W[1]-hardness of Outer Connected Dominating set in d-degenerate Graphs

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Abstract

A set \( D \subseteq V \) of a graph \( G = (V,E) \) is called an outer-connected dominating set of \( G \) if every vertex \( v \) not in \( D \) is adjacent to at least one vertex in \( D \), and the induced subgraph of \( G \) on \( V \setminus D \) is connected. The Minimum Outer-connected Domination problem is to find an outer-connected dominating set of minimum cardinality for the input graph \( G \). Given a positive integer \( k \) and a graph \( G = (V,E) \), the Outer-connected Domination Decision problem is to decide whether \( G \) has an outer-connected dominating set of cardinality at most \( k \). The Outer-connected Domination Decision problem is known to be NP-complete, even for bipartite graphs. We study the problem of outer-connected domination on sparse graphs from the perspective of parameterized complexity and show that it is W[1]-hard on d-degenerate graphs, while the original connected dominating set has FTP algorithm on d-degenerate graphs.

Keywords: outer-connected dominating set; parameterized complexity; sparse graphs

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1 Introduction

Domination problems are classical problems in computer science and are considered with many variations and for different classes of graphs. A set \( D \subseteq V \) is a dominating set if every vertex not in \( D \) is adjacent to (is dominated by) at least one vertex in \( D \). Most of the varieties of the dominating sets are shown to be NP-complete, even for simple classes of graphs such as bipartite graphs. A considerable part of the algorithmic study on this NP-complete problem has been focused on the design of parameterized algorithms. Formally, a parameterization of a problem is assigning an integer \( k \) to each input instance as a parameter. A parameterized problem is fixed-parameter tractable (FPT) if there exists an algorithm which solves the problem in time \( f(k)\cdot|\ell|^{O(1)} \), where \( |\ell| \) is the size of the input and \( f \) is an arbitrary computable function depending only on the parameter \( k \). Just as NP-hardness is used as evidence that a problem is not probably polynomial time solvable, there exists a hierarchy of complexity classes above FPT, and showing that a parameterized problem is hard for one of these classes is considered evidence that the problem is unlikely to be fixed-parameter tractable. The main classes in this hierarchy are:

\[ FPT \subseteq W[1] \subseteq W[2] \subseteq \cdots \subseteq W[P] \subseteq XP. \]
In general, finding a dominating set of size k is a canonical W[2]-complete problem, and therefore does not admit FPT algorithms unless an unexpected collapse occurs in the W-hierarchy \[4\]. Nevertheless, there are interesting classes of graphs for which the dominating set problem admits FPT algorithms, namely sparse graphs. For example, there are FPT algorithms for nowhere dense graphs \[3\] and d-degenerate graphs \[1\]. Also, an FPT algorithm was reported in \[8\] for t-biclique-free graphs, i.e., graphs that do not contain \(K_{t,t}\) as a subgraph. To the best of our knowledge, this is the largest class of graphs for which the Dominating Set problem is known to be fixed-parameter tractable, d-degenerate and nowhere dense graphs are subclasses of t-biclique-free graphs. Figure 1 shows the relationship between some well-known classes of sparse graphs.

The outer-connected domination problem in graphs was introduced by Cyman in \[2\] and is shown it is NP-complete for bipartite graphs \[2\]. A dominating set \(S \subseteq V\) is an outer-connected dominating set, abbreviated as OCD-set if the subgraph induced by \(V \setminus S\) is connected. The outer-connected domination number, denoted by \(\gamma_c(G)\), is the cardinality of a minimum outer connected dominating set. It is subsequently studied in different classes of graphs, for instance, it is shown to be NP-complete for doubly chordal graphs \[6\], and undirected path graphs \[6\]. Also, some algorithmic and approximation results are presented in \[7\].

In this paper, we explore the W-hardness of the Outer-connected Domination Decision problem on the d-degenerate graphs. More precisely, we show there is no fixed-parameter tractable (FPT) algorithm for the outer connected dominating set on degenerate graphs, while Alon and Gutner \[1\] proved that the problems of dominating set and connected dominating set become FPT when the input graph is d-degenerate.

