Abstract
We construct a delay functional \( d \) on an open subset of the space \( C^1_r = C^1((-r, 0], \mathbb{R}) \) and find \( h \in (0, r) \) so that the equation
\[
x'(t) = -x(t - d(x_t))
\]
defines a continuous semiflow of continuously differentiable solution operators on the solution manifold
\[
X = \{ \phi \in C^1_r : \phi'(0) = -\phi(-d(\phi)) \},
\]
and along each solution the delayed argument \( t - d(x_t) \) is strictly increasing, and there exists a solution whose short segments
\[
x_{t, \text{short}} = x(t + \cdot) \in C^2_h, \quad t \geq 0,
\]
are dense in an infinite-dimensional subset of the space \( C^2_h \). The result supplements earlier work on complicated motion caused by state-dependent delay with oscillatory delayed arguments.

Keywords Delay differential equation · State-dependent delay · Complicated motion

AMS Subject Classification 34 K 23

1 Introduction
The present paper continues the studies [6,10–14] of how time lags which are state-dependent affect the behaviour of feedback systems. The basic equation considered is
\[
x'(t) = -\alpha x(t - r)
\]
\((\alpha, r)\)

Dedicated to the memory of Geneviève Raugel.
with $\alpha > 0$ and constant time lag $r > 0$. This is the simplest delay differential equation
modelling negative feedback with respect to the zero solution. Let $C^0_r$ denote the Banach space
of continuous functions $[-r, 0] \to \mathbb{R}$ with the maximum norm, $|\phi|_{0,r} = \max_{-r \leq t \leq 0} |\phi(t)|$. The solutions $x : [-r, \infty) \to \mathbb{R}$ of Eq. (α, r), which are continuous and have differentiable
restrictions to $[0, \infty)$ which satisfy Eq. (α, r), define a strongly continuous semigroup on $C^0_r$ by the equations $T(t)x_0 = x_t$ with the solution segments
\[ x_t : [-r, 0] \ni s \mapsto x(t + s) \in \mathbb{R} \quad \text{for} \quad t \geq 0, \]
see [2]. Except for $\alpha = \frac{\pi}{2} + 2k\pi, k \in \mathbb{N}_0$ the zero solution is hyperbolic [2,15].

Let $C^1_r$ denote the Banach space of continuously differentiable functions $\phi : [-r, 0] \to \mathbb{R}$, with the norm given by $|\phi|_{1,r} = |\phi|_{0,r} + |\phi'|_{0,r}$. In [6,10–12] delay functionals $d : C^1_r \supset U \to [0, r]$ were constructed so that for certain $\alpha > 0$ the modified equation
\[ x'(t) = -\alpha x(t - d(x_t)) \quad (\alpha, d) \]
has homoclinic solutions, with chaotic motion nearby.

The results in [13,14] established another kind of complicated solution behaviour, namely, the existence of delay functionals $d$ and parameters $\alpha > 0$ so that for a positive number $h < r$ there are solutions whose short solution segments
\[ x_{t,\text{short}} : [-h, 0] \ni s \mapsto x(t + s) \in \mathbb{R}, t \geq 0, \]
are dense in open subsets of the space $C^1_h$.

In [13] density of short segments in the whole space $C^1_h$ was achieved for a continuous delay functional on a set $Y \subset C^1_r$ which is large in some sense but not open, nor a differentiable
submanifold. Because of this lack of regularity results from [8,9] on well-posedness of initial value problems and on differentiability of solutions with respect to initial data do not apply.

In [14] we constructed a continuously differentiable delay functional $d : U \to [0, r]$, $U \subset C^1_r$ open, so that the results from [8] apply, and found $h \in (0, r)$ so that the previous equation with $\alpha = 1$, namely,
\[ x'(t) = -x(t - d(x_t)) \quad (1.1) \]
has a solution $x : [-r, \infty) \to \mathbb{R}$ whose short segments are dense in an open subset of the space $C^1_h$. The construction involves that the delayed argument function
\[ [0, \infty) \ni t \mapsto t - d(x_t) \in \mathbb{R} \]
along the solution $x$ is not monotonic, and this oscillatory behaviour seems crucial for density of short segments in an open subset of the space $C^1_h$.

Before stating the result of the present paper let us mention that equations with non-
constant, state-dependent delay are not covered by the theory with state space $C^0_r$ which is familiar from monographs on delay differential equations [1–3]. We recall what was shown in [8] for delay differential equations in the general form
\[ x'(t) = f(x_t) \]
under hypotheses designed for applications to examples with state-dependent delay. Let $C^0_{r,n}$ and $C^1_{r,n}$ denote the analogues of the spaces $C^0_r$ and $C^1_r$, for maps $[-r, 0] \to \mathbb{R}^n$. Assume $f : U \to \mathbb{R}^n, U \subset C^1_{r,n}$ open, is continuously differentiable so that
\begin{enumerate}
  \item[(e)] each derivative $Df(\phi) : C^1_{r,n} \to \mathbb{R}^n, \phi \in U$, has a linear extension $D_e f(\phi) : C^0_{r,n} \to \mathbb{R}^n$ and the map
  \[ U \times C^0_{r,n} \ni (\phi, \chi) \mapsto D_e f(\phi) \chi \in \mathbb{R}^n \]
\end{enumerate}
is continuous.

The extension property (e) is a variant of the notion of being \textit{almost Fréchet differentiable} for maps \( C^0_{r,n} \supset V \rightarrow \mathbb{R}^n \) which was introduced in [7].

Suppose also there exists \( \phi \in U \) with \( \phi'(0) = f(\phi) \). Then the nonempty set

\[
X_f = \{ \phi \in U : \phi'(0) = f(\phi) \}
\]

is a continuously differentiable submanifold with codimension \( n \) in \( C^1_{r,n} \), and each initial value problem

\[
x'(t) = f(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \phi \in X_f,
\]

has a unique maximal solution \( x : [-r, t_\phi) \rightarrow \mathbb{R}^n, \; 0 < t_\phi \leq \infty \), which is continuously differentiable with \( x'(t) = f(x_t) \) for all \( t \in [0, t_\phi) \). The arrow

\[
( t, \phi ) \mapsto x_\phi^t,
\]

with the said maximal solution \( x = x^\phi \), defines a continuous semiflow of continuously differentiable solution operators

\[
\{ \phi \in X_f : t_\phi > t \} \ni \phi \mapsto x_\phi^t \in X_f, \quad t \geq 0.
\]

In the present paper we prove the following result on complicated motion caused by a delay functional so that the delayed argument functions along solutions of Eq. (1.1) are monotonic.

\textbf{Theorem 1.1} There exist \( r > h > 0 \) and a continuously differentiable delay functional \( d : N \rightarrow (0, r), \; N \subset C^1_r \) open, and an open subset \( A \) of a closed affine subspace of codimension \( 6 \) in \( C^2_h \) so that Eq. (1.1) has a twice continuously differentiable solution \( x^{(d)} : [-r, \infty) \rightarrow \mathbb{R} \) whose short segments \( x^{(d)}_{t, \text{short}}, \; t \geq 0 \), are dense in \( A \cup (-A) \).

The functional \( f : N \ni \phi \mapsto -d(-\phi(\phi)) \in \mathbb{R} \) is continuously differentiable and has property (e), and for each \( \phi \in X_f \) the delayed argument function

\[
[0, t_\phi) \ni t \mapsto t - d(x^\phi_t) \in \mathbb{R}
\]

along the maximal continuously differentiable solution \( x^\phi : [-r, t_\phi) \rightarrow \mathbb{R} \) of the initial value problem

\[
x'(t) = f(x_t) = -x(t - d(x_t)) \quad \text{for} \quad t > 0, \quad x_0 = \phi \in X_f,
\]

is strictly increasing.

Here \( C^2_h \) denotes the Banach space of twice continuously differentiable functions \( \psi : [-h, 0] \rightarrow \mathbb{R} \), with the norm given by \( \| \psi \|_{2,r} = \sum_{k=0}^{2} \max_{-r \leq t \leq 0} |\psi^{(k)}(t)| \).

A different result on complicated motion caused by state-dependent delay with monotonic delayed argument functions has recently been obtained in [5].

The proof of Theorem 1.1 begins in Sect. 2 below with the choice of subsets \( A = A_h \subset C^2_h \) as in the theorem, for arbitrary \( h > 0 \). For arbitrary \( s > 0 \) Sect. 3 prepares a sequence of twice continuously differentiable functions \( \kappa_{s,n} : [-s, s] \rightarrow \mathbb{R} \) so that certain translates of \( \kappa_{s,n} \) and \( \kappa_{s,k}, \; n \neq k \), keep a minimal distance from each other, in the sense that there is a constant \( a > 0 \) with

\[
| \langle \kappa_{s,n} \rangle'(t + u) - \langle \kappa_{s,k} \rangle'(u) | \geq \frac{a}{4}
\]

for small \( t \) and some \( u \).
Section 4 is the core of the proof of Theorem 1.1. For suitably chosen \( t_b < t < t_5, h > 0, s > 0 \), a sequence of continuously differentiable delay functions \( \Delta_n : [0, t_5] \to (0, \infty) \) together with a sequence of twice continuously differentiable functions \( x(n) : [t_b, t_5] \to \mathbb{R} \) and a subset \( A = A_h \subset C^2_r \) as in Sect. 2 are constructed so that for each \( n \in \mathbb{N} \) - the linear nonautonomous equation

\[
(x(n))'(t) = -x(n)(t - \Delta_n(t))
\]

holds for \( 0 \leq t \leq t_5 \),

- the delayed argument function \( [0, t_5] \ni t \mapsto t - \Delta_n(t) \in \mathbb{R} \) along the delay function \( \Delta_n \) is strictly increasing,
- on some subinterval of length \( h \) in \( [0, t_5] \) the function \( x(n) \) coincides with a translate of a member \( p_n \) of a sequence which is dense in \( A \),
- on some subinterval of length \( 2s \) in \( [0, t_5] \) the function \( x(n) \) coincides with a translate of \( \kappa_n = \kappa_{s,n} \).

In Sect. 5 shifted copies of the functions \( \Delta_n \) and of the functions \( \pm x(n) \) are concatenated, respectively, and yield a twice continuously differentiable function \( x : [t_b, \infty) \to \mathbb{R} \) and a continuously differentiable delay function \( \Delta \) on \( [0, \infty) \) which is bounded by some \( r > \max\{h, -t_b\} \). A twice continuously differentiable extension of the function \( x \) to the ray \([-r, \infty) \to \mathbb{R} \) satisfies the linear equation

\[
x'(t) = -x(t - \Delta(t)) \tag{1.2}
\]

for all \( t \geq 0 \). Proposition 5.1 states that the curve \( [r, \infty) \ni t \mapsto x_t \in C^1_r \) is injective, hence the equation

\[
d(x_t) = \Delta(t)
\]

converts the delay function into a delay functional \( d \) on the trace \( \{x_t \in C^2_r : t \geq r\} \).

Sections 6, 7, and 8 prepare the extension of this functional to an open neighbourhood \( N \) of the trace \( \{x_t \in C^2_r : (j_r - 1)t_5 \leq t \} \) in the space \( C^1_r \), with an integer \( j_r \geq 2 \) so that \( r < (j_r - 1)t_5 \). Section 6 contains an ingredient of the construction which will be used in the final Sect. 9, namely, separation of nonadjacent arcs

\[
\{x_t \in C^2_r : (n - 1)t_5 \leq t \leq nt_5\} \quad \text{and} \quad \{x_t \in C^2_r : (j - 1)t_5 \leq t \leq jt_5\}, \quad 2 \leq n \in \mathbb{N} \quad \text{and} \quad 2 \leq j \in \mathbb{N} \quad \text{with} \quad |n - j| > 1,
\]

in the space \( C^1_r \). The separation result is based on the properties of the functions \( \kappa_{s,n} \) from Sect. 3 whose translates appear as restrictions of \( x \) on a sequence of mutually disjoint intervals tending to infinity.

The constructions in Sects. 2, 3, 4, 5, and 6 are to some extent parallel to constructions in [14]. The next steps in Sects. 7 and 8 are rather different from their counterparts in [14]. The new tool, introduced in Sect. 7, is a bundle of transversal hyperplanes \( K_t, t > 0 \), along the curve \( (0, \infty) \ni t \mapsto x_t \in C^0_r \). Working with the bundle allows for an extension of the delay functional from an arc \( \{x_t \in C^2_r : (k - 1)t_5 \leq t \leq kt_5\}, j_r \leq k \in \mathbb{N} \), to a kind of tubular neighbourhood \( U_k \subset C^0_r \) (Sect. 8), and for the arrangement of compatibility relations on overlapping domains \( U_k \cap U_{k+1} \), in ways which are simpler than corresponding procedures in [14].

