The algebraic and Hamiltonian structure of the
dispersionless Benney and Toda hierarchies

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Abstract
The algebraic and Hamiltonian structures of the multicomponent dispersionless Benney
and Toda hierarchies are studied. This is achieved by using a modified set of variables
for which there is a symmetry between the basic fields. This symmetry enables formulae
normally given implicitly in terms of residues, such as conserved charges and fluxes, to
be calculated explicitly. As a corollary of these results the equivalence of the Benney and
Toda hierarchies is established. It is further shown that such quantities may be expressed
in terms of generalized hypergeometric functions, the simplest example involving Leg-
endre polynomials. These results are then extended to systems derived from a rational
Lax function and a logarithmic function. Various reductions are also studied.

1. Introduction

One of the most studied integrable systems is the Toda lattice equation

$$\frac{\partial^2 \rho_n}{\partial t \partial \tilde{t}} = e^{\rho_{n+1}} - 2e^{\rho_n} + e^{\rho_{n-1}},$$

(1)
together with its various generalisations. Besides the inherent interest in such integrable
systems, these Toda systems have a wide number of applications, form sorting theory to
quantum field theory and differential geometry. The one dimensional Toda chain

$$\frac{\partial^2 \rho_n}{\partial t^2} = e^{\rho_{n+1}} - 2e^{\rho_n} + e^{\rho_{n-1}},$$

(1)
may be written as a coupled system of difference equations

\[ S_{n,t} = (\Delta - 1)P_n, \]
\[ P_{n,t} = P(1 - \Delta^{-1})S_n, \]  

(2)

where \( \rho_n = \log P_n \) and \( \Delta \) is the shift operator defined by \( \Delta^\pm \rho_n = \rho_{n\pm 1} \), and the properties of such systems have been extensively studied, notably by Kuperscheidt [1].

This paper concerns a particular limit of these systems, the continuum or long wave limit. The idea is to look for solutions whose natural length scale is large compared with the lattice spacing, or, alternatively, one lets the lattice spacing tend to zero. In the limit the discrete label \( n \) becomes a continuous variable \( x \) and the operator \( \Delta^1 - 2 + \Delta^{-1} \) degenerates into a second derivative. Thus under this limit the two dimensional Toda lattice (1) becomes

\[ \frac{\partial^2 \rho}{\partial t \partial \tilde{t}} = \frac{\partial^2 e^\rho}{\partial x^2}. \]  

(3)

One intriguing property of these systems is that the notion of integrability is preserved in this limit, so (3) is in fact a multidimensional integrable system. In this paper this limit will be called the dispersionless limit, as mathematically it is the same as the limit which send the KdV equation into the Monge equation

\[ u_t = uu_x + \kappa u_{xxx}, \]
\[ \kappa \rightarrow 0 \]
\[ u_t = uu_x. \]

Many of the geometrical properties of these integrable systems have been derived in [3]. One of the more recent motivations for the study of such integrable systems has come from their role in topological field theory [3].

Under this limit the difference system (4) becomes

\[ S_t = P_x, \]
\[ P_t = PS_x. \]  

(4)

and in an earlier paper [4] this system was studied. This was achieved by transforming to a set of modified variables defined by

\[ S = u + v, \]
\[ P = uv. \]

In these new variables the Toda system (4) becomes symmetric:

\[ u_t = uv_x, \]
\[ v_t = uu_x. \]  

(5)

and various properties of this system, such as its associated hierarchy and Hamiltonian structure, were studied. In this paper these result are extended to the multicomponent Toda hierarchy.
The multicomponent Toda hierarchy is defined in terms of a Lax function

\[ \mathcal{L}(p) = p^{N-1} + \sum_{i=-1}^{N-2} p^i S_i(x,t), \quad t = \{t_1, t_2, \ldots \} \]

by the Lax equation

\[ \frac{\partial \mathcal{L}}{\partial t_n} = \{(\mathcal{L}^{\varpi_{n}})_+, \mathcal{L}\}. \]  

(6)

Here the bracket is defined by the formula

\[ \{f, g\} = p \frac{\partial f}{\partial p} \frac{\partial g}{\partial x} - p \frac{\partial f}{\partial x} \frac{\partial g}{\partial p}. \]

and, as usual, \((\mathcal{O})_+\) denotes the projection of the function \(\mathcal{O}\) onto non-negative powers of \(p\). More general Lax functions will be considered in section 6 and 7.

**Example 1.1** One obtains from the Lax equation (6) with \(N = 2, n = 1\) equation (4) and, with \(N = 3, n = 1\) the system (where \(\mathcal{L} = p^2 + Sp + P + Qp^{-1}\))

\[ S_t = P_x - \frac{1}{2} SS_x, \]
\[ P_t = Q_x, \]
\[ Q_t = \frac{1}{2} QS_x. \]

(7)

The modified variables alluded to above are defined by the factorization of the Lax function

\[ \mathcal{L} = \frac{1}{p} \prod_{i=1}^{N} [p + u_i] \]  

(8)

This transformation may be thought of as a dispersionless Miura map, and their use in the study of the dispersionless Toda hierarchy is due to Kuperschmidt [1]. Recall that the transformation from the KdV to the modified KdV equation comes from a similar factorization of the Lax operator for the KdV equation. These new variables \(u_i\) will be called the modified variables.

There are a number of advantages in moving to these modified variables. Firstly, it puts all the fields on an egalitarian footing and any formula (such as those for the conserved charges and fluxes and Hamiltonian structures) will be symmetric functions of these variables (explicitly, such formulae are invariant under the map \(u_i \mapsto u_{\sigma(i)}\) where \(\sigma\) is an element of the permutation group). Secondly, it enables such functions to be calculated explicitly, using simple combinatorial arguments. In fact, rather than the calculations being of the form

\[ (1 + \text{sum of N terms})^\alpha \]

the calculations are of the form

\[ \sum_N (1 + \text{single term})^\alpha \]
and the symmetry between the modified variables means that only a single calculation has to be performed. The existence of such quantities and some of their properties follows from the general theory of these Lax equations (see, for example [3], and the references therein). However, the explicit forms for these quantities are rarely given except, perhaps, for the lowest members of the hierarchy. The modified variables enables one to perform the general calculations with an arbitrary numbers of fields.

One further property of these modified variables is best illustrated by means of an example.

**Example 1.2** In these modified variables equation (7) becomes

\[
\begin{align*}
  u_t &= \frac{1}{2} u(-u_x + v_x + w_x), \\
  v_t &= \frac{1}{2} v(+u_x - v_x + w_x), \\
  w_t &= \frac{1}{2} w(+u_x + v_x - w_x).
\end{align*}
\]

(9)

where \( S = u + v + w \), \( P = uv + vw + wu \) and \( Q = uvw \).

It is clear from these equations that one possible reduction of this system comes from the constraint \( v = w \). In the original variables this constraint is

\[
4P^3 + 27Q^2 - 18PQS - P^2S^2 + 4QS^3 = 0,
\]

this being the condition for the cubic equation \( \mathcal{L}(p) = 0 \) to have a double root. In the extreme case where all the fields are equal one obtains, after some rescalings, the dispersionless KdV hierarchy

\[
u_{tn} = u^n u_x.
\]

Thus reductions are far easier to study using these modified variables.

Such reductions are studied in section 5.

