Invariance and attraction properties of Galton-Watson trees

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Abstract

We give a complete description of invariants and attractors of the critical and subcritical Galton-Watson tree measures under the operation of Horton pruning (cutting tree leaves with subsequent series reduction). The class of invariant measures consists of the critical binary Galton-Watson tree and a one-parametric family of critical Galton-Watson trees with offspring distribution \( \{ q_k \} \) that has a power tail \( q_k \sim k^{-(1+q_0)/q_0} \), where \( q_0 \in (1/2, 1) \). Each invariant measure has a non-empty domain of attraction under consecutive Horton pruning, completely specified by the tail behavior of the initial Galton-Watson offspring distribution. The invariant measures are characterized by Toeplitz property of their Tokunaga coefficients; they satisfy Horton law with exponent \( R = (1 - q_0)^{-1/q_0} \).

1 Introduction and motivation

The study of random trees invariant with respect to pruning (erasure) from leaves down to the root emerges in attempts to understand symmetries of natural trees observed in fields as diverse as hydrology, phylogenetics, or computer science. In addition, it provides a unifying framework for analysis of coalescence and annihilation dynamical models, including the celebrated Kingman’s coalescent, and self-similar stochastic processes on the real line; see a recent survey \[32\] for details. A special place in the invariance studies is occupied by the family of Galton-Watson trees, whose transparent generation mechanism makes it a convenient testbed for general theories and approaches. A Galton-Watson tree describes the trajectory of the Galton-Watson branching process \[4\] with single progenitor and offspring distribution \( \{ q_k \}, k = 0, 1, \ldots \). We write \( GW(q_k) \) for the probability measure that corresponds to this random tree. A tree is called critical if the expected progeny of a single member equals unity: \( \sum_{k=1}^{\infty} kq_k = 1 \). Similarly, a tree is subcritical if \( \sum_{k=1}^{\infty} kq_k < 1 \). In this paper we give a complete description of the invariance and attraction properties of critical and subcritical Galton-Watson trees under the operation of combinatorial Horton pruning — cutting tree leaves followed by series reduction.

1.1 Invariance

Prune invariance of critical and subcritical Galton-Watson trees was first examined by Burd et al. \[3\], under the assumption of a finite second moment for the offspring distri-

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bution, $\sum_{k=1}^{\infty} k^{2}q_{k} < \infty$. These authors have shown that the only invariant measure in this class corresponds to the critical binary Galton-Watson tree, $q_{0} = q_{2} = \frac{1}{2}$ \cite[Theorem 3.9]{Burd2013}. Here we complete the work started in Burd et al. \cite{Burd2013}. In Theorem \cite{Burd2013} we describe all the Galton-Watson tree measures invariant under the operation of Horton pruning. This infinite family of Invariant Galton-Watson (IGW) measures can be characterized by a single parameter – the probability $q_{0} \in [1/2, 1)$ of having no offsprings. An individual distribution from this family is denoted by $\mathcal{IGW}(q_{0})$; it is a critical distribution with the offspring generating function

$$Q(z) = \sum_{k=0}^{\infty} q_{k}z^{k} = z + q_{0}(1-z)^{1/q_{0}}.$$  

The case $q_{0} = 1/2$ gives $Q(z) = z + (1-z)^{2}/2 = (1+z^{2})/2$ and corresponds to the critical binary Galton-Watson tree, i.e., $\mathcal{IGW}(1/2) = \mathcal{GW}(q_{0} = q_{2} = 1/2)$. Every invariant Galton-Watson measure $\mathcal{IGW}(q_{0})$ with $q_{0} \in (\frac{1}{2}, 1)$ corresponds to an unbounded offspring distribution of Zipf type:

$$q_{k} \sim k^{-(1+q_{0})/q_{0}} \quad \text{as} \quad k \to \infty.$$  

1.2 Attraction

Burd et al. \cite[Theorem 3.11]{Burd2013} have shown, furthermore, that any critical Galton-Watson tree with bounded offspring number (there exists $b$ such that $q_{k} = 0$ for all $k \geq b$) converges to the critical binary Galton-Watson tree under iterative Horton pruning, conditioned on surviving the pruning. Our Theorem \cite{Burd2013} shows that the collection of $\mathcal{IGW}(q_{0})$ measures for $q_{0} \in [1/2, 1)$ and a point measure $\mathcal{GW}(q_{0} = 1)$ are the only possible attractors of critical and subcritical Galton-Watson measures under iterative Horton pruning. Specifically, all subcritical measures converge to $\mathcal{GW}(q_{0} = 1)$, and critical measures converge to $\mathcal{IGW}(q_{0})$. The domain of attraction of $\mathcal{IGW}(q_{0})$ for any $q_{0} \in [1/2, 1)$ is non empty and is characterized by the tail behavior of the offspring distribution $\{q_{k}\}$ of the initial Galton-Watson measure. In particular, Lemma \cite{Burd2013} implies that every critical measure with Zipf tail $q_{k} \sim k^{-(1/q+1)}$ for $q \in [1/2, 1)$ converges to $\mathcal{IGW}(q)$ as $k \to \infty$. We notice that the subcritical attractor $\mathcal{GW}(q_{0} = 1)$ is the limiting point of the IGW family for $q_{0} = 1$ with generating function $Q(z) = z + (1-z) = 1$. This distribution, however, is not prune-invariant.

Our results expand the attraction domain of the critical binary Galton-Watson tree $\mathcal{IGW}(1/2)$ initially described by Burd et al. \cite{Burd2013}. Specifically, Lemma \cite{Burd2013} shows that any critical offspring distribution with finite $2-\epsilon$ moment (but possibly infinite second moment) belongs to the attraction domain of $\mathcal{IGW}(1/2)$. We give an example of such a measure with $q_{k} \sim k^{-3}$.

1.3 Toeplitz characterization

The results of Burd et al. \cite{Burd2013} revealed an interesting characterization of the critical binary Galton-Watson distribution in terms of its Tokunaga sequence. Recall that the Horton pruning removes the leaf vertices and their parental edges from a finite tree $T$, with subsequent series reduction (removing degree-2 vertices). The Horton order of a tree $T$ is the minimal number of Horton prunings sufficient to eliminate $T$. Informally, a branch of Horton order $k$ is a contiguous part of a tree (a collection of vertices and their parental edges in the initial tree) eliminated at $k$-th iteration of Horton pruning (see Figs. \cite{Burd2013} for examples, and Sect. \cite{Burd2013} for a formal definition). Each leaf (i.e., leaf vertex with parental edge) is a branch
of order 1. Branches of higher orders may consist of lineages of vertices and their parental edges. The vertex farthest from the root is called the terminal vertex of a branch. Applied literature often examines the statistics of mergers of branches of distinct orders within a tree. Burd et al. [5] formalize this by considering the Tokunaga coefficients $T_{i,j}$, for $i < j$, equal to the number of instances when a branch of order $i$ joins a non-terminal vertex of the leftmost branch of order $j$ closest to the root within $T$, given that the tree order is greater than $j$. This definition is suitable for describing a generic branch structure within a Galton-Watson tree, given its symmetric iterative generation mechanism. It has been shown [3, Theorem 3.16] that the critical Galton-Watson distribution is characterized, among the bounded offspring distributions, by the Toeplitz property:

$$\mathbb{E}[T_{i,j}[T]] = T_{j-i} \quad \text{for a positive Tokunaga sequence } \{T_k\}_{k=1,2,...}. \quad (1)$$

Specifically, the critical binary Galton-Watson distribution corresponds to $T_k = 2^{k-1}$. In Lemma 8, we drop the boundedness constraint and show that the invariant measures $\mathcal{IGW}(q_0)$ are characterized by the Toeplitz property among all Galton-Watson measures. In this analysis, we adopt an alternative, more general, definition of the Tokunaga coefficient $T_{i,j}$, which (i) accounts for branching at the terminal vertices, and (ii) can be applied to general (non-Galton-Watson) trees. In our definition, the invariant measure $\mathcal{IGW}(q_0)$ corresponds to the Tokunaga sequence (Lemma 8)

$$T_1 = c^{1/q_0} - c - 1, \quad T_k = ac^{k-1}, \quad k = 2, 3, \ldots$$

with (Fig. 1)

$$c = (1 - q_0)^{-1} \quad \text{and} \quad a = (c - 1)(c^{1/q_0-1} - 1).$$
Figure 4: Series reduction: Example. Tree $T$ before (a) and after (b) series reduction.