Section 9 begins with the definition of the domain \( N \subset C^1_r \) and the functional \( d : N \to (0, r) \), and completes the proof of Theorem 1.1. The verification that the functional \( f : N \to \mathbb{R} \) in Theorem 1.1 has property (e) uses that the delay functional \( d : N \to (0, r) \) has
property (e). The latter is achieved by means of the following proposition whose statement involves the injective linear continuous inclusion map
\[ J : C^1_r \ni \phi \mapsto \phi \in C^0_r. \]

**Proposition 1.2** [14, Proposition 1.2] Suppose \( d : C^1_r \supset N \to \mathbb{R} \) is continuously differentiable and for every \( \phi \in N \) there exist an open neighbourhood \( V \) of \( J\phi \) in \( C^0_r \) and a continuously differentiable map \( d_V : C^0_r \supset V \to \mathbb{R} \) with \( d(\psi) = d_V(J\psi) \) for all \( \psi \in N \cap J^{-1}(V) \). Then \( d \) has property (e), with
\[ D_d(\phi) \chi = Dd_V(J\phi) \chi \quad \text{for all} \quad \phi \in N \cap J^{-1}(V) \quad \text{and} \quad \chi \in C^0_r. \]

**Notation, preliminaries.** A sequence in a metric space is called dense if each point of the metric space is an accumulation point of the sequence. A metric space is called separable if it contains a dense sequence.

For \( \epsilon > 0 \) the open \( \epsilon \)-neighbourhoods of a point \( x \) in a normed space \( X \) and of a subset \( S \subset X \) are given by
\[ U_\epsilon(x) = \{ y \in X : |y - x| < \epsilon \}. \]
and
\[ U_\epsilon(S) = \{ y \in X : \text{dist}(y, S) < \epsilon \}, \]
respectively, with
\[ \text{dist}(y, S) = \inf_{x \in S} |y - x|. \]

For \( a < b \) in \( \mathbb{R} \) and \( j \in \mathbb{N} \) let \( C^j_{a,b} \) denote the Banach space of \( j \) times continuously differentiable functions \( \phi : [a, b] \to \mathbb{R} \), with the norm given by
\[ |\phi|_{j,a,b} = \sum_{j=0}^{j} \max_{a \leq t \leq b} |\phi^j(t)|, \]
and let \( C^0_{a,b} \) denote the Banach space of continuous functions \( \phi : [a, b] \to \mathbb{R} \), with the norm given by
\[ |\phi|_{0,a,b} = \max_{a \leq t \leq b} |\phi(t)|. \]

In case \( a = -r \) and \( b = 0 \), the abbreviations
\[ C^j_r = C^j_{-r,0} \quad \text{and} \quad |\cdot|_{j,r} = |\cdot|_{j,-r,0} \]
are used. If functions \( \phi \in C^2_r \) and \( \phi \in C^1_r \) are considered as elements of the ambient space \( C^0_r \) then we use \( \phi \in C^0_r \) or \( J\phi \in C^0_r \), depending on which form makes an argument more transparent.

For \( r > 0 \) the evaluation map
\[ C^0_r \times [-r, 0] \ni (\phi, t) \mapsto \phi(t) \in \mathbb{R} \]
is continuous but not locally Lipschitz continuous, and the evaluation map
\[ ev^1_r : C^1_r \times (-r, 0) \ni (\phi, t) \mapsto \phi(t) \in \mathbb{R} \]
is continuously differentiable with
\[ D e v^1_1(\phi, t)\hat{\phi}(t) = D_1 e v^1_1(\phi, t)\hat{\phi} + D_2 e v^1_1(\phi, t)\hat{t} = \hat{\phi}(t) + \hat{t}\phi'(t), \]
see e. g. [4,8].

In Sect. 8 below the following is used.

**Proposition 1.3** Let \( B \) be a Banach space. Let reals \( a < b \), a continuous injective map \( c : [a, b] \to B \), some \( t \in (a, b) \), and \( \epsilon > 0 \) be given. Then there exists \( \rho > 0 \) with
\[ U_\rho(c([a, t])) \cap U_\rho(c([t, b])) \subseteq U_{\epsilon/2}(c(t)). \]

**Proof** By continuity there exists \( t_a \in (a, t) \) with \( c([t_a, t]) \subseteq U_{\epsilon/2}(c(t)) \). The compact sets \( c([a, t_a]) \) and \( c([t, b]) \) are disjoint, which gives
\[ 0 < \min_{a \leq u \leq t_a} \text{dist}(c(u), c([t, b])) \].
Choose \( \rho \in (0, \frac{\epsilon}{2}) \) with
\[ 2\rho < \min_{a \leq u \leq t_a} \text{dist}(c(u), c([t, b])). \]
Consider \( z \in U_\rho(c([a, t])) \cap U_\rho(c([t, b])) \). There exist \( u_a \in [a, t] \) and \( u_b \in [t, b] \) with
\[ |z - c(u_a)| < \rho \quad \text{and} \quad |z - c(u_b)| < \rho, \]
hence \( |c(u_a) - c(u_b)| < 2\rho \). The assumption \( u_a < t_a \) yields a contradiction to the inequality
\[ 2\rho < \min_{a \leq u \leq t_a} \text{dist}(c(u), c([t, b])), \]
It follows that \( u_a \in [t_a, t_b] \). Consequently,
\[ |z - c(t)| \leq |z - c(u_a)| + |c(u_a) - c(t)| < \rho + \frac{\epsilon}{2} < \epsilon, \]
which means \( z \in U_{\epsilon/2}(c(t)) \).

## 2 Separability

Let \( h > 0 \) be given. The restrictions of polynomials \( \mathbb{R} \to \mathbb{R} \) to the interval \([-h, 0]\) are dense in \( C^2_h \), which is an easy consequence of the Weierstraß approximation theorem. Let \( P_5 \subset C^2_h \) denote the subspace of restrictions of polynomials of degree not larger than 5 and let \( C^2_{h-0} \subset C^2_h \) denote the closed subspace given by the equations
\[ \phi^{(j)}(-h) = 0 = \phi^{(j)}(0) \quad \text{for} \quad j \in \{0, 1, 2\}. \]
Then \( \dim P_5 = 6 \) and
\[ C^2_h = C^2_{h-0} \oplus P_5, \]
which follows from the fact that given \( \phi \in C^2_h \) there exists a unique \( p \in P_5 \) satisfying
\[ p^{(j)}(-h) = \phi^{(j)}(-h) \quad \text{and} \quad p^{(j)}(0) = \phi^{(j)}(0) \quad \text{for} \quad j \in \{0, 1, 2\}, \]
or, \( \phi - p \in C^2_{h-0} \).

**Proposition 2.1** Let an open set \( U \subset C^2_h \) and \( p_\ast \in C^2_h \) with \( A = U \cap (p_\ast + C^2_{h-0}) \neq \emptyset \) be given. The open subset \( A \) of the affine space \( p_\ast + C^2_{h-0} \) contains a sequence which is dense in \( A \).
Proof The restricted polynomials with rational coefficients form a sequence which is dense in $C^2_h$. Projection along $P_5$ onto $C^2_{h-0}$ yields a sequence which is dense in $C^2_{h-0}$, and translation by adding $p_*$ results in a sequence which is dense in $p_* + C^2_{h-0}$. The members of this sequence which belong to $U$ form a sequence which is dense in $A$.

Example 2.2 For given reals $w_0 < u_0 < 0$, $u_1 < w_1 < 0$, $u_2 > 0$, $w_2 > 0$ let $p_* \in P_5$ denote the unique restricted polynomial which satisfies

$$
 p_*^{(j)}(-h) = u_j, \ p_*^{(j)}(0) = w_j \quad \text{for } j \in \{0, 1, 2\},
$$

and take

$$
 U = \{ \phi \in C^2_h : 0 < \phi''(t) \text{ on } [-h, 0] \}.
$$

Notice that

$$
 A = U \cap (p_* + C^2_{h-0})
 = \{ \phi \in C^2_h : 0 < \phi''(t) \text{ on } [-h, 0], \ 
 \phi^{(j)}(-h) = u_j, \ \phi^{(j)}(0) = w_j \text{ for } j \in \{0, 1, 2\} \}.
$$

We add the obvious fact that the dense sequence provided by Proposition 2.1 is dense in $A \subset C^2_h \subset C^1_h$ also with respect to the norm $|\cdot|_{1,h}$.

3 Differentiable Functions with Separated Shifted Copies

Let $s > 0$ be given. We construct a sequence of functions $\kappa_n \in C^2_{-s,s}$, $n \in \mathbb{N}$, so that shifted copies of these functions keep a positive minimal distance from each other with respect to the norm $|\cdot|_{1,-s,s}$.

Let also positive reals $a, \xi, \eta$ be given and choose $\epsilon \in (0, \frac{a}{4})$. There exists $\chi \in C^1_{-s,0}$ with

$$
 \chi(-s) = -a, \quad \chi([-s, 0]) \subset [-a, -a + \epsilon],
 \chi'(-s) = \eta, \quad \chi'(t) > 0 \text{ on } [-s, 0].
$$

For every $n \in \mathbb{N}$ there exists $\rho_n \in C^1_{-s,s}$ with

$$
 \rho_n(t) = -\rho_n(-t) \text{ on } [-s, s]
$$

and

$$
 \rho_n(t) = \chi(t) \text{ on } \left[-s, -\frac{s}{2n}\right],
 \rho_n\left(-\frac{s}{2n+1}\right) = -\frac{a}{2},
 \rho_n(0) = 0,
 (\rho_n)'(t) = (\rho_n)'(0) \text{ constant on } \left[-\frac{s}{2n+1}, 0\right],
 (\rho_n)'(t) > 0 \text{ on } [-s, 0].
$$
Fig. 1 The function $\rho_1$ for $-s \leq t \leq 0$

Proposition 3.1 For all integers $n \neq k$ in $\mathbb{N}$ and for each $t \in \left[ -\frac{s}{2}, 0 \right]$ there exists $u \in [-s, s]$ with $t + u \in [-s, s]$ and

$$|\rho_n(t + u) - \rho_k(u)| \geq \frac{a}{2} - \epsilon.$$

Proof Let positive integers $n \neq k$ and $t \in \left[ -\frac{s}{2}, 0 \right]$ be given. In case $n > k$ consider $u = -\frac{s}{2^{k+1}}$. Then $u \in \left[ -\frac{s}{2}, 0 \right]$ and

$$-s \leq -\frac{s}{2} - \frac{s}{2^{k+1}} \leq t + u \leq u = -\frac{s}{2^{k+1}} \leq -\frac{s}{2^n},$$

hence

$$\rho_n(t + u) - \rho_k(u) = \chi(t + u) - \left(-\frac{a}{2}\right) \in [-a, -a + \epsilon] + \frac{a}{2} = \left[-\frac{a}{2}, \frac{a}{2} + \epsilon\right].$$

In case $k > n$ set $u = -t + \frac{s}{2^{n+1}}$. Then

$$0 < \frac{s}{2^k} \leq \frac{s}{2^{n+1}} = u + t \leq u \left( \leq \frac{s}{2} + \frac{s}{2^{n+1}} \leq s \right),$$

hence

$$|\rho_n(t + u) - \rho_k(u)| = |\rho_n\left(\frac{s}{2^{n+1}}\right) - \rho_k\left(-t + \frac{s}{2^{n+1}}\right)| \leq \left|\rho_n\left(-\frac{s}{2^{n+1}}\right) + \rho_k\left(-\frac{s}{2^{n+1}} + t\right)\right| \geq \left|\chi\left(-\frac{s}{2^{n+1}} + t\right) - \frac{a}{2}\right| \geq a - \epsilon - \frac{a}{2} \geq \frac{a}{2} - \epsilon.$$

For $n \in \mathbb{N}$ define $\kappa_n \in C^2_{-s, s}$ by

$$\kappa_n(t) = -\xi + \int_{-s}^{t} \rho_n(u)du$$

and observe that

$$\kappa_n(-t) = \kappa_n(t) \text{ on } [-s, s].$$
\[
\begin{aligned}
\kappa_n(-s) &= -\dot{\xi} = \kappa_n(s), \\
(\kappa_n)'(t) &< 0 \text{ on } [-s, 0), \\
(\kappa_n)'(t) &> 0 \text{ on } (0, s], \\
(\kappa_n)'(-s) &= -a, \\
(\kappa_n)'(s) &= a, \\
(\kappa_n)''(t) &> 0 \text{ on } [-s, s], \\
(\kappa_n)''(-s) &= \eta.
\end{aligned}
\]

Using Proposition 3.1 and \(\epsilon < \frac{a}{4}\) we get the following result.

