Differential Harnack inequalities and Perelman type entropy formulae for subelliptic operators

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Abstract

In this paper, under the generalized curvature-dimension inequality recently introduced by F. Baudoin and N. Garofalo, we obtain differential Harnack inequalities (Theorem 2.1) for the positive solutions to the Schrödinger equation associated to subelliptic operator with potential. As applications of the differential Harnack inequality, we derive the corresponding parabolic Harnack inequality (Theorem 4.1). Also we define the Perelman type entropy associated to subelliptic operators and derive its monotonicity (Theorem 5.3).

Keywords: Differential Harnack inequalities, Perelman entropy, Curvature-dimension inequality, Subelliptic operators

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1 Introduction

Let $M$ be a $C^\infty$ connected finite dimensional compact manifold with a smooth measure $\mu$ and a second-order diffusion operator $L$ on $M$, locally subelliptic, satisfying $L1 = 0$,

$$\int_M fLgd\mu = \int_M gLf d\mu, \quad \int_M fLfd\mu \leq 0$$

for every $f, g \in C^\infty(M)$. In the neighborhood of every point $x \in M$, it can be written as

$$L = - \sum_{i=1}^{m} X_i^*X_i,$$

where $X_1, X_2, \cdots, X_m$ are Lipschitz continuous, see [14] [7]. For any function $f, g \in C^\infty(M)$, the carré du champ associated to $L$ is

$$\Gamma(f, g) = \frac{1}{2}(L(fg) - fLg - gLf) = \sum_{i=1}^{m} X_i^*X_ig,$$

and $\Gamma(f) = \Gamma(f, f)$.

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In addition, as in [7] we assume that \( M \) is endowed with another smooth symmetric bilinear differential form, indicated with \( \Gamma^Z \), satisfying for \( f, g \in C^\infty(M) \),
\[
\Gamma^Z(fg, h) = f\Gamma^Z(g, h) + g\Gamma^Z(f, h),
\]
and \( \Gamma^Z(f) = \Gamma^Z(f, f) \). We make the following assumption (see [7] for detailed explanation) holds throughout the paper:

**Assumption:** For any \( f \in C^\infty(M) \), one has
\[
\Gamma(f, \Gamma^Z(f)) = \Gamma^Z(f, \Gamma(f)).
\]

Denote
\[
\Gamma_2(f, g) = \frac{1}{2}(L\Gamma(f, g) - \Gamma(f, Lg) - \Gamma(g, Lf)),
\]
\[
\Gamma_2^Z(f, g) = \frac{1}{2}(L\Gamma^Z(f, g) - \Gamma^Z(f, Lg) - \Gamma^Z(g, Lf)),
\]
and \( \Gamma_2(f) = \Gamma_2(f, f), \Gamma_2^Z(f) = \Gamma_2^Z(f, f) \).

Now we are ready to recall

**Definition 1.1. (due to [7])** We say that \( L \) satisfies the generalized curvature-dimension inequality \( CD(p_1, \rho_2, k, d) \) if there exists constants \( p_1 \in \mathbb{R}, \rho_2 > 0, k \geq 0 \) and \( d \in [2, \infty) \) such that the inequality
\[
\Gamma_2(f) + \nu \Gamma_2^Z(f) \geq \frac{1}{d}(Lf)^2 + \left( \rho_1 - \frac{k}{\nu} \right) \Gamma(f) + \rho_2 \Gamma^Z(f),
\]
holds for all \( f \in C^\infty(M) \) and every \( \nu > 0 \).

The above definition generalizes to the curvature dimension inequality \( CD(K, m) \) introduced by D. Bakry and M. Émery, see for example [1]. Indeed, we only need to take \( p_1 = K, \rho_2 = k = 0, d = m \). Besides Laplace-Beltrami operators on compact Riemannian manifolds with Ricci curvature bounded below and sub-Laplacian \( \Delta_b \) in a closed pseudohermitian 3-manifold \( (M^3, J, \theta) \), there exists a wide class of examples satisfying the generalized curvature-dimension condition, e.g. see [7].

Denote the heat semigroup \( P_t = e^{tL} \), due to the hypo-ellipticity of \( L \), one has the function \((t, x) \to P_t f(x)\) is smooth on \( M \times (0, \infty) \) and \( P_t f(x) = \int_M p(x, y, t)f(y)d\mu(y) \), where \( p(x, y, t) = p(y, x, t) > 0 \) is the so-called heat kernel associated to \( P_t \).

Suppose the subelliptic operator \( L \) satisfies the generalized curvature-dimension inequality \( CD(\rho_1, \rho_2, k, d) \), Baudoin and his collaborators ([7] [4] [5] [6] [9]) have obtained that 1). the heat semigroup \( P_t \) is stochastically complete, i.e. \( P_t 1 = 1 \); 2). Li-Yau type inequalities: \( \Gamma(log P_t f) + \frac{4}{3} \rho_2 \Gamma^Z(log P_t f) \leq \left( 1 + \frac{3k}{3d} \right) \frac{L P_t f}{P_t f} + \frac{d \rho_1}{6} t - \frac{d \rho_1}{2} \left( 1 + \frac{3k}{3d} \right) + \frac{d}{2} \left( 1 + \frac{3k}{3d} \right)^2 \); 3). Scale-invariant parabolic Harnack inequality; 4). Off-diagonal Gaussian upper bounds for the heat kernel \( p(x, y, t); 5) \). Liouville type property; 6). Bonnet-Myers type theorem; 7). Volume comparison property, volume doubling property and distance comparison theorem etc.; 8). Functional inequalities such as Poincaré inequality, Log-Sobolev inequality; · · · · ·, which generalizes the works of S. T. Yau, D. Bakry, M. Ledoux etc., see [17] [1] [15] and references therein. See also [11] [23] [24] [26] for recent works on gradient estimate and curvature property for subelliptic operators.
In the fundamental work [17], P. Li and S. T. Yau established the well-known gradient estimates (so-called Li-Yau inequalities) for the positive solutions to the heat equation \( \partial_t u = \Delta u \) on the complete Riemannian manifolds, since it becomes a powerful tool in differential geometry, PDE, etc. It also plays an important role in the Perelman’s solution to the Poicaré conjecture. As mentioned in the above section, F. Baudoin and N. Garofalo [7] have obtained the Li-Yau type inequality, but the method (semigroup tools) exploited in [7] (see also [2] for three dimensional subelliptic models) is not adaptable for the diffusion operators with potential \( L^V \) (Schödinger operator). Since \( L^V \) does not commute with the Feynman-Kac semigroup \( P_t^V \) generated by \( L^V \). While the Li-Yau method (maximum principle) exploited in [17] works for the Schödinger case, hence it would be interesting to find certain method adaptable for the Schödinger equation. This is the start point of this paper.

