Time-dependent solution for the manufacturing line with unreliable machine and batched arrivals

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Abstract. Time-dependent queue-size distribution in a finite-buffer manufacturing line with unreliable machine is investigated. Successive jobs arrive in batches (groups) with sizes being generally distributed random variables, and are being processed individually with exponential service times. Applying the approach based on the memory less property of exponential distribution and the total probability law, a system of integral equations for the transient queue-size distribution conditioned by the initial level of buffer saturation is derived. The solution of the corresponding system written for Laplace transforms is found via linear-algebraic approach.

1. Introduction
The in-depth analysis of the operation of a manufacturing line with an unreliable service station at different traffic scenarios requires the knowledge about the behaviour of key stochastic characteristics of the system, like e.g. the queue-size distribution of waiting jobs. It is worth noting that in the literature analytical results obtained for queues with server breakdowns are dedicated mainly to the stationary state of the system (see e.g. [1] and [8]). In the article we study the time-dependent queue-size distribution in a single-machine manufacturing line with failures, modelled by a finite-buffer queuing system with server breakdowns (see e.g. [2-6] for transient queue-size distribution in different queuing models). A typical input stream of individual jobs is generalized by considering batched arrivals occurring according to a compound Poisson process. Jobs are processed individually with exponentially distributed service time. The evolution of the system can be observed during successive working periods followed by repair times. If the arriving job finds the machine busy, joins the buffer queue or, if the buffer is saturated, is being lost. Applying the approach based on the memory less property of exponential distribution and the formula of total probability we derive a system of integral equations for the queue-size distribution conditioned by the initial state of the buffer queue. The solution of the corresponding system written for LTs (=Laplace transforms) of conditional distributions we obtain by using the linear-algebraic approach.
2. Mathematical Description of the Model

Let us take into consideration a finite-buffer queuing model with the input stream of jobs described by a compound Poisson process with intensity $\lambda$. Sizes of successive batches are independent random variables, and the batch consists of exactly $k$ jobs with probability $p_k$, where $\sum_{k=1}^{\infty} p_k = 1$. We assume that jobs are being processed individually, with mean $\mu^{-1}$ of exponential processing time. The maximal number of jobs simultaneously present in the system is bounded by a non-random value $N$, i.e., we have $N - 1$ places in the buffer queue and one place in service station. If the size of the arriving batch of jobs exceeds the number of free places in the buffer, the buffer is being saturated and the remaining jobs are removed (so, we accept the so-called partial batch acceptance strategy (PBAS)).

According to the jobs waiting for service in the buffer queue, the FIFO discipline is used. We assume that the system may contain a number of jobs before the opening at time $t = 0$, and that the machine is working at this time. After exponentially distributed working period, with mean $\gamma^{-1}$, a failure occurs and the repair period is assumed to be generally distributed with a DF (=distribution function) $F(\cdot)$.

At the completion epoch of the repair time the machine renews the processing normally (next working time begins) and so on. It is allowed for a machine failure to occur only when the machine is busy with a processing of a job. Successive interarrival, processing, working and repair times are assumed to be independent random variables.

3. Integral Equations for Transient Conditional Queue-Size Distribution

Introduce the following notation for the transient queue-size distribution of jobs, conditioned by the initial level of buffer saturation:

\[ Q_n(t, m) = P\{X(t) = m \mid X(0) = n\}, \quad 0 \leq m, n \leq N, \quad t > 0, \quad (1) \]

where $X(t)$ denotes the number of jobs present in the system at time $t$. Let us consider, firstly, the case in which the buffer queue is empty before the opening of the system. The following integral equation is then true:

\[ Q_0(t, m) = \lambda \int_0^t e^{-\lambda t} \left[ \sum_{k=1}^{N-1} p_k Q_k(t-x, m) + Q_N(t-x, m) \sum_{k=N}^{\infty} p_k \right] + \delta_{m,0} e^{-\lambda t}, \quad (2) \]

where the notation $\delta_{i,j}$ stands for the Kronecker delta function. Indeed, if the first arrival occurs at time $x < t$ and the batch size equals $k \leq N - 1$, then the system renews the operation at time $x$ with $k$ jobs present and the queue size is measured after time $t - x$ (the first summand on the right side of equation (2)). If the buffer is being saturated simultaneously with the first arrival at time $x < t$, then the system renews the operation with $N$ jobs present (second summand). If the first batch of jobs arrives after $t$ (third summand on the right side of equation (2)), then the only possibility is $m = 0$.

