GLOBAL ENDPOINT STRICHARTZ ESTIMATES FOR SCHRÖDINGER EQUATIONS ON THE CYLINDER $\mathbb{R} \times T$

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1. Long-time, scaling-critical Strichartz estimates on $\mathbb{R} \times T$

Define the norm on $\mathbb{R} \times \mathbb{R} \times T = (\mathbb{Z} + [0, 1)) \times \mathbb{R} \times T$:

$$
\|u\|^2_{L^4(L^4(\mathbb{R} \times \mathbb{T}))} := \sum_{\gamma \in \mathbb{Z}} \left( \int_{x,y \in \mathbb{R} \times T} |u(\gamma + s, x, y)|^4 \, dx \, dy \right)^{\frac{1}{4}} \, ds.
$$

(1.1)

In this paper, we prove the following global in time Strichartz-type estimate:

**Theorem 1.1.** There exists $C < \infty$ such that for all $f \in L^2(\mathbb{R} \times T)$,

$$
\|e^{it\Delta_{\mathbb{R} \times T}} f\|_{L^4(\mathbb{R} \times \mathbb{T})} \leq C\|f\|_{L^2(\mathbb{R} \times T)}.
$$

(1.2)

This inequality is saturated\(^1\) by two different families of functions of $(x, y) \in \mathbb{R} \times T$:

$$
F_n(x, y) = nG(n\sqrt{x^2 + y^2})1_{(n^2 + y^2) \leq 1}, \quad f_n(x, y) = n^{-\frac{3}{4}}G(n^{-1}x),
$$

(1.3)

where $G(s) = e^{-s^2}$ is a Gaussian. These correspond respectively to saturators for Strichartz estimates in $2d$ and in $1d$ [11].

Interpolating with the simple estimate\(^2\) when $q = 4$ and $p = \infty$, we obtain the family of scaling invariant Strichartz estimates on $\mathbb{R} \times T$:

$$
\|e^{it\Delta_{\mathbb{R} \times T}} f\|_{L^q(L^p(\mathbb{R} \times \mathbb{T}))} \lesssim \|f\|_{H^s(\mathbb{R} \times \mathbb{T})}, \quad \frac{2}{q} + \frac{1}{p} = \frac{1}{2}, \quad 4 \leq q \leq 8, \quad s = 1 - \frac{4}{p}.
$$

(1.4)

Strichartz-type inequalities with mixed norms in the time variable of the form (1.1) were introduced in [9] to study the asymptotic behavior of solutions to critical NLS on product spaces $\mathbb{R}^n \times \mathbb{T}^d$ which are examples of manifolds where the global dimension is smaller than the local dimension. Similar cases were later explored in [6, 13, 14] and the sharp results when $s > 0$ was obtained in [11] using results from $\ell^2$-decoupling [4].

However, to study NLS with data in $L^2$, estimates with loss of derivatives are useless. This raised the question of whether a Strichartz-type inequality with no loss of derivatives could hold for Schrödinger equations on $d$-dimensional manifolds smaller at infinity than $\mathbb{R}^d$. For the torus $\mathbb{T}^d$, for instance, a lossless inequality like (1.2) does not hold, not even locally in time (that is, with $a = \infty$) as observed in [3]. In fact, for manifolds “smaller” than $\mathbb{R}^2$, the only estimate known to the authors is the result from [12] which obtains local version of (1.2) (with $a = \infty$ instead of $a = 8$). We refer e.g. to [2, 5, 7] for the study of Strichartz estimates without losses in the presence of trapped geodesic.

As for nonlinear applications of (1.2), one can easily show local well-posedness of the cubic NLS in $L^2(\mathbb{R} \times \mathbb{T})$, recovering the result in [12]. However, the long-time behavior is modified scattering as shown in [10], which requires more information (and stronger control on initial data) than $L^2$-Strichartz.

\(^1\)In the sense that the quotient of both sides converges to a nonzero constant as $n \to \infty$.

\(^2\)This follows from classical $TT^*$ estimates as in Ginibre-Velo [3].
estimates and it remains a challenging open question as to whether nonlinear solutions satisfy global bounds of the type \( (1.2) \).

