On the relative velocity distribution for general statistics and revised big-bang nucleosynthesis under Tsallis statistics

Motohiko Kusakabe\(^1\), Toshitaka Kajino\(^1,2,3\), Grant J. Mathews\(^2,4\), and Yudong Luo\(^2,3\)

\(^1\)School of Physics and Nuclear Energy Engineering, and International Research Center for Big-Bang Cosmology and Element Genesis, Beihang University 37, Xueyuan Rd., Haidian-qu, Beijing 100083 China
\(^2\)National Astronomical Observatory of Japan, 2-21-1 Osawa, Mitaka, Tokyo 181-8588, Japan
\(^3\)Graduate School of Science, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-0033, Japan and
\(^4\)Center for Astrophysics, Department of Physics, University of Notre Dame, Notre Dame, IN 46556, U.S.A.

(Dated: June 6, 2018)

The distribution function of the relative velocity in a two-body reaction of nonrelativistic uncorrelated particles is derived for general cases of given distribution functions of single particle velocities. The distribution function is then used in calculations of thermonuclear reaction rates. As an example, we take the Tsallis non-Maxwellian distribution, and show that the distribution function of the relative velocity is different from the Tsallis distribution. We identify an inconsistency in previous studies of nuclear reaction rates within Tsallis statistics, and derive revised nuclear reaction rates. Utilizing the revised rates, accurate results of big bang nucleosynthesis are obtained for the Tsallis statistics. It is more difficult to reduce the primordial \(^7\)Li abundance while keeping other nuclear abundances within the observational constraints. A small deviation from a Maxwell distribution can increase the D abundance and slightly reduce \(^7\)Li abundance. Although it is impossible to realize a \(^7\)Li abundance at the level of metal-poor stars, a significant decrease is possible while maintaining a consistency with the observed D abundance.

I. INTRODUCTION

Deviations of particle distribution functions from the Maxwell-Boltzmann (MB) distribution are often found in geophysical and astrophysical observations of systems out of equilibrium. For example, a power-law distribution has been observed in the electron spectra of the magnetosphere. The power-law distribution called the kappa-distribution is realized in Tsallis’s statistical model in which a generalization of the entropy is postulated to be \(S_q \equiv k (1 - \sum_{i=1}^{W} p_i^q)/(q - 1)\), where \(q\) is a real parameter, \(k\) is Boltzmann’s constant, and \(p_i\) are the probabilities of \(i\) with the total configuration number \(W\). See Ref. [3] for formulations of the Tsallis statistics and relations between the Tsallis distribution function and the power-law. The Tsallis distribution function for \(q < 1\) has been found to be realized by a special pattern of temperature fluctuation.

Usually we assume a MB distribution for nonrelativistic particles in calculations of thermonuclear reaction rates. Especially, during big bang nucleosynthesis (BBN) and stellar nuclear burning, the temperature is high enough that very frequent scatterings quickly realize the MB distribution of nuclei (e.g., [6, 7]). Therefore, the standard BBN (SBBN) theory is based upon the MB nuclear distribution. In nonstandard BBN models such as those involving the injection of nonthermal photons [8] and hadrons [9], however, the distribution function significantly deviates from a MB distribution, and the primordial nuclear abundances can be affected.

Recently Bertulani et al. [11] analyzed effects of the Tsallis distribution function for nuclear velocities on BBN. They found that the nuclear reaction rates strongly depend on the Tsallis parameter \(q\). They then concluded that the Tsallis parameter should be very close to unity which corresponds to the MB distribution in order to satisfy observational constraints on light element abundances. An extended parameter search was made with modifications to the 2-body reverse reaction rates taken into account [12]. It was found that a slightly softer spectra than the MB distribution results in a decrease of the \(^7\)Li abundance. Therefore, this non-MB distribution is a possible solution to the Li problem of metal-poor stars [13]. In the SBBN model, theoretical primordial abundances are consistent with observational constraints except for the Li abundance [14, 15]. The discrepancy between the theoretical prediction of the primordial Li abundance and astronomical observations of metal-poor stars [16, 17] is called the Li problem.

In this paper, we derive a new formulation of the relative velocity distribution function for general distributions for reacting nonrelativistic particles. In Sec. II the relative velocity distribution function is formulated, and an explicit function is shown for the case of Tsallis statistics. In Sec. III the relative velocity distribution is calculated for Tsallis statistics, and we show that our accurate result is different from the inconsistent one adopted in previous studies. In addition, thermonuclear reaction rates and primordial light element abundances are calculated under the Tsallis statistics. In Sec. IV we summarize this study.
II. MODEL

The thermal rate for the two-body reaction of nonrelativistic uncorrelated particles is given by

\[ \langle \sigma v \rangle = \int dv_1 f(v_1) \int dv_2 f(v_2) \sigma(E)v, \]  

(1)

where \( \sigma \) is the reaction cross section, \( v_i \) is the velocity vector of species \( i = 1 \) and \( 2 \), \( f(v_i) \) is the velocity distribution function of \( i \), \( v = |v_1 - v_2| \) is the relative velocity, and \( E = \mu v^2/2 \) is the center of mass (CM) energy.

We use the CM parameter transformations as follows:

\[ M = m_1 + m_2 \]  

(2)

\[ \mu = \frac{m_1 m_2}{m_1 + m_2} \]  

(3)

\[ v = v_1 - v_2 \]  

(4)

\[ V = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}. \]  

(5)

There is also a conservation of energy relation,

\[ m_1 v_1^2 + m_2 v_2^2 = MV^2 + \mu v^2. \]  

(6)

We use the variable transformations from \( v_1 \) and \( v_2 \) to \( v \) and \( V \) (e.g., [34]). We define the distribution function of the relative velocity \( v \) that satisfies

\[ \int dv_1 f(v_1) \int dv_2 f(v_2) = \int dV f(v_1) f(v_2) = \int dv f^{rel}(v). \]  

(7)

The distribution function \( f^{rel}(v) \) is then given by

\[ f^{rel}(v) = \int dV [f(v_1)f(v_2)]_v, \]  

(8)

where the quantity in brackets with the subscript \( v \) is estimated for a fixed \( v \).

