Power and exponential moments of the number of visits and related quantities for perturbed random walks

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December 13, 2011

Abstract

Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots$ be a sequence of i.i.d. copies of a random vector $(\xi, \eta)$ taking values in $\mathbb{R}^2$, and let $S_n := \xi_1 + \ldots + \xi_n$. The sequence $(S_{n-1} + \eta_n)_{n \geq 1}$ is then called perturbed random walk.

We study random quantities defined in terms of the perturbed random walk: $\tau(x)$, the first time the perturbed random walk exits the interval $(-\infty, x]$, $N(x)$, the number of visits to the interval $(-\infty, x]$, and $\rho(x)$, the last time the perturbed random walk visits the interval $(-\infty, x]$. We provide criteria for the a.s. finiteness and for the finiteness of exponential moments of these quantities. Further, we provide criteria for the finiteness of power moments of $N(x)$ and $\rho(x)$.

2010 Mathematics Subject Classification: Primary: 60G50
Secondary: 60G40

Keywords: first passage time, last exit time, number of visits, perturbed random walk, random walk, renewal theory, shot-noise process

1 Introduction

The purpose of this article is to study the moments of certain basic renewal-theoretic quantities for a class of perturbed random walks formally defined below. Such random sequences arise as derived processes in various areas of Applied Probability and we refer to Subsection 1.2 for a number of examples.

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A part of this work was done while A. Iksanov was visiting Münster in January/February and May 2011. Grateful acknowledgment is made for financial support and hospitality. The research of A. Iksanov was partly supported by a grant from Utrecht University, the Netherlands.

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1
It is an interesting question and in fact the main motivation behind this work to what extent classical moment results for ordinary random walks must be adjusted in the presence of a perturbating sequence.

1.1 Setup
Let \((\xi_1, \eta_1), (\xi_2, \eta_2), \ldots\) be a sequence of i.i.d. two-dimensional random vectors with generic copy \((\xi, \eta)\). For notational convenience, we assume that \((\xi, \eta)\) is defined on the same probability space as the \((\xi_k, \eta_k), k \geq 1\) and independent of this sequence. No condition is imposed on the dependence structure between \(\xi\) and \(\eta\). Let \((S_n)_{n \geq 0}\) be the zero-delayed ordinary random walk with increments \(\xi_n\) for \(n \in \mathbb{N}\), i.e., \(S_0 = 0\) and \(S_n = \xi_1 + \ldots + \xi_n, n \in \mathbb{N}\). Then define its perturbed variant \((T_n)_{n \geq 1}\), called perturbed random walk (PRW), by

\[ T_n := S_{n-1} + \eta_n, \quad n \in \mathbb{N}. \tag{1.1} \]

It has appeared in a number of recent publications, see for instance [4, 11, 21, 34]. Here we should mention that, motivated by certain problems in sequential statistics, a very different class of perturbations, which may roughly be characterized by having slowly varying paths in a stochastic sense, has been considered under the label “nonlinear renewal theory”, see [28, 29, 38] and [15, Section 6].

For \(x \in \mathbb{R}\), define the level \(x\) first passage time

\[ \tau(x) := \inf\{n \in \mathbb{N} : T_n > x\}, \tag{1.2} \]

the number of visits to \((-\infty, x]\)

\[ N(x) := \#\{n \in \mathbb{N} : T_n \leq x\}, \tag{1.3} \]

and the associated last exit time

\[ \rho(x) := \sup\{n \in \mathbb{N} : T_n \leq x\}. \tag{1.4} \]

with the usual conventions that \(\sup \emptyset := 0\) and \(\inf \emptyset := \infty\). Our aim is to find criteria for the a.s. finiteness of these quantities and for the finiteness of their power and exponential moments.

Let us further denote by \(\tau^*(x), N^*(x)\) and \(\rho^*(x)\) the corresponding quantities for the ordinary random walk \((S_n)_{n \geq 0}\) which is obtained in the special case \(\eta = 0\) a.s. after a time shift. If \(\xi = 0\), then \((T_n)_{n \geq 1}\) reduces to a sequence of i.i.d. r.v.’s. In this case \(N(x) = \rho(x) = \infty\) a.s. and \(\tau(x)\) has a geometric distribution whenever \(0 < P\{\eta \leq x\} < 1\). Neither of the two afore-mentioned cases will be subject of our analysis and therefore be ruled out by making the

**Standing Assumption:** \(P\{\xi = 0\} < 1\) and \(P\{\eta = 0\} < 1\).

1.2 Examples and applications
Functionals of PRW’s appear in several areas of Applied Probability as demonstrated by the following examples.
Example 1.1 (Perpetuities). Provided that $\sum_{n \geq 1} e^{T_n}$ is a.s. convergent, this sum is called *perpetuity* due to its interpretation as a sum of discounted payment streams in insurance and finance. Perpetuities have received an enormous amount of attention which by now has led to a more or less complete theory. A partial survey of the relevant literature may be found in [3], for more recent contributions see [9, 18, 19, 20]. Presumably one of the most challenging open problems in the area is to provide sufficient (and close to necessary) conditions for the absolute continuity of the law of a perpetuity. In the light of serious complications that already arise in the “simple” case $\xi = \text{const} < 0$ (see [3] for more information), there is only little hope for the issue being settled in the near future.

Example 1.2 (The Bernoulli sieve). The Bernoulli sieve is an infinite occupancy scheme in a random environment $(P_k)_{k \geq 1}$, where

$$P_k := W_1W_2 \cdots W_{k-1}(1 - W_k), \quad k \in \mathbb{N}, \quad (1.5)$$

and $(W_k)_{k \geq 1}$ are independent copies of a random variable $W$ taking values in $(0, 1)$. One may think of balls that, given $(P_k)_{k \geq 1}$, are independently placed into one of infinitely many boxes $1, 2, 3, \ldots$, the probability for picking box $k$ being $P_k$. Assuming that the number of balls equals $n$, denote by $K_n$ the number of nonempty boxes. If the law of $|\log W|$ is non-lattice, it was shown in [14] that the weak convergence of $K_n$, properly centered and normalized, is completely determined by the weak convergence of

$$N(x) := \# \{k \in \mathbb{N} : P_k \geq e^{-x} \} = \# \{k \in \mathbb{N} : W_1 \cdots W_{k-1}(1 - W_k) \geq e^{-x} \}, \quad x > 0,$$

again properly centered and normalized. Notice that $N(x)$ is the number of visits to $(-\infty, x]$ by the PRW generated by the couples $(|\log W_1|, |\log(1 - W_1)|), (|\log W_2|, |\log(1 - W_2)|), \ldots$. A summary of known results including relevant literature for the Bernoulli sieve can be found in the recent survey [10].

Example 1.3 (Regenerative processes). Let $(W(t))_{t \geq 0}$ be a càdlàg process starting at $W(0) = 0$ and drifting to $-\infty$ a.s. Suppose there exists a zero-delayed renewal sequence of random epochs $(\tau_n)_{n \geq 0}$ such that the segments (also called cycles)

$$(W(t))_{0 \leq t \leq \tau_1}, \ (W(\tau_1 + t) - W(\tau_1))_{0 \leq t \leq \tau_2 - \tau_1}, \ldots$$

are i.i.d. Then $(W(t))_{t \geq 0}$ is a (strong-sense) regenerative process, see [3]. For $n \in \mathbb{N}$, put

$$\xi_n := W(\tau_n) - W(\tau_{n-1}) \quad \text{and} \quad \eta_n := \sup_{\tau_{n-1} \leq t < \tau_n} W(t) - W(\tau_{n-1}).$$

Then $(\xi_1, \eta_1), (\xi_2, \eta_2), \ldots$ are i.i.d., and

$$\sup_{t \geq 0} W(t) = \sup_{n \geq 1} (\xi_1 + \ldots + \xi_{n-1} + \eta_n),$$
i.e., the supremum of the regenerative process can be represented as the supremum of an appropriate PRW. The supremum $M$, say, of a PRW is a relatively simple functional that has received considerable attention in the literature. For instance, the tail behavior of $M$ was investigated in [4, 12, 16, 21, 33, 34]. Some moment results on $M$ can be found in [2, 3].

Example 1.4 (Queues and branching processes). Suppose that $\xi$ and $\eta$ are both non-negative and define for $t \geq 0$

$$R(t) := \sum_{k=0}^{\infty} \mathbb{1}_{\{S_k \leq t < S_k + \eta_k + 1\}} = \tau^*(t) - N(t), \quad t \geq 0.$$ 

In a GI/G/$\infty$-queuing system, where customers arrive at times $S_0 = 0 < S_1 < S_2 < \ldots$ and are immediately served by one of infinitely many idle servers, the service time of the $k$th customer being $\eta_k + 1$, $R(t)$ gives the number of busy servers at time $t \geq 0$. Another interpretation of $R(t)$ emerges in the context of a degenerate pure immigration Bellman-Harris branching process in which each individual is sterile, immigration occurs at the epochs $S_1, S_2$ etc., and the lifetimes of the ancestor and the subsequent immigrants are $\eta_1, \eta_2, \ldots$. Then $R(t)$ gives the number of particles alive at time $t \geq 0$. The process $(R(t))_{t \geq 0}$ was also used to model the number of active sessions in a computer network [27, 32].

2 Main results

2.1 Almost sure finiteness

It is well-known that a non-trivial zero-delayed random walk $(S_n)_{n \geq 0}$ (i.e. a random walk starting at the origin with increment distribution not degenerate at 0) exhibits one of the following three regimes:

1) drift to $+\infty$ (positive divergence): $\lim_{n \to \infty} S_n = \infty$ a.s.;
2) drift to $-\infty$ (negative divergence): $\lim_{n \to \infty} S_n = -\infty$ a.s.;
3) oscillation: $\lim \inf_{n \to \infty} S_n = -\infty$ and $\lim \sup_{n \to \infty} S_n = \infty$ a.s.

PRW’s exhibit the same trichotomy. In order to state the result precisely some further notation is needed. As usual, let $\xi^+ = \max(\xi, 0)$ and $\xi^- = \max(-\xi, 0)$. Then, for $x > 0$, define

$$A_{\pm}(x) := \int_0^x \mathbb{P}\{\pm \xi > y\} \, dy = \mathbb{E} \min(\xi^{\pm}, x) \quad \text{and} \quad J_{\pm}(x) := \frac{x}{A_{\pm}(x)},$$

whenever the denominators are non-zero. Notice that $J_{\pm}(x)$ for $x > 0$ is well-defined iff $\mathbb{P}\{\pm \xi > 0\} > 0$. In this case, we define $J_{\pm}(0) := \mathbb{P}\{\pm \xi > 0\}^{-1}$. The following theorem, though not stated explicitly in [13], can be read off from the results obtained there.
Theorem 2.1. Any PRW \((T_n)_{n \geq 1}\) satisfying the standing assumption is either positively divergent, negatively divergent or oscillating. Positive divergence takes place iff
\[
\lim_{n \to \infty} S_n = \infty \quad \text{and} \quad \mathbb{E} J_+ (\eta^-) < \infty,
\]
while negative divergence takes place iff
\[
\lim_{n \to \infty} S_n = -\infty \quad \text{a.s.} \quad \text{and} \quad \mathbb{E} J_- (\eta^+) < \infty.
\]
Oscillation occurs in the remaining cases, thus iff either
\[
-\infty = \liminf_{n \to \infty} S_n < \limsup_{n \to \infty} S_n = \infty \quad \text{a.s.},
\]
or
\[
\lim_{n \to \infty} S_n = \infty \quad \text{a.s.} \quad \text{and} \quad \mathbb{E} J_+ (\eta^-) = \infty,
\]
or
\[
\lim_{n \to \infty} S_n = -\infty \quad \text{a.s.} \quad \text{and} \quad \mathbb{E} J_- (\eta^+) = \infty.
\]

Remark 2.2. As a consequence of Theorem 2.1 it should be observed that a PRW \((T_n)_{n \geq 1}\) may oscillate even if the corresponding ordinary random walk \((S_n)_{n \geq 0}\) drifts to \(\pm \infty\).

In view of the previous result it is natural to take a look at the a.s. finiteness of the first passage times \(\tau(x)\). Plainly, if \(\limsup_{n \to \infty} T_n = \infty \quad \text{a.s.}\), then \(\tau(x) < \infty \quad \text{a.s.} \quad \text{for all} \quad x \in \mathbb{R}\). On the other hand, one might expect in the opposite case, viz. \(\lim_{n \to \infty} T_n = -\infty \quad \text{a.s.}\), that \(\mathbb{P} \{\tau(x) = \infty\} > 0\) for all \(x \geq 0\), for this holds true for ordinary random walks. Namely, if \(\lim_{n \to \infty} S_n = -\infty \quad \text{a.s.}\), then \(\mathbb{P} \{\sup_{n \geq 1} S_n \leq 0\} = \mathbb{P} \{\tau^* = \infty\} > 0\). The following result shows that this conclusion may fail for a PRW. It further provides a criterion for the a.s. finiteness of \(\tau(x)\) formulated in terms of \((\xi, \eta)\).

Theorem 2.3. Let \((T_n)_{n \geq 1}\) be negatively divergent and \(x \in \mathbb{R}\). Then \(\tau(x) < \infty \quad \text{a.s.} \quad \text{iff} \quad \mathbb{P} \{\xi < 0, \eta \leq x\} = 0\). Furthermore, \(\mathbb{P} \{\eta \leq x\} < 1\) holds true in this case.

In order to establish a criterion for the a.s. finiteness of the r.v.’s \(N(x)\) and \(\rho(x)\), it only takes to observe that, if one of those is a.s. finite for some \(x\), then \(\liminf_{n \to \infty} T_n > -\infty \quad \text{a.s.}\). Hence, by Theorem 2.1 \((T_n)_{n \geq 1}\) must be positively divergent. Since the converse holds trivially true, we can state the following result analogous to the case of ordinary random walks.

Theorem 2.4. The following assertions are equivalent:

(i) \((T_n)_{n \geq 1}\) is positively divergent.

(ii) \(N(x) < \infty \quad \text{a.s.} \quad \text{for some/all} \quad x \in \mathbb{R}\).

(iii) \(\rho(x) < \infty \quad \text{a.s.} \quad \text{for some/all} \quad x \in \mathbb{R}\).
2.2 Finiteness of exponential moments

The following theorems are on finiteness of exponential moments of \( \tau(x) \), \( N(x) \) and \( \rho(x) \).

**Theorem 2.5.** Let \( a > 0 \) and \( x \in \mathbb{R} \).

(a) If \( P\{\xi < 0, \eta \leq x\} = 0 \), then \( \mathbb{E}\exp(a\tau(x)) < \infty \) iff
\[
e^a P\{\xi = 0, \eta \leq x\} < 1.
\]

(b) If \( P\{\xi < 0, \eta \leq x\} > 0 \), then
\[
\mathbb{E}\exp(a\tau(x)) < \infty, \quad (2.7)
\]
\[
\mathbb{E}\exp(a\tau(y)) < \infty \quad \text{for all} \ y \in \mathbb{R}, \quad (2.8)
\]
\[
\mathbb{E}\exp(a\tau^*) < \infty, \quad (2.9)
\]
\[
R := -\log \inf_{t \geq 0} \mathbb{E} e^{-t\xi} \geq a \quad (2.10)
\]

are equivalent assertions.

