GAUSSIAN QUADRATURE RULES FOR COMPOSITE HIGHLY OSCILLATORY INTEGRALS

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Abstract. Highly oscillatory integrals of composite type arise in electronic engineering and their calculations are a challenging problem. In this paper, we propose two Gaussian quadrature rules for computing such integrals. The first one is constructed based on the classical theory of orthogonal polynomials and its nodes and weights can be computed efficiently by using tools of numerical linear algebra. An interesting connection between the quadrature nodes and the Legendre points is proved and it is shown that the rate of convergence of this rule depends solely on the regularity of the non-oscillatory part of the integrand. The second one is constructed with respect to a sign-changing function and the classical theory of Gaussian quadrature can not be used anymore. We explore theoretical properties of this Gaussian quadrature, including the trajectories of the quadrature nodes and the convergence rate of these nodes to the endpoints of the integration interval, and prove its asymptotic error estimate under suitable hypotheses. Numerical experiments are presented to demonstrate the performance of the proposed methods.

1. Introduction

Highly oscillatory integrals arise in a wide range of practical applications, such as acoustic and electromagnetic scattering, optics, electronic circuits and fluid mechanics. The computation of such integrals with traditional quadrature rules like Newton-Cotes and Gauss quadrature becomes intractable due to the oscillatory feature of the integrands. In the recent two decades, highly oscillatory integrals of the Fourier-type, i.e.,

\[ I[f] = \int_{a}^{b} f(x) e^{i\omega h(x)} dx, \quad \omega \gg 1, \]

where \( i \) is the imaginary unit and \( f \) and \( h \) are referred to as the amplitude and phase, have attracted substantial attention and several effective methods, such as asymptotic and Filon-type methods, Levin-type methods, numerical steepest descent methods and complex Gaussian quadrature, have been developed (see, e.g., [1, 12, 14, 15, 16]). All these methods share a remarkable advantage that their accuracy improves rapidly as \( \omega \) increases. We refer the interested readers to the recent monograph [10] for a comprehensive survey. In the numerical simulation of electronic circuits, the governing equation for the diode rectifier circuit can be formulated as the following nonlinear ordinary
differential equation
\begin{equation}
C \frac{dv(t)}{dt} = I_0 \left[ e^{b(t) - v(t)} - 1 \right] - \frac{v(t)}{R}, \quad t \geq 0, \quad v(0) = v_0,
\end{equation}
where \(C\) is the capacitance, \(R\) is the resistance, \(I_0\) is the diode reverse bias current, \(b(t)\) is the input signal and the unknown \(v(t)\) is the voltage (see, e.g., [5, 6, 8, 13]).

When choosing the input signal \(b(t) = \kappa \sin(\omega t)\) or \(b(t) = \kappa \cos(\omega t)\) and using waveform relaxation to (1.2), it is necessary to compute integrals of the forms
\begin{equation}
\int_a^b f(x) e^{\kappa \sin(\omega x)} \, dx, \quad \int_a^b f(x) e^{\kappa \cos(\omega x)} \, dx.
\end{equation}
It is easily seen that the above integrals are highly oscillatory integrals of composite type, which do not fit into the pattern of highly oscillatory integrals of Fourier-type in (1.1). Indeed, the above composite highly oscillatory integrals behave like \(O(1)\) as \(\omega \to \infty\) [13] and the Fourier-type integrals in (1.1) behave like \(O(\omega^{-\alpha})\) for some \(\alpha > 0\), and thus the asymptotic behaviors of (1.4) and (1.1) are quite different.

In this paper, we are concerned with the composite highly oscillatory integrals of the form
\begin{equation}
I[f] := \int_a^b f(x) (g \circ \phi_\omega)(x) \, dx, \quad \omega \gg 1,
\end{equation}
where \(g : [-1, 1] \to \mathbb{R}\) is a nonnegative and smooth function and \(\phi_\omega(x) = \sin(\omega x)\) or \(\phi_\omega(x) = \cos(\omega x)\). Note that (1.4) includes (1.3) as a special case (i.e., \(g(x) = e^{\kappa x}\)).

For such integrals, to the best of our knowledge, only few works have been conducted in the literature on their asymptotic expansion and numerical computations. We mention the asymptotic and Filon-type methods developed in [13] and the nonlinear approximation methods developed in [2]. However, each of these methods has its own disadvantage, such as the error of the asymptotic method is uncontrollable for a fixed \(\omega\) and the Filon-type methods require the computation of higher order derivatives of \(g(x)\) which is a notoriously ill-conditioned problem (see [13] Section 2.2). As for the nonlinear approximation method, it is based on the nonlinear approximation to \(f(x)\) by sums of complex exponentials and then evaluating the resulting integral with generalized Gaussian quadrature, which is relatively expensive in applications.

The goal of this paper is to develop two Gaussian quadrature rules for computing the composite highly oscillatory integrals (1.4). The first one is guaranteed by the standard theory of orthogonal polynomials and it is optimal in the sense that an \(n\)-point quadrature rule integrates exactly whenever \(f \in P_{2n-1}\), where \(P_k\) denotes the space of polynomials of degree at most \(k\) (i.e., \(P_k = \text{span}\{1, x, \ldots, x^k\}\)). We show that this rule is spectrally accurate whenever \(f\) is sufficiently smooth. The second Gaussian quadrature rule is constructed with respect to a sign-changing function and thus the existence of this rule can not be guaranteed. We explore numerically several theoretical aspects of this Gaussian quadrature, including the trajectories of the quadrature nodes and the convergence rate of these nodes to the endpoints of the integration interval. Once the quadrature rule exists, we prove that it is optimal in the sense that the error of an \(n\)-point quadrature rule behaves like \(O(\omega^{-n-1})\) as \(\omega \to \infty\), and thus its accuracy improves rapidly as \(\omega\) increases.

The rest of the paper is organized as follows. In section 2 we propose the first Gaussian quadrature rule for computing (1.4). We provide a detailed description and present a rigorous convergence analysis of the quadrature rule. In section 3
we propose the second Gaussian quadrature rule for computing \([14]\) based on the asymptotic analysis of the integrals \([13]\). Finally, in section \([\text{I}]\) we present some conclusions of our study.

