Log-aesthetic curves and their relation to fluid flow patterns in terms of streamlines

Mei Seen Wo1, R. U. Gobithaasan1,*, Kenjiro T. Miura2, Kak Choon Loy1, Sadaf Yasmeen1 and Fatimah Noor Harun1

1Faculty of Ocean Engineering Technology and Informatics, Universiti Malaysia Terengganu, 21030 Kuala Nerus, Malaysia and 2Department of Information Science and Technology, Shizuoka University, 3-5-1 Jouhoku, Naka-ku, 432-8561, Hamamatsu, Shizuoka, Japan

*Corresponding author. E-mail: gr@umt.edu.my

Abstract

The log-aesthetic curve (LAC) is a family of aesthetic curves with linear logarithmic curvature graphs (LCGs). It encompasses well-known aesthetic curves such as clothoid, logarithmic spiral, and circle involute. LAC has been playing a pivotal role in aesthetic design. However, its application for functional design is an uncharted territory, e.g. the relationship between LAC and fluid flow patterns may aid in designing better ship hulls and breakwaters. We address this problem by elucidating the relationship between LAC and flow patterns in terms of streamlines at a steady state. We discussed how LAC pathlines form under the influence of pressure gradient via Euler’s equation and how LAC streamlines are formed in a special case. LCG gradient (α) for implicit and explicit functions is derived, and it is proven that the LCG gradient at the inflection points of explicit functions is always 0 when its third derivative is nonzero. Due to the complexity of the parametric representation of LAC, it is almost impossible to derive the general representation of LAC streamlines. We address this by analyzing the streamlines formed by incompressible flow around an airfoil-like obstacle generated with LAC having various shapes, αr = {−20, −5, −1, −0.5, −0.15, 0, 1, 2, 3, 4, 20}, and simulating the streamlines using FreeFem++ reaching a steady state. We found that the LCG gradient of the resultant streamlines is close to that of a clothoid. When the obstacle shape is almost the same as that of a circle (α = 20), the streamlines adjacent to the obstacles have numerous curvature extrema despite nearing steady state. The flow speed variation is the lowest for α = −1.43 and gets higher as α is increased or decreased from α = −1.43.

Keywords: log-aesthetic curves; aesthetic curve; streamlines

List of symbols

\( \vec{F} \) : Force acting on an object
\( m \) : Mass of an object
\( \vec{a} \) : Acceleration of an object
\( \vec{v} \) : The velocity of an object
\( v \) : The volume of fluid element
\( p \) : The pressure of fluid element
\( \rho \) : The density of fluid element
\( g \) : Gravitational acceleration
\( \kappa \) : Curvature function of a curve
\( \vec{v}_x \) : Velocity component of \( \vec{v} \) in the x-direction
\( \vec{v}_y \) : Velocity component of \( \vec{v} \) in the y-direction
\( P \) : The static pressure of fluid traveling on a streamline
\( r \) : The radius of curvature of LAC
\( \alpha \) : LCG gradient of LAC
\( \Lambda \) : The shape parameter of LAC
\( \psi \) : Stream function

Received: 3 February 2020; Revised: 11 July 2020; Accepted: 24 July 2020

© The Author(s) 2020. Published by Oxford University Press on behalf of the Society for Computational Design and Engineering. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted reuse, distribution, and reproduction in any medium, provided the original work is properly cited.

1
M: A constant that describes the rate of issue of the fluid
k: Strength of a vortex
\(\dot{\rho}\): The radius of curvature function of an arbitrary implicit curve equation
s: Speed of an arbitrary implicit curve equation
\(\lambda\): LCG gradient function of an arbitrary explicit curve equation
\(a_i\): The LCG gradients of a collection of LAC objects created
\(\alpha_i\): The approximated LCG gradients of a collection of streamlines simulated around the LAC object
\(i\): LCG gradient value of a LAC object, \(i \in \mathcal{R}\)
\(j\): Index of streamline, \(j = 1, 2, \ldots, m\)
\(\alpha_j\): LCG gradient of a specific \(j\)th streamline simulated around the LAC object with LCG gradient \(i\)
\(q_i\): Inflection points of an explicit curve, \(i \in \mathbb{N}\)

List of nomenclatures

- LAC: Log-aesthetic curve
- LCG: Logarithmic curvature graph
- LDDC: Logarithmic distribution diagram of curvature
- FreeFem++: Software that solves partial differential equations numerically and is based on the finite element method (Hecht, 2012)

1. Introduction

Log-aesthetic curve (LAC), first derived by Miura (2006), is a family of curves deemed highly aesthetic (Levien & Séquin, 2009). LAC is constructed by restricting the slope of its logarithmic curvature graph (LCG), which is the analytical form of the logarithmic distribution diagram of curvature (LDDC), to a constant, \(\alpha\). The construction of LAC is based on the fact that many of the curves of both aesthetic human-made and natural objects have an approximately linear LDDC (Yoshimoto & Harada, 2002). LDDC was introduced and used by Harada, Yoshimoto, and Moriyama (1999) to categorize aesthetic curves used in automobile design. Famous curves such as clothoid (\(\alpha = -1\)), Nielsen’s spiral \(\alpha = 0\), logarithmic spiral \(\alpha = 1\), and circle involute \(\alpha = 2\) are all part of the LAC family (Miura, 2006; Yoshida & Saito, 2006).

