Number-phase Wigner function on extended Fock space

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Abstract

On the basis of the phase states, we present the correct integral expressions of the two number-phase Wigner functions discovered so far. These correct forms are derived from those defined in the extended Fock space with negative number states. The analogous conditions to Wigner's original ones cannot lead to the number-phase function uniquely. To show this fact explicitly, we propose another function satisfying all these conditions. It is also shown that the ununiqueness of the number-phase Wigner function result from the phase-periodicity problem.

PACS numbers: 03.65.Ca, 42.50.-p
I. INTRODUCTION

The Wigner function, first introduced by Wigner, gives the phase-space formulation of quantum mechanics for a pair of position and momentum that have continuous spectra \[1, 2, 3, 4\]. Since there is a correspondence between quantum observables and their classical-like functions in this formalism, physical quantities such as the expectation value of any observable can be calculated using the quasiprobability distribution (Wigner function).

Requiring the analogous properties of Wigner’s original function \[4\] and introducing some additional assumptions, Vaccaro \[5\] has defined a number-phase Wigner function

\[
S_1(n, \theta) = \frac{1}{\pi} \int_0^{2\pi} d\xi e^{2in\xi} \left( \frac{1 + e^{-i\xi}}{2} \right) |\theta - \xi; p\rangle \langle \theta + \xi; p|.
\]

(1)

Here \(|\theta; p\rangle\) is the phase state

\[
|\theta; p\rangle = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} e^{in\theta} |n\rangle,
\]

where the symbol “p” stands for a physical state. The function \(S_1\) gives a representation of the state \(\hat{\rho}\) that displays the underlying photon-number and phase properties.

Several authors \[6, 7, 8, 9\] have studied the rotational Wigner function for rotation angle and angular momentum. In particular, Bizarro \[8\] derived the function starting from six “natural” conditions, which constitute an appropriate way for constructing the function. If we interpret rotation angle and angular momentum as (extended) phase and (extended) number operators, respectively, the rotational Wigner function becomes a number-phase distribution function in the extended space. Moreover, restricting the resulting function to the physical space, we obtain another number-phase Wigner operator:

\[
\hat{S}_2(n, \theta) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\xi e^{2in\xi} |\theta - \xi; p\rangle \langle \theta + \xi; p|.
\]

(3)

Introducing a superoperator (acting on any operator) and some relations for the phase-number pair corresponding to the position-momentum pair, Ban \[10\] has obtained an alternative number-phase distribution function, which is different from \(S_1\) and \(S_2\). We do not consider Ban’s function here, because this function is defined without the above analogous conditions and does not satisfy even phase-shift condition, one of basic requirements.

At present we have at least three different number-phase functions. All these functions are calculated by using the number states to avoid \(2\pi\)-periodicity problem of phase variable.
As a consequence, as is shown in Eqs. (1) and (3), the phase-periodicity structure of the functions is not clear. The relationship between these functions is not clear either. Moreover, the integral representations (1) and (3) have a problem called an endpoint problem [11]: that is, we have

\[
\hat{S}_1(n, \theta) |\theta; p\rangle = \frac{1}{2\pi} \int_0^{2\pi} d\xi e^{2in\xi} \left( \frac{1 + e^{-i\xi}}{2} \right) \left[ \delta(\xi) + \delta(\xi - 2\pi) \right] |\theta - \xi; p\rangle + \cdots,
\]

\[
\hat{S}_2(n, \theta) |\theta - \pi/2; p\rangle = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\xi e^{2in\xi} \delta(\xi + \pi/2) |\theta - \xi; p\rangle + \cdots,
\]

which are ambiguous. In deriving Eq. (4), we have used

\[
\langle \theta; p | \theta'; p \rangle = \frac{1}{2} \sum_{n=-\infty}^{\infty} \delta(\theta - \theta' - 2n\pi) + \frac{i}{4\pi} \cot \left( \frac{\theta - \theta'}{2} \right) + \frac{1}{4\pi}.
\]

This problem is due to the fact that it is not always possible to integrate a generalized function over a finite region. An integral over a finite region should be converted into an integral over \((-\infty, \infty)\). For this purpose, we must use a generalized periodic function and a unit function [11, 12]. Thanks to the unit function, the phase and number variables can be treated freely as the position and momentum variables without bothering with the phase-periodicity problem.

The purpose of the present article is to present the correct representations for the operators \(\hat{S}_1\) and \(\hat{S}_2\), and to find the relationship between them by using the six “natural” conditions. The correct integral expressions can be obtained by making use of the phase states. Then, it becomes clear that \(\hat{S}_1\) and \(\hat{S}_2\) cannot be defined uniquely, although Bizarro [8] states that \(\hat{S}_2\) is determined uniquely by six “natural” conditions. In fact, there are infinite possibilities of defining number-phase Wigner operators satisfying these conditions.

