ON RYSER’S CONJECTURE

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Abstract. Motivated by an old problem known as Ryser’s Conjecture, we prove that for \( r = 4 \) and \( r = 5 \), there exists \( \epsilon > 0 \) such that every \( r \)-partite \( r \)-uniform hypergraph \( \mathcal{H} \) has a cover of size at most \((r - \epsilon)\nu(\mathcal{H})\), where \( \nu(\mathcal{H}) \) denotes the size of a largest matching in \( \mathcal{H} \).

1. Introduction

In this paper we are concerned with a packing and covering problem in hypergraphs. A hypergraph consists of a vertex set \( V \) and a set \( \mathcal{H} \) of edges, where each edge is a nonempty subset of \( V = V(\mathcal{H}) \). We say \( \mathcal{H} \) has rank \( r \) if the largest size of an edge is \( r \), and that \( \mathcal{H} \) is \( r \)-uniform if every edge has size \( r \). The packing number (also called matching number) \( \nu(\mathcal{H}) \) of \( \mathcal{H} \) is the size of a largest matching in \( \mathcal{H} \), where a matching is a set of pairwise disjoint edges in \( \mathcal{H} \). The covering number \( \tau(\mathcal{H}) \) of \( \mathcal{H} \) is the size of a smallest cover of \( \mathcal{H} \), where a cover is a subset \( W \subseteq V \) such that every edge of \( \mathcal{H} \) contains a vertex of \( W \). It is clear that if \( \mathcal{H} \) has rank \( r \) then \( \tau(\mathcal{H}) \leq r\nu(\mathcal{H}) \), and this is attained for example by the complete \( r \)-uniform hypergraph \( K_{2r-1}^r \) with \( 2r - 1 \) vertices, which has \( \nu(K_{2r-1}^r) = 1 \) and \( \tau(K_{2r-1}^r) = r \).

Our focus here is on a long-standing open problem known as Ryser’s Conjecture, which states that if \( \mathcal{H} \) is an \( r \)-partite \( r \)-uniform hypergraph then \( \tau(\mathcal{H}) \leq (r - 1)\nu(\mathcal{H}) \) (see e.g. [4, 9]; a stronger version of the conjecture was proposed by Lovász [6]). Here \( \mathcal{H} \) being \( r \)-partite means that its vertex set has a partition \( V_1 \cup \cdots \cup V_r \) and every edge contains exactly one vertex of each \( V_i \). When \( r = 2 \) this is the classical theorem of König, and for \( r = 3 \), after a number of partial results [8, 10, 5], the conjecture was proved by Aharoni [1]. Apart from these two cases, very little is known about the problem. If true, the statement is best possible whenever \( r - 1 \) is a prime power (see e.g. [9]). Until now no nontrivial bound of the form \( \tau(\mathcal{H}) \leq (r - \epsilon)\nu(\mathcal{H}) \) for \( \epsilon > 0 \) and any \( r \geq 4 \) was known.

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A hypergraph \( H \) is said to be \textit{intersecting} if \( \nu(H) = 1 \). Even for intersecting hypergraphs, Ryser’s Conjecture is open for all \( r \geq 6 \). There are many examples showing the result would be best possible in this case, and they can be quite sparse (see [7]). For \( r \leq 5 \), however, the conjecture has been proved in the special case of intersecting hypergraphs.

**Theorem 1.1.** (Tuza [9]) If \( H \) is an intersecting \( r \)-partite hypergraph of rank \( r \) and \( r \leq 5 \) then \( \tau(H) \leq r - 1 \).

Our aim in this paper is to prove the following theorem, the proof of which depends on Theorem 1.1, and thus give a nontrivial upper bound for Ryser’s problem in the cases \( r = 4 \) and \( r = 5 \).

**Theorem 1.2.** For each of \( r = 4 \) and \( r = 5 \), there exists a positive constant \( \epsilon \) such that \( \tau(H) \leq (r - \epsilon)\nu(H) \) for every \( r \)-partite \( r \)-uniform hypergraph \( H \).

2. General \( r \)

We begin the proof of Theorem 1.2 in this section, arguing in terms of general \( r \). We then complete the proof for \( r = 4 \) and \( r = 5 \) respectively in the next two sections.

Let \( J \) be an \( r \)-partite \( r \)-uniform hypergraph, with a fixed partition \( V_1 \cup \ldots \cup V_r \). Let \( B \) be a matching of size \( \nu(J) \) in \( J \). It is clear that \( \nu(B) \) is a cover of \( J \) of size \( \nu(B) \). For \( B_j \in B \) we let \( H_j \) denote the set of edges of \( J \) that intersect \( \nu(B) \) only in vertices of \( B_j \). Note then that \( H_j \) is intersecting and \( B_j \in H_j \).