Turning to exponential moments of \( N(x) \), the number of visits of \( (T_n)_{n \geq 1} \) to \((-\infty, x]\), for \( x \in \mathbb{R} \), let us point out before-hand that these random variables are a.s. finite iff \( (T_n)_{n \geq 1} \) is positively divergent which in turn holds true iff \( (S_n)_{n \geq 0} \) is positively divergent and
\[
\mathbb{E} J_+(\eta^-) < \infty \quad (2.11)
\]
(see Theorem 2.1) which will therefore be assumed hereafter.

**Theorem 2.6.** Let \( (T_n)_{n \geq 1} \) be a positively divergent PRW.

(a) If \( \xi \geq 0 \ a.s. \), then the assertions
\[
\mathbb{E}\exp(aN(x)) < \infty, \quad (2.12)
\]
\[
e^a P\{\xi = 0, \eta \leq x\} + P\{\xi = 0, \eta > x\} < 1 \quad (2.13)
\]
are equivalent for each \( a > 0 \) and \( x \in \mathbb{R} \). As a consequence,
\[
\{a > 0 : \mathbb{E} e^{aN(x)} < \infty\} = (0, a(x)) \quad (2.14)
\]
for any \( x \in \mathbb{R} \), where \( a(x) \in (0, \infty] \) equals the supremum of all positive \( a \) satisfying (2.13). As a function of \( x \), \( a(x) \) is nonincreasing with lower bound \(-\log P\{\xi = 0\}\).

(b) If \( \xi > 0 \ a.s. \), then \( a(x) = \infty \) for all \( x \in \mathbb{R} \), thus \( \mathbb{E} e^{aN(x)} < \infty \) for any \( a > 0 \) and \( x \in \mathbb{R} \).

(c) If \( P\{\xi < 0\} > 0 \), then the following assertions are equivalent:
\[
\mathbb{E}\exp(aN(x)) < \infty \quad \text{for some/all} \ x \in \mathbb{R}, \quad (2.15)
\]
\[
\mathbb{E}\exp(aN^*(x)) < \infty \quad \text{for some/all} \ x \in \mathbb{R}, \quad (2.16)
\]
\[
R = -\log \inf_{t \geq 0} \mathbb{E} e^{-t\xi} \geq a. \quad (2.17)
\]

6
Theorem 2.7. Let \((T_n)_{n \geq 1}\) be a positively divergent PRW, \(a > 0\) and \(R = -\log \inf_{t \geq 0} E e^{-t\xi}\).

(a) Assume that \(P\{\xi \geq 0\} = 1\). Let \(x \in \mathbb{R}\) and assume that \(P\{\eta \leq x\} > 0\). Then the following assertions are equivalent:

\[
E \exp(a \rho(x)) < \infty; \tag{2.18}
\]

\[
V_a(y) := \sum_{n \geq 1} e^{an} P\{T_n \leq y\} < \infty \text{ for some/all } y \geq x; \tag{2.19}
\]

\[
a < -\log P\{\xi = 0\} \text{ and } E e^{-\gamma \eta} < \infty, \tag{2.20}
\]

where \(\gamma\) is the unique positive number satisfying \(E e^{-\gamma \xi} = e^{-a}\).

(b) If \(P\{\xi < 0\} > 0\), then the following assertions are equivalent:

\[
E \exp(a \rho(x)) < \infty \text{ for some/all } x \in \mathbb{R}; \tag{2.21}
\]

\[
V_a(x) = \sum_{n \geq 1} e^{an} P\{T_n \leq x\} < \infty \text{ for some/all } x \in \mathbb{R}; \tag{2.22}
\]

\[
a < R \text{ and } E e^{-\gamma \eta} < \infty \text{ or } a = R, \text{ } E \xi e^{-\gamma \xi} < 0 \text{ and } E e^{-\gamma \eta} < \infty \tag{2.23}
\]

where \(\gamma\) is the minimal positive number satisfying \(E e^{-\gamma \xi} = e^{-a}\).

Remark 2.8. Notice that in Theorem 2.7 the case \(P\{\xi \geq 0\} = 1, P\{\eta \leq x\} = 0\) is not treated. But this case is trivial since then \(\rho(x) = 0\) a.s., cf. Lemma 4.3.

2.3 Finiteness of power moments

Theorem 2.9. Let \((T_n)_{n \geq 0}\) be a positively divergent PRW and \(p > 0\). The following conditions are equivalent:

\[
E N(x)^p < \infty \text{ for some/all } x \in \mathbb{R}; \tag{2.24}
\]

\[
E N^*(x)^p < \infty \text{ for some/all } x \geq 0 \text{ and } E J_+(\eta^-) < \infty; \tag{2.25}
\]

\[
E J_+(\xi^-)^{p+1} < \infty \text{ and } E J_+(\eta^-) < \infty. \tag{2.26}
\]

Theorem 2.10. Let \((T_n)_{n \geq 0}\) be a positively divergent PRW and \(p > 0\). Then the following assertions are equivalent:

\[
E \rho(x)^p < \infty \text{ for some/all } x \in \mathbb{R}; \tag{2.27}
\]

\[
E \rho^*(y)^p < \infty \text{ for some/all } y \geq 0 \text{ and } E J_+(\eta^-)^{p+1} < \infty; \tag{2.28}
\]

\[
E J_+(\xi^-)^{p+1} < \infty \text{ and } E J_+(\eta^-)^{p+1} < \infty. \tag{2.29}
\]
Remark 2.11. According to Theorem 2.7, for fixed $a > 0$,
\[ E e^{an(x)} < \infty \quad \text{for some/all } x \in \mathbb{R} \quad \iff \quad \sum_{n \geq 1} e^{an} \mathbb{P}\{T_n \leq x\} < \infty \quad \text{for some/all } x \in \mathbb{R}. \]

According to [26, Theorem 2.1], for fixed $p > 0$,
\[ E \rho^*(x)^p < \infty \quad \text{for some/all } x \geq 0 \quad \iff \quad \sum_{n \geq 1} n^{p-1} \mathbb{P}\{S_n \leq x\} < \infty \quad \text{for some/all } x \geq 0. \]

In the light of these results it may appear to be unexpected that, in general, the finiteness of $E \rho(x)^p$ is not equivalent to the convergence of the series $\sum_{n \geq 1} n^{p-1} \mathbb{P}\{T_n \leq x\}$. Indeed, it can be checked (but we omit the details) that a criterion for the convergence of the latter series is as follows:
\[ E \rho^*(x)^p < \infty \quad \text{for some/all } y \geq 0 \quad \text{and} \quad E J_+(\eta)^p < \infty. \]

2.4 Notation and overview

At this point, we introduce some notation which is used throughout the article. First of all, whenever it is convenient, we write $\tau$, $N$ and $\rho$ for $\tau(0)$, $N(0)$ and $\rho(0)$, respectively. Analogously, we write $\tau^*$, $N^*$ and $\rho^*$ for $\tau^*(0)$, $N^*(0)$ and $\rho^*(0)$, respectively.

As usual, $f(t) \sim g(t)$ as $t \to \infty$ for functions $f$ and $g$, means that $f(t)/g(t) \to 1$ as $t \to \infty$. Similarly, $f(t) \asymp g(t)$ as $t \to \infty$ means that $0 < \lim inf_{t \to \infty} f(t)/g(t) \leq \lim sup_{t \to \infty} f(t)/g(t) < \infty$.

We finish this section with an overview over the further organization of the article. The proofs of the main results are given in Section 4. The proofs concerning finiteness of moments of $N(x)$, Theorems 2.6 and 2.9, are based on general results on finiteness of exponential moments of shot-noise processes. These results and their proofs can be found in Section 3. The appendix contains auxiliary results from random walk theory (Subsection A.1) and some elementary facts (Subsection A.2).

3 Shot-noise processes

Let $\xi$ be a real-valued random variable with $\mathbb{P}\{\xi = 0\} < 1$ and $(X(t))_{t \in \mathbb{R}}$ a doubly infinite non-negative stochastic process with non-decreasing paths such that $\lim_{t \to -\infty} X(t) = 0$ a.s. Any dependence between $(X(t))_{t \in \mathbb{R}}$ and $\xi$ is allowed. Further, given a sequence $((X_n(t))_{t \in \mathbb{R}}, \xi_n)_{n \geq 1}$ of independent copies of $((X(t))_{t \in \mathbb{R}}, \xi)$, define
\[ S_0 := 0, \quad S_n := \xi_1 + \ldots + \xi_n, \quad n \in \mathbb{N}, \]
and then the renewal shot-noise process $Z(\cdot)$ with random response functions $X_n(\cdot)$ by
\[ Z(t) := \sum_{n \geq 1} X_n(t - S_{n-1}), \quad t \in \mathbb{R}. \]
3.1 Examples of shot-noise processes

In this subsection, we give some examples of shot-noise processes.

Example 3.1. The current at time $t$ induced by an electron that arrives at time $s$ at the anode of a vacuum tube equals $f(t - s)$ for some appropriate deterministic response function $f$ vanishing on the negative halfline. Assuming that $X(t) = f(t)$ and the $S_n$ are the arrival times in a homogeneous Poisson process, the total current at time $t$ equals

$$Z(t) = \sum_{n \geq 1} f(t - S_{n-1}), \quad t \geq 0.$$ 

This is the classical shot-noise process [36].

Example 3.2. A very popular model in the literature has $X(\cdot)$ in multiplicative form $X(t) = \eta f(t)$ for a non-negative random variable $\eta$ and some deterministic $f$ (see [7, 24, 31, 35, 37] and the references therein). In the particular case $f(t) = e^{at}$ for some $a \neq 0$, the corresponding shot-noise process is a perpetuity, namely

$$Z(t) = e^{at} \sum_{n \geq 1} e^{-aS_{n-1}} \eta_n, \quad t \in \mathbb{R}.$$ 

The moment results for shot-noise processes we are going to derive hereafter will be a key in the analysis of the moments of $N(t)$, the number of visits to $(-\infty, t]$ of a PRW $(T_n)_{n \geq 1}$. The link between $N(t)$ and shot-noise processes is disclosed in the following example.

Example 3.3. If $X_n(t) = 1_{\{\eta_n \leq t\}}$ for a real-valued random variable $\eta_n$, $n \geq 1$, then $Z(t)$ equals the number of visits to $(-\infty, t]$ of the PRW $(S_{n-1} + \eta_n)_{n \geq 1}$, thus $Z(t) = N(t)$.

3.2 Finiteness of exponential moments of shot-noise processes

Our first moment result for shot-noise processes, assuming $\xi \geq 0$ a.s., provides two conditions which combined are necessary and sufficient for the finiteness of $\mathbb{E} e^{aZ(t)}$ for fixed $a > 0$ and $t \in \mathbb{R}$. As before, let $\tau^*(x) = \inf\{n \geq 1 : S_n > x\}$. Moreover, we denote by $U := \sum_{n \geq 0} \mathbb{P}\{S_n \in \cdot\}$ the renewal measure associated with $(S_n)_{n \geq 0}$.

Theorem 3.4. Let $\xi \geq 0$ a.s. Then, for any $a > 0$ and $t \in \mathbb{R}$,

$$\mathbb{E} e^{aZ(t)} < \infty$$

holds if and only if

$$r(t) := \int \left( \mathbb{E} e^{aX(t-y)} - 1 \right) U(dy) < \infty$$

and

$$l(t) := \mathbb{E} \left( \prod_{n=1}^{\tau^*} e^{aX_n(t-S_{n-1})} \right) < \infty.$$ 

Moreover, (3.2) alone implies $\mathbb{E} e^{aZ(t_0)} < \infty$ for some $t_0 \leq t$. 

9
Remark 3.5. It is easily seen from the proof given next that we may replace \( \tau^* \) in (3.3) by any other \((S_n)_{n \geq 0}\)-stopping time \( \tau \geq \tau^* \). Note also that, unlike the case when \( \mathbb{P}\{\xi < 0\} > 0 \) to be discussed later, \( \tau^* \) coincides with \( \tilde{\tau} := \inf\{n \geq 1 : \xi_n > 0\} \) and thus has a geometric distribution with parameter \( \mathbb{P}\{\xi > 0\} \). Finally, (3.3) is a trivial consequence of (3.2) if \( \xi > 0 \) a.s.

Proof. Observe that

\[
e^{aZ(t)} - 1 = \sum_{n \geq 1} \left( e^{aX_n(t-S_{n-1})} - 1 \right) \prod_{k \geq n+1} e^{aX_k(t-S_{k-1})}
\]

(3.4)

and

\[
e^{aZ(t)} \geq \prod_{n=1}^{\tau^*} e^{aX_n(t-S_{n-1})}
\]

(3.5)

hold whenever \( Z(t) < \infty \). Taking expectations in the above inequalities therefore gives the implications "(3.1) \( \Rightarrow \) (3.2)" and "(3.1) \( \Rightarrow \) (3.3)".