2. The first Gaussian quadrature rule

In this section we construct the first Gaussian quadrature rule for computing the composite highly oscillatory integrals \([1.4]\). The key idea is to treat the composite function \((g \circ \phi_\omega)(x)\) as the weight function and construct a quadrature rule of the form

\[
I[f] = \int_a^b f(x)(g \circ \phi_\omega)(x)dx = \sum_{k=1}^n w_k f(x_k) + R_n[f],
\]

such that \(R_n[f] = 0\) whenever \(f \in \mathcal{P}_{2n-1}\). More precisely, let \(\{p_n^\omega(x)\}_{n=0}^\infty\) be a sequence of polynomials orthogonal with respect to \((g \circ \phi_\omega)(x)\). Notice that if \(g(x)\) is smooth and nonnegative, the existence of the sequence of the orthogonal polynomials \(\{p_n^\omega(x)\}_{n=0}^\infty\) is always guaranteed. Moreover, from the standard theory of Gaussian quadrature rules we know that the nodes \(\{x_k\}_{k=1}^n\) are precisely the zeros of \(p_n^\omega(x)\) and they are all located inside the interval \((a, b)\) and the quadrature weights \(\{w_k\}_{k=1}^n\) are all positive (see, e.g., \([11]\)).

In the following subsections, we shall discuss the computational aspects and develop some sharp error bounds for the Gaussian quadrature rule \((2.1)\). Hereafter, we shall restrict our attention to the case of \([a, b] = [-1, 1]\) for the sake of simplicity. However, the generalization to a more general setting is mathematically straightforward.

2.1. Polynomials orthogonal with respect to \((g \circ \phi_\omega)\). Let \(\{T_0(x), T_1(x), \ldots\}\) be the sequence of the Chebyshev polynomials of the first kind, i.e., \(T_k(x) = \cos(k \arccos(x))\). We express \(p_n^\omega(x)\) in terms of Chebyshev polynomials as

\[
p_n^\omega(x) = T_n(x) + \sum_{k=0}^{n-1} a_k T_k(x).
\]

From the orthogonality of \(p_n^\omega(x)\) it follows that

\[
\int_{-1}^1 p_n^\omega(x) T_j(x)(g \circ \phi_\omega)(x)dx = 0, \quad j = 0, \ldots, n - 1,
\]
or equivalently,

\[
\begin{pmatrix}
\mu_{0,0} & \mu_{1,0} & \cdots & \mu_{n-1,0} \\
\mu_{0,1} & \mu_{1,1} & \cdots & \mu_{n-1,1} \\
\vdots & \vdots & \ddots & \vdots \\
\mu_{0,n-1} & \mu_{1,n-1} & \cdots & \mu_{n-1,n-1}
\end{pmatrix} \begin{pmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{n-1}
\end{pmatrix} = - \begin{pmatrix}
\mu_{n,0} \\
\mu_{n,1} \\
\vdots \\
\mu_{n,n-1}
\end{pmatrix},
\]

where

\[
\mu_{k,j} = \int_{-1}^1 T_j(x) T_k(x)(g \circ \phi_\omega)(x)dx.
\]

It is easily verified that the matrix on the left-hand side of \((2.4)\) is symmetric positive definite and direct calculations show that its condition number for a fixed \(\omega\) behaves like \(O(n)\) as \(n\) increases, and thus the linear system \((2.4)\) can be solved efficiently by standard linear solvers. After solving \((2.4)\), we obtain the coefficients...
\{(a_k)_{k=0}^{n-1}\}, and the zeros of \(p_n^e(x)\) (i.e., \(x_k\)) can be obtained by computing the eigenvalues of the following colleague matrix (see, e.g., [17, Theorem 18.1])

\[
C = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
& \ddots & \ddots & \ddots & \\
& & \cdots & \cdots & \frac{1}{2} \\
& & & \cdots & 0
\end{pmatrix},
\]

In order to compute the quadrature nodes \(\{x_k\}_{k=1}^n\) for \(k = 0, \ldots, n\) and \(j = 0, \ldots, n - 1\). To achieve this, we introduce the modified Chebyshev moments

\[
\nu_j := \int_{-1}^{1} T_j(x)(g \circ \phi)(x)dx, \quad j = 0, 1, \ldots
\]

Recalling that \(T_j(x)T_k(x) = (T_{j+k}(x) + T_{|j-k|}(x))/2\) for all \(j, k \geq 0\), we obtain \(\mu_{k,j} = (\nu_{j+k} + \nu_{j-k})/2\), and thus it is sufficient to compute the modified Chebyshev moments \(\nu_0, \ldots, \nu_{2n-1}\). We now present two approaches to compute these modified moments:

1. If \(g(x)\) is analytic inside the disc \(|z| < r\) for some \(r > 1\). Let

\[
\rho_m = \frac{1}{2^{m-1}} \sum_{k=0}^{\infty} \frac{g^{(m+2k)}(0)}{k!(m+k)!4^k}, \quad m = 0, 1, \ldots,
\]

and let

\[
U_k(x) = \sum_{j=1}^{\infty} \frac{(-1)^j}{(2j)2j+1} \sin(2j\omega x) - \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)2j+1} \cos((2j + 1)\omega x),
\]

\[
V_k(x) = \sum_{j=1}^{\infty} \frac{(-1)^j}{(2j)2j+2} \cos(2j\omega x) + \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)2j+2} \sin((2j + 1)\omega x).
\]

From [13] we obtain for \(\phi_{\omega}(x) = \sin(\omega x)\) that

\[
\nu_j = \frac{\rho_0}{2} \int_{-1}^{1} T_j(x)dx + \frac{|j/2|}{\omega^{2k+1}} \left[ T_j^{(2k)}(1)U_k(1) - T_j^{(2k)}(-1)U_k(-1) \right]
\]

\[
+ \sum_{k=0}^{\lfloor(j-1)/2\rfloor} \frac{(-1)^k}{\omega^{2k+2}} \left[ T_j^{(2k+1)}(1)V_k(1) - T_j^{(2k+1)}(-1)V_k(-1) \right],
\]

and the derivatives of Chebyshev polynomials at both endpoints can be calculated by (see, e.g., [3, Equation (A.13)])

\[
T_n^{(k)}(\pm 1) = (\pm 1)^{n+k} \prod_{j=0}^{k-1} \frac{n^2 - j^2}{2j + 1}.
\]

