Destabilization of rotating flows with positive shear by azimuthal magnetic fields

Frank Stefani\textsuperscript{1,} and Oleg N. Kirillov\textsuperscript{2,}†

\textsuperscript{1}Helmholtz-Zentrum Dresden - Rossendorf
P.O. Box 510119, D-01314 Dresden, Germany
\textsuperscript{2}Russian Academy of Sciences, Steklov Mathematical Institute
Gubkina st. 8, 119991 Moscow, Russia

(Dated: November 13, 2015)

According to Rayleigh’s criterion, rotating flows are linearly stable when their specific angular momentum increases radially outward. Since this criterion applies to the Keplerian rotation profiles which are typical for low-mass accretion disks, the growth mechanism of central objects, such as protostars and black holes, had been a conundrum for many decades. Nowadays, the magnetorotational instability (MRI)\textsuperscript{2} is considered the main candidate to explain turbulence and enhanced angular momentum in accretion disks. The standard version of MRI (SMRI), with a vertical magnetic field $B_z$ applied to the rotating flow, requires both the rotation period and the Alfvén crossing time to be shorter than the timescale for magnetic diffusion. This implies, for a disk of height $H$, that both the magnetic Reynolds number $R_m = \mu_0 \sigma H^2 \Omega$ and the Lundquist number $S = \mu_0 \sigma H v_A$ must be larger than one ($\Omega$ is the angular velocity, $v_A := B_z / \sqrt{\mu \rho}$ is the Alfvén velocity, with $\rho$ denoting the density). While these conditions are safely fulfilled in well-conducting parts of accretion disks, the situation is less clear in the “dead zones” of protoplanetary disks, in stellar interiors and liquid cores of planets, because of the small value of the magnetic Prandtl number $P_m = \nu/\eta$\textsuperscript{3}, i.e. the ratio of viscosity $\nu$ to magnetic diffusivity $\eta := (\mu \sigma)^{-1}$.

This low $P_m$ case is also the subject of intense theoretical and experimental research initiated by Hollerbach and Rüdiger\textsuperscript{3}. Adding an azimuthal magnetic field $B_\phi$ to $B_z$, the authors found a new version of MRI, now called helical MRI (HMRI). It was proved to work also in the inductionless limit, $P_m = 0$, and to be governed by the Reynolds number $R = R_m P_m$\textsuperscript{-1} and the Hartmann number $H_a = S P_m^{-1/2}$, quite in contrast to standard SMRI that is governed by $R_m$ and $S$.

A somewhat sobering limitation of HMRI was identified by Liu et al.\textsuperscript{2} who used a local approximation (also called short-wavelength, Wentzel-Kramers-Brillouin (WKB), or geometric optics approximation, see \textsuperscript{8}) to find a minimum steepness of the rotation profile $\Omega(r)$, expressed by the Rossby number $R_o := r (2 \sigma)^{-1} \partial \Omega / \partial r$, of $R_o_{\text{LLL}} = 2 (1 - \sqrt{2}) \approx -0.828$. This lower Liu limit (LLL) implies that, at least for $B_\phi(r) \propto 1/r$, HMRI does not extend to the most relevant Keplerian case, characterized by $R_o_{\text{Kepler}} = -3/4$. Surprisingly, in addition to the LLL, the authors found also a second threshold of $R_o$, which we call upper Liu limit (ULL), at $R_o_{\text{ULL}} = 2 (1 + \sqrt{2}) \approx +4.828$. For $R_o > R_o_{\text{ULL}}$ one expects a magnetic destabilization of those flows with strongly increasing angular velocity that would even be stable with respect to SMRI.

