Classification of Normal Modes for Multiskyrmions

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The normal mode spectra of multiskyrmions play a key role in their quantisation. We present a general method capable of predicting all the low-lying vibrational modes of known minimal energy multiskyrmions. In particular, we explain the origin of the higher multipole breathing modes, previously observed but not understood. We show how these modes may be classified according to the symmetry group of the static solution. Our results provide strong hints that the $N$-skyrmion moduli space, for $N > 3$, may have a richer structure than previously thought, incorporating $8N - 4$ degrees of freedom.

I. INTRODUCTION

The Skyrme model of nuclear physics describes baryons as topological solitons in a non-linear field theory of $\pi$-mesons. It has enjoyed qualitative success in describing both single nucleon properties and the nucleon-nucleon interaction. It is thus of interest to try to apply the model to larger nuclei. In theory, it should be possible to calculate the binding energies and gamma ray spectra of all nuclei, without needing to introduce additional free parameters. This is a very attractive prospect. However, solutions of the classical field theory must be quantised before any comparison to the real world can be made.

The Skyrme model is not renormalisable, so there is little hope of a full treatment as a quantum field theory. Instead, a semiclassical quantisation is usually attempted, based on a limited number of degrees of freedom about a given classical solution. The earliest efforts at quantisation included only zero mode collective coordinates: spin and isospin. For a single skyrmion, this produces a description of nucleons and the $\Delta$ resonance in modest agreement with experiment. However, for solutions of higher baryon number $B$, a simple collective coordinate quantisation includes effects of order $\hbar^2$, while ignoring effects of order $\hbar$. For this reason, there has recently been a growing acceptance that it is necessary to include at least some low-lying vibrational modes. This requires a better understanding of the structure and dynamics of multi-skyrmion configurations, at least in the immediate neighbourhood of the minimal energy solutions.

There has been considerable recent progress in this direction, based on Manton’s old idea of representing low energy solitonic excitations as motion on a finite dimensional moduli space. Studies by Leese et al. and Walet, employing an instanton approximation for the Skyrme fields, gave encouraging results. Recently, Barnes et al. directly computed the normal mode spectra of $B = 2$, 3 and 4 multiskyrmions. A remarkable structure emerged in all three spectra. The lowest frequencies corresponded to known attractive channel scatterings; the next mode up was the ‘breather’ (a trivial size fluctuation), followed by various higher multipole breathing modes.

Barnes et al. were also able to classify the vibrations according to the symmetry groups of the respective static solitons. Remarkably, the $B = 4$ vibrational modes below the breather fell into representations exactly corresponding to those for small zero-mode deformations of the BPS 4-monopole solution. The same phenomenon occurred for the low frequency vibrations of the deuteron. This led to further investigation of the connection between BPS monopoles and Skyrmions, which has now been understood by Houghton et al in terms of rational maps. In fact, Houghton et al. were able to show that the correspondence should continue for any baryon number (monopole charge), and were thus able to predict the lowest $4B - 7$ vibrational modes for $B = 3$ and 7. Their $B = 3$ predictions were then confirmed by Barnes et al.

The higher multipole breathing modes were not so well understood. In the current Letter, we shall present a simple geometrical explanation for these modes, which relies only on the symmetry of the static solutions. We predict $4B - 7$ such modes for a multiskyrmion of baryon number $B$, and show how these may be classified as representations of the symmetry group of the static soliton. Our predictions match exactly all known vibrations for $B = 2, 3$ and 4; plus we predict an additional triplet of modes for $B = 4$. We also make detailed predictions for $B = 5, 6$ and 7.

Our idea is extremely simple, but has far-ranging consequences. Perhaps the most startling of these is the implication that the $N$-skyrmion moduli space may be of a higher dimension than was previously thought. Interestingly, our theory is in agreement with the $B = 3$ results of Barnes et al., which already contradicted the “standard wisdom” in this regard. We are thus able to clarify an outstanding puzzle. Overall, when added to the already considerable progress made by Barnes et al., this represents a major new insight into the moduli space approach.
II. CLASSIFICATION OF MULTIPOLAR BREATHING MODES

Classical multiskyrmion solutions are now known up to baryon number $B = 9 \,[13,12]$. All display considerable symmetry: the $B = 1$ solution is spherical, while the deuteron has axial symmetry. For $B \geq 3$, the minimal energy multiskyrmions are polyhedra, with $2B - 2$ faces. For example, $B = 3$ is a tetrahedron and $B = 4$ a cube. Baryon density is peaked at the vertices of these polyhedra, and to a lesser extent along the edges joining them. Lines of zero baryon density run out from the origin through the midpoints of each face. These lines are rather special in another way. Consider the inverse map from the field space $SU(2)$ back to real space. In general, for a configuration of baryon number $B$, each point in field space maps to $B$ points in real space. But there are certain special field values which map to fewer points in real space. For a minimal energy multiskyrmion, the image of these points, which we shall refer to as the branch locus, corresponds to the zero baryon density lines (or branch lines) running through each face. Since there are $2B - 2$ faces, there are also $2B - 2$ branch lines. Note that for this purpose, a line of zero baryon density which runs straight through the origin counts as two branch lines.