A. the Maxwell-Boltzmann distribution

The MB distribution is given [4,11,12] by

\[ f_{MB}(v_i) = \left( \frac{m_i}{2\pi kT} \right)^{3/2} \exp \left( -\frac{m_i v_i^2}{2kT} \right). \]  

(9)

If both of the reacting particles are described by an MB distribution, i.e., \( f_{MB} \), the integration over the velocities is given by

\[
\int dv_1 f_{MB}(v_1) \int dv_2 f_{MB}(v_2) = (m_1m_2)^{3/2} \left( \frac{2\pi kT}{\mu^2} \right)^{3/2} \int dv \exp \left( -\frac{MV^2 + \mu v^2}{2kT} \right) dV.
\]  

(10)

Therefore, the CM distribution function of the relative velocity for the case of MB statistics has the familiar form

\[ f_{MB}^{rel}(v) = \left( \frac{\mu^3}{(2\pi kT)^{3/2}} \right) \exp \left( -\frac{\mu v^2}{2kT} \right). \]  

(11)

B. Tsallis distribution

The Tsallis distribution is, however, given by

\[ f_q(v_i) = B_q \left( m_i c^2/kT \right) \left( \frac{m_i}{2\pi kT} \right)^{3/2} \left[ 1 - (q - 1) \frac{m_i v_i^2}{2kT} \right]^{1/(q-1)}, \]  

(12)

where \( B_q \left( m_i c^2/kT \right) \) is a normalization constant determined from the requirement \( \int f_q(v_i) dv_i = 1 \). When both of the reacting particles are described by this distribution with a same \( q \) value, the product of the two distribution functions is given by

\[ f_q(v_1)f_q(v_2) = B_q \left( m_1 c^2/kT \right) B_q \left( m_2 c^2/kT \right) \left( \frac{m_1 m_2}{2\pi kT} \right)^{3/2} \times \left[ 1 - (q - 1) \frac{m_1 v_1^2}{2kT} \right]^{1/(q-1)} \times \left[ 1 - (q - 1) \frac{m_2 v_2^2}{2kT} \right]^{1/(q-1)} \]  

(13)

We then have the following transformation:

\[ I_q(v_1, v_2; m_1, m_2, T) = \left[ 1 - (q - 1) \frac{m_1 v_1^2}{2kT} \right]^{1/(q-1)} \left[ 1 - (q - 1) \frac{m_2 v_2^2}{2kT} \right]^{1/(q-1)} \]  

(14)

\[ = \left[ 1 - (q - 1) \frac{M v^2}{2kT} \right]^{1/(q-1)} \left[ 1 - (q - 1) \frac{\mu v^2}{2kT} \right]^{1/(q-1)} \]  

(15)

\[ = \left[ 1 - (q - 1) \frac{MV^2 + \mu v^2}{2kT} \right]^{1/(q-1)} \left[ 1 - (q - 1) \frac{m_1 m_2 v^2}{(2kT)^2} \right]^{1/(q-1)}. \]  

(16)
where $v_1^2$ and $v_2^2$ can be expressed as functions of $V$ and $v$ by

$$v_1^2 = V^2 + 2\frac{m_1^2}{M} V \cdot v + \frac{m_1^2}{M^2} v^2$$  \quad (17)$$

$$v_2^2 = V^2 - 2\frac{m_1}{M} V \cdot v + \frac{m_1^2}{M^2} v^2.$$  \quad (18)

The distribution function of the relative velocity for the Tsallis particles is then given by

$$f_q^{\text{rel}}(v) = \int [f_q(v_1)f_q(v_2)]_v dV.$$  \quad (19)

The integration ranges of $V$ and $v$ are derived as follows. For $q > 1$, the velocities for two species are limited to be

$$m_i v_i^2 \leq \frac{2kT}{q-1},$$  \quad (20)

The integration ranges are then given by

$$0 \leq V \leq \sqrt{\frac{2kT}{q-1}} \frac{\sqrt{m_1} + \sqrt{m_2}}{M}$$  \quad (21)

$$0 \leq v \leq v_{1,\text{max}} + v_{2,\text{max}} = \sqrt{\frac{2kT}{q-1}} \left( \frac{1}{\sqrt{m_1}} + \frac{1}{\sqrt{m_2}} \right).$$  \quad (22)

For a fixed $v$, the distribution function of $v$ is then transformed to

$$f_q^{\text{rel}}(v) = 2\pi \int_{-1}^{1} d\cos \theta \int_{0}^{V_{\text{max}}} V^2 dV f_q(v_1)f_q(v_2)$$

$$= 2\pi B_q(m_1 c^2/kT) B_q(m_2 c^2/kT) \frac{(m_1 m_2)^{3/2}}{(2\pi kT)^{3/2}} \int_{-1}^{1} d\cos \theta \int_{0}^{V_{\text{max}}} V^2 dV I_q(V, \cos \theta; m_1, m_2, T, v).$$  \quad (23)

where the function $I_q$ is given by

$$I_q(V, \cos \theta; m_1, m_2, T, v) = \begin{cases} 
1 - (q-1) \frac{m_1 v_1^2}{2kT} \right)^{1/(q-1)} & \left( 1 - (q-1) \right) \frac{m_1 v_1^2}{2kT} > 0 \text{ and } 1 - (q-1) \frac{m_2 v_2^2}{2kT} > 0 \\
0 & \text{(otherwise)}
\end{cases}$$  \quad (24)

$$v_1^2 = V^2 + 2\frac{m_1}{M} V v \cos \theta + \frac{m_1^2}{M^2} v^2.$$  \quad (25)

$$v_2^2 = V^2 - 2\frac{m_1}{M} V v \cos \theta + \frac{m_1^2}{M^2} v^2.$$  \quad (26)

This distribution function is manifestly different from that adopted in previous studies, which adopted exactly the same Tsallis function for single particle velocity distribution with the mass replaced by the reduced mass. Thus, there is an approximation in Eq. (17) of Ref. [11], in which $E_i E_2$ is erroneously replaced by $(\mu v^2/2)(MV^2/2)$. As a result, they separate the distribution functions of the relative velocity and the CM velocity. Their calculations below Eq. (17) and Eq. (13) which is based upon Eq. (17) are then inconsistent with momentum conservation in Tsallis statistics. Equations for reaction rates in a subsequent paper [12] [their Eqs. (3) and (4)] are based upon the same formulation [11]. Therefore, their results are also inconsistent. We find that while the forms of the distribution function are the same for $v_i$ and $v$ in the MB case, they are different in the case of general statistics.

