The boundedness and zero isolation problems for weighted automata over nonnegative rationals

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ABSTRACT
We consider linear cost-register automata (equivalent to weighted automata) over the semiring of nonnegative rationals, which generalise probabilistic automata. The two problems of boundedness and zero isolation ask whether there is a sequence of words that converge to infinity and to zero, respectively. In the general model both problems are undecidable so we focus on the copyless linear restriction. There, we show that the boundedness problem is decidable.

As for the zero isolation problem we need to further restrict the class. We obtain a model, where zero isolation becomes equivalent to universal coverability of orthant vector addition systems (OVAS), a new model in the VAS family interesting on its own. In standard VAS runs are considered only in the positive orthant, while in OVAS every orthant has its own set of vectors that can be applied in that orthant. Assuming Schanuel’s conjecture is true, we prove decidability of universal coverability for three-dimensional OVAS, which implies decidability of zero isolation in a model with at most three independent registers.

CCS CONCEPTS
• Theory of computation → Formal languages and automata theory: Models of computation.

KEYWORDS
Weighted automata, vector addition systems, boundedness problem, isolation problem.

1 INTRODUCTION
Weighted automata are a natural model of computation that generalise finite automata [17] and linear recursive sequences [3]. They have various equivalent presentations: e.g. finite automata, rational series, matrix representation [5, 34]; or recently linear cost-register automata (linear CRA) [2]. A typical example is a probabilistic automaton $A$ that assigns to each word $w$ its probability of acceptance, denoted $A(w)$ [16, 19, 22, 32]. More generally, weighted automata are defined with respect to a semiring: a domain with two binary operations. The star height problem for regular languages can for instance be reduced to the boundedness problem of such automata. Hashiguchi showed that this problem is decidable [26]. Due to Hashiguchi’s proof being difficult, many alternative proofs of this result appeared, among them: via Simon’s factorisation trees [36]; and via games [8].

Depending on the context, different semirings for weighted automata have been studied. For instance when considering learning, the semirings are usually fields, like the rationals or reals [4, 21]. Most results on learning weighted automata depend on Schützenberger’s polynomial time algorithm deciding the equivalence problem of weighted automata over fields [34]. On the other hand, when considering regular expressions, weighted automata are usually studied over the tropical semiring, i.e. $\mathbb{N} \cup \{+\infty\}$ with the operations: $\min$ and $\max$. The star height problem for regular languages can for instance be reduced to the boundedness problem of such automata. Hashiguchi showed that this problem is decidable [26]. Due to Hashiguchi’s proof being difficult, many alternative proofs of this result appeared, among them: via Simon’s factorisation trees [36]; and via games [8].

This paper is primarily interested in weighted automata over the semiring of nonnegative rationals with $+$ and $\cdot$, denoted $\mathbb{Q}_{\geq 0}, \mathbb{Q}_{\geq 0}(+, \cdot)$. This is the minimal weighted automata model that captures probabilistic automata, but does not impose any restrictions on the model.
We study existential variants of these problems, where only $A$ words), and by a polynomial (in the size of the input word), respectively. The aforementioned decidability results for zero isolation asks whether there exists $c > 0$ such that for all words $w$ it holds $A(w) > c$. More intuitively, the two problems ask for the existence of a sequence of words $w_1, w_2, \ldots$ such that $\lim_{n \to +\infty} A(w_n) = 0$.

Notice that most of the mentioned problems are well-defined for probabilistic automata. Moreover, since probabilistic automata are known to be closed under complement (it is easy to define $B(w) = 1 - A(w)$) the two threshold problems are equivalent and undecidable [32]. In probabilistic automata the zero isolation problem, due to complementation, is equivalent to the value-1 problem: this is also undecidable [22], but decidable for the special class of leaktight probabilistic automata [10]. The boundedness problem is not interesting for probabilistic automata (since the output is always bounded by 1), but a folklore argument shows that it is undecidable for $Q_{\geq 0}(+,-)$ (see Section 3).

Since the above problems are undecidable in general, we are interested in these problems on subclasses of weighted automata. A common restriction is bounding the ambiguity, i.e. the number of accepting runs. The two most interesting classes are finitely-ambiguous and polynomially-ambiguous automata; when the number of accepting runs is bounded by a constant (universal for all words), and by a polynomial (in the size of the input word), respectively. Both classes have nice characterisations, by excluding some simple patterns in the automata [39]. In particular, it is easy to check if an automaton is finitely-ambiguous or polynomially-ambiguous.

Both threshold problems are undecidable for polynomially-ambiguous probabilistic automata [16, 19]. In the finitely-ambiguous case they are decidable [19], and one can infer that they remain decidable in the general setting of finitely-ambiguous weighted automata over $Q_{\geq 0}(+,-)$ [16]. Unlike for probabilistic automata, the two threshold problems are different (the closure under complement is not true in general over $Q_{\geq 0}(+,-)$), and while one of the inequalities is trivial to decide, the other one is known to be decidable [16] only assuming Schanuel’s conjecture [27]. Similarly, for boundedness and zero isolation, even though one could suspect they are equivalent problems, we also see a difference. One can show that for finitely-ambiguous weighted automata over $Q_{\geq 0}(+,-)$ the boundedness problem is trivially decidable; and exploiting [11] we show that zero isolation is decidable subject to Schanuel’s conjecture (see Section 3). The argument in the latter case is more involved. The aforementioned decidability results for zero isolation on leaktight probabilistic automata do not hold over $Q_{\geq 0}(+,-)$.

The decidability border between the finitely-ambiguous and polynomially-ambiguous classes is not surprising. It is often the case that undecidable problems for weighted automata are decidable for the finitely-ambiguous class [20], and remain undecidable even for very restricted variants of polynomially-ambiguous automata, e.g. copyless linear CRA [1]. However, it is not always the case, for example the $\epsilon$-gap threshold problem is decidable for polynomially-ambiguous probabilistic automata [16], and undecidable in general [12]. For zero isolation and boundedness the undecidability reductions do not work for polynomially-ambiguous automata, which is the starting point of our paper.

**Our contributions and techniques.** We study boundedness and zero isolation for copyless linear CRA, introduced in [2], and known to be strictly contained in polynomially-ambiguous weighted automata [1]. We show that boundedness is decidable for copyless linear CRA. Our proof shows that unboundedness can be detected with simple patterns in the style of patterns for finitely-ambiguous and polynomially-ambiguous automata in [39]. Intuitively, an automaton is unbounded if and only if either there is a loop of value larger than 1 or there is a pattern that generates unboundedly many runs of the same value. Like in [39] the patterns are easy to detect even in polynomial time, the difficulty is to prove correctness of the characterisation. Similarly, as in one of the mentioned proofs of Hashiguchi’s theorem [36], we find a way to abstract the set of generated matrices into a finite monoid, that allows us to exploit Simon’s factorisation trees. Otherwise, the proof is rather different from [36], as we need to exploit the particular shapes of the matrices (imposed by the copyless restriction), while the proof in [36] works for the general class of matrices. We conjecture that our pattern characterisation works for the whole class of polynomially-ambiguous automata.

For the zero isolation problem we have to further restrict the class of copyless linear CRA to a class in which the registers do not interact, that we call Independent-CRA. A similar model of CRA with independent registers was already defined in [15]. We start with a chain of reductions to equivalent problems. Firstly, we show that zero-isolation over $Q_{\geq 0}(+,-)$ is essentially equivalent to the boundedness problem over the semiring $Z(\min,+)$, i.e. the same problem as in Hashiguchi’s theorem with the exception that the domain includes negative numbers. This problem is known to be undecidable for the full class of weighted automata [1], but for polynomially-ambiguous, or even copyless linear CRA, decidability was left as an open problem in the same paper. Secondly, we further reduce this problem to a variant of the coverability problem for a new class of orrthant vector addition systems (OVAS).

The OVAS class lies between the standard VAS [13] and its integer relaxation [25]. Intuitively, in the standard VAS runs are considered only in the positive orthant, while in the integer relaxation runs go through the whole space. In OVAS every orthant has its own set of vectors that can be applied in that orthant. The **universal coverability** problem asks whether from any starting point the positive orthant can be reached. We prove that universal coverability is decidable in dimension 3. The proof is nontrivial and relies on a notion of a separator between the reachability set and the positive orthant that can be expressed in the first order logic over the reals. Depending on the encoding of the numbers, we can either rely on Tarski’s theorem [23], or the formula might require the exponential function. In the latter case decidability depends on Schanuel’s conjecture [27]. Since most of the proof works in any dimension, we believe that this is an important step to prove the theorem for
arbitrary dimensions. Interestingly, the proof relies on results about reachability for continuous VAS [7]. From universal coverability we infer decidability of zero isolation for copyless linear CRA with 3 independent registers. More importantly, we establish a nontrivial connection between: zero isolation over \(Q_{\geq 0}(+, \cdot)\); boundedness over \(Z(\min, +)\); and our new model OVAS. We are convinced that the latter model is of independent interest. Interestingly, we show that the usual coverability problem (with a fixed initial point) in undecidable.

