Hydrodynamic description of ultrarelativistic heavy-ion collisions

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Abstract Recent theoretical developments of relativistic hydrodynamics applied to ultrarelativistic heavy-ion collisions are briefly reviewed. In particular, the concept of a formal gradient expansion is discussed, which is a tool to compare different hydrodynamic models with underlying microscopic theories.

1 Introduction

The data collected in heavy-ion experiments performed at RHIC and the LHC are interpreted as the evidence for formation of an equilibrated strongly interacting matter that exhibits collective, fluid-like behavior. This new state of matter has been named a strongly interacting quark-gluon plasma (QGP). Theoretical description of plasma space-time evolution is based on vast applications of relativistic viscous hydrodynamics whose methods and applicability range are now very intensively studied.

In these lectures, several new developments within relativistic hydrodynamics used in the context of heavy-ion physics are presented. These developments refer, in particular, to: 1) the very concept of relativistic hydrodynamics as a universal description of systems approaching local equilibrium, 2) different formulations of relativistic hydrodynamics and methods that can be used to make comparisons between such formulations, and 3) the physical concept of early hydrodynamization that has replaced recently the idea of early thermalization of matter produced in the collisions.

The field of relativistic hydrodynamics is very broad and very actively analyzed at the moment, so these lectures cover necessarily only a few topics. For a general

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text that tries to answer the question what relativistic hydrodynamics is we refer to [11]. Other recent reviews on relativistic hydrodynamics used for heavy-ion collisions can be found in [2, 3, 4, 5, 6]. The classical text on hydrodynamics is [7] and its modern version can be found in [8]. Several textbooks are also available now that discuss relativistic hydrodynamics as a part of heavy-ion physics [9, 10, 11]. The connections between hydrodynamics, heavy-ion physics and string theory are discussed in [12, 13, 14].

In the remaining part of Introduction we introduce the standard model of heavy-ion collisions, discuss basic hydrodynamic concepts, define perfect-fluid and Navier-Stokes hydrodynamics, and present the insights from the AdS/CFT correspondence and the kinetic theory in the relaxation time approximation (RTA).

1.1 Standard model of heavy-ion collisions

Nowadays, one speaks often about the standard model of ultrarelativistic heavy-ion collisions, which separates the space-time evolution of the produced matter into three stages.

The first stage, lasting for about 1–2 fm/c after the initial impact, describes the matter that is highly out of equilibrium. To large extent, during this early time, the initial conditions for further evolution of matter are fixed. They can be determined by using, for example, the Glauber model which is based on simple geometric concepts [11]. In this case, the initial energy density of the system in the transverse plane reflects the distribution of nucleons in the colliding nuclei. In the first stage the hard probes are emitted: heavy quarks, jets and energetic photons. From a new theoretical perspective, the first stage can be called the hydrodynamization stage, i.e., the stage where the produced matter becomes eventually well described by equations of viscous hydrodynamics.

The second stage describes the hydrodynamic expansion of matter and lasts for about 10 fm/c (for central collisions of large nuclei). During this stage, a phase transition from the quark-gluon plasma to a hadronic gas takes place. It is built into the hydrodynamic equations through the use of the appropriate equation of state [15]. The hydrodynamic expansion leads to local equilibration, namely, to the situation where dissipative processes become negligible. The evidence for local equilibration comes from numerous successes of so called thermal models which analyze the ratios of hadronic abundances, for example, see [16] and references therein.

The final, third stage of the evolution of matter is called the freeze-out and describes decoupling of strongly interacting gas of hadrons into individual particles that are finally detected in heavy-ion experiments. In this lecture we shall not discuss the freeze-out process which is analyzed in a separate lecture by Ryblewski. We shall not discuss the effects of finite baryon density either, just concentrating on the collisions at the top available energies at RHIC and the LHC.
1.2 Basic hydrodynamic concepts

In classical physics hydrodynamics deals with liquids in motion. It is a subdiscipline of fluid mechanics (fluid dynamics) which deals with both liquids and gases \[^2\]. Liquids, gases, solids and plasmas are states of matter, characterized locally by macroscopic quantities, such as energy density, temperature or pressure. States of matter differ typically by compressibility and rigidity – liquids are less compressible than gases, solids are more rigid than liquids. We learn at school that a typical liquid conforms to the shape of its container but retains a (nearly) constant volume independent of pressure. A classical, natural explanation of different properties of liquids, gases, solids and plasmas is achieved within atomic theory of matter.

It is important for modern applications that hydrodynamics, similarly to thermodynamics, may be formulated without explicit reference to microscopic degrees of freedom \[^1\]. This is important if we deal with a strongly interacting QGP interacting matter — in this case neither hadronic nor partonic degrees of freedom seem to be adequate degrees of freedom.

The information about the state of matter is, to large extent, encoded in the structure of its energy-momentum tensor (equation of state, kinetic (transport) coefficients). This structure may be a priori determined by modeling of heavy-ion collisions. We are in some sense lucky that this scenario has been indeed realized. This is so, because the created system indeed evolves toward local equilibrium state, as we pointed out in the previous section discussing the applicability of thermal models.

The current understanding is that hydrodynamics can be treated as an effective theory describing approach of physical systems toward local equilibrium \[^1\]. During such an equilibration process different modes in the system are excited, which can be classified as either transient or long-lived ones. The former are called the

\[^1\] This point of view will be explicitly presented below in Sec. 3.3.
non-hydrodynamic modes, while the latter are called hydrodynamic ones. Genuine hydrodynamic behavior is attributed to the hydrodynamic modes, whose lifetime can be made larger by lowering the spatial variations of various physical variables such as energy density or pressure.