In turn, assume that (3.2) and (3.3) hold. Let \((\tau_n^*)_{n \geq 0}\) be the zero-delayed renewal sequence of strictly ascending ladder epochs of \((S_n)_{n \geq 0}\), thus \( \tau_1^* = \tau^* \) and define

\[L(s) := \prod_{n=1}^{\tau^*} e^{aX_n(s-S_{n-1})}\]

for \( s \in \mathbb{R} \). Then \( \mathbb{E}L(s) \leq \mathbb{E}L(t) < \infty \) for all \( s \leq t \), for \( L(\cdot) \) is non-decreasing and (3.3) holds. Pick \( \varepsilon > 0 \) so small that

\[\mathbb{E}L(s) \mathbb{1}_{\{\tau^* \leq \varepsilon\}} \leq \beta := \mathbb{E}L(t) \mathbb{1}_{\{\tau^* \leq \varepsilon\}} < 1\]

for all \( s \leq t \). Next define\( Z_0(\cdot) = Z_0^*(\cdot) = 0 \)

and

\[Z_n(\cdot) := \sum_{k=1}^{n} X_k(\cdot - S_{k-1}), \quad Z_n'(\cdot) := \sum_{k=\tau^*+1}^{\tau^*+n} X_k(\cdot - (S_{k-1} - S_{\tau^*}))\]

for \( n \in \mathbb{N} \). Plainly, \( Z_n(\cdot) \uparrow Z(\cdot) \) and similarly

\[Z_n'(\cdot) \uparrow Z'(\cdot) := \sum_{n \geq \tau^*+1} X_n(\cdot - (S_{n-1} - S_{\tau^*}))\]

as \( n \to \infty \). Note that each \( Z_n'(\cdot) \) is a copy of \( Z_n(\cdot) \) and further independent of \((L(\cdot), S_{\tau^*})\). Now observe that

\[Z_n(t) = Z_{\tau^*}(t) + Z_{n-\tau^*}(t) \mathbb{1}_{\{\tau^* \leq \varepsilon\}} + Z_n'(t - \varepsilon) \mathbb{1}_{\{\tau^* \leq \varepsilon\}} + Z_n(t - \varepsilon) \mathbb{1}_{\{\varepsilon > \tau^*\}} + Z_n'(t - \varepsilon) \mathbb{1}_{\{\varepsilon > \tau^*\}}\]

and therefore, using the stated independence properties,

\[
\mathbb{E}e^{aZ_n(t)} \leq \mathbb{E}\left(L(t) \mathbb{1}_{\{\tau^* \leq \varepsilon\}} e^{aZ_n(t)} + L(t) \mathbb{1}_{\{\varepsilon > \tau^*\}} e^{aZ_n'(t-\varepsilon)}\right)
\]

\[
\leq \beta \mathbb{E}e^{aZ_n(t)} + \mathbb{E}L(t) \mathbb{E}e^{aZ_n(t-\varepsilon)}
\]

(3.6)
for any $n \in \mathbb{N}$. Now notice that by (3.2), $\mathbb{E} e^{aX_1(t)} < \infty$ and hence, by Fubini’s theorem,

$$\mathbb{E} e^{aZ_n(t)} \leq \mathbb{E} \prod_{k=1}^{n} e^{aX_k(t)} = \left( \mathbb{E} e^{aX_1(t)} \right)^n < \infty. \quad (3.7)$$

By solving (3.6) for $\mathbb{E} e^{aZ_n(t)}$ and letting $n \to \infty$, we arrive at

$$\mathbb{E} e^{aZ(t)} \leq (1 - \beta)^{-1} \mathbb{E} L(t) \mathbb{E} e^{aZ(t-\varepsilon)}$$

and then upon successively repeating this argument at

$$\mathbb{E} e^{aZ(t)} \leq (1 - \beta)^{-n} \mathbb{E} e^{aZ(t-n\varepsilon)} \prod_{k=0}^{n-1} \mathbb{E} L(t-k\varepsilon)$$

for any $n \in \mathbb{N}$. Hence $\mathbb{E} e^{aZ(t)} < \infty$ as claimed if we verify $\mathbb{E} e^{aZ(t_0)} < \infty$ for some $t_0 < t$.

To this end, pick $t_0$ such that $r(t_0) < 1$ which is possible because (3.2) in combination with the monotone convergence theorem entails $\lim_{t \to -\infty} r(t) = 0$. Note also that $r(t_0) < 1$ implies $\mathbb{E} e^{aX(t_0)} < \infty$. Define

$$b_n := \mathbb{E} e^{aZ_n(t_0)} \quad \text{and} \quad c_n := \sum_{k=1}^{n} \mathbb{E} \left( e^{aX_k(t_0 - S_{k-1})} - 1 \right)$$

for $n \in \mathbb{N}_0$, in particular, $b_0 = 1$, $c_0 = 0$. The $b_n$’s are finite by the same argument as in (3.7). Moreover, $\sup_{n \geq 1} c_n = r(t_0) < 1$. With this notation and for any $n \in \mathbb{N}$, we obtain (under the usual convention that empty products are defined as 1)

$$e^{aZ_n(t_0)} - 1 = \sum_{k=1}^{n} \left( e^{aX_k(t_0 - S_{k-1})} - 1 \right) \prod_{j=k+1}^{n} e^{aX_j(t_0 - S_{j-1})}$$

$$\leq \sum_{k=1}^{n} \left( e^{aX_k(t_0 - S_{k-1})} - 1 \right) \prod_{j=k+1}^{n} e^{aX_j(t_0 - S_{j-1} - S_k)}$$

$$\leq \sum_{k=1}^{n} \left( e^{aX_k(t_0 - S_{k-1})} - 1 \right) \prod_{j=k+1}^{k+n-1} e^{aX_j(t_0 - S_{j-1} + S_k)}.$$ 

For fixed $k, n \in \mathbb{N}$, the random variable $\prod_{j=k+1}^{k+n-1} e^{aX_j(t- S_{j-1} - S_k)}$ is independent of $e^{aX_k(t_0 - S_{k-1})}$ and has the same law as $e^{aZ_{n-1}(t_0)}$. Taking expectations, we get

$$b_n - 1 \leq c_n b_{n-1} \leq r(t_0)b_{n-1} \quad \text{for} \quad n \in \mathbb{N}$$

and thereupon at $b_n \leq (1 - r(t_0))^{-1}$ for all $n \in \mathbb{N}$. Finally letting $n \to \infty$, we conclude $\mathbb{E} e^{aZ(t_0)} < \infty$.

The previous argument has only used (3.2) and thus also shown the last assertion of the theorem. \hfill \Box
We will now carry over the previous result to the case when \((S_n)_{n \geq 0}\) is a positively divergent random walk taking negative values with positive probability. As before, let \(U\) be the pertinent intensity measure and \(U^>\) the renewal measure of the associated renewal process \((S_n^>)_{n \geq 0}\), say, of strictly ascending ladder epochs are denoted as \(\tau_n^>\) for \(n \in \mathbb{N}\), thus \(\tau_1^* = \tau^*\). Further, defining \(M^*\) to be a generic copy of \(\inf_{n \geq 0} S_n\) that is independent of any other occurring random variable, a well-known identity in the fluctuation theory of random walks (see e.g. [5, Theorem VIII.2.2] after a change of sign) states that

\[
U(B) = (E \tau^*) Q \ast U^>(B) = (E \tau^*) E U^>(B - M^*) \tag{3.8}
\]

for all Borel subsets \(B\) of \(\mathbb{R}\), where \(Q := \mathbb{P}\{M^* \in \cdot\}\) and \(\ast\) denotes convolution.

**Theorem 3.6.** Let \((S_n)_{n \geq 0}\) be positively divergent and \(\mathbb{P}\{\xi < 0\} > 0\). Then the following assertions are equivalent for any \(a > 0\):

\[
\mathbb{E} e^{al(t)} < \infty \text{ for some } t \in \mathbb{R}, \tag{3.9}
\]
\[
\mathbb{E} e^{al(t)} < \infty \text{ for all } t \in \mathbb{R}, \tag{3.10}
\]
\[
r^>(t) < \infty \text{ for all } t \in \mathbb{R}, \tag{3.11}
\]
\[
r^>(t) < \infty \text{ for some } t \in \mathbb{R}, \tag{3.12}
\]

where \(l(t)\) is defined as in (3.3) and

\[
r^>(t) := \int (l(t - u) - 1) \mathbb{U}^>(du)
\]

for \(t \in \mathbb{R}\). Furthermore, the conditions imply \(r(t) < \infty\) and \(l(t) < \infty\) for all \(t \in \mathbb{R}\).

**Proof.** The last assertion follows from (3.4) and \(l(t) - 1 \leq r^>(t)\).

“(3.9) \Rightarrow (3.11)”. Put \(g(t) := \mathbb{E} e^{al(t)}\) for \(t \in \mathbb{R}\) and use the first line of (3.4) to infer via conditioning and with the help of (3.8)

\[
g(t) - 1 = \sum_{n \geq 1} \mathbb{E} \left( \left( e^{aXn(t-S_{n-1})} - 1 \right) \prod_{k \geq n+1} e^{aXk(t-S_{k-1})} \right)
\]
\[
= \sum_{n \geq 1} \mathbb{E} \left( e^{aXn(t-S_{n-1})} - 1 \right) \mathbb{E} \left( \prod_{k \geq n+1} e^{aXk(t-S_{k-1})} \mid S_n \right)
\]
\[
= \sum_{n \geq 1} \mathbb{E} \left( e^{aXn(t-S_{n-1})} - 1 \right) g(t - S_n)
\]
\[
= \int_{[0,\infty)} \mathbb{E} \left( e^{aX(t-y)} - 1 \right) g(t - y - \xi) \mathbb{U}(dy)
\]
\[
= (E \tau^*) \int_{[0,\infty)} \mathbb{E} \left( e^{aX(t-y-M^*)} - 1 \right) g(t - y - \xi - M^*) \mathbb{U}^>(dy)
\]
\[
\geq \mathbb{E} \left( e^{aX(t-M^*)} - 1 \right) g(t - \xi - M^*)
\]
for any $t \in \mathbb{R}$. But $\mathbb{P}\{\xi < 0\} > 0$ implies $\mathbb{P}\{\xi + M^* < -x\} > 0$ for all $x > 0$ (notice that $\xi + M^* \overset{d}{=} \inf_{n \geq 1} S_n$). Consequently, $g(t + x) < \infty$ for any $x > 0$ if $g(t) < \infty$. By monotonicity, we also have $g(t + x) < \infty$ for $x < 0$.

“(3.10) $\Rightarrow$ (3.11)” Put

$$L_n(s) := \prod_{k=1}^{\tau_n - 1} \exp \left( aX_k(s - (S_{k-1} - S_{\tau_n-1})) \right)$$

for $n \in \mathbb{N}$ and $s \in \mathbb{R}$ which are i.i.d. with $L_1(s) = L(s)$ as defined in the proof of Theorem 3.4. If $\mathbb{E} e^{aZ(t)} < \infty$, then

$$e^{aZ(t)} - 1 = \sum_{n \geq 1} \left( L_n(t - S_{\tau_n-1}) - 1 \right) \prod_{k \geq n+1} L_k(t - S_{\tau_k-1})$$

$$\geq \sum_{n \geq 1} \left( L_n(t - S_{\tau_n-1}) - 1 \right). \quad (3.13)$$

Taking expectations on both sides of this inequality gives $r^>(t) < \infty$.

“(3.12) $\Rightarrow$ (3.11)” If $r^>(t)$ for some $t \in \mathbb{R}$, then also $l(t) < \infty$ and, therefore, $r^>(t_0) < 1$ and $l(t_0) - 1 < 1$ for some $t_0 \leq t$. Since

$$e^{aZ_{\tau_n}(s)} \leq \prod_{k=1}^{n} L_k(s),$$

we infer

$$b_n := \mathbb{E} e^{aZ_{\tau_n}(t_0)} \leq (\mathbb{E} L(t_0))^n = (t_0)^n < \infty$$

for any $n \in \mathbb{N}$. Putting

$$c_n := \mathbb{E} \sum_{k=1}^{n} \left( L_k(t - S_{\tau_{k-1}}) - 1 \right)$$

we have $\sup_{n \geq 1} c_n = r^>(t_0) < 1$ and thus find by a similar estimation as in the proof of Theorem 3.4 for non-negative $\xi$ that $b_n \leq 1 + c_n b_{n-1}$ and thus $b_n \leq (1 - r^>(t_0))^{-1}$ for all $n \in \mathbb{N}$. Hence, $\mathbb{E} e^{aZ(t_0)} < \infty$, for $Z_{\tau_n}(t_0) \uparrow Z(t_0)$. □

### 3.3 Finiteness of power moments of shot-noise processes

Turning to power moments, we consider the case $\xi \geq 0$ a.s. only.

**Theorem 3.7.** Let $\xi \geq 0$ a.s. Then for any $p \geq 1$ and $t \in \mathbb{R}$, the following assertions are equivalent:

$$\mathbb{E} Z(t)^p < \infty. \quad (3.14)$$

$$s_q(t) := \int \mathbb{E} X(t - y)^q \mathbb{U}(dy) < \infty \quad \text{for all } q \in [1, p]; \quad (3.15)$$


Proof. “(3.14) ⇒ (3.15)” : Let \( \mathbb{E} Z(t)^p < \infty \) and \( q \in [1, p] \). Using the superadditivity of the function \( x \mapsto x^q \) for \( x \geq 0 \), we then infer

\[
\infty > \mathbb{E} Z(t)^q \geq \mathbb{E} \sum_{k \geq 1} X_k(t - S_{k-1})^q = \int_0^\infty \mathbb{E} X(t - y)^q \, \text{d}y
\]

which is the desired conclusion.

“(3.15) ⇒ (3.14)” : To prove this implication, we write \( p = n + \delta \) with \( n \in \mathbb{N}_0 \), \( \delta \in (0, 1] \) and use induction on \( n \). When \( n = 0 \), then necessarily \( \delta = 1 \), i.e., \( p = 1 \). Then there is nothing to verify, for

\[
\mathbb{E} Z(t) = \int_0^\infty \mathbb{E} X(t - y) \, \text{d}y = s_1(t) < \infty.
\]

In the induction step, we assume that the asserted implication holds for \( p = n \) and conclude that it then also holds for \( p = n + \delta \) for all \( \delta \in (0, 1] \). To this end, assume that \( p = n + \delta \) for some \( n \in \mathbb{N} \) and \( \delta \in (0, 1] \) and that \( s_q(t) < \infty \) for all \( q \in [1, p] \). By induction hypothesis, \( \mathbb{E} Z(t)^n < \infty \). For \( k \in \mathbb{N} \) and \( t \in \mathbb{R} \), define

\[
Z_k(t) := \sum_{j \geq k+1} X_j(t - (S_{j-1} - S_k)).
\]

Then \( Z_k(\cdot) \) is a copy of \( Z_0(\cdot) := Z(\cdot) \) and also independent of \( \mathcal{F}_k := \sigma((X_j, \xi_j) : j = 1, \ldots, k) \). \( Z_k \) satisfies \( Z_k(t) = X_{k+1}(t) + Z_{k+1}(t - \xi_{k+1}) \) for all \( t \in \mathbb{R} \). Using (A.10), we get

\[
Z(t)^p = (X_1(t) + Z_1(t - \xi_1))^p \\
\leq X_1(t)^p + Z_1(t - \xi_1)^p \\
+ p 2^{p-1} (X_1(t) Z_1(t - \xi_1)^{p-1} + X_1(t)^n Z_1(t - \xi_1)^\delta).
\]

Iterating this inequality and using

\[
Z_k(t - S_k) = \sum_{j \geq k+1} X_j(t - S_{j-1}) \to 0 \text{ a.s. as } k \to \infty
\]

we obtain the following upper bound for \( Z(t)^p \):

\[
Z(t)^p \leq \sum_{j \geq 1} X_j(t - S_{j-1})^p \\
+ p 2^{p-1} \left( \sum_{j \geq 1} X_j(t - S_{j-1}) Z_j(t - S_j)^{p-1} \\
+ \sum_{j \geq 1} X_j(t - S_{j-1})^n Z_j(t - S_j)^\delta \right).
\]

\( \mathbb{E} Z(t)^n < \infty \) implies that \( \mathbb{E} Z(t)^q \) is finite for \( 0 < q \leq n \). Using this and the monotonicity of \( Z_j \), we conclude

\[
\mathbb{E} Z(t)^p \leq s_p(t) + p 2^{p-1} (s_1(t) \mathbb{E} Z(t)^{p-1} + s_n(t) \mathbb{E} Z(t)^\delta) < \infty.
\]

\( \square \)
4 Proofs of the main results

4.1 Proofs of the results on a.s. finiteness of $\tau(x)$, $N(x)$ and $\rho(x)$

Proof of Theorem 2.1. By Theorem 2.1 in [13], (2.2) is necessary and sufficient for $\lim_{n \to \infty} T_n = -\infty$ a.s. and thus, by symmetry, (2.1) is equivalent to $\lim_{n \to \infty} T_n = \infty$ a.s. On p. 1215 of [13] it is shown that $\limsup_{n \to \infty} T_n < \infty$ a.s. entails $\lim_{n \to \infty} T_n = -\infty$ a.s. This proves the remaining assertions.