This leaves open some interesting questions:

1. Can one extend this result to other semi-periodic settings, i.e., does an estimate like

\[
\| e^{i t \Delta R^d \times \mathbb{T}^n} f \|_{L^p(R, L^p(R^d \times \mathbb{T}^n))} \lesssim \| f \|_{L^2(R^d \times \mathbb{T}^n)},
\]

hold? This is settled for \( n + d \leq 2 \), but for higher values, \( p < 4 \) and the problem is much more challenging.

2. Can one understand and characterize optimizers of \( (1.2) \)? In principle, introducing a parameter for the length of the torus (or the local time interval), one may expect that optimizers should vary smoothly between the two families in \( (1.3) \).

3. Can one obtain a good profile decomposition, i.e., study the defect of compactness of bounded sequences in \( L^2(R \times \mathbb{T}) \)?

2. Proof of Theorem 1.2

Since the analysis is done purely in the frequency space, we pass to the Fourier transform and consider \( f \in L^2(R \times \mathbb{Z}) \), which corresponds to the Fourier transform of the function in \( (1.2) \). By homogeneity, we may choose \( f \) to be of unit \( L^2 \) norm and by density we may assume that \( f \) is compactly supported so that all integrals below converge absolutely. We let \( \mathbb{T} = R/2\pi \mathbb{Z} \) and we define the Fourier transform on \( R \times \mathbb{Z} \)

\[
\hat{f}(x, y) = \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} f(\xi, k) e^{ix\xi} e^{i ky} d\xi, \quad \hat{g}(\xi, k) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_{y=0}^{2\pi} g(x, y) e^{-ix\xi} e^{-iky} dy dx.
\]

Since we will take Fourier transforms, it will be convenient to replace the integral over \([0, 1)\) by an integral over \( R \). To do this, we introduce a Gaussian cutoff in time and let

\[
J_\gamma := \| e^{-\frac{1}{4}(t-\gamma)^2} e^{i t \Delta R \times \mathbb{T}} \hat{f} \|_{L^2_{t, y} (R \times \mathbb{T} \times \mathbb{R})}.
\]

To prove \( (1.2) \), it will suffice to control the \( \ell^6 \)-norm of \( J_\gamma \). For simplicity of presentation, we let

\[
\tilde{\xi} = (\xi_1, \xi_2, \xi_3, \xi_4), \quad \tilde{k} = (k_1, k_2, k_3, k_4),
\]

\[
\langle \xi \rangle = \xi_1 - \xi_2 + \xi_3 - \xi_4 = (\xi, (1, -1, 1, -1)), \quad \langle k \rangle = k_1 - k_2 + k_3 - k_4,
\]

\[
f_j = f(\xi_j, k_j), \quad j \in \{1, 3\}, \quad f_j = \overline{f}(\xi_j, k_j), \quad j \in \{2, 4\},
\]

\[
Q(\xi, k) = |\xi_1|^2 + |\xi_3|^2 - |\xi_2|^2 - |\xi_4|^2 + |k_1|^2 + |k_3|^2 - |k_2|^2 - |k_4|^2.
\]

We substitute \( t \to t + \gamma \) in \( (2.1) \) and expand \( J_\gamma^4 \) into

\[
J_\gamma^4 = \int_{x, y, t} \left[ \sum_{\sum_{k_1 \cdots k_4} \int_{\xi_1 \cdots \xi_4} \Pi_{j=1}^4 f_j \cdot e^{-it^2 e^{-i(t+\gamma)Q(\xi, k)}} \cdot e^{ix\xi e^{i y\langle k \rangle}} \xi \right] dxdydt
\]

\[
= 4\pi^2 \sum_{\sum_{k_1 \cdots k_4} \int_{\xi_1 \cdots \xi_4} \Pi_{j=1}^4 f_j \cdot e^{-\frac{1}{4}(Q(\xi, k))^2} e^{-i \gamma Q(\xi, k)} \cdot \delta(\langle \xi \rangle) \delta(\langle k \rangle) d\xi.
\]
An argument of Takaoka-Tzvetkov [12] shows that each individual $J_i^4$ is bounded, but we need to handle the sum in $\gamma$. We square $J_i^3$ and sum over $\gamma$ to get

