THE GRADED STRUCTURE OF ALGEBRAIC CUNTZ-PIMSNER RINGS

DANIEL LÄNNSTRÖM

Abstract. The algebraic Cuntz-Pimsner rings are naturally \(\mathbb{Z}\)-graded rings that generalize both Leavitt path algebras and unperforated \(\mathbb{Z}\)-graded Steinberg algebras. We classify strongly, epsilon-strongly and nearly epsilon-strongly graded algebraic Cuntz-Pimsner rings up to graded isomorphism. As an application, we characterize noetherian and artinian fractional skew monoid rings by a single corner automorphism.

1. Introduction

The Cuntz-Pimsner \(C^*\)-algebras were first introduced by Pimsner in \cite{12} and further studied by Katsura in \cite{7}. These \(C^*\)-algebras generalize the famous Cuntz-Krieger \(C^*\)-algebras. The Cuntz-Pimsner algebra \(O_X\) is constructed from a \(C^*\)-correspondence and comes equipped with a natural gauge action. In a recent paper, Chirvasitu \cite{7} obtained necessary and sufficient conditions for the gauge action to be free. The (algebraic) Cuntz-Pimsner rings were introduced by Carlsen and Ortega in \cite{5} as algebraic analogues of the Cuntz-Pimsner algebras. Simplicity of Cuntz-Pimsner rings were further studied in \cite{6}. These rings are interesting to us since they generalize some very famous families of rings. Indeed, Carlsen and Ortega originally gave two important examples of rings realizable as Cuntz-Pimsner rings: Leavitt path algebras (see \cite{5}, Expl. 5.8 and Section 2.4) and corner skew Laurent polynomial rings (see \cite{5}, Expl. 5.7 and Section 2.5). Recently, Clark, Fletcher, Hazrat and Li \cite{19} showed that unperforated \(\mathbb{Z}\)-graded Steinberg algebras are also realizable as Cuntz-Pimsner rings. The Cuntz-Pimsner rings do not come with a gauge action but instead a natural \(\mathbb{Z}\)-grading. This grading is the main object of study in this article.

In the case of Leavitt path algebras, the natural \(\mathbb{Z}\)-grading was systematically investigated by Hazrat \cite{10}. In particular, he obtained necessary and sufficient conditions for the Leavitt path algebra of a finite graph to be strongly graded (see \cite{10}, Thm. 3.15). The class of epsilon-strongly graded rings was first introduced by Nystedt, Öinert and Pinedo in \cite{17} as a generalization of unital strongly graded rings. This subclass of graded rings has been investigated further by the author in \cite{13, 14}. Interestingly, the Leavitt path algebra of a finite graph was proved to be epsilon-strong by Nystedt and Öinert (see \cite{18}, Thm. 24). Seeking to extend their result, they introduced the notion of a nearly epsilon-strongly graded ring (see Definition 2.2) and proved that every Leavitt path algebra (even for infinite graphs) is nearly epsilon-strongly \(\mathbb{Z}\)-graded (see \cite{18}, Thm. 24, Thm. 28). In other words, there are sufficient conditions in the literature for the natural \(\mathbb{Z}\)-grading of a Leavitt path algebra to be strong, epsilon-strong and nearly epsilon-strong respectively. These types of gradings

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have certain structural properties that help us understand the Leavitt path algebras. The present work began as an effort to generalize the previously mentioned results about Leavitt path algebras to a larger class of Cuntz-Pimsner rings. It turns out that we can obtain partial classifications of nearly epsilon-strongly and epsilon-strongly graded Cuntz-Pimsner rings (see Theorem 6.1 and Theorem 6.2). For unital strongly graded Cuntz-Pimsner rings we obtain a complete classification up to graded isomorphism (see Theorem 6.3). We also obtain sufficient conditions for a Cuntz-Pimsner ring to be strongly graded (see Corollary 4.10). As special cases, we recover Hazrat’s result on Leavitt path algebras (see Corollary 4.12) and corner skew Laurent polynomial ring (see Corollary 4.13).

Carlsen and Ortega [5] constructed the Cuntz-Pimsner rings using a categorical approach. Let $R$ be an associative but not necessarily unital ring. Recall (see [5, Def. 1.1]) that an $R$-system is a triple $(P, Q, \psi)$ where $P$ and $Q$ are $R$-bimodules and $\psi : P \otimes_R Q \to R$ is an $R$-bimodule homomorphism where $P \otimes_R Q$ denotes the balanced tensor product. A technical assumption called Condition (FS) (see Definition 2.8) is generally imposed on the $R$-system $(P, Q, \psi)$. We will introduce two special types of $R$-systems called $s$-unital and unital $R$-systems (see Definition 3.6). Given an $R$-system, Carlsen and Ortega considered representations of that system. This is the key definition in their construction:

**Definition 1.1.** ([5, Def. 1.2, Def. 3.3]) Let $R$ be a ring and let $(P, Q, \psi)$ be an $R$-system. A covariant representation is a tuple $(S, T, \sigma, B)$ such that the following assertions hold:

(a) $B$ is a ring;
(b) $S : P \to B$ and $T : Q \to B$ are additive maps;
(c) $\sigma : R \to B$ is a ring homomorphism;
(d) $S(pr) = S(p)\sigma(r), S(rp) = \sigma(r)S(p), T(qr) = T(q)\sigma(r), T(rq) = \sigma(r)T(q)$ for every $r \in R$, $q \in Q$ and $p \in P$;
(e) $\sigma(\psi(p \otimes q)) = S(p)T(q)$ for all $p \in P$ and $q \in Q$.

The covariant representation $(S, T, \sigma, B)$ is injective if the map $\sigma$ is injective. The covariant representation $(S, T, \sigma, B)$ is surjective if $B$ is generated by $\sigma(R) \cup S(P) \cup T(Q)$.

A surjective covariant representation $(S, T, \sigma, B)$ is called graded if there is a $\mathbb{Z}$-grading $\{B_i\}_{i \in \mathbb{Z}}$ of $B$ such that $\sigma(R) \subseteq B_0$, $T(Q) \subseteq B_1$ and $S(P) \subseteq B_{-1}$.

**Remark 1.2.** Let $(S, T, \sigma, B)$ be a covariant representation and assume that $B$ is $\mathbb{Z}$-graded. Note that $(S, T, \sigma, B)$ is a graded covariant representation if and only if the grading of $B$ is compatible with the representation structure.

Carlsen and Ortega [5] then considered the category of surjective covariant representations of $(P, Q, \psi)$ denoted by $\mathcal{C}_{(P,Q,\psi)}$. The maps between $(S, T, \sigma, B)$ and $(S', T', \sigma', B')$ are ring homomorphisms $\phi : B \to B'$ such that $\phi \circ S = S'$, $\phi \circ T = T'$ and $\phi \circ \sigma = \sigma'$. We write $(S, T, \sigma, B) \cong (S', T', \sigma', B')$ if the covariant representations are isomorphic as objects in $\mathcal{C}_{(P,Q,\psi)}$. In the case when $(P, Q, \psi)$ satisfies Condition (FS) (see Definition 2.8), they obtained a complete classification of injective, graded, surjective covariant representations up to isomorphism in $\mathcal{C}_{(P,Q,\psi)}$ (see [5, Sect. 7]). The Cuntz-Pimsner rings are defined as certain covariant representations (see Definition 2.12). Unlike in the $C^*$-setting, the Cuntz-Pimsner ring is not well-defined for all $R$-systems $(P, Q, \psi)$ (see [5, Expl. 4.11]).

Now, we let both $R$ and $(P, Q, \psi)$ vary. If a $\mathbb{Z}$-graded ring $B$ shows up in a graded covariant representation $(S, T, \sigma, B)$ of some $R$-system $(P, Q, \psi)$, then we call $B$ a representation.
ring. Following Clark, Fletcher, Hazrat and Li [19], we then say that $B$ is realized by the representation $(S, T, \sigma, B)$ of the $R$-system $(P, Q, \psi)$.

The key new technique of this article is to consider a special type of graded covariant representations:

**Definition 1.3.** Let $R$ be ring, let $(P, Q, \psi)$ be an $R$-system and let $(S, T, \sigma, B)$ be a graded covariant representation of $(P, Q, \psi)$. For $k \geq 0$ define,

$$I_{\psi, \sigma}^{(k)} = \text{Span}\{\sigma(\psi_k(p \otimes q)) \mid p \in P^\otimes k, q \in Q^\otimes k\}.$$

We call $(S, T, \sigma, B)$ a semi-full covariant representation if $B_{-k}B_k = I_{\psi, \sigma}^{(k)}$ for each $k \geq 0$.

**Remark 1.4.** A $C^*$-correspondence $(A, E, \phi)$ is called full if the closure of $\langle x, y \rangle$ for $x, y \in E$ spans $A$. One way to generalize this to the algebraic setting is to require that $\psi$ be surjective. Semi-fullness is a weaker condition. Indeed, if $R$ is unital and $\psi$ is surjective, then every graded covariant representation of $(P, Q, \psi)$ is semi-full.

Below is an outline of the rest of this article:

In Section 2, we recall the definitions of nearly epsilon-strongly graded rings and algebraic Cuntz-Pimsner rings.

In Section 3, we prove that certain nearly epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings can be realized from semi-full covariant representations (see Corollary 3.13). This is based on recent work by Clark, Fletcher, Hazrat and Li [19] and is the crucial reduction step in the classification.

In Section 4, we find sufficient conditions for an injective and graded covariant representation to be strongly $\mathbb{Z}$-graded (see Proposition 4.8).

In Section 5, we obtain sufficient conditions for an injective and semi-full covariant representation ring to be nearly epsilon-strongly $\mathbb{Z}$-graded (see Proposition 5.7). In particular, this provides sufficient conditions for a semi-full Cuntz-Pimsner representation to be epsilon-strongly $\mathbb{Z}$-graded.

In Section 6, we obtain a partial classification of epsilon-strongly graded and nearly epsilon-strongly graded Cuntz-Pimsner rings (see Theorem 6.2 and Theorem 6.1). For unital strongly graded Cuntz-Pimsner rings we obtain a complete classification (see Theorem 6.3).

In Section 7, we collect some important examples. Notably, we give an example of a Leavitt path algebra realizable as a Cuntz-Pimsner ring in two different ways (see Example 7.3). We also give an example of a trivial Cuntz-Pimsner ring that is not nearly epsilon-strongly $\mathbb{Z}$-graded (see Example 7.1).

In Section 8, we apply our classification results to characterize noetherian and artinian corner skew Laurent polynomials (see Corollary 8.3).

2. Preliminaries

All rings are assumed to be associative but not necessarily equipped with a multiplicative identity element.

2.1. Nearly epsilon-strongly graded rings. Recall that a ring $S$ is called $\mathbb{Z}$-graded if there exists a family of additive subsets $\{S_i\}_{i \in \mathbb{Z}}$ of $S$ such that $S = \bigoplus_{i \in \mathbb{Z}} S_i$ and $S_mS_n \subseteq S_{m+n}$ for all $m, n \in \mathbb{Z}$. If the stronger condition $S_mS_n = S_{m+n}$ holds for all $m, n \in \mathbb{Z}$, then the $\mathbb{Z}$-grading $\{S_i\}_{i \in \mathbb{Z}}$ is called strong. The subsets $S_i$ are called the homogeneous components of $S$. The component $S_0$ is called the principal component of $S$. It is straightforward to show
that $S_0$ is a subring of $S$. Next, let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ and $T = \bigoplus_{i \in \mathbb{Z}} T_i$ be two $\mathbb{Z}$-graded rings. A ring homomorphism $\phi: S \to T$ is called graded if $\phi(S_i) \subseteq T_i$ for each $i \in \mathbb{Z}$. If $\phi: S \cong T$ is a graded ring isomorphism, then we write $S \cong_{gr} T$ and say that $S$ and $T$ are graded isomorphic.

Let $R$ be a ring. Recall that a left (right) $R$-module $RM$ is called left (right) $s$-unital if for every $x \in M$ there exists some $r_x \in R$ such that $r_x \cdot x = x$ ($x \cdot r_x = x$). A left (right) $R$-module $RM$ is called left (right) $s$-unital if there exists some $r \in R$ such that $r \cdot x = x$ ($x \cdot r = x$) for every $x \in M$. Let $R, S$ be rings. A bimodule $RM_S$ is called $s$-unital if $RM$ is left $s$-unital (unital) and $MS$ is right $s$-unital (unital). A ring $R$ is an $s$-unital ring if and only if $R \bigcirc R$ is an $s$-unital $R$-bimodule.

**Remark 2.1.** Let $R$ be a ring. It follows from [18, Thm. 1] that if $M$ is a left (right) $s$-unital $R$-module, then for any positive integer $n$ and elements $x_1, x_2, \ldots, x_n \in M$ there exists some $r \in R$ such that $r \cdot x_i = x_i$ ($x_i \cdot r = x_i$) for all $i \in \{1, \ldots, n\}$.

If $S$ is a $\mathbb{Z}$-graded ring, then $S_i$ is an $S_0$-bimodule for every $i \in \mathbb{Z}$ (see e.g. [15, Rmk. 1.1.2]). Note that $S_i S_{-i}$ is a subring of $S_0$ for every $i \in \mathbb{Z}$. Hence, in particular, $S_i$ is an $S_{i-1} S_{i+1}$-bimodule for each $i \in \mathbb{Z}$. The following definitions was introduced by Nystedt and Öinert:

**Definition 2.2.** ([18, Def. 6, Def. 9]) Let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ be a $\mathbb{Z}$-graded ring.

(a) If $S_i$ is an $s$-unital $S_{i-1} S_{i+1}$-bimodule for each $i \in \mathbb{Z}$, then $S$ is called nearly epsilon-strongly $\mathbb{Z}$-graded.

(b) If $S_i$ is a unital $S_{i-1} S_{i+1}$-bimodule for each $i \in \mathbb{Z}$, then $S$ is called epsilon-strongly $\mathbb{Z}$-graded.

(c) ([17, Def. 6], [8, Def. 4.5]) If $S_i = S_i S_{-i}$ for every $i \in \mathbb{Z}$, then $S$ is called symmetrically $\mathbb{Z}$-graded.

