COMPATIBILITY OF QUANTIZATION FUNCTORS OF LIE BIALGEBRAS
WITH DUALITY AND DOUBLING OPERATIONS

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Abstract. We study the behavior of the Etingof-Kazhdan quantization functors under the natural duality operations of Lie bialgebras and Hopf algebras. In particular, we prove that these functors are "compatible with duality", i.e., they commute with the operation of duality followed by replacing the coproduct by its opposite. We then show that any quantization functor with this property also commutes with the operation of taking doubles. As an application, we show that the Etingof-Kazhdan quantization of some affine Lie superalgebras coincide with their Drinfeld-Jimbo-type quantizations.

We fix a field \( k \) of characteristic 0. Unless specified otherwise, “algebra”, “vector space”, etc., means “algebra over \( k \)”, etc.

Introduction

In [EK1, EK2], Etingof and Kazhdan solved the problem of quantization of Lie bialgebras. For each Drinfeld associator \( \Phi \), they constructed a quantization functor \( \tilde{Q}_\Phi : \{ \text{Lie bialgebras} \} \to \{ \text{quantized universal enveloping (QUE) algebras} \} \), right inverse to the semiclassical limit functor. This functor can in fact be viewed as a morphism of props \( Q_\Phi : \text{Bialg} \to S(LBA) \).

It was proved in [EK1] that \( \tilde{Q}_\Phi \) is compatible with natural operations on finite dimensional Lie bialgebras and QUE algebras. Namely, if \( a \) is a finite dimensional Lie bialgebra, then \( \tilde{Q}_\Phi(a^{\text{cop}}) \simeq \tilde{Q}_\Phi(a)^{\text{cop}} \) and \( \tilde{Q}_\Phi(\mathfrak{D}(a)) \simeq D(\tilde{Q}_\Phi(a)) \). Here for \( U \) a QUE algebra with finite-dimensional associated Lie bialgebra, \( U^* \) denotes the QUE algebra associated to the quantum formal series Hopf algebra \( \text{Hom}(U, k[[\hbar]]) \) (see [Dr1, Gav]); \( \text{cop} \) indicates the opposite cobracket or coproduct; \( \mathfrak{D}(a) \) is the double of \( a \) and \( D(U) \) is the double QUE algebra of \( U \).

In this paper, we prove that these isomorphisms also hold when \( a \) is infinite-dimensional. This will be proved as a consequence of statements about the prop morphism \( Q_\Phi : \text{Bialg} \to S(LBA) \).

One can explain the difference of our work with the relevant part of [EK1] as follows: the statements of [EK1] can be made into “propic” statements, involving the morphism \( \text{Bialg} \to S(LBA_{\text{fin}}) \), where \( LBA_{\text{fin}} \) is a “cyclic” prop (it has the same generators and relations as \( LBA \), except that diagrams with cycles are allowed). Even though compatibility with duality and doubling operations can be formulated for the prop \( LBA \), the proof of [EK1] uses diagrams involving cycles.

Our scheme of proof is the following. We first prove that the EK quantization functors are compatible with duality. More precisely, we show that the behavior of \( Q_\Phi \) w.r.t. duality functors is compatible with the natural duality operations on finite dimensional bialgebras and QUE algebras. We then show that any quantization functor \( Q \) which is compatible with duality (i.e., satisfies the propic analogue of \( \tilde{Q}(a^{\text{cop}}) \simeq \tilde{Q}(a)^{\text{cop}} \)) is automatically compatible with doubles. The proof is a propic version of the following argument. Let \( a \) be a finite dimensional Lie bialgebra and let \( Q(a) := \tilde{Q}^{-1}(D(\tilde{Q}(a))) \); here \( \tilde{Q}^{-1} \) is the dequantization functor \( \{ \text{QUE algebras} \} \to \{ k[[\hbar]] \text{-Lie bialgebras} \} \) inverse to \( \tilde{Q} \). Since we have morphisms \( Q(a) \to D(\tilde{Q}(a)) \), \( Q(a)^{\text{cop}} \to D(\tilde{Q}(a)) \) and \( Q \) is compatible with duals, we have Lie bialgebra morphisms \( a \to Q(a) \) and \( a^{\text{cop}} \to Q(a) \). By flatness, we then have \( Q(a) = a \oplus a^{\text{cop}} \); the structure of \( Q(a) \) is then uniquely determined.
by the brackets between $a$ and $a^\text{cop}$. While for a given $a$, these brackets could be such that $\mathcal{D}'(a) \not\cong \mathcal{D}(a)$, one can prove that the condition that these brackets are propic imply a propic statement, which implies that $\mathcal{D}'(a) \cong \mathcal{D}(a)$.

More generally, one can prove using [E1] that any quantization functor is compatible with duality; then Theorem 3.1 implies that it is also compatible with doubling operations.

The results of this paper have several corollaries. First, they are necessary ingredients to prove that when $a$ is a Kac-Moody Lie bialgebra, $\hat{Q}_a(a) \cong U^D_J(a)$, where $U^D_J(a)$ is the Drinfeld-Jimbo quantization of $a$ ([EK3]). In the final part of this paper, we show that these arguments can then be generalized to the case of some affine Lie superalgebras (these Lie superalgebras were introduced in [K, vL] and their quantum versions are due to [KhT, Yam]).

Similarly to [EK3], this can be used for proving a Kohno-Drinfeld-type theorem for these affine Lie superalgebras, namely the braid group representations arising from the monodromy of the KZ equations for with values in $\mathfrak{g}$-modules from category $\mathcal{O}$ are equivalent to those arising from the quantum $R$-matrix of $U^D_J(g)$.

1. Reminders on quantization functors

In this section, we recall the basic definitions in the theory of props, and the construction of Etingof-Kazhdan quantization functors (see [EH]). We then introduce some notions which we will use in this paper: prop bimodules, the dual of a prop, and the biprops $\text{LBA}_2$ and $\text{Bialg}_2$.

1.1. Schur functors. Let $\text{Sch}$ be the category whose objects are polynomial Schur functors, i.e., endofunctors $F$ of the category $\text{Vect}$ of finite dimensional vector spaces, of the form $F(V) = \bigoplus_{n \geq 0} \bigoplus_{\pi \in \hat{S}_n} F_{n,\pi} \otimes p_{\pi}(V^\otimes n)$, where for each $n \geq 0$ and irreducible $S_n$-module $\pi$, $p_{\pi} \in \mathbb{Q}S_n$ is a chosen rank one projector in $\text{End}(\pi) \subset \bigoplus_{\pi' \in \hat{S}_n} \text{End}(\pi') = \mathbb{Q}S_n$; in the above sum, the $F_{n,\pi}$ are finite dimensional vector spaces, which vanish for almost every $(n, \pi)$. Then $\text{Sch}$ is an abelian tensor category, where morphisms are natural transformations. The set $\text{Irr}(\text{Sch})$ of irreducible Schur functors is in bijection with $\{(n, \pi)|n \geq 0, \pi \in \hat{S}_n\}$; the bijection takes $(n, \pi)$ to $(V \mapsto p_{\pi}(V^\otimes n))$. The category $\text{Sch}$ is equipped with an involution, $F^*(V) := F(V^*)^*$. As a tensor category, $\text{Sch}$ is generated by the identity functor $\text{id}$, such that $\text{id}(V) = V$. The neutral object is $1$ defined by $1(V) = k$. We will also use the $n$th symmetric power functor $S^n$, the $n$th exterior power functor $\wedge^n$, and the $n$th tensor power functor $T_n = \text{id}^{\otimes n}$. We say that $F \in \text{Ob}(\text{Sch})$ has degree $n$ if $F_{n',\pi'} = 0$ for $n' \neq n$; we then denote by $|F|$ the degree of $F$.

Any $R \in \text{Ob}(\text{Sch})$ gives rise to an endofunctor of $\text{Sch}$ by $\text{Ob}(\text{Sch}) \ni F \mapsto F \circ R \in \text{Ob}(\text{Sch})$. For $f \in \text{Sch}(F,G)$, the corresponding morphism is $f(R) = f \circ R \in \text{Sch}(F \circ R,G \circ R)$. By functoriality, one also defines a (in general nonlinear) operation $\text{Sch}(F,G) \rightarrow \text{Sch}(R \circ F, R \circ G)$, $f \mapsto R(f)$.

We also define a completion $\text{Sch}$ of $\text{Sch}$, where objects are functors $\text{Vect} \rightarrow \text{Vect}$ (where $\text{Vect}$ is the category of vector spaces), of the same form as above, except that the condition that almost every $F_{n,\pi}$ vanishes is dropped. Then $\text{Sch}$ is again a tensor category equipped with an involution. The symmetric algebra functor $S = \bigoplus_{n \geq 0} S^n$ is an example of an object of $\text{Sch}$. The composition $R \circ F$ is an object of $\text{Sch}$ if: (a) $R, F \in \text{Ob}(\text{Sch})$, and $F$ has vanishing zero degree component; or if (b) $R \in \text{Ob}(\text{Sch})$, $F \in \text{Ob}(\text{Sch})$.

1.2. Props. A prop $P$ is a symmetric tensor category generated by a single object; equivalently, it is equipped with a tensor functor $\mathbf{1}_P : \text{Sch} \rightarrow P$ (also denoted $x \mapsto x^P$), inducing the identity on objects; so $\text{Ob}(\text{Sch}) = \text{Ob}(P)$. The prop $P$ is characterized by vector spaces $P(F,G)$ (for $F,G \in \text{Sch}$), composition maps $\circ : P(F,G) \otimes P(G,H) \rightarrow P(F,H)$, external product maps $\boxtimes : P(F,G) \otimes P(F',G') \rightarrow P(F \otimes F',G \otimes G')$ and $\mathbf{1}_P : \text{Sch}(F,G) \rightarrow P(F,G)$.

A $\text{Sch}$-prop $P$ is a symmetric tensor category with the same properties, except that the tensor functor is now $\mathbf{1}_P : \text{Sch} \rightarrow P$. 
If $P$ is a prop and $H \in \text{Ob}(\text{Sch})$, then we define a prop $H(P)$ by $H(P)(F, G) = P(F \circ H, G \circ H)$. If $P$ is a $\text{Sch}$-prop and $H \in \text{Ob}(\text{Sch})$, then one defines similarly a prop $H(P)$.

If $P, Q$ are props (or $\text{Sch}$-props), a morphism $f : P \to Q$ is a functor such that $f \circ i_P = i_Q$. If $P$ is a prop, then a prop ideal of $P$ is a collection $I(F, G)$, such that the projection $P/I$ defined by $(P/I)(F, G) := P(F, G)/I(F, G)$ is a prop and the canonical projection $P \to P/I$ is a prop morphism. If $I, J \subset P$ are prop ideals, then the product $IJ$ is the smallest prop ideal, such that $IJ(F, G)$ contains both $I(F, G)$ and $J(F, G)$. If $x \in P(F, G)$, we denote by $\langle x \rangle$ the smallest prop ideal of $P$ such that $x \in \langle x \rangle(F, G)$.

Similarly to algebras, props can be defined by generators and relations (see [EH]). The prop $\text{Coalg}$ of associative coalgebras is similarly generated by $\langle \xi \rangle$ (we call it an inner automorphism), whose action on $\xi$ setup in Section 1.10). We also have $\langle F, G \rangle$ (this construction, called inflation, is carried out in a the more general setup in Section 1.10). We have also $\langle \xi \rangle$ (recall that the $\text{LBA}$ prop and the canonical projection $P \to P/I$ defined by $i \in H, J \in \text{Ob}(\text{Sch})$ is a prop ideal, such that $IJ(F, G)$ contains both $I(F, G)$ and $J(F, G)$. If $x \in P(F, G)$, we denote by $\langle x \rangle$ the smallest prop ideal of $P$ such that $x \in \langle x \rangle(F, G)$.

The structure of some props. Let $P$ be a prop (or a $\text{Sch}$-prop). If $\xi \in P(\text{id}, \text{id})$, then there is a unique collection $(\xi_F)_{F \in \text{Ob}(\text{Sch})}$, where $\xi_F \in P(F, F)$ is such that $\xi_F \otimes G = \xi_{F \otimes G}$, and for $x \in \text{Sch}(F, G)$, $\xi_G \circ x^P = x^P \circ \xi_F$ (this construction, called inflation, is carried out in the more general setup in Section 1.10). We also have $\langle \xi \rangle$ (we call it an inner automorphism), whose action on $P(F, G)$ is $a \mapsto \xi_G \circ a \circ \xi_F^{-1}$. The map $P(\text{id}, \text{id})^\times \to \text{Aut}(P)$ is a group morphism.

1.3. The structure of some props. The prop $\text{LA}$ of Lie algebras is generated by $\mu \in \text{LA}(\text{id}, \text{id})$ and the Jacobi relation $\mu \circ (\mu \otimes \text{id}_{\text{id}}) \circ ((123) + (231) + (312)) = 0$. The prop $\text{LCA}$ of Lie coalgebras is generated by $\delta \in \text{LCA}(\text{id}, \text{id})$, the Jacobi relation $\delta \circ \mu = ((12) - (21)) \circ (\text{id} \otimes \text{id}_{\text{id}}) \circ ((12) - (21))$, with $a$ prop morphism.

We then have prop morphisms $\text{LA} \to \text{LBA}$ and $\text{LCA} \to \text{LBA}$, taking $\mu, \delta$ to their homonyms. Composing these inclusions with composition, we get a map

$$\bigoplus_{Z \in \text{Irr}(\text{Sch})} \text{LCA}(F, Z) \otimes \text{LA}(Z, G) \to \text{LBA}(F, G);$$

one shows that this map is a linear isomorphism ([E2, Pos]). More generally, one shows that if $(F_i)_{i \in I}$, $(G_j)_{j \in J}$ are finite families of Schur functors, then the map

$$(1) \quad \bigoplus_{(I, J) \in \text{Irr}(\text{Sch})} \text{LCA}(F_i, G_j) \otimes \text{LA}(Z_i, Z_j) \to \text{LBA}(Z_i, Z_j),$$

taking $\otimes_{i \in I} Z_i \otimes_{j \in J} Z_j$ to $\sigma_{I,J}^{\text{LBA}} \otimes_{i \in I} Z_i \otimes_{j \in J} Z_j$ (where $\sigma_{i,j} \in \text{Sch}(Z_i, Z_j)$, $\otimes_{i \in I} Z_i \otimes_{j \in J} Z_j)$ is the map $\otimes_{i \in I} Z_i \otimes_{j \in J} Z_j \rightarrow Z_i$), is a linear isomorphism.

The props $\text{LA}$ and $\text{LCA}$ give rise to $\text{Sch}$-props $\text{LA}$, $\text{LCA}$, where for $F, G \in \text{Ob}(\text{Sch})$, $\text{LA}(F, G) = \bigoplus_{\alpha, \beta} \text{LA}(F_\alpha, G_\beta)$ ($F_\alpha$ is the degree $\alpha$ component of $F$) and $\text{LCA}(F, G)$ is defined in the same way.

The prop $\text{LBA}$ is graded by $\mathbb{N}^2$ (with $|\mu| = (1, 0)$ and $|\delta| = (0, 1)$), which implies that it can be completed to a $\text{Sch}$-prop, given by

$$\text{LBA}(F, G) = \bigoplus_{Z \in \text{Irr}(\text{Sch})} \text{LCA}(F, Z) \otimes \text{LA}(Z, G);$$

(here $\hat{\oplus}$ is the direct product). We have also

$$\text{LBA}(F_i, G_j) = \bigoplus_{(I, J) \in \text{Irr}(\text{Sch})} \text{LCA}(F_i, G_j).$$

The prop $\text{Alg}$ of associative algebras is generated by $\eta \in \text{Alg}(1, \text{id})$ and $m \in \text{Alg}(T_2, \text{id})$, with relations $m \circ (\eta \otimes \text{id}_{\text{id}}) = m \circ (\text{id}_{\text{id}} \otimes \eta) = \text{id}_{\text{id}}$, and $m \circ (m \otimes \text{id}_{\text{id}}) = m \circ (\text{id}_{\text{id}} \otimes m)$ (associativity). The prop $\text{Coalg}$ of associative coalgebras is similarly generated by $\varepsilon \in \text{Coalg}(\text{id}, 1)$ and $\Delta \in \text{Coalg}$.
Coalg(id, T), with relations \((\varepsilon \boxtimes \text{id}_{\text{id}}) \circ \Delta = (\text{id}_{\text{id}} \boxtimes \varepsilon) \circ \Delta = \text{id}_{\text{id}} \) and \((\Delta \boxtimes \text{id}_{\text{id}}) \circ \Delta = (\text{id}_{\text{id}} \boxtimes \Delta) \circ \Delta \) (coassociativity). The prop Bialg of bialgebras is generated by Bialg with the same relations and the additional compatibility relation \(\Delta \circ m = (\text{id} \otimes m) (1234) \circ (\Delta \otimes \Delta) \).

Applying to \(\text{id}_{\text{id}}^\text{Bialg} \circ (\eta \circ \varepsilon) \in \text{Bialg}(\text{id}, \text{id})\). we have morphisms \(\text{Alg} \to \text{Bialg}\) and \(\text{Coalg} \to \text{Bialg}\), taking \(m, \eta\) and \(\Delta, \varepsilon\) to their homomorphs, which give rise to a map

\[ \oplus \in \text{Irr}(\text{Sch}) \text{Coalg}(F, Z) \otimes \text{Alg}(Z, G) \to \text{Bialg}(F, G) \]

by \(\oplus \in \text{Irr}(\text{Sch}) \otimes a \mapsto \sum \in \text{Irr}(\text{Sch}) \otimes (\text{id} \otimes (\eta \circ \varepsilon) \circ \Delta) \). One can show that this model is linear isomorphism.

More generally, if \((F_i)_{i \in I}\) and \((G_j)_{j \in J}\) are finite families of objects of Sch, the map

\[ \oplus(\in \text{Irr}(\text{Sch})) \otimes \otimes_{\in \text{Irr}(\text{Sch})} \text{Coalg}(F_i, \otimes_{\in \text{Irr}(\text{Sch})} Z_{ij})) \otimes (\otimes_{\in \text{Irr}(\text{Sch})} \text{Alg}(\otimes_{\in \text{Irr}(\text{Sch})} Z_{ij})) \to \text{Bialg}(\otimes_{\in \text{Irr}(\text{Sch})} F_i, \otimes_{\in \text{Irr}(\text{Sch})} G_j) \]

is a linear isomorphism.

We then define the completed prop \(\hat{\text{Bialg}} := \text{lim} \text{Bialg} (/(\text{id} \otimes (\eta \circ \varepsilon))^n \) (the prop ideal \(x^n\) has been defined in Section 1.2). Then one can prove that the isomorphism (2)

\[ \hat{\text{Bialg}}(F, G) \simeq \oplus \in \text{Irr}(\text{Sch}) \text{Coalg}(F, Z) \otimes \text{Alg}(Z, G). \]

More generally, the isomorphism (3) extends to an isomorphism

\[ \hat{\text{Bialg}}(\otimes_{\in \text{Irr}(\text{Sch})} F_i, \otimes_{\in \text{Irr}(\text{Sch})} G_j) \simeq \oplus(\in \text{Irr}(\text{Sch})) \otimes (\otimes_{\in \text{Irr}(\text{Sch})} \text{Coalg}(F_i, \otimes_{\in \text{Irr}(\text{Sch})} Z_{ij})) \otimes (\otimes_{\in \text{Irr}(\text{Sch})} \text{Alg}(\otimes_{\in \text{Irr}(\text{Sch})} Z_{ij})). \]

If \(S = \{\text{topologically free} k[[h]]\text{-modules}\}, then the category of \(S\text{-modules over} \text{Bialg}\) is \{QUE algebras\}; if \(S = \{k[[h]]\text{-modules of the form} V[[h]]\), where \(V\) is a vector space\}, then this category is \{QFSH algebras\}.

We define Hopf as a prop with the same generators as Bialg, with the additional generator \(a \in \text{Hopf}(id, id)\) (the antipode), and additional relations: \(a\) is invertible,

\[ a \circ (\text{id} \otimes \text{id}) \circ \Delta = a \circ (\text{id} \otimes \text{id}) \circ \Delta = \text{id}_{\text{id}}. \]

These relations imply \(a \circ a^{(2)} = a \circ (\text{id} \otimes \text{id}) \circ a = \text{id}_{\text{id}} \).

The natural morphism \(\text{Bialg} \to \text{Hopf}\) factors as \(\text{Bialg} \to \text{Hopf}\) where the morphism \(\text{Hopf} \to \text{Bialg}\) is such that \(a\) maps to \(\eta \circ \varepsilon + (\text{id} \otimes \text{id}) \circ \Delta \).

1.4 Definition of quantization functors. A quantization functor of Lie bialgebras is a prop morphism \(Q: \text{Bialg} \to S(\text{LBA})\) such that: (a) the composed morphism \(\text{Bialg} \to S(\text{LBA}) \to S(\text{Sch})\) (the second morphism is \(\mu \to 0, \delta \to 0\)) is \(m \to m^S, \Delta \to \Delta^S, \eta \to \text{pr}_0, \varepsilon \to \text{inj}_0\), where \(m^S \in S(\text{Sch})(T_2, id) = S(\text{Sch}^{S^2}, S) \) and \(\Delta^S \in S(\text{Sch})(T_2, id) = S(\text{Sch}(S^{(2)}), S^{(2)})\) are the universal versions of the product and coproduct map of the symmetric algebra \(S(V)\), where \(V\) is a vector space; and \(\text{pr}_1 \in S(\text{Sch}(S, 1))\), \(\text{inj}_0 \in S(\text{Sch}(1, S))\) are the canonical projection and injection; (b) the element \(\text{pr}_1^{\text{LBA}} \circ Q(m) \circ (\text{inj}_1^{\text{LBA}})^{S^2} \circ \text{inj}_{\lambda^2} \in \text{LBA}(\lambda^2, id)\) equals \(m\), the element \(\text{pr}_1^{\text{LBA}} \circ (\text{pr}_1^{\text{LBA}})^{S^2} \circ Q(\Delta) \circ \text{inj}_1^{\text{LBA}} \in \text{LBA}(id, \Lambda^2)\) equals \(\delta\) (here \(\text{pr}_1^{\text{LBA}} \in \text{Sch}(T_2, \Lambda^2)\)) and \(\text{inj}_{\lambda^2} \in \text{Sch}(\lambda^2, T_2)\) are the canonical morphisms corresponding to the decomposition \(T_2 = S^2 \otimes \Lambda^2\).

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1We denote by \(\sigma(1), \sigma(n)\) the permutation \(\sigma\), which we view as an element of \(\text{Sch}(F^n, F^n)\) is \(F\) is a Schur functor (here \(F = id)\) and we denote in the same way its image by \(i_P \in P(F^n, F^n)\) (here \(P = \text{Bialg}\).

2We denote by \(pr_k \in \text{Sch}(S, S^k)\) and \(inj_k \in \text{Sch}(S^k, S)\) the canonical projection and injection.
One proves that any quantization functor $Q : \text{Bialg} \to S(\text{LBA})$ factors uniquely as $\text{Bialg} \to \text{Bialg} \to S(\text{LBA})$, where the induced morphism $\text{Bialg} \to S(\text{LBA})$ (also denoted $Q$) is an isomorphism (see [EK2, EE]).

1.5. Schur multifunctors. If $I$ is a finite set, then $\text{Sch}_I$ is the category of polynomial functors $\text{Vec}^I \to \text{Vec}$, i.e., its objects are the functors of the form $(V_i)_{i \in I} \mapsto \otimes_{(n, \pi_i) \in \ell} F(n, \pi_i) \otimes ((\otimes_{i \in \ell} P_i(V_i^{\otimes n_i}))$, where the $F(n, \pi_i) \otimes ((\otimes_{i \in \ell} P_i(V_i^{\otimes n_i}))$, are almost all zero. We let $\text{Sch}_I$ be the category whose objects are functors $\text{Vec}^I \to \text{Vec}$ of the same form as above, except that the vanishing condition is dropped. If $I = \emptyset$, then $\text{Ob}(\text{Sch}_0) \subset \text{Ob}(\text{Vec})^{\text{Ob}(\text{Vec})^0} = \text{Ob}(\text{Vec})$, and by convention $\text{Ob}(\text{Sch}_0) = \text{Ob}(\text{Sch}_0) = \text{Ob}(\text{Vec})$; we denote by $1 \in \text{Ob}(\text{Sch}_0)$ the object corresponding to the base field $k$.

The category $\text{Sch}_I$ (as well as $\text{Sch}_I$) is equipped with a duality autofunctor $F^*(I) = F((V_i^*)_{i \in I})^*$. We set $\text{Sch}_n := \text{Sch}_{\{1, \ldots, n\}}$, $\text{Sch}_n := \text{Sch}_{\{1, \ldots, n\}}$.

Define a natural transformation $\mathbb{E} : \text{Sch}_I \times \text{Sch}_J \to \text{Sch}_{I \cup J}$, $(F \boxtimes G)((V_k)_{k \in I \cup J}) := F((V_i)_{i \in I}) \otimes G((V_j)_{j \in J})$. The irreducible objects of $\text{Sch}_I$ are then the $\mathbb{E}_{i \in I} Z_i$, where $(Z_i)_{i \in I} \in \text{Irr}(\text{Sch}^I)$. We define a functor $\otimes : \text{Sch}_I \to \text{Sch}$, taking $A : \text{Vec}^I \to \text{Vec} \times \text{diag}$, where $\text{diag} : \text{Vec} \to \text{Vec}^I$ is the diagonal embedding. Then $\otimes(\mathbb{E}_{i \in I} F_i) = \otimes_{i \in I} F_i$.

Define the tensor category of Schur multifunctors $\text{Sch}_{(1)}$ by $\text{Ob}(\text{Sch}_{(1)}) := \coprod_{n \geq 0} \text{Ob}(\text{Sch}_n)$, $\text{Sch}_{(1)}(F, G) = \text{Sch}_n(F, G)$ if $F, G \in \text{Ob}(\text{Sch}_n)$ and $\text{Sch}_{(1)}(F, G) = 0$ if $F \in \text{Ob}(\text{Sch}_n)$, $G \in \text{Ob}(\text{Sch}_n)$ with $n \neq n$, and the tensor product is $(F, G) \times F \boxtimes G$, where the identification $\{1, \ldots, n\} \cup \{1, \ldots, m\} \simeq \{1, \ldots, n + m\}$ is given by $i \mapsto i$ for $i \in \{1, \ldots, n\}$ and $i \mapsto n + i$ for $i \in \{1, \ldots, m\}$.

We define the tensor category of Schur multifunctors $\text{Sch}_{(1)}$ by $\text{Ob}(\text{Sch}_{(1)}) = \bigcup_{p, q \geq 0} \text{Sch}_{p+q}$. We denote by $\mathbb{E} : \text{Ob}(\text{Sch}_{(1)})^2 \to \text{Ob}(\text{Sch}_{(1)})$ the map obtained from $\mathbb{E} : \text{Sch}_p \times \text{Sch}_q \to \text{Sch}_{p+q}$. For $F, G \in \text{Ob}(\text{Sch}_{(1)})$, $\text{Sch}_{(1)}(F \boxtimes G, F' \boxtimes G') := \text{Sch}_{(1)}(F, F') \mathbb{E} \text{Sch}_{(1)}(G, G')$. The tensor product is defined by $(F \boxtimes G) \otimes (F' \boxtimes G') := (F \boxtimes G)(F' \boxtimes G')$. The tensor category $\text{Sch}_{(1)}$ is defined in the same way.

1.6. (Quasi)multibiprods. A multibiprop $P_{(1)}$ is a tensor category equipped with a tensor functor $\otimes_{P_{(1)}} : \text{Sch}_{(1)} \to P_{(1)}$, inducing the identity on objects. More explicitly, we have composition maps $P_{(1)}(F, G) \otimes P_{(1)}(G, H) = P_{(1)}(F, H)$, external product maps $P_{(1)}(F, G) \otimes P_{(1)}(F', G') = P_{(1)}(F \otimes F', G \otimes G')$ and maps $\otimes_{P_{(1)}} : \text{Sch}_{(1)}(F, G) \to P_{(1)}(F, G)$, satisfying some axioms.

In the case of a quasi-multibiprop, the composition maps are $P_{(1)}(F, G) \otimes P_{(1)}(G, H) \supset D(F, G, H) \to P_{(1)}(F, H)$, where $D(F, G, H)$ is a vector subspace of $P_{(1)}(F, G) \otimes P_{(1)}(G, H)$; the above axioms (e.g., associativity) should be satisfied when the involved expressions make sense.

1.7. Traces in some props. If $I, J$ are finite sets and $\Sigma \subset I \times J$, and for $A \in \text{Ob}(\text{Sch}_I)$, $B \in \text{Ob}(\text{Sch}_J)$, we define $\text{LBA}^\Sigma(A, B) \subset \text{LBA}(\otimes(A), \otimes(B))$

as follows. If $A = \mathbb{E}_i F_i$ and $B = \mathbb{E}_j G_j$, where $F_i, G_j \in \text{Irr}(\text{Sch})$, then

$$\text{LBA}^\Sigma(A, B) \simeq \otimes_{\Sigma^{-1}} \text{LCA}(F_i, \otimes_{j \in J} Z_{ij}) \otimes \text{LA}(\otimes_{i \in I} Z_{ij}, G_j).$$

under decomposition (1). We extend this definition to any $A, B$ by linearity. We define in the same way $\text{LBA}^\Sigma(A, B)$ for $A \in \text{Ob}(\text{Sch}_I)$, $B \in \text{Ob}(\text{Sch}_J)$.

If $I, J$ are finite sets, $a \in I \cap J$, and $\Sigma \subset I \times J$ is such that $(a, a) \not\in \Sigma$, define $I_a := I - \{a\}$, $J_a := J - \{a\}$ and $\Sigma(a) \subset I_a \times J_a$ as $\Sigma(a) := \{(i, j)| (i, j) \in \Sigma, \text{ or } (i, a) \text{ and } (a, j) \in \Sigma(a)\}$. 


Then for $A \in \text{Ob}(\text{Sch}_{I_a})$, $B \in \text{Ob}(\text{Sch}_{J_a})$, $F \in \text{Ob}(\text{Sch})$, we view $I$ as $I_a \sqcup \{a\}$, $J$ as $J_a \sqcup \{a\}$, $A \boxtimes F$ (resp., $B \boxtimes F$) as an object in $\text{Sch}_I$ (resp., $\text{Sch}_J$) and we define a trace map

$$\text{tr}_F : \text{LBA}^\Sigma (A \boxtimes F, B \boxtimes F) \to \text{LBA}^\Sigma(a)(A, B)$$

by the condition that the diagram

$$
\begin{array}{c}
\text{LBA}^\Sigma (A \boxtimes F, B \boxtimes F) \\
\downarrow \sim \\
\oplus \begin{pmatrix} i \in I \end{pmatrix} \text{LCA}(A_i, \oplus_{j \in J} Z_{a_j}) \otimes \text{LCA}(F, \oplus_{j \in J} Z_{a_j}) \\
\oplus \begin{pmatrix} j \in J \end{pmatrix} \text{LA}(\oplus_{i \in I} Z_{a_i}, B_j) \otimes \text{LA}(\oplus_{i \in I} Z_{a_i}, F)
\end{array}
\xrightarrow{\text{tr}_F} \text{LBA}^\Sigma(a)(A, B)
$$

commutes, where the diagonal map takes $(\begin{pmatrix} i \in I \end{pmatrix}) \mapsto \begin{pmatrix} i \in I \end{pmatrix} \text{LCA}(A_i, \oplus_{j \in J} Z_{a_j}) \otimes \text{LCA}(F, \oplus_{j \in J} Z_{a_j})$.

