TIME PERIODIC SOLUTIONS TO THE THREE–DIMENSIONAL EQUATIONS OF COMPRESSIBLE MAGNETOHYDRODYNAMIC FLOWS

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Abstract. In this paper, the compressible magnetohydrodynamic system with some smallness and symmetry assumptions on the time periodic external force is considered in \( \mathbb{R}^3 \). Based on the uniform estimates and the topological degree theory, we prove the existence of a time periodic solution in a bounded domain. Then by a limiting process, the result in the whole space \( \mathbb{R}^3 \) is obtained.

1. Introduction. In this paper, we focus on the existence of time periodic solution with time periodic external force for 3D compressible viscous magnetohydrodynamic equations (MHD), and the governing equations can be written as

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) - \mu \Delta u - (\mu + \lambda) \nabla \text{div}u - (\nabla \times H) \times H &= \rho f, \\
H_t - \nabla \times (u \times H) - \nu \Delta H &= 0, \\
\text{div} H &= 0.
\end{align*}
\]

Here \( x \in \mathbb{R}^3 \), \( \rho(x,t), u(x,t) = (u_1, u_2, u_3)(x,t) \) and \( H(x,t) = (H_1, H_2, H_3)(x,t) \) represent the density, the velocity and the magnetic field respectively, and the pressure \( P(\rho) = \rho^\gamma \) with \( \gamma > 1 \). The constants \( \mu, \lambda \) are the viscosity coefficients with the usual physical conditions

\[
\mu > 0, \lambda + \frac{2}{3} \mu \geq 0;
\]

the constants \( \nu > 0 \) is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. In addition, \( f(x,t) \) is a given external force, which is assumed to be periodic in time with period \( T \).

The aim of this paper is to show the problem (1) admits a time periodic solution around the constant state \( (\bar{\rho}, 0, 0) \) in \( \mathbb{R}^3 \), which has the same period as \( f \). Based on the uniform estimates and the topological degree theory, we are able to establish the existence of a time periodic solution in a bounded domain. And

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then the existence in the whole space is derived by several uniform bounds and the limiting process on the approximate solutions.

Our main result in this paper is stated as follows.

**Theorem 1.1.** Let the time periodic external force $f(x,t) \in L^2(0,T;L^\frac{6}{5}(\mathbb{R}^3)) \cap W^{1,1}_2((0,T) \times \mathbb{R}^3)$ with $f(-x,t) = -f(x,t)$. If

$$
\int_0^T \left( \|f(t)\|_{L^\frac{6}{5}}^2 + \|f(t)\|_{H^1}^4 \right) \, dt + \|f\|_{W^{1,1}_2}^2 \leq h,
$$

for some small constant $h > 0$, then the problem (1) admits a time periodic solution $(\rho, u, H) \in S_{\delta_0}$ with the same period as the external force $f(x,t)$, and $S_{\delta_0}$ is defined later.

There are extensive studies on the existence and uniqueness of solutions of the magnetohydrodynamic equations for the physical importance, complexity and wide range of applications, cf.\cite{3, 4, 5, 6, 7, 8, 9, 11, 12, 16, 17, 18, 21, 22, 24, 25, 26} and references therein. However, comparatively less works are received to the periodic solutions case. In particular, as for the incompressible case, in \cite{21}, the authors obtained the existence and uniqueness of the periodic strong solutions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 3$ or 4 by the spectral Galerkin method and compactness arguments. On the other hand, for the compressible case, Yan and Li \cite{25} proved the existence of time periodic solutions on a bounded domain of $\mathbb{R}^3$ in the framework of weak solution for large time periodic external forces. For the problem in an unbounded domain, Tan and Wang \cite{22} studied the existence, uniqueness and time–asymptotic stability of time periodic solutions in the whole space $\mathbb{R}^n$. It was shown in \cite{22} that only when the space dimension $n \geq 5$, there exists a time periodic solution for a sufficiently small time periodic external force $f(x,t) \in C(0,T;H^{N-1} \cap L^1), N \geq n+2$. However, how to handle the case when $n \leq 4$, especially the physical case when $n = 3$, there is no results available. Based on this, in this paper we will consider the existence of a periodic solution under some smallness and structure conditions on the time periodic external force in the whole space $\mathbb{R}^3$.

When there is no electromagnetic field, system (1) reduces to the compressible Navier–Stokes equations, cf. \cite{1, 2, 10, 13, 14, 15, 19, 20, 23} and references therein. Here we only mention some of them related to our paper. In \cite{19}, Ma, Ukai and Yang combined the linear decay analysis and the contraction mapping theorem to obtain the existence and stability of time periodic solutions when the space dimension $n \geq 5$. For recent works, in \cite{14}, Jin and Yang considered the existence of time periodic solutions to the whole space $\mathbb{R}^3$ through the topological degree theory. Also, by the spectral properties, the author in \cite{15} obtained a time periodic solution for sufficiently small and symmetry condition on the time periodic external force when the space dimension is greater than or equal to 3. Inspired by \cite{14}, in the present paper, we improve the result from \cite{22} in the following sense: we change the space dimension from $n \geq 5$ to $n = 3$.

The rest of the paper is organized as follows. We will reformulate the problem and give some preliminaries lemmas for later use in Section 2. In Section 3, we will obtain the existence of periodic solutions for (3) by uniform estimates and topological degree theory in a bounded domain. And the proof of the main result will be given in the last section.

**Notations.** Throughout this paper, for simplicity, we will omit the variables $t, x$ of functions if it does not cause any confusion. $C$ denotes a generic positive constant...
which may vary in different estimates. Moreover, we use $H^s$ to denote the usual $L^2$–Sobolev spaces with normal $\| \cdot \|_{H^s}$ and $L^p$, $1 \leq p \leq \infty$ to denote the usual $L^p$ spaces with norm $\| \cdot \|_{L^p}$. Finally, denote the $t$–anisotropic Sobolev spaces as $W_p^{m,k}((0, T) \times \Omega^L) = \{ u; D^\alpha u, D^\beta u \in L^p((0, T) \times \Omega^L), \text{ for any } |\alpha| \leq m, |\beta| \leq k \}$, with the norm

$$
\| u \|_{W_p^{m,k}} = \sum_{|\alpha| \leq m} \| D^\alpha u \|_{L^p} + \sum_{|\beta| \leq k} \| D^\beta u \|_{L^p}.
$$

And for $0 < \alpha < 1$, denote $C^{\alpha, \frac{\alpha}{2}}((0, T) \times \Omega^L)$ be the set of all functions $u$ such that $|u|_{\alpha, \frac{\alpha}{2}} < \infty$, where

$$
|u|_{\alpha, \frac{\alpha}{2}} = [u]_{\alpha, \frac{\alpha}{2}} + \| u \|_{L^\infty},
$$

here $[\cdot]_{\alpha, \frac{\alpha}{2}}$ is the semi–norm defined by

$$
[u]_{\alpha, \frac{\alpha}{2}} = \sup_{(x,t) \neq (y,s)} \frac{|u(x,t) - u(y,s)|}{(|x-y|^2 + |t-s|)^\frac{\alpha}{2}}.
$$

2. Preliminaries. In order to prove the existence of periodic solutions, we first substitute $\varrho = \rho - \bar{\rho}$, then the system (1) can be rewritten as

$$
\begin{cases}
\varrho_t + \bar{\rho} \text{div } \varrho = - \text{div}(\varrho \varrho), \\
(\bar{\rho} + \varrho) u_t - \mu \Delta u - (\mu + \lambda) \nabla \text{div } u + P'(\bar{\rho} + \varrho) \nabla \varrho = -(\bar{\rho} + \varrho)(u \cdot \nabla) u \\
\hfill + (\nabla \times H) \times H + (\bar{\rho} + \varrho) f, \\
H_t - \nu \Delta H = \nabla \times (u \times H), \quad \text{div } H = 0.
\end{cases}
$$

To solve the time periodic problem for (2) in $\mathbb{R}^3$, we now concern the following regularized problem in a bounded domain

$$
\begin{cases}
\varrho_t + \bar{\rho} \text{div } \varrho - \epsilon \Delta \varrho = - \text{div}(\varrho \varrho), \\
(\bar{\rho} + \varrho) u_t - \mu \Delta u - (\mu + \lambda) \nabla \text{div } u + P'(\bar{\rho} + \varrho) \nabla \varrho = -(\bar{\rho} + \varrho)(u \cdot \nabla) u \\
\hfill + (\nabla \times H) \times H + (\bar{\rho} + \varrho) f^L, \\
H_t - \nu \Delta H = \nabla \times (u \times H), \quad \text{div } H = 0, \\
\int_{\Omega^L} \varrho \, dx = 0,
\end{cases}
$$

where $\Omega^L = (-L, L)^3 \subset \mathbb{R}^3$, $f^L(x, t)$ is a time periodic function and an odd function on the space variable $x$ with periodic boundary, satisfying

$$
f^L \to f \text{ in } L^2(0, T; L^2(\mathbb{R}^3)) \cap W^{1,1}_2((0, T) \times \mathbb{R}^3)
$$

and

$$
\int_0^T \| f^L(t) \|_{L^2(\Omega^L)}^2 \, dt + \| f^L \|_{W^{1,1}_2((0, T) \times \Omega^L)}^2 \leq 2 \int_0^T \| f(t) \|_{L^2(\mathbb{R}^3)}^2 \, dt + 2 \| f \|_{W^{1,1}_2((0, T) \times \mathbb{R}^3)}^2.
$$

Precisely, denote the solution space in bounded domain $\Omega^L$ and the whole space $\mathbb{R}^3$ by

$$
\mathcal{S}^L = \{(\rho, v, B)(x, t) : (\rho, v, B) \in L^\infty(0, T; L^6(\Omega^L)) \};
$$
\((\rho, v) \in L^\infty(0, T; L^2(\Omega^L)) \cap L^2(0, T; H^1(\Omega^L));\)
\(B_t \in L^\infty(0, T; H^1(\Omega^L)) \cap L^2(0, T; H^2(\Omega^L));\)
\(\nabla \rho \in L^\infty(0, T; H^1(\Omega^L)) \cap L^2(0, T; H^1(\Omega^L));\)
\((\nabla v, \nabla B) \in L^\infty(0, T; H^1(\Omega^L)) \cap L^2(0, T; H^2(\Omega^L));\)
and \((\rho, v, B)\) satisfies (a), (b), (c),

and

\[
\mathcal{S} = \{ (\rho, v, B)(x, t) : (\rho, v, B) \in L^\infty(0, T; L^6(\mathbb{R}^3)); \\
(\rho, v) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)); \\
B_t \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)); \\
\nabla \rho \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)); \\
(\nabla v, \nabla B) \in L^\infty(0, T; H^1(\mathbb{R}^3)) \cap L^2(0, T; H^2(\mathbb{R}^3)); \\
and \ (\rho, v, B)\) satisfies (c),\}
\]

with

(a) \((\rho, v, B)\) is time periodic functions with periodic boundary condition;
(b) \(\int_{\Omega} \rho(x, t) \, dx = 0;\)
(c) \(\rho(x, t) = \rho(-x, t), v(x, t) = -v(-x, t), B(x, t) = -B(-x, t).\)

Set

\(S^L_\delta = \{ (\rho, v, B) \in S^L; \|((\rho, v, B)) < \delta \},\)

for some positive constant \(\delta\) and with the norm \(\| \cdot \|\) defined as

\[
\|((\rho, v, B))\|^2 = \sup_{0 \leq t \leq T} \left( \|((\rho, v, B)(t))\|^2_{L^6} + \|((\rho, v)_{\epsilon}(t))\|^2_{L^2} + \|B_t(t))\|^2_{H^1} + \|\nabla((\rho, v, B)(t))\|^2_{H^1} \right) \\
+ \int_0^T \left( \|((\rho, v)_{\epsilon}(t))\|^2_{H^1} + \|B_t(t))\|^2_{H^2} + \|\nabla\rho(t))\|^2_{H^1} + \|\nabla((v, B)(t))\|^2_{H^2} \right) \, dt.
\]

Then we have the following proposition for the problem (3), the proof of this proposition will be given in the end of next section.

