SOME RESULTS OF THE MARIÑO-VAFA FORMULA

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Abstract. In this paper we derive some new Hodge integral identities by taking the limits of Mariño-Vafa formula. These identities include the formula of $\lambda_{g-1}$-integral on $\overline{M}_{g,1}$, the vanishing result of $\lambda_g \text{ch}_2(E)$-integral on $\overline{M}_{g,1}$ for $1 \leq l \leq g - 3$. Using the differential equation of Hodge integrals, we give a recursion formula of $\lambda_{g-1}$-integrals. Finally, we give two simple proofs of $\lambda_g$ conjecture and some examples of low genus integral.

1. Introduction

Based on string duality, Mariño and Vafa [10] conjectured a closed formula on certain Hodge integrals in terms of representations of symmetric groups. Recently, C.C. Liu, K. Liu and J. Zhou [6] proved this formula and derived some consequences from it [7]. In this paper we follow their method to derive some new Hodge integral identities. One of the main results of this paper is the following identity: if $1 \leq m \leq 2g - 3$, then

\begin{equation}
-(2g - 2 - m)!(-1)^{2g-3-m} \int_{\overline{M}_{g,1}} \lambda_g \text{ch}_{2g-2-m}(E) \psi_1^m
\end{equation}

\begin{equation}
= b_g \sum_{k=0}^{m-1} \frac{(-1)^{2g-1-k}(2g-1-k)(2g-1-k)}{2g - 1 - k} B_{2g-1-m}
\end{equation}

\begin{equation}
+ \frac{1}{2} \sum_{g_1+g_2=g, g_1, g_2 \geq 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1-m-1)} \frac{(-1)^{2g_2-1-k}(2g_2-1-k)(2g-1-k)}{2g - 1 - k} B_{2g-1-m}.
\end{equation}

As a consequence, we find a new Hodge integral identity: if $g \geq 2$, then

\begin{equation}
\int_{\overline{M}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = \frac{1}{12} \left[g(2g - 3)b_g + b_1 b_{g-1}\right],
\end{equation}

and also a vanishing result: if $g \geq 2$, then for any $1 \leq t \leq g - 1$, we have

\begin{equation}
\int_{\overline{M}_{g,1}} \lambda_g \text{ch}_2(E) \psi_1^{2(g-1-t)} = 0.
\end{equation}

Recently, Liu-Xu [8] derived a generalized formula for Hodge integrals of type (2) by using the $\lambda_g$ conjecture.

The rest of this paper is organized as follows: In Section 2, we recall the Mariño-Vafa formula and the Mumford’s relations. In Section 3, we prove our main theorem and derive a new Hodge integral identity. In Section 4, we give another simple proof of $\lambda_g$ conjecture. In Section 5, we derive a recursion formula of $\lambda_{g-1}$-integrals. In the last section, we list some low genus examples.

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2. Preliminaries

2.1. Partitions. A partition $\mu$ of a positive integer $d$ is a sequence of integers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l(\mu) > 0$ such that

$$\mu_1 + \cdots + \mu_l(\mu) = d = |\mu|,$$

for each positive integer $i$, let

$$m_i(\mu) = |\{j | \mu_j = i, 1 \leq j \leq l(\mu)\}|.$$

The automorphism group $\text{Aut}(\mu)$ of $\mu$ consists of possible permutations among the $\mu_i$’s, hence its order is given by

$$|\text{Aut}(\mu)| = \prod_i m_i(\mu)!,$$

define the numbers

$$\kappa_\mu = \prod_{i=1}^{l(\mu)} \mu_i(\mu) - 2i + 1, \quad z_\mu = \prod_{i} m_j(\mu)! j^{m_j(\mu)}.$$

The Young diagram of $\mu$ has $l(\mu)$ rows of adjacent squares: the $i$-th row has $\mu_i$ squares. The diagram of $\mu$ can be defined as the set of points $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ such that $1 \leq j \leq \mu_i$, the conjugate of a partition $\mu$ is the partition $\mu'$ whose diagram is the transpose of the diagram $\mu$. Finally, we introduce the hook length of $\mu$ at the square $x \in (i, j)$:

$$h(x) = \mu_i + \mu_j' - i - j + 1.$$

Each partition $\mu$ of $d$ corresponds to a conjugacy class $C(\mu)$ of the symmetric group $S_d$ and each partition $\nu$ corresponds to an irreducible representation $R_\nu$ of $S_d$, let $\chi_\nu(C(\mu)) = \chi_{R_\nu}(C(\mu))$ be the value of the character $\chi_{R_\nu}$ on the conjugacy class $C(\mu)$.

2.2. Mariño-Vafa formula. Let $\overline{M}_{g,n}$ denote the Deligne-Mumford moduli stack of stable curves of genus $g$ with $n$ marked points. Let $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ be the universal curve, and let $\omega_\pi$ be the relative dualizing sheaf. The Hodge bundle

$$\mathbb{E} = \pi_* \omega_\pi$$

is a rank $g$ vector bundle over $\overline{M}_{g,n}$ whose fiber over $[\{C, x_1, \ldots, x_n\}] \in \overline{M}_{g,n}$ is $H^0(C, \omega_C)$, the complex vector space of holomorphic one forms on $C$. Let $s_i : \overline{M}_{g,n} \to \overline{M}_{g,n+1}$ denote the section of $\pi$ which corresponds to the $i$-th marked point, and let

$$\mathbb{L}_i = s_i^* \omega_\pi$$

be the line bundle over $\overline{M}_{g,n}$ whose fiber over $[\{C, x_1, \ldots, x_n\}] \in \overline{M}_{g,n}$ is the cotangent line $T_{x_i}C$ at the $i$-th marked point $x_i$. Consider the Hodge integral

$$\int_{\overline{M}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_1^{k_1} \cdots \lambda_g^{k_g}$$

where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$, and $\lambda_j = c_j(\mathbb{E})$ is the $j$-th Chern class of $\mathbb{E}$. The dimension of $\overline{M}_{g,n}$ is $3g - 3 + n$, hence (4) is equal to zero unless $\sum_{i=1}^{n} j_i + \sum_{i=1}^{g} ik_i = 3g - 3 + n$. Let

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g = \sum_{i=0}^{g} (-1)^i \lambda_i u^{g-i}$$

be the line bundle over $\overline{M}_{g,n}$ whose fiber over $[\{C, x_1, \ldots, x_n\}] \in \overline{M}_{g,n}$ is the cotangent line $T_{x_i}C$ at the $i$-th marked point $x_i$. Consider the Hodge integral