This map extends to a trace map $\text{tr}_F : \text{LBA}^\Sigma(A \boxtimes F, B \boxtimes F) \to \text{LBA}^\Sigma(a)(A, B)$ for $A \in \text{Ob}(\text{Sch}_I)$, $F \in \text{Ob}(\text{Sch}_J)$, $F \in \text{Ob}(\text{Sch})$.

Trace maps can also be constructed in the case of the prop Bialg. We define similarly Bialg$^\Sigma(a)(A, B) \subset \text{Bialg}(\otimes(a), \otimes(b))$ by selecting the analogous components in the decomposition (3). We then define a map

$$\text{tr}_F : \text{Bialg}^\Sigma(A \boxtimes F, B \boxtimes F) \to \text{Bialg}^\Sigma(a)(A, B);$$

the diagonal map is then

$$(\begin{pmatrix} i \in I \end{pmatrix} c_i) \otimes c_a \otimes \begin{pmatrix} j \in J \end{pmatrix} a_j \mapsto \begin{pmatrix} j \in J \end{pmatrix} \text{LCA}(A_j, a_j) \otimes \text{LCA}(F, a_j) \otimes \text{LCA}(\begin{pmatrix} i \in I \end{pmatrix} Z_{a_i}, \begin{pmatrix} j \in J \end{pmatrix} \text{LA}(\begin{pmatrix} i \in I \end{pmatrix} Z_{a_i}, B_j))$$

This map extends to a trace map

$$\text{tr}_F : \text{Bialg}^\Sigma(A \boxtimes F, B \boxtimes F) \to \text{Bialg}^\Sigma(a)(A, B),$$

where Bialg$^\Sigma(a)(A, B)$ is the direct product of the summands of Bialg$^\Sigma(a)(A, B)$.

If now $I, J, a$ are finite sets, $a, b, c \in I \times J$ are distinct, and $\Sigma \subset I \times J$ is such $(a, a) \notin \Sigma$ and $(b, b) \notin \Sigma(a)$, then $(b, b) \notin \Sigma$ and $(a, a) \notin \Sigma(b)$, so that $\Sigma(a)(b)$ and $\Sigma(b)(a)$ are both defined; moreover, $\Sigma(a)(b) = \Sigma(b)(a)$; we denote by $\Sigma(a, b)$ this set. If now $A \in \text{Ob}(\text{Sch}_{I_{a,b}})$, $B \in \text{Ob}(\text{Sch}_{J_{a,b}})$, $F, G \in \text{Ob}(\text{Sch})$, then $\text{tr}_F \circ \text{tr}_G = \text{tr}_G \circ \text{tr}_F$ (equality of maps LBA$^\Sigma(a)(A \boxtimes F, B \boxtimes F)$).

This property allows to generalize the trace map to the following setup. Let $I, J$ be finite sets and $K \subset I \cap J$. Let $\Sigma \subset I \times J$ be such that there is no finite sequence $(k_1, ..., k_p)$ of elements of $K$ such that $(k_1, k_2) \in \Sigma$, ..., $(k_{p-1}, k_p) \in \Sigma$, $(k_p, k_1) \in \Sigma$. Define $\Sigma(K) \subset (I - K) \times (J - K)$ be the set of $(i, j)$ such that there exists a (possibly empty) sequence $(k_1, ..., k_p)$ of elements of $K$, such that $(i, k_1) \in \Sigma$, $(k_1, k_2) \in \Sigma$, ..., $(k_p, j) \in \Sigma$.

If $A \in \text{Ob}(\text{Sch}_{I - K})$, $B \in \text{Ob}(\text{Sch}_{J - K})$, and $F_k \in \text{Irr}(\text{Sch})$ for each $k \in K$, we define a trace map

$$\text{tr}_{\otimes_k F_k} : \text{LBA}^\Sigma(A \otimes (\otimes_{k \in K} F_k), B \otimes (\otimes_{k \in K} F_k)) \to \text{LBA}^\Sigma(K)(A, B),$$
by \( \text{tr}_{\mathcal{K}} = \circ_{\mathcal{K}} \text{tr}_F \) (the order is irrelevant). This generalizes by linearity to a trace map \( \text{tr}_F : \text{LBA}^\Sigma(A \boxtimes F, B \boxtimes F) \rightarrow \text{LBA}^\Sigma(K) \) for any \( F \in \mathcal{K} \). One defines similar trace maps in the cases of \( \text{LBA} \), \( \text{Bialg} \) and \( \text{Bialg} \).

The above property then generalized as follows: if \( I, J \) are finite sets, if \( K, L \subseteq I \cap J \) are disjoint, and if \( \Sigma \subseteq I \times J \) is such that \( \Gamma(J \cup K) \) exists, then for \( A \in \text{Ob}(\text{Sch}_{I-(K \cup L)}) \), \( B \in \text{Ob}(\text{Sch}_{J-(K \cup L)}) \), \( F \in \text{Ob}(\text{Sch}_K) \), \( G \in \text{Ob}(\text{Sch}_L) \), we have the equality

\[
(4) \quad \text{tr}_F \circ \text{tr}_G = \text{tr}_G \circ \text{tr}_F = \text{tr}_{F \boxtimes G}
\]

of maps \( \text{LBA}^\Sigma(A \boxtimes F \boxtimes G, B \boxtimes F \boxtimes G) \rightarrow \text{LBA}^\Sigma(K \cup L)(A, B) \).

1.8. The quasi-multibiprop \( \Pi \). We define a quasi-multibiprop \( \Pi \) over \( \text{Sch}_{(1+1)} \) as follows. For \( F, ..., G' \in \text{Ob}(\text{Sch}_{(1)}) \), we set

\[
\Pi(F \boxtimes G, F' \boxtimes G') := \text{LBA}((F) \otimes (G')^\ast \otimes (F') \otimes (G)^\ast).
\]

The external product map and the functor \( i : \text{Sch}_{(1+1)} \rightarrow \Pi \) are obvious. Let us define the composition. If \( I, J, I', J' \) are disjoint sets, if \( F \in \text{Ob}(\text{Sch}_I) \), ..., \( G' \in \text{Ob}(\text{Sch}_J) \), and if \( \Sigma \subseteq (I \cup J') \times (I' \cup J) \), define

\[
\Pi^\Sigma(F \boxtimes G, F' \boxtimes G') := \text{LBA}((F \boxtimes G^\ast, F' \boxtimes G'^\ast).
\]

We attach an oriented graph \( \Gamma(\Sigma) \) to \( \Sigma \): vertices are the elements of \( I \cup \ldots \cup J' \); there is an edge from \( x \) to \( y \) iff \( (x, y) \in \Sigma \).

Then if \( I'', J'' \) are disjoint and disjoint of \( I, ..., J' \), and if \( \Sigma' \subseteq (I' \cup J'') \times (I'' \cup J) \), then \( \Sigma \) and \( \Sigma' \) are called composable if the graph obtained by gluing \( \Gamma(\Sigma) \) and \( \Gamma(\Sigma') \) has no loops. If this is the case, define \( \Sigma'' \subseteq (I \cup J') \times (I'' \cup J) \) as the set of all pairs of vertices \( (x, y) \) related by a sequence of edges of the big graph. We then define a linear map

\[
\Pi^\Sigma(F \boxtimes G, F' \boxtimes G') \otimes \Pi^\Sigma(F' \boxtimes G', F'' \boxtimes G'') \rightarrow \Pi^\Sigma(F \boxtimes G, F'' \boxtimes G'')
\]

as the map \( \text{LBA}^\Sigma(F \boxtimes G^\ast, F' \boxtimes G'^\ast) \otimes \text{LBA}^\Sigma(F' \boxtimes G'^\ast, F'' \boxtimes G''^\ast) \rightarrow \text{LBA}^\Sigma(F \boxtimes G^\ast, F'' \boxtimes G''^\ast) \),

\[
(x \otimes y) \mapsto \text{tr}_{G^\ast}((y \boxtimes \text{id}_{\text{LBA}(G)^\ast}) \otimes (\text{id}_{\text{LBA}(F') \boxtimes \text{LBA}(G') \otimes (G'^\ast))),
\]

where \( \sigma_{F,G} \in \text{Sch}(F \boxtimes G, G \boxtimes F) \) is \( x \otimes y \mapsto y \otimes x \). This partially defined composition \( \Pi \) with the structure of a quasi-multibiprop; in particular, the associativity of the composition follows from (4).

In the same way, one defines a quasi-multibiprop \( \Pi \) over \( \text{Sch}_{(1+1)} \) by \( \Pi(F \boxtimes G, F' \boxtimes G') = \text{LBA}((F) \otimes (G')^\ast \otimes (F') \otimes (G)^\ast) \) and quasi-multibiprops \( \Pi \) by

\[
\Pi(F \boxtimes G, F' \boxtimes G') := \text{Bialg}(F \otimes (G')^\ast \otimes (F') \otimes (G)^\ast),
\]

\[
\Pi(F \boxtimes G, F' \boxtimes G') := \text{Bialg}(F \otimes (G')^\ast \otimes (F') \otimes (G)^\ast).
\]

We need the linear isomorphism

\[
\Pi(F \boxtimes G, F' \boxtimes G') \rightarrow \Pi((F')^\ast \boxtimes (F')^\ast \otimes (G')^\ast \boxtimes (G')^\ast), \quad x \mapsto x^*,
\]

where \( F, ..., G' \in \text{Ob}(\text{Sch}_{(1)}) \), given by the sequence of maps \( \Pi(F \boxtimes G, F' \boxtimes G') \simeq \text{LBA}(F \otimes (G')^\ast \otimes (F') \otimes (G)^\ast \otimes (F') \otimes (G)^\ast) \simeq \Pi((F')^\ast \boxtimes (F')^\ast \otimes (G')^\ast \boxtimes (G')^\ast) \), where the middle map is \( x \mapsto \sigma_{(F')^\ast \otimes (F)^\ast} \circ x \circ \sigma_{(G')^\ast \otimes (G)^\ast} \), where \( \sigma_{A,B} \in \text{Sch}(A \otimes B, B \otimes A) \) is the permutation operator.

We have \( (x \otimes y)^* = y^* \otimes x^* \). The isomorphism \( x \mapsto x^* \) extends when \( \Pi \) is replaced by \( \Pi \).

We then define an element \( m_{\Pi} \in \Pi((S \boxtimes S)^{(2)}, (S \boxtimes S)) \); for \( a \) a finite dimensional Lie bialgebra, it specializes to the map \( (S(a) \otimes S(a)^*)^{(2)} \simeq U(\mathfrak{g}(a))^{(2)} \rightarrow U(\mathfrak{g}(a)) \simeq S(a) \otimes S(a)^* \), where \( S(a) \otimes S(a)^* \simeq U(\mathfrak{g}(a)) \) is the tensor product of inclusions followed by product and the middle map is the product of \( \mathfrak{g}(a) \). A graph for \( m_{\Pi} \) is given by the following set of edges, if the indices of the functors in the initial object \( (S \boxtimes S)^{(2)} \simeq (S \boxtimes S)\boxtimes (S \boxtimes S) \) are \((+,-)\) and those of the final object \( S \boxtimes S \) are \((+,-)\): \{edges of \( \Gamma \)\} = \{(1+,+), (2+,+), (-,1-), (-,2-), (2+,1-)\}.\}
It follows that both sides of the associativity identity $m_{II} \circ (m_{II} \boxtimes \text{id}_{\underline{S \otimes S}}) = m_{II} \circ (\text{id}_{\underline{S \otimes S}} \boxtimes m_{II})$ make sense; the identity holds. One shows that the possible definitions of the $n$fold iteration of $m_{II}$ all make sense and coincide; we denote by $m_{II}^{(n)} \in \Pi((S \otimes S)^{\otimes n}, S \otimes S)$ this $n$fold iteration.

We define $\Delta_0 \in \Pi(S \otimes S, (S \otimes S)^{\otimes 2}) \simeq LBA(S \otimes S \otimes S \otimes S)$ as the image of $(\Delta^S \otimes m^S)^{LBA}$, where $\Delta^S \in \text{Sch}(S \otimes S)$ and $m^S \in \text{Sch}(S \otimes S, S)$ are the universal versions of the coproduct and product maps of $S(V)$, $V$ a vector space. Then $\Delta_0$ specializes to the standard coproduct of $U(\mathfrak{g}(a))$, if $a$ is a finite dimensional Lie bialgebra. If the indices of the functors are as before, a graph for $\Delta_0$ is then given by the following set of edges (i.e. the indices for the factors of the source and target $(S \otimes S)^{\otimes 2} \simeq S \otimes S \otimes S \otimes S$ are 1+, 2+, 1-, 2- and $1^+, 2^+, 1^-, 2^-$): $(i+, j^+), (i^+, j^-), (i^+, j^+)$ for $i, j \in \{1, 2\}$.

It follows that the composition $\Delta_0 := Ad(J) \circ \Delta_0 \in \Pi(S \otimes S, (S \otimes S)^{\otimes 2})$

---

3Here and below, $\sigma_{i,j} \in \text{Sch}(\text{id}_{\underline{S}}^{(\otimes i)}, (\text{id}_{\underline{S}}^{(\otimes j)})^{(\otimes i)})$ corresponds to $\varphi_{i,j=1}^{(i)}(\varphi_{i'=1}^{(j)} x_{i'j'}) \mapsto \varphi_{i=1}^{(i)}(\varphi_{j=1}^{(j')} x_{i'j'})$. 

1.9. The Etingof-Kazhdan construction of quantization functors. Let $\Phi(A, B)$ be a Drinfeld associator: this is a series in noncommuting variables $A, B$, which may be identified with an element $\Phi(t_{12}, t_{23}) \in U(t_1)$. We recall the construction of a quantization functor $Q_\Phi : \text{Bial} \to S(LBA)$ attached to $\Phi$. We have $Q(\varepsilon) = pr_0 \in LBA(S, 1)$ and $Q(\eta) = i_{n=0} \in LBA(1, S)$. It remains to construct $Q(m)$ and $Q(\Delta)$.

One first constructs $J \in U_2$ of the form $J = 1 - r/2 + \ldots$, solution of

$$J^{1.2} \ast J^{1.23} = J^{2.3} \ast J^{1.23} \ast \Phi;$$

we denote by $J^{-1}$ the inverse of $J$ in the group of invertible elements of $(U_2, \ast)$. One then defines $\text{Ad}(J) \in \Pi((S \otimes S)^{\otimes 2}, (S \otimes S)^{\otimes 2})$ by

$$\text{Ad}(J) := (m_{II}^{(3)})^{\otimes 2} \circ \sigma_{2,3}(S \otimes S)^{\otimes 2} \circ (J \otimes \Delta_0 \otimes J^{-1}) \in \Pi(S \otimes S, (S \otimes S)^{\otimes 2})$$

A graph for $\text{Ad}(J)$ is then given by the following set of edges (if the indices for the factor of the source and target $(S \otimes S)^{\otimes 2} \simeq S \otimes S \otimes S \otimes S$ are 1+, 2+, 1-, 2- and $1^+, 2^+, 1^-, 2^-$): $(i+, j^+), (i^+, j^-), (i^+, j^+)$ for $i, j \in \{1, 2\}$.

It follows that the composition

$$\Delta_\Pi := \text{Ad}(J) \circ \Delta_0 \in \Pi(S \otimes S, (S \otimes S)^{\otimes 2})$$
is well defined and admits the graph (if the indices of the factors of the source $S \boxtimes S$ and target $(S \boxtimes S)^{\otimes 2} \simeq S \boxtimes S \boxtimes S$ are $+, -$ and $1+, 2+, 1-, 2-$: $(+, +), (i+, -), (i-, j++)$ for $i, j \in \{1, 2\}$.

We also construct $R := J^{21} \star \exp, (t/2) \star J^{-1}$ in $\exp \star (\exp)$, the exponential in the algebra $(\exp, \star)$. Then we have $R = (R_+ \boxtimes R_-) \circ \can_{S^{11}},$ where $\can_{S^{11}} \in \Pi(S \boxtimes S, S \boxtimes S) \simeq LBA(S, S)$ corresponds to $\id_s$, and $R_+ \in \Pi(S \boxtimes S, S \boxtimes S), R_- \in \Pi(S \boxtimes S, S \boxtimes S)$.

There exist $R_+^{(-1)} \in \Pi(S \boxtimes S, S \boxtimes S)$ and $R_-^{(-1)} \in \Pi(S \boxtimes S, S \boxtimes S)$, such that $R_+^{(-1)} \circ R_+ = \id_{S \boxtimes S},$ and $R_-^{(-1)} \circ R_- = \id_{S \boxtimes S}$. It follows from the quadrangular identities satisfied by $R$ that

$$R_+^{(-1)} \circ R_+^{(2)} = R_+^{(-1)} \circ R_+ \circ (21) \circ (R_-^{(-1)} \star)^{\otimes 2}, \quad (R_+^{(-1)})^{\otimes 2} \circ \Delta_1 \circ R_+ = \Delta_1 \circ R_+ = (R_+^{(-1)} \star)^{\otimes 2} \circ m_1 \circ (R_-^{(-1)} \star).$$

Then we define $m_a = \Pi(S \boxtimes S, (S \boxtimes S)^{\otimes 2})$ as the common value of both sides of the first identity, and $\Delta_1 \in \Pi((S \boxtimes S)^{\otimes 2}, S \boxtimes S)$ as the common value of both sides of the second identity.

We then define $Q(m) \in LBA(S^{\otimes 2}, S)$ as the element corresponding to $m_a,$ and $Q(\Delta) \in LBA(S, S^{\otimes 2})$ as the element corresponding to $\Delta_1.$

### 1.10. Prop bimodules

Let $P, Q$ be props, then a $(Q, P)$-prop bimodule $M$ is a symmetric tensor category bimodule over the symmetric tensor categories $Q$ and $P$, equipped with a tensor functor $i_M : Sch \to M$, such that $(i_Q, i_M, i_P)$ define a tensor category bimodule morphism from $Sch$, equipped with its obvious $(Sch, Sch)$-bimodule structure, to $M$, equipped with its $(Q, P)$-bimodule structure.

More explicitly, $M$ is a bimodule assignment $(F, G) \mapsto M(F, G)$ for any $F, G \in \Ob(Sch)$, equipped with composition maps $P(F, G) \otimes M(G, H) \to M(F, H)$ and $M(F, G) \otimes Q(H, K) \to M(F, K)$ all commuting.

One defines in the same way a prop left (or right) module over a prop $P$.

If $M$ is a prop module over a prop $P$, and $H$ is a Schur functor, then $H(M)$ is the prop module over $H(P)$ given by $H(P)(F, G) = P(F \circ H, G \circ H)$ and $\psi : M_1 \to M_2$ is a prop module morphism, then $H(\psi) : H(M_1) \to H(M_2)$ is defined by $H(\psi)(F, G) = \psi(F \circ H, G \circ H)$.

Let $M$ be a $(Q, P)$-prop bimodule.

**Proposition 1.1.** For any $\xi \in M(\id, \id)$, there is a unique system $(\xi_F)_{F \in \text{Sch}}$, such that $\xi_F \in M(F, F)$, and: if $x \in \text{Sch}(F, G)$, then $i_Q(x) \circ \xi_F = \xi_G \circ i_P(x)$, $\xi_{F \circ G} = \xi_{F \Delta G} \otimes \xi_G$, and $\xi_\Id = \xi$.

**Proof.** The construction of $\xi_F$ is a generalization of that of [EH], Section 1.11. If $Z \in \text{Irr}(Sch)$ has degree $n$, we denote by $pr_Z \in \text{Sch}(T_n, Z)$ and $inj_Z \in \text{Sch}(Z, T_n)$ morphisms such that $pr_Z \circ inj_Z = \Id_Z$. Then we set $\xi_Z := i_Q(pr_Z) \otimes \xi^Z \circ i_P(inj_Z)$. If $F \in \text{Sch}$ decomposes as $F = \bigoplus_{Z \in \text{Irr}(Sch)} F_Z \otimes Z$, where $F_Z$ is a multiplicity space, then $\xi_F := \sum_{Z \in \text{Irr}(Sch)} \Id_{F_Z} \otimes \xi_Z \in \bigoplus_{Z \in \text{Irr}(Sch)} \text{End}(F_Z) \otimes M(Z, Z) \subset M(F, F)$. $$

### 1.11. The dual prop (anti)automorphisms of a prop

If $P$ is a prop, define its dual prop $P^* := P^*(G^*, F^*)$. For $x \in P(F, G)$, we define $x^* \in P^*(G^*, F^*)$ as the element $x \in P(F, G)$.

The operation of $P^*$ are defined by $y^* \circ x^* := (y \star x)^*, y^* \otimes x^* := (y \otimes x)^*, i_{P^*} = i_P \circ t$, where $t : Sch \to Sch^*$ is induced by transposition; more precisely, for $F, G \in Sch$, $t_{F,G} : Sch(F, G) \to Sch^*(F, G) = Sch(G^*, F^*)$ is the map $\bigoplus_{Z \in \text{Irr}(Sch)} \text{Hom}(F_Z, G_Z) \to \bigoplus_{Z \in \text{Irr}(Sch)} \text{Hom}(G^*_Z, F^*_Z)$ induced by transposition.

A prop morphism $f : P \to Q$ induces a dual morphism $f^* : P^* \to Q^*$, defined by $f^*(x^*) = f(x)^*$, with $(g \circ f)^* = g^* \circ f^*$. We denote by $\text{Aut}(P) = \text{Aut}_+(P)$ the group of prop automorphisms of $P$, and by $\text{Aut}_-(P) := \text{Iso}(P, P^*)$ the set of prop isomorphisms from $P$.
to $P^*$. Set $\text{Aut}_\pm(P) := \text{Aut}_+(P) \cup \text{Aut}_-(P)$. Then elements of $\text{Aut}_\pm(P)$ can be composed by $g \cdot f := g \circ f$ if $f \in \text{Aut}_+(P)$, $g \cdot f := g'^{\ast} \circ f$ if $f \in \text{Aut}_-(P)$. Then $\text{Aut}_\pm(P)$ is a group, and we have an exact sequence $1 \to \text{Aut}_+(P) \to \text{Aut}(P) \to \{\pm 1\} \to 1$.

1.12. Biprops. We define $\text{Sch}_{1+1}$ as the symmetric tensor category whose objects are Schur bifunctors, i.e., objects of $\text{Sch}_2$, and

$$\text{Sch}_{1+1}(F \boxtimes G, F' \boxtimes G') := \text{Sch}(F, F') \otimes \text{Sch}^* (G, G') \simeq \text{Sch}(F, F') \otimes \text{Sch}(G, G').$$

as transposition gives rise to a prop isomorphism $\text{Sch} \simeq \text{Sch}^*$. We denote $\boxtimes : (\text{Sch})^2 \to \text{Sch}_{1+1}$ the natural functor, so the object $F \boxtimes G$ of $\text{Sch}_2$ is denoted $F \boxtimes G$ when viewed as an object of $\text{Sch}_{1+1}$ (and $\boxtimes : \text{Sch}(F, F') \otimes \text{Sch}(G, G') \to \text{Sch}_{1+1}(F \boxtimes G, F' \boxtimes G')$ is the natural map). The tensor product is given by $(F \boxtimes G) \otimes (F' \boxtimes G') := (F \boxtimes F') \boxtimes (G \boxtimes G')$. One defines similarly a tensor category $\text{Sch}_{1+1}$, where the objects are those of $\text{Sch}_2$. We define functors $\Delta, \Delta_\otimes : \text{Sch} \to \text{Sch}_{1+1}$, by $\Delta(F) := ((V, W) \mapsto F(V \oplus W))$ and $\Delta_\otimes(F) := ((V, W) \mapsto F(V \otimes W))$.

A $\text{Sch}_{1+1}$-biprop is a symmetric tensor category $\Pi_{1+1}$, equipped with a morphism $\text{Sch}_{1+1} \to \Pi_{1+1}$, inducing the identity on objects. One defines similarly $\text{Sch}_{1+1}$-biprops.

If $B \in \text{Ob}(\text{Sch}_{1+1})$ and $\Pi_{1+1}$ is a $\text{Sch}_{1+1}$-biprop, then $B(\Pi_{1+1})$ is a Sch-prop defined by $B(\Pi_{1+1})(F, G) := \Pi_{1+1}(F \circ B, G \circ B)$.

1.13. The biprop $P_2$. Let $P$ be one of the props (or Sch-prop) $\text{LBA}$, $\text{LBA}$, $\text{Bialg}$, $\text{Bialg}$. For $F, G, F', G' \in \text{Irr}(\text{Sch})$, we set

$$P_2(F \boxtimes G, F' \boxtimes G') := P_2^{{\boxtimes}}(F \boxtimes G^*, F' \boxtimes G'^*),$$

where $\Sigma = \{(F, F'), (F, G^*), (G'^*, G^*)\}$. We extend this definition by linearity to define $P_2(B, B')$ for $B, B' \in \text{Ob}(\text{Sch}_{1+1})$ (or $\text{Ob}(\text{Sch}_{1+1})$ in the case of $\text{LBA}$).

More explicitly, let $C \to P$, $O \to P$ be the prop morphisms and $e \in P(\text{id}, \text{id})$ the element such that for any $(F_i)_{i \in I}$, $(G_j)_{j \in J}$ in $\text{Irr}(\text{Sch})$, the map

$$\bigoplus_{i \in I} C(F_i, \otimes_{j \in J} Z_{ij}) \otimes \bigoplus_{j \in J} O(\otimes_{i \in I} Z_{ij}, G_j) \to P(\otimes_{i \in I} F_i, \otimes_{j \in J} G_j),$$

is a linear isomorphism (which is replaced by $\oplus$ in the case of $\text{LBA}$, $\text{Bialg}$). Then for $F, G, F', G' \in \text{Irr}(\text{Sch})$, $P_2(F \boxtimes G, F' \boxtimes G') \subset P(F \otimes G^*, F' \otimes G'^*)$ is given by

$$P_2(F \boxtimes G, F' \boxtimes G') := \bigoplus_{X,Y \in \text{Irr}(\text{Sch})} |Z_{X,Y}| P_2(B, B'),$$

where $\bigoplus_{X,Y \in \text{Irr}(\text{Sch})} |Z_{X,Y}| = 1$; this definition is extended by linearity.

In particular, $\text{Bialg}(F \boxtimes G, F' \boxtimes G')$ is the closure of $\text{Bialg}_{2}(F \boxtimes G, F' \boxtimes G') \subset \text{Bialg}(F \boxtimes G^*, F' \boxtimes G'^*)$ for the $\langle \text{id}_{\text{Bialg}} - \eta \circ \varepsilon \rangle$-adic topology of the latter space.

A structure of biprop is defined on $P_2$ as follows. We have $\text{Ob}(\text{Sch}_{1+1}) \subset \text{Ob}(\text{Sch}_{1+1})$ (and a similar inclusion replacing $\text{Sch}$ by $\text{Sch}$), and for any $B, B' \in \text{Ob}(\text{Sch}_{1+1})$ (resp., $B, B' \in \text{Ob}(\text{Sch}_{1+1})$) we have inclusions

$$\text{LBA}_2(B, B') \subset \Pi(B, B'), \quad \text{LBA}_2(B, B') \subset \Pi(B, B'),$$

$$\text{Bialg}_2(B, B') \subset \Pi(B, B'), \quad \text{Bialg}_2(B, B') \subset \Pi(B, B').$$

The composition of the diagrams $\{(F, F'), (F, G^*), (G'^*, G^*)\}$ and $\{(F', F''), (F', G'^*), (G''', G'^*)\}$ does not give rise to any loop, so the composition is well defined on $P_2(B, B') \otimes P_2(B', B'')$, and one checks that it takes its values in $P_2(B, B'')$. The external product also restricts to $P_2$. This defines a biprop structure on $P_2$.

If $P, Q$ are any props, define the biprop $P \boxtimes Q$ by $(P \boxtimes Q)(F \boxtimes G, F' \boxtimes G') := P(F, F') \otimes Q(G, G')$, for $F, ..., G' \in \text{Irr}(\text{Sch})$, which we extend by linearity.
Then for $P \in \{\text{LBA}, \text{LBA}, \text{Bialg}, \text{Bialg}\}$, we have a biprop morphism $P \Box P^{*} \to P_{2}$, given by $P(F, F') \otimes P^{*}(G, G') \subset P_{2}(F \Box F', G \Box G')$ for any $F, ..., G' \in \text{Sch}$.

When $P = S(\text{LBA})$, we set $P_{2} := S^2(\text{LBA})$.

2. Compatibility of quantization functors with duality

We first prove that the Etingof-Kazhdan functors are “almost compatible with duality”, i.e., the duality diagram commutes up to conjugation by an inner automorphism. We then prove that any such functor is equivalent to a functor (i.e., can be transformed into such a functor by composition with an inner automorphism) which is compatible with duality (i.e., for which the duality diagram commutes).

2.1. Almost compatibility with duality. The dihedral group $D_{4}$ can be presented as the group with generators $op, cop$ and $*$ and relations $op^{2} = cop^{2} = (op, cop) = *^{2} = 1, * \cdot cop \cdot * = op, * \cdot op \cdot * = cop$ (the product in $D_{4}$ is denoted by $\cdot$ and $(a, b) = a \cdot b \cdot a^{-1} \cdot b^{-1}$). We set $a_{1} \cdots a_{n} := a_{n} \cdots a_{1}$. So we have e.g. $*cop = cop \cdot *$.

We have two morphisms $\epsilon, \epsilon': D_{4} \to \mathbb{Z}/2\mathbb{Z}$, where $\epsilon$ is defined by $op, cop, * \mapsto 1$ and $\epsilon'$ by $op, cop \mapsto 1, * \mapsto -1$. Then $\text{Ker}(\epsilon) \simeq \mathbb{Z}/4\mathbb{Z}$, while $\text{Ker}(\epsilon') \simeq (\mathbb{Z}/2\mathbb{Z})^{2}$.

