Remarks on Existence of Proper Action for Reducible Gauge Theories

IGOR A. BATALIN\textsuperscript{a,b} and KLAUS BERING\textsuperscript{a,c}

\textsuperscript{a}The Niels Bohr Institute
The Niels Bohr International Academy
Blegdamsvej 17
DK–2100 Copenhagen
Denmark

\textsuperscript{b}I.E. Tamm Theory Division
P.N. Lebedev Physics Institute
Russian Academy of Sciences
53 Leninsky Prospect
Moscow 119991
Russia

\textsuperscript{c}Institute for Theoretical Physics & Astrophysics
Masaryk University
Kotlářská 2
CZ–611 37 Brno
Czech Republic

December 3, 2009

Abstract

In the field–antifield formalism, we review existence and uniqueness proofs for the proper action in the reducible case. We give two new existence proofs based on two resolution degrees called “reduced antifield number” and “shifted antifield number”, respectively. In particular, we show that for every choice of gauge generators and their higher stage counterparts, there exists a proper action that implements them at the quadratic order in the auxiliary variables.

PACS number(s): 11.10.-z; 11.10.Ef; 11.15.-q; 11.15.Bt.
Keywords: BV Field–Antifield Formalism; Open Reducible Lagrangian Gauge Theory; Koszul–Tate Complex.

\textsuperscript{b}E–mail: batalin@lpi.ru \quad \textsuperscript{c}E–mail: bering@physics.muni.cz
1 Introduction

This paper considers existence and uniqueness theorems for a proper action $S$ of a general open reducible gauge theory in the field–antifield formalism [1, 2]. We should stress that the term uniqueness theorem here should be understood in a generalized sense, i.e., as a theorem that specifies the natural arbitrariness/ambiguity of solutions (see Theorem 3.5). In our case, the uniqueness theorem states that all proper solutions to the classical master equation $(S, S) = 0$ can locally be reached from any other proper solution via a finite anticanonical transformation.

It is always possible to locally close or Abelianize the gauge algebra of a physical system by rotating and by adding off–shell contributions to the gauge generators [3, 4]. Moreover, closed and Abelian theories have well–known proper actions, so why do we need an other existence proof? The answer is that although closure and Abelianization are great theoretical tools, they have only limited practical use in field theory, where they often destroy space–time locality. Thus, ideally, one would like to establish the existence of the proper action $S$ without tampering in any way with the original gauge algebra, and its higher–stage counterparts. This raises the question: How many terms in the action can one preserve while constructing the proper action? Specifically, we shall prove that the most general gauge generators $R^{i\alpha}_{\alpha_0}$ and higher–stage counterparts $Z^{\alpha_{s-1}\alpha_s}$ locally fit in unaltered form into a proper solution

$$ S = S^{\text{quad}} + \mathcal{O} \left( (\Phi^*)^2, c^2 \right) , \quad S^{\text{quad}} := S_0 + \varphi_i^* R^{i\alpha_0} c^{\alpha_0} + \sum_{s=1}^{L} c^{\star}_{\alpha_{s-1}} Z^{\alpha_{s-1}\alpha_s} c^{\alpha_s} , \quad (1.1) $$

to the classical master equation, see Theorem 4.6. In other words, the Existence Theorem 4.6 states that any action $S^{\text{quad}}$, which is quadratic in auxiliary variables, and which satisfies maximal rank conditions and Noether identities, can locally be completed into a proper action $S = S^{\text{quad}} + \mathcal{O} \left( (\Phi^*)^2, c^2 \right)$.

With this said, we must admit, that our current treatment will still make use of space–time non–local items, at least behind the screen. In particular, we will use the existence of a set of transversal and longitudinal fields $\varphi_i \equiv \{\xi^I; \theta^{\Lambda_0}\}$, which are often non–local, see Section 4.1. Also we should say that we will for simplicity use DeWitt’s condensed index notation, where space–time locality is suppressed. So we work, strictly speaking, only with a finite number $2N$ of variables $\Gamma^A \equiv \{\Phi^\alpha; \Phi^{*\alpha}\}$.

There is of course an analogous story for the Hamiltonian/canonical formalism, which we omit for brevity. Also we do not discuss quantum corrections in this paper. Even today, there exists only a relatively limited number of general results in manifestly space–time local field–antifield formalism [5, 6]. See Ref. [7] and Ref. [8] for a treatment of Yang–Mills type theories.

The standard existence proof is bases on two key elements. Firstly, the use of antifield number

$$ \text{afn}(\Phi^\alpha) = 0 \quad \text{afn}(\Phi^{*\alpha}) = -\text{gh}(\Phi^{*\alpha}) , \quad (1.2) $$
as resolution degree see Table 1. (The antifield number is sometimes called antighost number, because it is just the negative part of the ghost number.) Secondly, the use of a nilpotent acyclic Koszul–Tate operator $s_{-1}$ of antifield number minus one,

$$ s_{-1} = V_0 \frac{\partial}{\partial \Phi^{*\alpha}} , \quad \text{afn}(s_{-1}) = -1 , \quad (1.3) $$

$$ s_{-1} \varphi_i^{*} = V_i = (S_0 \frac{\partial}{\partial \varphi_i}) , \quad (1.4) $$
Historically, the existence and uniqueness theorems for an arbitrary irreducible gauge theory in an arbitrary basis were established in Ref. [9] and Ref. [10]. In the reducible case, an $Sp(2)$ covariant proof (which at one point uses rotations of gauge generators $R^i_{\alpha_0}$ and higher–stage counterparts $Z^{a_{s-1}}_{\alpha_s}$) was given in Ref. [11]. A proof that does not change the gauge generators $R^i_{\alpha_0}$ and higher–stage counterparts $Z^{a_{s-1}}_{\alpha_s}$ was given in Ref. [12]. The heart of the proof consists in showing the existence of a nilpotent and acyclic Koszul–Tate operator. This was done in Ref. [12] by referring to the antisymmetric first–stage structure function $B_{\alpha_1}^{ij}$. It should be stressed that the nilpotency of the Koszul–Tate operator is a non-trivial statement. It is equivalent to a $L$-stage tower of higher–stage Noether identities (2.1)–(2.3), and their consistency relations [15]. The first consistency relation appears at stage 2. The first few consistency relations read

\begin{align}
    s_1 c^x_{\alpha_0} &= V_{\alpha_0} = \varphi_i^* R^i_{\alpha_0}, \quad (1.5) \\
    s_1 c^x_{\alpha_1} &= V_{\alpha_1} = c^x_{\alpha_0} Z^{\alpha_0}_{\alpha_1} + \frac{(-1)^{\alpha_1}}{2} \rho^j_{\alpha_1} \varphi_i^* B^{ij}_{\alpha_1}, \quad B^{ij}_{\alpha_1} = -(-1)^{\epsilon_i \epsilon_j} B^{ij}_{\alpha_1}, \quad (1.6) \\
    s_1 c^x_{\alpha_2} &= V_{\alpha_2} = c^x_{\alpha_1} Z^{\alpha_1}_{\alpha_2} - (-1)^{\alpha_1 \alpha_0} c^x_{\alpha_0} \varphi_i^* B^{i0}_{\alpha_2} + O \left( \Phi^{m^2} \right), \quad (1.7) \\
    s_1 c^x_{\alpha_3} &= V_{\alpha_3} = c^x_{\alpha_2} Z^{\alpha_2}_{\alpha_3} - (-1)^{\alpha_2 \alpha_1} c^x_{\alpha_1} \varphi_i^* B^{i0}_{\alpha_3} + \frac{1}{2} c^x_{\alpha_0} c^x_{\alpha_0} B^{i0}_{\alpha_3} + O \left( \Phi^{m^2} \right), \quad (1.8) \\
    B^{0}_{\alpha_4} &= (-1)^{\epsilon_0 \epsilon_0} B^{0}_{\alpha_4}, \quad (1.9) \\
    s_1 c^x_{\alpha_4} &= V_{\alpha_4} = c^x_{\alpha_3} Z^{\alpha_3}_{\alpha_4} - (-1)^{\alpha_3 \alpha_2} c^x_{\alpha_2} \varphi_i^* B^{i0}_{\alpha_4} + c^x_{\alpha_2} c^x_{\alpha_0} B^{i0}_{\alpha_4} + O \left( \Phi^{m^2} \right), \quad (1.10) \\
    &\vdots
\end{align}

where “$\approx$” means equality modulo equations of motion for $\varphi_i$. The number of consistency relations grows with the reducibility stage. It is natural to wonder if these consistency relations can be satisfied without changing the original gauge generators $R^i_{\alpha_0}$ and their higher–stage counterparts $Z^{a_{s-1}}_{\alpha_s}$ [16, 17]? This turns out to be possible, see Theorem 3 in Ref. [13] (or Theorem 10.2 in Ref. [14], or Lemma A.1 in this paper). The proof uses the acyclicity property of previous stages to prove the existence of a nilpotent extension of the Koszul–Tate operator up to a certain stage without changing $R^i_{\alpha_0}$ and $Z^{a_{s-1}}_{\alpha_s}$.

Besides the gauge generators $R^i_{\alpha_0}$ and $Z^{a_{s-1}}_{\alpha_s}$, $s \in \{1, \ldots, L\}$, it turns out as an added bonus, that the antisymmetric first–stage structure function $B^{ij}_{\alpha_1}$ in the first–stage Noether identity (2.2) can also be preserved. (This is because there is no consistency relations at first stage.)

On the other hand, the proof does reveal that it will in general be necessary to change the given $B^{ij}_{\alpha_{s-2}}$.
structure functions in the higher–stage Noether identities (2.3) as

\[ B_{\alpha s}^{j_{\alpha s} - 2} \rightarrow B_{\alpha s}^{j_{\alpha s} - 2} + R^i_{\alpha 0} X_{\alpha s}^{\alpha_0 \alpha s - 2} + (S_0 \frac{\partial^r}{\partial \phi^j}) Y_{\alpha s}^{j_{\alpha s} - 2}, \quad Y_{\alpha s}^{j_{\alpha s} - 2} = -(-1)^{\varepsilon I} Y_{\alpha s}^{j_{\alpha s} - 2}. \] (1.15)

The higher–stage Noether identities (2.3) are unaffected by such a change (1.15), due to the zero–stage Noether identity (2.1).

So what remains is to prove the acyclicity. Unfortunately, the treatments in Ref. [13] and Ref. [14] of acyclicity at higher stages are very brief. One of the main purposes to introduce reduced and shifted antifield number, is to properly spell out, in great detail and in a systematic way, an acyclicity proof for all stages.

It is very simple to motivate the construction of shifted antifield number “safn”. Consider what happens if one raises the resolution degree of all the antifields \( \Phi^*_\alpha \) by 1 unit. Then the \( R^i_{\alpha 0} \) and the \( Z_{\alpha s - 1} \) terms, which are linear in the antifields, will have their resolution degree raised by 1, while the \( B_{\alpha s}^{i_{\alpha s} - 2} \) terms and higher terms, which are at least quadratic in the antifields, will have their resolution degree raised by at least 2, and hence they become subleading, and can be dropped from the new “shifted” Koszul–Tate operator \( s_{(-1)} \). That is the good news! The bad news is that the very first term

\[ V_i \frac{\partial^r}{\partial \phi^i} \equiv (S_0 \frac{\partial^r}{\partial \phi^i}) \frac{\partial^r}{\partial \phi^*} \] (1.16)

in the Koszul–Tate operator now has resolution degree \(-2\), which would be devastating. But there is a remedy. The term (1.16) is proportional to the original equations of motion. The remainder of our constructions is concerned with somehow assigning shifted antifield number \( \geq 1 \) to the original equations of motion, so that the resolution degree of the term (1.16) becomes \( \geq -1 \). In practice, we implement this by assigning shifted antifield number, safn(\( \xi^I \)) = 1, to a set of so–called transversal fields \( \xi^I \), see Section 4.1.

Another idea is to democratically assign resolution degree 1 to all antifields \( \Phi^*_\alpha \). We call this degree for reduced antifield number. Then the terms

\[ \varphi^*_i R^i_{\alpha 0} \frac{\partial}{\partial \varphi^*_0} + \sum_{s=1}^{L} c^s_{\alpha_{s-1}} Z_{\alpha s - 1} \frac{\partial}{\partial c^s_{\alpha s}} \] (1.17)

will have resolution degree 0. Thus one would like the degree 0 sector to become the leading resolution sector. Again the \( B_{\alpha s}^{i_{\alpha s} - 2} \) terms and higher terms, which are at least quadratic in the antifields, will become subleading in resolution degree, and can be dropped from the new “reduced” Koszul–Tate operator \( s_{-1[0]} \). However, the very first term (1.16) now has resolution degree \(-1\), which would be devastating. Again we cure this by assigning reduced antifield number, rafn(\( \xi^I \)) = 1, to a set of transversal fields \( \xi^I \), see Section 4.1.

The paper is organized as follows. In Section 2 we review the starting point of an arbitrary gauge theory, and introduce the field–antifield formalism. In Section 3 we consider an existence and uniqueness proof via the standard methods of antifield number and Koszul–Tate operator. And finally, in Section 4 we consider a complete existence proof via the new methods of reduced (shifted) antifield number, and reduced (shifted) Koszul–Tate operator, respectively. Note that the proofs in Section 3 are incomplete in the sense that one needs the new technology of reduced or shifted antifield number (which is developed in Section 4) to prove the acyclicity of the Koszul–Tate operator.
In an appendix B, we show for completeness how the $B^{i\alpha_{s-2}}_a$ terms can be removed by (space–time non–local) change of the gauge generators $R^{i\alpha_0}_a$ and $Z^{\alpha_{s-1}}_a$.

**General Remarks about Notation:** An integer subindex without parenthesis refers to ordinary antifield number, while an integer subindex in round or square parenthesis refers to shifted or reduced antifield number, respectively. For instance, $s_{-1}$ denotes the ordinary Koszul–Tate operator, which carries antifield number $\text{afn}(s_{-1})=-1$, while $s_{(-1)}$ is the shifted Koszul–Tate operator, which on the other hand carries shifted antifield number $\text{afn}(s_{(-1)})=-1$. Strong equality “=” and weak equality “≈” refer to off–shell and on–shell equality with respect to the equations of motion for the original fields $\varphi^i$, respectively.

## 2 Gauge Theories

### 2.1 Starting Point

In its purest form, the starting point for quantization is just an action $S_0 = S_0(\varphi)$, which depends on a set of fields $\varphi^i, i \in \{1, \ldots, n \equiv m_{-1}\}$. We shall hereafter refer to $S_0$ and $\varphi^i$ as the original action and the original fields, respectively.

Under some mild regularity assumptions, it is possible to recast the starting point into a form that we will use in this paper. (For a collection of starting points in the irreducible case, see Postulates 2–2′ in Ref. [10].) Explicitly, we will assume that all the gauge–symmetries, and in the reducible case, all the gauge–symmetries, $s \in \{1, \ldots, L\}$, have been properly identified. This means, in terms of formulas, that there should be given gauge–generators $R^{i\alpha_0}_a \equiv Z^{\alpha_{s-1}}_a$, and gauge(–for–gauge)$^s$–generators $Z^{\alpha_{s-1}}_a, s \in \{1, \ldots, L\}$, such that

$$
(S_0 \frac{\partial}{\partial \varphi^i}) R^{i\alpha_0}_a = 0 ,
$$

$$(2.1)$$

$$
R^{i\alpha_0}_a Z^{\alpha_0_1} = (S_0 \frac{\partial}{\partial \varphi^j}) B^{j\alpha_1}_a ,
$$

$$(2.2)$$

$$
Z^{\alpha_{s-2}}_{a_{s-1}} Z^{\alpha_{s-1}}_a = (S_0 \frac{\partial}{\partial \varphi^i}) B^{i\alpha_s}_{a_s} ,
$$

$$(2.3)$$

where the indices runs over the following sets

$$
i \equiv \alpha_{-1} \in \{1, \ldots, n \equiv m_{-1}\} , \quad \alpha_0 \in \{1, \ldots, m_0\} , \quad \ldots , \quad \alpha_L \in \{1, \ldots, m_L\} .
$$

$$(2.4)$$

The letter $L$ denotes the number of reducibility stages of the theory. A theory with reducibility stage $L=0$ equal to zero is by definition an irreducible theory. For each stage $s \in \{0, \ldots, L\}$, the number of gauge(–for–gauge)$^s$–symmetries is denoted $m_s$. It is convenient to define multiplicities $m_s := 0$ for $s > L$. (Or, the other way around, $L = \min\{s | \forall r > s : m_r = 0\}$.) The rank conditions read

$$
0 \leq \text{rank}(\frac{\partial^2}{\partial \varphi^i^2} S_0 \frac{\partial}{\partial \varphi^j}) = M_{-1} , \quad 0 \leq \text{rank}(R^{i\alpha_0}_a) = M_0 ,
$$

$$(2.5)$$

$$
0 \leq \text{rank}(Z^{\alpha_{s-1}}_a) = M_s , \quad s \in \{0, \ldots, L\} ,
$$

$$(2.6)$$

near the stationary $\varphi$-surface, i.e., the $\varphi$-surface of extremals for $S_0$. Here we have defined

$$
M_s := \sum_{r=0}^{\infty} (-1)^r m_{r+s} = m_s - M_{s+1} \leq m_s .
$$

$$(2.7)$$
Remark B

antifields “Φ∗

It is tempting to call the original fields ϕ strictly positive ghost number. In particular, the symbol so. From now on, the letter “c” only the antifields for the ghost “ϕ∗

one may show that there locally always exists a i ↔ j skewsymmetric, since it arises from an action term \( \frac{1}{2} (-1)^{i+j} \varphi^i \varphi^j B_{α_1}^{ij} c^{α_1} \), cf. Section 2.2. We shall therefore for simplicity assume from now on, that the \( B_{α_1}^{ij} \) structure function, that appears in the first–stage Noether identity (2.2), is a i ↔ j skewsymmetric tensor, that is defined in at least some tubular ϕ-neighborhood of the stationary ϕ-surface.

