Single-facility Weber Location Problem based on the Lift Metric

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Abstract

The continuous single-facility min-sum Weber location problem based upon the lift metric is investigated. An effective algorithm is developed for its solution. Implementation for both the discrete and continuous location problems is developed in the programming package Mathematica.

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1 Introduction

Location problems represent a special class of optimization tasks, where coordinate of locations and distances between them are main parameters. In the general case, the task of location problem is to define positions of some new facilities from the actual space in which are already placed some other relevant objects (points). New facilities are centers that provide services and called suppliers. Existing facilities are the service users or clients, and called customers. Location problems occur frequently in real life. Many systems in the public and private sectors are characterized by facilities that provide homogeneous services at their locations to a given set of fixed points or customers. Examples of such facilities include warehouse location, positioning a computer and communication units, locating hospitals, police stations, locating fire stations in a city, locating base stations in wireless networks.

Different classifications of the location problems are known. The classification scheme from \textsuperscript{[14]} assumes five positions.

In the present article we pay attention to the selection of the distance function as the characterization criterion of the location problem. The distance between two points is the length of the shortest path connecting them. The metric by which the (generalized) distance between two points is measured may be different in various instances \textsuperscript{[2]}. In the calculating of distance between two points, the most common distance metrics in a continuous space are those known as the class of $l_p$ distance metrics, primarily rectangular ($l_1$), Euclidean ($l_2$) and Chebyshev ($l_{\infty}$) metric. Detailed explanation of various metrics one can find in Dictionary of distances \textsuperscript{[6]}. Many factors affect on the process of metrics choosing. The most important

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factor is the nature of the problem. For example, if it is possible to move rectilinearly between two points, the distance between them is exactly given by the Euclidean (or straight-line distance) metric. On the other hand, in the cities where streets intersect under the right angle mainly, the distance between two points will be the best approximated using the rectangular metric (also known as the Manhattan, “city block” distance, the right-angle distance metric or taxicab distance). Measures of distances in chess are a characteristic example. The distance between squares on the chessboard for rooks is measured in Manhattan distance; kings and queens use Chebyshev distance, and bishops use the Manhattan distance.

We emphasize the next main contribution of our paper.

The Weber location problem (also called the Fermat-Weber problem) is a basic model in the location theory which has received significant attention in the scientific literature. For a detailed review see, for example, [32]. The paper [24] investigated a reformulation of the unconstrained form of the classical Weber problem into an unconstrained minimum norm problem. The classical Weber problem is established with the Euclidean norm underlying in the definition of the distance function. But, other measures, principally \( l_p \) norms, also play an important role in the theory and practice of location problems. The norms are arbitrary, in general. The most popular method to solve the Weber problem with Euclidean distances is given by a one-point iterative procedure which was first proposed by Weiszfeld [31]. The procedure is readily generalized to \( l_p \) distances (see, for example, [19], Ch. 2). Solution of the continuous Weber problem in \( l_1 \) distance is described in [7]. The three-dimensional Fermat-Weber facility location problem with Tchebychev distance is investigated in [25]. The Weber location problem with squared Euclidean distances is considered in [7]; the same problem under the assumption that the weights are selected from a given set of intervals at any point, is studied in [9].

The \( l_p \) norms have received the most attention from location analysts. But, many other types of distances have been exploited in the facility location problem. A review of exploited metrics is presented in [7]:

- central metrics [26],
- distance functions based on altered norms [20, 21],
- weighted one-infinity norms [30],
- mixed norms [15],
- block and round norms [27],
- mixed gauges [10],
- asymptotic distances [16],
- weighted sums of order \( p \) [2, 29].

In the present article we solve the Weber problem in the plane, under the assumption that the distance is measured by the lift metric.

