Cutoff Regularization Method in Gauge Theories

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Abstract

In quantum field theories divergences generally turn up in loop calculations. Renormalization is a part of the theory which can be performed only with a proper regularization. In low energy effective theories there is a natural cutoff with well defined physical meaning, but the naive cutoff regularization is unsatisfactory. A Lorentz and gauge symmetry preserving regularization method is discussed in four dimension based on momentum cutoff. First we give an overview of various regularization methods then the new regularization is introduced. We use the conditions of gauge invariance or equivalently the freedom of shift of the loop momentum to define the evaluation of the terms carrying even number of Lorentz indices, e.g. proportional to $k_\mu k_\nu$. The remaining scalar integrals are calculated with a four dimensional momentum cutoff. The finite terms (independent of the cutoff) are free of ambiguities coming from subtractions in non-trivial cases. Finite parts of the result are equal with the results of dimensional regularization. The proposed method can be applied to various physical processes where the use of dimensional regularization is subtle or a physical cutoff is present. As a famous example it is shown that the triangle anomaly can be calculated unambiguously with this new improved cutoff. The anticommutator of $\gamma^5$ and $\gamma^\mu$ multiplied by five $\gamma$ matrices is proportional to terms that do not vanish under a divergent loop-momentum integral, but cancel otherwise.

1 Introduction

Several regularization methods are known and used in quantum field theory: three and four dimensional momentum cutoff, Pauli-Villars type, dimensional regularization, lattice regularization, Schwinger’s proper time method and others directly linked to renormalization like differential renormalization. Dimensional regularization (DREG) \cite{1} is the most popular and most appreciated as it respects the gauge and Lorentz symmetries. However DREG is not useful in all cases, for example it is not directly applicable to supersymmetric gauge theories as it modifies the number of bosons and fermions differently. DREG gets rid of (does not identify) naive quadratic divergences, which may

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be important in low energy effective theories or in the Wilson’s renormalization group method. Another shortcoming is that together with (modified) minimal subtraction DREG is a “mass independent” scheme, particle thresholds and decoupling are put in the theory by hand [2]. The choice of the ultraviolet regulator always depends on the problem.

In low energy effective field theories there is an explicit cut off, with well defined physical meaning. The cut off gives the range of the validity of the model. There are a few implementations in four dimensional theories: sharp momentum cut off in 3 or 4 dimensions, modified operator regularization (based on Schwinger proper time method [3]). In the Nambu-Jona-Lasinio model different regularizations proved to be useful calculating different physical quantities [4].

Using a naive momentum cut off the symmetries are badly violated. The calculation of the QED vacuum polarization function ($\Pi_{\mu\nu}(q)$) shows the problems. The Ward identity tells us that $q^\mu \Pi_{\mu\nu}(q) = 0$, e.g. in

$$\Pi_{\mu\nu}(q) = q_\mu q_\nu \Pi_L(q^2) - g_{\mu\nu} q^2 \Pi_T(q^2)$$

the two coefficients must be the same $\Pi(q^2)$. Usually the condition $\Pi(0) = 0$ is required to define a subtraction to keep the photon massless at 1-loop. However this condition is ambiguous when one calculates at $q^2 \neq 0$ in QED or in more general models. For example in the case of two different masses in the loop, it just fixes $\Pi(q^2, m_1, m_2)$ in the limit of degenerate masses at $q^2 = 0$. Ad hoc subtractions does not necessarily give satisfactory results.

There were several proposals how to define symmetry preserving cut off regularization. Usual way is to start with a regularization that respects symmetries and find the connection with momentum cut off. In case of dimensional regularization already Veltman observed [5] that the naive quadratic divergences can be identified with the poles in two dimensions (d=2) besides the usual logarithmic singularities in d=4. This idea turned out to be fruitful. Hagiwara et al. [6] calculated electroweak radiative corrections originating from effective dimension-six operators, and later Harada and Yamawaki performed the Wilsonian renormalization group inspired matching of effective hadronic field theories [7]. Based on Schwinger’s proper time approach Oleszczuk proposed the operator regularization method [8], and showed that it can be formulated as a smooth momentum cut off respecting gauge symmetries [8, 9]. A momentum cut off is defined in the proper time approach in [10] with the identification under loop integrals

$$k_\mu k_\nu \rightarrow \frac{1}{d} g_{\mu\nu} k^2$$

instead of the standard $d = 4$. The degree of the divergence determines $d$ in the result: $\Lambda^2$ goes with $d = 2$ and $\ln(\Lambda^2)$ with $d = 4$. This way the authors get correctly the divergent parts, they checked them in the QED vacuum polarization function and in the phenomenological chiral model.

1In what follows we denote the metric tensor by $g_{\mu\nu}$ both in Minkowski and Euclidean space.
Various authors formulated consistency conditions to maintain gauge invariance during the evaluation of divergent loop integrals. Finite \[11\] or infinite \[12, 13\] number of new regulator terms added to the propagators a’la Pauli-Villars, the integrals are tamed to have at most logarithmic singularities and become tractable. Pauli-Villars regularization technique were applied with subtractions to gauge invariant and chiral models \[14, 15, 16, 17\]. Differential renormalization can be modified to fulfill consistency conditions automatically, it is called constrained differential renormalization \[18\]. Another method, later proved to be equivalent with the previous one \[19\], is called implicit regularization, a recursive identity (similar to Taylor expansion) is applied and all the dependence on the external momentum (\(q\)) is moved to finite integrals. The divergent integrals contain only the loop momentum, thus universal local counter terms can cancel the potentially dangerous symmetry violating contributions \[20, 21\]. A strictly four dimensional approach to quantum field theory is proposed in \[22\]. They interpreted ultraviolet divergencies as a natural separation between physical and non-physical degrees of freedom providing gauge invariant and cutoff independent loop integrals, it was also applied to non-renormalizable theories \[23\]. Gauge invariant regularization is implemented in exact renormalization group method providing a cutoff without gauge fixing in \[24\]. Introducing a multiplicative regulator in the d-dimensional integral, the integrals are calculable in the original dimension with the tools of DREG \[25\].

In this chapter we give a definite method in four dimensions to use a well defined momentum cutoff. We show that there is a tension between naive application of Lorentz symmetry and gauge invariance. The core of the problem is that contraction with \(g^{\mu\nu}\) cannot necessarily be interchanged with the integration in divergent cases. The proper handling of the \(k_\mu k_\nu\) terms in divergent loop-integrals solves the problems of momentum cutoff regularizations. Working in strictly four dimensions we use the conditions of respecting symmetries to define the integrals with free Lorentz indices. Using our method loop calculations can be reduced to scalar integrals and those can be evaluated with a sharp momentum cutoff. We give a simple and well defined algorithm to have unambiguous finite and infinite terms \[26\] dubbed as improved momentum cutoff regularization. The method was successfully applied earlier to a non-renormalizable theory \[27, 28\]. The results respect gauge (chiral and other) symmetries and the finite terms agree with the result of DREG. There were various other proposals to modify the calculation with momentum cutoff to respect Lorentz and gauge symmetries \[8, 9, 11, 13, 24\].

