Renormalisation Group Improved
Thermal Coupling Constant
In An External Field

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Abstract

Starting from renormalised Effective Lagrangian, in the presence of an external Chromo-Electric field at finite temperature, the expression for thermal coupling constant \( \alpha = \frac{g^2}{4\pi} \) as a function of temperature and external field is derived, using finite temperature two parameter renormalisation group equation of Matsumoto, Nakano and Umezawa. For some values of the parameters, the coupling constant is seen to be approaching a value \( \sim \) unity.

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Study of coupling constant in finite temperature field theory (FTFT) \[\text{[1]}\] is of interest in various disciplines of physics, for the study of processes taking place in different physical situations varying from Early Universe Cosmology, Astro-physics, to Relativistic Heavy Ion Collision. From the strong interaction physics point of view, it has been extensively used in the description of Quark Gluon Plasma, the elusive state of matter, supposed to form in Relativistic Heavy Ion collision. Since the inception of the concept of asymptotic freedom, in the context of non-abelian (NA) gauge theories, it was argued in ref\[\text{[2]}\] that, as in the high momentum exchange processes, at high temperature/density or both of them, strong coupling constant \( \alpha_R \) also decreases making the system behave in an asymptotically free manner. Since then, from the point of view of strong interaction physics, i.e. Quantum Chromo Dynamics (QCD),
many studies of the running coupling constant, have been performed with different renormalisation schemes.

In this note of ours, we, instead of taking full QCD into account, and going into the details of those aspects, will be contented by showing the running of the strong coupling constant with respect to temperature in the presence of an external field with a SU(2) or U(1) internal symmetry present in it. This model was originally studied by Schwinger and has been applied successfully in RHIC also in some astrophysical as well as cosmological scenarios.

In this note, we show that, in the presence of heat bath and an external electric field, existence of a phase may be possible where the value of $\alpha_R$ can be around unity. In fact depending on the strength of the electric field, in the low temperature region, the running coupling constant shows an oscillatory behavior, where $\alpha_R(E, T)$ can have values of the order of unity. But interestingly as the temperature still increased there exists a critical "temperature" around which the coupling constant grows enormously and beyond that it (coupling constant) changes its sign; perhaps signaling a change of the phase of the system. Considering the limitations of this study, apriori it is not clear whether this analysis, can be extended to that domain of temperature, however we will discuss this point as we go along.

Having stated the motivation behind this study, we go about explaining the organisation of this document. That is as follows, in section II we provide the necessary background for this study, in section III we give the essential steps to get to the expression of the renormalised effective action. In section IV the computation of the coupling constant with the numerical results will be presented. In the last section we conclude by describing the future plans in this direction.

1 Thermal running coupling constant from Effective action

A formalism for the computation of the running coupling constant, at zero temperature, was developed by Coleman and Weinberg (CW) in the context of studying spontaneous symmetry breaking, in gauge theory, by radiative correction. For the purpose of computing the same quantity at nonzero temperature, one needs to generalise the method of CW to finite temperature by using the proper renormalisation group equations. At finite temperature, following Matsumoto, Nakano and Umezawa, we work with the two parameter renormalisation (RG) equations at finite temperature. In this formalism, in addition to the momentum variable $\mu$ one
introduces a dimensionful parameter $\tau$ at the corresponding renormalisation point (at nonzero temperature) and then writes a set of renormalisation group equations (RGE) w.r.t $\tau$. From the solution of these set of (two parameter) RGEs one recovers the corresponding flow of the coupling constant w.r.t temperature or external field or both.

Since the effective action is the generating functional of the one particle irreducible (1pI) Green functions, the RG equation satisfied by the Green functions should also be satisfied by the $L_{\text{eff}}$. So following the authors of ref [10] the flow equations can be written in terms of the effective Lagrangian as

$$
(\mu \partial \partial \mu + \beta_{\mu} \partial \partial g + 2\gamma_{\mu}(g)F \partial \partial F) \bar{L}_{\text{eff}} = 0 \tag{1}
$$

$$
(\tau \partial \partial \tau + \beta_{\tau} \partial \partial g + 2\gamma_{\tau}(g)F \partial \partial F) \bar{L}_{\text{eff}} = 0 \tag{2}
$$

