Deviations from the exponential decay law in relativistic quantum field theory: the example of strongly decaying particles

Francesco Giacosa\textsuperscript{a}, and Giuseppe Pagliara\textsuperscript{b,c}

\textsuperscript{a}Institut für Theoretische Physik, Johann Wolfgang Goethe Universität, Max-von-Laue-Str. 1, D–60438 Frankfurt am Main, Germany
\textsuperscript{b}Institut für Theoretische Physik, Ruprecht-Karls-Universität, Philosophenweg 16, D-69120, Heidelberg, Germany
\textsuperscript{c}Dip. di Fisica dell’Università di Ferrara and INFN Sez. di Ferrara, Via San Guglielmo 1, I-44100 Ferrara, Italy

We show that a short-time regime, in which a deviation from the exponential decay law occurs, exists also in the framework of a superrenormalizable relativistic quantum field theory. This, in turn, implies the possibility of a quantum Zeno effect also for elementary decays. The attention is then focused on the typical order of magnitude of strong decay rates of mesons: for these particles, strong deviations from the exponential decay law are present during a period of time comparable with their mean life time. As a concrete example, the case of the $\rho$ meson is studied.

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Since the discovery of radioactivity, the theoretical and experimental understanding of unstable particles has attracted much attention of physicists. The development of quantum mechanics in the beginning of the 20th century has been a crucial step in this direction. The decay is, in fact, subject to a fundamental quantum indetermination: only the probability that it happened or not can be calculated.

When an unstable state $|S\rangle$ is created/prepared at $t = 0$, the survival probability $p(t)$ is defined as the probability that the state has not decayed yet at the instant $t$. The usual, empirical exponential law of decay $p(t) = e^{-\gamma t} = e^{-t/\tau}$, where $\Gamma$ is the decay width and $\tau = \Gamma^{-1}$ the mean life time, describes to great accuracy the decay of unstable nuclei. It has found its theoretical derivation in quantum mechanics by means of the Fermi Golden rule and it can be easily obtained by assuming the Breit-Wigner distribution for the energy of the unstable state. However, such distribution, while very useful in describing data, is based on the assumption that $\Gamma$ is very small in comparison with the mass/energy of the state and, more important, suffers the problem of the absence of a minimum value for the energy, which in turn would correspond to an Hamiltonian unbounded from below.

It is actually a renowned fact that in quantum mechanics $p(t)$ does not exactly follow an exponential law. Deviations in the short- and long-time regimes take place, see the general discussion in Ref.\textsuperscript{[1]}. In particular, by writing $p(t) = e^{-\gamma(t)t}$, the function $\gamma(t)$ is not constant but decreases for short times and vanishes for $t \to 0$\textsuperscript{2,3}. This property is at the origin of the so-called quantum Zeno effect\textsuperscript{4}, according to which subsequent collapses of the wave function of the unstable state $|S\rangle$ during the non-exponential regime, generate a slower decay rate and, in the limit of a continuous observation, a complete inhibition of the decay. In fact, after $N$ measurements performed at time intervals $\Delta t$, the probability that the state has not decayed at the time $T = N\Delta t$ is given by $p(t_0)^N = e^{-\gamma(t_0)N}$, which is larger than the survival probability obtained for a single measurement performed at the instant $T$, $p(T) = e^{-\gamma(T)T}$, as long as $\gamma(t_0) < \gamma(T) \approx \Gamma$. Moreover, $p(t_0)^N \to 1$ for $t_0 \to 0$ (i.e. for $N \to \infty$ by keeping $T$ fixed) because $\lim_{t_0 \to 0} \gamma(t_0) = 0$, implying that $S$ does not decay at all. Interestingly, there can be also values of $t_0$ such that $\gamma(t_0) > \Gamma$, thus the measurements would originate a faster decay rate (anti-Zeno effect)\textsuperscript{5}. The quantum Zeno and anti-Zeno effects have indeed found experimental confirmation in cold atoms experiments\textsuperscript{6}, a discovery that renewed also the theoretical interest on these fascinating features of quantum systems.

