Parallel integrative learning for large-scale multi-response regression with incomplete outcomes *

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Abstract

Multi-task learning is increasingly used to investigate the association structure between multiple responses and a single set of predictor variables in many applications. In the era of big data, the coexistence of incomplete outcomes, large number of responses, and high dimensionality in predictors poses unprecedented challenges in estimation, prediction and computation. In this paper, we propose a scalable and computationally efficient procedure, called PEER, for large-scale multi-response regression with incomplete outcomes, where both the numbers of responses and predictors can be high-dimensional. Motivated by sparse factor regression, we convert the multi-response regression into a set of univariate-response regressions, which can be efficiently implemented in parallel. Under some mild regularity conditions, we show that PEER enjoys nice sampling properties including consistency in estimation, prediction, and variable selection. Extensive simulation studies show that our proposal compares favorably with several existing methods in estimation accuracy, variable selection, and computation efficiency.

Keywords — High dimensionality, Incomplete data, Latent factors, Multi-task learning, Singular value decomposition

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1 Introduction

Multi-task learning has been widely used in various fields, such as bioinformatics [Kim et al., 2009, Hilafu et al., 2020], econometrics [Fan et al., 2019], social network analysis [Zhu et al., 2020], and recommender systems [Zhu et al., 2016], when one is interested in uncovering the association between multiple responses and a single set of predictor variables. Multi-response regression is one of the most important tools in multi-task learning. For example, investigating the relationship between several measures of health of a patient (i.e., cholesterol, blood pressure, and weight) and eating habits of this patient, or simultaneously predicting asset returns for several companies via vector autoregression models, both result in multi-response regression problems.

In the high-dimensional setting where the number of predictors is large, it is challenging to infer the association between predictors and responses because the responses may depend on only a subset of predictors. To address this issue and recover sparse response-predictor associations, many regularization methods for multi-response regression models have been proposed; see, for example, Rothman et al. [2010], Bunea et al. [2011, 2012], Chen and Huang [2012], Chen et al. [2012], Chen and Chan [2016], Uematsu et al. [2019], and the references therein. In particular, Chen et al. [2012] and Chen and Chan [2016] have proposed sparse reduced-rank regression approaches, which combine the regularization and reduced-rank regression techniques [Izenman, 1975, Velu and Reinsel, 2013], and Uematsu et al. [2019] suggested the method of sparse orthogonal factor regression via the sparse singular value decomposition with orthogonality constrained optimization to find the underlying association networks.

In the era of big data, the coexistence of missing values, large number of responses, and high dimensionality in predictors is increasingly common in many applications. When both numbers of responses and predictors are large, the aforementioned methods may become inefficient because they are computationally intensive. In addition, these methods are not applicable to incomplete data because they mainly focus on full data problems. To obtain scalable estimation of sparse reduced-rank regression, some approaches based on sequential estimation techniques have been developed in recent years. To name a few, Mishra et al. [2017] proposed a sequential extraction procedure for model estimation, which extracts unit-rank factorization one by one in a sequential fashion, each time with the previously extracted components removed from the current response matrix. Although Mishra et al. [2017] also considered extensions to incomplete outcomes, they did not provide the theoretical justification for the case with incomplete outcomes. In addition, the sequential steps in their procedure may result in the error accumulation. Alternatively, Zheng et al. [2019] converted the sparse and low-rank regression problem to a sparse generalized eigenvalue problem and recovered the underlying coefficient matrix in a similar sequential fashion. Although this method has been shown to enjoy desirable theoretical properties, it cannot be applied directly to missing data.

In this paper, we propose a new methodology of parallel integrative learning regression (PEER) for large-scale multi-task learning with incomplete outcomes, where both responses and predictors are possibly of high dimensions. PEER is a novel two-step procedure, where in the first step we consider a constrained optimization and use an iterative singular value...
thresholding algorithm to obtain some initial estimates, and then in the second step we convert the multi-response regression into a set of univariate-response regressions, which can be efficiently implemented in parallel.

The major contributions of this paper are threefold. First, the proposed procedure PEER provides a scalable and computationally efficient approach to large-scale multi-response regression models with incomplete outcomes. PEER can uncover the association between multiple responses and a single set of predictor variables while simultaneously achieving dimension reduction and variable selection. Second, our procedure PEER addresses the error accumulation problem in existing sequential estimation approaches by converting the multi-response regression into a set of parallel univariate-response regressions. Third, we provide theoretical guarantees for PEER by establishing oracle inequalities in estimation and prediction. Our theoretical analysis shows that PEER can consistently estimate the singular vectors, latent factors as well as the regression coefficient matrix, and accurately predict the multivariate response vector under mild conditions. To the best of our knowledge, there is no existing theoretical result on large-scale multi-response regression with incomplete outcomes. Our theoretical results are new to the literature.

The rest of this paper is organized as follows. Section 2 introduces the model setting and our new procedure PEER. Section 3 establishes non-asymptotic properties of PEER in high dimensions. Section 4 illustrates the advantages of our method via extensive simulation studies. Section 5 presents the results of a real data example. Section 6 concludes with some discussions. All the proofs are relegated to the Appendix.

2 Model and Methodology

In this section, we first introduce our model setting and briefly review sparse orthogonal factor regression framework for high-dimensional multi-response regression models. We then present our new approach PEER.

2.1 Model setting and sparse orthogonal factor regression

Given $n$ observations of the vector of responses $\mathbf{y} \in \mathbb{R}^q$ and vector of predictors $\mathbf{x} \in \mathbb{R}^p$, we consider the following multi-response regression model

$$
\mathbf{Y} = \mathbf{X} \mathbf{C}^* + \mathbf{E},
$$

where $\mathbf{Y} = (y_1, \ldots, y_n)^T \in \mathbb{R}^{n \times q}$ is the response matrix, $\mathbf{X} = (x_1, \ldots, x_n)^T \in \mathbb{R}^{n \times p}$ is the design matrix, $\mathbf{C}^* \in \mathbb{R}^{p \times q}$ is the regression coefficient matrix, and $\mathbf{E} = (e_1, \ldots, e_n)^T \in \mathbb{R}^{n \times q}$ is the error matrix. We consider fixed design in this paper and assume that responses and predictors are centered so that there is no intercept term. Without loss of generality, we assume that each column of $\mathbf{X}$ is rescaled to have an $\ell_2$-norm $n^{1/2}$.

Similar to Mishra et al. [2017], Uematsu et al. [2019] and Zheng et al. [2019], we consider model (1) from a latent factor regression point of view. More specifically, assume the matrix rank of $\mathbf{C}^*$ is $r^*$ with $r^* \leq \min\{p, q\}$. We can write $\mathbf{C}^* = \mathbf{U}^* \mathbf{D}^* \mathbf{V}^*^T$ where $\mathbf{U}^* = (\mathbf{u}_1^*, \ldots, \mathbf{u}_{r^*}^*) \in \mathbb{R}^{p \times r^*}$, $\mathbf{V}^* = (\mathbf{v}_1^*, \ldots, \mathbf{v}_{r^*}^*) \in \mathbb{R}^{q \times r^*}$, and $\mathbf{D}^* = \text{diag}\{d_1^*, \ldots, d_{r^*}^*\}$ is an
sparsity assumption of can be predicted by a relatively small set of common factors. On the other hand, under the regression models with incomplete outcomes where the response matrix $Y$ admits the following representation

$$C^* = U^* D^* V^{*\text{T}} \quad \text{subject to} \quad U^{*\text{T}} \Gamma U^* = V^{*\text{T}} V^* = I_{r^*}.$$  

Note that the population covariance matrix $\Gamma$ is unknown. Using $n^{-1} X^T X$, the Gram matrix of the predictors, to replace its population counterpart $\Gamma$, we have the following decomposition

$$C^* = U^* D^* V^{*\text{T}} \quad \text{subject to} \quad \left( \frac{1}{\sqrt{n}} XU^* \right)^\text{T} \left( \frac{1}{\sqrt{n}} XU^* \right) = V^{*\text{T}} V^* = I_{r^*}. \quad (2)$$

Thus we can write the coefficient matrix $C^*$ as

$$C^* = \sum_{k=1}^{r^*} d_k^* u_k^* v_k^{*\text{T}} = \sum_{k=1}^{r^*} C_k^*,$$

where $C_k^* = d_k^* u_k^* v_k^{*\text{T}}$ is the unit rank matrix corresponding to the $k$th layer of $C^*$, $u_k^*$ and $v_k^*$ are the $k$th column of $U^*$ and $V^*$, respectively, and $d_k^*$ is the $k$th diagonal element of $D^*$.

The decomposition (2) gives a latent factor regression model

$$Y = XU^* D^* V^{*\text{T}} + E \quad \text{subject to} \quad \left( \frac{1}{\sqrt{n}} XU^* \right)^\text{T} \left( \frac{1}{\sqrt{n}} XU^* \right) = V^{*\text{T}} V^* = I_{r^*}. \quad (3)$$

with $r^*$ latent factors, where $X u_k^*$ is the $k$th latent factor, $u_k^*$ gives the weights for constructing the $k$th latent factor, $v_k^*$ describes the impacts of the $k$th latent factor on the response variables, and $d_k^*$ indicates the importance of the $k$th factor for $k = 1, 2, \ldots, r^*$. Each left singular vector $u_k^* \in \mathbb{R}^p$ is assumed to be sparse. Without loss of generality, here we assume that $\text{rank}(X C^*) = \text{rank}(C^*)$ since the redundant part of $C^*$ can be removed if $\text{rank}(X C^*) < \text{rank}(C^*)$ such that it reflects the true number of latent factors. Thanks to the orthogonality of $X U^*$, the sample latent factors are uncorrelated with each other. The low-rank structure imposed on the unknown coefficient matrix $C^*$ yields that all responses can be predicted by a relatively small set of common factors. On the other hand, under the sparsity assumption of $u_k^*$, each latent factor depends on only a subset of original predictors which facilitates the interpretation of model with the high-dimensional data.

However, unlike Mishra et al. [2017] and Uematsu et al. [2019], we do not require the right singular vectors $v_k^*$’s to be sparse. In this paper, we consider large-scale multi-response regression models with incomplete outcomes where the response matrix $Y$ may not be fully observed and both the numbers of responses and predictors can be high-dimensional. Denote by $\mathcal{M}$ the index set of all observed values in the response matrix $Y$, that is,

$$\mathcal{M} = \{(i, j) : y_{ij} \text{ is observed, } 1 \leq i \leq n, 1 \leq j \leq q\}. \quad (4)$$
We will focus on coefficient matrix estimation and variable selection.

Next, we introduce some notation and definitions which will be used throughout the paper. Denote by \( a \wedge b = \min\{a, b\} \) and \( a \vee b = \max\{a, b\} \). For any vector \( \mathbf{a} = (a_i) \), denote by \( \|\mathbf{a}\|_0 \) the number of non-zero entries in \( \mathbf{a} \), and let \( \|\mathbf{a}\|_1, \|\mathbf{a}\|_2, \) and \( \|\mathbf{a}\|_\infty \) be the \( \ell_1 \)-norm, \( \ell_2 \)-norm, and \( \ell_\infty \)-norm, respectively, which are defined as \( \|\mathbf{a}\|_1 = \sum_i |a_i|, \|\mathbf{a}\|_2 = \left( \sum_i a_i^2 \right)^{1/2}, \) and \( \|\mathbf{a}\|_\infty = \max_i |a_i| \). For any matrix \( \mathbf{A} = (a_{ij}) \), denote by \( \|\mathbf{A}\|_F = \left( \sum_{i,j} a_{ij}^2 \right)^{1/2} \), \( \|\mathbf{A}\|_{\text{op}} = \max_{\mathbf{u}\neq 0} \|\mathbf{A}\mathbf{u}\|_2/\|\mathbf{u}\|_2 \), and \( \|\mathbf{A}\|_{\max} = \max_{i,j} |a_{ij}| \) the Frobenius norm, the operator norm, and the entrywise maximum norm, respectively. In addition, we use \( \text{rank}(\mathbf{A}) \) to denote the rank of \( \mathbf{A} \), and \( d_k(\mathbf{A}) \) to denote the \( k \)th largest singular value of \( \mathbf{A} \). Let \( \mathbf{P}_\mathcal{M}(\mathbf{A}) \) denote the projection of \( \mathbf{A} \) onto \( \mathcal{M} \), which is the matrix with the observed elements of \( \mathbf{A} \) preserved, and the missing entries replaced with 0. Then \( \mathbf{A} = \mathbf{P}_\mathcal{M}(\mathbf{A}) + \mathbf{P}_{\mathcal{M}^c}(\mathbf{A}) \). For an index set \( J \subset \{1, \cdots, p\} \), denote by \( J^c \) the complement of a set \( J \) and \( \mathbf{d}_J \) the subvector of \( \mathbf{d} \in \mathbb{R}^p \) formed by components in \( J \). Let \( |J| \) be the cardinality of \( J \). Finally, \( a \lesssim b \) means that \( a \) is less than \( cb \) with some positive constant \( c \).

### 2.2 Parallel integrative learning via PEER

In this subsection, we will introduce our new method PEER. Recall that our goal is to accurately estimate not only the low-rank coefficient matrix \( \mathbf{C}^* \) but also \( \mathbf{u}_k^*, \mathbf{v}_k^* \) and \( d_k^* \) such that we can recover the latent factors, the significant predictors, and their impacts. Motivated by the decomposition in (3), we introduce a two-step procedure, where in the first step we consider a constrained optimization and propose an iterative singular value thresholding algorithm to obtain some initial estimates, and then in the second step we employ a scalable and efficient approach to estimate \( \mathbf{u}_k^*, \mathbf{v}_k^*, d_k^* \), and \( \mathbf{C}^* \) and select important predictors for each latent factor.

