On the Dirichlet and Neumann Evolution Operators in $\mathbb{R}^d_+$

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Abstract We prove some uniform and pointwise gradient estimates for the Dirichlet and the Neumann evolution operators $G_D(t, s)$ and $G_N(t, s)$ associated with a class of nonautonomous elliptic operators $A(t)$ with unbounded coefficients defined in $I \times \mathbb{R}^d_+$ (where $I$ is a right-halfline or $I = \mathbb{R}$). We also prove the existence and the uniqueness of a tight evolution system of measures $\{\mu^N_t\}_{t \in I}$ associated with $G_N(t, s)$, which turns out to be subinvariant for $G_D(t, s)$, and we study the asymptotic behaviour of the evolution operators $G_D(t, s)$ and $G_N(t, s)$ in the $L^p$-spaces related to the system $\{\mu^N_t\}_{t \in I}$.

Keywords Nonautonomous second-order elliptic operators · Unbounded coefficients · Evolution operators · Pointwise and uniform gradient estimates · Evolution systems of measures · Asymptotic behaviour.

Mathematics Subject Classifications (2010) 35K10 · 35K15 · 35B40 · 37L40

1 Introduction

The increasing interest in Kolmogorov equations is due to their relevant role in many branches of mathematics. In particular, these equations arise in a natural way from many applications in physics. For example in some free boundary problems in combustion theory and in the study of the Navier-Stokes equations in rotating exterior domains, simple changes of variables transform operators with bounded coefficients into operators with unbounded coefficients. Kolmogorov equations are also strongly connected to the study of many problems in population dynamics and in mathematical finance that lead to stochastic models.

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where it is quite natural to require that the unbounded coefficients are explicitly depending on time. Whereas the theory is already well developed in the autonomous case (see e.g., [6, 7] [8, 14] [24] and the monograph [9]), in the nonautonomous case some results have been proved very recently and a lot of significant problems are still open. To the best of our knowledge, all the literature in the nonautonomous setting is related to the case of the whole space $\mathbb{R}^d$. In such a case, many aspects of the Cauchy problem for nonautonomous parabolic equations have been studied in [3–5, 15, 16, 20, 22, 23].

This paper represents the first step to understand and analyze nonautonomous elliptic operators (and their associated evolution operators) in unbounded domains with homogeneous boundary conditions. Given a right halfline $I$ (possibly $I = \mathbb{R}$), we consider a class of linear nonautonomous second-order uniformly elliptic operators

$$A(t) = \sum_{i,j=1}^{d} q_{ij}(t, \cdot) D_{ij} + \sum_{i=1}^{d} b_{i}(t, \cdot) D_i - c(t, \cdot),$$

with sufficiently smooth and possibly unbounded coefficients defined in $I \times \mathbb{R}^d_+$, where $\mathbb{R}^d_+ := \mathbb{R}^{d-1} \times (0, +\infty)$. Under suitable assumptions on the coefficients of the operator $A(t)$, for any $s \in I$ the Cauchy-Dirichlet problem

$$\begin{array}{ll}
u(t, x) = A(t)u(t, x), & t \in (s, +\infty), \ x \in \mathbb{R}^d_+ \\
u(t, x) = 0, & t \in (s, +\infty), \ x \in \partial \mathbb{R}^d_+,
\end{array} \tag{P_D}$$

with $f \in C_b(\mathbb{R}^d_+)$, and the Cauchy-Neumann problem

$$\begin{array}{ll}
u(t, x) = A(t)u(t, x), & t \in (s, +\infty), \ x \in \mathbb{R}^d_+ \\
\frac{\partial u}{\partial v}(t, x) = 0, & t \in (s, +\infty), \ x \in \partial \mathbb{R}^d_+,
\end{array} \tag{P_N}$$

with $f \in C_b(\mathbb{R}^d_+)$, are governed by two evolution operators: the Dirichlet evolution operator $\{G_D(t, s) : t \geq s \in I\}$ and the Neumann evolution operator $\{G_N(t, s) : t \geq s \in I\}$. Our aim consists in investigating some properties of these evolution operators. In the first part of the paper we prove some pointwise gradient estimates satisfied by the functions $G_D(t, s)f$ and $G_N(t, s)f$. More precisely, for any $p > 1$ we prove that there exist two positive constants $c_p$ and $C_p$ such that

$$|\nabla_x G_D(t, s)f|^p \leq c_p e^{C_p(t-s)} G_N(t, s) \left( |f|^p + |\nabla f|^p \right), \tag{1.1}$$

for any $t > s \in I$ and $f \in C^1_b(\mathbb{R}^d_+)$ which vanishes on $\partial \mathbb{R}^d_+$, and

$$|\nabla_x G_N(t, s)f|^p \leq c_p e^{C_p(t-s)} G_N(t, s) \left( |f|^p + |\nabla f|^p \right), \tag{1.2}$$

for any $t > s \in I$, any $p > 1$, any $f \in C^1_b(\mathbb{R}^d_+)$. Clearly, in an estimate like (1.1) we cannot have $G_D(t, s)$ in the right-hand side, since $G_D(t, s)(|f|^p + |\nabla f|^p)$ vanishes on $\partial \mathbb{R}^d_+$, whereas, in general, $\nabla_x G_D(t, s)f$ does not. Our main assumptions (Hypotheses 1) are a dissipativity condition on the drift $b = (b_i)$, and some growth assumptions on the spatial derivatives of the diffusion coefficients $q_{ij}$ and on the potential term $c$. Under stronger assumptions we obtain (1.1) and (1.2) also for $p = 1$.
We also prove that, for any \( s \in I \), the estimate
\[
|\nabla_x G_\mathcal{I}(t, s) f|^p \leq \tau_p e^{\omega_p(t-s)} (t-s)^{-\frac{D}{2}} G_N(t, s) |f|^p, \quad \mathcal{I} \in \{D, \mathcal{N}\},
\]
holds in \( \mathbb{R}_+^d \) for any function \( f \in C_b(\mathbb{R}_+^d) \) (resp. \( f \in C_b(\mathbb{R}_+^d) \), if \( \mathcal{I} = \mathcal{N} \), any \( t \in (s, +\infty) \), any \( p \in (1, +\infty) \) and some constants \( \tau_p, \omega_p \in \mathbb{R} \).

To the best of our knowledge, the pointwise gradient estimates (1.1) and (1.3) are new also in the autonomous case when \( \mathcal{I} = D \). In this setting only uniform \( C^0-C^1 \) estimates are available for solutions to Dirichlet parabolic problems even in more general domains (see [14]). On the other hand, estimates similar to ours are available for solutions to Neumann parabolic problems in sufficiently smooth unbounded domains [6, 8].

Besides their own interest, the previous estimates represent a helpful tool both in studying the asymptotic behaviour of the evolution operators \( G_D(t, s) \) and \( G_N(t, s) \) and in establishing some summability improving results for such operators. As already noticed in the case of the whole space (see [4]), the usual \( L^p \)-spaces are not the appropriate setting where to study elliptic operators with unbounded coefficients and their associated evolution operators. On the contrary the \( L^p \)-spaces related to particular systems of measures, called evolution systems of measures (see Definition 2), seem to be more apt. Existence of such systems of measures have been proved in the case of the whole space, first for the Ornstein-Uhlenbeck evolution operator and, then, for more general nonautonomous elliptic operators with unbounded coefficients in [15, 20]. We also quote the related papers [10–13].

Here, in the case \( c \equiv 0 \), we prove that there exists an evolution system of measures \( \{\mu^N_t\}_{t \in I} \) associated with the evolution operator \( G_N(t, s) \), which turns out to be subinvariant for the Dirichlet evolution operator \( G_D(t, s) \) even if \( \inf_{t \times \mathbb{R}_+^d} c \geq 0 \). This family of measures is obtained as the weak* limit of the unique tight evolution system of measures for the evolution operators \( G^e(t, s) \) in the whole of \( \mathbb{R}_+^d \). Here, \( G^e(t, s) \) is the evolution operator associated with a suitably uniform evolution operator \( A^e(t) \), whose coefficients are defined in the whole of \( I \times \mathbb{R}_+^d \) starting from the coefficients of \( A(t) \).

Moreover, under suitable assumptions, the gradient estimate (1.2) implies both that \( \{\mu^N_t\}_{t \in I} \) is the unique tight evolution system of measures for \( G_N(t, s) \) and that the operators \( G_D(t, s) \) and \( G_N(t, s) \) are bounded from \( L^p \left( \mathbb{R}_+^d, \mu^N_t \right) \) into the Sobolev space \( W^{1,p} \left( \mathbb{R}_+^d, \mu^N_t \right) \) for any \( t > s \in I \).

As in the case of the whole space, the unique tight evolution system of measures appears naturally in the study of the asymptotic behaviour of \( G_N(t, s) \) and \( G_D(t, s) \) as \( t \) tends to infinity. More precisely, if \( m^N_s(f) \) denotes the average of \( f \) with respect to the measure \( \mu^N_s \), then, under suitable assumptions we prove that, for any \( R > 0 \) and any \( s \in I \), it holds that
\[
|(G_D(t, s) f)(x)| \leq c_{R, s} e^{\sigma_0(t-s)} \|f\|_\infty, \quad f \in C_b \left( \mathbb{R}_+^d \right)
\]
and
\[
|(G_N(t, s) f)(x) - m^N_s(f)| \leq c_{R, s} e^{\sigma_0(t-s)} \|f\|_\infty, \quad f \in C_b \left( \mathbb{R}_+^d \right),
\]
for any \( (t, x) \in (s, +\infty) \times B_R^+ \) and some constants \( \sigma_0 < 0 < c_{R, s} \). The previous pointwise estimates immediately give
\[
\lim_{t \to +\infty} \|G_D(t, s) f\|_{L^p \left( \mathbb{R}_+^d, \mu^N_t \right)} = 0,
\]
\[
\lim_{t \to +\infty} \|G_N(t, s) f - m^N_s(f)\|_{L^p \left( \mathbb{R}_+^d, \mu^N_t \right)} = 0,
\]
for any \( f \in L^p \left( \mathbb{R}^d_+, \mu^N_s \right) \) and any \( p \in (1, +\infty) \). The construction of the evolution system of measures \( \{ \mu^N_t \}_{t \in I} \), as the limit of the tight evolution system of measures associated with \( G^s(t, s) \), is the key tool to deduce many properties of \( G_D(t, s) \) and \( G_N(t, s) \) from the analogous ones of \( G^s(t, s) \). Assuming that the diffusion coefficients do not depend on \( x \), we prove both some exponential decay estimates for \( \| G_D(t, s) \|_{L^p} \left( \mathbb{R}^d_+, \mu^N_s \right) \) and \( \| G_N(t, s) f - m^N_s(f) \|_{L^p} \left( \mathbb{R}^d_+, \mu^N_s \right) \) and some logarithmic Sobolev inequalities with respect to the measures \( \{ \mu^N_t : t \in I \} \). Besides their own interest, the occurrence of logarithmic Sobolev inequalities allows to deduce notable properties such as compactness and hyper-contractivity for the evolution operators \( G_D(t, s) \) and \( G_N(t, s) \) as stated in Theorem 10. Note that, in some sense, the logarithmic Sobolev inequalities are the natural counterpart of the Sobolev embedding theorems that, in general, do not hold when the Lebesgue measure is replaced by evolution systems of measures: consider e.g., the case when \( \mathcal{A}(t) \) is the nonautonomous Ornstein-Uhlenbeck operator, where the tight evolution system of measures is of gaussian type.

The case of more general unbounded domains \( \Omega \) is under investigation and some partial results have been already obtained. We expect, as in the classical case of bounded coefficients, that the results proved in this paper could help to handle domains \( \Omega \) with a more complicated geometry.

The paper is organized as follows: in Section 2 we collect some preliminary results. In Section 3 we state and prove the pointwise and uniform gradient estimates for \( G_D(t, s) \) and \( G_N(t, s) \). In Section 4, we prove the existence and uniqueness of a tight evolution system of measures for \( G_N(t, s) \), we study the asymptotic behaviour of the evolution operators \( G_D(t, s) \) and \( G_N(t, s) \), we prove the logarithmic Sobolev inequality and some of its consequences. Section 5 contains examples of operators to which the results of this paper apply. Finally, in the Appendix we prove a result which is used in the proof of the pointwise gradient estimates.

1.1 Notations

For any \( k \geq 0 \), we consider the space \( C^k_b \left( \mathbb{R}^d_+ \right) \) (resp. \( C^k \left( \mathbb{R}^d_+ \right) \)) consisting of all the functions in \( C^k \left( \mathbb{R}^d_+ \right) \) which are bounded in \( \mathbb{R}^d_+ \) (resp. are bounded and can be extended by continuity to \( \mathbb{R}^d_+ \)) together with all their derivatives (up to the \([k]\)-th order). We use the subscript “\( b \)” instead of “\( c \)” for spaces of functions with compact support, whereas “\( \text{loc} \)” stands for locally. We also consider the space \( C^k_D \left( \mathbb{R}^d_+ \right), k = 0, 1, \) consisting of functions \( f \in C^k_b \left( \mathbb{R}^d_+ \right) \) vanishing on \( \partial \mathbb{R}^d_+ \).

We assume that the reader is familiar with the spaces \( C^{h,k}(I \times K) \) when \( h, k \in \mathbb{N} \cup \{0\} \), \( I \) is an interval and \( K \subset \mathbb{R}^N \) \((N \geq 1)\) is a domain or a closure of a domain. When \( \alpha \in (0, 1) \), we denote by \( C^{0,\alpha}(I \times K) \) the set of all continuous functions \( f \) such that \( f(t, \cdot) \) is \( \alpha \)-Hölder continuous in \( K \), uniformly with respect to \( t \in I \). Similarly, \( C^{b,j+\alpha}(I \times K) \) \((j = 1, 2, 3)\) is the set of all functions \( f \in C^{0,j}(I \times K) \) with \( j \)-th order spatial derivatives which belong to \( C^{0,\alpha}(I \times K) \). Further \( C^{a,1}(I \times K) \) denotes the set of all functions \( f \in C^{0,1}(I \times K) \) such that the function \( f(\cdot, x) \) is \( \alpha \)-Hölder continuous in \( I \), uniformly with respect to \( x \in K \). Finally, \( C^{\alpha,1}(I \times K) \) (resp. \( C^{1+\alpha/2}(I \times K) \)) is the set of all functions \( f \in C^{0,1}(I \times K) \) (resp. \( f \in C^{1,2}(I \times K) \)) such that the derivatives of maximum order are \( \alpha \)-Hölder continuous in \( I \times K \) with respect to the parabolic distance.

The partial derivatives \( \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x_i} \), and \( \frac{\partial^2 f}{\partial x_i \partial x_j} \) are denoted by \( D_t f, D_i f \) and \( D_{ij} f \), respectively. \( \text{Tr}(Q) \) and \( \langle x, y \rangle \) stand for the trace of the square matrix \( Q \) and the Euclidean scalar.
product of the vectors \( x, y \in \mathbb{R}^d \), respectively. By \( \chi_A \) we denote the characteristic function of the set \( A \subset \mathbb{R}^d \) and \( 1 := \chi_{\mathbb{R}^d} \). Given a probability measure \( \mu \) defined on the Borel \( \sigma \)-algebra \( B(\Omega) \), we write \( \langle \mu, f \rangle \) to denote the integral of \( f \in C^b(\Omega) \) with respect to the measure \( \mu \). Somewhere in the paper we find it convenient to split \( \mathbb{R}^d \ni x = (x', x_d) \) with \( x' \in \mathbb{R}^{d-1} \). Finally, the Euclidean ball with center at 0 and radius \( r > 0 \) is denoted by \( B_r \) and \( B_r^+ = B_r \cap \mathbb{R}^d_+ \).

2 Main Assumptions and Preliminary Results

This section is devoted to prove existence and uniqueness of a classical solution for the Cauchy problems (\( P_D \)) and (\( P_N \)). Here, the term classical has different meanings according to which problem we consider as it is pointed out in the following definition.

**Definition 1** A function \( u \in C^{1,2} ((s, +\infty) \times \mathbb{R}^d_+) \) is called a bounded classical solution

(i) of the problem (\( P_D \)) if it is bounded and continuous in \( (s, +\infty) \times \mathbb{R}^d_+ \) and satisfies (\( P_D \));

(ii) of the problem (\( P_N \)) if it is bounded and continuous in \( [s, +\infty) \times \mathbb{R}^d_+ \) and satisfies (\( P_N \)).

Throughout the paper we assume the following standing assumptions on the coefficients of the operators \( \{ A(t) : t \in I \} \), where \( I \) is an open right halfline or even \( I = \mathbb{R} \).

