Kinematic constraints on spatial curvature from Supernovae Ia and Hubble parameter data

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Abstract

We suggest a model independent approach to estimate the spatial curvature of the Universe. We use three kinematic parametrizations: a third degree polynomial on $z$ for comoving distance $D_C(z)$, a second degree polynomial on $z$ for Hubble parameter $H(z)$ and a first order polynomial expansion on $z$ for the deceleration parameter $q(z)$. We used as SNe Ia dataset, the Pantheon compilation, consisting on 1048 estimates of SNe apparent magnitudes in the range $0.01 < z < 2.3$, with statistical and systematic errors and we have considered the measurements of the Hubble parameter $H(z)$ in different redshifts with 51 observed data. We have found for the model-independent spatial curvature, for comoving distance $(D_C)$, $\Omega_k = 0.11^{+0.21}_{-0.24} + 0.48$, for $H(z)$ parametrization, $\Omega_k = -0.03^{+0.21}_{-0.24} + 0.46$ and for $q(z)$, $\Omega_k = -0.05^{+0.21}_{-0.25} + 0.45$. The results are compatible with each other, confirming their model independent nature, and are consistent with an spatially flat Universe, as predicted by most inflation models and estimated by Cosmic Microwave Background (CMB) data.

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I. INTRODUCTION

The idea of a late time accelerating universe is indicated by Supernovae Type Ia (SNe Ia) observations \cite{1-8} and confirmed by other independent observations such as Cosmic Microwave Background (CMB) radiation \cite{9-11}, Baryonic Acoustic Oscillations (BAO) \cite{12-16} and Hubble parameter, $H(z)$, measurements \cite{17-19}. The simplest theoretical model supporting such an accelerating phase is based on a cosmological constant $\Lambda$ term \cite{20,21} plus a Cold Dark Matter component \cite{22-24}, the so called $\Lambda$CDM model. The cosmological parameters of such a model have been constrained more and more accurately as new observations are added \cite{11,18,25}.

Beyond a constant $\Lambda$ based model, several other models have been also suggested recently in order to explain the accelerated expansion. The most popular ones are based on a dark energy fluid endowed with a negative pressure filling the whole universe \cite{26,27}. The nature of such an exotic fluid is unknown and sometimes it is attributed to one or more scalar fields, in quintessence models \cite{28-32}. There are also modified gravity theories that correctly describe the accelerated expansion of the Universe, as massive gravity theories \cite{33, f(R) and f(T) theories, with $R$ and $T$ being the Ricci and torsion scalars, that generalizes the general theory of relativity \cite{34-36}, models based on extra dimensions, such as brane world models \cite{37-41}, string \cite{42} and Kaluza-Klein theories \cite{43}, among many others. Having adopted a particular model, the cosmological parameters can be determined with basis on statistical analysis of observational data. That is the recipe to study cosmology nowadays.

However, some works have tried to explore the history of the universe without appealing to any specific cosmological model. Such approaches are sometimes called cosmography or cosmokinetic models \cite{44-49}. Here we will refer to them simply as kinematic models. This nomenclature comes from the fact that the complete study of the expansion of the Universe (or its kinematics) is described just by the Hubble expansion rate $H = \dot{a}/a$, the deceleration parameter $q = -\ddot{a}/\dot{a}^2$ and the jerk parameter $j = -\dddot{a}/(a\dot{a}^3)$, where $a$ is the scale factor in the Friedmann-Roberson-Walker (FRW) metric. The only assumption is that space-time is homogeneous and isotropic. In such parametrization, a simple dark matter dominated universe has $q = 1/2$ while the accelerating $\Lambda$CDM model has $j = -1$. The deceleration
parameter allows to study the transition from a decelerated phase to an accelerated one, while the jerk parameter allows to study departures from the cosmic concordance model, without restricting to a specific model.

All these parametrizations help to reconstruct the Universe evolution without mentioning the dynamics, that is, without the use of Einstein’s Equations. Furthermore, by assuming only the Cosmological Principle, we may relate these parametrizations \((H(z), q(z))\) to spatial curvature and cosmological distances, like luminosity distance and angular diameter distance. So, by using distance data, like the ones provided by SNe Ia, one may constrain spatial curvature, without assuming any particular Cosmology dynamics. This was first shown by [50].