Furthermore, it was shown in [13, Theorem 1] that the sequence \(\{\rho_m\}_{m=0}^{\infty}\) decays exponentially fast, and hence each \(U_k(x)\) and \(V_k(x)\) can be calculated accurately by truncating the infinite series properly. We remark that this approach is also applicable to the case \(\phi_{\omega}(x) = \cos(\omega x)\).
(II). We approximate the weight function with its Chebyshev expansion, i.e.,

\[(g \circ \phi_\omega)(x) \approx \sum_{k=0}^{m} c_k T_k(x)\]

and the coefficients \(\{c_k\}_{k=0}^{m}\) can be calculated efficiently by using the fast Fourier transform (FFT) in \(O(m \log m)\) operations, and then

\[\nu_j \approx \sum_{k=0}^{m} c_k \left[ \frac{1}{1 - (k + j)^2} + \frac{1}{1 - (k - j)^2} \right].\]

We remark that the approach (I) is advantageous whenever \(n\) is small and its computational cost does not increase as \(\omega\) increases. However, this approach requires the computation of higher order derivatives of \(g(x)\), which is a notoriously ill-conditioned problem. In contrast, the approach (II) is advantageous for all practical purposes since it can be implemented by means of the FFT. However, the parameter \(m\) in (2.11) will increase linearly with \(\omega\) to achieve a given target accuracy.

Remark 2.1. We can also apply Clenshaw-Curtis or Gauss-Legendre quadrature to calculate the modified Chebyshev moments \(\{\nu_j\}_{j=0}^{2n-1}\) directly and this strategy is advantageous when \(j\) is of small or moderate size. When \(j\) is large, however, a disadvantage of this strategy is that the number of quadrature nodes may be much larger than the number \(m\) in (2.12) when a prescribed accuracy for evaluating \(\nu_j\) is desired. This is what can be expected, since \(T_j(x)\) is also highly oscillatory for large \(j\).

Next, we consider the computation of the quadrature weights \(\{w_k\}_{k=1}^{n}\). Recalling that the Gaussian quadrature rule (2.1) is exact for \(f \in \mathcal{P}_{2n-1}\), we infer that

\[\sum_{k=1}^{n} w_k T_j(x_k) = \nu_j, \quad j = 0, \ldots, n-1,\]

or equivalently,

\[
\begin{pmatrix}
T_0(x_1) & T_0(x_2) & \cdots & T_0(x_n) \\
T_1(x_1) & T_1(x_2) & \cdots & T_1(x_n) \\
\vdots & \vdots & \ddots & \vdots \\
T_{n-1}(x_1) & T_{n-1}(x_2) & \cdots & T_{n-1}(x_n)
\end{pmatrix}
\begin{pmatrix}
w_1 \\
w_2 \\
\vdots \\
w_n
\end{pmatrix} =
\begin{pmatrix}
\nu_0 \\
\nu_1 \\
\vdots \\
\nu_{n-1}
\end{pmatrix},
\]

Hence, the quadrature weights \(\{w_k\}_{k=1}^{n}\) can be obtained from solving (2.14).

We now present some numerical experiments to illustrate the behavior of the nodes and weights of the Gaussian quadrature rule and the computations were performed using MATLAB on a laptop with an Intel(R) Core(TM) i5-8265U CPU with 8GB RAM. In all of the experiments, we apply the approach (II) to compute the modified moments \(\{\nu_0, \ldots, \nu_{2n-1}\}\) and the parameter \(m\) is chosen such that the last Chebyshev coefficient \(c_m\) is just above machine precision. Figure 1 illustrates the nodes \(\{x_k\}_{k=1}^{n}\) as a function of \(\omega\) for three values of \(n\) and Figure 2 illustrates the nodes \(\{x_k\}_{k=1}^{n}\) as a function of \(n\) for three values of \(\omega\) for the weight function \((g \circ \phi_\omega)(x) = \exp(2 \cos(\omega x))\). From Figure 1 we see that each node \(x_k\) converges to a limit value as \(\omega \to \infty\). A natural question to ask is what these limit values are? Interestingly, numerical experiments show that these limit values are exactly the
zeros of \( P_n(x) \), where \( P_n(x) \) is the Legendre polynomial of degree \( n \). To show this, let \( \{ x_k^{\text{GL}} \}_{k=1}^n \) be the zeros of \( P_n(x) \), we plot the absolute errors \(|x_k - x_k^{\text{GL}}|\) scaled by \( \omega \) in Figure 3. Clearly, we see that the nodes \( \{ x_k \}_{k=1}^n \) converge to the Legendre points \( \{ x_k^{\text{GL}} \}_{k=1}^n \) at the rate \( O(\omega^{-1}) \). Below we provide a rigorous proof for this observation.

**Theorem 2.2.** Let \( \{ x_k \}_{k=1}^n \) and \( \{ x_k^{\text{GL}} \}_{k=1}^n \) denote the zeros of \( p_n^\omega(x) \) and \( P_n(x) \), respectively, and assume that they are ordered according to the same rule. Then, for each \( k \in \{1, \ldots, n\} \), we have

\[
|x_k - x_k^{\text{GL}}| = O(\omega^{-1}), \quad \omega \to \infty.
\]

**Proof.** First, we define the following standard moments

\[
\lambda_k = \int_{-1}^{1} x^k (g \circ \phi_\omega)(x)dx, \quad k = 0, 1, \ldots.
\]

As \( \omega \to \infty \), by [13, Equation (2.3)] we know that

\[
\lambda_k = \frac{\rho_0}{2} \tau_k + O(\omega^{-1}),
\]

where \( \rho_0 \) is defined in [2,8] and \( \tau_k = 0 \) when \( k \) is odd and \( \tau_k = 2/(k+1) \) when \( k \) is even. Recalling the determinantal representation of orthogonal polynomials [7]
Chapter 1] and using the fact that the leading term of $T_n(x)$ is $2^{n-1}x^n$, we get

$$p_n^\phi(x) = 2^{n-1} \frac{\lambda_0 \lambda_1 \cdots \lambda_n}{\lambda_{n-1} \lambda_n \cdots \lambda_{2n-1}} \frac{\Delta_n}{x^n}, \quad \Delta_n = \begin{vmatrix} \lambda_0 & \cdots & \lambda_{n-1} \\ \vdots & \ddots & \vdots \\ \lambda_{n-1} & \cdots & \lambda_{2n-2} \end{vmatrix}.$$ 

