Research Article

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Asymptotic solution of the Cauchy problem for the singularly perturbed partial integro-differential equation with rapidly oscillating coefficients and with rapidly oscillating heterogeneity

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Abstract: In this paper, the regularization method of S. A. Lomov is generalized to integro-differential equations with rapidly oscillating coefficients and with a rapidly oscillating right-hand side. The main goal of the work is to reveal the influence of the oscillating components on the structure of the asymptotics of the solution of this problem. The case of coincidence of the frequencies of a rapidly oscillating coefficient and a rapidly oscillating inhomogeneity is considered. In this case, only the identical resonance is observed in the problem. Other cases of the relationship between frequencies can lead to so-called non-identical resonances, the study of which is nontrivial and requires the development of a new approach. It is supposed to study these cases in our further work.

Keywords: singularly perturbed, integro-partial differential equation, regularization of an integral, space of non-resonant solutions, iterative problems, solvability of iterative problems

MSC 2020: 35F10, 35R09

1 Introduction

Singularly perturbed integro-differential equations have been the subject of research for many decades, starting with the work of A. Vasilyeva, V. Butuzov [1–3], and M. Imanaliev [4,5]. These works argue the importance of such research for theory and applications. However, before the appearance of work related to the regularization method, S. Lomov [6–8], integro-differential equations were considered under the conditions of the spectrum of the matrix of the first variation (on a degenerate solution) lying in the open left half-plane, which significantly narrowed the scope of the above work in problems with purely imaginary points of the spectrum. And only after the development of the method of Lomov, it became possible to consider problems with a spectrum lying on an imaginary axis [9–24]. Note that the Lomov regularization method was mainly used for ordinary singularly perturbed differential and integro-differential equations [25–30]. The development of this method for integro-differential equations with partial derivatives was carried out by A. Bobodzhanov, V. Safonov, and B. Kalimbetov [31–38]. In the study of problems with a
slowly changing kernel, it turned out that the regularization procedure and the construction of a regularized asymptotic solution essentially depend on the type of integral operator. The most difficult case was when the upper limit of the integral is not a differentiation variable. For the integral operator with an upper limit coinciding with the differentiation variable, the scalar case is investigated. The case when the upper limit of the integral operator coincides with the differentiation variable is studied for equations of partial differential integro-differential equations with an integral operator, the kernel of which contains a rapidly changing exponential factor. Summarizing the results of work for integro-differential equations with one-dimensional integrals, the problem of constructing a regularized asymptotic solution of the problem for integro-differential equations with two independent variables is investigated. In the present paper, we consider a singularly perturbed partial differential integro-differential equation with high-frequency coefficients and with rapidly oscillating coefficients, and rapidly oscillating heterogeneity that generates essentially special singularities in the solution of the problem.

In this paper, we consider the Cauchy problem for the integro-differential equation with partial derivatives:

\[
L_\varepsilon y(x, t, \varepsilon) \equiv \varepsilon \frac{\partial y}{\partial x} - A(x)y - \int_{x_0}^{x} K(x, t, s)y(s, t, \varepsilon)\,ds - \varepsilon g(x)\cos\frac{\beta(x)}{\varepsilon}By \\
= \varepsilon h_1(x, t)\sin\frac{\beta(x)}{\varepsilon} + h_2(x, t),
\]

where \( A(x), g(x), h_1(x, t), h_2(x, t), y_0(t), \beta'(x) > 0 \) are known scalar functions, \( B = \text{const}, y = y(x, t, \varepsilon) \) is an unknown function, and \( \varepsilon > 0 \) is a small parameter.

Such an equation in the case \( \beta(x) = 2\beta(y), B \equiv 0 \) for ordinary equations in the absence of an integral term was considered in [39–44]. The limiting operator \( A(x) \) has a spectrum \( \lambda(x) = A(x), \beta'(x) \) is a frequency of rapidly oscillating \( \cos\frac{\beta(t)}{\varepsilon} \). In the following, functions \( \lambda_1(x) = -i\beta'(x), \lambda_2(x) = +i\beta'(x) \) will be called the spectrum of a rapidly oscillating coefficient.

We assume that the following conditions are fulfilled:

1. \( g(x), \beta(x), A(x) \in C^\infty([x_0, X], \mathbb{R}), h_1(x, t), h_2(x, t) \in C^\infty([x_0, X] \times [0, T], \mathbb{R}), y_0(t) \in C^\infty([0, T], \mathbb{R}), K(x, t, s) \in C^\infty ([x_0 \leq s \leq x \leq X, 0 \leq t \leq T], \mathbb{R}); \)
2. \( A(x) < 0 \) (\( \forall x \in [x_0, X] \)).

We will develop an algorithm for constructing a regularized asymptotic solution [6] of problem (1).

## 2 Regularization of problem (1)

Denote by \( \sigma_1 = \sigma_1(\varepsilon) \) independent of magnitude \( \sigma_1 = e^{\frac{i\beta(x_0)}{\varepsilon}}, \sigma_2 = e^{\frac{i\beta(x_0)}{\varepsilon}}, \) and rewrite equation (1) as