We have a unique morphism $D_{4} \to \text{Aut}_{\pm}(\text{LBA})$, $\theta \mapsto \theta_{\text{LBA}}$ such that $op_{\text{LBA}} : (\mu, \delta) \mapsto (-\mu, \delta), cop_{\text{LBA}} : (\mu, \delta) \mapsto (\mu, -\delta)$ and $*_{\text{LBA}} : (\mu, \delta) \mapsto (\delta^*, \mu^*)$. It is such that the diagram

$$
\begin{array}{c}
D_{4} \\
\epsilon' \\
\mathbb{Z}/2\mathbb{Z}
\end{array}
\xrightarrow{\text{Aut}_{\pm}(\text{LBA})}

$$

commutes. We also have a unique morphism $D_{4} \to \text{Aut}_{\pm}(\text{Bialg})$, $\theta \mapsto \theta_{\text{Bialg}}$ such that $op_{\text{Bialg}} : (m, \Delta) \mapsto (m \circ (21), \Delta), cop_{\text{Bialg}} : (m, \Delta) \mapsto (m, (21) \circ \Delta)$ and $*_{\text{Bialg}} : (m, \Delta) \mapsto (\Delta^*, m^*)$. It is such that the diagram

$$
\begin{array}{c}
D_{4} \\
\epsilon' \\
\mathbb{Z}/2\mathbb{Z}
\end{array}
\xrightarrow{\text{Aut}_{\pm}(\text{Bialg})}

$$

commutes.

Denote by $\text{Assoc}(k)$ the set of associators defined over $k$. This set consists of series $\Phi(A, B)$ in noncommutative variables $A, B$. It is equipped with an action of $\{\pm 1\}$, where $(-1) \cdot \Phi = \Phi$ given by $\Phi'(A, B) = \Phi(-A, -B)$. (The fixed points of this action are called the even associators.)

**Theorem 2.1.** For each $\theta \in D_{4}$, there exists $\xi_{\theta}$, where $\xi_{\theta} \in S(\text{LBA})(\text{id, id})^{\times}$ if $\epsilon'(\theta) = -1$ and $\xi_{\theta} \in S(\text{LBA}^*)(\text{id, id})^{\times}$ if $\epsilon'(\theta) = 1$ and such that:

$$
\begin{align*}
\theta_{S(\text{LBA})} \circ Q_{\Phi} &= \text{Inn}(\xi_{\theta}) \circ Q_{\epsilon(\theta) \cdot \Phi} \circ \theta_{\text{Bialg}} \text{ if } \epsilon'(\theta) = 1, \\
\theta_{S(\text{LBA})} \circ Q_{\Phi} &= \text{Inn}(\xi_{\theta}) \circ Q_{\epsilon(\theta) \cdot \Phi} \circ \theta_{\text{Bialg}} \text{ if } \epsilon'(\theta) = -1.
\end{align*}
$$

We say that a quantization functor $Q$ is almost compatible with duality if for some $\xi \in S(\text{LBA}^*)(\text{id, id})^{\times},$

$$
(5) \quad *_{\text{cop}} S(\text{LBA}) \circ Q = \text{Inn}(\xi) \circ Q^{*} \circ *_{\text{cop Bialg}}.
$$

So each $Q_{\Phi}$ is almost compatible with duality.

**Proof.** The subset of $D_{4}$ of elements $\theta$ for which the result holds is a subgroup. Since $D_{4}$ is generated by $cop$ and $*op$, it suffices to prove it for these elements. When $\theta = cop$, the
result was proved in [EH]. So we now prove it for $\theta = *op = (\text{cop})^{-1}$. We set in this section, $Q := Q_{*, \xi} := \xi_{*op}$, so we should find $\xi$ such that

\[ *op_{\text{S(LBA)}} \circ Q = \text{Im} (\xi) \circ Q^* \circ *op_{\text{Bialg}}. \]

This is written as follows

\[ *op_{\text{S(LBA)}} (Q(m)) = \xi \circ (Q(\Delta))^* \circ (\xi^{-1})^{522}, \quad *op_{\text{S(LBA)}} (Q(\Delta)) = \xi^{522} \circ (21) \circ Q(m)^* \circ \xi^{-1}, \]

i.e., applying $x \mapsto x^*$, as follows

\[ (6) \quad Q(m)^* = \xi^{522} \circ Q(\Delta) \circ \xi^{-1}, \quad Q(\Delta)^* = \xi \circ Q(m) \circ (21) \circ (\xi^{-1})^{522}, \]

where $x^* = (*op(x))^*$ and $\xi = (\xi^{-1})^{-1}$.

The map $x \mapsto x^* := (*op(x))^*$ is a linear map $\text{LBA}(A, B) \to \text{LBA}(B^*, A^*)$, such that $(y \circ x)^* = x^* \circ y^*$. It is uniquely defined by this condition, the assignments $(\mu, \delta) \mapsto (\delta, -\mu)$, and $(x \boxtimes y)^* = x^* \boxtimes y^*$, $i_{\text{LBA}}(z)^* = i_{\text{LBA}}(z^*)$ for $z \in \text{Sch}(A, B)$, where $z \mapsto z^*$ denotes the map $\text{Sch}(A, B) \to \text{Sch}(B^*, A^*)$ induced by the transposition.

We then construct a linear map

\[ \Pi(F \boxtimes G, F' \boxtimes G') \to \Pi(G \boxtimes F, G' \boxtimes F'), \quad x \mapsto x^*, \]

induced by the isomorphisms \( \Pi(F \boxtimes G, F' \boxtimes G') \simeq \text{LBA}((\otimes F) \otimes (\otimes G)^*, (\otimes F') \otimes (\otimes G)^*), \Pi(G \boxtimes F, G' \boxtimes F') \simeq \text{LBA}((\otimes G) \otimes (\otimes F^*), (\otimes G') \otimes (\otimes F^*)) \), and the map $x \mapsto x^*$ on $\Pi$-spaces. We then have (for $x, y$ in $\Pi$-spaces) $(y \circ x)^* = y^* \circ x^*$, $(x \boxtimes y)^* = x^* \boxtimes y^*$.

We finally define a linear map

\[ \Pi(F \boxtimes G, F' \boxtimes G') \to \Pi(F^* \boxtimes G^*, F^* \boxtimes G^*), \quad x \mapsto x^* := ((x)^*) = (x^*)^*; \]

then $(x \circ y)^* = y^* \circ x^*$. Let us denote by $x \mapsto x$ the canonical map \( \text{LBA}(S^{\otimes p}, S^{\otimes q}) \to \Pi((S \otimes 1)^{\otimes p}, (S \otimes 1)^{\otimes q}). \) Then one checks that $x^* = (x^*)^{522}$.

Applying the operation $x \mapsto x$ to (6), this condition translates in terms of $\Pi$-spaces as

\[ m_a^* = \xi^{522} \circ \Delta_a \circ \xi^{-1}, \quad \Delta_a = \xi \circ m_a \circ (21) \circ (\xi^{-1})^{522}. \]

Let us now compute $m_a^*, \Delta_a^*$.\n
**Lemma 2.1.** Define $\omega_S \in \text{Sch}(S, S^*)^\times$ as $\hat{\oplus}_{n \geq 0} (-1)^n \text{id}_{S^n}$. Then

\[ m_a = (\text{id}_{S \boxtimes \omega_S}) \circ m_{\Pi} \circ (21) \circ (\text{id}_{S \boxtimes \omega_S})^{522}. \]

**Proof of Lemma.** Let $a$ be a finite dimensional Lie bialgebra, and let $\mathfrak{g}$ be its double. Then $\mathfrak{g} = a \oplus a^{\text{cop}}$ (as Lie coalgebras; $a$ and $a^{\text{cop}}$ are also sub-Lie bialgebras of $\mathfrak{g}$). Then $m_{\Pi}$ is the propic version of the map $S(a) \otimes S(a^*)^{\otimes 2} \to S(a) \otimes S(a^*)$ induced by the product of $U(\mathfrak{g})$ and the isomorphism $S(a) \otimes S(a^*) \to U(\mathfrak{g})$ given by $x \otimes \xi \mapsto \text{sym}_a(x) \text{sym}_a(\xi)$, where \( \text{sym}_a : S(x) \to U(\mathfrak{x}) \) is the symmetrization map for the Lie algebra $\mathfrak{x}$.

Let now $\mathfrak{g}'$ be the double of $a^{\text{cop}}$. Then $\mathfrak{g}' = (a^*)^{\oplus} \oplus (a^{\text{cop}})^{\oplus} = a^{\text{cop}} \oplus a^{\text{op}, \text{cop}}$. Then $m_{\Pi}$ is the propic version of the map $S(a) \otimes S(a^*)^{\otimes 2} \to S(a) \otimes S(a^*)$ induced by the product of $U(\mathfrak{g}')$ and the isomorphism $S(a) \otimes S(a^*) \to U(\mathfrak{g}')$, $x \otimes \xi \mapsto \text{sym}_{a^*}(\xi) \text{sym}_{a^*}(x)$.

An isomorphism $\iota : \mathfrak{g} \to \mathfrak{g}'$ is given by $(a, \ell) \mapsto (\ell, -a)$. Then the map $S(U(a^*)^* \circ \iota : U(\mathfrak{g}) \to U(\mathfrak{g}')$ is a linear isomorphism, where $S(U(a^*))$ is the antipode of $U(\mathfrak{g}')$. Since the diagram

\[
\begin{array}{ccc}
S(a) & \xrightarrow{\text{sym}_a} & U(a) \\
\downarrow{S(- \text{id}_a)} & & \downarrow{U(\iota)} \\
S(a) & \xrightarrow{\text{sym}_{a^*}} & U(a^{\text{op}})
\end{array}
\]
commutes, the composed map \( S(a) \otimes S(a^*) \to U(g) \cong U(g') \leftarrow S(a) \otimes S(a^*) \) is \( x \otimes \xi \mapsto x \otimes \xi' \) (where \( v \mapsto \bar{v} \) denotes the automorphism \( S(- \text{id}_V) \) of \( S(V) \)).

In the commutative diagram

\[
\begin{array}{ccc}
x_1 \otimes \xi_1 \otimes x_2 \otimes \xi_2 & \xymatrix{ (S(a) \otimes S(a^*)) \otimes^2 & (S(a) \otimes S(a^*)) \otimes^2 } & x'_1 \otimes \xi'_1 \otimes x'_2 \otimes \xi'_2 \\
\coprod & \ar[rr]^\cong & \ar[rr]_\cong & \ar[ll]_{(m_{11})_a} \\
S(a) \otimes S(a^*) & \ar[r]_\prod & U(g) & \ar[l]^{(m_{11})_a} \\
\end{array}
\]

the image of \( x_1 \otimes \xi_1 \otimes x_2 \otimes \xi_2 \) in \( U(g') \) is \( \text{sym} \cdot (\xi_2) \text{sym}_{a^\circ p}(x_2) \text{sym} \cdot (\xi_1) \text{sym} \cdot p(x_1) \), while the image of \( x'_1 \otimes \xi'_1 \otimes x'_2 \otimes \xi'_2 \) in \( U(g') \) is \( \text{sym} \cdot (\xi'_1) \text{sym} \cdot p(x'_1) \text{sym} \cdot (\xi'_2) \text{sym} \cdot p(x'_2) \).

It follows that the diagram

\[
\begin{array}{ccc}
(S(a) \otimes S(a^*)) \otimes^2 & \xymatrix{ (S(a) \otimes S(a^*)) \otimes^2 } & S(a) \otimes S(a^*) \\
\ar[r]_\cong (m_{11})_a & \ar[r] & (m_{11})_a \\
S(a) \otimes S(a^*) & \ar[r] & S(a) \otimes S(a^*) \\
\end{array}
\]

commutes, where the horizontal maps are \( x_1 \otimes \xi_1 \otimes x_2 \otimes \xi_2 \mapsto x_2 \otimes \xi_2 \otimes x_1 \otimes \xi_1 \) and \( x \otimes \xi \mapsto x \otimes \xi' \).

The statement of the lemma is a propic version of the commutativity of this diagram; one shows that the proof can be carried to the propic setting. \( \square \)

**Lemma 2.2.** \( \Delta_0 = \Delta_0 \).

**Proof.** \( \Delta_0 \in \Pi(\text{Sch}(S,S) \otimes^2, \text{Sch}(S,S) \otimes^2) \subset \text{LBA}(S \otimes S \otimes^2, S \otimes^2 \otimes S) \) is \( \Delta^S \cong m^S \), where \( \Delta^S \in \text{Sch}(S,S) \otimes^2 \) and \( m^S \in \text{Sch}(S \otimes^2, S) \) are the propic versions of the coproduct and product of symmetric algebras. The statement then follows from the fact that \( m^S \) and \( \Delta^S \) are interchanged by the maps \( \text{Sch}(S,S) \otimes^2 \to \text{Sch}(S \otimes^2, S) \) and \( \text{Sch}(S \otimes^2, S) \to \text{Sch}(S,S) \otimes^2 \) defined by \( x \mapsto x^\tau \). \( \square \)

**Lemma 2.3.** For \( Y \) in the image of \( t_n \subset U(t_n) \to U_n = \Pi(1, \text{Sch}(S,S) \otimes^n) \), we have \( Y^\tau = -(\text{id}_S \otimes \omega_S)^{\otimes n} \circ Y \).

**Proof.** We prove this by induction on the degree of \( Y \) w.r.t. the generators \( t_{ij} \) of \( t_n \). If \( Y = t_{ij} \), then \( Y^\tau = Y \), while \((\text{id}_S \otimes \omega_S)^{\otimes n} \circ Y = -Y \). Let us assume that \( Y = [Y', Y''] = m_{11} \circ \sigma_{n,2}(S(S)^{\Pi} \circ (Y' \boxtimes Y'' - Y'' \boxtimes Y')). \) Then

\[
Y^\tau = (m_{11})^{\otimes n} \circ \sigma_{n,2}(S(S)^{\Pi} \circ (Y' \boxtimes Y'' - Y'' \boxtimes Y'))
\]

\[
= ((\text{id}_S \otimes \omega_S) \circ m_{11} \circ (21) \circ (\text{id}_S \otimes \omega_S)^{\otimes 2})^{\otimes n} \circ \sigma_{n,2}(S(S)^{\Pi} \circ (Y' \boxtimes Y'' - Y'' \boxtimes Y'))
\]

\[
= (\text{id}_S \otimes \omega_S)^{\otimes n} \circ m_{11} \circ \sigma_{n,2}(S(S)^{\Pi} \circ (Y'' \boxtimes Y' - Y' \boxtimes Y'')) = -(\text{id}_S \otimes \omega_S)^{\otimes n} \circ Y.
\]

\( \square \)

**Lemma 2.4.** For \( X \) in the image of \( \exp(t_n) \subset U(t_n) \to U_n = \Pi(1, \text{Sch}(S,S) \otimes^n) \), we have \( X^\tau = (\text{id}_S \otimes \omega_S)^{\otimes n} \circ X^{-1} \).
Proof. Let \( Y \in \mathfrak{t}_n \) and \( X := \exp(Y) \). Then \( X = \sum_{k \geq 0} (k!)^{-1} (m_{\Pi}^{(n)})^k \circ \sigma_{n,k}(S\mathcal{S}S)^\Pi \circ Y^\otimes k \). Then

\[
X^\tau = \sum_{k \geq 0} (k!)^{-1} (m_{\Pi}^{(k)})^n \circ \sigma_{n,k}(S\mathcal{S}S)^\Pi \circ (Y^\tau)^\otimes k
\]

\[
= \sum_{k \geq 0} (k!)^{-1} \left( (\text{id}_S \otimes \omega_S) \circ m_{\Pi}^{(k)} \circ (k, \ldots, 2, 1) \circ (\text{id}_S \otimes \omega_S)^\otimes k \right)^{\otimes n} \circ \sigma_{n,k}(S\mathcal{S}S)^\Pi \circ \left( (\text{id}_S \otimes \omega_S)^\otimes n \circ Y \right)^\otimes k
\]

\[
= \sum_{k \geq 0} (k!)^{-1} \left( (\text{id}_S \otimes \omega_S) \circ m_{\Pi}^{(k)} \right)^n \circ \sigma_{n,k}(S\mathcal{S}S)^\Pi \circ \left( (1 - Y)^\otimes k \right) = (\text{id}_S \otimes \omega_S)^\otimes n \circ X^{-1}.
\]

\[\square\]

Lemma 2.5. There exists \( u \in U_1^X \), such that \((\text{id}_S \otimes \omega_S)^\otimes 2 \circ J^\tau = u^{12} * (J^{2,1})^{-1} * (u^1 * u^2)^{-1}\).

Proof. \( J \) satisfies \( J^{1,2} \circ J^{1,2,3} = J^{2,3} \circ J^{1,23} \circ \Phi \), which is rewritten as

\[
m_{\Pi}^{(3)} \circ \sigma_{3,2}(S\mathcal{S}S)^\Pi \circ (J^{1,2} \otimes J^{1,23}) = (m_{\Pi}^{(3)})^\otimes \circ \sigma_{3,3}(S\mathcal{S}S)^\Pi \circ (J^{2,3} \otimes J^{1,2,3} \circ \Phi).
\]

Applying \( x \mapsto x^\tau \), we get

\[
(m_{\Pi}^{(3)} \circ \sigma_{3,2}(S\mathcal{S}S)^\Pi \circ (J^{1,2})^\tau \otimes (J^{1,23})^\tau) = (m_{\Pi}^{(3)})^\otimes \circ \sigma_{3,3}(S\mathcal{S}S)^\Pi \circ ((J^{2,3})^\tau \otimes (J^{1,2})^\tau \otimes ((\text{id}_S \otimes \omega_S)^\otimes 3 \circ \Phi^{-1}))
\]

So

\[
\left( (\text{id}_S \otimes \omega_S) \circ m_{\Pi}^{(3)} \circ \sigma_{3,2}(S\mathcal{S}S)^\Pi \circ ((\text{id}_S \otimes \omega_S)^\otimes (J^{1,2})^\tau \otimes (\text{id}_S \otimes \omega_S)^\otimes (J^{1,23})^\tau) \right)
\]

\[
= ((\text{id}_S \otimes \omega_S) \circ m_{\Pi}^{(3)} \circ \sigma_{3,3}(S\mathcal{S}S)^\Pi \circ ((\Phi^{-1} \otimes (\text{id}_S \otimes \omega_S)^\otimes (J^{2,3})^\tau \otimes (\text{id}_S \otimes \omega_S)^\otimes (J^{1,23})^\tau))
\]

Simplifying \((\text{id}_S \otimes \omega_S)^\otimes 3\), we get

\[
((\text{id}_S \otimes \omega_S)^\otimes (J^{2,3}) \circ J^\tau)^{1,23} = (\Phi^{-1} \circ (\text{id}_S \otimes \omega_S)^\otimes 2 \circ J^\tau)^{1,23} = ((\text{id}_S \otimes \omega_S)^\otimes (J^\tau)^{1,23})^\tau = (\Phi^{-1} \circ (\text{id}_S \otimes \omega_S)^\otimes 2 \circ J^\tau)^{1,2}.
\]

So \( J^\tau = (1 + r_2, 1 + r_{2,1}) \), therefore \( J^\tau = 1 + r_2 + r_{2,1} \), so according to [E3], Remark 6.6 (see also [E3], Thm. 2.1), \( J^\tau \) is gauge-equivalent to \( J^{2,1} \).

\[\square\]

Lemma 2.6. The elements \((\text{id}_S \otimes \omega_S)^\otimes 2 \circ J^\tau \) and \((\text{id}_S \otimes \omega_S)^\otimes (J^{-1})^\tau \) of \( U_2 \) are inverse of each other.

Proof. We have \( m_{\Pi}^{(3)} \circ (1324) \circ (J \circ J^{-1}) = 1 \). Applying \( x \mapsto x^\tau \), we get \((m_{\Pi}^{(3)} \circ (J^\tau \circ (J^{-1})^\tau))^\tau = 1 \). Therefore \((\text{id}_S \otimes \omega_S)^\otimes \circ m_{\Pi}^{(3)} \circ (1324) \circ (\text{id}_S \otimes \omega_S)^\otimes (J^{-1})^\tau \circ (\text{id}_S \otimes \omega_S)^\otimes (J^\tau) = 1 \). Simplifying \((\text{id}_S \otimes \omega_S)^\otimes 2\), we get the result.

\[\square\]

Lemma 2.7. \( R^\tau = (\text{id}_S \otimes \omega_S)^\otimes (u^1 * u^2 * (R_{2,1})^{-1} * (u^1 * u^2)^{-1}) \).
Lemma 2.8. \( \Pi(1 \boxtimes S, 1 \boxtimes S)^{\times} \), we have \( R^{-1} = (R_+ \boxtimes (R_- \circ \sigma)) \circ \text{can}_{S \boxtimes S}. \) Then \( (R^{-1})^2 = ((R_- \circ \sigma) \boxtimes R_+) \circ \text{can}_{S \boxtimes S}. \)
Therefore
\[
R^\tau = (\text{id}_S \boxtimes \sigma_S) \circ \text{Ad}(u) \circ \text{can}_{S \boxtimes S},
\]
where \( \text{Ad}(u) = m_{11}^{(3)} \circ (u \boxtimes \text{id}_{S \boxtimes S} \boxtimes u^{-1}) \in \text{Hom}(S \boxtimes 1, S \boxtimes 1)^{\times}. \) On the other hand, \( R = (R_+ \boxtimes R_-) \circ \text{can}_{S \boxtimes S} \) implies that
\[
R^\tau = (R^\tau_+ \boxtimes R^\tau_-) \circ \text{can}_{S \boxtimes S} = (R^\tau_+ \boxtimes R^\tau_-) \circ \text{can}_{S \boxtimes S}.
\]
Comparing these formulas, we get the existence of \( \psi \in \text{Hom}(1 \boxtimes S, 1 \boxtimes S)^{\times} \), such that
\[
R^\tau_+ = (\text{id}_S \boxtimes \sigma_S) \circ \text{Ad}(u) \circ R_+ \circ \sigma \circ \psi^{-1}, \quad R^\tau_- = (\text{id}_S \boxtimes \sigma_S) \circ \text{Ad}(u) \circ R_+ \circ \psi^*.
\]
It then follows that
\[
(R^{-1})^\tau = \psi \circ \sigma^{-1} \circ R^{-1} \circ \text{Ad}(u)^{-1} \circ (\text{id}_S \boxtimes \sigma_S), \quad (R^{-1})^\tau = (\psi^*)^{-1} \circ R^{-1} \circ \text{Ad}(u)^{-1} \circ (\text{id}_S \boxtimes \sigma_S).
\]

Lemma 2.8. \((\text{Ad}(J))^\tau = (\text{id}_S \boxtimes \sigma_S) \circ \text{Ad}(u^1 \times u^2 \times J^{2,1} \times (u^{12})^{-1}) \circ (\text{id}_S \boxtimes \sigma_S)^{\boxtimes 2}.
\)

Proof. We have
\[
(\text{Ad}(J))^\tau = (m_{11}^{(3)} \boxtimes m_{11}^{(3)}) \circ \sigma_{3,3}(S \boxtimes S)^{\boxtimes 1} \circ (J \boxtimes \text{id}_{S \boxtimes S} \boxtimes J^{-1})^\tau
\]
\[
= ((\text{id}_S \boxtimes \sigma_S) \circ (m_{11}^{(3)}) \circ (321) \circ (\text{id}_S \boxtimes \sigma_S) \boxtimes 2) \circ \sigma_{3,3}(S \boxtimes S)^{\boxtimes 1} \circ (J^\tau \boxtimes \text{id}_{S \boxtimes S} \boxtimes (J^{-1})^\tau)
\]
\[
= ((\text{id}_S \boxtimes \sigma_S) \circ (m_{11}^{(3)}) \boxtimes 2) \circ \sigma_{3,3}(S \boxtimes S)^{\boxtimes 1} \circ ((\text{id}_S \boxtimes \sigma_S)^{\boxtimes 2} \boxtimes J^\tau) \boxtimes (\text{id}_S \boxtimes \sigma_S)^{\boxtimes 2} \boxtimes (\text{id}_S \boxtimes \sigma_S)^{\boxtimes 2} \circ (J^{-1})^\tau
\]
\[
= ((\text{id}_S \boxtimes \sigma_S) \circ (m_{11}^{(3)}) \boxtimes 2) \circ \sigma_{3,3}(S \boxtimes S)^{\boxtimes 1}
\]
\[
= ((\text{id}_S \boxtimes \sigma_S) \circ (321) \circ (\text{id}_S \boxtimes \sigma_S) \boxtimes 2) \circ \sigma_{3,3}(S \boxtimes S)^{\boxtimes 1} \circ (J^\tau \boxtimes \text{id}_{S \boxtimes S} \boxtimes (J^{-1})^\tau)
\]
\[
= (u^1 \times u^2 \times J^{2,1} \times (u^{12})^{-1}) \circ (\text{id}_S \boxtimes \sigma_S)^{\boxtimes 2} \boxtimes (u^{12} \times (J^{2,1})^{-1} \times (u^1 \times u^2)^{-1})
\]
\[
= (\text{id}_S \boxtimes \sigma_S)^{\boxtimes 2} \circ \text{Ad}(u^1 \times u^2 \times J^{2,1} \times (u^{12})^{-1}) \circ (\text{id}_S \boxtimes \sigma_S)^{\boxtimes 2}.
\]
Lemma 2.9. $\Delta^*_\Pi = (\text{id}_S \otimes \omega_S)^{\otimes 2} \circ \text{Ad}(u)^{\otimes 2} \circ (21) \circ \Delta_\Pi \circ \text{Ad}(u)^{-1} \circ (\text{id}_S \otimes \omega_S)$.

Proof. We have

$$\Delta^*_\Pi = \text{Ad}(J)^\tau \circ \Delta^*_\Pi = (\text{id}_S \otimes \omega_S)^{\otimes 2} \circ \text{Ad}(u^1 \star u^2 \star J^{2,1} \star (u^{12})^{-1}) \circ (\text{id}_S \otimes \omega_S)^{\otimes 2} \circ \Delta_\Pi$$

$$= (\text{id}_S \otimes \omega_S)^{\otimes 2} \circ \text{Ad}(u^1 \star u^2 \star J^{2,1} \star (u^{12})^{-1}) \circ \Delta_\Pi \circ (\text{id}_S \otimes \omega_S)^{\otimes 2}$$

$$= (\text{id}_S \otimes \omega_S)^{\otimes 2} \circ \text{Ad}(u)^{\otimes 2} \circ (21) \circ \Delta_\Pi \circ \text{Ad}(u)^{-1} \circ (\text{id}_S \otimes \omega_S).$$

Now

$$m^*_\xi = (R^{(-1)}_+ \circ m_\Pi \circ R^{(-1)}_+)^\tau = (R^{(-1)}_+)^\tau \circ m_\Pi \circ (R^{(-1)}_+)^{\otimes 2}$$

$$= \psi \circ \sigma^{-1} \circ R^{(-1)}_+ \circ \text{Ad}(u)^{-1} \circ m_\Pi \circ (21) \circ (\text{Ad}(u) \circ R_- \circ \sigma \circ \psi^{-1})^{\otimes 2}$$

$$= \psi \circ \sigma^{-1} \circ R^{(-1)}_+ \circ m_\Pi \circ (21) \circ (R_- \circ \sigma \circ \psi^{-1})^{\otimes 2}$$

and

$$\Delta^*_\Pi = ((R^{(-1)}_+)^{\otimes 2} \circ \Delta_\Pi \circ R^{(-1)}_+) = ((R^{(-1)}_+)^\tau)^{\otimes 2} \circ \Delta_\Pi \circ R^{(-1)}_+$$

$$= (\psi \circ \sigma^{-1} \circ R^{(-1)}_+)^{\otimes 2} \circ (21) \circ \Delta_\Pi \circ R_- \circ \sigma \circ \psi^{-1}$$

$$= (\psi \circ \sigma^{-1})^{\otimes 2} \circ (21) \circ (R_-)^{\otimes 2} \circ \Delta_\Pi \circ R_- \circ (\sigma \circ \psi^{-1}).$$

Now

$$m^*_a = (m^*_\xi)^* = ((\sigma \circ \psi^{-1})^*^{\otimes 2} \circ (21) \circ ((R^{(-1)}_+)^{\otimes 2} \circ m^*_\Pi \circ R^{(-1)}_+)^* \circ (\psi \circ \sigma^{-1})^*$$

$$= ((\sigma \circ \psi^{-1})^*)^{\otimes 2} \circ (21) \circ \Delta_a \circ (\psi \circ \sigma^{-1})^*,$$

and

$$\Delta^*_a = (\Delta^*_\Pi)^* = (\sigma \circ \psi^{-1})^* \circ R^{(-1)}_+ \circ \Delta^*_\Pi \circ (R^{(-1)}_+)^* \circ (21) \circ ((\psi \circ \sigma^{-1})^*)^{\otimes 2}$$

$$= (\sigma \circ \psi^{-1})^* \circ m_a \circ ((\psi \circ \sigma^{-1})^*)^{\otimes 2}.$$

Let $S_a \in \Pi(S^{\otimes 2}, S^{\otimes 2})$ be the antipode for $(m_a, \Delta_a)$. Then $m_a \circ (21) = S_a \circ m_a \circ (S^{-1}_a)^{\otimes 2}$, $(21) \circ \Delta_a = S^{\otimes 2}_a \circ \Delta_a \circ S^{-1}_a$. Therefore

$$m^*_a = ((\sigma \circ \psi^{-1})^* \circ S_a)^{\otimes 2} \circ \Delta_a \circ ((\sigma \circ \psi^{-1})^* \circ S_a)^{-1},$$

$$\Delta^*_a = ((\sigma \circ \psi^{-1})^* \circ S_a) \circ m_a \circ (21) \circ (((\sigma \circ \psi^{-1})^* \circ S_a)^{-1})^{\otimes 2},$$

as wanted. $\square$

2.2. From almost compatible to compatible (with duality) quantization functors.

Let $Q$ be any quantization functor. Recall that is gives rise to a prop isomorphism $Q^* : \text{Bialg}^* \to S(\text{LBA})^*$. We say that $Q$ is almost compatible with duality iff there exists $\xi \in S(\text{LBA})(\text{id}, \text{id})^X$, of the form $\xi = \text{id}^{S(\text{LBA})}_\text{id} +$ higher order terms, such that $Q^* \circ \text{cop}_{\text{Bialg}} = \text{Im}((\xi^*) \circ S(\text{cop}_{\text{LBA}}) \circ Q$ (recall that $\xi \mapsto \xi^*$ is the tautological map $P(F, G) \to P^*(G^*, F^*)$), and $Q$ is compatible with duality if this identity holds with $\xi = \text{id}^{S(\text{LBA})}_\text{id}$, i.e., if the diagram of props

$$\text{Bialg} \xrightarrow{Q} S(\text{LBA})$$

$$\xrightarrow{\text{cop}_{\text{Bialg}}}$$

$$S(\text{cop}_{\text{LBA}}) \xrightarrow{Q^*} S(\text{LBA})^* \simeq S(\text{LBA})^*$$
commutes.

**Proposition 2.1.** If $Q$ is almost compatible with duality, then there exists $\xi_0 \in S(LBA)(\id, \id)^\times$ such that $\Inn(\xi_0) \circ Q$ is compatible with duality.