Note that $\Delta_n \neq 0$ for all $n \in \mathbb{N}$ since the sequence of orthogonal polynomials $\{p_n^\phi\}$ always exists. In the special case where $(g \circ \phi_j)(x) = 1$, it is clear that $p_n^\phi(x)$ reduces exactly to $P_n(x)$ up to a constant factor. Specifically, let $K_n = (2n)!/(2^n(n!)^2)$ be the leading coefficient of $P_n(x)$, then

$$P_n(x) = \frac{K_n}{L_n} \frac{\tau_0 \tau_1 \cdots \tau_n}{\tau_{n-1} \tau_n \cdots \tau_{2n-1}} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\lambda_{n-1} \cdots \lambda_{2n-2}} = \frac{K_n}{L_n} \frac{p_n^\phi(x)}{x^n}.$$ 

and $L_n \neq 0$ for all $n \in \mathbb{N}$. Moreover, from the asymptotic behavior of the standard moments $\lambda_k$ given above, we can deduce that

$$\Delta_n = \left(\frac{\rho_0}{2}\right)^n L_n + O(\omega^{-1}), \quad \omega \to \infty.$$ 

Next, we consider the asymptotic behavior of $p_n^\phi(x)$ as $\omega \to \infty$. Combining the above four results and using the elementary properties of determinants, we obtain

$$p_n^\phi(x) = \frac{(\rho_0)^n}{2 \Delta_n} \frac{\tau_0 \tau_1 \cdots \tau_n}{\tau_{n-1} \tau_n \cdots \tau_{2n-1}} + O(\omega^{-1}) = \frac{(\rho_0)^n L_n}{2 K_n \Delta_n} P_n(x) + O(\omega^{-1})$$

$$= \frac{2^{n-1} K_n}{L_n} P_n(x) + O(\omega^{-1}).$$

Hence $p_n^\phi(x)$ converges to a multiple of $P_n(x)$ for all $x \in [-1, 1]$ at the rate $O(\omega^{-1})$ as $\omega \to \infty$, and consequently, the zeros of $p_n^\phi(x)$ converge to the corresponding zeros of $P_n(x)$ as $\omega \to \infty$. For each $k \in \{1, \ldots, n\}$, if we assume that $x_k = x_k^{\text{GL}} + \varepsilon_k(\omega)$ for some function $\varepsilon_k(\omega)$, then we infer from the above equation that $\lim_{\omega \to \infty} \varepsilon_k(\omega) = 0$, and therefore

$$p_n^\phi(x_k^{\text{GL}}) = 2^{n-1} \prod_{k=1}^n (x_k - x_k^{\text{GL}})$$

$$= 2^{n-1} \prod_{k=1}^n (x_k - x_k^{\text{GL}}) + 2^{n-1} \sum_{k=1}^n \varepsilon_k(\omega) \prod_{j=1, j \neq k}^n (x_j - x_j^{\text{GL}}) + \text{HOT}$$

$$= \frac{2^{n-1} K_n}{L_n} P_n(x) + 2^{n-1} \sum_{k=1}^n \varepsilon_k(\omega) \prod_{j=1, j \neq k}^n (x_j - x_j^{\text{GL}}) + \text{HOT},$$

where HOT denotes higher order terms, from which it follows that

$$p_n^\phi(x_k^{\text{GL}}) = 2^{n-1} \varepsilon_k(\omega) \prod_{j=1, j \neq k}^n (x_k^{\text{GL}} - x_j^{\text{GL}}) + \text{HOT}.$$
Figure 3. The absolute errors $|x_k - x_{k}^{\text{GL}}|$ with $k = 1, \ldots, n$ scaled by $\omega$ for $n = 5, 10, 20$.

Note that $p_n^\omega(x) = 2^{n-1}P_n(x)/K_n + O(\omega^{-1})$ holds true for all $x \in [-1, 1]$, it follows immediately that $p_n^\omega(x_{k}^{\text{GL}}) = O(\omega^{-1})$ as $\omega \to \infty$. Combining this with the above equation and noting that the Legendre points $\{x_{k}^{\text{GL}}\}_{k=1}^n$ are independent of $\omega$, we conclude that $\varepsilon_k(\omega) = O(\omega^{-1})$. This ends the proof.

Regarding the quadrature weights $\{w_k\}_{k=1}^n$, numerical experiments show that each $w_k$ also converges to a limit value as $\omega \to \infty$, however, these limit values are no longer the corresponding Gauss-Legendre quadrature weights $\{w_{k}^{\text{GL}}\}_{k=1}^n$.

2.2. Convergence analysis of Gaussian quadrature. In this subsection we present an error analysis of the Gaussian quadrature rule defined in (2.1). Our main results are stated in the following theorem.

**Theorem 2.3.** Let $R_n[f]$ be the remainder of the Gaussian quadrature rule defined in (2.1) and let $K_\omega = \int_{-1}^{1} (g \circ \phi_\omega)(x)dx$.

(i) If $f$ is analytic with $|f(z)| \leq M_\rho$ in the region bounded by the ellipse with foci $\pm 1$ and major and minor semiaxis lengths summing to $\rho > 1$, then for each $n \geq 0$,

$$
|R_n[f]| \leq \frac{4K_\omega M_\rho}{\rho^{2n}(1 - \rho^{-1})}.
$$

(ii) If $f, f', \ldots, f^{(m-1)}$ are absolutely continuous on $[-1, 1]$ and $f^{(m)}$ is of bounded variation $V_m$ for some $m \in \mathbb{N}$. Then, for each $n \geq \lceil (m + 2)/2 \rceil$,

$$
|R_n[f]| \leq \frac{4K_\omega V_m}{\pi m(2n - 1) \cdots (2n - m)}.
$$

**Proof.** We first consider the proof of (2.16). Recalling the Chebyshev series of $f$, i.e.,

$$
f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j T_j(x), \quad a_j = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x)T_k(x)}{\sqrt{1 - x^2}} dx.
$$

