Tighter Bounds for the Discrepancy of Boxes and Polytopes

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Abstract

Combinatorial discrepancy is a complexity measure of a collection of sets which quantifies how well the sets in the collection can be simultaneously balanced. More precisely, we are given an n-point set \( P \), and a collection \( \mathcal{F} = \{F_1, \ldots, F_m\} \) of subsets of \( P \), and our goal is color \( P \) with two colors, red and blue, so that the largest absolute difference between the number of red elements and the number of blue elements (i.e. the discrepancy) in any \( F_i \) is minimized. Combinatorial discrepancy has many applications in mathematics and computer science, including constructions of uniformly distributed point sets, and lower bounds for data structures and private data analysis algorithms.

We investigate the combinatorial discrepancy of geometrically defined set systems, in which \( P \) is an n-point set in \( d \)-dimensional space, and \( \mathcal{F} \) is the collection of subsets of \( P \) induced by dilations and translations of a fixed convex polytope \( B \). Such set systems include sets induced by axis-aligned boxes, whose discrepancy is the subject of the well known Tusnády problem. We prove new discrepancy upper bounds for such set systems by extending the approach based on factorization norms previously used by the author and Matoušek. We improve the best known upper bound for the Tusnády problem by a logarithmic factor, using a result of Banaszczyk on signed series of vectors. We extend this improvement to any arbitrary convex polytope \( B \) by using a decomposition due to Matoušek.

1 Introduction

As usual, we define the combinatorial discrepancy of a system \( \mathcal{F} \) of subsets of some finite set \( P \) as:

\[ \text{disc} \mathcal{F} := \min_{\chi: P \to \{-1, 1\}} \max_{F \in \mathcal{F}} \sum_{p \in F} \chi(p). \]

We refer the reader to the book of Matoušek [Ms10] for background on combinatorial discrepancy. For a reference to the large number of applications of
combinatorial discrepancy to geometric discrepancy, numerical methods, and computer science, see the book of Chazelle [Cha00]. A recent application to private data analysis can be found in the thesis of the author [Nik14].

Let $\mathcal{A}_d$ be the family of anchored axis-aligned boxes in $\mathbb{R}^d$: $\mathcal{A}_d := \{A(x) : x \in \mathbb{R}^d\}$, where $A(x) := \{y \in \mathbb{R}^d : y_i \leq x_i \forall i \in [d]\}$ and $[d] := \{1, \ldots, d\}$. This is a slight abuse of terminology: $A(x)$ is not a box, but rather a polyhedral cone. Usually $A(x)$ is defined to be anchored at 0, i.e. it is defined as the set $\{y \in \mathbb{R}^d : 0 \leq y_i \leq x_i \forall i \in [d]\}$. However, we prefer to anchor $A(x)$ at $(-\infty, -\infty)$. This convention does not affect any of the results in the paper, and allows us to avoid some minor technicalities.

For an $n$-point set $P \subset \mathbb{R}^d$, we denote by $\mathcal{A}_d(P) := \{A(x) \cap P : x \in \mathbb{R}^d\}$ the set system induced by anchored boxes on $P$. (Note that this set system is finite, and, in fact, can have at most $n^d$ sets.) Tusnády’s problem asks for tight bounds on the largest possible combinatorial discrepancy of $\mathcal{A}_d(P)$ over all $n$-point sets $P$. Here we prove the following theorem, which gives a new upper bound on the discrepancy.

**Theorem 1.** For any $d \geq 2$, there exists a constant $C_d$ s.t. for any $n$-point set $P$ in $\mathbb{R}^d$

$$\text{disc} \mathcal{A}_d(P) \leq C_d(1 + \log n)^{d-1/2}.$$ 

It was shown in [MN15, MNT14] that for infinitely many $n$ there exists an $n$-point set $P$ such that $\text{disc} \mathcal{A}_d(P) = \Omega_d(\log n^{d-1})$, so the upper bound above is within a $O_d(\sqrt{\log n})$ factor from the lower bound. (Here, and in the rest of this paper, we use the notation $O_p(\cdot), \Theta_p(\cdot), \Omega_p(\cdot)$ to denote the fact that the implicit constant in the asymptotic notation depends on the parameter $p$.) The best previously known upper bound for the Tusnády problem was $O_d(\log d n)$, and was recently proved by Bansal and Garg [BG16]. Their result improved on the prior work of Larsen [Lar14] (see also the proof in [MNT14]), who showed an upper bound of $O_d(\log^{d+1/2} n)$.

