ON UNIVERSAL MINIMAL COMPACT G-SPACES

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ABSTRACT. For every topological group $G$ one can define the universal minimal compact $G$-space $X = M_G$ characterized by the following properties: (1) $X$ has no proper closed $G$-invariant subsets; (2) for every compact $G$-space $Y$ there exists a $G$-map $X \to Y$. If $G$ is the group of all orientation-preserving homeomorphisms of the circle $S^1$, then $M_G$ can be identified with $S^1$ (V. Pestov). We show that the circle cannot be replaced by the Hilbert cube or a compact manifold of dimension $> 1$. This answers a question of V. Pestov. Moreover, we prove that for every topological group $G$ the action of $G$ on $M_G$ is not 3-transitive.

1. Introduction

With every topological group $G$ one can associate the universal minimal compact $G$-space $M_G$. To define this object, recall some basic definitions. A $G$-space is a topological space $X$ with a continuous action of $G$, that is, a map $G \times X \to X$ satisfying $g(hx) = (gh)x$ and $1x = x$ ($g, h \in G$, $x \in X$). A $G$-space $X$ is minimal if it has no proper $G$-invariant closed subsets or, equivalently, if the orbit $Gx$ is dense in $X$ for every $x \in X$. A map $f : X \to Y$ between two $G$-spaces is $G$-equivariant, or a $G$-map for short, if $f(gx) = gf(x)$ for every $g \in G$ and $x \in X$.

All maps are assumed to be continuous, and ‘compact’ includes ‘Hausdorff’. The universal minimal compact $G$-space $M_G$ is characterized by the following property: $M_G$ is a minimal compact $G$-space, and for every compact minimal $G$-space $X$ there exists a $G$-map of $M_G$ onto $X$. Since Zorn’s lemma implies that every compact $G$-space has a minimal compact $G$-subspace, it follows that for every compact $G$-space $X$, minimal or not, there exist a $G$-map of $M_G$ to $X$.

The existence of $M_G$ is easy: consider the product of a representative family of compact minimal $G$-spaces, and take any minimal closed $G$-subspace of this product for $M_G$. It is also true that $M_G$ is unique, in the sense that any two universal minimal compact $G$-spaces are isomorphic \cite{1}. For the reader’s convenience, we give a proof of this fact in the Appendix.

If $G$ is locally compact, the action of $G$ on $M_G$ is free \cite{2} (see also \cite{3}, Theorem 3.1.1), that is, if $g \neq 1$, then $gx \neq x$ for every $x \in M_G$. On the other hand, $M_G$ is a singleton for many naturally arising non-locally compact groups $G$. This property of $G$ is equivalent to the following fixed point on compacta (f.p.c.) property: every compact $G$-space has a $G$-fixed point. (A point $x$ is $G$-fixed if $gx = x$ for all $g \in G$.) For example, if $H$ is a Hilbert space, the group $U(H)$ of all unitary operators on $H$, equipped with the pointwise convergence topology, has the f.p.c. property (Gromov

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another example of a group with this property, due to Pestov, is $H_+(\mathbb{R})$, the group of all orientation-preserving self-homeomorphisms of the real line. We refer the reader to beautiful papers by V. Pestov [3, 4, 5] on this subject.

Let $S^1$ be a circle, and let $G = H(S^1)$ be the group of all orientation-preserving self-homeomorphisms of $S^1$. Then $M_G$ can be identified with $S^1$, Theorem 6.6.

The question arises whether a similar assertion holds for the Hilbert cube $Q$. This question is due to V. Pestov, who writes in [3], Concluding Remarks, that his theorem “tends to suggest that the Hilbert cube $I^\omega$ might serve as the universal minimal flow for the group Homeo ($I^\omega$)”.

The aim of the present paper is to answer this question in the negative. Let us say that the action of a group $G$ on a $G$-space $X$ is 3-transitive if $|X| \geq 3$ and for any triples $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ of distinct points in $X$ there exists $g \in G$ such that $ga_i = b_i$, $i = 1, 2, 3$.

**Theorem 1.1.** For every topological group $G$ the action of $G$ on the universal minimal compact $G$-space $M_G$ is not 3-transitive.

Since the action of $H(Q)$ on $Q$ is 3-transitive, it follows that $M_G \neq Q$ for $G = H(Q)$. Similarly, if $K$ is compact and $G$ is a 3-transitive group of homeomorphisms of $K$, then $M_G \neq K$. This remark applies, for example, if $K$ is a manifold of dimension $> 1$ or a Menger manifold and $G = H(K)$.

