VIRTUAL CLASSES OF $\mathbb{G}_m$-GERBES

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ABSTRACT. We show that a perfect obstruction theory of a $\mathbb{G}_m$-gerbe determines a semi-perfect obstruction theory of its base, which is perfect if the gerbe is quasi-compact and affine-pointed. This allows us to relate virtual classes of the gerbe and its base.

1. INTRODUCTION

We discuss virtual classes of $\mathbb{G}_m$-gerbes with perfect obstruction theories. These gerbes appear naturally in the Donaldson-Thomas theory of smooth projective 3-folds as moduli stacks of perfect complexes which have no higher automorphisms and are simple, and their perfect obstruction theories come from derived enhancements.

Consider a $\mathbb{G}_m$-gerbe $\mathcal{G}$ with a perfect obstruction theory over a DM stack $B$. The main observation is that we can truncate its perfect obstruction theory from $[-1, 1]$ to $[-1, 0]$, then decompose it into moving and fixed parts. The moving part is given by a locally free sheaf $H$ in degree $-1$ and the fixed part determines a semi-perfect obstruction theory of $B$. The virtual class of $\mathcal{G}$ is obtained by pulling back the virtual class of $B$ then cap with the Euler class of the vector bundle associated to $H$. When $B$ is quasi-compact and affine-pointed, the semi-perfect obstruction of $B$ is actually a perfect obstruction theory. See Section 3 for details. We conclude the paper with a remark on virtual classes in DT theory of 3-folds.

2. PRELIMINARIES

We work over the field of complex numbers $\mathbb{C}$.

2.1. Notation. For an algebraic stack $X$, $\text{Qcoh}(X)$ denotes the abelian category of quasi-coherent sheaves on $X$, $D(\text{Qcoh}(X))$ its derived category, $D_{\text{qcoh}}(X)$ the derived category of $\mathcal{O}_X$-modules with quasi-coherent cohomology sheaves, and $D_{\text{coh}}(X)$ the derived category of $\mathcal{O}_X$-modules with coherent cohomology sheaves. For derived categories, superscripts are used to further specify the range of cohomology sheaves. The truncation functor for complexes is denoted by $\tau$, with a superscript to indicate the range of a truncation. A complex is perfect in $[a, b]$ if it is (smooth) locally quasi-isomorphic to a complex of locally free sheaves of finite rank in degrees $[a, b]$. The derived pullback of derived category objects along a map $f$ is also denoted by $f^*$. The superscript $\vee$ denotes taking dual.

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2.2. Quasi-coherent sheaves on $\mathbb{G}_m$-Gerbes and their derived categories. Let $B$ be a DM stack locally of finite type over $\mathbb{C}$, and $p: \mathcal{G} \to B$ a $\mathbb{G}_m$-gerbe over $B$. Any quasi-coherent sheaf $F$ on $\mathcal{G}$ has a decomposition $F = \bigoplus_{i \in \mathbb{Z}} F_i$ where $F_i$ has weight $i$. (See e.g., [7, Proposition 2.2.1.6].)

Remark 2.1. On the trivial gerbe $U \times B\mathbb{G}_m$, a quasi-coherent sheaf with weight $i$ is of the form $F \boxtimes \mathcal{O}(i)$, where $F$ is a quasi-coherent sheaf on $U$ and $\mathcal{O}(i)$ the line bundle on $B\mathbb{G}_m$ induced by the character of $\mathbb{G}_m$ with weight $i$.

Following [7], we call $F_0$ the fixed part of $F$ and $\bigoplus_{i \neq 0} F_i$ the moving part, denoted by $F^f$ and $F^m$ respectively.

We have a decomposition of abelian categories

$$\text{QCoh}(\mathcal{G}) \simeq \text{QCoh}(\mathcal{G})^f \times \text{QCoh}(\mathcal{G})^m,$$

where QCoh$(\mathcal{G})^f$ (resp. QCoh$(\mathcal{G})^m$) is the full subcategory generated by quasi-coherent sheaves with only fixed (resp. moving) parts. The pushforward $p_*$ induces an equivalence

$$\text{QCoh}(\mathcal{G})^f \simeq \text{QCoh}(B)$$

with inverse $p^*$.

