Variations of Checking Stack Automata: Obtaining Unexpected Decidability Properties *

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Abstract. We introduce a model of one-way language acceptors (a variant of a checking stack automaton) and show the following decidability properties:

1. The deterministic version has a decidable membership problem but has an undecidable emptiness problem.
2. The nondeterministic version has an undecidable membership problem and emptiness problem.

There are many models of accepting devices for which there is no difference with these problems between deterministic and nondeterministic versions, i.e., the membership problem for both versions are either decidable or undecidable, and the same holds for the emptiness problem. As far as we know, the model we introduce above is the first one-way model to exhibit properties (1) and (2). We define another family of one-way acceptors where the nondeterministic version has an undecidable emptiness problem, but the deterministic version has a decidable emptiness problem. We also know of no other model with this property in the literature. We also investigate decidability properties of other variations of checking stack automata (e.g., allowing multiple stacks, two-way input, etc.). Surprisingly, two-way deterministic machines with multiple checking stacks and multiple reversal-bounded counters are shown to have a decidable membership problem, a very general model with this property.

1 Introduction

The deterministic and nondeterministic versions of most known models of language acceptors exhibit the same decidability properties for each of the membership and emptiness problems. For example, for one-way models, it is easy to show (by coding the “transition rules” on the input string) that the emptiness problem is decidable for the deterministic version if and only if it is decidable

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for the nondeterministic version. For a formal proof of this, it is possible to ei-
ther use the model of Abstract Families of Acceptors (AFAs) from [1], or a type
of abstract store types used in this paper. For the membership problem, as far
as we know, no one-way model has been shown to exhibit different decidability
properties for deterministic and nondeterministic versions.

A one-way checking stack automaton [2] is similar to a pushdown automaton
that cannot erase its stack, but can enter and read the stack in two-way read-only
mode, but once this mode is entered, the stack cannot change. Here, we intro-
duce a new model of one-way language acceptors that exhibits the decidability
properties above. It is defined by augmenting a checking stack automaton with
reversal-bounded counters, and the deterministic and nondeterministic versions
are denoted by DCSACM and NCSACM, respectively. The models with two-way
input (with end-markers) are called 2DCSACM and 2NCSACM. These are gener-
ized further to models with \( k \) checking stacks: \( k \)-stack 2DCSACM and \( k \)-stack
2NCSACM.

We show the following results concerning membership and emptiness:

1. The membership and emptiness problems for NCSACMs are undecidable,
even when there are only two reversal-bounded counters.
2. The emptiness problem for DCSACM is decidable when there is only one
   reversal-counter but undecidable when there are two reversal-bounded coun-
ters.
3. The membership problem for \( k \)-stack 2DCSACMs is decidable for any \( k \).

We define another family of one-way acceptors where the deterministic ver-
sion has a decidable emptiness problem, but the nondeterministic version has an
undecidable emptiness problem. Further, we introduce a new family with deci-
dable emptiness, containment, and equivalence problems, which is one of the most
powerful families to have these properties (one-way deterministic machines with
one reversal-bounded counter and a checking stack that can only read from the
stack at the end of the input). We also investigate the decidability properties
of other variations of checking stack automata (e.g., allowing multiple stacks,
two-way input, etc.).

2 Preliminaries

This paper requires basic knowledge of automata and formal languages [4].

We use a variety of machine models here, mostly built on top of the checking
stack. It is possible to define each machine model directly. An alternate approach
is to define “store types” first, which describes just the behavior of the store,
including instructions that can change the store, and the manner in which the
store can be read. This can capture standard types of stores studied in the
literature, such as a pushdown, or a counter. Defined generally enough, it can also

\[ ^3 \] A counter is reversal-bounded if there is a bound on the number of changes between
increasing and decreasing.
define a checking stack, or an \( l \)-reversal-bounded counter (one that makes at most \( l \) alternations between increasing and decreasing). Then, machines using one or more store types can be defined, in a standard fashion. A \( \{ \Omega_1, \ldots, \Omega_k \} \) machine is a machine with \( k \) stores, where \( \Omega_i \) describes each store. This is the approach taken here, in a similar fashion to the one taken in [5] or [1] to define these same types of automata. This generality will also help in illustrating what is required to obtain certain decidability properties; see e.g. Lemma 1 and Proposition 1 which are proven generally for arbitrary store types.

First, store types, and machines using store types are defined formally.

A store type is a tuple \( \Omega = (I, I, f, g, c_0, L_I) \), where \( \Gamma^* \) is the store alphabet (potentially infinite available to all stores in these types of machines), \( I \) is the set of allowable sequences of instructions, \( c_0 \) is the initial configuration which is a word in \( \Gamma^* \), and \( L_I \subseteq \Gamma^* \) is the instruction language (over possibly an infinite alphabet) of allowable sequences of instructions, \( f \) is the read function, a partial function from \( \Gamma^* \) to \( I \), and \( g \) is the write function, a partial function from \( \Gamma^* \times I \) to \( \Gamma^* \).

We will study a few examples of store types. First, a stack store type is a machine with

