On Drinfel’d twist deformation of the super-Virasoro algebra

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Abstract. In this paper, we describe nonstandard quantum deformation of the super-Virasoro algebra. Using the Drinfel’d twist quantization technique, we obtain the deformed coproduct and antipode. Hence we get a family of noncommutative and noncocommutative Hopf superalgebras.

Key words: Quantization, Lie superbialgebra, Drinfel’d twist, the super-Virasoro algebra.

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1. Introduction

Quantum groups, mathematically carry the structures of noncommutative and noncocommutative Hopf algebras, were first introduced by Drinfel’d and Jimbo. One of the most important examples of quantum groups is deformation of the universal enveloping algebra of a Lie algebra. This deformation induces a Lie bialgebra structure on the underlying Lie algebra. In [6], Drinfel’d posed the quantization problem of the Lie bialgebra. Lately, Etingof and Kazhdan [7] settled this question. But a general formula of a quantization was not obtained. Many authors have made great efforts to quantize explicitly some Lie bialgebras.

Inspired by the discovery of quantum groups, quantum supergroups, i.e., Hopf superalgebras, have also been defined (c.f. [3, 4]), which provide a powerful tool for constructing trigonometric solutions of the $\mathbb{Z}_2$-graded Yang-Baxter equation. By extending Etingof and Kazhdan’s work [7], Geer [8] recently proved that there exist a general quantizations of Lie superbialgebras. Similar to Lie algebras cases, the deformations of Lie superalgebras are not unique. In [16], Zhang proved that there is a new Hopf superalgebra structure by the Drinfel’d twist. Using this method, some good Hopf superalgebras have been found in recent years, e.g., Aizawa [2] and Celeghini et al [5] studied the drinfel’d twist deformations of $sl(1|2)$ and $osp(1|2)$, respectively. By the Drinfel’d twist, [9] and [13] give two different Hopf algebra structures on the Witt algebra. The aim of this paper is to construct the quantization of the super-Virasoro algebra, which is generated by the same Drinfel’d twist of the Witt algebra, studied in [14] (see also [11, 12, 14]). As a by-product, we obtain two combinational identities (see (2.10) and (2.11)).

Throughout this paper, $\mathbb{F}$ denotes a field of characteristic zero, all vector space and tensor products are over $\mathbb{F}$. Let $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ denote the sets of all integers, nonnegative integers, positive integers, respectively. We use the convention that if an undefined term appears in an expression, we always treat it as zero; for instance, $L_{\frac{1}{2}} = 0$ if $\frac{1}{2} \notin \mathbb{Z}$.

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2. Main results

Now let us start by recalling some definitions and preliminary results. A supervector space $H$ is a $\mathbb{Z}_2$-graded vector space, i.e., $H = H_0 \oplus H_1$. If an element $x$ is in either $H_0$ or $H_1$, we say that it is $\mathbb{Z}_2$-homogeneous. We assume that all elements below are $\mathbb{Z}_2$-homogeneous, where $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. For $x \in H$, we always denote $[x] \in \mathbb{Z}_2$ to be its parity, i.e., $x \in H_{[x]}$. We say that $x$ is even (odd) if $x \in H_0$ (resp. $x \in H_1$). A superalgebra $(H, \mu, \tau)$ over a commutative ring $R$ is a supervector space equipped with an associative product $\mu : H \otimes H \to H$ respecting the grading and a unit element $1 \in H_0$. A Hopf superalgebra $(H, \mu, \tau, \Delta, \epsilon, S)$ is a superalgebra equipped with a coproduct $\Delta : H \to H \otimes H$, a counit $\epsilon : H \to \mathbb{F}$, and an antipode $S : H \to H$, satisfying compatibility conditions. Note that $S$ satisfies $S(xy) = (-1)^{[x][y]}S(y)S(x)$ for $x, y \in H$.

**Definition 2.1** A Drinfel’d twist $\mathcal{F}$ is an invertible element of $H \otimes H$ and satisfies

\[
(\mathcal{F} \otimes 1)(\Delta \otimes Id)(\mathcal{F}) = (1 \otimes \mathcal{F})(1 \otimes \Delta)(\mathcal{F}),
\]

\[
(\epsilon \otimes Id)(\mathcal{F}) = 1 \otimes 1 = (Id \otimes \epsilon)(\mathcal{F}).
\]

Write

\[
\mathcal{F} = \sum f_{(1)} \otimes f_{(2)}, \quad \mathcal{F}^{-1} = \sum f'_{(1)} \otimes f'_{(2)},
\]

and set

\[
u = \mu \cdot (S \otimes Id)(\mathcal{F}^{-1}) = \sum S(f'_{(1)})f'_{(2)}.
\]

The property of $S$ shows that $\nu$ is invertible with inverse

\[
\nu^{-1} = \mu \cdot (Id \otimes S)(\mathcal{F}) = \sum f_{(1)}S(f_{(2)}).
\]

The following result gives a method to construct new Hopf superalgebra from old ones (cf. [16]), the non-super case can be found in [6].

