Abstract

We use the spinor methods of the CALKUL collaboration, as realized by Xu, Zhang and Chang, to calculate the differential cross section for $e^+e^- \rightarrow e^+e^- + 2\gamma$ for c.m.s. energies in the SLC/LEP regime. An explicit complete formula for the respective cross section is obtained. The leading log approximation is used to check the formula. Applications of the formula to high precision luminosity calculations at SLC/LEP are discussed.

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1 Introduction

Currently, an unprecedented level of precision has been reached in both theory and experiment on the cross section $\sigma_l$ of the basic luminosity process $e^+e^- \rightarrow e^+e^- + n\gamma$ for SLC/LEP, and this precision level has made possible the strongest tests to date of the $SU_{2L} \times U_1$ theory in $Z^0$ physics\[1\]. The experimental precision is currently published\[2\] as .6% whereas the theoretical precision as calculated by Jadach et al.\[3\] is currently published generator as .25% and is based on the YFS Monte Carlo event generator BHLUMI.\[4\] For a detailed illustration of how the error in the SLC/LEP luminosity enters the various standard model parameter measurements, see the paper by F. Dydak in Ref.\[2\]. Thus, 1% checks of the standard model in $Z^0$ physics are now in progress.

In the near term, the experimental error on $\sigma_l$ at LEP is expected to improve to the .15% or better regime due to imminent hardware improvements\[4\]. Accordingly, it is important to improve the theoretical precision on $\sigma_l$ in Refs.\[3\] to the .05% regime in the same near term, so that we will get our first glimpse of the .2% tests of the $SU_{2L} \times U_1$ theory in $Z^0$ physics. Accordingly, the contributions to the error $\Delta \sigma_l$ in Table 3 of the third paper in Ref.\[3\] which violate our .05% requirement for the total error must be computed with an appropriately improved precision. The largest contribution to $\Delta \sigma_l$ in that table is due to the missing second order bremsstrahlung contribution, which is itself .15%. Hence, in this paper, we compute the exact expression for the differential cross section for the respective process $e^+e^- \rightarrow e^+e^- + 2\gamma$ in the SLC/LEP energy regime, with the understanding that the $e^+e^-$ scattering angles $\theta_{ee}$ are always much larger than $2m_e/\sqrt{s}$.

Specifically, we shall employ methods originally pioneered by the CALKUL collaboration in Refs.\[5,6\]. We use these methods in the manner of Xu, Zhang and Chang in Refs.\[7\]. Indeed, the CALKUL collaboration computed the process $e^+e^- \rightarrow \gamma \rightarrow \mu^+\mu^- + 2\gamma$ in Ref.\[5\] using their original formulation of the spinor product method. In Refs.\[6\], Jadach et al. extended the CALKUL computation to include $Z^0$ exchange in $e^+e^- \rightarrow \mu^+\mu^- + 2\gamma$, thereby arriving at a formula of direct importance to high precision $Z^0$ physics for the $e^+e^- \rightarrow Z^0 \rightarrow \mu^+\mu^- + n\gamma$ process. Entirely similar results, using the methods derived from those of Xu et al., were obtained essentially simultaneously by Kleiss and van der Marck in Ref.\[9\]. Hence, what we present in this paper is the natural extension of the work of Refs.\[5,6,9\] to the double radiative Bhabha scattering process $e^+e^- \rightarrow e^+e^- + 2\gamma$.

We should emphasize that the process which we compute has applications in the $Z^0$ regime beyond just the luminosity. For, recently, there has been interest in wide angle events with two hard photons\[10\] at $\sqrt{s} \sim M_{Z^0}$ in $e^+e^- \rightarrow f\bar{f} + 2\gamma$, with $f = e, \mu, \text{etc}$. Here, we present results which are directly relevant to these\[11\].
events.

Our work is organized as follows. In Sec. 2, we review the relevant spinor
product notation and set our kinematic and notation conventions. In Sec. 3, we
analyze $e^+e^- \rightarrow e^+e^- + 2\gamma$ and present a formula for the complete differential
cross section. In Sec. 4, we compare our results with known leading logarithm
approximations and thereby determine the size of the next-to-leading term in the
second order bremsstrahlung correction to the luminosity process at SLC/LEP.
Sec. 5 contains some concluding remarks.

2 Preliminaries

In order that our analysis is self-contained, we begin it in this section by stating
our notations and conventions. This will also facilitate comparisons of our work
with related efforts in the literature.

