NON-LOCAL SYMMETRIES
OF THE CLOSED N=2 STRING

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Abstract

By carefully analysing the picture-dependence of the BRST cohomology an infinite set of symmetry charges of the closed N=2 string is identified. The transformation laws of the physical vertex operators are shown to coincide with the linearised non-local symmetries of the Plebanski equation (which is the effective field theory of the closed N=2 string). Moreover, the corresponding Ward identities are powerful enough to allow for a rederivation of the well known vanishing theorem for the tree-level correlation functions with more than three external legs.
1 Introduction

There are hints that String/M-Theory has a large underlying symmetry whose improved understanding would certainly be a prerequisite for finding a general non-perturbative definition of the theory. A toy model that might be useful in this context is the closed $N=2$ string (general references are [1] - [3]). This is not a theory of realistic physics since local $(2,2)$ superconformal symmetry on the world sheet forces the target space to be a two complex dimensional Ricci-flat Kähler manifold. However, it has the remarkable property that contrary to most other string theories it possesses only a single massless scalar degree of freedom. Moreover, at tree level all its correlation functions can be calculated, either explicitly [5] or by the more sophisticated method of Berkovits and Vafa of embedding the theory into an $N=4$ topological string theory [6]. Both methods yield the result that all correlators beyond the three-point function vanish. This is certainly not a coincidence but suggests that a powerful target space symmetry must be at work. Such a connection is obvious from the point of view of the effective field theory, i.e. the field theory that reproduces all correlation functions (at tree level) of the string. In [1] it has been shown that this is a scalar theory which describes the deviation of the Kähler potential from flat space in suitably chosen coordinates. The corresponding equation of motion is known as the Plebanski equation. It possesses an infinite dimensional symmetry group (see [7] for a description and references) which, roughly speaking, is the loop group of symplectic diffeomorphisms in two real dimensions. This symmetry should also be present in the string theory and is surely in some implicit way contained in the approach of [6]. However, Berkovits and Vafa also stressed the importance of 'fleshing out this symmetry in a more conventional form'. This note is just an attempt in this direction.

Conventionally, unbroken symmetries in string theory should show up in the BRST cohomology at ghost number one. This is a rather general theorem of string field theory. The simplest example are target space translations whose charges can be constructed from the cohomology classes $c\partial X^\mu$. A more spectacular example is provided by two-dimensional string theory in a linear dilaton background (see [8] for a review and further references) where an infinite set of ghost number one cohomology classes were found at special values of the momenta. The corresponding charges were shown to form an infinite dimensional algebra in [9]. Moreover, it is important that these results have also been obtained by matrix-model techniques, which are completely independent from the BRST approach.

Despite some similarities between the 2D string and the $N=2$ string an analogous situation will certainly not hold in $N=2$ string theory in an uncompactified target space, for the very simple reason that the manifest rigid $SU(1,1)$

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1 Here we count as degrees of freedom the semi-relative cohomology classes at non-zero center-of-mass-momentum. For a different viewpoint, see [4].

2 In this paper we consider closed strings only and use the convention that physical states have ghost number two. Other conventions also exist, but what matters is that symmetries have one unit of ghost number less than physical states.
symmetry in target space rules out any phenomenon taking place at some dis-
tinguished non-zero value of the center-of-mass momentum. If symmetries show
up in the BRST cohomology they can do so only at zero momentum, which is
the case we should focus our attention at.

Theories of closed strings have the nice property that their Fock space factor-
ises into left-moving and right-moving sectors, which do not talk to each
other and are both isomorphic to the so-called chiral Fock space. Applying
the Künneth theorem to the closed string BRST operator (which is just the
sum of left- and right-moving parts) one easily sees that the BRST cohomology
also factorises. Thus, the most natural way to construct a ghost number one co-
homology class is to combine a left-moving ghost number one state (which
we know to exist since this is the ghost number of physical states in the chiral
cohomology) with a right-moving ghost number zero state, or vice versa. We
therefore arrive at the important conclusion that chiral cohomology classes of
ghost number zero signal the existence of symmetry charges for the closed string
theory. This has been pointed out most clearly in [9].

Summarising all the above, the first task in analysing the symmetries of
the \( N=2 \) string is to study the chiral cohomology for vanishing momentum
and ghost number. To do this we have to choose a picture (see [10] and the
appendix). Everything is very simple in the \((-1, -1)\) picture, which in many
aspects is the most natural. It is not hard to show that in this case the chiral
cohomology at ghost number zero is empty (the same happens in the \((0, -1)\)
and \((-1, 0)\) picture)! This seems disappointing at first but fortunately it is not
the full story: consider instead the \((0, 0)\) picture. Here, the ghost number zero
cohomology contains at least the \( sl(2) \) invariant ground state of the theory. This
state is certainly BRST invariant but not trivial, for otherwise we had serious
problems with our whole formalism. Thus, one sees that the zero-momentum
cohomology of the \( N=2 \) string is picture dependent [11]. An analogous property
for the Ramond sector of the \( N=1 \) string has been discussed in [12].

One may wonder what happens if one goes to still higher pictures. In this
case we do not know how to avoid direct computation of the cohomology which
becomes impractical very quickly\(^3\). The \((0, 1)\) and \((1, 0)\) pictures can, however,
be treated this way. One finds that their cohomologies contain two states each!
From a technical point of view this is the central result of our note. Since the
BRST cohomology is equipped with a natural multiplication law \( \square \), one can
take polynomials of the elements in the \((0, 1)\) and \((1, 0)\) cohomologies to create
more cohomology classes in higher pictures – a structure that is reminiscent of
Witten’s ground ring in 2D string theory. If one is willing to compare picture
number with Liouville momentum, this analogy becomes a rather close one;
recall that both of these quantum numbers are the momenta of a scalar field
coupled to a background charge. After a brief general introduction into \( N=2 \)
string theory in section 2 the detailed description of the cohomology will be

\(^3\)The usual method to relate the cohomologies at different pictures by the picture-changing
operation works for the \( N=1 \) string at zero momentum for the absolute, but not for the more
important relative cohomology [12]. In the \( N=2 \) theory it works neither for the relative nor
for the absolute cohomology [11].
presented in section 3.

Having found zero-momentum states at ghost number zero it is straightforward to combine them with ghost number one states from the same picture to form symmetry charges of the closed string. Using the formalism developed in the context of 2D string theory [13, 14] one then derives transformation laws for the physical vertex operators which can be compared with the symmetries of the Plebanski equation. This will be done in section 4. In section 5 we derive Ward identities, and it will be shown that they are strong enough to imply the vanishing of all tree-level amplitudes with more than three external legs. This constitutes an alternative proof of the vanishing theorem of [6]. It also unambiguously shows that the picture dependence of the cohomology is not just a bizarre side-effect of BRST quantisation, but can be used to obtain non-trivial information about the theory. In section 6 the results are summarised, and we make a couple of remarks concerning their interpretation. An appendix, finally, contains a summary of the chiral zero-momentum BRST cohomologies at ghost number zero and one and a brief description of the N=2 ghost system which plays an important role in all what follows.

2 The N=2 String

The N=2 string has a left- and a right-moving N=2 superconformal algebra as constraint algebra. The corresponding ghost system (see appendix) has central charge \(-6\) implying a critical dimension \(d = 4\). The underlying supergravity theory on the world sheet unfortunately requires the string coordinates to be complex so the target space is, in fact, two complex dimensional. A free field representation of the N=2 currents is

\[
T(z) = -\frac{1}{2} \partial \bar{Z} \cdot \partial Z - \frac{1}{4} \partial \psi^- \cdot \psi^+ - \frac{1}{4} \partial \psi^+ \cdot \psi^- ,
\]

\[
G^+(z) = \partial \bar{Z} \cdot \psi^+, \quad G^-(z) = \partial Z \cdot \psi^- ,
\]

\[
J(z) = \frac{1}{2} \psi^- \cdot \psi^+. \tag{1}
\]

Here \(Z^a, a = 0, 1\) are the complex string coordinates, \(\bar{Z}^\bar{a}\) their complex conjugates and \(\psi^{+a}, \psi^{-\bar{a}}\) the superpartners. 4

4Complex conjugation in target space and on the world sheet is denoted by a bar whereas antiholomorphic operators will be denoted by tilde. For the fermion fields a \(\pm\) index is used instead of a bar. This has the advantage that the world sheet \(U(1)\) charge of any field equals half the number of its \(\pm\) indices. The \(SU(1, 1)\) invariant scalar product is defined through \(\eta_{\bar{a} \bar{a}}\) with non-vanishing components \(\eta_{\bar{1} \bar{1}} = -\eta_{\bar{0} \bar{0}} = 1\). For example

\[
\partial Z \cdot \psi^- = \eta_{\bar{a} \bar{a}} \partial Z^\bar{a} \cdot \psi^{-\bar{a}} = -\partial Z^0 \cdot \psi^{-0} + \partial Z^1 \cdot \psi^{-1}.
\]

Parts of this notation are taken from 3

\[3\]
Using the operator product expansions

\[ Z^a(z)\bar{Z}^\bar{a}(w) \sim -2\eta^{\bar{a}a} \ln(z-w), \quad \bar{\psi}^a(z)\psi^{-\bar{a}}(w) \sim -2\frac{\eta^{\bar{a}a}}{z-w}, \quad (2) \]

one may check that the currents satisfy the \(N=2\) super Virasoro algebra with central charge \(c = 6\). Due to the spectral flow automorphism of this algebra it is no restriction to consider the NS sector only. Furthermore, one can consider the additional currents

\[ J^{++} = \frac{1}{4} \epsilon_{ab} \psi^{+a} \psi^{+b}, \quad J^{--} = -\frac{1}{4} \epsilon_{\bar{a}\bar{b}} \psi^{-\bar{a}} \psi^{-\bar{b}}, \quad (3) \]

\[ \hat{G}^+ = \epsilon_{ab} \partial Z^a \psi^{+b}, \quad \hat{G}^- = -\epsilon_{\bar{a}\bar{b}} \partial \bar{Z}^\bar{a} \psi^{-\bar{b}}. \quad (4) \]