**Proof.** Let $\xi$ be such that $Q^\ast \circ \copIB = \Inn(\xi^\ast) \circ \copS(LBA) \circ Q$. Since $\Inn(\xi^\ast) = \Inn(\xi^{-1})$, we get $Q \circ \copIB^\ast = \Inn(\xi^{-1}) \circ \copS(LBA)^\ast \circ Q^\ast$. Since $(\copS(LBA))^\ast \circ \copS(LBA) = \op \copS(LBA)$ and $(\copIB)^\ast \circ \copIB = \op \copIB$, we get $Q \circ \op \copIB = \Inn(\xi^{-1}) \circ (\copS(LBA))^\ast \circ \Inn(\xi^\ast) \circ \copS(LBA) \circ Q$, and since $(\copS(LBA))^\ast(\xi) = (\copS(LBA)(\xi))^\ast$, we get

$$Q \circ \op \copIB = \Inn(\xi_1) \circ \op \copS(LBA) \circ Q,$$

where $\xi_1 := \xi^{-1} \circ (\copS(LBA)(\xi))^\ast$. Since $(\op \cop)^2 = 1$, we get

$$Q = \Inn(\xi_1 \circ \op \copS(LBA)(\xi_1)) \circ Q,$$

so $\xi_1 \circ \op \copS(LBA)(\xi_1) = \exp(\alpha(\mu \circ \delta))$ for some scalar $\alpha$. Since $\op \copS(LBA)(\mu \circ \delta) = \mu \circ \delta$, if we set $\xi_2 := \exp(-\alpha(\mu \circ \delta)/2) \circ \xi_1$, we have $\xi_2 = (\op \copS(LBA)(\xi_2)) = \id_{\id S(LBA)}^\circ \copS(LBA)$, and

$$Q \circ \op \copIB = \Inn(\xi_2) \circ \op \copS(LBA) \circ Q.$$

Since $\xi_2 = \id_{\id S(LBA)}^\circ$ + higher order terms, there exists a unique $\xi_2^{1/2}$ of the same form, with $\xi_2 = (\xi_2^{1/2})^2$. Since $\xi_2 = \op \copS(LBA)(\xi_2^{-1})$, we have $\xi_2 = (\op \copS(LBA)((\xi_2^{1/2})^{-1}))^2$ so $
\xi_2^{1/2} = \op \copS(LBA)((\xi_2^{1/2})^{-1})$, hence

$$\xi_2 = \xi_2^{1/2} \circ \op \copS(LBA)((\xi_2^{1/2})^{-1}),$$

so if we set $Q_1 := \Inn(\xi_2^{1/2})^{-1} \circ Q$, we get

$$Q_1 \circ \op \copIB = \op \copS(LBA) \circ Q_1.$$

We also have

$$Q_1^\ast \circ \copIB = \Inn(\xi_3^\ast) \circ \copS(LBA) \circ Q_1,$$

where $\xi_3 = (\copS(LBA)(\xi_2^{1/2}))^\ast \circ \xi \circ \xi_2^{1/2}$. Then $(\copS(LBA)(\xi_3))^\ast = (\copS(LBA)(\xi_2^{1/2}))^\ast \circ \copS(LBA)(\xi_3)^\ast \circ \copS(LBA)(\xi_3^\ast) \circ \copS(LBA)(\xi_3)^\ast \circ \copS(LBA)(\xi_3^\ast) \circ \copS(LBA)(\xi_3)^\ast \circ \op \copS(LBA)(\xi_2^{1/2}) = (\copS(LBA)(\xi_2^{1/2}))^\ast \circ \xi \circ \xi_1 \circ \xi_2^{-1/2}$

$$= (\op \copS(LBA)(\xi_2^{1/2}))^\ast \circ \xi \circ \xi_2^{1/2} \circ \exp(\alpha(\mu \circ \delta)/2) = \xi_3 \circ \exp(\alpha(\mu \circ \delta)/2).$$

Set $\xi_4 := \xi_3 \circ \exp(\alpha(\mu \circ \delta)/4)$, then $(\copS(LBA)(\xi_4))^\ast = \xi_4$, as $(\copS(LBA)(\mu \circ \delta))^\ast = -\mu \circ \delta$, and we have

$$Q_1^\ast \circ \copIB = \Inn(\xi_4^\ast) \circ \copS(LBA) \circ Q_1.$$

Let $\xi_5 := \xi_4^{1/2}$. Then $\xi_5 = \copS(LBA)(\xi_5)^\ast$, so $\xi_5^\ast = \xi_5^\ast \circ \copS(LBA)(\xi_5)$, so $\Inn(\xi_5^{-1}) \circ Q_1^\ast \circ \copIB = \Inn(\copS(LBA)(\xi_5)) \circ \copS(LBA) \circ Q_1 = \copS(LBA) \circ \Inn(\xi_5) \circ Q_1$. So if we set $Q_2 := \Inn(\xi_5) \circ Q_1$, we have $Q_2^\ast \circ \copIB = \copS(LBA) \circ Q_2$, while $Q_2 = \Inn(\xi_0) \circ Q$, and $\xi_0 = \xi_5 \circ \xi_2^{-1/2}$. \qed
3. Compatibility of quantization functors with doubling operations

In this section, we formulate propic versions of the (Drinfeld) double constructions of Lie bialgebras and Hopf algebras. We then express a condition on a quantization functor \( Q : \text{Bialg} \to S(\text{LBA}) \), which we call “compatibility with doubling operations”. One then proves that if \( Q \) is compatible with doubling with duality, then it is compatible with doubling operations.

3.1. Doubles of Lie bialgebras. In this subsection, we set \( P := \text{LBA} \) or \( \text{LBA} \text{-LBA} \).

We define a prop \( D_{\text{add}}(P) \) by \( D_{\text{add}}(P)(F,G) := P_2(\Delta(F),\Delta(G)) \). We also define prop \((D_{\text{add}}(P), P)\)-bimodules \( M_{\text{add}}^\pm(P) \) by \( M_{\text{add}}^+(P)(F,G) := P_2(F \boxtimes 1, \Delta(G)) \) and \( M_{\text{add}}^-(P)(F,G) := P_2(1, \Delta(F) \cop P) \).

The structures of left prop \( D_{\text{add}}(P) \)-bimodules on \( M_{\text{add}}^\pm(P) \) are obvious. Let us define the structures of right prop \( P \)-modules on \( M_{\text{add}}^\pm(P) \).

In the case of \( M_{\text{add}}^+(P) \), the composition \( P(F,G) \otimes P_2(G \boxtimes 1, \Delta(H)) \to P_2(F \boxtimes 1, \Delta(H)) \) is \( x \otimes X \mapsto X \circ (x \boxtimes id_{\text{LBA}}) \).

In the case of \( M_{\text{add}}^-(P) \), the analogous composition is the map \( P(F,G) \otimes P_2(1, \Delta(H)) \to P_2(1, \Delta(F) \cop P) \) such that \( x \otimes X \mapsto X \circ (id_P \boxtimes \ast \cop x) \), where \( \ast \cop x = \cop \circ \ast : P(F,G) \to P^*(F,G) \).

\( M_{\text{add}}^P(P) \) also has a \( (D_{\text{add}}(P), P^*) \)-prop bimodule structure, induced by

\[
P^*(F,G) \otimes P_2(1, \Delta(H)) \to P_2(1, \Delta(F) \cop P), \quad x^* \otimes X \mapsto X \circ (id_P \boxtimes x^*).
\]

Note that if \( V \) is a finite dimensional module over \( P \), i.e., we have a prop morphism \( P \to \text{Prop}(V) \), then \( V \oplus V^* \) is a module over \( D_{\text{add}}(P) \).

We then have

\[
D_{\text{add}}(\text{LBA})(\wedge^2, id) = \text{LBA}_2(\wedge^2 \boxtimes 1, \text{id} \boxtimes 1) \oplus \text{LBA}_2(\text{id} \boxtimes \text{id}, \text{id} \boxtimes 1) \oplus \text{LBA}_2(\text{id} \boxtimes \text{id}, \text{id} \boxtimes \text{id}) \oplus \text{LBA}_2(1, \wedge^2, \text{id} \boxtimes \text{id}) \\
\approx \text{LBA}(\wedge^2, \text{id}) \oplus \text{LBA}(\text{id}, T_2) \oplus \text{LBA}(T_2, \text{id}) \oplus \text{LBA}(\text{id}, \wedge^2).
\]

Similarly,

\[
D_{\text{add}}(\text{LBA})(\text{id}, \wedge^2) = \text{LBA}_2(\text{id} \boxtimes 1, \wedge^2 \boxtimes 1) \oplus \text{LBA}_2(\text{id} \boxtimes 1, \text{id} \boxtimes \text{id}) \oplus \text{LBA}_2(1, \text{id} \boxtimes \text{id}, \text{id} \boxtimes \text{id}) \oplus \text{LBA}_2(1, \text{id} \boxtimes 1, \text{id} \boxtimes \text{id}) \\
\approx \text{LBA}(\text{id}, \wedge^2) \oplus \text{LBA}(\wedge^2, \text{id})
\]

as the two intermediate spaces are zero.

We have \( M_{\text{add}}^+(\text{LBA})(\text{id}, \text{id}) = \text{LBA}_2(\text{id} \boxtimes 1, \text{id} \boxtimes 1) \oplus \text{LBA}_2(\text{id} \boxtimes 1, \text{id} \boxtimes \text{id}) \approx \text{LBA}(\text{id}, \text{id}) \) as the second space is zero and \( M_{\text{add}}^-(\text{LBA})(\text{id}, \text{id}) = \text{LBA}_2(1 \boxtimes \text{id}, \text{id} \boxtimes 1) \oplus \text{LBA}_2(1 \boxtimes \text{id}, \text{id} \boxtimes \text{id}) \approx \text{LBA}(\text{id}, \text{id}) \) as the first space is zero.

**Lemma 3.1.** There are unique prop morphisms \( \text{double} : \text{LBA} \to D_{\text{add}}(\text{LBA}) \), left prop \( D_{\text{add}}(\text{LBA}) \)-module morphisms \( \alpha_{\text{can}}^\pm : D_{\text{add}}(\text{LBA}) \to M_{\text{add}}^\pm(\text{LBA}) \) and right prop \( \text{LBA} \)-module morphisms \( \beta_{\text{can}}^\pm : \text{LBA} \to M_{\text{add}}^\pm(\text{LBA}) \), such that

\[
\text{double}(\mu) \simeq \mu \mp \delta \mp (-\mu) \mp \delta, \quad \text{double}(\delta) \simeq \delta \mp (-\mu),
\]

\[
\alpha_{\text{can}}^+(id_{D_{\text{add}}(\text{LBA})}) = \beta_{\text{can}}^+(id_{\text{id}_{\text{LBA}}}) \simeq id_{\text{id}_{\text{LBA}}}, \quad \alpha_{\text{can}}^-(id_{D_{\text{add}}(\text{LBA})}) = \beta_{\text{can}}^-(id_{\text{id}_{\text{LBA}}}) \simeq id_{\text{id}_{\text{LBA}}}.
\]

The diagrams

\[
\begin{array}{ccc}
M_{\text{add}}^+(\text{LBA}) & \xrightarrow{\alpha_{\text{can}}^+} & D_{\text{add}}(\text{LBA}) \\
\beta_{\text{can}}^+ & \downarrow & \text{double} \\
\text{LBA} & \xrightarrow{\text{double}} & D_{\text{add}}(\text{LBA}) \\
\end{array}
\]
We define can\(\pm\) \(\in\) LBA\(_2\)(\(F\boxtimes 1, \Delta(F)\)) and can\(\mp\) \(\in\) LBA\(_2\)(\(1\boxtimes F, \Delta(F)\)) as follows: can\(\pm\) are the images of the elements of \(\text{Schr}_{1+1}(F\boxtimes 1, \Delta(F))\) and \(\text{Schr}_{1+1}(1\boxtimes F, \Delta(F))\), which are the universal versions of \(F(V) \to F(V \oplus V^*)\) induced by \(v \mapsto v \oplus 0\), resp. \(F(V^*) \to F(V \oplus V^*)\) induced by \(v^* \mapsto 0 \oplus v^*\).

Then \(\alpha_{\text{can}}^\pm\), \(\beta_{\text{can}}^\pm\) are given by \(\alpha_{\text{can}}^\pm(X) = X \circ \text{can}_F^\pm\), \(\beta_{\text{can}}(x) = \text{can}_F^+ \circ (x \boxtimes 1, \text{id}_{\text{LBA}}^+)\), \(\beta_{\text{can}}(x) = \text{can}_F^- \circ (\text{id}_{\text{LBA}}^- \boxtimes x, \text{id}_{\text{LBA}}^-)\).

The proof is a propic version of (a) the double Lie bialgebra construction \(a \mapsto \mathcal{D}(a)\) and (b) the Lie bialgebra morphisms \(a \to \mathcal{D}(a)\), \(a^{\ast\cop} \to \mathcal{D}(a)\), where \(a\) is a finite dimensional Lie bialgebra.

These morphisms have analogues when LBA is replaced by LBA or LBA.

3.2. Doubles of Hopf algebras. In this section, we will set \(P := \text{Bialg} or S(\text{LBA})\).

Define a prop \(D_{\text{mult}}(P)\) by \(D_{\text{mult}}(P)(F, G) := P_2(\Delta_{\text{op}}(F), \Delta_{\text{op}}(G))\), where \(\Delta_{\text{op}} : \text{Schr} \to \text{Schr}_{1+1}\) is given by \(\Delta_{\text{op}}(F)(V, W) := F(V \otimes W)\). We also define prop \((D_{\text{mult}}(P), P)\)-bimodules \(M_{\text{mult}}^\pm(P)\) by

\[
M_{\text{mult}}^+(P)(F, G) := P_2(F \boxtimes 1, \Delta_{\text{op}}(G)) \quad \text{and} \quad M_{\text{mult}}^-(P)(F, G) := P_2(1 \boxtimes F, \Delta_{\text{op}}(G)).
\]

The bimodule structures are defined as follows: the left prop module structure over \(D_{\text{mult}}(P)\) is obvious; the right prop module structures of \(M_{\text{mult}}^\pm(P)\) over \(P\) are defined as above, replacing \(\Delta\) by \(\Delta^-\), and in the case of \(M_{\text{mult}}^-(P), \ast\cop\) by \(\ast\cop_{\text{Bialg}}\). As above, \(M_{\text{mult}}^-(P)\) is also a \((D_{\text{mult}}(P), P^\ast\text{-prop bimodule})\).

We have \(D_{\text{mult}}(\text{Bialg})(T_2, 1\id) = \text{Bialg}(T_2 \boxtimes 1\id, 1\id \boxtimes 1\id) \subset \text{Bialg}(T_2 \boxtimes 1\id, 1\id \boxtimes 1\id)\) and similarly, \(D_{\text{mult}}(\text{Bialg})(1\id, T_2) \subset \text{Bialg}(1\id \boxtimes T_2, T_2 \boxtimes 1\id)\).

**Lemma 3.2.** There are unique prop morphisms \(\text{Double} : \text{Bialg} \to D_{\text{mult}}(\text{Bialg})\), \(\text{left prop} D_{\text{mult}}(\text{Bialg})\)-module morphisms \(\alpha_{\text{Bialg}}^\pm : D_{\text{mult}}(\text{Bialg}) \to M_{\text{add}}^\pm(\text{Bialg})\) and \(\text{right prop} \text{Bialg}\)-module morphisms \(\beta_{\text{Bialg}}^\pm : \text{Bialg} \to M_{\text{mult}}^\pm(\text{Bialg})\), such that

\[
\text{Double}(m) \simeq (m \boxtimes m(2) \boxtimes 1\id) \circ (152346) \circ a^{[2]} \circ (1\id \boxtimes \Delta(2) \boxtimes \Delta),
\]

\[
\text{Double}(\Delta) \simeq \Delta \boxtimes ((21) \circ m), \quad \text{Double}(\varepsilon) = \varepsilon \boxtimes \eta^*, \quad \text{Double}(\eta) = \eta \boxtimes \varepsilon^*,
\]

\[
\alpha_{\text{Bialg}}^+(1\id \boxtimes \text{id}_{\text{Bialg}}) = \beta_{\text{Bialg}}^+(1\id \boxtimes \text{id}_{\text{Bialg}}) = 1\id \boxtimes \text{id}_{\text{Bialg}}^*,
\]

\[
\alpha_{\text{Bialg}}^-(1\id \boxtimes \text{id}_{\text{Bialg}}) = \beta_{\text{Bialg}}^-(1\id \boxtimes \text{id}_{\text{Bialg}}) = \varepsilon \boxtimes \text{id}_{\text{Bialg}}^*.
\]

The diagrams commute.

Here \(a^{[2]} = 1\id \boxtimes \text{id}_{\text{Bialg}} \boxtimes \varepsilon \boxtimes 1\id \boxtimes \text{id}_{\text{Bialg}}\). The prop version of the map \(A^{\otimes 3} \to A^{\otimes 3}, x \otimes y \otimes z \mapsto x y^{(2)} \otimes y^{(3)} z^{(1)} a(y^{(1)}) \otimes z^{(2)}\); it is obtained by dualizing the formula of the multiplication formula of the Drinfeld double.

The prop bimodule properties are expressed as follows:

\[
\alpha_{\text{Bialg}}^+(F, H)(X) = H \circ \alpha_{\text{Bialg}}^+(F, G)(X)
\]
for \( X \in D_{\text{mult}}(\text{Bialg})(F,G), Y \in D_{\text{mult}}(\text{Bialg})(G,H), \)

\[
\beta_{\text{Bialg}}^+(F,H)(y \circ x) = \beta_{\text{Bialg}}^+(G,H)(y) \circ (x \otimes \text{id}_{1}^{\text{Bialg}}),
\]

and

\[
\beta_{\text{Bialg}}^-(F,H)(y \circ x) = \beta_{\text{Bialg}}^-(G,H)(y) \circ (\text{id}_{1}^{\text{Bialg}} \otimes \text{cop}_{\text{Bialg}}(x)).
\]

for \( x \in D_{\text{mult}}(\text{Bialg})(F,G), y \in D_{\text{mult}}(\text{Bialg})(G,H). \)

The proof is the universal version of the proof of the fact that the Drinfeld double \( D(A) \) of a Hopf algebra \( A \) is a bialgebra, equipped with bialgebra morphisms \( A \rightarrow D(A) \) and \( A^{* \text{cop}} \rightarrow D(A). \)

Since \( \text{Double}(\text{id}_{\text{Bialg}} - \eta \circ \varepsilon) = (\text{id}_{\text{Bialg}} - \eta \circ \varepsilon) \otimes \text{id}_{\text{Bialg}}^{*} + (\eta \circ \varepsilon) \otimes (\text{id}_{\text{Bialg}} - \eta \circ \varepsilon)^{*} \), \( \text{Double} \) extends to a prop morphism \( \text{Double} : \text{Bialg} \rightarrow D_{\text{mult}}(\text{Bialg}) \) with the same properties.

### 3.3. Compatibility of quantization functors with doubling operations

Let \( Q : \text{Bialg} \rightarrow S(\text{LBA}) \) be a quantization functor. As we have seen, this is a prop isomorphism.

**Proposition 3.1.** \( Q \) gives rise to an isomorphism of bipo\( \text{p}(\text{LBA}_2). \)

**Proof.** Let \( F, ..., G' \in \text{Sch}. \) The morphism \( Q \) gives rise to a continuous linear isomorphism

\[
Q(F \otimes G^{*}, F' \otimes G) : \text{Bialg}(F \otimes G^{*}, F' \otimes G) \rightarrow S(\text{LBA})(F \otimes G^{*}, F' \otimes G) = \text{LBA}(S(F) \otimes S(G^{*}), S(F') \otimes S(G))
\]

(denoted shortly by \( Q \)). Recall that \( \text{Bialg}_{2}(F \otimes G^{*}, F' \otimes G') \subset \text{Bialg}(F \otimes G^{*}, F' \otimes G) \) and

\[
S^{\otimes 2}(\text{LBA}_2)(F \otimes G, F' \otimes G') = \text{LBA}_2(S(F) \boxtimes S(G), S(F') \boxtimes S(G')) \subset \text{LBA}(S(F) \otimes S(G^{*}), S(F') \otimes S(G)).
\]

We will prove that \( Q \) maps \( \text{Bialg}_{2}(F \otimes G, F' \otimes G') \) bijectively to \( \text{LBA}_2(S(F) \boxtimes S(G), S(F') \boxtimes S(G')). \)
We first prove that $Q$ maps the first space into the second. Consider the diagram

\[
\begin{array}{c}
\hat{\psi}_{Z_{XY} \in \text{Irr}(\text{Sch})} \\
\text{Coalg}(F, Z_{FF'} \otimes Z_{FG}) \otimes \text{Coalg}(G^*, Z_{G^*G}) \\
\otimes \text{Alg}(Z_{FG} \otimes Z_{G^*G}, G^*) \\
\xrightarrow{Q'} \text{LBA}(S(F), S(Z_{FF'} \otimes S(Z_{FG}))) \\
\text{LCA}(S(F), Z_{FF'} \otimes Z_{FG}))
\end{array}
\xrightarrow{(a)}
\begin{array}{c}
\text{Bialg}(F \otimes G^*, F' \otimes G^*)
\end{array}
\]

\[
\begin{array}{c}
\hat{\psi}_{Z_{XY} \in \text{Irr}(\text{Sch})} \\
\text{LBA}(S(F), S(Z_{FF'} \otimes S(Z_{FG}))) \\
\xrightarrow{(b)}
\text{LBA}(S(F) \otimes S(G^*), S(F') \otimes S(G^*))
\end{array}
\]

\[
\begin{array}{c}
\hat{\psi}_{W_{XY}} \text{LCA}(S(F), W_{FF'} \otimes W_{FG}) \otimes \text{LCA}(S(G^*), W_{G^*G}) \\
\otimes \text{LA}(W_{FF'}, S(F')) \otimes \text{LA}(W_{FG}, W_{G^*G}, S(G^*))
\end{array}
\xrightarrow{(f)}
\begin{array}{c}
\hat{\psi}_{W_{XY}} \text{LCA}(S(F), Z_{FF'} \otimes Z_{FG}) \otimes \text{LCA}(S(G^*), Z_{G^*G}) \\
\otimes \text{LA}(Z_{FF'}, S(F'))
\end{array}
\]

\[
\begin{array}{c}
\hat{\psi}_{W_{XY}} \text{LCA}(S(F), Z_{FF'} \otimes Z_{FG}) \otimes \text{LCA}(S(G^*), Z_{G^*G}) \\
\otimes \text{LA}(Z_{FF'}, S(F'))
\end{array}
\xrightarrow{(d)}
\begin{array}{c}
\hat{\psi}_{W_{XY}} \text{LCA}(S(F), Z_{FF'} \otimes Z_{FG}) \otimes \text{LCA}(S(G^*), Z_{G^*G}) \\
\otimes \text{LA}(Z_{FF'}, S(F'))
\end{array}
\]

where the maps labeled $Q$, $Q'$ are those induced by $Q$, the map $(a)$ is $c_{F, FF'} \otimes c_{G^*, G} \otimes a_{FF', FF'} \otimes a_{FF', FF'} \mapsto (a_{FF'}, \text{ex}_{FF', FF'} \circ (\text{inj}_0 \circ \text{pr}_0))$, $\lambda_{F, FF'} \otimes \lambda_{F, FF'} \otimes \lambda_{FF', FF' \otimes FF'} \mapsto (\lambda_{FF', FF'} \otimes \lambda_{FF', FF'}) \circ (\text{inj}_0 \circ \text{pr}_0)^{LBA} \otimes (\text{inj}_0 \circ \text{pr}_0)^{LBA} \otimes (\text{inj}_0 \circ \text{pr}_0)$, and the analogous compositions with $\text{LA}$ instead of $\text{LCA}$. Finally, $(f)$ is the standard composition map.

The upper square of this diagram commutes because $Q$ is a prop morphism. One also checks that the lower square of this diagram commutes, so $Q \circ (a) = (f) \circ (c) \circ (d) \circ (c) \circ Q'$. The image
of (a) is $\text{Bialg}_\bullet(F \boxtimes G, F' \boxtimes G')$, so its image by $Q$ is the image of $Q \circ (a)$, which is therefore contained in the image of $(f)$, which is $\text{LBA}_2(S(F) \boxtimes S(G), S(F') \boxtimes S(G'))$.

Let us now check that $Q^{-1}$ maps $\text{LBA}_2(S(F) \boxtimes S(G), S(F') \boxtimes S(G'))$ to $\text{Bialg}_\bullet(F \boxtimes G, F' \boxtimes G')$.

Consider the diagram

\[
\begin{array}{c}
\hat{\phi}_{Z_{XY}} \in \text{Irr}(\text{Sch}) \\
\text{LCA}(S(F), Z_{FF'}, Z_{FG}) \\
\otimes \text{LCA}(S(G^*), Z_{G'G}) \\
\otimes \text{LA}(Z_{FG} \otimes Z_{G'G}, S(G^*)) \\
Q^{-1} \\
\hat{\phi}_{Z_{XY}} \in \text{Irr}(\text{Sch}) \\
\text{Bialg}(F, Z_{FF'} \otimes Z_{FG}) \\
\otimes \text{Bialg}(G^*, Z_{G'G}) \\
\otimes \text{Bialg}(Z_{FG} \otimes Z_{G'G}, G^*) \\
(c) \\
\hat{\phi}_{W_{XY}} \text{ Coalg}(F, W_{FF'} \otimes W_{FG}) \otimes \text{Coalg}(G^*, W_{G'G}) \\
\otimes \text{Alg}(W_{FF'}, F^*) \otimes \text{Alg}(W_{FG} \otimes W_{G'G}, G^*) \\
Q^{-1} \\
\hat{\phi}_{W_{XY}} \in \text{Irr}(\text{Sch}) \\
\text{Coalg}(F, Z_{FF'} \otimes Z_{FG}) \\
\otimes \text{Coalg}(S(G^*), Z_{G'G}) \\
\otimes \text{Alg}(Z_{FG} \otimes Z_{G'G}, G^*) \\
\otimes \text{Alg}(Z_{G'G} \otimes Z_{G'G}, G^*) \\
\otimes \text{Alg}(Z_{FG} \otimes Z_{G'G}, G^*) \\
\otimes \text{Alg}(Z_{FG} \otimes Z_{G'G}, G^*) \\
(d) \\
\end{array}
\]

Here $Q^{-1}$ is the composed map

\[
\hat{\phi}_{W_{XY}} \in \text{Irr}(\text{Sch}) \text{LCA}(S(F), Z_{FF'} \otimes Z_{FG}) \otimes \text{LCA}(S(G^*), Z_{G'G}) \otimes \text{LA}(Z_{FG} \otimes Z_{G'G}, S(G^*)) \\
-\hat{\phi}_{Z_{XY}} \in \text{Irr}(\text{Sch}) \text{LCA}(S(F), S(Z_{FF'}) \otimes S(Z_{FG})) \otimes \text{LCA}(S(G^*), S(Z_{G'G})) \otimes \text{LA}(S(Z_{FG}) \otimes S(Z_{G'G}), S(G^*)) \\
-\hat{\phi}_{Z_{XY}} \in \text{Irr}(\text{Sch}) \text{Bialg}(F, Z_{FF'} \otimes Z_{FG}) \otimes \text{Bialg}(G^*, Z_{G'G}) \otimes \text{Bialg}(Z_{FG} \otimes Z_{G'G}, S(G^*))
\]

in which the first map is the tensor product of maps $\text{LCA}(S(A), B) \rightarrow \text{LCA}(S(A), S(B))$, $c \mapsto \text{inj}_j(A) \text{LCA} \circ c$ and $\text{LA}(A, S(B)) \rightarrow \text{LA}(S(A), S(B))$, $\ell \mapsto \ell \circ pr_1(A) \text{LA}$, and the second map is the tensor product of the maps induced by $Q^{-1}$. Here the map $(a)$ is defined by $\kappa_{F | F'} \otimes \kappa_{G' | G} \otimes \alpha_{F | F'} \otimes \alpha_{G' | G} \mapsto (\alpha_{F | F'} \boxtimes \alpha_{G' | G}) \circ (\kappa_{F | F'} \otimes \kappa_{G' | G})$; the map $(b)$ is $b_{F | F'} \otimes b_{G' | G} \otimes b_{F | F'} \otimes b_{G' | G} \mapsto (b_{F | F'} \otimes b_{F | F'}) \circ (id - \eta \circ \varepsilon)_{Z_{FF'}} \otimes (id - \eta \circ \varepsilon)_{Z_{FG}} \otimes (id - \eta \circ \varepsilon)_{Z_{G'G}} \otimes (b_{F | F'} \otimes b_{G' | G})$; the map $(c)$ is induced by the isomorphisms (3); the map $(d)$ is a tensor product of composed maps

$$\text{Alg}(A, Z) \otimes \text{Coalg}(Z, B) \rightarrow \text{Bialg}(A, B) \rightarrow \hat{\phi}_{W_{XY}} \in \text{Irr}(\text{Sch}) \text{Coalg}(A, W) \otimes \text{Alg}(W, B),$$
where the first map is $a \otimes c \mapsto c \circ \text{id} - \eta \circ \varepsilon \circ Z \circ a$ and the second map is inverse to $\sum_{W} c_{W} \otimes a_{W} \mapsto \sum_{W} a_{W} \otimes \text{id} - \eta \circ \varepsilon \circ c_{W}$; the map (e) is a tensor product of the maps $\text{Coalg}(A, Z) \otimes \text{Coalg}(Z, W) \rightarrow \text{Coalg}(A, W), e \otimes e' \mapsto (\text{id} - \eta \circ \varepsilon)_{Z} \circ e \circ e'$, their analogues when $(Z, W)$ is replaced by two pairs $(Z_{i}, W_{j})$ $(i = 1, 2)$ or when $\text{Coalg}$ is replaced by $\text{Alg}$; the map (f) is the map $\sum_{XY} c_{F,G} \otimes c_{G'} \otimes a_{F,G'} \otimes a_{F,G'} \mapsto (a_{F,G} \boxtimes a_{G,G'}) \circ [(\text{id} - \eta \circ \varepsilon)_{W_{FF}} \circ (\text{id} - \eta \circ \varepsilon)_{W_{FG}} \boxtimes (c_{F,G} \boxtimes c_{G'})].$

The map (e) is well-defined, since nonzero elements of the the space labeled $(W_{XY}) \oplus (W_{FF}, W_{FG}, W_{GG})$ can be in the image only of the spaces labeled $(W_{XY}, Z_{X|Y}, Z_{XY}|Y)$, where $|Z_{X|Y}|, |Z_{XY}| \leq W_{XY}$, for each pair $(X, Y) \in \{(F, F'), (F, G), (G', G)\}.$

The upper square of this diagram commutes since it can be split into commuting triangle, sum over the $Z_{XY}$ of

$$
\begin{align*}
\text{LCA}(S(F), Z_{FF} \otimes Z_{FG}) \\
\otimes \text{LCA}(S(G''), Z_{GG}) \\
\otimes \text{LA}(Z_{FG} \otimes Z_{GG}, S(G^*)) \\
\otimes \text{LA}(Z_{FG} \otimes Z_{GG}, S(G^*)) \\
\end{align*}
$$

where $(g)$ is the tensor product of a map $\text{LCA}(S(A), Z) \rightarrow \text{LCA}(S(A), S(Z)), \kappa \mapsto \text{inj}_{1}(Z) \text{LCA}_{\otimes} \kappa$ with its analogues with $Z$ replaced by $Z, Z'$, and with $\text{LCA}$ replaced by $\text{LA}$ (and in $\text{inj}_{1}$ by $\text{pr}_{1}$); and $(h)$ is the map $\kappa_{F,G} \otimes \kappa_{G'} \otimes \lambda_{F,G'} \otimes \lambda_{G,G'} \mapsto (\text{id} - \eta \circ \varepsilon)_{Z} \circ \text{id} \circ \text{inj}_{1}(Z) \otimes \text{inj}_{2}(Z);$ this square commutes, since $Q^{-1}$ maps $\text{Bialg}(F \otimes G^*, F' \otimes G^*)$ to $\text{Bialg}(F \otimes G, F' \otimes G);$ and of the commuting square

$$
\begin{align*}
\text{Bialg}(F \otimes G, F' \otimes G^*) \\
\end{align*}
$$

where the map (i) is given by $\text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \mapsto \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \otimes \text{id} \circ \text{id} \circ \text{id} \circ \text{id} \circ \text{id} \circ \text{id}$ and the map (j) is given by the same expression, where (id $- \text{id} \circ \text{id} \circ \text{id}$) is replaced by (id $- \text{id} \circ \text{id} \circ \text{id}$); this square commutes, since $Q^{-1}$ replaces $\text{LBA}_{2}(S(F), S(F')) \rightarrow \text{Bialg}(F \otimes G, F' \otimes G^*)$ with the same argument as before, one obtains that $Q^{-1}$ maps $\text{LBA}_{2}(S(F), S(F')) \rightarrow \text{Bialg}(F \otimes G, F' \otimes G^*).$

\[\text{Recall that } |Z| \text{ is the degree of a homogeneous Schur functor.}\]
In the same way as one proved above that $Q : \mathrm{Bialg}(F \otimes G^*, F' \otimes G^*) \to S(\mathrm{LBA})(F \otimes G^*, F' \otimes G^*)$ restricts to a map $\mathrm{Bialg}_* (F \boxtimes G, F' \boxtimes G') \to S(\mathrm{LBA})_2(F \boxtimes G, F' \boxtimes G')$, one proves more generally that for any finite sets $I, J$, any $\Sigma \subset I \times J$, and any $A \in \mathrm{Sch}_I, B \in \mathrm{Sch}_J$, $Q : \mathrm{Bialg}(\otimes (A), \otimes (B)) \to S(\mathrm{LBA})(\otimes (A), \otimes (B))$ restricts to a linear map (in fact an isomorphism) $\mathrm{Bialg}_*(A, B) \to LBA^\Sigma(S(A), S(B))$ (where $S : \mathrm{Sch} \to \mathrm{Sch}$ takes the functor $A : \mathrm{Vec}^I \to \mathrm{Vec}^J$ to $\mathrm{Vec}^I \overset{S^I} \rightarrow \mathrm{Vec}^J \overset{A} \rightarrow \mathrm{Vec}$, where $A$ is the natural extension of $A$). If moreover $\bullet \notin I \cup J$ and $\bar{\Sigma} \subset \bar{I} \times \bar{J}$ is such that $(\bullet, \bullet) \notin \bar{\Sigma}$ (where $\bar{I} = I \cup \{\bullet\}$; $\bar{J} = J \cup \{\bullet\}$), then the trace maps $\text{tr} : \mathrm{Bialg}_*(A \boxtimes F, B \boxtimes F) \to \mathrm{Bialg}_*(A, B)$ and $\text{tr} : LBA^\Sigma(S(A) \boxtimes S(F), S(B) \boxtimes S(F)) \to LBA^\Sigma(S(A), S(B))$ are such that the diagram

\[
\begin{array}{ccc}
\mathrm{Bialg}_*(A \boxtimes F, B \boxtimes F) & \overset{Q} \longrightarrow & LBA^\Sigma(S(A) \boxtimes S(F), S(B) \boxtimes S(F)) \\
\downarrow \text{tr} & & \downarrow \text{tr} \\
\mathrm{Bialg}_*(A, B) & \overset{Q} \longrightarrow & LBA^\Sigma(S(A), S(B))
\end{array}
\]

commutes. Since the trace is the basic ingredient of the prop structures of $\mathrm{Bialg}_*$ and $S(\mathrm{LBA})_2$, it follows that $Q_2$ is a bieprop morphism, hence isomorphism. \hfill \Box

**Lemma 3.3.** $Q$ gives rise to a prop isomorphism $D_{\mathrm{mult}}(Q) : D_{\mathrm{mult}}(\mathrm{Bialg}) \to D_{\mathrm{mult}}(S(\mathrm{LBA}))$.

