The topology of local commensurability graphs

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Abstract

We initiate the study of the $p$-local commensurability graph of a group, where $p$ is a prime. This graph has vertices consisting of all finite-index subgroups of a group, where an edge is drawn between $A$ and $B$ if $[A:A \cap B]$ and $[B:A \cap B]$ are both powers of $p$. We show that any component of the $p$-local commensurability graph of a group with all nilpotent finite quotients is complete. Further, this topological criterion characterizes such groups. In contrast to this result, we show that for any prime $p$ the $p$-local commensurability graph of any large group (e.g. a nonabelian free group or a surface group of genus two or more or, more generally, any virtually special group) has geodesics of arbitrarily long length.

Keywords: commensurability, nilpotent groups, free groups, very large groups

Let $G$ be a group and $p$ a prime number. Recall that two subgroups $\Delta_1 \leq G$ and $\Delta_2 \leq G$ are commensurable if $\Delta_1 \cap \Delta_2$ is finite-index in both $\Delta_1$ and $\Delta_1$. We define the $p$-local commensurability graph of $G$ to be the graph with vertices consisting of finite-index subgroups of $G$ where two subgroups $A, B \leq G$ are adjacent if and only if $[A:A \cap B]$ and $[B:A \cap B]$ are both powers of $p$. We denote this graph by $\Gamma_p(G)$. For a warm-up example, see Figure 1.

The goal of this paper is to draw algebraic information of $G$ from the topology of $\Gamma_p(G)$.

Theorem 1. Let $G$ be a group. The following are equivalent:

1. For any prime $p$, every component of $\Gamma_p(G)$ is complete.

2. All of the finite quotients of $G$ are nilpotent.

The proof of Theorem 1 is in §2. The classification of finite simple groups and the structure theory of solvable groups play important roles in our proofs. Theorem 1 applies, for example, to Grigorchuk’s group [Gri83], which is a 2-group and therefore has only nilpotent finite quotients.

In contrast to the above theorem, we show that components of the local commensurability graphs of free groups are far from complete:

Theorem 2. Let $F$ be a rank two free group. For any prime $p$ and $N > 0$, there exist infinitely many geodesics $\gamma$, each in a different component of $\Gamma_p(F)$, such that the length of each $\gamma$ is greater than $N$.

We prove Theorem 2 in §3. A result of Robert Guralnick (which uses the classification of finite simple groups) concerning subgroups of prime power index in a nonabelian finite simple group is used in an essential way in our proof [Gur83]. Moreover, in our proof we get a clean description of an entire component of the $p$-local commensurability graph of many finite alternating groups. See Figure 2 for example.

Our next result demonstrates that arbitrarily long geodesics in the $p$-local commensurability graph of a free group cannot possibly all come from a single component. We prove this at the end of §4.

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Figure 1: Let $\text{Sym}_3$ be the symmetric group on 3 elements (note $\text{Sym}_3$ is solvable and not nilpotent). The figure above displays $\Gamma_2(\text{Sym}_3)$, $\Gamma_3(\text{Sym}_3)$, and $\Gamma_5(\text{Sym}_3)$ in that order. All $\Gamma_p(\text{Sym}_3)$ for primes $p > 3$ are discrete spaces.

**Proposition 3.** Let $G$ be any group. Let $\Omega$ be a connected component of $\Gamma_p(G)$. Then there exists $C > 0$ such that any two points in $\Omega$ are connected by a path of length less than $C$. That is, the diameter of $\Omega$ is finite. Moreover if any vertex of $\Omega$ is a normal subgroup of $G$ then the diameter of $\Omega$ is bounded above by 3.

As a consequence of Theorem 2 and Proposition 3 there exists components of the $p$-local commensurability graph of a nonabelian free group with no normal subgroups as vertices (see Corollary 22 at the end of §3).

Recall that a group is large if it contains a finite-index subgroup that admits a surjective homomorphism onto a non-cyclic free group. Such groups enjoy the conclusion of Theorem 2. See the end of §3 for the proof.

**Corollary 4.** Let $G$ be a large group. For any prime $p$ and $N > 0$, there exists infinitely many geodesics $\gamma$, each in a different component of $\Gamma_p(G)$, such that the length of each $\gamma$ is greater than $N$.

Experiments that led us to the above theorems were done using GAP [GAP15] and Mathematica [Res15].

This paper sits in the broader program of studying infinite groups through their residual properties, which is an area of much activity (see, for instance, [KT], [BRK12], [BRM11], [GK], [BRHP], [BRS], [KMT1], [Riv12], [Pat13], [LS03]). Specifically, a similar object is studied in the recent article [AAH+15]. There a graph is constructed with vertices consisting of subgroups of finite index, and an edge is drawn between two vertices if one is a prime-index subgroup (the prime is not fixed) of the other. They show that for every group $G$, their graph is bipartite with girth contained in the set $\{4, \infty\}$ and if $G$ is a finite solvable group, then their graph is connected.

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1 Preliminaries and basic facts

In this section we record some basic facts that will be used throughout. We start with a couple of elementary results.

**Lemma 5.** Let $\pi : G \to G/N$ be a quotient map. For subgroups $K \leq H \leq G$ we have

$$[H : K] = [\pi(H) : \pi(K)] [H \cap N : K \cap N].$$
Proof. We know that
\[ [H : K : K \cap N] = [H : K \cap N] \quad \text{and} \quad [H : H \cap N][H \cap N : K \cap N] = [H : K \cap N]. \]

Equating left hand sides and rearranging terms yields
\[ \frac{[H : H \cap N]}{[K : K \cap N]} = \frac{[H : K]}{[H \cap N : K \cap N]}. \]

Because \( \pi(K) = KN/N = K/(K \cap N) \), and similarly for \( H \), we see that
\[ [\pi(H) : \pi(K)] = \frac{[\pi(H)]}{[\pi(K)]} = \frac{[H : H \cap N]}{[K : K \cap N]} \cdot \]

The desired result follows. \[\square\]

Lemma 6. Let \( N \) be a normal subgroup of \( G \) and \( p \) a prime. If \( A \) and \( N \) are both subgroups of index a power of \( p \) in \( G \), then \( [G : A \cap N] \) is also a power of \( p \).