By relaxing the demand that the azimuthal field is current-free in the liquid, i.e. $B_\phi(r) \propto 1/r$, and allowing fields with arbitrary radial dependence, we have recently shown\textsuperscript{9-11} that the LLL and the ULL are just the endpoints of one common instability curve in a plane that is spanned by $R_o$ and a corresponding steepness of the azimuthal magnetic field, called magnetic Rossby number, $R_b := r (2 B_\phi(r) / r)^{-1} \partial (B_\phi(r)/r) / \partial r$. In the limit of large $R_o$ and $H_a$, this curve acquires the closed and simple form

\begin{equation}
R_b = -\frac{1}{8} \frac{(R_o + 2)^2}{R_o + 1}.
\end{equation}

A non-axisymmetric “relative” of HMRI, the azimuthal MRI (AMRI)\textsuperscript{10}, which appears for purely or dominantly $B_\phi$, has been shown to be governed by basically the same scaling behaviour, and the same Liu limits\textsuperscript{11}. Actually, the key parameter dependencies of HMRI and AMRI were confirmed in various liquid metal experiments at the PROMISE facility\textsuperscript{12,13}.

In the present paper, we focus exclusively on the case of positive $R_o$, i.e. on flows whose angular velocity (not only the angular frequency) is increasing outward. From the purely hydrodynamic point of view, such flows are linearly stable (while non-linear instabilities were actually observed in experiments\textsuperscript{11,14}). Flows with positive...
Ro are indeed relevant for the equator-near strip (approximately between ±30°) of the solar tachocline [13], which, is, interestingly, also the region of sunspot activity [16]. Up to present, the ULL at R_{0\text{ULL}} = +4.828 has only been predicted in the framework of various local approximations [9, 8], while attempts to confirm it in a 1-dimensional modal stability code on the basis of Taylor-Couette (TC) flows have failed so far [17]. Hence, the questions arise: Is the magnetically triggered flow instability for Ro > R_{0\text{ULL}} a real phenomenon (which would fundamentally modify the stability criteria for rotating flows in general), or just an artifact of the local approximation, and is there any chance to observe it in a TC experiment?

In order to tackle these problems we restrict our attention here to non-axisymmetric instabilities, which are the relevant ones for pure B\_0, and further assume Pm = 0. Under these assumptions, we had recently derived the closed equation

\[
Re^2 = \frac{1}{4} \left[ (1 + Ha^2 n^2)^2 - 4Ha^2 Rb(1 + Ha^2 n^2) - 4Ha^4 n^2 \right] \left[ (1 + Ha^2(n^2 - 2Rb))^2 - 4Ha^4 n^2 \right] [Ro + 1]
\]

for the marginal curves of the instability, where the following definitions for Re, Ha and the modified azimuthal wavenumber n are used:

\[
Re = \frac{\alpha \Omega(r)}{|k|^2 |\nu|}, \quad (3)
\]

\[
Ha = \frac{\alpha B_0(r)}{|k|^2 \tau(\mu_0 \rho \eta)^{1/2}}, \quad (4)
\]

\[
n = m/\alpha, \quad (5)
\]

with \(\alpha = k_z/|k|\) and \(|k|^2 = k_r^2 + k_z^2\) defined as functions of the axial and radial wavenumbers \(k_r\) and \(k_z\).

Because of its comparably simple form, and the absence of the ratio \(\beta\) of azimuthal to axial magnetic field (which would play a decisive role for HMRI), Equation (2) allows to easily visualize the transition from a shear-driven instability of the AMRI-type to the current-driven, kink-type Tayler instability (TI) [18], when going over from \(Rb = 1\) to \(Rb = 0\).

Let us start with the current-free case, \(Rb = 1\). Figure 1a shows, for varying values of Ro and the particular case \(n = 1.4\), the marginal curves in the Ha-Re plane. We see that the critical Re increases steeply for Ro below 6 which reflects the fact that we approach R_{0\text{ULL}} = 4.828 from above. We ask now for the dominant wavenumber ratio \(\alpha\) quite interesting. Figure 3b shows that the mode with \(n = 1\) (i.e. with \(k_r = 0\)), which is still dominant at Re = 0, is replaced by modes with higher values of n for increasing Re. The limits of the critical Ha for Re = 0 and Re \(\rightarrow\) \(\infty\) can be determined by setting to zero, in Equation 2, the nominator or denominator, respectively, which leads (for Rb = 0) to

\[
Ha_{Re=0} = 1/\sqrt{n(2-n)}, \quad (6)
\]

\[
Ha_{Re\rightarrow\infty} = \sqrt{\frac{(Ro + 1) + \sqrt{(Ro + 1)(Ro + 2)/n}}{Ro^2 + (Ro + 1)(4-n^2)}}. \quad (7)
\]

In the limit Ro \(\rightarrow\) \(\infty\) the limits of Ha converge slowly to zero according to Ha_{Re,Ro} \(\rightarrow\infty\) \(\approx n^{-1/2}Ro^{-1/4}\).