**Remark 2.3.** We make two remarks concerning Definition 2.2.

(a) Nystedt and Öinert made these definitions for general group graded rings graded by an arbitrary group. However, in this article we will only consider the special case of $\mathbb{Z}$-graded rings.

(b) If $S$ is epsilon-strongly $\mathbb{Z}$-graded, then $S$ is a unital ring (see [14, Prop. 3.8]). In other words, only unital rings admit an epsilon-strong grading.

We recall the following characterization of nearly and epsilon-strongly graded rings.

**Proposition 2.4.** ([18, Prop. 7, Prop. 10]) Let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ be a $\mathbb{Z}$-graded ring. The following assertions hold:

(a) $S$ is nearly epsilon-strongly graded if and only if $S$ is symmetrically $\mathbb{Z}$-graded and $S_i S_{-i}$ is an $s$-unital ideal for each $i \in \mathbb{Z}$;

(b) $S$ is epsilon-strongly graded if and only if $S$ is symmetrically $\mathbb{Z}$-graded and $S_i S_{-i}$ is a unital ideal for each $i \in \mathbb{Z}$.

The following implications hold (see [14, Rem. 3.4(a)]):

unital strongly graded $\Rightarrow$ epsilon strongly graded $\Rightarrow$ nearly epsilon-strongly graded.
2.2. The Toeplitz representation. Let \((P, Q, \psi)\) be an \(R\)-system. Put \(P^\otimes 0 = Q^\otimes 0 = R\) and \(\psi_0(r_1 \otimes r_2) = r_1 r_2\). Let \(\psi = \psi_1\). For \(n > 1\), recursively define \(Q^\otimes n = Q^\otimes n-1 \otimes Q\) and \(P^\otimes n = P \otimes P^\otimes n-1\). Let \(\psi_n : P^\otimes n \otimes Q^\otimes n \to R\) be defined by,

\[
\psi_n((p_1 \otimes p_2) \otimes (q_2 \otimes q_1)) = \psi(p_1 \cdot \psi_{n-1}(p_2 \otimes q_2), q_1),
\]

for \(p_1 \in P, p_2 \in P^\otimes n-1, q_1 \in Q\), and \(q_2 \in Q^\otimes n-1\). Then, \((P^\otimes n, Q^\otimes n, \psi_n)\) is an \(R\)-system for each \(n \geq 0\). Furthermore, by [5, Lem. 1.5], if \((S, T, \sigma, B)\) is a covariant representation of \((P, Q, \psi)\), then \((S^n, T^n, \sigma, B)\) is a covariant representation of \((P^\otimes n, Q^\otimes n, \psi_n)\) where \(S^n : P^\otimes n \to B\) and \(T^n : Q^\otimes n \to B\) are maps satisfying the equations \(S^n(p_1 \otimes \cdots \otimes p_n) = S(p_1)S(p_2) \cdots S(p_n)\) and \(T^n(q_1 \otimes \cdots \otimes q_n) = T(q_1)T(q_2) \cdots T(q_n)\) for \(q_i \in Q\) and \(p_j \in P\).

Carlsen and Ortega proved (see [5, Thm. 1.7]) that there is an injective, surjective and graded covariant representation that satisfies a certain universal property. This covariant representation is called the Toeplitz representation and is denoted by \((\iota_Q, \iota_P, \iota_R, (T_{(P,Q,\psi)})\)). The ring \(T_{(P,Q,\psi)}\) is called the Toeplitz ring.

Theorem 2.5. ([5, Thm. 1.7, Prop. 3.2]) Let \(R\) be a ring and let \((P, Q, \psi)\) be an \(R\)-system. Let \(T_{(P,Q,\psi)} = \bigoplus_{i \in \mathbb{Z}} T_i\) be the Toeplitz ring associated with \((P, Q, \psi)\) and let \((S, T, \sigma, B)\) be any graded covariant representation of \((P, Q, \psi)\). Then there is a unique \(
\mathbb{Z}\)-graded ring epimorphism \(\eta : T_{(P,Q,\psi)} \to B\) such that \(\eta \circ \iota_R = \sigma, \eta \circ \iota_Q = T, \) and \(\eta \circ \iota_P = S\).

Recall (see [5, Thm. 1.7, Prop. 3.1]) that the canonical \(
\mathbb{Z}\)-grading of the Toeplitz ring is given by first defining,

\[
T_{(m,n)} = \text{Span}\{\iota_Q^\otimes m(q)\iota_P^\otimes n(p) \mid q \in Q^\otimes m, p \in P^\otimes n\},
\]

for a pair of non-negative integers \((m, n)\) and then putting,

\[
T_i = \bigoplus_{m-n=i} T_{(m,n)}, \tag{1}
\]

for each \(i \in \mathbb{Z}\). In other words, \(T_i\) is the span of monomials of the form \(\iota_Q^\otimes m(q)\iota_P^\otimes n(p)\) where (i) \(m, n\) are non-negative integers such that \(m - n = i\) and (ii) \(q \in Q^\otimes m\) and \(p \in P^\otimes n\).

We relate morphisms in the category of graded covariant representations to morphisms in the category of \(\mathbb{Z}\)-graded rings:

Lemma 2.6. Let \(R\) be a ring and let \((P, Q, \psi)\) be an \(R\)-system. Suppose that \((S, T, \sigma, B)\) and \((S', T', \sigma', B')\) are two graded covariant representations of \((P, Q, \psi)\). If

\[
\phi : (S, T, \sigma, B) \to (S', T', \sigma', B')
\]

is a morphism in the category \(\mathcal{C}_{(P,Q,\psi)}\) (see the introduction), then \(\phi : B \to B'\) is a \(\mathbb{Z}\)-graded ring homomorphism.

Proof. Note that by Theorem 2.5 it follows that,

\[
B_i = \text{Span}\{T(q)S(p) \mid q \in Q^\otimes m, p \in P^\otimes n, \text{ where } m-n = i\},
\]

\[
B'_i = \text{Span}\{T'(q)S'(p) \mid q \in Q^\otimes m, p \in P^\otimes n \text{ where } m-n = i\},
\]

for every \(i \in \mathbb{Z}\). Since \(\phi \circ T = T'\) and \(\phi \circ S = S'\) it follows that \(\phi(B_i) \subseteq B'_i\). Thus, \(\phi\) is a \(\mathbb{Z}\)-graded ring homomorphism. \(\square\)

The following corollary is straightforward to prove:
Corollary 2.7. Let $R$ be a ring and let $(P,Q,\psi)$ be an $R$-system. Suppose that $(S,T,\sigma,B)\cong_r (S',T',\sigma',B')$ are two isomorphic graded covariant representations of $(P,Q,\psi)$. Then, in particular, $B\cong_{gr} B'$.

2.3. Adjointable operators, Condition (FS) and Cuntz-Pimsner representations.

Recall from the $C^*$-setting, that finite generation of the Hilbert module $E$ is equivalent to the ring of compact operators $B(E) = K(E)$ being unital. In the algebraic setting, the ring of compact operators $K(E)$ is replaced by $\mathcal{F}_P(Q)$ and $\mathcal{F}_Q(P)$ (see [5, Def. 2.1]). We now recall the definition of these rings. A right $R$-module homomorphism $T: Q_R \to Q_R$ is called adjointable if there exists a left $R$-module homomorphism $S: R_P \to R_P$ such that $\psi(p \otimes T(q)) = \psi(S(p) \otimes q)$ for all $q \in Q$ and $p \in P$. The set of adjointable homomorphisms is denoted by $\mathcal{L}_P(Q)$ and $\mathcal{L}_Q(P)$. Note that $\mathcal{L}_P(Q)$ and $\mathcal{L}_Q(P)$ are subrings of $\text{End}(Q_R)$ and $\text{End}(R_P)$ respectively. Given fixed elements $q \in Q$ and $p \in P$, define $\theta_{q,p}: Q_R \to Q_R$ and \( \theta_{p,q}: R_P \to R_P \) by $\theta_{q,p}(x) = q \cdot \psi(p \otimes x)$ and $\theta_{p,q}(y) = \psi(y \otimes q) \cdot p$ for $x \in Q$ and $y \in P$ respectively. The linear span of the homomorphisms $\{\theta_{q,p} \mid q \in Q, p \in P\}$ is denoted by $\mathcal{F}_P(Q)$. Similarly, the linear span of $\{\theta_{p,q} \mid q \in Q, p \in P\}$ is denoted by $\mathcal{F}_Q(P)$. It can be proved that $\mathcal{F}_P(Q)$ and $\mathcal{F}_Q(P)$ are two-sided ideals of $\mathcal{L}_P(Q)$ and $\mathcal{L}_Q(P)$ respectively (see [5, Lem. 2.3]).

The following technical condition was introduced by Carlsen and Ortega:

Definition 2.8. ([5, Def. 3.4]) Let $R$ be a ring. An $R$-system $(P,Q,\psi)$ is said to satisfy Condition (FS) if for all finite sets $\{q_1,q_2,\ldots,q_n\} \subseteq Q$ and $\{p_1,p_2,\ldots,p_m\} \subseteq P$ there exist $\Theta \in \mathcal{F}_P(Q)$ and $\Phi \in \mathcal{F}_Q(P)$ such that $\Theta(q_i) = q_i$ and $\Phi(p_j) = p_j$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$.

Note that we have the following inclusion of rings:

$$\mathcal{F}_P(Q) \subseteq \mathcal{L}_P(Q) \subseteq \text{End}(Q_R),$$
$$\mathcal{F}_Q(P) \subseteq \mathcal{L}_Q(P) \subseteq \text{End}(R_P).$$

Carlsen and Ortega (see [5, Def. 3.10]) defined maps $\Delta: R \to \mathcal{L}_P(Q)$ and $\Gamma: R \to \mathcal{L}_Q(P)$ by $\Delta(r)(q) = rq$ and $\Gamma(r)(p) = pr$ for all $r \in R, q \in Q, p \in P$.

In the $C^*$-setting, it turns out that there are always injective morphisms $\pi_n: K(E^\otimes n) \to T_E$ for each $n > 0$. In the algebraic setting, Carlsen and Ortega obtained something similar under the assumption that the system satisfies Condition (FS). Another way to put it is that if the $R$-system satisfies Condition (FS), then there are induced representations of $\mathcal{F}_P(Q)$ and $\mathcal{F}_Q(P)$. Recall that the opposite ring $R^{op}$ of a ring $R$ has the same additive structure but with a new product defined by $a \cdot b = ba$ for all $a,b \in R$.

Proposition 2.9. ([5 Prop. 3.11]) Let $R$ be a ring, let $(P,Q,\psi)$ be an $R$-system satisfying Condition (FS) and let $(S,T,\sigma,B)$ be a covariant representation of $(P,Q,\psi)$. Then there exist unique ring homomorphisms $\pi_{T,S}: \mathcal{F}_P(Q) \to B$ and $\chi_{T,S}: \mathcal{F}_Q(P) \to B^{op}$ such that $\pi_{T,S}(\theta_{q,p}) = T(q)S(p)$ and $\chi_{T,S}(\theta_{p,q}) = S(p)\ast T(q)$ for all $q \in Q, p \in P$. The maps satisfy the following equations for all $\Theta \in \mathcal{F}_P(Q)$ and $\Phi \in \mathcal{F}_Q(P)$:

$$\pi_{T,S}(\Delta(r)\Theta) = \sigma(r)\pi_{T,S}(\Theta), \quad \pi_{T,S}(\Theta \Delta(r)) = \pi_{T,S}(\Theta)\sigma(r)$$
$$\chi_{T,S}(\Gamma(r)\Phi) = \sigma(r) \ast \chi_{T,S}(\Phi), \quad \chi_{T,S}(\Phi \Gamma(r)) = \chi_{T,S}(\Phi) \ast \sigma(r)$$
$$\pi_{T,S}(\Theta)T(q) = T(\Theta(q)), \quad \chi_{T,S}(\Phi)S(p) = S(\Phi(p)).$$

Moreover, $\pi_{T,S}(\mathcal{F}_P(Q)) = \chi_{T,S}(\mathcal{F}_Q(P)) = \text{Span}\{T(q)S(p) \mid q \in Q, p \in P\} \subseteq B$. If $\sigma$ is injective, then the maps $\pi_{T,S}$ and $\chi_{T,S}$ are also injective.
Remark 2.10. We make some remarks concerning Proposition [2.9]
(a) The equation \( \chi_{T,S}(\Phi) \ast S(p) = S(p)\chi_{T,S}(\Phi) = S(\Phi(p)) \) is misprinted in [5 Prop. 3.11].
(b) Following Carlsen and Ortega, let \( \pi \) denote the map \( \bigcup_m \mathcal{F}_{P \otimes m}(Q^\otimes) \to \mathcal{T}_{(P,Q,\psi)} \).

We now recall the definition of the Cuntz-Pimsner invariant representations. If the \( R \)-system \((P,Q,\psi)\) satisfies Condition (FS), then the Cuntz-Pimsner invariant representations exhaust all injective, surjective graded covariant representations of \((P,Q,\psi)\) up to isomorphism in \( \mathcal{C}_{(P,Q,\psi)} \) (see [5 Rem. 3.30]).

Definition 2.11. ([5 Def. 3.15, Def. 3.16]) Let \( R \) be a ring and let \((P,Q,\psi)\) be an \( R \)-system satisfying Condition (FS). Let \( J \) be an ideal of \( R \). If \( J \subseteq \Delta^{-1}(\mathcal{F}_{P}(Q)) \), then the ideal \( J \) is called \( \psi \)-compatible. If \( \ker \Delta \cap J = \{0\} \), then \( J \) is called faithful. For a \( \psi \)-compatible ideal \( J \subseteq R \), let \( \mathcal{T}(J) \) be the ideal of \( \mathcal{T}_{(P,Q,\psi)} \) generated by \( \{\iota_R(x) - \pi(\Delta(x)) \mid x \in J\} \). The Cuntz-Pimsner ring relative to \( J \) is defined as the quotient ring \( \mathcal{O}_{(P,Q,\psi)} = \mathcal{T}_{(P,Q,\psi)}/\mathcal{T}(J) \). Let \( \rho: \mathcal{T}_{(P,Q,\psi)} \to \mathcal{O}_{(P,Q,\psi)} \) be the quotient map. Let \( \iota_Q^J, \iota_P^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J) \) be the Cuntz-Pimsner representation relative to \( J \).