This Toda hierarchy is related to the Benney moment equations [3] which take the form

\[
\frac{\partial \hat{S}_n}{\partial t} = \frac{\partial \hat{S}_{n+1}}{\partial x} + n\hat{S}_n \frac{\partial \hat{S}_0}{\partial x}, \quad n = 0, 1, \ldots.
\]

The corresponding Lax function for a truncation of these equations at finite \( N \) is given by

\[
\mathcal{L}_B = p + \sum_{n=0}^{N-1} \frac{\hat{S}_n(x, t)}{p^n}.
\]

and the corresponding Lax equation is

\[
\mathcal{L}_{B,t} = \{(\mathcal{L}_B)_+, \mathcal{L}_B\}.
\]
Example 1.3 For $N = 2$ the Benney system coincides with the dispersionless Toda equation (4). For $N = 3$ the Benney equations are (where $\mathcal{L}_B = p + \dot{S} + \dot{P} p^{-1} + \dot{Q} p^{-2}$)

\[
\begin{align*}
\dot{S}_t &= \dot{P}_x, \\
\dot{P}_t &= \dot{P} \dot{S}_x + \dot{Q}_x, \\
\dot{Q}_t &= 2\dot{Q} \dot{S}_x.
\end{align*}
\]

(10)

These equations possess an infinite number of conservation laws of the form

\[
\dot{Q}^{(n)}_t = \dot{F}^{(n)}_x
\]

Using the equations of motion one finds that

\[
\begin{align*}
\dot{Q}^{(n)}_S &= \dot{F}^{(n)}_P, \\
\dot{P} \dot{Q}^{(n)}_P + 2\dot{Q} \dot{Q}^{(n)}_Q &= \dot{F}^{(n)}_S, \\
\dot{Q}^{(n)}_P &= \dot{F}^{(n)}_Q,
\end{align*}
\]

and so the charge and flux may be written in terms of a potential $\hat{H}^{(n)}$ via

\[
Q^{(n)} = \frac{\partial H^{(n)}}{\partial P}, \quad P^{(n)} = \frac{\partial H^{(n)}}{\partial S},
\]

the first few being

\[
\begin{align*}
\hat{H}^{(0)} &= \dot{P}, \\
\hat{H}^{(1)} &= 2\dot{Q} + 2\dot{P} \dot{Q}, \\
\hat{H}^{(2)} &= 3\dot{P}^2 + 6\dot{Q} \dot{S}, \\
\hat{H}^{(3)} &= 12\dot{P} \dot{Q} + 12\dot{P}^2 \dot{S} + 12\dot{Q} \dot{S}^2 + 4\dot{P} \dot{S}^3.
\end{align*}
\]

With these one obtains equations satisfied by this potential, namely

\[
\frac{\partial^2 \hat{H}^{(n)}}{\partial P^2} = \frac{\partial^2 \hat{H}^{(n)}}{\partial S \partial Q},
\]

\[
\frac{\partial^2 \hat{H}^{(n)}}{\partial S^2} = \dot{P} \frac{\partial^2 \hat{H}^{(n)}}{\partial P^2} + 2\dot{Q} \frac{\partial^2 \hat{H}^{(n)}}{\partial P \partial Q}.
\]

It is also possible to derive many other equations satisfied by this potential, for example,

\[
\dot{Q} \frac{\partial^2 \hat{H}^{(n)}}{\partial Q^2} = \frac{1}{2} \frac{\partial^2 \hat{H}^{(n)}}{\partial S \partial P} - \frac{1}{2} \frac{\partial^2 (\dot{P} \hat{H}^{(n)})}{\partial P \partial Q}.
\]

Notice that if $H^{(n)}$ is a solution of the above equations, so is

\[
H^{(n-1)} = \frac{1}{n+1} \frac{\partial H^{(n)}}{\partial S}.
\]
This implies that all lower conserved densities may be obtained from the higher ones by
differentiation with respect to $\hat{S}$. A similar observation for the KdV conserved quantities
has been noted by Gerald Watts [7]. There is an extremely simple formula for polynomial
solutions for the potential $H^{(n)}$. It is simply given by the coefficient of $p^{-1}$ in the expansion of

$$(p + \hat{S} + \frac{\hat{P}}{p} + \frac{\hat{Q}}{p^2})^{n+1}$$

i.e. as the integral

$$\hat{H}^{(n)} = \frac{1}{2\pi i} \oint \mathcal{L}_B^{n+1} dp .$$

These results generalize to arbitrary $N$.

In terms of the modified variables (where $\hat{S} = \hat{u} + \hat{v} + \hat{w}$ etc.) this becomes

$$\begin{align*}
\hat{u}_t &= \hat{u}(\hat{v}_x + \hat{w}_x), \\
\hat{v}_t &= \hat{v}(\hat{w}_x + \hat{u}_x), \\
\hat{w}_t &= \hat{w}(\hat{u}_x + \hat{v}_x),
\end{align*}$$

(11)

and the equivalence between the Toda and Benney systems is given by the change of
variables

$$\hat{u} = vw, \quad \hat{v} = wu, \quad \hat{w} = uv .$$

With such a change the Benney system (11) transforms into

$$\begin{align*}
u_t &= u(vw)_x, \\
v_t &= v(wu)_x, \\
w_t &= w(uv)_x ,
\end{align*}$$

which is the $N = 3, n = 2$ flow of the Toda system (11). Without use of modified
variables such an equivalence is less clear.

Such changes of variable are discussed in greater depth in section 5. This will show the
equivalence between the Toda and Benney hierarchies. An alternative way of showing
this equivalence is given in section 6.

Throughout this paper the binomial coefficients $\binom{a}{b}$ should be interpreted in terms
of $\Gamma$-functions, i.e.

$$\binom{a}{b} = \frac{\Gamma(a + 1)}{\Gamma(a - b + 1)\Gamma(b + 1)} .$$
This enables the binomial coefficients to be defined for fractional values of $a$ and $b$. Such coefficients may also be defined for negative values if one manipulates the formula formally, using the basic definition $\Gamma(z + 1) = z\Gamma(z)$. For example
\[
\begin{align*}
\binom{-2}{+2} &= \frac{\Gamma(-1)}{\Gamma(-3)\Gamma(3)}, \\
&= \frac{(-2)\Gamma(-2)}{\Gamma(-3)\Gamma(3)}, \\
&= \frac{(-2)(-3)\Gamma(-3)}{(+2)(+1)\Gamma(-3)} = 3.
\end{align*}
\]

For notational simplicity binomial coefficients have been used, rather than the more correct $\Gamma$-function notation.

## 2. The multicomponent Toda hierarchy

It follows from the general theory behind such Lax equations \cite{4, 5} that the quantities

\[ Q^{(n)} = \frac{1}{2\pi i} \oint \frac{L_{N-1}}{p} dp \]  \hspace{1cm} (12)

are conserved with respect to the evolutions defined by the Lax equation (3).

**Proposition 2.1**

The conserved charges defined by (12) are given by the formula

\[ Q^{(n)} = \sum_{\{r_i; \sum_{i=1}^{N} r_i = n\}} \left\{ \prod_{i=1}^{N} \binom{n}{N-1}_{r_i} u_i^{r_i} \right\}. \hspace{1cm} (13) \]

**Proof**

It follows from (8) that

\[ L_{N-1}^{n} = p^n \prod_{i=1}^{N} \left( 1 + \frac{u_i}{p} \right)^{n/r_i} \]

and so, using the binomial expansion,

\[ L_{N-1}^{n} = p^n \prod_{i=1}^{N} \sum_{r_i = 0}^{\infty} \binom{n}{r_i} u_i^{r_i} p^{-r_i}. \]

Using the residue theorem the integral formula (12) may be easily evaluated,

\[ Q^{(n)} = \text{coefficient of } p^{-n} \text{ in } \prod_{i=1}^{N} \sum_{r_i = 0}^{\infty} \binom{n}{r_i} u_i^{r_i} p^{-r_i}, \]

\[ = \sum_{\{r_i; \sum_{i=1}^{N} r_i = n\}} \left\{ \prod_{i=1}^{N} \binom{n}{N-1}_{r_i} u_i^{r_i} \right\}. \]
Hence the result.

Example 2.2 Let \( N = 2 \). Then

\[
Q^{(n)} = \sum_{r+s=n} \binom{n}{r} \binom{n}{s} u^r v^s,
\]

\[
= \sum_{r=0}^{n} \binom{n}{r}^2 u^r v^{n-r}.
\]

This was calculated, using a different approach in [4].

These charges are conserved with respect to all the times, i.e.

\[
\frac{\partial Q^{(m)}}{\partial t_n} = \frac{\partial \Delta^{(m,n)}}{\partial x}
\]

for some function \( \Delta^{(m,n)} \). Before this function is calculated it is first necessary to find the explicit form of the evolution equations for the fields \( u_i \).

