REALIZATION OF INTERMEDIATE LINKS 
OF LINE ARRANGEMENTS

ARNAUD BODIN

Abstract. We investigate several topological and combinatorial properties of line arrangements. We associate to a line arrangement a link \( A \cap S^3_r(0) \) obtained by intersecting the arrangement with some sphere. Several topics are discussed: (a) some link configurations can be realized by complex line arrangements but not by real line arrangements; (b) if we intersect the arrangements with a vertical band instead of a sphere, what link configurations can be obtained? (c) relations between link configurations obtained by bands and spheres.

Introduction

The topic of this paper is the study of intermediate links. To an algebraic curve \((f(x, y) = 0)\) passing through the origin we classically associate a link \((f = 0) \cap S^3(0)\), which is independent of \(\epsilon\) for all sufficiently small \(\epsilon > 0\) (see Milnor [6]). Another well-known and studied situation are links at infinity when we consider the intersection with a sphere \(S^3_R(0)\) of radius \(R \gg 1\) sufficiently large. An intermediate link is the intersection \((f = 0) \cap S^3_r(0)\), with an arbitrary \(r > 0\). There is no much literature on that subject initiated by L. Rudolph (see the surveys [9], [1] and also [8], [2], [3]).

We will extend and compare several concepts of intermediate links in the case of line arrangements.

• We compare the configurations obtained by intersecting a complex line arrangement with a ball of \(\mathbb{C}^2\) and the configurations obtained by intersecting a real line arrangement with a ball of \(\mathbb{R}^2\).

• We compare the configurations obtained by intersecting a real line arrangement with a ball of \(\mathbb{R}^2\) and the configurations obtained by intersecting with a band of type \([-r, +r] \times \mathbb{R}\) in \(\mathbb{R}^2\).
To be more precise we define the intersection graph of an arrangement in a set. Let $\mathcal{A}$ be a real or complex line arrangement and let $B$ be a set (which will either be a ball $B^4_r(0)$ in $\mathbb{C}^2$; a disk $D^2_r(0)$ in $\mathbb{R}^2$; or a band $[-r, +r] \times \mathbb{R}$ in $\mathbb{R}^2$). The intersection graph of $\mathcal{A}$ in $B$ is the graph defined by:

- one vertex associated to one line;
- one edge connect two vertices if the corresponding lines have their intersection inside $B$.

**Theorem 1.**
*This graph can be realized as the intersection graph of some complex line arrangement with a ball, but cannot be realized as the intersection graph of a real line arrangement with a disk.*

Fix a realizable graph $G$: we can realize it by a real line arrangement $\mathcal{A}$ inside a real disk $D^2_1(0)$ of radius fixed to 1. We define the maximal radius $R_{\text{max}}$ to be the maximal $r \geq 1$ such that there is no intersection point in $D^2_r(0) \setminus D^2_1(0)$. In other words the intersection points not in $D^2_1(0)$ are as far as possible.

We also defined a maximal radius for bands (instead of disks). The second and third parts are devoted to a numerical algorithm to compute this maximal number and –among other things– prove the following results:

**Theorem 2.**

- The maximal radius for the band problem is an algebraic number;
- The maximal radius for the band problem is less or equal than the maximal radius for the disk problem.

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**Part 1. Real and complex intermediate links of arrangements**

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. An *arrangement* in $\mathbb{K}^n$ is a finite collection of lines $\mathcal{A} = \{L_i\}$. In this part we intersect a real or complex line arrangement with a sphere $S^3_r(0)$ of arbitrary radius $r$. We will detail a configuration feasible with complex lines but not feasible with real lines.
1. The problems

To a line arrangement $\mathcal{A}$ and a radius $r$ we associate its link $\mathcal{A} \cap S^3_r(0)$. We can also associate an intersection graph $G_{\mathcal{A} \cap \mathbb{B}}$, as follows: a vertex is associated to each line, and two vertices are connected by one edge if and only if the corresponding lines have intersection inside $\mathbb{B}_r = \{(x, y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 \leq r^2\}$. In other words $G_{\mathcal{A} \cap \mathbb{B}}$ is build as follows: a vertex for each knot, an edge between two linked knots.