We call an edge \( A \in J \) \textit{bad} if \( A \cap \nu(B) = \{v\} \) for some \( v \). The vertex \( v \) is also called \textit{bad}, and we say \( A \) is \textit{i-bad} where \( v \) is in the \( i \)th colour class \( V_i \) of the \( r \)-partition of \( J \). Note that each bad edge is in \( H_j \) for some \( j \). Let \( B_1 = \{B_j \in B : B_j \) has \( r \) bad vertices\}.

**Lemma 2.1.** If \( \tau(J) > (r - 1/2r)|B| \) then \( |B_1| > |B|/2 \).

\textit{Proof.} Suppose that \( |B_1| \leq |B|/2 \). Then there is a colour class \( i \) such that at least \( |B|/2r \) of the \( B_j \notin B_1 \) have no \( i \)-bad vertex. Let \( B^* \) denote the set of these \( B_j \). But then \( \bigcup_{B_j \notin B^*} B_j \cup \bigcup_{B_j \in B^*} B_j \forall V_i \) is a cover of \( J \) of size at most \( r(|B| - |B^*|) + (r - 1)|B^*| \leq (r - 1/2r)|B| \). \( \square \)

Lemma 2.1 indicates how our proof of Theorem 1.2 will proceed. Either \( J \) has a suitably small cover, or we can find a special subset of \( B \) whose size is a positive proportion of \( |B| \) (in this case \( B_1 \) which is at least half of \( B \)) about which we can make a further assumption. We may then cover all edges of \( J \) that intersect any edge of \( B \) that is not in the special subset by taking every vertex of every edge of \( B \).
not in the special subset. This will not change the hypergraphs $\mathcal{H}_j$, or the notion of bad, for the edges of $\mathcal{J}$ that remain. We then focus on showing that the remaining edges have a suitably small cover (in this case of size at most $(r - \alpha)|B_1|$ for some fixed positive $\alpha$). In our proof of Theorem 1.2 we will apply this procedure $r + 2$ times for $r = 4$, and $r + 3$ times for $r = 5$.

By Lemma 2.1 we may assume that $|B_1| > |B|/2$. As outlined in the previous paragraph, we let $\mathcal{J}_1 = \{A \in \mathcal{J} : A \cap B_j = \emptyset \text{ for all } B_j \in B \setminus B_1\}$. Then $\nu(\mathcal{J}_1) = |B_1|$, and $\tau(\mathcal{J}) \leq r(|B| - |B_1|) + \tau(\mathcal{J}_1)$.

**Lemma 2.2.** If $\tau(\mathcal{J}_1) > (r - 1/2)|B_1|$ then there is a matching of 1-bad edges in $\mathcal{J}_1$ of size at least $|B_1|/2r$.

**Proof.** Let $\mathcal{M} = \{M_1, \ldots, M_t\}$ be a maximum matching of 1-bad edges in $\mathcal{J}_1$. Note that since each $\mathcal{H}_j$ is intersecting, all edges of $\mathcal{M}$ are in distinct $\mathcal{H}_j$, say $\mathcal{H}_1, \ldots, \mathcal{H}_t$. Then

$$\bigcup_{j=1}^t (M_j \cup B_j) \cup \bigcup_{j>t} B_j \setminus V_1$$

is a cover of $\mathcal{J}_1$ of size at most $(2r - 1)|\mathcal{M}| + (r - 1)(|B_1| - |\mathcal{M}|) = (r - 1)|B_1| + r|\mathcal{M}|$. If $|\mathcal{M}| < |B_1|/2r$ then this is at most $(r - 1/2)|B_1|$. $\square$

By Lemma 2.2 we may assume that there is a matching $\mathcal{M}$ of 1-bad edges in $\mathcal{J}_1$ of size at least $|B_1|/2r$. Let $\mathcal{B}_2 = \{B_j \in \mathcal{B}_1 : B_j \cap M_k \neq \emptyset \text{ for some } M_k \in \mathcal{M}\}$. Then $|\mathcal{B}_2| = |\mathcal{M}| \geq |B_1|/2r$. Let $\mathcal{J}_2 = \{A \in \mathcal{J}_1 : A \cap B_j = \emptyset \text{ for all } B_j \in B_1 \setminus B_2\}$. Then $\nu(\mathcal{J}_2) = |\mathcal{B}_2|$, and $\tau(\mathcal{J}_1) \leq r(|B_1| - |\mathcal{B}_2|) + \tau(\mathcal{J}_2)$. We may repeat this argument another $r - 1$ times for colour classes $V_2, \ldots, V_r$ until we reach a hypergraph $\mathcal{J}_{r+1}$ and a matching $\mathcal{B}_{r+1}$ in $\mathcal{J}_{r+1}$, in which there exists a matching $\mathcal{M}_i$ of $i$-bad edges with $|\mathcal{M}_i| = |\mathcal{B}_{r+1}|$ for each $i$. Each edge of $\mathcal{M}_i$ is in a distinct $\mathcal{H}_j$, and $\nu(\mathcal{J}_{r+1}) = |\mathcal{B}_{r+1}|$. To prove Theorem 1.2 it will suffice to show that $\mathcal{J}_{r+1}$ has a cover of size at most $(r - \alpha)|\mathcal{B}_{r+1}|$ for some fixed positive $\alpha$.