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where $\psi_i = c_1(\mathbb{L}_i)$ is the first Chern class of $\mathbb{L}_i$, and $\lambda_j = c_j(\mathbb{E})$ is the $j$-th Chern class of $\mathbb{E}$. The dimension of $\overline{M}_{g,n}$ is $3g - 3 + n$, hence (4) is equal to zero unless $\sum_{i=1}^{n} j_i + \sum_{i=1}^{g} ik_i = 3g - 3 + n$. Let

$$\Lambda_g^\vee(u) = u^g - \lambda_1 u^{g-1} + \cdots + (-1)^g \lambda_g = \sum_{i=0}^{g} (-1)^i \lambda_i u^{g-i}$$
be the Chern polynomial of the dual bundle $E^\nu$ of $E$. For any partition $\mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{\ell(\mu)} > 0$, define

$$
C_{g,\mu}(\tau) = -\frac{\sqrt{-1}^{l(\mu)}}{|\text{Aut}(\mu)|} \left[ \tau(\tau + 1) \right]^{l(\mu) - 1} \prod_{i=1}^{l(\mu)} \frac{\prod_{k=1}^{\mu_i}(\mu_i \tau + a)}{(\mu_i - 1)!} \int_{\mathcal{M}_{g,n}} \frac{\Lambda_g^\nu(1) \Lambda_g^\nu(-\tau - 1) \Lambda_g^\nu(\tau)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)},
$$

$$
C_\mu(\lambda; \tau) = \sum_{g \geq 0} \lambda^{2g - 2 + l(\mu)} C_{g,\mu}(\tau),
$$

for a partition $\tau$ is a formal variable. Note that

$$
\int_{\mathcal{M}_{g,n}} \frac{\Lambda_g^\nu(1) \Lambda_g^\nu(-\tau - 1) \Lambda_g^\nu(\tau)}{\prod_{i=1}^{l(\mu)} (1 - \mu_i \psi_i)} = |\mu|^{l(\mu) - 3}
$$

for $l(\mu) \geq 3$, and we use this expression to extend the definition to the case $l(\mu) < 3$.

Introduce formal variables $p = (p_1, p_2, \cdots, p_n, \cdots)$, and define

$$
p_\mu = p_{\mu_1} \cdots p_{\mu_{\ell(\mu)}}
$$

for a partition $\mu$. Define generating functions

$$
C(\lambda; \tau, p) = \sum_{|\mu| \geq 1} C_\mu(\lambda; \tau)p_\mu,
$$

$$
C(\lambda; \tau, p)^* = e^{C(\lambda; \tau, p)}.
$$

In [6], Chiu-chu Melissa Liu, Kefeng Liu and Jian Zhou have proved the following formula which was conjectured by Mariño and Vafa in [10].

**Theorem 2.1. (Mariño-Vafa Formula)** For every partition $\mu$, we have

$$
C(\lambda; \tau, p) = \sum_{n \geq 1} (-1)^{n-1} \frac{n}{\mu} \sum_{n \geq 1} \prod_{i=1}^{n} \sum_{|\nu_i| = |\mu|} \frac{\chi_\nu(C(\mu_i))}{z_\mu} \frac{\psi(1)}{z_\mu} V_\nu(\lambda) p_\mu,
$$

$$
C(\lambda; \tau, p)^* = \sum_{|\mu| \geq 0} \left( \sum_{|\nu| = |\mu|} \frac{\chi_\nu(C(\mu))}{z_\mu} \frac{\psi(1)}{z_\mu} V_\nu(\lambda) \right) p_\mu,
$$

where

$$
V_\nu(\lambda) = \prod_{1 \leq a < b \leq l(\nu)} \frac{\sin[(\nu_a - \nu_b + b - a)\lambda/2]}{\sin[(b - a)\lambda/2]} \frac{1}{\prod_{i=1}^{l(\nu)} \prod_{v=1}^{\nu_i} 2\sin[(v - i + l(\nu))\lambda/2]}.
$$

It is known that

$$
V_\nu(\lambda) = \frac{1}{2^{l(\nu)} \prod_{x \in \nu} \sin[h(x)\lambda/2]}.
$$
2.3. Mumford's relations. Let \( c_t(E) = \sum_{i=0}^{g} t^i \lambda_i \), then we have
\[
c_{-t}(E) = t^g \Lambda^\vee \left( \frac{1}{t} \right).
\]
Mumford’s relations [11] are given by
\[
(2.7) \quad c_t(E)c_{-t}(E) = 1,
\]
equivalently
\[
(2.8) \quad \Lambda^\vee g(t) \Lambda^\vee g(-t) = (-1)^g t^{2g},
\]
then
\[
(2.9) \quad \lambda_k^2 = \sum_{i=1}^{k} (-1)^{i+1} 2 \lambda_{k-i} \lambda_{k+i},
\]
where \( \lambda_0 = 1 \) and \( \lambda_k = 0 \) for \( k > g \). Let \( x_1, \ldots, x_g \) be the formal Chern roots of \( E \), the Chern character is defined by
\[
\text{ch}(E) = \sum_{i=1}^{g} e^{x_i} = g + \sum_{n=1}^{+\infty} \sum_{i=1}^{g} \frac{x_i^n}{n!},
\]
we write
\[
(2.10) \quad \text{ch}_0(E) = g,
(2.11) \quad \text{ch}_n(E) = \frac{1}{n!} \sum_{i=1}^{g} x_i^n, \quad n = 1, 2, \ldots.
\]
From the above identities we have the relation between \( \text{ch}_n(E) \) and \( \lambda_n \):
\[
(2.12) \quad n! \text{ch}_n(E) = \sum_{i+j=n} (-1)^{i-1} i \lambda_i \lambda_j, \quad n < 2g,
(2.13) \quad \text{ch}_n(E) = 0, \quad n \geq 2g.
\]
It is easy to see that
\[
(2g - 1)! \text{ch}_{2g-1}(E) = (-1)^{g-1} \lambda_{g-1} \lambda_g,
(2g - 2)! \text{ch}_{2g-2}(E) = (-1)^{g-1} \left( (2g - 2) \lambda_{g-2} \lambda_g - (g - 1) \lambda_{g-1}^2 \right),
(2g - 3)! \text{ch}_{2g-3}(E) = (-1)^{g-1} (3 \lambda_{g-3} \lambda_g - \lambda_{g-1} \lambda_{g-2}).
\]

2.4. Bernoulli numbers. The Bernoulli numbers \( B_m \) are defined by the following series expansion:
\[
(2.14) \quad \frac{t}{e^t - 1} = \sum_{m=0}^{+\infty} B_m \frac{t^m}{m!},
\]
the first few terms are given by
\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = -\frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}.
\]
Finally we recall two formulas which will be used later:

\[
\frac{t/2}{\sin(t/2)} = 1 + \sum_{g=1}^{+\infty} \frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!} t^{2g},
\]

\[
\sum_{i=1}^{d-1} i^m = \sum_{k=0}^{m} \frac{(m+1)_k}{m+1} B_k d^{m+1-k},
\]

where \(m\) is a positive integer.

3. SOME NEW RESULTS FROM MARIÑO-VAFA FORMULA

In this section we derive some new results from the Mariño-Vafa formula, we will need two formulas in [7, 2.1 and 5.1].