The critical binary Galton-Watson case with $q_0 = 1/2$ corresponds to $c = 2$ and $a = 1$, which reconstructs the Burd et al. [3] result $T_k = 2^{k-1}$. Moreover, using the Tokunaga sequence definition from Burd et al. [3], we obtain a particularly simple Tokunaga sequence $T_k = c^{k-1}$ for $k = 1, 2, \ldots$.

1.4 Horton law

A ubiquitous empirical observation in the analysis of dendritic structures is the Horton law [7, 10, 13]. Informally, the law states that the numbers $N_k[T]$ of branches of order $k$ in a large tree $T$ decays geometrically:

$$\frac{N_k[T]}{N_{k+1}[T]} \approx R$$

for some Horton exponent $R \geq 2$. A formal definition of Horton law for tree measures is given in Sect. 2.5.

It has been shown by McConnell and Gupta [33] for a particular case of $T_k = ac^{k-1}$, and generalized by the authors of this paper [8] to an arbitrary Tokunaga sequence $\{T_k\}$, that Toeplitz property implies Horton law. Lemma 8 shows that the invariant Galton-Watson measure $IGW(q_0)$ for any $q_0 \in [1/2, 1)$ obeys Horton law with the Horton exponent $R = (1 - q_0)^{-1/q_0}$ (Fig. 1).

2 Preliminaries

2.1 Galton-Watson tree measures

Consider the space $\mathcal{F}$ of finite unlabeled rooted reduced trees. A tree is called rooted if one of its vertices, denoted by $\rho$, is selected as the tree root. The existence of root imposes a parent-offspring relation between each pair of adjacent vertices: the one closest to the root is called the parent, and the other the offspring. The space $\mathcal{F}$ includes the empty tree $\phi$ comprised of a root vertex and no edges. The tree root is the only vertex that does not have a parent. Let $\mathcal{F}^1$ denote a subspace of planted trees in $\mathcal{F}$; it contains $\phi$ and all the trees in $\mathcal{F}$ with the root vertex having exactly one offspring (see Figs. 2, 3). The degree of a vertex
is the number of its offspring plus one (for parent). The number of offspring at a vertex is called branching number. A tree from $T^\dagger$ is called reduced is it has no vertices of degree 2.

For a given probability mass function $\{q_k\}_{k=0,1,2,...}$, we let $GW(\{q_k\})$ denote the corresponding Galton-Watson tree measure. We assume that each tree begins with a single root vertex which itself produces a single offspring, so the resulting trees are in $T^\dagger$. In this renowned Markov chain construction, each non-root vertex produces $k$ offsprings with probability $q_k$, independently of other vertices. We assume $\sum_{k=0}^{\infty} kq_k \leq 1$ and $q_1 = 0$ as we need $GW(\{q_k\})$ to be a probability measure on $T^\dagger$ (the trees in $T^\dagger$ are required to be finite and reduced). The assumption of subcriticality or criticality implies

$$1 \geq \sum_{k=2}^{\infty} kq_k \geq 2 \sum_{k=2}^{\infty} q_k = 2(1 - q_0),$$

and therefore $q_0 \geq \frac{1}{2}$.

\section*{2.2 Horton pruning, orders}

Recall that series reduction on a tree $T$ removes each vertex of degree 2 and merges its two adjacent edges into one (Fig. 2). Horton pruning defined below is a surjection from $T$ onto itself.

\begin{definition}[Horton pruning] Horton pruning $R : T \to T$ is an onto function whose value $R(T)$ for a tree $T \neq \phi$ is obtained by removing the leaves and their parental edges from $T$, followed by series reduction. We also set $R(\phi) = \phi$.
\end{definition}

The trajectory of each tree $T$ under $R(\cdot)$ is uniquely determined and finite:

$$T \equiv R^0(T) \to R^1(T) \to \cdots \to R^k(T) = \phi,$$

with the empty tree $\phi$ as the (only) fixed point. The pre-image $R^{-1}(T)$ of any non-empty tree $T$ consists of an infinite collection of trees.

It is natural to think of the distance to $\phi$ under the Horton pruning map and introduce the respective notion of tree order \cite{7, 10, 13, 14}.

\begin{definition}[Horton-Strahler order] The Horton-Strahler order $\text{ord}(T) \in \mathbb{Z}^+$ of a tree $T \in T^\dagger$ is defined as the minimal number of Horton prunings necessary to eliminate the tree:

$$\text{ord}(T) = \min \left\{ k \geq 0 : R^k(T) = \phi \right\}.$$

In particular, the order of the empty tree is $\text{ord}(\phi) = 0$, because $R^0(\phi) = \phi$. This definition is illustrated in Fig. 3 for a tree $T$ of order $\text{ord}(T) = 4$. Most of our discussion will be focused on non-empty trees with orders $\text{ord}(T) > 0$. The probability measure $GW(\{q_k\})$ on $T^\dagger$, satisfying $\sum_{k=0}^{\infty} kq_k \leq 1$ and $q_1 = 0$, that we consider in this paper assigns probability zero to the empty tree $\phi$.

\begin{definition}[Horton-Strahler terminology] We introduce the following definitions related to the Horton-Strahler order of a tree (see Fig. 4):

\end{definition}
Figure 3: Horton-Strahler order: Example. Consecutive prunings $\mathcal{R}^k(T)$, $k = 0, 1, \ldots, 4$, of tree $T$. The order of tree is $\text{ord}(T) = 4$ since $\mathcal{R}^4(T) = \phi$. Different colors depict branches of different orders: $\text{ord} = 1$ (black), $\text{ord} = 2$ (green), $\text{ord} = 3$ (blue), and $\text{ord} = 4$ (red).

1. **(Subtree at a vertex)** For any non-root vertex $v$ in $T \neq \phi$, a subtree $T_v \subset T$ is the only planted subtree in $T$ rooted at the parental vertex $\text{parent}(v)$ of $v$, and comprised by $v$ and all its descendant vertices together with their parental edges. Figure 4 shows in black color the subtree $T_a$ at vertex $a$.

2. **(Vertex order)** For any vertex $v \in T \setminus \{\rho\}$ we set $\text{ord}(v) = \text{ord}(T_v)$. We also set $\text{ord}(\rho) = \text{ord}(T)$.

3. **(Edge order)** The parental edge of a non-root vertex has the same order as the vertex.

4. **(Branch)** A maximal connected component consisting of vertices and edges of the same order is called a branch. Figure 4 shows a branch $b$ of order 2 (blue) that consists of three vertices and their parental edges. Note that a tree $T$ always has a single branch of the maximal order $\text{ord}(T)$. In a stemless tree, the maximal order branch may consist of a single root vertex.

5. **(Initial and terminal vertex of a branch)** The branch vertex closest to the root is called the initial vertex of the branch. The branch vertex farthest from the root is called the terminal vertex of a branch. Figure 4 shows the terminal vertex of branch $b$ (blue) as a green circle.

The Horton-Strahler orders can be equivalently defined via hierarchical counting $[7, 14, 4, 13, 22]$. The first such definition beyond the binary case appeared in [3]. In this approach, each leaf is assigned order 1. If an internal vertex $p$ has $m \geq 1$ offspring with orders $i_1, i_2, \ldots, i_m$ and $r = \max \{i_1, i_2, \ldots, i_m\}$, then

$$\text{ord}(p) = \begin{cases} r & \text{if } \# \{s : i_s = r\} = 1, \\ r + 1 & \text{otherwise}. \end{cases}$$

(3)

The parental edge of a non-root vertex has the same order as the vertex. The Horton-Strahler order of a tree $T \neq \phi$ is $\text{ord}(T) = \max_{v \in T} \text{ord}(v)$, where the maximum is taken over all
vertices in $T$. This definition is most convenient for practical calculations, which explains its popularity in the literature.

Figures 5, 6 illustrate Horton-Strahler orders in trees with constant branching number $b$ ($q_0 + q_b = 1$) and bounded offspring distribution ($q_k = 0$ for $k > b$).

2.3 Horton self-similarity

Here we introduce the definition of self-similarity of a Galton-Watson measure with respect to Horton pruning $R$, which is the main operation on trees discussed in this work.

**Definition 4 (Horton self-similarity).** Consider a Galton-Watson measure $\mu$ on $T$ (or $T^1$) such that $\mu(\phi) = 0$. Let $\nu$ be the pushforward measure, $\nu = R_*(\mu)$, i.e.,

$$
\nu(T) = \mu \circ R^{-1}(T) = \mu(R^{-1}(T)).
$$

Measure $\mu$ is called invariant with respect to the Horton pruning (Horton prune-invariant) if for any tree $T \in T$ (or $T^1$) we have

$$
\nu(T | T \neq \phi) = \mu(T).
$$

(4)
Definition \textsuperscript{4} only requires a measure to be invariant with respect to Horton pruning (prune-invariant) to be called self-similar. The equivalence of prune-invariance and self-similarity is a particular property of Galton-Watson measures connected to their Markov structure. In a general case, prune-invariance happens to be a weak property that allows a multitude of obscure measures. A general prune-invariant measure on $\mathcal{T}$ has to satisfy an additional property, called \textit{coordination}, to be called self-similar. We refer to \textsuperscript{10} for a comprehensive discussion and examples.