A first test of this method was done by Mörtsell and Clarkson [51]. By using only SNe Ia data and 3 parametrizations of \(q(z)\), namely, constant, piecewise and linear on \(a\), they have shown that the Universe is currently accelerating regardless of spatial curvature, but could not conclude about an early expansion deceleration. By combining SNe Ia data with BAO, they concluded that the Universe could have early deceleration only for a flat or open Universe \((\Omega_k \geq 0)\).

It has been shown that future 21 cm intensity experiments can improve model-independent determinations of the spatial curvature [52].

Yu et al. [53] have compiled 36 data of \(H(z)\), where 31 are measured by the chronometric technique, while 5 come from BAO (Baryon Acoustic Oscillations) observations. They use Gaussian Processes (GP) to determine the continuous function of \(H(z)\) with values of \(H_0, z_t\) and \(\Omega_k\) to test the model ΛCDM. The value found by him for \(H_0 \sim 67\pm4\,\text{km}\,\text{s}^{-1}\text{Mpc}^{-1}\). The profile of the \(H(z)\) function in terms of redshift \(z\) is used to estimate limits for the curvature parameter \(\Omega_k\). The work shows that the deceleration redshift found is in \(0.33 < z_t < 1.0\) to \(1\sigma\) of significance and the value of \(\Omega_k = -0.03\pm0.11\) which is consistent with a flat universe.

In the present work we study the spatial curvature by means of a third order parametrization of the comoving distance, a second order parametrization of \(H(z)\) and a linear parametrization of \(q(z)\). By combining luminosity distances from SNe Ia [54] and \(H(z)\) measurements [55], it is possible to determine \(\Omega_k\) values in these cosmological model-independent frameworks. In this kind of approach, we obtain an interesting complementarity between
the observational data and, consequently, tighter constraints on the parameter spaces.

The paper is organized as follows. In Section II, we present the basic equations concerning the obtainment of $\Omega_k$ from comoving distance, $H(z)$ and $q(z)$. Section III presents the data set used and the analyses are presented in Section IV. Conclusions are left to Section V.

II. BASIC EQUATIONS

In principle, the spatial curvature could not be constrained from a simple parametrization of the cosmological observables. However, as curvature relates to geometry, if one parametrizes the dynamics, the geometry can be constrained through the relation among distances and dynamic observables. To realize this, let us assume the validity of the Cosmological Principle, which leads us to the Friedmann-Robertson-Walker metric:

$$ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right].$$  

(1)

In this context, as explained, e.g., in [56] we may obtain the transverse line-of-sight distance, which is the distance between two objects in the Universe that remains constant with epoch if the two objects are moving with the Hubble flow. The line-of-sight comoving distance between an object at redshift $z$ and us is given by

$$d_C = d_H \int_0^z d\zeta', \frac{dz'}{E(\zeta')},$$  

(2)

where $d_H \equiv \frac{c}{H_0}$ is the Hubble distance and the dimensionless Hubble parameter $E(z) \equiv \frac{H(z)}{H_0}$. As all cosmological distances scale with $d_H$, we shall adopt the notation where a distance written in upper case ($D_i$) is dimensionless, while a distance written in lower case ($d_i$) is dimensionful and $d_i \equiv d_H D_i$. So, we may write

$$D_C(z) = \int_0^z d\zeta' \frac{dz'}{E(\zeta')}.$$  

(3)

From this we may obtain the transverse comoving distance. The comoving distance between two events at the same redshift or distance but separated on the sky by some angle $\delta\theta$ is $d_M \delta\theta$ and the transverse comoving distance is related to the line-of-sight comoving
distance as:

\[
d_M = d_H \begin{cases} \\
\frac{1}{\sqrt{\Omega_k}} \sinh \left[ \sqrt{\Omega_k} D_C \right] & \text{for } \Omega_k > 0, \\
D_C & \text{for } \Omega_k = 0, \\
\frac{1}{\sqrt{-\Omega_k}} \sin \left[ \sqrt{-\Omega_k} D_C \right] & \text{for } \Omega_k < 0.
\end{cases}
\]  

(4)

where we have used the curvature parameter density \( \Omega_k \equiv -\frac{k}{a_0^2 H_0^2} \). By defining the following function

\[
sinn (x, \Omega_k) \equiv \begin{cases} \\
\frac{1}{\sqrt{\Omega_k}} \sinh \left[ x \sqrt{\Omega_k} \right] & \text{for } \Omega_k > 0, \\
x & \text{for } \Omega_k = 0, \\
\frac{1}{\sqrt{-\Omega_k}} \sin \left[ x \sqrt{-\Omega_k} \right] & \text{for } \Omega_k < 0,
\end{cases}
\]