**Hypotheses 1**

(i) The coefficients \( q_{ij}, b_j, c \) belong to \( C^{\alpha/2,1}_{loc} (I \times \mathbb{R}^d_+) \) for some \( \alpha \in (0, 1) \) and any \( i, j = 1, \ldots, d; \)

(ii) \( c_0 := \inf_{I \times \mathbb{R}^d_+} c(t, x) > 0; \)

(iii) \( q_{id} \equiv b_d \equiv 0 \) on \( I \times \partial \mathbb{R}^d_+ \) \((i = 1, \ldots, d - 1);\)

(iv) for every \( (t, x) \in I \times \mathbb{R}^d_+ \), the matrix \( Q(t, x) = [q_{ij}(t, x)] \) is symmetric and there exists a function \( \eta : I \times \mathbb{R}^d_+ \to \mathbb{R}^+ \) such that \( 0 < \eta_0 := \inf_{I \times \mathbb{R}^d_+} \eta \) and

\[
\langle Q(t, x)\xi, \xi \rangle \geq \eta(t, x)|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad (t, x) \in I \times \mathbb{R}^d_+;
\]

(v) there exists a continuous function \( r : I \times \mathbb{R}^d_+ \to \mathbb{R} \), satisfying \( r(t, x) \leq -L_0 \eta(t, x) + L_1 \chi_{B_{R_0}(x)} \) for any \( (t, x) \in I \times \mathbb{R}^d_+ \) and some positive constants \( L_0, L_1 \) and \( R_0 \), such that

\[
\langle \nabla_x b(t, x)\xi, \xi \rangle \leq r(t, x)|\xi|^2, \quad (t, x) \in I \times \mathbb{R}^d_+, \quad \xi \in \mathbb{R}^d; \tag{2.1}
\]

(vi) there exists a positive constant \( k_1 \) such that

\[
|\nabla_x q_{ij}(t, x)| \leq k_1 \eta(t, x), \quad (t, x) \in I \times \mathbb{R}^d_+, \quad i, j = 1, \ldots, d. \tag{2.2}
\]

**Remark 1** Note that Hypotheses 1 imply that, for any bounded set \( J \subset I \), there exists a positive constant \( \lambda = \lambda_J \) such that

\[
A(t)\varphi(x) \leq \lambda_J \varphi(x), \quad t \in J, \ x \in \mathbb{R}^d_+, \tag{2.3}
\]
where \( \varphi(x) = |x|^2 + 1 \) for any \( x \in \mathbb{R}^d \). Indeed, note that \( (A(t)\varphi)(x) = 2\text{Tr}(Q(t, x)) + 2\langle b(t, x), x \rangle - c(t, x) (|x|^2 + 1) \) for any \( (t, x) \in I \times \mathbb{R}^d_+ \). Thanks to (2.2) we can estimate

\[
\text{Tr}(Q(t, x)) = \text{Tr}(Q(t, 0)) + \sum_{i=1}^d \int_0^1 \frac{d}{d\sigma} q_{ii}(t, \sigma x) d\sigma
\]

\[
\leq \text{Tr}(Q(t, 0)) + |x| \sum_{i=1}^d \int_0^1 |\nabla_x q_{ii}(t, \sigma x)| d\sigma
\]

\[
\leq \text{Tr}(Q(t, 0)) + k_1 d |x| \int_0^1 \eta(t, \sigma x) d\sigma,
\]

for any \( (t, x) \in I \times \mathbb{R}^d_+ \) and any \( i = 1, \ldots, d \). Arguing similarly and taking Hypothesis 1(v) into account, we can prove that

\[
\langle b(t, x), x \rangle \leq \left| b(t, 0) \right| |x| - L_0 |x| \int_0^1 \eta(t, \sigma x) d\sigma
\]

\[
\leq \left| b(t, 0) \right| |x| - L_0 |x| \int_0^1 \eta(t, \sigma x) d\sigma + L_1 \min \left\{|x|^2, R_0 |x|\right\},
\]

for any \( (t, x) \in I \times \mathbb{R}^d_+ \). Summing up, we have

\[
(A(t)\varphi)(x) \leq 2\text{Tr}(Q(t, 0)) + 2\left| b(t, 0) \right| |x| + 2 d k_1 d |x| - L_0 |x| \int_0^1 \eta(t, \sigma x) d\sigma,
\]

for any \( (t, x) \in I \times \mathbb{R}^d_+ \). Observing that \( \int_0^1 \eta(t, \sigma x) d\sigma \geq \eta_0 \) for any \( (t, x) \in I \times \mathbb{R}^d_+ \), estimate (2.3) follows immediately.

### 2.1 Approximating Evolution Operators

In order to prove the announced existence and uniqueness theorem, we use an approximation procedure. Therefore, considering the standard reflection with respect to the \( x_d \)-variable, we define the extension operators \( \mathcal{E}, \mathcal{O} : L^\infty\left(\mathbb{R}^d_+\right) \to L^\infty\left(\mathbb{R}^d\right) \) by setting

\[
\mathcal{E} f(x) := \begin{cases} 
    f(x', x_d), & x_d \geq 0 \\
    f(x', -x_d), & x_d < 0
\end{cases}
\]

\[
\mathcal{O} f(x) := \begin{cases} 
    f(x', x_d), & x_d \geq 0 \\
    -f(x', -x_d), & x_d < 0
\end{cases}
\]

For any function \( \psi : I \times \mathbb{R}^d \to \mathbb{R} \) and any \( \varepsilon \in (0, 1] \), we denote by \( \psi^\varepsilon : I \times \mathbb{R}^d \to \mathbb{R} \) the convolution (with respect to \( x \)) of \( \psi \) with a standard mollifier \( \rho_\varepsilon \).

Let \( \mathcal{A}^\varepsilon(t) \) be the operator defined on smooth functions \( \zeta \) by

\[
\mathcal{A}^\varepsilon(t) \zeta = \text{Tr} \left( Q^\varepsilon(t, \cdot) D^2 \zeta \right) + \langle b^\varepsilon(t, \cdot), \nabla \zeta \rangle - c^\varepsilon(t, \cdot) \zeta,
\]

\( t \in I \),

where \( q^\varepsilon_{ij} = (\tilde{q}^\varepsilon_{ij})^\varepsilon, b^\varepsilon_j = (\tilde{b}^\varepsilon_j)^\varepsilon \) (\( i, j = 1, \ldots, d \)), \( c^\varepsilon = (\mathcal{E} c)^\varepsilon \) and

\[
\tilde{q}^\varepsilon_{ij} := \begin{cases} 
    \mathcal{E} q_{ij}, & (i, j < d) \lor (i = j = d), \\
    \mathcal{O} q_{ij}, & (i < d, j = d) \lor (i = d, j < d)
\end{cases},
\]

\[
\tilde{b}^\varepsilon_i := \begin{cases} 
    \mathcal{E} b_i, & i < d, \\
    \mathcal{O} b_i, & i = d.
\end{cases}
\]
Proposition 1 For any \( \varepsilon \in (0, 1] \), \( s \in I \) and \( f \in C_b(\mathbb{R}^d) \) the Cauchy problem

\[
\begin{aligned}
D_t u(t, x) &= \mathcal{A}^\varepsilon(t) u(t, x), \quad t > s, \quad x \in \mathbb{R}^d, \\
u(s, x) &= f(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]

admits a unique solution \( u^\varepsilon \in C_b([s, +\infty) \times \mathbb{R}^d) \cap C^{1+\alpha/2,2+\alpha}_{loc}((s, +\infty) \times \mathbb{R}^d) \). Moreover,

\[
\|u^\varepsilon(t, \cdot)\|_\infty \leq e^{-c_0(t-s)} \|f\|_\infty, \quad t > s. \tag{2.7}
\]

Proof We begin by observing that \( q_{ij}^\varepsilon, b_i^\varepsilon \) and \( c^\varepsilon \) belong to \( C^{\alpha/2,1+\alpha}_{loc}(I \times \mathbb{R}^d) \) for any \( i, j = 1, \ldots, d \) and they satisfy Hypotheses \( 1 \) in \( \mathbb{R}^d \), with the same constants \( c_0, \eta, k, L_0, L_1 \) and with \( r, \eta \) and \( R_0 \) replaced by \( r^\varepsilon := (\mathcal{E}r)^\varepsilon, \eta^\varepsilon := (\mathcal{E}\eta)^\varepsilon \) and \( R_0 + 1 \), respectively. We limit ourselves just to proving that

\[
Q^\varepsilon \geq \eta^\varepsilon I, \quad \nabla_x b^\varepsilon \leq r^\varepsilon I, \tag{2.8}
\]

in the sense of quadratic forms and that \( |\nabla_x q_{ij}^\varepsilon|^2 \leq k_1^2(\eta^\varepsilon)^2 \), since the other properties are straightforward to prove. For this purpose, we set \( \tilde{Q} = (\tilde{q}_{ij}) \) and observe that

\[
\langle \tilde{Q}(t, x, y)d\tilde{x}, \tilde{x} \rangle = \langle Q(t, x, -x_d)(\tilde{x}', -\xi_d), (\xi', -\xi_d) \rangle \geq \eta(t, x, \xi)\|\xi\|^2,
\]

for any \( t \in I, x \in \mathbb{R}^{d-1}, \) any \( x_d < 0 \) and \( \xi = (\xi', \xi_d) \in \mathbb{R}^d \). Therefore, we have

\[
\langle \tilde{Q}(t, x)\xi, \xi \rangle \geq \mathcal{E}\eta(t, x)|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad (t, x) \in I \times \mathbb{R}^d. \tag{2.9}
\]

Similarly, since \( D_t\tilde{b}_j = \mathcal{E}D_t b_j \), if \( i, j < d \) or \( i = j = d \), and \( D_t\tilde{b}_j = \mathcal{O}D_t b_j \), if \( i < d = j \) or \( j < d = i \), we conclude that

\[
|\nabla_x \tilde{b}(t, x)\xi, \xi| \leq \mathcal{E}r(t, x)|\xi|^2, \quad \xi \in \mathbb{R}^d, \quad (t, x) \in I \times \left(\mathbb{R}^d \setminus \{0\}\right). \tag{2.10}
\]

Estimates \( 2.9 \) and \( 2.10 \) immediately lead to the claimed properties on the matrices \( Q^\varepsilon \) and \( \nabla_x b^\varepsilon \).

Finally, since \( |D_k q_{ij}^\varepsilon| \leq (|\mathcal{E}D_k q_{ij}|)^\varepsilon \) for any \( i, j, k = 1, \ldots, d \), using \( 2.2 \) and Jensen inequality, we get

\[
|\nabla_x q_{ij}^\varepsilon(t, x)|^2 \leq k_1^2(\eta^\varepsilon(t, x))^2, \quad (t, x) \in I \times \mathbb{R}^d, \quad i, j = 1, \ldots, d. \tag{2.11}
\]

Thus, the arguments used in Remark 1 show that the function \( \varphi \), defined by \( \varphi(x) = 1 + |x|^2 \) for any \( x \in \mathbb{R}^d \), is a Lyapunov function for the operator \( \mathcal{A}^\varepsilon(t) \), i.e., for every bounded set \( J \subset I \), \( \lim_{|x| \to +\infty} \left( \frac{\mathcal{A}^\varepsilon(t) \varphi}{\varphi} \right)(x) \leq -c_j' \), \( c_j' \) being a positive constant, independent of \( t \in J \) and of \( \varepsilon \in (0, 1] \). Now, [4, Thm. 2.3] yields the assertion. \( \square \)

The family of bounded operators \( \{G^\varepsilon(t, s) : t \geq s \in I\} \), defined by \( G^\varepsilon(t, s)f := u^\varepsilon(t, \cdot) \) for any \( t \geq s \), where \( u^\varepsilon \) is the function in Proposition 1, is an evolution operator in \( C_b(\mathbb{R}^d) \). In view of [4, Thm. 2.3, Prop. 3.1] there exists a positive function \( g^\varepsilon \) such that

\[
\|g^\varepsilon(t, s, \cdot)\|_{L^1(\mathbb{R}^d)} \leq e^{-c_0(t-s)}, \quad (t, x) \in (s, +\infty) \times \mathbb{R}^d. \tag{2.12}
\]

and

\[
(G^\varepsilon(t, s)f)(x) = \int_{\mathbb{R}^d} f(y) g^\varepsilon(t, s, y) dy, \quad t > s, \quad x \in \mathbb{R}^d. \tag{2.13}
\]
for any $f \in C_b(\mathbb{R}^d)$. In particular, from (2.12) and (2.13) we deduce that
\[ |G^f(t, s)(\psi_1 \psi_2)| \leq (G^\varepsilon(t, s)|\psi_1|^r) \frac{1}{r} (G^\varepsilon(t, s)|\psi_2|^q) \frac{1}{q}, \]
for any $\psi_1, \psi_2 \in C_b(\mathbb{R}^d)$ and any $r, q \in (1, +\infty)$ such that $1/r + 1/q = 1$. \( \square \)

2.2 Existence and uniqueness of the solutions to $(P_D)$ and $(P_N)$

In this subsection, we construct by approximation the Dirichlet and the Neumann evolution operators $G_D(t, s)$ and $G_N(t, s)$ governing the Cauchy problems $(P_D)$ and $(P_N)$, respectively. We begin by stating two maximum principles which immediately yield uniqueness of the classical solutions to $(P_D)$ and $(P_N)$.

**Proposition 2** Fix $s \in I$ and $T > s$. Let $u \in C^{1,2}((s, T] \times \mathbb{R}^d_+)$ be such that
\[ Du(t, x) - \mathcal{L}(t)u(t, x) \leq 0, \quad (t, x) \in (s, T] \times \mathbb{R}^d_+, \]
where $\mathcal{L} = \mathcal{A}$ or $\mathcal{L} = \mathcal{A}^\varepsilon$ ($\varepsilon \in (0, 1]$). The following properties are satisfied.

(i) \[ u \in C_b \left( \left( [s, T] \times \mathbb{R}^d_+ \right) \setminus \left( \{s\} \times \partial \mathbb{R}^d_+ \right) \right) \text{ and } u \leq 0 \text{ in } (s, T] \times \partial \mathbb{R}^d_+, \text{ then } u \leq 0 \text{ in } [s, T] \times \mathbb{R}^d_+. \]

(ii) \[ u \in C_b((s, T] \times \mathbb{R}^d_+) \cap C^{0,1} \left( (s, T] \times \mathbb{R}^d_+ \right) \text{ and } \frac{\partial u}{\partial \nu} \leq 0 \text{ in } (s, T] \times \partial \mathbb{R}^d_+, \text{ then } u \leq 0 \text{ in } [s, T] \times \mathbb{R}^d_+. \]

**Proof** The assertions can be obtained adapting to the nonautonomous setting the proofs in [14, Thm. A.2] and in [6, Prop. 2.1], using $\psi(x) = 1 + |x|^2$ as a Lyapunov function. \( \square \)

**Theorem 2** For any $s \in I$ and $f \in C_b(\mathbb{R}^d_+)$ (resp. $f \in C_b(\mathbb{R}^d_+)$ the problem $(P_D)$ (resp. $(P_N)$) admits a unique bounded classical solution $u_D$ (resp. $u_N$). Moreover, $u_D$ and $u_N$ belong to $C^{1+\alpha/2,2+\alpha}_{loc}((s, +\infty) \times \mathbb{R}^d_+)$, they satisfy the estimates
\[ (i) \|u_D(t, \cdot)\|_{\infty} \leq e^{-c_0(t-s)} \|f\|_{\infty}, \quad (ii) \|u_N(t, \cdot)\|_{\infty} \leq e^{-c_0(t-s)} \|f\|_{\infty}, \]
for any $t > s$, and they are nonnegative if $f \geq 0$. Moreover, if $f \in C^{2+\alpha}_{c} (\mathbb{R}^d_+)$, then $u_D$ and $u_N$ belong to $C^{1+\alpha/2,2+\alpha}_{loc}([s, +\infty) \times \mathbb{R}^d_+)$.

**Proof** The uniqueness part and the non-negativity of $u_D$ and $u_N$, when $f \geq 0$, follow from Proposition 2. The existence of a solution will be proved in some steps. We begin by considering the Cauchy Dirichlet problem $(P_D)$.

**Step 1.** Here, we prove that, for any $f \in C_D(\mathbb{R}^d_+)$, the unique classical solution to the Cauchy-Dirichlet problem $(P_D)$ with $\mathcal{A}(t)$ replaced by $\mathcal{A}^\varepsilon(t)$, which we denote by $u_D^\varepsilon$, is the restriction to $\mathbb{R}^d_+$ of the function $G^\varepsilon(t, s)\mathcal{O}f$, i.e.,
\[ G_D^\varepsilon(t, s)f = (G^\varepsilon(t, s)\mathcal{O}f) |_{\mathbb{R}^d_+}, \quad t > s. \]

Clearly, the function in the right-hand side of (2.16) solves the differential equation and satisfies the initial condition in $(P_D)$. To prove that it vanishes on $(s, +\infty) \times \partial \mathbb{R}^d_+$, we show that, if $\psi \in C_b(\mathbb{R}^d)$ is odd with respect to the variable $x_d$, then, for any $s \in I$, $G^\varepsilon(t, s)\psi$ is odd with respect to the variable $x_d$. This
clearly implies that $G^e(t, s)\psi$ vanishes on $\partial \mathbb{R}^d_+$. To check this property, observe that the function $v \in C_b([s, +\infty) \times \mathbb{R}^d) \cap C^{1,2}((s, +\infty) \times \mathbb{R}^d)$, defined by $v(t, x) = (G^e(t, s)\psi)(x_1, \ldots, x_d-1, -x_d)$ for any $t > s$ and any $x \in \mathbb{R}^d$, solves the equation $v_t - A^e(t)v = 0$ in $(s, +\infty) \times \mathbb{R}^d$, due to the symmetry properties of the coefficients of the operator $A^e(t)$. Since $v(s, \cdot) = -\psi$ in $\mathbb{R}^d$, the uniqueness of the solution to the Cauchy problem

\[
\left\{ \begin{array}{l}
D_t w(t, x) = A^e(t)w(t, x), \ t > s, \ x \in \mathbb{R}^d, \\
w(s, x) = -\psi(x), \ x \in \mathbb{R}^d,
\end{array} \right.
\]

which follows from Proposition 1, guarantees that $v = -G^e(\cdot, s)\psi$, and this yields the claim.

**Step 2.**

Here, we prove the existence of a classical solution to $(P_D)$ in the case $f \in C^{2+\alpha}_c(\mathbb{R}^d_+)$. From the classical Schauder estimates we can infer that, for any $k \in \mathbb{N}$, there exists a positive constant $c_k$, depending only on $\eta^\varepsilon$, the $C^{\alpha/2,\alpha}$-norms of the coefficients of the operator $A^e(t)$ in $[s, s+k] \times B^+_{2k}$, such that

\[
\|u^e_d\|_{C^{1+\alpha/2,2+\alpha}([s, s+k] \times B^+_{2k})} \leq c_k \|f\|_{C^{2+\alpha}(\mathbb{R}^d_+)}.
\]

(2.17)

Note that the constant $c_k$ can be taken independent of $\varepsilon$ since $\eta^\varepsilon \geq \eta_0$ and the $C^{\alpha/2,\alpha}$-norms of the coefficients $a^e_{i,j}, b^e_j$ and $c^e(i, j = 1, \ldots, d) (s, s+k) \times B^+_{2k}$ can be estimated from above, uniformly with respect to $\varepsilon \in (0, 1)$, in terms of the $C^{\alpha,\alpha}$-norms of $q_{ij}, b_j$ and $c(i, j = 1, \ldots, d)$ in the same set.