An ideal application of the improved cutoff is the unambiguous calculation of the triangle anomaly in four dimensions presented in \[29\]. Dimensional regularization \[11\] respects Lorentz and gauge symmetries, but as it modifies the number of dimensions (at least in the loops) it is not directly applicable to chiral theories, such as the standard model or to supersymmetric theories. Continuation of \(\gamma_5\) to dimensions \(d \neq 4\) goes with a \(\gamma_5\) not anticommuting with the extra elements of gamma matrices, and it leads to “spurious anomalies”, see \[30, 31, 32, 33\], and references therein. The loop integrals using the novel improved momentum cutoff regularization are invariant to the shift of the loop momentum, therefore the usual derivation of the ABJ triangle anomaly
would fail in this case. We extend the method to graphs involving $\gamma_5$, and show that the proper handling of the trace of $\gamma_5$ and six gamma matrices provides the correct anomaly, the $\{\gamma_5, \gamma_\mu\}$ anticommutator does not vanish in special cases under divergent loop integrals.

The rest is organized as follows. In section 2 we present how to define a momentum cutoff using the method of DREG, then we give the gauge symmetry preserving conditions emerging during the calculation of the vacuum polarization amplitude. In section 4 we discuss the condition of independence of momentum routing in loop diagrams. Section 5 shows that gauge invariance and freedom of shift in the loop momentum have the same root. Next we show that the conditions are related to vanishing surface terms. In section 7 we give a definition of the new regularization method and in section 8 as an example we present the calculation of a general vacuum polarization function at 1-loop. In section 9 we show that the QED Ward-Takahashi identity holds at finite order using the new method. Section 10 deals with the famous triangle anomaly and the chapter is closed with conclusions.

## 2 Momentum cutoff via dimensional regularization

DREG is very efficient and popular, because it preserves gauge and Lorentz symmetries. Performing standard steps the integrals are evaluated in $d = 4 - 2\epsilon$ dimension. Generally the loop momentum integral is Wick rotated and with a Feynman parameter $(x)$ the denominators are combined, then the order of $x$ and momentum integrals are changed. Shifting the loop momentum does not generate surface terms and it leads to spherically symmetric denominator, terms linear in the momentum are dropped and (2) is used. Singularities are identified as $1/\epsilon$ poles, naive power counting shows that these are the logarithmic divergences of the theory. In DREG quadratic or higher divergences are set identically to zero. However Veltman noticed [5] that quadratic divergences can be calculated in $d = 2 - 2(\epsilon - 1)$ in the limit $\epsilon \to 1$. This observation led to a cutoff regularization based on DREG.

Carefully calculating the one and two point Passarino-Veltman functions in DREG and in 4-momentum cutoff the divergences can be matched as [6, 7]

$$
4\pi\mu^2 \left( \frac{1}{\epsilon - 1} + 1 \right) = \Lambda^2,
$$

$$
\frac{1}{\epsilon} - \gamma_E + \ln \left( 4\pi\mu^2 \right) + 1 = \ln \Lambda^2,
$$

where $\mu$ is the mass-scale of dimensional regularization and $\gamma_E$ is the Euler-Macheroni constant appearing always together with $1/\epsilon$. The finite part of a divergent quantity is defined as

$$
f_{\text{finite}} = \lim_{\epsilon \to 0} \left[ f(\epsilon) - R(0) \left( \frac{1}{\epsilon} - \gamma_E + \ln 4\pi + 1 \right) - R(1) \left( \frac{1}{\epsilon - 1} + 1 \right) \right],
$$

where $R(0), \quad R(1)$ are the residues of the poles at $\epsilon = 0, 1$ respectively. Note that
in the usual $\epsilon \to 0$ limit the left hand side (LHS) of (3) vanishes and no quadratic divergence appears in the original DREG.

The identifications above define a momentum cutoff calculation based on the symmetry preserving DREG formulae. This cutoff regularization is well defined, but still relies on DREG. Let us see the main properties in the calculation of the vacuum polarization function. In $\Pi_{\mu\nu}$ the quadratic divergence is partly coming from a $k_\mu k_\nu$ term via $\frac{1}{4} \cdot g_{\mu\nu} k^2$, which is evaluated at $d = 2$ instead of the $d = 4$ in the naive cutoff calculation. The $\Lambda^2$ terms cancel if and only if this term is evaluated at $d = 2$. This is a warning that the usual

$$k_\mu k_\nu \to \frac{1}{4} g_{\mu\nu} k^2$$

substitution during the naive cutoff calculation of divergent integrals might be too naive, especially as an intermediate step, the Wick rotation is legal only for finite integrals. A further finite term additional to the logarithmic singularity is coming from the well known expansion in $\frac{1}{4 - 2\epsilon} \approx \frac{1}{2} \left( \frac{1}{\epsilon} + \frac{1}{2} \right)$, and it is essential to retain gauge invariance. We stress that the shift of the loop momentum is allowed in DREG, an improved cutoff regularization should inherit it. In the next sections we derive consistency conditions for general regularizations.

3 Consistency conditions - gauge invariance

Calculation in a gauge theory ought to preserve gauge symmetries. Consider the QED vacuum polarization function with massive electrons. We start generally (see Fig. 1.) with two fermions with different masses in the loop [27] and restrict it to QED later,

$$i \Pi_{\mu\nu}(q) = - (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left( \gamma_\mu \frac{k + m_a}{k^2 - m_a^2} \gamma_\nu \frac{k + q + m_b}{(k + q)^2 - m_b^2} \right).$$

$\Pi_{\mu\nu}$ is calculated with the standard technique, only the $k_\mu k_\nu$ terms are considered with care. After performing the trace, Wick rotating and introducing the Feynman $x$-parameter the loop momentum is shifted $(k_\mu + x q_{E\mu}) \to l_{E\mu}$,

$$\Pi_{\mu\nu} = g^2 \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \frac{2l_{E\mu} l_{E\nu} - g_{\mu\nu} (l_E^2 + \Delta) - 2x(1-x)q_{E\mu}q_{E\nu} + 2x(1-x)g_{\mu\nu} q_E^2}{(l_E^2 + \Delta)^2},$$

$$\Delta = \frac{m_a^2 - m_b^2}{(k + q)^2 - m_b^2}.$$
where $\Delta = x(1 - x)q_E^2 + (1 - x)m_a^2 + xm_b^2$. In QED $m_a = m_b = m$ and $g = e$, it simplifies to $\Delta_1 = x(1 - x)q_E^2 + m^2$. Having a symmetric denominator and symmetric volume of integration the terms linear in $l_{E\mu}$ are dropped. After changing the order of momentum- and $x$-integration the loop momentum is shifted with $x$-dependent values, $xq_E\mu$ and sum up the results during the integration. Different shifts sums up to a meaningful result only if the shift does not modify the value of the momentum integral (it will be discussed in the next section).

In QED the Ward identity tells us, that

$$q^\nu \Pi_{\mu\nu} (q) = 0. \quad (9)$$

In (8) the terms proportional to $q_E$ fulfill the Ward-identity (9) and what remains is the condition of gauge invariance

$$\int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \frac{l_{E\mu}l_{E\nu}}{(l_E^2 + \Delta_1)^2} = \frac{1}{2} g_{\mu\nu} \int_0^1 dx \int \frac{d^4l_E}{(2\pi)^4} \frac{1}{(l_E^2 + \Delta_1)}. \quad (10)$$

This condition appeared already in [13, 20]. Any gauge invariant regulator should fulfill (10). It holds in dimensional regularization and in the momentum cutoff based on DREG of Section 2. In [11] a similar relation defined the finite or infinite Pauli-Villars terms to maintain gauge invariance.