In the middle of the 20th century Relativistic Quantum Field Theory (RQFT) has been developed. Since within this formalism the number of particles is not conserved, RQFT has been recognized to be the most natural theoretical framework for the study of decays. The fundamental randomness in the process of creating and annihilating particles is at the root of the indetermination of the lifetime of unstable particles. Electromagnetic decays of atoms can be driven back to the emission of photons in the context of the best known RQFT, Quantum Electrodynamics (QED). More in general, decay widths play nowadays an important role in many phenomenological studies of the Standard Model (SM), such as hadron decays in QCD and weak decays of leptons, heavy quarks and weak bosons. Also the recent search at LHC for the last missing particle of the SM, the Higgs boson, relies upon the predictions of its decay properties.

The evaluation of the decay width $\Gamma$ of an unstable state $|S\rangle$ in RQFT is now a technically well-defined task. However, in view of the present discussion, a natural question immediately emerges: Are short-time deviations from the exponential law $p(t) = e^{-\Gamma t}$ present also in the context of RQFT?

In the perturbative approach of Refs.\textsuperscript{5,8} no (or very much suppressed) short-time deviations from the exponential law, and thus no quantum Zeno effect, were found within RQFT. In this work, after a critical reconsideration of the issue of the survival probability in RQFT, we actually obtain the opposite answer: short-time devia-
tions from the exponential survival probability do occur in a genuine RQFT context in the case of superrenormalizable theories, to which we restrict our attention here. In fact, \( \gamma(t) < \Gamma \) for short times and \( \gamma(t \to 0^+)=0 \), implying that the quantum Zeno effect is possible also in RQFT. More in general, our results are compatible with the non-relativistic model of Ref. [10], in which a deeper understanding of the quantum Zeno and anti-Zeno effects has been achieved. Remarkably, the short-time deviation from the exponential law is strongly enhanced in the case of the short-living hadrons, such as in the case of the decay \( \rho \to \pi \pi \) which will be explicitly investigated later. This is an interesting result on its own, which might affect the dynamics of the fast expanding hadrons fireball in heavy ions collisions experiments in which these particles are abundantly produced.

We now turn to a concrete example by considering the following (superrenormalizable) RQFT Lagrangian with two scalar fields \( S \) and \( \varphi \):

\[
\mathcal{L} = \frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}M_S^2 S^2 + \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2 \varphi^2 + gS\varphi^2. \tag{1}
\]

The interaction term \( \mathcal{L}_{\text{int}} = gS\varphi^2 \) induces the decay process \( S \to \varphi\varphi \), whose tree-level decay rate reads

\[
\Gamma_{S\to\varphi\varphi}^{\text{tree}} = \frac{\sqrt{M_S^2 - m^2}}{8\pi M_S^2} (\sqrt{2}g)^2 \theta(M_S - 2m). \tag{2}
\]

The ‘naïve’, tree-level expression of the survival probability \( p(t) \) for the resonance \( S \) created at \( t = 0 \) is \( p_{\text{tree}}(t) = e^{-\Gamma_{S\to\varphi\varphi}^{\text{tree}} t} \) and the tree-level expression of the mean life time is \( \tau_{\text{tree}} = 1/\Gamma_{S\to\varphi\varphi}^{\text{tree}} \).

A crucial intermediate step toward the determination of the survival probability \( p(t) \) within the RQFT framework is the evaluation of the propagator \( \Delta_S(p^2) \) of the unstable resonance \( S \), which is obtained by (re)summing the one-particle irreducible self-energy contribution \( \Sigma(p^2) \):

\[
\Delta_S(p^2) = \left[ p^2 - M_S^2 + (\sqrt{2}g)^2 \Sigma(p^2) + i\varepsilon \right]^{-1}. \tag{3}
\]

To lowest order, \( \Sigma(p^2) \) corresponds to a bubble of two fields \( \varphi \):

\[
\Sigma(p^2) = i \int_q \frac{d^4q}{(2\pi)^4} \left[ \frac{e^{2\pi}}{(2\pi)^2} - m^2 + i\varepsilon \right] \left[ \frac{e^{2\pi}}{(2\pi)^2} - m^2 + i\varepsilon \right]. \tag{4}
\]

with \( \int_q = \int\frac{d^4q}{(2\pi)^4} \). The spectral function \( d_S(x) \) of the scalar field \( S \) is proportional to the imaginary part of the propagator:

\[
d_S(x) = \sqrt{x} \left[ \frac{d^4q}{(2\pi)^4} \left[ \frac{e^{2\pi}}{(2\pi)^2} - m^2 + i\varepsilon \right] \right]. \tag{5}
\]