Figure 5: Examples of Horton-Strahler ordering in trees with constant branching number $b (q_0 + g_b = 1)$. (a) $b = 2$, (b) $b = 3$, (c) $b = 5$, (d) $b = 10$. Each panel shows a tree of order $\text{ord} = 4$. Edges of different orders are shown in different colors, as indicated in the legend.
Figure 6: Examples of Horton-Strahler ordering in trees with bounded offspring distribution: $q_k = 0$ for $k > b$. (a) $b = 5$, (b) $b = 6$. Each panel shows a tree of order $\text{ord} = 4$. Edges of different orders are shown in different colors, as indicated in the legend.

### 2.4 Tokunaga coefficients, Toeplitz property

This section introduces *Tokunaga coefficients* that describe mergers of branches of different orders in a random tree. Empirically, a Tokunaga coefficient $T_{i,j}$ is the average number of branches of order $i$ that merge a branch of order $j$ within a tree $T$. The Markovian generation process ensures that all branches of a given order $j$ in a Galton-Watson tree have the same probabilistic structure. Hence, one can follow Burd et al. [3] and define $T_{i,j}$ as the mean number of order $i$ branches within a particular branch of order $j$, for instance – the leftmost branch closest to the root. We introduce below a more general definition, which is equivalent to that of Burd et al. [3] for Galton-Watson trees, and can extend to non-Markovian branching processes. This set up will also be needed to formulate the Horton law in Sect. 2.5.

Horton pruning partitions the underlying tree space into exhaustive and mutually exclusive collection of subspaces $H_k$ of trees in $\mathcal{T}$ (or $\mathcal{T}'$) of Horton-Strahler order $k \geq 0$ such that $\mathcal{R}(H_{k+1}) = H_k$. Here $H_0 = \{\phi\}$, $H_1$ consists of a single tree comprised of a root and a leaf descendant to the root, and all other subspaces $H_k$, $k \geq 2$, consist of an infinite number of trees. Naturally, $H_k \cap H_{k'} = \emptyset$ if $k \neq k'$, and $\bigcup_{k \geq 1} H_k = \mathcal{T}$ (or $\mathcal{T}'$). Consider a set of conditional probability measures $\{\mu_k\}_{k \geq 0}$ each of which is defined on $H_k$ by

$$
\mu_k(T) = \mu(T | T \in H_k)
$$

(5)
and let \( \pi_k = \mu(\mathcal{H}_k) \). Then \( \mu \) can be represented as a mixture of the conditional measures:

\[
\mu = \sum_{k=1}^{\infty} \pi_k \mu_k. \tag{6}
\]

Let \( N_k = N_k[T] \) be the number of branches of order \( k \) in a tree \( T \). For given integers \( 1 \leq i < j \), let \( n_{i,j} = n_{i,j}[T] \) denote the total number of vertices of order \( i \) that have parent of order \( j \) in a tree \( T \in \mathcal{T} \) (or \( \mathcal{T}' \)). We write \( \mathbb{E}_K[\cdot] \) for the expectation with respect to \( \mu_K \) of Eq. \((/five.oldstyle5)\).

We define the average Horton numbers for subspace \( \mathcal{H}_K \) as

\[
\mathcal{N}_k[K] = \mathbb{E}_K[N_k], \quad 1 \leq k \leq K, \quad K \geq 1.
\]

For subspace \( \mathcal{H}_K \), let

\[
t_{i,j}[K] = \frac{\mathbb{E}_K[n_{i,j}]}{\mathbb{E}_K[N_j]} = \mathbb{E}_K[N_{i,j}] / \mathbb{E}_K[N_j][K], \quad 1 \leq i < j \leq K, \tag{7}
\]

be the total Tokunaga merger statistics that is used to define the Tokunaga coefficients

\[
T_{i,j}[K] = t_{i,j}[K] - 2\delta_{i,j-1} \quad (1 \leq i < j). \tag{8}
\]

Finally, let \( n_{i,j}^o \) denote the total number of vertices of order \( i \) whose parent vertices are non-terminal vertices of order \( j \). Then,

\[
T_{i,j}^o[K] = \frac{\mathbb{E}_K[n_{i,j}^o]}{\mathbb{E}_K[N_j]} = \mathbb{E}_K[n_{i,j}^o] / \mathbb{E}_K[N_j][K] \quad (1 \leq i < j) \tag{9}
\]

are called the regular Tokunaga coefficients.

We observe that for a Galton-Watson measure \( \mu \) we have \((/three.oldstyle32)\):

\[
T_{i,j} := T_{i,j}[K] \quad \text{for all } K \geq 2 \text{ and } 1 \leq i < j \leq K. \tag{10}
\]

The respective Tokunaga matrix \( \mathbb{T}_K \) is a \( K \times K \) matrix

\[
\mathbb{T}_K = \begin{bmatrix}
0 & T_{1,2} & T_{1,3} & \cdots & T_{1,K} \\
0 & 0 & T_{2,3} & \cdots & T_{2,K} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & T_{K-1,K} \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix},
\]

which coincides with the restriction of any larger-order Tokunaga matrix \( \mathbb{T}_M, M > K \), to the first \( K \times K \) entries.

**Definition 5 (Toeplitz property).** A Galton-Watson measure \( \mu \) is said to satisfy the Toeplitz property if there exists a sequence \( T_k \geq 0, k = 1, 2, \ldots \) such that

\[
T_{i,j} = T_{j-i}. \tag{11}
\]

The elements of the sequences \( T_k \) are also referred to as Tokunaga coefficients, which does not create confusion with \( T_{i,j} \).
For a Galton-Watson measure that satisfies Toeplitz property, the corresponding Tokunaga matrices $T_K$ are Toeplitz:

$$T_K = \begin{bmatrix}
0 & T_1 & T_2 & \ldots & T_{K-1} \\
0 & 0 & T_1 & \ldots & T_{K-2} \\
0 & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 & T_1 \\
0 & 0 & \ldots & 0 & 0
\end{bmatrix}.$$ 

**Definition 6 (Tokunaga self-similarity).** A Galton-Watson measure $\mu$ on $\mathcal{T}$ is called Tokunaga self-similar with parameters $(a, c)$ if it satisfies Toeplitz property and its Tokunaga sequence $\{T_j\}_{j=1,2,...}$ is expressed as

$$T_j = a c^{j-1}, \quad k \geq 1$$

for some constants $a \geq 0$ and $c > 0$.

### 2.5 Horton law

Consider a measure $\mu$ on $\mathcal{T}$ (or $\mathcal{T}^\dagger$) and its conditional measures $\mu_K$, each defined on subspace $\mathcal{H}_K \subset \mathcal{T}$ of trees of Horton-Strahler order $K \geq 1$. We write $T \overset{d}{\sim} \mu_K$ for a random tree $T$ drawn from subspace $\mathcal{H}_K$ according to measure $\mu_K$.

**Definition 7 (Strong Horton law for mean branch numbers).** We say that a probability measure $\mu$ on $\mathcal{T}$ (or $\mathcal{T}^\dagger$) satisfies a strong Horton law for mean branch numbers if there exists such a positive (constant) Horton exponent $R \geq 2$ that for any $k \geq 1$,

$$\lim_{K \to \infty} \frac{N_k[K]}{N_1[K]} = R^{1-k}. \quad (13)$$

Here, the adjective strong refers to the type of geometric convergence; see [10] for details.

The work [8] establishes the strong Horton law in a binary tree that satisfies Toeplitz property. We observe that the results of [8] hold beyond the binary case, as the derivation steps are identical. Specifically, assume Toeplitz property with Tokunaga sequence $\{T_j\}$ and consider a sequence $t(k)$ defined by

$$t(0) = -1, \quad \text{and} \quad t(k) = T_k + 2\delta_{1,k} \quad \text{for} \quad k \geq 1.$$

Observe that $t_{i,j} = t_{i,j}[K] = t(j-i)$. The generating function of $t(k)$ is

$$\hat{t}(z) = \sum_{k=0}^{\infty} z^k t(k) = -1 + 2z + \sum_{k=1}^{\infty} z^k T_k.$$

**Theorem 1 (Strong Horton law in a mean self-similar tree, [8]).** Suppose $\mu$ is a Galton-Watson measure on $\mathcal{T}^\dagger$ that satisfies Toeplitz property with Tokunaga sequence $\{T_j\}_{j=1,2,...}$ such that

$$\limsup_{j \to \infty} T_j^{1/j} < \infty. \quad (14)$$

\footnote{Note that in [8], the Tokunaga sequence was set to satisfy $T_{i,j} = T_{i,j}[K] = T_{j-i}$. That is, the offsprings adjacent to the terminal vertex of order $j$ branch were not counted.}
Then the strong Horton law for mean branch numbers (Def. [2]) holds with the Horton exponent $R = 1/w_0$, where $w_0$ is the only real zero of the generating function $\hat{t}(z)$ in the interval $(0, \frac{1}{2}]$.

