An $SL_2(\mathbb{R})$-Casson invariant and Reidemeister torsions

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Abstract

We define an $SL_2(\mathbb{R})$-Casson invariant of closed 3-manifolds. Moreover, we describe a procedure for computing the invariant in terms of a Reidemeister torsion and discuss approaches to giving the Casson invariant some gradings.

Keywords

Casson invariant, Reidemeister torsion, 3-manifolds, Chern-Simons class

1 Introduction

In a series of lectures [Cas], Casson defined a $\mathbb{Z}$-valued topological invariant of an integral homology 3-sphere $M$. Choose a Heegaard splitting $M = W_1 \cup_\Sigma W_2$, where $\Sigma$ is a connected closed surface. Roughly speaking, the Casson invariant counts equivalent classes of irreducible representations $\pi_1(M) \to SU(2)$, in contrast to $\pi_1(\Sigma) \to SU(2)$. Several topologists (see, e.g., [Ati, BN]) have generalized the invariant to count representations in a number of other Lie groups $G$; see [Cur1, Cur2, BH] for the cases $G = SO(3), U(2), SO(4), SL_2(\mathbb{C}), SU(3)$. The Casson invariant is a landmark topic in low-dimensional topology, and it has been studied from many viewpoints, including through Chern-Simons theory; see, e.g., [AM, Sav].

This paper is inspired by the note of Johnson [John]. A difficult point of the Casson invariants is to explicitly determine appropriate weights appearing in the counts of representations. To solve this problem, he suggested a procedure for computing the weights from Reidemeister torsions under a certain condition; see Theorem B.2. Since the note is unpublished, we give a proof of the theorem, where we essentially use results of Stanford and Witten [SW, Wit]; see Appendix B.

In this paper, we mainly address the case $G = SL_2(\mathbb{R})$. Of particular interest to us is the relation to Reidemeister torsion and the Chern-Simons invariant. Since $SL_2(\mathbb{R}) = SU(1, 1)$ is over $\mathbb{R}$ and non-compact, we need a sensitive treatment, as in [Lab, SW, Wit]; e.g., we focus on the Zariski density instead of the irreducibility of representations. Then, in an analogous way to the previous Casson invariants, we define an $SL_2(\mathbb{R})$-Casson invariant for closed 3-manifolds (Definition 2.1). In addition, similar to Theorem B.2, we give an approach to determining the weight from the Reidemeister torsions of $M$ (Theorem 3.2); as an application, we compute the $SL_2(\mathbb{R})$-Casson invariants of some Brieskorn manifolds; see §3.3. In §3, we further discuss a grading of weights appearing in the counts of representations $\pi_1(M) \to SL_2(\mathbb{R})$, and define a graded $SL_2(\mathbb{R})$-Casson invariant; see Section 4.1. Here, the grading is obtained from Reidemeister torsions or the Chern-Simons 3-class of the Pontryagin class $p_1$; see Section 4.3 for some examples.

This paper is organized as follows. We introduce the $SL_2(\mathbb{R})$-Casson invariant in §2 and discuss some computations of the invariants in §3. In §4, we discuss approaches to giving the
Conventional notation. By $M$, we mean a connected closed 3-manifold with an orientation, and by $\Sigma$, we mean an oriented closed surface. Let $g \in \mathbb{N}$ denote the genus of $\Sigma$.

Acknowledgments

The author sincerely expresses his gratitude to Teruaki Kitano and Susumu Hirose for their valuable comments.

2 Definition: $SL_2(\mathbb{R})$-Casson invariant

We will define the $SL_2(\mathbb{R})$-Casson invariants by following the definition of the $SU(2)$-Casson invariant (see Appendix B for the definition).

As a preliminary, let us explain the diagram below (1). Let $(W_1, W_2, \Sigma)$ be a Heegaard splitting of $M$, where $W_i$ is a handlebody with $\partial W_i = \Sigma$ and $M = W_1 \cup_\Sigma W_2$. For a Lie group $G$ and a connected CW complex $Z$ of finite type, we mean by $\text{Hom}(\pi_1(Z), G)$ the set of homomorphisms $\pi_1(Z) \to G$ with compact-open topology, and by $\text{Hom}(\pi_1(Z), G)/G$ the quotient space of $\text{Hom}(\pi_1(Z), G)$ by the conjugate action. Then, the pushout diagram

$$
\begin{array}{ccc}
\pi_1(\Sigma) & \xrightarrow{i_1} & \pi_1(W_1) \\
\downarrow{i_2} & & \downarrow{j_1} \\
\pi_1(W_2) & \xrightarrow{j_2} & \pi_1(M)
\end{array}
$$

of surjections of fundamental groups induces a commutative diagram

$$
\begin{array}{ccc}
\text{Hom}(\pi_1(\Sigma), G)/G & \xrightarrow{i_1} & \text{Hom}(\pi_1(W_1), G)/G \\
\downarrow{i_2} & & \downarrow{j_1} \\
\text{Hom}(\pi_1(W_2), G)/G & \xrightarrow{j_2} & \text{Hom}(\pi_1(M), G)/G
\end{array}
$$

of inclusions. Here, we should notice that $\text{Hom}(\pi_1(M), G)/G = \bigcap_{i=1}^2 \text{Hom}(\pi_1(W_i), G)/G$.

In what follows, let $G$ be $SL_2(\mathbb{R})$, and $\mathfrak{g}$ be the Lie algebra of $G$.

Next, let us describe an open subset of $\text{Hom}(\pi_1(Z), G)/G$ in terms of the Zariski-density. Regarding $SL_2(\mathbb{R})$ as a real affine algebraic variety in $\mathbb{R}^4$, we canonically equip $SL_2(\mathbb{R})$ with a Zariski topology. Let $\Lambda$ be an infinite group of $SL_2(\mathbb{R})$ generated by $\{a_1, \ldots, a_n\}$. Then, as is known (see, e.g., [Lab, Proposition 5.3.4]), $\Lambda$ is Zariski-dense in $SL_2(\mathbb{R})$ if and only if

$$
\bigcap_{i: i \leq n} \{W \in \text{Gr}_k(\mathfrak{g}) \mid a_i.W = W\} = \emptyset
$$

for any $k < 3$, where $\text{Gr}_k(\mathfrak{g})$ denotes the Grassmannian manifold of $k$-planes in $\mathfrak{g}$. Thus, the subset

$$
\text{Hom}(\pi_1(Z), G)^{zd} := \{\rho \in \text{Hom}(\pi_1(Z), G) \mid \text{Im}(\rho) \subset G \text{ is Zariski-dense}\}
$$
is Zariski-open in $\text{Hom}(\pi_1(Z), G)$. It is known (see, e.g., [Lab, Theorem 5.2.6]) that if $Z$ is a manifold with $\text{Genus}(\Sigma) \geq 2$, then the conjugacy action of $PSL_2(\mathbb{R})$ on $\text{Hom}(\pi_1(\Sigma), G)$ is proper and free, and the quotient $\text{Hom}(\pi_1(\Sigma), G)/G$ is an open manifold of dimension $6g - 6$, and the tangent space at $\rho \in \text{Hom}(\pi_1(Z), G)$ is identified with the cohomology $H^1_\rho(\Sigma; g)$ with local coefficients by $\rho$. Here, we should notice that

$$H^0_\rho(\Sigma; g) = H^2_\rho(\Sigma; g) = 0, \quad H^1_\rho(\Sigma; g) \cong \mathbb{R}^{6g-6}, \quad \text{for any} \ \rho \in \text{Hom}(\pi_1(Z), G)$$

by considering the Euler characteristic. Further, recall from [G1] the symplectic structure on $\text{Hom}(\pi_1(\Sigma), G)/G$; precisely, the cohomology $H^1_\rho(\Sigma; g)$ admits the alternating non-degenerate bilinear form defined by the composite,

$$H^1_\rho(\Sigma; g)^2 \xrightarrow{\cup [\Sigma]} H^2_\rho(\Sigma; g \otimes g) \xrightarrow{\bullet \cap [\Sigma]} g \otimes g \xrightarrow{\text{Killing form}} \mathbb{R},$$

where $\cup$ is the cup product, and $\bullet \cap [\Sigma]$ is the pairing with the orientation class $[\Sigma] \in H_2(\Sigma; \mathbb{Z})$. In particular, $\text{Hom}(\pi_1(Z), G)/G$ is oriented.

Next, let us consider the case $Z = W_i$. Since $\pi_1(W_i)$ is the free group of rank $g$, $\text{Hom}(\pi_1(W_i), G)$ is identified with $G^g$, and the conjugacy action of $PSL_2(\mathbb{R})$ on $G^g$ is also proper and free. Furthermore, the action preserves the Haar measure of $G^g$; thus, it preserves the orientation as well. Therefore, the restricted action of the open set $\text{Hom}(\pi_1(W_i), G)/G$ is proper and free, and it preserves the orientation. In particular, the quotient $\text{Hom}(\pi_1(W_i), G)/G$ is an oriented open manifold of dimension $3g - 3$.

