Uncertainty and Certainty Relations for Pauli Observables in Terms of Rényi Entropies of Order $\alpha \in (0; 1]$

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Abstract We obtain uncertainty and certainty relations of state-independent form for the three Pauli observables with use of the Rényi entropies of order $\alpha \in (0; 1]$. It is shown that these entropic bounds are tight in the sense that they are always reached with certain pure states. A new result is the condition for equality in Rényi-entropy uncertainty relations for the Pauli observables. Upper entropic bounds in the pure-state case are also novel. Combining the presented bounds leads to a band, in which the rescaled average Rényi $\alpha$-entropy ranges for a pure measured state. A width of this band is compared with the Tsallis formulation derived previously.

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1 Introduction

Heisenberg’s uncertainty principle[1] is one of the most known results related to quantum incompatibility. Indeterminacy relations are still the subject of active researches.[2–3] Traditional relations pertain to uncertainties of several observables in the same state. The so-called state-extended uncertainty relations deal with one observable and two different states.[4] Due to the papers,[5–6] entropic functions are widely used in formulating the uncertainty principle.[7–8] A quantum state is generally characterized by the probabilities of the outcomes of a test.[9] In this sense, the entropic formulation deals with quantum-mechanical primary. Entropic uncertainty relations in the presence of quantum memory are formulated in Refs. [10–12]. Entropic lower bounds of the papers[13–14] pertain to a situation substantial in quantum optics. Entropic trade-off relations for a single quantum operation were examined in Refs. [15–16]. A majorization approach to entropic uncertainty relations has been proposed.[17–18]

Entropic uncertainties for more than two observables characterize a role of mutual unbiasedness.[7,19] Entropic uncertainty bounds are essential in analyzing the security of quantum cryptographic schemes.[20–22] Uncertainty relations for $(d + 1)$ mutually unbiased bases in $d$-dimensional Hilbert space were given in terms of the Shannon entropy.[23–25] Entropic uncertainty relations for mutually unbiased bases were considered.[26] The author of Ref. [24] derived exact bounds for the qubit case $d = 2$. Complementarity aspects of entropic relations are considered in Refs. [27–28]. Generalizations of the Shannon entropy are also used in quantum information theory.[29] The Rényi[30] and Tsallis entropies[31] form important families of one-parametric extensions. We have previously expressed uncertainty and certainty relations for the Pauli observables in terms of Tsallis’ entropies.[32] The writers of Ref. [33] examined uncertainty relations for two qubit observables in terms of Rényi’s entropies with arbitrary orders.

In this work, we study lower and upper bounds on the sum of Rényi’s $\alpha$-entropies, which quantify uncertainties in measurement of the Pauli observables. The present work is further development of the approach proposed in Refs. [32, 34]. The paper is organized as follows. The preliminary material is given in Sec. 2. In Sec. 3, tight lower bounds on the sum of three Rényi’s entropies of order $\alpha \in (0; 1]$ are obtained. The conditions for equality are developed as well. In Sec. 4, we examine upper bounds on the sum of three $\alpha$-entropies in the case of pure measured states. Tight upper bounds are derived for all real $\alpha \in (0; 1]$. Combining the lower and upper bounds gives a band, in which the rescaled average Rényi entropy ranges in the pure-state case for $\alpha \in (0; 1]$. In this regard, we also compare the Rényi and Tsallis formulations. In Sec. 5, we conclude the paper with a summary of results.

2 Preliminaries

In this section, the required material is reviewed. We will quantify uncertainties of quantum measurements by means of the Rényi entropy. Let $p = \{p_j\}$ be a probability distribution supported on $n$ points. For real $\alpha > 0 \neq 1$, the Rényi $\alpha$-entropy is defined by[30]

$$R_\alpha(p) := \frac{1}{1-\alpha} \ln \left( \sum_{j=1}^n p_j^\alpha \right).$$

(1)

