UNIVERSAL DEFORMATION RINGS AND QUIVERS OF FINITE REPRESENTATION TYPE

ROBERTO C. SOTO AND DANIEL J. WACKWITZ

Abstract. Let $k$ be a field of arbitrary characteristic and let $Q$ be a quiver of finite representation type. In this paper we prove that if $M$ is an indecomposable $kQ$-module then the universal deformation ring of $M$ over $kQ$ is isomorphic to $k$.

Introduction

Universal deformations rings of representations of profinite Galois groups have played an important role in proving long-standing problems in number theory. The most celebrated of these is Fermat’s Last Theorem (see [7]) but many other results have also been established such as the Taniyama-Shimura-Weil and Serre conjectures (see [4, 5]). In general, if $k$ is a field of arbitrary characteristic, $W$ is a complete local commutative Noetherian ring with residue field $k$, $G$ is a profinite group and $V$ is a finite dimensional $k$-vector space with continuous $G$ action, then it has been shown (see [8, 13]) that if $\text{End}_{kG}(V) \cong k$ and $H^1(G, \text{End}_k(V))$ is finite dimensional over $k$, that $V$ has a universal deformation ring, $R_W(G, V)$. Moreover, in [8] it was shown that $R_W(G, V)$ is isomorphic to the inverse limit of the universal deformation rings of $R_W(G_i, V)$, where the $G_i$ range over all discrete finite quotients of $G$ through which the action of $G$ on $V$ factors. As such it is important that we understand universal deformation rings for modules over finite dimensional algebras $kG$, where $G$ is a finite group. Indeed deformations of modules for finite dimensional algebras have been studied by many authors in different contexts (see [10, 12, 14]) and in particular we use the work of Velez in [3] to prove our result.

In this paper we consider $k$-algebras arising from quivers of finite representation type. In Section 1 we define universal deformation rings and when they exist. In Section 2 we give a brief overview of quivers of finite representation type. Finally in Section 3 we discuss Gabriel’s Theorem as it is a major result that allows us to prove our theorem below in Section 4.

Theorem 1. Let $Q$ be a quiver of finite representation type and $M$ an indecomposable $kQ$-module. Then the universal deformation ring of $M$ over $kQ$ is isomorphic to $k$.

1. Universal deformation rings

Let $k$ be a field of arbitrary characteristic. Let $\mathcal{C}$ be the category which has all complete local commutative Noetherian $k$-algebras with residue field $k$ as objects, and continuous $k$-algebra homomorphisms inducing the identity map on $k$ as morphisms.
Suppose $\Lambda$ is a finite dimensional $k$-algebra, $V$ is a finitely generated $\Lambda$-module and $R$ is an object in $\hat{\mathcal{C}}$. A lift of $V$ over $R$ is a pair $(M, \phi)$ where $M$ is a finitely generated $R \otimes_k \Lambda$-module which is free over $R$ and $\phi$ is a $\Lambda$-module isomorphism $\phi: k \otimes_R M \to V$. Two lifts $(M, \phi)$ and $(M', \phi')$ of $V$ over $R$ are said to be isomorphic if there exists an $R \otimes_k \Lambda$-module isomorphism $f: M \to M'$ such that $\phi' \circ (k \otimes_R f) = \phi$. A deformation $[M, \phi]$ of $V$ over $R$ is the isomorphism class of the lift $(M, \phi)$. Define $\text{Def}_\Lambda(V, R)$ to be the set of all such deformations of $V$ over $R$. Define the deformation functor $F_V : \hat{\mathcal{C}} \to \text{Sets}$ as the covariant functor which sends a ring $R$ in $\hat{\mathcal{C}}$ to $\text{Def}_\Lambda(V, R)$ and a morphism $\alpha : R \to R'$ in $\hat{\mathcal{C}}$ to the set map $F_V(\alpha) : \text{Def}_\Lambda(V, R) \to \text{Def}_\Lambda(V, R')$, which sends $[M, \phi]$ to $[R \otimes_{R, \alpha} M, \phi\alpha]$ where $\phi\alpha$ is the composition $k \otimes_{R'} (R' \otimes_{R, \alpha} M) \cong k \otimes_R M \to V$. Define the tangent space of $F_V$ to be the set $t_V = F_V(k[\varepsilon])$ where $k[\varepsilon]$ denotes the ring of dual numbers over $k$, ie $\varepsilon^2 = 0$.