The paper is organized as follows. Some basic definitions and algorithms are restated in the second section. In the third section we present an effective algorithm for the solution of the single-facility continuous Weber problem, assuming that distances are measured by the lift metric.
2 Preliminaries

The lift metric or the raspberry picker metric in the plane $\mathbb{R}^2$ is defined by

$$L(A, B) = \begin{cases} 
|x_A^1 - x_B^1|, & x_A^2 = x_B^2 \\
|x_A^1| + |x_A^2 - x_B^2| + |x_B^1|, & x_A^2 \neq x_B^2
\end{cases}$$

(2.1)

where $A(x_A^1, x_A^2)$ and $B(x_B^1, x_B^2)$ are given points. It can be defined as the minimum Euclidean length of all admissible connecting curves between two given points, where a curve is called admissible if it consists of only segments of straight lines parallel to $x$-axis, and of segments of $y$-axis (see, for example [6]). Therefore, under the assumption $x_A^2 \neq x_B^2$ the distance between two points $A$ and $B$ in the lift metric equals the sum of lengths $AA'$, $A'B'$ and $B'B$, where $A'$ and $B'$ are orthogonal projections of the points $A$ and $B$ to the $y$-axis, respectively (Figure 1,Left). In the opposite case, $x_A^2 = x_B^2$, the distance between $A$ and $B$ is simply the length of the segment $AB$ (Figure 1,Right).

![Figure 1. Left) The case $x_A^2 \neq x_B^2$ Right) The case $x_A^2 = x_B^2$](image)

This distance is appropriate for usage in cities which have one main street (corresponding to the $y$-axis), and the other side streets are normal to it (Figure 2). We are observed that in the main city of Zakynthos island in Greek-Zakynthos, the streets are deployed on this way. Similar situation also occurs in tier buildings where the lift (in the role of $y$-axis) connects tiers.

The 2-dimensional continuous Weber location problem can be briefly restated as follows (see, for example [8, 32]). Let $m$ demand centers $A_1, \ldots, A_m$ be given in the plane $\mathbb{R}^2$ (locations of given customers), where $A_i(a_i^1, a_i^2)$, $i = 1, \ldots, m$. It is necessarily to find a new point $X(x_1, x_2) \in \mathbb{R}^2$ which has minimal sum of weighted distances with respect to given points. Therefore, one needs to solve the unconstrained optimization problem (single-facility min-sum problem), where it is necessary to minimize the sum

$$\min_X f(X) = \sum_{i=1}^{m} w_i \cdot d(A_i, X).$$

(2.2)

The real quantity $w_i$ is a positive weighted coefficient of the point $A_i$. Essentially, the weight $w_i$ converts the distance $d(A_i, B_k)$ into a cost of serving the demand of customer $i$ considerate to $k$th offered facility location.

For the sake of completeness, we restate well-known method which gives solution of the Weber problem (2.2) in the case when the underlying distance function is defined by the $l_1$
metric, which means
\[ l_1(A_i, X) = |x_1 - a^i_1| + |x_2 - a^i_2|. \]

Then the goal function \( f(X) \) divides in two sums:
\[
\begin{align*}
    f_1(x_1) &= \sum_{i=1}^{m} w_i |x_1 - a^i_1|, \\
    f_2(x_2) &= \sum_{i=1}^{m} w_i |x_2 - a^i_2|.
\end{align*}
\]

Therefore, the initial optimization problem splits into two independent optimization tasks. It is sufficient to consider the optimization problem
\[\min_x f(x) = \sum_{i=1}^{m} w_i |x - a^i|. \tag{2.3}\]

We restate well-known algorithm for solving the problem (2.3) (one can find it in [23, 28]). The first step in this procedure is optional, but it can be used to accelerate the remaining two steps.

**Algorithm 2.1** Solve the single-facility continuous Weber location problem.

**Require:** Real quantities \( a^1, \ldots, a^m \).

1: Step I. For each subset of identical elements \( a^{i_1} = a^{i_2} = \cdots = a^{i_j} \) perform the following activities: put \( w_{i_1} = w_{i_2} = \cdots = w_{i_j} \), eliminate multiple elements \( a^{i_2}, \ldots, a^{i_j} \) as well as corresponding weights \( w_{i_2}, \ldots, w_{i_j} \) and later perform appropriate shifting of the indices of residual elements \( a^j \) and their weights \( w_j \).

2: Step II. If Step I is applied, denote by \( q \) the cardinal number of different elements in the set \( \{a^1, \ldots, a^m\} \); otherwise, use \( q = m \). Sort coordinates \( a^i, i = 1, \ldots, q \) in non-descending order. Let the sorted sequence of coordinates is
\[ a^{i_1'} \leq a^{i_2'} \leq \cdots \leq a^{i_q'}. \]

Rearrange the weighting coefficients \( \{w_1, \ldots, w_q\} \rightarrow \{w_{1'}, \ldots, w_{q'}\} \) applying identical replacements on the weights. For the sake of simplicity, let us denote partial sums of the array \( \{w_{1'}, \ldots, w_{q'}\} \) by
\[
S_2[0] = 0, \quad S_2[k] = \sum_{i=1}^{k} w_{i'}, \quad 1 \leq k \leq q. \tag{2.4}
\]

3: Step III. There are two possibilities, denoted by \( P_1 \) and \( P_2 \).

