Wave transmission and reflection at the boundary of phononic crystals

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Abstract. This paper deals with two problems concerning wave propagation in non-homogeneous media. The non-homogeneous media is modelled as one-dimensional phononic crystal. First problem is the determination of the low frequency passband. The second one is evaluation of the transmission ratio of the waves that propagate through the boundary of non-homogeneous periodic media. For the first problem low frequency passband boundaries are studied for the waves travelling along one-dimensional multicomponent periodic medium. And for the problem of wave transmission from the homogeneous medium into the one-dimensional multicomponent periodic medium dependencies of energy reflection on parameters of incident wave and medium properties are obtained and analysed.

1. Introduction
There are many studies showing that heterogeneous multicomponent composites can have promising vibration isolating and sound absorbing properties (see e.g. [1, 2]). This paper is a part of the study focusing on theoretical synthesis of the systems for broadband sound absorption and isolation starting from nano- and microstructure levels. Based on the results of works [3, 4], a vibration isolation and sound protection efficiency is illustrated through the study of the wave propagation in porous medium filled with the fluid and gas. Results of the study could be used to obtain new composite materials and finally to design the elements of interior and/or carrying structures of a transport vehicle. In the paper an approach is presented for mathematical modeling of such materials and numerical analysis of their behavior and protection properties under the broadband vibration and noise excitation.

There are some key points in design of the sound protection materials. It is well known that some material do not pass acoustic waves from certain frequency intervals [5]. In this paper we propose an approach to optimize the low frequency intervals of waves traveling through multicomponent periodic medium. Another approach is to operate the reflection properties of the sound protection materials. This feature of the multicomponent periodic medium is also discussed.

2. Problems definition and results

2.1. Waveguide problem for one-dimensional periodic medium
Let a heterogeneous one-dimensional periodic medium (Figure 1) consist of two components $M_1=\{\rho_1, c_1\}$ and $M_2=\{\rho_2, c_2\}$, where $\rho$ and $c$ are the density and the sound velocity of the corresponding medium. Let's accept that a chain of the inhomogeneities, consisting of the component $M_1$, is filled into the component $M_2$ (a matrix of the composite) and the linear concentration of coupled layers of
components are equal \( k_1 \) and \( k_2 \) respectively \( (k_1 + k_2 = 1) \). This chain is spatially periodic; the smallest spatial period of the medium is equal to \( L \).

![Figure 1. A monodisperse chain of inhomogeneities.](image_url)

The steady-state acoustic oscillations of the pressure in the media are described by the following equations:

\[
\begin{align*}
\rho_1 p^{(1)} + \lambda_1^2 p^{(1)} &= 0, \\
\rho_2 p^{(2)} + \lambda_2^2 p^{(2)} &= 0,
\end{align*}
\]

where \( p^{(1)} \) and \( p^{(2)} \) are the acoustic pressures in these components, \( \lambda_1 = \omega/c_1 \), \( \lambda_2 = \omega/c_2 \), and \( \omega = 2\pi f \) is the angular frequency. The conditions at the component boundary are:

\[
\begin{align*}
p^{(1)} &= p^{(2)}, \\
p^{(1)}_1 &= p^{(2)}_2.
\end{align*}
\]

Considering the symmetry properties of the medium the problem solution satisfy the Floquet theorem

\[
p(x + L) = \exp(i\xi) p(x).
\]

As shown in [3] non trivial solution of \( (1-3) \) for given \( \omega \) exist if and only if the angular frequency \( \omega \) belongs to the certain intervals. Boundaries of these intervals are defined by the dispersive properties of the periodic medium. The non-trivial solution of problem \( (1-3) \) will be called waveguide function. Frequency open intervals

\[
I_n = \left( \inf_{0 \leq \xi \leq \pi} \omega_n (\xi), \sup_{0 \leq \xi \leq \pi} \omega_n (\xi) \right), \quad n \in N
\]

are the passbands for the modes \( \omega_n = \omega_n (\xi) \).

The complete equations system of given problem consist of equations \( (2) \) and the following equations derived from the Floquet theorem

\[
\begin{align*}
p^{(1)}(-k_1L)\exp(i\xi) &= p^{(2)}(k_1L), \\
\rho_1 p^{(1)}(-k_1L)\exp(i\xi) &= \frac{p^{(2)}_1(k_2L)}{\rho_2}.
\end{align*}
\]

Functions \( p^{(i)}(x) \) satisfy \( (1) \) and have the form

\[
p^{(i)}(x) = A^{(i)} \exp(i\lambda_i x) + B^{(i)} \exp(-i\lambda_i x), \quad i = 1, 2.
\]

The homogeneous system of linear equations \( (2), (4) \) has non zero solution if the determinant of this system equals to zero. Thus the dispersion relation for all waveguide modes is

\[
4\cos(\xi) - (\theta + 1)^2 \cos(\lambda_1 k_1 + \lambda_2 k_2) + (\theta - 1)^2 \cos(\lambda_1 k_1 - \lambda_2 k_2) = 0,
\]

where \( \theta = \frac{\rho_1 c_1}{\rho_2 c_2} \) and we assume that \( \theta \leq 1 \).

