Topological Speed Limit

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Any physical system evolves at a finite speed that is constrained not only by the energetic cost but also by the topological structure of the underlying dynamics. In this Letter, by considering such structural information, we derive a unified topological speed limit for the evolution of physical states using an optimal transport approach. We prove that the minimum time required for changing states is lower bounded by the discrete Wasserstein distance, which encodes the topological information of the system, and the time-averaged velocity. The bound obtained is tight and applicable to a wide range of dynamics, from deterministic to stochastic, and classical to quantum systems. In addition, the bound provides insight into the design principles of the optimal process that attains the maximum speed. We demonstrate the application of our results to chemical reaction networks and interacting many-body quantum systems.

Introduction.—Investigating how fast a system can evolve is one of the central problems in classical and quantum mechanics. In a seminal work by Mandelstam and Tamm [1], a fundamental bound on the operational time required for the transformation between two orthogonal states for unitary dynamics was derived. Since then, generalizations of the bound for arbitrary states and nonunitary dynamics have been intensively studied [2–36], leading to the notion of speed limits (see Ref. [37] for a review). These speed limits establish the ultimate rate at which a system can evolve to a distinguishable state and have found diverse applications, for example, in quantum control [38–41], quantum metrology [42, 43], and thermodynamics of computation [33, 44–49].

Interacting systems generally form topological structures in their dynamics, such as chemical reaction networks that consist of several species (see the schematic in Fig. 1). In general, a state represented by a vector \( \mathbf{x}_t \) evolves over time and is significantly affected by the topology of the dynamics. For instance, a Markov jump process with dense connectivity may relax toward an equilibrium state faster than one with sparse connectivity. A many-body system with long-range interactions can change quantum states faster than one with short-range interactions [50]. Although speed limits for state transformations have been intensively investigated, the topological nature arising from the network structure in the dynamics has not been fully accounted for. Note that conventional speed limits, which read \( \tau \geq \mathcal{L}(\mathbf{x}_0, \mathbf{x}_\tau)/\mathfrak{v} \), employed non-topological metrics \( \mathcal{L} \), such as the Bures angle, trace norm, quantum Fisher information, etc., to quantify the distance between the initial and final states [37]. These metrics are always upper bounded by a constant that does not scale with the size of the system, whereas the dynamics strongly depends on the system size. Velocity \( \mathfrak{v} \) is determined by the entire dynamics of the system [51], and hence it is generally of the order of system size. Consequently, conventional speed limits become trivial (i.e., \( \tau \geq \mathcal{L}(\mathbf{x}_0, \mathbf{x}_\tau)/\mathfrak{v} \to 0 \)) as the system increases in terms of size [52]. This indicates that in order to derive meaningful bounds, metrics that capture the topological nature and are scalable with system size should be considered.

In this Letter, we derive a speed limit for arbitrary states \( \mathbf{x}_t \) using a topological metric defined through the network structure in the dynamics. The time evolution of such states is described by a graph in which each vertex exchanges flows with each other and may be pumped by an external flow. Examples include reactant concentrations in deterministic chemical reactions and boson numbers in interacting bosonic systems.

![Diagram](image-url)

**FIG. 1.** (a) Generic time evolution of a physical state \( \mathbf{x}_t = [x_1(t), \ldots, x_N(t)]^\top \) on a graph. \( x_i(t) \) is evolved because of the flows \( \{f_{ij}(t)\} \) exchanged between neighboring vertices and an external flow \( f_i(t) \). (b) Examples include reactant concentrations in deterministic chemical reactions and boson numbers in interacting bosonic systems.
increases. Moreover, it is applicable to a broad range of dynamics ranging from deterministic and stochastic classical systems to isolated and open quantum systems. For example, we apply the theory to chemical reaction networks using the Wasserstein distance applicable to any reversible chemical reaction and provide a reaction speed formula that can discriminate between different chemical reactions [54]. Another important application is the interacting bosonic transport for arbitrary initial (mixed) states with and without a thermal environment, which is relevant to the Lieb–Robinson velocity [55]. Through the examples, we demonstrate that considering topological metrics does not only provide quantitatively tight bounds but also qualitatively reveals the physical mechanism of state transformations, which cannot be obtained with speed limits reported thus far.

General setup.—We consider a time-dependent vector state \( x_t = [x_1(t), \ldots, x_N(t)] \) and an undirected graph \( G(V, E) \) with the vertex set \( V = \{1, \ldots, N\} \) and edge set \( E \). Each element \( x_i(t) \) corresponds to a vertex \( i \in V \). For example, \( x_t \) can be a vector of the probability distribution of chemical reaction networks, or physical observables in classical and quantum systems (examples are provided later). For each vertex \( i \), let \( B_i := \{ j \mid (i, j) \in E \} \) denote the set of neighboring vertices of \( i \). Assume that the time evolution of \( x_t \) is given by the following deterministic equation [see Fig. 1(a)]:

\[
\dot{x}_i(t) = f_i(t) + \sum_{j \in B_i} f_{ij}(t),
\]

where \( f_{ij}(t) = -f_{ji}(t) \) denotes the flow exchange between vertices \( i \) and \( j \) for \( i \neq j \) and \( f_i(t) \) is an arbitrary external flow. In the absence of external flows [i.e., \( f_i(t) = 0 \) for all \( i \)], \( \sum_{j=1}^N x_i(t) \) is invariant. Examples of Eq. (1) include the master equation of Markov jump processes, rate equation of chemical reaction networks, and time evolution of the observables in quantum systems. We define a time-dependent velocity [56], which is the sum of the absolute values of the external and exchanged flows, given by

\[
v_{t, \lambda} = \lambda \sum_i |f_i(t)| + \sum_{(i,j) \in E} |f_{ij}(t)|,
\]

where \( \lambda \geq 0 \) is a weighting factor, and the second summation is over all unordered pairs \( (i, j) \in E \). For simplicity, we denote \( v_{t,0} \) by \( v_t \). We also define the Manhattan norm for an arbitrary vector \( x \) as \( \|x\|_1 = \sum_i |x_i| \) and the time average of an arbitrary time-dependent quantity \( w_t \) as \( \langle w_t \rangle := \tau^{-1} \int_0^\tau w_t \, dt \).

Wasserstein distance.—Here we introduce the discrete \( L^1 \)-Wasserstein distance between two states \( x \) and \( y \) on the graph \( G(V, E) \). First, we consider the case in which \( x \) and \( y \) are balanced (that is, \( \sum_i x_i = \sum_i y_i \)), and then we generalize the distance to the unbalanced case (that is, \( \sum_i x_i \neq \sum_i y_i \)). Let \( d_{ij} \) denote the shortest path distance between the vertices \( i \) and \( j \) in the graph. In other words, \( d_{ij} \) is the minimum length of paths connecting \( i \) and \( j \). Graph \( G \) is assumed to be connected [57]; therefore, \( d_{ij} \) is always finite. Suppose that we have a transport plan that redistributes \( x \) to \( y \) by sending an amount of \( \pi_{ij} \) from \( x_j \) to \( y_i \) with a cost of \( d_{ij} \) per unit weight for all ordered pairs \( (i, j) \). The Wasserstein distance is then defined as the minimum transport cost for all feasible plans, given by

\[
W_1(x, y) := \min_{\pi \in \Pi(x, y)} \sum_{i,j} d_{ij} \pi_{ij}. \tag{3}
\]

Here, \( \Pi(x, y) \) denotes the set of all transport plans \( \pi = [\pi_{ij}] \in \mathbb{R}_{+}^{N \times N} \) that satisfy \( \sum_i \pi_{ij} = y_i \) and \( \sum_j \pi_{ji} = x_i \). Previous studies have shown that the Wasserstein distance plays a crucial role in statistics and machine learning [58], computer vision [59], linguistics [60], molecular biology [61], and stochastic thermodynamics [33, 62–64].

Next, we describe the generalized Wasserstein distance for the unbalanced case. Transport between two unbalanced states can be enabled by allowing add and remove operations in addition to transportation between vertices. More precisely, an infinitesimal mass \( \delta x \) of \( x \) can either be removed at cost \( \lambda \delta x_1 \) or moved from \( x \) to \( y \) at cost \( W_1(\delta x, \delta y) \). Mathematically, the generalized Wasserstein distance between unbalanced states can be defined as [65]

\[
W_{1,\lambda}(x, y) := \min \left\{ \lambda (\|x - \bar{x}\|_1 + \|y - \bar{y}\|_1) + W_1(\bar{x}, \bar{y}) \right\}, \tag{4}
\]

where the minimum is over all the states \( \bar{x} \) and \( \bar{y} \) such that \( \|\bar{x}\|_1 = \|\bar{y}\|_1 \). By definition (4), distance \( W_{1,\lambda} \) always satisfies the triangle inequality [65]. If \( x \) and \( y \) are balanced states, then \( W_{1,\lambda} \) is reduced to \( W_1 \) within the limit \( \lambda \to +\infty \). We also note that \( W_{1,\lambda} \) can be calculated numerically using the linear programming method [66].

Main results.—We now utilize the generalized Wasserstein distance (4) to derive a topological speed limit for any state \( x_t \) obeying the general dynamics (1). Specifically, we prove that the minimum time required to transform \( x_0 \) into \( x_\tau \) is lower bounded by the Wasserstein distance divided by the average velocity:

\[
\tau \geq \frac{W_{1,\lambda}(x_0, x_\tau)}{\langle v_{t,\lambda} \rangle}, \quad \forall \lambda \geq 0. \tag{5}
\]

In the case that the external flows are absent [i.e., \( f_i(t) = 0 \)], inequality (5) can be reduced to a simple bound by taking the limit \( \lambda \to +\infty \), which reads

\[
\tau \geq \frac{W_1(x_0, x_\tau)}{\langle v_t \rangle}. \tag{6}
\]

The inequalities (5) and (6) are our main results; the proof is postponed to the end of the Letter. These results have several physically critical properties. (i) First, these bounds can be derived as long as
the time evolution of $x_t$ is described by Eq. (1), which is a general setting for both the classical and quantum cases. Notably, the bounds can be saturated if the time evolution (1) realizes an optimal transport plan. (ii) Second, our bounds utilize topological information about the system dynamics to provide a stringent constraint on the speed of changing states. Topological information is encoded into the Wasserstein distance, and this distance term can be as large as the order of the system’s size.

(iii) Third, by further upper bounding the time-averaged velocity $\langle v_t, \lambda_t \rangle$, by relevant quantities, such as the thermodynamic and kinetic costs, we can derive more interpretable bounds, which clarify the physical mechanism of the speed of state transformations. (iv) Finally, the speed limit for an arbitrary scalar observable defined in terms of state $x_t$ can also be obtained as a consequence of Eq. (5) [66].

In the following, we illustrate the above remarks, especially (i)- (iii), through two applications to classical and quantum systems (see Ref. [66] for further applications in isolated and Markovian open quantum systems, measurement-induced quantum walk [67], and quantum communication [68, 69]).