In the next section, the six conditions for determining the Wigner operator are considered, using the phase states in the extended Fock space. Then, the nonuniqueness of the Wigner operator becomes clear. In section 3, we show that the basic properties which the Wigner operator should satisfy can be derived from a part of six conditions. The correct forms of the operators \(\hat{S}_1\) and \(\hat{S}_2\) are given in section 4. We end with a conclusion in section 5.

II. CONDITIONS FOR THE WIGNER OPERATOR

In this section, we first give a brief review of the quantum phase in the extended Fock space to overcome the endpoint problem. The six “natural” conditions and the nonuniqueness of
the Wigner operator are considered, using mainly the phase states in the extended space.

A. Phase states

The basis of the extended Fock space is given by \{|n\rangle\} \((n = 0, \pm 1, \pm 2, \cdots)\) [11], where the (extended) phase states and the number states are defined by

\[
|\theta\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\theta e^{-i\theta\theta} |n\rangle,
|n\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\theta e^{i\theta\theta} |\theta\rangle,
\]

where \(d\theta = U(\theta) d\theta\). Here \(U(\theta)\) is called a rapidly decreasing unit function [12] satisfying

\[
U(\theta) = 0 \ (|\theta| \geq 2\pi), \quad \sum_{n=-\infty}^{\infty} U(\theta + 2n\pi) = 1, \quad \sum_{n=-\infty}^{\infty} U^{(k)}(\theta + 2n\pi) = 0,
\]

where \(U^{(k)}\) is the \(k\)th derivative of \(U\). An example of a unit function could be given by [12]

\[
U(\theta) = \frac{1}{K} \int_{0}^{2\pi} dt \exp\left[\frac{-4\pi^2 t}{(2\pi - t)}\right], \quad (|\theta| \leq 2\pi)
\]

where

\[
K = \int_{0}^{2\pi} dt \exp\left[\frac{-4\pi^2 t}{(2\pi - t)}\right].
\]

The states \(|\theta\rangle\) and \(|n\rangle\) satisfy the orthonormality relations

\[
\langle n|m\rangle = \delta_{nm}, \quad \langle \theta|\theta'\rangle = \sum_{n=-\infty}^{\infty} \delta(\theta - \theta' + 2n\pi) \equiv \delta_{2\pi}(\theta - \theta').
\]

Here \(\delta_{2\pi}(\theta - \theta')\) is a periodic generalized function with a period of \(2\pi\). The phase operator \(\hat{\theta}\) and the number operator \(\hat{N}\) can be represented as [11]

\[
\hat{\theta} = \int_{-\infty}^{\infty} d\theta \frac{1}{|\theta\rangle\langle \theta|} [\theta|\theta\rangle, \quad \hat{N} = \sum_{n=-\infty}^{\infty} n|n\rangle\langle n|,
\]

where \([\theta]\) is a sawtooth wave, a periodic function with a period of \(2\pi\) of the phase variable \(\theta\): \([\theta] = \theta \ (0 \leq \theta < 2\pi)\) [see [11]].

Let us next consider the number-phase Wigner function \(W(n, \theta) = \text{Tr}[\hat{W}(n, \theta)\hat{\rho}]\) in the extended Fock space, where the number-phase Wigner operator can be expressed in terms of the number states and the phase states:

\[
\hat{W}(n, \theta) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} |k\rangle\langle k|\hat{W}(n, \theta)|\ell\rangle\langle \ell|
= \int_{-\infty}^{\infty} d\xi d\xi' \xi\langle \xi|\hat{W}(n, \theta)|\xi'\rangle\langle \xi'|,
\]

\(4\)
where \( du(\xi, \xi') = du(\xi)du(\xi') \). In Eq. \( \text{(12)} \), the completeness relations

\[
\sum_{n=-\infty}^{\infty} |n\rangle\langle n| = 1, \quad \int_{-\infty}^{\infty} du(\theta) |\theta\rangle\langle \theta| = 1
\]

has been used.

For later convenience, we give here some useful formulas:

\[
\int_{-\infty}^{\infty} du(\xi) \delta_{2\pi}(\xi - \theta) \langle \xi|\psi \rangle = \langle \theta|\psi \rangle,
\]

(14)

\[
\int_{-\infty}^{\infty} du(\xi) \delta'_{2\pi}(\xi - \theta) \langle \xi|\psi \rangle = -\frac{\partial}{\partial \theta} \langle \theta|\psi \rangle,
\]

(15)

\[
\int_{-\infty}^{\infty} du(\xi) |\xi + \Delta\rangle\langle \xi + \Delta| = \int_{-\infty}^{\infty} du(\xi) |\xi\rangle\langle \xi|,
\]

(16)

\[
\int_{-\infty}^{\infty} d\xi U(\theta - \xi) \langle \xi|\psi \rangle = \int_{-\infty}^{\infty} du(\xi) \langle \xi|\psi \rangle,
\]

(17)

where \( |\psi\rangle \) is any state, \( |\xi\rangle \) any phase state and \( \Delta \) a real constant. Proofs of the formulas \((16)\) and \((17)\) are given in the appendix. Using Eq. \((14)\), we easily obtain \( \hat{\theta}|\theta\rangle = [\theta]|\theta\rangle \); that is, \([\theta]\) \((0 \leq [\theta] < 2\pi)\) is an eigenvalue of the phase operator \( \hat{\theta} \). Note that we can treat the operators \( \hat{\theta} \) and \( \hat{N} \) quite easily in the extended space, as as seen in equations \((5)\) and \((10)\).