The choices of gauge(–for–gauge)s–generators \( Z^{α,s−1}_{α_s} \), \( s \in \{0, \ldots, L\} \), are not unique. An arbitrary other choice (with the same multiplicities) is locally given as

\[
\begin{align*}
\mathcal{R}^i_{β_0} &= R^i_{α_0} (Λ^{-1})^{α_0 β_0} + \left( S_0 \frac{∂}{∂ ϕ^i} \right) K^{ij}_{β_0}, & K^{ij}_{β_0} &= -(-1)^{ε_j} β_i K^{ji}_{β_0}, \\
\mathcal{Z}^{β,s−1}_{α_s} &\approx Λ^{β,s−1}_{α_s−1} Z^{α,s−1}_{α_s} (Λ^{-1})^{α_s β_s}, & s \in \{1, \ldots, L\}.
\end{align*}
\]

To summarize, we shall assume that some gauge(–for–gauge)s–generators, \( Z^{α,s−1}_{α_s} \), \( s \in \{0, \ldots, L\} \), have been given to us (perhaps outside the field–antifield formalism), and they are hereafter considered as a part of the starting point.

2.2 Field–Antifield Formulation

Let us now reformulate the problem in the field–antifield language [1, 2]. We shall for simplicity only consider the minimal sector. (The non–minimal sector, which is needed for gauge–fixing, can be treated by similar methods.) The minimal content of fields \( Φ^α \), \( α \in \{1, \ldots, N\} \), for a gauge theory of reducible stage \( L \), is

\[
Φ^α = \{ ϕ^i ≡ Φ^{α−1}; c^{α_0} ≡ Φ^{α_0}; \ldots; c^{α_L} ≡ Φ^{α_L}\}, \quad α \in \{1, \ldots, N\}.
\]

The \( c^{α_0} \) fields are the ghosts, or in a systematical terminology, the stage–zero ghosts. The \( c^{α_s} \) fields are the (ghost–for) s–ghosts, or stage-s ghost, \( s \in \{1, \ldots, L\} \). For Grassmann parity and ghost number assignments, see Table 1. To simplify notation, the stage \( s \) of a ghost \( c^{α_s} \) can only be identified though its index–variable \( α_s \). (Hopefully, this slight misuse of notation does not lead to confusion.)

It is tempting to call the original fields \( ϕ^i \) for stage–minus–one ghosts \( c^{α−1} ≡ ϕ^i \), but we shall not do so. From now on, the letter “c” without indices, and the word “ghost”, will always refer to a ghost with strictly positive ghost number. In particular, the symbol \( c^s \) will not refer to original antifields \( ϕ^* \), but only the antifields for the ghost “c”, i.e., \( gh(c^s) ≤ −2 \). We will collectively refer to ghosts “c” and antifields “Φ∗” as auxiliary variables. Auxiliary variables are characterized by non–zero ghost number.

Remark: Mathematically, the various structure functions, such as, e.g., \( R^i_{α_0} = R^i_{α_0} (ϕ) \) and \( B^{ij}_{α_1} = B^{ij}_{α_1} (ϕ) \), are tensors, or sections of appropriate vector bundles over the ϕ-basemanifold. They should
be defined in at least some tubular $\varphi$-neighborhood of the stationary $\varphi$-surface. The auxiliary variables can be thought of as a local basis for the corresponding vector bundle. For instance, the antifield

$$\varphi_i^* = \left(\frac{\partial}{\partial \varphi^j}\varphi_i^j\right) \varphi_j^* = \varphi_j^* \left(\frac{\partial}{\partial \varphi^j}\right),$$

transforms as a co–vector under general coordinate transformations $\varphi^i \to \varphi'^i$, and can thus be identified with a local basis $\partial_i$ for vector fields $X = X^i\partial_i$. On the other hand, the indices $\alpha_s, s \in \{0, \ldots, L\}$, do only transform under rigid $\varphi$-independent transformations, and the ghost $c^\alpha_s$ and their antifields $c^* \alpha_s$ can often be taken as global coordinates.

The total number $N$ of fields is

$$N := \sum_{s=-1}^L m_s = \sum_{s=-1}^{\infty} m_s.$$  

For each field $\Phi^\alpha$, one introduces an antifield $\Phi^* \alpha$ of opposite Grassmann number $\varepsilon(\Phi^* \alpha) = \varepsilon(\Phi^\alpha) + 1$ and of ghost number $\text{gh}(\Phi^* \alpha) = -\text{gh}(\Phi^\alpha) - 1$. The antibracket is defined as

$$(f, g) := f \left(\frac{\partial^s}{\partial \Phi^\alpha \partial \Phi^* \alpha} - \frac{\partial^s}{\partial \Phi^* \alpha \partial \Phi^\alpha}\right) g.$$  

The problem that we address in this paper is the existence of a proper solution $S$ with ghost number zero

$$\text{gh}(S) = 0.$$  

| Variable/Operator          | Symbol | Multiplicity | Grassmann parity | Ghost number | Antifield number | Shifted antifield number | Reduced antifield number |
|----------------------------|--------|--------------|------------------|--------------|------------------|--------------------------|--------------------------|
| Generic variable           | $\Gamma^A$ | 2N          | $\varepsilon_A$ | gh$_A$       | afn$_A$         | safn$_A$                 | rafn$_A$                 |
| Field                      | $\Phi^\alpha$ | N           | $\varepsilon_\alpha$ | gh$_\alpha$ | 0                | $\geq 0$                 | $\geq 0$                 |
| Original field             | $\varphi^i \equiv \Phi^\alpha_{-1}$ | n $\equiv m_{-1}$ | $\varepsilon_i \equiv \varepsilon_{\alpha_{-1}}$ | 0             | 0                | $\geq 0$                 | $\geq 0$                 |
| Transversal field          | $\zeta^I$ | $M_{-1}$    | $\varepsilon_I$ | 0             | 0                | 1                       | 1                        |
| Longitudinal field         | $\theta^a_0$ | $M_0$       | $\varepsilon_{A_0}$ | 0             | 0                | 0                       | 0                        |
| Stage-s ghost              | $c^\alpha_s \equiv \Phi^* \alpha$, | $m_s$ | $\varepsilon_{\alpha_s} + s + 1$ | $s + 1$       | 0                | 0                       | 0                        |
| Antifield                  | $\Phi^* \alpha$ | N           | $\varepsilon_\alpha + 1$ | $-\text{gh}_\alpha - 1$ | $\text{gh}_\alpha + 1$ | $\text{gh}_\alpha + 2$ | 1                        |
| Original antifield         | $\varphi^*_i \equiv \Phi^* \alpha_{-1}$ | n $\equiv m_{-1}$ | $\varepsilon_i + 1 \equiv \varepsilon_{\alpha_{-1}} + 1$ | $-1$           | 1                | 2                       | 1                        |
| Stage-s ghost antifield    | $c^*_\alpha_s \equiv \Phi^*_ \alpha_s$, | $m_s$ | $\varepsilon_{\alpha_s} + s$ | $-(s + 2)$ | $s + 2$ | $s + 3$ | 1                        |
| BRST operator              | $s = (S, \cdot)$ | 1           | 1                | $\leq -1$     | $\geq -1$       | $\geq 0$                 | $\geq 0$                 |
| Koszul-Tate operator       | $s_{-1}$ | 1           | 1                | $\leq -1$     | $\geq -1$       | $\geq 0$                 | $\geq 0$                 |
| Shifted/Reduced KT op.     | $\delta = s_{-1} = s_{-1[0]}$ | 1           | 1                | $\leq -1$     | $\geq -1$       | $\geq 0$                 | $\geq 0$                 |
| Contracting homotopy op.   | $\delta^{-1}$ | 1           | 1                | $\leq -1$     | $\geq -1$       | $\geq 0$                 | $\geq 0$                 |
to the classical master equation
\[ (S, S) = 0, \] (2.15)
such that \( S \) has the correct original limit
\[ S = S_0 + \mathcal{O}(\Phi^*) \overset{(2.14)}{=} S_0 + \mathcal{O}(c), \] (2.16)
and satisfies the properness condition
\[ \text{rank}(\left. \frac{\partial^T S}{\partial \Gamma^A} \right|_{\Phi^*=0=c}) = N \] (2.17)
at the stationary \( \varphi^i \)-surface, when all the auxiliary variables are put to zero, \( \Phi^*_s = 0 \) and \( e^{\alpha_s} = 0 \), \( s \in \{0, \ldots, L\} \). Here \( \Gamma^A \equiv \{\Phi^\alpha; \Phi^*_\alpha\}, A \in \{1, \ldots, 2N\} \), is a collective notation for both fields \( \Phi^\alpha \) and antifields \( \Phi^*_\alpha \).

The half rank \( N \) is the maximal possible for a solution \( S \) to the classical master equation. The properness condition (2.17) is important, because it implies (after the non–minimal sector has been included) that the gauge–fixed proper action \( S(\Phi, \Phi^* = \partial \Psi / \partial \Phi) \) is free of flat directions.

We will require one more condition besides eqs. (2.14)–(2.17). It will encode the gauge(–for–gauge)\(^s\) symmetries, \( s \in \{0, \ldots, L\} \), into the proper action \( S \). To explain it, let us divide an arbitrary proper solution \( S \) into two parts,
\[ S = S^{\text{quad}} + S^{\text{non–quad}}, \] (2.18)
The first part \( S^{\text{quad}} \) contains all terms that are at most quadratic in auxiliary variables, while the second part \( S^{\text{non–quad}} = \mathcal{O}\left( (\Phi^*)^2, c^2 \right) \) contains all terms that are at least cubic in auxiliary variables, which actually means all terms at least quadratic in ghosts or antifields, due to ghost number conservation (2.14). Because of ghost number conservation (2.14) and the original limit requirement (2.16), the action \( S^{\text{quad}} \) must be of the form
\[ S^{\text{quad}} = S_0 + \sum_{s=1}^L \Phi^*_\alpha \phi^{s-1} \alpha_s c^{\alpha_s} = S_0 + \varphi^i R^i \alpha_0 c^{\alpha_0} + \sum_{s=1}^L c^{s-1} \alpha_s Z^{\alpha_s-1} \alpha_s c^{\alpha_s}. \] (2.19)
In particular, there are no action terms that are linear in the auxiliary variables. \textit{A priori} the structure functions \( Z^{\alpha_s-1} \alpha_s = Z^{\alpha_s-1} \alpha_s(\varphi) \) in the action (2.19) could be different from the given gauge(–for–gauge)\(^s\)–generators, \( s \in \{0, \ldots, L\} \), specified in eqs. (2.1) and (2.3). However, the classical master equation (2.15) implies that the structure functions \( Z^{\alpha_s-1} \alpha_s \) in eq. (2.19) also satisfy the Noether identities (2.1)–(2.3), although perhaps with some other \( B^{i\alpha_s-2} \) structure functions. We will therefore demand that the structure functions \( R^i \alpha_0 \) and \( Z^{\alpha_s-1} \alpha_s \) in eq. (2.19) are the same as the given gauge generator \( R^i \alpha_0 \) and gauge(–for–gauge)\(^s\)–generators \( Z^{\alpha_s-1} \alpha_s \) in the Noether identities (2.1)–(2.3), respectively.

With the above identification, the rank of the \( S^{\text{quad}} \) part (2.19) of the Hessian is
\[ \text{rank}(\left. \frac{\partial^T S^{\text{quad}}}{\partial \Gamma^A} \right|_{\beta=0}) = \sum_{s=0}^\infty M_s + \sum_{s=-1}^\infty (M_s + M_{s+1}) = \sum_{s=1}^\infty m_s = N \] (2.20)
at the stationary \( \varphi \)-surface
\[ T_i \equiv \left. \frac{\partial^r}{\partial \varphi^i} \right|_{T=0} = 0, \] (2.21)
as a result of the rank conditions (2.5) and (2.6). So the full action \( S = S^{\text{quad}} + S^{\text{non-quad}} \) is then guaranteed to be proper. The properness condition for the solution \( S \) at the stationary \( \varphi \)-surface implies, by continuity, properness of the \( S \) solution in some sufficiently small \( \varphi \)-neighborhood of the stationary \( \varphi \)-surface.

### 2.3 Classical BRST Operator

If a proper solution \( S \) to eqs. (2.14)–(2.17) exists, the corresponding classical BRST operator is defined as

\[
S := (S, \cdot) .
\]

It is nilpotent

\[
s^2 = (S, (S, \cdot)) = \frac{1}{2}((S, S), \cdot) = 0 ,
\]

as a consequence of the Jacobi identity for the antibracket and the classical master equation (2.15).

**Theorem 2.1 (Acyclicity of the BRST Operator)** Let \( S \) be a proper solution defined in a tubular \( \varphi \)-neighborhood of the stationary \( \varphi \)-surface. Then the cohomology of the BRST operator \( S = (S, \cdot) \) is acyclic, i.e.,

\[
\forall \text{ functions } f : \quad s(f) = 0 \wedge \text{gh}(f) < 0 \quad \Rightarrow \quad \exists g : \quad f = s(g)
\]

in some tubular \( \varphi \)-neighborhood of the stationary \( \varphi \)-surface.

**Proof of Theorem 2.1:** Use Theorem 3.2, Theorem 3.3, Lemma D.1, and the fact that antifield number is (weakly) greater than minus ghost number, i.e., \( \text{afn}(f) \geq -\text{gh}(f) > 0 \), since \( \text{gh}(f) + \text{afn}(f) = \text{puregh}(f) \geq 0 \).

### 3 Standard Methods

#### 3.1 Introduction

In this Section 3 we try to attack the problem of existence of a field–antifield formulation for reducible theories by applying the standard method that is known to work in the irreducible case \([9, 10, 12, 14]\), i.e., generating a proper action \( S \) from a Koszul–Tate operator \( s_{-1} \) and keeping track of antifield number “afn”. Unfortunately, it is cumbersome to directly verify the existence and acyclicity of the Koszul–Tate operator \( s_{-1} \) in the reducible case using this standard approach. The standard method will nevertheless serve as a simplified template, on which we will develop a complete existence proof in the next Section 4, using a reduced (shifted) Koszul–Tate operator \( s_{(-1)} \) and a (shifted) antifield number “safn”, respectively.

#### 3.2 Antifield Number

The antifield number “afn” is defined as zero for fields \( \Phi^\alpha \), i.e., \( \text{afn}(\Phi^\alpha) = 0 \), and it is defined as minus the ghost number for antifields \( \Phi^*_\alpha \), i.e., \( \text{afn}(\Phi^*_\alpha) = -\text{gh}(\Phi^*_\alpha) \). See Table 1. Any action \( S \) of ghost number zero can be expanded with respect to antifield number.

\[
S = \sum_{r=0}^{\infty} S_r , \quad \text{afn}(S_r) = r .
\]
Let us also expand the antibracket \((\cdot, \cdot)\) according to antifield number.

\[
(f, g) = \sum_{s=1}^{L+2} (f, g)_{-s} = \sum_{s=-1}^{L} (f, g)_{-s-2}, \quad \text{afn}(f, g)_r = \text{afn}(f) + \text{afn}(g) + r, \quad (3.2)
\]

\[
(f, g)_{-s-2} : = f \left( \frac{\delta}{\delta \Phi} \frac{\delta^t}{\delta \Phi^*_s} - \frac{\delta^t}{\delta \Phi^*_s} \frac{\delta}{\delta \Phi} \right) g, \quad s \in \{-1, \ldots, L\}. \quad (3.3)
\]

Elementary considerations reveal the following useful Lemma 3.1.

**Lemma 3.1** Let \(f, g\) be two functions of definite antifield number. Then

\[
(f, g)_{-s} \neq 0 \implies \text{afn}(f) \geq s \vee \text{afn}(g) \geq s. \quad (3.4)
\]

Assume that \(f\) also has definite ghost number. Then

\[
(f, g)_{-s} \neq 0 \land \text{gh}(f) \geq -1 \implies \text{puregh}(f) \equiv \text{afn}(f) + \text{gh}(f) \geq s-1. \quad (3.5)
\]

In particular,

\[
(f, g)_{-s} \neq 0 \land \text{gh}(f) = 0 \implies \text{afn}(f) \geq s-1. \quad (3.6)
\]

**Proof of Lemma 3.1:** There are only two possibilities. The first case is

\[
\exists \alpha_{s-2} : \left( \frac{\delta^t}{\delta \Phi^*_s} f \right) \neq 0 \land \left( \frac{\delta^t}{\delta \Phi^*_{s-2}} g \right) \neq 0, \quad (3.7)
\]

and the second case is with the roles of \(f\) and \(g\) interchanged,

\[
\exists \alpha_{s-2} : \left( \frac{\delta^t}{\delta \Phi^*_{s-2}} f \right) \neq 0 \land \left( \frac{\delta^t}{\delta \Phi^*_s} g \right) \neq 0. \quad (3.8)
\]

In the first case, \(\text{afn}(f) \geq \text{afn}(\Phi^*_{s-2}) = s\). If also \(\text{gh}(f) \geq -1\), then \(\text{puregh}(f) \equiv \text{afn}(f) + \text{gh}(f) \geq s-1\). In the second case, \(\text{afn}(g) \geq \text{afn}(\Phi^*_{s-2}) = s\), and \(\text{puregh}(g) \geq \text{puregh}(\Phi^*_{s-2}) = s-1\).

\[
\square
\]

### 3.3 Koszul–Tate Operator \(s_{-1}\)

If the proper solution \(S\) exists, the Koszul–Tate operator \(s_{-1}\) is then defined as the leading antifield number sector for the classical BRST operator,

\[
s := (S, \cdot) = \sum_{p=-1}^{\infty} s_p, \quad (3.6)
\]

where

\[
s_p = \sum_{r=0}^{\infty} (S_r, \cdot)_{p-r}, \quad \text{afn}(s_p) = p, \quad p \in \{-1, 0, 1, 2, \ldots\}. \quad (3.10)
\]
The Koszul–Tate operator \( s_{-1} \) is of the form

\[
s_{-1} = \sum_{r=0}^{L+1} (S_r, \cdot)_{r-1} = V_\alpha \frac{\partial^r}{\partial \Phi_\alpha^r} = T_i \frac{\partial^r}{\partial \varphi_i^r} + \sum_{s=0}^L \Phi_{\alpha s-1}^* Z_{\alpha s-1} \alpha_s \frac{\partial^r}{\partial c_\alpha^s} + M_\alpha \frac{\partial^r}{\partial \Phi_\alpha^r},
\]

(3.11)

where the functions \( M_\alpha = \mathcal{O}((\Phi^*)^2) \) are at least quadratic in the antifields. It follows from the requirement

\[
\text{puregh}(V_\alpha) = \text{puregh}(s_{-1}) = \text{gh}(s_{-1}) + \text{afn}(s_{-1}) = 1 - 1 = 0
\]

(3.12)

that the functions \( V_\alpha = V_\alpha(\varphi, \Phi^*) \) cannot depend on the ghosts “\( c \)”. In more detail, the \( V_\alpha \) functions read

\[
V_{\alpha s-1} = T_i = (S_0 \frac{\partial^r}{\partial \varphi^r}), \quad V_{\alpha_0} = \varphi_i^r R_{\alpha_0}^r,
\]

(3.13)

\[
V_{\alpha_0} = \Phi_{\alpha s-1}^* Z_{\alpha s-1} \alpha_s + M_\alpha, \quad s \in \{0, \ldots, L\},
\]

(3.14)

\[
M_\alpha = \mathcal{O}((\Phi^*)^2), \quad \text{afn}(V_\alpha) = \text{afn}(M_\alpha) = \text{afn}(\Phi^*) - 1.
\]

(3.15)

(See also eqs. (1.3)–(1.8).) Phrased differently, the classical BRST operator “\( s \)” is a deformation of the Koszul–Tate operator, when one uses the antifield number “\( \text{afn} \)” as a resolution degree. The classical master eq. (2.15) implies that the classical BRST operator “\( s \)” is nilpotent, and hence that the Koszul–Tate operator \( s_{-1} \) is nilpotent,

\[
(s_{-1})^2 = 0 \iff V_\alpha \left( \frac{\partial^r}{\partial \Phi_\alpha^r} V_\beta \right) = 0 \iff s_{-1} \left( S_0 + \sum_{s=0}^L V_{\alpha s} c_\alpha^s \right) = 0.
\]

(3.16)

The last rewriting shows that the proper action \( S = S_0 + \sum_{s=0}^L V_{\alpha s} c_\alpha^s + \mathcal{O}(c^2) \) is \( s_{-1} \)-invariant modulo terms that are at least quadratic in the ghosts “\( c \)”. Explicitly, the nilpotency of \( s_{-1} \) implies the Noether identities (2.1)–(2.3), and their consistency relations, e.g., eqs. (1.11) and (1.12).