**Theorem 2.3**

The Lax equation (6) implies the following evolution equations for the fields

\[
u_{i,t_n} = A_i^{(n)} u_{i,x} + \sum_{j \neq i} u_j B_{ij}^{(n)} u_{j,x}
\]

where

\[
A_i^{(n)} = \left( \frac{n}{N-1} - 1 \right) \sum_{\{r_j : \sum_{j=1}^N r_j = n\}} \left[ \prod_{k=1 \atop k \neq i}^{N} \left( \frac{n}{r_k} \right) u_k^{r_k} \right] \left( \frac{n}{r_i} - 2 \right) u_i^{r_i}
\]

and

\[
B_{ij}^{(n)} = \frac{n}{N-1} \sum_{\{r_j : \sum_{j=1}^N r_j = n-1\}} \left[ \prod_{k=1 \atop k \neq i,j}^{N} \left( \frac{n}{r_k} \right) u_k^{r_k} \right] \left( \frac{n}{r_i} - 1 \right) u_i^{r_i} \left( \frac{n}{r_j} - 1 \right) u_j^{r_j}.
\]

**Proof**

Since

\[
\mathcal{L} = \frac{1}{p} \prod_{i=1}^{N} (p + u_i)
\]

we have, on differentiating,
\[ \frac{\partial \mathcal{L}}{\partial n} = \frac{1}{p} \sum_{j=0}^{N} \left[ \prod_{i=1, i \neq j}^{N} (p + u_i) \right] u_{j,t,n}, \]

\[ = \mathcal{L} \sum_{i=1}^{N} \frac{u_{i,t,n}}{p + u_i} \]

and similarly

\[ \frac{\partial \mathcal{L}}{\partial x} = \mathcal{L} \sum_{i=1}^{N} \frac{u_{i,x}}{p + u_i}, \]

\[ \frac{\partial \mathcal{L}}{\partial p} = \mathcal{L} \left\{ -\frac{1}{p} + \sum_{i=1}^{N} \frac{1}{p + u_i} \right\}. \]

The first of these derivatives implies that

\[ u_{i,t,n} = \text{Res}_{p=-u_i} [\mathcal{L}^{-1} \mathcal{L}_{t,n}], \]

\[ = \text{Res}_{p=-u_i} [\mathcal{L}^{-1} \{\mathcal{M}, \mathcal{L}\}], \]

where, for convenience, we have defined \( \mathcal{M} \) to be the quantity \( \mathcal{M} = (\mathcal{L}^{n_1})_{+} \). The explicit form of \( \mathcal{M} \) is easy to calculate.

\[ \mathcal{L}^{n_1} = p^n \prod_{i=1}^{N} \sum_{r_i=0}^{\infty} \left( \frac{n}{N-1} \right) u_i^{r_i} p^{-r_i}, \]

\[ = p^n \sum_{t=0}^{\infty} \sum_{\{r_i : \sum_{i=1}^{N} r_i = t\}} \left[ \prod_{i=1}^{N} \left( \frac{n}{N-1} \right) u_i^{r_i} \right] \frac{1}{p^t}. \]

So

\[ \mathcal{M} = (\mathcal{L}^{n_1})_{+}, \]

\[ = p^n \sum_{t=0}^{\infty} \sum_{\{r_i : \sum_{i=1}^{N} r_i = t\}} \left[ \prod_{i=1}^{N} \left( \frac{n}{N-1} \right) u_i^{r_i} \right] \frac{1}{p^t}. \]

The derivatives of \( \mathcal{M} \) are more complicated, but are easily evaluated. Using these we obtain

\[ u_{i,t} = \text{Res}_{p=-u_i} \sum_{t=0}^{n} p^{n-1} \sum_{\{r_i, \sum_{i=1}^{N} r_i = t\}} \left[ \prod_{i=1}^{N} \left( \frac{n}{N-1} \right) u_i^{r_i} \right] \left[ \sum_{i=1}^{N} (n-t) u_{i,x} \frac{1}{p + u_i} \right] \left\{ 1 - \sum_{k=1}^{N} \frac{1}{p + u_k} \right\} \]

and hence

\[ u_{i,t,n} = A^{(n)}_i u_{i,x} + \sum_{j \neq i} B^{(n)}_{ij} u_{j,x} \]

where
\[ A_i^{(n)} = \sum_{t=0}^{n} (-1)^{n-t} \sum_{\{r_j : \sum_{j=1}^{N} r_j = t\}} \left[ \prod_{k=1}^{N} \left( \frac{n}{r_k} \right) u_k^{r_k} \right] \left( \frac{n}{r_i} \right) (n-t) u_i^{n+r_i-t} , \]

\[ B_{ij}^{(n)} = \sum_{t=0}^{n} (-1)^{n-t} \sum_{\{r_j : \sum_{j=1}^{N} r_j = t\}} \left[ \prod_{k=1}^{N} \left( \frac{n}{r_k} \right) u_k^{r_k} \right] \left( \frac{n}{r_i} \right) \left( \frac{n}{r_j} \right) u_i^{n+r_i-t} u_j^{r_j-1} . \]

To obtain the explicit forms (15) and (16) of these one has to perform a complicated resummation. In the simplest \((N = 2)\) case this amounts to using the identity

\[ \sum_{t=0}^{n} (-1)^{n-t} \sum_{r+s=t} \left( \begin{array}{c} n \\ r \end{array} \right) \left( \begin{array}{c} n \\ s \end{array} \right) s u^{n+r-t} v^{s-1} = n \sum_{r+s=n-1} \left( \begin{array}{c} n-1 \\ r \end{array} \right) \left( \begin{array}{c} n-1 \\ s \end{array} \right) u^r v^s , \]

the proof of which has used various binomial identities.

\[ \square \]

**Example 2.4** For general \(N\) the first \((n = 1)\) flow is

\[ u_{i,t} = -u_i u_{i,x} + \frac{u_i}{N-1} \sum_{j=1}^{N} u_{j,x} \]

and the \((N-1)^{th}\) flow is:

\[ u_{i,t} = u_i \left( \prod_{j \neq i} u_j \right) x \]

For \(N = 2\) these are the same.

**Example 2.5** With \(N = 3\) and fields \(u, v\) and \(w\) the above formulae give:

\[ u_t = A u_x + u B v_x + u C w_x \]

and cyclically, where

\[ A_n = \frac{n}{2} - 1 \sum_{p+q+r=n} \left( \begin{array}{c} n \\ 2 \end{array} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) w^p v^q w^r , \]

\[ B_n = \frac{n}{2} \sum_{p+q+r=n-1} \left( \begin{array}{c} n \\ 2 \end{array} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) w^p v^q w^r , \]

\[ C_n = \frac{n}{2} \sum_{p+q+r=n-1} \left( \begin{array}{c} n \\ 2 \end{array} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) w^p v^q w^r . \]

**Example 2.6** A similar calculation may be performed for the Benney hierarchy. In terms of the modified variables the first flow is

\[ \hat{u}_{i,t} = \hat{u}_i \sum_{j \neq i} \hat{u}_{i,x} , \]
this being the multicomponent generalisation of (11).

Having found the explicit form of the evolution equations one can now derive the flux
\( \Delta^{(m,n)} \) corresponding to the conserved charge (12,13).