**B. Six conditions**

Following Bizarro \[8\] and Vaccaro \[5\], we introduce the six conditions to determine the Wigner operator \( \hat{W} \).

(i) The operator \( \hat{W}(n, \theta) \) should be Hermitian and so \( W(n, \theta) \) should be real. Thus we have

\[
\langle k|\hat{W}(n, \theta)|\ell\rangle = \langle \ell|\hat{W}(n, \theta)|k\rangle^*, \quad \langle \xi|\hat{W}(n, \theta)|\xi'\rangle = \langle \xi'|\hat{W}(n, \theta)|\xi\rangle^*.
\]

(18)

(ii) \( W(n, \theta) \) gives the probability distributions for \( n \) and \( \theta \)

\[
\int_{-\infty}^{\infty} du(\theta) W(n, \theta) = \langle \psi|n\rangle\langle n|\psi \rangle,
\]

(19)

\[
\sum_{n=-\infty}^{\infty} W(n, \theta) = \langle \psi|\theta\rangle\langle \theta|\psi \rangle.
\]

(20)

We thus require that

\[
\int_{-\infty}^{\infty} du(\theta) \hat{W}(n, \theta) = |n\rangle\langle n|,
\]

(21)

\[
\sum_{n=-\infty}^{\infty} \hat{W}(n, \theta) = |\theta\rangle\langle \theta|.
\]

(22)
(iii) $W(n, \theta)$ should be Galilei invariant with respect to displacements in phase $\theta$ and number $n$. For phase shift of $\Delta$ we have

$$\langle \xi | \hat{W}(n, \theta) | \xi' \rangle = \langle \xi + \Delta | \hat{W}(n, \theta + \Delta) | \xi' + \Delta \rangle,$$

which leads to

$$\langle k | \hat{W}(n, \theta) | \ell \rangle = \int_{-\infty}^{\infty} du(\xi, \xi') \langle k | \xi \rangle \langle \xi | \hat{W}(n, \theta) | \xi' \rangle \langle \xi' | \ell \rangle$$

$$= \int_{-\infty}^{\infty} du(\xi, \xi') e^{-i(k-\ell)\Delta} \langle k | \xi + \Delta \rangle \langle \xi | \hat{W}(n, \theta) | \xi' \rangle \langle \xi' + \Delta | \ell \rangle$$

$$= e^{-i(k-\ell)\Delta} \langle k | \hat{W}(n, \theta + \Delta) | \ell \rangle.$$

Setting $\Delta = -\theta$, we have

$$\langle k | \hat{W}(n, \theta) | \ell \rangle = e^{i(k-\ell)\theta} \langle k | \hat{W}(n, 0) | \ell \rangle.$$  

(24)

In Eq. (24) we have used the equality for the completeness relation (16).

From invariance with respect to the number shift $m$, it follows that

$$\langle k | \hat{W}(n, \theta) | \ell \rangle = \langle k + m | \hat{W}(n + m, \theta) | \ell + m \rangle.$$  

(26)

Hence we have

$$\langle k | \hat{W}(n, \theta) | \ell \rangle = \langle k - n | \hat{W}'(0, \theta) | \ell - n \rangle$$  

(27)

and

$$\langle \xi | \hat{W}'(n, \theta) | \xi' \rangle = e^{-i(n-\xi')} \langle \xi | \hat{W}'(0, \theta) | \xi' \rangle.$$  

(28)

(iv) The transition probability between the states $|\psi\rangle$ and $|\psi'\rangle$ should be given, in terms of the respective $W(n, \theta)$ and $W'(n, \theta)$, by

$$2\pi \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} du(\theta) W(n, \theta)W'(n, \theta) = |\langle \psi | \psi' \rangle|^2.$$  

(29)

(v) $W(n, \theta)$ should be invariant with respect to number-phase reflection. From this condition we have

$$\langle \xi | \hat{W}(n, \theta) | \xi' \rangle = \langle -\xi | \hat{W}(-n, -\theta) | -\xi' \rangle,$$

(30)

which is equivalent to

$$\langle k | \hat{W}(n, \theta) | \ell \rangle = \langle -k | \hat{W}(-n, -\theta) | -\ell \rangle.$$  

(31)
(vi) \( W(n, \theta) \) should be invariant with respect to time reversal, i.e., \( \langle k|\psi \rangle \rightarrow \langle k|\psi \rangle^* \) and \( \langle k|\hat{W}(n, \theta)|\ell \rangle \rightarrow (-\langle k|\hat{W}(-n, \theta)|-\ell \rangle \). We then have

\[
W(n, \theta) = \sum_{k, \ell} \langle \psi|k \rangle^* \langle -k|\hat{W}(-n, \theta)|-\ell \rangle \langle \ell|\psi \rangle^*
\]

\[
= \sum_{k, \ell} \langle \psi|k \rangle \langle -\ell|\hat{W}(-n, \theta)|-k \rangle \langle \ell|\psi \rangle
\]  