This paper is devoted to study differential Harnack inequalities (Li-Yau type gradient estimates) for the positive solutions to the heat equation (Schödinger equation) associated to subelliptic operators with potential, see section 2 for the statement of Li-Yau type inequalities and section 3 for the proof. Section 4 is devoted to the Harnack inequalities for positive solutions to the associated Schödinger equation. By applying the differential Harnack inequalities obtained in section 2 we study the Perelman type entropy and its monotonicity in section 5.

2 Differential Harnack inequalities

In this section, we shall study the Schrödinger equation \( \partial_t u = (L - V)u \), where \( L \) is the subelliptic diffusion operator as in Section 1 and the potential \( V = V(x, t) \geq 0 \) defined on \( M \times [0, \infty) \) is \( C^\infty \). Denote \( L^V = L - V \).

Throughout this section, we assume that \( L \) (rather than \( L^V \)) satisfies the generalized curvature-dimension inequality \( CD(\rho_1, \rho_2, k, d) \) and there exists some positive constants \( \gamma_1, \gamma_2, \theta \) such that

\[
\Gamma(V) \leq \gamma_1^2 \Gamma^Z(V) \leq \gamma_2^2, L^V \leq \theta. \tag{2.1}
\]

For a given \( C^1 \) function \( a(t) : [0, \infty) \to [0, \infty) \) such that \( a(0) = 0 \), \( \lim_{t \to 0} a(t) = 0 \), \( \frac{a'}{a} > 0 \),

\[
\frac{\int_0^t a(s)ds}{a(t)} > 0 \quad \text{and for any} \quad T > 0, \quad \frac{a^2}{a(t)}, \quad \frac{a^3}{a(t)}, \quad \frac{1}{a(t)} \left( \int_0^t \frac{a^2(s)}{a(u)du} ds \right)^2, \quad \frac{1}{a(t)} \left( \int_0^t a(s)ds \right)^2
\]

are continuous and integrable on the domain \( (0, T] \). We have the following Li-Yau type inequality for the Schrödinger equation:

**Theorem 2.1.** Let \( u \) be a positive solution to the Schrödinger equation

\[
\partial_t u = L^V u, \tag{2.2}
\]

such that \( u_0 := u(\cdot, 0) \) is \( C^\infty(M) \) and \( u_0 \geq 0 \), we have the following differential Harnack inequality: for any \( \varepsilon_1, \varepsilon_2 \in (0, 1) \), we have

\[
\Gamma(\log u) + b(t)\Gamma^Z(\log u) - \alpha(t) ((\log u)_t + V) \leq \varphi(t), \tag{2.3}
\]

where

\[
b(t) = \frac{2(1 - \varepsilon_2)\rho_2 \int_0^t a(s)ds}{a(t)}, \quad \eta(t) = \frac{d}{4} \left( (1 + \varepsilon_1) \frac{a'}{a} + \frac{ka(t)}{\rho_2 \int_0^t a(s)ds - 2\rho_1} \right),
\]
and
\[
\alpha(t) = \frac{4}{d \cdot a(t)} \int_0^t a(s) \eta(s) ds, \quad \varphi(t) = \frac{1}{a(t)} \int_0^t a(s) \left( \frac{\gamma^2 a(s) |\alpha - 1|^2}{e_1 a'(s)} + \frac{b^2 \gamma^2}{2 e_2 p_2} + \theta a(s) + \frac{2 \eta^2(s)}{d} \right) ds.
\]

**Remark 2.2.** (i) For the elliptic operators (possibly with potential) satisfying the curvature dimension inequality \(CD(\rho, m)\) introduced in [1], the corresponding result has been obtained in [25]. Moreover, the local differential Harnack inequalities have been derived in [25]. The method in this paper is motivated by the one in [16, 25], it works not only for the diffusion operators but also works for the Schrödinger operators (i.e. diffusion operators with potential), see Section 4. While the one in [7] does not work for the Schrödinger operator \(L^V\). But due to the validity of the parabolic comparison theorem (or maximum principle), we assume \(M\) is compact here.

(ii) In the case of sub-Laplacian \(\Delta_b\) in a closed pseudohermitian 3-manifold \((M^3, J, \theta)\), in fact, \(\Delta_b\) satisfies \(CD(k, \frac{1}{2}, 1, 2)\) where \(k\) is the lower bound of Tanaka-Webster curvature. The above result generalizes Theorem 1.1 in [12] by Chang et al.