Proof. Let \( \pi : G \to G/N \) be the quotient map. Then \( [A : A \cap N] = |\pi(A)| \). Because \( G/N \) is a \( p \)-group, it follows that \( [A : A \cap N] \) is a power of \( p \). Therefore \( [G : A \cap N] = [G : A][A : A \cap N] \) is a power of \( p \). \[\square\]

Our next couple of lemmas give control of local commensurability graphs under some maps.

Lemma 7. If \( G \) is a group, \( \pi : G \to Q \) is a surjection, and \( \gamma \) a path in \( \Gamma_p(G) \), then \( \pi(\gamma) \) is a path in \( \Gamma_p(Q) \) with length bounded above by the length of \( \gamma \).

Proof. If \( K \leq H \leq G \) then \( [\pi(H) : \pi(K)] \) divides \( [H : K] \) by Lemma 5. Therefore adjacent vertices in \( \gamma \) map to adjacent vertices in \( \pi(\gamma) \), or are possibly identified in \( \Gamma_p(Q) \). \[\square\]

Lemma 8. Suppose \( G \) is a group and \( p \) is prime.

1. If \( N \) is a normal subgroup of \( G \), then the quotient map \( \pi : G \to G/N \) induces an isometric graph embedding \( \Gamma_p(G/N) \to \Gamma_p(G) \) as an induced subgraph.

2. If \( H \) is a finite-index subgroup of \( G \), then the inclusion \( i : H \to G \) induces a graph embedding \( \Gamma_p(H) \to \Gamma_p(G) \) as an induced subgraph.

3. If \( N \) is a finite-index normal subgroup of \( G \), then the inclusion \( i : N \to G \) induces an isometric graph embedding \( \Gamma_p(N) \to \Gamma_p(G) \) as an induced subgraph.

Proof. For 1, if \( \pi : G \to G/N \) is a quotient map, then the assignment \( K \mapsto \pi^{-1}(K) \) defines a graph embedding \( \Gamma_p(G/N) \to \Gamma_p(G) \) whose image is an induced subgraph. This embedding is isometric by Lemma 2.

For 2, if \( H \leq G \) has finite-index, then the assignment \( K \mapsto i(K) \) defines a graph embedding \( \Gamma_p(H) \to \Gamma_p(G) \) whose image is an induced subgraph.

For 3, let \( N \triangleleft G \) be a finite-index subgroup, with assignment \( \phi : K \mapsto i(K) \) defined over all subgroups \( K \) in \( N \). Let \( H_1, H_2 \in \phi(\Gamma_p(N)) \) and let \( H_1 = J_1, \ldots, J_n = H_2 \) be a path in \( \Gamma_p(G) \) from \( H_1 \) to \( H_2 \). Then for each \( i = 1, \ldots, n - 1 \), we have that
\[ [J_i : J_i \cap J_{i+1}][J_{i+1} : J_i \cap J_{i+1}] \]
is a power of \( p \). By Lemma 7, \( \pi(J_1), \ldots, \pi(J_n) \) is a path in \( \Gamma_p(G/N) \). Because \( J_1 \leq N \), this is a path of \( p \)-subgroups of \( G/N \). Therefore \( [J_i : J_i \cap N] \) is a power of \( p \) for all \( i = 1, \ldots, n \). Thus, by Lemma 6 applied to \( J_i \cap N \) and \( J_{i+1} \cap J_i \), we have for \( i = 1, \ldots, n - 1 \),
\[ [J_i : (J_i \cap N) \cap (J_i \cap J_{i+1})][J_{i+1} : (J_{i+1} \cap N) \cap (J_i \cap J_{i+1})], \]
is a power of $p$. Hence, for $i = 1, \ldots, n - 1$,

$$[J_i : N \cap J_i][N : J_i \cap J_{i+1}] = [J_i : N \cap J_i \cap J_{i+1}]$$

is a power of $p$ giving that $[N \cap J_i : N \cap J_i \cap J_{i+1}]$ is a power of $p$, since above we showed that $[J_i : N \cap J_i]$ is a power of $p$. By a similar argument, we get that $[N \cap J_{i+1} : N \cap J_i \cap J_{i+1}]$ is a power of $p$, and thus $N \cap J_i$ and $N \cap J_{i+1}$ are adjacent in $\Gamma_p(G)$. It follows that the path $J_1, \ldots, J_n$ can be replaced by the path (which possibly has repeated vertices) $J_1 = J_1 \cap N, J_2 \cap N, \ldots, J_{n-1} \cap N, J_n \cap N = J_n$, which is entirely contained in $\Gamma_p(H)$. It follows that $\Gamma_p(H)$ is a geodesic metric space in the path metric induced from $\Gamma_p(G)$, as desired. \hfill $\square$

Note that the hypothesis of normality in \cite{3} cannot be removed. For example, suppose $S$ and $T$ are disjoint sets with $|S| = |T| = 5$ and consider the non-normal subgroup $\operatorname{Alt}_S \times \operatorname{Alt}_T \leq \operatorname{Alt}_{S \cup T}$. It can be shown using Lemma \cite{18} below that $\operatorname{Alt}_S$ and $\operatorname{Alt}_T$ are in the same component of $\Gamma_5(\operatorname{Alt}_{S \cup T})$ but in different components of $\Gamma_5(\operatorname{Alt}_S \times \operatorname{Alt}_T)$.

Our next lemma will lead us to proving our first result concerning free groups.

**Lemma 9.** Let $A$ be a vertex in $\Gamma_p(G)$. Suppose $B$ shares an edge with $A$. If $q^k$ divides $|G : A|$ for some prime $q \neq p$ then $d^k$ divides $|G : B|$.

