Sample Truncation for Scenario Approach to Closed-loop Chance Constrained Trajectory Optimization for Linear Systems

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Abstract—This paper studies closed-loop chance constrained control problems with disturbance feedback (equivalently state feedback) where state and input vectors must remain in a prescribed polytopic safe region with a predefined confidence level. We propose to use a scenario approach where the uncertainty is replaced with a set of random samples (scenarios). Although a standard form of scenario approach is applicable in principle, it typically requires a large number of samples to ensure the required confidence levels. To resolve this drawback, we propose a method to reduce the computational complexity by eliminating the redundant samples and, more importantly, by truncating the less informative samples. Unlike the prior methods that start from the full sample set and remove the less informative samples at each step, we sort the samples in a descending order by first finding the most dominant ones. In this process the importance of each sample is measured via a proper mapping. Then the most dominant samples can be selected based on the allowable computational complexity and the rest of the samples are truncated offline. The truncation error is later compensated for by adjusting the safe regions via properly designed buffers, whose sizes are functions of the feedback gain and the truncation error.

I. INTRODUCTION

Computing solutions of finite horizon optimal control problems (FHOP) is a key capability in controlling dynamical systems with input and state constraints [1]. Given the initial state of the system in a FHOP, the future state trajectory can be obtained for any input by propagating the states through an explicit model. The propagated trajectories that meet the constraints and minimize the objective cost are then selected as optimal input and state trajectories. In practice, however, uncertainties may impact the accuracy of the explicit model and consequently the predicted state trajectory. A robust approach leads to a solution that satisfies the constraints for every uncertainty realization [2]–[4]. Since uncertainties are typically of the stochastic nature, they can be large enough to make the robust control problem infeasible, i.e., a solution satisfying the constraints under all possible realizations of the uncertainty can not be found. Hence, this paper adopts a stochastic problem formulation, also referred to as Chance Constrained Trajectory Optimization (CCTO), that relaxes the control problem to a probabilistic one where the constraints are to be met by a prescribed confidence level $1 - \delta$, $\delta \in (0, 1)$ [5].

Solving CCTO, that is widely investigated in Stochastic Model Predictive Control (SMPC) context [6]–[8] (earlier work) and [9] (contains a comprehensive review), is typically challenging as it requires calculating multi-dimensional integrals, which can result in a non-convex optimization problem [10], [11]. Even CCTO problems with convex constraints are often computationally intractable [10]. When the uncertainty distribution is assumed to be known and there is additional structure in the problem (e.g., temporal spatial independence of uncertainty, affine dependence of constraints on the uncertainty), the original CCTO can be analytically approximated by a conservative but tractable problem (e.g. [11]–[16]). Nevertheless, existing approximation-based approaches either require restrictive assumptions to hold true, which may limit their applicability.

To tackle this issue, scenario approach suggests to replace the uncertainty with a finite number (say $N$) of so-called scenarios through a random sampling, and then find the robust solution for the sampled uncertainty set [17], [18]. A bound on the sufficient number of samples is given in [17], [19] based on the required confidence level. Since the samples are taken randomly, there is a risk of failure in achieving the desired confidence level unlike the analytical methods. The main advantages of scenario approach are its generality and tractability: it converts the original stochastic problem to a convex problem regardless of the probability distribution of the uncertainty. That is, there is no need to know what the distribution is; all we need is to be able to sample from the distribution. However, to achieve a reasonable risk of failure, a large number of samples is typically needed. The scenario approach may consequently result in a computationally expensive problem since the constraints must be checked at each sample. Although some studies offer fewer number of samples (for instance, [20]–[22]) as compared to the original scenario approach by either presenting a tighter bound or discarding the redundant scenarios (e.g. [19], [20], [23]), a fairly large number of samples are still needed to ensure the desired specification.

We have recently introduced an approximate convex hull-based framework to reduce the computational complexity of open-loop CCTO by truncating the samples without losing the desired confidence level [24]–[26]. Earlier scenario-based methods discard samples, hence reduce the number of samples, while increasing the risk of failure (which is denoted by $\beta$ in the current paper). On the contrary, our truncation approach preserves prescribed levels of risk of failure ($\beta$) and the confidence in constraint satisfaction (denoted by $\delta$) by buffering the feasible constraint sets to account for the truncation error. To this end, we inner approximate the state uncertainty region of scenario approach using a subset
of \( \hat{N} \) scenarios, referred to as truncated sample set, where \( \hat{N} \) can be chosen significantly smaller than \( N \) (sufficient number of samples needed for scenario approach). CCTO problem is then solved by checking the constraints only over the truncated sample set while the truncation error is compensated for by adjusting the safe region using a proper buffer. In our proposed method, the truncated sample set is initiated from an empty set as opposed to discarding samples that remove one sample at a time from the full sample set. Since the truncated sample set gives the best approximation of the convex hull of original sample set (with \( N \) samples) with an approximation error of \( \epsilon \), this method is referred to as \( \epsilon \)-approximate convex hull. We use truncation and approximation interchangeably in this paper.