**Corollary 3.2** For all integers \(n \neq k\) in \(\mathbb{N}\) and for each \(t \in \left[-\frac{\xi}{2}, 0\right]\) there exists \(u \in [-s, s]\) with \(t + u \in [-s, s]\) and

\[
| (\kappa_n)'(t + u) - (\kappa_k)'(u) | \geq \frac{a}{4}.
\]

### 4 The Delay Function on a Compact Interval

In this section we find \(h > 0\), a set \(A \subset C_h^2\), constants \(t_b < 0\) and \(t_5 < -t_b\), and functions

\[
\Delta_n : [0, t_5] \rightarrow (0, \infty) \quad \text{and} \quad x(n) : [t_b, t_5] \rightarrow \mathbb{R}, \quad n \in \mathbb{N},
\]

which in the next section will be used to form a solution of Eq. (1.2) whose short segments are dense in the set \(A \cup (-A)\). Choose reals

\[
\xi > b > a > 0 \quad \text{with} \quad \xi - a > b
\]

such that there exists \(t_2 > 1\) with

\[
bt_2 > \xi > at_2.
\]

and choose \(t_b \in (-1, 0)\) with

\[
b < (-t_b)\xi.
\]

Choose \(v \in C_{t_b, 0}^1\) with

\[
\begin{aligned}
v(t) &< 0 \text{ on } [t_b, 0], \\
v'(t) &> 0 \text{ on } [t_b, 0], \\
v(t_b) &= -\dot{\xi}, \\
v(0) &= -b, \\
v'(t_b) &= \frac{a}{2}.
\end{aligned}
\]

Because of \(v([t_b, 0]) = [-\dot{\xi}, -b]\) and

\[
b + t_bb > 0 > b + t_b\xi
\]

we can choose \(v\) in such a way that also

\[
b + \int_0^{t_b} v(t) dt = 0.
\]
The equation

\[ x(t) = b + \int_{t_b}^{t} v(u) \, du \]

defines a strictly decreasing function \( x \in C^2_{t_b,0} \) with

\[ x(t_b) = b \quad \text{and} \quad x(0) = 0. \]

Let \( t_a \in (t_b, 0) \) be given by \( x(t_a) = a \).

Extend \( v \in C^1_{t_b,0} \) to a function in \( C^1_{t_b,t_2} \) with

\[ v(t) < 0 \quad \text{on} \quad [0, t_2], \]
\[ v(t_2) = -a, \]
\[ v'(t) > 0 \quad \text{on} \quad [0, t_2]. \]

Because of \( v([0, t_2]) = [-b, -a] \) and

\[ -bt_2 < -\xi < -at_2 \]
we can choose \( v \in C_{ta,t_2}^1 \) in such a way that also
\[
\int_0^{t_2} v(t)dt = -\xi.
\]
Set
\[
\eta = v'(t_2) > 0.
\]
Extend \( x \in C_{tb,t_2}^2 \) to a strictly decreasing function in \( C_{tb,t_2}^2 \) by
\[
x(t) = \int_0^t v(u)du = b + \int_{tb}^t v(u)du \text{ on } (0, t_2],
\]
so that \( x(t_2) = -\xi \), and let \( t_1 \in (0, t_2) \) be given by \( x(t_1) = -a \).
Fix \( t_d \in (t_1, t_2) \) and
\[
h > 0 \text{ with } t_1 < t_d - h
\]
and define
\[
 u_j = x^{(j)}(t_d - h) \text{ and } w_j = x^{(j)}(t_d) \text{ for } j \in \{0, 1, 2\}.
\]
Then \( 0 > u_0 > w_0, u_1 < w_1 < 0, 0 < u_2, 0 < w_2 \). Consider the set \( A \subset C_h^2 \) from Example 2.2. The functions in \( A \) are negative and strictly decreasing, with the derivative strictly increasing. Proposition 2.1 guarantees a sequence \( (p_n)_{n \in \mathbb{N}} \) in \( A \) which is dense in \( A \). For \( n \in \mathbb{N} \) define \( x_{(n)} \in C_{tb,t_2}^2 \) by
\[
x_{(n)}(t) = x(t) \text{ on } [t_b, t_d - h] \cup [t_d, t_2],
\]
Then notice that
\[ x(0)(t) = p_{1/2} (t - t_d) \] on \([t_d - h, t_d]\)
in case \(n\) odd,
\[ x(n)(t) = p_{n/2} (t - t_d) \] on \([t_d - h, t_d]\)
in case \(n\) even.

Notice that \(x(n)\) is strictly decreasing on \([t_b, t_2]\) with
\[ x(n)(t_b) = b, \quad x(n)(0) = 0, \quad x(n)(t_1) = -a, \quad x(n)(t_2) = -\xi, \]
\((x(n))'(t) < 0 \) on \([t_b, t_2]\),
\((x(n))'(t_2) = x'(t_2) = v(t_2) = -a, \]
\((x(n))''(t) > 0 \) on \([t_b, t_2]\),
\((x(n))'(t) = v'(t_2) = \eta. \]

The inverse \(y_n = (x(n))^{-1} \in C^2_{\xi, b}\) maps its domain \([-\xi, b]\) onto the interval \([t_b, t_2]\), with
\[
(y_n)'(u) = \frac{1}{(x(n))'(y_n(u))} < 0 \quad \text{for all} \quad u \in [-\xi, b].
\]

Obviously,
\[
(x(n))'(t) = \begin{cases} (x(n))'(0), & (x(n))'(t_2)] = [-b, -a], \\
-(x(n))'(t_2) = [a, b] \subset [-\xi, b].
\end{cases}
\]

It follows that the equation
\[ y_n(-(x(n))'(t)) = t - \Delta_n(t) \]
defines a function \(\Delta_n \in C^1_{0, t_2}\) with
\[
1 - (\Delta_n)'(t) = (y_n)'(-(x(n))'(t))[-(x(n))''(t)] > 0 \quad \text{on} \quad [0, t_2],
\]
\[
0 - \Delta_n(0) = y_n(-(x(n))'(0)) = y_n(b) = t_b,
\]
\[
t_2 - \Delta_n(t_2) = y_n(-(x(n))'(t_2)) = y_n(a) = t_a,
\]
\[
(x(n))'(t) = -x(n)(t - \Delta_n(t)) \quad \text{on} \quad [0, t_2].
\]

In particular,
\[ (id - \Delta_n)([0, t_2]) = [t_b, t_a]. \]

The estimate \(t - \Delta_n(t) \leq t_a\) on \([0, t_2]\) yields
\[ \Delta_n(t) \geq t - t_a \geq t \geq 0 \quad \text{on} \quad [0, t_2]. \]

Fix some \(s > 0\) and recall \(\kappa_n \in C^2_{-\xi, a}\) from Sect. 3, with \(a, \xi, \eta\) from the present section. Then
\[ (\kappa_n)^{(j)}(-s) = (x(n))^{(j)}(t_2) \quad \text{for} \quad j \in \{0, 1, 2\}. \]

Set
\[ t_3 = t_2 + 2s \]
and define an extension of \(x(n)\) to a map in \(C^2_{t_2, t_3}\) by
\[ x(n)(t) = \kappa_n(t - t_3 + s) \quad \text{on} \quad [t_2, t_3]. \]
By the symmetry of $\kappa_n$,

$$\begin{align*}
x^{(n)}(t_3) &= x^{(n)}(t_2) = -\xi, \\
(x^{(n)})'(t_3) &= -(x^{(n)})'(t_2) = a, \\
(x^{(n)})''(t_3) &= (x^{(n)})''(t_2) = \eta,
\end{align*}$$

and

$$(x^{(n)})'([t_2, t_3]) = (\kappa_n)'([-s, s]) = [-a, a] = x^{(n)}([t_a, t_1]).$$

It follows that the equation

$$y_n(-(x^{(n)})'(t)) = t - \delta_n(t) \quad \text{on} \quad [t_2, t_3]$$

defines a map $\delta_n \in C^1_{t_2, t_3}$, with

$$\begin{align*}
t_2 - \delta_n(t_2) &= y_n(-(x^{(n)})'(t_2)) = y_n(a) = t_a, \\
t_3 - \delta_n(t_3) &= y_n(-(x^{(n)})'(t_3)) = y_n((x^{(n)})'(t_2)) = y_n(-a) = t_1, \\
1 - (\delta_n)'(t) &= (y_n)'(-(x^{(n)})'(t))[-(x^{(n)})''(t)] > 0 \text{ on } [t_2, t_3], \\
(x^{(n)})'(t) &= -x^{(n)}(t - \delta_n(t)) \quad \text{on} \quad [t_2, t_3].
\end{align*}$$

Notice that $\delta_n(t_2) = t_2 - t_a = \Delta_n(t_2)$ and

$$1 - (\delta_n)'(t_2) = (y_n)'(-(x^{(n)})'(t_2))[-(x^{(n)})''(t_2)] = 1 - (\Delta_n)'(t_2).$$
The estimate $t - \delta_n(t) \leq t_1$ on $[t_2, t_3]$ yields
\[ \delta_n(t) \geq t - t_1 \geq t_2 - t_1 > 0 \text{ on } [t_2, t_3]. \]

Setting
\[ \Delta_n(t) = \delta_n(t) \text{ on } [t_2, t_3] \]
we get an extension of $\Delta_n \in C_{0,t_2}^1$ to a nonnegative map in $C_{0,t_3}^1$, with
\[ 1 - (\Delta_n)'(t) > 0 \text{ on } [0, t_3] \text{ and } (id - \Delta_n)((t_2, t_3)) = [t_a, t_1]. \]

Because of $a < \xi - b$ there exists $t_4 > t_3$ with
\[ a(t_4 - t_3) < \xi - b < \xi(t_4 - t_3), \]
for example, $t_4 = t_3 + 1$.

**Proposition 4.1** There exists $\delta_{n*} \in C_{t_3,t_4}^1$ with
\[
1 - (\delta_{n*})'(t) > 0 \text{ in } [t_3, t_4], \\
t_3 - \delta_{n*}(t_3) = t_1, \\
t_4 - \delta_{n*}(t_4) = t_3, \\
1 - (\delta_{n*})'(t_3) = 1 - (\Delta_n)'(t_3), \\
1 - (\delta_{n*})'(t_4) = \frac{1}{2}, \text{ and} \\
\]
\[
-\xi + \int_{t_3}^{t_4} x_n(t - \delta_{n*}(t)) dt = -b.
\]

**Proof** Consider the discontinuous function $g_0 : [t_3, t_4] \to \mathbb{R}$ given by $g_0(t_3) = t_1$ and $g_0(t) = t_3$ for $t_3 < t \leq t_4$. There is a sequence of functions $g_j \in C_{t_3,t_4}^1$, $j \in \mathbb{N}$, with
\[
(g_j)'(t) > 0 \text{ on } [t_3, t_4], \\
g_j(t_3) = t_1, \\
g_j(t_4) = t_3, \\
(g_j)'(t_3) = 1 - (\Delta_n)'(t_3), \\
(g_j)'(t_4) = \frac{1}{2},
\]
which converge pointwise to $g_0$. For every $j \in \mathbb{N}$, $g_j([t_3, t_4]) = [t_1, t_3]$, and the Lebesgue dominated convergence theorem yields
\[
G_j = \int_{t_3}^{t_4} [-x_n(g_j(t))] dt \to -\int_{t_3}^{t_4} x_n(t_3) dt = \xi(t_4 - t_3) \text{ as } j \to \infty.
\]

Similarly there is a sequence of functions $h_j \in C_{t_3,t_4}^1$ with the same properties as $g_j$ which converge pointwise to $h_0 : [t_3, t_4] \to \mathbb{R}$ given by $h_0(t_4) = t_3$ and $h_0(t) = t_1$ for $t_3 \leq t < t_4$, and
\[
H_j = \int_{t_3}^{t_4} [-x_n(h_j(t))] dt \to -\int_{t_3}^{t_4} x_n(t_1) dt = a(t_4 - t_3) \text{ as } j \to \infty.
\]

The limits satisfy
\[
a(t_4 - t_3) < \xi - b < \xi(t_4 - t_3),
\]
due to the choice of \( t_4 \). So there exists \( j \in \mathbb{N} \) with
\[
H_j < \xi - b < G_j.
\]