One half of the proof of Theorem 2.3 is settled by the following lemma.

Lemma 4.1. Let $x \in \mathbb{R}$, $\mathbb{P}\{\xi < 0, \eta \leq x\} = 0$ and $p := \mathbb{P}\{\eta \leq x\} < 1$. Then $\mathbb{P}\{\tau(x) > n\} \leq p^n$ for $n \in \mathbb{N}$. If $p = 1$, then $\limsup_{n \to \infty} T_n = \infty$ a.s.

Proof. Let $x \in \mathbb{R}$ and $\mathbb{P}\{\xi < 0, \eta \leq x\} = 0$. Then $p = 1$ entails $\xi \geq 0$ a.s., thus $\lim_{n \to \infty} S_n = \infty$ a.s. (recalling our standing assumption) and thus, by Theorem 2.1, $\limsup_{n \to \infty} T_n = \infty$ a.s.

Now assume that $p < 1$. Then $\nu := \inf\{n \in \mathbb{N} : \eta_n > x\}$ has a geometric distribution, namely $\mathbb{P}\{\nu > n\} = p^n$ for $n \in \mathbb{N}$. By assumption, $\xi_k \geq 0$ a.s. for $k = 1, \ldots, n - 1$ on $\{\nu = n\}$ whence $T_n = \xi_1 + \ldots + \xi_{n-1} + \eta_n \geq \eta_n > x$ a.s. on $\{\nu = n\}$ and therefore

$$\mathbb{P}\{\tau(x) > n\} = \mathbb{P}\{T_k \leq x \text{ for } k = 1, \ldots, n\} \leq \mathbb{P}\{\nu > n\} = p^n.$$ for any $n \in \mathbb{N}$.

4.2 Proofs of the results on finiteness of exponential moments of $\tau(x)$, $N(x)$ and $\rho(x)$

Proof of Theorem 2.2. In view of the previous lemma it remains to argue that, given a negatively divergent PRW $(T_n)_{n \geq 1}$, the a.s. finiteness of $\tau(x)$ for some $x \in \mathbb{R}$ implies $\mathbb{P}\{\xi < 0, \eta \leq x\} = 0$ which will be done by contraposition:

If $\mathbb{P}\{\xi < 0, \eta \leq x\} > 0$, we can fix $\varepsilon > 0$ such that $\mathbb{P}\{\xi \leq -\varepsilon, \eta \leq x\} > 0$. By negative divergence, $\sup_{n \geq 1} T_n < \infty$ a.s. so that we can further pick $y \in \mathbb{R}$ such that $\mathbb{P}\{\sup_{n \geq 1} T_n \leq y\} > 0$. Define $m := \inf\{k \in \mathbb{N}_0 : k\varepsilon \geq y - x\}$. Then

$$\mathbb{P}\{\tau(x) = \infty\} = \mathbb{P}\{\sup_{n \geq 1} T_n \leq x\} \geq \mathbb{P}\left\{\max_{1 \leq k \leq m} \xi_k \leq -\varepsilon, \max_{1 \leq k \leq m} \eta_k \leq x, \sup_{j > m} T_j - S_m \leq y\right\} = \mathbb{P}\{\xi \leq -\varepsilon, \eta \leq x\}^m \mathbb{P}\{\sup_{n \geq 1} T_n \leq y\} > 0$$

yields the desired conclusion.

Recall that $\tau^*(x)$ denotes the counterpart of $\tau(x)$ for the ordinary random walk $(S_n)_{n \geq 0}$ and note also that, for any $a > 0$, $\mathbb{E} e^{a\tau^*(x)} < \infty$ for all $x \in \mathbb{R}$ is equivalent to $\mathbb{E} e^{a\tau^*} < \infty$. Put

$$\nu(x) := \inf\{n \geq 1 : \eta_n > x\}$$

for $x \in \mathbb{R}$. We make the observation that $\tau^* \wedge \nu(x) \leq \tau(x)$, for
either \( S_{\tau(x)-1} > 0 \) (\( \Rightarrow \tau^* < \tau(x) \)),

or \( S_{\tau(x)-1} \leq 0 \) and \( \eta_{\tau(x)} > x \) (\( \Rightarrow \nu(x) \leq \tau(x) \)).

**Lemma 4.2.** Let \( a > 0 \) and suppose that \( \mathbb{P}\{\xi < 0, \eta \leq x\} > 0 \) as well as \( \mathbb{E} e^{a\tau(x)} < \infty \) for some fixed \( x \in \mathbb{R} \). Then \( \mathbb{E} e^{a\tau(y)} < \infty \) for all \( y \in \mathbb{R} \).

**Proof.** By monotonicity, \( \mathbb{E} e^{a\tau(y)} < \infty \) for all \( y \leq x \). Now fix some \( \varepsilon > 0 \) such that \( \mathbb{P}\{\xi \leq -\varepsilon, \eta \leq x\} > 0 \). Then

\[
\mathbb{E} e^{a\tau(x)} \geq \mathbb{E} e^{a\tau(x+\varepsilon)} \mathbb{P}\{\xi \leq -\varepsilon, \eta \leq x\}
\]

implies \( \mathbb{E} e^{a\tau(x+\varepsilon)} < \infty \). By repeating this argument with \( x + \varepsilon, x + 2\varepsilon, \ldots \) and noting that \( \mathbb{P}\{\xi \leq -\varepsilon, \eta \leq x + n\varepsilon\} > 0 \), we infer \( \mathbb{E} e^{a\tau(x+nc)} < \infty \) for all \( n \in \mathbb{N} \).

**Proof of Theorem 4.2** (a) If \( \mathbb{P}\{\xi < 0, \eta \leq x\} = 0 \), then \( \tau(y) \leq \nu(y) \) for all \( y \leq x \) and therefore \( g(y) := \mathbb{E} e^{a\tau(y)} < \infty \) when \( e^a \mathbb{P}\{\eta \leq y\} < 1 \), because in this case

\[
g(y) \leq \mathbb{E} e^{a\tau(y)} = \mathbb{P}\{\eta > y\} \sum_{n \geq 1} e^{an} \mathbb{P}\{\eta \leq y\}^{n-1} < \infty.
\]

Turning to the asserted equivalence, note first that \( e^a \mathbb{P}\{\xi = 0, \eta \leq x\} \geq 1 \) implies \( \mathbb{E} e^{a\tau(x)} = \infty \) because

\[
\sum_{n \geq 1} e^{an} \mathbb{P}\left\{ \max_{1 \leq k \leq n} T_k \leq x \right\} \geq \sum_{n \geq 1} e^{an} \mathbb{P}\left\{ \max_{1 \leq k \leq n} T_k \leq x, S_{n-1} = 0 \right\} = \mathbb{P}\{\eta \leq x\} \sum_{n \geq 1} e^{an} \mathbb{P}\{\xi = 0, \eta \leq x\}^{n-1} = \infty.
\]

For the converse implication, assume \( e^a \mathbb{P}\{\xi = 0, \eta \leq x\} < 1 \). For \( n \in \mathbb{N} \), define \( \hat{\xi}_n := \xi_n 1_{\{\eta_n \leq x\}} + 1_{\{\eta_n > x\}} \), \( n \in \mathbb{N} \). Observe that \( \hat{\xi}_n \geq 0 \) a.s. since \( \mathbb{P}\{\xi_n < 0\} = \mathbb{P}\{\xi_n < 0, \eta_n \leq x\} = 0 \). Let \( \hat{S}_n := \hat{\xi}_1 + \ldots + \hat{\xi}_n \), and \( \hat{T}_n := \hat{S}_{n-1} + \eta_n \), \( n \in \mathbb{N} \), i.e., \( (\hat{T}_n)_{n \geq 1} \) is the PRW based on the sequence \( (\hat{\xi}_1, \eta_1), (\hat{\xi}_2, \eta_2), \ldots \). By construction, \( \hat{T}_n = T_n \) for all \( n \leq \nu(x) \). On the other hand, \( \tau(x) \leq \nu(x) \) a.s. due to the assumption \( \mathbb{P}\{\xi < 0, \eta \leq x\} = 0 \). Consequently, \( \hat{\tau}(x) := \inf\{n \geq 1 : \hat{T}_n > x\} = \tau(x) \) a.s. In particular, \( \mathbb{E} e^{a\tau(x)} \) is finite iff \( \mathbb{E} e^{a\hat{\tau}(x)} \) is finite. To see that the latter is finite in the given situation, let \( \hat{\tau}^*(y) := \inf\{n \geq 1 : \hat{S}_n > y\} \) for \( y \geq 0 \) and observe that \( \mathbb{E} e^{a\hat{\tau}^*(y)} < \infty \) for all \( y \geq 0 \) by Proposition 1.1 in [22] since \( e^a \mathbb{P}\{\hat{\xi}_1 = 0\} = e^a \mathbb{P}\{\xi = 0, \eta \leq x\} < 1 \). Pick \( u \in \mathbb{R} \) such that \( \mathbb{P}\{\eta \leq -u\} < e^{-a} \) and define \( \nu' := \inf\{n \geq 1 : \eta_{\hat{\tau}^*(u+x)+n} > -u\} \). Then

\[
\mathbb{E} e^{a\nu'} = \sum_{n \geq 1} e^{an} \mathbb{P}\{\nu' = n\} = \mathbb{P}\{\eta > -u\} \sum_{n \geq 1} e^{an} \mathbb{P}\{\eta \leq -u\}^{n-1} < \infty.
\]

Since \( \hat{S}_n \) is increasing in \( n \), we have \( \tau'(x) \leq \hat{\tau}^*(x + u) + \nu' \). Therefore, using the independence of \( \hat{\tau}^*(x + u) \) and \( \nu' \), we infer

\[
\mathbb{E} e^{a\tau(x)} \leq \mathbb{E} e^{a\hat{\tau}(x+u) + \nu'} = \mathbb{E} e^{a\hat{\tau}(x+u)} \mathbb{E} e^{a\nu'} < \infty.
\]
(b) Since Lemma 4.2 gives the equivalence of (2.7) and (2.8) and the equivalence of (2.9) and (2.10) has been shown as Theorem 1.2 in [22], we are left with a proof of "(2.8) ⇒ (2.10)" and "(2.9) ⇒ (2.7)".

"(2.8) ⇒ (2.10)" Suppose \( E e^{a\tau(y)} < \infty \) for all \( y \in \mathbb{R} \) and recall that \( \tau^* \wedge \nu(y) \leq \tau(y) \). Then it follows that

\[
E \left( e^{a\tau^*} \mathbb{1}_{(\nu(y) > \tau^*)} \right) < \infty
\]

for all \( y \in \mathbb{R} \). Let \( \tau^*_y \) denote the first strictly ascending ladder epoch of a standard random walk with increment distribution \( P \{ \xi \in \cdot | \eta \leq y \} \) for any \( y \) with \( e^{-\theta(y)} := P \{ \eta \leq y \} > 0 \). Then

\[
P\{\tau^*_y = n\} = P\{\tau^* = n | \nu(y) > n\}
\]

for each \( n \in \mathbb{N} \) and therefore

\[
\infty > E \left( e^{a\tau^*} \mathbb{1}_{(\nu(y) > \tau^*)} \right) = \sum_{n \geq 1} e^n P\{\tau^* = n, \nu(y) > n\}
\]

\[
= \sum_{n \geq 1} e^{a - \theta(y)n} P\{\tau^*_y = n\}
\]

\[
= E e^{a - \theta(y)\tau^*_y}.
\]

By invoking Theorem 1.2 in [22], we infer

\[
a - \theta(y) \leq - \log \inf_{t \geq 0} E(e^{-\xi} | \eta \leq y) = - \log \inf_{t \geq 0} E e^{-\xi} \mathbb{1}_{(\eta \leq y)} - \theta(y)
\]

and hence \( a \leq - \log \inf_{t \geq 0} E e^{-\xi} \mathbb{1}_{(\eta \leq y)} \) for all sufficiently large \( y \). It remains to show that \( \inf_{t \geq 0} E e^{-\xi} \mathbb{1}_{(\eta \leq y)} \rightarrow \inf_{t \geq 0} E e^{-\xi} \) as \( y \rightarrow \infty \). To this end, put \( \varphi_y(t) := E e^{-\xi} \mathbb{1}_{(\eta \leq y)} \) and notice that \( P\{\xi < 0, \eta \leq y \} > 0 \) for all sufficiently large \( y \). For these \( y \), \( \varphi_y \) assumes its infimum at some unique \( 0 \leq t_y < \infty \), say. Let \( t_\infty \) denote the unique minimizer of \( \varphi(t) = E e^{-\xi} \) on \([0, \infty)\). Then \( \varphi_y(t_y) \leq \varphi(t_\infty) \leq 1 \). On the other hand, \( P\{\xi < 0\} > 0 \) implies that \( \varphi(t) > 2 \) for some \( t \). Since \( \varphi_y(t) \uparrow \varphi(t) \) \( (y \uparrow \infty) \), \( \varphi_y(t) > 1 \geq \varphi_y(t_y) \) for all large enough \( y \). Using \( \varphi_y(0) < 1 \) and the convexity of \( \varphi_y \), we infer that \( t_y < t \) and therefore \( \limsup_{y \rightarrow \infty} \varphi_y(t) < \infty \). By compactness, any sequence increasing to \( +\infty \) has a subsequence \( y \uparrow \infty \) such that \( t_y \) converges. Using the continuity of \( \varphi \) and the fact that \( \varphi_y \) increases to \( \varphi \) it can easily be seen that the limit is \( t_\infty \). Therefore, we conclude that \( t_y \rightarrow t_\infty \) along any arbitrary sequence \( y \rightarrow \infty \). Using this, we get that

\[
0 \leq R - \inf_{t \geq 0} E e^{-\xi} \mathbb{1}_{(\eta \leq y)} = \varphi(t_\infty) - \varphi_y(t_y)
\]

\[
= (\varphi(t_\infty) - \varphi(t_y)) + (\varphi(t_y) - \varphi_y(t_y)).
\]

The first term tends to 0 because of the continuity of \( \varphi \) and \( t_y \rightarrow t_\infty \), the second term tends to 0 because of the (local) uniform convergence of \( \varphi_y \) to \( \varphi \) on \( \{ \varphi < \infty \} \). This implies (2.10).
“(2.3) ⇒ (2.7)” Suppose that \( \mathbb{E} e^{at^*} < \infty \) and consider the renewal sequence of strictly ascending ladder epochs \((\tau_n^*)_{n \geq 0}\) associated with \((S_n)_{n \geq 0}\), thus \( \tau_1^* = \tau^* \). Pick \( s \in \mathbb{R} \) such that

\[
\gamma := \mathbb{E} \left( e^{at^*} \mathbb{I}_{\{\eta^* \leq \xi^* + s\}} \right) < 1
\]

and then define \( \sigma := \inf \{ n \geq 1 : \eta_n^* > \xi_n^* + s \} \), which has a geometric distribution on \( \mathbb{N} \) with parameter \( \mathbb{P}\{\eta^* > \xi^* + s\} \). Since \( T_{\tau^*} = S_{\tau^*} + \eta_{\tau^*} - \xi_{\tau^*} > s \), we infer \( \tau(s) \leq \tau^* \). Finally, use that \( (\tau_n^* - \tau_{n-1}^*, \xi_{\tau_n^*}) \), \( n \in \mathbb{N} \) are i.i.d. to infer

\[
\mathbb{E} e^{a\tau^*} = \mathbb{E} \left( \prod_{k=1}^{\sigma} e^{a(\tau_k^* - \tau_{k-1}^*)} \right) = \sum_{n \geq 0} \gamma^n \mathbb{E} \left( e^{a\tau^*} \mathbb{I}_{\{\eta^* > \xi^* + s\}} \right) < \infty
\]

and therefore \( \mathbb{E} e^{at^*} < \infty \). If \( s \geq x \), this also proves \( \mathbb{E} e^{at^*(x)} < \infty \). Otherwise, consider the level 1 ladder epochs \( \tau_1^*(1), \tau_2^*(1), \ldots \) of \((S_n)_{n \geq 0}\) and pick \( m \) so large that \( s + m \geq x \). Observe that \( \tau(x) \leq \tau_n^*(1) + \tau'(s) \) where

\[
\tau'(s) := \inf \{ n \geq 1 : T_{\tau_n^*(1)+n} - S_{\tau_n^*(1)} > s \}.
\]

Then \( \tau'(s) \) is a copy of \( \tau(s) \) and independent of \((\tau_k^*(1))_{1 \leq k \leq m}\). In combination with \( \mathbb{E} e^{at^*(1)} < \infty \), this implies

\[
\mathbb{E} e^{at^*(x)} \leq \left( \mathbb{E} e^{at^*(1)} \right)^m \mathbb{E} e^{at^*(s)} < \infty.
\]

The proof is complete.