**Theorem 3.2 (Local acyclicity of the Koszul–Tate operator)** Assume that the K–T operator \( s_{-1} = V_\alpha \frac{\partial^r}{\partial \Phi_\alpha^r} \) is nilpotent in some \( \varphi \)-neighborhood of a \( \varphi \)-point on the stationary \( \varphi \)-surface, where the structure functions \( V_\alpha \) are of the form (3.13)–(3.15). Then the cohomology of the Koszul–Tate operator \( s_{-1} \) is acyclic in some \( \varphi \)-neighborhood of the \( \varphi \)-point, i.e.,

\[
\forall \text{ functions } f : \quad s_{-1} f = 0 \land \text{afn}(f) > 0 \quad \Rightarrow \quad \exists g : f = s_{-1} g.
\]

(3.17)

**Proof of Theorem 3.2 using reduced antifield number:** Use Theorem 4.3, Lemma D.2, and the fact that strictly positive antifield number implies strictly positive reduced antifield number \( \text{afn}(f) > 0 \Rightarrow \text{rafn}(f) > 0 \), see Section 4.

\( \square \)

**Proof of Theorem 3.2 using shifted antifield number:** Use Corollary 4.4, Lemma D.1, and the fact that shifted antifield number is (weakly) greater than antifield number \( \text{safn}(f) \geq \text{afn}(f) > 0 \), see Section 4.

\( \square \)
Theorem 3.3 (Globalization (Theorem 2 in Ref. [13])) Assume that for every $\varphi$-point on the stationary $\varphi$-surface there exists some nilpotent and acyclic Koszul–Tate operator in a local $\varphi$-neighborhood around the $\varphi$-point. Then there exists a smooth, nilpotent and acyclic Koszul–Tate operator in a tubular $\varphi$-neighborhood of the whole stationary $\varphi$-surface.

**Sketched Proof of Theorem 3.3**: Cover the stationary $\varphi$-surface with sufficiently small $\varphi$-neighborhoods $U_i$ such that the local Koszul–Tate operators (which may depend on the $\varphi$-neighborhoods $U_i$) are nilpotent and acyclic. Use a smooth partition of unity $\sum_i f_i = 1$, $f_i = f_i(\varphi)$, supp$f_i \subseteq U_i$, to define a global Koszul–Tate operator $s_{-1}$. Here it is used that the Koszul–Tate operators do not act on the original $\varphi$-variables.

### 3.4 Existence of $S$

Now we would like to deduce a proper solution from the Koszul–Tate operator.

**Theorem 3.4 (Existence theorem for $S$ action [10])** Let there be given a nilpotent, acyclic Koszul–Tate operator $s_{-1} = V_\alpha \frac{\delta}{\delta \alpha}$, that is defined in a $\varphi$-neighborhood $U$, with antifield number minus one, and where $V_\alpha = V_\alpha(\varphi, \Phi^*)$. This guarantees the existence of a proper solution $S = S_0 + \sum_{s=0}^{L} V_\alpha e^\alpha + \mathcal{O}(c^2)$ to the classical master equation (2.15) in $U$.

**Proof of Theorem 3.4**: Define the classical master expression

$$\text{CME} := \frac{1}{2} (S, S)$$

and the Jacobiator

$$J := \frac{1}{2} (S, (S, S)) = s(\text{CME}) .$$

It follows from Lemma 3.1 that the $r$th classical master expression $\text{CME}_r$ can be written as

$$\text{CME}_r := \frac{1}{2} \sum_{p,q \geq 0} (S_p, S_q)_{r-p-q} \overset{(3.6)}{=} \frac{1}{2} \sum_{0 \leq p,q \leq r+1} (S_p, S_q)_{r-p-q} \overset{(3.4)}{=} s_{-1} S_{r+1} + B_r ,$$

where

$$B_r := \frac{1}{2} \sum_{0 \leq p,q \leq r} (S_p, S_q)_{r-p-q} \overset{r-p-q<0}{=} \frac{1}{2} \sum_{1 \leq p,q \leq r} (S_p, S_q)_{r-p-q} = \mathcal{O}(c^2) , \quad r \geq 0 .$$

The second equality of eq. (3.21) follows because the antibracket itself has negative antifield number. The third equality of eq. (3.21) follows from the $s_{-1}$ nilpotency (3.16). The proof of the main statement is an induction in the antifield number $r$. Assume that there exist

$$S_0 , \quad S_1 = V_{\alpha_0} e^{\alpha_0} , \quad S_2 = V_{\alpha_1} e^{\alpha_1} + \mathcal{O}(c^2) , \quad \ldots , \quad S_r = V_{\alpha_{r-1}} e^{\alpha_{r-1}} + \mathcal{O}(c^2) , \quad (3.22)$$

such that

$$0 = \text{CME}_0 = \text{CME}_1 = \ldots = \text{CME}_{r-1} . \quad (3.23)$$
It follows that the $B_r$ function (3.21) exists as well, because $B_r$ only depends on the previous $S_{\leq r}$. We want to prove that there exists $S_{r+1}$ such that $CME_r = 0$. The Jacobi identity $J = 0$ gives
\[ 0 = J_{r-1} = \sum_{p=-1}^{\infty} s_p CME_{r-p-1} = s_{-1} CME_r = s_{-1} B_r, \tag{3.24} \]
so $B_r$ is $s_{-1}$-closed. If $r = 0$, one defines
\[ S_1 := V_{\alpha_0} c^{\alpha_0} = \varphi_i R^i_{\alpha_0} c^{\alpha_0}. \tag{3.25} \]
Then
\[ CME_0 = s_{-1} S_1 + B_0 = 0 + 0 = 0, \tag{3.26} \]
because of the Noether identity (2.1). If $r > 0$, then the acyclicity condition (3.17) shows that there exists a function $S_{r+1} = V_{\alpha_r} c^{\alpha_r} + \mathcal{O}(c^2)$ such that $-s_{-1} S_{r+1} = B_r = \mathcal{O}(c^2)$. Here we used that Koszul–Tate operator $s_{-1}$ does not depend on the ghosts $c^s$, $s \in \{0, \ldots, L\}$.

3.5 Anticanonical Transformations

Consider finite anticanonical transformations $e^{(\Psi, \cdot)}$, where $\Psi$ is a Grassmann–odd generator with ghost number minus one, $\text{gh}(\Psi) = -1$. Such transformations form a group under composition
\[ e^{(\Psi_1, \cdot)} e^{(\Psi_2, \cdot)} = e^{(\text{BCH}(\Psi_1, \Psi_2), \cdot)}, \tag{3.27} \]
where
\[ \text{BCH}(\Psi_1, \Psi_2) = \Psi_1 + \int_0^1 dt \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left[ e^{-t(\Psi_2, \cdot)} e^{-(\Psi_1, \cdot)} - 1 \right]^n \Psi_2 = \Psi_1 + \Psi_2 + \frac{1}{2} (\Psi_1, \Psi_2) + \frac{1}{12} (\Psi_1, (\Psi_1, \Psi_2)) + \frac{1}{12} ((\Psi_1, \Psi_2), \Psi_2) + \mathcal{O}(\Psi^4) \tag{3.28} \]
is the Baker–Campbell–Hausdorff series expansion (with Lie brackets replaced by antibrackets).

Note that an anticanonical transformation $e^{(\Psi, \cdot)} S$ of a solution $S$ to the classical master eq. (2.15) is again a solution to the classical master eq. (2.15). We shall now show that any two proper solutions are related via an anticanonical transformation modulo global obstructions.

**Theorem 3.5 (Natural arbitrariness/ambiguity of proper solution)** Let $S$ and $\overline{S}$ be two proper solutions to the classical master eq. (2.15) with ghost number zero, with correct original limit (2.16), and defined in some $\varphi$-neighborhood of a $\varphi$-point on the stationary $\varphi$-surface. Then there exists a Grassmann–odd generator $\Psi$ with ghost number minus one, $\text{gh}(\Psi) = -1$, and of order $\Psi = \mathcal{O}(c)$, such that
\[ \overline{S} = e^{(\Psi, \cdot)} S \tag{3.29} \]
in some $\varphi$-neighborhood of the $\varphi$-point. If $S$ and $\overline{S}$ also have the same gauge (–for–gauge) $s$–generators, $Z^{\alpha_{s-1}}_{\alpha_s} = \overline{Z}^{\alpha_{s-1}}_{\alpha_s}$, $s \in \{0, \ldots, L\}$, then the Grassmann–odd generator $\Psi$ can be chosen of the form $\Psi = \mathcal{O}((\Phi^*)^2, c^2)$.

**Remark:** The essential ingredient of the proof of Theorem 3.5, is the acyclicity of the Koszul–Tate operator $s_{-1}$ associated with the proper solution $S$, where $s = (S, \cdot) = s_{-1} + \ldots$. 

13
Proof of Theorem 3.5: The main proof is an induction in the number \( m \geq 1 \) of ghosts \( c \). The induction assumption is that there exists an anticanonical transformation \( e^{-(\Psi, \cdot)S} \) of \( S \), such that one has (after renaming \( e^{-(\Psi, \cdot)S} \rightarrow S) \)

\[
S - S = O(c^m) .
\]  

(3.30)

The induction assumption is true for \( m = 1 \), because one assumes that the correct original limit (2.16) is fulfilled, \( S_0 = S_0 \). Now consider \( m = 2 \). One has that

\[
S = S_0 + \sum_{s=0}^{L} V_{\alpha_s} c^{\alpha_s} + O(c^2) , \quad \overline{S} = S_0 + \sum_{s=0}^{L} \nabla_{\alpha_s} c^{\alpha_s} + O(c^2) .
\]  

(3.31)

One now uses (nested) induction in antifield number \( 0 \leq r \leq L \). (This is a finite induction, if the number \( L \) of stages is finite.) The induction assumption is that there exists an anticanonical transformation \( e^{-(\Psi, \cdot)S} \) of \( \overline{S} \), with \( \Psi = O(c) \), such that one has (after renaming \( e^{-(\Psi, \cdot)S} \rightarrow \overline{S} \))

\[
\nabla_{\alpha_{r-1}} = V_{\alpha_{r-1}} , \quad \nabla_{\alpha_0} = 0 _{\alpha_0} , \quad \nabla_{\alpha_1} = V_{\alpha_1} , \quad \ldots , \quad \nabla_{\alpha_{r-1}} = V_{\alpha_{r-1}} .
\]  

(3.32)

From nilpotency (3.16) of the two Koszul–Tate operators, one knows that

\[
s_{-1} V_{\alpha_r} \overset{(3.16)}{=} 0 , \quad \overline{s}_{-1} \nabla_{\alpha_r} \overset{(3.16)}{=} 0 .
\]  

(3.33)

Since the function \( V_{\alpha} = V_{\alpha}(\phi, \Phi^s) \) does not depend on the ghost variables \( c^{\alpha_s}, s \in \{0, \ldots, L\} \), the antifield number is

\[
\text{afn}(V_{\alpha_r}) = -gh(V_{\alpha_r}) = gh(c^{\alpha_r}) = r + 1 .
\]  

(3.34)

It follows that the structure function \( V_{\alpha_r} = V_{\alpha_r}(\phi^j; \phi^*_j, c^{\alpha_0}_0, \ldots, c^{\alpha_s}_s) \) cannot depend on antifields \( c^{\alpha_s}_s \), for \( r \leq s \leq L \), because their antifield number \( \text{afn}(c^{\alpha_s}_s) = s + 2 \) is too big. Similarly, for \( \nabla_{\alpha_r} = \nabla_{\alpha_r}(\phi^j; \phi^*_j, c^{\alpha_0}_0, \ldots, c^{\alpha_r}_{r-1}) \). From the induction assumption (3.32), one concludes that

\[
0 \overset{(3.16)}{=} \overline{s}_{-1} \nabla_{\alpha_r} = \sum_{s=-1}^{r-1} \nabla_{\alpha_s} \left( \frac{\partial}{\partial \Phi^*_s} \nabla_{\alpha_r} \right) \overset{(3.32)}{=} \sum_{s=-1}^{r-1} V_{\alpha_s} \left( \frac{\partial}{\partial \Phi^*_s} \nabla_{\alpha_r} \right) = s_{-1} \nabla_{\alpha_r} .
\]  

(3.35)

Hence the difference \( s_{-1}(V_{\alpha_r} - \nabla_{\alpha_r}) = 0 \) is zero, so by acyclicity (3.17) of the Koszul–Tate operator \( s_{-1} \), there exists

\[
U_{\alpha_r} = U_{\alpha_r}(\phi^j; \phi^*_j, c^{\alpha_0}_0, \ldots, c^{\alpha_r}_{r-1}) = c^{\alpha_r}_{\beta_r} \tilde{U}^{\beta_r}_{\alpha_r}(\phi^j) + O((\Phi^s)^2)
\]  

(3.36)

with \( \text{afn}(U_{\alpha_r}) = -gh(U_{\alpha_r}) = r + 2 \), such that

\[
V_{\alpha_r} - \nabla_{\alpha_r} = s_{-1} U_{\alpha_r} .
\]  

(3.37)

One is allowed to change

\[
U_{\alpha_r} \rightarrow U_{\alpha_r} + s_{-1} W_{\alpha_r} ,
\]  

(3.38)

where

\[
W_{\alpha_r} = W_{\alpha_r}(\phi^j; \phi^*_j, c^{\alpha_0}_0, \ldots, c^{\alpha_r}_{r+1}) = c^{\alpha_r}_{\beta_{r+1}} \tilde{W}^{\beta_{r+1}}_{\alpha_r}(\phi^j) + O((\Phi^s)^2)
\]  

(3.39)

with \( \text{afn}(W_{\alpha_r}) = -gh(W_{\alpha_r}) = r + 3 \). The leading contribution proportional to \( c^{\alpha_r}_{\alpha_{r-1}} \) in eq. (3.37) and \( c^{\alpha_r}_{\alpha_r} \) in eq. (3.38) read

\[
Z^{\alpha_{r-1}}_{\alpha_r} - \overline{Z}^{\alpha_{r-1}}_{\alpha_r} = Z^{\alpha_{r-1}}_{\beta_r} \tilde{U}^{\beta_r}_{\alpha_r} , \quad \tilde{U}^{\beta_r}_{\alpha_r} \rightarrow \tilde{U}^{\beta_r}_{\alpha_r} + Z^{\beta_r}_{\beta_{r+1}} \tilde{W}^{\beta_{r+1}}_{\alpha_r} .
\]  

(3.40)
Alternatively, if one defines $\Delta^{\beta}_{\alpha_r} := (1 - \bar{U})^{\beta}_{\alpha_r}$, one has

$$
\overline{Z}^{\alpha_r-1}_{\alpha_r} = Z^{\alpha_r-1}_{\alpha_r} \Delta^{\beta}_{\alpha_r} \quad \text{and} \quad \Delta^{\beta}_{\alpha_r} \rightarrow \Delta^{\beta}_{\alpha_r} - Z^{\beta}_{\beta_r+1} \bar{W}^{\beta_{r+1}}_{\alpha_{r+1}}.
$$

(3.41)

It follows from the rank conditions for $\overline{Z}^{\alpha_r-1}_{\alpha_s}$ and $Z^{\alpha_{r-1}}_{\alpha_r}$, that one may assume that $\Delta^{\beta}_{\alpha_r}$ is a regular invertible matrix, possibly after an allowed change (3.38). (This is the only place in the proof where one uses that the $\overline{S}$ solution is proper.) Therefore one may assume that the matrix $\bar{U}^{\beta}_{\alpha_r} = (1 - \Delta)^{\beta}_{\alpha_r}$ has no eigenvalues equal to 1. Next apply the anticanonical transformation $e^{-\Psi} \cdot \overline{S}$, with $\Psi = \Psi_{\alpha_r} \Theta_{\alpha_r}$, where

$$
\Psi_{\alpha_r} := U_{\beta_r} f(\bar{U})^{\beta}_{\alpha_r},
$$

(3.42)

$$
\Psi_{\alpha_r} = \Psi_{\alpha_r}(\varphi^j ; \varphi^j_r, c^j_{\beta_r}), \quad c^j_{\beta_r} = c^j_{\beta_r} \bar{\Psi}^{\beta}_{\alpha_r} (\varphi^j) + O\left((\varphi^j)^2 \right),
$$