For a given graph $G$ is such a configuration of lines exists? In other words:

*Question 1.* Can any graph $G$ (such that between any two vertices there is no or one edge) can be realized as the link of an arrangement?

One clue that it could be true is the following.

**Lemma 3.**

1. Any graph $G$ (such that between any two vertices there is no or one edge) can be realized as a quasipositive link.
2. We may moreover suppose that each component of the link is a trivial knot and any pair of knots make a trivial link or a positive Hopf link.
3. This link is the intersection of $S^3_r(0)$ with some complex curve $f(x, y) = 0$.

*Proof.* We use Rudolph’s theory of quasipositive link. By induction on the number of components: when adding a trivial link either it is unlinked with the other components so the braid word is unchanged, either it is linked with one (or more) component: it corresponds to the addition of a word (or several) of type: $w\sigma_i^2w^{-1}$. Then, by a deep result of [7], any quasipositive link is a transversal $\mathbb{C}$-link. □

2. An example and a counter-example

The following example is quite interesting. Let $G_6$ be the following graph:

**Proposition 4.** The graph $G_6$ can be realized as the intersection graph of some complex line arrangement but cannot be realized as the intersection graph of a real line arrangement.
Proof. Realization as a complex line arrangement. Let $\omega = \exp\left(\frac{2i\pi}{5}\right)$. Set $P_0 = (1,1)$ and $P_i = (\omega^i, \omega^{5-i})$. Consider the 5-lines arrangement $A$ composed of $(P_0P_2)$, $(P_2P_4)$, $(P_4P_1)$, $(P_1P_3)$, $(P_3P_0)$. Finally define the sixth line $L$ of equation $(6x - 4y = 1)$.

Let the 6-lines arrangement $A' = A \cup L$. It has the following picture: all bold curves are lines, including the bold circle! The thin circle is the sphere.

Fact: $A'$ has intersection graph $G_6$. The proof is just a computation of the intersection points.

Menelaus theorem for polygons. 
One of the key-point, due to Patrick Popescu-Pampu, is a Menelaus theorem for polygons in the real plane. The statement is given here for a pentagon, the proof for all polygons is the same as the one for triangles. Let a line $L$ that intersects the edge lines of a pentagon $P_1, \ldots, P_5$ at points $Q_1, \ldots, Q_5$: $Q_i$ is the intersection of $L$ with the line $(P_iP_{i+1})$. 

**Theorem 5** (Menelaus theorem for polygons).

\[
\frac{Q_1 P_1}{Q_1 P_2} \times \frac{Q_2 P_2}{Q_2 P_3} \times \cdots \times \frac{Q_5 P_5}{Q_5 P_1} = 1.
\]

The overline \( AB \) means the algebraic measure of \( AB \) with respect to an orientation of the line \( (AB) \). The ratio \( \frac{Q_i P_i}{Q_i P_{i+1}} \) is negative if and only if \( Q_i \) is in the segment \( [P_i, P_{i+1}] \) (this is independent of the chosen orientation of the line).

As a corollary we get:

**Corollary 6.** A line cannot intersect an odd number of segments of a pentagon.

Otherwise the product of the five ratios would be negative, that contradicts the fact that this product equals 1.

**Non-realization as a real line arrangement.**

We will apply this to our configurations. Suppose that 5 lines with real equations are disposed as follows: there are 5 intersection points \( P_1, \ldots, P_5 \) outside the ball \( B_r \) and 5 intersection points inside the ball. We denote the lines as follows \( \ell_i = (P_{i-1}P_{i+1}) \) and \( Q_i = \ell_i \cap \ell_{i+1} \) for \( i \) from 0, 1, \ldots, 4 (counting modulo 5).
The first remark is that the polygon $Q_0Q_1\cdots Q_4$ is convex otherwise two lines $\ell_j, \ell_k$ would have an intersection point, distinct from the $Q_i$, inside the convex hull of $Q_0Q_1\cdots Q_4$ and hence inside the ball $B_r$.