A covariant representation \((S,T,\sigma,B)\) is called invariant relative to \( J \) if \( \pi_{T,S}(\Delta(x)) = \sigma(x) \) holds in \( B \) for each \( x \in J \). The relative Cuntz-Pimsner representation \((\iota_Q^J, \iota_P^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J))\) is invariant relative to \( J \) and satisfies a universal property among invariant representations (see [5 Thm. 3.18]).

Definition 2.12. ([5 Def. 5.1]) Let \( R \) be a ring and let \((P,Q,\psi)\) be an \( R \)-system. Suppose that there exists a unique maximal \( \psi \)-compatible, faithful ideal \( J \) of \( R \). The Cuntz-Pimsner ring is defined as \( \mathcal{O}_{(P,Q,\psi)}(J) = \mathcal{T}_{(P,Q,\psi)}/\mathcal{T}(J) \) and the Cuntz-Pimsner representation \((\iota_Q^\mathcal{CP}, \iota_P^\mathcal{CP}, \iota_R^\mathcal{CP}, \mathcal{O}_{(P,Q,\psi)}(J))\) is defined to be \((\iota_Q^J, \iota_P^J, \iota_R^J, \mathcal{O}_{(P,Q,\psi)}(J))\).

2.4. Leavitt path algebras. The Leavitt path algebra associated to a directed graph was first considered by Ara, Moreno and Pardo [4] and independently using a different approach by Abrams and Aranda Pino [2]. The Leavitt path algebra can be defined as a universal object satisfying certain relations. For a thorough account of the theory of Leavitt path algebras, we refer the reader to the monograph by Abrams, Ara, and Siles Molina [1]. We now recall the realization of Leavitt path algebras as Cuntz-Pimsner rings by Carlsen and Ortega (see [5 Exempl. 1.10, Exempl. 5.9]). Let \( E = (E^0, E^1, s, r) \) be a directed graph consisting of a vertex set \( E^0 \), an edge set \( E^1 \) and maps \( s: E^1 \to E^0 \) and \( r: E^1 \to E^0 \) specifying the source vertex \( s(f) \) and range vertex \( r(f) \) for each edge \( f \in E^1 \). Let \( K \) be a unital ring that will serve as the coefficient ring. For vertices \( u, v \in E^0 \) let \( \delta_{u,v} = 1 \) if \( u = v \) and \( \delta_{u,v} = 0 \) if \( u \neq v \). Let \( \{\eta_v \mid v \in E^0\} \) be a copy of the set \( E^0 \) and similarly let \( \{\eta_f \mid f \in E^1\} \) and \( \{\eta_{f^*} \mid f \in E^1\} \) be copies of the set \( E^1 \).

(a) Put \( R := \bigoplus_{v \in E^0} K \eta_v \). Define a multiplication on \( R \) by linearly extending the rules \( \eta_u \cdot \eta_v = \delta_{u,v} \eta_0 \) for all \( u, v \in E^0 \).

(b) Put \( Q := \bigoplus_{f \in E^1} K \eta_f \). Let \( R \) act on the left by linearly extending the rules \( \eta_v \cdot \eta_f = \delta_{v,s(f)} \eta_f \) for all \( v \in E^0, f \in E^1 \). Let \( R \) act on the right by linearly extending the rules \( \eta_f \cdot \eta_v = \delta_{v,r(f)} \eta_f \).

(c) Put \( P := \bigoplus_{f \in E^1} K \eta_{f^*} \). Let \( R \) act on the left by linearly extending the rules \( \eta_v \cdot \eta_{f^*} = \delta_{v,s(f)} \eta_{f^*} \) for all \( v \in E^0, f \in E^1 \). Let \( R \) act on the right by linearly extending the rules \( \eta_{f^*} \cdot \eta_v = \delta_{v,r(f)} \eta_{f^*} \) for all \( v \in E^0, f \in E^1 \).
(d) Define an \( R \)-bimodule homomorphism,
\[
\psi: P \otimes_R Q \to R,
\]
by linearly extending the rules \( \eta_f \otimes \eta_{f'} \mapsto \delta_{f,f'} \eta_{r(f)} \) for all \( f, f' \in E \).

We will refer to the above \( R \)-system \( (P, Q, \psi) \) as the standard Leavitt path system associated to the directed graph \( E \). Carlsen and Ortega proved (see [3] Expl. 5.8]) that \( (P, Q, \psi) \) satisfies Condition (FS), that the Cuntz-Pimsner ring is well-defined and that \( \mathcal{O}_{(P,Q,\psi)} \cong_{gr} L_K(E) \). The covariant representation \( (\iota_Q^C, \iota_P^C, \iota_R^C, \mathcal{O}_{(P,Q,\psi)}) \) is called the standard Leavitt path algebra covariant representation. Clark, Fletcher, Hazrat and Li also obtained this fact using more general methods (see [19] Expl. 3.6]).

2.5. Corner skew Laurent polynomial rings. The general construction of fractional skew monoid rings was introduced by Ara, Gonzalez-Barroso, Goodearl and Pardo in [3] as algebraic analogues of certain \( C^* \)-algebras introduced by Paschke [20]. Here, we consider the special case of a fractional skew monoid ring by a corner isomorphism which are also called corner skew Laurent polynomial rings. Let \( R \) be a unital ring and let \( \alpha: R \to eRe \) be a corner ring isomorphism where \( e \) is an idempotent of \( R \). The corner skew Laurent polynomial ring \( R[t_+, t_-; \alpha] \) is defined to be the universal unital ring satisfying the following conditions:

(a) There is a unital ring homomorphism \( i: R \to R[t_+, t_-; \alpha] \);

(b) \( R[t_+, t_-; \alpha] \) is the \( R \)-algebra satisfying the following equations for every \( r \in R \):
\[
t_-t_+ = 1, \quad t_+t_- = i(e), \quad i(r)t_- = t_-i(\alpha(r)), \quad t_+i(r) = i(\alpha(r))t_+.
\]

Moreover \( R[t_+, t_-; \alpha] \) is \( \mathbb{Z} \)-graded with \( A_0 = R \), \( A_i = Rt_i^+ \) for \( i < 0 \) and \( A_i = t_i^- R \) for \( i > 0 \). Note the \( t_- \in A_1 \) and \( t_+ \in A_{-1} \). Carlsen and Ortega [5] Expl. 5.7 proved that the corner skew Laurent polynomial ring \( R[t_+, t_-; \alpha] \) can be realized as a Cuntz-Pimsner ring.

3. Nearly epsilon-strongly \( \mathbb{Z} \)-graded rings as Cuntz-Pimsner rings

In this section, we will see that a recent result by Clark, Fletcher, Hazrat and Li [19] will allow us to derive necessary conditions for certain Cuntz-Pimsner rings to be nearly epsilon-strongly graded. Inspired by Exel we make the following definition:

**Definition 3.1.** (cf. [9] Def. 4.9) Let \( A = \bigoplus_{i \in \mathbb{Z}} A_i \) be a \( \mathbb{Z} \)-graded ring. If \( A_n = (A_1)^n \) and \( A_{-n} = (A_{-1})^n \) for \( n > 0 \), then \( A \) is called semi-saturated.

We show that the Toeplitz ring and any graded covariant representation is semi-saturated.

**Proposition 3.2.** Let \( R \) be an arbitrary ring and let \( (P, Q, \psi) \) be an \( R \)-system.

(a) The Toeplitz ring \( \mathcal{T}_{(P,Q,\psi)} = \bigoplus_{i \in \mathbb{Z}} \mathcal{T}_i \) is semi-saturated.

(b) Let \((S, T, \sigma, B)\) be any graded covariant representation of \((P, Q, \psi)\). Then \( B = \bigoplus_{i \in \mathbb{Z}} B_i \) is semi-saturated.

**Proof.** (a): Take an arbitrary integer \( t > 0 \). Clearly, \( (\mathcal{T}_1)^t \subseteq \mathcal{T}_t \). We prove the reverse inclusion. Let \( \iota_{Q \otimes m}(q)\iota_{P \otimes n}(p) \in \mathcal{T}_t \) where \( q \in Q^{\otimes m}, p \in P^{\otimes n} \) and \( m-n=t \). We need to show that \( \iota_{Q \otimes m}(q)\iota_{P \otimes n}(p) \in (\mathcal{T}_1)^t \). Suppose \( q = f_1 \otimes f_2 \otimes \cdots \otimes f_{n+1} \) and \( p = g_1 \otimes g_2 \otimes \cdots \otimes g_{m+1} \). Then,
\[
\iota_{Q^{\otimes m}}(q)\iota_{P^{\otimes n}}(p) = \iota_Q(f_1)\iota_Q(f_2) \cdots \iota_Q(f_{t-1})\iota_Q(f_{t+1} \otimes f_{t+1} \otimes \cdots \otimes f_{n+1})\iota_{P^{\otimes n}}(p),
\]
is contained in \( (\mathcal{T}_1)^t \). Hence, \( \mathcal{T}_t = (\mathcal{T}_1)^t \) for \( t > 0 \). We similarly prove that \( \mathcal{T}_{-t} = (\mathcal{T}_{-1})^t \) for \( t > 0 \).
(b): By Theorem 2.5, there is a $\mathbb{Z}$-graded ring epimorphism $\eta: \mathcal{T}_{(P,Q,\psi)} \rightarrow B$. Hence, $B_n = \eta(T_n) = \eta(T_1)^n = (B_1)^n$ for any $n > 0$. Similarly, $B_{-n} = (B_{-1})^n$ for any $n > 0$.

Recall that if $M$ is a left $R$-module, then the left annihilator is an ideal of $R$ defined by $\text{Ann}_R(M) = \{r \in R \mid r \cdot m = 0 \quad \forall m \in M\}$. Moreover, if $J$ is an ideal of $R$, then let $J^\perp = \{r \in R \mid rx = xr = 0 \quad \forall x \in J\}$. The following result was recently obtained by Clark, Fletcher, Hazrat and Li. Their formulation of the theorem is weaker but they in fact prove the stronger statement below.

**Theorem 3.3.** ([19] Cor. 3.2) Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a $\mathbb{Z}$-graded ring satisfying the following assertions:

(a) $A$ is semi-saturated;
(b) For $\{a_1, a_2, \ldots, a_n\} \subseteq A_1$ there is $r \in A_1A_{-1}$ such that $ra_l = a_l$ for each $1 \leq l \leq n$, and for $\{b_1, b_2, \ldots, b_m\} \subseteq A_{-1}$ there is $s \in A_1A_{-1}$ such that $bs_l = b_l$ for each $1 \leq l \leq m$;
(c) $\text{Ann}_{A_0}(A_1) \cap \text{Ann}_{A_0}(A_{-1}) = \{0\}$.

Let $\psi: A_{-1} \otimes A_0 \rightarrow A_0$ be defined by $\psi(a' \otimes a) = a'a$. Then the $A_0$-system $(A_{-1}, A_1, \psi)$ satisfies Condition (FS). Let $i_{A_{-1}}, i_{A_1}, i_{A_0}$ denote the natural inclusions maps and let $J = A_1A_{-1}$. Then $(i_{A_{-1}}, i_{A_1}, i_{A_0}, A)$ is a surjective covariant representation of $(A_{-1}, A_1, \psi)$ and,

$$(i_{A_{-1}}, i_{A_1}, i_{A_0}, A) \cong_r (i_{A_{-1}}^J, i_{A_1}^J, i_{A_0}^J, \mathcal{O}_{A_{-1}, A_1, \psi^r}(J)).$$

Furthermore, $J$ is faithfully maximal, hence,

$$(i_{A_{-1}}^J, i_{A_1}^J, i_{A_0}^J, \mathcal{O}_{A_{-1}, A_1, \psi^r}(J)) = (i_{A_{-1}}^{CP}, i_{A_1}^{CP}, i_{A_0}^{CP}, \mathcal{O}_{A_{-1}, A_1, \psi^r}).$$

In particular, we have that $A \cong_{gr} \mathcal{O}_{A_{-1}, A_1, \psi^r}$.

**Proof.** Note that $(A_{-1}, A_1, \psi)$ is an $A_0$-system and $(i_{A_1}, i_{A_{-1}}, i_{A_0}, A)$ is a surjective covariant representation since $A$ is generated by $A_{-1} \cup A_1 \cup A_0$. In the proof of [19] Thm. 3.1, they show that $(A_{-1}, A_1, \psi)$ satisfies Condition (FS) and that the ideal $J = A_1A_{-1}$ is the maximal faithful, $\psi$-compatible ideal of $A_0$. Hence, the Cuntz-Pimsner representation is well-defined and equal to $(i_{A_{-1}}^J, i_{A_1}^J, i_{A_0}^J, \mathcal{O}_{A_{-1}, A_1, \psi^r}(J))$. Moreover, they show that the graded representation $(i_{A_{-1}}, i_{A_1}, i_{A_0}, A)$ is Cuntz-Pimsner invariant with respect to $J$. By the universal property of relative Cuntz-Pimsner rings (see [5] Thm. 3.18), there exists a surjective map $\eta: (i_{A_{-1}}^{CP}, i_{A_1}^{CP}, i_{A_0}^{CP}, \mathcal{O}_{A_{-1}, A_1, \psi^r}) \rightarrow (i_{A_1}, i_{A_{-1}}, i_{A_0}, A)$. It follows by Lemma 2.4 that $\eta: \mathcal{O}_{A_{-1}, A_1, \psi^r} \rightarrow A$ is $\mathbb{Z}$-graded. By the graded uniqueness theorem for Cuntz-Pimsner rings (see [5] Cor. 5.4), it follows that $\eta$ is also injective. Thus, 3.3 holds. Note that $A \cong_{gr} \mathcal{O}_{A_{-1}, A_1, \psi^r}$ follows from Corollary 2.7.