In view of Arzelà-Ascoli theorem and (2.17), for any $k \in \mathbb{N}$ there exist an infinitesimal sequence $(\varepsilon^k_n) \subset (0, 1)$ and a function $u_k \in C^{1+\alpha/2,2+\alpha}((s, s+k) \times B^+_{2k})$ such that $u^e_d$ converges to $u_k$ in $C^{1,2}([s, s+k] \times \overline{B}_{2k}^+)$ as $n \to +\infty$. Without loss of generality, we can assume that $(\varepsilon^k_{n+1}) \subset (\varepsilon^k_n)$ for any $k \in \mathbb{N}$. Hence, by a diagonal argument, we can find an infinitesimal sequence $(\varepsilon_n)$ such that $u^e_d$ converges to $u$ in $C^{1,2}([s, s+k] \times \overline{B}_{k}^+)$ as $n \to +\infty$, for any $k \in \mathbb{N}$, where $u : [s, +\infty) \times \mathbb{R}^d_+ \to \mathbb{R}$ is defined by $u(t, x) = u_k(t, x)$, $k$ being any integer such that $(t, x) \in [s, s+k] \times \overline{B}_{k}^+$. Clearly, $u \in C^{1+\alpha/2,2+\alpha}_{\text{loc}}([s, +\infty) \times \overline{\mathbb{R}^d_+})$ is a bounded classical solution to problem $(P_D)$ and it satisfies (2.15) thanks to (2.7).

**Step 3.**

We now fix $f \in C^\varepsilon_{\text{loc}}(\mathbb{R}^d_+)$ which tends to zero at infinity, and consider a sequence $(f_n) \subset C^{2+\alpha}(\mathbb{R}^d_+)$ converging to $f$ uniformly in $\mathbb{R}^d_+$. By Step 2, for any $n \in \mathbb{N}$, the Cauchy problem $(P_D)$, with $f$ replaced by $f_n$, admits a unique solution $u_n \in C^{1+\alpha/2,2+\alpha}_{\text{loc}}([s, +\infty) \times \overline{\mathbb{R}^d_+})$. Interior Schauder estimates show that, for any $R, T > 0$ and $\sigma < T - s$, there exists a positive constant $C$, independent of $n$, such that $\|u_n\|_{C^{1+\alpha/2,2+\alpha}([s+s+\sigma] \times \overline{B}_R^+)} \leq C \|f\|_{\infty}$ for any $n \in \mathbb{N}$. Arguing as in Step 2, we prove that $u_n$ converges to a function $u$ which belongs to $C^{1+\alpha/2,2+\alpha}_{\text{loc}}((s, +\infty) \times \overline{\mathbb{R}^d_+})$, vanishes on $[s, +\infty) \times \partial \mathbb{R}^d_+$ and satisfies $D_t u = A(t)u$ in $(s, +\infty) \times \mathbb{R}^d_+$. Moreover, since $\|u_n(t, \cdot) - u_m(t, \cdot)\|_{\infty} \leq e^{-c_0(t-s)}\|f_n - f_m\|_{\infty}$ for any $t > s$ and any $m, n \in \mathbb{N}$, $u$ actually belongs to $C_b([s, +\infty) \times \overline{\mathbb{R}^d_+})$ and $u(s, \cdot) = f$ since $u_n(s, \cdot) = f$ for any $n \in \mathbb{N}$. Hence, $u$ is a classical solution to problem $(P_D)$ and, of course, it satisfies estimate (2.15).
Step 4. Finally, we deal with the general case when \( f \in C_b(\mathbb{R}_+^d) \) and denote by \( u_g \) the unique solution to problem \((P_D)\) with initial datum \( g \in C_D(\mathbb{R}_+^d) \) which tends to zero at infinity. We consider a sequence of functions \( (f_n) \in C^{2+\alpha}(\mathbb{R}_+^d) \) converging to \( f \) locally uniformly in \( \mathbb{R}_+^d \) and such that \( M := \sup_{n \in \mathbb{N}} \|f_n\|_\infty < +\infty \). The already used compactness argument shows that, up to a subsequence, \( u_{f_n} \) converges in \( C^{1,2,\alpha}(s, +\infty) \times \mathbb{R}_+^d \) to a function \( u \in C^{1+\alpha/2,2+\alpha}(s, +\infty) \times \mathbb{R}_+^d \). Hence, \( u \) satisfies the differential equation, the boundary condition in \((P_D)\) and also the estimate (2.15).

To complete the proof, we show that \( u \) is continuous also on \([s] \times \mathbb{R}_+^d \) and \( u(s, \cdot) = f \) in \( \mathbb{R}_+^d \). For this purpose, we fix a compact set \( K \subset \mathbb{R}_+^d \) and a smooth and compactly supported function \( \psi \) such that \( 0 \leq \psi \leq 1 \) and \( \psi \equiv 1 \) in \( K \). Since \( f_n = \psi f_n + (1 - \psi) f_n \) for every \( n \in \mathbb{N} \), by linearity \( u_{f_n} = u\psi f_n + u(1-\psi) f_n \). We know that the functions \( u\psi f_n \) and \( u\psi \) are continuous up to \( s \) where they are equal to \( \psi f_n \) and \( \psi \) respectively. Proposition 2(ii) and the positivity of \( \varepsilon \) yield \( \|u(1-\psi)f_n\|_\infty \leq M(1-\psi) \) for any \( n \in \mathbb{N} \). Hence, \( |u f_n - f| \leq |u\psi f_n - \psi f| + M(1 - \psi) \) in \( (s, +\infty) \times K \). Letting \( n \to +\infty \) we obtain \( |u - f| = |u\psi - f| + M(1 - \psi) \) in the same set as above. Now, it follows that \( u \) can be extended by continuity at \( t = s \) by setting \( u(s, x) = f(x) \) for any \( x \in K \). By the arbitrariness of \( K \) we deduce that \( u \) is continuous on \([s] \times \mathbb{R}_+^d \) and \( u(s, \cdot) = f \).

The proof of the claim in the case of the Cauchy-Neumann problem \((P_N)\) follows essentially the same lines of the Dirichlet case, taking into account that, for any \( f \in C_b(\mathbb{R}_+^d) \), the unique classical solution to the Cauchy-Neumann problem \((P_N)\) with \( A(t) \) replaced by \( A^c(t) \) is the restriction to \( \mathbb{R}_+^d \) of the function \( G^c((\cdot, s)\mathcal{E} f) \), i.e.,

\[
G_N^c(t, s) f = (G^c(t, s)\mathcal{E} f)|_{\mathbb{R}_+^d}, \quad t > s. \tag{2.18}
\]

The starting point of the proof is the case \( f \in C^{2+\alpha}(\mathbb{R}_+^d) \) with null normal derivative on \( \partial\mathbb{R}_+^d \). Once the existence of a solution to problem \((P_N)\) is established for such functions \( f \), a density argument allows to cover the case when \( f \in C_b(\mathbb{R}_+^d) \) vanishing at infinity. The general case \( f \in C_b(\mathbb{R}_+^d) \) finally follows arguing as in Step 4 above. We stress that the continuity of the solution of problem \((P_N)\) on \([s] \times \mathbb{R}_+^d \) follows from observing that we can consider a sequence of functions \( f_n \in C^{2+\alpha}(\mathbb{R}_+^d) \), with null normal derivatives on \( \partial\mathbb{R}_+^d \), converging to \( f \) locally uniformly in \( \mathbb{R}_+^d \), and the compact set \( K \) can be a subset of \( \mathbb{R}_+^d \).

In view of Theorem 2 we can define two families of bounded linear operators \( \{G_D(t, s) : t \geq s \in I\} \subset C_b(\mathbb{R}_+^d) \) and \( \{G_N(t, s) : t \geq s \in I\} \subset C_b(\mathbb{R}_+^d) \), by setting

\[
(G_D(t, s) f)(x) := u_D(t, x), \quad (t, x) \in [s, +\infty) \times \mathbb{R}_+^d,
\]

\[
(G_N(t, s) f)(x) := u_N(t, x), \quad (t, x) \in [s, +\infty) \times \mathbb{R}_+^d.
\]

The evolution laws \( G_D(t, s) = G_D(t, r)G_D(r, s) \), \( G_N(t, s) = G_N(t, r)G_N(r, s) \), for any \( I \ni s < r < t \), are immediate consequence of the uniqueness of the solutions to problems \((P_D)\) and \((P_N)\). The families \( \{G_D(t, s) : t \geq s \in I\} \) and \( \{G_N(t, s) : t \geq s \in I\} \) are called
the evolution operator associated to problem \((P_D)\) and \((P_N)\), respectively. In the sequel, to simplify the notation, we simply write \(G_D(t, s)\) and \(G_N(t, s)\) to denote the previous two evolution operators.

In the following proposition we collect some useful properties of \(G_D(t, s)\) and \(G_N(t, s)\).

**Proposition 3** Fix \(s \in I\). The following statements are true:

(i) if \((f_n) \subseteq C_b (\mathbb{R}_+^d) \) (resp. \((f_n) \subseteq C_b (\overline{\mathbb{R}_+^d})\)) is a bounded sequence, with respect to the sup-norm, which converges to \(f \in C_b (\mathbb{R}_+^d)\) (resp. \(f \in C_b (\overline{\mathbb{R}_+^d})\)) locally uniformly in \(\mathbb{R}_+^d\), then \(G_D(\cdot, s)f_n\) (resp. \(G_N(\cdot, s)f_n\)) converges to \(G_D(\cdot, s)f\) (resp. \(G_N(\cdot, s)f\)) in \(C^{1,2}_{loc} ((s, +\infty) \times \mathbb{R}_+^d)\).

(ii) if \(f \in C_b (\overline{\mathbb{R}_+^d})\) is nonnegative, then \(G_D(t, s)f \leq G_N(t, s)f\) for every \(t > s\).

**Proof** Property (i) in the Dirichlet case follows essentially from the arguments of Step 4 in the proof of Theorem 2. Such arguments, with slight changes, can be used to prove (i) also in the Neumann case.

Let us prove property (ii). Fix a nonnegative function \(f \in C_b (\overline{\mathbb{R}_+^d})\). By Theorem 2, \(G_N(\cdot, s)f\) is nonnegative in \((s, +\infty) \times \mathbb{R}_+^d\). Now, we consider the function \(v = G_D(\cdot, s)f - G_N(\cdot, s)f\). Clearly, \(v \in C^{1,2} ((s, +\infty) \times \mathbb{R}_+^d) \cap C_b \left( (s, +\infty) \times \mathbb{R}_+^d \setminus \{s\times \partial \mathbb{R}_+^d\} \right)\), it solves \(D_t v - \mathcal{A}(t)v = 0\) in \((s, +\infty) \times \mathbb{R}_+^d\) and \(v \equiv 0\) in \(\{s\} \times \mathbb{R}_+^d\). Finally, since \(v = -G_N(\cdot, s)f \leq 0\) in \((s, +\infty) \times \partial \mathbb{R}_+^d\), we conclude, using Proposition 2, that \(v \leq 0\) in \((s, +\infty) \times \mathbb{R}_+^d\), i.e., \(G_D(\cdot, s)f \leq G_N(\cdot, s)f\).

### 3 Gradient Estimates

In this section we provide both pointwise and uniform (spatial) gradient estimates for the functions \(G_D(t, s)f\) and \(G_N(t, s)f\), when \(f \in C_b (\mathbb{R}_+^d)\) and \(f \in C_b (\overline{\mathbb{R}_+^d})\), respectively, and when \(f\) is even much smoother. If not otherwise specified, throughout this section we assume that the following conditions are satisfied.

**Hypotheses 3**

(i) Hypotheses 1 are satisfied;

(ii) there exists a continuous function \(\beta : I \times \mathbb{R}_+^d \rightarrow [0, +\infty)\) such that

\[
|\nabla_x c| \leq \beta, \quad \text{in } I \times \mathbb{R}_+^d, \quad \text{(3.1)}
\]

(iii) for every \(p \in (1, +\infty)\) there exists a positive constant \(C_p\) such that

\[
r + \left( \frac{k_2^2d^2}{4M_p} - M_p \right) \eta - \left( 1 - \frac{1}{p} \right) c + \frac{pk_2}{4(p-1)} \beta \leq C_p, \quad \text{(3.2)}
\]

in \(I \times \mathbb{R}_+^d\), where \(r\) is defined in Hypothesis 1(v) and \(M_p = \min \{1, p - 1\}\).
Remark 2 Note that the coefficients \( q_{ij}^e, b_i^e \) (i, j = 1, \ldots, d) and \( c^e \) satisfy Hypotheses 3(ii)-(iii) with \( \mathbb{R}^d_+ \) replaced by \( \mathbb{R}^d \), with the same constants \( k_1, k_2, M_p, C_p \), and with \( \eta, r \) and \( \beta \) replaced by \( \eta^e, r^e \) and \( \beta^e := (\mathcal{E}\beta)^e \), respectively.

3.1 \( C^1 - C^1 \) Uniform and Pointwise Estimates

As it has been already remarked in the introduction, we can not expect an estimate where \( |\nabla_x G_D(t, s)g|^p \) is controlled from above by \( G_D(t, s)(|g|^p + |\nabla g|^p) \) since this latter vanishes on \( \partial\mathbb{R}^d_+ \). However, for any function \( g \in C^1_D(\mathbb{R}^d_+) \), we can estimate \( |\nabla_x G_D(t, s)g|^p \) by means of \( G_N(t, s)(|g|^p + |\nabla g|^p) \), as the next theorem shows.

**Theorem 4** The pointwise gradient estimate

\[
|\nabla_x G_I(t, s) f|^p \leq 2^{\ell_p + p - 1} e^{C_p(t-s)} G_N(t, s)(|f|^p + |\nabla f|^p),
\]

holds in \( \mathbb{R}_+^d \) for any \( t > s \in I \), any \( p \in (1, +\infty) \) and any \( f \in C^1_b(\mathbb{R}_+^d) \), when \( I = N \), and any \( f \in C^1_D(\mathbb{R}_+^d) \) when \( I = D \). Here, \( C_p \) is the constant in (3.2) and \( \ell_p = \max\{p/2 - 1, 1\} \).

**Proof** The core of the proof consists in proving the gradient estimate

\[
|\nabla_x G^e_I(t, s) f|^p \leq 2^{\ell_p + p - 1} e^{C_p(t-s)} G^e_N(t, s)(|f|^p + |\nabla f|^p),
\]

in \( \mathbb{R}^d \) for any \( t > s \), any function \( f \in C^{3+\alpha}_c(\mathbb{R}^d) \) and any \( \varepsilon \in (0, 1] \). This estimate is obtained in Steps 1 and 2. More precisely, in Step 1 we prove that, for any \( t > s \in I \) and any \( n \in \mathbb{N} \),

\[
|\nabla_x G^e_{N, n}(t, s) f|^p \leq 2^{\ell_p} e^{C_p(t-s)} G^e_{N, n}(t, s)(|f|^p + |\nabla f|^p),
\]

in \( B_n \) for positive functions \( f \in C^{3+\alpha}(\mathbb{R}^d) \) which are constant outside a compact set contained in \( B_n \), where \( G^e_{N, n}(t, s) \) denotes the evolution operator associated to the restriction of the operator \( \mathcal{A}^\varepsilon(t) \) (see (2.5)) to \( B_n \), with homogeneous Neumann boundary conditions. In the second one, we complete the proof of (3.4). Finally, in Step 3, we prove (3.3).

**Step 1.** Fix \( s \in I \) and \( \varepsilon \in (0, 1] \); for any positive function \( f \in C^{3+\alpha}(\mathbb{R}^d) \), which is constant outside the ball \( B_{n_0} \), we set \( u_n^e := G^e_{N, n}(\cdot, s) f \) for any \( n > n_0 \) and consider the function \( w = \left( (u_n^e)^2 + |\nabla_x u_n^e|^2 \right)^{p/2} \) which belongs to \( C_b((s, +\infty) \times B_n) \cap C^{1,2}((s, +\infty) \times B_n) \) (see Proposition 6). Since \( u_n^e(t, x) \geq \delta \) for any \( t > s \), \( x \in B_n \) and some \( \delta > 0 \), \( w \) has positive infimum in \( (s, +\infty) \times B_n \). Moreover, \( w - \mathcal{A}(t) w = \psi_1, p + \psi_2, p + \psi_3, p + \psi_4, p \), where

\[
\psi_1, p = p w^{-\frac{2}{p}} \left( \sum_{i, j, k=1}^d D_k q_{ij}^e D_k u_n^e D_{ij} u_n^e - u_n^e (\nabla x e^e, \nabla_x u_n^e) \right),
\]

\[
\psi_2, p = - p(p - 2) w^{-\frac{4}{p}} |\sqrt{Q^e} (u_n^e, \nabla_x u_n^e + D_x^2 u_n^e, \nabla_x u_n^e)|^2,
\]

\[
\psi_3, p = p w^{-\frac{2}{p}} \left( (\nabla_x b^e, \nabla_x u_n^e, \nabla_x u_n^e) - |\sqrt{Q^e} \nabla_x u_n^e|^2 - \sum_{k=1}^d |\sqrt{Q^e} \nabla_x D_k u_n^e|^2 \right),
\]
\[ \psi_{4, p} = - (p - 1) c^\varepsilon w. \quad (3.9) \]

We are going to prove that

\[ w_t - \mathcal{A}^e(t) w \leq p C_p w. \quad (3.10) \]

For this purpose, we begin by observing that, from (2.11) and (3.1) we obtain

\[
\psi_{1, p} \leq p \omega \frac{1 - \frac{2}{p}}{\omega} \left( d k_1 \eta^\varepsilon | \nabla_x u_n^\varepsilon | | D_x^2 u_n^\varepsilon | + \beta^\varepsilon | u_n^\varepsilon | | \nabla_x u_n^\varepsilon | \right)
\]

\[
\leq p \omega \frac{1 - \frac{2}{p}}{\omega} \left[ a_1 d k_1 \eta^\varepsilon | D_x^2 u_n^\varepsilon |^2 + \left( \frac{d k_1}{4 a_1} \eta^\varepsilon + \frac{\beta^\varepsilon}{4 a_2} \right) | \nabla_x u_n^\varepsilon |^2 + a_2 \beta^\varepsilon (u_n^\varepsilon)^2 \right],
\]

(3.11)

for any \( a_1, a_2 > 0 \).

