A minimal coupling method for investigating one dimensional dissipative quantum systems

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November 12, 2018

Abstract
Quantum mechanics of a general one dimensional dissipative system investigated by it’s coupling to a Klein-Gordon field as the environment using a minimal coupling method. Heisenberg equation for such a dissipative system containing a dissipative term proportional to velocity obtained. As an example, quantum dynamics of a damped harmonic oscillator as the prototype of some important one dimensional dissipative models investigated consistently. Some transition probabilities indicating the way energy flows between the subsystems obtained.

1 Introduction

In classical mechanics dissipation can be taken into account by introducing a velocity dependent damping term into the equation of motion. Such an approach is no longer possible in quantum mechanics where a time-independent Hamiltonian implies energy conservation and accordingly we can not find a unitary time evolution operator for both states and observable quantities consistently.

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To investigate the quantum mechanical description of dissipating systems, there are some treatments, one can consider the interaction between two systems via an irreversible energy flow [1,2], or take a phenomenological treatment for a time dependent Hamiltonian which describes damped oscillations, here we can refer the interested reader to Caldirola-Kanai Hamiltonian for a damped harmonic oscillator [3].

\[ H(t) = e^{-\frac{2\beta t}{2m}} p^2 + e^{\frac{2\beta t}{2m}} \frac{1}{2} m \omega^2 q^2. \]  

(1)

There are significant difficulties about the quantum mechanical solutions of the Caldirola-Kanai Hamiltonian, for example quantizing such a Hamiltonian violates the uncertainty relations or canonical commutation rules and the uncertainty relations vanish as time tends to infinity.[4,5,6,7,8]

In 1931, Bateman [9] presented the mirror-image Hamiltonian which consists of two different oscillator, where one of them represents the main one-dimensional damped harmonic oscillator. Energy dissipated by the main oscillator completely will be absorbed by the other oscillator and thus the energy of the total system is conserved. Bateman Hamiltonian is given by

\[ H = \frac{p\bar{p}}{m} + \frac{\beta}{2m}(\bar{x}p - xp) + (k - \frac{\beta^2}{4m})x\bar{x}, \]

(2)

with the corresponding Lagrangian

\[ L = m\dot{x}\ddot{x} + \frac{\beta}{2}(\dot{x}\ddot{x} - \ddot{x}\dot{x}) - kx\bar{x}, \]

(3)
canonical momenta for this dual system can be obtained from this Lagrangian as

\[ p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} - \frac{\beta}{2}\bar{x}, \quad \bar{p} = \frac{\partial L}{\partial \dot{\bar{x}}} = m\dot{\bar{x}} + \frac{\beta}{2}x, \]

(4)
dynamical variables \( x, p \) and \( \bar{x}, \bar{p} \) should satisfy the commutation relations

\[ [x, p] = i, \quad [\bar{x}, \bar{p}] = i, \]

(5)

however the time-dependent uncertainty products obtained in this way, vanishes as time tends to infinity.[10]

Caldirola [3,11] developed a generalized quantum theory of a linear dissipative system in 1941 : equation of motion of a single particle subjected to a generalized non conservative force \( Q \) can be written as

\[ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}} \right) - \frac{\partial T}{\partial q} = -\frac{\partial V}{\partial q} + Q(q), \]

(6)
where $Q_r = -\beta(t) \sum a_{rj} \dot{q}_j$, and $a_{rj}$'s are some constants, changing the variable $t$ to $t^*$, using the following nonlinear transformation

$$t^* = \chi(t), \quad dt = \phi(t) dt^*, \quad \phi(t) = e^{\int_0^t \beta(t') dt'},$$

(7)

together with the definitions

$$\dot{q}^* = \frac{dq}{dt^*}, \quad L^* = L(q, \dot{q}^*, t^*), \quad p^* = \frac{\partial L^*}{\partial \dot{q}^*},$$

(8)

the Lagrangian equations, can be obtained from

$$\frac{d}{dt^*} \left( \frac{\partial L^*}{\partial \dot{q}^*} \right) - \frac{\partial L^*}{\partial q} = 0.$$

(9)

where $H^* = \sum p^* \dot{q}^* - L^*$. Canonical commutation rule and Schrödinger equation in this formalism are

$$[q, p^*] = i, \quad H^* \psi = i \frac{\partial \psi}{\partial t^*},$$

(10)

but unfortunately uncertainty relations vanish as time goes to infinity.\[10\]