One of the significant features of LAC is that its curvature increases or decreases monotonically. This property might aid in the design of ship hulls and breakwaters if the relationship between LAC and hydrodynamics was made clear. Recently, Wo, Gobithaan, and Miura (2014) have shown how the LAC is formed under the influence of the magnetic field and gave a physical insight of LAC in electromagnetism. Thus, this work is the outcome of an exploration of LAC patterns in the realm of hydrodynamics.

Streamlines, pathlines, and streaklines are three important flow fields examined in the flow visualization environment. Streamlines are used as indicators of the instantaneous direction of fluid motion throughout the flow field. A pathline is an actual path traveled by an individual fluid particle over a given time, whereas a streakline is the locus of the fluid particles that passed sequentially through a prescribed point in the flow. However, streamlines, pathlines, and streaklines are identical during steady flow.

We discuss LACs in the form of pathlines and how they are formed under the influence of pressure through Euler’s equation. LAC, in the form of a streamline, is also discussed briefly in this section. The relationship between a special case of LAC streamlines, where \(\alpha = 1\), and vortex-sink flow is clarified in Section 3. In Section 4, we presented the LCG gradient for implicit and explicit equations, which are used to determine if a stream function or implicit equation is aesthetic or otherwise, and proved that the LCG gradient for explicit curves is always 0 except for straight lines and some special cases. In Section 5, we analyzed the streamlines around a LAC obstacle obtained via numerical simulation. Results are discussed in the final section before ending the paper with a conclusion.

2. LAC as a Trajectory of Fluid Motion

The Euler's equation, shown by The Open University (2009), is as follows:

\[
\rho \frac{d\vec{r}}{dt} = -\nabla p + \rho\vec{g}.
\]

and when combined with the continuity equation, equation (1) becomes

\[
\rho \frac{d\vec{r}}{dt} + \rho(\vec{r} \cdot \nabla) \vec{v} = -\nabla p + \rho\vec{g}.
\]

where \(\vec{v}\) is the velocity vector field, \(t\) is the time parameter, \(g = (\frac{\partial g}{\partial y})\) is the gravitational acceleration (which can be a constant of an arbitrary function of \(x, y, \) and \(t\)), \(\rho\) is the density of the fluid element, and \(\nabla p\) is the pressure gradient (The Open University, 2009). For incompressible flow, \(\rho\) is a constant. Equation (2) describes the motion of the fluid particle driven by the pressure gradient developed in it, whereby the magnitude of the accelerations is proportional to the magnitude of the pressure gradients (Mattioli, 2010).

A LAC pathline, following the definition of a velocity vector field given in The Open University (2009), is as follows:

\[
\vec{v} = \left(\begin{array}{c}
\vec{v}_x \\
\vec{v}_y
\end{array}\right) = \left(\begin{array}{c}
\frac{dx}{dt} \\
\frac{dy}{dt}
\end{array}\right) = \left(\begin{array}{c}
r(t) \cos(t) \\
r(t) \sin(t)
\end{array}\right), \quad (3)
\]

where \(r(t) = \sqrt{(a - 1)\Lambda t + 1} \Lambda^2\), otherwise, \(\exp(\Lambda t)\), \(\alpha = 1\).

Integrating \(\frac{\partial p}{\partial x}\) we obtain the equation

\[
p(x, y, t) = \rho \left(\int g_x(x, y, t) \, dx + x r(t) \sin t - x' r(t) \cos t\right) + f(y, t).
\]

From the \(\frac{\partial p}{\partial y}\) of equation (6)

\[
\frac{\partial p}{\partial y} = \rho \left(k(x, y, t) - x(t) \cos t - r(t) \sin t\right) = \frac{\partial f}{\partial y}.
\]

Substituting \(f(y, t)\), which we obtained from integrating equation (8), we obtain the pressure field equation of the fluid flow as

\[
p(x, y, t) = h(t) + \rho \left(G(x, y, t) + x(r(t) \sin t - r'(t) \cos t)
\right) - y(\rho(t) \cos t + r(t) \sin t).
\]
where \( G(x, y, t) = \int g(x, y, t) \, dy \). The pressure field and \( g(x, y, t) \), together, influence the shape of the pathline. In the following examples, the pathline in the form of LAC does not change its shape because the flow velocity vector is preset to be that of a LAC. Nevertheless, we can observe how different combinations of gravitational force and pressure gradient can generate the same LAC pathline. An example of the pressure field (equation 9) of a LAC pathline is shown in Fig. 1. In this example, we set \( \alpha = 1.7, \Lambda = 1, \rho = 1, g = (0^1) \) and \( h(t) = t \). The black dot represents the fluid element at time \( t \), the arrows the streamlines, the dashed curve the LAC pathline \((t \in [0,10])\), and the color contour the pressure field. In the example shown in Fig. 2, we set \( \alpha = 1.7, \Lambda = 1, \rho = 1, g = (y^1) \) and \( h(t) = t \). In both Figs 1 and 2, the direction of the pressure gradient rotates with time. The pressure gradient drives the fluid element from a high-pressure area to a low-pressure area. The rotation and the variation in gravity cause the fluid element to follow the path of a LAC curve.

