Vacuum solutions which cannot be written in diagonal form*  

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Abstract 

The vacuum solution

$$ds^2 = dx^2 + x^2 dy^2 + 2 dz dt + \ln x dt^2$$

of the Einstein gravitational field equation follows from the general ansatz

$$ds^2 = dx^2 + g_{\alpha\beta}(x) dx^\alpha dx^\beta$$

but fails to follow from it if the symmetric matrix $g_{\alpha\beta}(x)$ is assumed to be in diagonal form.

KEY: Vacuum solution, Einstein field equation, symmetries, diagonalization

1 Introduction

The folklore reading “Every symmetric matrix can be brought into diagonal form by a suitable rotation” is strictly valid in the positive definite case only.

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In the Lorentz signature case, however, one needs additional assumptions to
get this result. It is generally believed that these assumptions do not repre-
sent a real restriction, and this is justified, e.g., for the energy-momentum
tensor: all physically sensible form of matter can be written with an energy-
momentum tensor in diagonal form.

It is the purpose of the present paper to show, that, nevertheless, impor-
tant examples exist where this folklore-statement leads to incorrect results.
Of course, some examples of this kind already exist. Most notably, the Kerr
metric, and all its generalizations, cannot be brought into diagonal form in
holonomic coordinates due to the fact that its timelike Killing field fails to
be hypersurface-orthogonal. However, there is a widespread feeling that, in
the case of the metric depending only on one coordinate, this diagonalization
can always be achieved. Here we give a class of solutions where this rough
diagonalization is not possible. This is essentially important if one wishes to
find all solutions of a prescribed symmetry type. In this respect, this is a
continuation of [6] and [7].

The paper is organized as follows: Section 2 presents the deduction of the
Bianchi type I vacuum solutions of Einstein’s gravitational field equations in
such a detailed form that it becomes clear, why the Kasner solution [8],
here cited from [5],\(^1\) is really possible in diagonal form.\(^2\) Section 3 gives the
analogous calculation as section 2, but now with a changed signature. In
the final section 4 we discuss the case of a metric with only one off-diagonal
term. We will show that the vacuum Einstein equations for this kind of
metric, that is a system of four nonlinear differential equations with four
unknown functions, can always be reduced to a system two equations with

\(^1\)According to [5], the Kasner solution published in 1921 was already known to Levi-
Civita in 1917, cf. also [4].

\(^2\)To the question when symmetric matrices cannot be diagonalized see [2].
two unknown functions. Moreover, in some cases we will be able to further
duce the system to only one equation in one unknown function. We will
also show one explicit solution of the system, that is a vacuum space-time
which cannot be diagonalized.

2 The Kasner solution

The general metric for a Bianchi type I model reads

\[ ds^2 = -dt^2 + g_{\alpha\beta}(t) \, dx^\alpha dx^\beta \]  

(1)

where \( g_{\alpha\beta}(x) \, dx^\alpha dx^\beta \) is the positive definite spatial metric. We want to find
out all vacuum solutions of the Einstein field equation of the form of met-
ric (1). The result reads, cf. section 11.3. of [5]:

\[ ds^2 = -dt^2 + t^2p \, dx^2 + t^2q \, dy^2 + t^2r \, dz^2 \]  

(2)

with

\[ p + q + r = p^2 + q^2 + r^2 \in \{0, 1\} \]  

(3)

Proof: This is a well-posed Cauchy problem. We take \([t = 0]\) as initial
space-like hypersurface. It is a 3-flat space. Therefore, we can take without
loss of generality

\[ g_{\alpha\beta}(0) = \delta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]  

(4)

otherwise a coordinate transformation involving only the 3 spatial coor-
dinates \( x^1 = x, \, x^2 = y, \, x^3 = z \) suffices to reach that.

\footnote{It is a fully standard proof, but here we need the details to see the distinction to the
other cases to be discussed below.}
The second fundamental tensor at \( [t = 0] \) is \( k_{\alpha\beta}(0) \), where

\[
k_{\alpha\beta} = \frac{d}{dt} g_{\alpha\beta}
\]

represents a symmetric matrix. The remaining freedom of spatial coordinate transformations keeping valid the equation (4) is just the orthogonal group \( O(3) \), see Appendix A for the proof that the 3 parameters of \( O(3) \) suffice to reach

\[
k_{12}(0) = k_{13}(0) = k_{23}(0) = 0.
\]

Therefore, due to the compactness of the rotation group we can assume without loss of generality that the second fundamental tensor at \( [t = 0] \) has diagonal form.