\[
L_\varepsilon y(x, t, \varepsilon) \equiv \varepsilon \frac{\partial y}{\partial x} - A(x)y - \int_{x_0}^{x} K(x, t, s)y(s, t, \varepsilon)\,ds - \varepsilon g(x)\cos\frac{\beta(x)}{\varepsilon}By \\
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= \varepsilon h_1(x, t)\sin\frac{\beta(x)}{\varepsilon} + h_2(x, t),
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where \( A(x), g(x), h_1(x, t), h_2(x, t), y_0(t), \beta'(x) > 0 \) are known scalar functions, \( B = \text{const}, y = y(x, t, \varepsilon) \) is an unknown function, and \( \varepsilon > 0 \) is a small parameter.
and instead of problem (2), consider the problem
\[
\varepsilon \frac{\partial \tilde{y}}{\partial x} + \sum_{j=1}^{3} \lambda_j(x) \frac{\partial \tilde{y}}{\partial t_j} - A(x) \tilde{y} = - \int_{x_0}^{x} K(x, t, s) \tilde{y} \left( s, t, \frac{\psi(s)}{\varepsilon} \right) ds + \varepsilon \frac{g(x)}{2} (e^{\sigma_1} + e^{\sigma_2}) B \tilde{y}
\]
(3)
for the function \( \tilde{y} = \tilde{y}(x, t, \tau, \varepsilon) \), where \( \tau = (\tau_1, \tau_2, \tau_3) \), \( \psi = (\psi_1, \psi_2, \psi_3) \). It is clear that if \( \tilde{y} = \tilde{y}(x, t, \tau, \varepsilon) \) is a solution of problem (3), then the function \( \tilde{y} = \tilde{y}(x, t, \tau, \varepsilon) \) is an exact solution to problem (2); therefore, problem (3) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral
\[
J \tilde{y} = \int_{x_0}^{x} K(x, t, s) \tilde{y} \left( s, t, \frac{\psi(s)}{\varepsilon} \right) ds.
\]

**Definition 1.** A class \( \mathcal{M}_\varepsilon \) is said to be asymptotically invariant (with \( \varepsilon \to +0 \)) with respect to an operator \( P_0 \) if the following conditions are fulfilled:

1. \( \mathcal{M}_\varepsilon \subset D(P_0) \) for each fixed \( \varepsilon > 0 \);
2. the image \( P_0 \mu(x, t, \varepsilon) \) of any element \( \mu(x, t, \varepsilon) \in \mathcal{M}_\varepsilon \) decomposes in a power series
\[
P_0 \mu(x, t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mu_n(x, t, \varepsilon) \quad (\varepsilon \to +0, \mu_n(x, t, \varepsilon) \in \mathcal{M}_\varepsilon, \; n = 0, 1, \ldots)
\]
convergent asymptotically for \( \varepsilon \to +0 \) (uniformly in \( (x, t) \in [x_0, X] \times [0, T] \)).

From this definition, it can be seen that the class \( \mathcal{M}_\varepsilon \) depends on the space \( U \), in which the operator \( P_0 \) is defined. In our case \( P_0 = J \).

Before describing the space \( U \), we introduce the sets of resonant multi-indices. We introduce the notations:
\[
\lambda(x) = (\lambda_1(x), \lambda_2(x), \lambda_3(x)), \quad (m, \lambda(x)) = \sum_{j=1}^{3} m_j \lambda_j(x),
\]
\[
|m| = \sum_{j=1}^{3} m_j, \quad \Gamma_0 = \{ m : (m, \lambda(x)) \equiv 0, \; \forall |m| \geq 2 \},
\]
\[
\Gamma_j = \{ m : (m, \lambda(x)) \equiv \lambda_j(x), \; \forall |m| \geq 2, \; j = 1, 2, 3, \}
\]
(the set \( \Gamma_0 \) corresponds to a point of the spectrum \( \lambda_0(x) \equiv 0 \), generated by the integral operator (see [45]).

For the space \( U \), we take the space of functions \( y(x, t, \tau, \sigma) \), represented by sums
\[
y(x, t, \tau, \sigma) = y_0(x, t, \sigma) + \sum_{j=1}^{3} y_j(x, t, \sigma) e^{\sigma_j} + \sum_{2 \leq |m| \leq N_y} y^m(x, t, \sigma) e^{(m, \tau)},
\]
(4)
y_0(x, t, \sigma), \; y_j(x, t, \sigma), \; y^m(x, t, \sigma) \in C^\infty([x_0, X] \times [0, T], \mathbb{C}), \quad j = 1, 3, 2 \leq |m| \leq N_y,
where asterisk * above the sum sign indicates that the summation for \( |m| = m_1 + m_2 + m_3 \geq 2 \) occurs only on the non-resonant multi-indexes, i.e., \( m \notin \bigcup_{j=0}^{3} \Gamma_j \), \( \sigma = (\sigma_1, \sigma_2) \).

Note that here the degree \( N_y \) of the polynomial \( y(x, t, \tau, \sigma) \) relative to the exponentials \( e^{\sigma_j} \) depends on the element \( y \). In addition, the elements of space \( U \) depend on bounded in \( \varepsilon > 0 \) terms of constants \( \sigma_1 = \sigma_1(\varepsilon) \) and \( \sigma_2 = \sigma_2(\varepsilon) \) which do not affect the development of the algorithm described below; therefore, in the record of element (4) of this space \( U \), we omit the dependence on \( \sigma = (\sigma_1, \sigma_2) \) for brevity. We show that the class \( \mathcal{M}_\varepsilon = U_{\{ y \in y(x, t, \sigma) \} \text{ is asymptotically invariant with respect to the operator } J } \) the image of the integral operator \( J \) on an arbitrary element \( y(x, t, \tau) \) of the space \( U \) has the form
\[ J(x, t, \tau) = \int_{x_0}^{x} K(x, t, s) y_0(s, t) \, ds + \sum_{j=1}^{3} \int_{x_0}^{x} K(x, t, s) y_j(s, t) \, e^{i \lambda_j(\theta) \frac{1}{\lambda_j(s)} \, ds} \]

\[ + \sum_{2 \leq |m| \leq N} \int_{x_0}^{x} K(x, t, s) y^m(s, t) \, e^{i \frac{1}{\lambda_j(\theta)} \, ds}. \]

Apply the operation of integration by parts to the second term

\[ J_f(x, t, \epsilon) = \epsilon \int_{x_0}^{x} \frac{K(x, t, s) y(s, t)}{\lambda_j(s)} \, ds \]

\[ = \epsilon \left[ \frac{K(x, t, x_0) y(x_0, t)}{\lambda_j(x)} - \int_{x_0}^{x} \left( \frac{\partial}{\partial s} \frac{K(x, t, s) y(s, t)}{\lambda_j(s)} \right) \, ds \right]. \]

Continuing this process, we obtain the series

\[ J_f(x, t, \epsilon) = \sum_{\nu=0}^{\infty} (-1)^{\nu+1} \left( I^\nu \left( K(x, t, s) y(s, t) \right) \right)_{x=x_0} e^{i \lambda_j(\theta) \frac{1}{\lambda_j(s)} \, ds}. \]

where the operators

\[ I^0 = \frac{1}{\lambda_j(s)}, \quad I^\nu = \frac{1}{\lambda_j(s)} \frac{\partial}{\partial s} I^{\nu-1} \quad (\nu \geq 1, \ j = 1, 3) \]

are introduced.