If \( p \geq 2 \), \( \psi_{2, p} \) is nonpositive in \( I \times B_n \). Moreover, taking (2.8) into account, we deduce that \( \psi_{3, p} \leq p \omega \frac{1 - \frac{2}{p}}{\omega} \left[ (r^\varepsilon - \eta^\varepsilon) | \nabla_x u_n^\varepsilon |^2 - \eta^\varepsilon | D_x^2 u_n^\varepsilon |^2 \right] \). Hence, from the above computations we obtain

\[
\psi_{1, p} + \psi_{2, p} + \psi_{3, p} + \psi_{4, p}
\leq w^{1 - \frac{2}{p}} \left[ p (a_1 d k_1 - 1) \eta^\varepsilon | D_x^2 u_n^\varepsilon |^2 + (p a_2 \beta^\varepsilon - (p - 1) c^\varepsilon)(u_n^\varepsilon)^2 \right.
\]

\[
+ \left( p r^\varepsilon + p \left( \frac{d k_1}{4 a_1} - 1 \right) \eta^\varepsilon + \frac{\beta^\varepsilon}{4 a_2} - (p - 1) c^\varepsilon \right) | \nabla_x u_n^\varepsilon |^2 \left].
\]

for \( p \geq 2 \) and any \( a_1, a_2 > 0 \). Choosing \( a_1 = M_p(d k_1)^{-1} \), \( a_2 = (p - 1)(p k_2)^{-1} \), recalling that \( \beta^\varepsilon \leq k_2 c^\varepsilon \) and observing that condition (3.2) implies that

\[
\frac{k_1^2 d^2}{4 M_p} - M_p \right) \eta^\varepsilon - \left( 1 - \frac{1}{p} \right) c^\varepsilon + \frac{p k_2}{4 (p - 1)} \beta^\varepsilon \leq C_p,
\]

we deduce that \( w_t - \mathcal{A}^e(t) w \leq p C_p w^{1 - \frac{2}{p}} | \nabla_x u_n^\varepsilon |^2 \leq p C_p w \), i.e., (3.10) follows.

If \( 1 < p < 2 \) we use the triangle inequality \( | \sqrt{Q^e} (\lambda + \mu) | \leq | \sqrt{Q^e} \lambda | + | \sqrt{Q^e} \mu | \), with \( \lambda = u_n^\varepsilon \nabla_x u_n^\varepsilon \) and \( \mu = D_x^2 u_n^\varepsilon \nabla_x u_n^\varepsilon \) to estimate

\[
\psi_{2, p} \geq p(2 - p) w^{1 - \frac{2}{p}}
\]

\[
\times \left[ u_n^\varepsilon | \sqrt{Q^e} \nabla_x u_n^\varepsilon | + \left( \sum_{h \neq k} D_h u_n^\varepsilon D_k u_n^\varepsilon (Q^e \nabla_x D_h u_n^\varepsilon, \nabla_x D_k u_n^\varepsilon) \right)^{\frac{1}{2}} \right]^2.
\]

Using twice the Cauchy-Schwarz inequality we get

\[
\psi_{2, p} \leq p(2 - p) w^{1 - \frac{2}{p}} \left[ u_n^\varepsilon | \sqrt{Q^e} \nabla_x u_n^\varepsilon | + | \nabla_x u_n^\varepsilon | \left( \sum_{h = 1}^d | \sqrt{Q^e} \nabla_x D_h u_n^\varepsilon |^2 \right)^{\frac{1}{2}} \right]^2
\]

\[
\leq p(2 - p) w^{1 - \frac{2}{p}} \left( | \sqrt{Q^e} \nabla_x u_n^\varepsilon |^2 + \sum_{h = 1}^d | \sqrt{Q^e} \nabla_x D_h u_n^\varepsilon |^2 \right).
\]
It thus follows that
\[
\psi_{2,p} + \psi_{3,p} \leq pw^{1-\frac{d}{p}} \left[ (\nabla b^f \nabla x u_n^f, \nabla x u_n^f) + (1 - p) \left( |\sqrt{Q^f} \nabla x u_n^f|^2 + \sum_{h=1}^{d} |\sqrt{Q^e} D_h u_n^e|^2 \right) \right] \\
\leq pw^{1-\frac{d}{p}} \left[ (r^e + (1 - p) \eta^e)|\nabla x u_n^e|^2 + (1 - p) \eta^e |D^2 u_n^e|^2 \right].
\]
(3.12)

From (3.11)-(3.12) we get inequality (3.10) also in the case \( p \in (1, 2) \).

To complete the proof of (3.5), we denote by \( v \) the unit exterior normal vector to \( \partial B_{n} \) and observe that
\[
\frac{\partial w}{\partial v}(t, x) = p(w(t, x))^{1-\frac{2}{p}} (D_{x}^{2} u_{n}^e(t, x) \nabla x u_n^f(t, x), v(x)),
\]
for any \((t, x) \in (s, +\infty) \times \partial B_{n}, \) which is nonpositive. Indeed, as it is immediately seen, \( \frac{\partial v}{\partial \tau}(x) = n^{-1} \) for any \( x \in \partial B_{n} \) and any unit vector \( \tau \) tangent to \( \partial \Omega \). Differentiating the identity \( \frac{\partial u_n^e}{\partial v} = 0 \) along the direction \( \tau = \nabla u_n^e(t, x) \) and taking the previous formula into account, we obtain
\[
0 = \langle D_{x}^{2} u_{n}^e(t, x) \nabla x u_n^f(t, x), v(x) \rangle + \frac{|\nabla x u_n^e(t, x)|^2}{n},
\]
for any \((t, x) \in (s, +\infty) \times \partial B_{n}. \) We thus conclude that \( \frac{\partial w}{\partial v} \leq 0 \) in \((s, +\infty) \times \partial B_{n}. \)

Summing up, the function \( v = w - e^{C_{p}(-s)}G_{N,n}^{e}(., s)(f^{2} + |\nabla f|^{2})^{\frac{p}{2}} \) satisfies
\[
v_{t} - (A^{e}(t) + C_{p})v \leq 0 \text{ in } (s, +\infty) \times B_{n}, \text{ vanishes on } \{s\} \times \bar{B}_{n} \text{ and its normal derivative is nonpositive in } (s, +\infty) \times \partial B_{n} \text{.}
\]
The classical maximum principle implies that \( v \leq 0 \), and estimate (3.5) follows at once.

Step 2. In view of [4, Thm. 2.3(ii)], \( G_{N,n}^{e}(t, s) \) converges to \( G^{e}(t, s) \) in \( C^{1,2}(D) \) for any compact set \( D \subset (s, +\infty) \times \mathbb{R}^{d}. \) Hence, letting \( n \to +\infty \) in (3.5) we get
\[
|\nabla_{x} G^{e}(t, s)|^{p} \leq 2C_{p}^{e} C_{p}(t-s) G^{e}(t, s)(|f|^{p} + |\nabla f|^{p}),
\]
(3.13)
for any \( t > s \) and any positive function \( f \) as in Step 1.

Estimate (3.13) can be extended easily to any nonnegative \( f \in C^{3+\alpha}_{c}(\mathbb{R}^{d}) \) by a density argument, approximating \( f \) by the sequence of function \((f_{n})\) defined by \( f_{n} = f + \frac{1}{n} \) for any \( n \in \mathbb{N}. \)

For a general function \( f \in C^{3+\alpha}_{c}(\mathbb{R}^{d}), \) we split \( f = f^{+} - f^{-} \), where \( f^{\pm} = \max(\pm f, 0). \) Clearly, \( f^{+} \) and \( f^{-} \) are bounded and Lipschitz continuous in \( \mathbb{R}^{d}. \) Moreover, \( \nabla f^{+} = \chi_{\{f > 0\}} \nabla f \) and \( \nabla f^{-} = \chi_{\{f < 0\}} \nabla f. \) In particular, \( |\nabla f^{\pm}| \leq |\nabla f|. \) Let us consider the sequences \((g_{n}^{+})\) and \((g_{n}^{-})\), where \( g_{n}^{\pm} := f^{\pm} \ast \rho_{1/n} \) for any \( n \in \mathbb{N}, \) and \((\rho_{1/n})\) is a standard sequence of mollifiers. Notice that \( g_{n}^{\pm} \in C_{0}^{\alpha}(\mathbb{R}^{d}) \) are nonnegative, for any \( n \in \mathbb{N}, \) and converge uniformly in \( \mathbb{R}^{d} \) to \( f^{\pm} \) as \( n \to +\infty. \) Moreover, up to a subsequence, we can assume that \( \nabla g_{n}^{\pm} \) converge pointwise a.e. in \( \mathbb{R}^{d} \) to \( \nabla f^{\pm} \) as \( n \to +\infty. \) Hence, we can write
\[
|\nabla_{x} G^{e}(t, s) g_{n}^{\pm}|^{p} \leq 2C_{p}^{e} C_{p}(t-s) G^{e}(t, s) \left[ (g_{n}^{\pm})^{p} + |\nabla g_{n}^{\pm}|^{p} \right],
\]
(3.14)
for any \( t > s \) and any \( n \in \mathbb{N} \). Arguing as above we can show that \(|\nabla_x G^e(t, s)g_{n}^{\pm}|\)
converges to \(|\nabla_x G^e(t, s) f^{\pm}|\), locally uniformly in \( \mathbb{R}^d \). Similarly, using (2.12) and
dominated convergence we conclude that also the right-hand side of (3.14) tends to
\[
2^{p-1}e^{C_p(t-s)} \int_{\mathbb{R}^d} \left( (f^{\pm}(y))^{p} + |\nabla f^{\pm}(y)|^{p} \right) g^{e}(t, s, x, y) dy,
\]
pointwise in \( \mathbb{R}^d \). Therefore,
\[
|\nabla_x G^e(t, s) f^{\pm}| \leq 2^{p}e^{C_p(t-s)} \int_{\mathbb{R}^d} \left( (f^{\pm}(y))^{p} + |\nabla f^{\pm}(y)|^{p} \right) g^{e}(t, s, x, y) dy
\leq 2^{p}e^{C_p(t-s)} G^e(t, s)(|f|^{p} + |\nabla f|^{p}),
\]
for any \( t > s \). This estimate yields (3.4).

**Step 3.**

Fix \( t > s \in I \). First we assume \( \mathcal{I} = \mathcal{D} \) and fix \( f \in C^{3+\alpha}_{c}(\mathbb{R}^d_{+}) \). Applying (3.13) with \( f \) replaced by \( \mathcal{O}f \in C^{3+\alpha}_{c}(\mathbb{R}^d_{+}) \) and taking (2.16) into account, we deduce that
\[
|\nabla_x G^e_{\mathcal{D}}(t, s) f| \leq 2^{p}e^{C_p(t-s)} G^e_{N}(t, s)(|f|^{p} + |\nabla f|^{p}).
\]
If \( p \geq 2 \), by Step 2 in the proof of Theorem 2, we can let \( \varepsilon \to 0^{+} \) in (3.15) and obtain (3.3) for functions in \( C^{3+\alpha}_{c}(\mathbb{R}^d_{+}) \). On the other hand, if \( p \in (1, 2) \), the function \(|f|^{p} + |\nabla f|^{p}\) is not in \( C^{2+\alpha}_{c}(\mathbb{R}^d) \). Anyway, \( G^e_{N}(t, s)(|f|^{p} + |\nabla f|^{p}) \) still converges to \( G_N(t, s)(|f|^{p} + |\nabla f|^{p}) \) as \( \varepsilon \to 0^{+} \). Indeed, we can approximate the function \(|f|^{p} + |\nabla f|^{p}\) uniformly in \( \mathbb{R}^d_{+} \) by a sequence of functions \( g_n \in C^{3+\alpha}_{c}(\mathbb{R}^d_{+}) \). Since
\[
\|G^e_{N}(t, s)(|f|^{p} + |\nabla f|^{p}) - G_{N}(t, s)(|f|^{p} + |\nabla f|^{p})\|_{L^{\infty}(K)}
\leq \|G^e_{N}(t, s)(|f|^{p} + |\nabla f|^{p}) - G_{N}(t, s)g_n\|_{L^{\infty}(K)}
+ \|G_{N}(t, s)(|f|^{p} + |\nabla f|^{p}) - G_{N}(t, s)g_n\|_{L^{\infty}(K)},
\]
for any \( n \in \mathbb{N} \) and any compact set \( K \subset \mathbb{R}^d_{+} \), from (2.7) and (2.15)(i), we can estimate
\[
\|G^e_{N}(t, s)(|f|^{p} + |\nabla f|^{p}) - G_{N}(t, s)(|f|^{p} + |\nabla f|^{p})\|_{L^{\infty}(K)}
\leq 2\|f|^{p} + |\nabla f|^{p} - g_n\|_{\infty} + \|G^e_{N}(t, s)g_n - G_{N}(t, s)g_n\|_{L^{\infty}(K)}.
\]
Taking first the limsup as \( \varepsilon \to 0^{+} \) and then the limit as \( n \to +\infty \) in the previous inequality, the claim follows. Estimate (3.3) follows also in this case for functions in \( C^{3+\alpha}_{c}(\mathbb{R}^d_{+}) \).

For a general function \( f \in C^1_{\mathcal{D}}(\mathbb{R}^d) \) we fix a sequence \( (f_n) \subset C^{3+\alpha}_{c}(\mathbb{R}^d_{+}) \) bounded in the \( C^1_{\mathcal{D}} \)-norm and converging to \( f \) in \( C^1_{\mathcal{D}}(\mathbb{R}^d_{+}) \). Writing (3.3) with \( f \) replaced by \( f_n \), using Proposition 3(i) and letting \( n \to +\infty \), we conclude the proof of the theorem when \( \mathcal{I} = \mathcal{D} \).

In the case \( \mathcal{I} = \mathcal{N} \), replacing the function \( \mathcal{O}f \) with \( \mathcal{E}f \) and using the same arguments as above (taking (2.18) into account), we can prove estimate (3.3) for any function \( f \in C^1_{b}(\mathbb{R}^d_{+}) \) with normal derivative vanishing on \( \partial \mathbb{R}^d_{+} \). To extend (3.3) to any function \( f \in C^1_{b}(\mathbb{R}^d_{+}) \), we consider the sequence of functions \( f_n = \mathcal{E}f \star \rho_{1/n} \) \((n \in \mathbb{N})\) which belong to \( C^\infty_{b}(\mathbb{R}^d) \) and have normal derivative on \( \partial \mathbb{R}^d_{+} \) which identically vanishes. We claim that \( (f_n) \) is a bounded
sequence in the $C^1_b$-norm which converges to $f$ in $C^1_{\text{loc}}(\mathbb{R}^d_+)$ as $n \to +\infty$. Indeed, since $\mathcal{E} f \in \text{Lip}(\mathbb{R}^d)$, $f_n$ converges to $f$ uniformly in $\mathbb{R}^d_+$. Moreover, $\nabla f_n = (\nabla \mathcal{E} f) \ast \rho_{1/n}$. Hence, $\nabla f_n$ converges locally uniformly in $\mathbb{R}^d_+$ to $\nabla f$ and $\|\nabla f_n\|_{C_b(\mathbb{R}^d_+)} \leq \|\nabla f\|_{C_b(\mathbb{R}^d_+)}$ for any $n \in \mathbb{N}$. Hence, as above, Proposition 3(i) allows us to complete the proof.

\[ \square \]

**Remark 3** We stress that estimate (3.3), with $\mathcal{I} = \mathcal{D}$, can not hold with $C^1_b(\mathbb{R}^d_+)$ replaced by $C^1_b(\mathbb{R}^d_+)$. Indeed, let $f \in C^1_b(\mathbb{R}^d_+)$ satisfy (3.3). By the mean value theorem we deduce that

\begin{equation}
|(G_{\mathcal{D}}(t, s)f)(x) - (G_{\mathcal{D}}(t, s)f)(y)| \leq K\|f\|_{C^1_b(\mathbb{R}^d_+)}|x - y|,
\end{equation}

for any $x, y \in \mathbb{R}^d_+$ and any $s, t \in I$, such that $s < t < s + 1$, with $K = 2^{e_p + p - 1}e^C_p$.

Now, taking $x = (x', x_d) \in \mathbb{R}^d_+$ and $y = (x', 0)$ in (3.16), we get $|(G_{\mathcal{D}}(t, s)f)(x)| \leq K\|f\|_{C^1_b(\mathbb{R}^d_+)}x_d$. Letting $t \to s^+$ in this inequality yields $|f(x)| \leq K\|f\|_{C^1_b(\mathbb{R}^d_+)}x_d$, which shows that $f(x', 0) = 0$. Hence, $f \in C^1_D(\mathbb{R}^d_+)$.

We now show that, when $c \equiv 0$, estimate (3.3) can be improved removing the dependence on $|f|^p$ from the right-hand side.

**Theorem 5** Let $c \equiv 0$ and $p \in (1, +\infty)$. Assume that Hypothesis 3(iii) is replaced by the following condition:

\begin{equation}
r(t, x) + \frac{k^2 d^2}{4M_p} \eta(t, x) \leq K_p, \quad (t, x) \in I \times \mathbb{R}^d_+,
\end{equation}

for some positive constant $K_p$. Then, it holds that

\begin{equation}
|\nabla_x G_{\mathcal{I}}(t, s)f)(x)|^p \leq e^{K_p(t - s)}(G_N(t, s)|\nabla f|^p)(x), \quad t > s, \ x \in \mathbb{R}^d_+,
\end{equation}

for any $f \in C^1_b(\mathbb{R}^d_+)$, when $\mathcal{I} = \mathcal{N}$, and for any $f \in C^1_D(\mathbb{R}^d_+)$, when $\mathcal{I} = \mathcal{D}$.

\[ \text{Proof} \]

The claim can be proved arguing as in the proof of Theorem 4 replacing the function $w$ therein defined with the function $w = \left( |\nabla_x G_{\mathcal{N}}^x(t, s)f|^2 + \tau \right)^{p/2}$, where $\tau$ is a positive constant. Notice that $w$ has positive infimum in $(s, +\infty) \times \mathbb{R}^d$. One can show that $w_t - \mathcal{A}^f(t)w \leq pK_pw$ and deduce (3.4) with $C_p$ and $(|f|^p + |\nabla f|^p)$ replaced, respectively, by $K_p$ and $(|\nabla f|^2 + \tau)^{p/2}$ Finally, letting $\varepsilon$ and $\tau$ tend to $0^+$ yields the assertion.

\[ \square \]

In the following theorem, under stronger assumptions, we extend estimates (3.3) and (17) to the case $p = 1$.