As there are infinitely many ways of forming LAC pathlines or streamlines, it is hard to generalize what each LAC shape parameter means for all possible scenarios. However, we know that if the pressure is the same as equation (9), LAC pathlines form, and then the shape parameters control the rate of rotation and the magnitude of the pressure gradient that accelerates the fluid particles.

When a steady state is achieved in a fluid flow, the pathlines, streaklines, and streamlines become the same (Munson, Young, Okishii, & Huebsch, 2009). As such, we can deduce that a LAC streamline at steady state has the following property.

Bernoulli’s principle states that

\[
P + \frac{1}{2} \rho v^2 + \rho gh = C, \tag{10}
\]

where \( P \) is the static pressure, \( v \) is the velocity of the fluid traveling on a streamline, \( h \) is the height or elevation of the fluid, and \( C \) is a constant. The components \( \frac{1}{2} \rho v^2 \) and \( \rho gh \) are kinetic energy density and hydrostatic pressure, respectively. Bernoulli’s principle is only applicable when the fluid flow is steady, the fluid density is constant along a streamline, and there is no friction. Assuming the flow is steady, and the streamline is in the form of a LAC

\[
x(t) = x_0 + \int_0^t r(u) \cos u \, du, \tag{11}
\]

\[
y(t) = y_0 + \int_0^t r(u) \sin u \, du. \tag{12}
\]
Log-aesthetic curves and their relation to fluid flow patterns in terms of streamlines

Figure 2: LAC ($\alpha = 1.7$) pathline with streamlines and pressure contours, where $g(x, y, t) = \frac{y}{x}$.

if the flow satisfies Bernoulli’s principle and hydrostatic pressure remains unchanged, the static pressure of the fluid will decrease increasingly rapidly along the LAC streamline of $\alpha < 2$ and increasingly otherwise. LAC streamline of $\alpha < 2$ has a constant acceleration; therefore, the static pressure decrease rate is a constant.

3. Streamline and Vector Fields of LAC of $\alpha = 1$ (Logarithmic Spiral)

Two special cases of LAC streamline of $\alpha = 1$ (also known as the logarithmic spiral) are presented in this section along with their respective velocity vector fields. Here, the fluid flow is assumed to be steady and inviscid (frictionless), and thus the velocity vector field does not change over time. It is shown that the LAC's ($\alpha = 1$) velocity vector field is in the form of a vortex.

3.1 Example 1: Compressible fluid flow

Derivation of the velocity vector field can also be done using the tangential angle parameterized LAC of $\alpha = 1$ as derived by Yoshida and Saito (2006). The equations of the streamline are

$$x(t) = \int_0^t \exp(\Lambda u) \cos \Lambda u \, du = \frac{\exp(\Lambda t) \sin(\Lambda t) + \cos(\Lambda t) - 1}{2\Lambda},$$  \hspace{1cm} (13)

$$y(t) = \int_0^t \exp(\Lambda u) \sin \Lambda u \, du = \frac{\exp(\Lambda t) \sin(\Lambda t) - \cos(\Lambda t) + 1}{2\Lambda}.$$  \hspace{1cm} (14)

Therefore, the field is defined by

$$\frac{d\psi}{dy} = \Lambda (x - y) + 1, \quad \frac{d\psi}{dx} = \Lambda (x + y).$$  \hspace{1cm} (15) \hspace{1cm} (16)

Solving equations (15) and (16), we obtain a set of streamline equations defined by

$$x(t) = \exp(\Lambda t) (c_1 \cos(\Lambda t) - c_2 \sin(\Lambda t)) - \frac{1}{2\Lambda},$$  \hspace{1cm} (17)

$$y(t) = \exp(\Lambda t) (c_1 \sin(\Lambda t) + c_2 \cos(\Lambda t)) + \frac{1}{2\Lambda}.$$  \hspace{1cm} (18)

Figure 3 shows the vector field and streamlines described by equations (15), (16), and (19). The stream function is

$$\psi(x, y) = y - \frac{\Lambda y^2}{2} - \Lambda + \Lambda xy.$$  \hspace{1cm} (19)

Since $\frac{\partial}{\partial x} (\frac{\partial \psi}{\partial y}) + \frac{\partial}{\partial y} (\frac{\partial \psi}{\partial x}) = 2\Lambda \neq 0$, this fluid is compressible.
for a source and negative for a sink. The velocity field is \( \mathbf{v} = \frac{-\lambda}{\Lambda^2} \mathbf{e}_x \). Incompressible flows: a sink, and a vortex flow (White, 2015).