Each line $\ell_i$ contains two points $Q_j$ and two points $P_k$. Two kinds of configurations for points on lines are possible. Type (A): $P_k - Q_j - Q_j' - P_{k'}$ (the ball separates the two $P$) or type (B): $P_k - P_{k'} - Q_j - Q_{j'}$ (the two $P$ are on the same side of the ball).

We now prove that configurations of type (B) are associated by pairs. Suppose for instance that we have the following configuration of type (B) for the line $\ell_2$ where $P_1$ is farthest point of the ball. Suppose now that the other line $\ell_0$ that contains $P_1$ is of type (A).
As $P_3 \in [P_1Q_1]$, the line $\ell_4 = (P_3Q_4)$ intersects $\ell_1 = (Q_0Q_1)$ in $[Q_0Q_1]$. Then $P_0 = \ell_1 \cap \ell_4$ is inside the ball, that gives a contradiction. (The same phenomenon arise if we exchange the role of $Q_1$ and $Q_2$ on the line $\ell_2$.) As a conclusion: there is an even number of type (B) line configurations.

Suppose now that there exists an additionnal line $L$ that intersects our five lines $\ell_i$ inside the ball (in order to realize the graph $G_6$). Consider the pentagon $P_0P_1 \ldots P_4$ and its 5 segments:

- For a segment $[P_iP_{i+2}]$ supported by a line of type (A), $L$ intersects $(P_iP_{i+2})$ in the ball, hence in the segment $[P_iP_{i+2}]$.
- For a segment $[P_iP_{i+2}]$ supported by a line of type (B), $L$ intersects $(P_iP_{i+2})$ in the ball, hence outside the segment $[P_iP_{i+2}]$.

As there is an even number of segment supported by lines of type (B) among the 5 segments, there is odd number of segments supported by a line of type (A), hence the line $L$ intersects an odd number of segments of the pentagon. By the Menelaus theorem it provides a contradiction. Then any line $L$ with real equation cannot intersect the 5 lines inside the ball. It is quite surprising that is the realm of complex number this is possible.

\[\square\]

Part 2. Arrangements and numerical experiments

We study in details a variation of a problem of realizability of intermediate links of real line arrangements. We get two problems: the
realisability and maximization of the radius. After replacing spheres by bands, we transcript the first problem into linear inequalities. For some examples we deduce (exact) lower bounds and (numerical) upper bounds for the maximum radius. We end by proving that this maximum radius is an algebraic number.

3. Statement of the problem

In this part we focus on the following geometric problem, dealing with lines in the real plane. Let $D_r = \{(x, y) \in \mathbb{R}^2 \mid |x| \leq r\}$ be the vertical band of radius $r$. Fix some $R \geq 1$. Given two lines, we will consider two conditions: the two lines have their intersection in $D_1$ (the band of radius 1); they do not have their intersection in $D_R$ (the band of radius $R$).

More precisely: fix $n$ and fix a graph $G$ with $n$ vertices. The problem is to find a set of $n$ distinct lines $\{\ell_i\}$ such that for each pair $(i, j)$ (with $i < j$): if an edge of $G$ connect the vertex $i$ to $j$ then $\ell_i \cap \ell_j \in D_1$ and if no edge connect the vertex $i$ to $j$ then $\ell_i \cap \ell_j \notin D_R$.

For a given graph $G$ and a given $R$ the first question is: is such a configuration of lines exists? If it exists for some $R$, what is the maximal $R$ that we can choose?

4. Linear programs

We denote by $(y = a_i x + c_i)$ an equation of $\ell_i$. The abscissa of the intersection $\ell_i \cap \ell_j$ is $x_{ij} = -\frac{c_i - c_j}{a_i - a_j}$. The condition $\ell_i \cap \ell_j \in D_1$ becomes

$$(E_{i,j}^\varepsilon) \quad |a_i - a_j| \geq |c_i - c_j|.$$ 

while the condition $\ell_i \cap \ell_j \notin D_R$ becomes

$$(E_{i,j}^\varepsilon) \quad R|a_i - a_j| < |c_i - c_j|.$$ 

These conditions can be seen as linear inequalities, after discussion on cases depending on the sign of $a_i - a_j$ and $c_i - c_j$ (see below, paragraph 6).