**Theorem 6** Assume that the diffusion coefficients are independent of $x$, Hypotheses 1 are satisfied and $|\nabla_x c| \leq \beta$ in $I \times \mathbb{R}^d_+$ for some continuous function $\beta$ such that $\kappa := \sup_{t \times \mathbb{R}^d_+} (|r|^{-1/2}\beta) < +\infty$. Then, estimate (3.3) holds true with $p = 1$ and $C_1 = \kappa^2$. In particular, if $c \equiv 0$, then estimate (3.17) holds for $p = 1$ with $K_1 = -L_0\eta_0$. 

\[ \copyright \ Springer \]
Proof The proof is similar to that of Theorem 4, hence we just sketch it, pointing out the main differences. The main step is the proof of the estimate

\[ |\nabla_x G^\varepsilon(t, s)f| \leq 2\varepsilon^{2(t-s)}G^\varepsilon(t, s)(|f| + |\nabla f|), \quad t > s \in I, \]

in \(\mathbb{R}^d\) for any \(f \in C_b^1(\mathbb{R}^d)\) with positive infimum. To prove it, let \(w_1 = (\varepsilon_n^e)^2 + |\nabla_x u_n^e|^2\)/2, where \(u_n^e = G_{\varepsilon_n}^\varepsilon(\cdot, s)f\). We observe that, in this case \(D_tw_1 - \mathcal{A}^\varepsilon(t)w_1 = \psi_{1,1} + \psi_{2,1} + \psi_{3,1}\), where \(\psi_{i,1}(i = 1, 2, 3)\) are given by \((3.6)-(3.8)\). Using \((3.1)\) to estimate \(\psi_{1,1}\) and \((3.12)\) we obtain

\[ D_tw_1 - \mathcal{A}^\varepsilon(t)w_1 \leq w_1^{-1}(\varepsilon r^\varepsilon|\nabla_x u_n^e|^2 + \varepsilon^2|u_n^e||\nabla_x u_n^e|), \quad (3.18) \]

where \(\varepsilon = (\mathcal{E}\varepsilon)^\varepsilon\). Since the Hölder inequality and the condition \(\varepsilon \leq \kappa \sqrt{|r|}\) imply that \(\varepsilon \leq \kappa \sqrt{|r|}\), we conclude that

\[ D_tw_1 - \mathcal{A}^\varepsilon(t)w_1 \leq w_1^{-1}\left\{[\varepsilon^2(\beta^\varepsilon)^2 + r^\varepsilon]|\nabla_x u_n^e|^2 + \kappa^2|u_n^e|^2\right\} \leq \kappa^2w_1. \]

Now, the proof of the first assertion can be completed arguing as in the proof of Theorem 4, so that the details are omitted.

Finally, if \(c \equiv 0\), \((3.18)\) reduces to \(D_tw_1 - \mathcal{A}^\varepsilon(t)w_1 \leq w_1^{-1}r^\varepsilon|\nabla_x u_n^e|^2 \leq -L_0\eta_0w_1\), by Hypothesis 1(v), and the claim can be easily proved.

Remark 4 The estimate \((3.17)\) holds with \(p = 1\) also when \(c \equiv 0\) and the diffusion coefficients depend on \(x\), provided they satisfy the following conditions:

\[ D_iq_{jk}(t, x) + D_jq_{ik}(t, x) + D_kq_{ij}(t, x) = 0, \]

\[ \langle \nabla_x b(t, x)\xi, \xi \rangle + \frac{1}{2\eta_0} \sum_{i,j=1}^d \langle \nabla_x q_{ij}(t, x), \xi \rangle^2 \leq d_0|\xi|^2, \]

for some \(d_0 \in \mathbb{R}\), any \((t, x) \in I \times \mathbb{R}^d_+,\) any \(\xi \in \mathbb{R}^d\) and any \(i, j, k = 1, \ldots, d\). Indeed, \(D_ig_{jk} + D_jg_{ik} + D_kg_{ij} = 0\) in \(I \times (\mathbb{R}^d \setminus \{0\})\), for any \(i, j, k\) as above (see \((2.6)\)). By convolution, it is immediate to check that \(D_iq_{jk} + D_jq_{ik} + D_kq_{ij} = 0\) in \(I \times \mathbb{R}^d\), for any \(i, j, k = 1, \ldots, d\) and any \(\varepsilon \in (0, 1]\). Similarly, arguing as in the proof of Proposition 1 we deduce that

\[ \langle \nabla_x b(t, x)\xi, \xi \rangle + \frac{1}{2\eta_0} \sum_{i,j=1}^d \langle \nabla_x q_{ij}(t, x), \xi \rangle^2 \leq d_0|\xi|^2, \]

for any \((t, x) \in I \times (\mathbb{R}^d \setminus \{0\})\), and any \(\xi \in \mathbb{R}^d\). Hence, by convolution and Hölder inequality, we obtain

\[ \langle \nabla_x b^\varepsilon(t, x)\xi, \xi \rangle + \frac{1}{2\eta_0} \sum_{i,j=1}^d \langle \nabla_x q^\varepsilon_{ij}(t, x), \xi \rangle^2 \leq d_0|\xi|^2, \]

for any \((t, x) \in I \times \mathbb{R}^d\), any \(\xi \in \mathbb{R}^d\) and any \(\varepsilon \in (0, 1]\). We can thus apply \([2, \text{Thm. 3.1}]\) to the operator \(\mathcal{A}^\varepsilon(t)\), which shows that

\[ |\nabla_x G^\varepsilon(t, s)f| \leq e^{d_0(t-s)}G^\varepsilon(t, s)|\nabla f|, \quad t > s \in I, \quad f \in C_b^1(\mathbb{R}^d). \quad (3.19) \]

Writing \((3.19)\) with \(f\) replaced with \(\mathcal{O}f\) (resp. \(\mathcal{E}f\)) and letting \(\varepsilon \to 0\) leads to \((3.17)\) with \(p = 1\), \(K_1 = d_0\) and \(T = D\) (resp. \(T = N\)).

As a consequence of Theorems 4, 6 and estimate \((2.15)\) we deduce the following uniform gradient estimates.
Corollary 1 For $\mathcal{I} \in \{\mathcal{D}, \mathcal{N}\}$ the uniform gradient estimate

$$\|\nabla_x G_{\mathcal{I}}(t, s)f\|_\infty \leq 2e^{C_2s/(t-s)}\|f\|_{C_b^1(\mathbb{R}^d_+)}^n, \quad t > s \in I,$$  \tag{3.20}

holds for any $f \in C_{\mathcal{D}}(\mathbb{R}^d_+)$, when $\mathcal{I} = \mathcal{D}$, and for any $f \in C_{\mathcal{N}}^1(\mathbb{R}^d_+)$, when $\mathcal{I} = \mathcal{N}$.

Assume, in addition, that the diffusion coefficients are independent of $x$. If the condition $\beta \leq \kappa \sqrt{|r|}$ is satisfied in $I \times \mathbb{R}^d_+$ for some positive constant $\kappa$, then (3.20) holds with $2e^{(C_2-c_0)(t-s)/2}$ replaced by $e^{(s^2-c_0)(t-s)}$. On the other hand, if $c \equiv 0$, then (3.20) is satisfied with $2e^{(C_2-c_0)(t-s)/2}$ and $\|f\|_{C_b^1(\mathbb{R}^d_+)}$ replaced with $e^{-\left(L\partial_0+c_0\right)(t-s)}$ and $\nabla f$, respectively.

3.2 $C^0$-$C^1$ Uniform and Pointwise Estimates

We now prove a second type of pointwise gradient estimates which, besides the interest in their own, will be used in Section 4 to study the asymptotic behaviour of $G_{\mathcal{D}}(t, s)f$ and $G_{\mathcal{N}}(t, s)f$ as $t \to +\infty$.

Theorem 7 For $\mathcal{I} \in \{\mathcal{D}, \mathcal{N}\}$, every $p \in (1, +\infty)$ and $s \in I$ the gradient estimate

$$\|\nabla_x G_{\mathcal{I}}(t, s)f\|^p \leq c_p e^{\alpha_p(t-s)}(t-s)^{-\frac{p}{2}} G_{\mathcal{I}}(t, s)|f|^p, \quad t > s \in I,$$  \tag{3.21}

holds in $\mathbb{R}^d_+$, for any $f \in C_b(\mathbb{R}^d_+)$ when $\mathcal{I} = \mathcal{D}$, and for any $f \in C_b^1(\mathbb{R}^d_+)$ when $\mathcal{I} = \mathcal{N}$. Here, $c_p$ is a positive constant and $\alpha_p$ is any constant larger than $\min\{C_p, 0\}$, where $C_p$ is given by (3.2). As a consequence, the following uniform gradient estimate

$$\|\nabla_x G_{\mathcal{I}}(t, s)f\|_\infty \leq \sqrt{c_2} e^{\alpha_2s/(t-s)}(t-s)^{-\frac{1}{2}} \|f\|_\infty$$

is satisfied for any $t > s$ and any $f$ as above.

Proof It suffices to prove (3.21) with $G_{\mathcal{I}}(t, s)$ replaced by $G^\varepsilon(t, s)$, $f \in C_{\mathcal{E}}^{3+\alpha}(\mathbb{R}^d)$ since, then, the same arguments as in the proof of Theorem 4 will allow us to conclude. Hence, let us prove that

$$\|\nabla_x G^\varepsilon(t, s)f\|^p \leq c_p e^{\alpha_p(t-s)}(t-s)^{-\frac{p}{2}} G^\varepsilon(t, s)|f|^p,$$  \tag{3.22}

for such $f$'s and any $\varepsilon \in (0, 1]$. Note that the case $p > 2$ follows from the case $p = 2$, writing $\|\nabla_x G^\varepsilon(t, s)f\|^p \leq \left(\|x G^\varepsilon(t, s)f\|^2\right)^{p/2}$ and using (2.14) to estimate the right-hand side of the previous inequality.

For $p \in (1, 2]$ and $f$ as above, the claim can be proved adapting the arguments in the proof of [22, Prop. 3.3], which are based on the gradient estimate (3.4). For the reader’s convenience we provide the ideas of the proof.

We introduce the function $g = G_{\mathcal{N},n}^\varepsilon(t, \cdot)|G_{\mathcal{N},n}^\varepsilon(\cdot, s)f\|$, where $G_{\mathcal{N},n}^\varepsilon(t, s)$ is the evolution operator introduced in (3.5). This function is differentiable in $(s, t)$ (see [1, Thm. 2.3(ix)]) and

$$g'(\sigma) = G_{\mathcal{N},n}^\varepsilon(t, \sigma)\left[-p(1-p)|G_{\mathcal{N},n}^\varepsilon(\sigma, s)f|^{p-2}Q(\sigma, \cdot)\nabla_x G_{\mathcal{N},n}^\varepsilon(\sigma, s)f|^2 + (1-p)c|G_{\mathcal{N},n}^\varepsilon(\sigma, s)f|^p\right]$$

$$\leq -p(1-p)\eta_0 G_{\mathcal{N},n}^\varepsilon(t, \sigma)\left[|G_{\mathcal{N},n}^\varepsilon(\sigma, s)f|^{p-2}|\nabla_x G_{\mathcal{N},n}^\varepsilon(\sigma, s)f|^2\right].$$  \tag{3.23}
Integrating the first and the last sides of (3.23) with respect to $\sigma$ in $[s + \delta, t - \delta]$ and then letting $n$ and $\delta$ tend to $+\infty$ and $0$, respectively, we get

$$G^\varepsilon(t, s)|f|^p \geq p(p - 1)\eta\int_s^t G^\varepsilon(t, \sigma)\left[|G^\varepsilon(\sigma, s)f|^{p-2} |\nabla_x G^\varepsilon(\sigma, s)f|^2\right]d\sigma. \quad (3.24)$$

Thanks to (3.4), we can estimate

$$|\nabla_x G^\varepsilon(t, s)f|^p = |\nabla_x G^\varepsilon(t, \sigma)G^\varepsilon(\sigma, s)f|^p \leq 2^{\ell_p + p-1}e^{C_p(t-\sigma)}|G^\varepsilon(t, \sigma)|G^\varepsilon(\sigma, s)f|^p + G^\varepsilon(t, \sigma)|\nabla_x G^\varepsilon(\sigma, s)f|^p \leq 2^{\ell_p + p-1}e^{C_p(t-\sigma)}[I_1(\sigma) + I_2(\sigma)]. \quad (3.25)$$

Applying estimate (2.14) with $r = 2/p$, $\psi_1 = |G^\varepsilon(\sigma, s)f|^{(p-2)/2}|\nabla_x G^\varepsilon(\sigma, s)f|^p$ and $\psi_2 = |G^\varepsilon(\sigma, s)f|^{(2-p)/2}$, and taking into account that $I_1(\sigma) \leq G^\varepsilon(t, s)|f|^p$ for any $\sigma \in (s, t)$, we get

$$I_2(\sigma) \leq \frac{p}{2} \gamma \frac{2}{p} G^\varepsilon(t, \sigma) \left(|G^\varepsilon(\sigma, s)f|^{p-2} |\nabla_x G^\varepsilon(\sigma, s)f|^2\right) + \left(1 - \frac{p}{2}\right) \gamma \frac{2}{p-2} G^\varepsilon(t, s)|f|^p,$$

for any $\gamma > 0$ and any $\sigma \in (s, t)$. Thus, estimate (3.25) becomes

$$|\nabla_x G^\varepsilon(t, s)f|^p \leq 2^{\ell_p + p-1}e^{C_p(t-\sigma)} \left[\frac{p}{2} \gamma \frac{2}{p} G^\varepsilon(t, \sigma) \left(|G^\varepsilon(\sigma, s)f|^{p-2} |\nabla_x G^\varepsilon(\sigma, s)f|^2\right) + \left(1 - \frac{p}{2}\right) \gamma \frac{2}{p-2} + 1\right] G^\varepsilon(t, s)|f|^p. \quad (3.26)$$

Now we multiply both sides of (3.26) by $e^{-C_p(t-\sigma)}$ and we integrate the so obtained inequality with respect to $\sigma \in (s, t)$ taking (3.24) into account. Finally, minimizing with respect to $\gamma > 0$ we deduce that

$$|\nabla_x G^\varepsilon(t, s)f|^p \leq \frac{\tau_p C_p}{1 - e^{-C_p(t-s)}}(t - s)^{1-\frac{p}{2}}[1 + (t - s)^{\frac{p}{2}}]G^\varepsilon(t, s)|f|^p,$$

for some positive constant $\tau_p$. Thus, estimate (3.22) follows. \hfill $\Box$

### 4 Evolution Systems of Measures and Asymptotic Behaviour

In this section we prove the existence of an evolution system of measures $\{\mu_t^N\}_{t \in I}$ associated to the Neumann evolution operator $G_N(t, s)$, which turns out to be subinvariant for the Dirichlet evolution operator $G_D(t, s)$. We study the asymptotic behaviour of both the evolution operators $G_D(t, s)$ and $G_N(t, s)$ in $L^p$-spaces with respect to the evolution system of measures $\{\mu_t^N\}_{t \in I}$. We also deduce logarithmic Sobolev inequalities with respect to the measures $\{\mu_t^N\}_{t \in I}$ and the hypercontractivity property for $G_D(t, s)$ and $G_N(t, s)$. Throughout this section, we assume that $I$ is bounded from below. Anyway, as it is easily seen, all our results hold true also when $I = \mathbb{R}$. 

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To begin with, let us recall the following definition.

**Definition 2** Let $\mathcal{O} \subset \mathbb{R}^d$ be an open set or the closure of an open set. Further, let $\{U(t, s) : t \geq s \in I\}$ be an evolution operator on $C_b(\mathcal{O})$. A family $\{\mu_t\}_{t \in I}$ of probability measures on $\mathcal{O}$ is called

(i) an evolution system of measures for $U(t, s)$, if

$$\int_{\mathcal{O}} U(t, s) f d\mu_t = \int_{\mathcal{O}} f d\mu_s,$$

for every $f \in C_b(\mathcal{O})$ and every $t > s \in I$;

(ii) an evolution system of subinvariant measures for $U(t, s)$ if,

$$\int_{\mathcal{O}} U(t, s) f d\mu_t \leq \int_{\mathcal{O}} f d\mu_s,$$

for every nonnegative $f \in C_b(\mathcal{O})$ and every $t > s \in I$.

**Remark 5** By virtue of Theorem 1, $0 \leq G_\tau(t, s) \mathbb{I} \leq \mathbb{I}$ for any $I \ni s < t$, where $\mathcal{I}$ is either $\mathcal{D}$ or $\mathcal{N}$. If $\{\mu_t\}_{t \in I}$ is an evolution system of measures for $G_\tau(t, s)$, then $\{\mu_t, G_\tau(t, s) \mathbb{I} \} = 1$ for any $I \ni s < t$. Hence, $G_\tau(t, s) \mathbb{I} = \mathbb{I}$ $\mu_t$-a.e. in $\mathbb{R}^d_+$. We claim that $G_\tau(t, s) \mathbb{I} = \mathbb{I}$ everywhere in $\mathbb{R}^d_+$. Indeed, by (2.15), $G_\tau(t, s) \mathbb{I} \leq \mathbb{I}$ in $\mathbb{R}^d_+$ for any $t > s$. Let $x_0 \in \mathbb{R}^d_+$ be such that $(G_\tau(t, s) \mathbb{I})(x_0) = 1$. Then, $(t, x_0)$ is a maximum point of the function $G_\tau(\cdot, s) \mathbb{I}$. The classical maximum principle (see e.g., [25, Thm. 3.3.5]) shows that $(G_\tau(t, s) \mathbb{I})(x) = 1$ for any $x \in \mathbb{R}^d_+$.

Clearly, the equality $G_\tau(t, s) \mathbb{I} = \mathbb{I}$ can not hold if $\mathcal{I} = \mathcal{D}$ since $G_D(t, s) \mathbb{I}$ vanishes on $\partial \mathbb{R}^d_+$. This shows that there exist no evolution systems of measures for the Dirichlet evolution operator $G_D(t, s)$. At the same time the equality $G_N(t, s) \mathbb{I} = \mathbb{I}$ implies that $c \equiv 0 \text{ since } (D_t - A(t)) \mathbb{I} = c(t, \cdot) \mathbb{I}$.

**Lemma 1** Any evolution system of measures $\{\mu_t\}_{t \in I}$ for the evolution operator $G_N(t, s)$ is an evolution system of subinvariant measures for the operator $G_D(t, s)$. Moreover, for any $I \ni s < t$, the operators $G_D(t, s)$ and $G_N(t, s)$ extend to contractions from $L^p(\mathbb{R}^d_+, \mu_t)$ into $L^p(\mathbb{R}^d_+, \mu_t)$ for any $p \in [1, +\infty)$.

