HOMOLOGICAL FINITENESS CONDITIONS FOR GROUPS, MONOIDS AND ALGEBRAS

STEPHEN J. PRIDE

Abstract
Recently Alonso and Hermiller [2] introduced a homological finiteness condition $bi-FP_n$ (here called weak $bi-FP_n$) for monoid rings, and Kobayashi and Otto [10] introduced a different property, also called $bi-FP_n$ (we adhere to their terminology). From these and other papers we know that: $bi-FP_n \Rightarrow$ left and right $FP_n \Rightarrow$ weak $bi-FP_n$; the first implication is not reversible in general; the second implication is reversible for group rings. We show that the second implication is reversible in general, even for arbitrary associative algebras (Theorem 1'), and we show that the first implication is reversible for group rings (Theorem 2). We also show that all four properties are equivalent for connected graded algebras (Theorem 4). A result on retractions (Theorem 3') is proved, and some questions are raised.

1. Introduction
Throughout the paper $K$ will denote a fixed but arbitrary commutative ring, and $R, S$ will denote (not necessarily commutative) rings. All rings will have an identity, and ring homomorphisms will be assumed to conserve the identity.

1.1 Groups and monoids
Let $B$ be a monoid, and let $KB$ be the corresponding monoid ring over $K$. We have the standard augmentation

$$\varepsilon : KB \to K \quad b \mapsto 1 \quad (b \in B),$$

and we can thus regard $K$ as a left $KB$-module $bK$ with the $KB$-action via $\varepsilon$:

$$a.k = \varepsilon(a)k \quad (a \in KB, k \in K).$$

Then $B$ is said to be of type left-$FP_n$ (over $K$) if there is a partial free resolution

$$0 \leftarrow bK \leftarrow P_0 \leftarrow P_1 \leftarrow \ldots \leftarrow P_n$$

where $P_0, P_1, \ldots, P_n$ are finitely generated free left $KB$-modules. Similarly, we can regard $K$ as a right $KB$-module $K_B$ via $\varepsilon$, and analogously define monoids of type right-$FP_n$ by requiring a partial resolution

$$0 \leftarrow K_B \leftarrow P'_0 \leftarrow P'_1 \leftarrow \ldots \leftarrow P'_n$$

by finitely generated free right $KB$-modules. These two properties are equivalent if there is an involution $*$ on $B$ (that is a mapping $*: B \to B$ satisfying $(bc)^* = c^*b^*, b^{**} = b$ for all $b, c \in B$). In particular, they are equivalent for groups, and more generally inverse monoids (and so in these cases we usually just use the term $FP_n$). However, in general
the left and right properties are different. In [6], an example is given of a monoid which
is left-$FP_{\infty}$ (i.e. $FP_n$ for all $n$) over $\mathbb{Z}$, but not even right-$FP_1$ over $\mathbb{Z}$ (and vice versa).

We remark that there are examples of groups which are of type $FP_n$ over $\mathbb{Z}$ but not of
type $FP_{n+1}$ over $\mathbb{Z}$ for all $n$ [4].

We can also regard $K$ as a $(KB, KB)$-bimodule $B_K$ with 2-sided action
\[ a.k.a' = \varepsilon(a)k\varepsilon(a') \quad (a, a' \in KB, k \in K). \]

We can then define a finiteness condition by requiring that there exists a partial free
bi-resolution
\[ 0 \leftarrow B_K \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_n \quad (3) \]
where $F_0, F_1, \ldots, F_n$ are finitely generated free $(KB, KB)$-bimodules. This property was
introduced in [2], and was called there $bi-FP_n$. However, in this paper we will call it weak
bi-$FP_n$, to distinguish it from another property discussed shortly.

As shown in [2],
\[ left-FP_n + right-FP_n \implies weak \ bi-FP_n \quad (4) \]
(at least in the case when $K$ is a PID). For, by the Künneth Theorem, the tensor
product over $K$ of the free partial resolutions in (1), (2) gives a free partial bi-resolution
of
\[ B_K \otimes_K B_K \cong B_K. \] (See [2, p.344] for details.)

