Geometric Spanners with Small Chromatic Number

Prosenjit Bose\textsuperscript{1}, Paz Carmi\textsuperscript{1}, Mathieu Couture\textsuperscript{1}, Anil Maheshwari\textsuperscript{1}, Michiel Smid\textsuperscript{1}, and Norbert Zeh\textsuperscript{2}

\textsuperscript{1} School of Computer Science, Carleton University
\textsuperscript{2} Faculty of Computer Science, Dalhousie University

Abstract. Given an integer $k \geq 2$, we consider the problem of computing the smallest real number $t(k)$ such that for each set $P$ of points in the plane, there exists a $t(k)$-spanner for $P$ that has chromatic number at most $k$. We prove that $t(2) = 3$, $t(3) = 2$, $t(4) = \sqrt{2}$, and give upper and lower bounds on $t(k)$ for $k > 4$. We also show that for any $\epsilon > 0$, there exists a $(1 + \epsilon)t(k)$-spanner for $P$ that has $O(|P|)$ edges and chromatic number at most $k$. Finally, we consider an on-line variant of the problem where the points of $P$ are given one after another, and the color of a point must be assigned at the moment the point is given. In this setting, we prove that $t(2) = 3$, $t(3) = 1 + \sqrt{3}$, $t(4) = 1 + \sqrt{2}$, and give upper and lower bounds on $t(k)$ for $k > 4$.

1 Introduction

Let $P$ be a set of $n$ points in the plane. A geometric graph with vertex set $P$ is an undirected graph whose edges are line segments that are weighted by their Euclidean length. For a real number $t \geq 1$, such a graph $G$ is called a $t$-spanner if the weight of the shortest path in $G$ between any two vertices $p$ and $q$ does not exceed $t|pq|$, where $|pq|$ is the Euclidean distance between $p$ and $q$. The smallest $t$ having this property is called the stretch factor of the graph $G$. Thus, a graph with stretch factor $t$ approximates the $\binom{n}{2}$ distances between the points in $P$ within a factor of $t$. The problem of constructing $t$-spanners with $O(n)$ edges for any given point set has been studied intensively; see the book by Narasimhan and Smid \cite{6} for an overview.

In this paper, we consider the problem of computing $t$-spanners whose chromatic number is at most $k$, for some given value of $k$. The goal is to minimize the value of $t$ over all finite sets $P$ of points in the plane. We call a spanner whose chromatic number is at most $k$ a $k$-chromatic spanner.

Problem 1. Given an integer $k \geq 2$, let $t(k)$ be the infimum of all real numbers $t$ with the property that for every finite set $P$ of points in the plane, a $k$-chromatic $t$-spanner for $P$ exists. Determine the value of $t(k)$.

\textsuperscript{*} Research partially supported by HPCVL, NSERC, MRI, CFI, and MITACS.
Observe that in the definition of \( t(k) \), there is no requirement on the number of edges of the chromatic spanner. This is not a restriction, because, as shown by Gudmundsson et al. [5], any \( t \)-spanner for \( P \) contains a subgraph with \( O(n) \) edges which is a \(((1+\epsilon)t)\)-spanner for \( P \).

We show how to obtain a 2-chromatic 3-spanner for any point set \( P \), thus showing that \( t(2) \leq 3 \). We also give an example of a point set \( P \) such that any 2-chromatic graph with vertex set \( P \) has stretch factor at least three. Thus, we have \( t(2) = 3 \).

Next, we show how to compute a 3-chromatic 2-spanner of any point set \( P \), thereby proving that \( t(3) \leq 2 \). We also show, by means of an example, that \( t(3) \geq 2 \). Thus, we obtain that \( t(3) = 2 \). For \( k = 4 \), we show how to compute a 4-chromatic \( \sqrt{2} \)-spanner of any point set \( P \); thus \( t(4) \leq \sqrt{2} \). Again by means of an example, we also show that \( t(4) \geq \sqrt{2} \). Therefore, we have \( t(4) = \sqrt{2} \).

For \( k > 4 \), we are not able to obtain the exact value of \( t(k) \). Inspired by the ordered \( \Theta \)-graph of Bose et al. [2], we show that \( t(k) \leq 1 + 2 \sin \frac{\pi}{2(k-1)} \). We also show that the vertex set of the regular \((k+1)\)-gon gives \( t(k) \geq 1/\cos \frac{\pi}{k+1} \).

In the second part of the paper, we consider an on-line variant of the problem where the points of \( P \) are given one after another, and the color of a point must be assigned at the moment when the point is given; thus, later on, the color of a point cannot be changed. This makes the problem more difficult. Consequently, the bounds are higher, but still tight for \( k = 2, 3, 4 \). All our bounds are summarized in Table 1.

**Problem 2.** Given an integer \( k \geq 2 \), let \( t'(k) \) be the infimum of all real numbers \( t \) with the property that for every finite set \( P \) of points in the plane, which is given on-line, a \( k \)-chromatic \( t \)-spanner for \( P \) exists. Determine the value of \( t'(k) \).

A simple variant of the ordered \( \Theta \)-graph shows that \( t'(k) \leq 1 + 2 \sin(\pi/k) \). Thus, we have \( t'(2) \leq 3, t'(3) \leq 1 + \sqrt{3} \) and \( t'(4) \leq 1 + \sqrt{2} \). Since \( t'(2) \geq t(2) = 3 \), it follows that \( t'(2) = 3 \). We also give examples showing that \( t'(3) \geq 1 + \sqrt{3} \) and \( t'(4) \geq 1 + \sqrt{2} \). We finally show that, for \( k \geq 5 \), \( t'(k) \geq 1/\cos \frac{\pi}{k} \).

The rest of this paper is organized as follows: in Section 2 we define the \( t \)-ellipse property and show its relationship to our problem. In Section 3 we give upper and lower bounds for the off-line problem (Problem 1). In Section 4 we give upper and lower bounds for the on-line problem (Problem 2). We conclude in Section 5. In Table 1 we summarize our results. We now motivate our work.

**Motivation:** In a recent paper, Raman and Chebrolu [7] proposed a new protocol, called 2P, allowing to address rural Internet connectivity in a low-cost manner using off-the-shelf 802.11 hardware. Since their infrastructure uses several directional antennae at one node rather than one single omnidirectional antenna, simultaneous communications are possible at one node. However, due to restrictions inherent in the 802.11 standard, backbone nodes have to communicate with each other using a single channel. While simultaneous transmissions and simultaneous receptions are possible, it is not physically possible for one node to both transmit and receive at the same time. Therefore, backbone nodes