**Application 1: Chemical reaction networks.**—We consider a chemical reaction system composed of several chemical species $X_i$ ($i \in \mathcal{S}$) that interact through reversible elementary reaction channels $\rho \in \mathcal{R}$. Here, $\mathcal{S}$ and $\mathcal{R}$ denote the set of indices of the species and reaction channels, respectively. Each reaction channel is represented as

$$ \sum \nu_i^\rho X_i = \kappa_i^\rho \sum \nu_i^{-\rho} X_i, $$

(7)

where $+\rho$ and $-\rho$ correspond to the forward and backward reactions, respectively, $\{\kappa_i^\rho\}$ are the macroscopic reaction rates, and $\{\nu_i^\rho\}$ are the stoichiometric coefficients. Let $x_t$ denote the vector of the mass concentrations of species. The molar concentration $c_i$ can be related as $c_i(t) = x_i(t)/m_i$, where $m_i$ denotes the molar mass of species $X_i$. The time evolution of $x_t$ can be described by the deterministic rate equation:

$$ \dot{x}_i(t) = \sum \rho m_i (\nu_i^\rho - \nu_i^{-\rho}) J_i^\rho, $$

(8)

where $J_i^\rho := J_i^\rho_+ - J_i^\rho_-$ is the net reaction current and $J_i^{\pm \rho} := \kappa_i^{\pm \rho} \prod c_i(t) \nu_i^{\pm \rho}$ are the reaction fluxes.

Next, we derive the speed limits for the system in terms of the Wasserstein distance defined on graph $G$. For simplicity, here we consider closed reaction networks, in which the total mass conservation is conserved [70].

The generalization for open reaction networks, wherein the total mass conservation may be violated, is presented in Ref. [66]. The total mass conservation law implies $\sum m_i (\nu_i^\rho - \nu_i^{-\rho}) = 0$ for any $\rho$. Due to these conditions, there always exist matrices $Z^\rho = [z_{ij}^\rho]$ such that the rate equation (8) can be expressed in the form of Eq. (1) with $f_{ij}(t) = \sum \rho z_{ij}^\rho J_i^\rho$ and $f_i(t) = 0$ [66]. The graph $G$ can be obtained by adding an undirected edge $(i, j)$ to $E$ for any $z_{ij}^\rho \neq 0$. After some simple manipulations [66], we can prove that

$$ v_t \leq \sum \rho \nu_i^\rho |J_i^\rho|, $$

(9)

where $\nu_i^\rho := (1/2) \sum \rho m_i (\nu_i^\rho - \nu_i^{-\rho})$. Combining Eqs. (6) and (9) yields the following speed limit:

$$ \tau \geq \frac{W_1(x_0, x_\tau)}{\sum \rho |J_i^\rho|}, $$

(10)

Equation (10) implies that the operational time is lower bounded by the Wasserstein distance and the net reaction currents.

A thermodynamic speed limit can also be obtained using Eq. (10). The entropy production rate of a chemical reaction system can be defined as [71]

$$ \sigma_t := \sum \rho \ln \left( \frac{J_i^\rho}{J_i^{-\rho}} \right), $$

(11)

where the gas constant is set to unity. We define the following kinetic quantity:

$$ \ell_t := \sum \rho (\nu_i^\rho)^2 \frac{J_i^{-\rho} - J_i^\rho}{\ln(J_i^\rho/J_i^{-\rho})}, $$

(12)

which is the sum of the microscopic Onsager coefficients [33, 72]. Applying the Cauchy–Schwarz inequality, we prove that $\langle |J_i^\rho| \rangle_\tau \leq \langle \sqrt{\sigma_t \ell_t} \rangle_\tau \leq \langle \sigma_t \rangle_\tau \langle \ell_t \rangle_\tau$. Consequently, we obtain the following thermodynamic speed limit:

$$ \tau \geq \frac{W_1(x_0, x_\tau)}{\sqrt{\langle \sigma_t \rangle_\tau \langle \ell_t \rangle_\tau}} = \tau_2. $$

(13)

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**FIG. 2.** Numerical demonstration of the speed limits in the cascade reaction network with $N = 10$. The operational time $\tau$, topological bounds $\tau_1$ and $\tau_2$, and non-topological bound $\tau_3$ are depicted by solid, dashed and dash-dotted, and dotted lines, respectively. The parameters are set to $k_f = 2$ and $k_b = 1$. The initial mass concentration is $x_0 = [1, 0.9, \ldots, 0.1]^T$. 

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Inequality (13) implies that the minimum time required to transform \(x_0\) into \(x_\tau\) is determined by the product of the thermodynamic and kinetic costs.

We numerically demonstrate the derived bounds in a cascade reaction network with \(|S| = 10\) species and \(|R| = 9\) reaction channels (see Fig. 2). We also compare the results with a non-topological bound reported in Ref. [73], which reads \(\tau \geq \tau_{\text{d}} := \mathcal{T}(c_0, c_0) / \sqrt{\langle \sigma_i \rangle} / (dt)\). Here, \(\mathcal{T}\) denotes the total variation distance and \(dt := (|S|/8) \sum_{\rho_1} (\tau^{b^\dagger b} - \tau^{b b^\dagger})^2 (J^b_{\rho} + J^b_{\rho}^\dagger)\) corresponds to the diffusion coefficient. We calculate and plot the lower bounds \(\tau_i(1 \leq i \leq 3)\) in Fig. 2. As shown, the topological speed limits \(\tau \geq \tau_1 \geq \tau_2\) are tight; especially, the bound \(\tau \geq \tau_1\) is always saturated. On the contrary, the non-topological bound \(\tau \geq \tau_3\) is loose and does not provide a meaningful bound for the speed of the system.

**Application 2: Interacting bosonic systems.**—Next, we describe an application for quantum many-body bosonic systems. We consider a model of bosons that hop on an arbitrary finite-dimensional lattice and interact with each other. Let \(\Lambda\) denote the set of all the sites in the lattice. The Hamiltonian can be expressed in the following generic form:

\[
H_i = -\gamma \sum_{(i,j)} (b^\dagger_i b_j + b^\dagger_j b_i) + \sum_{Z \subseteq \Lambda} h_Z. \tag{14}
\]

Here, the first summation is over neighboring lattice sites (which can be arbitrarily distant), \(\gamma > 0\) describes the boson mobility, \(b_i\) and \(b^\dagger_i\) are the bosonic creation and annihilation operators for site \(i\), respectively, \(\hat{n}_i := b^\dagger_i b_i\) is the number operator, and \(h_Z\) is an arbitrary function of \(\{\hat{n}_i\}_{i \in Z}\). Examples include the Bose–Hubbard model, given by \(h_Z = (U/2) \sum_i \hat{n}_i(\hat{n}_i - 1) - \mu \sum_i \hat{n}_i\), where \(U\) and \(\mu\) are real constants. Note that the graph \(G(\mathcal{V}, \mathcal{E})\) of the bosonic system is identical to the lattice topology (i.e., \(\mathcal{V}\) is the set of sites and \(\mathcal{E}\) is the set of edges that connect the two neighboring sites). The maximum vertex degree of the graph is denoted by \(d_G\).

We assume that the bosonic system is weakly coupled to a Markovian thermal reservoir and can exchange particles with the reservoir, where the time evolution of the reduced density matrix is described by the Lindblad equation [74]:

\[
\dot{\rho}_t = -i[H_i, \rho_t] + \sum_{i \in \Lambda} (D[L_{i,+}] + D[L_{i,-}]) \rho_t, \tag{15}
\]

where \(D[L] \rho := L \rho L^\dagger - (1/2)\{L^\dagger L, \rho\}\) is the dissipator, \(L_{i,+} = \sqrt{\hat{n}_i} b_i^\dagger\) and \(L_{i,-} = \sqrt{\hat{n}_i} b_i\) are the jump operators that characterize the absorption and emission of bosons at site \(i\), respectively. Hereafter, we set \(\hbar = 1\) for simplicity.

We consider the vector of boson numbers occupied at each site, \(x_i(t) = \text{tr} \{\hat{n}_i \rho_t\}\), and define the instantaneous total number of bosons as \(N_t := \sum_{i \in \Lambda} x_i(t)\). Using the relation \([b_i, \hat{n}_i] = b_i\), we can show that the time evolution of \(x_i(t)\) can be expressed in the form of Eq. (1) with \(f_i(t) = \text{tr} \{L_{i,+} \rho_i \} - \text{tr} \{L_{i,-} \rho_i \} + f_j(t) = 2\gamma \text{Im} [\text{tr} \{b_j^\dagger b_i \rho_t\}]\). By inserting these terms into \(v_{1,\lambda}\), we can immediately obtain the speed limit (5) for bosonic transport.

Next, we derive a more physically interpretable speed limit by upper bounding the velocity \(v_{1,\lambda}\). To this end, we introduce two relevant physical quantities. The first is the irreversible entropy production rate [75], which is the sum of the entropic changes in the system and environment, defined as \(\sigma_t := \sigma_t^{\text{sys}} + \sigma_t^{\text{env}}\). Here, \(\sigma_t^{\text{sys}} := -\text{tr} \{\dot{\rho}_t \ln \rho_t\}\) is the rate of von Neumann entropy of the bosonic system, and \(\sigma_t^{\text{env}}\) quantifies the heat dissipated to the environment as follows:

\[
\sigma_t^{\text{env}} := \sum_{i} (\text{tr} \{L_{i,+} \rho_t L_{i,+}^\dagger\} - \text{tr} \{L_{i,-} \rho_t L_{i,-}^\dagger\}) \ln \frac{\gamma_i^{+}}{\gamma_i^{-}}, \tag{16}
\]

where we have assumed the local detailed balance condition [that is, \(\ln(\gamma_i^+ / \gamma_i^-)\) is related to the heat dissipation of the boson exchange at site \(i\)]. The second is quantum dynamical activity [76, 77], which quantifies the boson exchange frequency between the system and reservoir, given by

\[
a_t := \sum_{i} (\text{tr} \{L_{i,+} \rho_t L_{i,+}^\dagger\} + \text{tr} \{L_{i,-} \rho_t L_{i,-}^\dagger\}). \tag{17}
\]

Using these quantities, we can prove that the velocity \(v_{1,\lambda}\) is upper bounded as [66]

\[
v_{1,\lambda} \leq \gamma d_G N_t + \lambda \frac{\sigma_t}{\Phi(\frac{\sigma_t}{2a_t})}, \tag{18}
\]

where \(\Phi(x)\) is the inverse function of \(x \tanh(x)\). By combining Eqs. (5) and (18), we obtain the following speed limit:

\[
\tau \geq \frac{W_{1,\lambda}(x_0, x_\tau)}{(\gamma d_G N_t + \lambda a_t \Phi(\sigma_t / 2a_t)^{-1} / \gamma^2)}. \tag{19}
\]

Equation (19) implies that the speed of bosonic transport is lower bounded by the lattice topology, boson mobility, and dissipation. The bound also indicates that dissipative controls can help accelerate the bosonic transport. The inequality (19) is valid for arbitrary initial states of the bosonic system.

