NEW APPROACH TO GENERALIZED MITTAG-LEFFLER FUNCTION VIA QUANTUM CALCULUS

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Abstract. We aim to introduce a new extension of Mittag-Leffler function via $q$-analogue and obtained their significant properties including integral representation, $q$-differentiation, $q$-Laplace transform, image formula under $q$-derivative operators. We also consider some particular cases to give the applications of our main results.

1. Overture

The Swedish mathematician Gösta Mittag-Leffler discovered a special function in 1903 (see, [3, 5]) defined as

$$E_\eta(u) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma(\eta m + 1) m!}, \ (\eta, u \in \mathbb{C}; \Re(\eta) > 0), \quad (1.1)$$

where $\Gamma(.)$ is a classical gamma function [9]. The special function defined in (1.1) is called Mittag Leffler function (MLf).

For the very first time, in 1905, A. Wiman [12] firstly proposed the generalization of the MLf $E_\eta(u)$ as

$$E_{\eta,\kappa}(u) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma(\eta m + \kappa) m!}, \ (\eta, \kappa \in \mathbb{C}; \Re(\eta) > 0, \Re(\kappa) > 0). \quad (1.2)$$

Subsequently, the generalized form of series (1.1) and (1.2) studied by Prabhakar [8] in 1971 as:

$$E^\sigma_{\eta,\kappa}(u) = \sum_{m=0}^{\infty} \frac{u^m(\sigma)_m}{\Gamma(\eta m + \kappa) m!}, \ (\eta, \kappa, \sigma \in \mathbb{C}; \Re(\eta) > 0, \Re(\kappa) > 0, \Re(\sigma) > 0), \quad (1.3)$$

where $(\sigma)_m = \frac{\Gamma(\sigma + m)}{\Gamma(\sigma)}$ denotes the Pochhammer symbol [9].

The Mittag-Leffler function plays a vital role in the solution of fractional order differential equations and fractional order integral equations. It has recently become a subject of rich interest in the field of fractional calculus and its applications and nowadays some mathematicians consider to refer the classical Mittag-Leffler function as the Queen Function in the Fractional Calculus. An enormous amount of research in the theory of Mittag-Leffler functions has

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been published in the literature. For detailed account of the various generalizations, properties and applications of the MLf readers may refer to the literatures \[1, 13, 14, 17, 18, 19, 20, 21\].

The \(q\)-calculus is the \(q\)-extension of the ordinary calculus. The theory of \(q\)-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problem, in finding solutions of the \(q\)-difference and \(q\)-integral equations and \(q\)-transform analysis.

In 2009, Mansoor [6] has proposed a new form of \(q\)-analogue of the Mittag-Leffler function is given as:

\[
e_{\eta, \kappa}(u; q) = \sum_{m=0}^{\infty} \frac{u^{m}}{\Gamma_q(\eta m + \kappa)} (|z| < (1 − q)^{-\alpha}), \tag{1.4}
\]

where \(\eta > 0, \kappa \in \mathbb{C}\).

Recently, Sharma and Jain [10] introduced the \(q\)-analogue of generalized MLf as given underneath:

\[
E_{\eta, \kappa, \sigma}^\sigma(u; q) = \sum_{m=0}^{\infty} \frac{(q^\sigma; q)_m}{(q; q)_m} \frac{u^{m}}{\Gamma_q(\eta m + \kappa)}, \tag{1.5}
\]

\((\eta, \kappa, \sigma \in \mathbb{C}; \Re(\eta) > 0, \Re(\kappa) > 0, \Re(\sigma) > 0, |q| < 1)\).

2. Prelude

In the theory of \(q\)-series (see [4]), for complex \(\lambda\) and \(0 < q < 1\), the \(q\)-shifted factorial is defined as follows:

\[
(\lambda; q)_m = \begin{cases} 
1 & ; m = 0, \\
(1 - \lambda)(1 - \lambda q) \ldots (1 - \lambda q^{m-1}) & ; m \in \mathbb{N}
\end{cases} \tag{2.1}
\]

which is equivalent to

\[
(\lambda; q)_m = \frac{(\lambda; q)_{\infty}}{((\lambda q^m); q)_{\infty}} \tag{2.2}
\]

and its extension naturally as:

\[
(\lambda; q)_\eta = \frac{(\lambda; q)_{\infty}}{((\lambda q^\eta); q)_{\infty}}, \; \eta \in \mathbb{C}, \tag{2.3}
\]

where the principal value of \(q^\eta\) is taken.