- \( \Gamma \) is an infinite set of store symbols available to stacks, where special symbols \( \downarrow \) are the position of the read/write head in the stack, \( Z_b \in \Gamma \) is the bottom-of-stack marker, and \( Z_t \in \Gamma \) is the top-of-stack marker, with \( I_0 = \Gamma - \{ \downarrow, Z_b, Z_t \} \),
- \( I = \{ \text{push}(y) \mid y \in I_0 \} \cup \{ \text{pop}, \text{stay} \} \cup \{ D, S, U \} \) is the set of instructions of the stack, where the first set are called the push instructions, the second set is the pop and stay instruction, and the third set are the move instructions (down, stay, or up),
- \( L_I = \Gamma^* \), \( c_0 = Z_b \downarrow Z_t \),
- \( f(xa \downarrow x') = a, a \in I_0 \cup \{ Z_t, Z_b \}, x, x' \in \Gamma^* \) with \( xax' \in Z_b \Gamma_0^* Z_t \),
- \( g \) is defined as:
  - \( g(Z_bx \downarrow Z_t, \text{push}(y)) = Z_bxy \downarrow Z_t \) for \( x \in \Gamma_0^*, y \in I_0 \),
  - \( g(Z_bxa \downarrow Z_t, \text{pop}) = Z_bx \downarrow Z_t \) for \( x \in \Gamma_0^*, a \in I_0 \),
  - \( g(Z_bx \downarrow Z_t, \text{stay}) = Z_bx \downarrow Z_t \) for \( x \in \Gamma_0^* \),
  - \( g(Z_bxa \downarrow x', D) = Z_bx \downarrow ax' \) for \( x, x' \in \Gamma^* \), \( a \in \Gamma_0 \cup \{ Z_t \}, \) with \( xax' \in \Gamma_0^* Z_t \),
  - \( g(Z_bx \downarrow x', S) = Z_bx \downarrow x' \) for \( x, x' \in \Gamma^* \), \( xx' \in \Gamma_0^* Z_t \),
  - \( g(Z_bx \downarrow x', U) = Z_bxa \downarrow x' \) for \( x, x' \in \Gamma^* \), \( a \in \Gamma_0 \cup \{ Z_t \}, xax' \in \Gamma_0^* Z_t \).

Many different types of stores can be represented by restricting the store type above. For example, the pushdown store type is obtained by restricting the instructions to only having the push and pop instructions (it is also then possible to remove the read/write head and the top of the stack symbol). Furthermore, a counter store tape can be obtained by restricting pushdowns to only having a single symbol \( c \in I_0 \) (plus the bottom-of-stack marker).

The instruction language \( L_I \) in the definition of \( \Omega \) restricts the allowable sequences of instructions available to the store type \( \Omega \). This restriction does not exist in the definition of AFAs, but can be used to define many classically studied machine models, while still preserving many useful properties. For example, an
l-reversal-bounded counter store type is a counter store type with \( L_I \) equal to the alternating concatenation of \( (\text{push}(c) \cup \text{stay})^* \) and \( (\text{pop} \cup \text{stay})^* \) with \( l \) applications of concatenation (this is more classically stated as, there are at most \( l \) alternations between increasing and decreasing). Also, the checking stack store type is a restriction of stack store type above where \( L_I \) is restricted to be in \{\text{push}(y), \text{stay} \mid y \in \Gamma_0\}\{(L, S, R)\}^*\). That is, a checking stack has two phases, a “writing phase”, where it can push or stay (no pop), and a “reading phase”, where it enters the stack in read-only mode. But once it starts reading, it cannot change the stack again.

Given store types \( (\Omega_1, \ldots, \Omega_k) \), with \( \Omega_i = (\Gamma_i, I_i, f_i, g_i, c_0, I_i, L_i) \), a two-way r-head \( k \)-tape \( (\Omega_1, \ldots, \Omega_k) \)-machine is a tuple \( M = (Q, \Sigma, \Gamma, \delta, \triangleright, \lessdot, <, q_0, F) \) where \( Q \) is the finite set of states, \( q_0 \in Q \) is the initial state, \( F \subseteq Q \) is the set of final states, \( \Sigma \) is the input alphabet, \( \Gamma \) is a finite subset of the store alphabets of \( I_1 \cup \cdots \cup I_k \), \( \delta \) is the finite transition relation from \( Q \times \Sigma^* \times I_1 \times \cdots \times I_k \) to \( Q \times I_1 \times \cdots \times I_k \times \{(-1, 0, +1)\}^* \).

An instantaneous description (ID) is a tuple \( (q, \triangleright w <, a_1, \ldots, a_r, x_1, \ldots, x_k) \), where \( q \in Q \) is the current state, \( w \) is the current input word (surrounded by left input end-marker and right input end-marker), \( 0 \leq a_j \leq |w| + 1 \) is the current position of tape head \( j \) (this can be thought of as \( 0 \) scanning \( \triangleright \), and \( |w| + 1 \) scanning \( < \)), for \( 1 \leq j \leq r \), and \( x_i \in I_i^{*} \) is the current word in the \( \Omega_i \) store, for \( 1 \leq i \leq k \). Then \( M \) is deterministic if \( \delta \) is a partial function (ie. it only maps each element to at most one element).

Then \( (q, \triangleright w <, a_1, \ldots, a_r, x_1, \ldots, x_k) \vdash_M (q', \triangleright w <, a'_1, \ldots, a'_r, x'_1, \ldots, x'_k) \), (two IDs) if there exists \( (q', a_1, \ldots, a_r, b_1, \ldots, b_k) \in \delta(q, a_1, \ldots, a_r, b_1, \ldots, b_k) \), where \( a_j \) is character \( a_j + 1 \) of \( \triangleright w < \), and \( a'_j = a_j + \gamma_j \), for \( 1 \leq j \leq r \), \( b_i = f_i(x_i) \), and \( g_i(x_i, \gamma_i) = x'_i \) for \( 1 \leq i \leq k \). We let \( M^* \) be the reflexive and transitive closure of \( M \). The language accepted by \( M \), \( L(M) \) is equal to

\[
\{w \mid (q_0, \triangleright w <, 1, \ldots, 1, c_{0,1}, \ldots, c_{0,k}) \vdash_M (q_f, \triangleright w <, a_1, \ldots, a_r, x_1, \ldots, x_k), q_f \in F\}.
\]

The different machine modes are combinations of either one-way or two-way, deterministic or nondeterministic, and r-head for some \( r \geq 1 \). For example, one-way, 1-head, deterministic, is a machine mode. Given a sequence of store types \( \Omega_1, \ldots, \Omega_k \) and a machine mode, one can study the set of all \( (\Omega_1, \ldots, \Omega_k) \) machines with this mode. The set of all such machines with a mode is said to be complete. Any strict subset is said to be incomplete. Given a set of (complete or incomplete) machines \( M \) of this type, the family of languages accepted by these machines is denoted \( L(M) \). For example, the set of all one-way deterministic pushdown automata is complete. But consider the set of all one-way deterministic pushdown automata that can only decrease the size of the stack when scanning the right end-marker. Then, the instructions available to such machines depend on the location of the input (whether it has reached the end of the input or not). Therefore, this is an incomplete set of automata. Later in the paper, we will consider variations of checking stack automata such as one called no-read, which means that they do not read from the inside of the checking stack.
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before hitting the right input end-marker. This is similarly an incomplete set of automata since the instructions allowed differ depending on the input position.