**Proposition 2.7**

\[
\Delta^{(m,n)} = \frac{N - 1}{m + n} \sum_{i=1}^{N} F_{u_i}^{(m)} \sum_{j \neq i} F_{u_j}^{(n)}
\]

where

\[
F^{(m)} = \Delta^{(m,1)},
\]

\[
= \sum_{\{r_k : \sum_{k=1}^{N} r_k = m+1\}} \left[ \left( \frac{m}{N-1} \right) u_k^r \right].
\]

**Proof**

It follows from the conservation law (14) that

\[
\sum_{i=1}^{N} Q_{u_i}^{(n)} u_{i,t} = \sum_{i=1}^{N} \Delta_{u_i}^{(n,m)} u_{i,x}
\]

and using the evolution equation this becomes, on equating the various coefficients of the derivatives,

\[
\sum_{i=1}^{N} u_i \Delta_{u_i}^{(m,n)} = \sum_{i=1}^{N} u_i Q_{u_i}^{(n)} [A_i^{(m)} + \sum_{j \neq i} B_{ij}^{(m)} u_j].
\]

It is clear, by power counting, that \( \Delta^{(m,n)} \) must be a homogeneous function of degree \((m + n)\) in the fields, so by Euler’s theorem

\[
\Delta^{(m,n)} = \frac{1}{m + n} \sum_{i=1}^{N} u_i Q_{u_i}^{(n)} [A_i^{(m)} + \sum_{j \neq i} B_{ij}^{(m)} u_j].
\]

Since the \( A_i \) and \( B_{ij} \) are known it is now just a matter of substituting these into the above formula and simplifying the results. In fact

\[
[A_i^{(m)} + \sum_{j \neq i} B_{ij}^{(m)} u_j] = \sum_{\{r_k : \sum_{k=1}^{N} r_k = m\}} m \left[ \prod_{k=1}^{N} \left( \frac{m}{N-1} \right) u_k^r \right] \left( \frac{m}{N-1} - 1 \right) u_i^r
\]

the proof of which, once again, involves the use of various binomial identities. Let \( F^{(m)} = \Delta^{(m,1)} \), i.e. the flux corresponding to the first time flow. The above formulae simplify to give

\[
F^{(m)} = \sum_{\{r_k : \sum_{k=1}^{N} r_k = m+1\}} \left[ \left( \frac{m}{N-1} \right) u_k^r \right],
\]
so
\[
F^{(m)}_{u_i} = \frac{1}{N-1} \left[ \sum_{\{r_k, k \geq 1 \}} m \left[ \prod_{k=1}^{N} \left( \frac{m}{r_k} \right) u_k^{r_k} \right] \left( \frac{m}{N-1} - 1 \right) u_i^{r_i} \right],
\]
\[
= \frac{1}{N-1} \left[ A_i^{(m)} + \sum_{j \neq i} B_{ij}^{(m)} \right]
\]
by using equations (17). Thus
\[
\Delta^{(m,n)} = \frac{N-1}{m+n} \sum_{i=1}^{N} u_i Q^{(n)} F^{(m)}_{u_i}
\]
The first order flow for general \( N \) has been calculated in an example and it follows from this that
\[
F^{(n)}_{u_i} = \frac{2-N}{N-1} u_i Q^{(n)} + \frac{1}{N-1} \sum_{j \neq i} u_j Q^{(n)}.
\]
Inverting this gives
\[
u_i Q^{(n)} = \sum_{j \neq i} F^{(n)}_{u_j}
\]
and so
\[
\Delta^{(m,n)} = \frac{N-1}{m+n} \sum_{i=1}^{N} F^{(m)}_{u_i} \sum_{j \neq i} F^{(n)}_{u_j},
\]
\[
= \frac{1}{m+n} \sum_{i,j} F^{(m)}_{u_i} G_{ij} F^{(n)}_{u_j}
\]
where \( G_{ij} \) is the matrix with zero entries along the diagonal and \( N - 1 \) everywhere else.

Note that \( \Delta^{(m,n)} = \Delta^{(n,m)} \), i.e.
\[
\frac{\partial Q^{(n)}}{\partial t_m} = \frac{\partial Q^{(m)}}{\partial t_n}.
\]
It follows from this, together with the commutativity of the flows, that
\[
\Delta^{(m,n)} = \frac{\partial}{\partial t_m} \frac{\partial}{\partial t_n} F
\]
for some function \( F \). This is called the free energy and is of great importance in topological field theory \[\mathbb{F} \mathbb{F}\]. There exist various algebraic relations between these \( \Delta^{(m,n)} \), these being derived from the differential Fay identities \[\mathbb{F} \mathbb{F}\]. The matrix \( G_{ij} \) is very closely connected to the components of the metric \( g \) which defined the Hamiltonian structure of the hierarchy (see section 4). Where this is accidental or indicative of some deeper result is unknown.
3. The algebraic structure of the charges

Consider the conserved charges for the $N = 2$ hierarchy

$$Q^{(n)} = \sum_{r+s=n} \binom{n}{r} \binom{n}{s} u^r v^s.$$

If one singles out one of the fields, say $u$, this may be written as

$$Q^{(n)} = u^n \sum_{r=0}^{n} \binom{n}{r}^2 \left(\frac{v}{u}\right)^r$$

and so the properties of these charges can be reduced to the study of the polynomials

$$f_n(x) = \sum_{r=0}^{n} \binom{n}{r}^2 x^r.$$

These polynomials are examples of hypergeometric functions,

$$f_n(x) = \binom{n}{2} F_1(-n, -n, 1; x)$$

(since $n$ is an integer this series will automatically truncate leaving a polynomial of degree $n$), and so the conserved charges may be written as

$$Q^{(n)} = u^n \binom{n}{2} F_1(-n, -n, 1; \frac{v}{u}).$$

Using the identities

$$\binom{n}{2} F_1(a, b, c; z) = (1-z)^{-b} \binom{n}{2} F_1(c-a, b, c; \frac{z}{z-1}),$$

$$P_n(z) = \binom{n}{2} F_1(n+1, -n, 1; \frac{1-z}{2})$$

(where $P_n(z)$ is the Legendre polynomial of degree $n$) one obtains

$$Q^{(n)} = (u-v)^n P_n\left(\frac{u+v}{u-v}\right),$$

and so a generating function for these charges may be constructed using the well known generating function for Legendre polynomials. Such a generating function was obtained in [4] using a different approach based on the construction of recursion relations.

In the multicomponent case one may perform a similar calculation and express the charges in terms of generalized hypergeometric functions [10]. Defining new variables $x_i$ by

$$x_i = \begin{cases} u_N, & i = N, \\ \frac{u_i}{u_N}, & i \neq N \end{cases}$$

equation (13) may be written as

$$Q^{(n)} = \left[ \binom{n}{N-1} \right]^N x_N^n \sum_{r \in \mathbb{Z}^{N-1}} \prod_{k=1}^{N} \frac{x^r}{\Gamma(1 + \mu_k(r) + \gamma_k)},$$

$$= \left(\frac{n}{N-1}\right)^N x_N^n \mathbf{F}(\mu, \gamma; \mathbf{x})$$
where, for convenience, \( x^r \) is defined to be
\[
x^r = x_1^r \ldots x_{N-1}^r,
\]
and \( r_i \) are the components of the vector \( r \). The linear map \( \mu : \mathbb{Z}^{N-1} \to \mathbb{Z}^{2N} \) is defined by
\[
\mu_k(r) = \sum_{j=1}^{N-1} a_{kj} r_j
\]
where
\[
a_{kj} = \begin{pmatrix}
+1 & 0 & \ldots & 0 \\
0 & +1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & +1 \\
+1 & +1 & \ldots & +1 \\
-1 & 0 & \ldots & 0 \\
0 & -1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & -1 \\
-1 & -1 & \ldots & -1
\end{pmatrix}
\]
and the constant vector \( \gamma \) is
\[
\gamma = \begin{pmatrix}
0 \\
0 \\
\vdots \\
N(\frac{2-N}{N-1}) \\
\frac{N}{N-1} \\
\frac{N}{N-1} \\
\vdots \\
n
\end{pmatrix}
\]
These manipulations are a manifestation of the Ore-Sato Theorem for generalized hypergeometric functions. Thus the charges are defined in terms of a linear map \( \mu \) and a constant vector \( \gamma \), or in terms of a lattice \( B \) inside \( \mathbb{Z}^{2N} \) as the image of \( \mathbb{Z}^{N-1} \) under the linear map \( \mu \). These generalized hypergeometric functions are related (but not identical) to those studied by Gelfand et al.\[10\].

The fluxes \( \mathbf{F}^{(n)} \) also have a similar description,
\[
\mathbf{F}^{(n)} = \left[ \left( \frac{n}{N-1} \right)! \right]^N x_N^n \mathbf{F}(\mu, \gamma'; \mathbf{x})
\]
where the linear map \( \mu \) is the same but the constant vector has changed to
\[
\gamma' = \gamma + \begin{pmatrix}
0 \\
0 \\
\vdots \\
1 \\
0 \\
\vdots \\
1
\end{pmatrix}.
\]
So both the charges and the flux are defined in terms of the same lattice $B \in \mathbb{Z}^{2N}$, the difference being in the constant vector. One remaining problem is to understand the structure of the more general fluxes $\Delta^{(m,n)}$ in terms of these structures.