**Proposition 1.** Assume that \(f^L(x, t)\) is a smooth function and \(f^L(x, t) \in L^2(0, T; L^6(\Omega^L)) \cap W^{1,1}_{2,2}(0, T \times \Omega^L)\) with \(f^L(-x, t) = -f^L(x, t).\) If

\[
\int_0^T (\|f^L(t))\|^2_{L^6} + \|f^L(t))\|^2_{H^1} \, dt + \|f^L\|^2_{W^{1,1}_{2,2}} \leq h^*,
\]

for some small constant \(h^* > 0,\) then the problem (3) admits a solution \((\rho^L, u^L, H^L)\) \(S^L_\delta,\) here \(\delta\) is a small constant independent of \(L\) and \(\epsilon.\)

Several elementary inequalities are needed later, cf. [14], here the coefficients independent of the domain play an important role in passing the limit of the approximate solutions in the last section.

**Lemma 2.1.** Assume that \(\Omega \subset \mathbb{R}^N\) is a bounded domain, and \(\partial \Omega\) is locally Lipschitz continuous. If \(u|_{\partial \Omega} = 0\) (or \(\int_{\Omega} u \, dx = 0\)), then for any \(1 \leq p < N, 1 \leq q \leq p^* = \frac{Np}{N-p},\)

\[
(\int_{\Omega} |u|^q \, dx)^{1/q} \leq C(N, p, q)|\text{mes}\Omega|^{\frac{1}{p^* - p}} \left( \int_{\Omega} |\nabla u|^{p} \, dx \right)^{1/p}.
\]
In particular, if $q = p^* = \frac{Np}{N-p}$, then
\[
\left( \int_\Omega |u|^p \, dx \right)^{1/p^*} \leq C(N,p,q) \left( \int_\Omega |\nabla u|^p \, dx \right)^{1/p}.
\]

**Lemma 2.2.** Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain, and $\partial \Omega$ is locally Lipschitz continuous. If $u|_{\partial \Omega} = 0$ (or $\int_\Omega u \, dx = 0$), then
\[
\|u\|_{L^1} \leq C\|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2},
\]
\[
\|u\|_{L^4} \leq C\|u\|_{L^2}^{1/4} \|\nabla u\|_{L^2}^{3/4},
\]
\[
\|u\|_{L^\infty} \leq C\|\nabla u\|_{H^1},
\]
where $C$ is independent of $\Omega$. Moreover, the above inequalities also hold in $\mathbb{R}^3$ if $u(x) \to 0$ as $|x| \to \infty$.

3. **Existence in bounded domain.** In this section, we are devoted to obtaining the existence of time periodic solutions for the regularized problem (3) in a bounded domain of Proposition 1. The proof is a combination of the uniform estimates and the topological degree theory. To do this, we begin with the introduction of a completely continuous operator $F$ to the linear parabolic problem (4).

3.1. **Introduction of an operator $F$.** For any $\tau \in [0,1]$, we define an operator
\[
F : S^L_\delta \times [0,1] \to S^L,
\]
\[
((\rho,v,B),\tau) \to (\rho,u,H),
\]
with $\delta$ being suitably small, here $(\rho,u,H)$ is the solution of the following system with periodic boundary
\[
\begin{aligned}
\rho_t + \bar{\rho} \text{div } u - \epsilon \Delta \rho &= Q_1(\rho,v,B,\tau), \\
(\bar{\rho} + \tau \rho)u_t - \mu \Delta u - (\mu + \lambda) \text{div } u + \frac{P'(\bar{\rho})}{\bar{\rho}} (\bar{\rho} + \tau \rho) \nabla \rho &= Q_2(\rho,v,B,\tau) + \tau(\bar{\rho} + \tau \rho) f^L, \\
H_t - \nu \Delta H &= Q_3(\rho,v,B,\tau), \quad \text{div } H = 0,
\end{aligned}
\]
(4)
where
\[
\begin{align*}
Q_1(\rho,v,B,\tau) &= -\tau \text{div}(\rho v), \\
Q_2(\rho,v,B,\tau) &= -\tau(\bar{\rho} + \tau \rho)(v \cdot \nabla) v + \left( \frac{P'(\bar{\rho})}{\bar{\rho}} (\bar{\rho} + \tau \rho) - P'(\bar{\rho} + \tau \rho) \right) \nabla \rho + \tau(\nabla \times B) \times B, \\
Q_3(\rho,v,B,\tau) &= \tau \nabla \times (v \times B).
\end{align*}
\]

**Remark 1.** To guarantee the uniqueness of solutions, we impose the condition
\[
\int_{\Omega^L} \rho \, dx = 0
\]
here. Since \(\frac{d}{dt} \int_{\Omega^L} \rho \, dx = 0\) implies that when $(\rho,u,H)$ is a solution of the system (4), then $(\rho + c,u,H)$ is also a solution for any constant $c$.

In what follows, we will concentrate on the properties of the operator $F$. To show $F$ is well defined, we shall establish the following lemma first.
Lemma 3.1. Assume that $\delta$ is sufficiently small, then for any $(\rho, v, B) \in S^L$, $\tau \in [0, 1]$, the problem (4) admits a unique time periodic solution $(\varrho, u, H) \in S^L$.

Proof. By Lemma 2.2, we have

$$\|\rho\|_{L^\infty} \leq C \|\nabla \rho\|_{H^1} \leq C\delta. \tag{5}$$

Then from the smallness of $\delta$, we obtain

$$\frac{\bar{\rho}}{2} \leq \bar{\rho} + \tau \rho \leq 2\bar{\rho}, \tag{6}$$

which implies that

$$\frac{1}{2\bar{\rho}} \leq \frac{1}{\bar{\rho} + \tau \rho} \leq \frac{2}{\bar{\rho}}. \tag{7}$$

Define

$$A = \begin{pmatrix} \epsilon \Delta & -\rho \nabla & 0 \\ -\frac{P'(\bar{\rho})}{\bar{\rho}} \nabla & \frac{\mu}{\bar{\rho} + \tau \rho} \Delta + \frac{\mu + \lambda}{\bar{\rho} + \tau \rho} \nabla \div & 0 \\ 0 & 0 & \nu \Delta \end{pmatrix},$$

and set $U = (\varrho, u, H)$, $W = (\rho, v, B)$, $Q(W) = (Q_1, Q_2, Q_3)$, $F = (0, \tau f^L, 0)$, then the system (4) takes the form

$$U_t = AU + Q(W) + F.$$

Now, we begin with the following initial value problem of (4) in $\Omega^L$, that is

$$\begin{cases} \\ \varrho_t + \bar{\rho} \div u - \epsilon \Delta \varrho = 0, \\ u_t - \frac{\mu}{\bar{\rho} + \tau \rho} \Delta u - \frac{\mu + \lambda}{\bar{\rho} + \tau \rho} \nabla \div u + \frac{P'(\bar{\rho})}{\bar{\rho}} \nabla \varrho = 0, \\ H_t - \nu \Delta H = 0, \quad \div H = 0, \\ (\varrho, u, H)(x, 0) = (\varrho_0, u_0, H_0)(x). \tag{8} \end{cases}$$

with periodic boundary condition and the initial data $\varrho_0(x)$ is an even function with $\int_\Omega \varrho_0 \, dx = 0$, $u_0, H_0$ are odd functions. Obviously, the solution $(\varrho, u, H)$ has the same properties as the initial data $(\varrho_0, u_0, H_0)$.

Multiplying (8) by $u$ and the integration over $\Omega^L$ yields

$$\int_{\Omega^L} \frac{1}{2} \frac{d}{dt} u^2 + \int_{\Omega^L} \left( \frac{\mu}{\bar{\rho} + \tau \rho} |\nabla u|^2 + \frac{\mu + \lambda}{\bar{\rho} + \tau \rho} \|\div u\|^2 \right) \, dx$$

$$= \int_{\Omega^L} \frac{\tau \mu}{(\bar{\rho} + \tau \rho)^2} \nabla u \nabla \rho u \, dx + \int_{\Omega^L} \frac{\tau (\mu + \lambda)}{(\bar{\rho} + \tau \rho)^2} \div u \nabla \rho u \, dx$$

$$\leq \frac{\tau (2\mu + \lambda)}{\bar{\rho} - \tau \|\rho\|_{L^\infty}} \|\nabla \rho\|_{L^\infty} \|\nabla u\|_{L^2} \|u\|_{L^6}$$

$$\leq C \frac{\tau (2\mu + \lambda)}{\bar{\rho} - \tau \|\rho\|_{L^\infty}} \|\nabla \rho\|_{H^1} \|\nabla u\|_{L^2}^2.$$
Similarly, multiplying \((8)_1, (8)_3\) by \(\frac{P'(\bar{\rho})}{\bar{\rho}^2} \varrho, H\) respectively, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} (\frac{P'(\bar{\rho})}{\bar{\rho}^2} \varrho^2 + H^2) \, dx + \int_{\Omega^L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}^2} \varrho \div u + \epsilon \frac{P'(\bar{\rho})}{\bar{\rho}^2} |\nabla \varrho|^2 + \nu |\nabla H|^2 \right) \, dx = 0.
\]