Logarithmic spiral streamlines is the result of superpositioning three incompressible flows: a sink, and a vortex, and \( \Psi \) is the strength of the vortex, and \( M \) is the sum of the stream functions of a sink and a vortex. The curvature of the stream function is determined by

\[
\kappa = \frac{k}{\sqrt{(k^2 + M^2)(x^2 + y^2)}} \tag{23}
\]

Figure 3: Logarithmic spiral velocity vector field and streamlines (LAC \( \alpha = 1, \lambda = 1 \)).

Figure 4: Velocity vector field and streamlines of a “swirling vortex” \( \alpha = 1, k = 2, m = -3 \) have a tornado pattern flow.

3.2 Example 2: Incompressible fluid flow

An incompressible, steady, and inviscid fluid flow that creates logarithmic spiral streamlines is the result of superpositioning two incompressible flows: a sink, and a vortex flow (White, 2015).

The logarithmic spiral stream function

\[
\psi(x, y) = M \tan^{-1} \left( \frac{2}{x} \right) - k \log \left( \sqrt{x^2 + y^2} \right) \tag{20}
\]

is the sum of the stream functions of a sink and a vortex. The constant \( k \) is the strength of the vortex, and \( M \) is a constant that describes the rate of issue of the fluid. The constant \( M \) is positive for a source and negative for a sink. The velocity field is

\[
d\psi = \frac{Mx - ky}{x^2 + y^2}, \tag{21}
\]

\[
d\psi = \frac{kx + My}{x^2 + y^2} \tag{22}
\]

Figure 4 shows the velocity field and streamlines of this phenomenon \( (M = -3, k = 2) \). The curvature of the stream functions is

\[
\kappa = \frac{k}{\sqrt{(k^2 + M^2)(x^2 + y^2)}} \tag{23}
\]

We can see that the constants \( k \) and \( M \) have direct effects on the curvature. Thus, \( k \) and \( M \) determine the value of \( \Lambda \) in the LAC’s logarithmic spiral generated through curve synthesis.

4. LCG Gradient of Implicit and Explicit Equations

4.1 LCG gradient of the implicit equation

The LCG gradient of implicit functions can be obtained by rewriting the LCG gradient, derived by Gobithaasan and Miura (2014), using the chain rule. The functions \( s(t) \) and \( \hat{\rho}(t) \) represent the arc length and radius of curvature, respectively. Setting \( s' = \frac{dA}{dt} \) and \( \hat{\rho} = -\frac{dA}{dt} \), we obtain

\[
s'(x, y) = \sqrt{f_x^2 + f_y^2} \tag{24}
\]

and

\[
\hat{\rho}(x, y) = \frac{f_{xy} f_x^2 + f_{x} (f_{xx} - f_{yy}) f_y - f_x f_{xy} f_{yy}}{f_x^2 + f_y^2} \tag{25}
\]

where \( f \) is an arbitrary function. The derivatives of \( s \) and \( \hat{\rho} \) are also obtained by applying the chain rule. The derivation of LCG of an implicit curve can be done using symbolic computation in software such as Mathematica. However, due to the length and complexity of the LCG gradient function, we omit it here. We were able to obtain the LCG gradient \( (\alpha = 1) \) of the stream function in equation (20) using this formulation. This function will be useful for determining if a stream function or implicit function is aesthetic or otherwise.

4.2 LCG gradient of the explicit equation

We derived the LCG gradient for explicit equations as follows:

\[
\lambda(x) = A/( -3b^2b^2 + (1 + b^2)^2 )^2 \tag{26}
\]

where

\[
A = (-3 + 6b^2)^2 b^4 - 4b (1 + b^2)^2 b^2b^2 - (1 + b^2)^2 b^2b^2 \tag{27}
\]

and \( b \) is an arbitrary explicit curve function of \( x \).

The derivation process is the same as that of the implicit equation. Equation (26) will be used to analyze the LCG and its gradient of the streamlines presented in Section 5. It can be proven in Theorem 1 that all explicit equations that satisfy \( b^{(i)} \neq 0 \) have \( \lambda = -1 \) at their inflection points. This is consistent with the findings presented by Yoshida, Fukuda, and Saito (2010), whereby the LCG gradient near the freeform curves’ inflection points approaches \( -1 \), except for a few special cases.