4. The Hamiltonian structure of the hierarchy

The evolution equations studied in this paper are specific examples of hydrodynamic equations, i.e. they are of the form

$$u_i^t = \sum_j V_j^i (u) u_x^j , \quad (19)$$

the simplest non-trivial example being the Monge equation $u_t = u u_x$. One must, in general, take care with the indices on the fields as they transform as tensors under an arbitrary transformation $u_i \rightarrow \tilde{u}_i (u)$. However, to avoid expressions such as $(u^i)_r$, all indices have been lowered for notational convenience. There is an extensive literature on such hydrodynamic equations (recently they have attracted much interest from their role in topological field theories). One important result concerns the Hamiltonian structure of these systems.

A system (19) of hydrodynamic type is said to be Hamiltonian if there exists a Hamiltonian $H = \int dx h(u)$ and a Hamiltonian operator

$$\hat{A}^{ij} = g^{ij}(u) \frac{d}{dx} + b^{ij}_k (u) u^k_x \quad (20)$$

which defines a skew-symmetric Poisson bracket on functionals

$$\{ I, J \} = \int dx \frac{\delta I}{\delta u^i(x)} \hat{A}^{ij} \frac{\delta J}{\delta u^j(x)}$$

which satisfies the Jacobi identity and which generates the system

$$u_i^t = \{ u^i(x), H \} .$$

Dubrovin and Novikov \[11\] proved necessary and sufficient conditions for $\hat{A}^{ij}$ to be a Hamiltonian operator in the case when $g^{ij}$ is not degenerate. These are:

a) $g = (g^{ij})^{-1}$ defines a Riemannian metric,

b) $b^{ij}_k = - g^{is} \Gamma^j_{sk}$, where $\Gamma^j_{sk}$ is the Christoffel symbol generated by $g$,

c) the Riemann curvature tensor of $g$ vanishes.

The system (19) may then be written as

$$u_i^t = (g^{is} \nabla_s \nabla_j h) u_x^j \quad (21)$$

where $\nabla$ is the covariant derivative generated by $g$. Expanding these in terms of the Christoffel symbols gives
\[ u^i_t = \left[ g^{ij} \partial_k \partial_j h - g^{is} \Gamma^j_{sk} \partial_j h \right] u^k_x, \]
\[ = \tilde{A}_i u^i_x + \sum_{k \neq i} u^i \tilde{B}_{ik} u^k_x \quad \text{(no sum)} \]

where

\[ \tilde{A}_i = \sum_j g^{ij} \partial_i \partial_j h - \sum_{s,j} g^{is} \Gamma^j_{si} \partial_j h, \quad (22) \]
\[ \tilde{B}_{ik} = \sum_j g^{ij} \partial_k \partial_j h - \sum_{s,j} g^{is} \Gamma^j_{sk} \partial_j h \quad (k \neq i), \quad (23) \]

Thus to find the Hamiltonian structure for the hierarchy \((3)\) one needs to find the metric and the Hamiltonians.

One approach is to first diagonalise the system, there being the following simple formulae for the metric coefficients for diagonal systems \([12]\):

\[ \frac{\partial_i V_j}{(V_i - V_j)} = \frac{1}{2} \partial_i \log g_{jj} \]

where \( V^i_j(u) = V_j(u) \delta^i_j \) (no sum). For \( N = 2 \) such an approach was adopted in \([3]\). However, to find the corresponding transformation for arbitrary \( N \) is difficult and another approach is needed, the \( N = 2 \) result suggesting the form of the general result.

**Theorem 4.1**

The Hamiltonian structure for the hierarchy \((3)\) is given by \((20)\), where the zero-curvature metric is given by

\[ g = \frac{1}{N - 1} \sum_{i \neq j} \frac{du_i}{u_i} \frac{du_j}{u_j} \quad (24) \]

and the Hamiltonians generating these flows are given by

\[ h^{(n)} = \frac{1}{n} Q^{(n)}, \]

where the \( Q^{(n)} \) are conserved charges given by \((12, 13)\).

**Proof**

This \( g \) defines a non-degenerate Riemannian metric, and by changing coordinates to \( \tilde{u}_i = \log(u_i) \) it is clear that the metric has vanishing Riemannian curvature tensor. The components of the inverse metric are

\[ g^{ii} = -(N - 2) u_i u_i \quad \text{(no sum)}, \]
\[ g^{ij} = u_i u_j \quad (i \neq j) \]
The only non-zero Christoffel symbols are $\Gamma_{ii}^i$ (no sum). This is easily proved by noting that in the $\tilde{u}_i$ coordinates the Christoffel symbols are all zero. The Christoffel symbols can then be calculated using transformation law for these symbols. This yields

$$\Gamma_{ii}^i = -\frac{1}{u_i} \quad \text{(no sum)}$$

with all other components being zero. Thus, by the theorem of Dubrovin and Novikov, this metric defines a Hamiltonian structure. To show that this is the dispersionless Toda hierarchy one has to verify that the $\tilde{A}_i$ given by (22) and the $\tilde{B}_{ij}$ given by (23) and $h^{(n)} = \frac{1}{n}Q^{(n)}$ are the same as given by equations (17) and (18) in theorem 2.3.

\[
\tilde{A}^i = \sum_j g^{ij} \partial_i \partial_j h - \sum_{s,j} g^{is} \Gamma_{si}^j \partial_j h,
\]

\[
= \left[ g^{ii} \partial_i \partial_i h + g^{ii} \frac{1}{u_i} \partial_i h \right] + \sum_{j \neq i} g^{ij} \partial_i \partial_j h.
\]

Now

\[
h^{(n)} = \frac{1}{n} \sum_{\{r_i: \sum_{i=1}^N r_i = n\}} \left\{ \prod_{i=1}^N \left( \frac{n}{N-1} \right) u_i^{r_i} \right\},
\]

and substitution of this into the above gives

\[
\tilde{A}^{(n)}_i = \left( \frac{n}{N-1} - 1 \right) \sum_{\{r_j: \sum_{j=1}^N r_j = n\}} \left[ \prod_{k=1}^N \left( \frac{n}{N-1} \right) u_k^{r_k} \right] \left( \frac{n}{N-1} - 2 \right) u_i^{r_i},
\]

\[= A^{(n)}_i.
\]

The proof that $\tilde{B}^{(n)}_{ij} = B^{(n)}_{ij}$ is similar, both using various binomial identities. Hence the result.

$\square$

This theorem thus determines the Hamiltonian structure of the dispersionless Toda hierarchy.

5. Miscellany

This section contains some miscellaneous results on the multicomponent Toda hierarchy.
5.1. Reductions

Besides the discrete symmetry \( u_i \mapsto u_{\sigma(i)} \), where \( \sigma \in S_N \) is an element of the permutation group, the multicomponent Toda hierarchy possesses many other symmetries, as befits an integrable system. For example, one has the continuous symmetries corresponding to time and space translations, and various scaling symmetries. These may be used to derive examples of integrable dynamical systems.

With the ansatz

\[
u_i(x, t) = (nx)^n z_i(t)\]

for the \( n^{th} \) flow (this being a particular example of a scaling symmetry) the \( x \)-dependence cancels and one is left with a dynamical system for the \( z_i \) variables.

**Example 5.1** With \( n = 1 \) the above ansatz reduces the system (11) to the dynamical system

\[
\begin{align*}
\dot{z}_1 &= z_1(z_2 + z_3), \\
\dot{z}_2 &= z_2(z_3 + z_1), \\
\dot{z}_3 &= z_3(z_1 + z_2).
\end{align*}
\]

This system has been studied and integrated by Bureau [13].

In general, such dynamical system are of the form

\[
\dot{z}_i = z_i \text{ (polynomial in the } z_j \text{ variables)}
\]

(since \( A_i^{(n)} \) given by equation (13) always has \( u_i \) as a factor), which suggest on should introduce variables \( \phi_i = \log z_i \). The resultant dynamical systems are then in the so-called Lotka-Volterra form, the nonlinear terms being sums of products of exponentials.

