Invariants of links with flat connections in their complements. II. Holonomy $R$-matrices related to quantized universal enveloping algebras at roots of 1.

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Abstract

Holonomy $R$-matrices parametrized by finite-dimensional representations are constructed for quantized universal enveloping algebras of simple Lie algebras at roots of 1.

1 Introduction

In the previous paper [KR] we gave the construction of invariants of tangles with flat $G$-connections in the complement. The construction was based on the notion of holonomy $R$-matrices. Such $R$-matrices are operator-valued functions on $G \times G$ which satisfy the holonomy Yang-Baxter equation [KR].

In this paper we show that quantized universal enveloping algebras at roots of 1 provide examples of such $R$-matrices.

In section 2 we present basic facts about quantized universal enveloping algebra when the quantization parameter is generic. In section 3 we show that the regular part of the universal $R$-matrix specialized at roots of 1 give examples of holonomy $R$-matrices. In section 4 we analyze the invariants of tangles with a flat connection in the complement derived from such holonomy $R$-matrices. This work was supported by the NSF grant DMS-0070931 and by the DMS grant RM1-2244.

2 Quantized Universal Enveloping Algebras

Let $\mathfrak{g}$ be a simple Lie algebra of rank $r$ with the root system $\Delta$. Denote by $P$ its weight lattice and by $Q$ its root lattice. Fix simple roots $\alpha_1, \ldots, \alpha_r \in \Delta$ and denote by $(a_{ij})_{i,j=1}^r$ the
corresponding Cartan matrix. Denote by $\omega_1, \ldots, \omega_r \in P$ the basis of fundamental weights (dual to the basis of simple roots $\{\alpha_i\}$) and let $d_i$ be the length of the $i$-th simple root.

### 2.1 Quantized universal enveloping algebras

Let $Q \leq M \leq P$ be a lattice. The quantized universal enveloping algebra $U_q^M(g)$ is the associative algebra with 1 over $\mathbb{C}(q)$ generated by $L_\mu, \mu \in M$, and $E_i, F_i, i = 1, \ldots, r$ with defining relations:

\[
L_\mu L_\nu = L_\nu L_\mu , \quad L_0 = 1 , \\
L_\mu E_i = q^{\alpha_i(\mu)} E_i L_\mu \\
L_\mu F_i = q^{-\alpha_i(\mu)} F_i L_\mu , \quad E_i F_j - F_j E_i = \delta_{ij} \frac{L_\alpha_i - L_{-\alpha_i}^{-1}}{q_i - q_i^{-1}} , \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} E_i^{1-k-a_{ij}} E_j E_k = 0 , \quad i \neq j \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right]_{q_i} F_i^{1-k-a_{ij}} F_j F_k = 0 , \quad i \neq j
\]

Here $q_i = q^{d_i}$,

\[
\left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[m-n]_q! [n]_q!} , \quad [n]_q! = [n]_q \ldots [2]_q [1]_q , \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} .
\]

The map $\Delta$ acting on generators as

\[
\Delta L_\mu = L_\mu \otimes L_\mu , \\
\Delta E_i = E_i \otimes 1 + L_\alpha_i \otimes E_i , \\
\Delta F_i = F_i \otimes L_{-\alpha_i} + 1 \otimes F_i
\]

extends to the homomorphism of algebras $\Delta : U_q^M(g) \rightarrow U_q^M(g) \otimes U_q^M(g)$. The pair $(U_q^M(g), \Delta)$ is a Hopf algebra with the counit $\varepsilon(L_\mu) = 1, \varepsilon(E_i) = \varepsilon(F_i) = 0$.

It is clear that if $P \leq M \leq M' \leq Q$ we have the embedding $U_q^M(g) \hookrightarrow U_q^{M'}(g)$ of Hopf algebras.

### 2.2 Quantum Weyl group

Let $B_W$ be the braid group associated with the Weyl group $W$,

\[
B_W = \{ \text{generated by } T_i \mid \text{with defining relations } T_i T_j T_i \ldots = T_j T_i T_j \ldots \}
\]
Here \( m_{ij} = 2 \) if \( i \) and \( j \) are not connected in the Dynkin diagram, \( m_{ij} = 3 \) if \( a_{ij}a_{ji} = 1 \), \( m_{ij} = 4 \) if \( a_{ij}a_{ji} = 2 \) and \( m_{ij} = 6 \) if \( a_{ij}a_{ji} = 3 \).

This group acts on \( U_q^M(\mathfrak{g}) \) by automorphisms \([L]:\)

\[
T_i(L_{\mu}) = L_{s_i(\mu)},
T_i(E_i) = -F_iL_{\alpha_i},
T_i(E_j) = \sum_{r+s=-a_{ij}} (-1)^r q_i^s E_i^{(r)} E_j E_i^{(s)},
T_i(F_i) = -L_{-1}^{-1} E_i,
T_i(F_j) = \sum_{r+s=-a_{ij}} (-1)^r q_i^s F_j E_i^{(r)} F_i^{(s)}
\]

where \( X_i^{(r)} = \frac{X_i^r}{[r]_q q_i^r} \).

