0.12) |
| $\sigma_{12} = 0$ | 0.00 (0.19) | 0.00 (0.19) | 0.00 (0.19) | 0.00 (0.08) |
| $\sigma_{22} = 1$ | 0.33 (0.44) | 0.33 (0.44) | 0.30 (0.43) | 0.00 (0.03) |
| $\pi_1 = 0.3$ | 0.00 (0.03) | 0.00 (0.03) | 0.00 (0.03) | 0.00 (0.03) |
| $\mu_1 = 0$ | -0.02 (0.28) | -0.02 (0.28) | -0.02 (0.28) | -0.01 (0.12) |
| $\mu_2 = -3$ | -0.01 (0.13) | -0.01 (0.13) | -0.01 (0.13) | -0.01 (0.12) |
| $\sigma_{11} = 5$ | -0.04 (0.93) | -0.04 (0.93) | -0.04 (0.93) | -0.01 (0.12) |
| $\sigma_{12} = 0$ | 0.00 (0.30) | 0.00 (0.30) | 0.00 (0.30) | 0.00 (0.08) |
| $\sigma_{22} = 1$ | -0.02 (0.19) | -0.02 (0.19) | -0.01 (0.12) | 0.00 (0.08) |
| $\pi_2 = 0.7$ | 0.00 (0.03) | 0.00 (0.03) | 0.00 (0.03) | 0.00 (0.03) |
| $\mu_1 = 0$ | 0.00 (0.09) | 0.00 (0.09) | 0.00 (0.09) | 0.00 (0.08) |
| $\mu_2 = 3$ | 0.00 (0.09) | 0.00 (0.09) | 0.00 (0.09) | 0.00 (0.08) |
| $\sigma_{11} = 1$ | -0.01 (0.12) | -0.01 (0.12) | -0.01 (0.12) | -0.01 (0.12) |
| $\sigma_{12} = 0$ | 0.00 (0.08) | 0.00 (0.08) | 0.00 (0.08) | 0.00 (0.08) |
| $\sigma_{22} = 1$ | 0.00 (0.12) | 0.00 (0.12) | 0.00 (0.12) | 0.00 (0.12) |
Table 3
Bias (std) under 3-component bivariate normal mixture models.

|                     | MLE         | PMLE1       | PMLE2       |
|---------------------|-------------|-------------|-------------|
| **Model II.1.1, component 1** |             |             |             |
| $\pi = 0.15$       | -0.10 (0.06)| -0.08 (0.07)| -0.04 (0.07)|
| $\mu_1 = 0$        | 0.69 (1.15) | 0.58 (1.28) | 0.25 (1.01) |
| $\mu_2 = -2$       | 1.17 (2.48) | 1.15 (2.32) | 1.24 (1.94) |
| $\sigma_{11} = 1$  | -0.33 (0.91)| -0.46 (0.60)| -0.33 (0.52)|
| $\sigma_{12} = 0$  | -0.04 (0.54)| -0.02 (0.46)| 0.02 (0.48) |
| $\sigma_{22} = 1$  | -0.22 (1.16)| -0.22 (1.01)| 0.12 (1.01) |
| **Model II.1.1, component 2** |             |             |             |
| $\pi_2 = 0.35$     | -0.02 (0.10)| -0.02 (0.10)| -0.03 (0.08)|
| $\mu_1 = 0$        | -0.10 (0.39)| -0.08 (0.38)| -0.06 (0.39)|
| $\mu_2 = 0$        | 0.61 (1.54) | 0.63 (1.53) | 0.56 (1.44) |
| $\sigma_{11} = 1$  | -0.13 (0.29)| -0.13 (0.30)| -0.14 (0.31)|
| $\sigma_{12} = 0$  | 0.02 (0.32) | 0.01 (0.33) | 0.02 (0.34) |
| $\sigma_{22} = 1$  | 0.24 (0.70) | 0.20 (0.71) | 0.22 (0.69) |
| **Model II.1.1, component 3** |             |             |             |
| $\pi_3 = 0.5$      | 0.11 (0.11) | 0.10 (0.12) | 0.06 (0.10) |
| $\mu_1 = 0$        | 0.02 (0.20) | 0.01 (0.21) | 0.01 (0.24) |
| $\mu_2 = 2$        | -1.23 (0.90)| -1.16 (0.89)| -1.02 (0.89)|
| $\sigma_{11} = 1$  | -0.08 (0.16)| -0.08 (0.17)| -0.10 (0.19)|
| $\sigma_{12} = 0$  | 0.03 (0.26) | 0.03 (0.27) | 0.00 (0.28) |
| $\sigma_{22} = 1$  | 0.86 (0.68) | 0.81 (0.70) | 0.65 (0.67) |
Table 4
Bias (std) under 3-component bivariate normal mixture models.