Moreover,
\[
\lim_{K \to \infty} \left( N_1[K] R^{-K} \right) = \text{const.} > 0.
\]

Conversely, if \( \lim_{j \to \infty} T_j^{1/j} = \infty \), then the limit \( \lim_{K \to \infty} \frac{N_1[K]}{N_1[R]} \) does not exist at least for some $k$.

3 Main results

3.1 Distribution of Horton orders and related functions

Consider a collection of critical or subcritical Galton-Watson measures $\mathcal{GW}(\{q_k\})$ ($q_1 = 0$) on $\mathcal{T}$. Let $Q(z) = \sum_{m=0}^{\infty} z^m q_m$ ($z \in [0, 1]$) denote the generating function of $\{q_k\}$, and for $T \sim \mathcal{GW}(\{q_k\})$, denote $\pi_j := P(\text{ord}(T) = j)$. Finally, let $\sigma_0 = 0$ and $\sigma_j := \sum_{i=1}^{j} \pi_i$ ($j \geq 1$).

Lemma 1 (Order distribution). Consider a Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$. Assume criticality or subcriticality, i.e., $\sum_{k=0}^{\infty} kq_k \leq 1$. Then,
\[
\pi_1 = q_0 \quad \text{and} \quad \pi_j = \frac{Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1} Q'(\sigma_{j-2})}{1 - Q'(\sigma_{j-1})} \quad (j \geq 2).
\]

Proof. The probability of tree $T$ consisting of a leaf is $\pi_1 = P(\text{ord}(T) = 1) = q_0$.

In general, for $j \geq 2$, the probability of the child vertex of the root being a terminal vertex in a branch of order $j$ is
\[
\sum_{m=2}^{\infty} q_m \sum_{\ell=2}^{m} \binom{m}{\ell} \pi_{j-1}^{\ell} \sigma_{j-2}^{m-\ell} = Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1} Q'(\sigma_{j-2}).
\]

Next, the probability of the child vertex of the root being a regular (non-terminal) vertex of order $j$ equals
\[
\sum_{k=2}^{\infty} q_k k \pi_j \sigma_{j-1}^{k-1} = \pi_j Q'(\sigma_{j-1}).
\]

Therefore,
\[
\pi_j = \pi_j Q'(\sigma_{j-1}) + \left( Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1} Q'(\sigma_{j-2}) \right),
\]

implying (16). \qed

Corollary 1. Consider a Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$. Assume criticality or subcriticality, i.e., $\sum_{k=0}^{\infty} kq_k \leq 1$. Then, $\sigma_j$ can be expressed via an iterated function (Fig. 7)
\[
\sigma_j = \underbrace{S \circ \ldots \circ S}_{j \text{ times}}(0) \quad \text{for} \quad j \geq 1,
\]

where
\[
S(z) = \frac{Q(z) - zQ'(z)}{1 - Q'(z)} , \quad |z| < 1.
\]
Figure 7: Illustration to Corollary 3. Function $S(z)$ is shown in red. Equation (17) implies that the values of $\sigma_j$ are obtained by iterative application of $S(t)$, starting with $\sigma_0 = 0$. These iterations are illustrated by blue lines with arrows. Vertical increments correspond to the values of $\pi_j$.

**Proof.** Equation (16) implies

$$\pi_j = \left[ Q(\sigma_{j-1}) + \pi_j Q'(\sigma_{j-1}) \right] - \left[ Q(\sigma_{j-2}) + \pi_{j-1} Q'(\sigma_{j-2}) \right] \quad \text{for } j \geq 2. \quad (19)$$

Hence, summing up the terms in (19), and substituting $\pi_1 = q_0$, we obtain

$$\sigma_j = \sum_{i=1}^{j} \pi_i = Q(\sigma_{j-1}) + \pi_j Q'(\sigma_{j-1}) = Q(\sigma_{j-1}) + (\sigma_j - \sigma_{j-1}) Q'(\sigma_{j-1})$$

for all $j \geq 1$. Thus, $\sigma_j = \frac{Q(\sigma_{j-1}) - \sigma_{j-1} Q'(\sigma_{j-1})}{1 - Q'(\sigma_{j-1})} = S(\sigma_{j-1})$.

Function $S(z)$ admits analytic continuation beyond $|z| < 1$, encircling the singularity at 1, where we set

$$S(1) = \lim_{x \in \mathbb{R}, x \to 1-} \frac{Q(x) - x Q'(x)}{1 - Q'(x)} = \lim_{x \in \mathbb{R}, x \to 1-} \frac{x Q''(x)}{Q''(x)} = 1. \quad (16)$$

Next, for the progeny variable $X \sim \{ q_k \}$, consider the following important function

$$g(x) = \sum_{m=0}^{\infty} \mathbb{E} [(X - m - 1)_+] x^m = \sum_{m=0}^{\infty} \sum_{k=m+1}^{\infty} (k - m - 1) q_k x^m, \quad (20)$$

where $x_+ = \max\{x, 0\}$.

**Proposition 1.** For a critical (i.e., $Q'(1) = 1$) Galton-Watson process $\mathcal{GW}\{ q_k \}$ with $q_1 = 0$, we have

$$Q(x) - x = (1 - x)^2 g(x)$$

for $g(x)$ as defined in (20).
Proof. Since \( \sum_{k=2}^{\infty} kq_k = Q'(1) = 1 \),

\[
Q(x) - x = q_0 + \sum_{k=2}^{\infty} q_k x^k - q_0 x - \sum_{k=2}^{\infty} q_k x = (1 - x) \left[ q_0 - \sum_{k=2}^{\infty} q_k \frac{1 - x^{k-1}}{1 - x} x \right]
\]

\[= (1 - x) \left[ \sum_{k=2}^{\infty} kq_k - \sum_{k=2}^{\infty} q_k - \sum_{k=2}^{\infty} q_k \frac{1 - x^{k-1}}{1 - x} \right] = (1 - x) \sum_{k=2}^{\infty} q_k \left( k - 1 - \sum_{j=1}^{k-1} x^j \right)\]

\[= (1 - x) \sum_{k=2}^{\infty} q_k \left( \sum_{j=1}^{k-1} (1 - x^j) \right) = (1 - x)^2 \sum_{k=2}^{\infty} q_k \sum_{j=1}^{k-1} \sum_{m=0}^{j-1} x^m
\]

\[= (1 - x)^2 \sum_{k=2}^{\infty} q_k \sum_{m=0}^{k-2} (k - m - 1)x^m = (1 - x)^2 \sum_{m=0}^{\infty} \sum_{k=m+2}^{\infty} (k - m - 1)q_k x^m
\]

\[= (1 - x)^2 g(x).
\]

The following is a complementary result.

**Proposition 2.** For a subcritical (i.e., \( Q'(1) < 1 \)) Galton-Watson process \( GW(\{q_k\}) \) with \( q_1 = 0 \), we have

\[
\lim_{x \to 1^-} \frac{Q(x) - x}{1 - x} = 1 - Q'(1).
\]

Here, when we write \( x \to 1^- \), we take \( x \in \mathbb{R} \).

**Proof.** Since \( \sum_{k=2}^{\infty} kq_k = Q'(1) < 1 \),

\[
\lim_{x \to 1^-} \frac{Q(x) - x}{1 - x} = \lim_{x \to 1^-} \left( q_0 - \sum_{k=2}^{\infty} q_k \frac{1 - x^{k-1}}{1 - x} x \right)
\]

\[= \lim_{x \to 1^-} \left( q_0 - \sum_{k=2}^{\infty} (k - 1)q_k \right) = 1 - \sum_{k=2}^{\infty} kq_k.
\]

\[\square\]

**Lemma 2.** For the progeny variable \( X \overset{d}{\sim} \{q_k\} \) and \( g(x) \) as defined in (20), if

\[
E[X^{2-\epsilon}] = \sum_{k=0}^{\infty} k^{2-\epsilon} q_k < \infty \quad \forall \epsilon > 0,
\]

then \( \lim_{x \to 1^-} \frac{\ln g(x)}{\ln(1-x)} = 0 \).