We first note that our metric will be that of Bjorken and Drell[11]. Our
notation for Dirac gamma matrices will follow that of Ref. [8] in the so-called
chiral basis. Similarly, our conventions for the left and right handed couplings
of the $Z^0$ to the electron are those of Ref. [8] so that

$$g_L = e \cot 2\theta_W, \quad g_R = -e \tan \theta_W,$$

(1)

where $e$ is the electric charge of the electron, so that it is negative. The rest
mass of the $Z^0$ is denoted by $M_Z$, which we take[1] as 91.187 GeV. The $Z^0$ width
will be denoted by $\Gamma_Z$ and will be taken[1] as 2.492 GeV.

Our conventions for the metric, Dirac $\gamma$ matrices and $Z^0$ charges we then
complete with our spinor notation from Ref. [7]. Specifically, a massless spinor
of four-momentum $p$ and helicity $\lambda$ is denoted by

$$|p, \lambda\rangle \equiv u_{\lambda}(p) = v_{-\lambda}(p),$$

$$\langle p, \lambda| \equiv \overline{u}_{\lambda}(p) = \overline{v}_{-\lambda}(p),$$

(2)

with the normalization

$$\langle p, \lambda| q, \lambda \rangle = 2p^\mu.$$

(3)

These spinors have a number of useful properties, a representative summary
of which may be found in Ref. [7]. Here, we finalize our notational discussion
of these objects by introducing the basic unit in which our amplitudes for our
Bhabha scattering process will be expressed, namely, the spinor product. For
two massless four-vectors $p$, $q$, we define the spinor product as

$$\langle p, q \rangle_\lambda = \langle p, -\lambda| q, \lambda \rangle.$$

(4)

It is sometimes convenient to introduce the triple product

$$\langle p, k, q \rangle_\lambda = \langle p, \lambda| k, q \rangle_\lambda.$$

(5)
where \( k \) is an arbitrary four-vector which need not be massless. This product is obviously linear in \( k \), and when \( k \) is massless, it factorizes into a pair of spinor products:

\[
\langle p, k, q \rangle_\lambda = \langle p, k \rangle_\lambda \langle k, q \rangle_\lambda \quad \text{if} \quad k^2 = 0.
\]  

(6)

Using the chiral basis for Dirac matrices in Ref. [8], the light-cone notation \( p^\pm = p^0 \pm p^3 \) with the 3-axis directed along the incoming positron beam direction, and the complex perpendicular variable \( p^\perp = p^1 + ip^2 \), it can be shown that

\[
|p, +\rangle = \frac{1}{\sqrt{p^+}} \begin{pmatrix} p^+ \\ p^\perp \\ 0 \\ 0 \end{pmatrix}, \quad |p, -\rangle = \frac{1}{\sqrt{p^+}} \begin{pmatrix} 0 \\ 0 \\ -p^\perp^* \\ p^+ \end{pmatrix},
\]

(7)

via standard Dirac equation manipulations. Under the crossing transformation, we need the analytic continuation of the square roots in (7–8) to negative values of \( p^+ \). We may use

\[
|\bar{p}, \lambda\rangle = i |p, \lambda\rangle, \quad \langle \bar{p}, \lambda| = i \langle p, \lambda|.
\]

(9)

Evidently, we may now express our basic spinor products as explicit functions of their four-vector arguments. We get

\[
\langle p, q \rangle_+ = \frac{1}{\sqrt{p^+\sqrt{q^+}}} (p^+ q^+ - p^+ q^+) ,
\]

\[
\langle p, q \rangle_- = \text{sgn} p^+ \text{sgn} q^+ \langle q, p \rangle^*,
\]

(10)

where the signs in the second expression are needed to account for the analytic continuation (9). For the triple product, we get

\[
\langle p, k, q \rangle_+ = \frac{1}{\sqrt{p^+\sqrt{q^+}}} (p^+ k^- q^+ - p^+ k^+ q^- - p^+ k^+ q^+ + p^+ k^+ q^+) ,
\]

\[
\langle p, k, q \rangle_- = \langle q, k, p \rangle_+.
\]

(11)

Finally, we note our convention for the photon polarizations, which we take from Ref. [9]. For a photon of helicity \( \lambda \) and four-momentum \( k \), we define the polarization four-vector

\[
e^\sigma(k, l, \lambda) = \frac{\lambda (l, -\lambda |\gamma^\sigma[k, -\lambda])}{\sqrt{2 \langle l, -\lambda |k, \lambda \rangle}},
\]

(12)

where \( l \) is a reference momentum such that \( l^2 = k^2 = 0 \).