**Proof** The first assertion follows immediately from Proposition 3(ii). As far as the other claim is concerned, we recall that, for any $f \in C_c^\infty(\mathbb{R}^d_+)$ and any $I \ni s < t$, $G_D(t, s)f$ is the pointwise limit, as $\varepsilon \to 0^+$, of the family of functions $G_\varepsilon(t, s) f$ (see Step 2 of the proof of Theorem 2). From (2.14) it follows that $|G_D(t, s)f|^p \leq G_N(t, s)|f|^p$. Therefore, using the subinvariance of the measures $\{\mu_t\}_{t \in I}$, we get

$$\langle \mu_t, |G_D(t, s)f|^p \rangle \leq \langle \mu_t, G_N(t, s)|f|^p \rangle = \langle \mu_s, |f|^p \rangle.$$

Since $C_c^\infty(\mathbb{R}^d_+)$ is dense in $L^p(\mathbb{R}^d_+, \mu_t)$ for any $t \in I$, $G_D(t, s)$ can be extended to a contraction from $L^p(\mathbb{R}^d_+, \mu_t)$ to $L^p(\mathbb{R}^d_+, \mu_t)$.

In the same way one can prove that also $G_N(t, s)$ extends to a contraction from $L^p(\mathbb{R}^d_+, \mu_s)$ to $L^p(\mathbb{R}^d_+, \mu_t)$. $\square$

In view of Remark 5 and Lemma 1, in the rest of this section we assume the following set of assumptions.

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Hypotheses 8

(i) Hypotheses 1(i), (iii) and (vi) are satisfied;
(ii) the coefficients \( q_{ii}(\cdot, 0) \) and \( b_i(\cdot, 0) \) \((i = 1, \ldots, d)\) are bounded in \([s, +\infty)\) for any \(s \in I\);
(iii) \( c \equiv 0; \)
(iv) for any \(p \in (1, +\infty)\) there exists a positive constant \(C_p\) such that

\[
r(t, x) + \left(\frac{k^2 d^2}{4M_p} - M_p\right) \eta(t, x) \leq C_p, \quad (t, x) \in I \times \mathbb{R}^d,
\]

where \(M_p = \min\{1, p - 1\} \).

We find it convenient to prove the existence of an evolution system of measures for the evolution operator \(G^t_N(t, s)\) as the weak limit of some evolution systems of measures associated to the evolution operator \(G^t(t, s)\) in the whole of \(\mathbb{R}^d\).

**Lemma 2** For any \(\epsilon \in (0, 1]\) there exists a unique evolution system of measures \(\{\mu^\epsilon_t\}_{t \in I}\) associated to \(G^t(t, s)\) such that the system \(\{\mu^\epsilon_t\}_{t \geq 0}\) is tight for any \(t_0 \in I\). Such a system is uniformly tight with respect to \(\epsilon \in (0, 1]\). Moreover, for any \(t \in I\), each measure \(\mu^\epsilon_t\) is absolutely continuous with respect the Lebesgue measure. More precisely, there exists a locally Hölder continuous function \(\rho_\epsilon : I \times \mathbb{R}^d \to \mathbb{R}\), which is even with respect to the variable \(x_d\), and \(\mu^\epsilon_t(dx) = \rho_\epsilon(t, x)dx\) for any \(t \in I\) and any \(\epsilon \in (0, 1]\).

**Proof** The existence of an evolution system of measures for the evolution operator \(G^t(t, s)\) and the absolute continuity of each measure \(\mu^\epsilon_t\) with respect the Lebesgue measure follow from [20, Prop. 5.2 & Thm. 5.4], which require the existence of a function \(\varphi \in C^2(\mathbb{R}^d)\) diverging to \(+\infty\) as \(|x| \to +\infty\) such that, for any \(s \in I\), \(A^\epsilon(t)\varphi(x) = a - c \varphi(x)\) for any \(t > s\), any \(x \in \mathbb{R}^d\) and some positive constants \(a\) and \(c\). In our situation, the function \(\varphi\), defined by \(\varphi(x) = 1 + |x|^2\) for any \(x \in \mathbb{R}^d\), has the previous property for any \(t \in I\), with the constant \(c\) being independent of \(\epsilon\). Moreover, \(\varphi\) is integrable with respect to the measure \(\mu^\epsilon_t\) for any \(t \in I\) and any \(\epsilon \in (0, 1]\) and, for any \(s \in I\),

\[
H_{s,1} := \sup\{\mu^\epsilon_t(\varphi) : t \geq s, \epsilon \in (0, 1]\} < +\infty. \tag{4.1}
\]

This fact and Chebyshev inequality imply that the family of measures \(\{\mu^\epsilon_t\}_{t \geq s}\) is tight for any \(s \in I\), uniformly with respect to \(\epsilon \in (0, 1]\). More precisely,

\[
\mu^\epsilon_t\left(\mathbb{R}^d \setminus B_r\right) \leq \frac{H_{s,1}}{r^2}, \quad t > s, \epsilon \in (0, 1]. \tag{4.2}
\]

In view of [12, Thm. 3.8] and [5, Lemma 2.4], \(\mu^\epsilon_t = \rho_\epsilon(t, \cdot)dx\) for any \(t \in I\), where the function \(\rho_\epsilon\) is locally \(\gamma\)-Hölder continuous in \(I \times \mathbb{R}^d\) for any \(\gamma \in (0, 1)\). To prove that \(\rho_\epsilon(t, \cdot)\) is even with respect to the variable \(x_d\) for any \(t \in I\), we need to recall briefly the construction of the tight evolution system of measures in [20, Thm. 5.4]. Let \(n_0 \in \mathbb{Z}\) be the smallest integer in \(I\) and fix \(x \in \mathbb{R}^d\). The family of measures \(\{\mu^\epsilon_{t, s, x}\}_{t > s \geq n_0}\), where

\[
\mu^\epsilon_{t, s, x}(A) = \frac{1}{t - s} \int_s^t (G^\epsilon(\tau, s)\mathbb{1}_A)(x)d\tau, \quad t > s \in I, \quad A \in \mathcal{B}(\mathbb{R}^d),
\]

is tight. Prokhorov theorem and a diagonal argument guarantee the existence of a sequence \(\{t^k\}\) diverging to \(+\infty\) such that \(\mu^\epsilon_{t, s, x}\) weakly converges to a measure \(\mu^\epsilon_{t, x}\) as \(k \to +\infty\) for any \(n \in \mathbb{Z}\) such that \(n \geq n_0\). For \(t \in I \setminus \{n \in \mathbb{Z} : n \geq n_0\}\) one defines the measure \(\mu^\epsilon_{t, x}\) by
setting $\mu_{t,x}^\varepsilon = (G^\varepsilon(n,t))^{*}\mu_{n,x}^\varepsilon$, where $(G^\varepsilon(n,t))^{*}$ is the operator adjoint to $G^\varepsilon(n,t)$, and $n$ is any integer greater than $t$. The family $\{\mu_{t,x}^\varepsilon\}_{t \in I}$ is a tight evolution system of measures for the evolution operator $G^\varepsilon(t,s)$. Clearly, the construction of the previous system of measures \textit{apriori} depends on the choice of $x \in \mathbb{R}^d$ as well as on the choice of the sequence $(t_k^\varepsilon)$, but the arguments in [5, Rem. 2.8] show that the tight evolution system of measures is unique whenever a gradient estimate of the type $|\nabla_x G^\varepsilon(t,s)f|_\infty \leq C_\varepsilon \|f\|_\infty$ is satisfied for any $f \in C^1_b(\mathbb{R}^d)$, any $t > s \in I$, and some $C, \sigma > 0$, which is actually our case in view of the proof of Theorem 5.

So, let us fix $x_0 = (0,\ldots,0,1)$ and $x_1 = -x_0$. Further, let $(t_k^\varepsilon)$ be a sequence diverging to $+\infty$ as $k \to +\infty$ such that both $\mu_{t_k,n,x_0}^\varepsilon$ and $\mu_{t_k,n,x_1}^\varepsilon$ weakly* converge to $\mu_n^\varepsilon$ for any $n \geq n_0$.

Fix a function $f \in C_b(\mathbb{R}^d)$, odd with respect to the $x_d$-variable. Then, as Step 1 in the proof of Theorem 2 shows, the function $G^\varepsilon(t,s)f$ is odd with respect to the $x_d$-variable as well. Therefore,

$$
\int_{\mathbb{R}^d} f d\mu_{t_k,n,x_1}^\varepsilon = \frac{1}{t_k - n} \int_n^{t_k} (G^\varepsilon(\tau,n)f(x_1))d\tau = -\frac{1}{t_k - n} \int_n^{t_k} (G^\varepsilon(\tau,n)f(0))d\tau.
$$

Letting $k \to +\infty$ gives $\langle \mu_n^\varepsilon, f \rangle = 0$ for any $n \geq n_0$. Using the definition of the evolution system of measures, we can extend the previous formula to any $t \in I$. Now a standard argument allows us to conclude that $\rho_\varepsilon(t,\cdot)$ is even with respect to the last variable: we fix $\psi \in C_b(\mathbb{R}^d)$ and write $\langle \mu_t, \psi \rangle = 0$ with $f = \psi$, where $\psi(x) := \psi(x_1,\ldots,x_{d-1},-x_d)$ for any $x \in \mathbb{R}^d$. Using a straightforward change of variables and the arbitrariness of $\psi$, the assertion follows at once.

\textbf{Lemma 3} There exist an infinitesimal sequence $(\varepsilon_n)$ and an evolution system of measures $\{\mu_{t,s}^{\varepsilon_n}\}_{t \in I}$ for the operator $G_{\varepsilon_n}(t,s)$ such that, for any bounded sequence $(f_n) \in C_b(\mathbb{R}^d_+)$ converging locally uniformly to $f$ as $n \to +\infty$,

$$
\lim_{n \to +\infty} \langle \tilde{\mu}_{t,s}^{\varepsilon_n}, f_n \rangle = \langle \mu_{t,s}^{\varepsilon_n}, f \rangle, \quad t \in I,
$$

where $\tilde{\mu}_{t,s} = 2\rho_\varepsilon(t,\cdot)dx$. Moreover, for any $s \in I$, the system $\{\mu_{t,s}^{\varepsilon_n}\}_{t \geq s}$ is tight.

\textit{Proof} Denote as usually by $G_{\varepsilon_n}(t,s)$ the evolution operator associated with the operator $\mathcal{A}(t)$ (see (2.5)) with homogeneous Neumann boundary conditions on $\partial \mathbb{R}^d_+$. Since, for any $f \in C_b(\mathbb{R}^d_+)$, $G_{\varepsilon_n}(t,s)\mathcal{E} f$ and $\rho_\varepsilon$ are even with respect to the $x_d$-variable, taking (2.18) into account, we can infer that

$$
\langle \tilde{\mu}_{t,s}^{\varepsilon_n}, G_{\varepsilon_n}(t,s)f \rangle = \langle \tilde{\mu}_{t,s}^{\varepsilon_n}, f \rangle,
$$

and each $\tilde{\mu}_{t,s}^{\varepsilon_n}$ is a probability measure in $\mathbb{R}^d_+$.

Using the tightness of the family of measures $\{\tilde{\mu}_{t,s}^{\varepsilon_n}\}_{t \geq n_0, \varepsilon \in (0,1)}$, where $n_0$ is the smallest integer in $I$, the same procedure as in the proof of Lemma 2 shows that there exist an infinitesimal sequence $(\varepsilon_k)$ and probability measures $\mu_{n,s}^{\varepsilon_k}$ ($n \in \mathbb{N}$, $n \geq n_0$) such that $\tilde{\mu}_{t,s}^{\varepsilon_k}$ weakly* converges to $\mu_{n,s}^{\varepsilon_k}$ as $k \to +\infty$. We claim that, for any $t, s \in \mathbb{Z}$ with $t > s \geq n_0$, we have that

$$
\langle \mu_{t,s}^{\varepsilon_k}, G_{\varepsilon_k}(t,s)f \rangle = \langle \mu_{t,s}^{\varepsilon_k}, f \rangle,
$$

or, equivalently, $\mu_{t,s}^{\varepsilon_k} = (G_{\varepsilon_k}(t,s))^{*}\mu_{t,s}^{\varepsilon_k}$. Formula (4.5) follows writing (4.4) with $\varepsilon_k$ replacing $\varepsilon$ and letting $k \to +\infty$. Clearly, the right-hand side of (4.4) converges to the right-hand
side of (4.5). As far as the convergence of the left-hand side is concerned, we observe that (4.2) implies that

\[
\langle \tilde{\mu}_t^{\varepsilon_k}, G_N^{\varepsilon_k}(t,s) f - G_N(t,s) f \rangle = \langle \tilde{\mu}_t^{\varepsilon_k}, G_N^{\varepsilon_k}(t,s) f - G_N(t,s) f | \chi_{B_t^+} \rangle \\
+ \langle \tilde{\mu}_t^{\varepsilon_k}, G_N^{\varepsilon_k}(t,s) f - G_N(t,s) f | \chi_{\mathbb{R}_+^d \setminus B_t^+} \rangle \\
\leq \| G_N^{\varepsilon_k}(t,s) f - G_N(t,s) f \|_{C(B_t^+)} \\
+ 2 \| f \|_\infty \tilde{\mu}_t^{\varepsilon_k} (\mathbb{R}_+^d \setminus B_t^+) \\
\leq \| G_N^{\varepsilon_k}(t,s) f - G_N(t,s) f \|_{C(B_t^+)} + \frac{2H_{s,1}}{r^2} \| f \|_\infty.
\]

Therefore,

\[
|\langle \tilde{\mu}_t^{\varepsilon_k}, G_N^{\varepsilon_k}(t,s) f - \langle \mu_t^{N'}, G_N(t,s) f \rangle | \\
\leq \| G_N^{\varepsilon_k}(t,s) f - G_N(t,s) f \|_{C(B_t^+)} + \frac{2H_{s,1}}{r^2} \| f \|_\infty \\
+ |\langle \tilde{\mu}_t^{\varepsilon_k}, G_N^{\varepsilon_k}(t,s) f - \langle \mu_t^{N'}, G_N(t,s) f \rangle |
\]

(4.6)

and the last side of the previous chain of inequalities vanishes letting first \(k\) and then \(r\) tend to \(+\infty\). Hence, (4.5) follows.

Now, for \(t \in I \setminus \mathbb{Z}\), we set \(\mu_t^{N'} := (G_N(n,t))_* \mu_n^{N}, \) where \(n = [t] + 1\). Clearly, \(\{\mu_t^{N'}\}_{t \in I}\) is an evolution system of measures for the operator \(G_N(t,s)\). Moreover, \(\tilde{\mu}_t^{\varepsilon_k}\) weakly* converges to \(\mu_t^{N'}\) as \(n \to +\infty\), for any \(t \in I\). Indeed, for any \(n \in \mathbb{Z}\) such that \(n > t\) and any \(f \in C_b(\mathbb{R}_+^d)\), from (4.6) it follows that

\[
\langle \mu_t^{N'}, f \rangle = \langle \mu_n^{N'}, G_N(n,t) f \rangle = \lim_{k \to +\infty} \langle \tilde{\mu}_t^{\varepsilon_k}, G_N^{\varepsilon_k}(n,t) f \rangle = \lim_{k \to +\infty} \langle \tilde{\mu}_t^{\varepsilon_k}, f \rangle.
\]

Further, observe that, for any \(t_0 \in I\), \(\{\mu_t^{N'}\}_{t \geq t_0}\) is a tight system. Indeed, (4.1) shows that \(\langle \tilde{\mu}_t^{\varepsilon_k}, \varphi \rangle \leq H_{t_0,1}\) for any \(t \geq t_0\) and any \(k \in \mathbb{R}\), where \(\varphi(y) = 1 + |y|^2\) for any \(y \in \mathbb{R}^d\). By monotonicity, \(\langle \tilde{\mu}_t^{\varepsilon_k}, \varphi \wedge m \rangle \leq H_{t_0,1}\) for any \(m \in \mathbb{N}\) and any \(t, k\) as above. Letting first \(k\) and then \(m\) tend to \(+\infty\) we deduce that \(\langle \mu_t^{N'}, \varphi \rangle \leq H_{t_0,1}\) for any \(t \geq t_0\) and then, by Chebyshev inequality, the system \(\{\mu_t^{N'}\}_{t \geq t_0}\) is tight.

Finally, we observe that formula (4.3) has been essentially already proved. Indeed, the same argument used to obtain (4.6) yields (4.3) for any \(t \in I\).

\(\square\)

**Remark 6** Note that, for any \(p > 1\) and any \(s \in I\), the function \(\varphi_p(x) = 1 + |x|^{2p}\) satisfies the condition \(A^e(t) \varphi_p(x) \leq a_p - c_p \varphi_p(x)\) for some positive constants \(a_p, c_p\), any \(t \in [s, +\infty), \) any \(x \in \mathbb{R}^d\) and any \(e \in (0, 1]\). Therefore, the same arguments used in the proof of Lemmas 2 and 3 show that each function \(\varphi_p\) is integrable with respect to the measure \(\mu_t^{N'}\) for any \(t \in I\), and there exists a positive constant \(H_{s,p}\) such that

\[
\langle \mu_t^{N'}, \varphi_p \rangle \leq H_{s,p}, \quad t \geq s \in I.
\]

(4.7)

The gradient estimates in the previous section show that each operator \(G_T(t,s)\) is bounded from \(L^p(\mathbb{R}_+^d, \mu_s^{N'})\) into the Sobolev space \(W^{1,p}(\mathbb{R}_+^d, \mu_t^{N'})\).
Proposition 4 The family \( \{ \mu_t^N \}_{t \in I} \) is an evolution system of subinvariant measures for the evolution operator \( G_D(t, s) \). Moreover, for any \( p \in (1, +\infty) \), any \( t > s \in I \) and \( I \in \{ \mathcal{D}, \mathcal{N} \} \) it holds that
\[
\left\| \nabla_x G_I(t, s) f \right\|_{L^p(\mathbb{R}^d_+, \mu_t^N)} \leq c_p e^{\omega_p(t-s)} (t-s)^{-\frac{d}{2}} \| f \|_{L^p(\mathbb{R}^d_+, \mu_s^N)}, \tag{4.8}
\]
for any \( f \in L^p(\mathbb{R}^d_+, \mu_s^N) \), where \( c_p \) and \( \omega_p \) are the constants in Theorem 7.