|               | Model II.2.4, component 1 | Model II.2.4, component 2 | Model II.2.4, component 3 |
|---------------|---------------------------|---------------------------|---------------------------|
| $\pi_1 = 0.15$| 0.00 (0.04)               | -0.01 (0.05)              | 0.00 (0.05)               |
| $\mu_1 = 0$   | 0.23 (0.86)               | -0.43 (1.12)              | 0.33 (0.88)               |
| $\mu_2 = -2$  | 0.12 (0.83)               | 0.14 (0.80)               | -0.19 (0.57)              |
| $\sigma_{11} = 1$ | 0.07 (0.69)          | 0.37 (1.12)              | -0.38 (0.57)              |
| $\sigma_{12} = 0$ | -0.05 (0.54)       | -0.01 (0.35)             | -0.38 (0.53)              |
| $\sigma_{22} = 1$ | 0.17 (0.99)            | 0.18 (0.95)              | -0.38 (0.34)              |

27
|                  | MLE  | PMLE1 | PMLE2 |
|------------------|------|-------|-------|
| Model III.1.1, component 1 |      |       |       |
| $\pi_1 = 0.3$    | -0.09 (0.15) | -0.08 (0.15) | -0.05 (0.14) |
| $\mu_1 = 0$     | -0.28 (0.61) | -0.26 (0.58) | -0.17 (0.51) |
| $\mu_2 = 0$     | -0.15 (0.58) | -0.14 (0.57) | -0.09 (0.52) |
| $\mu_3 = -1$    | 0.52 (0.09)  | 0.54 (0.11)  | 0.61 (0.09)  |
| $\sigma_{11} = 1$ | -0.12 (0.47) | -0.11 (0.46) | -0.11 (0.36) |
| $\sigma_{12} = 0$ | -0.01 (0.38) | 0.00 (0.35)  | 0.02 (0.27)  |
| $\sigma_{13} = 0$ | -0.10 (0.48) | -0.10 (0.47) | -0.07 (0.37) |
| $\sigma_{22} = 1$ | -0.09 (0.56) | -0.11 (0.47) | -0.13 (0.36) |
| $\sigma_{23} = 0$ | -0.04 (0.49) | -0.02 (0.47) | -0.01 (0.37) |
| $\sigma_{33} = 1$ | 0.22 (0.91)  | 0.18 (0.83)  | 0.12 (0.66)  |
| Model III.1.1, component 2 |      |       |       |
| $\pi_2 = 0.7$    | 0.09 (0.15)  | 0.08 (0.15)  | 0.05 (0.14)  |
| $\mu_1 = 0$     | 0.01 (0.15)  | 0.01 (0.15)  | 0.01 (0.16)  |
| $\mu_2 = 0$     | 0.02 (0.15)  | 0.02 (0.15)  | 0.02 (0.17)  |
| $\mu_3 = 1$    | -0.45 (0.41) | -0.44 (0.41) | -0.42 (0.44) |
| $\sigma_{11} = 1$ | -0.05 (0.13) | -0.05 (0.13) | -0.05 (0.14) |
| $\sigma_{12} = 0$ | 0.00 (0.10)  | 0.00 (0.10)  | 0.00 (0.10)  |
| $\sigma_{13} = 0$ | -0.02 (0.13) | -0.02 (0.13) | -0.02 (0.14) |
| $\sigma_{22} = 1$ | 0.03 (0.13)  | -0.03 (0.13) | -0.04 (0.14) |
| $\sigma_{23} = 0$ | 0.01 (0.14)  | 0.01 (0.14)  | 0.01 (0.15)  |
| $\sigma_{33} = 1$ | 0.44 (0.38)  | 0.43 (0.38)  | 0.39 (0.39)  |
Table 6
Bias (std) under 2-component trivariate normal mixture models.