We leave as an open problem decidability of zero isolation for polynomially ambiguous weighted automata over \(Q_{\geq 0}(+, \cdot)\). Nevertheless we show that the problem is undecidable for copyless CRA (nonlinear). The latter class is known to be: strictly between the polynomially-ambiguous class [30, 31]. The boundedness and zero isolation problems for weighted automata over nonnegative rationals

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2.1 Weighted automata

A weighted automaton \(\mathcal{A}\) over a semiring \(S(\oplus, \odot)\) is a tuple \(\mathcal{A} = (\Sigma, I, F, (M_a)_{a \in \Sigma})\), where: \(\Sigma\) is a finite alphabet; \(I, F \subseteq S^d\) are \(d\)-dimensional vectors; and \(M_a \in S^{d \times d}\) are \(d\)-dimensional square matrices for some fixed \(d \in \mathbb{N}\). For every word \(w = w_1 \cdots w_n \in \Sigma^*\) we define the matrix \(M_w = M_{w_1} M_{w_2} \cdots M_{w_n}\), where the matrices are multiplied with respect to the sum and the product of \(S\). If \(w\) is the empty word then \(M\) is the identity matrix. For every word \(w \in \Sigma\) the automaton outputs \(A(w) = I^T M_w F \in S\). Thus \(\mathcal{A}\) can be seen as a function \(\Sigma^* \rightarrow S\). Whilst formally \(\mathcal{A}\) does not have states, one can think that coordinates in \(I, F, M\), and the matrices \((M_a)_{a \in \Sigma}\) are indexed by states rather than natural numbers. In which case, we write \(q \mapsto q'\) if \(M_w[q, q'] = r\) (regardless of whether \(w\) is a word or character). We also say that \(l[q]\) and \(F[q]\) are the initial and the final value of \(q\) for every state \(q\).

A run \(\rho\) over a word \(w = w_1 \cdots w_n\) in \(\mathcal{A}\) is a sequence of states interleaved with values: \(q_0, v_1, q_1, \ldots, v_n, q_n\) such that \(v_i = q_i + 1\) \(q_i\) for \(w_i \in \{1, \ldots, n\}\). We then associate the value of the run \(\text{val}(\rho) = l[q_0] \oplus v_1 \oplus \cdots \oplus v_n \odot F[q_n]\). We say that \(\rho\) is an accepting run if \(\text{val}(\rho) \neq 0\). Equivalently all elements in the product \(l[q_0], v_1, \ldots, v_n, F[q_n]\) are different from \(0\) (for the semirings in this paper). We denote the set of all accepting runs of \(\mathcal{A}\) over \(w\) by \(\text{Acc}(\mathcal{A}, w)\). Then \(\mathcal{A}(w) = \bigoplus_{\rho \in \text{Acc}(\mathcal{A}, w)} \text{val}(\rho)\). The equivalence with the matrix definition is clear for all commutative semirings since runs that are not accepting contribute \(0\) to the sum.

Consider a weighted automaton \(\mathcal{A}\). We write that \(\mathcal{A}\) is:

- unambiguous if \(|\text{Acc}(\mathcal{A}, w)| < 1\) for all \(w \in \Sigma^*\);
- polynomially-ambiguous if there exists \(k \in \mathbb{N}\) such that \(|\text{Acc}(\mathcal{A}, w)| < k\) for all \(w \in \Sigma^*\);
- polynomially-ambiguous if there exists a polynomial function \(p\) such that \(|\text{Acc}(\mathcal{A}, w)| < p(|w|)\) for all \(w \in \Sigma^*\). If \(p\) is linear we also say that \(\mathcal{A}\) is linearly-ambiguous.

Below we show two examples of weighted automata over the semiring \(Q_{\geq 0}(+, \cdot)\).

**Example 2.1.** Consider \(\mathcal{A} = ([a], I, F, M_a)\), where \(I = (1, 0), F = (0, 1),\) and \(M_a = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\). Then \(\mathcal{A}(a^n) = n \mod 2\). The automaton \(\mathcal{A}\) is unambiguous (see Figure 1).

**Example 2.2.** Consider \(\mathcal{B} = ([a], I, F, M'_a)\), the same as \(\mathcal{A}\) except that \(M'_a = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\). Then \(\mathcal{B}(a^n) = n\). The automaton \(\mathcal{B}\) is linearly-ambiguous (see Figure 1). Moreover, it can be shown that the function defined by \(\mathcal{B}\) cannot be defined by a linearly-ambiguous automaton (see e.g. [3, Lemma 12]).

2.2 Cost-register automata and its restrictions

For a semiring \(S\) and a set of registers \(X\) we write affine(\(S, X\)) for the set of affine expressions, i.e., expressions of the form \(c \oplus \bigoplus_{x \in \Sigma} f_x \cdot x\), where \(c, f_x \in S\) and \(x \in X\). A different presentation of weighted automata are linear cost-register automata (linear CRA).

A linear CRA \(\mathcal{B}\) is defined as a tuple \((\Sigma, Q, q_0, I, F, A, \delta)\), where: \(\Sigma\) is a finite alphabet; \(Q\) is a finite set of states, with \(q_0 \in Q\) the designated initial state; \(X\) is a finite set of registers; \(I : X \rightarrow S\) and \(F : Q \times X \rightarrow S\) are, respectively, the initial values and final
coefficients of registers; and \( \delta : Q \times \Sigma \rightarrow Q \times (X \rightarrow \text{affine}(\mathbb{S}, X)) \)

is a deterministic transition function.\(^3\)

A configuration of a CRA is a pair \((q, \sigma)\), consisting of a state and a valuation of the registers \(\sigma \in \mathbb{S}^X\). The initial configuration is \((q_0, I)\). For every letter \(a \in \Sigma\) we write \((q, \sigma) \xrightarrow{a} (q', \sigma')\) if \(\delta(q, a) = (q', \sigma_q(a))\) and \(\sigma'(x) = \sigma_q(x)(\sigma)\), that is \(\sigma_q(x)\), where every register is substituted with its previous valuation \(\sigma\). Since \(\mathcal{B}\) is deterministic for every input word \(w = w_1 \cdots w_n\) there is a unique run, defined as a sequence of configurations:

\[(q_0, \sigma_0), \ldots, (q_n, \sigma_n)\]

where \((q_0, \sigma_0)\) is the initial configuration and \((q_{i-1}, \sigma_{i-1}) \xrightarrow{w_i} (q_i, \sigma_i)\) for every \(i \in \{1, \ldots, n\}\). In such a case, we write \((q_0, \sigma_0) \xrightarrow{w} (q_n, \sigma_n)\). Finally, if the configuration after reading \(w\) is \((q, \sigma_w)\) then the output of \(\mathcal{B}(w)\) is \(\bigoplus_{x \in X} F[q, x] \circ \sigma_w(x)\). Thus \(\mathcal{B}\) is a function \(\Sigma^* \rightarrow \mathbb{S}\).

Our linear CRA are defined with states \(Q\), as per their first introduction \([2]\). However, in the general case it is not hard to see that the stateless (or single state) model is equivalent. Indeed, it suffices to encode states into registers and consider \(Q \times X\) as registers. Note that this construction does not have to hold for restricted CRAs. Thus a stateless linear CRA is defined as a tuple \(\mathcal{B} = (\Sigma, I, F, X, \delta)\). All the notations are the same as for CRAs, but we will omit states in the stateless case, e.g. a configuration of \(\mathcal{B}\) is a valuation of the registers \(\sigma : X \rightarrow \mathbb{S}\).

We say that two automata are equivalent if they define the same function \(\Sigma^* \rightarrow \mathbb{S}\). Note that for every weighted automaton \(\mathcal{A}\) there exists an equivalent (stateless) linear cost-register automaton \(\mathcal{B}\), and conversely too. Indeed, it suffices to identify the dimension \(d\) in weighted automata with the size \(|X|\). Then \(\sigma(x)\) in \(\delta\) can be seen as rows of matrices in \(\mathbb{S}^{|X| \times |X|}\). Formally, this is proved, e.g. in \([2, \text{Theorem 9}]\). In the remainder of the paper we will work with linear CRA and its subclasses.

We are also interested in automata where the linear update functions are copyleft. A valuation \(\sigma : X \rightarrow \text{affine}(\mathbb{S}, X)\) is copyleft if every register \(x \in X\) occurs at most once across all affine expressions. Formally, using the notation \(\sigma(x) = \bigoplus_{z \in X} (s_x z \otimes z) \circ c_x z\),

for every \(z \in X\) at most one \(s_{x,z}\) is different from \(0\). A CRA \(\mathcal{B}\) is copyleft if \(\sigma_q(a)\) is copyleft for every transition \(\delta(q, a) = (q', \sigma_q(a))\).

Example 2.3. In Figure 1 we show that for both \(\mathcal{A}(a^n) = n \mod 2\) in Example 2.1 and \(\mathcal{B}(a^n) = n\) in Example 2.2 there are equivalent stateless copyleft linear CRA.

In general it is known that the classes are related as follows: copyleft linear CRA are contained in copyleft CRA, and the latter is contained in linear CRA \([29]\). Moreover, copyleft linear CRA are contained in the class of linearly-ambiguous weighted automata \([1, \text{Remark 4}]\). A detailed presentation showing how CRA and weighted automata compare in terms of expressiveness is in Figure 2.

2.3 Independent-CRA, a more restricted CRA

We say that \(\mathcal{B} = (\Sigma, I, F, X, \delta)\) is an independent-CRA if \(\mathcal{B}\) is a stateless linear CRA such that \(\sigma(a)(x) = (c_{a,x} \otimes x) + d_{a,x}\), where \(c_{a,x}, d_{a,x} \in \mathbb{S}\) for every \(\delta(a) = c_a\). In other words the new value of every register does not depend on other registers. Observe that Independent-CRA are a subclass of stateless copyleft linear CRA. A similar model was studied in \([15]\).

