Rost nilpotence and higher unramified cohomology

H. Anthony Diaz

Abstract
We develop an approach to proving the Rost nilpotence principle involving higher unramified cohomology. We use this to prove the principle for certain varieties of dimension \( \leq 3 \) over a perfect field.

Introduction
Let \( X \) be a smooth projective variety over a field \( k \). Also, let \( \mathcal{M}_k \) be the category of Chow motives over \( k \). The Rost nilpotence principle predicts that for any field extension \( E/k \), the kernel of the extension of scalars map

\[
\text{End}_{\mathcal{M}_k}(\mathcal{h}(X)) \to \text{End}_{\mathcal{M}_E}(\mathcal{h}(X_E))
\]

consists of nilpotent correspondences. Rost first proved this for a smooth projective quadric over a field of characteristic \( \neq 2 \) in [31] (see also [6]). A consequence is that the Chow motive of a smooth quadric can be decomposed as a direct sum of (twisted) motives of anisotropic quadrics, and this played an important role in the proof of the Milnor conjecture by Voevodsky [32].

The Rost nilpotence principle is desirable, as it would imply for instance that Chow motives do not vanish upon passage to field extensions (in this sense, it may be viewed as a torsion analogue of a well-known nilpotence conjecture for rational correspondences [21]). It is conjectured to hold in general and has been proved in several other important cases. Chernousov, Gille and Merkurjev [7] proved that it holds for projective homogeneous varieties. Moreover, using Rost cycle modules [30], Gille showed that the Rost nilpotence principle holds for geometrically rational surfaces [13] and for smooth, projective, geometrically integral surfaces (char \( k = 0 \)) [14]. (Using different methods, he also proved it for smooth projective, geometrically integral threefolds which are birationally isomorphic to toric models [15].)

Another approach to proving Rost nilpotence was developed by Rosenschon and Sawant [28]. Their approach involves étale motivic cohomology groups \( H'^p_c(X,\mathbb{Z}(n)) \) which in many ways are better behaved than the usual motivic cohomology groups (see, for instance, [29]). In particular, for any finite Galois extension \( E/k \) there is a Hochschild-Serre spectral sequence:

\[
E_2^{p,q} = H^p(\text{Gal}(E/k), H'^q_c(X_E,\mathbb{Z}(n))) \Rightarrow H'^{p+q}_c(X,\mathbb{Z}(n))
\]

Moreover, using the triangulated category of étale mixed motives, this spectral sequence is functorial for the action of correspondences after inverting the exponential characteristic of \( k \) (see [28] §3). Using this, they are able to prove that if \( \gamma \in CH^d(X \times X) \) lies in the kernel of \( \mathcal{h} \), then the action of \( \gamma \) on the étale motivic cohomology groups of \( X \) (and its products) is nilpotent. As a consequence, they obtain the Rost nilpotence principle for smooth projective surfaces in
characteristic 0 (and with some more work, birationally ruled threefolds in characteristic 0).

Our goal in this paper will be to explore the relationship between Rost nilpotence and higher unramified cohomology. The usual unramified cohomology groups $H^n_{nr}(X)$ are well-known and since [8] have been quite useful in proving that certain varieties are not rational (or stably rational). The right derived analogues of these groups are the higher unramified cohomology groups. (It’s worth noting that these latter have also been considered as birational invariants; see, for instance, [27] and [20].) The naïve view is that the higher unramified cohomology groups control the extent to which motivic cohomology and étale motivic cohomology fail to coincide. Thus, given the main result of [28], one would expect the Rost nilpotence principle to follow from a statement about correspondences acting on unramified cohomology. In this direction, our main result (Theorem 3.1) shows that a correspondence $\gamma$ that lies in the kernel of $\mathbf{1}$ is nilpotent, provided that the action of $\gamma$ on certain higher unramified cohomology groups is nilpotent. The use of unramified cohomology groups allows one to avoid the issue of $p$-primary torsion in characteristic $p > 0$, an issue which arises in the motivic context because of non-homotopy-invariant phenomena. In particular, since the assumption of Theorem 3.1 is automatically satisfied for surfaces, we are able to extend the Rost nilpotence principle to surfaces over a perfect field. Then, using a Bloch-Srinivas-type argument, we are able to prove the following as a consequence:

**Corollary 0.1** (─Corollary [3.3]. Suppose that $X$ is a smooth projective variety of dimension $\leq 3$ over a perfect field $k$ whose Chow group $CH_0$ is universally supported in dimension $\leq 2$ (in the sense of [3]; see also Definition 3.1). Then, $X$ satisfies the Rost nilpotence principle.

The proof of Theorem 3.1 is rather short and exploits certain functoriality properties of the local-to-global spectral sequence for higher unramified cohomology, as well as that of the Hochschild-Serre spectral sequence for étale cohomology groups. The idea will be to prove that any cycle in the kernel of $\mathbf{1}$ is nilpotent in étale cohomology and then to use the local-to-global spectral sequence to obtain a nilpotence statement in the Chow group. Since the cycle class map is not guaranteed to be injective even in low degree, we make use of a Bockstein-type sequence that overcomes this difficulty. The proofs of the applications that follow Theorem 3.1 are familiar argument in the spirit of [5].

**Acknowledgements**

The author would like to thank Bruno Kahn for taking the time to read several drafts of an earlier paper that inspired this one; his comments were indispensable. The author also thanks Anand Sawant for his interest and Frédéric Déglise for a clarification.

**Conventions and notations**

Throughout, we will assume that all schemes are separated and defined over a field $k$. The term *variety* will be used when the scheme is of finite type, reduced and equidimensional. In the case that $X$ is irreducible, we let $k(X)$ denote the function field and $\eta$ the generic point. For any field extension $L/k$ and $X$ a scheme over $k$, $X_L = X \times_k L$. We also let $\overline{k}$ the algebraic closure and $\overline{X} = X \times_k \overline{k}$. Unless otherwise specified, all Chow groups will have integral coefficients.

The notation $\ell$ will be reserved for a prime number. For an Abelian group $A$ and $n \geq 1$ an integer, $A[n]$ denotes the $n$-torsion of $A$; in particular, for a prime number $\ell$, $A[\ell^\infty]$ denotes the $\ell$-primary torsion of $A$. 