Let us denote $\text{Hom}(\pi_1(Z), G)/G$ by $R^{ad}(Z)$. Then, the restriction of $R^{ad}(\Sigma)$ can be written as

$$R^{ad}(\Sigma) \xrightarrow{i^*_1} R^{ad}(W_1) \xrightarrow{i^*_2} R^{ad}(W_2) \xrightarrow{j^*_2} R^{ad}(W_1) \cap R^{ad}(W_2) \subset R^{ad}(M).$$

Let us consider the union of 0-dimensional components in the intersection $\text{Im}(i^*_1) \cap \text{Im}(i^*_2)$ and denote the union by $\mathcal{I}_{0\text{-dim}}$, which is not always of finite order (This problem appears in $SL_2(\mathbb{C})$-case; see [CurI]). Notice that the inclusion $SL_2(\mathbb{R}) \hookrightarrow SL_2(\mathbb{C})$ canonically gives rise to $\iota: R^{ad}(\Sigma) \hookrightarrow \text{Hom}(\pi_1(\Sigma), SL_2(\mathbb{C}))/SL_2(\mathbb{C})$. Define

$$\mathcal{I}_{\text{comp}} := \{ P \in \mathcal{I}_{0\text{-dim}} \mid \iota(P) \text{ is a 0-dimensional component in } \text{Im}(i^*_1 \otimes \mathbb{C}) \cap \text{Im}(i^*_2 \otimes \mathbb{C}) \}.$$

We claim that $\mathcal{I}_{\text{comp}}$ is of finite order, and there is its open tubular neighborhood of $\mathcal{I}_{\text{comp}}$ which does not meet any other higher dimensional components of $\text{Im}(i^*_1) \cap \text{Im}(i^*_2)$. Indeed, as is shown in [CurI, §2], the complexification of $\mathcal{I}_{0\text{-dim}}$ is of finite order and admits its open tubular neighborhood that does not meet any higher dimensional component of the intersection, and $\iota(R^{ad}(W_1))$ meets $R^{ad}(W_2)$ transversally in $\text{supp}(h)$.
**Definition 2.1.** Let \((W_1, W_2, \Sigma)\) be a Heegaard decomposition of \(M\) with \(g > 1\), and \(h\) be the isotopy as above. Then, we define the \(SL_2(\mathbb{R})\)-Casson invariant by the formula,

\[
\lambda_{SL_2(\mathbb{R})}(M) := \sum (-1)^g \varepsilon_f \in \mathbb{Z},
\]

where the sum runs over \(f\) of \(h(R^{ed}(W_1)) \cap R^{ed}(W_2) \cap \mathcal{I}_{\text{comp}}\). In addition, \(\varepsilon_f\) equals \(\pm 1\), depending on whether the orientations of the spaces \(T_f h(R^{ed}(W_1)) \oplus T_f(R^{ed}(W_2))\) and \(T_f(R^{ed}(\Sigma))\) agree. If \(g \leq 1\), we define \(\lambda_{SL_2(\mathbb{R})}(M)\) to be zero.

In §5.1 we later show the topological invariance of \(\lambda_{SL_2(\mathbb{R})}(M)\). To be precise,

**Theorem 2.2.** The invariant \(\lambda_{SL_2(\mathbb{R})}(M) \in \mathbb{Z}\) depends only on the homeomorphism class of the 3-manifold \(M\).

### 3 Computation of \(SL_2(\mathbb{R})\)-Casson invariants

The purpose of this section is to give a procedure for computing the \(SL_2(\mathbb{R})\)-Casson invariant by means of Reidemeister torsions. As indicated in Appendix [John] the idea basically arises from the the work of [John] in the case \(G = SU(2)\). We will begin by reviewing the torsions in §3.1.

#### 3.1 Review: Reidemeister torsions

Let us review algebraic torsions for cochain complexes. Let \(\mathbb{F}\) be a commutative field of characteristic zero. Consider a cochain complex of length \(m\),

\[
C^*: 0 \to C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^{m-2}} C^{m-1} \xrightarrow{\partial^{m-1}} C^m \to 0,
\]

where \(C^i\) is a vector \(\mathbb{F}\)-space of finite dimension. Let us select a basis \(c_i\) for \(C^i\), a basis \(b_i\) for the boundaries \(B^i\), and a basis \(h_i\) for the cohomology \(H^i\), where we sometimes regard \(h_i\) as elements, \(\tilde{h}_i\), of \(C^i\) by lifts. In addition, we choose a lift, \(\tilde{b}_{i+1} \in C^i\), of \(b_{i+1}\), with respect to \(\partial_i : C^i \to B^{i+1}\). By \(\tilde{b}_i \tilde{h}_i \tilde{b}_{i+1}\), we mean the collection of elements given by \(b_i, \tilde{h}_i, \text{ and } \tilde{b}_{i+1}\). This set, \(b_i h_i b_{i+1}\), is indeed a basis for \(C^i\). For bases \(d, e\) of a finite-dimensional \(\mathbb{F}\)-space, we denote the invertible matrix of a basis change by \([d/e]\), i.e. \([d/e] = (a_{ij})\) where \(d_i = \sum_j a_{ij} e_j\). Then, the *algebraic torsion* (of the based complex \((C^*, c_i, h_i)\)) is defined to be the alternating product,

\[
\mathcal{T}(C^*, c, h) := \frac{\prod_{i} \det [b_2 \tilde{h}_2 \tilde{b}_{2i+1}/c_{2i}]}{\prod_{i} \det [b_{2i-1} \tilde{h}_{2i-1} \tilde{b}_{2i}/c_{2i-1}]} \in \mathbb{F}^\times.
\]

It is well-known that \(\mathcal{T}(C^*, c, h)\) is independent of the choices of \(b_i\) and \(\tilde{b}_{i+1}\), but it does depend on the choices of \(c_i\) and \(h_i\). More precisely, if we select such other bases \(c'_i\) and \(h'_i\), we can verify that

\[
\mathcal{T}'(C^*, c', h') = \mathcal{T}(C^*, c, h) \prod_{j \geq 0} (\det [c_j/c'_j] \det [h'_j/h_j])^{(-1)^{j+1}} \in \mathbb{F}^\times.
\]

If \(C^*\) is acyclic, we will often write \(\mathcal{T}(C^*, c)\) instead of \(\mathcal{T}(C^*, c, h)\).

Next, let us review Reidemeister torsions. Let \(X\) be a connected finite CW-complex. Take an \(SL_n\)-representation \(\rho : \pi_1(X) \to SL_n(\mathbb{F})\), and regard \(\mathbb{F}^n\) as a left \(\mathbb{Z}[\pi_1(X)]\)-module. Let \(\tilde{X}\)
be the universal covering space of $X$ as a CW complex and $C_*(\tilde{X};\mathbb{Z})$ be the cellular complex associated with the CW complex structure. This $C_*(\tilde{X};\mathbb{Z})$ can be considered to be a left free $\mathbb{Z}[\pi_1(X)]$-module by Deck transformations. The cochain complex with local coefficients is defined on

$$C^*_\rho(X;\mathbb{F}^n) := \text{Hom}_{\mathbb{Z}[\pi_1(X)]-\text{mod}}(C_*(\tilde{X};\mathbb{Z}),\mathbb{F}^n).$$

Let us choose orientations, $c_X$, of the cells of $X$ and take the canonical basis of $\mathbb{F}^n$. If we regard a lift of $c_X$ as a basis of $C_*(\tilde{X};\mathbb{Z})$, then $C^*_\rho(X;\mathbb{F}^n)$ is a based chain complex over $\mathbb{F}$. Furthermore, by choosing a basis $h_i$ of the cohomology $H^i_\rho(X;\mathbb{F}^n)$, the Reidemeister torsion of $(X,\rho)$ is defined to be

$$\mathcal{T}(C^*_\rho(X;\mathbb{F}^n),c_X,h) \in \mathbb{F}^\times.$$

From [5], if two representations $\rho$ and $\rho'$ are conjugate, the resulting torsions are equal. However, the discussion of the signs is subtle, and this torsion does depend on the CW-complex.

Before we obtain the topological invariants, let us review the refined torsions by Turaev [Tur Chapter 3] or [Dub1 Dub2]. Let $H^*(X;\mathbb{R})$ be the ordinary cohomology over $\mathbb{R}$. Suppose an orientation of $\oplus_{i\geq 0} H^i(X;\mathbb{R})$. Moreover, choose a basis $h_i^\mathbb{R} \subset H^i(X;\mathbb{R})$ such that the sequence $(h_0^\mathbb{R}, h_1^\mathbb{R}, \ldots)$ is a positive basis in the oriented vector space $H^*(X;\mathbb{R})$. Now let us define

$$\tilde{\tau}(C^*(X;\mathbb{R}), c_X, h^\mathbb{R}) := (-1)^{N(X)} \mathcal{T}(C^*(X;\mathbb{R}), c_X, h^\mathbb{R}) \in \mathbb{R}^\times,$$

where

$$N(X) = \sum_{i=0}^{\dim(X)} \left( \sum_{j=0}^{\dim(X) - i} \dim H^{\dim(X) - j}(X;\mathbb{R}) \sum_{j=0}^{\dim(C^{\dim(X) - j}(X;\mathbb{R}))} \right) \in \mathbb{Z}/2\mathbb{Z}.$$  

(6)

Then, the refined torsion is defined to be

$$\tau^0_\rho(X, h) := \text{sign}(\tilde{\tau}(C^*(X;\mathbb{R}), c_X, h^\mathbb{R})) \cdot \mathcal{T}(C^*_\rho(X;\mathbb{F}^n), c_X, h) \in \mathbb{F}^\times.$$

**Theorem 3.1** (see [Tur Chapter 18] or [Dub1 Chapter 2]). If $n$ is even (resp. odd), the torsion $\mathcal{T}(C^*_\rho(X;\mathbb{F}^n), c_X, h)$ (resp. refined torsion $\tau^0_\rho(X, h)$) is independent of the order of the cells of $X$, their orientation, and choice of $h^\mathbb{R}$ (however, it does depend on the choice of $h$). Moreover, the torsion is invariant under simple homotopy equivalences preserving the homology orientation.

Recall that any two triangulations of an oriented $C^\infty$-manifold $N$ are simple homotopy equivalent (see, e.g., [Tur §II.8]); consequently, if $X$ is a triangulation of $N$, the refined torsion gives a topological invariant of $N$ associated with $\rho : \pi_1(N) \to SL_n(\mathbb{F})$ and $h$.

**3.2 Statement**

In this subsection, we give a procedure for computing the $SL_2(\mathbb{R})$-Casson invariants. For this, we shall develop methods of analyzing the computation of $\varepsilon_f$, as an analog to Theorem 3.2.