It is a non-increasing function of order $\alpha$.[30] The Shannon entropy $H_1(p) = -\sum_j p_j \ln p_j$ is recovered in the limit $\alpha \to 1$. In the limit $\alpha \to \infty$, the right-hand side of (1)
leads to the min-entropy
\[ R_\infty(p) = -\ln(\max p_j), \] (2)
where \(\max p_j\) is the maximal probability. For \(\alpha \in (0; 1)\),
the right-hand side of (1) is a concave function of probability distribution. Namely, for all \(\lambda \in [0; 1]\) and two probability distributions \(p = \{p_j\}\) and \(q = \{q_j\}\), we have
\[ R_\alpha(\lambda p + (1 - \lambda)q) \geq \lambda R_\alpha(p) + (1 - \lambda)R_\alpha(q), \] (3)
whenever \(0 < \alpha < 1\). We merely note that: (i) the function \(\xi \mapsto \xi^\alpha\) is concave for \(\alpha \in (0; 1)\), and (ii) the function \(\xi \mapsto (1 - \alpha)^{-1} \ln \xi\) is increasing and concave for \(\alpha \in (0; 1)\).

Using Jensen’s inequality twice, these facts immediately lead to the property (3). The standard case \(\alpha = 1\) also deals with the concave entropy. For \(\alpha > 1\), however, the Rényi \(\alpha\)-entropy is neither purely convex nor purely concave.[35] We also see that the min-entropy (2) is convex, as the function \(\xi \mapsto -\ln \xi\). Applications of entropic measures in quantum information theory are discussed in the book.[29] The standard entropies of partitions on quantum logic were considered in Refs. [36–37]. Within the Rényi and Tsallis formulations, this issue was developed in Ref. [38].

Following [34], we further put the quantity
\[ \Phi_\alpha(p) := \sum_{j=1}^n p_j^\alpha. \] (4)
When \(\alpha < \beta\), we have \(\Phi_\beta(p) \geq \Phi_\beta(p)\). With respect to the distribution \(p = \{p_j\}\), the function (4) is concave for \(\alpha \in (0; 1)\) and convex for \(\alpha \in (1; \infty)\). The Rényi entropy (1) can be rewritten as
\[ R_\alpha(p) := (1 - \alpha)^{-1} \ln \Phi_\alpha(p). \] (5)

The maximum \(\ln n\) is reached with the equiprobable distribution, when \(p_j = 1/n\) for all \(1 \leq j \leq n\). The minimal zero value is reached with any deterministic distribution, for which one of probabilities is 1 and other are all zeros. We will also use the Tsallis entropy. In terms of the function (4), the Tsallis \(\alpha\)-entropy of degree \(\alpha > 0 \neq 1\) is written as
\[ H_\alpha(p) = \frac{\Phi_\alpha(p) - 1}{1 - \alpha}. \] (6)

With the equiprobable distribution, the entropy (6) reaches its maximal value \(\ln_\alpha(n)\). Here, the \(\alpha\)-logarithm of positive \(\xi\) is defined as
\[ \ln_\alpha(\xi) := \frac{\xi^{1-\alpha} - 1}{1 - \alpha}. \] (7)

In the limit \(\alpha \to 1\), the \(\alpha\)-logarithm is reduced to the usual logarithm. The Shannon entropy is obtained from (6) also in this limit.

In the following, we will deal with the qubit case \(n = 2\). Here, three complementary observables are usually represented by the Pauli matrices \(\sigma_x, \sigma_y, \sigma_z\), namely
\[ \sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \] (8)

These matrices traditionally used for describing spin-1/2 observables. Each of the matrices has the eigenvalues \(\pm 1\). By \{0\}, \{1\}\., we mean the eigenbasis of \(\sigma_z\), that is
\[ |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (9)

The eigenstates of \(\sigma_x\) and \(\sigma_y\) are written as
\[ |x_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}, \quad |y_\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \end{pmatrix}. \] (10)

Here, we have \(\sigma_x |x_\pm\rangle = \pm |x_\pm\rangle\) and \(\sigma_y |y_\pm\rangle = \pm |y_\pm\rangle\). The three bases given by (9) and (10) are mutually unbiased. Measurements in these eigenbases are used in six-state cryptographic protocols.[21–22]