As mentioned in section 2, one may obtain other systems by a conflation of the fields (or equivalently, by having multiple roots in the equation \( \mathcal{L}(p) = 0 \)). As an example, the system (3)

\[
\begin{align*}
u_t &= \frac{1}{2}u(-u_x + v_x + w_x), \\
v_t &= \frac{1}{2}v(-u_x - v_x + w_x), \\
w_t &= \frac{1}{2}w(u_x + v_x - w_x).
\end{align*}
\]

becomes, one setting \( v = w \),

\[
\begin{align*}
u_t &= \frac{1}{2}u(2v_x - u_x), \\
v_t &= \frac{1}{2}vu_x
\end{align*}
\]
Figure 1: Possible reductions of the $N = 4$ and $N = 5$ Toda hierarchies

and finally, on setting $u = v$,

$$u_t = \frac{1}{2} uu_x.$$  

The Hamiltonian structure of these later systems is obtained by restricting the metric (which defined the Hamiltonian structure of the original system) in the first case to the surface $v = w$ and in the second case to the line $u = v = w$. Schematically these systems may be written as $\{1, 1, 1\}, \{2, 1\}$ and $\{3\}$ respectively to show the confluence of the fields (the discrete permutation symmetry ensuring that $\{2, 1\}$ is the same as $\{1, 2\}$), and hence the possible reductions may be summarised in the diagram

$$\{1, 1, 1\} \rightarrow \{2, 1\} \rightarrow \{3\}.$$

With more fields there are many more possible reductions and a simple combinatorial argument gives the number of such subsystems (including the original system) to be $\frac{2N^2 + 7 + (-1)^N}{8}$ for the $N$-component Toda hierarchy. Figure 1 shows the possible reductions for $N = 4$ and $N = 5$.

5.2. Special flows

It is clear from the Lax equation (8) and the integral for the conserved charges (12) that if $n = (N - 1)m$ for some integer $m$ then some simplification might occur, since for such a value of $n$ the Lax equation

$$\frac{\partial \mathcal{L}}{\partial t_m} = \{(\mathcal{L}^m)_+, \mathcal{L}\}$$
(where \( \hat{t}_m = t_{(N-1)m} \)) and the integral form for the conserved charges

\[
\hat{Q}^{(m)} = \frac{1}{2\pi i} \oint \mathcal{L}^m \frac{dp}{p}
\]

do not involve fractional powers of the Lax function. These flows are best studied by a change of variables

\[
\hat{u}_i = \prod_{k \neq i} u_i,
\]

this being the generalisation of the one used in example 1.3. In these variables the evolution equations for these flows take the form

\[
\hat{u}_{i,m} = \hat{A}_i^{(m)} \hat{u}_{i,x} + \sum_{j \neq i} \hat{u}_i \hat{B}_{ij}^{(m)} \hat{u}_{j,x}
\]

(25)

where

\[
\hat{A}_i^{(m)} = (m-1) \sum_{\{r_j : \sum_{j=1}^N r_j = m\}} \left[ \prod_{k=1}^N \binom{m}{r_k} \hat{u}_r \right] \binom{m}{r_i-1} \hat{u}_i,
\]

\[
\hat{B}_{ij}^{(m)} = m \sum_{\{r_j : \sum_{j=1}^N r_j = m-1\}} \left[ \prod_{k=1}^N \binom{m}{r_k} \hat{u}_r \right] \binom{m-1}{r_i} \hat{u}_r \binom{m-1}{r_j} \hat{u}_j.
\]

and the corresponding conserved charges are

\[
\hat{Q}^{(m)} = \sum_{\{r_i : \sum_{j=1}^N r_j = m\}} \left[ \prod_{i=1}^N \binom{m}{r_i} \hat{u}_r \right].
\]

Superficially these equations look very similar to their unhatted counterparts. However, it is important to note that \( N \) only appears in a subordinate role, i.e. instead of the summation being over the set \( \{ r_i : \sum r_i = (N-1)m \} \) the change of variable enables this to be written as a summation over the set \( \{ r_i : \sum r_i = m \} \) irrespective of the value of \( N \). Similarly \( N \) does not enter the binomial coefficients that appear in the above formulae. One consequence of this is, if one denotes the evolution equation (25) by \( \Gamma(N,m) \), then a possible reduction is to set \( \hat{u}_N = 0 \) and the resultant system is just the \((N-1)\)-component system, i.e.

\[
\Gamma(N,m) \big|_{\hat{u}_N = 0} = \Gamma(N-1,m)
\]

for all values of \( m \).

The Hamiltonian structure of this system is easily derived by performing the change of variable on the metric (24), the result being

\[
\hat{g} = \frac{1}{(N-1)^2} \left[ (2-N) \sum_{i=1}^N \left( \frac{d\hat{u}_i}{\hat{u}_i} \right)^2 + \sum_{r \neq s} \frac{d\hat{u}_r}{\hat{u}_r} \frac{d\hat{u}_s}{\hat{u}_s} \right]
\]
(actually, the inverse metric
\[ \hat{g}^{ij} = \begin{cases} (N - 1)\hat{u}_i\hat{u}_j & i \neq j, \\ 0 & \text{otherwise}, \end{cases} \]
is simpler).

These calculations are not, perhaps, very illuminating, and the motivation for this subsection is best seen by means of the following examples.

**Example 5.2** For general \( N \) the \( (N - 1)^{th} \) flow (i.e. \( m = 1 \)) has been calculated in example 2.4, namely,
\[ u_{i,t} = u_i \left( \prod_{j \neq i} u_j \right)_x. \]
In the hatted variables these simplify to
\[ \hat{u}_i = \hat{u}_i \sum_{j \neq i} \hat{u}_{j,x}. \]
This is the first flow of the Benney hierarchy (see example 2.6) written in terms of the modified variables. Note that this contains only quadratic nonlinearities irrespective of the value of \( N \), unlike its unhatted counterpart whose nonlinearities depend on the value of \( N \).

**Example 5.3** For \( N = 3 \) the Lax equation (3) gives the evolution equations for \( n = 2 \)
\[ u_{t_2} = uwv_x + uvw_x \quad \text{and cyclically} \]
and for \( n = 4 \)
\[ u_{t_4} = 2uvw(v + w)u_x + 2uw(2uv + uw + vw)v_x + 2uv(uv + 2uw + vw)w_x \quad \text{and cyclically} \]

(together with the conserved charges)
\[ Q^{(2)} = uv + vw + wu, \quad Q^{(4)} = u^2v^2 + v^2w^2 + w^2u^2 + 4uvw(u + v + w). \]

In terms of the new hatted variables \( \hat{u} = vw \) etc. these flows become
\[ \hat{u}_{t_1} = \hat{u}(\hat{v}_x + \hat{w}_x), \]
\[ \hat{u}_{t_2} = (2\hat{u}\hat{v} + 2\hat{u}\hat{w})\hat{u}_x + (2\hat{u}^2 + 2\hat{u}\hat{v} + 4\hat{u}\hat{w})\hat{v}_x + (2\hat{u}^2 + 4\hat{u}\hat{v} + 2\hat{u}\hat{w})\hat{w}_x \]
(taken with their cyclic permutations) and the conserved charges become
\[ \hat{Q}^{(1)} = \hat{u} + \hat{v} + \hat{w}, \quad \hat{Q}^{(2)} = \hat{u}^2 + \hat{v}^2 + \hat{w}^2 + 4(\hat{u}\hat{v} + \hat{v}\hat{w} + \hat{w}\hat{u}). \]

These flows are thus simpler in the hatted variables. Note that it is now possible to set, for example, \( \hat{w} = 0 \) and these formulae reduce to those for \( N = 2 \) (for which there is no distinction between hatted and unhatted variables).
6. Rational Lax functions

The Lax function so far studied in this paper has been, apart from an overall $p^{-1}$, a polynomial function. However, the Lax formalism is valid for much more general functions such as rational functions:

$$L = \prod_{i=1}^{N} (p + u_i)^{\epsilon_i}, \quad \epsilon_i = \pm 1$$

or

$$L = \prod_{i=1}^{N} (p + u_i)^{\epsilon_i}$$

provided $N > M$. Here $\epsilon_i = \pm 1$ but this can be extended to cope with repeated roots and zeros. The $N > M$ condition enables this rational Lax function to be written as

$$L = \text{polynomial of degree } (N - M) + \sum_{i=N+1}^{N+M} \text{ simple poles}.$$ 

It should be clear from the earlier proofs that they may be extended to such rational potentials with little change. The results in this section will therefore be given without proof.