This completes the notation needed for our analysis. We turn now to computation of the amplitudes of interest to us in the next section.
In this section, we give exact expressions for all $O(\alpha^2)$ terms in the amplitude $e^+e^- \rightarrow e^+e^- + 2\gamma$, in the limit where the electron mass is negligible. The spinor conventions described in Sec. 2 allows these amplitudes to be expressed in a very compact form. The case of initial state radiation in the $s$ channel has already been obtained using these methods by Kleiss and van der Marck [9]. The complete $s$ channel result, with muons in the final state, has also been obtained by Jadach et al. [8], but in a much less compact form.

The following kinematic conventions will be used. The momenta of the incoming and outgoing electron will be denoted $p$ and $q$, while the corresponding helicities will be denoted $\lambda$ and $\mu$, respectively. The positron variables will be the same, but with a prime. The photon momenta and helicities will be denoted $k_i$ and $\rho_i$, respectively.

The total amplitude may be written as a sum $M = M_s + M_t$ of $s$ and $t$ channel contributions, with

$$M_s = M_{pp} + M_{ee} + M_{pe},$$

$$M_t = M_{ii} + M_{ff} + M_{ifi}.$$  

(13)

Here, the subscripts $p$, $e$, $i$ and $f$ indicate that a photon is emitted from the positron line, electron line, an initial state line, and a final state line, respectively. Helicity conservation plays an important role in simplifying the individual terms, as it is explained in Refs. [7, 9].

We begin with the $t$ channel. Helicity conservation requires $\lambda = \mu$, $\lambda' = \mu'$ in all nonzero amplitudes. It is convenient to express these amplitudes in terms of helicity-dependent momentum variables

$$h_i = \begin{cases} p \\ q \end{cases} \text{ if } \rho_i = \pm \lambda, \quad h_i' = \begin{cases} p' \\ q' \end{cases} \text{ if } \rho_i = \pm \lambda',$$

(14)

and to define the $t$-variables

$$t = (p - q)^2, \quad t' = (p' - q')^2, \quad t_i = (p - q - k_i)^2.$$  

(15)

When the photon helicities are identical, i.e. $\rho_1 = \rho_2 \equiv \rho$, the three terms in the $t$ channel amplitude are given by

$$M_{pp} = 4ie^4 G_{\lambda,-\lambda'}(t) \frac{\langle h, h' \rangle^2_{\rho} \langle p', q' \rangle_{\rho} \langle p, q \rangle_{-\rho}}{\langle k_1, q' \rangle_{\rho} \langle k_1, p' \rangle_{\rho} \langle k_1, k_2 \rangle_{\rho}},$$

(16)

$$M_{ee} = 4ie^4 G_{\lambda,-\lambda'}(t') \frac{\langle h, h' \rangle^2_{\rho} \langle q, p \rangle_{\rho} \langle q', p' \rangle_{-\rho}}{\langle k_1, p \rangle_{\rho} \langle k_1, q \rangle_{\rho} \langle k_1, k_2 \rangle_{\rho}},$$

(17)

$$M_{pe} = -4ie^4 \langle h, h' \rangle_{\rho}^2 \left\{ \frac{t_1 G_{\lambda,-\lambda}(t_1)}{\langle p, k_1, q \rangle_{\rho} \langle k_1, k_2, q' \rangle_{\rho}} + (1 \leftrightarrow 2) \right\}.$$  

(18)
The propagator factor for photon and \( Z^0 \) exchange is defined by

\[
G_{\lambda,\mu}(z) = \frac{1}{z} + \frac{[(1 - \lambda) - 4 \sin^2 \theta_W] [(1 - \mu) - 4 \sin^2 \theta_W]}{4 \sin^2 2\theta_W \left[ z (1 + i \Gamma_Z/M_Z) - M_Z^2 \right]}.
\]

The \( Z^0 \) width \( \Gamma_Z \) is to be omitted in the \( t \) channel, but is present in the \( s \) channel expressions below.