The following result is one of the key insights of this article:

**Proposition 3.4.** The covariant representation

$$(i_{A_{-1}}, i_{A_1}, i_{A_0}, A) \cong_r (i_{A_{-1}}^{I_{A_1}^{(k)}}, i_{A_1}^{I_{A_1}^{(k)}}, i_{A_0}^{I_{A_0}^{(k)}}, \mathcal{O}_{A_{-1}, A_1, \psi^{I_{A_0}^{(k)}}}(J)) = (i_{A_{-1}}^{CP}, i_{A_1}^{CP}, i_{A_0}^{CP}, \mathcal{O}_{A_{-1}, A_1, \psi^r})$$

in Theorem 3.3 is a semi-full covariant representation of $(A_{-1}, A_1, \psi)$.

**Proof.** Note that $A$ comes equipped with a $\mathbb{Z}$-grading which trivially satisfies $i_{A_{-1}}(A_{-1}) \subseteq A_{-1}$, $i_{A_1}(A_1) \subseteq A_1$ and $i_{A_0}(A_0) \subseteq A_0$. Hence, $(i_{A_{-1}}, i_{A_1}, i_{A_0}, A)$ is a graded representation of $(A_{-1}, A_1, \psi)$. It is enough to show that $A_{-k}A_k = I_\psi^{(k)}i_{A_0}$ for every $k \geq 0$. Take an arbitrary integer $k > 0$. Clearly, $I_\psi^{(k)}i_{A_0} \subseteq A_{-k}A_k$. Recall that $A$ is semi-saturated by
algebra is unital. We analogously introduce the following notions for

Let \( \psi \) be a unital ring belonging to a semi-full representation. 

We will see that, for our purposes, we will only need to consider s-unital and unital \( R \)-systems. In the \( C^* \)-setting, Chirvasitu [7] only considers unital \( C^* \)-correspondences (i.e. the coefficient \( C^* \)-algebra \( A \) is unital). This assumption guarantees that the Cuntz-Pimsner \( C^* \)-algebra is unital. We analogously introduce the following notions for \( R \)-systems:

**Definition 3.6.** Let \( R \) be a ring and let \((P,Q,\psi)\) be an \( R \)-system. The \( R \)-system \((P,Q,\psi)\) is called s-unital if \( R \) is an s-unital ring and \( P,Q \) are s-unital \( R \)-bimodules. The \( R \)-system \((P,Q,\psi)\) is called unital if \( R \) is a unital ring and \( P,Q \) are unital \( R \)-bimodules.

**Remark 3.7.** Note that we explicitly require that \( R \) is an s-unital (unital) ring for the \( R \)-system \((P,Q,\psi)\) to be s-unital (unital) (cf. Example 7.1). This is needed since the trivial module \( \{0\} \) is a unital \( R \)-bimodule for any ring \( R \).

**Remark 3.8.** Let \( R \) be a unital ring, let \((P,Q,\psi)\) be a unital \( R \)-system and let \((S,T,\sigma,B)\) be a covariant representation of \((P,Q,\psi)\). If \( 1_R \) is the multiplicative identity element of \( R \), then \( 1_B = \sigma(1_R) \) is the multiplicative identity element of \( B \).

We show that a certain type of semi-saturated, nearly epsilon-strongly \( \mathbb{Z} \)-graded rings can be realized as Cuntz-Pimsner rings coming from an s-unital \( R \)-systems.

**Definition 3.9.** If \( A = \bigoplus_{i \in \mathbb{Z}} A_i \) is a semi-saturated, nearly epsilon-strongly \( \mathbb{Z} \)-graded ring that satisfies \( \text{Ann}_{A_0}(A_1) \cap (\text{Ann}_{A_0}(A_1))^\perp = \{0\} \), then \( A \) is called pre-CP.

**Remark 3.10.** It is not clear to the author if there are semi-saturated, nearly epsilon-strongly \( \mathbb{Z} \)-graded rings such that \( \text{Ann}_{A_0}(A_1) \cap (\text{Ann}_{A_0}(A_1))^\perp \neq \{0\} \). In other words, the last condition in Definition 3.9 might not be necessary.

As a special case of Theorem 3.3 we obtain the following:

**Corollary 3.11.** Let \( A = \bigoplus_{i \in \mathbb{Z}} A_i \) be a pre-CP ring. Let \( \psi : A_{-1} \otimes A_1 \to A_0 \) be defined by \( a \otimes b \mapsto ab \). Then \((A_{-1},A_1,\psi)\) is an s-unital \( A_0 \)-system that satisfies Condition (FS) and

\[
(i_{A_{-1}},i_{A_1},i_{A_0},A) \cong_r (\ell_{A_{-1}}^{CP},\ell_{A_1}^{CP},\ell_{A_0}^{CP},O_{(A_{-1},A_1,\psi)}).
\]

(4)

In particular, \( A \cong_{gr} O_{(A_{-1},A_1,\psi)} \). Furthermore, the covariant representation in (4) is semi-full.

**Proof.** Note that condition (a) and (c) in Theorem 3.3 are satisfied by definition. Moreover, note that \( A_1 \) is an s-unital \( A_1 A_{-1} - A_{-1} A_1 \)-bimodule by the assumption that \( A \) is nearly epsilon-strongly \( \mathbb{Z} \)-graded (see Definition 2.2). From this (b) follows directly. In the same
manner, we see that \((A_{-1}, A_1, \psi)\) is an s-unital \(R\)-system. The conclusion now follows by applying Theorem 3.3 and Proposition 3.4.

Recall that a ring is called semi-prime if it has no nonzero nilpotent ideals. We give sufficient conditions for a ring to be pre-CP.

**Lemma 3.12.** The following assertions hold:

(a) Let \(A = \bigoplus_{i \in \mathbb{Z}} A_i\) be \(\mathbb{Z}\)-graded ring. If \(A_0\) is semi-prime, then \(\text{Ann}_{A_0}(A_1) \cap (\text{Ann}_{A_0}(A_1))^\perp = \{0\}\). If \(A\) is semi-saturated, nearly epsilon-strongly \(\mathbb{Z}\)-graded and \(A_0\) is semi-prime, then \(A\) is pre-CP.

(b) If \(A = \bigoplus_{i \in \mathbb{Z}} A_i\) is a unital strongly \(\mathbb{Z}\)-graded ring, then \(A\) is pre-CP. In particular, \(\text{Ann}_{A_0}(A_1) \cap (\text{Ann}_{A_0}(A_1))^\perp = \{0\}\).

**Proof.** (a): Note that \(\text{Ann}_{A_0}(A_1) \oplus (\text{Ann}_{A_0}(A_1))^\perp\) is a nilpotent ideal of \(A_0\).

(b): A unital strongly \(\mathbb{Z}\)-graded ring is nearly epsilon-strongly \(\mathbb{Z}\)-graded and semi-saturated. Moreover, \(\text{Ann}_{A_0}(A_1) \subseteq \text{Ann}_{A_0}(A_1A_{-1}) = \text{Ann}_{A_0}(A_0) = \{0\}\) since \(A_0\) is unital. Thus, \(\text{Ann}_{A_0}(A_1) \cap (\text{Ann}_{A_0}(A_1))^\perp = \{0\}\).

From Corollary 3.11, we obtain necessary conditions for certain Cuntz-Pimsner rings to be nearly epsilon-strongly \(\mathbb{Z}\)-graded.

**Corollary 3.13.** Let \((P, Q, \psi)\) be an \(R\)-system such that (i) \(O_{(P, Q, \psi)} = \bigoplus_{i \in \mathbb{Z}} O_i\) exists and is nearly epsilon-strongly \(\mathbb{Z}\)-graded and (ii) \(\text{Ann}_{O_0}(O_1) \cap (\text{Ann}_{O_0}(O_1))^\perp = \{0\}\).

Let \(\psi': O_{-1} \otimes O_1 \rightarrow O_0\) be defined by \(\psi'(a \otimes a') = aa'\). Then \((O_{-1}, O_1, \psi')\) is an s-unital \(O_0\)-system such that

\[(i_{O_{-1}}, i_{O_1}, i_{O_0}, O_{(P, Q, \psi)}) \cong_r (i_{O_{-1}}^{CP}, i_{O_1}^{CP}, i_{O_0}^{CP}, O_{(O_{-1}, O_1, \psi')}).\]

In particular, \(O_{(P, Q, \psi)} \cong_r O_{(O_{-1}, O_1, \psi')}\) Furthermore, the following assertions hold:

(a) \((O_{-1}, O_1, \psi')\) is an s-unital \(O_0\)-system that satisfies Condition (FS);

(b) \((i_{O_{-1}}^{CP}, i_{O_1}^{CP}, i_{O_0}^{CP}, O_{(O_{-1}, O_1, \psi')})\) is a semi-full covariant representation of \((O_{-1}, O_1, \psi')\);

(c) \(I_{\psi', i_{O_0}^{CP}}^{(k)} = O_{-k}O_k\) is s-unital for \(k \geq 0\).

**Proof.** By Proposition 3.2, \(O_{(P, Q, \psi)}\) is semi-saturated. Hence, with (i) and (ii), it follows that \(O_{(P, Q, \psi)}\) is pre-CP. Thus, Corollary 3.11 establishes the isomorphism of covariant representations and conclusions (a), (b). Since the covariant representation is semi-full we have that \(I_{\psi', i_{O_0}^{CP}}^{(k)} = O_{-k}O_k\) for each \(k \geq 0\). By (i) and Proposition 2.4 we see that \(O_{-k}O_k\) is s-unital for every \(k \geq 0\). Thus, (c) is established.

**Remark 3.14.** It is not clear to the author if condition (ii) in Corollary 3.13 is necessary. It follows from Lemma 3.12 that condition (ii) in Corollary 3.13 is satisfied if either \(O_0\) is semi-prime or \(O_{(P, Q, \psi)}\) is strongly \(\mathbb{Z}\)-graded.

**Remark 3.15.** Let \(R\) be a unital ring and let \(E\) be a finite graph without any loops with exits. In this case, \((L_R(E))_0\) is Morita equivalent to \(R\) (see [13, Cor. 4.12]). Since semi-primeness is preserved by Morita equivalence, it follows that \((L_R(E))_0\) is semi-prime if and only if \(R\) is semi-prime. Note that \(L_R(E)\) is nearly epsilon-strongly \(\mathbb{Z}\)-graded (see [13, Thm. 28]) and semi-saturated. Hence, if \(R\) is a unital semi-prime ring and \(E\) is a finite graph without any loops with exits, then \(L_R(E)\) is pre-CP by Lemma 3.12.
4. Strongly graded Cuntz-Pimsner rings

Throughout this section, we assume that $R$ is a unital ring and that $(P,Q,\psi)$ is a unital $R$-system. We will provide sufficient conditions for the Toeplitz and Cuntz-Pimsner rings to be strongly $\mathbb{Z}$-graded. This is an algebraic analogue of a recent result by Chirvasitu \cite{7} where he gave sufficient conditions for the gauge action of a Cuntz-Pimsner $C^*$-algebra to be free. We begin by introducing the following new condition that is stronger than Condition (FS):

**Definition 4.1.** Let $R$ be a ring. An $R$-system $(P,Q,\psi)$ is said to satisfy *Condition (FS')* if there exist some $\Theta \in \mathcal{F}_P(Q)$ and $\Phi \in \mathcal{F}_Q(P)$ such that $\Theta(q) = q$ and $\Phi(p) = p$ for every $q \in Q$ and $p \in P$.

We will later give an example (Example 4.4) that shows that Condition (FS) and Condition (FS') are in fact different. We omit the proof of the following proposition as it is an easy analogue of the corresponding statement for Condition (FS).

**Proposition 4.2.** (cf. \cite{5} Lem. 3.8) Let $R$ be a ring and let $(P,Q,\psi)$ be an $R$-system. If $(P,Q,\psi)$ satisfies condition (FS'), then $(P^\otimes n, Q^\otimes n, \psi_n)$ satisfies condition (FS') for every integer $n \geq 1$.

The following gives a characterization of Condition (FS'):

**Lemma 4.3.** Let $R$ be a unital ring and let $(P,Q,\psi)$ be an arbitrary unital $R$-system. Condition (FS') holds if and only if $\text{id}_Q = \Delta(1_R) \in \mathcal{F}_P(Q)$ and $\text{id}_P = \Gamma(1_R) \in \mathcal{F}_Q(P)$. In this case, $\mathcal{L}_P(Q) = \mathcal{F}_P(Q)$ and $\mathcal{L}_Q(P) = \mathcal{F}_Q(P)$.

**Proof.** Consider the inclusions in \cite{2}. If $1_R$ is the multiplicative identity element of $R$, then $\text{id}_Q = \Delta(1_R) \in \mathcal{L}_P(Q)$ is the multiplicative identity element for the ring $\mathcal{L}_P(Q)$. First assume that $(P,Q,\psi)$ satisfies Condition (FS'). Then, $\Theta \in \mathcal{F}_P(Q)$ is a multiplicative identity element of the ring $\mathcal{L}_P(Q)$. Hence, $\Theta = \Delta(1_R) = \text{id}_Q$ which implies that $\mathcal{L}_P(Q) = \mathcal{F}_P(Q)$. Similarly, $\Phi = \Gamma(1_R) = \text{id}_P$ which implies that $\mathcal{L}_Q(P) = \mathcal{F}_Q(P)$. The converse statement follows by noting that $\Delta(1_R)(q) = 1_R \cdot q = q$ and $\Gamma(1_R)(p) = p \cdot 1_R = p$ for all $q \in Q$ and $p \in P$. 

**Example 4.4.**

Let $E$ consist of one vertex $v$ with countably infinitely many loops $f_1, f_2, \ldots$. This is sometimes called a rose with countably many pedals. The standard system $(P,Q,\psi)$ attached to the graph $E$ satisfy Condition (FS) (see \cite{5} Expl. 5.8). Furthermore, $(P,Q,\psi)$ is a unital $R$-system with multiplicative identity element $1_R = \eta_e$.