It is worthwhile discussing the vanishing coupling limit (i.e., the case where the system becomes isolated). In this case, \(\sigma_t = a_t = 0\) and \(N_t = N\) for all times. Defining the boson concentration \(\bar{x}_i(t) := N^{-1} x_i(t)\), we obtain \(\sum_i \bar{x}_i(t) = 1\). By taking the \(\lambda \to +\infty\) limit, Eq. (19) is reduced to a simple speed limit for an isolated bosonic system:

\[
\tau \geq \frac{W_1(\bar{x}_0, \bar{x}_\tau)}{\gamma d_G}. \tag{20}
\]

Bound (20) has a remarkable implication for bosonic transport. Assume that all bosons are initially concentrated in a region \(R_1\), and we want to transport all of
them to a distinct region $R_2$ within a finite time $\tau$. In this case, $W_1(\bar{x}_0,\bar{x}_\tau) \geq \text{dist}(R_1,R_2)$, where $\text{dist}(R_1,R_2)$ denotes the length of the shortest path connecting the regions $R_1$ and $R_2$. Therefore, Eq. (20) implies that transporting bosons always takes at least a time proportional to the distance between the two regions: $\tau \geq \text{dist}(R_1,R_2)/(\gamma d_G)$, which cannot be obtained with conventional speed limits. This statement holds for arbitrary initial states, including the pure states considered in Ref. [78]. While the Lieb–Robinson bounds [79–83] imply a linear light cone for the operator spreading, Eq. (20) provides a useful bound for the operational time required for bosonic transport.

**Proof of Eq. (5).**—We consider the time discretization of Eq. (1) with time interval $\delta t = \tau/K$. For each $k \in [0, K - 1]$ and $t = k\delta t$, we have

$$x_i(t + \delta t) = x_i(t) + \delta t \left[ f_i(t) + \sum_{j \in B_i} f_{ij}(t) \right].$$  

Equation (21) indicates that we can transform $x_i$ to $x_i(t + \delta t)$ by adding $f_i(t)\delta t$ to $x_i(t)$ with cost $\lambda |f_i(t)|\delta t$ and exchanging $f_{ij}(t)\delta t$ between neighboring vertices $i$ and $j$ with cost $|f_{ij}(t)|\delta t$. Such the transport plan takes the total cost of

$$\lambda \sum_i |f_i(t)| + \sum_{(i,j) \in \mathcal{E}} |f_{ij}(t)| \delta t = v_{t,\lambda} \delta t,$$

which should be larger than or equal to $W_1(\mathbf{x}_t, \mathbf{x}_{t+\delta t})$. Therefore, taking the sum of Eq. (22) from $k = 0$ to $k = K - 1$ and applying the triangle inequality for $W_1,\lambda$ yield

$$\sum_{k=0}^{K-1} v_{t,\lambda} \delta t \geq W_1,\lambda(\mathbf{x}_0, \mathbf{x}_\tau).$$

By taking the $\delta t \to 0$ limit in Eq. (23), we obtain $\tau(v_{t,\lambda}) \geq W_1,\lambda(\mathbf{x}_0, \mathbf{x}_\tau)$, from which Eq. (5) is immediately derived.

**Conclusion.**—In this Letter, we derived the topological speed limit for vector states that accounts for the network structure in the underlying dynamics [84]. The speed limit provides a tight bound for the operational time and insight into the system speed from a topological perspective. We showed that the bound is applicable to various dynamics as long as the time evolution of the physical state can be described in terms of a graph. Because our speed limit is derived in a general setting, we expect that it can be applied to obtain fundamental bounds for several other dynamics.

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Supplemental Material for “Topological Speed Limit”
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This Supplemental Material describes the details of the analytical calculations presented in the main text and further applications of the topological speed limit for isolated and open quantum systems, measurement-induced quantum walk, and quantum communication. The equations and figure numbers are prefixed with S [e.g., Eq. (S1) or Fig. S1]. The numbers without this prefix [e.g., Eq. (1) or Fig. 1] refer to the items in the main text.

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S1. NUMERICAL CALCULATION OF THE GENERALIZED WASSERSTEIN DISTANCE

The Wasserstein distance can be calculated using the linear programming method. Mathematically, it can be formulated as the following minimization problem:

\[
\begin{align*}
\text{minimize} \quad & \operatorname{tr}(C \pi) \\
\text{subject to} \quad & \pi \mathbf{1} = \mathbf{y}, \quad \pi^\top \mathbf{1} = \mathbf{x}, \\
& \pi \geq 0,
\end{align*}
\]

where \( C = [d_{mn}] \) is the matrix of transport cost and \( \mathbf{1} \) is the all-ones vector.

The generalized Wasserstein distance can be calculated in a similar way. Note that it has been shown that the generalized Wasserstein distance can be achieved using only remove and transport operations (i.e., we do not need add operation) \[1\]. Therefore, \( \mathcal{W}_{1, \lambda} \) can be expressed as

\[
\mathcal{W}_{1, \lambda}(\mathbf{x}, \mathbf{y}) = \min \left\{ \lambda (\| \mathbf{x} - \bar{x} \|_1 + \| \mathbf{y} - \bar{y} \|_1) + \mathcal{W}_1(\bar{x}, \bar{y}) \right\},
\]

\[(S2)\]

where the minimum is over all states \( \bar{x} \) and \( \bar{y} \) such that \( \bar{x} \leq \mathbf{x} \), \( \bar{y} \leq \mathbf{y} \), and \( \| \bar{x} \|_1 = \| \bar{y} \|_1 \). Here, \( \mathbf{x} \leq \mathbf{y} \) means that \( x_n \leq y_n \) for all \( n \). Consequently, computing \( \mathcal{W}_{1, \lambda} \) is equivalent to solving the following minimization problem:

\[
\begin{align*}
\text{minimize} \quad & \operatorname{tr}(C \pi) + \lambda \left[ \sum_n (x_n - \bar{x}_n) + \sum_n (y_n - \bar{y}_n) \right] \\
\text{subject to} \quad & \pi \mathbf{1} = \bar{y}, \quad \pi^\top \mathbf{1} = \bar{x}, \\
& \bar{x} \leq \mathbf{x}, \quad \bar{y} \leq \mathbf{y}, \\
& \pi \geq 0.
\end{align*}
\]

\[(S3)\]

The linear programming problems in Eqs. (S1) and (S3) can be efficiently solved using programming languages such as Python, Julia, or Mathematica. A Mathematica code that computes the (generalized) Wasserstein distance can be found on GitHub \[2\].

S2. SPEED LIMIT FOR SCALAR OBSERVABLES

Here we derive a speed limit for a scalar observable \( \mathcal{O}_t \), defined in terms of state \( \mathbf{x}_t \) as

\[
\mathcal{O}_t = \sum_i o_i \mathbf{x}_i(t) = \mathbf{o}^\top \mathbf{x}_t,
\]

\[(S4)\]

where \( \mathbf{o} = [o_1, \ldots, o_N]^\top \) is a vector of real coefficients. For convenience, we define the spectral norm and Lipschitz constant of vector \( \mathbf{o} \) as follows:

\[
\| \mathbf{o} \|_\infty := \max_i |o_i|,
\]

\[(S5)\]

\[
\| \mathbf{o} \|_{\text{Lip}} := \max_{(i,j) \neq \bar{2}} |o_i - o_j|.
\]

\[(S6)\]

We first consider the general case where external flows are present. According to Prop. 1, we obtain the following speed limit for observable \( \mathcal{O}_t \) from Eq. (5):

\[
\tau \geq \frac{|\mathcal{O}_t - \mathcal{O}_0|}{\max \{ \lambda^{-1} |\mathbf{o}|_\infty, |\mathbf{o}|_{\text{Lip}} \} \{ v_{t, \lambda} \tau \}}.
\]

\[(S7)\]

For the case that the external flows are absent, we can obtain a more compact speed limit by taking the \( \lambda \to +\infty \)
limit in Eq. (S7), which reads
\[
\tau \geq \frac{\mathcal{O}_t - \mathcal{O}_0}{\mathcal{O}_{\text{Lip}}(v_t)}.
\] (S8)

This bound recovers the result reported in Ref. [3] [see Eq. (19) therein].

**Proposition 1.** For arbitrary states \( \mathbf{x}, \mathbf{y} \) and a real vector \( \mathbf{o} \), the following inequality holds:
\[
|\mathbf{o}^T(\mathbf{x} - \mathbf{y})| \leq \lambda^{-1} ||\mathbf{o}||_{\infty} ||\mathcal{O}_{\text{Lip}}|| \mathcal{W}_{1,\lambda}(\mathbf{x}, \mathbf{y}).
\] (S9)

**Proof.** Let \( \tilde{\mathbf{x}} \) and \( \tilde{\mathbf{y}} \) be two states that realize \( \mathcal{W}_{1,\lambda}(\mathbf{x}, \mathbf{y}) \). In other words, we have
\[
\mathcal{W}_{1,\lambda}(\mathbf{x}, \mathbf{y}) = \lambda(||\mathbf{x} - \tilde{\mathbf{x}}|| + ||\mathbf{y} - \tilde{\mathbf{y}}||) + \mathcal{W}_1(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}).
\] (S10)

Then, by applying the triangle inequality and the Kantorovich–Rubinstein duality formula [4]
\[
\mathcal{W}_1(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \max \left\{ \lambda^{-1} ||\mathbf{o}||_{\infty} ||\mathcal{O}_{\text{Lip}}|| \mathcal{W}_{1,\lambda}(\mathbf{x}, \mathbf{y}) \right\},
\] (S11)

we can prove Eq. (S9) as follows:
\[
|\mathbf{o}^T(\mathbf{x} - \mathbf{y})| \\
\leq |\mathbf{o}^T(\mathbf{x} - \tilde{\mathbf{x}})| + |\mathbf{o}^T(\mathbf{y} - \tilde{\mathbf{y}})| + |\mathbf{o}^T(\tilde{\mathbf{x}} - \tilde{\mathbf{y}})| \\
\leq ||\mathbf{o}||_{\infty}(||\mathbf{x} - \tilde{\mathbf{x}}|| + ||\mathbf{y} - \tilde{\mathbf{y}}||) + ||\mathcal{O}_{\text{Lip}}|| \mathcal{W}_1(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) \\
\leq \lambda^{-1} ||\mathbf{o}||_{\infty} ||\mathcal{O}_{\text{Lip}}|| \mathcal{W}_{1,\lambda}(\mathbf{x}, \mathbf{y}).
\] (S12)

\[\square\]