The first step of our method PEER is to consider the following constrained optimization problem

\[
(\tilde{\mathbf{D}}, \tilde{\mathbf{Z}}, \tilde{\mathbf{V}}) = \arg \min_{\mathbf{D}, \mathbf{Z}, \mathbf{V}} m^{-1} \|\mathbf{P}_\mathcal{M}(\mathbf{Y}) - \mathbf{P}_\mathcal{M}(\tilde{\mathbf{Z}} \mathbf{D} \mathbf{V}^\mathsf{T})\|_F^2,
\]

subject to \( \mathbf{Z}^\mathsf{T} \mathbf{Z} = \mathbf{V}^\mathsf{T} \mathbf{V} = \mathbf{I}_r \), \( \mathbf{D} \) is an \( r \times r \) diagonal matrix with \( 1 \leq r \leq \min\{p, q\} \), \( \tilde{\mathbf{Z}} = (\tilde{z}_1, \ldots, \tilde{z}_r) \in \mathbb{R}^{n \times r} \) and \( \tilde{\mathbf{V}} = (\tilde{v}_1, \ldots, \tilde{v}_r) \in \mathbb{R}^{q \times r} \). Without loss of generality, we assume that the singular values in \( \mathbf{D} \) are placed in descending order. As pointed out by Uematsu et al. [2019], when prior knowledge of the rank \( r^* \) is not available, it is often sufficient in practice to take an \( r \) such that it is slightly larger than the expected rank (estimated by some similar procedure such as in Bunea et al. [2011]). The solution \( (\tilde{\mathbf{D}}, \tilde{\mathbf{Z}}, \tilde{\mathbf{V}}) \) from (5) will be used as our initial estimates to estimate \( \mathbf{u}_k^*, \mathbf{v}_k^*, d_k^* \), and \( \mathbf{C}^* \) in the second step of PEER.

We use an iterative singular value thresholding algorithm to solve the optimization problem (5). To ease the presentation, for any matrix \( \mathbf{A} \in \mathbb{R}^{n \times p} \), let \( \mathbf{A} = \mathbf{L} \mathbf{S} \mathbf{R}^\mathsf{T} \) be the singular value decomposition of \( \mathbf{A} \) with a diagonal matrix \( \mathbf{S} \) including all singular values of \( \mathbf{A} \). We define \( \mathbf{T}(\mathbf{A}; r) = \mathbf{L} \mathbf{T}(\mathbf{S}; r) \mathbf{R}^\mathsf{T} \), where \( \mathbf{T}(\mathbf{S}; r) \) is the diagonal matrix with the first \( r \) largest
Then we can estimate regression coefficient matrix \( \mathbf{r} \) using a thresholding procedure to estimate the true rank \( \hat{r} \). Use cross validation or other criterion to estimate the true rank \( \hat{r} \). References therein. Once each \( \hat{r} \) is obtained, we can estimate \( \mathbf{D}^* \) and \( \mathbf{V}^* \) by \( \hat{\mathbf{D}} = n^{-1/2}\mathbf{D} \) and \( \hat{\mathbf{V}} = \mathbf{V} \), respectively. In other words, \( \hat{d}_k \) and \( \hat{v}_k \) are estimated by \( \hat{d}_k = n^{-1/2}\tilde{d}_k \) and \( \hat{v}_k = \tilde{v}_k \), respectively, for \( k = 1, \cdots, r \). Here we rescale \( \hat{\mathbf{D}} \) because the singular values of \( n^{-1/2}\mathbf{Y} \) are \( n^{-1/2} \) times of the singular values of \( \mathbf{Y} \). Note that \( \tilde{z}_k \) is an estimate of the matrix \( n^{-1/2}\mathbf{Xu}_k^* \). We can estimate \( \mathbf{u}_k^* \) by solving the univariate response Lasso regression

\[
\hat{\mathbf{u}}_k = \arg \min_{\mathbf{u}_k \in \mathbb{R}^p} n^{-1} \| \sqrt{n}\tilde{z}_k - \mathbf{Xu}_k \|_2^2 + \lambda_k \| \mathbf{u}_k \|_1,
\]

where \( \lambda_k \) is a regularization parameter and can be tuned by cross-validation or certain information criterion. The univariate response Lasso regression [Tibshirani, 1996] has been studied extensively in the literature and many efficient algorithms have been proposed for solving it. See, for examples, Efron et al. [2004], Zhao and Yu [2006], Friedman et al. [2007], Bunea et al. [2007], Van de Geer [2008], Wu and Lange [2008], Bickel et al. [2009], and the references therein. Once each \( (\hat{d}_k, \hat{v}_k, \tilde{v}_k) \) is obtained for \( k = 1, \cdots, r \) with a given \( r \), one can use cross validation or other criterion to estimate the true rank \( r^* \). In this paper, we propose a thresholding procedure to estimate \( r^* \) in Theorem 3.3. Denote by \( \hat{r} \) the estimated rank. Then we can estimate regression coefficient matrix \( \mathbf{C}^* \) by \( \hat{\mathbf{C}} = \sum_{k=1}^{r} \hat{\mathbf{C}}_k \) with \( \hat{\mathbf{C}}_k = \hat{d}_k \hat{\mathbf{u}}_k \hat{\mathbf{v}}_k^T \). This leads to our complete algorithm for PEER, which is described in Algorithm 2.

We remark that although the \( L_1 \) penalty is used in (6), one can use any favorite variable selection method in the second step of PEER, for example, Adaptive Lasso [Zou, 2006], SCAD [Fan and Li, 2001], SICA [Lv and Fan, 2009], and MCP [Zhang, 2010], among many others. See also Fan and Lv [2013] for the asymptotic equivalence of various regularization methods.

Recall that there are \( p \) predictors and \( q \) responses in our model (3). Thus, the original problem of model fitting and variable selection in our model involves a large-scale optimiza-
Algorithm 2 PEER

Input: response matrix $Y \in \mathbb{R}^{n \times q}$, design matrix $X \in \mathbb{R}^{n \times p}$, initial rank $r$, and tolerance parameter $\epsilon$

1: obtain $\bar{Z}, \bar{D}$ and $\bar{V}$ from (5) using Algorithm 1. \hspace{1cm} \triangleright$\text{Step one}$
2: for all $k = 1, \cdots, r$ do \hspace{1cm} \triangleright$\text{Step two}$
3: obtain $\hat{u}_k$ from (6) using $\bar{z}_k$ and $X$ where $\bar{z}_k$ is the $k$th column of $\bar{Z}$
4: $\hat{v}_k \leftarrow \bar{v}_k$ where $\bar{v}_k$ is the $k$th column of $\bar{V}$
5: $\hat{d}_k \leftarrow \frac{n^{-1/2} \bar{d}_k}{\sqrt{\hat{d}_k}}$ where $\bar{d}_k$ is the $k$th main diagonal element of $\bar{D}$
6: end for
7: use (9) to estimate the true rank $r^*$ and obtain $\hat{C}$
8: return $\hat{C} = \sum_{k=1}^{r^*} \hat{C}_k$ with $\hat{C}_k = \hat{d}_k \hat{u}_k \hat{v}_k^T$

Our method PEER can be simplified to handle the case with full data where the response matrix $Y$ is fully observed. In this case, there is no need to use Algorithm 1 in the first step to obtain initial estimates $\bar{Z}$, $\bar{D}$ and $\bar{V}$. In fact, when the response matrix $Y$ is fully observed, one can directly consider the singular value decomposition $Y = \bar{Z} \bar{D} \bar{V}^T$ and use $\bar{Z}$, $\bar{D}$ and $\bar{V}$ as initial estimates for the second step of PEER to estimate $u_k^*$, $v_k^*$, $d_k^*$, and $C^*$.
3 Theoretical properties

In this section, we investigate the theoretical properties of PEER. We first list some mild regularity conditions that facilitate our technical analysis.

3.1 Technical conditions

Condition 3.1. The error matrix $E$ has independent sub-Gaussian entries $e_{ij}$ with $\mathbb{E}(e_{ij}) = 0$ and $\text{Var}(e_{ij}) = \sigma^2 > 0$.

Condition 3.2. There exists some constant $\gamma_d > 0$ such that non-zero singular values satisfy $q^{-1/2}(d^*_k - d^*_{k+1}) \geq \gamma_d$ for all $1 \leq k \leq r^* - 1$. In addition, $d^*_k = O(\sqrt{q})$ for $1 \leq k \leq r^*$.

Condition 3.3. There exists certain sparsity level $s$ with a positive constant $\rho_l$ such that

$$\inf_{\delta \in \mathbb{R}} \left\{ \frac{\|X\delta\|_2^2}{n(\|\delta_J\|_2^2 \vee \|\delta_{J^c}\|_2^2)} : |J| \leq s, \|\delta_{J^c}\|_1 \leq 3\|\delta_J\|_1 \right\} \geq \rho_l,$$

where $\delta_{J^c}^{(1)}$ is a subvector of $\delta_{J^c}$ consisting of the $s$ largest components in magnitude.

Condition 3.4. Let $\pi_{ij}$ be the probability of $y_{ij}$ being observed for all $1 \leq i \leq n$ and $1 \leq j \leq q$. Then there exist some constants $\mu \geq 1$ and $\nu \geq 1$ such that

$$\min_{1 \leq i \leq n, 1 \leq j \leq q} \pi_{ij} \geq 1/(\mu n q) \quad \text{and} \quad \max_{1 \leq i \leq n, 1 \leq j \leq q} \left\{ \sum_{i=1}^n \pi_{ij}, \sum_{j=1}^q \pi_{ij} \right\} \leq \nu/(n \wedge q).$$

Condition 3.5. There exists a positive constant $L$ such that $\|XC^*\|_{\max} \leq L$. We also assume $m \geq \max\{\nu^{-1}(n \wedge q) \log^3(n + q), r(n \vee q) \log(n + q)\}$.

Condition 3.1 is a common assumption to control the tail behavior of the random errors. Gaussian distribution and distributions with bounded support are two special examples of sub-Gaussian distribution.

Condition 3.2 requires strict separation among the singular values which can ensure that the first $r^*$ left singular vectors are distinguishable. Condition 3.2 also assumes a spiked eigen-structure which is $d^*_k = O(\sqrt{q})$. This rate is reasonable since we do not impose sparsity on the columns of $C^*$. Similar assumptions can be found in the literature; see, for example, Zheng et al. [2019, 2021].

Condition 3.3 combines the restricted eigenvalue (RE) assumptions in Bickel et al. [2009], which has been been commonly used to establish the oracle inequalities for the Lasso and Dantzig selector [Candes and Tao, 2007].

Condition 3.4 puts constraints on the probabilities of entries of the response matrix being observed. To be specific, the first inequality requires the sampling probability of each entry of $Y$ is bounded below by a positive constant while the second one ensures that neither a row nor a column should be sampled far more frequently than the others. When $\mu = \nu = 1$, the condition corresponds to the special case of uniform sampling. The same condition has also been used in Klopp [2014], Lafond [2015], and Luo et al. [2018].
Condition 3.5 is assumed mainly for theoretical analysis. The first part of Condition 3.5 is not restrictive because we consider fixed design $X$ in this paper. The second part of Condition 3.5 imposes a lower bound on the number of observed entries of $Y$. Intuitively, the estimation may fail when the number of observed entries is too small.

### 3.2 Main results

For the theoretical analysis purpose, we reformulate the optimization (5) as

$$
\tilde{Y} = \arg\min_{A \in \mathcal{Y}} m^{-1}\|P_M(Y) - P_M(A)\|_F^2, \tag{7}
$$

where $\mathcal{Y} = \{A \in \mathbb{R}^{n \times q} : \|A\|_{\max} \leq L, r^* \leq \text{rank}(A) = r\}$. With $\tilde{Y}$, we consider its singular value decomposition $Y = ZDV^T$, where $Z = (\tilde{z}_1, \ldots, \tilde{z}_r) \in \mathbb{R}^{n \times r}$, $\tilde{V} = (\tilde{v}_1, \ldots, \tilde{v}_r)$ and $\tilde{D} = \text{diag}\{\tilde{d}_1, \ldots, \tilde{d}_r\}$. Compared with (5), the only difference is the constraint $\|A\|_{\max} \leq L$ in (7). This constraint is mainly for theoretical analysis since it is not used in our practical implementation for PEER; see Algorithm 2 in Section 2.2 for details. We first introduce the following lemma, which establishes the consistency of top-$r^*$ singular vectors, and top-$r^*$ singular values in the first step of PEER.

**Lemma 1.** Under Conditions 3.1, 3.2, 3.4, and 3.5, we have that

$$
\|\tilde{z}_k - n^{-1/2}Xu_k^*\|_2 \lesssim \gamma_d^{-1}B_n, \quad \|\tilde{v}_k - v_k^*\|_2 \lesssim \gamma_d^{-1}B_n, \quad \frac{|\tilde{d}_k - \sqrt{n}d_k^*|}{\sqrt{nq}} \lesssim B_n,
$$

hold uniformly over $k = 1, \cdots, r^*$ with probability at least $1 - (n + q)^{-1}$, where $B_n$ is given by

$$
B_n = \max \left[ L\mu^{1/2}\left\{ \frac{\log(n + q)}{m} \right\}^{1/4}, \mu(\sigma \vee L)^{1/2}\sqrt{\frac{r(n \vee \sigma)\log(n + q)}{m}} \right]. \tag{8}
$$

The results of Lemma 1 are the bases of our two-step procedure for estimating $u_k^*$, $v_k^*$, $d_k^*$, and $C^*$. The following theorem establishes the estimation and prediction bounds of PEER with incomplete outcomes. It also demonstrates that our method PEER enjoys oracle inequalities with non-asymptotic convergence rates for top-$r^*$ layers.

