A sufficient condition of violating the SPA conjecture

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Based on the general form of entanglement witnesses constructed from separable states, we first show a sufficient condition of violating the structural physical approximation (SPA) conjecture [\textit{Phys. Rev. A} \textbf{78}, 062105 (2008)]. Then we discuss the SPA conjecture for decomposable entanglement witnesses. Moreover, we make geometric illustrations of the connection between entanglement witnesses and the sets of quantum states, separable states, and entangled states comparing with planes and vectors in Euclidean space.

\section{I. INTRODUCTION}

Quantum entanglement is considered as the central resource for quantum information processing [1-2], such as quantum computation, quantum dense coding, quantum teleportation, quantum cryptography, etc. However, quantum entanglement is still not fully known by researchers. It is one of the main research topics in the theory of entanglement that how to detect a given state entangled or separable.

To the best of our knowledge, positive (linear) maps [3] up to date may be the most powerful method among various separability criteria. For any entangled state, there exists at least a positive but not completely positive map to detect it. As a consequence of the Jamiołkowski-Choi isomorphism [4], entanglement witnesses (EWs) [5] play the role to detect entanglement equivalently. An observables \( W = W^\dagger \) is said an EW if (i) the expectation value of \( W \) is non-negative for any separable state; and (ii) it is negative for at least an entangled state. Naturally, (iii) \( W \) keeps all properties of \( W \) as an EW for a non-negative number \( \gamma \). In this case, we say that \( \gamma W \) is the same EW as \( W \). An EW which detects a maximal set of entanglement is defined to be optimal in Ref. [6]. An EW \( \sigma \) whose expectation value vanishes on at least one product vector is said to be weakly optimal [7]. A necessary condition for an EW \( W \) to be optimal is that there must exist a separable \( \sigma \) with \( \text{tr}(W\sigma) = 0 \) [2]. It is, however, a sufficient condition for the weakly optimal EW (WOEW) but not for the optimal EW (OEW). We say \( W_2 \) is finer than \( W_1 \) if the entangled state detected by \( W_2 \) is more than the one by \( W_1 \) [8].

For our purpose, we can only consider the quantum states on the finite dimensional Hilbert space \( \mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B \). We let \( \dim(\mathcal{H}_A) = d_A \), \( \dim(\mathcal{H}_B) = d_B \) and \( \dim(\mathcal{H}_{AB}) = d_{AB} \).

\section{II. THE ENTANGLEMENT WITNESSES AND STRUCTURAL PHYSICAL APPROXIMATIONS}

Our recent work [8] showed that any EW \( W \) can be written as

\[ W = \sigma - c_\sigma I \]

where \( \sigma \) is a (normalized) separable density matrix and \( \lambda_{0\sigma} < c_\sigma \leq c_\sigma^{\max} \) is a real number related to \( \sigma \), \( \lambda_{0\sigma} \) is the minimum eigenvalue of \( \sigma \) and

\[ c_\sigma^{\max} = \inf_{||\mu_A||=1,||\nu_B||=1} \langle \mu_A \nu_B | \sigma | \mu_A \nu_B \rangle \]

is the maximum number in \( c_\sigma \) which makes \( W = \sigma - c_\sigma I \) an EW and \( \{\mu_A \nu_B\} \) is any unit product vector. Clearly, an EW \( W = \sigma - c_\sigma I \) is weakly optimal if and only if \( c_\sigma = c_\sigma^{\max} \), and there exists a separable density matrix \( \sigma \) with \( W^{\text{opt}} = \sigma - c_\sigma^{\max} I \) for any OEW \( W^{\text{opt}} \).

### A. A sufficient condition of violating the SPA conjecture

Recently, a conjecture that SPA to optimal positive maps correspond to entanglement-breaking maps (channels) has been posed in [9]. An equivalent presentation of the conjecture (called SPA conjecture) is that SPA to optimal entanglement witnesses correspond to separable (unnormalized) states, i.e., the SPA to an optimal EW \( W^{\text{opt}} \),

\[ \tilde{W}^{\text{opt}} = W^{\text{opt}} + sI, \]

where \( s > 0 \) is the smallest parameter for which \( \tilde{W} \) is a positive operator (possibly unnormalized state) [8].

Based on this result, we found that the SPA conjecture does not depend on the optimality of EWs. We have found that the separability problems of SPA to both optimal EWs and non-optimal EWs become the same problem, that is, whether any

\[ \tilde{W} = \sigma - \lambda_{0\sigma} \]

is separable with \( W = \sigma - c_\sigma I \) being an EW (\( W = \sigma - c_\sigma^{\max} I \) being an OEW) [8].
Theorem 1. If $W = \sigma - c_\sigma I$ is an EW with $\sigma$ being not full rank, its SPA defines an entanglement-breaking channel (EBC) (the output is just $\sigma$).