Perhaps one of the effective approaches in quantum mechanics of dissipative systems is the idea of considering an environment coupled to the main system and doing calculations for the total system but at last for obtaining observables related to the main system, the environment degrees of freedom must be eliminated. The interested reader is referred to the Caldeira-Legget model [12,13]. In this model the dissipative system is coupled with an environment made by a collection of $N$ harmonic oscillators with masses $m_n$ and frequencies $\omega_n$, the interaction term in Hamiltonian is as follows

$$H' = -q \sum_{n=1}^N c_n x_n + q^2 \sum_{n=1}^N \frac{c_n^2}{2m_n \omega_n^2},$$

(11)

where $q$ and $x_n$ denote coordinates of system and environment respectively and the constants $c_n$ are called coupling constants.

The above coupling is not suitable for dissipative systems containing a dissipative term proportional to velocity. In fact with above coupling we can not obtain a Heisenberg equation like $\ddot{q} + \omega^2 q + \beta \dot{q} = \xi(t)$, for a damped harmonic oscillator, consistently. In this paper we generalize the Caldeira-Legget model to an environment with continuous degrees of freedom by a
coupling similar to the coupling between a charged particle and the electromagnetic field known as the minimal coupling. In sections 2, the idea of a minimal coupling method is introduced. In section 3 the quantum dynamics of a damped harmonic oscillator is investigated completely and some transition probabilities indicating the way energy flows between subsystems, are obtained. In section 4, quantum dynamics of the oscillator environment is investigated.

2 Quantum dynamics of a one dimensional damped system

Quantum mechanics of a one dimensional damped system can be investigated by introducing a reservoir or an environment that interacts with the system through a new kind of minimal coupling term. For this purpose let the damped system be a particle with mass $m$ influenced by a external potential $v(q)$. we take the total Hamiltonian, i.e., system plus environment like this

$$H = \frac{(p - R)^2}{2m} + v(q) + H_B,$$

(12)

where $q$ and $p$ are position and canonical conjugate momentum operators of the particle respectively and satisfy the canonical commutation rule

$$[q, p] = i,$$

(13)

and $H_B$ is the reservoir Hamiltonian

$$H_B(t) = \int_{-\infty}^{+\infty} d^3k \omega_{\vec{k}} b_{\vec{k}}(t) b_{\vec{k}}^\dagger(t), \quad \omega_{\vec{k}} = |\vec{k}|.$$

(14)

Annihilation and creation operators $b_{\vec{k}}, b_{\vec{k}}^\dagger$, in any instant of time, satisfy the following commutation relations

$$[b_{\vec{k}}(t), b_{\vec{k}'}^\dagger(t)] = \delta(\vec{k} - \vec{k}'),$$

(15)

and we will show later that reservoir is a Klein-Gordon type equation with a source term. Operator $R$ have the basic role in interaction between the system and reservoir and is defined by

$$R(t) = \int_{-\infty}^{+\infty} d^3k [f(\omega_{\vec{k}}) b_{\vec{k}}(t) + f^*(\omega_{\vec{k}}) b_{\vec{k}}^\dagger(t)],$$

(16)
let us call the function $f(\omega_k)$, the coupling function. It can be shown easily that Heisenberg equation for $q$ and $p$ leads to

$$\dot{q} = i[H, q] = \frac{\vec{p} - \vec{R}}{m},$$

$$\dot{p} = i[H, p] = -\frac{\partial v}{\partial q},$$

where after omitting $p$, gives the following equation for the damped quantum system

$$m\ddot{q} = -\frac{\partial v}{\partial q} - \dot{\vec{R}}.$$  \hspace{1cm} (18)

Using (15) the Heisenberg equation for $b_k$, is

$$\dot{b}_k = i[H, b_k] = -i\omega_k b_k + i\dot{q}f^*(\omega_k),$$

with the following formal solution

$$b_k(t) = b_k(0)e^{-i\omega_k t} + if^*(\omega_k)\int_0^t dt' e^{-i\omega_k(t-t')}\dot{q}(t'),$$

(substituting $b_k(t)$ from (20) into (18) one can obtain

$$m\ddot{q} + \int_0^t dt' \dot{q}(t')\gamma(t-t') = -\frac{\partial v}{\partial q} + \xi(t)$$

$$\gamma(t) = 8\pi \int_0^\infty d\omega_k |f(\omega_k)|^2 \omega_k^3 \cos \omega_k t$$

$$\xi(t) = i\int_{-\infty}^{+\infty} d^3k \omega_k f(\omega_k)b_k(0)e^{-i\omega_k t} - f^*(\omega_k)b_k^\dagger(0)e^{i\omega_k t}).$$