|                | Model III.2.4, component 1 |                |                |
|----------------|-----------------------------|----------------|----------------|
| $\pi_1 = 0.3$  | 0.00 (0.04)                 | 0.00 (0.04)    | 0.00 (0.04)    |
| $\mu_1 = 0$   | 0.01 (0.13)                 | 0.01 (0.13)    | 0.01 (0.13)    |
| $\mu_2 = 0$   | 0.01 (0.22)                 | 0.01 (0.22)    | 0.01 (0.22)    |
| $\mu_3 = -3$  | -0.03 (0.52)                | -0.03 (0.52)   | -0.04 (0.52)   |
| $\sigma_{11} = 1$ | -0.01 (0.17)            | -0.01 (0.17)   | -0.01 (0.17)   |
| $\sigma_{12} = 0$ | -0.01 (0.20)            | -0.01 (0.20)   | -0.01 (0.19)   |
| $\sigma_{13} = 0$ | 0.03 (0.45)             | 0.03 (0.45)    | 0.03 (0.45)    |
| $\sigma_{22} = 3$ | -0.05 (0.49)            | -0.05 (0.49)   | -0.04 (0.49)   |
| $\sigma_{23} = 0$ | 0.00 (0.75)             | 0.00 (0.75)    | 0.01 (0.75)    |
| $\sigma_{33} = 10$ | -0.36 (2.10)           | -0.36 (2.11)   | -0.38 (2.09)   |
|                | Model III.2.4, component 2 |                |                |
| $\pi_2 = 0.7$  | 0.00 (0.04)                 | 0.00 (0.04)    | 0.00 (0.04)    |
| $\mu_1 = 0$   | 0.00 (0.15)                 | 0.00 (0.15)    | 0.00 (0.15)    |
| $\mu_2 = 0$   | -0.01 (0.19)                | -0.01 (0.19)   | -0.01 (0.19)   |
| $\mu_3 = 3$   | -0.01 (0.11)                | -0.01 (0.11)   | -0.01 (0.11)   |
| $\sigma_{11} = 4.87$ | -0.03 (0.47)           | -0.03 (0.48)   | -0.03 (0.47)   |
| $\sigma_{12} = -3.23$ | 0.03 (0.49)            | 0.03 (0.49)    | 0.03 (0.48)    |
| $\sigma_{13} = -0.5$ | 0.01 (0.23)            | 0.01 (0.23)    | 0.01 (0.23)    |
| $\sigma_{22} = 7.2$ | -0.07 (0.71)           | -0.07 (0.72)   | -0.07 (0.71)   |
| $\sigma_{23} = 2.16$ | -0.02 (0.30)           | -0.02 (0.30)   | -0.02 (0.30)   |
| $\sigma_{33} = 1.94$ | -0.01 (0.22)           | -0.01 (0.22)   | 0.00 (0.22)    |

29
Table 7
Bias (std) under 2-component trivariate normal mixture models.

|                  | Model III.3.6, component 1 | Model III.3.6, component 2 |
|------------------|-----------------------------|-----------------------------|
| $\pi_1 = 0.3$    | 0.00 (0.03)                 | 0.00 (0.03)                 |
| $\mu_1 = 0$     | 0.00 (0.10)                 | 0.00 (0.10)                 |
| $\mu_2 = 0$     | 0.01 (0.19)                 | 0.01 (0.19)                 |
| $\mu_3 = -5$    | 0.01 (0.37)                 | 0.01 (0.37)                 |
| $\sigma_{11} = 1$ | -0.01 (0.15)                  | -0.01 (0.15)                 |
| $\sigma_{12} = 0$ | 0.01 (0.18)                 | 0.01 (0.18)                 |
| $\sigma_{13} = 0$ | 0.02 (0.36)                 | 0.02 (0.36)                 |
| $\sigma_{22} = 3$ | -0.05 (0.45)                  | -0.05 (0.45)                 |
| $\sigma_{23} = 0$ | -0.02 (0.64)                 | -0.02 (0.64)                 |
| $\sigma_{33} = 10$ | -0.06 (1.81)                 | -0.06 (1.81)                 |
| $\pi_2 = 0.7$    | 0.00 (0.03)                 | 0.00 (0.03)                 |
| $\mu_1 = 0$     | 0.00 (0.15)                 | 0.00 (0.15)                 |
| $\mu_2 = 0$     | 0.00 (0.19)                 | 0.00 (0.19)                 |
| $\mu_3 = 5$     | 0.00 (0.10)                 | 0.00 (0.10)                 |
| $\sigma_{11} = 4.87$ | -0.05 (0.46)                  | -0.05 (0.46)                 |
| $\sigma_{12} = 3.23$ | -0.03 (0.46)                 | -0.03 (0.46)                 |
| $\sigma_{13} = -0.5$ | 0.00 (0.22)                 | 0.00 (0.22)                 |
| $\sigma_{22} = 7.2$ | -0.02 (0.70)                  | -0.02 (0.70)                 |
| $\sigma_{23} = -2.16$ | -0.01 (0.29)                 | -0.01 (0.29)                 |
| $\sigma_{33} = 1.94$ | -0.01 (0.20)                 | -0.01 (0.20)                 |
Table 8
Bias (std) under 3-component trivariate normal mixture models.