(The antisymmetric \(\epsilon\) symbol is defined as \(\epsilon_{01} = \epsilon_{\bar{0}\bar{1}} = -\epsilon^{01} = -\epsilon^{\bar{0}\bar{1}} = 1\). Together with the currents (1) they satisfy the small \(N = 4\) super-conformal algebra [12]. In particular the \(J\)-currents form an affine \(SU(2)\) algebra:

\[ J(z)J^{\pm\pm}(w) \sim \pm \frac{2}{z-w} J^{\pm\pm}(w), \]
\[ J^{--}(z)J^{++}(w) \sim -\frac{1}{(z-w)^2} + \frac{1}{z-w} J(w), \quad (5) \]
\[ J(z)J(w) \sim \frac{2}{(z-w)^2}. \]

The only physical state of the theory is the massless ground state. In the \((-1,-1)\) picture the holomorphic part of the corresponding vertex operator is

\[ V_{-1,-1}(k,z) = ce^{\varphi^+} e^{-\varphi^-} e^{\frac{1}{2}(k \cdot \bar{Z} + \bar{k} \cdot Z)}(z), \quad k \cdot \bar{k} = 0. \quad (6) \]

Using picture-changed versions of this operator, it is not hard to calculate the three-point function \(A_3(k_i) = A_3(k_i) \delta(k_1 + k_2 + k_3)\) at tree level:

\[ \bar{A}_3(k_i) = (k_1 \cdot k_2 - k_2 \cdot k_1)^2. \quad (7) \]

All \(N\)-point functions with \(N > 3\) vanish at tree level [4, 5]. These amplitudes are reproduced by a scalar field \(\phi(Z, \bar{Z})\) with equation of motion

\[ \eta^{\bar{a}a} \partial_a \bar{\partial}_{\bar{a}} \phi = \frac{1}{2} \epsilon_{ab} \epsilon^{\bar{a}\bar{b}} \partial_a \bar{\partial}_{\bar{b}} \phi \partial_b \bar{\partial}_{\bar{b}} \phi. \quad (8) \]

The geometrical meaning of this equation can be understood [1] by considering the Plebanski equation,

\[ \epsilon_{ab} \epsilon^{\bar{a}\bar{b}} \partial_a \bar{\partial}_{\bar{b}} \Omega \partial_b \bar{\partial}_{\bar{b}} \Omega = -2, \quad (9) \]

which describes the Kähler potential \(\Omega(Z, \bar{Z})\) of a Ricci-flat Kähler metric in a suitably chosen coordinate system. The field \(\phi\) parametrises deviations of \(\Omega\) from flat space, since inserting the expression

\[ \Omega = \eta^{\bar{a}a} Z^a \bar{Z}^\bar{a} + \phi(Z, \bar{Z}) \quad (10) \]
into (9) yields (8). An obvious symmetry of these equations are the usual Kähler transformations.

Interestingly the Plebanski equation is equivalent to the consistency condition \([\mathcal{L}_0, \mathcal{L}_1] = 0\) of the linear system (10)
\[
\mathcal{L}_0 = \partial_0 + \lambda W_0, \quad \mathcal{L}_1 = \partial_1 + \lambda W_1
\]
with
\[
W_a = \bar{\epsilon}^{\bar{a} \bar{b}} \partial_a \partial_{\bar{a}} \Omega \partial_{\bar{b}}
\]
and an arbitrary complex parameter \(\lambda\). This structure is familiar from the theory of integrable models and usually leads to a large symmetry which will be further explored in subsection 4.3. It is the main purpose of this paper to investigate how these symmetries are realized in the \(N=2\) string.

3 Chiral BRST cohomology

As has been mentioned in the introduction and will be further explained in section 4, an important technical tool to study unbroken symmetries in string theory is the chiral BRST cohomology (i.e. the cohomology of only the left-moving part of the Fock space). There exists a powerful method to solve the cohomology problem for non-zero momentum [15, 16]. The result of this analysis for the \(N=2\) string [11] is that the chiral, relative (see below) cohomology contains precisely one physical state in each picture. This state has ghost number one, and in the \((-1, -1)\) picture it is represented by the vertex operator (6). Unfortunately, the method of [15, 16] fails for vanishing momentum. This is unimportant if one is only interested in the spectrum of the theory, since the behaviour of the dynamical degrees of freedom at isolated points in momentum space is irrelevant. For the symmetry structure, however, the zero-momentum cohomology is very important.

We do not know of a systematic method to completely determine the cohomology for zero momentum except for explicit computation, which is the subject of the present chapter. For this the following elementary and well-known observation is crucial: The spectrum of the zero modes \(L_0\) and \(J_0\) of the bosonic currents is discrete so that representatives of non-trivial cohomology classes can always be chosen to be annihilated by \(L_0\) and \(J_0\). To prove this one uses the relations
\[
\{Q, b_0\} = L_0, \quad \{Q, b'_0\} = J_0
\]
to show that a BRST-invariant state with non-vanishing eigenvalue is always trivial. These relations imply that the space of states involved in the cohomology problem can be further constrained by the requirement that all states be annihilated by \(b_0\) and \(b'_0\). We therefore need to consider only the relative Fock space \(F_{rel}\), defined as
\[
F_{rel} := \left\{ |\psi\rangle \mid L_0 |\psi\rangle = J_0 |\psi\rangle = b_0 |\psi\rangle = b'_0 |\psi\rangle = 0 \right\}.
\]
The BRST cohomology of this space is called relative cohomology; in this section, however, the term `relative' will be dropped. What happens to the cohomology when the $b_0$ and $b'_0$ conditions are relaxed is described in great detail in section 3 of [13].

Moreover, we only need to consider pictures $\pi^\pm \geq -1$. The other half of the cohomology can be found by Poincaré duality (see section 3 of [16], for example) once the first half is known.

### 3.1 Ghost number zero

Let us first consider the $(-1, -1)$ picture. At ghost number zero it is simply impossible to write down a state obeying the conditions in (14). The relative Fock space is empty and so is the cohomology. Similarly, one shows that there is no cohomology at ghost number zero in the $(0, -1)$ and $(-1, 0)$ picture.

In the $(0, 0)$ picture the situation is different. The relative Fock space contains two candidate states at zero ghost number, namely

$$|0, 0, k = 0\rangle$$  \quad \text{and}  \quad $$\begin{aligned} c_1 b'_{-1} &|0, 0, k = 0\rangle. \end{aligned}

(15)

The first of these states is just the $sl(2)$ invariant ground state which is BRST invariant, whereas the second state is not invariant. Since the relative Fock space at ghost number $-1$ is empty in this picture, we need not worry about the image of $Q$ and have thus proven that the ground state (or the unit operator in the language of vertex operators) spans the ghost number zero cohomology.

It is instructive also to consider the $(-1, 1)$ picture. Candidate states are

$$d_{-1/2}^{-\bar{a}} d_{-1/2}^{-\bar{b}} |-1, 1, k = 0\rangle,$$

$$c_1 b'_{-1} d_{-1/2}^{-\bar{a}} d_{-1/2}^{-\bar{b}} |-1, 1, k = 0\rangle,$$

(16)

where the Fourier modes of the matter fermions $\psi^\pm$ appear:

$$i\psi^{+a} = \sum_{r \in \mathbb{Z}+1/2} d^{+a}_r z^{-r-1/2},$$

$$i\psi^{-\bar{a}} = \sum_{r \in \mathbb{Z}+1/2} d^{-\bar{a}}_r z^{-r-1/2}.\quad (17)$$

A small calculation shows that the combination

$$\left(1 - c_1 b'_{-1} d_{-1/2}^{-\bar{a}} d_{-1/2}^{-\bar{b}} \right) -1, 1, k = 0\rangle$$

(18)

is invariant. Again there is no state at ghost number $-1$ so that (18) spans the cohomology. The corresponding vertex operator is

$$A(z) := (1 - c_1 b'_{-1}) J^- e^{\varphi^+} e^{-\varphi^-} (z).$$

(19)

In exactly the same way one shows that the cohomology in the $(1, -1)$ picture is represented by

$$\hat{A}(z) := (1 + c_1 b'_{-1}) J^+ e^{-\varphi^+} e^{\varphi^-} (z).$$

(20)

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As in [17] ghost number is defined to commute with picture number, see also the appendix.
The operators \( A \) and \( \hat{A} \) are nothing but spectral flow operators with spectral parameter 1 and \(-1\). They induce an isomorphism between the cohomologies at picture \((\pi^+, \pi^-)\) and \((\pi^+ - 1, \pi^- + 1)\). To show this one uses the fact that the BRST cohomology possesses a natural multiplication rule \([9]\): given two BRST invariant but non-trivial vertex operators \( O_1 \) and \( O_2 \), their normal ordered product

\[
(O_1 \cdot O_2)(w) = \oint \frac{dz}{2\pi i (z - w)} O_1(z) O_2(w)
\]

defines a new cohomology class. This product\(^6\) has the extremely important property that on cohomology classes it is graded commutative and associative \([14]\). Using \((5)\) one finds that the product of \( A \) and \( \hat{A} \) is

\[
A \cdot \hat{A} = 1 \quad \Rightarrow \quad \hat{A} = A^{-1}.
\]

Multiplication by \( A \) is thus an invertible map between the cohomologies at \((\pi^+, \pi^-)\) and \((\pi^+ - 1, \pi^- + 1)\) which shows that they are isomorphic. This is how the spectral flow automorphism of the \(N=2\) super Virasoro algebra acts on its BRST cohomology.