For \(s, t \in \mathbb{R}\) the \(q\)-analogue of the exponent \((s - t)^m\) is

\[
(s - t)^{(m)} = \begin{cases} 
1 & ; m = 0, \\
\prod_{i=0}^{m-1} (s - t q^i) & ; m \neq 0
\end{cases} \tag{2.4}
\]

and connected by the following relationship

\[
(s - t)^{(m)} = s^m(t/s; q)_{m}, \; (s \neq 0). \tag{2.5}
\]

Obviously, its expansion for \(\tau \in \mathbb{R}\) as

\[
(s - t)^{(m)} = s^m \frac{(t/s; q)_{\infty}}{(q^\tau t/s; q)_{\infty}}, \; (s; q)_{\tau} = \frac{(s; q)_{\infty}}{(sq^\tau; q)_{\infty}}. \tag{2.5}
\]
Note that
\[(s - t)^{\tau} = s^\tau(t/s; q)_\tau.\]
The $q$-analogue of binomial coefficient is defined for $s, t > 0$ as
\[
\binom{s}{t}_q = \frac{(q; q)_s}{(q; q)_t(q; q)_{s-t}} = \binom{s}{s-t}_q.
\] (2.6)

The definition can be generalized in the following way. For arbitrary complex $\tau$ we have
\[
\binom{\tau}{m}_q = \frac{(q^{-\tau}; q)_m(-1)^m q^{\tau m - \frac{m(m-1)}{2}}}{(q; q)_m} = \frac{\Gamma_q(\tau + 1)}{\Gamma_q(m+1)\Gamma_q(\tau - m + 1)}
\] (2.7)
where $\Gamma_q(u)$ is the $q$-gamma function.

The $q$-gamma and $q$-beta functions (4) are defined by
\[
\Gamma_q(u) = \frac{(q; q)_\infty}{(q^u; q)_\infty}(1 - q)^{1-u} = (1 - q)^{(u-1)(1-q)}^{1-u},
\] (2.8)
for $u \in \mathbb{R} \setminus \{0, -1, -2, -3, \ldots\}; |q| < 1$.

Clearly,
\[
\Gamma_q(u + 1) = [u]_q \Gamma_q(u)
\]
and
\[
B_q(\eta, \kappa) = \frac{\Gamma_q(\eta)\Gamma_q(\kappa)}{\Gamma_q(\eta + \kappa)} = \int_0^1 u^{\eta-1} \frac{(qu; q)_\infty}{(q^\eta u; q)_\infty} d_q u = \int_0^1 u^{\eta-1}(uq; q)_{\kappa-1} d_q u,
\] (2.9)
Further, the $\Gamma_q(u)$ satisfies the functional equation
\[
\Gamma_q(u + 1) = \frac{1 - qu}{1 - q}\Gamma_q(u)
\] (2.10)
Also, the $q$-difference operator and $q$-integration of a function $f(u)$ defined on a subset of $\mathbb{C}$ are given by [4] respectively.
\[
D_q f(u) = \frac{f(u) - f(uq)}{u(1-q)} (u \neq 0, q \neq 1), (D_q f)(0) = \lim_{u \rightarrow 0} (D_q f)(u)
\] (2.11)
and
\[
\int_0^u f(t)d(t; q) = u(1-q) \sum_{m=0}^\infty q^m f(uq^k).
\] (2.12)

3. Extended $q$-Mittag-Leffler function and their properties

In this section, we extend the definition (1.5) by introducing the following relation for $(q^c; q)_m$
\[
\frac{(q^c; q)_m}{(q^\sigma; q)_m} = \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)}.
\] (3.1)
Now, we define the extension of generalized Mittag-Leffler function (1.5) using above relation as:
\[
E^{(\sigma, c)}_{\eta, \kappa}(u; q) = \sum_{m=0}^\infty \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^c; q)_m}{(q; q)_m} \frac{u^m}{\Gamma_q(qm + \kappa)}
\] (3.2)
(\Re(c) > \Re(\sigma) > 0, |q| < 1),

where \(B_q(.)\) is the \(q\)-analog of beta function.