Notice that $R_n[f] = 0$ for $f \in \mathcal{P}_{2n-1}$. Substituting the above Chebyshev series into the remainder yields

$$
R_n[f] = \sum_{j=2n}^{\infty} a_j R_n[T_j] = \sum_{j=2n}^{\infty} a_j \left( I[T_j] - \sum_{k=1}^{n} w_k T_j(x_k) \right).
$$
For the term inside the bracket, using the inequality $|T_j(x)| \leq 1$ we find that

\begin{equation}
I[T_j] - \sum_{k=1}^{n} w_k T_j(x_k) \leq K_\omega + \sum_{k=1}^{n} w_k = 2K_\omega,
\end{equation}

where we have used the fact that $K_\omega = \sum_{k=1}^{n} w_k$ in the last equality. On the other hand, recall that the Chebyshev coefficients of $f$ satisfy $|a_j| \leq 2M_\rho/\rho^j$ for each $j \geq 0$ (see, e.g., [17, Theorem 8.1]). Combining this with (2.19) and (2.18) we obtain that

\begin{equation}
|R_n[f]| \leq \sum_{j=2n}^{\infty} \frac{4K_\omega M_\rho}{\rho^j} = \frac{4K_\omega M_\rho}{\rho^{2n}(1 - \rho^{-1})}.
\end{equation}

This proves (2.16). Next, we consider the proof of (2.17) and the idea is similar to that of (2.16). Recall from [17, Theorem 7.1] that the Chebyshev coefficients of $f$ satisfy $|a_j| \leq \frac{2V_m \pi^j}{j! \cdots (j-m)!}$. Combining this with (2.18) and (2.19) yields

\begin{equation}
|R_n[f]| \leq \sum_{j=2n}^{\infty} \frac{4K_\omega V_m}{\pi^j (j-1)! \cdots (j-m)!}
= \frac{4K_\omega V_m}{\pi^m (2n-1)! \cdots (2n-m)!}.
\end{equation}

This proves (2.17) and the proof of Theorem 2.3 is complete. \qed

Some remarks on Theorem 2.3 are in order.

Remark 2.4. From Theorem 2.3 we can see that the rate of convergence of the proposed Gaussian quadrature rule in (2.1) depends solely on the regularity of $f$. More precisely, the proposed Gaussian quadrature rule converges at an exponential rate whenever $f$ is analytic in a neighborhood of $[-1, 1]$ and at an algebraic rate whenever $f$ is differentiable on $[-1, 1]$; see Figure 4 for an illustration.

Remark 2.5. If the weight function $(g \circ \phi_\omega)(x)$ is even, then due to the parity of Chebyshev polynomials we have

\begin{equation}
R_n[f] = \sum_{j=0}^{\infty} a_{2n+2j} R_n[T_{2n+2j}],
\end{equation}

and hence the error bounds in (2.16) and (2.17) can be further improved.

We now present several numerical experiments to demonstrate the performance of the Gaussian quadrature rule in (2.1). We consider the following functions

\begin{equation}
f(x) = e^x, \quad x \sin(x), \quad \sqrt{x + 2}, \quad (1 + x^2)^{-1}, \quad |x|^3, \quad |\sin(x)|,
\end{equation}

and we consider the weight function $(g \circ \phi_\omega)(x) = \exp(2\sin(\omega x))$. For these functions of $f(x)$, it is easily seen that the first two functions are entire and thus the Gaussian quadrature will converge super-exponentially. For the third and fourth functions, it is easily seen that they are analytic in a neighborhood of $[-1, 1]$ and
Figure 4. Absolute errors of the Gaussian quadrature rule as a function of $n$ for $\omega = 50, 100, 200, 500, 1000$. The first and second rows are plotted in a log scale and the last row is plotted in a log-log scale.

thus the Gaussian quadrature converges exponentially. For the last two test functions, they are differentiable functions and thus the Gaussian quadrature converges algebraically. In Figure 4, we plot the absolute errors of the Gaussian quadrature rule as a function of $n$ for several values of $\omega$. Clearly, we see that numerical results are consistent with our theoretical analysis.
3. **The second Gaussian quadrature rule**

Although we have constructed an efficient Gaussian quadrature rule for computing the composite highly oscillatory integrals (1.4), its accuracy depends solely on the regularity of $f$ and does not improve as the parameter $\omega$ increases. A natural question to ask is: Can we construct a Gaussian quadrature rule for computing (1.4) such that its accuracy improves as the parameter $\omega$ increases? In this section we shall consider this issue and explore an alternative Gaussian quadrature by extending the idea in [1] for computing the standard Fourier-type integrals to the current setting.

Before proceeding, it is instructive to revisit the asymptotic expansion of the composite highly oscillatory integrals (1.4) which has been studied in [13] in detail. Suppose that $g(x)$ is analytic inside the disc $|z| < r$ for some $r > 1$, it was shown in [13] Equation (2.3) that the integrals (1.4) with $\phi_\omega(x) = \sin(\omega x)$ admit the following asymptotic expansion

$$I[f] \sim \frac{\rho_0}{2} \int_a^b f(x)dx + \sum_{k=0}^\infty \frac{(-1)^k}{\omega^{2k+1}} \left[ f^{(2k)}(b)V_k(b) - f^{(2k)}(a)V_k(a) \right]$$

$$+ \sum_{k=0}^\infty \frac{(-1)^k}{\omega^{2k+2}} \left[ f^{(2k+1)}(b)V_k(b) - f^{(2k+1)}(a)V_k(a) \right],$$

where the constant $\rho_0$ is defined in (2.8) and $U_k(x)$ and $V_k(x)$ are defined in (2.9). As a direct consequence of (2.3) we see immediately that $I[f] \to \frac{\rho_0}{2} \int_a^b f(x)dx$ as $\omega \to \infty$. Moreover, it is easily verified that this asymptotic result still holds whenever $\phi_\omega(x) = \cos(\omega x)$. In the following, we shall explore a new Gaussian quadrature rule for computing the value of $I[f]$.

We first split the integral $I[f]$ into two parts

$$I[f] = \int_a^b f(x) \left[ (g \circ \phi_\omega)(x) - \frac{\rho_0}{2} \right] dx + \frac{\rho_0}{2} \int_a^b f(x)dx.$$

From (3.1) we know that the first integral on the right hand side behaves like $O(\omega^{-1})$ as $\omega \to \infty$ and the second integral on the right hand side is independent of $\omega$. In the sequel, we denote the first and second terms on the right hand side of (3.2) by $I_O[f]$ and $I_S[f]$, respectively. As for $I_S[f]$, notice that its integrand is non-oscillatory, which can be evaluated efficiently by using Gauss-Legendre or Clenshaw-Curtis quadrature. Hence, we shall restrict our attention to the computation of $I_O[f]$.