To describe precisely the behavior of the system being non-empty before the opening, we need some auxiliary notation. Introduce the following random events:

- $E_1$ – the first arrival occurs before $t$ and precedes the first departure and failure occurrences;
- $E_2$ – the first departure occurs before $t$ and precedes the first arrival and failure occurrences;
- $E_3$ – the first failure occurs before $t$ and precedes the first arrival and departure epochs;
- $E_4$ – the first arrival, departure and failure occur after time $t$.

Evidently, random events $E_1, E_2, E_3$ and $E_4$ are separable in pairs and moreover

\[ Q_n(t, m) = \sum_{i=1}^{4} Q_n^{(i)}(t, m) \equiv \sum_{i=1}^{4} P((X(t) = m) \cap E_i \mid X(0) = n). \quad (3) \]

Assume now that the system starts working at $t = 0$ with $1 \leq n \leq N - 1$ jobs present in the buffer queue. Due to memoryless property of exponential distributions of interarrival, service and machine working times, moments of arrival, departure and failure occurrences are Markov epochs in the evolution of the system. Basing on this conclusion we can write the following integral equations:
\[ Q_n^{(1)}(t, m) = \lambda \int_0^t e^{- (\lambda + \mu + \gamma)x} \left[ \sum_{k=1}^{N-n} p_k Q_n^{(k)}(t - x, m) + Q_N(t - x, m) \sum_{k=N-n}^{+\infty} p_k \right] dx; \]  
\[ Q_n^{(2)}(t, m) = \mu \int_0^t e^{- (\lambda + \mu + \gamma)x} Q_{n-1}(t - x, m) dx; \]  
\[ Q_n^{(3)}(t, m) = \gamma \int_{x=0}^t e^{- (\lambda + \mu + \gamma)x} \left[ \int_{y=0}^{t-x} \left[ \sum_{k=0}^{N-n-1} \left( \frac{\lambda y}{k!} \right)^k e^{-\lambda y} \sum_{j=1}^{N-n-1} p_j^k \right] Q_n^{(j)}(t - x - y, m) + Q_N(t - x - y, m) \left( \sum_{k=N-n}^{+\infty} \left( \frac{\lambda y}{k!} \right)^k e^{-\lambda y} \sum_{j=N-n}^{+\infty} p_j^k \right) \right] dF(y) + \int_{y=t-x}^{+\infty} I\{ n \leq m \} dF(y) \]  
\[ Q_n^{(4)}(t, m) = \delta_{m,N} e^{- (\lambda + \mu + \gamma)t}; \]

where \( I\{ A \} \) and \( p_j^k \) denote the indicator of a random event \( A \) and the \( j \)-th term of the \( i \)-fold convolution of the sequence \( (p_k) \) with itself, respectively.

Finally, if the buffer is saturated at the opening, we obtain similarly

\[ Q_n^{(1)}(t, m) = \lambda \int_0^t e^{- (\lambda + \mu + \gamma)x} Q_N(t - x, m) dx; \]  
\[ Q_n^{(2)}(t, m) = \mu \int_0^t e^{- (\lambda + \mu + \gamma)x} Q_{n-1}(t - x, m) dx; \]  
\[ Q_n^{(3)}(t, m) = \gamma \int_{x=0}^t e^{- (\lambda + \mu + \gamma)x} \left[ \int_{y=0}^{t-x} Q_n(t - x - y, m) dF(y) + \delta_{m,N} \bar{F}(t - x) \right] dx; \]  
\[ Q_n^{(4)}(t, m) = \delta_{m,N} e^{- (\lambda + \mu + \gamma)t}, \]

where \( \bar{F}(t) = 1 - F(t) \).

Introduce now the LT (=Laplace transform) of the functional \( Q_n(t, m) \) on the argument \( t \) in the following way:

\[ q_n(s, m) = \int_0^{+\infty} e^{-st} Q_n(t, m) dt, \quad 0 \leq m \leq N, \quad Re(s) > 0. \]

Now, from the representations equation (2)-equation (11), we get

\[ q_n(s, m) = \frac{\lambda}{\lambda + s} \left[ \sum_{k=1}^{N-n-1} p_k Q_k(s, m) + Q_N(s, m) \sum_{k=N-n}^{+\infty} p_k \right] + \frac{\delta_{m,0}}{\lambda + s}; \]

\[ q_n(s, m) = b(s) \left[ \sum_{k=1}^{N-n-1} p_k Q_k(s, m) + Q_N(s, m) \sum_{k=N-n}^{+\infty} p_k \right] + \mu q_{n-1}(s, m) + \gamma \left[ \sum_{k=0}^{N-n-1} a_k(s) \sum_{j=0}^{N-n-1} p_j^k \right] q_n^{(j)}(s, m) + \frac{\delta_{m,n}}{\lambda + s} \left[ \sum_{k=N-n}^{+\infty} p_k \right] + \delta_{m,N} \left[ \sum_{k=N-n}^{+\infty} \beta_k(s) \sum_{j=N-n}^{+\infty} p_j^k \right] \]

\[ 1 \leq n \leq N - 1 \]

and
where we utilize the following notation:

\[ a_k(s) = \int_0^\infty e^{-\lambda s}y \frac{(\lambda y)^k}{k!} dF(y); \]

\[ b(s) = (\lambda + \mu + \gamma + s)^{-1} \]

\[ \beta_k(s) = \int_0^\infty e^{-\lambda s}y \frac{(\lambda y)^k}{k!} \tilde{F}(y) dy. \]