We enumerate the relations and particular cases of \(q\)-analogue of extended generalized Mittag-Leffler function with other special functions as given below.

(i) If we put \(c = 1\) in (3.2), we obtain

\[
E_{\eta,\kappa}^{(\sigma;1)}(u; q) = \sum_{m=0}^{\infty} \frac{(q^\sigma; q)_m}{(q; q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)} = E_{\eta,\kappa}^\sigma(u; q), \tag{3.3}
\]

where the function \(E_{\eta,\kappa}^\sigma(u; q)\) is the \(q\)-analogue of Mittag-Leffler function defined in (1.3).

(ii) Again, if we take \(\sigma = 1\) in (3.2), we get

\[
E_{\eta,\kappa}^{(1;\sigma)}(u; q) = \sum_{m=0}^{\infty} \frac{u^m}{\Gamma_q(\eta m + \kappa)} = e_{\eta,\kappa}(u; q), \tag{3.4}
\]

the function \(e_{\eta,\kappa}(u; q)\) can be termed as \(q\)-analogue of Mittag-Leffler function defined in (1.4).

(iii) If we consider \(\eta = \kappa = \sigma = 1\), in (3.2), we find

\[
E_{1,1}^{(1;1)}(u; q) = \sum_{m=0}^{\infty} \frac{(q^\sigma; q)_m}{(q; q)_m} u^m = (q^\sigma u; q)_\infty \frac{(q^\sigma; q)_\infty}{(q; q)_\infty} = 1 \phi_0(q^\sigma; -; q, u), \tag{3.5}
\]

where the function \(2 \phi_0(q^\sigma; -; q, u) = (1 - u)^{-\sigma}\) can be termed as \(q\)-binomial function.

(iv) On setting \(c = c + \sigma\), in (3.2), then similarly, we obtain \(q\)-analogue of Mittag-Leffler function \(E_{\eta,\kappa}^{(\sigma;\sigma)}(u; q)\) defined in (1.5).

4. CONVERGENCE OF \(E_{\eta,\kappa}^{(\sigma;\sigma)}(u; q)\)

**Theorem 4.1.** The \(q\)-analogue of the extended generalized Mittag-Leffler function defined by the summation formula (3.2) converges absolutely for \(|u| < (1 - q)^{-\eta}\) provided that \(0 < q < 1, \eta > 0, \Re(c) > \Re(\sigma), c, \sigma \in \mathbb{C}\).

**Proof.** Writing the summation formula (3.2) as \(E_{\eta,\kappa}^{(\sigma;\sigma)}(u; q) = \sum_{m=0}^{\infty} s_m\), and by applying ratio formula, we find

\[
\lim_{m \to \infty} \left| \frac{s_{m+1}}{s_m} \right| = \left| \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma + m, c - \sigma)} \right| \left| \frac{(q^\sigma; q)_m}{(q^\sigma; q)_m} \right| \left| \frac{(q; q)_m}{(q; q)_m} \right| \left| \frac{\Gamma(\eta m + \kappa)}{\Gamma(\eta m + \eta + \kappa)} \right| u
\]

\[
= \lim_{m \to \infty} \left| \frac{(q^\sigma; q)_\infty}{(q^\sigma; q)_\infty} \right| \left| \frac{(q^\sigma; q)_\infty}{(q^\sigma; q)_\infty} \right| \left| \frac{(q^\sigma; q)_\infty}{(q^\sigma; q)_\infty} \right| \left| \frac{(q^\sigma; q)_\infty}{(q^\sigma; q)_\infty} \right| \left| \frac{(1 - q)^{-\eta} u}{(1 - q)^{-\eta} u} \right|
\]

\[
= \lim_{m \to \infty} \left| \frac{(1 - q^\sigma m)}{(1 - q^\sigma m)} \right| \left| \frac{(1 - q^\sigma m)}{(1 - q^\sigma m)} \right| \left| \frac{(1 - q^\sigma m)}{(1 - q^\sigma m)} \right| \left| \frac{(1 - q^\sigma m)}{(1 - q^\sigma m)} \right| \left| \frac{(1 - q)^{-\eta} u}{(1 - q)^{-\eta} u} \right|
\]

\[
= |(1 - q)^{-\eta}| u, \quad \text{for} \quad 0 < |q| < 1. \tag{4.1}
\]