A possible function \(a(t)\) is \(a(t) = t^\gamma\) with \(\gamma > 1\). In this case, Theorem 2.1 reduces to

**Corollary 2.3.** Under the assumption of Theorem 2.1, there exists positive constants \(C_i (i = 1, 2, 3, 4)\) depends on \(\rho_1, \rho_2, k, d, \gamma_1, \gamma_2, \gamma, \theta\), we have for all \(t > 0\),

\[
\Gamma(\log u) + \frac{\rho_2}{1 + \gamma} t \Gamma^Z(\log u) \leq \left( \frac{3}{2} + \frac{(1 + \gamma)k}{\gamma \rho_2} - \frac{2 \rho_1}{1 + \gamma} \right) ((\log u)_t + V) + \frac{d \left( 1.5 \gamma \rho_2 + (1 + \gamma)k \right)^2}{8(\gamma - 1) \rho_2^2 t} + C_1 t + C_2 t^2 + C_3 t^3 + C_4 t^4.
\]

Another possible function \(a(t)\) is \(a(t) = e^{\rho_1 t} (e^{\rho_1 t} - 1)^2\) with \(\gamma \cdot \text{sgn}(\rho_1) > 0\), taking \(\varepsilon_1 = \varepsilon_2 = \frac{1}{2}\), Theorem 2.1 reduces to

**Corollary 2.4.** Under the assumption of Theorem 2.1, there exist positive constants \(C_1, C_2\) depends on \(\rho_1, \rho_2, k, d, \gamma_1, \gamma_2, \gamma, \theta\), we have for all \(t > 0\),

\[
\Gamma(\log u) + \frac{\rho_2}{3 \gamma \rho_1} \left( e^{\gamma \rho_1 t} - 1 \right) e^{-\gamma \rho_1 t} t \Gamma^Z(\log u) \leq \alpha(t) ( ((\log u)_t + V) + C_1 + \frac{C_2}{e^{\gamma \rho_1 t} - 1} ),
\]

where

\[
\alpha(t) = \frac{3}{2} + \frac{k}{\rho_2} - \frac{2}{3 \gamma} + \left( \frac{2}{3 \gamma} + \frac{k}{2 \rho_2} \right) e^{-\gamma \rho_1 t},
\]

Comparing with the Corollary 2.3 (2.5) works for both small time and large time \(t\), while (2.4) works for the finite time \(t\).

As an application of the above Corollary, we have

**Corollary 2.5.** Assume \(L\) satisfies curvature inequality \(CD(\rho_1, \rho_2, k, d)\) with \(\rho_1 \neq 0\). Let \(u\) be a positive solution to the equation \(L^V u = 0\) with \(V\) defined on \(M\), there exists a positive constant \(C\) such that

\[
\Gamma(\log u) + \frac{\rho_2}{4 |\rho_1|} \Gamma^Z(\log u) \leq \left( 2 + \frac{k}{\rho_2} \right) V + C.
\]
In particular, for a given $C^1$ function $a(t) : [0, \infty) \to [0, \infty)$ such that $a(0) = 0$, $\lim_{t \to 0} \frac{a(t)}{a'(t)} = 0$, $\frac{a'}{a} > 0$, $\int_0^T \frac{a(s)ds}{a(t)} > 0$ and for any $T > 0$, $\frac{a^2}{a} \cdot \frac{a^3(t)}{\int_0^T a(s)ds^2}$ are continuous and integrable on the domain $(0, T]$, we have Li-Yau type differential Harnack inequalities for subelliptic diffusion operator $L$ (i.e. $V \equiv 0$).

**Proposition 2.6.** Let $u$ be a positive solution to the heat equation $\partial_t u = Lu$ such that $u_0 := u(\cdot, 0) \in C^{\infty}(M)$ with $u_0 \geq 0$, we have the following differential Harnack inequality:

$$
\Gamma(\log u) + \frac{2\rho_2}{1 + \gamma} \frac{\int_0^t a(s)ds}{a(t)} \Gamma^Z(\log u) + \alpha(t)(\log u) \leq \varphi(t),
$$

(2.7)

where

$$
\eta(t) = \frac{d}{4} \left( \frac{a'}{a} + \frac{ka(t)}{\rho_2 \int_0^t a(s)ds} - 2\rho_1 \right), \quad \alpha(t) = \frac{4 \int_0^t a(s)\eta(s)ds}{d \cdot a(t)}, \quad \varphi(t) = \frac{2 \int_0^t a(s)\eta^2(s)ds}{d \cdot a(t)}.
$$

Taking $a(t) = t^\gamma$ with $\gamma > 1$, then $\eta(t) = \frac{d}{4} \left( (\gamma \rho_2 + (\gamma + 1)k) \frac{1}{\rho_2 t} - 2\rho_1 \right)$, Proposition 2.6 reduces to

**Corollary 2.7.** Let $u$ be a positive solution to the heat equation $\partial_t u = Lu$ such that $u_0 = u(\cdot, 0) \in C^{\infty}(M)$ with $u_0 \geq 0$, we have for $\gamma > 1$,

$$
\Gamma(\log u) + \frac{2\rho_2}{1 + \gamma} \Gamma^Z(\log u) \leq \left( 1 + \frac{(1 + \gamma)k}{\gamma \rho_2} \right) (\log u)_t + \frac{d \rho_1^2}{2(1 + \gamma)} - \frac{d \rho_1}{2} \left( 1 + \frac{(1 + \gamma)k}{\gamma \rho_2} \right) + \frac{d \gamma^2}{8(\gamma - 1)t} \left( 1 + \frac{(1 + \gamma)k}{\gamma \rho_2} \right)^2.
$$

(2.8)

If $\rho_1 \geq 0$, it is easy to see (2.8) reduces to

$$
\Gamma(\log u) + \frac{2\rho_2}{1 + \gamma} \Gamma^Z(\log u) \leq \left( 1 + \frac{(1 + \gamma)k}{\gamma \rho_2} \right) (\log u)_t + \frac{d \gamma^2}{8(\gamma - 1)t} \left( 1 + \frac{(1 + \gamma)k}{\gamma \rho_2} \right)^2.
$$

(2.9)