More generally, let $\mathcal{F}$ be a collection of subsets of $\mathbb{R}^d$, and, for an $n$-point set $P \subset \mathbb{R}^d$, let $\mathcal{F}(P)$ be the set system induced by $\mathcal{F}$ on $P$, i.e. $\mathcal{F}(P) := \{F \cap P : F \in \mathcal{F}\}$. We are interested in how the worst-case combinatorial discrepancy of such set systems grows with $n$. This is captured by the function $\text{disc}(n, \mathcal{F}) := \sup\{\text{disc} \mathcal{F}(P) : P \subset \mathbb{R}^d, |P| = n\}$. In this notation, Theorem 1 shows that $\text{disc}(n, \mathcal{A}_d) = O_d(\log^{d-1/2} n)$.

Let $K \subset \mathbb{R}^d$, and let’s define $\mathcal{T}_K$ to be the family of images of $K$ under translations and homothetic transformations: $\mathcal{T}_K := \{tK + x : t \in \mathbb{R}_+, x \in \mathbb{R}^d\}$, where $\mathbb{R}_+$ is the set of positive reals. If we take $Q := [0, 1]^d$, then it’s well-known that $\text{disc}(n, \mathcal{T}_Q) \leq 2^d \text{disc}(n, \mathcal{A}_d)$ and, therefore, Theorem 1 implies $\text{disc}(n, \mathcal{T}_Q) = O_d(\log^{d-1/2} n)$. Our next result shows that this bound holds for any polytope $B$ in $\mathbb{R}^d$.

**Theorem 2.** For any $d \geq 2$, and any closed convex polytope $B \subset \mathbb{R}^d$,

$$\text{disc}(n, \mathcal{T}_B) = O_{d,B}(\log^{d-1/2} n).$$
We note that with the same proof we can establish the stronger fact that 
\[ \text{disc}(n, \text{POL}(\mathcal{H})) = O_d,\mathcal{H}(\log^{d-1/2} n), \]
where \( \mathcal{H} \) is a family of hyperplanes in \( \mathbb{R}^d \), and \( \text{POL}(\mathcal{H}) \) is the set of all polytopes which can be written as \( \bigcap_{i=1}^m \gamma_i \), with each \( \gamma_i \) a halfspace whose boundary is parallel to some \( h \in \mathcal{H} \). The best previously known upper bound in this setting is also due to Bansal and Garg [BG16], and is equal to \( O_{d,B}(\log n) \).

Our approach builds on the connection between the \( \gamma_2 \) norm and hereditary discrepancy shown in [NT15, MNT14]. The new idea which enables the tighter bounds is to use a result of Banaszczyk, proved in the context of improving the constants in the Steinitz lemma, in order to get one dimension “for free”. The proof of Theorem 2 combines the ideas in the proof of Theorem 1 with a decomposition due to Matoušek [Mat99].

1.1 Connection to Geometric Discrepancy

Geometric discrepancy measures the irregularity of a distribution of \( n \) points in \( [0,1]^d \) with respect to a family of distinguishing sets. In particular, for an \( n \)-point set \( P \subseteq [0,1]^d \) and a family of measurable subsets \( \mathcal{F} \) of \( \mathbb{R}^d \), we define the discrepancy
\[
D(P, \mathcal{F}) := \sup_{F \in \mathcal{F}} |P \cap F| - \lambda_d(F \cap [0,1]^d),
\]
where \( \lambda_d \) is the Lebesgue measure on \( \mathbb{R}^d \). The smallest achievable discrepancy over \( n \) point sets with respect to \( \mathcal{F} \) is denoted \( D(n, \mathcal{F}) := \inf\{D(P, \mathcal{F}) : P \subseteq \mathbb{R}^d, |P| = n\} \). A famous result of Schmidt [Sch72] shows that \( D(n, A_d) = \Theta(\log n) \). The picture is much less clear in higher dimensions. In a seminal paper [Rot54], Roth showed that \( D(n, A_d) = \Omega_d(\log^{d-1/2} n) \) for any \( d \geq 2 \); the best known lower bound in \( d \geq 3 \) is due to Bilyk, Lacey, and Vagharshakyan [BLV08] and is \( D(n, A_d) = \Omega_d(\log^{d-1/2} n) \), where \( \eta_d \) is a positive constant depending on \( d \) and going to 0 as \( d \) goes to infinity. On the other hand, the best known upper bound is \( D(n, A_d) = O_d(\log^{d-1} n) \) and can be achieved in many different ways, one of the simplest being the Halton-Hammersley construction [Hal60, Ham60]. The book by Beck and Chen [BC08] calls the problem of closing this significant gap “the Great Open Problem” (in geometric discrepancy theory). See the book of Matoušek [Ms10] for further background on geometric discrepancy.