(21)

It is clear that the expectation value of $\xi(t)$ in any eigenstate of $H_B$, is zero. For the following special choice of coupling function

$$f(\omega_k) = \frac{\beta}{4\pi^2 \omega_k^3},$$

equation (21) takes the form

$$m\ddot{q} + \beta\dot{q} = -\frac{\partial v}{\partial q} + \tilde{\xi}(t)$$

$$\tilde{\xi}(t) = i\beta \sqrt{\frac{4\omega_k^3}{\omega_k^3}} \int_{-\infty}^{+\infty} \frac{d^3k}{\omega_k^3}(b_k(0)e^{-i\omega_k t} - b_k^\dagger(0)e^{i\omega_k t}),$$

(23)

In the following we investigate for example one dimensional harmonic oscillator.
3 quantum mechanics of one dimensional damped harmonic oscillator

3.1 quantum dynamics

For a one dimensional harmonic oscillator with mass \( m \) and frequency \( \omega \) we have \( v(q) = \frac{1}{2} m \omega^2 q^2 \) and therefore we can write (23) as

\[
\ddot{q} + \frac{\beta}{m} \dot{q} + \omega^2 q = \frac{\tilde{\xi}(t)}{m} \tag{24}
\]

with the following solution

\[
q(t) = e^{-\frac{\beta t}{2m}} (\hat{A} e^{i \omega_1 t} + \hat{B} e^{-i \omega_1 t}) + M(t),
\]

\[
M(t) = i \int_{-\infty}^{+\infty} d^3k \sqrt{\frac{\beta}{4\pi^2 m^2 \omega_k^2}} \left[ \frac{b_k(0)}{\omega^2 - \omega_k^2 - i \frac{\beta}{m} \omega_k} e^{-i \omega_k t} - \frac{b_k^\dagger(0)}{\omega^2 - \omega_k^2 + i \frac{\beta}{m} \omega_k} e^{i \omega_k t} \right],
\]

where \( \omega_1 = \sqrt{\omega^2 - \frac{\beta^2}{4m^2}} \). Operators \( \hat{A} \) and \( \hat{B} \), are specified by initial conditions

\[
\hat{A} + \hat{B} = q(0) - M(0),
\]

\[
\left( \frac{-\beta}{2m} + i \omega_1 \right) \hat{A} + \left( \frac{-\beta}{2m} - i \omega_1 \right) \hat{B} = \dot{q}(0) - \dot{M}(0) = \frac{p(0) - R(0)}{m} - \dot{M}(0),
\]

solving above equations and substituting \( \hat{A} \) and \( \hat{B} \) in (25) one obtains

\[
q(t) = e^{-\frac{\beta t}{2m}} \left\{ \frac{p(0)}{m \omega_1} \sin \omega_1 t + q(0) \cos \omega_1 t + \frac{\beta}{2 m \omega_1} q(0) \sin \omega_1 t \right\}
\]

\[
- \frac{R(0)}{m \omega_1} \sin \omega_1 t - \frac{\beta M(0)}{2 m \omega_1} \sin \omega_1 t - M(0) \cos \omega_1 t - \frac{\dot{M}(0)}{\omega_1} \sin \omega_1 t \right\} + M(t),
\]

\[
(27)
\]
also substituting \( q(t) \) from (27) in (20) we can obtain a stable solution for \( b_{k}(t) \) in \( t \to \infty \) as

\[
b_{k}(t) = b_{k}(0)e^{-i\omega_{k}t} - i\sqrt{\frac{\beta}{4\pi^{2}\omega_{k}^{3}}} \left[ e^{-i\omega_{k}t}\left(\omega^{2}q(0) + i\omega_{k}p(0) - R(0)\right) - M(0)\omega^{2} - i\omega_{k}\dot{M}(0)\right]
\]

\[
+ \frac{i\beta}{4\pi^{2}m\sqrt{\omega_{k}^{3}}} \int_{-\infty}^{+\infty} d^{3}\omega_{k}^{\prime} \left\{ \frac{1}{\omega^{2} - \omega_{k}^{2} - \frac{i\beta}{m}\omega_{k}^{\prime}} \sin\left(\frac{(\omega_{k}^{\prime} - \omega_{k})t}{2}\right) e^{-i\left(\omega_{k}^{\prime} + \omega_{k}\right)t} \right\} ,
\]

now substituting \( b_{k}(t) \) from (28) in (16) and using (17), one obtains \( p = m\ddot{q} + R \).

