POLYNOMIAL BASINS OF INFINITY

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ABSTRACT. We study the projection \( \pi : \mathcal{M}_d \to \mathcal{B}_d \) which sends an affine conjugacy class of polynomial \( f : \mathbb{C} \to \mathbb{C} \) to the holomorphic conjugacy class of the restriction of \( f \) to its basin of infinity. When \( \mathcal{B}_d \) is equipped with a dynamically natural Gromov-Hausdorff topology, the map \( \pi \) becomes continuous and a homeomorphism on the shift locus. Our main result is that all fibers of \( \pi \) are connected. Consequently, quasiconformal and topological basin-of-infinity conjugacy classes are also connected.

1. Introduction

Let \( f : \mathbb{C} \to \mathbb{C} \) be a complex polynomial of degree \( d \geq 2 \). Iterating \( f \) yields a dynamical system. The plane then decomposes into the disjoint union of its open, connected basin of infinity defined by

\[
X(f) = \{ z \in \mathbb{C} : f^n(z) \to \infty \text{ as } n \to \infty \}
\]

and its complement, the compact filled Julia set \( K(f) \).

Many naturally defined loci in parameter space (such as the connectedness locus, external rays, their impressions, and parapuzzles) are defined by constraints on the dynamics of \( f \) on \( X(f) \). Motivated by this, we study the forgetful map sending a polynomial \( f : \mathbb{C} \to \mathbb{C} \) to its restriction \( f : X(f) \to X(f) \). The basin \( X(f) \) is equipped with a dynamically natural translation surface structure. In this work and its sequels \cite{DP1, DP2} we exploit this Euclidean perspective to analyze the global structure of moduli spaces of complex polynomials.

1.1. Connected fibers. The moduli space \( \mathcal{M}_d \) of complex affine conjugacy classes of degree \( d \) polynomials inherits a natural topology from the coefficients of representatives \( f \). Let \( \mathcal{B}_d \) denote the set of conformal conjugacy classes of maps \( f : X(f) \to X(f) \), and let \( \pi : \mathcal{M}_d \to \mathcal{B}_d \) be the map sending a polynomial \( f \) to its restriction \( f | X(f) \). For each \( f \in \mathcal{M}_d \), the basin of infinity \( X(f) \) is equipped with a canonical harmonic Green’s function \( G_f \) and hence a flat conformal metric \( | \partial G_f | \) with isolated singularities. We endow the space \( \mathcal{B}_d \) with the Gromov-Hausdorff topology on the metric spaces \( (X(f), 2 | \partial G_f |) \) equipped with the self-map \( f : X(f) \to X(f) \); see \cite{H4}. With respect to this topology, the space \( \mathcal{B}_d \) becomes a locally compact Hausdorff
Figure 1.1. For $d = 2$, the moduli space $\mathcal{M}_2$ is isomorphic to the complex plane, as each conjugacy class is uniquely represented by polynomial $f(z) = z^2 + c$ for $c \in \mathbb{C}$. The connectedness locus $\mathcal{C}_2$ is the much-studied Mandelbrot set. The projection $\pi : \mathcal{M}_2 \to \mathcal{B}_2$ collapses the Mandelbrot set to a point and is one-to-one elsewhere.

Recall that a continuous map between topological spaces is monotone if it has connected fibers. Our main result is

**Theorem 1.1.** The projection $\pi : \mathcal{M}_d \to \mathcal{B}_d$ is continuous, proper, and monotone. Furthermore, $\pi$ is a homeomorphism on the shift locus and, more generally, on the Cantor locus.

The key part of Theorem 1.1 is the connectedness of fibers, which is already well known in certain important cases. The fiber over $(z^d, \mathbb{C} \setminus \mathbb{D})$ is precisely the connectedness locus $\mathcal{C}_d$, the set of maps $f$ with connected filled Julia set $K(f)$. The set $\mathcal{C}_d$ is known to be cell-like (see [DH] for a proof in degree 2, [BH1] for degree 3, and [La] for general degrees), thus connected. Our theorem gives an alternate proof of its connectedness. The other extreme is also well known: for a polynomial in the shift locus, the basin $X(f)$ is a rigid Riemann surface, so such a polynomial is uniquely determined by its basin dynamics. We exploit this rigidity in the proof of Theorem 1.1. In the course of the proof we show that the shift locus is connected (Corollary 5.3), a fact which we could not find explicitly stated elsewhere.

1.2. **Topological conjugacy.** It was observed in [McS, §8] that any two polynomials $f, g$ which are topologically conjugate on their basins of infinity are in fact quasiconformally conjugate there. It follows that there is an (analytic) path of polynomials
$g_t, 0 \leq t \leq 1$ such that (i) $g_0 = g$, (ii) $g_t$ is quasiconformally conjugate on $X(g_t)$ to $g$ on $X(g)$ for all $0 \leq t \leq 1$, and (iii) $g_1$ is conformally conjugate on $X(g_1)$ to $f$ on $X(f)$, i.e. $\pi(g_1) = \pi(f)$. Since the fiber of $\pi$ containing $f$ is connected, we obtain the following corollary to Theorem 1.1:

**Corollary 1.2.** Topological or quasiconformal conjugacy classes of basins $(f,X(f))$ are connected in $M_d$.

1.3. **Local models and sketch of proof.** By recording the data of the holomorphic 1-form $\partial G$ on a basin of infinity, the space $B_d$ becomes a space of abelian differentials (or translation structures) of a special type. We examine this structure in pieces we call local models: branched covers between translation surfaces which model the restriction of $f$ to certain subsets of $X(f)$. See §3.

The idea of the proof of monotonicity in Theorem 1.1 is the following. For each $f$, the Green’s function $G_f : X(f) \to (0, \infty)$ is a harmonic and satisfies $G_f(f(z)) = d \cdot G_f(z)$. For $t > 0$, let $X(f,t) = \{z : G_f > t\}$. Then $f$ maps $X(f,t)$ into itself, and we may consider the restriction $f|X(f,t)$ up to conformal conjugacy. For each $f \in M_d$ and $t > 0$, we define

$$B(f,t) = \{g \in M_d : (g,X(g,t)) \text{ is conformally conjugate to } (f,X(f,t))\}.$$  

Clearly $B(f,t_1) \subset B(f,t_2)$ if $t_1 < t_2$, and the fiber of $\pi$ containing $f$ is the nested intersection $\bigcap_{t>0} B(f,t)$. We shall show that $B(f,t)$ is connected for all (suitably generic) $t$.

The intersection of $B(f,t)$ with the shift locus contains a distinguished subset $S(f,t)$ consisting of maps $g$ whose critical points all have height at least $t$. We deduce the connectedness of $B(f,t)$ from that of $S(f,t)$, and we show $S(f,t)$ is connected by proving that it is a finite quotient of a product of finitely many connected spaces of local models. Because polynomials in the shift locus have rigid basins of infinity, the abstract local model structure for the basins determines the structure of the subset $S(f,t)$ in $M_d$.

In fact, the proof of monotonicity of $\pi$ begins like the known proof of connectedness of the connectedness locus $C_d$. When $f$ has connected Julia set, the set $B(f,t)$ coincides with

$$B(t) = \{g \in M_d : G_g(c) \leq t \text{ for all critical points } c \text{ of } g\}$$

for every $t > 0$. It follows from [BH1] and [La] that $B(t)$ is topologically a closed ball, and its boundary $S(t) = \{g : \max_c G_g(c) = t\}$ is a topological sphere. The connectedness locus is the nested intersection $C_d = \bigcap_t B(t)$, showing that $C_d$ is cell-like. By contrast, for general $f$, the structure of $B(f,t)$ depends on $f$ and can change as $t$ decreases.
1.4. Remarks. Intuitively, one might expect that the fibers of \( \pi : \mathcal{M}_d \to \mathcal{B}_d \) are homeomorphic to products of connectedness loci \( \mathcal{C}_{d_i} \) of degrees \( d_i \leq d \), each of which is connected. That is, the affine conjugacy class of a polynomial \( f \) should be determined by the conformal conjugacy class of its restriction \( (f, X(f)) \) together with a finite amount of “end-data”: the restriction of \( f \) to non-trivial periodic components of the filled Julia set \( K(f) \). It is easily seen to hold in degree 2, and it follows in degree 3 by the results of Branner and Hubbard in \([BH1]\), \([BH2]\), where every fiber of \( \pi \) in \( \mathcal{M}_3 \) is either a point, a copy of the Mandelbrot set \( \mathcal{C}_2 \), or the full connectedness locus \( \mathcal{C}_3 \). However, discontinuity of straightening should imply that this intuitive expectation fails in higher degrees; see \([In]\).

The set \( S(f, t) \) introduced in \( \S 1.3 \) has been studied by other authors in the special case when all critical points of \( f \) have height \( \leq t \). In this case, the set \( S(f, t) \) is a set \( \mathcal{G}(t) \) independent of \( f \); it is the collection of polynomials in \( \mathcal{M}_d \) where all critical points escape at the same rate \( t \). Further, the set \( \mathcal{G}(t) \) is homeomorphic to a finite quotient of the compact, connected space of degree \( d \) critical orbit portraits \([Ki]\). The set \( \mathcal{G}(t) \) is equipped with a natural measure \( \mu \) inherited from the external angles of critical points. The Branner-Hubbard stretching operation deforms a polynomial in the escape locus along a path accumulating on the connectedness locus. For \( \mu \)-almost every point of \( \mathcal{G}(t) \), this path has a limit, and the measure \( \mu \) pushes forward to the natural bifurcation measure supported in the boundary of the connectedness locus \([DF]\).

As described above, we use spaces of local models to describe the structure of our sets \( S(f, t) \). In a sequel to this article, we give alternative descriptions of spaces of local models in terms of branched coverings of laminations. In this way, spaces of local models may be viewed as a generalization of the space of critical portraits. Using this extra combinatorial structure, we address in \([DP2]\) the classification, presently unknown, of the countable set of globally structurally stable conjugacy classes in the shift locus.