Our idea is to consider vibrations of the branch locus, while leaving the baryon density distribution of the multiskyrmion roughly intact. We leave the branch locus fixed at the origin, but allow the locations at which the branch lines intersect a sphere of fixed radius to vary. There are $2B - 2$ branch lines, each of which has two angular degrees of freedom, giving a total of $4B - 4$ modes. Three of these will correspond to global rotation, so we are left with $4B - 7$ non-trivial vibrations. The latter clearly correspond to complex breathing motions: two branch lines moving towards each other will compress the baryon density between them, whereas motion away from baryon density peaks will cause expansion.

The non-trivial vibrational modes must lie in multiplets (of degenerate frequency), transforming under irreducible representations of the symmetry group of the static soliton. These representations can be found by decomposing the $(4B - 4)$-dimensional representation formed by the angles of the branch lines in spherical polar coordinates. The irreducible component(s) corresponding to the rotational zero modes can then be removed, leaving only the true vibrational modes. Note that while the spherical polar angles form the most convenient general definition of the $(4B - 4)$-dimensional representation, in practice it is usually possible to find a simpler parametrisation for the movement of the branch lines. This is the case for most examples we will consider here.

Let us now consider detailed predictions for individual multiskyrmions, beginning with the deuteron. The minimal energy $B = 2$ solution is a torus: with axial symmetry, plus a reflection symmetry in the plane of the torus. The symmetry group is $D_{\infty h}$, axial symmetry extended by inversion. The two branch lines occupy the axis of symmetry; take this to be the $z$-axis. Then the $x$- and $y$-axes at some points $z = \pm \alpha$ are equivalent to the $4$-dimensional angular basis. A simple computation reveals that the characters of this representation are uniformly zero, other than the identity with value $4$, and rotations by angle $\theta$ with value $4 \cos \theta$. So using the notation of $[8]$, we have two $2$-dimensional irreducible representations: $1^+$ and $1^-$. The former corresponds to rotations around axes perpendicular to $z$, so we are left with $1^-$ as a true vibration. This is exactly the mode found by $[8]$. It is a “dipole” breathing motion, where one side of the torus inflates while the other is compressed. The two-fold degeneracy corresponds to motion aligned along the $x$- or $y$-axes.

The classical $B = 3$ multiskyrmion is a tetrahedron; its branch lines pass through a (dual) tetrahedron. There is no obviously simple way to parametrise the $8$-dimensional angular representation in this case (although we have performed this calculation and checked the results below). However, the possible motions of the branch lines are quite limited, and are easy to see. One can have a dipole motion, where three of the branch lines move towards the fourth. There are four obvious directions for this motion, but exciting all four at once gives a trivial size fluctuation, so only three are independent. A quick check of the symmetry of this vibration under the tetrahedral group $T_d$ shows its representation to be $F_2$, using the notation of Hamermesh $[14]$. (We will use this notation throughout, except where otherwise explicitly noted.) It is also possible to split the branch locus into two pairs of lines, with the lines in each pair moving towards each other. This creates a “quadrupole” breathing motion, where two opposite edges of the multiskyrmion are compressed, while the other four inflate; and vice versa. This motion has an obvious threefold basis, but only two directions are independent, so the corresponding representation is $E$. Again, these two modes correspond to those observed by Barnes et al $[10]$.

The $B = 4$ multiskyrmion is a cube; symmetry group $O_h$. The branch lines are just the Cartesian axes. We label the irreducible representations by those of the group of rotations of a cube ($O$), with superscripts indicating parity. Decomposition of the $12$-dimensional branch line angle representation gives $F_1^+, F_1^-, F_2^+$ and $F_2^-$, all $3$-dimensional. The $F_1^+$ corresponds to rotational zero-modes. $F_1^-$ is a dipole breathing motion, and $F_2^+$ a quadrupole, similar to the modes described above for $B = 2$ and $B = 3$. Both of these modes were observed, with the right symmetries, by $[12]$. The remaining vibration, $F_2^-$, corresponds to a twisting motion: grab diagonally opposite corners of the cube, and twist them in opposite directions. This mode was not seen by $[12]$, but it may well have considerably higher energy than the other
two. Finding it would provide striking confirmation of our theory.