The function
\[
k : [0, 1] \times [t_3, t_4] \ni (\theta, t) \mapsto g_j(t) + \theta (h_j(t) - g_j(t)) \in \mathbb{R}
\]
is continuous. Using the intermediate value theorem we find some \( \theta \in (0, 1) \) with
\[
\int_{t_3}^{t_4} x_n(k(\theta, t))dt = (1 - \theta)G_j + \theta H_j = \xi - b.
\]
Notice that the convex combination \( k(\theta, \cdot) \in C_{t_3, t_4}^{1} \) shares the properties of \( g_j \) and \( h_j \). Define \( \delta_{n*} \) by
\[
t - \delta_{n*}(t) = k(\theta, t).
\]
The estimate \( t - \delta_{n*}(t) \leq t_3 \) on \([t_3, t_4]\) yields
\[
\delta_{n*}(t) \geq t - t_3 \geq 0 \quad \text{on} \quad [t_3, t_4].
\]
It follows that the equation
\[
\Delta_n(t) = \delta_{n*}(t) \quad \text{for} \quad t_3 < t \leq t_4
\]
extends \( \Delta_n \in C_{0, t_3}^{1} \) to a nonnegative function in \( C_{0, t_4}^{1} \) which satisfies
\[
1 - (\Delta_n)'(t) > 0 \quad \text{on} \quad [0, t_4],
\]
\[
t_4 - \Delta_n(t_4) = t_3,
\]
\[
t - \Delta_n(t) \in [t_1, t_3] \quad \text{for} \quad t_3 \leq t \leq t_4,
\]
\[
1 - (\Delta_n)'(t_4) = \frac{1}{2}.
\]
The function \( x_{n*} \in C_{t_3, t_4}^{2} \) given by
\[
x_{n*}(t) = -\xi + \int_{t_3}^{t} [-x_n(u - \Delta_n(u))]du
\]
satisfies
\[
x_{n*}(t_3) = -\xi = x_n(t_3),
\]
\[
x_{n*}(t_4) = -b,
\]
\[
(x_{n*})'(t) = -x_n(t - \Delta_n(t)) \quad \text{on} \quad [t_3, t_4],
\]
\[
(x_{n*})'(t_3) = -x_n(t_3 - \Delta_n(t_3)) = -x_n(t_1) = a = x_n'(t_3),
\]
\[
(x_{n*})'(t_4) = -x_n(t_4 - \Delta_n(t_4)) = -x_n(t_3) = \xi,
\]
\[
(x_{n*})''(t_3) = -(x_n)'(t_3 - \Delta_n(t_3))[1 - (\Delta_n)'(t_3)] = (x_n)''(t_3)
\]
\[
(x_{n*})''(t_4) = -(x_n)'(t_4 - \Delta_n(t_4))[1 - (\Delta_n)'(t_4)]
\]
\[
-\frac{1}{2} \left[(x_n)'(t_3)\right] = -a.
\]
Therefore the equation
\[
x_n(t) = x_{n*}(t) \quad \text{for} \quad t_3 < t \leq t_4
\]
The function \( x(n) \in C^2_{tb, t_4} \) defines a continuation of \( x(n) \in C^2_{tb, t_3} \) to a function in \( C^2_{tb, t_4} \) which satisfies Eq. (1.2) on \([0, t_4]\) and maps the interval \([t_3, t_4]\) onto \([-\xi, -b]\), with positive derivative and

\[
\begin{align*}
  x(n)(t_4) &= -b = -x(n)(tb), \\
  (x(n))'(t_4) &= \xi = -v(tb) = -(x(n))'(tb), \\
  (x(n))''(t_4) &= -a/2 = -v'(tb) = -(x(n))''(tb).
\end{align*}
\]

We set \( t_5 = t_4 - tb \) and extend \( x(n) \in C^2_{tb, t_4} \) to a function in \( C^2_{tb, t_5} \) by

\[
x(n)(t) = -x(n)(t - t_5) \quad \text{on} \quad [t_4, t_5].
\]

Then

\[
-(x(n))'(t_4, t_5) = (x(n))'(tb, 0) = [-\xi, -b] = x(n)([t_3, t_4]).
\]

The derivative of the function

\[
y_{n,5} = (x(n)|_{[t_3, t_4]})^{-1} \in C^2_{-\xi, -b}
\]

is strictly positive, due to \((x(n))'(t) > 0\) on \([t_3, t_4]\). The equation

\[
y_{n,5}(-(x(n))'(t)) = t - \delta_{n,5}(t) \quad \text{for} \quad t_4 \leq t \leq t_5
\]

defines a function \( \delta_{n,5} \in C^1_{t_4, t_5} \) which satisfies

\[
\begin{align*}
  t_4 - \delta_{n,5}(t_4) &= y_{n,5}(-(x(n))'(t_4)) = y_{n,5}((x(n))'(t_4 - t_5)) \\
  &= y_{n,5}((x(n))'(tb)) = y_{n,5}(-\xi) = t_3 = t_4 - \Delta_n(t_4), \\
  t_5 - \delta_{n,5}(t_5) &= y_n(-(x(n))'(t_5)) = y_n((x(n))'(0)) = y_n(-b) = t_4, \\
  1 - (\delta_{n,5})'(t) &= (y_{n,5})'(...)(-(x(n))'')'(t)
\end{align*}
\]
\[ (y_{n,5})'((x_n)'(t-t_5)) > 0 \quad \text{on} \quad [t_4, t_5], \]
\[ 1 - (\delta_{n,5})(t_4) = (y_{n,5})'(-(x_n)'(t_4))[-(x_n)''(t_4)] \]
\[ = (y_{n,5})'((x_n)'(t_b))[(x_n)''(t_b)] = (y_{n,5})'(-\xi) \frac{a}{2} \]
\[ = \frac{1}{2} = 1 - (\Delta_n)'(t_4). \]

The estimate \( t - \delta_{n,5}(t) \leq t_4 \) on \([t_4, t_5]\) yields

\[ \delta_{n,5}(t) \geq t - t_4 \geq 0 \quad \text{on} \quad [t_4, t_5]. \]

It follows that the equation

\[ \Delta_n(t) = \delta_{n,5}(t) \quad \text{for} \quad t_4 < t \leq t_5 \]

defines a continuation of \( \Delta_n \in C_{0,t_4}^1 \) to a nonnegative function in \( C_{0,t_5}^1 \) so that we have

\[ t_4 - \Delta_n(t_4) = t_3, \]
\[ t_5 - \Delta_n(t_5) = t_4, \quad \text{or equivalently,} \]
\[ \Delta_n(t_5) = t_5 - t_4 = -t_b = \Delta_n(0), \]
\[ 1 - (\Delta_n)'(t) > 0 \quad \text{on} \quad [0, t_5], \]
\[ (x_n)'(t) = -x_n(t - \Delta_n(t)) \quad \text{on} \quad [0, t_5]. \]

Also,

\[ (\Delta_n)'(t_5) = (\Delta_n)'(0) \]

because of

\[ 1 - (\Delta_n)'(t_5) = 1 - (\delta_{n,5})(t_5) = (y_{n,5})'(-(x_n)'(t_5))[-(x_n)''(t_5)] \]
\[ = (y_{n,5})'((x_n)'(0))(x_n)''(0) = (y_{n,5})'(-b)(x_n)''(0) \]
\[ = \frac{1}{(x_n)'(t_4)}(x_n)''(0) = \frac{1}{-(x_n)'(t_b)}(x_n)''(0) \]

and

\[ (x_n)''(0) = -(x_n)'(0 - \Delta_n(0)[1 - (\Delta_n)'(0)] = -(x_n)'(t_b)[1 - (\Delta_n)'(0)]. \]

**5 Concatenation**

All functions \( x_n(t) \in C_{t_4,t_5}^2, n \in \mathbb{N}, \) coincide on the set

\[ [t_b, t_d - h] \cup [t_d, t_2] \cup [t_4, t_5], \]

we have \( t_4 = t_5 + t_b, \) and for every \( n \in \mathbb{N}, \)

\[ x_n(t) = -x_n(t - t_5) \quad \text{for all} \quad t \in [t_4, t_5]. \]

Moreover, for every \( n \in \mathbb{N} \) the nonnegative function \( \Delta_n \in C_{0,t_5}^1 \) satisfies

\[ \Delta_n(t_5) = \Delta_n(0) = -t_b. \]
The function $x(n) \in C^2_{tb,t_5}$

$$(\Delta_n)'(t_5) = (\Delta_n)'(0),$$

$$1 - (\Delta_n)'(t) > 0 \quad \text{for all} \quad t \in [0, t_5],$$

and we have

$$(x(n))'(t) = -x(n)(t - \Delta_n(t)) \quad \text{for all} \quad t \in [0, t_5].$$

Therefore the relations

$$x(t) = (-1)^{n-1}x(n)(t - (n - 1)t_5) \quad \text{for} \quad n \in \mathbb{N}, \quad (n - 1)t_5 + tb \leq t \leq nt_5,$$

$$\Delta(t) = \Delta_n(t - (n - 1)t_5) \quad \text{for} \quad n \in \mathbb{N}, \quad (n - 1)t_5 \leq t \leq nt_5$$

define a twice continuously differentiable function $x : [tb, \infty) \to \mathbb{R}$ and a continuously differentiable nonnegative function $\Delta : [0, \infty) \to \mathbb{R}$ so that Eq. (1.2) holds for all $t \geq 0$, $\Delta(0) = -tb$, and

$$1 - \Delta'(t) > 0 \quad \text{for all} \quad t \geq 0.$$

The short segments $x_{(n-1)t_5+td,short} = p_n^{n+1} \in C^2_h$, $n \in \mathbb{N}$ odd, which are given by

$$x_{(n-1)t_5+td,short}(u) = x((n-1)t_5 + td + u) \quad \text{for} \quad -h \leq u \leq 0,$$

are dense in the infinite-dimensional set $A \subset C^2_h \subset C^1_h$ with respect to the norm $|\cdot|_{1,h}$.

Recall

$$t_b \leq t - \Delta_n(t) \quad \text{in} \quad [0, t_2],$$

$$t_d \leq t - \Delta_n(t) \quad \text{in} \quad [t_2, t_3],$$

$$t_1 \leq t - \Delta_n(t) \quad \text{in} \quad [t_3, t_4],$$

$$t_3 \leq t - \Delta_n(t) \quad \text{in} \quad [t_4, t_5] = [t_4, t_4 - tb]$$

for each $n \in \mathbb{N}$ and set

$$r = \max\{t_2 - t_b, t_3 - t_d, t_4 - t_1, t_4 - t_b - t_3, t_5 + 3s\}.$$
Then

$$\Delta(t) \leq r \quad \text{for all} \quad t \geq 0.$$ 

Extend $x : [t_b, \infty) \to \mathbb{R}$ backward to a twice continuously differentiable function $x : [-r, \infty) \to \mathbb{R}$, with long segments $x_t \in C^2_r \subset C^1_r$, $t \geq 0$, given by

$$x_t(u) = x(t + u) \quad \text{for} \quad -r \leq u \leq 0.$$ 

The curve

$$\hat{x} : (0, \infty) \ni t \mapsto x_t \in C^1_r$$

is continuously differentiable with

$$D\hat{x}(t)1 = (x_t)' = (x')_t \in C^1_r \quad \text{for all} \quad t > 0,$$

compare [13, Proposition 4.1]. As $\frac{t_k + t_n}{2}$ is the only zero of $(x(n))' : [t_b, t_5] \to \mathbb{R}$, for any $n \in \mathbb{N}$, we have

$$D\hat{x}(t)1 = (x_t)' \neq 0 \quad \text{for all} \quad t > 0.$$

**Proposition 5.1** The restriction of the curve $\hat{x}$ to the ray $[r, \infty)$ is injective.

**Proof** Assume $r \leq t \leq u$ and $\hat{x}(t) = \hat{x}(u)$. Then

$$x(t + v) = x(u + v) \quad \text{for all} \quad v \in [-r, 0].$$

There are $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with

$$(n - 1)t_5 \leq t < nt_5 \quad \text{and} \quad (k - 1)t_5 \leq u < kt_5.$$

From $t_5 < r \leq t$ we have $n \geq 2$, and from $t \leq u$ we have $n \leq k$.

1. Proof of $t - (n - 1)t_5 = u - (k - 1)t_5$. The argument $w = (n - 1)t_5 - t$ is contained in $(-t_5, 0) \subset [-r, 0)$, and

$$0 = x((n - 1)t_5) = x(t + w) = x(u + w).$$

As the interval $(u - t_5, u]$ contains exactly one zero of $x$, situated at $(k - 1)t_5$, we get $u + w = (k - 1)t_5$, hence

$$u - (k - 1)t_5 = -w = t - (n - 1)t_5.$$

2. The case $(n - 1)t_5 + t_3 \leq u$ ($< nt_5$). Using Part 1 of the proof we get

$$(k - 1)t_5 + t_3 \leq u.$$ 

For every $w \in [-s, s]$ we obtain

$$\kappa_n(w) = (-1)^{n-1} x(n)(t_3 - s + w) = x((n - 1)t_5 + t_3 - s + w)$$

$$= x(t + [-t + (n - 1)t_5 + t_3 - s + w])$$

$$= x(u + [-t + (n - 1)t_5 + t_3 - s + w])$$

(with $[-t + (n - 1)t_5 + t_3 - s + w] \in [-t_5, 0] \subset [-r, 0]$)

$$= x(u + [-u + (k - 1)t_5 + t_3 - s + w])$$

(with Part 1)

$$= x((k - 1)t_5 + t_3 - s + w) = (-1)^{k-1} x(k)(t_3 - s + w) = \kappa_k(w),$$
and it follows that \( n = k \). By Part 1, \( t = u \).