**Proof.** For $F, G \in \mathrm{Sch}$, we have isomorphisms $D_{\mathrm{mult}}(\mathrm{Bialg})(F, G) = \mathrm{Bialg}(\Delta_{\otimes}(F), \Delta_{\otimes}(G)) \overset{Q} \cong S_{\boxtimes}(\mathrm{LBA}_2)(\Delta_{\otimes}(F), \Delta_{\otimes}(G)) \cong S(\mathrm{LBA})_2(\Delta_{\otimes}(F), \Delta_{\otimes}(G))$ for $Q = D_{\mathrm{mult}}(S(\mathrm{LBA}))(F, G)$. One checks that they are compatible with the prop operations. \hfill \Box

**Lemma 3.4.** We have a canonical prop isomorphism $D_{\mathrm{mult}}(S(\mathrm{LBA})) \cong S(D_{\mathrm{add}}(\mathrm{LBA}))$.

**Proof.** This is given by $D_{\mathrm{mult}}(S(\mathrm{LBA}))(F, G) = S(\mathrm{LBA})_2(\Delta_{\otimes}(F), \Delta_{\otimes}(G)) = \mathrm{LBA}_2(\Delta_{\otimes}(F), \Delta_{\otimes}(G)) \overset{\Delta_{\otimes}(S(F)), \Delta_{\otimes}(S(G))} = S(D_{\mathrm{add}}(\mathrm{LBA}))(F, G) = D_{\mathrm{add}}(\mathrm{LBA})(S(F), S(G))$;

the third equality uses the canonical isomorphism $S_{\boxtimes}(\mathrm{LBA}_2)(\Delta_{\otimes}(F), \Delta_{\otimes}(G)) \cong \Delta(S(F), S(G))$, which follows from the isomorphism $F(S(V) \otimes S(V)^*) \cong F(S(V \oplus V^*))$ for any $V \in \mathrm{Vec}$. \hfill \Box

We say that $Q$ is compatible with the doubling operations iff there exists an inner automorphism $\text{Inn}(\Lambda)$ of $S(D_{\mathrm{add}}(\mathrm{LBA}))$, where $\Lambda \in S(D_{\mathrm{add}}(\mathrm{LBA}))(\text{id}, \text{id})^*$, such that the diagram

\[
\begin{array}{ccc}
\mathrm{Bialg} & \overset{Q} \longrightarrow & S(\mathrm{LBA}) \\
\downarrow \text{Double} & & \downarrow \text{Double} \\
S(\mathrm{LBA}) & \overset{S(\text{double})} \longrightarrow & S(D_{\mathrm{add}}(\mathrm{LBA})) \overset{\text{Inn}(\Lambda)} \longrightarrow S(D_{\mathrm{add}}(\mathrm{LBA})) \overset{\cong} \longrightarrow D_{\mathrm{mult}}(S(\mathrm{LBA}))
\end{array}
\]

commutes.

Recall that $Q$ gives rise to a prop morphism $Q^* : \mathrm{Bialg}^* \to S(\mathrm{LBA})^*$, and $Q$ is called compatible with duality iff the diagram of props

\[
\begin{array}{ccc}
\mathrm{Bialg}^* & \overset{Q^*} \longrightarrow & S(\mathrm{LBA})^* \\
\downarrow \text{\*cop} & & \downarrow \text{\*cop} \\
\mathrm{Bialg}^* & \overset{Q^*} \longrightarrow & S(\mathrm{LBA})^* \cong S(\mathrm{LBA})^*
\end{array}
\]

commutes.
Theorem 3.1. If a quantization functor $Q$ is compatible with duality, then it is also compatible with the doubling operations. In particular, the EK quantization functors are compatible with the doubling operations.

The rest of this section is devoted to the proof of this theorem.

3.4. Construction of commuting triangles of prop modules based on $S(LBA)$. We have shown that the prop isomorphism $Q : \mathbb{Bialg} \rightarrow S(LBA)$ induces a prop isomorphism $D_{mult}(Q) : D_{mult}(\mathbb{Bialg}) \rightarrow D_{mult}(S(LBA))$.

Lemma 3.5. $Q$ induces isomorphisms

$$M^\pm_{mult}(Q)(F,G) : M^\pm_{mult}(\mathbb{Bialg})(F,G) \rightarrow M^\pm_{mult}(S(LBA))(F,G),$$

(shortly denoted $M^\pm_{mult}(Q)$) with the following compatibilities with $Q$, $Q^*$ and $D_{mult}(Q)$:

$$M^\pm_{mult}(Q)(X \circ m) = D_{mult}(Q)(X) \circ M^\pm_{mult}(Q)(m)$$

for $m \in M^\pm_{mult}(\mathbb{Bialg})(F,G)$, $X \in D_{mult}(\mathbb{Bialg})(G,H)$,

$$M^+_{mult}(Q)(m \circ (\alpha^1 \id_1^{\mathbb{Bialg}})) = M^+_{mult}(Q)(m) \circ (Q(x) \id_1^{S(LBA)^*}),$$

$$M^-_{mult}(Q)(m \circ (\beta^1 \id_1^{\mathbb{Bialg}})) = M^-_{mult}(Q)(m) \circ (id_1^{S(LBA)} \boxtimes Q^*(x^*)},$$

for $x^* \in \mathbb{Bialg}^*(F,G)$, $m \in M^\pm_{mult}(\mathbb{Bialg})(G,H)$.

So $M^\pm_{mult}(Q) : M^\pm_{mult}(\mathbb{Bialg}) \rightarrow M^\pm_{mult}(S(LBA))$ are prop bimodule isomorphisms compatible with the prop morphisms and with $Q$, $Q^*$ and $D_{mult}(Q)$.

Proof. We have isomorphisms

$$M^+_{mult}(\mathbb{Bialg})(F,G) = \mathbb{Bialg}_2(F \boxtimes 1, \Delta_\otimes(G))$$

and

$$M^-_{mult}(\mathbb{Bialg})(F,G) = \mathbb{Bialg}_2(1 \boxtimes F, \Delta_\otimes(G)) = M^+_{mult}(S(LBA))(F,G),$$

and

$$M^\pm_{mult}(\mathbb{Bialg})(F,G) \simeq M^\pm_{mult}(S(LBA))(F,G) \text{ by replacing } F \boxtimes 1 \text{ by } 1 \boxtimes F.$$

The properties of these isomorphisms follow from the fact that $Q_2$ is a bprop morphism such that

$$Q_2(x \boxtimes \id_1^{\mathbb{Bialg}}) = Q(x) \id_1^{S(LBA)^*},$$

and

$$\alpha^1 \id_1^{\mathbb{Bialg}} \boxtimes x^* = \id_1^{S(LBA)} \boxtimes Q^*(x^*).$$

We now transport $\alpha^\pm_{\mathbb{Bialg}}$ and $\beta^\pm_{\mathbb{Bialg}}$ using the isomorphisms $Q$, $D_{mult}(Q)$ and $M^\pm_{mult}(Q)$ as follows.

For $F, G \in \text{Sch}$, define $\hat{\alpha}^\pm(F,G) : D_{mult}(S(LBA))(F,G) \rightarrow M^\pm_{mult}(S(LBA))$ by

$$\hat{\alpha}^\pm(F,G) := M^\pm_{mult}(Q)(F,G) \circ \alpha^\pm_{\mathbb{Bialg}}(F,G) \circ D_{mult}(Q)(F,G)^{-1},$$

define $\hat{\beta}^\pm(F,G) : S(LBA)(F,G) \rightarrow M^\pm_{mult}(S(LBA))$ by

$$\hat{\beta}^\pm(F,G) := M^\pm_{mult}(Q)(F,G) \circ \beta^\pm_{\mathbb{Bialg}}(F,G) \circ Q(F,G)^{-1},$$

and define $\hat{\text{Double}}(F,G) : S(LBA)(F,G) \rightarrow D_{mult}(S(LBA))$ by

$$\hat{\text{Double}}(F,G) := D_{mult}(Q)(F,G) \circ \hat{\text{Double}}(F,G) \circ Q(F,G)^{-1}.$$

Proposition 3.2. $\hat{\text{Double}}$ is a prop morphism. If $Q$ is compatible with duality, then we have

$$\hat{\alpha}^\pm(F,H)(Y \circ X) = Y \circ \hat{\alpha}^\pm(F,G)(X),$$

for $X \in D_{mult}(S(LBA))(F,G)$, $Y \in D_{mult}(S(LBA))(G,H)$,

$$\hat{\beta}^+(F,H)(y \circ x) = \hat{\beta}^+(G,H)(y) \circ (x \boxtimes \id_1^{S(LBA)^*}),$$

$$\hat{\beta}^-(F,H)(y \circ x) = \hat{\beta}^-(G,H)(y) \circ (id_1^{S(LBA)} \boxtimes \text{cop}_{S(LBA)}(x)).$$
for $x \in S(LBA)(F, G)$, $y \in S(LBA)(G, H)$, and $\tilde{\alpha}^\pm(F, G) \circ \widehat{\text{Double}}(F, G) = \tilde{\beta}^\pm(F, G)$ for any $F, G$.

Proof. The first statement follows from the fact that Double is a composition of prop morphisms. The first (resp., second) identity follows from the prop isomorphism property of $D_{\text{mult}}(Q)$ (resp., $Q$) and the prop module properties of $\alpha^\pm_{\text{Bialg}}$ (resp., $\beta^\pm_{\text{Bialg}}$) and $M^\pm_{\text{mult}}(Q)$ (resp., $M^\pm_{\text{mult}}(Q)$). Using the prop isomorphism property of $Q$ and the prop module properties of $\beta^\pm_{\text{Bialg}}$ and $M^\pm_{\text{mult}}(Q)$, we prove that $\tilde{\beta}^-(F, H)(y \circ x) = \tilde{\beta}^-(G, H)(y) \circ (\text{id}^1_{LBA}(Q^* \circ \ast \circ \text{Bialg} \circ Q^{-1})(x))$, which implies the third identity since $Q$ is compatible with duality.

The last identity follows from $\alpha^\pm_{\text{Bialg}}(F, G) \circ \widehat{\text{Double}}(F, G) = \beta^\pm_{\text{Bialg}}(F, G)$ for any $F, G$. $\square$

These identities mean that we have a prop morphism $\widehat{\text{Double}} : S(LBA) \to D_{\text{mult}}(S(LBA))$, left prop $D_{\text{mult}}(S(LBA))$-module morphisms $\tilde{\alpha}^\pm : D_{\text{mult}}(S(LBA)) \to M^\pm_{\text{mult}}(S(LBA))$ and right prop $S(LBA)$-module morphisms $\tilde{\beta}^\pm : S(LBA) \to M^\pm_{\text{mult}}(S(LBA))$, such that $\tilde{\alpha}^\pm \circ \widehat{\text{Double}} = \tilde{\beta}^\pm$.

Now using the canonical isomorphisms $X_{\text{mult}}(S(LBA)) \simeq S(X_{\text{add}}(LBA))$ for $X \in \{D, M^\pm\}$, we view these morphisms as: a prop morphism $\widehat{\text{Double}} : S(LBA) \to D_{\text{add}}(LBA))$, left prop $S(D_{\text{add}}(LBA))$-module morphisms $\tilde{\alpha}^\pm : S(D_{\text{add}}(LBA)) \to S(M^\pm_{\text{add}}(LBA))$ and right prop $S(LBA)$-module morphisms $\tilde{\beta}^\pm : S(LBA) \to S(M^\pm_{\text{add}}(LBA))$, such that the diagrams

$$
\begin{array}{ccc}
S(M^\pm_{\text{add}}(LBA)) & \xrightarrow{\tilde{\beta}^\pm} & S(D_{\text{add}}(LBA)) \\
\downarrow & & \downarrow \\
\uparrow & & \uparrow \\
S(LBA) & \xrightarrow{\tilde{\alpha}^\pm} & D_{\text{add}}(LBA)
\end{array}
$$

commute. The prop module morphism properties are expressed by the relations of Proposition 3.2, where now $X \in S(D_{\text{add}}(LBA))(F, G)$, $Y \in S(D_{\text{add}}(LBA))(G, H)$.

3.5. Construction of a prop morphism $\varphi : LBA \to D_{\text{add}}(LBA)$. The diagram of prop modules

$$
\begin{array}{ccc}
S(Sch) & \xrightarrow{S(\text{id})} & S(LBA) \\
\downarrow & & \downarrow & \leftarrow & \uparrow
\\
S(\text{Sch}) & \xrightarrow{S(\text{id})} & S(D_{\text{add}}(LBA))
\end{array}
$$

\begin{array}{ccc}
S(\text{id}) & \xrightarrow{S(\text{id})} & S(LBA) \\
\downarrow & & \downarrow & \leftarrow & \uparrow
\\
S(\text{id}) & \xrightarrow{S(\text{id})} & S(D_{\text{add}}(LBA))
\end{array}
$$

does not necessarily commute.

In this section, we will construct an inner automorphism $\text{Inn}(\Xi)$ of $S(D_{\text{add}}(LBA))$ (where $\Xi \in S(LBA)(\text{id}, \text{id})^\infty$), and a prop morphism $\varphi : LBA \to D_{\text{add}}(LBA)$, such that $\text{Inn}(\Xi)^{-1} \circ \text{Double} \circ S(\text{LBA}) = S(i_{D_{\text{add}}(\text{LBA})})$ and $\text{Inn}(\Xi)^{-1} \circ \text{Double} = S(\varphi)$. The existence of $\Xi$ and $\varphi$ follows from Proposition 3.3 below.

We first define generators $m^S, \Delta^S, p^S_n (n \geq 0)$ of the prop $S(Sch)$. Recall that $m^S \in S(Sch)(T_2, \text{id})$ and $\Delta^S \in S(Sch)(\text{id}, T_2)$ are the universal versions of the product $S(V)^{\otimes 2} \to S(V)$ and the coproduct $S(V) \to S(V)^{\otimes 2}$ (where $V$ is a vector space). We define $p^S_n \in S(Sch)(\text{id}, \text{id}) = S(Sch, S, S)$ as $p^S_n : = \text{id}_{jn} \circ \text{pr}_n$, i.e., $p^S_n$ is the universal version of the projector $p^S_n \in \text{End}(S(V))$ onto $S^n(V) \subset \oplus_{n \geq 0} S^n(V) = S(V)$. One can prove that $S(Sch)$ is generated by the elements $m^S, \Delta^S, p^S_n$.

Proposition 3.3. Let $P, Q$ be Sch-props graded by $\mathbb{N}$, complete and separated for these gradings. Assume that $\Phi : S(P) \to S(Q)$ is a prop morphism (not necessarily compatible with the
gradings), such that for any of the generators $x \in \{p_n^S, m_n^S, \Delta_n^S\}$ of $S(Sch)$, $(S(i_Q) - \Phi \circ S(i_P))(x)$ has positive degree ($i_P : Sch \to P$, $i_Q : Sch \to Q$ are the canonical morphisms).

Then there exists a prop morphism $\varphi : P \to Q$ and an invertible element $\Xi \in S(Q)(\text{id}, \text{id})^\times$, such that $\Phi = \text{Inn}(\Xi) \circ S(\varphi)$.

Proof. We first prove that if the smallest degree of all $(S(i_Q) - \Phi \circ S(i_P))(x)$ is $N > 0$ (for $x \in S(Sch)(F, G)$ and $F, G \in \text{Ob}(Sch)$), then one can construct $\Xi_N \in S(Q)(\text{id}, \text{id})$ of degree $N$, such that the degree $N$ part of $(S(i_Q) - \text{Inn}(\text{id}) + \Xi_N) \circ \Phi \circ S(i_P))(x)$ vanishes for any $x$.

Defining $\Xi$ as the product of all the id$_{n_0} := (\Xi_{n_0})_{n_0 \geq 0}$, we will then have: $S(i_Q) = \text{Inn}(\Xi)^{-1} \circ \Phi \circ S(i_P)$.

For $x^S \in S(Sch)(F, G)$ and $F, G \in \text{Ob}(Sch)$, define $\dot{x}^Q \in S(Q)(F, G)$ as the degree $N$ part of $(S(i_Q) - \Phi \circ S(i_P))(x^S)$ and set $\dot{x}^Q := S(i_Q)(x^S)$. The condition on $\Xi_N$ is that for any such $x^S$,

$$\dot{x}^Q = (d/dt)_{t=0}(\text{Inn}(\text{id}) + t\Xi_N)(x^Q);$$

it suffices that this holds when $x^S$ belongs to a set of generators of $S(Sch)$.

We set $\Xi_N = \sum_{n_0, n_1 \geq 0}(\Xi_{n_0})_{n_0, n_1}$ according to the decomposition $S(Q)(\text{id}, \text{id}) = \hat{\otimes}_{n_0, n_1 \geq 0}Q(S^n, S^m)$; we will first construct the off-diagonal part of $\Xi_N$, then its diagonal part.

The idempotent relations between $p_n^S$ imply $p_n^Q \circ p_n^Q + p_n^Q \circ p_n^Q = \delta_{n, n} p_n^Q$ and $\sum_n p_n^Q = 0$.

For $n \neq m$, we set $(\Xi_{n, m}) := p_n^Q \otimes p_m^Q$ and $(\Xi_N)_{n, m} := \sum_{n_0, m_0 \neq n, m}(\Xi_{n_0})_{n_0, m_0}$. We will prove that for any $i \geq 0$,

$$p_n^Q = ([\Xi_N]_{n, m} \circ p_n^Q) = \sum_{n_0, m_0 \neq n, m}(\Xi_{n_0})_{n_0, m_0} \circ p_n^Q = 0.$$

We have $([\Xi_N]_{n, m} \circ p_n^Q) = \sum_n q_n^Q \circ p_n^Q = \sum_n p_n^Q \circ q_n^Q = \sum_n p_n^Q \circ p_n^Q - \sum_n p_n^Q \circ p_n^Q = 0$.

For $n = m$, we set $(\Xi_N)_{n, n} := (\Xi_{n_0})_{n_0} \circ (\Xi_{n_0})_{n_0}$. Therefore $([\Xi_N]_{n, n} \circ p_n^Q) = \sum_n p_n^Q \circ p_n^Q = 0$.

We will prove that for any $i \geq 0$,

$$p_n^Q = ([\Xi_N]_{n, n} \circ p_n^Q) = \sum_{n_0} p_n^Q \circ p_{n_0}^Q = 0.$$

We have $([\Xi_N]_{n, n} \circ p_n^Q) = \sum_n q_n^Q \circ p_n^Q = \sum_n p_n^Q \circ q_n^Q = \sum_n p_n^Q \circ p_n^Q = 0$.

Let for $n \geq 0$, $m_n^S := (m^{(n)} \circ (p_1^S)\otimes^n) \in S(Sch)(T_n, \text{id})$ and $\Delta_n^S := (p_1^S)\otimes^n \circ (\Delta^S)^{n) \in S(Sch)(id, T_n)$, where $(m^{(n)} = m^{S \circ \cdots \circ S{id}^n - 2})$ and $(\Delta^S)^{n) \in (\Delta^S \otimes \text{id}^n - 2) \cdots \otimes \Delta^S$.

These elements generate $S(Sch)$, as $m_n^S$ and $\Delta_n^S$ can be constructed from them by $m_n^S = \sum_{p, q \geq 0} m_{p+q}^S \circ (\Delta_p^S \otimes \Delta_q^S)$ and $\Delta_n = \sum_{p, q \geq 0} \frac{1}{p!q!}(m_p^S \otimes m_q^S) \circ \Delta_{p+q}^S$.

They satisfy the relations

$$m_1^S = \Delta_1^S = p_1^S,$$

$$m_n^S \circ m_m^S = \delta_{n, m} m_{n+m}^S,$$

$$\Delta_n^S \circ m_m^S = \delta_{n, m} m_{n+m}^S,$$

$$\Delta_n^S = \frac{p_n^S}{n},$$

where $\text{Sym}_n \in S(Sch)(T_n, T_n)$ is the total symmetrization. (8) implies

$$m_n^S \circ \text{Sym}_n = m_n^S,$$

$$\text{Sym}_n \circ \Delta_n^S = \Delta_n^S.$$
(7) implies that $(\Xi_N)_{11} = 0$, so the r.h.s. is $(\Xi_N)_{n,n} \circ m^Q_n = \frac{1}{m^n} \tilde{m}^Q_n \circ \Delta^Q_n \circ m^Q_n = \tilde{m}^Q_n \circ \text{Sym}_n \circ (p^Q_1)^{\otimes n} \circ \text{Sym}_n = (m^Q_n \circ (p^Q_1)^{\otimes n} \circ \text{Sym}_n) = \tilde{m}^Q_n$, where the second equality follows from the first part of (10), the third equality follows from the equality $\text{Sym}_n \circ (p^Q_1)^{\otimes n} = (p^Q_1)^{\otimes n} \circ \text{Sym}_n$ in $S(\text{Sch})$, the fourth equality follows from $p^Q_1 = 0$, and the last equality follows from the second part of (8) and the first part of (11).

The second part of (10) implies that

$$(\Xi_N)_{nn} = -m^S_n \circ \Delta^S_n.$$ 

We then prove that $\Delta^Q_n = \sum_{i=1}^n (\Xi_N)_{1i} \circ \Delta^Q_n - \Delta^Q_n \circ (\Xi_N)_{nn}$ as above.

We also have $p^Q_1 = [(\Xi_N)_{11}, p^Q_2]$, as $p^Q_2 = 0$ and $(\Xi_N)_{11} = 0$. It follows that if $(\Xi_N)_{\text{diag}} := \sum_{n\geq 1} (\Xi_N)_{nn}$, then $x^Q = \left. (d/dt)_{t=0} \text{Im}(id_{id} + t(\Xi_N)_{\text{diag}})(x^Q) \right|$ for $x^S \in \{m^S_n, \Delta^S_n, p^S_n\}$, hence for any $x^S \in S(\text{Sch})(F,G)$ and any $F,G \in \text{Ob}(\text{Sch})$.

If we now set $\Xi_N := (\Xi_N)_{\text{diag}} + (\Xi_N)_{ij}$, we get $x^Q = \left. (d/dt)_{t=0} \text{Im}(id_{id} + t(\Xi_N)(x^Q)) \right|$ for any $x^S \in S(\text{Sch})(F,G)$, as wanted.

The morphism $\Phi^i := \text{Im}(\Xi)^{-1} \circ \Phi : S(P) \to S(Q)$ now satisfies $\Phi^i \circ S(id_P) = S(id_Q)$. To prove the proposition, it remains to prove:

**Lemma 3.6.** Let $\Phi^i : S(P) \to S(Q)$ be a prop morphism such that $\Phi^i \circ S(id_P) = S(id_Q)$, then there exists a prop morphism $\varphi : P \to Q$ such that $\Phi^i = S(\varphi)$.

**Proof of Lemma.** We have morphisms of Schur functors $\text{inj}_j : \text{id} \to S$ and $pr_1 : S \to \text{id}$, corresponding to the direct sum $S = \text{id} \oplus \oplus_{i \neq 1} S^i$. These induce morphisms of props $P = \text{id}(P) \to S(P)$ and $S(Q) \to \text{id}(Q) = Q$. We then define $\varphi$ as the composed morphism $P \to S(P) \xrightarrow{\Phi^i} S(Q) \to Q$. If $F,G$ are any Schur functors, we want to prove that the diagram

$$
\begin{array}{ccc}
P(F,G) & \overset{\Phi^i(F,G)}{\longrightarrow} & S(Q)(F,G) \\
\oplus_{i,j} P(S^i(F), S^j(G)) & \overset{\oplus_{i,j} \varphi(S^i(F), S^j(G))}{\longrightarrow} & \oplus_{i,j} Q(S^i(F), S^j(G))
\end{array}
$$

commutes, i.e., that for any pair $(i,j)$ of integers, the diagram

$$
\begin{array}{ccc}
P(S^i(F), S^j(G)) & \overset{\varphi(S^i(F), S^j(G))}{\longrightarrow} & Q(S^i(F), S^j(G)) \\
S(P)(F,G) & \overset{\Phi^i(F,G)}{\longrightarrow} & S(Q)(F,G)
\end{array}
$$

commutes. If $F,G$ are any Schur functors, then the map $\varphi(F,G) : P(F,G) \to Q(F,G)$ is the composition $P(F,G) \to P(S(F), S(G)) \approx S(P)(F,G) \xrightarrow{\Phi^i(F,G)} S(Q)(F,G) \approx Q(S(F), S(G)) \to Q(F,G)$, where the initial map is $x \mapsto i_{P}(\text{inj}_1(G)) \circ x \circ i_P(pr_1(F))$ and the final map is $y \mapsto i_Q(pr_1(G)) \circ y \circ i_Q(\text{inj}_1(F))$. So we want to prove that the diagram

$$
\begin{array}{cccc}
S(P)(S^i(F), S^j(G)) & \overset{\Phi^i(S^i(F), S^j(G))}{\longrightarrow} & S(Q)(S^i(F), S^j(G)) \\
S(P)(F,G) & \overset{\Phi^i(F,G)}{\longrightarrow} & S(Q)(F,G) \\
Q(S^i(F), S^j(G)) & \overset{\Phi^i(S^i(F), S^j(G))}{\longrightarrow} & S(Q)(S^i(F), S^j(G)) \\
S(P)(F,G) & \overset{\Phi^i(F,G)}{\longrightarrow} & S(Q)(F,G)
\end{array}
$$
commutes, where the map (a) is $x \mapsto i_p (\text{inj}_1 (S^j(G))) \circ x \circ i_p (\text{pr}_1 (S^i(F)))$, the map (b) is $y \mapsto i_q (\text{pr}_1 (S^j(G))) \circ y \circ i_q (\text{inj}_1 (S^i(F)))$, the map (c) is $x \mapsto \text{inj}_j(G) \circ x \circ \text{pr}_i(F)$, and the map (d) is $x \mapsto \text{pr}_j(G) \circ x \circ \text{inj}_j(F)$. Here $\text{inj}_j : S^j \hookrightarrow S$ and $\text{pr}_i : S \twoheadrightarrow S^i$ are the canonical injection and projection attached to $S = \oplus_{j \geq 0} S^j$.

To prove the commutativity of this diagram, we construct a surjective map $S(P)(F^{\oplus i}, G^{\oplus j}) \to P(S^i, S^j)$ and an injective map $Q(S^i, S^j) \hookrightarrow S(Q)(F^{\oplus i}, G^{\oplus j})$, and prove that the external diagram in

\[
\begin{array}{ccc}
S(P)(F^{\oplus i}, G^{\oplus j}) & \xrightarrow{\Phi'(S^j(F), S^j(G))} & S(Q)(F^{\oplus i}, G^{\oplus j}) \\
\downarrow v & & \downarrow u' \\
S(P)(S^i(F), S^j(G)) & & S(Q)(S^i(F), S^j(G)) \\
\downarrow u & & \downarrow \Phi'(F, G) \\
S(P)(F, G) & & S(Q)(F, G)
\end{array}
\]

commutes, where in this diagram the diagonal arrows are defined by the condition that the triangles commute and the inner rectangle is the above diagram. The commutativity of the external diagram then implies that of the inner rectangle.

