Registering Seconds with a *Conic Clock*

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Abstract

The feasibility of registering seconds using the frictionless motion of a point-like particle that slides under gravity on an inverted conical surface is studied. Depending on the integer part of the relation between the angular and radial frequencies of the particle trajectory, only an angular interval for the cone is available for this purpose. For each one of these possible angles, there exists a unique trajectory that has the capability of registering seconds. The method to obtain the geometrical properties of these trajectories and the necessary initial conditions to reach them are then established.

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1 Introduction

A clock, by definition, is an instrument that requires some kind of periodic process that registers the passing of the time. This periodicity can be typically obtained using an oscillating system such as a pendulum or a balance wheel. It also needs a trigger mechanism connecting the oscillating structure with a source of energy, such as a weight or spring, in order to compensate for the dissipation [1]. However, in a first approach, one could think of different ideal configurations in which the bounded frictionless dynamics of a particle is used for registering time. The simplest example of this type is a particle bouncing on the ground under the sole effect of gravity. In this case, the relation between the height, \( h \), from where the ball is released and the time of a complete oscillation, \( T \), is \( h = gT^2/8 \), with \( g \) being gravity. Hence, if seconds is what we want to register, advice of at least 1.2 metres height is necessary. In this case the oscillations are performed in a parallel direction to gravity. On the contrary, the usual simple pendulum oscillates in a perpendicular direction to the gravitational field due to the effect of the motionless point where the pendulum is fixed. The relation, \( l = gT^2/(4\pi^2) \), between the length of the pendulum, \( l \), and the small-amplitude oscillation period, \( T \), indicates that a 0.25 metre pendulum-length is required for registering seconds. In both cases the particle always bounces on the same point. Then an additional mechanism should be inserted in these ideal clocks in order to have, for instance, a moving light signal giving us the visual impression of the passing of time.

A particle sliding on an inverted cone is an alternative system that combines both oscillations, in the parallel and in the perpendicular directions to gravity, and that can put in evidence the passing of time without additional mechanisms. Think, for instance, of a phosphorescent circular limb at the top of the cone where the recurrent trajectory of the particle is bouncing. One second elapses between consecutive bounces, and thus the reading of time is manifest. This hypothetical machine will be called a conic clock. It can be inferred from the other cases, and by dimensional
analysis, that the height, $H$, of the cone trunk where the particle motion develops, follows a similar law, $H = cgT^2$, with $c$ a constant of order 1 depending on the opening angle of the cone.

In Section 2 we will recall the non-dimensional form of the evolution equations of a particle sliding on a conical surface. Section 3 is devoted to explaining the method to obtain the trajectories with a recurrence time of seconds. Finally, our conclusions are stated in Section 4.

## 2 Dynamics of the particle

The detailed analysis of the frictionless motion of a particle on a conical surface was developed in Ref. [2]. Here we sketch the way to obtain the non-dimensional equations of its dynamical evolution. The dynamics of the particle inside the cone can be expressed in the generalized coordinates $(r, \theta)$, where $r$ is the coordinate in the direction of cone generatrix and $\theta$ is the angular variable around the vertical axis. The Lagrangian function $\mathcal{L}$ for this system in these particular coordinates appears as

$$\mathcal{L} = \frac{1}{2} m(r^2 + r^2 \dot{\theta}^2 \sin^2 \phi_0) - mgr \cos \phi_0,$$

and the Hamiltonian, $\mathcal{H}$, is

$$\mathcal{H} = \frac{p_r^2}{2m} + \frac{p_{\theta}^2}{2mr^2 \sin^2 \phi_0} + mgr \cos \phi_0.$$  

The equations of motion of the particle are determined by the Lagrange equations,

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial q_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0,$$

with $q_i = (r, \theta)$, $i = 1, 2$. The first of these equations for $q_1 = r$ gives us the evolution of the particle in the radial direction. It yields

$$\ddot{r} - r\dot{\theta}^2 \sin^2 \phi_0 + g \cos \phi_0 = 0.$$  

And the second for $q_2 = \theta$ puts in evidence an invariant of motion because $\mathcal{L}$ is independent of the angular variable $\theta$:

$$mr^2 \dot{\theta} \sin^2 \phi_0 = cte. = L_z.$$  

This dynamical constant, $L_z$, is the vertical component of the angular momentum. If we substitute the value of $\dot{\theta} = L_z/(mr^2 \sin^2 \phi_0)$ in Eq. (3), the radial evolution is uncoupled from its angular dependence, which remains present only through the constant $L_z$,

$$\ddot{r} - \left(\frac{L_z}{m \sin \phi_0}\right)^2 \frac{1}{r^3} + g \cos \phi_0 = 0.$$  

(5)

This last equation corresponds to an integrable nonlinear oscillator in the radial direction. After integrating this motion, the angular part of the dynamics is obtained through Eq. (4). Note that the trajectory is independent of the mass of the particle.