Outline. In section 2 we summarize background from polynomial dynamics. In section 3 we develop the theory of local models. In section 4 we define the Gromov-Hausdorff topology on \( \mathcal{B}_d \) and prove that the projection \( \pi \) is continuous and a homeomorphism on the shift locus. In section 5, the connectedness of \( S(f, t) \) is proved, and then applied to complete the proof of Theorem 1.1.

Acknowledgement. We would like to thank Curt McMullen for his helpful suggestions, and we thank Hiroyuki Inou, Chris Judge, and Yin Youcheng for useful conversations.
2. Spaces of polynomials

In this section we introduce the moduli spaces \( \mathcal{M}_d \) and give some background on polynomial dynamics.

2.1. Polynomial dynamics. Let \( f \) be a complex polynomial of degree \( d \geq 2 \). The filled Julia set
\[
K(f) = \{ z : \text{the sequence } f^n(z) \text{ is bounded} \}
\]
is compact, and its complement \( X(f) = \mathbb{C} \setminus K(f) \) is open and connected. For \( t \in [0, \infty) \) define \( \log^+(t) = \max\{0, \log t\} \). The function
\[
G_f : \mathbb{C} \to [0, \infty)
\]
given by
\[
G_f(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|
\]
measures the rate at which the point \( z \) escapes to infinity under iteration of \( f \). It vanishes exactly on \( K(f) \), is harmonic on \( X(f) \), and on all of \( \mathbb{C} \) it is continuous, subharmonic, and satisfies the functional equation \( G_f(f(z)) = d \cdot G_f(z) \) (see e.g. [Mi]).

2.2. Monic and centered polynomials. Every polynomial
\[
f(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0,
\]
with \( a_i \in \mathbb{C} \) and \( a_d \neq 0 \), is conjugate by an affine transformation \( A(z) = az + b \) to a polynomial which is monic (\( a_d = 1 \)) and centered (\( a_{d-1} = 0 \)). The monic and centered representative is not unique, as the space of such polynomials is invariant under conjugation by \( A(z) = \zeta z \) where \( \zeta^{d-1} = 1 \). In this way, we obtain a finite branched covering
\[
\mathcal{P}_d \to \mathcal{M}_d
\]
from the space \( \mathcal{P}_d \simeq \mathbb{C}^{d-1} \) of monic and centered polynomials to the moduli space \( \mathcal{M}_d \) of conformal conjugacy classes of polynomials. Thus, \( \mathcal{M}_d \) has the structure of a complex orbifold of dimension \( d-1 \). It is sometimes convenient to work in a space with marked critical points. Let \( \mathcal{H} \subset \mathbb{C}^{d-1} \) denote the hyperplane given by \( \{(c_1, \ldots, c_{d-1}) : c_1 + \cdots + c_{d-1} = 0\} \). Then the map
\[
\rho : \mathcal{H} \times \mathbb{C} \to \mathcal{P}_d
\]
given by
\[
(2.1) \quad \rho(c_1, \ldots, c_{d-1}; a) = \int_0^z d \cdot \prod_{i=1}^{d-1} (\zeta - c_i) \, d\zeta + a
\]
gives a polynomial parameterization of \( \mathcal{P}_d \) by the location of the critical points and the image of the origin. Setting \( \mathcal{P}_d^\times = \mathcal{H} \times \mathbb{C} \), we refer to \( \mathcal{P}_d^\times \) as the space of critically marked polynomials.
The function

\[ M(f) = \max \{ G_f(c) : f'(c) = 0 \} \]

assigning to \( f \in P_d \) the maximal escape rate of a critical point of \( f \) is continuous, proper, and invariant under affine conjugation \([BH1, \text{Prop. 3.6}]\).

2.3. External rays and angles. Fix a monic and centered polynomial \( f \) of degree \( d > 1 \). Near infinity, there is a conformal change of coordinates which conjugates \( f \) to \( z \mapsto z^d \). The local conjugating isomorphism is unique up to multiplication by a \((d - 1)\)st root of unity and is therefore uniquely determined if required to have derivative 1 at infinity (see e.g. \([BH1]\)). It extends to an isomorphism \( \varphi_f : \{ G_f > M(f) \} \to \{ z \in \mathbb{C} : |z| > e^{M(f)} \} \) called the Böttcher map, satisfying \( G_f = \log |\varphi_f| \). For each fixed \( \theta \in \mathbb{R}/2\pi\mathbb{Z} \), the preimage under \( \varphi_f \) of the ray \( \{ re^{i\theta} : r > e^{M(f)} \} \) is called the external ray of angle \( \theta \) for \( f \). There are exactly \( d - 1 \) fixed external rays mapped to themselves under \( f \); their arguments are asymptotic to \( 2\pi k/(d - 1) \) near infinity for \( k = 0, \ldots, d - 2 \).

On \( \{ |z| > e^{M(f)} \} \), for each angle \( \theta \), the external ray of angle \( \theta \) coincides with a gradient flow line of \( G_f \). This ray can be extended uniquely to all radii \( r > 1 \) provided that when flowing downward, the trajectory does not meet any of the critical points of \( G_f \), i.e. critical points of \( f \) or any of their iterated inverse images. It follows that for all but countably many \( \theta \), the external ray of angle \( \theta \) admits such an extension, i.e. is nonsingular. We see then that the external rays of \( f \) define a singular vertical foliation on \( X(f) \) which is orthogonal to the singular horizontal foliation defined by the level sets of \( G_f \). These foliations coincide with the vertical and horizontal foliations associated to the holomorphic 1-form

\[ \omega_f = 2i \partial G_f \]

on \( X(f) \). We will exploit this point of view further in the next section.

We emphasize that, by definition, \( M_d \) is a quotient of \( P_d \) by the cyclic group of order \( d - 1 \) acting by conjugation via rotations of the plane centered at the origin. Therefore, given an element of \( M_d \), it defines a conjugacy class of a dynamical system on a Riemann surface isomorphic to the plane, and it defines an identification of this surface with the plane, up to this rotational ambiguity. Thus, given an element of \( M_d \), together with a choice of fixed external ray, there is a unique such identification sending this chosen fixed external ray to the external ray whose asymptotic argument is zero.

2.4. Critical values. The following two lemmas have nothing to do with dynamics and will be used in the proof of the connectedness and compactness of the space of local models (§3.8). The proof of Lemma 2.1 is a non-dynamical version of the proof in \([BH1]\) showing properness of \( f \mapsto M(f) \).
Lemma 2.1. Let \( \tilde{\nu} : \mathcal{P}_d^\times \to \mathbb{C}^{d-1} \) be the map sending a critically marked polynomial to its ordered list of critical values and \( \nu : \mathcal{P}_d \to \mathbb{C}^{d-1}/S_{d-1} \simeq \mathbb{C}^{d-1} \) the map sending a polynomial to its unordered set of critical values. Then \( \tilde{\nu} \) and \( \nu \) are proper. Moreover, \( \tilde{\nu} \) is a polynomial map.

Proof. Equation (2.1) shows the map \( \tilde{\nu} \) is polynomial. Fix \( R > 0 \). Suppose \( f(z) = z^d + a_{d-2}z^{d-2} + \ldots + a_1z + a_0 \) belongs to \( \mathcal{P}_d^\times \) and the critical values of \( f \) lie in \( D_{R^1/d} \). There is a unique univalent analytic map \( \psi_f : \mathbb{C} \setminus D_{R^1/d} \to \mathbb{C} \) tangent to the identity at infinity and satisfying \( f \circ \psi_f(w) = w^d \). By standard results from the theory of univalent functions, \( \psi_f(\mathbb{C} \setminus D_{R^1/d}) \supset \mathbb{C} \setminus D_{2R^1/\mu} \), so the critical points of \( f \) are contained in \( D_{2R^1/\mu} \). It follows that the coefficients \( a_{d-2}, \ldots, a_1 \) are bounded in modulus by a constant \( C_1(R) \). Since in addition \( f \) is assumed monic, the map \( f \) is Lipschitz on \( D_{2R^1/\mu} \) with constant \( C_2(R) \), so the image \( f(D_{2R^1/\mu}) \) has diameter less than a constant \( C_3(R) \). Since the critical values of \( f \) lie in \( D_{R^1/d} \), the image \( f(D_{2R^1/\mu}) \) meets \( D_{R} \), and so \( |a_0| = |f(0)| \) is bounded by a constant \( C_4(R) \) as well. Hence \( \tilde{\nu} \) and \( \nu \) are proper.

Lemma 2.2. Let \( C \) be any compact and path-connected subset of \( \mathbb{C} \). The subset of \( \mathcal{P}_d \) with all critical values in \( C \) is compact and path-connected.

Proof. Consider the diagram

\[
\begin{array}{ccc}
\mathcal{P}_d^\times & \xrightarrow{\tilde{\nu}} & \mathbb{C}^{d-1} \\
\rho \downarrow & & \downarrow \\
\mathcal{P}_d & \xrightarrow{\nu} & \mathbb{C}^{d-1}/\sim
\end{array}
\]

The right-hand vertical map is proper, and Lemma 2.1 implies the horizontal maps are proper, so the compactness conclusion holds. In addition, \( \tilde{\nu} \) is a polynomial map, so it has the property that any path in the range can be lifted (not necessarily uniquely) to a path in the domain; see [GR].

Fix now \( v \in C \). There is a unique monic and centered polynomial with a single critical value at \( v \) of multiplicity \( d - 1 \). It is \( f_1(z) = z^d + v \). For any other \( f \) with all critical values in \( C \), we can construct a path to \( f_1 \). Let \( (v_1, \ldots, v_{d-1}) \in \mathbb{C}^{d-1} \) be a labelling of the critical values of \( f \) (listed with multiplicity). Choose a continuous deformation of these points \( v_i(t) \in C \) for \( t \in [0, 1] \) so that

(i) \( v_i(0) = v_i \) for all \( i \),

(ii) \( v_i(1) = v \) for all \( i \).