A vector in fock space of reservoir is a linear combination of basis vectors

\[
|N(\vec{k}_{1}), N(\vec{k}_{2}), \ldots)_{B} = \frac{(b_{k_{1}}^{N(\vec{k}_{1})}(b_{k_{2}}^{N(\vec{k}_{2})}\ldots |0)_{B}}{\sqrt{N(\vec{k}_{1})!N(\vec{k}_{2})!\ldots}}
\]

where are eigenstates of \( H_{B} \) and the operators \( b_{k} \) and \( b_{k}^\dagger \) act on them as

\[
b_{k}|N(\vec{k}_{1}), N(\vec{k}_{2}), \ldots N(\vec{k}), \ldots)_{B} = \sqrt{N(\vec{k})}|N(\vec{k}_{1}), N(\vec{k}_{2}), \ldots N(\vec{k}) - 1, \ldots)_{B}
\]

\[
b_{k}^\dagger|N(\vec{k}_{1}), N(\vec{k}_{2}), \ldots N(\vec{k}), \ldots)_{B} = \sqrt{N(\vec{k}) + 1}|N(\vec{k}_{1}), N(\vec{k}_{2}), \ldots N(\vec{k}) + 1, \ldots)_{B}
\]

If the state of system in \( t = 0 \) is taken to be \( |\psi(0)\rangle = |0\rangle_{B} \otimes |n\rangle_{\omega} \) where \( |0\rangle_{B} \) is vacuum state of reservoir and \( |n\rangle_{\omega} \) an excited state of the Hamiltonian \( H_{s} = \frac{p^{2}}{2m} + \frac{1}{2}m\omega^{2}q^{2} \), then it is clear that

\[
\langle\psi(0)|\frac{p^{2}(0)}{2m} + \frac{1}{2}m\omega^{2}q^{2}(0)|\psi(0)\rangle = (n + \frac{1}{2})\omega,
\]
On the other hand from (27), (28) and (16) we find

\[
\lim_{t \to \infty} [\langle \psi(0) | : \frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \omega^2 q^2 : | \psi(0) \rangle] = 0,
\]

\[
\lim_{t \to \infty} [\langle \psi(0) | : p^2 : | \psi(0) \rangle] = \frac{\beta^2 \omega^4}{2 \pi^2 m} \lim_{t \to \infty} \left| \int_{-\infty}^{+\infty} \frac{dx}{x} \frac{e^{ixt}}{x^2 + \frac{\beta^2}{m} x^2} \right|^2 \langle q^2(0) \rangle_n
\]

\[
\simeq \frac{\beta^2}{2m \omega} (n + \frac{1}{2})
\]  

(32)

where : : denotes normal ordering operator. Now by substituting \(b_k(t)\) from (28) into (14), it is easy to show that

\[
\lim_{t \to \infty} [\langle \psi(0) | : H_B(t) : | \psi(0) \rangle]
\]

\[
= \frac{\beta \omega^4}{\pi} \int_{0}^{\infty} \frac{dx}{(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2} \langle q^2(0) \rangle_n + \frac{\beta}{\pi m^2} \int_{0}^{\infty} \frac{x^2 dx}{(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2} \langle p^2(0) \rangle_n
\]

\[
= \frac{\beta \omega^3}{\pi m} \left( n + \frac{1}{2} \right) \int_{0}^{\infty} \frac{dx}{(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2} + \frac{\beta \omega}{\pi m} \left( n + \frac{1}{2} \right) \int_{0}^{\infty} \frac{x^2 dx}{(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2}
\]

(33)

For sufficiently weak damping that is when \(\beta\) is very small, the integrands in (33) have singularity points \(x = \pm (\omega_1 \pm \frac{\beta k}{2m})\) and by using residual calculus we find

\[
\lim_{t \to \infty} [\langle \psi(0) | : n : | \psi(0) \rangle] = \langle n + \frac{1}{2} \rangle \omega
\]  

(34)

Comparing (31) and (34), one can show that the total energy of oscillator has been transmitted to the reservoir and according to (32), the kinetic energy of oscillator tends to zero.