There is also another “natural” $(KB, KB)$-bimodule associated with $KB$, namely $KB$
itself, regarded as a bimodule by left and right multiplication. In [10], the authors defined
$B$ to be of type $bi-FP_n$ (and we will adhere to their terminology in this paper) if there
exists a partial bi-resolution
\[ 0 \leftarrow KB \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_n \quad (5) \]
where $F_0, F_1, \ldots, F_n$ are finitely generated free $(KB, KB)$-bimodules.

As shown in [10],
\[ bi-FP_n \implies left-FP_n + right-FP_n. \quad (6) \]
For if we apply $- \otimes_{KB} K$ to (5) then the sequence remains exact and gives a partial
resolution of
\[ KB \otimes_{KB} K \cong B_K \] by finitely generated free left $KB$-modules. (See [10, p.338] for details.)

It was proved in [2] that the implication (4) is reversible for groups. However, in private
correspondence with the second author of [2], it emerged that no example was known to
show in general that the implication (4) is not reversible. We will prove that no such
example can exist.

**Theorem 1** If a monoid is weak bi-$FP_n$ then it is both left- and right-$FP_n$.

As regards the reverse of the implication (6), an example is given in [9] of a monoid
which is left- and right-$FP_\infty$ but is not bi-$FP_3$. However, it has been an open question
whether (6) is reversible for groups. We will show that this is the case.

**Theorem 2** If a group is $FP_n$ then it is bi-$FP_n$.

Thus for groups the four properties weak bi-$FP_n$, left-$FP_n$, right-$FP_n$, bi-$FP_n$ all co-
incide.
Question. Is Theorem 2 true for inverse monoids?

A monoid $C$ is called a retract of a monoid $B$ if there are monoid homomorphisms

$$B \xrightarrow{\psi} C \quad \psi\phi = \text{id}_C.$$ 

Theorem 3 Each of the properties left-$FP_n$, right-$FP_n$, bi-$FP_n$, weak bi-$FP_n$ is closed under retractions.

(In the case of groups, this is proved in [1] for the more general concept of quasi-retracts.)

1.2 Algebras

Let $A$ be a $K$-algebra with an augmentation, that is, a $K$-algebra epimorphism

$$\varepsilon : A \to K.$$ 

Then we can regard $K$ as a left $A$-module $AK$, or a right $A$-module $KA$, or an $(A,A)$-bimodule $AKA$ via $\varepsilon$. Also we can regard $A$ as an $(A,A)$-bimodule $AAA$ by left and right multiplication. Then we can define $A$ to be left-$FP_n$, right-$FP_n$, weak bi-$FP_n$, bi-$FP_n$ if there is a partial free resolution analogous to (1), (2), (3), (5) respectively.

The argument in [2] shows that the implication (4) holds provided $K$ is a PID and $A$ is free (or, more generally, flat) as a $K$-module. Also, the argument in [10] shows that the implication (6) holds provided $A$ is free (or, more generally, projective) as a $K$-module. We will show that the reverse of (4) holds without restrictions.

Theorem 1' If $A$ is weak bi-$FP_n$, then $A$ is left-$FP_n$ and right-$FP_n$.

We remark that Anick [3] showed that if $A$ can be presented as a quotient of a finitely generated free $K$-algebra by an ideal generated by a finite Gröbner base, then $A$ is left- and right-$FP_\infty$. Recently Kobayashi [8] has improved this to show that such an algebra is bi-$FP_\infty$.

A $K$-algebra $D$ is a retract of $A$ if there are $K$-algebra homomorphisms

$$A \xrightarrow{\kappa} D \quad \kappa\theta = \text{id}_D.$$ 

Moreover, if $D$ has augmentation

$$\varepsilon_D : D \to K$$ 

then $D$ is an augmented retract if there exist $\kappa, \theta$ as above such that $\varepsilon_D\kappa = \varepsilon$ (and thus $\varepsilon\theta = \varepsilon_D$).

Theorem 3' (i) The property bi-$FP_n$ for algebras is closed under retractions.

(ii) The properties left-$FP_n$, right-$FP_n$, weak bi-$FP_n$ are closed under augmented retractions.
Theorem 3 follows from this, because if a monoid \( C \) is a retract of a monoid \( B \), then the monoid algebra \( KC \) is an augmented retract of \( KB \).

We will also consider connected graded algebras (definitions will be given in §5).