Seeking a contradiction, suppose that $(P,Q,\psi)$ satisfies Condition (FS'). By Lemma 4.3 $\text{id}_Q \in \mathcal{F}_P(Q)$. Thus, $\text{id}_Q = \sum_i \Theta_{\eta_{f_i}, \eta_{f_i}^*}$ for some edges $f_i \in E^1$ where the sum is finite. Let $f_k$ be some edge not included in the sum. Recall that $\psi(\eta_{f'} \otimes \eta_{f''}) = \eta_{r(f)}$ if $f = f'$ and 0 otherwise. Hence,

$$\eta_{f_k} = \text{id}_Q(\eta_{f_k}) = \sum_i \Theta_{\eta_{f_i}, \eta_{f_i}^*}(\eta_{f_k}) = \sum_i \eta_{f_i} \cdot \psi(\eta_{f_i}^* \otimes \eta_{f_k}) = 0,$$

which is a contradiction. Thus, $(P,Q,\psi)$ does not satisfy Condition (FS'). In other words, $(P,Q,\psi)$ is an example of an $R$-system satisfying Condition (FS) but not Condition (FS').
We will later see that Condition (FS') implies that the ideals $T_iT_{-i}$ are unital for $i \geq 0$ (see Section 5). To prove that the Toeplitz ring is strongly $\mathbb{Z}$-graded, we need the following even stronger condition.

**Definition 4.5.** Let $R$ be a unital ring, let $(P,Q,\psi)$ be an $R$-system satisfying Condition (FS') and let $(S,T,\sigma,B)$ be a covariant representation of $(P,Q,\psi)$. Then $(S,T,\sigma,B)$ is called **faithful** if $\pi_{T,S}(\Delta(1_R)) = \sigma(1_R)$.

We will consider a graded covariant representation and derive sufficient conditions for it to be strongly $\mathbb{Z}$-graded.

**Lemma 4.6.** Let $R$ be a unital ring. Suppose that $(P,Q,\psi)$ is an $R$-system and that $(S,T,\sigma,B)$ is a graded, injective, surjective and faithful representation of $(P,Q,\psi)$. Then,

$$\pi_{T^n,S^n}(\Delta^{(n)}(1_R)) = \sigma(1_R) = 1_B \in B_nB_{-n}$$

for $n > 0$.

**Proof.** Take an arbitrary $n > 0$. By Proposition 4.2, $(P^{\otimes n},Q^{\otimes n},\psi_n)$ satisfies Condition (FS'). This means that $\Delta^{(n)}(1_R) \in F_{P^{\otimes n}}(Q^{\otimes n})$. Furthermore, by faithfulness, $\pi_{T,S}(\Delta(1_R)) = \sigma(1_R) = \sum_i T(q_i)S(p_i)$ for some $q_i \in Q, p_i \in P$. Hence,

$$1_B = \sigma(1_R) = \sum_{i,j} T(q_i)T(q_j)S(p_j)S(p_i)$$

$$\in \text{Span}\{T^2(q)S^2(p) \mid q \in Q^{\otimes 2}, p \in P^{\otimes 2}\} \subseteq B_2B_{-2}.$$

By an induction argument, we get that

$$1_B \in \text{Span}\{T^n(q)S^n(p) \mid q \in Q^{\otimes n}, p \in P^{\otimes n}\} \subseteq B_nB_{-n}$$

for any $n > 0$. By Proposition 4.2 and the assumption that the covariant representation is injective, it follows that the map $\pi_{T^n,S^n}: F_{P^{\otimes n}}(Q^{\otimes n}) \to \text{Span}\{T(q)S(p) \mid q \in Q^{\otimes n}, p \in P^{\otimes n}\}$ is a ring isomorphism. Hence, $\pi_{T^n,S^n}(\Delta^{(n)}(1_R)) = 1_B = \sigma(1_R) \in B_nB_{-n}$ for $n > 0$. □

**Lemma 4.7.** Let $R$ be a unital ring and let $(P,Q,\psi)$ be a unital $R$-system such that the map $\psi: P \otimes Q \to R$ is surjective. Let $(S,T,\sigma,B)$ be a surjective, graded covariant representation of $(P,Q,\psi)$. Then, $1_B \in B_nB_{-n}$ for $n > 0$.

**Proof.** We prove that if $\psi: P \otimes Q \to R$ is surjective, then $\psi_n: P^{\otimes n} \otimes Q^{\otimes n} \to R$ is surjective for $n > 1$. The proof goes by induction on $n$. Suppose that $\psi_{n-1}$ is surjective. Then there is some $p \in P^{\otimes(n-1)}$ and $q \in Q^{\otimes(n-1)}$ such that $\psi_{n-1}(p \otimes q) = 1_R$. Then, since $1_R$ acts trivially on $Q$, it follows that $\psi_n((p' \otimes p) \otimes (q \otimes q')) = \psi(p' \otimes \psi_{n-1}(p \otimes q) \cdot q') = \psi(p' \otimes q') = 1_R$ if we choose $p'$ and $q'$ appropriately. Thus, the claim follows from the induction principle.

Take an arbitrary integer $n > 0$. We have that $1_R = \psi_n(p \otimes q)$ for some $p \in P^{\otimes n}$ and $q \in Q^{\otimes n}$. Hence, $\sigma(1_R) = \sigma(\psi_n(p \otimes q)) = S^n(p)T^n(q) \in B_nB_{-n}$ which proves that $1_B = \sigma(1_R) \in B_nB_{-n}$ for every $n > 0$. □

We have now found sufficient conditions for a representation ring to be strongly $\mathbb{Z}$-graded.

**Proposition 4.8.** Let $R$ be a unital ring and let $(P,Q,\psi)$ be a unital $R$-system that satisfies Condition (FS'). Let $(S,T,\sigma,B)$ be an injective, surjective and graded covariant representation of $(P,Q,\psi)$. Furthermore, suppose that the following assertions hold:

(a) $(S,T,\sigma,B)$ is a faithful representation of $(P,Q,\psi)$;

(b) $\psi$ is surjective.
Then $B$ is strongly $\mathbb{Z}$-graded.

**Proof.** By assumption (a), it follows from Lemma 4.6 that $1_B \in B_n B_{-n}$ for $n > 0$. By assumption (b) and Lemma 4.7, it follows that $1_B \in B_{-n} B_n$ for $n > 0$. Furthermore, since $1_B = \sigma(1_R) \in B_0$, it follows that $B_0$ is a unital subring of $B$. Thus, $1_B \in T_i T_{-i}$ for every $i \in \mathbb{Z}$. It follows that $B$ is strongly $\mathbb{Z}$-graded (see e.g. [15, Prop. 1.1.1]). \hfill $\square$

Note that since the Toeplitz representation $(\iota_P, \iota_Q, \iota_R, \mathcal{T}(P,Q,\psi))$ is injective, surjective and graded, Proposition 4.8 gives sufficient conditions for the Toeplitz ring to be strongly $\mathbb{Z}$-graded.

**Corollary 4.9.** Let $R$ be a unital ring and let $(P, Q, \psi)$ be a unital $R$-system that satisfies Condition (FS'). Consider the Toeplitz ring $\mathcal{T}(P,Q,\psi) = \bigoplus_{i \in \mathbb{Z}} T_i$. If $\pi(\Delta(1_R)) = \iota_R(1_R)$ and $\psi$ is surjective, then $\mathcal{T}(P,Q,\psi)$ is strongly $\mathbb{Z}$-graded.

The requirement of faithfulness is more easily formulated when considering the relative Cuntz-Pimsner representations.

**Corollary 4.10.** Let $R$ be a unital ring and let $(P, Q, \psi)$ be a unital $R$-system that satisfies Condition (FS'). Let $J \subseteq R$ be a $\psi$-compatible ideal. Furthermore, suppose that the following assertions hold:

(a) $1_R \in J$;
(b) $\psi$ is surjective.

Then the relative Cuntz-Pimsner ring $\mathcal{O}(P,Q,\psi)(J)$ is strongly $\mathbb{Z}$-graded.

**Proof.** Recall that the Cuntz-Pimsner representation $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}(P,Q,\psi)(J))$ is injective, surjective and graded. Furthermore, note that (a) implies that the identity $\iota_R^J(1_R) = \pi_{\iota_Q^J, \iota_P^J}(\Delta(1_R))$ holds in the Cuntz-Pimsner ring. This implies that the representation $(\iota_P^J, \iota_Q^J, \iota_R^J, \mathcal{O}(P,Q,\psi)(J))$ is faithful. By Proposition 4.8 and the additional assumption that $\psi$ is surjective, we have that $\mathcal{O}(P,Q,\psi)(J)$ is strongly $\mathbb{Z}$-graded. \hfill $\square$

For the rest of this section, we apply the above theorems to the special cases of Leavitt path algebras and fractional skew monoid rings. We begin by proving that the conditions in Corollary 4.10 are satisfied for any Leavitt path algebra associated with a finite graph without sinks.

**Remark 4.11.** The Leavitt path algebra of a graph $E$ is the Cuntz-Pimsner ring relative to the ideal of $R$ generated by $\text{Reg}(E) \subseteq E^0$, i.e. the regular vertices of $E$ (see [5, Expl. 5.8]). Suppose that $E$ is a finite graph without any sinks. We note that the conditions (a) and (b) in Corollary 4.10 are satisfied.

(a) Since a singular vertex (non-regular vertex) is either an infinite emitter or a sink, by the requirement on $E$, it follows that $\text{Reg}(E) = E^0$. This implies that $1_R = \sum_{v \in E^0} \eta_v \in J$.

(b) Since $E$ does not contain any sinks, we have that for any $v \in E^0$ there is some $f \in E^1$ such that $r(f) = v$. Hence, $\eta_v = \eta_{r(f)} = \psi(\eta_f \otimes \eta_f)$. This proves that $\psi$ is surjective.

We now prove that the Leavitt path algebra system (see Section 2.4) satisfies Condition (FS') when $E$ is a finite graph. Recall that $R = \bigoplus_{v \in E^0} K \eta_v$, $P = \bigoplus_{f \in E^1} K \eta_f$, and $Q = \bigoplus_{f \in E^1} K \eta_f$. Since $E^0$ is finite, $R$ is a unital ring with multiplicative identity element $1_R = \eta_\text{reg}$.
\[
\sum_{v \in E^0} \eta_v. \text{ Let } \Theta = \sum_{f \in E^1} \theta_{f, \eta_f, \eta_f'} \in \mathcal{F}_P(Q). \text{ Note that, for any } \eta_f' \in Q, \\
\Theta(\eta_f') = \sum_{f \in E^1} \eta_f \cdot \psi(\eta_{f'} \otimes \eta_f') = \sum_{f \in E^1} \eta_f \cdot \delta_{f, f'} \eta_{\psi(f')} = \eta_{f'}.
\]

Similarly, \( \Phi = \sum_{f \in E^1} \theta_{f, \eta_f} \in \mathcal{F}_Q(P) \) is proved to satisfy \( \Phi(\eta_{f'}) = \eta_{f'} \) for all \( \eta_{f'} \in P \). Thus, \((P, Q, \psi)\) satisfies Condition (FS').

We can now recover the result obtained by Hazrat (see [10, Thm. 3.15]) on when a Leavitt path algebra of a finite graph is strongly \( \mathbb{Z} \)-graded.

**Corollary 4.12.** Let \( E \) be a finite graph without any sinks. Then the Leavitt path algebra \( L_K(E) \) is strongly \( \mathbb{Z} \)-graded.

We will now consider the fractional skew monoid ring by a corner isomorphism. Recall that we need to specify a unital ring \( R \), an idempotent \( e \in R \) and a corner isomorphism \( \alpha : R \to eRe \). Moreover, recall that an idempotent \( e \in R \) is called full if \( ReR = R \). Hazrat showed (see [11 Prop. 1.6.6]) that \( R[t_+, t_-; \alpha] \) is strongly \( \mathbb{Z} \)-graded if and only if \( e \) is a full idempotent.

**Corollary 4.13.** Let \( R \) be a unital ring and let \( \alpha : R \to eRe \) be a ring isomorphism where \( e \) is an idempotent of \( R \). The fractional skew group ring \( R[t_+, t_-; \alpha] \) is strongly \( \mathbb{Z} \)-graded if \( e \) is a full idempotent.

**Proof.** Let \((P, Q, \psi)\) denote the \( R \)-system in [5 Expl. 5.6]. Assume that \( e \) is a full idempotent. Then \( \psi \) is surjective. Furthermore, it can be shown that \((P, Q, \psi)\) satisfies Condition (FS'). By [5 Expl. 5.7] we have that \( J = R \) and that the Cuntz-Pimsner ring exists and moreover \( R[t_+, t_-; \alpha] \cong_{gr} \mathcal{O}(P, Q, \psi) \). By Corollary [4.10] it follows that \( R[t_+, t_-; \alpha] \) is strongly \( \mathbb{Z} \)-graded.

\[
5. \text{ Epsilon-strongly graded Cuntz-Pimsner rings}
\]

We will show that Condition (FS) and Condition (FS') correspond to local unit properties of the rings \( T_iT_{-i} \) for \( i > 0 \). This allows us to find sufficient conditions for the Cuntz-Pimsner rings to be nearly epsilon-strongly and epsilon-strongly graded.

**Proposition 5.1.** Let \( R \) be an \( s \)-unital ring and let \((P, Q, \psi)\) be an \( s \)-unital \( R \)-system that satisfies Condition (FS). Consider the Toeplitz ring \( T_{(P, Q, \psi)} = \bigoplus_{i \in \mathbb{Z}} T_i \). The following assertions hold:

(a) For \( i \geq 0 \), \( T_i \) is a left \( s \)-unital \( T_iT_{-i} \)-module;
(b) For \( i \geq 0 \), \( T_{-i} \) is a right \( s \)-unital \( T_iT_{-i} \)-module;
(c) \( T_iT_{-i} \) is an \( s \)-unital ring for \( i \geq 0 \);
(d) \( T_i = T_iT_{-i}T_i \) for every \( i \in \mathbb{Z} \).