Fix a reduced decomposition of the longest element \( w_0 \in W \). If \( w_0 = s_{j_1} \ldots s_{j_N}, N = |\Delta_+|, \)

\[
\beta_a = s_{j_1} \ldots s_{j_a-1} \alpha_{j_a}, a = 1, \ldots, N,
\]

were \( \alpha_1 \ldots \alpha_r \) are simple roots. This gives a total convex ordering \( \beta_1 < \cdots < \beta_N \) on the set of roots \( \Delta \) of \( \mathfrak{g} \). Such construction give all convex orderings and vice versa.

According to [Lu] define root elements of \( U_q(\mathfrak{g}) \) as

\[
E_{\beta_a} = T_{j_1} \ldots T_{j_{a-1}} E_{j_a},
F_{\beta_a} = T_{j_1} \ldots T_{j_{a-1}} F_{j_a}
\]

The elements

\[
E_{\beta_1}^{k_1} \ldots E_{\beta_N}^{k_N} L_{\mu} F_{\beta_N}^{\ell_N} \ldots F_{\beta_1}^{\ell_1}
\]

form a linear basis in the algebra \( U_q(\mathfrak{g}) \).

### 2.3 Integral form of quantized universal enveloping algebras

Define the \( \mathbb{C}[q, q^{-1}] \)-subalgebra \( U_q^M(\mathfrak{g}) \subset U_q^M(\mathfrak{g}) \) as the smallest \( B_W \)-stable \( \mathbb{C}[q, q^{-1}] \)-subalgebra of \( U_q^M(\mathfrak{g}) \) containing the elements

\[
\overline{E}_i = (q_i - q_i^{-1}) E_i, \quad \overline{F}_i = (q_i - q_i^{-1}) F_i.
\]
Set $E_\alpha = (q_\alpha - q_\alpha^{-1}) E_\alpha$, $F_\alpha = (q_\alpha - q_\alpha^{-1}) F_\alpha$, then monomials

$$\bar{E}_{\beta_1}^{k_1} \cdots \bar{E}_{\beta_N}^{k_N} L_\mu \bar{F}_{\beta_1}^{l_1} \cdots \bar{F}_{\beta_N}^{l_N}$$

form a linear basis in $U_q^M(g)$. Here we used the enumeration of positive roots corresponding to a reduced decomposition of the longest element of the Weyl group (see above).

### 2.4 Poisson Lie groups $G$ and $G^*$

It is well known that the algebra $U_q^Q(g)$ can be regarded as a Hopf algebra deformation of the algebra of polynomial functions on the Poisson Lie group $G^* = \{(b_+, b_-) \in B_+ \times B_- | [b_+]_0 = [b_-]_0^{-1}\}$. Notice that as a Lie group $G^*$ is naturally isomorphic to the semidirect product $H \ltimes (N_+ \times N_-)$ where $H$ act naturally on $N_\pm$. The tangent Lie bialgebra for this Poisson Lie group is dual to the standard Lie bialgebra structure on $g$ [CP]. In this sense the Poisson Lie group $G^*$ is dual to the Poisson Lie group $G$.

Similarly for any lattice $M$, $Q \leq M \leq P$ the covering group $G_M^*$ of $G^*$ is also a Poisson Lie group which is dual to the standard Poisson Lie group structure on $G$ in a sense that their tangent Lie bialgebras are dual.

The algebra $C(G_M^*)$ of algebraic functions on the Poisson Lie group $G_M^*$ is a Poisson Hopf algebra. As a Poisson algebra it is generated by elements $k_\mu, e_i, f_i, \mu \in M, i = 1, \ldots, r$ with defining relations

$$\{k_\mu, k_\nu\} = 0, \{k_\mu, e_j\} = \mu(\alpha_i) k_\mu e_j$$

$$\{e_i, f_j\} = \delta_{ij} (k_{\alpha_i} - k_{\alpha_i}^{-1})$$

$$\underbrace{\{e_i, \ldots, e_j\}^{(d_i(a_{ij})) \cdots}}_{-a_{ij} + 1} \underbrace{(-d_i(a_{ij}))}_{-d_i(a_{ij})} = 0$$

$$\underbrace{\{f_i, \ldots, f_j\}^{(d_i(a_{ij})) \cdots}}_{-a_{ij} + 1} \underbrace{(-d_i(a_{ij}))}_{-d_i(a_{ij})} = 0$$

where $\{X, Y\}^{(n)} = \{X, Y\} - nXY$.

The comultiplication acts on generators as in (1). The elements $k_\mu$ are coordinate functions on the Cartan subgroup of $G_M^*$, which is a finite cover of the Cartan subgroup of $G$. The elements $e_i$ and $f_i$ are coordinate functions on the nilpotent subgroups $N^\pm \subset G^*$ corresponding to the simple roots.
The braid group $B_W$ acts on $C(G_M^*)$ by Poisson automorphisms.

$$
\tau_i(k_\mu) = k_{s_i(\mu)}, \\
\tau_i(e_i) = -f_i k_{a_i}^{-1}, \\
\tau_i(f_i) = e_i k_{a_i}, \\
\tau_i(e_j) = \frac{(-1)^{a_{ij}}}{(-a_{ij})^2} \{e_i, \ldots, \{e_i, e_j\}^{(a_{ij}d_i)}\}^{(d_i(a_{ij}+2))} \cdots d_i(-a_{ij}-2), \\
\tau_i(f_j) = \frac{1}{(-a_{ij})^2} \{f_i, \ldots, \{f_i, f_j\}^{(a_{ij}d_i)}\}^{(d_i(a_{ij}+2))} \cdots d_i(-a_{ij}-2)
$$

One can define coordinates corresponding to all positive and negative roots on $G_M^*$ similarly to how it was done for $U_q^M(\mathfrak{g})$.