So far the $x$ integrals were not performed. Expanding the denominator in $q_E^2$, the $x$-integration can be easily done and we arrive at a condition for gauge invariance at each order of $q^2$. At order $q^{2n}$ we get (omitting the factor $(2\pi)^4$)

$$\int d^4l_E \frac{l_{E\mu}l_{E\nu}}{(l_E^2 + m^2)^{n+1}} = \frac{1}{2n} g_{\mu\nu} \int d^4l_E \frac{1}{(l_E^2 + m^2)^n}, \quad n = 1, 2, ... \quad (11)$$

The conditions (11) are valid for arbitrary $m^2$ mass, so it holds for any function $\Delta$ independent of the loop momentum in 1-loop two or n-point functions with arbitrary masses in the propagators. These conditions mean that in any gauge invariant regularization the two sides of (11) should give the same result. We will use this condition to define the LHS of (11) in the new improved cutoff regularization. This is the novelty of our regularizations method.

4 Consistency conditions - momentum routing

Evaluating any loops in QFT one encounters the problem of momentum routing. The choice of the internal momenta should not affect the result of the loop calculation. The simplest example is the 2-point function. In (7) there is a loop momentum $k$, and the external momentum $q$ (see Fig. 1.) is put on one line $(k + q, k)$, but any partition of the external momentum $(k + q + p, \ k + p)$ must be as good as the original. The arbitrary shift of the loop momentum should not change the physics. This independence of the choice of the internal momentum gives a conditions. We will impose it on a very simple
loop integral
\[ \int d^4 k \frac{k_\mu}{k^2 - m^2} - \int d^4 k \frac{k_\mu + p_\mu}{(k + p)^2 - m^2} = 0 \]  \tag{12}
which turns up during the calculation of the 2-point function. Expanding \ref{12} in powers of \( p \) we get a series of conditions, meaningful at \( p, p^3, p^5 \ldots \). At linear order we arrive at
\[ \int d^4 k \left( \frac{p_\mu}{k^2 - m^2} - 2 \frac{k_\mu k \cdot p}{(k^2 - m^2)^2} \right) = 0, \]  \tag{13}
which is equivalent to \ref{11} for \( n = 1 \). At order \( p^3 \) a linear combination of two conditions should vanish
\[ p_\rho p_\alpha p_\beta \int d^4 k \left[ \frac{4 k_\alpha k_\beta}{(k^2 - m^2)^3} \right. \left. - \frac{g_{\alpha \beta}}{(k^2 - m^2)^2} \right] g_{\mu \rho} - 4 k_\mu \left( \frac{2 k_\alpha k_\beta k_\rho}{(k^2 - m^2)^4} - \frac{g_{\alpha \beta} k_\rho}{(k^2 - m^2)^3} \right) \]  \tag{14}
These two conditions get separated if the freedom of the shift of the loop momentum is considered in \( \int d^4 k \frac{k_\mu}{(k^2 - m^2)^2} \). At leading order it provides
\[ p_\nu \int d^4 k \left( \frac{g_{\mu \nu}}{(k^2 - m^2)^2} - 4 \frac{k_\mu k_\nu}{(k^2 - m^2)^3} \right) = 0, \]  \tag{15}
equivalent with \ref{11} for \( n = 2 \). Using \ref{15} twice the second part of the condition \ref{14} connects 4 loop momenta numerators to 2 \( k \)'s. Symmetrizing the indices we get
\[ \int d^4 k \frac{k_\alpha k_\beta k_\mu k_\rho}{(k^2 - m^2)^4} = \frac{1}{24} \int d^4 k \frac{g_{\alpha \beta} g_{\mu \rho} + g_{\alpha \mu} g_{\beta \rho} + g_{\alpha \rho} g_{\beta \mu}}{(k^2 - m^2)^2}. \]  \tag{16}
Invariance of momentum routing provides conditions for symmetry preserving regularization and these conditions are equivalent with the conditions coming from gauge invariance.

## 5 Gauge invariance and loop momentum shift

We show at one loop level that gauge invariance of the vacuum polarization function is equivalent to invariance of a special loop integrand against shifting the loop momentum \ref{12}. Consider \( \Pi_{\mu \nu} \) defined in \ref{7}, performing the trace we get
\[ i \Pi_{\mu \nu}(q) = -g^2 \int \frac{d^4 k}{(2\pi)^4} \frac{k_\mu (k_\nu + q_\nu) + k_\nu (k_\mu + q_\mu) - g_{\mu \nu} (k^2 + k \cdot q - m_a m_b)}{(k^2 - m_a^2) ((k + q)^2 - m_b^2)}. \]  \tag{17}
Specially in QED \( m_a = m_b = m \), gauge invariance requires \ref{12}, which simplifies to
\[ i q^\nu \Pi_{\mu \nu}(q) = g^2 \int \frac{d^4 k}{(2\pi)^4} \left( \frac{k_\mu + q_\mu}{((k + q)^2 - m^2)} - \frac{k_\mu}{(k^2 - m^2)} \right) = 0. \]  \tag{18}
This example shows that the Ward identity is fulfilled only if the shift of the loop momentum does not change the value of the integral, like in \ref{12}. 

7
In [21] based on the general diagrammatic proof of gauge invariance it is shown that the Ward identity is fulfilled if the difference of a general n-point loop and its shifted version vanishes

\[-i \int d^4p_1 Tr \left[ \frac{i}{p_n - m} \gamma^{\mu_2} \ldots \frac{i}{\not{p}_1 - m} \gamma^{\mu_1} - \frac{i}{p_n + \not{p}_1 - m} \gamma^{\mu_2} \ldots \frac{i}{\not{p}_1 + \not{p}_1 - m} \gamma^{\mu_1} \right] = 0.\]  

(19)

We interpret (18) and (19) as a necessary condition for gauge invariant regularizations.

6 Consistency conditions - vanishing surface terms

All the previous conditions are related to the volume integral of a total derivative

\[\int d^4k \frac{\partial}{\partial k^\nu} \left( \frac{k^\mu k^\nu}{(k^2 + m^2)^n} \right) = \int d^4k \left( \frac{k^\mu k^\nu}{(k^2 + m^2)^{n+1}} - \frac{1}{2n} g_{\mu\nu} \frac{1}{(k^2 + m^2)^n} \right), \quad n = 1, 2, \ldots\]  

(20)

The total derivative on the LHS leads to surface terms [34], which vanish for finite valued integrals and should vanish for symmetry preserving regularization. In our improved regularization this will follow from new definitions. The LHS is in connection with an infinitesimal shift of the loop momentum \( k \), it should be zero if the integral of the term in the delimiter is invariant against the shift of the loop momentum. The vanishing of this surface terms reproduces on the RHS the previous conditions (13) and (11). In (20) starting with any odd number of \( k \)'s in the numerator we end up with some conditions, three \( k \)'s for \( n = 3 \) provide (16) after some algebra. Starting with even number of \( k_\mu \)'s in the numerator on the LHS in (20) we get relations between odd number of \( k_\mu \)'s in the numerators, which vanish separately.