There is a known connection between combinatorial and geometric discrepancy. Roughly speaking, combinatorial discrepancy is an upper bound on geometric discrepancy. More precisely, we have the following transference lemma, which goes back to the work of Beck on Tusnády’s problem [Bec81] (see [Ms10] for a proof).

**Lemma 3.** Let \( \mathcal{F} \) be a family of measurable sets in \( \mathbb{R}^d \) such that there is some \( F \in \mathcal{F} \) which contains \( [0,1]^d \). Assume that \( \frac{D(n,F)}{n} \) goes to 0 as \( n \) goes to infinity, and that \( \text{disc}(n,F) \leq f(n) \) for a function \( f \) that satisfies \( f(2n) \leq (2-\delta)f(n) \) for all \( n \) and some fixed \( \delta > 0 \). Then there exists a constant \( C_\delta \) that only depends on \( \delta \), for which \( D(n, \mathcal{F}) \leq C_\delta f(n) \).
Lemma 3 and Theorem 2 imply that
\[ D(n, T_B) = O_d,B(\log^{d-1/2} n) \]
for any convex polytope \( B \) in \( \mathbb{R}^d \). This bound gets within an \( O_d,B(\sqrt{\log n}) \) factor from the best bound known for boxes in \( d \) dimensions. See the remarks after Section 4.6. of \([Ms10]\) for references to prior work on bounding \( D(n, T_B) \) for a convex polytope \( B \).

\[ \gamma_2 \] factorization norm

The \( \gamma_2 \) norm was introduced in functional analysis to study operators that factor through Hilbert space. We say that an operator \( u : X \to Y \) between Banach spaces \( X \) and \( Y \) factors through a Hilbert space if there exists a Hilbert space \( H \) and bounded operators \( u_1 : X \to H \) and \( u_2 : H \to Y \) such that \( u = u_2 u_1 \).

Then the \( \gamma_2 \) norm of \( u \) is
\[ \gamma_2(u) := \inf \| u_1 \| \| u_2 \|, \]
where the infimum is taken over all Hilbert spaces \( H \), and all operators \( u_1 \) and \( u_2 \) as above. Here \( \| u_1 \| \) and \( \| u_2 \| \) are the operator norms of \( u_1 \) and \( u_2 \), respectively. The book of Tomczak-Jaegermann \([TJ89]\) is an excellent reference on factorization norms and their applications in Banach space theory.

In this work we will use the \( \gamma_2 \) norm of an \( m \times n \) matrix \( A \), which is defined as the \( \gamma_2 \) norm of the linear operator \( u : \ell^1_n \to \ell^\infty_m \) with matrix \( A \) (in the standard bases of \( \mathbb{R}^m \) and \( \mathbb{R}^n \)). In the language of matrices, this means that
\[ \gamma_2(A) := \inf \{ \| U \|_2 \| V \|_1 : A = UV \}, \]
where the infimum is taken over matrices \( U \) and \( V \), \( \| V \|_1 \) equals the largest \( \ell_2 \) norm of a column of \( V \), and \( \| U \|_2 \) equals the largest \( \ell_2 \) norm of a row of \( U \). By a standard compactness argument, the infimum is achieved; moreover, we can take \( U \in \mathbb{R}^{m \times r} \) and \( V \in \mathbb{R}^{r \times n} \), where \( r \) is the rank of \( A \). Yet another equivalent formulation, which will be convenient for us, is that \( \gamma_2(A) \) is the smallest non-negative real \( t \) for which there exist vectors \( u_1, \ldots, u_m, v_1, \ldots, v_n \in \mathbb{R}^r \) such that for any \( i \in [m], j \in [n], a_{ij} = \langle u_i, v_j \rangle \) and \( \| u_i \|_2 \leq t, \| v_j \|_2 \leq 1 \).