### 2.1 Radial and angular frequency relation

Given a $L_z$ there exists a unique circular orbit with this vertical component of the angular momentum. The energy of this orbit is $E_0 = \frac{3}{2}mgr_0 \cos \phi_0$, with $r_0$ its radial dimension,

$$r_0 = \left[\frac{L_z^2}{m^2g \sin^2 \phi_0 \cos \phi_0}\right]^{\frac{1}{3}},$$

(6)

and its angular frequency, $\omega_0$, given by

$$\omega_0^2 = \frac{g \cos \phi_0}{r_0 \sin^2 \phi_0}.$$  

(7)

A radial perturbation of this trajectory, that is, an increase in the energy, $E$, of the particle leaving $L_z$ constant, provokes an additional radial oscillation superimposed on the angular oscillation. If $E$ is only slightly bigger than $E_0$, then the radial frequency is

$$\omega_{r,E_0}^2 = \frac{3g \cos \phi_0}{r_0},$$

(8)

and the relation between these two frequencies is

$$\frac{\omega_{r,E_0}}{\omega_0} = \sqrt{3} \sin \phi_0.$$  

(9)

The trajectory of the perturbed circular orbit in the plane $(\theta, r)$ covers a longer $\theta$-angular distance when its radial coordinate is under the value $r = r_0$ than when it is over $r_0$. When the energy increases, in the limit $E \to \infty$, the dynamics for
$r > r_0$ is projected onto a tight peak in the $(\theta, r)$ plane. Thus most of the part of the $\theta$-coordinate is covered when the particle is under the circular orbit $r = r_0$ although the system spends its time essentially over the circular orbit. In this limit the ratio of frequencies is:

$$\frac{\omega_{r,E=\infty}}{\omega_\theta} = 2 \sin \phi_0. \quad (10)$$

The general behaviour of this ratio, for an arbitrary value of $E$, was shown to be [2]:

$$\frac{\omega_r}{\omega_\theta} = k \sin \phi_0 \text{ with } k \text{ in the range } \left\{ \begin{array}{c} \sqrt{3} < k < 2 \\
\n\uparrow \quad \uparrow \quad \uparrow \\
E_0 < E < \infty \end{array} \right. . \quad (11)$$

2.2 The non-dimensional equations

The non-dimensional equations are obtained by rescaling the radial, angular and time variables. If we substitute the value of the vertical component of the angular momentum, $L_z^2$, given by Eq. (6) and perform the change of variables: $\tilde{r} = r/r_0$, $\tilde{\theta} = \theta \sin \phi_0$ and $\tilde{t} = \frac{\omega_r E_0}{\sqrt{3}} t$, the radial and angular equations are reduced to

$$\ddot{\tilde{r}} + (1 - \tilde{r}^{-3}) = 0, \quad (12)$$

$$\dot{\tilde{\theta}} - \tilde{r}^{-2} = 0. \quad (13)$$

In this adimensional form, the equations apparently lose every characteristic of the system. Every possible trajectory on the conical surface is projected into a solution of these equations and, in that sense, we say that they are universal. They contain all the information about the dynamical behaviour of the particle.

In particular, $\tilde{r} = 1$ is the singularity representing the circular orbit and any other orbit runs between the extreme values, $\tilde{r}_{\min}$ and $\tilde{r}_{\max}$, which satisfy: $0 < \tilde{r}_{\min} < 1$ and $1 < \tilde{r}_{\max} < \infty$. These extreme points verify the relation

$$\frac{2 \tilde{r}_{\min}^2 \tilde{r}_{\max}^2}{\tilde{r}_{\min} + \tilde{r}_{\max}} = 1. \quad (14)$$
3 Trajectories clocking seconds

An important result in Hamiltonian dynamics is the Poincaré recurrence theorem. It states that a bounded Hamiltonian system returns systematically to an arbitrarily small neighborhood of its initial condition [3, 4]. Apparently this microscopic reversibility would imply the same behaviour in the macroscopic scale, and this is the case, but this theorem does not give any information about the time that the system takes to return close to its initial state. As this recurrence time is extremely long in most of the real systems, the usual irreversible macroscopic behaviour is observed. In spite of the lack of a general rule to calculate the recurrence time, the success of statistical mechanics has consisted in converting the ignorance of the recurrence times of a system in equilibrium into a probabilistic value: that correspondent to the distribution or the ensemble associated to the Hamiltonian physical situation under study.