**Theorem 3.2.** We assume $H_*(M;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$, i.e., $M$ is a rational homology 3-sphere, and that, for any $f \in \mathcal{I}_{\text{comp}}$, the intersection of $\text{Im}(i_1^*)$ and $\text{Im}(i_2^*)$ at $f$ is transverse. Then,
the equality $\varepsilon_f = (-1)^q \cdot \sign(\tau_J^0(M))$ holds for any $f \in I_{\text{comp}}$. In particular,

$$\lambda_{SL_2(\mathbb{R})}(M) = \sum_{f \in I_{\text{comp}}} \sign(\tau_J^0(M)). \quad (7)$$

Since the proof is technical, we will put it in §5.3. The assumption is characterized by the following lemma.

**Lemma 3.3.** Take $f \in I_{\text{comp}}$. Then, the intersection of $\im(i_1^*)$ and $\im(i_2^*)$ at $f$ is transverse if and only if $H_f^1(M; g) = H_f^2(M; g) = 0$.

**Proof.** Consider the Mayer-Vietoris sequence from $(\Sigma, W_1, W_2)$:

$$H_f^0(\Sigma; g) \to H_f^1(M; g) \to H_f^1(W_1; g) \oplus H_f^1(W_2; g) \xrightarrow{i_1^* \oplus i_2^*} H_f^1(\Sigma; g) \to H_f^2(M; g) \to 0.$$  

Notice that $H_f^0(\Sigma; g) = 0$ because of $f \in R^{ad}(\Sigma)$. Since the intersection is transverse if and only if $i_1^* \oplus i_2^*$ is an isomorphism, we get the desired result.  

### 3.3 Examples; some Brieskorn manifolds

Using Theorem 3.2, we will compute the $SL_2(\mathbb{R})$-Casson invariants of some Brieskorn 3-manifolds.

Let us review the Brieskorn 3-manifolds. Fix integers $m, p, q, d \in \mathbb{N}$ such that $m, p, q$ are relatively prime and $m, p \geq 3$, $q = dp + 1$. Then, the Brieskorn 3-manifold

$$\Sigma(m, p, q) := \{(x, y, z) \in \mathbb{C}^3 \mid x^m + y^p + z^q = 0, \ |x|^2 + |y|^2 + |z|^2 = 1\}$$

is a homology 3-sphere, and $\Sigma(m, p, q)$ is an Eilenberg-MacLane space if $1/m + 1/p + 1/q < 1$. Furthermore, consider the group presentation $\langle x_1, x_2, \ldots, x_m \mid r_1, \ldots, r_m \rangle$ with

$$r_i := x_i x_{i+q} x_{i+2q} \cdots x_{i+(q-1)dq} x_{i+1} x_{i+q+1} \cdots x_{i+(q-1)dq} x_{i+2q+1} x_{i+(q-1)dq+1}^{-1},$$

where the subscripts are taken by mod $m$. According to [CHK], this group is isomorphic to $\pi_1(\Sigma(m, p, q))$, and this presentation is derived from a genus $m$ Heegaard decomposition of $\Sigma(m, p, q)$.

Let $\tilde{X}$ be the universal covering of $\Sigma(m, p, q)$ as a contractible space, and let $\pi_1$ be $\pi_1(\Sigma(m, p, q))$ for short. We now address the cellular complex of $\tilde{X}$. By the Heegaard decomposition, the cellular complex is described as

$$C_*(\tilde{X}; \mathbb{Z}) : 0 \to \mathbb{Z}[\pi_1] \xrightarrow{\partial_3} \mathbb{Z}[\pi_1]^m \xrightarrow{\partial_2} \mathbb{Z}[\pi_1]^m \xrightarrow{\partial_1} \mathbb{Z}[\pi_1] \to 0 \quad \text{(exact).}$$

Then, through a similar discussion to the one in [Ko], we can verify that the boundary maps $\partial_*$ have matrix presentations of the forms,

$$\partial_3 = (1 - x_1^{+d_q - d_q} - x_1^{+d_q - d_q} \cdots x_1, 1 - x_2^{+d_q - d_q} - x_2^{+d_q - d_q} \cdots x_2, \ldots, 1 - x_m^{+d_q - d_q} - x_m^{+d_q - d_q} \cdots x_m) \in \text{Mat}(m \times 1; \mathbb{Z}[\pi_1]),$$

$$\partial_2 = \left\{ \left[ \frac{\partial r_i}{\partial x_j} \right]_{1 \leq i, j \leq m} \right\} \in \text{Mat}(m \times m; \mathbb{Z}[\pi_1]), \quad (8)$$
\[ \partial_i = (1 - x_1, 1 - x_2, 1 - x_3, \ldots, 1 - x_m)^\text{transpose}. \]  

Here, \( \frac{\partial_r}{\partial x_i} \) is the Fox derivative of \( r_j \) with respect to \( x_i \). As is known \cite{Cur, KY, Sav}, if \( G = SL_2(\mathbb{R}) \), then \( R^{ad}(W_1) \cap R^{ad}(W_2) = R(\Sigma(m, p, q)) \) is true as a finite set, and it satisfies the assumption in Theorem 3.2. Given concrete \( m, p, q \in \mathbb{N} \) and a non-trivial Zariski-dense representation \( f : \pi_1(\Sigma(m, p, q)) \to SL_2(\mathbb{R}) \), by the definition of torsion, we can compute the torsion \( \tau_f^0(\Sigma(m, p, q)) \) (Here, Theorem 2.2 in \cite{Tur} makes the computation easier). When \( m, p, q \leq 9 \) or \( (m, p, q) = (m, 2, 3) \) with \( m < 25 \), we can verify that \( \tau_f^0(\Sigma(m, p, q)) < 0 \) with the help of a computer program in Mathematica. Therefore, we suggest a conjecture.

**Conjecture 3.4.** Let \( m, p, q \in \mathbb{Z} \) be as above. Then, \((-1)^g \text{sign}(\tau_f^0) = \epsilon_f \in \{\pm 1\} \) would be negative for any \( f \in R^{ad}(W_1) \cap R^{ad}(W_2) = R^{ad}(\Sigma(m, p, q)) \). In particular, Theorem 3.2 implies that the invariant \( \lambda_{SL_2(\mathbb{R})}(\Sigma(m, p, q)) \in \mathbb{Z} \) would be \(-|R^{ad}(\Sigma(m, p, q))|\).

**Remark 3.5.** If we replace \( SL_2(\mathbb{R}) \) by \( SU(2) \), then \( \epsilon_f = -1 \) is known; see \cite{Sav}. Furthermore, as has been shown \cite{KY} Corollary 1.4, the order of \( R^{ad}(\Sigma(m, p, q)) \) is equal to

\[
\frac{(m - 1)(p - 1)(q - 1)}{4} - 2\# \left\{ (s, t, u) \in \mathbb{N}_{>0}^3 \mid s < m, t < p, u < q, \frac{s}{m} + \frac{t}{p} + \frac{u}{q} < 1 \right\}.
\]

## 4 Invariants graded by the Chern-Simons invariant

Now let us discuss graded \( SL_2(\mathbb{R}) \)-Casson invariants.

### 4.1 Discussion; grading the invariant

In order to give a grading of the \( SL_2(\mathbb{R}) \)-Casson invariant, we first reconsider the isotopy \( h \) in \( \mathcal{I} \). Since \( 3g - 3 \geq 3 \), we can apply a Whitney trick when constructing \( h \). Hence, for any \( f \in \mathcal{I}_{\text{comp}} \), we can choose \( h \) such that \( h(f) = f \) if the local intersection number at \( f \) is \( \pm 1 \), and \( h(f) \) is not contained in \( \mathcal{I}_{\text{comp}} \) if the intersection number is 0. Therefore, if we have a map \( F : \text{Hom}(\pi_1(M), SL_2(\mathbb{R}))/SL_2(\mathbb{R}) \to K \) for some group \( K \), we can verify that the sum

\[
\lambda_{SL_2(\mathbb{R})}^F(M) := (-1)^g \sum_{f \in h(R^{ad}(W_1)) \cap R^{ad}(W_2) \cap \mathcal{I}_{\text{comp}}} \epsilon_f F(f) \in \mathbb{Z}[K]
\]

in the group ring is a topological invariant; the proof is similar to that of Theorem 2.2. As examples of \( F \), the Reidemeister torsion and the Chern-Simons invariant are invariant with respect to the conjugacy action.

Now we will explain the definition of the Chern-Simons invariant in detail. For a group \( G \), let \( BG \) be the Eilenberg-MacLane space. The classifying map \( c_M : M \to B\pi_1(M) \) gives rise to \((c_M)_* : H_3(M; \mathbb{R}) \to H_3(B\pi_1(M); \mathbb{Z}) \). As is shown \cite{Dup}, the \((p_1-)\)-Chern-Simons class, \( P_1 \), is a representative 3-cocycle in the third cohomology \( H^3(BSL_2(\mathbb{R}); \mathbb{R}/\mathbb{Z}) \); see Theorem 4.1 below. Let \([M] \in H_3(M; \mathbb{Z})\) be the orientation 3-class of \( M \). Then, given a representation \( f : \pi_1(M) \to SL_2(\mathbb{R}) \), the **Chern-Simons invariant** is defined to be the pairing,

\[
\langle P_1, f_* \circ (c_M)_*[M] \rangle \in \mathbb{R}/\mathbb{Z}.
\]
Moreover, as is well-known, the Chern-Simons invariant is invariant with respect to the conjugacy action and is locally constant on $\text{Hom}(\pi_1(M), SL_2(\mathbb{R}))/SL_2(\mathbb{R})$.