This concept is illustrated in Figure [1]. Assume that state \( x \) at time \( t \) is to remain in the safe region \( X \) with some confidence level. Let \( N \) be the sufficient number of samples proposed by scenario approach for the desired \( \delta \) and let \( X_t = \text{co}\{x_1^{(1)}, \ldots, x_N^{(N)}\} \) be the convex hull of state trajectories due to the \( N \) scenarios. \( \tilde{X}_t = \text{co}\{\tilde{x}_1^{(1)}, \ldots, \tilde{x}_N^{(N)}\} \) is an inner approximation of \( X_t \) obtained by \( N \) samples after truncation. \( X \) and \( \tilde{X} \) are also referred to as original and truncated uncertainty envelopes, respectively. The buffered constraint set at time \( t \) is shown as \( \tilde{X}_t \) and is calculated such that the original uncertainty envelope at time \( t \) remains in the safe region \( X \) when the truncated uncertainty envelope \( \tilde{X}_t \) is kept in \( \tilde{X}_t \subseteq \tilde{X} \). Consequently, we will impose that \( \tilde{X}_t \subseteq X_t \) to ensure the specified levels of confidence and risk of failure for constraint satisfaction.

In open-loop scheme, sample truncation and buffer computation are performed once and offline [26]. In addition, since the inputs are deterministic, buffers only need to be calculated for the state constraints. However, closed-loop scheme with state feedback is preferred due to the benefits of using the potential knowledge of future states as they become available. When the state feedback gain is itself a decision variable, the closed-loop CCTO becomes a non-convex problem. To resolve this issue and set up a convex optimization problem, disturbance feedback can be equivalently used [27]–[29], which is the approach adopted in this paper.

In this paper, we extend our sample truncation framework to the closed-loop CCTO with disturbance feedback, where the feedback gain is also an optimization variable and can be selected from a set of stabilizing gains with a common Lyapunov function. In this case, unlike the open-loop scheme, the input is also stochastic and needs buffering to compensate for the truncation error. Furthermore, buffers are dynamic variables and are functions of the feedback gain.

**Notation:** \( \mathbb{R} \) and \( \mathbb{N} \) are sets of real and natural numbers, with \( \mathbb{R}^n \) a length \( n \) vector of real numbers and \( \mathbb{N}_{\leq n} \) the set of natural numbers up to \( n \). For \( x \in \mathbb{R}^n \), \( x^T \) denotes the transpose of \( x \) and \( x_i \) is the \( i \)th element of \( x \). Also we define \( \mathbb{C} = \{ x \in \mathbb{R}^n | x_i \geq 0, \sum_i x_i = 1 \} \) as the set of convex coefficients. Given a set \( \mathbb{W} \) with elements \( w \) and function \( f(w) \), let \( f(\mathbb{W}) \) denote the set \( \{ f(w) | w \in \mathbb{W} \} \). \( I_n \in \mathbb{R}^{n \times n} \) is the identity matrix. \( \mathbf{1} \) is a vector of ones of compatible dimension. Given a set \( \mathbb{W} \), \( W = \text{co} \mathbb{W} \) is the convex hull of \( \mathbb{W} \). \( \mathbb{W}_1 \setminus \mathbb{W}_2 \) is the set obtained by removing the elements in \( \mathbb{W}_2 \) from \( \mathbb{W}_1 \). Let \( M_1 \) to \( M_m \) be \( m \) matrices from the same size. \( \text{vec}(M_1, \cdots, M_m) \) shows the vectorized form of the concatenated matrix \( M = [M_1, \cdots, M_m] \).

**II. Problem Description**

We consider the following discrete-time LTI system,

\[
x_{t+1} = Ax_t + Bu_t + B_ww_t
\]

(1)

where \( x_t \in \mathbb{R}^{n_x} \) and \( u_t \in \mathbb{R}^{n_u} \) are the system state and input vectors at sample time \( t \), respectively, and \( w_t \in \mathbb{W} \subseteq \mathbb{R}^{n_w} \), where \( \mathbb{W} \) represents an unbounded disturbance set which its probability distribution is not necessarily known, but it can be sampled from. Given the system dynamics (1), the design goal in a CCTO problem is to determine control inputs belonging to set \( U \) over a \( p \)-length future horizon such that the resulting state trajectory remains in a desired set \( X \) with a prescribed confidence. In this paper, \( X \) and \( U \) are assumed to be polytopes,

\[
X = \{ x | fx \leq 1 \},
\]

(2a)

\[
U = \{ u | fu \leq 1 \},
\]

(2b)

where \( f^x \in \mathbb{R}^{n_{x^2} \times n_x} \) and \( f^u \in \mathbb{R}^{n_{u^2} \times n_u} \). Let \( p \) be a positive integer and without loss of generality assign \( t = 0 \) to the current, decision, time and \( t > 0 \) are time epochs in the future. By stacking input, output, and state as,

\[
X = \begin{bmatrix} x_1 & \cdots & x_p \end{bmatrix},
U = \begin{bmatrix} u_0 & \vdots & u_{p-1} \end{bmatrix},
W = \begin{bmatrix} w_0 & \vdots & w_{p-1} \end{bmatrix},
\]

(3)

we define the stacked system as

\[
X = G_x x_0 + G_u U + G_w W,
\]

(4)

where \( x_0 \) is the initial state and,

\[
G_x = \begin{bmatrix} A & \vdots & \vdots \end{bmatrix},
G_u = \begin{bmatrix} B_w & \vdots & 0 \end{bmatrix},
G_w = \begin{bmatrix} A^{-1}B_w & \vdots & B_w \end{bmatrix}.
\]

(5)
Furthermore, $G_u$ is defined similar to $G_w$ by replacing $B_w$ with $B_u$.