For an algebraic stack $X$, the inclusion map $\text{QCoh}(X) \to \text{Mod}(\mathcal{O}_X)$ of quasi-coherent sheaves on $X$ into the category of $\mathcal{O}_X$-modules induces a map between derived categories. Recall $X$ is affine pointed if for every morphisms $\text{Spec } k \to X$ from a field $k$ to $X$ is affine. For instance, if $X$ has affine diagonal, then it is affine-pointed.

Proposition 2.2 ([8, Theorem C.1]). Let $X$ be an algebraic stack. If $X$ is either quasi-compact with affine diagonal or noetherian and affine-pointed, then the natural map between derived categories $D^+(\text{QCoh}(X)) \to D^+_{\text{qcoh}}(X)$ is an equivalence.

2.3. Perfect obstruction theory (POT).

Definition 2.3 ([4, 13, 21]). Let $f : X \to Y$ be a map between algebraic stacks locally of finite type over $\mathbb{C}$, an obstruction theory for $f$ is a map $\phi : E^\bullet \to L_f$ in $D_{\text{qcoh}}^{\leq 1}(X)$ such that $h^0(\phi), h^1(\phi)$ are isomorphisms and $h^{-1}(\phi)$ surjective in $\text{QCoh}(X)$. Here $L_f = \tau^{-1}_X \mathbb{L}_f$ is the truncated cotangent complex, with $\mathbb{L}_f \in D_{\text{coh}}^{\leq 1}(X)$ being the cotangent complex of $f$.

The obstruction theory $\phi$ is perfect if $E^\bullet$ is perfect in $[-1, 1]$. The obstruction sheaf $\text{ob}_f$ is defined as $h^1(E^\bullet)$.

Remark 2.4. For a summary of $\mathbb{L}_f$, see e.g., [2, 2.4]. POTs defined using $\mathbb{L}_f$ in place of $L_f$ can be truncate to $[-1, 1]$ and give rise to POTs defined above. We use $L_f$ in this paper so that we can replace $D^{\leq 1}_{\text{qcoh}}(X)$ by $D^{\leq 1}(\text{QCoh}(X))$ when $D^b(\text{QCoh}(X)) \simeq D^b_{\text{qcoh}}(X)$.

When the map $f$ is DM, $L_f \in D_{\text{coh}}^{[0,1]}(X)$, and a POT for $f$ induces a closed embedding of the intrinsic normal sheaf $\mathcal{N}_f = h^1/h^0(L_f)$ into the vector bundle stack $\mathcal{E}_f = h^1/h^0(E^\bullet)$ ([4, Theorem 4.5] [21, Theorem 2.3]).

\footnote{Here $L_f$ also denote its extension to the big fppf site, see [4, Section 2].}
Remark 2.5. When \( f \) is not DM, similar to \([4]\), \( L_f \) determines a Picard 2-stack \( \mathcal{N}_f \), and a POT \( \phi \) induces a closed embedding of \( \mathcal{N}_f \) into the vector bundle 2-stack associated to \( E^* \). To define a virtual class, we only need a closed embedding \( \pi_0(\mathcal{N}_f) \to \pi_0(\mathcal{E}_f) \) between their coarse sheaves (See e.g., [3] Section 2), this observation goes back to [15] and is recasted into semi-perfect obstruction theory. In particular, only the truncation of \( \phi \) to \([-1, 0]\) should matter, and this is the approach of [18] [21].

\textbf{Definition 2.6 ([6], [13]).} Let \( X \) be a DM stack over a pure dimensional base \( S \), a semi-perfect obstruction theory for \( X \) over \( S \) consists of a collection of étale locally defined POTs \( (U_i, \phi_i) \), where \( \{U_i\} \) is an étale cover of \( X \) and \( \phi_i : E^*_i \to L_{U_i/S} \) is a POT for \( U_i \), and they satisfy the following conditions (1) and (2).