From this perspective it may come as surprise that hydrodynamic models used at the moment to analyze the heavy-ion data are based on the systems of equations which include both non-hydrodynamic and hydrodynamic modes. This is so because such models are constructed as approximations to microscopic theories which naturally include all kinds of excitations. The hydrodynamic modeling reveals the authentic hydrodynamic behavior of the systems only if it is not sensitive to the transient modes [17, 18, 19].

1.3 From global to local equilibrium

As hydrodynamical behavior is characteristic for system approaching equilibrium let us introduce first the concepts of global and local equilibrium which are defined by the specific forms of the energy-momentum tensor. The global equilibrium energy-momentum tensor in the fluid rest-frame is given by the expression [7]

\[
T_{\mu\nu}^{\text{EQ}} = \begin{pmatrix}
E_{\text{EQ}} & 0 & 0 & 0 \\
0 & P(E_{\text{EQ}}) & 0 & 0 \\
0 & 0 & P(E_{\text{EQ}}) & 0 \\
0 & 0 & 0 & P(E_{\text{EQ}})
\end{pmatrix}.
\] (1)

Here we assume that the equation of state is known, so the equilibrium pressure \( P \) is a given function of the energy density \( E_{\text{EQ}} \). In an arbitrary frame of reference we have

\[
T_{\mu\nu}^{\text{EQ}} = E_{\text{EQ}} u^\mu u^\nu - P(E_{\text{EQ}}) \Delta_{\mu\nu},
\] (2)

where \( u^\mu \) is a constant velocity, and \( \Delta_{\mu\nu} \) is the operator that projects on the space orthogonal to \( u^\mu \), namely

\[
\Delta_{\mu\nu} = g_{\mu\nu} - u^\mu u^\nu, \quad \Delta_{\mu\nu} u^\nu = 0.
\] (3)

The concept of local equilibrium is introduced by allowing the variables \( \mathcal{E} \) and \( u^\mu \) to depend on the spacetime point \( x \)

\[
T_{\mu\nu}^{\text{eq}}(x) = \mathcal{E}(x) u^\mu u^\nu(x) - P(\mathcal{E}(x)) \Delta_{\mu\nu}(x).
\] (4)

Here, the subscript “eq” refers to local thermal equilibrium. The energy-momentum tensor \( T_{\mu\nu}(x) \) describes perfect fluid. Local effective temperature \( T(x) \) is determined by the condition that the equilibrium energy density at this temperature agrees with

\[\text{We use the metric tensor } g_{\mu\nu} = \text{diag}(+,-,-,-).\]
the non-equilibrium value of the energy density, namely
\[ \mathcal{E}_{\text{EQ}}(T(x)) = \mathcal{E}_{\text{eq}}(x) = \mathcal{E}(x). \]

The variables \( T(x) \) and \( u^\mu(x) \) are fundamental fluid/hydrodynamic variables. The relativistic perfect-fluid energy-momentum tensor (4) is the most general symmetric tensor which can be expressed in terms of these variables without using derivatives.

Dynamics of the perfect fluid is determined by the conservation equations of the energy-momentum tensor
\[ \partial_\mu T^{\mu\nu}_{\text{eq}} = 0. \]

These are four equations for the four independent hydrodynamic fields \( T(x) \) and three independent components of \( u^\mu(x) \). In this way one obtains a self-consistent theoretical framework. It is important to notice that dissipation does not appear in the perfect-fluid dynamics. The projection of Eq. (6) along \( u_\nu(x) \) leads to the entropy conservation law
\[ \partial_\mu (\mathcal{S} u^\mu) = 0. \]

The quantity \( \mathcal{S} \) is the entropy density. The other three components in Eq. (6) correspond to the non-relativistic Euler equation.

### 1.3.1 Landau and Bjorken models

The two famous perfect-fluid hydrodynamic models of particle production were formulated in the past by Landau [20] and Bjorken [21]. Till now we refer frequently to them discussing different initial conditions used for hydrodynamic equations. In the Landau model [20], the matter produced in a collision forms initially a highly compressed disk. The equations of perfect fluid are then used to determine a one-dimensional expansion of matter along the collision axis. The Landau initial conditions correspond to so called full stopping scenario with high initial baryon number density in the central part of the collision region. On the other hand, in the Bjorken model we deal with so called transparency regime and typically negligible baryon number density in the central region. The Bjorken model [21] is motivated by the observation that fast particles are produced later and further away from the collision center than the slow ones. It is possible to account for this effect in the hydrodynamic description by imposing special initial conditions. In the Bjorken model they are implemented by the assumption that hydrodynamic expansion is invariant with respect to longitudinal Lorentz boosts (commonly known as boost invariance).

A popular expectation is that the Landau model is more appropriate for low-energy collisions, while the Bjorken model is suitable for description of high-energy processes. In practice one encounters sometimes Landau-like features at high energies and Bjorken-type features at low energies. For example, the rapidity distribution of the produced pions has usually (at low and high energies) a Gaussian shape that
naturally follows from the Landau model. This means that real modeling should be done with advanced three-dimensional hydro codes rather than with simple analytic models. The latter can be used to make simple estimates. In particular, the Bjorken model is commonly used to make an estimate of the initial energy density in the central region at the time when matter becomes equilibrated. Such estimates usually indicate that this energy is much larger than the critical energy corresponding to the phase transition.