**Remark 2.8.** Choose $\gamma = 2$, Corollary 2.4 reduces to

$$
\Gamma(\log u) + \frac{2\rho_2}{3} \Gamma^Z(\log u) \leq \left( 1 + \frac{3k}{2\rho_2} - \frac{2\rho_1t}{3} \right) (\log u)_t + \frac{d \rho_1^2}{6} - \frac{d \rho_1}{2} \left( 1 + \frac{3k}{2\rho_2} \right) + \frac{d \left( 1 + \frac{3k}{2\rho_2} \right)^2}{2t},
$$

which is nothing less than theorem 5.1 in [7]. In particular, if $\rho_1 \geq 0$, we have

$$
\Gamma(\log u) + \frac{2\rho_2}{3} \Gamma^Z(\log u) \leq \left( 1 + \frac{3k}{2\rho_2} \right) (\log u)_t + \frac{d \left( 1 + \frac{3k}{2\rho_2} \right)^2}{2t}.
$$

(2.10)

As a consequence, we can get the Liouville property if $\rho_1 \geq 0$, i.e. there is no positive constant solution to $Lu = 0$. 

5
Taking $a(t) = e^{\gamma \rho_1 t} (1 - e^{\gamma \rho_1 t})^\beta$ with $\beta > 1$ even, $\gamma \geq \frac{2\rho_2}{(\rho_2 + k)(1 + \beta)}$, then $a' = \gamma \rho_1 ((1 + \beta) e^{\gamma \rho_1 t} - 1)$, hence Proposition 2.6 reduces to

Corollary 2.9. Let $u$ be a positive solution to the heat equation $\partial_t u = Lu$ such that $u_0 := u(\cdot, 0) \in C^\infty(M)$ with $u_0 \geq 0$, we have for $\beta > 1$ even, $\gamma \geq \frac{2\rho_2}{(\rho_2 + k)(1 + \beta)}$,

$$
\Gamma(\log u) - \frac{2\rho_2}{(1 + \beta) \gamma \rho_1} (1 - e^{\gamma \rho_1 t}) e^{-\gamma \rho_1 t} \Gamma^Z(\log u) \leq \alpha(t)(\log u)_t + \varphi(t), \quad (2.11)
$$

where

$$
\alpha(t) = 1 + \frac{(1 + \beta) k}{\beta \rho_2} e^{-\gamma \rho_1 t} + \left(\frac{2}{(1 + \beta) \gamma} - \frac{k}{\rho_2}\right) (1 - e^{\gamma \rho_1 t}) e^{-\gamma \rho_1 t},
$$

and

$$
\varphi(t) = \frac{d \rho_1 (\gamma(p_2 + k)(1 + \beta) - 2\rho_2)^2}{8(1 + \beta) \rho_2^2} (1 - e^{-\gamma \rho_1 t}) e^{-\gamma \rho_1 t} + \frac{d \rho_1 \gamma(p_2 \beta + k + k \beta)^2}{8 \rho_2^2 (\beta - 1)} e^{-\gamma \rho_1 t} - 1.
$$

In particular, let $\gamma = \frac{2\rho_2}{(\rho_2 + k)(1 + \beta)}$ ($\beta > 1$ even), we have

$$
\Gamma(\log u) + \frac{\rho_2 + k}{\rho_1} \left(1 - e^{-\frac{2\rho_1 \rho_2 t}{(\rho_2 + k)(1 + \beta)}}\right) \Gamma^Z(\log u) \leq \frac{k + \beta k + \beta \rho_2}{\beta \rho_2} e^{-\frac{2\rho_1 \rho_2 t}{(\rho_2 + k)(1 + \beta)}} (\log u)_t
$$

$$
+ \frac{d \rho_1 (\beta \rho_2 + k + k \beta)^2}{4 \rho_2 (\rho_2 + k)(\beta + 1)(\beta - 1)} e^{-\frac{2\rho_1 \rho_2 t}{(\rho_2 + k)(1 + \beta)}} - 1. \quad (2.12)
$$

It yields the following lower bound for $(\log u)_t$:

$$
(\log u)_t \geq - \frac{d(\beta \rho_2 + k + k \beta)}{4(\rho_2 + k)(\beta + 1)(\beta - 1)} \frac{\rho_1}{e^{\frac{2\rho_1 \rho_2 t}{(\rho_2 + k)(1 + \beta)}} - 1}.
$$

Comparing with Corollaries 2.7, 2.11 works for both small time and large time $t$ while (2.8) only works for finite time $t$ in the case of $\rho_1 \geq 0$.

### 3 Proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1. The method of the proof is inspired by [25, 16].

Assume $u$ is a positive solution to the Schrödinger equation $\partial_t u = L^V u$. Let $f = \log u$, we have

$$
L f + \Gamma(f) = f_t + V, \quad (3.1)
$$

where $\Gamma$ is induced by $L$ rather than $L^V$. For some positive function $b(t)$ and some functions $\alpha(t), \varphi(t)$, define the parameter function:

$$
F = \Gamma(f) + b(t) \Gamma^Z(f) - \alpha(t)(f_t + V) - \varphi(t),
$$

and
it follows that
\[(L - \partial_t)F = L\Gamma(f) - 2\Gamma(f, f_t) + b(t) (L\Gamma^Z(f) - 2\Gamma^Z(f, f_t)) - \alpha(t)(Lf_t - f_t) \]
\[\quad - \alpha(t)(LV - V_t) - b'(t)\Gamma^Z(f) + \alpha'(t)(f_t + V) + \varphi'(t).\]

Applying (3.1) and assumption (H.2), we have
\[(L - \partial_t)F = 2\Gamma_2(f) + 2b(t)\Gamma^Z(f) - 2\Gamma(f, F) - 2(\alpha(t) - 1)\Gamma(f, V) + 2b(t)\Gamma^Z(f, V) \]
\[\quad - \alpha(t)LV - b'(t)\Gamma^Z(f) + \alpha'(t)f_t + \alpha'(t)V + \varphi'(t),\]