The motion of labelled critical values can be first lifted under \( \tilde{\nu} \) and then projected under \( \rho \) to obtain a path from \( f \) to \( f_1 \) for which the corresponding maps all have critical values lying in \( C \). Hence this set of polynomials is path-connected. \( \square \)
3. Local models

3.1. Translation surfaces. Let $X$ be a Riemann surface, possibly with boundary, and $\omega$ a holomorphic 1-form on $X$. Away from the zeros of $\omega$, the collection of locally defined functions of the form $\psi(z) = \int_{z_0}^{z} \omega$ provide a compatible atlas of charts into $\mathbb{C}$. The ambiguity in the definition of these charts is a complex translation. It follows that away from the zeroes of $\omega$, the norm $|\omega|$ defines a flat Riemannian metric, and this metric extends to a length metric on $X$ with conical singularities at the zeros of $\omega$. Conversely, given an atlas $\{(U, \psi_U)\}$ on a Riemann surface where the overlap maps between charts differ by translations, the 1-form $\omega$ on $X$ defined by $\omega = \psi_U^*(dz)$ is globally well-defined. We call such a pair $(X, \omega)$ a translation surface.

A translation surface has natural horizontal and vertical foliations given by the inverse images of horizontal and vertical lines under the above defined local charts. These foliations have singularities at the zeros of $\omega$.

3.2. Horizontal and local model surfaces. A horizontal translation surface is a translation surface $(X, \omega)$ for which the overlap maps between charts are translations of the form $z \mapsto z + c$, $c \in \mathbb{R}$. On such a surface, there is a globally defined harmonic height function $G_X : X \to \mathbb{R}$, well-defined up to an additive constant, given by $G_X(x) = \int_{x_0}^{x} \Im \omega$. The leaves of the horizontal foliation are then connected components of level sets of the height function.

In our applications, we will only encounter horizontal translation surfaces with additional properties. We single them out as follows. A model surface is a horizontal translation surface $(X, \omega)$ with the following properties:

- $X$ is planar (genus 0);
- the level sets of the height function $G_X$ are compact and have constant length $\int_{\{G_X=c\}} |\omega| = 2\pi$ for every $c$ in the range of $G_X$;
- for all sufficiently large height values $c$, the level set $\{G_X = c\}$ is connected.

Two model surfaces $(X, \omega), (Y, \eta)$ are isomorphic if there is a conformal isomorphism $f : X \to Y$ such that $\omega = f^*(\eta)$.

A local model surface is a model surface $X$ with at most one singular horizontal leaf. We also require that such a surface be equipped with a distinguished horizontal leaf $L_X$, called its central leaf, which is either the unique singular leaf, if it exists, or else is some specified nonsingular leaf. The outer annulus of a local model surface is the annulus formed by points whose heights are greater than that of the central leaf.

Two local model surfaces $(X, \omega)$ and $(Y, \eta)$ are isomorphic if there exists a conformal isomorphism $f : X \to Y$ satisfying $\omega = f^*\eta$ and $f(L_X) = L_Y$. Note in particular that an isomorphism preserves outer annuli. If both $X$ and $Y$ have singular central leaves, then the condition $\omega = f^*\eta$ implies that $f$ preserves central leaves.
3.3. **Example: local model surfaces without singular leaves.** A local model surface without singular leaves is isomorphic to the punctured disk \( \{0 < |z| < 1\} \), the punctured plane \( \mathbb{C} \setminus \{0\} \), or an annulus \( \{r < |z| < R\} \), with the 1-form \( \omega = i dz/z \). In the case of the punctured plane, any choice of central leaf is equivalent to any other, while in the cases of the annulus or punctured disk, the (finite) modulus of the outer annulus is an invariant of equivalence of the local model surface. The metric \( |\omega| \) makes the surface isometric to a Euclidean cylinder with circumference \( 2\pi \).

3.4. **Example: polynomial pull-back.** Let \( f \) be a polynomial of degree \( k > 1 \) such that all critical values lie in \( S^1 \cup \{0\} \). Let \( X = f^{-1}(\mathbb{C} \setminus \{0\}) \) and \( \omega = i k^{-1} f^*(dz/z) \). Then \((X, \omega)\) is a local model surface with central leaf \( L_X = f^{-1}(S^1) \). Writing

\[
 f(z) = a \prod_j (z - q_j)^{m_j},
\]

we see that

\[
 \omega = i k^{-1} d(\log f(z)) = i \sum_j \frac{m_j}{z - q_j} dz.
\]

In particular, \( \omega \) is a meromorphic 1-form with only simple poles; it has residue \( i m_j/k \) at each finite pole \( q_j \in \mathbb{C} \); note \( \sum_j m_j/k_j = 1 \). The central leaf \( L_X \) is singular if and only if \( f \) has a nonzero critical value.

3.5. **Uniformizing local model surfaces.** Fix a local model surface \((X, \omega)\) and a point \( x_0 \in X \) lying on a nonsingular vertical leaf. Delete the vertical leaf through \( x_0 \), as well as the portion of any singular vertical leaves at height below or equal to that of the central leaf. The result is a simply-connected domain \( W \). Taking suitable
paths in $W$ along which to integrate, the map $\psi(x) = \int_{x_0}^{x} \omega$ defines an isomorphism from $W$ onto a slit rectangle

$$R = \{\theta + ih : 0 < \theta < 2\pi, a < h < b\} - \Sigma$$

where $\Sigma$ is a finite collection of vertical segments

$$\Sigma = \bigcup_k \{\theta_k\} \times \{a < h \leq c\};$$

here $c$ is the corresponding height of the central leaf. It follows that every local model surface is a quotient of a such a slit rectangle $R$, where the quotient maps identify the left side of a slit with real part $\theta_k$ with the right side of slit with real part $\theta_{\sigma(k)}$ for some permutation $\sigma$ on the set of slits. In these coordinates, the one-form $\omega$ is given by $dz$. The metric $|\omega|$ is simply the flat Euclidean metric on $R$.

**Remark:** The identification of vertical edges in $R$ must satisfy some obvious planarity conditions which will be treated in more detail in a sequel to this paper.

The following alternative point of view is more immediately useful.

**Lemma 3.1.** Every local model surface $(X, \omega)$ embeds uniquely into a maximal local model surface whose underlying Riemann surface is isomorphic to a finitely punctured plane. By uniformization, this extension can be represented by

$$\left(\mathbb{C} \setminus \{q_1, \ldots, q_n\}, i \sum_j \frac{r_j}{z - q_j} \, dz\right)$$

for some finite set $\{q_1, \ldots, q_n\}$ in $\mathbb{C}$ and real numbers $r_j > 0$ such that $\sum r_j = 1$. Such a representation is unique up to an affine transformation $A \in \text{Aut} \mathbb{C}$.

**Proof.** Glue half-infinite cylinders to each boundary component using isometries to obtain a new Riemann surface $\hat{X}$. The height function and 1-form extend uniquely to $\hat{X}$. Moduli estimates show that the ends of $\hat{X}$ are isomorphic to punctured disks. The complex structure extends over the punctures to yield a compact Riemann surface $\overline{X}$. Since $X$ is assumed planar, $\overline{X}$ is homeomorphic to the sphere, hence by the Uniformization Theorem is isomorphic to the $\hat{\mathbb{C}}$ via an isomorphism carrying the unique end containing points with unbounded and positive height to the point at infinity. Transporting the defining 1-form $\omega$ to $\hat{\mathbb{C}}$ via this isomorphism, infinity is a simple pole, and the remaining finite poles $q_j$ are also simple with purely imaginary residues $ir_j$, $r_j > 0$. Since in the definition of model surface the sum of the lengths of the leaves at a given height is required to be $2\pi$, we have that the $r_j$ sum to 1. $\square$
3.6. **External angles.** Suppose \((X, \omega)\) is a local model surface, and fix an embedding \(e : (X, \omega) \to \mathbb{C}\) as in Lemma 3.1. Near the end of \(\hat{X}\) corresponding to the point at infinity, each vertical leaf is nonsingular. For \(z\) large we have, since \(\sum_j r_j = 1\), that
\[
e^*(\omega) = i \left(1 + O(z^{-1})\right) \frac{dz}{z}.
\]
Hence each vertical leaf has a limiting asymptotic argument, \(\theta \in \mathbb{R}/2\pi\mathbb{Z}\), at infinity. Thus, once the embedding \(e\) has been chosen, one may speak meaningfully of the *external ray of angle* \(\theta\) of a model surface \((X, \omega)\). These angles coincide with the \(\theta\)-coordinate of a suitable rectangular representation \(R\) as in §3.5.

3.7. **Local model maps.** A branched cover of translation surfaces
\[
f : (Y, \eta) \to (X, \omega)
\]
is a holomorphic branched cover \(f : Y \to X\) such that
\[
\eta = \frac{1}{\deg f} f^* \omega.
\]

A local model map\(^1\) is a branched cover of local model surfaces
\[
f : (Y, \eta) \to (X, \omega)
\]
such that \(f\) maps the central leaf \(L_Y\) of \(Y\) to the central leaf \(L_X\) of \(X\). It follows that any critical values of \(f\) must lie in \(L_X\) and that the outer annulus of \(Y\) is mapped by a degree \(\deg f\) covering map onto the outer annulus of \(X\). Dividing by the degree ensures that with respect to the natural local Euclidean coordinates determined by \(\eta\) and \(\omega\), away from singular points the map \(f\) takes the form \(w = (\deg f) z + c\) where \(c\) is a constant.

Two local models \(f : (Y, \eta) \to (X, \omega)\) and \(g : (Z, \alpha) \to (X, \omega)\) over the same base \((X, \omega)\) are equivalent if there exists an isomorphism of local model surfaces \(i\) such that

\[
\begin{array}{ccc}
(Y, \eta) & \xrightarrow{i} & (Z, \alpha) \\
\downarrow f & & \downarrow g \\
(X, \omega) & \xrightarrow{g} & (X, \omega)
\end{array}
\]

commutes. In what follows, we think of local models as covering spaces, not as dynamical systems.