**Lemma 2.2** The superalgebra $(H, \mu, \tau, \tilde{\Delta}, \tilde{\epsilon}, \tilde{S})$ is a new Hopf superalgebra with

\[
\tilde{\Delta} = \mathcal{F}\Delta\mathcal{F}^{-1}, \quad \tilde{\epsilon} = \epsilon, \quad \tilde{S} = \nu^{-1}Su.
\]

Now let us recall that the classical super-Virasoro algebra $\mathcal{L}$ without central extension over $\mathbb{F}$ is defined as an infinite-dimensional Lie superalgebra generated by the generators $\{L_i, G_k \mid i \in \mathbb{Z}, k \in \frac{1}{2}\mathbb{Z}\}$ satisfying the defining relations

\[
[L_i, L_j] = (j - i)L_{i+j}, \quad [L_i, G_k] = (k - i\alpha)L_{i+k}, \quad [G_k, G_l] = 2L_{k+l},
\]

for all $i, j \in \mathbb{Z}, k, l \in \frac{1}{2}\mathbb{Z}$. Obviously, $\mathcal{L}$ contains the Witt algegra $\mathcal{W}$ as subalgebra. In the following, we fix $m \in \mathbb{Z}/\{0\}$, $\alpha \in \mathbb{F}$. Denote

\[
X = \frac{1}{m}(L_0 + \alpha mL_{-m}), \quad Y = \exp(\alpha \text{ad}L_{-m})(L_m).
\]
In [11], Su and Zhao proved that $X$ and $Y$ span a two-dimensional subalgebra of the Virasoro algebra, i.e., $[X, Y] = Y$.

Denote by $\mathcal{U}(\mathcal{L})$ the universal enveloping algebra of $\mathcal{L}$. For any $x \in \mathcal{U}(\mathcal{L}), a \in \mathbb{F}, r, k \in \mathbb{Z}_+, i \in \mathbb{Z}$, we set

$$x_a^{<r>} = (x + a)(x + a + 1) \cdots (x + a + r - 1), \quad x_a^{[r]} = (x + a)(x + a - 1) \cdots (x + a - r + 1),$$

$$\binom{a}{r} = \frac{a(a - 1) \cdots (a - r + 1)}{r!}, \quad \left[ \begin{array}{c} a \\ r \end{array} \right]_k = \frac{a(a - k)(a - 2k) \cdots (a - (r - 1)k)}{r!}.

Especially, we have $\left[ \begin{array}{c} a \\ r \end{array} \right]_1 = \binom{a}{r}$ and $\left[ \begin{array}{c} a \\ r \end{array} \right]_{-1} = \binom{1 - a - r}{r}$. Denote $x_a^{<r>} = x_0^{<r>}, x_a^{[r]} = x_0^{[r]}$. Note that $\mathcal{U}(\mathcal{L})$ has a natural $\mathbb{Z}_2$-graded Hopf superalgebra structure. On the generators $x \in \mathcal{L}$ and the unit element 1, we define

$$\Delta_0(x) = x \otimes 1 + 1 \otimes x, \quad \Delta_0(1) = 1 \otimes 1,$$

$$\epsilon_0(x) = 0, \quad \epsilon_0(1) = 1,$$

$$S_0(x) = -x, \quad S_0(1) = 1.$$

Obviously, the Hopf superalgebra $(\mathcal{U}(\mathcal{L}), \mu, \tau, \Delta_0, S_0, \epsilon_0)$ is cocommutative. Denote by $\mathcal{U}(\mathcal{L})[[t]]$ an associative $\mathbb{F}$-algebra of formal power series with coefficients in $\mathcal{U}(\mathcal{L})$. Then $\mathcal{U}(\mathcal{L})[[t]]/i\mathcal{U}(\mathcal{L})[[t]] \cong \mathcal{U}(\mathcal{L})$. Naturally, $\mathcal{U}(\mathcal{L})[[t]]$ equips with an induced Hopf superalgebra structure arising from that on $\mathcal{U}(\mathcal{L})$, denoted still by $(\mathcal{U}(\mathcal{L})[[t]], \mu, \tau, \Delta_0, S_0, \epsilon_0)$, called the quantized universal enveloping superalgebra. As the non-super case [6], Andruskiewitsch [1] proved that the Lie superalgebra $\mathcal{L}$ has a natural Lie superbialgebra structure. Thus $\mathcal{U}(\mathcal{L})[[t]]$ is also called the quantization of the Lie superbialgebra $\mathcal{L}$. Define the coproduct $\Delta$ and the antipode $S$ on $\mathcal{U}(\mathcal{L})[[t]]$ as follows:

$$\Delta(L_i) = \sum_{r=0}^{\infty} \alpha^r \left[ \begin{array}{c} (r - 2)m - i \\ r \end{array} \right]_m (L_{i-rm} \otimes (1 - Yt)^{\frac{r}{m}-r}Y^rt^r) \quad (2.3)$$

$$+ \sum_{r=0}^{\infty} (-1)^r \sum_{s=0}^{2r} \alpha^s a_s(r, i) (X^{<r>} \otimes (1 - Yt)^{-r}L_{i+(r-s)m}t^r),$$

$$\Delta(G_k) = \sum_{r=0}^{\infty} \alpha^r \left[ \begin{array}{c} (r - \frac{3}{2})m - k \\ r \end{array} \right]_m (G_{k-rm} \otimes (1 - Yt)^{\frac{r}{m}-r}Y^rt^r) \quad (2.4)$$

$$+ \sum_{r=0}^{\infty} (-1)^r \sum_{s=0}^{2r} \alpha^s b_s(r, k) (X^{<r>} \otimes (1 - Yt)^{-r}G_{k+(r-s)m}t^r),$$

$$\epsilon(L_i) = 0, \quad \epsilon(G_k) = 0, \quad (2.5)$$
The main result of this paper is the following theorem which gives the quantization of the super-Virasoro algebra.

**Theorem 2.3.** The superalgebra $U(L)_m$ under the coproduct $\Delta$, the counit $\epsilon$, and the antipode $S$ defined by (2.3-2.7) is a noncommutative and noncocommutative Hopf superalgebra.

As a by-product of our proof, we obtain the following two combinatorial identities, which do not seem to be obvious to us:

$$a_s(r, i) = \sum_{p=0}^{s} (-1)^p \binom{m}{i + m} \binom{m}{i + (r - p - 2)m} \binom{m}{i + (r - p + 1)m},$$

$$b_s(r, k) = \sum_{p=0}^{s} (-1)^p \binom{m}{k + \frac{m}{2} + p} \binom{m}{k + (r - p - \frac{3}{2})m} \binom{m}{k + (r - p + \frac{1}{2})m}.$$  

(2.10)  

(2.11)

where $i \in \mathbb{Z}$, $k \in \frac{1}{2} \mathbb{Z}$, and for $r \in \mathbb{Z}^+; 0 \leq s \leq 2r$;

$$\sum_{p=0}^{s} (-1)^p \binom{m}{i + m} \binom{m}{i + (r - p - 2)m} \binom{m}{i + (r - p + 1)m} = 0,$$  

(2.10)

$$\sum_{p=0}^{s} (-1)^p \binom{m}{k + \frac{m}{2} + p} \binom{m}{k + (r - p - \frac{3}{2})m} \binom{m}{k + (r - p + \frac{1}{2})m} = 0,$$  

(2.11)

where $i \in \mathbb{Z}, k \in \frac{1}{2} \mathbb{Z}, r \in \mathbb{Z}^+, s > 2r$.

### 3. Proof of Theorem 2.3

We shall divide the proof of Theorem 2.3 into several lemmas. Set

$$X' = \frac{1}{m}L_0, \quad Y' = L_m.$$  

Let

$$\mathcal{W} = \{ \sum_{i \in \mathbb{Z}} a_i L_i | a_i \in \mathbb{F}, \text{ and } a_i = 0 \text{ for } i \ll 0 \}$$

be the completed Witt Lie algebra. Then $\exp(\text{ad } L_{-m}) \in \text{Aut}(\mathcal{W})$. Evidently, we have

$$\exp(\text{ad } L_{-m})(X') = X, \quad \exp(\text{ad } L_{-m})(Y') = Y.$$
Lemma 3.1 For any $r \in \mathbb{Z}_+, i \in \mathbb{Z}$, we have

$$\begin{align*}
(ad Y)^r(L_i) &= \sum_{q=0}^{2r} \alpha^q r! a_q(r, i) L_{i+r-(q-2)m}, \\
(ad Y)^r(G_k) &= \sum_{q=0}^{2r} \alpha^q r! b_q(r, i) G_{k+r-(q-2)m}.
\end{align*}$$