We now write the probabilities corresponding to measurement of each of the observables \(\sigma_x, \sigma_y, \sigma_z\). Up to a unimodular factor, we can represent a normalized pure state in the form
\[ |\psi\rangle = \cos \tau |0\rangle + e^{i\varphi} \sin \tau |1\rangle = \begin{pmatrix} \cos \tau \\ e^{i\varphi} \sin \tau \end{pmatrix}, \] (11)

where \(\tau\) and \(\varphi\) are real numbers. Assuming \(\varphi \in [0; 2\pi)\), we will take \(\tau \in [0; \pi/2]\), since a global phase in the state vector has no physical relevance. For the observables \(\sigma_x\) and \(\sigma_y\), the probabilities are respectively given as \(|x_\pm |\psi\rangle|^2\) and \(|y_\pm |\psi\rangle|^2\). The final expressions are obtained in the form[32]
\[ p_\pm = \frac{1 - \cos 2\tau}{2}, \] (12)
\[ q_\pm = \frac{1 - \sin 2\tau}{2}, \] (13)
\[ r_\pm = \frac{1 - \cos 2\tau}{2}. \] (14)

Substituting (12), (13), (14) into the right-hand side of (1), one gives the three entropies \(R_\alpha (\sigma_x |\psi\rangle), R_\alpha (\sigma_y |\psi\rangle), R_\alpha (\sigma_z |\psi\rangle)\) for the state (11). We will study lower and upper bounds on the sum of such entropies for \(\alpha \in (0; 1]\).

### 3 Tight Lower Bounds on the Sum of Entropies of Degree \(\alpha \in (0; 1]\)

In this section, we derive tight lower bounds on the sum of three Rényi entropies of order \(\alpha \in (0; 1]\). A desired bound will firstly be obtained for pure states of the form (11), when the probabilities are given by (12), (13), and (14). Using the concavity properties, we then extend the result to all mixed states of a qubit. For \(\alpha > 0 \neq 1\), we introduce the function
\[ F_\alpha (\tau, \varphi) = \Phi_\alpha(p) \Phi_\alpha(q) \Phi_\alpha(r), \] (15)
in which we substitute (12), (13), and (14). Using this function, the entropic sum is rewritten as
\[ R_\alpha (\sigma_x |\psi\rangle) + R_\alpha (\sigma_y |\psi\rangle) + R_\alpha (\sigma_z |\psi\rangle) = \frac{1}{1 - \alpha} \ln F_\alpha (\tau, \varphi). \] (16)
Since the function $\xi \mapsto (1 - \alpha)^{-1} \ln \xi$ increases for $\alpha \in (0; 1)$, we aim to minimize (15) in the domain of interest. As was already noted, the variables are initially in the intervals $\tau \in [0; \pi/2]$ and $\varphi \in [0; 2\pi]$. In the task of optimization, however, we can restrict a consideration to the rectangular domain(32)

$$D := \{(\tau, \varphi) : \tau \in [0; \pi/4], \varphi \in [0; \pi/4]\}. \quad (17)$$

Here, we claim that in the total domain $\{(\tau, \varphi) : \tau \in [0; \pi/2], \varphi \in [0; 2\pi]\}$, the function (15) takes the same range of values as in the domain (17). The reasons are the following. Taking $\tau \in [0; \pi/2]$ and $\varphi \in (\pi; 2\pi)$, one first uses $\varphi \mapsto \varphi - \pi$. The latter merely swaps two values in each of the pairs (12) and (13), without altering $\Phi_\alpha(p)$ and $\Phi_\alpha(q)$. So, we restrict to $\varphi \in [0; \pi]$. Taking further $\tau \in [0; \pi/2]$ and $\varphi \in (\pi/2; \pi]$, one applies $\varphi \mapsto \pi - \varphi$. Then the probabilities $p_\pm$ are swapped and the probabilities $q_\pm$ are the same, with no changes in (15). Hence, we restrict to $\varphi \in [0; \pi/2]$. Acting in $\varphi \in (\pi/4; \pi/2]$, mapping $\varphi \mapsto \pi/2 - \varphi$ implies swapping $p_j$ and $q_j$ for $j = \pm$, without altering the product $\Phi_\alpha(p)\Phi_\alpha(q)$. So, we can focus on $\tau \in [0; \pi/2]$ and $\varphi \in [0; \pi/4]$. Finally, mapping $\tau \mapsto \pi/2 - \tau$ for $\tau \in (\pi/4; \pi/2]$ does not alter $\sin 2\tau$ and reverses the sign of $\cos 2\tau$. So, each of the three multipliers in the right-hand side of (15) remains unchanged. The following statement takes place.

