SUBSPACES DISCERNING NULLCONTINUITY

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Abstract. Given positive linear functional \( \ell \) on a vector lattice \( \mathcal{L} \) of real functions, and a vector subspace \( M \) of \( \mathcal{L} \), we construct a vector subspace \( P(M) \) of \( M \) in such a way that 1) \( \ell \) is nullcontinuous on \( P(M) \), and 2) if \( \ell \) is nullcontinuous on \( M \) then \( P(M) \) is all of \( M \). We mention here that this result continues to hold for quite general modes of convergence, including \( \tau \)-continuity. Our construction uses a new method involving the “kernel” of a seminorm.

1. Basic notation and terminology

We consider a fixed structure \((X, \mathcal{L}, \ell, \rho)\) consisting of a non-empty set \( X \), a vector sublattice \( \mathcal{L} \) of \( \mathbb{R}^X \), a positive linear functional \( \ell \) on \( \mathcal{L} \), and a set \( \rho \) of non-empty downward directed subsets \( I \) of \( \mathcal{L}_+ \) with \( \bigwedge I = 0 \) in \( \mathbb{R}^X \).

The symbols \( M, M_1, M_2 \) shall henceforth denote variable vector subspaces of \( \mathcal{L} \).

Definition 1. We shall say that \( \ell \) is \( \rho \)-continuous on \( M \), if

\[
\bigwedge_{f \in I} \ell(f) = 0 \text{ for all } I \in \rho \text{ with } I \subset M.
\]

If \( \rho = \{ \{ f_n \} \subset \mathcal{L}_+ : f_n \downarrow 0 \text{ in } \mathbb{R}^X \} \) then \( \rho \)-continuity is the same as nullcontinuity. If \( \rho \) is the set of all non-empty downward directed subsets \( I \) of \( \mathcal{L}_+ \) with \( \bigwedge I = 0 \) in \( \mathbb{R}^X \), then \( \rho \)-continuity is also known as \( \tau \)-continuity.

2. Review of terminology concerning vector lattices

The vector space \( M \) is called solid in \( \mathcal{L} \), if the conditions \( f \in \mathcal{L}, g \in M, \) and \( |f| \leq |g| \) together imply that \( f \) belongs to \( M \), cf. e.g. [1, 1.3.9]. In this case \( M \) is a vector sublattice of \( \mathcal{L} \), because then \( |f| \in M \) whenever \( f \in M \).

A seminorm \( q \) on \( \mathcal{L} \) is called a lattice seminorm, if for all \( f, g \in \mathcal{L} \) with \( |f| \leq |g| \) one has \( q(f) \leq q(g) \), cf. [1, 1.10.1]. (An equivalent requirement is that \( q(f) = q(|f|) \) for all \( f \in \mathcal{L} \).) In the affirmative
case, the “kernel” \( \{ f \in \mathcal{L} : q(f) = 0 \} \) of the lattice seminorm \( q \) is a solid subspace of \( \mathcal{L} \), and thereby a solid vector sublattice of \( \mathcal{L} \).

The vector lattice \( \mathcal{L} \) is called Stonean, if it contains with each function \( f \) the function \( f \wedge 1_X \), cf. e.g. [1, 2.5.14]. This property is inherited to all solid vector subspaces.

3. The notion of \( \rho \)-regularity: definition and simple properties

**Definition 2.** Denote by \( T \) the set of functions \( g : X \to [0, 1] \), such that \( gf \in \mathcal{L}_+ \) for all \( f \in \mathcal{L}_+ \). (The plus signs are a matter of convenience.)

One checks that \( T \) is a convex sublattice of \([0, 1]^X\). Clearly \( 0, 1_X \in T \). If \( g \in T \), so is \( 1_X - g \), and both \( g \) and \( 1_X - g \) are positive, so multiplication by them maps \( \mathcal{L}_+ \) to itself in an order preserving way.

**Definition 3.** We put

\[ S(M) := \{ g \in T : \bigwedge_{f \in I} \ell(gf) = 0 \text{ for all } I \in \rho \text{ with } I \subset M \}. \]

This is a convex sublattice of \([0, 1]^X\). If \( M_1 \subset M_2 \), then \( S(M_2) \subset S(M_1) \).