To rewrite the equations of the system equation (14) in a specific form, let us define the following functional sequences:

\[ \alpha_0(s) = \mu b(s), \alpha_1(s) = \gamma a_0(s)b(s), \alpha_{k+1}(s) = b(s) \left[ \lambda p_k + \gamma \sum_{i=0}^k p_k^i a_i(s) \right], k \geq 1; \]

\[ \varphi_n(s, m) = -b(s) [q_n(s, m)A_n(s, m) + B_n(s, m)], \]

where

\[ A_n(s, m) = \lambda \sum_{k=0}^N p_k + \gamma \left( \sum_{k=0}^N a_k(s) + \sum_{k=0}^{N-n-1} a_k(s) \sum_{j=N-n}^\infty p_j^{k^*} \right) \]

and

\[ B_n(s, m) = \gamma \left[ \left\lfloor n \leq m \leq N - 1 \right\rfloor \sum_{k=0}^{m-n} p_k^{k^*} \beta_k(s) + \delta_{m,n} \left( \sum_{k=0}^N \beta_k(s) + \sum_{k=0}^{N-n-1} \beta_k(s) \sum_{j=N-n}^\infty p_j^{k^*} \right) \right] + \delta_{m,n}. \]

Applying now equation (19)-equation (22), we can transform the system equation (14) to the following one:

\[ q_n(s, m) = \sum_{k=1}^{N-n-1} \alpha_{k+1}(s)q_{n+k}(s, m) - \varphi_n(s, m), \quad 1 \leq n \leq N - 1. \]

4. Compact-Form Solution for the LT of the Queue-Size Distribution

Let us use in equation (13), equation (15) and equation (23) the following substitutions:

\[ d_n(s, m) = q_{N-n}(s, m), \quad \psi_n(s, m) = \varphi_{N-n}(s), \quad 0 \leq n \leq N. \]

Now, the system of equation (13), equation (15) and equation (23) can be rewritten as follows:

\[ d_0(s, m) \left[ 1 - b(s)(\lambda + \gamma f(s)) \right] = b(s) \left[ \mu d_1(s, m) + \delta_{m,N}(\gamma s^{-1}(1 - f(s)) + 1) \right]; \]

\[ \sum_{k=1}^{N-n-1} \alpha_{k+1}(s) d_{n-k}(s, m) - d_n(s, m) = \psi_n(s, m), \quad 1 \leq n \leq N - 1 \]

and

\[ d_N(s, m) = \frac{\lambda}{\lambda + s} \left[ \sum_{k=1}^{N-1} p_{N-k} d_k(s, m) + d_0(s, m) \sum_{k=N}^\infty p_k \right] + \frac{\delta_{m,0}}{\lambda + s}. \]
In [7] one can find the following lemma:

4.1. Lemma 1
Let \((\alpha_k), k \geq 0,\) and \((\psi_k), k \geq 1,\) be two given sequences, where additionally the condition \(\alpha_0 \neq 0\) is assumed. Define the following infinite-sized system of linear equations with unknowns \(y_1, y_2, \ldots:\)

\[
\sum_{k=1}^{n-1} \alpha_{k+1} y_{n-k} - y_n = \psi_n, \quad n \geq 1. \tag{28}
\]

Successive components of arbitrary solution of (28) can be represented in the form

\[
y_n = CR_n + \sum_{k=1}^{n} R_{n-k} \psi_k, \quad n \geq 1, \tag{29}
\]

where \(C\) is a constant and the sequence \((R_k), k \geq 0,\) is defined recursively by means of two given sequences \((\alpha_k), k \geq 0,\) and \((\psi_k), k \geq 1,\) in the following way:

\[
R_0 = 0, \quad R_1 = \alpha_0^{-1}, \quad R_{k+1} = R_1 \left( R_k - \sum_{i=0}^{k} \alpha_{i+1} R_{k-i} \right), \quad k \geq 1. \tag{30}
\]