\qed
5. Recurrence Relations

Theorem 5.1. If $\eta, \kappa, \sigma \in \mathbb{C}$, $\Re(\eta) > 0, \Re(\kappa) > 0, \Re(\sigma) > 0$ and $\sigma \neq c$, then

$$E^{(\sigma;c)}_{\eta,\kappa}(u; q) = E^{(\sigma+1;c+1)}_{\eta,\kappa}(u; q) - u^c E^{(\sigma+1;c+1)}_{\eta,\kappa+c}(u; q).$$

Proof. By the definition (3.2), we have

$$E^{(\sigma;c)}_{\eta,\kappa}(u; q) = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^c q)_m}{(q; q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)};$$

$$= \frac{1}{\Gamma(\kappa)} \sum_{m=1}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(1 - q^c)(q^{c+1}; q)_{m-1}}{(q; q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)}.$$ 

Since $(1 - q^c) = (1 - q^{c+m}) - q^c(1 - q^m)$, the above equation reduces to

$$E^{(\sigma;c)}_{\eta,\kappa}(u; q) = \frac{1}{\Gamma(\kappa)} \sum_{m=1}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(1 - q^{c+m})(q^{c+1}; q)_{m-1}}{(q; q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)} - q^c \sum_{m=1}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(1 - q^m)(q^{c+1}; q)_{m-1}}{(q; q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)}.$$ 

On replacing $m$ with $m + 1$ in the second summation, it becomes

$$E^{(\sigma;c)}_{\eta,\kappa}(u; q) = \frac{1}{\Gamma(\kappa)} + \sum_{m=1}^{\infty} \frac{B_q(\sigma + m, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^{c+1}; q)_m}{(q; q)_m} \frac{u^m}{\Gamma_q(\eta m + \kappa)} - q^c \sum_{m=1}^{\infty} \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^{c+1}; q)_m}{(q; q)_m} \frac{u^{m+1}}{\Gamma_q(\eta m + \eta + \kappa)},$$

which leads to the required result $\square$.

6. Some Elementary Properties of Extended $q$-Mittag-Leffler Function

We begin with the underlying theorem, which shows the integral representation of extended $q$-Mittag-Leffler function:

Theorem 6.1. (Integral representation) For the extended $q$-Mittag-Leffler function, we have

$$E^{(\sigma;c)}_{\eta,\kappa}(u; q) = \frac{1}{B_q(\sigma, c - \sigma)} \int_0^1 t^{\sigma-1} \frac{(tq; q)_\infty}{(tq^{c-\sigma}; q)_\infty} E^{(c)}_{\eta,q}(tu; q) \, dq \, dt, \quad (6.1)$$

provided that, $\eta, \kappa, \sigma \in \mathbb{C}$, $\Re(\eta) > 0, \Re(\kappa) > 0, \Re(\sigma) > 0$ and $\sigma \neq c$.

Proof. By the definition of $q$-analogue of beta function, we can rewrite equation (3.2) as follows:

$$E^{(\sigma;c)}_{\eta,\kappa}(u; q) = \sum_{m=0}^{\infty} \left\{ \int_0^1 t^{\sigma+m-1} \frac{(tq; q)_\infty}{(tq^{c-\sigma}; q)_\infty} \, dt \right\} \frac{1}{B_q(\sigma, c - \sigma)} \times \frac{(q^c q)_m}{\Gamma_q(\eta m + \kappa)} \frac{u^m}{(q; q)_m}.$$
Proof. By considering the function

\[ f(u) = u^{\kappa - 1} E_{\eta,\kappa}^{(\sigma; c)} (\lambda u^n; q) \]

and using the definition (3.2), then, in view of (2.11), we obtain

\[
D_q^{m} [u^{\kappa - 1} E_{\eta,\kappa}^{(\sigma; c)} (\lambda u^n; q)] = u^{\kappa - m - 1} E_{\eta,\kappa-m}^{(\sigma; c)} (\lambda u^n; q).
\] (6.2)

Proof. By considering the function

\[ f(u) = u^{\kappa - 1} E_{\eta,\kappa}^{(\sigma; c)} (\lambda u^n; q) \]

and using the definition (3.2), then, in view of (2.11), we obtain

\[
D_q^{m} [u^{\kappa - 1} E_{\eta,\kappa}^{(\sigma; c)} (\lambda u^n; q)] = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^n; q)_m}{(q; q)_m} \frac{m! u^{m \eta + m + \kappa - 2}}{1 - q^{m \eta + m + \kappa - 2}}
\]