If the state of system in \(t = 0\) is \(\rho(0) = \rho_B^T \otimes |S\rangle_\omega\) where \(\rho_B^T = \frac{e^{-H_B T}}{Tr_B(e^{-H_B T})}\) is the Maxwell-Boltzman distribution and \(|S\rangle_\omega\) is an arbitrary state of harmonic oscillator, then by using of \(Tr_B[b_k^\dagger(0)b_k(0)\rho_B^T] = \frac{\delta(k-k')}{e^{\frac{\pi k}{\beta}} - 1}\) one can show the expectation value of kinetic energy of oscillator in \(t \to \infty\) tends to

\[
\lim_{t \to \infty} [\langle \frac{1}{2} m \dot{q}^2(t) + \frac{1}{2} m \omega^2 q^2(t) : \rangle] = \frac{2 \beta}{\pi m^2} \int_{0}^{\infty} \frac{x}{[(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2]} (e^{\frac{\pi k}{\beta}} - 1) dx
\]

\[
+ \frac{2 \beta}{\pi m^2} \int_{0}^{\infty} \frac{x^3}{[(\omega^2 - x^2)^2 + \frac{\beta^2}{m^2} x^2]} (e^{\frac{\pi k}{\beta}} - 1) dx
\]  

(35)
3.2 Transition probabilities

We can write the Hamiltonian (12) as

\[ H = H_0 + H', \]
\[ H_0 = (a^\dagger a + \frac{1}{2})\omega + H_B, \]
\[ H' = -\frac{p}{m}R + \frac{R^2}{2m}, \]

(36)

where \( a \) and \( a^\dagger \) are annihilation and creation operators of the Harmonic oscillator. In interaction picture we can write

\[ a_I(t) = e^{iH_0t}a(0)e^{-iH_0t} = ae^{-i\omega t}, \]
\[ b_{\vec{k}I}(t) = e^{iH_0t}b_{\vec{k}}(0)e^{-iH_0t} = b_{\vec{k}}(0)e^{-i\omega_{\vec{k}}t}, \]

(37)

the terms \( \frac{R}{m}p \) and \( \frac{R^2}{2m} \) are of the first order and second order of damping respectively, therefore, for a sufficiently weak damping, \( \frac{R}{m}p \) is small in comparison with \( \frac{R^2}{2m} \). Furthermore \( \frac{R^2}{2m} \) has not any role in those transition probabilities where initial and final states of harmonic oscillator are different, hence we can neglect the term \( \frac{R^2}{2m} \) in \( H' \). Substituting \( a_I \) and \( b_{\vec{k}I} \) from (37) in \( -\frac{R}{m}p \), one can obtain \( H'_I \) in interaction picture as

\[ H'_I = -i\sqrt{\frac{\omega}{2m}} \int_{-\infty}^{+\infty} d^3k [f(\omega_{\vec{k}})a^\dagger b_{\vec{k}}(0)e^{i(\omega-\omega_{\vec{k})}t} - f^*(\omega_{\vec{k}})ab_{\vec{k}}(0)e^{-i(\omega-\omega_{\vec{k}})t} \]
\[ -f(\omega_{\vec{k}})ab_{\vec{k}}(0)e^{-i(\omega_{\vec{k}}+\omega)t} + f^*(\omega_{\vec{k}})a^\dagger b_{\vec{k}}(0)e^{i(\omega_{\vec{k}}+\omega)t}], \]

(38)

the terms containing just \( ab_{\vec{k}}(0) \) and \( a^\dagger b_{\vec{k}}^\dagger(0) \) violate the conservation of energy in the first order perturbation, because \( ab_{\vec{k}}(0) \) destroys an excited state of harmonic oscillator while at the same time destroying a reservoir excitation state and \( a^\dagger b_{\vec{k}}^\dagger(0) \) creates an excited state of harmonic oscillator, while creating an excited reservoir state at the same time, therefore we neglect the terms involving \( ab_{\vec{k}}(0) \) and \( a^\dagger b_{\vec{k}}^\dagger(0) \) because of energy conservation and write \( H'_I \) as

\[ H'_I = -i\sqrt{\frac{\omega}{2m}} \int_{-\infty}^{+\infty} d^3k [f(\omega_{\vec{k}})a^\dagger b_{\vec{k}}(0)e^{i(\omega-\omega_{\vec{k}})t} - f^*(\omega_{\vec{k}})ab_{\vec{k}}^\dagger(0)e^{-i(\omega-\omega_{\vec{k}})t}]. \]

(39)

Now the time evolution of density operator in interaction picture is [14]

\[ \rho_I(t) = U_I(t, t_0)\rho_I(t_0)U_I^\dagger(t, t_0), \]