(32)

for any state \(|\psi\rangle\). Hence

\[
\langle k|\hat{W}(n, \theta)|\ell \rangle = (-\langle \ell|\hat{W}(-n, \theta)|-k \rangle \),
\]

(33)

which leads to

\[
\langle \xi|\hat{W}(n, \theta)|\xi' \rangle = \langle \xi'|\hat{W}(-n, \theta)|\xi \rangle.
\]  

(34)

C. Nonuniqueness of the Wigner operator

We next apply the above conditions to the quantity \( \langle \xi|\hat{W}(n, \theta)|\xi' \rangle \). It is sufficient to treat \( \langle \xi|\hat{W}(0, \theta)|\xi' \rangle \), because of number-shift condition (28). First expand \( \langle \xi|\hat{W}(0, \theta)|\xi' \rangle \) as the Fourier series:

\[
\langle \xi|\hat{W}(0, \theta)|\xi' \rangle = \sum_{k, \ell} \langle \xi - \theta|k \rangle \langle k|\hat{W}(0, 0)|\ell \rangle \langle \ell|\xi' - \theta \rangle
\]

\[
= \frac{1}{2\pi} \sum_k C_k(\xi' - \xi) e^{ik(\theta - \xi)},
\]  

(35)

where

\[
C_k(\omega) = \sum_\ell \langle k + \ell|\hat{W}(0, 0)|\ell \rangle e^{i\ell\omega}.
\]  

(36)

In deriving Eq. (35), we have used phase-shift condition (23). It should be noted that the Fourier coefficient \( C_k(\omega) \) is periodic with a period of \( 2\pi \). Integrating the both sides of equation (35) with respect to \( \theta \) and using marginal condition (21), we find

\[
C_0(\omega) = \frac{1}{2\pi}.
\]  

(37)

Since the other marginal condition (22) and number-shift condition (28) lead to

\[
\sum_{n=-\infty}^{\infty} \langle \xi|\hat{W}(n, \theta)|\xi' \rangle = \delta_{2\pi}(\xi - \theta)\delta_{2\pi}(\xi - \xi'),
\]

\[
\sum_{n=-\infty}^{\infty} e^{-in(\xi - \xi')} \langle \xi|\hat{W}(0, \theta)|\xi' \rangle = 2\pi\delta_{2\pi}(\xi - \xi')\langle \xi|\hat{W}(0, \theta)|\xi' \rangle,
\]  

(38)
we arrive at
\[ \langle \xi | \hat{W}(n, \theta) | \xi \rangle = \frac{\delta_{2\pi}(\theta - \xi)}{2\pi}. \] (39)

Similarly, from conditions (21) and (25), it follows that
\[ \langle k | \hat{W}(n, \theta) | k \rangle = \frac{\delta_{nk}}{2\pi}. \] (40)

Equation (39) gives
\[ C_k(0) = \frac{1}{2\pi}. \] (41)

Now let us rewrite overlap condition (29) to obtain another condition for the coefficient \( C_k(\omega) \). Noting that
\[ \int_{-\infty}^{\infty} du(\theta) \int_{-\infty}^{\infty} d\xi, d\xi' \langle \xi | \hat{W}(n, \theta) | \xi' \rangle \langle \eta | \hat{W}(n, \theta) | \eta' \rangle \langle \psi | \xi \rangle \langle \psi' | \eta \rangle = \frac{1}{2\pi} \langle \xi | 0 \rangle, \] (44)

where \( |0\rangle = |\theta\rangle|_{\theta=0} \). It follows from equation (44) that
\[ C_k(\omega)C_{-k}(\omega) = \frac{1}{(2\pi)^2} e^{-ik\omega}. \] (45)

From equation (45) it is convenient to rewrite \( C_k(\omega) \) as
\[ C_k(\omega) = \frac{1}{(2\pi)^2} e^{-ik\omega/2} g_k(\omega). \] (46)

Moreover, equations (37), (41) and (46) lead to
\[ g_0(\omega) = g_k(0) = g_k(\omega)g_{-k}(\omega) = 1. \] (47)

It should be noted that we cannot take \( g_k(\omega) = 1 \), because \( e^{-ik\omega/2} \) does not have a period of \( 2\pi \) if \( k \) is odd. That is, if \( k = 2m + 1 \) \((m = 0, \pm 1, \pm 2, \cdots)\), from equation (46) it follows that \( g_{2m+1}(\omega) \) has period \( 4\pi \):
\[ g_{2m+1}(\omega + 2\pi) = -g_{2m+1}(\omega), \quad g_{2m+1}(\omega + 4\pi) = g_{2m+1}(\omega). \] (48)
Then, the quantity \( e^{-i\omega/2} g_{2m+1}(\omega) \) has a period of \( 2\pi \) and, as a result, the coefficient \( C_{2m+1}(\omega) \) has also a period of \( 2\pi \).