Let us now turn to the \((0,1)\) and \((1,0)\) pictures. What do we expect to find? Recall that there exist picture-changing operators \( X^\pm \). They act on physical states by the above multiplication rule \((21)\). According to Friedan, Martinec and Shenker \([10]\) they are constructed rather ingeniously as

\[
X^\pm = \{ Q, \xi^\pm \}\]

which are non-trivial since the zero modes of \( \xi^\pm \) are not part of the theory (bosonization of the \( \beta \) ghosts involves only \( \partial \xi^\pm \)). Their explicit form is

\[
X^\pm = -c_\pm \partial \xi^\pm \pm (G^\pm - 4G^\pm b \pm 2\delta^\pm \partial b') e^{\pm} + 4b' e^\pm \partial e^\pm.\]

\(X^+\) has picture number \((1,0)\) and \(X^-\) has \((0,1)\). The states \(X^\pm(z = 0)|0\rangle\) are ordinary cohomology classes with ghost number zero and vanishing momentum. There exists, however, an alternative way to construct a cohomology class in the \((0,1)\) picture, say: just consider the operator \(Y^- := A \cdot X^+\) \([18]\). Its explicit form is

\[
Y^- = \left[ c^+ e^+ - 4(1 - cb')(b + \frac{1}{2} \partial b') e^+ + 4b' e^+ \partial e^+ \right] J^-
\]

\[+ (1 - cb') e^+ \hat{G}^-.
\]

The important observation is that \(Y^-\) is BRST inequivalent to \(X^-\). If they were equivalent some linear combination of \(X^-\) and \(Y^-\) had to be trivial. But this cannot be the case since one may easily convince oneself that the relative Fock space in the \((0,1)\) picture at ghost number \(-1\) consists only of the state

\[6\] There is a little problem with this product if one considers general vertex operators with momentum \(k\), since then the OPE between the vertex operators typically contains singularities of the form \((z - w)^{k_1 k_2}\). Therefore the product makes sense only between operators whose momenta are constrained to have integer scalar product. In this paper we only consider the case where at least one operator involved has zero momentum and so this difficulty disappears.\]
$b'_1 d^{-\frac{3}{2}} | 0, 1, k = 0 \rangle$. Obviously the image of this state under $Q$ is not equal to a linear combination of $X^-(0)|0\rangle$ and $Y^-(0)|0\rangle$, which shows that they are inequivalent. Alternatively, one could find $X^-$ and $Y^-$ by explicitly writing down a basis of the relative Fock space at ghost number zero in this picture and look for BRST-invariant combinations. In this way one can prove that there are no further cohomology classes in this picture besides $X^-$ and $Y^-$. In complete analogy one shows that the cohomology in the $(1,0)$ picture is represented by $X^+$ and $Y^+ := X^- \cdot A^{-1}$.

It is instructive to work out how these operators act on physical states. Let us start with the vertex operator in the $(-1, -1)$ picture, given in (6), and define vertex operators in higher pictures as

$$V_{\pi^+, \pi^-}(k) = (X^+)_{\pi^+ + 1} \cdot (X^-)_{\pi^- + 1} \cdot V_{-1, -1}(k).$$

To see what $A$ and $A^{-1}$ do, consider the operators

$$V_{-1, 0}(k) = c k \cdot \phi^- e^{-\phi^-} e^{\frac{i}{2}(k \cdot \bar{Z} + \bar{k} \cdot Z)},$$
$$V_{0, -1}(k) = c \bar{k} \cdot \phi^+ e^{-\phi^+} e^{\frac{i}{2}(k \cdot \bar{Z} + \bar{k} \cdot Z)}.$$ (26)

Using the explicit expressions (19) and (20) it is easy to check that

$$A \cdot V_{0, -1}(k) = h(k) V_{-1, 0}(k), \quad A^{-1} \cdot V_{-1, 0}(k) = h(k)^{-1} V_{0, -1}(k)$$ (27)

with $h(k)$ defined as

$$h(k) = \frac{k^0}{k^1} = \frac{k^1}{k^0}. \quad (28)$$

Note that $|h| = 1$ so that $h^* = 1/h$. This particular function of the momenta features prominently in [6] (see also [19]) and will also be very important in what follows. In position space $h$ translates into a non-local expression, indicating that we are on the right track to discover the non-local symmetries of the Plebanski equation on the string theory side. Using the commutativity of the product (21) and the definition (25) one sees that $A$ and $A^{-1}$ act on all higher vertex operators as in equation (27).

Ghost number zero cohomology classes in higher pictures can now be constructed by simply considering positive powers of $X^\pm$ and integer powers of $A$. For a given picture $(\pi^+, \pi^-)$ one can write down $\pi^+ + \pi^- + 1$ operators:

$$O_{\pi^+, \pi^-, n} := (X^+)_{\pi^+ + n} \cdot (X^-)_{\pi^- - n} \cdot A^n, \quad n = -\pi^+, \ldots, \pi^-. \quad (29)$$

The range of $n \in \mathbb{Z}$ is restricted because negative powers of $X^\pm$ do not exist. The operators $O_{\pi^+, \pi^-, n}$ are all non-trivial. To see this assume that some linear combination of the $O_{\pi^+, \pi^-, n}$ were BRST trivial, i.e.

$$\sum_n \alpha_n O_{\pi^+, \pi^-, n} = \{Q, A\} \quad (30)$$
for some $\Lambda$ and suitably chosen coefficients $\alpha_n$. Then its product with any vertex operator $V(k)$ necessarily had to be trivial, as well. On the other hand one has, using (27),

$$\left( \sum_n \alpha_n O_{\pi^+, \pi^-} \right) \cdot V_{0,0}(k) = \left( \sum_n \alpha_n h(k)^n \right) V_{\pi^+, \pi^-}(k).$$

(31)

The right hand side can only be BRST trivial if the sum $\sum_n \alpha_n h(k)^n$ vanishes for any momentum $k$ with $k \cdot \bar{k} = 0$. This can certainly not be the case, which proves that any of the operators in (29) represents a distinct cohomology class. This does not prove, however, that the operators (29) span the full cohomology. It may well be that there exist additional cohomology classes that cannot be constructed this way.

In order to summarize the above findings let us introduce the notation $\pi = \pi^+ + \pi^-$. Then the following facts about the ghost number zero cohomologies at vanishing momentum hold:

- The cohomologies at picture $(\pi^+, \pi^-)$ and $(\bar{\pi}^+, \bar{\pi}^-)$ are isomorphic if $\pi = \bar{\pi}$ (this is just the spectral flow automorphism and is also true for other ghost numbers and any momentum).

- There is no cohomology for $\pi = -2$ or $\pi = -1$.

- There is one cohomology class for $\pi = 0$. For $(\pi^+, \pi^-) = (-n, n)$ this class is represented by the operator $A^n$.

- There are two classes for $\pi = 1$. For $(\pi^+, \pi^-) = (1 - n, n)$ they are represented by $X^+ \cdot A^n$ and $X^- \cdot A^{n-1}$.

- For $\pi > 1$ the cohomology contains the states of (29). All these states represent distinct cohomology classes but may not exhaust the full cohomology.

Analogous statements for $\pi < -1$ can be obtained by Poincaré duality.

### 3.2 Ghost number one

The zero momentum cohomology at ghost number one contains one state in the $(-1, -1)$ picture and two states in the $(-1, 0)$ and $(0, -1)$ pictures respectively [11]. In this paper, however, we are mainly interested in the case $\pi \geq 0$. In the $(0, 0)$ picture, the cohomology at ghost number one is spanned by the four operators [24]

$$-iP_{0,0}^a = c\partial Z^a - 2\gamma^-\psi^a, \quad -i\bar{P}_{0,0}^\bar{a} = c\bar{\partial}Z^{\bar{a}} - 2\gamma^+\psi^-\bar{a}.$$  

(32)

Multiplication with the operators of equation (23) yields new cohomology classes in higher pictures,

$$P_{\pi^+, \pi^-}^a \cdot O_{\pi^+, \pi^-} = O_{\pi^+, \pi^-} \cdot P_{0,0}^a,$$

$$\bar{P}_{\pi^+, \pi^-}^{\bar{a}} \cdot O_{\pi^+, \pi^-} = O_{\pi^+, \pi^-} \cdot \bar{P}_{0,0}^{\bar{a}}.$$  

(33)
For a given picture \((\pi^+, \pi^-)\) these are \(4(\pi + 1)\) states. To show that they all represent different cohomology classes we need the so-called bracket operation \([14]\), defined as

\[
\{O_1, O_2\}(w) := \oint w \frac{dz}{2\pi i} O_1^{(1)}(z) O_2(w) \tag{34}
\]

with

\[
O^{(1)}(w) := \oint w \frac{dz}{2\pi i} b(z) O(w). \tag{35}
\]

\(O_1\) and \(O_2\) are arbitrary operators and \(b\) is the anti-ghost field. In \([14]\) it has been shown that on cohomology classes the bracket operation is graded commutative although the operators \(O_1\) and \(O_2\) enter quite differently into the definition \((34)\). Moreover, it acts as a graded derivation on the normal ordered product in the cohomology:

\[
\{O_1, O_2 \cdot O_3\} = \{O_1, O_2\} \cdot O_3 + (-)^{(\langle O_1 \rangle | - 1)} O_2 \cdot \{O_1, O_3\}. \tag{36}
\]

As usual, the symbol \(|O|\) is zero if \(O\) is a bosonic operator and one if it is fermionic. The BRST cohomology together with the normal ordered product and the bracket operation has been called ‘Gerstenhaber algebra’ in \([14]\) where more details and references to the original work of Gerstenhaber can be found. Obviously the bracket carries ghost number \(-1\), so it is the appropriate operation for the ghost number one states \((33)\) to define an operator map within each sector of fixed ghost number. The operators in \((32)\) act on the physical vertex operators as momentum operators,

\[
\{P^a_{\pi^+ \pi^-}, V_{\pi^+ \pi^-}(k)\} = k^a V_{\pi^+ \pi^-}(k),
\]

\[
\{\bar{P}^{\dot{a}}_{\pi^+ \pi^-}, V_{\pi^+ \pi^-}(k)\} = \bar{k}^{\dot{a}} V_{\pi^+ \pi^-}(k). \tag{37}
\]