**Proof of Theorem 2.6.**

(a) Fix any \( a > 0 \) and \( x \in \mathbb{R} \). For \( y \geq 0 \), define

\[
\tilde{\tau}(y) := \inf \{ n \geq 1 : \xi_n > y \}.
\]

Consider the renewal shot-noise process \( Z(\cdot) \) with generic response function \( X(t) := \sum_{k=1}^{\tilde{\tau}(0)} \mathbb{I}_{\{\eta_k \leq t\}} \) and generic renewal increment \( \xi'(0) := S_{\tilde{\tau}(0)} > 0 \) having distribution \( \mathbb{P}\{\xi \in \cdot | \xi > 0\} \). Then it is easily seen that \( N(x) = Z(x) \) for all \( x \in \mathbb{R} \) and therefore, by Theorem 3.3 and Remark 3.3 that \( \mathbb{E} e^{aN(x)} < \infty \) iff

\[
\int_{0}^{\infty} \left( \mathbb{E} \exp \left( a \sum_{k=1}^{\tilde{\tau}(0)} \mathbb{I}_{\{\eta_k \leq x-u\}} \right) - 1 \right) U'(du) < \infty, \tag{4.1}
\]

where \( U' \) denotes the renewal measure associated with \( \xi' \) and satisfies

\[
J_+(y) \mathbb{P}\{\xi > 0\} \leq U'(y) \leq 2J_+(y) \mathbb{P}\{\xi > 0\} \tag{4.2}
\]

for all \( y > 1 \), see e.g. (4.1) in [3]. Since

\[
\mathbb{E} e^{a(\eta \leq x)} \mathbb{I}_{\{\xi = 0\}} = e^{a} \mathbb{P}\{\xi = 0, \eta \leq x\} + \mathbb{P}\{\xi = 0, \eta > x\},
\]

18
we see that (2.13) is equivalent to

\[ E \exp \left( a \sum_{k=1}^{\bar{\tau}(0)} 1_{\{n_k \leq x\}} \right) = \frac{E e^{a \overline{1}_{\{n \leq x\}} 1_{\{\xi > 0\}}} \leq \infty}{1 - E e^{a \overline{1}_{\{n \leq x\}} 1_{\{\xi = 0\}}} < \infty} \quad (4.3) \]

Validity of (4.3) further implies (4.1) because

\[ \int_0^\infty \left( E \exp \left( a \sum_{k=1}^{\bar{\tau}(0)} 1_{\{n_k \leq x-u\}} \right) - 1 \right) U'(du) \]

\[ = \int_0^\infty \frac{E e^{a \overline{1}_{\{n \leq x-u\}} - 1}}{1 - E e^{a \overline{1}_{\{n \leq x-u\}} 1_{\{\xi = 0\}}} U'(du) \quad (e^a - 1) P\{\eta \leq x-u\} U'(du) \]

\[ \leq \frac{(e^a - 1)}{1 - E e^{a \overline{1}_{\{n \leq x\}} 1_{\{\xi = 0\}}} U'(du) \quad \int_0^\infty P\{\eta - x \geq u\} U'(du) \]

\[ \leq \frac{2(e^a - 1) P\{\xi > 0\}}{1 - E e^{a \overline{1}_{\{n \leq x\}} 1_{\{\xi = 0\}}} E J_{+}(\eta - x^-) \]

where (4.2) has been utilized for the last line and $E J_{+}(\eta - x^-) < \infty$ by (2.11).

Since, conversely, (2.13) follows directly from (4.1), we have thus proved the equivalence of (2.12) and (2.13). To check the remaining assertions is easy and therefore omitted.

(c) First observe that (2.16) is equivalent to (2.17) by Theorem 1.2 in [22]. Next we show that $E e^{aN(x)} < \infty$ for some $x \in \mathbb{R}$ implies $E e^{aN(x)} < \infty$ for all $x \in \mathbb{R}$. Indeed, since $P\{\xi < 0\} > 0$, for any given $y > x$ we find $n \in \mathbb{N}$ such that $P\{S_n \leq x-y\} > 0$ and hence

\[ \infty > E e^{aN(x)} \geq E 1_{\{S_n \leq x-y\}} e^{a \sum_{k>n} 1_{\{\tau_k - S_n \leq y\}}} \]

\[ \geq P\{S_n \leq x-y\} E e^{aN(y)} \]

Now we show “(2.15) ⇒ (2.16)”. Since $P\{\xi < 0, \eta \leq x\} \rightarrow P\{\xi < 0\} > 0$ as $x \rightarrow \infty$, we can choose $x \in \mathbb{R}$ so large such that $P\{\xi < 0, \eta \leq x\} > 0$. Using that $N(x) \geq \tau(x) - 1$, we infer from (2.15) that $E e^{a\tau(x)} < \infty$. By Theorem (2.17), this implies $E e^{a\tau} < \infty$ which is equivalent to (2.16) by Theorem 1.2 in [22].

(2.17) ⇒ (2.15). By (2.17), there exists a minimal $\gamma > 0$ such that $E e^{-a\tau} = e^{-a}$. $\gamma$ can be used to define a new probability measure $P_\gamma$ by

\[ E_\gamma h(S_0, \ldots, S_n) = e^{an} E e^{-\gamma S_n} h(S_0, \ldots, S_n), \quad n \in \mathbb{N} \quad (4.4) \]

for each non-negative Borel measurable function $h$ on $\mathbb{R}^{n+1}$ where $E_\gamma$ denotes the expectation with respect to $P_\gamma$.

Recall that $\tau_n^*$ denotes the $n$ strictly increasing ladder index of the process $(S_n)_{n \geq 0}$ and that $U^\gamma(\cdot) := \sum_{n \geq 0} P\{S_{\tau_n^*} \in \cdot\}$ denotes the renewal measure

19
of the corresponding ladder height process. Then, according to Theorem 3.6 (with \(X(t) = \mathbb{1}_{\{t \leq t\}}\)) it suffices to prove that

\[
r^>(0) := \int (l(-u) - 1) U^>(du) < \infty,
\]

(4.5)

where \(l(x) := \mathbb{E} \left( \prod_{n=1}^{\tau^*} e^{\alpha (T_n \leq x)} \right), x \in \mathbb{R}\).

For \(x \in \mathbb{R}\), set

\[
\beta(x) := \sup\{n \leq \tau^* : T_n \leq x\}
\]

if \(\min_{1 \leq n \leq \tau^*} T_n \leq x\), and let \(\beta(x) := 0\), otherwise. Then \(l(x) \leq \mathbb{E} e^{a \beta(x)}\). Therefore, (4.5) follows from

\[
\int \left( \mathbb{E} \exp(a \beta(-u)) - 1 \right) U^>(du) < \infty.
\]

(4.6)

Now

\[
\mathbb{E} e^{a \beta(x)} - 1 \leq \sum_{n \geq 0} e^{an} \mathbb{P}\{\tau^* \geq n, T_n \leq x\} - \mathbb{P}\{\min_{1 \leq n \leq \tau^*} T_n > x\}
\]

\[
\leq \sum_{n \geq 1} e^{an} \mathbb{P}\{\tau^* \geq n, T_n \leq x\}
\]

\[
= \sum_{n \geq 1} e^{an} \mathbb{E} F(x - S_{n - 1}) \mathbb{1}_{\{\tau^* \geq n\}}
\]

\[
= e^a \sum_{n \geq 0} \mathbb{E}_\gamma e^{\gamma S_n} F(x - S_n) \mathbb{1}_{\{\tau^* < \infty\}}
\]

(4.7)

where (4.4) has been utilized in the last step. Now let \(\sigma_0^* := 0\) and \(\sigma_n^* := \inf\{k > \sigma_{n-1}^* : S_k \leq S_{\sigma_n^* - 1}\}\) for \(n \geq 1\) where \(\inf \emptyset = \infty\). We now make use of the following duality, see e.g. [5, Theorem VIII.2.3(b)],

\[
\sum_{n \geq 0} \mathbb{P}\gamma \{S_n \in \cdot, \tau^* > n\} = \sum_{n \geq 0} \mathbb{P}\gamma \{S_{\sigma_n^*} \in \cdot, \sigma_n^* < \infty\}
\]

(4.8)

Using this in (4.7) gives

\[
\mathbb{E} e^{a \beta(x)} - 1 \leq e^a \sum_{n \geq 0} \mathbb{E}_\gamma e^{\gamma S_{\sigma_n^*}} F(x - S_{\sigma_n^*}) \mathbb{1}_{\{\sigma_n^* < \infty\}}.
\]
Integrating with \( x \) replaced by \(-u\) w.r.t. \(\mathbb{U}^>(du)\) gives

\[
\int \mathbb{E}(e^{\alpha(-u)} - 1) \mathbb{U}^>(du)
\]

\[
\leq e^{a} \int \sum_{n \geq 0} \mathbb{E}_e e^{\gamma S_{\sigma^*_n}} F\left(-u - S_{\sigma^*_n}\right) \mathbb{1}_{\{\sigma^*_n < \infty\}} \mathbb{U}^>(du)
\]

\[
\leq e^{a} \sum_{n \geq 0} \mathbb{E}_e e^{\gamma S_{\sigma^*_n}} \mathbb{1}_{\{\sigma^*_n < \infty\}} \mathbb{U}^>\left((\eta + S_{\sigma^*_n})^-\right)
\]

\[
\leq e^{a} \sum_{n \geq 0} \mathbb{E}_e e^{\gamma S_{\sigma^*_n}} \mathbb{1}_{\{\sigma^*_n < \infty\}} \left(\mathbb{U}^>(\eta^-) + \mathbb{U}^>(S_{\sigma^*_n})\right)
\]

(4.9)

where in the last step we have used the subadditivity of \( y \mapsto y^-\), \( y \in \mathbb{R}\) and \(\mathbb{U}^>(y)\), \( y \geq 0\). Recall that by definition \( \eta \) is copy of \( \eta_1 \) independent of \((\xi_1, \eta_1), (\xi_2, \eta_2), \ldots\). Using this, we get \(\mathbb{E}_e e^{\gamma S_{\sigma^*_n}} \mathbb{1}_{\{\sigma^*_n < \infty\}} \mathbb{U}^>(\eta^-)\). Consequently, \(\mathbb{E}_e \mathbb{U}^>(\eta^-) < \infty\) due to (2.11) and the fact that \(\mathbb{U}^>(y) \propto J_+(y)\) as \(y \to \infty\) (see (A.6)). Again by the subadditivity of \(\mathbb{U}^>(y)\), we have \(\mathbb{U}^>(y) = O(y)\) as \(y \to \infty\). In view of this, in order to conclude the finiteness of the term in (4.9) it suffices to show that

\[
\sum_{n \geq 0} \mathbb{E}_e e^{\theta S_{\sigma^*_n}} \mathbb{1}_{\{\sigma^*_n < \infty\}} < \infty
\]

(4.10)

for some \(0 < \theta < \gamma\). When \(\mathbb{P}_e \{\sigma^*_1 < \infty\} < 1\), the term on the left-hand side of (4.10) is bounded by \(1/\mathbb{P}_e \{\sigma^*_1 = \infty\}\). If, on the other hand, \(\mathbb{P}_e \{\sigma^*_1 < \infty\} = 1\), then we can drop the indicators in (4.10) and get

\[
\sum_{n \geq 0} \mathbb{E}_e e^{\theta S_{\sigma^*_n}} = \sum_{n \geq 0} (\mathbb{E}_e e^{\theta S_{\sigma^*_1}})^n = \frac{1}{1 - \mathbb{E}_e e^{\theta S_{\sigma^*_1}}} < \infty.
\]

The proof is complete. \(\square\)

**Proof of Theorem 2.7** This proof is based on the two inequalities

\[
\mathbb{P}\{\rho(x) = n\} \leq \mathbb{P}\{T_n \leq x\}, \quad x \in \mathbb{R}
\]

(4.11)

and

\[
\mathbb{P}\{\rho(x) \geq n\} \geq \mathbb{P}\{T_n \leq x\}, \quad x \in \mathbb{R}.
\]

(4.12)

We can write \(\mathbb{E} e^{\alpha \rho(x)}\) in the following two ways:

\[
\mathbb{E} e^{\alpha \rho(x)} = \sum_{n \geq 0} e^{an} \mathbb{P}\{\rho(x) = n\}
\]

(4.13)

\[
= e^{-a} \left( e^a - 1 \right) \sum_{n \geq 0} e^{an} \mathbb{P}\{\rho(x) \geq n\} + 1.
\]

(4.14)

The implications “\((2.19) \Rightarrow (2.18)\)” and “\((2.22) \Rightarrow (2.21)\)” follow from (4.11) and (4.13). In turn, the implications “\((2.18) \Rightarrow (2.19)\)” (for fixed \(y = x\)) and “\((2.21) \Rightarrow (2.22)\)” follow from (4.12) and (4.14).
Next assume that (2.20) holds in case \(\mathbb{P}\{\xi \geq 0\} = 1\) and that (2.23) holds in case \(\mathbb{P}\{\xi < 0\} > 0\). Then, by Proposition A.1, \(V^*_a(y) := \sum_{n \geq 0} e^{an} \mathbb{P}\{S_n \leq y\}\) is finite for all \(y \in \mathbb{R}\) and \(V^*_a(y) \leq Ce^{\gamma y}\) for some constant \(C > 0\) and all \(y \geq 0\). Further, by assumption, \(\mathbb{E} e^{-\gamma \eta} < \infty\). Taking all this into account, we infer, for \(x \in \mathbb{R}\) (condition \(\mathbb{P}\{\eta \leq x\} > 0\) is not required),

\[
V_a(x) = e^a \mathbb{E} V^*_a(x - \eta) \leq e^a V^*_a(0) + e^a Ce^{\gamma x} \mathbb{E} e^{-\gamma \eta} < \infty.
\]

Thus, the implications “(2.20) \(\Rightarrow\) (2.19)” and “(2.23) \(\Rightarrow\) (2.20)” hold.