The same amplitudes for opposite photon helicities are given by the more complicated expressions

\[
\mathcal{M}_{pp} = \frac{4ie^4 \delta_{\rho,\lambda} \lambda'}{\langle k_i, q', k_j \rangle \langle k_i, p', k_j \rangle} \left\{ \langle h_i, (q' + k_j), p' \rangle_{\lambda',\lambda} \langle (p' - k_i), h_j \rangle_{\lambda'} + \Delta^{-1} \langle q', k_j \rangle_{\lambda} \langle h_i, (q' + k_j), k_i \rangle_{\lambda'} \langle (p' - k_i), h_j \rangle_{\lambda'} + \Delta' \langle p', k_j \rangle_{\lambda} \langle q', k_i \rangle_{\lambda'} \langle h_i, (q' + k_j), k_j \rangle_{\lambda'} \langle (p' - k_i), h_j \rangle_{\lambda'} \right\},
\]

\[
\mathcal{M}_{ee} = \frac{4ie^4 \delta_{\rho,\lambda} \lambda'}{\langle k_j, p, k_i \rangle \langle k_j, q, k_i \rangle} \left\{ \langle h_j', (q + k_i), p \rangle_{\lambda,\lambda'} \langle (p - k_j), h_i \rangle_{\lambda} + \Delta \langle q, k_i \rangle_{\lambda} \langle h_j', (q + k_i), k_i \rangle_{\lambda'} \langle (p - k_j), h_i \rangle_{\lambda} + \Delta' \langle p, h_i \rangle_{\lambda} \langle q, k_i \rangle_{\lambda'} \langle h_j', (q + k_i), k_j \rangle_{\lambda'} \langle (p - k_j), h_i \rangle_{\lambda'} \right\},
\]

\[
\mathcal{M}_{pc} = 4ie^4 \left\{ G_{\lambda,\lambda'}(t_1) \frac{\langle h_{2}', (\lambda h_2 + \rho_1 k_1), h_1 \rangle_{\rho_1}^2}{\langle p, k_1, q \rangle_{\rho_1} \langle p', k_2, q' \rangle_{\rho_2}} + (1 \leftrightarrow 2) \right\}.
\]

The indices \((i, j) = (1, 2)\) or \((2, 1)\) are chosen so that \((20, 21)\) are nonzero, and the denominators are defined by

\[
\Delta = (p - k_1 - k_2)^2, \quad \Delta' = (p' - k_1 - k_2)^2, \quad \Delta = (q + k_1 + k_2)^2, \quad \Delta' = (q' + k_1 + k_2)^2.
\]

The \( s \) channel results are analogous. In this case, helicity conservation requires \( \lambda' = -\lambda \) and \( \mu' = -\mu \) for nonzero amplitudes. We define the helicity-dependent momentum variables

\[
l_i = \left\{ \begin{array}{ll} p & \text{if } \rho_i = \pm \lambda, \\ q & \text{if } \rho_i = \mp \mu, \end{array} \right. \quad \widehat{l}_i = \left\{ \begin{array}{ll} q & \text{if } \rho_i = \pm \lambda, \\ p & \text{if } \rho_i = \mp \mu, \end{array} \right.
\]

and the \( s \) variables

\[
s = (p + p')^2, \quad \widehat{s} = (q + q')^2, \quad s_i = (p + p' - k_i)^2.
\]

The three terms in the \( s \) channel amplitude are given by

\[
\mathcal{M}_{ff} = 4ie^4 \lambda \mu G_{\lambda,\mu}(s) \frac{\langle l, \widehat{l} \rangle_{\rho}^2 \langle q, q' \rangle_{\rho} \langle p, p' \rangle_{-\rho}}{\langle k_1, q, k_2 \rangle_{\rho} \langle k_1, q', k_2 \rangle_{-\rho}}.
\]
\[ M_{ii} = 4ie^4 \lambda \mu G_{\lambda, \mu}(s) \frac{\langle \hat{l}, \hat{l}' \rangle^2 \langle p, p' \rangle_\rho \langle q, q' \rangle_\rho}{\langle k_1, p, k_2 \rangle_\rho \langle k_1, p', k_2 \rangle_\rho} \]  
(27)

\[ M_{if} = 4ie^4 \lambda \mu \langle \hat{l}, \hat{l}' \rangle^2 \rho \left\{ \frac{s_1 G_{\lambda, \mu}(s_1)}{\langle p, k_1, p' \rangle_\rho \langle q, k_2, q' \rangle_\rho} + (1 \leftrightarrow 2) \right\} \]  
(28)