**Proof.** We note first that the “\( \bigwedge \)” in the defining relation for \( S(M) \) actually is a limit since each \( I \in \rho \) is downward directed. This implies that \( S(M) \) is convex and thence a sublattice of \([0, 1]^X\). \( \square \)

**Definition 4.** We shall say that \( \ell \) is \( \rho \)-regular on \( M \), if for every \( h \in M \) one has

\[ \ell(|h|) = \bigvee_{g \in S(M)} \ell(g|h|). \]

(This can be put in terms of \( h \in M_+ \) in case \( M \) is a vector sublattice of \( \mathcal{L} \).) We shall reformulate this condition in the next two items.

**Definition 5.** A lattice seminorm \( q_M \) is defined on \( \mathcal{L} \) by putting

\[ q_M(h) := \bigwedge_{g \in S(M)} \ell((1_X - g)|h|) \]

for every \( h \in \mathcal{L} \). If \( M_1 \subset M_2 \), then \( q_{M_1} \leq q_{M_2} \).

**Proof.** The “\( \bigwedge \)” is a limit as \( S(M) \) is upward directed, so \( q_M \) is a seminorm. The last part follows from the last part of definition. \( \square \)

**Theorem 6.** The functional \( \ell \) is \( \rho \)-regular on \( M \) if and only if \( q_M \) vanishes identically on \( M \).
Proof. Let \( h \in \mathcal{M} \). Using that \( S(\mathcal{M}) \) is upward directed, one finds
\[
q_{\mathcal{M}}(h) = \bigwedge_{g \in S(\mathcal{M})} \ell((1_X - g)|h|)
= \lim_{g \in S(\mathcal{M})} \ell((1_X - g)|h|)
= \ell(|h|) - \lim_{g \in S(\mathcal{M})} \ell(g|h|)
= \ell(|h|) - \bigvee_{g \in S(\mathcal{M})} \ell(g|h|).
\]
It follows that \( q_{\mathcal{M}}(h) = 0 \) if and only if
\[
\ell(|h|) = \bigvee_{g \in S(\mathcal{M})} \ell(g|h|),
\]
whence the statement. \( \square \)

4. EQUIVALENCE OF \( \rho \)-CONTINUITY AND \( \rho \)-REGULARITY

Proposition 7. The functional \( \ell \) is \( \rho \)-continuous on \( \mathcal{M} \) if and only if \( S(\mathcal{M}) \) contains \( 1_X \).

Corollary 8. If \( \ell \) is \( \rho \)-continuous on \( \mathcal{M} \), then \( \ell \) is \( \rho \)-regular on \( \mathcal{M} \).

Proof. If \( \ell \) is \( \rho \)-continuous on \( \mathcal{M} \), then \( S(\mathcal{M}) \) contains \( 1_X \) by the preceding proposition. Then \( q_{\mathcal{M}} \) vanishes identically on \( \mathcal{M} \), from which \( \ell \) is \( \rho \)-regular on \( \mathcal{M} \) by virtue of theorem 6. \( \square \)

Proposition 9. For each \( I \in \rho \) with \( I \subset \mathcal{M} \), one has
\[
\bigwedge_{f \in I} \ell(f) = \bigwedge_{f \in I} q_{\mathcal{M}}(f).
\]

Proof. Let \( g \in S(\mathcal{M}) \) be arbitrary. Since \( I \) is downward directed, one finds
\[
\bigwedge_{f \in I} \ell(f) = \lim_{f \in I} \ell(f) - \lim_{f \in I} \ell(gf)
= \lim_{f \in I} \ell((1_X - g)f)
= \bigwedge_{f \in I} \ell((1_X - g)f).
\]
Since \( g \in S(\mathcal{M}) \) is arbitrary, one also has
\[
\bigwedge_{f \in I} \ell(f) = \bigwedge_{g \in S(\mathcal{M})} \bigwedge_{f \in I} \ell((1_X - g)f)
= \bigwedge_{f \in I} q_{\mathcal{M}}(f).
\]
This allows us to reformulate definition \ref{def:rho-continuous} in the following way.