(3.43)

and where the holomorphic function

$$
f(z) := -\frac{\ln(1 - z)}{z} = \sum_{n=0}^{\infty} \frac{z^n}{n + 1}
$$

(3.44)

has a logarithmic singularity at $z = 1$. One may place the branch–cut of $f$ away from the eigenvalue spectrum of the matrix $\bar{U}^{\beta}_{\alpha_r}$, so that $f(\bar{U})$ is well-defined. The leading tilde piece of the $\Psi$ function (3.43) is

$$
\bar{\Psi}^{\beta}_{\alpha_r} = -\ln(1 - \bar{U})^{\beta}_{\alpha_r} \leftrightarrow \bar{U}^{\beta}_{\alpha_r} = (1 - e^{-\Psi})^{\beta}_{\alpha_r}.
$$

(3.45)

One calculates

$$
(\cdot, \Psi) = (\cdot, \Psi) - \sum_{s=0}^{r} \sum_{\alpha} \frac{\partial}{\partial c^s_{\alpha}}(\frac{\partial}{\partial c^s_{\beta}}) \Psi_{\alpha_r} + (\cdot, \frac{\partial}{\partial c^s_{\beta}})\Psi_{\alpha_r},
$$

(3.46)

so that

$$
(\overline{S}, \Psi) + O(c^2) = (S_0 + \sum_{s=0}^{r} \overline{V}_{\alpha_s} c^s_r, \Psi) = \overline{s}_1 \Psi_{\alpha_r} c^\alpha_r + O(c^2),
$$

(3.47)

and hence

$$
\left[ e^{-\Psi} \cdot \overline{S} + O(c^2) \right] = \left[ e^{-\Psi} \cdot - 1 \right] \left[ S_0 + \sum_{s=0}^{r} \overline{V}_{\alpha_s} c^s_r \right] + O(c^2)
$$

(3.32)

$$
\overline{s}_1 \Psi_{\beta_r} E(\bar{\Psi})^{\beta}_{\alpha_r} c^\alpha_r + \overline{V}_{\alpha_s} - V_{\alpha_s} \overline{V}_{\alpha_r} \frac{\partial^2}{\partial c^s_{\beta}} \Psi_{\beta_r} E(\bar{\Psi})^{\beta}_{\alpha_r} c^\alpha_r
$$

(3.37)+(3.43)

$$
= \overline{s}_1 \Psi_{\beta_r} E(\bar{\Psi})^{\beta}_{\alpha_r} c^\alpha_r - \overline{s}_1 U_{\gamma_r} \overline{V}_{\alpha_r} \frac{\partial^2}{\partial c^s_{\beta}} \Psi_{\beta_r} E(\bar{\Psi})^{\beta}_{\alpha_r} c^\alpha_r
$$

(3.42)+(3.45)

$$
= \overline{s}_1 \Psi_{\beta_r} f(1 - e^{-\Psi})^{\gamma_r}_{\beta_r} E(\bar{\Psi})^{\beta}_{\alpha_r} c^\alpha_r - \overline{s}_1 U_{\gamma_r} (e^{-\Psi} - 1)^{\gamma_r}_{\alpha_r} c^\alpha_r
$$

(3.48)

where $E(z) := \frac{e^z - 1}{z} = \sum_{m=0}^{\infty} \frac{z^n}{(n+1)!}$. One has (after renaming $e^{-\Psi} \overline{S} \rightarrow \overline{S}$)

$$
\overline{V}_{\alpha_r} = V_{\alpha_r}, \quad \overline{V}_{\alpha_0} = V_{\alpha_0}, \quad \overline{V}_{\alpha_1} = V_{\alpha_1}, \ldots, \quad \overline{V}_{\alpha_r} = V_{\alpha_r},
$$

(3.49)

which is the nested induction assumption (3.32) with $r \rightarrow r + 1$. This finishes the proof that the induction assumption (3.30) is true for $m = 2$, and that the two Koszul–Tate operators $\overline{s}_1 = s_1$ are equal.
Now assume that $m \geq 3$ and that the induction assumption (3.30) is true up to previous number "$m-1$" of ghosts. We would like to prove eq. (3.30) for "$m$". One now uses nested induction in antifield number $0 \leq r \leq m(L+2)$. Assume that there exists an anticanonical transformation $e^{-(\Psi_{r+2} \cdot)}\mathcal{S}$ of $\mathcal{S}$, such that one has (after renaming $e^{-(\Psi_{r+2} \cdot)}\mathcal{S} \rightarrow \mathcal{S}$)

$$
\mathcal{S}_0 - S_0 = \mathcal{O}(c^m) \ , \ \mathcal{S}_1 - S_1 = \mathcal{O}(c^m) \ , \ ... \ , \ \mathcal{S}_r - S_r = \mathcal{O}(c^m) ,
$$

(3.50)

while $\mathcal{S}_{r+1} - S_{r+1} = \mathcal{O}(c^{m-1})$. The two classical master equations yield

$$
0 = \text{CME}_r \overset{(3.20)}{=} s_{-1} S_{r+1} + B_r , \quad 0 = \overline{\text{CME}_r} \overset{(3.20)}{=} \overline{s_{-1} S_{r+1} + B_r} ,
$$

(3.51)

where

$$
B_r \overset{(3.21)}{=} \frac{1}{2} \sum_{1 \leq p,q \leq r} (S_p, S_q)_{r-p-q} , \quad \overline{B_r} \overset{(3.21)}{=} \frac{1}{2} \sum_{1 \leq p,q \leq r} (\overline{S}_p, \overline{S}_q)_{r-p-q} .
$$

(3.52)

Hence the difference is

$$
s_{-1}(S_{r+1} - \mathcal{S}_{r+1}) = B_r - \overline{B}_r = \frac{1}{2} \sum_{1 \leq p,q \leq r} \left[ (S_p - S_p, S_q)_{r-p-q} + (\overline{S}_p - \overline{S}_p, \overline{S}_q)_{r-p-q} \right] = \mathcal{O}(c^m) .
$$

(3.53)

Since the Koszul–Tate operator $s_{-1}$ is acyclic (3.17) and preserves the number of c’s, there exists a $\Psi_{r+2} = \mathcal{O}(c^{m-1})$ with $\text{afn}(\Psi_{r+2}) = r + 2$ such that $S_{r+1} - \mathcal{S}_{r+1} - s_{-1} \Psi_{r+2} = \mathcal{O}(c^m)$. Next apply the anticanonical transformation $e^{-(\Psi_{r+2} \cdot)}$ to $\mathcal{S}$. One calculates

$$
\left[ e^{-(\Psi_{r+2} \cdot)} - 1 \right] \mathcal{S} = \left[ e^{-(\Psi_{r+2} \cdot)} - 1 \right] \left[ S_0 + \sum_{s=0}^{L} V_{\alpha_s} c^{\alpha_s} + \mathcal{O}(c^2) \right] = \mathcal{O}(c^m) ,
$$

for $m \geq 3$

$$
\left( S_0 + \sum_{s=0}^{L} V_{\alpha_s} c^{\alpha_s} , \Psi_{r+2} \right) + \mathcal{O}(c^m) = s_{-1} \Psi_{r+2} + \mathcal{O}(c^m) .
$$

(3.54)

One has (after renaming $e^{-(\Psi_{r+2} \cdot)}\mathcal{S} \rightarrow \mathcal{S}$) that $\mathcal{S} - S = \mathcal{O}(c^{m-1})$, and

$$
\mathcal{S}_0 - S_0 = \mathcal{O}(c^m) , \ \mathcal{S}_1 - S_1 = \mathcal{O}(c^m) , \ ... \ , \ \mathcal{S}_{r+1} - S_{r+1} = \mathcal{O}(c^m) .
$$

(3.55)

\[ \square \]

### 4 Existence of Proper Action

#### 4.1 Transversal and Longitudinal Fields

Because of the Noether identity (2.1), there exist locally $M_{-1}$ independent on–shell gauge–invariants $\xi^I = \xi^I(\phi)$, which we will call the transversal fields. They satisfy in terms of formulas

$$
(\xi^I \frac{\partial}{\partial \phi^J}) R^J_{\alpha_0} = T_i K^I_{\alpha_0} \approx 0 \ , \quad I \in \{1, \ldots, M_{-1}\} ,
$$

(4.1)

near the stationary $\phi$-surface, where “$\approx$” means equality modulo equations of motion

$$
T_i \equiv (S_0 \frac{\partial}{\partial \phi^I}) ,
$$

(4.2)

and where $K^I_{\alpha_0} = K^I_{\alpha_0}(\phi)$ are some structure functions. Consider now an arbitrary $\phi$-neighborhood $U$, where the transversal fields $\xi^I$ are defined, and where $U$ is sufficiently close so that it intersects
the stationary \( \varphi \)-surface. It follows from eq. (4.1) that the values of the gauge–invariants \( \xi^I(\varphi) = \xi^I_{cl} \) do not depend on the point \( \varphi \) if one only varies the point \( \varphi \) within the \( M_0 \)-dimensional stationary subsurface \( T_1(\varphi) = 0 \). By a redefinition \( \xi^I \to \xi^I - \xi^I_{cl} \) it is possible to assume (and we will do so from now on), that \( \forall \varphi \in U : T_1(\varphi) = 0 \Rightarrow \xi^I(\varphi) = 0 \). Since transversal fields \( \xi^I \) are independent, it follows that the equations of motion are equivalent to the vanishing of the transversal fields

\[
\forall \varphi \in U : T_1(\varphi) = 0 \Leftrightarrow \xi^I(\varphi) = 0 \ .
\] (4.3)

The transversal fields \( \xi^I \) can locally be complemented with so–called \emph{longitudinal} fields \( \theta^{A_0} = \theta^{A_0}(\varphi) \), such that the change of coordinates

\[
\varphi^i \longrightarrow \overline{\varphi}^i \equiv \begin{bmatrix} \xi^I \\ \theta^{A_0} \end{bmatrix}
\] (4.4)
is a non-singular coordinate transformation. Here the indices runs over

\[
i \in \{1, \ldots, n \} \ , \quad I \in \{1, \ldots, M_1 \} \ , \quad A_0 \in \{1, \ldots, M_0 \} \ , \quad M_1 + M_0 = n \ .
\] (4.5)

By definition

\[
(S_0 \frac{\partial^r}{\partial \xi^I}) = (S_0 \frac{\partial^r}{\partial \varphi^i})(\varphi^i \frac{\partial^r}{\partial \xi^I}) \equiv \mathcal{O}(T) \ .
\] (4.6)

The rectangular matrix

\[
R^{A_0 \alpha_0} := (\theta^{A_0} \frac{\partial}{\partial \varphi^i})R^i_{\alpha_0} \ , \quad \alpha_0 \in \{1, \ldots, m_0 \} \ , \quad m_0 \geq M_0 \ ,
\] (4.7)
must have maximal rank near the stationary \( \varphi \)-surface

\[
0 \leq \text{rank}(R^{A_0 \alpha_0}) = M_0 \ .
\] (4.8)

(If it didn’t have maximal rank, it would signal the possibility to define at least one more transversal coordinate that satisfies eq. (4.1).) Therefore there exists a right inverse matrix \( N^{\alpha_0 A_0} \), such that

\[
R^{A_0 \alpha_0}N^{\alpha_0 B_0} = \delta^{A_0}_{B_0} \ .
\] (4.9)

The Noether identity (2.1) yields

\[
0 = (S_0 \frac{\partial^r}{\partial \varphi^i})R^i_{\alpha_0} = (S_0 \frac{\partial^r}{\partial \theta^{A_0}})(\theta^{A_0} \frac{\partial^r}{\partial \varphi^i})R^i_{\alpha_0} + (S_0 \frac{\partial^r}{\partial \xi^I})(\xi^I \frac{\partial^r}{\partial \varphi^i})R^i_{\alpha_0}
\]

\[
\equiv (S_0 \frac{\partial^r}{\partial \theta^{A_0}})R^{A_0 \alpha_0} + T_1K^{i j}_{\alpha_0}(\xi^I \frac{\partial^r}{\partial \xi^J}S_0)(-1)^{\varepsilon_j \varepsilon_{\alpha_0}} \quad (4.10)
\]

\[
= (S_0 \frac{\partial^r}{\partial \theta^{A_0}})\overline{R}^{A_0 \alpha_0} + (S_0 \frac{\partial^r}{\partial \xi^I})(\xi^I \frac{\partial^r}{\partial \xi^J}K^{i j}_{\alpha_0}(-1)^{\varepsilon_j \varepsilon_{\alpha_0}} \approx \overline{R}^{A_0 \alpha_0} \ ,
\] (4.11)

where

\[
\overline{R}^{A_0 \alpha_0} := R^{A_0 \alpha_0} + (\theta^{A_0} \frac{\partial}{\partial \varphi^i})K^{i j}_{\alpha_0}(-1)^{\varepsilon_j \varepsilon_{\alpha_0}} \approx R^{A_0 \alpha_0} .
\] (4.12)

The expression (4.11) is for later convenience. From the expression (4.10), one sees that

\[
(S_0 \frac{\partial^r}{\partial \theta^{A_0}}) \ (4.6)+(4.10) \mathcal{O}(T^2) .
\] (4.13)
By differentiating the Noether identity (2.1) with respect to $\varphi^i$, one derives that

$$0 \approx \left( \frac{\partial^\ell}{\partial \varphi^i} S_0 \right) R^j_{\alpha_0} \approx \left( \frac{\partial^r}{\partial \theta^A_0} S_0 \right) R^j_{\alpha_0},$$

and hence,

$$\left( \frac{\partial^r}{\partial \varphi^i} S_0 \right) \approx 0 .$$

Therefore the rank condition (2.5) implies that

$$0 \leq \text{rank} \left( \frac{\partial^r}{\partial \xi^I} S_0 \right) = M - 1$$

near the stationary $\varphi$-surface. The transversal and longitudinal fields are not uniquely defined.

If the $K^i_{\alpha_0}$ structure functions additionally satisfy the integrability condition

$$K^i_{\alpha_0} := \left( \xi^I \frac{\partial^r}{\partial \varphi^i} \right) K^i_{\alpha_0} = -(-1)^{\varepsilon_i \varepsilon_j} (I \leftrightarrow J),$$

then the second term of the expression (4.11) vanishes identically, so that

$$\left( \frac{\partial^r}{\partial \theta^A_0} S_0 \right) = 0 ,$$

instead of just eq. (4.13). Therefore, in case of the integrability condition (4.17), one has the following.

**Principle 4.1 (The Gauge Principle)** _Locally near the stationary $\varphi$-surface, the original action $S_0 = S_0(\xi)$ depends on only $M - 1$ independent quantities $\xi^I = \xi^I(\varphi)$, $I \in \{1, \ldots, M - 1\}$. _

In fact, the Gauge Principle 4.1 is precisely equivalent to the pair of eqs. (4.1) and (4.17). It was shown in Ref. [4] by integrating the generalized Lie equations that finite gauge transformations do exist, and in particular, that the gauge principle (4.1) holds, and hence that there exist $\xi^I$ such that both eqs. (4.1) and (4.17) are satisfied. This implies, among other things, that the set of stationary points for $S_0$ constitute a smooth submanifold. However, since $\xi^I$ generically will be space–time non–local, and since we will not actually need the integrability condition (4.17) in the following, we are reluctant to unnecessarily enforce eq. (4.17) on $\xi^I$ in what follows.

### 4.2 Reduced and Shifted Antifield Number

We would like to redefine the Koszul–Tate operator and the resolution degree, so that the Koszul–Tate operator is more directly related to the $S^\text{quad}$ part (2.19) of the action $S$ in the reducible case. The new resolution degrees will be the so–called reduced and shifted antifield number. Transversal fields $\xi^I$ and antifields $\Phi^*_{\alpha}$ are charged under reduced and shifted antifield number

$$\text{rafn}(\xi^I) = 1 = \text{safn}(\xi^I), \quad \text{rafn}(\Phi^*_{\alpha}) = 1, \quad \text{safn}(\Phi^*_{\alpha}) = 1 + \text{afn}(\Phi^*_{\alpha}) = 1 - \text{gh}(\Phi^*_{\alpha}),$$

$$\text{safn}(\Phi^*_{\alpha_s}) = s + 3 , \quad s \in \{-1, \ldots, L\} .$$

18
All the other variables, i.e., the longitudinal fields $\theta^A$, and ghosts $c^\alpha, s \in \{0, \ldots, L\}$, carry no shifted or reduced antifield number, see Table 1. Reduced and shifted antifield number are not independent, because in general
\[
\text{safn} = \text{rafn} + \text{afn} .
\]
(4.21)

Notice also that
\[
\forall \text{ functions } f = f(\Gamma) : \quad \text{safn}(f) = 0 \iff \text{rafn}(f) = 0 .
\]
(4.22)

On one hand, when considering only the Koszul–Tate operator, $s_{-1} = V_0 \frac{\partial}{\partial x^0}$, $V_0 = V_0(\varphi, \Phi^*)$, where ghosts “$c$” are passive spectators, then the reduced antifield number “rafn” is the simplest resolution degree to work with, cf. Theorem 4.5. Reduced antifield number is also easy to transcribe into the Hamiltonian framework. On the other hand, when considering the whole BRST operator “$s$”, where ghosts “$c$” are active, then one needs the shifted antifield number “safn”, cf. Theorem 4.6. (For instance, some parts of the longitudinal derivative [14] would have leading resolution degree, if one only uses reduced antifield number as resolution degree, which would be devastating. On the other hand, shifted antifield number appropriately pushes the longitudinal derivative down the resolution hierarchy.)

Reduced and shifted antifield number will depend on the local choice of transversal fields, so they are not globally defined.