Example 2.4. The right automaton in Figure 1 is an example Independent-CRA. It is not hard to show that there is no Independent-CRA that is equivalent to the automaton \(\mathcal{A}(a^n) = n \mod 2\). Indeed, it follows immediately from the definition that if \(C\) is an Independent-CRA and \(C(\epsilon) = C(a^2) = 0\) then \(C(a^n) = 0\) for all \(n \in \mathbb{N}\).

2.4 Decision problems

We define decision problems with respect to semirings. The problems are well-defined for functions \(\Sigma^* \rightarrow \mathbb{S}\), in particular for weighted automata, linear CRA and Independent-CRA. The decision problems are well-defined with respect to all considered semirings, but we will mostly focus on the semiring \(\mathbb{S}_{\geq 0}(\epsilon, \cdot)\).

The \(\leq\)-threshold problem: given an automaton \(\mathcal{A}\) and a number \(c\) (from the domain of the semiring) is it the case that \(\mathcal{A}(w) \leq c\) for all \(w \in \Sigma^*\). The \(\geq\)-threshold problem is defined similarly, where \(\mathcal{A}(w) \geq c\) is replaced with \(\mathcal{A}(w) \geq c\).

Footnote: \(\text{Linear CRA were originally defined with linear updates (rather than affine). Affine updates can be simulated by linear updates by introducing one extra register with value fixed to 1. We use affine updates because the register constraints we introduce later do not apply to this special register.}\)
The \textbf{boundedness} problem: given an automaton \( \mathcal{A} \) does there exist a finite number \( c \) such that \( \mathcal{A}(w) \leq c \) for all \( w \in \Sigma^* \). A sequence \( w_1, w_2, \ldots \) of words with \( \lim_{i \to \infty} \mathcal{A}(w_i) = +\infty \) is a witness of unboundedness.

The \textbf{zero isolation} problem: given an automaton \( \mathcal{A} \) is it the case that there exists a positive rational number \( c > 0 \) such that \( \mathcal{A}(w) \geq c \) for all \( w \in \Sigma^* \). A sequence \( w_1, w_2, \ldots \) of words with \( \lim_{i \to \infty} \mathcal{A}(w_i) = 0 \) is a witness of nonisolated zero.

3 DETAILED STATE OF THE ART AND OUR RESULTS

We already remarked that for probabilistic automata both the \textless{}- and \textgreater{}-threshold problems are well-known to be undecidable [32], even when the model is restricted to linearly ambiguous [16, Theorem 2]. The zero isolation problem is also undecidable for probabilistic automata [22]. Hence, these three problems are also undecidable for weighted automata over \( \mathbb{Q} \).

We remarked that the boundedness problem is not interesting for probabilistic automata, since all words have value bounded by 1. For weighted automata over \( \mathbb{Q}(+,-) \) the problem is undecidable; we are not aware whether this fact is stated in the literature. Nevertheless, it can be proven within this paragraph (a similar argument appears e.g. in the proof of [6, Theorem 1]). Consider the undecidable \textless{}-threshold problem for probabilistic automata: given a probabilistic automaton \( \mathcal{A} \) is it the case that \( \mathcal{A}(w) \leq \frac{1}{2} \) for all \( w \in \Sigma^* \).

One can easily define \( \mathcal{B} \) (which is no longer probabilistic, but over \( \mathbb{Q}(+,-) \)) such that \( \mathcal{B}(w_1 \# \ldots \# w_n) = 2\mathcal{A}(w_1) \ldots 2\mathcal{A}(w_n) \), where \# is some fresh symbol, which intuitively restarts the automaton. Then \( \mathcal{B} \) is bounded if and only if \( \mathcal{A}(w) \leq \frac{1}{2} \) for all words \( w \).

\textbf{Corollary 3.1.} The \textless{}-threshold, \textgreater{}-threshold, \textbf{zero isolation}, and \textbf{boundedness problems} are undecidable for weighted automata over the semiring \( \mathbb{Q}(+,-) \). The first two problems are undecidable even for linearly ambiguous models.

On the positive side, when ambiguity is restricted to be finitely ambiguous we can infer some decidability results for the \textless{}-threshold and \textgreater{}-threshold problems from [16].

\textbf{Proposition 3.2.} For \textbf{finitely ambiguous} weighted automata over \( \mathbb{Q}(+,-) \) the \textless{}-threshold problem and the boundedness problem are \textbf{decidable}, and the \textgreater{}-threshold problem and the \textbf{zero isolation problem} are \textbf{decidable} assuming Schanuel’s conjecture is true.

In this paper we are mostly interested in the boundedness and zero isolation problems over \( \mathbb{Q}(+,-) \) for copyless linear CRA and Independent-CRA. Below we state our main results.

\textbf{Theorem 3.3.} \textbf{Boundedness} for \textbf{copyless linear CRA over} \( \mathbb{Q}(+,-) \) is \textbf{decidable in polynomial time}.

\textbf{Theorem 3.4.} \textbf{Zero isolation} for \textbf{Independent-CRA in dimension 3 over} \( \mathbb{Q}(+,-) \) is \textbf{decidable}, subject to Schanuel’s conjecture. For \textbf{copyless CRA zero isolation is undecidable}.

As mentioned in the introduction, the main contribution of the results are: the techniques in the decidability result that we believe might generalise to arbitrary dimension; and the nontrivial connections with other problems. To prove Theorem 3.4 we will show that the zero isolation problem is essentially equivalent to the boundedness problem over \( \text{LogQ}(\min,+) \). Thus the positive part of Theorem 3.4 will be a corollary of the following.

\textbf{Theorem 3.5.} \textbf{Zero isolation for Independent-CRA in dimension 3 over} \( \text{LogQ}(\min,+) \) \textbf{is decidable}, subject to Schanuel’s conjecture. For \textbf{Independent-CRA in dimension 3 over} \( \mathbb{Z}(\min,+) \) \textbf{the boundedness problem is decidable in \textit{ExpTime} (independent of Schanuel’s conjecture)}.

\textbf{Proof of Proposition 3.2.} \textbf{Threshold problems:} Consider the following containment problem: given two probabilistic automata \( \mathcal{A} \) and \( \mathcal{B} \) is it the case that \( \mathcal{A}(w) \leq \mathcal{B}(w) \) for all words \( w \). When \( \mathcal{A} \) is finitely ambiguous and \( \mathcal{B} \) is unambiguous then the problem is decidable [16, Proposition 16]. When \( \mathcal{A} \) is unambiguous and \( \mathcal{B} \) is finitely ambiguous then the problem is decidable, assuming Schanuel’s conjecture is true [16, Theorem 17]. Consider an input for one of the threshold problems: a \textbf{finitely weighted automaton} \( \mathcal{A} = (\Sigma, I, F, (M_a)_{a \in \Sigma}) \) over \( \mathbb{Q}(+,-) \) and \( c \in \mathbb{Q} \). Let \( N \) be the sum of all constants that appear in \( I, F, \) and \( M_a \) for all \( a \in \Sigma \) and let \( C = \max(c, N) \). We define the automaton \( \mathcal{A}/C = (\Sigma, I', F', (M'_a)_{a \in \Sigma}) \), where \( I'(q) = I(q)/C, F(q) = F(q)/C, \) and \( M'_a(p, q) = M_a(p, q)/C \). It is easy to see that \( \mathcal{A}/C \) is a probabilistic automaton and that \( \mathcal{A}(C(w)) = \mathcal{A}(w)/C^{\mid w \mid^2} \) for all \( w \in \Sigma^* \). It remains to observe that it is easy to define an unambiguous probabilistic automaton \( \mathcal{B} \) such that \( \mathcal{B}(w) = c/C^{\mid w \mid^2} \) for all \( w \in \Sigma^* \). Thus the threshold problems can be reduced to the containment problems between \( \mathcal{A} \) and \( \mathcal{B} \). We conclude by the mentioned results from [16].

\textbf{Boundedness:} Since there are \textbf{finitely many runs}, check that at least one run is unbounded, which occurs if and only if some accessible cycle has weight greater than one.

\textbf{Zero isolation:} We reduce to the Big-O problem, which asks whether there exists \( C > 0 \) such that for all \( w \in \Sigma^* \) \( \mathcal{A}(w) \leq C \cdot \mathcal{B}(w) \). The problem is decidable for \textbf{finitely-ambiguous} \( \mathcal{A} \) assuming Schanuel’s conjecture is true [11, Theorem 2]. Let \( \mathcal{A}(w) = 1 \) for all \( w \in \Sigma^* \). Then there exists \( C > 0 \) such that \( \mathcal{B}(w) \geq \frac{1}{C} \) for all \( w \in \Sigma^* \). \( \square \)

\textbf{Organisation.} In the following sections we will prove Theorems 3.3, 3.4 and 3.5. Section 4 proves decidability of the boundedness problem for copyless linear CRA over \( \mathbb{Q}(+,-) \) (Theorem 3.3). Section 5 shows the chain of reductions from zero isolation for weighted automata over \( \mathbb{Q}(+,-) \), through boundedness for weighted automata over \( \text{LogQ}(\min,+) \), up to universal coverability in OVAS. Finally, Section 6 shows that universal coverability is decidable in dimension three, proving Theorem 3.4 and Theorem 3.5. Figure 2 presents the results also explaining how Independent-CRA and copyless linear CRA relate to other classes of weighted automata in terms of expressiveness. Omitted proofs (including the negative part of Theorem 3.4) can be found in the full version [14].

