THE NON-EXISTENCE OF (104, 22, 3)-ARCS

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Abstract. Using some recent results about multiple extendability of arcs and codes, we prove the nonexistence of (104, 22)-arcs in PG(3, 5). This implies the non-existence of Griesmer [104, 4, 82]_5-codes and settles one of the four remaining open cases for the main problem of coding theory for q = 5, k = 4, d = 82.

1. Introduction

A central problem in coding theory is to optimize one of the three main parameters of a linear code over a fixed field F_q given the other two [7]. The most popular version of this problem is to find the minimal length, denoted by n_q(k, d), of a linear code for fixed values of the dimension k and the minimum distance d. There is a well-known lower bound on n_q(k, d), due to Griesmer [5, 15]:

\[ n_q(k, d) \geq g_q(k, d) := \sum_{i=0}^{k-1} \left\lfloor \frac{d}{q^i} \right\rfloor. \]

Codes meeting the Griesmer bound are called Griesmer codes. For fixed k and q, Griesmer codes exist for all sufficiently large d [6]. Thus the problem of determining the exact value of n_q(k, d) for fixed q and k for all d is a finite one. It has been solved in the following cases:

- for q = 2, k ≤ 8 for all d;
- for q = 3, k ≤ 5 for all d;
- for q = 4, k ≤ 4 for all d;
- for q = 5, 7, 8, 9, k ≤ 3 for all d.

In the case of q = 5, k = 4, there exist four values of d for which n_q(k, d) is not decided. They are given in the table below.

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In this paper, we present a new method for proving the non-existence of arcs and codes with prescribed parameters which provides a solution of certain instances of the main problem of coding theory. The method is based on the extendability of a special family of codes called quasidivisible codes [10]. In particular, we prove the non-existence of [104, 4, 82]₅-codes, which implies that \( n_5(4, 82) = 105 \) and settles one of the four open cases for \( q = 5, k = 4 \).

The approach to this problem is geometric. It is known that the existence of a linear \([n, k, d]_q\) code is equivalent to that of a (multi)arc with parameters \((n, n - d)\) in \( PG(k - 1, q) \) [3]. We consider a hypothetical \((104, 22)\)-arc \( K \) in \( PG(3, 5) \). It is known (cf. [13]) that an arc with these parameters is not extendable. This implies certain restrictions on a special arc in the dual space related to \( K \). The existence of the latter is ruled out using recent results from [9, 10].

The paper is structured as follows. In section 2, we recall some basic definitions on arcs in \( PG(k - 1, q) \). We introduce the notions of induced and dual arc and prove a necessary condition for the existence of an arc with given parameters based on analogues of the MacWilliams identities for arcs. At the end of the section, we present the spectra of some arcs in \( PG(2, 5) \) that are needed in the proof. In section 3, we introduce the notion quasidivisible arc and relate the structure of a quasidivisible arc \( K \) to the structure of a certain dual arc \( \tilde{K} \) of \( K \). We prove a theorem which relates the size of \( \tilde{K} \) to the structure of the restriction of \( K \) to a hyperplane. We present a characterization of the \((3 \text{ mod } 5)\)-arcs of small sizes in \( PG(2, 5) \) and \( PG(3, 5) \). The non-existence of \((104, 22)\)-arcs is proved in section 4. We show that such an arc is necessarily quasidivisible. Furthermore, we rule out the existence of planes of certain multiplicities and show that \( \tilde{K} \) is not “too big”. This implies that the only valid possibility for \( \tilde{K} \) is a sum of three planes, which in turn is a contradiction to earlier classification results.

2. Preliminary results

Let \( \mathcal{P} \) be the set of points of the projective geometry \( \Sigma = PG(k - 1, q) \). Every mapping \( K : \mathcal{P} \rightarrow \mathbb{N}_0 \) is called a multiset in \( PG(k - 1, q) \). This mapping is extended additively to the subsets \( Q \) of \( \mathcal{P} \) by \( K(Q) = \sum_{P \in Q} K(P) \). The integer \( n := K(\mathcal{P}) \) is called the cardinality of \( K \). The support of \( K \) is the set of all points of positive multiplicity. The sequence \((a_i)\), where \( a_i \) is the number of hyperplanes \( H \) with \( K(H) = i \), is called the spectrum of \( K \).

A multiset \( K \) in the geometry \( PG(k - 1, q) \) is called an \((n, w)\)-multiset (or simply an \((n, w)\)-arc) if (a) \( K(\mathcal{P}) = n \), (b) \( K(H) \leq w \) for every hyperplane \( H \), and (c) there exists a hyperplane \( H_0 \) with \( K(H_0) = w \). A multiset \( K \) in \( PG(k - 1, q) \) is called an \((n, w)\)-blocking set (or \((n, w)\)-minihyper) if (a) \( K(\mathcal{P}) = n \), (b) \( K(H) \geq w \) for every hyperplane \( H \), and (c) there exists a hyperplane \( H_0 \) with \( K(H_0) = w \).