1 Higher unramified cohomology

Throughout this section, consider a smooth variety $X$ over a perfect field $k$. Then, for $n, r \geq 1$, there are the étale sheaves over $X$:

$$\left(\mathbb{Z}/\ell^r\right)_X(n) = \begin{cases} \mu_{\ell^n}^\otimes & \ell \neq \text{char } F \\ \nu_r(n)[-n] & \ell = \text{char } F = p \text{ (if } p > 0) \end{cases}$$

where $\nu_r(n)$ denotes the sheaf of logarithmic Hodge-Witt differentials defined by Illusie [19]. We will abuse notation and use $\mathbb{Z}/\ell^r(n)$ instead of $(\mathbb{Z}/\ell^r)_X(n)$, when there is no danger of ambiguity. Moreover, we let $\mathcal{H}_X^m(\mathbb{Z}/\ell^r(n))$ denote the Zariski sheaf associated to the presheaf $U \mapsto H^m_{\acute{e}t}(U, \mathbb{Z}/\ell^r(n))$. This is the unramified cohomology sheaf.

1.1 Gersten resolution and operations

Now, let $X^{(i)}$ denote the irreducible closed subsets of $X$ of codimension $i$ and $i_x : \text{Spec } k(x) \hookrightarrow X$ be the corresponding inclusion. For $\ell \neq \text{char } k$, the sheaf $\mathcal{H}_X^m(\mathbb{Z}/\ell^r(n))$ fits into the following short exact sequence of sheaves involving Galois cohomology:

$$0 \to \mathcal{H}_X^m(\mathbb{Z}/\ell^r(n)) \to \bigoplus_{x \in X^{(0)}} i_{x*}(H^m(k(x), \mu_{\ell^n}^\otimes)) \xrightarrow{\partial^0} \bigoplus_{x \in X^{(1)}} i_{x*}(H^{m-1}(k(x), \mu_{\ell^n}^\otimes)) \tag{2}$$

where the arrows are the sums of residue maps for Galois cohomology. By the results of [2], [2] extends to the well-known Gersten resolution:

$$\ldots \to \bigoplus_{x \in X^{(i)}} i_{x*}(H^{m-i}(k(x), \mu_{\ell^n}^\otimes)) \xrightarrow{\partial^i} \bigoplus_{x \in X^{(i+1)}} i_{x*}(H^{m-i-1}(k(x), \mu_{\ell^n}^\otimes)) \to \ldots \tag{3}$$

When $\ell = \text{char } k = p > 0$ and $m = n$, Gros and Suwa show (using Gabber’s effacement theorem) that there is a Gersten resolution ([17] Corollaire 1.6):

$$0 \to \mathcal{H}_X^m(\mathbb{Z}/p^n(n)) \to \bigoplus_{x \in X^{(0)}} i_{x*}(\nu_r(n)(k(x))) \xrightarrow{\partial^0} \bigoplus_{x \in X^{(1)}} i_{x*}(\nu_r(n-1)(k(x))) \xrightarrow{\partial^1} \ldots \tag{4}$$

where $\nu_r(n)(k(x))$ is the module of logarithmic Hodge-Witt differentials and the boundary maps are explained in the proof of Lemme 4.11 of op. cit. Denote by $\mathcal{C}_X^m(m, n)$ either of the complexes [3] or [4], where it is understood that $m = n$ in case $\ell = \text{char } k = p > 0$. Then, as a consequence, the corresponding Zariski cohomology groups are computed as:

$$H^\bullet_{\text{Zar}}(X, \mathcal{H}_X^m(\mathbb{Z}/\ell^r(n))) = R^\bullet \Gamma(\mathcal{C}_X^m(m, n))$$

We call these groups the higher unramified cohomology groups.

As in [30] (3.4) and (3.5), one can use the Gersten resolution to define push-forward and pull-back operations. Indeed, for $f : X \to Y$ a proper morphism of relative dimension $d$, there is an induced map of complexes:

$$f_* : R^q \Gamma(\mathcal{C}_X^m(m, n)) \to R^q \Gamma(\mathcal{C}_Y^m(m-d, n-d))$$

induced in the following way. When $\ell \neq \text{char } k$ and $x \in X^{(i)}$, there is the map

$$H^m(k(x), \mu_{\ell^n}^\otimes) \to H^m(k(f(x)), \mu_{\ell^n}^\otimes)$$
is the co-restriction map on Galois cohomology when \([k(x) : k(f(x))] < \infty\) and is 0 otherwise. When \(\ell = \text{char } k = p > 0\), the map

\[\nu_r(n)(k(x)) \to \nu_r(n)(k(f(x)))\]

is the trace map when \([k(x) : k(f(x))] < \infty\) and is 0 otherwise. (The trace map is given explicitly \[12\] p. 313; note that this exists even when \(k(x)/k(f(x))\) is an inseparable extension.) When \(f : X \to Y\) is a flat morphism, there is an induced map of complexes:

\[f_* : R^\ell \Gamma(C^\ell_X(m, n)) \to R^\ell \Gamma(C^\ell_Y(m, n))\]

induced in the following way. For every \(y \in Y^{(i)}\), let \(x\) be an irreducible component of \(f^{-1}(y)\). When \(\ell \neq \text{char } k\), the map is induced by

\[H^m(k(y), \mu_{r^n}^{\otimes n}) \to H^m(k(x), \mu_{r^n}^{\otimes n})\]

which is the restriction map on Galois cohomology. When \(\ell = \text{char } k = p > 0\), the map is induced by the pull-back

\[\nu_r(n)(k(y)) \to \nu_r(n)(k(x))\]

on differential forms.