The thermonuclear reaction rate for Tsallis statistics is then given by

$$\langle \sigma v \rangle = \int dv f_q^{\text{rel}}(v) \sigma v.$$  \quad (27)

When the particles are nonrelativistic, i.e., $x_i = m_i c^2/(kT) \gg 1$, the $B_q(x_i)$ factor does not depend on $x_i$. In the present case of the nuclear distribution during the BBN epoch, nuclei are nonrelativistic. In this case the distribution function of $v$ reduces to
\[ f_{q}^{\text{rel}}(v) = \frac{B_{q}^{2}}{(2\pi)^{2}} \left( x_{1}x_{2} \right)^{3/2} \int_{-1}^{1} d\cos \theta \int_{0}^{V_{\text{max}}} V^{2} dV I_{q}(V, \cos \theta; m_{1}, m_{2}, T, v), \]  

(28)

\[ I_{q}(V, \cos \theta; m_{1}, m_{2}, T, v) = \begin{cases} 
1 - (q - 1) \frac{x_{1}v_{1}^{2}}{2} & 1/(q-1) \left[ 1 - (q - 1) \frac{x_{2}v_{2}^{2}}{2c^{2}} \right] > 0 \text{ and } 1 - (q - 1) \frac{x_{2}v_{2}^{2}}{2c^{2}} > 0 \\
0 & \text{(otherwise)}
\end{cases} \]  

(29)

\[ v_{1}^{2} = V^{2} + 2 \frac{m_{2}}{M} V \cos \theta + \frac{m_{2}^{2}}{M^{2}} \theta^{2} \]  

(30)

\[ v_{2}^{2} = V^{2} - 2 \frac{m_{1}}{M} V \cos \theta + \frac{m_{1}^{2}}{M^{2}} \theta^{2}, \]  

(31)

where we adopt the natural units of \( h = k = c = 1 \).

We then normalize all velocity variables in terms of the thermal velocity as follows:

\[ y = \frac{V}{v_{\text{th}}}, \]  

(33)

\[ y_{\text{max}} = \frac{V_{\text{max}}}{v_{\text{th}}} = \frac{m_{1} \sqrt{m_{2} + m_{2} \sqrt{m_{1}}}}{\sqrt{q - 1} M^{3/2}} \]  

(34)

\[ r = \frac{v}{v_{\text{th}}}, \]  

(35)

\[ r_{1} = \frac{v_{1}}{v_{\text{th}}}, \]  

(36)

\[ r_{1}(y, \cos \theta; m_{1}, m_{2}, r)^{2} = y^{2} \frac{2 m_{m}}{M} y r \cos \theta + \frac{m_{1}^{2}}{M^{2}} r^{2}, \]  

(37)

\[ r_{2}(y, \cos \theta; m_{1}, m_{2}, r)^{2} = y^{2} - 2 \frac{m_{m}}{M} y r \cos \theta + \frac{m_{1}^{2}}{M^{2}} r^{2}, \]  

(38)

Using this transformation, we derive the general distribution function for Tsallis statistics, i.e.,

\[ f_{q}^{\text{rel}}(v) = \frac{B_{q}^{2}}{(2\pi)^{2}} \left( \frac{4m_{1}m_{2}}{\mu^{2}} \right)^{3/2} v_{\text{th}}^{3} \int_{-1}^{1} d\cos \theta \int_{0}^{y_{\text{max}}} y^{2} dy I_{q}(y, \cos \theta; m_{1}, m_{2}, r), \]  

(39)

\[ I_{q}(y, \cos \theta; m_{1}, m_{2}, r) = \left[ 1 - (q - 1) \frac{m_{1}r_{1}^{2}}{\mu} \right]^{1/(q-1)} \left[ 1 - (q - 1) \frac{m_{2}r_{2}^{2}}{\mu} \right]^{1/(q-1)}. \]  

(40)

C. Reduced equations

1. Tsallis statistics

We defined a thermal velocity \( v_{\text{th}} \) such that the distribution function for the relative velocity amplitude \( f^{\text{rel}}(v) \) for MB statistics is maximal at that velocity. The distribution function \( f(v) \) satisfies \( \int f(v) dv = \int f(v) dv \). The thermal velocity is given by

\[ v_{\text{th}} = \sqrt{\frac{2kT}{\mu}}. \]  

(32)

\[ f_{q}^{\text{rel}}(v) = \frac{B_{q}^{2}}{(2\pi)^{2}} \left( \frac{4m_{1}m_{2}}{\mu^{2}} \right)^{3/2} v_{\text{th}}^{3} \int_{-1}^{1} d\cos \theta \int_{0}^{y_{\text{max}}} y^{2} dy I_{q}(y, \cos \theta; m_{1}, m_{2}, r), \]  

(39)

\[ I_{q}(y, \cos \theta; m_{1}, m_{2}, r) = \left[ 1 - (q - 1) \frac{m_{1}r_{1}^{2}}{\mu} \right]^{1/(q-1)} \left[ 1 - (q - 1) \frac{m_{2}r_{2}^{2}}{\mu} \right]^{1/(q-1)}. \]  

(40)

2. The 1 particle Tsallis distribution for the reduced mass

Equation (12) with the mass replaced with \( \mu \) is given by

\[ f_{q}(v) = B_{q} \left( \frac{1}{v_{\text{th}}^{2}} \right)^{3/2} \left[ 1 - (q - 1) \frac{E}{kT} \right]^{1/(q-1)} \]  

(12)

\[ = B_{q} \frac{1}{\pi^{1/2} v_{\text{th}}} \left[ 1 - (q - 1) \frac{E}{kT} \right]^{1/(q-1)}. \]  

(41)

For comparison, the MB distribution function can be written as

\[ f_{\text{MB}}(v) = \frac{1}{\pi^{1/2} v_{\text{th}}} \exp \left( - \frac{E}{kT} \right). \]  

(42)
The quantity corresponding to \( I_q \) in the Tsallis statistical case for the MB case is given by

\[
I_{MB}(y; m_1, m_2, r) = \exp \left[ - \left( \frac{M}{\hbar^2} y^2 + r^2 \right) \right].
\]

(43)

We find that the normalized distribution functions \( v_{th}^3 f^{rel}(v) \) only depend on \( E/T \) and \( m_1 \) and \( m_2 \) for the Tsallis case: Eq. (39) and do not depend on \( T \) alone.