**Proof.** Suppose (21) holds, then by the Dominated Convergence Theorem,

\[
(m + 1)^{1-\epsilon} E\left[(X - m - 1)_{+}\right] \leq E\left[X^{1-\epsilon}(X - m - 1)_{+}\right] \leq E\left[X^{2-\epsilon} 1_{\{X \geq m+1\}}\right] \to 0 \quad (22)
\]

as \( m \to \infty \).
Next, for $\epsilon > 0$, the $m$th coefficient of the power series expansion of $(1 - x)^{-\epsilon}$ is
\[
a((1 - x)^{-\epsilon}; m) = \frac{\prod_{i=0}^{m-1} (\epsilon + i)}{m!} = \frac{\Gamma(\epsilon + m)}{\Gamma(\epsilon) m!} \sim m^{\epsilon-1}. \tag{23}
\]
Together, (22) and (23) imply
\[
\limsup_{x \to 1^-} \frac{\ln g(x)}{- \epsilon \ln(1 - x)} \leq 1 \quad \forall \epsilon > 0.
\]
Hence, $\lim sup_{x \to 1^-} \frac{\ln g(x)}{- \ln(1 - x)} \leq 0$ while obviously, $\lim inf_{x \to 1^-} \frac{\ln g(x)}{- \ln(1 - x)} \geq 0$. \hfill \Box

**Lemma 3.** Consider a Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$. Assume criticality and the existence and finiteness of the limit
\[
\lim_{x \to 1^-} \left( \frac{\ln g(x)}{\ln(1 - x)} \right) = L,
\]
where $g(x)$ is as defined in (20). Then $S'(1) = \lim_{x \to 1^-} \frac{1 - S(x)}{1 - x} = \frac{1 - L}{2 - L}$.

**Proof.** Prop. 2 implies
\[
S'(1) = 1 - \lim_{x \to 1^-} \frac{Q(x) - x}{(1 - x)(1 - Q'(x))} = 1 - \lim_{x \to 1^-} \frac{\frac{d}{dx} \ln(1 - x)}{\frac{d}{dx} \ln(Q(x) - x)}
\]
\[
= 1 - \lim_{x \to 1^-} \frac{\ln(1 - x)}{\ln(Q(x) - x)} = 1 - \lim_{x \to 1^-} \frac{\ln(1 - x)}{2 \ln(1 - x) + \ln g(x)} = 1 - \frac{1}{2 - L}. \tag{24}
\]
\hfill \Box

**Proposition 3.** For a subcritical Galton-Watson process $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$, we have
\[
S'(1) = \lim_{x \to 1^-} \frac{1 - S(x)}{1 - x} = 0.
\]

**Proof.** By Prop. 2 and subcriticality,
\[
S'(1) = 1 - \lim_{x \to 1^-} \frac{Q(x) - x}{(1 - x)(1 - Q'(x))} = 1 - \frac{1 - Q'(1)}{1 - Q'(1)} = 0.
\]
\hfill \Box

**Lemma 4 (Zipf distribution).** Consider a critical Galton-Watson process $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$, with probability mass function $q_k$ of Zipf type,
\[
q_k \sim k^{-(\alpha+1)} \quad \text{with} \quad \alpha \in (1, 2]. \tag{25}
\]
Then,
\[
L = \lim_{x \to 1^-} \left( \frac{\ln g(x)}{- \ln(1 - x)} \right) = 2 - \alpha \quad \text{and} \quad S'(1) = \frac{\alpha - 1}{\alpha}. \tag{26}
\]
Proof. First notice that (25) implies \( E[(X - m - 1)_+] \sim m^{1 - \alpha} \). Next, recall the power series expansion of \((1 - x)^{-\epsilon}\) in (23). Hence,

\[
\limsup_{x \to 1-} \frac{\ln g(x)}{-\epsilon \ln(1 - x)} < 1 \quad \text{if } \epsilon > 2 - \alpha
\]

and

\[
\liminf_{x \to 1-} \frac{\ln g(x)}{-\epsilon \ln(1 - x)} > 1 \quad \text{if } \epsilon < 2 - \alpha,
\]

implying \( L = 2 - \alpha \). Finally, by Lemma 3, \( S'(1) = \frac{1}{2 - L} = \frac{\alpha}{\alpha - 1} \).

\[\square\]

Example 1 (Infinite second moment, \( L = 0 \)). Consider a critical Galton-Watson process \( \mathcal{GW}(\{q_k\}) \) with \( q_0 = \frac{2}{3} \), \( q_1 = 0 \), and

\[
q_k = \frac{4/3}{k(k^2 - 1)} \quad (k \geq 2).
\]

Observe that the probability mass function \( q_k \) is of Zipf type (25) with \( \alpha = 2 \). This offspring distribution has infinite second moment. Here,

\[
Q(x) - x = (1 - x)^2 g(x) \quad \text{with} \quad g(x) = -\frac{2/3}{x} \ln(1 - x),
\]

and therefore, \( L = \lim_{x \to 1-} \left( \frac{\ln g(x)}{-\ln(1 - x)} \right) = \lim_{x \to 1-} \left( \frac{\ln(-\ln(1 - x))}{-\ln(1 - x)} \right) = 0 \) and \( S'(1) = \frac{1}{2} \).

### 3.2 Tokunaga coefficients in recursive form

Here we derive a recursive expression for Tokunaga coefficients of a Galton-Watson measure in the form \( T_{i,j} = \pi_i f(\sigma_{j-2}, \pi_{j-1}, \pi_j) \). The recursive nature of this representation is connected to the recursive expression (16) for \( \pi_i \) of Lemma 4.

Lemma 5 (Tokunaga coefficients). Consider a Galton-Watson measure \( \mathcal{GW}(\{q_k\}) \) with \( q_1 = 0 \). Assume criticality or subcriticality, i.e., \( \sum_{k=0}^{\infty} kq_k \leq 1 \). Then, for all \( 1 \leq i < j - 1 \), we have

\[
T_{i,j} = \pi_i \frac{Q'(\sigma_{j-1}) - Q'(\sigma_{j-2}) - \pi_{j-1}Q''(\sigma_{j-2})}{Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1}Q'(\sigma_{j-2})} + T_{i,j},
\]

and for \( 1 \leq i = j - 1 \),

\[
T_{j-1,j} = \frac{\pi_{j-1}Q'(\sigma_{j-1}) + \pi_{j-1}Q'(\sigma_{j-2}) - 2Q(\sigma_{j-1}) + 2Q(\sigma_{j-2})}{Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1}Q'(\sigma_{j-2})} + T_{j-1,j},
\]

where \( T_{j-1,j} = \pi_i \frac{Q''(\sigma_{i-1})}{1 - Q'(\sigma_{j-1})} \) is the expected number of order \( i \) offsprings descendant to all regular (non-terminal) vertices in a branch of order \( j \).

Note that (27) can be rewritten as

\[
T_{i,j} = \pi_i \frac{d}{dx} \ln \left( \frac{Q(x + \pi_{j-1}) - Q(x) - \pi_{j-1}Q'(x)}{1 - Q'(x + \pi_{j-1})} \right) \bigg|_{x=\sigma_{j-2}}.
\]
Proof. Recall that \( \sum_{m=0}^{k} m \binom{k}{m} a^m b^{k-m} = ka(a+b)^{k-1} \). First, we find the expected number \( M_{i,j}^{i\text{term}} \) of order \( i \) offsprings in the terminal vertex of a branch of order \( j \). If \( i \leq j - 2 \), then

\[
M_{i,j}^{i\text{term}} = \sum_{m=2}^{\infty} \frac{q_m \sum_{\ell=2}^{m} \binom{m}{\ell} \pi_j^{\ell-1} \sum_{k=0}^{m-\ell} k \binom{m-\ell}{k} \pi_i (\sigma_{j-2} - \pi_i)^{m-\ell-k}}{\sum_{m=2}^{\infty} q_m \sum_{\ell=2}^{m} \binom{m}{\ell} \pi_j^{\ell-1} \sigma_{j-2}^{m-\ell}}
\]

\[
= \frac{\pi_i \sum_{m=2}^{\infty} q_m \sum_{\ell=2}^{m} (m-\ell) \binom{m}{\ell} \pi_j^{\ell-1} \sigma_{j-2}^{m-\ell}}{Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1} Q'(\sigma_{j-2})}
\]

\[
= \frac{\pi_i Q'(\sigma_{j-1}) - Q'(\sigma_{j-2}) - \pi_{j-1} Q''(\sigma_{j-2})}{Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1} Q'(\sigma_{j-2})}
\]

\[
= \frac{\pi_i Q'(\sigma_{j-1}) - Q'(\sigma_{j-2}) - \pi_{j-1} Q''(\sigma_{j-2})}{Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1} Q'(\sigma_{j-2})}
\]

\[
\text{(29)}
\]