Finally, we observe that there is a natural product structure:

\[H_X^m(Z/\ell^n(n)) \otimes H_X^{m'}(Z/\ell^n(n')) \to H_X^{m+m'}(Z/\ell^n(n+n'))\]  

This is defined sheaf-theoretically and is induced by the cup product operation in étale cohomology. This induces a product map on the corresponding Zariski cohomology groups:

\[H^p_{\text{Zar}}(X, H^m_X(Z/\ell^n(n))) \otimes H^q_{\text{Zar}}(X, H^{m'}_X(Z/\ell^n(n'))) \to \bigcup H^{p+q}_{\text{Zar}}(X, H^{m+m'}_X(Z/\ell^n(n+n'))}\]

Moreover, there is the Bloch-Quillen formula

\[CH^n(X)/\ell^n \cong H^p_{\text{Zar}}(X, H_X^m(Z/\ell^n(n)))\]

which is well-known when \(\ell \neq \text{char } k\) and is Theoreme 4.13 of \[17\] when \(\ell = \text{char } k = p > 0\). Thus, there is a natural action of the Chow group given by:

\[H^p_{\text{Zar}}(X, H^m_X(Z/\ell^n(n))) \otimes CH^q(X) \to H^{p+q}_{\text{Zar}}(X, H^{m+q}_X(Z/\ell^n(n+q)))\]

There is also an action of correspondences on the higher unramified cohomology groups in the familiar way. Indeed, assume that \(X\) and \(Y\) are both smooth and projective and let

\[\text{Cor}^*(X, Y) = \bigoplus_{i \in \mathbb{Z}} CH^{i+\dim(X)}(X \times Y)\]

be the ring of correspondences, where multiplication is given by the composition operation \(\circ\) (see, for instance, \[11\] Chapter 16). With the above definitions of pull-back, pushforward and \(\cdot\), for any \(\Gamma \in \text{Cor}^q(X, Y)\) we define a corresponding map

\[
\Gamma_* : H^p_{\text{Zar}}(X, H^m_X(Z/\ell^n(n))) \to H^{p+q}_{\text{Zar}}(X, H^{m+q}_X(Z/\ell^n(n+q))), \quad \Gamma_*(\alpha) = \pi_Y^*(\pi_X^\star \alpha \cdot \Gamma)
\]

where \(\pi_X : X \times Y \to X\) and \(\pi_Y : X \times Y \to Y\) are the projection maps. As in \[9\] §9, we have \((\Gamma' \circ \Gamma)_* = \Gamma'_* \circ \Gamma_*\).
1.2 Local-to-global spectral sequence

The unramified cohomology sheaves may also be viewed in the following way. Let $\mathcal{F}$ be an étale sheaf of Abelian groups on $X$ and $\alpha : X^{\text{ét}} \to X^{\text{Zar}}$ be the forgetful functor from the étale site over $X$ to the Zariski site over $X$. Then, there is a Grothendieck spectral sequence:

$$E_2^{p,q} = H^{p}_\text{Zar}(X, R^q\alpha_* \mathcal{F}) \Rightarrow H^{p+q}_\text{ét}(X, \mathcal{F})$$

Moreover, by the usual formalism, $R^q\alpha_* \mathcal{F} = \mathcal{H}^q_X(\mathcal{F})$, the Zariski sheaf associated to the presheaf $U \mapsto H^q_X(U, \mathcal{F})$. In particular, if $\mathcal{F} = \mathbb{Z}/\ell^r(n)$, the Grothendieck spectral sequence becomes the Bloch-Ogus spectral sequence:

$$H^p_\text{Zar}(X, \mathcal{H}^q_X(\mathbb{Z}/\ell^r(n))) \Rightarrow H^{p+q}_\text{ét}(X, \mathbb{Z}/\ell^r(n)) \quad (6)$$

When $\ell \neq \text{char } k$, the Gersten resolution shows that $E_2^{p,q} = 0$ for $p > q$, so there are edge maps:

$$e^{m,n} : H^m_\text{Zar}(X, \mathcal{H}^n_X(\mathbb{Z}/p^r(n))) \to H^{m+n}_\text{ét}(X, \mathbb{Z}/p^r(n))$$

for $m = n - 1, n$. When $m = n$ one may take this to be the definition of the cycle class map:

$$CH^n(X)/\ell^r \to H^{2n}_\text{ét}(X, \mathbb{Z}/\ell^r(n))$$

given the Bloch-Quillen formula. When $\ell = \text{char } k = p > 0$, this spectral sequence degenerates to a long exact sequence ([17] (1.14)):

$$0 \to H^1_\text{Zar}(X, \mathcal{H}^n_X(\mathbb{Z}/p^r(n))) \to H^{n+1}_\text{ét}(X, \mathbb{Z}/p^r(n)) \to H^0_\text{Zar}(X, \mathcal{H}^{n+1}_X(\mathbb{Z}/p^r(n))) \to \ldots$$

$$\to H^n_\text{Zar}(X, \mathcal{H}^n_X(\mathbb{Z}/p^r(n))) \to H^n_\text{ét}(X, \mathbb{Z}/p^r(n)) \to H^{n-1}_\text{Zar}(X, \mathcal{H}^{n+1}_X(\mathbb{Z}/p^r(n))) \to 0 \quad (7)$$

since we have by (1.11) and (1.12) of op. cit.:

$$\mathcal{H}^m_X(\mathbb{Z}/p^r(n)) = \begin{cases} 
\alpha_*\nu_r(n) & m = n \\
R^1\alpha_*\nu_r(n) & m = n + 1 \\
0 & \text{else}
\end{cases}$$

Thus, the edge maps $e^{m,n}$ are defined in the case that $\ell = \text{char } k > 0$ (and one checks that $e^{n,n}$ coincides with the cycle class map defined in [16]). We also have the following basic lemma:

**Lemma 1.1.** With the above notation,

(a) When $\ell \neq \text{char } k$, the local-to-global spectral sequence (6) is functorial with respect to flat pull-back, proper push-forward and the action of the Chow group. In particular, it is functorial with respect to correspondences.