Since, according to the functional equation (2.10), the r.h.s of the above expression can be written as

\[
\sum_{m=0}^{\infty} \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^n; q)_m}{(q; q)_m} \frac{m! u^{m \eta + m + \kappa - 2}}{1 - q^{m \eta + m + \kappa - 2}} = u^{\kappa-2} E_{\eta,\kappa-1}^{(\sigma; c)} (\lambda u^n; q).
\]

Conclusively, we obtain

\[
D_q^{m} [u^{\kappa - 1} E_{\eta,\kappa}^{(\sigma; c)} (\lambda u^n; q)] = u^{\kappa-2} E_{\eta,\kappa-1}^{(\sigma; c)} (\lambda u^n; q).
\]

Iterating above result \( m - 1 \) times, we obtain the required result (6.2). \( \square \)

Theorem 6.3. Let \( \xi, \zeta, \sigma, \kappa \in \mathbb{R}; \mathbb{R}(\xi), \mathbb{R}(\kappa), \mathbb{R}(\sigma) > 0; \zeta \neq 0, -1, -2, \ldots \) then

\[
\int_{0}^{1} u^{\xi-1} (1 - qu)(\zeta-1) E_{\eta,\kappa}^{(\sigma; c)} (xu^{\rho}; q) du = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^n; q)_m}{(q; q)_m} \frac{x^m \Gamma_q(\xi + pm) \Gamma_q(\xi)}{\Gamma_q(\eta m + \kappa + pm) \Gamma_q(\zeta + \xi + pm)}.
\] (6.3)

In particular,

\[
\int_{0}^{1} u^{\xi-1} (1 - qu)(\zeta-1) E_{\eta,\kappa}^{(\sigma; c)} (xu^{\rho}; q) du = \Gamma_q(\zeta) E_{\eta,\kappa+\zeta}^{(\sigma; c)} (x; q)
\] (6.4)

Proof. By using the definition (3.2), the l.h.s of equation (6.3) can be written as

\[
\int_{0}^{1} u^{\xi-1} (1 - qu)(\zeta-1) \sum_{m=0}^{\infty} \frac{B_q(\sigma + m + 1, c - \sigma)}{B_q(\sigma, c - \sigma)} \frac{(q^n; q)_m}{(q; q)_m} u^{\eta m + \kappa m} du
\]

Interchanging the order of summation and integration and in view of equation (2.9), we obtain the required result (6.3).

In equation (6.3) replacing \( \rho = \eta, \kappa = \zeta \), then in view of equation (3.2), we can clearly obtain (6.4). \( \square \)
Theorem 6.4. (q-Laplace transform) For q-analogue of the extended generalized Laplace transform is defined as follows:

$$q_L \{E_{\eta,\kappa}^{(\sigma,c)}(xu^\rho; q)\} = \frac{1}{s} \sum_{m=0}^{\infty} \frac{B_q(\sigma + m, c - \sigma)(q^\sigma; q)_m}{B_q(\sigma, c - \sigma)(q; q)_m} \frac{\Gamma_q(1 + \rho m)}{\Gamma_q(\eta m + \kappa)}$$

(6.5)

provided that $\kappa, \sigma, s \in \mathbb{C}; \Re(\beta), \Re(\kappa), \Re(s) > 0$.

Proof. The q-Laplace transform of a suitable function is given by means of following q-integral [11]

$$q_L \{f(u)\} = \frac{1}{(1 - q)} \int_0^{s^{-1}} E_q^{q s u} f(u) d_q u.$$  

(6.6)

The q-extension of the exponential function [4] is given by

$$E_q^u = \phi_0(-, -; q, -u) = \sum_{m=0}^{\infty} q^{q^m} \frac{u^m}{(q; q)_m} = (-u; q)_\infty$$

(6.7)

and

$$e_q^u = 0 \phi_0(0, -; q, -u) = \sum_{m=0}^{\infty} \frac{u^m}{(q; q)_m} = \frac{1}{(u; q)_\infty}, \quad |u| < 1.$$  

(6.8)