|                  | MLE       | PMLE1     | PMLE2     |
|------------------|-----------|-----------|-----------|
| **Model IV.1.1, component 1** |           |           |           |
| $\pi_1 = 0.15$  | -0.05 (0.07) | -0.06 (0.07) | -0.01 (0.07) |
| $\mu_1 = 0$     | 0.10 (0.64)  | 0.28 (0.97)  | 0.12 (0.69)  |
| $\mu_2 = 0$     | -0.08 (0.64) | 0.11 (0.97)  | -0.04 (0.65) |
| $\mu_3 = -2$    | 3.07 (2.16)  | 2.65 (2.17)  | 2.16 (1.89)  |
| $\sigma_{11} = 1$ | -0.05 (0.73) | -0.25 (0.63) | -0.19 (0.47) |
| $\sigma_{12} = 0$ | 0.07 (0.50)  | 0.05 (0.40)  | 0.04 (0.35)  |
| $\sigma_{13} = 0$ | 0.00 (0.58)  | 0.00 (0.51)  | 0.00 (0.48)  |
| $\sigma_{22} = 1$ | -0.04 (0.74) | -0.23 (0.63) | -0.16 (0.47) |
| $\sigma_{23} = 0$ | 0.03 (0.51)  | 0.03 (0.47)  | 0.04 (0.43)  |
| $\sigma_{33} = 1$ | -0.01 (1.16) | 0.01 (1.19)  | 0.31 (1.05)  |
| **Model IV.1.1, component 2** |           |           |           |
| $\pi_2 = 0.35$  | -0.05 (0.09) | -0.07 (0.11) | -0.05 (0.09) |
| $\mu_1 = 0$     | 0.04 (0.33)  | 0.02 (0.43)  | 0.01 (0.34)  |
| $\mu_2 = 0$     | 0.00 (1.47)  | 0.02 (1.52)  | 0.26 (1.42)  |
| $\sigma_{11} = 1$ | -0.09 (0.26) | -0.12 (0.32) | -0.11 (0.29) |
| $\sigma_{12} = 0$ | 0.02 (0.20)  | 0.01 (0.23)  | 0.02 (0.21)  |
| $\sigma_{13} = 0$ | -0.05 (0.32) | -0.05 (0.41) | -0.03 (0.35) |
| $\sigma_{22} = 1$ | -0.09 (0.28) | -0.11 (0.30) | -0.11 (0.28) |
| $\sigma_{23} = 0$ | 0.02 (0.33)  | -0.01 (0.37) | 0.01 (0.33)  |
| $\sigma_{33} = 1$ | 0.46 (0.83)  | 0.48 (0.93)  | 0.46 (0.84)  |
| **Model IV.1.1, component 3** |           |           |           |
| $\pi_3 = 0.5$   | 0.10 (0.12)  | 0.13 (0.15)  | 0.06 (0.12)  |
| $\mu_1 = 0$     | 0.01 (0.19)  | 0.00 (0.18)  | 0.00 (0.21)  |
| $\mu_2 = 0$     | -0.01 (0.18) | -0.01 (0.17) | 0.00 (0.21)  |
| $\mu_3 = 2$     | -0.96 (0.81) | -1.00 (0.79) | -0.97 (0.86) |
| $\sigma_{11} = 1$ | -0.07 (0.17) | -0.07 (0.17) | -0.08 (0.19) |
| $\sigma_{12} = 0$ | 0.01 (0.12)  | 0.00 (0.11)  | 0.01 (0.13)  |
| $\sigma_{13} = 0$ | -0.04 (0.22) | -0.04 (0.22) | -0.04 (0.24) |
| $\sigma_{22} = 1$ | -0.06 (0.16) | -0.06 (0.16) | -0.07 (0.18) |
| $\sigma_{23} = 0$ | 0.04 (0.22)  | 0.03 (0.22)  | 0.03 (0.25)  |
| $\sigma_{33} = 1$ | 0.76 (0.72)  | 0.88 (0.77)  | 0.75 (0.76)  |
Table 9
Bias (std) under 3-component trivariate normal mixture models.