The functor $F_V$ is said to be represented by a ring $R(\Lambda, V)$ in $\hat{\mathcal{C}}$ if $F_V$ is naturally isomorphic to the functor $\text{Hom}_\Lambda(R(\Lambda, V), -)$. In other words, there exists a lift $(U(\Lambda, V), \phi_U)$ of $V$ over $R(\Lambda, V)$ such that for any $R$ in $\hat{\mathcal{C}}$, the map $\nu_R : \text{Hom}_\Lambda(C(\Lambda, V), R) \to F_V(R)$, which sends $\alpha \in \text{Hom}_\Lambda(C(\Lambda, V), R)$ to $F_V(\alpha)(U(\Lambda, V), \phi_U)$, is bijective. In this case, $R(\Lambda, V)$ is said to be the universal deformation ring of $V$ over $R$.

**Theorem 1.1.** There is a $k$-vector space isomorphism $t_V \cong \text{Ext}^1_\Lambda(V, V)$. Furthermore, when $\text{End}_\Lambda(V) \cong k$, then $V$ has a universal deformation ring $R(\Lambda, V)$.

**Proof.** See [1, Proposition 2.1].

**Theorem 1.2.** If $\text{End}_\Lambda(V) \cong k$ and $\dim_k \text{Ext}^1_\Lambda(V, V) = r$, then there exists a surjective homomorphism $\lambda : k[[t_1, \cdots, t_r]] \to R(\Lambda, V)$ in $\hat{\mathcal{C}}$, and $r$ is minimal with this property.

**Proof.** Since $\text{End}_\Lambda(V) \cong k$, $F_V$ is representable by Theorem 1.1. Therefore $t_V = F_V(k[\varepsilon]) \cong \text{Hom}_\Lambda(C(\Lambda, V), k[\varepsilon])$. Furthermore, by Theorem 1.1 $t_V \cong \text{Ext}^1_\Lambda(V, V)$ as a $k$-vector space, thus giving the desired result.

## 2. Quivers of Finite Representation Type

A quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, where $Q_0$ is a finite set of vertices, $Q_1$ is a finite set of arrows, and $s, t : Q_1 \to Q_0$ are maps assigning to each arrow it source, resp. target. A representation $M$ of a quiver $Q$ consists of a family of vector spaces $V_i$ indexed by the vertices $i \in Q_0$, together with a family of linear maps $f_{\alpha} : V_{s(\alpha)} \to V_{t(\alpha)}$ indexed by the arrows $\alpha \in Q_1$. If $M = (V_i, f_{\alpha})$ is a representation of $Q$, then its dimension vector $\underline{n} = \dim M = (\dim V_i)_{i \in Q_0}$.

Given two representations $M = ((V_i)_{i \in Q_0}, (f_{\alpha})_{\alpha \in Q_1})$, $N = ((W_i)_{i \in Q_0}, (g_{\alpha})_{\alpha \in Q_1})$ of a quiver $Q$, a morphism $u : M \to N$ is a family of linear maps $(u_i : V_i \to W_i)_{i \in Q_0}$ such that for any $\alpha \in Q_1$ we have $u_{t(\alpha)} \circ f_{\alpha} = g_{\alpha} \circ u_{s(\alpha)}$. For a quiver $Q$ and a field $k$ we can form the category $\text{Rep}_k(Q)$ whose objects are representations of $Q$ with the morphisms as defined above. A morphism $\phi : M = (V_i, f_{\alpha}) \to N = (W_i, g_{\alpha})$ is an isomorphism if $\phi_i$ is invertible for every $i \in Q_0$. As can be expected, we wish to classify all representations of a given quiver $Q$ up to isomorphism.
If $M$ and $N$ are two representations of the same quiver $Q$, we define their direct sum $M \oplus N$ by $(M \oplus N)_i = V_i \oplus W_i$ for all $i \in Q_0$, and $(M \oplus N)_\alpha : V_{s(\alpha)} \oplus W_{s(\alpha)} \to V_{t(\alpha)} \oplus W_{t(\alpha)}$ for all $\alpha \in Q_1$. We say that $M$ is a trivial representation if $V_i = 0$ for all $i \in Q_0$. If $M$ is isomorphic to a direct sum $M_1 \oplus M_2$, where $M_1$ and $M_2$ are nontrivial representations, then $M$ is called decomposable. Otherwise $M$ is called indecomposable. Every representation has a unique decomposition into indecomposable representations, up to isomorphism and permutation of components. Thus the classification problem reduces to classifying the indecomposable representations. We say that a quiver is of finite representation type, or just finite type, if it has only finitely many indecomposable representations; otherwise, it is of infinite representation type. The following theorem classifying the quivers of finite representation type is due to Gabriel (see [9], [11]).

