COHOMOLOGICAL CHARACTERIZATION OF T-LAU PRODUCT ALGEBRAS

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Abstract. Let $A$ and $B$ be Banach algebras and let $T$ be an algebra homomorphism from $B$ into $A$. The Cartesian product space $A \times B$ by $T$-Lau product and $\ell^1$-norm becomes a Banach algebra $A \times_T B$. We investigate the notions such as injectivity, projectivity and flatness for the Banach algebra $A \times_T B$. We also characterize Hochschild cohomology for the Banach algebra $A \times_T B$.

1. Introduction and Preliminaries

Suppose that $A$ and $B$ are Banach algebras and $T : B \to A$ is an algebra homomorphism. Then we consider the Cartesian product space $A \times B$ with the following multiplication

$$(a, b) \times_T (c, d) = (ac + T(b)c + aT(d), bd) \quad ((a, b), (c, d) \in A \times B),$$

which is denoted by $A \times_T B$. Let $\|T\| \leq 1$. Then we consider $A \times_T B$ with the following norm

$$\|(a, b)\| = \|a\| + \|b\| \quad ((a, b) \in A \times_T B).$$

We note that $A \times_T B$ is a Banach algebra with this norm and it is called $T$-Lau product algebras.

Whenever the Banach algebra $A$ is commutative, Bhatt and Dabshi [1] have investigated the properties of the Banach algebra $A \times_T B$, such as Gelfand space, Arens regularity and amenability.

Whenever $A$ is unital with unit element $e$ and $\varphi : B \to \mathbb{C}$ is a character on $B$, assume $T : B \to A$ is defined by $T(b) = \varphi(b)e$. In this case the multiplication $\times_T$ corresponds with the product studied by Lau [10]. Lau product was extended by Sangani Monfared for the general case and many basic properties of this product are studied in [13].

In the definition of $T$-Lau product, we can replace condition $\|T\| \leq 1$ with a bounded algebra homomorphism $T$, because if we consider the following norms

$$\|a\|_T = \|T\| \|a\| \quad (a \in A)$$
$$\|b\|_T = \|T\| \|b\| \quad (b \in B)$$
$$\|(a, b)\|_T = \|a\|_T + \|b\|_T \quad (a, b) \in A \times_T B,$$

then all these norms are equivalent with the original norms. Clearly all results of this paper hold when we consider these equivalent norms.

The authors in [12] for every Banach algebras $A$ and $B$ and for an algebra homomorphism $T : B \to A$ with $\|T\| \leq 1$ have investigated some homological properties of $T$-Lau product algebra $A \times_T B$ such as

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approximate amenability, pseudo amenability, $\phi$-pseudo amenability, $\phi$- biflatness and $\phi$-biprojectivity and have presented the characterization of the double centralizer algebra of $A \times_T B$.

Following [12], in this paper we studied the homological notions such as injectivity, projectivity and flatness for the Banach algebra $A \times_T B$. We also characterize the Hochschild cohomology for the Banach algebra $A \times_T B$.

2. Injectivity, Flatness and Projectivity

Let $A$ be a Banach algebra. In this paper, the category of Banach left $A$-modules and Banach right $A$-modules is denoted by $A\text{-mod}$ and $\text{mod}-A$, respectively. We denote by $B(E,F)$ the Banach space of all bounded operators from $E$ into $F$. In the category of $A$-mod, we denote the space of bounded morphisms from $E$ into $F$ by $AB(E,F)$. A function $S \in B(E,F)$ is called admissible if there exists $S' \in B(F,E)$ such that $S \circ S' \circ S = S$.

A. Ya. Helemskii introduced the concepts of injectivity and flatness for Banach algebras [5] and these concepts have been investigated for different classes of Banach modules in [3, 4, 11, 14].

Definition 2.1. A Banach left $A$-module $K$ is called projective if for every admissible epimorphism $S : E \to F$ in $A\text{-mod}$, the induced map $S_A : AB(K,E) \to AB(K,F)$ defined by $S_A(R_A) = R_A \circ S \quad (R_A \in AB(K,E))$

is surjective.