We denote by $\mathcal{C}_j$ the hypergraph consisting of the $r$ edges of $\bigcup_{i=1}^r \mathcal{M}_i$ in $\mathcal{J}_{r+1}$ that intersect $B_j$, together with the edge $B_j$ itself. Then $\mathcal{C}_j \subset \mathcal{H}_j$.

**Lemma 2.3.** For each $\mathcal{C}_j$ we have $\tau(\mathcal{C}_j) \geq 2$, and no cover of $\mathcal{C}_j$ of size two consists of vertices from distinct colour classes.

**Proof.** If on the contrary $\tau(\mathcal{C}_j) = 1$ then without loss of generality we may assume that the vertex of $B_j$ of colour 1 covers $\mathcal{C}_j$. But then the $\mathcal{M}_2$-edge in $\mathcal{C}_j$ is not covered. Thus $\tau(\mathcal{C}_j) \geq 2$. 


Suppose now that vertices \( v \in V_1 \) and \( w \in V_2 \) form a cover of \( C_j \). We may assume without loss of generality that \( v \) is in \( B_j \). Then the \( M_3 \) edge in \( C_j \) is not covered by \( v \), hence \( w \) must not be in \( B_j \). But then the \( M_2 \) edge in \( C_j \) is not covered by \( \{v, w\} \).

Next we would like to restrict to a hypergraph in which \( V(\mathcal{H}_j) \cap V(C_k) \neq \emptyset \) if and only if \( j = k \). To do this we will need to consider a more general setting in which our \( r \)-uniform hypergraph is replaced with a hypergraph of rank \( r \).

A sunflower with centre \( C \) in a hypergraph is a set \( S \) of edges such that \( S \cap S' = C \) for all \( S \neq S' \) in \( S \). Each edge of \( S \) is called a petal. A classical theorem of Erdős and Rado [3] tells us that every hypergraph of rank \( r \) with more than \( (t - 1)^r r! \) edges contains a sunflower of size \( t \).

Let \( \mathcal{H} \) be a hypergraph of rank \( r \). We call a set \( S \) of \( t \) edges in \( \mathcal{H} \) a giant sunflower if it forms a sunflower and \( t \geq r(2r - 4) + 1 \). Note that since \( t > r \), if an intersecting hypergraph \( \mathcal{H} \) contains a giant sunflower \( S \) with centre \( C \), then \( \mathcal{H}' = \mathcal{H} \setminus S \cup \{C\} \) is also intersecting. We refer to the hypergraph \( \mathcal{H}' \) as the hypergraph obtained by picking the sunflower \( S \).

We apply the following procedure to each \( \mathcal{H}_j \) where \( B_j \in \mathcal{B}_{r+1} \). If \( \mathcal{H}_j = \mathcal{H}_j^0 \) contains a giant sunflower \( S_0 \), we pick it to obtain \( \mathcal{H}_j^1 \).

We repeat this process with the current hypergraph \( \mathcal{H}_j^k \) to get \( \mathcal{H}_j^{k+1} \), until for some \( u \) we obtain a hypergraph \( \mathcal{D}_j = \mathcal{H}_j^u \) that is free of giant sunflowers. Then in particular each \( \mathcal{D}_j \) is intersecting. Let \( J' = (J_{r+1} \setminus \bigcup_j \mathcal{H}_j) \cup \bigcup_j \mathcal{D}_j \). For every edge \( A \in \mathcal{H}_j \) there exists a unique edge \( \hat{A} \in J' \) and a sequence of edges \( A = A^0, \ldots, A^u = \hat{A} \) with \( A^k \in \mathcal{H}_j^k \) such that for \( i = 1, \ldots, u \), either \( A^i = A^{i-1} \) or \( A^i = A^{i-1} - 1 \) is a petal of \( S_{r-1} \) and \( A^i \) is its centre. We extend this definition to every \( A \in J_{r+1} \) by setting \( \hat{A} = A \) for each \( A \in J_{r+1} \) that is not in any \( \mathcal{H}_j \).

Note that \( J' \) has rank at most \( r \) but may not be \( r \)-uniform. Also, we do not know that \( \nu(J') \leq \nu(J_{r+1}) \).

**Lemma 2.4.** Any cover of \( J' \) is also a cover of \( J_{r+1} \).

*Proof.* Every edge \( A \) of \( J_{r+1} \) has a subset \( \hat{A} \) that is an edge of \( J' \). \( \Box \)

Thus to prove Theorem 1.2 it will suffice to find a cover of \( J' \) of size \( (r - \alpha)|\mathcal{B}_{r+1}| \) for some \( \alpha > 0 \).