Inspired by the Gaussian quadrature rule for computing the standard Fourier-type integrals, i.e., $\int_a^b f(x)e^{i\omega x}dx$, developed in [1], we extend the idea to the current setting. Specifically, let $\{q^n_\omega\}_{n=0}^\infty$ be the sequence of orthogonal polynomials with respect to the function $(g \circ \phi_\omega)(x) - \rho_0/2$, i.e.,

$$\int_a^b q^n_\omega(x)x^j \left[ (g \circ \phi_\omega)(x) - \frac{\rho_0}{2} \right] dx = 0, \quad j = 0, \ldots, n - 1.$$

Notice that $(g \circ \phi_\omega)(x) - \rho_0/2$ may be a sign-changing function on the interval $[a,b]$, the existence of $\{q^n_\omega\}_{n=0}^\infty$ can not be guaranteed anymore. However, once $q^n_\omega(x)$ exists, we can construct a Gaussian quadrature of the form

$$I_O[f] = \sum_{k=1}^n w_k f(x_k) + R_n^O[f],$$
where the quadrature nodes \( \{x_k\}_{k=1}^n \) are the zeros of \( q_n^\omega(x) \) and \( R_n^f = 0 \) for \( f \in \mathcal{P}_{2n-1} \). To gain some insight into the Gaussian quadrature (3.4), we consider the example \( g(x) = \ln(x + 4), \phi_\omega(x) = \cos(\omega x) \) and we restrict our attention to the case where \( n \) is an integer multiple of 4, i.e., \( n = 4, 8, \ldots \). For this example, it is easily checked from (2.8) that \( \rho_0 = 2 \ln(2 + \sqrt{15})/2 \). The zeros of \( q_n^\omega(x) \) are computed by using steps similar to the computation of the zeros of \( p_n^g(x) \) in subsection 2.1.

In Figure 5 we display the trajectories of the zeros of \( q_n^\omega(x) \) as a function of \( \omega \) for \( \omega \in [20, 100] \). Clearly, we see that the zeros of \( q_n^\omega(x) \) are distributed in the complex plane and half the zeros converge towards to the left endpoint and half the zeros converge towards to the right endpoint as \( \omega \to \infty \). To clarify the rate of convergence of these zeros to the endpoints, we further display the absolute errors of these zeros with the corresponding endpoint scaled by \( \omega \) and the computations were performed by using the multiprecision computing toolbox available at www.advanpix.com with 20-digit arithmetic. Numerical results are illustrated in Figure 5 and we can see that the zeros of \( q_n^\omega(x) \) converge to the corresponding endpoint at the rate \( O(\omega^{-1}) \) as \( \omega \to \infty \). We also examined more examples, including \( g(x) = e^x \), \((x + 2)^4\), \(1/(4 - x)\), \(1/(1 + x^2)\), and the computed results are quite similar to those displayed in Figures 5 and are thus omitted. We summarize our experimental observations when \( n \) is an integer multiple of 4 as follows:

- The zeros of \( q_n^\omega(x) \) are distributed in the complex plane and half the zeros converge toward the left endpoint and half the zeros converge toward the right endpoint as \( \omega \to \infty \).
- The trajectories of the zeros of \( q_n^\omega(x) \) are symmetric with respect to both the real and imaginary axes whenever \( (g \circ \phi_\omega)(x) \) is even or odd.
- The rates of convergence of the zeros of \( q_n^\omega(x) \) to one of the endpoints are \( O(\omega^{-1}) \) as \( \omega \to \infty \).

We remark that the behavior of the zeros of \( q_n^\omega(x) \) is quite similar to that of the zeros of the polynomials orthogonal with respect to the function \( e^{\omega x} \) (see [1] for details). Based on the above observations, we now prove the asymptotic error estimate of the Gaussian quadrature rule for computing \( I_\omega[f] \) for large \( \omega \) under the hypotheses on the asymptotic behavior of the quadrature nodes.

**Theorem 3.1.** Suppose that \( f \) is analytic in a region containing \([a, b]\). Let \( n \) be an even positive integer and let \( \{x_k, w_k\}_{k=1}^n \) be the nodes and weights of the Gaussian quadrature rule in (3.4). If the quadrature nodes \( \{x_k\}_{k=1}^n \) can be splitted into two groups \( \{\hat{x}_k, \hat{x}_k\}_{k=1}^{n/2} \) which satisfy

\[
\hat{x}_k = a + O(\omega^{-1}), \quad \hat{x}_k = b + O(\omega^{-1}), \quad k = 1, \ldots, n/2.
\]

Then,

\[
I_\omega[f] - \sum_{k=1}^n w_k f(x_k) = O(\omega^{-n-1}), \quad \omega \gg 1.
\]

**Proof.** We follow the line as the proof of Theorem 4.1 in [1] and we only prove the case of \( \phi_\omega(x) = \sin(\omega x) \) since the proof of the case \( \phi_\omega(x) = \cos(\omega x) \) is similar. Let \( h_{2n-1}(x) \) be the Hermite interpolation polynomial of degree \( 2n - 1 \) which interpolates \( f \) and its derivatives up to order \( n - 1 \) at both endpoints \( \{a, b\} \). By the Hermite’s integral formula [9, Theorem 3.6.1] we have that \( f(x) = \)
h_{2n-1}(x) + \zeta(x)(x-a)^n(x-b)^n$, where $\zeta(x)$ is analytic inside the analyticity region of $f$. Recalling the Gaussian quadrature rule in (3.4) satisfies $R_n^O[f] = 0$ for $f \in \mathbb{P}_{2n-1}$, we obtain that
\[
I_O[f] - \sum_{k=1}^{n} w_k f(x_k) = \int_{a}^{b} \zeta(x)(x-a)^n(x-b)^n \left[ (g \circ \phi_\omega)(x) - \frac{\rho_n}{2} \right] dx
\]
(3.7)
\[-\sum_{k=1}^{n} w_k \zeta(x_k)(x_k-a)^n(x_k-b)^n.
\]
For the first term on the right hand side of (3.7), note that $\{a, b\}$ are the $n$ multiple zeros of $\zeta(x)(x-a)^n(x-b)^n$. Invoking the asymptotic expansion of the composite highly oscillatory integrals in (3.1), we deduce that the first term behaves like $O(\omega^{-n-1})$ as $\omega \to \infty$. Next, we consider the asymptotic estimate of the second term on the right hand side of (3.7). Since the quadrature nodes satisfy $x_k = a + O(\omega^{-1})$ or $x_k = b + O(\omega^{-1})$, we deduce immediately that $(x_k-a)^n(x_k-b)^n = O(\omega^{-n})$ for
all $k = 1, \ldots, n$. Now we consider the asymptotic estimate of the quadrature weight \( w_k \). Recall that the Gaussian quadrature is an interpolatory quadrature rule, we infer that