**Theorem 3.1.** We have the following results:

\[
\sum_{g \geq 0} \lambda^{2g} \int_{\mathcal{M}_{g,1}} \frac{d\tau}{1 - d\psi} \left[ (\Lambda^\vee_g(1)\Lambda^\vee_g(\tau))\Lambda^\vee_g(-\tau - 1) \right] = -\lambda_{g-1} - \lambda_g \sum_{k \geq 0} k!(-1)^{k-1} c_k(E).
\]

3.1. **The coefficient of** \(\lambda^{2g}\). Introduce the notation

\[
b_g = \begin{cases} 
1, & g = 0, \\
\frac{2^{2g-1} - 1}{2^{2g-1}} \frac{|B_{2g}|}{(2g)!}, & g > 0,
\end{cases}
\]

then the coefficient of \(\lambda^{2g}\) in \(-\sum_{a=1}^{d-1} \frac{1}{a} \frac{d\lambda/2}{a \sin(d\lambda/2)}\) is

\[
(\sum_{a=1}^{d-1} \frac{1}{a}) b_g d^{2g-1}.
\]

If \(g_1, g_2 \geq 0\) and \(g_1 + g_2 = g\), define

\[
F_{g_1,g_2}(d) = \sum_{i+j=d,i,j \neq 0} i^{2g_1-1} j^{2g_2-1}.
\]

In [6] it is showed that if \(g_1, g_2 \geq 1\), then

\[
F_{g_1,g_2}(d) = \sum_{k=0}^{2g_2-1} \sum_{l=0}^{2g_2-2-k} \frac{(-1)^{2g_2-1-k}}{2g_1-1-k} \binom{2g_2-1}{k} \binom{2g-1-k}{l} B_{l} d^{2g_1-1-k}.
\]
for the rest case we have

\begin{equation}
F_{0,g}(d) = \sum_{i+j=d, i,j \neq 0} i^{-1} j^{2g-1}
\end{equation}

\begin{equation}
= \sum_{i=1}^{d-1} i^{-1} (d-i)^{2g-1}
\end{equation}

\begin{equation}
= \sum_{k=0}^{2g-1} (-1)^{2g-1-k} \binom{2g-1}{k} d^{k} \sum_{i=1}^{2g-k-1} i^{2g-2-k}
\end{equation}

\begin{equation}
= \sum_{k=0}^{2g-3} (-1)^{2g-1-k} \binom{2g-1}{k} d^{k} \sum_{l=0}^{2g-k-2} \frac{(2g-1-k)}{2g-1-k} B_{l} d^{2g-1-k-l}
\end{equation}

\begin{equation}
- (2g-1) d^{2g-2} (d-1) + d^{2g-1} \sum_{i=1}^{d-1} \frac{1}{i}
\end{equation}

\begin{equation}
+ (2g-1) d^{2g-2} + d^{2g-1} \sum_{i=1}^{d-1} \frac{1}{i}
\end{equation}

Note that $F_{0,g}(d) = F_{g,0}(d)$ and

\begin{equation}
\sum_{i+j=d, i,j \neq 0} \frac{\lambda^{2}}{8 \sin(i \lambda/2) \sin(j \lambda/2)} = \frac{1}{2} \sum_{g \geq 0} \lambda^{2g} \left( \sum_{g_{1}+g_{2}=g} b_{g_{1}} b_{g_{2}} F_{g_{1},g_{2}}(d) \right).
\end{equation}

3.2. The Main Theorem. Let

$$
\sum_{g \geq 0} \lambda^{2g} \text{LHS} = \sum_{g \geq 0} \lambda^{2g} \int_{\mathcal{M}_{g,1}} \frac{d\tau}{\tau} \left[ \Lambda_{g}^{\nu}(1) \Lambda_{g}^{\nu}(\tau) \Lambda_{g}^{\nu}(-\tau - 1) \right] / (1 - d \psi_{1}),
$$

and

\begin{equation}
\sum_{g \geq 0} \lambda^{2g} \text{RHS} = - \sum_{a=1}^{d-1} \frac{d \lambda/2}{a \sin(d \lambda/2)} + \sum_{i+j=d, i,j \neq 0} \frac{\lambda^{2}}{8 \sin(i \lambda/2) \sin(j \lambda/2)},
\end{equation}

then we have

\begin{equation}
\text{LHS} = - \int_{\mathcal{M}_{g,1}} (\lambda_{g-1} \psi_{1}^{2g-1}) d^{2g-1}
\end{equation}

\begin{equation}
- \sum_{k=0}^{2g-2} \frac{(2g-2-k)! (2g-3-k)}{2g-1-k} \int_{\mathcal{M}_{g,1}} \lambda_{g} \text{ch}_{2g-2-k}(\lambda_{g}) \psi_{1}^{k} d^{k},
\end{equation}

\begin{equation}
\text{RHS} = - \sum_{a=1}^{d-1} b_{g} a d^{2g-1} + b_{g} F_{0,g}(d) + \frac{1}{2} \sum_{g_{1}+g_{2}=g, g_{1}, g_{2} > 0} b_{g_{1}} b_{g_{2}} F_{g_{1},g_{2}}(d).
\end{equation}

Hence we can derive our main theorem:
Theorem 3.2. If $1 \leq m \leq 2g - 3$ and $g \geq 2$, then
\[-(2g - 2 - m)!(-1)^{2g-3-m}\int_{\mathcal{M}_{g,1}} \lambda_g \chi_{2g-2-m}(\mathbb{E}) \psi_1^m\]
\[= b_g \sum_{k=0}^{m-1} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m} B_{2g-1-m}\]
\[+ \frac{1}{2} \sum_{g_1 + g_2 = g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^\min(2g_2-1,m-1) \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g_1-1}{2g-1-k} B_{2g-1-m}.\]

Remark 3.3. Liu-Liu-Zhou [7] have only considered the cases $m = 2g - 1$ and $m = 1$.

3.3. The case of $m = 2g - 3$. If $m = 2g - 3$, we find that $1! \chi_1(\mathbb{E}) = \lambda_1$, then
\[LHS = -\int_{\mathcal{M}_{g,1}} \lambda_g \lambda_1 \psi_1^{2g-3},\]
\[RHS = b_g \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m} B_2\]
\[+ \frac{1}{2} \sum_{g_1 + g_2 = g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^\min(2g_2-1,2g-4) \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g_1-1}{2g-1-k} B_2.\]

From the above formula we obtain a new result of the Hodge integral.