These surface terms all vanish in DREG and give the basis of DREG respecting Lorentz and gauge symmetries. Vanishing of the surface term is inherited to any regularization, like improved momentum cutoff, if the identification (10) is understood to evaluate integrals involving even number of free Lorentz indices, e.g. numerators alike \( k_\mu k_\nu \). The value of integrals with odd number of \( k \)'s in the numerator are similarly dictated by symmetry, these are required to vanish by the symmetry of the integration volume.

7 Improved momentum cutoff regularization

We propose a new symmetry preserving regularization based on 4-dimensional momentum cutoff. During this improved momentum cutoff regularization method a simple sharp momentum cutoff is introduced to calculate the divergent scalar integrals in the end. The evaluation of loop-integrals starts with the usual Wick rotation, Feynman parametrization and loop momentum shift. The only crucial modification is that the potentially symmetry violating loop integrals containing explicitly the loop momenta with free Lorentz indices are calculated with the identification
\[
\int d^4 l_E \frac{l_{E\mu}l_{E\nu}}{(l_E^2 + \Delta)^{n+1}} \rightarrow \frac{1}{2n} g_{\mu\nu} \int d^4 l_E \frac{1}{(l_E^2 + \Delta)^n} \tag{21}
\]

under the loop integrals or with more momenta using the condition (16) or generalizations of it, like

\[
\int d^4 l_E \frac{l_{E\mu}l_{E\nu}l_{E\rho}l_{E\sigma}}{(l_E^2 + \Delta)^{n+1}} \rightarrow \frac{g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho} + g_{\mu\nu}g_{\rho\sigma}}{4n(n-1)} \cdot \int d^4 l_E \frac{1}{(l_E^2 + \Delta)^n}. \tag{22}
\]

The momentum integrals containing further the loop momentum with indices summed up (e.g. \(l_E^2\)) in the numerator are simplified in a standard way cancelling a factor in the denominator

\[
\int d^4 l_E \frac{l_{E\mu}l_{E\nu}l_{E\rho}l_{E\sigma}}{(l_E^2 + \Delta)^{n+1}} = \int d^4 l_E \frac{l_{E\mu}l_{E\nu}l_{E\rho}l_{E\sigma}}{(l_E^2 + \Delta)^n} - \int d^4 l_E \frac{\Delta l_{E\mu}l_{E\nu}l_{E\rho}l_{E\sigma}}{(l_E^2 + \Delta)^{n+1}}. \tag{23}
\]

Integrals with odd number of the loop momenta vanish identically. These identifications guarantee gauge invariance and freedom of shift in the loop momentum. Under any regularized momentum integrals the identifications (21) or generalizations like (22) are understood as a part of the regularization procedure for \(n = 1, 2, \ldots\). For finite integrals (non divergent, for high enough \(n\) the standard calculation automatically fulfills (21, 22). The connection with the standard substitution of free indices is discussed in Appendix A.

Fulfilling the condition (11) via the substitution (21) the results of momentum cutoff based on DREG of section 2 are completely reproduced performing the calculation in the physical dimensions \(d = 4\) [27, 35]. The next three examples show that the new regularization provides a robust framework for calculating loop integrals and respects symmetries.

8 Vacuum polarization function

As an example let us calculate the vacuum polarization function of Fig. 1. in a general gauge theory with fermion masses \(m_a, m_b\). Performing the calculation in 4 dimensions generally the Ward identities (required by the theory) are restored by ambiguous and ad hoc subtractions. The finite terms of different calculations do not match each other in the literature, see [36], papers citing it and [37]. For sake of simplicity we consider only vector couplings. Performing the trace in (7) we get (17). Now we can introduce a Feynman x-parameter, shift the loop momentum and get [43] after dropping the linear terms. Generally we are interested in low energy observables like the precision electroweak parameters and need the first few terms in the power series of \(\Pi_{\mu\nu}(q)\).

Using the rule (21) for \(n = 1\) and expanding the denominator in \(q^2\) the scalar loop and x-integrals can be easily calculated with a 4-dimensional momentum cutoff (\(\Lambda\)). The result in this construction is automatically transverse

\[
\Pi_{\mu\nu}(q) = \frac{g^2}{4\pi^2} \left( q^2 g_{\mu\nu} - q_\mu q_\nu \right) \left[ \Pi(0) + q^2 \Pi'(0) + \ldots \right]. \tag{24}
\]
The terms independent of the cutoff completely agree with the results of DREG [35]; the logarithmic singularity can be matched with the $1/\epsilon$ terms using (4). Up to $O \left( \frac{m^2}{\Lambda^2} \right)$ we get

$$\Pi(0) = \frac{1}{4}(m_a^2 + m_b^2) - \frac{1}{2}(m_a - m_b)^2 \ln \left( \frac{\Lambda^2}{m_a m_b} \right) - \frac{m_a^4 + m_b^4 - 2m_a m_b (m_a^2 + m_b^2)}{4(m_a^2 - m_b^2)} \ln \left( \frac{m_b^2}{m_a^2} \right).$$

The first derivative is

$$\Pi'(0) = -\frac{2}{9} - \frac{4m_a^2 m_b^2 - 3m_a m_b (m_a^2 + m_b^2)}{6(m_a^2 - m_b^2)^2} + \frac{1}{3} \ln \left( \frac{\Lambda^2}{m_a m_b} \right) + \frac{1}{6} \ln \left( \frac{m_b^2}{m_a^2} \right).$$

The photon remains massless in QED, as in the limit, $m_a = m_b$ we get $\Pi(0) = 0$.

The proposed regularization is robust and gives the same result if the calculation is organized in a different way. Introducing Feynman parameters and shifting the loop momentum can be avoided if we need only the first few terms in the Taylor expansion of $q$. For small $q$ the second denominator in (17) can be Taylor expanded, for simplicity we give the expanded integrand for equal masses, up to $O(q^4)$

$$\Pi_{\mu\nu}(q) = -g^2 \int \frac{d^4k_E}{(2\pi)^4} \left[ 2k_{\mu}k_{\nu} \left( \frac{1}{(k_E^2 + m^2)^2} - \frac{q_E^2}{(k_E^2 + m^2)^3} + \frac{4(k_E^2 q_E^2)^2}{(k_E^2 + m^2)^4} \right) - \frac{2(k_{E\mu}q_{E\nu} + k_{E\nu}q_{E\mu})}{(k_E^2 + m^2)^3} k_E \cdot q_E - g_{\mu\nu} \left( \frac{1}{(k_E^2 + m^2)^2} - \frac{q_E^2}{(k_E^2 + m^2)^3} + \frac{2(k_E \cdot q_E)^2}{(k_E^2 + m^2)^4} \right) \right].$$

Taking into account that $k_E \cdot q_E = k_{E\mu}q_{E\mu}$, (21) and (22) can be used and the remaining scalar integrals can be easily calculated. The result agrees with (25) and (26) and the finite terms with DREG if and only if we use the proposed symmetry preserving substitutions. Applying the naive $k_{E\mu}k_{E\nu} \rightarrow \frac{1}{2}g_{\mu\nu}k_E^2$ substitution [6] in both approaches the finite terms will differ from each other and also from the result of DREG. This is why finite terms differ from each other in [36] and [37].