We are now left with the proofs of the implications “(2.19) \(\Rightarrow\) (2.21)” and “(2.22) \(\Rightarrow\) (2.23)” Assume that (2.19) holds. We have to show that \(a < - \log \beta\) with \(\beta := \mathbb{P}\{\xi = 0\}\). This is trivial in case \(\beta = 0\), and is a consequence of the chain of inequalities

\[
\infty > \sum_{n \geq 1} e^{an} \mathbb{P}\{T_n \leq x\} \\
\geq \sum_{n \geq 1} e^{an} \mathbb{P}\{\xi_1 = \ldots = \xi_{n-1} = 0, \eta_n \leq x\} \\
= e^a \mathbb{P}\{\eta \leq x\} \sum_{n \geq 0} (\beta e^a)^n
\]

in case \(\beta \in (0, 1)\), since \(\mathbb{P}\{\eta \leq x\} > 0\) by the assumption. The inequality \(\mathbb{E} e^{-\gamma \eta} < \infty\) will be established at the end of the proof.

Assume now that \(\mathbb{P}\{\xi < 0\} > 0\) and that (2.22) holds. Thus,

\[
\infty > \sum_{n \geq 1} e^{an} \mathbb{P}\{T_n \leq x\} = e^a \mathbb{E} V^*_a(x - \eta).
\]

In particular, \(V^*_a(y)\) is finite for some \(y \in \mathbb{R}\). This yields \(a < R\) or \(a = R\) and \(\mathbb{E} \xi e^{-\gamma \xi} < 0\) in view of Proposition A.1.

It remains to prove that \(\mathbb{E} e^{-\gamma \eta} < \infty\) under the assumption (2.19) as well as under the assumption (2.22). To this end, notice that by what we have already shown, in both cases, \(V^*_a(y)\) is finite for all \(y \in \mathbb{R}\) and \(0 < c := \inf_{y \geq 0} e^{-\gamma y} V^*_a(y) < \infty\) by Proposition A.1. Thus, in view of (4.15), we obtain

\[
\mathbb{E} V^*_a(x - \eta) \geq ce^{\gamma x} \mathbb{E} e^{-\gamma \eta} 1_{\{\eta \leq x\}},
\]

which immediately leads to the conclusion that \(\mathbb{E} e^{-\gamma \eta} < \infty\). The proof is herewith complete.

4.3 Proofs of the results on finiteness of power moments of \(N(x)\) and \(\rho(x)\)

Proof of Theorem 2.6 Assume first that \(\xi \geq 0\) a.s. and fix an arbitrary \(x \in \mathbb{R}\). According to parts (a) and (b) of Theorem 2.6 whenever \(N(x) < \infty\) a.s. it has some finite exponential moments. In particular, \(\mathbb{E} N(x)^p < \infty\) for every \(p > 0\). Therefore, from now on, we assume that \(\mathbb{P}\{\xi < 0\} > 0\).

“(2.23) \(\iff\) (2.20)” To prove this equivalence, it suffices to show that \(\mathbb{E} N^*(x)^p < \infty\) iff \(\mathbb{E} J_\dagger(\xi^-)^{p+1} < \infty\). This follows from the discussion on p. 27 in [26].
For the first two conditions in (2.26) imply that $E J_+(\eta^-) < \infty$ is equivalent to $E J_+((\eta - x)\ - < \infty$. Further, (by the equivalence (2.25) $\iff (2.26)$) we know that $E N^*(x)^p < \infty$ for some $x \geq 0$ implies $E N^*(x)^p < \infty$ for all $x \geq 0$. Thus replacing $\eta$ by $\eta - x$ it suffices to prove that $E N(0)^p < \infty$ if $E N^*(0)^p < \infty$ and $E J_+(\eta^-) < \infty$.

**Case 1:** $p \in (0, 1)$. Using the subadditivity of the function $x \mapsto x^p$, $x \geq 0$ we obtain

$$
N(0)^p \leq \left( \sum_{k \geq 1} 1\{T_k \leq 0, S_{k-1} \leq 0\} \right)^p + \left( \sum_{k \geq 1} 1\{T_k \leq 0, S_{k-1} > 0\} \right)^p \\
\leq N^*(0)^p + \sum_{k \geq 1} 1\{0 < S_{k-1} \leq \eta^-\} \text{ a.s.}
$$

Since $E N^*(0)^p < \infty$ by assumption, it remains to check that

$$
\sum_{k \geq 1} \mathbb{P}\{0 < S_{k-1} \leq \eta^-\} < \infty. \quad (4.16)
$$

$\lim_{n \to \infty} T_n = \infty$ implies $\lim_{n \to \infty} S_n = +\infty$ a.s. The latter ensures $E \tau^* < \infty$. Let $U^>(\cdot)$ be the renewal function of the renewal process of strict ladder heights. For $x \geq 0$ we have

$$
\sum_{k \geq 1} \mathbb{P}\{0 < S_{k-1} \leq x\} = \int_0^\infty \mathbb{E}\left( \sum_{k=0}^{\tau^*-1} 1\{-y < S_k \leq x-y\} \right) dU^>(y) \\
= \mathbb{E}\sum_{k=0}^{\tau^*-1} \left( U^>(x - S_k) - U^>(-S_k) \right) \leq \mathbb{E}\tau^*U^>(x),
$$

where in the last step the subadditivity of the function $x \mapsto U^>(x)$, $x \geq 0$ has been utilized. Now (4.16) follows from the last inequality, the fact that

$$
U^>(x) \asymp J_+(x) \quad \text{as } x \to \infty \quad (4.17)
$$

(see (A.6)) and the assumption $E J_+(\eta^-) < \infty$.

**Case 2:** $p \geq 1$. According to [26, Theorem 2.1 and formulae (2.9) and (2.10)], the first two conditions in (2.26) imply

$$
E(\tau^*)^{p+1} < \infty. \quad (4.18)
$$

Let $\tau^*_0 := 0$ and $\tau^*_n := \inf\{k > \tau^*_{n-1} : S_k > \tau^*_{n-1}\}$. Retaining the notation of Section 3 let $X_n(x) := \sum_{k=\tau^*_n}^{\tau^*_n+1} 1\{T_k \leq x\}$ and $\xi_n := S_{\tau^*_n} - S_{\tau^*_{n-1}}$ and observe that $Z(x) = N(x)$. Since the so defined $\xi_n$ are a.s. positive, we can apply Theorem 3.7 to conclude that it is enough to show that, for every $q \in [1, p]$,

$$
\int_0^\infty \mathbb{E}\left( \sum_{k=1}^{\tau^*_n} 1\{T_k \leq -y\} \right)^q dU^>(y) < \infty, \quad (4.19)
$$

23
where, as above, $U^>(\cdot)$ is the renewal function of $(S_{\tau_k})_{n \geq 0}$. Fix any $q \in [1,p]$. For $x \leq 0$, it holds that

$$
\left( \sum_{k=1}^{\tau^*} \mathbb{1}_{\{T_k \leq x\}} \right)^q \\
\leq \left( \sum_{k=1}^{\tau^*} \left( \mathbb{1}_{\{S_{k-1} - \eta_k^* \leq x, -\eta_k \leq x\}} + \mathbb{1}_{\{S_{k-1} - \eta_k \leq x, -\eta_k^* > x\}} \right) \right)^q \\
\leq 2^{q-1} \left( \left( \sum_{k=1}^{\tau^*} \mathbb{1}_{\{-\eta_k \leq x\}} \right)^q + \left( \sum_{k=1}^{\tau^*} \mathbb{1}_{\{S_{k-1} - \eta_k^* \leq x, -\eta_k > x\}} \right)^q \right) \\
=: 2^{q-1}(I_1(x) + I_2(x)).
$$

By [15], Theorem 5.2 on p. 24, there exists a positive constant $B_q$ such that

$$
\mathbb{E} \int_0^\infty I_1(-y) \, dU^>(y) \leq B_q \mathbb{E}(\tau^*)^q \int_0^\infty \mathbb{P}\{\eta^- \geq y\} \, dU^>(y) \\
\leq B_q \mathbb{E}(\tau^*)^q \mathbb{E}U^>(\eta^-).
$$

Here, $\mathbb{E}U^>(\eta^-) < \infty$ in view of (1.17) and the last condition in (2.26). $\mathbb{E}(\tau^*)^q < \infty$ is a consequence of (1.18).

Turning to the term involving $I_2$, notice that from the inequality $(x_1 + \ldots + x_m)^q \leq m^{q-1}(x_1^q + \ldots + x_m^q)$, $x_1, \ldots, x_m \geq 0$ and the subadditivity of the function $x \mapsto U^>(x)$, $x \geq 0$ it follows that

$$
\int_0^\infty I_2(-y) \, dU^>(y) \leq (\tau^*)^{q-1} \sum_{k=1}^{\tau^*} \int_0^{\infty} \mathbb{1}_{\{S_{k-1} - \eta_k \leq -y, -\eta_k^* > -y\}} \, dU^>(y) \\
= (\tau^*)^{q-1} \sum_{k=1}^{\tau^*} \left( U^>(\eta_k^- - S_{k-1}) - U^>(\eta_k^-) \right) \\
\leq (\tau^*)^{q-1} \sum_{k=0}^{\tau^*-1} U^>(-S_k) \\
\leq (\tau^*)^{q-1} \sum_{k=0}^{\tau^*-1} U^>(\xi_1^- + \ldots + \xi_k^-) \\
\leq (\tau^*)^{q-1} \left( 1 + \sum_{k=1}^{\tau^*-1} \left( U^>(\xi_1^-) + \ldots + U^>(\xi_k^-) \right) \right) \\
= (\tau^*)^{q-1} \left( 1 + \sum_{k=1}^{\tau^*-1} (\tau^* - k)U^>(\xi_k^-) \right) \\
\leq (\tau^*)^{q-1} + (\tau^*)^q \sum_{k=1}^{\tau^*} U^>(\xi_k^-).$

By Hölder’s inequality,

$$
\mathbb{E}(\tau^*)^q \sum_{k=1}^{\tau^*} U^>(\xi_k^-) \leq (\mathbb{E}(\tau^*)^{q+1})^{q/(q+1)} \left( \mathbb{E} \left( \sum_{k=1}^{\tau^*} U^>(\xi_k^-) \right)^{q+1} \right)^{1/(q+1)}.
$$
The finiteness of the first factor is secured by \([4.18]\). According to \([15, \text{Theorem } 5.2 \text{ on p. } 24]\), the second factor is finite provided \(\mathbb{E}(t^*)^{q+1} < \infty\) and \(\mathbb{E} \cup^>(\xi)^{q+1} < \infty\). The former follows from \([4.17]\), the latter from \([4.17]\) and \([2.26]\). Thus we have proved that \(\mathbb{E} \int_0^\infty I_2(-y) d \cup^>(y) < \infty\), hence \([4.19]\).

\((2.24) \Rightarrow (2.25)\): Assume that \(\mathbb{E} N(x)^p < \infty\). We only have to prove that \(\mathbb{E} N^*(y)^p < \infty\) for some \(y \geq 0\).

**Case 1:** \(p \in (0, 1)\). By \([17, \text{Theorem } 2]\), without loss of generality, we can assume that \(\xi\) and \(\eta\) are independent. We will briefly explain how this reduction can be justified. Let \((\eta'_n)_{n \in \mathbb{N}}\) be a sequence of i.i.d. copies of \(\eta\) and assume that this sequence is independent of the sequence \((\xi_n, \eta_n)_{n \in \mathbb{N}}\). Define \(T'_n := S_{n-1} + \eta'_n\), \(n \in \mathbb{N}\) and \(F'_n := \sigma((\xi_k, \eta_k) : k = 1, \ldots, n)\). Then

\[
\mathbb{P}(T_n \leq x | F'_{n-1}) = \mathbb{P}(\eta_n \leq x - S_{n-1} | F'_{n-1}) = G(x - S_{n-1}) \quad \text{a.s.}
\]

where \(G(t) := \mathbb{P}\{\eta \leq t\}, t \in \mathbb{R}\) and, analogously,

\[
\mathbb{P}(T'_n \leq x | F'_{n-1}) = \mathbb{P}(\eta'_n \leq x - S_{n-1} | F'_{n-1}) = G(x - S_{n-1}) \quad \text{a.s.,}
\]

that is, the sequences \((1_{\{T_n \leq x\}})_{n \in \mathbb{N}}\) and \((1_{\{T'_n \leq x\}})_{n \in \mathbb{N}}\) are tangent. Moreover, \((\xi_k)_{k \in \mathbb{N}}\) and \((\eta'_k)_{k \in \mathbb{N}}\) are independent. This means that we may work under the additional assumption of independence between the random walk and the perturbing sequence. In the following, we do not introduce a new notation to indicate this feature.

Let \(y \geq x\) be such that \(\mathbb{P}\{\eta \leq y\} > 0\) and let \(A := \{N^*(x - y) > 0\}\). Observe that \(\mathbb{P}(A) > 0\) since we assume that \(\mathbb{P}\{\xi < 0\} > 0\). The following inequality holds a.s. on \(A\):

\[
N(x)^p \geq \left( \sum_{k \geq 1} 1_{\{S_{k-1} \leq x-y, \eta_k \leq y\}} \right)^p
\]

\[
= N^*(x - y)^p \left( \sum_{k \geq 1} 1_{\{S_{k-1} \leq x-y\}} 1_{\{\eta_k \leq y\}} / N^*(x - y) \right)^p
\]

\[
\geq N^*(x - y)^p \sum_{k \geq 1} 1_{\{S_{k-1} \leq x-y\}} 1_{\{\eta_k \leq y\}},
\]

where for the second inequality the concavity of \(t \mapsto t^p, t \geq 0\) has been used. Taking expectations gives

\[
\infty > \mathbb{E} N(x)^p \geq \mathbb{E} \left( 1_A N^*(x - y)^p \sum_{k \geq 1} 1_{\{S_{k-1} \leq x-y\}} 1_{\{\eta_k \leq y\}} \right)
\]

\[
= \mathbb{P}\{\eta \leq y\} \mathbb{E} N^*(x - y)^p.
\]

An appeal to Lemma \([A.2]\) completes the proof of this case.