**Theorem 2.1** (Gabriel). A quiver is of finite representation type if and only if each connected component of its underlying undirected graph is a Dynkin graph of type $A, D,$ or $E$, shown below:

- $A_n$:
  \[ \bullet \cdots \bullet \cdots \bullet \cdots \bullet \cdots \bullet \quad (n\text{ vertices, } n \geq 1) \]

- $D_n$:
  \[ \bullet \cdots \bullet \cdots \bullet \cdots \bullet \cdots \bullet \quad (n\text{ vertices, } n \geq 4) \]

- $E_6$:
  \[ \bullet \cdots \bullet \cdots \bullet \cdots \bullet \cdots \bullet \]

- $E_7$:
  \[ \bullet \cdots \bullet \cdots \bullet \cdots \bullet \cdots \bullet \]

- $E_8$:
  \[ \bullet \cdots \bullet \cdots \bullet \cdots \bullet \cdots \bullet \cdots \bullet \]
3. Gabriel’s Theorem

The algebra of a quiver $Q$ is the associative algebra $kQ$ determined by the generators $e_i$ and $\alpha$, where $i \in Q_0$ and $\alpha \in Q_1$, and the relations

$$e_i^2 = e_i, e_i e_j = 0 (i \neq j), e_{t(\alpha)} \alpha = \alpha e_{s(\alpha)} = \alpha.$$ 

Note that the category of representations of any quiver $Q$ is equivalent to the category of left $kQ$-modules (see [1, Theorem III.1.5]).

Recall, if $M$ and $N$ are arbitrary $kQ$-modules we define the groups $\text{Ext}^i_Q(M, N)$ as follows: first choose a projective resolution

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0.$$ 

Then take morphisms to $N$ yielding a complex

$$\text{Hom}_Q(P_0, N) \rightarrow \text{Hom}_Q(P_1, N) \rightarrow \text{Hom}_Q(P_2, N) \rightarrow \cdots$$

The homology groups of this complex are independent of the choice of a projective resolution of $M$; the $i$th homology group is denoted $\text{Ext}^i_Q(M, N)$ (see [2, Section 2.4]).

Note that $\text{Ext}^0_Q(M, N) = \text{Hom}_Q(M, N)$ and recall that $\text{Ext}^1_Q(M, N)$ is the set of equivalence classes of extensions of $M$ by $N$, i.e., of exact sequences of $kQ$-modules

$$0 \rightarrow N \rightarrow E \rightarrow M \rightarrow 0$$

up to isomorphisms that induce the identity maps on $N$ and $M$ (see [2, Section 2.6]).

The Euler form of the quiver $Q$ is the bilinear form $\langle \cdot, \cdot \rangle_Q$ on $\mathbb{R}^{Q_0}$ given by

$$\langle m, n \rangle_Q = \sum_{i \in Q_0} m_i n_i = \sum_{\alpha \in Q_1} m_{s(\alpha)} n_{t(\alpha)}$$

for any $m = (m_i)_{i \in Q_0}$ and $n = (n_i)_{i \in Q_0}$. The quadratic form associated to the Euler form is called the Tits form $q_Q$, i.e.

$$q_Q(n) = \langle n, n \rangle_Q = \sum_{i \in Q_0} n_i^2 - \sum_{\alpha \in Q_1} n_{s(\alpha)} n_{t(\alpha)}.$$

Note that the Tits form depends only on the underlying undirected graph of $Q$ and determines the graph uniquely (see [6]).

Finally a representation $M$ of the quiver $Q$ is called a Schur representation (also known as a brick), if $\text{End}_Q(M) \cong k$. Clearly, any Schur representation is indecomposable, but the converse only holds if the Tits form of $Q$ is positive definite.