The proof of the following lemma is a straightforward application of the ideas in the proof of Lemma 3.1.

**Lemma 3.2.** Via the extension of Lemma 3.1 applied to both domain and range, every local model extends to a polynomial \(f : \mathbb{C} \to \mathbb{C}\). The extension is unique up to affine changes of coordinates in domain and range.

\(^1\)When the context is clear, we will use the terms local model and local model map interchangeably.
Note in particular that the number of critical values in $X$ of a local model $f : (Y, \eta) \to (X, \omega)$ is at most $(\deg f) - 1$.

3.8. **Spaces of local models.** Fix a local model surface $(X, \omega)$ and an extension to $\mathbb{C}$, as in Lemma 3.1. By Lemma 3.2 every local model over $(X, \omega)$ is the restriction of a polynomial; by precomposing with an automorphism of $\mathbb{C}$, we can assume that the polynomial is monic and centered. Via this representation, the set of equivalence classes of local models over $(X, \omega)$ inherits a topology from the space of monic and centered polynomials.

In detail, let $LM_k(X, \omega)$ be the set of equivalence classes of local models of degree $k$ over $(X, \omega)$. Fix an embedding $e : (X, \omega) \to (\mathbb{C} \setminus \{q_1, \ldots, q_n\}, i \sum \frac{r_j}{z - q_j} \, dz)$ as given by Lemma 3.1. Recall that $e$ is uniquely determined up to postcomposition by an affine transformation. Let $L_X \subset \mathbb{C}$ denote the image of the central leaf under the embedding $e$. Let $P_k(X, \omega)$ be the collection of monic and centered polynomials of degree $k$ with all critical values contained in the set $L_X \cup \{q_1, \ldots, q_n\}$.

Note that the location of the critical values implies that the preimage $p^{-1}(e(X))$ is connected for any $p$ in $P_k(X, \omega)$.

**Lemma 3.3.** Restriction of polynomials defines a bijection

$$P_k(X, \omega)/\langle \zeta : \zeta^k = 1 \rangle \to LM_k(X, \omega)$$

where the $k$-th roots of unity act on polynomials by precomposition: $\zeta \cdot p(z) = p(\zeta z)$.

**Proof.** For each polynomial $p$ in $P_k(X, \omega)$, its restriction to the connected subset $p^{-1}(e(X))$ defines a local model

$$p : \left(p^{-1}(e(X)), \frac{1}{k} p^* \omega \right) \to (X, \omega)$$

of degree $k$. Precomposing $p$ by a rotation of order $k$ produces another element of $P_k(X, \omega)$ which is clearly an equivalent local model. Surjectivity follows from Lemma 3.2 and injectivity follows from the uniqueness (up to conformal automorphism) of the extension in Lemma 3.2. \hfill $\square$

The bijection of Lemma 3.3 induces a topology on $LM_k(X, \omega)$, as a quotient space of the subset $P_k(X, \omega)$ of $\mathcal{P}_k$. While the set $P_k(X, \omega)$ depends on the choice of embedding $e$, the quotient sets $LM_k(X, \omega)$ are canonically homeomorphic for any two such choices. Indeed, suppose $e_1$ and $e_2$ are two embeddings and let $P_k^{(1)}(X, \omega)$ and $P_k^{(1)}(X, \omega)$ be the corresponding sets of polynomials. The composition $e_2 \circ e_1^{-1}$ extends to an affine
automorphism $z \mapsto az + b$ of $\mathbb{C}$. It follows that $p(z) \in P_k^{(1)}(X, \omega)$ if and only if $e_2 \circ e_1^{-1} \circ p(a^{-1/k}z) \in P_k^{(2)}(X, \omega)$ for any choice of root $a^{-1/k}$.

**Lemma 3.4.** Fix a local model surface $(X, \omega)$. The subset $\text{LM}_k^{k-1}(X, \omega) \subset \text{LM}_k(X, \omega)$, consisting of local model maps with all $k-1$ critical values in the central leaf of $X$, is compact and path-connected.

**Proof.** Let $S$ be the subset of $P_k(X, \omega)$ consisting of polynomials with all $k-1$ critical values in the connected set $L_X$. By Lemma 2.2, $S$ is compact and path-connected. By Lemma 3.3, the subset $\text{LM}_k^{k-1}(X, \omega)$ is homeomorphic to the quotient $S/\langle \zeta : \zeta^k = 1 \rangle$, hence is also compact and path-connected. \(\square\)

### 3.9. Pointed local model surfaces

Let $(X, \omega)$ be a local model surface, and let $x$ be a point in the outer annulus of $X$. A **pointed local model surface** is a triple $(X, x, \omega)$. We now consider pointed local model maps

$$f : (Y, y, \eta) \rightarrow (X, x, \omega),$$

i.e. maps where $f : (Y, \eta) \rightarrow (X, \omega)$ is a local model map and $f(y) = x$. Two pointed local model maps $f, g$ with the same image are **equivalent** if there exists an isomorphism $i$ of pointed local model surfaces such that

$$
\begin{array}{ccc}
(Y, y, \eta) & \xrightarrow{i} & (Z, z, \alpha) \\
\downarrow f & & \downarrow g \\
(X, x, \omega) & & \\
\end{array}
$$

commutes. We let $\text{LM}_k(X, x, \omega)$ denote the set of equivalence classes of these pointed local models. As in the non-pointed case, the set can be topologized via an identification with monic and centered polynomials. Let $P_k(X, \omega)$ be the set of monic and centered polynomials defined in §3.8. Compare the statement of the following lemma to that of Lemma 3.3.

**Lemma 3.5.** Let $(X, x, \omega)$ be a pointed local model surface. The canonical projection $\text{LM}_k(X, x, \omega) \rightarrow \text{LM}_k(X, \omega)$ factors through a bijection $b$ such that the diagram

$$
\begin{array}{ccc}
\text{LM}_k(X, x, \omega) & \xrightarrow{b} & P_k(X, \omega) \\
\downarrow & & \downarrow \\
\text{LM}_k(X, \omega) & \leftarrow & P_k(X, \omega)/\langle \zeta : \zeta^k = 1 \rangle \\
\end{array}
$$

commutes, where $r$ is the restriction map of Lemma 3.3.

**Proof.** Fix an embedding

$$e : (X, \omega) \rightarrow \left( \mathbb{C} \setminus \{ q_1, \ldots, q_n \}, \ i \sum_j \frac{r_j}{z - q_j} \ dz \right)$$
so that the marked point $x$ lies on a vertical leaf with external angle 0. For each element $f : (Y, y, \eta) \to (X, x, \omega)$ of $\text{LM}_k(X, x, \omega)$, choose an extension of the domain so that the marked point $y$ lies on a vertical leaf of external angle 0. This uniquely determines an element of $P_k(X, \omega)$; denote this element by $b(f)$. If two pointed local model maps extend to the same polynomial, then they are clearly isomorphic, via an isomorphism which preserves the marked points; this proves injectivity of $b$. For surjectivity of $b$, note that the restriction of any element $p \in P_k(X, \omega)$ to $p^{-1}(e(X))$ determines an element of $\text{LM}_k(X, x, \omega)$ with marked point chosen as the unique preimage of $x$ on the external ray of angle 0. Consequently $b$ is a bijection.

The diagram commutes by construction. □

The bijection $b$ of Lemma 3.5 induces a topology on the set $\text{LM}_k(X, x, \omega)$, making the projection $\text{LM}_k(X, x, \omega) \to \text{LM}_k(X, \omega)$ continuous. The following is then an immediate consequence of Lemma 2.2:

**Lemma 3.6.** Fix a pointed local model surface $(X, x, \omega)$. The subset $\text{LM}_k^{-1}(X, x, \omega) \subset \text{LM}_k(X, x, \omega)$, consisting of pointed local models with all $k - 1$ critical values in the central leaf of $X$, is compact and path-connected.

In applications, pointed local models are used to keep track of external angles; a point in the outer annulus of $X$ marks a unique vertical leaf in the foliation of $\omega$.

**Remark.** We have chosen to define the topology on spaces of local models with Lemmas 3.3 and 3.5, via the uniformization by polynomial branched covers of $\mathbb{C}$. One can also present a Gromov-Hausdorff topology (as we do for polynomial basins of infinity in §4.2), using the rectangular representation from §3.5. In [DP2], we introduce yet another perspective on local models, in terms of branched covers of laminations, which gives a third way to view these spaces of local models and their topology. The various viewpoints produce equivalent topologies.

## 4. Restricting to the basins of infinity

Recall that $\mathcal{B}_d$ denotes the space of conformal conjugacy classes of $(f, X(f))$. Here we introduce the **Gromov-Hausdorff topology** on $\mathcal{B}_d$ and begin the analysis of the restriction map

$$\pi : \mathcal{M}_d \to \mathcal{B}_d.$$  

### 4.1. The metric $|\omega|$ on the basin of infinity.

Fix a polynomial $f$ of degree $d \geq 2$. On its basin of infinity $X(f)$, recall that $G_f$ denotes the harmonic escape rate function and

$$\omega_f = 2i \partial G_f$$

the corresponding holomorphic 1-form, so that $|\omega_f|$ is the associated singular flat metric. In this way, the pair $(X(f), \omega_f)$ becomes a model surface with height function
Note that the height $G_f(z)$ of any point $z \in X(f)$ coincides with its $|\omega_f|$-distance to the lower end of $X(f)$. Recall that

$$M(f) = \max\{G_f(c) : f'(c) = 0\}$$

denotes the maximal critical height of $f$. We will write $G$ for $G_f$ and $\omega$ for $\omega_f$ when the dependence on $f$ is clear.