Theorem 3.4. If $g \geq 2$, then
\[
\int_{\mathcal{M}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = \frac{1}{12} \left[ g(2g-3)b_g + b_1 b_{g-1} \right].
\]

Proof. Note that
\[
\int_{\mathcal{M}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = -b_g B_2 \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m}
\]
\[-\frac{B_2}{2} \sum_{g_1 + g_2 = g, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^\min(2g_2-1,2g-4) \frac{(-1)^{2g_2-1-k}}{2g-1-k} \binom{2g_2-1}{k} \binom{2g_1-1}{2g-1-k},\]

let us write
\[
A_1 = \sum_{k=0}^{2g-4} \frac{(-1)^{2g-1-k}}{2g-1-k} \binom{2g-1}{k} \binom{2g-1-k}{2g-1-m},
\]
\[
f_1(x) = \sum_{k=0}^{2g-3} (-1)^{2g-1-k} \binom{2g-1}{k} (2g - 2 - k)x^{2g-3-k},
\]
\[
g_1(x) = \sum_{k=0}^{2g-2} (-1)^{2g-1-k} \binom{2g-1}{k} x^{2g-2-k},
\]
then
\[ A_1 = \frac{1}{2} \sum_{k=0}^{2g-4} (-1)^{2g-1-k} \binom{2g-1}{k} (2g-2-k), \quad \text{for} \quad xg_1(x) = (1-x)^{2g-1} - 1, \quad f_1(x) = g'_1(x). \]

Hence
\[ f_1(x) = \frac{(2g-1)x(1-x)^{2g-2} - (1-x)^{2g-1} + 1}{x^2}, \quad f_1(1) = 1, \]
and we obtain
\[ A_1 = \frac{1}{2} \left[ f_1(1) - \binom{2g-1}{2g-3} \right] = \frac{1}{2} \left[ \binom{2g-1}{2g-3} - 1 \right]. \]

Similarly, we write
\[ A_2 = \sum_{k=0}^{\min(2g_2-1,2g-4)} \frac{(-1)^{2g_2-1-k} \binom{2g_2-1}{2g_2-1-k} (2g_2-1) (2g_2-1-k)}{2g_2-1-k} \]
then
\[ A_2 = \begin{cases} \frac{1}{2} \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} (2g_2-2-k) (2g_2-1) (2g_2-1-k) & \text{if} g_2 \leq g-2, \\ \frac{1}{2} \sum_{k=0}^{2g_2-4} (-1)^{2g_2-1-k} (2g_2-2-k) (2g_2-3) (2g_2-1-k) & \text{if} g_2 = g-1. \end{cases} \]

**Case 1:** \( g_2 \geq g-2. \) Let
\[ f_2(x) = \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} (2g_2-2-k) \binom{2g_2-1}{k} x^{2g_2-3-k}, \]
\[ g_2(x) = \sum_{k=0}^{2g_2-1} (-1)^{2g_2-1-k} \binom{2g_2-1}{k} x^{2g_2-2-k}. \]
Since \( g \geq g_2 + 2, \) then \( 2g-3 - (2g_2-1) \geq 2 > 0 \) and \( g'_2(x) = f_2(x). \) On the other hand
\[ g_2(x) = (1-x)^{2g_2-1} x^{2g_2-1}, \]
hence
\[ g'_2(x) = -(2g_2-1)(1-x)^{2g_2-2} x^{2g_2-1} + (2g_1-1)(1-x)^{2g_2-1} x^{2g_2-2} \]
and
\[ f_2(1) = \begin{cases} -1, & g_2 = 1, \\ 0, & 1 < g_2 \leq g-2. \end{cases} \]

**Case 2:** \( g_2 = g-1. \) let
\[ f_3(x) = \sum_{k=0}^{2g-4} (-1)^{2g-1-k} (2g-2-k) \binom{2g-3}{k} x^{2g-3-k}, \]
\[ g_3(x) = \sum_{k=0}^{2g-4} (-1)^{2g-1-k} \binom{2g-3}{k} x^{2g-2-k}. \]
Since \( 2g-3 - (2g-4) = 1 > 0, \)
\[ \frac{g_3(x)}{x} = \sum_{k=0}^{2g-4} (-1)^{2g-3-k} \binom{2g-3}{k} x^{2g-3-k} = (1-x)^{2g-3} - 1 \]
and
\[ g'_3(x) = (1-x)^{2g-3} - 1 - (2g-3)(1-x)^{2g-4}x, \]
therefore we have
\[ f_3(1) = \begin{cases} -2, & g = 2, \\ -1, & g > 2. \end{cases} \]
From the values of \( f_1(1), f_2(1), f_3(1) \), we obtain
\[
\int_{\mathcal{M}_{g,1}} \lambda_1 \lambda_g \psi_1^{2g-3} = -b_g B_2 A_1 - \frac{B_2}{2} \left[ \frac{1}{g_1+g_2=g,1 \leq g_2 \leq g-2} \sum b_{g_1} b_{g_2} f_2(1) + \frac{1}{2} b_1 b_{g-1} f_3(1) \right] \\
= \frac{B_2}{2} [-b_g A_1 + b_1 b_{g-1}] \\
= \frac{1}{12} \left[ g(2g-3)b_g + b_1 b_{g-1} \right]
\]
\[ \square \]

Since \( B_n = 0 \) for \( n \) odd and \( n > 1 \), we also have the following vanishing result.

**Theorem 3.5.** If \( g \geq 2 \), then for any \( 1 \leq t \leq g-1 \), we have
\[
(3.11) \quad \int_{\mathcal{M}_{g,1}} \lambda_g \text{ch}_{2t}(E) \psi_1^{2(g-1-t)} = 0.
\]

**4. Another Simple Proof of \( \lambda_g \) Conjecture**

Let \( |\mu| = d, l(\mu) = n \), denote by \([C_{g,\mu}(\tau)]_k\) the coefficient of \( \tau^k \) in the polynomial \( C_{g,\mu}(\tau) \), and let
\[
J^0_{g,\mu}(\tau) = \sqrt{-1}^{|\mu|-l(\mu)} C_{g,\mu}(\tau), \\
J^1_{g,\mu}(\tau) = \sqrt{-1}^{|\mu|-l(\mu)-1} \left( \sum_{\nu \in J(\mu)} I_1(\nu) C_{g,\mu}(\tau) + \sum_{\nu \in C(\mu)} I_2(\nu) C_{g-1,\nu}(\tau) + \sum_{g_1+g_2=g,\mu_1+\mu_2 \in C(\mu)} I_3(\nu^1,\nu^2) C_{g_1,\nu^1}(\tau) C_{g_2,\nu^2}(\tau) \right).
\]
The set \( J(\mu) \) consists of partitions of \( d \) of the form
\[ \nu = (\mu_1, \cdots, \tilde{\mu}_i, \cdots, \mu_{l(\mu)}, \mu_i + \mu_j) \]
and the set \( C(\mu) \) consists of partitions of \( d \) of the form
\[ \nu = (\mu_1, \cdots, \tilde{\mu}_i, \cdots, \mu_{l(\mu)}, j, k) \]
where \( j + k = \mu_i \). The definitions of \( I_1, I_2 \) and \( I_3 \) can be found in [5]. Liu-Liu-Zhou [6] have proved the following differential equation:
\[
(4.1) \quad \frac{d}{d\tau} J^0_{g,\mu}(\tau) = -J^1_{g,\mu}(\tau).
\]
It is straightforward to check that