The quantity \( d_S(x)dx \) represents the probability that in the rest frame of \( S \) the state \( S \) has a mass between \( x \) and \( x + dx \). It is correctly normalized for each \( g \),

\[
\int_0^\infty d_S(x)dx = 1 \text{ and, in the limit } q \to 0, \text{ the expected spectral function } d_S(x) = \delta(x - M_B) \text{ is obtained [10,11].}
\]

This fact allows to determine the probability amplitude \( a(t) \), and therefore the survival probability \( p(t) \):

\[
a(t) = \int_{-\infty}^{+\infty} dx \ d_S(x)e^{-ixt}, \quad p(t) = |a(t)|^2. \tag{6}
\]

The condition \( p(0) = 1 \) is fulfilled in virtue of the normalization of \( d_S(x) \). This property is, in turn, a consequence of the 1-loop resummation and the validity of the Källen-Lehman representation. (The integral in Eq. (6) is actually limited to the interval \((2m, \infty) \) in virtue of the step-function \( \theta(x - 2m) \) arising in \( d_S(x) \), see the optical theorem below. The extension to the integration range \((-\infty, \infty) \) allows to express \( a(t) \) as the Fourier-transform of \( d_S(x) \), which represents a technical help in a variety of applications). The general discussion on \( p(t) \) in Refs. [1,12] in the framework of quantum mechanics is applicable in the present RQFT theory.

An important aspect concerning the definition of the properties of unstable states has been raised in Ref. [12] where it has been pointed out that physical measurable quantities must be invarinant under field redefinitions, as the S-matrix elements, leading to the so called “complex mass renormalization scheme”. A natural question arises whether the survival probability of Eq. (6) is invariant under field redefinition. In Appendix A the issue is discussed in more detail: a redefinition of the fields corresponds to a change of the initial state \( |S\rangle \). A related subtle point concerns the state formation at \( t = 0 \): in Ref. [8] the full scattering process \( \varphi\varphi \to S \to \varphi\varphi \) is computed to second order in perturbation theory and also the “formation time” of the resonance has been modeled. Even if there is no instant of time at which the state of the system corresponds to the state \( S \), the survival amplitude \( a(t) \) directly enters in the calculation of the temporal evolution of the system and could therefore lead to “observable” effects (see Appendix B for a detailed discussion).

For what concerns the measurability of the spectral function \( d_S(x) \), we devise the following situation: we introduce two scalar fields \( A \) and \( B \), the first massless and the second with mass \( M_B > M_S \) and write down the interaction Lagrangian \( \mathcal{L}_{\text{int}} = cB\bar{A}\varphi\varphi \). We suppose that the interaction strength \( c \) is small enough to allow for a tree-level analysis of the process \( B \to AS \), which reads \( \Gamma_{B\to AS}^{\text{tree}}(M_B) = \frac{c^2 B\bar{A}\varphi\varphi}{M_B^2} \). When \( g \neq 0 \) the state \( S \) decays into \( \varphi\varphi \). Physically, one observes a tree-body decay \( B \to A\varphi\varphi \), whose decay-rate reads:

\[
\Gamma_{B\to AS}^{\text{tree}}(M_B) = \int_0^{M_B} \Gamma_{B\to AS}^{\text{tree}}(M_B) d_S(x) dx. \tag{7}
\]

The tree-body decay is decomposed into two steps: \( B \to AS \) and \( S \to \varphi\varphi \). The quantity \( \Gamma_{B\to AS}^{\text{tree}}(M_B) \) represents the decay rate for \( B \to AS \) (at a given mass \( x \) for the state \( S \)) and \( d_S(x)dx \) is the corresponding weight, i.e. the probability that the resonance \( S \) has a mass between
of the loop integral. In the rest frame theorem: A general property for \( \Sigma(x) \) is perfectly possible in the present RQFT context. This, in turn, implies that the quantity \( dS(x) \) can be ‘measured’. Interestingly, there are experimental situations which are conceptually similar to the here presented case: the decay \( \phi \rightarrow \gamma \pi^0 \pi^0 \) through the intermediate \( a_0(980) \) and \( f_0(980) \) mesons [14], the similar decay of the \( J/\psi \) lepton into \( \nu \pi \pi \), dominated by the \( \rho \) meson for an invariant \( \pi \pi \) mass close to \( \rho \) mass [15]. It should be clearly stressed that the mentioned experiments are by far not so clean as our depicted toy model due to the presence of many possible intermediate states and background interactions. Moreover, the exact theory of hadrons, being not derivable from QCD, is unknown and therefore the determination of hadronic spectral functions is in most cases model dependent. Our attention to hadrons, specifically to the example of the \( \rho \) meson later on, is thus limited to simple hadronic models. However, here we are not interested to a precision study of hadronic spectral functions, but only to the order of magnitude involved in the deviation from the exponential decay law, for which a simplified treatment of hadrons is –at least as a first step– justifiable.