1.4 Navier-Stokes hydrodynamics

In order to include dissipation in hydrodynamics, one adds a dissipative part $\Pi^{\mu\nu}$ to the perfect-fluid form of the energy-momentum tensor and constructs the complete $T^{\mu\nu}$ as

$$T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + \Pi^{\mu\nu},$$

(8)

where $\Pi^{\mu\nu}u_{\nu} = 0$, which corresponds to the Landau definition of the hydrodynamic flow $u^{\mu}$

$$T^{\mu}_{\nu}u^{\nu} = \varepsilon u^{\mu} = \varepsilon_{\text{eq}} u^{\mu}.$$  

(9)

It is useful to further decompose $\Pi^{\mu\nu}$ into two components,

$$\Pi^{\mu\nu} = \pi^{\mu\nu} + \Pi^{\Delta\mu\nu},$$

(10)

where $\Pi$ is the bulk viscous pressure (the trace part of $\Pi^{\mu\nu}$) and $\pi^{\mu\nu}$ is the shear stress tensor. The latter is symmetric, $\pi^{\mu\nu} = \pi^{\nu\mu}$, traceless, $\pi^{\mu}_{\mu} = 0$, and orthogonal to $u^{\mu}$, $\pi^{\mu\nu}u_{\nu} = 0$.

In the Navier-Stokes\(^3\) theory, the bulk pressure and shear stress tensor are given by the gradients of the flow vector

$$\Pi = -\zeta \partial_{\mu} u^{\mu}, \quad \pi^{\mu\nu} = 2\eta \sigma^{\mu\nu}.$$  

(11)

Here $\zeta$ and $\eta$ are the bulk and shear viscosity coefficients, respectively, and $\sigma^{\mu\nu}$ is the shear flow tensor defined as

$$\sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \partial_{\alpha} u^{\beta},$$

(12)

where the projection operator $\Delta^{\mu\nu}_{\alpha\beta}$ has the form

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\(^3\) Claude-Louis Navier, 1785–1836, French engineer and physicist, Sir George Gabriel Stokes, 1819–1903, Irish physicist and mathematician. The relativistic versions of their hydrodynamic equations were introduced by Eckart and Landau.
\[ \Delta_{\mu \beta} = \frac{1}{2} \left( \Delta_{\alpha \beta} \Delta_{\mu \alpha} + \Delta_{\mu \alpha} \Delta_{\beta \alpha} \right) - \frac{1}{3} \Delta^{\mu \nu} \Delta_{\mu \beta}. \] (13)

The shear viscosity describes the fluid’s reaction to the local change of its shape, while the bulk viscosity describes the reaction to a change of volume. For conformal systems, the bulk viscosity vanishes

\[ 0 = T_{\mu}^{\mu} = \varepsilon - 3 \phi - 3 \Pi + \pi_{\mu}^{\mu} = -3 \Pi, \quad \Pi = 0. \] (14)

In the Navier-Stokes theory the complete energy-momentum tensor has the structure

\[ T^{\mu \nu} = T_{\text{eq}}^{\mu \nu} + \Pi \Delta^{\mu \nu} = T_{\text{eq}}^{\mu \nu} + 2 \eta \sigma^{\mu \nu} - \zeta \theta \Delta^{\mu \nu} \] (15)

and we use again four equations,

\[ \partial_{\mu} T^{\mu \nu} = 0, \] (16)

for four unknowns: \( T(x) \) and three independent components of \( u^{\mu}(x) \). It turns out, however, that the use of (15) in the conservation laws (16) leads to problems connected with causality and stability of the solutions [22, 23]. Therefore, the framework based on (15) and (16) should be abandoned in most of practical applications.

At this place we note that the form (15) can be treated as an expansion of the energy-momentum tensor in gradients of \( T \) and \( u^{\mu}(x) \) around local equilibrium up to the terms of the first order in gradients

\[ T^{\mu \nu} = T_{\text{eq}}^{\mu \nu} + \pi^{\mu \nu} + \Pi \Delta^{\mu \nu} = T_{\text{eq}}^{\mu \nu} + 2 \eta \sigma^{\mu \nu} - \zeta \theta \Delta^{\mu \nu}. \] (17)

One can try to generalise this approach by adding further gradients

\[ T^{\mu \nu} = T_{\text{eq}}^{\mu \nu} + 2 \eta \sigma^{\mu \nu} - \zeta \theta \Delta^{\mu \nu} + \ldots \] (18)

It turns out, however, that this strategy does not lead to any improvements. As the matter of fact, as indicated by Heller, Janik, and Witaszczyk [24], this type of the gradient expansion is an asymptotic series with the convergence radius zero. We shall come back to the formal aspects of the gradient expansion in Sec. 4.

Although the approach based on (15) and (16) suffers from many conceptual and practical problems, the form (15) turns out to be a very good approximation to many energy-momentum tensors of the systems approaching local equilibrium. The resolution of this paradox lies in the observation that Eqs. (15) and (16) include only hydrodynamic modes, while microscopic theories include both. Therefore, hydrodynamic models used for interpretation of the data should go beyond the simple scheme based on (15) and (16).
1.5 Insights from AdS/CFT

By replacing the quark sector of QCD by a matter sector consisting of 6 scalar fields and 4 Weyl spinor fields one obtains a Yang-Mills theory which is conformal and finite $\mathcal{N}=4$ SYM theory. In 1990s Maldacena [25] and other authors (Gubser et al. [26], Witten [27]) realized that this quantum field theory, taken in the ’t Hooft limit, is a string theory that can be studied with the methods of classical gravity.

Although QCD and $\mathcal{N}=4$ SYM is quite different (apart from the gluon sector), at sufficiently high temperatures these differences become less prominent. In particular, the two theories seem to have a small value of the shear viscosity to entropy density ratio. For heavy-ion physics the important point is that $\mathcal{N}=4$ SYM provides a reliable means of observing how hydrodynamic behavior appears in a strongly coupled nonequilibrium system.