In addition, when $M$ is an integral homology 3-sphere, we can give an $\mathbb{R}$-valued lift of the invariant as follows. Let $\tilde{G} \to PSL_2(\mathbb{R})$ be the universal covering of $SL_2(\mathbb{R})$ associated with $\pi_1(SL_2(\mathbb{R})) \cong \mathbb{Z}$, which is a central extension of the fiber $\mathbb{Z}$. Notice that every homomorphism $f \in \pi_1(M) \to SL_2(\mathbb{R})$ uniquely admits a lift $\tilde{f} : \pi_1(M) \to \tilde{G}$, since $H_1(M; \mathbb{Z}) = H_2(M; \mathbb{Z}) = 0$. Moreover, as in [Dup], §1 and §4 (and this has also been noted by others), as a lift of $P_1$, there is a 3-cocycle $\tilde{P}_1 \in H^3(B\tilde{G}; \mathbb{R})$. To summarize, the sum

$$\sum \epsilon_f \{\langle \tilde{P}_1, \tilde{f} \ast (c_M)_s[M] \rangle \} \in \mathbb{Z}[\mathbb{R}]$$

(12)

\[ \sum_{i \leq j \leq n} \epsilon_f \{\langle \tilde{P}_1, \tilde{f} \ast (c_M)_s[M] \rangle \} \in \mathbb{Z}[\mathbb{R}] \]

gives a topological invariant of integral homology 3-spheres, as a graded $SL_2(\mathbb{R})$-Casson invariant.

### 4.2 Computation of the graded invariant

Here, we give a procedure for computing the $\mathbb{R}/\mathbb{Z}$-valued invariant (11), if $M$ is an Eilenberg-MacLane space.

First, let us recall the (normalized) definition of group (co-)homology. For a group $G$, the group homology, $H_\ast(G; \mathbb{Z})$, is defined to be $\text{Tor}_n^G(\mathbb{Z}, \mathbb{Z})$. For example, if we let $C^n_{\text{Nor}}(G; \mathbb{Z})$ be the quotient $\mathbb{Z}$-free module of $\mathbb{Z}\langle G^{n+1} \rangle$ subject to the relation $(g_0, \ldots, g_n) \sim 0$ if $g_i = g_{i+1}$ for some $i$, the complex $C^n_{\text{Nor}}(G; \mathbb{Z})$ with boundary map,

$\partial(g_0, \ldots, g_n) = \sum_{i=0}^n (-1)^i(g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n)$,

is acyclic and the homology of $C^n_{\text{Nor}}(G; \mathbb{Z}) \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ is isomorphic to $H_n(G; \mathbb{Z})$. Dually, for an abelian group $M$, we can define a coboundary map on $\text{Map}(G^{n+1}, M)$ and define the cohomology $H^\ast(G; M)$. Any cohomology class of $H^n(G; M)$ can be represented by a map $G^{n+1} \to M$. As is well-known, $H_\ast(G; \mathbb{Z}) \cong H_\ast(BG; \mathbb{Z})$ and $H^\ast(BG; M) \cong H^\ast(G; M)$.

Let us recall from [Dup] the 3-cocycle, which represents the $P_1$ in detail. Given 4-tuples of distinct points $\{a_0, a_1, a_2, a_3\}$ in $P\mathbb{R}^1$, the cross ratio is defined by

$$\{a_0, a_1, a_2, a_3\} := \frac{a_0 - a_2}{a_0 - a_3} \cdot \frac{a_1 - a_3}{a_1 - a_2} \in \mathbb{R} \setminus \{0, 1\}.$$ 

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, we define $g \infty$ by $b/d$ if $d \neq 0$, and by $a/c$ if $d = 0$. In addition, consider the real Rogers’ $L$-function,

$$L(x) := -\frac{\pi^2}{6} - \frac{1}{2} \int_0^x \left( \frac{\log(1-t)}{t} + \frac{\log(1-t)}{1-t} \right) dt$$

for $0 \leq x \leq 1$, which is extended to $\mathbb{R}$ by

$$L(x) := \begin{cases} -L(1/x) & \text{for } x > 1, \\ L(1 - 1/x) & \text{for } x < 0. \end{cases}$$
Theorem 4.1 ([Dup] Theorem 1.11). Take the map $l : SL_2(\mathbb{R})^4 \to \mathbb{R}/\mathbb{Z}$ defined by

$$l(g_0, g_1, g_2, g_3) := -\frac{1}{4\pi^2}L(\{0, g_0^{-1}g_1 \infty, g_0^{-1}g_2 \infty, g_0^{-1}g_3 \infty\}).$$

Here, we put $l(\{a_0, a_1, a_2, a_3\}) = 0$ whenever there are two equal among $a_0, a_1, a_2, a_3 \in P\mathbb{R}^1$.

Then, $l$ is a 3-cocycle, and it coincides with the Chern-Simons 3-class associated with the first Pontryagin class modulo 1/24. That is, $24l$ and $24P_1$ are equal in $H^3(SL_2(\mathbb{R}); \mathbb{R}/\mathbb{Z})$.

Next, we will discuss an algorithm to describe the fundamental 3-class in the group complex $C_3(\pi_1(M); \mathbb{Z})$. Take a genus-$g$ Heegaard decomposition of $M$. Since the 1-skeleton consists of $g$ one-handles, we have a presentation $\langle x_1, \ldots, x_g | r_1, \ldots, r_g \rangle$ of $\pi_1(M)$. Then, since $M$ is an Eilenberg-MacLane space, the cellular complex of the universal cover $\widetilde{M}$ is described as

$$C_\ast(\widetilde{M}) : 0 \to \mathbb{Z}[\pi_1(M)] \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(M)]^g \xrightarrow{\partial_2} \mathbb{Z}[\pi_1(M)]^g \xrightarrow{\partial_3} \mathbb{Z}[\pi_1(M)] \to \mathbb{Z} \quad \text{(exact)}.$$

Here, according to [Lyn], the boundary maps $\partial_2$ and $\partial_3$ are given by (S) and (D), respectively. Denote the basis of $C_3(\widetilde{M})$ by $\{O_{g, n}\}$. Then, if we can construct a chain map $c_* : C_\ast(\widetilde{M}) \to C_\ast(\pi_1(M); \mathbb{Z})$ as a $\mathbb{Z}[\pi_1(M)]$-homomorphism which is unique up to homotopy, then $[c_3(O_M)] \in C_3^{\text{Nor}}(\pi_1(M); \mathbb{Z}) \otimes \mathbb{Z}^{[\pi_1(M)]} \mathbb{Z}$ means the fundamental 3-class.

The chain map $c_*$ can be constructed as follows. Let $A \in G$ be any element. Define $c_1(Ax_i) := (A, Ax_i)$. If $r_i$ is expanded as $x_{i_1}^{\epsilon_1}x_{i_2}^{\epsilon_2} \cdots x_{i_m}^{\epsilon_m}$ for some $\epsilon_k \in \{\pm 1\}$, we define $c_2(Ar_i)$ to be

$$\sum_{m:1 \leq m \leq n} \epsilon_m(A, Ax_{i_1}^{\epsilon_1}x_{i_2}^{\epsilon_2} \cdots x_{i_{m-1}}^{\epsilon_{m-1}}x_{i_m}^{(\epsilon_m-1)/2}, Ax_{i_1}^{\epsilon_1}x_{i_2}^{\epsilon_2} \cdots x_{i_{m-1}}^{\epsilon_{m-1}}x_{i_m}^{(\epsilon_m+1)/2}) \in C_2^{\text{Nor}}(\pi_1(M); \mathbb{Z}).$$

Then, we can easily verify $\partial_3^A \circ c_1 = c_0 \circ \partial_1$ and $\partial_2^A \circ c_2 = c_1 \circ \partial_2$. Notice that $\partial_3^A \circ c_2 \circ \partial_3(O_M) = \partial_3^A \circ \partial_2 \circ \partial_3(O_M) = 0$, that is, $\partial_3 \circ \partial_3(O_M)$ is a 2-cycle. If we expand $c_2 \circ \partial_3(O_M)$ as $\sum n_i(g_0^i, g_1^i, g_2^i)$ for some $n_i \in \mathbb{Z}, g_j^i \in G$, then $O_M := -\sum n_i(1, g_0^i, g_1^i, g_2^i)$ satisfies $\partial_3^O(O_M) = c_2 \circ \partial_3(O_M)$. Therefore, the correspondence $O_M \mapsto O'_M$ gives rise to a chain map $c_* : C_\ast(\widetilde{M}) \to C_\ast(\pi_1(M); \mathbb{Z})$, as desired. In conclusion, the above discussion can be summarized as follows:

Proposition 4.2. For $f : \pi_1(M) \to SL_2(\mathbb{R})$, the composite $l(f_\ast(O'_M)) \in \mathbb{R}/\mathbb{Z}$ is equal to the pairing $(P_1, f_\ast(c_M)_\ast[M])$ modulo 1/24. In particular, the graded $SL_2(\mathbb{R})$-Casson invariant is computed as

$$\lambda^{24P_3}_{SL_2(\mathbb{R})}(M) = \sum \varepsilon_f \{24l(f_\ast(O'_M))\} \in \mathbb{Z}[\mathbb{R}/\mathbb{Z}].$$

4.3 Examples; some Seifert manifolds

For odd numbers $m, n \in \mathbb{Z}$, let us consider the Seifert manifolds $M_{m,n} := \Sigma((m, 1), (n, 1), (2, -1))$ over $S^2$, where the three singular fibers are characterised by the integral surgery coefficients $(m, 1), (n, 1)$ and $(2, -1)$. Then, if $1/m + 1/n < 1/2$, the manifold is an Eilenberg-MacLane space and admits a genus-two Heegaard diagram; see, e.g., [Sav] §6. The fundamental group is presented as

$$\langle x, y | r_1 := y^n(xy)^{-2}, r_2 := x^m(yx)^{-2} \rangle.$$

Furthermore, we can verify that $\partial_3(O_M)$ is given by $(1 - y)r_1 + (1 - x)r_2 \in C_2(\widetilde{M})$. Therefore, by the above construction of $O'_M$, we can easily verify that

$$O'_M = -(1, x, 1, y) - (1, x, y, xy) - (1, x, yx, xyx) - (1, y, 1, x) - (1, y, x, xy)$$
\[-(1, y, xy, xyx) + \sum_{j: 0 \leq j \leq m - 2} (1, x, x^j, x^{j+1}) + \sum_{j: 0 \leq j \leq n - 2} (1, y, y^j, y^{j+1}).\]