A. State Feedback vs. Disturbance Feedback

In open-loop scheme, at any time $t \in \mathbb{N}_{\leq p}$, the input $u_t$ is computed only based on the initial state $x_0$. When a full state feedback is available, one can also use the knowledge of states $x_t$ to $x_{t-1}$ to calculate $u_t$ at the future times $t \in \mathbb{N}_{\leq p}$. For instance, the input policy in a typical tracking error feedback with respect to the nominal state trajectory is defined as $u_t = L_i e_t + \hat{u}_t$ with $e_t = x_t - \hat{x}_t$ where $\hat{x}$ and $\hat{u}$ denote the nominal state and input vectors, and are calculated via the nominal system $\dot{x}_{t+1} = A \dot{x}_t + B_w \hat{u}_t$. This feedback policy results in the error dynamics $e_{t+1} = (A + B_p L_i) e_t + B_w \hat{u}_t$. This closed-loop input policy can be written as $u_t = L_i x_t + g_t$ with $g_t = L_i \hat{x}_t + \hat{u}_t$, which is also captured within the general from,

$$u_t = \sum_{j=0}^{i} L_{i,j} x_j + g_t, \quad i = 0, \ldots, p-1. \quad (6)$$

In [9], the input at time $t$ is parametrized as a time varying affine function of states up to time $t$ while $L_{i,j}$ and $g_t$ are to be calculated online. It can be easily investigated that the CFTTO problem for system [1] with the input policy (6) is non-convex [28], [29].

An alternative to state feedback policy (6) is disturbance feedback as suggested in [27]–[29]. Note that for any $j \leq i$ in [6] $x_j = A x_{j-1} + B_w y_{j-1} + B_w w_{j-1}$, the input can be alternatively parametrized as an affine function of previous disturbances,

$$u_t = \sum_{j=0}^{i-1} K_{i,j} w_j + v_t, \quad i = 0, \ldots, p-1. \quad (7)$$

This can be shown in the stacked form $U = KW + V$ with,

$$V = \begin{bmatrix} v_0 \\ \vdots \\ v_{p-1} \end{bmatrix}, \quad K = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ K_{p-1,0} & \cdots & K_{p-1,p-2} \end{bmatrix}. \quad (8)$$

The input policy (7) is easy to implement since it has a causal structure, i.e., the input at time $t$ only depends on disturbances up to time $t - 1$. Note that the disturbance sequences can be calculated as the difference between actual and nominal states. In addition, since the disturbance is not a decision variable as opposed to state, policy (7) for a convex $\mathcal{W}$ set results in a convex optimization problem. Furthermore, disturbance feedback is in fact equivalent to the state feedback (see Theorem 9 in [29]), i.e., given $K$ and $V$, one can easily derive equivalent gains for policy (6) (transformation mapping is given in [29], Section 5).

B. Chance Constrained Trajectory Optimization

Here we present a standard chance constrained control problem formulation that ensures state and control constraint satisfaction by a predefined probability $1 - \delta$, also called confidence level, for some $\delta \in [0, 1]$ in exchange for reducing conservatism and achieving feasibility. Therefore, the original closed-loop CFTTO problem with disturbance feedback $U = KW + V$ is defined as

**Problem 1: Original CFTTO**

$$\min_{K,V} \mathbb{E}(J(X,U))$$

s.t. $X = G_a x_0 + G_a V + (G_a K + G_w)W$

$$\mathbb{P}(F^x X \leq 1) \geq 1 - \delta \quad W \in \mathcal{W}^p \quad (9)$$

where $\mathbb{P}$ denotes the probability measure, $F^x = I_p \otimes f^x$, $F^u = I_p \otimes f^u$, and $\mathcal{W}^p = \mathcal{W} \times \cdots \times \mathcal{W} (p \text{ times}).$

C. Sampling based approach

Problem [1] is quite often nonconvex since a multidimensional integration is typically required to compute the probability measure. In order to convert Problem [1] to a convex problem, scenario approach suggests to check the constraints over a finite set of random disturbances (sampled from the distribution $\mathbb{W}_N = \{W^{(1)}, \ldots, W^{(N)}\}$, instead of $\mathcal{W}^p$.