The zeroes of $\omega$ coincide with the critical points of $f$ in $X(f)$ and all of their preimages by the iterates $f^n$. The neighborhood $\{z : G(z) > M(f)\}$ of infinity is isometric to a half-infinite cylinder of radius 1. In fact, if $L$ is any horizontal leaf of $\omega$, meaning that $L$ is a connected component of a level set $\{G = c\}$, and if the level of $L$ is the integer

$$(4.1) \quad l(L) = \min\{n \geq 0 : d^n c \geq M(f)\},$$

then

$$\int_L |\omega| = \frac{\deg(f^l|L)}{\ell(L)} 2\pi.$$ 

Further, if $A$ is a connected component of $\{a < G < b\}$ which is topologically an annulus, then it is isometric to a cylinder of height $(b - a)$ and circumference $\int_L |\omega|$ for any horizontal leaf $L$ in $A$.

If $L$ is a (possibly singular) horizontal leaf of $\omega$ of level $l$, and if $X$ is a connected component of $\{a < G < b\}$ containing $L$ and no other singular leaves, then the pair $(X, \omega_X)$ defines a local model surface, where

$$(4.2) \quad \omega_X = \frac{\ell}{\deg(f^l|L)} \omega_f = \frac{2\pi}{\int_L |\omega|} \omega_f,$$

with central leaf $L_X = L$.

On the basin of infinity, conformal and isometric conjugacies are the same thing.

**Lemma 4.1.** Two polynomials $f$ and $g$ are conformally conjugate on their basins of infinity if and only if they are isometrically conjugate with respect to the metrics $|\omega_f|$ and $|\omega_g|$. In particular, the escape rates of the critical points are isometric conjugacy invariants.

**Proof.** If the polynomials are conformally conjugate, then the conjugacy sends $\omega_f$ to $\omega_g$, and therefore their basins are isometrically conjugate. Conversely, the metric $|\omega|$ determines the complex structure, so an isometry must be a conformal isomorphism. \qed

4.2. The topology of $\mathcal{B}_d$. We define here the Gromov-Hausdorff topology on the space of triples $(f, X(f), |\omega|)$ in $\mathcal{B}_d$. We will see that this topology is fine enough to capture the topology of the shift locus in $\mathcal{M}_d$, but coarse enough to allow various continuity arguments to be fruitfully applied.
A basis of open sets \( U_{t,\varepsilon}(f) \) for this topology is given as follows. For each small \( t > 0 \) such that \( M(f) < 1/t \), let

\[
X_t(f) = \{ z \in X(f) : t \leq G(z) \leq 1/t \}.
\]

Let \( \rho_f(\cdot, \cdot) \) denote the distance function on \( X_t(f) \) induced by the metric \(|\omega|\). An \( \varepsilon \)-conjugacy between \( f|X_t(f) \) and \( g|X_t(g) \) is a relation which is nearly the graph of an isometric conjugacy. That is, it is a subset \( \Gamma \subset X_t(f) \times X_t(g) \) such that

1. (1) **nearly surjective:**
   - (a) for every \( a \in X_t(f) \), there exists a pair \( (x, y) \in \Gamma \) such that \( \rho_f(a, x) < \varepsilon \),
   - (b) for every \( b \in X_t(g) \), there exists a pair \( (x, y) \in \Gamma \) such that \( \rho_g(b, y) < \varepsilon \),

2. (2) **nearly isometric:** if \( (x, y) \) and \( (x', y') \) are in \( \Gamma \), then
   \[
   |\rho_f(x, x') - \rho_g(y, y')| < \varepsilon,
   \]

and

3. (3) **nearly conjugacy:** for each \( (x, y) \in \Gamma \) such that \( (f(x), g(y)) \) lies in \( X_t(f) \times X_t(g) \), there exists \( (x', y') \in \Gamma \) such that \( \rho_f(x', f(x)) < \varepsilon \) and \( \rho_g(y', g(y)) < \varepsilon \).

The neighborhood basis element \( U_{t,\varepsilon}(f) \) consists of all triples \((g, X(g), |\omega|)\) for which there is an \( \varepsilon \)-conjugacy between \( f|X_t(f) \) and \( g|X_t(g) \).

**Lemma 4.2.** \( \pi \) is continuous, surjective, and proper.

**Proof.** Surjectivity holds by definition. If \( f_k \to f \) in \( \mathcal{M}_d \), then there are polynomial representatives which converge uniformly on compact subsets of \( \mathbb{C} \), and the escape-rate functions \( G_k \) converge to \( G \) by [BH1, Proposition 1.2]. For small \( t > 0 \), consider the compact subset

\[
Y_t(f) = \{ z \in \mathbb{C} : t \leq G_f(z) \leq 1/t \}
\]

of the plane. The sets \( Y_t(f_k) \) converge to \( Y_t(f) \) in the Hausdorff topology on compact subsets of \( \mathbb{C} \), and the action of \( f_k \) on \( Y_t(f_k) \) converges to that of \( f \) on \( Y_t(f) \) (with respect to the Euclidean metric on \( \mathbb{C} \)). Moreover, the escape-rate functions \( G_k \) and \( G \) are harmonic on these sets \( Y_t \), so uniform convergence implies also the convergence of their derivatives. Therefore, the 1-forms \( \omega_k \) on \( Y_t(f_k) \) converge to \( \omega_f \) on \( Y_t(f) \) and so the metrics \( |\omega_k| \) on \( Y_t(f_k) \) converge to the metric \(|\omega|\) on \( Y_t(f) \). More precisely: let \( \Gamma_k \) be the graph of the nearest-point projection from \( Y_t(f_k) \) to \( Y_t(f) \). For all large enough \( k \), the graph \( \Gamma_k \) defines an \( \varepsilon \)-conjugacy between \( f_k|X_t(f_k) \) and \( f|X_t(f) \). Therefore \( \pi \) is continuous. Properness of the map \( \pi \) follows from the known fact that \( f \mapsto M(f) \) is proper [BH1].

**Lemma 4.3.** The space \( \mathcal{B}_d \) equipped with the Gromov-Hausdorff topology is Hausdorff, locally compact, second-countable, and metrizable. Moreover, it is homeomorphic to the quotient space of \( \mathcal{M}_d \) obtained by identifying the fibers of \( \pi \) to points.
Proof. A standard application of the definitions and a diagonalization argument shows that $B_d$ is Hausdorff. By definition, the topology is first-countable. By [Da, Thm. 5, p. 16] it follows that the Gromov-Hausdorff and quotient topologies coincide. Local compactness follows from continuity and properness of the projection $\pi$. Metrizability follows from [Da, Prop. 2, p. 13] and second-countability follows. □

4.3. The shift locus and rigid Riemann surfaces. Recall that the shift locus $S_d$ is the collection of polynomials in $M_d$ where all critical points escape to $\infty$.

A planar Riemann surface $X$ is rigid if a holomorphic embedding $X \hookrightarrow \hat{\mathbb{C}}$ is unique up to the conformal automorphisms of $\hat{\mathbb{C}}$. Equivalently, the complement of $X \subset \mathbb{C}$ has absolute area zero; that is, the spherical area of $\hat{\mathbb{C}} \setminus X$ is 0 under any embedding. Further, an absolute area zero subset of the plane is removable for locally bounded holomorphic functions with finite Dirichlet integral [AS]. In particular, if the complement of $X \subset \mathbb{C}$ has absolute area zero, then any proper holomorphic degree $d$ self-map $X \to X$ extends uniquely to a degree $d$ rational function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. In [Mc, §2.8], McMullen showed that an open subset $X \subset \mathbb{C}$ is rigid if it satisfies the infinite-modulus condition: for each $n \in \mathbb{N}$, there is a finite union of disjoint unnested annuli $E_n \subset X$, contained in the bounded components of $\mathbb{C} \setminus E_{n-1}$, such that for each nested sequence of connected components $\{A_n \subset E_n\}_n$, we have $\sum_n \text{mod } A_n = \infty$, and the nested intersection of the bounded components of $\hat{\mathbb{C}} \setminus E_n$ is precisely $\mathbb{C} \setminus X$.

In the case of the shift locus, the following lemma and its proof were known (see e.g. [BH2], [BDK], [Br], [Mc]), though never stated in quite this way. The proof in the more general case of the Cantor locus uses different methods, so we keep the statements and proofs separate.

Lemma 4.4. The projection $\pi$ is a homeomorphism on the shift locus $S_d$ and, more generally, on the Cantor locus.

Proof. When $f$ is in the shift locus, it is easy to see that the basin $X(f)$ satisfies the infinite-modulus condition. Consider an annulus $A = \{a < G(z) < b\}$ with $M(f) < a < b < dM(f)$ and disjoint from the critical orbits. Since $f$ is in the shift locus, the iterated preimages of this annulus map with uniformly bounded degree onto $A$. Hence each such preimage has modulus at least $m > 0$. It follows that there is a unique embedding of $X(f)$ into $\hat{\mathbb{C}}$, up to an affine transformation, sending infinity to infinity. Furthermore, the complement $\hat{\mathbb{C}} \setminus X(f)$ is removable for $f$, so $f : X(f) \to X(f)$ extends uniquely to a rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is totally ramified at infinity. In other words, up to affine conjugacy, we can reconstruct the polynomial $f : \mathbb{C} \to \mathbb{C}$ from its restriction $f : X(f) \to X(f)$.

This is also possible for maps in the Cantor locus. Suppose $K(f), K(g)$ are Cantor sets and $h : X(f) \to X(g)$ is a holomorphic conjugacy. Then $h$ extends to a homeomorphism $h : \mathbb{C} \to \mathbb{C}$. By the main result of [Z], $h$ is quasiconformal. By the main result of [YZ], $h$ is affine and so $f = g$ in $MP_d$. 
Using Lemma 4.2, we see that $\pi$ is a continuous bijection on the shift and Cantor loci. The images are Hausdorff, and the domains are locally compact. It follows that on each locus, the map $\pi$ is a local homeomorphism and therefore a global homeomorphism since $\pi$ is proper.