**Proof.** In this case, we have

$$|G : A \cap B| = |G : A||A : A \cap B| = |G : B||B : A \cap B|.$$  

Hence, if $q^k$ divides $|G : A|$, then $q^k$ must divide $|G : B|$ because $|B : A \cap B|$ is a power of $p$. \hfill $\square$

**Proposition 10.** The $p$-commensurability graph of a free group has infinitely many components.

**Proof.** Any free group has subgroups $N_1, N_2, \ldots$ with distinct prime indices $q_1, q_2, \ldots$. By the previous lemma, any vertex that is in the connected component of $N_i$ has index divisible by $q_i$. Thus, no path exists between $N_i$ and $N_j$ for distinct $i, j$. \hfill $\square$

We finish this section by proving a general result: for any group $G$, any component of $\Gamma_p(G)$ has finite diameter.

**Proof of Proposition 5**. Let $G$ be any group and $\Omega$ a component of $\Gamma_p(G)$. Take any vertex $A$ in $\Omega$ and let $N$ be the normal core of $A$. Let $\pi : G \to G/N$ be the quotient map. Let $D = \{BN : B \in \Omega\}$. We claim that the diameter of $\Gamma_p(G)$ is less than $|D| + 2$.

Let $B$ be a subgroup in $\Omega$. Let $V_1, \ldots, V_m$ be a path in $\Gamma_p(G)$ connecting $A$ to $B$. Then by Lemma \cite{5} $\pi(V_1), \ldots, \pi(V_m)$ is a path in $\Gamma_p(G/N)$ connecting $\pi(A)$ to $\pi(B)$. Hence

$$\pi(V_1)N, \ldots, \pi(V_m)N$$

is a path connecting $A$ to $BN$, and so $BN$ is an element of $\Omega$. Further, if $[G : B] = np^r$ where $\gcd(n, p^r) = 1$, then $[G : BN] = np^r$ by Lemma \cite{9} Since $B \leq BN$ and $[G : BN][BN : B] = [G : B]$, we get

$$np^r[BN : B] = np^k$$

and therefore $[BN : B] = p^{k-r}$. Hence $BN$ and $B$ are adjacent in $\Gamma_p(G)$. It follows that there is an edge from any element in $\Omega$ to one in $D$, and so the diameter of $\Omega$ is bounded above by the diameter of the subgraph induced by $D$ plus 2. This gives the desired bound $|D| + 2$.

If $\Omega$ contains a normal subgroup as a vertex then we can pick $A = N$ in the above argument. Therefore $D$ is the set of $p$-subgroups of $G/N$. Any two such subgroups are connected by an edge, so the diameter of $\Omega$ is bounded above by 3. \hfill $\square$
2 Nilpotent groups: The Proof of Theorem 1

We will prove Theorem 1 in two steps, as Propositions 12 and 13 below. For a finite nilpotent group $G$ let $S_p(G)$ denote the unique Sylow $p$-subgroup of $G$. Recall that $G$ is the direct product of its Sylow subgroups.

Lemma 11. Suppose $G = S_{p_1}(G) \times \cdots \times S_{p_k}(G)$ for primes $p_1, \cdots, p_k$. Let $\pi_i : G \to S_{p_i}(G)$ be the quotient map for each $i$. Then any subgroup $H \leq G$ has the form $H = \pi_1(H) \times \cdots \times \pi_k(H)$.

Proof. Choose $\ell_1, \ldots, \ell_k$ so that $g^{\ell_i} = 1$ for all $g \in S_{p_i}(G)$. Choose $N$ so that

$$Np_1^{\ell_1} \cdots p_k^{\ell_k} \equiv 1 \pmod{p_k^{\ell_k}}.$$ 

Take any $h \in H$ and write $h = (h_1, \ldots, h_k)$ for $h_i \in S_{p_i}(G)$ for all $i$. Then

$$h^Np_1^{\ell_1} \cdots p_k^{\ell_k} = (1, \ldots, 1, h_k).$$

Therefore $(1, \ldots, 1, h_k) \in H$, and so we may identify $\pi_k(H)$ with a subgroup of $H$. Applying this argument to each other factor, the result follows. \hfill \Box

Proposition 12. If $G$ is a finitely generated group such that every finite quotient of $G$ is nilpotent, then every component of $\Gamma_p(G)$ is complete for all $p$.

Proof. Suppose $A$ and $B$ are subgroups of $G$ in the same component of $\Gamma_p(G)$ for some prime $p$ and take any path $A = P_0, P_1, \ldots, P_n = B$ from $A$ to $B$. Let $N$ be a normal, finite-index subgroup of $G$ contained in $P_i$ for every $i$. Then $G/N$ is a nilpotent group and $\pi(P_0), \pi(P_1), \ldots, \pi(P_n)$ is a path in $\Gamma_p(G/N)$, where $\pi : G \to G/N$ is the quotient map.

Let $\mathcal{P}$ be a finite set of primes so that $G/N = \prod_{q \in \mathcal{P}} S_q(G/N)$. By Lemma 11 we have decompositions $\pi(P_i) = \prod_{q \in \mathcal{P}} S_q(\pi(P_i))$ for each $i$. It is straightforward to see that

$$\pi(P_i) \cap \pi(P_{i+1}) = \prod_{q \in \mathcal{P}} S_q(\pi(P_i)) \cap S_q(\pi(P_{i+1}))$$

for any $i$, and so for $j = i$ or $j = i + 1$ we have

$$[\pi(P_i) : \pi(P_i) \cap \pi(P_{i+1})] = \prod_{q \in \mathcal{P}} [S_q(\pi(P_i)) : S_q(\pi(P_i)) \cap S_q(\pi(P_{i+1}))].$$

Since $\pi(P_i)$ and $\pi(P_{i+1})$ are adjacent in the $p$-local commensurability graph of $G/N$, it follows that $S_q(\pi(P_i)) = S_q(\pi(P_{i+1}))$ for all $i$ and all $q \neq p$. Therefore $S_q(\pi(A)) = S_q(\pi(B))$ for all $q \neq p$, and so $[\pi(A) : \pi(A) \cap \pi(B)][\pi(B) : \pi(A) \cap \pi(B)]$ is a power of $p$. Because $[K : L] = [\pi(K) : \pi(L)]$ for any subgroups $L \leq K \leq G$ containing $N$, this shows that $A$ and $B$ are adjacent in $\Gamma_p(G)$. \hfill \Box

Lemma 13. If $Q$ is a finite solvable group that is not nilpotent then there is some prime $p$ so that a connected component of $\Gamma_p(Q)$ is not complete.