for equal photon helicities, and by

\[ M_{ff} = \frac{-4ie^4 \delta_{\rho, \mu}}{\langle k_j, q, k_i \rangle_\mu \langle k_j, q, k_i \rangle_\mu} \left\{ \langle q, (q' + k_j), l_i \rangle_\mu \langle l_j, (q + k_i), q' \rangle_\mu \right. \]
\[ + \Delta^{-1} \langle q', k_j \rangle_\mu \langle q, l_j \rangle_{-\mu} \langle k_i, (q' + k_j), l_i \rangle_\mu \langle k_j, q, k_i \rangle_\mu \]
\[ + \Delta^{-1} \langle q, k_i \rangle_{-\mu} \langle q', l_i \rangle_\mu \langle l_j, (q + k_i), k_j \rangle_\mu \langle k_j, q', k_i \rangle_\mu \}, \]  
(29)

\[ M_{ii} = \frac{-4ie^4 \delta_{\rho, \lambda}}{\langle k_j, p, k_i \rangle_\lambda \langle k_j, p', k_i \rangle_\lambda} \left\{ \langle \hat{l}_j, (p' - k_i), p \rangle_\lambda \langle p', (p - k_j), \hat{l}_i \rangle_\lambda \right. \]
\[ + \Delta^{-1} \langle p, k_j \rangle_\lambda \langle p', \hat{l}_j \rangle_{-\lambda} \langle k_i, (p - k_j), \hat{l}_i \rangle_\lambda \langle k_j, p', k_i \rangle_\lambda \]
\[ + \Delta^{-1} \langle p, \hat{l}_i \rangle_\lambda \langle p', k_i \rangle_{-\lambda} \langle \hat{l}_j, (p' - k_i), k_j \rangle_\lambda \langle k_j, p, k_i \rangle_\lambda \}, \]
\[ M_{if} = \left\{ 4ie^4 \lambda \mu \left\{ G_{\lambda, \mu}(s_1) \frac{\langle \hat{l}_2, (l_2 - k_1), l_1 \rangle^2_{\rho_1}}{\langle p, k_1, p' \rangle_{\rho_1} \langle q, k_2, q' \rangle_{\rho_2}} + (1 \leftrightarrow 2) \right\} \right\} \]  
(30)

for opposite photon helicities.

The amplitudes \( M \) above are related by crossing symmetries. Interchanging the incoming positron and outgoing electron, so that

\[ p' \leftrightarrow -q, \quad \lambda' \leftrightarrow -\mu, \]  
(31)

interchanges the \( s \) and \( t \) channel amplitudes

\[ M_{pp} \leftrightarrow M_{ff}, \quad M_{ee} \leftrightarrow M_{ii}, \quad M_{pe} \leftrightarrow M_{if}, \]  
(32)

while interchanging the incoming electron and outgoing positron, so that

\[ p \leftrightarrow -q', \quad \lambda \leftrightarrow -\mu', \]  
(33)

gives

\[ M_{pp} \leftrightarrow M_{ii}, \quad M_{ee} \leftrightarrow M_{ff}, \quad M_{pe} \leftrightarrow M_{if}. \]  
(34)

These expressions are useful in practice, since, together, they allow all of the amplitudes to be obtained from only four expressions, for example, (16), (18), (20), and (22). The form of the expression depends only on whether the photon
helicities are equal or opposite, and whether they are emitted from the same or different fermion lines.

This completes our derivation of the exact second order matrix element for the process $e^+e^- \rightarrow e^+e^- + 2\gamma$. We turn now to some of its applications. This we do in the next section.

4 Sample Results

In this section we illustrate our exact result in the context of its main purpose, which is to check the non-leading bremsstrahlung correction for two hard photons in the low angle regime of Bhabha scattering in our SLC/LEP luminosity calculations in Ref. [3]. Thus, we will compare our exact results with the leading log expectations in the low angle Bhabha scattering luminosity regime. We begin with a discussion of the differential cross section for associated with our exact result.

Using entirely standard manipulations [11], we get the following expression for the exact differential cross section for $e^+e^- \rightarrow e^+e^- + 2\gamma$ in the $Z^0$ resonance

$$d\sigma = \frac{1}{2!} \delta^4(p + p' - q - q' - k_1 - k_2) \frac{|\mathcal{M}|^2}{(2\pi)^8 s} \frac{d^3q d^3q' d^3k_1 d^3k_2}{24 q^0 q'^0 k_1^0 k_2^0},$$

(35)

where the averaged and summed squared matrix element may be expressed as a sum over helicities

$$|\mathcal{M}|^2 \equiv \frac{1}{4} \sum_{\lambda, \lambda', \mu, \mu', \rho_1, \rho_2 = \pm 1} |\mathcal{M}(\lambda, \lambda', \mu, \mu', \rho_1, \rho_2)|^2,$$

(36)

using the total exact amplitude derived in the previous section. We will compare the exact cross section (35) with the leading log expectations. This comparison requires a leading log approximation to the exact cross section, namely

$$d\sigma_{LL} = \frac{1}{2!} \delta^4(p + p' - q - q' - k_1 - k_2) \frac{|\mathcal{M}_{LL}|^2}{(2\pi)^8 s} \frac{d^3q d^3q' d^3k_1 d^3k_2}{24 q^0 q'^0 k_1^0 k_2^0},$$

(37)

where the leading log summed squared matrix element is given by (38) below. It is to the derivation of the latter equation that we now turn.