**Theorem 10.** The functional $\ell$ is $\rho$-continuous on $\mathcal{M}$ if and only if
\[
\bigwedge_{f \in I} q_{\mathcal{M}}(f) = 0 \text{ for all } I \in \rho \text{ with } I \subset \mathcal{M}.
\]

**Corollary 11.** If $\ell$ is $\rho$-regular on $\mathcal{M}$, then $\ell$ is $\rho$-continuous on $\mathcal{M}$.

*Proof.* If $\ell$ is $\rho$-regular on $\mathcal{M}$, then $q_{\mathcal{M}}$ vanishes identically on $\mathcal{M}$ by theorem \ref{thm:rho-regular}. Theorem \ref{thm:rho-continuous} implies that $\ell$ is $\rho$-continuous on $\mathcal{M}$. $\square$

**Theorem 12.** The functional $\ell$ is $\rho$-continuous on $\mathcal{M}$ if and only if it is $\rho$-regular on $\mathcal{M}$.

In the light of theorems \ref{thm:rho-continuous} and \ref{thm:rho-regular}, we can now see that theorem \ref{thm:rho-continuous} is a vast improvement on definition \ref{def:rho-continuous}.

5. The main result

**Definition 13.** We denote the “kernel” of the lattice seminorm $q_{\mathcal{M}}$ by
\[
\mathcal{K}(\mathcal{M}) := \{ h \in \mathcal{L} : q_{\mathcal{M}}(h) = 0 \}.
\]

This is a solid vector subspace of $\mathcal{L}$, and thus a vector sublattice of $\mathcal{L}$. Also, if $\mathcal{M}_1 \subset \mathcal{M}_2$ then $\mathcal{K}(\mathcal{M}_2) \subset \mathcal{K}(\mathcal{M}_1)$.

*Proof.* This follows from the statements in definition \ref{def:rho-continuous}. $\square$

**Theorem 14.** The functional $\ell$ is $\rho$-regular on $\mathcal{M}$ if and only if $\mathcal{M} \subset \mathcal{K}(\mathcal{M})$.

*Proof.* Theorem \ref{thm:rho-regular} and definition \ref{def:rho-continuous}. $\square$

**Definition 15.** Let $\mathcal{P}(\mathcal{M}) := \mathcal{M} \cap \mathcal{K}(\mathcal{M})$. This is a vector subspace of $\mathcal{M}$.

**Theorem 16.** The functional $\ell$ is $\rho$-regular on $\mathcal{P}(\mathcal{M})$.

*Proof.* Put $\mathcal{N} := \mathcal{P}(\mathcal{M}) = \mathcal{M} \cap \mathcal{K}(\mathcal{M})$, and let $f \in \mathcal{N}$. By theorem \ref{thm:rho-regular} we have to prove that $q_{\mathcal{N}}(f) = 0$. One one hand, one has $\mathcal{N} \subset \mathcal{M}$, and so $q_{\mathcal{N}} \leq q_{\mathcal{M}}$ by definition \ref{def:rho-continuous}. On the other hand, $f \in \mathcal{K}(\mathcal{M})$, so that $q_{\mathcal{M}}(f) = 0$ by definition \ref{def:rho-continuous}. It follows that $q_{\mathcal{N}}(f) \leq q_{\mathcal{M}}(f) = 0$. $\square$

**Theorem 17.** The functional $\ell$ is $\rho$-regular on $\mathcal{M}$ precisely when $\mathcal{P}(\mathcal{M}) = \mathcal{M}$.

*Proof.* If $\ell$ is $\rho$-regular on $\mathcal{M}$, then $\mathcal{M} \subset \mathcal{K}(\mathcal{M})$ by theorem \ref{thm:rho-regular}. It follows that $\mathcal{P}(\mathcal{M}) = \mathcal{M}$ by definition \ref{def:rho-continuous}. Conversely, if $\mathcal{P}(\mathcal{M}) = \mathcal{M}$, then $\ell$ is $\rho$-regular on $\mathcal{M}$ by the preceding theorem \ref{thm:rho-continuous}. $\square$

**Theorem 18.** The vector subspace $\mathcal{P}(\mathcal{M})$ has the following properties:

(i) $\ell$ is $\rho$-continuous on $\mathcal{P}(\mathcal{M})$,
(ii) $\ell$ is $\rho$-continuous on $\mathcal{M}$ if and only if $\mathcal{P}(\mathcal{M})$ is all of $\mathcal{M}$.