\[
[C_{g,\mu}(\tau)]_{n-1} = -\frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \int_{\mathcal{M}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^{n}(1 - \mu_i \psi_i)},
\]

\[
\left[ \sum_{\nu \in J(\mu)} I_1(\nu) C_{g,\nu}(\tau) \right]_{n-2} = -\frac{\sqrt{-1}^{d+n-1}}{|\text{Aut}(\nu)|} \int_{\mathcal{M}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1}(1 - \mu_i \psi_i)},
\]

\[
\left[ \sum_{\nu \in C(\mu)} I_2(\nu) C_{g-1,\nu}(\tau) \right]_{n-2} = 0,
\]

\[
\left[ \sum_{g_1 + g_2 = g, \nu^1, \nu^2 \in C(\mu)} I_3(\nu^1, \nu^2) C_{g_1,\nu^1}(\tau) C_{g_2,\nu^2}(\tau) \right]_{n-2} = 0,
\]

hence, from (31) we have the identity

(4.2) \[ \frac{n-1}{|\text{Aut}(\mu)|} \int_{\mathcal{M}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^{n}(1 - \mu_i \psi_i)} = \sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \int_{\mathcal{M}_{g,n-1}} \frac{\lambda_g}{\prod_{i=1}^{n-1}(1 - \mu_i \psi_i)}. \]

**Theorem 4.1.** For any partition \( \mu : \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n > 0 \) of \( d \) and \( g > 0 \), then

(4.3) \[ \int_{\mathcal{M}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^{n}(1 - \mu_i \psi_i)} = d^{2g+n-3} b_g. \]

**Proof.** Recall the definition of \( I_1(\nu) \), where \( \nu = (\mu_1, \cdots, \tilde{\mu}_i, \cdots, \tilde{\mu}_j, \cdots, \mu_n, \mu_i + \mu_j) \):

\[ I_1(\nu) = \frac{\mu_i + \mu_j}{1 + \delta_{\mu_i j}} m_{\mu_i + \mu_j}(\nu), \]

and it is easy to see that

\[ m_{\mu_i + \mu_j}(\nu) \frac{|\text{Aut}(\nu)|}{|\text{Aut}(\mu)|} = m_{\mu_i}(\mu)(m_{\mu_j}(\mu) - \delta_{\mu_i j}). \]

Let

\[
\mu : \mu_{k_1} = \cdots = \mu_{k_1} > \mu_{k_2} = \cdots = \mu_{k_2} > \cdots > \mu_{k_s} = \cdots = \mu_{k_s} > 0,
\]

where

\[
\sum_{i=1}^{s} t_i = n, \quad \sum_{i=1}^{s} t_i \mu_{k_i} = d,
\]
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then

\[
\sum_{\nu \in J(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|}
= \frac{1}{|\text{Aut}(\mu)|} \sum_{\nu \in J(\mu)} \frac{\mu_i + \mu_j}{1 + \delta_{ij}} [m_{\mu_i}(\mu)m_{\mu_j}(\mu) - \delta_{ij}]
\]

\[
= \frac{1}{|\text{Aut}(\mu)|} \left[ \frac{1}{2} \sum_{i=1}^s \sum_{j \neq t} (\mu_{k_i} + \mu_{k_j}) m_{\mu_{k_i}}(\mu)m_{\mu_{k_j}}(\mu) + \sum_{i=1}^s \mu_{k_i} m_{\mu_{k_i}}(\mu)(m_{\mu_{k_i}}(\mu) - 1) \right]
\]

\[
= \frac{1}{|\text{Aut}(\mu)|} \left[ \frac{1}{2} \sum_{i=1}^s \sum_{j \neq t} (\mu_{k_i} + \mu_{k_j}) t_i t_j + \sum_{i=1}^s \mu_{k_i} t_i (t_i - 1) \right]
\]

\[
= \frac{1}{|\text{Aut}(\mu)|} \left[ \sum_{i=1}^s \sum_{j \neq t} \mu_{k_j} t_j t_i + \sum_{i=1}^s \mu_{k_i} t_i^2 - d \right]
\]

\[
= \frac{1}{|\text{Aut}(\mu)|} \left[ \sum_{i=1}^s \mu_{k_i} t_i (d - \mu_{k_i} t_i) + \sum_{i=1}^s \mu_{k_i} t_i^2 - d \right]
\]

\[
= (n - 1)d
\]

By the induction of \(n\) and the initial value of the Mariño-Vafa formula

\[
\int_{\mathcal{M}_{g,1}} \frac{\lambda_g}{1 - \mu_1 \psi_1} = d^{2g-2} b_g,
\]

we have

\[
\int_{\mathcal{M}_{g,n}} \frac{\lambda_g}{\prod_{i=1}^n (1 - \mu_i \psi_1)} = d \cdot d^{2g+n-1-3} b_g = d^{2g+n-3} b_g.
\]

\(\square\)

**Corollary 4.2.** The following \(\lambda_g\) conjecture is true:

\[
(4.4) \quad \int_{\mathcal{M}_{g,n}} \lambda_g \prod_{l=1}^n \psi_i^{k_l} = \binom{2g + n - 3}{k_1, \ldots, k_n} b_g,
\]

where \(g > 0\).

5. A RECURSION FORMULA OF THE \(\lambda_{g-1}\) INTEGRAL

E.Getzler, A.Okounkov and R.Pandharipande have derived explicit formula for the multipoint series of \(\mathbb{C}P^1\) in degree 0 from the Toda hierarchy [2], then they obtained certain formulas for the Hodge integrals \(\int_{\mathcal{M}_{g,n}} \lambda_{g-1} \psi_1^{k_1} \cdots \psi_n^{k_n}\). In this section we give an effective recursion formula of the \(\lambda_{g-1}\) integrals using Mariño-Vafa formula. It is straightforward to check the following lemma.
**Lemma 5.1.** We have the following identities

\[
\left[ \prod_{i=1}^{n} \frac{\prod_{a=1}^{\mu_{i} - 1}(\mu_{i} \tau + a)}{\mu_{i} - 1)!} \right]_{0} = 1,
\]

\[
\left[ \prod_{i=1}^{n} \frac{\prod_{a=1}^{\mu_{i} - 1}(\mu_{i} \tau + a)}{\mu_{i} - 1)!} \right]_{1} = \sum_{i=1}^{n} \sum_{a=1}^{\mu_{i} - 1} \mu_{i} a,
\]

\[
[\Lambda_{g}^{\psi}(1)\Lambda_{g}^{\psi}(-\tau - 1)\Lambda_{g}^{\psi}(\tau)]_{0} = \lambda_{g},
\]

\[
[\Lambda_{g}^{\psi}(1)\Lambda_{g}^{\psi}(-\tau - 1)\Lambda_{g}^{\psi}(\tau)]_{0} = -\lambda_{g-1} - \sum_{k \geq 0} k!(-1)^{k-1}ch_{k}(E)\lambda_{g},
\]