Let us note that, since equation (26) has the same form as equation (28), then we can write

\[
d_n(s, m) = C(s, m)R_n(s) + \sum_{k=1}^{n} R_{n-k}(s) \psi_k(s, m), \quad n \geq 1, \tag{31}
\]

where \((R_k(s))\) is given by equation (30) with \((\alpha_k(s))\) given by equation (19). The remaining to do is to find the representation for \(d_0(s, m)\) (occurring also in the definition of \(\psi_k(s, m),\) see equation (20)) and \(C(s, m).\) Substituting \(n = 1\) in equation (31) we obtain \(d_1(s, m) = C(s, m) \alpha_0^{-1}(s).\) Comparing the right side of this representation with equation (25), after eliminating \(d_1(s, m),\) we get \(C(s, m)\) in a function of \(d_0(s, m),\) namely

\[
C(s, m) = \alpha_0(s)\left[ \Gamma(s) d_0(s, m) + \Delta(s, m) \right], \tag{32}
\]

where

\[
\Gamma(s) = (\mu b(s))^{-1} [1 - b(s) (\lambda + \gamma f(s))], \quad \Delta(s, m) = -\mu^{-1} \delta_{mN} (\gamma s^{-1} (1 - f(s)) + 1). \tag{33}
\]

Now, similarly, substituting \(n = N\) into equation (31) and comparing the right side to the right side of equation (27), applying equation (20), equation (24) and equation (33), we eliminate \(d_0(s, m)\) explicitly as follows:

\[
d_0(s, m) = q_N(s, m) = [H_1(s, m) + H_2(s, m)] [H_3(s, m) + H_4(s, m)]^{-1}, \tag{34}
\]

\[
H_1(s, m) = \alpha_0(s) \Delta(s, m) R_N(s) - b(s) \sum_{k=1}^{N} R_{N-k}(s) B_{N-k}(s, m) - \frac{\delta_{mN}}{\lambda + s}, \tag{35}
\]

\[
H_2(s, m) = \frac{\lambda}{\lambda + s} \left[ \sum_{k=1}^{N-1} p_{N-k} \left( b(s) \sum_{i=1}^{k} R_{k-i}(s) B_{N-i}(s, m) - \alpha_0(s) \Delta(s, m) R_k(s) \right) \right]; \tag{36}
\]

\[
H_3(s, m) = \frac{\lambda}{\lambda + s} \left[ \sum_{k=1}^{N} p_{N-k} \left( b(s) \Gamma(s) R_k(s) - b(s) A_{N-k}(s, m) \sum_{i=1}^{k} R_{k-i}(s) \right) \right]; \tag{37}
\]

\[
H_4(s, m) = \frac{\lambda}{\lambda + s} \sum_{k=N}^{\infty} p_k - \alpha_0(s) \Gamma(s) R_N(s) + b(s) \sum_{k=1}^{N} R_{N-k}(s) A_{N-k}(s, m). \tag{38}
\]

Collecting now the formulae equation (20), equation (24), equation (31)-equation (34), we can state the following:
4.2. Theorem 1
The representation for the LT $q_n(s, m)$ of conditional queue-size distribution in the considered finite-buffer manufacturing model with unreliable server is following:

$$q_n(s, m) = \left( a_0(s) \Gamma(s) R_{N-n}(s) - b(s) \sum_{k=1}^{N-n} R_{N-n-k}(s) A_{N-k}(s, m) \right) q_N(s, m) - a_0(s) \Delta(s, m) R_{N-n}(s) + b(s) \sum_{k=1}^{N-n} R_{N-n-k}(s) B_{N-k}(s, m), \quad 1 \leq n \leq N - 1,$$

where the formulae for $b(s), a_0(s), A_k(s, m), B_k(s, m), R_k(s), \Gamma(s), \Delta(s, m)$ and $q_N(s, m)$ are found in equation (17), equation (19), equation (21), equation (22), equation (30), equation (33) and equation (34), respectively.

The result obtained in Theorem 1 may be used in the analysis of the effect of frequently changing traffic parameters of the production line (e.g. the intensity of job arrivals) on the process of accumulation of jobs in the buffer, and on their waiting times. Such an information makes it possible to build a preventive algorithm for redirecting jobs to other machines in the case of risk of buffer overflow or too long delay.

5. Conclusions
In the paper transient queue-size distribution in a finite-buffer model of manufacturing line with unreliable machine and batched job arrivals is studied. Using the approach based on the memoryless property of exponential distributions of interarrival, processing and failure-free times and the total probability law, a system of integral equations for time-dependent queue-size distribution conditioned by the number of jobs present in the system initially is built. The compact-form solution of the corresponding system written for Laplace transforms is derived via linear-algebraic approach.

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