The vacuum Einstein field equations are equivalent to \( R_{ij} = 0 \), where \( R_{ij} \) represents the Ricci tensor, and the index of the coordinates \( x^i \) covers the values \( i = 0, 1, 2, 3, \) and \( t = x^0 \). Then it turns out that the 3 equations

\[
R_{12} = R_{13} = R_{23} = 0
\]

suffice to maintain the diagonal form of the metric \( g_{\alpha\beta}(t) \) for all times.

Up to now we have shown that all vacuum metrics of the form (1) can be written as

\[
ds^2 = -dt^2 + e^{2\alpha(t)} dx^2 + e^{2\beta(t)} dy^2 + e^{2\gamma(t)} dz^2.
\]

Inserting this metric into \( R_{ij} = 0 \) it turns out that up to trivial rescalings, the one-parameter set of solutions defined by eqs. (2) (3) cover the set of all solutions.

\[4\] In fact, we have proven even more: already the connected component of the unity element of \( O(3) \), namely the \( SO(3) \), is enough to get that result, but we do not need this additional property here.
In the final step one observes, that $-dt^2 + dx^2$ and $-dt^2 + t^2 dx^2$ are both locally flat, and therefore, one may omit the case $p = q = r = 0$ from eq. (3) without loosing any solutions.

Result: Every cosmological Bianchi type I solution of the Einstein vacuum field equations, i.e., every solution of the form (1) can be written as (2) with

$$p + q + r = p^2 + q^2 + r^2 = 1.$$  \hspace{1cm} (9)

3 The signature changed Kasner solution

In this section we want to deduce the consequences of another signature in the metric. First of all, one would be tempted to go just the same way as before: Looking at eqs. (2, 3), one can perform an imaginary rotation of $x$ and $y$. After rewriting $ds^2$ as $-ds^2$ and renaming the coordinates one gets:

$$ds^2 = dx^2 + x^2 r dy^2 + x^2 t dz^2 - x^2 p dt^2$$ \hspace{1cm} (10)

It holds: If eq. (9) is fulfilled, then metric (10) represents a vacuum solution of the Einstein field equations. Contrary to the positive definite case, a permutation between $p$, $q$, and $r$ is no more generally possible. It remains only the permutation between $q$ and $r$. So, the set of solutions in metric (10) can be parametrized by eqs. (10, 11, 12) with $0 \leq \phi \leq \pi$.

We will now carefully look for the question whether these solutions represent all solutions of the form of metric (1).

3.1 The diagonal ansatz

The diagonal ansatz analogous to eq. (5) reads

$$ds^2 = dx^2 + e^{2\gamma(x)} dy^2 + e^{2\beta(x)} dz^2 - e^{2\alpha(x)} dt^2.$$ \hspace{1cm} (11)

\footnote{A geometric parametrization of this set is given in Appendix B.}
Inserting this metric into the equation $R_{ij} = 0$, one gets as expected, again just the known solutions \([10]\) with \([9]\).

### 3.2 The non-diagonal ansatz

We are now only interested to show that truly non-diagonal metrics really exist, and we do not intend to exhaust all of them in the present paper. Therefore, we restrict to those metrics, where only one off-diagonal element of $g_{\alpha\beta}(x)$ is different from zero.

An off-diagonal component between two space-like directions can be made vanish by the procedure shown in Appendix A. So, this essential off-diagonal component must exist between one space-like and one time-like direction. This leads us to the following ansatz for the metric:

$$ ds^2 = dx^2 + A(x)dy^2 + g_{mn}(x)dx^m dx^n $$

(12)

where $A(x) > 0$ and $g_{mn}(x)$ is negative definite. We count the coordinates $x^3 = z$ and $x^4 = t$, so the indices $m, n$ run from 3 to 4. We write the 2-dimensional metric

$$ g_{mn}(x) = \begin{pmatrix} B(x) & P(x) \\ P(x) & -C(x) \end{pmatrix} $$

(13)

and use the abbreviation: $- \det g_{mn} = \Gamma = P^2 + BC$. The conditions for the negative definiteness are:

$$ A > 0 \quad \Gamma > 0. $$

The inverse reads

$$ g^{mn}(x) = -\frac{1}{\Gamma} \begin{pmatrix} -C(x) & P(x) \\ P(x) & -B(x) \end{pmatrix}. $$