Unless otherwise specified, EWs with $W = \sigma - c_\sigma I$ discussed below refer to EWs with $\sigma$ being full rank.

Following the definition in Ref. 9, if an EW can be written in the form $W = P + Q^2$ with $P, Q \geq 0$, we say it decomposable, otherwise we say it indecomposable. It is well known that the division of EWs to decomposable and indecomposable is translated from positive maps via the Jamiołkowski-Choi isomorphism 41.

For simplicity, we consider optimal nondecomposable EWs (ONEWS).

**Lemma 1** 41. 9. $W$ is an ONEW if and only if $W^T$ is an optimal nondecomposable EW, where $\Gamma$ denotes the partial transposition.

**Theorem 2.** If there exist an ONEW $W = \sigma - c_\sigma^{\max} I$ with $\lambda_{0W} < \lambda_\sigma$, $W$ violates the SPA conjecture.

**Proof.** For simplicity, let $0 < \lambda_{00} < \lambda_\sigma$ and $W = \sigma - \lambda_{00} I$, $W^T = \sigma - \lambda_{00} I$. By $\lambda_{00} < \lambda_\sigma$, $W^T = \sigma - \lambda_{00} I < 0$, and $W = \sigma - \lambda_\sigma I$ is not separable by positive partial transposition (PPT) criterion 10.

**Corollary 1.** If $W = \sigma - c_\sigma^{\max} I$ is an ONEW with $\lambda_{0W} < \lambda_{0W}$, the SPA of $W$ or the SPA of $W^T$ violates the SPA conjecture.

Following Corollary 1, the SPA conjecture is not true if there exists an EW with its minimum eigenvalue being not equal to the minimum eigenvalue of its partial transposition. Since the result of Theorem 2 is followed with the PPT criterion and the PPT criterion is a necessary but not a sufficient separable condition for separability, it is not easy to find the necessary condition of violating the SPA conjecture by our result.

**B. An example and discussion**

Since all structural approximations to positive maps of low dimensions define entanglement-breaking channels 8, it is not easy to construct the counterexample of the SPA conjecture. Very recently, Ha and Kye 11 exhibited counterexamples of indecomposable EWs violating the SPA conjecture. Stormer 12 gave a sufficient condition of violating the SPA conjecture by their theory.

Consider the ONEW in 11

$$W[a, b, c; \theta] = \begin{pmatrix}
  a & \cdots & -e^{i\theta} & \cdots & -e^{-i\theta} \\
  \cdot & c & \cdots & \cdot & \cdot \\
  \cdot & b & \cdot & \cdot & \cdot \\
  \cdot & \cdot & b & \cdot & \cdot \\
  -e^{i\theta} & \cdot & a & \cdot & -e^{-i\theta} \\
  \cdot & \cdot & c & \cdot & \cdot \\
  \cdot & \cdot & c & \cdot & \cdot \\
  \cdot & \cdot & \cdot & b & \cdot \\
  -e^{i\theta} & \cdot & -e^{-i\theta} & \cdot & a
\end{pmatrix}$$

Let $\theta = \pi/12, a = \frac{4}{3} \cos \frac{\pi}{12}, b = \frac{2}{3} \cos \frac{\pi}{12}$, and $c = 0$. We compute $\lambda_{0W} \approx -0.6440, \lambda_{0W} \approx -0.7286$, and $\lambda_{0W} \neq \lambda_{0W}$. By Corollary 1, $W[\frac{4}{3} \cos \frac{\pi}{12}, \frac{2}{3} \cos \frac{\pi}{12}; 0; \pi/12]$ violates the SPA conjecture.

Let $W = \sigma - c_\sigma^{\max} I$ be an ONEW. Suppose the spectral decomposition of $\sigma_\Gamma$, $\sigma_\Gamma = \sum_{i=0}^{d_{AB}-1} \lambda_{i\sigma} |f_i\rangle \langle f_i|$, the spectral decomposition of $\sigma$, $\sigma = \sum_{i=0}^{d_{AB}-1} \lambda_{i\sigma} |f_i\rangle \langle f_i|$, $W = \sigma - c_\sigma^{\max} I$ is an EW of $|f_0\rangle \langle f_0|$ and $|e_0\rangle \langle e_0|$: $W^T = \sigma^T - c_\sigma^{max} I$ is an EW of $|f_0\rangle \langle f_0|$ and $|e_0\rangle \langle e_0|$.