Fix a linear isomorphism between $U_q^M(\mathfrak{g})$ and $C(G_M^*)$ by identifying monomials (5) with corresponding monomials in $k_\mu$, $e_\alpha$, $f_\alpha$. Then it is clear that the Hopf algebra structure on $U_q^M(\mathfrak{g})$ is a Hopf algebra deformation of the Poisson Hopf algebra structure on $C(G_M^*)$ described above.

### 2.5 Symplectic leaves of $G_M^*$

According to the general structural facts about Poisson Lie groups symplectic leaves of $G^*$ are orbits of the (local) dressing action of the dual Poisson Lie group $G$ [STS]. This action can be describe as follows.

Let $I : G^* \to G$ be the natural map $(x_+, x_-) \mapsto x_+(x_-)^{-1}$. This map intertwines the dressing action with the adjoint action of $G$ on $G$.

The map $I$ brings the dressing action of $G$ on $G^*$ to the action of $G$ on itself by conjugations, i.e. if we will write $g : x \mapsto g(x)$ for the dressing action of $g \in G$ on $x \in G^*$ we have:

$$I(g(x)) = gI(x)g^{-1}$$

Thus, orbits of dressing action in $G^*$ are connected components of orbits of adjoint action of $G$ on itself.

The image of the map $I$ is open dense in $G$. Over generic point in $G$ it is a branched cover map with $2^r$-fibers and it gives an isomorphism between a neighborhood of 1 in $G$ and neighborhoods of points $(\sigma, \sigma^{-1}) \in G^*$ where $\sigma \in H$, $H$ is a Cartan subgroup in $G$ and $\sigma^2 = 1$. Using this ismorphisms we can identify these neighborhoods of $G^*$ and $G$. We will call it a realization of $G^*$ on $G$. 

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The natural projection $G^*_{M} \to G^*$ is Poisson and is a finite cover. Therefore symplectic leaves of $G^*_{M}$ are connected components of preimages of symplectic leaves in $G^*$.

2.6 Formal Poisson Lie group $G^*_{M}$

Let $\Gamma_M$ be the finite subgroup in $G^*_{M}$ which is the pre-image of $1 \in G^*$ with respect to the natural projection $G^*_{M} \to G^*$. Let $\varepsilon_\mu \in \Gamma_M$ be the element corresponding to the weight $\mu \in M/Q$.

Denote by $F(G^*_{M})$ the completion of $C(G^*_{M})$ by formal power series in $k_\mu \varepsilon_\mu^{-1} - 1$, $e_\alpha$ and $f_\alpha$. This Poisson Hopf algebra is the formal Poisson Lie group $G^*_{M}$. Instead of formal variables $k_\mu$ we can work with $z_\mu$ such that $k_\mu = \varepsilon_\mu \exp(z_\mu)$.

3 Quantized universal enveloping algebras at roots of 1

3.1 The algebra $\mathcal{U}^M_{\varepsilon}(\mathfrak{g})$ and its center

Let $\ell$ be an odd integer such that $\ell > \max_i(d_i)$ and $\varepsilon \in \mathbb{C}$ be a primitive $\ell$-th root of 1. Denote by $\mathcal{U}^M_{\varepsilon}(\mathfrak{g})$ be the quotient algebra

$$
\mathcal{U}^M_{\varepsilon}(\mathfrak{g}) = \frac{\mathcal{U}^M_q(\mathfrak{g})}{(q - \varepsilon)\mathcal{U}^M_q(\mathfrak{g})}.
$$

The center $Z^M_{\varepsilon} = Z(\mathcal{U}^M_{\varepsilon}(\mathfrak{g}))$ has natural structure of Poisson algebra and, as a Poisson algebra, it acts on $\mathcal{U}^M_{\varepsilon}(\mathfrak{g})$ by derivations [DCP].

Denote by $Z^M_0$ the subalgebra in $Z^M_{\varepsilon}$ generated by $\bar{E}_\alpha^\ell$, $\bar{F}_\alpha^\ell$, $L_\mu^\ell$, $\alpha \in \Delta_+$, $\mu \in P$.

The following is known (see [DP] and references therein):

**Proposition 1**

- The subalgebra $Z^M_0$ is a Hopf subalgebra in $\mathcal{U}^M_{\varepsilon}(\mathfrak{g})$.
- It is also a Poisson subalgebra in $Z^M_{\varepsilon}$ and together with the Hopf algebra structure is a Poisson-Hopf algebra.
- $Z^M_{\varepsilon}$ is integrally closed
- $Z^M_{\varepsilon}$ is a free $Z^M_0$ module of the rank $\ell^r$.
- $\mathcal{U}^M_{\varepsilon}(\mathfrak{g})$ is finite-dimensional over $Z^M_0$ with $\dim_{Z^M_0}(\mathcal{U}^M_{\varepsilon}(\mathfrak{g})) = \ell^{\dim \mathfrak{g}}$. 