4.4. Local models and gluing. Fix a polynomial $f \in \mathcal{M}_d$ and let as usual $G$ and $\omega$ denote its escape rate function and 1-form. Fix $t > 0$ and choose real numbers $0 < a < t < b$ so that each connected component of $\{a < G < b\}$ defines a local model surface with central leaf $\{G = t\}$. Let $X$ be a connected component of $\{da < G < db\}$ and let $\omega_X$ be the restriction of the rescaled 1-form $\omega$ to $X$ as defined in (4.2) so that $(X, \omega_X)$ forms a local model surface with central leaf at height $dt$. If we pick a point $x$ in the outer annulus of $X$, we obtain a pointed local model surface $(X, x, \omega_X)$. Suppose $Z$ is a connected component of the preimage $f^{-1}(X)$ mapping by degree $k$ onto $X$; it is a connected component of $\{a < G < b\}$. If we choose a preimage $z$ of $x$ in $Z$ and set $\mu = (d/k) \omega_Z = k^{-1} f^* \omega_X$, then

$$f : (Z, z, \mu) \to (X, x, \omega_X)$$

is a pointed local model map; see Figure 4.1.

In what follows, we repeatedly use the procedure of gluing a local model to a polynomial basin. In the setup of the previous paragraph, let now

$$p : (Y, y, \eta) \to (X, x, \omega_X)$$
be any pointed local model map of degree $k$ over $(X, x, \omega_X)$. The restrictions of $p$ and of $f$ to the outer annuli of $Y$ and of $Z$, respectively, are covering maps of degree $k$ onto the outer annulus of $X$, so there is a unique conformal isomorphism identifying these annuli which sends $y$ to $z$ and which identifies their outer boundaries. This isomorphism identifies $\eta$ and $\mu$, so the Riemann surface $\{G > t\} \cup Y$ becomes a horizontal translation surface. The original height function $G$ extends down over $Y$ to a height function $\tilde{G} : \{G > t\} \cup Y \to \mathbb{R}$, and the map $f|\{G > t\}$ extends holomorphically to

\[ \tilde{f} : \{G > t\} \cup Y \to \{G > t\} \]

so that $\tilde{f}$ agrees with $p$ on $Y$. We say that $\tilde{f}$ is obtained from $f$ by \textit{gluing in the local model} $p$ at height $t$. We can repeat this construction for each component $Z_i$ of $\{a < G < b\}$.

In summary, we note the following. On the one hand, by restriction, any polynomial $f$ determines a collection of pointed local model maps $\tilde{f}_i : (Z_i, z_i, \mu_i) \to (X_i, x_i, \omega_i)$, where $\omega_i = \omega_{X_i}$; the collection is indexed by the set of connected components of $\{G = t\}$. On the other hand, given arbitrary pointed local model maps $p_i : (Y_i, y_i, \eta_i) \to (X_i, x_i, \omega_i)$, we can glue in these local models to obtain a new horizontal translation surface $\{G > t\} \cup (\cup_i Y_i)$ and a new map $\tilde{f} : \{G > t\} \cup (\cup_i Y_i) \to \{G > t\}$ from this surface to itself, now defined at all heights above some height $s$, where $s < t$.

Using these ideas, we can deduce that rigidity of the basins in the shift locus also implies that the restriction $f|\{G > t\}$ for any small enough $t > 0$ uniquely determines the polynomial $f$:

**Lemma 4.5.** Let $f \in S_d$ and fix $t > 0$ such that $t < \min\{G_f(c) : f'(c) = 0\}$. If $g$ is any polynomial such that $(g, \{G_g > t\})$ is conformally conjugate to $(f, \{G_f > t\})$, then $g$ and $f$ are conformally conjugate on $\mathbb{C}$.

**Proof.** It suffices to show that the affine conjugacy class of the polynomial $f$ is determined by the conformal conjugacy class of the restriction $f : \{G > t\} \to \{G > dt\}$. The choice of $t$ guarantees that if $a, b$ are sufficiently close to $t$ and satisfy $a < t < b$ then the components of $\{a < t < b\}$ are mapped under $f$ by degree one onto their images. In the terminology of the previous paragraphs, then, there are no choices for how to glue in local models. This observation and induction imply that by iterating the gluing procedure, there is a unique extension of $f : \{G > t\} \to \{G > dt\}$ to a Riemann surface $f : X \to X$ which is a proper holomorphic map of degree $d$. The same argument that shows that basins of maps in the shift locus are rigid shows that $X$ satisfies the infinite modulus condition and is therefore rigid. Since up to affine maps there is a unique embedding of $X$ into $\mathbb{C}$ and $f : X \to X$ extends uniquely and holomorphically to $f : \mathbb{C} \to \mathbb{C}$, the affine conjugacy class of the polynomial $f$ is uniquely determined. \hfill \Box

**Proposition 4.6.** The shift locus is dense in $B_d$. 
Proof. Suppose the polynomial $f$ represents an element of $B_d$, and let $t > 0$. For each boundary component of $\{G > t\}$, glue in a pointed local model map with the special property that if the degree of this local model map is $k > 1$, then all $k - 1$ critical points are contained in the central leaf. We obtain an extended holomorphic map

$$\tilde{f} : \{\tilde{G} > s\} \to \{\tilde{G} > ds\}$$

for some $s < t$ which is proper, of degree $d$, and has the maximum possible number $d - 1$ of critical points, each of which has height at least $t$. On any leaf with height between $s$ and $t$, the extended map $\tilde{f}$ has degree 1. The proof of Lemma 4.5 shows that $\tilde{f}$ extends uniquely first to a map $\tilde{f} : X \to X$ where $X$ is a rigid Riemann surface and finally to a polynomial $\tilde{f} : \mathbb{C} \to \mathbb{C}$ whose affine conjugacy class is uniquely determined.

By construction, the polynomial $\tilde{f}$ is in the shift locus, and $\tilde{f}|\{G > t\}$ is conformally conjugate to $f|\{G > t\}$. Letting $t \to 0$, one easily checks that $\tilde{f}$ approximates $f$ in the Gromov-Hausdorff topology on $B_d$. □

5. Proof that $\pi$ is monotone

5.1. The sets $B(f, t)$ and $S(f, t)$. Fix a polynomial $f$ and a positive real number $t$. Let $B(f, t)$ be the collection of all polynomials $g$ in $M_d$ such that $g|\{G_g > t\}$ is conformally conjugate to $f|\{G_f > t\}$. Let $S(f, t)$ be the polynomials $g \in B(f, t)$ such that $G_g \geq t$ at all critical points. Note that $S(f, t)$ is contained in the shift locus.

Recall that $M(f)$ denotes the maximal critical escape rate of $f$. If $t$ is large enough so that $t \geq M(f)$, then $B(f, t)$ consists of all polynomials $g$ with $t \geq M(g)$. In this case, $S(f, t)$ is the collection of polynomials $g$ with $G_g(c) = t$ at all critical points $c$ of $g$; that is, all critical points of $g$ escape at the same rate $t$. In degree 2, for the family $z^2 + c$, we have the following dichotomy:

- If $t < M(f)$, then $B(f, t) = S(f, t) = \{f\}$.
- If $t \geq M(f)$, then $S(f, t)$ is the equipotential curve $\{c : G_c(0) = t\}$ around the Mandelbrot set, and $B(f, t)$ is the closed ball it bounds.

In every degree, when $f$ is in the shift locus and $t$ is small enough that $t < G(c)$ for all critical points $c$ of $f$, then $B(f, t) = S(f, t) = \{f\}$. The general structure of $B(f, t)$ and $S(f, t)$ is somewhat more complicated, but it can be analyzed in terms of spaces of local models.

The following lemma is the key ingredient in the proof that $\pi : M_d \to B_d$ is monotone.

Lemma 5.1. For every $f \in M_d$ and any $t > 0$, the set $S(f, t)$ is path-connected.

Before turning to the details of the proof, we make some preliminary observations to keep the main thread of the argument clean.
It will be more convenient to work with the set $\mathcal{P}_d$ of monic and centered polynomials, so that each basin of infinity has well-defined external angles. In particular, any $f \in \mathcal{P}_d$ fixes exactly $d-1$ distinct external rays. Let $\mathcal{P}_d^\infty$ be the set of conformal conjugacy classes of monic, centered polynomials restricted to their basins of infinity, where now the conjugacy is required to have derivative 1 at infinity. As a set, $\mathcal{P}_d^\infty$ is the set of equivalence class of triples $(f, X, P)$, where $\theta_f$ is one of the $d-1$ external rays that are fixed under $f$, and where two triples $(f, X(f), \theta_f), (g, X(g), \theta_g)$ are equivalent if there is a holomorphic conjugacy from $f$ on $X(f)$ to $g$ on $X(g)$ sending $\theta_f$ to $\theta_g$. We equip $\mathcal{P}_d^\infty$ with the smallest topology such that the natural projection $\mathcal{P}_d^\infty \to \mathcal{B}_d$ is continuous. More concretely: an $\epsilon$-conjugacy in this setting has the same definition as for the Gromov-Hausdorff topology in §4.2, with the following additional requirement. Observe that if $1/t > M(f)$ then the set $\{G_f = 1/t\} \cap \theta_f$ is a singleton $x_f$. We require that an $\epsilon$-conjugacy $\Gamma$ send $x_f$ to $x_g$, i.e. $(x_f, x_g) \in \Gamma$. We refer to this topology as the Gromov-Hausdorff topology on $\mathcal{P}_d^\infty$. The arguments showing that the projection $\pi : \mathcal{M}_d \to \mathcal{B}_d$ is a homeomorphism on the shift locus (Lemma 4.4) immediately show that the projection $\tilde{\pi} : \mathcal{P}_d \to \mathcal{P}_d^\infty$ is also a homeomorphism on the corresponding shift locus $\tilde{S}_d \subset \mathcal{P}_d$ equipped with its algebraic topology (from the polynomial coefficients).