**S3. CHEMICAL REACTION NETWORKS**

**A. Graph construction for closed reaction networks**

Here we describe in detail the construction of an undirected graph \( G(V, E) \) for the closed chemical system from which the Wasserstein distance can be defined immediately. Notice the total mass conservation \( \sum_i m_i(\nu_i^+ - \nu_i^-) = 0 \) for any \( \rho \). The set of vertices is defined as \( V = S \), where the vertex \( i \) corresponds to the species \( X_i \). For each reaction channel \( \rho \), define \( S^\rho_+ := \{ i | \nu_i^+ > \nu_i^- \} \) and \( S^\rho_- := \{ i | \nu_i^+ < \nu_i^- \} \). Evidently, \( \sum_{i \in S^\rho_+} m_i(\nu_i^+ - \nu_i^-) = \sum_{i \in S^\rho_-} m_i(\nu_i^+ - \nu_i^-) \). According to Prop. 2, there exists a matrix \( Z^\rho = [z_{ij}^\rho] \in \mathbb{R}^{|S| \times |S|} \) such that
\[
\sum_{j \in S^\rho_+} z_{ij}^\rho = m_i(\nu_i^+ - \nu_i^-), \quad \forall i \in S^\rho_+,
\] (S13)
\[
\sum_{i \in S^\rho_+} z_{ij}^\rho = m_j(\nu_j^+ - \nu_j^-), \quad \forall j \in S^\rho_-,
\] (S14)
\[
z_{ij}^\rho = -z_{ji}^\rho \leq 0, \quad \forall i \in S^\rho_+, \quad \forall j \in S^\rho_-.
\] (S15)
\[
z_{ij}^\rho = 0, \quad \text{otherwise.}
\] (S16)

For each \( z_{ij}^\rho \neq 0 \), we add an undirected edge \( \langle i, j \rangle \) to \( E \) (see Fig. S1 for illustration). Note that the existence of

\[
2X_1 + 3X_2 \Rightarrow X_3 + 4X_4
\]

\[
[z_{ij}^\rho] = \begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 3 \\
-1 & 0 & 0 & 0 \\
-1 & -3 & 0 & 0
\end{bmatrix}
\]

**FIG. S1.** Illustration of the process of adding edges to the graph \( G \). Here, for simplicity, the molar mass is set to unity for all species (i.e., \( m_i = 1 \) for any \( i \)). For each \( z_{ij}^\rho \neq 0 \), an edge between \( i \) and \( j \) is added to \( G \).

**B. Derivation of Eq. (9)**

Furthermore, by applying the triangle inequality, we can prove that the velocity is upper bounded as follows:
\[
\nu_t = \sum_{\langle i, j \rangle \in E} |f_{ij}(t)|
\]
\[
= \frac{1}{2} \sum_i \sum_{j \in B_i} |f_{ij}(t)|
\]
\[
\leq \frac{1}{2} \sum_i \sum_{j \in B_i} \sum_{\rho} |z_{ij}^\rho||J_{ij}^\rho|
\]
\[
= \frac{1}{2} \sum_{\rho} |J_{ij}^\rho| \sum_i \sum_{j \in B_i} |z_{ij}^\rho|
\]
\[
= \frac{1}{2} \sum_{\rho} |J_{ij}^\rho| \sum_i |\nu_i^+ - \nu_i^-|
\]
\[
= \sum_{\rho} \nu_t^\rho |J_{ij}^\rho|,
\] (S18)

where \( \nu_t^\rho = (1/2) \sum_i m_i(\nu_i^+ - \nu_i^-) \).

**C. Generalization to generic reaction networks**

Here we derive the topological speed limit for generic (open) chemical reaction networks, in which the total
mass concentration may not be conserved. To this end, we construct the graph corresponding to the time evolution of $\rho_i$. Once done, the topological speed limit Eq. (5) is immediately obtained. We need only consider reaction channels that do not conserve the total mass concentration because the conservative reactions have already been considered above.

Analogously, for each reaction channel $\rho$, we define $S^\rho _i := \{ i | v_i^+ > v_i^- \}$ and $S^\rho _j := \{ i | v_i^+ < v_i^- \}$. A non-conservative reaction channel implies $\sum_{i \in S^\rho _i} m_i (v_i^+ - v_i^-) \neq \sum_{i \in S^\rho _j} m_i (v_i^+ - v_i^-)$. Then, according to Prop. 2, there exist a matrix $Z^\rho = \{ z_{ij}^\rho \} \in \mathbb{R}^{|S| \times |S|}$ and a vector $\mu^\rho \in \mathbb{R}^{|S|}$ such that

$$\sum_{i \in S^\rho _i} z_{ij}^\rho = \mu_i^\rho \leq m_i (v_i^+ - v_i^-), \quad \forall i \in S^\rho _i,$$

$$\sum_{i \in S^\rho _j} z_{ij}^\rho = -\mu_j^\rho \leq m_i (v_i^+ - v_i^-), \quad \forall j \in S^\rho _j,$$

$$z_{ij}^\rho = -z_{ji}^\rho \leq 0, \quad \forall i \in S^\rho _i, \ j \in S^\rho _j,$$

$$z_{ij}^\rho = 0, \quad \forall i \in S^\rho _i, \ j \in S^\rho _j,$$

$$\mu_i^\rho \geq 0, \quad \forall i \in S^\rho _i,$$

$$\mu_j^\rho \leq 0, \quad \forall j \in S^\rho _j,$$

$$\mu_i^\rho = 0, \quad \forall i \in S^\rho _i,$$

$$\mu_j^\rho = 0, \quad \forall j \in S^\rho _j,$$

$$\sum_{i \in S^\rho _i} \mu_i^\rho = \min \left\{ \sum_{i \in S^\rho _i} m_i (v_i^+ - v_i^-), \sum_{i \in S^\rho _j} m_i (v_i^+ - v_i^-) \right\}.$$

For each $z_{ij}^\rho \neq 0$, we add an undirected edge $\{ i, j \}$ to $E$. Again, the existence of $Z^\rho$ and $\mu^\rho$ may not be unique; nevertheless, one can always find coefficients such that at most $|S^\rho _i| + |S^\rho _j| - 1$ edges are added to the graph for each reaction channel $\rho$. Repeating this process for all $\rho \in \mathcal{R}$, we readily obtain the graph $G$.

With the above construction, the time evolution of the reaction network can be expressed in the form of Eq. (1) with

$$f_i(t) = \sum_{\rho} \left[ m_i (v_i^+ - v_i^-) - \mu_i^\rho \right] J_i^\rho,$$

$$f_{ij}(t) = \sum_{\rho} z_{ij}^\rho J_i^\rho.$$

For convenience, we define

$$\chi^\rho := \min \left\{ \sum_{i \in S^\rho _i} m_i (v_i^+ - v_i^-), \sum_{i \in S^\rho _j} m_i (v_i^+ - v_i^-) \right\},$$

$$\eta^\rho := \max \left\{ \sum_{i \in S^\rho _i} m_i (v_i^+ - v_i^-), \sum_{i \in S^\rho _j} m_i (v_i^+ - v_i^-) \right\}.$$

Using these quantities, the velocity term $v_{i,\lambda}$ can be upper bounded as

$$v_{i,\lambda} = \chi^\rho \sum_{\rho} \left| \sum_{i \in S^\rho _i} m_i (v_i^+ - v_i^-) - \mu_i^\rho \right| J_i^\rho + \sum_{\rho} \left| \sum_{i \in S^\rho _j} z_{ij}^\rho J_i^\rho \right|$$

$$\leq \sum_{\rho} |J_i^\rho| \left( \lambda \sum_{i} |m_i (v_i^+ - v_i^-) - \mu_i^\rho| + \sum_{i,j} |z_{ij}^\rho| \right)$$

$$= \sum_{\rho} (\lambda \eta^\rho + \chi^\rho) |J_i^\rho|.$$  

(S31)

Consequently, we obtain the following speed limit for generic reaction networks:

$$\tau \geq \tau_1 := \frac{W_{1,\lambda}(x_0, x_\tau)}{\left( \sum_{\rho} (\lambda \eta^\rho + \chi^\rho) |J_i^\rho| \right)_{\tau}}.$$

(S32)

Define the generalized kinetic quantity $\ell_{i,\lambda}$ as

$$\ell_{i,\lambda} := \sum_{\rho} (\lambda \eta^\rho + \chi^\rho)^2 J_i^\rho |J_i^\rho| \ln(J_i^\rho / J_i^\rho).$$

(S33)

we can prove that

$$\left( \sum_{\rho} (\lambda \eta^\rho + \chi^\rho) |J_i^\rho| \right)_{\tau} \leq \sqrt{(\sigma_1\ell_{i,\lambda})},$$

$$\tau \leq \tau_2 := \frac{\sqrt{(\sigma_1\ell_{i,\lambda})}}{(\lambda \eta^\rho + \chi^\rho) J_i^\rho},$$

(S34)

Combining Eqs. (S32) and (S34) yields the following thermodynamic speed limit:

$$\tau \geq \tau_3 := \frac{\omega_{1,\lambda}(c_0, c_\tau)}{(\lambda \eta^\rho + \chi^\rho) J_i^\rho},$$

(S36)

Next, we show that the $\lambda = 1/2$ speed limit is always tighter than a thermodynamic bound reported in Ref. [5], which reads

$$\tau \geq \tau_3 := \frac{T(c_0, c_\tau)}{(\sigma_1)(t_{i,\lambda})},$$

(S37)

where $T(c_0, c_\tau) := \| c_0 - c_\tau \| / 2$ is the total variation distance and $d_i := (|S|/8) \sum_{\rho} m_i (v_i^+ - v_i^-)^2 (J_i^\rho + J_i^\rho)$ corresponds to the diffusion coefficient. Note that $(1/2) \eta^\rho + \chi^\rho = (1/2) \sum |v_i^+ - v_i^-|$. Applying the Cauchy–Schwarz inequality and the inequality $(x - y) / \ln(x/y) \leq (x + y)/2$, we have

$$\ell_{i,1/2} = \frac{1}{4} \sum_{\rho} \left( \sum_{i} |v_i^+ - v_i^-| \right)^2 |J_i^\rho - J_i| \ln(J_i^\rho / J_i^\rho)$$

$$\leq \frac{|S|}{8} \sum_{\rho} (v_i^+ - v_i^-)^2 (J_i^\rho + J_i^\rho)$$

$$= d_i.$$
By combining the above inequality and the relation
\( W_{1,1/s}(c_0,c_\tau) = T(c_0,c_\tau) \), we readily obtain the following
hierarchical relationship:
\[ \gamma_1' \geq \tau_2' \geq \tau_3. \]  
(S39)

**Proposition 2.** Let \( \mathbf{a} = [a_1, \ldots, a_n]^T \) and \( \mathbf{b} = [b_1, \ldots, b_m]^T \) be two vectors of positive numbers. Then there exist matrix \( Z = [z_{ij}] \in \mathbb{R}^{n \times m} \) and nonnegative vectors \( \mathbf{a} = [a_1, \ldots, a_n]^T \) and \( \mathbf{b} = [b_1, \ldots, b_m]^T \) such that the following conditions are satisfied:
\[
\begin{align*}
\sum_{j=1}^{m} z_{ij} &= \bar{a}_i \leq a_i, \quad \forall 1 \leq i \leq n, \quad (S40) \\
\sum_{i=1}^{n} z_{ij} &= \bar{b}_j \leq b_j, \quad \forall 1 \leq j \leq m, \quad (S41) \\
\sum \bar{a}_i &= \min \left\{ \sum_i a_i, \sum_j b_j \right\}. \quad (S42)
\end{align*}
\]