Furthermore, it is not difficult to classify all the \(SL_2\)-representations of \(M_{m,n}\). Precisely,

**Lemma 4.3.** For \(k, \ell \in \mathbb{N}\) with \(k \leq n/2\) and \(\ell \leq m/2\), take

\[
\beta_k := \exp(2\pi k \sqrt{-1}/n) + \exp(-2\pi k \sqrt{-1}/n), \quad \gamma_\ell := \exp(2\pi \ell \sqrt{-1}/m) + \exp(-2\pi \ell \sqrt{-1}/m).
\]

When \(\beta_k^2 + \gamma_\ell^2 > 4\), let us consider the correspondence,

\[
f_{k,\ell}(y) = \left(\frac{\beta_k/2}{(\gamma_\ell + \sqrt{\beta_k^2 + \gamma_\ell^2 - 4})/2}, \frac{(-\gamma_\ell + \sqrt{\beta_k^2 + \gamma_\ell^2 - 4})/2}{\beta_k/2}\right), \quad f_{k,\ell}(xy) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\]

This gives rise to a homomorphism \(f_{k,\ell} : \pi_1(M_{m,n}) \to SL_2(\mathbb{R})\). Furthermore, the map \((k, \ell) \mapsto f_{k,\ell}\) yields a bijection,

\[
\{ (k, \ell) \in \mathbb{Z}_>^2 \mid k \leq \frac{n}{2}, \ell \leq \frac{m}{2}, \beta_k^2 + \gamma_\ell^2 > 4 \} \longleftrightarrow \text{Hom}(\pi_1(M_{m,n}), SL_2(\mathbb{R}))^{zd}/SL_2(\mathbb{R}).
\]

In summary, since \(O'_M\) and \(f_{k,\ell}\) are explicitly described, for small \(k, \ell\) we can numerically compute the pairings \(24l(f_*(O'_M))\) with the help of a computer program. Here, in a similar fashion to §3.3 we can verify that \(\varepsilon_{f_{k,\ell}} < 0\) for any \(k, \ell\). We give some examples below.

**Example 4.4.** (I) The case of \(m = 3\) and \(n \leq 15\). The set consists of \(\{f_{1,(n-1)/2}\}\). With the help of a computer program, the resulting computations of the pairing are listed as

| \(n\) | 7 | 9 | 11 | 13 | 15 |
|---|---|---|---|---|---|
| Pairing \(\in \mathbb{R}/\mathbb{Z}\) | 0.100637\ldots | 0.826310\ldots | 0.660662\ldots | 0.549320\ldots | 0.950164\ldots |

(II) The case of \(m = 5\) and \(n \leq 11\). The resulting computations of the pairing \(24l(f_*(O'_M))\) are listed as

| \((n, k, \ell)\) | (7,2,1) | (7,2,3) | (9,2,1) | (9,2,4) |
|---|---|---|---|---|
| Pairing \(\in \mathbb{R}/\mathbb{Z}\) | 0.562345\ldots | 0.275253\ldots | 0.906666\ldots | 0.979077\ldots |

| \((n, k, \ell)\) | (11,1,5) | (11,2,1) | (11,2,4) | (11,2,5) |
|---|---|---|---|---|
| Pairing \(\in \mathbb{R}/\mathbb{Z}\) | 0.658563\ldots | 0.456043\ldots | 0.111275\ldots | 0.942540\ldots |

As the examples imply, one may hope that if \(\pi_1(M)\) has a non-trivial \(SL_2(\mathbb{R})\)-representation, the graded invariant \(\lambda_{SL_2(\mathbb{R})}^{P_1/24}(M)\) is a strong invariant. In addition, the author [Nos] gave many examples of other 3-manifolds such that the boundary maps \(\partial_*\) are concretely described; therefore, we can compute the \(SL_2(\mathbb{R})\)-invariants in a similar way.

## 5 Proofs of the theorems

Here, we give the proofs of Theorems 2.2 and 3.2. Throughout this section, we let \(G = SL_2(\mathbb{R})\).
5.1 Proofs of Theorem 2.2

Proof of Theorem 2.2: The proof is almost the same as the discussion in [AM], Chapter IV or [Sav], §16.3. First, consider the case where $M$ is one of the lens spaces, $S^3$ and $S^1 \times S^2$. Then, $R^{zd}(M)$ is empty for any Heegaard decomposition of $M$. Hence, $\lambda_{SL_2(\mathbb{R})}(M) = 0$ by definition, and we may assume $M \neq S^3$ and $g > 1$ in what follows.

Let $(W'_1, W'_2, \Sigma')$ be another Heegaard decomposition of $M$. If $(W_1, W_2, \Sigma)$ and $(W'_1, W'_2, \Sigma')$ are isotopic, we can easily verify the invariance of $\lambda_{SL_2(\mathbb{R})}(M)$. Thanks to the famous theorem of Reidemeister, it is enough to show the invariance of $\lambda_{SL_2(\mathbb{R})}$ if $(W_1, W_2, \Sigma)$ is a Heegaard decomposition obtained from $(W_1, W_2, \Sigma)$ by attaching an unknotted handle; see Figure IV. Then, we have the identifications $\pi_1(W'_1) = \mathbb{Z} \ast \pi_1(W_1)$ and $\pi_1(W'_2) = \mathbb{Z} \ast \pi_1(W_2)$, where the $\mathbb{Z}$ are generated by the loops $a_0$ and $b_0$ in Figure IV.

Let $\Sigma_0 = \Sigma \setminus D^2$ and $\Sigma'_0 := \Sigma' \setminus D^2$, where $D^2$ is the 2-disc removed in the handle-attaching; see Figure IV. Then, $\pi_1(\Sigma'_0) = \mathbb{Z} \ast \mathbb{Z} \ast \pi_1(\Sigma_0)$, where the factor $\mathbb{Z} \ast \mathbb{Z}$ is freely generated by $a_0, b_0$. Accordingly, we get the identifications

$$R(W'_k) = G \times R(W_k), \quad R(\Sigma'_0) = G \times G \times R(W_k).$$

Consider the inclusions,

$$G \times \text{Hom}(\pi_1(W_1), G) \hookrightarrow G \times G \times \text{Hom}(\pi_1(\Sigma_0), G); \quad (a, \alpha) \mapsto (a, 1, \alpha),$$

$$G \times \text{Hom}(\pi_1(W_2), G) \hookrightarrow G \times G \times \text{Hom}(\pi_1(\Sigma_0), G); \quad (b, \alpha) \mapsto (1, b, \alpha),$$

which factor through $\text{Hom}(\pi_1(\Sigma'), G)$. Then, we have the following identifications:

$$\text{Hom}(\pi_1(W'_1), G) \cap \text{Hom}(\pi_1(W'_2), G) = 1 \times 1 \times \text{Hom}(\pi_1(W_1), G) \cap \text{Hom}(\pi_1(W_2), G)$$

$$= 1 \times 1 \times \text{Hom}(\pi_1(M), G).$$

We see that

$$\text{Hom}(\pi_1(W'_1), G)^{zd} \cap \text{Hom}(\pi_1(W'_2), G)^{zd} = 1 \times 1 \times (\text{Hom}(\pi_1(W_1), G)^{zd} \cap \text{Hom}(\pi_1(W_2), G)^{zd}).$$

Since these identifications are equivariant with respect to the conjugacy $\text{PSL}_2(\mathbb{R})$-action, we have

$$R^{zd}(W'_1) \cap R^{zd}(W'_2) = 1 \times 1 \times (R^{zd}(W_1) \cap R^{zd}(W_2)).$$

Now let us discuss the isotopy $h$. A similar discussion to the one on [AM], pages 70–78 enables us to verify that there is an isotopy $\tilde{h} : R^{zd}(\Sigma') \rightarrow R^{zd}(\Sigma')$ such that

$$\tilde{h}(R^{zd}(W'_1)) \cap R^{zd}(W'_2) = 1 \times 1 \times (h(R^{zd}(W_1)) \cap R^{zd}(W_2)).$$

Therefore, we have

$$\lambda_{SL_2(\mathbb{R})}(M)' = (-1)^{g+1} \sum_f \varepsilon'_f, \quad \lambda_{SL_2(\mathbb{R})}(M) = (-1)^g \sum_f \varepsilon_f. \quad (13)$$

Hence, it is enough to show $\varepsilon'_f = -\varepsilon_f$ for any $f$. However, the proof is the same as in the case $G = SU(2)$; see, e.g., [Sav], pages 155–156. Thus, we will omit the details.

$\square$
5.2 Proof of Theorem 3.2

Next, to prove Theorem 3.2, let us review a theorem of Milnor [Mil1]. Consider a short exact sequence $0 \rightarrow C^* \rightarrow k \rightarrow C^* \rightarrow 0$, in the category of bounded chain complexes and chain mappings over $\mathbb{F}$. Then, the long exact homology sequence

$$H^*: H^0 \rightarrow H^1 \rightarrow \cdots \rightarrow H^m \rightarrow H^{m+1} \rightarrow \cdots$$

(14)

can be thought of as an acyclic chain complex of length $3m + 3$. Hence, if we fix bases $h, h, h$ of $H^*$, $H^*$, $H^*$, respectively, we can define the torsion $T(H_*, h \cup h \cup h)$. Theorem 5.1 ([Mil1, Theorem 3.2]). Now let us assume that $C^*, C^*, C^*$ have distinguished bases $c_i, c_i, c_i$ such that $\det[c_i/c, c_i] = 1$ for all $i$. Then, there is $\eta \in \mathbb{Z}/2$ such that

$$(-1)^\eta T(C^*, c, h) = T(C^*, h) T(C^*, c, h) T(H_*, h \cup h \cup h) \in \mathbb{F}^\times.$$

Remark 5.2. The original paper does not clarify $\eta$. However, by thoughtfully following the proofs of [Tur, Theorem 1.5] and [Mil1, Theorem 3.2], we can verify that $\eta$ is formulated as

$$\eta = \sum_{i=0}^m \dim(\text{Im}(j_i)) \dim B_i + \dim(\text{Im}(k_i)) \dim B_{i+1} + \dim B_{i+1} \dim B_i \in \mathbb{Z}/2.$$

The same equality is also written in [Dub2, Chapter 7].