\( P_1 \). If the condition
\[ S_2[k^* - 1] < \frac{1}{2} S_2[q] < S_2[k^*] \tag{2.5} \]

is satisfied for some \( k^* \in \{1, \ldots, q\} \), then we end the algorithm without any possible solution. Indeed, formal solution \( x = a^{k^*} \) is eliminated according to assumption (3.9) in this case.

\( P_2 \). If the condition
\[ \frac{1}{2} S_2[q] = S_2[k^*] \tag{2.6} \]

holds for some \( k^* \in \{1, \ldots, q\} \), then the solution is multiple, i.e. the searched coordinate \( x_2 \) can to have any value from the interval \( (a^{k^*}, a^{k^*+1}) \). As agreed, we use the midpoint value \( x = (a^{k^*} + a^{k^*+1})/2 \).
3 Continuous Weber problem and lift metric

In the sequel we solve the single-facility min-sum Weber problem \( \text{(2.2)} \) applying the lift metric \( \text{(2.1)} \). Therefore, the distance function is defined by

\[
d(A_i, X) = L(A_i, X) \left\{ \frac{|x_1 - a_1^i|}{|x_1| + |x_2 - a_2^i| + |a_1^i|}, \quad x_2 = a_2^i, \quad x_1 \neq a_1^i, \quad i = 1, \ldots, m. \right.
\]  

(3.1)

Two major steps (denoted as \textbf{Step 1} and \textbf{Step 2}) are separated in our algorithm, as in the following.

\textbf{Step 1.} Generate the list \( X \) of permissible solutions of the problem. Its initial value is the empty set \( X = \emptyset \). Two different procedures are separated during the construction of the set \( X \), in accordance with the definition \( (3.1) \).

\textbf{Procedure 1.} Let us consider the quotient set \( S/\sim \) of the set \( S = \{a_1^1, \ldots, a_m^n\} \) into the equivalence classes \( S = \{S_1 = [a_1^1], \ldots, S_d = [a_1^d]\} \), where the class \( S_j \) contains elements from \( S \) whose values are \( a_1^i \) for each \( j = 1, \ldots, d \). For each \( j = 1, \ldots, d \) seek the second coordinate of the optimal point \( X(x_1, x_2) \) in the form

\[
x_2 \in S_j \Leftrightarrow x_2 = a_2^i.
\]  

(3.2)

So, as we know value of the coordinate \( x_2 (x_2 = a_2^i) \), it is necessary to determine value of the coordinate \( x_1 \). Denote by \( Q_j \) the set of indices corresponding to points whose second coordinates are contained in the set \( S_j \). According to \( (3.1) \) and \( (3.2) \), the objective function \( f(X) \) consists of two separated sums

\[
f(X) = \sum_{i \in Q_j} w_i (|x_1| + |x_2 - a_2^i| + |a_1^i|) + \sum_{i \in Q_j} w_i |x_1 - a_1^i| 
\]  

(3.3)

Taking into account \( x_2 = a_2^i \) and grouping the first term in the first sum with the second sum, we obtain

\[
f(X) = \sum_{i \notin Q_j} w_i \left(|a_2^i - a_2^i| + |a_1^i|\right) + \sum_{i=1}^m w_i |x_1 - a_1^\beta(i)|, 
\]  

(3.4)

where

\[
a_1^\beta(i) = \begin{cases} a_1^i, & \beta(i) = i_p \in I \\ 0, & \beta(i) \notin I. \end{cases} 
\]  

(3.5)

As the first sum in the expression \( (3.4) \) is constant, the problem is reduced on determining the minimum of the function

\[
f_1(x_1) = \sum_{i=1}^m w_i |x_1 - a_1^\beta(i)|. 
\]  

(3.6)

We apply Algorithm \( \text{(2.1)} \) in adapted form for our specific situation \( (3.5), (3.6) \). That process consists of three major steps.