**Problem 2: CFTTO via scenario approach**

$$\min_{K,V} \mathbb{E}(J(X,U))$$

s.t. $X^{(i)} = G_a x_0 + G_a V + (G_a K + G_w)W^{(i)}$

$$F^x X^{(i)} \leq 1 \quad i = 1, \ldots, N \quad (10)$$

$$F^u (K W^{(i)} + V) \leq 1$$

Let $K^*_N, V^*_N$ be the optimal solutions of Problem [2]. If the number of random disturbances is sufficient, any feasible $K^*_N, V^*_N$ will also be feasible for Problem [1] with a probability (referred to as risk of failure) $1 - \beta$. Specifically, for a new random disturbance $W \in \mathcal{W}^p$ and consequently for the resulting optimal input $U_N = K^*_N W + V^*_N$,

$$\mathbb{P}(\mathbb{P}(X(U_N) \in \mathcal{X}) \not\succ 1 - \delta) \leq \beta, \quad \mathbb{P}(\mathbb{P}(U_N \in \mathcal{U}) \not\succ 1 - \delta) \leq \beta. \quad (11)$$

The required number of samples for guaranteeing this requirement differs based on the problem, but in general case the following bound on $N$ is provided by the scenario approach [17]

$$N \geq \frac{2}{\delta} \ln \frac{1}{\beta} + \frac{2n_u}{\delta} \ln \frac{2}{\delta} \quad (12)$$

where $n_u$ is the number of decision variables.

**Remark 1:** For a quadratic cost function, Problem [2] is a QP since the disturbance has been characterized by a polytope with at most $N$ extreme points.

The computational complexity of Problem [2] increases with the number of samples since each sample imposes a new set of constraints. Typically a large number of samples is required for a reasonably prescribed confidence level and risk of failure. The original scenario approach proposes sample reduction using a greedy method by removing one sample at a time. In detail, at step $k$, $N - k + 1$ QP problems with $N - k$ scenarios are solved to remove the less dominant...
D. CCTO using Approximate Convex Hull

In order to decrease the online computational complexity, we have recently suggested a greedy truncation method to select the best \( N \ll N \) samples and discard the rest [26]:

\[
\mathcal{W}_N = \begin{cases}
\text{selected: } \mathcal{W}^*_N & \text{truncated: } \mathcal{W}^o_N
\end{cases} = \left\{ W^{(1)}, \ldots, W^{(N)}, W^{(N+1)}, \ldots, W^{(E)} \right\},
\]

(13)

removed due to redundancy

The main idea behind this method is to approximate the original uncertainty envelope by a polytope with \( N \) extreme points and then account for the approximation error by adjusting the constraints using proper buffers. As a consequence, the sample truncation problem is simplified to a geometrical problem of computing the best approximate convex hull of the original uncertainty envelope. The truncation is based on using a mapping \( S(W) \) that reflects the uncertain parts of state equation (4).

**Definition 1:** Let \( S(W) \in \mathbb{R}^n \) be any mapping of \( W \) and for arbitrary sets \( \mathcal{W}_{N_1} = \{ W^{(1)}_{N_1}, \ldots, W^{(N_1)}_{N_1} \} \) and \( \mathcal{W}_{N_2} = \{ W^{(1)}_{N_2}, \ldots, W^{(N_2)}_{N_2} \} \) where \( \mathcal{W}_{N_1} \subseteq \mathcal{W}_{N_2} \) let

\[
S_1 = \text{co } S(\mathcal{W}_{N_1}) = \text{co } \{ S(W^{(1)}_{N_1}), \ldots, S(W^{(N_1)}_{N_1}) \},
\]

\[
S_2 = \text{co } S(\mathcal{W}_{N_2}) = \text{co } \{ S(W^{(1)}_{N_2}), \ldots, S(W^{(N_2)}_{N_2}) \}.
\]

Then the Hausdorff distance from \( S_1 \) to \( S_2 \) is given by [26]

\[
d_H(S_1, S_2) = \max_{W \in \mathcal{W}_{N_2}} \min_{\alpha \in \mathbb{C}} \left\| S(W) - \sum_{i=1}^{N_1} \alpha_i S(W^{(i)}_{N_1}) \right\|_{\infty}.
\]

(15)

Due to infinity norm definition and since \( S(W) \) is a vector, (15) is simplified to a search over max and min of some scalars. Here, let \( [S(W)]_k \) indicate the \( k \)th row of \( S(W) \) and for \( k \in \mathbb{N} \leq n \) define

\[
[\varepsilon]_k = \max_{W \in \mathcal{W}_N} \left\{ \max_{W \in \mathcal{W}_N} [S(W)]_k - \max_{W \in \mathcal{W}_N} [S(\hat{W})]_k, \right\}
\]

\[
\min_{W \in \mathcal{W}_N} [S(\hat{W})]_k - \min_{W \in \mathcal{W}_N} [S(W)]_k.
\]

Hence (15) simplifies to

\[
d_H(S_1, S_2) = \| \varepsilon \|_{\infty}.
\]

(17)

**Definition 2** (\( \epsilon \)-ACH): \( \hat{S} \) is the best \( \epsilon \)-approximate convex hull of \( S \) iff \( \hat{S} \) is the smallest subset of \( S \), with fewest number of elements, so that \( d_H(\hat{S}, S) \leq \epsilon \).