In the studies of thermalization, one considers commonly the difference of pressures components for a one-dimensional, boost-invariant expansion,

$$R \equiv \frac{P_T - P_L}{P}.$$  \hfill (19)

Here $P_L$ ($P_T$) is the pressure acting along (transversely to) the beam direction and $P$ is the equilibrium pressure corresponding to the effective temperature $T$ of the system. Introducing a dimensionless variable

$$w = \tau T(\tau),$$  \hfill (20)

where $\tau = \sqrt{t^2 - z^2}$ is the proper time and $T$ is the effective temperature, and the dimensionless function

$$f(w) = \frac{\tau}{w} \frac{dw}{d\tau},$$  \hfill (21)

we find [28, 29]

$$f(w) = \frac{2}{3} + \frac{R}{18}. \hfill (22)$$

Computing $f(w)$ within $\mathcal{N}=4$ SYM at late times one finds [30]

$$f(w) = \frac{2}{3} + \frac{1}{9\pi w} + \frac{1 - \log 2}{27\pi^2 w^2} + \frac{15 - 2\pi^2 - 45\log 2 + 24\log^2 2}{972\pi^3 w^3} + \ldots.$$  \hfill (23)

The first term in this expansion corresponds obviously to the perfect-fluid case with $R = 0$. The second term describes a correction due to the shear viscosity that corresponds to the famous Kovtun-Son-Starinets lower bound [31]

$$\frac{\eta}{\mathcal{F}} = \frac{1}{4\pi}. \hfill (24)$$
We note that this value is smaller than that of any other known substance, including superfluid helium.

The expansion of \( f(w) \) contains only inverse integer powers of \( w \)

\[
f(w) = \sum_{n=0}^{\infty} f_n w^{-n}.
\] (25)

The computation of further coefficients \( f_n \) was done and showed that this series has zero convergence radius \([24]\). Thus, the series (25) cannot be treated as a good approximation for the exact functions \( f(w) \). Nevertheless, the first terms become a very good approximation for the exact solutions at late times. This agreement may be attributed to the real hydrodynamic behavior manifested by the system evolving toward local equilibrium. Interestingly, the agreement between exact solutions and the first terms in (25) sets in at rather early times (inversely proportional to the initial effective temperature) when the system is anisotropic in the momentum space \([28, 29]\). This observation led to the concept of early hydrodynamization of matter that replaced the idea of early thermalization.

Let us note that the idea of early thermalization was introduced in the context of perfect-fluid hydrodynamics used to describe the RHIC data. The hydrodynamic fits favoured early starting time (\( \sim 0.5 \) fm/c) of the evolution. Since perfect-fluid hydrodynamics assumes local equilibrium, successful perfect-fluid fits with early starting time suggested early thermalization. The situation has changed with the use of viscous codes. Although the shear viscosity (divided by the entropy density) is small, the initial flow gradients are large and this leads to large corrections to the equilibrium pressure.

### 1.6 RTA kinetic equation

Several results discussed below refer to the kinetic-theory approach in the relaxation time approximation. Thus, before we continue our discussion of heavy-ion phenomenology and various hydrodynamic concepts, it is useful to define this framework.

The Boltzmann kinetic equation in the relaxation time approximation has the form

\[
p^\mu \partial_\mu f(x, p) = C[f(x, p)],
\] (26)

where the collision term is given by the expression \([32]\)

\[
C[f] = p^\mu u_\mu \frac{f^{\text{eq}} - f}{\tau_{\text{rel}}}.
\] (27)

In the case of Boltzmann (classical) statistics, the equilibrium background distribution is
\[ f^{eq} = \frac{gs}{(2\pi)^3} \exp \left( -\frac{pT u u}{T} \right), \]  
\[ (28) \]

where \( T \) is an effective temperature; \( T \) is chosen locally in such a way that \( f^{eq}(x, p) \) yields the same energy density as \( f(x, p) \), which is consistent with Eq. (5).

For boost invariant systems, Eq. (26) is reduced to a simple differential equation 
\[ \frac{\partial f(\tau, u, v)}{\partial \tau} = f^{eq}(\tau, u, v) - f(\tau, u, v), \]  
\[ (29) \]

where we have introduced the variables \( u = t p_L - zE \) and \( v = tE - z p_L \) [37, 38], and the equilibrium distribution function equals
\[ f^{eq}(\tau, u, p_T) = \frac{gs}{(2\pi)^3} \exp \left( -\frac{\sqrt{(u/\tau)^2 + p_T^2}}{T} \right). \]  
\[ (30) \]

The formal solution of Eq. (30) is
\[ f(\tau, u, p_T) = D(\tau, \tau_0) f_0(u, p_T) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_{eq}(\tau')} D(\tau, \tau') f^{eq}(\tau', u, p_T). \]  
\[ (31) \]

where \( D \) is the damping function
\[ D(\tau_2, \tau_1) = \exp \left[ -\int_{\tau_1}^{\tau_2} \frac{d\tau''}{\tau_{eq}(\tau'')} \right] \]  
\[ (32) \]

and \( f_0 \) denotes the initial distribution.

The RTA kinetic theory has become a popular tool in many theoretical studies because: 1) the simple form of the collision term allows for straightforward calculations of the kinetic coefficients [39, 40, 41], 2) the knowledge of exact solutions can be used to verify which hydrodynamic model is the best approximation of the kinetic-theory results [35, 36, 42, 43]. 3) the results described in this section can be generalised to the case of finite particle mass [44], which allows for studies of the bulk pressure and shear-bulk coupling effects [45]. 4) the formal gradient expansion can be done for this model [46] and compared with similar expansions done for hydrodynamic approaches.
2 Basic dictionary for phenomenology

In this section we very briefly discuss how hydrodynamic modeling of heavy-ion collision may bring us information about properties of strongly interacting QGP such as the viscosity coefficients $\eta$ and $\zeta$, and the equation of state.