Summing up the above two inequalities, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}^2} \varrho^2 + u^2 + H^2 \right) \, dx + \int_{\Omega^L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}^2} |\nabla \varrho|^2 + \nu |\nabla H|^2 \right) \, dx + \int_{\Omega^L} \left( \frac{\mu}{\bar{\rho}} |\nabla u|^2 + \frac{\mu + \lambda}{2\bar{\rho}} |\div u|^2 \right) \, dx \leq 0. \tag{9}
\]

On the other hand, multiplying \((8)_2\) by \(\mu \Delta u + (\mu + \lambda) \div u\), we deduce that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\div u|^2 \right) \, dx - (2\mu + \lambda) \frac{P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega^L} \nabla \varrho \div u \, dx + \int_{\Omega^L} \frac{1}{\bar{\rho} + \tau \rho} (\mu \Delta u + (\mu + \lambda) \div u)^2 \, dx = 0.
\]

Applying \(\nabla\) to \((8)_1, (8)_3\) and multiplying the resultant identities by \((2\mu + \lambda) \frac{P'(\bar{\rho})}{\bar{\rho}^2} \nabla \varrho, \nabla H\) respectively, we have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( (2\mu + \lambda) \frac{P'(\bar{\rho})}{\bar{\rho}^2} |\nabla \varrho|^2 + \nabla H^2 \right) \, dx + \int_{\Omega^L} (2\mu + \lambda) \frac{P'(\bar{\rho})}{\bar{\rho}} \nabla \varrho \div u \, dx + \int_{\Omega^L} \left( \epsilon (2\mu + \lambda) \frac{P'(\bar{\rho})}{\bar{\rho}^2} |\Delta \varrho|^2 + \nu |\Delta H|^2 \right) \, dx = 0.
\]

From the above two estimates, we arrive at

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( (2\mu + \lambda) \frac{P'(\bar{\rho})}{\bar{\rho}^2} |\nabla \varrho|^2 + \mu |\nabla u|^2 + (\mu + \lambda) |\div u|^2 + |\nabla H|^2 \right) \, dx + \nu \int_{\Omega^L} |\Delta H|^2 \, dx + \epsilon (2\mu + \lambda) \frac{P'(\bar{\rho})}{\bar{\rho}^2} \int_{\Omega^L} |\Delta \varrho|^2 \, dx + \int_{\Omega^L} \frac{1}{\bar{\rho} + \tau \rho} (\mu \Delta u + (\mu + \lambda) \div u)^2 \, dx = 0. \tag{10}
\]

This together with \((9)\) and the poincaré inequality imply

\[
\| (\varrho, u, H)(x, t) \|_{H^1} \leq \| (\varrho_0, u_0, H_0) \|_{H^1} e^{-Ct},
\]

that is,

\[
\| e^{\Delta t} U_0 \|_{H^1} \leq \| U_0 \|_{H^1} e^{-Ct}.
\]

Then by Duhamel’s principle, the solution to the system \((4)\) can be written in a mild form as

\[
U(t) = \int_{-\infty}^{t} e^{(t-s)\Delta} \left( Q(W)(s) + F(s) \right) \, ds,
\]

and it satisfies that

\[
\| U(t) \|_{H^1} \leq \int_{-\infty}^{t} \| e^{(t-s)\Delta} \left( Q(W)(s) + F(s) \right) \|_{H^1} \, ds \leq \int_{-\infty}^{t} e^{-C(t-s)} \| Q(W)(s) + F(s) \|_{H^1} \, ds.
\]
Lemma 3.2. Assume that the proof is similar to Lemma 2.2 and Lemma 2.3 in [13].

We have used the time periodic property of $W$ and $F,$ also we have

$$U(t + T) = \int_{-\infty}^{t+T} e^{(t+T-s)\lambda} (Q(W)(s) + F(s)) \, ds$$

$$= \int_{-\infty}^{t+T} e^{(t-(s-T))\lambda} (Q(W)(s-T) + F(s-T)) \, ds$$

$$= \int_{-\infty}^{t} e^{(t-s)\lambda} (Q(W)(s) + F(s)) \, ds = U(t).$$

That is, $U(t) \in L^\infty(0, T; H^1(\Omega^2))$ is a time periodic solution of (4) with time period $T.$

Moreover, by the classical theory of parabolic equations, we have that for any $(\rho, v, B) \in S^k, \tau \in [0, 1],$ the problem (4) admits a time periodic solution $(\rho, u, H) \in S^k.$ To prove the uniqueness, assume that there exists two solutions $U_1 = (\varrho_1, u_1, H_1), U_2 = (\varrho_2, u_2, H_2)$ for some $(\rho, v, B) \in S^k, \tau \in [0, 1],$ then we have

$$(U_1 - U_2)_t = \mathcal{A}(U_1 - U_2).$$

Similar to the first half of this proof, Let $\rho = \varrho_1 - \varrho_2, u = u_1 - u_2, H = H_1 - H_2,$ we have

$$\int_0^T \int_{\Omega^2} \left( e^{\frac{\lambda - \mu}{\rho^2}} |\nabla \rho|^2 + \nu |\nabla H|^2 + \frac{\mu}{4\rho} |\nabla u|^2 + \frac{\mu + \lambda}{2\rho} |\text{div} \, u|^2 \right) \, dx \, dt \leq 0.$$

By Poincaré inequality, $(\rho, u, H) = (0, 0, 0)$ which implies the uniqueness. Finally, if $(\rho(x, t), u(x, t), H(x, t))$ is the periodic solution of (4), then $(\rho(-x, t), -u(-x, t), -H(-x, t))$ is also the solution of (4), by the uniqueness, we easily obtain $(\rho(x, t), u(x, t), H(x, t)) = (\rho(-x, t), -u(-x, t), -H(-x, t)).$ This completes the proof of Lemma 3.1.

Now, in the following lemma, we see that the operator $\mathcal{F}$ is completely continuous. The proof is similar to Lemma 2.2 and Lemma 2.3 in [13].

**Lemma 3.2.** Assume that $\delta$ is sufficiently small, then the operator $\mathcal{F}$ is compact and continuous.

3.2. **Uniform estimates.** The task of this subsection is to give several uniform estimates for $(\varrho, u, H)$ of the following system. This is crucial for the proof of
Proposition 1 and the main theorem.

\[
\begin{aligned}
&\rho_t + \rho \nabla u - \epsilon \Delta \rho = -\tau \nabla (\rho u), \\
&(\rho + \tau \varrho) u_t - \mu \Delta u - (\mu + \lambda) \nabla \nabla \nabla u + P'(\rho + \tau \varrho) \nabla \varrho = -\tau (\rho + \tau \varrho) (u \cdot \nabla) u \\
&+ \tau (\nabla \times H) \times H + \tau (\rho + \tau \varrho) f^L, \\
&H_t - \nu \Delta H = \tau \nabla \times (u \times H), \quad \text{div} H = 0, \\
&\int_{\Omega^L} \varrho \, dx = 0,
\end{aligned}
\]

(11)

where \( \tau \in (0, 1] \). Since when \( \tau = 0 \), similar to the proof given in the section 3 of [14], it is easy to obtain that \( F((\rho, v, B), 0) = 0 \).

By elaborate calculation, the uniform estimates for \( (\varrho, u, H) \) independent of \( L \) and \( \epsilon \) are derived as follows.

**Lemma 3.3.** Let \( \tau \in (0, 1] \), if \( |\varrho| \leq \frac{\bar{\varrho}}{2} \), then the solution \( (\varrho, u, H) \in S^L \) to the system (11) satisfies

\[
\frac{d}{dt} \int_{\Omega^L} \left( \frac{2}{\tau^2(\gamma - 1)} P(\rho + \tau \varrho) + (\rho + \tau \varrho) u^2 + H^2 \right) \, dx + \mu \int_{\Omega^L} |\nabla u|^2 \, dx \\
+ 2 \int_{\Omega^L} ((\mu + \lambda)|\nabla u| + \nu|\nabla H|^2) \, dx + \frac{\epsilon}{\gamma - 1} \int_{\Omega^L} P''(\rho + \tau \varrho)|\nabla \varrho|^2 \, dx \\
\leq C \tau \epsilon \|\nabla u\|_{L^2}^3 \|\Delta u\|_{L^2} + C \tau \|f^L\|_{L^2}^2,
\]

where \( C \) is a constant independent of \( L \) and \( \epsilon \).

**Proof.** By multiplying (11)_2 with \( u \) and integrating over \( \Omega^L \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} (\rho + \tau \varrho) u^2 \, dx + \int_{\Omega^L} (\mu |\nabla u|^2 + (\mu + \lambda)|\nabla u|^2) \, dx \\
+ \int_{\Omega^L} P'(\rho + \tau \varrho) \nabla \varrho \cdot u \, dx \\
= \frac{\tau \epsilon}{2} \int_{\Omega^L} u^2 \Delta \rho \, dx + \tau \int_{\Omega^L} (\nabla \times H) \times H \cdot u \, dx + \tau \int_{\Omega^L} (\rho + \tau \varrho) f^L u \, dx \\
= \frac{\tau \epsilon}{2} \int_{\Omega^L} u^2 \Delta \rho \, dx - \tau \int_{\Omega^L} (H^T \nabla u H + \frac{1}{2} \nabla (H^2) \cdot u) \, dx + \tau \int_{\Omega^L} (\rho + \tau \varrho) f^L u \, dx,
\]

where we have used the (11)_1 and the periodic boundary conditions.