2 Degenerate Graphs

A graph G is d-degenerate if every subgraph of G contains a vertex of degree at most d. Equivalently, a graph G is d-degenerate if and only if there exists an elimination ordering on its vertices such that every vertex has at most d neighbors appearing later in the ordering. The following parameterized problem, proved to be W[1]-hard in \[5\], is used in our proof. In the Multicolored Independent Set problem, we are given a graph G and a coloring of V with k colors. The problem is parameterized by considering k, which equals to the number of colors, as the parameter of the input instance, and the goal is to find a k-sized independent set in G containing exactly one vertex of each color. We will reduce the Multicolored Independent Set problem (with parameter k) to the problem of finding an outer connected dominating set of size at most \(2k + 2\) in a graph of degeneracy at most 3.

The reduction. Let k be an integer and G be a k-colored graph where its vertices are partitioned into k groups \(V_1, V_2, \ldots, V_k\) and each group corresponds to an independent set of the same color, because it is a valid coloring. For every edge \(e = \{u, v\} \in E(G)\), we replace it by a 2-path \(u, v_e, v\), where \(v_e\) is a new
vertex corresponding to the edge $e$. The set $S_E$ is the collection of all vertices $v_e$ which are added to the graph $G'$. We connect all the vertices in the set $S_E$, that is the newly added vertices, to a new vertex $r$. For $1 \leq i \leq k$, we add new vertices $x_{i1}, x_{i2}$ and $u_i$ to each group $V_i$ and connect them to all the old vertices in $V_i$. Also, we build a path of length 4 for each group and connect all the vertices in the path to the vertex $u_i$ and connect $x_{i1}$ and $x_{i2}$ to the first vertices in the path. We also add a new vertex $r'$ and add edge between the last vertex of each path of length 4 and $r'$. Then, we connect the vertex $r'$ to a new pendent vertex $r''$.

This concludes the construction of $G'$. Figure 2 shows the constructed graph $G'$ in general.

**Lemma 2.1.** The constructed graph $G'$ has degeneracy of at most 3.

**Proof.** The proof is by constructing a degeneracy ordering, which is an ordering on the vertices that we get from repeatedly removing a vertex of minimum degree in the remaining subgraph. First, we put the degree-one vertex in the degeneracy ordering and delete them. In the remaining subgraph, we select all the vertices in $S_E$ as well as all the vertices of degree three and put them in the ordering, because every vertex has degree 3. After removing all vertices of $S_E$ and paths from the graph, each vertex in any block $V_i$, for $1 \leq i \leq k$, is of degree three. This happens since after removing $S_E$, such vertices are only connected to $x_{i1}, x_{i2}$, and $u_i$. So, we can obtain this ordering. Finally, we add all of the remaining vertices.

**Lemma 2.2.** If there exists a $k$-colored independent set in $G$, then there exists an outer connected dominating set of size $2k + 2$ in $G'$.

**Proof.** Suppose that $S$ is a $k$-colored independent set in $G$. We construct the outer connected dominating set $D$ for $G'$ which contains $r, r''$, all the vertices in $S$ as well as all of the $k$ vertices $u_i$, for $1 \leq i \leq k$. Clearly, the size of $D$ is $2k + 2$. It remains to argue that $D$ is an outer connected dominating set. Each vertex $v_j \in V_i$ and each vertex in the paths of length 4 which is connected to $u_i$ is dominated by $u_i$. Moreover, each pair of $x_{i1}, x_{i2}$ vertices is dominated by a single vertex in $V_i$. All the vertices in $S_E$ are dominated by $r$. It is easy to investigate outer connectivity of this dominating set.

**Lemma 2.3.** If there exists an outer connected dominating set of size $2k + 2$ in $G'$, then there exists a $k$-colored independent set in $G$. 
Proof. Let $D$ be an outer connected dominating set of size at most $2k + 2$. First, note that the set $D$ must contain the vertices $r''$ as well as all $u_i$ vertices. Otherwise, $D$ must contain at least two vertices of each path. So, $k + 1$ vertices are needed and only $k + 1$ vertices remain to dominate the remaining vertices. To dominate the vertices in $S_E$, we select the vertex $r$. To dominate all the vertices $x^1_i$ and $x^2_i$ for each $1 \leq i \leq k$, either all of these vertices for each $i$ must be chosen or one vertex in each group $V_i$ needs to be chosen. Again, the first case is impossible, so, we must select one vertex from each group. The $k$ selected vertices in each group must not have any neighbors in $S_E$, otherwise, a vertex in $S_E$ is not connected to the vertices in $V \setminus D$. Therefore, all of these $k$ vertices are independent in the original graph.

Theorem 2.4. The Minimum Outer-connected Domination problem, parameterized by solution size, is $W[1]$-hard on graphs of degeneracy 3.

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