\textbf{Step 1.} For each subset of identical elements \( a_1^\beta(i_1) = a_1^\beta(i_2) = \ldots = a_1^\beta(i_j) \) perform the following: put \( w_i = w_{i_1} + w_{i_2} + \ldots + w_{i_j} \), eliminate multiple elements \( a_1^\beta(i_2), \ldots, a_1^\beta(i_j) \) as well as their weights \( w_{i_2, \ldots, w_{i_j}} \) and then perform appropriate renumeration of the indices of the remainder elements \( a_1^\beta(i) \) and their weights \( w_j \).
Step II. If Step I is applied, denote by \( p \) the cardinal number of different elements in the set \( \{a_1^{\beta(1)}, \ldots, a_1^{\beta(m)}\} \); otherwise, use \( p = m \). Sort coordinates \( a_1^{\beta(1)}, \ldots, a_1^{\beta(p)} \) in non-descending array. Furthermore we suppose that the ordered sequence is 

\[
a_1^{\beta'} \leq a_1^{\beta''} \leq \cdots \leq a_1^{\beta^p}.
\]

Rearrange the corresponding weighting coefficients \( w_1, \ldots, w_p \) analogously in the sequence 

\[
w_1', w_2', \ldots, w_p'.
\]

For the sake of simplicity, let us denote partial sums of the array \( \{w_1', \ldots, w_p'\} \) by 

\[
S_1[0] = 0, \quad S_1[k] = \sum_{i=1}^{k} w_i', \quad 1 \leq k \leq p.
\]

Step III. There are two possible cases capable to produce permissible minimizers for \( f_1 \), denoted by \( C_1 \) and \( C_2 \).

\( C_1 \). If the inequalities

\[
S_1[k' - 1] < \frac{1}{2}S_1[p] < S_1[k'],
\]

are satisfied for some \( k' \in \{1, \ldots, p\} \), then the searched coordinate is \( x_1 = a_1^{k'} \). Later, we use \( X(x_1, x_2) \) as the possible optimal point: \( X = X \cup \{(a_1^{k'}, a_2^i)\} \).

\( C_2 \). If the condition

\[
\frac{1}{2}S_1[p] = S_1[k']
\]

is satisfied for some \( k' \in \{1, \ldots, p\} \), then the solution is multiple, i.e. the searched coordinate \( x_1 \) can to have any value from the interval \([a_1^{k'} + a_1^{k'+1}]/2\). Thus, we found the additional possible solution of the starting problem \((2.2)\), which implies: \( X = X \cup \{((a_1^{k'} + a_1^{k'+1})/2, a_2^i)\} \).

Procedure 2. Compute \( x_2 \) under the assumption

\[
x_2 \notin S \iff x_2 \neq a_2^i \quad \text{for each} \quad i \in \{1, \ldots, m\}
\]

(under the assumptions opposite with respect to \((3.2)\)), the function \( f(X) \) is reduced to

\[
f(X) = \sum_{i=1}^{m} w_i \left(|x_1| + |x_2 - a_2^i| + |a_1^1|\right).
\]

It is necessary to minimize that function. Since the third term \( w_i|a_1^i| \) in the function \( f(X) \) defined in \((3.10)\) is constant, one needs to minimize the next two objectives:

\[
\min_{x_1} f_1(x_1) = \sum_{i=1}^{m} w_i |x_1| \quad \text{(3.11)}
\]

\[
\min_{x_2} f_2(x_2) = \sum_{i=1}^{m} w_i |x_2 - a_2^i|.
\]

Thus, solving the problem \((3.10)\) with two variables was reduced to solving two independent tasks of unconstrained optimization \((3.11)\) and \((3.12)\) with one variable \((x_1 \text{ and } x_2, \text{ respectively})\).
Solution of the optimization problem (3.11) is evidently $x_1 = 0$. The optimization problem (3.12) is the classical continuous Weber location model with underlying $l_1$ metric, where only the additional assumption (3.9) is imposed. Therefore, in order to find optimal value for $x_2$ it suffices to apply Algorithm 2.1 assuming that the input sequence is $a^1 = a^1_1, \ldots , a^m = a^m_2$ and taking into account conditions (3.9).

Step I. For each subset of identical elements $a^{i_1}_2 = a^{i_2}_2 = \cdots = a^{i_j}_2$ perform the following activities: put $w_{i_1} = w_{i_1} + w_{i_2} + \cdots + w_{i_j}$, eliminate multiple elements $a^{i_2}_2, \ldots , a^{i_j}_2$ as well as corresponding weights $w_{i_2}, \ldots , w_{i_j}$ and later perform appropriate shifting of the indices of residual elements $a^{j}_2$ and their weights $w_j$.