Then multiplying (11)_3 by \( H \), integrating over \( \Omega^L \), we can arrive at

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} H^2 \, dx + \nu \int_{\Omega^L} |\nabla H|^2 \, dx \\
= \tau \int_{\Omega^L} \nabla \times (u \times H) \cdot H \, dx \\
= \tau \int_{\Omega^L} \left( H^T \nabla u H + \frac{1}{2} \nabla (H^2) \cdot u \right) \, dx.
\]

Similarly, multiplying (11)_1 by \( P'(\rho + \tau \varrho) \) and integrating over \( \Omega^L \) by part to conclude that

\[
\frac{1}{\tau} \frac{d}{dt} \int_{\Omega^L} P(\rho + \tau \varrho) \, dx + \epsilon \tau \int_{\Omega^L} P''(\rho + \tau \varrho)|\nabla \varrho|^2 \, dx
\]
and Young inequalities that

\[ \sum_{\text{above identities}} \text{we derive from Lemma 2.1, Lemma 2.2, the Hölder and Young inequalities that} \]

\[ \begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} & \left( \frac{2}{\tau^2(\gamma - 1)} P(\bar{\rho} + \tau \bar{g}) + (\bar{\rho} + \tau \bar{g}) u^2 + H^2 \right) \, dx + \mu \int_{\Omega^L} |\nabla u|^2 \\
+ & \int_{\Omega^L} (|\mu + \lambda| \div u^2 + \nu |\nabla H|^2) \, dx + \frac{\epsilon}{\gamma - 1} \int_{\Omega^L} P''(\bar{\rho} + \tau \bar{g}) |\nabla \bar{g}|^2 \, dx \\
= & \frac{\epsilon}{\tau} \int_{\Omega^L} \nabla u^2 \nabla \bar{g} \, dx + \tau \int_{\Omega^L} (\bar{\rho} + \tau \bar{g}) f^L u \, dx \\
\leq & \frac{\epsilon}{2(\gamma - 1)} \int_{\Omega^L} P''(\bar{\rho} + \tau \bar{g}) |\nabla \bar{g}|^2 \, dx + C \tau \epsilon \|\nabla u\|^2_2 \|\Delta u\|_{L^2} + C \tau \|f^L\|^2_2 \\
+ & \frac{H_t}{2} \|\nabla u\|^2_2. \\
\end{align*} \]

This gives the estimate (12) and completes the proof of the lemma. \( \square \)

**Lemma 3.4.** Under the same conditions in Lemma 3.3, we have

\[ \begin{align*}
\frac{d}{dt} \int_{\Omega^L} & \left( |\mu |\nabla u|^2 + (|\mu + \lambda| \div u^2 + \nu |\nabla H|^2 - \frac{2}{\tau} P(\bar{\rho} + \tau \bar{g}) \div u \right) \, dx \\
+ & \int_{\Omega^L} (|\bar{\rho} + \tau \bar{g}| u^2 + H^2) \, dx \\
\leq & C \|\nabla u\|^2_2 + C \tau \|\nabla u\|^2_2 \|\nabla \bar{g}\|^2_2 + C \tau \|\Delta \bar{g}\|^2_2 + C \tau \|\nabla u\|^2_2 \|\nabla \bar{g}\|^2_2 + C \tau \|f^L\|^2_2, \\
\end{align*} \]

where \( C \) is a constant independent of \( L \) and \( \epsilon \).

**Proof.** Multiplying (11)_(2), (11)_(3) by \( u_t, H_t \) respectively, summing the two resultant equations, and the integration over \( \Omega^L \) yield

\[ \begin{align*}
\int_{\Omega^L} (|\bar{\rho} + \tau \bar{g}| u_t^2 + H_t^2) \, dx + & \frac{1}{\tau} \int_{\Omega^L} (|\mu |\nabla u|^2 + (|\mu + \lambda| \div u^2 + \nu |\nabla H|^2) \, dx \\
+ & \frac{1}{\tau} \int_{\Omega^L} \nabla P(\bar{\rho} + \tau \bar{g}) \cdot u_t \, dx \\
= & -\int_{\Omega^L} \tau (\bar{\rho} + \tau \bar{g})(u \cdot \nabla) u_t \, dx + \tau \int_{\Omega^L} \nabla \times (\nabla \times H) \times H \cdot u_t \, dx \\
+ & \tau \int_{\Omega^L} \nabla \times (u \times H) \cdot H_t \, dx + \tau \int_{\Omega^L} (\bar{\rho} + \tau \bar{g}) f^L u_t \, dx. \\
\end{align*} \]

Then multiplying (11)_(1) by \( P'(\bar{\rho} + \tau \bar{g}) \div u \), integrating it over \( \Omega^L \), we have

\[ \begin{align*}
\frac{1}{\tau} \int_{\Omega^L} P_t(\bar{\rho} + \tau \bar{g}) \div u \, dx - & \epsilon \int_{\Omega^L} P'(\bar{\rho} + \tau \bar{g}) \Delta \bar{g} \div u \, dx \\
= & -\int_{\Omega^L} (|\bar{\rho} + \tau \bar{g}| P'(\bar{\rho} + \tau \bar{g}) \div u^2 + \tau P'(\bar{\rho} + \tau \bar{g}) \div u \nabla \bar{g} \cdot u) \, dx. \\
\end{align*} \]
The above two estimates imply that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div} \, u|^2 + \nu |\nabla H|^2 - \frac{1}{\tau} P(\bar{\rho} + \tau \varrho) \text{div} \, u \right) \, dx
+ \int_{\Omega^L} \left( (\bar{\rho} + \tau \varrho) u_t^2 + H_t^2 \right) \, dx
= \int_{\Omega^L} \left( \gamma P(\bar{\rho} + \tau \varrho) |\text{div} \, u|^2 + \tau P'(\bar{\rho} + \tau \varrho) \text{div} \, u \nabla \varrho \cdot u \right) \, dx
- \epsilon \int_{\Omega^L} P'(\bar{\rho} + \tau \varrho) \Delta \varrho \text{div} \, u \, dx - \int_{\Omega^L} \tau (\bar{\rho} + \tau \varrho)(u \cdot \nabla) u \cdot u_t \, dx
+ \tau \int_{\Omega^L} \left( (\nabla \times H) \times H - u_t + \nabla \times (u \times H) \cdot H + (\bar{\rho} + \tau \varrho) f^L u_t \right) \, dx
\leq C \|\text{div} \, u\|_{L^2}^2 + C \tau \|\nabla \varrho\|_{L^2} \|\nabla \varrho\|_{L^2} + C \epsilon \|\text{div} \, u\|_{L^2} \|\Delta \varrho\|_{L^2}
+ C \tau \|u\|_{L^2}^2 \|\nabla \varrho\|_{L^2} + C \tau \|\nabla H\|_{L^2} \|H\|_{L^2} + C \tau \|\nabla H\|_{L^2} \|u\|_{L^2}
+ \frac{1}{2} \int_{\Omega^L} (\bar{\rho} + \tau \varrho) u_t^2 \, dx + \frac{1}{2} \int_{\Omega^L} H_t^2 \, dx,
\]
which yields (13). This completes the proof of the lemma. \qed

**Lemma 3.5.** Under the same conditions in Lemma 3.3, we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}} |\nabla \varrho|^2 + \bar{\rho} |\nabla u|^2 + |\nabla H|^2 \right) \, dx + \frac{\mu}{2} \int_{\Omega^L} |\Delta u|^2 \, dx
+ \epsilon \int_{\Omega^L} |\Delta \varrho|^2 \, dx
\leq C \|\nabla \varrho\|_{L^2}^2 \|\nabla \varrho\|_{L^2} + C \tau \|\nabla \varrho\|_{H^1}^2 \|u_t\|_{L^2}^2 + C \tau \|\nabla u\|_{H^1}^2 \|\nabla u\|_{L^2}^2
+ C \tau \|\nabla u\|_{H^1}^2 \|\nabla H\|_{L^2}^2 + C \tau \|\nabla u\|_{H^1}^2 \|\nabla H\|_{H^1}^2 + C \tau \|f^L\|_{L^2}^2
\]
where \( C \) is a constant independent of \( L \) and \( \epsilon \).

**Proof.** Multiplying (11) \(_2\), (11) \(_3\) by \( \Delta u, \Delta H \) respectively, and then integrating the resulting equalities over \( \Omega^L \) give that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} (\bar{\rho} |\nabla u|^2 + |\nabla H|^2) \, dx + \int_{\Omega^L} (\mu |\Delta u|^2 + (\mu + \lambda) |\text{div} \, u|^2 + \nu |\nabla H|^2) \, dx
= P'(\bar{\rho}) \int_{\Omega^L} \nabla \varrho \Delta u \, dx + \tau \int_{\Omega^L} \varrho u_t \Delta u \, dx + \int_{\Omega^L} \left( P'(\bar{\rho} + \tau \varrho) - P'(\bar{\rho}) \right) \nabla \varrho \Delta u \, dx
+ \tau \int_{\Omega^L} (\bar{\rho} + \tau \varrho)(u \cdot \nabla) u \Delta u \, dx - \tau \int_{\Omega^L} (\nabla \times H) \times H \cdot \Delta u \, dx
- \tau \int_{\Omega^L} \nabla \times (u \times H) \cdot \Delta H \, dx - \tau \int_{\Omega^L} (\bar{\rho} + \tau \varrho) f^L \Delta u \, dx.
\]
For the first term at right–hand side of the above equality, due to \( \Delta \text{div} \, u = \text{div} \, \Delta u \), it holds that
\[
\int_{\Omega^L} \nabla \varrho \Delta u \, dx = - \int_{\Omega^L} \varrho \Delta \text{div} \, u \, dx = \int_{\Omega^L} \nabla \varrho \nabla \text{div} \, u \, dx.
\]
Then applying $\nabla$ to (11)$_1$, and multiplying the resulted equation by $\nabla \varrho$, one can get that

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} |\nabla \varrho|^2 \, dx + \bar{\rho} \int_{\Omega^L} \nabla \varrho \nabla \varrho \, dx + \tau \int_{\Omega^L} |\Delta \varrho|^2 \, dx + \epsilon \int_{\Omega^L} |\Delta \varrho|^2 \, dx
= -\tau \int_{\Omega^L} \nabla \varrho \nabla \varrho \, dx
= -\tau \int_{\Omega^L} \left( \frac{1}{2} |\nabla \varrho|^2 \, dx + \nabla \varrho \nabla \varrho \right) \, dx.
$$

Plugging it into the first estimate yields

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}} |\nabla \varrho|^2 + \bar{\rho} |\nabla u|^2 + |\nabla H|^2 \right) \, dx + \mu \int_{\Omega^L} |\Delta u|^2 \, dx
+ (\mu + \lambda) \int_{\Omega^L} |\nabla u|^2 \, dx + \nu \int_{\Omega^L} |\nabla H|^2 \, dx + \epsilon \int_{\Omega^L} |\Delta \varrho|^2 \, dx
= -\frac{\tau P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega^L} \left( \frac{1}{2} |\nabla \varrho|^2 \, dx + \nabla \varrho \nabla \varrho + \bar{\rho} \nabla \varrho \nabla \varrho \right) \, dx + \tau \int_{\Omega^L} \varrho_1 \Delta u \, dx
+ \int_{\Omega^L} \left( P'(\bar{\rho}) + \tau \varrho \right) \nabla \varrho \Delta u \, dx + \tau \int_{\Omega^L} (\bar{\rho} + \tau \varrho) (u \cdot \nabla \varrho) u \, dx
- \tau \int_{\Omega^L} (u \times \Delta u) \cdot (u \times H) \cdot \Delta u + \nabla (u \times H) \cdot \Delta H + (\bar{\rho} + \tau \varrho) f^L \Delta u \, dx
\leq C \tau \|\nabla \varrho\|_{L^2} \|\nabla \varrho\|_{L^2} \|\Delta u\|_{L^2} + C \tau \|\nabla \varrho\|_{H^1} \|u_i\|_{L^2} \|\Delta u\|_{L^2}
+ C \tau \|\nabla u\|_{H^1} \|\nabla u\|_{L^2} \|\Delta u\|_{L^2} + C \tau \|\nabla H\|_{H^1} \|\nabla H\|_{L^2} \|\Delta u\|_{L^2}
+ C \tau \|\nabla u\|_{H^1} \|\nabla H\|_{L^2} \|\Delta H\|_{L^2} + C \tau \|\nabla H\|_{H^1} \|\nabla H\|_{L^2} \|\Delta H\|_{L^2}
+ C \frac{\mu}{2} \|\Delta u\|_{L^2}^2 + \frac{\nu}{2} \|\Delta H\|_{L^2}^2,
$$

where we have used the fact that $(P'(\bar{\rho} + \tau \varrho) - P'(\bar{\rho})) \sim \tau \varrho$. Hence, (14) follows from the above inequality immediately. This completes the proof of the lemma. 