(5)

Eq. (4) can be simplified as

\[
d_M = d_H \sin (D_C, \Omega_k).
\]

(6)

The luminosity distance \( d_L \) is defined by the relationship between bolometric flux \( S \) and bolometric luminosity \( L \):

\[
d_L = \sqrt{\frac{L}{4\pi S}}.
\]

(7)

We may relate it to the transverse comoving distance by

\[
D_L(z) = (1 + z) D_M(z).
\]

(8)

We shall briefly mention the dynamics here just to show how the curvature density parameter definition emerges. As it is well known, the Friedmann equations can be written as:

\[
H^2 = \frac{8\pi G \rho_T}{3} - \frac{k}{a^2},
\]

(9)

\[
\frac{\dot{a}}{a} = \dot{H} + H^2 = -\frac{4\pi G}{3} (\rho_T + 3p_T),
\]

(10)

where \( \rho_T \) represents the total energy density and \( p_T \) the total pressure. As it can be seen, the spatial curvature contributes to the Hubble parameter through Eq. (9), while it does not contribute to acceleration (\( \ddot{a} \)) explicitly (10). The Friedmann equation shows that if we know the matter-energy content of the Universe, we can estimate its spatial curvature. This can be seen clearer if we rewrite Eq. (9) as

\[
\Omega_T + \Omega_k = 1
\]

(11)
where $\Omega_T \equiv \frac{8\pi G \rho_T}{3H^2}$ is the total energy density parameter and $\Omega_k \equiv -\frac{k}{a^2 H^2}$ is the curvature parameter.

Here, we intend to obtain constraints over spatial curvature without making any assumptions about the matter-energy content of the Universe. Thus, we shall assume kinematic expressions for the observables like $H(z)$, $q(z)$ and $D_C(z)$.

Assuming this kinematic approach, we can see that $H(z)$ data alone cannot constrain spatial curvature, but luminosity distances from SNe Ia can constrain it through the $\Omega_k$ dependence in Eq. (4). Concerning the deceleration parameter $q(z)$, it can be given as

$$q(z) = -\frac{\ddot{a}}{a H^2} = 1 + \frac{z}{H} \frac{dH}{dz} - 1,$$

(12)

So, as expected from Eq. (10), a $q(z)$ kinematical parametrization will not depend explicitly on spatial curvature, however, the spatial curvature can be constrained through the distance relation (4).

Therefore, from a formal point of view, we may access the value of $\Omega_k$ through a parametrization of both $q(z)$ and $H(z)$. As a third method we can also parametrize the line-of-sight comoving distance, which is directly related to the luminosity distance, in order to obtain the spatial curvature. In which follows we present the three different methods considered here.

A. $\Omega_k$ from line-of-sight comoving distance, $D_C(z)$

In order to put limits on $\Omega_k$ by considering the line-of-sight comoving distance, we can write $D_C(z)$ as a third degree polynomial such as:

$$D_C = z + d_2 z^2 + d_3 z^3,$$

(13)

where $d_2$ and $d_3$ are free parameters. From Eq.(2), we may write

$$E(z) = \left[ \frac{dD_C(z)}{dz} \right]^{-1}.$$

(14)

Naturally, from Eqs.(14) and (13), one obtains

$$E(z) = \frac{1}{1 + 2d_2 z + 3d_3 z^2}.$$

(15)
Finally, from Eqs. (8), (13) and (6) the dimensionless luminosity distance is

\[ D_L(z) = (1 + z) \sin n(z + d_2 z^2 + d_3 z^3, \Omega_k). \]  

(16)

Equations (15) and (16) shall be compared with \( H(z) \) measurements and luminosity distances from SNe Ia, respectively, in order to determine \( z_t \) and \( d_2 \).

B. \( \Omega_k \) from \( H(z) \)

In order to assess \( \Omega_k \) by means of \( H(z) \) we need an expression for \( H(z) \). If one wants to avoid dynamical assumptions, one must resort to kinematical methods which use an expansion of \( H(z) \) over the redshift.