Moreover, let us discuss the refined torsions on closed surfaces. Recall from (3) that, for any representation $\rho \in R^{zd}(\Sigma)$, the cohomology $H^1(\Sigma; g) \cong \mathbb{R}^{6g-6}$ admits a symplectic structure; we can choose a symplectic basis $h_{\text{sym}} \subset H^1(\Sigma; g)$. Moreover, concerning ordinary cohomology, choose a symplectic basis $h_{\text{sym}} \subset H^1(\Sigma; R) \cong \mathbb{R}^{2g}$, which is compatible with the orientation of $H^*(\Sigma; R)$. Then, we can define the refined torsion,

$$\tau_\rho^0(\Sigma, h_{\text{sym}}) \in \mathbb{R}^\times.$$

By (5) and the symplecticity of $h_{\text{sym}}$, this torsion does not depend on the choice of $h_{\text{sym}}$. 
Remark 5.3. In [SW, Section 3.4.4] (see also [Lab, Proposition 4.3.6] or [Wit, §4.5]), the function $R^{sd}(\Sigma) \to \mathbb{R}^\times$ which takes $\rho$ to $\tau^0_\rho(\Sigma, h_{\text{sym}})$ is mathematically shown to be constant on each connected component of $R^{sd}(\Sigma)$. The following proposition lets us finish the proof of Theorem 5.2.

**Proposition 5.4** (cf. [SW, Wit]). For any $\rho \in R^{sd}(\Sigma)$, the torsion $\tau^0_\rho(\Sigma, h_{\text{sym}})$ equals $1/2^{g-1}$.

**Proof of Theorem 5.2**. Let $G = SL_2(\mathbb{R})$. We will apply two situations to Theorem 5.1. The first one is

$$C^* := C^*_f(\Sigma; g), \quad \overline{C}^* := C^*_f(W_1; g) \oplus C^*_f(W_2; g), \quad C^* := C^*_f(M; g).$$

Here, let $c, \overline{c}, c$ be the basis obtained from the orientations of the cellular structure of $M, W_1 \cup W_2, \Sigma$, respectively. Then, from the proof of Lemma 5.3, the acyclic complex $\mathcal{H}_\kappa$ in (14) is equivalent to the isomorphism $i_1^* \oplus i_2^* : H_f^1(W_1; g) \oplus H_f^1(W_2; g) \to H_f^1(\Sigma; g)$. Let $h$ be $\emptyset, \overline{h}$ be $h_{\text{sym}}, h_1 \in H_f^1(W_i; g)$ be bases which gives the orientation of $R^{sd}(W_i)$, and $\overline{h}$ be $h_1 \cup h_2$. Then, by definition of $\varepsilon_f$, we have

$$\varepsilon_f = \text{sign}(\det(i_1^* \oplus i_2^*)) = \text{sign}(\mathcal{T}(\mathcal{H}_\kappa, h \cup \overline{h} \cup \overline{\overline{h}})) \in \{\pm 1\}.$$

The other situation is given by the ordinary cellular complexes of the forms,

$$C^* := C^*(\Sigma; \mathbb{R}), \quad \overline{C}^* := C^*(W_1; \mathbb{R}) \oplus C^*(W_2; \mathbb{R}), \quad C^* := C^*(M; \mathbb{R}).$$

Here, let the 1-dimensional parts of $h^\mathbb{R}, \overline{h}^\mathbb{R}$ be the dual bases represented by the curves $a_1, b_1, \ldots, a_g, b_g$ in Figure 1. Then, since $H^*(M; \mathbb{R}) \cong H^*(S^3; \mathbb{R})$, we can easily check that $\mathcal{T}(\mathcal{H}_\kappa, h^\mathbb{R} \cup \overline{h}^\mathbb{R} \cup \overline{\overline{h}}^\mathbb{R})$ is equal to 1. Furthermore, we give some examples of the number $N(X)$ in (6):

$$N(D) = N(S^1) = 1, \quad N(\Sigma) = N(\Sigma_0) = 0, \quad N(W_i) = N(M) = g \in \mathbb{Z}/2,$$

where the cellular complexes of $\Sigma, \Sigma_0, W_i, M$ are canonically obtained from the Heegaard decomposition of $M$.

Next, by considering the ratio of the applications from the two situations of Theorem 5.1, we have

$$(-1)^g \cdot \tau^0_f(W_1, h_1) \tau^0_f(W_2, h_2) = \tau^0_f(\Sigma, h_{\text{sym}}) \tau^0_f(M) \det(i_1^* \oplus i_2^*) \in \mathbb{R}^\times. \quad (15)$$

Note $\tau^0_f(\Sigma, h_{\text{sym}}) = 1/2^{g-1} > 0$ from Proposition 5.4. Therefore, if $\text{sign}(\tau^0_f(W_1, h_1)) = \text{sign}(\tau^0_f(W_2, h_2))$, the signs of (15) lead to the the desired result, (7).

Finally, it suffices to show $\text{sign}(\tau^0_f(W_1, h_1)) = \text{sign}(\tau^0_f(W_2, h_2))$. Note that the function $\tau^0_*(W_i, h_i)$ is a continuous one on the connected space $R^{sd}(W_i)$ by Lemma 5.5 below. From the duality of the handle attaching of $M$, there are $f_1 \in R^{sd}(W_1)$ and $f_2 \in R^{sd}(W_2)$ such that $\tau^0_f(W_1, h_1) = \tau^0_f(W_2, h_2)$, which implies the desired $\text{sign}(\tau^0_f(W_1, h_1)) = \text{sign}(\tau^0_f(W_2, h_2))$ by connectivity.

**Lemma 5.5**. $\text{Hom}(\pi_1(W_i), G)^{sd}$ is connected.
Proof. Let \( C \subset \text{Hom}(\pi_1(W_i), G) = G^g \) be the complement of \( \text{Hom}(\pi_1(W_i), G)^{ad} \). For the proof, it is enough to show that \( C \) is of codimension > 1 over \( \mathbb{R} \).

For this, recall the classification theorem of algebraic subgroups \( K \) of \( SL_2(\mathbb{R}) \) with \( \dim(K) < 3 \). More precisely,

- If \( \dim(K) = 2 \), \( K \) is isomorphic to either \( \mathbb{R} \times \mathbb{R}^\times \) or \( \mathbb{R} \times \mathbb{R}^\times_0 \).
- If \( \dim(K) = 1 \), \( K \) is either abelian or isomorphic to \( \mathbb{R} \times \{ \pm 1 \} \).
- If \( \dim(K) = 0 \), \( K \) is a cyclic group.

The conjugacy action of \( G \) on \( K \) has a stabilizer subgroup whose dimension is more than zero. Therefore, if \( f \in C \), the orbits of \( f \) in \( \text{Hom}(\pi_1(W_i), G) \) are of dimension < 3. Notice that the quotient \( C/G \) by the conjugacy action is a union of real varieties of dimension < \( 3g - 3 \). Hence, the dimension of \( C \) is at most \( 3g - 1 \), as required. \( \square \)

5.3 Proof of Proposition 5.4

In this proof, we will often use theorem 5.6 below. To describe the theorem, for a \( PSL_2(\mathbb{R}) \)-representation \( \phi : \pi_1(\Sigma) \to PSL_2(\mathbb{R}) \), consider the associated \( P^1\mathbb{R} \)-bundle over \( \Sigma \), and let \( e(\phi) \in H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z} \) be the Euler class. Furthermore, let \( p : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R}) \) be the projection. Then, for an \( SL_2 \)-representation \( f : \pi_1(\Sigma) \to SL_2(\mathbb{R}) \), the Euler class \( e(p \circ f) \) is known to be even (see (16) below).

**Theorem 5.6 ([G2 Theorems A, B, and D]).** The connected components of \( \text{Hom}(\pi_1(\Sigma), PSL_2(\mathbb{R}))/PSL_2(\mathbb{R}) \) are in 1:1-correspondence with \( \{ m \in \mathbb{Z} | 2g - 2 \geq |m| \} \) through the map \( \phi \mapsto e(\phi) \).

Moreover, \( |e(\phi)| = 2g - 2 \) if and only if \( \phi : \pi_1(\Sigma) \to PSL_2(\mathbb{R}) \) is a discrete and faithful representation.

Furthermore, we will explain how to compute the Euler classes \( e(\phi) \), for \( \phi : \pi_1(\Sigma) \to PSL_2(\mathbb{R}) \). Let \( \tilde{G} \to PSL_2(\mathbb{R}) \) be the universal covering associated with \( \pi_1(PSL_2(\mathbb{R})) \cong \mathbb{Z} \), which is a central extension of the fiber \( \mathbb{Z} = \{ z^m \}_{m \in \mathbb{Z}} \). Choose a set-theoretical lift \( \phi(a_i) \in \tilde{G} \) of \( \phi(a_i) \). Then, Milnor [Mi2, p. 218–220] showed the equality,

\[
[\phi(b_1), \phi(b_2)] [\phi(b_3), \phi(b_4)] \cdots [\phi(b_g), \phi(b_g)] = z^{e(\phi)}.
\]

(16)

In particular, the left hand side is independent of the choice of the lifts.

**Proof of Proposition 5.4** The proof will be divided into four steps.

(Step 1) First, we consider the cases \( g = 2, 3 \). Then, given even \( N \) with \( |N| \leq 2g - 2 \), we can concretely construct \( \phi_N : \pi_1(\Sigma) \to SL_2(\mathbb{R}) \) with \( e(p \circ \phi_N) = N \). For such \( \phi_N \), we can use Proposition 5.2 to verify that \( \tau_{\phi_N}^0(\Sigma, h_{\text{sym}}) = 1/2^{g-1} \) with the help of a computer, although the program is a bit intricate. Thanks to Remark 5.3 for any \( \rho : \pi_1(\Sigma) \to SL_2(\mathbb{R}) \), we directly have \( \tau_{\rho}^0(\Sigma, h_{\text{sym}}) = 1/2^{g-1} \), as required.