Step II. If Step I is applied, denote by $q$ the cardinal number of different elements in the set $\{a^{1}_2, \ldots , a^{m}_2\}$; otherwise, use $q = m$. Sort coordinates $a^{j}_2$, $i = 1, \ldots , q$ in non-descending order. Let the sorted sequence of coordinates is

$$a^{1'}_2 \leq a^{2'}_2 \leq \cdots \leq a^{q'}_2.$$ 

Rearrange the weighting coefficients $\{w_1, \ldots , w_q\} \rightarrow \{w_{1'}, \ldots , w_{q'}\}$ applying identical replacements on the weights. Subsequently, generate the partial sums $S^{q}_{2,i}$, $i = 0, \ldots , q$ of the array $\{w_{1'}, \ldots , w_{q'}\}$ as in (2.4).

Step III. There are two possibilities, denoted by $P_1$ and $P_2$.

$P_1$. If the condition (2.5) is satisfied for some $k^* \in \{1, \ldots , q\}$, then the algorithm is finished without any solution. Indeed, the formal solution $x_2 = a^{k^*}_2$ is eliminated according to assumption (3.9), actual for this case.

$P_2$. If the condition (2.6) holds for some $k^* \in \{1, \ldots , q\}$, then the solution is multiple, i.e. the searched coordinate $x_2$ can to have any value from the interval $(a_2^{k^*}, a_2^{k^*+1})$. We use the midpoint value $x_2 = (a_2^{k^*} + a_2^{k^*+1})/2$, so that the possible optimal point is $X(0, x_2)$. Place the point $X$ at the end of the list $\mathcal{X}$ by $\mathcal{X} = \mathcal{X} \cup \{(0, (a_2^{k^*} + a_2^{k^*+1})/2))\}$.

Step 2. Thus, we got one or more permissible solutions of the starting problem (2.2). For all obtained values $X$ from $\mathcal{X}$ we determine the values of the function $f(X)$ defined in (3.10). Solution of the Weber problem will be the point $X^*(x_1^*, x_2^*)$ for which the function $f(X)$ has a minimal value. Actually in this step we are solving generated discrete location problem, where the set $\mathcal{X}$ contains in advance defined feasible locations of the supplier.

Let $X_1, \ldots , X_r$ be $r$ locations on which it is possible to set a new desired object (supplier). The sum of weighted distances from the permissible location $X_k$, $k \in \{1, \ldots , r\}$ of the supplier to the customers is equal to

$$W_k = \sum_{i=1}^{m} w_i \cdot L(A_i, X_k). \quad (3.13)$$

The task is to determine the location $B_{k^*}$ for which the sum of weighted distances is minimal, i.e.

$$W_{k^*} = \min \{W_k \mid 1 \leq k \leq r\}.$$ 

In accordance with the previous considerations, we state the following general algorithm.

**Algorithm 3.2** Solution of the single-facility min-sum Weber problem in the lift metric.

**Require:** List $lp = \{(a^{1}_2, a^{1'}_2), \ldots , (a^{m}_2, a^{m'}_2)\}$ and the list of weights $lt = \{w_1, \ldots , w_m\}$.

1. **Step 1:** Form the quotient set $S = \{a^1_2, \ldots , a^m_2\}$ in the form $\{S_1, \ldots , S_d\}$, where each equivalence class $S_j$ contains identical elements from $S$ with the value $a^{ij}_2$. 

2. **Step 1:** Generate the list \( X \) applying the procedure included into the possibilities \( C_1, C_2 \) (included in Procedure 1.) to all distinctive values \( a_{ij} \) of the set \( S \), i.e. using \( x_2 = a_{ij} \), \( j = 1, \ldots, d \).

3. **Step 1:** Extend the list \( X \) applying the method defined in the case \( P_2 \) (included in Procedure 2.).

4. **Step 2:** Solve the discrete location problem using given locations \( lp \), discrete set \( X \) of possible solutions and the weights \( lt \).