4 \textbf{BOUNDEDNESS FOR COPYLESS LINEAR CRA OVER} \( \mathbb{Q}(+,-) \)

The goal of this section is to establish that the boundedness problem for copyless linear CRA over \( \mathbb{Q}(+,-) \) is decidable in polynomial time, that is, Theorem 3.3.
Our first step is to translate copyless linear CRA into WA with certain properties. More precisely, the WA will be nearly deterministic, except for a single state introducing ambiguity. The resulting automaton can build in polynomial time a simple linearly-ambiguous WA $\mathcal{B}$ over $\mathbb{Q}_{>0}(+,\cdot)$ such that $\mathcal{A}$ is bounded if and only if $\mathcal{B}$ is bounded.

Relying on Lemma 4.2 we may focus on simple linearly-ambiguous WA. We prove that unboundedness of such an automaton is characterised by certain patterns occurring in it. Lemma 4.3 shows what happens when such patterns are not present, and it is the key technical contribution in the proof. Then to prove Theorem 3.3 we only need to detect patterns violating the assumptions of Lemma 4.3.

We define specific sets of runs based on whether they exceed a given threshold: given $r \in \mathbb{Q}_{>0}$ we set

$$\text{Runs}_{>r}(w) = \{ \rho \mid \rho \text{ is a run over } w, \text{val}_{\omega}(\rho) > r \}.$$

**Lemma 4.3.** Let $\mathcal{B} = (\Sigma, I, F, (M_\omega)_{\omega \in \Sigma})$ be a simple linearly-ambiguous WA with distinguished state $p$. Assume that for every word $u$ and every $q$-cycle $\rho$ over $u$, where $q \neq p$, both conditions hold:

(i) $\text{val}_{\omega}(p) < 1$;

(ii) if $\text{val}_{\omega}(p) = 1$, then $M_u[p,q] = 0$.

Then $|\text{Runs}_{>1/k}(w)| \leq \text{poly log}(k)$ for every $k \geq 2$ and every word $w$.

The constants implied by the poly log in Lemma 4.3 depend on the rational numbers occurring in the transitions of $\mathcal{B}$. However, it is crucial that the bound on $|\text{Runs}_{>1/k}(w)|$ does not depend on $w$. To get some intuition we show how the lemma concludes the proof of Theorem 3.3.

**Sketch of Theorem 3.3.** If the automaton violates the assumptions of Lemma 4.3, one can construct a witness for unboundedness. Conversely, divide all runs into $P_i = \{ \rho \mid \frac{1}{c_i} < \text{val}_{\omega}(\rho) \leq \frac{1}{c_i} \}$ for $i \in \{1, \ldots, n\}$ and some constant $c$. By Lemma 4.3 we have...
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$|P_i| \leq \text{poly}(\log \epsilon^i) = \text{poly}(i \log c) = \text{poly}(i)$. We obtain

$$B(w) \leq \sum_{\rho \in \text{Runs}_{\Sigma^\omega}(w)} \text{val}(\rho) \leq \sum_{i=1}^n \frac{|P_i|}{e^{i-1}} \leq \sum_{i=1}^\infty \frac{\text{poly}(i)}{e^{i-1}}.$$  

Notice that the series converges, independent of $w$. $\square$

Before establishing Lemma 4.3 we need to introduce some notation and intermediary results. Roughly, our goal is to obtain a finite representation of the set of matrices $M_w$. This will allow us to invoke Simon’s Factorisation Forest Theorem that gives a tree representation on runs on $w$, such that nodes (corresponding to subwords of $w$) have height independent on $w$. Then, intuitively, the degree of poly log in Lemma 4.3 corresponds to the height of the node.

Let $B$ be as in Lemma 4.3. As usual we will identify the dimensions of the vectors with the set of states $Q = \{p, q_1, \ldots, q_n\}$, where $p$ is the distinguished state. Recall that: $B$ is deterministic when restricted to $Q \setminus \{p\}$; $M_w[p, p] = 1$; and $M_w[q, p] = 0$ for every $q \in Q \setminus \{p\}$ and every word $w \in \Sigma^*$. Thus, for every $q \neq p$ and every $w \in \Sigma^*$, there exists at most one $q'$ such that $M_w[q, q'] > 0$. We further observe that for every pair of states $q, q' \in Q \setminus \{p\}$ and every word $w \in \Sigma^*$ there is at most one run $\rho$ over $w$ starting in $q$ and ending in $q'$ such that $\text{val}_{\alpha}(\rho) > 0$. If there is such a run then we will denote $\rho = (q, w, q')$ and $\text{val}_{\alpha}(\rho) = M_w[q, q']$. We say that $\rho \in \mathcal{Q} > 0$ is an admissible weight if there exists a word $w$ and a run $\rho$ over $w$ such that $\text{val}_{\alpha}(\rho) > 0$. For every $q \in Q \setminus \{p\}$ let

$$s_q = \min_{w \in \Sigma^*} \left\{ \frac{1}{\text{val}_{\alpha}(\rho)} \mid \rho \in \text{Runs}_{\Sigma^\omega}(w), \rho \text{ starts in } q \right\};$$

$$e_q = \min_{w \in \Sigma^*} \left\{ \frac{1}{\text{val}_{\alpha}(\rho)} \mid \rho \in \text{Runs}_{\Sigma^\omega}(w), \rho \text{ ends in } q \right\}.$$

Claim 4.4. $0 < s_q, e_q \leq 1$ and both are computable rationals.

Claim 4.5. Let $x > 0$. There are finitely many admissible weights larger than $x$.

Recall that $M_w \in (\mathcal{Q} > 0)^{Q \times Q}$ for every $w \in \Sigma^*$. Let $\epsilon \in \mathcal{Q} > 0$ be a fresh symbol. We define the abstraction $\overline{M} \in (\mathcal{Q} > 0 \cup \{\epsilon\})^{Q \times Q}$ as follows

$$\overline{M}[q, q'] = \begin{cases} 0 & \text{if } M[q, q'] = 0 \\ M[q, q'] & \text{if } q \neq p \text{ and } M[q, q'] > e_q \cdot s_q \\ \epsilon & \text{otherwise} \end{cases}$$

for all $q, q' \in Q$, and $e_q, s_q$ as defined in Equation (1). In words, $\overline{M}$ is the same as $M$, but some positive entries are replaced with $\epsilon$. Notice that by Claim 4.5 this set of matrices is finite, as intended. The special symbol $\epsilon$ appears within $\overline{M}$ in two cases: it replaces $M_w[q, q']$ if $q \neq p$ and this value is small enough; and it is used to indicate whether there are any positive runs from $p$ to $q'$ (their exact values are not important). In particular, all non-zero weights of transitions from $p$ are set to $\epsilon$. An example translation is in Figure 3. The claim below states the property of $\epsilon$ formally.

Claim 4.6. For every $w \in \Sigma^*$ and $q, q' \in Q \setminus \{p\}$:

1. if $\overline{M}_w[q, q'] \neq 0$, then $\overline{M}_w[q, q'] = \epsilon$ if and only if, for every run $\rho$ such that $(q, w, q')$ is its subrun, $\text{val}_{\alpha}(\rho) < 1$.
2. $\overline{M}_w[p, q'] = \epsilon$ if and only if $M_w[p, q'] > 0$.

We define the sum, product and order of $\epsilon$ with rationals. One can think that $\epsilon$ represents a number above zero but ‘smaller’ than the positive rationals. The only operation where this intuition breaks is addition, where $\epsilon$ is an absorbing element. This will be explained later. Formally: $0 < \epsilon < \epsilon + \rho = 0 = \epsilon \cdot 0 = \epsilon$, $\epsilon + \epsilon = \epsilon$, $\epsilon \cdot \epsilon = \epsilon$, $\epsilon + x = x + \epsilon = \epsilon + x = x + \epsilon + \epsilon = \epsilon$ for every $x \in \mathcal{Q} > 0$ and $\epsilon \in \mathcal{Q} > 0$.

We define the product of abstracted matrices. For every $M, N \in (\mathcal{Q} > 0 \cup \{\epsilon\})^{Q \times Q}$ let $MN$ be the usual product of matrices, i.e. $MN[q, q'] = \sum_{q'' \in Q} M[q, q''] \cdot N[q'', q']$. Then we define

$$M \otimes N[q, q'] = \begin{cases} \epsilon & \text{if } 0 < MN[q, q'] < e_q \cdot s_q \\ MN[q, q'] & \text{otherwise} \end{cases}.$$  

For matrices $\overline{M}_w, \overline{M}_u$, the states in $Q \setminus \{p\}$ are deterministic, thus to define $\overline{M}_w \otimes \overline{M}_u$, for $q \neq p$ we will need to sum elements at most one of which is nonzero. In case of $q = p$, we sum several positive elements. However, in this case we will only be interested in whether the transition is positive or zero; this explains our definition of addition with $\epsilon$.

Claim 4.7. The set $\{\overline{M}_w \mid w \in \Sigma^*\}$ is finite and $\overline{M}_{\text{run}(w)} = \overline{M}_w \otimes \overline{M}_u$ for every $w, u \in \Sigma^*$. Thus $\{\overline{M}_w \mid w \in \Sigma^*\}$ is a finite monoid with the product $\otimes$.

We let $M = \{\overline{M}_w \mid w \in \Sigma^*\}$ denote the finite monoid of Claim 4.7. An element $M \in M$ is idempotent if $M \otimes M = M$.