This can easily be shown by performing the computation explicitly in the \((-1, -1)\) picture and then using equations \((22)\), \((36)\) and the relations

\[
\{P^a_{\pi^+ \pi^-}, X^{\pm}\} = \{\bar{P}^{\dot{a}}_{\pi^+ \pi^-}, A\} = \{P^a_{\pi^+ \pi^-}, X^{\pm}\} = \{\bar{P}^{\dot{a}}_{\pi^+ \pi^-}, A\} = 0, \tag{38}
\]

which obviously hold since \(X^{\pm}\) and \(A\) are operators with vanishing momentum. Similarly, one shows that

\[
\{P^a_{\pi^+ \pi^- \pi^-, \pi^-}, \pi^+ \pi^- \pi^-(k)\} = h(k)^a k^a V_{\pi^+ \pi^- \pi^-(k)}.
\]

\[
\{\bar{P}^{\dot{a}}_{\pi^+ \pi^- \pi^-, \pi^-}, \pi^+ \pi^- \pi^-(k)\} = h(k)^{\dot{a}} k^{\dot{a}} V_{\pi^+ \pi^- \pi^-(k)}. \tag{39}
\]

With the same argument that established the BRST inequivalence of the operators in equation \((29)\) one now proves that the operators in \((33)\) all represent distinct cohomology classes. But, as in the ghost number zero case, there may well be more cohomology in higher pictures.
3.3 Ghost number two and higher

There are plenty of cohomology classes at arbitrarily high ghost number when \( \pi^+ + \pi^- \geq -1 \). They can be obtained by acting with \( X^\pm \) and \( A \) on the states \( A_N, B_N \) and \( C_N \) of equation (15) in [11], but their role within the symmetry structure of the theory is not clear to us.

4 Symmetries

We can now use the results of the previous section to study the symmetries of the \( N=2 \) string. The machinery to systematically analyse symmetries and Ward identities of a theory – once its BRST cohomology is known – has been developed in the context of 2D string theory. For completeness we briefly review some of the material following [13].

4.1 Some generalities

A current with components \( J_z \) and \( J_{\bar{z}} \) in a two dimensional theory (complex coordinates \( z \) and \( \bar{z} \)) is conserved, i.e. satisfies \( \bar{\partial} J_z + \partial J_{\bar{z}} = 0 \), when the one-form

\[
\Omega^{(1)} = J_z dz - J_{\bar{z}} d\bar{z}
\]

is closed. The corresponding charge

\[
\mathcal{A} = \oint_C \Omega^{(1)}
\]

is conserved if it has the same value for contours \( C \) and \( C' \) that are homologous, i.e. are the boundaries of some surface \( M, \partial M = C - C' \). Current conservation implies charge conservation by Stokes’ Theorem.

In BRST quantisation these relations are required to hold only up to BRST commutators. Current conservation then reads\footnote{Depending on whether two operators are both fermionic or not, one must consider their anti-commutator or commutator. In this section we generally denote this by \([\cdot,\cdot]\). It should always be clear from the context which is meant. We use this notation in order to avoid confusion with the Gerstenhaber bracket which, strictly speaking, is not an anticommutator.}

\[
d\Omega^{(1)} = [Q, \Omega^{(2)}]_{\pm}
\]

for some two form \( \Omega^{(2)} = \Omega^{(2)}_{\pm} dz \wedge d\bar{z} \). BRST invariance of the charge requires \([Q, \Omega^{(1)}]_{\pm} = d\Omega^{(0)} \). Applying \( Q \) to this relation implies that \([Q, \Omega^{(0)}]_{\pm} \) is a constant. Furthermore, this constant must vanish, since otherwise the unit operator were BRST trivial. Summarising, we have the descent equations

\[
\begin{align*}
[Q, \Omega^{(0)}]_{\pm} &= 0, \\
[Q, \Omega^{(1)}]_{\pm} &= d\Omega^{(0)}, \\
[Q, \Omega^{(2)}]_{\pm} &= d\Omega^{(1)}.
\end{align*}
\]
Moreover, the whole formalism is unaffected by the replacements

\[ \Omega^{(0)} \rightarrow \Omega^{(0)} + [Q, \alpha], \]
\[ \Omega^{(1)} \rightarrow \Omega^{(1)} + d\alpha, \quad (44) \]

where \( \alpha \) is some form of degree zero. The equations for \( \Omega^{(0)} \) are precisely the defining relations for the BRST cohomology. Taking as \( \Omega^{(0)} \) some cohomology class of ghost number \( g \) one can then construct the forms \( \Omega^{(1)} \) and \( \Omega^{(2)} \) of ghost numbers \( g - 1 \) and \( g - 2 \). It is most natural to choose \( \Omega^{(0)} \) to have ghost number one. This results in a charge of ghost number zero that can map physical states to physical states. In principle one could also consider charges of different ghost number (see [21] for a discussion), but they would annihilate the physical states.

The fact that symmetries sit at ghost number one can also be seen in a string field approach, in which the equation of BRST invariance \( Q|\Psi\rangle = 0 \) can be regarded as the linearized equation of motion for the string field \( |\Psi\rangle \).

Schematically, gauge symmetries have the form

\[ \delta|\Psi\rangle = Q|\Lambda\rangle + |\Psi \ast \Lambda\rangle + \text{higher order terms.} \quad (45) \]

Since the string field has ghost number two (it contains the physical states) the symmetry parameter \( |\Lambda\rangle \) has ghost number one. If \( |\Lambda\rangle \) is not killed by \( Q \) we have a gauge symmetry that starts with a field-independent term. Such a symmetry can often be used to simply gauge away some fields (see section 4 of [22]). It is not an unbroken symmetry of the chosen background. Moreover, a BRST-trivial parameter \( |\Lambda\rangle = Q|\Sigma\rangle \) leads to a trivial symmetry that cannot be used to derive conserved currents. We thus see again that ghost number one cohomology is the right tool for studying symmetries in string theory.

As a simple example let us consider target space translations in bosonic string theory. The only ghost number zero chiral cohomology class is the unit operator, which we may take as the right-moving piece of the closed string cohomology. As left-moving piece we must take a ghost number one state that also has vanishing momentum — the only candidate is \( c\partial X^\mu \). The forms \( \Omega^{(i)} \) take the form (suppressing the right-moving unit operator)

\[ \Omega^{(0)} = c\partial X^\mu, \quad \Omega^{(1)} = \partial X^\mu dz, \quad \Omega^{(2)} = 0. \quad (46) \]

The charge is just the center-of-mass momentum operator \( P^\mu = \oint dz \partial X^\mu \).

4.2 Transformation laws of physical states

In this section the formalism just described will be applied to the \( N=2 \) string. To construct the symmetry charges we first have to specify the ghost number one cohomology classes \( \Omega^{(0)} \) of the closed strings. Without loss of generality

\footnote{We have neglected the effects of the anti-ghost zero modes here. Taking them into account properly leads the semi-relative cohomology. We leave it to future work to do this for the \( N=2 \) string.}
one can take a chiral cohomology class of ghost number one as left-mover and one of ghost number zero as right-mover (denoted by \( \gamma \)). Exchanging left- and right-movers does not lead to anything new. Moreover left- and right-movers are chosen to have the same picture number. Using the expressions in equations (29) and (33) leads to
\[
\Omega^{(0)}_{\pi^+, \pi^-, m, n} = \mathcal{P}^{\alpha}_{\pi^+, \pi^-, m}(z) \tilde{O}_{\pi^+, \pi^-, n}(\tilde{z}), \quad m, n = -\pi^+, \ldots, \pi^-.
\] (47)

An analogous zero form can be constructed from \( \tilde{\mathcal{P}}^{\alpha} \). \( \Omega^{(1)} \) and \( \Omega^{(2)} \) can be found using the fact that, given a chiral cohomology class \( V \), the ‘current’ \( V^{(1)} \) defined as in equation (55) BRST-transforms into the derivative of \( V \),
\[
[Q, V^{(1)}]_\pm = \partial V.
\] (48)

It is then not hard to check that \( \Omega^{(0)} \) together with
\[
\Omega^{(1)}_{\pi^+, \pi^-, m, n} = -\mathcal{P}^{(1)}_{\pi^+, \pi^-, m}(z) \tilde{O}_{\pi^+, \pi^-, n}(\tilde{z}) dz + \mathcal{P}^{(1)}_{\pi^+, \pi^-, m}(z) \tilde{O}^{(1)}_{\pi^+, \pi^-, n}(\tilde{z}) d\tilde{z}
\] (49)
and
\[
\Omega^{(2)}_{\pi^+, \pi^-, m, n} = \mathcal{P}^{(1)}_{\pi^+, \pi^-, m}(z) \tilde{O}^{(1)}_{\pi^+, \pi^-, n}(\tilde{z}) dz \wedge d\tilde{z}
\] (50)
satisfy the descent equation.