We first construct the maps $S(P)(F^{\oplus i}, G^{\oplus j}) \to P(S^i(F), S^j(G))$ and $Q(S^i(F), S^j(G)) \to S(Q)(F^{\oplus i}, G^{\oplus j})$. There is a unique morphism of Schur functors $a_i : S^i \hookrightarrow S \circ \text{id}^{\oplus i}$, given by the composition $S^i \hookrightarrow \text{id}^{\oplus i} \text{inj}_{\oplus i} S^i = S \circ \text{id}^{\oplus i}$ and $b_i : S \circ \text{id}^{\oplus i} \twoheadrightarrow S^i$, given by the composition $S \circ \text{id}^{\oplus i} = S^{\oplus i} \xrightarrow{\text{pr}_{\oplus i}} \text{id}^{\oplus i} \to S^i$. Then $S(P)(F^{\oplus i}, G^{\oplus j}) \to P(S^i(F), S^j(G))$ is $x \mapsto b_j(G) \circ x \circ a_i(F)$ and $Q(S^i(F), S^j(G)) \twoheadrightarrow S(Q)(F^{\oplus i}, G^{\oplus j})$ is $y \mapsto a_j(G) \circ y \circ b_i(F)$.

We now compute the diagonal maps. We define $a_i \in S(Sch)(\text{id}^{\oplus i}, S^i) \simeq Sch(S \circ \text{id}^{\oplus i}, S \circ S^i)$ as the composition $S \circ \text{id}^{\oplus i} \simeq S^{\oplus i} \xrightarrow{\text{pr}_{\oplus i}} \text{id}^{\oplus i} \to S^{\text{inj}_{\oplus i}} S$. Similarly, we define:

- $\beta_i \in S(Sch)(\text{id}^{\oplus i}, \text{id})$ as the composition $S \circ \text{id}^{\oplus i} \simeq S^{\oplus i} \xrightarrow{\text{pr}_{\oplus i}} \text{id}^{\oplus i} \to S^{\text{inj}_{\oplus i}} S$;
- $\alpha_i \in S(Sch)(S^i, \text{id}^{\oplus i})$ as the composition $S \circ S^i \twoheadrightarrow S^i \hookrightarrow \text{id}^{\oplus i} \text{inj}_{\oplus i} S^{\oplus i} \simeq S \circ \text{id}^{\oplus i}$;
- $\beta_i' \in S(Sch)(\text{id}, \text{id}^{\oplus i})$ as the composition $S \circ \text{id} \simeq S \twoheadrightarrow S^i \hookrightarrow \text{id}^{\oplus i} \text{inj}_{\oplus i} S^{\oplus i} \simeq S \circ \text{id}^{\oplus i}$.

The diagonal map $u$ is then $x \mapsto (S(i_p)(\alpha_j(G)) \circ x \circ S(i_p)(\alpha_i'(F)))$; the map $v$ is $x \mapsto (S(i_p)(\beta_j(G)) \circ x \circ S(i_p)(\beta_i'(F)))$; the map $v'$ is $y \mapsto S(i_q)(\alpha_j'(G) \circ y \circ S(i_q)(\alpha_i(F)))$; and the map $u'$ is $u' \circ \Phi'(S^j(F), S^j(G)) \circ u = u' \circ \Phi'(F, G) \circ v$.

Let $x \in S(P)(F^{\oplus i}, G^{\oplus j})$. Then

\[
(u' \circ \Phi'(S^j(F), S^j(G)) \circ u)(x)
= (u' \circ \Phi'(S^j(F), S^j(G)))(S(i_p)(\alpha_j(G)) \circ x \circ S(i_p)(\alpha'_i(F)))
= u'(S(i_q)(\alpha_j(G)) \circ \Phi'(F^{\oplus i}, G^{\oplus j})(x) \circ S(i_q)(\alpha'_i(F)))
= S(i_q)(\alpha'_j(G) \circ S(i_q)(\alpha_j(G)) \circ \Phi'(F^{\oplus i}, G^{\oplus j})(x) \circ S(i_q)(\alpha'_i(F)) \circ S(i_q)(\alpha_i(F))
= S(i_q)(\alpha'_j \circ \alpha_j(G)) \circ \Phi'(F^{\oplus i}, G^{\oplus j})(x) \circ S(i_q)(\alpha'_i \circ \alpha_i(F)),
\]
where the second equality follows from $\Phi' \circ S(i_q) = S(i_q)$. On the other hand,
\[
(v' \circ \Phi'(F, G) \circ v)(x) 
= (v' \circ \Phi'(F, G))(S(i_{ip})(\beta_i(G))) \circ x \circ S(i_{ip})(\beta'_i(F))) 
= v'(S(i_{iq})(\beta'_i(G))) \circ \Phi'(F^{\otimes 1}, G^{\otimes 2})(x) \circ S(i_{iq})(\beta'_i(F))) 
= S(i_{iq})(\beta'_i(G)) \circ S(i_{iq})(\beta_i(G)) \circ \Phi'(F^{\otimes 1}, G^{\otimes 2})(x) \circ S(i_{iq})(\beta_i(F)) 
= S(i_{iq})(\beta'_i \circ \beta_i(G)) \circ \Phi'(F^{\otimes 1}, G^{\otimes 2})(x) \circ S(i_{iq})(\beta_i(F)).
\]

To prove that these terms are equal, we will prove that
\[
\alpha'_i \circ \alpha_i = \beta'_i \circ \beta_i
\]
(an equality of endomorphisms of the Schur functor $S \circ \text{id}^\otimes(i_{ip})$). We have $\alpha_i = \text{inj}_1(S^i) \circ b_i$, $\beta_i = \text{inj}_1 \circ b_i$, $\alpha'_i = \text{inj}_1 \circ \text{pr}_i(S^i)$, $\beta'_i = \text{inj}_1 \circ \text{pr}_i$, so it suffices to show the equality $\text{pr}_1(S^i) \circ \text{inj}_1(S^i) = \text{pr}_1 \circ \text{inj}_1$. But $\text{pr}_1 \circ \text{inj}_1 = \text{id}_{S^i}$, and $\text{pr}_1 \circ \text{inj}_1 = \text{id}_{S^i}$, which implies that $\text{pr}_1(S^i) \circ \text{inj}_1(S^i) = \text{id}_{S^i}$. This ends the proof of the lemma, and hence of the proposition.

3.6. Construction of commuting triangles of prop modules based on $\text{LBA}$. Let us then define $\hat{\alpha}^\pm : S(D^\text{add}(\text{LBA})) \to S(M^\pm_{\text{add}}(\text{LBA}))$ by $\hat{\alpha}^\pm(X) := \Xi_G^{-1} \circ \hat{\alpha}^\pm(\text{Inn}(\Xi)(X))$ for $X \in S(D^\text{add}(\text{LBA}))(F, G)$, and $\hat{\beta}^\pm : S(\text{LBA}) \to S(M^\pm_{\text{add}}(\text{LBA}))$ by $\hat{\beta}^\pm(x) := \hat{\alpha}^\pm(\psi(x))$ for $x \in S(\text{LBA})(F, G)$.

**Lemma 3.7.** $\hat{\alpha}^\pm$ is a prop left $S(D^\text{add}(\text{LBA}))$-module morphism and $\hat{\beta}^\pm$ is a prop right $S(\text{LBA})$-module morphism.

**Proof.** We have for $X \in S(D^\text{add}(\text{LBA}))(F, G)$ and $Y \in S(D^\text{add}(\text{LBA}))(G, H)$, $\hat{\alpha}^\pm(Y \circ X) = Y \circ \hat{\alpha}^\pm(X)$. Then $\hat{\alpha}^\pm(Y \circ X) = \Xi_H^{-1} \circ \hat{\alpha}^\pm(\Xi_H \circ Y \circ X \circ \Xi_F^{-1}) = \Xi_H^{-1} \circ \hat{\alpha}^\pm(\Xi_H \circ Y' \circ X') = \Xi_H^{-1} \circ Y' \circ \hat{\alpha}^\pm(X') = Y \circ \Xi_G^{-1} \circ \hat{\alpha}^\pm(\Xi_G \circ X \circ \Xi_F^{-1}) = Y \circ \hat{\alpha}^\pm(X)$, where $Y' = \Xi_H \circ Y \circ \Xi_G^{-1}$, $X' = \Xi_G \circ X \circ \Xi_F^{-1}$.

On the other hand, we have for $x \in S(\text{LBA})(F, G)$, $\hat{\beta}^\pm(x) = \Xi_G^{-1} \circ \hat{\beta}^\pm(x)$, so the prop module properties of $\hat{\beta}^\pm$ imply that for $y \in S(\text{LBA})(F, G)$, we have $\hat{\beta}^\pm(y \circ x) = \hat{\beta}^\pm(y) \circ (x \Xi \text{id}_{\text{LBA}}^\pm \circ \text{cop}(x))$.

So we have commuting triangles of prop morphisms

\[
\begin{array}{ccc}
S(M^\pm_{\text{add}}(\text{LBA})) & \xrightarrow{\beta^\pm} & S(\text{LBA}) \\
\xleftarrow{\hat{\alpha}^\pm} & S(\psi) & \xrightarrow{\alpha^\pm} S(D^\text{add}(\text{LBA}))
\end{array}
\]

For $F \in \text{Sch}$, set $\hat{\alpha}^\pm_F := \hat{\alpha}^\pm(\text{id}_{F}^{S(D^\text{add}(\text{LBA}))})$, $\hat{\beta}^\pm_F := \hat{\beta}^\pm(\text{id}_{F}^{S(\text{LBA})})$; then $\hat{\alpha}^\pm_F, \hat{\beta}^\pm_F \in S(M^\pm_{\text{add}}(\text{LBA}))(F, F)$.

The lemma implies that if $x \in S(\text{LBA})(F, G)$, then $\hat{\beta}^\pm(x) = \hat{\beta}_G^\pm \circ (x \Xi \text{id}_{\text{LBA}}^\pm \circ \text{cop}(x))$, $\hat{\beta}^- (x) = \hat{\beta}_G^- \circ (\text{id}_{\text{LBA}}^\pm \circ \text{cop}(x))$, and if $X \in S(D^\text{add}(\text{LBA}))(F, G)$, then $\hat{\alpha}^\pm(X) = X \circ \hat{\alpha}^\pm_F$.

We have therefore
\[
(12) \quad \hat{\beta}^\pm_X \circ (x \Xi \text{id}_{\text{LBA}}^\pm) = S(\psi)(x) \circ \hat{\alpha}^\pm_F, \quad \hat{\beta}^-_X \circ (\text{id}_{\text{LBA}}^\pm \circ \text{cop}(x)) = S(\psi)(x) \circ \hat{\alpha}^-_F,
\]
for $x \in S(\text{LBA})(F, G)$.

For $F = G$ and $x = \text{id}_{F}^{S(\text{LBA})}$, we get $\hat{\beta}_F^- = \hat{\alpha}^\pm_F$.

For $X_i \in S(D^\text{add}(\text{LBA}))(F, G_i)$ ($i = 1, 2$), $\hat{\beta}^\pm(X_1 \Xi X_2) = \hat{\beta}^\pm(X_1) \Xi \hat{\beta}^\pm(X_2)$ so $(X_1 \Xi X_2) \circ \hat{\alpha}^\pm_F \otimes G = (X_1 \circ \hat{\alpha}^\pm_F) \Xi (X_2 \circ \hat{\alpha}^\pm_G)$, so $\hat{\alpha}^\pm_F \Xi \hat{\alpha}^\pm_G$. Moreover, for $f \in \text{Sch}(F, G)$, (12) together
with \( S(\varphi) \circ S(i_{LBA}) = S(i_{D_{add}(LBA)}) \) implies \( \hat{\alpha}^\pm G \circ i_{S(LBA)}(f) = i_{S(D_{add}(LBA))}(f) \circ \hat{\alpha}^\pm G \). This implies that the \( \hat{\alpha}_id^\pm \) are obtained from \( \alpha_id^\pm \) by inflation (see Proposition 1.1).

We now study \( \hat{\alpha}_id^\pm \in S(M_{add}^+(LBA))(id, id) = M_{add}^+(LBA)(S, S) \).

Let \( inj_1 \in \text{Sch}(id, S) \) and \( pr_j \in \text{Sch}(S, id) \) be the canonical morphisms. Set \( \alpha_id^\pm := pr_1 S(D_{add}(LBA)) \circ \hat{\alpha}_id^\pm \circ (inj_1 S(LBA))^1 \circ \text{id}^1 LBA^* \in M_{add}^+(LBA)(id, id) \). Then:

Lemma 3.8. We have
\[
\hat{\alpha}_id^\pm = (\alpha_id^\pm)_{S},
\]
i.e.,
\[
\hat{\alpha}_id^\pm = \sum_{k \geq 0} (inj_1 k \circ pr_k D_{add}(LBA)) \circ (\alpha_id^\pm)^{\otimes k} \circ (id^1 \otimes \text{id}^1 LBA^*),
\]
\[
\hat{\alpha}_id^- = \sum_{k \geq 0} (inj_1 k \circ pr_k D_{add}(LBA)) \circ (\alpha_id^\pm)^{\otimes k} \circ (\text{id}^1 LBA \otimes (id_1 \circ pr_k LBA^*),
\]
where \( \text{id}^1 \otimes pr_k \xrightarrow{id} S \) is the natural morphisms in \( \text{Sch} \) (with \( pr_k \circ inj_1 = id_{S^k} \)).

Proof. We prove (14). For \( x = S(i_{LBA})(p^S_\alpha) \in S(LBA)(id, id), \) (12) together with \( S(\varphi) \circ S(i_{LBA}) = S(i_{D_{add}(LBA)}) \) gives \( \hat{\alpha}_id^\pm \circ (S(i_{LBA})(p^S_\alpha) \otimes \text{id}^1 LBA^*) = S(i_{D_{add}(LBA)})(p^S_\alpha) \circ \hat{\alpha}_id^\pm \). Recall that \( \alpha_id^\pm \) decomposes uniquely as \( \sum_{n,m \geq 0} inj_{D_{add}(LBA)}^n \circ \alpha_n \circ (\text{id}^1 LBA \otimes \text{id}^1 LBA^*), \) where \( \alpha_n \in M_{add}^+(LBA)(S^n, S^m) \); then the above identity implies \( \sum_{q \geq 0} inj_{D_{add}(LBA)}^q \circ \alpha_n \circ (pr_n LBA \otimes \text{id}^1 LBA^*), \) where \( \alpha_n \in M_{add}^+(LBA)(S^n, S^m) \).

For \( x = S(i_{LBA})(m^S_n) \), (12) gives \( \hat{\alpha}_id^\pm \circ S(i_{LBA})(m^S_n) = S(i_{D_{add}(LBA)})(m^S_n) \circ (\hat{\alpha}_id^\pm)_{S^n} \). Composing this equality with \( inj_1 \otimes \text{id}^1 LBA^* \) from the right and with \( pr_n D_{add}(LBA) \) from the left, we get
\[
\alpha_n \circ (p^S_\alpha \otimes \text{id}^1 LBA^*) = p^n_{D_{add}(LBA)} \circ (\hat{\alpha}_id^\pm)_{S^n}
\]
(an identity in \( M_{add}^+(LBA)(id^S, S^n) \)). Composing from the right by \( (id_n \circ pr_n) LBA \otimes \text{id}^1 LBA^* \) and from the left by \( inj_n D_{add}(LBA) \), we find that \( inj_n D_{add}(LBA) \circ \alpha_n \circ (pr_n LBA \otimes \text{id}^1 LBA^*) \) is the \( n \)th term of the r.h.s. of (14), which implies this identity. The identity on \( \hat{\alpha}_id^\pm \) is proved in the same way, which implies (13).

By computing classical limits, one shows that \( \alpha_id^\pm = \text{can}_{id} \pm + \text{terms of higher degree} \), and similarly \( \beta_id^\pm = \text{can}_{id} \pm + \text{terms of higher degree} \) (\text{can}_{id}^\pm are defined after Lemma 3.1).

Define \( \alpha_F^\pm \in M_{add}^+(LBA)(F, F) \) by applying inflation (Proposition 1.1) to \( \alpha_id^\pm \in M_{add}^+(LBA)(id, id) \). Since we have \( \alpha_F^\pm G = \alpha_F^\pm G \), the maps \( D_{add}(LBA)(F, G) \ni X \mapsto X \circ \alpha_F^\pm \in M_{add}^+(LBA)(F, G) \) and \( LBA(F, G) \ni x \mapsto \alpha_F^\pm \circ (x \circ \text{id}^1 LBA^*) \in D_{add}(LBA)(F, G) \), resp., \( LBA(F, G) \ni x \mapsto \alpha_F^\pm \circ (\text{id}^1 LBA \otimes \text{id}^1 LBA^*), \) define a left prop \( D_{add}(LBA) \)-module morphism \( \alpha^\pm : D_{add}(LBA) \rightarrow M_{add}^+(LBA) \) and a right prop \( LBA \)-module morphism \( \beta^\pm : LBA \rightarrow M_{add}^+(LBA) \).

Lemma 3.9. We have \( \hat{\alpha}^\pm = S(\alpha^\pm), \hat{\beta}^\pm = S(\beta^\pm) \).

Proof. This follows from the fact that if \( F \in \text{Ob}(\text{Sch}) \), the identification \( S(M_{add}^+(LBA))(F, F) \simeq M_{add}^+(LBA)(F \circ S, F \circ S) \) takes \( \hat{\alpha}_F^\pm \) to \( \alpha^\pm \). Let \( \gamma_F^\pm \in M_{add}^+(LBA)(F \circ S, F \circ S) \) be the image of \( \hat{\alpha}_F^\pm \). Then (13) implies that \( \gamma^\pm = \alpha^\pm S \). On the other hand, the relations satisfied by
the \( \hat{\alpha}^\pm \) imply that \( \gamma^\pm_{FoS} = \gamma^\pm_F \boxtimes \gamma^\pm_G \), and if \( f \in \text{Sch}(F,G) \), then \( (f \circ S) \text{D}_{\text{add}}(\text{LBA}) \circ \gamma^+_F = \gamma^+_G \circ ((f \circ S) \text{LBA} \boxplus \text{id}_1^{\text{LBA}^+}), (f \circ S) \text{D}_{\text{add}}(\text{LBA}) \circ \gamma^-_F = \gamma^-_G \circ (\text{id}_1^{\text{LBA}^-} \boxtimes (f \circ S) \text{LBA}^+) \). All these conditions are satisfied by the family \((\alpha^\pm_{FoS})_{F \in \text{Ob}(\text{Sch})}\), and since they determine the family \((\gamma^\pm_F)_{F \in \text{Ob}(\text{Sch})}\) uniquely, we obtain \( \gamma^\pm_F = \alpha^\pm_{FoS} \), as wanted. \( \square \)

**Lemma 3.10.** The diagrams of prop module morphisms

\[
\begin{array}{ccc}
M^\pm_{\text{add}}(\text{LBA}) & \xrightarrow{\beta^\pm} & D_{\text{add}}(\text{LBA}) \\
\text{LBA} & \xrightarrow{\varphi} & D_{\text{add}}(\text{LBA}) \\
\end{array}
\]

commute.

**Proof.** Let \( F, G \in \text{Ob}(\text{Sch}) \). We know that the diagram

\[
\begin{array}{ccc}
S(M^\pm_{\text{add}}(\text{LBA}))(F,G) & \xrightarrow{\beta^\pm(F,G)} & S(D_{\text{add}}(\text{LBA}))(F,G) \\
S(\text{LBA})(F,G) & \xrightarrow{S(\varphi)(F,G)} & S(D_{\text{add}}(\text{LBA}))(F,G) \\
\end{array}
\]

commutes, i.e., the inner triangle of the following diagram

\[
\begin{array}{ccc}
M^\pm_{\text{add}}(\text{LBA})(F,G) & \xrightarrow{\beta^\pm(F,G)} & D_{\text{add}}(\text{LBA})(F \circ S, G \circ S) \\
\text{LBA}(F \circ S, G \circ S) & \xrightarrow{\varphi(FoS,GoS)} & D_{\text{add}}(\text{LBA})(F \circ S, G \circ S) \\
\end{array}
\]

commutes. Here the maps between the triangles are

\[
\begin{align*}
\text{LBA}(F,G) & \to \text{LBA}(F \circ S, G \circ S), \quad x \mapsto G(inj_1)^{\text{LBA}} \circ x \circ (F(pr_1))^{\text{LBA}}, \\
D_{\text{add}}(\text{LBA})(F,G) & \to D_{\text{add}}(\text{LBA})(F \circ S, G \circ S), \quad X \mapsto G(inj_1)^{D_{\text{add}}(\text{LBA})} \circ X \circ (F(pr_1))^{D_{\text{add}}(\text{LBA})}, \\
M^+_{\text{add}}(\text{LBA})(F,G) & \to M^+_{\text{add}}(\text{LBA})(F \circ S, G \circ S), \quad m \mapsto G(inj_1)^{D_{\text{add}}(\text{LBA})} \circ m \circ ((F(pr_1))^{\text{LBA} \boxplus \text{id}_1^{\text{LBA}^-})), \\
M^-_{\text{add}}(\text{LBA})(F,G) & \to M^-_{\text{add}}(\text{LBA})(F \circ S, G \circ S), \quad m \mapsto G(inj_1)^{D_{\text{add}}(\text{LBA})} \circ m \circ (\text{id}_1^{\text{LBA}^-} \boxtimes (F(pr_1))^{\text{LBA}^-}).
\end{align*}
\]

Since these maps are injective, the commutativity of the outer triangle follows from that of the three side quadrilaterals. We now check the commutativity of these diagrams. The bottom diagram obviously commutes. The commutativity of the upper right diagram follows from the identities

\[
\begin{align*}
(F(pr_1))^{D_{\text{add}}(\text{LBA})} \circ \alpha^-_{FoS} & = \alpha^-_F \circ ((F(pr_1))^{\text{LBA} \boxplus \text{id}_1^{\text{LBA}^-}}), \\
(F(pr_1))^{D_{\text{add}}(\text{LBA})} \circ \alpha^-_{FoS} & = \alpha^-_F \circ (\text{id}_1^{\text{LBA}^-} \boxtimes (F(pr_1))^{\text{LBA}^-}).
\end{align*}
\]
and the commutativity of the upper left diagram follows from the identities
\begin{equation}
(G(\text{inj}_1))^{D_{add}(\text{LBA})} \circ \alpha_G^+ = \alpha_{G\circ S} \circ ((G(\text{inj}_1))^{\text{LBA}} \otimes \text{id}_1^{\text{LBA}^*}),
\end{equation}
\begin{equation}
(G(\text{inj}_1))^{D_{add}(\text{LBA})} \circ \alpha_G^- = \alpha_{G\circ S} \circ (\text{id}_1^{\text{LBA}} \otimes (G(\text{inj}_1))^{\text{LBA}^*}).
\end{equation}

We now prove (16), (17). When $F = \text{id}$ or $G = \text{id}$, these identities follow from the equations (14), (15) relating $\alpha_S^\pm$ and $\alpha_{\text{id}}^\pm$. Let us prove the first identity of (16). We denote its l.h.s. by $\xi_F$ and its r.h.s. by $\eta_F$. Both are elements of $M_{add}^{+}(\text{LBA})(F \circ S, F)$: $(\xi_F)_{F \in \text{Sch}}$ satisfies the relations $\xi_F \otimes G = \xi_F \otimes \xi_G$ and for $f \in \text{Sch}(F, G)$, $f^{D_{add}(\text{LBA})} \circ \xi_F = \xi_G \circ (f^{\text{LBA}} \otimes \text{id}_1^{\text{LBA}^*})$, and $(\eta_F)_{F \in \text{Sch}}$ satisfies the same relations; we also have $\xi_{\text{id}} = \eta_{\text{id}}$. Applying the uniqueness proposition 1.1 to the biprop $(F, G) \to M_{add}^{+}(\text{LBA})(F \circ S, G)$ (over the props $D_{add}(\text{LBA})$ and $S(\text{LBA})$), we get $\xi_F = \eta_F$ for any $F$. The other identities are proved in the same way. \hfill $\Box$

3.7. The relation between $\varphi$ and double. In particular, we have for $x \in \text{LBA}(F, G)$,
\begin{equation}
\varphi(x) \circ \alpha_F^+ = \alpha_G^+ \circ (x \otimes \text{id}_1^{\text{LBA}^*}), \quad \varphi(x) \circ \alpha_F^- = \alpha_G^- \circ (\text{id}_1^{\text{LBA}} \otimes \text{cop}(x)).
\end{equation}

We construct $\gamma \in D_{add}(\text{LBA})(\text{id}, \text{id})^\times$ such that $\gamma \circ \alpha_{\text{id}}^\pm = \text{can}_{\text{id}}^\pm$, where $\text{can}_{\text{id}}^\pm \in M_{add}(\text{LBA})(\text{id}, \text{id})$ have been defined in Section 3.1.

We have canonical identifications $M_{add}^{+}(\text{LBA})(\text{id}, \text{id}) \simeq \text{LBA}(\text{id}, \text{id})$ and $M_{add}^{-}(\text{LBA})(\text{id}, \text{id}) \simeq \text{LBA}^{*}(\text{id}, \text{id})$; they send $\text{can}_{\text{id}}^+ \text{id}_{\text{id}}^{\text{LBA}}$ to $\text{id}_{\text{id}}^{\text{LBA}^*}$, resp., $\text{id}_{\text{id}}^{\text{LBA}^*}$ to $\text{id}_{\text{id}}^{\text{LBA}}$. Let us again denote by $\alpha_{\text{id}}^\pm$ the images of $\text{can}_{\text{id}}^\pm$ in $\text{LBA}(\text{id}, \text{id})$, resp., in $\text{LBA}^{*}(\text{id}, \text{id})$. These elements expand as $\text{id}_{\text{id}}^{\text{LBA}^*}$ + higher terms, resp., $\text{id}_{\text{id}}^{\text{LBA}}$ + higher terms, hence they are invertible. Let us set
\begin{equation}
\gamma := (\alpha_{\text{id}}^+)^{-1} \otimes \text{id}_1^{\text{LBA}^*} + \text{id}_1^{\text{LBA}} \otimes (\alpha_{\text{id}}^-)^{-1},
\end{equation}
then $\gamma \in (\text{LBA} \otimes \text{LBA}^*)(\Delta(\text{id}), \Delta(\text{id})) \subset D_{add}(\text{LBA})(\text{id}, \text{id})$ is invertible, and satisfies $\gamma \circ \alpha_{\text{id}}^\pm = \text{can}_{\text{id}}^\pm$.

We then set $\hat{D} := \text{Im}(\gamma) \circ \varphi$. Then $\hat{D} : \text{LBA} \to D_{add}(\text{LBA})$ is a prop morphism. For any $x \in \text{LBA}(F, G)$, $\hat{D}(x) \in \text{LBA}_2(\Delta(F), \Delta(G))$ and
\begin{equation}
\hat{D}(x) \circ \text{can}_x^+ = \text{can}_x^- \circ (x \otimes \text{id}_1^{\text{LBA}^*}), \quad \hat{D}(x) \circ \text{can}_x^- = \text{can}_x^+ \circ (\text{id}_1^{\text{LBA}} \otimes \text{cop}(x)),
\end{equation}
(the first identity is in $\text{LBA}_2(F \otimes 1, \Delta(G))$ and the second identity is in $\text{LBA}_2(1 \otimes F, \Delta(F))$).

Indeed, $\hat{D}(x) \circ \text{can}_x^+ = \gamma_G \circ \varphi(x) \circ \gamma_F^{-1} \circ \text{can}_x^+ = \gamma_G \circ \varphi(x) \circ \alpha_F^+ = \gamma_G \circ \alpha_G^- \circ (x \otimes \text{id}_1^{\text{LBA}^*}) = \text{can}_x^+ \circ (x \otimes \text{id}_1^{\text{LBA}^*})$; the second identity is proved in the same way.

This means that we have a commutative diagram of prop modules
\begin{equation}
\begin{array}{ccc}
\text{LBA} & \xrightarrow{\beta_{can}^\pm} & \text{LBA}^* \\
\downarrow{\alpha_{can}^\pm} & & \downarrow{\alpha_{can}^\pm} \\
D_{add}(\text{LBA}) & \xrightarrow{\alpha_{\text{id}}^\pm} & D_{add}(\text{LBA})
\end{array}
\end{equation}

We will show:

**Proposition 3.4.** For $\lambda$ a scalar, set $Z_\lambda := \exp(\lambda(\mu \circ \delta)) \in \text{LBA}(\text{id}, \text{id})$. There exists a scalar $\lambda$ such that $\hat{D} = \text{Im}(\text{id}_1^{\text{LBA}} \otimes \text{id}_1^{\text{LBA}^*}) \otimes (Z_\lambda) \circ \text{double}$.

**Proof of Proposition.** $\hat{D}$ is uniquely determined by $\hat{D}(\mu) \in D_{add}(\text{LBA})(\wedge^2, \text{id}) = \text{LBA}_2(\wedge^2 \otimes 1 \otimes \text{id} \otimes 1 \otimes \wedge^2, 1 \otimes 1 \otimes \text{id} \otimes 1 \otimes \wedge^2) = \text{LBA}_2(1 \otimes 1 \otimes \wedge^2, 1 \otimes 1 \otimes 1 \otimes 1 \otimes \wedge^2) = \text{LBA}_2(1 \otimes 1 \otimes \wedge^2, 1 \otimes 1 \otimes 1 \otimes 1 \otimes \wedge^2)$.

In the finite dimensional case, the semiclassical limit of the Drinfeld double of the quantization of a Lie bialgebra $a$ is $\mathcal{O}(a)$. The propic counterpart of this fact is that we have the
Lemma 3.11. Proposition (bracket expansions) gives a realization of $\hat{\mu}$ with an element $\hat{\sigma}$.

We decompose $\hat{\mu}$ as $\mu_\lambda \circ \text{id}_{1,2,3} + \mu_{1,2,3} \circ \text{id}_{1,2,3} + \mu_{1,2,3} \circ \text{id}_{1,2,3} \circ \text{id}_{1,2,3}$, where $\mu_\lambda \circ \text{id}_{1,2,3} \in \text{LBA}_\lambda(\Lambda^2 1, \text{id}\Lambda21)$, etc. Similarly, we decompose $\hat{\sigma}$ as $\sigma_\lambda \circ \text{id}_{1,2,3} + \sigma_{1,2,3} \circ \text{id}_{1,2,3} \circ \text{id}_{1,2,3}$.

The relation $\hat{\mu} \circ \text{can}_{1,2} = \text{can}_{1,2} \circ \mu$ gives $\mu_\lambda \circ \text{id}_{1,2,3} = \mu$. Similarly, $\hat{\sigma} \circ \text{can}_{1,2} = \text{can}_{1,2} \circ \sigma$ gives $\sigma_{1,2,3} \circ \text{id}_{1,2,3} \circ \text{id}_{1,2,3} = \sigma$.