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There is an isomorphism of Poisson Hopf algebras $\mathbb{Z}_0^M \simeq C(G^*_M)$. Here $G^*_M$ is the finite covering of the Poisson Lie group dual to the Poisson Lie group $G$ (see section 2.4). The isomorphism $\mathbb{Z}_0^M \simeq C(G^*_M)$ is given by the map $\phi$:

$$\phi(L^{\ell}_\mu) = k_\mu, \quad \phi(\bar{E}^{\ell}_i) = e_i, \quad \phi(\bar{F}^{\ell}_i) = f_i.$$ 

**Remark 1** Geometrically, the algebra $U^M_M(\mathfrak{g})$ can be regarded as a sheaf of algebras over $G^*_M$ such that it is a bundle of algebras over each symplectic leaf of $G^*_M$ with a flat connection over each simplectic leaf.

### 3.2 The completion $\overline{U}_\varepsilon M(\mathfrak{g})$ and its center

Let $\ell$ be a positive integer.

**Proposition 2** There exists a unique algebra structure $\overline{U}_\varepsilon M(\mathfrak{g})$ over $C[q^{\pm 1}][q^{\ell} - 1]$ on the space formal power series

$$\sum_{\substack{k_1, \ldots, k_N \geq 0 \\
m_{1, \ldots, m_{N+1}} \geq 0 \\
_1, \ldots, n \geq 0 \sum_{i=1}^N L^{\ell}_{\mu_i} - 1}^N \prod \bar{E}^{k_1}_{\beta_1} \cdots \bar{E}^{k_N}_{\beta_N} \cdot \prod \bar{F}^{m_{N+1}}_{\beta_N} \cdots \bar{F}^{m_1}_{\beta_1}$$

such that restricted to polynomials in $\bar{E}_{\beta}, \bar{F}_{\beta}, L_{\mu_i}, q^{\ell} - 1, q^{\pm 1}$ it coincides with $U^M_M(\mathfrak{g})$. Here $C_{\{k\}}^{\{m\}} \in C[L^{\pm 1}, q^{\pm 1}][[q^{\ell} - 1]]$, $N = |\Delta_+|$, $\beta_1, \ldots, \beta_N$ is a convex ordering on $\Delta_+$, $\mu_1, \ldots, \mu_r$ are generators of $M$ and $\bar{E}_{\beta_i}$ and $\bar{F}_{\beta_i}$ are as in (2).

Specializing $q$ to $\varepsilon$ as in the previous section we obtain the completion $\overline{U}_\varepsilon M(\mathfrak{g})$ of $U^M_M(\mathfrak{g})$. The center $Z^M_M(\mathfrak{g}) = Z(\overline{U}_\varepsilon M(\mathfrak{g}))$ has a natural Poisson algebra structure. The following proposition is the formal version of the proposition [1].

**Proposition 3**

- The subalgebra $Z^M_0 \subset Z^M_\varepsilon$ generated by formal power series in $\bar{E}_{\alpha}, \bar{F}_{\alpha}, L^{\ell}_{\mu} - 1$ is a Hopf-Poisson subalgebra in $\overline{U}_\varepsilon M(\mathfrak{g})$.
- $Z^M_0$ is isomorphic to the formal group $F(G^*_M)$.
- $Z^M_\varepsilon$ is a free $Z^M_0$-module of rank $\ell^r$.
- $\overline{U}_\varepsilon M(\mathfrak{g})$ is f.d. over $Z^M_0$ with $\dim_{Z^M_0}(\overline{U}_\varepsilon M(\mathfrak{g})) = \ell^\dim \mathfrak{g}$

Let us introduce formal variables $z_\mu$ as $L^{\ell}_{\mu} = \varepsilon_\mu \exp(z_\mu)$ where $\varepsilon_\mu$ are elements of the finite order which generate the group of automorphisms of the covering map $G^*_M \to G^*$ in a neighborhood of 1 (the same $\varepsilon_\mu$ that were used in section 2.6). Then $L^{\ell}_{\mu} = L_{\mu} \exp\left(-\frac{z_\mu}{\ell}\right)$ are elements of finite order.
3.3 The universal $R$-matrix for $\overline{U}_\varepsilon^P(\mathfrak{g})$

For each positive root $\beta$ let $z_\beta, \bar{E}_\beta, \bar{F}_\beta$ be corresponding elements of $\overline{U}_\varepsilon^P(\mathfrak{g})$. Let $\overline{U}_\varepsilon^P(\mathfrak{g})^{\otimes 2}$ be the completion of the tensor product with respect to the gradation given by the degree function $\text{deg}(z_\beta) = \text{deg}(\bar{E}_\beta) = \text{deg}(\bar{F}_\beta) = 1$. Define the outer automorphism $\mathcal{R}_\beta^{(n)}$ of $\overline{U}_\varepsilon^P(\mathfrak{g})^{\otimes 2}$ as

$$\mathcal{R}_\beta^{(n)}(x) = \exp\left(\frac{1}{\ell^2} L_{i2}(\bar{E}_\beta^\ell \otimes \bar{F}_\beta^\ell)\right)(x)$$

where $\exp(y) \circ x = \sum_{n=0}^{\infty} \frac{1}{n!} \{y\{y\{y, x\}\ldots\}}$ and

$$L_{i2}(x) = -\int_0^x \frac{\log(1-s)}{s} ds = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Define the outer automorphism $\mathcal{R}^{(n)} = \mathcal{R}_\beta^{(n)} \circ \cdots \circ \mathcal{R}_\beta^{(n)}$.