The class of one-way deterministic (resp. nondeterministic) checking stack automata is denoted by DCSA (resp., NCSA) \[2\]. The class of deterministic (resp. nondeterministic), finite automata is denoted by DFA (resp., NFA) \[4\].

For \(k, l \geq 1\), the class of one-way deterministic (resp. nondeterministic) \(l\)-reversal-bounded \(k\)-counter machines is denoted by DCM\((k, l)\) (resp. NCM\((k, l)\)). If only one integer is used, e.g. NCM\((k)\), this class contains all \(l\)-reversal-bounded \(k\) counter machines, for some \(l\), and if the integers is omitted, e.g., NCM and DCM, they contain all \(l\)-reversal-bounded \(k\) counter machines, for some \(k, l\). Note that a counter that makes \(l\) reversals can be simulated by \(\lceil \frac{l+1}{2} \rceil\) 1-reversal-bounded counters \[7\]. Closure and decidable properties of various machines augmented with reversal-bounded counters have been studied in the literature (see, e.g., \[7\]). For example, it is known that the membership and emptiness problems are decidable for NCM \[7\].

Also, here we will study the following new classes of machines that have not been studied in the literature: one-way deterministic (resp. nondeterministic) machines defined by stores consisting of one checking stack and \(k l\)-reversal-bounded counters, denoted by DCSACM\((k, l)\) (resp. NCSACM\((k, l)\)), those with \(k\)-reversal-bounded counters, denoted by DCSACM\((k)\) (resp. NCSACM\((k)\)), and those with some number of reversal-bounded counters, denoted by DCSACM (resp. NCSACM).

All models above also have two-way versions of the machines defined, denoted by preceding them with 2, e.g. 2DCSA, 2NCSA, 2NCM\((1)\), 2DFA, 2NFA, etc.

We will also define models with \(k\) checking stacks for some \(k\), which we will precede with the phrase “\(k\)-stack”, e.g. \(k\)-stack 2DCSA, \(k\)-stack 2NCSA, \(k\)-stack 2DCSACM, \(k\)-stack 2NCSACM, etc. When \(k = 1\), then this corresponds with omitting the phrase “\(k\)-stack”.

3 A Checking Stack with Reversal-Bounded Counters

Before studying a new type of store and machine model, we determine several properties that are equivalent for any complete set of machines. This helps to demonstrate what is required to potentially have a machine model where the deterministic version has a decidable membership problem with an undecidable emptiness problem, while both problems are undecidable for the nondeterministic version.

First, we examine a machine’s behavior on one word.

Lemma 1. Let \(M\) be a one- or two-way, \(r\)-head, for some \(r \geq 1\), \((\Omega_1, \ldots, \Omega_k)\)-machine, and let \(w \in \Sigma^*\). We can effectively construct another \((\Omega_1, \ldots, \Omega_k)\)-machine \(M_w\) that is one-way and 1-head which accepts \(\lambda\) if and only if \(M\) accepts \(w\). Furthermore, \(M_w\) is deterministic if \(M\) is deterministic.

Proof. The input \(w\) is encoded in the state of \(M_w\), and \(M_w\) on input \(\lambda\), simulates the computation of \(M\) and accepts \(\lambda\) if and only if \(M\) accepts \(w\). This uses a
subset of the sequence of transitions used by $M$. Since $M_w$ is only reading $\lambda$, two-way input is not needed in $M_w$, and the $r$-heads are simulated completely in the finite control. \hfill \Box

Then, for all machines with the same store types, the following decidability problems are equivalent:

**Proposition 1.** Consider store types $(\Omega_1, \ldots, \Omega_k)$. The following problems are equivalently decidable, for the stated complete sets of automata:

1. the emptiness problem for one-way deterministic $(\Omega_1, \ldots, \Omega_k)$-machines,
2. the emptiness problem for one-way nondeterministic $(\Omega_1, \ldots, \Omega_k)$-machines,
3. membership problem for one-way nondeterministic $(\Omega_1, \ldots, \Omega_k)$-machines,
4. acceptance of $\lambda$, for one-way nondeterministic $(\Omega_1, \ldots, \Omega_k)$-machines,
5. the membership problem for two-way $r$-head (for $r \geq 1$) nondeterministic $(\Omega_1, \ldots, \Omega_k)$-machines.

**Proof.** The equivalence of 1) and 2) can be seen by taking a nondeterministic machine $M$. Let $T = \{t_1, \ldots, t_m\}$ be labels in bijective correspondence with the transitions of $M$. Then construct $M'$ which operates over alphabet $T$. Then $M'$ reads each input symbol $t$ and simulates $t$ of $M$ on the store, while always moving right on the input. However, if it is a stay transition on the input of $M$, then $M'$ also checks that the next input symbol read (if any) $t'$ is defined on the same letter of $\Sigma$ in $M$. Then $M'$ is deterministic, and changes its stores identically in sequence to $M$, and $L(M')$ is therefore empty if and only if $L(M)$ is empty.

To show that 4) implies 2), notice that any complete set of nondeterministic one-way automata are closed under homomorphisms $h$ where $h(a) \leq 1$, for all letters $a$. Then considering the homomorphism that erases all letters, the resulting language is empty if and only if $\lambda$ is accepted by the original machine.

It is immediate that 5) implies 4), and 4) implies 5) from Lemma [1]. Similarly, 3) implies 4), and 4) implies 3) from Lemma [1].