Our goal now consists in calculating exactly in this particular system the motions of the particle with a recurrence time of seconds, $T = 1$, and advancing a $2\pi \delta$ clockwise polar angle in each new bouncing on the top of the conic clock. The first step is to set up the interval of angles in the opening of the cone, $[\phi_{\min}, \phi_{\max}]$, that can handle a trajectory verifying these conditions. Thus, the frequency relation must verify:

$$\frac{\omega_\theta}{\omega_r} = n + \delta, \quad (15)$$

where $n$ is the number of complete angular rounds for each radial oscillation and $\delta$ is the $2\pi$-clockwise ratio shift between two successive particle marks on the top surface of the conic clock. From Eq. (11) we find that the extreme angles are given by the relation

$$k \sin \phi_0 = 1/(n + \delta), \quad (16)$$

when $k = \sqrt{3}$ and $k = 2$. The result is

$$\sin \phi_{\max} = \frac{1}{\sqrt{3}(n + \delta)}, \quad (17)$$
\[ \sin \phi_{\min} = \frac{1}{2(n + \delta)}. \] 

(18)

Thus, we see now that there exists a unique orbit for each \( \phi_0 \in [\phi_{\min}, \phi_{\max}] \), when \( T, n \) and \( \delta \) are fixed. For this \( \phi_0 \) there is a \( k_0 \) satisfying the relation (16). Because \( k \) is a monotone increasing function of the non-dimensional energy parameter \( \tilde{E} = \frac{3E}{2E_0} \), there exists a concrete value of \( \tilde{E}_0 \) which determines the trajectory in the non-dimensional Eqs. (12-13) or, equivalently, by undoing the change of variables, the real motion in the Eqs. (4-5). The function \( k(\tilde{E}) \) can be calculated by computational methods as follows.

Firstly, we observe that the energy conservation condition, in the non-dimensional coordinates, becomes

\[ \frac{\tilde{r}^2}{2} + \tilde{V}(\tilde{r}) = \tilde{E}, \] 

(19)

with \( \tilde{V}(\tilde{r}) = \tilde{r} + \frac{1}{2\tilde{r}^2} \). Hence, in this representation, the dynamics settles in the circular orbit when \( \tilde{E} = 3/2 \), and, small or large oscillations are obtained when the value of the normalized energy \( \tilde{E} \) runs on the interval \( \frac{3}{2} < \tilde{E} < \infty \). For an arbitrary energy \( \tilde{E} \), the extreme values, \( \tilde{r}_{\min} \) and \( \tilde{r}_{\max} \), of the radial oscillation are deduced from the equality \( \tilde{V}(\tilde{r}_{\min}) = \tilde{V}(\tilde{r}_{\max}) = \tilde{E} \). Then the functions \( \tilde{r}_{\min}(\tilde{E}) \) and \( \tilde{r}_{\max}(\tilde{E}) \) are obtained. The angle \( \tilde{\theta} \) covered by the particle during a radial semi-period is given by

\[ \tilde{\theta}(\tilde{E}) = \int_{\tilde{r}_{\min}(\tilde{E})}^{\tilde{r}_{\max}(\tilde{E})} \frac{d\tilde{r}}{\tilde{r} \sqrt{2\tilde{E}\tilde{r}^2 - 2\tilde{r}^3 - 1}}, \] 

(20)

and \( k(\tilde{E}) \) is

\[ k(\tilde{E}) = \frac{\pi}{\tilde{\theta}(\tilde{E})}. \] 

(21)