Define the element $\tilde{R}_\beta^{(n)} \in \overline{U}_\varepsilon^M(\mathfrak{g})^{\otimes 2}$ as

$$\tilde{R}_\beta^{(n)} = \prod_{m=0}^{\ell-1} (1 - \varepsilon^m \bar{E}_\beta^\ell \otimes \bar{F}_\beta^\ell)^{-\frac{n}{\ell^2}}.$$

Define the element $R^{(n)} \in \overline{U}_\varepsilon^P(\mathfrak{g})^{\otimes 2}$ as

$$R^{(n)} = R_{\beta_N}^{(n)} \cdots R_{\beta_1}^{(n)}$$

where

$$R_{\beta_i}^{(n)} = \mathcal{R}_{\beta_1}^{(n)} \circ \cdots \circ \mathcal{R}_{\beta_{i-1}}^{(n)}(\tilde{R}_{\beta_i}^{(n)}).$$

Define the outer automorphism $\mathcal{R}^{(c)}$ of $\overline{U}_\varepsilon^P(\mathfrak{g})^{\otimes 2}$ as

$$\mathcal{R}^{(c)}(x) = \exp\left(\frac{1}{2\ell^2} \sum_{i,j=1}^{r} (b^{-1})_{ij} z_{\alpha_i} \otimes z_{\alpha_j}\right)(x)$$

where $b_{ij} = a_{ij}d_j$ is the symmetrized Cartan matrix.

Define the element $R^{(c)} \in \overline{U}_\varepsilon^P(\mathfrak{g})^{\otimes 2}$ as

$$R^{(c)} = \sum_{\lambda,\mu \in \mathcal{P}/d\mathcal{P}} \varepsilon^{(\lambda,\mu)} P_\lambda \otimes P_\mu.$$
Here $d$ is the degree of the covering map $G^* \to G^*$ and $P_\lambda$ are idempotents in the subalgebra generated by $\bar{L}_\mu$ such that

$$\bar{L}_\mu P_\lambda = P_\lambda \bar{L}_\mu = \varepsilon(\lambda, \mu) P_\lambda$$

**Theorem 1**

1. The outer automorphism $\mathcal{R}$ restricted to $\tilde{Z}_0 \hat{\otimes} \tilde{Z}_0$ is a Poisson automorphism.

2. The automorphism $\mathcal{R} = \mathcal{R}^{(c)} \circ \mathcal{R}^{(n)}$ restricted to $\tilde{Z}_0 \hat{\otimes} \tilde{Z}_0$ satisfies the Yang-Baxter equation

$$\mathcal{R}_{12} \circ \mathcal{R}_{13} \circ \mathcal{R}_{23} = \mathcal{R}_{23} \circ \mathcal{R}_{13} \circ \mathcal{R}_{12}$$

(6)

Here $\mathcal{R}_{ij}$ act on $\tilde{Z} \hat{\otimes}^3$, $\mathcal{R}_{12} = \mathcal{R} \otimes \text{id}$, $\mathcal{R}_{23} = \text{id} \otimes \mathcal{R}$, $\mathcal{R}_{13} = \text{id} \otimes \sigma_{23} \circ \mathcal{R}_{12} \circ \text{id} \otimes \sigma_{23}$ and $\sigma_{23}(x \otimes y \otimes z) = x \otimes z \otimes y$.

3. The element $R = R^{(c)} R^{(n)}$ satisfies the twisted Yang-Baxter relation

$$(\mathcal{R}_{12}^{-1} \circ \mathcal{R}_{13})(R_{23}) \cdot \mathcal{R}_{12}^{-1}(R_{13}) \cdot \mathcal{R}_{12} = (\mathcal{R}_{23}^{-1} \circ \mathcal{R}_{13})(R_{12}) \mathcal{R}_{23}^{-1}(R_{13}) R_{23},$$

(7)

4. $\Delta'(a) = \mathcal{R}(R \Delta(a) R^{-1})$ for all $a \in \mathcal{U}_\varepsilon(g)$.

This theorem follows from the asymptotical behavior of the universal $R$-matrix for $\text{sl}_2 \mathbb{R}^2$, from the multiplicative formula for the universal $R$-matrix and from the Campbell-Hausdorff formula. For arbitrary simple Lie algebra it was proven in [Ga]. The Poisson automorphism $\mathcal{R}$ was studied in [WX] and [R-1].

4 Representations of $\mathcal{U}_\varepsilon^P(g)$ and holonomy $R$-matrices

4.1 Representations of $\mathcal{U}_\varepsilon^M(g)$

We will denote by $(\pi^V_x, V)$ a representation $\pi^V_x : \mathcal{U}_\varepsilon^M(g) \to \text{End}(V)$ of $\mathcal{U}_\varepsilon^M(g)$ in the vector space $V$ with $Z_0^M$-central character $x \in G^*_M$. Here we used a natural identification of Poisson-Hopf algebras $Z_0^M \simeq C(G^*_M)$ (see proposition [3]).