The next two multiskyrmions have rather less symmetry: \( D_{2d} \) and \( D_{4d} \) respectively. Pictures of them may be found in [2]. The \( B = 5 \) solution has four square and four pentagonal faces. The branch lines form two slightly distorted tetrahedral configurations; see Figure 1 for a schematic diagram. The symmetry group \( D_{2d} \) maps each of these “tetrahedra” to themselves; they cannot be interchanged. The 16-dimensional angular representation can therefore be reduced to two identical 8-dimensional representations. Each of these can then be decomposed into \( A_1, A_2, B_1, B_2 \) and \( E \) (twice). The rotational zero modes are \( E \) and \( A_2 \). Again, we have dipole and quadrupole type breathing motions. Since the \( B = 5 \) multiskyrmion is slightly elongated about one axis, both the dipole and quadrupole motions split into a singlet and a doublet: \( B_2 \) and \( E \) for the dipole, \( A_1 \) and \( E \) for the quadrupole. There are three 1-dimensional twisting motions, two \( B_1 \)'s and an \( A_2 \). The remaining motions correspond to two groups of four axes each vibrating in a dipole or quadrupole, but with the two groups out of phase. Hence the remaining \( B_2 \) is an axial dipole anti-dipole, \( E \) a transverse dipole anti-dipole, and \( A_1 \) an axial quadrupole anti-quadrupole.

The \( B = 6 \) multiskyrmion has two square faces and eight pentagonal faces. Arrange four pentagons around each square, then rotate one of the squares by 45° so that the two halves can be fitted together. This configuration, like \( B = 5 \), is slightly elongated in one direction. The symmetry group \( D_{4d} \) is not included in Hamermesh [4], so we write out its character table (see Table I) in order to define notation for the representations. Our standard decomposition of the 20-dimensional representation gives \( 1^+, 1^-, A^+, A^-, C \) (twice), \( B^+ \) (three times) and \( B^- \) (three times). The rotational modes to be discarded are \( A^+ \) and \( B^- \). The situation is now too complicated for us to be confident of identifying the kind of motion corresponding to each representation. However, we can make a few obvious assignments. There will be axial and transverse dipole motions \((1^- \text{ and } A^-)\); also axial and transverse quadrupole modes \((1^+ \text{ and } C)\). \( B^+ \) and the remaining \( C \) could be twisting modes. Bending or wobbling the \( z \)-axis (the branch lines passing through the two squares) while leaving the other axes fixed should give \( B^+ \) and \( B^- \) respectively. This leaves two modes unidentified; it will be interesting to examine them when they are computed.

Finally, we consider \( B = 7 \). This is a perfect dodecahedron, with symmetry group \( I_h \). We label the representations of this group by those of \( Alt_5 \), the even permutations of five objects, with superscripts to indicate parity. We make no attempt to identify the physical appearances of these modes. We can, however, predict their symmetries. Our decomposition gives \( 5^+, 5^-, 4^+, 4^- \), \( 3^+ \) and \( 3^- \). The triplet \( 3^+ \) corresponds to the discarded zero-modes. Interestingly, these representations correspond to those predicted by [1] for the \( 4B - 7 \) lower frequency scattering modes, although these authors do not give parity assignments. This also happens for the \( B = 3 \) tetrahedron, which we attribute to the self-duality of a tetrahedron. In all other cases, however, the \( 4B - 7 \) higher breathing modes have quite different symmetries to the \( 4B - 7 \) scattering modes.

### Table I. Character table for \( D_{4d} \)

| \( I \) | \( E \) | \( C_4 \) | \( C_2 \) | \( S_8 \) | \( S_4 \) | \( sC \) | \( sS \) |
|---|---|---|---|---|---|---|---|
| 1^+ | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1^- | 1 | 1 | 1 | -1 | -1 | 1 | 1 |
| \( A^+ \) | 1 | 1 | 1 | 1 | -1 | 1 | -1 |
| \( A^- \) | 1 | 1 | 1 | -1 | 1 | -1 | 1 |
| \( B^+ \) | 2 | 0 | -2 | \( \sqrt{2} \) | \(-\sqrt{2} \) | 0 | 0 |
| \( B^- \) | 2 | 0 | -2 | \(-\sqrt{2} \) | \( \sqrt{2} \) | 0 | 0 |
| \( C \) | 2 | -2 | 2 | 0 | 0 | 0 | 0 |

### III. DISCUSSION AND CONCLUSIONS

We have proposed a simple explanation of the origin of the higher multipole breathing modes observed in multiskyrmions; namely, that they correspond to vibrations of the branch locus, or lines of zero baryon density. Our results are summarised in Table I. We predict \( 4B - 7 \) such modes for a multiskyrmion of baryon number \( B \) (\( 4B - 6 \) for the deuteron). Together with the \( 4B - 7 \) lower frequency scattering modes, plus nine zero modes and one trivial breather, this gives a total of \( 8B - 4 \) modes \((8B - 3 \text{ for the deuteron})\). This would appear to resolve a long-standing “counting problem”.

