A New 2 + 1 Dimensional Gravity Solution
Coupled to Non Linear Electrodynamics with a
Cosmological Constant

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Abstract
A solution for the Einstein gravity coupled with non linear electrodynamics is introduced in 2 + 1 dimensions. Especially, in the case with a non-vanishing cosmological constant, we obtain a novel black hole solution. To find fundamental characters of the solution, radial null rays in the space-time described by the solution are investigated.

1 Introduction

The most standard theory of gravitation is Einstein’s general theory of relativity in 3+1 space-time. However, since general theory of relativity is classical theory, this theory needs to be quantized. This problem is one of the most biggest issue in modern physics. There are many consideration to quantize the theory of gravitation in the past half century. Especially, more simpler theory like a 2 + 1 Einstein gravity has been considered since around 1980’s. They are easier than higher dimensional ones, so it is hoped that it will be quantized.

The advantages of the lower dimensional gravity are to be able to obtain exact solutions easier than the higher dimensional one. In addition, 2 + 1 gravity has some characteristic structure. First, the Weyl tensor always vanishes in 2 + 1 dimensions. Second, there is no propagating mode in 2 + 1 dimensional Einstein gravity. Besides, geometry itself differs from that in the higher dimensional case; for instance, the asymptotic behavior of the black hole space-time is peculiar in 2 + 1 dimensions. The review article [1] summarized study on 2 + 1 dimensional gravity.

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Gravity theory in 2+1 dimensions came to attract much attention, especially after a black hole solution has been found. This black hole is called BTZ black hole [2], and it is in the space-time that presence of a negative cosmological constant $\Lambda < 0$. In the limit of zero mass and zero angular momentum, there appears a naked singular point. The geometric structure and thermodynamical property of the BTZ space-time is also well investigated.

We note that the existence of a black hole solution in 2+1 dimensional world attract much attention to theory of 2+1 gravity. That is why to find a new solution in 2+1 dimensional Einstein gravity is important. In fact, recently new solution is considered in the case with other classical matter fields. As the classical field, a real scalar field [3], a field of nonlinear electrodynamics (NED) [4] (for example, the Born-Infeld type electrodynamics [5] is one of the NED, which attract much attention in string theory [6]), and the other fields are adopted and the solution with circular symmetry is investigated. The common feature of these substances is a non-vanishing radial component of the stress tensor $T^{rr}$. The authors of Ref. [4] claim that the solution with non-vanishing cosmological constant and NED describes a flat space-time because of the stress tensor from NED and the cosmological constant cancel out. However, component of the stress tensor from NED is $T^{rr}$, on the other hand, the cosmological constant generates the stress tensor not only $T^{rr}$ but also $T^t_t$ and $T^\theta_\theta$. That is why we doubt the result of Ref. [3] with the cosmological constant.

In the present paper, we derive a new solution for 2+1 gravity with non-vanishing cosmological constant and NED. In this paper, we use the natural system of unit. In addition, we consider the space-time metric with the signature $(-++$).

2 NED Lagrangian and Einstein field equation in 2+1 dimensional gravity

The following action which governs the Einstein gravity and NED is considered:

$$I = \frac{1}{2} \int d^3x \sqrt{-g} \left( R + \frac{2}{\ell^2} + \alpha \sqrt{|F|} \right),$$

(1)

where $F = F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Maxwell invariant, $\alpha$ is a coupling constant and $\ell$ is related to the cosmological constant $\Lambda$ with $\ell = \sqrt{-1/\Lambda}$. Let us take a circular symmetric metric ansatz:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + R(r)d\theta^2,$$

(2)

where $A(r)$, $B(r)$, $R(r)$ are arbitrary functions depending on $r$. Next, we assume that the field strength tensor $F_{\mu\nu}$ is calculated from the vector potential $\mathbf{A}$

$$\mathbf{A} = E_0(-b\theta, 0, at),$$

(3)

where $a$, $b$ are constants. In Ref. [3], they are taken as $a + b = 1$. We thus take the same condition in the present paper. Under this condition, non-vanishing
component of $F_{\mu\nu}$ is

$$F_{t\theta} = \partial_t A_\theta - \partial_\theta A_t = E_0$$

and

$$F^{t\theta} = g^{tt} g^{\theta\theta} F_{t\theta} = -\frac{E_0}{A(r) R(r)}.$$  (4)

From Eqs. (4) and (5), $\mathcal{F}$ becomes

$$\mathcal{F} = 2F_{t\theta} F^{t\theta} = -\frac{2E_0^2}{A(r) R(r)}.$$  (6)