In the following, we compare our WKB results with recent findings [19] obtained for a TC flow with inner and outer radii \(r_i\) and \(r_o\) rotating with the angular velocities \(\Omega_i\) and \(\Omega_o\), respectively. The corresponding ratios are defined as \(\hat{\eta} = r_i/r_o\) and \(\hat{\mu} = \Omega_o/\Omega_i\). For this TC configuration, the following modified definitions of the Reynolds and Hartmann number were used: Re = \(\Omega_o r_i(r_o - r_i)/\nu\), Ha = \(B_0(r_i)(r_o(r_o - r_i))^{1/2}/(\mu_0 \rho \eta)\)^{1/2}. The non-trivial point is now how to translate the \(\hat{\mu}\) of a TC flow, characterized by \(\Omega(r) = a + b/r^2\), to the Ro of a flow with
\[ \Omega(r) \sim r^{2R_0}. \] An often used correspondence, based on equalizing the corresponding angular velocities at \( r_i \) and \( r_o \) [20], leads to

\[ \text{Ro}^* \simeq -1/2 \log_\hat{\eta} \hat{\mu} \quad (8) \]

while an alternative, more shear-oriented version leads to

\[ \text{Ro}^{**} \simeq \frac{1}{2} \frac{(1 + \hat{\eta})(\hat{\mu} - 1)}{(1 - \hat{\eta})(\hat{\mu} + 1)}. \quad (9) \]

Actually, for comparably small (positive or negative) values of Ro, the differences are not very significant, but they increase for steeper profiles. This is a key point for the adequateness of TC flows to "emulate" steep power function flows. In [19], the destabilizing effect of positive shear had been studied for TC flows (with \( R_b = 0 \) only), both for a wide gap with \( \hat{\eta} = 0.5 \) as well as a narrow gap with \( \hat{\eta} = 0.95 \). In either case, for large values of \( \hat{\mu} \), the critical Ha converged to some non-zero constant, which is not compatible with the translation to \( \text{Ro}^* \) since the latter should lead to a zero critical Ha (according to

\[ \text{Ro}^{**} \approx - \frac{1}{2} \log_\hat{\eta} \hat{\mu} \quad (8) \]

while an alternative, more shear-oriented version leads to

\[ \text{Ro}^{**} \approx \frac{1}{2} \frac{(1 + \hat{\eta})(\hat{\mu} - 1)}{(1 - \hat{\eta})(\hat{\mu} + 1)}. \quad (9) \]

Actually, for comparably small (positive or negative) values of Ro, the differences are not very significant, but they increase for steeper profiles. This is a key point for the adequateness of TC flows to "emulate" steep power function flows. In [19], the destabilizing effect of positive shear had been studied for TC flows (with \( R_b = 0 \) only), both for a wide gap with \( \hat{\eta} = 0.5 \) as well as a narrow gap with \( \hat{\eta} = 0.95 \). In either case, for large values of \( \hat{\mu} \), the critical Ha converged to some non-zero constant, which is not compatible with the translation to \( \text{Ro}^* \) since the latter should lead to a zero critical Ha (according to

\[ \text{Ro}^{**} \approx - \frac{1}{2} \log_\hat{\eta} \hat{\mu} \quad (8) \]

while an alternative, more shear-oriented version leads to

\[ \text{Ro}^{**} \approx \frac{1}{2} \frac{(1 + \hat{\eta})(\hat{\mu} - 1)}{(1 - \hat{\eta})(\hat{\mu} + 1)}. \quad (9) \]