(40)
where $U_I$ is the time evolution operator, which in first order perturbation is

$$U_I(t, t_0 = 0) = 1 - i \int_0^t dt_1 H'_I(t_1) =$$

$$1 - \sqrt{\frac{\omega}{2m}} \int_{-\infty}^{+\infty} d^3k [f(\omega_k) a^\dagger b_k(0) e^{i(\omega - \omega'_k)t} - f^*(\omega_k) a b^\dagger_k(0) e^{-i(\omega - \omega'_k)t}] \sin \left(\frac{(\omega - \omega'_k)t}{2}\right).$$

(41)

Let $\rho_I(0) = |n\rangle_\omega \omega \langle n| \otimes |0\rangle_B B \langle 0|$ where $|0\rangle_B$ is the vacuum state of the reservoir and $|n\rangle_\omega$, an excited state of the harmonic oscillator, then by substituting $U_I(t, 0)$ from (41) in (40) and taking trace over reservoir parameters we obtain

$$\rho_{sI}(t) := Tr_B(\rho_I(t)) = |n\rangle_\omega \omega \langle n| + \frac{n\omega}{2m} |n - 1\rangle_\omega \omega \langle n - 1| \int_{-\infty}^{+\infty} d^3p |f(\omega_p)|^2 \frac{\sin^2 \left(\frac{(\omega - \omega'_p)t}{2}\right)}{\left(\frac{(\omega - \omega'_p)}{2}\right)^2},$$

(42)

where we have used the formula $Tr_B[|1\rangle_B B \langle 1|] = \delta(\vec{k} - \vec{k}')$. For very large time, we can write $\frac{\sin^2 \left(\frac{(\omega - \omega'_p)t}{2}\right)}{\left(\frac{(\omega - \omega'_p)}{2}\right)^2} = 2\pi t \delta(\omega_p - \omega)$ which leads to the following relation for density matrix

$$\rho_{sI}(t) = |n\rangle_\omega \omega \langle n| + 4\pi^2 \omega^3 nt |f(\omega)|^2 \frac{m}{n - 1} |n - 1\rangle_\omega \omega \langle n - 1|,$$

(43)

from density matrix we can calculate the probability of transition $|n\rangle_\omega \rightarrow |n - 1\rangle_\omega$ as

$$\Gamma_{n\rightarrow n-1} = Tr[|n - 1\rangle_\omega \omega \langle n - 1| \rho(t)] =$$

$$Tr_s[|n - 1\rangle_\omega \omega \langle n - 1| \rho_{sI}(t)] = 4\pi^2 \omega^3 nt |f(\omega)|^2 \frac{m}{n - 1} |n - 1\rangle_\omega \omega \langle n - 1|,$$

(44)

where $Tr_s$ denotes taking trace over harmonic oscillator eigenstates. For the special choice (22), above transition probability becomes

$$\Gamma_{n\rightarrow n-1} = \frac{n\beta t}{m}.$$

(45)
Now consider the case where the reservoir is an excited state in \( t = 0 \) for example \( \rho_I(0) = |n_\omega \rangle \otimes |\vec{p}_1, \ldots, \vec{p}_j \rangle_B \) where \( |\vec{p}_1, \ldots, \vec{p}_j \rangle_B \) denotes a state of reservoir that contains \( j \) quanta with corresponding momenta \( \vec{p}_1, \ldots, \vec{p}_j \), then by making use of

\[
Tr_B[b^\dagger_\vec{k} |\vec{p}_1, \ldots, \vec{p}_j \rangle_B \langle \vec{p}_1, \ldots, \vec{p}_j | b_\vec{k}'] = \delta(\vec{k} - \vec{k}'),
\]

\[
Tr_B[b_\vec{k} |\vec{p}_1, \ldots, \vec{p}_j \rangle_B \langle \vec{p}_1, \ldots, \vec{p}_j | b^\dagger_\vec{k}'] = \sum_{l=1}^j \delta(\vec{k} - \vec{p}_l) \delta(\vec{k}' - \vec{p}_l),
\]

and long time approximation, we find

\[
\rho_{sI}(t) = |n_\omega \rangle \langle n_\omega | + \frac{(n + 1)\omega}{2m} |n + 1_\omega \rangle \langle n + 1_\omega | + \sum_{l=1}^j |f(\omega_{\vec{p}_l})|^2 \sin^2 \left( \frac{\omega_{\vec{p}_l} - \omega}{2} \right) t
\]