Thanks to the generalized curvature dimension inequality $CD(\rho_1, \rho_2, k, d)$,
\[(L - \partial_t)F \geq -2\Gamma(f, F) + \frac{2}{d}(Lf)^2 + \left(2\rho_1 - \frac{2k}{b(t)}\right)\Gamma(f) + (2\rho_2 - b'(t))\Gamma^Z(f) \]
\[\quad - 2(\alpha(t) - 1)\Gamma(f, V) + 2b(t)\Gamma^Z(f, V) - \alpha(t)LV + \alpha'(t)(V + f_t) + \varphi'(t) \]
\[\geq -2\Gamma(f, F) - \frac{4\eta^2(t)}{d}Lf - \frac{2\eta^2(t)}{d} + \left(2\rho_1 - \frac{2k}{b(t)}\right)\Gamma(f) + (2\rho_2 - b'(t))\Gamma^Z(f) \]
\[\quad - 2(\alpha(t) - 1)\Gamma(f, V) + 2b(t)\Gamma^Z(f, V) - \alpha(t)LV + \frac{2\eta^2(t)}{d} + \varphi'(t),\]

where $\eta = \eta(t)$ is some function to be determined later. It follows by (3.1),
\[(L - \partial_t)F \geq -2\Gamma(f, F) + \left(\frac{4\eta}{d} + 2\rho_1 - \frac{2k}{b(t)}\right)\Gamma(f) + (2\rho_2 - b'(t))\Gamma^Z(f) - \left(\frac{4\eta}{d} - \alpha'(t)\right)(f_t + V) \]
\[\quad - 2(\alpha(t) - 1)\Gamma(f, V) + 2b(t)\Gamma^Z(f, V) - \alpha(t)LV - \frac{2\eta^2(t)}{d} + \varphi'(t).\]

Applying (2.1) and the basic fact $2ax \leq \frac{a^2}{\varepsilon} + \varepsilon x^2$, $\forall a, \varepsilon \in \mathbb{R}^+$, we have for any positive functions $\epsilon_1(t), \epsilon_2(t)$,
\[(L - \partial_t)F + 2\Gamma(f, F) \geq \left(\frac{4\eta}{d} + 2\rho_1 - \frac{2k}{b(t)} - \epsilon_1(t)\right)\Gamma(f) + (2\rho_2 - b'(t))\Gamma^Z(f) - \left(\frac{4\eta}{d} - \alpha'(t)\right)(f_t + V) \]
\[\quad - 2|\alpha(t) - 1|\Gamma(f)^{1/2} - 2b(t)\gamma_2(\Gamma^Z(f))^{1/2} - \alpha(t)\theta - \frac{2\eta^2(t)}{d} + \varphi'(t) \]
\[\geq \left(\frac{4\eta}{d} + 2\rho_1 - \frac{2k}{b(t)} - \epsilon_1(t)\right)\Gamma(f) + (2\rho_2 - b'(t) - \epsilon_2(t))\Gamma^Z(f) \]
\[\quad - \left(\frac{4\eta}{d} - \alpha'(t)\right)(f_t + V) - \left|\frac{\alpha(t) - 1^2}{\epsilon_1(t)}\right| - \frac{b^2(t)\gamma_2^2}{\epsilon_2(t)} - \alpha(t)\theta - \frac{2\eta^2(t)}{d} + \varphi'(t).\]
For any \( \epsilon_1, \epsilon_2 \in (0, 1) \), Take

\[
\epsilon_1(t) = \frac{\epsilon_1 a'}{a}, \\
\epsilon_2(t) = 2\epsilon_2 \rho_2, \\
b(t) = \frac{2(1 - \epsilon_2) \rho_2 \int_0^t a(s) ds}{a(t)}, \\
\eta(t) = \frac{4}{d} \left( (1 + \epsilon_1) \frac{a'}{a} + \frac{2k}{b(t)} - 2\rho_1 \right), \\
\alpha(t) = \frac{4}{d} a(t) \int_0^t a(s) \eta(s) ds, \\
\varphi(t) = \frac{1}{a(t)} \int_0^t a(s) \left( \frac{\gamma_2^2 a(s) |\alpha - 1|^2}{\epsilon_1 a'(s)} + \frac{b^2 \gamma_2^2}{2\epsilon_2 \rho_2} + \theta a(s) + \frac{2\eta^2(s)}{d} \right) ds.
\]

such that

\[
\frac{4\eta}{d} + 2\rho_1 - \frac{2k}{b(t)} = (1 + \epsilon_1) \frac{a'}{a}, \\
2(1 - \epsilon_2) \rho_2 - b'(t) = \frac{a'b}{a}, \\
\frac{4\eta}{d} - \alpha'(t) = \frac{a'\alpha}{a}, \\
\varphi'(t) - \frac{|\alpha - 1|^2 \gamma_1^2}{\epsilon_1} - \frac{b^2 \gamma_2^2}{\epsilon_2} - \alpha\theta - \frac{2\eta^2}{d} = -\varphi a'
\]

With this choice, we have

**Lemma 3.1.** For the parameter function \( F \), we have

\[
(L - \partial_t) F + 2\Gamma(f, F) \geq \frac{a'}{a} F.
\]

Now we are ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** Note that

\[
(L - \partial_t)(a(t)uF) = a(t)u((L - \partial_t)F + 2\Gamma(f, F)) - a'uF(u) + a(t)F(L - \partial_t)u \\
\geq a(t)uFV,
\]

it implies

\[
(L^V - \partial_t)(a(t)uF) \geq 0.
\]

Notice that \( a(0) = 0 \), the desired result follows by the maximum principle, see [13, 10].
4 Harnack inequalities

In this section, we shall derive the Harnack inequalities for the positive solutions to the Schrödinger equation (2.2), which are the consequence of the Li-Yau type inequality (2.7). Harnack inequality is one of the main techniques in the regularity theory of PDEs over the past several decades.