**Theorem 4** If a connected graded algebra is left-\( FP_n \) or right-\( FP_n \) then it is bi-\( FP_n \).

Thus for connected graded algebras, the four properties weak bi-\( FP_n \), left-\( FP_n \), right-\( FP_n \), bi-\( FP_n \) all coincide. We remark that the answer to the following seems to be unknown:

**Question.** For any \( n \), is there a graded algebra of type \( FP_n \) but not of type \( FP_{n+1} \)?

To prove Theorems 1’ and 3’ we first obtain a result concerning what we call retractive pairs (see §2). Theorem 3’ follows directly from this, and Theorem 1’ follows by working with the enveloping algebra \( E = A \otimes_K A^{opp} \) of \( A \) (see §3).

The proof of Theorem 2 is given in §4, and the proof of Theorem 4 is given in §5.

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### 2. Retractive pairs and the Property \( FP_n \)

The following consequence of the Generalised Schanuel Lemma is useful [5, p 193].

**Lemma 1** Let \( M \) be a left \( R \)-module. For \( n \geq 0 \), if

\[
0 \xleftarrow{\partial_{n-1}} M \xleftarrow{\partial_n} P_0 \xleftarrow{} P_1 \cdots \xleftarrow{\partial_{n-1}} P_{n-1}
\]

is a partial free resolution of \( M \) of length \( n-1 \), with \( P_0, \ldots, P_{n-1} \) finitely generated free modules, then \( M \) is of type \( FP_n \) if and only if \( \ker \partial_{n-1} \) is finitely generated.

Suppose we have ring homomorphisms

\[
R \xrightarrow{\rho} S \quad \rho \iota = \text{id}_S.
\]

ie, \( \rho \) is a retraction of \( R \) onto \( S \), with section \( \iota \).

A (left) **retractive pair** consists of a left \( R \)-module \( M \), a left \( S \)-module \( L \), and abelian group homomorphisms

\[
M \xrightarrow{\alpha^+} L \quad \alpha^+ \alpha^- = \text{id}_L,
\]

where \( \alpha^+ \) is an \( R \)-module homomorphism (regarding \( L \) as an \( R \)-module via \( \rho \)), and \( \alpha^- \) is an \( S \)-module homomorphism (regarding \( M \) as an \( S \)-module via \( \iota \)). We will denote such a retractive pair by \( M\alpha L \).

Retractive pairs form a category, where a **mapping**

\[
(\phi, \psi) : M\alpha L \rightarrow M'\beta L'
\]

consists of an \( R \)-module homomorphism

\[
\phi : M \rightarrow M'
\]
and an $S$-module homomorphism
\[ \psi : L \rightarrow L', \]
such that the diagram
\[ \begin{array}{ccc}
M & \overset{\phi}{\longrightarrow} & M' \\
\downarrow{\alpha^+} & & \downarrow{\beta^+} \\
L & \overset{\psi}{\longrightarrow} & L'
\end{array} \]
commutes. It is then easily checked that $\alpha^+(\ker \phi) \subseteq \ker \psi$ and $\alpha^-(\ker \psi) \subseteq \ker \phi$, so by restriction, we get the retractive pair $\ker(\phi, \psi)$:
\[ \begin{array}{ccc}
\ker \phi & \overset{\alpha^+}{\longrightarrow} & \ker \psi \\
\downarrow{\alpha^-} & & \\
\ker \phi & \overset{\beta^+}{\longrightarrow} & \ker \psi
\end{array} \]
Similarly, we get the retractive pair $\text{Im}(\phi, \psi)$:
\[ \begin{array}{ccc}
\text{Im} \phi & \overset{\beta^+}{\longrightarrow} & \text{Im} \psi \\
\downarrow{\beta^-} & & \\
\text{Im} \phi & \overset{\beta^+}{\longrightarrow} & \text{Im} \psi
\end{array} \]

**Proposition 1** Let $M \alpha L$ be a retractive pair. If $M$ is of type $FP_n$ then there is a sequence
\[ 0 \leftarrow M \alpha L \overset{(\partial_0, \delta_0)}{\leftarrow} P_0 \beta_0 F_0 \overset{(\partial_1, \delta_1)}{\leftarrow} P_1 \beta_1 F_1 \leftarrow \cdots \leftarrow P_n \beta_n F_n \]
with $P_i, F_i$ finitely generated free $R$-modules, $S$-modules respectively ($0 \leq i \leq n$), $\text{Im}(\partial_0, \delta_0) = M \alpha L$, and $\text{Im}(\partial_{i+1}, \delta_{i+1}) = \ker(\partial_i, \delta_i)$ ($0 \leq i < n$).