and

\[
[C_{g,\mu}(\tau)]_{n-1} = -\frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \int_{\mathcal{M}_{g,n}} \frac{\lambda_{g}}{\prod_{i=1}^{n}(1 - \mu_{i}\psi_{i})},
\]

\[
[C_{g,\mu}(\tau)]_{n} = -\frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \left[ n - 1 + \sum_{i=1}^{n} \sum_{a=1}^{\mu_{i}-1} \frac{\mu_{i}}{a} \right] \int_{\mathcal{M}_{g,n}} \frac{\lambda_{g}}{\prod_{i=1}^{n}(1 - \mu_{i}\psi_{i})}
\]

\[
+ \frac{\sqrt{-1}^{d+n}}{|\text{Aut}(\mu)|} \int_{\mathcal{M}_{g,n}} \frac{\lambda_{g-1} + \sum_{k \geq 0} k!(-1)^{k-1}ch_{k}(E)\lambda_{g}}{\prod_{i=1}^{n}(1 - \mu_{i}\psi_{i})},
\]

Now, we can state our main theorem in this section using equation (31).

**Theorem 5.2.** For any partition \( \mu \) with \( l(\mu) = n \), we have the following recursion formula

\[
\frac{n}{|\text{Aut}(\mu)|} \left[ n - 1 + \sum_{i=1}^{n} \sum_{a=1}^{\mu_{i}-1} \frac{\mu_{i}}{a} \right] \int_{\mathcal{M}_{g,n}} \frac{\lambda_{g}}{\prod_{i=1}^{n}(1 - \mu_{i}\psi_{i})}
\]

\[
- \frac{n}{|\text{Aut}(\mu)|} \int_{\mathcal{M}_{g,n}} \frac{\lambda_{g-1} + \sum_{k \geq 0} k!(-1)^{k-1}ch_{k}(E)\lambda_{g}}{\prod_{i=1}^{n}(1 - \mu_{i}\psi_{i})}
\]

\[
= \sum_{\nu \in J(\mu)} \frac{I_{1}(\nu)}{|\text{Aut}(\nu)|} \left[ n - 2 + \sum_{i=1}^{n-1} \sum_{a=1}^{\nu_{i}-1} \frac{\nu_{i}}{a} \right] \int_{\mathcal{M}_{g,n-1}} \frac{\lambda_{g}}{\prod_{i=1}^{n-1}(1 - \nu_{i}\psi_{i})}
\]

\[
- \sum_{\nu \in J(\mu)} \frac{I_{1}(\nu)}{|\text{Aut}(\nu)|} \int_{\mathcal{M}_{g,n-1}} \frac{\lambda_{g-1} + \sum_{k \geq 0} k!(-1)^{k-1}ch_{k}(E)\lambda_{g}}{\prod_{i=1}^{n-1}(1 - \nu_{i}\psi_{i})}
\]

\[
+ \sum_{g_{1}+g_{2}=g_{1}, g_{2} \geq 0} \sum_{\nu_{1}, \nu_{2} \in C(\mu)} \frac{I_{3}(\nu_{1}, \nu_{2})}{|\text{Aut}(\nu_{1})||\text{Aut}(\nu_{2})|} \int_{\mathcal{M}_{g_{1},n_{1}}} \frac{\lambda_{g_{1}}}{\prod_{i=1}^{n_{1}}(1 - \nu_{i}^{1}\psi_{i})} \int_{\mathcal{M}_{g_{2},n_{2}}} \frac{\lambda_{g_{2}}}{\prod_{i=1}^{n_{2}}(1 - \nu_{i}^{2}\psi_{i})}.
\]

5.1. **The \( \lambda_{g} \)-Integral.** In this subsection, we re-derive the \( \lambda_{g} \)-integral from theorem 5.2. Let \( \mu_{i} = Nx_{i} \) for some \( N \in \mathbb{N} \) and \( x_{i} \in \mathbb{R} \), from Kim-Liu[4]’s method and consider the
coefficients of $\ln NN^{2g+n-2}$ in theorem 5.2., then
\[ n(x_1 + \cdots + x_n) \prod_{l=1}^{n} x_l^{k_l} \int_{\mathcal{M}_{g,n}} \lambda_g \prod_{l=1}^{n} \psi_l^{k_l} \]
\[ = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} (x_i + x_j)^{k_i+k_j} (x_1 + \cdots + x_n) \prod_{l \neq i,j} x_l^{k_l} \int_{\mathcal{M}_{g,n-1}} \lambda_g \psi_l^{k_l} \]
\[ + (x_1 + \cdots + x_n) \prod_{l=1}^{n} x_l^{k_l} \int_{\mathcal{M}_{g,n}} \lambda_g \prod_{l=1}^{n} \psi_l^{k_l}, \]

i.e.
\[(n-1) \prod_{l=1}^{n} x_l^{k_l} \int_{\mathcal{M}_{g,n}} \lambda_g \prod_{l=1}^{n} \psi_l^{k_l} \]
\[ = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} (x_i + x_j)^{k_i+k_j} \prod_{l \neq i,j} x_l^{k_l} \int_{\mathcal{M}_{g,n-1}} \lambda_g \psi_l^{k_l} \]
\[ \cdot \prod_{l \neq i,j} \psi_l^{k_l}. \]

After introducing the formal variables $s_i \in \mathbb{R}^+$ and applying the Laplace transformation
\[ \int_{0}^{+\infty} x^k e^{-x/2s} \, dx = k!(2s)^{k+1}, \quad s > 0, \]
we select the coefficient of $\prod_{l=1}^{n} (2s_l)^{k_l+1}$ from the transformation of (35), then we derive
\[(n-1) \int_{\mathcal{M}_{g,n}} \lambda_g \prod_{l=1}^{n} \psi_l^{k_l} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} (k_i + k_j)! \prod_{l \neq i,j} k_l! k_j ! \int_{\mathcal{M}_{g,n-1}} \lambda_g \psi_l^{k_l} \cdot \prod_{l \neq i,j} \psi_l^{k_l} \]
\[ \cdot \prod_{l \neq i,j} \psi_l^{k_l}. \]

By the induction of $n$, we obtain the $\lambda_g$ conjecture
\[ \int_{\mathcal{M}_{g,n}} \lambda_g \prod_{l=1}^{n} \psi_l^{k_l} = (2g+n-3) \left( k_1, \ldots, k_n \right) b_g, \]
in fact, in (36) we have
\[ \text{RHS} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{(k_i + k_j)!}{k_i! k_j !} \frac{(2g+n-4)!}{\prod_{l \neq i,j} k_l!(k_i + k_j - 1)!} b_g \]
\[ = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \frac{k_i + k_j}{2g + n - 3} \left( 2g + n - 3 \right) \left( k_1, \ldots, k_n \right) b_g, \]

note that $k_1 + \cdots + k_n = 2g + n - 3$, therefore
\[ \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} (k_i + k_j) = \frac{1}{2} \sum_{i=1}^{n} [(n-1)k_i + (2g+n-3-k_i)] \]
\[ = \frac{1}{2} \left[ (n-2)(2g+n-3) + (2g+n-3)n \right] \]
\[ = \frac{1}{2} \left[ (2n-2)(2g+n-3) \right] \]
\[ = (n-1)(2g + n - 3). \]
5.2. **The Recursion Formula of \( \lambda_{g-1} \)-integral.** We have found the **singular part** \( \sum_{i=1}^{n} \sum_{a=1}^{\mu_{i}-1} \frac{\mu_i}{a} \) in theorem 5.2., using the following theorem, we can eliminate this part and derive the recursion formula of \( \lambda_{g-1} \)-integral. The notation \([F]_{\text{sing}}\) means the singular part of \( F \).