$$\mathcal{J} := \sum_{\gamma \in \mathbb{Z}} J_\gamma^3$$

$$= 16\pi^3 \sum_{k_1, \ldots, k_4} \prod_{j=1}^4 \Pi_j \int_{\xi_{j-1}, \ldots, \xi_j} e^{-\frac{i}{2}(Q(\xi, k))^2} e^{-\frac{i}{2}(Q(\xi, k')^2)} \cdot \sum_{\gamma} e^{-i\gamma[Q(\xi, k) - Q(\xi', k')]} \cdot \delta(\langle \xi \rangle) \delta(\langle \xi' \rangle) \delta(\langle k \rangle) \delta(\langle k' \rangle) d\xi d\xi' \tag{2.2}$$

Using Poisson summation in $\gamma$ we observe that

$$\sum_{\gamma \in \mathbb{Z}} e^{-i\gamma[Q(\xi, k) - Q(\xi', k')]} = 2\pi \sum_{\mu \in 2\pi \mathbb{Z}} \delta(\mu - Q(\xi, k) + Q(\xi', k')).$$

Introducing the new notations

$$\Xi := (\xi_1, \xi_3, \xi_2, \xi_4), \quad \Xi' := (\xi_2, \xi_4, \xi_1, \xi_3),$$

$$K := (k_1, k_3, k_2, k_4), \quad K' := (k_2, k_4, k_1, k_3),$$

$$F(\Xi, K) := f(\xi_1, k_1) f(\xi_3, k_3) f(\xi_2, k_2) f(\xi_4, k_4), \quad F(\Xi', K') := f(\xi_2, k_2) f(\xi_4, k_4) f(\xi_1, k_1) f(\xi_3, k_3),$$

$$\phi_\mu := \mu - Q(\xi, k) + Q(\xi', k') = \mu - |\Xi|^2 - |K|^2 + |\Xi'|^2 + |K'|^2,$$

we arrive at

$$\mathcal{J} = 32\pi^6 \sum_{K, K' \in \mathbb{Z}^4} \int_{\Xi, \Xi'} F(\Xi, K) F(\Xi', K') \cdot K(\Xi, K; \Xi', K') \cdot d\Xi d\Xi'$$

$$K(\Xi, K; \Xi', K') := e^{-\frac{i}{4}[Q(\xi, k)^2] + Q(\xi', k')^2]} \sum_{\mu \in 2\pi \mathbb{Z}} \delta(\phi_\mu) \delta(\langle \xi \rangle) \delta(\langle \xi' \rangle) \delta(\langle k \rangle) \delta(\langle k' \rangle).$$

Using the Schur test, the inequality (1.2) follows from the next lemma.

**Lemma 2.1.** With the notations above,

$$\sup_{(\Xi, K) \in \mathbb{R}^4 \times \mathbb{Z}^4} \sum_{K' \in \mathbb{Z}^4} \int K(\Xi, K; \Xi', K') d\Xi' < \infty.$$
A similar identity holds for $Q'$ when $\langle \xi' \rangle = 0 = \langle k' \rangle$. Indeed, on the support of $\delta(\langle \xi' \rangle)\delta(\langle k' \rangle)$ we can substitute

$$
\xi'_4 = -\xi'_1 + \xi'_2 + \xi'_3 \quad \text{and} \quad k'_3 = -k'_1 + k'_2 + k'_4
$$

into $Q'$ and factor to obtain

$$
Q(\xi', k') = 2 \left[ |(\xi'_1 - c'_x, k'_1 - c'_y)|^2 - (R')^2 \right],
$$

$$
(c'_x, c'_y) = \left( \frac{\xi'_1 + \xi'_2}{2}, \frac{k'_2 + k'_4}{2} \right), \quad (R')^2 = \left( \frac{\xi'_1 - \xi'_4}{2} \right)^2 + \left( \frac{k'_2 - k'_4}{2} \right)^2.
$$