5. Numerical results

We will give some examples and results for several graphs. For the graph $G = A_4$ we conjecture numerically that $R_{\text{max}}(A_4) = 3 + 2\sqrt{2}$. More precisely: we found a configuration of lines realizable for $R = 3 + 2\sqrt{2}$ and we numerically compute that no such configuration exists for $R = 3 + 2\sqrt{2} + \varepsilon$ with $\varepsilon = 10^{-6}$.

\begin{center}
\begin{tikzpicture}
\node at (0,0) {$A_4$};
\end{tikzpicture}
\end{center}
Here are the graph, lines and equations. The red dots are the intersections whose abscissa verify $|x| \leq 1$ (here all $|x| = 1$), the blue squares are the intersections whose abscissa have maximal $|x| \geq R$, for this example $|x| \geq R_{\text{max}} = 3 + 2\sqrt{2}$ (here all $|x| = 3 + 2\sqrt{2}$).

\begin{align*}
(\ell_1) \quad y &= 0 \\
(\ell_2) \quad y &= (1 + \frac{\sqrt{2}}{2})x - 1 - \frac{\sqrt{2}}{2} \\
(\ell_3) \quad y &= -\frac{\sqrt{2}}{2}x - 2 - 3\frac{\sqrt{2}}{2} \\
(\ell_4) \quad y &= x - 3 - 2\sqrt{2}
\end{align*}
For the graph $G = A_5$ we conjecture numerically that $R_{\text{max}}(A_5) = 2 + \sqrt{3}$.

\[
\begin{align*}
(\ell_1) & \quad y = 0 \\
(\ell_2) & \quad y = 3 + \sqrt{5} \\
(\ell_3) & \quad y = x \\
(\ell_4) & \quad y = x + 2 + \sqrt{5} \\
(\ell_5) & \quad y = -\frac{\sqrt{5} - 1}{2} x + 3 + \frac{\sqrt{5}}{2}
\end{align*}
\]

We also find for $G = G_1$, $R_{\text{max}}(G_1) = 3$ and for $G = G_2$, $R_{\text{max}}(G_2) = 3 + 2\sqrt{2}$. But for both theses graphs the bound is obtained by a sequence of configuration that tends to a "degenerate" configuration with two lines that are equal.

For $G = G_3$, $R_{\text{max}}(G_3) = \alpha = 2.60 \ldots$, where $\alpha$ is an algebraic number of degree 3, that is a root of $x^3 + x^2 - 9x - 1 = 0$.

\textit{Question 2.} It would be interesting to know the value of $R_{\text{max}}(A_n)$ (where $A_n$ is the line-graph with $n$ vertices). Conjecturally $R_{\text{max}}(A_n) \to 3$ as $n \to +\infty$.

6. Implementation

An algorithm has been implemented in MATLAB to decide whether for a given graph $G$ and a given $R$ a corresponding configuration of lines exists. Moreover–if it exists–it gives a numeric solution.

The first step is to separate the situation in several linear problems. To each pair $(i, j)$ with $i < j$ we have 4 possibilities for the two signs of $a_i - a_j$ and $c_i - c_j$. The number of pairs being $\frac{n(n-1)}{2}$. After reduction of the case by symmetry it yields $4^{n(n-1)/2-1}$ cases.

The second step is to study each case: for a fixed condition of sign for $a_i - a_j$ and $c_i - c_j$, the condition $(E_{i,j})$ or the condition $(E_{i,j}^\ell)$ yields a linear problem that can be solved numerically by standard tools.

This algorithm enables to find numerically $R_{\text{max}}(G)$, by testing several $R$. Rigorously: it first gives a value $R_0$ such that $R_0 - \epsilon \leq R_{\text{max}}(G) < R_0 + \epsilon$ where $\epsilon$ is a numerical value (say $\epsilon = 10^{-6}$ in practise).

Then it is possible to conjecture a value $R_1$ and the coefficients of the limit configuration and then check that this configuration works. We then rigorously have proved $R_1 \leq R_{\text{max}}(G) < R_1 + \epsilon$. 
Due to the exponential growth of the number of cases, we were only able to deal examples with 4 or 5 lines.