Proof The first part of the statement follows from Lemma 1. As far as the second part of the statement is concerned, we observe that, since \( \{ \mu_t^N \}_{t \in I} \) is an evolution system of measures (resp. of subinvariant measures) for \( G_N(t, s) \) (resp. for \( G_D(t, s) \)), we get formula (4.8) as a consequence of (3.21) and of the density of the space \( C_c^\infty(\mathbb{R}^d_+) \) in \( L^p(\mathbb{R}^d_+, \mu_s) \) for any \( s \in I \).

Remark 7 Let \( A(t) = \sum_{i,j=1}^d q_{ij}(t, \cdot)D_{ij} + \sum_{i=1}^d b_i(t, \cdot)D_i - c(t, \cdot) \) with \( c \) nonnegative and such that the diffusion and drift coefficients satisfy Hypotheses 8. Let \( \{ \mu_t^N \}_{t \in I} \) be the evolution system of invariant measures for the evolution operator \( G_N(t, s) \), associated with the operator \( A(t) + c(t, \cdot) \). Then, such a system of measures turns out to be subinvariant for the evolution operator \( G_I(t, s) (I \in \{ \mathcal{D}, \mathcal{N} \}) \) associated to the operator \( A(t) \). This fact follows from observing that these two latter evolution operators are controlled from above by \( G_N(t, s) \) on the set of all the nonnegative functions \( f \in C_b(\mathbb{R}^d_+) \).

The following proposition and theorem deal with the asymptotic behaviour of the Dirichlet and Neumann evolution operators.

Proposition 5 Suppose that
\[
2\sigma_0 := \sup_{(t,x) \in I \times \mathbb{R}^d_+} \left[ r(t, x) + \left( \frac{k_1^2q^2}{4} - 1 \right) \eta(t, x) \right] < 0,
\]
and fix \( s \in I \). Then, the following properties are satisfied.

(i) For any \( f \in C_b(\mathbb{R}^d_+) \), \( G_N(t, s) f \) tends to \( m^N_s( f ) \) locally uniformly in \( \mathbb{R}^d_+ \) as \( t \to +\infty \). More precisely, for any \( K > 0 \) there exists a positive constant \( c_{K, s} \) such that
\[
| (G_N(t, s) f)(x) - m^N_s( f ) | \leq c_{K, s} e^{\sigma_0(t-s)} \| f \|_{L^\infty}, \quad t > s, \ x \in B_K^+.
\tag{4.9}
\]
As a byproduct, \( \| G_N(t, s) f - m^N_s( f ) \|_{L^p(\mathbb{R}^d_+, \mu_t^N)} \) tends to 0 as \( t \to +\infty \) for any \( f \in L^p(\mathbb{R}^d_+, \mu_s^N) \) and any \( p \in [1, +\infty) \).

(ii) For any \( f \in C_b(\mathbb{R}^d_+) \), \( G_D(t, s) f \) tends to 0 as \( t \to +\infty \), locally uniformly in \( \mathbb{R}^d_+ \). More precisely, for any \( K > 0 \) there exists a positive constant \( c_{K, s}^{'} \) such that
\[
| (G_D(t, s) f)(x) | \leq c_{K, s}^{'} e^{\sigma_0(t-s)} \| f \|_{L^\infty}, \quad t > s, \ x \in B_K^+.
\]
As a byproduct, \( \| G_D(t, s) f \|_{L^p(\mathbb{R}^d_+, \mu_t^N)} \) tends to 0 as \( t \to +\infty \) for any \( f \in L^p(\mathbb{R}^d_+, \mu_s^N) \) and any \( p \in [1, +\infty) \).

(iii) The evolution system of measures \( \{ \mu_t^N \}_{t \in I} \) is the unique evolution family associated with the operator \( G_N(t, s) f \), such that, for any \( s \in I \), the family \( \{ \mu_t^N \}_{t \geq s} \) is tight, and, for any \( t \in I \), \( \mu_t^N \) weakly* converges to \( \mu_t^N \) as \( \varepsilon \to 0^+ \).
Proof

(i) We fix \( f \in C^1_\mathcal{b}(\mathbb{R}_+^d) \), and observe that

\[
(G_N(t, s) f)(x) - m^N_s(f) = (\mu^N_t \ast (G_N(t, s) f)(x) - G_N(t, s) f),
\]

for any \( t > s \in \mathbb{I} \) and any \( x \in \mathbb{R}_+^d \). We now set \( R(t) = e^{-\sigma_0(t-s)/2} \), \( A_t = \mathbb{R}_+^d \setminus B_{R(t)} \).

Thanks to (4.7) and Chebyshev inequality, we can estimate \( \mu^N_t(A_t) \leq H_{s,1} e^{\sigma_0(t-s)} \) for any \( t > s \). Moreover,

\[
\int_{\mathbb{R}_+^d} |y| d\mu^N_t \leq \int_{\mathbb{R}_+^d} \varphi(y) d\mu^N_t \leq H_{s,1}, \quad t > s,
\]

where \( H_{s,1} \) is given by (4.1). Therefore, using Corollary 1, where we can take \( C_2 = 2\sigma_0 \), we deduce that

\[
|(G_N(t, s) f)(x) - m^N_s(f)| \leq \int_{A_t} |(G_N(t, s) f)(x) - G_N(t, s) f| d\mu^N_t
\]

\[
+ \int_{B_{R(t)}} |(G_N(t, s) f)(x) - G_N(t, s) f| d\mu^N_t
\]

\[
\leq 2\|f\|_{\infty} \mu^N_t(A_t) + \|\nabla_x G_N(t, s) f\|_{\infty} \int_{B_{R(t)}} |x - y| d\mu^N_t
\]

\[
\leq 2H_{s,1} e^{\sigma_0(t-s)} \|f\|_{\infty} + 2e^{\sigma_0(t-s)} \|f\|_{C^1_\mathcal{b}(\mathbb{R}_+^d)} \|x + H_{s,1}\|,
\]

for any \( x \in \mathbb{R}_+^d \) and any \( t > s \). Now, let \( f \) be a general function in \( C_\mathcal{b}(\mathbb{R}_+^d) \). Splitting \( G_N(t, s) f = G_N(t, s + 1) G_N(s + 1, s) f \) for any \( t > s + 1 \), and observing that, by Theorem 7, \( G_N(s + 1, s) f \in C^1_\mathcal{b}(\mathbb{R}_+^d) \) and \( m^N_{s+1}(G_N(s + 1, s) f) = m^N_s(f) \), we get

\[
|(G_N(t, s) f)(x) - m^N_s(f)| \leq K_s e^{\sigma_0(t-s)} \|x + H_{s,1} + 1\| \|f\|_{\infty}, \quad (4.10)
\]

for any \( t > s \) and some positive constant \( K_s \), which yields (4.9). Raising both sides of (4.10) to the power \( p \) and, then, integrating in \( \mathbb{R}_+^d \) with respect to the measure \( \mu^N_t \), we deduce that

\[
\|G_N(t, s) f - m^N_s(f)\|_{L^p(\mathbb{R}_+^d, \mu^N_s)}^p \leq K_s, p e^{\sigma_0p(t-s)} \|f\|_{C^1_\mathcal{b}(\mathbb{R}_+^d)}^p, \quad (4.11)
\]

for any \( t > s \), any \( f \in C_\mathcal{b}(\mathbb{R}_+^d) \) and some positive constant \( K_{s,p} \) where Remark 6 is taken into account. Since \( C_\mathcal{b}(\mathbb{R}_+^d) \) is dense in \( L^p(\mathbb{R}_+^d, \mu^N_s) \) for any \( s \in I \), from (4.11) we deduce that \( \|G_N(t, s) f - m^N_s(f)\|_{L^p(\mathbb{R}_+^d, \mu^N_s)} \) tends to 0 as \( t \to +\infty \), for any \( f \in L^p(\mathbb{R}_+^d, \mu^N_s) \).

(ii) The proof is similar to the above one, and even simpler. Indeed, from the mean value theorem and Corollary 1 we deduce that

\[
|(G_D(t, s) f)(x)| = |(G_D(t, s) f)(x) - (G_D(t, s) f)(0)| \leq 2e^{\sigma_0(t-s)} |x| \|f\|_{C^1_D(\mathbb{R}_+^d)},
\]

for any \( f \in C^1_D(\mathbb{R}_+^d) \) and any \( x \in \mathbb{R}_+^d \). Now, the proof follows the same lines as the proof of property (i). Hence, the details are omitted.
(iii) We observe that the tools used to get (4.9) are the gradient estimate in Corollary 1 and the tightness of the family of measures \( \{\mu_N^t\}_{t \in I} \). Hence, if \( \{\mu_t\}_{t \in I} \) is another tight evolution system of measures for the operator \( G_N(t, s) \), then, for any \( f \in C_b(\mathbb{R}^d) \), \( G_N(t, s)f \) converges to the average of \( f \) with respect to the measure \( \mu_s \), as \( t \to +\infty \). It thus follows that \( \langle \mu_N^t, f \rangle = \langle \mu_t, f \rangle \) for any \( t \in I \) and any \( f \in C_b(\mathbb{R}^d) \), i.e., \( \mu_N^t = \mu_t \) for any \( t \in I \).

The arguments in the proof of Lemma 3 now show that, for any \( f \in C_b(\mathbb{R}^d) \) and any infinitesimal sequence \( (\varepsilon_n) \), there exists a subsequence \( (\varepsilon_{n_k}) \) such that \( \langle \tilde{\mu}_{\varepsilon_{n_k}}^t, f \rangle \) tends to \( \langle \mu_N^t, f \rangle \) as \( k \to +\infty \). This implies that \( \langle \tilde{\mu}_\varepsilon^t, f \rangle \) converges to \( \langle \mu_N^t, f \rangle \) as \( \varepsilon \to 0^+ \).

\[\square\]

The previous proposition does not provide any information on the decay rate of \( \|G_N(t, s) f - m_s^N(f)\|_{L^p(\mathbb{R}^d, \mu_N^t)} \) and \( \|G_D(t, s) f\|_{L^p(\mathbb{R}^d, \mu_N^t)} \) to zero as \( t \to +\infty \). However, Lemma 3 is the key tool to prove that any estimate satisfied by \( G_s(t, s) \) in the \( L^p(\mathbb{R}^d, \mu_s^t) \) norm, which is uniform with respect to \( \varepsilon > 0 \), can be extended to \( G_D(t, s) \) and \( G_N(t, s) \) in the \( L^p(\mathbb{R}^d, \mu_N^t) \) norm. Therefore, we are able to give a more precise information about the decay rate of the previous norms assuming that the diffusion coefficients are independent of \( x \), as the following theorem shows.

**Theorem 9** Suppose that the diffusion coefficients are independent of \( x \). Then, for any \( p \in [1, +\infty) \) there exists a positive constant \( k_p \) such that

\[
\|G_N(t, s)f - m_s^N(f)\|_{L^p(\mathbb{R}^d, \mu_N^t)} \leq k_pe^{-L_0|t-s|} \|f\|_{L^p(\mathbb{R}^d, \mu_N^t)},
\]

(4.12)

\[
\|G_D(t, s)f\|_{L^p(\mathbb{R}^d, \mu_N^t)} \leq k_pe^{-L_0|t-s|} \|f\|_{L^p(\mathbb{R}^d, \mu_N^t)},
\]

(4.13)

for any \( t > s \in I \).

**Proof** To prove estimates (4.12) and (4.13), we observe that [5, Cor. 5.4] shows that for every \( p > 1 \) there exists a constant \( k_p > 0 \) (depending on \( p, \|q_{ij}\|_\infty, L_0 \) and \( \eta_0 \)) such that

\[
\|G_s(t, s)g - m_s^\varepsilon(g)\|_{L^p(\mathbb{R}^d, \mu_s^t)} \leq k_pe^{-L_0\eta_0|t-s|} \|g\|_{L^p(\mathbb{R}^d, \mu_s^t)},
\]

(4.14)

for any \( t > s \in I \) and \( g \in L^p(\mathbb{R}^d, \mu_s^t) \), where \( m_s^\varepsilon(g) \) denotes the average of \( g \) with respect to the measure \( \mu_s^\varepsilon \). Actually, in [5] the case \( I = \mathbb{R} \) is considered but the same arguments can be applied in our situation and lead to (4.14).

We fix \( f \in C^\infty_c(\mathbb{R}^d) \) and write (4.14) with \( g = \mathcal{E} f \) and \( g = \mathcal{O} f \), respectively. Taking (2.16) and (2.18) into account and observing that \( m_s^\varepsilon(\mathcal{E} f) = \langle \tilde{\mu}_s^\varepsilon, f \rangle \) and \( m_s^\varepsilon(\mathcal{O} f) = 0 \), see Lemmas 2 and 3, we get

\[
\|G_N(t, s)f - m_s^\varepsilon(\mathcal{E} f)\|_{L^p(\mathbb{R}^d, \mu_s^t)} \leq k_pe^{-L_0\eta_0|t-s|} \|f\|_{L^p(\mathbb{R}^d, \mu_s^t)},
\]

\[
\|G_D(t, s)f\|_{L^p(\mathbb{R}^d, \mu_s^t)} \leq k_pe^{-L_0\eta_0|t-s|} \|f\|_{L^p(\mathbb{R}^d, \mu_s^t)}.
\]

Letting \( \varepsilon \to 0^+ \) from Step 2 in the proof of Theorem 2 and Proposition 5(iii) we get (4.12) and (4.13) for such a function \( f \). A straightforward density argument allows us to extend the previous estimate to any \( f \in L^p(\mathbb{R}^d, \mu_N^t) \).

\[\square\]
Again, using Proposition 5(iii), we conclude this section extending some results proved in [5] to this setting. The following theorem proves some logarithmic Sobolev inequalities with respect to the tight evolution system of measures \( \{ \mu^\epsilon_t \}_{t \in I} \) and some remarkable properties of the Dirichlet and Neumann evolution operators such as hypercontractivity.

**Theorem 10** Assume that the diffusion coefficients \( q_{ij} \) are independent of \( \epsilon \). Then the following properties hold true:

(i) for any \( f \in C_b^1(\mathbb{R}^d_+) \), any \( p \in (1, +\infty) \) and \( s \in I, \)

\[
\langle \mu^\epsilon_s, |f|^p \log |f| \rangle \leq \frac{1}{p} \langle \mu^\epsilon_s, |f|^p \rangle \log \left( \langle \mu^\epsilon_s, |f|^p \rangle \right) + \frac{p\Lambda}{2L_0\gamma_0} \langle \mu^\epsilon_s, |f|^{p-2} |\nabla f|^2 \chi_{(f \neq 0)} \rangle,
\]

where \( \Lambda := \sup \{ \langle Q(t)\xi, \xi \rangle : t \in I, \xi \in \mathbb{R}^d, |\xi| = 1 \} \);

(ii) for any \( s \in I \) and \( p \in [2, +\infty) \), the Sobolev space \( W^{1,p}(\mathbb{R}^d_+, \mu^\epsilon_s) \) is compactly embedded in \( L^p(\mathbb{R}^d_+, \mu^\epsilon_s) \). As a consequence, for any \( t > s \in I \) and \( p \in (1, +\infty) \), \( G_D(t,s) \) and \( G_N(t,s) \) are compact from \( L^p(\mathbb{R}^d_+, \mu^\epsilon_s) \) into \( L^p(\mathbb{R}^d_+, \mu^\epsilon_t) \);

(iii) for any \( p, q \in (1, +\infty) \) with \( p \leq e^{2L_0\gamma_0\Lambda^{-1}(t-s)}(q-1) + 1 \), \( G_D(t,s) \) and \( G_N(t,s) \) map \( L^q(\mathbb{R}^d_+, \mu^\epsilon_t) \) to \( L^p(\mathbb{R}^d_+, \mu^\epsilon_t) \) for every \( t > s \) and, for \( \mathcal{I} \in \{ \mathcal{D}, \mathcal{N} \}, \)

\[
\| G_{\mathcal{I}}(t,s)f \|_{L^p(\mathbb{R}^d_+, \mu^\epsilon_t)} \leq \| f \|_{L^q(\mathbb{R}^d_+, \mu^\epsilon_t)}, \quad t > s, \quad f \in L^q(\mathbb{R}^d_+, \mu^\epsilon_t).
\]

**Proof** (i) From [5, Thm. 3.3] we know that, for any \( g \in C_b^1(\mathbb{R}^d) \), any \( p \in (1, +\infty) \) and any \( s \in I, \)

\[
\langle \mu^\epsilon_s, |g|^p \log |g| \rangle \leq \frac{1}{p} \langle \mu^\epsilon_s, |g|^p \rangle \log \left( \langle \mu^\epsilon_s, |g|^p \rangle \right) + \frac{p\Lambda}{2L_0\gamma_0} \langle \mu^\epsilon_s, |g|^{p-2} |\nabla g|^2 \chi_{(g \neq 0)} \rangle,
\]

since \( \langle Q^\epsilon(s)\xi, \xi \rangle \leq \Lambda|\xi|^2 \) for any \( s \in I \) and any \( \xi \in \mathbb{R}^d. \)

Now, let \( f \in C_b^1(\mathbb{R}^d) \); writing (4.16) with \( g \) replaced with \( \mathcal{E}f \) and using the symmetry of \( \rho^\epsilon(t, \cdot) \) with respect the last variable, we get

\[
\langle \tilde{\mu}^\epsilon_s, |f|^p \log |f| \rangle \leq \frac{1}{p} \langle \tilde{\mu}^\epsilon_s, |f|^p \rangle \log \left( \langle \tilde{\mu}^\epsilon_s, |f|^p \rangle \right) + \frac{p\Lambda}{2L_0\gamma_0} \langle \tilde{\mu}^\epsilon_s, |f|^{p-2} |\nabla f|^2 \chi_{(f \neq 0)} \rangle.
\]

Hence the claim follows applying Proposition 5(iii).

(ii) Once (4.15) is established, the proof follows as in [5, Thm. 3.4].