From condition (v), we have

\[
g_{-k}(-\omega) = g_k(\omega). \tag{49}\]

Also, it follows from condition (vi) that the function \( g_k(\omega) \) must be an even function:

\[
g_k(-\omega) = g_k(\omega). \tag{50}\]

All six conditions give equations (47)–(50) for the function \( g_k(\omega) \). However, we cannot determine the function \( g_k(\omega) \) uniquely even if we use these conditions. Consequently, against Ref. [8], the number-phase Wigner operator (function) cannot be determined uniquely from the above six conditions. If the function \( g_k(\omega) \) would not be periodic, then from equations (47) and (49), the function \( g_k(\omega) \) can be determined uniquely; that is, \( g_k(\omega) = 1 \). In fact, Bizarro [8] has been used a solution corresponding to \( g_k(\omega) = 1 \). The origin of the nonuniqueness of definition of the Wigner operator is thus the \( 2\pi \)–periodicity of the phase.

III. BASIC PROPERTIES OF THE WIGNER OPERATOR

It is shown that the first four conditions, (i) to (iv), are sufficient to get basic properties of the Wigner operator; that is, these four conditions are fundamental for obtaining the Wigner operator. Here, in this section, we do not consider the last two conditions, (v) and (vi).

The number-phase Wigner representation of an operator \( \hat{A} \) is defined by

\[
A(n, \theta) \equiv [\hat{A}](n, \theta) = 2\pi \text{Tr}[\hat{W}(n, \theta)\hat{A}]. \tag{51}\]

For the phase operator, from equations (14) and (39) and the eigenvalue equation \( \hat{\theta} |\xi\rangle = [\xi]|\xi\rangle \), we get

\[
[\hat{\theta}](n, \theta) = 2\pi \int_{-\infty}^{\infty} du(\xi) \langle \xi | \hat{W}(n, \theta)\hat{\theta} |\xi\rangle = [\theta]. \tag{52}\]

Similarly,

\[
[\hat{N}](n, \theta) = 2\pi \sum_{k=\infty}^{\infty} \langle k | \hat{W}(n, \theta)\hat{N} |k\rangle = n, \tag{53}\]

where we have used equation (10). These two Wigner representations (52) and (53) are quite natural.
The trace of a product of any two operators \( \hat{A} \) and \( \hat{B} \) can be represented in terms of their Wigner representations:

\[
\text{Tr}[\hat{A}\hat{B}] = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} du(\theta) A(n, \theta) B(n, \theta). \tag{54}
\]

Indeed, this formula can be derived in a straightforward way:

\[
\begin{align*}
\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} du(\theta) & A(n, \theta) B(n, \theta) \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} du(\theta, \xi', \eta') \langle \xi | \hat{A} | \xi' \rangle \langle \eta | \hat{B} | \eta' \rangle \langle \xi' | \hat{W}(n, \theta) | \xi \rangle \langle \eta' | \hat{W}(n, \theta) | \eta \rangle \\
&= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} du(\xi, \xi', \eta) \langle \xi | \hat{A} | \xi' \rangle \langle \eta | \hat{B} | \xi - \xi' + \eta \rangle C_k(\xi - \xi') C_{-k}(\xi' - \xi) e^{ik(\xi - 2\xi' + \eta)} \\
&= \int_{-\infty}^{\infty} du(\xi, \xi', \eta) \langle \xi | \hat{A} | \xi' \rangle \langle \eta | \hat{B} | \xi - \xi' + \eta \rangle \delta_{2\pi}(\eta - \xi') \\
&= \text{Tr}[\hat{A}\hat{B}]. \tag{55}
\end{align*}
\]

Setting \( \hat{B} = |\xi'\rangle \langle \xi| \) in Eq. (54), we get

\[
\langle \xi | \hat{A} | \xi' \rangle = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} du(\theta) A(n, \theta) \text{Tr}(\hat{W}(n, \theta)|\xi') \langle \xi | \\
= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} du(\theta) \langle \xi | \hat{W}(n, \theta)|\xi' \rangle A(n, \theta), \tag{56}
\]

which implies

\[
\hat{A} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} du(\theta) \hat{W}(n, \theta) A(n, \theta). \tag{57}
\]

From the trace formula (54), it also follows that the expectation value of the operator \( \hat{A} \) in a state \( \hat{\rho} \) becomes

\[
\text{Tr}[\hat{\rho}\hat{A}] = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} du(\theta) W(n, \theta) A(n, \theta). \tag{58}
\]

Consider finally the number-phase Wigner function for some simple states. For the number state \( \hat{\rho} = |k\rangle \langle k| \), the number-phase Wigner function is given by equation (40), whereas the Wigner function for the phase state \( \hat{\rho} = |\xi\rangle \langle \xi| \) is given by equation (39).