For our leading log representation of $|\mathcal{M}_{LL}|^2$, we follow the development given by Jadach and Ward in Ref. [12]. Specifically, we generalize the initial state $s$-channel double bremsstrahlung leading log differential cross section in Ref. [12] to include analogous contributions from the final state and all other channels containing collinear singularities. This leads to an expression for the leading log version of the squared matrix element (38) given as a sum of six
The pure Born amplitude with no photons is

$$|\mathcal{M}|^2_{LL} = |\mathcal{L}_i + \mathcal{L}_f + \mathcal{L}_p + \mathcal{L}_m - \mathcal{L}_{m'}|.$$  (38)

The individual terms in (38) are given by

$$\mathcal{L}_i = \frac{2 e^8}{s^2 D(p, p')} \left[f(p, p') |\mathcal{M}_B(\bar{s}, \bar{t}')|^2 + f(p', p') |\mathcal{M}_B(\bar{s}, t'_i)^2\right],$$

$$\mathcal{L}_f = \frac{2 e^8}{s^2 D(q, q')} \left[f(-q', -q) |\mathcal{M}_B(s, t_f)|^2 + f(-q, -q') |\mathcal{M}_B(s, t'_f)|^2\right],$$

$$\mathcal{L}_p = \frac{2 e^8}{\tilde{u}^2 D(p', q')} \left[f(-q', p') |\mathcal{M}_B(s_p, t)|^2 + f(p', -q') |\mathcal{M}_B(s_p, t)|^2\right],$$

$$\mathcal{L}_c = \frac{2 e^8}{\tilde{u}^2 D(p, q')} \left[f(p, -q) |\mathcal{M}_B(\bar{s}_c, t')|^2 + f(-q, p) |\mathcal{M}_B(\bar{s}_c, t')|^2\right],$$

$$\mathcal{L}_m = \frac{2 e^8}{\tilde{u} u' D(p, q')} \left[f(-q', p) |\mathcal{M}_B(s_m, t_m)|^2 + f(p, -q') |\mathcal{M}_B(s_m, t_m)|^2\right],$$

$$\mathcal{L}_{m'} = \frac{2 e^8}{\tilde{u} u' D(p', q')} \left[f(p', -q) |\mathcal{M}_B(s_{m'}, t_{m'})|^2 + f(-q, p') |\mathcal{M}_B(s_{m'}, t_{m'})|^2\right]$$  (39)

where the notation is defined as follows. The basic s, t, u invariants are defined by (13), (25) and

$$u = (p - q')^2, \quad u' = (p' - q)^2.$$  (40)

The pure Born amplitude with no photons is $\mathcal{M}_B(s_B, t_B)$, where the effective Born parameters $s_B, t_B$ are as shown, with

$$t_i = \frac{-\bar{s} t}{t + u}, \quad t'_i = \frac{-\bar{s} t'}{t' + u'}, \quad t_f = \frac{-s t}{t + u'}, \quad t'_f = \frac{-s t'}{t' + u},$$

$$s_p = \frac{-\bar{s} t}{s + u'}, \quad \tilde{s}_p = \frac{-\bar{s} t'}{s + u}, \quad s_c = \frac{-s t'}{s + u}, \quad \tilde{s}_c = \frac{-\bar{s} t}{s + u},$$

$$s_m = \frac{-s u'}{s + t'}, \quad \tilde{s}_m = \frac{-\bar{s} u'}{s + t}, \quad t_m = \frac{-t u'}{s + t}, \quad t'_m = \frac{-t' u'}{s + t'},$$

$$s_{m'} = \frac{-s u}{s + t}, \quad \tilde{s}_{m'} = \frac{-\bar{s} u}{s + t'}, \quad t_{m'} = \frac{-t u}{s + t}, \quad t'_{m'} = \frac{-t' u}{s + t'}.$$  (41)