Definition 2.2. A Banach left $A$-module $K$ is called injective if for every admissible monomorphism $S : E \to F$ in $A\text{-mod}$, the induced map $S_A : AB(F,K) \to AB(E,K)$ defined by $S_A(R_A) = R_A \circ S \quad (R_A \in AB(F,K))$

is surjective.

Definition 2.3. A Banach left $A$-module $K$ is called flat if the dual module $K^*$ in $\text{mod}-A$ is injective with the action defined by $(f \cdot a)(x) = f(a \cdot x)$, where $a \in A$, $x \in K$ and $f \in K^*$.

Let $A, B,$ and $C$ be Banach algebras and let $T : B \to A$ be an algebra homomorphism with $\|T\| \leq 1$. We note that if $A$ is a Banach left $C$-module, then $A \times_T B$ is a Banach left $C$-module via the following action $c \cdot (a,b) = (c \cdot a + c \cdot T(b),0) \quad ((a,b) \in A \times_T B, c \in C)$.

Similarly if $B$ is a Banach left $C$-module, then $A \times_T B$ is a Banach left $C$-module via the following action $c \cdot (a,b) = (-T(c \cdot b),c \cdot b)$.

Theorem 2.4. Suppose that $A$ and $B$ are Banach algebras and $T : B \to A$ is an algebra homomorphism with $\|T\| \leq 1$. Suppose that $C$ is Banach algebra and $A \times_T B$ is injective as $C$-module. Then $A$ and $B$ are injective as $C$-module.
Proof. Let $A$ be a Banach left $C$-module and let $F, K \in \mathcal{C}$-mod. Suppose that $S \in \mathcal{C}B(F, K)$ is admissible and monomorphism. We will show that the induced map $S_A : \mathcal{C}B(K, A) \to \mathcal{C}B(F, A)$ is onto.

We conclude that $A \times_T B \in \mathcal{C}$-mod via the following action
\[
c \cdot (a, b) = (c \cdot a + c \cdot T(b), 0).
\]

Since $A \times_T B$ is injective, the induced map $S_{A \times T B} : \mathcal{C}B(K, A \times_T B) \to \mathcal{C}B(F, A \times_T B)$ is onto. Let $\lambda \in \mathcal{C}B(F, A)$ and $f \in F$. We define $\tilde{\lambda} : F \to A \times_T B$ by $\tilde{\lambda}(f) = (\lambda(f), 0)$. Hence we have $\tilde{\lambda} \in \mathcal{C}B(F, A \times_T B)$. Since $S_{A \times T B} : \mathcal{C}B(K, A \times_T B) \to \mathcal{C}B(F, A \times_T B)$ is onto, there exists $R_{A \times T B} : K \to A \times_T B$ such that $R_{A \times T B}(T(f)) = \tilde{\lambda}(f) = (\lambda(f), 0)$. We define $R_A : K \to A$ by $R_A = P_A \circ R_{A \times T B}$, where $P_A : A \times_T B \to A$ is defined by $p_A(a, b) = a$. Clearly $R_A \in \mathcal{C}B(K, A)$ and $R_A \circ T' = \lambda$. So $A$ is injective.

For injectivity of $B$, suppose that $B$ is a Banach left $C$-module, $F$ and $K \in \mathcal{C}$-mod and $S \in \mathcal{C}B(F, K)$ is admissible and monomorphism. We must show that the induced map $S_B : \mathcal{C}B(K, B) \to \mathcal{C}B(F, B)$ is onto. We have $A \times_T B \in \mathcal{C}$-mod with the following actions
\[
c \cdot (a, b) = (-T(c), c \cdot b).
\]