*Proof.* Theorems \ref{thm:rho-continuous} \ref{thm:rho-regular} and \ref{thm:rho-regular}. $\square$
The preceding theorem is our main result. It suggests that the subspace \( P(\mathcal{M}) \) of \( \mathcal{M} \) is a “large” subspace of \( \rho \)-continuity.

(A largest subspace of \( \rho \)-continuity need not exist in the present generality, as is shown by an argument communicated to me by Torben Maack Bisgaard.)

6. Properties of the map \( \mathcal{M} \mapsto P(\mathcal{M}) \)

**Theorem 19.** One has \( P(P(\mathcal{M})) = P(\mathcal{M}) \).

**Proof.** This follows from theorem by replacing \( \mathcal{M} \) with \( P(\mathcal{M}) \). □

We shall denote by \( \| \cdot \| \) the usual lattice seminorm on \( \mathcal{L} \) given by \( \| f \| = \ell(|f|) \) for all \( f \in \mathcal{L} \).

**Proposition 20.** The seminorm \( q_\mathcal{M} \) is dominated by \( \| \cdot \| \), and thereby is continuous on \( (\mathcal{L}, \| \cdot \|) \). It follows that \( \mathcal{K}(\mathcal{M}) \) is a closed subspace of \( (\mathcal{L}, \| \cdot \|) \).

**Corollary 21.** The set \( \mathcal{K}(\mathcal{M}) \) is a closed solid vector sublattice of \( (\mathcal{L}, \| \cdot \|) \).

**Theorem 22.** The following inheritance properties hold.
If \( \mathcal{M} \) is a vector sublattice of \( \mathcal{L} \), so is \( P(\mathcal{M}) \).
If \( \mathcal{M} \) furthermore is Stonean, so is \( P(\mathcal{M}) \).
If \( \mathcal{M} \) is solid in \( \mathcal{L} \), so is \( P(\mathcal{M}) \).
If \( \mathcal{M} \) is closed in \( (\mathcal{L}, \| \cdot \|) \), so is \( P(\mathcal{M}) \).

**Proof.** Definition 15 and the preceding corollary □

**Corollary 23.** The vector space \( P(\mathcal{L}) = \mathcal{K}(\mathcal{L}) \) is the “kernel” of the lattice seminorm \( q_\mathcal{L} \). It is a closed solid vector sublattice of \( (\mathcal{L}, \| \cdot \|) \).

If \( \mathcal{L} \) is Stonean, so is \( P(\mathcal{L}) \).

**Proof.** Definitions 15 and 13, corollary 21 and theorem 22 □

**Theorem 24.** If \( \mathcal{M}_1 \subset \mathcal{M}_2 \), then \( P(\mathcal{M}_2) \cap \mathcal{M}_1 \subset P(\mathcal{M}_1) \).

**Proof.** One has \( P(\mathcal{M}_2) \cap \mathcal{M}_1 \subset \mathcal{K}(\mathcal{M}_2) \cap \mathcal{M}_1 \subset \mathcal{K}(\mathcal{M}_1) \cap \mathcal{M}_1 = P(\mathcal{M}_1) \) by definition 15 and the last statement in definition 13 □

Whence, as special cases, the following three corollaries:

**Corollary 25.** One has \( P(\mathcal{L}) \cap \mathcal{M} \subset P(\mathcal{M}) \).

**Corollary 26.** If \( \mathcal{M}_1 \) is a subspace between \( P(\mathcal{M}_2) \) and \( \mathcal{M}_2 \), that is, if \( P(\mathcal{M}_2) \subset \mathcal{M}_1 \subset \mathcal{M}_2 \), then \( P(\mathcal{M}_2) \subset P(\mathcal{M}_1) \subset \mathcal{M}_1 \subset \mathcal{M}_2 \).

**Corollary 27.** If \( P(\mathcal{L}) \subset \mathcal{M} \), then \( P(\mathcal{L}) \subset P(\mathcal{M}) \subset \mathcal{M} \).

**References**

[1] C. Constantinescu, W. Filter, K. Weber, *Advanced Integration Theory*, Kluwer Academic Publishers, 1998.

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