1. The local obstruction sheaves \( \text{ob}_i = h^1(E^*_i) \) are isomorphic over \( U_{ij} \) and descends to an obstruction sheaf \( \text{ob} \) on \( X \).
2. The restrictions of \( \phi_i \) and \( \phi_j \) to \( U_{ij} \) give the same obstruction assignment under the identification of local obstruction sheaves in (1).

A semi-perfect obstruction theory is symmetric if its local POTs \( \phi_i \) are symmetric ([3], [5]) and the induced isomorphisms \( \text{ob}_i \simeq \Omega_{U_i} \) descend to \( \text{ob} \simeq \Omega_X \).

Condition (2) is equivalent to the following by the proof of [6, Proposition 2.1]. For any closed point \( x : \text{Spec} \ C \to X \), there exists a well-defined map
\[
h^1((x^*L_X)^\vee) \to x^*\text{ob}
\]
obtained using a factorization of \( x \) as the composition of \( y : \text{Spec} \ C \to U_i \) and \( U_i \to X \). For any choice of \( y \), the POT \( \phi_i \) induces a map
\[
h^1((y^*\phi_i)^\vee) : h^1((y^*L_{U_i})^\vee) \to h^1((y^*E^*_i)^\vee) \simeq y^*\text{ob}_i,
\]
which can be identified as a map
\[
h^1((x^*L_X)^\vee) \to x^*\text{ob}.
\]

For any étale map \( U \to X \), the pullback of a POT \( \phi \) for \( X \) to \( U \) determines a POT for \( U \). In this way, a (symmetric) POT \( \phi \) for \( X/S \) induces a (symmetric) semi-perfect obstruction theory.

\textbf{Remark 2.7.} It is clear from [6] that (1) and (2) imply that
\[
\pi_0(\mathcal{N}^{\text{red}}_i) \to \pi_0(\mathcal{E}_i) \simeq \text{ob}^{[3]}
\]
descends to
\[
\pi_0(\mathcal{N}^{\text{red}}_{X/S}) \to \text{ob}.
\]
Here \( \mathcal{N}^{\text{red}}_i \) denote the reduced stack associated to the intrinsic normal sheaf \( \mathcal{N}_i \) of \( U_i/S \), \( \mathcal{E}_i \) the vector bundle stack associated with \( E^*_i \), and \( \pi_0(\mathcal{N}^{\text{red}}_i) \to \pi_0(\mathcal{E}_i) \) is the map between coarse sheaves of the composition of the closed embedding \( \mathcal{N}^{\text{red}}_i \to \mathcal{N}_i \) and the embedding \( \mathcal{N}_i \to \mathcal{E}_i \) induced by \( \phi_i \). The map \( \pi_0(\mathcal{N}^{\text{red}}_{X/S}) \to \text{ob} \) is used to construct the virtual class.

\textbf{Remark 2.8.} If a semi-perfect obstruction theory is obtain from a POT on \( X \) then the vector bundle stacks \( \mathcal{E}_i = h^1/h^0(E^*_i) \) descends to \( X \), which would require an isomorphism \( a_{ij} : \mathcal{E}_i|_{U_{ij}} \simeq \mathcal{E}_j|_{U_{ij}} \) on each \( U_{ij} \), and a two arrow \( b_{ijk} \) between \( \phi_{jk}|_{U_{ijk}} \) and \( a_{ij}|_{U_{ijk}} \circ a_{jk}|_{U_{ijk}} \) on each \( U_{ijk} \), and compatibilities between \( \{b_{ijk}\} \) on each

\footnotetext{2}{Here \( \text{ob}_i \) denote its extension to the big étale site of \( U_i \) as in [3].}
In terms of complexes, $b_{ijk}$ correspond to chain homotopies, which are invisible in the derived category.

3. VIRTUAL CLASS OF $\mathcal{G}_M$ GERBES

In this section, we prove the results sketched in the introduction.

Let $p: \mathcal{G} \to B$ be a $\mathcal{G}_m$-gerbe over $B$ as in the last section, and $\phi: E^\bullet \to \mathcal{L}_G$ a POT of $\mathcal{G}$.