Theorem 1. Given an explicit curve \( b \), and its inflection point(s) \( q_i, i \in N, \lambda(q_i) = -1 \) if \( b^{(i)} \neq 0 \).
Table 1: Start angle $\theta$ variations for $\alpha_r$.

| $\alpha_r$ | $\theta$ (radians) |
|------------|---------------------|
| -20        | 1.48349             |
| -5         | 1.27237             |
| -1.43      | 0.864188            |
| -1         | 0.732727            |
| -0.5       | 0.47683             |
| -0.15      | 0.206596            |
| 0          | 0.161258            |
| 1          | 0.525628            |
| 2          | 0.829296            |
| 3          | 1.02451             |
| 4          | 1.14656             |
| 20         | 1.48008             |

Proof: Since the curvature at inflection points, $\kappa(q_i) = \frac{\psi'(q_i)}{(1+\psi(q_i))^\frac{3}{2}} = 0$, thus $b'(q_i) = 0$, and $\lambda(q_i) = -1$, $b''(0) \neq 0$. □

5. Numerically Simulated Streamlines Around an Obstacle Built with LAC

This section is dedicated to identifying the types of streamlines created by LAC objects. Let a collection of LAC objects created with LCG gradients, $\alpha_r = \{\alpha_i \in R \mid i \in \mathbb{R}\}$. For each $\alpha_r$, there exists a collection of streamlines of which the approximated values of LCG gradients are denoted as $\alpha_s = \{\alpha_j^i \in R \mid j = 1, 2, \ldots, 20\}$. Hence, $\alpha_i^j$ represents the $j$th streamline of LAC constructed with LAC object having $\alpha_r$ LCG gradient. The average of this collection is denoted as $\alpha_i^j$.

For numerical simulation, we construct LAC objects with $\alpha_r = \{-20, -5, -1, -0.5, -0.15, 0, 1, 2, 3, 4, 20\}$ that have distinct shapes, as we hope to identify the patterns of streamlines, $\alpha_s$, using LCG gradients. The LAC objects are constructed symmetrically, like an airfoil, and set at $0^\circ$ angle of attack, as illustrated in Fig. 5. The airfoil shape is chosen because it can be easily built with LAC. The angle between the tangent at the trailing edge and the $x$-axis is denoted as $\theta$. The $\theta$ values for each $\alpha_i$ are listed in Table 1. The tangent angle at the tip of the leading edge is $\frac{\pi}{2}$. The curvature at the leading edge (facing the inlet) and the length of the object (obstacle in the flow domain) are set the same for all $\alpha_r$, with the exception $\alpha_r = \{-5, -1, -0.5\}$, due to their shape restriction. The obstacles are $C^2$ continuous everywhere except for their trailing edge.

Numerical simulations were carried out in FreeFem++ (Hecht, 2012) by employing a second-order semi-implicit backward difference formula, to obtain velocity vectors of the flow around the obstacle when the incompressible flow achieves steady state. The vector field is then imported into Mathematica, and the streamlines are plotted using ListStreamPlot. The streamlines are extracted from the plot in the form of points and then fitted with a cubic spline. The $\alpha_s$ values and gradient plot of the splines are then analyzed with equation (26). To plot the streamlines, we set the coordinates of the “stream points,” which the streamlines pass through. For the streamlines nearest to the obstacle, we set it 0.2 unit apart (in the $y$-direction) from the highest or lowest point on the obstacle. For the next two streamlines, we set the “stream points” to be 0.5 unit apart from the previous one. The following ones will be 1.0 unit apart from each other. There will be a total of 20 streamlines or “stream points” for each $\alpha_r$. The “stream points” setup is illustrated in Fig. 6. Black dots represent “stream points,” and the blue curves represent streamlines. This is done to ensure that the comparison of streamlines around objects of different $\alpha$ values is fair.

5.1 Domain and computational setups

The domains each consists of a rectangular boundary and an obstacle (LAC object) in it. The top, bottom, and left boundaries are set as $u=1$, $v=0$, and the right boundary is open (boundary condition not set).
have the boundary condition $u = 0$, $v = 1$, whereas the obstacle has $u = 0$, $v = 0$ (no-slip condition). The finite element meshes of the domains have, on average, around 50,000 nodes. The mesh grid size on the boundary is 0.3 unit and 0.1 unit on the obstacle. Figure 7 shows the setup of the numerical simulation. The blue arrows are the boundary condition velocity vectors. Figure 8 shows the domain plots with nodes (gray dots) in the mesh for $\alpha_1 = 0$, $\alpha_0 = -0.5$. Reynolds number is a dimensionless number defined as the ratio of inertial forces to viscous forces within a fluid (White, 2015). The arbitrarily chosen
Log-aesthetic curves and their relation to fluid flow patterns in terms of streamlines

Figure 10: Case $\alpha_{2.15}$: LCG gradient of the second streamline above the obstacle $\alpha_{2.15}$.  

Figure 11: Case $\alpha_{2.0}$.  

Figure 12: Case $\alpha_{1.5}$.  

Figure 13: Case $\alpha_{1.43}$.  

Figure 14: Case $\alpha_{1}$.  