Here the effective Lagrangian i.e. $L_{\text{eff}}=L_{\text{eff}}(gE,T,\mu,\tau)$ and $F$ corresponds to $\frac{E^2-B^2}{2}$; though for our purpose we set $B = 0$ We rewrite these equations (1) and (2) in terms of dimensionless quantity $\bar{L}_{\text{eff}} = \frac{L_{\text{eff}}}{\partial L_{\text{eff}} / \partial F}$ so as to get

$$
\left( \mu \partial \partial \mu + \beta_{\mu} \partial \partial g + 2\gamma_{\mu}(g) \left( 1 + F \partial \partial F \right) \right) \bar{L}_{\text{eff}} = 0 \tag{3}
$$

$$
\left( \tau \partial \partial \tau + \beta_{\tau} \partial \partial g + 2\gamma_{\tau}(g) \left( 1 + F \partial \partial F \right) \right) \bar{L}_{\text{eff}} = 0 \tag{4}
$$

Here, $\beta_{\mu}$, $\gamma_{\mu}$, $\beta_{\tau}$, and $\gamma_{\tau}$ are defined as $\beta_{\mu} = \frac{dg}{d\mu}$, $\gamma_{\mu} = \frac{dlnZ}{d\mu}$ and $\beta_{\tau} = \frac{dg}{d\tau}$, $\gamma_{\tau} = \frac{dlnZ}{d\tau}$.

Subtracting equation (3) from equation (4) and introducing another new variable $\zeta$, such that $ln\frac{\tau}{\mu} = ln\zeta$, we rewrite the corresponding equations to arrive at

$$
\left( \zeta \partial \partial \zeta + \beta_{\zeta} \partial \partial g + 2\gamma_{\zeta}(g) \left( 1 + F \partial \partial F \right) \right) \bar{L}_{\text{eff}} = 0 \tag{5}
$$

At this stage, if we introduce a dimensionless quantity $\kappa = 2ln \left[ \frac{2\xi_{\tau}^{1/2}}{\zeta} \right]$, equation (5) takes the form

$$
\left( - \partial \partial \kappa + \beta_{\kappa} \partial \partial g + 2\gamma_{\kappa}(g) \right) \bar{L}_{\text{eff}} = 0 \tag{6}
$$

The quantities $\beta_{\kappa}$, $\gamma_{\kappa}$ are defined to be $\beta_{\kappa} = \frac{\beta_{\zeta}}{(1-\gamma_{\kappa})}$ and $\gamma_{\kappa} = \frac{\gamma_{\zeta}}{(1-\gamma_{\kappa})}$. This is the basic equation, that when solved with the boundary condition,

$$
\left( \frac{\partial L_{\text{eff}}}{\partial F} \right)_{\kappa=0} = 1 \tag{7}
$$
will provide us with the information, how the coupling constant changes with the change of the corresponding scale defined by the strength of the external field and/or temperature. This boundary condition is chosen at a particular scale, such that; 
\[ \frac{2gE_T}{\mu} = 1. \] The method of solving equation (6) quite standard, and the solution is,
\[ \bar{L}_{\text{eff}}(\kappa, g) = -\exp\left[2\int_0^\kappa \bar{g}(x, g) \, dx\right] \]
as can be checked by direct substitution also. Since \( \beta = -g\gamma \) (This can be derived from the relation between the bare and the renormalised coupling constant i.e. \( g_b = Z_r g \) and differentiate it w.r.t \( \kappa \), then set the right side to zero and use standard results) and as
\[ \frac{dg(\kappa)}{d\kappa} = \beta(g(\kappa)) \]
using these two results one can arrive at the relation
\[ \bar{L}_{\text{eff}} = -\frac{1}{g^2(\kappa)} \] (8)
Using the relation, \( \bar{L}_{\text{eff}} = \frac{\partial L_{\text{eff}}}{\partial F} \), as defined before, the equation for the flow of running coupling constant becomes
\[ \bar{L}_{\text{eff}} = \frac{\partial L_{\text{eff}}}{\partial F} = -\frac{1}{g^2(\kappa)} \] (9)
We will employ this relation to see the flow of coupling at finite temperature and nonzero external field.

2 Renormalisation of the Effective Lagrangian

We start from the “partition function” in Minkowski space defined by
\[ Z[A] = \int D\bar{\psi} D\psi e^{i\int L d^4x} \]
where \( L = \bar{\psi}(i\gamma_\mu \partial^\mu - g\gamma_\mu A_{\mu}^a \tau_a) \psi - m\bar{\psi}\psi \) is the fermionic Lagrangian in the presence of external vector field \( A_{\mu}^a \) with a \( SU(2) \) color symmetry present in it, \( \tau_a \)’s are the Pauli matrices and \( L_0 = \bar{\psi}(i\gamma_\mu \partial^\mu - m) \psi \) is the free fermionic Lagrangian so that \( Z[0] = 1 \).