To this end, we introduce the metric associated to Schrödinger operator $L_V$, see [17] for elliptic operators on Riemannian manifolds. For $\delta > 1$, denote

$$\rho_0(x, y, t) = \inf_{\gamma \in \Gamma_0} \left\{ \frac{\delta}{4t} \int_0^1 \sum_{i=1}^m a_i^2(s) ds + t \int_0^1 V(\gamma(s), (1-s)t_2 + st_1) ds \right\}$$

where $\Gamma_0$ is a set of admissible curves $\gamma : [0, 1] \to M$ satisfying $\gamma(0) = x, \gamma(1) = y$, and $\gamma'(s) = \sum_{i=1}^m a_i(s)X_i(\gamma(s))$ a.e. s. Since $\sum_{i=1}^m a_i^2(s)$ does not depend on the choice of the $X_j$'s (see section 2 in [14]), nor does the distance function $\rho_0$. In the case of $V \equiv 0$, $\rho_0$ is nothing less than the Carnot-Carathéodory distance $d_{CC}$ induced by the subelliptic operator $L$.

**Theorem 4.1.** Let $u$ be a positive solution to the Schrödinger equation (2.2). There exist positive constants $C'_i, i = 1, \cdots, 6$ and $\delta_0 = \delta_0(\rho_2, k) > 1$, we have for all $0 < t_1 < t_2, x, y \in M$ and $\delta > \delta_0$,

$$u(x, t_1) \leq u(y, t_2) \left( \frac{t_2}{t_1} \right)^{C'_i} \exp \left( \rho_0(x, y, t_2 - t_1) + \sum_{i=1}^5 \frac{C'_i + 1}{t} \left( t_2 - t_1 \right) \right).$$

**Proof.** Let $\gamma$ be any admissible curve given by $\gamma : [0, 1] \to M$, with $\gamma(0) = y$ and $\gamma(1) = x$. We define $\eta : [0, 1] \to M \times [t_1, t_2]$ by $\eta(s) = (\gamma(s), (1-s)t_2 + st_1)$, clear that $\eta(0) = (y, t_2)$ and $\eta(1) = (x, t_1)$. Integrating $\frac{d}{ds}(\log u)$ along $\gamma$, we get

$$\log u(x, t_1) - \log u(x, t_2) = \int_0^1 \frac{d}{ds} \log u ds = \int_0^1 \sum_{i=1}^m a_i X_i \log u(x, s) - (t_2 - t_1)(\log u)_s ds$$

Applying (2.4) in Corollary 2.3 there exist some constants $C'_i, i = 1, \cdots, 6$ and $\delta_0 = \delta_0(\rho_2, k) > 1$ such that for $\delta > \delta_0, t = (1-s)t_2 + st_1$, we have

$$\log \frac{u(x, t_1)}{u(x, t_2)} \leq \int_0^1 \sum_{i=1}^m |a_i||X_i \log u| + (t_2 - t_1) \left( -\frac{1}{\delta} \Gamma(\log u) + \sum_{i=1}^6 C'_i t^{i-2} + V(\gamma(s), t) \right) ds$$

$$\leq \int_0^1 \sum_{i=1}^m \left( |a_i(s)||X_i \log u| - \frac{t_2 - t_1}{\delta} |X_i \log u|^2 \right) + (t_2 - t_1)V(\gamma(s), t) ds$$

$$+ C'_2(t_2 - t_1) + C'_3 \log \frac{t_2}{t_1} + \sum_{i=1}^4 \frac{C'_{i+2}}{i+1} \left( t_2 - t_1 \right)$$

$$\leq \int_0^1 \frac{\delta}{4(t_2 - t_1)} \sum_{i=1}^m a_i^2(s) + (t_2 - t_1)V(\gamma(s), (1-s)t_2 + st_1) ds$$

$$+ C'_1 \log \frac{t_2}{t_1} + \sum_{i=1}^5 \frac{C'_{i+1}}{i} \left( t_2 - t_1 \right),$$
taking exponentials of the above inequality gives the desired result.

Applying Corollary 2.5 in the above proof, we have

**Theorem 4.2.** Let \( u \) be a positive solution to the Schrödinger equation \( L^V u = 0 \). There exists positive constant \( C = C(\theta, \gamma_1, \gamma_2, \rho_1, \rho_2, k, d) \), for any \( x, y \in M \), we have

\[
u(x) \leq u(y) \exp(Cd(x, y)),
\]

where \( d(x, y) = \inf_{\gamma \in \Gamma_0} \left\{ \int_0^1 \sum_{i=1}^m a_i^2(s)ds + \int_0^1 V(\gamma(s))ds \right\} \).

5 Perelman type Entropy and its monotonicity

Throughout this section, we assume that the operator \( L \) satisfies the generalized curvature-dimension inequality \( CD(\rho_1, \rho_2, k, d) \) for some \( \rho_1 \geq 0, \rho_2 > 0, k \geq 0 \) and \( d < \infty \). Consequently, the invariant measure is finite, see [4] Theorem 1.3. Without loss any generality, we suppose \( \mu \) is a probability measure. Let \( u(x, t) \) be the positive solution of the heat equation \( \partial_t u = Lu \) on \( M \times [0, \infty) \), we can assume \( \int_M u d\mu = 1 \). Denote \( g(x, t) \) the function

\[
u(x, t) = \frac{e^{-g(x, t)}}{(4\pi t)^{\frac{d}{2}}}
\]

with \( D = d \left( 1 + \frac{3k}{2\rho_2} \right)^2 \), where \( \tau \) is some positive constant determined later. We define the so-called Nash-type entropy, see [20, 21],

\[
N(u, t) = -\int_M u \log u d\mu
\]

and

\[
\tilde{N}(u, t) = N(u, t) - \frac{\tau D}{2} (\log(4\pi t) + 1). \tag{5.2}
\]