**Proof.** (a): Take an arbitrary integer \( i \geq 0 \) and an arbitrary element \( s \in T_i \). Then, \( s = \sum_{j \geq 0} t_{Q^\otimes m_j}^j \cdot q_j \) for some non-negative integers \( \{m_j\}, \{n_j\} \) and elements \( q_j \in Q^{\otimes m_j} \) and \( p_j \in Q^{\otimes n_j} \). Note that \( m_j - n_j = i \) for all indices \( j \). Furthermore, since \( i \) is non-negative, we have that \( 0 \leq i \leq m_j \) for all \( j \).

For each \( j \), let \( q_j' \) denote the \( i \)th initial segment of \( q_j \). In other words, \( q_j' = q_j' \otimes q_j'' \) where \( q_j' \in Q^{\otimes i} \) and \( q_j'' \in Q^{\otimes (m_j-i)} \). By Condition (FS) there is some \( \Theta \in \mathcal{F}_{P\otimes(Q^{\otimes i})} \) such that \( \Theta(q_j') = q_j' \) for all \( j \). Put \( e(s) = \pi(\Theta) \). By Proposition [2.9] we have that \( \pi(\Theta) \in \)
Span\{t Q \otimes (q) t \otimes p \mid q \in Q \otimes i, p \in P \otimes i\} \subseteq T_i T_{-i}. Then, by using the relations in Proposition 2.9

\[ \epsilon(s)s = \pi(\Theta) \sum_j t Q \otimes m_j(q_j)t \otimes p \otimes n_j(p_j) = \pi(\Theta) \sum_j t Q \otimes (q_j') t Q \otimes (m_j-i)(q_j'') t \otimes p \otimes n_j(p_j) \]

\[ = \sum_j (\pi(\Theta)t Q \otimes (q_j'))t Q \otimes (m_j-i)(q_j'') t \otimes p \otimes n_j(p_j) = \sum_j (t Q \otimes (\Theta(q_j'))t Q \otimes (m_j-i)(q_j'') t \otimes p \otimes n_j(p_j) \]

\[ = \sum_j (t Q \otimes (q_j')) t Q \otimes (m_j-i)(q_j'') t \otimes p \otimes n_j(p_j) = s. \]

(b): Analogous to (a)

(c): Let \( i \geq 0 \) be an arbitrary non-negative integer. Any element of \( T_i T_{-i} \) is a finite sum \( s = \sum_j a_j b_j \) where \( a_j \in T_i \) and \( b_j \in T_{-i} \). Since \( T_i \) is a \( s \)-unital \( T_i T_{-i} \)-module by (a), Remark 2.1 implies that we can find some element \( t_1 \in T_i T_{-i} \) such that \( t_1a_j = a_j \) for all indices \( j \). Similarly, (b) and Remark 2.1 implies that there is some element \( t_2 \in T_i T_{-i} \) such that \( b_j t_2 = b_j \) for all indices \( j \). Hence, \( t_1 s = s \) and \( s t_2 = s \). This implies that \( T_i T_{-i} \) is a \( s \)-unital \( T_i T_{-i} \)-module, and a right \( s \)-unital \( T_i T_{-i} \)-module. Thus, \( T_i T_{-i} \) is an \( s \)-unital ring.

(d): Take an arbitrary integer \( i \in \mathbb{Z} \). From the grading, it is clear that \( T_i T_{-i} T_i \subseteq T_i \). It remains to show that \( T_i \subseteq T_i T_{-i} T_i \). Let \( s \in T_i \) be an arbitrary element. First suppose that \( i \geq 0 \), then by (a) there is some \( \epsilon(s) \in T_i T_{-i} \) such that \( s = \epsilon(s) \in T_i T_{-i} T_i \). On the other hand, if \( i < 0 \), then by (b) there is some \( \epsilon(s) \in T_i T_{-i} T_i \) such that \( s = \epsilon(s) \in T_i T_{-i} T_i \). Thus, \( T_i = T_i T_{-i} T_i \) for every \( i \in \mathbb{Z} \).

Recall that for idempotents \( e, f \) we define the idempotent order by \( e \leq f \iff ef = fe = e \).

Remark 5.2. Let \( A \) be an epsilon-strongly \( \mathbb{Z} \)-graded ring. Let \( \epsilon_i \in A_i A_{-i} \) denote the multiplicative identity element of \( A_i A_{-i} \) for \( i \in \mathbb{Z} \) (see Proposition 2.4). If the gradation on \( A \) is semi-saturated, then \( \epsilon_0 \geq \epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq \ldots \) and \( \epsilon_0 \geq \epsilon_{-1} \geq \epsilon_{-2} \geq \epsilon_{-3} \geq \ldots \).

For the next section, let \( (P, Q, \psi) \) be a unital \( R \)-system. Suppose that \( (P, Q, \psi) \) satisfies Condition (FS'). Consider the Toeplitz representation \( (t Q, t P, t R, T) \). We define, \( \epsilon_0 := t R(1_R), \epsilon_i := \pi(t Q \otimes \psi \otimes t P)\Delta^i(1_R) = \chi(t Q \otimes \psi \otimes t P)\Gamma^i(1_R) \in T_i T_{-i}, \) for \( i > 0 \).

Lemma 5.3. The sequence \( \{\epsilon_i\}_{i \geq 0} \) consists of idempotents such that \( \epsilon_0 \geq \epsilon_1 \geq \epsilon_2 \geq \epsilon_3 \geq \epsilon_4 \geq \ldots \) holds in the idempotent ordering.

Proof. Fix an arbitrary integer \( i \geq 0 \) and let \( \epsilon_i = \pi(\Delta^i(1_R)) = \sum_j t Q \otimes (q_j)t \otimes p \otimes (p_j) \) for some \( q_j \in Q \otimes i \) and \( p_j \in P \otimes i \). Then, using Proposition 2.9

\[ \epsilon_i^2 = \sum_j \epsilon_i t Q \otimes (q_j)t \otimes p \otimes (p_j) = \sum_j (\pi(\Delta^i(1_R))t Q \otimes (q_j))t \otimes p \otimes (p_j) \]

\[ = \sum_j t Q \otimes (\Delta^i(1_R)(q_j))t \otimes p \otimes (p_j) = \sum_j t Q \otimes (q_j)t \otimes p \otimes (p_j) = \epsilon_i. \]

Hence, \( \epsilon_i \) is an idempotent.

It is clear that \( t R(1_R) = \epsilon_0 \geq \epsilon_1 \). Take an arbitrary integer \( m > 0 \). We will prove that \( \epsilon_m \geq \epsilon_{m+1} \). This is equivalent to \( \epsilon_{m+1} = \epsilon_{m+1} \epsilon_m = \epsilon_m \epsilon_{m+1} \). We first prove that \( \epsilon_m \epsilon_{m+1} = \epsilon_m \).
Let \( \epsilon_{m+1} = \sum_j t_{Q^\otimes m+1}(q_j)t_{P^\otimes m+1}(p_j) \). Write \( q_j = q_j' \otimes q_j'' \) where \( q_j' \in Q^\otimes m \) and \( q_j'' \in Q \). Then, by Proposition 2.9,

\[
\epsilon_m \epsilon_{m+1} = \sum_j \epsilon_m t_{Q^\otimes m+1}(q_j)t_{P^\otimes m+1}(p_j) = \sum_j \epsilon_m t_{Q^\otimes m}(q_j')t_{Q^\otimes m}(q_j'')t_{P^\otimes m+1}(p_j) = \sum_j t_{Q^\otimes m}(q_j')t_{Q^\otimes m}(q_j'')t_{P^\otimes m+1}(p_j) = \epsilon_m.
\]

Again, let \( \epsilon_{m+1} = \sum_j t_{Q^\otimes m+1}(q_j)t_{P^\otimes m+1}(p_j) \). This time write \( p_j = p_j' \otimes p_j'' \) for some \( p_j' \in P \) and \( p_j'' \in P^\otimes m \). Then, again by using Proposition 2.9,

\[
\epsilon_{m+1} \epsilon_m = \sum_j t_{Q^\otimes m+1}(q_j)t_{P^\otimes m+1}(p_j) \epsilon_m = \sum_j t_{Q^\otimes m+1}(q_j)t_{P^\otimes m}(p_j) \epsilon_{m+1} = \sum_j t_{Q^\otimes m+1}(q_j)t_{P^\otimes m}(p_j) = \epsilon_m.
\]

Proposition 5.4. Let \( R \) be a unital ring and let \((P,Q,\psi)\) be a unital \( R \)-system that satisfies Condition (FS'). Let \( \epsilon_i \) be the idempotents defined above. The following assertions hold for every non-negative integer \( i \geq 0 \):

(a) For any \( s \in T_i \) we have that \( \epsilon_i s = s \);

(b) For any \( t \in T_{-i} \) we have that \( t \epsilon_i = t \).

Consequently, \( T_i T_{-i} \) is a unital ideal with multiplicative identity element \( \epsilon_i \) for every \( i \geq 0 \).

Proof. The statements are clear for \( i = 0 \).

(a): Take an arbitrary positive integer \( i > 0 \). Consider a monomial \( t_{Q^\otimes m}(q)t_{P^\otimes n}(p) \) where \( m,n \) are non-negative integers such that \( m - n = i \). Then, \( 0 < i \leq m \). By Lemma 5.3, \( \epsilon_m \geq \epsilon_i \). Hence,

\[
t_{Q^\otimes m}(q)t_{P^\otimes n}(p) = t_{Q^\otimes m}(\Delta^m(1_R)(q))t_{P^\otimes n}(p) = \pi(\Delta^m(1_R))t_{Q^\otimes m}(q)t_{P^\otimes n}(p) = \epsilon_m t_{Q^\otimes m}(q)t_{P^\otimes n}(p) = \epsilon_i t_{Q^\otimes m}(q)t_{P^\otimes n}(p).
\]

Since any element \( s \in T_i \) is a finite sum of elements of the above form, it follows that \( \epsilon_i s = s \).

(b): Take an arbitrary positive integer \( i > 0 \). Consider a monomial \( t_{Q^\otimes m}(q)t_{P^\otimes n}(p) \) where \( m,n \) are non-negative integers such that \( m - n = -i \). Then \( 0 < i \leq n \). By Lemma 5.3, \( \epsilon_n \geq \epsilon_i \). Hence, \( t_{Q^\otimes m}(q)t_{P^\otimes n}(p) = t_{Q^\otimes m}(q)t_{P^\otimes n}(\Gamma^n(1_R)(p)) = t_{Q^\otimes m}(q)t_{P^\otimes n}(p)\epsilon_n = t_{Q^\otimes m}(q)t_{P^\otimes n}(p)\epsilon_i = t_{Q^\otimes m}(q)t_{P^\otimes n}(p)\epsilon_i = t_{Q^\otimes m}(q)t_{P^\otimes n}(p)\epsilon_i \). Since any element \( t \in T_{-i} \) is a finite sum of elements of the above form, it follows that \( t \epsilon_i = t \). \( \square \)

We will see that restricting our attention to semi-full covariant representations \((S,T,\sigma,B)\) makes life easier. This special type of graded covariant representations have the property that the image of \( \psi_k \) is enough to generate the ideal \( B_{-k}B_k \) for \( k \geq 0 \). We first prove that the property of being semi-full is invariant under isomorphism in the category \( C_{(P,Q,\psi)} \).
**Proposition 5.5.** Let $R$ be a ring, let $(P, Q, \psi)$ be an $R$-system and suppose that $(S, T, \sigma, B) \cong_r (S', T', \sigma', B')$ are two isomorphic covariant representations of $(P, Q, \psi)$. If $(S, T, \sigma, B)$ is semi-full, then $(S', T', \sigma', B')$ is semi-full.

**Proof.** Let $\phi: B \to B'$ be the $\mathbb{Z}$-graded isomorphism coming from Lemma 2.6. Hence,

$$B'_{-k}B_k = \phi(B_{-k})\phi(B_k) = \phi(B_{-k}B_k) = \phi(I^{(k)}_{\psi,\sigma})$$

$$= \text{Span}\{\phi \circ \sigma(\psi(p \otimes q)) | p \in P^{\otimes k}, q \in Q^{\otimes k}\} = I^{(k)}_{\psi,\sigma}.$$ 

Thus, $(S', T', \sigma', B')$ is semi-full. \qed

We now establish sufficient conditions for a semi-full covariant representation to be nearly epsilon-strongly $\mathbb{Z}$-graded.

**Proposition 5.6.** Let $R$ be an s-unital ring and let $(P, Q, \psi)$ be an s-unital R-system. Suppose that $(S, T, \sigma, B)$ is a semi-full covariant representation of $(P, Q, \psi)$ and that the following assertions hold:

(a) $(P, Q, \psi)$ satisfies Condition (FS),

(b) $I^{(k)}_{\psi,\sigma}$ is s-unital for $k \geq 0$.

Then, $B$ is nearly epsilon-strongly $\mathbb{Z}$-graded.

**Proof.** Let $T_{(P, Q, \psi)} = \bigoplus_{i \in \mathbb{Z}} T_i$ be the Toeplitz ring associated with the $R$-system $(P, Q, \psi)$. By Proposition 5.1 (a) implies that the ring $T_{i}T_{-i}$ is s-unital for $i \geq 0$. By Theorem 2.3 there is a $\mathbb{Z}$-graded ring epimorphism $\eta: T_{(P, Q, \psi)} \to B$. Since the image of an s-unital ring under a ring homomorphism is in turn s-unital, it follows that $B_iB_{-i} = \eta(T_i)\eta(T_{-i}) = \eta(T_iT_{-i})$ is s-unital for $i \geq 0$. Furthermore, by Proposition 5.1 (a) implies that $T_{(P, Q, \psi)}$ is symmetrically $\mathbb{Z}$-graded, i.e., $T_i = T_iT_{-i}T_i$. Applying $\eta$ to both sides yields $B_i = B_iB_{-i}B_i$. Hence, $B$ is symmetrically $\mathbb{Z}$-graded.