Let us further overload the meaning of \( \gamma_2 \) by defining \( \gamma_2(F) = \gamma_2(A) \) for a set system \( F \) with incidence matrix \( A \). We recall that the incidence matrix of a system \( F = F_1, \ldots, F_m \) of subsets of a set \( P = p_1, \ldots, p_n \) is defined as
\[ a_{ij} := \begin{cases} 1 & p_j \in F_i, \\ 0 & p_j \not\in F_i. \end{cases} \]
In other worse, \( \gamma_2(\mathcal{F}) \) is the smallest non-negative real \( t \) such that there exist vectors \( u_1, \ldots, u_m, v_1, \ldots, v_n \) satisfying
\[
\langle u_i, v_j \rangle = \begin{cases} 
1 & p_j \in F_i \\
0 & p_j \notin F_i
\end{cases},
\]
and \( \|u_i\|_2 \leq t, \|v_j\|_2 \leq 1 \) for all \( i \in [m], j \in [n] \).

In \cite{MN15} it was shown that \( \gamma_2(\mathcal{F}) \) is, up to logarithmic factors, equivalent to a hereditary version of combinatorial discrepancy. We do not directly use this connection here. In \cite{MN15} it was also shown that \( \gamma_2 \) satisfies a number of nice properties which help in estimating the norm of specific matrices or set systems. Here we only need the following inequality, which holds for a set system \( \mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k \), where \( \mathcal{F}_1, \ldots, \mathcal{F}_k \) are set systems over the same set \( P \):
\[
\gamma_2(\mathcal{F}) \leq \sqrt[k]{\sum_{i=1}^{k} \gamma_2(\mathcal{F}_i)^2}.
\]

We also need the following simple lemma, which follows, e.g. from the results in \cite{MN15, MNT14}, but also appears in a similar form in \cite{Lar14}, and follows from standard dyadic decomposition techniques.

**Lemma 4.** For any \( d \geq 1 \) there exists a constant \( C_d \) such that the following holds. For any \( n \)-point set \( P \subset \mathbb{R}^d \), \( \gamma_2(\mathcal{A}_d(P)) \leq C_d(1 + \log n)^d \).

### 2.2 Signed Series of Vectors

We will use the following result of Banaszczyk:

**Lemma 5 \((\text{Ban12})\).** Let \( v_1, \ldots, v_n \in \mathbb{R}^m \), \( \forall i : \|v_i\|_2 \leq 1 \), and let \( K \subset \mathbb{R}^m \) be a convex body symmetric around the origin. If \( \mu_m(K) \geq 1 - 1/(2n) \), then there exists an assignment of signs \( \chi : [n] \rightarrow \{-1, 1\} \) so that
\[
\forall j \in \{1, \ldots, n\} : \sum_{i=1}^{j} \chi(i)v_i \in 5K.
\]

This lemma was proved in the context of the well-known Steinitz problem: given vectors \( v_1, \ldots, v_n \), each of Euclidean norm at most 1, such that \( v_1 + \ldots + v_n = 0 \), find a permutation \( \pi \) on \([n]\) such that for all integers \( i, 1 \leq i \leq n \), \( \|v_{\pi(1)} + \ldots + v_{\pi(i)}\|_2 \leq C\sqrt{m} \), where \( C \) is an absolute constant independent of \( m \) or \( n \). Lemma 5 gives the best partial result in this direction: it can be used to show a bound of \( C(\sqrt{m} + \sqrt{\log n}) \) in place of \( C\sqrt{m} \).

Lemma 5 follows relatively easily from another powerful result Banaszczyk proved in \cite{Ban98}. Unfortunately, the proof of the latter does not suggest any efficient algorithm to find the signs \( \chi(i) \), and no such algorithm is yet known.
3 Proof of Theorem 1

In this section we give the proof of Theorem 1. Let us fix the $n$-point set $P \subset \mathbb{R}^d$ once and for all. Without loss of generality, assume that each $p \in P$ has a distinct last coordinate, and order the points in $P$ in increasing order of their last coordinate as $p_1, \ldots, p_n$. Write each $p_i$ as $p_i = (q_i, r_i)$, where $q_i \in \mathbb{R}^{d-1}$ and $r_i \in \mathbb{R}$. With this notation, and the ordering we assumed, we have that $r_i < r_j$ whenever $i < j$.