Legend:
- $-1.5$
- $-1.4$
- $-1.3$
- $-1.2$
- $-1.1$
- $-1.0$
- $-0.9$
- $-0.8$
Reynolds number is 30, which makes the kinematic viscosity of the fluid $0.211036 \text{cm}^2/\text{s}$ (assuming 1 unit = 1 m/s), similar to that of olive oil at 54.4 °C (The EngineeringToolBox.com, 2003). Since the obstacle is much shorter for $\alpha_i \leq -0.5$, we reduced the outlet’s endpoint on the $x$-axis to 18 from the original 24 units.

We assume that the flow achieves steady state when the mean square error $e$ of the flow speed is less than $\text{tol} = 0.0005$, $\alpha_r \leq 1$.

5.2 LCG of streamlines around the obstacle

The $\alpha$ of streamlines are mostly values near $-1$, except for places near curvature extrema of the streamline, which occurs around the leading edge. The $\alpha$ values are large negative numbers. Figure 9 shows the LCG gradient graph of two streamlines: $\alpha_{10}$, which is located right below the rectangular grid, and $\alpha_{11}$, which is located immediately on top of the obstacle. Figure 10 shows the LCG gradient graph of the second streamline above.
Log-aesthetic curves and their relation to fluid flow patterns in terms of streamlines

Figure 19: Case: $\alpha_j$

Figure 20: Case: $\alpha_j$

Figure 21: Case: $\alpha_j$

Figure 22: Case: $\alpha_j$

the obstacle of case $\alpha_{0.15}$, which is $\alpha_{0.15}^{12}$. We observe that the largest $\alpha_s$ values occur when the obstacle shape is almost the same as that of a circle. It can also be seen that the $\alpha_s$ gets nearer to $-1$ after the first half of the obstacle. Curve segments that have almost constant LCG gradient but with value further away from $-1$ can be seen in $x \in [5, 12]$ in Fig. 10. Each point on the streamline is colored according to its $\alpha_s$ value, as indicated in Figs 11–22. The lowest class of values is $\alpha_s < -5$, colored in black; these $\alpha_s$ values that tend to $-\infty$ usually occur for curves with curvature extrema. We can also observe $\alpha_s \approx -1$ at places...
where the curvature is smaller or where the streamline is almost a straight line. This makes the streamlines a combination of curve segments that are very similar to clothoid, where some are actual LAC with constant LCG gradient, as shown in Fig. 23. Each LCG gradient graph represents the LCG gradient of a segment of the $j$th streamline.

We derived the streamline points and analyzed the LCG gradient of the streamlines. Table 2 shows the percentage of points on the streamline with $\alpha_s \in [-1.1, -0.9]$. As $\alpha_r$ increases, the number of streamline segments with $\alpha_s \in [-1.1, -0.9]$ decreases, but not for $\alpha_4$ as shown in Table 2. However, it must be noted that for cases $\alpha_r \geq 2$, the criteria for the flow to be assumed having reached steady state were not met. It is indicated by a high number of black dots representing $\alpha_s \rightarrow \mp \infty$.

Figures 24 and 25 show the box whisker chart of the LCG gradient distributions for each case of $\alpha_r$. The dots represent the outliers that have extreme $\alpha_s$ values whereas the star symbol represents the average, $\overline{\alpha_s}$. It is evident from this boxplot that $\overline{\alpha_s}$ cannot be used to represent the distribution of $\alpha_s$ as it is left-skewed with the presence of extreme $\alpha_s$ values. The top and bottom ends of the gray bar represent 25% and 75% percentile, respectively. The white horizontal line in the middle of the gray bar represents the median. It is consistent with the finding that the median LCG gradient remains very near to $-1$ even when the distributions are quite different for each case, indicating the streamlines produced are in the form of clothoid. Furthermore, the 75% quartile value increases as the $\alpha_r$ increases. We show only the range with a significant number of outliers since $\alpha_s \rightarrow \pm \infty$ in line with the fitted cubic curve having curvature extrema.

Table 3 shows the mean LCG gradient $\overline{\alpha_s}$ of streamline points with no LCG gradient data removed, the two lowest data removed, and the four lowest data removed. We observed that the cases $\alpha_{-0.5}$ and $\alpha_{-1}$ have the least $\alpha_s$ variation as it is the clos-
Log-aesthetic curves and their relation to fluid flow patterns in terms of streamlines

Figure 25: Box whisker chart showing $\alpha_j^r$, zoomed in and without extreme values.