D. Reverse reaction rates

The detailed balance relations [31] between cross sections of forward and reverse reactions for \( 1(2,\gamma)3 \) and \( 1(2,3)4 \) are given by

\[
\sigma_{3(3,2)1}^{\gamma} = \frac{g_1 g_2}{(1 + \delta_{12}) g_3} \left( \frac{\mu_{12} E_{12}^2}{E_3^2} \right) \sigma_{1(2,\gamma)3}^{\gamma},
\]

(44)

\[
\sigma_{4(3,2)1}^{\gamma} = \frac{(1 + \delta_{12}) g_3 g_5 y_{m_1} m_2 E_{12}}{(1 + \delta_{12}) g_3 g_5 y_{m_3} m_4 E_{34}} \sigma_{1(2,3)4}^{\gamma},
\]

(45)

where the \( g_i \) are the statistical weights of the respective nuclear species \( i \), while \( \mu_{ij} \) and \( E_{ij} \) are respectively the reduced mass and the CM energy of the \( i+j \) system.

1. Photodisintegration reactions

Under the assumption that nuclei are nonrelativistic and that photons have a Planckian energy distribution, the photodisintegration rates do not depend on the nuclear distribution function. The photodisintegration rate is then given by

\[
\langle \sigma v \rangle = \int dE \sigma(E) f(E) v c,
\]

(46)

\[
f(E) = \exp \left[ \frac{E_{\gamma}^2}{E_{\gamma}^2} - 1 \right],
\]

\[
\sigma(E) = \frac{E_{\gamma}^2}{E_{\gamma}^2} \left[ \exp \left( \frac{E_{\gamma}}{E_{\gamma}/kT} \right) - 1 \right] \cdot \frac{1}{2 \zeta(3)(kT)^3},
\]

(47)

where \( \zeta(3) = 1.2021 \) is the Riemann zeta function of 3. As far as the photon distribution follows the Planck distribution, the photodisintegration rate is the same as that of the SBBN. Therefore, we adopt the standard rates.

2. Two-nuclear reactions

The thermal reaction rate for the reverse reaction of the type \( 4(3,2)1 \) is

\[
\langle \sigma v \rangle_{34} = \int d\v_3 f(\v_3) \int d\v_4 f(\v_4) \sigma_{34(3,2)1}(E_{34}) v_{34},
\]

(48)

where the subscript 34 indicates physical quantities of the \( 3+4 \) system, and the subscripts 3 and 4 indicate physical quantities of particles 3 and 4, respectively. When the distribution functions for all nuclei are the Tsallis distribution, this rate is reduced with Eqs. (39) and (40) to

\[
\langle \sigma v \rangle_{34} = \int d\v_{34} f^{rel}_{q}(\v_{34}) \sigma_{34(3,2)1}(E_{34}) v_{34}.
\]

(49)

We calculate reverse reaction rates using this equation and the detailed balance relation [Eq. (35)].

III. RESULTS

A. Relative velocity distribution

Figure 1 shows normalized distribution functions for the CM kinetic energy \( E \). Solid and dash-dotted lines are for Tsallis statistics, i.e., \( (A_1, A_2) = (1,1), (2,2) \) (dash-dotted line), \( (4,3), (3,2), (2,1), (3,1), \) and \( (7,1) \) from the top to the bottom, respectively. The dashed line corresponds to the erroneous distribution taken from the one-particle Tsallis distribution. The dotted line is the MB distribution. Upper vertical lines show the ratios \( E_{th}/E_{34} \) for the freeze-out of \(^{2}H\) and \(^{3}He \) nuclides (see Table I).
pared with the previously assumed function. The maximum velocity [Eq. (22)] is always larger than that of the one particle Tsallis distribution for the same reduced mass. Therefore, the extended high energy tails are realized in the exact distribution function.

B. Thermnuclear reaction rates

Table I shows the eleven important reactions of BBN (the first column), and references to the available cross section data that we adopted in this study (the second column).

We calculated the freeze-out temperature \( T_f \) for one chosen nuclide \( i \) participating in one reaction \( a \) which satisfies the freezeout condition that its abundance rate of change equals the cosmic expansion rate \( H(T) \), i.e.,

\[
H(T_i) = \frac{\langle \langle n_i \rangle \rangle_{i}}{n_i} = \frac{n_k n_l \langle \sigma v(T_i) \rangle_{kl}}{n_i},
\]

where \( n_j \) is the number density of nuclide \( j \), while \( k \) and \( l \) are nuclides in the initial state of the reaction \( a \), and \( \langle \sigma v(T) \rangle_{kl} \) is the average reaction rate as a function of \( T \). The third and fourth columns in Table I show the nuclide whose abundance freezes out and its freeze-out temperature, respectively. The fifth column shows the ratio of the peak energy to \( T_f \), where the peak energy has the largest contribution to the integrand in deriving the average reaction rate. We checked that the change in the peak energy is small if the Tsallis \( q \) value is not changed significantly, i.e., \( |q - 1| \lesssim 0.1 \).

Figure 2 shows contours of the functions \( I_q = 1.075 \) [Eq. (10)] (solid lines) and \( I_{MB} \) [Eq. (13)] (dashed lines) in the \( (y, cos \theta) \) plane for the \(^3\)He \(^4\)He system at \( T_9 = 0.4 \) for the energies of \( r = 1 \) (a) and 3 (b), respectively. From this figure, a difference in the contributions of parameter regions to the reaction rate between the Tsallis and MB cases is apparent. We considered the \(^7\)Be production reaction and its freeze-out temperature for this figure. In panel (a), it is seen that \( I_q \) values are hindered compared to the MB case. In addition, the \( I_q \) value significantly depends on the angle \( \theta \) between the nuclear momenta, which is different from the function \( I_{MB} \). Near parallel or antiparallel scatterings are hindered as seen from the curved contours of \( I_q \). In the regions of \( cos \theta \gtrsim -1 \) and \( cos \theta \lesssim 1 \), \( r_1^2 \) or \( r_2^2 \) becomes maximally large leading to small values of \( I_q \) [Eq. (10)]. For intermediate \( cos \theta \) values, both of \( r_1^2 \) and \( r_2^2 \) values are intermediate, and the hindrance of \( I_q \) is minimal. In panel (b), vertical dashed lines correspond to \( I_{MB} = e^{-10}, e^{-13}, \ldots, e^{-28} \), from the left to right, respectively, that is a hindrance factor depending on \( y \), exactly the same as those in panel (a). The solid lines show the contours of \( I_q = 1.075 = e^{-13}, e^{-18}, \) and \( e^{-28} \), respectively. Curvatures are larger than in panel (a), which indicate that the contribution from the intermediate \( cos \theta \) satisfying \( m_1 r_1^2 = m_2 r_2^2 \) is exclusively important.