**Question 1.2.** Let $G = H(Q)$. Is $M_G$ metrizable?

A similar question can be asked when $Q$ is replaced by a compact manifold or a Menger manifold.

Let $P$ be the pseudoarc (= the unique hereditarily indecomposable chainable continuum) and $G = H(P)$. The action of $G$ on $P$ is transitive but not 2-transitive, and the following question remains open:

**Question 1.3.** Let $P$ be the pseudoarc and $G = H(P)$. Can $M_G$ be identified with $P$?

**2. Proof of the main theorem**

The proof of Theorem 1.1 depends on the consideration of the space of maximal chains of closed sets. For a compact space $K$ let $\text{Exp}K$ be the (compact) space of all non-empty closed subsets of $K$, equipped with the Vietoris topology. A subset $C \subset \text{Exp}K$ is a chain if for any $E, F \in C$ either $E \subset F$ or $F \subset E$. If $C \subset \text{Exp}K$ is a chain, so is the closure of $C$. It follows that every maximal chain is a closed subset of $\text{Exp}K$ and hence an element of $\text{Exp}\text{Exp}K$. Let $\Phi \subset \text{Exp}\text{Exp}K$ be the space of all maximal chains. Then $\Phi$ is closed in $\text{Exp}\text{Exp}K$ and hence compact. Let us sketch a proof. Clearly the closure of $\Phi$ consists of chains. Assume $C \in \text{Exp}\text{Exp}K$ is a non-maximal chain. We construct a neighbourhood $\mathcal{U}$ of $C$ in $\text{Exp}\text{Exp}K$ which is disjoint from $\Phi$. One the following cases holds: (1) the first member of $C$ has more than one point, or (2) the last member of $C$ is not $K$, or (3) the chain $C$ contains “big gaps”: there are $F_1, F_2 \in C$ such that $|F_2 \setminus F_1| \geq 2$ and for every
F ∈ C either F ⊂ F_1 or F_2 ⊂ F. For example, consider the third case (the first two cases are simpler). Find open sets U, V_1, V_2 in K with pairwise disjoint closures such that F_1 ⊂ U and F_2 meets both V_1 and V_2. Let \( \mathcal{W} = \{ D \in \text{Exp Exp} K : \) every member of \( D \) either is contained in \( U \) or meets both \( V_1 \) and \( V_2 \}. Then \( \mathcal{W} \) is a neighbourhood of \( C \) which does not meet \( \Phi \). Indeed, suppose \( D \in \mathcal{W} \cap \Phi \). Let \( E_1 \) be the largest member of \( D \) which is contained in \( \hat{U} \). Let \( E_2 \) be the smallest member of \( D \) which meets both \( V_1 \) and \( V_2 \). For every \( E \in D \) we have either \( E \subset E_1 \) or \( E \subset E \), and \( |E_2 \setminus E_1| \geq 2 \). Pick a point \( p \in E_2 \setminus E_1 \). The set \( E_1 \cup \{ p \} \) is comparable with every member of \( D \) but is not a member of \( D \). This contradicts the maximality of \( D \). We have proved that \( \Phi \) is compact.

Suppose \( G \) is a topological group and \( K \) is a compact \( G \)-space. Then the natural action of \( G \) on \( \text{Exp} \text{Exp} K \) is continuous, hence \( \text{Exp} \text{Exp} K \) is a compact \( G \)-space, and so is \( \text{Exp} \text{Exp} K \). Since the closed set \( \Phi \subset \text{Exp} \text{Exp} K \) is \( G \)-invariant, \( \Phi \) is a compact \( G \)-space, too.

**Proposition 2.1.** Let \( G \) be a topological group. Pick \( p \in M_G \), and let \( H = \{ g \in G : gp = p \} \) be the stabilizer of \( p \). There exists a maximal chain \( C \) of closed subsets of \( M_G \) such that \( C \) is \( H \)-invariant: if \( F \subset C \) and \( g \in H \), then \( gF \subset C \).

Note that members of an \( H \)-invariant chain need not be \( H \)-invariant.

**Proof.** Every compact \( G \)-space \( X \) has an \( H \)-invariant point. Indeed, there exists a \( G \)-map \( f : M_G \rightarrow X \), and since \( p \) is \( H \)-invariant, so is \( f(p) \in X \).