A single skyrmion has 6 degrees of freedom, which leads to the naive expectation that an \( N \)-nucleon system should have \( 6N \) degrees of freedom, and hence be described by a \( 6N \)-dimensional moduli space. It can be argued that this dimensionality should be increased by one, since all Skyrme configurations have a trivial size fluctuation. Since minimal energy solutions for \( B > 1 \) are single large solitons, they therefore have a maximum of 9 zero modes. This gave rise to the hope that multiskyrmions would have exactly \( 6B - 9 \) low-lying vibrational modes; that these vibrations might in fact correspond directly to the “broken zero modes” of well-separated skyrmions. The results of Barnes et al for \( B = 2 \) and \( B = 4 \) seemed to support this notion; they found just exactly the right number of vibrations in each case. However, they found one too many mode for \( B = 3 \), and in a multiplet which prevented separating this “extra mode” from the others. Since the lower half of this spectrum perfectly matched the predictions of [1], however, it was hard to discard their results. We now predict exactly the multiplets Barnes et al observed. This is strong evidence against
a \( (6B + 1) \)-dimensional moduli space.

So what is going on? It would seem either that the moduli space approach is wrong, or that Skyrme configurations (for \( B \geq 3 \)) have \( 2B - 5 \) more degrees of freedom than was previously thought. This means that knowing the position and orientations of \( N \) skyrmions is not sufficient information to determine the field everywhere in space. What other structure could there be? One possibility is that the branch locus contains additional information. Another is that additional dynamical variables are required, for example arising from interactions between angular velocities of different skyrmions. Further speculation is beyond the scope of this Letter, but the current results certainly indicate that further investigation of the structure of the branch loci would be worthwhile.

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| \( B \) | Symmetry | Mode | Degeneracy | Description                        |
|-------|----------|------|------------|------------------------------------|
| 2     | \( D_{\infty h} \) | \( 1^- \) | 2          | dipole                             |
| 3     | \( T_d \) | \( F_2 \) | 3          | dipole                             |
|       |          | \( E \) | 2          | quadrupole                         |
| 4     | \( O_h \) | \( F_1^+ \) | 3          | dipole                             |
|       |          | \( F_2^- \) | 3          | quadrupole                         |
| 5     | \( D_{2d} \) | \( B_2 \) | 1          | axial dipole                        |
|       |          | \( E \) | 2          | transverse dipole                   |
|       |          | \( A_1 \) | 1          | axial quadrupole                    |
|       |          | \( E \) | 2          | transverse quadrupole               |
|       |          | \( B_1 \) | 1          | twist                              |
|       |          | \( B_1 \) | 1          | twist                              |
|       |          | \( A_2 \) | 1          | twist                              |
|       |          | \( B_2 \) | 1          | axial ‘anti-dipole’                 |
|       |          | \( E \) | 2          | transverse ‘anti-dipole’            |
|       |          | \( A_1 \) | 1          | axial ‘anti-quadrupole’             |
| 6     | \( D_{4d} \) | \( 1^+ \) | 1          | axial dipole                        |
|       |          | \( B^+ \) | 1          | transverse dipole                   |
|       |          | \( 1^+ \) | 1          | axial quadrupole                    |
|       |          | \( C \) | 2          | transverse quadrupole               |
|       |          | \( A^+ \) | 1          | twist                              |
|       |          | \( C \) | 2          | twist                              |
|       |          | \( B^+ \) | 2          | z-axis bend                        |
|       |          | \( B^- \) | 2          | z-axis wobble                      |
| 7     | \( I_h \) | \( 5^- \) | 5          | dipole                             |
|       |          | \( 5^+ \) | 5          | quadrupole                         |
|       |          | \( 4^+ \) | 4          |                                    |
|       |          | \( 4^- \) | 4          |                                    |
|       |          | \( 3^- \) | 3          |                                    |

**TABLE II.** Summary of Predictions for \( B = 2 \) to \( B = 7 \)

\[ \text{FIG. 1. Schematic diagram of branch lines of } B = 5 \text{ multi-skyrmion. Solid lines pass through square faces; dotted lines through pentagons.} \]

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