After this digression on the spectral functions, we turn to the main subject of this work, which is the behavior of the survival probability \( p(t) \). The first derivative of \( p(t) \) is well defined and vanishes, \( p'(t = 0) = 0 \) as a consequence of the fact that the integral \( \int_0^\infty x \, dS(x) \, dx \) is finite and real (it is the mean mass \( M \), a reasonable definition for the mass of a resonance [11]). This, in turn, implies that the function \( \gamma(t) = \frac{1}{t} \ln p(t) \) vanishes for \( t \to 0^+ \):

\[
\lim_{t \to 0^+} \gamma(t) = - \lim_{t \to 0^+} \frac{p'(t)}{p(t)} = 0. \tag{8}
\]

We can therefore conclude that the quantum Zeno effect is perfectly possible in the present RQFT context.

In order to explicitly calculate the function \( p(t) \) one has first to evaluate the loop integral. In the rest frame of the \( S \) particle \((p = (x, 0))\) one first solves the integral over \( q^0 \) by calculating the residues and then introduces a cutoff \( \Lambda \) on the remaining integral over \( d^3q \) obtaining:

\[
\Sigma(x) = \frac{-\sqrt{4m^2 - x^2}}{8\pi^2 x} \arctan \left( \frac{\Lambda x}{\sqrt{\Lambda^2 + x^2 \sqrt{4m^2 - x^2}}} \right) - \frac{1}{8\pi^2} \log \left( \frac{m}{\Lambda + \sqrt{\Lambda^2 + m^2}} \right). \tag{9}
\]

A general property for \( \Sigma(x) \) follows from the optical theorem:

\[
I(x) = (\sqrt{2}g)^2 \text{Im}[\Sigma(x)] = x \Gamma_{S \phi \phi}^{\text{d}1}(x) \theta \left( \sqrt{\Lambda^2 + m^2} - \frac{x}{2} \right).
\]

FIG. 1: The survival probability \( p(t) \) of Eq. [6] is shown in the case of infinite (thin solid line) and finite, \( \Lambda = 1 \text{ GeV} \), (thick gray line) cutoff. In both cases the non-exponential behavior at short times is clearly visible. The exponential tree-level decay is shown for comparison (dashed line). The quantity \( |p(t) - e^{-t^2/2}| \) is also displayed by the thin dot-dashed and thick gray dot-dashed lines for the two cases respectively.

The function \( I(x) \) is zero for \( 0 < x < 2m \) and for \( x > 2\sqrt{\Lambda^2 + m^2} \) and -in between- does not depend on the cutoff \( \Lambda \). The quantity \( R(x) = (\sqrt{2}g)^2 \text{Re}[\Sigma(x)] \) is nonzero below and above threshold and depends explicitly on \( \Lambda \). The physical (Breit-Wigner) mass \( M \) of the scalar field \( S \) is modified by the 1-loop corrections and is determined by the equation: \( M_f^2 - M_0^2 + R(M) = 0 \).

In general, \( M \neq M_0 \). However, the requirement \( M = M_0 \) can be fulfilled by introducing a counterterm in Eq. [11]: \( \mathcal{L} \rightarrow \mathcal{L} - \frac{1}{2} CS^2 \) with \( C = R(M_0) \). (Note, one could well work with a physical mass \( M \neq M_0 \), provided that the tree-level decay width in Eq. [6] is evaluated at the physical mass \( M \).)

There are basically two different approaches to deal with the described set of equations: (i) the theory is regarded as a fundamental theory valid up to -say- the Planck energy; (ii) the theory is regarded as an effective, low-energy manifestation of some other theory and the cutoff \( \Lambda \) is a finite number of the same order of magnitude of the masses. In the following we study separately these two cases.