Now we obtain a more precise form of Gabriel’s Theorem with important consequences for our result.

Theorem 3.1. Assume that the Tits form $q_Q$ is positive definite. Then:

(i) Every indecomposable representation is Schur and has no non-zero self-extensions.

(ii) The dimension vectors of the indecomposable representations are exactly those $\underline{n} \in \mathbb{N}^{Q_0}$ such that $q_Q(\underline{n}) = 1$.

(iii) Every indecomposable representation is uniquely determined by its dimension vector, up to isomorphism.
There are only finitely many isomorphism classes of indecomposable representations of $Q$.

Proof. See [6, Theorem 2.4.3].

Note that, as a result of [6, Proposition 1.4.6], the quivers $Q$ with positive definite Tits form are exactly the quivers which are of finite representation type. In particular we have that if $Q$ is a quiver of finite representation type and $M$ is an indecomposable representation of the quiver $Q$, i.e. an indecomposable $kQ$-module, then $\text{End}_Q(M) \cong k$ and $\text{Ext}^1_Q(M, M) = 0$.

4. Proof of Theorem 1

Now we can prove our result. Assume the hypotheses of Theorem 1.

Proof. By Theorem 3.1 we have that $\text{End}_Q(M) \cong k$ and that $\text{Ext}^1_Q(M, M) = 0$. Thus by Theorem 1.1 a universal deformation ring exists and by Theorem 1.2 we have a surjective map $\lambda : k \to R(kQ, M)$. Since $\ker \lambda = 0$ we obtain our isomorphism.

References

1. M. Auslander, I. Reiten, and S.O. Smalø, Representation theory of Artin algebras, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1995.
2. D. J. Benson, Representations and cohomology. I, second ed., Cambridge Studies in Advanced Mathematics, vol. 30, Cambridge University Press, Cambridge, 1998, Basic representation theory of finite groups and associative algebras.
3. F. M. Bleher and J. A. Vélez-Marulanda, Universal deformation rings of modules over Frobenius algebras, J. Algebra 367 (2012), 176–202.
4. G. Böckle, Presentations of universal deformation rings, $L$-functions and Galois representations, London Math. Soc. Lecture Note Ser., vol. 320, Cambridge Univ. Press, Cambridge, 2007, pp. 24–58.
5. C. Breuil, B. Conrad, F. Diamond, and R. Taylor, On the modularity of elliptic curves over $\mathbf{Q}$: wild 3-adic exercises, J. Amer. Math. Soc. 14 (2001), no. 4, 843–939 (electronic).
6. M. Brion, Representations of quivers, Geometric methods in representation theory. I, Sémin. Congr., vol. 24, Soc. Math. France, Paris, 2012, pp. 103–144.
7. G. Cornell, J. H. Silverman, and G. Stevens (eds.), Modular forms and Fermat’s last theorem, Springer-Verlag, New York, 1997, Papers from the Instructional Conference on Number Theory and Arithmetic Geometry held at Boston University, Boston, MA, August 9–18, 1995.
8. B. de Smit and H. W. Lenstra, Jr., Explicit construction of universal deformation rings, Modular forms and Fermat’s last theorem (Boston, MA, 1995), Springer, New York, 1997, pp. 313–326.
9. P. Gabriel, Unzerlegbare Darstellungen I, Manuscripta Math. 6 (1972), 71–103.
10. R. Ile, Change of rings in deformation theory of modules, Trans. Amer. Math. Soc. 356 (2004), no. 12, 4873–4896 (electronic).
11. H. Kraft and Ch. Riedtmann, Geometry of representations of quivers, Representations of algebras (Durham, 1985), 109-145, London Math. Soc. Lecture Note Ser., vol. 116, Cambridge Univ. Press, Cambridge, 1986.
12. O. A. Laudal, Noncommutative deformations of modules, Homology Homotopy Appl. 4 (2002), no. 2, part 2, 357–396, The Roos Festschrift volume, 2.
13. B. Mazur, Deforming Galois representations, Galois groups over $\mathbf{Q}$ (Berkeley, CA, 1987), Math. Sci. Res. Inst. Publ., vol. 16, Springer-Verlag, New York, 1989, pp. 385–437.
14. D. Yau, Deformation theory of modules, Comm. Algebra 33 (2005), no. 7, 2351–2359.