By using the above q-exponential series and the q-integral equation (2.12), we can write equation (6.6) as

$$q_L \{f(u)\} = \frac{(q; q)_\infty}{s} \sum_{j=0}^{\infty} \frac{q^j f(s^{-1}q^j)}{(q; q)_j}.$$  

(6.9)

Using the definition [3.2] and the definition of q-Laplace transform, we obtain

$$q_L \{E_{\eta,\kappa}^{(\sigma,c)}(xu^\rho; q)\} = \frac{(q; q)_\infty}{s} \sum_{j=0}^{\infty} \frac{q^j}{(q; q)_j} \times \sum_{m=0}^{\infty} \frac{B_q(\sigma + m, c - \sigma)(q^\sigma; q)_m}{B_q(\sigma, c - \sigma)(q; q)_m} \frac{[u(s^{-1}q^j)^\rho]^m}{\Gamma_q(\eta m + \kappa)}.$$  

On interchanging the order of summation and writing the j series as $1 \phi_0$, which can be summed up as $\frac{1}{(q^{\rho m}; q)_\infty}$ and after some simplifications, we obtain the required result (6.5). □

7. Kober type fractional q- calculus operators

Agarwal [2] established Kober type fractional q-integral operator in the following manner

$$(I_q^{\nu,\mu} f)(u) = \frac{u^{-\nu-\mu}}{\Gamma_q(u)} \int_0^u (u - tq)^{\mu-1} t^\nu f(t) d_q t,$$  

(7.1)

where $\Re(\mu) > 0$. Also, Garg et al. [7] introduced Kober fractional q-derivative
operator given by

\[(D_q^{\nu,\mu})(u) = \prod_{i=0}^{m} \left( [\nu+j]_q + uq^{\nu+j}D_q \right) (P_q^{\nu+m+\mu})(u), \quad (7.2)\]

where \(m = [\Re(\mu)] + 1, m \in \mathbb{N}.\)

The image formula of the power function \(u^m\) under the above operators \([7]\) are given as:

\[I^{\nu,\mu}_{q}\{u^m\} = \frac{\Gamma_q(\nu + m + 1)}{\Gamma_q(\nu + \mu + m + 1)} u^m \quad (7.3)\]

\[D^{\nu,\mu}_{q}\{u^m\} = \frac{\Gamma_q(\nu + \mu + m + 1)}{\Gamma_q(\nu + m + 1)} u^m \quad (7.4)\]

**Theorem 7.1.** The underlying assumption holds true:

\[I^{\nu,\mu}_{q}\{E^{(\sigma,c)}(\nu;u)\} = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m,c - \sigma)(q^c;q)_m}{B_q(\sigma,c - \sigma)(q;q)_m} \frac{\Gamma_q(\nu + m + 1)}{\Gamma_q(\nu + \mu + m + 1)} \frac{u^m}{\Gamma_q(\eta m + \kappa)} \quad (7.5)\]

particularly,

\[I^{\nu,\mu}_{q}\{E^{(\nu+1;1)}(\nu;u)\} = \frac{\Gamma_q(\nu + 1)}{\Gamma_q(\nu + \mu + 1)} E^{(\nu+1;1)}_{\eta,c}(u;\eta) \quad (7.6)\]

provided that if \(\eta, c > 0, \kappa, \sigma, u \in \mathbb{C}; \Re(\kappa), \Re(\sigma) > 0.\)

**Proof.** The proof of \((7.5)\) can easily be obtained by making use of the definition \((3.2)\) and the result \((7.3)\).

Now, on setting \(\sigma = \nu + \mu\) in the definition \((3.2)\), we obtain the result \((7.6)\). \(\square\)

**Theorem 7.2.** The underlying assumption holds true:

\[D^{\nu,\mu}_{q}\{E^{(\sigma,c)}(\nu;u)\} = \sum_{m=0}^{\infty} \frac{B_q(\sigma + m,c - \sigma)(q^c;q)_m}{B_q(\sigma,c - \sigma)(q;q)_m} \frac{\Gamma_q(\nu + m + 1)}{\Gamma_q(\nu + \mu + m + 1)} \frac{u^m}{\Gamma_q(\eta m + \kappa)} \quad (7.7)\]

particularly,

\[D^{\nu,\mu}_{q}\{E^{(\nu+1;1)}(\nu;u)\} = \frac{\Gamma_q(\nu + 1)}{\Gamma_q(\nu + \mu + 1)} E^{(\nu+1;1)}_{\eta,c}(u;\eta) \quad (7.8)\]

provided that if \(\eta, c > 0, \kappa, \sigma, u \in \mathbb{C}; \Re(\kappa), \Re(\sigma) > 0.\)