**Theorem 3.1 (Estimation and prediction bounds with incomplete outcomes).** Suppose that Condition 3.1–3.5 holds and the sparsity level $s \geq \max_{1 \leq k \leq r^*} s_k$ with $s_k = \|u_k^*\|_0$. Choose $\lambda_k = 4c\gamma_d^{-1}B_n$ with $c$ some positive constant, where $B_n$ is defined in (8). Then with probability at least $1 - (n + q)^{-1}$, the following inequalities

$$
\|\hat{u}_k - u_k^*\|_2 \lesssim (\gamma_d \mu)^{-1}\sqrt{s_k}B_n, \quad \|\hat{v}_k - v_k^*\|_2 \lesssim \gamma_d^{-1}B_n,
$$

$$
\frac{q^{-1/2}|\tilde{d}_k - d_k^*|}{\sqrt{n}} \lesssim B_n, \quad \frac{q^{-1/2}\|\tilde{C}_k - C_k^*\|_F}{\sqrt{nq}} \lesssim (\gamma_d \mu)^{-1}\sqrt{s_k}B_n,
$$

$$
\frac{\|X(\hat{u}_k - u_k^*)\|_2}{\sqrt{n}} \lesssim \frac{\sqrt{s_k}B_n}{\gamma_d \sqrt{\mu}}, \quad \frac{\|X(\tilde{C}_k - C_k^*)\|_F}{\sqrt{nq}} \lesssim \frac{\sqrt{s_k}B_n}{\gamma_d \sqrt{\mu}}
$$

hold uniformly over $k = 1, \cdots, r^*$. 

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Theorem 3.1 presents the oracle inequalities and establishes non-asymptotic convergence rates for top-$r^*$ layers when the response matrix $Y$ may not be fully observed. To be specific, it gives the uniform estimation error bounds for top-$r^*$ left singular vectors $u^*_k$, right singular vectors $v^*_k$, singular values $d^*_k$, unit rank matrices $C^*_k$, latent factors $Xu^*_k$, and the uniform prediction error bounds of the top-$r^*$ layers $XC^*_k$. The factor $\sqrt{s_k}$ in the estimation error bounds for $u^*_k$ and $Xu^*_k$ reflects the sparsity constraint as there are $s_k$ non-zero components in $u^*_k$ for each $k = 1, \cdots, r^*$. Note that the bounds on right singular vectors $v^*_k$ do not involve the factor $\sqrt{s_k}$ since there is no sparsity constraint on $v^*_k$.

Our results in Theorem 3.1 are new to the literature. As mentioned in the Introduction, there is no existing theoretical result when the response matrix $Y$ cannot be fully observed. Although Mishra et al. [2017] considered extensions to incomplete data, they did not provide corresponding theoretical justification. To the best of our knowledge, Theorem 3.1 provides the first formal theoretical result on large-scale multi-response regression with incomplete outcomes. In addition, our error bounds for incomplete outcomes are all non-asymptotic while most existing results for complete data are asymptotic. For example, Mishra et al. [2017] and Zheng et al. [2019] have focused on complete data where the response matrix $Y$ is fully observed and all corresponding results are asymptotic except for one non-asymptotic estimation error bound for unit rank matrices $C^*_k$ in Mishra et al. [2017]. However, the non-asymptotic error bound in Mishra et al. [2017] does not admit an explicit form and is given in a recursive fashion, where the error bound for the $k$th unit rank matrix $C^*_k$ is bounded by the sum of estimation errors for the first $k - 1$ unit rank matrices and four additional terms. Of these four terms, one involves the Frobenius norm of true $C^*_k$ and another one measures the size of the left-over signal in the model.

The following proposition shows that our bounds in Theorem 3.1 can be further improved when the response matrix $Y$ is fully observed.

**Proposition 3.2** (Estimation and prediction bounds with full data). Suppose that Condition 3.1–3.3 holds and the sparsity level $s \geq \max_{1 \leq k \leq r^*} s_k$ with $s_k = \|u^*_k\|_0$. Choose $\lambda_k = 4\tilde{c}^{-1}\tilde{B}_n$ with $\tilde{c}$ some positive constant and $\tilde{B}_n = \sigma \left( \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{q}} \right)$. Then with probability at least $1 - 2e^{-(\sqrt{n} + \sqrt{q})^2}$, the following inequalities

$$\|\mathbf{u}_k - u^*_k\|_2 \lesssim (\gamma_d\rho_k)^{-1/2}\sqrt{s_k}\tilde{B}_n,$$

$$\|\mathbf{v}_k - v^*_k\|_2 \lesssim \gamma_d^{-1}\tilde{B}_n,$$

$$\frac{q^{-1/2}|\tilde{a}_k - d^*_k|}{\sqrt{n}} \lesssim \tilde{B}_n,$$

$$\|X(\mathbf{u}_k - u^*_k)\|_2 \lesssim \sqrt{s_k}\tilde{B}_n,$$

$$\|\mathbf{C}_k - C^*_k\|_F \lesssim \gamma_d^{-1}\sqrt{s_k}\tilde{B}_n,$$

$$\|X(\mathbf{C}_k - C^*_k)\|_F \lesssim \frac{\sqrt{nq}}{\gamma_d\sqrt{nq}} \tilde{B}_n,$$

hold uniformly over $k = 1, \cdots, r^*$.

To see the difference between the bounds of PEER with incomplete outcomes and with full data, write $\alpha = m/(nq)$. Then $0 < \alpha \leq 1$ and $1 - \alpha$ is the missing rate of the response matrix $Y$. It follows from the definition of $B_n$ in (8) that the second term in (8) will dominate the first term if $n \geq 2$ by noting that $\nu \geq 1$, $\mu \geq 1$ and $m \leq nq$. Without loss of generality, assume $n \geq 2$. Thus we have $B_n = O \left( \sqrt{m^{-1}r(n \lor q) \log(n + q)} \right) = $
\[ O \left( \sqrt{1/(n \wedge q)} \cdot \sqrt{r \log(n + q)/\alpha} \right) \] since \( nq = (n \wedge q)(n \lor q) \). Note that the term \( \sqrt{1/(n \wedge q)} \) has the same order as \( \tilde{B}_n \). Therefore, the factor \( \sqrt{r \log(n + q)/\alpha} \) reflects the price we pay in dealing with incomplete outcomes, implying that smaller \( \alpha \) leads to larger estimation and prediction errors. It is reasonable since smaller \( \alpha \) means that more entries of \( Y \) cannot be observed and thus corresponds to more challenging case. This has also been observed in our simulations.

When the response matrix \( Y \) is fully observed, Zheng et al. [2019] showed that the error bounds for \( u^*_k, C^*_k, Xu^*_k, \) and \( XC^*_k \) are all in the same order of \( O \left( \sqrt{s/n \log(pq/\delta)} \right) \) with some constant \( \delta \in (0, 1) \) while our error bounds for those are in the order of \( O \left( \sqrt{s_k(1/\sqrt{n} + 1/\sqrt{q})} \right) \) under the same assumption \( \max_{1 \leq k \leq r^*} s_k \leq s \) as used in Zheng et al. [2019]. The factor \( \log(pq) \) in the error bounds of Zheng et al. [2019] can become large and may not be negligible when either the number of predictors \( p \) or the number of responses \( q \) grows rapidly with sample size \( n \). This indicates that our procedure PEER is preferred for big data applications.

The results in Theorem 3.1 indicate that the regression coefficient matrix \( \tilde{C}^* \) can be accurately recovered and prediction error \( (nq)^{-1/2} \| \hat{X}(\hat{C} - C^*) \|_F \) can be controlled if the true rank \( r^* \) is correctly identified. In particular, we use the following method to tune the true rank and have established its consistency.

**Theorem 3.3 (Consistency of rank recovery).** Suppose Conditions 3.1, 3.2, 3.4 and 3.5 hold and \( m^{-1}r(n \lor q)(\log^2 n) \log(n + q) = O(1) \). We estimate the true rank \( r^* \) by

\[
\hat{r} = \arg \max_k \{ 1 \leq k \leq r : (nq)^{-1/2} (\tilde{d}_k - \tilde{d}_{k+1}) > \tau_n \},
\]

where \( \tau_n = (\log n)^{-1} \log \log n \). Then for sufficiently largely \( n \), we have \( \hat{r} = r^* \) with probability at least \( 1 - (n + q)^{-1} \).

Since the response matrix \( Y \) may not be fully observed, the GIC-type [Fan and Tang, 2013] information criterion proposed in Zheng et al. [2019] cannot be used to tune the true rank here. In addition, compared with tuning the true rank via cross validation, our method in (9) enjoys much lower computational cost.

### 4 Simulation Studies

In this section, we evaluate the finite-sample performance of the proposed approach PEER through two simulation studies. The main difference between these two studies lies in right singular vectors \( \tilde{v}^*_k \)'s. The right singular vectors \( \tilde{v}^*_k \)'s are sparse in the second study but not necessarily sparse in the first study.

#### 4.1 Study 1

We first state some model setups and simulation settings used in our numerical studies. For each \( k = 1, \cdots, r^* \), the sparse left singular vector \( u^*_k \) is generated with \( u^*_k = \tilde{u}_k / \| \tilde{u}_k \|_2 \), where
\( \hat{u}_k = (\text{rep}(0, sk - s), \text{unif}(Q_u, s), \text{rep}(0, p - sk))^T \). Here \( \text{unif}(Q, s) \) denotes a vector of length \( s \) whose entries are i.i.d. uniformly distributed on set \( Q \) and \( \text{rep}(a, b) \) represents a vector of length \( b \) whose entries are all equal to \( a \). Then we generate a matrix \( \bar{V} = (\bar{v}_1, \ldots, \bar{v}_{r^*}) \in \mathbb{R}^{q \times r^*} \) where \( \bar{v}_k = \text{unif}(Q_v, q) \) for \( k = 1, \ldots, r^* \) and compute the QR decomposition \( \bar{V} = QR \) where \( Q \in \mathbb{R}^{q \times r^*} \) with \( Q^T Q = I_{r^*} \) and \( R \in \mathbb{R}^{r^* \times r^*} \) is a triangular matrix. Take the \( k \)-th column of \( Q \) as \( \bar{v}_k \) for \( k = 1, \ldots, r^* \). For the singular values, let \( d_k^* = 5 + 5(r^* - k + 1) \) for \( k = 1, \ldots, r^* \). The true coefficient matrix \( C^* \) is constructed as \( C^* = U^* D^* V^T \) with \( U^* = (u_1^*, \ldots, u_r^*), V^* = (v_1^*, \ldots, v_r^*) \) and \( D^* = \text{diag}(d_1^*, \ldots, d_r^*) \).

We use the similar procedure described in Mishra et al. [2017] to generate the design matrix \( X \). More specifically, let \( x \sim N(0, \Gamma) \), where \( \Gamma = (\gamma_{ij})_{p \times p} \) with \( \gamma_{ij} = 0.5|j-i| \). Given \( U^* = (u_1^*, \ldots, u_r^*) \), we can find \( U_1^* \in \mathbb{R}^{p \times (p - r^*)} \) such that \( P = (U^*, U_1^*) \in \mathbb{R}^{p \times p} \) and \( \text{rank}(P) = p \). Denote \( x_1 = U_1^T x \) and \( x_2 = U^T x \). We first generate a matrix \( X_1 \in \mathbb{R}^{n \times r^*} \) whose entries are from \( N(0, \Gamma_{r^*}) \) and then we generate \( X_2 \in \mathbb{R}^{n \times (p - r^*)} \) by drawing \( n \) random samples from the conditional distribution of \( x_2 \) given \( x_1 \). The design matrix is then set as \( X = (X_1, X_2) P^{-1} \).

The entries of the error matrix \( E \) are generated as i.i.d. samples from \( N(0, \sigma^2) \). Here \( \sigma^2 \) is chosen such that the signal to noise ratio (SNR), defined as \( \text{SNR} = \|d^*_k X u_k^* V^T\|_F^2 / \|E\|_F^2 \), is equal to a given value. Finally, the response matrix \( Y \) is generated by \( Y = XC^* + E \).

To obtain the data with incomplete outcomes, we randomly remove some entries in \( Y \) such that \( (1 - \alpha) \times 100\% \) percentage of entries in \( Y \) are unobserved. We consider four different values for the missing rate \( 1 - \alpha \): 0, 0.05, 0.1, and 0.15. The case of missing rate 0 means that \( Y \) is fully observed.

We consider four different settings: \( (p, \text{SNR}) = (200, 0.25), (400, 0.25), (200, 0.5), \) and \( (400, 0.5) \). In all settings, we take \( r^* = 3, s = 4, q = 100 \), and \( Q_u = \{1, -1\}, Q_v = [-1, -0.3] \cup [0.3, 1] \). In all simulations, the regularization parameter \( \lambda_k \) in (6) is tuned by the following GIC-type [Fan and Tang, 2013] information criterion

\[
\hat{\lambda}_k = \arg\min_{\lambda_k} \left\{ \log(n^{-1})\sqrt{n} \| \tilde{z}_k - X \hat{u}_k(\lambda_k) \|_2^2 + n^{-1}\| \hat{u}_k(\lambda_k) \|_0 \log(p) \log \log n \right\}.
\]

The experiment under each setting is repeated 200 times.