(3.1)

(3.2)

Proof. Note that $\exp(\alpha \text{ad} L) \in \text{Aut}(\mathcal{W})$ with the inverse $\exp(-\alpha \text{ad} L)$. We have

$$\begin{align*}
(ad Y)^r(L_i) &= \exp(\alpha \text{ad} L - m) \exp(-\alpha \text{ad} L - m) ((ad Y)^r(L_i)) \\
&= \exp(\alpha \text{ad} L - m) (ad Y')^r \exp(-\alpha \text{ad} L - m)(L_i) \\
&= \exp(\alpha \text{ad} L - m) \sum_{p=0}^{\infty} (-\alpha)^p \left[ \begin{array}{c} i + m \\ p \end{array} \right]_{m} \left[ \begin{array}{c} i + (r - p - 2)m \\ r \end{array} \right]_{m} L_{i+(r-q)m} \\
&= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (-1)^p \alpha^{q+p} r! \left[ \begin{array}{c} i + m \\ p \end{array} \right]_{m} \left[ \begin{array}{c} i + (r - p - 2)m \\ r \end{array} \right]_{m} \times \\
& \quad \times \left[ \begin{array}{c} i + (r - p + 1)m \\ q \end{array} \right]_{m} L_{i+(r-q)m} \\
&= \sum_{q=0}^{\infty} \sum_{p=0}^{q} (-1)^p \alpha^{q+p} r! \left[ \begin{array}{c} i + m \\ p \end{array} \right]_{m} \left[ \begin{array}{c} i + (r - p - 2)m \\ r \end{array} \right]_{m} \times \\
& \quad \times \left[ \begin{array}{c} i + (r - p + 1)m \\ q - p \end{array} \right]_{m} L_{i+(r-q)m},
\end{align*}$$

(3.3)

where the last equality follows by first setting $q = q' - p$ and exchanging summands over $q'$ and $p$ and then replacing $q'$ by $q$. From the fact that $(ad Y)^r(L_i) \in \bigoplus_{q=0}^{i+r-m} \mathbb{F} L_q$, we prove (3.1). Note that

$$\begin{align*}
\exp(\alpha \text{ad} L - m)(G_k) &= \sum_{p=0}^{\infty} (-\alpha)^p \left[ \begin{array}{c} k + \frac{1}{2}m \\ p \end{array} \right]_{m} G_{k-pm}, \\
(ad Y')^r(G_k) &= r! \left[ \begin{array}{c} k + (r - \frac{3}{2})m \\ r \end{array} \right]_{m} G_{k+rm}.
\end{align*}$$
We have

\[(\text{ad} Y)^r(G_k) = \sum_{q=0}^{\infty} \sum_{p=0}^{q} (-1)^p \alpha^q r! \left[ \begin{array}{c} k + \frac{m}{r} \\ p \end{array} \right] \left[ \begin{array}{c} k + (r - p - \frac{3}{2}m) \\ r \end{array} \right] m \times \left[ \begin{array}{c} k + (r - p - \frac{1}{2}m) \\ q - p \end{array} \right] m G_{k+(r-q)m}, \quad (3.4)\]

Then (3.2) follows. \qed

As a by product of (3.3) and (3.4), we immediately obtain the combinatorial identities (2.10) and (2.11).

**Lemma 3.2** For any \( a \in \mathbb{F}, r, s \in \mathbb{Z}_+, i \in \mathbb{Z}, \) and \( k \in \frac{1}{2}\mathbb{Z}, \) the following equations hold in \( \mathcal{U}(\mathcal{L}). \)

\[
L_i X_a^{< r >} = \sum_{p=0}^{r} \frac{\alpha^p r!}{(r - p)!} \left( \begin{array}{c} p - 2 \m - i \\ p \end{array} \right) m X_{a + p - \frac{i}{m}}^{< r - p >} L_{i - p m}, \quad (3.5)
\]

\[
G_k X_a^{< r >} = \sum_{p=0}^{r} \frac{\alpha^p r!}{(r - p)!} \left( \begin{array}{c} p - \frac{3}{2} \m - k \\ p \end{array} \right) m X_{a + p - \frac{k}{m}}^{< r - p >} G_{k - p m}, \quad (3.6)
\]