It is well known that decomposable EWs cannot detect PPT entangled states. If $W = \sigma - c_\sigma^{\max} I$ is a decomposable EW on $Q^T = Q \geq 0$ with $Q$ supporting by an entangled subspace and $W^T$ is entangled. Moreover, we have $\lambda_{0\sigma} < \lambda_{0\sigma}^T$ and $W^T = \sigma^T - \lambda_{0\sigma} I > 0$ for the decomposable EW $W = \sigma - c_\sigma I$.

Clearly, if $W = \sigma - \lambda_{0\sigma} I$ is entangled with $W = \sigma - c_\sigma I$ being a decomposable EW, it is a PPT entangled state and there exists an indecomposable EW detect it. If $W = \sigma - c_\sigma I$ is a decomposable EW with the spectral decomposition $\sigma = \sum_{i=0}^{d_{AB}-1} \lambda_{i\sigma} |f_i\rangle \langle f_i|$, $\pi = |f_0\rangle \langle f_0| > 0$ is entangled and is not a PPT entangled state, and one of its EW is $W = \sigma - c_\sigma I$, $\pi^T = |f_0\rangle \langle f_0|$ is an EW 8. By $\pi^T = |f_0\rangle \langle f_0|$ is optimal 13 and $tr(W^T \pi^T) = tr(W \pi) = 0$. Although we cannot obtain that $W^T$ is separable by $tr(W^T \pi^T) = 0$ for the optimal EW $\pi^T$, it seems that the SPA conjecture is true for decomposable EWs as stated in 11.

### III. GEOMETRIC ILLUSTRATIONS OF ENTANGLEMENT WITNESSES AND QUANTUM STATES

Considering the finite-dimensional Hilbert space $H_{AB} = C^{d_{AB}}$ as in 14, where observables $W$ are represented by all Hermitian matrices and states $\rho$ by density matrices, we can regard these quantities as elements of a real Hilbert space $H_r = R^{d_{AB}}$ with scalar product

$$\langle \rho | W \rangle = tr \rho W$$

and corresponding norm

$$\|W\|_2 = (tr W^2)^{1/2}.$$ (7)

Both density matrices with trace unity and observables are represented by vectors in $H_r$.

By Hahn-Banach theorem, we can view an EW $W$ as a hyperplane, which has dimension $d_{AB} - 1$ 15. We can view $W$ as a separating hyperplane with entangled state $\pi$ on one side and all separable states on the other 16. We can rewrite Eq. 1 as

$$W = \sigma - c_\sigma \tau_0,$$ (8)

where $\tau_0 = \frac{1}{d_{AB}} I$ denotes the maximally mixed state (separable). Therefore, any EW is equal to the difference between a separable state $\sigma$ and the product of a non-negative real number and the maximally mixed.
that tr(W^(wopt) σ') = 0, and tr(W^(wopt) σ) > 0.

The relation between separable states, entangled states and different EWs in the form W = σ − c_σ I is shown in Fig. 2. EWs are denoted as planes in Hermitian operator space. Sets Q, S, and E of quantum states, separable states, and entangled states are such that Q = S ∪ E. W^(wopt) = σ − c^(max) _σ _τ_0 and W'(wopt) = σ' − c'(σ) ^max _τ_0 are weakly optimal entanglement witnesses. W_2 = σ − c_2 σ I is finer than W_1 = σ − c_1 σ I. The “boundary” of EWs, W = σ − λ_0 σ I is not an entanglement witness, but a quantum state, entangled or separable.

FIG. 1. Geometric illustration of (a) The weakly optimal entanglement witness W^(wopt) = σ − c^(max) _σ _τ_0 and the “weakly optimal” plane W'(wopt); (b) The non-weakly-optimal and weakly optimal entanglement witnesses W^(wopt), W'(wopt) = σ − c_σ _τ_0 and the “non-weakly-optimal” and “weakly-optimal” planes W^(wopt), W'(wopt).

FIG. 2. By the general form of EWs constructed from separable states, schematic representation of entanglement witnesses and sets of quantum states, separable states, and entangled states.

IV. SUMMARY

In summary, we give a sufficient condition of violating the SPA conjecture [Phys. Rev. A 78, 062105 (2008)] following the general form of EWs constructed from separable states. Comparing with planes and vectors in Euclidean space, we make geometric illustrations of the connection between entanglement witnesses and the sets of quantum states, separable states, and entangled states.

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