**Question 3.** Find an algorithm for the feasibility of band and ball problems over $\mathbb{R}$ and $\mathbb{C}$ that is efficient up to $n = 10$ lines. If a graph $G$ is feasible then compute a configuration.

**Question 4.** Have a rigorous proof (other than numerical) for the upper bounds of $R_{\text{max}}$.

7. **The maximum radius is an algebraic number**

Consider the coefficients of the lines $(a_1, c_1, a_2, c_2, \ldots, a_n, c_n) \in \mathbb{R}^{2n}$ as parameters. For a given graph $G$, the condition $(E_{i,j}^R)$ and the condition $(E_{i,j}^\in)$ for $R = 1$ define a semi-algebraic set $S \subset \mathbb{R}^{2n}$. First define a function $F_1 : S \to \mathbb{R}^{n(n-1)/2}$ by $(a_1, c_1, \ldots) \mapsto (x_{ij})_{i<j}$ where $x_{ij} = -\frac{c_i - c_j}{a_i - a_j}$. Secondly define $F_2 : \mathbb{R}^{2n(n-1)} \to \mathbb{R}$ by $(x_{ij}) \mapsto \min_{|x_{ij}| > 1} |x_{ij}|$ (equivalently the minimum runs over the pairs $(i, j)$ such that no edge of $G$ goes from $i$ to $j$).

Let $F = F_2 \circ F_1 : S \to \mathbb{R}$. Then by definition $R_{\text{max}}(G) = \sup_S F$. By general results in semi-algebraic geometry it implies:

**Proposition 7.** Fix a graph $G$. If $R_{\text{max}}(G)$ exists and is finite then it is an algebraic number.

**Part 3. Combinatorics**

In this last part we will compare two problems of realisability and end with questions.

8. **Two real problems**

We consider two real problems. Firstly the problem already considered in section 3, dealing with the realisability of a graph as the configuration of lines within two bands. We define a similar problem for balls, by replacing a band $D_r$ by the ball $B_r = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq r^2\}$.

The questions are the same. Given a graph $G$ with $n$ vertices and a real number $R \geq 1$, find a set of $n$ distinct lines $\{\ell_i\}$ such that for each pair $(i, j)$ (with $i < j$): if an edge of $G$ connect the vertex $i$ to $j$ then $\ell_i \cap \ell_j \in B_1$ and if no edge connect the vertex $i$ to $j$ then $\ell_i \cap \ell_j \notin B_R$.

For a given graph $G$ and a given $R$ the questions are: is such a configuration of lines exists? If it exists for some $R$, what is the maximal
$R$ that we can choose? We will compare the two problems from the combinatorial point of view.

9. From bands to spheres

Lemma 8. If $G$ is feasible for the bands $(\mathbb{D}_1, \mathbb{D}_R)$ (in $\mathbb{R}^2$) then $G$ is feasible for the balls $(\mathbb{B}_1, \mathbb{B}_{R'})$ (in $\mathbb{R}^2$) with $R' = R(1 - \epsilon)$ (for all $\epsilon > 0$). In particular the maximal radius $R_{\max}$ for the band problem is less or equal than the maximal radius $R'_{\max}$ for the ball problem.

Remark 9. In general the reciprocal is false. For example let $G = C_5$. This graph is feasible for the ball problem $(\mathbb{B}_1, \mathbb{B}_R)$ for some $R \geq 1$ but not feasible for the band problem $(\mathbb{D}_1, \mathbb{D}_R)$ for any $R \geq 1$.

Drawing a 5-star proves the feasibility for balls. To prove that $G$ is not feasible for bands, a first step is to remark that the five points of intersection in $\mathbb{D}_1$ draw a convex pentagon (otherwise there would be a sixth point of intersection inside $\mathbb{D}_1$). The second step is to notice that for the 5 points of intersection not in $\mathbb{D}_1$, at least 3 of them are on the same side. So that, among this 3 points, you can choose $Q_1$ and $Q_2$ that are on a same line of the configuration. On this line two intersection points of the configuration are in $\mathbb{D}_1$ but by convexity of the pentagon they should also be in the segment $[Q_1, Q_2]$ which is entirely out of $\mathbb{D}_1$. It yields a contradiction.