Let us now decompose the action and the antibracket with respect to shifted antifield number. First rewrite the Jacobian matrix of the transformation (4.4),
\[
\left(\frac{\partial^p}{\partial \varphi^j}\right) =: \Lambda^i_j
\]
(4.23)
as functions $\Lambda^i_j = \Lambda^i_j(\varphi)$ of transversal and longitudinal fields $\varphi_i \equiv \{\xi^I; \theta^A\}$. One can expand these functions in shifted antifield number.
\[
\Lambda^i_j = \sum_{r=0}^{\infty} \Lambda^i_{(r)j}, \quad \text{safn}(\Lambda^i_{(r)j}) = r .
\]
(4.24)

One next expands the classical master action $S$ according to the shifted antifield number.
\[
S = \sum_{r=0}^{\infty} S_{(r)} , \quad \text{safn}(S_{(r)}) = r ,
\]
(4.25)

\[
S_{(0)} = S_{0(0)} \overset{(4.13)}{=} \text{const.} , \quad S_{(1)} = S_{0(1)} \overset{(4.6)}{=} 0 , \quad S_{(2)} = S_{0(2)} + \varphi^*_i R^i_{(0)\alpha_0} c^{\alpha_0} .
\]
(4.26)

Here $S_{0(2)}$ is of the form
\[
S_{0(2)} = \frac{1}{2} \xi^I H_{IJ} \xi^J , \quad H_{IJ} = -(-1)^{(\varepsilon_I+1)(\varepsilon_J+1)} H_{IJ} , \quad H_{IJ} = H_{IJ}(\theta) .
\]
(4.27)

The antibracket $(\cdot, \cdot)$ expands as
\[
(f, g) = (f, g)^\xi + (f, g)^{\theta} + (f, g)^c = \sum_{r=-(L+3)}^{\infty} (f, g)_{(r)} ,
\]
(4.28)

\[
(f, g)_{(r)} = (f, g)^\xi_{(r)} + (f, g)^{\theta}_{(r)} + (f, g)^c_{(r)} , \quad \text{safn}(f, g)_{(r)} = \text{safn}(f) + \text{safn}(g) + r ,
\]
(4.29)

\[
(f, g)^\xi = \sum_{r=-3}^{\infty} (f, g)^\xi_{(r)} , \quad (f, g)^{\theta} = \sum_{r=-2}^{\infty} (f, g)^{\theta}_{(r)} , \quad (f, g)^c = \sum_{s=0}^{L} (f, g)^c_{(-s-3)} .
\]
(4.30)
\( (f, g)_{(r-3)}^{\xi} := (f \frac{\partial^r}{\partial \xi^r}) \Lambda_{(r)j}^{I} (\frac{\partial^f}{\partial \varphi^f} g) - (-1)^{\varepsilon_f+1}(\varepsilon_g+1) (f \leftrightarrow g) , \quad r \in \{0, 1, 2, \ldots\} \),  
(4.31)

\( (f, g)_{(r-2)}^{q} := (f \frac{\partial^r}{\partial \varphi^f} \Lambda_{(r)j}^{A} (\frac{\partial^q}{\partial \varphi^q} g) - (-1)^{(\varepsilon_f+1)(\varepsilon_g+1)} (f \leftrightarrow g) , \quad r \in \{0, 1, 2, \ldots\} \),  
(4.32)

\( (f, g)_{(-s-3)}^{c} := f \left( \frac{\partial^r}{\partial c_\alpha} \frac{\partial^f}{\partial c_{\alpha_s}} - \frac{\partial^r}{\partial c_\alpha} \frac{\partial^f}{\partial c_{\alpha_s}} \right) g , \quad s \in \{0, \ldots, L\} \).  
(4.33)

Elementary considerations reveal the following useful Lemma 4.2 for the ghost sector \((\cdot, \cdot)^c\).

**Lemma 4.2** Let \(f, g\) be two functions of definite shifted antifield number. Then

\[ (f, g)_{(-s)}^{c} \neq 0 \quad \Rightarrow \quad \text{safn}(f) \geq s \lor \text{safn}(g) \geq s . \]  
(4.34)

Assume that \(f\) also has definite ghost number. Then

\[ (f, g)_{(-s)}^{c} \neq 0 \land \text{gh}(f) \geq -1 \quad \Rightarrow \quad \text{safn}(f) + \text{gh}(f) \geq s-1 . \]  
(4.35)

In particular,

\[ (f, g)_{(-s)}^{c} \neq 0 \land \text{gh}(f) = 0 \quad \Rightarrow \quad \text{safn}(f) \geq s-1 . \]  
(4.36)

The equations of motion expand as

\[ T_i = \sum_{r=0}^{\infty} T_{(r)i} , \quad \text{safn}(T_{(r)i}) = r . \]  
(4.37)

Eq. (4.3) implies that

\[ T_{(0)i}^{(4.3)} = 0 . \]  
(4.38)

Together with the Noether identity (2.1), this implies that

\[ T_{(1)i} R_{(0)a_0}^{i} = 0 . \]  
(4.39)

We also have

\[ \Lambda_{(0)i} R_{(0)a_0}^{i} \overset{(4.1)}{=} 0 . \]  
(4.40)

Eq. (4.40) implies that

\[ \frac{1}{2} (S_{(2)}, S_{(2)})_{(-3)} = \frac{1}{2} (S_{(2)}, S_{(2)})_{(0)_{(-3)}}^{\xi} = (S_{0(2)} \frac{\partial^r}{\partial \xi^r}) \Lambda_{(0)i} R_{(0)a_0}^{i} c_{\alpha_0} \overset{(4.40)}{=} 0 . \]  
(4.41)
4.3 Transversal and Longitudinal Ghosts and Antifields

The only purpose of this Section 4.3 is to device rotated auxiliary variables, in which the acyclicity of a shifted Koszul–Tate operator (4.51) becomes apparent, see Section 4.4. We stress that we do not use the rotated auxiliary variables thereafter. In particular, the rotated auxiliary variables do not enter the existence proofs in Sections 4.5–4.6.

We will first show by induction in the stage \( s \in \{0, \ldots, L\} \), that there exists a sequence of invertible rotation matrices \( \Lambda^\beta_s \alpha_s = \Lambda^\beta_s \alpha_s (\varphi) \), such that the first \( M_{s-1} \) rows in the rotated gauge(–for–gauge)\(^s\)-generator
\[
\Lambda^\beta_{s-1} \alpha_{s-1} Z^{\alpha_{s-1}} \approx \begin{bmatrix} 0 & 0 \\ 0 & M_{s-1} \times M_s \end{bmatrix} m_{s-1} \times m_s,
\]
vanishes weakly in some \( \varphi \)-neighborhood of the stationary \( \varphi \)-surface. The rotation matrix \( \Lambda^\beta_{s-1} \alpha_{s-1} \) in eq. (4.42) is fixed at the previous stage \( s - 1 \) of the induction proof. Eq. (4.42) for \( s = 0 \) is just eq. (4.1). From the rank condition (2.6), it follows that it is possible to find an invertible rotation matrix \( \Lambda^\beta_s \alpha_s = \Lambda^\beta_s \alpha_s (\varphi) \), such that the rotated gauge(–for–gauge)\(^s\)-generator is
\[
\bar{Z}^\beta_{s-1} \beta_s := \Lambda^\beta_{s-1} \alpha_{s-1} Z^{\alpha_{s-1}} \Lambda^{-1} \alpha_s \beta_s \approx \begin{bmatrix} 0 & 0 \\ 0 & M_{s-1} \times M_s \end{bmatrix} m_{s-1} \times m_s,
\]
(4.43)
Due to eq. (4.43) and the Noether relation (2.3), the induction assumption (4.42) is then satisfied for the next stage, and so forth.

One next defines rotated ghosts and antifields
\[
\begin{aligned}
\begin{bmatrix}
\xi^B_B \\
\delta^B_{B+1}
\end{bmatrix}
\equiv \begin{bmatrix}
\xi^B \\
\delta^B_{B+1}
\end{bmatrix} := \Lambda^\beta_{(0)} \alpha_s c^\alpha_s , \quad s \in \{0, \ldots, L\}, \\
\begin{bmatrix}
\xi^B B \infty \\
\delta^B B \infty +1
\end{bmatrix}
\equiv \begin{bmatrix}
\xi^B_j \\
\delta^B_{B_0}
\end{bmatrix} \equiv \varphi^s_j (\Lambda^{-1} \delta^i_{(0)} j), \\
\begin{bmatrix}
\xi^B B \infty \\
\delta^B B \infty +1
\end{bmatrix}
\equiv \begin{bmatrix}
\xi^B j \\
\delta^B_{B_0}
\end{bmatrix} := \Phi^s_j (\Lambda^{-1} \delta^i_{(0)} j), \quad s \in \{-1, \ldots, L\}. 
\end{aligned}
\]
(4.44)
(4.45)
(4.46)
The first \( M_s \) rotated ghosts are called transversal ghosts \( \bar{\xi} \) and the last \( M_{s+1} \) rotated ghosts are called longitudinal ghosts \( \bar{\delta} \), where \( M_s + M_{s+1} = m_s \). (Please do not confuse the notation for rotated ghosts \( \bar{\xi} \) with antighosts, which we do not consider here.) The antibrackets of rotated auxiliary variables are not completely on standard Darboux form, so the rotated antifields are not antifields in the strict sense of the word. In more detail, the antibrackets of rotated variables read
\[
(\bar{\xi}^i, \bar{\xi}^j) \approx \delta^i_j , \quad (\bar{\xi}^\alpha_s, \bar{\delta}^\beta_s) = \delta^\alpha_s \delta^\beta_s ,
\]
(4.47)
and other fundamental antibrackets vanish, except for the \( \theta^s_{(0)} \) sector, which can have non-trivial antibrackets with rotated auxiliary variables \( \bar{\xi} \) and \( \bar{\delta} \), because of \( \theta \)-dependence of the rotation matrix \( \Lambda^\alpha_{(0) \beta_s} = \Lambda^\alpha_{(0) \beta_s} (\theta) \). Rotated auxiliary variables \( \bar{\xi} \) and \( \bar{\theta} \) carry the same shifted and reduced antifield number as the unrotated auxiliary variables \( c \) and \( \Phi^s \), respectively.
4.4 Shifted Koszul–Tate Operator $s_{(-1)}$ and Reduced Koszul–Tate Operator $s_{-1[0]}$

The shifted Koszul–Tate operator $s_{(-1)}$ is defined as the leading shifted antifield number sector for the classical BRST operator

$$s := (S, \cdot) \overset{(4.36)}{=} \sum_{p=-1}^{\infty} s_p,$$

where

$$s_p = \sum_{r=0}^{\infty} (S^{(r)}, \cdot)_{(p-r)} \overset{(4.26)}{=} \sum_{r=2}^{\infty} (S^{(r)}, \cdot)_{(p-r)}, \quad \text{afn}(s_p) = p, \quad p \in \{-1, 0, 1, 2, \ldots\}.$$  

Similarly, the reduced Koszul–Tate operator $s_{-1[0]}$ is defined as the leading reduced antifield number sector for the Koszul–Tate operator

$$s_{-1} = \sum_{p=0}^{\infty} s_{-1[p]}.$$  

The shifted and reduced Koszul–Tate operator $s_{(-1)}$ and $s_{-1[0]}$ are equal, and of the form

$$s_{(-1)} = \sum_{r=0}^{\infty} (S^{(r)}, \cdot)_{(-r-1)} \overset{(4.26)}{=} \sum_{r=2}^{\infty} (S^{(r)}, \cdot)_{(-r-1)} = \sum_{s=1}^{L} V^{(s+2)\alpha_s} \frac{\partial^\ell}{\partial \Phi^*_{\alpha_s}}$$

$$= T_{(1)i} \frac{\partial^\ell}{\partial \varphi_i^*} + \sum_{s=0}^{L} \Phi^*_{\alpha_s} Z^{\alpha_s-1}_{(0)\alpha_s} \frac{\partial^\ell}{\partial \Phi^*_{\alpha_s}} = T_{[1]} \frac{\partial^\ell}{\partial \varphi_i^*} + \sum_{s=0}^{L} \Phi^*_{\alpha_s} Z^{\alpha_{s-1}}_{(0)\alpha_s} \frac{\partial^\ell}{\partial \Phi^*_{\alpha_s}}$$

$$\overset{(3.11)}{=} s_{-1[0]}.$$  

Here we have introduced shifted structure functions $V^{(s+2)\alpha_s}$ with

$$\text{gh}(V^{(s+2)\alpha_s}) = -(s+1), \quad \text{afn}(V^{(s+2)\alpha_s}) < \text{afn}(V^{(s+2)\alpha_s}) = s+2 \geq 1, \quad s \in \{-1, \ldots, L\}.  

It follows from the requirement

$$0 \leq \text{puregh}(V^{(s+2)\alpha_s}) = \text{gh}(V^{(s+2)\alpha_s}) + \text{afn}(V^{(s+2)\alpha_s}) < \text{gh}(V^{(s+2)\alpha_s}) + \text{afn}(V^{(s+2)\alpha_s}) = 1,$$

that the $V^{(s+2)\alpha_s}$ functions cannot depend on the ghosts “$c$”, and must contain precisely one antifield or transversal field in each term. In more detail, the $V^\alpha$ functions read

$$V_{(1)\alpha_{s-1}} = V_{(1)i} = T_{(1)i}, \quad V_{(2)\alpha_0} = \varphi_i^* R_{(0)\alpha_0},$$

$$V_{(s+2)\alpha_s} = \Phi^*_{\alpha_s} Z^{\alpha_{s-1}}_{(0)\alpha_s}, \quad s \in \{0, \ldots, L\}.  

Let us use the notation $\delta$ for the shifted (=reduced) Koszul–Tate operator $\delta := s_{(-1)} = s_{-1[0]}$. It is nilpotent because of the Noether identities (2.1)–(2.3).

Theorem 4.3 (Local acyclicity of the reduced Koszul–Tate operator) The operator cohomology of the reduced Koszul–Tate operator $\delta := s_{-1[0]}$ is acyclic in a local $\varphi$-neighborhood $U$ of each $\varphi$-point on the stationary $\varphi$-surface, i.e.,

$$\forall \text{ operators } X : \quad [\delta, X] = 0 \land \text{ afn}(X) \neq 0 \Rightarrow \exists Y : \ X = [\delta, Y].$$
Corollary 4.4 (Local acyclicity of the shifted Koszul–Tate operator) The cohomology of the shifted Koszul–Tate operator $\delta := s_{(-1)}$ is acyclic in a local $\varphi$-neighborhood $U$ of each $\varphi$-point on the stationary $\varphi$-surface, i.e.,

$$\forall \text{ functions } f : \quad (\delta f) = 0 \land \text{safn}(f) > 0 \Rightarrow \exists g : f = (\delta g). \quad (4.57)$$

Remark: Note that $(\delta f)$ or $\delta f$ are shorthand notations for $[\delta, f]$. The function $g$ in eq. (4.57) is unique up to an $\delta$-exact term, because of $\text{safn}(g) = \text{safn}(f) + 1 > 1$ and the acyclicity condition (4.57).

Proof of Corollary 4.4: Use that one can identify a function $f$ with the operator $L_f$ that multiplies with $f$ from the left, $L_f g := f g$. Now use Theorem 4.3 and the fact that strictly positive shifted antifield number implies strictly positive reduced antifield number $\text{safn}(f) > 0 \Rightarrow \text{rafn}(f) > 0$.

Proof of Theorem 4.3: In rotated variables, the shifted (=reduced) Koszul–Tate operator $\delta$ reads

$$\delta = \xi_I \frac{\partial^\ell}{\partial \xi_I} + \sum_{s=0}^{L} \tau^s_A \frac{\partial^\ell}{\partial \tau^s_A}, \quad (4.58)$$

where we have defined rotated transversal fields

$$\xi_I := (S_{0(2)} \rightarrow \frac{\partial}{\partial \xi_I}) = \xi^I H_{JI}, \quad \text{rafn}(\xi_I) = 1. \quad (4.59)$$

The dual operator reads

$$\bar{\delta} = \xi^I \frac{\partial^\ell}{\partial \xi^I} + \sum_{s=0}^{L} \tau_A^s \frac{\partial^\ell}{\partial \tau_A^s}, \quad \text{rafn}(\bar{\delta}) = 0. \quad (4.60)$$

The commutator is a Euler/conformal vector field

$$K := [\delta, \bar{\delta}] = \xi_I \frac{\partial^\ell}{\partial \xi_I} + \frac{\partial^\ell}{\partial \varphi_A^s}, \quad \text{rafn}(K) = 0. \quad (4.61)$$

The commutator $K$ commutes with both $\delta$ and $\bar{\delta}$. For instance, $[\delta, K] = [\delta, [\delta, \bar{\delta}]] = \frac{1}{2}[[\delta, \delta], \bar{\delta}] = 0$, due to the Jacobi identity. It is useful to define the operator

$$k := [K, \cdot], \quad [K, X] = \text{rafn}(X) X, \quad k(\ln f) = \frac{1}{f} k(f) = \text{rafn}(f). \quad (4.62)$$

Moreover, the kernel of the $k$ operator is precisely the sector with zero reduced antifield number,

$$k(X) = 0 \iff \text{rafn}(X) = 0. \quad (4.63)$$

The contracting homotopy operator $\delta^{-1}$ is defined as

$$\delta^{-1} X := \begin{cases} k^{-1} [\bar{\delta}, X] = [\delta, k^{-1} X] & \text{for } \text{rafn}(X) \neq 0, \\ 0 & \text{for } \text{rafn}(X) = 0. \end{cases} \quad (4.64)$$
In contrast to the dual operator \( \bar{\delta} \), the contracting homotopy operator \( \delta^{-1} \) is \textit{not} a linear derivation, \textit{i.e.}, \( \delta^{-1} \) does \textit{not} satisfy a (Grassmann–graded) Leibniz rule. One has

\[
\delta^{-1}[\delta, X] + [\delta, \delta^{-1} X] = \begin{cases} X & \text{for } \text{rafn}(X) \neq 0 , \\ 0 & \text{for } \text{rafn}(X) = 0 . \end{cases}
\] (4.65)

In particular, one may choose \( Y := \delta^{-1} X \) in eq. (4.56).

\( \square \)

Remark: The operator \( k \), when applied to an operator \( X \), does not change the auxiliary variables in \( X \). The operators \( \delta \) and \( \bar{\delta} \) act trivially on ghosts \( c \), but may annihilate or create a single original antifield \( \varphi^* \) (or its derivative).