To see that 2) implies 4), then take a one-way nondeterministic machine and make a new one that cannot accept if there is an input letter. Then this new machine is empty if and only if $\lambda$ is accepted in the original machine. \hfill \Box

It is important to note that this proposition is not necessarily true for incomplete sets of automata, as the machines constructed in the proof need to be present in the set. We will see some natural restrictions later where this is not the case, such as sets of machines where there is a restriction on what instructions can be performed on the store based on the position on the input. And indeed, to prove the equivalence of 1) and 2) above, the deterministic machine created reads a letter for every transition of the nondeterministic machine applied. So, consider a set of automata that is only allowed to apply a strict subset of store instructions before the end-marker. Let $M$ be a nondeterministic machine of this type, and say that $M$ applies some instruction on the end-marker that is not available to the machine before the end-marker. But the deterministic machine $M'$ created from $M$ in Proposition [1] reads an input letter when every instruction
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is applied, even those applied on the end-marker of \( M \). But since \( M' \) is reading an input letter during this operation, it would violate the instructions allowed by \( M' \) before the end-marker.

The above proposition indicates that for complete sets of one-way machines, membership for nondeterminism, emptiness for nondeterminism, and emptiness for determinism are equivalent. Thus, the only one that can potentially differ is membership for deterministic machines. Yet we know of no existing model where it differs from the other three properties. We examine one next.

Next, we will study NCSACMs and DCSACMs, which are NCSAs and DCSAs (nondeterministic and deterministic checking stack automata) respectively, augmented by reversal-bounded counters. First, two examples will be shown each demonstrating a language that can be accepted by a DCSACM.

**Example 1.** Consider the language \( L = \{(a^n\#)^n \mid n \geq 1\} \). A DCSACM \( M \) with one 1-reversal-bounded counter can accept \( L \) as follows: \( M \) when given an input \( w \) (we may assume that the input is of the form \( w = a^n\# \cdots a^1\# \) for some \( k \geq 1 \) and \( n_i \geq 1 \) for \( 1 \leq i \leq k \), since the finite control can check this), copies the first segment \( a^{n_1} \) to the stack while also storing number \( n_1 \) in the counter. Then \( M \) goes up and down the stack comparing \( n_1 \) to the rest of the input to check that \( n_1 = \cdots = n_k \) while decrementing the counter by 1 for each segment it processes. Clearly, \( L(M) = L \) and \( M \) makes only 1 reversal on the counter.

**Example 2.** Let \( L = \{a^ib^jc^k \mid i, j \geq 1, k = i \cdot j\} \). We can construct a DCSACM(1) \( M \) to accept \( L \) as follows. \( M \) reads \( a^i \) and stores \( a^i \) in the stack. Then it reads \( b^j \) and increments the counter by \( j \). Finally, \( M \) reads \( c^k \) while moving up and down the stack containing \( a^i \) and decrementing the counter by 1 every time the stack has moved \( i \) cells, to verify that \( k \) is divisible by \( i \) and \( k/i = j \). Then \( M \) accepts \( L \), and \( M \) needs only one 1-reversal counter.

The following shows that, in general, NCSACMs and DCSACMs are computationally more powerful than NCSAs and DCSAs, respectively.

**Proposition 2.** There are languages in \( \mathcal{L}(\text{DCSACM}(1,1)) - \mathcal{L}(\text{NCSA}) \). Hence, \( \mathcal{L}(\text{DCSA}) \subseteq \mathcal{L}(\text{DCSACM}(1,1)) \), and \( \mathcal{L}(\text{NCSA}) \subseteq \mathcal{L}(\text{NCSACM}(1,1)) \).

**Proof.** Consider the language \( L = \{(a^n\#)^n \mid n \geq 1\} \) from Example 1. \( L \) cannot be accepted by an NCSA; otherwise, \( L' = \{a^{n^2} \mid n \geq 1\} \) can also be accepted by an NCSA (since NCSA languages are closed under homomorphism), but it was shown in \([2]\) that \( L' \) cannot be accepted by any NCSA. However, Example 1 showed that \( L \) can be accepted by a DCSACM.

We now proceed to show that the membership problem for DCSACMs is decidable. In view of Lemma 1 our problem reduces to deciding, given a DCSACM \( M \), whether it accepts \( \lambda \). So we only need to worry about the operation of the checking stack and the counters. For acceptance of \( \lambda \), the next lemma provides a normal form.

**Lemma 2.** Let \( M \) be a DCSACM. We can effectively construct a DCSACM \( M' \) such that:
-- all counters of $M'$ are 1-reversal-bounded and each must return to zero before accepting,
-- $M'$ always writes on the stack at each step during the writing phase,
-- the stack head returns to the left end of the stack before accepting,

whereby $M'$ accepts $\lambda$ if and only if $M$ accepts $\lambda$.

Proof. It is evident that all counters can be assumed to be 1-reversal-bounded as with DCM, and that each counter can be forced to return to zero before accepting. Similarly, the checking stack can be forced to return to the left end before accepting. We introduce a dummy symbol $\$ to the stack alphabet so that if in a step, $M$ does not write on the stack, then $M'$ writes $\$. When $M'$ enters the reading phase, $M'$ simulates $M$ but ignores (i.e., skips over) the $\$'s. Then $M'$ accepts $\lambda$ if and only if $M$ accepts $\lambda$. $\square$

In view of Lemma 2, we may assume that a DCSACM writes a symbol at the end of the stack at each step during the writing phase. This is important for deciding the following problem.

Lemma 3. Let $M$ be a DCSACM satisfying the assumptions of Lemma 2. We can effectively decide whether or not $M$ on $\lambda$ input has an infinite writing phase (i.e., will keep on writing).

Proof. Let $s$ be the number of states of $M$. We construct an NCM $M'$ which, when given an input $w$ over the stack alphabet of $M$, simulates the computation of $M$ on $\lambda$ input while checking that $w$ could be written by $M$ on the stack at some point during the computation of the writing phase of $w$, while also verifying that there is a subword $x$ of $w$ of length $s + 1$ such that $x$ was written by $M$ without:

1. incrementing a counter that has so far been at zero, and
2. decrementing a non-zero counter.