The relation $\text{can}_{1,2} \circ \delta = \hat{\sigma} \circ \text{can}_{1,2}$ gives $\delta_{1,2,3} \circ \text{id}_{1,2,3} \circ \text{id}_{1,2,3} = \delta$; similarly, $\text{can}_{1,2} \circ \delta = \hat{\sigma} \circ \text{can}_{1,2}$ gives $\delta_{1,2,3} \circ \text{id}_{1,2,3} \circ \text{id}_{1,2,3} = -\mu$.

Let us set $\rho := \mu_{1,2,3} \circ \text{id}_{1,2,3}, \hat{\rho} := \mu_{1,2,3} \circ \text{id}_{1,2,3} \circ \text{id}_{1,2,3}$; we identify $-\rho$ with an element $\sigma \in \text{LBA}(T_2, \text{id}_{1,2,3})$ and $\hat{\rho}$ with an element $\hat{\sigma} \in \text{LBA}(T_3, \text{id}_{1,2,3})$. We will write some equations satisfied by $\sigma$ and $\hat{\sigma}$.

If $(a, \mu_\lambda, \delta_a)$ is a finite dimensional Lie bialgebra, then its double is $D(a) = a \oplus a^*$ with the bracket

$$\{x, y\}_D(a) = \{x, y\}_a + \{\xi, \eta\}_D(a) = \{\xi, \eta\}_a^*, \quad \{x, \xi\}_D(a) = \text{ad}_a^*\{\xi\} = -\delta_a^*\{\xi\}$$

where $\{x, y\}_a = \mu_a(x, y)$, $\{-, -\}_a^* = \delta_a$, $\text{ad}^*$ is the coadjoint action of $(a, \{-, -\}_a)$ on $a^*$, and with the cobracket

$$\delta_D(a)(x) = \delta_a(x), \quad \delta_D(a)(\xi) = -\mu_a(\xi);$$

here $x, y \in a$ and $\xi, \eta \in a^*$.

On the other hand, $D$ gives rise to a composed prop morphism $\text{LBA} \to D_{\text{red}}(\text{LBA}) \to \text{Prop}(a \oplus a^*)$, hence to a Lie bialgebra structure on $a \oplus a^*$, denoted $(\mu_D(a), \delta_D(a))$. The brackets and cobrackets for this structure are the same as above, except for

$$\mu_D(a)(x, \xi) = \rho_x^a(\xi) + \rho_x^a(\xi),$$

where $\rho : a \otimes a^* \to a^*$ and $\rho : a \otimes a^* \to a$, $x \otimes \xi \mapsto \rho^a(\xi)$ and $\rho^a : a \otimes a^* \to a$, $x \otimes \xi \mapsto \rho^a(\xi)$ are the realizations of $\rho$ and $\hat{\rho}$. The realization of $\sigma$ is then $\sigma^a : a^{\otimes 2} \to a, x \otimes y \mapsto \sigma_x^a(y)$, such that for any $x \in a$,

$$\sigma_x^a = -\delta^a(\rho_x^a).$$

The fact that $\mu_D(a)$ satisfies the Jacobi identity implies that $\sigma_x^a = [\sigma_x^a, \sigma_y^a]$ (identity in $\text{End}(a)$) and the cocycle identity for the bracket $\mu_D(a)(x, \xi)$ implies $\sigma_x^a(y, \xi) = [\sigma_x^a(y), z] + [y, \sigma_x^a(z)]$ for any $x, y, z \in a$.

At the propic level, one shows that the Jacobi identity satisfied by $\hat{\mu}$ and the cocycle identity satisfied by $(\mu_D, \delta_D)$ imply that $\sigma$ satisfies the universal versions of these identities

$$\sigma \circ (\mu \otimes \text{id}_{1,2}) = \sigma \circ (\text{id}_{1,2} \otimes \text{id}_{1,2} \otimes \text{id}_{1,2} \circ (213),$$

$$\sigma \circ (\text{id}_{1,2} \otimes \mu) = \mu \circ (\sigma \otimes \text{id}_{1,2} \otimes \text{id}_{1,2} \circ (132).$$

$\sigma \in \text{LBA}(T_2, \text{id}_{1,2})$ has the expansion $\mu + \text{terms of higher degree}$ of $\sigma - \mu$ if $\sigma - \mu \neq 0$, or 0 if $\sigma = \mu$. Then $\sigma'$ satisfies the equations

$$\sigma' \circ (\text{id}_{1,2} \otimes \text{id}_{1,2} \otimes \mu) = \mu \circ (\sigma' \otimes \text{id}_{1,2} \otimes \text{id}_{1,2} \circ (132),$$

$$\sigma'(\mu \otimes \text{id}_{1,2}) = \mu \circ (\text{id}_{1,2} \otimes \text{id}_{1,2} \otimes \text{id}_{1,2} \circ (213) - \sigma' \circ (\text{id}_{1,2} \otimes \text{id}_{1,2} \circ (132).$$

Lemma 3.11. Let $\sigma' \in \text{LBA}(\text{id} \otimes 2, \text{id}_{1,2})$ satisfy the above identities; then $\sigma'$ is proportional to $\mu \circ ((\mu \circ \delta) \otimes \text{id}_{1,2}).$

Proof of Lemma. These identities imply

$$\sigma' \circ (\mu \otimes \text{id}_{1,2}) = \mu \circ ((\sigma' \circ (21)) \otimes \text{id}_{1,2}).$$
The proof is a propic version of the following argument. The above identities are the universal versions of
\[(19)\]
\[\sigma^{\alpha}_x([y, z]) = [\sigma^{\alpha}_x(y), z] + [y, \sigma^{\alpha}_x(z)]\]
(identity of maps \(\mathfrak{a} \circ \mathfrak{a} \to \mathfrak{a}\)) and
\[\sigma^{\alpha}_{[x, y]} = [\sigma^{\alpha}_x, \text{ad}_y] - [\sigma^{\alpha}_y, \text{ad}_x]\]
(identity in \(\text{End}(\mathfrak{a})\), where \(\text{ad}_x(y) = [x, y]\)) which is written
\[\sigma^{\alpha}_{[x, y]}(z) = \sigma^{\alpha}_x([y, z]) - [y, \sigma^{\alpha}_x(z)] - (\sigma^{\alpha}_y([x, z]) - [x, \sigma^{\alpha}_y(z)])\]
Taking into account (19), this gives
\[\sigma^{\alpha}_{[x, y]}(z) = [\sigma^{\alpha}_x(y) - \sigma^{\alpha}_y(x), z].\]

We have:

**Fact 3.1.** If \(A\) is any Schur functor, the map \(\text{LBA}(\text{id} \otimes A, \text{id}) \to \text{LBA}(T_2 \boxtimes A, \text{id})\), \(x \mapsto x \circ (\mu \otimes \text{id}^{\text{LBA}}_{\text{id}})\) is injective.

**Proof of Fact.** The composed map \(\oplus_{Z, W \in \text{Irr}(\text{Sch})} \text{LCA}(\text{id}, W) \otimes \text{LCA}(A, Z) \otimes \text{LA}(W \otimes Z, \text{id}) \cong \text{LBA}(\text{id} \otimes A, \text{id}) \to \text{LBA}(T_2 \otimes A, \text{id}) \cong \oplus_{Z, W \in \text{Irr}(\text{Sch})} \text{LCA}(\text{id}, W) \otimes \text{LCA}(A, W) \otimes \text{LA}(\text{id} \otimes Z \otimes W, \text{id})\), where the last projection is onto the sum of components with \(Z = \text{id}\), is the direct sum over \(Z, W\) of the maps \(\text{LA}(Z \otimes W, \text{id}) \to \text{LA}(\text{id} \otimes Z \otimes W, \text{id})\), \(\lambda \mapsto \lambda \circ (\mu \otimes \text{id}^{\text{LBA}}_{\text{id}})\), with the identity; here \(\mu_Z \in \text{LA}(\text{id} \otimes Z, Z)\) is induced by \(\mu_{Z | T[Z]} \in \text{LA}(\text{id} \otimes T[Z], T[Z])\) given by \(x \otimes x_1 \otimes \ldots \otimes x_{|Z|} \mapsto \sum_{i=1}^{|Z|} x_1 \otimes \ldots \otimes [x, x_i] \otimes \ldots \otimes x_{|Z|}\). It follows from the first statement of [EH], Prop. 3.2, that this map is injective. It follows that the map \(\text{LBA}(\text{id} \otimes A, \text{id}) \to \text{LBA}(T_2 \otimes A, \text{id})\) is injective.

We now prove the following fact:

**Fact 3.2.** Assume that \(\psi_1, \psi_2 \in \text{LBA}(T_2, \text{id})\) are such that \(\psi_1 \circ (\mu \otimes \text{id}^{\text{LBA}}_{\text{id}}) = \mu \circ (\psi_2 \otimes \text{id}_{\text{id}})\), then there exists \(\psi \in \text{LBA}(\text{id}, \text{id})\), such that \(\psi = \mu \circ (\psi \otimes \text{id}^{\text{LBA}}_{\text{id}})\) and \(\psi_2 = \psi \circ \mu\).

**Proof of Fact.** We decompose \(\psi_i = \sum_{Z, W \in \text{Irr}(\text{Sch})} \psi^{i}_{Z, W},\) where \(\psi^{i}_{Z, W}\) belongs to the image of \(\text{LCA}(\text{id}, Z) \otimes \text{LCA}(\text{id}, W) \otimes \text{LA}(Z \otimes W, \text{id}) \to \text{LBA}(T_2, \text{id})\). Then \(\psi^{i}_{Z, W} \circ (\mu \otimes \text{id}^{\text{LBA}}_{\text{id}})\) belongs to the image of \(\oplus_{Z_1, Z_2 \in \text{Irr}(\text{Sch})} \text{LCA}(\text{id}, Z_1) \otimes \text{LCA}(\text{id}, Z_2) \otimes \text{LA}(Z_1 \otimes Z_2 \otimes W, \text{id}) \to \text{LBA}(T_3, \text{id})\).

Since this map is injective, this implies that for \(W \neq \text{id}\), \((\sum_{Z} \psi^1_{Z, W}) \circ (\mu \otimes \text{id}^{\text{LBA}}_{\text{id}}) = 0\). Fact 3.1 then implies that for such \(W\), \(\psi^1_{Z, W} = 0\), so \(\psi = \sum_{Z} \psi^1_{Z, W}\).

Let us denote by \(\text{std} \in \text{LA}(S \otimes \text{id}, \text{id})\) the direct sum of all prop elements corresponding to \(x_1 \ldots x_k \otimes x = \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} [x_{\sigma(1)}, \ldots, x_{\sigma(k)}]\). The PBW theorem for free algebras implies that the map
\[\text{LA}(Z, S) \to \text{LA}(Z \otimes \text{id}, \text{id})\]
\(\lambda \mapsto \text{std} \circ (\lambda \otimes \text{id}^{\text{LBA}}_{\text{id}})\) is a linear isomorphism. The decomposition of \(\text{LBA}(F, G)\) using LA and LCA then implies that the map
\[(20)\]
\[\text{LBA}(A, S) \to \text{LBA}(A \otimes \text{id}, \text{id}), \ x \mapsto \text{std} \circ (x \otimes \text{id}^{\text{LBA}}_{\text{id}})\]
is injective.
Then \( \psi_{Z,\id}^1 \in \text{LCA}(\id, Z) \otimes \text{LA}(Z \otimes \id, \id) \simeq \text{LCA}(\id, Z) \otimes \text{LA}(Z, S) \); we denote by \( \psi_{Z,S}^1 \) the image of \( \psi_{Z,\id}^1 \) in the latter space (and by \( \psi_{Z, S}^k \) its \( S^k \) component), which may be viewed as the element of \( \text{LBA}(id, S) \) such that
\[
\text{std} \circ (\psi_{Z,S}^1 \otimes \id_{id}^{\text{LBA}}) = \psi_{Z,\id}^1.
\]
We now have
\[
\text{std} \circ ((\psi_{Z,S}^1 \otimes \mu) \otimes \id_{id}^{\text{LBA}}) = \text{std} \circ (\psi_2 \otimes \id_{id}^{\text{LBA}}).
\]
It follows from the injectivity of (20) that \( \psi_{Z, k}^1 \circ \mu = 0 \) for \( k \neq 1 \), and \( \psi_{Z,1}^1 \circ \mu = \psi_2 \). Now Fact 3.1 implies that \( \psi_{Z, k}^1 = 0 \) for \( k \neq 1 \), so if we set \( \psi := \psi_{Z,1} \), we have \( \psi_1 = \mu \circ (\psi \otimes \id_{id}^{\text{LBA}}) \) and \( \psi_2 = \psi \circ \mu \).

According to Fact 3.2, (18) implies that there exists \( f \in \text{LBA}(id, id) \), such that
\[
(21)
\]
\[
\sigma' = \sigma' \circ (21) = f \circ \mu, \quad \sigma' = \mu \circ (f \otimes \id_{id}^{\text{LBA}}).
\]
These identities imply that \( f \) satisfies the derivation identity \( f \circ \mu = \mu \circ (f \otimes \id_{id}^{\text{LBA}} + \id_{id}^{\text{LBA}} \otimes f) \).
We proved in [E2] that this implies that \( f \) is proportional to \( \mu \circ \delta \). Together with (21), this implies Lemma 3.11.

It follows that \( \sigma \) has the expansion \( \mu \circ (1 - \lambda(\mu \circ \delta)) \otimes \id_{id}^{\text{LBA}} \) + terms of higher degree, where \( \lambda \) is a scalar.

Since \( \sigma \circ (\exp(\lambda(\mu \circ \delta))) \otimes \id_{id}^{\text{LBA}} \) satisfies the same identities as \( \sigma \), and has the expansion \( \mu + \text{terms of } \delta \)-degree \( \geq 2 \), it is equal to \( \mu \), hence
\[
\sigma = \mu \circ (\exp(\lambda(\mu \circ \delta))) \otimes \id_{id}^{\text{LBA}}.
\]
One proves similarly that for some scalar \( \lambda' \),
\[
\hat{\sigma} = (\id_{id}^{\text{LBA}} \otimes \exp(-\lambda'(\mu \circ \delta))) \circ \delta.
\]
The Jacobi identity for the bracket \( \hat{D}(\mu) \) then implies that \( \lambda = \lambda' \).

To give an idea of the proof, we write conditions for the bracket \([-, -]_{\lambda, \lambda'}\) on \( \text{D}(a) \) (where \( a \) is a finite dimensional Lie bialgebra) to be Jacobi, where \([-,-]_{\lambda, \lambda'} \) is given by the same formulas as above, except for
\[
[x, \xi]_{\lambda, \lambda'} = \text{ad}^*_{\text{e}^{-\lambda D_a}(x)} \xi - \text{ad}^*_{\text{e}^{\lambda D_a}(\xi)} x,
\]
where \( D_a \) is the derivation given by \( D_a = \mu_a \circ \delta_a + [-^t(\mu_a \circ \delta_a)] \in \text{End}(a) \oplus \text{End}(a^*) \subset \text{End}(\text{D}(a)) \).

The Jacobi identity for \( \text{D}(a) \) yields
\[
\text{ad}^*_{\text{e}^{\lambda D_a}(x_1, x_2)} \xi = \text{ad}^*_{\text{e}^{\lambda D_a}(x_2)} \text{ad}^*_{\text{e}^{\lambda D_a}(x_1)} \xi - \text{ad}^*_{\text{e}^{\lambda D_a}(\xi)} x_1 - \text{ad}^*_{\text{e}^{\lambda D_a}(\xi)} x_2
\]
for \( \xi \in a^* \), \( x_1, x_2 \in a \). On the other hand, the Jacobi identity for \([-, -]_{\lambda, \lambda'} \) implies that
\[
\text{ad}^*_{\text{e}^{\lambda D_a}(x_1, x_2)} \xi = \text{ad}^*_{\text{e}^{\lambda D_a}(x_2)} \text{ad}^*_{\text{e}^{\lambda D_a}(x_1)} \xi - \text{ad}^*_{\text{e}^{\lambda D_a}(\xi)} x_1 - \text{ad}^*_{\text{e}^{\lambda D_a}(\xi)} x_2
\]
where \( \xi' = \text{e}^{\lambda D_a}(\xi) \); replacing \( \xi' \) by \( \xi \), the difference of these identities is
\[
\text{ad}^*_{\text{e}^{\lambda D_a}(x_1, x_2)} \xi - \text{ad}^*_{\text{e}^{\lambda D_a}(x_2)} \text{ad}^*_{\text{e}^{\lambda D_a}(x_1)} \xi = \text{ad}^*_{\text{e}^{\lambda D_a}(\xi)} x_1 - \text{ad}^*_{\text{e}^{\lambda D_a}(\xi)} x_2
\]
which translates as an identity in \( \text{End}(a^{\otimes 2}) \)

\[
(\id_a \otimes \mu_a) \circ (\delta_a \otimes \id_a) \circ (\id_a \otimes (\text{e}^{\lambda D_a} - 1)) = (\id_a \otimes \mu_a) \circ (\delta_a \otimes \id_a) \circ (\id_a \otimes (\text{e}^{\lambda D_a} - 1)) \circ (21).
\]

One shows that at the proper level, the Jacobi identity for \( \hat{D}(\mu) \) implies similarly \( (\id_{id}^{\text{LBA}} \otimes \mu) \circ (\delta \otimes \id_{id}^{\text{LBA}}) \circ (\id_{id}^{\text{LBA}} \otimes (\text{e}^{\lambda D_a} - 1)) = (\id_{id}^{\text{LBA}} \otimes \mu) \circ (\delta \otimes \id_{id}^{\text{LBA}}) \circ (\id_{id}^{\text{LBA}} \otimes (\text{e}^{\lambda D_a} - 1)) \circ (21) \).

The lowest degree term of this identity is the product of \( \lambda' - \lambda \) by
\[
\left( (\id_{id}^{\text{LBA}} \otimes \mu) \circ (\delta \otimes \id_{id}^{\text{LBA}}) \circ (\id_{id}^{\text{LBA}} \otimes (\delta \circ \mu)) \right) \circ ((12) - (21));
\]
once checks that this is a nonzero element of \( \text{LBA}(T_2, T_2) \), so \( \lambda = \lambda' \).
We now prove that \( \hat{D} = \text{Im}(\text{id}_{\text{id}_{\text{LBA}}} \boxtimes \text{id}_{\text{LBA}} \ast (\text{Z}_\Lambda)) \circ \text{double}. \) For this, one has to check that
\[
\hat{D}(\mu) = (\text{id}_{\text{id}_{\text{LBA}}} \boxtimes \text{id}_{\text{LBA}} \ast (\text{Z}_\Lambda)) \circ \text{double}(\mu)
\]
\[
\circ (\text{id}_{\text{id}_{\text{LBA}}} \boxtimes \text{id}_{\text{LBA}} \ast (\text{Z}_\Lambda) + \text{id}_{\text{LBA}} \boxtimes (\text{Z}_\Lambda) + \text{id}_{\text{LBA}} \boxtimes \ast (\text{Z}_\Lambda))^2)^{-1},
\]
\[
\hat{D}(\delta) = \left(\text{id}_{\text{id}_{\text{LBA}}} \boxtimes \text{id}_{\text{LBA}} \ast (\text{Z}_\Lambda) + \text{id}_{\text{LBA}} \boxtimes \ast (\text{Z}_\Lambda) + \text{id}_{\text{LBA}} \boxtimes \ast \ast (\text{Z}_\Lambda)^2\right)^{-1}.
\]
The second identity, as well as the components \( \wedge^2 \boxtimes 1 \rightarrow \text{id} \boxtimes 1 \) and \( 1 \boxtimes \wedge^2 \rightarrow \text{id} \boxtimes 1 \) of the first identity, follow from the fact that \( \text{Z}_\Lambda \) is central in \( \text{LBA} \), i.e., \( \text{Im}(\text{Z}_\Lambda) \) is the identity.

We now show that
\[
(22)
\]
\[
\mu_{\text{id}_{\text{id}_{\text{id}_{\text{LBA}}}}} = \text{double}(\mu) \circ (\text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \boxtimes (\text{Z}_\Lambda^{-1})),
\]
\[
(23)
\]
\[
\mu_{\text{id}_{\text{id}_{\text{id}_{\text{LBA}}}}} = (\text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \boxtimes \ast (\text{Z}_\Lambda)) \circ \text{double}(\mu) \circ (\text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \boxtimes (\text{Z}_\Lambda^{-1})).
\]

Let us denote by \( \xi \mapsto \tilde{\xi} \) the canonical map \( \text{LBA}_2(F \otimes G, F \otimes G') = \text{LBA}(F \otimes G^*, F' \otimes G^*) \). Then for \( x \in \text{LBA}(G^*, G^*) \) and \( y \in \text{LBA}(G', G^*) \), we have
\[
[(\text{id}_{\text{LBA}} \boxtimes \ast (y)) \circ \xi \circ (\text{id}_{\text{LBA}} \boxtimes \ast (x))] = (\text{id}_{\text{id}_{\text{LBA}}} \boxtimes \ast \ast (\text{Z}_\Lambda)^{-1}).
\]
Then \( \mu_{\text{id}_{\text{id}_{\text{id}_{\text{LBA}}}}} = \tilde{\sigma} \circ (\text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \boxtimes \ast \ast (\text{Z}_\Lambda)^{-1}) \), on the other hand, \( \{\text{double}(\mu) \circ \text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \} = \delta \), so \( \{\text{double}(\mu) \circ \text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \} = \delta \), which implies (22).

Similarly, \( \mu_{\text{id}_{\text{id}_{\text{id}_{\text{LBA}}}}} = \text{id} \circ \text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \), on the other hand, \( \{\text{double}(\mu) \circ \text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \} = \delta \), so \( \{\text{double}(\mu) \circ \text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \} = \delta \), which implies (23).

3.8. Compatibility of \( Q \) with doubling operations. We now summarize our results. We have
\[
D_{\text{mult}}(Q) \circ \text{Double} \circ Q^{-1} = \text{Im}(\text{Z}) \circ S(\varphi),
\]
\[
\varphi = \text{Im}(\gamma^{-1}) \circ \hat{D}, \quad D = \text{Im}(\text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \boxtimes \text{id}_{\text{LBA}} \ast (\text{Z}_\Lambda)) \circ \text{double},
\]
which implies that
\[
D_{\text{mult}}(Q) \circ \text{Double} \circ Q^{-1} = \text{Im}(\Lambda) \circ S(\text{double}),
\]
where
\[
\Lambda = \text{Im}(\gamma^{-1}) \circ \text{id}_{\text{id}_{\text{id}_{\text{LBA}}}} \boxtimes \text{id}_{\text{LBA}} \ast (\text{Z}_\Lambda) \circ S(\text{double}) \circ S(\text{Z}_\Lambda).
\]
which means that \( Q \) is compatible with the doubling operations, as wanted.

4. Applications

4.1. Finite dimensional Lie bialgebras. Let us say that the QUE algebra \( U \) is of finite type if \( \text{Prim}(U/hU) \) is finite dimensional. Then a quantization functor gives rise to a category equivalence \( \hat{Q} \colon \{ \text{finite dimensional QUE bialgebras} \} \rightarrow \{ \text{QUE algebras of finite type} \} \).

We also have an action of \( D \) on \{ finite dimensional Lie bialgebras \} and on \{ QUE algebras of finite type \}. We have also self-maps of \{ finite dimensional Lie bialgebras \} and of \{ QUE algebras of finite type \}, \( \Lambda \rightarrow \text{D}(\Lambda) \) and \( U \rightarrow \text{D}(U) \).

Then it follows from Theorem 2.1 that if %Q is an Etingof-Kazhdan functor, then \( Q(a \ast \text{cop}) \simeq Q(a) \ast \text{cop} \) for any finite dimensional Lie bialgebra \( a \), and from Theorem 3.1 that if \( Q \) is an Etingof-Kazhdan functor, then \( Q(D(a)) \simeq D(Q(a)) \) for any finite dimensional Lie bialgebra \( a \). We recover in this way the result of [EK1].

\(^5\text{Prim}(H) \) is the Lie algebra of primitive elements of a bialgebra \( H \).
4.2. ±N-graded Lie bialgebras. Let us say that a ±N-graded Lie bialgebra is of finite type if each fixed degree component is finite dimensional. Let us say that a ±N-graded QUE algebra \(U = \oplus_{n \in \pm N} U_n\) is of finite type if: (a) \(U_0\) is of finite type, and (b) each \(U_n\) is finitely generated as a module over \(U_0\).

The permutation \(\ast\) of \{finite dimensional Lie bialgebras\} extends to a bijection \{N-graded Lie bialgebras of finite type\} \(\rightarrow\) \{-N-graded Lie bialgebras of finite type\}. The self-map \(\mathcal{D}\) of \{finite dimensional Lie bialgebras\} extends to a map \{N-graded Lie bialgebras of finite type\} \(\rightarrow\) \{Z-graded Lie bialgebras of finite type\}.

Then the permutation \(\ast\) of \{QUE algebras of finite type\} extends to a bijection \(\ast\) : \{N-graded QUE algebras of finite type\} \(\rightarrow\) \{-N-graded QUE algebras of finite type\}. Moreover, the self-map \(\tilde{D}\) of \{QUE algebras of finite type\} extends to a map \(\tilde{D} : \{N-graded QUE algebras of finite type\} \rightarrow \{Z-graded QUE algebras\}\).

A quantization functor \(Q\) gives rise to functors \(\tilde{Q}\) : \{N-graded Lie bialgebras of finite type\} \(\rightarrow\) \{N-graded QUE algebras of finite type\} and \(\tilde{Q} : \{Z-graded bialgebras\} \rightarrow \{Z-graded QUE algebras\}\).

Then Theorem 2.1 implies that if \(a\) is a \(N\)-graded Lie bialgebra of finite type and \(Q\) is an Etingof-Kazhdan functor, then \(\tilde{Q}(a^{\ast\ast\ast}) \simeq \tilde{Q}(a)^{\ast\ast\ast}\) and Theorem 3.1 implies that with the same assumptions, \(\tilde{Q}(\mathcal{D}(a)) \simeq \tilde{D}(\tilde{Q}(a))\).

Theses result and those of the previous Subsection extend immediately to the super case.

4.3. Affine superalgebras. In the rest of this section, \(k = \mathbb{C}\). Lie (bi)superalgebras, QUE superalgebras, etc., mean Lie (bi)superalgebras, etc., in the category of vector superspaces, i.e., the category whose objects are \(\mathbb{Z}/2\mathbb{Z}\)-graded vector spaces; morphisms are even linear maps; the symmetry constraint is given by the usual sign rule.

Let \(g\) be an affine Lie superalgebra with symmetrized Cartan matrix \((A, \tau)\), where \(A = (a_{ij})_{1 \leq i, j \leq s}\) is a matrix with coefficients in \(\mathbb{C}\) and \(\tau\) is a subset of \(I := \{1, \ldots, s\}\) determining the parity of the generators (see [vL]). Let \(d_1, \ldots, d_s\) be nonzero rational numbers such that \(d_ia_{ij} = d_ja_{ji}\) for \(i, j \in I\).

Let \(h\) be the Cartan sub-superalgebra of \(g\). There exists linearly independent sets \(\{\alpha_i\}_{i \in I} \subset h^*\) and \(\{h_i\}_{i \in I} \subset h\) such that \(a_{ij}(h_i) = \alpha_j\); up to isomorphism these sets are determined by \(A\). Let \((-,-)\) be the non-degenerate supersymmetric invariant bilinear form on \(g\) defined in Proposition 4.2 of [vL]. The restriction of this form to the Cartan sub-superalgebra \(h\) is non-degenerate. Furthermore, \((a, h_i) = d_i^{-1}a_i(a)\) for all \(i \in I\) and \(a \in h\).

Let \(\tilde{g}\) be the Lie superalgebra presented by generators \(h_i, e_i, f_i\) for \(i \in I\) (which are all even, except for the \(e_i, f_i\) for \(i \in \tau\), which are odd) and relations:

\[
\[h_i, h_j\] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad [e_i, f_j] = \delta_{ij}h_i, \quad i, j \in I.
\]

Then \(\tilde{g}\) has a unique maximal ideal \(\mathfrak{r}\) which intersects \(h\) trivially. We set \(g := \tilde{g}/\mathfrak{r}\). When \(g\) is not of type \(A(n, n)^{(i)}\), Yamane ([Yam]) wrote down explicit Serre-type relations generating \(\tau\).

The Lie algebra \(g\) is equipped with a Lie bi-superalgebra structure, which we now describe. Let \(\mathfrak{n}_+\) (resp., \(\mathfrak{n}_-\)) be the nilpotent Lie sub-superalgebra of \(g\) generated by \(e_i\)'s (resp., \(f_i\)'s). Let \(\mathfrak{b}_+ := \mathfrak{n}_+ \oplus h\) be the Borel Lie sub-superalgebras of \(g\). Let \(\eta_{\pm} : \mathfrak{b}_{\pm} \rightarrow g \oplus h\) be defined by \(\eta_{\pm}(x) = x \oplus (\pm \tilde{x})\), where \(\tilde{x}\) is the image of \(x\) in \(h\). Using this embedding we can regard \(\mathfrak{b}_+\) and \(\mathfrak{b}_-\) as Lie sub-superalgebras of \(g \oplus h\). Let \((-,-)_{g \oplus h} := (-,-) \oplus (-,-)_{h}\), where \((-,-)_{h}\) is the restriction of \((-,-)\) to \(h\).

**Proposition 4.1.** \((g \oplus h, \mathfrak{b}_+, \mathfrak{b}_-)\) is a super Manin triple (where \(g\) is equipped with \((-,-)_{g \oplus h}\)).

**Proof.** Under the embedding \(\eta_{\pm}\), the Lie subsuperalgebras \(\mathfrak{b}_{\pm}\) are isotropic with respect to \((-,-)_{g \oplus h}\). Since \((-,-)\) and \((-,-)_{h}\) both are invariant super-symmetric nondegenerate bilinear forms, then so is \((-,-)_{g \oplus h}\). Therefore the proposition follows. \(\square\)
The proposition implies that $g \oplus h, b_+ \text{ and } b_-$ are Lie bi-superalgebras. Moreover, we have that $b_+^* \cong b_+^\text{op}$ as Lie bi-superalgebras, where $^*$ means the graded dual (for the principal grading on $g$, given by $|a| = 0$ for $a \in h$ and $|e_i| = 1$ for $i \in I$) and $^\text{op}$ means changing the cobracket into its negative. The cobrackets of these Lie bi-superalgebras are given by the following formulas (see [G2]):

$$
\delta(e_i) = \frac{d_i}{2}(e_i \otimes h_i - h_i \otimes e_i) = \frac{d_i}{2} e_i \wedge h_i, \quad \delta(f_i) = -\frac{d_i}{2} f_i \wedge h_i, \quad \delta(a) = 0
$$

for any $i \in I$ and $a \in h \subset b_{\pm}$.