Proof. Let $\mathcal{P}$ be the set of prime divisors of the order of the finite solvable group $Q$. For any prime $q \in \mathcal{P}$ there is a Hall subgroup $H_q$ so that $[Q : H_q] = q^k$ for some $k$ and $q$ does not divide the order of $H_q$. Because $Q$ is not nilpotent, there is some prime $p$ and a Hall subgroup $H_p$ so that $g^{-1}H_pg \neq H_p$ for some $g \in Q$. Then both $H_p$ and $g^{-1}H_pg$ are adjacent to $Q$ in $\Gamma_p(Q)$, but there is no edge between $H_p$ and $g^{-1}H_pg$ in $\Gamma_p(Q)$. \hfill \Box

Lemma 14. If $Q$ is a non-abelian finite simple group then $Q$ contains a non-nilpotent solvable subgroup.
Proof. By Theorem 1 in [BW97] any non-abelian finite simple group contains a minimal simple group, a non-abelian simple group all of whose proper subgroups are solvable. By the Main Theorem of [Tho73], minimal simple groups come from the following list:

\[ \text{PSL}_2(q), \text{Sz}(q), \text{PSL}_3(3), M_{11}, 2F_4(2), \text{and } \text{Alt}_7. \]

We show that each of these subgroups contains a solvable and non-nilpotent subgroup:

1. The groups \( \text{PSL}_2(2) \cong \text{SL}_2(2) \cong \text{Sym}_3 \) and \( \text{PSL}_2(3) \cong \text{Alt}_4 \) are non-nilpotent and solvable. Suppose now that \( q - 1 > 2 \). As the group of units in a finite field of order \( q \) form a cyclic subgroup of order \( q - 1 \), there exists \( a \) such that \( a^2 \neq 1 \). Hence, there exists \( a, b \) in any finite field of order \( q \) such that \( ab = 1 \) and \( a \neq b \). The group in \( \text{SL}_2(q) \) generated by

\[
A := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

is solvable. Moreover, the order of \( A \) divides \( (q - 1) \) and the order of \( B \) divides \( q \). Moreover, we have

\[
ABA^{-1}B^{-1} = \begin{pmatrix} 1 & a/b - 1 \\ 0 & 1 \end{pmatrix},
\]

which is not central (the only central elements in \( \text{SL}_2(q) \) are diagonal matrices). Hence, \( \langle A, B \rangle \) has non-nilpotent image in \( \text{PSL}_2(q) \), as elements of coprime order in a finite nilpotent group must commute.

2. Any Suzuki group by [Suz60, 4.] has a cyclic subgroup that is maximal nilpotent and of index 4 in its normalizer. Because every group of order 4 is nilpotent, this normalizer is solvable. Since the cyclic subgroup is maximal nilpotent, this normalizer cannot be nilpotent.

3. \( \text{PSL}_3(3) \cong \text{SL}_3(3) \) contains \( \text{SL}_2(3) \), which is solvable and not nilpotent.

4. \( \text{M}_{11} \) contains \( \text{Sym}_5 \) as a maximal subgroup by [Hun80] and hence contains \( \text{Sym}_3 \).

5. \( 2F_4(2) \) contains \( \text{PSL}_2(25) \) as a maximal subgroup by [Hun80] and hence contains a subgroup that is solvable and not nilpotent by the above.

6. \( \text{Alt}_7 \) contains \( \text{Alt}_4 \).

\[\square\]

Proposition 15. Suppose \( G \) is a finitely generated group with a finite-index, normal subgroup \( N \) such that \( G/N \) is not nilpotent. Then there is some \( p \) so that a component of \( \Gamma_p(G) \) is not complete.

Proof. Take \( G \) and \( N \) as above, let \( Q = G/N \) and let \( \pi : G \to Q \) be the quotient map. If \( Q \) is solvable, then by Lemma 13 there is a prime \( p \) and subgroups \( A, B \leq Q \) in the same component of \( \Gamma_p(Q) \) that are not adjacent. Then \( \pi^{-1}(A) \) and \( \pi^{-1}(B) \) are non-adjacent vertices in the same component of \( \Gamma_p(G) \) by Lemma 8, so \( \Gamma_p(G) \) is not complete.

Now consider the case that \( Q \) is not solvable. Let \( Q = N_0 \geq N_1 \geq \cdots \geq N_{k-1} \geq N_k = \{1\} \) be any chain of subgroups such that \( N_{i+1} \) is a maximal normal subgroup of \( N_i \) for all \( i \). Because \( Q \) is not solvable, there is some \( j \) so that \( N_j/N_{j+1} \) is not abelian. Then by Lemma 14 there is a non-nilpotent solvable subgroup \( S \leq N_j/N_{j+1} \). By Lemma 13 there is some prime \( p \) with a component \( \Omega \) of \( \Gamma_p(S) \) that is not complete. By Lemma 8 the component \( \Omega \) fully embeds in a component of \( \Gamma_p(G) \), which is therefore not complete. \[\square\]
3 Free groups: The Proof of Theorem 2

Let $F$ be the free group of rank two. Let $p$ be a prime and $N \in \mathbb{N}$ be given. By Lemma 8 to prove Theorem 2 it suffices to find a finite quotient $Q$ of $F$ with subgroups $A, B \leq Q$ such that the length of any geodesic in $\Gamma_p(Q)$ connecting $A$ to $B$ is greater than $N$. Our candidate for $Q$ is $\text{Alt}_X$, the alternating group on a set $X$ of more than $p^k > N$ elements, and our candidates for $A$ and $B$ are conjugates of $\text{Alt}_S$ for a subset $S \subseteq X$ with $p^k$ elements.