(b) When $\ell = \text{char } k = p > 0$, the edge map

$$e^{m,n} : H^m_\text{Zar}(X, \mathcal{H}^n_X(\mathbb{Z}/p^r(n))) \to H^{m+n}_\text{ét}(X, \mathbb{Z}/p^r(n))$$

is functorial with respect to flat pull-back, proper push-forward and the action of the Chow group. In particular, it is functorial with respect to correspondences.
Proof. For $[a]$, the functoriality with respect to pull-back and push-forward follow as in [2] Prop. 3.7, thanks to the Gersten resolution. For functoriality with respect to the action of the Chow group, we note that for $\gamma \in CH^n(X)$, $H^m_{Zar}(X, H^n_X(\mathbb{Z}/\ell^r(n)))$, we may view
\[ \cup \gamma : R\alpha_*(\mathbb{Z}/\ell^r)X(m) \to R\alpha_*(\mathbb{Z}/\ell^r)X(m+n)[n] \]
as a map in the bounded derived category of Zariski sheaves of Abelian groups over $X$, from which we obtain a corresponding map of spectral sequences. For $[b]$, the statement is straightforward since the edge map in this case is the obvious map:
\[ H^m_{Zar}(X, H^n_{Zar}(\mathbb{Z}/\ell^r(n))) \to H^m_{\text{et}}(X, H^n_{\text{et}}(\mathbb{Z}/\ell^r(n))) \]

Remark 1.1. Note that in the case $\ell = \text{char} k = p > 0$, it is not clear how to define the pull-back and push-forward for the Zariski cohomology groups of $H^m_{Zar}(\mathbb{Z}/p^r(n))$ for $m = n + 1$ since there is no Gersten resolution in this case. So, there is no functoriality of the spectral sequence (i.e., the long exact sequence (7)) with respect to these operations.

Corollary 1.1. Assume that $\ell \neq \text{char} k$ and $\Gamma \in \text{Cor}^q(X, Y)$ and suppose that the maps
\[ \Gamma_* : H^m_{Zar}(X, H^{2n-2-m}_X(\mathbb{Z}/\ell^r(n))) \to H^m_{Zar}(Y, H^{2n-2-m+q}_Y(\mathbb{Z}/\ell^r(n+q))) \]
for all $m < n - 2$ and $\Gamma_* : H^{2n-1}_{\text{et}}(X, \mathbb{Z}/\ell^r(n)) \to H^{2n+2q-1}_{\text{et}}(Y, \mathbb{Z}/\ell^r(n+q))$ vanish. Then,
\[ \Gamma_* : H^{n-1}_{Zar}(X, H^n_X(\mathbb{Z}/\ell^r(n))) \to H^{n+q-1}_{Zar}(Y, H^n_{Y}(\mathbb{Z}/\ell^r(n+q))) \]
also vanishes.

Proof. This follows directly from the functoriality of the local-to-global spectral sequence with respect to correspondences. Indeed, given the assumption on the action of $\Gamma$ on étale cohomology, it suffices to show that $\Gamma_*$ kills the kernel of the edge map:
\[ e^{n-1,n} : H^{n-1}_{Zar}(X, H^n_X(\mathbb{Z}/\ell^r(n))) \to H^{2n-1}_{\text{et}}(X, \mathbb{Z}/\ell^r(n)) \]
which is determined by (8).

1.3 A Bockstein exact sequence

Now, for a field $k$ let $K^M_k$ be the Milnor $K$-theory ring of $k$. We also let $K^M_{*,X}$ be the corresponding Zariski sheaf over $X$; i.e., the Zariski sheaf occurring in the short exact sequence of Zariski sheaves:
\[ 0 \to K^M_{*,X} \to \bigoplus_{x \in X^{(0)}} i_x(K^M_*(k(x))) \to \bigoplus_{x \in X^{(1)}} i_{x*}(K^M_{*+1}(k(x))) \]
using the notation from earlier. Here, the rightmost arrow is the sum of the residue maps first defined in [26]. This short exact sequence extends to a Gersten resolution similar to (4), using the fact that Milnor $K$-theory is a Rost cycle module.
Proposition 1.1. Let $X$ be a smooth variety over a perfect field $k$, $\ell$ a prime and $n, r \geq 1$. Then, there exists a natural short exact sequence:

$$0 \to H^{n-1}_{\text{Zar}}(X, K_{n,X}^M)/\ell^r \to H^{n-1}_{\text{Zar}}(X, H^n_X(\mathbb{Z}/\ell^r(n))) \xrightarrow{\delta_X^n} CH^n(X)[\ell^r] \to 0$$

for which $\delta^*$ commutes with proper push-forward and flat pull-back. Moreover, it is compatible with the action of the Chow group; i.e., it fits into the commutative diagram below:

$$\begin{array}{ccc}
H^{n-1}_{\text{Zar}}(X, H^n_X(\mathbb{Z}/\ell^r(n))) & \xrightarrow{\delta_X^n} & H^{n+n'-1}_{\text{Zar}}(X, H^n_X(\mathbb{Z}/\ell^r(n))) \\
\downarrow \delta_X^n \otimes \text{id} & & \downarrow \delta_X^{n+n'} \\
CH^n(X) \otimes CH^{n'}(X) & \longrightarrow & CH^{n+n'}(X)
\end{array}$$

Proof. Consider the commutative diagram below:

$$\begin{array}{ccc}
\bigoplus_{x \in X^{(n-2)}} K_2(k(x)) & \xrightarrow{\ell^r} & \bigoplus_{x \in X^{(n-1)}} K_2(k(x)) \\
\downarrow \delta^{n-2} & & \downarrow \delta^{n-2} \\
\bigoplus_{x \in X^{(n-1)}} k(x)^* & \xrightarrow{\ell^r} & \bigoplus_{x \in X^{(n-1)}} k(x)^* \\
\downarrow \delta^{n-1} & & \downarrow \delta^{n-1} \\
\bigoplus_{x \in X^{(n)}} \mathbb{Z} & \xrightarrow{\ell^r} & \bigoplus_{x \in X^{(n)}} \mathbb{Z}/\ell^r \\
\end{array}$$

A diagram chase then gives a short exact sequence:

$$0 \to \frac{\text{Ker} \ \partial^{n-1}}{\text{Im} \ \partial^{n-2}} \otimes \mathbb{Z}/\ell^r \to \frac{\text{Ker} \ \partial^{n-1}}{\text{Im} \ \partial^{n-2}} \to CH^n(X)[\ell^r] \to 0$$

By the Merkurjev-Suslin Theorem [23] (when $\ell \neq \text{char} \ k$) and the Bloch-Gabber-Kato Theorem [1] (when $\ell = \text{char} \ k > 0$), the middle term is naturally identified with $H^{n-1}_{\text{Zar}}(X, H^n_X(\mathbb{Z}/\ell^r(n)))$

The functoriality statements are straightforward and verified as in [3]. Observe that since proper push-forward and flat pull-back are defined on the level of cycle complexes, it follows that the above diagram is functorial with respect to these operations and hence so is the above exact sequence. In particular, these operations commute with the boundary map. Moreover, the proof of [3] Prop. 3.5 shows that the boundary map is also compatible with the action of the Chow group.