The group $G$ acts on $G^*_M$ locally by dressing transformations. This $G$-action on $G^*_M \simeq \text{Spec}(Z_0)$ lifts to the action of $G$ on $\mathcal{U}_\varepsilon^M(g)$ by outer automorphisms. We will write $g : a \mapsto \tau_g(a)$ for this action.

We will say that the representation $(\pi^V_x, V)$ is $G$-equivalent to the representation $(\pi^W_y, W)$ if
• $x, y \in G_M^*$ belong to the same $G$-orbit, i.e. if there exists $g \in G$ with $y = g(x)$

• if there exists a linear map $\varphi_{V,W}(x,g) : V \to W$ such that

$$\pi_g^W(x) = \varphi_{V,W}(x,g) \pi_x^V(\pi_g(a)) \varphi_{V,W}(x,g)^{-1}.$$

The representation of $U^M(\mathfrak{g})$ dual to $(\pi_x^V, V)$ is the representation in the dual vector space $V^*$ with the algebra acting as $a \mapsto \pi_x^V(S(a)^*)$. Here $S$ is the antipode and $f^*$ is the linear map dual to $f : V \to V$. It is clear that if $(\pi_x^V, V)$ has $Z_0^M$-character $x \in G_M^*$ then the dual representation will have the $Z_0^M$-central character $i(x)$ where $i$ is the operation of taking the inverse in the group $G^*$.

### 4.2 Irreducible representations

Because the algebra $U^M(\mathfrak{g})$ is finite-dimensional over its center, there exists a non-empty Zariski open subset $S^M_{\epsilon} \subset Spec(Z^M_{\epsilon})$ such that $U^M(\mathfrak{g})/\langle (c-\chi(c))U^M(\mathfrak{g}) \mid c \in Z^M_{\epsilon} \rangle$ is isomorphic to a matrix algebra for any $\chi \in S^M(\mathfrak{g})$; for more details see [39]. We will call such elements generic. Thus, for each generic $\chi \in Spec(Z^M_{\epsilon})$ we have unique isomorphism class of irreducible representations.

Denote by $S^M_0 \subset Spec(Z^M_0)$ the image of $S^M_{\epsilon}$ with respect to the projection $Spec(Z^M_{\epsilon}) \to Spec(Z^M_0)$. The variety $S^M_{\epsilon}$ is a finite cover of $S^M_0$. Points of $Spec(Z^M_0)$ are ”common level surfaces” of all central elements of $U^M(\mathfrak{g})$. Points of $Spec(Z^M_0)$ are ”common level surfaces” of elements of central subalgebra generated by $L_\ell, F_\ell, E_\ell$ and by their Poisson brackets. The number of branches of the projection $S^M_{\epsilon} \to S^M_0$ over generic point is $\ell^\epsilon$. The number of branches of $S^M_0 \to S^Q_0$ is $\ell^{|M/Q|}$. So, over generic point of $G^*$ we expect $\ell^{\epsilon+|M/Q|}$ irreducible representations of $U^M(\mathfrak{g})$. All these irreducible representations have dimension $\ell^{|M/Q|}$.

Central elements of $U^M(\mathfrak{g})$ which also belong to the Poisson center of $Z^M_{\epsilon}$ are constant on dressing orbits. We will call this central subalgebra the Casimir subalgebra.

Let $\mathcal{O} \subset G^*$ be a dressing orbit and $U \subset G^*$ be a neighborhood of $1$ on which the local action of $G$ integrates to an action. Let $\{(\pi_x^V, V) \mid x \in U \subset \mathcal{O}\}$ be a family of irreducible representations of $U^M(\mathfrak{g})$. Assume that these representations have the same central character with respect to the Casimir subalgebra. Because Poisson Casimirs are constant on $G$-orbits and the specter of primitive ideals is a finite cover over the specter of primitive ideals of the Poisson center of $Z^M_0$, this assumption will hold for sufficiently small $U$. Let $g \in G$ and $x \in \mathcal{O} \cap U \subset G^*$ be such that $x$ and $g(x)$ are generic. Representations $\pi_x^V \circ \pi_g$ and $\pi_{g(x)}^V$ have the same central characters and therefore isomorphic. Thus, we have a family $\{T(g|x)\}$ of
linear automorphisms of $V$ such that

$$\pi^V_x(\tau^x(g)(a)) = T(g|x)\pi^V_{g(x)}(a)T(g|x)^{-1}$$ (8)

Considering formal neighborhood of 1 in $G$ and $G^*$ and representations of $\mathcal{U}^M_\epsilon(g)$ over such neighborhood we get what is called formal representations. Such representations are homomorphisms from $\bar{\mathcal{U}}^M_\epsilon(g)$ to the algebra $End(V)[[x]]$ where $V$ is the space of representations and $x$ are formal coordinates in a neighborhood of 1.

4.3 Holonomy $R$-matrices

From now on we will use the map $I$ to identify neighborhoods of identities in $G^*$ and $G$. After this the dressing action of $G$ on $G^*$ is identified with the adjoint action of $G$ on itself.

Let $(\pi^V_x, V)$ and $(\pi^W_y, W)$ be two generic formal representations of $\mathcal{U}^P_\epsilon(g)$.