**Proof.** Without loss of generality, we can assume that \( a_1 \leq \cdots \leq a_n \) and \( b_1 \leq \cdots \leq b_m \). We prove by induction on \( k = m + n \geq 2 \). In the case of \( k = 2 \) (i.e., \( m = n = 1 \)), we can set \( z_{11} = \bar{a}_1 = \bar{b}_1 = \min(a_1, b_1) \). Suppose that the statement holds for all \( k \leq k \). We consider an arbitrary case with \( k = k + 1 \). Assume that \( a_1 \leq b_1 \). If \( n = 1 \) then we can choose \( z_{11} = a_1, \ z_{1j} = 0 \) for all \( j > 1, \mathbf{a} = [a_1] \), and \( \mathbf{b} = [a_1, 0, \ldots, 0]^T \). If \( n \geq 2 \), then consider two vectors \( \mathbf{a}' = [a_2, \ldots, a_n]^T \) and \( \mathbf{b}' = [b_1 - a_1, b_2, \ldots, b_m]^T \). There exist \( \mathbf{Z}', \mathbf{a}', \) and \( \mathbf{b}' \) such that
\[
\begin{align*}
\sum_{j=1}^{m} z_{ij}' &= a_i' \leq a_i, \quad \forall 1 \leq i \leq n - 1, \quad (S43) \\
\sum_{i=1}^{n} z_{ij}' &= b_j' \leq b_j, \quad \forall 1 \leq j \leq m, \quad (S44) \\
\sum a_i' &= \min \left\{ \sum_i a_i, \sum_j b_j \right\}. \quad (S45)
\end{align*}
\]

We construct \( z_{11} = a_1, \ z_{1j} = 0 \) for all \( j > 1, \mathbf{a} = [a_1, a_1', \ldots, a_{n-1}]^T \), and \( \mathbf{b} = [a_1 + \bar{b}_1', b_2, \ldots, b_m]^T \). It is easy to verify that this combination satisfies all conditions (S40), (S41), and (S42).

For example, for two vectors \( \mathbf{a} = [4, 5]^T \) and \( \mathbf{b} = [1, 2, 3]^T \), we can construct an instance of matrix \( \mathbf{Z} \) and vectors \( \mathbf{a}, \mathbf{b} \) as follows:
\[ \mathbf{Z} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \end{bmatrix}, \quad \mathbf{a} = [4, 2]^T, \quad \mathbf{b} = [1, 2, 3]^T. \]  
(S46)

### S4. BOSONIC TRANSPORT

#### A. Time evolution of boson number

Using the relation \([b_i, \hat{n}_i] = b_i \) and \([\hat{n}_i, h_Z] = 0 \), we can calculate as follows:
\[ -i \text{tr} \{ \hat{n}_i [H_t, \varrho_t] \} = -i \text{tr} \{ [\hat{n}_i, H_t] \varrho_t \} = i \gamma \sum_{j \in B_i} \text{tr} \{ [\hat{n}_i, b_j b_j + b_j^\dagger b_j] \varrho_t \} = i \gamma \sum_{j \in B_i} \text{tr} \{ [b_j b_j + b_j^\dagger b_j] \varrho_t \} = 2 \gamma \sum_{j \in B_i} \text{Im} \{ [b_j b_j^\dagger \varrho_t] \}. \]  
(S47)

Similarly, by noting that \([b_i, b_j^\dagger] = 1 \), we have
\[
\begin{align*}
\text{tr} \{ \hat{n}_i [D(L_{j,\downarrow}) + D(L_{j,\uparrow})] \varrho_t \} &= \text{tr} \{ \hat{n}_i \sum_{j \in \Lambda} [\hat{\gamma}_j + \{b_j, \varrho_t b_j\} / 2] \} \\
&= \text{tr} \{ \hat{n}_i \sum_{j \in \Lambda} \{b_j^\dagger \varrho_t b_j - \{b_j^\dagger b_j, \varrho_t\} / 2\} \} \\
&= \text{tr} \{ \hat{n}_i \sum_{j \in \Lambda} \{b_j^\dagger \varrho_t b_j - \{b_j^\dagger b_j, \varrho_t\} / 2\} \} \\
&= \text{tr} \{ \hat{n}_i \sum_{j \in \Lambda} \{b_j^\dagger \varrho_t b_j - \{b_j^\dagger b_j, \varrho_t\} / 2\} \} \\
&= \text{tr} \{ L_{i,\downarrow} \varrho_t L_{i,\downarrow}^\dagger \} - \text{tr} \{ L_{i,\uparrow} \varrho_t L_{i,\uparrow}^\dagger \}. \quad (S48)
\end{align*}
\]

Taking the time derivative of \( x_i(t) = \text{tr} \{ \hat{n}_i \varrho_t \} \), we can calculate the time evolution of \( x_i(t) \) as follows:
\[ \dot{x}_i(t) = \text{tr} \{ \hat{n}_i \dot{\varrho}_t \} = \text{tr} \{ -i \hat{n}_i [H_t, \varrho_t] + \hat{n}_i \sum_{j \in \Lambda} [D(L_{j,\downarrow}) + D(L_{j,\uparrow})] \varrho_t \} = 2 \gamma \sum_{j \in B_i} \text{Im} \{ [b_j b_j^\dagger \varrho_t] \} \\
+ \text{tr} \{ L_{i,\downarrow} \varrho_t L_{i,\downarrow}^\dagger \} - \text{tr} \{ L_{i,\uparrow} \varrho_t L_{i,\uparrow}^\dagger \} = f_i(t) + \sum_{j \in B_i} f_{ij}(t), \quad (S49)
\]
where
\[
\begin{align*}
f_i(t) &= \text{tr} \{ L_{i,\downarrow} \varrho_t L_{i,\downarrow}^\dagger \} - \text{tr} \{ L_{i,\uparrow} \varrho_t L_{i,\uparrow}^\dagger \}, \quad (S50) \\
f_{ij}(t) &= 2 \gamma \text{Im} \{ [b_j b_j^\dagger \varrho_t] \}. \quad (S51)
\end{align*}
\]

It can be verified that \( f_{ij}(t) = -f_{ji}(t) \).

#### B. Derivation of Eq. (18)

Applying the Cauchy–Schwarz inequality, we can upper bound \( |f_{ij}(t)| \) as follows:
\[ |f_{ij}(t)| \leq 2 \gamma \sqrt{\text{tr} \{ b_j^\dagger b_j \varrho_t \} \text{tr} \{ b_j^\dagger b_j \varrho_t \}} \leq \gamma \left( \text{tr} \{ b_j^\dagger b_j \varrho_t \} + \text{tr} \{ b_j^\dagger b_j \varrho_t \} \right) = \gamma [x_i(t) + x_j(t)]. \]  
(S52)
Taking the sum over all edges \( (i, j) \in \mathcal{E} \), we obtain
\[
\sum_{(i,j)} |f_{ij}(t)| \leq \sum_{(i,j)} \gamma [x_i(t) + x_j(t)]
\leq \gamma d_G \sum_i x_i(t) = \gamma d_G N_t. \quad (S53)
\]
In addition, according to Prop. 3, we have
\[
\sum_i |f_i(t)| = \sum_i |\text{tr}\{L_{i,+} \hat{\rho}_t L_{i,+}^\dagger - \text{tr}\{L_{i,-} \hat{\rho}_t L_{i,-}^\dagger\}| \leq \frac{\sigma_t}{2} \Phi \left( \frac{\sigma_t}{2 \alpha_t} \right)^{-1}. \quad (S54)
\]
By combining the inequalities (S53) and (S54), the velocity \( v_{t,\lambda} \) can be upper bounded as
\[
v_{t,\lambda} \leq \gamma d_G N_t + \lambda \frac{\sigma_t}{2} \Phi \left( \frac{\sigma_t}{2 \alpha_t} \right)^{-1}. \quad (S55)
\]

**Proposition 3.** The following inequality holds:
\[
\sum_i |\text{tr}\{L_{i,+} \hat{\rho}_t L_{i,+}^\dagger - \text{tr}\{L_{i,-} \hat{\rho}_t L_{i,-}^\dagger\}| \leq \frac{\sigma_i}{2} \Phi \left( \frac{\sigma_i}{2 \alpha_t} \right)^{-1},
\]
where \( \Phi(x) \) is the inverse function of \( x \tanh(x) \).

**Proof.** Let \( \hat{\rho}_t = \sum_{n} p_n(t) |n\rangle \langle n| \) be the spectral decomposition of the density operator \( \hat{\rho}_t \). Define \( r_{mn}^\dagger(t) := |\langle n|L_{i,+}|m\rangle|^2 \), then the rates of entropy production and dynamical activity can be expressed as
\[
\sigma_t = \sum_{i, m, n} r_{mn}^\dagger(t)p_n(t) - r_{nm}^\dagger(t)p_m(t) \ln \frac{r_{mn}^\dagger(t)p_n(t)}{r_{nm}^\dagger(t)p_m(t)}
= : \sum_{i, m, n} \sigma_{mn}^i(t),
\]
\[
a_t = \sum_{i, m, n} \left[ r_{mn}^\dagger(t)p_n(t) + r_{nm}^\dagger(t)p_m(t) \right]
= : \sum_{i, m, n} a_{mn}^i(t). \quad (S57)
\]
Note that \( x \Phi(x/y)^{-1} \) is a is a concave function over \( (0, +\infty) \times (0, +\infty) \). Applying Jensen’s inequality yields
\[
\sum_i |\text{tr}\{L_{i,+} \hat{\rho}_t L_{i,+}^\dagger - \text{tr}\{L_{i,-} \hat{\rho}_t L_{i,-}^\dagger\}| \leq \sum_{i, m, n} \sigma_{mn}^i(t) \Phi \left( \frac{\sigma_{mn}^i(t)}{2 \alpha_t} \right)^{-1}
\leq \frac{\sigma_t}{2} \Phi \left( \frac{\sigma_t}{2 \alpha_t} \right)^{-1}, \quad (S59)
\]
which completes the proof.

