MODULAR SYMMETRY, TWISTED SECTORS AND
FLAVOUR

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We investigate the implications for fermion mass models in heterotic orbifolds of
the modular symmetry mixing twisted states localized at different fixed points. We
show that, unlike in the case of continuous gauge symmetries, the mass eigenstates
do not mix under the symmetry; thus coupling constants in the low-energy theory
are invariant functions of the moduli.

Introduction

Twisted sectors of heterotic orbifolds have long been of phenomenologi-
cal interest, since their massless states are automatically localized at the
fixed points, resulting in effective couplings exponentially suppressed by
the distance between different points. In the field theory limit of large
compactification radii, such nonlocal couplings vanish, but they acquire
nonzero values through nonperturbative effects on the worldsheet involv-
ing stretched strings. Attempts have been made to explain the hierarchy
of fermion masses (which generally requires some Yukawa couplings to be
orders of magnitude smaller than unity) and the CKM charged current
mixing of quarks, by using this suppression.

The usual route to explaining small Yukawa couplings in a natural way is
a (broken) flavour symmetry, so one might ask whether there is a symmetry
associated with this suppression of nonlocal interactions. In fact, the target-



space modular invariance of heterotic strings might play such a role. This
symmetry can be thought of as the extension of T-duality to a \( d \)-dimensional
torus, where \( d \geq 2 \). In the simplest case \( d = 2 \), the path integral for
strings propagating on a background with modulus \( T \sim V + iB \), where \( V \)
is the torus volume in string units and \( B \) is the antisymmetric tensor field
\( B_{MN} \equiv B\epsilon_{MN} \) where \( M, N \) label the torus coordinates), is equivalent to
that on a background \( T' \), where \( T' = (\alpha T - i\beta)/(i\gamma T + \delta) \), \( \alpha, \beta, \gamma, \delta \in \mathbb{Z} \),
\( \alpha\delta - \beta\gamma = 1 \). This \( \text{(P)SL}(2, \mathbb{Z}) \) symmetry is generated by
\( S: T \mapsto 1/T \),
\( T: T \mapsto T + i \). For 6-dimensional orbifolds, the full duality group may be
larger but includes three \( \text{PSL}(2,\mathbb{Z}) \) subgroups acting on diagonal moduli \( T^I, I = 1,\ldots,3 \). Correlation functions involving twisted states, which are related to Yukawa couplings, are then invariant, provided that the states in a given sector have the nontrivial transformation

\[
\sigma_i \mapsto U_{ij} \sigma_j, \quad U^\dagger U = 1
\]

where \( U \) depends on \( \alpha, \beta, \gamma, \) and \( \delta \) but not on \( T \). This discrete non-abelian symmetry relates states with the same quantum numbers, acting as a horizontal symmetry, which may be relevant for the flavour problem.

Previous investigations of mass textures used models with more than three matter generations massless at tree level, and exploited the freedom to pick a subset of the fixed point states to correspond to the MSSM matter fields, under the assumption that the remaining states became unobservably heavy. Physical observables in such a framework are, in general, not invariant functions of \( T \) since MSSM states transform into heavy states under some group elements. However, it can be shown that such behaviour is inconsistent with modular invariance of the mass term in the Lagrangian, since there must be off-diagonal masses in a basis where supposedly “light” and “heavy” states are mixed by the symmetry. One can find a basis where some elements of \( \text{SL}(2,\mathbb{Z}) \) act diagonally, but other elements inevitably mix states within a twisted sector. We find that (except in very special cases) the resulting mass eigenstates are \( \text{SL}(2,\mathbb{Z}) \) invariant, \( T \)-dependent linear combinations of the original fixed point states. Thus, even when the modular symmetry is spontaneously broken, the couplings of the light states are invariant functions of moduli.

**Invariance of mass eigenstates**

Our starting points are the existence of the unitary transformation \( U \) and the invariance of the part of the \( d = 4 \) effective Lagrangian bilinear in matter fields \( C_i \), which will result in mass terms:

\[
-\mathcal{L}_m = \frac{1}{2} C_i C_j y_{ijkl}(T) H_k \prod_m X_m^{a_{im}} + \text{h.c.} \equiv C_i C_j M_{ij}(T, H, X) + \text{h.c.} \quad (2)
\]

where \( i \) runs over complete twisted sectors. Here, \( y_{ijkl} \) are \( T \)-dependent Yukawa couplings, and we include an arbitrary number of Higgses \( H_k \) and

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\(^a\)Up to a overall complex phase within each twisted sector.