Corollary 1.2. The boundary map $H^{n-1}_{\text{Zar}}(X, H^n_X(\mathbb{Z}/\ell^r(n))) \xrightarrow{\delta_X^n} CH^n(X)[\ell^r]$ commutes with the action of correspondences.
2 Refined homological equivalence

Again, fix $\ell$ a prime number and $r \geq 1$ and suppose that $k$ is a perfect field. There is a commutative diagram for which the vertical arrows are extension-of-scalars maps

$$
\begin{array}{ccc}
CH^n(X)/\ell^r & \xrightarrow{e_{X}^n} & H^2_{\text{et}}(X, \mathbb{Z}/\ell^r(n)) \\
\downarrow & & \downarrow \\
CH^n(\overline{X})/\ell^r & \xrightarrow{e_{\overline{X}}^n} & H^2_{\text{et}}(\overline{X}, \mathbb{Z}/\ell^r(n))
\end{array}
$$

To avoid confusion, we write $cl^d_X$ for the composition:

$$
CH^n(X)/\ell^r \to CH^n(\overline{X})/\ell^r \xrightarrow{e_{\overline{X}}^n} H^2_{\text{et}}(\overline{X}, \mathbb{Z}/\ell^r(n))
$$

for the other cycle class map. We will need the following auxiliary result which shows that homologically trivial correspondences have nilpotent image under the cycle class map.

**Proposition 2.1.** Let $X$ be a smooth, projective variety of dimension $d$ over a perfect field $k$ and let

$$
\gamma \in \text{Cor}^0(X, X) = CH^d(X \times X)
$$

Suppose that

$$
e^d_{X \times X}(\gamma) = 0 \in H^2_{\text{et}}(\overline{X} \times \overline{X}, \mathbb{Z}/\ell^r(d))
$$

Then for all $N \geq 2d + 1$,

$$
e^d_{X \times X}(\gamma \circ N) = 0 \in H^2_{\text{et}}(X \times X, \mathbb{Z}/\ell^r(d))
$$

The proof will exploit some basic properties of the Hochschild-Serre spectral sequence for étale cohomology:

$$H^p_{\text{et}}(G_k, H^q_{\text{et}}(X, \mathbb{Z}/\ell^r(n))) \Rightarrow H^{p+q}_{\text{et}}(X, \mathbb{Z}/\ell^r(n)) \tag{9}$$

where $G_k = \text{Gal}(\overline{k}/k)$ denotes the absolute Galois group of $k$. The induced descending filtration $F^*$ on $H^*_{\text{et}}(X, \mathbb{Z}/\ell^r(n))$ satisfies the following well-known properties:

**Properties.** Suppose that $X$ is a smooth variety over a perfect field $k$.

(a) $F^0H^m_{\text{et}}(X, \mathbb{Z}/\ell^r(n)) = H^m_{\text{et}}(X, \mathbb{Z}/\ell^r(n))$ and $F^{m+1}H^m_{\text{et}}(X, \mathbb{Z}/\ell^r(n)) = 0$.

(b) There is a natural short exact sequence:

$$0 \to F^1H^m_{\text{et}}(X, \mathbb{Z}/\ell^r(n)) \to H^m_{\text{et}}(X, \mathbb{Z}/\ell^r(n)) \to H^m_{\text{et}}(\overline{X}, \mathbb{Z}/\ell^r(n))G_k
$$

(c) The filtration is functorial with respect to pull-backs; i.e., for any morphism $f : Y \to X$ of smooth varieties,

$$f^*(F^iH^m_{\text{et}}(X, \mathbb{Z}/\ell^r(n))) \subset F^iH^m_{\text{et}}(Y, \mathbb{Z}/\ell^r(n))$$

(d) The filtration is functorial with respect to proper push-forwards; i.e., for every proper morphism of smooth varieties of relative dimension $d$, $f : Y \to X$,

$$f_*(F^iH^m_{\text{et}}(Y, \mathbb{Z}/\ell^r(n))) \subset F^iH^{m-2d}_{\text{et}}(X, \mathbb{Z}/\ell^r(n - d))$$
(e) The filtration is additive with respect to the cup product; i.e.,

\[ F^i \mathcal{H}_\text{ét}^m(X, \mathbb{Z}/\ell^n(\pi)) \cdot F^{i'} \mathcal{H}_\text{ét}^{m'}(X, \mathbb{Z}/\ell^{n'}(\pi')) \subset F^{i+i'} \mathcal{H}_\text{ét}^{m+m'}(X, \mathbb{Z}/\ell^{n+n'}(\pi)) \]

**Proof of Properties.** The first two properties are clear. For the other three, let \( D_X \) be the bounded derived category of étale sheaves of Abelian groups over \( X \), then any map in \( D^F \to G \) gives rise to a corresponding map of spectral sequences (and this just follows by general non-sense). In particular, the induced map on the abutment, \( \mathcal{H}_\text{ét}^m(X, \mathcal{F}) \to \mathcal{H}_\text{ét}^m(X, \mathcal{G}) \), respects the filtration. The compatibility with respect to pull-back and push-forward then follows from the fact that they are induced (respectively) by maps in \( D_X \):

\[ \mathbb{Z}/\ell^n(n)_X \to Rf_* \mathbb{Z}/\ell^n(n)_Y, \quad Rf_* \mathbb{Z}/\ell^n(n)_Y \to \mathbb{Z}/\ell^n(n-d)_X[-d] \]

When \( \ell \neq \text{char } k \) this is well-known; when \( \ell = \text{char } k > 0 \), these maps were defined in [16].