The gauge invariance of the improved momentum cutoff regularization can be checked directly by the well known identity in QED.

### 9 The Ward-Takahashi identity

In this section we show by explicit calculation that the QED Ward-Takahashi identity is fulfilled for infinite and finite terms using the proposed regularization at 1-loop. The identity reflects the gauge invariance of the underlying theory.
Following the notation of \[38\] it has to be proved that
\[
\frac{d\Sigma}{dp} = -\delta\Gamma^\mu(p, p) \bigg|_{p'=m},
\] (28)
where \(\Sigma\) is the electron self-energy (see Fig. 2. left panel)
\[
-\I\bar{u}(p)\Sigma u(p) = -e^2 \int_0^1 dx \int \frac{d^4l}{(2\pi)^4} \bar{u}(p) \frac{-2x \not{p} + 4m}{(l^2 - \Delta_2 + i\epsilon)^2} u(p),
\] (29)
here \(\Delta_2 = -x(1-x)p^2 + (1-x)m^2 + x\mu^2, l = k - xp, m\) is the mass of the electron and \(\mu\) is the infrared regulator.
\[
\frac{d\Sigma}{dp} \bigg|_{p'=m} = \frac{\alpha}{2\pi} \int_0^1 dx \left[ -x\gamma^\mu \left( \ln \left( \frac{\Lambda^2}{(1-x)^2m^2 + x\mu^2} \right) - 1 + \frac{2(2-x)(1-x)}{(1-x)^2m^2 + x\mu^2} \right) \right],
\] (30)
\(\delta\Gamma^\mu\) is the electron vertex correction (see Fig. 2. right panel)
\[
\bar{u}(p')\delta\Gamma^\mu u(p) = 2ie^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}(p') \left[ \gamma^\mu(k + \not{l}) + m^2\gamma^\mu - 2m(k + (k+q))^\mu \right] u(p)
\] (31)
After using the Dirac equation in the limit \(p = p'\) and \(q = 0\) we get
\[
-\I\bar{u}(p)\delta\Gamma^\mu u(p) = 2e^2 \int_{0}^{1} \int_0^1 dx dy dz \delta(x + y + z - 1) \int \frac{d^4l}{(2\pi)^4} \bar{u}(p) \frac{[\gamma^\mu \not{l} + (z^2 - 4z + 1)m^2\gamma^\mu] u(p)}{(l^2 - \Delta_3 + i\epsilon)^3},
\] (32)
where \(\Delta_3 = (1-z^2)m^2 + z\mu^2\) and \(l = k - zp\). Here \(\gamma^\mu \not{l} = 2l^\nu\gamma_\nu - \gamma^\mu l^2\), for the first term \([21]\) should be used for \(n = 2\) or directly \([61]\) from Appendix B. After the momentum and \(x, y\) integration
\[
\delta\Gamma^\mu \bigg|_{p'=m} = \frac{\alpha}{2\pi} \int_0^1 dz \left[ (1-z) \left( \ln \left( \frac{\Lambda^2}{(1-z)^2m^2 + z\mu^2} \right) - 1 + \frac{(1-4z + z^2)}{(1-z)m^2 + z\mu^2} \right) \right].
\] (33)
The result of the new method is the constant $-1$ after the log, with the naive calculation using (50) one would get $-1/2$. Calculating the Feynman-parameter integral taking care of the infrared regulator the identity (28) holds up to $m^2/\Lambda^2$ terms at 1-loop

$$- \frac{d\Sigma}{dp^\mu} \bigg|_{p=m} = \delta\Gamma^\mu(p,p) \bigg|_{p=m} = \frac{\alpha}{2\pi} \left( \frac{1}{2} \ln \left( \frac{\Lambda^2}{m^2} \right) + \ln \left( \frac{\mu^2}{m^2} \right) + 2 \right) + O \left( \frac{m^2}{\Lambda^2} \right).$$

(34)

We have seen that the proposed method provides regularized 1-loop electron self-energy and vertex correction in QED which fulfill the Ward-Takahashi identity.

10 Triangle anomaly

In the improved momentum cutoff framework the triangle anomaly has to be recalculated. The loop integrals using the novel improved momentum cutoff regularization are invariant to the shift of the loop momentum, therefore the usual derivation of the ABJ triangle anomaly would fail in this case (cannot pick up a finite term shifting the linear divergence). We extend our regularization to graphs involving $\gamma_5$, and show that the proper handling of the trace of $\gamma_5$ and six gamma matrices provides the correct anomaly, the $\{\gamma_5, \gamma_\mu\}$ anticommutator does not vanish under divergent loop integrals. In this section we show that the new method provides a well defined result for the famous triangle anomaly.

Consider the 1-loop triangle graph on the left on Fig. 3.

$$T^{\mu\nu\rho}_1 = e^2 \int \frac{d^4k}{(2\pi)^4} Tr \left( \gamma_5 \frac{k - \beta_1 + m}{(k - q_1)^2 - m^2} \gamma^\mu \frac{k + m}{k^2 - m^2} \gamma^\nu \frac{k + \beta_2 + m}{(k + q_2)^2 - m^2} \gamma^\rho \right).$$

(35)

The amplitude of the crossed graph $T^{\mu\nu\rho}_2$ is similar with $(q_1, \mu)$ and $(q_2, \nu)$ interchanged ($T^{\mu\nu\rho} = T^{\mu\nu\rho}_1 + T^{\mu\nu\rho}_2$). The Ward identities require

$$q_{1\mu} T^{\mu\nu\rho}_1 = 0,$$

(36)

$$q_{2\nu} T^{\mu\nu\rho}_1 = 0,$$

(37)

$$-(q_1 + q_2)_\rho T^{\mu\nu\rho}_1 = 2m T^{5\mu\nu},$$

(38)
where $T^{\mu\nu}$ corresponds to the same graphs with a pseudoscalar current instead of the axialvector one. There is a formal proof of (38). Replace

$$-(q_1 + q_2)_\rho \gamma^\rho \gamma^5 = -(k + \gamma_2 - m) \gamma^5 + (k - \gamma_1 - m) \gamma^5.$$  \hfill (39)

The first term combines with the numerator of the last term in (35) and cancels the denominator. If

$$\{\gamma^\mu, \gamma^5\} = 0 \hfill (40)$$

assumed, then the second term in (39) is

$$-(k - \gamma_1 - m) \gamma^5 = +\gamma^5 (k - \gamma_1 - m) + 2m\gamma^5.$$  

Here the first term cancels the adjacent fraction in (35) and the second term gives the right hand side of (38). The

$$-((q_1 + q_2)_\rho T^{\mu\nu\rho}) = m\gamma^5.$$  \hfill (41)

Shifting $k \rightarrow k + q_1$ in the first term and passing $\gamma^\mu$ through $\gamma^5$ (using again (40)) to the back of the second term we arrive to a formula that is totally antisymmetric under the interchange of $(q_1, \mu)$ and $(q_2, \nu)$, and thus adding the crossed graph $(T^{\mu\nu\rho}_2)$ the result vanishes. Similarly (36) and (37) can be proven but in this case (40) is not needed to apply, because the terms leading to cancellation are not separated by a factor of $\gamma^5$.