(iii) Theorem 4.1 in [5] shows that if \( p, q \in (1, +\infty) \) and \( p \leq e^{2L_0\gamma_0\Lambda^{-1}(t-s)}(q-1) + 1 \) then \( G^\epsilon(t,s) \) map \( L^q(\mathbb{R}^d_+, \mu^\epsilon_t) \) to \( L^p(\mathbb{R}^d_+, \mu^\epsilon_t) \) for every \( t > s \) and

\[
\| G^\epsilon(t,s)g \|_{L^p(\mathbb{R}^d_+, \mu^\epsilon_t)} \leq \| g \|_{L^q(\mathbb{R}^d_+, \mu^\epsilon_t)}, \quad t > s, \quad g \in L^q(\mathbb{R}^d_+, \mu^\epsilon_t).
\]

Now, arguing as in the proof of Theorem 9 we get the claim. \( \square \)

5 Examples

In this section, we exhibit two classes of nonautonomous elliptic operators \( A(t) \) to which the main results of this paper apply.
Let $\mathcal{A}(t)$ be defined on smooth functions $\zeta$ by
\[
\mathcal{A}(t)\zeta(x) = (1 + |x|^2)^k \text{Tr}(B(t, x)D^2\zeta(x)) - b_0(t) \left(1 + |x|^2\right)^m \langle x, \nabla\zeta(x) \rangle + g(t, x_d)D_d\zeta(x) - \gamma(t) \left(1 + |x|^2\right)^q \zeta(x),
\]
for any $(t, x) \in I \times \mathbb{R}^d_+$, where $I$ is an open halfline (possibly $I = \mathbb{R}$). We assume the following conditions on the coefficients of the operator $\mathcal{A}(t)$.

**Hypotheses 11**

(i) $k, m, q \in (1, +\infty)$ with $k \leq m < q$;
(ii) for any $i, j = 1, \ldots, d$, $b_{ij} = b_{ji}$ belongs to $C^{\alpha/2, 1}_{\text{loc}}(I \times \mathbb{R}^d_+)$. Moreover, the $b_{ij}$'s and their first-order spatial derivatives are bounded in $I \times \mathbb{R}^d_+$, and $b_{id}(\cdot, 0) \equiv 0 (i = 1, \ldots, d - 1)$;
(iii) there exist a positive constant $\eta_0$ such that $\langle B(t, x)\xi, \xi \rangle \geq \eta_0|\xi|^2$, for any $t \in I$, $x \in \mathbb{R}^d_+$ and $\xi \in \mathbb{R}^d$;
(iv) $b_0 \in C^{\alpha/2}_{\text{loc}}(I)$, $g \in C^{\alpha/2, 1}_{\text{loc}}(I \times \mathbb{R}^d_+)$, $g(\cdot, 0) \equiv 0$, $D_d g(t, x_d) \leq \vartheta(t) \left(1 + x_d^2\right)^m$ for any $(t, x) \in I \times \mathbb{R}^d_+$ and some positive function $\vartheta$ such that $b_0(t) - \vartheta(t) \geq \beta_0$ for any $t \in I$;
(v) $\gamma \in C^{\alpha/2}_{\text{loc}}(I) \cap C_b(I)$ and there exists a positive constant $\gamma_0$ such that $\gamma(t) \geq \gamma_0$ for any $t \in I$.

Under such assumptions, Hypotheses 3 are satisfied. Indeed, we can take
\[
r(t, x) = -(b_0(t) - \vartheta(t)) \left(1 + |x|^2\right)^m, \quad (t, x) \in I \times \mathbb{R}^d_+, \quad L_0 = \beta_0 \eta_0^{-1}, \quad L_1 = 0.
\]
Moreover, if we set $q_{ij}(t, x) = b_{ij}(t, x) \left(1 + |x|^2\right)^k$, it holds that
\[
|\nabla_x q_{ij}(t, x)| \leq \left(\|\nabla_x b_{ij}\|_\infty + k\|b_{ij}\|_\infty\right) \left(1 + |x|^2\right)^k, \quad (t, x) \in I \times \mathbb{R}^d_+,
\]
so that we can take $k_1 = \sup_{i, j \leq d}(\|\nabla_x b_{ij}\|_\infty + k\|b_{ij}\|_\infty)\eta_0^{-1}$.

Finally, Hypothesis 3(ii) is satisfied with $\beta(t, x) = 2q \|\gamma\|_\infty \left(1 + |x|^2\right)^{q-1/2}$, for any $(t, x) \in I \times \mathbb{R}^d$, and $k_2 = 2q \|\gamma\|_\infty \gamma_0^{-1}$. It thus follows that
\[
r(t, x) + \left(\frac{k_1^2 \eta_0^2}{M_\rho} - \frac{M_\rho}{4}\right) \eta(t, x) - \left(1 - \frac{1}{p}\right) c(t, x) + \frac{pk_2}{4(p - 1)} \beta(t, x)
\]
\[
\leq -\beta_0(1 + |x|^2)^m + \left(\sup_{i, j \leq d}(\|\nabla_x b_{ij}\|_\infty + k\|b_{ij}\|_\infty)^2d^2\right) 4M_\rho \eta_0^2 \left(1 + |x|^2\right)^k
\]
\[
- \left(1 - \frac{1}{p}\right) \gamma_0(1 + |x|^2)^q + \frac{pq^2}{(p - 1)\gamma_0} \|\gamma\|_\infty^2 \left(1 + |x|^2\right)^{q-1/2}.
\]
If $\gamma \equiv 0$ and the other conditions in Hypotheses 11 are satisfied, then estimates (3.17) and (4.8) hold true, respectively with

$$K_p = \sup_{y \geq 1} \left( -\beta_0 y^m + \frac{\sup_{i,j \leq d} (\|\nabla_x b_{ij}\|_\infty + k\|b_{ij}\|_\infty)^2 d^2}{4\eta_0 M_p} y^k \right),$$

$$C_p = \sup_{y \geq 1} \left[ -\beta_0 y^m + \left( \frac{\sup_{i,j \leq d} (\|\nabla_x b_{ij}\|_\infty + k\|b_{ij}\|_\infty)^2 d^2}{4\eta_0 M_p} - \eta_0 \right) y^k \right],$$

for any $p \in (1, +\infty)$.

Further, if $\sup_{i,j \leq d} (\|\nabla_x b_{ij}\|_\infty + k\|b_{ij}\|_\infty)^2 d^2 < 4\eta_0^2 (\eta_0 + \beta_0)$, then, the results in Proposition 5 hold true with

$$\sigma_0 = -\beta_0 + \frac{\sup_{i,j \leq d} (\|\nabla_x b_{ij}\|_\infty + k\|b_{ij}\|_\infty)^2 d^2}{4\eta_0^2} - \eta_0.$$

Finally, we consider the case when the diffusion coefficients of the operators $A(t)$ in (5.1) are independent of $x$, and the following conditions are satisfied.

**Hypotheses 12**

(i) $m, q \in \mathbb{N}$ with $2q - 1 \leq m$;

(ii) $b_{ij} = b_{ji}$ belongs to $C^{\alpha/2}_{\text{loc}}(I)$ for any $i, j = 1, \ldots, d$;

(iii) $b_{id}(t) = 0$ for any $t \in I$ and $i = 1, \ldots, d - 1$;

(iv) there exist a positive constant $\eta_0$ such that $\langle B(t)\xi, \xi \rangle \geq \eta_0 |\xi|^2$, for any $t \in I$ and $\xi \in \mathbb{R}^d$;

(v) Hypotheses 11(iv)-(v) are satisfied.

In this case estimate (3.3) is satisfied with $p = 1$ and $C_1 = 4q^2 \|\gamma\|_\infty^2 / \beta_0$.

If $\gamma \equiv 0$ and the other conditions in Hypotheses 12 are satisfied, then (3.17), (4.12) and (4.13) hold true and we can take $L_0 = \beta_0 \eta_0^{-1}$.

**Appendix: An Auxiliary Result**

Here, we prove a result which is used in the proof of Theorem 4 and provides us with a (local) higher spatial Hölder regularity of the third-order derivatives of the solution of the Cauchy-Neumann problem

$$\begin{cases}
  u_t(t, x) = A(t)u(t, x), & t \in (0, T), \ x \in \Omega \\
  \frac{\partial u}{\partial v}(t, x) = 0, & t \in (0, T), \ x \in \partial\Omega, \\
  u(s, x) = f(x), & x \in \overline{\Omega},
\end{cases}$$

in a bounded domain $\Omega$ of class $C^{2+\alpha}$ (for some $\alpha \in (0, 1)$). We do not assume any Hölder regularity in $t$ of the spatial gradient of the coefficients of the uniformly elliptic operator $A(t)$, defined on smooth functions $\psi$ by

$$(A(t)\psi)(x) = \text{Tr} \left( Q(t, x)D^2\psi(x) \right) + \langle b(t, x), \nabla\psi(x) \rangle - c(t, x)\psi(x),$$

for any $t \in [0, T]$ and any $x \in \Omega$. Even if it seems quite predictable we did not find any reference for this result. Hence, for the sake of completeness we provide a proof of it.
Proposition 6 Assume that the coefficients of the operator \( A(t) \) belong to \( C^{\alpha/2,\alpha}((0, T) \times \Omega) \cap C^{0,1+\alpha}([0, T] \times \Omega) \). Then, for any \( f \in C^{3+\alpha}((0, T) \times \Omega) \) with normal derivative identically vanishing on \( \partial \Omega \), problem (5.3) has a unique solution \( u \in C^{1+\alpha/2,2+\alpha}((0, T) \times \Omega) \cap C^{0,3+\alpha}_{loc}([0, T] \times \Omega) \).

Proof By Theorem IV.5.3 in [21], problem (5.3) admits a unique solution \( u \) which belongs to \( C^{1+\alpha/2,2+\alpha}((0, T) \times \Omega) \) and there exists a positive constant \( C_1 \) such that

\[
\|u\|_{C^{1+\alpha/2,2+\alpha}((0,T) \times \Omega)} \leq C_1 \|f\|_{C^{2+\alpha}((0,T) \times \Omega)}. \tag{5.4}
\]

Moreover, we claim that \( D_j u \) is \((1+\alpha)/2\)-Hölder continuous in \( t \), uniformly with respect to \( x \in \mathbb{R}^d \), for any \( j = 1, \ldots, d \). Indeed, writing

\[
u(t, x) - u(s, x) = \int_s^t D_\sigma u(\sigma, x) d\sigma, \quad t, s \in [s, T], \ x \in \Omega,
\]
we can easily show that

\[
\|u(t, \cdot) - u(s, \cdot)\|_{C^\alpha(\Omega)} \leq \|D_t u\|_{C^{0,\alpha}([0,T] \times \Omega)} |t - s| \text{ for any } t, s \in [0, T].
\]

Since \( C(\Omega) \) belongs to the interpolation class \( J_{(1-\alpha)/2} \) between \( C^\alpha_b(\Omega) \) and \( C^2_b(\Omega) \) (see e.g., [26, Sect. 4.5]), there exists a constant \( K \), independent of \( s, t \), such that

\[
\|u(t, \cdot) - u(s, \cdot)\|_{C^\alpha(\Omega)} \leq K \|u(t, \cdot) - u(s, \cdot)\|_{C^{1/2,\alpha}((0,T) \times \Omega)} \leq 2^{1-\alpha/2} K C_1 \|f\|_{C^{2+\alpha}((0,T) \times \Omega)}, \quad j = 1, \ldots, d. \tag{5.5}
\]

To prove that \( u \) admits third-order spatial derivatives in \( C^{0,\alpha}_{loc}((0, T) \times \Omega) \), we fix an open set \( \Omega' \), compactly contained in \( \Omega \), and a function \( \vartheta \in C^\infty(\mathbb{R}^d) \) such that \( \chi_{\Omega'} \leq \vartheta \leq \chi_{\Omega''} \) for some open set \( \Omega'' \) compactly contained in \( \Omega \). Let \( v \in C^{1+\alpha/2,2+\alpha}_b([0, T] \times \mathbb{R}^d) \) denote the trivial extension of the function \( u \vartheta \) to the whole of \( \mathbb{R}^d \). The function \( v \) solves the Cauchy problem

\[
\begin{cases}
\nu_t(t, x) = \hat{A}(t) \nu(t, x) + \hat{g}(t, x), & t \in (0, T), \ x \in \mathbb{R}^d, \\
v(s, x) = \vartheta(x) f(x), & x \in \mathbb{R}^d,
\end{cases}
\]

where, with a slight abuse of notation, we still denote by \( \vartheta f \) the trivial extension of \( \vartheta f \) to the whole of \( \mathbb{R}^d \). Here, \( \hat{A}(t) = \text{Tr}(\hat{Q} D^2 + \langle \hat{b}, \nabla \rangle - \hat{c}), \hat{Q} = \eta \hat{Q} + (1-\eta) I, \hat{b} = \eta b \) and \( \hat{c} = \eta c \), where \( \eta \in C^\infty(\Omega) \) is a smooth function satisfying \( \eta \equiv 1 \) in \( \Omega'' \). Finally, \( \hat{g} \) denotes the trivial extension to the whole of \( [0, T] \times \mathbb{R}^d \) of the function \( -u(\hat{A}(t) + \hat{c}) \vartheta + 2(\hat{Q} \nabla \vartheta, \nabla u) \).

For any \( j \in \{1, \ldots, d\} \), any \( h > 0 \) and any \( \psi : [0, T] \times \mathbb{R}^d \to \mathbb{R} \), we denote by \( \tau_h \psi \) the function defined by \( \tau_h \psi = h^{-1}(\psi(\cdot, \cdot + he_j) - \psi) \). Clearly, \( \nu_h := \tau_h \nu \) belongs to \( C^{1+\alpha/2,2+\alpha}_b([0, T] \times \mathbb{R}^d) \) and solves the Cauchy problem

\[
\begin{cases}
D_t \nu(t, x) = \hat{A}(t) \nu(t, x) + \tau_h \hat{g}(t, x) + F_h(t, x), & t \in (0, T), \ x \in \mathbb{R}^d, \\
\nu_h(s, x) = \tau_h(\vartheta f)(x), & x \in \mathbb{R}^d,
\end{cases}
\]

where

\[
F_h = \sum_{i,j=1}^d (\tau_h \hat{q}_{ij}) D_{ij} v(\cdot, \cdot + he_j) + \sum_{j=1}^d (\tau_h \hat{b}_j) D_j v(\cdot, \cdot + he_j) - (\tau_h \hat{c}) v(\cdot, \cdot + he_j).
\]
By the results in [17–19] the \(C^{0,2+\alpha}_b([0, T] \times \mathbb{R}^d)\)-norm of \(v_h\) can be estimated from above by a positive constant, independent of \(h\), times the sum of the \(C^{2+\alpha}_b(\mathbb{R}^d)\)-norm of \(\tau_h(\partial f)\) and the \(C^{0,\alpha}_b([0, T] \times \mathbb{R}^d)\)-norm of \(\tau_h \hat{g}\) and \(F_h\). More precisely, taking (5.4) into account, we can write
\[
\|v_h\|_{C^{0,2+\alpha}_b} \leq C_2 \left( \|\tau_h(\partial f)\|_{C^{2+\alpha}_b} + \|\tau_h \hat{g}\|_{C^{0,\alpha}_b} + \|f\|_{C^{2+\alpha}_b} \|\tau_h \hat{c}\|_{C^{0,\alpha}_b} + \|f\|_{C^{2+\alpha}_b} \sum_{i,j=1}^d \|\tau_h \hat{q}_{ij}\|_{C^{0,\alpha}_b} \right),
\]
with \(C_2\) (as all the forthcoming constants) independent of \(h\) and \(f\), and where \(C^{2+\alpha}_b\) and \(C^{0,\alpha}_b\) stand for \(C^{2+\alpha}_b(\mathbb{R}^d)\) and \(C^{0,\alpha}_b([0, T] \times \mathbb{R}^d)\), respectively.

Since \(\tau_h \psi(x) = \int_0^1 D_j \psi(x + h s \varepsilon_j) ds, \quad x \in \mathbb{R}^d\), (5.6)
for any \(\psi \in C^{1}(\mathbb{R}^d)\), it follows that
\[
\|v_h\|_{C^{0,2+\alpha}_b} \leq C_2 \left( \|\partial f\|_{C^{3+\alpha}_b} + \|\hat{g}\|_{C^{0,1+\alpha}_b} + \|f\|_{C^{2+\alpha}_b} \|\hat{c}\|_{C^{0,1+\alpha}_b} + \|f\|_{C^{2+\alpha}_b} \sum_{i,j=1}^d \|\hat{q}_{ij}\|_{C^{0,1+\alpha}_b} \right).
\]

The definition of \(\hat{g}\), estimate (5.4) and the assumptions on \(f\) and the coefficients of the operator \(A(t)\) allow us to conclude that
\[
\|v_h\|_{C^{0,2+\alpha}_b} \leq C_3 \|f\|_{C^{3+\alpha}(\Omega)}.
\]

Further, taking advantage of (5.5) and (5.6), we can easily check that \(v_h\) belongs to \(C_{b}^{(1+\alpha)/2,0}([0, T] \times \mathbb{R}^d)\) and \(\|v_h\|_{C_{b}^{(1+\alpha)/2,0}([0, T] \times \mathbb{R}^d)}\) can be estimated by a constant independent of \(h\). Again, since \(C_{b}^{2} (\mathbb{R}^d)\) is of class \(J_{2/(2+\alpha)}\) between \(C_{b} (\mathbb{R}^d)\) and \(C_{b}^{2+\alpha} (\mathbb{R}^d)\), from (5.7) we deduce that
\[
\|v_h(t, \cdot) - v_h(s, \cdot)\|_{C_{b}^{2}(\mathbb{R}^d)} \leq C_4 |t - s|^\theta, \quad s, t \in [0, T],
\]
where \(\theta = \alpha(1 + \alpha)/(4 + 2\alpha).\) Combining (5.7) and (5.8) it follows that \(v_h \in C_{b}^{\theta/2,2+\theta}([0, T] \times \mathbb{R}^d)\) and \(\|v_h\|_{C_{b}^{\theta/2,2+\theta}([0, T] \times \mathbb{R}^d)} \leq C_5.\) Now, we can use a compactness argument and conclude that \(D_j v\) belongs to \(C_{b}^{0,2+\alpha}([0, T] \times \mathbb{R}^d).\) Recalling that \(\theta = 1\) in \(\Omega',\) we deduce that \(u \in C^{0,3+\alpha}([0, T] \times \Omega').\)

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