|                  | MLE     | PMLE1   | PMLE2   |
|------------------|---------|---------|---------|
| **Model IV.2.4, component 1** |          |         |         |
| \(\pi_1 = 0.15\) | 0.00 (0.05) | 0.00 (0.04) | 0.01 (0.04) |
| \(\mu_1 = 0\)  | 0.04 (0.43) | 0.04 (0.37) | 0.02 (0.29) |
| \(\mu_2 = 0\)  | 0.20 (0.96) | 0.20 (0.90) | 0.24 (0.88) |
| \(\mu_3 = -2\) | 0.19 (0.86) | 0.17 (0.80) | 0.20 (0.80) |
| \(\sigma_{11} = 1\) | 0.05 (0.63) | 0.02 (0.52) | 0.01 (0.38) |
| \(\sigma_{12} = 0\) | -0.03 (0.54) | -0.01 (0.41) | -0.01 (0.34) |
| \(\sigma_{13} = 0\) | 0.04 (0.79) | 0.01 (0.58) | 0.01 (0.35) |
| \(\sigma_{22} = 1\) | 0.18 (1.06) | 0.13 (0.81) | 0.18 (0.73) |
| \(\sigma_{23} = 0\) | -0.15 (1.09) | -0.10 (0.65) | -0.09 (0.62) |
| \(\sigma_{33} = 1\) | 0.65 (2.52) | 0.53 (2.17) | 0.68 (2.31) |
| **Model IV.2.4, component 2** |          |         |         |
| \(\pi_2 = 0.35\) | -0.01 (0.06) | -0.01 (0.06) | -0.02 (0.06) |
| \(\mu_1 = 0\)  | 0.01 (0.19) | 0.01 (0.19) | 0.01 (0.18) |
| \(\mu_2 = 3\)  | -0.51 (1.25) | -0.46 (1.21) | -0.34 (1.13) |
| \(\mu_3 = 0\)  | 0.24 (0.94) | 0.21 (0.91) | 0.13 (0.86) |
| \(\sigma_{11} = 1\) | 0.56 (1.54) | 0.50 (1.47) | 0.35 (1.27) |
| \(\sigma_{12} = 0\) | -0.49 (1.32) | -0.44 (1.26) | -0.32 (1.10) |
| \(\sigma_{13} = 0\) | 0.09 (0.42) | 0.08 (0.42) | 0.05 (0.41) |
| \(\sigma_{22} = 3\) | 0.48 (1.78) | 0.41 (1.71) | 0.20 (1.53) |
| \(\sigma_{23} = 0\) | -0.33 (0.98) | -0.30 (0.96) | -0.25 (0.88) |
| \(\sigma_{33} = 10\) | -1.40 (3.55) | -1.26 (3.45) | -1.03 (3.31) |
| **Model IV.2.4, component 3** |          |         |         |
| \(\pi_3 = 0.5\) | 0.01 (0.05) | 0.01 (0.05) | 0.00 (0.05) |
| \(\mu_1 = 0\)  | -0.02 (0.18) | -0.02 (0.18) | -0.01 (0.19) |
| \(\mu_2 = 0\)  | 0.37 (0.87) | 0.34 (0.86) | 0.27 (0.79) |
| \(\mu_3 = 2\)  | -0.28 (0.72) | -0.25 (0.68) | -0.17 (0.58) |
| \(\sigma_{11} = 4.87\) | -0.57 (1.42) | -0.51 (1.36) | -0.39 (1.22) |
| \(\sigma_{12} = -3.23\) | 0.45 (1.24) | 0.41 (1.20) | 0.30 (1.07) |
| \(\sigma_{13} = 0.5\) | -0.07 (0.33) | -0.06 (0.33) | -0.04 (0.32) |
| \(\sigma_{22} = 7.2\) | -0.46 (1.48) | -0.42 (1.46) | -0.33 (1.38) |
| \(\sigma_{23} = -2.16\) | 0.31 (0.95) | 0.27 (0.89) | 0.18 (0.77) |
| \(\sigma_{33} = 1.94\) | 0.88 (2.23) | 0.79 (2.16) | 0.58 (1.90) |
Table 10
Bias (std) under 3-component trivariate normal mixture models.