\[
L_i X_a^{[ r ]} = \sum_{p=0}^{r} \frac{\alpha^p r!}{(r - p)!} \left( \begin{array}{c} p - 2 \m - i \\ p \end{array} \right) m X_{a - \frac{i}{m}}^{[ r - p ]} L_{i - p m}, \quad (3.7)
\]

\[
G_k X_a^{[ r ]} = \sum_{p=0}^{r} \frac{\alpha^p r!}{(r - p)!} \left( \begin{array}{c} p - \frac{3}{2} \m - k \\ p \end{array} \right) m X_{a - \frac{k}{m}}^{[ r - p ]} G_{k - p m}; \quad (3.8)
\]

\[
Y^s X_a^{< r >} = X_{a + s}^{< r >} Y^s, \quad Y^s X_a^{[ r ]} = X_{a + s}^{[ r ]} Y^s, \quad (3.9)
\]

\[
L_i Y^r = \sum_{p=0}^{2p} \sum_{q=0}^{p} \frac{(-1)^p \alpha^q r! a_q(p, i)}{(r - p)!} a_q(p, i) Y^{r - p} L_{i + (p - q)m}, \quad (3.10)
\]

\[
G_k Y^r = \sum_{p=0}^{2p} \sum_{q=0}^{p} \frac{(-1)^p \alpha^q r! b_q(p, k)}{(r - p)!} b_q(p, k) Y^{r - p} G_{k + (p - q)m}. \quad (3.11)
\]

**Proof.** We prove (3.5) and (3.6) by induction on \( r. \) The case of \( r = 1 \) follows from the formula 
\( L_i X = (X - \frac{i}{m}) L_i - \alpha(m + i) L_{i - m} \) and \( G_k X = (X - \frac{k}{m}) G_k - \alpha(\frac{m}{2} + k) G_{k - m}. \) Let 
\[
a^r_p = \frac{\alpha^p r!}{(r - p)!} \left( \begin{array}{c} p - 2 \m - i \\ p \end{array} \right) m X_{a + p - \frac{i}{m}}^{< r - p >}, \quad b^r_p = \frac{\alpha^p r!}{(r - p)!} \left( \begin{array}{c} p - \frac{3}{2} \m - k \\ p \end{array} \right) m X_{a + p - \frac{k}{m}}^{< r - p >}. \]
Suppose that (3.5) and (3.6) holds for \( r \). As for the case \( r + 1 \), we have

\[
L_i X^\langle r+1 \rangle = L_i X\langle r \rangle (X + a + r)
\]

\[
= \sum_{p=0}^{r} a^r_p ((X + a + r + p - \frac{i}{m}) L_{i-pm} + \alpha (pm - m - i) L_{i-(p+1)m})
\]

\[
= \sum_{p=0}^{r+1} (a^r_p (X + a + r + p - \frac{i}{m}) + \alpha a^r_{p-1} (pm - 2m - i)) L_{i-pm}
\]

\[
= \sum_{p=0}^{r+1} a^{r+1}_p L_{i-pm}
\]

and

\[
G_k X^\langle r+1 \rangle = G_k X\langle r \rangle (X + a + r)
\]

\[
= \sum_{p=0}^{r} b^r_p ((X + a + r + p - \frac{k}{m}) G_{k-pm} + \alpha (pm - \frac{1}{2}m - k) G_{k-(p+1)m})
\]

\[
= \sum_{p=0}^{r+1} (b^r_p (X + a + r + p - \frac{k}{m}) + \alpha b^r_{p-1} (pm - \frac{3}{2}m - k)) G_{k-pm}
\]

\[
= \sum_{p=0}^{r+1} b^{r+1}_p G_{k-pm}.
\]

Then (3.5) and (3.6) hold. Let

\[
c^r_p = \frac{\alpha^r p!}{(r-p)!} \left[ \begin{array}{c} (p-2)m - i \\ p \end{array} \right] X^{[r-p]}_{a-\frac{i}{m}}, \quad d^r_p = \frac{\alpha^r p!}{(r-p)!} \left[ \begin{array}{c} (p-\frac{3}{2})m - k \\ p \end{array} \right] X^{[r-p]}_{a-\frac{k}{m}}.
\]