**Lemma 2.5.** Let \( \{A'_1, \ldots, A'_s\} \) be a matching of size \( s \leq 2r - 3 \) in \( J' \). Then there exists a matching \( \{A_1, \ldots, A_s \in J_{r+1}\} \) such that

- \( A'_i \subseteq A_i \) for each \( i \),
- if \( A'_i \in D_j \) then \( A_i \in \mathcal{H}_j \).
Proof. If every $A'_i \in \mathcal{J}_{r+1}$ then we set $A_i = A'_i$ for each $i$. Otherwise, since each $D_j$ is intersecting, we may assume that $A'_1, \ldots, A'_{c-1} \in \mathcal{J}_{r+1}$, and that there are distinct $D_i$ for $c \leq i \leq s$ such that $A'_i \not\in D_i$. Set $A_i = A'_i$ for each $1 \leq i \leq c - 1$.

Let $A_i$ for $c \leq i \leq s$ be such that the following hold.

- $A'_i \subseteq A_i$ for each $i$,
- $A_i \in \mathcal{H}_{k_i}^i$ for some $k_i$,
- $A_1, \ldots, A_s$ are all disjoint,
- $\sum_{i=c}^s k_i$ is as small as possible.

Such a choice of $A_i$ exists because $A'_c, \ldots, A'_s$ satisfy the conditions. We claim that $k_i = 0$ for each $i$, which implies the lemma.

Suppose on the contrary that $A_i \in \mathcal{H}_{k_i}^i$ for some $i$, where $k_i \geq 1$. Since $\sum_{i=c}^s k_i$ is as small as possible we know that $A_i \not\in \mathcal{H}_{k_i-1}^i$, which implies that it is the centre of a giant sunflower $S$ in $\mathcal{H}_{k_i-1}^i$. Let $A^*_i \in \mathcal{H}_{k_i-1}^i$ be a petal of $S$ that is disjoint from all of $A_1, \ldots, A_{i-1}$ and all of $A_{i+1}, \ldots, A_s$. This is possible because the union of these edges has size at most $r(s-1) \leq r(2r-4)$, and $S$ has at least $r(2r-4) + 1$ petals. But then replacing $A_i$ by $A^*_i$ gives a new family satisfying the conditions, contradicting the fact that $\sum_{i=c}^s k_i$ was as small as possible. Thus $k_i = 0$ for each $i$, completing the proof. □

In fact it follows from the proof of Lemma 2.5 that $A'_i = \hat{A}_i$ for each $i$.

Lemma 2.6. Each $D_j$ has at most $r^{r+1}(2r-4)^{r!}$ vertices.

Proof. In particular there is no sunflower of size $r(2r-4) + 1$ in $D_j$, so by the Erdős-Rado theorem $D_j$ has at most $(r(2r-4))^{r!}$ edges, and hence at most $r^{r+1}(2r-4)^{r!}$ vertices. □

Lemma 2.7. For each $B_j \in \mathcal{B}_{r+1}$ we have $\hat{B}_j = B_j$.

Proof. Suppose the contrary. Then for some $k$ we have that $B_j$ is a petal of a sunflower $S_k$ in $\mathcal{H}_k^j$. We may assume without loss of generality that the centre $C$ of $S_k$ does not contain a vertex of colour 1. Let $M$ be the $M_1$-edge in $C_j$. Then $M \cap C = \emptyset$, contradicting the fact that $D_j$ is intersecting. □

Lemma 2.7 implies that if an edge $A \in \mathcal{J}'$ intersects exactly one $B_j \in \mathcal{B}_{r+1}$ then $A \in D_j$.

Lemma 2.8. $V(\mathcal{B}_{r+1})$ is a cover of $\mathcal{J}'$.

Proof. Suppose on the contrary that an edge $A \in \mathcal{J}'$ is disjoint from $V(\mathcal{B}_{r+1})$. Since each $D_j$ is intersecting and $B_j \in D_j$, we know that
A \notin D_j \text{ for any } j, \text{ so } A \in J_{r+1}. \text{ But then since } V(B_{r+1}) \text{ is a cover of } J_{r+1} \text{ we find a contradiction.}\quad \Box

For each } j \text{ let } C'_j = \{ A : A \in C_j \}, \text{ so } C'_j \subseteq D_j \text{ for each } j. \text{ To restrict to our hypergraph in which } C'_j \text{ shares a vertex with } D_k \text{ if and only if } j = k, \text{ for convenience we define an auxiliary directed graph } G \text{ as follows. The vertex set of } G \text{ is } B_{r+1}. \text{ We put an arc from } B_k \text{ to } B_j \text{ if and only if } D_k \text{ and } C'_j \text{ share a vertex.}

\textbf{Lemma 2.9.} The graph } G \text{ has an independent set } B'' \text{ of vertices of size at least } |B_{r+1}| / (2r^{r+3}(2r - 4)^r! + 1). \text{ Thus for any } B_j, B_k \in B'', \text{ if } C'_j \text{ shares a vertex with } D_k \text{ then } j = k.