Consider a sequence of elements $e_1, e_2, \ldots, e_n$ from $M$. A factorisation of these elements is a labelled tree whose set of nodes is a subset of $\{(i, j) \mid 1 \leq i \leq j \leq n\}$. Intuitively, a node $(i, j)$ corresponds to an infix $e_i, \ldots, e_j$. Formally, the leaves are $(1, 1), \ldots, (n, n)$; the root is $(1, n)$; and for every $(i, j)$ its children are $(i_0 + 1, i_1), (i_1 + 1, i_2), \ldots, (i_{j-1} + 1, i_j)$, where $i_0 = 1 = i_1 < i_2 \leq \ldots < i_j = j$. The index $i_0 = 1$ is chosen so that even the first pair $(i_0 + 1, i_1) = (i, i_1)$ can be expressed as $(i_{j-1} + 1, i_1)$. Every node $(i, j)$ is labelled with $e_i \otimes e_{i+1} \otimes \ldots \otimes e_j$. Each node $e_i \otimes e_{i+1} \otimes \ldots \otimes e_j \in \mathcal{P}$. Notice that the label of every parent is equal to the product of the labels of the children in its right order. We say that a node is idempotent if its label is idempotent. We will use the following result from [35].

Lemma 4.8 (Simon’s Factorisation Forest Theorem). Consider a sequence of elements $e_1, e_2, \ldots, e_n$ from a finite monoid $S$. There exists a factorisation into a tree of height at most $9|S|$ such that every inner node has either two children, or all its children are idempotents with the same label.

We can now establish Lemma 4.3. Fix $k \geq 2$, $w = w_1 \ldots w_n \in \Sigma^*$ and let $s_q, e_q$ be defined as in Equation (1) for all $q \in Q \setminus \{p\}$. Given $1 \leq s \leq t \leq n$ we will denote the infix $w_{s:t} = w_s \ldots w_t$. Let $a = \max\left\{ \frac{1}{e_q \cdot s_q} \mid q, q' \in Q \setminus \{p\} \right\}$. Notice that $a \geq 1$ and that all admissible weights are bounded by $a$. Let $b$ be some rational number such that $b < 1$; and for all $w \in \Sigma^*$ and $q, q' \in Q \setminus \{p\}$ if $\overline{M}_w[q, q'] = \epsilon$ then $\overline{M}_w[q, q'] < b$. Notice that by Claim 4.5 $b$ is well-defined,
We conclude since \( i \) hence means that val
node. Let \((a, \leq)\) a leaf and \(w\) that the number of runs starting in \(Q\) exists \(\rho\) be words such that \(\{w_1, \ldots, w_m\}\) the sequence of states where the \(\rho_1, \ldots, \rho_m\) end, respectively, then there exists \(i \in \{1, \ldots, m\}\) such that \(q_1 = q_2 = \ldots = q_{i-1} = p\) and \(q_{i+1} = q_{i+2} = \ldots = q_m\). Moreover, either \(i = m\) or \(M[q_i+1, q_{i+1}] = \epsilon\).

Claim 4.10. Let \((s, t)\) be a node in the factorisation of height \(i\), \(0 \leq i < H\). Then
\[
\left|\text{Runs}_{\leq \epsilon}^{q_i} (w_s, t)\right| \leq \left(\theta(1 + \log \frac{1}{\epsilon} k)^{i+1} + |Q|\right).
\]

Intuitively, either a node has not many children, then the number of runs cannot increase by a lot; or if there are many children then most runs will have a small value.

Proof of claim: For simplicity we will write \(\log\) for \(\log \frac{1}{\epsilon}\). Since the automaton restricted to \(Q \setminus \{p\}\) is deterministic, it suffices to prove that the number of runs starting in \(p\) is bounded by \((\theta(1 + \log k)^{i+1} + 1\).

We proceed by induction on \(i\). In the base case, when \(i = 0, (s, t)\) is a leaf and \(w_s, t\) is a letter. Then there are at most \(|Q|\) runs from \(p\).

We conclude since \((\theta(1 + \log k)^{i} \geq |Q|\) for \(k \geq 2\) by the choice of \(\theta\).

For the induction step assume that the claim holds for all \(0, \ldots, i\) and we prove it true for \(i + 1\). Since \(i + 1 > 0\) \((s, t)\) is an inner node. Let \((s_0 + 1, s_1), (s_1 + 1, s_2), \ldots, (s_{m-1} + 1, s_m)\) be the children of \((s, t)\) such that \(s - 1 = s_0 < s_1 < s_2 < \ldots < s_m = t\) and the height of every child is at most \(i\). Consider a run \(\rho \in \text{Runs}_{\leq \epsilon}^{q_i} (w_s, t)\) starting in \(p\). Then \(\rho\) can be decomposed into \(m\) runs \(\rho_1, \ldots, \rho_m\) over \(w_{s_0+1, s_1}, \ldots, w_{s_{m-1}+1, s_m}\), respectively. Notice that \(\text{val}_\omega(\rho) = \prod_{j=1}^m \text{val}_\omega(\rho_j)\). As \(\text{val}_\omega(\rho) > \frac{1}{k} a^{i+1-H}\) this means that \(\rho_x \in \text{Runs}_{\leq \epsilon}^{q_i} (w_{s_0+1, s_1})\) for all \(x \in \{1, \ldots, m\}\). Indeed, by the choice of \(a\) we know that
\[
a \geq \prod_{j=1}^{x-1} \text{val}_\omega(\rho_j) \cdot \prod_{j=x}^m \text{val}_\omega(\rho_j),
\]

hence
\[
\text{val}_\omega(\rho_x) a \geq \text{val}_\omega(\rho) > \frac{1}{k} a^{i+1-H}.
\]

We denote by \(q_j\) the ending state of \(\rho_j\) for \(j \in \{1, \ldots, m\}\) (which is also the starting state of \(\rho_{j+1}\) for \(j < m\)). We consider two cases depending on the number of children \(m\).

First, suppose there are two children, i.e. \(m = 2\). Let us count the number of possible \(\rho\), depending on whether \(q_1 = p\) or \(q_1 \neq p\). In the first case since there is exactly one run from \(p\) to \(\rho\), the runs \(\rho\) differ only on \(p_2\) and thus the number of such runs is bounded by \(|\text{Runs}_{\leq \epsilon}^{q_1} (w_{s_1+1, s_2})|\). In the second case since the transitions from \(Q \setminus \{p\}\) are deterministic the number of runs is bounded by \(|\text{Runs}_{\leq \epsilon}^{q_1} (w_{s_1+1, s_2})|\). By the induction assumption altogether this is bounded \((\theta(1 + \log k))^{i+1} \leq (\theta(1 + \log k))^{i+2}\) by the choice of \(\theta\).

Second, by Lemma 4.8 suppose that all children are idempotents with the same label, denote it \(M\). By Claim 4.9, there is an index \(x \in \{1, \ldots, m\}\) such that \(q_1 = q_2 = \ldots = q_x = p\) and \(\rho_{x+1} = q_{x+2} = \ldots = q_m\) and if \(x < m\) then \(M(q_{x+1}, q_{x+1}) = \epsilon\).

By definition of \(b\) we get that \(\text{val}_\omega(\rho_y) \leq b\) for \(x < y < m\). Thus \(\text{val}_\omega(\rho) \leq a \cdot b^{m-x}\) and since \(\rho \in \text{Runs}_{\leq \epsilon}^{q_i} (w_s, t)\) we get
\[
a \cdot b^{m-x} \geq a^{i+1-H},
\]

which implies \(m - x < H \log a + \log k\). Thus there are at most \(\theta + \log k \leq (1 + \log k)^{i+1}\) valid indices for \(x\). Let us count all possible \(\rho\), depending on the value \(x\). For a fixed \(x\) the number of possible \(\rho\) is bounded by \(|\text{Runs}_{\leq \epsilon}^{q_i} (w_{s_1+1, s_x})|\).

This is because the automaton is deterministic on \(Q \setminus \{p\}\). Thus by the induction assumption the number of all possible \(\rho\) is bounded by \((\theta(1 + \log k))^{i+1} = (\theta(1 + \log k))^{i+2}\).

Lemma 4.3 follows by applying Claim 4.10 with \(i = H\).

We conjecture that the results can be generalised to polynomially ambiguous weighted automata.

5 FROM INDEPENDENT-CRA TO OVAS

THEOREM 5.1. For Independent-CRA the problems of zero-isolation over \(Q_{\geq 0}(+, \cdot)\) and boundedness over \(\log \cap \{\min, +\}\) are interreducible in polynomial time.

The detailed proof can be found in the full version [14]. The rough intuition is that for a weighted automaton \(A\) over \(Q_{\geq 0}(+, \cdot)\) one can define \(A_{\log}\), where every weight \(c\) is replaced with \(-\log c\). Notice that \(c_j \rightarrow 0\) iff \(-\log c_j \rightarrow +\infty\).