We first need a couple technical group theoretic results. First, we give a description of a connected component in $\Gamma_p(\text{Alt}_X)$. This requires a simple lemma.

Lemma 16. If $T_1 \cap T_2$ has more than one element and $|T_1|, |T_2| \geq 4$, then $\langle \text{Alt}_{T_1}, \text{Alt}_{T_2} \rangle = \text{Alt}_{T_1 \cup T_2}$.

Proof. We prove this by induction on $|T_1 \cup T_2|$. The case that $T_1 = T_2$ is clear, so suppose $T_1 \neq T_2$. The base case, when $|T_1| = |T_2| = 4$ and $|T_1 \cap T_2| \in \{2, 3\}$, follows by computation (we did this in GAP15). For the inductive step, suppose without loss of generality that $x \in T_1 \setminus T_2$. By inductive hypothesis $\langle \text{Alt}_{T_1 \setminus \{x\}}, \text{Alt}_{T_2} \rangle = \text{Alt}_{T_1 \cup T_2 \setminus \{x\}}$. Arguing similarly if $T_2 \setminus T_1$ is nonempty, we reduce to the case when $T_1 \setminus T_2 \setminus T_1 \cap T_2$ consists of at most two points. To finish, we claim that any 3-cycle on points in $T_1 \cup T_2$ is in $\langle \text{Alt}_{T_1}, \text{Alt}_{T_2} \rangle$. Let $v_1, v_2, v_3$ be distinct points in $T_1 \cup T_2$. If $\{v_1, v_2, v_3\} \subseteq T_1$ or $\{v_1, v_2, v_3\} \subseteq T_2$, then we are done. Thus, by suitably relabeling, we may assume $v_1, v_2 \in T_1$ and $v_3 \in T_2$. Further, since $T_1 \cup T_2 \setminus T_1 \cap T_2$ consists of at most two points, then by relabeling again, we may assume $v_2 \in T_2$. Select $w_1, w_2 \in T_1 \cap T_2$ that are distinct from $v_1, v_2$, and $v_3$. Then, by the base case applied to $\langle v_1, v_2, v_3, w_1 \rangle \leq \text{Alt}_{T_1}$ and $\langle v_1, v_2, v_3, w_2 \rangle \leq \text{Alt}_{T_2}$, we obtain that $\langle v_1, v_2, v_3, w_1, w_2 \rangle$ is contained in $\langle \text{Alt}_{T_1}, \text{Alt}_{T_2} \rangle$, and hence the desired 3-cycle is found. This completes the proof. □

For any subset $S \subseteq X$, we denote the symmetric group on $S$ by $\text{Sym}_S$ and the alternating group on $S$ by $\text{Alt}_S$. For a subgroup $P \leq \text{Sym}_S$ we define the support to be the complement of the fixed point set of the action of $P$ on $S$.

Lemma 17. Let $p$ be a prime number and $k$ an integer so that $p^k > 4$. Let $X$ be a finite set, $S \subseteq X$, and $P \leq \text{Sym}_S$ a $p$-group with support disjoint from $S$. Let $E$ be an index $p^k$ subgroup of $\text{Alt}_S \times P$. If $|S| = p^k$ or $|S| = p^k - 1$, then we have the decomposition $E = \text{Alt}_T \times P'$ for some $P' \leq P$ and some $T \subseteq S$ with $|T| = p^k$ or $|T| = p^k - 1$.

Proof. Let $\pi : \text{Alt}_S \times P \rightarrow \text{Alt}_S$ be the projection map. By Lemma 5 we have

$$[\text{Alt}_S \times P : E] = [\text{Alt}_S : \pi(E)][1 \times P : E \cap (1 \times P)].$$

The left hand side of this equation is a power of $p$, so $[\text{Alt}_S : \pi(E)]$ is a power of $p$. Because $|S| = p^k$ or $|S| = p^k - 1$ by assumption, Theorem 1(a) in [Our88] immediately implies that either $\pi(E) = \text{Alt}_S$ or $|S| = p^k$ and $\pi(E) = \text{Alt}_{S \setminus \{v\}}$ for some $v \in S$. Let $T$ denote the set such that $\pi(E) = \text{Alt}_T$. Let $q$ be 3 if $p \neq 3$ and $q$ be 2 if $p = 3$. For the case $p \neq 3$, recall that $\text{Alt}_T$ is generated by 3-cycles by elementary properties of alternating groups. In the case $p = 3$, note that $p^k > 6$. Because $\text{Alt}_S$ is generated by an element of order 2 and one of order 4, Lemma 16 implies that $\text{Alt}_T$ is generated by elements of order 2 or 4 in this case. Therefore in either case it follows that $\text{Alt}_T$ is generated by elements $g_1, \ldots, g_k$ each with order dividing a power of $q$. Since $\pi$ maps onto $\text{Alt}_T$, we have that for each $i = 1, \ldots, k$, there exists $v_i \in P$ such that $(g_i, v_i) \in E$. Since $v_i \in P$, we have that the order of $v_i$ is coprime with $g_i$, hence as $q \neq p$, there exists $\ell$ such that

$$(g_i, v_i)^\ell = (g_i, 1).$$

It follows then that $E$ contains all of $\text{Alt}_T \times 1$, and hence $E = \text{Alt}_T \times P'$ where $P' \leq P$, as desired. □

Let $\Omega_{S,X}$ be the component of $\Gamma_p(\text{Alt}_X)$ containing $\text{Alt}_S$, and let $B_{S,X}$ denote the set of subgroups in $\Omega_{S,X}$ isomorphic to $\text{Alt}_T$ for some $|T| \in \{p^k, p^k - 1\}$. For odd primes $p$, we get the following description:
Lemma 18. Let \( S \subseteq X \) be a set of cardinality \( p^k \) for some odd prime \( p \) such that \( p^k > 4 \). Vertices of the component \( \Omega_{S,X} \) in \( \Gamma_p(Alt_X) \) consist of two classes of subgroups:

**Type 1.** subgroups of the form \( (Alt_T, P) \), where \( |T| = p^k \) and \( P \leq Alt_X \), and

**Type 2.** subgroups of the form \( (Alt_T, P) \), where \( |T| = p^k - 1 \) and \( P \leq Alt_X \).

In either case, the subgroup is \( Alt_T \times P \), where \( P \) is a \( p \)-group with support in \( T^c \). Moreover, for all primes \( p \), if \( V \) is of Type 1 or Type 2, the set \( T \) is uniquely determined by \( V \).