Let us recall that \( \sqrt{3} < k(\tilde{E}) < 2 \), for any trajectory of arbitrary energy, \( \frac{3}{2} < \tilde{E} < \infty \). In order to find the computational solution of the equation \( k(\tilde{E}) = k_0 \), we can start an iteration process using the initial values: \( k = \sqrt{3} \) and \( \tilde{E} = 3/2 \). By increasing \( \tilde{E} \) in a fixed small quantity \( \Delta \tilde{E} \), we calculate for each step a new \( k \) through Eqs. (20-21). When the condition \( k > k_0 \) is reached, the iteration process stops and this value of \( \tilde{E} \) is retained as the solution for \( \tilde{E}_0 \). If more precision is required then it is sufficient to take a smaller \( \Delta \tilde{E} \).
Secondly, we proceed to find the period of the radial oscillation, $\tilde{T}_r$, for the particular motion given by $\tilde{E} = \tilde{E}_0$. This is obtained by integrating the expression:

$$\tilde{T}_r(\tilde{E}_0) = 2 \int_{\tilde{r}_{\text{min}}(\tilde{E}_0)}^{\tilde{r}_{\text{max}}(\tilde{E}_0)} \frac{\tilde{r} \, d\tilde{r}}{\sqrt{2\tilde{E}_0 \tilde{r}^2 - 2\tilde{r}^3 - 1}}. \quad (22)$$

Now the change from the non-dimensional to the dimensional variables is performed. Undoing the time change we obtain

$$r_0 = g \cos \phi_0 \left( \frac{T}{\tilde{T}_r(\tilde{E}_0)} \right)^2 \quad (23)$$

where $T = 1$ second in our concrete case. From this value the radial and height lengths of the trajectory are obtained:

$$r_{\text{min}} = \tilde{r}_{\text{min}}(\tilde{E}_0) \cdot r_0, \quad (24)$$
$$r_{\text{max}} = \tilde{r}_{\text{max}}(\tilde{E}_0) \cdot r_0, \quad (25)$$
$$H = (r_{\text{max}} - r_{\text{min}}) \cdot \cos \phi_0, \quad (26)$$

with $H = H_{\text{max}} - H_{\text{min}}$ the height of the trunk of the cone where the dynamics develops. The length, $L$, that the particle covers on the conical surface, in a radial semi-oscillation, is

$$L(\tilde{E}_0) = r_0 \cdot \int_{\tilde{r}_{\text{min}}(\tilde{E}_0)}^{\tilde{r}_{\text{max}}(\tilde{E}_0)} \sqrt{\frac{2\tilde{E}_0 \tilde{r}^2 - 2\tilde{r}^3}{2\tilde{E}_0 \tilde{r}^2 - 2\tilde{r}^3 - 1}} \, d\tilde{r}. \quad (27)$$

If we take the $H$ and $L$ dependence on $r_0$, then, as we have advanced in the Introduction, it is found that

$$H \sim cte_H(\phi_0) \cdot g T^2, \quad (28)$$
$$L \sim cte_L(\phi_0) \cdot g T^2, \quad (29)$$

where $cte_H$ and $cte_L$ depend only on the opening angle of the cone.

Finally, the velocity of release at the highest point of the trajectory, $v_\theta$, can be obtained from the relation

$$v_\theta = \sqrt{\frac{2\gamma g H}{1 - \gamma}}, \quad (30)$$
with \( \gamma = \left( \frac{r_{\text{min}}}{r_{\text{max}}} \right)^2 \).

The results of this method, for different values of \( n \) and \( \delta \), are collected in Tables 1 – 3. These calculations show that the fact of clocking seconds, \( T = 1 \), in the gravitational field imposes strong restrictions in the dimensions of the conic clock. If the mean value of \( L \) is obtained in each table, we observe that an approximate constant value of \( c_L = L/(gT^2) \) appears in the three tables. In Table 1, \( c_L \sim 0.109 \), in Table 2, \( c_L \sim 0.108 \), and in Table 3, \( c_L \sim 0.124 \). These non-dimensional constants are similar to those commented on in Section I. In the case of the particle bouncing on the ground, \( c_1 = 1/8 = 0.125 \), and that of the simple pendulum, \( c_2 = 1/(4\pi^2) = 0.025 \).

4 Conclusions

It is known that the calculation of the recurrence time in a bounded Hamiltonian system is, in general, not an easy task. In fact, the consequence of the difficulty for its computability gave rise to the well-known polemics on the "recurrence paradox" at the end of the nineteenth century [5, 6]. This discussion was concluded by Boltzmann who finally admitted that recurrences are completely consistent with the statistical viewpoint and there is no contradiction with the second law of thermodynamics. Thus recurrences can be interpreted as fluctuations, which are almost certain to occur if a long enough time is waited.