For \( i = j - 1 \), we have

\[
M_{j-1,j}^{j\text{term}} = \sum_{m=2}^{\infty} q_m \sum_{\ell=2}^{m} (\ell - 2) \binom{m}{\ell} \pi_j^{\ell-1} \sigma_{j-2}^{m-\ell}
\]

\[
= \frac{Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1} Q'(\sigma_{j-2})}{Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1} Q'(\sigma_{j-2})}
\]

\[
= \frac{\pi_{j-1} Q'(\sigma_{j-1}) + \pi_{j-1} Q'(\sigma_{j-2}) - 2Q(\sigma_{j-1}) + 2Q(\sigma_{j-2})}{Q(\sigma_{j-1}) - Q(\sigma_{j-2}) - \pi_{j-1} Q'(\sigma_{j-2})}
\]

\[
\text{(30)}
\]

The expected number \( V_j^o \) of regular (non-terminal) vertices a branch of order \( j \) is computed as follows:

\[
V_j^o = \frac{\sum_{r=0}^{\infty} r \left( \sum_{k=2}^{\infty} q_k k \sigma_{j-1}^{k-1} \right)^r}{\sum_{r=0}^{\infty} \left( \sum_{k=2}^{\infty} q_k k \sigma_{j-1}^{k-1} \right)^r} = \frac{Q'(\sigma_{j-1})}{1 - Q'(\sigma_{j-1})}
\]

\[
\text{(31)}
\]

Finally, the expected number \( M_{i,j}^o \) of order \( i \) offsprings in a regular (non-terminal) vertex in a branch of order \( j \) is

\[
M_{i,j}^o = \frac{1}{\sum_{k=2}^{\infty} q_k k \sigma_{j-1}^{k-1}} \sum_{k=0}^{\infty} q_k k \sum_{s=0}^{k-1} s \binom{k-1}{s} \pi_i (\sigma_{j-2} - \pi_i)^{k-1-s}
\]

\[
= \frac{1}{Q'(\sigma_{j-1})} \pi_i \sum_{k=2}^{\infty} q_k k (k-1) \sigma_{j-1}^{k-2} = \pi_i Q''(\sigma_{j-1}) Q'(\sigma_{j-1})
\]

\[
\text{(32)}
\]

for \( 1 \leq i < j \).

The statement of the lemma follows from equations (29), (30), (31), (32) as \( T_{i,j} = M_{i,j}^{i\text{term}} + T_{i,j}^o \) with \( T_{i,j}^o = V_j^o M_{i,j}^o \) by Wald’s equation.

\[
\square
\]

Corollary 2 (Critical binary Galton-Watson tree). Distribution \( GW(q_0 = q_2 = 1/2) \) is Tokunaga self-similar with Tokunaga sequence \( T_j = 2^{j-1}, j \geq 1 \), and \( \pi_i = 2^{-i}, i \geq 1 \).
Proof. For \( q_0 = q_2 = \frac{1}{2} \), we have \( Q(z) = \frac{1}{2} + \frac{1}{2} z^2 \). Thus, Corollary \([5]\) yields \( \sigma_j = S(\sigma_{j-1}) = 1 + \frac{q_{j-1}}{2} \), which, by induction, implies \( \sigma_j = 1 - 2^{-j} \). Hence, \( \pi_j = 2^{-j} \). Equations \([27]\) and \([28]\) give \( T_{i,j} = \frac{\pi_i}{1 - \sigma_{j-1}} = 2^{j-i-1} \) for all \( 1 \leq i < j \). \( \square \)

### 3.3 Invariant Galton-Watson measures

The following result was originally proved in \([3]\). We state and prove it here since the expression \([35]\) will be used in the proof of Theorem \([2]\) later in the paper.

**Lemma 6 (Pruning Galton-Watson tree, \([3]\)).** Consider a critical or subcritical Galton-Watson measure \( \mu \equiv \mathcal{GW}(\{q_k\}) \) \( (q_1 = 0) \) on \( \mathcal{T}^\dagger \) with generating function \( Q(z) \), and the corresponding pushforward probability measure induced by the Horton pruning operator \( \mathcal{R} \),

\[
\nu(T) = \mu \circ \mathcal{R}^{-1}(T) = \mu(\mathcal{R}^{-1}(T)).
\]

Then, \( \nu(T \mid T \neq \phi) \) is a Galton-Watson measure \( \mathcal{GW}(\{q_k^{(1)}\}) \) on \( \mathcal{T}^\dagger \) with offspring probabilities

\[
q_0^{(1)} = \frac{Q(q_0) - q_0}{(1 - q_0)(1 - Q'(q_0))}, \quad \text{and}
\]

\[
q_1^{(1)} = 0, \quad \text{and}
\]

\[
q_k^{(0)} = \frac{(1 - q_0)^{k-1}Q^{(k)}(q_0)}{k!(1 - Q'(q_0))} \quad (k \geq 2),
\]

and generating function

\[
Q_1(z) = z + \frac{Q(q_0 + (1 - q_0)z) - q_0 - z(1 - q_0)}{(1 - q_0)(1 - Q'(q_0))}.
\]

Moreover, if \( \mu(T) \) is critical, then so is \( \nu(T \mid T \neq \phi) \). If \( \mu(T) \) is subcritical, then the first moment is decreasing with pruning, i.e., \( \sum_{k=2}^{\infty} kq_k^{(1)} < \sum_{k=2}^{\infty} kq_k < 1 \).

**Proof.** The standard thinning argument (with \( \pi_1 = q_0 \) being the probability of eliminating an offspring) implies that \( \mathcal{R}(T) \) is distributed as a Galton-Watson tree, i.e., \( \mathcal{R}(T) \stackrel{d}{\sim} \mathcal{GW}(\{q_m^{(1)}\}) \). Indeed, think of a random tree obtained as a result of the auxiliary branching process defined in the following way. We trace the branching process that starts with one generation zero progenitor vertex (the root) that produces exactly one offspring. From generation one on, the branching process evolves according to the offspring probability mass function \( \{q_k\}_{k=1}^{\infty} \). Next, the process is thinned: once an offspring is produced (in each generation, including generation zero), it is either instantaneously eliminated with probability \( q_0 \) or is left untouched with probability \( 1 - q_0 \), where these Bernoulli trials are performed independently of each other and the branching history. Naturally, this generates a Galton-Watson branching process with branching probabilities \( \{p_m\} \) calculated as follows

\[
p_m = \sum_{k=m/2}^{\infty} \binom{k}{m} q_0^{k-m}(1 - q_0)^m \frac{q_k}{1 - q_0}.
\]
The above defined *thinned Galton-Watson process* can be equivalently formulated by tracking the original branching process with branching probabilities \( \{q_k\} \). Here, for each offspring, it is instantaneously decided whether the offspring is a leaf or not via a Bernoulli trial with probabilities \( q_0 \) and \( 1-q_0 \) for ‘leaf’ and ‘no leaf’ outcomes respectively. If the offspring is decided to be a leaf, it is pruned instantaneously. If not a leaf, it will branch according to the derivation of (55). This branching process induces the tree measure \( \nu(T) \). Note that there is an alternative derivation of (33) as by Cor. 1 \( q_0^{(1)} = \frac{\pi_2}{1-\sigma_1} = \frac{S(q_0) q_0}{1-q_0} = \frac{Q(q_0) q_0}{(1-q_0)(1-Q(q_0))} \).

We notice that the corresponding generating function can be computed as follows

\[
Q_1(z) = \sum_{m=0}^{\infty} z^m q_m^{(1)} = \frac{(1-q_0)^{-1}}{1 - \sum_{k=2}^{\infty} k q_k^{k-1} q_k} \left( \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} \sum_{k=m}^{\infty} (zq_k^{k-1}(1-q_0))^m \binom{k}{m} q_k^{k-1} q_k \right)
\]

\[
= \frac{(1-q_0)^{-1}}{1 - Q'(q_0)} \left( \sum_{k=2}^{\infty} k q_k^{k-1} q_k + \sum_{k=2}^{\infty} \binom{k}{m}(zq_k^{k-1}(1-q_0))^m q_k^{k-1} q_k \right)
\]

\[
= \frac{(1-q_0)^{-1}}{1 - Q'(q_0)} \left( Q(z + (1-z)q_0) - q_0 - z(1-q_0)Q'(q_0) \right)
\]

by the binomial theorem, implying (35). We proceed by differentiating \( \frac{d}{dz} \) in (35), obtaining

\[
Q_1'(z) = \frac{Q'(q_0 + z(1-q_0)) - Q'(q_0)}{1 - Q'(q_0)}.
\]  

Next, we observe that if \( \mu(T) \) is critical, (37) implies \( \sum_{k=2}^{\infty} k q_k^{(1)} = Q_1'(1) = \frac{Q'(1-Q(q_0))}{1-Q'(q_0)} = 1 \). That is, the critical process stays critical after a Horton pruning. Finally, in the subcritical case, \( Q'(1) < 1 \), and by formula (37), \( Q_1'(1) = \frac{Q'(1-Q(q_0))}{1-Q'(q_0)} < Q'(1) \).