When there is no feedback \((K = 0)\), the input constraints are deterministic and can be satisfied if a feasible solution of the state constrained optimization problem exists. Define \( S(W) = F^x G_x W \) and let \( \hat{S} = \text{co } S(\mathcal{W}_N) \). Find the best \( N \) samples such that \( d_H(\hat{S}, S) \leq \epsilon \) where \( \hat{S} = \text{co } S(\mathcal{W}_N) \). \( \mathcal{W}_N \) can simply be obtained using a greedy approach, which is initiated by a random extreme point of \( S(\mathcal{W}_N) \) and continued by recursively adding the sample that minimizes the Hausdorff distance (see [26] for details). Given \( \mathcal{W}_N \), the following problem with only \( \hat{N} \) set of constraints and the state buffer \( \varepsilon^x \) is suggested as a replacement for Problem 2.

**Problem 3:** Open-loop CCTO via \( \epsilon \)-ACH

\[
\min_V \quad E(J(X, V))
\]

s.t.

\[
X^{(i)} = G_x x_0 + G_u V + G_w W^{(i)}
\]

\[
F^x X^{(i)} \leq 1 - \varepsilon^x, \quad i = 1, \ldots, \hat{N}
\]

\[
F^u V \leq 1
\]

\( \varepsilon^x \in \mathbb{R}^{n_x \times p} \) is simply calculated through (16) where each element of \( \varepsilon^x \) indicates the buffer on a specific constraint at some specific time.

**Remark 2:** Although buffering may reduce the feasible set of solutions, it preserves the original confidence level and failure risk. Let \( V_{N_1}^N \) and \( V_{N_2}^N \) be the optimal solutions of Problem 3 and Problem 2 when \( K = 0 \), respectively. For any \( W \in \mathcal{W}_N \) the following conditions hold [26]:

\[
P(\mathbb{P}(X(V_{N_1}^N) \in \mathcal{X}_p) \neq 1 - \delta) \leq P(\mathbb{P}(X(V_{N_2}^N) \in \mathcal{X}_p) \neq 1 - \delta) \leq \beta,
\]

(19)

It is noted that the truncated sample set and buffers are computed once offline, although they may be updated online for time-varying constraints. Truncating more samples speeds up the online computation by effectively decreasing the number of constraints. However, a bigger buffer may be imposed to the constraints, which may decrease the size of the feasible region (and hence increase the conservativeness of the solution). This trade-off can be selected by user and based on the application.

III. ACH-CCTO with Disturbance Feedback

With the inclusion of feedback control, the input is also a stochastic variable and needs to remain in its safe region by a prescribed confidence level. Consequently, sample truncation also impacts the input constraint satisfaction. Therefore, similar to state constraints, input constraints need to be adjusted using proper buffers to preserve the probabilistic requirements. In addition, since the uncertainty envelope is a function of the feedback gain \( K \), input and state buffers must be also synthesized as functions of \( K \).

In this section, we extend our sample truncation method for CCTO problem with disturbance feedback (Problem 1) while the feedback gain \( K \) is defined as in (8). In Section III-B, we present a proper truncation mapping \( S(W) \) to extract \( N \) samples. Given the truncation mapping we define the state and input buffers in Section III-B and finally set up the truncated control problem as an alternative to the original CCTO problem.

A. Sample Truncation

Similar to the open-loop case, we define the truncation mapping \( S(W) \) using the uncertain terms of state and input trajectories to capture the uncertainty in the direction of
normal vectors that describe the polytopes for the state and control constraints.

The contribution of uncertainties to state constraints are captured by the following mappings $F^x G_u KW$ and $F^y G_u W$ while $F^w KW$ captures the contribution of uncertainty to input constraints. Note that $K$ is a decision variable and cannot be used in defining the truncation mapping, and consequently sample truncation, since the truncation is performed offline and $K$ is not given a priori. On the other hand, $F^x G_u KW$ and $F^u KW$ contain $K$. To resolve this problem, we suggest to reorder these two mappings by using Lemma 1. This reordering gives us the flexibility to truncate the samples and compute preliminary buffers based on the parts of these mappings that do not depend on $K$, offline, and then update the buffers according to $K$, online.

**Lemma 1 (reordering matrix multiplication):** Let $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_z \end{bmatrix} \in \mathbb{R}^{n \times z}$ and $B = \begin{bmatrix} b_1^T & b_2^T & \cdots & b_z^T \end{bmatrix}^T \in \mathbb{R}^{z \times m}$ be two matrices with $a_i$ and $b_j$ denoting the $i$th column of $A$ and $j$th row of $B$, respectively. It is proven that $AB = BA$ for

$$
\begin{align*}
\hat{B} &= [b_1 & b_2 & \cdots & b_z] \odot I_n \\
\hat{A} &= [(I_m \odot a_1)^T (I_m \odot a_2)^T \cdots (I_m \odot a_z)^T]^T.
\end{align*}
\tag{20}
$$

**Proof:** For any $a_i \in \mathbb{R}^{n \times 1}$ and $b_\ell \in \mathbb{R}^{1 \times m}$, $\ell \in \mathbb{N}_{\leq z}$, one can show $a_i b_\ell = (b_\ell \odot I_n)(I_m \odot a_\ell)$. Thus,

$$
AB = \sum_{\ell=1}^{z} a_ib_\ell = \sum_{\ell=1}^{z}(b_\ell \odot I_n)(I_m \odot a_\ell)
$$

which can be written in the vector form $\hat{B} \hat{A}$ with $\hat{B}$ and $\hat{A}$ defined as in (20).