**C. Numerical demonstration**

Here we numerically demonstrate the speed limits for bosonic transport in a one-dimensional Bose–Hubbard model with the size of \( N = 15 \) [see Fig. S2(a) for illustration]. Bosons can hop from site \( i \) to site \( j \) if and only if \( |i-j| = 1 \). The Hamiltonian is given by
\[
\hat{H}_t = -\gamma \sum_{i=1}^{N-1} (b_i^\dagger b_{i+1} + b_{i+1}^\dagger b_i) + \sum_{i=1}^{N} U_i(t) \hat{n}_i(\hat{n}_i - 1)/2. \quad (S60)
\]
For this isolated bosonic system, the flows are determined as follows:
\[
\dot{f}_i(t) = 0, \quad (S61)
\]
\[
\dot{f}_i(t) = \delta_{i-j,1} 2\gamma \text{Im} [\text{tr}\{b_i b_i^\dagger \}], \quad (S62)
\]
and the maximal vertex degree is \( d_G = 2 \). Here \( \delta_{x,y} \) denotes the Kronecker delta. The velocity \( v_t \) is thus given by
\[
v_t = 2\gamma \sum_{i=1}^{N-1} \text{Im} [\text{tr}\{b_i b_i^\dagger \}]. \quad (S63)
\]
Consequently, the derived speed limit reads
\[
\tau \geq \frac{\mathcal{W}_1(x_0, x_\tau)}{(v_t)_\tau} = \tau_1. \quad (S64)
\]
For comparison, we also examine two other non-topological speed limits. The first one is the generalization of the Mandelstam–Tamm bound [6], which reads
\[
\tau \geq \frac{\mathcal{L}(\hat{\rho}_0, \hat{\rho}_\tau)}{(\Delta_{\hat{H}_\tau})_\tau} =: \tau_{MT}, \quad (S65)
\]
where \( \mathcal{L}(\rho, \sigma) = \arccos \text{tr}\{\rho^{1/2} \sigma^{1/2}\} \) is the Bures angle. The second one is a speed limit using the total variation distance \( \mathcal{T}(x, y) = \|x - y\|_2 \), which does not consider the topological nature and is given by
\[
\tau \geq \frac{\mathcal{T}(x_0, x_\tau)}{(v_t)_\tau} =: \tau_{TV}. \quad (S66)
\]
Notice that \( \tau_1 \geq \tau_{TV} \) since \( \mathcal{W}_1(x_0, x_\tau) \geq \mathcal{T}(x_0, x_\tau) \).

We set the initial state to the ground state of the Hamiltonian \( \hat{H}_0 \), which can be calculated using the algorithm of density matrix renormalization group [7]. Initially, bosons are almost uniformly distributed. The time-dependent Hamiltonian is modulated as \( \hat{H}_t = (1 - t/\tau) \hat{H}_0 + (t/\tau) \hat{H}_\tau \), where \( \hat{H}_\tau \) is the final Hamiltonian. More specifically, the interaction coefficients \( U_i(t) \) (1 \( \leq i \leq N \)) are given by
\[
U_i(t) = \left(1 - \frac{t}{\tau}\right) U_0 + \frac{t}{\tau} U_{\tau,i}, \quad (S67)
\]
where \( \{U_{\tau,i}\} \) are the interaction coefficients of the final Hamiltonian \( \hat{H}_\tau \). By this protocol, bosons tend to gather.
at the central site after a period of time $\tau$ [see Fig. S2(b)]. We employ the time dependent variational principle algorithm [8] to simulate the time evolution of the bosonic system. For each time $t \leq \tau$ ($=10$), we calculate all relevant quantities and plot the bounds in Fig. S2(c). As shown, the topological bound $t \geq t_1$ is tight and can be saturated. In contrast, the non-topological bound $t_{TV}$ cannot accurately capture the operational time of bosonic transport. This is because the total variation distance does not consider topological information as the Wasserstein distance does. Although the Mandelstam–Tamm bound $t \geq t_{MT}$ yields good prediction for $t \leq 6$, it subsequently becomes trivial as $t$ increases.

**S5. ISOLATED QUANTUM SYSTEMS**

Here we derive a topological speed limit for isolated systems whose dynamics is described by the von Neumann equation:

$$\dot{\rho}_t = -i[H_t, \rho_t], \quad (S68)$$

where $H_t$ is a time-dependent Hamiltonian. We consider a complete orthogonal set of projection operators $\{P_n\}_n$, i.e., $P_m P_n = \delta_{mn} P_n$ and $\sum_n P_n = \mathbb{1}$. Since the dynamics is invariant under transformation $H_t \rightarrow H_t + \alpha \mathbb{1}$, we can assume that $\text{tr}(H_t \dot{\rho}_t) = 0$, where $\dot{\rho}_t := \sum_n \rho_t P_n$ is a density matrix projected on the space of $\{P_n\}_n$. Let $G(\mathcal{V}, \mathcal{E})$ be the graph, in which $\mathcal{V} = \{n\}_n$ and $\mathcal{E} = \{(m,n) \mid P_m H_t P_n \neq 0, m < n\}$, and $d_G$ be the maximum vertex degree of the graph. Considering a vector of projective measurements, $x_n(t) = \text{tr}(P_n \rho_t)$, we obtain the time evolution of $x_n(t)$ in the following equation:

$$\dot{x}_n(t) = -i \sum_{m(z_n)} \left( \text{tr}(P_n H_t P_m \rho_t) - \text{tr}(P_m H_t P_n \rho_t) \right) \quad (S69)$$

where $f_{mn}(t) = -f_{nm}(t)$ is given by

$$f_{mn}(t) = -i(\text{tr}(P_n H_t P_m \rho_t) - \text{tr}(P_m H_t P_n \rho_t)). \quad (S70)$$

Applying the speed limit in Eq. (6), we obtain the following bound on the operational time:

$$\tau \geq \frac{W_1(x_0, x_\tau)}{\langle v_t \rangle_\tau}, \quad (S71)$$

where $v_t$ can be explicitly expressed as

$$v_t = \sum_{(m,n) \in \mathcal{E}} |\text{tr}(P_n H_t P_m - P_m H_t P_n \rho_t)|. \quad (S72)$$

By further upper bounding $v_t$, we can also obtain another speed limit that resembles the Mandelstam–Tamm bound. By applying the Cauchy–Schwarz inequality, we have

$$|\text{tr}(P_n H_t P_m \rho_t)| \leq \left( \text{tr}(P_m H_t P_m P_n \rho_t) \right)^{1/2} \left( \text{tr}(P_n \rho_t) \right)^{1/2}, \quad (S73)$$

$$|\text{tr}(P_m H_t P_n \rho_t)| \leq \left( \text{tr}(P_m H_t P_m P_n \rho_t) \right)^{1/2} \left( \text{tr}(P_n \rho_t) \right)^{1/2}. \quad (S74)$$

Combining the above inequalities and applying the Cauchy–Schwarz inequality we have

$$|f_{mn}(t)| \leq \left[ \text{tr}(\{(P_m H_t P_n H_t P_m + P_n H_t P_m H_t P_n) \rho_t\}) \right]^{1/2} \times \left[ \text{tr}(\{(P_n + P_m) \rho_t\}) \right]^{1/2}. \quad (S75)$$

Therefore, using the fact $\sum_n P_n = \mathbb{1}$, we have

$$v_t \leq \sum_{(m,n) \in \mathcal{E}} \left[ \text{tr}(\{(P_m H_t P_n H_t P_m + P_n H_t P_m H_t P_n) \rho_t\}) \right]^{1/2} \quad (S76)$$
\[ \times \left[ \operatorname{tr}(P_n + P_m) \theta_t \right]^{1/2} \]
\[ \leq \left\{ \sum_{(m,n) \in E} \operatorname{tr}(P_m H_t P_n H_t P_n + P_n H_t P_m H_t P_n) \theta_t \right\}^{1/2} \]
\[ \times \left\{ \sum_{(m,n) \in E} \operatorname{tr}(P_m + P_n) \theta_t \right\}^{1/2} \]
\[ \leq \left\{ \sum_n \operatorname{tr}(H_t^2 P_n \theta_t P_n) \right\}^{1/2} \sqrt{d_G} \]
\[ = \sqrt{d_G} \Delta \delta H_t, \quad (S76) \]

where \( \Delta \delta H_t = \sqrt{\operatorname{tr}(H_t^2 \theta) - (\operatorname{tr}(H_t \theta))^2} \) is the energy fluctuation of the Hamiltonian \( H_t \) with respect to \( \theta \). Consequently, we obtain the following speed limit:

\[ \tau \geq \frac{\mathcal{W}_1(x_0, x_r)}{\sqrt{d_G} \langle \Delta \delta, H_t \rangle \tau}. \quad (S77) \]

We note that this speed limit is similar but different from the generalization of the Mandelstam–Tamm bound \([6]\), which reads

\[ \tau \geq \frac{\mathcal{L}(\theta_0, \theta_2)}{\langle \Delta \theta, H_t \rangle \tau}, \quad (S78) \]

where \( \mathcal{L}(\theta, \sigma) = \arccos \left\{ \frac{\theta^{1/2} \sigma^{1/2}}{2} \right\} \) is the Bures angle. Notably, the new speed limit can be applied to quantum systems under measurement, making it more applicable than the generalized Mandelstam–Tamm bound.

We now consider a specific case where \( P_n = |n\rangle \langle n| \). In this case, \( |f_{mn}(t)| \) can be upper bounded as

\[ |f_{mn}(t)| = |\langle n| H_t |m \rangle \langle m| \theta_t |n \rangle - \langle n| H_t |n \rangle \langle n| \theta_t |m \rangle| \ni 2 |\langle n| H_t |m \rangle| \langle n| \theta_t |m \rangle|. \quad (S79) \]

According to the Cauchy–Schwarz inequality, we have

\[ |\langle n| \theta_t |m \rangle| = |\langle n| \theta_t |m \rangle^{1/2} \theta_t^{1/2} |m \rangle| \leq \sqrt{\langle n| \theta_t |n \rangle} \langle m| \theta_t |m \rangle. \quad (S80) \]

Therefore,

\[ |f_{mn}(t)| \leq 2 |\langle m| H_t |n \rangle| \sqrt{\langle n| \theta_t |n \rangle} \langle m| \theta_t |m \rangle. \quad (S81) \]

Consequently, the velocity can be upper bounded in terms of the Hamiltonian and diagonal terms of density matrix as

\[ v_t \leq 2 \sum_{(m,n) \in E} |h_{mn}(t)| \sqrt{p_n(t)} p_m(t) = \ddot{v}_t, \quad (S82) \]

where we have defined \( h_{mn}(t) = \langle m| H_t |n \rangle \) and \( p_n(t) = \langle n| \theta_t |n \rangle \) for simplicity. Then, Eq. (S71) yields the following speed limit:

\[ \tau \geq \frac{\mathcal{W}_1(x_0, x_r)}{\langle \ddot{v}_t \rangle \tau}, \quad (S83) \]

which is easier to compute than the original bound (S71).