Figure 4 shows the average rates for the reactions \(^2\)H\((d,p)\)^3H (upper panel) and \(^3\)He\((\alpha,\gamma)\)^7Be (lower panel) as a function of temperature \( T_9 \equiv T/(10^9 \text{ K}) \). The former reaction is one of two most important reactions for D destruction. We note that the reaction rate of the other reaction, i.e., \(^2\)H\((d,n)\)^3He is very similar to that of \(^2\)H\((d,p)\)^3H. The \(^3\)He\((\alpha,\gamma)\)^7Be reaction is the most important \(^7\)Be production reaction. Therefore, the two reactions in Fig. 3 determine the final freeze-out abundances of D and \(^7\)Be, respectively. The Tsallis parameter is set to \( q = 1.075 \) [12]. That value has been suggested to be an interesting value for which the Li problem is solved. The solid line is for the Tsallis statistics, while the dashed line is erroneously based on the one-particle distribution function. The dotted line corresponds to the MB distribution.

For this specific \( q \) value and the temperature range \( T_9 = [10,0.1] \), the \(^2\)H\((d,p)\)^3H reaction rate for the Tsallis statistics is smaller than that of the MB statistics. Since the distribution function at high energies is smaller than that of the MB function, the reaction rate is hindered by the Coulomb penetration factor. The difference between cases of the Tsallis statistics and the previously assumed function is small.

The reaction rate of \(^3\)He\((\alpha,\gamma)\)^7Be for the Tsallis statistics is also smaller than that of the MB statistics because of the more effective Coulomb suppression factor. We find a significant difference between the reaction rates of the Tsallis statistics and the previously assumed case. For a fixed CM energy, the Coulomb penetration factor is suppressed more than that of the reaction \(^2\)H\((d,p)\)^3H because of larger atomic numbers and reduced mass. Therefore, the thermal reaction rate is contributed from higher energies, where the difference in the distribution function \( f^{\text{TS}}(\alpha) \) between the Tsallis and the previous case is largest (see Fig. 1). The averaged reaction rates of \(^3\)He\((\alpha,\gamma)\)^7Be then differ more than those of \(^2\)H\((d,p)\)^3H.
FIG. 2. Contours of the functions $I_{q=1.075}$ and $I_{MB}$ in the $(y, \cos \theta)$ parameter plane for the $^3\text{He}+^4\text{He}$ system at $T_9 = 0.4$. Panel (a) corresponds to the energy $r = 1$. The solid and dashed lines from left to right show contours of $e^{-2}$, $e^{-5}$, ... $e^{-20}$, as labeled. Panel (b) is for $r = 3$. The solid lines are for $I_{q=1.075}$ values of $e^{-13}$, $e^{-18}$, and $e^{-28}$, respectively, while the dashed lines are for $I_{MB} = e^{-10}$, $e^{-13}$, ..., $e^{-28}$ from the left to right, respectively.

C. BBN calculation

We adopt the SBBN code [35, 36] and have updated reaction rates of nuclei with mass numbers $\leq 10$ using the JINA REACLIB Database [39] (December, 2014). The neutron lifetime is the central value of the Particle Data Group, $\tau = 880.2 \pm 1.0 \pm 1.0$ s [37]. The baryon-to-photon ratio is taken to be $(6.094 \pm 0.063) \times 10^{-10}$ calculated using a conversion [40] of the baryon density in the standard $\Lambda$CDM model (TT+lowP+lensing) determined from Planck observation of cosmic microwave background, $\Omega_{m,0} h^2 = 0.02226 \pm 0.00023$ [38]. For eleven important reactions of BBN, the reaction rates are calculated for the different distribution cases. Two-body reverse reaction rates are calculated with the detailed balance relation using Eqs. (45)–(47) and (49). Since the effect of the Planckian distribution is small [46], we approximate the distribution by an exponential.

Figure 4 shows the evolution of nuclear abundances as a function of $T_9$. $X_p$ and $Y$ are mass fractions of $^1\text{H}$ and $^4\text{He}$ in total baryon matter, respectively. For other nuclear abundances, the number ratios to $^1\text{H}$, i.e., $A/H$ are shown. In the upper panel, the solid lines are for the Tsallis statistics, while the dashed lines are for the previously assumed relative velocity distribution function. The dotted lines are results for the MB statistics, i.e., SBBN. The Tsallis parameter is set to $q = 1.075$. For $q > 1$, the high energy region of the velocity distribution functions of nuclei is suppressed. Therefore, rates of charged-particle reactions are significantly reduced since the cross sections are larger at high energies. For $q < 1$, the opposite situation is realized. On the other hand, rates of neutron reactions are unaffected because there is no Coulomb penetration factor.

In the upper panel of Fig. 4 since the reaction rate
of deuteron destruction via $^2\text{H}(d,p)^3\text{H}$ and $^2\text{H}(d,n)^3\text{He}$ is smaller than in SBBN (Fig. 3, upper panel), the freeze-out D abundance is larger [11]. Since $^3\text{He}$ and $^3\text{H}$ are produced via the same reactions, the higher D abundance results in higher production rates of $^3\text{He}$ and $^3\text{H}$. Therefore, the final abundances of $^3\text{He}$ and $^3\text{H}$ are higher than in SBBN. The neutron abundance is slightly larger at late times for which $T_9 \gtrsim 0.7$. This is because of the larger D abundance. The neutron abundance is determined from the relation of forward and reverse reactions of $^3\text{H}(n,\gamma)^3\text{H}$. The forward rate is slightly larger and the reverse rate is the same as that of the SBBN. The $^1\text{H}$ abundance is almost the same in the Tsallis case and SBBN. Therefore, after the D destruction freezes out at a higher level, the $n$ abundance is kept higher via the photodisintegration $^3\text{H}(\gamma,n)^3\text{H}$. The final $^7\text{Li}$ abundance is smaller than in SBBN because of the smaller reaction rate for $^3\text{He}((\alpha,\gamma)^7\text{Be}$ (Fig. 3, lower panel). Since the reaction rate is underestimated in the previous calculations [11, 12], our $^7\text{Be}$ abundance is larger than the previous estimate. The $^7\text{Li}$ destruction rate via $^7\text{Li}(p,\alpha)^4\text{He}$ is also smaller than in SBBN. Therefore, the freeze-out $^7\text{Li}$ abundance is larger.