Let us now compute how the symmetry charges
\[
\mathcal{A}_{\pi^+, \pi^-, m, n} = \oint_{2\pi i} dz \mathcal{P}^{(1)}_{\pi^+, \pi^-, m}(z) \tilde{O}_{\pi^+, \pi^-, n}(\tilde{z}) - \oint_{2\pi i} d\tilde{z} \mathcal{P}^{(1)}_{\pi^+, \pi^-, m}(z) \tilde{O}^{(1)}_{\pi^+, \pi^-, n}(\tilde{z})
\] (51)
act on the closed string vertex operators
\[
V_{\pi^+, \tilde{\pi}^-}(k, \tilde{z}, \tilde{z}) = V_{\pi^+, \tilde{\pi}^-}(k, z) \tilde{V}_{\pi^+, \tilde{\pi}^-}(k, \tilde{z})
\] (52)
(the holomorphic and antiholomorphic pieces are defined in eq. (51)). The calculation can be done by rewriting the contour integrals in terms of the product \( \mathcal{P} \) and the Gerstenhaber bracket \( \{ \cdot , \cdot \} \).

\[\delta_{\pi^+, \pi^-, m, n} V_{\pi^+, \tilde{\pi}^-}(k, w, \tilde{w}) := [\mathcal{A}_{\pi^+, \pi^-, m, n}, V_{\pi^+, \tilde{\pi}^-}(k, w, \tilde{w})]
\]
\[= \oint_{2\pi i} \mathcal{P}^{(1)}_{\pi^+, \pi^-, m}(z) V_{\pi^+, \tilde{\pi}^-}(w) \tilde{O}_{\pi^+, \pi^-, n}(\tilde{z}) \tilde{V}_{\pi^+, \tilde{\pi}^-}(\tilde{w})
\]
\[+ \oint_{2\pi i} \mathcal{P}^{(1)}_{\pi^+, \pi^-, m}(z) V_{\pi^+, \tilde{\pi}^-}(w) \tilde{O}^{(1)}_{\pi^+, \pi^-, n}(\tilde{z}) \tilde{V}_{\pi^+, \tilde{\pi}^-}(\tilde{w})
\]
\[= \{ \mathcal{P}^{(1)}_{\pi^+, \pi^-, m}, V_{\pi^+, \tilde{\pi}^-}(w) \} \tilde{O}_{\pi^+, \pi^-, n} \tilde{V}_{\pi^+, \tilde{\pi}^-}(\tilde{w})
\]
\[+ \mathcal{P}^{(1)}_{\pi^+, \pi^-, m} V_{\pi^+, \tilde{\pi}^-}(w) \{ \tilde{O}_{\pi^+, \pi^-, n}, \tilde{V}_{\pi^+, \tilde{\pi}^-}(\tilde{w}) \}.
\] (53)

\[\text{For notational simplicity the vector indices \( a \) and \( \bar{a} \) are suppressed in eqs. (49) - (53).}
\]
\[\text{A completely analogous computation which is described in some more detail is the derivation of equation (5.19) in [43].}
\]
The term in the fourth line is known from section 3 and the last term vanishes because of ghost number counting. To see this consider the expression $P_{\pi^+,\pi^-} V_{\pi^+,\pi^-}(k)(w)$ which is an operator with momentum $k \neq 0$ and ghost number two. However, from the general analysis it is known that there is no chiral cohomology at this ghost number. We thus proved that this operator must be BRST trivial. The final result then reads

$$\delta^a_{\pi^+,\pi^-} m V^\pi_\pi + \hat{m} V^\pi_\pi (k, w, \bar{w}) = h(k)^m n k a V^\pi_\pi + \hat{m} V^\pi_\pi (k, w, \bar{w}). \quad (54)$$

An analogous expression with $k^a$ replaced by $\bar{k}^\bar{a}$ also holds.

### 4.3 Symmetries of the Plebanski equation

In order to propose a possible interpretation of the transformations (54), we study in this subsection the symmetries of the Plebanski equations (8) and (9), hoping to find something resembling (54). The symmetry structure of self-dual gravity has been intensively investigated (see [11] for further references) and can be described in a precise and mathematically rigorous way in terms of twistor theory. However, this formalism is unnecessarily abstract for our purposes, and we prefer a more pedestrian but explicit approach. So instead of trying to describe the symmetries in full generality we only work out specific transformation laws, keeping in mind that they are part of a deep underlying mathematical structure.

Recall that the Plebanski equation

$$\epsilon^{ab} \epsilon^{\bar{a}\bar{b}} \partial_\bar{b} \partial_{\bar{b}} \Omega \partial_b \partial_{\bar{b}} \Omega = -2 \quad (55)$$

describes a Ricci-flat Kähler potential in a specific coordinate system. It is only invariant under antiholomorphic (or holomorphic) coordinate transformations generated by divergence-free vector fields with components $v^a = v^a(Z)$. $\Omega$ transforms as a scalar,

$$\delta \Omega = v^a \partial_a \Omega \quad \text{with} \quad \partial_a v^a = 0. \quad (56)$$

However, the Plebanski equation possesses more symmetries. Consider a vector field $\rho^a(Z, \bar{Z}) \partial_{\bar{a}}$ that still has vanishing divergence but can depend on both $Z$ and $\bar{Z}$. Then (55) is invariant under a transformation $\delta \Omega$ satisfying

$$\partial_a \delta \Omega = \rho^{a\bar{c}} \partial_{\bar{b}} \partial_b \Omega. \quad (57)$$

To check this statement we compute

$$\epsilon^{ab} \epsilon^{\bar{a}\bar{b}} \partial_\bar{b} \partial_{\bar{b}} \Omega \partial_b \partial_{\bar{b}} \Omega = \epsilon^{ab} \epsilon^{\bar{a}\bar{b}} \partial_\bar{b} \Omega \partial_b \partial_{\bar{b}} \Omega$$

$$= \epsilon^{ab} \epsilon^{\bar{a}\bar{b}} \partial_\bar{b} \Omega \partial_b \partial_{\bar{b}} \Omega + \frac{1}{2} \rho^{a\bar{c}} \partial_{\bar{b}} \partial_b \Omega. \quad (58)$$

The first term in the second line vanishes since $\epsilon^{\bar{a}\bar{b}} \partial_\bar{b} \rho^{\bar{c}}$ is symmetric in $\bar{b} \leftrightarrow \bar{c}$ and the second term vanishes due to the equation of motion. For antiholomorphic
components $\rho^\hat{a}(\bar{Z})$ the transformation (57) reduces to (56) and corresponds to a diffeomorphism. If however, the functions $\rho^\hat{a}$ also depend on the holomorphic coordinates, i.e. $\partial_a \rho^\hat{a} \neq 0$, the transformation (57) should not be thought of as due to a coordinate change of the underlying complex manifold. It has no direct connection to diffeomorphisms.

Of course, this is not the full story. (57) is a differential equation for $\delta \Omega$ which has a solution only if the right hand side obeys the consistency condition

$$0 = \epsilon^{ab} \partial_a \partial_b \delta \Omega = \epsilon^{ab} \partial_a \rho^\hat{a} \partial_b \partial_c \Omega. \tag{59}$$

An infinite set of solutions $\rho^\hat{a}_n$, $n = 0,1,2,\ldots$ can be found iteratively as follows. Let us assume there is a solution $\rho^\hat{a}_n$. Then one may verify that $\rho^\hat{a}_{n+1}$, defined as

$$\partial_a \rho^\hat{a}_{n+1} = \tilde{\partial}_a (\rho^\hat{a}_n \epsilon^{\hat{a} \hat{b}} \partial_b \partial_c \Omega), \tag{60}$$

is a new one. To complete the iterative description we have to provide a vector field $\rho^\hat{a}_0$, which we take to be purely antiholomorphic, i.e. $\partial_a \rho^\hat{a}_0 = 0$. This evidently satisfies (60). The transformations generated by the $\rho^\hat{a}_n$ via (57) are denoted by $\delta_n$ in the following. Then only $\delta_0$ is a gauge symmetry as in (56), while the other transformations are of the type (57). It is important to note that the determination of the $\rho^\hat{a}_n$ according to (60) involves integrating the right hand side. Thus $\delta_n$ is in general a non-local symmetry.

As already mentioned in section two, in order to compare the results of this section to the $N=2$ string we need to reformulate the symmetry structure in terms of the field $\phi$, defined as

$$\Omega(Z, \bar{Z}) = \eta_{\hat{a} \hat{b}} Z^\hat{a} \bar{Z}^\hat{b} + \phi(Z, \bar{Z}). \tag{61}$$

$\phi$ describes deviations of the Kähler potential from flat space. For convenience we recall its equation of motion

$$\eta^{\hat{a} \hat{b}} \partial_a \partial_b \phi = \frac{1}{2} \epsilon^{ab} \epsilon^{\hat{c} \hat{d}} \partial_a \partial_b \partial_c \partial_d \phi. \tag{62}$$

The transformations $\delta_n$ read in terms of $\phi$:

$$\partial_a \delta_n \phi = \rho^\hat{a}_n \tilde{\partial}_a \partial_c \Omega = \eta_{\hat{a} \hat{b}} \rho^\hat{b}_n + \rho^\hat{a}_n \tilde{\partial}_a \partial_d \phi, \tag{63}$$

and one may check that they leave (62) invariant. Moreover, (60) becomes

$$\partial_a \rho^\hat{a}_{n+1} = \epsilon^{\hat{a} \hat{b}} \partial_a \rho^\hat{b}_n + \partial_d (\rho^\hat{a}_n \epsilon^{\hat{c} \hat{b}} \partial_b \partial_d \phi) \tag{64}$$

The iterative construction just described is in fact well known from the theory of integrable models in two dimensions. To see this rewrite the Plebanski equation as $[W_0, W_1] = 0$ with $W_0$ from (12). Moreover, $W_0$ satisfies $\epsilon^{ab} \partial_a W_0 = 0$. These two equations are the equations of a two dimensional model derived from a Wess-Zumino action [23]. Such models are known to possess an infinite set of non-local symmetries that are constructed as above [24].
with $\epsilon^a\phi = \epsilon^a c_n$. It is instructive to work out the first few transformations explicitly:

\[
\begin{align*}
\delta_0 \phi &= \eta_{a} Z^a \rho^c + \rho_0 \delta_c \phi, \\
\rho^c_1 &= \epsilon^c_0 Z^a \bar{\partial}_a \rho_0 + \epsilon^c \bar{\partial}_a (\rho_0 \delta_c \phi), \\
\partial_a \delta_1 \phi &= \eta_{a} \epsilon^a_0 Z^b \bar{\partial}_a \rho^c + \epsilon^a \bar{\partial}_a (\rho_0 \delta_c \phi) + \epsilon^a Z^b \bar{\partial}_a \rho_0 \delta_c \partial_a \phi \\
&\quad+ \epsilon^a \bar{\partial}_a (\rho_0 \delta_c \phi) \bar{\partial}_c \partial_a \phi.
\end{align*}
\]