3.1. Virtual class of $\mathcal{G}$. For the distinguished triangle between cotangent complexes $p^*\mathbb{L}_B \to \mathbb{L}_G \to \mathbb{L}_p$, we have $p^*\mathbb{L}_B \simeq \tau_{\leq 0}(\mathbb{L}_G)$ and $h^1(\mathbb{L}_G) = h^1(\mathbb{L}_p)$, since $\mathbb{L}_B \in D_{\text{qcoh}}(B)$ and $\mathbb{L}_p[1] \simeq h^1(\mathbb{L}_p)$ is locally free. Therefore we have a distinguished triangle between truncated cotangent complexes

$$p^*\mathbb{L}_B \to \mathbb{L}_G \to \mathbb{L}_p.$$ 

We first truncate $E^\bullet$ to $[-1, 0]$. There exists a map between distinguished triangles

$$\begin{array}{ccc}
F^\bullet & \to & E^\bullet \\
\downarrow \phi & & \downarrow \text{Id} \\
p^*\mathbb{L}_B & \to & \mathbb{L}_G \\
\end{array}$$

where $E^\bullet \to \mathbb{L}_p$ is the composition of $\phi$ and $\mathbb{L}_G \to \mathbb{L}_p$. Denote $\psi$ the first vertical map $F^\bullet \to p^*\mathbb{L}_B$, then it can be identified with $\tau_{\leq 0}(\phi)$. Note that $h^0(\psi)$ is an isomorphism, $h^{-1}(\psi)$ is surjective, and $F^\bullet$ is perfect in $[-1, 0]$.

Since $p$ is flat, $h^{-1}(p^*\mathbb{L}_B) = p^*h^{-1}(\mathbb{L}_B)$ has no moving part. We can remove the moving part of $h^{-1}(F^\bullet)$ from $\psi$ as follows. The map

$$h^{-1}(F^\bullet)[1] \simeq \tau_{\leq -1}F^\bullet \to F^\bullet$$

and the inclusion map $h^{-1}(F^\bullet)^m \to h^{-1}(F^\bullet)$ induces

$$h^{-1}(F^\bullet)^m[1] \to h^{-1}(F^\bullet)[1] \to F^\bullet,$$

denote $F^\bullet f$ the cone of this map. As $h^{-1}(F^\bullet)^m$ maps to 0 in $h^{-1}(p^*\mathbb{L}_B)$, we have an induced map

$$\psi^f: F^\bullet f \to p^*\mathbb{L}_B,$$

and $h^0(\psi^f)$ is an isomorphism, $h^{-1}(\psi^f)$ is surjective.

Lemma 3.1. The sheaf $h^{-1}(F^\bullet)^m$ is locally free.

Proof. Locally, we can assume $F^\bullet$ is given by a two term complex of locally free sheaves in $[-1, 0]$, then its moving part is also a two term complex of locally free sheaves and has vanishing $h^0$, so $h^{-1}(F^\bullet)^m$ is the kernel of a surjective map between locally frees, hence locally free.

As $h^{-1}(F^\bullet)^m$ is locally free, $F^\bullet f$ is perfect, and $\psi^f$ induces a closed imbedding

$$\iota: p^*\mathcal{N}_B \to \mathcal{F}^f$$

where $\mathcal{N}_B = h^1/h^0(\mathcal{L}_B^\vee)$ is the intrinsic normal sheaf of $B$ and $\mathcal{F}^f = h^1/h^0(F^\bullet f^\vee)$ the vector stack associated with $F^\bullet f$.

Let $C_B \subset \mathcal{N}_B$ be the intrinsic normal cone of $B$, then we can view $p^*C_B$ as a closed substack of $\mathcal{F}^f$ via $\iota$.  

Lemma 3.2. The virtual class $[G]^{\text{vir}}$ determined by $\phi$ is given by

$$0!^\phi(p^*C_B) \cap e(V(h^{-1}(F^*)^m)),$$

here $0! : A_*(\mathcal{F}) \to A_*(G)$ denotes Gysin pullback along the zero section of $\mathcal{F}$, $V(h^{-1}(F^*)^m)$ the vector bundle $\text{Spec} \text{Sym} h^{-1}(F^*)^m$, and $e$ the Euler class.