\textbf{Proof.} Since each } M_i \text{ is a matching, no vertex can be in more than } r + 1 \text{ edges of } \bigcup_j C'_j = \bigcup_j \{ B_j \} \cup \{ M : M \in M_i \text{ for some } 1 \leq i \leq r \}. \text{ By Lemma 2.6 each } D_k \text{ has fewer than } r^{r+1}(2r - 4)^r! \text{ vertices, and } B_k \text{ can share a vertex with at most } r^{r+3}(2r - 4)^r! \text{ of } C_j \text{'s. Thus the outdegree of } G \text{ is at most } r^{r+3}(2r - 4)^r!, \text{ which implies that it has an independent set of size at most } |V(G)| / (2r^{r+3}(2r - 4)^r! + 1).\quad \Box

Let } J'' = \{ A \in J' : A \cap B_j = \emptyset \text{ for all } B_j \in B_{r+1} \setminus B'' \}. \text{ Then } B'' \text{ is a matching in } J'' \text{ such that } V(B'') \text{ covers } J'', \text{ and to prove Theorem 1.2 it suffices to prove that } \tau(J'') < (r - \alpha)|B''| \text{ for some fixed positive } \alpha. \text{ One important consequence of the definition of } B'' \text{ is the fact that if } B_j, B_k \in B'' \text{ then } V(C'_j) \cap V(C'_k) = \emptyset.

\textbf{Lemma 2.10.} Every edge of } J'' \text{ contains a cover of } C'_j \text{ for some } j.

\textbf{Proof.} Suppose not. \text{ Then since the } C'_j \text{ are all vertex-disjoint, some edge } A \text{ together with an edge } A_j \text{ in } C'_j \text{ for each } j \text{ forms a matching of size } |B''| + 1 \text{ in } J''\text{. Except for the set } I \text{ of at most } r \text{ indices } j \text{ for which } A \cap V(C'_j) \neq \emptyset, \text{ we may assume } A_j = B_j. \text{ Then Lemma 2.5 applied to } A \text{ together with } \{ A_j : j \in I \} \text{ gives a matching in } J_{r+1} \text{ of size } |I| + 1, \text{ which by our construction of } J'' \text{ consists of edges that do not intersect any edge of } B_{r+1} \text{ except } \{ B_j : j \in I \}. \text{ But then together with } \{ B_j : j \notin I \} \text{ this forms a matching in } J_{r+1} \text{ of size } |B_{r+1}| + 1, \text{ a contradiction.}\quad \Box

Lemma 2.10 tells us that for every edge } A \in J'' \text{ there exists } j \text{ such that } A \text{ contains a cover of } C'_j. \text{ Since every cover of } C'_j \text{ is a cover of } C_j, \text{ Lemma 2.3 tells us that this cover is of size at least } 3. \text{ Thus } j \text{ is unique for } r = 4 \text{ and } r = 5. \text{ Let } C'_r = \{ A \in J'' : A \text{ contains a cover of } C'_j \}, \text{ so since } C'_j \text{ is intersecting we have } C'_j \subseteq C'_r. \text{ Then } J'' = \bigcup_j C'_j, \text{ where the union is a disjoint union.
Lemma 2.11. Suppose that $A \cap A' = \emptyset$ for $A, A' \in C_j^*$. Then there exists $k \neq j$ such that $A \cup A'$ contains a cover of $C_k^*$.

Proof. Suppose the contrary. Let $I$ denote the set of at most $2(r-3)+1$ indices such that $(A \cup A') \cap V(C_j^*) \neq \emptyset$. Then $A$ and $A'$ together with an edge of $C_k^*$ for all $k \in I \setminus \{j\}$ forms a matching of size $|I| + 1$, consisting of edges that are disjoint from each $B_j$ with $j \notin I$. Then as in the proof of Lemma 2.10 this leads to a matching in $J_{r+1}$ that is larger than $B_{r+1}$. This contradiction completes the proof. □

3. $r = 4$

We have now done essentially all the required work to prove Theorem 1.2 for $r = 4$.

Lemma 3.1. Suppose $r = 4$. Then each $C_j^*$ is intersecting.

Proof. Suppose on the contrary that $A \cap A' = \emptyset$ where $A, A' \in C_j^*$. By Lemma 2.3, each of $A$ and $A'$ must have three vertices in $V(C_j^*)$. By Lemma 2.11 we know $A \cup A'$ covers $C_k^*$ for some $k \neq j$. Since every cover of $C_k^*$ is a cover of $C_j$, and $V(C_j^*) \cap V(C_k^*) = \emptyset$, we may assume that the vertices of colour 1 in $A$ and $A'$ form a cover of $C_k^*$. But then one of these vertices is not in $B_k$, so one of the edges, say $A$, contains 3 vertices of $C_j^*$ and one vertex of $C_k^*$ that is not in $B_k$. Thus $A \in H_j$, which implies $A \in D_j$. But then $A$ cannot intersect $C_k^*$ by Lemma 2.9. □

We close this section with the $r = 4$ case of Theorem 1.2.