These properties now easily imply the following lemma:

**Lemma 2.1.** For \( X, Y, Z \) smooth and projective, \( \gamma \in F^i \mathcal{H}_\text{ét}^m(X \times Y, \mathbb{Z}/\ell^n(n)), \gamma' \in F^{i'} \mathcal{H}_\text{ét}^m(Y \times Z, \mathbb{Z}/\ell^{n'}(n)) \), we have

\[ \gamma' \circ \gamma = \pi_{XZ}^* (\pi_{XY}^* \gamma \cdot \pi_{YZ}^* \gamma') \in F^{i+i'} \mathcal{H}_\text{ét}^{m+m'-2d_Y}(X \times Z, \mathbb{Z}/\ell^{n'}(n)) \]

where \( d_Y \) is the dimension of \( Y \).

**Proof of Proposition 2.1.** The result follows rather quickly from the above lemma. Indeed, for any homologically trivial \( \gamma \in CH^d(X \times X) \), we have

\[ [\gamma] := e^{d,d}_{X \times X}(\gamma) \in F^1 \mathcal{H}_\text{ét}^{2d}(X \times X, \mathbb{Z}/\ell^n(d)) \]

from which it follows that \( [\gamma]^0 \in F^N \mathcal{H}_\text{ét}^{2d}(X \times X, \mathbb{Z}/\ell^n(n)) \), which vanishes for \( N \geq 2d + 1 \).

### 3 Rost nilpotence principle

#### 3.1 A reduction

**Notation 3.1.** For \( E/k \) a field extension, consider the extension of scalars map:

\[ CH^i(X) \to CH^i(X_E) \]

For any \( \alpha \in CH^i(X) \), we will denote by \( \alpha_E \) its image under this map.

We begin with a lemma that is essentially folklore:

**Lemma 3.1.** Suppose that \( k \) is perfect and that \( E/k \) is a field extension. If \( \alpha_E = 0 \), then \( \alpha \in CH^i(X)_{\text{tors}} \).

**Proof.** Since the natural map

\[ \lim_{\to F} CH^i(X_F) \to CH^i(X_E) \]
(where $F$ ranges over all finitely generated extensions of $k$ in $E$) is surjective, we may assume that $E$ is finitely generated over $k$. Then, by [18] Theorem I.4.8A, $E$ is separably generated; i.e., there is some finitely generated purely transcendental extension $F/k$ for which $E/F$ is separable. Now, by a standard transfer argument:

$$\ker \{ CH^i(X_F) \to CH^i(X_E) \} \subset CH^i(X_E)_{\text{tors}}$$

Since $F$ is purely transcendental, $CH^i(X) \to CH^i(X_F)$ is injective, from which it follows that $\alpha \in CH^i(X)_{\text{tors}}$, as desired.

**Corollary 3.1.** Suppose that $k$ is perfect and that $E/k$ is a field extension. If $\alpha_E = 0$, then $\alpha = 0$.

**Proof.** By Lemma 3.1, $\alpha_E = 0 \Rightarrow \alpha \in CH^i(X)_{\text{tors}}$. So, let $\overline{E}$ be the algebraic closure of $E$. Then, it suffices to show that (in the obvious commutative diagram below), the kernel of the top horizontal arrow is contained in the kernel of the left vertical arrow.

$$\begin{align*}
CH^i(X)_{\text{tors}} & \longrightarrow CH^i(X_E) \\
\downarrow & \quad \downarrow \\
CH^i(X_{\overline{E}})_{\text{tors}} & \longrightarrow CH^i(X_{\overline{E}})
\end{align*}$$

By Theorem 3.11 of [22], the bottom horizontal arrow is injective. Thus, the kernel of the left vertical arrow contains the kernel of the top horizontal arrow, as desired.

### 3.2 Main result

To state the result below, let $X$ be a smooth projective variety over a perfect field $k$ and suppose that $\gamma \in Cor^0(X, X)$ lies in the kernel of

$$CH^d(X \times X) \to CH^d(X_F \times X_F)$$

for some field extension $F/k$. By Corollary 3.1, we can assume without loss of generality that $F = \overline{F}$. Moreover, by Lemma 3.1 $\gamma$ is torsion. Since

$$CH^d(X \times X)_{\text{tors}} = \bigoplus_{\ell \text{ prime}} CH^d(X \times X)[\ell^{\infty}]$$

we can write

$$CH^d(X \times X)_{\text{tors}} = CH^d(X \times X)' \oplus CH^d(X \times X)[p^{\infty}]$$

To prove a Rost nilpotence statement, it will suffice to handle the cases of $CH^d(X \times X)'$ and $CH^d(X \times X)[p^{\infty}]$ separately.

**Theorem 3.1.** With the notation above, suppose that either of the following holds:

(a) $\gamma \in CH^d(X \times X)'$ and for all primes $\ell \neq \text{char } k$ and $r \geq 1$,

$$\gamma \times \Delta_X : H^m_{\text{zar}}(X \times X, \mathcal{H}^{2d-2-m}(\mathbb{Z}/\ell^r(d))) \to H^m_{\text{zar}}(X \times X, \mathcal{H}^{2d-2-m}(\mathbb{Z}/\ell^r(d)))$$

vanishes for $m < d - 2$. 

10
(b) For \( p = \text{char } k > 0 \), \( \gamma \in CH^d(X \times X)[p^\infty] \) and \( H_{Zar}^{d-3}(X \times X, \mathcal{H}_{X \times X}^{d+1}(\mathbb{Z}/p^r(d))) = 0 \). Then, \( \gamma^n = 0 \in Cor^0(X, X) \) for \( n >> 0 \).

**Proof.** For any \( \ell \), we can in fact assume that \( \gamma \) is \( \ell \)-torsion for some prime \( \ell \) and \( r \geq 1 \). Now, for convenience set \( Y = X \times X \) and let \( \tilde{\gamma} \in H_{Zar}^{d-1}(Y, \mathcal{H}_Y^d(\mathbb{Z}/\ell^r(d))) \) be a lift of \( \gamma \) via the boundary map:

\[
\delta_Y : H_{Zar}^{d-1}(Y, \mathcal{H}_Y^d(\mathbb{Z}/\ell^r(d))) \to CH^d(Y)[\ell^r]
\]

Then, consider the correspondence \( \beta_N = \gamma^N \times \Delta_X \in Cor^0(Y, Y) \). We note that \( \beta_N \ast (\gamma) = \gamma \circ \gamma^N = \gamma^N \) using Liebermann’s lemma for correspondences. Now, it follows from Corollary 1.2 that

\[
\delta_Y(\beta_N \ast (\tilde{\gamma})) = \beta_N \ast (\delta_Y(\tilde{\gamma})) = \gamma^N \in CH^d(Y)[\ell^r]
\]