**Proposition 4** Let $a, b \in \mathcal{U}^P_\epsilon(g)$ and $\mathcal{R}$ be the outer automorphism of $\mathcal{U}^P_\epsilon(g) \otimes 2$ defined in the theorem 4. Then

$$(\pi^V_x \otimes \pi^W_y)(\mathcal{R}(a \otimes b)) = \pi^V_x(\tau_{xL(x,y)}(a)) \otimes \pi^W_y(\tau_{x-1}^{-1}(b))$$ (9)

where $x_L(x,y) = x - yx^{-1}$

Proof. This proposition follows from the definition of $\mathcal{R}$ and from results of [WX].

For generic formal $x$ and $y$ define the element $R^{V,W}(x,y) \in End(V \otimes W)[[x,y]]$ as

$$R^{V,W}(x,y) = (T(x_L(x,y) + |x)\otimes T(x_+|y)^{-1})(\pi^V_x \otimes \pi^W_y)(R)$$ (10)

Here $x$ and $y$ are formal coordinates on a formal neighborhood of 1 in $G$.

**Theorem 2** Linear maps (10) satisfy the holonomy Yang-Baxter equation:

$$R_{12}^{V,W}(x_{R}(x,xL(y,z)), x_R(y,z))R_{13}^{V,U}(x,xL(y,z))R_{23}^{W,U}(y,z) = R_{23}^{W,U}(xL(x,y), xL(xR(x,y),z))R_{13}^{V,U}(xR(x,y),z)R_{12}^{V,W}(x,y)$$ (11)

$$R_{12}^{V,W}(xR(x,xL(y,z)), xR(y,z))R_{23}^{W,U}(y,z) = R_{23}^{W,U}(xL(x,y), xL(xR(x,y),z))R_{12}^{V,W}(x,y)$$ (12)

Proof. We can choose linear isomorphisms $T(g|x)$ such that $T(g_1g_2|x)$ is proportional to $T(g_1|x)T(g_2|x)$ and

$$T(g^{-1}|x)T(g|gxg^{-1}) = 1$$ (13)

11
From Proposition 4 we can evaluate the both sides of the equation (7) in the tensor product of three $G^*_P$ evaluation representations. For the left side we have:

\[
(\pi^V_x \otimes \pi^W_y \otimes \pi^U_z)((R^{-1}_{12} \circ R_{13})(R_{23}) \cdot R^{-1}_{12}(R_{13}) \cdot R_{12}) = (14)
\]

\[
(\pi^V_x \otimes \pi^W_y \otimes \pi^U_z)(\tau_{x_L(x,x_L(y,z))_+} \otimes \tau_{x_L(y,z)_+})(R)_{12}(id \otimes \tau_{y_+})(R)_{13}R_{23} = (15)
\]

Similarly one can evaluate the right side:

\[
(\pi^V_x \otimes \pi^W_y \otimes \pi^U_z)((R_{23}^{-1} \circ R_{13})(R_{12})R_{23}^{-1}(R_{13})R_{23}) = (16)
\]

\[
(\tau^{-1}_{x_+} \otimes \tau^{-1}_{x_R(x,y)_+})(R)_{23}(\tau^{-1}_{x_L(x,y)_+} \otimes id)(R)_{13}R_{12} (17)
\]

where $x_R(x,y) = x_L(x,y)^{-1}xx_L(x,y)_+$.

The holonomy Yang-Baxter equation for linear maps (10) follows from the identities (8) and the identities for $T(x|y)$.

Thus we constructed solutions to formal holonomy Yang-Baxter equation. Let $S$ and $S'$ be two generic symplectic leaves in $G^*_P$. Linear operators (10) admit analytical continuation to sections of vector bundles over $S \times S'$.

5 Invariants of tangles

5.1 d-matrix

In the construction of invariants of tangles with flat connection given in [KR] an important role played linear operators $d^V(x) \in \text{End}(V)$ defined in terms of holonomy $R$-matrices as

\[
d^X(a) = (\text{tr} \otimes id)(P((R^a_1(a,i(a)^{-1})^{-1})^{t_1})
\]

Since we constructed holonomy $R$-matrices for irreducible $U^P_\epsilon(g)$-modules, we have such $d$ operators for each generic irreducible representation $(\pi^V_x, V)$.

Lemma 1 Let $(\pi^V_x, V)$ be an irreducible representation of $U^P_\epsilon(g)$ with generic $x \in G^*$, then

\[
d^V(x) = c_V(x)\pi^V_x(L_\rho)
\]

where $c_V(x)$ is a non-zero complex number and $\rho = 1/2 \sum_{\alpha \in \Delta_+} \alpha$.

Proof. For each pair of representations $(\pi^V_x, V)$ and $(\pi^W_x, W)$ of $U^P_\epsilon(g)$ we have:

\[
R_{12}^{XY}(x,y) = d_2(a)^{-1}(((R_{12}^{XY}(x,y)^{-1})^{t_2})^{-1})^{t_2})d_2(y)
\]
R_{12}^{X,Y}(a, b) = d_1(y)^{-1}((R_{12}^{X,Y}(a, b)^t)^{-1}t_1)^{-1})d_1(a) \tag{19}

These equations imply that for generic \(x \in G^*\) and a representation \((\pi^V, V)\) we have:

\[
\pi^V_x(S^2(a)) = d^V(x)\pi^V_x(a)d^V(x)^{-1}
\]

On the other hand from the definition of the antipode it is easy to see that

\[
S^2(a) = L_\rho aL_\rho^{-1}
\]

The lemma now follows from the Schur’s lemma.