**Theorem 1** Let qubit state be described by density matrix $\rho$. For all $\alpha \in (0; 1]$, the entropic sum satisfies

$$R_\alpha(\sigma_x|\rho) + R_\alpha(\sigma_y|\rho) + R_\alpha(\sigma_z|\rho) \geq 2\ln 2, \quad (18)$$

with equality if and only if the qubit state is an eigenstate of either of the observables $\sigma_x$, $\sigma_y$, $\sigma_z$.

**Proof** Since the case of the Shannon entropy was already given in theorem 1 of the paper[12] we further assume $\alpha \in (0; 1)$. We will show that the right-hand side of (18) is equal to the minimum of (16) in the domain (17). Differentiating (15) with respect to $\varphi$, we have

$$\frac{\partial}{\partial \varphi} F_\alpha(\tau, \varphi) = F_\alpha(\tau, \varphi) \left( \frac{1}{\Phi_\alpha(p)} \frac{\partial \Phi_\alpha(p)}{\partial \varphi} + \frac{1}{\Phi_\alpha(q)} \frac{\partial \Phi_\alpha(q)}{\partial \varphi} \right). \quad (19)$$

Usual calculations show that

$$\frac{1}{\Phi_\alpha(p)} \frac{\partial \Phi_\alpha(p)}{\partial \varphi} = -\frac{\alpha}{2} \frac{p_{\alpha}^{\alpha-1} - p_{\alpha}^{\alpha-1}}{p_+^{\alpha} + p_-^{\alpha}} \sin 2\tau \sin \varphi, \quad (20)$$

$$\frac{1}{\Phi_\alpha(q)} \frac{\partial \Phi_\alpha(q)}{\partial \varphi} = \frac{\alpha}{2} \frac{q_{\alpha}^{\alpha-1} - q_{\alpha}^{\alpha-1}}{q_+^{\alpha} + q_-^{\alpha}} \sin 2\tau \cos \varphi. \quad (21)$$

Introducing the variables $u = \sin 2\tau \cos \varphi$ and $v = \sin 2\tau \sin \varphi$, we rewrite (19) as

$$\frac{1}{F_\alpha(\tau, \varphi)} \frac{\partial}{\partial \varphi} F_\alpha(\tau, \varphi) = \alpha uv \left( \frac{f_\alpha(u)}{g_\alpha(u)} - \frac{f_\alpha(v)}{g_\alpha(v)} \right). \quad (22)$$

Here, we use the functions

$$f_\alpha(u) = u^{-1}(1 - u)^{\alpha-1} - (1 + u)^{\alpha-1}, \quad (23)$$

$$g_\alpha(u) = (1 + u)^{\alpha} + (1 - u)^{\alpha}. \quad (24)$$

For $\alpha \in (0; 1)$, the function $f_\alpha(u)$ monotonically increases, whereas the function $g_\alpha(u)$ monotonically decreases with $u \in [0; 1]$. To prove the claim, these functions are expanded into power series about the origin. Using the binomial theorem and properties of the binomial coefficients, we have

$$f_\alpha(u) = 2(1 - \alpha) + 2 \sum_{k=1}^{\infty} \left( \frac{2k + 1 - \alpha}{2k + 1} \right) u^{2k}. \quad (25)$$

We stress that this series contains only strictly positive coefficients. Indeed, for $k \geq 1$ and $\alpha \in (0; 1)$ we have

$$\left( \frac{2k + 1 - \alpha}{2k + 1} \right) \left( \frac{2k + 1 - \alpha - 1}{2k + 1} \right) \cdots \left( \frac{2(1 - \alpha)}{2k + 1} \right) > 0. \quad (26)$$

Hence, the function (25) monotonically increases. Further, we obtain the expansion

$$g_\alpha(u) = 2 - 2 \sum_{k=1}^{\infty} c_{2k} u^{2k}. \quad (27)$$

For $k \geq 1$ and $\alpha \in (0; 1)$, the coefficients $c_{2k}$ are strictly positive, i.e.

$$c_{2k} = (-1)^{\frac{\alpha}{2k}} \left( \frac{\alpha}{2k} \right). \quad (28)$$

So, the function (27) monotonically decreases. Hence, the ratio $f_\alpha(u)/g_\alpha(u)$ is monotonically increasing function of $u \in [0; 1]$. The inequality $v < u$ then gives that the right-hand side of (22) is strictly positive. In the interior of the domain (17), the function $F_\alpha(\tau, \varphi)$ increases with $\varphi$. On the boundary lines $\tau = 0$ and $\tau = \pi/4$, we respectively have $\partial F_\alpha/\partial \varphi = 0$ and $\partial F_\alpha/\partial \varphi \geq 0$. These points imply that the minimal and maximal values of $F_\alpha(\tau, \varphi)$ in the domain (17) are reached on the lines $\varphi = 0$ and $\varphi = \pi/4$, respectively.