Denote by \( \mathcal{H} \) the set of all hyperplanes in \( \Sigma \) and let \( K \) be a multiset in \( \Sigma \). Consider a function \( \sigma \) such that \( \sigma(K(H)) \) is a non-negative integer for all hyperplanes \( H \). The arc

\[
\bar{K}_{\sigma} : \begin{cases}
\mathcal{H} \rightarrow \mathbb{N}_0 \\
H \mapsto \sigma(K(H))
\end{cases}
\]
is called the $\sigma$-dual of $K$. If $\sigma$ is a linear function, the parameters of $\tilde{K}_\sigma$ are easily computed from the parameters of $K$ [11].

An $(n, w)$-arc $K$ in $PG(k - 1, q)$ is called $t$-extendable, if there exists an $(n + t, w)$-arc $K'$ in $PG(k - 1, q)$ with $K'(P) \geq K(P)$ for every point $P \in P$. An arc is called simply extendable if it is $1$-extendable. An $(n, w)$-blocking set $K$ in $PG(k - 1, q)$ is called reducible (also minimal) if there exists an $(n - 1, w)$-blocking set $K'$ with $K'(P) \leq K(P)$.

An $(n, w)$-arc $K$ with spectrum $(a_i)$ is said to be divisible with divisor $\Delta > 1$ if
\[ a_i = 0 \quad \text{for all } i \not\equiv n \pmod{\Delta}. \]

The $(n, w)$-arc $K$ with $w \equiv n + t \pmod{\Delta}$ is called $t$-quasidivisible with divisor $\Delta > 1$ (or $t$-quasidivisible modulo $\Delta$) if $a_i = 0$ for all $i \not\equiv n, n + 1, \ldots, n + t \pmod{\Delta}, 1 \leq t \leq q - 1$.

Let $K$ be an $(n, w)$-arc in $PG(k - 1, q)$. Let $U$ and $V$ be subspaces of $PG(k - 1, q)$ with $\dim U + \dim V = k - 2, U \cap V = \emptyset$. We can define a projection from $U$ onto $V$ by
\[
\varphi = \varphi_{U, V} : \begin{cases}  
P \setminus P(U) & \mapsto V \\  P & \mapsto V \cap \langle U, P \rangle \end{cases}
\]

where $P$ is the pointset of $PG(k - 1, q)$. Obviously, $\varphi$ maps $(u + s)$-dimensional subspaces of $PG(k - 1, q)$ containing $U$ to $(s - 1)$-dimensional subspaces of $V$. For an $(n, w)$-arc $K$ in $PG(k - 1, q)$, we define the induced arc $K^\varphi$ on the set of points of $V$ by
\[
K^\varphi : \begin{cases}  \mathcal{P}(V) & \mapsto \mathbb{N}_0 \\  P & \mapsto \sum_{Q \in \mathcal{P} \setminus U, \varphi(Q) = P} K(Q). \end{cases}
\]

If $T$ is a $t$-dimensional subspace of $V$, then $K^\varphi(T) = K(\langle T, U \rangle) - K(U)$. Clearly, $K^\varphi$ is a multiarc in $V$ with parameters $(n - K(U), w - K(U))$. In this paper, the subspace $U$ will be always a point.

Let $K$ be an $(n, w)$-arc in $PG(k - 1, q)$ with spectrum $(a_i)$. Denote by $\lambda_j$ the number of points $P$ with $K(P) = j$. Simple counting arguments yield the following identities, which are equivalent to the first three MacWilliams identities for linear codes:
\[
\begin{align*}
\sum_{i=0}^{w} a_i &= q^k - 1 \quad / q - 1, \\
\sum_{i=1}^{w} i a_i &= n \cdot q^{k-1} - 1 \quad / q - 1, \\
\sum_{i=2}^{w} \binom{i}{2} a_i &= \binom{n}{2} q^{k-2} - 1 \quad / q - 1 + q^{k-2} \cdot \sum_{i \geq 2} \binom{i}{2} \lambda_i.
\end{align*}
\]

Set $v_i = (q^i - 1)/(q - 1)$. The above identities imply:
\[
(2) \quad \sum_{i=0}^{w} \binom{w-i}{2} a_i = \binom{w}{2} v_k - n(w - 1)v_{k-1} + \binom{n}{2} v_{k-2} + q^{k-2} \cdot \sum_{i \geq 2} \binom{i}{2} \lambda_i.
\]

Note that the sum on the left-hand side can be written as $\sum_H \binom{w-K(H)}{2}$, where the sum is taken over all hyperplanes $H$ of $PG(k - 1, q)$. Fix a hyperplane $H_0$ and consider all hyperlines $S$ (i.e. over all subspaces of co-dimension 2) contained in $H_0$. For such a subspace $S$, denote by $H_1, \ldots, H_q$ the remaining hyperplanes through $S$.
and set
\( \eta_i = \max_{S \subset K(S) = i} \frac{1}{2} \sum_{j=1}^{q} \left( w - K(H_j) \right) \).