2.1 Glauber model

Glauber model treats a nucleus-nucleus collision as a multiple nucleon-nucleon collision process (for a review of this approach see, for example, Ref. [11]). In this approach, the nucleon distributions in nuclei are random and given by the nuclear density profiles (Woods-Saxon densities for large nuclei), while the elementary nucleon-nucleon collision is described by the total inelastic cross section $\sigma_{NN}$. In the original formulation, the Glauber model was applied to elastic collisions only. In this case a nucleon does not change its properties in individual collisions, so all nucleon interactions can be well described by the same cross section. In subsequent applications of the Glauber model to inelastic collisions, one assumes that after a single inelastic collision an excited nucleon-like object is created that interacts basically with the same inelastic cross section with other nucleons. The starting point for the Glauber model is the eikonal approximation – the classical approximation to the angular momentum $l$, that is applied to the standard expansion of the elastic scattering amplitude.

The Glauber model can be used to determine the probability of having $n$ inelastic binary nucleon-nucleon collisions in a nucleus-nucleus collision at the impact parameter $b$. The Glauber model can be also used to calculate the number of nucleons that participate in a collision. To be more precise one distinguishes between the participants which may interact elastically and the participants which interact only inelastically. The latter are called the wounded nucleons [47].

The results obtained within the Glauber model can be directly used as an input for hydrodynamics. In this case, one assumes that the initial energy density (or entropy density) is proportional to the density of sources (for particle production) that are identified with binary collisions and wounded nucleons. In many calculations one simply uses a linear combinations of the density of binary collisions and wounded nucleons to define the initial entropy density in hydrodynamic codes. This procedure assumes implicitly equilibration (in the case where the perfect-fluid hydrodynamics is used) or hydrodynamization (if viscous hydrodynamics is used) of matter at the starting time when hydrodynamic equations are initialized.
2.2 Harmonic flows

At high energies one usually distinguishes between the longitudinal direction (along the beam axis) and the transverse plane (orthogonal to the beam). Then, one introduces the transverse mass of the produced particles as

$$m_T = \sqrt{m^2 + p_T^2} = \sqrt{m^2 + p_x^2 + p_y^2}.$$  \hfill (33)

The measure of the longitudinal momentum of a particle is rapidity

$$y = \frac{1}{2} \ln \left( \frac{E + p_L}{E - p_L} \right) = \text{arctanh} \left( \frac{p_L}{E} \right) = \text{arctanh} (v_L).$$  \hfill (34)

The region $y \approx 0$ in the center-of-mass frame of the colliding nuclei is called the central region. At ultrarelativistic collisions, in this region the energy density of the produced particles is the highest, while the baryon number density is the lowest (this feature can be easily explained in the color-flux-tube models in which the baryon number is carried by the ends of the tubes, while the baryon free matter is produced by the tube decays, see also our comments about the Bjorken model in Sec. 1.3.1). The central region is a very suitable place for making basic comparisons between various model predictions and the data. Since the baryon number density in this region may be neglected, the equation of state characterising matter in this region can be obtained directly from lattice simulations of QCD [15].

The produced particles are characterized by their spectra in rapidity and transverse momentum, which are commonly written in the form

$$\frac{dN}{dy d^2 p_T} = \frac{dN}{2 \pi p_T dp_T dy} \left[ 1 + \sum_{k=1}^{\infty} 2 v_k \cos \left( k (\phi_p - \Psi_k) \right) \right].$$  \hfill (35)

The angle $\phi_p$ is the azimuthal angle of the three-momentum in the transverse plane, whereas the angles $\Psi_k$ define reaction planes (different for each value of $k$). Equation (35) is nothing but the Fourier decomposition of the transverse-momentum distribution, with the harmonic flow coefficients $v_k$ characterising the strength of different types of the transverse-momentum anisotropy ($v_1$, $v_2$, and $v_3$ are called the directed, elliptic [48], and triangular flow [49], respectively).

A great accomplishment of hydrodynamic modeling of heavy-ion collisions is the explanation of the measured values of the harmonic flows [6], in particular of the elliptic flow [48]. In the hydrodynamic approach, the non-zero values of $v_k$ reflect asymmetries in the initial distribution of the energy density — hydrodynamic evolution transforms asymmetry of the initial shape of the collision region into asymmetry of the measured momentum distribution. Very important physical role is played here by fluctuations in the initial state, which lead to non-zero values of odd harmonic coefficients.

It turns out, that the hydrodynamic predictions of the magnitude of the elliptic flow are sensitive to the assumed value of the shear viscosity [50]. This allows for
quantitative determination of the shear viscosity to the entropy density ratio that lies in the range $1/(4\pi) \leq \eta/s \leq 2.5/(4\pi)$ \[51\], which is very close to the AdS/CFT lower bound \[24\]. The measurements of the correlations between produced pions give information about the space-time extensions of the produced system, which are affected by the system’s equation of state. This allows for experimental selection of the proper equation of state, which turns out to be consistent with the lattice simulations of QCD \[52, 53\]. In the coming years, the bulk viscosity can be also estimated by the experiment data, so that the two main kinetic coefficient of QGP will be determined \[54, 55\].

3 Viscous fluid dynamics

3.1 Müller-Israel-Stewart theory

Let us now turn to discussion of different hydrodynamic frameworks. In the Navier-Stokes theory, the bulk pressure and the shear stress tensor are defined directly by the effective temperature and the form of the hydrodynamic flow, see Eq. (11). As we have mentioned above, this leads to conceptual and practical problems if such a framework is applied to model physical processes (problems with causality and stability).