To find the minimum, we substitute $\varphi = 0$ and obtain probabilities

$$p_\pm = \frac{1}{2} \pm \sin 2\tau, \quad q_\pm = \frac{1}{2} \pm \cos 2\tau, \quad (29)$$

whence $\Phi_\alpha(q) = 2^{1-\alpha}$. Differentiating with respect to $\tau$, we further write

$$\frac{\partial}{\partial \tau} F_\alpha(\tau, 0) = F_\alpha(\tau, 0) \left( \frac{1}{\Phi_\alpha(p)} \frac{\partial \Phi_\alpha(p)}{\partial \tau} + \frac{1}{\Phi_\alpha(r)} \frac{\partial \Phi_\alpha(r)}{\partial \tau} \right). \quad (30)$$

Using (29), we easily obtain

$$\frac{1}{\Phi_\alpha(p)} \frac{\partial \Phi_\alpha(p)}{\partial \tau} = \frac{\alpha}{2} \frac{p_{\alpha}^{\alpha-1} - p_{\alpha}^{\alpha-1}}{p_+^{\alpha} + p_-^{\alpha}} \cos 2\tau, \quad (31)$$

$$\frac{1}{\Phi_\alpha(r)} \frac{\partial \Phi_\alpha(r)}{\partial \tau} = \frac{\alpha}{2} \frac{r_{\alpha}^{\alpha-1} - r_{\alpha}^{\alpha-1}}{r_+^{\alpha} + r_-^{\alpha}} \sin 2\tau. \quad (32)$$
Denoting \( u = \cos 2\tau \) and \( v = \sin 2\tau \), we rewrite (30) as
\[
\frac{1}{F_\alpha(\tau, 0)} \frac{\partial}{\partial \tau} F_\alpha(\tau, 0) = 2\alpha uv \left( \frac{f_\alpha(u)}{g_\alpha(u)} - \frac{f_\alpha(v)}{g_\alpha(v)} \right). \tag{33}
\]
As \( u > v \) for \( \tau \in (0; \pi/8) \) and \( u < v \) for \( \tau \in (\pi/8; \pi/4) \), the derivative (33) is strictly positive in the former interval and strictly negative in the latter one. So, the minimal value of \( F_\alpha(\tau, 0) \) is reached at the end points of the interval \( \tau \in [0; \pi/4] \). In both the points, the function (15) is equal to
\[
F_\alpha(0, 0) = F_\alpha(\pi/4, 0) = 2^{2(1-\alpha)}. \tag{34}
\]
Combining this with (16) immediately leads to the inequality (18) for all pure states. This bound remains valid for mixed states due to concavity of the Rényi entropy of order \( \alpha \in (0; 1) \).

Let us prove conditions for equality. We first prove the claim for the case of pure measured state (11). In the domain (17), the function \( F_\alpha(\tau, \varphi) \) takes its minimum (34) only at the points \( \tau = \varphi = 0 \) and \( \tau = \pi/4, \varphi = 0 \). In both the points, one of the distributions \( \{p_\pm\}, \{q_\pm\}, \{r_\pm\} \) is deterministic and other two are equiprobable. This is the only situation, in which the minimum of \( F_\alpha(\tau, \varphi) \) holds. It is seen from (29) that the distribution \( \{q_\pm\} \) is equiprobable for the above two points. The total domain \((\tau, \varphi) : \tau \in [0; \pi/2], \varphi \in [0; 2\pi]\) for the state (11) contains also points, in which the distribution \( \{r_\pm\} \) is deterministic and other two are equiprobable. In any case, the minimum is reached only if one of the three distributions is deterministic. Clearly, this condition is sufficient as well. To saturate the inequality (18), the state \(|\psi\rangle\) should be an eigenstate of one of the observables \( \sigma_x, \sigma_y, \sigma_z \).