**Proof.** The proof of \((7.7)\) can easily be obtained by making use of the definition \((3.2)\) and the result \((7.4)\). Similarly, on setting \(\sigma = \nu + 1\) in the definition \((3.2)\), we obtain the result \((7.8)\). \(\square\)

**References**

[1] Agarwal, P., Chand, M. and Jain, S. Certain Integrals Involving Generalized Mittag-Leffler Functions. Proceedings of the National Academy of Sciences, India - Section A (2015)85(3).

[2] Agarwal, R.P., Certain fractional \(q\)-integrals and \(q\)-derivatives. Proc. Camb. Phil. Soc. 66 (1969), 365-370.
[3] Mittag-Leffler, G., Sur la nouvelle fonction $E_\eta(u)$, C. R. Acad. Sci. Paris 137 (1903), 554-558.

[4] Gasper, G., Rahman, M., Basic Hypergeometric Series, 2nd ed, Encyclopedia of Mathematics and its Applications 96, Cambridge University Press, Cambridge, 2004.

[5] Mittag-Leffler, G., Une generalisation de l’integrale de Laplace-Abel, Comptes Rendus de l’Academie des Sciences Serie 137 (1903), 537-539.

[6] Mansour, Z.S.I., Linear sequential q-difference equations of fractional order, Fract. Calc. Appl. Anal., 12(2) (2009), 159-178.

[7] Garg, M., Chanchlani, L., Kober fractional q-derivative operators, Le Matematiche, 66(1) (2011), 13-26.

[8] Prabhakar, T.R. A singular equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. 19 (1971), No.4, 7-15.

[9] Rainville, E. D. Special Functions, Macmillan Company, New York, (1960); Reprinted by Chelsea Publ.co. Bronx, New York, (1971).

[10] Sharma, S.K. and Jain, R. On Some properties of generalized q-Mittag-Leffler function, Mathematica Aeterna, 4 (2014) No. 6, 613-619.

[11] Hahn, W., Beitrage Zur Theorie der Heineschen Reihen, die 24 Integrale der hypergeometrischen $q$-Differenzengleichung, das $q$-Analogon der Laplace Transformation. Math. Nachr., 2 (1949), 340-379.

[12] Wiman, A. Über de fundamental satz in der theorie der funktionen $E_\eta(u)$, Acta Math. 29 (1905), 191-201.

[13] Kilbas, A.A., Saigo, M. and Saxena, R.K., Generalized Mittag-Leffler function and fractional calculus operators, Integral Transform. Spec. Funct., 15(1) (2004), 31-49.

[14] Podlubny, I., Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to methods of their solution and some of their Applications, Academic Press, San Diego, C.A. 1999.

[15] Rajković, P.M., Marinković, S.D. and Stanković, M.S., On q-analogues of Caputo derivative and Mittag-Leffler function, Fract. Calc. Appl. Anal., 10(4) (2007), 359-374.

[16] Rajković, P.M., Marinković, S.D. and Stanković, A generalization of the concept of q-fractional integrals, Acta Mathematica Sinica, 25(10) (2009), 1635-1646.

[17] Saxena, R.K., Kalla, S.L. and Saxena, R., Multivariate analogue of generalized Mittag-Leffler function, Integral Transforms. Spec. Funct., 22(7) (2011), 533-548.

[18] Soubhia, A.L., Camargo, R.F., Oliveira, E.C. de and Jr J.V., Theorem for series in threeparameter Mittag-Leffler function, Fract. Calc. Appl. Anal., 13(1) (2010), 9-20.

[19] Podlubny, I., Geometric and physical interpretation of fractional integration and fractional differentiation. Fract. Calc. Appl. Anal. 5(4), 367-386 (2002).

[20] Mainardi, F., Goreno, R., On Mittag-Leffer-type functions in fractional evolution processes. J. Comput. Appl. Math. 118(1-2), 283-299 (2000).

[21] Kilbas, A. A., Fractional calculus of the generalized Wright function. Fract. Calc. Appl. Anal. 8(2) (2005), 113-126.

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