Case (i): \( \mathcal{L} \) as ‘fundamental’ theory: When the cutoff \( \Lambda \) is much larger than the other scales of the model, as in the case \( \Lambda \approx M_{\text{Planck}} \), it is convenient and numerically exhaustive to perform the limit \( \Lambda \to \infty \). In order to have a finite physical mass \( M = M_0 \), the counterterm \( C = R(M_0) \) needs to be very large (formally divergent, \( \Lambda \to \infty \)). Once this divergence has been subtracted, all the results -including the survival probability \( p(t) \)- are finite and well defined.

Now we turn to a quantitative estimate of the short-time interval in which the deviations from the exponential decay law are non-negligible. In the Literature a Taylor expansion of the function \( p(t) \) is often performed and the so-called Zeno time \( \tau_Z = \sqrt{-2/p(0)} \) is introduced as a measure of the short-time interval with a non-exponential behavior. This procedure is, however,
FIG. 2: $\tau_M/\tau_{\ell}$ as function of $\tau_{\ell}$ in the cases of infinite (thin line) and finite ($\Lambda = 1$, thick gray line) cutoffs. For short living particles, such as hadronic resonances, the non-exponential regime lasts for a time scale of the same order of magnitude of the mean life time of the particle.

not general: it is in fact a priori not obvious that the second derivative $p''(0)$ exists. In the present case, for instance, the latter diverges for $\Lambda \to +\infty$ since $d_S(x)$ behaves asymptotically as $1/x^3$. We thus introduce a more general definition, which does not depend on the higher derivatives of $p(t)$ at $t = 0$. The time $\tau_M$ is defined as the instant of time at which the deviation of the function $p(t)$ from the exponential behavior $e^{-\Gamma t}$ is maximal:

$$\max (p(t) - e^{-\Gamma t}) \to t = \tau_M. \quad (10)$$

Clearly, $\frac{d}{dt} (p(t) - e^{-\Gamma t})_{t=\tau_M} = 0$. In all practical cases $\tau_M$ corresponds to the first root of the derivative of the function $p(t) - e^{-\Gamma t}$.

We now turn to a numerical example. We choose the physical mass as $M = M_0 = 1$ GeV and $m = m_{\pi} = 0.139$ GeV (typical values of hadronic particles). In Fig. 1 the function $p(t)$ is plotted (solid thin line) for the choice $g = 2\sqrt{2}$ GeV, which corresponds to a tree-level decay width $\Gamma_{S_{\rho\pi\pi}} = 305.7$ MeV (on the high side of a typical hadronic decay) and to a tree-level lifetime $\tau_{\ell} = 3.27$ GeV$^{-1}$. The existence of a non-exponential behavior is clearly visible: numerically, one obtains $\tau_M/\tau_{\ell} = 0.48$, implying that the non-exponential regime lasts an amount of time comparable with the mean life time. Also the function $|p(t) - e^{-\Gamma t}|$ is displayed (thin dot-dashed line) to clearly show the existence of $\tau_M$.

In Fig. 2 the ratio $\tau_M/\tau_{\ell}$ (thin solid line) is plotted as a function of $\tau_{\ell}$ (i.e., as a function of the coupling constant $g^{-2}$). The ratio $\tau_M/\tau_{\ell}$ increases for decreasing $\tau_{\ell}$: the non-exponential regime is always present but is enhanced for short living particles (lifetime typical of a strong decay), while it decreases for long-living particles (i.e., in the regimes of electromagnetic and weak decays). For instance, decreasing the coupling $g$ to 1.15 GeV implies a decay width of about 12 MeV, which is on the low side of a typical hadronic decay (as, for instance, the meson $f_0(980)$). In this case, $\tau_{\ell} \approx 20$ GeV$^{-1}$, corresponding to a ratio $\tau_M/\tau_{\ell} \approx 0.16$, which is still a sizable quantity. We thus conclude that the non-exponential regime for a typical hadronic decays amounts to 15-50% of the mean lifetime.