Applying the integration operation in parts to integrals

\[ J_m(x, t, \epsilon) = \int_{x_0}^{x} K(x, t, s) y^m(s, t) \, e^{i \frac{1}{(m, \lambda(s)) \frac{1}{(m, \lambda(s))} \, ds}, \]

we note that for all multi-indices \( m = (m_1, m_2, m_3), m \notin \bigcup_{j=0}^{3} I_f \) inequalities

\[(m, \lambda(x)) \neq 0 \quad \forall x \in [x_0, X] \]

are satisfied. Therefore, integration by parts in integrals \( J_m(x, t, \epsilon) \) is possible. Performing it, we will have:

\[ J_m(x, t, \epsilon) = \epsilon \int_{x_0}^{x} \frac{K(x, t, s) y^m(s, t)}{(m, \lambda(s))} \, ds \]

Continuing this process, we obtain the series

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where the operators

\[ I_m^0 = \frac{1}{(m, \lambda(s))}, \quad I_m^v = \frac{1}{(m, \lambda(s))} \frac{\partial}{\partial s} I_m^{v-1} \quad (v \geq 1, |m| \geq 2) \]

are introduced. Therefore, the image of the operator \( J \) on the element (4) of the space \( U \) is represented as a series

\[
f_y(x, t) = \int_{x_0}^x K(x, t, s)y_j(s, t) \, ds
\]

\[ + \sum_{j=1}^3 \sum_{v=0}^{\infty} (-1)^v e^{v+1} \left[ (I_j^v (K(x, t, s)y_j(s, t))_{s=x} e^{v+1} - (I_j^v (K(x, t, s)y_j(s, t)))_{s=x_0} \right]
\]

\[ + \sum_{2 \leq |m| \leq N_p} \sum_{v=0}^{\infty} (-1)^v e^{v+1} \left[ (I_m^v (K(x, t, s)y^m(s, t))_{s=x} e^{v+1} - (I_m^v (K(x, t, s)y^m(s, t)))_{s=x_0} \right].\]

It is easy to show (see, for example, [46, pp. 291–294]) that this series converges asymptotically for \( \varepsilon \to +0 \) (uniformly in \( (x, t) \in [x_0, X] \times [0, T] \)). This means that the class \( M_{\varepsilon} \) is asymptotically invariant (for \( \varepsilon \to +0 \)) with respect to the operator \( J \).

We introduce operators \( R_{\varepsilon} : U \to U \), acting on each element \( y(x, t, \tau) \in U \) of the form (4) according to the law:

\[
R_{\varepsilon} y(x, t, \tau) = \int_{x_0}^x K(x, t, s)y_j(s, t) \, ds,
\]

\[
R_{\varepsilon} y(x, t, \tau) = \sum_{j=1}^3 \left[ ([I_j^v (K(x, t, s)y_j(s, t))_{s=x} e^{v} - (I_j^v (K(x, t, s)y_j(s, t)))_{s=x_0}]\right]
\]

\[
+ \sum_{1 \leq |m| \leq N_p} \left[ ([I_m^v (K(x, t, s)y^m(s, t))_{s=x} e^{v} - (I_m^v (K(x, t, s)y^m(s, t)))_{s=x_0}]\right],
\]

\[
R_{\varepsilon v_1} y(x, t, \tau) = \sum_{j=1}^3 (-1)^v [([I_j^v (K(x, t, s)y_j(s, t))_{s=x} e^{v} - (I_j^v (K(x, t, s)y_j(s, t)))_{s=x_0}]\right]
\]

\[
+ \sum_{2 \leq |m| \leq N_p} (-1)^v [([I_m^v (K(x, t, s)y^m(s, t))_{s=x} e^{v} - (I_m^v (K(x, t, s)y^m(s, t)))_{s=x_0}]\right].
\]

Now let \( \hat{y}(x, t, \tau, \varepsilon) \) be an arbitrary continuous function on \( (x, t, \tau, \varepsilon) \in G = [x_0, X] \times [0, T] \times \{\tau : \Re \tau_1 < 0, \Re \tau_j \leq 0, j = 1, 3\} \), with asymptotic expansion

\[
\hat{y}(x, t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \hat{y}_k(x, t, \tau), \quad y_k(x, t, \tau) \in U
\]
converging as $\varepsilon \to +0$ (uniformly in $(x, t, \tau) \in G$). Then, the image $f_\varepsilon(x, t, \tau, \varepsilon)$ of this function is decomposed into an asymptotic series

$$f_\varepsilon(x, t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k f_k(x, t, \tau).$$

This equality is the basis for introducing an extension of an operator $J$ on series of the form (6):

$$J\tilde{y}(x, t, \tau, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \tilde{y}_k(x, t, \tau).$$

Although the operator $J$ is formally defined, its utility is obvious; since in practice, it is usual to construct the $N$th approximation of the asymptotic solution of problem (2), which impose only $N$th partial sums of the series (6), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2):

$$L \tilde{y}(x, t, \tau, \varepsilon) = \varepsilon \frac{\partial \tilde{y}}{\partial \tau} + \sum_{j=1}^{3} \lambda_j(x) \frac{\partial \tilde{y}}{\partial \tau_j} - A(x)\tilde{y} - \frac{h(x, t)}{2i} (e^{\sigma_1} - e^{\sigma_2}) B\tilde{y} = \varepsilon \frac{h(x, t)}{2i} (e^{\sigma_1} - e^{\sigma_2}) + h_2(x, t), \quad \tilde{y}(x_0, t, 0, \varepsilon) = y_0(t), \quad ((x, t) \in [x_0, X] \times [0, T]).$$