Einstein field equation derived from the action (1) is written as

$$G_{\mu\nu} + \frac{1}{\ell^2} \delta_{\mu\nu} = T_{\mu\nu},$$

where $G_{\mu\nu}$ is the Einstein tensor defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R \delta_{\mu\nu}$ and $T_{\mu\nu}$ denotes the stress tensor

$$T_{\mu\nu} = \frac{\alpha}{2} \sqrt{|\mathcal{F}|} \left( \delta_{\mu\nu} - \frac{2F_{\nu\lambda} F^{\mu\lambda}}{\mathcal{F}} \right).$$  (8)

If $\mu \neq \nu$, Eq. (8) indicates that $T_{\mu\nu}$ vanishes. Let us write the diagonal components of the stress tensor explicitly. They are found to be

$$T^t_t = T^\theta_\theta = \frac{\alpha}{2} \sqrt{|\mathcal{F}|} \left( 1 - \frac{2F_{t\theta} F^{t\theta}}{2F_{t\theta} F^{t\theta}} \right) = 0,$$  (9)

$$T^r_r = \frac{\alpha}{2} \sqrt{\frac{2E_0^2}{A(r) R(r)}}.$$  (10)

Therefore, only non-zero component of the stress tensor is $T^r_r$. It is the same result as the case with a real scalar field, which depends only on the radial coordinate $r$.

The non-zero components of the Einstein tensor obtained from the metric (2) are

$$G^t_t = \frac{2R'' BR - R'^2 B - R'B'R}{4B^2 R^2},$$

$$G^r_r = \frac{4B^2 R^2}{4ARB},$$

$$G^{\theta}_\theta = \frac{2A'' BA - A'^2 B - A'B'A}{4B^2 A^2},$$

where $'$ means the derivative with respect to the radial coordinate $r$.

Finally, we get the field equation as follows:

$$\frac{2R'' BR - R'^2 B - R'B'R}{4B^2 R^2} - \frac{1}{\ell^2} = 0,$$  (14)
\[
\frac{A'R'}{4ARB} - \frac{1}{\ell^2} = \frac{\alpha}{2} \sqrt{\frac{2E_0^2}{A(r)R(r)}},
\]
(15)

\[
\frac{2A''BA - A'^2B - A'B'A}{4B^2A^2} - \frac{1}{\ell^2} = 0.
\]
(16)

As seen from Eqs. (14) and (16), we can set \( A = KR \), where \( K \) is a constant. Since \( K \) is taken to be unity in Ref. [4], we also take \( K = 1 \) in the following discussion. Then, Eq. (15) becomes

\[
\frac{A'^2}{AB} - \frac{4}{\ell^2}A = \frac{4\alpha E_0}{\sqrt{2}}.
\]
(17)

In Ref. [4], the corresponding term to the second term in Eq. (17) seems to be misplaced. Therefore they obtained the wrong solution for \( \Lambda \neq 0 \).

On the other hand, the condition \( A = R \) makes Eqs. (14) and (16) be

\[
\left( \frac{A'^2}{AB} \right)' - \frac{4}{\ell^2}A' = 0.
\]
(18)

We find that Eq. (18) are derived from Eq. (17). This fact will yield a new result with \( \Lambda \neq 0 \). The solutions with and without a cosmological constant will be exhibited in the next section.

3 The solution of the field equation

3.1 \( \Lambda = 0 (\ell \to \infty) \) case

In this subsection, we focus on the \( \Lambda = 0 \) case. Then, it is shown that the same result as Ref. [4] is re-derived. If \( \Lambda = 0 \), Eq. (17) turns out to be

\[
B = \frac{A'^2}{2\sqrt{2\alpha E_0}}.
\]
(19)

Let us take the following new coordinate:

\[
\bar{r} = \int \sqrt{AB}dr.
\]
(20)

Using the new coordinate, the metric (2) takes the form

\[
d\bar{s}^2 = -f(\bar{r})dt^2 + \frac{d\bar{r}^2}{f(\bar{r})} + f(\bar{r})d\theta^2,
\]
(21)

where \( f(\bar{r}) \equiv A(r(\bar{r})) \). This implies that it can be taken as \( AB = 1 \) without a loss of generality. When \( AB = 1 \), Eq. (19) turns out to be