Theorem 3.2. Suppose $r = 4$. Then there exists $\epsilon > 0$ such that $\tau(J) \leq (4 - \epsilon)\nu(J)$.

Proof. Since $J'' = \bigcup_j C_j^*$, by Lemma 3.1 we may apply Theorem 1.1 to conclude that each $C_j^*$ has a cover of size 3. Therefore $\tau(J'') \leq 3|B''|$, completing the proof. □

4. $r = 5$

Our approach for the case $r = 5$ will be to start with the hypergraph $J''$ and the matching $B''$ as defined in Section 2, and restrict once more to a portion of $J''$ in which all the hypergraphs $C_j^*$ are intersecting.

We begin by fixing $B_j \in B''$, and considering how the edges in $C_j^*$ can intersect other sets $C_k^*$. In particular, we will need some technical information on pairs of disjoint edges in $C_j^*$. We will make use of the following classical theorem of Bollobás [2].
Theorem 4.1. (Bollobás [2]) Suppose sets $F_1, \ldots, F_m$ and $F'_1, \ldots, F'_m$ satisfy $F_i \cap F'_i = \emptyset$ if and only if $i = h$. Then

$$\sum_{i=1}^{m} \left( \frac{|F_i| + |F'_i|}{|F_i|} \right) \leq 1.$$ 

We say that a set of vertices is multicoloured if no two of its elements come from the same partition class $V_i$. For $B_j \in \mathcal{B}''$, suppose $(S, S')$ is a pair of disjoint multicoloured covers of $C'_j$. Since every cover of $C'_j$ is a cover of $C_j$, by Lemma 2.3 we know each of $S$ and $S'$ has size at least three. Let

$$A(S, S') = \{(A, A') : A, A' \in C'_j, A \cap A' = \emptyset, A \cap V(C'_j) = S, A' \cap V(C'_j) = S'\}.$$ 

Our key lemma in this section is the following.

Lemma 4.2. Let $B_j \in \mathcal{B}''$, and suppose $(S, S')$ is a fixed pair of disjoint multicoloured covers of $C'_j$. Let

$$U = \{B_k \in \mathcal{B}'' \setminus \{B_j\} : A \cup A' \text{ covers } C'_k \text{ for some } (A, A') \in A(S, S')\}.$$ 

Then there exist $B, B' \in \mathcal{B}'' \setminus \{B_j\}$ such that for all but at most 42 elements $B_k \in U$, if $A \cup A'$ covers $C'_k$ where $(A, A') \in A(S, S')$ then $(A \cup A') \cap (B \cup B') \neq \emptyset$.

Proof. Note that $|S|, |S'| \geq 3$, for any $(A, A') \in A(S, S')$ we know that each of $A$ and $A'$ has at most two vertices outside $V(C'_j)$.

Let $U_0$ be the set of $B_k$ in $U$ for which there is some $(A, A') \in A(S, S')$ with $A \cup A'$ covering $C'_k$, such that $A \cup A'$ has at least 3 vertices in $C'_k$. Let $U_1 = U \setminus U_0$.

Suppose that $|U_0| \geq 3$. For each $B_k \in U_0$ pick $(A_k, A'_k) \in A(S, S')$ with $|(A_k \cup A'_k) \cap V(C'_k)| \geq 3$. Then one of $A_k, A'_k$ must have 2 vertices in $C'_k$ and the other must have at least 1. Without loss of generality, we may assume that there are at least two sets $A_k$, say $A_1, A_2$, such that $A_k$ has 2 vertices in $C'_k$. In particular, for $i = 1, 2, A_i$ is contained in $S \cup V(C'_j)$. Now consider $A'_3$: if it has no vertex in $C'_j$ then $A'_3$ and $A_i$ are disjoint and contradict Lemma 2.11. On the other hand, $A'_3$ has at most one vertex outside $B_j \cup V(C'_j)$. So we must have $|U_0| \leq 2$.

Now we consider $U_1$. For each $B_k \in U_1$ and $(A_k, A'_k) \in A(S, S')$ that covers $C'_k$, by Lemma 2.3 we know that the vertices $y_k$ and $y'_k$ are of the same colour, where $A_k \cap V(C'_k) = \{y_k\}$ and $A'_k \cap V(C'_k) = \{y'_k\}$.