### 3 Iterative problems and their solvability in the space $U$

Substituting the series (6) into (7) and equating the coefficients of the same powers of $\varepsilon$, we obtain the following iterative problems:

$$L_0 y_0(x, t, \tau, \varepsilon) = \sum_{j=1}^{3} \lambda_j(x) \frac{\partial y_0}{\partial \tau_j} - A(x)y_0 - R_0 y_0 = h_2(x, t), \quad y_0(x_0, t, 0, \varepsilon) = y_0(t);$$

$$L_1 y_1(x, t, \tau, \varepsilon) = -\frac{\partial y_0}{\partial x} + \frac{g(x)}{2} (e^{\sigma_1} + e^{\sigma_2}) B y_0 + R_1 y_0 + \frac{h_1(x, t)}{2i} (e^{\sigma_1} - e^{\sigma_2}), \quad y_1(x_0, t, 0) = 0;$$

$$L_k y_k(x, t, \tau, \varepsilon) = -\frac{\partial y_{k-1}}{\partial x} + \frac{g(x)}{2} (e^{\sigma_1} + e^{\sigma_2}) B y_{k-1} + R_k y_k + R_{k-1} y_{k-1}, \quad y_k(x_0, t, 0) = 0; \quad k \geq 1.$$ 

Each iterative problem (8k) has the form

$$L y(x, t, \tau, \varepsilon) = \sum_{j=1}^{3} \lambda_j(x) \frac{\partial y}{\partial \tau_j} - A(x)y - R_0 y = H(x, t, \tau), \quad y(x_0, t, 0) = y(t),$$

where $H(x, t, \tau) = H_0(x, t) + \sum_{j=1}^{3} H_j(x, t) e^{\xi_j} + \sum_{2|\xi| \leq N_H} H^{m}(x, t) e^{i m \cdot \tau}$, is the known function of space $U$, $y(t)$ is the known function of the complex space $C$, and the operator $R_0$ has the form (see (50))

$$R_0 y(x, t, \tau) = R_0 \left[ y_0(x, t) + \sum_{j=1}^{3} y_j(x, t) e^{\xi_j} + \sum_{2|\xi| \leq N_H} y^{m}(x, t) e^{i m \cdot \tau} \right] = \int_{x_0}^{x} K(x, t, s) y(s, t) \, ds.$$

We introduce scalar (for each $x \in [x_0, X], \quad t \in [0, T]$) product in space $U$:

$$\langle u, w \rangle \equiv \left( u_0(x, t) + \sum_{j=1}^{3} u_j(x, t) e^{\xi_j} + \sum_{2|\xi| \leq N_u} u^{m}(x, t) e^{i m \cdot \tau}, w_0(x, t) + \sum_{j=1}^{3} w_j(x, t) e^{\xi_j} + \sum_{2|\xi| \leq N_u} w^{m}(x, t) e^{i m \cdot \tau} \right)$$

$$\equiv \left( u_0(x, t), w_0(x, t) \right) + \sum_{j=1}^{3} \left( u_j(x, t), w_j(x, t) \right) + \sum_{2|\xi| \leq \min(N_H, N_u)} \left( u^{m}(x, t), w^{m}(x, t) \right),$$
where we denote by \((\ast, \ast)\) the usual scalar product in the complex space \(\mathbb{C} : (u, v) = u \cdot \overline{v}\). Let us prove the following statement.

**Theorem 1.** Let conditions (1) and (2) be fulfilled and the right-hand side \(H(x, t, \tau) = H_0(x, t) + \sum_{j=1}^{3} H_j(x, t) e^{\tau_j} + \sum_{2 \leq |m| \leq N_H} H^m(x, t) e^{m \cdot \tau}\) of equation (9) belongs to the space \(U\). Then, equation (9) is solvable in \(U\), if and only if

\[
\langle H(x, t, \tau), e^\tau \rangle \equiv 0, \quad \forall (x, t) \in [x_0, X] \times [0, T].
\]

(10)

**Proof.** We will determine the solution of equation (9) as an element (4) of the space \(U\):

\[
y(x, t) = y_0(x, t) + \sum_{j=1}^{3} y_j(x, t) e^{\tau_j} + \sum_{2 \leq |m| \leq N_H} y^m(x, t) e^{m \cdot \tau}.
\]

(11)

Substituting (11) into equation (9), we will have

\[
\sum_{j=1}^{3} [\lambda_j(x) - A(x)] y_j(x, t) e^{\tau_j} + \sum_{2 \leq |m| \leq N_H} [(m, \lambda(x)) - A(x)] y^m(x, t) e^{m \cdot \tau} - A(x) y_0(x, t) - \int_{x_0}^{x} K(x, t, s) y_0(s, t) ds = H_0(x, t) + \sum_{j=1}^{3} H_j(x, t) e^{\tau_j} + \sum_{2 \leq |m| \leq N_H} H^m(x, t) e^{m \cdot \tau}.
\]

Equating here the free terms and coefficients separately for identical exponents, we obtain the following equations:

\[
-A(x) y_j(x, t) - \int_{x_0}^{x} K(x, t, s) y_j(s, t) ds = H_0(x, t),
\]

(12)

\[
[\lambda_j(x) - A(x)] y_j(x, t) = H_j(x, t), \quad j = 1, 3,
\]

(12j)

\[
[(m, \lambda(x)) - A(x)] y^m(x, t) = H^m(x, t), \quad 2 \leq |m| \leq N_H.
\]

(12m)