\[
A' = \sqrt{2\sqrt{2\alpha E_0}}.
\]
(22)
This equation can easily be solved and the solution is
\[ A = \xi r + C, \quad (23) \]
where \( C \) is an integration constant and \( \xi \equiv \sqrt{2\alpha E_0} \). Then, the metric becomes
\[ ds^2 = -(\xi r + C)dt^2 + \frac{dr^2}{\xi r + C} + (\xi r + C)d\theta^2. \quad (24) \]
Since the constant \( C \) can be removed by the transformation \( r \to r - \frac{C}{\xi} \), we obtain
\[ ds^2 = -\xi rdt^2 + \frac{dr^2}{\xi r} + \xi r d\theta^2. \quad (25) \]
In order to compare this with the result of Ref. [3], we introduce new variables:
\[ r = \frac{\xi}{4} \tilde{r}^2, \quad \theta = \frac{2\tilde{\theta}}{\xi}, \quad t = \frac{2\tilde{t}}{\xi}. \quad (26) \]
Consequently, the solution in the 2 + 1 dimensional gravity coupled to NED is given by
\[ ds^2 = -\tilde{r}^2 dt^2 + d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2. \quad (27) \]
This form of the metric is the same as in Ref. [3]. Moreover, the solution (27) is not a “usual” black hole, but a singularity arises at \( \tilde{r} = 0 \); so it should be called “black point” (BP). For example, in Ref. [7], BP appears in 3 + 1 dimensions when a logarithmic \( U(1) \) gauge theory couples to gravity. In Ref. [8], the case with not only \( U(1) \) theory but also a dilatonic field is studied and a similar result has been found. Of course, this solution agrees with that in Ref. [3].

3.2 \( \Lambda \neq 0 \) case

In this subsection, we consider the field equation (17) for a \( \Lambda \neq 0 \) case. Again, without a loss of generality, it can be taken as \( AB = 1 \) (see subsection 3.1). Then, Eq. (17) becomes
\[ (A')^2 - \frac{4}{\ell^2} A = \frac{4\alpha E_0}{\sqrt{2}}. \quad (28) \]
This differential equation can be solved analytically, and the general solution is
\[ \xi^2 + \frac{4}{\ell^2} A = \frac{4}{\ell^2} (r - C)^2, \quad (29) \]
where \( C \) is an integration constant and we define \( \xi \equiv \sqrt{2\alpha E_0} \), again. When \( \Lambda \) approaches to zero, the solution (29) should coincide with (25). This condition implies that \( C = \xi/(2\Lambda) \). Then, the metric reduces to
\[ ds^2 = -(\xi r + \frac{r^2}{\ell^2})dt^2 + \frac{dr^2}{\xi r + \frac{r^2}{\ell^2}} + (\xi r + \frac{r^2}{\ell^2})d\theta^2. \quad (30) \]
Let us take the following variables

\[ r + \frac{1}{\xi} \frac{r^2}{\ell^2} = \frac{\xi}{4} r^2, \quad \theta = \frac{2\tilde{\theta}}{\xi}, \quad t = \frac{2\tilde{t}}{\xi}. \]  

(31)

Using them, the metric (30) can be rewritten as

\[ ds^2 = -\tilde{r}^2 d\tilde{t}^2 + \frac{dr^2}{1 + \frac{r^2}{\ell^2}} + \tilde{r}^2 d\tilde{\theta}^2. \]  

(32)

This is the new solution of 2 + 1 gravity coupled with NED. Of course, it can be seen when \( \Lambda \) reaches zero, the metric (32) becomes (27). Moreover, in the solution (32), we notice that if the spatial part of the metric reduce \( 1 + \frac{r^2}{\ell^2} \to \frac{r^2}{\ell^2} - M \), it seems to be a “black hole” solution. In fact, we introduce new variables as

\[ \tilde{r} = L\theta, \quad \tilde{\theta} = \frac{\theta}{L}, \quad t = \frac{t}{L}, \]  

(33)

where \( L \) is some arbitrarily constant. Then, the metric (32) reduce to

\[ ds^2 = -\tilde{r}^2 d\tilde{t}^2 + \frac{dr^2}{L^2 + \frac{r^2}{\ell^2}} + \tilde{r}^2 d\tilde{\theta}^2. \]  

(34)

Finally, let us take a constant \( L \) as \( M = -1/L^2 \), we get

\[ ds^2 = -r^2 dt^2 + \frac{dr^2}{(r^2/\ell^2) - M} + r^2 d\theta^2. \]  

(35)

Where \( \tilde{t}, \tilde{r}, \tilde{\theta} \) are replaced by \( t, r, \theta \) for simplicity. The metric has the same form of the ansatz (2), so the solution (35) must satisfy the field equation (14), (15) and (16). It can be shown the solution (35) satisfy (14) and (16) easily. Furthermore, (16) becomes

\[ \frac{4}{4r^2} \left( r^2 - M \right) - \frac{1}{r^2} = \frac{\alpha}{2} \sqrt{\frac{2E_0^2}{r^4}}. \]  

(36)

This equation means that coupling constant \( \alpha \) or \( E_0 \) must be negative if the “black hole mass” \( M \) is lager than zero, and the “mass” is

\[ M = -\frac{\alpha E_0}{\sqrt{2}}. \]  

(37)

So we conclude the meaning of “black hole mass” of the space time (35) is different from the BTZ black hole mass, because \( M \) is determined by the coupling constant \( \alpha \) and \( E_0 \).