With these substitutions made, notice that

$$
\phi_{\mu} = \mu + 2|\langle \xi_2 - c_x, k_2 - c_y \rangle|^2 + 2\langle \xi'_1 - c'_x, k'_1 - c'_y \rangle|^2 - (R')^2
$$

and therefore

$$
\delta(\phi_{\mu}) = \frac{1}{2} \delta\left(|\langle \xi_2 - c_x, k_2 - c_y \rangle|^2 + |\langle \xi'_1 - c'_x, k'_1 - c'_y \rangle|^2 - A_{\mu}\right), \quad A_{\mu} = \frac{R^2 + (R')^2 - \mu}{2}.
$$

Using these observations to estimate (2.8) we arrive at

$$
\int_{|\xi| = 2\pi} e^{-\frac{1}{2} \sum_{k_2, k_1'} \int_{\mathbb{R}^2} e^{-\frac{1}{2} \left[ |(\xi_2 - c_x, k_2 - c_y)|^2 - R^2 \right] + |\langle \xi'_1 - c'_x, k'_1 - c'_y \rangle|^2 - (R')^2} \delta\left(|\langle \xi, \kappa \rangle - \tilde{C}|^2 + |\langle \xi', \kappa' \rangle - \tilde{C}'|^2 - A\right) d\xi d\xi' \quad (2.4)
$$

uniformly in $\tilde{C}, \tilde{C}' \in \mathbb{R}^2$, $A, R, R' \in \mathbb{R}$. Moreover, since $2c_y$ and $2c'_y$ are both integers we can assume the second components of $\tilde{C}, \tilde{C}'$ are in $\frac{1}{2}\mathbb{Z}$.

The integral in (2.4) is invariant with respect to translation on $(\mathbb{R} \times \mathbb{Z}) \times (\mathbb{R} \times \mathbb{Z})$, and we may therefore assume that $\tilde{C} = (0, c), \tilde{C}' = (0, c')$ for $c, c' \in \{0, \frac{1}{2}\}$. To control $I$ we introduce sets where the exponential factors behave nicely. When $R \geq 50$, we let

$$
S_0 := \{|(\zeta, \kappa) - \tilde{C}| - R| \leq R^{-1}\},
$$

$$
S_j := \{|(\zeta, \kappa) - \tilde{C}| - R| \in \mathbb{N}^2[j, j + 1]\}, \quad 1 \leq j \leq R^2 + 1,
$$

$$
S_\infty := \{|(\zeta, \kappa) - \tilde{C}| - R| \geq R^{-\frac{1}{2}}\}
$$

and when $R \leq 50$, we let $S_j = \emptyset$ and $S_\infty = \mathbb{R} \times \mathbb{Z}$. These satisfy

$$
1_{S_j}(\zeta, \kappa)e^{-\frac{1}{2} |\langle \zeta, \kappa \rangle - \tilde{C}|^2 - R^2} \leq e^{-\frac{1}{2} j^2} 1_{S_j}(\zeta, \kappa), \quad 0 \leq j \leq R^2 + 1,
$$

$$
1_{S_\infty}(\zeta, \kappa)e^{-\frac{1}{2} |\langle \zeta, \kappa \rangle - \tilde{C}|^2 - R^2} \leq e^{-\frac{1}{2} |\langle \zeta, \kappa \rangle - \tilde{C}|} 1_{S_\infty}(\zeta, \kappa).
$$

Indeed, the estimate on $S_j$ in (2.6) follows by factoring the term in the exponential. To prove the estimate on $S_\infty$ note that if $|(\zeta, \kappa) \in S_\infty$ and $R \geq 50$ then

$$
|\langle \zeta, \kappa \rangle - \tilde{C}|^2 - R^2 \geq |R^{-\frac{1}{2}}|\langle \zeta, \kappa \rangle - \tilde{C}| + R| \geq |\langle \zeta, \kappa \rangle - \tilde{C}| + R.
$$

On the other hand

$$
|\langle \zeta, \kappa \rangle - \tilde{C}|^2 - R^2 \geq |\langle \zeta, \kappa \rangle - \tilde{C}| - 2,
$$

and the estimate in (2.6) in $S_\infty$ follows if $R \leq 50$. 


