What is the definition of two meromorphic functions sharing a small function?

ANDREAS SCHWEIZER
Department of Mathematics,
Korea Advanced Institute of Science and Technology (KAIST),
Daejeon 305-701
South Korea
e-mail: schweizer@kaist.ac.kr

Abstract
Two meromorphic functions $f(z)$ and $g(z)$ sharing a small function $\alpha(z)$ usually is defined in terms of vanishing of the functions $f - \alpha$ and $g - \alpha$. We argue that it would be better to modify this definition at the points where $\alpha$ has poles. Related to this issue we also point out some possible gaps in proofs in the published literature.

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The question in the title might surprise. After all, most papers on meromorphic functions that share a small function give a definition. Usually it is

Definition 1. Let $f(z)$, $g(z)$ and $\alpha(z)$ be meromorphic functions on a domain $D$. We say that the functions $f$ and $g$ share the function $\alpha$ in the sense of vanishing if on $D$ the functions $f(z) - \alpha(z)$ and $g(z) - \alpha(z)$ have the same zeroes. More precisely we call this sharing the function $\alpha$ IM (ignoring multiplicities) in the sense of vanishing.

If the zeroes of $f - \alpha$ and $g - \alpha$ not just coincide in location but also in multiplicity, we say that $f$ and $g$ share $\alpha$ CM (counting multiplicities) in the sense of vanishing.

So in short, $f$ and $g$ share $\alpha$ IM resp. CM in the sense of vanishing if and only if the functions $f - \alpha$ and $g - \alpha$ share the value 0 IM resp. CM.

The words ’in the sense of vanishing’ are not standard and were added by us to be able to distinguish this from another definition of sharing, which we will define below.

Definition 1 ties in nicely with the generalizations of the Second Main Theorem that involve the counting function of the zeroes of $f - \alpha$. See for example [7, Section 1.5, Theorem 1.36]. We refer to this book for all background information desired. The above definition of sharing is also used in this book in Sections 3.1.4 and 3.1.5.

The smallness of $\alpha(z)$ is irrelevant for the definition of the sharing. In general the purpose of the smallness is to allow Nevanlinna Theory to extract nontrivial consequences from the sharing.
When we give examples in this paper, we always construct them in such a way that $f$ and $g$ are meromorphic functions of finite order and $\alpha$ is a small function with respect to $f$ and $g$.

**Example 1.** The functions

$$f = \frac{1}{z} + e^z \quad \text{and} \quad g = \frac{1}{z} - \frac{e^z}{z}$$

share $\frac{1}{z}$ CM in the sense of vanishing. This is at odds with intuition, because $\alpha$ and $f$ have a simple pole with residuum 1 at $z = 0$, whereas $g$ is entire.

We shall see that this is not the only problem.

A useful fact is that if $f$ and $g$ share a value $a$ and if $M$ is a Möbius transformation, then $M(f)$ and $M(g)$ share the value $M(a)$. In other words, sharing of values behaves well under Möbius transformations. Obviously, translations and scaling do also respect sharing of a small function in the sense of vanishing. However, the inversion $z \mapsto \frac{1}{z}$ causes problems.

**Example 2.** The functions

$$f = z + z^2 e^z \quad \text{and} \quad g = z + z^3 e^z$$

share $\alpha = z$ IM in the sense of vanishing, but $\frac{1}{f}$ and $\frac{1}{g}$ do not share $\frac{1}{\alpha} = \frac{1}{z}$ in the sense of vanishing.

However, the only point that causes trouble is $z = 0$. And since both, $\frac{1}{f}$ and $\frac{1}{g}$, have a simple pole with residuum 1 at $z = 0$, one sort of feels that they should share $\frac{1}{z}$ also at this point.

Even starting with two functions that share a small function CM does not help.

**Example 3.** The functions

$$f = \frac{1}{z} + e^z \quad \text{and} \quad g = \frac{1}{z} - \frac{e^z}{z}$$

from Example 1 share $\alpha = \frac{1}{z}$ CM in the sense of vanishing, but $\frac{1}{f}$ and $\frac{1}{g}$ do not share $\frac{1}{\alpha} = z$ in the sense of vanishing, not even IM. Note that $\frac{1}{f} - z = \frac{-ez^2}{1+ze^z}$ vanishes at $z = 0$, but $\frac{1}{g} - z = \frac{ze^z}{1-ze^z}$ takes the value $-1$ at $z = 0$.