In particular,
\[ 0 \leftarrow L \overset{\delta_0}{\leftarrow} F_0 \overset{\delta_1}{\leftarrow} F_1 \leftarrow \cdots \leftarrow F_n \]
is a partial free resolution of $L$, so $L$ is of type $FP_n$.

**Proof.** Suppose $M$ is of type $FP_0$ (i.e. finitely generated), and let $\{m_e : e \in e\}$ be a finite set of $R$-module generators for $M$. Then $\{\alpha^+(m_e) : e \in e\}$ is a set of $S$-module generators for $L$. Let $P_0$ be the free $R$-module (of rank $|e|\rangle$)
\[ P_0 = (\oplus_{e\in e} R e) \oplus (\oplus_{e\in e} R e'), \]
and let $F_0$ be the free $S$-module (of rank $|e|$)
\[ F_0 = \oplus_{e\in e} S \bar{e}. \]
Then
\[ P_0 \overset{\beta_0}{\leftarrow} F_0 \]
is a retractive pair. We have the surjective $R$-module homomorphism
\[ \partial_0 : P_0 \rightarrow M, \quad \partial_0(e) = \alpha^- \alpha^+(m_e), \delta_0(e') = m_e - \alpha^- \alpha^+(m_e) (e \in e) \]
and the surjective $S$-module homomorphism
\[ \delta_0 : F_0 \rightarrow L, \quad \delta_0(\bar{e}) = \alpha^+(m_e) (e \in e), \]
and it is easily checked that $(\partial_0, \delta_0)$ is a mapping of retractive pairs.
Let $M_1\beta_0L_1 = \text{Ker}(\partial_0, \delta_0)$. By Lemma 1, if $M$ is if type $FP_1$, then $M_1$ is finitely generated. We can then repeat the above procedure to obtain a finitely generated free retractive pair $P_1\beta_1F_1$ and a surjective map

$$P_1\beta_1F_1 \to M_1\beta_0L_1.$$ 

Composing this with the inclusion of $M_1\beta_0L_1$ into $P_0\beta_0F_0$ we obtain a mapping

$$(\partial_1, \delta_1) : P_1\beta_1F_1 \to P_0\beta_0F_0.$$ 

Continuing in this way, after $n + 1$ steps we get the required sequence.

### 3. Bi-Resolutions and Enveloping Algebras

Recall that an $(A, A)$-bimodule $M$ is an abelian group on which $A$ acts on the left and right, with the condition that: $(am)b = a(mb)$ for all $a, b \in A, m \in M$; $km = mk$ for all $m \in M, k \in K$.

The $(A, A)$-bimodule $A \otimes_K A$ (with bi-action given by $a.(u \otimes v).b = au \otimes vb$ for all $a, b, u, v \in A$) is free on the generator $1 \otimes 1$. Thus a direct sum of $r$ copies of $A \otimes_K A$ is a free $(A, A)$-bimodule of rank $r$ $(r$ may be infinite).

An $(A, A)$-bimodule $M$ is said to be of type $bi-FP_n$ if there is a partial resolution

$$0 \leftarrow M \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_n$$

(7)

where $F_0, F_1, \ldots, F_n$ are finitely generated free $(A, A)$-bimodules.

It is obvious that Proposition 1 extends to bimodules.

**Proof of Theorem 3’.** Let $A$, $D$ be as in the paragraph before the statement of Theorem 3’.

(i) Regarding $A$, $D$ as $(A, A)$-, $(D, D)$-bimodules, respectively, $A \xrightarrow{\theta} D$ is a retractive pair.

(ii) Regarding $K$ as a left (respectively, right, bi) $A$-module, and a left (respectively, right, bi) $D$-module, $K \xrightarrow{id} K$ is a retractive pair.

Recall that for a $K$-algebra $A$, there is the opposite algebra $A^{opp}$. This has the same underlying set as $A$, and the same addition and scalar multiplication. The product of two elements $a, b$ (in that order) in $A^{opp}$ is defined to be the product $ba$ in $A$. When regarding an element $a$ of $A$ as an element of $A^{opp}$, we will denote it by $a^{opp}$.