In the Müller-Israel-Stewart theory \[56, 57, 58\], the bulk pressure, $\Pi$, and the shear stress tensor, $\pi^{\mu\nu}$, are promoted to independent dynamic variables which satisfy the following two differential equations

$$\dot{\Pi} + \frac{\Pi}{\tau_{\Pi}} = -\beta_{\Pi} \theta,$$
$$\dot{\pi}^{\langle\mu\nu\rangle} + \frac{\pi^{\mu\nu}}{\tau_{\pi}} = 2\beta_{\pi} \sigma^{\mu\nu}.$$

Here $\tau_{\Pi}$ and $\tau_{\pi}$ are called the relaxation times and the coefficients $\beta_{\Pi}$ and $\beta_{\pi}$ are chosen in such a way that $\eta = \beta_{\pi} \tau_{\pi}$ and $\zeta = \beta_{\Pi} \tau_{\Pi}$. The dots on the left-hand sides of (36) and (37) denote the convective time derivative $u^{\mu} \partial_{\mu}$ and the angular brackets denote contraction with the projector $\Delta^{\mu\nu}_{ab}$ defined by Eq. (13). In the case where the terms with the dots are negligible compared to the terms containing the relaxation times, one reproduces the Navier-Stokes limit.

It is interesting to study initial dynamics of a system that is uniform in space but anisotropic in the momentum space. In this case $\theta = 0$ and $\sigma^{\mu\nu} = 0$ at the initial time, however, initial values of $\Pi$ and $\pi^{\mu\nu}$ are, in general, different from zero. According to Eqs. (36) and (37) the initial dynamics of such a system is described by exponential decays of $\Pi$ and $\pi^{\mu\nu}$. The timescales for these decays are set by $\tau_{\Pi}$ and $\tau_{\pi}$, respectively. This example illustrates a common situation, where initial dynamics of a system is dominated by transient, fast decaying modes. Such modes are known in the literature as non-hydrodynamic modes — the modes whose
frequency $\omega(k)$ does not vanish in the limit where the wave vector $k$ vanishes. The modes satisfying the condition $\omega(k) \to 0$ for $k \to 0$ are known as hydrodynamic modes.

During the time evolution of a system toward local and possibly global equilibrium, at first the non-hydrodynamic modes play a role and later the hydrodynamic modes become important. The behavior dominated by hydrodynamic modes corresponds to genuine hydrodynamic behavior that can be described by a few effective parameters such as the viscosity coefficients $\eta$ and $\zeta$.

Structures such as Eqs. (36) and (37) appear naturally if hydrodynamic equations are derived from the kinetic theory. As the matter of fact, in such derivations more terms are found than those appearing in (36) and (37). For example, many viscous hydrodynamic models of heavy-ion collisions have the structure [59, 60, 61, 62]

$$\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta_\Pi \theta - \frac{\zeta T}{2\tau_\Pi} \Pi \partial_k \left( \frac{\tau_\Pi}{\zeta T} u^k \right),$$

(38)

$$\dot{\pi}^{(\mu\nu)} + \frac{\pi^{(\mu\nu)}}{\tau_\pi} = 2\beta_\pi \sigma^{\mu\nu} - \frac{\eta T}{2\tau_\pi} \pi^{(\mu\nu)} \partial_k \left( \frac{\tau_\pi}{\eta T} u^k \right).$$

(39)

In the following we shall refer to Eqs. (38) and (39) by the acronym MIS.

### 3.2 DNMR theory

In order to check which terms should appear in the hydrodynamic equations one should introduce certain expansion parameters and count their powers. In the context of the kinetic theory (treated as an underlying microscopic theory for an effective, hydrodynamic description) such an expansion has been rigorously performed by Denicol, Niemi, Molnar and Rischke (DNMR) [63, 64, 65].

In their works, DNMR derive a general expansion of the phase space distribution function $\delta f(x, p) = f(x, p) - f_{eq}(x, p)$ in terms of its irreducible moments. In the second step, exact equations of motion for these moments are derived. In general, there is an infinite number of such equations and one deals with infinite number of coupled differential equations in order to determine the time evolution of the system. However, substantial reduction of the number of equations is possible, if the terms are classified according to a systematic power-counting scheme in the Knudsen and inverse Reynolds numbers [4]. As long as one keeps terms of second order (in both parameters) the equations of motion can be closed and expressed in terms of only 14 dynamical variables.

The DNMR formalism can be applied for a general collision term. In the case where one uses a simplified RTA form of the collision term, see Sec. 1.6 and ne-

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[4] The Knudsen number is the ratio of the characteristic microscopic and macroscopic scales describing the system. The inverse Reynolds numbers count the corrections in powers of $\sqrt{\tilde{P}_\mu^\nu \Pi^{\mu\nu} / \mathcal{P}}$ and $\Pi / \mathcal{P}$. 
glects vorticity, the DNMR equations have the following structure

\[
\dot{\Pi} + \frac{\Pi}{\tau_\Pi} = -\beta \Pi \theta - \delta_{\Pi \Pi} \Pi \theta + \lambda_{\Pi \pi} \Pi \mu^\nu \sigma_{\mu \nu},
\]

\[
\dot{\pi}^{(\mu \nu)} + \frac{\pi^{(\mu \nu)}}{\tau_\pi} = 2\beta \pi \sigma^{\mu \nu} - \delta_{\pi \pi} \pi^{(\mu \nu)} \theta - \tau_{\pi \pi} \pi^{(\mu \nu)} \sigma^{\nu \gamma} + \lambda_{\pi \pi} \Pi \sigma^{\mu \nu}.
\]

The terms \(\delta_{\Pi \Pi}, \lambda_{\Pi \pi}, \delta_{\pi \pi}, \tau_{\pi \pi}, \lambda_{\pi \pi}\) are new kinetic coefficients (compared to MIS). Their form follows directly from the RTA collision term. An interesting feature of Eqs. (40) and (41) is the presence of coupling between the bulk and shear sectors — terms proportional to \(\lambda_{\Pi \pi}\) and \(\lambda_{\pi \Pi}\) [45].