We first examine the impact of the missing rate on the performance of our method PEER through the following four measures. The estimation accuracy is measured by \( \text{Er}(\hat{C}) = \| \hat{C} - C^* \|_F^2 / (\mu q) \) while the prediction performance is measured by \( \text{Er}(X \hat{C}) = \| X (\hat{C} - C^*) \|_F^2 / (\mu q) \).

The variable selection performance is characterized by the false positive rate (FPR) and false negative rate (FNR) in recovering the sparsity patterns of the left singular vectors \( u_k \), where \( \text{FPR} = \text{FP} / (\text{TN} + \text{FP}) \) and \( \text{FNR} = \text{FN} / (\text{TP} + \text{FN}) \) with TP, FP, TN and FN being the numbers of true nonzeros, false nonzeros, true zeros, and false zeros of \( \{u_1, \ldots, u_r\} \), respectively.

Figure 2 presents the boxplots of the estimation error, prediction error, false positive rate and false negative rate for our method PEER when sample size \( n = 100 \). It shows that the estimation error and prediction error increase with the missing rate, which is consistent with our theory in Theorem 3.1. It is also clear that the variable selection performance of PEER is robust to the missing rate since both false positive rates and false negative rates are stabilized.
Figure 2: Impact of the missing rate on the performance of our method PEER when sample size $n = 100$. First row: $(p, SNR) = (200, 0.25)$; second row: $(p, SNR) = (400, 0.25)$; third row: $(p, SNR) = (200, 0.5)$; fourth row: $(p, SNR) = (400, 0.5)$.

Next we compare our proposed method PEER with other competing approaches, including sequential factor extraction via co-sparse unit-rank estimation (SeCURE) [Mishra et al., 2017], and mixed-response reduced-rank regression (mRRR) [Luo et al., 2018]. We do not include the sequential estimation with eigen-decomposition (SEED) [Zheng et al., 2019] in our comparison because SEED is not applicable for incomplete outcomes. In addition to the four measures $\text{Er}(\mathbf{X}_C)$, $\text{Er}(\mathbf{X}_C)$, FPR, and FNR, we also compare these methods in terms of the computational cost, which is evaluated by average running time (in seconds) of 200 replicates and denoted by Time. We employ the R packages \texttt{rrpack} [Chen, 2019] and
secure [Mishra and Chen, 2017] to implement mRRR and SeCURE, respectively. We use 5-fold cross validation to select the rank of mRRR and the threshold \( \tau_n = (\log \log n)/\log n \) to tune the rank of PEER. The maximum rank of SeCURE is set as 4. We consider two different sample sizes \( n = 100 \) and \( n = 200 \).

| Method | \( \text{Er}(\hat{C}) \times 10^3 \) | \( \text{Er}(\hat{X}\hat{C}) \) | FPR (%) | FNR (%) | Time (s) |
|--------|---------------------------------|-----------------|---------|--------|---------|
| \( p = 200, \text{SNR} = 0.25 \) |
| mRRR   | 21.65 (1.41)                    | 1.20 (0.20)     | 96.83 (9.80) | 3.17 (9.80) | 366.91 (116.29) |
| SeCURE | 36.15 (0.61)                    | 7.23 (0.69)     | 0.00 (0.01)  | 99.00 (4.77) | 3.10 (2.12)  |
| PEER   | 12.81 (9.32)                    | 2.19 (1.86)     | 1.38 (0.97)  | 35.63 (19.98) | 1.06 (0.12)  |
| \( p = 400, \text{SNR} = 0.25 \) |
| mRRR   | 14.40 (0.42)                    | 1.33 (0.27)     | 91.00 (14.84) | 9.00 (14.84) | 765.45 (267.19) |
| SeCURE | 18.10 (0.19)                    | 7.17 (0.67)     | 0.00 (0.00)  | 99.25 (3.36) | 10.81 (6.02) |
| PEER   | 7.28 (4.61)                     | 2.43 (1.83)     | 0.49 (0.36)  | 38.67 (19.27) | 1.07 (0.12)  |
| \( p = 200, \text{SNR} = 0.5 \) |
| mRRR   | 18.42 (1.28)                    | 0.29 (0.07)     | 99.83 (2.36) | 0.17 (2.36)  | 654.67 (153.86) |
| SeCURE | 34.80 (1.48)                    | 6.74 (0.70)     | 0.02 (0.13)  | 90.50 (9.56) | 2.90 (1.68)  |
| PEER   | 2.20 (0.78)                     | 0.27 (0.07)     | 3.41 (1.42)  | 0.29 (2.27)  | 1.05 (0.12)  |
| \( p = 400, \text{SNR} = 0.5 \) |
| mRRR   | 13.61 (0.43)                    | 0.32 (0.16)     | 98.50 (6.93) | 1.50 (6.93)  | 1323.48 (272.69) |
| SeCURE | 17.30 (0.82)                    | 6.68 (0.72)     | 0.01 (0.06)  | 89.63 (10.55) | 10.57 (5.20) |
| PEER   | 1.65 (0.74)                     | 0.39 (0.15)     | 1.27 (0.60)  | 1.25 (4.40)  | 1.07 (0.12)  |

Table 1: The means and standard errors (in parentheses) of various performance measures for different approaches in Study 1 with \( n = 100 \) and missing rate 0.1.

Table 1 reports the comparison results with different \( p \) and SNR when sample size \( n = 100 \) and 10% of entries of \( Y \) are not observed. It can be seen that PEER has better performance than mRRR and SeCURE. Firstly, we can see that PEER has smaller estimation error \( \text{Er}(\hat{C}) \) than mRRR and SeCURE across all settings. In terms of prediction error \( \text{Er}(\hat{X}\hat{C}) \), PEER is superior to SeCURE under all settings and comparable to mRRR (especially when SNR=0.5). Secondly, compared to PEER, other approaches result in either larger false positive rates or larger false negative rates for all settings. Thirdly, PEER is much faster than other approaches. We can see that PEER can achieve a speed up of about 3-10 times in runtime compared with SeCURE and be more than 1000 times faster than mRRR.

The comparison results with sample size \( n = 200 \) are reported in Table 2, which shows the similar results as Table 1. In particular, we can see that PEER has smaller estimation error \( \text{Er}(\hat{C}) \) and prediction error \( \text{Er}(\hat{X}\hat{C}) \) than other approaches across all settings. By comparing the results for the same approach in Tables 1 and 2, it is clear that larger sample size can lead to smaller estimation error \( \text{Er}(\hat{C}) \) and prediction error \( \text{Er}(\hat{X}\hat{C}) \) for both mRRR and PEER.

We should remark that in this study the true right singular vectors \( \nu_k^* \)’s are not necessarily sparse. However, SeCURE assumes right singular vectors \( \nu_k^* \)’s to be sparse. This may be the reason why SeCURE has larger estimation error \( \text{Er}(\hat{C}) \) and prediction error \( \text{Er}(\hat{X}\hat{C}) \) in this.
Table 2: The means and standard errors (in parentheses) of various performance measures for different approaches in Study 1 with \( n = 200 \) and missing rate 0.1.

| Method | \( \text{Er}(\hat{C}) \times 10^3 \) | \( \text{Er}(X\hat{C}) \) | FPR (%) | FNR (%) | Time (s) |
|--------|----------------------------------|----------------|--------|---------|---------|
| mRRR   | 11.49 (1.12)                     | 0.71 (0.07)    | 100.00 (0.00) | 0.00 (0.00) | 333.99 (41.32) |
| SeCURE | 35.74 (1.17)                     | 7.09 (0.58)    | 0.02 (0.12) | 96.58 (7.76) | 5.83 (3.70) |
| PEER   | 3.30 (1.26)                      | 0.51 (0.21)    | 2.21 (1.00) | 2.58 (8.82) | 2.08 (0.22) |

| Method | \( \text{Er}(\hat{C}) \times 10^3 \) | \( \text{Er}(X\hat{C}) \) | FPR (%) | FNR (%) | Time (s) |
|--------|----------------------------------|----------------|--------|---------|---------|
| mRRR   | 10.36 (0.48)                     | 0.80 (0.09)    | 99.83 (2.36) | 0.17 (2.36) | 556.37 (145.22) |
| SeCURE | 17.45 (0.79)                     | 6.78 (0.56)    | 0.01 (0.05) | 91.04 (10.37) | 16.69 (9.56) |
| PEER   | 1.90 (0.65)                      | 0.56 (0.21)    | 0.85 (0.43) | 2.71 (9.07) | 2.01 (0.23) |

| Method | \( \text{Er}(\hat{C}) \times 10^3 \) | \( \text{Er}(X\hat{C}) \) | FPR (%) | FNR (%) | Time (s) |
|--------|----------------------------------|----------------|--------|---------|---------|
| mRRR   | 6.51 (0.82)                      | 0.19 (0.02)    | 100.00 (0.00) | 0.00 (0.00) | 497.60 (200.45) |
| SeCURE | 34.04 (1.77)                     | 6.58 (0.65)    | 0.25 (0.48) | 89.00 (11.63) | 6.73 (3.72) |
| PEER   | 0.90 (0.20)                      | 0.13 (0.02)    | 3.71 (1.36) | 0.00 (0.00) | 1.97 (0.21) |

| Method | \( \text{Er}(\hat{C}) \times 10^3 \) | \( \text{Er}(X\hat{C}) \) | FPR (%) | FNR (%) | Time (s) |
|--------|----------------------------------|----------------|--------|---------|---------|
| mRRR   | 9.05 (0.53)                      | 0.20 (0.02)    | 100.00 (0.00) | 0.00 (0.00) | 1132.99 (352.77) |
| SeCURE | 16.47 (0.67)                     | 6.16 (0.54)    | 0.07 (0.13) | 79.08 (15.27) | 18.37 (10.02) |
| PEER   | 0.55 (0.16)                      | 0.15 (0.03)    | 1.56 (0.68) | 0.00 (0.00) | 1.96 (0.19) |

4.2 Study 2

Following Mishra et al. [2017], we generate sparse right singular vectors \( \mathbf{v}^*_k \)'s in this study. To be specific, we generate \( \mathbf{u}^*_k = \bar{\mathbf{u}}_k / \| \bar{\mathbf{u}}_k \|_2 \) and \( \mathbf{v}^*_k = \bar{\mathbf{v}}_k \), where \( \bar{\mathbf{u}}_k = (\text{rep}(0, ks_u - s_u), \text{unif}(Q_u, s_u), \text{rep}(0, p - ks_u)) \) and \( \bar{\mathbf{v}}_k = (\text{rep}(0, ks_v - s_v), \text{unif}(Q_v, s_v), \text{rep}(0, q - ks_v)) \) with \( s_u = 4 \) and \( s_v = 5 \), meaning that \( \| \mathbf{u}^*_k \|_0 = 4 \) and \( \| \mathbf{v}^*_k \|_0 = 5 \) for each \( k = 1, \ldots, r^* \). We take \( n = 200 \) and consider four different settings \((p, \text{SNR}) = (200, 0.5), (400, 0.5), (200, 1), \) and \((400, 1)\). The other settings are the same as those in Study 1. Then we follow the same procedure as in Study 1 to generate the true coefficient matrix \( \mathbf{C}^* \), the design matrix \( \mathbf{X} \) and the response matrix \( \mathbf{Y} \).

Table 3 records the simulation results for Study 2 under different settings. As expected, SeCURE has good performance when the true right singular vectors \( \mathbf{v}^*_k \)'s are also sparse. It is also interesting to see that PEER still has good performance in this study. This is not surprising because PEER does not require the sparsity assumption on \( \mathbf{v}^*_k \) and can deal with both sparse \( \mathbf{v}^*_k \) and non-sparse \( \mathbf{v}^*_k \). Table 3 also shows that the performance of both methods improves when SNR becomes higher.
Table 3: The means and standard errors (in parentheses) of various performance measures for SeCURE and PEER in Study 2 with missing rate 0.1.

| Method | $\text{Er}(\hat{C}) \times 10^3$ | $\text{Er}(X\hat{C})$ | FPR (%) | FNR (%) | Time (s) |
|--------|-------------------------------|----------------------|---------|---------|---------|
|        | $p = 200, \text{SNR} = 0.5$  |                      |         |         |         |
| SeCURE | 6.32 (8.54)                  | 0.90 (1.58)          | 9.17 (7.31) | 19.54 (28.03) | 15.11 (4.64) |
| PEER   | 0.81 (0.18)                  | 0.12 (0.02)          | 2.41 (1.30) | 0.00 (0.00) | 2.36 (0.34) |
|        | $p = 400, \text{SNR} = 0.5$  |                      |         |         |         |
| SeCURE | 0.04 (0.04)                  | 0.01 (0.01)          | 0.05 (0.16) | 0.00 (0.00) | 92.58 (18.31) |
| PEER   | 0.49 (0.13)                  | 0.14 (0.03)          | 1.07 (0.55) | 0.00 (0.00) | 2.38 (0.37) |
|        | $p = 200, \text{SNR} = 1$   |                      |         |         |         |
| SeCURE | 2.18 (5.37)                  | 0.29 (0.93)          | 5.48 (6.21) | 8.83 (21.15) | 22.43 (4.55) |
| PEER   | 0.24 (0.06)                  | 0.03 (0.01)          | 3.87 (1.37) | 0.00 (0.00) | 2.38 (0.40) |
|        | $p = 400, \text{SNR} = 1$   |                      |         |         |         |
| SeCURE | 0.03 (0.04)                  | 0.01 (0.01)          | 0.15 (0.28) | 0.00 (0.00) | 108.23 (19.46) |
| PEER   | 0.15 (0.04)                  | 0.04 (0.01)          | 1.64 (0.64) | 0.00 (0.00) | 2.38 (0.35) |

5 Yeast cell cycle data analysis

In this section, we apply the proposed method to a multivariate Yeast cell cycle data, in which our goal is to identify the association between transcription factors (TFs) and RNA transcript levels within the Eukaryotic cell cycle. The dataset that we used includes the yeast cell cycle data originally collected by Spellman et al. [1998] and the chromatin immunoprecipitation (ChIP) data in Lee et al. [2002]. The yeast cell cycle data in Spellman et al. [1998] consist of RNA levels measured every 7 minutes for 119 minutes with a total of 18 time points covering two cell cycle of 6178 genes while the ChIP data in Lee et al. [2002] contain complete binding information of only a subset of these 6178 genes of size 1790 for a total of 113 TFs. In addition, the RNA levels corresponding to these 1790 genes in the yeast cell cycle data contain about 2% missing values.