**IV. CORRECT EXPRESSIONS FOR \( \hat{S}_1(n, \theta) \) AND \( \hat{S}_2(n, \theta) \)**

We present three examples satisfying the first four conditions (i) to (iv); the first example corresponds to Vaccaro’s operator \( \hat{S}_1 \) and the second corresponds to \( \hat{S}_2 \). These two examples
give the correct integral forms for $\hat{S}_1$ and $\hat{S}_2$. To show explicitly that the Wigner operator cannot be defined uniquely from all six conditions, we consider the third example.

As the three examples of solutions for $g_k(\omega)$, consider the following:

(a) $g_{2m}(\omega) = 1$, $g_{2m+1}(\omega) \equiv f_1(\omega) = e^{-i\omega/2}$; (59)
(b) $g_{2m}(\omega) = 1$, $g_{2m+1}(\omega) \equiv f_2(\omega) = \begin{cases} 1, & (-\pi < \omega < \pi) \\ -1, & (\pi < \omega < 3\pi) \end{cases}$; (60)
(c) $g_{4m}(\omega) = 1$, $g_{4m+2}(\omega) = f_2(2\omega)$, $g_{2m+1}(\omega) = f_2(\omega)$, (61)

where $m$ is any integer and $f_2(\omega)$ a square wave with a period of $4\pi$. Note that the first example does not satisfy the last two conditions (v) and (vi), whereas the other two ones satisfy all conditions (i) to (vi).

Let us obtain the Wigner operators corresponding to these solutions. First consider case (a). Rewriting $\hat{W}(n,\theta)$ as $\hat{W}_1(n,\theta)$ and substituting equation (59) into equation (35), we find
$$\langle \xi | \hat{W}_1(0,\theta) | \xi' \rangle = \frac{1}{\pi} \delta_{2\pi}(2\theta - \xi - \xi') h_1(\theta - \xi),$$ (62)
where
$$h_1(\xi) = \frac{1 + f_1(2\xi)}{2}. \quad (63)$$
Substituting equations (28) and (62) into equation (12) and using equation (14), we get
$$\hat{W}_1(n,\theta) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\xi e^{2in\xi h_1(\xi) | \theta - \xi |}.$$ (64)

Taking equation (17) into account, we arrive at the first Wigner operator in the extended Fock space:
$$\hat{W}_1(n,\theta) = \frac{1}{\pi} \int_{-\infty}^{\infty} d\xi e^{2in\xi h_1(\xi) | \theta - \xi |}.$$ (65)

By using the projection operator
$$P = \sum_{n=0}^{\infty} |n\rangle \langle n| \quad (66)$$
ono onto the physical space spanned by $|n\rangle$ ($n \geq 0$), the correct form of Vaccaro’s operator can be derived:
$$\hat{S}_1(n,\theta) = P \hat{W}_1(n,\theta) P = \frac{1}{\pi} \int_{-\infty}^{\infty} d\xi e^{2in\xi h_1(\xi) | \theta - \xi; p |} | \theta + \xi; p \rangle.$$ (67)
Note that \( P[\theta] = |\theta; p \) [see equation (21)]. Vaccaro’s operator (11) should be written as equation (67). Since any physical state \( \hat{\rho} \) satisfies \( P \hat{\rho} P = \hat{\rho} \), we can always use \( \hat{W}_1 \) for any physical state:

\[
W_1(n, \theta) \equiv \text{Tr}[\hat{W}_1(n, \theta)\hat{\rho}] = \text{Tr}(\hat{S}_1(n, \theta)\hat{\rho}) \equiv S_1(n, \theta),
\]

where \( S_1(n, \theta) \) is the correct Vaccaro’s number-phase Wigner function.

Similarly, in case (b), we have

\[
\hat{W}_2(n, \theta) = \frac{1}{\pi} \int_{-\infty}^{\infty} du(\xi) e^{2in\xi} h_2(\xi)|\theta - \xi\rangle\langle \theta + \xi|,
\]

where \( h_2(\xi) \) is also a square wave, a periodic function with a period of \( 2\pi \):

\[
h_2(\xi) = \frac{1 + f_2(2\xi)}{2} = \begin{cases} 1, & (-\pi/2 < \theta < \pi/2) \\ 0, & (\pi/2 < \omega < 3\pi/2). \end{cases}
\]

In deriving equation (69), we have used the equality

\[
d_{2\pi}(2\theta - \xi - \xi')e^{-i(\xi' - \xi)/2}f_2(\xi' - \xi) = d_{2\pi}(2\theta - \xi - \xi')e^{-i(\theta - \xi)}f_2(2(\theta - \xi)).
\]

Note here that \( e^{-i\omega/2}f_2(\omega) \) has a period of \( 2\pi \). The physical part of the operator \( \hat{W}_2 \) gives the correct expression for the second number-phase Wigner operator (3):

\[
\hat{S}_2(n, \theta) = P\hat{W}_2(n, \theta)P = \frac{1}{\pi} \int_{-\infty}^{\infty} du(\xi) e^{2in\xi} h_2(\xi)|\theta - \xi; p\rangle\langle \theta + \xi; p|.
\]

In fact, if we consider any ordinary (not generalized) periodic function, then equation (72) reduces to equation (61) [11, 12]. That is, for any normalizable states \( |\psi\rangle \) and \( |\phi\rangle \), we have

\[
\langle \psi|\hat{S}_2(n, \theta)|\phi\rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} du(\xi) e^{2in\xi} h_2(\xi)\langle \psi|\theta - \xi; p\rangle\langle \theta + \xi; p|\phi\rangle.
\]

Hence, the operator (3) should be written as equation (72).