The denominator factor $D(p_1, p_2)$ is defined to be

$$D(p_1, p_2) = (p_1 \cdot k_1)(p_1 \cdot k_2)(p_2 \cdot k_1)(p_2 \cdot k_2)(p_1 \cdot p_2)^{-4}.$$  (45)

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and if the indices \((i, j) = (1, 2)\) or \((2, 1)\) are chosen so that \(p_1 \cdot k_i + p_2 \cdot k_i \geq p_1 \cdot k_j + p_2 \cdot k_j\), the form factors \(f(p_1, p_2)\) are given by

\[
    f(p_1, p_2) = Y \left( \frac{p_1 \cdot k_i}{p_1 \cdot p_2}, \frac{p_1 \cdot k_j}{p_1 \cdot (p_2 - k_i)}, \frac{p_2 \cdot k_j}{p_2 \cdot (p_1 - k_i)} \right) + Y \left( \frac{p_1 \cdot k_i}{p_1 \cdot (p_2 - k_j)}, \frac{p_1 \cdot k_j}{p_1 \cdot p_2}, \frac{p_2 \cdot k_j}{p_2 \cdot (p_1 - k_i)} \right),
\]

where \(Y(x, u, v) = (1 - x)^2 \left[ (1 - u)^2 + (1 - v)^2 \right]\).

We should note that, unlike the original LL expression in Ref. [12], the result (37) does not control the next-to-leading-log corrections. We will now discuss the comparison of exact and leading log differential cross sections (35) and (37), respectively, in the SLC/LEP luminosity regime.

We have made a detailed comparison of the exact and leading log versions of the cross section (35) in the luminosity regime. We illustrate these comparisons in Figs. 1 and 2. In Fig. 1, the two photons are emitted near the incoming \(e^+\) line in direction, with angles as given in the figure whereas in Fig. 2, one photon is emitted along the direction of each incoming charged particle, with the respective angles as given in the figure. The final particles in the events associated with Figs. 1 and 2 are all taken to fall within the typical LEP-type trigger region as described in Ref. [3] and as we indicate in the figures. What we see from Figs. 1 and 2, cases 1 and 2, is that the LL result is always within 20% of the exact result in the relevant region of the phase space. We have verified that this is true throughout the wide range of the kinematic variables in the respective region of phase space. This indicates that the error estimate for the next-to-leading (NLL) \(\mathcal{O}(\alpha^2)\) double bremsstrahlung effects not included in BHLUMI 2.01 in Ref. [3] is rather conservative. The case 3 in Figs. 1 and 2 is relevant for next-to-next-to-leading (NNLL) \(\mathcal{O}(\alpha^2)\) corrections which are at the level of \(10^{-4}\) of the integrated cross section. The present calculation opens a path to implementation of the NLL \(\mathcal{O}(\alpha^2)\) double hard bremsstrahlung effect into BHLUMI in order to move it closer to the desired .05% precision tag needed to support the .15% experimental errors expected in the near term high precision \(Z^0\) physics program at LEP. This implementation will appear elsewhere [13].

The last curves in the figures show the size of the big logarithm \(L = \ln(|t|/m^2) - 1\) in the respective regions of the phase space. We see that \(L\) varies significantly and, hence, that giving it the proper argument \(|t|\) instead of \(s\) in the low angle regime suppresses possible next-to-leading-log corrections. This then is consistent with the implementation of YFS exponentiation in BHLUMI , where two of us (S. J. and B. F. L. W.) have also found that the proper argument for \(L\) in the respective radiation probability is \(|t|\).
We should also point out that our exact $2\gamma$ emission results in low angle Bhabha scattering are directly relevant to the QED expectations for high mass $2\gamma$ states in wide angle Bhabha scattering at $Z^0$ energies. Thus, recent interest in such events warrants a detailed assessment of such phenomena using our results in this paper. This assessment will appear elsewhere. Our main objective in this paper is to present the exact results for $e^+e^- \rightarrow e^+e^- + 2\gamma$ at order $\alpha^2$ and to assess the error estimate of the respective missing part of the order $\alpha^2$ bremsstrahlung effect in BHLUMI as it is given in Ref. [3].

5 Conclusions

In this paper we have used the methods of the CALKUL collaboration to compute the important process of two photon bremsstrahlung in Bhabha scattering in the $Z^0$ resonance region at $O(\alpha^2)$. We have compared our exact results for the corresponding differential cross section with leading log (LL) expectations as a check of the results in Ref. [3] on the total precision of the Monte Carlo program BHLUMI2.01 [4].