Proof of lemma 8. Fix $0 < \epsilon \ll 1$. Suppose that a configuration of lines $\mathcal{L}$ realizes a graph $G$ for the band problem $(\mathbb{D}_1, \mathbb{D}_R)$. The transformation $(x, y) \mapsto (x, \lambda y)$ preserves equations $(E_{i,j}^e)$ and $(E_{i,j}^f)$. So that by choosing a sufficiently small $0 < \lambda \ll 1$ we get a “flat” configuration of lines $\mathcal{L}'$. On the picture below the original configuration is on the left, the flattened one on the right.
Now let \( h : (x, y) \mapsto (1 - \epsilon) \cdot (x, y) \) be the homothety centred at the origin of ratio \( 1 - \epsilon \). Let \( P \) be a point of intersection of two lines of \( \mathcal{L}' \). Due to the flatness if \( P \in \mathbb{D}_1 \) then \( h(P) \in \mathbb{B}_1 \) and if \( P \notin \mathbb{D}_R \) then \( P \notin \mathbb{B}_{R(1-\epsilon)} \). So that the configuration \( \mathcal{L}' \) proves the feasibility for the problem \( (\mathbb{B}_1, \mathbb{B}_{R(1-\epsilon)}) \).

\( \square \)

10. From spheres to bands

Lemma 10. Let \( G \) be a graph. If \( G \) is feasible for the balls \( (\mathbb{B}_1, \mathbb{B}_R) \) for all \( R \gg 1 \), then \( G \) is feasible for the bands \( (\mathbb{D}_1, \mathbb{D}_R) \) for all \( R \gg 1 \).

Proof. We give a heuristic proof, and start with the case where \( \overline{CG} \) is a connected graph. Then for a big \( R \), all lines of the configurations are nearly equal (see picture below): firstly any line of the configuration should pass through the ball \( \mathbb{B}_1 \). Pick a line \( L_0 \); any other line \( L \) connected to \( L_0 \) in \( \overline{CG} \) should pass through \( \mathbb{B}_1 \) that is very small compare to \( \mathbb{B}_R \), so that we think of \( \mathbb{B}_1 \) as (nearly) a point. \( L \) should also intersect \( L_0 \) outside \( \mathbb{B}_R \) at \( Q \) (because \( \overline{CG} \) is a connected). So the two “points” of intersection \( \mathbb{B}_1 \) and \( Q \) define (nearly) the same line \( L \) and \( L_0 \). Because \( \overline{CG} \) is supposed to be a connected set, it proves that all lines are (nearly) equal.
We may have supposed that $L_0$ was an horizontal line, then replacing $B_1$ by $D_1$ and $B_R$ by $D_R$ proves the feasibility.

If $\mathcal{G}$ is no longer connected, then each connected component of $\mathcal{G}$ yields a bundle of lines with (nearly) the same direction, any two bundles intersecting each other only in $B_1$. After choosing all directions sufficiently horizontal and replacing balls by bands, it gives the conclusion.

Question 5. For each class of problem (over $\mathbb{R}$ or $\mathbb{C}$) characterize feasible graphs.

Question 6. More specifically for the complex problem with spheres, each component of a link of arrangement is in fact a true circle (a circle in the Euclidean meaning). For instance it is known that a Borromean ring cannot be obtained with true circles. See [4, Lemma 3.2] and [5]. The following questions seem to be open:

- Let two links of arrangements $L_1$ and $L_2$ with the same dual graph $G_1 = G_2$. Does it imply $L_1$ isotopic to $L_2$?
• Given a graph $G$, is it possible to construct a link $L$ in $S^3$ whose components are true circles and whose dual graph is $G$?
• Given a link $L$ in $S^3$ whose components are true circles, is it possible to realize $L$ as the link of a line arrangement?

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E-mail address: Arnaud.Bodin@math.univ-lille1.fr

Laboratoire Paul Painlevé, Mathématiques, Université Lille 1, 59655 Villeneuve d’Ascq Cedex, France