Since we are interested in evaluating the effective action, in the presence of external chromo-electric field, we choose \( A_0^a = -E^a z \) and other components of \( A \).
to be equal to zero. The expression for the effective action is calculated, from the euclidean action,

\[ S_{\beta} = \frac{1}{\beta^2} \sum_{n=-\infty}^{\infty} \int \bar{\psi}_n(x) \left[ (\omega_n \gamma^0 + gA^a_o \tau^a \gamma^0) + i\gamma^j \partial_j - m \right] \psi_n(x) d^3x \]  

(11)

that is gotten after wick rotating the time component of the original Minkowski space action and compactifying the imaginary time direction over a length scale \( \beta = \frac{1}{T} \) (for details see ref [1]). Here the fermion fields, i.e the \( \psi \)'s are in the fundamental representation of SU(2) defined as

\[ \psi(x) = \frac{1}{\beta} \sum_n \int d^3x \frac{2\pi}{3} e^{-i(\omega_n x - p \bar{A}o \gamma^0)} \bar{\psi}_n(\omega_n, p) \]

\[ S_{o\beta} = \frac{1}{\beta^2} \sum_{n=-\infty}^{\infty} \int \bar{\psi}_n(x) [\omega_n \gamma^0 + i\gamma^j \partial_j - m] \psi_n(x) d^3x \]

Then one arrives at the finite temperature partition function from the Euclidean action \( S_{\beta} \) as

\[ Z[A] = \prod_{n=-\infty}^{\infty} \int D\bar{\psi}_n D\psi_n e^{-S_{\beta}} \]

\[ \prod_{n=-\infty}^{\infty} \int D\bar{\psi}_n D\psi_n e^{-S_{o\beta}} \]

(12)

From this partition function one arrives at the expression for the effective Lagrangian, (details will be reported elsewhere ref[8].)

\[ \text{Re} \mathcal{L}^Q_{\text{eff}} = -\frac{1}{4\pi} p.v \int_0^\infty ds \frac{ds}{s^3} e^{-sm^2} \left[ (gEs \cot(gEs) - 1) - \frac{1}{2\pi^2} \sum_{n=1}^\infty (-1)^n \right] e^{-s^2 - \frac{s^2 \beta^2}{4s}} \]

\[ \cosh \left( ngA_o \right) (gEs \cot(gEs) \cos(n^2 \beta^2 gE/4) - 1) e^{-s^2 - \frac{s^2 \beta^2}{4s}} \]

\[ + \pi \sum_{l=1}^\infty \cosh \left( ngA_o \right) \left( \frac{gE}{l\pi} \right)^2 \sin \left( \frac{n^2 \beta^2 gE}{4l} \right) e^{-s^2 - \frac{s^2 \beta^2}{4s}} \]

(13)

with \( \bar{A}_o = -Ez \).

As one can see that the divergence of this effective Lagrangian is contained in the \( T = 0 \) piece; following Schwinger one can remove this divergence by subtracting out a quantity (that is found by analysing the \( s \to 0 \) behavior of the zero temperature piece of the effective Lagrangian) defined as

\[ \mathcal{L}_d = -\left( \frac{(gE)^2}{12\pi^2} \right) \int_0^\infty ds \frac{ds}{s^3} e^{-sm^2} \]

(14)

In order to get a finite answer, from this divergent \( \mathcal{L}_{\text{eff}} \) one absorbs these infinities in the redefinition of the field and the coupling constant, by redefining them in terms of some multiplicative renormalisation constant such that \( g_R E_R = g_{un} E_{un} \) where subscript \( R/Un \) refers to whether the quantity is renormalised or unrenormalised.

\[ g_R = \frac{g_{un}}{(1+Z)^{1/2}} \quad \text{and} \quad E_R = (1+Z)^{1/2} E_{un} \]