To show that this is 0, it suffices to show that

\[
\beta_N : H_{Zar}^{d-1}(Y, \mathcal{H}_Y^d(\mathbb{Z}/\ell^r(d))) \to H_{Zar}^{d-1}(Y, \mathcal{H}_Y^d(\mathbb{Z}/\ell^r(d)))
\]

vanishes for \( N >> 0 \). Indeed, note that

\[
ed_Y^d(\gamma^N) = 0 \in H_{\et}^{2d}(Y, \mathbb{Z}/\ell^r(d))
\]

for \( N \geq 2d + 1 \) by Proposition 2.1, from which we deduce that

\[
\beta_N : H_{\et}^*(Y, \mathbb{Z}/\ell^r(d)) \to H_{\et}^*(Y, \mathbb{Z}/\ell^r(d))
\]

vanishes for \( N >> 0 \). To verify (a) it follows from Corollary 1.1 that (10) vanishes for \( N >> 0 \). To obtain \( (b) \) an examination of the long exact sequence (7) shows that the kernel of the edge map \( e_Y^{d-1,d} \) is \( H_{Zar}^{d-3}(Y, \mathcal{H}_Y^{d+1}(\mathbb{Z}/p^r(d))) \). Since the edge map commutes with correspondences by Lemma 1.1, it follows from (11) that (10) vanishes for \( N >> 0 \).

In the case that \( d = 2 \), we obtain the following as an immediate consequence.

**Corollary 3.2.** The Rost nilpotence principle holds for a smooth projective surface over a perfect field.

### 3.3 Application

**Definition 3.1.** Given a variety \( X \) over a field \( k \), we say that the Chow group of \( X \) is universally supported in dimension \( \leq i \) if there exists a closed subset \( V \subset X \) of dimension \( \leq i \) for which the push-forward

\[
CH_0(V_F) \to CH_0(X_F)
\]

is surjective for all field extensions \( F/k \).

The Bloch-Srinivas decomposition method [5] gives the following result directly:
Lemma 3.2. Suppose that $X$ is a smooth projective variety whose Chow group is universally supported in dimension $\leq i$. Then, there exists a divisor $D \subset X$ and a closed subset $V \subset X$ of dimension $\leq i$ such that for every $\gamma \in CH^{\dim(X)}(X \times X)$

$$\gamma = \gamma_1 + \gamma_2 \in CH^{\dim(X)}(X \times X)$$

(13)

for some $\gamma_1$ is supported on $D \times X$ and $\gamma_2$ is supported on $X \times V$. Moreover, if for some field extension $F/k$ we have

$$\gamma_F = 0 \in CH^{\dim(X)}(X_F \times X_F)$$

then one can take $\gamma_i$ to be such that $\gamma_{i,F} = 0$ for $i = 1, 2$.

Proof. Only the second statement requires justification, given op. cit. For this, we observe by Liebermann’s lemma that

$$\gamma = (\gamma \times \Delta_X) \ast (\Delta_X) \in CH^{\dim(X)}(X \times X)$$

Then, $\Delta_X$ decomposes as $\Delta_1 + \Delta_2$ as in (13) so that we can take $\gamma_i = (\gamma \times \Delta_X) \ast (\Delta_i)$, for which we have $\gamma_{i,F} = 0$.

Lemma 3.3. Assume that the Rost nilpotence principle holds for smooth projective varieties of dimension $\leq d - 1$ and let $X$ be a smooth projective variety of dimension $d$ whose Chow group is universally supported in dimension $\leq d - 1$. Then, $X$ satisfies the Rost nilpotence principle.

Proof. Suppose that $\gamma \in Cor^0(X, X)$ is a correspondence for which $\gamma_F = 0$. By Lemma 3.2, there is a decomposition (13) for which $\gamma_i,F = 0$. Since a sum of nilpotent correspondences is again nilpotent, we reduce to showing that $\gamma_i$ is nilpotent. So, we may assume that $\gamma$ is such that $\gamma_F = 0$ and that there is some (not necessarily smooth) subset $V \subset X$ of dimension $\leq d - 1$ for which $\gamma$ lies in the image of the pushforward:

$$CH^{d-1}(X \times V) \to CH^d(X \times X)$$

Now, by de Jong’s alterations theorem [10], there exists a smooth projective variety $\tilde{V}$ and a dominant morphism $\tilde{V} \to V$. Let $\phi: \tilde{V} \to V \hookrightarrow X$ be the composition. Then, we can write

$$\gamma = (\Delta_X \times \phi) \ast (\alpha) = \Gamma_\phi \circ \alpha \in CH^d(X \times X)$$

for $\alpha \in CH^{d-1}(X \times \tilde{V})$, again using Liebermann’s lemma. Now, we consider

$$\beta = \alpha \circ \Gamma_\phi \in Cor^0(\tilde{V}, \tilde{V})$$

Then, we have $\beta^{\circ 2} = \alpha \circ \gamma \circ \Gamma_\phi$. Since $\gamma_F = 0$, it follows that $\beta^{\circ 2}_F = 0$. Since the Rost nilpotence principle holds for varieties of dimension $\leq d - 1$ by assumption, we deduce that $\beta^{\circ 2}$ is nilpotent and, hence, so is $\beta$. Since for $n \geq 0$ we have

$$\gamma^{\circ n+1} = \Gamma_\phi \circ \beta^{\circ n} \circ \alpha$$

it follows that $\gamma$ is also nilpotent, as desired. □
Remark 3.1. The proof of the above lemma may easily be modified to an induction argument for the Rost nilpotence principle in general, provided that one is able to prove that for $F/k$ a field extension and

$$\gamma \in \text{Ker} \{ \text{Cor}^0(X, X) \to \text{Cor}^0(X_F \times X_F) \}$$

some power of $\gamma$ admits a decomposition such as $[13]$.

We obtain the following as an immediate consequence of Lemma 3.3 and Corollary 3.2:

Corollary 3.3. Suppose that $X$ is a smooth projective variety of dimension $\leq 3$ over a perfect field $k$ whose Chow group is universally supported in dimension $\leq 2$. Then, $X$ satisfies the Rost nilpotence principle.