Since \(A(x) \neq 0\), equation (12) can be written as

\[
y_j(x, t) = \int_{x_0}^{x} (-A^{-1}(x) K(x, t, s)) y_j(s, t) ds - A^{-1}(x) H_0(x, t).
\]

(12o)

Due to the smoothness of the kernel \(-A^{-1}(x) K(x, t, s)\) and heterogeneity \(-A^{-1}(x) H_0(x, t)\), this Volterra integral equation has a unique solution \(y_j(x, t) \in C^\infty([x_0, X] \times [0, T], \mathbb{C})\). The equations (12j) and (12m) also have unique solutions

\[
y_j(x, t) = [\lambda_j(x) - A(x)]^{-1} H_j(x, t) \in C^\infty([x_0, X] \times [0, T], \mathbb{C}), \quad j = 2, 3,
\]

(13)

since \(\lambda_j(x), \lambda_j(x)\) are not equal to \(A(x)\). The equation (12) is solvable in space \(C^\infty([x_0, X] \times [0, T], \mathbb{C})\) if and only \(\langle H(x, t, \tau), e^\tau \rangle \equiv 0 \forall (x, t) \in [x_0, X] \times [0, T]\) hold. It is not difficult to see that these identities coincide with identities (10). Furthermore, since \((m, \lambda(x)) \neq \lambda_j(x), \forall m \notin \bigcup_{j=0}^{3} \Gamma_j, j = 0, 3, |m| \geq 2\) (see (4)), the equations (12m) has a unique solution

\[
y^m(x, t) = [(m, \lambda(x)) - A(x)]^{-1} H^m(x, t) \in C^\infty([x_0, X] \times [0, T], \mathbb{C}), \quad \forall |m| \geq 2, m \notin \bigcup_{j=0}^{3} \Gamma_j.
\]

Thus, condition (10) is necessary and sufficient for the solvability of equations (9) in the space \(U\).

The Theorem 1 is proved.

\[\square\]

**Remark 1.** If identity (10) holds, then under conditions (1), (2), equation (9) has the following solution in the space \(U\):
\[ y(x, t, \tau) = y_0(x, t) + \sum_{j=1}^{3} y_j(x, t) e^{5_j} + \sum_{2\leq |m| \leq N_{H}} y^m(x, t) e^{(m, \tau)} \]
\[ \equiv y_0(x, t) + \alpha_0(x, t) e^{5_0} + h_{20}(x, t) e^{5_0} + h_{30}(x, t) e^{5_1} + \sum_{2\leq |m| \leq N_{H}} p^m(x, t) e^{(m, \tau)}, \]

where \( \alpha_0(x, t) \in C^\infty([x_0, X] \times [0, T], C) \) are arbitrary function, \( y_0(x, t) \) is the solution of an integral equation (12\( _{0} \)), and introduced notations
\[ h_{20}(x, t) \equiv \frac{H_{0}(x, t)}{\lambda_0(x) - \lambda_1(x)}, \quad h_{30}(x, t) \equiv \frac{H_{0}(x, t)}{\lambda_1(x) - \lambda_2(x)}, \quad p^m(x, t) \equiv [(m, \lambda(x)) - \lambda_0(x)]^{-1}H^m(x, t). \]

4 The unique solvability of the general iterative problem in the space \( U \): residual term theorem

Let us proceed to the description of the conditions for the unique solvability of equation (9) in the space \( U \). Along with problem (9), we consider the following equation:
\[ L y(x, t, \tau) = -\frac{\partial y}{\partial x} + \frac{g(x)}{2}(e^{5_0} \sigma_1 + e^{5_1} \sigma_2) B y + R_{1} y + Q(x, t, \tau), \]
where \( y = y(x, t, \tau) \) is the solution (14) of equation (9), \( Q(x, t, \tau) \in U \) is the well-known function of the space \( U \). The right part of this equation:
\[ G(x, t, \tau) \equiv -\frac{\partial y}{\partial x} + \frac{g(x)}{2}(e^{5_0} \sigma_1 + e^{5_1} \sigma_2) B y + R_{1} y + Q(x, t, \tau) \]
\[ = -\frac{\partial}{\partial x} \left[ y_0(x, t) + \sum_{j=1}^{3} y_j(x, t) e^{5_j} + \sum_{2\leq |m| \leq N_{H}} y^m(x, t) e^{(m, \tau)} \right] \]
\[ + \frac{g(x)}{2}(e^{5_0} \sigma_1 + e^{5_1} \sigma_2) \left[ y_0(x, t) + \sum_{j=1}^{3} y_j(x, t) e^{5_j} + \sum_{2\leq |m| \leq N_{H}} y^m(x, t) e^{(m, \tau)} \right] \]
\[ + R_{1} \left[ y_0(x, t) + \sum_{j=1}^{3} y_j(x, t) e^{5_j} + \sum_{2\leq |m| \leq N_{H}} y^m(x, t) e^{(m, \tau)} \right] + Q(x, t, \tau), \]
may not belong to the space \( U \), if \( y = y(x, t, \tau) \in U \). Indeed, taking into account the form (14) of the function \( y = y(x, t, \tau) \in U \), we consider in \( G(x, t, \tau) \), for example, the terms
\[ Z(x, t, \tau) \equiv \frac{g(x)}{2}(e^{5_0} \sigma_1 + e^{5_1} \sigma_2) B \left[ y_0(x, t) + \sum_{j=1}^{3} y_j(x, t) e^{5_j} + \sum_{2\leq |m| \leq N_{H}} y^m(x, t) e^{(m, \tau)} \right] \]
\[ = \frac{g(x)}{2} B y_0(x, t) \left( e^{5_0} \sigma_1 + e^{5_1} \sigma_2 \right) + \frac{g(x)}{2} B y_1(x, t) \left( e^{5_1} \sigma_1 + e^{5_1+5_0} \sigma_2 \right) \]
\[ + \frac{g(x)}{2} B y_2(x, t) \sum_{2\leq |m| \leq N_{H}} y^m(x, t) e^{(m, \tau)}. \]