(15)
Here Z is a function of \( L_d \). So with this redefinition of the field and the coupling constant the real part of the effective Lagrangian can be expressed as

\[
\mathcal{L}_{\text{eff}} = \frac{E^2}{2} - \left[ \frac{1}{4g^2} \int_0^\infty \frac{ds}{s^3} e^{-sm^2} \left[ (gEs) \cot(gEs) - 1 + \left( \frac{gE^2}{3} \right) \right] \right] + \frac{1}{2\pi^2} \sum_{n=1}^\infty (-1)^n \left[ \int_0^\infty \frac{ds}{s^3} \right] 
\times \left[ \cosh \left( n\beta \bar{A}_o \right) (gEs) \cot(gEs) \cos \left( n^2 \beta^2 gE/4 \right) - 1 \right] e^{-sm^2 - n^2 \beta^2/4s} \\
\times \left[ \pi \sum_{\ell=1}^\infty \cosh \left( n\beta \bar{A}_o \right) \left( \frac{2n^2 \beta^2 E}{\ell^2} \right) \sin \left( \frac{n^2 \beta^2 E}{4 \ell^2} \right) e^{-n^2 \beta^2 gE/4s} \right] 
\] 
(16)

The first term here is the tree level term, and since this satisfies the R.G equation trivially, the quantity \( \mathcal{L}_{\text{eff}} \) also satisfies the R.G equation. As a next step we will scale \( E \to \frac{E}{g} \) to get

\[
\mathcal{L}_{\text{eff}} = \frac{E^2}{2g^2} - \left[ \frac{1}{4\pi^2} \int_0^\infty \frac{ds}{s^3} e^{-sm^2} \left[ (Es) \cot(Es) - 1 + \left( \frac{(Es)^2}{3} \right) \right] \right] + \frac{1}{2\pi^2} \sum_{n=1}^\infty (-1)^n \left[ \int_0^\infty \frac{ds}{s^3} \right] 
\times \left[ \cosh \left( n\beta \bar{A}_o \right) (Es) \cot(Es) \cos \left( n^2 \beta^2 E/4 \right) - 1 \right] e^{-sm^2 - n^2 \beta^2/4s} \\
\times \left[ \pi \sum_{\ell=1}^\infty \cosh \left( n\beta \bar{A}_o \right) \left( \frac{2n^2 \beta^2 E}{\ell^2} \right) \sin \left( \frac{n^2 \beta^2 E}{4 \ell^2} \right) e^{-n^2 \beta^2 gE/4s} \right] 
\] 
(17)

Now employing equation (9) we finally come to the expression for running coupling constant at any temperature or electric field defined as

\[
\frac{1}{4\pi^2 g_R^2} = \frac{1}{4\pi^2 g^2} = [DF_1 + DF_2 + DF_3] 
\] 
(18)

where the quantities \( DF_1 \) and \( DF_2 \) are the finite temperature pieces and \( DF_3 \) is the zero temperature piece. These quantities are defined as

\[
DF_1 = \frac{1}{(2\pi^2 E^2)} \sum_{n=1}^\infty (-1)^n \left[ \left( n\beta \bar{A}_o \right) \sinh(n\beta \bar{A}_o) \cos \left( \frac{n^2 \beta^2 E}{4} \right) \right. \\
- \left( \frac{n^2 \beta^2 E}{4} \right) \cosh(n\beta \bar{A}_o) \sin \left( \frac{n^2 \beta^2 E}{4} \right) \\
+ \cosh(n\beta \bar{A}_o) \cos \left( \frac{n^2 \beta^2 E}{4} \right) \right] \int_0^\infty \frac{ds}{s^3} (Es) \cot(Es) e^{-sm^2 - n^2 \beta^2/4s} \\
- \cosh(n\beta \bar{A}_o) \cos(n^2 \beta^2 E/4) \int_0^\infty \frac{ds}{s^3 \sin^2(Es)} e^{-sm^2 - n^2 \beta^2/4s} \right] 
\] 
(19)

\[
DF_2 = \frac{1}{\pi} \sum_{n=1}^\infty (-1)^n \sum_{\ell=1}^\infty \left[ \left( \frac{1}{\ell \pi} \right)^2 e^{-n^2 \beta^2 E/4\ell^2 g} \right] \left[ \cosh(n\beta \bar{A}_o) \sin \left( \frac{n^2 \beta^2 E}{4} \right) \right] 
\]
\[ + \left( \frac{n^2 \beta \bar{A}_0}{2} \right) \sinh(n \beta \bar{A}_0) \sin \left( \frac{n^2 \beta^2 E}{4} \right) \]

\[ + \left( \frac{n^2 \beta^2 E}{8} \right) \cosh(n \beta \bar{A}_0) \cos \left( \frac{n^2 \beta^2 E}{4} \right) \]

\[ + \frac{1}{2} \cosh(n \beta \bar{A}_0) \sin \left( \frac{n \beta^2 E}{4} \right) \left( \frac{\pi \ell m^2}{E} - \frac{n^2 \beta^2 E}{4 \pi \ell} \right) \] \hspace{1cm} (20)

\[ DF_3 = \frac{1}{4\pi^2 E^2} \int_{0}^{\infty} ds \frac{e^{-sm^2}}{s^3} \left[ (Es) \cot(ES) - \frac{(Es)^2}{\sin^2(ES)} + \frac{2(ES)^2}{3} \right] \] \hspace{1cm} (21)