3. The case \((n-1)t_5 \leq t < (n-1)t_5 + t_3\). Using Part 1 of the proof we get
\[
((k-1)t_5 \leq) u < (k-1)t_5 + t_3.
\]

For every \( w \in [-s, s] \) we have
\[
-t + (n-2)t_5 + t_3 - s + w > -(n-1)t_5 + t_3 + (n-2)t_5 + t_3 - s + w
\]
\[
= t_5 - s + w \geq -t_5 - 2s \geq -r
\]
and
\[
-t + (n-2)t_5 + t_3 - s + w \leq -(n-1)t_5 + (n-2)t_5 + t_3 - s + w
\]
\[
\leq -t_5 + t_3 - s + s \leq 0,
\]

hence \([-t + (n-2)t_5 + t_3 - s + w] \in [-r, 0]\). It follows that
\[
\kappa_{n-1}(w) = (-1)^{n-2}x_{(n-1)}(t_3 - s + w) = x((n-2)t_5 + t_3 - s + w)
\]
\[
= x(t + [-t + (n-2)t_5 + t_3 - s + w])
\]
\[
= x(u + [-u + (n-2)t_5 + t_3 - s + w])
\]
(with Part 1)
\[
= x((k-2)t_5 + t_3 - s + w) = (-1)^{k-2}x_{(k-1)}(t_3 - s + w) = \kappa_{k-1}(w).
\]

Hence \( n-1 = k-1 \), and by Part 1, \( t = u \).

6 Separation of Arcs

**Proposition 6.1** There exists \( \hat{a} > 0 \) so that for all integers \( n \geq 2, j \geq 2 \) with \( |n-j| > 1 \) and for all \( t \in [(n-1)t_5, nt_5], u \in [(j-1)t_5, jt_5] \) we have
\[
|\hat{x}(t) - \hat{x}(u)|_{1,r} \geq \hat{a}.
\]

**Proof** 1. Recall from Sect. 4 the function \( v \in C^1_{t_5,t_2} \). Let \( n \in \mathbb{N} \). Notice that
\[
(x_m)'(t) = v(t) < 0 \quad \text{on} \quad [t_a, t_1].
\]

With \( v_m = -\max_{t_a \leq t \leq t_1} v(t) \) and \( x_m(0) = 0 \) we obtain
\[
|x_m(t)| \geq |t|v_m \quad \text{on} \quad [t_a, t_1].
\]

On \([t_1, t_5 + t_a] \) we have \( x_m(t) \leq -a \).

2. Let \( n \in \mathbb{N}, j \in \mathbb{N} \) and \( t \in [(n-1)t_5, nt_5], u \in [(j-1)t_5, jt_5] \) be given. Then
\[
t = (n-1)t_5 + t_* \quad \text{with} \quad 0 \leq t_* \leq t_5 \quad \text{and} \quad u = (j-1)t_5 + u_* \quad \text{with} \quad 0 \leq u_* \leq t_5.
\]

We may assume \( u_* \leq t_* \). Set \( w = u_* - t_* \in [-t_5, 0] \).

3. In case \( t_1 \leq -w \leq t_5 + t_a \) Part 1 yields the estimate
\[
|\hat{x}(t) - \hat{x}(u)|_{1,r} \geq |x_1(-u_* - u_*(-u_*))|
\]
\[
= |x((n-1)t_5 + t_* - u_* - x((j-1)t_5 + u_* - u_*))|
\]
\[
= |x((n-1)t_5 - w)| = |x_m(-w)| \geq a.
\]
4. In case \( \min \{ t_1, \frac{s}{2} \} \leq -w \leq t_1 \) Part 1 yields the estimate
\[
|\dot{x}(t) - \dot{x}(u)|_{1,r} \geq |x_t(-u_a) - x_u(-u_a)|
= |x((n - 1)t_5 - w)| = |x(n_t_5 - w)| \geq (-w)v_m \geq v_m \cdot \min \left\{ t_1, \frac{s}{2} \right\}.
\]

5. In case \( t_5 + t_a \leq -w \leq t_5 - \min \left\{ -t_a, \frac{s}{2} \right\} \) we have \( t_a \leq -w - t_5 \leq - \min \left\{ -t_a, \frac{s}{2} \right\} \).

Using Part 1 we infer
\[
|\dot{x}(t) - \dot{x}(u)|_{1,r} \geq |x_t(-u_a) - x_u(-u_a)|
= |x((n - 1)t_5 + t_a - t_5 - w)|\]
\[
\geq | -w - t_5 | v_m \geq v_m \cdot \min \left\{ -t_a, s \right\}.
\]

6. The case \( n - j \in 2\mathbb{Z} + 1 \), \( -w \leq s \), and \( t_3 \leq u_a \). Then \( t_2 + s - t_a \in [-t_5, 0] \subset [-r, 0] \) since
\[
-t_5 \leq -t_a \leq t_2 + s - t_a \leq t_3 - t_a \leq t_3 - u_a \leq 0.
\]

Using \( x_{(n)}(t) \leq -\frac{x}{2} \) for all \( m \in \mathbb{N} \) and all \( t \in [t_2, t_3] = [t_2, t_2 + 2s] \) we infer
\[
|\dot{x}(t) - \dot{x}(u)|_{1,r} \geq |x_t(t_2 + s - t_a) - x_u(t_2 + s - t_a)|
= |x((n - 1)t_5 + t_a + t_2 + s - t_a) -
\]
\[
x((j - 1)t_5 + u_a + t_2 + s - t_a)|
= |x((n - 1)t_5 + t_2 + s) - x((j - 1)t_5 + t_2 + s + w)|
= |(-1)^{n-1}x_{(n)}(t_2 + s) - (-1)^{j-1}x_{(j)}(t_2 + s + w)|
= |(-1)^{n-j}x_{(n)}(t_2 + s) - x_{(j)}(t_2 + s + w)|
\geq 2|x|.
\]

7. The case \( 0 \neq n - j \in 2\mathbb{Z}, -w \leq \frac{s}{2} \), and \( t_3 \leq u_a \). Corollary 3.2 yields some \( v \in [-s, s] \) so that \( w + v \in [-s, s] \) and
\[
|\left( \kappa_{j} \right)'(w + v) - \left( \kappa_{n} \right)'(v)| \geq \frac{a}{4}.
\]

We have \( t_2 + s + v - t_a \in [-t_5, 0] \subset [-r, 0] \) since
\[
-t_5 \leq -t_a \leq t_2 + s + v - t_a \leq t_3 - t_a \leq t_3 - u_a \leq 0.
\]

Hence
\[
|\dot{x}(t) - \dot{x}(u)|_{1,r} \geq |(x_t)'(t_2 + s + v - t_a) - (x_u)'(t_2 + s + v - t_a)|
= |x'((n - 1)t_5 + t_a + t_2 + s + v - t_a) - 
\]
\[
x'((j - 1)t_5 + u_a + t_2 + s + v - t_a)|
= |x'((n - 1)t_5 + t_2 + s + v) - x'((j - 1)t_5 + t_2 + s + w + v)|
= |(-1)^{n-1}(x_{(n)})'(t_2 + s + v) - (-1)^{j-1}(x_{(j)})'(t_2 + s + w + v)|
= |(-1)^{n-j}(x_{(n)})'(t_2 + s + v) - (x_{(j)})'(t_2 + s + w + v)|
= |(\kappa_{j})'(v) - (\kappa_{n})'(w + v)| \geq \frac{a}{4}.
\]

8. The case \( n - j \in 2\mathbb{Z} + 1 \), \( 2 \leq n, 2 \leq j \), \( -w \leq s \), and \( u_a < t_3 \). Then \( t_5 + t_a - t_2 - s \in [0, t_5 + 2s] \subset [0, r] \) since
\[
0 \leq t_5 - t_3 + u_a \leq t_5 - (t_2 + s) + t_a = t_5 + t_a - t_2 - s
\]
\[ \leq t_5 + (u_* + s) - t_2 - s \leq t_5 + t_3 - t_2 = t_5 + 2s \leq r. \]

Hence
\[
|\hat{x}(t) - \hat{x}(u)|_{1,r} \geq |x_t(-(t_5 - t_6 + t_2 + s) - x_u(-(t_5 - t_6 + t_2 + s)) - \\
\quad (n-2)x(x_{(n-1)}(t_2 + s) - (-1)^{j-2}x_{(j)}(w + t_2 + s))|
\]
\[
\geq |(\kappa_{j-1})'(w + v) - (\kappa_{n-1})'(v)| \geq \frac{a}{4}. \]

We have \( t_5 + t_6 - t_2 - s - v \in [0, t_5 + 3s] \subset [0, r] \) since
\[
0 \leq t_5 - t_3 + u_* \leq t_5 - (t_2 + 2s) + t_6 \leq t_5 + t_6 - t_2 - s - v \leq t_5 + \left( u_6 + \frac{s}{2} \right) - t_2 - s - v \leq t_5 + 2s - v \leq r.
\]

Hence
\[
|\hat{x}(t) - \hat{x}(u)|_{1,r} \geq |(x_t)'(-(t_5 - t_6 + t_2 + s + v)) - \\
\quad (x_u)'(-(t_5 - t_6 + t_2 + s + v)) - \\
\quad x'((n-2)x(x_{(n-1)}(t_2 + s) - (-1)^{j-2}x_{(j)}(w + t_2 + s))|
\]
\[
\geq |(\kappa_{j-1})'(v) - (\kappa_{n-1})'(w + v)| \geq \frac{a}{4}. \]

10. The case \( 0 \neq n - j \in 2\mathbb{Z}, 2 \leq j, t_5 - \min\{-t_0, s\} \leq -w = t_6 - u_* \leq t_5 \). Then
\[
u_* \leq t_6 - t_5 + s \leq s,
\]
and \( w_* = t_6 - u_* - t_5 \) satisfies \( w_* \in [-s, 0] \). We have \( t_5 + u_* - t_2 - s \in [0, t_5] \subset [0, r] \) since
\[
0 \leq t_5 - t_3 \leq t_5 - t_2 - s \leq t_5 + u_* - t_2 - s \leq t_5 + s - t_2 - s \leq t_5 \leq r.
\]

Hence
\[
|\hat{x}(t) - \hat{x}(u)|_{1,r} \geq |x_t(-(t_5 - u_* + t_2 + s) - x_u(-(t_5 - u_* + t_2 + s)) - \\
\quad (n-1)x(x_{(n)}(w_2 + t_2 + s) - (-1)^{j-2}x_{(j)}(w + t_2 + s))|
\]
\[
\geq |(\kappa_{j-1})'(w + v) - (\kappa_{n-1})'(w + v)| \geq 2\xi. \]
11. The case \( n - j \in 2\mathbb{Z} + 1, 2 \leq j, j - 1 \neq n, t_5 - \min \{-t_a, \frac{s}{2}\} \leq -w = t_\ast - u_\ast \leq t_5. \) Now

\[
 u_\ast \leq t_\ast - t_5 + \frac{s}{2} \leq \frac{s}{2},
\]

and \( w_\ast = t_\ast - u_\ast - t_5 \) belongs to \([-\frac{s}{2}, 0]\). Corollary 3.2 yields \( v \in [-s, s] \) so that \( w_\ast + v \in [-s, s] \) and

\[
|\xi - x| \leq \frac{a}{4}.
\]

We have \( t_5 + u_\ast - t_2 - s - v \in [0, t_5 + s] \subset [0, r] \) since

\[
0 \leq t_5 - t_3 = t_5 - t_2 - 2s \leq t_5 + u_\ast - t_2 - 2s \leq t_5 + \frac{s}{2} - t_2 - s - v
\]

\[
\leq t_5 - v \leq t_5 + s \leq r.
\]

Hence

\[
|\hat{x}(t) - \hat{x}(u)|_{1,r} \geq |(x_\ast'(-t_5 - u_\ast + t_2 + s + v) - (x_\ast'(-t_5 - u_\ast + t_2 + s + v))| - x'(n - 1)t_5 + t_\ast - t_5 - u_\ast + t_2 + s + v) - x'(j - 1)t_5 + u_\ast - t_5 - u_\ast + t_2 + s + v)\]

\[
= |(\xi - x)'(t_2 + s + w_\ast + v) - x'((j - 2))t_5 + t_2 + s + v)| - |(-1)^{n-1}(x'(n))'((j - 1))t_5 + t_2 + s + v)\]

\[
= |(-1)^{n-1}(x'(n))'(t_2 + s + w_\ast + v) - (-1)^{j-2}(x'(j-1))'(t_2 + s + v)| - |(-1)^{n-1+1}(x'(n))'(t_2 + s + w_\ast + v) - (x'(j-1))'(t_2 + s + v)|\]

\[
= |(\xi - x)'(t_2 + s + w_\ast + v) - (\xi - x)'(t_2 + s + v)| \geq \frac{a}{4}.
\]