Here the maximum is taken over all hyperlines \( S \) and multiplicity \( i \) contained in \( H_0 \).

Assume that the spectrum \((b_i)\) of the restriction \( K|_{H_0} \) is known. Then we obtain from (2)
\[
\sum_{H} \left( w - K(H) \right) \leq \sum_{j} b_j \eta_j + \left( w - K(H_0) \right),
\]
which implies
\[
\sum_{j} b_j \eta_j + \left( w - K(H_0) \right) \geq \left( w \right) v_k - n(w-1)v_{k-1} + \left( \frac{n}{2} \right) n_k + q^{k-2} \sum_{i \geq 2} \left( i \right) \lambda_i.
\]

For projective arcs, i.e. arcs with point multiplicities 0 and 1, this becomes
\[
\sum_{j} b_j \eta_j + \left( w - K(H_0) \right) \geq \left( w \right) v_k - n(w-1)v_{k-1} + \left( \frac{n}{2} \right) n_k.
\]
Note that the right-hand side depends only on the parameters of \( K \). Inequality (6) is a necessary condition for the existence of an \((n, w)\)-arc in \( \text{PG}(k-1, q) \).

We finish by presenting the spectra of the arcs in \( \text{PG}(2, 5) \) with parameters \((9,3)\), \((10,3)\), \((11,3)\), and \((22,5)\) which will be needed in the sequel. The classification of the arcs with the first three parameter sets is given in [8]. The \((22,5)\)-arcs are obtained as complements of the \((9,1)\)-blocking sets. There exist three non-equivalent \((9,1)\)-blocking sets: two reducible consisting of a line and three further points and one irreducible – the projective triangle.

| Parameters | Type | \( a_0 \) | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \# arcs |
|------------|------|----------|----------|----------|----------|----------|---------|
| \((11, 3)\) | A1   | 4        | 4        | 7        | 16       | 1        | 1       |
|            | A2   | 5        | 1        | 10       | 15       |          |         |
| \((10, 3)\) | B1   | 3        | 9        | 6        | 13       | 1        | 1       |
|            | B2   | 4        | 6        | 9        | 12       | 4        |         |
|            | B3   | 5        | 3        | 12       | 11       | 1        |         |
|            | B4   | 6        | 0        | 15       | 10       | 1        |         |
| \((9, 3)\)  | C1   | 3        | 12       | 6        | 10       | 2        |         |
|            | C2   | 4        | 9        | 9        | 9        | 7        |         |
|            | C3   | 5        | 6        | 12       | 8        | 5        |         |
|            | C4   | 6        | 3        | 15       | 7        | 2        |         |
| \((22, 5)\) | D1   | 1        | 0        | 1        | 0        | 15       | 14      | 1       |
|            | D2   | 1        | 0        | 0        | 3        | 12       | 15      | 1       |
|            | D3   | 0        | 0        | 3        | 4        | 6        | 18      | 1       |

### 3. Quasidivisible arcs and the extendability of arcs

Let \( K \) be a \( t \)-quasidivisible \((n, w)\)-arc with divisor \( q \) in \( \Sigma = \text{PG}(k-1, q) \), \( t < q \).
Denote by \( \bar{K} \) the \( \sigma \)-dual to \( K \) in the dual geometry \( \bar{\Sigma} \), where \( \sigma(x) = n + t - x \mod q \). In other words, we have
\[
\bar{K} : \begin{cases} \mathcal{H} & \rightarrow \{0, 1, \ldots, t\} \\ H & \mapsto \bar{K}(H) \equiv n + t - K(H) \mod q \end{cases}
\]
where $\mathcal{H}$ is the set of all hyperplanes in $\Sigma$. This means that hyperplanes of multiplicity congruent to $n + a \pmod{q}$ become $(t - a)$-points in the dual geometry. In particular, maximal hyperplanes are 0-points with respect to $\tilde{K}$. Let us note that in the general case the cardinality of $\tilde{K}$ cannot be obtained from the parameters of $K$. Define the sum of two multisets $K'$ and $K''$ in the same geometry by $(K' + K'')(P) = K'(P) + K''(P)$. For a set of points $Q \subseteq P$, $\chi_Q$ denotes the characteristic function of $Q$. The following theorem is straightforward.

**Theorem 3.1** ([9, 10]). Let $K$ be an $(n, w)$-arc in $\Sigma = \text{PG}(k - 1, q)$ which is $t$-quasidivisible modulo $q$ with $t < q$. Let $\tilde{K}$ be defined by (7). If

$$\tilde{K} = \sum_{i=1}^{c} \chi_{\tilde{P}_i} + K'$$

for some multiset $K'$ in $\tilde{\Sigma}$ and $c$ not necessarily different hyperplanes $\tilde{P}_1, \ldots, \tilde{P}_c$ in $\tilde{\Sigma}$, then $K$ is $c$-extendable. In particular, if $\tilde{K}$ contains a hyperplane in its support, then $K$ is extendable.