We will further prove that the inequality (18) cannot be saturated with impure states. Let the spectral decomposition of impure \( \rho \) be written as
\[
\rho = \lambda_+ |\psi_+\rangle \langle \psi_+ | + \lambda_- |\psi_-\rangle \langle \psi_- |. \tag{35}
\]
Here, the eigenstates are mutually orthogonal and the strictly positive eigenvalues obey the condition \( \lambda_+ + \lambda_- = 1 \). Since the entropy (1) is concave for \( \alpha \in (0; 1) \), we obtain
\[
\sum_{\nu=x,y,z} R_\alpha(\sigma_\nu | \rho) \geq \lambda_+ \sum_{\nu=x,y,z} R_\alpha(\sigma_\nu | \psi_+) + \lambda_- \sum_{\nu=x,y,z} R_\alpha(\sigma_\nu | \psi_-). \tag{36}
\]
If for any of the states \( |\psi_\pm\rangle \) the entropic sum does not reach the lower bound \(2\ln 2\), the left-hand-side of (36) does not reach this bound as well. Hence, the question is quite reduced to the case, when the matrix \( \rho \) is diagonal with respect to eigenbasis of either of the \( \{\sigma_x, \sigma_y, \sigma_z\} \). For definiteness, we assume that \( |\psi_\pm\rangle = |x_\pm\rangle \). Measuring each of the observables \( \sigma_y \) and \( \sigma_z \) then results in the equiprobable distribution, whence \( R_\alpha(\sigma_y | \rho) = R_\alpha(\sigma_z | \rho) = \ln 2 \). For the measurement of \( \sigma_x \), we have outcomes \pm 1 with probabilities \( \lambda_\pm \), respectively. For \( \alpha \in (0; 1) \), the first derivative of the function \( x \mapsto x^\alpha + (1-x)^\alpha \) is strictly positive for \( 0 < x < 1/2 \) and strictly negative for \( 1/2 < x < 1 \). Except for \( \lambda_0 = 0 \) and \( \lambda_+ = 1 \), we then have \( \lambda_0^\alpha + \lambda_+^\alpha = \Phi_\alpha(\lambda) > 1 \) and \( R_\alpha(\sigma_x | \rho) > 0 \). The latter implies that the sum of three entropies is strictly larger than \( 2\ln 2 \).

Theorem 1 provides a lower bound on the sum of three Rényi’s entropies for all \( \alpha \in (0; 1) \). This bound is tight in the sense that it is certainly reached with an eigenstate of one of the Pauli observables. Previously, the standard case \( \alpha = 1 \) has been considered in Ref. [24] and, as a particular case of the Tsallis formulation, also in Ref. [32]. Namely, we have the lower bound
\[
H_1(\sigma_x | \rho) + H_1(\sigma_y | \rho) + H_1(\sigma_z | \rho) \geq 2 \ln 2. \tag{37}
\]
For \( \alpha \in (0; 1) \), the inequality (18) could be derived from (37) due to the fact that the Rényi \( \alpha \)-entropy does not increase with \( \alpha \). In this way, however, we cannot resolve conditions for equality. The above method allows to formulate such conditions. Thus, we obtained tight uncertainty relations for the Pauli observables in terms of Rényi’s entropies of order \( \alpha \in (0; 1) \). A utility of entropic bounds with a parametric dependence was noted in [6]. In particular, this dependence allows to find more exactly the domain of acceptable values for unknown probabilities with respect to known ones. Some studies were devoted to uncertainty relations of state-dependent form. For example, the writers of [39] have derived a stronger bound, in which the right-hand side of (37) is added by the von Neumann entropy of \( \rho \). For mutually unbiased bases, state-dependent uncertainty relations in terms of the Shannon entropies were derived in [26]. Such uncertainty relations have been extended to both the Rényi and Tsallis formulations. A dependence of lower entropic bounds on a degree of impurity of the measured density matrix deserves further investigations.