12. Combining the results of Parts 3-11 and the relation \( \xi > a \) we arrive at the estimate

\[
|\hat{x}(t) - \hat{x}(u)|_{1,r} \geq \min \left\{ \frac{a}{4}, v_m \cdot \min \{t_1, s\}, v_m \cdot \min \{-t_a, s\} \right\}
\]

for all integers \( n \geq 2, j \geq 2 \) with \( |n - j| > 1 \) and all \( t \in [(n - 1)t_5, nt_5], u \in [(j - 1)t_5, jt_5]. \)

7 Delay Functionals on \( C^0_r \)-Neighbourhoods of Compact Arcs

For \( t > 0 \) define \( x_i' \in C^0_r \) by \( x_i'(u) = x'(t + u), -r \leq u \leq 0. \) Then

\[
x_i' = J(x') = JD\hat{x}(t).
\]

The curve

\[
\hat{x}' : (0, \infty) \ni t \mapsto x_i' \in C^0_r
\]

is continuously differentiable since the derivative \( x' : [-r, \infty) \to \mathbb{R} \) is continuously differentiable, compare [13, Proposition 4.1]. Consider the map

\[
L : (0, \infty) \times C^0_r \to \mathbb{R}
\]

given by

\[
L(t, \phi) = \phi(0)x'(t) + \phi(t_b)x'(t + t_b).
\]
We have
\[ L = m \circ ((ev_0 \circ \dot{x} \circ pr_1) \times (ev_0 \circ pr_2)) + m \circ ((ev_{tb} \circ \dot{x} \circ pr_1) \times (ev_{tb} \circ pr_2)) \]
with the projections
\[ pr_1 : (0, \infty) \times C^0_r \to \mathbb{R}, \quad pr_2 : (0, \infty) \times C^0_r \to C^0_r \]
onto the first and second component, respectively, with the continuous linear evaluation maps
\[ ev_0 : C^0_r \ni \phi \mapsto \phi(0) \in \mathbb{R}, \quad ev_{tb} : C^0_r \ni \phi \mapsto \phi(t_b) \in \mathbb{R}, \]
and with the multiplication \( m : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \). So \( L \) is continuously differentiable.

Each map \( L(t, \cdot) : C^0_r \to \mathbb{R}, t > 0 \), is linear. For the nullspace
\[ K_t = \{ \phi \in C^0_r : L(t, \phi) = 0 \} \]
of \( L(t, \cdot) \) we have
\[ x'_t \notin K_t \]
since
\[ L(t, \dot{x}(t)) = (x'(t))^2 + (x'(t + t_b))^2 > 0, \]
which follows from the fact that the zeros of \( x' \) in \([t_b, \infty)\) are given by \( \frac{1}{2}(t_2 + t_3) + jt_5, j \in \mathbb{N}_0 \). We infer
\[ C^0_r = \mathbb{R}x'_t \oplus K_t \quad \text{for all} \quad t > 0. \]

In the sequel we show that every compact arc \( J\dot{x}([u, v]) \subset C^0_r, r < u < v \), has a neighbourhood \( U \) in \( C^0_r \) on which the representation
\[ \phi = x_t + \kappa \quad \text{with} \quad \kappa \in K_t, \quad t \text{ close to } [u, v], \quad \text{and} \quad \kappa = \phi - x_t \text{ small in } C^0_r \]
is unique. Knowing this we shall define a delay functional \( d_U : C^0_r \supset U \to \mathbb{R} \) by
\[ d(\phi) = \Delta(x_t). \]
Then \( d \) is constant along each fibre \( (x_t + K_t) \cap U \), with \( t \) close to \([u, v]\).

Obviously,
\[ \phi - x_t \in K_t \iff L(t, \phi - x_t) = 0 \]
for all \( \phi \in C^0_r \) and all \( \sigma > 0 \).

**Proposition 7.1** [Local fibre representation] For every \( t > 0 \) there exist \( \delta \in (0, t), \epsilon \in (0, \delta] \), and a continuously differentiable map
\[ \tau : C^0_r \supset U_\epsilon(x_t) \to (t - \delta, t + \delta) \subset \mathbb{R} \]
with \( \tau(x_t) = t \) so that for every \((\sigma, \phi) \in (t - \delta, t + \delta) \times U_\epsilon(x_t)\),
\[ L(\sigma, \phi - x_\sigma) = 0 \iff \sigma = \tau(\phi). \]

For every \( \phi \in U_\epsilon(x_t) \) and for \( \sigma = \tau(\phi) \),
\[ |\phi - x_\sigma|_0, r \leq \left(1 + \sup_{t - \delta \leq u \leq t + \delta} |x'_u|_0, r\right) \delta. \]
Proof Let \( t > 0 \) be given. The map
\[
f : (0, \infty) \times C^0_r \ni (\sigma, \phi) \mapsto L(\sigma, \phi - J\hat{x}(\sigma)) \in \mathbb{R}
\]
is continuously differentiable and satisfies \( f(t, x_t) = 0 \). Using the formula defining the map \( L \) we infer
\[
D_1 f(t, \phi) 1 = \phi(0)x''(t) + \phi(t_b)x''(t + t_b) - ((x'(t))^2 + x(t)x''(t)) - ((x'(t + t_b))^2 + x(t + t_b)x''(t + t_b)),
\]
hence
\[
D_1 f(t, x_t) 1 = -(x'(t))^2 - (x'(t + t_b))^2 < 0.
\]
Apply the Implicit Function Theorem and obtain \( \delta \in (0, t), \epsilon > 0 \), and a continuously
differentiable map \( \tau \) with the properties stated in the first sentence of the proposition. Notice
that one can achieve \( \epsilon \leq \delta \). For \( \phi \in U^\epsilon(x_t) \) and \( \sigma = \tau(\phi) \) we get
\[
|\phi - x_\sigma|_{0, r} \leq |\phi - x_t|_{0, r} + |x_t - x_\sigma|_{0, r}
\]
\[
= |\phi - x_t|_{0, r} + |J\hat{x}(t) - J\hat{x}(\sigma)|_{0, r}
\]
\[
\leq \epsilon + \sup_{t - \delta \leq u \leq t + \delta} |DJ\hat{x}(u)|_{0, r} |t - \sigma|
\]
\[
= \epsilon + \sup_{t - \delta \leq u \leq t + \delta} |JD\hat{x}(u)|_{0, r} |t - \sigma|
\]
\[
= \epsilon + \sup_{t - \delta \leq u \leq t + \delta} |x'_u|_{0, r} |t - \sigma|
\]
\[
\leq (1 + \sup_{t - \delta \leq u \leq t + \delta} |x'_u|_{0, r}) \delta.
\]

Proposition 7.2 (Fibre representation along compact arcs) Let reals \( u < v \) in \((r, \infty)\) and \( n \in \mathbb{N} \) be given. There exist positive \( \rho = \rho(u, v, n) \leq \frac{1}{n} \) so that for every \( \phi \in U^\rho(J\hat{x}([u, v])) \) there is one and only one
\[
\sigma \in \left[u - \frac{1}{n}, v + \frac{1}{n}\right] \cap (0, \infty)
\]
such that
\[
L(\sigma, \phi - x_\sigma) = 0 \quad \text{and} \quad |\phi - x_\sigma|_{0, r} \leq \frac{1}{n}.
\]
In case \( \phi = x_t \) with \( t \in [u, v] \) we have \( \sigma = t \).

Proof 1. Let reals \( u < v \) in \((r, \infty)\) be given. As the curve \( J \circ \hat{x} \) is continuously differentiable
with \( DJ\hat{x}(w) 1 = x'_w \in C^0_r \) for all \( w > 0 \) we obtain
\[
|x_t - x_\sigma|_{0, r} \leq c|t - \sigma| \quad \text{for all} \quad t, \sigma \in \left[u - \frac{1}{n}, v + \frac{1}{n}\right]
\]
with
\[
c = \max_{\frac{1}{2} \leq w \leq v + 1} |x'_w|_{0, r}.
\]
2. Apply Proposition 7.1 to each \( w \in [u, v] \), and obtain \( \epsilon = \epsilon_w \) and \( \delta = \delta_w \) and \( \tau = \tau_w \) according to Proposition 7.1. Notice that one may assume
\[
u \leq w - \delta_w, \quad (1 + c)\delta_w \leq \frac{1}{n}.
\]
Using the compactness of $J \hat{x}([u, v]) \subset C^0_r$ one finds a strictly increasing finite sequence $(w_j)_j$ in $[u, v]$ so that the associated neighbourhoods $U_{\epsilon_{w_j}}(\hat{x}(w_j))$, $j \in \{1, \ldots, j\}$, form a covering of $J \hat{x}([u, v])$. There exists a positive real number

$$\rho = \rho(u, v, n) \leq \min_{j=1,\ldots,j} \epsilon_{w_j}$$

with

$$U_{\rho}(J \hat{x}([u, v]) \subset \bigcup_{j=1}^{\bar{j}} U_{\epsilon_{w_j}}(\hat{x}(w_j)).$$

Notice that

$$\rho \leq \min_{j=1,\ldots,j} \epsilon_{w_j} \leq \max_{j=1,\ldots,j} \delta_{w_j} \leq \frac{1}{n}.$$  

For every $\phi \in U_{\rho}(J \hat{x}([u, v])$ we obtain (at least one)

$$\sigma \in \bigcup_{j=1}^{\bar{j}} (w_j - \delta_{w_j}, w_j + \delta_{w_j})$$

$$\subset \left[ \max \left\{ \frac{u}{2}, u - \frac{1}{n} \right\}, v + \frac{1}{n} \right]$$

with

$$L(\sigma, \phi - x_\sigma) = 0 \quad \text{and} \quad |\phi - x_\sigma|_{0,r} \leq (1 + c) \max_{j=1,\ldots,j} \delta_{w_j} \leq \frac{1}{n}.$$  

Or, the set $R_n \subset (0, \infty)$ of all $\rho \in (0, \frac{1}{n}]$ such that for every $\phi \in U_{\rho}(J \hat{x}([u, v]))$ there exist $\sigma \in [u - \frac{1}{n}, v + \frac{1}{n}] \cap (0, \infty)$ with

$$L(\sigma, \phi - x_\sigma) = 0 \quad \text{and} \quad |\phi - x_\sigma|_{0,r} \leq \frac{1}{n}$$

is nonempty. Observe that

$$\rho_n = \frac{1}{2} \sup R_n$$

belongs to $R_n$.

3. Assume that the set $I$ of all $n \in \mathbb{N}$ such that $U_{\rho_n}(J \hat{x}([u, v]))$ contains $\phi$ with

$$2 \leq \# \left\{ \sigma \in [u - \frac{1}{n}, v + \frac{1}{n}] \cap (0, \infty) : L(\sigma, \phi - x_\sigma) = 0 \right. \quad \text{and} \quad \left. |\phi - x_\sigma|_{0,r} \leq \frac{1}{n} \right\}$$

is unbounded. We derive a contradiction. The elements of $I$ form a strictly increasing sequence $(n_k)_{k=1}^\infty$. For every $k \in \mathbb{N}$ select some $\phi_k$ in $U_{\rho_n}(J \hat{x}([u, v]))$ with $\rho = \rho_{n_k}$ and $\sigma_k^{(1)} < \sigma_k^{(2)}$ in $[u - \frac{1}{n_k}, v + \frac{1}{n_k}] \cap (0, \infty)$ with

$$L(\sigma_k^{(m)}, \phi_k - x_{\sigma_k^{(m)}}) = 0 \quad \text{and} \quad |\phi_k - x_{\sigma_k^{(m)}}|_{0,r} \leq \frac{1}{n_k} \quad \text{for} \ m \in \{1, 2\}.$$
Using the compactness of, say, \([0, v + 1]\), and successively choosing subsequences we find a strictly increasing sequence \((k_\kappa)_{\kappa}^{\infty}\) so that the equations

\[ z^{(m)}_\kappa = \sigma^{(m)}_{k_\kappa} \quad \text{for} \quad \kappa \in \mathbb{N} \quad \text{and} \quad m \in \{1, 2\} \]

define two sequences which converge to \(z^{(1)} \leq z^{(2)}\) in \([0, v + 1]\), respectively. Necessarily, \(u \leq z^{(1)} \leq z^{(2)} \leq v\). The continuity of \(J \circ \hat{x}\) yields \(x^{(m)}_{k_\kappa} \to x^{(m)}_\kappa\) in \(C_r^0\) as \(\kappa \to \infty\), for \(m \in \{1, 2\}\). Using the inequalities

\[ |\phi_k - x^{(m)}_{\kappa}|_{0,r} \leq \frac{1}{n_k} \quad \text{for} \quad m \in \{1, 2\} \quad \text{and} \quad k \in \mathbb{N} \]

we obtain \(\phi_{k_\kappa} \to x^{(1)}_\kappa = x^{(2)}_\kappa\) as \(\kappa \to \infty\). As \(\hat{x}\) is injective on \([r, \infty) \supset [u, v]\), \(z^{(1)} = z^{(2)}\).