The lower panel shows abundances for Tsallis statistics with $q = 0.9$ (dashed line), 1 (dotted), and 1.1 (solid), respectively. When the $q$-value increases, rates of charged-particle reactions are smaller. For the same reason explained for the left panel, the abundances of D, $^3\text{H}$, $^3\text{He}$, $n$, and $^7\text{Li}$ increase while the $^7\text{Be}$ abundance decreases.

Calculated BBN results are compared to observational constraints on light element abundances. Constraints on the primordial $^4\text{He}$ abundance come from observations of metal-poor extragalactic H II regions. We use two different determinations of $Y_p = 0.2446 \pm 0.0029$ [12] and $Y_p = 0.2551 \pm 0.0022$ [11]. The primordial D abundance is constrained with observations of metal-poor Lyman-α absorption systems towards quasi-stellar objects. We use the weighted mean value of $D/H = (2.527 \pm 0.030) \times 10^{-5}$ [13]. $^3\text{He}$ abundances in Galactic H II regions are determined using the 8.665 GHz hyperfine transition of $^3\text{He}^+$ ion. The primordial $^3\text{He}$ abundance can evolve during Galactic chemical evolution. However, the net effect of Galactic chemical evolution is uncertain since stars can both destroy and synthesize $^3\text{He}$. Moreover, it is not expected that the $^3\text{He}$ abundance has decreased significantly over Galactic history as this would require that a large fraction of Galactic baryonic material has been absorbed in stars that destroy $^3\text{He}$, while the present interstellar deuterium abundance limits the amount of astration to not more than about a factor of two. We then only adopt the $2\sigma$ upper limit from the abundance $^3\text{He}/H = (1.9 \pm 0.6) \times 10^{-5}$ [14] in Galactic H II regions. We also use the abundance $\log(7\text{Li}/H) = -12 + (2.199 \pm 0.086)$ derived by observations of Galactic metal-poor stars using a 3D nonlocal thermal equilibrium model [15].

Figure 3 shows calculated light element abundances as a function of the parameter $q$. Boxes show the $2\sigma$ observational limits on D/H and $^7\text{Li}/H$. The dashed and dotted lines show abundances of $^7\text{Be}$ and $^7\text{Li}$, respectively, immediately after the BBN. Long after the BBN, $^7\text{Be}$ nuclei electron capture to produce $^7\text{Li}$ nuclei. Therefore, the sum of $^7\text{Be}$ and $^7\text{Li}$ abundances becomes the primordial Li abundance. The $^3\text{He}$ abundance is predominantly contributed by $^3\text{He}$, plus a small abundance of $^3\text{H}$ produced during BBN (see Fig. 4) has been added. The vertical line is at $q = 1$ and corresponding to the SBBN case. The plotted range is allowed by the $2\sigma$ limit on the $^4\text{He}$ abundance of Ref. [12] and excluded by that of Ref. [11]. Also this region is allowed by the $2\sigma$ upper limits on $^3\text{He}/H$ [14].

The reasons for the abundance changes of D, $^3\text{He}$, $^7\text{Be}$, and $^7\text{Li}$ have been explained above. The one percent level of change for the $^4\text{He}$ mass fraction is caused by different

![Figure 4](https://example.com/figure4.png)
neutron abundances during the $^4\text{He}$ synthesis. For larger $q$, the D destruction rate is smaller and the D abundance is larger. As a result of the balance of forward and reverse reactions of $^4\text{He}(n,\gamma)^7\text{Be}$ (see above), the $n$ abundance is kept higher and more neutrons are lost by $\beta$-decay before $^4\text{He}$ synthesis is completed. The final $^4\text{He}$ abundance is therefore smaller.

It is seen that the D abundance increases and the $^7\text{Li}$ abundance decreases with increasing $q$ value. At $q \approx 1.01$–1.02, the theoretical result for the D abundance is consistent with the observation. On the other hand, for $q \gtrsim 1.055$, the $^7\text{Li}$ abundance agrees with the observation. However, in this parameter region, the D abundance is enhanced to above D/H = $3 \times 10^{-5}$, which requires an additional mechanism for later D destruction. Because of their fragility, deuterons can be destroyed easily if there is a source of nonthermal photons in the early universe (e.g., $^4\text{He}$ [47]). Then, the D destruction by non-thermal photons can reproduce the primordial elemental abundances consistent with observations of all light nuclei in a model including a photon cooling by an axion [48], for example.

Unless a later D destruction mechanism is induced, the D enhancement is very problematic. Therefore, the observed D abundance places an upper limit on $q$. We find that in the range of $q \approx 1.01$–1.02 where the observed D abundance is reproduced, the $^7\text{Li}$ abundance is smaller than in the SBBN by 0.1–0.2 dex. This level of Li reduction is significant but not enough. On the other hand, there are other astrophysical processes which can further reduce the Li abundance, i.e., a chemical separation of $^7\text{Li}^+\text{ions}$ during structure formation [49] or a depletion on stellar surfaces [50, 52].

IV. SUMMARY

We formulated the thermal rate of two-body reactions of a gas of nonrelativistic uncorrelated particles with general velocity distribution functions. Taking the Tsallis distribution as an example of non-MB distribution, we derived the distribution function of the relative velocity, i.e., $f_q^{\text{rel}}(v)$. It was found that in general the distribution function $f_q^{\text{rel}}(v)$ contains a complicated integration over the CM velocity $V$. By defining a normalized distribution function $v_\text{th}^{\text{rel}}f_q^{\text{rel}}(v)$, the MB distribution can be expressed in terms of the ratio of the CM energy to the temperature, i.e., $E/T$. However, we found that the normalized distribution function for the Tsallis statistics has additional dependences on the nuclear masses (Sec. III).