### 3.3 BRSSS theory

The DNMR approach constructs hydrodynamic equations in a direct relation to the kinetic theory. It is possible, however, to construct hydrodynamic equations without reference to any microscopic model or theory. An example of such a formulation for conformal systems is the method worked out by Baier, Romatschke, Son, Starinets, and Stephanov (BRSSS) [66]. In this formulation one constructs first the shear stress tensor out of gradients of the effective temperature \(T\) and the flow vector \(u^\mu\). This construction is based on the gradient expansion and includes all terms up to the second order which are allowed by the Lorentz and conformal symmetry. Among several allowed terms, the expression for \(\pi^{\mu \nu}\) contains the term \(\sigma^{\mu \nu}\). Using the first order relation between \(\pi^{\mu \nu}\) and \(\sigma^{\mu \nu}\), namely the Navier-Stokes relation \(\pi^{\mu \nu} = 2\eta \sigma^{\mu \nu}\), one can replace the term \(\sigma^{\mu \nu}\) by an expression containing \(\pi^{\mu \nu}\). In this way, one obtains a dynamic equation for the shear stress tensor, which has the form familiar from the MIS or DNMR theories.

What is important in the BRSSS approach is that it does not refer directly to any microscopic theory. The kinetic coefficients that appear in this approach should be matched to any underlying theory or the experiment. If one neglects the terms including vorticity and space-time curvature, the BRSSS equation take the form [66]

\[
\Pi = 0,
\]

\[
\dot{\pi}^{(\mu \nu)} + \frac{\pi^{(\mu \nu)}}{\tau_\pi} = 2\beta \pi \sigma^{\mu \nu} - \frac{4}{3} \pi^{(\mu \nu)} \theta + \frac{\lambda_1}{\tau_\pi} \pi^{(\mu \nu)} \pi^{(\nu \lambda)}.
\]

Equation (42) reflects the fact that we deal with conformal systems. The term \(\lambda_1\) is a new kinetic coefficient.
3.4 Anisotropic hydrodynamics

As we have discussed above, viscous hydrodynamics can be derived as expansion of the underlying microscopic kinetic theory in the Knudsen and inverse Reynolds numbers around the local equilibrium state [65]. This type of expansion may be questioned in the situation where space-time gradients and/or deviations from the local equilibrium are large. The goal of the anisotropic hydrodynamics program is to create a dissipative hydrodynamics framework that is better suited to deal with such cases (for a recent review see [5]).

The initial ideas of anisotropic hydrodynamics were introduced in Refs. [67, 68], see also [69, 70]. They were restricted to the boost-invariant, one-dimensional case. The framework of [67] was based on the energy-momentum conservation law and used a specific ansatz for the entropy source term that defined the off-equilibrium dynamics. The approach of [68] was based on the kinetic theory, and employed the zeroth and first moments of the Boltzmann kinetic equation in the relaxation time approximation.

Subsequent developments of anisotropic hydrodynamics were based mainly on the kinetic theory and can be classified as perturbative and non-perturbative schemes. In the perturbative approach [71, 72, 73] one assumes that the phase-space distribution function has the form \( f = f_{RS} + \delta f \), where \( f_{RS} \) is the leading order described by the Romatschke-Strickland form [74], accounting for the difference between the longitudinal and transverse pressures, while \( \delta f \) represents a correction. In this case, advanced methods of traditional viscous hydrodynamics (following DNMR) are used to restrict the form of \( \delta f \) and to derive hydrodynamic equations. In this way non-trivial dynamics may be included in the transverse plane and, more generally, in the full (3+1)D case. In the non-perturbative approach one starts with the decomposition \( f = f_{aniso} + \delta f \), where \( f_{aniso} \) is the leading order distribution function given by the generalised RS form. In this case, all effects due to anisotropy are included in the leading order, while the correction term \( \delta f \) is typically neglected. The latest development in this direction is known as the anisotropic matching principle [75].

Anisotropic and viscous hydrodynamics predictions have been checked against exact solutions available for the Boltzmann kinetic equation in the relaxation time approximation. Such studies have been done for the one-dimensional Bjorken geometry [35, 36, 44] and for the Gubser flow that includes transverse expansion [42, 43]. Such studies showed that anisotropic hydrodynamics better reproduces the results of the underlying kinetic theory than the standard viscous hydrodynamics. In addition, important constraints on the structure of the hydrodynamic equations and the form of the kinetic coefficients have been obtained within such studies.
Fig. 2 Schematic illustration of the gradient expansion as a formal tool that can be used for comparisons between microscopic theories and phenomenological hydrodynamic models. Such comparisons can be used to transfer the knowledge about the kinetic coefficients from the microscopic theory to hydrodynamic frameworks.

4 Gradient expansion

4.1 Formal aspects

It is very much convenient to have a formal method that allows for making comparisons between different hydrodynamical models and, more generally, between hydrodynamic models and close-to-equilibrium behavior of microscopic theories for which hydrodynamic models are regarded as good approximations. Such method exists and is based on the formal expansion of the energy-momentum tensor in gradients of $T$ and $u^\mu$ (around its local equilibrium form),

$$T^{\mu\nu} = T^{\mu\nu}_{\text{eq}} + \text{powers of gradients of } T \text{ and } u^\mu. \quad (44)$$

It is important to emphasise that elements of the gradient expansion are present in various methods used to derive hydrodynamic equations from the underlying microscopic theories. For example, the counting in the Knudsen number or the first step in the BRSSS method refer to the number of gradients of $T$ and $u^\mu$. However, such derivations include usually other arguments or assumptions that allow for construction of a self-consistent system of equations. In contrast to such derivations, the gradient expansion of the energy-momentum tensor (44) is not useful for finding approximate solutions of the microscopic theory. It should be regarded rather as a formal tool to make comparisons between different theories and to check their close-to-equilibrium behavior. This is illustrated schematically in Figs. 2 and 3.