We refer to $A$ and $B$ as lifted matrices of $A$ and $B$ since we are lifting $A$ and $B$ to higher dimensions.

Now using Lemma 1, we can define lifted matrices $F^x G_u$, $F^y$, $K^x$, and $K^u$, such that $F^x G_u K = K^x F^y G_u$ and $F^u K = K^u F^y$. Note after the lifting the matrix multiplications are reordered as suggested by Lemma 1. $K^x$ and $K^u$ are the design variables since they are constructed from the elements of $K$. Therefore, we define the truncation mappings based on the constant parts of these mappings as follows

$$
S(W) = \begin{bmatrix} S^{cl} \\ S^{ol} \\ S^{cl}^T \\ S^{ol}^T \end{bmatrix} = \begin{bmatrix} F^x G_u \\ F^y G_u \\ F^y \\ F^y \end{bmatrix} W. \tag{21}
$$

Let $\mathbb{W}_N$ be a set of $N$ random samples satisfying the desired confidence level and risk of failure. Also let $S = co S(\mathbb{W}_N)$ where $S(W)$ is defined as (21). The truncated sample set $\mathbb{W}_N$ can be computed so that $\hat{S} = co S(\mathbb{W}_N)$ is the $\varepsilon$-ACH of $S$, i.e., $d_H(\hat{S}, S) \leq \varepsilon$ where $d_H$ is defined in (17). A greedy algorithm to compute $\mathbb{W}_N$ is later given in Section IV.

Next, we will explain the buffering process to compensate for the truncation error due to fewer number of samples than suggested by scenario approach and removing $K$ from the uncertainty mappings.

**B. State/Input Buffer Computation**

Given $\mathbb{W}_N$ (as computed in Section III-A) and the truncation mapping (21), using (16) we compute

$$
\varepsilon = \begin{bmatrix} \varepsilon^{clT} & \varepsilon^{clT} & \varepsilon^{ol} \end{bmatrix}^T
\tag{22}
$$

and define

$$
\begin{align*}
\varepsilon^{cl} &= d_H(\hat{S}^{cl}, S^{cl}) = \|\varepsilon^{cl}\|_{\infty}, \\
\varepsilon^{ol} &= d_H(\hat{S}^{ol}, S^{ol}) = \|\varepsilon^{ol}\|_{\infty}, \\
\varepsilon^{u} &= d_H(\hat{S}^{u}, S^{u}) = \|\varepsilon^{u}\|_{\infty}.
\end{align*}
\tag{23}
$$

Note that since $d_H(\hat{S}, S) = \|\varepsilon\|_{\infty} = \varepsilon$, it is concluded from (22) that $\varepsilon = \max\{\varepsilon^{cl}, \varepsilon^{ol}, \varepsilon^{u}\}$, and consequently $\varepsilon^{cl}, \varepsilon^{ol}, \varepsilon^{u}$ are upper bounded by $\varepsilon$. Also define,

$$
\kappa_t = vec(K_{1,0}, K_{2,0}, \cdots K_{t-1,2}). \tag{24}
$$

For $t \in \mathbb{N}_{\leq p}$, $\kappa_t$ is the vectorized form of block rows of $K$ associated with time instances up to $t$. For instance, $\kappa_1$ is a zero vector, $\kappa_2$ is the vectorized form of $K_{1,0}$, and $\kappa_3$ is a vector that consists of all elements of $K_{1,0}$, $K_{2,0}$ and $K_{2,1}$. We claim that by defining the following buffered constraint sets at time $t$, the prescribed confidence level is preserved. This will be proven later in Theorem 1.

$$
\hat{X}_t = \{x | fx \leq 1 - \varepsilon^{cl}\|\kappa_t\| - \varepsilon^{ol}\}, \tag{25a}
$$

$$
\hat{U}_t = \{u | fu \leq 1 - \varepsilon^{u}\|\kappa_t\|\}, \tag{25b}
$$

**C. Approximated Problem**

Given the truncated sample set (as computed in Section III-A) and buffer coefficients calculated using (23), we suggest the following truncated problem. The theoretical properties of the solutions of this problem are summarized in Theorem 1.

**Problem 4:** closed loop CCTO via $\varepsilon$-ACH

$$
\min_{K, V, \zeta} \mathbb{E}(J(X, U))
$$

s.t. $X^{(i)} = G_x x_0 + G_u V + (G_u K + G_w)W^{(i)}$

$$
F^x X^{(i)} \leq 1_p - \varepsilon^{cl}\zeta^{x} - \varepsilon^{ol} i = 1, \cdots, \hat{N}
$$

$$
F^u (K W^{(i)} + V) \leq 1_p - \varepsilon^{u}\zeta^{u} t = 1, \cdots, p
$$

where $\hat{N}$ is calculated from (11) while $\hat{N}$ is calculated from (17). A greedy algorithm to compute $\mathbb{W}_N$ is later given in Section IV.

It is important to note that Problem 4 is convex for any convex $J$, $\varepsilon^{cl} \in \mathbb{R}$, $\varepsilon^{ol} \in \mathbb{R}^{n_{\theta} \times n_{\theta}}$, and $\varepsilon^{u} \in \mathbb{R}$ are computed offline using (23) and $\zeta$ adjusts the buffers based on the optimal $K$.