**Lemma 3.6.** Under the same conditions in Lemma 3.3, we have

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( \frac{d_0 P'(\bar{\rho})}{\bar{\rho}} \varrho_i^2 + d_0 (\bar{\rho} + \tau \varrho) u_i^2 + d_0 H_i^2 + \frac{P'(\bar{\rho})}{\bar{\rho}} |\Delta \varrho|^2 + \bar{\rho} |\nabla \varrho|^2 \right) \, dx
+ \frac{1}{2} \frac{d}{dt} \int_{\Omega^L} |\Delta H|^2 \, dx + d_0 \int_{\Omega^L} \left( \frac{\mu}{4} \nabla u_i^2 + \frac{\lambda}{2} |\nabla \varrho|^2 \right) \, dx
+ \frac{\mu + \lambda}{2} \int_{\Omega^L} |\Delta u|^2 \, dx + \frac{\lambda}{2} \int_{\Omega^L} |\nabla \Delta u|^2 \, dx
+ d_0 \int_{\Omega^L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}} \varrho_i^2 + \frac{P'(\bar{\rho})}{\bar{\rho}} |\Delta \varrho|^2 + \bar{\rho} |\nabla \varrho|^2 \right) \, dx
\leq C \tau \|\varrho_i\|_{L^2}^2 \|\nabla \varrho\|_{H^1}^2 + C \tau \|\varrho_i\|_{L^2}^4 \|u_i\|_{L^2}^2 + C \tau \|\varrho_i\|_{L^2}^2 \|\nabla u_i\|_{L^2}^2 \|u_i\|_{H^1}^2
+ C \tau \|\varrho_i\|_{L^2}^2 \|\nabla u_i\|_{H^1}^2 + C \tau \|\nabla H_i\|_{H^1}^2 \|u_i\|_{L^2}^2 + C \tau \|\nabla H_i\|_{L^2}^2 \|u_i\|_{L^2}^2
+ C \tau \|\nabla u_i\|_{H^1}^2 \|H_i\|_{L^2}^2 + C \tau \|\nabla \varrho\|_{H^1}^2 \|\nabla u_i\|_{L^2}^2 + C \tau \|\nabla \varrho\|_{L^2}^2 \|\nabla u_i\|_{H^1}^2 \|u_i\|_{L^2}^2 \|u_i\|_{H^1}^2.
\[ + C_\tau \| \Delta u \|_{L^2} ^2 \| \nabla u \|_{H^1} ^2 + C_\tau \| \nabla H \|_{H^1} ^2 \| \Delta H \|_{L^2} ^2 + C_\tau \| \nabla u \|_{H^1} ^2 \| \nabla H \|_{L^2} ^2 \]
\[ + C_\tau \| \nabla \theta \|_{H^1} ^2 \| u \|_{L^2} ^2 + C_{\eta_1} \| \theta \|_{L^2} ^2 + C_{\eta_1} \| \nabla \theta \|_{H^1} ^2 + \eta_1 \| \nabla u \|_{H^1} ^2 + \eta_2 \| u \|_{L^2} ^2 \]
\[ + C_{\eta_2} \| f \|_{L^2} ^2 + C_\tau \| f \|_{L^2} ^2 + C_\tau \| \nabla f \|_{L^2} ^2 , \]

(15)

where \( C, d_0, \eta_1, \eta_2, C_{\eta_1}, C_{\eta_2} \) are constants independent of \( L \) and \( \epsilon \). Moreover, \( d_0 \) can be chosen to be suitably large, \( \eta_1, \eta_2 \) can be chosen to be arbitrarily small. And the constants \( C_{\eta_1}, C_{\eta_2} \) are depending on \( \eta_1, \eta_2 \) respectively.

Proof. Applying \( \partial_t \) to (11) \(_1\) \(-\) (11) \(_3\), and then multiplying the resultant equations by \( \frac{P'(\bar{\rho})}{\bar{\rho}} \| \partial_t \| u \| H_s \) respectively, integrating by part over \( \Omega_L \) to have

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega_L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}} \| \partial_t \| + (\bar{\rho} + \tau \theta) \| u \| ^2 + H_t ^2 \right) dx + \epsilon \frac{P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega_L} | \nabla \partial_t | ^2 dx
\]
\[
+ \int_{\Omega_L} (\mu | \nabla u | ^2 + (\mu + \lambda) | \text{div} \ u | ^2 + \nu | \nabla H | ^2 ) dx
\]
\[
= -\frac{\tau \frac{P'(\bar{\rho})}{\bar{\rho}}}{\rho} \int_{\Omega_L} \left( \| \theta \| ^2 \frac{1}{2} \frac{d}{dt} \| u \| + \nabla \theta \| u \| _{L^2} + \theta \| \nabla u \| _{L^2} \right) dx - \frac{\tau}{2} \int_{\Omega_L} \theta | u \| _{L^2} ^2 dx
\]
\[
+ \int_{\Omega_L} \left( P' \bar{\rho} + \tau \theta \right) | \partial_t \| u \| _{L^2} \right) dx - \tau \int_{\Omega_L} \theta \| u \| _{L^2} \right) dx
\]
\[
- \tau \int_{\Omega_L} (\| u \|_2 \| u \| _{L^2} - \| \nabla \| H | \| u \| _{L^2} \right) dx + \tau \int_{\Omega_L} (\| \nabla \| H | \| H \| _{L^2} \right) dx
\]
\[
+ \tau \int_{\Omega_L} (\| \nabla \| H | \| H \| _{L^2} \right) dx
\]
\[
\leq C_\tau \| \partial_t \| _{L^2} \| u \| _{L^2} + C_\tau \| \partial_t \| _{L^2} \| \nabla \theta \| _{H_1} \| \nabla u \| _{L^2} + C_\tau \| \nabla \theta \| _{H_1} \| \theta \| _{L^2} \| \text{div} \ u \| _{L^2} + C_\tau \| \theta \| _{L^2} \| \nabla \| u \| _{L^2}
\]
\[
+ C_\tau \| \nabla \| H \| _{L^2} \| \nabla \| H \| _{H_1} \| \text{div} \ u \| _{L^2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| u \| _{L^2} + C_\tau \| \nabla \| H \| _{L^2} \| \nabla \| H \| _{H_1} \| H \| _{L^2}
\]
\[
+ C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| \text{div} \ u \| _{L^2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| u \| _{L^2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| u \| _{L^2}
\]
\[
\leq C_\tau \| \partial_t \| _{L^2} \| u \| _{H_2} + C_\tau \| \partial_t \| _{L^2} \| \nabla \theta \| _{H_2} + C_\tau \| \theta \| _{L^2} \| \nabla \theta \| _{H_2} + C_\tau \| \theta \| _{L^2} \| \nabla \theta \| _{H_2}
\]
\[
+ C_\tau \| \nabla \| H \| _{H_2} \| \nabla \| H \| _{H_1} \| \text{div} \ u \| _{L^2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| \text{div} \ u \| _{L^2}
\]
\[
+ C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| \text{div} \ u \| _{L^2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| u \| _{L^2}
\]
\[
+ C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| \text{div} \ u \| _{L^2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| u \| _{L^2}
\]
\[
+ C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| \text{div} \ u \| _{L^2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| u \| _{L^2}
\]
\[
+ C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| \text{div} \ u \| _{L^2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| u \| _{L^2}
\]
\[
+ C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_1} \| \text{div} \ u \| _{L^2}
\]

which implies that

\[
\frac{d}{dt} \int_{\Omega_L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}} \| \partial_t \| + (\bar{\rho} + \tau \theta) \| u \| ^2 + H_t ^2 \right) dx + 2 \epsilon \frac{P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega_L} \| \nabla \partial_t \| ^2 dx
\]
\[
+ \int_{\Omega_L} (\mu | \nabla u | ^2 + (\mu + \lambda) | \text{div} \ u | ^2 + \nu | \nabla H | ^2 ) dx
\]
\[
\leq C_\tau \| u \| _{L^2} \| \text{div} \ u \| _{H_2} + C_\tau \| \theta \| _{L^2} \| \nabla \theta \| _{H_2} + C_\tau \| \theta \| _{L^2} \| \nabla \theta \| _{H_2}
\]
\[
+ C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_2}
\]
\[
+ C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_2} + C_\tau \| \text{div} \ u \| _{L^2} \| \text{div} \ u \| _{H_2}
\]
\[ + C \tau \| \nabla H \|_{L^2}^2 \| H_t \|_{L^2}^2 + C \tau \| \nabla u \|_{H^1}^2 \| H_t \|_{L^2}^2 + C \tau \| f^L \|_{L^2} \| u \|_{L^2} \]

Now, applying the operator \( \Delta \) to (11)_1 and (11)_3, then multiplying the resultant
identities by \( \frac{P'(\bar{\rho})}{\bar{\rho}} \Delta \varrho, \Delta H \) respectively, it holds that

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}} |\Delta \varrho|^2 + |\Delta H|^2 \right) dx + \epsilon \frac{P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega^L} |\nabla \Delta \varrho|^2 dx + \nu \int_{\Omega^L} |\nabla \Delta H|^2 dx
\]

\[
= -\frac{\tau P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega^L} \left( \frac{1}{2} |\Delta \varrho|^2 \text{div } u + \nabla \varrho \Delta u \Delta \varrho + 2 \nabla^2 \varrho \nabla u \Delta \varrho + \varrho \Delta \text{div } u \Delta \varrho \right) dx
\]

\[
- 2 \frac{\tau P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega^L} \nabla \varrho \nabla \text{div } u \Delta \varrho dx - P'(\bar{\rho}) \int_{\Omega^L} \Delta \text{div } u \Delta \varrho dx
\]

\[
- \tau \int_{\Omega^L} \nabla \nabla \times (u \times H) \nabla \Delta H dx.
\]