(Step 2) Recall the notation \( D \subset \Sigma, \Sigma' \), and \( \Sigma_0, \Sigma_0' \) in 5.1. Let \( T \) be the torus with two circle boundaries such that \( \Sigma_0 = T \cup S^1; \Sigma_0' \); see Figure 1. Let \( h_{S^1} \cup h_{S^1} \) be a basis of \( H^*(\partial T; \mathbb{R}) \otimes g \cong H^*(S^1; g)^2 \cong H^*(S^1; \mathbb{R}) \otimes g^2 \) as 6-copies of the dual of the orientation class \( [S^1] \). When \( \rho_T : \pi_1(T) \to SL_2(\mathbb{R}) \) is trivial, we will show that \( \tau_{\rho_T}^0(T, h_{\text{sym}}|T \cup h_{S^1} \cup h_{S^1}) = 1/2 \).
Consider $\Sigma = \Sigma_0 \cup S_1 \cdot D$. By the Mayer-Vietoris argument and Theorem 5.1, we notice that
\[
\tau^0_f(D, h_D)\tau^0_f(\Sigma, h_{sym}) = \tau^0_f(\Sigma_0, h_{sym} \cup h_{S_1})\tau^0_f(S^1, h_{S_1})T(\mathcal{H}_*, h \cup h \cup \overline{h}),
\]
\[
\tau^0_f(D', h_{D'})\tau^0_f(\Sigma', h'_{sym}) = \tau^0_f(\Sigma_0', h'_{sym} \cup h_{S_1})\tau^0_f(S^1, h_{S_1})T(\mathcal{H}'_*, h \cup h \cup \overline{h}).
\]
We can easily see that $\tau^0_f(D, h_D) = \tau^0_f(S^1, h_{S_1}) = -1$ by definition. Furthermore, we can verify that $\mathcal{T}(\mathcal{H}_*)$ and $\mathcal{T}(\mathcal{H}'_*)$ are equal to 1 from the choice of the bases $h_{sym}, h_{S_1}, h_D$. Therefore, the two equalities can be rewritten as
\[
\tau^0_f(\Sigma, h_{sym}) = \tau^0_f(\Sigma_0, h_{sym} \cup h_{S_1}), \quad \tau^0_f(\Sigma', h'_{sym}) = \tau^0_f(\Sigma_0', h'_{sym} \cup h_{S_1}).
\]
By Step 1, these terms are 1/2 and 1/4, respectively, if $g = 2$. Let $\rho_T$ be the restriction $f_T$. Therefore, the Mayer-Vietoris sequence together with Theorem 5.1 gives rise to
\[
\tau^0_f(T, h_T)/2 = \tau^0_f(\Sigma_0, h_{sym} \cup h_{S_1})\tau^0_{\rho_T}(T, h_T \cup h_{S_1} \cup h_{S_1})
\]
\[
= -\tau^0_f(\Sigma_0', h'_{sym} \cup h_{S_1} \cup h_{S_1})\tau^0_f(S^1, h_{S_1}) = \tau^0_f(\Sigma_0', h'_{sym} \cup h_{S_1}) = 1/4.
\]
Hence, $\tau^0_{\rho_T}(T, h_{sym}|_T \cup h_{S_1} \cup h_{S_1}) = 1/2$ as required.

**Step 3** We suppose that Proposition 5.4 is true if $k = g$. First, consider the case of $|e(p \circ f')| \leq 2k - 4$, where $f' : \pi_1(\Sigma') \to SL_2(\mathbb{R})$. By Theorem 5.6 there is $f_0 : \pi_1(\Sigma') \to SL_2(\mathbb{R})$ such that $f_0$ and $f'$ lie in the same connected components of $\text{Hom}(\pi_1(\Sigma'), SL_2(\mathbb{R}))$ and the restriction $f_0|_{\pi_1(T)}$ is constant. Since $\tau^0_f(S^1, h_{S_1} \cup h_{S_1}) = 1/2$ by Step 2, a similar Mayer-Vietoris argument shows that
\[
\tau^0_{f_0}(\Sigma', h_{sym}) = \tau^0_f(\Sigma, h_{sym})\tau^0_f(T, h_{sym}|_T) = \frac{1}{2^k}\frac{1}{2} = \frac{1}{2^{k+1}}.
\]
Hence, we can complete the proof with $|e(p \circ f')| \leq 2g - 4$ by induction on $g$.

**Step 4** Here, consider the case $g = k + 1$ and $|e(p \circ f)| = 2g - 2$. By Theorem 5.6 again, $f : \pi_1(\Sigma') \to SL_2(\mathbb{R})$ is a faithful discrete representation. Let $h'_{S_1} \cup h''_{S_1}$ be a basis of $H^*(\partial T, g)$ $\cong H^*(S^1; \mathbb{R})$ as 2-copies of the dual of the orientation class $[S^1]$. In a similar way to Step 2, we can show that $\tau^0_{\rho_T}(T, h_{sym}|_T \cup h'_{S_1} \cup h''_{S_1}) = 1/2$ as well. Hence, as in Step 3, we can prove Proposition 5.4 with $|e(p \circ f)| = 2g - 2$ by induction on $g$.

## A Computation of the symplectic structures on flat moduli spaces

Here, we give an algebraic description of the non-degenerate alternating 2-form in (4). Although a similar discussion is presented in [GT] §3.10, it contains minor errors; here, we reformulate the description in a simplified way.

Take the standard presentation $\pi_1(\Sigma) = \langle a_1, b_1, \ldots, a_g, b_g \rangle$, where $r = [a_1, b_1] \cdots [a_g, b_g]$. Since $\Sigma$ is an Eilenberg-MacLane space, the cellular complex of the universal covering space can be expressed by a complex of group homology:
\[
C_* : 0 \to \mathbb{Z}[\pi_1(\Sigma)] \xrightarrow{\partial_2} \mathbb{Z}[\pi_1(\Sigma)]^{2g} \xrightarrow{\partial_1} \mathbb{Z}[\pi_1(\Sigma)] \xrightarrow{\epsilon} \mathbb{Z} \to 0 \quad \text{(exact)}.
\]
Let us fix the canonical basis of $C_2$ and $C_1$ by $R$ and $x_1, y_1, \ldots, x_g, y_g$, respectively. Then, the boundary maps are known to be
\[
\partial_1(x_i) = 1 - a_i, \quad \partial_1(y_i) = 1 - b_i \quad \text{and} \quad \partial_1(aR) = a \sum_{i=1}^g \frac{\partial r}{\partial a_i} x_i + \frac{\partial r}{\partial b_i} y_i,
\]
for \( a \in \mathbb{Z}[\pi_1(\Sigma)] \). Here, \( \frac{\partial \psi}{\partial y} \) is the Fox derivative. Then, given a left \( \mathbb{Z}[\pi_1(\Sigma)] \)-module \( M \), any 1-cocycle in a local coefficient \( M \) can be regarded as a left \( \mathbb{Z}[\pi_1(\Sigma)] \)-homomorphism \( f : \mathbb{Z}[\pi_1(\Sigma)]^2g \rightarrow M \) satisfying \( \sum_{i=1}^g \frac{\partial \psi}{\partial y_i} f(x_i) + \frac{\partial \psi}{\partial y_i} f(y_i) = 0 \).

Next, we will give a description of the cup product \( H^1 \otimes H^1 \rightarrow H^2 \). Let \( F \) be the free group \( \langle a_1, b_1, \ldots, a_g, b_g \rangle \). Consider the function,

\[
\kappa : F \times F \rightarrow \mathbb{Z}[\pi_1(\Sigma)]^2g \otimes \mathbb{Z}[\pi_1(\Sigma)]^2g; \quad (u, v) \mapsto \alpha(u) \otimes u\alpha(v).
\]

Here, \( \alpha(w) \) is defined as \( \sum_{i=1}^g \frac{\partial \psi}{\partial y_i} x_i + \frac{\partial \psi}{\partial y_i} y_i \). Then, according to [Tro], Lemma in §2.3, there is a unique map \( \Upsilon : F \rightarrow \mathbb{Z}[\pi_1(\Sigma)]^2g \otimes \mathbb{Z}[\pi_1(\Sigma)]^2g \) satisfying

\[
\Upsilon(uv) = \Upsilon(u) + u\Upsilon(v) + \kappa(u, v), \quad \Upsilon(1) = \Upsilon(a_i) = \Upsilon(b_i) = 0, \quad \text{for any } u, v \in F.
\]

**Proposition A.1** (A special case of [Tro] §2.4). Let \( \rho : \mathbb{Z}[\pi_1(\Sigma)] \rightarrow \text{End}(M) \) be a homomorphism, and regard \( M \) as a left \( \mathbb{Z}[\pi_1(\Sigma)] \)-module. For any two 1-cocycles \( f, f' : \mathbb{Z}[\pi_1(\Sigma)]^2g \rightarrow M \), the cup product \( f \smile f' \) as a 2-cocycle is represented by a map \( \mathbb{Z}[\pi_1(\Sigma)] \rightarrow M \otimes M \) give by

\[
f \smile f' = f(a \cdot R) = (f \otimes f')((a \otimes a) \cdot \Upsilon(r)), \quad \text{for } a \in \mathbb{Z}[\pi_1(\Sigma)].
\]

As a special case, consider a bilinear map \( \psi : M \otimes M \rightarrow A \) which is diagonally invariant with respect to \( \mathbb{Z}[\pi_1(\Sigma)] \). Then,