Theorem 3.1 links the extendability of $t$-quasidivisible arcs with the structure of the multiset $\tilde{K}$ defined in the dual geometry. Note that this theorem gives a sufficient but not a necessary condition since 0-points with respect to $\tilde{K}$ can correspond to hyperplanes that are not of maximal multiplicity.

**Theorem 3.2** ([9, 10]). Let $K$ be an $(n, w)$-arc in $\Sigma = \text{PG}(k - 1, q)$ which is $t$-quasidivisible modulo $q$ with $t < q$. For every subspace $\tilde{S}$ of $\tilde{\Sigma}$, with $\dim \tilde{S} \geq 1$,

$$\tilde{K}(\tilde{S}) \equiv t \pmod{q}.$$

An arc $\mathcal{F}$ in $\text{PG}(k - 1, q)$ is called a $(t \pmod{q})$-arc if (a) all points have multiplicity at most $t$; (b) for all subspaces $S$ of positive dimension $\mathcal{F}(S) \equiv t \pmod{q}$. The importance of $(t \pmod{q})$-arcs is due to the fact that every $t$-quasidivisible arc $K$ gives rise to a unique $(t \pmod{q})$-arc $\tilde{K}$.

There exist three straightforward constructions for $(t \pmod{q})$ arcs:

1. the sum of $t$ hyperplanes;
2. arcs lifted from $(t \pmod{q})$-arcs in lower dimensions (cf. [10]).
3. the $\sigma$-dual of an $(m - t)q + m, m - t)$-blocking set in $\text{PG}(2, q)$ with line multiplicities $m - t, \ldots, m$, and with $\sigma(x) = (x - t)/q$ (cf. [10]).

Lifted arcs are arcs obtained by the following construction. Let $\mathcal{F}_0$ be a $(t \pmod{q})$-arc in $\text{PG}(k - 2, q)$. Let $H$ be a hyperplane in $\text{PG}(k - 1, q)$. For a fixed point $P$ in $\text{PG}(k - 1, q)$, not incident with $H$, define an arc $\mathcal{F}$ in $\text{PG}(k - 1, q)$ as follows: $\mathcal{F}(P) = t$; for each point $Q \neq P$, $\mathcal{F}(Q) = \mathcal{F}_0(R)$, where $R = (P, Q) \cap H$. It turns out that $\mathcal{F}$ is a $(t \pmod{q})$-arc in $\text{PG}(k - 1, q)$ of size $q \cdot |\mathcal{F}_0| + t$. We call this arc a lifted arc from $\mathcal{F}_0$.

A plane $(1 \pmod{q})$-arc is either a hyperplane or the complete space. The $(2 \pmod{q})$-arcs with $q \geq 5$, $q$ odd, in the plane were characterized by Maruta [12]. Such an arc is either a lifted arc from a $(2 + iq)$-line, $i = 0, 1, 2$, or the sum of an oval, a tangent to the oval, and twice the internal points to the oval. It can be easily proved by induction that for higher dimensions $\tilde{K}$ is a lifted arc and hence contains a hyperplane without 0-points. This implies that the arc $K$ is again extendable (in fact, in many cases even 2-extendable).

For $(3 \pmod{q})$-arcs in $\text{PG}(2, q)$, the situation is far more complicated. We have many $(3 \pmod{q})$-arcs obtained as the sum of a $(2 \pmod{q})$- and a $(1 \pmod{q})$-arc,
but also some non-trivial indecomposable arcs. For instance, in the case of \( q = 5 \), a \((3 \mod 5)\)-arc of size 18 is the sum of three lines. There exists a unique \((3 \mod 5)\)-arc of size 23 which has as 2-points the vertices of quadrangle, as 3-points – its diagonal points and as 1-points the intersections of the diagonal lines with the sides of the quadrangle \([14]\). There exists also a unique \((3 \mod 5)\)-arc of size 28. It has as 3-points the points of an oval and as 1-points – the internal points of this oval \([14]\). Note that in the last two cases there is always a 3-line incident with one 3-point only. This fact will be used later in Lemma 4.3.

**Theorem 3.3** (\([14]\)). Every \((3 \mod 5)\)-arc \( F \) in \( PG(3,5) \) with \(|F| \leq 158 \) is a lifted arc or the sum of hyperplanes.

Consider a Griesmer \( t \)-quasidivisible arc \( K \), \( t < q \), with parameters \((n, w)\) in \( PG(k-1, q) \). Set \( d = n - w \) and write \( d \) as

\[
d = s q^{k-1} - \sum_{i=0}^{k-2} \varepsilon_i q^i, \quad 0 \leq \varepsilon_i < q.
\]

Denote by \( w_j \) the maximal multiplicity of a subspace \( S \) of co-dimension \( j \) of \( PG(k-1, q) \): \( w_j = \max_{\text{codim } S = j} \) \( \lambda(S) \), \( j = 1, \ldots, k-1 \). We have

\[
w_j = \sum_{i=j}^{k-1} \left\lfloor \frac{d}{q^i} \right\rfloor = s v_{k-j} - \sum_{i=j}^{k-2} \varepsilon_i v_{1-j+1}.
\]

By convention, \( w_0 = n \).