From the fact that

\[
c^{r+1}_p = c^r_p (X + a + r + p - \frac{i}{m}) + \alpha c^r_{p-1} (pm - 2m - i)
\]

and

\[
d^{r+1}_p = d^r_p (X + a + r + p - \frac{k}{m}) + \alpha d^r_{p-1} (pm - \frac{3}{2}m - k),
\]

we can prove (3.7) and (3.8) by induction on \( r \). The proof of (3.9) is similar. Using (3.1), together with the fact that

\[
L_i Y^r = \sum_{p=0}^{r} (-1)^p \binom{r}{p} Y^{r-p} (adY)^p (L_i),
\]

we immediately get (3.10). The proof of (3.11) is similar with (3.10). \( \square \)

The following lemma belongs to [9].
Lemma 3.3 For any \( x \in \mathcal{U}(\mathcal{L}) \), \( a, d \in \mathbb{F} \) and \( r, s, m \in \mathbb{Z}_+ \), we have
\[
\begin{align*}
x^{<r+s>}_a &= x^{<r>}_a x^{<s>}_a, \quad (3.12) \\
x^{[r+s]}_a &= x^{[r]}_a x^{[s]}_a, \quad (3.13) \\
x^{<r>}_a &= x^{<r-1>}_{a-r}, \quad (3.14) \\
\sum_{r+s=m} \frac{(-1)^s}{r!s!} x^{[r]}_a x^{<s>}_d &= \binom{a-d}{m}, \quad (3.15) \\
\sum_{r+s=m} \frac{(-1)^s}{r!s!} x^{[r]}_a x^{[s]}_d &= \binom{a-d+m-1}{m}. \quad (3.16)
\end{align*}
\]

For \( a \in \mathbb{F} \), we set
\[
\begin{align*}
\mathcal{F}_a &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X^r_a \otimes Y^r t^r, \\
F_a &= \sum_{r=0}^{\infty} \frac{1}{r!} X^{<r>}_a \otimes Y^r t^r, \\
u_a &= \mu \cdot (S_0 \otimes \text{Id})(F_a), \\
v_a &= \mu \cdot (\text{Id} \otimes S_0)(F_a).
\end{align*}
\]
Since \( S_0(X^{<r>}_a) = (-1)^r X^r_a \) and \( S_0(Y^r) = (-1)^r Y^r \), we have
\[
\begin{align*}
u_a &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X^r_a Y^r, \\
v_a &= \sum_{r=0}^{\infty} \frac{1}{r!} X^r_a Y^r t^r.
\end{align*}
\]

Denote \( \mathcal{F} = \mathcal{F}_0, F = F_0, u = u_0, v = v_0 \). Following the results in [10] and [14], \( \mathcal{F} \) is a Drinfel’d twist of the Witt algebra. The observation of \( \mathcal{W} \subset \mathcal{L} \) implies that \( \mathcal{F} \) is also a Drinfel’d twist of \( \mathcal{L} \).

Lemma 3.4. For \( a, d \in \mathbb{F} \), we have
\[
\mathcal{F}_a F_d = 1 \otimes (1 - Y t)^{a-d}, \quad v_a u_d = (1 - Y t)^{-(a+d)}.
\]

Therefore \( F_a, \mathcal{F}_a, u_a, v_a \) are all invertible, and \( \mathcal{F}_a^{-1} = F_a, u_a^{-1} = v_a^{-1} \).

Proof. Using (3.15), we have
\[
\begin{align*}
\mathcal{F}_a F_d &= \left( \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} X^r_a \otimes Y^r t^r \right) \cdot \left( \sum_{s=0}^{\infty} \frac{1}{s!} X^{<s>}_d \otimes Y^s t^s \right) \\
&= \sum_{r,s=0}^{\infty} \frac{(-1)^r}{r!s!} X^r_a X^{<s>}_d \otimes Y^{r+s} t^{r+s} \\
&= \sum_{p=0}^{\infty} (-1)^p \binom{a-d}{p} \otimes Y^p t^p \\
&= 1 \otimes (1 - Y t)^{a-d}.
\end{align*}
\]
Using (3.16) and the second formula of (3.9), we have

\[ v_u u_d = \left( \sum_{r=0}^{\infty} \frac{1}{r!} X^{[r]} Y^{r} t^r \right) \cdot \left( \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} X^{[s]} Y^s t^s \right) \]

\[ = \sum_{r,s=0}^{\infty} \frac{1}{r!} \frac{(-1)^s}{s!} X^{[r]} Y^{r} X^{[s]} Y^s t^{r+s} \]

\[ = \sum_{p=0}^{\infty} \left( \frac{a + d + p - 1}{p} \right) Y^p t^p \]

\[ = (1 - Y t)^{-(a+d)}. \]