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**Example 3.1.** Solve Weber problem using the specified algorithm with the next data:

\[
A_1(4, 4), \ w_1 = 4, \ A_2(3, 1), \ w_2 = 1, \ A_3(6, 4), \ w_3 = 2, \ A_4(6, 2), \ w_4 = 3.
\]

We have \( S = \{4, 1, 4, 2\} \). The quotient set of \( S \) is defined as \( S_1 = [4], S_2 = [1], S_3 = [2] \). Therefore, it is necessary to consider three possibilities for the cases \( C_1 \) and \( C_2 \).

1. Let be \( x_2 = 4 \). Then the function \( f(X) \) has the following form:

\[
f(x) = w_2(|x_1| - a_1^2) + w_4(|x_1 + |x_2 - a_2^2| + |a_1^2|) + w_1|x_1 - a_1^1| + w_3|x_1 - a_1^3|.
\]

According to the constant value \( x_2 = 4 \) of the coordinate \( x_2 \), the function \( f \) just depend on \( x_1 \), so we can consider the next function

\[
f_1(x_1) = w_1|x_1 - a_1^1| + w_2|x_1 - 0| + w_3|x_1 - a_1^3| + w_4|x_1 - 0|
\]

\[
= \sum_{i=1}^{4} w_i |x_1 - a_1^{\beta(i)}|
\]

where \( a_1^{\beta(1)} = a_1^4 = 4, a_1^{\beta(3)} = a_1^3 = 6, a_1^{\beta(2)} = a_1^{\beta(4)} = 0 \).

Let us sort the coordinates \( a_1^{\beta(i)} \rightarrow a_1^1 \) and rearrange corresponding weights \( w_i \rightarrow w_i' \), using the same replacements:

| \( a_1^{\beta(i)} \) | 0 | 0 | 4 | 6 |
|---|---|---|---|---|
| \( w_i' \) | 1 | 3 | 4 | 2 |

Let \( k = 1 \). Then conditions (3.7) are satisfied for \( k = 2, 3 \), so \( x_1 = a_1^0 = 4 \). Therefore, one possible solution is \( X_1(4, 4) \).

In the case when **Step 1** is applied, data from Table 1 reduce to

| \( a_1^{\beta(i)} \) | 0 | 4 | 6 |
|---|---|---|---|
| \( w_i' \) | 4 | 4 | 2 |

Then conditions (3.7) are satisfied for \( k = 2 \), so that the same possible solution is generated.

The list of permissible solutions is now equal to \( X = \{X_1\} \).

2. In this case it is assumed \( x_2 = 1 \). Now the function \( f_1(x_1) \) looks like:

\[
f_1(x_1) = w_1|x_1 - 0| + w_2|x_1 - 3| + w_3|x_1 - 0| + w_4|x_1 - 0|.
\]

On the similar procedure as in the case 1, we get the table:

| \( a_1^{\beta(i)} \) | 0 | 4 | 6 |
|---|---|---|---|
| \( w_i' \) | 1 | 2 | 3 |

| \( a_1^{\beta(i)} \) | 4 | 8 | 10 |
|---|---|---|---|
| \( k \) | 1 | 2 | 3 |
| \( S_1[k] = \sum_{i=1}^{4} w_i' \) | 4 | 8 | 10 |
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Table 3.

| coordinates \((a_i')\) | 0 | 0 | 0 | 3 |
| weights \((w_i')\)    | 4 | 2 | 3 | 1 |
| \(k\)                | 1 | 2 | 3 | 4 |
| \(S_1[k] = \sum_{i=1}^{k} w_i'\) | 4 | 6 | 9 | 10 |

Inequalities (3.7) are valid for \(k' = 2\), so \(x_1 = a_1'' = 0\), i.e. we got the second possible solution \(X_2(0, 1)\).

In the case when Step I is applied, Table 3 transforms to Table 4.

Table 4.

| coordinates \((a_i')\) | 0 | 3 |
| weights \((w_i')\)    | 9 | 1 |
| \(k\)                | 1 | 2 |
| \(S_1[k] = \sum_{i=1}^{k} w_i'\) | 9 | 10 |

Condition (3.8) holds for \(k = 1\), so that \(X_2(0, 2)\) is again the second eventual solution.

We have \(\mathcal{X} = \{X_1, X_2\}\).