As already noted, the size of \( \overline{K} \) cannot be obtained from the parameters of \( K \). Nevertheless knowledge of the structure of the restriction of \( K \) to some hyperplane gives an upper bound on the size of \( \overline{K} \).

**Lemma 3.4** (\([10]\)). Let \( K \) be a \( t \)-quasidivisible \((n, n-d)\)-Griesmer arc with \( d \) given by \((8)\). Let \( S \) be a hyperline contained in the hyperplane \( H_0 \) with \( K(H_0) = w_1 - aq \) where \( a \geq 0 \) is an integer.

- (i) If \( K(S) = w_2 - a - b \), \( 0 \leq b \leq t - 2 \), then \( \overline{K}(\overline{S}) \leq t + bq \);
- (ii) If \( K(S) = w_2 - a - b \), \( b \geq t - 1 \), then \( \overline{K}(\overline{S}) \leq t + (t-1)q \).

4. The nonexistence of \((104,22)\)-arcs

Assume that \( K \) is a \((104, 22)\)-arc in \( PG(3,5) \). Let us denote by \( \gamma_i \), \( i = 0, 1, 2 \), the maximal multiplicity of an \( i \)-dimensional subspace in \( PG(3,5) \). In the following lemma, we summarize the straightforward properties of \((104, 22)\)-arcs.

**Lemma 4.1.** Let \( K \) be a \((104,22)\)-arc with spectrum \((a_i)\). Then

- (a) \( \gamma_0 = 1 \), \( \gamma_1 = 5 \), \( \gamma_2 = 22 \);
- (b) The maximal multiplicity of a line in an \( m \)-plane is \([(6 + m)/5]\);
- (c) There do not exist planes with 2, 3, 7, 8, 12, 13, 17, 18 points.
- (d) \( a_0 = 0 \).
- (e) \( a_1 = 0 \).
- (f) \( a_4 = a_5 = 0 \)
- (g) \( \sum_{i=0}^{20} \binom{22-i}{2} a_i = 468 \).

**Proof.** (a) and (b) are obtained by easy counting.

(c) follows by the nonexistence of \((2,1)\)-, \((7,2)\)-, \((12,3)\)- and \((17,4)\)-arcs in \( PG(2,5) \).
(d) This follows by the fact that there is no \((21, 3)\)-blocking set in \(AG(3, 5)\) with respect to the planes (cf. Corollary 2.3 in [1])

(e) This follows by counting the number of points through a 1-line in a 1-plane and using the fact that a \((22, 5)\)-arc does not have a 1-line (cf. the spectra of \((22, 5)\)-arcs).

(f) Assume \(H_0\) is a 4-plane in \(PG(3, 5)\). By Lemma 4.1(b), each line from \(H_0\) meets \(K\) in at most two points, i.e. \(K|_{H_0}\) is a plane \((4, 2)\)-arc. It is easily seen that such an arc can be extended to an oval \(- (6, 2)\)-arc. In other words, there exist at least two points which is incident only with 1-lines from \(K|_{H_0}\). Let \(X\) be such a point.

Let the points of multiplicity 1 in \(H_0\) be \(Y_0, Y_1, Y_2, Y_3\). The lines \(XY_0, \ldots, XY_3\) are 1-lines. Denote by \(H_i, i = 1, \ldots, 5\), the other five planes through \(XY_0\). Since a 22-plane does not have a 1-line, all these planes have multiplicity 21.

Consider a projection \(\varphi\) from \(X\) onto some plane \(\delta\) that does not contain \(X\). Set \(L_i = \varphi(H_i), i = 0, \ldots, 5\), and \(Z = \varphi(XY_0)\). The points on the line \(L_0\) have multiplicities \((1, 1, 1, 1, 0, 0)\) with respect to \(K^\varphi\), by construction. Moreover \(K^\varphi(L_i) = 21\) for \(i = 1, \ldots, 5\).

Assume there is a point \(U\) on some of the lines \(L_1, \ldots, L_5\) with \(K^\varphi(U) = 5\). Since \(U\) is the image of a 5-line with respect to the original arc \(K\) and since it is contained in a 21-plane (the preimage of one of \(L_1, \ldots, L_5\)) there must exist at least three lines through \(U\) in \(\delta\), \(M_0, M_1, M_2\) say, with \(K^\varphi(M_i) = 22\). At least one of these lines meets \(L_0\) in a 1-point. This means that there exists a 22-plane with a 1-line, a contradiction. Hence \(K^\varphi(L) = 4\) for every point \(U\) in \((\cup_{i=1}^{5}L_i) \setminus \{Z\}\). In other words, each one of the lines \(L_i, i = 1, \ldots, 5\), has points of multiplicities \((4, 4, 4, 4, 1)\) under \(K^\varphi\).