Since $A \times_T B$ is injective, the induced map $S_{A \times T B} : \mathcal{C}B(K, A \times_T B) \to \mathcal{C}B(F, A \times_T B)$ is onto. Let $\mu \in \mathcal{C}B(F, B)$ and $f \in F$. We define $\tilde{\mu} : F \to A \times_T B$ by $\tilde{\mu}(f) = (0, \mu(f))$. Then we have $\tilde{\mu} \in \mathcal{C}B(F, A \times_T B)$. Since $S_{A \times T B} : \mathcal{C}B(K, A \times_T B) \to \mathcal{C}B(F, A \times_T B)$ is onto, there exists $R_{A \times T B} : K \to A \times_T B$ such that $R_{A \times T B}(T(f)) = \tilde{\mu}(f) = (0, \mu(f))$. We define $R_A : K \to A$ by $R_A = P_A \circ R_{A \times T B}$. Clearly $R_A \in \mathcal{C}B(K, B)$ and $R_A \circ S = \mu$. Hence $B$ is injective. This completes the proof. \hfill \square

Let $A, B$ and $C$ be Banach algebras and let $T : B \to A$ be an algebra homomorphism with $\|T\| \leq 1$. We note that if $A \times_T B$ is the Banach left $C$-module, then $A$ and $B$ can be the Banach left $C$-modules with the following actions
\[
c \cdot a = c \cdot (a, 0) \quad \text{and} \quad c \cdot b = c \cdot (0, b),
\]
where $c \in C, a \in A$ and $b \in B$.

**Theorem 2.5.** Suppose that $A$ and $B$ are Banach algebras and $T : B \to A$ is an algebra homomorphism with $\|T\| \leq 1$. Suppose that $C$ is a Banach algebra and $A$ and $B$ are injective as $\mathcal{C}$-mod. Then $A \times_T B$ is injective as $\mathcal{C}$-mod.

**Proof.** Let $A \times_T B$ be a Banach left $C$-module and $F, K \in \mathcal{C}$-mod. Let $S \in \mathcal{C}B(F, K)$ such that $S$ is admissible and monomorphism. We must show that the induced map $S_{A \times T B} : \mathcal{C}B(K, A \times_T B) \to \mathcal{C}B(F, A \times_T B)$ is onto. We have $A, B \in \mathcal{C}$-mod with the following actions
\[
c \cdot a = c \cdot (a, 0) \quad \text{and} \quad c \cdot b = c \cdot (0, b),
\]
where $c \in C, a \in A$ and $b \in B$. Since $A$ and $B$ are injective, the induced maps $S_A : \mathcal{C}B(K, A) \to \mathcal{C}B(F, A)$ and $S_B : \mathcal{C}B(K, B) \to \mathcal{C}B(F, B)$ are onto. Suppose that $\lambda \in \mathcal{C}B(F, A \times T B)$ and $(a, b) \in A \times_T B$ such that $\lambda(f) = (a, b)$ for $f \in F$.

We define $\tilde{\lambda} : F \to A$ by $\tilde{\lambda}(f) = a + T(b)$ and $\tilde{\mu} : F \to B$ by $\tilde{\mu}(f) = b$. Hence we have $\tilde{\lambda} \in \mathcal{C}B(F, A)$ and $\tilde{\mu} \in \mathcal{C}B(F, B)$. Since $S_A : \mathcal{C}B(k, A) \to \mathcal{C}B(F, A)$ and $S_B : \mathcal{C}B(K, B) \to \mathcal{C}B(F, B)$ are onto, there exist $R_A : K \to A$ and $R_B : K \to B$ such that $R_A \circ S(f) = \tilde{\lambda}(f) = a + T(b)$ and $R_B \circ S(f) = \tilde{\mu}(f) = b$. 


We define $R_{A \times T} : K \to A \times T$ by $R_{A \times T} = q_A \circ R_A + \eta_B \circ R_B$, where $\eta_B : B \to A \times T$ such that $\eta_B(b) = (-T(b), b)$. Clearly $R_{A \times T} \in \mathcal{C}(K, A \times T)$ and $R_{A \times T} \circ S = \lambda$. So $A \times T$ is injective. □

Let $A, B$ and $C$ be Banach algebras. We note that if $A^* \times B^*$ is a Banach left $C$-module, then $A^*$ and $B^*$ can be consider as Banach left $C$-modules via the following actions

$$c \cdot a^* = c \cdot (a^*, 0) \quad \text{and} \quad c \cdot b^* = c \cdot (0, b^*),$$

where $c \in C$, $a^* \in A^*$ and $b^* \in B^*$.