Then multiply (11)_2 by \( \nabla \Delta \text{div } u \) and integrating it over \( \Omega^L \) lead to

\[
\frac{\dot{\bar{\rho}}}{\bar{\rho}} \frac{d}{dt} \int_{\Omega^L} |\nabla \text{div } u|^2 dx + (2\mu + \lambda) \int_{\Omega^L} |\Delta \text{div } u|^2 dx
\]

\[
= P'(\bar{\rho}) \int_{\Omega^L} \Delta \varrho \Delta \text{div } u dx + \tau \int_{\Omega^L} \varrho \text{div } u \Delta \text{div } u dx + \tau \int_{\Omega^L} \nabla \varrho \nabla u \Delta \text{div } u dx
\]

\[
+ \int_{\Omega^L} \left( P'(\bar{\rho} + \tau \varrho) - P'(\bar{\rho}) \right) \Delta \varrho \Delta \text{div } u dx + \tau \int_{\Omega^L} P''(\bar{\rho} + \tau \varrho) |\nabla \varrho|^2 \Delta \text{div } u dx
\]

\[
+ \tau \int_{\Omega^L} \text{div } ((\bar{\rho} + \tau \varrho) (u \cdot \nabla) u \Delta \text{div } u dx - \tau \int_{\Omega^L} \text{div } ((\nabla \times H) \times H) \Delta \text{div } u dx
\]

Summing up the above two estimates, we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( \frac{P'(\bar{\rho})}{\bar{\rho}} |\Delta \varrho|^2 + \bar{\rho} |\nabla \text{div } u|^2 + |\Delta H|^2 \right) dx + \epsilon \frac{P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega^L} |\nabla \Delta \varrho|^2 dx
\]

\[
+ (2\mu + \lambda) \int_{\Omega^L} |\Delta \text{div } u|^2 dx + \nu \int_{\Omega^L} |\nabla \Delta H|^2 dx
\]

\[
= -\frac{\tau P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega^L} \left( \frac{1}{2} |\Delta \varrho|^2 \text{div } u + \nabla \varrho \Delta u \Delta \varrho + 2 \nabla^2 \varrho \nabla u \Delta \varrho + \varrho \Delta \text{div } u \Delta \varrho \right) dx
\]

\[
- 2 \frac{\tau P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega^L} \nabla \varrho \nabla \text{div } u \Delta \varrho dx + \tau \int_{\Omega^L} \varrho \text{div } u \Delta \text{div } u dx + \tau \int_{\Omega^L} \nabla \varrho \nabla u \Delta \text{div } u dx
\]

\[
+ \int_{\Omega^L} \left( P'(\bar{\rho} + \tau \varrho) - P'(\bar{\rho}) \right) \Delta \varrho \Delta \text{div } u dx + \tau \int_{\Omega^L} P''(\bar{\rho} + \tau \varrho) |\nabla \varrho|^2 \Delta \text{div } u dx
\]

\[
+ \tau \int_{\Omega^L} \text{div } ((\bar{\rho} + \tau \varrho) (u \cdot \nabla) u \Delta \text{div } u dx - \tau \int_{\Omega^L} \text{div } ((\nabla \times H) \times H) \Delta \text{div } u dx
\]

\[
- \tau \int_{\Omega^L} \text{div } ((\bar{\rho} + \tau \varrho) f^L) \Delta \text{div } u dx - \tau \int_{\Omega^L} \nabla \nabla \times (u \times H) \nabla \Delta H dx
\]

\[
\leq C \tau \| (\Delta \varrho \|_{L^2}^2 \| \nabla u \|_{H^1} + \| \nabla \varrho \|_{H^1} \| \Delta u \|_{H^1} \| \Delta \varrho \|_{L^2} + \| \nabla \varrho \|_{H^1} \| \Delta \text{div } u \|_{L^2} \| \Delta \varrho \|_{L^2})
\]

\[
+ C \tau \| \nabla \varrho \|_{H^1} \| \text{div } u \|_{L^2} \| \Delta \text{div } u \|_{L^2} + C \tau \| \nabla \varrho \|_{H^1} \| \nabla u \|_{L^2} \| \nabla u \|_{H^1} \| \Delta \text{div } u \|_{L^2}
\]

\[
+ C \tau \| \nabla \varrho \|_{L^2} \| \Delta \text{div } u \|_{L^2} + C \tau \| \nabla \varrho \|_{H^1} \| \nabla u \|_{L^2} \| \nabla u \|_{H^1} \| \Delta \text{div } u \|_{L^2}
\]
Lemma 3.7. which gives

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega^L} \left( \frac{P(\bar{\rho})}{\rho} |\Delta \varphi|^2 + \rho |\nabla \varphi|^2 + |\Delta H|^2 \right) dx + \epsilon \frac{P'(\bar{\rho})}{\rho} \int_{\Omega^L} |\nabla \varphi|^2 dx \\
+ (\mu + \lambda) \int_{\Omega^L} |\nabla u|^2 dx + \frac{\nu}{2} \int_{\Omega^L} |\nabla \Delta H|^2 dx \leq C \int_{\Omega^L} \left( |\nabla u|^2 + C \nabla \varphi ||\Delta \varphi||^2 \right) dx + C \int_{\Omega^L} |\nabla u|^2 dx \\
+ C \tau ||\Delta u||^2 + C \tau ||\nabla \varphi||^2 + C \tau ||\Delta \varphi||^2 \\
+ C \tau ||\nabla u||^2 + C \tau ||\nabla \varphi||^2 + C \tau ||\Delta \varphi||^2 \\
+ C \tau ||\nabla u||^2 + C \tau ||\nabla \varphi||^2 + C \tau ||\Delta \varphi||^2. \]

(17)

Note that

\[ ||\nabla \Delta u||^2 \leq C ||\nabla u||^2 + ||\text{curl} \Delta u||^2. \]

We apply the operator curl to (11) to have

\[ \text{curl} ((\bar{\rho} + \tau \varphi) u) - \mu \text{curl} u + \tau \text{curl} ((\bar{\rho} + \tau \varphi) (u \cdot \nabla) u) \]

\[ - \tau \text{curl} ((\nabla \times H) \times H) = \tau \text{curl} ((\bar{\rho} + \tau \varphi) f^L). \]

Multiplying the above equation by \text{curl} \Delta u, then the integration over \( \Omega^L \) yields

\[ \mu \int_{\Omega^L} |\text{curl} \Delta u|^2 dx \]

\[ \leq C ||\nabla u||^2 + C \tau ||\nabla \varphi||^2 + C \tau ||\Delta \varphi||^2 + C \tau ||\nabla u||^2 + C \tau ||\nabla \varphi||^2 + C \tau ||\Delta \varphi||^2. \]

(18)

Multiplying (16) by a suitably large constant \( d_0 \) and combining it with (17), (18) yield (15). This completes the proof of the lemma.

Now, we will use the equations (11)\_1, (11)\_2 to give the uniform estimates for \( \varphi_t, \nabla \varphi_t, \nabla \varphi \) and \( \Delta \varphi \). The proof is similar to Lemma 3.5 given in [14].

**Lemma 3.7.** Under the same conditions in Lemma 3.3, we have

\[ \int_{\Omega^L} \varphi_t^2 dx + \frac{d}{dt} \int_{\Omega^L} |\nabla \varphi|^2 dx \leq C ||\text{div} u||^2 + C \tau ||\nabla \varphi||^2 + ||\nabla u||^2, \]

(19)

\[ \int_{\Omega^L} |\nabla \varphi_t|^2 dx + \frac{d}{dt} \int_{\Omega^L} |\Delta \varphi|^2 dx \leq C ||\text{div} u||^2 + C \tau ||\nabla \varphi||^2 + ||\nabla u||^2, \]

(20)
\[ \int_{\Omega} |\nabla \varrho|^2 \, dx \leq C \| u_t \|^2_{L^2} + C \| \Delta u \|^2_{L^2} + C \| \nabla \text{div} u \|^2_{L^2} + C \| \nabla u \|^2_{L^2} \]
\[ + C \| \nabla H \|^2_{L^2} \| \Delta H \|_{L^2} + C \| f \|^2_{L^2}, \]

(21)

\[ \int_{\Omega} |\Delta \varrho|^2 \, dx \leq C \| \text{div} u \|^2_{L^2} + C \| \Delta \text{div} u \|^2_{L^2} + C \| \nabla \varrho \|^2_{L^2} \| \nabla u \|^2_{L^2} \]
\[ + C \| \nabla u \|^2_{L^2} \| \Delta u \|_{L^2} + C \| \nabla \varrho \|^2_{L^2} \| \nabla u \|^2_{L^2}, \]

(22)

where \( C \) is a constant independent of \( L \) and \( \epsilon \).

**Lemma 3.8.** Under the same conditions in Lemma 3.3, we have

\[ \frac{d}{dt} \int_{\Omega} |\nabla H_t|^2 \, dx + \nu \int_{\Omega} |\Delta H_t|^2 \, dx \]
\[ \leq C \| \nabla u_t \|^2_{L^2} \| \nabla H \|^2_{H^1} + C \| \nabla u \|^2_{L^2} \| \nabla H_t \|^2_{L^2}, \]

(23)

and

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \mu |\Delta u|^2 + (\mu + \lambda) |\nabla \text{div} u|^2 + \frac{d_1 P'(\bar{\rho})}{\bar{\rho}} |\Delta \varrho|^2 + d_1 \bar{\rho} |\nabla \text{div} u|^2 \right) \, dx \]
\[ + \frac{d_1}{2} \frac{d}{dt} \int_{\Omega} |\nabla H|^2 \, dx + P'(\bar{\rho}) \frac{d}{dt} \int_{\Omega} \Delta \varrho \, dx + \epsilon d_1 \frac{d}{dt} \int_{\Omega} |\nabla \varrho|^2 \, dx \]
\[ + \frac{\rho}{2} \int_{\Omega} |\nabla u_t|^2 \, dx + d_1 \int_{\Omega} \varrho_t^2 \, dx + \epsilon d_1 \frac{P'(\bar{\rho})}{\bar{\rho}} \int_{\Omega} |\nabla \varrho|^2 \, dx \]
\[ + (\mu + \lambda) d_1 \int_{\Omega} |\Delta u|^2 \, dx + \frac{d_1 \nu}{2} \int_{\Omega} |\nabla \Delta H|^2 \, dx \]
\[ \leq C \| \nabla u \|^2_{L^2} + C \| \nabla \varrho \|^2_{H^1} \| \nabla u \|^2_{H^1} + C \| \nabla \varrho \|^2_{L^2} \| \Delta \varrho \|^2_{L^2} \]
\[ + C \| \nabla \varrho \|^2_{H^1} \| \nabla u \|^2_{L^2} \| \nabla u \|^2_{H^1} + C \| \nabla \varrho \|^2_{L^2} \| \Delta \varrho \|^2_{L^2} \]
\[ + C \| \nabla \varrho \|^2_{H^1} \| \nabla u \|^2_{L^2} + C \| \nabla u \|^2_{H^1} \| \nabla u \|^2_{H^1}, \]

(24)

where \( d_1 \) and \( C \) are constants independent of \( L \) and \( \epsilon \), and \( d_1 \) can be chosen to be suitably large.