If \(A\) were to be considered over \(Q_{\geq 0}(\max, \cdot)\) (i.e. when accepting runs are aggregated with max instead of +) then this theorem is essentially a syntactic translation. Thus the crux of Theorem 5.1 is to show that it is equivalent to consider the maximum run, rather than the aggregation with +.
We recall some definitions to define a new VAS model. Given a positive integer \(d \in \mathbb{N}\) an orthant in \(\mathbb{R}^d\) is a subset of the form \(x = (x_1, \ldots, x_d) : e_1 x_1 \geq 0, \ldots, e_d x_d \geq 0\) for some \(e_i \in \{-1, 1\}\). We write \(O_d\) for the set of all orthants. Notice that \(|O_d| = 2^d\). For example when \(d = 2\) there are four orthants also called quadrants. Let \(A_0, A_\infty \in O_d\), where \(A_\infty = \{x \mid e_i x_1 \geq 0, \ldots, e_d x_d \geq 0\}\) for \(s \in \{0, \infty\}\). We write \(A_0 \preceq A_\infty\) if \(e_i x_1 \leq e_i x_\infty\) for all \(i \in \{1, \ldots, d\}\). This is a partial order on \(O_d\), where the negative orthant, defined by \(x_1 < 0, \ldots, x_d < 0\), is the smallest element; and the positive orthant, defined by \(x_1 > 0, \ldots, x_d > 0\), is the largest element. Given an orthant \(A = \{x \mid e_i x_1 > 0, \ldots, e_d x_d > 0\}\) we will be often interested in points \(A^c_\prec = \{x \mid \exists A_x \in O_d \text{ s.t. } A \preceq A_x \text{ and } x \in A_x\}\). Let \(i_1, \ldots, i_k\) be all indices such that \(e_{ij} = 1\). Notice that \(A_\prec = \{x \mid e_i x_i > 0, \ldots, e_{ij} x_j > 0\}\).

Notice that some vectors belong to more than one orthant, when some of their coordinates are zero. Given a vector \(v\) we denote by \(A^*_v\) and \(A^-v\) the largest and the smallest orthants that contain \(v\), respectively. Notice that these are well-defined since \(\Delta\) induces a lattice on \(O_d\).

We define a model related to vector addition systems over integers [25]. Consider a positive integer \(d \in \mathbb{N}\). A \(d\)-dimensional orthant vector addition system (\(d\)-OVAS or OVAS if \(d\) is irrelevant) is \(V = (T_A)_{A \in O_d}\) where every \(T_A\) is a finite set of vectors \(T_A \subseteq \mathbb{R}^d\) with the following property. If \(A \preceq B\) then \(T_A \subseteq T_B\). We will refer to this property as monotonicity of \(V\). It will be convenient to denote \(T = \bigcup_{A \in O_d} T_A\). We define the norm of \(V\) as \(\|v\| = \max\{\|v\| : v \in T\}\). The transitions in \(V\) are encoded efficiently, i.e. for every \(v \in T\) it suffices to store the minimal orthants \(A\) such that \(v \in T_A\). Note that \(v\) may be minimal for multiple incomparable orthants.

A run from \(v_0\) over \(V\) is a sequence \(v_0, v_1, \ldots, v_n\) such that \(v_{i+1} = v_i + \delta_{v_i}\) for all \(i \in \{0, \ldots, n-1\}\). If such a run exists then we write \(v_0 \rightarrow^* v_n\). We allow \(n = 0\) and thus \(v \rightarrow^* V\) for every vector \(v\).

The universal coverability problem is defined as follows. Given a \(d\)-VAS \(V\) decide if for every vector \(v \in \mathbb{R}^d\), there exists \(w\) in the positive orthant (i.e. \(w \in \mathbb{R}^d_\geq\)) such that there is a run \(v \rightarrow^* w\). If there are such runs then we say that \(V\) is a positive instance of universal coverability.

The coverability problem is similar but the initial point is fixed. Formally, given a \(d\)-VAS \(V\) and a vector \(v \in \mathbb{R}^d\) decide if there is a run \(v \rightarrow^* w\) for some \(w \in \mathbb{R}^d_\geq\).

**Theorem 5.2.** The boundedness problem for Independent-CRA over \(\mathbb{L}\log \mathbb{Q}(\min, +)\) and the universal coverability problem for OVAS are interreducible in polynomial time.

**Proof Sketch.** We only give an intuition. Given an Independent-CRA \(\mathcal{A} = (\Sigma, I, F, X, (\delta_a)_{a \in \Sigma})\) the dimension of the OVAS \(V\) is \(|X| = d\). For every letter \(a \in \Sigma\) let \(\delta_a(x) = \min(x + c_a, x, da)\). The idea is that \(c_a, x\) are the coordinates of a corresponding vector \(c_a\) in \(V\), while \(da\) determine the orthants in which it is available. Intuitively, \(\mathcal{A}\) consumes letters in the reversed order compared to applying the corresponding vectors in \(V\). Then \(da, x = +\infty\) does not impose any restrictions, while \(da, x < +\infty\) means that the value of register \(x\) needs to be big enough for the transition to be fired.

## 5.1 OVAS with continuous semantics

Let \(V = (T_A)_{A \in O_d}\) be a \(d\)-OVAS. A continuous run from \(v\) over \(V\) is a sequence \(v_0, v_1, \ldots, v_n\) such that for every \(i \in \{0, \ldots, n-1\}\) there exists an orthant \(A_i\), where \(v_i, v_{i+1} \in A_i\); and there exists \(\delta_i \in \mathbb{R}^d_\geq\) such that \(\delta_i(v_{i+1} - v_i) \in T_{A_i}\). Notice that the former implies that orthants are crossed only by passing on the boundaries. If such a run exists then we write \(v_0 \rightarrow^*_c v_n\).

We say \(A_i\) is the orthant witnessing the transition if \(A_i\) is the maximal orthant such that \(v_i, v_{i+1} \in A_i\), and then we write that \(A_0, A_1, \ldots, A_{n-1}\) is the witnessing sequence of orthants.

We remark that it would be possible to drop the additional restriction of passing at the boundaries. One would have to require \(\delta_i \geq 1\) (otherwise, the vector \(\delta_i(v_{i+1} - v_i)\) essentially becomes available in \(T_{A_{i+1}}\)). Moreover, with the restriction of passing at the boundaries the behaviour within an orthant is similar to the standard continuous VAS model [7]. This will be convenient in Section 6, in particular to invoke Proposition 6.9.

**Remark 5.3.** A continuous run, where \(\delta_i = 1\) for all \(i\), is also a run.

The universal continuous coverability problem is defined as the universal coverability problem, where \(v \rightarrow^*_c w\) is replaced with \(v \rightarrow^*_c w\). Similarly, we will say e.g. that \(V\) is a positive instance of universal continuous coverability. In this subsection we will prove the following theorem.

**Theorem 5.4.** Let \(V = (T_A)_{A \in O_d}\) be a \(d\)-OVAS. Then \(V\) is a positive instance of universal coverability if and only if it is a positive instance of universal continuous coverability.

To prove the theorem we require several auxiliary lemmas about continuous runs over a \(d\)-OVAS \(V = (T_A)_{A \in O_d}\). Figure 4 shows geometric intuitions. In the following we assume that all vectors are over \(\mathbb{R}^d_\geq\), unless specified otherwise.

Given two vectors \(v, w\), we define the set of maximal orthants on the path from \(v\) to \(w\):

\[ O_{v,w} = \{A^+_x(v_\infty - v_0) : \delta \in [0, 1]\}\]

**Lemma 5.5.** Fix some vectors \(v, w\). Suppose that, for all \(C \in O_v, w\), there exists \(\delta_C \in \mathbb{R}_{\geq 0}\) such that \(\delta_C(v - w) \in T_C\). Then \(v \rightarrow^*_c w\) for every \(v' = \lambda_1 v + (1 - \lambda_1)w\), \(w' = \lambda_2 v + (1 - \lambda_2)w\), where \(0 \leq \lambda_2 \leq \lambda_1 \leq 1\).

**Lemma 5.6.** Let \(v \rightarrow^*_c w\). Suppose, for all \(C \in O_v, w\), there exists \(\delta_C \in \mathbb{R}_{\geq 0}\) such that \(\delta_C(v - w) \in T_C\). Let \(v' \geq v\) and \(w' \geq w\) be such that \(w' - v' = \delta'(v - w)\) for some \(\delta' \in \mathbb{R}_{\geq 0}\). Then \(v' \rightarrow^*_c w'\).

**Lemma 5.7.** Let \(v_0, v_1, \ldots, v_n\) be a continuous run such that \(v_0 \in \mathbb{R}^d_\geq\) and let \(a_i = v_i - v_{i-1}\) for \(i \in \{1, \ldots, n\}\). Let \(m \geq 1\) and consider the sequence \(v'_0, v'_1, \ldots, v'_n\) defined by \(v'_0 = v_0, v'_i = v'_{i-1} + m a_i\). Then \(v'_i \rightarrow^*_c v'_j\) for all \(i \in \{0, \ldots, n-1\}\).

We also need the following technical lemma, which is a direct consequence of the simultaneous version of the Dirichlet’s approximation theorem. Intuitively, it says that given a finite set of reals we can multiply them all with the same natural number so that all resulting numbers are arbitrarily close to integers.
Let $V' = (T_A)_{A \in O}$ be a $d$-OVAS. We start with the implication that if $V' = (T_A)_{A \in O}$ is a positive instance of universal coverability then it is also a positive instance of universal continuous coverability. Thus, fix any $v \in \mathbb{R}^d$. We aim to prove that there is a continuous run $v \to w$ for some $w \in \mathbb{R}^d_{>0}$.

Let $u = (\|V'\|, \ldots, \|V'\|) \in \mathbb{R}^d$. By assumption there is a run $v_0, \ldots, v_n$, where $v_0 = (v-u)$ and $v_n \in \mathbb{R}_{>0}$. Let $a_{i+1} = v_{i+1} - v_i$ for $i \in \{0, \ldots, n-1\}$, i.e., the differences between consecutive elements in the run. Since $a_{i+1}$ is a transition, $\|a_{i+1}\| \leq \|u\|$. To conclude the proof we need to show that $v_i + u - a_{i+1} + u$ for every $i \in \{0, \ldots, n-1\}$; indeed, $v_0 + u = v$ and $v_n + u \in \mathbb{R}^d_{>0}$.