(65)

In general these are inhomogeneous transformations, containing $\phi$-independent terms. They are spontaneously broken by our choice of ground state $\phi = 0$ which corresponds to a flat background. We cannot expect to see this kind of symmetry in first-quantised string theory since there all symmetries are connected to BRST invariant operators. The latter can only generate unbroken symmetries, that do not contain field-independent terms, as is clear from the discussion at the end of subsection 4.1. Yet if we take $\rho_0$ to be constant, i.e. consider global translations in the $Z$ coordinates, it turns out that the truncated transformation

\[
\tilde{\delta}_0 \phi = \rho_0 \partial_c \phi
\]

(66)

still is a symmetry of the Plebanski equation since $\delta_0$ and $\tilde{\delta}_0$ differ by a Kähler transformation. Moreover, in $\rho^c_1$ and $\delta_1$ the $\phi$-independent terms disappear:

\[
\begin{align*}
\rho^c_1 &= \epsilon^c \rho^c_0 \bar{\partial}_a \partial_a \phi, \\
\partial_a \delta_1 \phi &= \epsilon^a \rho^c_0 \bar{\partial}_a \partial_c \phi + \epsilon^a \rho_0 \bar{\partial}_a \partial_c \partial_a \phi.
\end{align*}
\]

(67)

\[\delta_1\] contains a linear and a non-linear term. But the best we can do is to compare the string transformations (64) to the linearised symmetries of the field $\phi$. We therefore focus in the following on the linear part of the non-local transformations $\delta_1$. Let us first note that the iterative construction (64) does not create $\phi$-independent terms in any $\rho^c_n$ derived from a constant $\rho_0$. This leads to the very important conclusion that the corresponding transformations $\delta_n$ (one might call them affinisations of translations) are unbroken symmetries. Moreover, working to lowest order in $\phi$ we may drop the second term on the right hand side of (63). Differentiating this equation $n - 1$ times leads to

\[
\partial_1 \ldots \partial_n \delta_n \phi = \eta_{a} \epsilon^a \rho^c_0 \bar{\partial}_a \partial_c \phi + \mathcal{O}(\phi^2).
\]

(68)

Repeated application of (64) and again dropping non-linear terms gives

\[
\partial_1 \ldots \partial_n \delta_n \phi = \epsilon^a \ldots \epsilon^a_n \bar{\partial}_a \ldots \bar{\partial}_a (\rho_0 \delta_c \phi) + \mathcal{O}(\phi^2).
\]

(69)

After a Fourier transformation

\[
\phi(Z, \bar{Z}) = \int d^4k e^{\frac{i}{\hbar}(k \cdot Z + \bar{k} \cdot \bar{Z})} \hat{\phi}(k)
\]

(70)

we may impose the mass-shell condition $k \cdot \bar{k} = 0$. Using the relation

\[
\eta_{ab} \bar{k}^a = -\hbar(k) \epsilon_{ab} k^b
\]

(71)

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the $n$-th transformation for $\tilde{\phi}(k)$ reads
\[ \delta_n \tilde{\phi}(k) = \frac{i}{2} h(k)^{-n} \eta_{c\bar{c}} \rho_{\bar{c}}^c k^c \tilde{\phi}(k) + \mathcal{O}(\tilde{\phi}^2). \] (72)

These are the non-local transformations derived from global translations $\tilde{Z}^\alpha \rightarrow \tilde{Z}^\alpha + \rho_{\bar{c}}^c$. The similarity to (54) is evident since one may drop the constant prefactor $\frac{i}{2} \eta_{c\bar{c}} \rho_{\bar{c}}^c$. We discuss in more detail what we think about the relation of equations (54) and (72) in section six\textsuperscript{12}.

Let us finally add that the unbroken linearised symmetries of equation (62) can be derived more quickly in the following fashion. To find the linear part of a transformation $\delta \phi$ one may simply drop the right hand side of (62) and study the equation
\[ \eta^{a\bar{a}} \partial_a \delta_{\bar{a}} \delta \phi = 0. \] (73)
Starting from a rigid translation $\delta_0 \phi = \rho_{\bar{c}}^c \partial_c \phi$ with constant $\rho_{\bar{c}}^c$ one can iteratively define
\[ \partial_a \delta_n \phi = \epsilon_a {}^\alpha \partial_\alpha \delta_{n-1} \phi \] (74)
which again leads to (73). Thus, finding the linearised symmetries amounts to the simple task of solving (73), e.g. by Fourier transformation. Moreover, we see that different pictures correspond to different solutions of (73). We have nevertheless preferred to investigate the exact symmetries of the Plebanski equation in some more detail in this section, hoping to better understand its connection to the $N=2$ string.

5 Ward identities

Symmetries in string and field theories manifest themselves in relations between correlation functions, known as Ward identities, which can be extremely useful either for studying general properties of the theory or for explicitly figuring out physically interesting quantities. Therefore, the Ward identities following from the symmetries uncovered in the previous section will now be investigated. Before turning to the $N=2$ string, however, the first subsection will briefly review the general formalism following [13], [25] and [21].

5.1 More generalities

For simplicity we restrict ourselves to lowest order (genus zero and no $U(1)$ instantons) in string perturbation theory in this paper, but higher orders can be treated in a similar way. Moreover, in this subsection we consider bosonic string theory, in which perturbative evaluation of correlation functions involves only integration over metric moduli. We defer to the next subsection the discussion\textsuperscript{12} One may also compare this to equation (4.1) from [4].
of how the more complicated perturbative structure of the \(N=2\) string affects the derivation of Ward identities.

In the operator formalism a correlation function

\[
\Theta = \langle V_1 \ldots V_{N+1} \rangle
\]

(75)
of \(N+1\) BRST invariant operators \(V_i\) with ghost numbers \(g_i\) can be regarded as a differential form of degree \(\sum_{i=1}^{N+1} g_i - 6\) on the moduli space \(M_{N+1}\) of the sphere with \(N+1\) marked points (which is just the configuration space of \(N-2\) distinct points on the sphere). This can be understood heuristically since \(\sum_{i=1}^{N+1} g_i - 6\) is the number of anti-ghost insertions and therefore of integrations needed to obtain a number from \(\Theta\). Moreover, it is an important general result that this form is closed. For scattering amplitudes one takes each of the \(V_i\) to have ghost number two. Then \(\Theta\) is a top form and can be integrated over the full moduli space to produce a number – the scattering amplitude. To derive a Ward identity for the \(N\)-point function one takes \(N\) physical vertex operators \(V_i\) plus one ghost number one operator that is connected to the symmetries of the theory. Then \(\Theta\) is a form of codimension one and can only be integrated over a submanifold of \(M_{N+1}\) of codimension one. The natural candidate is the boundary \(\partial M_{N+1}\). Because of \(d\Theta = 0\) and Stokes' Theorem this integral vanishes, so the Ward identity reads

\[
\int_{\partial M_{N+1}} \Theta = \int_{M_{N+1}} d\Theta = 0.
\]

(76)
The boundary of the moduli space of a sphere with \(N+1\) marked points corresponds to configurations where the sphere splits into two spheres (one containing \(N+1-p\) and the other \(p\) of the points for some \(2 \leq p \leq N-1\)) connected by an infinitely long tube. This tube can be represented by a complete set of physical states propagating between the spheres. The twist angle of the tube is one of the moduli leading to an insertion of \(b(z) - \tilde{b}(\bar{z})\). The complete set of states therefore takes the form

\[
\sum_i |\hat{O}_i\rangle \langle O_i|,
\]

(77)
where \(i\) labels a basis of the absolute BRST cohomology and

\[
(O_j | O_i) = \delta^i_j, \quad (\hat{O}_i) = (b_0 - \tilde{b}_0) | O_i\rangle.
\]

(78)
The Ward identity for a correlation function involving \(N\) ghost number two vertex operators \(V_i := V(k_i, z_i, \tilde{z}_i)\) and one ghost number one operator \(\Omega^{(0)}\) thus reads:

\[
\sum_{i,\alpha} \langle \langle V_{u_1} \ldots V_{u_p} \Omega^{(0)} \hat{O}_i \rangle \rangle |O_i V_{u_{p+1}} \ldots V_{u_N} \rangle = 0.
\]

(79)
The sum over \(\alpha\) runs over all possible ways to divide the \(N\) physical vertex operators into a subset \(\{V_{u_1} \ldots V_{u_p}\}\) on the sphere of \(\Omega^{(0)}\) and the remainder \(\{V_{u_{p+1}} \ldots V_{u_N}\}\) located on the other sphere. Moreover, we have adopted the notation from \[\text{[13]}\] and indicate by a double bracket that the integration over moduli space has already been performed.
Before trying to apply the Ward identity (79) let us briefly comment on the moduli of the $N=2$ string at tree level and zero $U(1)$ instanton number. The integration over the fermionic moduli can be explicitly performed, resulting in the prescription that the picture numbers $\pi^+ = \tilde{\pi}^+$ and $\pi^- = \tilde{\pi}^-$ of the vertex operators both have to add up to $-2$ in order to get a non-vanishing correlator. The $U(1)$ moduli enter via the homology of the punctured Riemann surface. At genus zero they simply correspond to the non-trivial monodromies of the $U(1)$ charged fields around the punctures. However, spectral flow identifies the correlators for different monodromies and we can restrict ourselves to the NS sector at each puncture. The integration over the $U(1)$ moduli merely contributes to the normalisation. Moreover, the $U(1)$ moduli space has no boundaries. It can play no role in this subsection as one sees from the way the Ward identity has been derived above. We will therefore not write down the $U(1)$ ghosts explicitly.