The conserved charges, now defined by the integral formula

$$Q^{(n)} = \frac{1}{2\pi i} \int L^{n-M} \frac{dp}{p},$$

are

$$Q^{(n)} = \sum_{\{r_i : \sum_{i=1}^{N+M} r_i = n\}} \left\{ \prod_{i=1}^{N+M} \left( \frac{\epsilon_i n}{N-M} \right) u_i^{r_i} \right\},$$

and the evolution of the fields, given by the Lax equation

$$\frac{\partial L}{\partial t} = \{ L^{n-M}, L \},$$

are

$$u_{i,t} = A_i^{(n)} u_{i,x} + \sum_{j \neq i} u_j B_{ij}^{(n)} u_{j,x}$$

where

$$A_i^{(n)} = \left( \frac{\epsilon_i n}{N-M} - 1 \right) \sum_{\{r_j : \sum_{j=1}^{N+M} r_j = n\}} \left[ \prod_{k=1}^{N+M} \frac{\epsilon_k n}{N-M} r_k \right] u_k^{r_k} \left( \frac{\epsilon_i n}{N-M} r_i - 1 \right) u_i^{r_i}$$

and

$$B_{ij}^{(n)} = \frac{\epsilon_j n}{N-M} \sum_{\{r_j : \sum_{j=1}^{N+M} r_j = n\}} \left[ \prod_{k=1}^{N+M} \frac{\epsilon_k n}{N-M} r_k \right] u_k^{r_k} \left( \frac{\epsilon_i n}{N-M} r_i - 1 \right) u_i^{r_i} \left( \frac{\epsilon_j n}{N-M} - 1 \right) u_j^{r_j}. $$
Example 6.1 The simplest case is \( N = 2, M = 1 \), and the Lax function will be written as

\[
\mathcal{L} = \frac{(p + u)(p + v)}{p + w}.
\]

Hence a consistent reduction is to set \( w = 0 \), with which the results simplify further. The conserved charges are

\[
Q^{(n)} = \sum_{r+s+t=n} \binom{n}{r} \binom{n}{s} \binom{n}{t} u^r v^s w^t.
\]

The binomial coefficients have to be interpreted formally, as indicated in section 1. For example, the first two charges are

\[
Q^{(1)} = u + v - w,
\]
\[
Q^{(2)} = u^2 + v^2 + 3w^2 + 4(uv - vw - uw),
\]

and the first two flows are given by

\[
\left( \begin{array}{c} u \\ v \\ w \end{array} \right)_{t_1} = \left( \begin{array}{ccc} 0 & u & -u \\ v & 0 & -v \\ w & w & -2w \end{array} \right) \left( \begin{array}{c} u \\ v \\ w \end{array} \right)_x,
\]

and

\[
\left( \begin{array}{c} u \\ v \\ w \end{array} \right)_{t_2} = \left( \begin{array}{ccc} 2u(v - w) & 2u(u + v - 2w) & -2u(u + 2v - 3w) \\ 2v(u + v - 2w) & 2v(u - w) & -2v(2u + v - 3w) \\ 2w(u + 2v - 3w) & 2w(2u + 2v - 3w) & -6w(u + v - 2w) \end{array} \right) \left( \begin{array}{c} u \\ v \\ w \end{array} \right)_x.
\]

Once again, care has to be taken in evaluating the binomial coefficients.

Example 6.2 Let \( M = N - 1 \) and \( u_i = 0 \) for \( i = N + 1, \ldots, N + M \). With these the Lax function takes the form

\[
\mathcal{L} = \frac{\prod_{i=1}^{N}(p + u_i)}{p^{N-1}},
\]

\[
= p \prod_{i=1}^{N} \left( 1 + \frac{u_i}{p} \right),
\]

\[
= p + \sum_{r=0}^{N} S_r p^r,
\]

\[
= \mathcal{L}_B.
\]

Thus the Lax function \( \mathcal{L}_B \) for the Benney hierarchy is a special case of the rational Lax function. Thus one obtains an explicit family of conservation laws and evolution equations for the Benney hierarchy.
In fact, the rational Lax function (26) itself may be considered as a reduction of the infinite component Lax function

\[ L_\infty = p^{N'} + \sum_{n=-\infty}^{N'-1} S_i p^n. \]

By formally expanding (26) one obtains the above Lax function, with various conditions of the fields. This and some other reductions of this Lax function, and the resulting systems, have been studied by [14].

7. Logarithmic charges

In section 4 it was shown that the conserved charges (for \( N = 2 \)) may be written in terms of Legendre polynomials

\[ Q^{(n)} = (u-v)^n P_n \left( \frac{u+v}{u-v} \right). \]  

This suggests that one should introduce the new variables

\[ r = u - v, \quad \cos \theta = \frac{u+v}{u-v}. \]

In these new variables the differential equation for the conserved charges

\[(uQ_u)_u = (vQ_v)_v\]

becomes the axially symmetric Laplace equation, and so one obtains the conserved charges (27). However there is another family of charges given by Legendre functions of the second kind, the first few conservation laws begin:

\[
\left[ \frac{1}{2} \log(uv) \right]_t = \left[ \frac{1}{2} (u+v) \right]_x, \\
\left[ \frac{1}{2} (u+v) \log(uv) \right]_t = \left[ \frac{1}{2} uv \log(uv) + \frac{1}{4} (u^2 + v^2) \right]_x.
\]

This raises the question of how these charges are related to the Lax formalism. The answer is that they are given by the formula

\[ \tilde{Q}^{(n)} = \frac{1}{2\pi i} \oint L^n (\log L - c_n) \frac{dp}{p} \]

where

\[ c_n = \sum_{j=1}^{n} \frac{1}{j}, \quad c_0 = 0. \]

This results in the scaling symmetry \[15, 16\]

\[ \frac{\partial}{\partial L} [L^n (\log L - c_n)] = n [L^{n-1} (\log L - c_{n-1})], \]
since $c_n = c_{n-1} + 1/n$. Once again using the modified variables this integral may be performed for all values of $n$ and $N$. For simplicity we restrict attention to the $N = 2$ case. Conceptually the general case is the same, just notationally more complex.

**Theorem 7.1**

The logarithmic charges defined by (28) are

$$\tilde{Q}^{(n)} = \frac{1}{2} \log(uv) \sum_{r+s=n} \binom{n}{r} \binom{n}{s} u^r v^s - \sum_{r+s=n} \binom{n}{r} \binom{n}{s} c(r, s) u^r v^s$$

(29)

where

$$c(r, s) = \sum_{j=1}^{r} \frac{1}{j} - \sum_{j=s+1}^{r+s} \frac{1}{j}$$

and $c(0, r) = 0$ (so $c(r, s) = c(s, r)$ for all $r$ and $s$).