Let us try a simple \( H(z) \) expansion, namely, the quadratic expansion:

\[ \frac{H(z)}{H_0} = E(z) = 1 + h_1 z + h_2 z^2. \]  

(17)

In order to constrain the model with SNe Ia data, we obtain the luminosity distance from Eqs. (8), (3) and (17). We have

\[ D_C = \int_0^z \frac{dz'}{E(z')} = \int_0^z \frac{dz'}{1 + h_1 z' + h_2 z'^2}, \]  

(18)

which gives three possible solutions, according to the sign of \( \Delta \equiv h_1^2 - 4 h_2 \), such as

\[ D_C = \begin{cases} \frac{2}{\sqrt{-\Delta} 2z} \left[ \arctan \left( \frac{2h_2 z + h_1}{\sqrt{-\Delta}} \right) - \arctan \frac{h_1}{\sqrt{-\Delta}} \right], & \Delta < 0, \\ \frac{h_1 z + 2}{\sqrt{\Delta}}, & \Delta = 0, \\ \frac{1}{\sqrt{\Delta}} \ln \left( \frac{\sqrt{\Delta} + h_1}{\sqrt{\Delta} - h_1} \right) \left( \frac{\sqrt{\Delta} - h_1 - 2h_2 z}{\sqrt{\Delta} + h_1 + 2h_2 z} \right), & \Delta > 0, \end{cases} \]  

(19)

from which follows the luminosity distance \( D_L(z) = (1 + z) \sin n(D_C, \Omega_k) \).

C. \( \Omega_k \) from \( q(z) \)

Now let us see how to assess \( \Omega_k \) by means of a parametrization of \( q(z) \). From (12) one may find \( E(z) \) as

\[ E(z) = \exp \left[ \int_0^z \frac{1 + q(z')}{1 + z'} dz' \right]. \]  

(20)
FIG. 1: a) SNe Ia apparent magnitude $m_B$ from Pantheon. The error bars shown correspond only to statistical errors, but we use the full covariance matrix (statistical+systematic errors) in the analysis. b) 51 $H(z)$ data compilation. The lines represent the best fit from SNe+$H(z)$ data for each model.

If we assume a linear $z$ dependence in $q(z)$, as

$$q(z) = q_0 + q_1 z,$$

which is the simplest $q(z)$ parametrization that allows for an acceleration transition as required by SNe Ia data, one may find

$$E(z) = e^{q_1 z}(1 + z)^{1+q_0 - q_1},$$

while the line-of-sight comoving distance $D_C(z)$ (3) is given by

$$D_C(z) = e^{q_1 q_0 - q_1} [\Gamma(q_1 - q_0, q_1) - \Gamma(q_1 - q_0, q_1 (1 + z))],$$

where $\Gamma(a, x)$ is the incomplete gamma function defined in [57] as $\Gamma(a, x) \equiv \int_x^\infty e^{-t} t^{a-1} dt$, with $a > 0$, from which follows the luminosity distance as $D_L(z) = (1 + z) \sinh(D_C; \Omega_k)$, which can be constrained from observational data.

III. SAMPLES

A. $H(z)$ data

In order to constrain these free parameters we have considered the measurement of the Hubble parameter $H(z)$ in different redshifts. These kind of observational data are quite
reliable because in general such observational data are independent of the background cosmological model, just relying on astrophysical assumptions. We have used the currently most complete compilation of $H(z)$ data, with 51 measurements [55].

Hubble parameter data as function of redshift yields one of the most straightforward cosmological tests because it is inferred from astrophysical observations alone, not depending on any background cosmological models.

At the present time, the most important methods for obtaining $H(z)$ data are\(^1\) (i) through “cosmic chronometers”, for example, the differential age of galaxies (DAG) [59–64], (ii) measurements of peaks of acoustic oscillations of baryons (BAO) [65–70] and (iii) through correlation function of luminous red galaxies (LRG) [71, 72].

Among these methods for estimating $H(z)$, the 51 data compilation as grouped by [55], consists of 20 clustering (BAO+LRG) and 31 differential age $H(z)$ data.

Differently from [55], we choose not to use $H_0$ in our main results here, due to the current tension among $H_0$ values estimated from different observations [73, 75].