Here, we use these 1790 genes with RNA transcript levels at 18 time points and binding information of 113 TFs to examine the association between the TFs and the RNA transcript levels. Thus, our response matrix $Y$ is a $1709 \times 18$ matrix, recording RNA levels of 1709 genes at 18 time points, and there are about 2% missing values in $Y$. Our design matrix $X$ is a $1709 \times 113$ matrix, corresponding to complete binding information of these 1790 genes for a total of 113 TFs. In other words, our data set has sample size $n = 1709$, number of covariates $p = 113$, and number of responses $q = 18$.

The same dataset has been analyzed in Mishra et al. [2017] and can be accessed in the R package secure [Mishra and Chen, 2017]. Following Mishra et al. [2017], we consider the first four latent factors. Table 4 presents the variable selection results for these four latent factors, where TFs in bold have been confirmed to be related to the cell cycle regulation in Wang et al. [2007]. Our approach PEER selected 26 distinct TFs in total, of which 10 are among the confirmed TFs. SeCURE selected 23 distinct TFs with 10 confirmed TFs.
Table 4: TFs selected by PEER and SeCURE where those TFs in bold are confirmed to related to the cell cycle regulation in Wang et al. [2007].

| 1th factor | PEER | SeCURE |
|------------|------|--------|
| ABF1, CIN5, FHL1, HIR2, MTH1, RGM1, SKO1, SOK2, USV1, YAP5, REB1, STE12, NDD1, FKH2 | | NDD1 |
| 2th factor | FHL1, GAT3, HSF1, MBP1, NDD1, STE12, SWI4, YAP5 | ACE2, CUP9, DAL81, STB1, SWI4, NDD1, PDR1, RTG1, MET31, SKN7, SWI5, SW16, USV1, MBP1, GAT3, HI1, HIR2, SKO1, STE12, SIP4 |
| 3th factor | DAL82, HSF1, INO4, SFL1, SOK2, MCM1, SWI5, RAP1, SW14 | | SWI5 |
| 4th factor | ACE2, CEN5, FHL1, STE12, FKH2, NDD1, RFX1, SW14 | ACE2, HIR1, MBP1, SWI4, NDD1, STE12, STP1, YAP5 |

Of these confirmed TFs, 6 (ACE2, MBP1, NDD1, STE12, SWI4, SWI5) were selected by both methods. The 4 confirmed TFs (ABF1, FKH2, MCM1, REB1) were selected by PEER but missed by SeCURE. However, these four TFs missed by SeCURE can be also important in cell cycle. For example, ABF1 and FKH2 play an important role in DNA/RNA/protein biosynthesis, and REB1 works for environmental response in cell cycle [Lee et al., 2002]. We remark that our method PEER also missed 4 confirmed TFs (MET31, SKN7, STB1, SWI6), which were selected by SeCURE. Therefore, our method can be used to uncover genes that are important but missed by SeCURE. For variable selection purpose, we can use both methods in practice to avoid missing important features.

6 Discussion

In this paper, we have proposed a new and efficient approach PEER to achieve scalable and accurate estimation for large-scale multi-response regression with incomplete outcomes, where both responses and predictors are possibly of high dimensions. It has been shown through our theoretical properties and numerical studies that PEER achieves nice estimation and prediction accuracy.

Here we have focused on multi-response linear models with incomplete outcomes, where all the responses are continuous variables. In many applications, linear models can become restrictive. In addition, the responses can be categorical, counts, or functional in many real-world problems. For more flexible modeling, it is also of practical importance to extend the idea of PEER to more general model settings with incomplete outcomes, such as multi-response generalized linear models [Dette et al., 2013], partial linear multiplicative models [Zhang et al., 2018], multivariate varying coefficient models [He et al., 2018], semiparametric quantile factor models [Ma et al., 2021], and multivariate functional responses [Liu et al., 2020]. These possible extensions are beyond the scope of the current paper and will be interesting topics for future research.
A Proofs of main results

To ease the presentation, we further introduce some notation which will be used later. Let $\langle A, B \rangle$ be the Frobenius inner product of two matrices $A$ and $B$. We also use $\langle a, b \rangle$ to denote the inner product of two vectors $a$ and $b$. For each $k = 1, \ldots, r^*$, let $J_k = \{1 \leq j \leq p : u^*_k,j \neq 0\}$ be the support of $u^*_k$ where $u^*_k,j$ is the $j$th element of $u^*_k$. In addition, define $\hat{\delta}_k = \hat{u}_k - u^*_k$, where $\hat{u}_k$ is the estimation of $u^*_k$. Hereafter we use $c$ to denote a generic positive constant whose value may vary from place to place.

We first present two additional lemmas. These results can also be of independent interest. Lemmas 1 and 2 will be used in the proof of Theorem 3.1 while lemma 3 will be used in the proof of Theorem 3.2. The proofs of all lemmas are provided in B.

Lemma 2. Assume that $\|XC^*\|_{\max} \leq L$ and $m \geq n^{-1}(n \wedge q) \log^3(n + q)$. Then, under Conditions 3.1 and 3.2, we have that $\|\Delta\|_F/\sqrt{nq} \leq cB_n$ holds with probability at least $1 - (n + q)^{-1}$, where $c$ is a positive constant, $\Delta = \hat{Y} - XC^*$, and $\hat{Y}$ and $B_n$ are defined in (7) and (8), respectively.

Lemma 3. Under Conditions 3.1 and 3.2, we have that, with probability at least $1 - 2e^{-nq}2^{-nq}$

$$\max \{|\tilde{z}_k - n^{-1/2}Xu^*_k|_2, |\tilde{v}_k - v^*_k|_2\} \lesssim \gamma^{-1}_d \tilde{b}_n,$$

$$|\tilde{d}_k - \sqrt{n}\tilde{d}_k|/\sqrt{nq} \lesssim \tilde{B}_n,$$

hold uniformly over $k = 1, \ldots, r^*$, where $\tilde{B}_n = \sigma \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{q}}\right)$.

A.1 Proof of Theorem 3.1

Define the event $\Omega = \left\{||\Delta||_F/\sqrt{nq} \leq cB_n\right\}$. Then it follows from Lemma 2 that the event $\Omega$ holds with probability at least $1 - (n + q)^{-1}$, that is, $P(\Omega) \geq 1 - (n + q)^{-1}$. Thus, to prove Theorem 3.1, it suffices to show that, conditional on the event $\Omega$, the following six inequalities

$$n^{-1/2}\|X(\hat{u}_k - u^*_k)\|_2 \lesssim \frac{\sqrt{s_k}}{\gamma_d \sqrt{\rho_l}} B_n, \quad \|\hat{u}_k - u^*_k\|_2 \lesssim \frac{\sqrt{s_k}}{\rho_l \gamma_d} B_n,$$

$$|\tilde{v}_k - v^*_k|_2 \lesssim \gamma^{-1}_d B_n, \quad |\tilde{d}_k - d^*_k|/\sqrt{q} \lesssim B_n,$$

$$\frac{1}{\sqrt{nq}}\|X(\hat{C}_k - C^*_k)\|_F \lesssim \frac{\sqrt{s_k}}{\gamma_d \sqrt{\rho_l}} B_n, \quad \frac{1}{\sqrt{q}}\|\hat{C}_k - C^*_k\|_F \lesssim \frac{\sqrt{s_k}}{\rho_l \gamma_d} B_n$$

hold uniformly over $k = 1, \ldots, r^*$.

Hereafter our analysis will be conditional on the event $\Omega$. By (49), we have that, conditional on the event $\Omega$, $|\tilde{v}_k - v^*_k|_2 \lesssim \gamma^{-1}_d B_n$ holds uniformly over $k = 1, \ldots, r^*$. Hence it remains to prove other five inequalities. To enhance readability, we split the proof into five parts.
An upper bound for $n^{-1/2}\|X(\hat{u}_k - u_k^*)\|_2$. Since $\hat{u}_k$ is the minimizer of of (6), we have

$$n^{-1}\|n^{1/2}\tilde{z}_k - X\hat{u}_k\|_2^2 + \lambda_k\|\hat{u}_k\|_1 \leq n^{-1}\|n^{1/2}\tilde{z}_k - Xu_k^*\|_2^2 + \lambda_k\|u_k^*\|_1.$$  

Recall that $\tilde{\delta}_k = \hat{u}_k - u_k^*$. Thus $n^{1/2}\tilde{z}_k - X\hat{u}_k = (n^{1/2}\tilde{z}_k - Xu_k^*) - X\tilde{\delta}_k$. Substituting this into the above inequality, after some simple algebra, we obtain

$$n^{-1}\|X\tilde{\delta}_k\|_2^2 + \lambda_k\|\hat{u}_k\|_1 \leq 2n^{-1}\langle n^{1/2}\tilde{z}_k - Xu_k^*, X\tilde{\delta}_k \rangle + \lambda_k\|u_k^*\|_1 = 2n^{-1}\langle X^T(n^{1/2}\tilde{z}_k - Xu_k^*), \tilde{\delta}_k \rangle + \lambda_k\|u_k^*\|_1. \quad (10)$$

An application of the triangle inequality yields

$$n^{-1}\|X^T(n^{1/2}\tilde{z}_k - Xu_k^*), \tilde{\delta}_k\| \leq n^{-1}\|X^T(n^{1/2}\tilde{z}_k - Xu_k^*)\|_\infty \|\tilde{\delta}_k\|_1. \quad (11)$$

Write $X = (\tilde{x}_1, \ldots, \tilde{x}_p)$, where $\tilde{x}_j$ is the $j$th column of $X$ for $j = 1, \ldots, p$. By the Cauchy-Schwarz inequality, we have

$$n^{-1}\|X^T(n^{1/2}\tilde{z}_k - Xu_k^*)\|_\infty = \max_{1 \leq j \leq p} \left| n^{-1/2}\langle \tilde{x}_j, n^{-1/2}Xu_k^* \rangle \right| \leq \max_{1 \leq j \leq p} n^{-1/2}\|\tilde{x}_j\|_2 \|\tilde{z}_k - n^{-1/2}Xu_k^*\|_2 \leq \|\tilde{z}_k - n^{-1/2}Xu_k^*\|_2,$$

where the last inequality holds since each column of $X$ is rescaled to have an $\ell_2$-norm $n^{1/2}$. Conditional on the event $\Omega$, this together with (48) yields

$$n^{-1}\|X^T(n^{1/2}\tilde{z}_k - Xu_k^*)\|_\infty \leq c\gamma_d^{-1}B_n, \quad (12)$$

where $c$ is some positive constant. Taking $\lambda_k = 4c\gamma_d^{-1}B_n$ in (10) and combining it with (11) and (12) lead to

$$n^{-1}\|X\tilde{\delta}_k\|_2^2 + \lambda_k\|\hat{u}_k\|_1 \leq \frac{\lambda_k}{2}\|\tilde{\delta}_k\|_1 + \lambda_k\|u_k^*\|_1, \quad (13)$$

Let $\hat{u}_{J_k}$ and $u_{J_k}^*$ be the subvectors of $\hat{u}_k$ and $u_k^*$ formed by components in $J_k$, respectively. Similarly, let $\tilde{\delta}_{J_k}$ and $\tilde{\delta}_{J_k'}$ be the subvectors of $\tilde{\delta}_k$ formed by components in $J_k$ and $J_k'$, respectively. The inequality (13) entails

$$\|\hat{u}_k\|_1 = \|\hat{u}_{J_k}\|_1 + \|\hat{\delta}_{J_k'}\|_1, \quad (14)$$