(14)
4 The Einstein equations

Since we are dealing with the vacuum case, the Einstein equations reduce to

\[ R_{ij} = 0, \]

where \( R_{ij} \) is the Ricci tensor of the metric (12). It is well known that, due to the Bianchi identities, not all the Einstein equations are independent. In this case it is convenient to take, as our basic equations:

\[ R_{yy} = 0, \quad R_{zz} = 0, \quad R_{tt} = 0, \quad R_{tz} = 0. \]

It is a straightforward calculation to compute the Ricci components. The explicit expressions read:

\[ R_{yy} = 0 \Rightarrow -2 \Gamma A \dot{A} + \Gamma \left( \frac{\dot{A}}{A} \right)^2 - A \ddot{A} = 0, \quad (15) \]

\[ R_{zz} = 0 \Rightarrow 2 \Gamma \dot{B} + 2B \left( \dot{\rho} \right)^2 - \dot{B} \left( 2P \dot{\rho} + C \dot{B} - B \dot{C} - \Gamma \frac{\dot{A}}{A} \right) = 0, \quad (16) \]

\[ R_{tt} = 0 \Rightarrow 2 \Gamma \dot{C} + 2C \left( \dot{\rho} \right)^2 - \dot{C} \left( 2P \dot{\rho} + B \dot{C} - C \dot{B} - \Gamma \frac{\dot{A}}{A} \right) = 0, \quad (17) \]

\[ R_{tz} = 0 \Rightarrow 2 \Gamma \dot{\rho} + 2P \dot{B} \dot{C} - \dot{\rho} \left( B \dot{C} + C \dot{B} - \Gamma \frac{\dot{A}}{A} \right) = 0, \quad (18) \]

where \( \dot{f} = \frac{df}{dx} \). First of all, let us notice that eq. (15) can be explicitly solved for \( A \):

\[ 2A^\frac{\dot{A}}{A} = \kappa^2 \int_0^x \frac{dx'}{\sqrt{1 + \frac{I_1}{\Gamma A}}} + I_1 \Rightarrow \]

\[ \Rightarrow \frac{\kappa}{A} = \Gamma \frac{\dot{A}}{A} \quad (20) \]

where \( I_1 \) and \( \kappa \) are integration constants. Now, since \( A \) is expressed in terms of \( \Gamma \), it is clear that, thanks to the identity (20), we are left with a system of three equations in the three unknown functions \( B, C \) and \( P \). This system
looks highly nontrivial due to the nonlinearities. Nevertheless, it is possible to further reduce it by rewriting the equations (16), (17) and (18) in a more symmetric way. It is important to stress here that, thanks to (16) and (17), $P^2$ is a symmetric function in the exchange of $B$ and $C$. Hence, $P$ can be either symmetric or antisymmetric in the exchange of $B$ and $C$. In the following $B$, $C$ and $P$ will be supposed to be different from zero. In fact, if $P$ is zero one obtains a Kasner-like solution, while if $B$, or $C$, is zero then it is always possible to make a coordinate transformation in such a way that $B \neq 0$, the same holds for $C$. Now, it is easy to show that the system of equations (16), (17) and (18) is equivalent to the following system:

$$2\Gamma \frac{\dot{Y}_i}{Y_i} + 2\Sigma - \frac{\dot{Y}_i}{Y_i} \left( \Gamma - \kappa \right) = 0,$$

(21)

where $Y_1 = B$, $Y_2 = C$, $Y_3 = P$ and $\Sigma = (\dot{P})^2 + \dot{B}\dot{C}$. It could look that in these equations we cannot put $\dot{A} = 0$. However, by remembering the identity (20), it is obvious that one obtains $\dot{A} = 0$ by taking in these equations $\kappa = 0$. Thus, we rewrote the system of the equations (16), (17) and (18) in a manifestly symmetric form: all the $Y_i$’s obey the same equation. By introducing the variables $\eta_{ij} = Y_i \dot{Y}_j - \dot{Y}_i Y_j$, i.e. the Wronskian of $Y_i$ and $Y_j$, and supposing $\dot{\eta}_{ij} \neq 0 \forall i, j$ (otherwise, if, for some $i$ and $j$, $\dot{\eta}_{ij} = 0 \Rightarrow Y_i \sim Y_j$ and the system is immediately reduced) we arrive at the following system of equations:

$$2\Gamma \dot{\eta}_{ij} - \left( \Gamma - \kappa \right) \eta_{ij} = 0.$$

(22)