If so, $M$ accepts $w$. Next, it will be argued that $L(M')$ is not empty if and only if $M$ has an infinite writing phase on $\lambda$, and indeed this is decidable since emptiness for NCM is decidable [7].

If $L(M')$ is not empty, then there is a sequence of $s + 1$ transitions during the writing phase where no counter during this sequence is increased from zero, and no counter is decreased. Thus, there must be some state $q$ hit twice by the pigeonhole principle, and the sequence of transitions between $q$ and itself must repeat indefinitely in $M$. Thus, $M$ has an infinite writing phase on $\lambda$ input.

Conversely, assume $M'$ has an infinite writing phase. Then there must be a sequence of $s + 1$ transitions where no counter is decreased, and no counter is increased from zero. Thus, $L(M')$ must be non-empty. $\square$

From this, decidability of acceptance of $\lambda$ is straightforward.

Lemma 4. It is decidable, given a DCSACM $M$ satisfying the assumptions of Lemma 2, whether or not $M$ accepts $\lambda$. 

Proof. From Lemma 3 we can decide if \( M \) has an infinite writing phase. If so, \( M \) will not accept \( \lambda \) (as the stack must return to the bottom before accepting).

If \( M \) does not have an infinite writing phase, the (final) word \( w \) written in the stack is unique and hence has a unique length \( d \). In this case, we can simulate faithfully the computation of \( M \) (on \( \lambda \) input) and determine \( d \).

We then construct a DCM \( M_d \), which on \( \lambda \) input, encodes the stack in the state and simulates \( M \). Thus, \( M_d \) needs a buffer of size \( d \) to simulate the operation of the stack, and \( M_d \) accepts if and only if \( M \) accepts. The result follows, since the membership problem for DCM is decidable [7]. ⊓ ⊔

From Lemmas 1, 2, and 3:

**Proposition 3.** For \( r \geq 1 \), the membership problem for \( r \)-head 2DCSACM is decidable.

We now give some undecidability results. The proofs will use the following result in [7]:

**Proposition 4.** [7] It is undecidable, given a 2DCM(2) \( M \) over a letter-bounded language, whether \( L(M) \) is empty.

**Proposition 5.** The membership problem for NCSACM(2) is undecidable.

Proof. Let \( M \) be a 2DCM(2) machine over a letter-bounded language. Construct from \( M \) an NCSACM \( M' \) which, on \( \lambda \) - input (i.e. the input is fixed), guesses an input \( w \) to \( M \) and writes it on its stack. Then \( M' \) simulates the computation of \( M \) by using the stack and two reversal-bounded counters and accepts if and only if \( M \) accepts. Clearly, \( M' \) accepts \( \lambda \) if and only if \( L(M) \) is not empty which is undecidable by Proposition 4. ⊓ ⊔

By Propositions 1 and 5, the following is true:

**Corollary 1.** The emptiness problem for DCSACM(2) is undecidable.

The next restriction serves to contrast this undecidability result. Consider an NCSACM where during the reading phase, the stack head crosses the boundary of any two adjacent cells on the stack at most \( d \) times for some given \( d \geq 1 \). Call this machine a \( d \)-crossing NCSACM. Then we have:

**Proposition 6.** It is decidable, given a \( d \)-crossing NCSACM \( M \), whether or not \( L(M) = \emptyset \).

Proof. Define a \( d \)-crossing NTMCM to be an nondeterministic Turing machine with a one-way read-only input tape and a \( d \)-crossing read/write worktape (i.e., the worktape head crosses the boundary between any two adjacent worktape cells at most \( d \) times) augmented with reversal-bounded counters. Note that a \( d \)-crossing NCSACM can be simulated by a \( d \)-crossing NTMCM. It was shown in [3] that it is decidable, given a \( d \)-crossing NTMCM \( M \), whether \( L(M) = \emptyset \). The proposition follows. ⊓ ⊔
Although we have been unable to resolve the open problem as to whether the emptiness is decidable for both NCSACM and DCSACM with one reversal-bounded counter, as with membership for the nondeterministic version, we show they are all equivalent to an open problem in the literature.

**Proposition 7.** The following are equivalent:

1. the emptiness problem is decidable for $2NCM(1)$,
2. the emptiness problem is decidable for NCSACM(1),
3. the emptiness problem is decidable for DCSACM(1),
4. the membership problem is decidable for $r$-head $2NCSACM(1)$,
5. it is decidable if $\lambda$ is accepted by a NCSACM(1).

**Proof.** The last four properties are equivalent by Proposition 1.

It can be seen that 2) implies 1) because a NCSACM(1) machine can simulate a $2NCM(1)$ machine by taking the input, copying it to the stack, then simulating the $2NCM(1)$ machine.

Furthermore, it can be seen that 1) implies 5) as follows: given a NCSACM(1) machine $M$, assume without loss of generality, that $M$ immediately and nondeterministically sets the stack and returns to the bottom of the stack in read-only mode in some special state $q$ before changing any counter (as it can verify that $M$ would have pushed the stack contents). Then, build a $2NCM(1)$ machine that on some input over the stack alphabet, simulates the stack using the input, and the counter using the counter starting at state $q$. Then $L(M')$ is non-empty if and only if $\lambda$ is accepted by $M$. □

It is indeed a longstanding open problem as to whether the emptiness problem for $2NCM(1)$ is decidable [7].

Now consider the following three restricted models, with $k$ counters: For $k \geq 1$, a DCSACM($k$) (NCSACM($k$)) machine is said to be:

- no-read/no-counter if it does not read the checking stack nor use any counter before hitting the right input end-marker,
- no-read/no-decrease if it does not read the checking stack nor decrease any counter before hitting the right input end-marker,
- no-read if it does not read the checking stack before hitting the right input end-marker.

We will consider the families of DCSACM($k$) (NCSACM($k$)) machines satisfying each of these three conditions.