Case 1. Suppose that there exist $B_k \in U_1$ and associated $(A_k, A'_k)$ such that for some $B_i \in \mathcal{B}'' \setminus \{B_j, B_k\}$, the vertices $x_k$ and $x'_k$ exist and are both in $C'_i$, where $\{x_k\} = A_k \setminus (V(C'_k) \cup V(C'_j))$ and $\{x'_k\} = A'_k \setminus (V(C'_k) \cup V(C'_j))$. We claim that $B = B_k$ and $B' = B_i$ satisfy the lemma in this case. To verify this, we first observe that by Lemma 2.3,
one of $A_k$ and $A'_k$ (say $A_k$) does not contain a vertex of $B_k$. If $x_k \in A_k$ is not a vertex of $B_k$, then since its other three vertices are in $C'_j$, and the $C'_k$ are all vertex-disjoint, we find $A_k \in D_j$. But this contradicts Lemma 2.9. Therefore $x_k \in A_k \cap B_l$, so $\{x_k, x'_k\} \cap B_l \neq \emptyset$. We know $\{y_k, y'_k\} \cap B_l \neq \emptyset$ since $\{y_k, y'_k\}$ covers $C'_l$. Then to prove our claim we show that for every $B_l \in U_1$ and every associated $(A_l, A'_l)$, if the colour of $\{y_l, y'_l\}$ is the same as the colour of $\{y_k, y'_k\}$ then $\{x_k, x'_k\} \subset A_l \cup A'_l$, and if the colour of $\{y_l, y'_l\}$ is not the same as the colour of $\{y_k, y'_k\}$ then either $\{y_k, y'_k\} \subset A_l \cup A'_l$ or $\{x_k, x'_k\} \cap B_l \subset A_l \cup A'_l$.

Let $B_l \neq B_k$ in $U_1$ be given, and first assume that the colour of $\{y_l, y'_l\}$ (say 2) is the same as the colour of $\{y_k, y'_k\}$. Then $A_k$ and $A'_k$ are both in $C'_j$. If they are not disjoint then $A'_l$ must contain $x_k$. Suppose they are disjoint. Then by Lemma 2.11 the vertex $x'_l$ where $A'_l = S' \cup \{y'_k\} \cup \{x'_k\}$ must exist and $\{x_k, x'_k\}$ must cover $C'_l$, and hence $x_k$ and $x'_l$ are the same colour (say 1). (Note that $\{y_k, x'_k\}$ cannot cover $C'_l$ because they are different colours, contradicting Lemma 2.3.) But then since $A'_l = S' \cup \{y'_k\} \cup \{x'_k\}$ and $y'_k$ has colour 2, we see that $x'_l$ has colour 1. Therefore $x'_l = x'_k$, since otherwise there is an edge of $C'_l$ containing $x'_l \in V(C'_l)$ that is not covered by $\{x_k, x'_k\}$. Thus $x'_l \in A'_l$. Now the same argument applies to the pair $A'_k$ and $A_l$. Therefore since $A_l \cap A'_l = \emptyset$ we find that $\{x_k, x'_l\} \subset A_l \cup A'_l$.

If the colour of $\{y_l, y'_l\}$ (say 2) is not the same as the colour of $\{y_k, y'_k\}$ (say 1) then both elements of $\{x_k, x'_l\}$ also have colour 2. If $C'_l \neq C'_j$ then consider $A_k$ and $A'_l$. If they are disjoint then, since $A_k \cap V(C'_l) = \emptyset$, by Lemma 2.11 they must cover $C'_k$. Thus $y'_k \in A'_l$. If they are not disjoint then $y_k \in A'_l$. The same argument applies to $A'_k$ and $A_l$, then since $A_l \cap A'_l = \emptyset$ we conclude $\{y_k, y'_l\} \subset A_l \cup A'_l$. If $C'_l = C'_j$, recall that one of $x_k$ and $x'_l$ is the vertex of colour 2 in $B_l$. But then since $\{y_l, y'_l\}$ covers $C'_l$ it must contain the vertex of colour 2 in $B_l$. Therefore $\{x_k, x'_l\} \cap B_l \subset \{y_l, y'_l\} \subset A_l \cup A'_l$. This finishes the proof for Case 1.

Case 2. Suppose that for each $B_k \in U_1$ and associated $(A_k, A'_k)$, the vertices $x_k$ and $x'_k$ (if they exist) do not lie in a common $C'_l$. To finish the proof we will show that $|U_1| \leq 40$. Suppose not, then there is a subset $U_2$ of $U_1$ of size at least 21 in which all $\{y_k, y'_k\}$ are the same colour. For each $x_k$ that exists and lies in a cover of size two of the $C'_l$ it is in, set $z_k$ to be the other vertex of the cover. Note that $z_k$ is unique by Lemma 2.3. Define $z'_k$ similarly for each $x'_k$. Define $F_k = (A_k \setminus S) \cup \{z_k\}$ and $F'_k = (A'_k \setminus S') \cup \{z'_k\}$ for each $k$ (if $z_k$ or $z'_k$ do not exist then simply set $F_k = (A_k \setminus S)$, $F'_k = (A'_k \setminus S')$). We claim that these pairs of sets satisfy the conditions for Theorem 4.1. Since $x_k$ and $x'_k$ do not lie in a common $B_l$, we have that $F_k \cap F'_k = \emptyset$ for each $k$. Suppose that
that $C_i$ follows. Consider a vertex $B$ of size at least three (and at most four), and let $U$ be the set defined in Lemma 4.2 for this choice of $B$. If $|U| \leq 42$ then we put an arc $(B, B_k)$ for each $B_k \in U$. If $|U| \geq 43$ then, for $B, B'$ guaranteed by Lemma 4.2, we put arcs $(B_j, B)$ and $(B_j, B')$, and an arc $(B_j, B_k)$ for each $B_k \in U$ that fails to satisfy the conclusion of Lemma 4.2. We do this for each $B_j$ and each pair $(S, S')$ of disjoint multicoloured covers of $C_j$.