4 Tight Upper Bounds in the Pure-State Case for \( \alpha \in (0; 1) \)

In this section, we will study upper bounds on the sum of three Rényi’s entropies for the Pauli observables. Certainly, such bounds essentially depend on a type of considered states. The completely mixed state is described by density operator \( \rho_* = \frac{1}{2} \), where \( \frac{1}{2} \) is the identity 2 × 2-matrix. Measuring each of the observables \( \sigma_x, \sigma_y, \sigma_z \) in this state will lead to the equiprobable distribution. With this distribution, the entropy (1) takes its maximal possible value \( \ln 2 \). For all \( \alpha > 0 \) and arbitrary \( \rho \), therefore, we have the upper bound
\[
\sum_{\nu=x,y,z} R_\alpha(\sigma_\nu | \rho) \leq \sum_{\nu=x,y,z} R_\alpha(\sigma_\nu | \rho_*) = 3 \ln 2. \tag{38}
\]
Further, we ask for upper entropic bounds in the case of pure states. Modifying the method of previous section, one will obtain tight bounds from above for real \( \alpha \in (0; 1) \). Following [32], we recall an intuitive reason that makes the result physically reasonable. Mathematically, we aim to
maximize the function (16) in the domain (17). According to the proof of Theorem 1, the maximum is reached on the line $\varphi = \pi/4$. Substituting the latter into the formulas (12), (13), and (14), we obtain the probabilities

$$p_\pm = q_\pm = \frac{1 \pm v}{2}, \quad r_\pm = \frac{1 \pm u}{2},$$

(39)

where $u = \cos 2\tau$, $v = \sin 2\tau/\sqrt{2}$. Here, the variables $u$ and $v$ satisfy the condition

$$u^2 + 2v^2 = 1.$$  

(40)

Due to (39), the distributions $\{p_\pm\}$ and $\{q_\pm\}$ should concur for maximizing the entropic sum in the case of pure states and considered values of $\alpha$. For impure states, the maximum (38) is reached only if the probability distributions are all equiprobable and herewith identical. In the case of distributions (39), we can assume the following.\[32\]

The maximum takes place, when the distribution $\{r_\pm\}$ also concurs with $\{p_\pm\} = \{q_\pm\}$, i.e. $u = v$. Combining the latter with (40) gives $u = v = 1/\sqrt{3}$. The Rényi $\alpha$-entropy of each of three probability distributions is equal to

$$\tilde{R}_\alpha = \frac{1}{1 - \alpha} \ln \left\{ \frac{1 + \sqrt{3}}{2} + \frac{1 - \sqrt{3}}{2} \right\}.$$  

(41)

This hint is fruitful in the case of Tsallis’ entropies.\[32\] It can be adapted for the Rényi case as well.

**Theorem 2** Let qubit state be described by the ket $|\psi\rangle$. For all $\alpha \in (0; 1)$, the entropic sum obeys

$$R_\alpha(\sigma_x|\psi\rangle) + R_\alpha(\sigma_y|\psi\rangle) + R_\alpha(\sigma_z|\psi\rangle) \leq 3 \tilde{R}_\alpha.$$  

(42)

Equality holds if and only if the three probability distributions are all, up to swapping, the pair $(1 \pm \sqrt{3})/2$.

**Proof** We again assume $\alpha \in (0; 1)$, since the case of the Shannon entropy was fully analyzed in theorem 5 of \[32\]. Using the probabilities (39), we rewrite the product (15) in the form

$$\tilde{F}_\alpha(u, v) = 2^{-3\alpha}g_\alpha(u)g_\alpha(v)^2.$$  

(43)

When $\tau \in [0; \pi/4]$, the variables $u$ and $v$ lie in the interval $[0; 1]$. The function (43) should be maximized in this interval under the condition (40). By (40), we have that $du/dv = -2v/u$. Differentiating (43) with respect to $v$, we then obtain

$$2^{-3\alpha} \left( \frac{2}{u} g_\alpha'(u) g_\alpha(v)^2 + 2g_\alpha(u)g_\alpha(v)g_\alpha'(v) \right)$$

$$= 2\alpha v \tilde{F}_\alpha(u, v) \left( \frac{f_\alpha(u)}{g_\alpha(u)} - \frac{f_\alpha(v)}{g_\alpha(v)} \right).$$  

(44)

We used here that $u^{-1}g_\alpha'(u) = -\alpha f_\alpha(u)$ due to (23) and (24). As was already shown, the ratio $f_\alpha(u)/g_\alpha(u)$ is monotonically increasing function of $u \in [0; 1]$. So, the derivative (44) vanishes only for $u = v$. Together with (40), we get $u = v = 1/\sqrt{3}$. For $v < 1/\sqrt{3} < u$, the derivative (44) is strictly positive. For $u < 1/\sqrt{3} < v$, the derivative (44) is strictly negative. At the point $u = v = 1/\sqrt{3}$, therefore, the function (43) reaches its conditional maximum

$$\max \tilde{F}_\alpha = \left( 2^{-\alpha}g_\alpha(1/\sqrt{3}) \right)^3.$$  

(45)