Since $v_0, \ldots, v_n$ is a run, then $(v_i + v + u) \in T_{A_{i+1}}$ for every $i \in \{0, \ldots, n-1\}$. Let $v_{i+1} = v((v_{i+1} + u) + (1 - \epsilon)(v_{i+1} + u))$, for $\epsilon \in [0, 1]$. We argue that $\delta(v_{i+1} - v_i) = \delta((v_{i+1} + u) - (v_i + u)) \in T_{A_{i+1}}$ for some $\delta$. Indeed

$v_{i+1} = v((v_{i+1} + u) + (1 - \epsilon)(v_{i+1} + u)) = (v + (1 - \epsilon)(v_{i+1} + u) + v_i + u + a_{i+1}$.

Since $\|u\| > \|a_{i+1}\| > \|a_{i+1}\|$, we have $v_{i+1} \geq v_i$, thus, by monotonicity $\delta(v_{i+1} - v_i) \in T_{A_{i+1}}$. Hence $v_i + u \to v_{i+1}$ by Lemma 5.5.

Conversely, we fix $v \in \mathbb{R}^d$ and aim to prove that there is a run $v \to w$ for some $w \in \mathbb{R}^d_{>0}$. Take any vector $v_0 \in \mathbb{R}_{>0}$ such that $v_0 + 1 \in \mathbb{R}$, where $1 = \{1, \ldots, 1\}$. By assumption there is a continuous run $v_0, v_1, \ldots, v_n$ for some $v_0 \in \mathbb{R}_{>0}$. Let $A_0, \ldots, A_{n-1}$ be the corresponding witness sequence of orths: $a_{i+1} = v_{i+1} - v_i$; $A_{i+1} \in \mathbb{R}_{>0}$ be such that $\delta = a_{i+1} \in T_A$, for all $i \in \{0, \ldots, n-1\}$, and $b_{i+1} = a_{i+1} \in T_A$. By Lemma 5.8 there exists $m \in \mathbb{N} \setminus \{0\}$ such that for every $i \in \{0, \ldots, n-1\}$ there exist $n_i \in \mathbb{N}$ such that $\frac{n_i}{\delta} + n_i < \epsilon$.

Consider the scaled run $v_0', \ldots, v_n'$ defined by $v_0' = v_0$, $v_i' = v_{i+1} + n_i b_i$. By Lemma 5.7 $v_i' \to v_{i+1}'$ for all $i \in \{0, \ldots, n-1\}$. Moreover, it is easy to see that $v_0' \in \mathbb{R}_{>0}$. We define the sequence of points $v_0', \ldots, v_n'$ as follows: $v_0' = v$ and $v_n' = v_{n-1} + n_i b_i$. Intuitively, this is like the $v_i'$ run approximated to integers, and shifted by $1$ to compensate for errors. The rest of the proof is dedicated to show first that $v_n' \to v_{n+1}'$ and then that $v_i' \to v_i''$, which will conclude the proof. The first step is depicted in Figure 4.

We prove that $v_i' \to v_i''$ for every $i \in \{0, \ldots, n-1\}$. Notice that $\frac{n_i}{n_i} (v_{i+1}' - v_i') = b_{i+1} \in T_A$. Moreover,

$v_i'' = v + \sum_{j=1}^{i} n_i b_j \geq v_0' + 1 + \sum_{j=1}^{i} n_i b_j$

by the choice of $v_0 = v_0'$

$\geq v_0' + 1 + \sum_{j=1}^{i} \left( \frac{m_j}{\delta_j} - \frac{1}{n} \|b_j\| \|b_j\| \right)$

by the choice of $m$

$\geq v_0' + 1 + \sum_{j=1}^{i} \left( m_j - \frac{1}{n} \right)$

$\geq v_0' + 1 + \sum_{j=1}^{i} m_j b_j = \delta_j a_j$ and $\|b_j\| < \|V\|$

$\Rightarrow \frac{n_i}{n_i} (v_{i+1}' - v_i') = b_{i+1} \in T_A$. Moreover,

Lemma 5.8. [33, Theorem 1A] Let $r_1, \ldots, r_k \in \mathbb{R}$. For every $\epsilon > 0$ there exists $m \in \mathbb{N} \setminus \{0\}$ such that for every $i \in \{1, \ldots, k\}$ there exists $z_i \in \mathbb{Z}$, where $|mr_i - z_i| < \epsilon$.

Proof of Theorem 5.4. We show how to find a continuous witnessing run from a discrete run, and visa versa. A discrete run is not a continuous run, because a continuous run must pause at the boundary, and can only go further into the adjacent orthant if this direction remains available. We take a run from a sufficiently smaller starting point and shift it to our desired starting point. After the shift, if we cross orhtants then directions will be always available due to monotonicity.

Conversely, when converting a continuous run to a discrete run, we face a continuous run using non-integer multiples of vectors. We use Lemma 5.7 to scale-up the run, using Lemma 5.8 to find a multiple so that the scaled real coefficients are sufficiently close to an integer.
Since $v_i^* \rightarrow^* v_{i+1}^*$ by Lemma 5.6 we get $v_i^* \rightarrow^* v_{i+1}^*$.

To conclude we need to prove that $v_i^* \rightarrow^* v_{i+1}^*$, for every $i \in \{0, \ldots, n - 1\}$. Recall that $v_{i+1}^* = n_{i+1}b_{i+1}$ and that $n_{i+1}$ is a natural number. Let $u_0, \ldots, u_{n_{i+1}}$ be defined as $u_0 = v_i^*$ and $u_{j+1} = u_j + b_{j+1}$. Notice that $u_{n_{i+1}} = v_{i+1}^*$. By Lemma 5.5 and Remark 5.3 $u_j \rightarrow^* u_{j+1}$ for every $j \in \{0, \ldots, n_{i+1} - 1\}$. Therefore, $v_i^* \rightarrow^* v_{i+1}^*$, which concludes the proof. □

6 COVERABILITY FOR OVAS

In this section we present our undecidability and decidability results for coverability and universal coverability, respectively. In the definition of $d$-OVAS for each orthant $A$ the set of transitions $T_A$ is a subset of $\mathbb{R}^d$. For a set $S \subseteq \mathbb{R}$ an OVAS over $S$ is an OVAS using numbers only from $S$ in its transitions, namely for each orthant $A$ we have $T_A \subseteq S^d$.

**Theorem 6.1.** The coverability problem for OVAS over $\mathbb{Z}$ is undecidable.

Our two decidability results are the following.

**Theorem 6.2.** The universal continuous coverability problem for 3-OVAS over $\mathbb{Q}$ is decidable in $\text{ExpTime}$.  

**Theorem 6.3.** Assuming Schanuel’s conjecture the universal continuous coverability problem for 3-OVAS over $\text{LogQ}$ is decidable.

Together with Theorem 5.1, Theorem 6.3 completes the proof of Theorem 3.5.

Remark 6.4. Computations with elements from $\text{LogQ}$ require considerable care. While they are easy to represent (e.g. by storing $2^x \in \mathbb{Q}$ in place of $x \in \text{LogQ}$), note that $(\text{LogQ}, +, \cdot)$ is not a semiring. In particular, the product of two elements, e.g. $\log_2(a) \log_2(b)$, is not necessarily an element of $\text{LogQ}$. In general, such computations are curiously difficult; for example (unconditionally) deciding whether $\log_2(a) \log_2(b) \leq \log_2(c) \log_2(d)$ for $a, b, c, d \in \mathbb{Q}_{>0}$, is, to the best of our knowledge, open and related to the four exponential conjectures which ask if they can ever be equal (see e.g.,[38, Sec. 1.3 and 1.4]). However, Schanuel’s conjecture implies decidability of the first order theory of the reals with exponential operations $\text{FO}(+, \cdot, \exp, <)$.[27] In particular, this allows arithmetic operations between elements of $\text{LogQ}$.

The rest of this section is devoted to the proofs of Theorems 6.2 and 6.3. We slowly introduce required notions and at the end we show how the developed techniques allow to prove both theorems. Most of our steps will work for a $d$-OVAS in any dimension $d$.

We define a notion of a separator with a property that an OVAS $V$ is a negative instance of the universal coverability problem if and only if there exist a separator for $V$. Finally we show that the existence of a separator in 3-OVAS over $\mathbb{Q}$ can be expressed in the first order logic $\text{FO}(+, \cdot, <)$ with bounded quantifier alternation which is decidable in $\text{ExpTime}$ due to Tarski’s theorem [23]. The existence of a separator in 3-OVAS over $\text{LogQ}$ can be expressed in the first order logic $\text{FO}(+, \cdot, \exp, <)$ which is decidable, subject to Schanuel’s conjecture [27].

Given a $d$-OVAS we define the walls set $W = \{v \in \mathbb{R}^d \mid \exists i \in [1, d] : v_i = 0\}$, i.e. vectors in $\mathbb{R}^d$ with some coordinate equal to zero. The set $W$ contains all of the faces of $d$-dimensional orthants. Recall that the negative orthant is defined as $\mathbb{R}_{<0}^d = \mathbb{R}_{<0} \setminus W$. For a $d$-OVAS $V$ let

$$\text{reach}(V) = \{v \mid \exists u \in \mathbb{R}_{<0}^d : u \rightarrow^* v\}$$

be the set of all vectors reachable from the strictly negative orthant. We observe the following.

**Claim 6.5.** A $d$-OVAS $V$ is positive instance of universal continuous coverability problem if and only if $\text{reach}(V) \cap \mathbb{R}_{\geq 0}^d \neq \emptyset$.