The *enveloping algebra* of $A$ is the tensor product $E = A \otimes_K A^{opp}$, with multiplication defined by

$$(a \otimes b^{opp})(c \otimes d^{opp}) = ac \otimes b^{opp}d^{opp} = ac \otimes db \quad (a, b, c, d \in A).$$

(8)

There is the induced augmentation

$$\varepsilon_E : E \to K \quad \varepsilon_E(a \otimes b^{opp}) = \varepsilon(a) \varepsilon(b) \quad (a, b \in A).$$

If $M$ is an $(A, A)$-bimodule we can regard it as a left $E$-module (denoted $\mathcal{E}(M)$) with $E$-action given by

$$(a \otimes b^{opp})m = amb \quad (a, b \in A, m \in M).$$

Also, if $\phi : M \to M'$ is a bimodule homomorphism, then it can be regarded as a left $E$-module homomorphism $\mathcal{E}(\phi) : \mathcal{E}(M) \to \mathcal{E}(M')$. Then $\mathcal{E}$ is an (exact) functor. This
functor has an inverse $\mathcal{A}$, where for a left $E$-module $N$, $\mathcal{A}(N)$ is $N$ regarded as an
$(A, A)$-bimodule, with left and right actions given by

$$an = (a \otimes 1)n, \quad nb = (1 \otimes b^{opp})n \quad (a, b \in A, n \in N).$$

It is easily shown that $\mathcal{E}(A \otimes_K A)$ is $E$ acting on itself by left multiplication. In other words, $\mathcal{E}(A \otimes_K A)$ is a free left $E$-module of rank 1. Thus, if $F$ is a free $(A, A)$-bimodule
of rank $r$, then $\mathcal{E}(F)$ is a free left $E$-module of rank $r$. Applying $\mathcal{E}$ to a partial resolution
as in (7), we thus see that if $M$ is an $(A, A)$-bimodule of type bi-$FP_n$, then $\mathcal{E}(M)$ is of
type $FP_n$. By considering the inverse functor $\mathcal{A}$, the converse is also true. Thus we have:

**Lemma 2** An $(A, A)$-bimodule $M$ is of type bi-$FP_n$ if and only if $M$, regarded as a left
$E$-module, is of type $FP_n$.

**Proof of Theorem 4'.** Regarding $K$ as an $(A, A)$-bimodule via $\varepsilon$, $\mathcal{E}(K)$ is easily seen to
be $F$ regarded as a left $E$-module via $\varepsilon_E$. Thus, by Lemma 2, $A$ is weak bi-$FP_n$ if and
only if $E$ is left-$FP_n$. Then, since $A$ is an augmented retract of $E$ under the maps

$$E \xrightarrow{\rho} A \quad \rho(a \otimes b^{opp}) = \varepsilon(b)a, \quad \iota(a) = a \otimes 1 \quad (a, b \in A), \quad \rho \iota = \text{id}_A,$$

if $E$ is left-$FP_n$ then so is $A$, by Theorem 3'(ii).

4. **Proof of Theorem 2**

For a group $G$ we define a functor $- \otimes KG$ from the category of left $KG$-modules to the
category of $(KG, KG)$-bimodules as follows. For $M$ a left $KG$-module, $M \otimes KG$ is the
tensor product $M \otimes_K KG$ with bi-$KG$-action given by

$$g \cdot (m \otimes x) \cdot h = gm \otimes gxh \quad (g, h, x \in G, \ m \in M).$$

For a left $KG$-module homomorphism $\alpha : M_1 \rightarrow M_2$ we define $\hat{\alpha} : M_1 \otimes KG \rightarrow M_2 \otimes KG$
to be

$$\alpha \otimes \text{id}_{KG} : M_1 \otimes_K KG \rightarrow M_2 \otimes_K KG,$$

regarded as a $(KG, KG)$-bimodule homomorphism.

**Lemma 3** (i) $K \otimes KG$ is isomorphic to $KG$ (regarded as a $(KG, KG)$-bimodule by left
and right multiplication)

(ii) If $P$ is a free left $KG$-module of rank $r$, then $P \otimes KG$ is a free $(KG, KG)$-bimodule
of rank $r$.