\[
+ \frac{n\omega}{2m} |n - 1_\omega \rangle \langle n - 1_\omega | \int_{-\infty}^{+\infty} d^3k |f(\omega_{\vec{k}})|^2 \sin^2 \left( \frac{\omega_{\vec{k}} - \omega}{2} \right) t,
\]

which gives the transition probability for \( |n_\omega \rangle \to |n - 1_\omega \rangle \) and \( |n_\omega \rangle \to |n + 1_\omega \rangle \), respectively as follows

\[
\Gamma_{n\to n-1} = Tr_s[|n - 1_\omega \rangle \langle n - 1_\omega | \rho_{sI}(t)] = \frac{4\pi^2 \omega^3 nt}{m} |f(\omega)|^2,
\]

\[
\Gamma_{n\to n+1} = Tr_s[|n + 1_\omega \rangle \langle n + 1_\omega | \rho_{sI}(t)] = \frac{(n + 1)\pi t \omega}{m} |f(\omega)|^2 \sum_{l=1}^j \delta(\omega_{\vec{p}_l} - \omega).
\]

Specially for the choice (22), we have

\[
\Gamma_{n\to n-1} = \frac{n\beta t}{m},
\]

\[
\Gamma_{n\to n+1} = \frac{\beta(n + 1)t}{4\pi m \omega^2} \sum_{l=1}^j \delta(\omega_{\vec{p}_l} - \omega).
\]

Another important case is when the reservoir has a Maxwell-Boltzmann distribution so let \( \rho_I(0) = |n_\omega \rangle \otimes \rho^T_B \) where
\[ \rho_B^T = \frac{e^{-\frac{\mu_B}{e^{\frac{\mu_B}{kT}}}}}{TR_B(e^{\frac{\mu_B}{kT}})}, \]

then by making use of following relations

\[ Tr_B[b_k \rho_B^T b_{k'}] = Tr_B[b_k^T \rho_B^T b_{k'}^T] = 0, \]
\[ Tr_b[b_k \rho_B^T b_{k'}^T] = \frac{\delta(\vec{k} - \vec{k}')}{e^{\frac{\epsilon}{kT}} - 1}, \]
\[ Tr_B[b_k^T \rho_B^T b_{k'}] = \frac{\delta(\vec{k} - \vec{k}')e^{\frac{\epsilon}{kT}}}{e^{\frac{\epsilon}{kT}} - 1}, \]

(50)

we can obtain the density operator \( \rho_{sl}(t) \) in interaction picture as

\[ \rho_{sl}(t) := Tr_B[\rho_I(t)] = |n\rangle_\omega \langle n| \]
\[ + \frac{(n+1)\omega}{2m} |n+1\rangle_\omega \langle n+1| \int_{-\infty}^{+\infty} d^3k \frac{|f(\omega_k)|^2 \sin^2 \left( \frac{\omega - \omega_k}{2} \right) t}{e^{\frac{\epsilon_k}{kT}} - 1 \left( \frac{\omega - \omega_k}{2} \right)^2} \]
\[ + \frac{n\omega}{2m} |n-1\rangle_\omega \langle n-1| \int_{-\infty}^{+\infty} d^3k \frac{|f(\omega_k)|^2 e^{\frac{\epsilon_k}{kT}} \sin^2 \left( \frac{\omega - \omega_k}{2} \right) t}{e^{\frac{\epsilon_k}{kT}} - 1 \left( \frac{\omega - \omega_k}{2} \right)^2}, \]

(51)

which accordingly gives the following transition probabilities in very long time as

\[ \Gamma_{n\rightarrow n-1} = Tr_s[|n-1\rangle_\omega \langle n-1| \rho_{sl}(t)] = \frac{4\pi^2 \omega^3 n t |f(\omega)|^2 e^{\frac{\epsilon_k}{kT}}}{m e^{\frac{\epsilon_k}{kT}} - 1}, \]
\[ \Gamma_{n\rightarrow n+1} = Tr_s[|n+1\rangle_\omega \langle n+1| \rho_{sl}(t)] = \frac{4\pi^2 \omega^3 (n+1) t |f(\omega)|^2}{m e^{\frac{\epsilon_k}{kT}} - 1}, \]

(52)

substituting (22) in these recent relations we find

\[ \Gamma_{n\rightarrow n-1} = \frac{n \beta t e^{\frac{\epsilon_k}{kT}}}{m(e^{\frac{\epsilon_k}{kT}} - 1)}, \]
\[ \Gamma_{n\rightarrow n+1} = \frac{(n+1) \beta t}{m(e^{\frac{\epsilon_k}{kT}} - 1)}. \]