Case (ii): $\mathcal{L}$ as effective hadronic theory: When the toy model is interpreted as a prototype of an hadronic effective theory, the cutoff is a further parameter entering in the model with a typical value of about $\sim 1$ GeV. (For definiteness we set $\Lambda = 1$ GeV [17]). The results are qualitatively similar to the case $\Lambda \to +\infty$, but the existence of a finite cutoff increases the size of the short-time deviations from the exponential law, as clearly visible in Fig. 1 for $g = 2\sqrt{2}$ GeV (thick gray line). Indeed in this case the second derivative of $p(t)$ is finite at $t = 0$ and the usual quadratic approximation for $p(t)$ at short times could be adopted. In Fig. 2 the ratio $\tau_M/\tau_{\ell}$ is shown as a function of $\tau_{\ell}$ for $\Lambda = 1$ GeV and is quite similar to the previous case. However, while the value of $\tau_M$ is almost independent from the choice of the cutoff, the difference between $p(t)$ and the exponential decay law is instead larger in the case of a finite cutoff.

Bearing in mind all the previously mentioned caveats of hadronic spectral functions, it is anyhow interesting to conclude the present study with a physical example. To this purpose scalar states are not suitable because their masses and decay widths are often affected by large errors, see [18]. We consider instead the $\rho$ meson, whose mass and (by far dominant) decay into two pions are very well measured: $M_\rho = 775 \pm 1$ MeV, $\Gamma_{\rho \to \pi\pi} = 149 \pm 0.5$ MeV [18]. The mass distribution reads:

$$d_\rho(x) = \frac{2x}{\pi} \frac{x_\Gamma_{\rho \to \pi\pi}(x)}{x^2 - M_\rho^2 + x^2 \Gamma_{\rho \to \pi\pi}(x)^2}, \quad (11)$$

where $\Gamma_{\rho \to \pi\pi}(x) = \left(\frac{m_\rho^2 - m_{\pi\pi}^2}{4\langle x\rangle}\right)^{3/2} g_\rho^2$ and $g_\rho = 5.98$. The ratio $\tau_M/\tau_{\ell}$ = 0.16 implies that, also in this concrete case, a sizable interval of non-exponential regime holds. Note, the function $p(t)$ crosses the exponential function $e^{-\Gamma_{\rho \to \pi\pi} t}$, thus indicating the existence also of an anti-Zeno regime [12]. It will be interesting to study to which...
extent our results affect the evolution of the hot and expanding gas of hadrons produced in heavy ions collisions and the spectra of the particles emitted by the plasma. In the presently available transport simulations indeed a simple exponential decay law is assumed for the hadronic resonances whereas during the very short characteristic time scale of the evolution of the plasma sizable deviations from the exponential law are present.

The present work is based on the (resummed) 1-loop approximation. Future studies should go beyond this scheme and include higher order terms, the first one being the ‘sunset’ diagram, in which a particle $S$ is exchanged by the two particles $\varphi$ circulating in the loop. The non-exponential nature of the decay does not depend on the truncation and would take place also when of higher order contributions are included. The numerical influence of the latter is, however, not expected to be large: higher contributions are included. The numerical influence of field redefinitions on the results of the redefinition with unit Jacobian:

$$S \rightarrow \tilde{S} = S - \alpha \varphi^2 , \varphi \rightarrow \tilde{\varphi} = \varphi ,$$

where $\alpha$ is a dimensionful constant. In this way we go from the representation 1 (in terms of the fields $\{S, \varphi\}$, whose Lagrangian $L_{\text{repr}1}(S, \varphi) = \mathcal{L}$ is given by Eq. (11) to the representation 2 (in terms of the fields $\{\tilde{S}, \tilde{\varphi}\}$). In the representation 2 the Lagrangian $L_{\text{repr}2}(\tilde{S}, \tilde{\varphi})$ reads

$$L_{\text{repr}2}(\tilde{S}, \tilde{\varphi}) = L_{\text{repr}1}(S = \tilde{S} + \alpha \varphi^2, \varphi = \tilde{\varphi}) .$$

(13)

Also in term of the Hamiltonians, $H_{\text{repr}1}(S, \varphi)$ in representation 1 and $H_{\text{repr}2}(\tilde{S}, \tilde{\varphi})$ in representation 2 one has

$$H_{\text{repr}2}(\tilde{S}, \tilde{\varphi}) = H_{\text{repr}1}(S = \tilde{S} + \alpha \varphi^2, \varphi = \tilde{\varphi}) .$$