Then this Lemma follows. \( \square \)

**Lemma 3.5.** For \( a \in \mathbb{F}, i \in \mathbb{Z} \), we have

\[ (L_i \otimes 1) F_a = \sum_{s=0}^{\infty} \alpha^s \left[ \begin{array}{c} (s-2)m - i \\ s \end{array} \right] F_{a - \frac{m}{m} + s} (L_{i-sm} \otimes Y^{s} t^{s}), \] (3.17)

\[ (G_k \otimes 1) F_a = \sum_{s=0}^{\infty} \alpha^s \left[ \begin{array}{c} (s-3) \frac{m}{m} - k \\ s \end{array} \right] F_{a - \frac{m}{m} + s} (G_{k-sm} \otimes Y^{s} t^{s}), \] (3.18)

\[ (1 \otimes L_i) F_a = \sum_{s=0}^{\infty} (-1)^s F_{a+s} \left( \sum_{p=0}^{2s} \alpha^p a_p(s,i) X^{[p]} \otimes L_{i+(s-p)m} t^a \right), \] (3.19)

\[ (1 \otimes G_k) F_a = \sum_{s=0}^{\infty} (-1)^s F_{a+s} \left( \sum_{p=0}^{2s} \alpha^p b_p(s,i) X^{[p]} \otimes G_{k+(s-p)m} t^a \right), \] (3.20)

\[ L_i u_a = u_{a + \frac{r}{m}} \sum_{s=0}^{\infty} \sum_{p=0}^{2p} \sum_{q=0}^{2p} (-1)^s \alpha^{s+q} \left[ \begin{array}{c} (s-2)m - i \\ s \end{array} \right] \times \] (3.21)

\[ \times a_q(p, i - sm) X^{[p]}_{-a-m} L_{i+(s-p-q)m} Y^{s} t^{s+p}, \]

\[ G_k u_a = u_{a + \frac{r}{m}} \sum_{s=0}^{\infty} \sum_{p=0}^{2p} \sum_{q=0}^{2p} (-1)^s \alpha^{s+q} \left[ \begin{array}{c} (s-3) \frac{m}{m} - k \\ s \end{array} \right] \times \] (3.22)

\[ \times b_q(p, k - sm) X^{[p]}_{-a-m} G_{k+(s-p-q)m} Y^{s} t^{s+p}. \]

**Proof.** From (3.5) and the definition of \( F_a \), we have

\[ (L_i \otimes 1) F_a = \sum_{r=0}^{\infty} \frac{1}{r!} L_i X^{[r]} \otimes Y^{r} t^r \]

\[ = \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{s=0}^{r} \frac{\alpha^s r!}{(r-s)!} \left( \begin{array}{c} (s-2)m - i \\ s \end{array} \right) X^{[r-s]}_{a - \frac{m}{m} + s} L_{i-sm} \otimes Y^{r} t^r \]
we have

This proves (3.17). Similarly, (3.18) follows from (3.6). For (3.19), using (3.12) and (3.10), we get

So (3.19) is right. (3.20) is obtained from (3.12) and (3.11). Now we prove (3.21). From (3.13), (3.7), (3.9) and (3.10), we have

So (3.19) is right. (3.20) is obtained from (3.12) and (3.11). Now we prove (3.21). From (3.13), (3.7), (3.9) and (3.10), we get
Proof of Theorem 2.3. Since Lemma 3.4, we have

\[ Sd \text{ determine the action of } \Delta \text{ and } S \]

Now we give the proof of Theorem 2.3.

Proof of Theorem 2.3. Since \( F \) is a Drinfel’d twist, according to Lemma 2.2, we only need to determine the action of \( \Delta \) and \( S \) on \( L_i, G_k \in \mathcal{L}, i \in \mathbb{Z}, k \in \frac{1}{2} \mathbb{Z} \). From (3.17), (3.19), and Lemma 3.4, we have

\[
\Delta(L_i) = F\Delta_0(L_i)F^{-1} = F(L_i \otimes 1)F^{-1} + F(1 \otimes L_i)F^{-1} = F(L_i \otimes 1)F + F(1 \otimes L_i)F
\]

\[
= \mathcal{F} \sum_{r=0}^{\infty} \alpha^r \left[ \frac{(r-2)m-i}{r} \right] \left( F_{\frac{1}{m}+r} (L_{i-rm} \otimes Y^r t^r) \right)
\]