The loop momentum can be shifted, this is a fundamental property of the improved momentum cutoff regularization. However (36-38) cannot be all true. Pauli-Villars regularization or careful simple momentum cutoff calculation identifies a finite anomaly term when shifting the linearly divergent integral. There is still a remaining ambiguity in connection with momentum routing and which identity contains the anomaly term in (36-38). At the same time in improved momentum cutoff or DREG (36) and (37) holds but the proof of (38) is false, it relies additionally on (40). This is the first sign that the naive anticommutator (40) cannot be used in all situations.

The explicit calculation of the triangle diagram (35) is based on the evaluation of the trace of $\gamma^5$ with six $\gamma$’s. There are various methods to calculate this trace with superficially different terms at the end. The different results of the trace can be transformed to each other using the Schouten identity, involving two loop momenta it reads

$$-k^2 \epsilon_{\mu\nu\lambda\rho} + k^\alpha k_\mu \epsilon_{\alpha\nu\lambda\rho} + k_\nu k^\alpha \epsilon_{\mu\alpha\lambda\rho} + k_\lambda k^\alpha \epsilon_{\mu\nu\alpha\rho} + k_\rho k^\alpha \epsilon_{\mu\nu\lambda\rho} = 0.$$  \hfill (42)

In the present method this identity cannot be used for the loop momentum $k$ of a divergent integral before applying the identifications (21) or (22), because it would mix free Lorentz indices and contracted indices, which must be evaluated in a different way (DREG faces the same difficulty). After performing the identifications (21) and (22) the quadratic loop momenta factors cancel with the denominators. The remaining formula contains the loop momentum in the numerators at maximum linearly, the corresponding Schouten identity can be applied. The root of the problem is that in

\footnote{Functional integral derivation of the anomaly shows that the Ward identity corresponding to the axial vector current \cite{38} must be anomalous \cite{39}.}
case of divergent integrals the totally antisymmetric tensor $\epsilon_{\mu\nu\lambda\rho}$ cannot taken out of the integral, similarly to the case of $g_{\mu\nu}$ in the previous section. No such problem emerges for finite integrals.

The breakdown of the early application of the Schouten identity forces us to choose one dedicated calculation of the trace. The trace is calculated not using the anticommutator (40), only

$$\{\gamma_5, \gamma_{\nu}\} = 2 g_{\mu\nu},$$

and general properties of the trace. The unambiguous result is

$$\frac{1}{4} \text{Tr} \left[ \gamma_5 \gamma_\alpha \gamma_\mu \gamma_\beta \gamma_\nu \gamma_\rho \gamma_\lambda \right] = \epsilon_{\alpha\mu\beta\nu} g_{\rho\lambda} - \epsilon_{\alpha\mu\beta\rho} g_{\nu\lambda} - \epsilon_{\alpha\beta\nu\rho} g_{\mu\lambda} + \epsilon_{\mu\beta\nu\rho} g_{\alpha\lambda} - \epsilon_{\lambda\alpha\mu\beta} g_{\rho\nu} + \epsilon_{\lambda\alpha\mu\nu} g_{\beta\rho} - \epsilon_{\lambda\beta\nu\rho} g_{\mu\alpha} + \epsilon_{\lambda\rho\nu\beta} g_{\alpha\mu}.$$ (44)

It reflects the complete Lorentz structure of the $\gamma$ matrices in the trace. This choice of the trace also appeared in earlier papers without detailed argumentations [40, 41]. All different calculations of the trace are in agreement with each other and with (44) if (40) is modified. $\gamma_5$ and $\gamma_\mu$ does not always anticommute (rather the anticommutator picks up terms proportional to the sum of a few Schouten identities.) Explicitly, the following definition will eliminate all the ambiguities burdening the calculation of the trace of $\gamma_5$ and six $\gamma$’s

$$\text{Tr} \left[ \{\gamma_\mu, \gamma_5\} \gamma_\lambda \gamma_\alpha \gamma_\gamma_5 \gamma_\beta \gamma_\nu \right] = 2 \text{Tr} \left[ g_{\mu\nu} \gamma_5 \gamma_\lambda \gamma_\alpha \gamma_\gamma_5 \gamma_\beta \gamma_\nu - g_{\beta\nu} \gamma_5 \gamma_\lambda \gamma_\alpha \gamma_\gamma_5 \gamma_\mu \gamma_\beta \gamma_\nu + g_{\mu\rho} \gamma_5 \gamma_\lambda \gamma_\alpha \gamma_\gamma_5 \gamma_\beta \gamma_\nu \right].$$ (45)

The above anticommutator is defined only under the trace. (45) can be understood as the $\{\gamma_5, \gamma_\rho\}$ anticommutator is defined by picking up all the terms when moving $\gamma_\rho$ all the way round through the other five $\gamma$’s. Evaluating the trace the right hand side is proportional to Schouten identities. Under a divergent loop integral it will not vanish in the present method (nor in DREG). The nontrivial anticommutator contributes to the triangle anomaly but vanishes in non-divergent cases and for less $\gamma$’s. The amplitude of the triangle diagrams can be calculated with the definition of the trace (44) and the identifications (21), (22). Finally we arrive at the extra anomaly term in (38).

In what follows we calculate directly the anomaly term missing in (38). We use (39) and move $(k - \not{q}_1 - m)$ from the back to the front in (35) using (43). Without this trick the same result is obtained evaluating the trace of six $\gamma$’s and $\gamma_5$ using (44),
which is consistent with non-anticommuting $\gamma^5$ in this special case, see \([15]\).

\[- (q_1 + q_2)_\rho T_{1}^{\mu\nu} = e^2 \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ -\gamma^5 \frac{1}{k - \ell_1} - m \gamma^\mu \frac{1}{k - m \gamma^\nu} \frac{1}{k + m \gamma^\nu} \frac{1}{k + \ell_2 + m} + +2\gamma^5 \frac{1}{k - \ell_1 - m} \gamma^\mu \frac{1}{k - m} (k^\nu - q_i^\nu) + -2\gamma^5 \frac{1}{k - \ell_1 - m} \gamma^\mu \gamma^\nu \frac{1}{k + m} \cdot (k^\mu - q_i^\mu) + +2\gamma^5 \frac{1}{k - \ell_1 - m} \gamma^\mu \gamma^\nu \frac{1}{k + \ell_2 + m} \cdot k^\nu - -2\gamma^5 \frac{1}{k - \ell_1 - m} \gamma^\mu \gamma^\nu \frac{1}{k + m} \cdot (k^\mu - q_i^\mu) \right] \] (46)