**Proof.** We first show uniqueness of \( T \). This implies that Type 1 and Type 2 are disjoint classes. Let \( V \) be a vertex with distinct decompositions \( Alt_T \times P_i \) with \( |T_i| > 3 \) and \( p \)-group \( P_i \) with support in \( T_i^c \) for \( i = 1, 2 \) such that \( T_1 \neq T_2 \). If \( T_i \cap T_j \) is empty, then

\[
[V : Alt_{T_1} \times Alt_{T_2} \times 1 : Alt_{T_1} \times 1] = |V : Alt_{T_1} \times 1| = |P_1|,
\]

and thus \( [Alt_{T_1} \times Alt_{T_2} \times 1 : Alt_{T_1} \times 1] = |Alt_{T_2}| \) must be a power of \( p \). But this is impossible as \( |Alt_{T_2}| \) is either \( (p^k)!/2 \) or \( (p^k - 1)!/2 \) for \( p^k > 4 \). Thus, \( T_1 \) and \( T_2 \) overlap. If \( T_1 \neq T_2 \) then \( Alt_{T_1} \times 1 \) cannot be normal because \( Alt_{T_2} \) acts transitively on \( T_2 \). But \( Alt_{T_1} \times 1 \) is clearly normal in \( Alt_{T_1} \times P_1 \), so this is a contradiction. Therefore \( T_1 = T_2 \).

Since elements in \( B_{S,X} \) are of Type 1 or 2, it suffices to show that any \( E \) that is adjacent to an element of Type 1 or 2 must itself be of Type 1 or 2.

Let \( E \) be adjacent to \( V = Alt_T \times P \) where \( P \) is a \( p \)-group with support in \( T^c \) and \( |T| = p^k \) or \( |T| = p^k - 1 \). Then \( E \cap V \) is a subgroup of \( Alt_T \times P \) of index a power of \( p \). By Lemma 17, \( E \cap V = Alt_T \times P' \) or \( E \cap V = Alt_T \times P' \) where \( P' \leq P \) and \( v \in T \). We will therefore assume without loss of generality that \( E \) contains \( Alt_T \times 1 \) and \( Alt_T \) as a subgroup of \( p \)-power index.

Suppose that \( E \) does not leave \( T \) invariant. Let \( T_1, T_2, \ldots, T_k \) be the orbit of \( T \) acting on \( T \) and note that \( E \) contains \( Alt_T \) for each \( i \). Suppose \( T_i \cap T_{i+1} \) has fewer than two elements for some \( i \). The group \( Alt_T \) contains \( Alt_{T \setminus T_i \cup T_{i+1}} \), which includes a permutation of order 2 since \( |T_i| > 4 \). Hence \( E \) contains \( Alt_T \times \mathbb{Z}/2 \mathbb{Z} \). But this is impossible, as \( Alt_T \) is of index \( p^k \) in \( E \) for an odd prime \( p \). We therefore know that \( T_i \cap T_{i+1} \) has more than two elements for every \( i \). Then by applying Lemma 16, we conclude that \( E \) contains \( Alt_{T_1 \cup T_2 \cup \cdots \cup T_k} \). Since \( E \) contains \( Alt_T \) as a subgroup of prime power index and \( T_1 \cup T_2 \cup \cdots \cup T_k \neq T_1 \), it follows that \( |T_1 \cup T_2 \cup \cdots \cup T_k| = p^k \) and in fact \( E \) contains \( Alt_{T_1 \cup T_2 \cup \cdots \cup T_k} \) as a subgroup of index \( p^k \) for some \( \ell \).

We may therefore assume, after replacing \( T \) with \( T_1 \cup T_2 \cup \cdots T_k \) if necessary, that \( E \) leaves \( T \) invariant. Then \( E \leq Sym_T \times Q \) where \( Q \) is a group with support disjoint from \( T \). Let \( \pi : Sym_T \times Q \rightarrow Sym_T \) be the projection onto the first coordinate. By Lemma 5, \( [\pi(E) : Alt_T] \) divides \( |E : Alt_T| \) and hence is a power of \( p \). It follows that \( \pi(E) = Alt_T \), as \( Alt_T \) is a maximal subgroup of \( Sym_T \) of index two. Further, since \( Alt_T \) is normal, we apply Lemma 5 to the map \( \psi : Alt_T \times Q \rightarrow Q \) to see that \( |\psi(E)| \) is a power of \( p \). Applying Lemma 17, we obtain the desired conclusion.

The prime \( p = 2 \) requires relaxing the conclusion of Lemma 18 since any symmetric group on three or more elements contains an alternating group of index 2.

Lemma 19. Let \( S \subseteq X \) be a set of cardinality \( 2^k \) such that \( k > 2 \). Vertices of the component \( \Omega_{S,X} \) in \( \Gamma_p(Alt_X) \) is at least one of two types:

**Type 1'.** subgroups \( V \) such that \( Alt_T \times 1 \leq V \leq Sym_T \times P \), where \( |T| = 2^k \) and \( P \leq Alt_X \), and

**Type 2'.** subgroups \( V \) such that \( Alt_T \times 1 \leq V \leq Sym_T \times P \), where \( |T| = 2^k - 1 \) and \( P \leq Alt_X \).

In either case, \( P \) is a \( 2 \)-group with support in \( T^c \).