Let us now consider the Ward identities involving the ghost number one states (47) and suppress the picture numbers for notational simplicity. For the case of the three-point function involving three closed-string vertex operators $V_i(k_i)$ with lightlike momenta $k_1 + k_2 + k_3 = 0$ the identity (79) becomes:

$$
\sum_i \langle V_1 \Omega_m^{(0)} \hat{O}^i \rangle \langle O_i V_2 V_3 \rangle + \sum_i \langle V_2 \Omega_m^{(0)} \hat{O}^i \rangle \langle O_i V_3 V_1 \rangle + \sum_i \langle V_3 \Omega_m^{(0)} \hat{O}^i \rangle \langle O_i V_1 V_2 \rangle = 0.
$$

The picture and ghost numbers and the momenta of the operators $O_i$ and $\hat{O}^i$ are uniquely fixed. For example, the only operator that contributes in the first sum is $O_i = V_1$ with conjugate states $O^i = \partial c \bar{\partial} \tilde{c} V_1$ and $\hat{O}^i = (\partial c + \bar{\partial} \tilde{c}) V_1$. The second factor in the first term is simply the ordinary three-point function $A_3$, while the first factor splits into holomorphic and antiholomorphic parts

$$
\langle V_1(k_1) \Omega_m^{(0)} (\partial c + \bar{\partial} \tilde{c}) V_1(-k_1) \rangle = \langle V_1(k_1) \mathcal{P}_m V_1(-k_1) \rangle_L \langle \hat{V}_1(k_1) \bar{\partial} \tilde{c} V_1(-k_1) \rangle_R.
$$

The subscripts $L$ and $R$ indicate that the two correlators are meant with respect to the chiral left- and right-moving theories (the term $\partial c V_1$ does not contribute because of ghost number counting). Note that the whole discussion does not depend on the pictures chosen, so that there is no restriction on the values of $m$ and $n$. The correlators in (81) can be evaluated using the relations from section 3 and then yield $k_i^2 h(k_i)^n$. The explicit Ward identity for the three-point function finally takes the form

$$
A_3 \sum_{i=1}^3 k_i^2 h(k_i)^n = 0
$$

For the three-point function on the sphere there is no difference between single or double brackets since the corresponding moduli space consists of a single point.
for any \( m \in \mathbb{Z} \). An analogous relation holds with \( k_i^a \) replaced by \( \bar{k}_i^a \). The amplitude can be non-zero only if the sum above vanishes for all \( m \). The unique solution is

\[
h(k_1) = h(k_2) = h(k_3).
\] (83)

This exactly coincides with the result of Berkovits and Vafa in [6] to which we refer for further discussion.

Next, we discuss the Ward identities for general \( N \)-point functions involving \( N \) lightlike momenta \( k_i \) with \( k_1 + k_2 + \ldots + k_N = 0 \). The \( \alpha \) sum in (79) now runs over a large number of possible degenerations of the Riemann surface. In particular, in the splitting the ghost number zero operator \( \Omega^{(0)} \) may be accompanied by more than one vertex operator, leading to a correlator

\[
\langle \langle V_{u_1} V_{u_2} \ldots V_{u_p} \Omega^{(0)}_{m,n} \hat{O}^i \rangle \rangle \langle \langle O_i V_{u_{p+1}} \ldots V_{u_N} \rangle \rangle.
\] (84)

\( O_i \) has momentum \( k_{u_1} + k_{u_2} + \ldots + k_{u_p} \) and is generally not an on-shell vertex operator. The standard way to deal with this situation is to invoke the canceled propagator argument: Evaluate the above expression in a kinematical region where the intermediate states have positive scaling dimension. The contributions from the boundary of moduli space vanish in this case. By virtue of analytic continuation the correlator must then also vanish in other regions of momentum space. However, if \( \Omega^{(0)} \) splits off with only one vertex operator \( V_i \) the canceled propagator argument does not apply since the intermediate state in this situation has momentum \( k_i \) and is always on-shell. The Ward identity therefore receives contributions only from this type of degeneration of the Riemann surface and reads

\[
\sum_{i=1}^{N} \langle \langle V(k_i) \Omega^{(0)}_{m,n} \hat{V}(-k_i) \rangle \rangle A_N = A_N \sum_{i=1}^{N} k_i^a h(k_i)^{m+n} = 0
\] (85)

which holds for any \( m + n \in \mathbb{Z} \). It does not look very exciting (after all, it stems from affinisations of translations) but implies that \( A_N(k) \) vanishes unless

\[
\sum_{i=1}^{N} k_i^a h(k_i)^m = 0 \quad \text{for any } m \in \mathbb{Z}.
\] (86)

To study the solutions of this equation it is useful to recall that \( h \) is only a phase and can be rewritten as \( h(k_i) = e^{i\gamma_i} \). Dividing (86) by \( h(k_1)^m \) and summing over \( m \) leads to

\[
0 = \sum_{i=1}^{N} k_i^a \sum_{m \in \mathbb{Z}} e^{im(\gamma_i - \gamma_1)} = \sum_{i=1}^{N} k_i^a \delta(\gamma_i - \gamma_1).
\] (87)

\[\text{We are grateful to Helge Dennhardt for this suggestion.}\]
The first term, $i = 1$, in the sum is non-vanishing. Without loss of generality we can neglect kinematical situations where a true subset of the momenta sums to zero, so the only possibility for (87) to hold is

$$\gamma_1 = \gamma_2 \ldots = \gamma_N.$$  \hfill (88)

Then (87) is satisfied because of momentum conservation. The Ward identity (86) therefore leads to the final conclusion that the $N$-point function vanishes unless

$$h(k_1) = h(k_2) = \ldots = h(k_N)$$  \hfill (89)

which implies that all scalar products $k_i \bar{k}_j + k_j \bar{k}_i$ are zero and exactly reproduces the general vanishing theorem for tree-level amplitudes of $\mathcal{N}$. In the discussion above we neglected special kinematical situations and invoked analytic continuation. Whether or not one should require on-shell correlation functions to be analytic in a spacetime of signature $(2,2)$ depends on one’s interpretation. For example, Parkes argues in [19] that $\delta$-function contributions to the $S$-matrix of the $\mathcal{N}=2$ string are important with respect to the role of self-dual gravity and $\mathcal{N}=2$ string theory in the theory of integrable models. Clearly, such subtleties are not accessible by the above methods.

6 Concluding remarks

The purpose of this paper is to provide evidence for our belief that the non-trivial picture structure of the BRST cohomology is not just an irrelevant detail of the BRST approach but important for a deeper understanding of the theory (see [12] for similar remarks). The key points are:

- The symmetry transformation of the first-quantised string theory coincides with the symmetry transformation of the field theory at the linearised level.
- The Ward identity correctly implies the vanishing of all correlation functions with more than three external legs, at least for generic values of the momenta.

The first point deserves further discussion. It is, of course, not a simple task to discover the full symmetry group of a string model. Doing so would roughly correspond to having found a useful non-perturbative definition of the theory. In this paper we have worked in the standard first-quantised formalism. To see how much insight into the symmetry structure one may gain within such an approach, it is instructive to recall the situation in closed bosonic string theory. The latter includes gravity and should therefore be a theory invariant under general coordinate transformations. The standard perturbative expansion, however, is around flat space which breaks the symmetry down to the
Poincaré group. The zero momentum cohomology at ghost number one contains only translations (see (46)). One can certainly write down currents that generate Lorentz transformations,

\[ J^{\mu\nu} = X^\mu \partial X^\nu - X^\nu \partial X^\mu, \tag{90} \]

but they do not show up in the cohomology since \( X^\mu \) is not a legal operator that takes part in the operator-state correspondence of the conformal field theory.

Equivalently, the field \( \phi \) that encodes the degrees of freedom of the \( N=2 \) string describes deviations from the Kähler potential of flat space. Hence we find translations in the \((0,0)\) picture (see (32)). This is the most natural picture in the sense that it is the only one where currents derived from cohomology classes do not contain contributions from the ghost sector. They can thus be integrated over the world sheet to generate deformations of the conformal field theory.

We have seen in subsection 4.3 that the non-local transformations derived from translations also leave the flat metric invariant. We should therefore find them somewhere in the zero momentum cohomology of the \( N=2 \) string. The similarity between equations (54) and (72) suggests that they show up via the picture dependence of the zero momentum BRST cohomology, but it is needless to say that at the moment this is just a tentative statement. There are further unbroken symmetries in the field theory. One can check, for example, that the non-local transformation \( \delta_2 \) derived from a rotation, i.e. \( \rho_0^0 = \bar{Z}^e \), is free of field-independent parts. However, the target space coordinates appear explicitly in the transformation law, and we should not expect to find a counterpart in the cohomology – just as explained above in the context of bosonic string theory.

One further aspect should be mentioned. The transformations \( \delta_n \) are gauge symmetries only for \( n = 0 \). On the other hand, it is a general theorem that string theory does not admit continuous rigid symmetries in target space (see chapter 18 of [26]). The proof of this statement is based on the fact that for any such symmetry there are associated conserved currents on the world-sheet from which one can construct vertex operators for gauge bosons. One may therefore wonder how the unbroken non-local symmetries can be realised in \( N=2 \) string theory. The point is that the above theorem typically assumes that the world-sheet currents are chirally conserved, which is not always true. One counter example are the currents (10) of Lorentz transformations (‘string theory gauges only translations’, [27]). In the case of the \( N=2 \) string the currents associated to the symmetry charges are given in (49). They are chirally conserved only for \( (\pi^+, \pi^-) = (0,0) \) since then the right moving part is just the unit operator. The corresponding gauge boson is the Plebanski field \( \phi \). For other values of \( (\pi^+, \pi^-) \) with \( m = n = 0 \) in (10) the currents are not chirally conserved but nevertheless give rise to gauge bosons which are picture changed versions of the \((0,0)\) vertex operator. For non-zero values of \( m \) and \( n \), however, the currents do not allow one to construct a gauge boson vertex operator, which explains why they correspond only to global symmetries in target space.