Next, we define a single parameter family of critical Galton-Watson measures \( GW(\{q_k\}) \) \( (q_1 = 0) \) over \( T^l \).
Definition 8 (Invariant Galton-Watson measures). For a given \( q \in [1/2, 1) \), the critical Galton-Watson measure \( GW(\{q_k\}) \) is said to be an invariant Galton-Watson (IGW) measure with parameter \( q \) and denoted by \( IGW(q) \) if its generating function has the following expression:

\[
Q(z) = z + q(1-z)^{1/q}.
\]

The respective branching probabilities are \( q_0 = q, q_1 = 0, q_2 = \frac{1-q}{2q} \), and

\[
q_k = \frac{1-q}{k!q} \prod_{i=2}^{k-1} (i-1/q) \quad (k \geq 3).
\]

Here, if \( q = \frac{1}{2} \), then the distribution is critical binary, i.e., \( GW(q_0 = 1/2, q_2 = 1/2) \). If \( q \in (1/2, 1) \), the distribution is of Zipf type with

\[
q_k = \frac{(1-q)\Gamma(k-1/q)}{q\Gamma(2-1/q)k!} \sim k^{-(1+q)/q}.
\]

Theorem 2 (Self-similar Galton-Watson measures). Consider a critical or subcritical Galton-Watson measure \( GW(\{q_k\}) \) with \( q_1 = 0 \). It is Horton self-similar (Def. 4) if and only if it is the invariant Galton-Watson (IGW) measure \( IGW(q_0) \) with \( q_0 \in [1/2, 1) \).

Proof. Recall that Horton self-similarity for Galton-Watson trees is equivalent to Horton prune-invariance. Equation (35) states that

\[
Q_1(z) = z + q_0 \frac{Q(q_0 + (1-q_0)z)}{Q(q_0) - q_0} - (q_0 + (1-q_0)z).
\]

If the Galton-Watson measure is Horton prune-invariant, then \( Q_1(z) = Q(z) \), and

\[
R(z) = \frac{R(q_0 + (1-q_0)z)}{R(q_0)} \quad \text{for} \quad R(z) = \frac{Q(z) - z}{q_0}
\]

for \( z \in [0, 1) \). Hence, letting \( \ell(z) = \ln R(1-z) \) for \( z \in (0, 1) \), we have \( \ell(z) + \ell(1-q_0) = \ell((1-q_0)z) \). Finally, for \( r(y) = \ell(e^{-y}) = \ln R(1-e^{-y}) \) for \( y \in [0, \infty) \) and \( \ell_0 = -\ln(1-q_0) \),

\[
r(y + \ell_0) = r(y) + r(\ell_0) \quad \forall y \in [0, \infty).
\]

Therefore, \( r'(y + \ell_0) = r'(y) \) and \( r(y) = \alpha y \) for some scalar \( \alpha \in \mathbb{R} \). Thus,

\[
Q(z) = z + q_0 R(z) = z + q_0 e^{\ell(1-z)} = z + q_0 e^{r(-\ln(1-z))} = z + q_0(1-z)^{-\alpha}.
\]

Finally,

\[
0 = q_1 = Q'(0) = 1 + \alpha q_0
\]

implying \( \alpha = -\frac{1}{q_0} \). The statement of the theorem follows by plugging \( \alpha = -\frac{1}{q_0} \) into (42).

Observe that for an invariant Galton-Watson measure \( IGW(q) \) with any \( q \in [1/2, 1) \), equation (38) implies

\[
S(z) = q + (1-q)z.
\]
3.4 Toeplitz property

Lemma 7 (Toeplitz implies IGW). Consider a Galton-Watson measure $GW(\{q_k\})$ with $q_1 = 0$. Suppose it is either subcritical or critical. If it satisfies Toeplitz property, then it must be the invariant Galton-Watson measure $IGW(q_0)$.

Proof. Suppose $GW(\{q_k\})$ ($q_1 = 0$) is mean Horton self-similar with Tokunaga sequence $\{T_k\}_{k \in \mathbb{N}}$. Equation $[27]$ implies, there is a scalar $c > 0$ such that

$$\frac{T_{k+1}}{T_k} = \frac{\pi_i}{\pi_{i+1}} = c \quad \forall k \geq 2, i \geq 1.$$  

Thus, as $\pi_1 = q_0$, we have $\pi_j = q_0 c^{1-j}$. Now, since $\sum_j \pi_j = 1$, we have $c = \frac{1}{1-q_0}$. Hence, $\sigma_j = 1 - c^{-j}$. Consequently, observe that for $s(z) = S(z) - q_0 - (1-q_0)z$,

$$s(\sigma_j) = \sigma_{j+1} - q_0 - (1-q_0)\sigma_j = 0 \quad \text{for all } j = 0, 1, 2, \ldots,$$

and $s(1) = 0$. Hence, as $\sigma_j \uparrow 1$, we have $s \equiv 0$. See the corresponding uniqueness theorem in Ahlfors [6] (Ch. 4, Sect. 3.2). Therefore, $S(z) = q_0 + (1-q_0)z$, where equation $[18]$ implies the following ODE

$$(z - S(z))Q'(z) - Q(z) + S(z) = 0$$  

with $Q(0) = 1$ as the initial condition. Next, we solve $[44]$, obtaining $Q(z) = z + q_0 (1-z)^{1/q_0}$ as the unique solution. \[\square\]

First, we notice that Lemma 7 yields an alternative proof of ‘only if’ part in Theorem 2 as Horton prune-invariance implies Toeplitz property. Next, observe that Lemmas 2 and 7 imply the following corollary.

Corollary 3. Consider a critical or subcritical Galton-Watson measure $GW(\{q_k\})$ with $q_1 = 0$, and suppose $E[X^{2-\epsilon}] < \infty$ for the progeny variable $X \overset{d}{\sim} \{q_k\}$ and all $\epsilon > 0$. If it satisfies Toeplitz property, then $q_0 = q_2 = \frac{1}{2}$, i.e., it is critical binary Galton-Watson tree $GW(q_0 = q_2 = 1/2)$.

3.5 Attractors and basins of attraction

Theorem 3 (Attraction property of critical Galton-Watson trees). Consider a critical Galton-Watson measure $\rho_0 \equiv GW(\{q_k\})$ ($q_1 = 0$) on $\mathcal{T}$. Starting with $k = 0$, and for each consecutive integer, let $\nu_k = R_*(\rho_k)$ denote the pushforward probability measure induced by the pruning operator, i.e., $\nu_k(T) = \rho_k \circ R^{-1}(T) = \rho_k(R^{-1}(T))$, and set $\rho_{k+1}(T) = \nu_k(T \mid T \neq \phi)$. Suppose the limit

$$\lim_{x \to 1^-} \left( \frac{\ln g(x)}{-\ln(1-x)} \right) = L$$

exists and is finite, where $g(x)$ is as defined in $[26]$. Then, for any $T \in \mathcal{T}$,

$$\lim_{k \to \infty} \rho_k(T) = \rho^*(T),$$

where $\rho^*$ denotes the invariant Galton-Watson measure $IGW\left(\frac{1}{2-L}\right)$.

Finally, if the Galton-Watson measure is subcritical, then $\rho_k(T)$ converges to a point mass measure, $GW(q_0 = 1)$. 

21
Proof of Theorem 3. Let $q^{(k)}_m$ denote the branching probability mass function corresponding to the critical Galton-Watson tree measure $\mu_k$, where $q^{(k)}_1 = 0$ by series reduction. First, we observe that

$$
\lim_{k \to \infty} q^{(k)}_0 = \lim_{k \to \infty} \frac{\pi_k}{1 - \sigma_{k-1}} = \lim_{k \to \infty} \frac{S(\sigma_{k-1}) - \sigma_{k-1}}{1 - \sigma_{k-1}} = \lim_{k \to \infty} \frac{1 + S'(1)(\sigma_{k-1} - 1) + o\left(1 - \sigma_{k-1}\right)}{1 - \sigma_{k-1}} = 1 - S'(1) = \frac{1}{2 - \ell}, \quad (45)
$$

where $S'(1) = \frac{1-\ell}{2-\ell}$ by Lemma 3.