We first use (2.7) from Lemma 2.2 to control the contribution of $S_\infty$ to (2.4). In particular
\[
I_{\infty} := \sum_{\kappa, \kappa'} \int \mathbf{1}_{S_\infty}(\zeta, \kappa) \mathbf{1}_{S_\infty}(\zeta', \kappa') e^{-\frac{i}{2} \left[ |(\zeta, \kappa-c)^2 - R^2|^2 + |(\zeta', \kappa'-c')^2 - (R')^2|^2 \right]}
\cdot \delta(|\zeta|^2 + |\kappa - c|^2 + |\zeta'|^2 + |\kappa' - c'|^2 - A) d\zeta d\zeta'
\lesssim \sum_{\kappa, \kappa'} e^{-\frac{i}{8}(|\zeta'| + |\kappa' - c'|)} \int \delta(|\zeta|^2 + |\kappa - c|^2 + |\zeta'|^2 + |\kappa' - c'|^2 - A) d\zeta d\zeta'.
\]

Next, we consider
\[
I_j = \sum_{\kappa, \kappa'} \int \mathbf{1}_{S_j}(\zeta, \kappa) \mathbf{1}_{S_\infty}(\zeta', \kappa') e^{-\frac{i}{2} \left[ |(\zeta, \kappa-c)^2 - R^2|^2 + |(\zeta', \kappa'-c')^2 - (R')^2|^2 \right]}
\cdot \delta(|\zeta|^2 + |\kappa - c|^2 + |\zeta'|^2 + |\kappa' - c'|^2 - A) d\zeta d\zeta'.
\]
We can split the integral above into two regions: (i) when $|\kappa| \in [R-10, R+2]$, the sum is only over a uniformly bounded number of $\kappa$ and we can use (2.7); and (ii) when $|\kappa| \leq R-10$, in which case we use (2.8) and the rapid decay of $e^{-|\zeta'|}$. In both cases, we obtain a bounded contribution after summing over $j$.

Finally, by symmetry, it remains to consider:
\[
I_{jp} = \sum_{\kappa, \kappa'} \int \mathbf{1}_{S_j}(\zeta, \kappa) \mathbf{1}_{S_p}(\zeta', \kappa') \mathbf{1}_{\{\zeta' \leq |\zeta|\}} e^{-\frac{i}{2} \left[ |(\zeta, \kappa-c)^2 - R^2|^2 + |(\zeta', \kappa'-c')^2 - (R')^2|^2 \right]}
\cdot \delta(|\zeta|^2 + |\kappa - c|^2 + |\zeta'|^2 + |\kappa' - c'|^2 - A) d\zeta d\zeta'.
\]
Note that we may assume $R, R' \geq 50$ since otherwise $S_j$ or $S_p$ is empty. Using (2.7) we estimate
\[
I_{jp} \leq 2e^{-\frac{i}{8}(j^2+p^2)} \left[ I_{jp} + J_{jp}^2 \right],
\]
\[
J_{jp}^1 = \sum_{R-10 \leq |\kappa|, |\kappa'| \leq R+10} \int \mathbf{1}_{S_j} \mathbf{1}_{S_p} \delta(|\zeta|^2 + |\kappa - c|^2 + |\zeta'|^2 + |\kappa' - c'|^2 - A) d\zeta d\zeta',
\]
\[
J_{jp}^2 = \sum_{\kappa'} \int \mathbf{1}_{S_p} \left( \sum_{|\kappa| \leq R-10} \int \mathbf{1}_{S_j} \delta(|\zeta|^2 + |\kappa - c|^2 + |\zeta'|^2 + |\kappa' - c'|^2 - A) d\zeta \right) d\zeta'.
\]
For $J_{jp}^1$, we observe that the sum is only over a uniformly bounded number of $\kappa, \kappa'$ and we can use (2.7). For $J_{jp}^2$, we can use (2.8) followed by Lemma 5.1. Summing over $j, p$, we obtain an acceptable contribution.