B. SNe Ia

We choose to work with one of the largest SNe Ia sample to date, namely, the Pantheon sample [54]. This sample consists of 279 SNe Ia from Pan-STARRS1 (PS1) Medium Deep Survey (0.03 $< z < 0.68$), combined with distance estimates of SNe Ia from Sloan Digital Sky Survey (SDSS), SNLS and various low-$z$ and Hubble Space Telescope samples to form the largest combined sample of SNe Ia, consisting of a total of 1048 SNe Ia in the range of 0.01 $< z < 2.3$.

IV. ANALYSES AND RESULTS

In our analyses, we used flat priors over the parameters, so always the posteriors are proportional to the likelihoods. For $H(z)$ data, the likelihood distribution function is given

\(^1\) See [58] for a review.
by $\mathcal{L}_H \propto e^{-\frac{L}{2}}$, where

$$\chi^2_H = \sum_{i=1}^{51} \frac{(H_{\text{obs},i} - H(z_i, S))^2}{\sigma_{H_{\text{obs}}}}$$

(24)

As explained on [54], the PS1 light-curve fitting has been made with SALT2 [76], as it has been trained on the JLA sample [77]. Three values are determined in the light-curve fit that are needed to derive a distance: the color $c$, the light-curve shape parameter $x_1$ and the log of the overall flux normalization $m_B$.

The SALT2 light-curve fit parameters are transformed into distances using a modified version of the Tripp formula [78],

$$\mu = m_B - M + \alpha x_1 - \beta c + \Delta M + \Delta B,$$

(25)

where $\mu$ is the distance modulus, $\Delta M$ is a distance correction based on the host galaxy mass of the SN, and $\Delta B$ is a distance correction based on predicted biases from simulations. As can be seen, $\alpha$ is the coefficient of the relation between luminosity and stretch, while $\beta$ is the coefficient of the relation between luminosity and color, and $M$ is the absolute $B$-band magnitude of a fiducial SN Ia with $x_1 = 0$ and $c = 0$.

Differently from previous SNe Ia samples, like JLA [77], Pantheon uses a calibration method named BEAMS with Bias Corrections (BBC), which allows to determine SNe Ia distances without one having to fit SNe parameters jointly with cosmological parameters. Thus, Pantheon provide directly corrected $m_B$ estimates in order for one to constrain cosmological parameters alone.

The systematic uncertainties were propagated through a systematic uncertainty matrix. An uncertainty matrix $C$ was defined such that

$$C = D_{\text{stat}} + C_{\text{sys}}.$$  

(26)

The statistical matrix $D_{\text{stat}}$ has only a diagonal component that includes photometric errors of the SN distance, the distance uncertainty from the mass step correction, the uncertainty from the distance bias correction, the uncertainty from the peculiar velocity uncertainty and redshift measurement uncertainty in quadrature, the uncertainty from stochastic gravitational lensing, and the intrinsic scatter.
The $\chi^2$ function for Pantheon can then be given as

$$\chi^2 = \Delta m^T \cdot C^{-1} \cdot \Delta m,$$

where $\Delta m = m_B - m_{\text{mod}}$, and

$$m_{\text{mod}} = 5 \log_{10} D_L(z) + \mathcal{M},$$

where $\mathcal{M}$ is a nuisance parameter which encompasses $H_0$ and $M$. We choose to project over $\mathcal{M}$, which is equivalent to marginalize the likelihood $\mathcal{L} \propto e^{-\chi^2/2}$ over $\mathcal{M}$, up to a normalization constant. In this case we find the projected $\chi^2_{\text{proj}}$:

$$\chi^2_{\text{proj}} = S_{mm} - \frac{S_m^2}{S_A},$$

where $S_{mm} = \sum_{i,j} \Delta m_i \Delta m_j A_{ij} = \Delta m^T \cdot A \cdot \Delta m$, $S_m = \sum_{i,j} \Delta m_i A_{ij} = \Delta m^T \cdot A \cdot 1$, $S_A = \sum_{i,j} A_{ij} = 1^T \cdot A \cdot 1$ and $A \equiv C^{-1}$.