**Proposition 8.** For any $k \geq 1$, every $2DCM(k)$ machine can be effectively converted to an equivalent no-read/no-decrease DCSACM($k$) machine, and vice-versa.

**Proof.** First, a $2DCM(k)$ machine $M$ can be simulated by a no-read/no-decrease DCSACM($k$) machine $M'$ that first copies the input to the stack, and simulates the input of $M$ using the checking stack, while simulating the counters faithfully.
Indeed, the checking stack is not read and counters are not decreased until $M'$ reads the entire input.

Next we will prove the converse. Let $M$ be a no-read/no-decrease DCSACM($k$) machine with input alphabet $\Sigma$ and stack alphabet $\Gamma$.

A two-way deterministic gsm, 2DGSM, is a deterministic generalized sequential machine with a two-way input (surrounded by end-markers), accepting states, and output. It is known that if $L$ is a language accepted by a two-way $k$-head deterministic machine augmented with some storage/memory structure (such as a pushdown, checking stack, $k$ checking stacks, etc.), then $T^{-1}(L)$ is also accepted by the same type of machine [5] (where $T^{-1}(L) = \{ x \mid T(x) = y, y \in L \}$).

Let $T$ be 2DGSM which, on input $x \in \Sigma^*$, first outputs $x\#$. Then it moves to the left end-marker and on the second sweep of input $x$, simulates $M$ and outputs the string $z$ written on the stack during the writing phase of $M$. Note that $T$ can successfully do this as $M$ generates the checking stack contents from left-to-right, and does not read the contents during the writing phase; and because the counters of $M$ are not decreased during the writing phase of $M$, the counters can never empty during the writing phase, thereby affecting the checking stack contents created. When $T$ reaches its right end-marker, it outputs the state $s$ of $M$ at that time, and then $T$ enters an accepting state. Thus, $T(x) = x\#zs$.

Now construct a 2DCM($k$) $M'$ which when given a string $x\#zs$, reads $x$, and while doing so, $M'$ simulates the writing phase of $M$ on $x$ by only changing the counters as $M$ would do. Then, $M'$ moves to the right and stores the state $s$ in the finite control. Then $M'$ simulates the reading phase of $M$ on string $z$, starting in state $s$ and the current counter contents, and accepts if and only if $M$ accepts.

It is straightforward to see that $T^{-1}(L(M')) = L$, which can therefore be accepted by a 2DCM($k$) machine. \hfill $\Box$

From this, the following is immediate, since emptiness for 2DCM(1) is known to be decidable [8].

**Corollary 2.** The emptiness problem for no-read/no-decrease DCSACM(1) is decidable.

In the first part of the proof of Proposition 8, the DCSACM($k$) machine created from a 2DCM($k$) machine was also no-read/no-counter. Therefore, the following is immediate:

**Corollary 3.** For $k \geq 1$, the family of languages accepted by the following three sets of machines coincide:

- all no-read/no-decrease DCSACM($k$) machines,
- all no-read/no-counter DCSACM($k$) machines,
- 2DCM($k$).

One particularly interesting corollary of this result is the following:
Corollary 4. 1. The family of languages accepted by no-read/no-decrease (respectively no-read/no-counter) DCSACM(1) is effectively closed under union, intersection, and complementation.

2. Containment and equivalence are decidable for languages accepted by no-read/no-decrease DCSACM(1) machines.

This follows since this family is equal to 2DCM(1), and these results hold for 2DCM(1) [8]. Something particularly noteworthy about closure of languages accepted by no-read/no-decrease 2DCSACM(1) under intersection, is that, the proof does not follow the usual approach for one-way machines. Indeed, it would be usual to simulate two machines in parallel, each requiring its own counter (and checking stack). But here, only one counter is needed to establish intersection, by using a result on two-way machines. Later, we will show that Corollary 4, part 2 still holds for no-read DCSACM(1) s.

Also, since emptiness is undecidable for 2DCM(2), even over letter-bounded languages [7], the following is true:

Corollary 5. The emptiness problem for languages accepted by no-read/no-counter DCSACM(2) is undecidable, even over letter-bounded languages.

Turning now to the nondeterministic versions, from the first part of Proposition 8, it is immediate that for any \( k \geq 1 \), every \( 2NCM(k) \) can be effectively converted to an equivalent no-read/no-decrease NCSACM(k). But, the converse is not true combining together the following two facts:

Proposition 9. 1. For every \( k \geq 1 \), the emptiness problem for languages accepted by \( 2NCM(k) \) over a unary alphabet is decidable.

2. The emptiness problem for languages accepted by no-read/no-counter (or no-read/no-decrease) NCSACM(2) over a unary alphabet is undecidable.

Proof. The first part was shown in [8]. For the second part, it is known that the emptiness problem for \( 2DCM(2) \) \( M \) (even over a letter-bounded language) is undecidable by Proposition [4]. We construct a no-read/no-counter NCSACM(2) \( M' \) which, on a unary input, nondeterministically writes some string \( w \) on the stack. Then \( M' \) simulates \( M \) using \( w \). The result follows since \( L(M') = \emptyset \) if and only if \( L(M) = \emptyset \).

In contrast to part 2 of this proposition:

Proposition 10. For any \( k \geq 1 \), the emptiness problem for languages accepted by no-read/no-decrease DCSACM(\( k \)) machines over a unary alphabet, is decidable.

Proof. If \( M \) is a no-read/no-decrease DCSACM(\( k \)) over a unary alphabet, we can effectively construct an equivalent \( 2DCM(\( k \)) \) \( M \) (over a unary language) from Proposition [8]. The result follows since the emptiness problem for \( 2NCM(\( k \)) \) over unary languages is decidable [8].

Combining these two results yields the following somewhat strange contrast:
Corollary 6. Over a unary input alphabet and for all $k \geq 2$, the emptiness problem for no-read/no-counter NCSACM($k$)s is undecidable, but decidable for no-read/no-counter DCSACM($k$)s.