“Black hole” solution (35) has the horizon at

\[ r_H = \sqrt{M\ell}. \]  

(38)
This point is not a singular point, because the Kretchmann invariant $R = R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$ from the metric (35) does not diverge at $r_H$. However, $R$ diverge at $r = 0$, this implies that spacetime (35) has a singular point at $r = 0$.

4 Radial null rays

In the previous section, we get the new solution (35) in $2+1$ gravity. Then we wonder that, a solution (35) describe black hole or not? That is why we consider radial null rays in this section to answer the above question.

A radial world line for a photon in space-time (35) is

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2 - r_H^2} = 0.$$  \hspace{1cm} (40)

We then have

$$dt = \pm \frac{\ell}{r} \sqrt{r^2 - r_H^2} dr.$$  \hspace{1cm} (41)

Where + sign stands for the outgoing, and the − sign for the incoming. This equation can integrate analytically, and the solution is

$$t + t_0 = \begin{cases} \pm \sqrt{M} \arctan \left( \frac{r^2 - r_H^2}{1 - r^2 / r_H^2} \right) & (r > r_H) \\ \pm \sqrt{M} \ln \left( \frac{1 + \sqrt{1 - r^2 / r_H^2}}{1 - \sqrt{1 - r^2 / r_H^2}} \right) & (r < r_H) \end{cases}.$$  \hspace{1cm} (42)

Light like geodesic described by Eqs. (42) in polar coordinates is shown in Fig.1. From Fig.1 light cones in the space-time (35) behave like Fig.2. This implies that light cones spread out when $r$ reaches to infinity. Action (1) represent asymptotically AdS space-time, this result is consistent. On the other hand, when $r$ reaches to horizon $r_H$, light cones become narrower. At the horizon $r = r_H$, the slope $dt/dr$ reaches to infinity, so light cones collapse. Next, we attend light cones that lies inside the horizon. Light cones are tipped over and null rays go to the singular point $r = 0$. This implies that any particle (except tachyon) can not escape from that lies in side of the horizon.

Above behavior of space-time (35) is similar to the well known black holes (For example, Schwarzschild black hole in AdS space-time). That is why we suggest our new solution (35) describes one of the black hole space-time.
Figure 1: Radial null rays in space-time (35). The horizontal axis represents the particle position $r$, and the vertical axis represents the coordinate time $t$. We set $\ell = 1, \sqrt{M} = 1$ for simplicity. The black solid line represents the horizon $r = r_H$. Outgoing rays are described by the blue lines, and incoming rays are red lines.

Figure 2: The behaviour of light cones of space-time (35) in the polar coordinates $(r, t)$. 

$r = 0$ $r = r_H$
5 Conclusion and remarks

In this paper, the solution of the field equation for $2 + 1$ dimensional gravity coupled to NED has been drawn. In the $\Lambda = 0$ case, it agrees with the solution derived in Ref. [4]. On the hand, in the $\Lambda \neq 0$ case, we have obtained the new solution (32). We also get the new black hole solution (35) from the field equation (14), (15) and (16). We notice that the black hole mass $M$ is related to the coupling constant $\alpha$ and the amplitude of the field strength $E_0$. Addition, the new solution (35) has the horizon at $r = \sqrt{M\ell}$. However, from the Kretchamann invariant, physical singular point of this space-time lies at $r = 0$, only. We also investigate radial null rays of the space-time (35), and the feature of rays are similar to Schwarzschild one. That is why we suggest that the new solution (35) from the action (1) describe one of the black hole solution.

By the way, although the sign in front of $dr^2$ is reversed by $r < r_c$ in the line element (35), the sign of the coefficient of $dt^2$ is unchanged. Thus it seems that the $r = r_c$ is not a genuine horizon. The singular behavior should be further investigated with an interest.

Moreover, in the BTZ black hole, thermodynamical consideration has been performed and the behavior of thermodynamic quantities are investigated. The space-time described the solution (35) should be studied along with the similar thermodynamical context. It is a future issue to survey thermodynamics of the space-time.

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