**Theorem 2.6.** Suppose that $A$ and $B$ are Banach algebras and $T : B \to A$ is an algebra homomorphism with $\|T\| \leq 1$. Suppose that $C$ is a Banach algebra. Then $A \times T$ is flat as $\text{mod-} C$ if and only if $A$ and $B$ are flat as $\text{mod-} C$.

**Proof.** Let $A \times T$ be flat as $\text{mod-} C$. With a simple argument we can show that $(A \times T)^* \cong A^* \times B^*$. Hence by similar argument as in Theorem 2.4 one can show that $A$ and $B$ are flat Banach algebras as $\text{mod-} C$.

Conversely, let $A$ and $B$ be flat Banach algebras as $\text{mod-} C$, let $F, K \in \mathcal{C}\text{-mod}$ and let $S \in \mathcal{C}(B(F, K))$ such that $S$ is admissible and monomorphism. Then we show that the induced map $S_{A^* \times B^*} : \mathcal{C}(B(K, A^* \times B^*)) \to \mathcal{C}(B(F, A^* \times B^*))$ is onto. We have $A^*, B^* \in \mathcal{C}\text{-mod}$ with the following actions

$$c \cdot a^* = c \cdot (a^*, 0) \quad \text{and} \quad c \cdot b^* = c \cdot (0, b^*),$$

where $c \in C$, $a^* \in A^*$ and $b^* \in B^*$.

Since $A^*$ and $B^*$ are injective as left $C$-module, the induced maps $S_{A^*} : \mathcal{C}(B(K, A^*)) \to \mathcal{C}(B(F, A^*))$ and $S_{B^*} : \mathcal{C}(B(K, B^*)) \to \mathcal{C}(B(F, B^*))$ are onto. Suppose that $\lambda^* \in \mathcal{C}(B(F, A^* \times B^*))$ and $(a^*, b^*) \in A^* \times B^*$ such that $\lambda^* (f) = (a^*, b^*)$ for $f \in F$.

We define $\tilde{\lambda}^* : F \to A^*$ by $\tilde{\lambda}^*(f) = a^*$ and $\tilde{\mu}^* : F \to B^*$ by $\tilde{\mu}^*(f) = b^*$. Hence we have $\tilde{\lambda}^* \in \mathcal{C}(B(F, A^*))$ and $\tilde{\mu}^* \in \mathcal{C}(B(F, B^*))$. Since $S_{A^*} : \mathcal{C}(B(K, A^*)) \to \mathcal{C}(B(F, A^*))$ and $S_{B^*} : \mathcal{C}(B(K, B^*)) \to \mathcal{C}(B(F, B^*))$ are onto, there exist $R_{A^*} : K \to A^*$ and $R_{B^*} : K \to B^*$ such that $R_{A^*} \circ S (f) = \tilde{\lambda}^*(f) = a^*$ and $R_{B^*} \circ S (f) = \tilde{\mu}^*(f) = b^*$.

We define $R_{A^* \times B^*} : K \to A^* \times B^*$ by $R_{A^* \times B^*} = q_{A^*} \circ R_{A^*} + q_{B^*} \circ R_{B^*}$, where $q_{A^*} : A^* \to A^* \times B^*$ and $q_{B^*} : B^* \to A^* \times B^*$ are defined by $q_{A^*}(a^*) = (a^*, 0)$ and $q_{B^*}(b^*) = (0, b^*)$, respectively. Clearly $R_{A^* \times B^*} \in \mathcal{C}(B(K, A^* \times B^*))$ and $R_{A^* \times B^*} \circ S = \lambda^*$. So $A^* \times B^*$ is injective as left $C$-module. This completes the proof. □

Let $A, B$ and $C$ be Banach algebras and let $T : B \to A$ be an algebra homomorphism with $\|T\| \leq 1$. If $A \times T$ is a Banach left $C$-module, as we have seen before, $A$ and $B$ are Banach left $C$-modules.