With algebraic manipulations using the antisymmetry of the trace including $\gamma_5$ and four $\gamma$'s we can group the terms

\[- (q_1 + q_2)_\rho T_{1}^{\mu\nu} = e^2 \int \frac{d^4k}{(2\pi)^4} \text{Tr} \gamma^5 \left[ \frac{k}{N_1 N_2} f_1 \gamma^\mu \gamma^\nu + \frac{k}{N_2 N_3} f_2 \gamma^\mu \gamma^\nu + \frac{2}{N_1 N_2 N_3} \left\{ - f_1 f_2 \gamma^\mu \gamma^\nu \cdot k^2 + f_1 f_2 \gamma^\mu \gamma^\nu \cdot k q_2 + - f_1 f_2 \gamma^\mu \gamma^\nu \cdot k q_1 - f_1 f_2 \gamma^\mu \gamma^\nu \cdot q_i^2 - f_1 f_2 \gamma^\mu \gamma^\nu \cdot q_i q_j - f_1 f_2 \gamma^\mu \gamma^\nu \cdot q_i q_j \right\} \right] \] (47)

where $N_1 = ((k - q_1)^2 - m^2)$, $N_2 = (k^2 - m^2)$ and $N_3 = ((k + q_2)^2 - m^2)$. The first two terms vanish after performing the trace and the integral (they are proportional to $\epsilon^{\mu\rho\sigma} q_{1a} q_{1b}$ and $\epsilon^{\mu\rho\sigma} q_{2a} q_{2b}$ respectively). The third one gives $2m$ times the pseudoscalar amplitude $T_5^{\mu\nu} = T_1^{5\mu\nu} + T_2^{5\mu\nu}$,

\[T_1^{5\mu\nu} = - m \epsilon^{\mu\rho\sigma} q_{1a} q_{2b} e^2 \int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{N_1 N_2 N_3} \right], \] (48)

we get $T_2^{5\mu\nu}$ interchanging $(q_1, \mu) \leftrightarrow (q_2, \nu)$ in the integrand.

The last five terms in \([47]\) contain one factor of the loop momentum $(k)$ and after tracing vanish by the Schouten identity, the loop integration does not spoil the cancellation. The contribution of the one but last five terms in the curly bracket does not vanish. It contains two factor of the loop momentum, and it is proportional to the Schouten identity \([12]\) broken under the divergent loop integral. Calculating it with
the improved momentum cutoff of Section 2 using the formulas of the Appendix (or with DREG) we get the anomaly term

$$- (q_1 + q_2)_\mu T^{\mu \nu} = 2mT^{\gamma_\nu} - i \frac{e^2}{2\pi^2} \epsilon^{\mu \nu \alpha \beta} q_1_\alpha q_2_\beta. \quad (49)$$

In the case of the naive substitution (6) the Schouten identity (42) is satisfied, the curly bracket vanishes. (In that case with simple momentum cutoff the anomaly term originates from shifting the linearly divergent first two terms in (47), but the result depends on momentum routing.) The presented method identifies without ambiguity the value of the anomaly in the axial-vector current and leaves the vector currents anomaly free without any further assumptions.

11 Conclusions

We have presented in this chapter a new method for the reliable calculation of divergent 1-loop diagrams (even involving $\gamma_5$) with four dimensional momentum cutoff. Various conditions were derived to maintain gauge symmetry, to have the freedom of momentum routing or shifting the loop momentum. These conditions were known by several authors [11, 13, 20, 21]. Our new proposal is that these conditions will be satisfied during the regularization process if terms proportional to loop momenta with even number of free Lorentz indices (e.g. $\sim k_\mu k_\nu$) are calculated according to the special identifications (21) and (22) or generalizations thereof. In the end the scalar integrals are calculated with a simple momentum cutoff. The calculation is robust - at least at 1-loop level - as we have shown via the fermionic contribution to the vacuum polarization function. The finite terms agree with the one in dimensional regularization in all examples. The connection with DREG is more transparent if one uses alternatively the $k_\mu k_\nu \rightarrow \frac{1}{d} g_\mu \nu k^2$ or (55) substitution and $d$ takes different values determined by the degree of divergence in each term (52, 53, 54). We stress that this new regularization stands without DREG as the substitutions (21), (22) and scalar integration with a cutoff are independent of DREG. The success of both regularizations based on the property that they fulfill the consistency conditions of gauge invariance and momentum shifting.

At 1-loop the finite terms in the improved momentum cutoff are found to be equivalent with DREG. For practical calculations we propose to use the same renormalization scheme, $\overline{MS}$ or $MS$ subtractions plus BPHZ forest formula as with DREG. DREG is not just the generally used method, but it is proved to be a mathematically rigorous regularization within the Epstein-Glaser framework [42]. The equivalence of the results of the proposed method and DREG gives a hint that the improved cutoff method with e.g. $MS$ subtraction and BPHZ can be used as a renormalization scheme for more complicated diagrams.

Regularization schemes based on consistency conditions have been applied to more involved cases. Constrained differential renormalization is useful in supersymmetric [43] and non-Abelian gauge theories, it fulfills Slavnov-Taylor identities at one and
two loops [44]. Implicit regularization [20, 21] requires the same conditions as we used and it was successfully applied to the Nambu-Jona-Lasinio model [20] and to higher loop calculations in gauge theory. It was shown that the conditions guarantee gauge invariance generally and the Ward identities are fulfilled explicitly in QED at two-loop order [21]. In an effective composite Higgs model, the Fermion Condensate Model [15, 28] oblique radiative corrections (S and T parameters) were calculated in DREG and with the improved cutoff, too, the finite results completely agree. The calculation involved vacuum polarization functions with two different fermion masses and no ambiguity appeared [27, 35].

As an application the triangle anomaly was calculated within the 4 dimensional improved momentum cutoff framework. The property that the loop-integrals are invariant under the shift of the loop momentum spoils the usual derivation of the ABJ anomaly in the presence of a cutoff. We calculated the trace (44) corresponding to the triangle graphs of Fig. 3. ($\gamma_5$ and six $\gamma$’s) and the Ward identity (49) ($\gamma_5$ and four $\gamma$’s) exploiting only the standard anticommutators of the $\gamma$ matrices (43) and not using the $\gamma^\mu, \gamma^5$ anticommuting relation. It turns out that different evaluations of the trace agree with each other if and only if $\gamma^5$ does not always anticommute with $\gamma^\mu$, rather $\{\gamma^\mu, \gamma^5\}$ picks up terms proportional to the Schouten identity (45) if it is multiplied with five more $\gamma$’s under the trace. The trace of the $\{\gamma^\mu, \gamma^5\}$ anticommutator multiplied with three $\gamma$’s vanishes, as the definition of $\text{Tr}(\gamma^5\gamma_\alpha\gamma_\beta\gamma_\mu\gamma_\nu)$ is unambiguous. The right hand side of (45) is only non-vanishing if it is under a divergent loop momentum integral. In four dimensional field theory the nontrivial properties of $\gamma^5$ and $\gamma$’s appear first time in the divergent triangle diagram.

Traces involving $\gamma^5$ and even number of $\gamma$’s can be calculated in the same manner avoiding the anticommutation of $\gamma^\mu$ and $\gamma^5$. First the order of $\gamma^\nu$’s are reversed applying (43) then using the cyclicity of the trace we get back the original trace in the reversed order, the difference gives the trace twice. This way the $\{\gamma^\mu, \gamma^5\}$ anticommutator can be defined, it will not vanish generally. If it is multiplied with (2n+1) $\gamma$’s it is equal to the sum of (2n+1) trace involving $\gamma^5$ and (2n) $\gamma$’s, see (45). It is well known that the general properties of the trace and $\{\gamma^\mu, \gamma^5\}=0$ are in conflict with each other, this led to the ’t Hooft-Veltman scheme [11, 30]. Our proposal similarly modifies $\{\gamma^\mu, \gamma^5\}$ but works in four dimensions and the modifications come into action only under divergent loop integrals involving enough number of $\gamma$ matrices. There were attempts to preserve $\{\gamma^\mu, \gamma^5\}=0$, but in that case the cyclicity of the trace was lost [33]. We have shown with the new method that the vector currents are conserved and the axial vector current is anomalous, and no ambiguity appears.