**Case 2:** \(p \geq 1\). It holds that

\[
\infty > \mathbb{E} N(x)^p \geq \mathbb{E} \left( \sum_{k \geq 1} 1_{\{S_{k-1} \leq x-y, \eta_k \leq y\}} \right)^p
\]

\[
\geq \text{const} \mathbb{E} N^*(x - y)^p (\mathbb{P}\{\eta \leq y\})^p,
\]

25
where at the last step the convex function inequality \[6, \text{Theorem 3.2}\], applied to \(\Phi(t) = t^p\), has been utilized. An appeal to Lemma \[A.2\] completes the proof.

Turning to the proof of Theorem \[2.10\] we start with a simple lemma.

**Lemma 4.3.** Let \(x \in \mathbb{R}\). Then the following assertions are equivalent:

(i) \(\rho(x) = 0\) a.s.;

(ii) \(\inf_{k \geq 1} T_k > x\) a.s.;

(iii) \(\mathbb{P}\{\xi \geq 0\} = 1\) and \(\mathbb{P}\{\eta > x\} = 1\).

**Proof.** The equivalence of (i) and (ii) follows just from the definition of \(\rho(x)\).

If (iii) holds, then \(T_n > x\) a.s. for all \(n \in \mathbb{N}\), which is equivalent to (ii). Conversely, since \(\{\xi_n \leq x\} \subseteq \{\inf_{n \geq 1} T_n \leq x\}\), the condition \(\eta > x\) a.s. is necessary for (ii) to hold. It remains to show that the condition \(\xi \geq 0\) a.s. is also necessary for (ii) to hold. To this end, assume that \(\mathbb{P}\{\xi_1 \leq -\varepsilon\} > 0\) for some \(\varepsilon > 0\). Further pick \(y \in \mathbb{R}\) with \(\mathbb{P}\{\eta \leq y\} > 0\) and choose \(n\) so large such that \(y - n\varepsilon \leq x\). Then

\[
\mathbb{P}\{\inf_{k \geq 1} T_k \leq x\} \geq \mathbb{P}\{T_{n+1} \leq x\} \geq \mathbb{P}\{T_{n+1} \leq y - n\varepsilon\} \geq \mathbb{P}\{\xi_1 \leq -\varepsilon, \ldots, \xi_n \leq -\varepsilon, \eta_{n+1} \leq y\} > 0,
\]

which completes the proof. \(\square\)

**Proof of Theorem \[2.10\]** (\(2.28 \Leftrightarrow 2.29\)) was proved in \[26, \text{Theorem 2.1 and formulae (2.9) and (2.10)}\].

From the representation

\[
\mathbb{E}\rho(x)^p = \sum_{n \geq 1} n^p \mathbb{P}\{\rho(x) = n\} = \sum_{n \geq 1} n^p \mathbb{P}\{T_n \leq x, \inf_{k \geq n+1} T_k > x\} = \sum_{n \geq 1} n^p(\mathbb{P}\{\inf_{k \geq n} T_k \leq x\} - \mathbb{P}\{\inf_{k \geq n+1} T_k \leq x\}),
\]

and Lemma \[A.7\] it follows that \(\mathbb{E}\rho(x)^p < \infty\) iff

\[
\sum_{n \geq 1} n^{p-1} \mathbb{P}\{\inf_{k \geq n} T_k \leq x\} = \mathbb{E}\mathbb{U}_{p-1}(x - \inf_{k \geq 1} T_k) < \infty, \tag{4.20}
\]

where \(\mathbb{U}_{p-1}(y) := \sum_{n \geq 0} n^{p-1} \mathbb{P}\{S_n \leq y\}\) is the power renewal function of \((S_n)_{n \geq 0}\) at \(y \in \mathbb{R}\). Indeed, with \(b_n = \mathbb{P}\{\inf_{k \geq n} T_k \leq x\} - \mathbb{P}\{\inf_{k \geq n+1} T_k \leq x\}\) in Lemma \[A.7\] we have \(\sum_{k=n}^{\infty} b_k = \mathbb{P}\{\inf_{k \geq n} T_k \leq x\}\), since \(\lim_{n \to \infty} \mathbb{P}\{\inf_{k \geq n} T_k \leq x\} = 0\) due to the assumption that \(T_n \to \infty\) a.s.

(\(2.27 \Rightarrow 2.28\)) Suppose (4.20) holds for some \(x \in \mathbb{R}\). We distinguish two cases.
There exists since for any fixed \( J \subseteq E \) we conclude that also \( E \) and (4.20) holds. Case 1: for all \( y \geq 0 \). From [26, Theorem 2.1], we infer that \( U_{p-1}(y) \) must be finite for some \( y \geq 0 \). From [26, Theorem 2.1], we infer that \( U_{p-1}(y) \) must be finite for all \( y \geq 0 \). Further, by (A.6), \( U_{p-1}(y) \approx J_{+}(y)^p \) as \( y \to \infty \). Consequently, since for any fixed \( z \in \mathbb{R}, J_{+}(y+z)^p \sim J_{+}(y)^p \) as \( y \to \infty \), [14.20] implies that \( \mathbb{E} J_{+}((\inf_{k \geq 1} T_k)^{-})^p < \infty \). From

\[
T_k = \xi_1 + \ldots + \xi_{k-1} + \eta_k \leq \xi_1^+ + \ldots \xi_{k-1}^+ + \eta_k =: \widehat{T}_k, \quad k \in \mathbb{N},
\]

we conclude that also \( \mathbb{E} J_{+}((\inf_{k \geq 1} \widehat{T}_k)^{-}) < \infty \). Thus, it suffices to show that \( \mathbb{E} J_{+}((\inf_{k \geq 1} \widehat{T}_k)^{-}) < \infty \) implies \( \mathbb{E} J_{+}(\eta)^{-p+1} < \infty \). To a large extent, this follows from the proof of [21, Lemma 3.4], although some details have to be explained.

Pick \( \varepsilon > 0 \) such that \( \alpha := \mathbb{P}\{\inf_{k \geq 1} \widehat{T}_k \geq -\varepsilon\} > 0 \). Such an \( \varepsilon \) exists since we assume that \( T_\alpha \to \infty \) a.s. Let \( (M_k, Q_k), k \geq 1 \) be independent copies of a random vector \( (M, Q) := (e^{-\xi^+, e^{-\eta}}) \), and set

\[
\Pi_k := e^{-(\xi_1^+ + \ldots + \xi_{k-1}^+)} = \prod_{j=1}^{k} M_j, \quad k \in \mathbb{N}_0.
\]

Using this notation the function \( J \) defined after (2) in [2] coincides with the function \( J_{+} \) defined after (2.1) if we use the convention that \( J_{+}(x) = 0 \) for \( x < 0 \). In the cited work it was proved that, for \( \delta > \varepsilon \) and for every non-decreasing and absolutely continuous function \( f : [0, \infty) \to [0, \infty) \), we have

\[
\mathbb{E} f\left( \sup_{k \geq 1} Q_k \right) \geq \alpha \mathbb{E} \left( \mathbb{1}_{\{Q > e^{2\delta}\}} f\left( \frac{\log Q}{2} \right) \right).
\]

The idea now is to choose \( f(x) := J_{+}(\log^+ x)^p \) for \( x > 0, f(0) = 0 \). Then (4.21) becomes

\[
\mathbb{E} J_{+}\left( \inf_{k \geq 1} \widehat{T}_k \right)^p \geq \alpha \mathbb{E} \left( \mathbb{1}_{\{\eta < 2\delta\}} J_{+}(\eta / 2)^p J_{+}(\eta / 2) \right) \geq \alpha 2^{-p+1} \mathbb{E} \left( \mathbb{1}_{\{\eta < 2\delta\}} J_{+}(\eta)^{-p+1} \right)
\]

where in the last step, we have used that, by Lemma (A.4), \( J_{+}(x/2) \geq 2\lambda J_{+}(x) \) for all \( x \geq 0 \). So in order to make the argument rigorous it remains to show that \( f \) has the properties needed. The latter follows from Lemma (A.4) + (2.28) \( \Rightarrow (2.27) \): We have to prove that the inequality in (4.20) holds for any \( x \in \mathbb{R} \). By [26, Theorem 2.1], \( \mathbb{E} \rho^+(y)^p < \infty \) for some \( y \geq 0 \) ensures that \( U_{p-1}(y) \) ensures that \( \mathbb{E} J_{+}(y)^p \), \( y \to \infty \). Case 1: There exists \( y \in \mathbb{R} \) such that \( \inf_{k \geq 1} T_k > y \) a.s. Then

\[
\mathbb{E} U_{p-1}(x - \inf_{k \geq 1} T_k) \leq U_{p-1}(x - y) < \infty,
\]

and (4.20) holds.
Subcase 2b. \( \xi \geq 0 \) a.s. Set \( f(x) := J_+(x)^p \). \( f \) is absolutely continuous, in particular a.e. differentiable with derivative \( f' \). Therefore, it is sufficient to show that
\[
K := \int_0^\infty f'(u) \mathbb{P}\{ - \inf_{k \geq 1} T_k > u \} \, du < \infty.
\]
Since
\[
\mathbb{P}\{ - \inf_{k \geq 1} T_k > u \} \leq \sum_{k \geq 1} \mathbb{P}\{ T_k \leq -u \} = \mathbb{E} \mathbb{U}_0(-u - \eta),
\]
we have
\[
K = \mathbb{E} \int_0^{\eta^-} f'(u) \mathbb{U}_0(-u - \eta) \, du \leq \mathbb{E} f(\eta^-) \mathbb{U}_0(\eta^-) < \infty.
\]
The assertion follows in view of the asymptotics (4.6) and the assumption \( \mathbb{E} J_+(\eta^-)^{p+1} < \infty \).

Subcase 2b. \( \mathbb{P}\{ \xi < 0 \} > 0 \). Define the stopping times
\[
\tau_0^* := 0, \quad \tau_{n+1}^* := \inf\{ k > \tau_n^* : S_k > S_{\tau_n^*} \}, \quad n \in \mathbb{N}_0.
\]
By assumption, \( \lim_{n \to \infty} S_n = \infty \) a.s. Hence, each \( \tau_n^* \) is a.s. finite. For \( k \in \mathbb{N}_0 \), define new random variables as follows:
\[
\hat{\eta}_k := \min\{ \eta_{\tau_{k-1}^*+1}, \xi_{\tau_{k-1}^*+1} + \eta_{\tau_{k-1}^*+2}, \ldots, \xi_{\tau_{k-1}^*} + \ldots + \xi_{\tau_{k-1}^*+1} + \eta_{\tau_{k-1}^*+1} \};
\]
\[
\hat{\xi}_k := \xi_{\tau_{k-1}^*+1} + \ldots + \xi_{\tau_{k-1}^*}.
\]
The random vectors \( (\hat{\xi}_k, \hat{\eta}_k), k \in \mathbb{N}, \) are independent copies of the random vector \( (S_{\tau_n^*}, \min_{1 \leq k \leq \tau_n^*} T_k) \). Denote by \( (\tilde{T}_k)_{k \in \mathbb{N}} \) the perturbed random walk generated by the vectors \( (\hat{\xi}_k, \hat{\eta}_k), k \in \mathbb{N}, \) i.e.,
\[
\tilde{T}_k := \hat{S}_{k-1} + \hat{\eta}_k, \quad k \in \mathbb{N},
\]
where
\[
\hat{S}_0 := 0, \quad \hat{S}_k := \hat{\xi}_1 + \ldots + \hat{\xi}_k, \quad k \in \mathbb{N}.
\]
Note that, by construction, \( \hat{S}_k > 0 \) for all \( k \in \mathbb{N} \). Finally,
\[
\inf_{k \geq 1} \tilde{T}_k = \inf_{k \geq 1} T_k.
\]
According to the already established Subcase 2a it suffices to prove that
\[
\mathbb{E} J_+(\hat{\eta}^-)^{p+1} = \mathbb{E} J_+ \left( \left( \min_{1 \leq k \leq \tau_n^*} T_k \right)^- \right)^{p+1} < \infty. \tag{4.22}
\]
To this end, obtain that, a.s.,
\[
\left( \min_{1 \leq k \leq \tau_1^*} T_k \right)^- \leq \left| \min_{0 \leq k \leq \tau_1^*-1} S_k \right| + \left( \min_{1 \leq k \leq \tau_1^*} \eta_k \right)^- \leq \left| \min_{0 \leq k \leq \tau_1^*-1} S_k \right| + \sum_{k=1}^{\tau_1^*} \eta_k^-.
\]
Hence, using the monotonicity and subadditivity of \( x \mapsto J_+ (x) \) we conclude that a.s.
\[
J_+ \left( \left( \min_{1 \leq k \leq \tau_1^*} T_k \right)^- \right) \leq J_+ \left( \left| \min_{0 \leq k \leq \tau_1^*-1} S_k \right| \right) + \sum_{k=1}^{\tau_1^*} J_+ (\eta_k^-) \leq 2^{p+1} \left( J_+ \left( \left| \min_{0 \leq k \leq \tau_1^*-1} S_k \right| \right) + \left( \sum_{k=1}^{\tau_1^*} J_+ (\eta_k^-) \right) \right)^{p+1}.
\]
Using the already proved equivalence \((2.28) \Leftrightarrow (2.29)\), \( \mathbb{E} \rho_1^* (y)^p < \infty \) for some \( y \geq 0 \) implies that \( \mathbb{E} (\tau_1^*)^{p+1} < \infty \) and \( \mathbb{E} (\xi^-)^{p+1} < \infty \). Hence
\[
\mathbb{E} J_+ \left( \left| \min_{0 \leq k \leq \tau_1^*-1} S_k \right| \right)^{p+1} < \infty,
\] (4.23)
by virtue of Lemma \([A.5]\). Further, by \([15]\) Theorem 5.2 on p. 24],
\[
\mathbb{E} \left( \sum_{k=1}^{\tau_1^*} J_+ (\eta_k^-) \right)^{p+1} \leq \text{const} \mathbb{E} (\eta^-)^{p+1} \mathbb{E} (\tau_1^*)^{p+1} < \infty,
\]
and (4.22) follows. The proof is complete. \( \square \)

A Appendix: Auxiliary results

A.1 Auxiliary results from classical random walk theory

This subsection contains some facts from classical random walk theory that are either reformulations or slight extensions of known results. The first result is a combination of Theorems 2.1 and 2.2 in [23].

Proposition A.1. For \( a > 0 \), let \( V_a^* (I) := \sum_{n \geq 0} e^{an} \mathbb{P} \{ S_n \in I \}, I \subseteq \mathbb{R} \) Borel and \( V_a^* (x) := V_a^* ((-\infty, x]), x \in \mathbb{R} \). Further, let \( R := -\log \inf_{t \geq 0} \mathbb{E} e^{-t\xi} \).