By applying the Li-Yau type inequality established in section 2, we have the monotonicity of the Nash entropy formulae:

**Theorem 5.1.** For all \( t > 0 \) and \( \tau \geq 1 \), we have

\[
\frac{d}{dt} \tilde{N}(u, t) = \int_M u \left( \Gamma(g) + \left( 1 + \frac{3k}{2\rho_2} \right) g_t + \frac{3k\tau D}{4\rho_2 t} \right) d\mu \leq 0.
\]

**Proof.** Denote \( f = \log u \) as above, we have \( f = -g - \frac{\tau D}{2t} \log(4\pi t) \), it yields

\[
f_t = -g_t - \frac{\tau D}{2t}, \quad \Gamma(f) = \Gamma(g). \tag{5.3}
\]

Thanks to the fact \( \int_M u d\mu = 1 \), the integration by parts formula gives

\[
\frac{d}{dt} \tilde{N}(u, t) = \int_M u\Gamma(\log u)d\mu - \frac{\tau D}{2t} = \int_M u\Gamma(g)d\mu - \frac{\tau D}{2t},
\]
\[
\int_M u g_t d\mu = -\frac{\tau D}{2t} - \int_M u f_t d\mu = -\frac{\tau D}{2t} - \int_M u_t d\mu = -\frac{\tau D}{2t}.
\]

It follows
\[
\frac{d}{dt} \tilde{N}(u,t) = \int_M u (\Gamma(g)) d\mu + \left(1 + \frac{3k}{2\rho_2}\right) \left(\int_M u g_t d\mu + \frac{\tau D}{t}\right) - \frac{\tau D}{2t}
\]
\[
= \int_M u \left(\Gamma(g) + \left(1 + \frac{3k}{2\rho_2}\right) g_t + \frac{3k\tau D}{4\rho_2 t}\right) d\mu
\]

By (2.10) and (5.3), we have
\[
\Gamma(g) + \left(1 + \frac{3k}{2\rho_2}\right) g_t + \left(1 + \frac{3k}{2\rho_2}\right) \frac{\tau D}{2t} - \frac{D}{2t} \leq 0.
\]

For \(\tau \geq 1\), clearly we have \(\frac{d}{dt} \tilde{N}(u,t) \leq 0\).

The desired conclusion follows.

Following Perelman [22], we define
\[
W(u,t) = \int_M u \left(t \Gamma(g) + g - \tau D\right) d\mu = \frac{d}{dt} \left(t \tilde{N}(u,t)\right), \tag{5.4}
\]
and
\[
W_\varsigma(u,t) = W(u,t) + \varsigma t^2 \int_M u \Gamma_Z (\log u) d\mu, \tag{5.5}
\]
where \(\varsigma\) is some positive constant to be determined.

To study the monotonicity of the functional \(W_\varsigma\), let us give the following useful lemma.

**Lemma 5.2.** For any positive solution to the heat equation \(\partial_t u = Lu\), we have
\[
\frac{d^2}{dt^2} N(u,t) = -2 \int_M u \Gamma_2 (\log u) d\mu,
\]
this gives
\[
\frac{d}{dt} W(u,t) = 2 \int_M u \Gamma (\log u) d\mu - 2t \int_M u \Gamma_2 (\log u) d\mu - \frac{\tau D}{2t}.
\]
Moreover, denote \(B(u,t) = \int_M u \Gamma_Z (\log u) d\mu\), we have
\[
B'(u,t) = -2 \int_M u \Gamma_Z (\log u) d\mu.
\]

**Proof.** Compute
\[
\frac{d^2}{dt^2} N(u,t) = \frac{d^2}{dt^2} N(u,t)
\]
\[
= -\frac{d}{dt} \int_M u L \log u d\mu
\]
\[
= -\int_M L u L \log u d\mu - \int_M u \partial_t (L \log u) d\mu
\]
\[
= -2 \int_M L u \log u d\mu - \int_M u (\partial_t - L) (L \log u) d\mu.
\]
Notice that 
\[(\partial_t - L)\log u = L(\partial_t - L)\log u = L\Gamma(\log u),\]
this yields
\[
\frac{d^2}{dt^2} N(u, t) = -\int_M uL\Gamma(\log u) d\mu + 2\int_M u\Gamma(\log u, L\log u) d\mu
\]
\[
= -2\int_M u\Gamma_2(\log u) d\mu.
\]
Hence
\[
\frac{d}{dt} W(u, t) = \frac{d^2}{dt^2} \left( t\tilde{N}(u, t) \right)
\]
\[
= tN''(u, t) + 2N'(u, t) - (t\tau D(1 + \log(4\pi t)))''
\]
\[
= -2t\int_M u\Gamma_2(\log u) d\mu + 2\int_M u\Gamma(\log u) d\mu - \frac{\tau D}{2t}.
\]
Meanwhile
\[
B'(u, t) = \int_M \partial_t u\Gamma^Z(\log u) d\mu + 2\int_M u\Gamma(\log u, \partial_t \log u) d\mu
\]
\[
= \int_M Lu\Gamma^Z(\log u) d\mu + 2\int_M u\Gamma^Z \left( \log u, L\log u + \Gamma(\log u) \right) d\mu
\]
\[
= \int_M uL\Gamma^Z(\log u) d\mu + 2\int_M u\Gamma^Z \left( \log u, L\log u \right) d\mu + 2\int_M u\Gamma \left( \log u, \Gamma^Z(\log u) \right) d\mu
\]
\[
= \int_M uL\Gamma^Z(\log u) d\mu + 2\int_M u\Gamma^Z \left( \log u, L\log u \right) d\mu + 2\int_M u\Gamma \left( \log u, \Gamma^Z(\log u) \right) d\mu,
\]
where in the last equality we apply the assumption (H. 2). Notice that
\[
\int_M uL\Gamma^Z(\log u) d\mu = -\int_M \Gamma(u, \Gamma^Z(\log u)) d\mu = -\int_M u\Gamma(\log u, \Gamma^Z(\log u)) d\mu,
\]
it follows
\[
B'(u, t) = -\int_M uL\Gamma^Z(\log u) d\mu + 2\int_M u\Gamma^Z \left( \log u, L\log u \right) d\mu
\]
\[
= -2\int_M u\Gamma^Z_2(\log u) d\mu.
\]
Hence we complete the proof. \hfill \Box