3. Let us now start from the assumption \(x_2 = 2\).

\[
f_1(x_1) = w_1|x_1 - 0| + w_2|x_1 - 0| + w_3|x_1 - 0| + w_4|x_1 - 2|.
\]

The relational table is:

Table 5.

| coordinates \((a_i')\) | 0 | 0 | 0 | 2 |
| weights \((w_i')\)    | 4 | 1 | 2 | 3 |
| \(k\)                | 1 | 2 | 3 | 4 |
| \(S_1[k] = \sum_{i=1}^{k} w_i'\) | 4 | 5 | 7 | 10 |

Since the equality of the form (3.8) are satisfied for \(k' = 2\), the solution \(x_1\) is from the interval \([a_1''', a_1'''] = [0, 0]\), i.e. \(x_1 = 0\), which implies \(X_3(0, 2)\).

The list \(\mathcal{X}\) is expanded: \(\mathcal{X} = \{X_1, X_2, X_3\}\).

Let us observe that Step I transforms Table 5 into the next table

Table 6.

| coordinates \((a_i')\) | 0 | 2 |
| weights \((w_i')\)    | 5 | 3 |
| \(k\)                | 1 | 2 |
| \(S_1[k] = \sum_{i=1}^{k} w_i'\) | 7 | 10 |

Now, condition (3.7) is satisfied for \(k = 1\), so that \(X_3(0, 2)\) is possible optimal point.

4. Let be \(x_2 \neq 1, 2, 4\). In this case we take \(x_1 = 0\) and then seek for the minimum of the function

\[
f_2(x_2) = \sum_{i=1}^{4} w_i|x_2 - a_2''|
\]

Let sort the coordinates \(a_2'' \rightarrow a_2''\) and perform analogous rearrangement \(w_i \rightarrow w_i'.\) The relational table is

Table 7.

| coordinates \((a_i')\) | 1 | 2 | 4 | 4 |
| weights \((w_i')\)    | 1 | 3 | 4 | 2 |
| \(k\)                | 1 | 2 | 3 | 4 |
| \(S_2[k] = \sum_{i=1}^{k} w_i'\) | 1 | 4 | 8 | 10 |
Inequalities of the form $k^* = 3$. We stop algorithm. This case has no solution, since the assumption $x_2 \neq 4$ is made.

Let us mention that Step I gives the next Table 8.

| coordinates ($a_{i}'$) | 1 | 2 | 4 |
|------------------------|---|---|---|
| weights ($w_{i}'$)     | 1 | 3 | 6 |
| $k$                    | 1 | 2 | 3 |
| $S_2[k] = \sum_{i=1}^k w_{i}'$ | 1 | 4 | 10 |

Therefore, the conclusion is the same as from Table 7.

At the end, in order to solve Step 2 of Algorithm we must compute and compare the values of the function $f$ at each point $X_i$, $i = 1, 2, 3$. We get

$f(X_1) = 50$, $f(X_2) = 55$, $f(X_3) = 62$.

Therefore, the solution of the Weber problem will be the point in which the function $f$ has a minimal value, i.e. $X^* = X_1 = A_1 = (4,4)$.

4 Conclusion and future work

Our paper is the first attempt to solve the discrete and the single-facility min-sum continuous location problem with the lift metric as the measure of distances.

A couple of variants and extensions of continuous location problems have been investigated in literature. Let us mention main between them. More complex problems include the placement of multiple facilities. Problems with barriers are the subject in [5, 13, 17, 18]. The location of undesirable (obnoxious) facilities requires to maximize minimum distances (see, e.g., [1, 11, 12, 22]. Location models with both desirable and undesirable facilities have been analyzed in [4]. It seems interesting to investigate these extensions in the sense of the lift metric or in the more general nonconvex case, where the shortest length of arc is used as distance instead of a particular metric.