Let $Q_k(z) := \sum_{m=0}^{\infty} z^m q^{(k)}_m$ denote the generating function corresponding to the Galton-Watson measure $\mu_k$. Next, let $S_k(z) = \frac{Q_k(z) - z Q_k'(z)}{1 - Q_k'(z)}$ denote the function corresponding to $S(z) = \frac{Q(z) - z Q'(z)}{1 - Q'(z)}$. Equation (41) implies

$$
S_1(z) = \frac{1}{1 - q_0} S\left(q_0 + (1 - q_0)z\right) - \frac{q_0}{1 - q_0}.
$$

(46)

For a given $z \in [0, 1)$, we iterate (46), obtaining

$$
S_k(z) = \prod_{i=0}^{k-1} \frac{1}{1 - q^{(i)}_0} S\left((1 - \prod_{i=0}^{k-1} (1 - q^{(i)}_0)) + z (1 - \prod_{i=0}^{k-1} (1 - q^{(i)}_0))\right) + \left(1 - \prod_{i=0}^{k-1} (1 - q^{(i)}_0)\right), \quad (47)
$$

where $\prod_{i=0}^{k-1} (1 - q^{(i)}_0) \leq 2^{-k} \rightarrow 0$ as $k \rightarrow \infty$. Next, we substitute

$$
S\left((1 - \prod_{i=0}^{k-1} (1 - q^{(i)}_0)) + z (1 - \prod_{i=0}^{k-1} (1 - q^{(i)}_0))\right) = 1 + (z - 1)S'(1) \prod_{i=0}^{k-1} (1 - q^{(i)}_0) + o\left(\prod_{i=0}^{k-1} (1 - q^{(i)}_0)\right)
$$

into (46), getting

$$
S_k(1) = 1 + (z - 1)S'(1) + o(1).
$$

Hence, for a given $z \in [0, 1)$, we have

$$
\frac{d}{dz} \ln (Q_k(z) - z) = \frac{1}{z - S_k(z)} \rightarrow \frac{1}{(1 - S'(1))(z - 1)} \quad \text{as } k \rightarrow \infty.
$$

Also, we notice that $Q_k(x) - x$ is a decreasing function ($Q'_k(x) < Q'_k(1) = 1$) and

$$
q^{(k)}_0 \geq Q_k(x) - x \geq Q_k(z) - z > 0 \quad \forall x \in [0, z].
$$

Therefore, letting $k \rightarrow \infty$, we have

$$
\ln (Q_k(z) - z) = \ln q^{(k)}_0 + \int_0^z \frac{d}{dx} \ln (Q_k(x) - x) \, dx \rightarrow \ln(1 - S'(1)) + \frac{1}{1 - S'(1)} \ln(1 - z)
$$

as $\lim_{k \to \infty} q^{(k)}_0 = 1 - S'(1) = \frac{1}{2 - \ell}$ by (45). We conclude that

$$
\lim_{k \to \infty} Q_k(z) = z + (1 - S'(1))(1 - z)^{1/(1 - S'(1))} = z + \frac{1}{2 - \ell} (1 - z)^{2 - \ell}
$$

where the right hand side is the generating function for $\mathcal{IGW}\left(\frac{1}{2 - \ell}\right)$.

Finally, if $\mu \equiv \mathcal{GW}(\{q_k\})$ is subcritical, Prop. 3 and (45) imply $\lim_{k \to \infty} q^{(k)}_0 = 1 - S'(1) = 1$. □
Lemma 4 immediately implies the following corollary.

**Corollary 4 (Attraction property of critical Galton-Watson trees of Zipf type).** Consider a critical Galton-Watson process \( \rho_0 \equiv \mathcal{GW}(\{q_k\}) \) with \( q_1 = 0 \), with probability mass function \( q_k \) of Zipf type, i.e., \( q_k \sim k^{-(\alpha + 1)} \), with \( \alpha \in (1, 2] \). Starting with \( k = 0 \), and for each consecutive integer, let \( \nu_k = \mathcal{R}_*(\rho_k) \) denote the pushforward probability measure induced by the pruning operator, and set \( \rho_{k+1}(T) = \nu_k(T \mid T \neq \phi) \). Then, for any \( T \in \mathcal{T} \),

\[
\lim_{k \to \infty} \rho_k(T) = \rho^*(T),
\]

where \( \rho^* \) is the invariant Galton-Watson measure \( IGW\left(\frac{1}{\alpha}\right) \).

Next, under finiteness of “2−” (second minus) moment assumption, as stated in (43), Lemma 2 implies the following result from [5] as a corollary of Theorem 3.

**Corollary 5 (Attraction property of critical binary Galton-Watson tree, [3]).** Consider a critical Galton-Watson process \( \rho_0 \equiv \mathcal{GW}(\{q_k\}) \) with \( q_1 = 0 \), such that the “2−” moment assumption is satisfied, i.e.,

\[
\sum_{k=2}^{\infty} k^{2-\epsilon} q_k < \infty \quad \forall \epsilon > 0.
\]

Starting with \( k = 0 \), and for each consecutive integer, let \( \nu_k = \mathcal{R}_*(\rho_k) \) denote the pushforward probability measure induced by the pruning operator, and set \( \rho_{k+1}(T) = \nu_k(T \mid T \neq \phi) \). Then, for any \( T \in \mathcal{T} \),

\[
\lim_{k \to \infty} \rho_k(T) = \rho^*(T),
\]

where \( \rho^* \) is the critical binary Galton-Watson measure \( IGW(1/2) \).

### 3.6 Explicit Tokunaga coefficients, Horton law

In the next lemma we establish the expressions for Tokunaga coefficients and Horton exponent in an invariant Galton-Watson tree measure.

**Lemma 8 (Tokunaga coefficients).** Consider an invariant Galton-Watson measure \( IGW(q_0) \) for \( q_0 \in [1/2, 1) \). Then,

\[
\pi_i = q_0 \cdot c^{1-i} \quad \text{with} \quad c = \frac{1}{1-q_0}.
\]

The measure satisfies Toeplitz property with the Tokunaga coefficients

\[
T^o_{i,j} = T^o_{j-i}, \quad \text{where} \quad T^o_k = c^{k-1} \quad (k = 1, 2, \ldots),
\]

and

\[
T_{i,j} = T_{j-i}, \quad \text{where} \quad T_1 = c^{1/q_0} - c - 1 \quad \text{and} \quad T_k = a c^{k-1} \quad (k = 2, 3, \ldots)
\]

with \( a = (c-1)\left(c^{1/q_0}-1\right) \). Finally, the strong Horton law [13] holds with Horton exponent \( R = c^{1/q_0} \).
Figure 8: Binary attractor: Illustration. The tree $T$ (panel a) has bounded offspring distribution ($q_k = 0$ for $k > b$) with maximal branching number $b = 6$. Its first pruning (panel b) $R(T)$ has maximal branching number $b = 3$. Its second pruning (panel c) $R^2(T)$ has maximal branching number $b = 2$. This convergence to binary branching is generic in Galton-Watson trees that have offspring distribution with a finite $2 - \epsilon$ moment, as discussed in Corollary 5.

Proof. Equations (17) and (43) imply $\sigma_i = 1 - (1 - q_0)^i + (1 - q_0)^i z$. Hence, $\pi_i = \sigma_i - \sigma_{i-1} = q_0(1 - q_0)^{i-1}$. Equations (48) and (49) are obtained via substituting $\pi_i$ and $\sigma_i$ into Lemma 5.

Finally, the results of [8] apply. Theorem 1 implies the strong Horton law with the Horton exponent $R = 1/w_0$, where $w_0$ is the only real zero of the generating function $\hat{t}(z)$ in the interval $(0, \frac{1}{2}]$. Since here,

$$\hat{t}(z) = -1 + (T_1 + 2)z + \frac{acz^2}{1 - cz},$$

one obtains $w_0 = c^{-1/q_0}$. Hence, $R = c^{1/q_0}$.

4 Concluding remarks

In this paper we described the invariance and attractor properties of combinatorial Galton-Watson trees with respect to Horton pruning. A similar approach can be applied to a broader class of generalized dynamical prunings on trees with edge lengths introduced and analyzed in [9, 10] and to the pruning operation studied in Evans [6] and in Duquesne and Winkel [5]. This will be done in a follow-up paper.

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