\section{4.5 Existence of Koszul–Tate Operator \( s_{-1} \)}

Let there be given an original action \( S_0 \), gauge(–for–gauge)\(^n\)-generators \( Z^{\alpha_{s-1}} \), \( s \in \{0, \ldots, L\} \), an antisymmetric first–stage structure function \( B^i_{\alpha_1} \), and higher–stage structure functions \( B^i_{\alpha_s} \), \( s \in \{2, \ldots, L\} \), that satisfy the Noether identities (2.1)–(2.3). In particular, there are given action parts

\[
S^{\text{Noether}} := S^{\text{fixed}} - \sum_{s=2}^{L} (-1)^{s \epsilon(c^{i}_{s-2})} c^{*}_{s-2} \varphi^{*}_{i} B^{i}_{\alpha_s} \varphi^{*}_{s-2} c^{\alpha_s} ,
\] (4.66)

\[
S^{\text{fixed}} := S^{\text{quad}} + \frac{(-1)^{\ell \epsilon}}{2} \varphi^{*}_{j} \varphi^{*}_{i} B^{ij}_{\alpha_1} c^{\alpha_1} ,
\] (4.67)

\[
S^{\text{quad}} := S_0 + \sum_{s=0}^{L} \Phi^{*}_{\alpha_{s-1}} Z^{\alpha_{s-1}} c^{\alpha_s} .
\] (4.68)

Define

\[
V^{\text{Noether}}_{\alpha_s} := (S^{\text{Noether}} \frac{\partial}{\partial c^{\alpha_s}}) = V^{\text{fixed}}_{\alpha_s} - \begin{cases} (-1)^{s \epsilon(c^{i}_{s-2})} c^{*}_{s-2} \varphi^{*}_{i} B^{i}_{\alpha_s} \varphi^{*}_{s-2} c^{\alpha_s} & \text{for } 2 \leq s \leq L , \\ 0 & \text{for } 0 \leq s \leq 1 , \end{cases}
\] (4.69)

\[
V^{\text{fixed}}_{\alpha_s} := (S^{\text{fixed}} \frac{\partial}{\partial c^{\alpha_s}}) = V^{\text{quad}}_{\alpha_s} + \begin{cases} (-1)^{\ell \epsilon} \varphi^{*}_{j} \varphi^{*}_{i} B^{ij}_{\alpha_1} c^{\alpha_1} & \text{for } s = 1 , \\ 0 & \text{for } s \neq \pm 1 , \end{cases}
\] (4.70)

\[
V^{\text{quad}}_{\alpha_s} := (S^{\text{quad}} \frac{\partial}{\partial c^{\alpha_s}}) = \Phi^{*}_{\alpha_{s-1}} Z^{\alpha_{s-1}} c^{\alpha_s} \text{ for } 0 \leq s \leq L ,
\] (4.71)

\[
V^{\text{Noether}}_{\alpha_1} := V^{\text{fixed}}_{\alpha_s} := V^{\text{quad}}_{\alpha_1} := T := (S_0 \frac{\partial}{\partial \varphi^i}) .
\] (4.72)

\textbf{Theorem 4.5 (Local existence of Koszul–Tate operator \( s_{-1} \))} Let there be given an acyclic, nilpotent, reduced Koszul–Tate operator

\[
\delta := s_{-1[0]} = T[1] \frac{\partial}{\partial \varphi^s} + \sum_{s=0}^{L} \Phi^{*}_{\alpha_{s-1}} Z^{s_{-1}} \alpha_{s-1} \frac{\partial}{\partial c^{\alpha_s}} ,
\] (4.73)

that is defined in some \( \varphi \)-neighborhood of a \( \varphi \)-point on the stationary \( \varphi \)-surface, and with reduced antifield number zero. This guarantees the local existence (in some \( \varphi \)-neighborhood of the \( \varphi \)-point) of a nilpotent, acyclic Koszul–Tate operator

\[
\delta := s_{-1} = V_{\alpha} \frac{\partial}{\partial \Phi^i} ,
\] (4.74)
with antifield number minus one, and that satisfies the boundary condition

$$\nabla_\alpha = \nabla^{\text{fixed}}_\alpha + \mathcal{O}\left(\Phi^*e^*, (\Phi^*)^3\right).$$  \hspace{1cm} (4.75)

All such operators are of the form

$$\nabla^{\text{fixed}}_\alpha s = \nabla^{\text{fixed}}_\alpha s - (-1)^{\varepsilon(c^\alpha s - 2)} c^{s} \left( B_{ia s}^{\alpha} + R_{i \alpha}^{\alpha} X_{\alpha s}^{\alpha} + T_{j \alpha}^{\alpha} Y_{ia s}^{\alpha} + \mathcal{O}\left((e^*)^2, (\Phi^*)^3\right)\right),$$  \hspace{1cm} (4.76)

where $Y_{ia s}^{\alpha} = -(-1)^{\varepsilon_i \varepsilon_j} Y_{ij \alpha s}^{a}.$

**Proof of Theorem 4.5:** Let us, for notational reasons, put a bar on top of the sought–for Koszul–Tate operator $\bar{\pi}_{-1},$ and no bar on quantities associated with the given boundary conditions (4.66)–(4.72). We use $\Delta$ to denote differences, e.g., $\Delta V^{\text{Noether}} = \nabla^{\text{Noether}}_\alpha - V^{\text{Noether}}_\alpha,$ $\Delta V^{\text{fixed}} = \nabla^{\text{fixed}}_\alpha - V^{\text{fixed}}_\alpha,$ and so forth. We shall below inductively define the bar solution $\bar{s}_{-1}$ to all orders in the reduced antifield number, but initially, we only fix the zeroth–order part $\bar{s}_{-1[0]} = \delta$ to be equal to the reduced Koszul–Tate operator $\delta.$ The rth nilpotency expression $\bar{\nabla}_{[r]}$ for a bar solution $\bar{s}_{-1}$ can then be written as

$$\bar{\nabla}_{[r]} := \frac{1}{2} \sum_{p=0}^{r} \left[ \bar{s}_{-1[p]}, \bar{s}_{-1[r-p]} \right] = \left\{ \begin{array}{ll}
\frac{1}{2} [\delta, \delta] = 0 & \text{for } r = 0 , \\
[\delta, \bar{s}_{-1[r]}] + \bar{B}_{[r]} & \text{for } r \geq 1 ,
\end{array} \right.$$  \hspace{1cm} (4.77)

where

$$\bar{B}_{[r]} := \frac{1}{2} \sum_{p=1}^{r-1} \left[ \bar{s}_{-1[p]}, \bar{s}_{-1[r-p]} \right], \quad r \geq 1 .$$  \hspace{1cm} (4.78)

The $\bar{B}_{[r]}$ operator (4.78) is a linear derivation, since it is a commutator of linear derivations. It cannot contain derivatives with respect to original antifields $\varphi^*_i,$ since $\text{afn}(\bar{B}_{[r]}) = -2.$ Hence the $\bar{B}_{[r]}$ operator (4.78) is of the form

$$\bar{B}_{[r]} = \sum_{s=0}^{L} \bar{B}_{[r+1][s]} \frac{\partial^{\hat{q}}}{\partial \Phi^*_s}. $$  \hspace{1cm} (4.79)

The proof of the main statement is an induction in the reduced antifield number $r \geq 1.$ Assume that there exists a bar solution

$$\bar{s}_{-1[p]} = \nabla^{\text{Noether}}_{[p+1][s]} \frac{\partial^{\hat{q}}}{\partial \Phi^*_s}, \quad \text{rafn}(\bar{s}_{-1[p]}) = p , \quad p \in \{0,1,\ldots,r-1\} ,$$  \hspace{1cm} (4.80)

$$\nabla^{\text{Noether}}_{[p][s]} = \nabla^{\text{fixed}}_{[p][s]} + \mathcal{O}\left((e^*)^2, (\Phi^*)^3\right), \quad \text{rafn}(\nabla^{\text{fixed}}_{[p][s]}) = p , \quad p \in \{1,2,\ldots,r\} ,$$  \hspace{1cm} (4.81)

such that the boundary condition

$$\Delta V^{\text{fixed}}_{[p][s]} = 0 , \quad p \in \{1,2,\ldots,r\} ,$$  \hspace{1cm} (4.82)

is fulfilled, such that

$$\Delta B^{ia s}_{[p][s]} = \sum_{q=0}^{p} R^{i}_{[p-q][q]} X_{[q][s]}^{\alpha} + \sum_{q=0}^{p-1} T_{[p-q][q]}^{ij s} Y_{ia s}^{j}, \quad p \in \{0,1,\ldots,r-2\} ,$$  \hspace{1cm} (4.83)

$$Y_{[p][s]}^{ij s} = -(-1)^{\varepsilon_i \varepsilon_j} Y_{[p][s]}^{ijs}, \quad p \in \{0,1,\ldots,r-3\} , \quad s \in \{2,3,\ldots,L\} ,$$  \hspace{1cm} (4.84)

and such that the nilpotency holds up to the order $r-1$ in reduced antifield number

$$0 = \bar{\nabla}_{[0]} = \bar{\nabla}_{[1]} = \ldots = \bar{\nabla}_{[r-1]} .$$  \hspace{1cm} (4.85)
It follows from the induction assumption that the $\mathcal{B}_{[r]}$ operator (4.78) exists. The Jacobi identity $\mathcal{J}$ gives

$$0 = \mathcal{J}_{[r]} = \frac{1}{2} \mathcal{R}_{[1]}, [\mathcal{R}_{[1]}, \mathcal{R}_{[1]}]_{[r]} = \sum_{p=0}^{\infty} [\mathcal{R}_{[p]}], \mathcal{R}_{[1-p]}]_{[r]} \quad (4.85) = [\delta, \mathcal{R}_{[r]}] \quad (4.77) = [\delta, \mathcal{B}_{[r]}] \ . \quad (4.86)$$

Hence the $\mathcal{B}_{[r]}$ operator (4.78) is $\delta$-closed. Let us tentatively define

$$\mathcal{R}_{[1]} := -\delta^{-1} \mathcal{B}_{[r]} \ , \quad (4.87)$$

cf. definition (4.64). It follows from eq. (4.65) that the $r$th nilpotency relation (4.77) is fulfilled

$$\mathcal{R}_{[r]} = \mathcal{B}_{[r]} + [\delta, \mathcal{R}_{[1]}] \quad (4.87) = \mathcal{B}_{[r]} - [\delta, \delta^{-1} \mathcal{B}_{[1]}] \quad (4.65) = \delta^{-1}[\delta, \mathcal{B}_{[r]}] \quad (4.86) = 0 \ , \quad (4.88)$$

since $r \neq 0$. It is easy to see that the $\mathcal{R}_{[1]}$ operator (4.87) is a linear derivation. It cannot contain derivatives with respect to the rotated transversal coordinates $\bar{\xi}_i$, since $\text{afn}(\mathcal{R}_{[1]}) = -1$. Hence the $\mathcal{R}_{[1]}$ operator (4.87) is of the form

$$\mathcal{R}_{[1]} = V_{[r+1]}^\alpha \frac{\partial}{\partial \varphi_{\alpha}} \ . \quad (4.89)$$

This choice $\mathcal{R}_{[1]}$ may not meet the prescribed boundary condition (4.75). Let us probe the difference in terms of cohomology.

$$[\delta, \Delta V_{[r+1]}^\alpha] \frac{\partial}{\partial \varphi_{\alpha}} = 0 \ , \quad (4.90)$$

$$\Delta V_{[r+1]}^\alpha \frac{\partial}{\partial \varphi_{\alpha}} = [\delta, \Delta V_{[r+1]}^\alpha] \frac{\partial}{\partial \varphi_{\alpha}} \nonumber$$

$$= (\Delta T_{[r+1]}^i R_{[0]}^{i, \alpha_0} + T_{[1]}^i \Delta R_{[0]}^i, \alpha_0) \frac{\partial}{\partial \varphi_{\alpha}} \quad (2.1) = -\Delta \sum_{p=2}^r T_{[p]}^i R_{[r+1-p]}^i, \alpha_0 \frac{\partial}{\partial \varphi_{\alpha}} \quad (4.82) = 0 \ , \quad (4.91)$$

$$\Delta V_{[r+1]}^\alpha \frac{\partial}{\partial \varphi_{\alpha}} = [\delta, \Delta V_{[r+1]}^\alpha] \frac{\partial}{\partial \varphi_{\alpha}} \nonumber$$

$$= \varphi^T \left( \Delta R_{[0]}^i, \alpha_0 Z_{[0]}^i, \alpha_0 + R_{[0]}^i, \alpha_0 \Delta Z_{[r]}^i, \alpha_0 - T_{[1]}^i \Delta B_{[r]}^{i, 1, \alpha_0} \right) \frac{\partial}{\partial \varphi_{\alpha}} \quad (4.82) = 0 \ , \quad (4.92)$$

$$\Delta V_{[r+1]}^\alpha \frac{\partial}{\partial \varphi_{\alpha}} = [\delta, \Delta V_{[r+1]}^\alpha] \frac{\partial}{\partial \varphi_{\alpha}} + O \left( (\Phi^*)^2 \frac{\partial}{\partial \varphi_{\alpha}} \right) \nonumber$$

$$= c_{\alpha-2} \Delta Z_{[r]}^i, \alpha_0 \frac{\partial}{\partial \varphi_{\alpha}} + [\delta, \Delta V_{[r+1]}^\alpha] \frac{\partial}{\partial \varphi_{\alpha}} \nonumber$$

$$= c_{\alpha-2} \Delta Z_{[r]}^i, \alpha_0 \frac{\partial}{\partial \varphi_{\alpha}} \quad (2.3) \ . \quad (4.93)$$

$$= c_{\alpha-2} \Delta \left( \sum_{p=2}^r T_{[p]}^i B_{[r-p]}^i, \alpha_0 - \sum_{p=1}^{r-1} T_{[p]}^i Z_{[r-p]}^i, \alpha_0 \right) \frac{\partial}{\partial \varphi_{\alpha}} = c_{\alpha-2} \sum_{p=2}^r T_{[p]}^i B_{[r-p]}^i, \alpha_0 \frac{\partial}{\partial \varphi_{\alpha}}$$

$$= \Delta \left( \sum_{p=2}^r T_{[p]}^i B_{[r-p]}^i, \alpha_0 - \sum_{p=1}^{r-1} T_{[p]}^i Z_{[r-p]}^i, \alpha_0 \right) \frac{\partial}{\partial \varphi_{\alpha}} = 0 \ . \quad (4.94)$$
\[ (4.83) \quad c^*_{\alpha s - 2} \left( \sum_{p=2}^{r} T[p] \sum_{q=0}^{r-p} R[q] \alpha \theta^{\alpha_{s-2}} + \sum_{p=2}^{r-1} T[p] \sum_{q=0}^{r-1-p} T[q] \alpha \theta^{j_{\alpha_{s-2}}} - \sum_{p=2}^{r-1} T[p] \sum_{q=0}^{r-1-p} T[q] \alpha \theta^{j_{\alpha_{s-2}}} \right) \frac{\partial \ell}{\partial c^*_{\alpha s}} = \]
\[ c^*_{\alpha s - 2} \left( \sum_{p=2}^{r} T[p] \sum_{q=0}^{r-p} R[q] \alpha \theta^{\alpha_{s-2}} + \sum_{p=2}^{r-3} T[p] \sum_{q=0}^{r-3-p} T[q] \alpha \theta^{j_{\alpha_{s-2}}} \right) \frac{\partial \ell}{\partial c^*_{\alpha s}} \]
\[ = -c^*_{\alpha s - 2} T[1] B[i] \frac{\partial \ell}{\partial c^*_{\alpha s}} = -c^*_{\alpha s - 2} \left[ \delta, \varphi_{[r-1] \alpha s} B[i] \right] \frac{\partial \ell}{\partial c^*_{\alpha s}} , \] (4.93)

where
\[ B[i] := \sum_{q=0}^{r-2} R[i-1] q \alpha \theta^{\alpha_{s-2}} + \sum_{q=0}^{r-3} T[q] \alpha \theta^{j_{\alpha_{s-2}}} \] (4.94)

If one adds together eqs. (4.90)–(4.93), one gets
\[ \left[ \delta, \Delta s^{\text{Noether}} + B[i] \right] = \mathcal{O} \left( \Phi^* \right)^2 \frac{\partial \ell}{\partial c^*} , \] (4.95)
\[ \Delta s^{\text{Noether}} := \Delta V^{\text{Noether}} + B[i] \]
\[ \Delta s := \Delta V^{\text{Noether}} + B[i] \]
\[ B[i] := \sum_{s=2}^{L} (-1)^{s} c_{\alpha s - 2} J_{s} \varphi_{[r-1] \alpha s} B[i] \frac{\partial \ell}{\partial c^*_{\alpha s}} \] (4.96)

Now redefine the \( R_{-1}[r] \) solution by a \( \delta \)-exact amount
\[ R_{-1}[r] \rightarrow R_{-1}[r] - \left[ \delta, \delta^{-1} \left( \Delta s^{\text{Noether}} + B[i] \right) \right] , \] (4.97)

cf. definition (4.64). The new \( R_{-1}[r] \) operator (4.97) is still a linear derivation that satisfies eq. (4.88), but now
\[ \Delta s^{\text{Noether}} \rightarrow \Delta s^{\text{Noether}} - \left[ \delta, \delta^{-1} \left( \Delta s^{\text{Noether}} + B[i] \right) \right] \]
\[ = \delta^{-1} \left[ \delta, \Delta s^{\text{Noether}} + B[i] \right] - B[i] \frac{\partial \ell}{\partial c^*} = \mathcal{O} \left( \Phi^* \right)^2 \frac{\partial \ell}{\partial \Phi^*} , \] (4.98)

so now the following boundary condition is satisfied as well
\[ \Delta V^{\text{quad}} := V^{\text{quad}}_{[r+1] \alpha} - V^{\text{quad}}_{[r+1] \alpha} = 0 \] (4.99)
or equivalently,
\[ \Delta T_{[r+1] i} = 0 , \quad \Delta R[i] = 0 , \quad \Delta Z[i] = 0 , \quad s \in \{ 1, \ldots, L \} , \] (4.100)

cf. eq. (4.65). It still remains to show that the boundary condition \( \Delta B[i]_{-1}[r] = 0 \) can be achieved (after an appropriate \( \delta \)-exact shift of the \( R_{-1}[r] \) solution (4.97)). To this end, repeat the calculation (4.92) with the knowledge (4.100), and conclude that
\[ \left[ \delta, \Delta V_{[r+1] \alpha} \right] = 0 , \quad \Delta V_{[r+1] \alpha} = \frac{(-1)^{s_j}}{2} \phi_{j} \phi_{i} \Delta B[i]_{[r-1] \alpha} , \] (4.101)

Now redefine the \( R_{-1}[r] \) solution (4.97) by a \( \delta \)-exact amount
\[ R_{-1}[r] \rightarrow R_{-1}[r] - \left[ \delta, \delta^{-1} \left( \Delta V_{[r+1] \alpha} \right) \right] \]
Table 2: Table over the antibracket \((f, g)\) of various functions \(f\) and \(g\) with ghost number zero, \(\text{gh}(f) = 0 = \text{gh}(g)\).