(a) (i) Assume that \( \mathbb{P} \{ \xi \geq 0 \} = 1 \) and let \( \beta := \mathbb{P} \{ \xi = 0 \} \in [0, 1) \). Then for \( a > 0 \) the following conditions are equivalent:
\[
V_a^* (x) < \infty \text{ for some/all } x \geq 0; \quad (A.1)
\]
\[
0 < V_a^* (I) < \infty \text{ for some bounded interval } I \subseteq \mathbb{R}; \quad (A.2)
\]
\[
a < -\log \beta \quad (A.3)
\]
where \( -\log \beta \) := \( \infty \) if \( \beta = 0 \).
(ii) Assume that \( P\{\xi < 0\} > 0 \). Then for \( a > 0 \) condition (A.1) (with \( x \in \mathbb{R} \)) is equivalent to
\[
a < R \quad \text{or} \quad a = R \quad \text{and} \quad E\xi e^{-\gamma_0 \xi} > 0,
\]
where \( \gamma_0 \) is the unique positive value defined by \( E e^{-\gamma_0 \xi} = e^{-R} \).

(b) Whenever \( V^*_a(x) \) is finite,
\[
0 < \liminf_{x \to \infty} e^{-\gamma x} V^*_a(x) \leq \limsup_{x \to \infty} e^{-\gamma x} V^*_a(x) < \infty.
\]

Part (a) of the Proposition contains more equivalent criteria for the finiteness of the exponential renewal function of a random walk than Theorem 2.1 in [2]. For this reason, we decided to include a proof.

**Proof.** We begin with part (a)(i) and assume that \( P\{\xi \geq 0\} = 1 \). Then the equivalence between (A.1) and (A.3) follows from [23, Theorem 2.1(b) and (c)]. Moreover, the implication “(A.1) \( \Rightarrow \) (A.2)” is trivial. It remains to prove that \( 0 < V^*_a(I) < \infty \) implies that \( a < -\log \beta \). We will use contraposition and assume that \( a \geq -\log \beta \), in particular, \( \beta > 0 \). Then let \( I \subseteq [0, \infty) \) denote an arbitrary bounded interval with \( V^*_a(I) > 0 \). We have to show that \( V^*_a(I) = \infty \).

To this end, notice that \( V^*_a(I) > 0 \) implies that \( P\{S_n \in I\} > 0 \) for some \( n \in \mathbb{N} \). Then, for any \( k \geq 0 \), we infer
\[
P\{S_{n+k} \in I\} \geq P\{S_n \in I, \xi_{n+1} = \ldots = \xi_{n+k} = 0\} = P\{S_n \in I\} \beta^k.
\]
In conclusion,
\[
V^*_a(I) = \sum_{k \geq 0} e^{ak} P\{S_k \in I\} = \sum_{k \geq 0} e^{a(n+k)} P\{S_{n+k} \in I\} \geq e^{an} P\{S_n \in I\} \sum_{k \geq 0} (e^a \beta)^k = \infty.
\]

Part (a)(ii) follows from Theorem 2.1(a) in [23]. Part (b) follows from Theorem 2.2 in [23].

**Lemma A.2.** Let \( p > 0 \) and \( I \subseteq \mathbb{R} \) be an open interval such that \( 0 < E \left( \sum_{n \geq 0} 1_{\{S_n \in I\}} \right)^p < \infty \). Then \( E \left( \sum_{n \geq 0} 1_{\{S_n \in I\}} \right)^p < \infty \) for any bounded interval \( J \subseteq \mathbb{R} \). In particular, \( E N^*(x)^p < \infty \) for some \( x \in \mathbb{R} \) entails \( E N^*(y)^p < \infty \) for every \( y \in \mathbb{R} \).

**Remark A.3.** In the case that \( x \geq 0 \) the second assertion was known from [26].

**Proof.** Let \( I = (a, b) \) such that \( 0 < E \left( \sum_{n \geq 0} 1_{\{S_n \in I\}} \right)^p < \infty \). We assume w.l.o.g. that \( -\infty < a < b < \infty \). We first show that
\[
E \left( \sum_{n \geq 0} 1_{\{|S_n| \leq \varepsilon\}} \right)^p < \infty \quad \text{for some} \quad \varepsilon > 0.
\]
Pick \( \varepsilon > 0 \) so small that \( I_\varepsilon := (a + \varepsilon, b - \varepsilon) \) satisfies \( \mathbb{E} \left( \sum_{n \geq 0} \mathbb{1}_{\{S_n \in I_\varepsilon\}} \right)^p > 0. \) Then \( \mathbb{P}\{S_n \in I_\varepsilon\} > 0 \) for some \( n \in \mathbb{N}. \) In particular, \( \mathbb{P}\{\tau^*(I_\varepsilon) < \infty\} > 0, \) where \( \tau^*(I_\varepsilon) = \inf\{n \geq 0 : S_n \in I_\varepsilon\}. \) Using the strong Markov property at \( \tau^*(I_\varepsilon), \) we get

\[
\infty > \mathbb{E} \left( \sum_{n \geq 0} \mathbb{1}_{\{S_n \in I_\varepsilon\}} \right)^p
\geq \mathbb{E} \left( \mathbb{1}_{\{\tau^*(I_\varepsilon) < \infty\}} \sum_{n \geq \tau^*(I_\varepsilon)} \mathbb{1}_{\{|S_n - S_{\tau^*(I_\varepsilon)}| < \varepsilon\}} \right)^p
= \mathbb{P}\{\tau^*(I_\varepsilon) < \infty\} \mathbb{E} \left( \sum_{n \geq 0} \mathbb{1}_{\{|S_n| < \varepsilon\}} \right)^p.
\]

Hence, \((A.3)\) holds. Now let \( J \) be a non-empty bounded interval \( \subseteq \mathbb{R}, \) and \( J_1, \ldots, J_m \) open intervals of length at most \( \varepsilon \) such that \( J \subseteq J_1 \cup \ldots \cup J_m. \) Using the inequality \( (x_1 + \ldots + x_m)^p \leq (m^{p-1} \lor 1)(x_1^p + \ldots + x_m^p), x_j \geq 0 \) for \( j = 1, \ldots, m, \) leads to

\[
\mathbb{E} \left( \sum_{n \geq 0} \mathbb{1}_{\{S_n \in J\}} \right)^p \leq \mathbb{E} \left( \sum_{k=1}^m \sum_{n \geq 0} \mathbb{1}_{\{S_n \in J_k\}} \right)^p
\leq (m^{p-1} \lor 1) \sum_{k=1}^m \mathbb{E} \left( \sum_{n \geq 0} \mathbb{1}_{\{S_n \in J_k\}} \right)^p.
\]

Therefore, it suffices to prove the result under the additional assumption that the length of \( J \) is at most \( \varepsilon. \) Using the strong Markov property at \( \tau^*(J) := \inf\{n \geq 0 : S_n \in J\} \) gives

\[
\mathbb{E} \left( \sum_{n \geq 0} \mathbb{1}_{\{S_n \in J\}} \right)^p \leq \mathbb{E} \left( \mathbb{1}_{\{\tau^*(J) < \infty\}} \sum_{n \geq \tau^*(J)} \mathbb{1}_{\{|S_n - S_{\tau^*(J)}| < \varepsilon\}} \right)^p
= \mathbb{P}\{\tau^*(J) < \infty\} \mathbb{E} \left( \sum_{n \geq 0} \mathbb{1}_{\{|S_n| < \varepsilon\}} \right)^p < \infty.
\]

This proves the first assertion of the lemma. Concerning the second, assume that \( \mathbb{E} N^*(x)^p < \infty \) for some \( x \in \mathbb{R}. \) Then, for any \( y > x, \)

\[
\mathbb{E} N^*(y)^p \leq (2^{p-1} \lor 1) \left( \mathbb{E} N^*(x)^p + \mathbb{E} \left[ \sum_{n \geq 0} \mathbb{1}_{\{x < S_n \leq y\}} \right]^p \right) < \infty,
\]

where the last term is finite by the first part of the lemma.

The following lemma summarizes properties of the functions \( A_+ \) and \( J_+ \) that are frequently used throughout the proofs. These properties were known before and are stated here only for the reader’s convenience. Recall from \((2.1)\) that

\[
A_+(x) := \int_0^x \mathbb{P}\{\xi > y\} \, dy = \mathbb{E} \min(\xi^+, x) \quad \text{and} \quad J_+(x) := \frac{x}{A_+(x)}
\]

31
and, analogously, $U^>_{\tau}(x) = \sum_{n \geq 1} n^{p-1} \mathbb{P}\{S_n \leq x\}$ where $\tau^*_n$ is the $n$th strictly ascending ladder index of the random walk $(S_n)_{n \geq 0}$.

**Lemma A.4.** Assume that $S_n \rightarrow \infty$ a.s. Then the following assertions are true:

(a) $A_+(x) > 0$ for all $x > 0$; $A_+$ and $J_+$ are non-decreasing.

(b) $\lim_{x \rightarrow \infty} J_+(x) = \infty$.

(c) $J_+$ is subadditive, i.e., $J_+(x + y) \leq J_+(x) + J_+(y)$ for all $x, y \geq 0$. In particular, $J_+(x + y) \sim J_+(x)$ as $x \rightarrow \infty$ for any $y \in \mathbb{R}$.

(d) For any $p > 0$ if $U_{p-1}(0) < \infty$, then

$$
U_{p-1}(x) \asymp U^>_{p-1}(x) \asymp J_+(x)^p \quad (A.6)
$$

as $x \rightarrow \infty$. Moreover, with $W(\cdot)$ denoting either $\tau^*(\cdot)$, $N^*(\cdot)$ or $\rho^*(\cdot)$, then $\mathbb{E} W(x)^p \asymp J_+(x)^p$ whenever $\mathbb{E} W(0)^p < \infty$.

**Proof.** (a) Since $S_n \rightarrow \infty$ a.s. is assumed, $\mathbb{P}\{\xi^+ > 0\} = \mathbb{P}\{\xi > 0\} > 0$ and therefore $\mathbb{P}\{\xi > y\} > 0$ in a right neighborhood of 0. The monotonicity of $A_+$ follows from its definition. The monotonicity of $J_+$ and assertion (b) follow from the following representation

$$
J_+(x) = \left( \int_0^1 \mathbb{P}\{\xi > xy\} \, dy \right)^{-1}, \quad x > 0.
$$

Regarding (c) notice that the subadditivity of $J_+$ follows from the monotonicity of $A_+$, $J_+(x+y) \sim J_+(x)$ as $x \rightarrow \infty$ immediately follows from the subadditivity of $J_+$ together with (b).

(d) follows from equations [26, Theorems 2.1 and 2.2, Eq. (4.5)] and one of the displayed formulas on p. 28 of the cited reference. \hfill \Box

**Lemma A.5.** Let $p > 0$ and assume $\lim_{k \rightarrow \infty} S_k = +\infty$ a.s. Then the following assertions are equivalent:

$$
\mathbb{E} J_+\left( \min_{0 \leq k \leq \tau^*_1} S_k \right)^{p+1} < \infty; \quad (A.7)
$$

$$
\mathbb{E} J_+\left( \inf_{k \geq 0} S_k \right)^p < \infty; \quad (A.8)
$$

$$
\mathbb{E} J_+(\xi^-)^{p+1} < \infty \quad (A.9)
$$

where $\tau^* := \inf\{k \in \mathbb{N} : S_k > 0\}$.

Lemma A.5 has several predecessors, e.g. [25, Theorem 1], [1, Theorem 3], [26, Proposition 4.1], [23, Lemma 3.5]. Even though Lemma A.5 does not follow directly from either of these results, the proofs given in [25] and [26] can be adopted to treat the present case after the observation that the function $x \mapsto J_+(x)$ is non-decreasing and subadditive. Therefore, we omit a proof.
A.2 Elementary facts

Lemma A.6. Let \( 1 \leq p = n + \delta \) with \( n \in \mathbb{N}_0 \) and \( \delta \in (0, 1] \). Then, for any \( x, y \geq 0 \),
\[
(x + y)^p \leq x^p + y^p + p2^{p-1}(xy^{p-1} + x^n y^\delta). \tag{A.10}
\]

This estimate is a variant of an estimate we have learned from [14]. For the reader’s convenience, we include a brief proof which is a slight modification of the argument given in the cited reference.

Proof. For any \( 0 \leq r \leq 1 \), we have \((1 + r)^p = 1 + p \int_0^r (1 + t)^{p-1} dt\). By the mean value theorem for integrals, for some \( \gamma \in (0, r) \),
\[
(1+r)^p = 1 + pr(1+\gamma)^{p-1} \leq 1 + p2^{p-1}r \leq 1 + p2^{p-1}r^{\delta}, \tag{A.11}
\]
where in the last step we have used that \( 0 \leq r \leq 1 \). Now let \( x, y \geq 0 \). When \( x \leq y \), use the first estimate in (A.11) to get \((x + y)^p \leq y^p + p2^{p-1}xy^{p-1}\). When \( y \leq x \) use the second estimate in (A.11) to infer \((x + y)^p \leq x^p + p2^{p-1}x^n y^\delta\). Thus, in any case, (A.10) holds.

The next auxiliary result is an elementary consequence of a version of the summation by parts formula.

Lemma A.7. Let \( b_n \geq 0 \) for all \( n \in \mathbb{N} \) and \( p > 1 \). Then
\[
\sum_{n \geq 1} n^{p-1} \sum_{k=n}^{\infty} b_k < \infty \quad \text{iff} \quad \sum_{n \geq 1} n^p b_n < \infty.
\]

Proof. For arbitrary \( m \in \mathbb{N} \),
\[
\sum_{n=1}^{m} n^{p-1} \sum_{k=n}^{\infty} b_k = \sum_{n=1}^{m} n^{p-1} \sum_{k=m}^{\infty} b_k + \sum_{n=1}^{m-1} b_n \sum_{k=1}^{n} k^{p-1}. \tag{A.12}
\]
In particular,
\[
\sum_{n=1}^{m} n^{p-1} \sum_{k=n}^{\infty} b_k \geq \sum_{n=1}^{m-1} b_n \sum_{k=1}^{n} k^{p-1}.
\]
Consequently, if \( \sum_{n \geq 1} n^{p-1} \sum_{k=n}^{\infty} b_k \) converges, then so does \( \sum_{n \geq 1} b_n \sum_{k=1}^{n} k^{p-1} \) and thus also \( \sum_{n \geq 1} b_n n^p \). Conversely, if the series \( \sum_{n \geq 1} n^p b_n \) converges, then \( \sum_{n \geq 1} b_n \sum_{k=1}^{n} k^{p-1} \) converges and, in particular,
\[
\lim_{m \to \infty} \sum_{n=m}^{\infty} b_n \sum_{k=1}^{n} k^{p-1} = 0.
\]
Further,
\[
0 \leq \sum_{k=1}^{m} k^{p-1} \sum_{n=m}^{\infty} b_n \leq \sum_{n=m}^{\infty} b_n \sum_{k=1}^{n} k^{p-1} \to 0 \quad \text{as} \quad m \to \infty.
\]
Letting \( m \) tend to \( \infty \) in (A.12), we conclude that \( \sum_{n \geq 1} n^{p-1} \sum_{k=n}^{\infty} b_k \) converges.\qed
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