\[
+ \mathcal{F} \left( -1 \right)^r F_r \left( \sum_{s=0}^{2r} \alpha^s a_s(r, i) X^{r>s} \otimes L_{i+s-r} t^r \right)
\]

\[
= \sum_{r=0}^{\infty} \alpha^r \left[ \frac{(r-2)m-i}{r} \right] \left( (1 \otimes (1 - Yt)\frac{1}{m} - r) (L_{i-rm} \otimes Y^r t^r) \right)
\]

\[
+ \sum_{r=0}^{\infty} \left( -1 \right)^r (1 \otimes (1 - Yt)^r) \left( \sum_{s=0}^{2r} \alpha^s a_s(r, i) X^{r>s} \otimes L_{i+s-r} t^r \right)
\]

\[
= \sum_{r=0}^{\infty} \alpha^r \left[ \frac{(r-2)m-i}{r} \right] \left( L_{i-rm} \otimes (1 - Yt)\frac{1}{m} - r) Y^r t^r \right)
\]

\[
+ \sum_{r=0}^{\infty} \left( -1 \right)^r \sum_{s=0}^{2r} \alpha^s a_s(r, i) (X^{r>s} \otimes (1 - Yt)^r L_{i+s-r} t^r) \right).
\]

From (3.18), (3.20), and Lemma 3.4, we have

\[
\Delta(G_k) = F\Delta_0(G_k)F^{-1} = F(G_k \otimes 1)F^{-1} + F(1 \otimes G_k)F^{-1} = F(G_k \otimes 1)F + F(1 \otimes G_k)F
\]

\[
= \mathcal{F} \sum_{r=0}^{\infty} \alpha^r \left[ \frac{(r-3/2)m-k}{r} \right] \left( F_{\frac{1}{m}+r} (G_{k-rm} \otimes Y^r t^r) \right)
\]

\[
+ \mathcal{F} \left( -1 \right)^r F_r \left( \sum_{s=0}^{2r} \alpha^s b_s(r, k) X^{r>s} \otimes G_{k+s-r} t^r \right)
\]

\[
= \sum_{r=0}^{\infty} \alpha^r \left[ \frac{(r-3/2)m-k}{r} \right] \left( (1 \otimes (1 - Yt)\frac{1}{m} - r) (G_{k-rm} \otimes Y^r t^r) \right)
\]

\[
+ \sum_{r=0}^{\infty} \left( -1 \right)^r (1 \otimes (1 - Yt)^r) \left( \sum_{s=0}^{2r} \alpha^s b_s(r, k) X^{r>s} \otimes G_{k+s-r} t^r \right).
\]
\[
= \sum_{r=0}^{\infty} \alpha^r \left[ (r - \frac{3}{2})m - k \right]_m (G_{k-rm} \otimes (1 - Yt)^{k-r}Y^r t^r) \\
+ \sum_{r=0}^{\infty} (-1)^r \sum_{s=0}^{2r} \alpha^s b_s(r, k) \left( X^{<r>} \otimes (1 - Yt)^{-r} G_{k+(r-s)m} t^r \right).
\]

Using (3.21) and Lemmas 3.4, we have
\[
S(L_i) = u^{-1} S_0(L_i) u = -v L_i u \\
= -vu \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{2p} (-1)^r \alpha^{r+q} \left[ (r - 2)m - i \right]_m a_q(p, i - rm) X_{\frac{i}{m}}^{[p]} L_{i+(p-r-q)m} Y^r t^{r+p} \\
= -(1 - Yt)^{-\frac{k}{m}} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{2p} (-1)^r \alpha^{r+q} \left[ (r - 2)m - i \right]_m \times \\
a_q(p, i - rm) X_{\frac{i}{m}}^{[p]} L_{i+(p-r-q)m} Y^r t^{r+p}.
\]

Using (3.22) and Lemmas 3.4 we have
\[
S(G_k) = u^{-1} S_0(G_k) u = -v G_k u \\
= -vu \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{2p} (-1)^r \alpha^{r+q} \left[ (r - \frac{3}{2})m - k \right]_m b_q(p, k - rm) X_{\frac{i}{m}}^{[p]} G_{k+(p-r-q)m} Y^r t^{r+p} \\
= -(1 - Yt)^{-\frac{k}{m}} \sum_{r=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{2p} (-1)^r \alpha^{r+q} \left[ (r - \frac{3}{2})m - k \right]_m \times \\
b_q(p, k - rm) X_{\frac{i}{m}}^{[p]} G_{k+(p-r-q)m} Y^r t^{r+p}.
\]

The proof is completed. \qed

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