Actually, for comparably small (positive or negative) values of Ro, the differences are not very significant, but they increase for steeper profiles. This is a key point for the adequateness of TC flows to "emulate" steep power function flows. In [19], the destabilizing effect of positive shear had been studied for TC flows (with \( R_b = 0 \) only), both for a wide gap with \( \hat{\eta} = 0.5 \) as well as a narrow gap with \( \hat{\eta} = 0.95 \). In either case, for large values of \( \hat{\mu} \), the critical Ha converged to some non-zero constant, which is not compatible with the translation to \( \text{Ro}^* \) since the latter should lead to a zero critical Ha (according to

\[ \text{Ro}^{**} \approx - \frac{1}{2} \log_\hat{\eta} \hat{\mu} \quad (8) \]

while an alternative, more shear-oriented version leads to

\[ \text{Ro}^{**} \approx \frac{1}{2} \frac{(1 + \hat{\eta})(\hat{\mu} - 1)}{(1 - \hat{\eta})(\hat{\mu} + 1)}. \quad (9) \]

Actually, for comparably small (positive or negative) values of Ro, the differences are not very significant, but they increase for steeper profiles. This is a key point for the adequateness of TC flows to "emulate" steep power function flows. In [19], the destabilizing effect of positive shear had been studied for TC flows (with \( R_b = 0 \) only), both for a wide gap with \( \hat{\eta} = 0.5 \) as well as a narrow gap with \( \hat{\eta} = 0.95 \). In either case, for large values of \( \hat{\mu} \), the critical Ha converged to some non-zero constant, which is not compatible with the translation to \( \text{Ro}^* \) since the latter should lead to a zero critical Ha (according to

\[ \text{Ro}^{**} \approx - \frac{1}{2} \log_\hat{\eta} \hat{\mu} \quad (8) \]

while an alternative, more shear-oriented version leads to

\[ \text{Ro}^{**} \approx \frac{1}{2} \frac{(1 + \hat{\eta})(\hat{\mu} - 1)}{(1 - \hat{\eta})(\hat{\mu} + 1)}. \quad (9) \]
the two ways of translation: the use of Ro∗∗ for TC flows, while the use of Ro∗ for assuming a translation to Ro∗∗, leads to a zero limit value.

Results, both for assuming a translation to Ro∗∗ (dashed lines) and to Ro∗ (full lines). For Re = 0, our result Ha(Re,Ro) → ∞ ≈ n−1/2Ro−1/4, see above). It turns out that the translation to Ro∗∗ is physically more adequate.

With the reasonable choice k_z = k_r = π/(r_i − r_o) we obtain the translations Re = π²2⁵/²µh/(1 + µh)(1 − ²/³) Re and Ha = π²(1 + µh)/(2µh)²(1 − ³/⁴) Ha. For ²/³ = 0.95 this amounts to Re = 1061/(1 + 1/²)Re and Ha = 2435Ha. Figure 4 shows the corresponding WKB results, both for assuming a translation to Ro∗ (dashed lines) and to Ro∗∗ (full lines). For Re = 0, our result Ha = 2670 agrees reasonably well with the exact value Ha = 3060 of the modal stability analysis [19]. What is more, the typical bend of the marginal curve to the left for increasing Re, and the limit values of Ha for large Re, are also confirmed. Yet, subtle differences show up for the two ways of translation: the use of Ro∗∗ confirms the existence of a finite limit value for the critical Ha, as typical for TC flows, while the use of Ro∗ would ultimately lead to a zero limit value.

This encouraging consistency of the local approximation and the modal stability analysis, evidenced for Rb = 0, brings us back to the point whether, for Rb = −1, the ULL can be confirmed in a TC experiment. Assuming Ro∗∗ as more physical than Ro∗, in the limit µ → ∞ we obtain Ro∗∗(µ →∞) = 1/(1 + µh)/(1 − ²/³). This means, in turn, that to emulate some Ro in a TC-flow, ²/³ has to fulfill the relation ³/² = µ/(2µ − 1). With view on the ULL, this implies that for Ro = 6, say, a minimum value of ³/² = 11/13 = 0.846 is needed. For TC-flows with wider gaps, such as ³/² = 1/2 the necessary shear could simply not be realized.