It is also interesting to note the similarity between the \( N=2 \) string and 2D strings: Both theories have the same spectrum (one scalar) and possess a ring
of ghost number zero states. This leads in both cases to an infinite dimensional symmetry which completely fixes the dynamics. The ground ring elements of the 2D theory are labelled by quantised values of the momenta. On physical vertex operators (tachyons) they act by multiplication of a momentum dependent function and by a shift of the momenta. This is completely analogous to the relation

$$O_{\pi^+,\pi^-} \cdot V_{\hat{\pi}^+,\hat{\pi}^-} (k) = h(k)^n V_{\pi^+,\pi^-;\pi^+,\pi^-} (k),$$

which suggests that the fields $\varphi^\pm$ appearing in the bosonisation of the spinor ghosts play a role similar to the Liouville field in two dimensions. Both are scalar fields coupled to a background charge, and their momenta are quantized. For the $N=2$ string this suggests the appearance of an additional complex dimension. Such extra dimensions in connection with the $\beta$, $\gamma$ ghost system have also been proposed in [28].

Finally, it would be interesting to find further examples in which the picture structure yields non-trivial information about a theory. In [12] the picture dependence of the relative zero-momentum cohomology of the Ramond sector of the $N=1$ string in flat space has been discussed. Using the picture independence of the absolute cohomology it is, however, not hard to show that no new ghost number zero cohomology classes appear in higher pictures (we have checked this for the $1/2$ and $3/2$ picture, and it seems unlikely that new states appear at still higher pictures). To find such a phenomenon in a 10D superstring theory it is certainly necessary to consider non-trivial backgrounds. For example, in [29] an investigation of string propagation on a manifold that includes $AdS_3$ has been initiated. Because of the recently proposed CFT/$AdS$ correspondence an infinite symmetry is expected in this type of theory, which might possibly show up in some cases through a picture dependence of the BRST cohomology.

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A Zero-momentum states

For convenience we summarise in this subsection the chiral zero-momentum cohomology classes of ghost number zero and one in various pictures:

### Ghost number zero

| picture number $(\pi^+, \pi^-)$ | cohomology classes | dimension of cohomology |
|----------------------------------|--------------------|-------------------------|
| $(-2 - p, p)$                    | $\pi^+ \cdot A^p$, $\pi^- \cdot A^{p-1}$ | 0                       |
| $(-1 - p, p)$                    | $\pi^+ \cdot A^p$, $\pi^- \cdot A^{p-1}$ | 0                       |
| $(p, p)$                         | $\pi^+ \cdot A^p$, $\pi^- \cdot A^{p-1}$ | 1                       |
| $(1 - p, p)$                     | $\pi^+ \cdot A^p$, $\pi^- \cdot A^{p-1}$ | 2                       |
| $\pi^+ + \pi^- > 1$             | $\pi^+ \cdot A^p$, $\pi^- \cdot A^{p-1}$ | $\geq \pi^+ + \pi^- + 1$ |

Here $p \in \mathbb{Z}$ is an arbitrary integer, the normal ordered product is defined in (21), and the explicit expressions for the operators $A$, $A^{-1}$ and $X^\pm$ are

$$A = (1 - cb')J^- e^{\varphi^+} e^{-\varphi^-},$$

$$A^{-1} = (1 + cb')J^+ e^{-\varphi^+} e^{\varphi^-},$$

$$X^\pm = -c \partial \xi^\pm + (G^\pm - 4 \gamma^\pm b \pm 4 \partial \gamma^\pm b' \pm 2 \gamma^\pm \partial b') e^{\varphi^\pm}. \quad (91)$$

### Ghost number one

| picture number $(\pi^+, \pi^-)$ | cohomology classes | dimension of cohomology |
|----------------------------------|--------------------|-------------------------|
| $(-2 - p, p)$                    | $A^{p+1} \cdot c e^{-\varphi^+} e^{-\varphi^-}$ | 1                       |
| $(-1 - p, p)$                    | $A^p \cdot c \psi^a$, $A^p \cdot \bar{\psi}^\bar{a}$ | 2                       |
| $(p, p)$                         | $X^+ \cdot A^p \cdot \bar{P}^\bar{a}$, $X^- \cdot A^{p-1} \cdot \bar{P}^\bar{a}$ | 4                       |
| $(1 - p, p)$                     | $X^+ \cdot A^p \cdot \bar{P}^\bar{a}$, $X^- \cdot A^{p-1} \cdot \bar{P}^\bar{a}$ | $\geq 8$                |
| $\pi^+ + \pi^- > 1$             | $O_{\pi^+, \pi^-, n} \cdot \bar{P}^\bar{a}$, $O_{\pi-, \pi^+, n} \cdot \bar{P}^\bar{a}$ | $\geq 4(\pi^+ + \pi^- + 1)$ |

The momentum operators are

$$-i \bar{P}^\bar{a} = -i \bar{P}^\bar{a}_{0,0} = c \partial \bar{Z}^a - 2 \gamma^+ \psi^a,$$

$$-i \tilde{P}^\alpha = -i \tilde{P}^\alpha_{0,0} = c \partial Z^a - 2 \gamma^+ \psi^a. \quad (92)$$

Due to Poincaré duality one obtains similar tables for $\pi^+ + \pi^- \leq -2$. 

24
Only the currents in (1) arise as constraints from gauge-fixing the \( N=2 \) supergravity theory on the world-sheet. They have to be used to construct the chiral BRST operator. We furthermore need the standard \( b,c \) ghosts of weight (2, -1), spinor ghosts \( \beta^\pm, \gamma^\mp \) of weight (3/2, -1/2) and \( U(1) \) ghosts \( b', c' \) of weight (1, 0). Their mode expansions and commutation relations are standard (the spinor ghosts are half-integer moded since we work in the NS sector). The ground state \( |\pi^+, \pi^-\rangle \) with picture number \((\pi^+, \pi^-)\) is defined by dividing the spinor ghost modes into annihilators and creators:

\[
\beta^\pm_r |\pi^+, \pi^-\rangle = 0 \quad \text{when} \quad r \geq -\pi^+ - \frac{1}{2},
\]

\[
\gamma^\mp_r |\pi^+, \pi^-\rangle = 0 \quad \text{when} \quad r \geq \pi^+ + \frac{3}{2},
\]  

(93)

It is useful to bosonise the spinor ghosts:

\[
\gamma^\pm \rightarrow \eta^\pm e^{\pi^\pm}, \quad \beta^\pm \rightarrow e^{-\varphi^\mp} \partial \xi^\pm.
\]  

(94)

\( \eta^\pm \) and \( \xi^\pm \) are fermionic fields with weight 1 and 0 respectively and OPE \( \eta^\pm(z)\xi^\pm(w) \sim (z-w)^{-1} \). \( \varphi^\pm \) are bosonic scalars that couple to a background charge. Their OPE is \( \varphi^\pm(z)\varphi^\pm(w) \sim -\ln(z-w) \). With these variables the state \( |\pi^+, \pi^-\rangle \) can be created from the \( sl(2) \) invariant ground state \( |0\rangle \) as

\[
|\pi^+, \pi^-\rangle = e^{\pi^+\varphi^- + \pi^-\varphi^+}|0\rangle|0\rangle.
\]  

(95)

As in [17] we define the ghost number current in a slightly unusual way as

\[
j_{gh} = -bc - b'c' + \eta^+ \xi^- + \eta^- \xi^+.
\]  

(96)

This assigns the correct ghost number to all ghost fields, but commutes with the operators \( X^\pm \) and \( A \) defined in section 3. Moreover, all states \( |\pi^+, \pi^-\rangle \) have zero ghost number.

The chiral BRST operator \( Q = Q^{\text{mat}} + Q^{\text{gh}} \) splits into two pieces:

\[
Q^{\text{mat}} = \sum_n (c_n^L L_n + c'_n J_n) + \sum_r (\gamma^+_r G^-_r + \gamma^-_r G^+_r),
\]

\[
Q^{\text{gh}} = -\frac{1}{2} \sum (m-n) : c_{-m} c_{-n} b_{m+n} : - \sum m : c'_{-m} c_{-n} b'_{m+n} :
\]

\[
+ \frac{1}{2} \sum (n-2s) c_{-n} : (\gamma^-_s \beta^+_n + \gamma^+_s \beta^-_n ) :
\]

\[
- \sum c'_{-n} : (\gamma^-_s \beta^-_n - \gamma^+_s \beta^+_n ) :
\]

\[
- 4 \sum \gamma^-_s \gamma^+_{-r} b_{r+n} - 2 \sum (s-r) \gamma^-_s \gamma^+_{-r} b'_{r+n}.
\]  

(97)

Normal ordering is defined with respect to the \( sl(2) \) invariant ground state, i.e. the spinor ghosts are normal ordered with respect to the \( (0,0) \) picture. The
ghost parts of the super Virasoro generators can be obtained by anticommuting $Q$ with the anti-ghost modes. The explicit expressions for $L_{gh}^0$ and $J_{gh}^0$ are

\begin{align*}
L_{gh}^0 &= \sum_m m : c_m b_m : + \sum_m m : c'_m b'_m : - \sum_s s : (\gamma^- s \beta^+_s + \gamma^+ s \beta^-_s) :, \\
J_{gh}^0 &= - \sum_s s : (\gamma^+_s \beta^-_s - \gamma^-_s \beta^+_s) :.
\end{align*}

(98)

On the states $|\pi^+, \pi^-\rangle$ these operators act as

\begin{align*}
L_{gh}^0 |\pi^+, \pi^-\rangle &= \left[ \pi^+ \left( \frac{\pi^+}{2} + 1 \right) + \pi^- \left( \frac{\pi^-}{2} + 1 \right) \right] |\pi^+, \pi^-\rangle, \\
J_{gh}^0 |\pi^+, \pi^-\rangle &= (\pi^- - \pi^+) |\pi^+, \pi^-\rangle.
\end{align*}

(99)

These equations are useful to explicitly construct states in the relative Fock space (14).

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