Apply Proposition 7.1 to \(t = z^{(1)} = z^{(2)}\) and choose positive \(\epsilon \leq \delta\) according to this proposition. For \(\kappa \in \mathbb{N}\) sufficiently large we have

\[ \phi_{k_\kappa} \in U_\epsilon(x_t), \]

both \(z^{(1)}_\kappa < z^{(2)}_\kappa\) belong to \((t - \delta, t + \delta)\), and

\[ L(\sigma, \phi_{k_\kappa} - x_\sigma) = 0 \quad \text{for} \quad \sigma = z^{(1)}_\kappa \quad \text{and} \quad \sigma = z^{(2)}_\kappa. \]

This yields a contradiction to the first part of Proposition 7.1.

4. Combining the results of Parts 1 and 2 we obtain \(n(u, v) \in \mathbb{N}\) such that for every integer \(n \geq n(u, v)\) and for every \(\phi \in U_{\hat{r}_n}(J \hat{x}([u, v]))\) there exists one and only one \(\sigma \in [u - \frac{1}{n}, v + \frac{1}{n}] \cap (0, \infty)\) with \(L(\sigma, \phi - x_\sigma) = 0\) and \(|\phi - x_\sigma|_{0,r} \leq \frac{1}{n}\). Now the assertion of Proposition 7.2 follows easily.

Proposition 7.2 yields that for \(u < v\) in \((r, \infty)\) and \(n \in \mathbb{N}\) there exists \(\rho \leq \frac{1}{n}\) so that the relations

\[ \phi \in U_\rho(J \hat{x}([u, v])), \quad \sigma \in \left[ u - \frac{1}{n}, v + \frac{1}{n} \right] \cap (0, \infty), \]

\[ L(\sigma, \phi - x_\sigma) = 0, \quad |\phi - x_\sigma|_{0,r} \leq \frac{1}{n} \]

define a map

\[ s_{u,v,\rho} : C_r^0 \supset U_{\rho}(J \hat{x}([u, v])) \to (0, \infty) \]

with

\[ |\phi - x_{s_{u,v,\rho}((\phi))}|_{0,r} \leq \frac{1}{n} \quad \text{for all} \quad \phi \in U_{\rho}(J \hat{x}([u, v])). \]

**Proposition 7.3** Let reals \(u < v\) in \((r, \infty)\) and \(n \in \mathbb{N}\) be given and choose \(\rho = \rho(u, v, n)\) according to Proposition 7.2. There exist \(\eta = \eta(u, v, n) \in (0, \rho]\) so that the restriction \(s_{u,v,\eta}\) of \(s_{u,v,\rho}\) to \(U_{\eta}(J \hat{x}([u, v]))\) is continuously differentiable.

For every \(\phi \in U_{\eta}(J \hat{x}([u, v]))\) and for every \(\sigma \in [u - \frac{1}{n}, v + \frac{1}{n}] \cap (0, \infty)\),

\[ \sigma = s_{u,v,\eta}(\phi) \iff \left( L(\sigma, \phi - x_\sigma) = 0 \quad \text{and} \quad |\phi - x_\sigma|_{0,r} \leq \frac{1}{n} \right). \]

For every \(\sigma \in [u, v]\), \(s_{u,v,\eta}(x_\sigma) = \sigma\).
Proof For each $t \in [u, v]$ choose $\epsilon = \epsilon_t \leq \delta_t = \delta$ and $\tau = \tau_t$ according to Proposition 7.1. Observe that we may assume that $\delta_t$ satisfies

$$
\max\left\{0, u - \frac{1}{n}\right\} < t - \delta_t, \quad t + \delta_t < v + \frac{1}{n}
$$

and

$$
(1 + \sup_{t - \delta_t \leq w \leq t + \delta_t} |x'_w|_{0,r})^{\delta_t} < \frac{1}{n}.
$$

For every $\phi \in U_{\rho}(J\hat{x}([u, v])) \cap U_{\epsilon_t}(x_t)$ we have that

$$
\sigma = \tau_t(\phi) \in (t - \delta_t, t + \delta_t) \subset \left[ u - \frac{1}{n}, v + \frac{1}{n} \right] \cap (0, \infty)
$$

satisfies $L(\sigma, \phi - x_\sigma) = 0$ and

$$
|\phi - x_\sigma|_{0,r} \leq (1 + \sup_{t - \delta_t \leq w \leq t + \delta_t} |x'_w|_{0,r})^{\delta_t} < \frac{1}{n}.
$$

By the definition of $s_{u,v,\rho}$,

$$
s_{u,v,\rho}(\phi) = \sigma = \tau_t(\phi).
$$

It follows that the restriction of $s_{u,v,\rho}$ to $U_{\rho}(J\hat{x}([u, v])) \cap U_{\epsilon_t}(x_t)$ is continuously differentiable. There exists $\eta \in (0, \rho)$ with

$$
U_{\eta}(J\hat{x}([u, v])) \subset \bigcup_{u \leq t \leq v} U_{\rho}(J\hat{x}([u, v])) \cap U_{\epsilon_t}(x_t).
$$

The last statement in Proposition 7.3 is obvious from Proposition 7.2.

Using continuous differentiability of the delay function $\Delta$ we infer that the delay functional

$$
d_{u,v,\eta} = \Delta \circ s_{u,v,\eta}
$$

defined on the open neighbourhood $U_{\eta}(J\hat{x}([u, v]))$ of the arc $J\hat{x}([u, v])$ is continuously differentiable (with respect to the topology of $C_r^0$). For every $\sigma \in [u, v]$ we have $s_{u,v,\eta}(x_\sigma) = \sigma$, hence

$$
d_{u,v,\eta}(x_\sigma) = \Delta(s_{u,v,\eta}(x_\sigma)) = \Delta(\sigma).
$$

8 Compatibility on $C_r^0$-Neighbourhoods of Adjacent Arcs

Let $j = j_r \geq 2$ denote the smallest integer with $r < (j - 1)t_5$. For $j \leq k \in \mathbb{N}$ set

$$
X_k = \hat{x}([(k - 1)t_5, kt_5]) \subset C_r^0.
$$

In the sequel we construct open neighbourhoods $U_k$ of $JX_k$ in $C_r^0$ and continuously differentiable delay functionals $d_k : C_r^0 \supset U_k \to (0, r)$ with $d_k(x_t) = \Delta(t)$ for all $t \in [(k - 1)t_5, kt_5]$ so that for every integer $k \geq j$ we have

$$
d_k(\phi) = d_{k+1}(\phi) \quad \text{for all} \quad \phi \in U_k \cap U_{k+1}.
$$

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The construction is iterative. We carry out the initial step and the step thereafter. This second step is the model for the step from statements for general \( k \geq j \) to statements for \( k + 1 \).

1. The initial step for \( k = j \).

1.1. Apply Proposition 7.1 with \( t = j t_5 \) at \( \hat{x}(t) \), choose \( \delta = \delta(j) > 0, \epsilon = \epsilon(j) \in (0, \delta] \), and a map \( \tau = \tau_j \) from \( U_\epsilon(\hat{x}(t)) \subseteq C^0_t \) into \( (t - \delta, t + \delta) \) accordingly. By continuity there are \( n = n(j) \in \mathbb{N} \) with

\[
\hat{x}\left(\left[t - \frac{1}{n}, t + \frac{1}{n}\right]\right) \subseteq U_\epsilon(\hat{x}(t)) \quad \text{and} \quad r < (j - 1)t_5 - \frac{1}{n},
\]

and \( \epsilon_j \in (0, \epsilon(j)) \) with

\[
\tau(U_{\epsilon_j}(\hat{x}(t))) \subseteq \left[t - \frac{1}{n}, t + \frac{1}{n}\right].
\]

An application of Proposition 1.3 with \( a = (j - 1)t_5, b = (j + 1)t_5, t = jt_5 \) yields \( \rho = \rho(j) > 0 \) with

\[
U_\rho(JX_j) \cap U_\rho(JX_{j+1}) \subseteq U_{\epsilon_j}(\hat{x}(t));
\]

notice that \( X_j = \hat{x}([a, t]) \) and \( X_{j+1} = \hat{x}([t, b]) \).

1.2. We apply Proposition 7.3 twice, first with \( u = (j - 1)t_5, v = jt_5, \) and \( n = n(j) \). This yields \( \eta > 0 \) and a continuously differentiable map

\[
s_{u,v,\eta} : U_\eta(JX_j) \to \left[u - \frac{1}{n}, u + \frac{1}{n}\right] \subseteq \mathbb{R}
\]

so that for every \( \phi \in U_\eta(JX_j) \) we have

\[
\left(\sigma \in \left[u - \frac{1}{n}, u + \frac{1}{n}\right] \text{ and } L(\sigma, \phi - x_\sigma) = 0\right) \iff \sigma = s_{u,v,\eta}(\phi).
\]

Also, \( s_{u,v,\eta}(x_w) = w \) for all \( w \in [u, v] \). We may assume

\[
\eta < \rho = \rho(j).
\]

Set

\[
U_j = U_\eta(JX_j) \quad \text{and} \quad s_j = s_{u,v,\eta}.
\]

The map

\[
d_j : U_j \ni \phi \mapsto \Delta(s_j(\phi)) \in (0, r)
\]

is continuously differentiable with \( d_j(x_w) = \Delta(w) \) for all \( w \in [(j - 1)t_5, jt_5] \).

The second application of Proposition 7.3, with \( \hat{u} = (j + 1 - 1)t_5 = jt_5, \hat{v} = (j + 1)t_5, \) and \( n = n(j) \) yields \( \hat{\eta} > 0 \) and a continuously differentiable map \( s_{\hat{u},\hat{v},\hat{\eta}} : U_{\hat{\eta}}(JX_{j+1}) \to \left[\hat{u} - \frac{1}{n}, \hat{v} + \frac{1}{n}\right] \subseteq \mathbb{R} \) such that for every \( \phi \in U_{\hat{\eta}}(JX_{j+1}) \) we have

\[
\left(\sigma \in \left[\hat{u} - \frac{1}{n}, \hat{v} + \frac{1}{n}\right] \text{ and } L(\sigma, \phi - x_\sigma) = 0\right) \iff \sigma = s_{\hat{u},\hat{v},\hat{\eta}}(\phi).
\]

Also, \( s_{\hat{u},\hat{v},\hat{\eta}}(x_w) = w \) for all \( w \in [\hat{u}, \hat{v}] \). We may assume

\[
\hat{\eta} < \rho = \rho(j).
\]

Set

\[
\hat{U}_{j+1} = U_{\hat{\eta}}(JX_{j+1}) \quad \text{and} \quad \hat{s}_{j+1} = s_{\hat{u},\hat{v},\hat{\eta}}.
\]
1.3. Let \( \phi \in U_j \cap \hat{U}_{j+1} \). Proof of \( s_j(\phi) = \hat{s}_{j+1}(\phi) \).

We have \( \phi \in U_{\epsilon_j}(\hat{x}(t)) \), due to Part 1.1 and to \( \max\{\eta, \hat{\eta}\} \leq \rho(j) \). Hence

\[
\tau(\phi) \in \left[ t - \frac{1}{n}, t + \frac{1}{n} \right].
\]

Notice that \( t = v = \hat{u} \), and thereby

\[
\left[ t - \frac{1}{n}, t + \frac{1}{n} \right] \subseteq \left[ u - \frac{1}{n}, v + \frac{1}{n} \right] \cap \left[ \hat{u} - \frac{1}{n}, \hat{v} + \frac{1}{n} \right].
\]

For \( \sigma = \tau(\phi) \) we have \( L(\sigma, \phi - x_\sigma) = 0 \), see Proposition 7.1. Now the properties of \( s_j \) and of \( s_{\hat{u}, \hat{v}, \hat{\eta}} \) from Part 1.2 yield

\[
s_j(\phi) = \sigma = s_{\hat{u}, \hat{v}, \hat{\eta}}(\phi) = \hat{s}_{j+1}(\phi).
\]

2. The second step, which includes the definitions of \( U_{j+1} \subset \hat{U}_{j+1} \), of \( s_{j+1} \), and of \( d_{j+1} \), and contains the proof of \( d_j(\phi) = d_{j+1}(\phi) \) on \( U_j \cap U_{j+1} \).