Combining this with (16) completes the proof. \[\blacksquare\]

**Fig. 1** The upper bounds $A(\alpha)$ and $B(\alpha)$ as functions of $\alpha$.

In the case of pure measured state, the statement of Theorem 2 gives tight upper bounds on the entropic sum for all real $\alpha \in (0; 1]$. Note that these bounds can be motivated by some plausible reasons. Let us consider the entropic value, which is averaged over the individual ones. It is also useful to rescale each entropy by its maximal possible value in $2$. Combining (18) and (42), we finally obtain

$$\frac{2}{3} \leq \frac{1}{3 \ln 2} \sum_{\nu=x,y,z} R_\alpha(\sigma_\nu|\psi\rangle) \leq B(\alpha) = \frac{\tilde{R}_\alpha}{\ln 2}.$$  

(46)

These bounds hold for $\alpha \in (0; 1]$ and arbitrary pure state $|\psi\rangle$ of qubit. The lower and upper bounds of (46) are both tight in the sense that they are reached under the certain conditions for equality. In the paper,\[32\] we have derived the bounds with Tsallis’ entropies. Denoting

$$\tilde{H}_\alpha = \frac{1}{1 - \alpha} \left\{ \frac{1 + \sqrt{3}}{2} + \frac{1 - \sqrt{3}}{2} \right\}^{-1},$$

(47)

for real $\alpha \in (0; 1]$ and integer $\alpha \geq 2$ we have

$$\frac{2}{3} \leq \frac{1}{3 \ln_\alpha(2)} \sum_{\nu=x,y,z} H_\alpha(\sigma_\nu|\psi\rangle) \leq A(\alpha) = \frac{\tilde{H}_\alpha}{\ln_\alpha(2)}.$$  

(48)

In both the Tsallis and Rényi cases, the lower bound on the rescaled average $\alpha$-entropy is equal to $2/3$. It is instructive to compare the corresponding upper bounds $A(\alpha)$ and $B(\alpha)$. In Fig. 1, these bounds are shown as functions of $\alpha \in (0; 1]$. The upper bounds $A(\alpha)$ and $B(\alpha)$ are very close to each other, though $A(\alpha) < B(\alpha)$ for all $\alpha \in (0; 1]$. The difference between them does not exceed 2.5% in a relative scale. Although the value $\alpha = 0$ itself is not considered, we have $A(\alpha) \rightarrow 1^-$ and $B(\alpha) \rightarrow 1^-$ in the limit $\alpha \rightarrow 0^+$. In the case $\alpha = 1$, we obtain $A(1) = B(1) \approx 0.744$. At the constant lower bound, the upper bounds $A(\alpha)$ and $B(\alpha)$ monotonically decrease.
with $\alpha$. So, the bands are reducing with growth of $\alpha$. A width of the corresponding band may be interpreted as a measure of sensitivity in quantifying the complementarity. From this viewpoint, there is no significant distinction between the Tsallis and Rényi formulations for $\alpha \in (0; 1)$. In particular problems, we often have specific reasons for choosing an appropriate entropic measure.

5 Conclusions

We have examined uncertainty and certainty relations for the Pauli observables in terms of the Rényi $\alpha$-entropies of order $\alpha \in (0; 1]$. These relations are respectively formulated as lower and upper bounds on the sum of three $\alpha$-entropies. The bounds are tight in the sense that they can certainly be saturated. Explicit conditions for equality are obtained as well. As the Rényi $\alpha$-entropy is a non-increasing function of $\alpha$, the lower bounds for $\alpha \in (0; 1]$ could be derived from the previous results on the Shannon entropies. However, conditions for equality cannot be obtained in this way. In the case of pure measured states, tight upper bounds on the sum of three Rényi’s entropies are further derived. We also discussed the interval, in which the rescaled average Rényi entropy of order $\alpha \in (0; 1]$ ranges in the pure-state case. This interval has been compared with the Tsallis formulation.

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