Since $\|\hat{u}_k\|_1 = \|\hat{u}_{J_k}\|_1 + \|\hat{\delta}_{J_k'}\|_1$, the above inequality yields

$$\|\tilde{\delta}_{J_k'}\|_1 \leq \|\tilde{\delta}_{J_k}\|_1 + 2\|u_{J_k}^*\|_1 - \|\tilde{\delta}_{J_k'}\|_1 \leq 3\|\tilde{\delta}_{J_k}\|_1, \quad (14)$$

where the last inequality follows from the reverse triangle inequality for $\| \cdot \|_1$. Using (13) and the reverse triangle inequality again gives

$$n^{-1}\|X\tilde{\delta}_k\|_2^2 \leq \lambda_k\|\tilde{\delta}_k\|_1/2 + \lambda_k\|u_k^*\|_1 - \|\hat{u}_k\|_1 \leq 3\lambda_k\|\tilde{\delta}_k\|_1/2.$$
Thus, by (14) and the Cauchy–Schwarz inequality, we have
\[
\frac{1}{n} \| X \hat{\delta}_k \|_2^2 \leq 3 \lambda_k (\| \hat{\delta}_{j_k} \|_1 + \| \hat{\delta}_{\hat{J}_k} \|_1) / 2 \leq 6 \lambda_k \| \hat{\delta}_{j_k} \|_1 \leq 6 \lambda_k \sqrt{s_k} \| \hat{\delta}_{j_k} \|_2, \tag{15}
\]
where \( s_k = \| u_k^\ast \|_0 \) is equal to the cardinality of the set \( J_k \). This, together with (14) and Condition 3.3, yields
\[
\max (\| \hat{\delta}_{j_k} \|_2^2, \| \hat{\delta}_{J_k}^{(1)} \|_2^2) \leq \rho^{-1} n^{-1} \| X \hat{\delta}_k \|_2 \leq 6 \rho^{-1} \lambda_k \sqrt{s_k} \| \hat{\delta}_{j_k} \|_2,
\]
where \( \hat{\delta}_{J_k}^{(1)} \) is a subvector of \( \hat{\delta}_{J_k} \) consisting of the \( s_k \) largest components in magnitude. This leads to
\[
\| \hat{\delta}_{J_k} \|_2 \leq 6 \rho^{-1} \lambda_k \sqrt{s_k}, \tag{16}
\]
\[
\| \hat{\delta}_{J_k}^{(1)} \|_2 \leq 6 \rho^{-1} \lambda_k \sqrt{s_k}. \tag{17}
\]
Recall that \( \hat{\delta}_k = \hat{u}_k - u_k^\ast \) and \( \lambda_k = 4 c \gamma_d^{-1} B_n \). In view of (15) and (16), we have
\[
n^{-1/2} \| X(\hat{u}_k - u_k^\ast) \|_2 = n^{-1/2} \| X \hat{\delta}_k \|_2 \leq \left( 6 \lambda_k \sqrt{s_k} \| \hat{\delta}_{j_k} \|_2 \right)^{1/2} \leq 6 \rho^{-1/2} \lambda_k \sqrt{s_k} = 24 c \sqrt{\gamma_d / \rho} B_n \lesssim \sqrt{\lambda_k} B_n. \tag{18}
\]

**An upper bound for \( \| \hat{u}_k - u_k^\ast \|_2 \).** Let \( \hat{\delta}_{J_k}^{(2)} \) be a subvector of \( \hat{\delta}_{J_k} \) excluding those components with the \( s_k \) largest magnitude. Since the \( j \)th largest absolute component of \( \hat{\delta}_{J_k} \) is bounded from above by \( \| \hat{\delta}_{J_k} \|_1 / j \), we have
\[
\| \hat{\delta}_{J_k}^{(2)} \|_2 \leq \sum_{j=s_k+1}^{p} \| \hat{\delta}_{J_k} \|_2^2 / j^2 \leq s_k^{-1} \| \hat{\delta}_{J_k} \|_2^2.
\]
This inequality, together with (14) and the Cauchy–Schwarz inequality, entails that \( \| \hat{\delta}_{J_k}^{(2)} \|_2 \leq s_k^{-1/2} \| \hat{\delta}_{J_k} \|_2 \leq 3 s_k^{-1/2} \| \hat{\delta}_{J_k} \|_2 \leq 3 \| \hat{\delta}_{J_k} \|_2 \). Combining this with (16) and (17), we have
\[
\| \hat{\delta}_k \|_2 \leq \| \hat{\delta}_{J_k} \|_2 + \| \hat{\delta}_{J_k}^{(1)} \|_2 + \| \hat{\delta}_{J_k}^{(2)} \|_2 \leq 4 \| \hat{\delta}_{J_k} \|_2 + \| \hat{\delta}_{J_k}^{(1)} \|_2 \leq 30 \rho^{-1} \lambda_k \sqrt{s_k},
\]
which entails that
\[
\| \hat{u}_k - u_k^\ast \|_2 \leq 120 c \sqrt{\gamma_d} B_n \leq \frac{\sqrt{s_k}}{\rho t \gamma_d} B_n. \tag{19}
\]

Since \( \hat{\delta}_k = \hat{u}_k - u_k^\ast \) and \( \lambda_k = 4 c \gamma_d^{-1} B_n \).

**An upper bound for \( | \hat{d}_k - d_k^\ast | / \sqrt{q} \).** Note that \( \hat{d}_k = \hat{d}_k / \sqrt{n} \). By (50) from Lemma 1, conditional on the event \( \Omega \), we have
\[
| \hat{d}_k - d_k^\ast | / \sqrt{q} \leq | \hat{d}_k - \sqrt{n} d_k^\ast | / \sqrt{n q} \leq B_n. \tag{20}
\]
An upper bound for $\|X(\hat{C}_k - C^*_k)\|_F/\sqrt{nq}$. Recall that $\hat{C}_k = \hat{d}_k \hat{\mu}_k \hat{v}_k^T$ and $C^*_k = d^*_k u^*_k v^*_k$. We can write

$$X(\hat{C}_k - C^*_k) = \hat{d}_k X \hat{\mu}_k \hat{v}_k^T - d^*_k X u^*_k v^*_k = T_1 + d^*_k (T_2 + T_3 + T_4) + (\hat{d}_k - d^*_k)(T_2 + T_3 + T_4),$$

where

$$T_1 = (\hat{d}_k - d^*_k)X u^*_k v^*_k, \quad T_2 = X(\hat{\mu}_k - u^*_k)v^*_k, \quad T_3 = Xu^*_k(v_k - v^*_k)^T, \quad T_4 = X(\hat{\mu}_k - u^*_k)(\hat{v}_k - v^*_k)^T.$$ 

Thus, we have

$$\frac{\|X(\hat{C}_k - C^*_k)\|_F}{\sqrt{nq}} \leq \frac{\|T_1\|_F}{\sqrt{nq}} + \left( \frac{d^*_k}{\sqrt{q}} + \frac{|\hat{d}_k - d^*_k|}{\sqrt{q}} \right) \left( \frac{\|T_2\|_F}{\sqrt{n}} + \frac{\|T_3\|_F}{\sqrt{n}} + \frac{\|T_4\|_F}{\sqrt{n}} \right).$$

(21)

We next find the bounds for $\|T_1\|_F$, $\|T_2\|_F$, $\|T_3\|_F$, and $\|T_4\|_F$ separately. Since $\|\tilde{n}^{-1/2}Xu_k\|_2 = 1$ and $\|v_k\|_2 = 1$, we have

$$\frac{\|T_1\|_F}{\sqrt{nq}} = \frac{|\hat{d}_k - d^*_k|}{\sqrt{q}} \|n^{-1/2}Xu_k\|_2 \|v^*_k\|_2 = \frac{|\hat{d}_k - d^*_k|}{\sqrt{q}} \leq B_n,$$

(22)

where the last inequality follows from (20). Similarly, using $\|n^{-1/2}Xu_k\|_2 = 1$, $\|v^*_k\|_2 = 1$, (18) and (49), we obtain

$$\frac{\|T_2\|_F}{\sqrt{n}} = n^{-1/2}\|X(\hat{\mu}_k - u^*_k)\|_2 \|v^*_k\|_2 \leq \frac{\sqrt{s_k}}{\gamma_d \sqrt{p_l}} B_n,$$

(23)

$$\frac{\|T_3\|_F}{\sqrt{n}} = n^{-1/2}\|Xu_k\|_2 \|\hat{v}_k - v^*_k\|_2 \leq \gamma_d^{-1} B_n,$$

(24)

$$\frac{\|T_4\|_F}{\sqrt{n}} = n^{-1/2}\|X(\hat{\mu}_k - u^*_k)\|_2 \|\hat{v}_k - v^*_k\|_2 \leq \frac{\sqrt{s_k}}{\gamma_d \sqrt{p_l}} B_n^2.$$ 

(25)

If follows from Condition 3.2 and (20) that

$$\frac{d^*_k}{\sqrt{q}} + \frac{|\hat{d}_k - d^*_k|}{\sqrt{q}} \leq c_3 + c_4 B_n,$$

(26)

where $c_3$ and $c_4$ are some positive constants. Combining this with (47) and (21)-(25) entails

$$\frac{1}{\sqrt{nq}}\|X(\hat{C}_k - C^*_k)\|_F \leq B_n + (c_3 + c_4 B_n) \left( \frac{\sqrt{s_k}}{\gamma_d \sqrt{p_l}} B_n + \frac{B_n}{\gamma_d} + \frac{\sqrt{B_n}}{\gamma_d \sqrt{p_l}} + \frac{\sqrt{B_n}}{\gamma_d \sqrt{p_l}} \right) + \frac{c_4 \sqrt{s_k}}{\gamma_d \sqrt{p_l}} B_n^2 + \frac{c_4 \sqrt{s_k}}{\gamma_d^2 \sqrt{p_l}} B_n^3 \leq \frac{c_3 \sqrt{s_k}}{\gamma_d \sqrt{p_l}} B_n + \frac{c_6 \sqrt{s_k}}{\gamma_d \sqrt{p_l}} B_n^2 + \frac{c_4 \sqrt{s_k}}{\gamma_d \sqrt{p_l}} B_n^3 \leq \frac{c_3 \sqrt{s_k}}{\gamma_d \sqrt{p_l}} B_n,$$ 

(27)
where $c_5$ and $c_6$ are some positive constants.

**An upper bound for $\|\hat{\mathbf{C}}_k - \mathbf{C}_k\|_F/\sqrt{q}$.** Similar to (21), we can show that

$$
\frac{\|\hat{\mathbf{C}}_k - \mathbf{C}_k\|_F}{\sqrt{q}} \leq \frac{\|T_5\|_F}{\sqrt{q}} + \left( \frac{d_k^*}{\sqrt{q}} + \frac{\hat{d}_k - d_k^*}{\sqrt{q}} \right) (\|T_6\|_F + \|T_7\|_F + \|T_8\|_F),
$$

where

$$
T_5 = (\hat{d}_k - d_k^*)\mathbf{u}_k^*\mathbf{v}_k^*^T, \quad T_6 = (\hat{\mathbf{u}}_k - \mathbf{u}_k^*)\mathbf{v}_k^T, \\
T_7 = \mathbf{u}_k^*(\hat{\mathbf{v}}_k - \mathbf{v}_k^*)^T, \quad T_8 = (\hat{\mathbf{u}}_k - \mathbf{u}_k^*)(\hat{\mathbf{v}}_k - \mathbf{v}_k^*)^T.
$$

Recall that $J_k$ is the support of $\mathbf{u}_k^*$, and $\mathbf{u}_{j_k}^*$ and $\mathbf{u}_{j_k}^*$ are the subvectors of $\mathbf{u}_k^*$ formed by components in $J_k$ and $J_k^c$, respectively. Then we have $0 = \|\mathbf{u}_{j_k}^*\|_1 \leq 3\|\mathbf{u}_{j_k}^*\|_1$. It follows from $s_k = \|\mathbf{u}_k^*\|_0 \leq s$ and Condition 3.3 that $\|\mathbf{u}_{j_k}^*\|_2 \leq \rho_l^{-1/2}n^{-1/2}\|\mathbf{X}\mathbf{u}_k^*\|_2$. Since $\|n^{-1/2}\mathbf{X}\mathbf{u}_k^*\|_2 = 1$ and $\|\mathbf{u}_k^*\|_2 = \|\mathbf{u}_{j_k}^*\|_2$, we have

$$
\|\mathbf{u}_k^*\|_2 = \|\mathbf{u}_{j_k}^*\|_2 \leq \rho_l^{-1/2}n^{-1/2}\|\mathbf{X}\mathbf{u}_k^*\|_2 = \rho_l^{-1/2}.
$$

Using similar arguments for bounding $\|\mathbf{X}(\hat{\mathbf{C}}_k - \mathbf{C}_k)\|_F/\sqrt{nq}$, we can obtain a similar bound

$$
\frac{1}{\sqrt{q}}\|\hat{\mathbf{C}}_k - \mathbf{C}_k\|_F \leq \frac{\sqrt{s_k}}{\rho_l\gamma_d}\bar{B}_n. \quad (28)
$$

It concludes the proof of Theorem 3.1.