Given an element $f$ in $\mathcal{P}_d$, consider its restriction to $\{G_f > t\}$. Define $\tilde{S}(f, t)$ in $\mathcal{P}_d$ to be the set of polynomials $g \in \mathcal{P}_d$ with $g|\{G_g > t\}$ conjugate to $f|\{G_f > t\}$ via a conformal isomorphism with derivative 1 at infinity, and such that $G_g(c) \geq t$ for all critical points $c$ of $g$. Then, for each polynomial $g \in \tilde{S}(f, t)$, there is a unique isomorphism $\varphi_g : \{G_g > t\} \to \{G_f > t\}$ conjugating $g$ to $f$ and sending the ray of angle 0 for $g$ to that of $f$. It is a straightforward consequence of the definitions that the set $\tilde{S}(f, t)$ maps surjectively onto $S(f, t)$ under the natural projection $\mathcal{P}_d \to \mathcal{M}_d$.

So, Lemma 5.1 follows once we establish that the corresponding set $\tilde{S}(f, t)$ in $\mathcal{P}_d$ is path-connected. This is what we prove below.

**Proof.** Fix $f \in \mathcal{P}_d$ and let $t > 0$ be arbitrary. Consider now the process of restricting $f$ to heights near $t$, as described in §4.4, to define local model maps

$$f_i : (Z_i, z_i, \mu_i) \to (X_i, x_i, \omega_i)$$

of degree $k_i$, indexed by the connected components of $\{G_f = t\}$. For each $g \in \tilde{S}(f, t)$, the domains $\{G_f > t\}$ and $\{G_g > t\}$ are canonically identified via $\varphi_g$, so we may think that the $X_i$ belong to $\{G_g > t\}$. By the definition of the set $\tilde{S}(f, t)$, for each component $i$, the corresponding local model map $g_i$ has all $k_i - 1$ critical values contained in the central leaf of $X_i$. We obtain a well-defined map

$$L : \tilde{S}(f, t) \to \prod_i L_{k_i}^{k_i-1}(X_i, x_i, \omega),$$
defined by $g \mapsto (g_i)$. The right-hand side is compact and path-connected by Lemma 3.6. The remainder of the proof is devoted to establishing that $\mathcal{L}$ is in fact a homeomorphism.

It suffices to show that $\mathcal{L}$ is a bijection, and that the inverse $\mathcal{L}^{-1}$ is continuous.

Surjectivity is clear: if $p_i \in L_{k_i}^{k_i-1}$ for each $i$, then gluing the $p_i$ to $\{G_f > t\}$ yields a map of a horizontal surface to itself all of whose lower boundary components map by degree one. Applying the argument used in the proof of Proposition 4.6 (that is, extending and appealing to rigidity and uniformization) shows that the new map obtained by gluing extends to a polynomial in $\tilde{S}(f,t)$.

For injectivity, suppose $\mathcal{L}(g_1) = \mathcal{L}(g_2)$. The isomorphisms of pointed local models glue to the isomorphism $\varphi_{g_2}^{-1} \circ \varphi_{g_1}$ from $\{G_{g_1} > t\}$ to $\{G_{g_2} > t\}$; therefore above some height $s < t$ the polynomials $g_1, g_2$ are conjugate via a conformal isomorphism with derivative 1 at infinity. By Lemma 4.5 the polynomials are then affine conjugate on $\mathbb{C}$, again via an isomorphism of derivative 1 at infinity. Because they are monic and centered, they must be equal.

It remains to establish continuity.

Because $\tilde{r} : \mathcal{P}_d \to \mathcal{P}_d^\infty$ is a homeomorphism on the shift locus, it suffices to establish continuity of the map $\mathcal{L}_1 : \prod_i L_{k_i}^{k_i-1}(X_i, x_i, \omega) \to \tilde{S}_d$ where the shift locus $\tilde{S}_d$ is equipped with the Gromov-Hausdorff topology of $\mathcal{P}_d^\infty$. The argument is essentially the same one given in the proof of Lemma 4.2 showing that $\pi$ is continuous.

Fix $\varepsilon, s > 0$ such that $s < t$, and fix $g \in S(f,t)$. We aim to show that the preimage of the Gromov-Hausdorff neighborhood $U_{s,\delta}(g)$ is open in $\prod_i L_{k_i}^{k_i-1}(X_i, x_i, \omega_i)$. Take $n$ so that $s > t/d^n$. For any tuple $(g_1, \ldots, g_m)$ in $\prod_i L_{k_i}^{k_i-1}(X_i, x_i, \omega_i)$ which is sent to $g$, and any $\varepsilon' > 0$, the gluing construction clearly allows us to choose a neighborhood of $(g_1, \ldots, g_m)$ which is sent into a neighborhood $U_{t-\delta, \varepsilon'}(g)$ of $g$ in $\mathcal{B}_d$ for some $\delta > 0$. The uniquely determined extensions to the next level $\{G > (t-\delta)/d\}$ are therefore contained in $U_{(t-\delta)/d, \varepsilon' + \varepsilon'}(g)$. By induction, the neighborhoods are sent to $U_{(t-\delta)/d^n, (n+1)\varepsilon'}(g)$. Choose $\varepsilon' < \varepsilon/(n+1)$, and we conclude that this neighborhood of $(g_1, \ldots, g_m)$ in $\prod_i L_{k_i}^{k_i-1}(X_i, x_i, \omega_i)$ is sent into $U_{s,\varepsilon}(g)$. Therefore $\mathcal{L}_1$ is continuous. \qed

**Remark.** An alternative, more intrinsic proof of Lemma 5.1 may be given along the following lines. There is a natural Gromov-Hausdorff topology on $\mathcal{L}^k(X_i, x_k, \omega)$. The surgery constructions in [EMZ Section 8] are affinely natural. This shows that branch values of local model maps can be pushed through zeros of $\omega$ in $X_i$, and these branch values can be pushed so as to coalesce together to a single branch value as in the proof of Lemma 2.2.

5.2. Deforming the basin of infinity. In the next lemma, we use a “pushing deformation” to show that $B(f, t) \cap S_d$ is connected. The construction is similar to the pushing deformation of [BDK]; in their case, they push critical values down to smaller heights, while we push critical values up along external rays. Certain
deformations require a change in the local topology of the translation structure, like moving through a stratum of $\mathcal{B}_g$ governed by the multiplicities of zeroes of the 1-form $\omega$; compare [EMZ] Section 8.

**Lemma 5.2.** For any $f \in S_d$ and any $t > 0$, there is a path from $f$ to $S(f,t)$ contained in $B(f,t)$. Further, we may choose the path $\{f_h : s \leq h \leq t\}$ so that $f = f_s$ and $f_h \in S(f,h)$ for all $h$.

**Proof.** If all critical points of $f$ have height at least $t$ then already $f \in S(f,t)$. So suppose $f$ has critical points below height $t$, and let

$$0 < s = \min \{G(c) : f'(c) = 0\} < t$$

be the height of the lowest critical point. We will “push” the lowest critical values from the level curves $\{G = ds\}$ up along their external rays in a continuous fashion, without changing $f|\{G > t\}$, until all critical values have height $\geq dt$.

For each component $L_i$ of $\{G = s\}$, let $k_i \geq 1$ be the degree of the restriction of $f$ to $L_i$. Consider the restriction $f_i$ of $f$ to a narrow neighborhood $Z_i$ of $L_i$. As in §4.4 setting $\omega_i = \omega_{X_i}$ as defined by (4.2) and $\mu_i = k_i^{-1}f_i^*(\omega_i)$, we obtain a pointed local model map

$$f_i : (Z_i, z_i, \mu_i) \to (X_i, x_i, \omega_i),$$

with all $k_i - 1$ critical values contained in the central leaf $L_{X_i}$.

Suppose first that $\omega$ has no zeroes at height $ds$, so for each $i$, the base $X_i$ is an annulus and there is a unique vertical leaf passing through each of the critical values of $f_i$. Let $s'$ be the smallest height with $s < s' \leq t$ at which $\omega$ has zeroes; if no such height exists, let $s' = t$. For each height $s \leq h < s'$, and for each $i$, we use the construction of §4.4 to glue in the local model $f_i$ to $f|\{G > h\}$ at height $h$, so that the critical values lie on the same vertical leaf for all $h$. The extended map has degree 1 along each boundary component below height $h$ and therefore extends uniquely to a polynomial in $S(f,h)$. For $h = s$, we recover $f$. As $h$ increases from $s$ to $s'$, we see that this defines a continuous deformation of basins with respect to the Gromov-Hausdorff topology, just as in the proof of Lemma 5.1. By Lemma 4.4, $\pi$ is a homeomorphism on the shift locus, so we obtain a continuous path $f_h \in \mathcal{M}_d$ with $f_s = f$. Furthermore, as $h \to s'$, the path accumulates on $S(f,s')$. In fact, the path converges to a unique element of $S(f,s')$: the vertical leaf of each critical value is constant along the path, so the set of accumulation points is compact, discrete, and connected, therefore a singleton. We have thus joined $f$ to an element of $S(f,s')$ by a path contained in $B(f,t)$. Either $s' = t$ and we’re done, or $\omega$ had a zero at height $s'$ and we begin again (taking $s'$ for the value of $s$ and $f_{s'}$ for the initial $f$).