\(^b\)The \( C_i \) are left-handed Weyl fermions, with \( M_{ij} = M_{ji} \) a complex symmetric matrix: this notation allows for both Dirac and Majorana masses. Gauge indices are suppressed.
SM singlets \(X_m\) which may acquire v.e.v.’s in the stabilized vacuum; all fields may have nontrivial modular transformation properties.

To determine whether it is possible to separate the states into two sets, light (observable) and heavy, which intermix under \(\text{SL}(2, \mathbb{Z})_T\), we must find the modular transformation of the mass eigenstates, given the matrices \(U\) acting in the fixed point basis. When general off-diagonal mass matrices are considered, such transformations quickly become unwieldy: so instead of the fixed point basis, we start with a basis where the modular transformation that we wish to consider acts as a diagonal matrix of phases: \(\Gamma \in \text{SL}(2, \mathbb{Z})\): \(C_i \mapsto e^{i\zeta} C_i\). This can always be done, because unitary matrices are normal, and does not introduce additional \(T\)-dependence. Then, since \(\mathcal{L}_m\) necessarily transforms into itself, we have in this basis

\[
M_{ij} \mapsto e^{-i(\zeta_i + \zeta_j)} M_{ij}, \quad V_{jp} \mapsto e^{i\zeta_j} V_{jp}
\]

where \(V\) is \(T\), \(H_k\), and \(X_m\)-dependent unitary matrix satisfying \(V^T M V \equiv \text{diag}(m_i(T, H, X)) \). Then the mass eigenstates \(\Psi_p = V_p^j C_j\) are invariant when the \(C_i\), \(T\), \(H_k\) and \(X_m\) are all transformed under \(\Gamma\). If we rewrite the theory in the \(\Psi\) basis, all coupling constants must be invariant functions of \(T\), \(H_k\), and \(X_m\), following from the invariance of the 4d effective Lagrangian.

There might be exceptions to this result if the off-diagonal “light-heavy” mass-terms vanish exactly, or when the modular transformation acts on the \(C_i\) as a permutation. However, the first possibility could only occur at isolated points in moduli space\(^d\), and the second is not realised in any known orbifold.

The case of antigenerations which acquire masses by pairing up with matter generations at energies somewhat below the string scale can be treated by the above method, but one can also think of it in two stages at widely-separated mass scales. First, consider a \(N_g + M\)-by-\(M\)-dimensional mass matrix, where \(M\) is the number of paired (anti)generations, with \(M\) large mass eigenvalues. Then the diagonalising matrices are determined only up to an \(N_g\)-dimensional unitary redefinition of the remaining massless generations. The main result remains: if a modular symmetry mixed massless and heavy fields, then there would necessarily be \(T\)-dependent off-diagonal mass terms, which could not be set to zero except at isolated values of \(T\), hence we could not actually be in the basis of light and heavy fields.

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\(^c\)The invariant functions \(m_i\) may be complex, depending on the structure of \(M\).

\(^d\)As discussed later, off-diagonal terms cannot in general be removed by redefinition.
Discussion

Our result, which generalises that of\cite{6}, may appear unexpected compared to the analogous situation for continuous gauge symmetry, where for example in the case of $SU(2)_W$ the mass eigenstates $t$ and $b$ (neglecting the CKM mixing) are mixed by some group elements and $SU(2)$ appears to be explicitly broken below the top mass. In that case, starting from an arbitrary Higgs v.e.v. of magnitude $v$, we are free to change basis by a global $SU(2)$ redefinition to obtain a diagonal mass matrix (i.e. unitary gauge), and the group acts in an identical way in the new basis. But for $SL(2,\mathbb{Z})_T$, the fixed point basis is special, and the group acts differently if one changes basis. One cannot in general redefine either $T$ or the $C_i$ by a group element to obtain a (block-)diagonal mass matrix: the diagonalisation requires $V$ to be explicitly modulus-dependent, which affects the transformation properties of the mass eigenstates.

The main lesson is that the freedom to assign MSSM fields to particular fixed points, and discard other states, is an illusion. For self-consistency, one should take the light fields to be invariant $T$-dependent combinations of twist states, as happens automatically in most cases. Modular invariance is thus a useful tool for checking the consistency of toy model results. More realistic models with three light generations, in which there is need to decouple extra states, most likely would require more complicated string constructions which would break part or all of the modular symmetry, uniquely determining the states that remain. One might argue that choosing a non-modular invariant set of states from a larger spectrum could mimic this situation, but the Yukawa couplings are unlikely to take the same form in the more realistic case.

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