Proof. This follows directly from the definition of virtual classes in [21]. In this case, the virtual class of $\mathcal{G}$ is obtained by pulling back $[p^*C_B]$ along the zero section of $\mathcal{F}$, which is isomorphic to $\mathcal{F}^\sharp \times_G V(h^{-1}(F^*)^m)$. □

Remark 3.3. As $G$ has affine stabilizers, it is stratified by global quotients by Proposition 3.5.9 and $0! : A_*(\mathcal{F}) \to A_*(G)$ is defined.

3.2. Semi-perfect obstruction theory of $B$. For any étale map $U \to B$ with (smooth) section $s : U \to G$, note that $s^*p^*L_B \simeq L_U$ and $s^*\psi^f$ is a POT for $U$, denote it $\psi_U$.

Proposition 3.4. $\{\psi_U\}$ determines a semi-perfect obstruction theory $\psi_B$ for $B$.

Proof. We verify conditions (1) and (2) for semi-perfect obstruction theories.

The local obstruction sheaf on $U$ is obtained by pullback along $s$ of $h^1(F^*\psi^f)$. As smooth locally $F^*\psi^f$ is given by a complex of locally free sheaves, and its cohomology sheaves have no moving parts, we see that $h^1(F^*\psi^f)$ has no moving parts. Then (1) is satisfied with $\text{ob} = p^*h^1(F^*\psi^f)$.

Condition (2) is satisfied because there is a unique section over any closed point of $B$.

Remark 3.5. If there exists a nondegenerate symmetric bilinear pairing $F^* \simeq F^*\psi^f[1]$, then $\psi^f = \psi$ and $\psi_B$ is symmetric.

Theorem 3.6. Assume $B$ is proper, let $[B]^{\text{vir}}$ be the virtual class determined by $\psi_B$, then

$$[G]^{\text{vir}} = p^*[B]^{\text{vir}} \cap e(V(h^{-1}(F^*)^m)).$$

Proof. We show that

$$p^*[B]^{\text{vir}} = 0!^\phi(p^*C_B),$$

and the theorem follows from Lemma 3.2.

As $\psi_B$ is a semi-perfect obstruction theory, it determines a map

$$\pi_0(N^\text{red}_B) \to \text{ob}.$$

By the construction of $\psi_B$, its pullback along $p$ can be identified with the map

$$\pi_0(p^*N^\text{red}_B) \to \pi_0(\mathcal{F}) \simeq p^*\text{ob}$$

induced by $\iota$, as the two maps match when pulled back to any $U$ étale over $B$. Then it is easy to see

$$p^*[B]^{\text{vir}} = 0!^\phi(p^*C_B)$$

from the construction of $[B]^{\text{vir}}$ in [6], since the flat pullback $p^*$ commutes with the operations used in defining $[B]^{\text{vir}}$. □

Remark 3.7. A cosection $\text{ob}_0 \to O_G$ induces a cosection of $\psi_B$, and the theorem also holds for localized virtual cycles ([12] [11]).

Theorem 3.8. When $B$ is quasi-compact and affine pointed, the semi-perfect obstruction theory $\psi_B$ is a POT.
Proof. If $B$ is quasi-compact and affine pointed, so is $G$.

As now $D^+(\text{QCoh}(\mathcal{G})) \simeq D^+_{\text{qcoh}}(\mathcal{G})$, we view $\psi$ as a map in $D^+(\text{QCoh}(\mathcal{G}))$. Under the equivalence $D^+(\text{QCoh}(\mathcal{G})) \simeq D^+(\text{QCoh}(\mathcal{G}^\dagger)) \times D^+(\text{QCoh}(\mathcal{G}^m))$, the component in $D^+(\text{QCoh}(\mathcal{G}^\dagger))$ of $\psi$ is $\psi^\dagger$, and $\psi_B$ is obtained from $\psi^\dagger$ under the equivalences $D^+(\text{QCoh}(\mathcal{G}^\dagger)) \simeq D^+(\text{QCoh}(B)) \simeq D^+_{\text{qcoh}}(B)$.