**Proof.** (i) As an abelian group, $K \otimes KG$ is just $K \otimes_K KG$, which is isomorphic to $KG$
by the isomorphism

$$\theta : K \otimes_K KG \rightarrow KG \quad k \otimes x \mapsto kx \quad (k \in K, x \in G).$$

It is easily checked that for $g, h, x \in G, \ k \in K$

$$\theta(g \cdot (k \otimes x) \cdot h) = g \theta(k \otimes x)h.$$

(ii) Since $P$ is the direct sum of $r$ copies of $KG$, it suffices to show that $KG \otimes KG$ is a
free $(KG, KG)$-module of rank 1. The free $(KG, KG)$-bimodule of rank 1 is $KG \otimes_K KG$
with action

$$x(g \otimes h)y = xg \otimes hy \quad (x, y, g, h \in G).$$
As a $K$-module, $KG \otimes_K KG$ is free with basis $g \otimes h$ ($g, h \in G$). For convenience write $g \hat{\otimes} h$ for $g \otimes h$ when considered as an element of $KG \otimes KG$.

Since $KG \otimes_K KG$ is free on $1 \otimes 1$, we get a bi-module homomorphism

$$\beta : KG \otimes_K KG \to KG \otimes_K KG \quad 1 \otimes 1 \mapsto 1 \otimes 1.$$ 

Thus $\beta(g \hat{\otimes} h) = \beta(g)(1 \otimes 1 \cdot h) = g \hat{\otimes} gh$ ($g, h \in G$). Also, we have a $K$-module homomorphism

$$\alpha : KG \otimes_K KG \to KG \otimes_K KG \quad g \hat{\otimes} h \mapsto g \otimes g^{-1}h$$

It is easily checked that $\alpha$ and $\beta$ are mutually inverse (as $K$-maps), so $\beta$ is a $(KG, KG)$-bimodule isomorphism.

To prove Theorem 2, suppose $G$ is left-$FP_n$ over $K$. Then there is a partial free resolution as in (1). Tensoring by $- \otimes_K KG$ is exact, since $KG$ is free as a $K$-module, so we obtain the exact sequence

$$0 \leftarrow K \otimes_K KG \leftarrow P_0 \otimes_K KG \leftarrow P_1 \otimes_K KG \leftarrow \cdots \leftarrow P_n \otimes_K KG.$$ 

Regarding this as a sequence of $(KG, KG)$-bimodules and using Lemma 3, we obtain an exact sequence

$$0 \leftarrow KG \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_n$$

where $F_0, F_1, \ldots, F_n$ are finitely generated free $(KG, KG)$-bimodules, so $G$ is bi-$FP_n$.

### 5. Connected Graded Algebras and Proof of Theorem 4

Suppose $K$ is a field, and that $A$ is a graded $K$-algebra. Thus $A$ is a direct sum $\oplus_{i \geq 0} A_i$ of $K$-modules such that $A_i A_j \subseteq A_{i+j}$ ($0 \leq i, j$). Elements of $A_i$ are said to be of degree $i$. In the context of graded algebras, modules will also be graded. Thus a left $A$-module is a directed sum $M = \oplus_{i \geq 0} M_i$ of $K$-modules such that $A_i M_j \subseteq M_{i+j}$ ($0 \leq i, j$). Right modules and bimodule are defined analogously. A module is concentrated in dimension $n$ if $M_i = 0$ for $i \neq n$. We regard $K$ as a graded module concentrated in degree 0. A mapping $\phi : M \to L$ of left modules consists of a family $\phi_i : M_i \to L_i$ ($i \geq 0$) of $K$-maps such that $\phi_{i+j}(a_i m_j) = a_i \phi_j(m_j)$ ($a_i \in A_i, m_j \in M_j, i, j \geq 0$).

We will assume that $A$ is connected, that is, $A_0$ has basis the identity $1_A$. Then we have the standard augmentation

$$\varepsilon : A \to K \quad 1_A \mapsto 1_K, \quad A_i \to 0 \ (i > 0)$$

with kernel $A^+ = \oplus_{i > 0} A_i$.