**Theorem 2.7.** Suppose that $A$ and $B$ are Banach algebras and $T : B \to A$ is an algebra homomorphism with $\|T\| \leq 1$. Suppose that $C$ is a Banach algebra. Then $A \times T$ is projective as $\text{C-mod}$ if and only if $A$ and $B$ are projective as $\text{C-mod}$. 
Proof. Let $A \times T B$ be projective as Banach left $C$-module and $F, K \in C\text{mod}$. Let $S \in cB(K, F)$ be admissible and epimorphism. We show that the induced map $S_{A \times T B} : cB(A \times T B, K) \to cB(A \times T B, F)$ is onto.

Since $A, B \in C\text{mod}$ are projective, the induced map $S_A : cB(A, K) \to cB(A, F)$ and $S_B : cB(B, K) \to cB(B, F)$ are onto. Let $\lambda \in cB(A \times T B, K)$ and $f_1, f_2 \in F$ such that $\lambda(a, 0) = f_1, \lambda(0, b) = f_2$ for $(a, 0), (0, b) \in A \times T B$. We define $\tilde{\lambda} : A \to F$ by $\tilde{\lambda}(a) = \lambda(a, 0) = f_1$ and $\tilde{\mu} : B \to F$ by $\tilde{\mu}(b) = \lambda(0, b) = f_2$. Hence we have $\tilde{\lambda} \in cB(A, F)$ and $\tilde{\mu} \in cB(B, F)$. Since $S_A : cB(A, K) \to cB(A, F)$ and $S_B : cB(B, K) \to cB(B, F)$ are onto, there exist $R_A \in cB(A, K)$ and $R_B \in cB(B, K)$ such that $S \circ R_A(a) = \tilde{\lambda}(a) = f_1$ and $S \circ R_B(b) = \tilde{\mu}(b) = f_2$. We define $R_{A \times T B} : A \times T B \to K$ by $R_{A \times T B} = R_A \circ P_A + R_B \circ P_B$. Clearly $R_{A \times T B} \in cB(A \times T B, K)$ and $T' \circ R_{A \times T B} = \lambda$. Hence $A \times T B$ is projective as left $C$-module.

Conversely, let $A$ be a Banach left $C$-module and $F, K \in C\text{mod}$ and let $S \in cB(K, F)$ such that $S$ be admissible and epimorphism. We show that the induced map $S_A : cB(A, K) \to cB(A, F)$ is onto. We have $A \times T B \in C\text{mod}$ with the following action

$$c \cdot (a, b) = (c \cdot a + c \cdot T(b), 0).$$

Since $A \times T B$ is projective, the induced map $S_{A \times T B} : cB(A \times T B, K) \to cB(A \times T B, F)$ is onto. Let $\lambda \in cB(A, F)$ and $f \in F$ such that $\lambda(a) = f$ for $a \in A$. We define $\tilde{\lambda} : A \times T B \to F$ by $\tilde{\lambda}(a, b) = \lambda(a + T(b))$. Hence we have $\tilde{\lambda} \in cB(A \times T B, F)$. Since $S_{A \times T B} : cB(A \times T B, K) \to cB(A \times T B, F)$ is onto, there exists $R_{A \times T B} \in B(A \times T B, K)$ such that $S \circ R_{A \times T B} = \tilde{\lambda}$. We define $R_A : A \to K$ by $R_A = R_{A \times T B} \circ q_A$, where $q_A : A \to A \times T B$ is defined by $q_A(a) = (a, 0)$. Clearly $R_A \in cB(A, K)$ and $S \circ R_A = \lambda$. Hence $A$ is projective as left $C$-module.