From this system, it immediately follows that

$$\frac{\dot{\eta}_{ij}}{\eta_{ij}} = \left( \frac{\Gamma - \kappa}{A} \right) = \frac{\left( \Gamma - \kappa \right)}{2\Gamma},$$

so, after a trivial integration, it comes out that the $\eta_{ij}$’s are all proportional:

$$\eta_{12} \sim \eta_{13} \sim \eta_{23}.$$  

(23)
As it is well known from the theory of the linear system of ordinary differential equations, that (23) implies that one of the three unknown functions is a linear combination of the other two. It is convenient to choose $P$ as dependent function. In fact, since we know that $P^2$ is symmetric in the exchange of $B$ and $C$, then the only possibilities for $P$ are:

$$P = \alpha (B \pm C),$$

where $\alpha$ is an arbitrary nonzero constant. In this way, we reduced the initial system of four nonlinear equations in four unknown functions to a system of two equations in the unknown functions $B$ and $C$. In general, due to the term $\frac{\kappa}{A}$ that couples in a non-trivial way $B$ and $C$, it is not possible either to decouple the two equations or to further reduce the system. However, if one takes

$$P = \pm \frac{1}{2} (B - C),$$  \hspace{1cm} (24)

then the system (21) can be reduced to one equation in one unknown function. In fact, if eq. (24) holds, then

$$\Gamma = \frac{1}{4} (B + C)^2, \Sigma = \frac{1}{4} (\dot{B} + \dot{C})^2.$$  \hspace{1cm} (25)

Now, if we introduce

$$u = B + C, \hspace{0.2cm} v = B - C,$$

then, from eq. (21), $u$ and $v$ satisfy the following two equations:

$$\frac{u^2}{2} \ddot{u} + \frac{\dot{u}^2}{2} u + u \left( \frac{\kappa}{A} - \frac{\dot{u}}{2} \right) = 0,$$  \hspace{1cm} (25)

$$\frac{v^2}{2} \ddot{v} + \frac{\dot{v}^2}{2} v + v \left( \frac{\kappa}{A} - \frac{\dot{u}}{2} \right) = 0.$$  \hspace{1cm} (26)

Since eq. (26) is a linear homogeneous ordinary differential equation, $v$ can be expressed in a closed form in terms of $u$, so the system (21) is reduced to the only eq. (25). Hence, once eq. (25) is solved for $u$, the other metric coefficients immediately follow.
4.1 An explicit example

An interesting explicit example is the following:

\[ ds^2 = dx^2 + x^2 dy^2 + \left( 1 + \frac{1}{2} \ln |x| \right) dz^2 - \left( 1 - \frac{1}{2} \ln |x| \right) dt^2 - \ln |x| dz \, dt , \]

where \( \frac{1}{e} < \sqrt{|x|} < e \).

(27)

In this case we have:

\[ B + C = 2, \quad B - C = \ln |x|, \quad P = - \frac{1}{2} (B - C). \]

Then it is trivial to show that \( u = B + C \) satisfies eq. (25), while \( v = B - C \) satisfies eq. (26) for \( \kappa = 4 \) and \( A \) follows from eq. (19) by taking \( I_1 = 0 \).

It is interesting, at this point, to make a comparison with the Kasner case. In particular, one could ask: why the procedure to diagonalize the metric in the Kasner case works and in this case does not work? In this case, the first part of the exercise is the same as the Kasner one: Let us take the initial hypersurface \([x = 0] \), and then without loss of generality let

\[ a(0) = 1 \quad \text{and} \quad g_{mn}(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

(28)

Then we define

\[ k_{mn} = \frac{d}{dx} g_{mn} \]

and try to diagonalize \( k_{mn}(0) \). However, due to the non-compactness of the Lorentz group, the arguments of Appendix A no more apply; moreover, see Appendix C, one can really find examples of matrices \( k_{mn}(0) \) which cannot be brought into diagonal form. The calculation to be done is straightforward and will be omitted here. Then, it is clear that the differences between the two cases are group theoretical in nature.
Appendix A

Let
\[ k = k_{\alpha\beta} = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix} \] (30)
be any symmetric matrix, i.e. \( k = k^T \). We want to show that in the positive definite case, this matrix can be diagonalized; the 3-dimensional real orthogonal group is denoted by \( O(3) \), and the superscript \( T \) denotes the transposed matrix. Then \( U \in O(3) \) acts continuously on \( k \) to give
\[ U^T \cdot k \cdot U. \] (31)

We have to show that one can always choose \( U \in O(3) \) such that the matrix eq. (31) has diagonal form. To this end we define the quantity
\[ J(k) = d^2 + e^2 + f^2. \] (32)

Due to the compactness of \( O(3) \), the minimum of \( J \) exists; we have to prove that this minimum leads to \( J = 0 \). Assumed, this is not the case. Without loss of generality we may assume that this is due to \( d \neq 0 \), for otherwise, a permutation of the coordinate axes would lead to this inequality.