Next, we show that $B_iB_{-i}$ is s-unital for $i < 0$. Since $(S, T, \sigma, B)$ is semi-full, we have that $B_{-k}B_k = I^{(k)}_{\psi,\sigma}$ for $k \geq 0$. Hence, (b) implies that $B_iB_{-i}$ is s-unital for $i < 0$. Thus, we have showed that $B_iB_{-i}$ is s-unital for $i \in \mathbb{Z}$ and that $B$ is symmetrically $\mathbb{Z}$-graded. By Proposition 2.4 it follows that $B = \bigoplus_{i \in \mathbb{Z}} B_i$ is nearly epsilon-strongly $\mathbb{Z}$-graded. \qed

The proof of the following proposition is entirely analogous to the proof of Proposition 5.6.

**Proposition 5.7.** Let $R$ be a unital ring and let $(P, Q, \psi)$ be a unital R-system. Suppose that $(S, T, \sigma, B)$ is a semi-full covariant representation of $(P, Q, \psi)$ and that the following assertions hold:

(a) $(P, Q, \psi)$ satisfies Condition (FS'),

(b) $I^{(k)}_{\psi,\sigma}$ is unital for $k \geq 0$.

Then, $B$ is epsilon-strongly $\mathbb{Z}$-graded.

**6. Classification up to Graded Isomorphism**

In this section, we finally give classifications of unital strongly, nearly epsilon-strongly and epsilon-strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings up to $\mathbb{Z}$-graded isomorphism.

**Theorem 6.1.** Let $\mathcal{O}_{(P, Q, \psi)}$ be a Cuntz-Pimsner ring of some system $(P, Q, \psi)$. If $\mathcal{O}_{(P, Q, \psi)}$ is nearly epsilon-strongly $\mathbb{Z}$-graded and $\text{Ann}_{\mathcal{O}}(\mathcal{O}_1) \cap (\text{Ann}_{\mathcal{O}}(\mathcal{O}_1))^\perp = \{0\}$, then $\mathcal{O}_{(P, Q, \psi)} \cong_{gr} \mathcal{O}_{(P', Q', \psi')}$ where $(P', Q', \psi')$ is an $R'$-system and the following assertions are satisfied:
(a) \((P', Q', \psi')\) is an s-unital \(R'\)-system;
(b) \((\iota'_{CP}, \iota'_{CQ}, \iota'_{CR}, \Omega_{(P', Q', \psi')})\) is a semi-full covariant representation of \((P', Q', \psi')\);
(c) \((P', Q', \psi')\) satisfies Condition \((FS)\);
(d) \(I^{(k)}_{\psi' \iota'_{CP}}\) is s-unital for \(k \geq 0\).

Conversely, if \((P', Q', \psi')\) is an \(R'\)-system satisfying assertions (a)-(d), then \(\Omega_{(P', Q', \psi')}\) is nearly epsilon-strongly \(Z\)-graded.

**Proof.** If the Cuntz-Pimsner ring \(\Omega_{(P, Q, \psi)}\) is nearly epsilon-strongly \(Z\)-graded and \(\text{Ann}_{\Omega_0}(O_1) \cap (\text{Ann}_{\Omega_0}(O_1))^\perp = \{0\}\), then it follows from Corollary 3.13 that the Cuntz-Pimsner ring is graded isomorphic to \(\Omega_{(S_{-1}, O_1, \psi)}\) and that (a)-(d) are satisfied.

Conversely, if \((P', Q', \psi')\) is an \(R'\)-system satisfying (a)-(d), then it follows from Proposition 5.6 that \(\Omega_{(P', Q', \psi')}\) is nearly epsilon-strongly \(Z\)-graded. \(\square\)

For epsilon-strongly \(Z\)-graded Cuntz-Pimsner rings, we obtain the following theorem.

**Theorem 6.2.** Let \(\Omega_{(P, Q, \psi)}\) be a Cuntz-Pimsner ring of some system \((P, Q, \psi)\). If \(\Omega_{(P, Q, \psi)}\) is epsilon-strongly \(Z\)-graded and \(\text{Ann}_{\Omega_0}(O_1) \cap (\text{Ann}_{\Omega_0}(O_1))^\perp = \{0\}\), then \(\Omega_{(P, Q, \psi)} \cong_{gr} \Omega_{(P', Q', \psi')}\) where \((P', Q', \psi')\) is an \(R'\)-system and the following assertions are satisfied:

(a) \((P', Q', \psi')\) is a unital \(R'\)-system;
(b) \((\iota'_{CP}, \iota'_{CQ}, \iota'_{CR}, \Omega_{(P', Q', \psi')})\) is a semi-full covariant representation of \((P', Q', \psi')\);
(c) \((P', Q', \psi')\) satisfies Condition \((FS')\);
(d) \(I^{(k)}_{\psi' \iota'_{CP}}\) is unital for \(k \geq 0\).

Conversely, if \((P', Q', \psi')\) is an \(R'\)-system satisfying assertions (a)-(d), then \(\Omega_{(P', Q', \psi')}\) is epsilon-strongly \(Z\)-graded.

**Proof.** Assume that \((P', Q', \psi')\) satisfies the properties in (a)-(d). Then Proposition 5.7 implies that \(\Omega_{(P', Q', \psi')}\) is epsilon-strongly \(Z\)-graded.

Conversely, assume that \(\text{Ann}_{\Omega_0}(O_1) \cap (\text{Ann}_{\Omega_0}(O_1))^\perp = \{0\}\) and that \(\Omega_{(P, Q, \psi)}\) is epsilon-strongly \(Z\)-graded. Then, in particular, \(\Omega_{(P, Q, \psi)}\) is nearly epsilon-strongly \(Z\)-graded. Hence, by Theorem 6.1, \(\Omega_{(P, Q, \psi)} \cong_{gr} \Omega_{(S_{-1}, O_1, \psi)}\) where \((O_{-1}, O_1, \psi)\) is an s-unital \(O_1\)-system that satisfies Condition \((FS)\) and such that (b) is satisfied. Furthermore (see Corollary 3.13),

\[
(i_{O_{-1}}, i_{O_1}, i_{O}, \Omega_{(P, Q, \psi)}) \cong_{i} (i_{O_{-1}}^{CP}, i_{O_1}^{CP}, i_{O}^{CP}, \Omega_{(S_{-1}, O_1, \psi)}).
\]  

(5)

First note that since the \(Z\)-grading is assumed to be epsilon-strong it follows that \(O_i\) is a unital \(O_{-1} \cap O_1 \cap O\)-bimodule for each \(i \in \mathbb{Z}\) (see Definition 2.2.2). This implies that \((O_{-1}, O_1, \psi)\) is a unital \(O_0\)-system. Hence, (a) is satisfied.

Next, we prove that the \(O_0\)-system \((O_{-1}, O_1, \psi)\) satisfies Condition \((FS')\). Let \(\epsilon_1\) be the multiplicative identity element of \(O_1 O_1^{-1}\). Write \(\epsilon_1 = \sum_j q_j p_j\) for some elements \(q_j \in O_1\) and \(p_j \in O_{-1}\). Let \(\Theta = \sum_j \theta_{q_j, p_j} \in \mathcal{F}_{O_{-1}}(O_1)\). Note that by Proposition 2.9 \(\pi_{\iota^{CP}_{O_{-1}} \iota^{CP}_{O_1}}(\Theta) = \epsilon_1\).

We claim that \(\Theta(q) = q\) for all \(q \in O_1\). Take an arbitrary \(q \in O_1\). By Proposition 2.9 we have that,

\[
\iota_{O_1}^{CP}(\Theta(q)) = \pi_{\iota^{CP}_{O_{-1}} \iota^{CP}_{O_1}}(\Theta) \iota_{O_1}^{CP}(q) = \epsilon_1 \iota_{O_1}^{CP}(q) = \iota_{O_1}^{CP}(q).
\]

By (5), this implies that \(i_{O_1}(\Theta(q)) = i_{O_1}(q)\). Since \(i_{O_1}\) is the inclusion map, it follows that \(\Theta(q) = q\). This shows the claim. Similarly, let \(\Phi \in \mathcal{F}_{O_1}(O_{-1})\) such that \(\chi(\Phi) = \epsilon_1\). Then,

\[
\iota_{O_1}^{CP} \mathcal{F}_{O_1}(O_{-1}) \mathcal{F}_{O_1}(O_{-1}) \cong_{i} \mathcal{F}_{O_1}(O_{-1}) \mathcal{F}_{O_1}(O_{-1}) 
\]

(6)

Theorem 6.2 is satisfied. \(\square\)
analogously, $\Phi(p) = p$ for all $p \in O_{-1}$. Hence, $(O_{-1}, O_1, \psi)$ satisfies Condition (FS'). In other words, (c) holds.

Moreover, since we assume that $O_{(P,Q,\psi)}$ is epsilon-strongly graded, it follows by Proposition 2.4(a) that, in particular, $O_{-k}O_k$ is unital for $k \geq 0$. Hence, $O_{-k}O_k = I^{(k)}_{\psi, \epsilon CP}$ is unital for $k \geq 0$. This establishes (d).

For unital strongly $\mathbb{Z}$-graded Cuntz-Pimsner rings we actually obtain a complete classification up to graded isomorphism.

**Theorem 6.3.** Let $O_{(P,Q,\psi)}$ be an algebraic Cuntz-Pimsner ring of some system $(P,Q,\psi)$. Then, $O_{(P,Q,\psi)}$ is unital strongly $\mathbb{Z}$-graded if and only if

$$O_{(P,Q,\psi)} \cong_{gr} O_{(P',Q',\psi')}$$

where $(P', Q', \psi')$ is an $R'$-system satisfying the following assertions:

(a) $(P', Q', \psi')$ is a unital $R'$-system;

(b) $(\iota'_{CP}, \epsilon'_{CP}, \iota'_R, O_{(P',Q',\psi')})$ is a semi-full and faithful covariant representation of $(P', Q', \psi')$;

(c) $\psi'$ is surjective.

**Proof.** By Proposition 4.8, (a) and (c) are sufficient for the ring $O_{(P',Q',\psi')}$ to be strongly $\mathbb{Z}$-graded.

Conversely, assume that $O_{(P,Q,\psi)}$ is unital strongly $\mathbb{Z}$-graded. By Lemma 3.12(b), it follows that $\text{Ann}_{O_0}(O_1) \cap (\ker(\text{Ann}_{O_0}(O_1)))^\perp = \{0\}$. Then, by Theorem 6.2 $O_{(P,Q,\psi)} \cong_{gr} O_{(O_{-1}, O_1, \psi)}$ where $(O_{-1}, O_1, \psi)$ satisfies Condition (FS'), (b) is satisfied and $I^{(k)}_{\psi, \epsilon CP}$ is unital for $k \geq 0$.

Since $O_{(O_{-1}, O_1, \psi)}$ is unital strongly $\mathbb{Z}$-graded,

$$1_{O_{(O_{-1}, O_1, \psi)}} = \iota'_{CP}(1_{O_0}) \in O_0 = O_{-1}O_1 = I^{(1)}_{\psi, \epsilon CP}.$$  

Since $\iota'_{CP}$ is injective, we get that $1_{O_0} \in \text{Im}(\psi')$. Hence, $\psi'$ is surjective.

Furthermore, since $O_{(O_{-1}, O_1, \psi)}$ is an epsilon-strongly $\mathbb{Z}$-graded ring that is also strongly $\mathbb{Z}$-graded, we must have $\epsilon_1 = 1$ (see [17] Prop. 8) where $\epsilon_1$ is the multiplicative identity element of the ring $O_1O_{-1}$. On the other hand, by Condition (FS'), it follows that $\Delta(1_O) \in F_P(Q)$. But $\pi_{O_{-1}O_1} \iota'_{CP} (\Delta(1_O)) \in O_1O_{-1}$ is a multiplicative identity element of $O_1O_{-1}$ by Proposition 2.9. Thus, $\pi_{O_{-1}O_1} \iota'_{CP} (\Delta(1_O)) = 1 = 1_{O_{(O_{-1}, O_1, \psi')}}$ and therefore $(\iota'_{CP}, \epsilon'_{CP}, \iota'_R, O_{(O_{-1}, O_1, \psi')})$ is a faithful representation of $(O_{-1}, O_1, \psi')$. 

7. Examples

In this section, we collect some important examples.

**Example 7.1.** (Non-nearly epsilon-strongly graded Cuntz-Pimsner ring) Let $R$ be an idempotent ring that is not s-unital (see e.g. [10] Expl. 5]). Put $P = Q = \{0\}$ and let $\psi \equiv 0$ be the zero map. Note that $(P,Q,\psi)$ is an $R$-system that satisfies Condition (FS') trivially. It is not hard to see that the Toeplitz ring is given by $T_0 = R$, and $T_i = \{0\}$ for all $i \neq 0$. Furthermore, note that $\ker \Delta = R$. Recall that an ideal $J$ of $R$ is called faithful if $J \cap \ker \Delta = \{0\}$. Clearly, $J := \{0\}$ is the maximal faithful and $\psi$-compatible ideal of $R$. It follows that the Cuntz-Pimsner ring $O_{(P,Q,\psi)}$ is well-defined and coincides with the Toeplitz ring. Since $T_0 = R = R^2 = T_0T_0$ is not s-unital it follows by Proposition 2.4(a)
that the Cuntz-Pimsner ring $O_{(P,Q,\psi)} = \mathcal{T}_{(P,Q,\psi)}$ is not nearly epsilon-strongly graded. This shows that the assumption of $(P,Q,\psi)$ being an s-unital system in Proposition 5.6 cannot be removed.

The following example shows that for some graphs, the standard Leavitt path algebra covariant representation is a semi-full covariant representation of the Leavitt path algebra system $(P,Q,\psi)$ attached to the graph $E$ (see Section 2.4).

**Example 7.2.**

\[ \bullet_{v_1} \xrightarrow{f_1} \bullet_{v_2} \]

Let $K$ be an unital ring and let $E$ consist of two vertices $v_1, v_2$ connected by a single edge $f$. Consider the associated Leavitt path algebra system $(P,Q,\psi)$ and the standard covariant representation $(\iota^CP, \iota^CP, \iota^CP, O_{(P,Q,\psi)})$. To save space we write $I_k = \iota^{(k)}_{\psi, R}$ for $k \geq 0$. Note that $I_0 = \text{Span}\{v_1, v_2\}$, $I_1 = \text{Span}\{v_1\}$, and $I_k = \{0\}$ for $k > 2$. Furthermore, since $f_1f_1^* = v_1$ we have that $(L_K(E))_0 = \text{Span}\{v_1, v_2\} = I_0 = R$.