**Proof.** Since elements in \( B_{S,X} \) are of Type 1’ or 2’, it suffices to show that any \( E \) that is adjacent to an element of one of the types must itself be of one of the types.
Let $E$ be adjacent to some $V$ with $\text{Alt}_T \times 1 \leq V \leq \text{Sym}_T \times P$ where $P$ is a 2-group. Because $V$ has index a power of 2 in $\text{Sym}_T \times P$, we know that $E \cap V$ also has index a power of 2 in $\text{Sym}_T \times P$. Since $\text{Alt}_T \times P$ is a normal subgroup of $\text{Sym}_T \times P$, we have by Lemma 6 that $(\text{Alt}_T \times P) \cap E \cap V$ has index a power of 2 in $\text{Sym}_T \times P$, and hence in $\text{Alt}_T \times P$. By Lemma 17 $E \cap V \cap (\text{Alt}_T \times P) = \text{Alt}_T \times P'$ or $E \cap V \cap (\text{Alt}_T \times P) = \text{Alt}_{T \setminus \{v\}} \times P'$ where $P' \leq P$ and $v \in T$. We conclude that $\text{Alt}_T \times 1$ or $\text{Alt}_{T \setminus \{v\}} \times 1$ has index a power of 2 in $E \cap V \cap (\text{Alt}_T \times P)$, and hence has index a power of 2 in $E$. We will therefore assume without loss of generality that $E$ contains $\text{Alt}_T \times 1 = \text{Alt}_T$ as a subgroup with index a power of 2, where $|T'| = 2^k$ or $|T'| = 2^k - 1$.

Suppose that $E$ does not leave $T$ invariant. Let $T_1, T_2, \ldots, T_k$ be the orbit of $E$ acting on $T$ and note that $E$ contains $\text{Alt}_T$ for each $i$. Suppose $T_1 \cap T_{i+1}$ has fewer than two elements for some $i$. The group $\text{Alt}_T$ contains $\text{Alt}_{T_1 \cap T_{i+1}}$, which includes a permutation of order 3 because $|T_i| > 3$. Hence $E$ contains $\text{Alt}_T \times \mathbb{Z}/3\mathbb{Z} \geq \text{Alt}_T$. This is impossible, as $\text{Alt}_T$ is of 2 power index in $E$. We therefore know that $T_1 \cap T_{i+1}$ has more than two elements for every $i$. Then by applying Lemma 16 we conclude that $E$ contains $\text{Alt}_{T_1 \cup T_2 \cup \cdots \cup T_k}$. Since $E$ contains $\text{Alt}_T$ as a subgroup of prime power index and $T_1 \cup T_2 \cup \cdots \cup T_k \neq T_1$, it follows that $|T_1 \cup T_2 \cup \cdots \cup T_k| = 2^k$ and in fact $E$ contains $\text{Alt}_{T_1 \cup T_2 \cup \cdots \cup T_k}$ as a subgroup of index $2^k$ for some $\ell$.

We may therefore assume, after replacing $T$ with $T_1 \cup T_2 \cup \cdots \cup T_k$ if necessary, that $E$ leaves $T$ invariant. Then $\text{Alt}_T \times 1 \leq E \leq \text{Sym}_T \times Q$ where $Q$ is a 2-group with support disjoint from $T$, as desired. 

Note that groups of Type 1 and Type 2 are of Type 1’ and Type 2’ respectively. The next result allows us to restrict attention to geodesics in $B_{S,X}$ when computing distances there.

**Lemma 20.** Let $S \subseteq X$ be a set of cardinality $p^k > 4$ for some prime $p$ and integer $k$. Then $B_{S,X}$ is a geodesic metric space in the path metric induced from $\Omega_{S,X}$.

**Proof.** We first need a local fact. Let $V_1, V_2$ be two adjacent vectors in $\Omega_{S,X}$. If $p$ is odd, then by Lemma 18 we have $V_i = \text{Alt}_{T_i} \times P_i$, where $|T_i| = p^k$ or $|T_i| = p^k - 1$ and the support of $P_i$ is disjoint from $T_i$ for $i = 1, 2$. If $p = 2$, then by Lemma 19 $\text{Alt}_T \times 1 \leq V_i \leq \text{Sym}_T \times P_i$, where $|T_i| = p^k$ or $|T_i| = p^k - 1$ and the support of $P_i$ is disjoint from $T_i$ for $i = 1, 2$. We claim that in either case, $\text{Alt}_{T_1}$ and $\text{Alt}_{T_2}$ are connected by an edge in $\Gamma_p(\text{Alt}_X)$.

Since $V_1$ and $V_2$ are adjacent, we have that

$$[V_1 : V_1 \cap V_2][V_2 : V_1 \cap V_2]$$

is a power of $p$. Thus, when $p$ is odd, Lemma 17 applied twice along with the uniqueness in Lemma 18 gives that $V_1 \cap V_2$ is $\text{Alt}_T \times P$ where $S \subseteq T_1 \cap T_2$ satisfies $|S| = p^k$ or $|S| = p^k - 1$ and $P \leq P_1 \cap P_2$. Thus it is straightforward to see that $\text{Alt}_T$ is adjacent to $\text{Alt}_{T_2}$.

When $p = 2$, set $H_i = \text{Alt}_{T_i} \times 1$ and $\Lambda = V_1 \cap V_2$. Then $H_i$ is normal in $V_i$, thus $H_i \cap \Lambda$ is normal in $\Lambda$. Since $[\text{Sym}_T \times P_i : H_i]$ is a power of 2 and

$$[\text{Sym}_T \times P_i : V_i][V_i : H_i] = [\text{Sym}_T \times P_i : H_i],$$

we get $[V_i : H_i]$ is a power of 2. Since $H_i$ is normal in $V_i$, Lemma 6 implies that $[V_i : H_i \cap \Lambda]$ is a power of 2. Further, as $[V_i : \Lambda]$ is a power of 2 and

$$[V_i : \Lambda][\Lambda : H_i \cap \Lambda] = [V_i : H_i \cap \Lambda]$$

we conclude that $[\Lambda : H_i \cap \Lambda]$ is a power of 2 for $i = 1, 2$. Thus, applying Lemma 6 to $H_1 \cap \Lambda \triangleleft \Lambda$ and $H_2 \cap \Lambda \triangleleft \Lambda$, we have that $H_1 \cap H_2 \cap \Lambda$ has index a power of 2 in $\Lambda$. As