As far as we know, this demonstrates the first known example of a family of one-way acceptors where the nondeterministic version has an undecidable emptiness problem, but the deterministic version has a decidable emptiness problem. This presents an interesting contrast to Proposition 1 where it was shown that for complete sets of automata for any store types, the emptiness problem of the deterministic version is decidable if and only if it is decidable for the nondeterministic version. However, the set of unary no-read/no-counter NCSACM($k$) machines can be seen to not be a complete set of automata, as a complete set of machines contains every possible machine involving a store type. This includes those machines that read input letters while performing read instructions on the checking stack. And indeed, to prove the equivalence of 1) and 2) in Proposition 1, the deterministic machine created reads a letter for every transition applied, which can produce machines that are not of the restriction no-read/no-counter.

When there is only one counter, decidability of the emptiness problem for no-read/no-decrease NCSACM(1), and for no-read/no-counter NCSACM(1) can be shown to be equivalent to all problems listed in Proposition 7. This is because 2) of Proposition 7 implies each immediately, and each implies 1) of Proposition 7 as a 2NCM(1) machine $M$ can be converted to a no-read/no-decrease, or no-read/no-counter NCSACM(1) machine where the input is copied to the stack, and then the 2NCM(1) machine simulated.

Therefore, it is open as to whether the emptiness problem for no-read/no-decrease (or no-read/no-counter) NCSACM(1) is decidable, as this is equivalent to the emptiness problem for 2NCM(1). One might again suspect that decidability of emptiness for no-read/no-decrease DCSACM(1) implies decidability of emptiness for no-read/no-decrease NCSACM(1) by Proposition 7. However, it is again important to note that Proposition 7 only applies to complete sets of machines, including those machines that read input letters while performing read instructions on the checking stack, again violating the ‘no-read/no-decrease’ condition.

Even though it is open as to whether the emptiness problem is decidable for no-read/no-decrease NCSACM(1) s, we have the following result, which contrasts Corollary 3 part 2:

**Proposition 11.** The universe problem is undecidable for no-read/no-counter NCSACM(1) s. (Thus, containment and equivalence are undecidable.)

*Proof.* It is known that the universe problem for a one-way nondeterministic 1-reversal-bounded one-counter automaton $M$ is undecidable [9]. Clearly, we can construct a no-read/no-counter NCSACM(1) $M'$ to simulate $M$. \hfill $\Box$

In the definition of a no-read/no-decrease DCSACM, we imposed the condition that the counters can only decrement when the input head reaches the end-marker. Consider the weaker condition no-read, i.e., the only requirement
is that the machine can only enter the stack when the input head reaches the end-marker, but there is no constraint on the reversal-bounded counters. It is an interesting open question about whether no-read DCSACM($k$) languages are also equivalent to a 2DCM($k$) (we conjecture that they are equivalent). However, the following stronger version of Corollary 2 can be proven.

**Proposition 12.** The emptiness problem is decidable for no-read DCSACM(1)s.

**Proof.** Let $M$ be a no-read DCSACM(1). Let $T = \{t_1, \ldots, t_m\}$ be symbols in bijective correspondence with transitions of $M$ that can occur in the writing phase. Then, build a 2DCM(1) machine $M'$ that, on input $w$ over $T$, reads $w$ while changing states as $w$ does, and changing the counter as the transitions do. Let $q$ be the state where the last transition symbol ends. Then, at the end of the input, $M'$ simulates the reading phase of $M$ starting in $q$ by scanning $w$, and interpreting a letter $t$ of $w$ as being the stack letter written by $t$ in $M$ (while always skipping over a letter $t$ if $t$ does not write to the stack in $M$). Then $L(M')$ is empty if and only if $L(M)$ is empty. □

We can further strengthen Proposition 12 somewhat. Define a restricted no-read NCSACM(1) to be a no-read NCSACM(1) which is only nondeterministic during the writing phase. Then the proof of Proposition 12 applies to the following, as the sequence of transition symbols used in the proof can be simulated deterministically:

**Proposition 13.** The emptiness problem is decidable for languages accepted by restricted no-read NCSACM(1)s.

While we are unable to show that the intersection of two no-read DCSACM(1) languages is a no-read DCSACM(1) language, we can prove:

**Proposition 14.** It is decidable, given two no-read DCSACM(1)s $M_1$ and $M_2$, whether $L(M_1) \cap L(M_2) = \emptyset$.

**Proof.** Let $M_1$ and $M_2$ be no-read DCSACM(1) over input alphabet $\Sigma$. Let $T_i$ be symbols in bijective correspondence with transitions of $M_i$ that can occur in the writing phase, for each $i$. Let $T'$ be the set of all pairs of symbols $(r, s)$, where $r$ is a transition of $M_1$, $s$ is a transition of $M_2$, and where both $r$ and $s$ read the same input letter of $\Sigma$. Let $T''$ be all those symbols $(r, \$)$ where $r$ is a transition of $M_1$ that reads $\lambda$, and let $T'''$ be all those symbols $(\$, s)$ where $s$ is a transition of $M_2$ that reads $\lambda$.

Build a 2DCM(1) machine $M'$ operating over alphabet $T' \cup T'' \cup T'''$. On input $w$, $M'$ verifies that the first component changes states as $M_1$ does (skipping over any $\$ symbol), and changing the counter as $M_1$ does. Let $q$ be the state where the last transition symbol ends. Then, at the end of the input, $M'$ simulates the reading phase of $M_1$ starting in $q$ by scanning $w$, and interpreting a letter $t \neq \$ in the first component of $w$ as being the stack letter written by $t$ in $M$, and skipping over $\$ or any $t$ that does not write to the stack. After completion, $M'$ does the same thing with $M_2$ using the second component. Notice that
the alphabet is structured such that a transition of $M_1$ on a letter $a \in \Sigma$ is used exactly when a transition of $M_2$ using $a \in \Sigma$ is used, since $M_1$ and $M_2$ are both no-read (so their entire input is used before the reading phases starts). For example, a word $w = (s_1,r_1)(s_2,\$)(s_3,\$)(\$ r_2)\(s_4,r_3)$ implies $s_1$ reads the same input letter in $M_1$ as does $r_1$ in $M_2$, similarly with $s_4$ and $r_2$, $s_2$ and $s_3$ are $\lambda$ transitions in $M_1$, and $r_2$ is a $\lambda$ transition in $M_2$. Hence, $L(M')$ is empty if and only if $L(M_1) \cap L(M_2)$ is empty. \qed