For the proof of projectivity of $B$, let $B$ be a Banach left $C$-module and $F, K \in C\text{mod}$ and let $S \in cB(K, F)$ such that $S$ is admissible and epimorphism. We show that the induced map $S_B : cB(B, K) \to cB(B, F)$ is onto. We have $A \times T B \in C\text{mod}$ with the following action

$$c \cdot (a, b) = (-T(c \cdot b), c \cdot b).$$

Since $A \times T B$ is projective as left $C$-module, the induced map $S_{A \times T B} : cB(A \times T B, K) \to cB(A \times T B, F)$ is onto. Let $\lambda \in cB(B, F)$. We define $\tilde{\lambda} : A \times T B \to F$ by $\tilde{\lambda}(a, b) = \lambda(b)$ for $(a, b) \in A \times T B$. Hence $\tilde{\lambda} \in cB(A \times T B, F)$. Since $S_{A \times T B} : cB(A \times T B, K) \to cB(A \times T B, F)$ is onto, there exists $R_{A \times T B} \in cB(A \times T B, K)$ such that $S \circ R_{A \times T B}(a, b) = \tilde{\lambda}(a, b) = \lambda(b)$. We define $R_B : B \to K$ by $R_B = R_{A \times T B} \circ q_B$, where $q_B : B \to A \times T B$ is defined by $q_B(b) = (0, b)$ for $b \in B$. Clearly $R_B \in cB(B, K)$ and $S \circ R_B = \lambda$. Hence $B$ is projective as left $C$-module.

\hfill $\Box$

3. Hochschild cohomology for the Banach algebra $A \times T B$

The concept of Hochschild cohomology for Banach algebras has been studied by Kamowitz \cite{9}, Johnson \cite{7}, \cite{11} and others. Recall that let $A$ be a Banach algebra and let $X$ be a Banach $A$-bimodule. We denote the space of bounded $n$-linear maps from $A$ into $X$ by $C^n(A, X)$. For $T \in C^n(A, X)$ we define the map
\( \delta^n : C^n(A, X) \to C^{n+1}(A, X) \) by
\[
(\delta^n T)(a_1, \ldots, a_{n+1}) = a_1 \cdot T(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i T(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) = (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1}.
\]

\( T \) is called an \( n \)-cocycle if \( \delta^n T = 0 \) and it is called \( n \)-coboundary if there exists \( S \in C^{n-1}(A, X) \) such that \( T = \delta^{n-1} S \). We denote the linear space of all \( n \)-cocycles by \( Z^n(A, X) \) and the linear space of all \( n \)-coboundaries by \( B^n(A, X) \). Clearly \( Z^n(A, X) \) includes \( B^n(A, X) \). We also recall that the \( n \)-th Hochschild cohomology group \( \mathcal{H}^n(A, X) \) is defined by the following quotient,
\[
\mathcal{H}^n(A, X) = \frac{Z^n(A, X)}{B^n(A, X)},
\]
for more details, see [3]. We remark that a left (right) Banach \( A \)-module \( X \) is called left (right) essential if the linear span of \( A \cdot X = \{ a \cdot x : a \in A, x \in X \} \) \( (X \cdot A = \{ x \cdot a : x \in X, a \in A \} \) is dense in \( X \). A Banach \( A \)-module \( X \) is called essential, if it is left and right essential.

Let \( A \) and \( B \) be Banach algebras and \( T : B \to A \) be an algebra homomorphism with \( \| T \| \leq 1 \). Let \( E \) be a Banach \( A \)-bimodule. Then \( E \) is also a Banach \( B \)-bimodule and a Banach \( A \times_T B \)-bimodule with the following actions, respectively
\[
b \cdot x = T(b) \cdot x \quad \text{and} \quad x \cdot b = x \cdot T(b),
\]
where \( b \in B, x \in E \) and
\[
(a, b) \cdot x = T(b) \cdot x \quad \text{and} \quad x \cdot (a, b) = x \cdot T(b),
\]
where \( (a, b) \in A \times_T B, x \in E \).

**Lemma 3.1.** Let \( E \) be an essential Banach \( A \times_T B \)-bimodule. Then \( E \) is an essential Banach \( A \)-bimodule and a Banach \( B \)-bimodule.