Here, for example, terms with exponents
\[ e^{5_0+5_1} = e^{(m, \tau)} \text{ if } m_1 = 0, m_2 + 1 = m_3, \]
\[ e^{5_1+5_0} \text{ if } m_1 = 0, m_3 + 1 = m_2, \]
\[ e^{5_1+5_1} \text{ if } m_1 = 0, m_2 = m_3, \]
\[ e^{5_2+5_0} \text{ if } m_1 = 0, m_2 = m_3, \]
\[ e^{5_0+5_1} \text{ if } m_1 = 0, m_2 = m_3, \]
\[ e^{5_2+5_1} \text{ if } m_1 = 0, m_2 = m_3, \]
\[ e^{5_2+5_0} \text{ if } m_1 = 0, m_2 = m_3. \]
do not belong to the space $U$, since multi-indexes 

$$(0, n, n) \in \Gamma_0, \quad (0, n + 1, n) \in \Gamma_1, \quad (0, n, n + 1) \in \Gamma_2, \quad \forall n \in N$$

are resonant. Then, according to the well-known theory (see [6, p. 234]), we embed these terms in the space $U$ according to the following rule (see (*)): 

$$e^{\xi (m, r)} = e^{0} = 1, \quad e^{\xi (m, r)} = e^{0} = 1 \text{ if } m_1 = 0, m_2 + 1 = m_3, \quad e^{\xi (m, r)} = e^{0} = 1 \text{ if } m_1 = 0, m_3 + 1 = m_2,$$

$$e^{\xi (m, r)} = e^{0} = 1 \text{ if } m_1 = 0, m_2 + 1 = m_3, \quad e^{\xi (m, r)} = e^{0} = 1 \text{ if } m_1 = 0, m_2 = m_3, \quad e^{\xi (m, r)} = e^{0} = 1 \text{ if } m_1 = 1, m_2 = m_3.$$

In other words, terms with resonant exponentials $e^{(m, r)}$ replaced by members with exponents $e^{0}, e^{5}, e^{5}, e^{5}$ according to the following rule:

$$e^{(m, r)}|_{m \in \Gamma_0} = e^{0} = 1, \quad e^{(m, r)}|_{m \in \Gamma_1} = e^{5}, \quad e^{(m, r)}|_{m \in \Gamma_2} = e^{5}, \quad e^{(m, r)}|_{m \in \Gamma_3} = e^{5}.$$

After embedding, the right-hand side of equation (15) will look like

$$\widehat{G}(x, t, \tau) = -\frac{\partial}{\partial x} \left[ y_0(x, t) + \sum_{j=1}^{9} y_j(x, t) e^{5j} + \sum_{2 \leq |m| \leq N_0} m^9(x, t) e^{(m, r)} + \sum_{j=0}^{3} \sum_{m \in \Gamma_j} y_m(x) e^{5j} \right] + Q(x, t, \tau).$$

As indicated in [6], the embedding $G(x, t, \tau) \rightarrow \widehat{G}(x, t, \tau)$ will not affect the accuracy of the construction of asymptotic solutions of problem (2), since $G(x, t, \tau)$ at $\tau = \frac{y(x)}{\epsilon}$ coincides with $G(x, t, \tau)$.

**Theorem 2.** Let conditions (1) and (2) be fulfilled and the right-hand side $H(x, t, \tau) = H_0(x, t) + \sum_{j=1}^{3} H_j(x, t) e^{5j} + \sum_{2 \leq |m| \leq N_0} H^m(x, t) e^{(m, r)} \in U$ of equation (9) satisfy condition (10). Then, problem (9) under additional conditions

$$(\widehat{G}(x, t, \tau), e^{5}) \equiv 0 \quad \forall (x, t) \in [x_0, X] \times [0, T],$$

where $Q(x, t, \tau) = Q_0(x, t) + \sum_{k=1}^{3} Q_k(x, t) e^{5k} + \sum_{2 \leq |m| \leq N_0} Q^m(x, t) e^{(m, r)}$ is the known function of the space $U$, is uniquely solvable in $U$.

**Proof.** Since the right-hand side of equation (9) satisfies condition (10), this equation has a solution in the space $U$ in the form (14), where $\alpha(x, t) \in C^{0}([x_0, X] \times [0, T], C)$ is the arbitrary function. Submit (14) to the initial condition $y(x_0, t, 0) = y^{*}(t)$. We get $\alpha(x_0, t) = \chi(t)$, where denoted

$$\chi(t) = y^{*}(t) + A^{-1}(x_0) H_0(x_0, t) - \frac{H_0(x_0, t)}{\lambda_0(x_0) - \lambda_0} - \frac{H_2(x_0, t)}{\lambda_2(x_0) - \lambda_2} - \sum_{2 \leq |m| \leq N_0} \left[(m, \lambda(x_0)) - A(x_0) \right]^{-1} H^m(x_0, t).$$

Now we subordinate the solution (14) to the orthogonality condition (16). We write $G(t, \tau)$ in more detail the right side of equation (9):

$$G(x, t, \tau) \equiv -\frac{\alpha}{\partial x} \left[ y_0(x, t) + \alpha(x, t) e^{5} + h_2(x, t) e^{5} + h_3(x, t) e^{5} + \sum_{2 \leq |m| \leq N_0} P^m(x, t) e^{(m, r)} \right] + \frac{g(x)}{2} (e^{5} \sigma_1 + e^{5} \sigma_2) B \left[ y_0(x, t) + \alpha(x, t) e^{5} + h_2(x, t) e^{5} + h_3(x, t) e^{5} + \sum_{2 \leq |m| \leq N_0} P^m(x, t) e^{(m, r)} \right] + R \left[ y_0(x, t) + \alpha(x, t) e^{5} + h_2(x, t) e^{5} + h_3(x, t) e^{5} + \sum_{2 \leq |m| \leq N_0} P^m(x, t) e^{(m, r)} \right] + Q(x, t, \tau).$$