Lemma 4.3. The directed graph $G$ has outdegree less than $44(5)^{16}$, and hence has an independent set $\mathcal{B}^\dagger$ of size at least $|\mathcal{B}''|/100(5)^{16}$.

Proof. Since $|V(C_j)| \leq r^2$, the number of distinct choices of $(S, S')$ in $C_j$ is less than $(|V(C_j)|)^2 < (\frac{r^2}{4})^2 < r^{16} = 5^{16}$. Thus the outdegree of $G$ is less than $49(5)^{16}$. Therefore $G$ has an independent set of size at least $|V(G)|/(98(5)^{16} + 1) < |\mathcal{B}''|/100(5)^{16}$. 

Let $\mathcal{J}^\dagger = \{A \in \mathcal{J}'' : A \cap B_j = \emptyset \text{ for all } B_j \in \mathcal{B}'' \setminus \mathcal{B}^\dagger\}$. Then $\mathcal{B}^\dagger$ is a matching in $\mathcal{J}^\dagger$ such that $V(\mathcal{B}^\dagger)$ covers $\mathcal{J}^\dagger$, and to prove Theorem 1.2 for $r = 5$ it suffices to prove that $\tau(\mathcal{J}^\dagger) < (r - \alpha)|\mathcal{B}^\dagger|$ for some fixed positive $\alpha$.

Lemma 4.4. Each $C_j^* \cap \mathcal{J}^\dagger$ is intersecting.

Proof. Suppose on the contrary that $A$ and $A' \in C_j^*$ are edges of $\mathcal{J}^\dagger$ that do not intersect. We know by Lemma 2.11 that $A \cup A'$ covers some $C_k^*$, $k \neq j$. Since then $(A \cup A') \cap V(C_j^*) = \emptyset$, it must be true that $B_k \in \mathcal{B}^\dagger$. Let $S = A \cap V(C_j^*)$ and $S' = A' \cap V(C_j^*)$. Since $B_j, B_k \in \mathcal{B}^\dagger$, there cannot be an arc $(B_j, B_k)$ in $G$. The construction of $G$ implies then that for this choice of $B_j$ and $(S, S')$, the set $U$ satisfies $|U| \geq 47$ and that $B$ and $B'$ exist satisfying the conclusion of Lemma 4.2. Since $\mathcal{B}^\dagger$ is an independent set in $G$ and $B_j \in \mathcal{B}^\dagger$ we know that $B, B' \notin \mathcal{B}^\dagger$. But then by Lemma 4.2 one of $A$ and $A'$ intersects $B$ or $B'$, and hence it is not an edge of $\mathcal{J}^\dagger$ by definition. This contradiction completes the proof.

The $r = 5$ case of Theorem 1.2 follows.

Theorem 4.5. Suppose $r = 5$. Then there exists a fixed $\epsilon > 0$ such that $\tau(\mathcal{H}) \leq (5 - \epsilon)\nu(\mathcal{H})$. 

\[ F_k \cap F'_k = \emptyset. \] Then $A_k$ and $A'_k$ are disjoint edges in $C_j^*$ that do not cover any $C_k^*$, contradicting Lemma 2.11. Therefore by Theorem 4.1 we find that $|U_2| \leq (\frac{5}{2}) = 20$. This contradiction completes the proof. \[ \square \]

We define an auxiliary directed graph $G$ on the vertex set $\mathcal{B}''$ as follows. Consider a vertex $B_j$ and a pair $(S, S')$ of disjoint multicoloured covers of $C_j^*$ of size at least three (and at most four), and let $U$ be the set defined in Lemma 4.2 for this choice of $B_j$ and $(S, S')$. If $|U| \leq 42$ then we put an arc $(B_j, B_k)$ for each $B_k \in U$. If $|U| \geq 43$ then, for $B, B'$ guaranteed by Lemma 4.2, we put arcs $(B_j, B)$ and $(B_j, B')$, and an arc $(B_j, B_k)$ for each $B_k \in U$ that fails to satisfy the conclusion of Lemma 4.2. We do this for each $B_j$ and each pair $(S, S')$ of disjoint multicoloured covers of $C_j^*$.
Proof. Since $J^\dagger = \bigcup_j C_j^* \cap J^\dagger$, by Theorem 1.1 we conclude that each $C_j^* \cap J^\dagger$ has a cover of size 4. Therefore $\tau(J^\dagger) \leq 4|B^\dagger|$, completing the proof. □

We end with the remark that for each of $r = 4$ and $r = 5$, an explicit lower bound for $\epsilon$ could be computed by following the steps of our proof. However, as this value is probably very far from the truth we make no attempt to do this here.

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