**S6. Measurement-Induced Quantum Walk**

Here we demonstrate an application of the speed limits (S71), (S77), and (S83) for a quantum system under measurement. We consider a model of the continuous-time quantum walk \([9]\), which is induced by measurements performed at discrete times. The system Hamiltonian is given by

\[ H = \sum_{n=1}^{N-1} \gamma_n (|n\rangle\langle n+1| + |n+1\rangle\langle n|), \quad (S84) \]

which describes hops between nearest neighbors on a finite line lattice. The projective measurements with operators \( \{ |n\rangle\langle n| \} \) are performed at times \( t_k = k \Delta t \) for \( k = 0, \ldots, K = \tau / \Delta t \). Between each measurement event, the system unitarily evolves according to the von Neumann equation. The density matrix after the \( k \)th measurement is given by

\[ \theta_t^k = \sum_n \langle n| \theta_t |n \rangle |n\rangle \langle n|. \quad (S85) \]

For this system, we consider the set of projection operators \( \{ P_n = |n\rangle\langle n| \} \). The graph \( G \) thus has \( N \) vertices and \( N-1 \) edges that connect \( n \) and \( n+1 \) for all \( 1 \leq n \leq N-1 \). The maximum degree of the graph is \( d_G = 2 \). Applying Eqs. (S71) and (S77) to the time evolution of the system between the \( k \)th and \( (k+1) \)th measurements yields the following result:

\[ \sqrt{2} \int_{t_k}^{t_{k+1}} \Delta \delta, H dt \geq \int_{t_k}^{t_{k+1}} v_t dt \geq \mathcal{W}_1(x_t, x_{t+1}). \quad (S86) \]

Taking the sum of both sides of Eq. (S86) for \( k = 0, \ldots, K-1 \) and applying the triangle inequality for \( \mathcal{W}_1 \), we obtain

\[ \sqrt{2} \int_0^\tau \Delta \delta, H dt \geq \int_0^\tau v_t dt \geq \mathcal{W}_1(x_0, x_r). \quad (S87) \]

Consequently, the following speed limits are derived:

\[ \tau \geq \frac{\mathcal{W}_1(x_0, x_r)}{\langle v_t \rangle \tau} \geq \frac{\mathcal{W}_1(x_0, x_r)}{\sqrt{2} \langle \Delta \delta, H \rangle \tau}. \quad (S88) \]

We note that the system is measured at discrete times; therefore, the generalized Mandelstam–Tamm bound cannot be applied directly.

By applying the speed limit (S83), we can also obtain another meaningful bound. Note that \( \mathcal{E} = \{ \{n, n+1\} \mid 1 \leq n \leq N-1 \} \) and \( h_{mn}(t) = \gamma_{\min(m,n)} \delta_{m-n,1} \) in this system. The quantity \( \ddot{v}_t \) can be upper bounded as follows:

\[ \ddot{v}_t = 2 \sum_{n=1}^{N-1} \gamma_n \sqrt{p_n(t)} p_{n+1}(t) \]
\[ \leq \sum_{n=1}^{N-1} \gamma_n [p_n(t) + p_{n+1}(t)] \]

which is easier to compute than the original bound (S71).
\[ \leq 2 \max_n \gamma_n. \]  
\text{(S89)}

Therefore, the operational time is lower bounded as
\[ \tau \geq \frac{W_1(x_0, x_\tau)}{2 \max_n \gamma_n}. \]  
\text{(S90)}

## S7. QUANTUM COMMUNICATION USING SPIN SYSTEMS

Here we derive a topological speed limit for quantum communication through an arbitrary graph \( G(\mathcal{V}, \mathcal{E}) \) of spins with ferromagnetic Heisenberg interactions \([10]\). The vertices \( \{ n \in \mathcal{V} \} \) represent spins and the edges \( \{ (n, m) \in \mathcal{E} \} \) connect interacting spins. The Hamiltonian is given by

\[ H_t = -\gamma \sum_{(n, m) \in \mathcal{E}} \hat{\sigma}_n \cdot \hat{\sigma}_m + \sum_{n \in \mathcal{V}} B_n(t) \sigma_n^z, \]  
\text{(S91)}

where \( \gamma > 0 \) denotes the coupling strength, \( \hat{\sigma}_n = [\sigma_n^x, \sigma_n^y, \sigma_n^z] \) is the vector of Pauli spin operators for the \( n \)th spin, and \( B_n(t) \) is an external magnetic field in the \( z \) direction.

We assume that the sender, Alice, has a quantum state encoded in spin 1 and wants to relay it to the receiver, Bob, who can access and read out spin \( N \). By manipulating the external magnetic field (which Alice can control), quantum information can be transmitted through the graph of spins. After a predetermined time when the state of spin 1 is transferred to spin \( N \), Bob reads out the state of this site. The minimum time required for high-fidelity information transmission is the quantity of interest.

Consider state \( x_t \) with \( x_n(t) = (\text{tr} \{ \sigma_n^z \rho_t \} + 1)/2 \geq 0 \). Notice that \( x_n(0) = 0 \) (\( x_n(t) = 1 \)) corresponds to spin down (up) with respect to the \( z \) direction. Taking the time derivative of \( x(t) \), we can calculate as follows:

\[ \dot{x}_n(t) = \frac{\text{tr} \{ \sigma_n^z \dot{\rho}_t \}}{2} = \frac{(i/2) \text{tr} \{ [\sigma_n^z, H_t] \dot{\rho}_t \}}{2} = \frac{-(i\gamma/4) \sum_{m \in \mathcal{B}_n} \text{tr} \{ \sigma_n^z \cdot \sigma_m \} \dot{\rho}_t}{2} = \frac{(\gamma/2) \sum_{m \in \mathcal{B}_n} \text{tr} \{ (\sigma_n^x \sigma_m - \sigma_n^y \sigma_m^y) \} \dot{\rho}_t}{2} = \frac{\sum_{m \in \mathcal{B}_n} f_{nm}(t)}{2}, \]  
\text{(S92)}

where \( f_{nm}(t) = -\dot{f}_{mn}(t) \) is given by

\[ f_{nm}(t) = \frac{(\gamma/2) \text{tr} \{ (\sigma_n^y \sigma_m^x - \sigma_n^x \sigma_m^y) \} \} \dot{\rho}_t}. \]  
\text{(S93)}

Therefore, the velocity can be expressed as

\[ v_t = \frac{\gamma}{2} \sum_{(m, n) \in \mathcal{E}} |\text{tr} \{ (\sigma_n^y \sigma_m^x - \sigma_n^x \sigma_m^y) \} \dot{\rho}_t|. \]  
\text{(S94)}

Consequently, the derived topological speed limit reads

\[ \tau \geq \frac{W_1(x_0, x_\tau)}{\langle v_t \rangle}. \]  
\text{(S95)}

It can be easily verified that

\[ \sigma_n^y \sigma_m^x - \sigma_n^x \sigma_m^y \leq (\sigma_n^z + 1_2) \otimes 1_2 + 1_2 \otimes (\sigma_m^z + 1_2), \]  
\text{(S96)}

\[ \sigma_n^x \sigma_m^y - \sigma_n^y \sigma_m^x \leq (\sigma_n^z + 1_2) \otimes 1_2 + 1_2 \otimes (\sigma_m^z + 1_2), \]  
\text{(S97)}

where \( 1_2 \) is the two-dimensional identity matrix. Here, \( A \leq B \) means that \( B - A \) is positive semi-definite. Therefore, \( |f_{nm}(t)| \) can be upper bounded as

\[ |f_{nm}(t)| \leq \gamma |x_n(t) + x_m(t)|. \]  
\text{(S98)}

Using Eq. (S98), we can show that the velocity \( u_t \) is bounded from above as

\[ u_t \leq \sum_{(m, n) \in \mathcal{E}} \gamma |x_n(t) + x_m(t)| \leq \gamma d_G \langle |x_t| \rangle. \]  
\text{(S99)}

It should be noted that the total spin \( \| x_t \|_1 \) is invariant for all times. For convenience, we define \( M = \| x_t \|_1 \). Using the inequality (S99) and the topological speed limit (S95), we obtain the following bound on the operational time required for transmitting information:

\[ \tau \geq \frac{W_1(x_0, x_\tau)}{\gamma d_G M}. \]  
\text{(S100)}

This inequality implies that the speed of information transmission is constrained by the topology of the graph \( G \) and the coupling strength.

Now let us discuss a particular case where the graph \( G \) is a spin chain of length \( N \geq 2 \), and the spins \( n \) and \( n+1 \) interact with each other for all \( 1 \leq n \leq N-1 \) \([10, 11]\). Alice prepares the spin chain in the initial state \( \rho_0 = |\varphi_0 \rangle \langle \varphi_0| \), with spin 1 in the excited state \( |1 \rangle \), and all other spins in the ground state \( |0 \rangle \). Specifically, \( |\varphi_0 \rangle \) is given by

\[ |\varphi_0 \rangle = |1 \rangle \otimes |0 \rangle \otimes \cdots \otimes |0 \rangle. \]  
\text{(S101)}

In this setup, the maximum degree of the graph is \( d_G = 2 \), and the total spin is \( M = 1 \). The initial and target vectors are given by \( x_0 = [1, 0, \ldots, 0]^T \) and \( x_\tau = [0, \ldots, 0, 1]^T \), respectively. The Wasserstein distance can also be analytically calculated as \( W_1(x_0, x_\tau) = N - 1 \). Then, according to Eq. (S100), the minimum time required for transmitting the quantum state is lower bound as

\[ \tau \geq \frac{N - 1}{2\gamma}. \]  
\text{(S102)}

Intriguingly, Eq. (S102) implies that it takes at least a time proportional to the distance between spins to reliably transfer a quantum state. The longer the distance, the more time is required. This is in agreement with the numerical result in Ref. [11], wherein an optimal control of the external magnetic field was used. It is noteworthy that this implication cannot be obtained from the conventional speed limits such as the Mandelstam–Tamm and Margolus–Levitin bounds.
S8. MARKOVIAN OPEN QUANTUM SYSTEMS

Here we derive a topological speed limit for Markovian open quantum systems. We consider a finite-dimensional quantum system that is weakly coupled to thermal reservoirs. The dynamics of the system’s reduced density matrix is governed by the local Lindblad equation:

\[ \dot{\rho}_t = -i[H + V_t, \rho_t] + \sum_k D[L_k] \rho_t, \]  

(S103)

where \( H = \sum_n \epsilon_n |\epsilon_n\rangle \langle \epsilon_n| \) is the system Hamiltonian with \( \epsilon_n \neq \epsilon_m \) for \( n \neq m \), \( V_t \) is an external driving field, and \( \{ L_k \} \) are jump operators that characterize jumps between energy eigenstates with the same energy change \( \omega_k \) (i.e., \([L_k, H] = \omega_k L_k\)). We assume that the Hamiltonian has no energy degeneracy. Each jump operator \( L_k \) has a counterpart \( L_k' \), which corresponds to the reversed jump and satisfies the local detailed balance condition \( L_k = e^{\Delta_k/2} L_k^\dagger \). Here, \( s_k = -s_k' \) denotes the change in environmental entropy due to the jump \( L_k \).