Using these expressions the running coupling constant is defined as (at some temperature T and electric field E)

\[ \alpha_R(E, T) = \frac{\alpha_0}{1 - \frac{4\pi^2 g_o^2 [DF_1 + DF_2 + DF_3]}{\alpha_0}} \] \hspace{1cm} (22)

Here \( \alpha_0 \) is the strong coupling constant at the renormalisation point.

3 Analysis Of The Result

Equation (22) provides one with the relation, how the coupling constant varies with the external parameters present in the theory. A close inspection, of the aforementioned relation shows that if the denominator, on the right hand side of the equation becomes less than one and tends to zero, the value of \( \alpha_R \) increases. In view of the fact that the expression \( DF_1 \) is has terms with a relative sign difference between them, it might be justified to imagine that for some value of the temperature or the electric field the denominator may become less than one. Since the terms there are highly oscillating, it is difficult to come to a compact analytical expression for them. In any case we have tried to evaluate them numerically and have plotted the value of the corresponding coupling constant with the variation in the temperature;(the corresponding \( \alpha_R(E, 0) \sim 0.1 \)). There are several features that emerges from this analysis namely; i) the magnitude of the coupling constant seems to peak around \( \sim \) unity or even more, for some value of the temperature, ii)for distances away from the center (i.e. \( A_o \neq 0 \)), even though it (thermal coupling constant) continues to peaks around the same value of the temperature but it’s magnitude increases. A qualitative interpretation of this phenomena may be given like this; at zero temperature, in the presence of an external electric field the particles and the antiparticles will try to move away from each other [7] and naturally the coupling constant appears to be smaller in magnitude. But at finite temperature, because of the thermal pressure( this balances
the force due to the external field), the particles and the antiparticles are held close to each other in a metastable state and hence the magnitude of $\alpha_R(E,T)$ appears to have increased.

The space dependence of $\alpha_R(E,T)$ is consistent with the fact that, at finite temperature, the Effective Lagrangian itself is position dependent. It is worth remembering that, the particle production rate - as follows from the imaginary part of the Effective Lagrangian - increases as one moves away from the center of the system, therefore the number density of particles, away from the center, are more. Other than thermal pressure, this effect additionally contributes to hold the particle, anti-particle pairs, close to each other, hence there is an increase in $\alpha_R(E,T)$ at distances $d \neq 0$ than the same at $d = 0$. Any further increment of the temperature, shows the existence of a critical temperature ‘$T_c$’ (it depends on the magnitude of other parameters present in the theory), around which $\alpha_R(E,T)$ grows enormously and beyond that it changes its sign. We feel that around this temperature, one is ought to take the quantum back reaction into account. This effect has been studied before\cite{11} at zero temperature, where the external field was seen to admit a complex value.

4 Conclusion

We believe that there might be some observational consequence of this phenomena in the physical situations considered earlier, for instance during the production part of QGP in RHIC as well as studying properties of mesons in a hot medium \cite{12}. It is also reasonable to assume that, this can be tested in the future Heavy Ion Experiments to be held at CERN and BNL. Lastly it will also be of interest to compute the screening of the external field in the plasma, taking the thermal coupling constant into account. Some work along this direction is under progress, and will be reported elsewhere.

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Figure Captions:

Fig. 1: Behavior Of Running Coupling Constant with Temperature. Distance= 0.1 fm, Mass= 1.0 GeV, , gE=1.15 GeV$^2$ and $\alpha_0(0,0) = 0.3$

Fig. 2: Behavior Of Running Coupling Constant with Temperature. Distance= 0.0 fm, Mass= 1.0 GeV, , gE=1.15 GeV$^2$ and $\alpha_0(0,0) = 0.3$. 
mass = 1.0 GeV
\( g_E = 1.15 \text{ GeV}^2 \)
\( \alpha \Phi(0) = 0.3 \)
\( \text{Cut} = 0.1 \text{ fm} \)
mass = 1.0 GeV
$g \kappa = 1.15 \text{ GeV}^2$
$\alpha_{200} = 0.3$
$\alpha_{00} = 0.0$

Fig. 2