**Proof**

Care has to be taken with the interpretation of $\log \mathcal{L}$ to avoid $\log p$-terms. Since

$$\log \mathcal{L} = + \log p + \log \left(1 + \frac{u}{p}\right) + \log \left(1 + \frac{v}{p}\right)$$

and

$$\log \mathcal{L} = - \log p + \log \left(1 + \frac{p}{u}\right) + \log \left(1 + \frac{p}{v}\right) + \log(uv)$$

then

$$\log \mathcal{L} = \frac{1}{2} \left[ \log \left(1 + \frac{u}{p}\right) + \log \left(1 + \frac{v}{p}\right) + \log(uv) + \log \left(1 + \frac{p}{u}\right) + \log \left(1 + \frac{p}{v}\right) \right].$$

To evaluate the integral one Taylor expands the function as power series in $p$ and uses the residue theorem. Since

$$\mathcal{L}^n \log \left(1 + \frac{u}{p}\right) = p^n \sum_{r=0}^{n} \sum_{s=0}^{n} \sum_{t=0}^{\infty} \binom{n}{r} \binom{n}{s} \binom{(-1)^t}{t+1} u^{r+t+1} v^s p^{-(r+s+t+1)}$$

then

$$\frac{1}{2\pi i} \oint \mathcal{L}^n \log \left(1 + \frac{u}{p}\right) \frac{dp}{p} = \sum_{r+s+t=n-1} \binom{n}{r} \binom{n}{s} \binom{(-1)^t}{t+1} u^{n-s} v^s.$$

The other parts of the integral may be evaluated in a similar way and so the final result is

$$\tilde{Q}^{(n)} = + \frac{1}{2} \log(uv) \sum_{r+s=n} \binom{n}{r} \binom{n}{s} u^r v^s + \sum_{r+s+t=n-1} \binom{n}{r} \binom{n}{s} \frac{(-1)^t}{t+1} (u^{n-s} v^s + v^s u^{n-s})$$

$$- c_n \sum_{r+s=n} \binom{n}{r} \binom{n}{s} u^r v^s.$$

Note that there is an ambiguity in the definition of $\tilde{Q}^{(n)}$, since $\tilde{Q}^{(n)} + \lambda Q^{(n)}$ is also a conserved charge. The numbers $c_n$ have the property that the coefficient of $(u^n + v^n)$
in the terms not involving \( \log(uv) \) in the above expression is zero and this removes this ambiguity. Explicitly this coefficient is

\[
\sum_{r+t=n-1} \binom{n}{r} \frac{(-1)^t}{t+1} - c_n
\]

and this is, using the identity

\[
\sum_{t=0}^{n-m} \binom{n}{m+t} \frac{(-1)^t}{t+1} = \binom{n}{m-1} \sum_{j=m}^{n} \frac{1}{j}, \quad n \geq m,
\]

zero, as required. Rearranging and resuming (using the above identity) results in the expression (28).

\[\square\]

These logarithmic charges define a new set of evolution equations defined by the Lax equation

\[
\frac{\partial \mathcal{L}}{\partial \tau_n} = \{[\mathcal{L}^n(\log \mathcal{L} - c_n)]_+, \mathcal{L}\}, \quad n = 0, 1, 2, \ldots ,
\]

or, equivalently, by the Hamiltonian equations

\[
u_{i, \tau_n} = \{u_i, \tilde{H}^{(n)}\}
\]

with \( \tilde{H}^{(n)} \) proportional to the logarithmic charges \( \tilde{Q}^{(n)} \). These flows all commute, both with themselves and those defined by (30), and the corresponding Hamiltonians are in involution with respect to the Hamiltonian structure defined by (20). The techniques developed in sections 2 and 4 may easily be extended to derive these more complicated evolution equations explicitly. However, for \( N = 2 \) one may sidestep these calculations. It is straightforward to show that any flow that commutes with the basic equation (5) must be of the form

\[
u_{x, \tau} = F u_x + u Q v_x ,
\]

where \( F \) and \( Q \) are any functions of \( u \) and \( v \) (not necessarily symmetric) which satisfy the equations

\[
F_u = v Q_v ,
\]

\[
F_v = u Q_u ,
\]

or equivalently, the conservation law \( Q_{t_1} = F_x \). Thus to find the evolution equations associated with the logarithmic charges given by (28) one only has to solve the above equations for the flux \( F \).

\textbf{Lemma 7.2} The logarithmic charges (28) give rise to the evolution equations

\[
\begin{align*}
\bar{u}_{\tau_n} &= \bar{F}^{(n)} u_x + u \bar{Q}^{(n)} v_x , \\
\bar{v}_{\tau_n} &= \bar{F}^{(n)} v_x + v \bar{Q}^{(n)} u_x ,
\end{align*}
\]
where the $\tilde{Q}^{(n)}$ are the charges defined by (28) and the $\tilde{F}^{(n)}$ are the corresponding fluxes with respect to the $t_1$ flow, Explicitly

$$\tilde{F}^{(n)} = \frac{1}{2} \log(uv) \sum_{r+s=n+1} \begin{pmatrix} n \\ r \end{pmatrix} \begin{pmatrix} n \\ s \end{pmatrix} u^r v^s - \sum_{r+s=n+1} \tilde{c}(r, s) u^r v^s \quad (31)$$

where

$$\tilde{c}(r, s) = \begin{pmatrix} n \\ r \end{pmatrix} \begin{pmatrix} n \\ s \end{pmatrix} \left[ c(r, s) + \frac{1}{1+n} \right] - \frac{1}{2(1+n)} \begin{pmatrix} n+1 \\ r \end{pmatrix} \begin{pmatrix} n+1 \\ s \end{pmatrix}.$$

**Proof** Straightforward. One just has to integrate equation (30).

**Example 7.3** The simplest such logarithmic flow is

$$u_{\tau_0} = \frac{1}{2}(u + v)u_x + \frac{1}{2}u \log(uv) v_x,$$

$$v_{\tau_0} = \frac{1}{2}(u + v)v_x + \frac{1}{2}v \log(uv) u_x,$$

or, in the original variables,

$$S_{\tau_0} = \frac{1}{2}SS_x + \frac{1}{2} \log P P_x,$$

$$P_{\tau_0} = \frac{1}{2}SP_x + \frac{P}{2} \log P S_x.$$

Calculations of this type were first performed (for $N = 2$ and small values of $n$) in [15, 16] and then (for $N = 3$ and small values of $n$) in [17].

### 8. Comments

The results of this paper stem from the simple observation that by factorizing the Lax function the calculations simplify dramatically. As an example, this enables the metric which defines the Hamiltonian structure to be found for arbitrary $N$. Even for $N = 3$ the metric, when written in terms of the original, unmodified, variables is somewhat unmanageable and the degree of complexity increases rapidly as $N$ increases. Not all of the possible formulae have been calculated in this paper, though it should be clear how these may be derived. For example, one may regard the $\tau_n$ flows as negative $t_n$ flows (i.e. $t_{-n} \equiv \tau_n$), and so one may derive explicit forms for the function $\Delta^{(m,n)}$ for all $m, n \in \mathbb{Z}$. Similarly, the Hamiltonian structure for the evolution equations derived from the rational Lax function (26) has not been calculated; to do this should be straightforward.
The use of generalized hypergeometric functions to study the conservation laws follows from an earlier result [4] where it was shown that the $N = 2$ conservation laws were expressible in terms of Legendre polynomials. This results was used to construct new families of integrable (or at least soluble) hierarchies with similar properties to the original Toda system, namely

\[ u_t = u^{a-c}v^{c-1}v_x, \]
\[ v_t = u^{a-c-1}v^c u_x, \]

where $a$ and $c$ are free parameters. The conservation laws of this system depend on the hypergeometric function $2F1(-n, n + a - 1, c; z)$. One interesting possibility is to use the properties of the generalized hypergeometric series to perform a similar calculation for higher values of $N$.

As remarked at the very beginning of this paper, these systems come from the dispersionless limit of integrable dispersive systems. The Lax formalism for such dispersive systems comes from the replacements

\[ p \longrightarrow \Delta, \]
\[ \{f, g\} \longrightarrow [f, g], \]

where $\Delta$ is the shift operator introduced in section 1 and $[f, g] = fg - gf$ where now $f$ and $g$ are operators depending on $\Delta$. Alternatively, one may use a deformation of the Poisson bracket [18], this having the advantage that one retains a geometrical description of these hierarchies [4]. While all this clearly works for polynomial Lax functions, there is a problem of how to find dispersionless analogues of the systems derived from the rational Lax function (26), the difficulty coming form the interpretation of the operator $(\Delta + u)^{-1}$. One may to resolve this problem has been recently proposed in [19].

Finally, it must be stressed again that the existence of the results in this paper have been known for many years. However, apart from some trivial examples, the precise forms of these results have not been calculated. We have adopted an utilitarian view that such implicit formulae should be calculated explicitly, and the use of modified variables, as advocated in this paper, provides a simple way to do this.

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