Now it is easily checked that \(K^\varphi(L) \leq 21\), for every line \(L\) in \(\delta\). This means that the projection point \(X\) is not contained in a 22-plane. Therefore \(K\) can be extended to a \((105, 22)\)-arc in \(PG(3, 5)\). But such an arc does not exist (cf. [13]). This is a contradiction which rules out the existence of 4-planes.

The impossibility of 5-planes is proved analogously.

(g) This follows from (2) by taking \(n = 104, w = 22, k = 4, q = 5\). \(\square\)

By Lemma 4.1, a \((104, 22)\)-arc \(K\) is 3-quasidivisible. Moreover, 0-points with respect to the dual arc \(\bar{K}\) must come necessarily from maximal planes. This forces certain restrictions on the structure of \(\bar{K}\) described in the lemma below.

**Lemma 4.2.** Let \(K\) be a \((104, 22)\)-arc in \(PG(3, 5)\). Then there exists no plane \(\bar{P}\) in the dual space such that \(\bar{K}|_{\bar{P}}\) is \(3_{\bar{X}_{\bar{L}}}\) for some line \(\bar{L}\) in the dual space.

**Proof.** Let \(X\) be a point in \(PG(3, 5)\). Summing up the multiplicities of all planes through \(X\), we have

\[
\sum_{H : H \ni X} K(H) = 6|K| + 25K(X).
\]

On the other hand, a point \(\bar{H}\) in the dual space with \(K(\bar{H}) = 0\) comes necessarily from a maximal plane. The points on the line \(\bar{L}\) with \(K(\bar{L}) = 18\) correspond to planes \(H\) for which we have

\[
\sum_{H : H \ni \bar{L}} K(H) = |K| + 5K(L).
\]

**Advances in Mathematics of Communications**

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This implies that for \( X = P \):
\[
6|K| + 25K(P) = 25 \cdot 22 + |K| + 5K(L),
\]
which gives
\[
25K(P) = 30 + 5K(L),
\]
a contradiction.

**Lemma 4.3.** Let \( K \) be a \((104, 22)\)-arc in \( \text{PG}(3, 5) \). Then \(|\tilde{K}| \geq 163\).

**Proof.** The arc \( K \) is not extendable. Otherwise there would be a \((105, 22)\)-arc and such an arc does not exist [13]. Hence \( \tilde{K} \) does not contain a plane in its support whence \(|\tilde{K}| \leq 158\). Then by Theorem 3.3 \( \tilde{K} \) is lifted from a plane \((3 \mod 5)\)-arc with 23 or 28 points. But such arcs have a 3-line incident with just one 3-point. After lifting, this line leads to an 18-plane ruled out in Lemma 4.2. \( \square \)

We can use Lemma 4.3 together with the necessary condition (6) to restrict further the possible multiplicities of planes. Our key observation is that if a 5-tuple of planes through a line \( L \) in \( H_0 \) gives a high contribution to the left-hand side of (6) then \( \tilde{K}(L) \) is small.

**Lemma 4.4.** Let \( K \) be a \((104, 22)\)-arc in \( \text{PG}(3, 5) \). Then \( a_6 = 0 \).

**Proof.** Let \( H_0 \) be a 6-plane. the \( K|_{H_0} \) is a \((6, 2)\)-arc and has spectrum \( a_2 = 15, a_1 = 6, a_0 = 10 \). Consider an arbitrary line \( L \) in \( H_0 \). By Lemma 3.4, if \( L \) is a 2-line with respect to \( K \), then it is an 8-line with respect to \( \tilde{K} \); similarly, if \( L \) is a 1-line it is a 3-line with respect to \( \tilde{K} \) (since 22-planes do not have 1-lines) and, finally, if it is a 0-line with respect to \( K \), it is a 3-, 8- or 13-line with respect to \( \tilde{K} \).

In the case of \( K(L) = 0 \) and \( \tilde{K}(L) = 3 \) the maximal contribution of the planes through \( L \) is 66 obtained for
\[
(K(H_0), \ldots, K(H_5)) = (6, 22, 22, 22, 22, 10);
\]
if \( \tilde{K}(L) = 8 \) the maximal contribution of the planes through \( L \) is 31 obtained for
\[
(K(H_0), \ldots, K(H_5)) = (6, 22, 22, 21, 19, 14);
\]
and if \( \tilde{K}(L) = 13 \) the maximal contribution of the planes through \( L \) is 12 obtained for
\[
(K(H_0), \ldots, K(H_5)) = (6, 22, 19, 19, 19, 19).
\]
Let us denote by \( x \) the number of 0-lines \( L \) of \( H_0 \) with \( \tilde{K}(H_0) = 3 \) and by \( y \) the number of such lines with \( \tilde{K}(H_0) = 8 \). Counting the contribution of the different planes through the lines of \( L \) we get
\[
\left( \begin{array}{c} 16 \\ 2 \end{array} \right) + 15 \cdot 1 + 6 \cdot 3 + 66x + 31y + 12(10 - x - y) \geq 468,
\]
whence \( 54x + 19y \geq 195 \). On the other hand, we have
\[
|\tilde{K}| = 1 + 21 \cdot 2 + 2x + 7y + 12(10 - x - y) = 163 - 10x - 5y.
\]
Since \( K \) is not extendable we have \(|\tilde{K}| \geq 163\), and hence \( 2x + y \leq 0 \), i.e. \( x = y = 0 \), a contradiction to \( 54x + 19y \geq 195 \). \( \square \)