Let \( \Phi \subset \text{Exp} \text{Exp} M_G \) be the compact space of all maximal chains of closed subsets of \( M_G \). We saw that \( \Phi \) is a compact \( G \)-space. Thus \( \Phi \) has an \( H \)-invariant point. \( \square \)

Theorem 1.1 follows from Proposition 2.1.

**Proof of Theorem 1.1.** Assume that the action of \( G \) on \( X = M_G \) is 3-transitive. Pick \( p \in X \), and let \( H = \{ g \in G : gp = p \} \). According to Proposition 2.1, there exists an \( H \)-invariant maximal chain \( C \) of closed subsets of \( X \). The smallest member of \( C \) is an \( H \)-invariant singleton. Since \( G \) is 2-transitive on \( X \), the only \( H \)-invariant singleton is \( \{ p \} \). Thus \( \{ p \} \subset C \), and all members of \( C \) contain \( p \). Our definition of 3-transitivity implies that \( |X| \geq 3 \). Thus there exists \( F \subset C \) such that \( F \neq \{ p \} \) and \( F \neq X \). Pick \( a \in F \setminus \{ p \} \) and \( b \in X \setminus F \). The points \( p, a, b \) are all distinct. Since \( G \) is 3-transitive on \( X \), there exists \( g \in G \) such that \( gp = p \), \( ga = b \) and \( gb = a \). Since \( a \in F \) and \( b \notin F \), we have \( b = ga \in gF \) and \( a = gb \notin gF \). Thus \( a \in F \setminus gF \) and \( b \in gF \setminus F \), so \( F \) and \( gF \) are not comparable. On the other hand, the equality \( gp = p \) means that \( g \in H \). Since \( C \) is \( H \)-invariant and \( F \subset C \), we have \( gF \subset C \). Hence \( F \) and \( gF \) must be comparable, being members of the chain \( C \). We have arrived at a contradiction. \( \square \)

**Example 2.2.** Consider the group \( G = H_+(S^1) \) of all orientation-preserving self-homeomorphisms of the circle \( S^1 \). According to Pestov's result cited above, \( M_G = S^1 \). This example shows that the action of \( G \) on \( M_G \) can be 2-transitive. Pick \( p \in S^1 \), and let \( H \subset G \) be the stabilizer of \( p \). Proposition 2.1 implies that there must exist \( H \)-invariant maximal chains of closed subsets of \( S^1 \). It is easy to see that there are precisely two such chains. They consist of the singleton \( \{ p \} \), the whole circle and of all arcs that either "start at \( \hat{p} \)" or "end at \( \hat{p} \)", respectively.
Remark 2.3. Let $P$ be the pseudoarc, and let $G = H(P)$. Pick a point $p \in P$, and let $H \subset G$ be the stabilizer of $p$. Then there exists an $H$-invariant maximal chain $C$ of closed subsets of $P$. Namely, let $C$ be the collection of all subcontinua $F \subset P$ such that $p \in F$. Since any two subcontinua of $P$ are either disjoint or comparable, it follows that $C$ is a chain. The chain $C$ can be shown to be maximal, and it is obvious that $C$ is $H$-invariant. Thus Proposition 2.4 does not contradict the conjecture that $M_G = P$. This observation motivates our question 1.3.

3. Appendix: Uniqueness of $M_G$

We sketch a proof of the uniqueness of $M_G$ up to a $G$-isomorphism.

Let $G$ be a topological group. The greatest ambit $X = \mathcal{S}(G)$ for $G$ is a compact $G$-space with a distinguished point $e$ such that for every pointed compact $G$-space $(Y, e')$ there exists a unique $G$-morphism $f : X \to Y$ such that $f(e) = e'$. The greatest ambit is defined uniquely up to a $G$-isomorphism preserving distinguished points. We can take for $\mathcal{S}(G)$ the Samuel compactification of $G$ equipped with the right uniformity, which is the compactification of $G$ corresponding to the algebra of all bounded right uniformly continuous functions. The distinguished point is the unity of $G$. See [3, 4, 5] for more details.