Remark 3.9. For gerbes banded by the cyclic group $\mu_r$ with POTs, their POTs have no $h^1$ and we can also decompose them into moving and fixed part with weights in $\mathbb{Z}/r\mathbb{Z}$, then all results above hold.

3.3. Let $X$ be a smooth projective 3-fold, the moduli stack of perfect complexes which are simple and without higher automorphisms,\footnote{Negative self Ext groups are zero.} is a $\mathbb{G}_m$-gerbe locally of finite type (\cite{16} Corollary 4.3.3), and has a derived enhancement (\cite{24} Definition 5.1), which has a $(1)$-symplectic structure (\cite{19}) when $X$ is Calabi-Yau. These derived enhancements induce obstruction theories (\cite{24} Proposition 1.2)). Alternatively, obstruction theories can be constructed as in \cite{6} using the deformation-obstruction results in \cite{9}. When truncated obstruction theories are perfect, results in this section apply, and hopefully complement the perspectives in existing literatures, e.g., \cite{6 20 22}.

Remark 3.10. There are two choices to fix the determinant of perfect complexes. Denote $\text{Perf}(X)$ the stack of perfect complexes on $X$ (\cite{24,23}), $\mathcal{G}$ the open sub-stack of $\text{Perf}(X)$ whose objects are simple and without higher automorphisms, $\text{Pic}(X)$ the Picard stack of $X$, $[\text{Pic}(X)/B\mathbb{G}_m]$ the Picard scheme of $X$, and let $L$ be a line bundle on $X$. Consider the cartesian diagrams

$$
\begin{array}{ccc}
\mathcal{G}_L & \longrightarrow & \text{Spec} \mathbb{C} \\
\downarrow & & \downarrow \\
\mathcal{G}'_L & \longrightarrow & B\mathbb{G}_m \\
\downarrow & & \downarrow \\
\mathcal{G} & \longrightarrow & \text{Pic}(X) \\
\end{array}
$$

where $\mathcal{G}_L$ and $\mathcal{G}'_L$ denote fiber products, the vertical arrows from $\text{Spec} \mathbb{C}$ are determined by $L$, $\mathcal{G} \to \text{Pic}(X)$ induced by the perfect determinant morphism (\cite{24} Definition 3.1).

Note that $\mathcal{G}_L$ is a gerbe banded by cyclic groups and $\mathcal{G}'_L$ a $\mathbb{G}_m$-gerbe. It is not hard to check that the map $\mathcal{G}_L \to \mathcal{G}'_L$ induces an isomorphism after rigidification.

The obstruction theory for $\mathcal{G} \to \text{Pic}(X)$ comes from the tangent complex of $\mathbb{R}\text{Perf}(X) \to \mathbb{R}\text{Pic}(X)$ (\cite{24} Proposition 3.2), which is only perfect in $[0,2]$. By base change, we obtain obstruction theories for $\mathcal{G}_L \to \text{Spec} \mathbb{C}$ and $\mathcal{G}'_L \to B\mathbb{G}_m$. From the obstruction theory of $\mathcal{G}'_L \to B\mathbb{G}_m$, we obtain an obstruction theory of $\mathcal{G}'_L$ (\cite{17} Appendix B). As the tangent complex of $B\mathbb{G}_m$ is perfect in degree $-1$, the obstruction theory on $\mathcal{G}_L$ is perfect if and only if the obstruction theory on $\mathcal{G}'_L$ is, and in that case, the induced semi-perfect obstruction theories on their rigidifications are identical. In fact, the truncation of $\phi$ to $\psi$ in the beginning of the section reverses the process of obtaining the POT of $\mathcal{G}'_L$ from $\mathcal{G}'_L \to B\mathbb{G}_m$.

\footnote{See \cite{2} C.3 for viewing rigidification as taking quotient.}
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