One can show that no-read $\text{DCSACM}(1)$ languages are effectively closed under complementation. Thus, from Proposition 14:

**Corollary 7.** The containment and equivalence problems are decidable for no-read $\text{DCSACM}(1)$s.

No-read $\text{DCSACM}(1)$ is indeed quite a large family for which emptiness, equality, and containment are decidable. The proof of Proposition 14 also applies to the following:

**Proposition 15.** It is decidable, given two restricted no-read $\text{NCSACM}(1)$s $M_1$ and $M_2$, whether $L(M_1) \cap L(M_2) = \emptyset$.

Finally, consider the general model $\text{DCSACM}(1)$ (i.e., unrestricted). While it is open whether no-read $\text{DCSACM}(1)$ is equivalent to $2\text{DCM}(1)$, we can prove:

**Proposition 16.** $L(2\text{DCM}(1)) \subsetneq L(\text{DCSACM}(1))$.

*Proof.* It is obvious that any $2\text{DCM}(1)$ can be simulated by a $\text{DCSACM}(1)$ (in fact by a no-read/no-counter $\text{DCSACM}(1)$). Now let $L = \{a^i b^j c^k \mid i, j \geq 1, k = i \cdot j\}$. We can construct a $\text{DCSACM}(1)$ $M$ to accept $L$ by Example 2. However, it was shown in [10] that $L$ cannot be accepted by a $2\text{DCM}(1)$ by a proof that shows that if $L$ can be accepted by a $2\text{DCM}(1)$, then one can use the decidability of the emptiness problem for $2\text{DCM}(1)$s to show that Hilbert’s Tenth Problem is decidable. \qed

4 Multiple Checking-Stacks with Reversal-Bounded Counters

In this section, we will study deterministic and nondeterministic $k$-checking-stack machines. These are defined by using multiple checking stack stores. Implied from this definition is that each stack has a “writing phase” followed by a “reading phase”, but these phases are independent for each stack.

A $k$-stack $\text{DCSA}$ ($\text{NCSA}$ respectively) is the deterministic (nondeterministic) version of this type of machine. The two-way versions (with input end-markers) are called $k$-stack $2\text{DCSA}$ and $k$-stack $2\text{NCSA}$, respectively. These $k$-stack models can also be augmented with reversal-bounded counters and are called $k$-stack $\text{DCSACM}$, $k$-stack $\text{NCSACM}$, $k$-stack $2\text{DCSACM}$, and $k$-stack $2\text{NCSACM}$. 
Consider a \( k \)-stack DCSACM \( M \). By Lemma 1, for the membership problem, we need only investigate whether \( \lambda \) is accepted. Also, as in Lemma 2, we may assume that each stack pushes a symbol at each move during its writing phase, and that all counters are 1-reversal-bounded.

We say that \( M \) has an infinite writing phase (on \( \lambda \) input) if no stack enters a reading phase. Thus, all stacks will keep on writing a symbol at each step. If \( M \) has a finite writing phase, then directly before a first such stack enters its reading phase, all the stacks would have written strings of the same length.

**Lemma 5.** Let \( k \geq 1 \) and \( M \) be a \((k+1)\)-stack DCSACM satisfying the assumption of Lemma 2.

1. We can determine if \( M \) has an infinite writing phase. If so, \( M \) does not accept \( \lambda \).
2. If \( M \) has a finite writing phase, we can construct a \( k \)-stack DCSACM \( M'' \) such that \( M'' \) accepts \( \lambda \) if and only if \( M \) accepts \( \lambda \).

**Proof.** Let \( M \) have \( s \) states and stack alphabets \( \Gamma_1, \ldots, \Gamma_{k+1} \). Let \( \Gamma = \{ [a_1, \ldots, a_{k+1}] | a_i \in \Gamma_i, 1 \leq i \leq k+1 \} \). By assumption, each stack of \( M \) writes a symbol during its writing phase.

We can determine if \( M \) has a finite writing phase as follows: As in Lemma 3, we construct an NCM \( M' \) which, when given an input \( w \in \Gamma^* \), simulates the computation of \( M \) on \( \lambda \) such that the input \( w \) was written by \( M \) (in a component-wise fashion on each checking stack) and there is a subword \( x \) of \( w \) of length \( s+1 \) such that the subword was written by \( M \) without:

1. incrementing a counter that has so far been at zero, and
2. decrementing a non-zero counter.

If so, \( M \) accepts \( w \). So we need only check if \( L(M') \) is not empty, which is decidable since emptiness is decidable for NCM [7]. Then, \( M \) does not accept \( \lambda \) if and only if \( M \) has an infinite writing phase, and if and only if \( L(M') \) is not empty, which is decidable.

If \( L(M') \) is empty, we then simulate \( M \) faithfully to determine the unique word \( w \in \Gamma^* \) and its length \( d \) just before the reading phase of at least one of the stacks, say \( S_i \), is entered. Note that by construction, no stack entered its stack earlier.

We then construct a \( k \)-stack DCSACM \( M'' \) which, on \( \lambda \) input, encodes the operation of stack \( S_i \) in the state and simulates \( M \). Thus, \( M'' \) needs a buffer of size \( d \) to simulate the operation of stack \( S_i \). \( M'' \) accepts if and only if \( M \) accepts, and has one less stack than \( M \).

Notice that \( M'' \) has fewer stacks than \( M \). Then, from Proposition 3 (the result for a single stack) and using Lemma 5 recursively:

**Proposition 17.** The membership problem for \( k \)-stack DCSACMs is decidable.

Then, by Lemma 1:

**Corollary 8.** The membership problem for \( r \)-head \( k \)-stack 2DCSACM is decidable.
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