The new regularization is advantageous in special loop-calcualtions where one wants to remain in four dimensions, keep the cutoff of the model, like in effective theories, in derivation of renormalization group equations, in extra dimensional scenarios or in models explicitly depending on the space-time dimensions, like supersymmetric theories. Similar approaches succeeded in the calculation of the anomalous decay of the Higgs boson to two photons in four dimensions, where gauge invariance is crucial [46, 47]. We argue that the method can be successfully used in higher order calculations containing terms up to quadratic divergences in (non-Abelian) gauge theories,
as it allows for shifts in the loop momenta, which guarantees the ’t Hooft identity [21]. This symmetry preserving method can be used also in automatized calculations (similar to [49]) as even the Veltman-Passarino functions [50] can be defined with the improved cutoff. The strength of the improved momentum cutoff method is that it can be used in theories with quadratic divergencies important for example in gauge theories including gravitational interactions [51]. The calculation in the Einstein-Maxwell system was presented in [52] and quadratic contributions to the photon 2-point function were identified but after renormalization they vanished and did not change the original running of the gauge coupling.

A Connection with simple momentum cutoff

What is the relation of the new method with the standard (textbook) substitution? We have to modify it in case of divergent integrals to respect gauge symmetry, i.e to fulfill (11). Lorentz invariance dictates that in (11) the LHS must be proportional to the only available tensor $g_{\mu\nu}$, i.e.

$$l_E l_E \rightarrow \frac{1}{d} g_{\mu\nu} t_E^2$$

(51)

can be used, where $d$ is a number to determine3. Now both sides of equation (11) can be calculated with simple 4-dimensional momentum cutoff. The different powers of $\Lambda$ can be matched on the two sides, and for $n = 1$ we get the following conditions (from gauge invariance) for the value of $d$,

$$\frac{1}{d} \Lambda^2 \quad \rightarrow \quad \frac{1}{2} \Lambda^2,$$

(52)

$$\frac{1}{d} \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) \quad \rightarrow \quad \frac{1}{4} \left( \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) + \frac{1}{2} \right),$$

(53)

$$\frac{1}{d} \quad \rightarrow \quad \frac{1}{4} \quad \text{for finite terms.}$$

(54)

We see that for finite valued integrals when the Wick-rotation is legal, the condition (11) and the rule (21) gives the usual substitution (50), but for divergent cases we get back the identification partially found by [6, 7, 10] and others. Quadratic divergence goes with $d = 2$, logarithmic divergence goes with $d = 4$ plus a finite term (a shift), it is the $+1$ in equation (11). For more than 2 even number of indices generalizations of (51) should be used, for example in case of 4 indices the

$$l_E l_E l_E l_E \rightarrow \frac{1}{d(d + 2)} \cdot (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) t_E^4.$$

(55)

3The usual method is to calculate the trace (and get $d=4$), but interchanging the order of tracing (multiplication with $g^{\mu\nu}$) and calculating the divergent integrals cannot be proven to be valid.
substitution works.

We emphasize again that for non-divergent integrals the rules (21) and (22) give the same result as the usual calculation (50).

B Basic integrals

In this appendix we list the basic divergent integrals calculated by the regularization proposed in this paper. In the following formulae $m^2$ can be any loop momentum $k$ independent expression depending on the Feynman $x$ parameter, external momenta, etc., e.g. $\Delta(x, q, m_a, m_b)$. The regulated integrals are denoted by $\int_{\Lambda_{reg}}$ meaning $\int_{|k| \leq \Lambda}$; the integration is understood for Euclidean momenta with absolute value below $\Lambda$. The integrals (56) and (60) are just given for comparison, those calculated with a simple momentum cutoff.

\[
\int_{\Lambda_{reg}} \frac{d^4k}{i(2\pi)^4} \frac{1}{k^2 - m^2} = -\frac{1}{(4\pi)^2} \left( \Lambda^2 - m^2 \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) \right) \tag{56}
\]

\[
\int_{\Lambda_{reg}} \frac{d^4k}{i(2\pi)^4} k_\mu k_\nu \frac{1}{k^2 - m^2} = -\frac{1}{(4\pi)^2} \frac{g_{\mu\nu}}{2} \left( \Lambda^2 - m^2 \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) \right) \tag{57}
\]

\[
\int_{\Lambda_{reg}} \frac{d^4k}{i(2\pi)^4} k_\mu k_\nu k_\rho k_\sigma \frac{1}{k^2 - m^2} = -\frac{1}{(4\pi)^2} \frac{g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}}{8} \left( \Lambda^2 - m^2 \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) \right) \tag{58}
\]

\[
\int_{\Lambda_{reg}} \frac{d^4k}{i(2\pi)^4} k^2 k_\mu k_\nu \frac{1}{k^2 - m^2} = -\frac{1}{(4\pi)^2} \frac{g_{\mu\nu}}{4} \left( 2\Lambda^2 - 3m^2 \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) + m^2 - \frac{m^4}{\Lambda^2 + m^2} \right) \tag{59}
\]

\[
\int_{\Lambda_{reg}} \frac{d^4k}{i(2\pi)^4} k^4 k_\mu \frac{1}{k^2 - m^2} = \frac{1}{(4\pi)^2} \left( \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) + \frac{m^2}{\Lambda^2 + m^2} - 1 \right) \tag{60}
\]

\[
\int_{\Lambda_{reg}} \frac{d^4k}{i(2\pi)^4} k^4 k_\mu k_\nu \frac{1}{k^2 - m^2} = \frac{1}{(4\pi)^2} \frac{g_{\mu\nu}}{4} \left( 3\ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) + \frac{5m^2}{\Lambda^2 + m^2} - \frac{m^4}{(\Lambda^2 + m^2)^2} - 1 \right) \tag{61}
\]

\[
\int_{\Lambda_{reg}} \frac{d^4k}{i(2\pi)^4} k^4 k_\mu k_\nu k_\rho k_\sigma \frac{1}{k^2 - m^2} = \frac{1}{(4\pi)^2} \frac{g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}}{24} \left( \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) + \frac{m^2}{\Lambda^2 + m^2} - 1 \right) \tag{62}
\]

\[
\int_{\Lambda_{reg}} \frac{d^4k}{i(2\pi)^4} k^4 k_\mu k_\nu k_\rho k_\sigma \frac{1}{k^2 - m^2} = \frac{1}{(4\pi)^2} \frac{g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}}{24} \left( \ln \left( \frac{\Lambda^2 + m^2}{m^2} \right) + \frac{m^2}{\Lambda^2 + m^2} - 1 \right) \tag{63}
\]

(56), (58) depend on the same function of $\Lambda$. (57), (58) are traced back to (56) via (21) and (22). (59) and (60) have a different $\Lambda$ dependence. Evaluating these integrals at first step (23) is used, then (21) or (22) can be applied to the remaining free indices.

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