By Claim 6.5 it is enough to focus on deciding whether $\text{reach}(V)$ intersects the positive orthant. For $S \subseteq \mathbb{R}^d$ we define its downward closure as $S^\downarrow = \{v \mid \exists s \in S : v < s\}$. Suppose $S \subseteq T \subseteq \mathbb{R}^d$. We say that $S$ is downward closed inside $T$ if $S = S^\downarrow \cap T$. Notice that $\text{reach}(V)$ intersects the positive orthant if and only if $\text{reach}(V)^\downarrow$ intersects the positive orthant. Furthermore, let

$$\text{reach}^\downarrow(V) = \text{reach}(V) \cap \mathbb{R}_{\geq 0}^d.$$

**Claim 6.6.** For every $d$-OVAS $V$ the following are equivalent: $\text{reach}^\downarrow(V) \cap \mathbb{R}_{\geq 0}^d \neq \emptyset$ if and only if $\text{reach}(V) \cap \mathbb{R}_{\geq 0}^d \neq \emptyset$.

We will focus on deciding whether $\text{reach}^\downarrow(V)$ intersects the positive orthant. For each orthant $A$ and $u_0, u_n \in A$ we write $u_0 \rightarrow^* u_n$ if there is a continuous run $u_0, \ldots, u_n$ such that $u_i \in A$ for all $0 \leq i \leq n$.

**Definition 6.7.** Given a $d$-OVAS $V$ we say that $S \subseteq \mathbb{R}^d$ is a separator for $V$ if the following conditions are satisfied:

1. $S$ is closed under scaling, namely for every $\lambda > 0$ and $s \in S$, we have $\lambda \cdot s \in S$;
2. $S$ is downward closed inside $W$, namely $S = S^\downarrow \cap W$;
3. For every orthant $A$ if $u, v \in S$ and $u \rightarrow^*_c A$ then $v \in S$;
4. $S \cap \mathbb{R}_{\geq 0}^d = \emptyset$.

**Lemma 6.8.** For every OVAS $V$: $\text{reach}^\downarrow(V) \cap \mathbb{R}_{\geq 0}^d = \emptyset$ if and only if there exists a separator for $V$.

We aim to show that the existence of a separator in 3-OVAS can be expressed in appropriate first order logics. It is helpful to use the following observation about continuous VAses from [7], which helps us to construct the needed first order sentences.

**Proposition 6.9 (Reformulation of Proposition 3.2 in [7]).** Fix a $d$-OVAS $V$ and an orthant $A \in \mathbb{Q}_d$. Consider two vectors of variables $x = (x_1, \ldots, x_d)$ and $y = (y_1, \ldots, y_d)$. There is an existential formula $\varphi_A(x, y)$ such that

$$[\varphi_A] = \left\{(v, w) \in \mathbb{A}^2 \mid v \rightarrow^*_{c, A} w\right\}.$$

If $V$ is over $\mathbb{Q}$ then $\varphi_A \in \text{FO}(+,-,<)$, and if $V$ is over $\text{LogQ}$ then $\varphi_A \in \text{FO}(+,-,\exp,<)$.

Notice that in our setting we work over reals, while in [7] they work over rationals. The results in [7] are stated for the logic $\text{FO}(+,-,<)$ over $\mathbb{Q}$, but it is easy to see that the same formulas work for our logics over $\mathbb{R}$. The reason why we need to consider irrational numbers is that whenever we deal with a number of the form $\log_2\left(\frac{r}{q}\right)$ then we express this by $\exp(x) = \frac{r}{q}$.

The following lemma concludes the proofs in this section.
Lemma 6.10. The existence of a separator in 3-OVAS:
(1) over $Q$ is expressible in $FO(+,\cdot, <)$ over reals with fixed number of quantifier alternations;
(2) over $LogQ$ is expressible in $FO(+,\cdot, \exp, <)$ over reals.

Proof. The key observation is that a set $S \subseteq W_3$, which is downward closed and closed under scaling, can be described by at most 18 real numbers. Notice first that the set $W_3$ is a union of 12 quarters of a plane. Indeed, $W_3$ consists of three planes (defined by $x[1] = 0$, $x[2] = 0$ and $x[3] = 0$). Each of the three planes is divided into exactly four quarters. Thus a quarter $Q$ is described by the choice of $i \in \{1,2,3\}$ such that $x \in Q$ if $x[i] = 0$ and two signs for the other coordinates. For example consider a quarter $Q$ such that if $x \in Q$ then $x[3] = 0$. Then $Q$ is determined by $e_1, e_2 \in \{-1,1\}$, defining $Q = \{x \in \mathbb{R}^3 \mid x[1] \cdot e_1 > 0, x[2] \cdot e_2 > 0, x[3] = 0\}$. So every quarter is determined by a triple $(s_1, s_2, s_3) \in \{+,-,0\}$ such that there is exactly one zero among $s_1, s_2$ and $s_3$. We will show that for every quarter $Q$ the set $S \cap Q$ can be described using either one or two real numbers. To simplify the notation whenever $Q$ is fixed we will think of quarters $Q$ and $S$ as subsets of $\mathbb{R}^3$ (projecting on the coordinates that are not fixed to 0 in $Q$).

Given a quarter $Q$ consider two cases: 1) $e_1 = e_2$, 2) $e_1 \neq e_2$. We show that in the first case if $S \cap Q \neq \emptyset$ then $Q \subseteq S$. Indeed, assume $x \in S \cap Q$ and let $y \in Q$. We aim to show that $y \in S$. Since $e_1 = e_2$, there are $\lambda \in \mathbb{R}_{\geq 0}$ and $v \in \mathbb{R}^2_{\geq 0}$ such that $y = \lambda x + v$. As $x \in S$ and $S$ is closed under scaling we have $\lambda x \in S$. As $S$ is downward closed and $v \in \mathbb{R}^2_{\geq 0}$ we have $y = \lambda x + v \in S$. To conclude either $S \cap Q$ is empty or it is the full quarter. Thus such quarters can be described by one variable $b \in \{0,1\}$ (one bit of information: 0 for empty set, 1 for full set).

Consider the second case when $e_1 \neq e_2$ and assume without loss of generality that $e_1 = 1$ and $e_2 = -1$. Thus $Q = \{(x_1, x_2) \mid x_1 > 0, x_2 \leq 0\}$. We observe that $Q \cap S$ is actually a part of $Q$ which lies below some line $\alpha$. This will be the only step where we use the assumption that our OVAs is in 3-dimensions (which implies that $Q$ is in 2 dimensions). Formally, a line $\{(x_1, x_2) \mid \alpha x_1 = \alpha x_2\}$ in $Q$ is described by $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \in \mathbb{R}_{\geq 0}$, $\alpha_2 \in \mathbb{R}_{< 0}$, such that at least one $\alpha_1 \neq 0$ (the sign of $\alpha_1$ comes from $e_1$).

Claim 6.11. $Q \cap S = \{(x_1, x_2) \in Q \mid \alpha x_1 < \alpha_2 x_2\}$, for some line $\alpha$ in $Q$, where $< \equiv \leq <$. We note that this claim is not true in higher dimensions, and it is the main obstacle for the proof to work in general. By Claim 6.11 the set $Q \cap S$ is described by two real numbers (recall that $\alpha_1$ also determines the value of $e_2$). Summarising: 6 quarters need 1 bit of description, and 6 quarters need 2 real numbers. In total 18 numbers are needed to describe the downward closed set $S$, which is also closed under scaling.

Note that our description will satisfy conditions 1 and 2 in Definition 6.7. In order to check that given 18 real numbers describe a separator it remains to check that:

- the descriptions of quarters are consistent on the intersections of quarters;
- conditions 3 and 4 in Definition 6.7 are satisfied.

It is not hard to see that the first item can be described in $FO(+,\cdot, <)$, we just need to guarantee that quarters are consistent on the intersecting lines. In order to check condition 4 it is enough to guarantee that the quarters $(+,+,0)$, $(+,0,0)$ and $(0,+,+)\) are all described by the bit 0. The most involved part is to check condition 3. Here, we invoke Proposition 6.9 that defines the reachability formula $\phi_A(x,y)$.

We express condition 3 as follows: for every orthant $A$ if $x \in S$ and $\phi_A(x,y)$ then $y \in S$. It is easy to transform the above description to sentences of first order logic. Moreover, observe that these sentences have quantifier alternation at most two: there exists a separator $S$, such that for all $x \in S$ there exists $y \in S$ fulfilling $\phi_A(x,y)$, where $\phi_A$ has only existential quantifiers. We have proved Lemma 6.10.

Remark 6.12. Notice that in 4-OVAS for $d > 3$ it is not clear whether a separator can be described by a bounded number of real numbers as Claim 6.11 is no longer true. A natural generalisation of techniques used in the proof of Lemma 6.10 would result in expressing the existence of a separator in monadic second-order logic $MSO(+,\cdot, <)$ over the reals. However, the validation problem for $MSO(+,\cdot, <)$ is undecidable as it is easy to express natural numbers in $MSO(+,\cdot)$ as follows: $N$ is the smallest set of numbers containing 1 and closed under adding 1. Thus decidability of $MSO(+,\cdot, <)$ over $\mathbb{R}$ would imply decidability of $MSO(+,\cdot, <)$ over $\mathbb{N}$ and in particular decidability of $FO(+,\cdot, <)$ over $\mathbb{N}$, which is well known to be undecidable [24]. Extending the techniques used in the proof of Lemma 6.10 would probably require showing that we can describe separators in higher dimensional spaces using a bounded number of real number.

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