In this work, the orbits with a recurrence time of one second have been obtained for the frictionless dynamics of a particle on an inverted cone. Depending on the angular and radial frequency ratio, an interval of opening angles of the cone is available for this purpose. For each one of these angles, only a trajectory embodies the possibility of registering seconds. The method to calculate these trajectories has been established. We have also calculated the characteristic lengths of the conical surface for different values of the frequency ratio and for different opening angles.
It is worth remarking the strong restrictions that the gravitational field imposes on the possible dimensions of the hypothetical device registering seconds: if $L$ is the length of the conic clock, $T$ is the time to be registered and $g$ is gravity, then the non-dimensional quantity $c_L = L/(gT^2)$ varies only a little for each interval of possible angles. The value of this constant indicates that the dimensions of a conic clock are of the same order as the dimensions of a clock-device based on the dynamics of a particle bouncing on the ground or, more easily, based on the classical simple pendulum. The advantage of the conic clock with respect to the other cases is that the particle does not bounces on the same point in each new recurrence. Therefore, it is not necessary to insert here an additional mechanism to obtain the visual impression of the passing of time.
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### TABLE 1

| $\phi_0$ | $E$  | $H_{\text{min}}$ | $H_{\text{max}}$ | $H$  | $L$  | $v_\theta$ |
|---------|------|------------------|------------------|------|------|------------|
| 29.5    | 108.2| 0.0006           | 0.928            | 0.927| 1.066| 0.003      |
| 30      | 15.5 | 0.011            | 0.919            | 0.909| 1.062| 0.049      |
| 31      | 6    | 0.044            | 0.894            | 0.849| 1.054| 0.202      |
| 32      | 3.6  | 0.092            | 0.859            | 0.767| 1.055| 0.419      |
| 33      | 2.5  | 0.165            | 0.805            | 0.639| 1.065| 0.744      |
| 34      | 1.8  | 0.287            | 0.706            | 0.419| 1.089| 1.276      |
| 34.5    | 1.55 | 0.412            | 0.594            | 0.181| 1.107| 1.820      |

Table 1: $T = 1$, $n = 1$ and $\delta = 1/60$, then $\phi_{\text{min}} = 29.46^\circ$ and $\phi_{\text{max}} = 34.60^\circ$. The lengths are in metres and the velocity in metres/second.

### TABLE 2

| $\phi_0$ | $E$  | $H_{\text{min}}$ | $H_{\text{max}}$ | $H$  | $L$  | $v_\theta$ |
|---------|------|------------------|------------------|------|------|------------|
| 30.2    | 34.6 | 0.003            | 0.916            | 0.912| 1.058| 0.015      |
| 30.6    | 14.2 | 0.012            | 0.908            | 0.896| 1.056| 0.056      |
| 31      | 9.1  | 0.023            | 0.898            | 0.874| 1.051| 0.108      |
| 32      | 4.8  | 0.061            | 0.870            | 0.809| 1.048| 0.281      |
| 33      | 3.1  | 0.117            | 0.829            | 0.713| 1.051| 0.531      |
| 34      | 2.2  | 0.200            | 0.764            | 0.564| 1.065| 0.902      |
| 35      | 1.6  | 0.358            | 0.629            | 0.271| 1.093| 1.595      |
| 35.2    | 1.53 | 0.426            | 0.565            | 0.139| 1.100| 1.894      |

Table 2: $T = 1$, $n = 1$ and $\delta = 1/3600$, then $\phi_{\text{min}} = 30^\circ$ and $\phi_{\text{max}} = 35.3^\circ$. The lengths are in metres and the velocity in metres/second.
Table 3: $T = 1$, $n = 2$ and $\delta = 1/60$, then $\phi_{\text{min}} = 14.35^\circ$ and $\phi_{\text{max}} = 16.63^\circ$. The lengths are in metres and the velocity in metres/second.

| $\phi_0$ | $E$ | $H_{\text{min}}$ | $H_{\text{max}}$ | $H$ | $L$ | $v_\theta$ |
|----------|-----|------------------|------------------|-----|-----|-----------|
| 14.4     | 61.7| 0.0017           | 1.150            | 1.148| 1.186| 0.007     |
| 14.8     | 9.2 | 0.029            | 1.142            | 1.113| 1.186| 0.120     |
| 15.2     | 5   | 0.074            | 1.128            | 1.054| 1.190| 0.299     |
| 15.6     | 3.4 | 0.134            | 1.102            | 0.968| 1.202| 0.535     |
| 16       | 2.3 | 0.236            | 1.047            | 0.811| 1.226| 0.921     |
| 16.5     | 1.66| 0.462            | 0.885            | 0.423| 1.275| 1.763     |