(14)

$H_{\text{repr}1}$ is written as $H_{\text{repr}1} = H_{0,\text{repr}1} + H_{1,\text{repr}1}$ where as usual the ‘non interacting part’ $H_{0,\text{repr}1}$ is given by

$$H_{0,\text{repr}1} = \int d^3x \left[ \frac{1}{2} (\partial_0 S)^2 + \frac{1}{2} (\nabla S)^2 + \frac{M^2}{2} S^2 + \frac{1}{2} (\partial_0 \varphi)^2 + \frac{1}{2} (\nabla \varphi)^2 + m^2 r^2 .
$$

(15)

Similarly, $H_{\text{repr}2}$ is written as $H_{\text{repr}2} = H_{0,\text{repr}2} + H_{1,\text{repr}2}$ where by definition:

$$H_{0,\text{repr}2}(\tilde{S}, \tilde{\varphi}) = H_{0,\text{repr}1}(\tilde{S}, \tilde{\varphi}) .$$

(16)

This implies that $H_{0,\text{repr}1}$ and $H_{0,\text{repr}2}$ have the same functional form. However, it is important to stress that the two operators are different: $H_{0,\text{repr}2}(\tilde{S}, \tilde{\varphi}) \neq H_{0,\text{repr}1}(S, \varphi)$. This can be easily proven by plugging Eqs. (12) into $H_{0,\text{repr}2}(\tilde{S}, \tilde{\varphi})$: one obtains the operator $H_{0,\text{repr}2}(\tilde{S}, \tilde{\varphi}) = H_{0,\text{repr}2}(S - \alpha \varphi^2, \varphi)$ which is indeed -in terms of $S$ and $\varphi$- a complicated Hamiltonian.

In this work we have calculated the survival probability

$$a(t) = \langle S | e^{-i H_{\text{repr}1}(S, \varphi) t} | S \rangle$$

(17)

where $|S\rangle$ is an eigenstate of $H_{0,\text{repr}1}$ with eigenvalue $M_0$ (and with three-momentum $\vec{P} = 0$). This quantity is indeed, in virtue of Eq. (14), invariant under the choice of representation.

However, if one would repeat the calculation of the survival probability in representation 2 using the same mathematical approach leading to Eq. (11), one would calculate the quantity

$$a(t) = \langle \tilde{S} | e^{-i H_{\text{repr}2}(\tilde{S}, \tilde{\varphi}) t} | \tilde{S} \rangle ,$$

(18)

where $|\tilde{S}\rangle$ is the eigenstate with energy $M_0$ (and $\vec{P} = 0$) of the operator $H_{0,\text{repr}2}(\tilde{S}, \tilde{\varphi}) \neq H_{0,\text{repr}1}(S, \varphi)$. It should be stressed that

$$| \tilde{S} \rangle \neq | S \rangle .$$

(19)

Naively, the state $| \tilde{S} \rangle \approx | S \rangle + \alpha | \varphi \varphi \rangle$ (with proper normalizations and also including the proper regularizations) is a superposition of the state $| S \rangle$ with the two-body state $| \varphi \varphi \rangle$ (which includes a sum over internal momenta, which we do not specify here). It is then clear that

I. APPENDIX A

Field redefinitions do not change the physical content of the theory, as e.g. the $S$-matrix elements for asymptotic initial and final states. However, the Green-functions, and in particular the propagator of an unstable particle, are not invariant under field redefinitions, see Ref. [13] and refs. therein. It is then important to study the influence of field redefinitions on the results of the present work. For definiteness we consider the following transformation with unit Jacobian:

$$S \rightarrow \tilde{S} = S - \alpha \varphi^2 , \varphi \rightarrow \tilde{\varphi} = \varphi ,$$

where $\alpha$ is a dimensionful constant. In this way we go from the representation 1 (in terms of the fields $\{S, \varphi\}$,
and its imaginary part are modified. This is also the reason why the mass distributions \( d_S(x) \) and \( d_S(x) \) (which are the imaginary part of the propagators of \( S \) and \( S \) in the first and second representations, respectively) do not coincide. In order to be consistent and to calculate the same quantity in the second representation, one should not start from the initial state \( \hat{S} \), but from the state \( |S\rangle \cong |\hat{S}\rangle - \alpha |\varphi\rangle \).