(53)

So in very low temperatures the energy flows from oscillator to the reservoir by the rate \( \Gamma_{n\rightarrow n-1} \mapsto \frac{n \beta t}{m} \) and no energy flows from reservoir to oscillator.
4 Quantum field of reservoir

Let us define the operators $Y(x, t)$ and $\Pi_Y(x, t)$ as follows

$$Y(\vec{x}, t) = \int_{-\infty}^{+\infty} \frac{d^3k}{\sqrt{2}(2\pi)^3} \omega_\vec{k} \left( b_{\vec{k}}(t)e^{i\vec{k}.\vec{x}} + b_{\vec{k}}^\dagger(t)e^{-i\vec{k}.\vec{x}} \right),$$

$$\Pi_Y(\vec{x}, t) = i \int_{-\infty}^{+\infty} \frac{d^3k}{\sqrt{2}(2\pi)^3} \omega_\vec{k} \left( b_{\vec{k}}^\dagger(t)e^{-i\vec{k}.\vec{x}} - b_{\vec{k}}(t)e^{i\vec{k}.\vec{x}} \right),$$

(54)

then using commutation relations (15), one can show that $Y(\vec{x}, t)$ and $\Pi_Y(\vec{x}, t)$ satisfy the equal time commutation relations

$$[Y(\vec{x}, t), \Pi_Y(\vec{x}', t)] = i\delta(\vec{x} - \vec{x}'),$$

(55)

furthermore by substituting $b_{\vec{k}}(t)$ from (20) in (54), we obtain

$$\frac{\partial\Pi_Y(\vec{x}, t)}{\partial t} = \nabla^2 Y + 2\dot{q}(t)P(\vec{x}), \quad P(\vec{x}) = Re\int_{-\infty}^{+\infty} d^3k \sqrt{\frac{\omega_\vec{k}}{2(2\pi)^3}} f(\omega_\vec{k}) e^{-i\vec{k}.\vec{x}},$$

$$\Pi_Y(\vec{x}, t) = \frac{\partial Y}{\partial t} - 2\dot{q}(t)Q(\vec{x}), \quad Q(\vec{x}) = Im\int_{-\infty}^{+\infty} d^3k \frac{f(\omega_\vec{k})}{\sqrt{2(2\pi)^3} \omega_\vec{k}} e^{-i\vec{k}.\vec{x}},$$

(56)

so $Y(\vec{x}, t)$ satisfies the following source included Klein-Gordon equation

$$\frac{\partial^2 Y}{\partial t^2} - \nabla^2 Y = 2\dot{q}(t)Q(\vec{x}) + 2\dot{q}(t)P(\vec{x}),$$

(57)

with the corresponding Lagrangian density

$$\mathcal{L} = \frac{1}{2} \left( \frac{\partial Y}{\partial t} \right)^2 - \frac{1}{2} \nabla Y \cdot \nabla Y - 2\dot{q}Q(\vec{x}) \frac{\partial Y}{\partial t} + 2\dot{q}P(\vec{x})Y.$$ 

(58)

Therefore the reservoir is a massless Klein-Gordon field with source $2\dot{q}Q(\vec{x}) + 2\dot{q}P(\vec{x})$. The Hamiltonian density for (57) is as follows

$$\mathcal{H} = \frac{(\Pi_Y + 2\dot{q}Q)^2}{2} + \frac{1}{2} |\nabla Y|^2 - 2\dot{q}YP,$$

(59)

and equations (56) are Heisenberg equations for $Y$ and $\Pi_Y$. If we obtain $b_{\vec{k}}$ and $b_{\vec{k}}^\dagger$ from (24) in terms of $Y$ and $\Pi_Y$ and substitute them in $H_B$ defined in (14) we obtain

$$H_B = \int_{-\infty}^{+\infty} d^3k \omega_\vec{k} b_{\vec{k}}^\dagger b_{\vec{k}} = \frac{\Pi_Y^2}{2} + \frac{1}{2} |\nabla Y|^2.$$ 

(60)
5 Concluding remarks

By generalizing Caldeira-Legget model to an environment with continuous degrees of freedom, for example a Klein-Gordon field, a new minimal coupling method introduced which can be extended and applied to a large class of dissipative systems consistently. Such method applied to a quantum damped harmonic oscillator as a prototype of these models, with a dissipation term proportional to velocity. Some transition probabilities explaining the way energy flows between subsystems obtained. Choosing different coupling functions in (16) we could investigate another classes of dissipative systems.

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