2.1. Apply Proposition 7.1, now at \( \hat{x}(t) \) with \( t = (j + 1)t_5 \), and choose \( \delta = \delta(j + 1) > 0 \), \( \epsilon = \epsilon(j + 1) \in (0, \delta) \), and a map \( \tau = \tau_{j+1} \) from \( U_\epsilon(\hat{x}(t)) \) into \( (t - \delta, t + \delta) \) accordingly. By continuity there is an integer \( n = n(j + 1) \geq n(j) \) with

\[
J\hat{x} \left[ \left[ t - \frac{1}{n}, t + \frac{1}{n} \right] \right] \subseteq U_\epsilon(\hat{x}(t)) \quad \text{and} \quad r < ((j + 1) - 1)t_5 - \frac{1}{n},
\]

and there exists \( \epsilon_{j+1} \in (0, \epsilon(j + 1)) \) with

\[
\tau(U_{\epsilon_{j+1}}(\hat{x}(t))) \subseteq \left[ t - \frac{1}{n}, t + \frac{1}{n} \right].
\]

An application of Proposition 1.3 with \( a = ((j + 1) - 1)t_5 = jt_5, b = ((j + 1) + 1)t_5 = (j + 2)t_5, t = (j + 1)t_5 \) yields \( \rho = \rho(j + 1) > 0 \) with

\[
U_\rho(JX_{j+1}) \cap U_\rho(JX_{j+2}) \subseteq U_{\epsilon_{j+1}}(\hat{x}(t));
\]

notice that \( X_{j+1} = \hat{x}([a, b]) \) and \( X_{j+2} = \hat{x}([t, b]) \).

2.2. First we restrict \( \hat{s}_{j+1} \) from Part 1.2. As \( \hat{s}_{j+1} \) maps \( JX_{j+1} \) onto \( [(jt_5, (j + 1)t_5] \) continuity yields \( \hat{\eta} \in (0, \rho(j + 1)) \) such that

\[
U_{j+1} = U_{\hat{\eta}}(JX_{j+1})
\]

is contained in \( \hat{U}_{j+1} \) and

\[
\hat{s}_{j+1}(U_{j+1}) \subseteq \left[ jt_5 - \frac{1}{n}, (j + 1)t_5 + \frac{1}{n} \right],
\]

with \( n = n(j + 1) \). Set \( s_{j+1} = \hat{s}_{j+1}|_{U_{j+1}} \). Part 1.3 gives

\[
s_{j+1}(\phi) = s_j(\phi) \quad \text{for all} \quad \phi \in U_{j+1} \cap U_j,
\]

and it follows that the continuously differentiable map

\[
d_{j+1} : U_{j+1} \ni \phi \mapsto \Delta(s_{j+1}(\phi)) \in (0, r)
\]

satisfies \( d_{j+1}(\phi) = \Delta(s_{j+1}(\phi)) = \Delta(s_j(\phi)) = d_j(\phi) \) for all \( \phi \in U_{j+1} \cap U_j \). Also, \( d_{j+1}(x_w) = \Delta(s_{j+1}(x_w)) = \Delta(w) \) for all \( w \in [jt_5, (j + 1)t_5] \).
Next we apply Proposition 7.3, with \( \tilde{u} = (j + 2) - 1 \) and \( \tilde{v} = (j + 1)t_5 \), and \( n = n(j + 1) \). This yields \( \tilde{\eta} > 0 \) and a continuously differentiable map \( s_{\tilde{u}, \tilde{v}, \tilde{\eta}} : U_{\tilde{\eta}}(JX_{j+2}) \rightarrow [\tilde{u} - \frac{1}{n}, \tilde{v} + \frac{1}{n}] \subset \mathbb{R} \) such that for every \( \phi \in U_{\tilde{\eta}}(JX_{j+2}) \) we have

\[
(\sigma \in [\tilde{u} - \frac{1}{n}, \tilde{v} + \frac{1}{n}] \text{ and } L(\sigma, \phi - x_\sigma) = 0) \iff \sigma = s_{\tilde{u}, \tilde{v}, \tilde{\eta}}(\phi).
\]

Also, \( s_{\tilde{u}, \tilde{v}, \tilde{\eta}}(x, w) = w \) for all \( w \in [\tilde{u}, \tilde{v}] \). Again we may assume

\[ \tilde{\eta} < \rho = \rho(j + 1). \]

Set

\[ \hat{U}_{j+2} = U_{\tilde{\eta}}(JX_{j+2}) \quad \text{and} \quad \hat{s}_{j+2} = s_{\tilde{u}, \tilde{v}, \tilde{\eta}}. \]

2.3. Proof of \( s_{j+1}(\phi) = \hat{s}_{j+2}(\phi) \) for all \( \phi \in U_{j+1} \cap \hat{U}_{j+2} \). Such \( \phi \) belong to \( U_{e_{j+1}}(\hat{x}((j + 1)t_5)) \), due to Part 2.1 and to the inequality \( \max{\tilde{\eta}, \eta} \leq \rho(j + 1) \). Hence \( \sigma = \tau(\phi) \) is contained in \([t - \frac{1}{n}, t + \frac{1}{n}]\), for \( n = n(j + 1) \). Notice that \( t = (j + 1)t_5 = \tilde{u} \), and thereby

\[
\begin{align*}
[t - \frac{1}{n}, t + \frac{1}{n}] & \subset [jt_5 - \frac{1}{n}, (j + 1)t_5 + \frac{1}{n}] \cap \left[ \tilde{u} - \frac{1}{n}, \tilde{v} + \frac{1}{n} \right] \\
& \subset [jt_5 - \frac{1}{n(j)}, (j + 1)t_5 + \frac{1}{n(j)}] \cap \left[ \tilde{u} - \frac{1}{n(j + 1)}, \tilde{v} + \frac{1}{n(j + 1)} \right].
\end{align*}
\]

We also have \( L(\sigma, \phi - x_\sigma) = 0 \), see Proposition 7.1. Now the properties of \( \hat{s}_{j+1} \) from Part 1.2 and of \( \hat{s}_{j+2} = s_{\tilde{u}, \tilde{v}, \tilde{\eta}} \) from Part 2.2 yield

\[ \hat{s}_{j+1}(\phi) = \sigma = \hat{s}_{j+2}(\phi), \]

which is \( s_{j+1}(\phi) = \hat{s}_{j+2}(\phi) \).

This ends the second step.

9 A Functional on a \( C^1 \)-Neighbourhood of the Trace \( \hat{x}((j_r - 1)t_5, \infty)) \)

In this section the constructions from Sects. 2–8 are used to prove Theorem 1.1. Let an integer \( k \geq j_r \) be given. On the open set of all reals \( t > 0 \) with \( J\hat{x}(t) \in U_k \) we have that the map given by \( t \mapsto d_k(J\hat{x}(t)) \) is continuously differentiable, with the derivatives given by

\[ Dd_k(Jx_t)J D\hat{x}(t)1 = Dd_k(Jx_t)x'_t \in \mathbb{R}. \]

On \([(k - 1)t_5, kt_5]\) we have \( \Delta(t) = d_k(J\hat{x}(t)) \). It follows that on this interval,

\[ 1 > \Delta'(t) = Dd_k(Jx_t)x'_t. \]

Recall the constant \( \hat{a} \) from Proposition 6.1. The subset

\[ N_k = \{ \phi \in C^1_r \cap J^{-1}(U_k) : Dd_k(J\phi) \phi' < 1 \quad \text{and there exists} \}

\[ t \in [(k - 1)t_5, kt_5] \quad \text{with} \quad |\phi - x_t|_{1, r} < \frac{\hat{a}}{2} \}

of the space \( C^1_r \) is open. Proposition 6.1 yields \( N_k \cap N_m = \emptyset \) for all integers \( k \geq j_r \) and \( m \geq j_r \) with \(|k - m| > 1\). Also, \( N_k \cap N_{k+1} \subset J^{-1}(U_k) \cap J^{-1}(U_{k+1}) \) for \( j_r \leq k \in \mathbb{N}. \) Using
the relations (8.1) we obtain that on the open set
\[
N = \bigcup_{k \geq j_r} N_k \supset \hat{x}([(j_r - 1)t_5, \infty))
\]
the equations
\[
d(\phi) = d_k(J\phi) \quad \text{for} \quad \phi \in N_k \quad \text{and} \quad j_r \leq k \in \mathbb{N}
\]
define a map \(d : C^1_r \supset N \rightarrow (0, \tau)\). It follows that
\[
d(x_t) = \frac{\Delta(t)}{\Delta_1(t)} \quad \text{for all} \quad t \geq (j_r - 1)t_5
\]
since for such \(t\) there exists \(k \geq j_r\) with \(t \in [(k - 1)t_5, kt_5]\), hence \(x_t \in N_k\), and thereby \(d(x_t) = d_k(Jx_t) = d_k(x_t) = \Delta(t)\), see Sect. 8.

Proposition 1.2 applies and yields that the functional \(d\) is continuously differentiable and has property (e).

**Proposition 9.1** The functional
\[
f : C^1_r \supset N \ni \phi \mapsto -\phi(-d(\phi)) \in \mathbb{R}
\]
is continuously differentiable and has the extension property (e).

This is analogous to [14, Proposition 11.1]. We include the proof for convenience.

**Proof** We have
\[
f(\phi) = -e_{v^1_r}(\phi, -d(\phi)) = -(e_{v^1_r} \circ (i \times (-d)))(\phi) \quad \text{for all} \quad \phi \in N,
\]
which shows that \(f\) is continuously differentiable. Recall \(D_1e_{v^1_r}(\phi, t)\hat{\phi} = \hat{\phi}(t)\) and \(D_2e_{v^1_r}(\phi, t)\hat{t} = t\hat{\phi}'(t)\). The chain rule yields
\[
Df(\phi)\hat{\phi} = -\hat{\phi}(-d(\phi)) - \phi'(-d(\phi))[-Dd(\phi)\hat{\phi}] = \phi'(-d(\phi))Dd(\phi)\hat{\phi} - \hat{\phi}(-d(\phi)).
\]
For \(\phi \in N\) the equation
\[
D_{\hat{\phi}}f(\phi)\hat{\chi} = \phi'(-d(\phi))D_{\hat{\phi}}d(\phi)\hat{\chi} - \chi(-d(\phi)).
\]
defines a linear extension \(D_{\hat{\phi}}f(\phi) : C^0_r \rightarrow \mathbb{R}\) of the derivative \(Df(\phi) : C^1_r \rightarrow \mathbb{R}\). Using the continuity of the evaluation map \(C^0_r \times [-r, 0] \ni (\chi, t) \mapsto \chi(t) \in \mathbb{R}\) and property (e) of \(d\) one finds that the map \(N \times C^0_r \ni (\phi, \chi) \mapsto D_{\hat{\phi}}f(\phi)\hat{\chi} \in \mathbb{R}\) is continuous.

For \(t \geq j_r t_5\) we have \(x_t \in N\) and, due to Eq. (9.1),
\[
x'(t) = -x(t - \Delta(t)) = -x(t - d(x_t)) = f(x_t).
\]
This implies that the twice continuously differentiable function
\[
x^{(d)} : [-r, \infty) \ni t \mapsto x(t + j_r t_5) \in \mathbb{R}
\]
is a solution of the equation
\[
y'(t) = f(y_t)
\]
with the flowline \([0, \infty) \ni t \mapsto x^{(d)}_t \in C^1_r\) in the solution manifold
\[
X_f = \{ \phi \in N : \phi'(0) = f(\phi) \}.
\]
Recall the non-empty set $A \subset C^2_h$ chosen in Sect. 4 as a special case of the sets from Example 2.2. The set $A$ is open in the affine space $p_* + C^2_{h-0}$ of codimension 6 in $C^2_{h-0}$. Recall the choice of $x$ on $[t_d - h, t_d] \subset [0, t_5]$ in Sect. 4. The short segments $x^{(d)}_{t_d+(n-1)t_5,short} \in C^2_h$, $n \in \mathbb{N}$, are dense in $A \cup (-A)$.

Finally we show that for each $\phi \in X_f$ the delayed argument function

$$[0, t_{\phi}) \ni t \mapsto t - d(x^\phi_t) \in \mathbb{R}$$

is strictly increasing. Let $\phi \in X_f$ and $t \in (0, t_{\phi})$ be given and set $y = x^\phi$. As $y : [-r, t_{\phi}) \to \mathbb{R}$ is continuously differentiable the curve $\tilde{y} : [0, t_{\phi}) \ni t \mapsto Jy_t \in C^0$ is continuously differentiable with $D\tilde{y}(u)1 = y_u'$ for all $u > 0$, compare [13, Proposition 4.1]. The segment $y_t \in X_f \subset N$ is contained in $N_k$ for some integer $k \geq j_r$. By continuity of the flowline $[0, t_{\phi}) \ni u \mapsto y_u \in X_f \subset N \subset C^1$, there is $\epsilon > 0$ with $y_u \in N_k$ for all $u \in (t - \epsilon, t + \epsilon)$. Then $d(y_u) = d_k(Jy_u) = d_k(\tilde{y}(u))$ on $(t - \epsilon, t + \epsilon)$. It follows that the curve

$$(t - \epsilon, t + \epsilon) \ni u \mapsto d(y_u) \in \mathbb{R}$$

is differentiable with derivatives given by $Dd_k(Jy_u)y_u' < 1$. This implies that on $(0, t_{\phi})$ the delayed argument function is differentiable with positive derivative, from which the assertion follows.

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