Embedding this function into the space $U$, we will have
Given the initial condition $\gamma(x, t) = \gamma_0(t)$, we also conclude that after the embedding operation the function $\hat{G}(t, \tau)$ will linearly depend on the scalar function $a(x, t)$. Given that in condition (16) scalar multiplication by functions $e^{\sigma_i}$, containing only the exponent $e^{\sigma_i}$, in the expression for $\hat{G}(t, \tau)$, it is necessary to keep only the term with the exponent $e^{\sigma_i}$. Then, condition (16) takes the form

$$\left\{ -\frac{\partial}{\partial x}(a(x, t)e^{\sigma_i}) + \left( \sum_{|m| \geq 2; m^{\alpha} \in \Gamma_{m}} w_m^{\alpha}(a(t), t) \right) e^{\sigma_i} + Q_2(x, t) e^{\sigma_i}, e^{\sigma_i} \right\} = 0 \quad \forall (x, t) \in [x_0, X] \times [0, T],$$

where $w_m^{\alpha}(a(t), t)$ are some functions linearly dependent on $a(x, t)$.

Performing scalar multiplication here, we obtain a linear ordinary differential equation (relative $x$) for a function $a(x, t)$. Given the initial condition $a(x_0, t) = \gamma(t)$, found above, we find uniquely the function $a(x, t) \in C^\infty([x_0, X] \times [0, T])$, and therefore, we will uniquely construct a solution to equation (9) in the space $U$. The theorem is proved.
As mentioned earlier, the right-hand sides of iterative problems (8k) (if solved sequentially) may not belong to the space $U$. Then, according to [6, p. 234], the right-hand sides of these problems must be embedded into $U$, according to the above rule. As a result, we obtain the following problems:

\[
L_0y_0(x, t, \tau) = \sum_{j=1}^{\infty} \lambda_j(x) \frac{\partial y_0}{\partial \tau_j} - A(x)y_0 - R_0y_0 = h_2(x, t), \quad y_0(x_0, t, 0, \varepsilon) = y^0(t); \quad (8_0)
\]

\[
Ly(x, t, \tau) = -\frac{\partial y}{\partial x} + \left[\frac{g(x)}{2}(e^{1}_1 + e^{1}_2)By_0\right] + R_1y + \frac{h_1(x, t)}{2\varepsilon} (e^{1}_1 - e^{1}_2), \quad y(x_0, t, 0) = 0; \quad (8_1)
\]

\[
Ly(x, t, \tau) = -\frac{\partial y_1}{\partial x} + \left[\frac{g(x)}{2}(e^{n}_1 + e^{n}_2)By_1\right] + R_2y_1 + R_2y_0, \quad y_2(x_0, t, 0) = 0; \quad (8_2)
\]

\[
Ly(x, t, \tau) = -\frac{\partial y_{k-1}}{\partial x} + \left[\frac{g(x)}{2}(e^{n}_1 + e^{n}_2)By_{k-1}\right] + R_ky_0 + \cdots + R_1y_{k-1}, \quad y_k(x_0, t, 0) = 0, \quad k \geq 1 \quad (8_k)
\]

(images of linear operators $\frac{\partial}{\partial \tau}$ and $R_\tau$ do not need to be embedding in the space $U$, since these operators operate from $U$ to $U$). Such a change will not affect the construction of the asymptotic solution of the original problem (1) (or the equivalent problem (2)), so on the restriction $\tau = \frac{\psi x}{\varepsilon}$ series of problems (8k) will coincide with a series of problems (8k) (see [6, pp. 234–235]).

Applying Theorems 1 and 2 to iterative problems (8k), we find uniquely their solutions in the space $U$ and construct series (6). Just as in [6], we prove the following statement:

**Theorem 3.** Suppose that conditions (1) and (2) are satisfied for equation (2). Then, when $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 > 0$ is sufficiently small), equation (2) has a unique solution $y(x, t, \varepsilon) \in C^2([x_0, X] \times [0, T], C)$, in this case, the estimate

\[
\|y(x, t, \varepsilon) - y_{N\varepsilon}(x, t)\|_{Q([x_0, X] \times [0, T])} \leq c_N e^{N+1}, \quad N = 0, 1, 2, \ldots,
\]

holds true, where $y_{N\varepsilon}(x, t)$ is the restriction (for $\tau = \frac{\psi x}{\varepsilon}$) of the N-partial sum of series (6) with coefficients $y_N(x, t, \tau) \in U$, satisfying the iteration problems (8k), and the constant $c_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_0]$.

### 5 Construction of the solution of the first iteration problem

Using Theorem 1, we will try to find a solution to the first iteration problem (80). Since the right side $h_2(x, t)$ of the equation (80) satisfies condition (10), this equation has (according to (14)) a solution in the space $U$ in the form:

\[
y_0(x, t, \tau) = y_0^{(0)}(x, t) + a_1^{(0)}(x, t)\varepsilon,
\]

where $a_1^{(0)}(x, t) \in C^\infty([x_0, X] \times [0, T], C)$ are arbitrary function and $y_0^{(0)}(x, t)$ is the solution of the integral equation

\[
y_0^{(0)}(x, t) = \int_{x_0}^{x} (-A^{-1}(x)K(x, t, s))y_0^{(0)}(s, t)ds - A^{-1}(x)h_2(x, t). \quad (18)
\]

Subordinating (17) to the initial condition $y_0(x_0, t, 0) = y^0(t)$, we have

\[
y_0^{(0)}(x_0, t) + a_1^{(0)}(x_0, t) = y^0(t) \Leftrightarrow a_1^{(0)}(x_0, t) = y^0(t) - y_0^{(0)}(x_0, t) \Leftrightarrow a_1^{(0)}(x_0, t) = y^0(t) + A^{-1}(x_0)h_2(x_0, t).
\]