Moreover, note that $(L_K(E))_1 = \text{Span}\{f_1\}$, $S_1 = \text{Span}\{f_1^*\}$ and hence, $(L_K(E))_1 (L_K(E))_1 = \text{Span}\{v_2\} = I_1$. Thus, $(\iota_P, \iota_Q, \iota_R, O_{(P,Q,\psi)})$ is a semi-full covariant representation of $(P,Q,\psi)$. Furthermore, $(P,Q,\psi)$ satisfies Condition (FS') and $I_k$ is unital for $k \geq 0$. Thus, $L_K(E)$ is epsilon-strongly $\mathbb{Z}$-graded by Theorem 6.2.

In general, however, it is not true that the standard Leavitt path algebra covariant representation is a semi-full representation of the Leavitt path algebra system as the following example shows.

**Example 7.3.** (cf. [8, Expl. 26]) Let $K$ be a field and consider the following finite directed graph $E$.

\[ \bullet_{v_1} \xrightarrow{f_1} \bullet_{v_2} \xrightarrow{f_2} \bullet_{v_3} \xrightarrow{f_3} \bullet_{v_4} \xrightarrow{f_4} \bullet_{v_5} \]

Let $(P,Q,\psi)$ be the standard Leavitt path algebra system and consider standard Leavitt path algebra covariant representation,

\[ (\iota^CP, \iota^CP, \iota^CP, O_{(P,Q,\psi)}). \]  

(6)

We write $S_i = (L_K(E))_i$ and $I_i = \iota^{(i)}_{\psi, R}$ to save space. Note that,

$S_0 = \text{Span}\{v_1, v_2, v_3, v_4, v_5, f_1f_1^*, f_2f_2^*, f_3f_3^*, f_4f_4^*\}$,

$S_1 = \text{Span}\{f_1, f_2, f_3, f_4, f_4f_4f_4^2\}$, $S_{-1} = \text{Span}\{f_1^*, f_2^*, f_3^*, f_4^*, f_4f_4^*f_4^*\}$,

$S_2 = \text{Span}\{f_4^*f_3\}$, $S_{-2} = \text{Span}\{f_4^*f_4^*\}$, and $S_n = \{0\}$, for $|n| > 2$.

Furthermore,

$I_0 = \text{Span}\{v_1, v_2, v_3, v_4, v_5\}$, $I_1 = \text{Span}\{v_1, v_3, v_4\}$, $I_2 = \text{Span}\{v_3\}$, $I_k = \{0\}$, $k > 2$.

In particular, we have that $S_{-1}S_1 = \text{Span}\{v_1, v_3, v_4, f_2f_2^*\} = \epsilon_1 S_0 \supseteq I_1$. Hence, the standard covariant representation is not semi-full. However, $O_{(P,Q,\psi)} \cong_{gr} L_K(E)$ (see Section 2.4). On the other hand, since $E$ is finite acyclic and $K$ is semi-prime it follows by Remark 3.13 that $L_K(E)$ is pre-CP. Thus, $L_K(E)$ is realized by the Cuntz-Pimsner representation,

\[ (\iota^CP_{(L_K(E))_1}, \iota^CP_{(L_K(E))_1}, \iota^CP_{(L_K(E))_0}, O((L_K(E))_1, (L_K(E))_1, \psi)) \]  

(7)
of the \((L_K(E))_0\)-system \((L_K(E))_{-1}, (L_K(E))_1, \psi')\) by Corollary 3.11. Moreover, the corollary implies that \((7)\) is semi-full and \(O((L_K(E))_{-1}, (L_K(E))_1, \psi')) \cong_{fr} L_K(E)\). Since \((6)\) is not semi-full and \((7)\) is semi-full, it follows by Proposition 4.3 that the covariant representations \((6)\) and \((7)\) cannot be isomorphic. Thus, \(L_K(E)\) is realizable as a Cuntz-Pimsner ring in two different ways!

The following example shows that (a) is crucial in Theorem 6.2. It also gives an example of a nearly epsilon-strongly \(\mathbb{Z}\)-graded ring that is not epsilon-strongly \(\mathbb{Z}\)-graded.

**Example 7.4.** (cf. [14, Expl. 4.5]) Let \(K\) be a field and consider the infinite discrete graph \(E\) consisting of countably infinitely many vertices but no edges.

\[
\bullet_{v_1} \cdot v_2 \cdot v_3 \cdot v_4 \cdot v_5 \cdot v_6 \cdot v_7 \cdot v_8 \cdot v_9 \cdot v_{10} \cdots
\]

The standard system is given by \(R = \bigoplus_{v \in E^0} \eta_v, P = Q = \{0\}\). The \(R\)-system \((P, Q, \psi)\) trivially satisfies Condition (FS'). However, \((P, Q, \psi)\) is not unital as \(R\) does not have a multiplicative identity element. On the other hand, note that \((P, Q, \psi)\) is \(s\)-unital.

We show that the standard covariant representation is semi-full. Since \(P = Q = \{0\}\) and \(\psi = 0\) it follows that the grading is given by \(O_0 = R\) and \(O_i = \{0\}\) for \(i \neq 0\) (see Example 7.1). Furthermore, \(I(k)_{\psi,CP} = \{0\}\) for \(k > 0\). Thus, the standard covariant representation satisfies (b)-(d) in Theorem 6.2 but not (a). Since \(E\) contains infinitely many vertices, \(L_K(E)\) is not unital (see [1, Lem. 1.2.12]). By Remark 2.3 \(L_K(E)\) is not epsilon-strongly \(\mathbb{Z}\)-graded (cf. [14, Expl. 4.5]). Thus, (a) in Theorem 6.1 cannot be removed. On the other hand, it follows from Theorem 6.1 that \(L_K(E)\) is nearly epsilon-strongly \(\mathbb{Z}\)-graded. Thus, \(L_K(E)\) is an example of a Cuntz-Pimsner ring that is nearly epsilon-strongly \(\mathbb{Z}\)-graded but not epsilon-strongly \(\mathbb{Z}\)-graded.

### 8. Noetherian and Artinian Corner Skew Laurent Polynomial Rings

We end this article by characterizing noetherian and artinian corner skew Laurent polynomial rings. The following proposition can be proved in a straightforward manner using direct methods, but we show it as an application of our result.

**Proposition 8.1.** Let \(R\) be a unital ring, let \(e \in R\) be an idempotent and let \(\alpha : R \to eRe\) be a corner ring isomorphism. The corner skew Laurent polynomial ring \(R[t_+, t_-; \alpha] = \bigoplus_{i \in \mathbb{Z}} A_i\) is epsilon-strongly \(\mathbb{Z}\)-graded.

**Proof.** Let \(\psi : \mathbb{A}_- \otimes \mathbb{A}_1 \to \mathbb{A}_0\) be the map defined by \(\psi(a' \otimes a) = a'a\) for \(a' \in A_{-1}\) and \(a \in A_1\). In [15, Expl. 3.4], they show that \(R[t_+, t_-; \alpha] = \bigoplus_{i \in \mathbb{Z}} A_i\) satisfies the conditions in Theorem 3.3. Hence,

\[
(i_{A_{-1}}, i_{A_1}, i_{A_0}, R[t_+, t_-; \alpha]) \cong_{fr} (\mathcal{F}^{CP}_{A_{-1}}, \mathcal{F}^{CP}_{A_1}, \mathcal{F}^{CP}_{A_0}, O_{(A_{-1}, A_1, \psi')}).
\]

By Proposition 3.4 this is a semi-full covariant representation of the unital \(R\)-system \((A_{-1}, A_1, \psi')\). Furthermore, it is injective by definition. Note that \(1_R[t_+, t_-; \alpha] = t_-t_+ \in A_1A_{-1}\). The map,

\[\pi_{iA_1, i_{A_{-1}}}: \mathcal{F}_{A_{-1}}(A_1) \to \operatorname{Span}\{i_{A_1}(q)i_{A_{-1}}(p) \mid q \in A_{1}, p \in A_{-1}\} = A_{1}A_{-1},\]

is a ring isomorphism by Proposition 2.9. Hence, it follows that \(\mathcal{F}_{A_{-1}}(A_1)\) is unital. By Lemma 4.3 \((A_{-1}, A_1, \psi')\) satisfies Condition (FS'). It can furthermore be shown that \(I(k)^{CP}_{\psi} = A_{-k}A_k\) is unital with multiplicative identity element \(e\) for each \(k > 0\). By Theorem 6.2 it follows that \(R[t_+, t_-; \alpha]\) is epsilon-strongly \(\mathbb{Z}\)-graded. \(\square\)
We recall the following Hilbert basis theorem for epsilon-strongly $\mathbb{Z}$-graded rings.

**Theorem 8.2.** ([13, Thm. 1.1, Thm. 1.2]) Let $S = \bigoplus_{i \in \mathbb{Z}} S_i$ be an epsilon-strongly $\mathbb{Z}$-graded ring. The following assertions hold:

(a) If $S_0$ is left (right) noetherian, then $S$ is left (right) noetherian;

(b) If $S_0$ is left (right) artinian and there exists some positive integer $n$ such that $S_i = \{0\}$ for all $|i| > n$, then $S$ is left (right) artinian.

Applying Theorem 8.2 to the special case of corner skew Laurent polynomial rings, we obtain the following result.

**Corollary 8.3.** Let $R$ be a unital ring and let $\alpha : R \to eRe$ be a ring isomorphism where $e$ is an idempotent of $R$. Consider the corner skew Laurent polynomial ring $R[t_+, t_-; \alpha]$. The following assertions hold:

(a) $R[t_+, t_-; \alpha]$ is left (right) noetherian if and only if $R$ is left (right) noetherian;

(b) $R[t_+, t_-; \alpha]$ is neither left nor right artinian.

**Proof.** (a): Straightforward.

(b): By Theorem 8.2, $R[t_+, t_-; \alpha]$ is left (right) artinian if and only if $A_0$ is left (right) artinian and $|\text{Supp}(R[t_+, t_-; \alpha])| < \infty$. However, since $t_n^+ \neq 0$ and $t_n^- \neq 0$ for all $n > 0$, it follows that $A_n = R[t_n^+; \alpha] \neq \{0\}$ and $A_{-n} = t_n^- R \neq \{0\}$ for $n > 0$. Hence, $\text{Supp}(R[t_+, t_-; \alpha]) = \mathbb{Z}$ and $R[t_+, t_-; \alpha]$ is neither left nor right artinian. □

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**References**

[1] G. Abrams, P. Ara, and M. S. Molina. *Leavitt Path Algebras*, volume 2191. Springer, 2017.

[2] G. Abrams and G. A. Pino. The Leavitt path algebra of a graph. *Journal of Algebra*, 293(2):319–334, 2005.

[3] P. Ara, M. A. González-Barroso, K. R. Goodearl, and E. Pardo. Fractional skew monoid rings. *Journal of Algebra*, 278(1):104–126, 2004.

[4] P. Ara, M. A. Moreno, and E. Pardo. Nonstable K-theory for graph algebras. *Algebras and representation theory*, 10(2):157–178, 2007.

[5] T. M. Carlsen and E. Ortega. Algebraic Cuntz–Pimsner rings. *Proceedings of the London Mathematical Society*, 103(4):601–653, 2011.

[6] T. M. Carlsen, E. Ortega, and E. Pardo. Simple Cuntz–Pimsner rings. *Journal of Algebra*, 371:367–390, 2012.

[7] A. Chirvasitu. Gauge freeness for Cuntz-Pimsner algebras. *arXiv e-prints*, page arXiv:1805.12318, May 2018.

[8] L. O. Clark, R. Exel, and E. Pardo. A generalized uniqueness theorem and the graded ideal structure of Steinberg algebras. In *Forum Mathematicum*, volume 30, pages 533–552. De Gruyter, 2018.

[9] R. Exel. Partial dynamical systems, Fell bundles and applications, volume 224. American Mathematical Soc, 2017.

[10] R. Hazrat. The graded structure of Leavitt path algebras. *Israel Journal of Mathematics*, 195(2):833–895, 2013.

[11] R. Hazrat. *Graded rings and graded Grothendieck groups*, volume 435. Cambridge University Press, 2016.

[12] T. Katsura. On C*-algebras associated with C*-correspondences. *Journal of Functional Analysis*, 217(2):366–401, 2004.
[13] D. Länström. Chain conditions for epsilon-strongly graded rings with applications to leavitt path algebras. arXiv preprint arXiv:1808.10163, 2018.

[14] D. Länström. Induced quotient group gradings of epsilon-strongly graded rings. arXiv preprint arXiv:1809.04935, 2018.

[15] C. Nastasescu and F. van Oystaeyen. Methods of Graded Rings. Lecture Notes in Mathematics. Springer, 2004.

[16] P. Nystedt. Unital rings, rings with enough idempotents, rings with sets of local units, locally unital rings, s-unital rings and idempotent rings. arXiv preprint arXiv:1809.02117, 2018.

[17] P. Nystedt, J. Öinert, and H. Pinedo. Epsilon-strongly graded rings, separability and semisimplicity. Journal of Algebra, 514:1 – 24, 2018.

[18] P. Nystedt and J. Öinert. Leavitt path algebras are nearly epsilon-strongly graded. ArXiv e-prints, Mar. 2017.

[19] L. Orloff Clark, J. Fletcher, R. Hazrat, and H. Li. Z-graded rings as Cuntz-Pimsner rings. ArXiv e-prints, Aug. 2018.

[20] W. L. Paschke. The crossed product of a C*-algebra by an endomorphism. Proceedings of the American Mathematical Society, 80(1):113–118, 1980.

[21] M. V. Pimsner. A class of C*-algebras generalizing both Cuntz-Krieger algebras and crossed products by Z. In Free probability theory (Waterloo, ON, 1995), volume 12 of Fields Inst. Commun., pages 189–212. Amer. Math. Soc., Providence, RI, 1997.

[22] H. Tominaga. On s-unital rings. Mathematical Journal of Okayama University, 18(2), 1976.