Let $Q := \{q_i : 1 \leq i \leq n\}$. Notice that this is an $n$-point set in $\mathbb{R}^{d-1}$. Denote the sets in $A_{d-1}(Q)$ as $A_1, \ldots, A_m$ (in no particular order). By Lemma 4, there exist vectors $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ such that

$$
\langle u_i, v_j \rangle = \begin{cases} 1 & q_j \in A_i \\ 0 & q_j \notin A_i \end{cases},
$$

and $\|u_i\|_2 \leq C_d(1 + \log n)^{(d-1)}$, $\|v_j\|_2 \leq 1$ for all $i$ and $j$. Define the symmetric polytope

$$
K := \{x \in \mathbb{R}^m : \langle u_i, x \rangle \leq C_d'(1 + \log n)^{(d-1)/2} \forall i \in \{1, \ldots, m\},
$$

where $C_d' > C_d$ is a constant large enough that $\mu_m(K) \geq 1 - 1/(2n)$. The fact that such a constant exists follows from standard concentration of measure results in Gaussian space. Indeed, using a Bernstein-type inequality for Gaussian measure, we can show that, for $C_d'$ big enough, $\mu_m(S_i) \geq 1 - 1/(2n^{d+1})$ for all $i \in [m]$, where $S_i := \{x : \langle u_i, x \rangle \leq C_d'(1 + \log n)^{(d-1)/2}\}$. By the union bound, since $m \leq n^d$, this implies $\mu_m(\cap_{i=1}^m S_i) \geq 1 - 1/2n$.

The body $K$ and the vectors $v_1, \ldots, v_n$ then satisfy the assumptions of Lemma 5 and, therefore, there exists an assignment of signs $\chi : [n] \to \{-1, 1\}$ such that, for any $k$, $1 \leq k \leq n$,

$$
\sum_{j=1}^k \chi(j)v_j \in 5K.
$$

Expanding the definition of $K$, we see this is equivalent to

$$
\forall i \in \{1, \ldots, m\} : \left| \sum_{j=1}^k \chi(j)\langle u_i, v_j \rangle \right| \leq 5C_d'(1 + \log n)^{(d-1)/2}.
$$

For each $i$, $1 \leq i \leq m$, let us define $A'_i = \{p_j : q_j \in A_i\}$. We claim that for any $x \in \mathbb{R}^d$, we can write $A(x) \cap P$ as $A'_i \cap \{p_1, \ldots, p_k\}$ for some $i$ and $k$. (Here we assume that $A(x) \cap P$ is non-empty: the other case is irrelevant to the proof.) To see this, let $x = (y, x_d)$, where $y \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$. Let $i$ be such that $A(y) \cap Q = A_i$, and let $k$ be the largest integer such that $r_k \leq x_d$. Then:

$$
A(x) \cap P = \{p_j : q_j \in A(y), j \leq k\} = A'_i \cap \{p_1, \ldots, p_k\}.
$$
It follows that

\[
\left| \sum_{j : p_j \in A(x)} \chi(j) \right| = \left| \sum_{j : q_j \in A, j \leq k} \chi(j) \right| = \sum_{j \leq k} \chi(j)(u_i, v_j) \leq 5C_d(1 + \log n)^{d-1/2},
\]

where the penultimate equality follows from (2), and the final inequality is (3). Since \(x\) was arbitrary, we have shown that \(\text{disc } A_d(P) \leq 5C_d(1 + \log n)^{d-1/2}\), as was required.

4 Proof of Theorem

The main ingredient in extending Theorem 1 to arbitrary polytopes is a geometric decomposition due to Matoušek. To describe the decomposition we define the \(k\)-composition \(Q_k(F)\) of sets from a (finite) set system \(F\) as follows. For \(k = 0\), \(Q_k(F) = \emptyset\); for an integer \(k > 0\), we have

\[
Q_k(F) := \{ F_1 \cup F_2 : F_1 \in Q_{k_1}(F), F_2 \in Q_{k_2}(F), F_1 \cap F_2 = \emptyset, k_1 + k_2 = k \} \cup \{ F_1 \setminus F_2 : F_1 \in Q_{k_1}(F), F_2 \in Q_{k_2}(F), F_2 \subseteq F_1, k_1 + k_2 = k \}.
\]

By an easy induction on \(k\), we see that

\[
\text{disc } Q_k(F) \leq k \text{ disc } F. \tag{4}
\]

We also extend the notion of anchored boxes to “corners” whose bounding hyperplanes are not necessarily orthogonal. Let \(W = \{w_1, \ldots, w_d\}\) be a basis of \(\mathbb{R}^d\). Then we define \(A_W := \{A_W(x) : x \in \mathbb{R}^d\}\), where \(A_W(x) = \{y \in \mathbb{R}^d : \langle w_i, y \rangle \leq \langle w_i, x \rangle \forall i \in [d]\}\).