Specifically, we have presented new formulas for the process $e^+e^- \rightarrow e^+e^- + 2\gamma$ at $O(\alpha^2)$ in the $Z^0$ resonance region. We have compared our exact results with the LL expectations for the respective differential cross section and we find that the two differential cross sections are within 20% of each other throughout the relevant region of the phase space. This indicates that the estimate in Ref. [3] for the contribution of the missing NLL $O(\alpha^2)$ bremsstrahlung in BHLUMI2.01 [4] to its total precision of 25% is rather conservative. The implementation of this missing part of the $O(\alpha^2)$ bremsstrahlung into BHLUMI and will be reported elsewhere [13].

We emphasize that our new formulas are valid both at low and wide electron, positron scattering angles as defined by the typical LEP/SLC trigger in the luminosity regime, i.e., low refers to scattering angles in the luminosity regime and wide refers to such angles larger than the trigger angles. This means that our results have an immediate application to the recent discussion of large two-photon mass wide angle lepton pair production events in Ref. [10]. Such applications will appear elsewhere [13].
In summary, the exact results which we have presented for the two photon bremsstrahlung process in Bhabha scattering in the $Z^0$ resonance region allow us to make an important step in reducing the error on the theoretical prediction for the LEP/SLC luminosity process to below .1% regime. We look forward to the attendant refinement in the precision of the respective Standard Model tests in $Z^0$ physics.

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Figure captions

Fig. 1. Comparison of LL and exact results for low angle Bhabha scattering in the SLC/LEP luminosity regime. The functions $F$ are the normalized differential cross-sections defined in Ref. [8]. We parametrize 8-dimensional $e^+e^-\gamma\gamma$ phase space in terms of seven angles and the energy of one photon $E_1$. We plot dependence on photon energy $E_1$ for fixed values of angles which are listed in the following table. Here, in the table, the subscripts 1 and 2 refer to the photons and where the photon angles are relative to the incoming positron direction so that, in this figure, both photons are emitted along the incoming positron direction:

| case | $\theta_1$ | $\phi_1$ | $\theta_2$ | $\phi_2$ | $\theta_{e^+}$ | $\phi_{e^+}$ | $\theta_{e^-}$ |
|------|------------|-------|-----------|-------|----------------|-------|--------|
| 1    | $6.4 \times 10^{-3}$ | 0°   | $6.4 \times 10^{-3}$ | 180° | 4.9°           | 0°    | 2.7°    |
| 2    | $8 \times 10^{-2}$   | 0°   | $8 \times 10^{-2}$  | 180° | 4.9°           | 0°    | 2.7°    |
| 3    | 1°          | 0°   | 1°            | 180° | 4.9°           | 0°    | 2.7°    |

Note that the energy of the outgoing charged particles is required to remain above 0.5 of the beam energy $E_B$. For the three cases 1, 2, 3 of kinematics shown in the table, the plots show respectively the individual exact and LL cross sections, their ratio. The cases 1 and 2 are relevant to NLL correction while case 3 covers NNLL region of the phase space. The size of the dominant big logarithm $\ln(|t|/m_e^2) - 1$ which determines the probability to radiate in the process is also plotted.

Fig. 2. The same plots as those given in Fig. 1 with photon 1’s spherical angles measured relative to the incoming positron direction and with photon 2’s angles measured relative to the incoming electron direction. Thus, the two photons are emitted in generally opposite directions. The specific kinematic input is summarized in the following table:

| case | $\theta_1$ | $\phi_1$ | $\theta_2$ | $\phi_2$ | $\theta_{e^+}$ | $\phi_{e^+}$ | $\theta_{e^-}$ |
|------|------------|-------|-----------|-------|----------------|-------|--------|
| 1    | $6.4 \times 10^{-3}$ | 0°   | $6.4 \times 10^{-3}$ | 180° | 2.7°           | 0°    | 2.7°    |
| 2    | $8 \times 10^{-2}$   | 0°   | $8 \times 10^{-2}$  | 180° | 2.7°           | 0°    | 2.7°    |
| 3    | 1°          | 0°   | 1°            | 180° | 2.7°           | 0°    | 2.7°    |

The cut on the charged particles’ outgoing energies is the same as in Fig. 1.
Figure 1
Exact vs. Leading Log Cross Sections

Ratio of Leading Log to Exact Cross Section

Logarithm value $L = \log(|t|/m^2) - 1$

Figure 2