The passband boundaries can be obtained from dispersion relation by substituting \( \xi = 0 \) or \( \xi = \pi \) into (6).
From the equation (6) we obtain
\[ \tan \left( \frac{\lambda_1 k_1}{2} \right) \tan \left( \frac{\lambda_2 k_2}{2} \right) + \left( \tan \left( \frac{\lambda_1 k_1}{2} \right) \tan \left( \frac{\lambda_2 k_2}{2} \right) \right)^{-1} = \theta + \theta^{-1}, \]
\[ \tan^{-1} \left( \frac{\lambda_1 k_1}{2} \right) \tan \left( \frac{\lambda_2 k_2}{2} \right) + \tan \left( \frac{\lambda_1 k_1}{2} \right) \tan^{-1} \left( \frac{\lambda_2 k_2}{2} \right) = \theta + \theta^{-1}, \]

\[ \tan \left( \frac{\omega L k_1}{2c_1} \right) \tan \left( \frac{\omega L k_2}{2c_2} \right) = \frac{\rho_1 c_1}{\rho_2 c_2}. \]

Solution of this equation satisfies inequality \( \omega_{pass} \leq \frac{2c_1}{L} \left( \frac{\rho_1}{\rho_2 k_1 k_2} \right)^{1/2} \) or \( f_{pass} \leq \frac{c_1}{\pi L} \left( \frac{\rho_1}{\rho_2 k_1 k_2} \right)^{1/2} \), which provides a good approximation in case of medium components are very contrast.

Analysis and computations of the equation (8) allow us to make the following conclusions:

- The width \( \omega_{pass} \) of the first passband depends on the components density ratio and does not depend on the density magnitudes. Examples of this dependency are depicted on Figure 2.
- The concentration of the component with lower acoustic resistance increase from 0 to 0.5 implies decreasing of passband width. Further concentration increase may cause enlarging of the passband width as well as its reduction (see Figure 3).
- Sound velocity of the component with the lowest acoustic resistance has the most sufficient influence on passband width. This property is illustrated on Figure 4. One can notice a knee on the graphs at the argument value corresponding to \( \rho_1 c_1/\rho_2 c_2 = 1 \). The examples of passband width dependence on component sound velocity are given on Figures 4 and 5.
- The width of the first passband inversely proportional to the period length of inhomogeneous medium. For the computations presented on Figures 2 – 5 the period \( L \) was equal to unit.
2.2. Wave transmission from the homogeneous medium into the composite.

The following problem is to evaluate the energy transmission of the wave traveling from homogeneous medium into the one-dimensional periodic composite (Figure 6), described in previous paragraph. As long as there is no energy dissipation, the sum of reflected and transmitted energy flow is equal to the energy flow of the incident wave:

\[ W_R + W_T = W_I. \]

Let \( p(x) = \exp(i\lambda x) + R \exp(-i\lambda x) \) be the acoustic pressure in the homogeneous medium \( M=\{\rho, c\} \), \( \lambda = \omega \sigma / c \). So the value \( R^2 \) signifies reflected energy flow, since energy flow of the incident wave is equal to unit. At the boundary of the homogeneous media and composite we have

\[ p = p^{(i)} \quad \rho \frac{p_x}{\rho_1}. \]  

Since \( p^{(i)}(x) \) is the waveguide function (5) we have \( B^{(i)} = \tau(\omega)A^{(i)} \). For given \( \omega \) coefficient \( \tau(\omega) \) is determined using dispersion relation (6) and equations (2) and (4). Solution of the system (9) provides

\[ R = \frac{\tau(1+\theta)(1+\theta_0)+(1-\theta)(1-\theta_0)}{\tau(1+\theta)(1-\theta)+(1-\theta)(1+\theta_0)}, \]

where \( \theta_0 = \rho \xi / \rho_1 c_1 \). It should be noticed that \( R \) will not change if we use \( p^{(2)} \) instead of \( p^{(i)} \) under condition that parameters of periodic medium are the same. In other words wave propagation in composite does not depend on the composite orientation.

Results of the computations \( R^2 \) allow us to denote some properties of the wave propagation from homogeneous medium into composite. By default the following parameters of the medium were used for the computations: \( L=0.01, \) Air \( \rho = 1.2, c = 340, \) Water \( \rho = 1000, c = 1400, \) Aluminum \( \rho = 2700, c = 5500, \) Quartz \( \rho = 2650, c = 5760, \) Polypropylene \( \rho = 950, c = 950, \) Iron \( \rho = 7800, c = 5900 \) and \( k = 0.5. \)
Graphs on Figures 7 and 8 shows that the reflection ratio depends on parameters of the composite as well as on parameters of the homogeneous media. The reflected energy flow is increasing while frequency of incident wave grows. When frequency reach passband boundary we obtain full reflection. The minimum of reflection ratio reaches at \( \omega \to 0 \).

Examples displaying reflection dependence on concentration of the composite components are presented on Figures 9 and 10. It shows that increase of component concentration with smaller acoustic resistance may do not change, decrease or even enlarge reflection ratio.

**Figure 7.** The reflected energy flow dependence on frequency for some composites. Homogeneous media is Air.

**Figure 8.** The reflected energy flow dependence on frequency for different homogeneous media. Composite is Air-Quarts, \( k_1=0.5 \).

Formula (10) shows that reflection ratio depends on the acoustic resistance of the homogeneous media. Graphs of the minimum of reflected energy (\( \omega \to 0 \)) flow are shown on Figure 11. On this figure we can see that behavior of the reflection ratio is similar to the case of reflection at the boundary of two homogeneous media. In the case of two homogeneous media minimum of the reflection ratio is reached when the media have the same acoustic resistance. So using this analogy we can suggest the acoustic resistance of composite to be equal to the value of homogeneous medium acoustic resistance that corresponds to the minimal value of reflection ratio \( R \).

**Figure 9.** Examples of the concentration influence on reflected energy flow for some composites. Homogeneous media is Air.

**Figure 10.** Examples of the concentration influence on reflected energy flow for different homogeneous media. Composite is Quartz-Air.
The minimal point of the reflection ratio $R$ does not depend on frequency, but $R$ at this point grows when frequency increases.

![Figure 11. The reflected energy flow dependence on acoustic resistance of homogeneous media.](image)

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