We consider the time evolution of the energetic population \( x_n(t) = \langle \epsilon_n | \rho | \epsilon_n \rangle \), which can be described by the following equation:

\[ \dot{x}_n(t) = -i \langle \epsilon_n | [V_t, \rho_t] | \epsilon_n \rangle + \sum_{m(x_n)} \sum_{k} \left[ r_{nm} x_m(t) - r_{mn}' x_n(t) \right], \]

(S104)

where \( r_{nm}^k = |\langle \epsilon_n | L_m | \epsilon_m \rangle|^2 \geq 0 \) is the transition rate satisfying the local detailed balance \( r_{nm}^k = e^{\Delta_k} r_{mn}^k \). Notice that Eq. (S104) can be expressed in the form of Eq. (1) with

\[ f_n(t) = -i \langle \epsilon_n | [V_t, \rho_t] | \epsilon_n \rangle, \]

\[ f_{nm}(t) = \sum_k \left[ r_{nm}^k x_m(t) - r_{mn}'^k x_n(t) \right]. \]

(S105)

(S106)

The graph \( G(\mathcal{V}, \mathcal{E}) \) is thus defined by \( \mathcal{V} = \{1, \ldots, n, \ldots\} \) and \( \mathcal{E} = \{(m, n) | m < n, \exists k \text{ s.t. } r_{mn}^k \neq 0\} \). According to Eq. (5), we obtain the following speed limit:

\[ \tau \geq \frac{\mathcal{W}_{1,\lambda}(x_0, x_\tau)}{\langle v_{t,\lambda} \rangle}. \]

(S107)

Next, we derive an upper bound for \( v_{t,\lambda} \). First, according to Prop. 4, we have

\[ \sum_n |f_n(t)| \leq \|[V_t, \rho_t]\|_1 \leq 2\Delta_{\theta_t} V_t, \]

(S108)

where \( \|X\|_p = \text{tr}[|X|^p]^{1/p} \) is the Schatten \( p \)-norm of operator \( X \). Furthermore, according to Prop. 5, we can prove that

\[ \sum_{(m,n) \in \mathcal{E}} |f_{mn}(t)| \leq \frac{\sigma_t}{2} \Phi\left( \frac{\sigma_t}{2\alpha_t} \right)^{-1}. \]

(S109)

Here, \( \sigma_t = \sum_k \text{tr}\{L_k \rho_t L_k^\dagger\} \) is the dynamical activity rate, \( \sigma_t^{\text{pop}} + \sigma_t^{\text{env}} \) is the total entropy production rate, \( \sigma_t^{\text{env}} = \sum_k \text{tr}\{L_k \rho_t L_k^\dagger\} s_k \) is the environmental entropy rate and \( \sigma_t^{\text{pop}} = -\sum_n \dot{x}_n(t) \ln x_n(t) \) is the sum of the Shannon entropy rate of the population distribution and the entropic change contributed by the external Hamiltonian \( V_t \). Combining Eqs. (S108) and (S109) yields

\[ v_{t,\lambda} \leq 2\lambda \Delta_{\theta_t} V_t + \frac{\sigma_t}{2} \Phi\left( \frac{\sigma_t}{2\alpha_t} \right)^{-1}. \]

(S110)

Consequently, we obtain the following thermodynamic speed limit:

\[ \tau \geq \frac{\mathcal{W}_{1,\lambda}(x_0, x_\tau)}{\langle 2\lambda \Delta_{\theta_t} V_t + \sigma_t \Phi(\sigma_t/2\alpha_t)^{-1}/2 \rangle}. \]

(S111)

**Proposition 4.** The following inequality holds for arbitrary Hermitian matrix \( V \) and density matrix \( \rho \):

\[ \sum_n |\langle \epsilon_n | [V, \rho] | \epsilon_n \rangle| \leq \|[V, \rho]\|_1 \leq 2\Delta_{\theta} V. \]

(S112)

**Proof.** Note that \( \langle \epsilon_n | [V, \rho] | \epsilon_n \rangle \) is pure imaginary. Let \( U = \sum_n u_n \epsilon_n \) be a diagonal matrix with elements \( u_n = \text{sgn}(\langle \epsilon_n | [V, \rho] | \epsilon_n \rangle) \). Here, \( \text{sgn}(x) = 1 \) if \( x \geq 0 \) and \( \text{sgn}(x) = -1 \) if \( x < 0 \). It is evident that all singular values of matrix \( U \) equal 1; thus, \( \|U\|_\infty = 1 \). According to von Neumann’s trace inequality, the first inequality in Eq. (S112) can be proved as

\[ \sum_n |\langle \epsilon_n | [V, \rho] | \epsilon_n \rangle| = |\text{tr}\{U[V, \rho]\}| \leq \|U\|_\infty \|[V, \rho]\|_1 \]

= \|[V, \rho_t]\|_1. \]

(S113)

Next, we need only prove the second inequality in Eq. (S112). To this end, we follow the idea in Ref. [12]. Let \( \theta = \sum_i p_i |i\rangle \langle i| \) be the spectral decomposition of the density matrix \( \rho \) acting on the \( d \)-dimensional Hilbert space \( \mathcal{H} \). Let \( \mathcal{H}' \) be another copy of the Hilbert space \( \mathcal{H} \) with an orthonormal basis \( \{|i'\}\}. \) Then, \( |\theta\rangle = \sum_i \sqrt{p_i} |i\rangle \otimes |i'\rangle \in \mathcal{H} \otimes \mathcal{H}' \) is the purification of the density matrix \( \rho \).

We also define \( \tilde{V} = V \otimes 1 \in \mathcal{H} \otimes \mathcal{H}' \), where 1 is the identity matrix acting on the Hilbert space \( \mathcal{H}' \). Let \( \Phi(\cdot) = \text{tr}_{\mathcal{H}'}(\cdot) \) denote the map of the partial trace with respect to \( \mathcal{H}' \).

By simple algebraic calculations, we can verify that

\[ \Phi(\tilde{V}|\rho\rangle\langle\rho|) = |V, \rho\rangle. \]

(S114)

Since the trace norm is contractive under a completely positive and trace-preserving map, we have

\[ \|[V, \rho]\|_1 = \|\Phi([V, \rho]|\rho\rangle\langle\rho|)\|_1 \]

\[ \leq \|[V, \rho]|\rho\rangle\langle\rho|\|_1 \]

\[ = \Delta_{\theta|\rho\rangle} \tilde{V} \text{tr} \left( \sqrt{|\rho\rangle\langle\rho|} \otimes |\rho\rangle\langle\rho| \right) \]

\[ = 2\Delta_{\theta|\rho\rangle} \tilde{V}. \]

(S115)

Here, \( |\rho\rangle \) is a state orthogonal to \( |\theta\rangle \), given by

\[ |\rho\rangle := \frac{(V - \langle \theta | V | \theta \rangle |\theta\rangle)}{\Delta_{\theta|\rho\rangle} \tilde{V}}. \]

(S116)
In addition, the energy fluctuation $\Delta_{\rho|\rho} \tilde{V}$ can be simplified as follows:

$$\Delta_{\rho|\rho} \tilde{V} = \left( \langle \rho | \tilde{V}^2 | \rho \rangle - \langle \rho | \tilde{V} | \rho \rangle^2 \right)^{1/2} = \left( \text{tr} \{ \tilde{V}^2 \rho \} - \text{tr} \{ \tilde{V} \rho \}^2 \right)^{1/2} = \Delta \rho \tilde{V}.$$  \hspace{1cm} (S17)

Combining Eqs. (S115) and (S117) yields the desired second inequality in Eq. (S112).

Proposition 5. The following inequality holds:

$$\sum_{(m,n) \in E} \left| \sum_k [r_{nm}^k x_m(t) - r_{nm}^{k'} x_n(t)] \right| \leq \frac{\sigma_t}{2} \Phi \left( \frac{\sigma_t}{2a_t} \right)^{-1},$$  \hspace{1cm} (S118)

where $\Phi(x)$ is the inverse function of $x \tanh(x)$.

Proof. Note that $[L_k^L, L_k, H] = 0$ and $\{ \epsilon_m, L_k^L L_k \} = 0$ for $m \neq n$. Therefore,

$$\text{tr} \{ L_k^L L_k \} = \sum_m \{ \epsilon_m, L_k^L L_k \} \epsilon_m = \sum_m x_m(t) \sum_n \{ \epsilon_m, L_k^L \} \epsilon_n = \sum_m r_{mn}^k x_m(t).$$  \hspace{1cm} (S119)

We first calculate $\sigma_{t, \text{env}}$ as follows:

$$\sigma_{t, \text{env}}^2 = \sum_k \text{tr} \{ L_k^L T_k^L \} s_k$$

$$= \frac{1}{2} \sum_k \left( \text{tr} \{ L_k^L T_k^L \} - \text{tr} \{ L_k L_k^L \} \right) s_k$$

$$= \frac{1}{2} \sum_k \sum_{m,n} [r_{nm}^k x_m(t) - r_{nm}^{k'} x_n(t)] s_k$$

$$= \frac{1}{2} \sum_k \sum_{m,n} [r_{nm}^k x_m(t) - r_{nm}^{k'} x_n(t)] \ln \frac{r_{nm}^k}{r_{nm}^{k'}}. \hspace{1cm} (S120)$$

Consequently, we can calculate

$$\sigma_t = \sigma_{t, \text{env}} + \sigma_{t, \text{pop}}$$

$$= - \sum_k \sum_{m,n} r_{nm}^k x_m(t) \ln x_n(t)$$

$$+ \frac{1}{2} \sum_k \sum_{m,n} r_{nm}^k x_m(t) \ln \frac{r_{nm}^k}{r_{nm}^{k'}}$$

$$= - \sum_k \sum_{m,n} r_{nm}^k x_m(t) \ln x_n(t)$$

$$+ \frac{1}{2} \sum_k \sum_{m,n} r_{nm}^k x_m(t) \ln \frac{r_{nm}^k}{r_{nm}^{k'}}$$

$$= - \sum_k \sum_{m,n} r_{nm}^k x_m(t) \ln x_n(t)$$

$$+ \frac{1}{2} \sum_k \sum_{m,n} r_{nm}^k x_m(t) \ln \frac{r_{nm}^k}{r_{nm}^{k'}}.$$

Similarly, the dynamical activity rate can also be calculated as

$$a_t = \sum_k \text{tr} \{ L_k T_k^L \}$$

$$= \frac{1}{2} \sum_k \left( \text{tr} \{ L_k T_k^L \} + \text{tr} \{ L_k L_k^L \} \right)$$

$$= \frac{1}{2} \sum_k \sum_{m,n} [r_{nm}^k x_m(t) + r_{nm}^{k'} x_n(t)]$$

$$= \sum_k \sum_{m,n} a_{nm}^k(t). \hspace{1cm} (S122)$$

By applying the triangle inequality and Jensen’s inequality, we obtain the desired inequality (S118) as

$$\sum_{(m,n) \in E} \left| \sum_k [r_{nm}^k x_m(t) - r_{nm}^{k'} x_n(t)] \right| \leq \frac{\sigma_t}{2} \Phi \left( \frac{\sigma_t}{2a_t} \right)^{-1}.$$

$$\leq \frac{1}{2} \sum_k \sum_{m,n} \left| r_{nm}^k x_m(t) - r_{nm}^{k'} x_n(t) \right| \leq \sum_k \sum_{m,n} \sigma_{nm}^k(t) \Phi \left( \frac{\sigma_{nm}^k(t)}{2a_{nm}^k} \right)^{-1}.$$

$$\leq \frac{1}{2} \sum_k \sum_{m,n} \sigma_{nm}^k(t) \Phi \left( \frac{\sigma_{nm}^k(t)}{2a_{nm}^k} \right)^{-1}.$$

\hspace{1cm} (S123)
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