First, in theorem 5.2., we have

\[
\left[ \frac{\text{LHS}}{d^{2g+n-4}b_g} \right]_{\text{sing}} = n \sum_{i=1}^{n} \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} d,
\]

\[
\left[ \frac{\text{RHS}}{d^{2g+n-4}b_g} \right]_{\text{sing}} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} (\mu_i + \mu_j) \left( \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} + (\mu_i + \mu_j) \sum_{a=1}^{\mu_i+\mu_j-1} \frac{1}{a} \right)
\]

\[+ \sum_{i=1}^{n} \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} d - \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_j (\mu_i + \mu_j)}{a} .\]

**Theorem 5.3.** Under the above notation, we have

\[
\left[ \frac{\text{RHS}}{d^{2g+n-4}b_g} \right]_{\text{sing}} = \left[ \frac{\text{LHS}}{d^{2g+n-4}b_g} \right]_{\text{sing}} + 2(n - 1)d.
\]

**Proof.** Since

\[
\sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_j (\mu_i + \mu_j)}{a}
\]

\[= \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_i (\mu_i + \mu_j)}{a}
\]

\[= \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_i (\mu_i + \mu_j)}{a}
\]

\[= \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a} - \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{\mu_j (\mu_i + \mu_j)}{a}
\]

\[+ \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=\mu_j+1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)}{a}.
\]

where we use the identity

\[
\sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)\mu_i}{a} = \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)\mu_j}{a}
\]

\[= \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{a=1}^{\mu_i+\mu_j-1} \frac{(\mu_i + \mu_j)^2}{a}.
\]
Note that $\sum_{i=1}^{n} \sum_{j \neq i} (\mu_i + \mu_j) = 2(n - 1)d$, hence

\[
\left[ \frac{RHS}{d^{2g+n-4}b_g} \right]_{\text{sing}}
\]

\[= \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} (\mu_i + \mu_j) \sum_{l \neq i,j} \frac{\mu_l}{a} + \sum_{i=1}^{n} \sum_{a=1}^{\mu_i-1} \frac{\mu_i d}{a}
\]

\[+ \sum_{i=1}^{n} \sum_{j \neq i} \frac{\mu_j}{a} (\mu_i + \mu_j)^2 - \sum_{i=1}^{n} \sum_{j \neq i} \frac{\mu_i \mu_j}{a} + \sum_{i=1}^{n} \sum_{j \neq i} (\mu_i + \mu_j)
\]

\[= \left( \sum_{i=1}^{n} \frac{\mu_i}{a} \right) d + \frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} (\mu_i + \mu_j) \sum_{l \neq i,j} \frac{\mu_l}{a} + 2(n - 1) d,
\]

it is straightforward to check that

\[
\frac{1}{2} \sum_{i=1}^{n} \sum_{j \neq i} (\mu_i + \mu_j) \sum_{l \neq i,j} \frac{\mu_l}{a} = (n - 2) \sum_{i=1}^{n} \frac{\mu_i \mu_j - 1}{a},
\]

\[
\sum_{i=1}^{n} \sum_{j \neq i} \frac{\mu_j}{a} (\mu_i + \mu_j) = \sum_{i=1}^{n} \frac{\mu_i \mu_j}{a} + \sum_{i=1}^{n} \sum_{j \neq i} \frac{\mu_j}{a},
\]

\[
\sum_{i=1}^{n} \sum_{j \neq i} \frac{\mu_j}{a} = (n - 1) \sum_{i=1}^{n} \frac{\mu_i}{a}.
\]

Finally, we obtain

\[
\left[ \frac{RHS}{d^{2g+n-4}b_g} \right]_{\text{sing}}
\]

\[= \left( \sum_{i=1}^{n} \frac{\mu_i}{a} \right) d + (n - 2) \sum_{i=1}^{n} \frac{\mu_i \mu_j - 1}{a}
\]

\[+ \sum_{i=1}^{n} \frac{\mu_i}{a} \sum_{j \neq i} \frac{\mu_j}{a} + (n - 1) \sum_{i=1}^{n} \frac{\mu_i \mu_j}{a} + 2(n - 1) d
\]

\[= \left( \sum_{i=1}^{n} \frac{\mu_i}{a} \right) d + (n - 1) \sum_{i=1}^{n} \mu_i \left( \sum_{j \neq i} \frac{\mu_j}{a} + \sum_{a=1}^{\mu_i-1} \frac{\mu_i}{a} \right) + 2(n - 1) d
\]

\[= \left( \sum_{i=1}^{n} \frac{\mu_i}{a} \right) d + (n - 1) \left( \sum_{i=1}^{n} \frac{\mu_i}{a} \right) d + 2(n - 1) d
\]

\[= \left[ \frac{LHS}{d^{2g+n-4}b_g} \right]_{\text{sing}} + 2(n - 1) d.
\]
Let $\mathbb{R}[\mu_1, \cdots, \mu_n]$ be the space of all homogeneous polynomials with real coefficients in $\mu_1, \cdots, \mu_n$ of degree $k$, then it is the subring of $\mathbb{R}[\mu_1, \cdots, \mu_n]$. From the Theorem 5.3, we obtain the recursion formula of $\lambda_{g-1}$ Hodge integral.

**Theorem 5.4.** For any partition $\mu$ with $l(\mu) = n$ and $|\mu| = d$, we have the recursion formula

$$
\frac{n}{|\text{Aut}(\mu)|} \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n} (1 - \mu_i \psi_i) \lambda_{g-1} = \sum_{\nu \in \mathcal{J}(\mu)} \frac{I_1(\nu)}{|\text{Aut}(\nu)|} \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n-1} (1 - \nu_i \psi_i) \lambda_{g-1} - \sum_{g_1+g_2=g, \nu_1, \nu_2 \in C(\mu)} I_2(\nu_1, \nu_2) \int_{\mathcal{M}_{g,n}} \prod_{i=1}^{n-1} (1 - \nu_i \psi_i) \lambda_{g-1} \prod_{i=1}^{n} (1 - \nu_i \psi_i)
$$

under the ring $\mathbb{R}_{2g-2+n}[\mu_1, \cdots, \mu_n]$, where $l(\nu_i) = n_i$ and $|\nu_i| = d_i$ for $i = 1, 2$.