Let \( A_\phi \in O(3) \) be defined by
\[ A_\phi = \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (33)

It holds: The inverse matrix to \( A_\phi \) equals \( A_{-\phi} \) which is nothing but \( A_\phi^T \).

Analogously to expression (31), we define
\[ k_\phi = A_\phi^T \cdot k \cdot A_\phi, \] (34)
and then we get with eqs. (32) and (33) up to linear order in the Taylor expansion with respect to φ

\[ J(k_\phi) = J(k) + 2(a - b) \cdot d \cdot \phi. \tag{35} \]

For \( a \neq b \) we are already finished: A small change of \( \phi \) will change the value of \( J \) linearly with \( \phi \), so there cannot be a minimum at \( \phi = 0 \). For \( a = b \), this linear expansion does not suffice to decide, but here, the exact value is easy to evaluate, it reads:

\[ J(k_\phi) = J(k) - 4 \cdot d^2 \cdot \sin^2 \phi \cdot \cos^2 \phi. \tag{36} \]

There is a local maximum at \( \phi = 0 \), and therefore, this cannot be a minimum. Result: the assumption \( d \neq 0 \) leads to a contradiction.

**Appendix B**

Equation (9), i.e.,

\[ p + q + r = p^2 + q^2 + r^2 = 1 \tag{37} \]

represents the intersection of the plane \( p + q + r = 1 \) with the unit sphere in the \( p-q-r \)-space. Thus, it must be a circle. We parametrize it by the angular coordinate \( \phi \). The following 3 points

\[ P = (1, 0, 0), \quad Q = (0, 1, 0), \quad R = (0, 0, 1) \tag{38} \]

are obviously on this circle; in turn, this circle is uniquely determined by them. The center \( M \) of this circle is given by the arithmetic mean of \( P, Q \) and \( R \), i.e.

\[ M = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right). \tag{39} \]
Its radius equals the distance from $M$ to $P$, i.e. $\sqrt{2/3}$. So, we get the parametrization of eq. (37) as

\[
\begin{align*}
p &= \frac{1}{3} + \frac{2}{3} \cos \phi \\
q &= \frac{1}{3} (1 - \cos \phi) + \sqrt{1/3} \sin \phi \\
r &= \frac{1}{3} (1 - \cos \phi) - \sqrt{1/3} \sin \phi
\end{align*}
\]

Obviously, it suffices to restrict to the $\phi$-interval $0 \leq \phi < 2\pi$. However, a permutation between the 3 numbers $p$, $q$ and $r$ can be compensated by a coordinate transformation (namely a related permutation of the spatial axes of metric (2)), therefore, to get a one-to-one correspondence it proves useful to require additionally $r \leq q \leq p$. Comparing eqs. (41) and (40) one can see that the inequality $r \leq q$ is fulfilled for $0 \leq \phi \leq \pi$ only. The other inequality, $q \leq p$ further reduces this interval via the identity

\[
p - q = \cos \phi - (\sin \phi)/\sqrt{3}
\]

\[
0 \leq \phi \leq \pi/3 .
\]

Clearly, as there are six possible permutations, the length of this interval is $2\pi/6$. The boundary of this interval consists of two points. The point related to $\phi = 0$ is the already discussed flat space-time. The other one, related to $\phi = \pi/3$, i.e. that point where $p = q = 2/3$, $r = -1/3$ is the other axially symmetric solution for Bianchi type I. Here one can see what one also meets in other circumstances: The solutions with higher symmetry (here: axial symmetry) are at the boundary of the space of solutions.
Appendix C

Let
\[ \eta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  
\hfill (45)

The Lorentz group \( O(1, 1) \) is the group of all those transformations leaving the matrix eq. (45) invariant. For a given symmetric matrix
\[ k = \begin{pmatrix} A & D \\ D & B \end{pmatrix} \]  
\hfill (46)
defined by 3 parameters, the one-parameter group \( O(1, 1) \) acts continuously, so, from counting the degrees of freedom one could be tempted to assume, that one can always choose an element of \( O(1, 1) \) such that \( D \) becomes zero. However, this is not the case. For our purposes it suffices to give an example: For
\[ A = B = D = \frac{1}{2} \]  
\hfill (47)
put into eq. (46), no diagonalization is possible.