In this way one would obtain the quantity \( a(t) \) also in representation 2. It is then evident from the present discussion that the representation choice is intimately connected with the definition of the initial state of the system at \( t = 0 \). More about this is discussed in the next Appendix.

Notice that if we perform the field transformation \( \mathcal{L}_{\text{int}} = c \mathcal{B} \mathcal{A} \mathcal{S} \) on the toy Lagrangian \( \mathcal{L}_{\text{int}} = c \mathcal{B} \mathcal{A} \mathcal{S} \) we obtain \( \mathcal{L}_{\text{int}} = c \mathcal{B} \mathcal{A} \mathcal{S} + c B \mathcal{A} \mathcal{S}^2 \). Now, in the evaluation of the three-body decay \( B \rightarrow A \varphi \), there is not only the intermediate state \( S \) because the new interaction \( B \mathcal{A} \mathcal{S}^2 \) has emerged. For this reason the theoretical result for the line shape is not \( \Gamma_{\mathcal{B} \mathcal{A} \mathcal{S}^2}(MB) / d_S(x) \). The new term \( B \mathcal{A} \mathcal{S}^2 \) generates an interference with the amplitude given from the exchange of \( S \), in such a way that the final result -in agreement with the equivalence theorem- coincides with Eq. \( (5) \). Thus, when speaking about the mass distribution we should always be aware that the discussion is valid in a given representation. A change of representation generates a change of the state \( S \) and therefore also the propagator and its imaginary part are modified.

II. APPENDIX B

Since we are dealing with unstable and short living particles one should also consider the mechanism by which these resonances are created. The most complete framework is the scattering \( \varphi \rightarrow S \rightarrow \varphi \). A full treatment implies the consideration of the wave packets with proper initial conditions leading to some non-negligible spatial overlap at -say- the time \( t = 0 \). In the framework of plane waves, the full state of the system can be expressed in terms of the eigenstates of the Hamiltonian \( \mathcal{H}_0 \):

\[
|s(t)\rangle = \sum_k c_k(t) |\varphi_k \varphi_{-k}\rangle + c_S(t) |S\rangle.
\]

The coefficient \( c_S(t) \) is practically zero for \( t < < 0 \) and only for \( t \approx 0 \) it becomes significant. The way in which this happens can be possibly considered as the formation process. If it were possible to tune the starting conditions in such a way that \( c_S(0) = 1 \), we would have \( |s(t = 0)\rangle = |S\rangle \). From this point on, the evolution is obtained by applying the time evolution operator. However, in general the state at \( t = 0 \) is a superposition:

\[
|s(0)\rangle = \sum_k c_k(0) |\varphi_k \varphi_{-k}\rangle + c_S(0) |S\rangle.
\]

Further evolution implies:

\[
e^{-iHt} |s(0)\rangle = \sum_k c_k(0) e^{-iHt} |\varphi_k \varphi_{-k}\rangle + c_S(0) e^{-iHt} |S\rangle = \sum_k c_k(0) e^{-iHt} |\varphi_k \varphi_{-k}\rangle + c_S(0) (a(t) |S\rangle + |\varphi\rangle).
\]

Clearly, the amplitude \( a(t) \) is part of a more general expression. The situation is of course more complicated, because we cannot evaluate properly the quantity \( e^{-iHt} |\varphi_k \varphi_{-k}\rangle \). It is indeed interesting to observe that, if \( e^{-iHt} |\varphi_k \varphi_{-k}\rangle \) does not contain the state \( |S\rangle \) (for instance, if the two wave packets are already far apart at \( t > 0 \), then (up to a phase): \( a(t) = e^{-iKt/2} \), i.e. the exponential regime is realized. As also discussed in Ref. \( [1] \), the rescattering processes, which can occur if the two wave packets are close to each other, are responsible for the non-exponential behavior.

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Formally, one can directly introduce the cutoff in the Lagrangian by rendering it nonlocal and, moreover, one could use smooth cutoff functions [11].

In Eq. (11) the real part is neglected. The spectral function is normalized for $\Lambda_\rho \simeq 0.94$ GeV. The $\rho\pi\pi$ interaction, being of the kind $g_\rho\rho\partial^\mu\pi\cdot\pi$, is only renormalizable. However, the use of finite cutoff guarantees the convergence and the previous discussion holds.

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