**Lemma 4.5.** Let \( K \) be a \((104, 22)\)-arc in \( \text{PG}(3, 5) \). Then \( a_9 = a_{10} = a_{11} = 0 \).
Proof. We will demonstrate only the non-existence of 9-planes of type C4. The non-existence of 9-planes of the other three types, as well as the non-existence of 10- and 11-planes, is done analogously.

Let \( H_0 \) be a 9-plane and let \( \mathcal{K}|_{H_0} \) be a \((9,3)\)-arc of type C4. For an arbitrarily fixed line \( L \) in \( H_0 \) we denote by \( H_1, \ldots, H_5 \) the other 5 planes through \( L \). We have the following possibilities:

\[
\begin{array}{c|c|c|c|c}
\mathcal{K}(L) & \mathcal{K}(L) & \eta_i & (\mathcal{K}(H_0), \ldots, \mathcal{K}(H_5)) \\
3 & 3 & 0 & (9,22,22,22,22,22) \\
2 & 8 & 4 & (9,22,22,22,22,19) \\
1 & 8 & 15 & (9,21,21,21,16) \\
1 & 13 & 7 & (9,21,21,19,19) \\
0 & 8 & 79 & (9,22,22,20,9) \\
0 & 13 & 34 & (9,22,21,19,14) \\
0 & 18 & 15 & (9,19,19,19,19) \\
\end{array}
\]

Denote by \( x \) the number of lines \( L \) in \( H_0 \) such that \( \mathcal{K}(L) = 1 \) and \( \tilde{\mathcal{K}}(\tilde{L}) = 8 \); similarly, denote by \( u \), resp. \( v \), the number of lines \( L \) with \( \mathcal{K}(L) = 0 \), \( \tilde{\mathcal{K}}(\tilde{L}) = 8 \), resp. \( \mathcal{K}(L) = 0 \), \( \tilde{\mathcal{K}}(\tilde{L}) = 13 \). Counting the contribution of the planes through the different lines in \( H_0 \) to the left-hand side of (6), we get

\[
\begin{align*}
(\binom{13}{2}) + 7 \cdot 0 + 15 \cdot 4 + 15x + 7 \cdot (3 - x) + 79u + 34v + 15(6 - u - v) & \geq 468, \\
\end{align*}
\]
whence \( 8x + 64u + 19v \geq 219 \).

On the other hand, computing the cardinality of \( \tilde{\mathcal{K}} \) and taking into account that \( \tilde{\mathcal{K}}(\tilde{H}) = 3 \), we get

\[
3 + 7 \cdot 0 + 15 \cdot 5 + 5x + 10(3 - x) + 5u + 10v + 15(6 - u - v) \geq 163,
\]
whence \( x + 2u + v \leq 7 \). Now we have the chain of inequalities

\[
224 \geq 32x + 64u + 32v \geq 8x + 64u + 19v \geq 219.
\]
This implies that \( x = v = 0 \), which in turn gives \( 224 \geq 64u \geq 219 \), a contradiction since \( u \) is an integer. \( \square \)

Now we are ready to prove the nonexistence of \((104,22)\)-arcs.

**Theorem 4.6.** There is no \((104,22)\)-arc in \( \text{PG}(3,5) \).

*Proof.* Assume there exists a \((104,22)\)-arc \( \mathcal{K} \) in \( \text{PG}(3,5) \). As already mentioned, such an arc is not extendable and by Lemma 4.3 we have \( |\tilde{\mathcal{K}}| \geq 163 \). We are going to show that on the other hand \( |\tilde{\mathcal{K}}| \) cannot be greater than 158 which will provide the contradiction needed to rule out the existence of \((104,22)\)-arcs.