The greatest ambit $X$ has a natural structure of a left-topological semigroup. This means that there is an associative multiplication $(x, y) \mapsto xy$ on $X$ (extending the original multiplication on $G$) such that for every $y \in X$ the self-map $x \mapsto xy$ of $X$ is continuous. Let $x, y \in X$. There is a unique $G$-map $r_y : X \to X$ such that $r_y(e) = y$. Define $xy = r_y(x)$. If $y$ is fixed, the map $x \mapsto xy$ is equal to $r_y$ and hence is continuous. If $y, z \in X$, the self-maps $r_x r_y$ and $r_y z$ of $X$ are equal, since both are $G$-maps sending $e$ to $y z = r_z(y)$. This means that the multiplication on $X$ is associative. The distinguished element $e \in X$ is the unity of $X$: we have $e x = r_x(e) = x$ and $x e = r_e(x) = x$. If $g \in G$ and $x \in X$, the expression $g x$ can be understood in two ways: in the sense of the exterior action of $G$ on $X$ and as a product in $X$; these two meanings agree. If $f : X \to X$ is a $G$-self-map and $a = f(e)$, then $f(x) = f(x e) = x f(e) = x a = r_a(x)$ for all $x \in X$. Thus the semigroup of all $G$-self-maps of $X$ coincides with the semigroup $\{r_y : y \in X\}$ of all right multiplications.

A subset $I \subset X$ is a left ideal if $XI \subset I$. Closed $G$-subspaces of $X$ are the same as closed left ideals of $X$. An element $x$ of a semigroup is an idempotent if $x^2 = x$. Every closed $G$-subspace of $X$, being a left ideal, is moreover a left-topological compact semigroup and hence contains an idempotent, according to the following fundamental result of R. Ellis (see [3], Proposition 2.1 or [2], Theorem 3.11):

**Theorem 3.1.** Every non-empty compact left-topological semigroup $K$ contains an idempotent.

**Proof.** Zorn’s lemma implies that there exists a minimal element $Y$ in the set of all closed non-empty subsemigroups of $K$. Fix $a \in Y$. We claim that $a^2 = a$ (and hence $Y$ is a singleton). The set $Y a$, being a closed subsemigroup of $Y$, is equal to $Y$. It follows that the closed subsemigroup $Z = \{x \in Y : xa = a\}$ is non-empty. Hence $Z = Y$ and $xa = a$ for every $x \in Y$. In particular, $a^2 = a$. \qed
Let $M$ be a minimal closed left ideal of $X$. We have just proved that there is an idempotent $p \in M$. Since $Xp$ is a closed left ideal contained in $M$, we have $Xp = M$. Thus the $G$-map $r_p : X \to M$ defined by $r_p(x) = xp$ is a retraction of $X$ onto $M$. In particular, $xp = x$ for every $x \in M$.

**Proposition 3.2.** Every $G$-map $f : M \to M$ has the form $f(x) = xy$ for some $y \in M$.

**Proof.** The composition $h = fr_p : X \to M$ is a $G$-map of $X$ into itself, hence it has the form $h = r_y$, where $y = h(e) \in M$. Since $r_p \upharpoonright M = \text{Id}$, we have $f = h \upharpoonright M = r_y \upharpoonright M$.

**Proposition 3.3.** Every $G$-map $f : M \to M$ is bijective.

**Proof.** According to Proposition 3.2, there is $a \in M$ such that $f(x) = xa$ for all $x \in M$. Since $Ma$ is a compact $G$-space contained in $M$, we have $Ma = M$ by the minimality of $M$. Thus there exists $b \in M$ such that $ba = p$. Let $g : M \to M$ be the $G$-map defined by $g(x) = xb$. Then $fg(x) = xba = xp = x$ for every $x \in M$, therefore $fg = 1$ (the identity map of $M$). We have proved that in the semigroup $S$ of all $G$-self-maps of $M$, every element has a right inverse. Hence $S$ is a group. (Alternatively, we first deduce from the equality $fg = 1$ that all elements of $S$ are surjective and then, applying this to $g$, we see that $f$ is also injective.)

We are now in a position to prove that every universal compact minimal $G$-space is isomorphic to $M$. First note that the minimal compact $G$-space $M$ is itself universal: if $Y$ is any compact $G$-space, there exists a $G$-map of the greatest ambit $X$ to $Y$, and its restriction to $M$ is a $G$-map of $M$ to $Y$. Now let $M'$ be another universal compact minimal $G$-space. There exist $G$-maps $f : M \to M'$ and $g : M' \to M$. Since $M'$ is minimal, $f$ is surjective. On the other hand, in virtue of Proposition 3.3 the composition $gf : M \to M$ is bijective. It follows that $f$ is injective and hence a $G$-isomorphism between $M$ and $M'$.

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