Let us look from another side: The trace of \( k \), namely the expression\(^6\)

\[ A - B, \]  
and the determinant, namely \( AB - D^2 \), are invariants of it with respect to \( O(1, 1) \)-actions. Again, the counting is misleading: 3 free parameters in (46), a one-parameter gauge group, so these two invariants should suffice for an invariant characterization. But this is not true: both \( A = B = D = 1/2 \) and \( A = B = D = 0 \) lead to a vanishing of both invariants, whereas no element of \( O(1, 1) \) can be given that transforms the one into the other. This is analogous to the discussion in \[8\]: two objects are different, but no continuous invariant exists to distinguish between them. Here it holds: every continuous invariant of \( k \) can be written as a function of trace and determinant only.

\(^6\)The minus sign in front of \( B \) is due to the minus 1 in eq. (45).
Appendix D

Let a metric be given as
\[ ds^2 = dx^2 + x^2 dy^2 + 2 dz \, dt + a(x) \, dt^2 \] (48)
where \( a(x) \) is any free function. Due to the off-diagonal term \( dz \, dt \), \( a(x) \) may have zeroes without leading to a singularity there.

We denote \((x, y, z, t)\) by \( x^i, i = 1, \ldots, 4 \). The only component of the Ricci tensor \( R_{ij} \) which does not vanish identically, is
\[ R_{44} = -\frac{1}{2} \left( \frac{d^2 a}{dx^2} + \frac{1}{x} \cdot \frac{da}{dx} \right). \] (49)

The only components of the Riemann tensor \( R_{ijkl} \) which do not vanish identically, are
\[ R_{1414} = -\frac{1}{2} \cdot \frac{d^2 a}{dx^2}, \quad R_{2424} = -\frac{x}{2} \cdot \frac{da}{dx}. \] (50)
This statement is meant, of course, only “up the usual antisymmetries”.

As a result of eq. (50) we find: Metric (48) is flat if and only if the function \( a \) is a constant.

To find out all non-flat vacuum solutions of the Einstein field equation of the form (48), one has therefore to solve \( R_{44} = 0 \) using eq. (49) with a non-constant \( a(x) \). The result is, after a possible redefinition of the coordinates \( t \) and \( z \), be expressible as
\[ a(x) = c \pm \ln x \] (51)
where \( c \) is a given constant of integration.

Let us calculate the curvature invariants of metric (48): Let \( I \) be any polynomial invariant like \( R^{ij} R_{ij} \). Then \( I \) depends on the one coordinate \( x \)...
only. To calculate one special value $I(x)$ we make the following construction: We replace, for any positive real $\epsilon$, the coordinate $t$ by $\epsilon t$ and the coordinate $z$ by $z/\epsilon$. This does not change the form of metric (48), only the function $a(x)$ is now replaced by $\epsilon^2 \cdot a(x)$.

In the limit $\epsilon \to 0$, we meet the flat spacetime having $I \equiv 0$. But $I(x)$ is a continuous function, and as invariant it does not change with $\epsilon$, therefore: Every polynomial curvature invariant for metric (48) identically vanishes.

**Note added**

The paper [9] and ours are different in scope, but the discussed metrics have much overlap: [9] presents the most general metric that depends on just one coordinate and cannot be diagonalized. The metric is a generalization of the Levi-Civita, or Kasner metrics. The authors of [9] also analyzed the global structure of the spacetime described by this metric – it has closed timelike curves, without a Cauchy horizon, and the question of whether such metric can represent the “exterior” of some “tube of matter”, the answer is that in general an energy condition is violated.

In [10], several results on diagonalization procedures can be found, too. According to M. MacCallum, the solution in the abstract of the present paper is just a special pp-wave and appears as eq. (22.5) in the new edition of ref. [5] (with $a = 1$ and $\rho = \ln x$). It also represents a special case of solutions already given in refs. [1] and [4]. In refs. [11] and [12], similar solutions have been discussed, too.
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