This lemma implies that the invariant of a knot defined by the functor \(F\) constructed in [KR] is identically zero. The situation is similar to invariants studied in [Ro] (see also the references therein).

### 5.2 Invariants knots of string knots

Recall that a string knot is a tangle with one connected component and with two boundary components. If \(D_t\) is a diagram of a string knot a generic \(G\)-coloring of the whole diagram is determined by the corresponding \(G\)-coloring of one of its boundary component.

**Proposition 5** Let \(t\) is a string knot and \(F_V(t, x)\) is the value of the functor \(F\) on it. Here we assume that the lower and upper boundaries of \(t\) are \(G\)-colored by \(x\) and decorated by \(U_{\varepsilon}^M(\mathfrak{g})\)-module \((\pi^V_x, V)\). Then the element \(F_V(t, x) \in \text{End}(V)\) is invariant with respect to gauge transformations, i.e. \(F_V(t^g, g(x)) = F_V(t, x)\) where \(t^g\) is the result of the gauge action of \(g \in G\) (see [KR]) on the \(G\)-colored tangle \(t\) and \(g(x)\) is the result of the dressing action of \(g \in G\) on \(x\).

Proof. The gauge invariance of the functor \(F\) (see [KR]) implies that for any other representation \((\pi^W_x, W)\) of \(U_{\varepsilon}^M(\mathfrak{g})\) we have:

\[
(1 \otimes F_V(t^y+, y_+(x)))R_{21}^{V,W}(x_R(y, x), x_L(y, x))^{-1} = R_{21}^{V,W}(x_R(y, x), x_L(y, x))^{-1}(1 \otimes F_V(t, x))
\]

\[
(1 \otimes F_V(t^{y-}, y_-(x)))R_{12}^{W,V}(y, y_-(x))^{-1} = R_{21}^{W,V}(y, y_-(x))^{-1}(1 \otimes F_V(t, x))
\]

These equations with \(y = 1\) imply that elements \(F_V(t, x)\) are central. Same equations for generic \(y \in G\) imply that \(F_V(t^y, y(x)) = F_V(t, x)\).

For generic \(q\) the functor \(F\) (see [RT]) applied to a string knot defines an element of a completion of the center of \(U_q(\mathfrak{g})\). This element can be evaluated in a finite-dimensional
representation and up to a scalar factor (which is equal to the quantum dimension of the representation) coincides with the corresponding invariant of the knot obtained by closing the string knot. This means that the central element itself is not only an invariant of a string knot but is also an invariant of the knot obtained by closing the string knot.

One can argue that the same happens in our case. The "limit" of this central element when \( q \to \varepsilon \) according to the asymptotical behavior of the universal \( R \)-matrix \([R-2]\) has an essential singularity and a regular part. The essential singularity will give the invariant related to the Poisson \( R \)-matrix (see \([WX]\) and \([R-1]\)). The regular part will give the invariant discussed here. As it was explained above for generic \( q \) the central element which is the invariant of a string knot is also an invariant of the knot obtained by the closing the string knot. Therefore, we should expect the same for roots of 1. We will return to the detailed discussion of this question in a separate publication.

6 Conclusion

We constructed invariants of tangles with flat connections over the complement using representation theory of quantized universal enveloping algebras at roots of 1.

We conjecture that these invariants for \( SL_2 \) coincide with the invariants constructed in \([BB]\) in case when the 3-manifold is a complement to a tangle. When \( G = SL_2 \) and the flat connection is trivial they coincide with the invariant constructed in \([Ka]\).

Since the invariant for \( G = SL_2 \) and trivial flat connection in the complement gives the hyperbolic volume of the complement when \( l \to \infty \). It would be very interesting to describe the asymptotic of our new invariants in terms of corresponding geometrical invariants.

Let \( \phi \) be a flat connection in the complement to a tangle. We expect that in the limit \( \phi \to 1 \), where 1 is the trivial flat connection, our new invariant becomes the invariant constructed in \([RJ]\) for roots of 1 and reducible but indecomposable representations of dimension \( \ell^{\Delta_+} \). For \( SL_2 \) this gives the relation between the invariant constructed in \([Ka]\) and the Jones polynomial at roots of 1, which was observed in \([MM]\).

What has been done in our two papers is a first step in the larger program. Here we will outline of some further steps.

First question is whether there is a topological quantum field theory which can give a geometric description of these invariants. In case of Jones polynomials such phenomenological quantum field theory (Chern-Simons theory for compact simple Lie groups) was proposed by Witten and allowed to describe the invariants in geometrical terms. One can guess that appropriate version of Chen-Simons theory for complex simple Lie groups will describe the large \( l \) asymptotic of our invariants.
There is another description of our invariant in terms of triangulated manifolds which is based on "6j-symbols" for the category of modules over $\mathcal{U}_q(g)$. It generalizes the construction from [TV] [Ka] and [BB]. This construction also gives invariants of more general 3-manifolds with flat connections. We will do it in a separate publication.

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