**Theorem 1:** Let $\mathbb{W}_N$ be a set of $N$ random disturbances where $N$ is calculated from (11) while $n_{\theta}$ indicates the
the number of decision variables in Problem 4. Let the truncation bound be computed such that $d_H(S, S) \leq \epsilon$, where $S = \cos(W_N)$ and $S = \cos(W_N)$. Let $K_N^{*}, V_N^{*}, \zeta_N^{*}$ be the optimal solution of Problem 4. For the closed-loop system given by (1) with control input $U_N^{*} = K_N^{*}W + V_N^{*}$ and $W$ with the probability distribution $\mathbb{P}$, the following probabilistic guarantees hold true:

\begin{align}
\mathbb{P}(\mathbb{P}(X(U_N^{*}, W) \in \mathcal{X}) \neq 1 - \delta) \leq \beta, \\
\mathbb{P}(\mathbb{P}(U_N^{*} \in \mathcal{U}) \neq 1 - \delta) \leq \beta.
\end{align}

**Proof:** We already know that (27) holds for $\hat{N} = N$ for any $K$ and $V$ (including $K_N^{*}$ and $V_N^{*}$), which is implied from Corollary 3.4 in [19] where this bound on the number of sufficient samples is presented, given in (12), for any prescribed confidence level and failure risk. Note that this bound is independent of the trajectories, while it depends on the number of decision variables $n_{\theta}$.

Next we prove that $1e^{cl}G_t + \epsilon^{ol}_t$ and $1e^{u}G_t$ represent guaranteed upper bounds at time instant $t$ on the state and the input violations due to truncation, respectively. Equivalently we prove that keeping the state/input truncated uncertainty envelope in the buffered state/input constraint set ensures that the original state/input uncertainty envelope remains in the original state/input constraint set.

Violation on the $\ell^{th}$ state constraint at time instant $t$ can be calculated as in (29c) where $X(W)$ is defined as in (1), and subscript $t, \ell$ denotes the row associated with time instant $t$ and constraint $\ell$. (29c) simplifies to (29b) by using the fact that $\max_{W \in \mathcal{W}} \min_{\alpha \in \mathcal{C}} \|\ast\| = 0$ for any $\ast$ that is independent of $W$. (29b) is resulted using triangle inequality. According to Lemma 1 there always exist at least one $K^\pi$ and one $F^\pi G_u K$ multiplication can be reordered as $F^\pi G_u W$ and $F^\pi G_u W$ in (29b) denote $S^\pi(W)$ and $S^{ol}(W)$, respectively. Thus, according to the definition of the Hausdorff distance $\|1 - \hat{N}\|$, and since the violation on $S^{ol}(W)$ and $S^{ol}(W)$ are bounded by $\epsilon^{ol}$ and $\epsilon^{ol}$ (see (23), (24), (25), (26)) is concluded. The last statement of (29d) is concluded by using the fact that $F^\pi G_u K$ and consequently their multiplication, are block lower triangular matrices where blocks are divided timewise from 1 to $p$. Therefore, by setting the upper triangular part of $K^{\pi}$ including the main diagonal to zero, $F^\pi G_u K = K^{\pi}F^\pi G_u$ still remains valid. Thus, for any $\ell \in N_{\leq n_{\epsilon}}$, $\|K_{t,\ell}\| = 0$ and for $2 \leq t \leq p$ one can show

$$\|K_{t,\ell}\| = \|\text{vec}(K_{1,0}, \ldots, K_{t-1,t-2})\| = \|K_{t}\|.$$

Since $\zeta_t$ is an upper bound on $\|K_{t}\|$, $1e^{cl}G_t + \epsilon^{ol}_t$ provides an upper bound on the state violation due to truncation, and consequently presents a proper buffer on the state constraint set.

Since $K^{u}_u$ has the same structure as $K^{\pi}$ (with different number of rows), one can similarly prove $\epsilon^{u}G_t$ provides an upper bound on the input constraint violation due to truncation at time $t$. Hence, the proof is completed.

**Remark 3:** Problem 4 has $p$ decision variables more than Problem 2 due to the addition of the slack variable $\zeta$. Therefore, more original samples are required for constructing Problem 4. However, these samples are only used to define the buffers and only $\hat{N}$ of them are directly used in the online computation of the closed-loop input.

**IV. IMPLEMENTATION**

The implementation of the proposed truncated CTO problem with disturbance feedback is simply executed in two offline and online steps.

**offline:** Using (12) one can find the required number of samples based on the desired confidence level $\delta$ and risk of failure $\beta$ for Problem 4 with variables $K$, $V$, and $\zeta$ (typically another slack variable may be also used to convert the quadratic cost function to a linear one) and generate $\mathbb{W}_N$ through randomly sampling the disturbance set $\mathbb{W}$.

Given $F^\pi$, $F^u$, and $G^u$, one can use Lemma 1 to construct $F^\pi G_u W$ and $F^u$ and then map $\mathbb{W}_N$ to $S(\mathbb{W}_N)$ using (21). It is noted that $S(\mathbb{W}_N)$ is a set of $N$ vectors corresponding to $N$ different scenarios (disturbances). To select the dominant $N$ scenarios, the following greedy algorithm can be used.