\[
    w_k = \int_a^b \ell_k(x) \left[ (g \circ \phi_\omega)(x) - \frac{p_0}{2} \right] dx,
\]

where \( \ell_k(x) \) is the $k$th Lagrange basis polynomial associated with the nodes \( \{x_k\}_{k=1}^n \). Furthermore, using the asymptotic expansion in (3.1) to \( w_k \) and noting that \( \ell_k(x) \) is a polynomial of degree \( n - 1 \), we obtain that

\[
    w_k = \sum_{j=0}^{n/2-1} \frac{(-1)^j}{\omega^{2j+1}} \left[ \ell_k^{(2j)}(b) U_k(b) - \ell_k^{(2j)}(a) U_k(a) \right] + \sum_{j=0}^{n/2-1} \frac{(-1)^j}{\omega^{2j+2}} \left[ \ell_k^{(2j+1)}(b) V_k(b) - \ell_k^{(2j+1)}(a) V_k(a) \right],
\]

where \( U_k(x) \) and \( V_k(x) \) are defined as in (2.6) and \( U_k(x) = O(1) \) and \( V_k(x) = O(1) \) as \( \omega \to \infty \). In the following we shall consider the estimates of \( \ell_k^{(j)}(a) \) and \( \ell_k^{(j)}(b) \) for $k = 1, \ldots, n$ and $j = 0, \ldots, n - 1$ as \( \omega \to \infty \). Assuming that \( x_k = a + O(\omega^{-1}) \) for $k = 1, \ldots, n/2$ and \( x_k = b + O(\omega^{-1}) \) for $k = n/2 + 1, \ldots, n$ and letting \( \Lambda_1 \) and \( \Lambda_2 \) denote the sets \( \{0, \ldots, n/2 - 1\} \) and \( \{n/2, \ldots, n - 1\} \), respectively. From the definition of \( \ell_k(x) \) it follows that

\[
    \ell_k(x) = \frac{n}{n} \prod_{j=1, j \neq k}^{n} \frac{(x - x_j)}{(x_k - x_j)}, \quad \ell_k'(x) = \frac{n}{n} \prod_{j=1, j \neq k}^{n} \frac{(x - x_j)}{(x_k - x_j)},
\]

and hence, by direct calculation, we find that \( \ell_k(a) = O(1) \) and \( \ell_k'(a) = O(\omega) \) for $k = 1, \ldots, n/2$ and \( \ell_k(a) = O(\omega^{-1}) \) and \( \ell_k'(a) = O(1) \) for $k = n/2 + 1, \ldots, n$, and \( \ell_k(b) = O(\omega^{-1}) \) and \( \ell_k'(b) = O(1) \) for $k = 1, \ldots, n/2$ and \( \ell_k(b) = O(\omega) \) for $k = n/2 + 1, \ldots, n$. Further, it can be shown by induction that, for $k = 1, \ldots, n/2$,

\[
    \ell_k^{(j)}(a) = \begin{cases} O(\omega^j), & j \in \Lambda_1, \\ O(\omega^{n/2-1}), & j \in \Lambda_2, \end{cases} \quad \ell_k^{(j)}(b) = \begin{cases} O(\omega^{j-1}), & j \in \Lambda_1, \\ O(\omega^{n/2-1}), & j \in \Lambda_2, \end{cases}
\]

and for $k = n/2 + 1, \ldots, n$,

\[
    \ell_k^{(j)}(a) = \begin{cases} O(\omega^{-1}), & j \in \Lambda_1, \\ O(\omega^{n/2-1}), & j \in \Lambda_2, \end{cases} \quad \ell_k^{(j)}(b) = \begin{cases} O(\omega^j), & j \in \Lambda_1, \\ O(\omega^{n/2-1}), & j \in \Lambda_2, \end{cases}
\]

and thus we can deduce that \( w_k = O(\omega^{-1}) \) for each $k = 1, \ldots, n$. Combining this with the fact \( (x_k - a)^n(x_k - b)^n = O(\omega^{-n}) \), we deduce that the second term on the right hand side of (3.7) also behaves like \( O(\omega^{-n-1}) \) as \( \omega \to \infty \). Thus, we conclude that both terms on the right hand side of (3.7) are \( O(\omega^{-n-1}) \), which completes the proof.

To show the sharpness of our asymptotic error estimates derived in Theorem 3.1, we consider two examples: \( f(x) = 1/(x+2) \), \( (g \circ \phi_\omega)(x) = \exp(\sin(\omega x)) \) and \( f(x) = 1/(x^2+1) \), \( (g \circ \phi_\omega)(x) = \ln(4 + \cos(\omega x)) \). In each example we evaluate
the integral $I_2[f]$ directly and compute the integral $I_O[f]$ by applying the Gaussian quadrature rule in $\mathbb{R}^d$. In Figure 6 we display the absolute errors of the $n$-point Gaussian quadrature rule for $n = 4$ and $n = 8$. It is easily seen that the error of the Gaussian quadrature rule behaves like $O(\omega^{-5})$ for $n = 4$ and $O(\omega^{-9})$ for $n = 8$, and these results are consistent with our theoretical analysis.