**A.2 Proof of Proposition 3.2**

Recall that the event $\mathcal{B}$, defined in (53) in the proof of Lemma 3, holds with probability at least $1 - 2e^{-(\sqrt{n^2} + \sqrt{q})^2}$, that is, $P(\mathcal{B}) \geq 1 - 2e^{-(\sqrt{n^2} + \sqrt{q})^2}$. Thus, to prove Proposition 3.2, it suffices to show that, conditional on the event $\mathcal{B}$, the following six inequalities

$$
n^{-1/2}\|\mathbf{X}(\hat{\mathbf{u}}_k - \mathbf{u}_k^*)\|_2 \lesssim \frac{\sqrt{s_k}}{\gamma_d\sqrt{\rho_l}}\bar{B}_n, \quad \|\hat{\mathbf{u}}_k - \mathbf{u}_k^*\|_2 \lesssim \frac{\sqrt{s_k}}{\rho_l\gamma_d}\bar{B}_n, \\
\|\hat{\mathbf{v}}_k - \mathbf{v}_k^*\|_2 \lesssim \gamma_d^{-1}\bar{B}_n, \quad |\hat{d}_k - d_k^*|/\sqrt{q} \lesssim \bar{B}_n, \\
\frac{1}{\sqrt{nq}}\|\mathbf{X}(\hat{\mathbf{C}}_k - \mathbf{C}_k)\|_F \lesssim \frac{\sqrt{s_k}}{\gamma_d\sqrt{\rho_l}}\bar{B}_n, \quad \frac{1}{\sqrt{q}}\|\hat{\mathbf{C}}_k - \mathbf{C}_k\|_F \lesssim \frac{\sqrt{s_k}}{\rho_l\gamma_d}\bar{B}_n
$$

hold uniformly over $k = 1, \ldots, r^*$.  

Hereafter our analysis will be conditional on the event $\mathcal{B}$. By (56), we have that, conditional on the event $\mathcal{B}$, $\|\hat{\mathbf{v}}_k - \mathbf{v}_k^*\|_2 \lesssim \gamma_d^{-1}\bar{B}_n$ holds uniformly over $k = 1, \ldots, r^*$. Using similar arguments for proving (18), (19), (20), (27), and (28), we can show that conditional on the event $\mathcal{B}$, other five inequalities also holds uniformly over $k = 1, \ldots, r^*$. So the details are omitted here to save space. This completes the proof of Proposition 3.2.
A.3 Proof of Theorem 3.3

Under Conditions 3.1, 3.4, and 3.5, the event \( \Omega = \left\{ \| \tilde{\Delta} \|_F / \sqrt{nq} \leq cB_n \right\} \) following from Lemma 2 holds with probability at least \( 1 - (n + q)^{-1} \), where \( c \) is a positive constant. Thus it suffices to show that \( \tilde{r} = r^* \) conditional on \( \Omega \).

Without loss of generality, \( B_n \) can be simplified as \( O \left( \sqrt{m^{-1}r(n \vee q) \log(n + q)} \right) \) when assume \( n \geq 2 \). Since \( \tilde{d}_k / \sqrt{nq} \) and \( d^*_k / \sqrt{nq} \) are the singular values of \( \tilde{Y} / \sqrt{nq} \) and \( XC^* / \sqrt{nq} \), respectively, applying Weyl’s theorem [Stewart, 1998, Theorem 2] leads to

\[
|\tilde{d}_k - \sqrt{nq}d_k^*| / \sqrt{nq} = |\tilde{d}_k / \sqrt{nq} - d^*_k / \sqrt{nq}| \leq \| \tilde{\Delta} \|_F / \sqrt{nq} \quad \text{for} \ 1 \leq k \leq r^* \tag{29}
\]

and

\[
|\tilde{d}_k| / \sqrt{nq} = |\tilde{d}_k / \sqrt{nq} - 0| \leq \| \tilde{\Delta} \|_F / \sqrt{nq} \quad \text{for} \ k > r^*. \tag{30}
\]

From now on, we condition on the event \( \Omega \). Then we have

\[
\frac{\| \tilde{\Delta} \|_F}{\sqrt{nq}} \leq cB_n = O \left( \sqrt{m^{-1}r(n \vee q) \log(n + q)} \right) = O \left( \frac{1}{\log n} \right), \tag{31}
\]

where the last identity follows from the assumption that \( m^{-1}r(n \vee q)(\log^2 n) \log(n + q) = O(1) \). We can show that conditional on the event \( \Omega \), for sufficiently largely \( n \), we have that \( (nq)^{-1/2}(\tilde{d}_k - \tilde{d}_{k+1}) \) is greater than \( \tau_n \) for all \( 1 \leq k \leq r^* \) and smaller than \( \tau_n \) for \( k > r^* \). To this end, we consider three cases.

Case 1: \( 1 \leq k \leq r^* - 1 \). Note that

\[
(nq)^{-1/2}(\tilde{d}_k - \tilde{d}_{k+1}) = q^{-1/2}(d^*_k - d^*_k) + (nq)^{-1/2}(\tilde{d}_k - \sqrt{nq}d_k^*) + (nq)^{-1/2}(\sqrt{nq}d_{k+1}^* - \tilde{d}_{k+1}).
\]

Thus, it follows from Condition 3.2, (29) and (31) that

\[
(nq)^{-1/2}(\tilde{d}_k - \tilde{d}_{k+1}) \geq q^{-1/2}(d^*_k - d^*_k) - 2\| \tilde{\Delta} \|_F / \sqrt{nq} \geq \gamma_d - O \left( \frac{1}{\log n} \right) > \frac{\log \log n}{\log n}
\]

for sufficiently largely \( n \).

Case 2: \( k = r^* \). In view of Condition 3.2, (30) and (31), we have

\[
(nq)^{-1/2}(\tilde{d}_k - \tilde{d}_{k+1}) = q^{-1/2}d^*_k + (nq)^{-1/2}(\tilde{d}_k - \sqrt{nq}d_k^*) - (nq)^{-1/2}\tilde{d}_{k+1}
\]

\[
\geq O(1) - 2\| \tilde{\Delta} \|_F / \sqrt{nq} \geq O(1) - O \left( \frac{1}{\log n} \right) > \frac{\log \log n}{\log n}
\]

for sufficiently largely \( n \).

Case 3: \( k > r^* \). It follows from (30) and (31) that

\[
(nq)^{-1/2}(\tilde{d}_k - \tilde{d}_{k+1}) \leq |\tilde{d}_k| / \sqrt{nq} + |\tilde{d}_{k+1}| / \sqrt{nq} \leq 2\| \tilde{\Delta} \|_F / \sqrt{nq} \leq O \left( \frac{1}{\log n} \right) < \frac{\log \log n}{\log n}
\]

for sufficiently largely \( n \).
Combining Cases 1-3 above along with \( \tau_n = (\log n)^{-1} \log \log n \) yields that, conditional on the event \( \Omega \), the following bounds hold for sufficiently large \( n \):

\[
(nq)^{-1/2}(\tilde{d}_k - \tilde{d}_{k+1}) > \tau_n \text{ for } 1 \leq k \leq r^*;
(nq)^{-1/2}(\tilde{d}_k - \tilde{d}_{k+1}) < \tau_n \text{ for } k > r^*.
\]

Therefore, by choosing \( \hat{r} = \arg \max_k \{1 \leq k \leq r : (nq)^{-1/2}(\tilde{d}_k - \tilde{d}_{k+1}) > \tau_n\} \), we have \( r = r^* \) with probability at least \( 1 - (n + q)^{-1} \) for sufficiently large \( n \), which concludes the proof of Theorem 3.3.

## B Proofs of Lemmas

### B.1 Proof of Lemma 2

Recall that \( \tilde{Y} \) is the minimizer of (7). Thus, we have

\[
m^{-1}\|P_M(Y) - P_M(\tilde{Y})\|_F^2 \leq m^{-1}\|P_M(Y) - P_M(XC^*)\|_F^2. \tag{32}
\]

Note that \( P_M(Y) - P_M(XC^*) = P_M(E) \) and

\[
P_M(Y) - P_M(\tilde{Y}) = [P_M(Y) - P_M(XC^*)] - [P_M(\tilde{Y}) - P_M(XC^*)] = P_M(E) - P_M(\tilde{\Delta}),
\]

where \( \tilde{\Delta} = \tilde{Y} - XC^* \). Substituting these two identities into (32) yields

\[
m^{-1}\|P_M(\tilde{\Delta})\|_F^2 \leq 2m^{-1}\langle P_M(E), P_M(\tilde{\Delta}) \rangle.
\]

For simplicity, we write \( \tilde{\Delta}_M = P_M(\tilde{\Delta}) \) and \( E_M = P_M(E) \). Thus the above inequality can be written as

\[
m^{-1}\|\tilde{\Delta}_M\|_F^2 \leq 2m^{-1}\langle E_M, \tilde{\Delta}_M \rangle. \tag{33}
\]

Let \( \tilde{\Delta}_{ij} \) be the \((i, j)\) entry of the matrix \( \tilde{\Delta} \). By the definition of \( \tilde{Y} \), we have \( \|\tilde{Y}\|_{\text{max}} \leq L \).

Under the assumption that \( \|XC^*\|_{\text{max}} \leq L \), we further have

\[
\|\tilde{\Delta}\|_{\text{max}} = \|\tilde{Y} - XC^*\|_{\text{max}} \leq \|\tilde{Y}\|_{\text{max}} + \|XC^*\|_{\text{max}} \leq 2L.
\]

Recall that \( M = \{(i, j) : y_{ij} \text{ is observed}, 1 \leq i \leq n, 1 \leq j \leq q\} \). Denote by \( \{\omega_t\}_{t=1}^m \) the sampled sequence of entries, where \( \omega_t = (i_t, j_t) \in M \) for all \( t \). Then we can define a sequence of matrices \( \{W_t\}_{t=1}^m \), where the entries of \( W_t \in \mathbb{R}^{n \times q} \) are all zeros except for 1 at the location \( \omega_t \) (i.e. \( (W_t)_{i_t,j_t} = 1 \)). Let \( \{\epsilon_t\}_{t=1}^m \) be a Rademacher sequence independent of \( (\omega_t, W_t)_{t=1}^m \). Define \( \Sigma_R = m^{-1} \sum_{t=1}^m \epsilon_t W_t \).

In order to proceed, we first show that

\[
\frac{1}{nq}\|\tilde{\Delta}\|_F^2 \leq c \max \left\{ \mu L^2 \frac{\sqrt{\log(n + q)}}{m}, \frac{rnq}{m^2} \mu^2 \|d_1^2(E_M)\| + \frac{rnq \mu^2 L^2 \|E^2\|}{m^2} [d_1(\Sigma_R)] \right\}, \tag{34}
\]

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where $c$ is some positive constant. To prove this, we consider two cases.

Case 1: $\sum_{i,j} \pi_{ij} \tilde{\Delta}_{ij}^2 < 4L^2 \sqrt{\frac{64 \log(n+q)}{m \log(6/5)}}$. Then by Condition 3.4, we have

$$\frac{1}{\mu_{nq}} \|\tilde{\Delta}\|_F^2 \leq \frac{1}{m} \sum_{i,j} \pi_{ij} \tilde{\Delta}_{ij}^2 < 4L^2 \sqrt{\frac{64 \log(n+q)}{m \log(6/5)}}. \tag{35}$$

Case 2: $\sum_{i,j} \pi_{ij} \tilde{\Delta}_{ij}^2 \geq 4L^2 \sqrt{\frac{64 \log(n+q)}{m \log(6/5)}}$. Note that $\text{rank}(\tilde{\Delta}) \leq \text{rank}(\tilde{Y}) + \text{rank}(XC^*) \leq 2r$. Applying Lemma 12 of Klopp [2014] entails

$$\frac{1}{m} (2L)^{-2} \|\tilde{\Delta}_m\|_F^2 \geq \frac{1}{2} (2L)^{-2} \sum_{i,j} \pi_{ij} \tilde{\Delta}_{ij}^2 - 8\mu_{nq}E^2 [d_1(\Sigma_R)]$$

$$\geq \frac{1}{2\mu_{nq}} (2L)^{-2} \|\tilde{\Delta}\|_F^2 - 8\mu_{nq}E^2 [d_1(\Sigma_R)],$$

where the last inequality follows from Condition 3.4. Multiplying both sides of the above inequality by $4L^2$ yields

$$\frac{1}{m} \|\tilde{\Delta}_m\|_F^2 \geq \frac{1}{2\mu_{nq}} \|\tilde{\Delta}\|_F^2 - 352L^2 \mu_{nq}E^2 [d_1(\Sigma_R)]. \tag{36}$$

This inequality, together with (33), gives

$$\frac{1}{2\mu_{nq}} \|\tilde{\Delta}\|_F^2 \leq 2m^{-1} \langle E_M, \tilde{\Delta}_m \rangle + 352L^2 \mu_{nq}E^2 [d_1(\Sigma_R)]$$

$$= 2m^{-1} \langle E_M, \tilde{\Delta} \rangle + 352L^2 \mu_{nq}E^2 [d_1(\Sigma_R)].$$

After some algebra, we obtain

$$\frac{1}{2\mu_{nq}} \|\tilde{\Delta}\|_F^2 \leq 2m^{-1} d_1(E_M, \tilde{\Delta}) + 352L^2 \mu_{nq}E^2 [d_1(\Sigma_R)]$$

$$\leq 2m^{-1} d_1(E_M) \sqrt{2r} \|\tilde{\Delta}\|_F + 352L^2 \mu_{nq}E^2 [d_1(\Sigma_R)],$$

where $\|\tilde{\Delta}\|_*$ is nuclear norm of $\tilde{\Delta}$ (i.e., the sum of the singular values of $\tilde{\Delta}$). Using the fact that $2xy \leq a^{-1}x^2 + ay^2$ for any $a > 0$, we have

$$\frac{1}{2\mu_{nq}} \|\tilde{\Delta}\|_F^2 \leq a^{-1} d_1^2(E_M) + 2m^{-1} ar \|\tilde{\Delta}\|_F^2 + 352L^2 \mu_{nq}E^2 [d_1(\Sigma_R)].$$

Taking $a = m/(8r\mu_{nq})$ in the inequality above leads to

$$\frac{1}{4\mu_{nq}} \|\tilde{\Delta}\|_F^2 \leq \frac{8r\mu_{nq}}{m^2} d_1^2(E_M) + 352L^2 \mu_{nq}E^2 [d_1(\Sigma_R)]. \tag{37}$$

Thus (34) holds by combining (35) in Case 1 and (37) in Case 2.
Next, we derive the bounds for $d_1^2(E_M)$ and $\mathbb{E}[d_1(\Sigma_R)]$, respectively. Under Conditions 3.1 and 3.4, it follows from Lemma 5 of Klopp [2014] that the event

$$A = \left\{ \frac{1}{\sigma m} d_1(E_M) \leq c \max \left[ \sqrt{\frac{\nu(t + \log(n + q))}{m(n \land q)}}, \frac{\log(n \land q)(t + \log(n + q))}{m} \right] \right\}$$

holds with probability at least $1 - e^{-t}$ for all $t > 0$. Taking $t = \log(n + q)$ gives that

$$\frac{1}{m} d_1(E_M) \leq c \max \left\{ \sigma \sqrt{\frac{\log(n + q)}{m(n \land q)}}, \frac{\sigma \log(n \land q) \log(n + q)}{m} \right\} \leq c \sigma \sqrt{\frac{\nu \log(n + q)}{m(n \land q)}} \quad (38)$$

holds with probability at least $1 - (n + q)^{-1}$, where the second inequality follows from the assumption that $m \geq \nu^{-1}(n \land q) \log^3(n + q)$.