**Theorem 3.2.** Let \( A \) and \( B \) be Banach algebras with bounded approximate identity and let \( T : B \to A \) be an algebra homomorphism with \( \| T \| \leq 1 \). Let \( E \) be an essential Banach \( A \times_T B \)-bimodule. Then
\[
\mathcal{H}^1(A \times_T B, E^*) \simeq \mathcal{H}^1(A, E^*) \times \mathcal{H}^1(B, E^*),
\]
where \( \simeq \) denotes the vector space isomorphism.

**Proof.** By [2] Theorem 2.9.53] we have \( \mathcal{H}^1(A \times_T B, E^*) \simeq \mathcal{H}^1(M(A \times_T B), E^*) \), where \( M(A \times_T B) \) denotes the double centralizer algebra of \( A \times_T B \). But from [12] Theorem 4.3] we have
\[
M(A \times_T B) \cong M(A) \times M(B),
\]
where \( \cong \) denotes the algebra isomorphism. this implies that
\[
\mathcal{H}^1(M(A \times_T B), E^*) \simeq \mathcal{H}^1(M(A) \times M(B), E^*),
\]
where \( M(A) \) and \( M(B) \) denote the double centralizer algebra of \( A \) and \( B \), respectively.

Hence we have
\[
\mathcal{H}^1(A \times_T B, E^*) \simeq \mathcal{H}^1(M(A) \times M(B), E^*),
\]
Using [6, Theorem 4] we obtain $\mathcal{H}^1(\mathcal{M}(A) \times \mathcal{M}(B), E^*) \simeq \mathcal{H}^1(\mathcal{M}(A), E^*) \times \mathcal{H}^1(\mathcal{M}(B), E^*)$, thus we have
\[
\mathcal{H}^1(A \times_T B, E^*) \simeq \mathcal{H}^1(\mathcal{M}(A), E^*) \times \mathcal{H}^1(\mathcal{M}(B), E^*).
\]
Since $E$ is an essential Banach $A$-bimodule and an essential Banach $B$-bimodule, we have
\[
\mathcal{H}^1(\mathcal{M}(A), E^*) \simeq \mathcal{H}^1(A, E^*)
\]
and
\[
\mathcal{H}^1(\mathcal{M}(B), E^*) \simeq \mathcal{H}^1(B, E^*).
\]
This completes the proof. □

We can extend the previous theorem for the $n$-th Hochschild cohomology for the Banach algebra $A \times_T B$.

**Corollary 3.3.** Let $A$ and $B$ be Banach algebras with bounded approximate identity and let $T : B \to A$ be an algebra homomorphism with $\|T\| \leq 1$. Let $E$ be an essential $A \times_T B$-bimodule. Then
\[
\mathcal{H}^n(A \times_T B, E^*) \simeq \mathcal{H}^n(A, E^*) \times \mathcal{H}^n(B, E^*)
\]
for every $n \geq 1$.

**Proof.** By [12, Lemma 3.1] $A$ and $B$ have bounded approximate identities if and only if $A \times_T B$ has a bounded approximate identity. Using [2, Theorem 2.9.54] one can show that if $A \times_T B$ has a bounded approximate identity, then for every essential $A \times_T B$-bimodule $E$, we have $\mathcal{H}^n(A \times_T B, E^*) \simeq \mathcal{H}^n(M(A \times_T B), E^*)$. In [12, Theorem 4.3] the authors showed that $M(A \times_T B) \cong M(A) \times M(B)$. On the other hand Hochschild [6, Theorem 4] showed that $\mathcal{H}^n(M(A) \times M(B), E^*) \simeq \mathcal{H}^n(M(A), E^*) \times \mathcal{H}^n(M(B), E^*)$. Hence we have $\mathcal{H}^n(A \times_T B, E^*) \simeq \mathcal{H}^n(A, E^*) \times \mathcal{H}^n(B, E^*)$ where $n \geq 1$. □

Note that Bhatt and Dabshi in [1] showed that $A \times_T B$ is amenable if and only if $A$ and $B$ are amenable. In the essential case this is an immediate corollary of Theorem 3.2.

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