1. Initiate $\mathbb{W}_N$ with the argument of an extreme point of $S(\mathbb{W}_N)$ (with some abuse of notation, say $W_1$). For instance, the farthest element from any element in a set is an extreme point. Find $\epsilon$ from (16) by setting $\mathbb{W}_N = W_1 = W_1$.

2. At step $\ell$ given $\mathbb{W}_{\ell-1}$ and $\epsilon$ find $W_\ell$ so that $d_H(\cos(W_{\ell-1} \cup W_\ell), \cos(S(\mathbb{W}_N)))$ is minimized. According to the Hausdorff distance definition given in (16)–(17) this is simply to select the $W_\ell \in \mathbb{W}_N \setminus W_{\ell-1}$ that minimizes the largest element of $\epsilon$. This procedure is fast since only a search over the elements of a vector is required!

3. Repeat step 2 while $\ell \leq \hat{N}$ and $\epsilon \neq 0$.

After finding the best $\hat{N}$ samples, $\epsilon^{ol}$, $\epsilon^{cl}$, $\epsilon^{u}$ and consequently $\epsilon^{cl}$ and $\epsilon^{u}$ can be calculated offline from $\epsilon$.

**online:** Given $\mathbb{W}_N$, $\epsilon^{cl}$, $\epsilon^{u}$ and $\epsilon^{ol}$, the given convex problem (Problem 4) is solved for $K_N^{\pi}$, $V_N^{\pi}$, and $\zeta_N^{\pi}$. Note that for any disturbance sequence $W \in \mathbb{W}$, $U_N^{*} = K_N^{\pi}W + V_N^{\pi}$ guarantees the state and input constraints satisfaction by confidence level of $1 - \delta$ and failure risk of $\beta$.

**V. ILLUSTRATIVE EXAMPLE**

To show the application of the suggested method for closed-loop CTO, we simulate the trajectory performance of the following double-integrator robot (in a 2-D plane). In this example we assume $T_s = 1 s$.

$$\begin{bmatrix}
x_x \\
v_x \\
x_y \\
v_y
\end{bmatrix} =
\begin{bmatrix}
1 & T_s & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & T_s \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_x \\
v_x \\
x_y \\
v_y
\end{bmatrix} +
\begin{bmatrix}
0 & 0 \\
T_s & 0 \\
0 & 0 \\
T_s & 0
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y
\end{bmatrix}.$$

We assume $B_w = I_4$ and the states are disturbed by $w \sim \mathcal{N}(0, \sigma_w^2)$ where $\sigma_w^2 = \text{diag}(1e - 3, 4e - 4, 1e -$
\[
\max_{W \in \mathbb{W}} \min_{\alpha \in \mathbb{C}} \left\| F_{t,\ell}^x X(W) - \sum_{i=1}^{\hat{N}} \alpha_i F_{t,\ell}^x X(W^{(i)}) \right\| \leq \max_{W \in \mathbb{W}} \min_{\alpha \in \mathbb{C}} \left( \left\| F_{t,\ell}^x G_u KW + G_u W - \sum_{i=1}^{\hat{N}} \alpha_i F_{t,\ell}^x G_u KW^{(i)} \right\| + \left\| F_{t,\ell}^x G_u W - \sum_{i=1}^{\hat{N}} \alpha_i F_{t,\ell}^x G_u W^{(i)} \right\| \right) \leq \max_{W \in \mathbb{W}} \min_{\alpha \in \mathbb{C}} \left( \left\| F_{t,\ell}^x G_u KW - \sum_{i=1}^{\hat{N}} \alpha_i F_{t,\ell}^x G_u KW^{(i)} \right\| + \left\| F_{t,\ell}^x G_u W - \sum_{i=1}^{\hat{N}} \alpha_i F_{t,\ell}^x G_u W^{(i)} \right\| \right) \]

\[
\leq \left\| K_{t,\ell}^x \|d_H(\hat{S}^{cl}, S^{cl}) + d_H(\hat{S}^{ol}, S^{ol}) = \|K_{t,\ell}^x\|e^{cl} + \epsilon_{t,\ell} = \|\epsilon^{cl} + \epsilon_{t,\ell}\| \right\|.
\]

and truncated problems results.

**VI. Conclusion**

In this paper, we presented a method to reduce the number of samples in scenario approach for a closed-loop chance constrained control problem with disturbance feedback. In the proposed method, the most dominant samples are selected by the user and the rest are truncated. The truncation error is later compensated for by adjusting the constraint set using proper dynamic buffers. We showed that the new problem with the truncated sample set and constraints is convex and its solution satisfies the problem specifications (the required confidence level and risk of failure). The proposed method was successfully implemented through simulations on an illustrative 2-D robot example.

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Fig. 3. State trajectory tracking results with 6 and 20 samples (out of 5564 samples) while states are subject to remain in the safe region. The original and truncated uncertainty envelopes have been distinguished by bright and dark polytopes for $t = 1, \cdots, 5$.

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