The Wigner operator \( \hat{W}_3 \) corresponding to case (c) has two terms:

\[
\hat{W}_3(n, \theta) = \frac{1}{\pi} \int_{-\infty}^{\infty} du(\xi) e^{2in\xi} h_3(\xi)|\theta - \xi\rangle\langle \theta + \xi|
+ \frac{1}{\pi} \int_{-\infty}^{\infty} du(\xi) e^{2in\xi} \tilde{h}_3(\xi)|\theta - \pi/2 - \xi\rangle\langle \theta - \pi/2 + \xi|,
\]

where \( h_3(\xi) \) and \( \tilde{h}_3(\xi) \) are, respectively, given by

\[
h_3(\xi) = \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{4}f_2(4\xi) + f_2(2\xi) \right], \quad \tilde{h}_3(\xi) = \frac{1}{4} \left[ 1 - f_2(4\xi) \right].
\]

\[12\]
In equation (74) we have used the equality
\[ \delta_{2\pi}(2\theta) = \frac{1}{2} \left[ \delta_{2\pi}(\theta) + \delta_{2\pi}(\theta + \pi) \right]. \] (76)

The operator \( \hat{W}_1 \) does not satisfy conditions (v) and (vi), whereas \( \hat{W}_2 \) and \( \hat{W}_3 \) satisfy both of them. Thus the Wigner operators \( \hat{W}_2 \) and \( \hat{W}_3 \) have higher symmetry than \( \hat{W}_1 \). It should be mentioned that there exist infinite Wigner operators satisfying all six conditions (i) to (vi). The operator \( \hat{W}_2 \) has the simplest integral form.

To show that the Wigner operators \( \hat{W}_2 \) and \( \hat{W}_3 \) are better than \( \hat{W}_1 \), we consider the Wigner representation of the operator \( \hat{\theta}\hat{N} \). For this purpose, we first get
\[
\left( \frac{\partial}{\partial \xi} | \hat{\theta}(n,\theta)| \xi \right)_{\xi' = \xi} = \frac{i}{2\pi} \left( n - \frac{i}{2} \frac{\partial}{\partial \theta} \right) \delta_{2\pi}(\theta - \xi) + \frac{1}{(2\pi)^2} \sum_{k=-\infty}^{\infty} g'_k(0) e^{ik(\theta - \xi)}.
\] (77)

Then we arrive at
\[
[\hat{N}\hat{\theta}](n,\theta) = -2\pi i \int_{-\infty}^{\infty} du(\xi) \left( \frac{\partial}{\partial \xi} | \hat{W}(n,\theta)| \xi \right)_{\xi' = \xi} [\xi] = \left( n + \frac{i}{2} \frac{\partial}{\partial \theta} \right) [\theta] + \sum_{k=-\infty \atop k \neq 0}^{\infty} \frac{g'_k(0)}{k} e^{ik\theta} - \pi ig'_0(0).
\] (78)

Since \( f_1'(0) = -i/2 \) and \( f_2'(0) = 0 \), the number-phase Wigner representations for three cases are given by
\[
[\hat{N}\hat{\theta}](n,\theta) = \begin{cases} 
(n + \frac{i}{2} \frac{\partial}{\partial \theta}) [\theta] + R(\theta), & \text{case (a)} \\
(n + \frac{i}{2} \frac{\partial}{\partial \theta}) [\theta], & \text{cases (b) and (c)}
\end{cases}
\] (79)

where
\[
R(\theta) = \sum_{m=0}^{\infty} \frac{\sin(2m+1)\theta}{2m+1} = \frac{\pi}{4} f_2(2\theta - \pi/2).
\] (80)

The operator \( \hat{W}_1 \) leads to a more complex expression for \([\hat{N}\hat{\theta}](n,\theta)\) than the others.