Now suppose $\omega$ has zeroes in $X_i$ at height $ds$. By Lemma 3.5, each $f_i$ is represented by a polynomial of degree $k_i$ in $P_{k_i}(X_i, x_i, \omega_i)$. When so represented, the surface $X_i$ is a subset of the plane. Each critical value $v$ of $f_i$ lies on the singular central leaf of $X_i$ and lies on one or more external rays. For each critical value $v$, choose one such ray
θv. Now consider a continuous deformation \( f_{i,h, s \leq h \leq s+\varepsilon} \) of \( f_i = f_{i,s} \) constructed as follows. The critical values move upwards along the chosen external rays such that for each parameter \( h \), the height of the critical values increases monotonically from \( ds \) to \( dh \). Such a deformation exists by the path-lifting property of the map associating a polynomial to its critical values; cf. the proof of Lemma 2.2. Let \( L \) be the leaf containing the critical values of the perturbed map. The restriction of this perturbed map, over a narrow foliated annular neighborhood of \( L \), determines a new local model of degree \( k_i \) which can be glued to \( f \) at a height \( h > s \). This defines a continuous deformation in \( M_d \) from \( f \) to an element of \( S(f,h) \). Note that the base of each new local model \( f_{i,h} \) is an annulus. After this deformation, we then use the pushing deformation of the previous paragraph to increase the heights of the critical points to the next height at which occur zeroes of \( \omega \).

Continuing inductively until all critical values have been pushed up to height \( dt \) defines a continuous path in the shift locus from \( f \) to \( S(f,t) \). By construction, the path is contained in \( B(f,t) \). □

Remark. It can be seen from the proof of Lemma 5.2 that the “pushing-up” deformation is canonical unless the moving critical values encounter zeroes of \( \omega \). That is, the path is uniquely determined except when the lowest critical values are pushed up through critical points of \( f \) or any of their iterated preimages. Note, however, that if a choice is made at height \( t_0 < t \), the path-connectedness of \( S(f, h) \) by Lemma 5.1 implies that different choices can themselves be connected by paths within \( B(f,t) \cap S_d \).

Corollary 5.3. For any \( f \) in \( M_d \) and \( t > 0 \), the intersection of \( B(f,t) \) with the shift locus \( S_d \) is path-connected. In particular, the shift locus is connected.

Proof. Fix \( f \). It follows immediately from the definition that \( B(g,t) = B(f,t) \) if and only if \( g \in B(f,t) \). Similarly, \( S(f,t) = S(g,t) \) if and only if \( g \in B(f,t) \). Thus, we may choose any element \( g \in B(f,t) \cap S_d \) and apply Lemma 5.2 to find a path from \( g \) to \( S(g,t) = S(f,t) \) contained in \( B(g,t) = B(f,t) \). As \( S(f,t) \) is path-connected by Lemma 5.1, we conclude \( B(f,t) \cap S_d \) is path-connected. Since the shift locus is an increasing union of sets of the form \( B(f,t) \cap S_d \) for \( t > M(f) \), the shift locus is connected. □

Recall that a Gromov-Hausdorff basis neighborhood of a polynomial \( f \) is denoted \( U_{t,\varepsilon}(f) \), where \( 1/t > M(f) \).

Lemma 5.4. For any \( f \in M_d \) and \( t > 0 \) such that \( M(f) < 1/\varepsilon t \), we have

\[
B(f,t) = \bigcap_{\varepsilon > 0} \pi^{-1} U_{t,\varepsilon}(f).
\]

Proof. The set \( B(f,t) \) is clearly contained in the nested intersection, because a conformal conjugacy to \( f|\{G > t\} \) is an isometry with respect to the metric \( |\omega| \). We now prove the other inclusion. Any polynomial \( g \in \cap_{\varepsilon > 0} \pi^{-1} U_{t,\varepsilon}(f) \) is, on \( \{t < G_g < 1/\varepsilon t\} \),
isometrically conjugate to $f$ on $\{t < G_f < 1/dt\}$. The condition on $t$ guarantees that higher up on the domains $\{G_f > 1/dt\}$, $\{G_g > 1/dt\}$, the maps $f$ and $g$ are ramified only at the point at infinity. It follows that this conjugacy extends uniquely to a isometric conjugacy between $f$ and $g$ on the unions $\{t < G_g < \infty\}$ and $\{t < G_f < \infty\}$.

As the metrics $|\omega_f|$, $|\omega_g|$ determine the complex structures, $g$ must in fact be conformally conjugate to $f|\{G > t\}$, and therefore in $B(f,t)$. □

5.3. Completing the proof that $\pi$ has connected fibers. Let $\mathcal{M}_d(n) \subset \mathcal{M}_d$ be the polynomials with at least $n$ escaping critical points. Then $\mathcal{M}_d(d-1) = \mathcal{S}_d$ is the shift locus, $\mathcal{M}_d(1) = \mathcal{E}_d$ is the escape locus, and $\mathcal{M}_d(0) = \mathcal{M}_d$. The bifurcation locus $\text{Bif}_d$ is the subset of $\mathcal{M}_d$ consisting of maps $f$ with the following property: when suitably parameterized so as to depend on $f$, the family of iterates of some critical point $f^n(c_i(f)), n = 1, 2, 3, \ldots$ fails to form a normal family, i.e. the critical point $c_i$ is active (see [Mc, DF]).

Below, we say that a value $t > 0$ is generic for $f$ if the grand orbits of the critical points do not intersect $\{G_f = t\}$.

**Lemma 5.5.** For every $f$ and each generic value $t$ such that $0 < t < 1/dM(f)$, the set $B(f,t)$ is connected.

**Proof.** Fix $f$ and a generic value of $0 < t < 1/dM(f)$. Fix $g_0 \in B(f,t)$, and let $U_\varepsilon$ be the connected component of $\pi^{-1}U_{t,\varepsilon}(f)$ containing $g_0$. We will show that $g_0$ can be connected by a path in $U_\varepsilon$ to $S(f,t)$. Because $S(f,t)$ is connected (Lemma 5.1) and $g_0$ is arbitrary, it follows that $B(f,t)$ is contained in $U_\varepsilon$. From Lemma 5.4 we have

$$B(f,t) = \bigcap_{\varepsilon > 0} U_\varepsilon$$

and therefore $B(f,t)$ is connected.

If $g_0$ is structurally stable but not in the shift locus, we begin by taking a small perturbation $g_1$ (within $U_\varepsilon$) which has no critical orbit relations in its filled Julia set $K(g_1)$; in particular, it has no superattracting cycles. We may perform quasiconformal deformations on $K(g_1)$, leaving unchanged the dynamics on the basin of infinity, which limit upon a point $g_2 \in \text{Bif}_d$; indeed, the Teichmüller space of $g_1$ factors as a product, so deformations on $K(g_1)$ are independent from those of $X(g_1)$ [McS]. Because $U_\varepsilon$ is open and contains the closure of these deformations of $K(g_1)$, there is a path joining $g_1$ and $g_2$ contained in $U_\varepsilon$.

Suppose $g_2 \in \mathcal{M}_d(n)$, so it has at least $n$ escaping critical points. The polynomial $g_2$ has an active critical point and $\mathcal{M}_d(n)$ is open, so there is a polynomial $g_3$ in $U_\varepsilon \cap \mathcal{M}_d(n+1)$. By induction we obtain a path in $U_\varepsilon$ from $g_0$ to a polynomial $g \in \mathcal{S}_d$. By Lemma 5.2 we can continue this path to a polynomial $g' \in S(g,t)$, so the path remains in $U_\varepsilon$.

It remains to find a path from $g'$ to $S(f,t)$ contained in $U_\varepsilon$. Since $t$ is generic, a small neighborhood of the level set $\{G_f = t\}$ contains no singular leaves. It follows
that if $\varepsilon > 0$ is sufficiently small, every polynomial $g$ in $U_{t,\varepsilon}(f)$ has the same number of connected components of \{\(G_g = t\}\, mapping to \{\(G_g = dt\}\ with the same local degrees. The path from $g_0$ to $g$ induces a deformation of the restricted maps from $g_0|\{G > t\} \simeq f|\{G > t\}$ to $g|\{G > t\}$. By our choice of small $\varepsilon > 0$, for all polynomials along this path, the components of $\{G = t\}$ can be consistently indexed by the components of $\{G_f = t\}$ and the local degrees of the restrictions remain constant. As in §4.4, choose markings for $f$ which determine a collection of pointed local models

\[
\tilde{f}_i : (Z_i, z_i, \mu_i) \rightarrow (X_i, x_i, \omega_i).
\]

The markings can be transported along the path to define natural isomorphisms between bases $(X_i, x_i, \omega_i)$ at height $dt$ for all polynomials along the path. In this way, the polynomial $g' \in S(g, t)$ also determines local models over the same bases,

\[
g'_i : (Y_i, y_i, \eta_i) \rightarrow (X_i, x_i, \omega_i),
\]

where now each $g'_i$ lies in the local model space $L^{k_i-1}(X_i, x_i, \omega_i)$. That is, all $k_i - 1$ critical values of the local model $g'_i$ lie in the central leaf of $X_i$. Following the procedure of §4.4, we glue the local models $g'_i$ to each restricted polynomial along the path from $g|\{G > t\}$ back to $g_0|\{G > t\}$, ending with a polynomial in $S(f, t)$.

By continuity of the gluing procedure, the gluing determines a path from polynomial $g'$ to $S(f, t)$. (The continuity argument follows the same line of reasoning as in the proof of Lemma 5.1) Joining all paths, we have constructed a path from $g_0$ to $S(f, t)$ contained in $U_{\varepsilon}$. □

**Proof of Theorem 1.1.** Continuity and properness of $\pi : \mathcal{M}_d \rightarrow \mathcal{B}_d$ are included in the statement of Lemma 4.2. For each point $P = (f, X(f))$ in $\mathcal{B}_d$, its fiber is exactly

\[
\pi^{-1}(P) = \bigcap_{\text{generic } t > 0} B(f, t) = \bigcap_{t > 0} B(f, t),
\]

because the sets $B(f, t)$ are nested and generic $t$ are dense. For generic $t$ small enough, the set $B(f, t)$ is connected by Lemma 5.5, therefore $\pi^{-1}(P)$ is connected. Finally, Lemma 4.4 states that $\pi$ is a homeomorphism on the shift locus. □

Though the fibers of $\pi$ are connected, our methods do not show that they are path-connected. For example, it is not known if the Mandelbrot set is path-connected.

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