Now we are ready to state the main result in this section.

**Theorem 5.3.** For \(\rho_2 \leq \varsigma \leq \frac{5}{3}\rho_2\) and \(\tau \geq 2 + \frac{2k}{\rho_2}\), we have
\[
\frac{d}{dt} \mathcal{W}_\varsigma(u, t) \leq -\frac{2t}{d} \int_M u(L\log u)^2 d\mu + \frac{D}{2t} \left( \frac{2(k + \varsigma)}{\varsigma} - \tau \right) \leq 0
\]
holds for all \(t > 0\).
Proof. By Lemma 5.2, we have
\[
\frac{d}{dt} W_\varsigma(u,t) = \frac{d}{dt} W(u,t) + 2\varsigma tB(t) + \varsigma^2 \frac{d}{dt} B(t)
\]
\[
= -2t \int_M u\Gamma_2(\log u)\,d\mu - 2\varsigma t^2 \int_M u\Gamma_2^2(\log u)\,d\mu + 2t \int_M u\Gamma(\log u)\,d\mu
\]
\[
+ 2\varsigma t \int_M u\Gamma(\log u)\,d\mu - \frac{t}{2t} \int_M u\Gamma(\log u)\,d\mu = -2t \int_M u(L\log u)^2\,d\mu + 2\varsigma t \int_M u\Gamma Z(\log u)\,d\mu - \frac{\tau D}{\varsigma t}.
\]

The last inequality follows from the generalized curvature-dimension inequality \(CD(\rho_1, \rho_2, k, d)\) with \(\rho_1 \geq 0\). Applying the Li-Yau type estimate (2.10), we have for \(\rho_2 \leq \varsigma \leq \frac{3}{\gamma_1} \rho_2\),
\[
\frac{2(k + \varsigma)}{\varsigma} \int_M u\Gamma(\log u)\,d\mu + 2t(\varsigma - \rho_2) \int_M u\Gamma Z(\log u)\,d\mu \leq \frac{(k + \varsigma)(2\rho_2 + 3k)}{\varsigma\rho_2} \int_M u\,d\mu + \frac{(k + \varsigma)D}{\varsigma t} \leq \frac{(k + \varsigma)D}{\varsigma t}.
\]

This gives
\[
\frac{d}{dt} W_\varsigma(u,t) \leq -2t \int_M u(L\log u)^2\,d\mu + \frac{D}{2t} \left( \frac{2(k + \varsigma)}{\varsigma} - \tau \right),
\]

hence for \(\tau \geq 2 + \frac{2k}{\rho_2} \geq 2 + \frac{2k}{\varsigma}\), we have
\[
\frac{d}{dt} W_\varsigma(u,t) \leq 0.
\]

We complete the proof. \(\square\)

Remark 5.4. (i). Theorem 5.1 and Theorem 5.3 generalize Theorem 1.12 and Theorem 1.13 in [12] respectively, where they consider the case of sub-Laplacians on a closed pseudohermitian 3-manifold with Tanaka-Webster curvature bounded below, i.e. \(CD(k, \frac{1}{2}, 1, 2)\).

(ii). For the diffusion operators satisfying the curvature-dimension inequality \(CD(k, m)\) on a complete noncompact Riemannian manifold, the monotonicity of the Perelman entropy has been obtained in [18], see also [8, 19]. It would be very interesting to derive monotonicity of the Perelman type entropies for subelliptic operator on complete but noncompact Riemannian manifolds, which will be studied in near future.

(iii). In the definitions of (5.1), (5.2) and (5.4), if we replace the parameter \(D = d \left( 1 + \frac{3k}{2\rho_2} \right)^2\) by \(\widetilde{D} = \frac{d^2}{4(\gamma-1)} \left( 1 + \frac{(1+\gamma)k}{\gamma\rho_2} \right)^2 \) (\(\gamma > 1\)), using the Li-Yau type inequality (2.10) instead of (2.10) in the above proofs, we have for all \(t > 0, \gamma > 1\) and \(\tau \geq 1\),
\[
\frac{d}{dt} \tilde{N}(u,t) = \int_M u \left( \Gamma(g) + \left( 1 + \frac{(1+\gamma)k}{\gamma\rho_2} \right) g_t + \frac{(\gamma + 1)k\tau \widetilde{D}}{2\gamma\rho_2 t} \right) \,d\mu \leq 0.
\]
Moreover, for $\gamma > 1$, $\rho_2 \leq \varsigma \leq \frac{3+\gamma}{1+\gamma}\rho_2$ and $\tau \geq 2 + \frac{2k}{\rho_2}$, we have

$$\frac{d}{dt} W_{\varsigma}(u, t) \leq -\frac{2t}{d} \int_M u(L \log u)^2 d\mu + \frac{\bar{D}}{2t} \left( \frac{2(k + \varsigma)}{\varsigma} - \tau \right) \leq 0.$$ 

If we choose $\gamma = 2$, the above two results become Theorem 5.1 and Theorem 5.3 respectively.

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