Next we show that the operators \( \hat{W}_2 \) and \( \hat{W}_3 \) correspond to Wigner’s original operators \( \hat{W}(q,p) \) for position \( q \) and momentum \( p \). To this end, using \([\hat{N}\hat{\theta}]^\dagger(n,\theta) = [\hat{N}\hat{\theta}](n,\theta)^\ast\), we find
\[
\left[ (\hat{N}\hat{\theta} + \hat{\theta}\hat{N})/2 \right](n,\theta) = \begin{cases} 
n[\theta] + R(\theta), & \text{case (a)} \\
n[\theta], & \text{cases (b) and (c)}
\end{cases}
\] (81)

The symmetric operator \((\hat{N}\hat{\theta} + \hat{\theta}\hat{N})/2\) has then its correspondence \(n[\theta]\) in cases (a) and (b). The operators \( \hat{W}_2 \) and \( \hat{W}_3 \) thus lead to a “symmetric” representation. This fact results
from \( g_k(0) = 0 \) in the neighborhood of the origin. Recall here that, in the original Wigner function, the symmetric operator \( (\hat{q}\hat{p} + \hat{p}\hat{q})/2 \) has its correspondence \( qp \), where \( \hat{q} \) and \( \hat{p} \) are the position and momentum operators, respectively. However, the operator \( \hat{W}_1 \) does not have such a property.

V. CONCLUSION

We have investigated the problem of defining the number-phase Wigner operator. We first presented the correct integral expressions for the two Wigner operators \( \hat{S}_1(n, \theta) \) and \( \hat{S}_2(n, \theta) \), which were derived, respectively, from the Wigner operators \( \hat{W}_1(n, \theta) \) and \( \hat{W}_2(n, \theta) \) in the extended Fock space. The operator \( \hat{W}_2(n, \theta) \) satisfies all six “natural” conditions, whereas \( \hat{W}_1(n, \theta) \) satisfies only four ones. As a result, \( \hat{W}_1(n, \theta) \) does not correspond to a symmetric representation; it leads to an unnecessary term \( R(\theta) \), as shown in equation (81). The Wigner operator cannot be derived uniquely from the six conditions, because of the periodic property of the phase. To show this fact explicitly, we have obtained another Wigner operator \( \hat{W}_3(n, \theta) \) satisfying all six conditions, which, however, has more complex integral form than \( \hat{W}_2(n, \theta) \). The operator \( \hat{W}_2(n, \theta) \) has the simplest integral form in all other Wigner operators. We need more “natural” conditions to define the number-phase Wigner operator uniquely, which are not clear at present.

APPENDIX

Since it is easy to verify the formulas (14) and (15), we give here proofs of (16) and (17). Consider first the relation (16). For any states \( |\psi\rangle \) and \( |\varphi\rangle \), we have

\[
\int_{-\infty}^{\infty} du(\xi) \langle \psi|\xi + \Delta \rangle \langle \xi + \Delta|\varphi \rangle = \int_{-\infty}^{\infty} d\xi U(\xi - \Delta) \langle \psi|\xi \rangle \langle \xi|\varphi \rangle
\]

\[
= \sum_{k, \ell = -\infty}^{\infty} \langle \psi|k \rangle \langle \ell|\varphi \rangle \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi U(\xi - \Delta) e^{i(k-\ell)\xi}
\]

\[
= \sum_{k, \ell = -\infty}^{\infty} \langle \psi|k \rangle \langle \ell|\varphi \rangle \frac{1}{2\pi} \sum_{m = -\infty}^{\infty} \int_{2\pi m} \int_{2\pi(m+1)} d\xi U(\xi - \Delta) e^{i(k-\ell)\xi}
\]

\[
= \sum_{k, \ell = -\infty}^{\infty} \langle \psi|k \rangle \langle \ell|\varphi \rangle \frac{1}{2\pi} \sum_{m = -\infty}^{\infty} \int_{0}^{2\pi} d\eta U(\eta - \Delta + 2\pi m) e^{i(k-\ell)\eta}
\]

\[
= \sum_{k = -\infty}^{\infty} \langle \psi|k \rangle \langle k|\varphi \rangle,
\]
where we have used \( \sum_{m=-\infty}^{\infty} U(\eta - \Delta + 2\pi m) = 1 \). Equation (82) implies the relation

\[
\int_{-\infty}^{\infty} d\xi \ |\xi + \Delta\rangle\langle \xi + \Delta| = \sum_{k=-\infty}^{\infty} |k\rangle\langle k| = 1,
\]

which is independent of the constant \( \Delta \), so that equation (16) holds.

Next proceed to prove equation (17). Using the Fourier expansion \( \langle \xi|\psi\rangle = \sum_{k=-\infty}^{\infty} \langle \xi|k\rangle\langle k|\psi\rangle \), the left-hand side of equation (17) becomes

\[
\int_{-\infty}^{\infty} d\xi U(\theta - \xi) \langle \xi|\psi\rangle = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi U(\theta - \xi) \langle \xi|k\rangle\langle k|\psi\rangle
\]

\[
= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \int_{/2\pi\ell}^{2\pi(\ell+1)} d\xi U(\theta - \xi) \langle \xi|k\rangle\langle k|\psi\rangle
\]

\[
= \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} \int_{0}^{2\pi} d\eta U(\theta - \eta - 2\pi\ell) \langle \eta|k\rangle\langle k|\psi\rangle
\]

\[
= \sqrt{2\pi} \langle 0|\psi\rangle.
\]

Similarly, it is easily shown that the right-hand side of equation (17) is also \( \sqrt{2\pi} \langle 0|\psi\rangle \).

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