Recall that a birationally ruled variety $X$ is a smooth projective variety which is birational to a variety of the form $Y \times \mathbb{P}^1$. Rosenschon and Sawant in [28] prove the Rost nilpotence principle for birationally ruled threefolds in characteristic 0. Their proof does not generalize so easily to positive characteristic, since it invokes a non-trivial result in birational geometry known as the Weak Factorization Theorem. As an application of Corollary 3.3, we can prove the following generalization of their result:

Corollary 3.4. Suppose that $X$ is a birationally ruled threefold over a perfect field. Then, $X$ satisfies the Rost nilpotence principle.

Proof. By the projective bundle formula, any variety of the form $Y \times \mathbb{P}^1$ (where $Y$ is a surface) certainly has Chow group universally supported in dimension $\leq 2$. Moreover, $CH_0$ is a birational invariant of smooth projective varieties (in any characteristic). Indeed, if $X_1$ and $X_2$ are smooth projective varieties and $\phi : X_1 \to X_2$ is a rational map. Then, it is a folklore argument (using the moving lemma) that the push-forward $\phi_* : CH_0(X_1) \to CH_0(X_2)$ is well-defined. Thus, any birational map $\phi : X_1 \to X_2$ induces an isomorphism on $CH_0$; moreover, if the Chow group of $X_1$ is universally supported in dimension $\leq i$, so is that of $X_2$. We deduce that if $X$ is a birationally ruled threefold, its Chow group is universally supported in dimension $\leq 2$. Hence, we may apply Corollary 3.3 to obtain the desired result.

References

[1] S. Bloch. K. Kato. $p$-adic etale cohomology, Publ. math. IHÉS 63 (1986), 107-152.

[2] S. Bloch, A. Ogus. Gersten Conjecture and the homology of schemes, Ann. Sci. l’ÉNS, 4 (7) (1974), 181-202.

[3] S. Bloch. Torsion algebraic cycles and a theorem of Roitman, Comp. Math. 39 (1) (1979), 107-127.

[4] S. Bloch. Algebraic cycles and higher K-theory, Adv. Math. 61 (1986), 267-304.

[5] S. Bloch, V. Srinivas. Remarks on Correspondences and Algebraic Cycles, Am. J. Math. 105 (5) (1983), 1235-1253.

[6] P. Brosnan. A short proof of Rost nilpotence via refined correspondences, Doc. Math. 8 (2003), 69-78.
[7] V. Chernousov, S. Gille, A. Merkurjev. Motivic decomposition of isotropic projective homogeneous varieties, Duke Math. J., 126 (1) (2005) 137-159.

[8] J.-L. Colliot-Thélène, M. Ojanguren. Variétés unirationnelles non rationnelles: au-delà de l’exemple d’Artin et Mumford, Invent. math. 97 (1989), 141-158.

[9] J.-L. Colliot-Thélène, C. Voisin. Cohomologie non ramifiée et conjecture de Hodge entière, Duke Math. J. 161 (5) (2012), 735-801.

[10] A. J. de Jong. Smoothness, semi-stability and alterations, Publ. Math. IHÉS 83 (1996), 51-93.

[11] W. Fulton. Intersection Theory, Ergebnisse, 3. Folge, Band 2, Springer Verlag (1984).

[12] P. Gille, T. Szamuely. Central Simple Algebras and Galois Cohomology, Cambridge Studies in Advanced Mathematics 101, Cambridge University Press, Cambridge (2006).

[13] S. Gille. The Rost nilpotence theorem for geometrically rational surfaces, Invent. Math. 181 (1) (2010), 1-19.

[14] S. Gille. On Chow motives of surfaces, J. Reine Angew. Math. 686 (2014), 149-166.

[15] S. Gille. On the Rost nilpotence theorem for threefolds, Bull. London Math. Soc. 50 (2018), 63-72.

[16] M. Gros. Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique, Mém. de la S. M. F. 2e série, 21 (1985), 1-87.

[17] M. Gros, S. Suwa. La conjecture de Gersten pour les faisceaux de Hodge-Witt logarithmiques, Duke Math. J. 57 (1988), 615-628.

[18] R. Hartshorne. Algebraic Geometry, Graduate Texts in Math., Springer (1977).

[19] L. Illusie. Complexe de de Rham-Witt et cohomologie cristalline, Ann. Sci. l’ÉNS 12 (1979), 501-661.

[20] B. Kahn, R. Sujatha. The derived functors of unramified cohomology, Selecta Math. 24(2) (2018), 1121-1146.

[21] S-I. Kimura. Chow groups are finite dimensional, in some sense, Math. Ann. 331 (1) (2005), 173-201.

[22] F. Lecomte. Rigidité des groupes de Chow, Duke Math. J. 53 (1986), 405-426.

[23] A. S. Merkurjev, A. Suslin. K-cohomology of Severi-Brauer varieties and the norm residue homomorphism, Math USSR Izv. 21 (1983), 307-340.

[24] J. Milne. Étale cohomology, Princeton University Press (1980).

[25] J. Milne. Motivic cohomology and values of zeta functions, Comp. Math. 68 (1988), 59-102.

[26] J. Milnor. Algebraic K-theory and quadratic forms, Invent. Math. 9 (1970), 318-344.
[27] A. Pirutka. *Invariants birationnels dans la suite spectrale de Bloch-Ogus*, Journal of K-theory 10 (2012), 565-582.

[28] A. Rosenschon, A. Sawant. *Rost nilpotence and étale motivic cohomology*, to appear in Adv. Math.

[29] A. Rosenschon, V. Srinivas. *Étale motivic cohomology and algebraic cycles*, J. Inst. Math. Jussieu 15 (3) (2016), 511-537.

[30] M. Rost. *Chow groups with coefficients*, Doc. Math. 1 (1996), 316-393.

[31] M. Rost. *The motive of a Pfister form*, preprint, (1998).

[32] V. Voevodsky. *Motivic cohomology with Z/2-coefficients*, Publ. Math. IHÉS 98 (2003), 59-104.

Department of Mathematics, University of California Riverside, Riverside, CA 92521

Email address: humbertoadiaziii@gmail.com