In order to obtain the constraints over the free parameters, we have sampled the likelihood $\mathcal{L} \propto e^{-\chi^2/2}$ through Monte Carlo Markov Chain (MCMC) analysis. A simple and powerful MCMC method is the so called Affine Invariant MCMC Ensemble Sampler by [79], which was implemented in Python language with the emcee software by [80]. This MCMC method has the advantage over simple Metropolis-Hastings (MH) methods of depending on only one scale parameter of the proposal distribution and on the number of walkers, while MH methods in general depend on the parameter covariance matrix, that is, it depends on $n(n+1)/2$ tuning parameters, where $n$ is dimension of parameter space. The main idea of the Goodman-Weare affine-invariant sampler is the so called “stretch move”, where the position (parameter vector in parameter space) of a walker (chain) is determined by the position of the other walkers. Foreman-Mackey et al. modified this method, in order to make it suitable for parallelization, by splitting the walkers in two groups, then the position of a walker in one group is determined by only the position of walkers of the other group\(^2\).

We used the freely available software emcee to sample from our likelihood in $n$-dimensional parameter space. We have used flat priors over the parameters. In order to plot all the constraints on each model in the same figure, we have used the freely available software

\(^2\) See [81] for a comparison among various MCMC sampling techniques.
FIG. 2: All constraints from Pantheon and $H(z)$ for $D_C(z) = z + d_2 z^2 + d_3 z^3$. The contours correspond to 68.3% c.l. and 95.4% c.l.

getdist$^3$, in its Python version. The results of our statistical analyses can be seen on Figs. 2-8 and on Table I.

In Figs. 2-4 we show explicitly the independent constraints, in order to see the comp-

\footnote{getdist is part of the great MCMC sampler and CMB power spectrum solver COSMOMC, by \cite{82}.}
FIG. 3: All constraints from Pantheon and $H(z)$ for $H(z) = H_0(1 + h_1 z + h_2 z^2)$. The contours correspond to 68.3% c.l. and 95.4% c.l.

The complementarity between SNe Ia and $H(z)$ data. First of all, as expected, SNe Ia does not constrain $H_0$. In SNe confidence level contours, $H_0$ is only limited by our prior, but $H(z)$ data gives good constraints over $H_0$. We can see also, that in general, SNe Ia alone does not constrain well $\Omega_k$, but by combining with $H(z)$, which constrain the other parameters, good constraints over the curvature are found. In the planes not containing $\Omega_k$ ($d_2 - d_3$, $h_1 - h_2$ and $q_0 - q_1$) we can see that $H(z)$ also helps to reduce a lot the allowed parameter space.

In Figs. 5-7, we have the combined results for each parametrization, where we can clearly see how the combination SNe Ia+$H(z)$ yield good constraints over $\Omega_k$, as well as the other kinematic parameters.
FIG. 4: All constraints from Pantheon and $H(z)$ for $q(z) = q_0 + q_1 z$. The contours correspond to 68.3% c.l. and 95.4% c.l.

For all parametrizations, the best constraints over the spatial curvature comes from $H(z)$ and $H(z)$ models, as can be seen on Fig. 8. We can also see that all constraints are compatible at 1$\sigma$ c.l.
FIG. 5: Combined constraints from Pantheon and $H(z)$ for $D_C(z) = z + d_2 z^2 + d_3 z^3$. The contours correspond to 68.3% c.l. and 95.4% c.l.

Table I shows the full numerical results from our statistical analysis.

Comparing with previous results in the literature, Li et al. [83] have combined 22 $H(z)$ data from cosmic chronometers with Union 2.1 SNe Ia data and JLA SNe Ia data. The combination with Union 2.1 yielded $\Omega_k = -0.045^{+0.176}_{-0.172}$ and they found $\Omega_k = -0.140^{+0.161}_{-0.158}$ from JLA combination. Wang et al. [84] have put model independent constraints over $\Omega_k$
FIG. 6: Combined constraints from Pantheon and $H(z)$ for $H(z) = H_0(1 + h_1 z + h_2 z^2)$. The contours correspond to 68.3% c.l. and 95.4% c.l.

and opacity from JLA SNe Ia data and 30 $H(z)$ data. They have used Gaussian Processes method and have obtained $\Omega_k = 0.44 \pm 0.64$, with a high uncertainty, due to degeneracy with opacity. It is worth to mention that, although model-independent, both [83] and [84] have followed a different approach from the present paper. They do not parametrize any cosmological observable, instead they obtain a distance modulus from $H(z)$ data, and compare with distance modulus from SNe Ia, which are dependent on spatial curvature. As already mentioned, Yu et al. [53] have used $H(z)$ and BAO, with the aid of Gaussian Processes and have found $\Omega_k = -0.03 \pm 0.21$, consistent with our results. By combining CMB data with BAO, in the context of $\Lambda$CDM, the Planck collaboration [85] have found
FIG. 7: Combined constraints from Pantheon and $H(z)$ for $q(z) = q_0 + q_1 z$. The contours correspond to 68.3% c.l. and 95.4% c.l.