Table 3: $\alpha_j^r$ of streamline points.

| $\alpha_j$ | $\bar{\alpha}_j^r$ of original data | $\bar{\alpha}_j^r$ of data with two lowest valued data removed | $\bar{\alpha}_j^r$ of data with four lowest valued data removed |
|------------|--------------------------------------|---------------------------------------------------------------|---------------------------------------------------------------|
| $-20$      | $-3.27806$                           | $-1.31235$                                                   | $-1.15088$                                                   |
| $-5$       | $-5529.49$                           | $-2.06502$                                                   | $-1.54343$                                                   |
| $-1.43$    | $-1319.48$                           | $-2.93921$                                                   | $-1.09713$                                                   |
| $-1$       | $-2.41617$                           | $-1.27463$                                                   | $-1.08009$                                                   |
| $-0.5$     | $-2.29977$                           | $-1.20108$                                                   | $-1.10677$                                                   |
| $-0.15$    | $-2.63759$                           | $-1.31632$                                                   | $-1.1797$                                                    |
| $0$        | $-3.02866$                           | $-2.66839$                                                   | $-1.5782$                                                    |
| $1$        | $-2.52659$                           | $-1.76439$                                                   | $-1.51579$                                                   |
| $2$        | $-286.411$                            | $-8.36347$                                                   | $-2.56668$                                                   |
| $3$        | $-17.8912$                            | $-5.68054$                                                   | $-3.48089$                                                   |
| $4$        | $-45.8257$                            | $-6.93369$                                                   | $-3.31262$                                                   |
| $20$       | $-1268.68$                            | $-5.5712$                                                    | $-3.68533$                                                   |

Table 4: Maximum, mean, and median flow speed.

| $\alpha_j$ | Maximum | Mean  | Median |
|------------|---------|-------|--------|
| $-20$      | 1.179   | 0.8176| 0.9988 |
| $-5$       | 1.181   | 0.8006| 0.9916 |
| $-1.43$    | 1.188   | 0.8277| 0.9949 |
| $-1$       | 1.185   | 0.9945| 0.8246 |
| $-0.5$     | 1.199   | 0.9915| 0.8234 |
| $-0.15$    | 1.206   | 0.9866| 0.8108 |
| $0$        | 1.209   | 0.9709| 0.7874 |
| $1$        | 1.283   | 0.9894| 0.8160 |
| $2$        | 1.355   | 0.9683| 0.7945 |
| $3$        | 1.409   | 0.9832| 0.8366 |
| $4$        | 1.448   | 0.9749| 0.8240 |
| $20$       | 1.580   | 0.9702| 0.8329 |

To the median ($\approx 1.00002$) in the original data and the data with the two lowest values removed. However, cases $\alpha_{-1}$ and $\alpha_{-1.43}$ have the least variation when the four lowest LCG gradient values are removed from the data. Since $\alpha_j$ is not a normal distribution, a relation between $\alpha_j$ and $\alpha_s$ cannot be established. Furthermore, there is no clear pattern that it follows. The $\alpha_j$ values are directly influenced by extreme $\alpha_s$ values, which might explain why we cannot observe any trend across the cases. It is also the reason why $\alpha_{-1.43}$ has such low value in the original LCG gradient data. Nevertheless, it is safe to say that the case $\alpha_{-1}$ has the least variation since its mean is consistently the lowest (nearest to the median) or the second lowest in all three sets of data. The black spots that we see in the streamlines have a distinct pattern, which indicates turbulence due to resistance flow creating a drag effect, thus increasing the time to reach a steady state.
5.3 Speed of flow around the obstacle

Table 4 illustrates the maximum, mean, and median flow speed of all $\alpha_i$. Even though the maximum speed increases as $\alpha_i$ increases, the mean and median stay around 0.9 and 0.8, respectively. Figure 26 shows the flow speed plot for $\alpha_{-1}$. Figure 27 shows the speed distribution for each case of $\alpha_i$. The variation of speed is the highest for $\alpha_{20}$, as expected. However, $\alpha_{-1.43}$ has the least variation instead of $\alpha_{-20}$, the second least being $\alpha = -1$. Although case $\alpha_{-1.43}$ did not have the least variation of LCG gradients in the original data, it has the second least variation after removing the extreme outlier LCG gradient values. Hence, we conjectured that a LAC curve with the shape of a clothoid ($\alpha = -1$) and a LAC curve of $\alpha = -1.43$ may minimize the drag. LAC curve of $\alpha = -0.5$ may have the potential in minimizing the drag too. The uniqueness of clothoid compared to other members of LAC is that clothoid has a linear curvature profile, whereas other family members of LAC are in the form of a monotonic curvature profile. This outcome is similar to the path planning problem tackled in Gobithaasan, Yip, Miura, and Madhavan (2020b) and Gobithaasan, Yip, and Miura (2020a), where clothoid tends to possess least curvature variation energy for total turning angle variation between $[0, \pi/2]$.

6. Conclusions

Through Euler’s equation, we find out how LAC pathlines are formed under the influence of pressure gradient and gravitational pull. In a special case of LAC ($\alpha = 1$) where the stream...
function can be found, the LAC shape parameter \( \alpha \) affects the location of the source of the logarithmic spiral flow and the density of the fluid changes the rate of change of curvature of the flow. We have also derived the LCG gradient for implicit and explicit equations and applied them in the streamline LCG analysis. Due to the parametric LAC equations’ integrals (for cases \( \alpha \neq 1 \)), their vector field cannot be derived analytically. Thus, we employed numerical simulations to analyze the streamlines formed around a LAC obstacle.