References

[1] J. Brimberg, A. Mehrez, Multi-facility location using a maximin criterion and rectangular distances, Location Science 2 (1994), 11–19.
[2] J. Brimberg, R.F. Love, Properties of ordinary and weighted sums of order $p$ used for distance estimation, Recherche Opérationnelle, 29 (1995), 59–72.
[3] R. Chen, Location problems with costs being sums of powers of Euclidean distances, Comput. & Ops. Res. 11 (1984), 285–294.
[4] P.C. Chen, P. Hansen, B. Jaumard, H. Tuy, Weber’s problem with attraction and repulsion, Journal of Regional Science 32 (1992), 467–486.
[5] P.M. Dearing, K. Klamroth, R. Sears Jr., Planar location problems with block distance and Barriers, Ann. Oper. Res. 136 (2005), 117–143.
[6] E. Deza and M.M. Deza, Dictionary of Distances, Elsevier, Boston, 2006.
[7] Z. Drezner and H. Hawacher, Facility location: applications and theory, Springer-Verlag, Berlin, Heidelberg, 2004.
[8] Z. Drezner, A note on the Weber location problem, Ann. Oper. Res. 40 (1992), 153–161.
[9] Z. Drezner, C.H. Scott, On the feasible set for the squared Euclidean Weber problem and applications, Europ. J. Oper. Res. 118 (1999), 620–630.
[10] R. Durier, C. Michelot, Geometrical properties of the Fermat-Weber problem, J. Math. Anal. Appl. 117 (1985), 506–528.
[11] E. Erkut, S. Neuman, Analytical models for locating undesirable facilities, Europ. J. Oper.Res. 40 (1989), 275–291.
[12] F. Follert, E. Schomer, J. Sellen, Subquadratic Algorithms for the Weighted Maximum Facility Location Problem, http://www.mpi-sb.mpg.de/∼schoemer/publications/WM, 1995.
[13] H.W. Hamacher, S. Nickel, Combinatorial algorithms for some 1-facility median problems in the plane, Europ. J. Oper. Res. 79 (1994), 340–351.
[14] H.W. Hamacher, S. Nickel, Classification of location models, Location Science 6 (1998), 229–242.
[15] P. Hansen, J. Perreur, J.F. Thisse, Location theory, dominance and convexity: Some further results, Oper. Res. 28 (1980), 1285–1295.
[16] M.J. Hodgson, R.T. Wong, J. Honsaker, The P-Centroid Problem on an Inclined Plane, Operations Research 35 (1987), 221–233.
[17] B. Käfer, S. Nickel, 2001. Error bounds for the approximate solution of restricted planar location problems, Europ. J. Oper. Res. 135 (2001), 67–85.
[18] K. Klamroth, Planar Weber location problems with line barriers, Optimization 49 (2001), 517–527.
[19] R.F. Love, J.G. Morris, G.O. Wesolowsky, Facilities Location: Models and Methods, North-Holland, New York, 1988.
[20] R.F. Love, J.G. Morris, Computation procedure for the exact solution of location-allocation problems with rectangular distances, Naval Research Logistics Quarterly 22 (1975), 441–453.
[21] R.F. Love, W.G. Truscott, J. Walker, Terminal Location Problem: A Case Study Supporting the Status Quo, J. Opl Res. Soc. 36 (1985), 131–136.
[22] E. Melachrinoudis, An efficient computational procedure for the rectilinear maximin location problem, Transportation Science 22 (1998), 217–223.
[23] N. Mladenović, Continual location problems, Matematički institut, SANU, Beograd, 2004, In Serbian.
[24] I.A. Osinuga and O.N. Bamigbola, On the Minimum Norm Solution to Weber problem, SANSA Conference Proceedings (2007), 27–30.
[25] M. Parthasarathy, T. Hale, J. Blackhurst and M. Frank, The three-dimensional Fermat-Weber problem with Tchebychev distances, Advances Modeling and Optimization (2006), 65–71.
[26] J. Perreur, J. Thisse, Central Metrics and Optimal Location, Journal of Regional Science 14 (1974), 411–421.
[27] J.F. Thisse, J.E. Ward, R.E. Wendell, Some Properties of Location Problems with Block and Round Norms, Operations Research, 32 (1984), 1309–1327.
[28] V.A. Trubin, Effective algorithm for the Weber problem with a rectangular metric, Cybernetics and Systems Analysis, 14(6), DOI:10.1007/BF01070282, Translated from Kibernetika, No. 6, pp. 67–70, November–December, 1978.
[29] H. Uster, Weighted sum of order p and minisum location models, Ph.D. Thesis, McMaster University, Canada, 1999.
[30] J. Ward, R. Wendell, A New Norm for Measuring Distance Which Yields Linear Location Problems, Operations Research, 28 (1980), 836–844.
[31] E. Weiszfeld, Sur le point pour lequel la somme des distances de n points donnés est minimum, Tôhoku Math. J. 43 (1937), 355–386.
[32] G. Wesolowsky, The Weber problem: History and perspectives, Location Science 1 (1993), 5–23.