We showed differences in the relative velocity distribution function between the Tsallis and MB statistics (Sec. IIIA). Using calculated distribution functions, reaction rates that are important for BBN were derived (Sec. IIIB). Finally, we performed the BBN nuclear reaction network calculation, and analyzed effects of changing the Tsallis $q$ parameter. An increase of $q$ results in softer nuclear spectra, and an upper cutoff of the CM energy appears for $q > 1$. We observed that the increase of $q$ reduces rates of reactions between charged particles, and explained reasons that abundances of D, $^3\text{He}$, $^7\text{Li}$, $n$, and $^7\text{Be}$ increase while the Li abundance decrease. Predicted abundances as a function of the parameter $q$ were calculated. We found the following points: (1) A slight deviation from the MB statistics, i.e., $q \approx 1.01$–1.02 can lead to D abundances consistent with observations, which are larger than the SBBN prediction; (2) The D observation provides the most stringent constraint on the $q$ parameter; (3) In that $q$ region, the primordial Li abundance is reduced from the SBBN value by 0.1–0.2 dex (Sec. III C).

ACKNOWLEDGMENTS

MK acknowledges support by the visiting scholar program of NAOJ during his stay there. This work was supported by Grants-in-Aid for Scientific Research of JSPS (15H03665, 17K05459). Work of GJM supported in part by DOE nuclear theory grant DE-FG02-95-ER40934 and in part by the visitor program at NAOJ.
[1] S. J. Bame, J. R. Asbridge, H. E. Felthauer, E. W. Hones, I. B. Strong, “Characteristics of the plasma sheet in the Earth’s magnetotail,” J. Geophys. Res. 72, 113 (1967).

[2] V. M. Vasyliunas, “A survey of low-energy electrons in the evening sector of the magnetosphere with OGO 1 and OGO 3,” J. Geophys. Res. 73, 2839 (1968).

[3] G. Livadiotis and D. J. McComas, “Beyond kappa distributions: Exploiting Tsallis statistical mechanics in space plasmas,” J. Geophys. Res. (Space Phys.) 114, A13 (2009).

[4] C. Tsallis, “Possible generalization of Boltzmann-Gibbs statistics,” J. Stat. Phys. 52, 479 (1988).

[5] G. Wilk and Z. Wlodarczyk, “On the interpretation of nonextensive parameter q in Tsallis statistics and Levy distributions,” Phys. Rev. Lett. 84, 2770 (2000).

[6] V. T. Voronchev, Y. Nakao and M. Nakamura, “Nonthermal processes in standard big bang nucleosynthesis. I: In-flight nuclear reactions induced by energetic protons,” JCAP 0805, 010 (2008).

[7] V. T. Voronchev, M. Nakamura and Y. Nakao, “Nonthermal processes in standard big bang nucleosynthesis: II: Two-body disintegration of D, Li-7, Be-7 nuclei by fast neutrons,” JCAP 0905, 001 (2009).

[8] D. Lindley, “Radiative decay of massive neutrons and cosmic element abundances,” Mon. Not. R. Astron. Soc. 188, 15P (1979).

[9] S. Dimopoulos, R. Esmailzadeh, L. J. Hall and G. D. Starkman, “Is the Universe Closed by Baryons? Nucleosynthesis With A Late Decaying Massive Particle,” Astrophys. J. 330, 545 (1988).

[10] M. H. Reno and D. Seckel, “Primordial Nucleosynthesis: The Effects of Injecting Hadrons,” Phys. Rev. D 37, 3441 (1988).

[11] C. A. Bertulani, J. Fuqua and M. S. Hussein, “Big Bang nucleosynthesis with a non-Maxwellian distribution,” Astrophys. J. 767, 67 (2013).

[12] S. Q. Hou, J. J. He, A. Parikh, D. Kahl, C. A. Bertulani, T. Kajino, G. J. Mathews and G. Zhao, “Non-extensive Statistics to the Cosmological Lithium Problem,” Astrophys. J. 834, no. 2, 165 (2017).

[13] A. Coc, J. P. Uzan and E. Vangioni, “Standard big bang nucleosynthesis and primordial CNO Abundances after Planck,” JCAP 1410, 050 (2014).

[14] R. H. Cyburt, B. D. Fields, K. A. Olive and T. H. Yeh, “Big Bang Nucleosynthesis: 2015,” Rev. Mod. Phys. 88, 015004 (2016).

[15] G. J. Mathews, M. Kusakabe and T. Kajino, “Introduction to Big Bang Nucleosynthesis and Modern Cosmology,” Int. J. Mod. Phys. E 26, no. 08, 1741001 (2017).

[16] F. Spite and M. Spite, “Abundance of lithium in unevolved halo stars and old disk stars: Interpretation and consequences,” Astron. Astrophys. 115, 357 (1982).

[17] S. G. Ryan, T. C. Beers, K. A. Olive, B. D. Fields and J. E. Norris, “Primordial Lithium and Big Bang Nucleosynthesis,” Astrophys. J. 530, L57 (2000).

[18] J. Meléndez and I. Ramirez, “Reappraising the Spite Lithium Plateau: Extremely Thin and Marginally Consistent with WMAP,” Astrophys. J. 615, L33 (2004).

[19] M. Asplund, D. L. Lambert, P. E. Nissen, F. Primas and V. V. Smith, “Lithium isotopic abundances in metal-poor halo stars,” Astrophys. J. 644, 229 (2006).

[20] P. Bonifacio et al., “First stars. 7. Lithium in extremely metal poor dwarfs,” Astron. Astrophys. 462, 851 (2007).

[21] J. R. Shi, T. Gehren, H. W. Zhang, J. L. Zeng, J. L. and G. Zhao, “Lithium abundances in metal-poor stars,” Astron. Astrophys. 465, 587 (2007).

[22] W. Aoki, P. S. Barklem, T. C. Beers, N. Christlieb, S. Inoue, A. E. G. Perez, J. E. Norris and D. Carollo, “Lithium Abundances of Extremely Metal-Poor Turn-off Stars,” Astrophys. J. 698, 1803 (2009).

[23] J. I. G. Hernandez, P. Bonifacio, E. Caffau, M. Steffen, H. -G. Ludwig, N. T. Behara, L. Sbordone and R. Cayrel et al., “Lithium in the Globular Cluster NGC 6397: Evidence for dependence on evolutionary status,” Astron. Astrophys. 505, L13 (2009).