| \(f\) | \(g\) | \(\mathcal{O}(\Phi^c)\) | \(\mathcal{O}((\Phi^c)^2)\) | \(\mathcal{O}(c^2)\) |
|-------|-------|-----------------|-----------------|-----------------|
| \(S_0\) | 0 | \(\mathcal{O}(c)\) | \(\mathcal{O}(\Phi^\xi c)\) | \(\mathcal{O}(c^2)\) |
| \(\mathcal{O}(\Phi^c)\) | \(\mathcal{O}(\Phi^c)\) | \(\mathcal{O}((\Phi^c)^2)\) | \(\mathcal{O}(c^2)\) |
| \(\mathcal{O}((\Phi^c)^2)\) | \(\mathcal{O}((\Phi^c)^2)\) | \(\mathcal{O}((\Phi^c)^2c^2)\) | \(\mathcal{O}(c^2)\) |
| \(\mathcal{O}(c^2)\) | | | | |

\[ \delta := s_{(-1)} = T_{(1)_s} \frac{\partial}{\partial \varphi_i^s} + \sum_{s=0}^{L} \Phi_{\alpha_{s-1}} \frac{\partial}{\partial c_{\alpha_s}}, \quad (4.107) \]

The second and third term on the right-hand side of eq. (4.102) will change the structure functions \(\overline{B}_{[r-1]_\alpha}^j\) and \(\overline{B}_{[r-1]_\alpha}^{\alpha_0}\), respectively. In detail,

\[ \Delta V_{[r+1]_\alpha} \longrightarrow \Delta V_{[r+1]_\alpha} - \left[ \delta, \delta^{-1} \Delta V_{[r+1]_\alpha} \right] (4.65) \]

so now at least \(\Delta B_{[r-1]_\alpha}^{ij} = 0\), and therefore the boundary condition (4.75) is fulfilled,

\[ \Delta V_{[r+1]_\alpha}^{\text{fixed}} := \overline{V}_{[r+1]_\alpha}^{\text{fixed}} - V_{[r+1]_\alpha}^{\text{fixed}} = 0. \]

Finally, repeat calculation (4.93) with the knowledge (4.100), and conclude that

\[ [\delta, \varphi_i^s \Delta B_{[r-1]_\alpha}^{\alpha_{s-2}}] = [\delta, \varphi_i^s \overline{B}_{[r-1]_\alpha}^{\alpha_{s-2}}], \quad s \in \{2, \ldots, L\}. \]

In other words, there exist structure functions \(X_{[r-1]_\alpha}^{\alpha_{s-2}}\) and \(Y_{[r-2]_\alpha}^{\beta_{s-2}}\) such that

\[ \Delta B_{[r-1]_{\alpha_s}} = \overline{B}_{[r-1]_{\alpha_s}} + R^i_{[r-1]_{\alpha_s}} X_{[r-1]_{\alpha_s}} + T^j_{[r-1]_{\beta_s}} Y_{[r-1]_{\beta_s}} = \sum_{q=0}^{r-1} R^i_{[r-1-q]_{\alpha_s}} X_{[q]_{\alpha_s}} + \sum_{q=0}^{r-2} T_{[r-1-q]_{\beta_s}} Y_{[q]_{\beta_s}}, \]

which is induction assumption (4.83) for the next step \(p = r - 1\).

\[ \square \]

### 4.6 Existence of Proper Action \(S\)

Let there be given an original action \(S_0\), gauge-(for-gauge)\(^s\)-generators \(Z^\alpha_{\alpha_{s-1}}\), \(s \in \{0, \ldots, L\}\), an antisymmetric first-stage structure function \(B_{ij}^\alpha\), and higher-stage structure functions \(B_{\alpha_{s-2}}^\alpha\), \(s \in \{2, \ldots, L\}\), that satisfy the Noether identities (2.1)–(2.3).

**Theorem 4.6 (Local existence of proper action \(S\))** Let there be given a nilpotent, acyclic shifted Koszul–Tate operator

\[ \delta := s_{(-1)} = T_{(1)_s} \frac{\partial}{\partial \varphi_i^s} + \sum_{s=0}^{L} \Phi_{\alpha_{s-1}} \frac{\partial}{\partial c_{\alpha_s}}, \quad (4.107) \]
The proof of the main statement is an induction in the shifted antifield number. All such solutions are of the form

\( \mathcal{S} = S^{\text{fixed}} + \mathcal{O}\left((\Phi^*)^2, (\Phi^*)^3, c^2\right) \) .

(4.108)

First proof of Theorem 4.6 using shifted antifield number: Combine Theorem 3.4 and Theorem 4.5.

\[ \square \]

Second proof of Theorem 4.6 using shifted antifield number: Let us, for notational reasons, put a bar on top of the sought–for proper action \( \mathcal{S} \), and no bar on quantities associated with the given boundary conditions (4.66)–(4.68). We use \( \Delta \) to denote differences, e.g., \( \Delta S^{\text{Noether}} := S^{\text{Noether}} - S^{\text{Noether}}, \Delta S^{\text{fixed}} := S^{\text{fixed}} - S^{\text{fixed}}, \) and so forth. We shall below inductively define the bar solution \( \mathcal{B} \) to all orders in the shifted antifield number, but initially, we only fix the zeroth–order, first–order and second–order part as

\( \mathcal{S}(0) := S(0), \quad \mathcal{S}(1) := S(1), \quad \mathcal{S}(2) := S(2), \)

(4.110)

cf. eq. (4.26). The \( r \)th classical master expression \( \overline{\text{CME}}(r) \) for a bar solution \( \mathcal{S} \) can be written as

\( \overline{\text{CME}}(r) := \frac{1}{2} \sum_{p,q \geq 0} (\mathcal{S}(p), \mathcal{S}(q))_{(r-p-q)} \)

(4.111)

\[ \overline{\text{CME}}(r) = \frac{1}{2} \sum_{p,q \geq 0} (\mathcal{S}(p), \mathcal{S}(q))_{(r-p-q)} \]

where

\( \overline{\mathcal{B}}(r) := \frac{1}{2} \sum_{2 \leq p, q \leq r} (\mathcal{S}(p), \mathcal{S}(q))_{(r-p-q)} , \quad r \geq 0 \).

(4.112)

The proof of the main statement is an induction in the shifted antifield number \( r \geq 2 \). Assume that there exist a bar solution

\( \overline{\mathcal{S}}(p) = \overline{\mathcal{S}}(p) + \mathcal{O}\left((c^*), (\Phi^*)^3, c^2\right) , \quad p \in \{0, 1, \ldots, r\} ,

(4.113)

such that the boundary condition

\( \Delta S^{\text{fixed}}_{(p)} = 0 , \quad p \in \{0, 1, \ldots, r\} ,

(4.114)

is fulfilled, such that

\( \Delta B^{(p)}_{(p)\alpha_s} = \sum_{q=0}^p R_{(p-q)0} X_{(q)\alpha_s} + \sum_{q=0}^{p-1} T_{(p-q)j} Y_{(q)\alpha_s} , \quad p \in \{0, 1, \ldots, r-s-3\} ,

(4.115)
\[ Y_{(p)}^{i,j\alpha_{s-2}} = -(-1)^{\frac{r_c r_j}{2}} Y_{(p)}^{i,j\alpha_{s-2}}, \quad p \in \{0, 1, \ldots, r-s-4\}, \quad s \in \{2, 3, \ldots, L\}, \]

and such that the classical master equation holds up to a order \( r-1 \) in shifted antifield number

\[ 0 = \text{CME}_{(0)} = \text{CME}_{(1)} = \ldots = \text{CME}_{(r-1)}. \]

The action \( S(p) = S(p)(\Phi^\alpha; \varphi^*_r, c^*_a, \ldots, c^*_a) \) cannot depend on antifields \( c^*_a \), for \( p-3 < q \leq L \), because their shifted antifield number \( \text{sf}(c^*_a) = q+3 \) is too big. It follows from the induction assumption that the \( \mathcal{B}(r) \) function (4.112) exists, and that it is a function \( \mathcal{B}(r) = \mathcal{B}(r)(\Phi^\alpha; \varphi^*_r, c^*_a, \ldots, c^*_a) \). We want to prove that there exists \( \mathcal{S}(r+1) = \mathcal{S}(r+1)(\Phi^\alpha; \varphi^*_r, c^*_a, \ldots, c^*_a) \), such that \( \text{CME}_{(r)} = 0 \). The Jacobi identity \( \mathcal{J} \) gives

\[ 0 = \mathcal{J}_{(r-1)} = \sum_{p=1}^{\infty} S(p) \text{CME}_{(r-p-1)} = \mathcal{S}_{(-1)} \text{CME}_{(r)} = \mathcal{S}_{(-1)} \mathcal{B}_{(r)} = \delta \mathcal{B}_{(r)} \cdot \]

In the last equality of eq. (4.118) is used that the two shifted Koszul–Tate operators \( \delta \equiv s_{(-1)} \) and \( \mathcal{S}_{(-1)} \) agree on functions \( f = f(\Phi^\alpha; \varphi^*_r, c^*_a, \ldots, c^*_a) \), due to the induction assumption (4.114).

\[ \Delta s_{(-1)} = \sum_{p=1}^{\infty} (\Delta S_{(p)}^{\text{quad}}, \cdot )_{(-p-1)} = \sum_{p=r+1}^{\infty} (\Delta S_{(p)}^{\text{quad}}, \cdot )_{(-p-1)} = \sum_{p=r+1}^{\infty} (\Delta S_{(p)}^{\text{quad}}) \frac{\partial^p}{\partial c^{\alpha_{p-2}}} \frac{\partial^p}{\partial c^{\alpha_{p-2}}} \cdot \]

Hence the \( \mathcal{B}(r) \) function (4.112) is \( \delta \)-closed. The acyclicity condition (4.57) then shows that there exists a function \( \mathcal{S}(r+1) \) such that

\[ -\delta \mathcal{S}_{(r+1)} = \mathcal{B}_{(r)}, \]

because \( r > 0 \). The \( r \)th classical master equation is then satisfied

\[ \text{CME}_{(r)} = \mathcal{S}_{(-1)} \mathcal{S}_{(r+1)} + \mathcal{B}_{(r)} = \delta \mathcal{S}_{(r+1)} + \mathcal{B}_{(r)} = 0. \]

This choice \( \mathcal{S}_{(r+1)} \) may not meet the prescribed boundary condition (4.108). Let us probe the difference in terms of cohomology.

\[ \delta \Delta S_{0(r+1)} = 0, \]

\[ \delta \Delta S_{1(r+1)} = T_{(1)} \delta R^i_{(r+1)} a_0 c^{\alpha_0} = -\Delta \sum_{p=2}^{r} T_{(p)}^i R^i_{(r-p-1)} a_0 c^{\alpha_0} = 0, \]

\[ \delta \Delta S_{2(r+1)} = \varphi^*_r \left( R^i_{(r-2)} \partial Z^{\alpha_0}_{(r-2)} a_1 - T_{(1)} \partial B^i_{(r-3)} a_1 \right) c^{\alpha_1} = 0, \]

\[ \delta \Delta S_{\text{Noether}}_{s(r+1)} + O((\Phi^*)^2) = \delta \Delta S_{\text{fixed}}_{s(r+1)} - c_{\alpha_{s-3}} \delta \varphi^*_r \Delta B^i_{(r-s-1)a_1} c^{\alpha_{s-1}} = 0, \]

\[ c_{\alpha_{s-3}} \left( Z^{\alpha_{s-3}}_{(0)} \partial Z^{\alpha_{s-2}}_{(r-s)} a_{s-1} - T_{(1)} \partial B^i_{(r-s-1)} a_1 \right) c^{\alpha_{s-1}} = 0, \]

\[ c_{\alpha_{s-3}} \left( \sum_{p=2}^{r-s} T_{(p)}^i B^i_{(r-s-p)} a_{s-1} - \sum_{p=1}^{r-s} Z^{\alpha_{s-3}}_{(r-s-p)} a_{s-1} \right) c^{\alpha_{s-1}} = 0, \]

\[ c_{\alpha_{s-3}} \sum_{p=2}^{r-s} T_{(p)}^i B^i_{(r-s-p)} a_{s-1} c^{\alpha_{s-1}} = 0. \]
which is induction assumption (4.115) for the next step $p = r - s - 2$. That is, one now repeats the calculation (4.125) with $\Delta_i^{\alpha_2-3}$.

One may now repeat the calculation (4.125) with $\Delta Z_{(s-2)\alpha^{a+1}}^{\alpha s}$.

In other words, there exist structure functions $X_{\alpha^{a+2}a}$ and $Y_{(s-3)\alpha^{a+1}}^{ij\alpha s-2}$ such that

$$\Delta B_{(s-2)\alpha^{a+1}}^{\alpha s} = \tilde{B}_{(s-2)\alpha^{a+1}}^{\alpha s} + R_{(0)\alpha\alpha_0} X_{(s-2)\alpha^{a+1}}^{\alpha s_0} + T_{(1)j} Y_{(s-3)\alpha^{a+1}}^{ij\alpha s-2}$$

$$= \sum_{q=0}^{r-s-2} R_{(s-2-q)\alpha\alpha_0} X_{(s-2)\alpha^{a+1}}^{\alpha s_0} + \sum_{q=0}^{r-s-3} T_{(s-2-q)j} Y_{(s-3)\alpha^{a+1}}^{ij\alpha s-2},$$

which is induction assumption (4.115) for the next step $p = r - s - 2$. \qed
ACKNOWLEDGEMENT: We thank Poul Henrik Damgaard and Marc Henneaux for discussion. We would like to thank Poul Henrik Damgaard, the Niels Bohr Institute and the Niels Bohr International Academy for warm hospitality. I.A.B. would like to thank Marc Henneaux, Glenn Barnich and Université Libre de Bruxelles for warm hospitality. K.B. would also like to thank M. Vasiliev and the Lebedev Physics Institute for warm hospitality. The work of I.A.B. is supported by grants RFBR 08–01–00737, RFBR 08–02–01118 and LSS–1615.2008.2. The work of K.B. is supported by the Ministry of Education of the Czech Republic under the project MSM 0021622409.

A From Acyclicity to Nilpotency of Koszul–Tate Operator

Let the $r$th complex be the set of functions $f = f(\varphi^i; \varphi^*_i, c^*_a, \ldots, c^*_r)$ that does not depend on $c^*_{a+1}$, $c^*_{a+2}$, $c^*_{a+3}$, ... .

Lemma A.1 (Nilpotent extension to the next stage) Let there be given a positive integer $r \geq 1$. Let there be given a nilpotent Koszul–Tate operator $s_{-1}$ on the $r$th complex in a tubular $\varphi$-neighborhood of the stationary $\varphi$-surface, such that it is acyclic on the $(r-2)$th complex. Let there also be given a $(r+1)$th stage Noether identity

$$Z^{a_{r-1}}_r Z^{a_r}_r = T_i B_{i_{a_r}}^{a_{r-1}}, \quad T_i := (S_0 \frac{\partial}{\partial \varphi^i}). \quad (A.1)$$

Then there exists a nilpotent extension of the Koszul–Tate operator $s_{-1}$ to the $(r+1)$th complex in some tubular $\varphi$-neighborhood of the stationary $\varphi$-surface, so that

$$s_{-1} c^*_r = c^*_r Z^{a_r}_r + O \left( (\Phi^*)^2 \right). \quad (A.2)$$

All such extensions must be of the form

$$s_{-1} c^*_r = c^*_r Z^{a_r}_r - (-1)^{\varepsilon(c^*_r)} \varepsilon c^*_{a_r-1} \varphi^*_i B_{i_{a_r}}^{a_{r-1}} + O \left( (c^*)^2, (\Phi^*)^3 \right), \quad (A.3)$$

where $Y^{ij}_{a_{r+1}} = -(-1)^{\varepsilon \varepsilon} Y^{ij}_{a_{r+1}}$.

Remark The Lemma A.1 says nothing about if the nilpotent Koszul–Tate extension is also acyclic on the $(r-1)$th complex. Instead, we find that the shifted Koszul–Tate operator is better suited to address the issue of acyclicity. Nevertheless, the Lemma A.1 makes perfectly clear that nilpotency is not the bottleneck in the Koszul–Tate construction (acyclicity is!), and that the higher–stage gauge(–for–gauge)$(r+1)$–generators $Z^{a_r}_r$ can be preserved in the Koszul–Tate operator $s_{-1}$. The latter point has received very little attention in the literature, see e.g., Theorem 3 and Theorem 4 in Ref. [13].