**Proof.** Applying \( \partial_t \) to (11)_3, and multiplying the resulting identity by \( \Delta H_t \), we see that

\[ \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla H_t|^2 \, dx + \nu \int_{\Omega} |\Delta H_t|^2 \, dx = -\tau \int_{\Omega} \partial_t \nabla \times (u \times H) \cdot \Delta H_t \, dx \]
\[ \leq C \| u_t \|^2_{L^2} \| \nabla H \|^2_{L^2} + C \| u \|^2_{L^\infty} \| \nabla H \|^2_{L^2} + C \| H_t \|^2_{L^6} \| \nabla u \|^2_{L^3} \]
\[ + C \| H \|^2_{L^6} \| \nabla u_t \|^2_{L^2} + \frac{\nu}{2} \| \Delta H_t \|^2_{L^2} \]

Then it follows from the above inequality that the estimate (23) holds.

On the other hand, multiplying (11)_2 by \( \Delta u_t \) and integrating by parts, we find

\[ \int_{\Omega} (\bar{\rho} + \tau \varrho) |\nabla u_t|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} (\mu |\Delta u|^2 + (\mu + \lambda) |\nabla \text{div} u|^2) \, dx \]
\[ + P'(\bar{\rho}) \frac{d}{dt} \int_{\Omega} \Delta \varrho \, dx \]
3.3. Proof of Proposition 2.1. With the uniform estimates at hand, we can apply the topological degree theory to prove the existence of time period solutions of Proposition 1. To this end, we first need to combine the above uniform estimates together.

Proof of Proposition 1. To prove the existence of a solution \((\rho, u, H) \in S_{b_0}^L\) in (3) is equivalent to solving the equation

\[
U - \mathcal{F}(U, 1) = 0, \quad U = (\rho, u, H) \in S_{b_0}^L.
\]

That is, by the topological degree theory, to show

\[
\deg \left( I - \mathcal{F}(\cdot, 1), B_{b_0}(0), 0 \right) \neq 0,
\]

where \(B_{b_0}(0)\) is a ball of radius \(b_0\) centered at the origin in \(S^L\). To do this, we may show that there exists \(\delta_0 > 0\) such that

\[
(I - \mathcal{F}(\cdot, \tau))(\partial B_{b_0}(0)) \neq 0, \text{ for any } \tau \in [0, 1]
\]

by the topological degree theory.

By the fact that \(\|\varrho\|_{L^{\infty}} \leq C\|\nabla \varrho\|_{H_1} \leq \frac{\bar{\rho}}{2}\), if \(\delta_0\) is small enough. We can choose \(d_2, d_3\) suitably large, then consider \(d_2d_3 \times (12) + d_3 \times (13) + d_2d_3 \times (14) + d_4 \times (15) + d_3 \times (19) + (20) + (21) + (22) + (23)\), the integration from 0 to \(T\) yields

\[
\begin{align*}
&\frac{d_2d_3}{4} \int_0^T \int_{\Omega^L} \left( \mu|\nabla u|^2 + (\mu + \lambda)|\nabla u|^2 + 2\nu|\nabla H|^2 + 2(\mu + \lambda)|\nabla \nabla u|^2 \right) \, dx \, dt \\
&+ \int_0^T \int_{\Omega^L} \left( \frac{\mu d_2d_3}{2} \Delta u^2 + \frac{d_2d_3\nu}{4} |\Delta H|^2 + \frac{\mu + \lambda}{4} |\nabla \nabla u|^2 \right) \, dx \, dt \\
&+ \frac{d_3}{2} \int_0^T \int_{\Omega^L} \left( \mu|\nabla u|^2 + \nu|\nabla \Delta H|^2 + \frac{\bar{\rho}}{2} u_1^2 + H_1^2 \right) \, dx \, dt \\
&+ \frac{d_0d_3}{4} \int_0^T \int_{\Omega^L} \left( \mu|\nabla u|^2 + 2(\mu + \lambda)|\nabla \nabla u|^2 + 2\nu|\nabla H|^2 \right) \, dx \, dt \\
&+ \frac{\nu}{2} \int_0^T \int_{\Omega^L} |\Delta H|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_{\Omega^L} \left( d_3 \rho_t^2 + |\nabla \varrho_t|^2 + |\nabla \varrho|^2 + |\Delta \varrho|^2 \right) \, dx \, dt \\
&+ \frac{e_3d_3P'(\bar{\rho})}{\bar{\rho}} \int_0^T \int_{\Omega^L} \left( d_0|\nabla \varrho_t|^2 + |\nabla \Delta \varrho|^2 \right) \, dx \, dt
\end{align*}
\]
Thus, there exists a time \( t' \in (0, T) \) such that

\[
\begin{align*}
\int_{\Omega_t} & \left( \frac{\partial^2 q}{\partial t^2} + |\nabla q_t|^2 + u_t^2 + |\nabla u_t|^2 + |\Delta H_t|^2 + |\Delta \rho|^2 \right) (x, t') \, dx \\
+ \int_{\Omega_t} & \left( |\nabla u_t|^2 + |\Delta u_t|^2 + |\nabla | \Delta \rho ||^2 \right) (x, t') \, dx \\
+ \int_{\Omega_t} & \left( |\nabla q_t|^2 + |\Delta \theta|^2 \right) (x, t') \, dx \\
+ \epsilon & \int_{\Omega_t} (|\nabla \theta_t|^2 + |\nabla \Delta \rho|^2) (x, t') \, dx \\
\leq C_1 \tau \delta_0^3 + C_2 \tau \delta_0^3 + C_3 \tau \left( \int_0^T (\|L^2 \|_{L^2}^2 + \|f^2 \|_{H^1}^2) \, dt + \|f^2 \|_{W^{1,1}}^2 \right).
\end{align*}
\]  

\[ (27) \]

In addition, from (14), (16), (17), (23), (24), we derive that

\[
\begin{align*}
\frac{1}{2} & \frac{d}{dt} \int_{\Omega_t} \left( \frac{P(\rho)}{\rho} |\nabla q|^2 + \rho |\nabla u|^2 + |\nabla H|^2 + \frac{P(\rho)}{\rho} \bar{q}^2 + (\rho + \tau \rho) u_t^2 + H_t^2 \right) \, dx \\
+ \frac{1}{2} & \frac{d}{dt} \int_{\Omega_t} \left( \mu |\Delta u|^2 + (\mu + \lambda) |\nabla u|^2 + \frac{d_1 P(\rho)}{\rho} |\nabla \rho|^2 + d_1 \rho |\nabla | \Delta \rho ||^2 \right) \, dx \\
+ \frac{1}{2} & \frac{d}{dt} \int_{\Omega_t} \left( d_1 |\Delta H|^2 + |\nabla H|^2 \right) dx + 2P(\bar{\rho}) |\Delta \rho| \Delta \rho \, dx \\
+ cd_1 & \frac{d}{dt} \int_{\Omega_t} |\nabla \rho|^2 \, dx \\
\leq C_1 \|u_t\|_{L^2}^2 + C_1 |\nabla u_t|^2 \|_{L^2}^2 + C_7 \|\theta_t\|_{L^2}^2 \| \nabla u \|_{H^2} + C_7 |\nabla \theta_t|_{L^2}^2 \|\nabla u\|_{H^2} + C_7 \|\theta_t\|_{L^2}^2 \\
+ C_7 |\nabla u_t|_{L^2}^2 + C_7 |\nabla H_t|_{L^2}^2 + C_7 |\nabla \theta_t|_{L^2}^2 + C_7 |\nabla u|_{H^2}^2 + C_7 |\nabla \rho|_{H^2}^2 + C_7 |\nabla \theta|_{H^2}^2 \\
+ C_7 |\nabla \rho|_{H^2}^2 + C_7 |\nabla u_t|_{L^2}^2 + C_7 |\nabla H_t|_{L^2}^2 + C_7 |\nabla \theta_t|_{L^2}^2 + C_7 |\nabla u_t|_{L^2}^2 \\
+ C_7 |\nabla u_t|_{H^2}^2 + C_7 |\nabla H_t|_{H^2}^2 + C_7 |\nabla \theta_t|_{H^2}^2 + C_7 |\nabla u|_{H^2}^2 + C_7 |\nabla H|_{H^2}^2 + C_7 |\nabla \theta|_{H^2}^2 .
\end{align*}
\]

\[ (29) \]
By the fact that
\[ 2P'(\bar{\rho}) \int_{\Omega_t} \Delta \varrho \, \text{div} \, u \, dx \leq \frac{\bar{\rho}}{2} \int_{\Omega_t} |\nabla u|^2 \, dx + \frac{d_1 P'(\bar{\rho})}{2\rho} \int_{\Omega_t} |\Delta \varrho|^2 \, dx, \]
we have from integrating (29) from \( t' \) to \( t \) for \( t \in [t', t' + T] \) that
\[
\sup_{0 \leq t \leq T} \int_{\Omega_t} (|\nabla \varrho|^2 + |\nabla u|^2 + |\nabla H|^2 + |\Delta u|^2 + |\nabla \text{div} \, u|^2 + |\Delta \varrho|^2 + |\Delta H|^2) \, dx
\]
\[
+ \sup_{0 \leq t \leq T} \int_{\Omega_t} (\varrho_t^2 + u_t^2 + H_t^2 + |\nabla H_t|^2) \, dx
\]
\[
\leq \int_{\Omega_t} (|\nabla \varrho|^2 + |\nabla u|^2 + |\nabla H|^2 + |\Delta u|^2 + |\nabla \text{div} \, u|^2 + |\Delta \varrho|^2 + |\Delta H|^2) \, dx
\]
\[
+ \int_{\Omega_t} (\varrho_t^2 + u_t^2 + H_t^2 + |\nabla H_t|^2) \, dx + C \sup_{0 \leq t \leq T} \left( \|\nabla u\|_{L^2}^2 + \int_0^T \|\varrho_t\|_{L^2}^2 \, dt \right)
\]
\[
+ C \sup_{0 \leq t \leq T} \left( \|H_t\|_{L^2}^2 + \int_0^T \|H_t\|_{H^1}^2 \, dt \right) + C \sup_{0 \leq t \leq T} \left( \|\nabla u\|_{H^1}^2 + \|\varrho_t\|_{L^2}^2 \right)
\]
\[
+ C \sup_{0 \leq t \leq T} \left( \|\nabla \varrho\|_{L^2}^2 + \|\Delta H\|_{H^1}^2 \right) \int_0^T \|\nabla u_t\|_{L^2}^2 \, dt + C \int_0^T \|f^L\|_{W^{1,1}}^2 \, dt
\]
\[
+ C \sup_{0 \leq t \leq T} \left( \|u_t\|_{L^2}^2 + \|\varrho_t\|_{L^2}^2 + \|\nabla \varrho\|_{H^1}^2 \right) \int_0^T \|u_t\|_{L^2}^2 \, dt + C \int_0^T \|f^L\|_{H^1}^2 \, dt
\]
\[
\leq C_4 \tau \delta_4 + C_5 \tau \delta_5 + C_6 \tau \left( \int_0^T (\|f^L\|_{L^2}^2 + \|f^L\|_{H^1}^2) \, dt + \|f^L\|_{W^{1,1}}^2 \right).
\]