Applying Lemma 6 of Klopp [2014] with $m \geq \nu^{-1}(n \land q) \log^3(n + q)$ yields

$$\mathbb{E}[d_1(\Sigma_R)] \leq c \sqrt{\frac{\nu \log(n + q)}{m(n \land q)}} \quad (39)$$

In view of (34), (38) and (39), we have

$$\frac{1}{nq} \| \tilde{\Delta} \|_F^2 \leq c \max \left\{ \mu L^2 \sqrt{\frac{\log(n + q)}{m}}, \frac{rnq \mu^2 (\sigma^2 + L^2) \nu \log(n + q)}{m} \right\} \leq c \max \left\{ \mu L^2 \sqrt{\frac{\log(n + q)}{m}}, \frac{\nu \mu^2 (\sigma^2 + L^2) \log(n + q)}{m} \right\} = cB_n^2$$

holds with probability at least $1 - (n + q)^{-1}$, where second inequality follows from the fact that $nq = (n \land q)(n \lor q)$ and $\sigma^2 + L^2 \leq 2(\sigma^2 \lor L^2)$. This completes the proof of Lemma 2.

### B.2 Proof of Lemma 1

Recall that $\tilde{z}_k$ and $\tilde{v}_k$ are the kth left and right singular vectors of $\tilde{Y}/\sqrt{nq}$, respectively. In addition, $n^{-1/2}Xu_k^*$ and $v_k^*$ are the kth left and right singular vectors of $XC^*/\sqrt{nq}$. Using the fact that $\| \tilde{\Delta} \|_{op} \leq \| \tilde{\Delta} \|_F$ and Theorem 3 of Yu et al. [2015] yields

$$\| \tilde{z}_k - n^{-1/2}Xu_k^* \|_2 \leq \frac{2^{3/2}(2d_1^*/\sqrt{q} + \| \tilde{\Delta} \|_F/\sqrt{nq})}{\min(d_{k-1}^2 - d_k^2, d_k^2 - d_{k+1}^2)/q}$$

$$= \frac{2^{3/2}}{\min(d_{k-1}^2 - d_k^2, d_k^2 - d_{k+1}^2)/q} \left( \frac{2d_1^*/\sqrt{q} + \| \tilde{\Delta} \|_F/\sqrt{nq}}{nq} \right)$$

(40)

for each $k = 1, \ldots, r^*$, where $d_0^2 = +\infty$, $d_{r^*+1}^2 = -\infty$, and $\tilde{\Delta} = \tilde{Y} - XC^*$. This together with Condition 3.2 entails

$$\| \tilde{z}_k - n^{-1/2}Xu_k^* \|_2 \approx \gamma_a^{-1} \left[ \| \tilde{\Delta} \|_F/\sqrt{nq} + \| \tilde{\Delta} \|_F^2/(nq) \right]$$

(41)

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Similarly to above, we have
\[
\|\tilde{v}_k - v_k\|_2 \lesssim \gamma_d^{-1} \left[ \|\tilde{\Delta}\|_F / \sqrt{mq} + \|\tilde{\Delta}\|_F^2 / (nq) \right].
\]

(42)

Note that \(\tilde{d}_k / \sqrt{mq}\) and \(d_k^* / \sqrt{q}\) are the singular values of \(\tilde{Y} / \sqrt{mq}\) and \(XC^* / \sqrt{mq}\), respectively. Thus, an application of Weyl’s theorem [Stewart, 1998, Theorem 2] leads to
\[
|\tilde{d}_k - \sqrt{nd_k^*}| / \sqrt{mq} = |d_k / \sqrt{mq} - d_k^* / \sqrt{q}| \leq \|\tilde{\Delta}\|_F / \sqrt{mq}.
\]

(43)

Combining (41), (42), and (43) together yields that the following three inequalities
\[
\|\tilde{z}_k - n^{-1/2} Xu_k^*\|_2 \lesssim \gamma_d^{-1} \left[ \|\tilde{\Delta}\|_F / \sqrt{mq} + \|\tilde{\Delta}\|_F^2 / (nq) \right],
\]

(44)

\[
\|\tilde{v}_k - v_k^*\|_2 \lesssim \gamma_d^{-1} \left[ \|\tilde{\Delta}\|_F / \sqrt{mq} + \|\tilde{\Delta}\|_F^2 / (nq) \right],
\]

(45)

\[
|\tilde{d}_k - \sqrt{nd_k^*}| / \sqrt{mq} \leq \|\tilde{\Delta}\|_F / \sqrt{mq},
\]

(46)

hold uniformly for all \(k = 1, \cdots, r^*\).

Recall that the event \(\Omega = \left\{ \|\tilde{\Delta}\|_F / \sqrt{mq} \leq cB_n\right\}\). It follows from Lemma 2 that \(\|\tilde{\Delta}\|_F / \sqrt{mq} \leq cB_n\) holds with probability at least \(1 - (n + q)^{-1}\), where \(c\) is a positive constant. Then we have \(P(\Omega) \geq 1 - (n + q)^{-1}\).

Under the assumption that \(m \geq \max\{\nu^{-1}(n \wedge q) \log^3(n + q), r(n \lor q) \log(n + q)\}\), we have
\[
\frac{\log(n + q)}{m} \leq \frac{\nu \log(n + q)}{(n \wedge q) \log^3(n + q)} \leq \frac{\nu}{\log^2(2)} \quad \text{and} \quad \sqrt{\frac{r(n \lor q) \log(n + q)}{m}} \leq 1.
\]

Thus, by the definition of \(B_n\), combining these two bounds yields
\[
B_n \leq \max \left\{ L\mu^{1/2} \nu^{1/4} / \log^{1/2}(2), \mu(\sigma \lor L)^{1/2} \right\} \leq c,
\]

(47)

where \(c\) is some positive constant. This, together with (44), (45), and (46), entails that, conditional on the event \(\Omega\), the following inequalities
\[
\|\tilde{z}_k - n^{-1/2} Xu_k^*\|_2 \lesssim \gamma_d^{-1}(B_n + B_n^2) = \gamma_d^{-1}(1 + B_n)B_n \lesssim \gamma_d^{-1}B_n,
\]

(48)

\[
\|\tilde{v}_k - v_k^*\|_2 \lesssim \gamma_d^{-1}(B_n + B_n^2) = \gamma_d^{-1}(1 + B_n)B_n \lesssim \gamma_d^{-1}B_n,
\]

(49)

\[
|\tilde{d}_k - \sqrt{nd_k^*}| / \sqrt{mq} \leq B_n,
\]

(50)

hold uniformly for all \(k = 1, \cdots, r^*\). Since \(P(\Omega) \geq 1 - (n + q)^{-1}\), we have that these three inequalities hold uniformly for all \(k = 1, \cdots, r^*\) with probability at least \(1 - (n + q)^{-1}\). This concludes the proof of Lemma 1.

### B.3 Proof of Lemma 3

Let \(d_1(E)\) be the the largest singular value of \(E\). Note that the operator norm of \(E\) is equal to its largest singular value, that is, \(\|E\|_{op} = d_1(E)\). Recall that \(\tilde{d}_k / \sqrt{mq}, \tilde{z}_k\) and \(\tilde{v}_k\) are
the $k$th singular value, left and right singular vectors of $Y/\sqrt{nq}$, respectively. In addition, $n^{-1/2}Xu_k^*$ and $v_k^*$ are the $k$th left and right singular vectors of $XC\ast/\sqrt{nq}$. Using Theorem 3 of Yu et al. [2015] and Condition 3.2 yields

$$
\|\tilde{z}_k - n^{-1/2}Xu_k^*\|_2 \leq \frac{2^{3/2}(2d_1^*/\sqrt{q} + \|E\|_{op}/\sqrt{nq})}{\min(d_{k-1}^2, d_k^2 - d_{k+1}^2)/q} < \frac{\sqrt{3}}{2} \frac{1}{\|E\|_{op}/\sqrt{nq}}
$$

for each $k = 1, \ldots, r^*$, where $d_0^2 = +\infty$, $d_{r^*+1}^2 = -\infty$ and we use $Y - XC\ast = E$. This together with Condition 3.2 entails

$$
\|\tilde{z}_k - n^{-1/2}Xu_k^*\|_2 \lesssim \gamma_d^{-1} [d_1(E) / \sqrt{nq} + d_1^2(E) / (nq)]. \tag{51}
$$

Similarly, we have

$$
\|\tilde{v}_k - v_k^*\|_2 \lesssim \gamma_d^{-1} [d_1(E) / \sqrt{nq} + d_1^2(E) / (nq)] \tag{52}
$$

for each $k = 1, \ldots, r^*$.

We next find an upper bound for $d_1(E)$. It follows from Condition 3.1 and Proposition 2.4 of Rudelson and Vershynin [2010] that

$$
P(\sigma^{-1}d_1(E) > C_1(\sqrt{n} + \sqrt{q}) + t) \leq 2e^{-C_2t^2},
$$

where $C_1$ and $C_2$ are some positive constants. Taking $t = (\sqrt{n} + \sqrt{q})/\sqrt{C_2}$, we have that the event

$$
B = \{d_1(E) \leq C_3\sigma(\sqrt{n} + \sqrt{q})\} \tag{53}
$$

holds with probability at least $1 - 2e^{-\sqrt{nq}/\sqrt{C_2}^2}$, where $C_3 = C_1 + C_2^{-1/2}$ is a positive constant. Thus, to prove Lemma 3, it suffices to show that, conditional on the event $B$, the following three inequalities

$$
\|\tilde{z}_k - n^{-1/2}Xu_k^*\|_2 \lesssim \gamma_d^{-1} \tilde{B}_n,
$$

$$
\|\tilde{v}_k - v_k^*\|_2 \lesssim \gamma_d^{-1} \tilde{B}_n,
$$

$$
\frac{|d_k - \sqrt{nd_k^*}|}{\sqrt{nq}} \lesssim \tilde{B}_n,
$$

hold uniformly over $k = 1, \ldots, r^*$.

It follows from the definition of $\tilde{B}_n$ that

$$
\tilde{B}_n = \sigma \left( \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{q}} \right) \leq 2\sigma, \tag{54}
$$

where we use the facts that $q \geq 1$ and $n \geq 1$. Thus, conditional on the event $B$, we have

$$
\frac{d_1(E)}{\sqrt{nq}} \leq \frac{\sigma C_3(\sqrt{n} + \sqrt{q})}{\sqrt{nq}} = C_3\sigma \left( \frac{1}{\sqrt{q}} + \frac{1}{\sqrt{n}} \right) = C_3 \tilde{B}_n \leq c,
$$

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where \( c \) is some positive constant. This together with (51) and (52) yields
\[
\| \tilde{z}_k - n^{-1/2} \mathbf{X}u_k^* \|_2 \lesssim \gamma_d^{-1} \left[ 1 + \frac{d_1(\mathbf{E})}{\sqrt{nq}} \right] \frac{d_1(\mathbf{E})}{\sqrt{nq}} \lesssim \gamma_d^{-1} \tilde{B}_n, \tag{55}
\]
\[
\| \tilde{v}_k - v_k^* \|_2 \lesssim \gamma_d^{-1} \left[ 1 + \frac{d_1(\mathbf{E})}{\sqrt{nq}} \right] \frac{d_1(\mathbf{E})}{\sqrt{nq}} \lesssim \gamma_d^{-1} \tilde{B}_n \tag{56}
\]
for all \( k = 1, \ldots, r^* \). An application of Weyl’s theorem [Stewart, 1998, Theorem 2] entails
\[
\left| \frac{d_k - \sqrt{n}d_k^*}{\sqrt{nq}} \right| = \left| \frac{\tilde{d}_k}{\sqrt{nq}} - \frac{d_k^*}{\sqrt{nq}} \right| \leq \frac{d_1(\mathbf{E})}{\sqrt{nq}} \lesssim \tilde{B}_n \tag{57}
\]
for all \( k = 1, \ldots, r^* \). It concludes the proof of Lemma 3.

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