Proof of Lemma A.1: Define

$$U_{a_{r+1}} := c^*_r Z^{a_r}_r - (-1)^{\varepsilon(c^*_r)} \varepsilon c^*_{a_r-1} \varphi^*_i B_{i_{a_r}}^{a_{r-1}}, \quad (A.4)$$

$$B_{a_{r+1}} := s_{-1} U_{a_{r+1}} \quad (A.4)$$

$$= s_{-1} \left( c^*_r Z^{a_r}_r - (-1)^{\varepsilon(c^*_r)} \varepsilon c^*_{a_r-1} \varphi^*_i B_{i_{a_r}}^{a_{r-1}} \right) \quad (A.1)$$

$$= \left( c^*_r Z^{a_r}_r + M_{a_r} \right) Z^{a_r}_r - c^*_{a_r-1} T_i B_{i_{a_r}}^{a_{r-1}} - (-1)^{\varepsilon(c^*_r)} V_{a_{r-1}} \varphi^*_i B_{i_{a_r}}^{a_{r-1}}$$

$$= M_{a_r} Z^{a_r}_r - (-1)^{\varepsilon(c^*_r)} V_{a_{r-1}} \varphi^*_i B_{i_{a_r}}^{a_{r-1}} = O \left( (\Phi^*)^2 \right). \quad (A.5)$$
The $B_{α+1}$ function (A.5) belongs to the $r+1$th complex, since both the functions
\[ M_{αr} = M_{αr}(ϕ; ϕ^*, c^*_0, ..., c^*_r) \quad \text{and} \quad V_{αr-1} = V_{αr-1}(ϕ; c^*_0, ..., c^*_r) \] (A.6)
belong to that. Moreover $\text{afn}(B_{α+1}) = r+3$. Because of acyclicity, there exists a function $\tilde{M}_{α+1} = \tilde{M}_{α+1}(ϕ; c^*, c^*_0, ..., c^*_r)$ in the $(r+1)$th complex, such that $s_1 \tilde{M}_{α+1} = B_{α+1}$. It follows moreover that $\text{afn}(\tilde{M}_{α+1}) = r+2 = \text{afn}(c^*_r)$, so that $\tilde{M}_{α+1}$ cannot contain terms that are first or zeroth order in antifields. Hence $\tilde{M}_{α+1} = \mathcal{O}(c^*_r)$ is of the form
\[ \tilde{M}_{α+1} = (-1)^{ε^*_r} c^*_r ϕ^*_r B_{α+1}^0 + \mathcal{O}(c^*_r, (ϕ^*)^3). \] (A.7)
Therefore
\[ B_{α+1} = s_1 \tilde{M}_{α+1} = c^*_r T_1 B_{α+1}^0 + \mathcal{O}(ϕ^*). \] (A.8)
Since $B_{α+1} = \mathcal{O}(ϕ^*)$, it follows that
\[ T_1 B_{α+1}^0 = 0, \] (A.9)
and therefore there exist functions $X_{α+1}^0$ and $Y_{α+1}^0 = -(-1)^{ε_3} Y_{α+1}$ such that
\[ B_{α+1}^0 = R_{α+1}^0 X_{α+1}^0 + T_1 Y_{α+1}. \] (A.10)
Now define
\[ s_1 c^*_r := U_{α+1} - \tilde{M}_{α+1}. \] (A.11)
It is nilpotent, because
\[ s_1^2 c^*_r = s_1 (U_{α+1} - \tilde{M}_{α+1}) = B_{α+1} - B_{α+1} = 0. \] (A.12)

**B** Elimination of $B$-Terms by Off–Shell Change of Generators

If the number of stages is finite $L < ∞$, it is possible to apply off-shell changes to the gauge(–for–gauge)$^s$–generators
\[ Z^{αs-1}_β := Z^{αs-1}_β, \quad s ∈ \{0, ..., L\}, \] (B.1)
so that the higher–stage Noether identities (2.3) becomes strong
\[ Z^{αs-1}_β Z^{αs-1}_α = 0, \quad s ∈ \{1, ..., L\}, \] (B.2)
or equivalently, that the $B_{αs-2} = 0$ vanish from the higher–stage Noether identities (2.3). We should stress that changes to the gauge(–for–gauge)$^s$–generators (B.1) goes against the paper’s main policy of preserving the original gauge algebra as it is.

**Proof of Strong Noether Identities (B.2):** Define that the new $s$-stage generator $Z^{αs-1}_β := 0$, the $s$-stage gauge condition matrix $X^{αs-1}_α := 0$, and the $s$-stage Faddeev–Popov propagator $D^{αs}_β := 0$ vanish for $s>L$. Define $P^{αs}_β := δ^{αs}_β$, so that the top stage is unchanged $Z^{αL-1}_β := Z^{αL-1}_β$. 33
If the structure function

\[ \text{rank}(\chi^\alpha_{\alpha_{s-1}}) = M_s, \]  

(B.3)

Define the s-stage Faddeev–Popov propagator \( D^{\alpha_{s}}_{\beta_s} = D^{\alpha_{s}}_{\beta_s}(\varphi) \) so that

\[ D^{\alpha_{s}}_{\beta_s} \chi^{\beta_{s+1}}_{\beta_s+1} \chi^{\beta_{s+1} \gamma_s} = \delta^{\alpha_{s-1}}_{\beta_{s-1}} D^{\alpha_{s}}_{\beta_s} \chi^{\beta_{s+1} \gamma_s} \]  

(B.4)

The feasibility to meet condition (B.4) for \( s < L \) can be seen from the induction assumption (B.2) (with substitution \( s \to s+1 \)). The Faddeev–Popov propagator \( D^{\alpha_{s}}_{\beta_s} \) is typically space–time non-local. Define

\[ P^{\alpha_{s-1}}_{\beta_{s-1}} := \delta^{\alpha_{s-1}}_{\beta_{s-1}} D^{\alpha_{s}}_{\beta_s} \chi^{\beta_{s+1} \gamma_s} \]  

(B.5)

Next calculate the new s-stage generator \( \tilde{Z}^{\alpha_{s+2}}_{\beta_{s-1}} \) from the definition (B.1). The weak Noether identities (2.3) guarantee that the generator only changes off-shell, \( \tilde{Z}^{\alpha_{s+2}}_{\beta_{s-1}} \approx Z^{\alpha_{s+2}}_{\beta_{s-1}} \). Calculate

\[ \tilde{Z}^{\alpha_{s+2}}_{\beta_{s-1}} \tilde{Z}^{\beta_{s-1} \gamma_s} \]

(B.1)\]

\[ = Z^{\alpha_{s+2}}_{\beta_{s-1}} P^{\alpha_{s-1}}_{\beta_{s-1}} \tilde{Z}^{\beta_{s-1} \gamma_s} \]

(B.5)

\[ = Z^{\alpha_{s+2}}_{\beta_{s-1}} (\delta^{\alpha_{s-1}}_{\beta_{s-1}} - \tilde{Z}^{\alpha_{s+2}}_{\beta_{s-1}} D^{\alpha_{s}}_{\beta_s} \chi^{\beta_{s+1} \gamma_s}) \]

(B.4)

\[ = Z^{\alpha_{s+2}}_{\beta_{s-1}} \tilde{Z}^{\beta_{s-1} \gamma_s} - Z^{\alpha_{s+2}}_{\beta_{s-1}} \tilde{Z}^{\beta_{s-1} \gamma_s} (\delta^{\alpha_{s-1}}_{\beta_{s-1}} - \tilde{Z}^{\alpha_{s+2}}_{\beta_{s-1}} D^{\alpha_{s}}_{\beta_s} \chi^{\beta_{s+1} \gamma_s}) \]

(B.2)

where the induction assumption (B.2) (with substitution \( s \to s+1 \)) was used in the last step.

\( \Box \)

\section{C Antisymmetric \( \tilde{B}^{ij}_{\alpha_1} \) Exists.}

\textbf{Proposition C.1} If the structure function \( \tilde{B}^{ij}_{\alpha_1} \) locally satisfies the first–stage Noether identity (2.2), then there exists an antisymmetric structure function \( \tilde{B}^{ij}_{\alpha_1} = \tilde{B}^{ij}_{\alpha_1} \) that does the same.

Proof of Proposition C.1: Recall that there exists transversal and longitudinal fields \( \tilde{\phi} = \{\xi^I; \theta^{A_0}\} \) so that the original action \( S_0 \) only depends on \( \xi^I \), cf. the Gauge Principle 4.1. Define

\[ R^{\alpha}_{\alpha_0} := \Lambda^I \tilde{R}^{\alpha_0}_{\alpha_0}, \quad \tilde{B}^{ij}_{\alpha_1} := \Lambda^I \tilde{B}^{ij}_{\alpha_1} \Lambda^J (-1)^{\epsilon_j + \epsilon_{\alpha_1}}(\epsilon_j + \epsilon_{\alpha_1}), \]  

(C.1)

where we have used the Jacobian \( \Lambda^I \) matrix (4.23). Then

\[ (S_0 \frac{\partial}{\partial \tilde{\phi}^I}) \tilde{B}^{ij}_{\alpha_1} = (S_0 \frac{\partial}{\partial \tilde{\phi}^I}) \tilde{B}^{ij}_{\alpha_1} \equiv R^{\alpha}_{\alpha_0} Z^{\alpha_0}_{\alpha_1}, \]  

(C.2)

\[ R^{\alpha}_{\alpha_0} \equiv (S_0 \frac{\partial}{\partial \tilde{\phi}^I}) \tilde{R}^{\alpha}_{\alpha_0} \equiv (S_0 \frac{\partial}{\partial \tilde{\phi}^I}) K^{\alpha_0}_{\alpha_0} = (-1)^{\epsilon_j + \epsilon_{\alpha_1}} K^{\alpha_0}_{\alpha_0}, \]  

(C.3)

Now define antisymmetric tilde structure functions \( \tilde{B}^{ij}_{\alpha_1} = \tilde{B}^{ij}_{\alpha_1} \) as follows.

\[ \tilde{B}^{IJJ}_{\alpha_1} := K^{IJJ}_{\alpha_0} Z^{\alpha_0}_{\alpha_1}, \quad \tilde{B}^{I0}_{\alpha_1} := B^{I0}_{\alpha_1} =: -(-1)^{\epsilon_j + \epsilon_{\alpha_1}} \tilde{B}^{I0}_{\alpha_1}, \quad \tilde{B}^{A0}_{\alpha_1} := 0. \]  

(C.4)

It is easy to see that the antisymmetric structure functions \( \tilde{B}^{ij}_{\alpha_1} \) also satisfies the first–stage Noether identity (C.2).
Remark: If the structure functions $B^{ij}_{\alpha i}$ is a tensor under change of coordinates (as is normally assumed), the antisymmetric $\tilde{B}^{ij}_{\alpha i}$ in the proof of Proposition C.1 is not necessarily also a tensor.

D Deformation of Acyclicity

Acyclicity is stable under deformations in the following sense.

Lemma D.1 (Function version) Let there be given a nilpotent Grassmann–odd operator $\delta$, $\delta^2 = 0$, with resolution expansion $\delta = \sum_{k=-1}^{\infty} \delta_k$, deg($\delta_k$) = $k$, with respect to an integer resolution degree “deg”. Assume the leading nilpotent operator $\delta_{(-1)}$ is acyclic, i.e.,

$$\forall \text{ functions } f: \delta_{(-1)}f = 0 \land \text{deg}(f) > 0 \Rightarrow \exists g: f = \delta_{(-1)}g . \quad (D.1)$$

Then the operator $\delta$ itself is acyclic as well, i.e.,

$$\forall \text{ functions } f: \delta f = 0 \land \text{deg}(f) > 0 \Rightarrow \exists g: f = \delta g . \quad (D.2)$$

Proof of Lemma D.1: The $n$th nilpotency relation reads

$$0 = (\delta^2)(n) = \sum_{k=-1}^{n+1} \delta_k \delta(n-k) . \quad (D.3)$$

Let there be given $\delta$-closed function $f = \sum_{k=1}^{\infty} f(k)$, $(\delta f) = 0$, with deg($f$) > 0. We would like to find a function $\tilde{g} = \sum_{k=2}^{\infty} g(k)$, such that $f = (\delta \tilde{g})$, i.e.,

$$f(n) = \sum_{k=-1}^{m-2} \delta_k g(m-k) = \sum_{k=2}^{m+1} \delta(m-k)g(k) = \sum_{k=0}^{m-1} \delta(m-k-2)g(k+2) . \quad (D.4)$$

The proof of the main statement is an induction in the resolution degree. Assume that there exist functions $g(2), g(3), g(4), \ldots, g(n+1)$, such that $f(1), f(2), f(3), \ldots, f(n)$, satisfy $\delta$-exactness relation (D.4), where $n \geq 0$. We would like to find a function $g(n+2)$, such that $f(n+1)$ satisfies $\delta$-exactness relation (D.4). The following two functions $A_{(n+1)}$ and $B_{(n)}$ are well-defined by the induction assumption.

$$A_{(n+1)} := \sum_{k=0}^{n-1} \delta_k g(n+1-k) , \quad (D.5)$$

$$B_{(n)} := \sum_{k=-1}^{n-1} \delta_k f(n-k) \overset{(D.4)}{=} \sum_{k=0}^{n-1} \delta_k \sum_{\ell=0}^{n-k-1} \delta(n-k-\ell-2)g(\ell+2) = \sum_{k=0}^{n-1} \sum_{\ell=0}^{n-k-1} \delta_k \delta(n-\ell-k-2) g(\ell+2) \overset{(D.3)+(D.5)}{=} -\delta_{(-1)} A_{(n+1)} . \quad (D.6)$$

The $n$th closeness relation is

$$0 = (\delta f)(n) = \sum_{k=-1}^{n-1} \delta_k f(n-k) = \delta_{(-1)} f(n+1) + B_{(n)} \overset{(D.6)}{=} \delta_{(-1)} [f(n+1) - A_{(n+1)}] , \quad (D.7)$$

so $f(n+1) - A_{(n+1)}$ is $\delta_{(-1)}$-closed. Because $n \geq 0$, and because of the acyclicity (D.1), there exists a function $g(n+2)$ such that

$$f(n+1) = \delta_{(-1)} g(n+2) + A_{(n+1)} = \sum_{k=-1}^{n} \delta_k g(n+1-k) , \quad (D.7)$$

which is just the sought–for $\delta$-exactness relation (D.4).
Lemma D.2 (Operator version) Let there be given a nilpotent Grassmann–odd operator \( \delta \), \( [\delta, \delta] = 0 \), with resolution expansion \( \delta = \sum_{k=0}^{\infty} \delta[k] \), \( \deg(\delta[k]) = k \), with respect to an integer resolution degree “deg”. Assume the leading nilpotent operator \( \delta[0] \) is acyclic, i.e.,

\[
\forall \text{ operators } X : \ [\delta[0], X] = 0 \land \deg(X) > 0 \Rightarrow \exists Y : X = [\delta[0], Y]. \quad (D.8)
\]

Then the operator \( \delta \) itself is acyclic as well, i.e.,

\[
\forall \text{ operators } X : \ [\delta, X] = 0 \land \deg(X) > 0 \Rightarrow \exists Y : X = [\delta, Y]. \quad (D.9)
\]

Proof of Lemma D.2: The \( n \)th nilpotency relation reads

\[
0 = [\delta, \delta][n] = \sum_{k=0}^{n} [\delta[k], \delta[n-k]]. \quad (D.10)
\]

Let there be given \( \delta \)-closed operator \( X = \sum_{k=1}^{\infty} X[k] \), \( [\delta, X] = 0 \), with \( \deg(X) > 0 \). We would like to find an operator \( Y = \sum_{k=1}^{\infty} Y[k] \), such that \( X = [\delta, Y] \), i.e.,

\[
X[m] = \sum_{k=0}^{m-1} [\delta[k], Y[m-k]] = \sum_{k=1}^{m} [\delta[m-k], Y[k]]. \quad (D.11)
\]

The proof of the main statement is an induction in the resolution degree. Assume that there exist operators \( Y[1], Y[2], Y[3], \ldots, Y[n-1] \), such that \( X[1], X[2], X[3], \ldots, X[n-1] \), satisfy \( \delta \)-exactness relation (D.11), where \( n \geq 1 \). We would like to find an operator \( Y[n] \), such that \( X[n] \) satisfies \( \delta \)-exactness relation (D.11). The following two operators \( A[n] \) and \( B[n] \) are well-defined by the induction assumption.

\[
A[n] := \sum_{k=1}^{n-1} [\delta[n-k], Y[k]], \quad (D.12)
\]

\[
B[n] := \sum_{k=1}^{n-1} [\delta[k], X[n-k]] = \sum_{k=1}^{n-1} \left[ \delta[k], \sum_{\ell=1}^{k} \left[ \delta[n-k-\ell], Y[\ell] \right] \right]
= \sum_{\ell=1}^{n-1} \left[ \int_{k=0}^{n-\ell} \left[ \delta[k], \delta[n-k-\ell], Y[\ell] \right] - \delta[0], \delta[n-\ell], Y[\ell] \right]
= \sum_{\ell=1}^{n-1} \left[ \frac{1}{\ell} [\delta, \delta][n-\ell], Y[\ell] \right] \quad (D.10)+(D.12) \quad \Rightarrow \quad [\delta[0], A[n]]. \quad (D.13)
\]

The \( n \)th closeness relation is

\[
0 = [\delta, X][n] = \sum_{k=0}^{n-1} [\delta[k], X[n-k]] = [\delta[0], X[n]] + B[n] \quad (D.13) \quad \Rightarrow \quad [\delta[0], X[n] - A[n]]. \quad (D.14)
\]

so \( X[n] - A[n] \) is \( \delta[0] \)-closed. Because \( n \geq 1 \), and because of the acyclicity (D.8), there exists an operator \( Y[n] \) such that \( X[n] = [\delta[0], Y[n]] + A[n] = \sum_{k=0}^{n-1} [\delta[k], Y[n-k]] \), which is just the sought–for \( \delta \)-exactness relation (D.11).
References

[1] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. **102B** (1981) 27.

[2] I.A. Batalin and G.A. Vilkovisky, Phys. Rev. **D28** (1983) 2567 [E: **D30** (1984) 508].

[3] I.A. Batalin, J. Math. Phys. **22** (1981) 1837.

[4] I.A. Batalin and G.A. Vilkovisky, Nucl. Phys. **B234** (1984) 106.

[5] M. Henneaux, Commun. Math. Phys. **140** (1991) 1.

[6] G. Barnich, F. Brandt and M. Henneaux, Commun. Math. Phys. **174** (1995) 57, arXiv:hep-th/9405109.

[7] O. Piguet and S.P. Sorella, Lect. Notes Phys. **M28** (1995) 1.

[8] G. Barnich, F. Brandt and M. Henneaux, Phys. Rept. **338** (2000) 439, arXiv:hep-th/0002245.

[9] B.L. Voronov and I.V. Tyutin, Theor. Math. Phys. **50** (1982) 218.

[10] I.A. Batalin and G.A. Vilkovisky, J. Math. Phys. **26** (1985) 172.

[11] I.A. Batalin, P.M. Lavrov and I.V. Tyutin, J. Math. Phys. **31** (1990) 1487 [E: **32** (1991) 1970]; ibid **32** (1991) 532; ibid **32** (1991) 2513.

[12] J.M.L. Fisch and M. Henneaux, Commun. Math. Phys. **128** (1990) 627;

[13] J.M.L. Fisch, M. Henneaux, J. Stasheff and C. Teitelboim, Commun. Math. Phys. **120** (1989) 379.

[14] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton, Univ. Pr. 1992.

[15] D. Bashkirov, G. Giachetta, L. Mangiarotti and G. Sardanashvily, J. Math. Phys. **46** (2005) 103513, arXiv:math-ph/0506034.

[16] S. Vandoren and A. Van Proeyen, Nucl. Phys. **B411** (1994) 257, arXiv:hep-th/9306147.

[17] S. Vandoren, PhD thesis, K.U. Leuven, 1995, arXiv:hep-th/9601013.