**Proposition A.2.** Let \( M \) be as above and \( f, f' \) be 1-cocycles. Suppose \( \psi(a, b) = \psi(\rho(g)a, \rho(g)b) \) for any \( a, b \in M \) and \( g \in \pi_1(\Sigma) \). Then, the composite of \( \psi \) and the cup product,

\[
H^1(\pi_1(\Sigma); M)^{\otimes 2} \xrightarrow{\sim} H^2(\pi_1(\Sigma); M^{\otimes 2}) \cong M^{\otimes 2} \xrightarrow{\psi} A
\]

are represented by the map \( \mathbb{Z}[\pi_1(\Sigma)]^{2g} \otimes \mathbb{Z}[\pi_1(\Sigma)]^{2g} \rightarrow A \) which sends \( (\sum_{i=1}^g k_i x_i + \ell_i y_i) \otimes (\sum_{j=1}^g k'_j x'_j + \ell'_j y'_j) \) to

\[
\sum_{i=1}^g \psi(f(k_i x_i), \rho(a_i + b_i - a_i b_i a_i^{-1} b_i^{-1}) f'(\ell'_i y_i)) - \psi(f(\ell_i y_i), \rho(b_i a_i^{-1} f(k'_i x_i)))
\]

\[
+ \psi(f(k_i x_i), \rho(1 - a_i b_i a_i^{-1}) f'(k'_i x_i)) + \psi(f(\ell_i y_i), \rho(1 - b_i a_i^{-1} b_i^{-1}) f'(\ell'_i y_i))
\]

\[
+ \sum_{m: 1 \leq m < i} \psi(\rho(I_m - I_m a_m b_m a_m^{-1}) f(k_m x_m) + \rho(I_m a_m - I_{m+1}) f(\ell_m y_m), \rho(I_i - I_i a_i b_i a_i^{-1}) f'(k'_i x_i) + \rho(I_i a_i - I_{i+1}) f'(\ell'_i y_i)),
\]

where \( k_i, k'_i, \ell_j, \ell'_j \in \mathbb{Z}[\pi_1(\Sigma)] \) and \( I_i = [a_1, b_1] \cdots [a_{i-1}, b_{i-1}] \).

Thanks to Proposition A.1, this proposition can be proven by directly computing \( \Upsilon \).

**B The work of Johnson [John]**

In this appendix, we explain the work of Johnson [John] (see Theorem B.2), which gives a way of computing the \( SU(2) \)-Casson invariant under a certain assumption. Let \( G \) be \( SU(2) \) hereafter. We will suppose that the reader has read §§2 and 3.1.

First, we will briefly review the \( SU(2) \)-Casson invariant of an integral homology 3-sphere \( M \). Let \( \text{Hom}(\pi_1(Z), G)^{\text{irr}} \) be the open subset consisting of irreducible representations \( \pi_1(Z) \rightarrow SU(2) \).
Denote the conjugacy quotient $\operatorname{Hom}(\pi_1(Z), G)^{irr}/G$ by $R^{irr}(Z)$. It is known (see, e.g., [AM] and [Sav]) that if $Z$ is $\Sigma$ with $g \geq 2$, then the conjugacy action of $\operatorname{PSU}(2)$ on $\operatorname{Hom}(\pi_1(\Sigma), G)^{irr}$ is proper and free, the quotient $R^{irr}$ is an oriented manifold of dimension $6g - 6$, and the tangent space at $p \in R^{irr}$ is identified with the first cohomology $H^1_\rho(\Sigma; g)$ with local coefficients by $\rho$. Furthermore, $R^{irr}(W_k)$ is known to be an oriented manifold of dimension $3g - 3$. To summarize, the restriction of (1) can be written as

$$
\begin{array}{c}
R^{irr}(\Sigma) \\
\downarrow \iota^1 \\
R^{irr}(W_1) \\
\downarrow \iota^2 \\
R^{irr}(W_2) \\
\downarrow j^2 \\
\end{array}
\quad \begin{array}{c}
\iota^1 \quad \iota^2 \\
\downarrow \\
\uparrow \\
\end{array}
\begin{array}{c}
R^{irr}(W_1) \cap R^{irr}(W_2) \subset R^{irr}(M),
\end{array}
$$

of $C^\infty$-embeddings. The intersection $R^{irr}(W_1) \cap R^{irr}(W_2)$ is compact, but not always transverse. If it is not transverse, by the Transversality theorem, we can choose an isotopy $h : R^{irr}(\Sigma) \to R^{irr}(\Sigma)$ such that $h$ is supported in a compact neighborhood of $R^{irr}(W_1) \cap R^{irr}(W_2)$ and $h(R^{irr}(W_i))$ meets $R^{irr}(W_2)$ transversally in $\operatorname{supp}(h)$.

Then, the $\operatorname{SU}(2)$-Casson invariant, $\lambda_{\operatorname{SU}(2)}(M)$, is defined to be $(-1)^g \sum \varepsilon_f$. Here, the sum runs over $h(R^{irr}(W_1)) \cap R^{irr}(W_2)$, and the number $\varepsilon_f$ equals $\pm 1$ depending on whether the orientations of the spaces $T_f h(R^{irr}(W_1)) \oplus T_f R^{irr}(W_2)$ and $T_f R^{irr}(\Sigma)$ agree. It is known that $\lambda_{\operatorname{SU}(2)}(M)$ is a topological invariant of $M$.

While it is not so easy to compute $\varepsilon_f$, Johnson [John] suggested a procedure for computing $\varepsilon_f$ from Reidemeister torsions. For this, we shall mention a proposition similar to that of Lemma 3.3.

**Proposition B.1 ([Sav] Theorem 16.4).** Take $f \in R^{irr}(W_1) \cap R^{irr}(W_2)$. Then, the intersection of $R^{irr}(W_1) \cap R^{irr}(W_2)$ at $f$ is transverse if and only if $C^*_f(\rho; g)$ is acyclic, i.e., $H^1_f(\rho; g) = 0$.

Thus, under the transversality, by definition, in order to compute the invariants, it is enough to compute the sign $\varepsilon_f$ with respect to $f \in R^{irr}(M)$, since the isotopy $h$ may be the identity. In the note [John], Johnson gave the following theorem:

**Theorem B.2 ([John]).** Suppose $H_\ast(M; \mathbb{Z}) \cong H_\ast(S^3; \mathbb{Z})$ and that, for any $f \in R^{irr}(W_1) \cap R^{irr}(W_2)$, the intersection of $\operatorname{Im}(i^1_f)$ and $\operatorname{Im}(i^2_f)$ at $f$ is transverse.

Then, the $\operatorname{SU}(2)$-Casson invariant is formulated as

$$\lambda_{\operatorname{SU}(2)}(M) = \sum_{f \in R^{irr}(W_1) \cap R^{irr}(W_2)} \operatorname{sign}(\tau^0_f(M)).$$

(17)

This theorem might be a folklore; however, since the note [John] is unpublished, we now give a proof of this theorem:

**Proof.** The proof is almost the same as the proof of Theorem 3.2, so we will suppose that the reader has read 3.2. Let $G$ be $\operatorname{SU}(2)$, and $g$ be $\mathfrak{su}(2)$.

By (3), we have a symplectic structure on $H^1_\rho(\Sigma; g) \cong \mathbb{R}^{6g-6}$ for any $\rho \in R^{irr}(\Sigma)$. With the choice of a symplectic basis $h_{\text{sym}} \subset H^1_\rho(\Sigma; g)$, we can also define the refined torsion: $\tau^0_\rho(\Sigma, h_{\text{sym}}) \in \mathbb{R}^\times$. By (5) and symplecticity of $h_{\text{sym}}$, this torsion does not depend on the choice of $h_{\text{sym}}$. 


In a similar way to (15), we obtain
\[ (-1)^g \cdot \tau_f^0(W_1, h_1)\tau_f^0(W_2, h_2) = \tau_f^0(\Sigma, h_{\text{sym}})\tau_f^0(M)\det(i_1^* \oplus i_2^*) \in \mathbb{R}^\times. \] (18)

We can show that \( \text{sign}(\tau_f^0(W_1, h_1)) = \text{sign}(\tau_f^0(W_2, h_2)) \) as in Lemma 5.5. These equalities can be proven in the same way as in §5.2 so we will omit the details.

From the construction, the function \( R^{\text{irr}}(\Sigma) \to \mathbb{R}^\times \) which takes \( \rho \) to \( \tau_f^0(\Sigma, h_{\text{sym}}) \) is continuous. Hence, it is sufficient to show \( \tau_{f_0}^0(\Sigma, h_{\text{sym}}) > 0 \) in the case \( G = SU(2) \). For this, note a well known fact that the open set \( R^{\text{irr}}(\Sigma) \) is connected; see, e.g., [GX]. Therefore, we may show \( \tau_{f_0}^0(\Sigma, h_{\text{sym}}) > 0 \) for appropriate \( f_0 \in R^{\text{irr}}(\Sigma) \). Moreover, a discussion similar to the one in §5.3 means that we only have to consider the case \( g = 2 \). For this, let us consider \( f_0 \) defined by

\[
\begin{align*}
\tau_{f_0}^0(a_1) &= \left( \frac{2\sqrt{-1} + 2\sqrt{10}}{6}, \frac{-2 + \sqrt{12}}{\sqrt{6}} \right), \\
\tau_{f_0}^0(b_1) &= \left( \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right), \\
\tau_{f_0}^0(a_2) &= \left( \frac{1 - \sqrt{-1}}{2}, \frac{-1 + \sqrt{-1}}{2} \right), \\
\tau_{f_0}^0(b_2) &= \left( \frac{\sqrt{2}}{2}, \frac{-\sqrt{2}}{2} \right).
\end{align*}
\]

By Proposition [A.2] and with the help of a computer, we can verify that \( \tau_{f_0}^0(\Sigma, h_{\text{sym}}) > 0 \), as desired.

**Remark B.3.** In the \( SU(2) \) case, we can show that \( \tau_f^0(\Sigma, h_{\text{sym}}) = 1 \) for any \( g > 1 \) and any irreducible representation \( f : \pi_1(\Sigma) \to SU(2) \). The proof follows that of Proposition 5.4.

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