To fully compute the function $a_1^{(0)}(x, t)$, we proceed to the next iteration problem (81). Substituting into it the solution (17) of the equation (80), we arrive at the following equation:
Let us analyze the exponents of the second dimension included here for their resonance:

\[ e^{\eta_1} = e^{i\beta'(\theta) + \Lambda(\theta)} \]

Thus, exponents \( e^{\eta_1} \) and \( e^{\eta_2} \) are not resonant. Then, for solvability, equation (18), it is necessary and sufficient that the condition

\[ -\frac{\partial}{\partial x} \left( a_1^{(0)}(x, t) \right) + \frac{K(x, t, x) a_1^{(0)}(x, t)}{\lambda(x)} = 0 \]

is satisfied. Attaching the initial condition

\[ a_1^{(0)}(x_0, t) = y_0^{(0)}(t) + A^{-1}(x_0) h_2(x_0, t), \]

to this equation, we find

\[ a_1^{(0)}(x, t) = a_1^{(0)}(x_0, t) e^{\int_{x_0}^{x} \frac{K(s, t)}{\lambda(s)} ds}, \]

and therefore, we uniquely calculate the solution (17) of the problem (80) in the space \( U \). Moreover, the main term of the asymptotic of the solution to problem (2) has the form

\[ \gamma_0(x, t) = \gamma_0^{(0)}(x, t) + a_1^{(0)}(x_0, t) e^{\int_{x_0}^{x} \frac{K(s, t)}{\lambda(s)} ds} + \int_{x_0}^{x} \frac{\Lambda(s, t)}{\lambda(s)} ds + \int_{x_0}^{x} A(s, t) ds, \]  

(19)

where \( a_1^{(0)}(x_0, t) = y_0^{(0)}(t) + A^{-1}(x_0) h_2(x_0, t), \gamma_0^{(0)}(x, t) \) is the solution of the integrated equation (18). From expression (19) for \( \gamma_0(x, t) \), it is clear that \( \gamma_0(x, t) \) is independent of rapidly oscillating terms. However, already in the next approximation, their influence on the asymptotic solution of problem (1) is revealed. Indeed, in view of condition (10), equation (81) will be written as

\[ L\gamma_1 = -\frac{\partial y_0^{(0)}(x, t)}{\partial x} + \left[ \frac{g(x)}{2} \left( e^{\eta_1} + e^{\eta_2} \right) B(y_0^{(0)}(x, t) + a_1^{(0)}(x, t) e^{\eta_1}) \right] \]

\[ + \frac{K(x, t, x) a_1^{(0)}(x_0, t)}{\lambda(x_0)} - \frac{h_2(x, t)}{2i} \left( e^{\eta_1} - e^{\eta_2} \right) \]

\[ = -\frac{\partial y_0^{(0)}(x, t)}{\partial x} + \left[ \frac{g(x)}{2} B(y_0^{(0)}(x, t) e^{\eta_1} + y_0^{(0)}(x, t) \sigma_1 e^{\eta_1} + y_0^{(0)}(x, t) \sigma_2 e^{\eta_2}) \right] \]

\[ + \frac{g(x)}{2} B(a_1^{(0)}(x, t) \sigma_1 e^{\eta_1} + a_1^{(0)}(x, t) \sigma_2 e^{\eta_2}) \]

\[ - \frac{K(x, t, x) a_1^{(0)}(x_0, t)}{\lambda(x_0)} + \frac{\sigma_1}{2i} h_2(x, t) e^{\eta_1} = \frac{\sigma_2}{2i} h_1(x, t) e^{\eta_2}, \]  

(20)
Since there are no resonance exponents on the right side $H^{(1)}(x, t, \tau)$ of this equation, then $H^{(1)}(x, t, \tau) \in U$.

By Theorem 1, equation (20) has the following solution in the space $U$ (see (14)):

$$y_1(x, t, \tau) = y_1^{(0)}(x, t) + a_1^{(1)}(x, t) e^{h} + \frac{H_2^{(1)}(x, t)}{\lambda_2(x) - \lambda_1(x)} e^{h} + \frac{H_3^{(1)}(x, t)}{\lambda_3(x) - \lambda_1(x)} e^{h} + \frac{g(x)}{2\lambda_2(x)} B_0 a_1^{(0)}(x, t) e^{h+\tau} + \frac{g(x)}{2\lambda_3(x)} B_0 a_1^{(0)}(x, t) e^{h+\tau},$$

(21)

where $a_1^{(1)}(x, t) \in C^\infty([x_0, X] \times [0, T])$ is an arbitrary function determined from the solvability condition (10) of the equation (5) in the space $U$, the function $y_1^{(0)}(x, t)$ and the function

$$H_2^{(1)}(x, t) = \frac{\sigma_1 [g(x) B_0 y_0^{(0)}(x, t) - i h_0(x, t)]}{2},$$
$$H_3^{(1)}(x, t) = \frac{\sigma_2 [g(x) B_0 y_0^{(0)}(x, t) + i h_0(x, t)]}{2},$$

are computed uniquely. From (21), it is seen that $y_1(x, t, \frac{\psi(x)}{\epsilon})$ depends on rapidly oscillating exponents $e^{\frac{\psi(x)}{\epsilon}}$, i.e., an already asymptotic solution

$$y_{1\epsilon}(x, t) = y_0(x, t, \frac{\psi(x)}{\epsilon}) + \epsilon y_1(x, t, \frac{\psi(x)}{\epsilon})$$

of the first order depends on rapidly oscillating terms in equation (1).

### 6 Conclusion

The function $y_{1\epsilon}(x, t)$ shows that when passing from a differential equation of type (1) ($K(x, t, s) \equiv 0$) to an integro-differential one ($K(x, t, s) \neq 0$), the main term of the asymptotic is influenced by the kernel $K(x, t, s)$ of the integral operator. Their effects are detected when constructing the next approximation $y_{1\epsilon}(x, t)$.

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