$\Omega_k = 0.001 \pm 0.002$. It is consistent with our result, but it is dependent on the chosen dynamical model, $\Lambda$CDM.

Another interesting result that can be seen on Table II is the $H_0$ constraint. As one may see, the constraints over $H_0$ are consistent among the three different parametrizations, with a little smaller uncertainty for $D_C(z)$, $H_0 = 67.8 \pm 1.4$ km/s/Mpc. The constraints over $H_0$ are quite stringent today from many observations [85, 86]. However, there is some tension among $H_0$ values estimated from Cepheids [86] and from CMB [85]. While Riess et al. advocate $H_0 = 74.03 \pm 1.42$ km/s/Mpc, the Planck collaboration analysis, in the context of $\Lambda$CDM, yields $H_0 = 67.4 \pm 0.5$ km/s/Mpc, a $4.4\sigma$ lower value. It is interesting to note,
FIG. 8: Likelihoods for spatial curvature density parameter from Pantheon and $H(z)$ data combined. Blue solid line corresponds to $D_C(z)$ parametrization, orange long-dashed line corresponds to $H(z)$ parametrization and green dotted line corresponds to $q(z)$ parametrization.

from our Table I that, although we are working with model independent parametrizations and data at intermediate redshifts, our result is in better agreement with the high redshift result from Planck. In fact, all our results are compatible within 1σ with the Planck’s result, while, for the Riess’ result, our $D_C(z)$ result is incompatible at 3.1σ, and $H(z)$ and $q(z)$ are marginally compatible at 2.4σ.

V. CONCLUSION

In the present work, we wrote the comoving distance $D_C$, the Hubble parameter $H(z)$ and the deceleration parameter $q(z)$ as third, second and first degree polynomials on $z$, respectively (see equations (13), (17) and (21)), and obtained, for each case, the $\Omega_k$ value. We have shown that by combining Supernovae type Ia data and Hubble parameter measurements, nice constraints are found over the spatial curvature, without the need of assuming any particular dynamical model. Our results can be found in Figures 2-7. As one may see
| Parameter | $D_C(z)$ | $H(z)$ | $q(z)$ |
|-----------|----------|--------|--------|
| $H_0$     | 67.8 ± 1.4^{+2.9}_{-2.8} | 68.8 ± 1.7^{+3.4}_{-3.3} | 68.8 ± 1.7 ± 3.4 |
| $\Omega_k$ | 0.11^{+0.21+0.48}_{-0.24-0.44} | -0.03^{+0.21+0.46}_{-0.24-0.45} | -0.05^{+0.21+0.48}_{-0.25-0.45} |
| $d_2$     | -0.274 ± 0.019^{+0.038}_{-0.036} | - | - |
| $d_3$     | 0.0356 ± 0.0050^{+0.0097}_{-0.010} | - | - |
| $h_1$     | - | 0.523 ± 0.071^{+0.15}_{-0.14} | - |
| $h_2$     | - | 0.194 ± 0.023^{+0.044}_{-0.046} | - |
| $q_0$     | - | - | -0.440 ± 0.073^{+0.15}_{-0.14} |
| $q_1$     | - | - | 0.453 ± 0.062^{+0.12}_{-0.13} |

TABLE I: Constraints from Pantheon+$H(z)$ for $D_C(z)$, $H(z)$ and $q(z)$ parametrizations. The central values correspond to the mean and the 1 $\sigma$ and 2 $\sigma$ c.l. correspond to the minimal 68.3% and 95.4% confidence intervals.

from Figs. 2-4 the analyses by using SNe Ia and $H(z)$ data are complementary to each other, providing tight limits in the parameter spaces. As result, the values obtained for the spatial curvature in each case were $\Omega_k = 0.11^{+0.21}_{-0.24}$, $-0.03^{+0.21}_{-0.24}$ and $-0.05^{+0.21}_{-0.25}$ at 1$\sigma$ c.l., respectively (see Fig. 8), all compatible with an spatially flat Universe, as predicted by most inflation models and confirmed by CMB data, in the context of $\Lambda$CDM model.

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