$$[V_i : \Lambda][\Lambda : H_i \cap H_2 \cap \Lambda] = [V_i : H_i \cap H_2 \cap \Lambda],$$

it follows that $[V_i : H_1 \cap H_2 \cap \Lambda]$ is a power of 2. Because $[V_i : H_i]$ is also a power of 2 (shown above) and

$$[V_i : H_i][H_i : H_1 \cap H_2 \cap \Lambda] = [V_i : H_i \cap H_2 \cap \Lambda]$$
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we have $|H_i : H_1 \cap H_2 \cap \Lambda|$ is a power of 2 for each $i$. By applying Theorem 1(a) in [Gur83] and the uniqueness in Lemma 18, we have $H_1 \cap H_2 \cap \Lambda$ is $\text{Alt}_S$ for some $S \subseteq T_1 \cap T_2$ with $|S| = p^k$ or $|S| = p^k - 1$. Thus, $\text{Alt}_{T_i}$ is adjacent to $\text{Alt}_{T_j}$, as claimed.

Now let $\gamma$ be a path in $\Omega_{S,X}$ that, except for its endpoints, is entirely in the complement of $B_{S,X}$. Enumerate the vertices of $\gamma$ in the order they are traversed,

$$V_1, V_2, \ldots, V_m,$$

where $\text{Alt}_{T_i} \times 1 \leq V_i \leq \text{Sym}_{T_i} \times P_i$ for all $i = 1, \ldots, m$.

Then by the previous claim, we may form a new path (after throwing out repeated vertices)

$$\text{Alt}_{T_1}, \text{Alt}_{T_2}, \ldots, \text{Alt}_{T_m},$$

that is entirely contained in $B_{S,X}$ and has the same endpoints as $\gamma$. It follows that $B_{S,X}$ is geodesic in $\Omega_{S,X}$, as desired.

**Proposition 21.** Let $S \subseteq X$ be a set of cardinality $p^k > 4$ for some prime $p$ and integer $k$. There exists $V, W \in B_{S,X}$ such that any path in $\Omega_{S,X}$ connecting $V$ to $W$ has length at least $p^k - \max\{0, 2p^k - |X|\}$. 

Figure 2: $\Omega_{S,X}$ with $|S| = 5$ and $|X| = 7$. The coloring gives the types and the numbers give the valence of each vertex. This figure was generated using GAP [GAP15] and Mathematica [Res15].
Proof. By Proposition 20, it suffices to show that there exists $V, W \in B_{S,Y}$ such that any path in $B_{S,Y}$ has length greater than $|X| - p^k$. Let $O_1, O_2 \subseteq X$ with $|O_1 \cap O_2| \leq \max\{0, 2p^k - |X|\}$. Let $E_1, E_2, \ldots, E_m$ be distinct vertices in a non-back-tracking path in $B_{S,Y}$ connecting $Alt_{O_1}$ to $Alt_{O_2}$. Let $T_1, T_2, \ldots, T_m$ be subsets of $X$ such that $E_i = Alt_{T_i}$ for $i = 1, \ldots, m$. For each $i = 1, \ldots, m$, we have one of three cases:

1. $E_i$ is Type 1 and $E_{i+1}$ is Type 1: In this case, $|T_{i+1} \cap T_i| = |T_i| - 1 = p^k - 1$.
2. $E_i$ is Type 1 and $E_{i+1}$ is Type 2: In this case, $T_{i+1} \subseteq T_i$ and $|T_{i+1}| = |T_i| - 1 = p^k - 1$.
3. $E_i$ is Type 2 and $E_{i+1}$ is Type 1: In this case, $T_{i+1} \supseteq T_i$ and $|T_{i+1}| = |T_i| + 1 = p^k$.
4. $E_i$ is Type 2 and $E_{i+1}$ is Type 2: This case never occurs, as $[Alt_T : Alt_U]$ is not a power of $p$ for any proper subset $U \subset T$ with $|T| = p^k - 1$.

Thus, we see that for each $i$, we see that $T_i$ and $T_{i+1}$ differ by moving, adding, or removing at most one element. It follows that $m \geq p^k - |O_1 \cap O_2| \geq p^k - \max\{0, 2p^k - |X|\}$. \hfill \qed

Proof of Theorem 2. Let $F$ be a rank two free group and $p$ a prime. Given $N > 0$, choose $k$ so that $p^k > N$ and $p^k > 4$. For any finite set $X$ with $|X| > 2p^k$, let $\gamma_k$ be a path of length $p^k$ in $\Gamma_p(Alt_X)$ guaranteed by Proposition 21. Then pulling back $\gamma_k$ over any surjection $\pi : G \to Alt_X$ produces a path of length $p^k$ in $\Gamma_p(F)$ by Lemma 8. By Lemma 8, sets $X_1$ and $X_2$ with relatively prime cardinalities will produce geodesics in different components of $\Gamma_p(F)$. \hfill \qed

Proof of Corollary 4. Let $G$ be a large group, $p$ a prime, and $N > 0$. Since a finite-index subgroup of a nonabelian free group is nonabelian, there exists a normal finite-index subgroup $H \leq G$ that surjects onto $F$, the free group of rank 2. By Lemma 7 and Theorem 2, there exists vertices $V, W \in \Gamma_p(H)$ such that any path connecting them in $\Gamma_p(G)$ has length greater than $N$. The result now follows from Lemma 8 as $\Gamma_p(H)$ isometrically embeds into $\Gamma_p(G)$. \hfill \qed

Corollary 22. Let $G$ be a large group and $p$ a prime. There exists a connected component of $\Gamma_p(G)$ that does not contain any normal subgroup.

Proof. By Proposition 4, any component of $\Gamma_p(G)$ containing a normal subgroup as a vertex has diameter at most 3. By Corollary 4, there are components of $G$ with arbitrarily long geodesics. \hfill \qed

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