The following lemma gives the decomposition result we need.

Lemma 6 ([Mat99]). Let \(B \subset \mathbb{R}^d\) be a convex polytope. There exists a constant \(k\) depending on \(d\) and \(B\), and \(k\) bases \(W_1, \ldots, W_k\) of \(\mathbb{R}^d\) such that every \(B' \in T_B\) belongs to \(Q_k(A_{W_1} \cup \cdots \cup A_{W_k})\). Moreover, \(e_1 \in W_1 \cap W_2 \cap \cdots \cap W_k\), where \(e_1\) is the first standard basis vector of \(\mathbb{R}^d\).

Matoušek does not state the condition after “moreover”; nevertheless, it is easy to verify this condition holds for the recursive decomposition in his proof of Lemma 6.

We will also need a bound on \(\gamma_2(A_W(P))\) for a basis \(W\) and an \(n\)-point set \(P\).

Lemma 7. For any \(d \geq 1\) there exists a constant \(C_d\) such that the following holds. For any basis \(W\) of \(\mathbb{R}^d\), and any \(n\)-point set \(P\), \(\gamma_2(A_W(P)) \leq C_d(1 + \log n)^d\).

Proof. Let \(W = w_1, \ldots, w_d\), and let \(u\) be the linear map that sends the \(i\)-th standard basis vector \(e_i\) to \(w_i\) for each \(i \in [d]\). Let \(Q = u(P) := \{u^*(p) : p \in P\},\)
where $u^*$ is the adjoint of $u$. It is easy to verify that $A_W(P) = A_d(Q)$, and, therefore,
\[
\gamma_2(A_W(P)) = \gamma_2(A_d(Q)) \leq C_d(1 + \log n)^d,
\]
where the final inequality follows from Lemma 4.

We are now ready to finish the proof of Theorem 2. As in the proof of Theorem 1, we fix the $n$-point set $P \subset \mathbb{R}^d$ once and for all, and we order $P$ as $p_1, \ldots, p_n$ in increasing order of the last coordinate. We write each $p_i$ as $p_i = (q_i, r_i)$ for $q_i \in \mathbb{R}^d$ and $r_i \in \mathbb{R}$, and define $Q := \{q_i : 1 \leq i \leq n\}$.

Let $W_1, \ldots, W_k$ be as in Lemma 6, and let $W_i' = W_i \setminus \{e_1\}$ for each $i$, $1 \leq i \leq k$. Observe that $W_i'$ is a basis of $\mathbb{R}^{d-1}$. By (1) and Lemma 7,
\[
\gamma_2(A_{W_1'}(Q) \cup \ldots \cup A_{W_k'}(Q)) \leq \sqrt{kC_d(1 + \log n)^d - 1}.
\]

By an argument using Lemma 5 analogous to the one used in the proof of Theorem 1, we can then show that there exists a constant $C_{B,d}'$ depending on $B$ and $d$ and a coloring $\chi : [n] \to \{-1, 1\}$, such that for any integer $i$, $1 \leq i \leq k$, and any $x \in \mathbb{R}^d$,
\[
\left| \sum_{j : p_j \in A_{W_i}(x)} \chi(j) \right| \leq C_{B,d}'(1 + \log n)^{d-1/2}.
\]

Here $C_{B,d}'$ is implicitly assumed to depend on $k$ as well, which depends on $B$ and $d$. This establishes that $\text{disc} F \leq C_{B,d}'(1 + \log n)^{d-1/2}$, where $F = A_{W_1}(P) \cup \ldots \cup A_{W_k}(P)$. Because, by Lemma 6 $T_B(P) \subset C_k(F)$, (4) implies
\[
\text{disc} T_B(P) \leq \text{disc} C_k(F) \leq k \text{disc} F \leq kC_{B,d}'(1 + \log n)^{d-1/2}.
\]

This finishes the proof of the theorem. The same asymptotic bound with $T_B$ replaced by $\text{POL}(H)$ for a family of hyperplanes $H$ can be proved by replacing Lemma 6 with an analogous decomposition lemma for $\text{POL}(H)$, also proved in [Mat99].

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