In the proof above, we have used two simple bounds that allow us to cancel two integrals.

**Lemma 2.2.** We have
\[
\sup_{A \in \mathbb{R}} \int_{\mathbb{R}^2} \delta(\zeta^2 + \eta^2 - A) d\zeta d\eta = \pi
\] (2.7)
and, for $S_j$ defined as in (2.5) and $R \geq 50$, 

$$
\sup_{A \in R} \sum_{|\kappa| \leq R-10} \int_R 1_{S_j}(\zeta, \kappa)\delta(\zeta^2 - A)d\zeta \lesssim 1, \quad 0 \leq j \leq R^{1/2} + 1.
$$

(2.8)

Proof of Lemma 2.2. The first bound is direct after passing to polar coordinates. To prove (2.8), we may assume $R \geq 50$. We first claim that 

$$(\zeta, \kappa) \in S_j, \quad |\kappa| \leq R - 10 \Rightarrow |\zeta| \geq R^{1/2}(R - |\kappa| - 1)^{1/2}$$

(2.9)

Indeed, on $S_j$, we see that 

$$
\zeta^2 + (\kappa - c)^2 \geq R^2 - 3\sqrt{R} \quad \text{for some } c \in \{0, \frac{1}{2}\} \text{ and }
$$

$$
\zeta^2 \geq (R + |\kappa - c|)(R - |\kappa - c|) - 3\sqrt{R} \geq R(R - |\kappa| - 1) + R/2 - 3\sqrt{R}.
$$

Eliminating some terms and taking square roots give the result. To prove (2.8) we then apply a change of variables along with (2.9) to estimate 

$$
\sum_{|\kappa| \leq R-10} \int_R 1_{S_j}(\zeta, \kappa)\delta(\zeta^2 - A)d\zeta \lesssim \sum_{|\kappa| \leq R-10} R^{-1/2} (R - |\kappa| - 1)^{-1/2} \lesssim 1,
$$

which gives (2.8). \Box

3. On volumes of annuli in $R \times Z$

As we saw in the last section, the contribution of the integral $I_{jp}$ is controlled by the following geometric lemma which says that the volume of a (large and thin) annulus in $R \times Z$ is proportional to its volume in $R^2$. The result is essentially Lemma 2.1 from [12].

Lemma 3.1. For $0 \leq w \leq 20 \leq R$ and $0 \leq |x| \leq 1/2$, 

$$
V(R, w) = |R_x \times Z_\kappa \cap \{R^2 \leq \zeta^2 + (\kappa + x)^2 \leq (R + w)^2\}| \lesssim \sqrt{Rw} + Rw.
$$

As a consequence, for the sets in (2.5) we have $|S_j| \lesssim 1$ for $0 \leq j \leq R^{1/2} + 1$.

Proof of Lemma 3.1. Let 

$$
\ell(y) = \begin{cases} 
\sqrt{(R+w)^2 - y^2} & \text{if } R \leq |y| \leq R + w \\
(R+w)^2 - y^2 - \sqrt{R^2 - y^2} & \text{if } 0 \leq |y| \leq R 
\end{cases}
$$

be the length of the horizontal segment in the annulus under consideration at ordinate $y$. This is maximized at $|y| = R$ when it is at most $\sqrt{3Rw}$. In addition, for $2^p \leq ||\kappa + x| - R| \leq 2^p + 1$ and $32 \leq 2^p \leq R$, we can estimate 

$$
\ell(\kappa + x) \leq \frac{2Rw}{\sqrt{R^2 - \kappa - 21}} \leq 4R^{\frac{1}{2}}2^{-\frac{1}{2}}w.
$$

Summing a bounded number of contributions when $\kappa + x \geq R - 50$ and the above bound otherwise, we conclude that the volume under consideration is at most 

$$
V \lesssim \sqrt{Rw} + R^{\frac{1}{2}}w \sum_p 2^\frac{1}{2} \lesssim \sqrt{Rw} + Rw.
$$

\Box
GLOBAL ENDPOINT STRICHARTZ ESTIMATES ON $\mathbb{R} \times \mathbb{T}$

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