The opposite algebra $A^{opp}$ is also a connected graded algebra with the same grading, and so $E = A \otimes_K A^{opp}$ inherits a connected graded algebra structure with grading

$$E_i = \oplus_{p+q=i} A_p \otimes_K A_q$$

and multiplication as in (8).

**Remark** If $M$ is a graded left $A$-module then there are two possible definitions of $FP_n$, according to whether we consider free resolutions

$$0 \leftarrow M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots \leftarrow P_i \leftarrow \cdots$$

where the $P_i$'s are free modules which are graded and the $\partial_i$'s are graded maps (“$FP_n$ in the graded sense”), or whether we just consider ungraded resolutions (“$FP_n$ in the ungraded sense”). Clearly, if $M$ is $FP_n$ in the graded sense then it is $FP_n$ in the ungraded sense. The converse is also true. For if $M$ is $FP_n$ in the ungraded sense then
applying $K \otimes_A -$ to an ungraded resolution (9) with $P_0, P_1, \ldots, P_n$ finitely generated, we see that $\text{Tor}^A_i(K, M)$ is a finitely generated $K$-module for $0 \leq i \leq n$. Now associated with any graded module is a canonical (minimal) graded resolution [7], where the $i$th term is $A \otimes_K \text{Tor}^A_i(K, M)$, so $M$ is of type $FP_n$ in the graded sense. Similar remarks also hold for right modules, and bimodules.

**Proposition 2** Let $M$ be an $(A, A)$-bimodule which as a right $A$-module is free. If the left $A$-module $M \otimes A K$ is of type $FP_n$, then the bimodule $M$ is of type $FP_n$. (An analogous result holds if we interchange left and right.)

Taking $M =_A A_A$ in Proposition 2 we obtain Theorem 4.

**Proof.** Consider the left $E$-module $\mathcal{E}(M)$. By standard theory [7], there is a unique (up to isomorphism) minimal resolution

$$0 \leftarrow \mathcal{E}(M) \leftarrow \partial_0 \longrightarrow P_0 \leftarrow \partial_1 \longrightarrow P_1 \leftarrow \ldots$$

where $P_i$ is a free $E$-module, and $\text{Ker} \partial_i \subseteq E^+ P_i$. Applying the functor $A$, we then get a bi-resolution

$$0 \leftarrow M \leftarrow \overline{\partial}_0 \longrightarrow \overline{P}_0 \leftarrow \overline{\partial}_1 \longrightarrow \overline{P}_1 \leftarrow \ldots$$

where $\overline{P}_i$ is a free $(A, A)$-bimodule and $\text{Ker} \overline{\partial}_i \subseteq A(E^+ P_i) = A^+ \overline{P}_i + \overline{P}_i A^+$. Note that since a free $(A, A)$-bimodule is also free as a right $A$-module, the above sequence is also a right free resolution of $M$ regarded as a right $A$-module.

Now applying $- \otimes_A K$ we obtain

$$0 \leftarrow M \otimes_A K \leftarrow \overline{\partial}_0 \otimes 1 \longrightarrow \overline{P}_0 \otimes_A K \leftarrow \overline{\partial}_1 \otimes 1 \longrightarrow \overline{P}_1 \otimes_A K \leftarrow \ldots$$

Then for $i \geq 1$

$$\frac{\text{Ker} \overline{\partial}_i \otimes 1}{\text{Im} \overline{\partial}_{i+1} \otimes 1} = \text{Tor}^A_i(M, K) = 0$$

since $M$ is free as a right $A$-module. So the sequence is exact. Since $\text{Ker}(\overline{\partial}_1 \otimes 1) \subseteq A^+ (\overline{P}_1 \otimes 1)$ the sequence is, in fact, the minimal resolution of $M \otimes_A K$. Thus if $M \otimes_A K$ is of type $FP_n$, then the left $A$-modules $\overline{P}_j \otimes_A K$ ($0 \leq j \leq n$) are finitely generated. Thus the $(A, A)$-bimodules

$$(\overline{P}_j \otimes_A K) \otimes_K A \cong \overline{P}_j \quad (0 \leq j \leq n)$$

are finitely generated, so the bimodule $M$ is of type $FP_n$.

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Address of author:
Department of Mathematics
University of Glasgow
15 University Gardens
Hillhead
Glasgow
G12 8QW

e-mail address:
sjp@maths.gla.ac.uk