|                  | MLE     | PMLE1   | PMLE2   |
|------------------|---------|---------|---------|
| **Model IV.3.6, component 1** |         |         |         |
| $\pi_1 = 0.15$   | 0.00 (0.05) | 0.00 (0.05) | 0.00 (0.05) |
| $\mu_1 = 0$      | 0.05 (0.41) | 0.05 (0.41) | 0.05 (0.40) |
| $\mu_2 = 0$      | -0.01 (0.64) | -0.01 (0.64) | -0.01 (0.61) |
| $\mu_3 = -2$     | -0.21 (1.23) | -0.21 (1.23) | -0.23 (1.20) |
| $\sigma_{11} = 1$ | 0.28 (1.24) | 0.28 (1.24) | 0.24 (1.12) |
| $\sigma_{12} = 0$ | -0.19 (1.16) | -0.19 (1.16) | -0.15 (1.05) |
| $\sigma_{13} = 0$ | 0.14 (1.04) | 0.14 (1.03) | 0.13 (0.99) |
| $\sigma_{22} = 3$ | 0.21 (1.48) | 0.21 (1.48) | 0.18 (1.40) |
| $\sigma_{23} = 0$ | -0.42 (1.54) | -0.42 (1.54) | -0.39 (1.50) |
| $\sigma_{33} = 10$ | -1.37 (3.73) | -1.37 (3.73) | -1.34 (3.64) |
| **Model IV.3.6, component 2** |         |         |         |
| $\pi_2 = 0.35$   | -0.01 (0.06) | -0.01 (0.06) | -0.01 (0.06) |
| $\mu_1 = 0$      | -0.01 (0.33) | -0.01 (0.33) | 0.00 (0.32) |
| $\mu_2 = 3$      | -0.20 (0.61) | -0.2 (0.61) | -0.19 (0.60) |
| $\mu_3 = 0$      | 0.25 (0.96) | 0.25 (0.96) | 0.26 (0.94) |
| $\sigma_{11} = 4.87$ | -0.15 (1.18) | -0.15 (1.18) | -0.13 (1.14) |
| $\sigma_{12} = -3.2$ | 1.23 (2.89) | 1.23 (2.89) | 1.2 (2.87) |
| $\sigma_{13} = 0.5$ | -0.16 (0.62) | -0.16 (0.62) | -0.15 (0.62) |
| $\sigma_{22} = 7.2$ | -0.84 (2.15) | -0.24 (1.56) | -0.21 (1.52) |
| $\sigma_{23} = -2.16$ | 0.21 (1.97) | 0.21 (1.97) | 0.19 (0.73) |
| $\sigma_{33} = 1.94$ | 0.21 (1.61) | 0.21 (1.61) | 0.18 (1.52) |
| **Model IV.3.6, component 3** |         |         |         |
| $\pi_3 = 0.5$    | 0.02 (0.07) | 0.02 (0.07) | 0.02 (0.07) |
| $\mu_1 = 0$      | -0.02 (0.22) | -0.02 (0.22) | -0.02 (0.22) |
| $\mu_2 = 0$      | 0.16 (0.43) | 0.17 (0.43) | 0.16 (0.43) |
| $\mu_3 = 2$      | -0.33 (0.68) | -0.33 (0.68) | -0.32 (0.68) |
| $\sigma_{11} = 4.87$ | -0.18 (0.66) | -0.18 (0.66) | -0.17 (0.65) |
| $\sigma_{12} = 3.23$ | -1.06 (2.14) | -1.06 (2.15) | -1.04 (2.15) |
| $\sigma_{13} = -0.5$ | 0.17 (0.47) | 0.17 (0.47) | 0.16 (0.47) |
| $\sigma_{22} = 7.2$ | -0.21 (0.97) | -0.21 (0.98) | -0.20 (0.98) |
| $\sigma_{23} = -2.16$ | 0.03 (0.45) | 0.03 (0.45) | 0.03 (0.46) |
| $\sigma_{33} = 1.94$ | 0.03 (0.39) | 0.03 (0.38) | 0.03 (0.38) |