**Remark 5.5.** When we consider the simplest case $n = 1$, the above identity become the formula used in [6].

### 6. Some Examples of The Main Theorem

In this section we give some examples of theorem 3.2.

#### 6.1. The case of $g = 3$.

If $g = 3$, then $1 \leq m \leq 3$. We consider three cases.

**6.1.1. $m=1$.** $LHS = -3 \int_{\mathcal{M}_{3,1}} \lambda_3 \text{ch}^3(\mathcal{E}) \psi_1$, and $3 \text{ch}_3(\mathcal{E}) = \sum_{i+j=3} (-1)^{i-j} \lambda_i \lambda_j = 3 \lambda_3 - \lambda_1 \lambda_2$, then we get

$$
LHS = \int_{\mathcal{M}_{3,1}} \lambda_3 (\lambda_1 \lambda_2 - 3 \lambda_3) \psi_1 = \int_{\mathcal{M}_{3,1}} \lambda_1 \lambda_2 \lambda_3 \psi_1,
$$

$$
RHS = b_3 \frac{(-1)^5}{5} \binom{5}{0} \binom{4}{5} B_4 + \frac{1}{2} \sum_{g_1+g_2=3, g_1, g_2 > 0} b_1 b_2 \frac{1}{5} \binom{2g_2-1}{0} \binom{5}{4} B_4
$$

$$
= -B_4 (b_3 + b_1 b_2).
$$

Since $b_1 = \frac{1}{24}, b_2 = \frac{7}{3700}, b_3 = \frac{31}{96000}$, we have

$$
(6.1) \int_{\mathcal{M}_{3,1}} \lambda_1 \lambda_2 \lambda_3 \psi_1 = \frac{1}{362880}.
$$
6.1.2. $m=2$. In this case we have $LHS = 2 \int_{\mathcal{M}_{3,1}} \lambda_3 \text{ch}_2(\mathcal{E}) \psi_1^2$, and $2! \text{ch}_2(\mathcal{E}) = 2\lambda_2 - \lambda_1^2$, $B_3 = 0$. Then we have

\[
LHS = \int_{\mathcal{M}_{3,1}} (2\lambda_2 \lambda_3 - \lambda_3 \lambda_1^2) \psi_1^2,
\]

\[
RHS = b_3 \sum_{k=0}^{1} \frac{(-1)^{4-k}}{5-k} \binom{5}{3} \left(\frac{5-k}{3}\right) B_3
\]

\[
+ \frac{1}{2} \sum_{g_1+g_2=3, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{1} \frac{(-1)^{5-k}}{5-k} \binom{2g_2 - 1}{3} \left(\frac{5-k}{3}\right) B_3
\]

\[
= 0,
\]

hence

\[
\int_{\mathcal{M}_{3,1}} \lambda_3 \lambda_1^2 \psi_1^2 = 2 \int_{\mathcal{M}_{3,1}} \lambda_3 \lambda_1^2 \psi_1^2.
\]

Using the formula $\int_{\mathcal{M}_{3,1}} \lambda_2 \lambda_3 \psi_1^2 = \frac{1}{120960}$, we get

\[
(6.2) \quad \int_{\mathcal{M}_{3,1}} \lambda_3 \lambda_1^2 \psi_1^2 = \frac{1}{60480}.
\]

6.1.3. $m=3$. In this case $LHS = - \int_{\mathcal{M}_{3,1}} \lambda_3 \text{ch}_1(\mathcal{E}) \psi_1^3$ and $\text{ch}_1(\mathcal{E}) = \lambda_1$, hence

\[
LHS = - \int_{\mathcal{M}_{3,1}} \lambda_1 \lambda_3 \psi_1^3,
\]

\[
RHS = b_3 \sum_{k=0}^{2} \frac{(-1)^{5-k}}{5-k} \binom{5}{2} \left(\frac{5-k}{2}\right) B_2
\]

\[
+ \frac{1}{2} \sum_{g_1+g_2=3, g_1, g_2 > 0} b_{g_1} b_{g_2} \sum_{k=0}^{\min(2g_2-1,2)} \frac{(-1)^{2g_2-1-k}}{5-k} \binom{2g_2 - 1}{2} \left(\frac{5-k}{2}\right) B_2
\]

\[
= \frac{9}{2} b_3 B_2 - \frac{1}{2} b_1 b_2 B_2
\]

\[
= - \frac{41}{1451520},
\]

so

\[
(6.3) \quad \int_{\mathcal{M}_{3,1}} \lambda_1 \lambda_3 \psi_1^3 = \frac{41}{1451520}.
\]

**Remark 6.1.** The values of (37) and (39) match with the results in [9], the identity (38) is a new result.

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REFERENCES

[1] C. Faber, R. Pandharipande. *Hodge integrals and Gromov-Witten theory*, Invent. Math. **139** (2000), 173-199.

[2] C. Faber, R. Pandharipande. *Hodge integrals, partition matrices, and the \( \lambda_g \) conjecture*, Ann. Math. (2) **157** (2003), 97-124.

[3] E. Getzler, A. Okounkov, R. Pandharipande. *Multipoint series of Gromov-Witten invariants of \( \mathbb{C}P^1 \)*, preprint: math.AG/0207106

[4] Y.-S. Kim, K. Liu. *A simple proof of Witten conjecture through localization*, preprint: math.AG/0508384

[5] A.M. Li, G. Zhao, Q. Zheng. *The number of ramified coverings of a Riemann surface by Riemann surface*, Comm. Math. Phys. **213** (2000), 685-696.

[6] C.-C. Liu, K. Liu, J. Zhou. *A proof of a conjecture of Mariño-Vafa on Hodge Integrals*, J. Differential Geom. **65** (2003), 289-340.

[7] W. Lu. *A note of the Mariño-Vafa formula*, Science in China Ser. A Math. (11) **35** (2005), 1276-1287. (in chinese)

[8] K. Liu, H. Xu. *Estimate of denominators of Hodge integrals*, preprint, 2006.

[9] M. Mariño, C. Vafa. *Framed knots at large \( N \)*, Orbifolds in mathematics and Physics (Madison, WI, 2001), 185-204, Contemp. Math., **310**, Amer. Math. Soc., Providence, RI, 2002.

[10] D. Mumford. *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and geometry, Vol. II, Progr. Math., **36**, Birkhäuser Boston, Boston, 1983, pp. 271-328.

[11] J. Zhou. *Some closed formulas and conjectures for Hodge integrals*, Math. Res. Lett. **10** (2003), 275-286.

[12] **Hodge integrals, Hurwitz numbers, and symmetric group**, preprint: math.AG/0308024

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