In the first case, see Fig. 2 one performs the gradient expansion of the energy-momentum tensor for a specified microscopic theory and a phenomenological hydrodynamic model (the latter can be taken, for example, in the BRSSS formulation that has unspecified values of the kinetic coefficients). The comparison of the two expansions allows for fixing the values of kinetic coefficients in the hydrodynamic model that can be later used to approximate the close-to-equilibrium behavior of the microscopic theory. This strategy has been used in heavy-ion modeling by using...
the value of the shear viscosity obtained from the SYM theory in the hydrodynamic
codes. In the second case, see Fig. 3 one can use the gradient expansion to check
the internal consistnsccy between the underlying theory and the hydrodynamic model
which has been constructed as its close-to-equilibrium approximation.

**4.2 RTA kinetic model with Bjorken geometry**

Performing the gradient expansion in a general case is quite complicated but the
calculations become doable in the cases where the flow pattern is restricted by cer-
tain symmetries. This is the case of the Bjorken-type geometry, where expansion of
matter is one-dimensional and boost invariant.

In this section we consider the case where the underlying theory corresponds to
the conformal RTA kinetic model. For this model the gradient expansion has been
performed recently by Heller, Kurkela and Spalinski [46]. Their results can be com-
pared with the expansions done for the hydrodynamic models, for which we use the
MIS, DNMR and BRSSS versions. For the Bjorken geometry the only independent
component of the shear stress tensor is its longitudinal component $\phi = -\pi^{zz}$. The
equation expressing the energy and momentum conservation is common to all the
hydrodynamic frameworks and reads

$$\tau \dot{\varepsilon} = -\frac{4}{3} \varepsilon + \phi. \quad (45)$$

On the other hand, the dynamic equation for $\phi$ depends on the hydrodynamic frame-
work. For the MIS case with the RTA kinetic equation one finds

$$\tau \pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{4\tau \pi \phi}{3\tau} - \phi. \quad (46)$$

For the DNMR framework, again with the RTA kinetic equation, we have
\[ \tau_\pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{38}{21} \tau_\pi \dot{\phi} - \phi. \]  
(47)

Finally, for the BRSSS approach one finds

\[ \tau_\pi \dot{\phi} = \frac{4\eta}{3\tau} - \frac{\lambda_1 \phi^2}{2\eta^2} - \frac{4\tau_\pi \phi}{3\tau} - \phi. \]  
(48)

For conformal systems, the relaxation time should scale inversely with the temperature, namely

\[ \tau_{\text{rel}} = \frac{c}{T}, \]  
(49)

where \( c \) is a constant. For the RTA kinetic equation one can connect the relaxation time with the ratio of the shear viscosity to the entropy density \( \frac{\eta}{S} \), thus

\[ c = \frac{5\eta}{S}, \]  
(50)

Adopting the value of the \( \frac{\eta}{S} \) ratio (for example as \( \frac{1}{(4\pi)} \), see Eq. (24)) we fix \( c \).

Constructing the gradient expansion for hydrodynamic models we expand \( T, \dot{\varepsilon} \) and \( \phi \) around the Bjorken solution. For example, in the case of \( T \) we use the series

\[ T = T_0 \left( \frac{\tau_0}{\tau} \right)^{1/3} \left( 1 + \sum_{n=1}^{\infty} \left( \frac{c}{T_0 \tau_0} \right)^n \tau_\pi \left( \frac{\tau_0}{\tau} \right)^{2n/3} \right), \]  
(51)

where \( \tau \) is the proper time and \( \tau_0 \) is the initial proper time for which the initial Bjorken temperature is \( T_0 \). Using similar expansions for \( \dot{\varepsilon} \) and \( \phi \) we find the expansion coefficients and may construct the gradient expansion of \( T^{\mu \nu} \). However, as demonstrated in [28, 29] it is better to analyze the expansion of the function \( f(w) \) defined in Sec. 1.5.

The results for the coefficients \( f_n \) obtained for the RTA kinetic model and various hydrodynamic formulations are shown in Table 1. We can see that both BRSSS and DNMR agree with the exact results up to the second order in \( n \). There are, however, differences in obtaining such agreement in the two cases. For the BRSSS framework the agreement has been achieved by using the same value of the viscosity (as in the RTA model) and by fitting the \( \lambda_1 \) parameter in (48) — we remember that BRSSS does not specify the kinetic coefficients, thus, they can be adjusted to any microscopic model. For the DNMR approach the agreement is a consequence that the kinetic coefficients used in (47) follow from the RTA model. Table 1 shows also the results for MIS, which agree only up to the first order. This indicates that the MIS framework is incomplete (in the second order).

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5 We note that it is possible to get the agreement up to the third order if the third-order dissipative hydrodynamics formulated in [76] is used.
\begin{table}[h]
\begin{tabular}{|c|c|c|c|c|}
\hline
\(n\) & RTA & BRSSS & DNMR & MIS \\
\hline
0 & 2/3 & 2/3 & 2/3 & 2/3 \\
1 & 4/45 & 4/45 & 4/45 & 4/45 \\
2 & 16/945 & 16/945 & 16/945 & 8/135 \\
3 & \(-208/4725\) & \(-1712/99225\) & \(-304/33075\) & \(112/2025\) \\
4 & \(-0.044\) & \(-0.017\) & \(-0.009\) & \(0.055\) \\
\hline
\end{tabular}
\caption{Expansion coefficients \(f_n\) for the RTA kinetic model and various hydrodynamic frameworks.}
\end{table}

5 Closing remarks

Success of relativistic viscous hydrodynamics as a basic theoretical tool used to model heavy-ion collisions triggered broad interest in formal aspects of this framework and relations between hydrodynamic models and underlying microscopic theories. In these lectures we discussed a few topics related to formal gradient expansion and mutual relations between different hydrodynamic models. For a discussion of other issues we refer to the articles mentioned in Introduction.

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