We apply the technique from Lemma 4.5 to the three non-isomorphic \((22,5)\)-arcs. Let \( H_0 \) be a fixed 22-plane. For a line \( L \) in \( H_0 \) we have the possibilities as shown in the table below. Now we are going to rule out the three cases where \( \mathcal{K}|_{H_0} \) is a \((22,5)\)-arc of type \((D1)\), \((D2)\) or \((D3)\).

\[\begin{array}{c|c|c|c|c|c|c|c|c}
K(L) & K(L) & \eta_i & (K(H_0), \ldots, K(H_5)) \\
3 & 3 & 0 & (9,22,22,22,22,22) \\
2 & 8 & 4 & (9,22,22,22,22,19) \\
1 & 8 & 15 & (9,21,21,21,16) \\
1 & 13 & 7 & (9,21,21,19,19) \\
0 & 8 & 79 & (9,22,22,20,9) \\
0 & 13 & 34 & (9,22,21,19,14) \\
0 & 18 & 15 & (9,19,19,19,19) \\
\end{array}\]

Denote by \( x \) the number of lines \( L \) in \( H_0 \) such that \( \mathcal{K}(L) = 1 \) and \( \tilde{\mathcal{K}}(\tilde{L}) = 8 \); similarly, denote by \( u \), resp. \( v \), the number of lines \( L \) with \( \mathcal{K}(L) = 0 \), \( \tilde{\mathcal{K}}(\tilde{L}) = 8 \), resp. \( \mathcal{K}(L) = 0 \), \( \tilde{\mathcal{K}}(\tilde{L}) = 13 \). Counting the contribution of the planes through the different lines in \( H_0 \) to the left-hand side of (6), we get

\[
14 \cdot 3 + 28x + 7(15 - x) + 1 \cdot 57 + 1 \cdot 87 \geq 468,
\]
whence $21x \geq 177$, i.e. $x \geq 9$. On the other hand,

$$|\tilde{K}| \leq 14 \cdot 3 + x \cdot 3 + (15 - x) \cdot 8 + 13 + 13,$$

whence $|\tilde{K}| \leq 188 - 5x$. This implies $188 - 5x \geq 163$, i.e. $x \leq 5$, a contradiction.

| $\kappa(L)$ | $\kappa(L)$ | $\eta_i$ | $(\kappa(H_0), \ldots, \kappa(H_5))$ |
|------------|------------|---------|-----------------------------------|
| 5          | 3          | 3       | (22,22,22,22,22,19)               |
| 4          | 3          | 28      | (22,22,22,22,22,14)               |
| 4          | 8          | 7       | (22,22,22,20,19,19)               |
| 3          | 3          | 36      | (22,22,22,22,16,15)               |
| 3          | 8          | 32      | (22,22,22,20,19,14)               |
| 3          | 13         | 12      | (22,21,19,19,19,19)               |
| 2          | 3          | 45      | (22,22,22,16,16,16)               |
| 2          | 8          | 57      | (22,22,22,20,14,14)               |
| 2          | 13         | 37      | (22,21,19,19,19,19)               |
| 0          | 8          | 86      | (22,22,22,16,16,14,14)            |
| 0          | 13         | 87      | (22,22,22,19,14,14,14)            |

(D2) Denote by $x$ the number of 4-lines $L$ with $\tilde{K}(\tilde{L}) = 3$; by $u$ – the number of 3-lines $L$ with $\tilde{K}(\tilde{L}) = 3$, and by $v$ – the number of 3-lines $L$ with $\tilde{K}(\tilde{L}) = 8$. Again counting the contribution to the left-hand side of (6), we have

$$15 \cdot 3 + x \cdot 28 + (12 - x) \cdot 7 + u \cdot 36 + v \cdot 32 + (3 - u - v)12 + 1 \cdot 87 \geq 468,$$

whence $21x + 24u + 20v \geq 216$. On the other hand,

$$|\tilde{K}| = 15 \cdot 3 + 3x + 8(12 - x) + 3u + 8v + (3 - u - v) \cdot 13 + 13 \geq 163,$$

$x + 2u + v \leq 6$. Now we get

$$126 \geq 21x + 42u + 21v \geq 21x + 24u + 20v \geq 216,$$

a contradiction.

(D3) Let $x$, $u$ and $v$ be as above. Denote also by $s$ the number of 2-lines $L$ with $\tilde{K}(\tilde{L}) = 3$, and by $t$ – the number of 2-lines $L$ with $\tilde{K}(\tilde{L}) = 8$. Once again:

$$18 \cdot 3 + x \cdot 28 + (6 - x) \cdot 7 + u \cdot 36 + v \cdot 32 + (4 - u - v)12 + s \cdot 45 + t \cdot 57 + (3 - s - t)37 \geq 468,$$

whence $21x + 24u + 20v + 8s + 20t \geq 213$. On the other hand

$$|\tilde{K}| = 18 \cdot 3 + 3x + 8(6 - x) + 3u + 8v + 13(4 - u - v) + 3s + 8t + 13(3 - s - t) \geq 163,$$

hence $x + 2u + v + 2s + t \leq 6$. This implies

$$126 \geq 21x + 42u + 21v + 42s + 21t \geq 21x + 24u + 21v + 42s + 21t \geq 213,$$

a contradiction.

**Corollary 1.** A linear code with parameters $[104, 4, 82]_5$ does not exist. In particular, $n_5(4, 82) = 105$.

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