This together with (27) yields
\[
\sup_{0 \leq t \leq T} \int_{\Omega_t} (|\nabla \varrho|^2 + |\nabla u|^2 + |\nabla H|^2 + |\Delta u|^2 + |\nabla \text{div} \, u|^2 + |\Delta \varrho|^2 + |\Delta H|^2) \, dx
\]
\[
+ \sup_{0 \leq t \leq T} \int_{\Omega_t} (\varrho_t^2 + u_t^2 + H_t^2 + |\nabla H_t|^2) \, dx + \int_0^T \int_{\Omega_t} (|\nabla u|^2 + |\text{div} \, u|^2) \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega_t} (|\nabla H|^2 + |\Delta u|^2 + |\nabla \text{div} \, u|^2 + |\Delta H|^2 + |\Delta \text{div} \, u|^2 + |\text{curl} \, \Delta u|^2) \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega_t} (|\nabla \Delta H|^2 + u_t^2 + H_t^2 + |\nabla u_t|^2 + |\text{div} \, u_t|^2 + |\nabla H_t|^2 + |\Delta H_t|^2) \, dx \, dt
\]
\[
+ \int_0^T \int_{\Omega_t} (\varrho_t^2 + |\nabla \varrho_t|^2 + |\Delta \varrho|^2 + |\Delta \varrho|^2 + \epsilon|\nabla \varrho_t|^2 + \epsilon|\Delta \varrho|^2) \, dx \, dt
\]
\[
\leq C_7 \tau \delta_4 + C_8 \tau \delta_5 + C_9 \tau \left( \int_0^T (\|f^L\|_{L^2}^2 + \|f^L\|_{H^1}^2) \, dt + \|f^L\|_{W^{1,1}}^2 \right).
\]
In view of Lemma 2.2 that \( \|(\rho, u, H)\|_{L^6} \leq C\|\nabla(\rho, u, H)\|_{L^2}. \) Hence, when \( \delta_0 \) and \( C\tau \int_0^T (\|f^L\|_{L^6}^2 + \|f^L\|_{H^1}^4) \, dt + \|f^L\|_{W^{1,1}}^2 \) are suitably small, we immediately obtain
\[
\|(\rho, u, H)\|^2 + \epsilon \int_0^T \int_{\Omega^L} (|\nabla \rho|^2 + |\nabla \Delta \rho|^2) \, dx \, dt \\
\leq C_1 \tau \delta_0^4 + C_2 \tau \delta_0^6 + \tilde{C}_3 \tau \left( \int_0^T (\|f^L\|_{L^6}^2 + \|f^L\|_{H^1}^4) \, dt + \|f^L\|_{W^{1,1}}^2 \right)
\] (32)
\[
\leq \frac{1}{2} \delta_0^2.
\]
Thus, (26) holds. Since \( \mathcal{F}(\cdot, 0) = 0 \), then
\[
\deg \left( I - \mathcal{F}(\cdot, 1), \tilde{B}_{\delta_0}(0), 0 \right) = \deg \left( I - \mathcal{F}(\cdot, 0), \tilde{B}_{\delta_0}(0), 0 \right) = \deg \left( I, \tilde{B}_{\delta_0}(0), 0 \right) = 1.
\]
Consequently, we have proved (25) which implies the problem (3) admits a solution \((\rho, u, H) \in S_{\delta_0}^L\). This completes the proof of Proposition 1.

4. Existence in \( \mathbb{R}^3 \). Based on Proposition 1, one can show the existence of time periodic solutions stated in Theorem 1.1 by a limiting process.

Proof of Theorem 1.1. First, we denote \((\rho^L, u^L, H^L)\) be the solution of the regularized problem (3). By Sobolev imbedding theorem, we see that \((\rho^L, u^L, H^L) \in C^{\alpha, \frac{\alpha}{2}}((0, T) \times \Omega^L)\), and
\[
[\rho^L, u^L, H^L]_{\alpha, \frac{\alpha}{2}} \leq C\delta_0.
\]
Now, let \( \epsilon \to 0 \), and then let \( L \to \infty \), for any fixed \( \ell > 0 \), there exists a subsequence \((\rho_n, u_n, H_n)_{n=1}^\infty\) and \((\rho, u, H) \in S_{\delta_0}^L\), such that
\[
(\rho_n, u_n, H_n) \to (\rho, u, H) \quad \text{uniformly in } \Omega^\ell;
\]
\[
(\rho_n, u_n, H_n) \to (\rho, u, H) \quad \text{strongly in } L^2((0, T); L^6(\Omega^\ell));
\]
\[
(\rho_{n\ell}, u_{n\ell}) \to (\rho_t, u_t) \quad \text{in } L^\infty((0, T); L^2(\Omega^\ell));
\]
\[
H_{n\ell} \to H_t \quad \text{in } L^\infty((0, T); H^1(\Omega^\ell));
\]
\[
(\nabla \rho_n, \nabla u_n, \nabla H_n) \to (\nabla \rho, \nabla u, \nabla H) \quad \text{in } L^\infty((0, T); H^1(\Omega^\ell));
\]
\[
(\rho_{n\ell}, u_{n\ell}) \to (\rho_t, u_t) \quad \text{in } L^2((0, T); H^1(\Omega^\ell));
\]
\[
H_{n\ell} \to H_t \quad \text{in } L^2((0, T); H^2(\Omega^\ell));
\]
\[
(\nabla \rho_n, \nabla H_n) \to (\nabla u, \nabla H) \quad \text{in } L^2((0, T); H^2(\Omega^\ell)).
\]

On the other hand, integrating (29) from \( t \) to \( t + \eta \) and then integrating the resulting inequality from 0 to \( T \) over \( t \) to obtain
\[
\int_0^T \int_{\Omega^L} (|\nabla \rho|^2 + |\nabla u|^2 + |\nabla H|^2 + \rho_t^2 + u_t^2 + H_t^2 + \nabla H_t + |\Delta \rho|^2) \, (x, t + \eta) \, dx \, dt \\
+ \int_0^T \int_{\Omega^L} (|\Delta u|^2 + |\nabla \Delta u|^2 + |\Delta H|^2) \, (x, t + \eta) \, dx \, dt \\
- \int_0^T \int_{\Omega^L} (|\nabla \rho|^2 + |\nabla u|^2 + |\nabla H|^2 + \rho_t^2 + u_t^2 + H_t^2 + \nabla H_t + |\Delta \rho|^2) \, (x, t) \, dx \, dt \\
- \int_0^T \int_{\Omega^L} (|\Delta u|^2 + |\nabla \Delta u|^2 + |\Delta H|^2) \, (x, t) \, dx \, dt \\
\leq C\eta,
\]
where \( \eta \) is a constant suitably small and \( C \) is a constant independent of \( L \). Thus, we have

\[
(\varrho_{nt}, u_{nt}) \to (\varrho, u) \quad \text{strongly in } L^2((0, T); L^2(\Omega^L));
\]

\[
H_{nt} \to H_t \quad \text{strongly in } L^2((0, T); H^1(\Omega^L));
\]

\[
\nabla \varrho_{nt} \to \nabla \varrho \quad \text{strongly in } L^2((0, T); L^2(\Omega^L));
\]

\[
(\nabla u_{nt}, \nabla H_{nt}) \to (\nabla u, \nabla H) \quad \text{strongly in } L^2((0, T); H^1(\Omega^L)).
\]

Now, choosing a sequence \( L_n \) with \( L_n \to +\infty \) as \( n \to \infty \), let \( \{(\varrho_{n}^{k}, u_{n}^{k}, H_{n}^{k})\} \) be the convergent function sequence in \( \Omega^{L_{k}} \) given in the above sense. Then, let \( \{(\varrho_{n}^{k+1}, u_{n}^{k+1}, H_{n}^{k+1})\} \) be a subsequence of \( \{(\varrho_{n}^{k}, u_{n}^{k}, H_{n}^{k})\} \), which convergence in \( \Omega^{L_{k+1}}, (k = 1, 2, \ldots, n, \ldots) \). Repeating the argument as follows:

\[
(\varrho_{1}^{1}, u_{1}^{1}, H_{1}^{1}) \quad (\varrho_{2}^{1}, u_{2}^{1}, H_{2}^{1}) \quad \ldots \quad (\varrho_{n}^{1}, u_{n}^{1}, H_{n}^{1}) \quad \text{converges in } \Omega^{L_{1}}
\]

\[
(\varrho_{1}^{2}, u_{1}^{2}, H_{1}^{2}) \quad (\varrho_{2}^{2}, u_{2}^{2}, H_{2}^{2}) \quad \ldots \quad (\varrho_{n}^{2}, u_{n}^{2}, H_{n}^{2}) \quad \text{converges in } \Omega^{L_{2}}
\]

\[
\vdots \quad \vdots \quad \ldots \quad \vdots
\]

\[
(\varrho_{1}^{n}, u_{1}^{n}, H_{1}^{n}) \quad (\varrho_{2}^{n}, u_{2}^{n}, H_{2}^{n}) \quad \ldots \quad (\varrho_{n}^{n}, u_{n}^{n}, H_{n}^{n}) \quad \text{converges in } \Omega^{L_{n}}
\]

\[
\vdots \quad \vdots \quad \ldots \quad \vdots
\]

Hence, we get a Cantor diagonal subsequence \( \{(\varrho_{n}^{n}, u_{n}^{n}, H_{n}^{n})\} \) which converges to \( (\varrho, u, H) \) in \( \Omega^{L} \) for any \( L > 0 \). By the fact that \( L > 0 \) is arbitrary, we see that \( (\varrho, u, H) \in S_{\delta_{k}} \) is the time periodic solution of (2) in \( \mathbb{R}^{3} \). This completes the proof of Theorem 1.1. \( \square \)

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