Then the map $b_+ \to b_+^*$ given by $e_i \mapsto e_i^*$ and $h_i \mapsto -\frac{d_i}{2} \sum_{j=1}^{n} a_{ji} h_j^*$ is an isomorphism of Lie bi-superalgebras (where $h_j^*, e_i^* \in b_+^*$ are defined by: $h_j^*$ has degree 0, $h_j^*(h_i) = \delta_{ij}$ and $e_i^*$ has degree $-1$, $e_i^*(e_j) = \delta_{ij}$, and in $b_+^*$ the $\mathbb{N}$-grading is changed into its opposite). This isomorphism restricts to the identification $h \to b^*$ coming from the form $-2(-,-)_h$.

Define $b_{\pm} \subset \hat{g}$ as the Lie sub-superalgebras generated by $h_i, e_i$ (resp., $h_i, f_i$), $i \in I$. One checks that the above formulas define Lie bialgebra structures on $b_{\pm}$. We have Lie bi-superalgebra morphisms $b_{\pm} \to b_{\pm}$.

**4.4. Quantized affine Lie superalgebras.** In this subsection we use the previous results to show that Yamane’s [Yam] Drinfeld-Jimbo type superalgebras associated to certain affine Lie superalgebras are isomorphic to the corresponding Etingof-Kazhdan quantizations. The methods of this section are inspired by [EK3].

Let $Q_\Phi : \text{Bialg} \to S(\text{LBA})$ be an Etingof-Kazhdan functor. As above, $Q_\Phi$ gives rise to a functor $Q_\Phi : \{\pm \mathbb{N}\text{-graded Lie bi-superalgebras of finite type, topologically free over } \mathbb{C}[[h]]\} \to \{\pm \mathbb{N}\text{-graded QUE superalgebras of finite type}\};$ we then define $U^{\text{EK}}_h : \{\mathbb{N}\text{-graded Lie bi-superalgebras of finite type over } \mathbb{C}\} \to \{\text{QUE superalgebras over } \mathbb{C}\}$, by $U^{\text{EK}}_h(a, \mu_a, \delta_a) := Q_\Phi(a[[h]], \mu_a, h\delta_a)$.

**Theorem 4.1.** The quantized universal enveloping (QUE) superalgebra $U^{\text{EK}}_h(b_+)$ is isomorphic to the QUE superalgebra generated over $\mathbb{C}[[h]]$ by $b_+$ and the elements $E_i$, $i \in I$ (all generators are even except for $E_i$, $i \in \tau$ which are odd) satisfying the relations

$$
[a, a'] = 0, \quad [a, E_i] = \alpha_i(a) E_i,
$$

with coproduct

$$
\Delta(a) = a \otimes 1 + 1 \otimes a, \quad \Delta(E_i) = E_i \otimes q^{d_i h_i} + 1 \otimes E_i,
$$

for all $a, a' \in b_+$ and $i, j \in I$. (Here $q = e^{h/2}$.)

The theorem follows from the following two lemmas.

**Lemma 4.1.** The QUE superalgebra $U^{\text{EK}}_h(b_+)$ is isomorphic to the QUE superalgebra generated over $\mathbb{C}[[h]]$ by $b_+$ and the elements $E_i$, $i \in I$ (all generators are even except for $E_i$, $i \in \tau$ which are odd) satisfying the relations

$$
[a, a'] = 0, \quad [a, E_i] = \alpha_i(a) E_i,
$$

with coproduct

$$
\Delta(a) = a \otimes 1 + 1 \otimes a, \quad \Delta(E_i) = E_i \otimes q^{d_i h_i} + 1 \otimes E_i,
$$

for all $a, a' \in b_+$ and $i, j \in I$ and suitable elements $\gamma_i \in b[[h]]$.

**Proof.** An analogous lemma was proved in [G2] for finite dimensional Lie superalgebras of type A-G. The proof in [G2] does not use the finite dimensional condition. Thus, the proof of Lemma 4.1 follows word for word as in the analogous lemma of [G2].

**Lemma 4.2.** $\gamma_i = d_i h_i$. 


Proof. By the definition of $\mathfrak{b}_+$, we have a (surjective) Lie bi-superalgebra morphism $\pi: \tilde{\mathfrak{b}}_+ \to \mathfrak{b}_+$. Since $\mathcal{U}^{EK}_h$ is a functor, this induces a morphism of Hopf superalgebras $\mathcal{U}^{EK}_h(\mathfrak{b}_+) \to \mathcal{U}^{EK}_h(\mathfrak{b}_+)$. This morphism is onto as it is onto modulo $\hbar$. Therefore, $\mathcal{U}^{EK}_h(\mathfrak{b}_+)$ is generated by $\mathfrak{h}$ and $E_\ell$ satisfying the relations (26)-(27) (and possibly other relations). So it suffices to show that the images of $\gamma_1, h_1$ by $\mathcal{U}^{EK}_h(\pi)$ (again denoted $\gamma_1, h_1$) satisfy $\gamma_1 = d_i h_i$ in $\mathcal{U}^{EK}_h(\mathfrak{b}_+)$.  

As mentioned above, $\mathfrak{b}_+$ is self dual in the graded sense, i.e., $\mathfrak{b}_+ \cong \mathfrak{b}_+^*$ (as graded Lie bi-superalgebras, where one of the gradings is changed to its negative). Since $\mathcal{U}^{EK}_h$ is a functor, we get a $\mathbb{N}$-graded isomorphism

$$\mathcal{U}^{EK}_h(\mathfrak{b}_+) \cong \mathcal{U}^{EK}_h(\mathfrak{b}_+^*)$$

(again one of the gradings is changed to its negative). It follows from the discussion in Subsection 4.3 that the restriction of this isomorphism to $\mathcal{U}^{EK}_h(\mathfrak{h})$ comes from the identification $\mathfrak{h} \to \mathfrak{h}^*$ using the form $-2((-,-))$ on $\mathfrak{h}$. Using Theorem 2.1 with $\theta = \ast \text{cop}$, we have

$$\mathcal{U}^{EK}_h(\mathfrak{b}_+^*) \cong \mathcal{U}^{EK}_h(\mathfrak{b}_+^*)^{\ast \text{cop}}.$$ 

Moreover, this theorem says that if $V$ is a finite dimensional commutative and cocommutative Lie bialgebra, then the isomorphism $\mathcal{U}^{EK}(V)^* \simeq \mathcal{U}^{EK}(V^*)$ is induced by the bialgebra pairing $\mathcal{U}^{EK}_h(V) \otimes \mathcal{U}^{EK}_h(V^*) \simeq S(V)[[\hbar]] \otimes S(V^*[[\hbar]] \to \mathbb{C}((\hbar))$ given by $V \otimes V^* \in \xi \otimes x \mapsto \hbar^{-1}\xi(x)$. Combining Equations (28) and (29) with $\mathcal{U}^{EK}_h(\mathfrak{b}_+^*) \cong \mathcal{U}^{EK}_h(\mathfrak{b}_+^*)^{\ast \text{cop}}$, we have

$$\mathcal{U}^{EK}_h(\mathfrak{b}_+) \cong \mathcal{U}^{EK}_h(\mathfrak{b}_+)^{\ast \text{cop}}.$$ 

This isomorphism gives rise to a nondegenerate bilinear form $B: \mathcal{U}^{EK}_h(\mathfrak{b}_+) \otimes \mathcal{U}^{EK}_h(\mathfrak{b}_+) \to \mathbb{C}((\hbar))$ which satisfies the following conditions

$$B(xy, z) = B(x \otimes y, \Delta(z)), \quad B(x, yz) = B(\Delta(x), y \otimes z)$$

$$B(q^a, q^b) = q^{-(a,b)},$$

where $a, b \in \mathfrak{h}$ and $x, y, z \in \mathcal{U}^{EK}_h(\mathfrak{b}_+)$ (the second line follows from the properties of the pairing $\mathcal{U}^{EK}_h(\mathfrak{h}) \otimes \mathcal{U}^{EK}_h(\mathfrak{h}^*) \to \mathbb{C}((\hbar))$ recalled above).

Let $a \in \mathfrak{h}$ and $i \in I$. Set $B_i = B(E_i, E_i)$, which is nonzero. Using the above properties of $B$ we have

$$B_i q^{\alpha_i(a)} = B(E_i, q^a E_i q^{-a}).$$

From relation $[a, E_i] = \alpha_i(a) E_i$ it follows that $q^{b_i} E_i q^{-b_i} = q^{\alpha_i(h_i)} E_i$ and so

$$q^{\alpha_i(a)} E_i q^{-a} = q^{\alpha_i(a)} E_i.$$ 

Combining (30) and (31) we have

$$B_i q^{\alpha_i(a)} = B(E_i, q^a E_i q^{-a}) = B(E_i, q^{\alpha_i(a)} E_i) = B_i q^{\alpha_i(a)}.$$ 

Hence, $(a, \gamma_i) = \alpha_i(a)$, but $\alpha_i(a) = d_i(a, h_i)$, and so $\gamma_i = d_i h_i$, which completes the proof. 

Lemma 4.3. Let $I_h := \text{Ker}(\mathcal{U}^{EK}_h(\pi): \mathcal{U}^{EK}_h(\tilde{\mathfrak{b}}_+) \to \mathcal{U}^{EK}_h(\mathfrak{b}_+))$. Then $I_h \subset \mathcal{U}^{EK}_h(\mathfrak{b}_+)$ is maximal with the following properties: is a graded bialgebra ideal, $h$-divisible (i.e., the quotient by it is $h$-torsion free), whose components of degree 0 and 1 are zero.

Proof. This is equivalent to saying that the maximal ideal of $\mathcal{U}^{EK}_h(\mathfrak{b}_+)$ with the same properties is 0. Indeed, such an ideal $J_h \subset \mathcal{U}^{EK}_h(\mathfrak{b}_+)$ gives rise to a QUE superalgebra $\mathcal{U}^{EK}_h(\mathfrak{b}_+)/J_h$ (because this is $h$-torsion free, and by the Milnor-Moore theorem any quotient of an enveloping superalgebra is again an enveloping superalgebra) and to a QUE superalgebra morphism $\mathcal{U}^{EK}_h(\mathfrak{b}_+) \to \mathcal{U}^{EK}_h(\mathfrak{b}_+)/J_h$ (because a bi-superalgebra morphism between universal enveloping superalgebras is induced by a morphism of Lie superalgebras). The classical limit of this morphism is a surjective morphism $\mathfrak{b}_+ \to \mathfrak{r}$ of $\mathbb{N}$-graded Lie superalgebras, which is the identity in
degrees 0 and 1. Let \( i := \text{Ker}(b_+ \to \mathfrak{x}) \) is a Lie ideal and coideal. Then \( i^\perp \subset b_+ \simeq b_- \) is a Lie sub-superalgebra, containing \( \mathfrak{h} \) and the \( c_i^+ \). Since \( b_- \) is generated by these elements, \( i = 0 \), so \( b_+ \to \mathfrak{x} \) is the identity of \( b_+ \), so \( J_h = 0 \).

**Theorem 4.2.** There exists a unique bilinear form on \( \mathcal{U}_h^{EK}(\hat{b}_+) \), taking values in \( \mathbb{C}((\mathfrak{h})) \) and with the following properties:

\[
C(xy, z) = C(x \otimes y, \Delta(z)), \quad C(x, yz) = C(\Delta(x), y \otimes z), \quad C(q^a, q^b) = q^{-(a,b)}, \quad a, b \in \mathfrak{h},
\]

\[
C(E_i, E_j) = \delta_{ij} B(E_i, E_j),
\]

where \( B \) is given in the proof of Lemma 4.2. Moreover, \( \mathcal{U}_h^{EK}(b_+) \cong \mathcal{U}_h^{EK}(b_+)/\text{Ker}(C) \) as QUE superalgebras.

**Proof.** The existence and uniqueness follows from the fact that the superalgebra generated by the \( E_i \) is free (see Theorem 4.1).

We will show that there is a nondegenerate bilinear form on \( \mathcal{U}_h^{EK}(b_+) \) with the same properties as \( B \).

We first construct an isomorphism \( \mathcal{U}_h^{EK}(b_+)^{\text{cop}} \to \mathcal{U}_h^{EK}(b_+) \) of \( \mathbb{N} \)-graded QUE algebras. Let \( T \) be the superalgebra presented by (25). We have an isomorphism \( \mathcal{U}_h^{EK}(\hat{b}_+) \simeq T \) of \( \mathbb{N} \)-graded superalgebras. Let \( \Delta_h : T \to T \otimes T \) be the coproduct on \( T \) obtained from the coproduct of \( \mathcal{U}_h^{EK}(b_+) \). Let \( c \in \text{Aut}(T) \) be the conjugation by \( q^{\sum x_i^2/2} \), where \( x_i \) is an orthonormal basis for \( \mathfrak{h} \). Then \( \Delta_h^{\pm 1} = (c \otimes c) \Delta_h \circ c^{-1} \). Let \( I_h := \text{Ker}(T \to \mathcal{U}_h^{EK}(b_+)) \). We have \( \Delta_h(I_h) \subset I_h \otimes T + T \otimes I_h \). Therefore \( \Delta_h(c(I_h)) \subset c(I_h) \otimes T + T \otimes c(I_h) \), so by the maximality of \( I_h \) w.r.t. \( \Delta_h \), \( c(I_h) \subset I_h \). Replacing \( h \) by \( -h \), we get similarly \( c^{-1}(I_h) \subset I_h \), so \( c(I_h) = -I_h \). It follows that \( c \) induces an isomorphism \( T/I_h \to T/I_h \), intertwining the coproduct maps \( T/I_h \to T/I_h \otimes T/I_h \) induced by \( \Delta_h \) and \( \Delta_h^{\pm 1} \). This is an isomorphism \( \mathcal{U}_h^{EK}(b_+)^{\text{cop}} \to \mathcal{U}_h^{EK}(b_+) \).

From the proof of Lemma 4.2, we have that \( \mathcal{U}_h^{EK}(b_+)^{\text{cop}} \cong \mathcal{U}_h^{EK}(b_+)^{\text{cop}} \). Combining this with the above isomorphism \( \mathcal{U}_h^{EK}(b_+)^{\text{cop}} \to \mathcal{U}_h^{EK}(b_+) \), we obtain an isomorphism \( \mathcal{U}_h^{EK}(b_+) \cong \mathcal{U}_h^{EK}(b_+) \). This isomorphism gives rise to the desired form on \( \mathcal{U}_h^{EK}(b_+) \).

It follows that \( C \) is the pull-back of the bilinear form \( B \) constructed in Lemma 4.2. Thus the image of the kernel of \( C \) is contained in the kernel of \( B \) under the natural projection. But the kernel of \( B \) is zero since the form is nondegenerate. Thus we have \( \mathcal{U}_h^{EK}(b_+) \cong \mathcal{U}_h^{EK}(b_+)/\text{Ker}(C) \).

In [Yam], Yamane introduced what we will call here the Drinfeld-Jimbo type quantization \( \mathcal{U}_h^{DJ}(\mathfrak{g}) \) of \( \mathfrak{g} \) as follows: \( \mathcal{U}_h^{DJ}(\mathfrak{b}_+) := \mathcal{U}_h^{EK}(\hat{b}_+)/\text{Ker}(C) \) (where \( C_+ = C \) and \( C_- \) is its analogue for \( \mathcal{U}_h^{EK}(b_-) \)); we have a non-degenerate pairing \( \mathcal{U}_h^{DJ}(\mathfrak{b}_+) \otimes \mathcal{U}_h^{DJ}(\mathfrak{b}_-) \to \mathbb{C}((\mathfrak{h})) \), which allows to identify \( \mathcal{U}_h^{DJ}(\mathfrak{b}_-) = \mathcal{U}_h^{DJ}(\mathfrak{b}_+)^{\text{cop}} \); then \( \mathcal{U}_h^{DJ}(\mathfrak{g}) := D(\mathcal{U}_h^{DJ}(\mathfrak{b}_+))/\mathfrak{h} \simeq \mathfrak{h}^* \). The superalgebra \( \mathcal{U}_h^{DJ}(\mathfrak{g}) \) is a braided (i.e., quasitriangular) quantized universal enveloping superalgebra where the braiding is given by the universal R-matrix. An immediate corollary of the above theorem is:

**Corollary 4.1.** \( \mathcal{U}_h^{EK}(\mathfrak{b}_+) \simeq \mathcal{U}_h^{DJ}(\mathfrak{b}_+) \) as QUE superalgebras.

We also have:

**Theorem 4.3.** There exists an isomorphism of braided quantized universal enveloping (QUE) superalgebras:

\[
\alpha : \mathcal{U}_h^{DJ}(\mathfrak{g}) \to \mathcal{U}_h^{EK}(\mathfrak{g})
\]

such that \( \alpha|_{\mathfrak{h}} = \text{id}_{\mathfrak{h}} \).
Proof. We first construct an isomorphism between these QUE superalgebras which is the identity on \( \mathfrak{h} \) (disregarding the braiding). We have \( U^{cD}_{\mathfrak{h}}(g) \cong D(U^{cD}_{\mathfrak{h}}(b_+))/ (\mathfrak{h} \simeq \mathfrak{h}^*) \cong U^{cD}_{\mathfrak{h}}(b_+) \otimes U^{cD}_{\mathfrak{h}}(b_+)^{\text{cop}}/ (\mathfrak{h} \simeq \mathfrak{h}^*) \). Corollary 4.1 yields an isomorphism \( U^{cD}_{\mathfrak{h}}(b_+) \cong U^{EK}_{\mathfrak{h}}(b_+) \) inducing the identity on \( \mathfrak{h} \), hence an isomorphism \( D(U^{cD}_{\mathfrak{h}}(b_+))/ (\mathfrak{h} \simeq \mathfrak{h}^*) \simeq D(U^{EK}_{\mathfrak{h}}(b_+))/ (\mathfrak{h} \simeq \mathfrak{h}^*) \) inducing the identity on \( \mathfrak{h} \). Since quantization commutes with the double, we have an isomorphism \( D(U^{EK}_{\mathfrak{h}}(b_+)) \cong U^{EK}_{\mathfrak{h}}(\mathcal{D}(b_+)) \). The fact that this isomorphism induces the identity on \( \mathfrak{h} \oplus \mathfrak{h}^* \) is a consequence of the following property of the isomorphism \( D(U^{EK}_{\mathfrak{h}}(a)) \cong U^{EK}_{\mathfrak{h}}(\mathcal{D}(a)) \):

**Lemma 4.4.** For N-graded Lie bi(supera)algebra morphisms \( b \overset{i}{\rightarrow} a \) and \( a \overset{p}{\rightarrow} b \) such that \( b \rightarrow a \rightarrow b \) is the identity, we get a morphism \( \mathcal{D}(b) \rightarrow \mathcal{D}(a) \); in the same way, if \( A,B \) are N-graded QUE algebra of finite type, QUE algebra morphisms \( B \rightarrow A \) and \( A \rightarrow B \) such that \( B \rightarrow A \rightarrow B \) is the identity give rise to a morphism \( D(B) \rightarrow D(A) \); taking \( (B \rightarrow A) := U^{EK}_{\mathfrak{h}}(b \rightarrow a), (A \rightarrow B) := U^{EK}_{\mathfrak{h}}(a \rightarrow b) \), we get a morphism \( D(U^{EK}_{\mathfrak{h}}(b)) \rightarrow D(U^{EK}_{\mathfrak{h}}(a)) \): the diagram \( \frac{D(U^{EK}_{\mathfrak{h}}(b))}{D(U^{EK}_{\mathfrak{h}}(a))} \) commutes.

**Proof of Lemma.** We have \( U^{EK}_{\mathfrak{h}}(a) = (\mathcal{S}(a)[[h]], m(\mu_\alpha, h\delta_\alpha), \Delta(\mu_\alpha, h\delta_\alpha)) \). The isomorphism \( U^{EK}_{\mathfrak{h}}(a) \otimes U^{EK}_{\mathfrak{h}}(a)^{\text{cop}} \cong D(U^{EK}_{\mathfrak{h}}(a)) \cong U^{EK}_{\mathfrak{h}}(\mathcal{D}(a)) \) is induced by a map \( \lambda^{-1}(\mu_\alpha, h\delta_\alpha) : \mathcal{S}(a) \otimes \mathcal{S}(a)^*[h] \rightarrow \mathcal{S}(a \otimes a^*[h]) \), where \( \lambda \in \text{LBA}_2(\mathbb{S}^{\otimes 2}, \mathbb{S}^{\otimes 2}) \). We thus have to show that \( \lambda^{-1}(\mu_\alpha, h\delta_\alpha) \circ S(i \otimes p') = S(i \otimes p') \circ \lambda^{-1}(\mu_\alpha, h\delta_\alpha) \) (equality of maps \( S(b \otimes b^*)[[h]] \rightarrow S(a \oplus a^*)[[h]] \)).

Let \( \lambda \in \text{LBA}(\mathbb{S}^{\otimes 2}, \mathbb{S}^{\otimes 2}) \) be the element corresponding to \( \lambda^{-1} \), then we should show that \( (\text{id}_{\mathcal{S}(a)} \circ S(p)) \circ \lambda(\mu_\alpha, h\delta_\alpha) \circ S(i) \circ \text{id}_{\mathcal{S}(a)} = (\text{id}_{\mathcal{S}(a)} \circ S(p)) \circ \lambda(\mu_\alpha, h\delta_\alpha) \circ (\text{id}_{\mathcal{S}(a)} \circ S(p)) \) (equality of maps \( S(b \oplus a)[[h]] \rightarrow S(a \oplus b)[[h]] \)).

\[ \lambda = \sum_{Z_1, Z_2, Z_3 \in \text{Irr}(\text{Sch})} (\lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3}) \circ (\kappa_d^{Z_1} \otimes \kappa_d^{Z_2} \otimes \kappa_d^{Z_3}), \]

where \( \lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3} \in \text{LCA}(S, Z_1 \otimes Z_2) \otimes \text{LCA}(S, Z_3) \otimes \text{LA}(Z_1, S) \otimes \text{LA}(Z_2, Z_3, S) \).

Then

\[
(id_{\mathcal{S}(a)} \circ S(p)) \circ \lambda(\mu_\alpha, h\delta_\alpha) \circ (S(i) \circ \text{id}_{\mathcal{S}(a)})
\]

\[
= (id_{\mathcal{S}(a)} \circ S(p)) \circ \left( \sum_{Z_1, Z_2, Z_3} (\lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3} \circ (\kappa_d^{Z_1} \otimes \kappa_d^{Z_2} \otimes \kappa_d^{Z_3})) \circ (S(i) \circ \text{id}_{\mathcal{S}(a)}) \right)
\]

\[
= \sum_{Z_1, Z_2, Z_3} \left( \lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3} \circ (\kappa_d^{Z_1} \otimes \kappa_d^{Z_2} \otimes \kappa_d^{Z_3} \circ \lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3}) \circ (\kappa_d^{Z_1} \otimes \kappa_d^{Z_2} \otimes \kappa_d^{Z_3} \circ \lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3}) \right)
\]

\[
= \sum_{Z_1, Z_2, Z_3} \left( \lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3} \circ (\kappa_d^{Z_1} \otimes \kappa_d^{Z_2} \otimes \kappa_d^{Z_3} \circ \lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3}) \circ (\kappa_d^{Z_1} \otimes \kappa_d^{Z_2} \otimes \kappa_d^{Z_3} \circ \lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3}) \right)
\]

\[
= (S(i) \circ \text{id}_{\text{Sch}}(Z_1)) \circ \left( \sum_{Z_1, Z_2, Z_3} (\lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3} \circ (\kappa_d^{Z_1} \otimes \kappa_d^{Z_2} \otimes \kappa_d^{Z_3} \circ \lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3}) \circ (\kappa_d^{Z_1} \otimes \kappa_d^{Z_2} \otimes \kappa_d^{Z_3} \circ \lambda_d^{Z_1} \otimes \lambda_d^{Z_2} \otimes \lambda_d^{Z_3})) \right)
\]

\[
= (S(i) \circ \text{id}_{\text{Sch}}(Z_1)) \circ \lambda(\mu_\alpha, h\delta_\alpha) \circ (S(i) \circ \text{id}_{\text{Sch}}(Z_1))
\]

as wanted; the second and fourth equalities follow from the fact that \( p \) and \( i \) are Lie bialgebra morphisms, the third one follows from \( p \circ i = \text{id}_a \).

We thus get an isomorphism \( D(U^{EK}_{\mathfrak{h}}(b_+))/ (\mathfrak{h} \simeq \mathfrak{h}^*) \cong U^{EK}_{\mathfrak{h}}(\mathcal{D}(b_+))/ (\mathfrak{h} \simeq \mathfrak{h}^*) = U^{EK}_{\mathfrak{h}}(g) \) inducing the identity on \( \mathfrak{h} \). We thus have an QUE isomorphism \( U^{cD}_{\mathfrak{h}}(g) \cong U^{EK}_{\mathfrak{h}}(g) \), inducing the isomorphism on \( \mathfrak{h} \).

This isomorphism is actually an isomorphism of braided QUE superalgebras. This follows from the fact that the R-matrix is uniquely determined by its degree 1 part and its relations with the comultiplication of the Hopf superalgebra. For a proof of this fact see [KhT].
When $\mathfrak{g}$ is not of type $A(n,n)^{(i)}$, Yamane wrote down generators for $\mathrm{Ker}(C_{\pm})$ in terms of the $E_i,F_i$. This gives explicit presentations of $U_h^{D,J}(\mathfrak{b}_{\pm})$ and $U_h^{D,J}(\mathfrak{g})$. He also gave an explicit formula for the $R$-matrix. The above corollaries then imply that these QUE (braided) superalgebras identify with $U_h^{EK}(\mathfrak{b}_{\pm}), U_h^{EK}(\mathfrak{g})$.

4.5. **Category $O$.** In this subsection we show that the category $O$ for $\mathfrak{g}$ can be deformed to an analogous category over $U_h^{D,J}(\mathfrak{g})$. We then give a Drinfeld-Kohno type theorem for affine Lie superalgebras. This subsection follows [EK3] where similar results are proved for Kac-Moody groups.

Let $O[[\hbar]]$ be the deformed category $O$ for $\mathfrak{g}$, i.e., representations of $\mathfrak{g}$ on topologically free $\mathbb{C}[[\hbar]]$-modules which are $\mathfrak{h}$-diagonalizable, have finite dimensional weight spaces and whose weights belong to a union of finitely many cones $\lambda - \sum \mathbb{N} \alpha_i, \lambda \in \mathfrak{h}^*[[\hbar]]$.

In a similar way one defines $O_\hbar$ to be the category of topologically free $U_h^{D,J}(\mathfrak{g})$-modules whose weights satisfy the above conditions. The category $O_{\hbar}$ is a braided tensor category where the braiding comes from the universal $R$-matrix of $U_h^{D,J}(\mathfrak{g})$.

Let $\Omega \in \mathfrak{g}^{\otimes 2}$ be the inverse element corresponding to the bilinear form $(-,-)$ $(\mathfrak{g}^{\otimes 2} = \oplus_{a,b} \mathfrak{g}_a \otimes \mathfrak{g}_b$, where $\mathfrak{g}$ decomposes as $\oplus_a \mathfrak{g}_a$ for the principal grading). The action of $\Omega$ is well-defined on the tensor product of two modules from the category $O[[\hbar]]$. Following Drinfeld, we give $O[[\hbar]]$ a braided tensor structure with associator $\Phi(\hbar \Omega_{12}, \hbar \Omega_{23})$ and braiding $\sigma^\Omega$.

**Theorem 4.4.** There exists a functor $F : O[[\hbar]] \to O_\hbar$ which is an equivalence of braided tensor categories and commuting with the forgetful functors $O[[\hbar]], O_\hbar \to \{\hbar\text{-graded} \mathbb{C}[[\hbar]]\text{-modules}\}$.

**Proof.** Theorem 4.3 allows us to replace $U_h^{D,J}(\mathfrak{g})$ with $D(U_h^{EK}(\mathfrak{b}_{\pm}))/(\mathfrak{h} = \mathfrak{h}^*)$. Therefore, it is enough to construct a functor between the corresponding categories for the superalgebras $O(\mathfrak{b}_{\pm})$ and $D(U_h^{EK}(\mathfrak{b}_{\pm}))$.

In the non-super case such a functor was defined in Theorem 4.1 of [EK3]. Since Theorem 4.1 of [EK3] is formulated in the general categorical language of [EK2] it is easy to check it generalizes to the super setting and so the theorem follows.

Let $V$ be a module of $O[[\hbar]]$ and let $V_q$ be its image under $F$ in $O_\hbar$. Let $B_n = \langle \sigma_i \rangle$ be the braid group. Define $\rho_n$ to be the representation of $B_n$ on $V_q^{\otimes n}$ given by

$$\sigma_i \mapsto \tau_{i,i+1} R_{i,i+1}$$

where $\tau_{i,i+1}$ is the super permutation of the $i$th and the $(i+1)$th component and $R$ is the universal $R$-matrix of $U_h^{D,J}(\mathfrak{g})$.

On the other hand, consider the Knizhnik-Zamolodchikov system of differential equations with respect to a function $\omega(z_1,\ldots,z_n)$ of complex variables $z_1,\ldots,z_n$ with values in $V^{\otimes n}$:

$$\frac{\partial \omega}{\partial z_i} = \frac{\hbar}{2\pi i} \sum_{i \neq j} \frac{\Omega_{ij} \omega}{z_i - z_j}$$

(32)

This system of equations defines a flat connection on the trivial bundle $Y_n \times V^{\otimes n}$, where $Y_n = \{(z_1,\ldots,z_n)| i \neq j \text{ implies } z_i \neq z_j\} \subset \mathbb{C}^n$. This connection determines a monodromy representation from $\pi_1(Y_n)$ to $\text{Aut}_{\mathbb{C}[[\hbar]]}(V^{\otimes n})$ (as it is a direct sum of connections in the weight spaces, which are finite dimensional). Moreover, since the system of equations (32) is invariant under the action of the symmetric group we obtain a monodromy representation

$$\rho_n^{KZ} : \pi_1(X_n,p) \to \text{Aut}_{\mathbb{C}[[\hbar]]}(V^{\otimes n})$$

where $X_n = Y_n/S_n$ and $p = (1,2,\ldots,n) \in \mathbb{C}^n$. Finally, we identify $\pi_1(X_n,p)$ with the braid group $B_n$ to get a monodromy representation of $B_n$. 

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Theorem 4.5. (The Drinfeld-Kohno theorem for affine Lie superalgebras) The representations \( \rho_n \) and \( \rho^K_{nZ} \) are equivalent.

Proof. Let \( \rho^{R_{KZ}}_n \) be the representation of \( B_n \) on \( V^\otimes n \) induced by the \( R \)-matrix \( R_{KZ} = e^{\hbar \Omega/2} \) in the category \( \mathcal{O}[[h]] \) with the Knizhnik-Zamolodchikov associator \( \Phi_{KZ} \). From Theorem 4.4 we have that \( \rho^{R_{KZ}}_n \) and \( \rho_n \) correspond to each other under the braided tensor functor \( F \). The theorem follows since \( \rho^{KZ}_n \) coincides with \( \rho^{R_{KZ}}_n \) when \( \Phi = \Phi_{KZ} \).

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