[24] L. Sbordone, P. Bonifacio, E. Caffau, H. -G. Ludwig, N. T. Behara, J. I. G. Hernandez, M. Steffen and R. Cayrel et al., “The metal-poor end of the Spite plateau. 1: Stellar parameters, metalicities and lithium abundances,” Astron. Astrophys. 522, A26 (2010).

[25] L. Monaco, P. Bonifacio, L. Sbordone, S. Villanova and E. Pancino, “The lithium content of omega Centauri. New clues to the cosmological Li problem from old stars in external galaxies,” Astron. Astrophys. 519, L3 (2010).

[26] L. Monaco, S. Villanova, P. Bonifacio, E. Caffau, D. Geisler, G. Marconi, Y. Momany and H. -G. Ludwig, “Lithium and sodium in the globular cluster M4. Detection of a Li-rich dwarf star: preservation or pollution?,” Astron. Astrophys. 539, A157 (2012).

[27] A. Mucciarelli, M. Salari and P. Bonifacio, “Giants reveal what dwarfs conceal: Li abundance in lower RGB stars as diagnostic of the primordial Li,” Mon. Not. R. Astron. Soc. 419, 2195 (2012).

[28] W. Aoki, H. Ito and A. Tajitsu, “Examination of the mass-dependent Li depletion hypothesis by the Li abundances of the very metal-poor double-lined spectroscopic binary G166-45,” Astrophys. J. 751, L6 (2012).

[29] W. Aoki, “Li abundances in very metal-poor, main-sequence turn-off stars,” Memorie della Societa Astronomica Italiana Supplementi 22, 35 (2012).

[30] D. D. Clayton, “Principles of Stellar Evolution and Nucleosynthesis,” (University of Chicago Press, 1984).

[31] J. M. Blatt and V. F. Weisskopf, “Theoretical Nuclear Physics,” (Dover, New York, 1991).

[32] P. Descouvemont, A. Adahchour, C. Angulo, A. Coc and E. Vangioni-Flam, “Compilation and R-matrix analysis of Big Bang nuclear reaction rates,” Atom. Data Nucl. Data Tabl. 88, 203 (2004).

[33] A. Coc, P. Petitjean, J. P. Uzan, E. Vangioni, P. Descouvemont, C. Iliadis and R. Longland, “New reaction rates for improved primordial D/H calculation and the cosmic evolution of deuterium,” Phys. Rev. D 92, no. 12, 123526 (2015).

[34] S. Ando, R. H. Cyburt, S. W. Hong and C. H. Hyun, “Radiative neutron capture on a proton at BBN energies,” Phys. Rev. C 74, 025809 (2006).

[35] L. Kawano, “Let’s go: Early universe 2. Primordial nucleosynthesis the computer way,” NASA STI/Recon Technical Report N 92 25163 (1992).

[36] M. S. Smith, L. H. Kawano and R. A. Malaney, “Experimental, computational, and observational analysis of
primordial Astrophys. J. Suppl. 85, 210 (1993).

[37] C. Patrignani, & Particle Data Group 2016, “Review of Particle Physics,” Chin. Phys. C, 40, 100001 (2016).

[38] P. A. R. Ade et al. [Planck Collaboration], “Planck 2015 results. XIII. Cosmological parameters,” Astron. Astrophys. 594, A13 (2016).

[39] R. H. Cyburt et al., “The JINA REACLIB Database: Its Recent Updates and Impact on Type-I X-ray Bursts,” Astrophys. J. Suppl. Ser. 189, 240 (2010).

[40] H. Ishida, M. Kusakabe and H. Okada, “Effects of long-lived 10 MeV-scale sterile neutrinos on primordial elemental abundances and the effective neutrino number,” Phys. Rev. D 90, 083519 (2014).

[41] Y. I. Izotov, T. X. Thuan and N. G. Guseva, “A new determination of the primordial He abundance using the HeI 10830A emission line: cosmological implications,” Mon. Not. Roy. Astron. Soc. 445, 778 (2014).

[42] A. Peimbert, M. Peimbert and V. Luridiana, “The primordial helium abundance and the number of neutrino families,” Rev. Mex. Astron. Astrofis. 52, 419 (2016).

[43] R. J. Cooke, M. Pettini and C. C. Steidel, “One Percent Determination of the Primordial Deuterium Abundance,” Astrophys. J. 855, no. 2, 102 (2018).

[44] T. M. Bania, R. T. Rood and D. S. Balser, “The cosmological density of baryons from observations of $^3$He in the Milky Way,” Nature 415, 54 (2002).

[45] L. Sbordone et al., “The metal-poor end of the Spite plateau. I. Stellar parameters, metallicities, and lithium abundances,” Astron. Astrophys. 522, A26 (2010).

[46] G. J. Mathews, Y. Pehlivan, T. Kajino, A. B. Balantekin and M. Kusakabe, “Quantum Statistical Corrections to Astrophysical Photodisintegration Rates,” Astrophys. J. 727, 10 (2011).

[47] M. Kusakabe, A. B. Balantekin, T. Kajino and Y. Pehlivan, “Big-bang nucleosynthesis limit on the neutral fermion decays into neutrinos,” Phys. Rev. D 87, no. 8, 085045 (2013).

[48] M. Kusakabe, A. B. Balantekin, T. Kajino and Y. Pehlivan, “Solution to Big-Bang Nucleosynthesis in Hybrid Axion Dark Matter Model,” Phys. Lett. B 718, 704 (2013).

[49] M. Kusakabe and M. Kawasaki, “Chemical separation of primordial Li$^+$ during structure formation caused by nanogauss magnetic field,” Mon. Not. Roy. Astron. Soc. 446, 1597 (2015).

[50] O. Richard, G. Michaud and J. Richer, “Implications of WMAP observations on Li abundance and stellar evolution models,” Astrophys. J. 619, 538 (2005).

[51] A. J. Korn, F. Grundahl, O. Richard, L. Mashonkina, P. S. Barklem, R. Collet, B. Gustafsson and N. Piskunov, “Atomic Diffusion and Mixing in Old Stars. 1. VLT/FLAMES-UVES Observations of Stars in NGC 6397,” Astrophys. J. 671, 402 (2007).

[52] K. Lind, F. Primas, C. Charbonnel, F. Grundahl and M. Asplund, “Signatures of intrinsic Li depletion and Li-Na anti-correlation in the metal-poor globular cluster NGC6397,” Astron. Astrophys. 503, 545 (2009).