Finally, we consider comparing the performance of the Gaussian quadrature rule in (3.4) with the Filon-type methods developed in [13]. Let $a = c_1 < c_2 < \cdots < c_\nu = b$ be a set of points and let $\mu_1, \mu_2, \cdots, \mu_\nu \in \mathbb{N}$ be the corresponding multiplicities. Let $\psi$ be a polynomial of degree $N = \sum_1^\nu \mu_k - 1$ which satisfies

$$\psi^{(j)}(c_k) = f^{(j)}(c_k), \quad j = 0, 1, \cdots, \mu_k - 1, \quad k = 1, 2, \cdots, \nu.$$ 

The Filon-type method for computing $I[f]$ is defined by (see [13] Equation (3.11))

$$Q^F_\mathbf{F}[f] = \frac{P_0}{2} \int_a^b f(x)dx + \sum_{k=0}^{|N/2|} \frac{(-1)^k}{\omega^{2k+1}} \left[ \psi^{(2k)}(b)U_k(b) - \psi^{(2k)}(a)U_k(a) \right]$$

$$+ \sum_{k=0}^{|(N-1)/2|} \frac{(-1)^k}{\omega^{2k+2}} \left[ \psi^{(2k+1)}(b)V_k(b) - \psi^{(2k+1)}(a)V_k(a) \right],$$

where $r = \min\{\mu_1, \mu_\nu\}$. Furthermore, it was proved in [13] Theorem 2] that the asymptotic error estimate of $Q^F_\mathbf{F}[f]$ is $O(\omega^{-r-1})$ as $\omega \to \infty$. We first consider the test functions: $g(x) = e^x$, $1/(4 - x)$, $\sin(x)$ and we choose $f(x) = \sin(x)$, $\phi(x) = \sin(\omega x)$ and $[a, b] = [-1, 1]$. In our comparison, we choose $n = 4$ in the Gaussian quadrature (3.4) and $c_1 = -1, c_2 = 1$ and $\mu_1 = 2, \mu_2 = 2$ in the Filon-type method (3.8) and thus both methods are implemented with the same number of function evaluations. Figure 7 illustrates the absolute errors of the Gaussian quadrature rule scaled by $\omega^5$ and the absolute errors of the Filon-type method scaled by $\omega^4$. Clearly, we see that the asymptotic order of the Gaussian quadrature rule is higher than that of the Filon-type method. Next, we consider another set of test functions:
Figure 7. Absolute errors of the Gaussian quadrature rule scaled by $\omega^5$ and the absolute errors of the Filon-type method scaled by $\omega^3$ for the integrals: (a) $\int_{-1}^{1} \sin(x)e^{\sin(\omega x)}dx$, (b) $\int_{-1}^{1} \sin(x)/(4 - \sin(\omega x))dx$ and (c) $\int_{-1}^{1} \sin(x)\sin(\omega x))dx$.

Figure 8. Absolute errors of the Gaussian quadrature rule scaled by $\omega^5$ and the absolute error of the Filon-type method scaled by $\omega^3$ for the integrals: (a) $\int_{-1}^{1} e^{\cos(\omega x))/(x + 2)dx$, (b) $\int_{-1}^{1} (1 - \cos(\omega x))/(x + 2)dx$ and (c) $\int_{-1}^{1} \ln(4 + \cos(\omega x))/(x + 2)dx$.

$g(x) = e^x$, $1 - x$, $\ln(4 + x)$ and we choose $f(x) = 1/(x + 2)$ and $\phi_\omega(x) = \cos(\omega x)$. Figure 8 illustrates the absolute errors of the Gaussian quadrature rule scaled by $\omega^5$ and the absolute errors of the Filon-type method scaled by $\omega^3$. We observe again that the asymptotic order of the Gaussian quadrature rule is higher than that of the Filon-type method.

4. Concluding remarks

In this paper, we proposed two Gaussian quadrature rules for computing composite highly oscillatory integrals of the form $\int_{-1}^{1} g(x)\phi_{\omega}(x)dx$. The idea of the first Gaussian quadrature rule is to treat the oscillatory part $(g \circ \phi_{\omega})(x)$ as the weight function and then construct a Gaussian quadrature rule for computing the integrals. We
showed that this quadrature is guaranteed by the classical theories of orthogonal polynomials and its nodes and weights can be computed efficiently using tools of numerical linear algebra. We proved that the quadrature nodes have an interesting property that they converge to the Legendre points at the rate $O(\omega^{-1})$ as $\omega \to \infty$. Moreover, we presented a rigorous convergence analysis of this Gaussian quadrature rule and showed that its convergence rate depends solely on the regularity of $f(x)$. To develop a method for computing (1.4) such that its accuracy improves as $\omega$ increases, we further proposed the second Gaussian quadrature rule based on the asymptotic analysis of the integrals (1.4) developed in [13]. In this case, however, the Gaussian quadrature rule is constructed with respect to a sign-changing function and its existence cannot be guaranteed. We explored numerically the behavior of the quadrature nodes and presented some experimental observations, including the trajectories of the zeros and the convergence rate of the zeros to the endpoints as $\omega \to \infty$. Based on these observations, we showed that the asymptotic error estimate of an $n$-point Gaussian quadrature rule is $O(\omega^{-n-1})$ and thus the accuracy of the Gaussian quadrature rule improves rapidly as $\omega$ increases.

Finally, we point out that the properties of orthogonal polynomials with respect to the complex exponential function $e^{i\omega x}$ were extensively studied in [4] and it was proved that the orthogonal polynomials with respect to $e^{i\omega x}$ with even degree always exist for $\omega > 0$ and their zeros tend to one of the endpoints at the rate $O(\omega^{-1})$ as $\omega \to \infty$. Note that similar properties on the behavior of the zeros of $q^\omega_n(x)$ with $n$ being an integer multiple of four were observed, it is still unclear whether the proof can be extended to the case of $q^\omega_n(x)$. Another challenging problem is the design of efficient algorithms for Gaussian quadrature rules developed in this work. Note that the algorithms presented in subsection 2.1 for the computation of the modified Chebyshev moments $\nu_j$ $\omega$ were not satisfactory and the algorithm for the second Gaussian quadrature rule in section 3 requires high-precision arithmetic for moderate and large values of $n$. On the other hand, we point out that the three-term recurrence coefficients of $\nu_j(x)$ can actually be calculated by using the modified Chebyshev algorithm [11, Section 2.1.7], and therefore the first Gaussian quadrature rule can also be achieved by the well-known Golub-Welsch algorithm (see [11, Section 3.1]). For the second Gaussian quadrature rule, however, the modified Chebyshev algorithm for computing the three-term recurrence coefficients of $q^\omega_n(x)$ will suffer from numerical instability due to the fact that the polynomials $q^\omega_n(x)$ are orthogonal with respect to a sign-changing function. We leave these issues for future work.

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