It is found that the streamlines have \( \alpha \) values of a clothoid (\( \alpha = -1 \)) for most segments of the streamlines. Clothoids possess a linear curvature profile and tend to minimize curvature variation energy for a segment with a total turning angle of \( \pi/2 \). LAC with \( \alpha = -1.43 \) has the least curvature variation energy for this setup. As the total turning angle varies, so does curvature variation energy; as the total turning angle increases, least curvature variation energy is obtained with optimum \( \alpha \rightarrow -\infty \) (Gobithaasan et al., 2020a).

Even though this is the first step toward the functional analysis of LAC, we believe further work is essential to understand how the physical meaning is related to the aesthetic appeal. As the number of variables increases, the data set tends a point cloud of \( \mathbb{R}^n \) dimension, e.g. the analysis of lift and drag for various LAC designed shapes and the exploration of Reynolds number variations of fluid flow on the LCG of the streamlines. It is recommended that topological data analysis can be handy to handle such a data set (Tierny, 2017).

**Acknowledgements**

This research was supported by Ministry of Education (MOE) through Fundamental Research Grant Scheme (FRGS/1/2016/STG06/UMT/02/3). The authors would also like to thank the anonymous reviewers for their careful reading of our manuscript and insightful comments that helped us to improve the manuscript.

**References**

Gobithaasan, R. U., & Miura, K. T. (2014). Logarithmic curvature graph as a shape interrogation tool. *Applied Mathematical Sciences*, 8(13–16), 755–765. https://doi.org/10.12988/ams.2014.312709.

Gobithaasan, R. U., Yip, S. W., & Miura, K. T. (2020a). An analysis of length, energy and variation energy of log-aesthetic curves. *Submitted for Publication*.

Gobithaasan, R. U., Yip, S. W., Miura, K. T., & Madhavan, S. (2020b). Optimal path smoothing with log-aesthetic curves based on shortest distance, minimum bending energy or curvature variation energy. *Computer-Aided Design & Applications*, 17(3), 639–658. https://doi.org/10.14733/cdadaps.2020.639-658.

Harada, T., Yoshimoto, F., & Moriyama, M. (1999). An aesthetic curve in the field of industrial design. In Proceedings of the IEEE Symposium on Visual Language (pp. 38–47). https://doi.org/10.1109/vl.1999.795873.

Hecht, F. (2012). New development in FreeFem++. *Journal of Numerical Mathematics*, 20(3–4), 251–265. https://doi.org/10.1515/jnum-2012-0013.

Levien, R., & Séquin, C. H. (2009). Interpolating splines: Which is the fairest of them all? *Computer-Aided Design and Applications*, 6(1), 91–102. https://doi.org/10.3722/cadaps.2009.91-102.

Mattioni, F. (2010). The Euler equations. In *Principles of fluid dynamics*. Vol. 1. (pp. 73–78). http://www.fluidynamics.it/capitoli/Eule.pdf.

Miura, K. T. (2006). A general equation of aesthetic curves and its self-affinity. *Computer-Aided Design and Applications*, 3(1–4), 457–464. https://doi.org/10.1016/j.cada.2006.07.004.

Munson, B. R., Young, D. F., Okishii, T. H., & Huebsch, W. W. (2009). *Fundamentals of fluid mechanics* (6th edn). New Jersey, Wiley.

The EngineeringToolBox.com. (2003). Liquids - Kinematic viscosities. The Engineering ToolBox.Com. https://www.engineeringtoolbox.com/kinematic-viscosity-d_397.html.

The Open University. (2009). *Mathematical methods and fluid mechanics*: Block 2: Course MST 326. Milton Keynes, UK, Open University Worldwide. https://www.open.edu/openlearn/openlearn/owc/pluginfile.php/111686/mod_resource/content/3/Kinematics офluids.pdf.

Tierny, J., (2017). Topological Data Analysis for Scientific Visualization. Topological Data Analysis for Scientific Visualization, Paris, Springer.

White, F. M. (2015). *Fluid mechanics*(8th edn). Boston, McGraw-Hill Education.

Wo, M. S., Gobithaasan, R. U., & Miura, K. T. (2014). Log-aesthetic magnetic curves and their application for CAD systems. *Mathematical Problems in Engineering*, Vol. 2014, Article ID: 504610, 16 pages. https://doi.org/10.1155/2014/504610.

Yoshida, N., Fukuda, R., & Saito, T. (2010). Logarithmic curvature and torsion graphs. Lecture Notes in Computer Science (Including Subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), 5862 LNCS (pp. 434–443). https://doi.org/10.1007/978-3-642-11620-9_28.

Yoshida, N., & Saito, T. (2006). Interactive aesthetic curve segments. *Visual Computer*, 22(9–11), 896–905. https://doi.org/10.1007/s00371-006-0076-5.

Yoshimoto, F., & Harada, T. (2002). Analysis of the characteristics of curves in natural and factory products. In *Proceedings of the 2nd IASTED International Conference on Visualization, Imaging and Image Processing* (pp. 276–281).