What are, then, the prospects for a corresponding experiment? Evidently, we need a rather narrow gap flow. Let us stick, for a first estimate, to the safe value ³/² = 0.95, and take the typical values Ro = 6, Ha = 2 and Re = 12 as read off from Figure 1a. This translates to µ = 1.89, Re = 8324 and Ha = 4870. For a prospective TC experiment with Na at 150°C, with ρ = 910 kg/m³, ν = 5.94 × 10⁻⁷ m²/s, σ = 9 × 10⁶ S/m, and an outer diameter of r_o = 0.25 m, this would amount to a rather moderate rotation frequency of ³/² = 0.26 Hz, yet a huge magnetic field B_o(r_i) = 0.69 T that requires a central current of I = 8.6 × 10⁵ A. Exhausting the shear resources, by choosing µ → ∞ and ³/² = 0.85 ≈ 11/13, those values would drop to Re = 3796, Ha = 892 or, physically, to ³/² = 0.044 Hz, B_o(r_i) = 77 mT, I = 8.2 × 10⁴ A. Any real TC experiment, however, would need more detailed simulations with a 1D marginal stability code to confirm and optimize the parameters.

This work was supported by German Helmholtz Association in frame of the Helmholtz Alliance LIMTECH. F.S. gratefully acknowledges fruitful discussions with Günther Rüdiger.

\* Electronic address: f.stefani@hzdr.de

\† Electronic address: kirillov@mi.ras.ru
[1] Lord Rayleigh, Proc. R. Soc. London A 93, 148 (1917).
[2] E.P. Velikhov, JETP 9, 995 (1959); S.A. Balbus, J.F. Hawley, Astrophys. J. 376, 214 (1991)
[3] W. Liu, J. Goodman, H. Ji, Astrophys. J. 643, 306 (2006)
[4] S.A. Balbus, P. Henri, Astrophys. J. 674, 408 (2008)
[5] R. Hollerbach, G. Rüdiger, Phys. Rev. Lett. 95, 124501 (2005)
[6] J. Priede, Phys. Rev. E 84, 066314 (2011)
[7] W. Liu, J. Goodman, I. Herron, H. Ji, Phys. Rev. E 74, 056302 (2006)
[8] O.N. Kirillov, F. Stefani, Y. Fukumoto, J. Fluid Mech. 760, 591 (2014)
[9] O.N. Kirillov, F. Stefani, Phys. Rev. Lett. 111, 061103 (2013)
[10] R. Hollerbach, V. Teeluck, G. Rüdiger, Phys. Rev. Lett. 104, 044502 (2010)
[11] O.N. Kirillov, F. Stefani, Y. Fukumoto, Astrophys. J. 756, 83 (2012)
[12] F. Stefani et al., Phys. Rev. Lett. 97, 184502 (2006); F. Stefani et al., Phys. Rev. E. 80, 066303 (2009)
[13] M. Seilmayer et al., Phys. Rev. Lett. 113, 024505 (2014)
[14] T. Tsukahara, N. Tillmark, P.H. Alfredsson, J. Fluid Mech. 648, 5 (2010)
[15] K.P. Parfrey, K. Menou, Astrophys. J. Lett. 667, L207 (2007)
[16] P. Charbonneau, Liv. Rev. Sol. Phys. 7, 3 (2010)
[17] G. Rüdiger, personal communication
[18] G. Rüdiger, M. Schultz, Astron. Nachr. 331, 121 (2010); M Seilmayer et al., Phys. Rev. Lett. 108, 244501 (2012)
[19] G. Rüdiger et al., Phys. Fluids, subm. (2015); arXiv:1505.05320
[20] G. Rüdiger et al., Mon. Not. R. Astron. Soc, 438, 271 (2014)