And even if the sharing survives the inversion, the multiplicities might not.

**Example 4.** The functions

$$f = \frac{1}{z} + e^z \quad \text{and} \quad g = \frac{1}{z} + \frac{e^z}{z}$$
share $\alpha = \frac{1}{z}$ CM in the sense of vanishing, but $\frac{1}{f}$ and $\frac{1}{g}$ share $\frac{1}{\alpha} = z$ only IM in the sense of vanishing.

We will come back to this. But let us first point out yet another problem.

One argument that we have seen in several papers is the following. Let $f$ and $g$ be two meromorphic functions and $\alpha$ a small function that is neither constant 0 nor constant $\infty$. If $f$ and $g$ share $\alpha$ CM (in the sense of vanishing), then $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ share the value 1 CM. See for example the proofs of [2, Theorems 3, 4, and 5], [3, Theorems 1.1 and 1.2], [8, Theorems 1.1, 1.2, and 1.5], and [9, Theorem 2].

However, in general this claim is not true.

**Example 5.** Let $f = \frac{1}{z} + e^z$, $g = \frac{1}{z} + \frac{e^z}{z}$, and $\alpha = \frac{1}{z}$ as in Example 4. Then $f$ and $g$ share $\alpha$ CM in the sense of vanishing. But $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ do not share the value 1, not even IM. (On $\mathbb{C}^*$ they would share 1 CM.)

One can even give examples where the sharing gets lost at infinitely many points.

**Example 6.** Let $f = \frac{1}{\sin z} + e^{z^2}$, $g = \frac{1 + e^{z^2}}{\sin z}$, and $\alpha = \frac{1}{\sin z}$.

Then $f$ and $g$ share $\alpha$ CM in the sense of vanishing, but $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ do not share the value 1, not even IM.

To overcome some of these problems, let us look at another possible (and in our opinion better) definition of sharing a meromorphic function.

As is already evoked by the expression ‘moving target’, one can think of $\alpha(z)$ as a value that is changing with $z$.

**Definition 2.** Let $f(z)$, $g(z)$ and $\alpha(z)$ be meromorphic functions on a domain $D$. We say that the functions $f$ and $g$ share the function $\alpha$ (IM) in the sense of value if for every $z_0 \in D$ we have

$$f(z_0) = \alpha(z_0) \iff g(z_0) = \alpha(z_0).$$

At the points $z_0$ where $\alpha$ has a pole, $f(z_0) = \alpha(z_0)$ simply means that $f$ also has a pole at $z_0$.

If $\alpha$ is constant (including the possibility $\alpha \equiv \infty$) this definition specializes to the usual definition of sharing a value IM.

By the new definition, the functions $f$ and $g$ from Example 1 do indeed not share the function $\frac{1}{z}$. 

In the first version of this paper (arXiv:1705.05048v1) we had also given an ad-hoc definition of sharing a function $\alpha$ CM in the sense of value. But that definition had several drawbacks; so we omit it here.

Instead, we now give what we believe is the good definition of sharing a function CM. We will back up this conviction by showing that this definition has some desirable properties that Definition 1 is lacking.

We start from the observation that if $\alpha$ has a pole at $z_0$, then the condition $f(z_0) = \alpha(z_0)$ is equivalent to the vanishing of $\frac{1}{f} - \frac{1}{\alpha}$ at $z_0$. This is the only modification made to get from Definition 1 to Definition 2. Refining this, towards sharing CM, we take into account multiplicities, namely: outside the poles of $\alpha$ the order of vanishing of $f - \alpha$ and $g - \alpha$ as before, and at the poles of $\alpha$ the order of vanishing of $\frac{1}{f} - \frac{1}{\alpha}$ and $\frac{1}{g} - \frac{1}{\alpha}$.

**Definition 3.** Let $f(z)$, $g(z)$ and $\alpha(z)$ be meromorphic functions on a domain $D$. We say that the functions $f$ and $g$ share the function $\alpha$ CM in the sense of value if for every $z_0 \in D$ the following conditions hold:

- At points $z_0$ with $\alpha(z_0) \neq \infty$: The function $f - \alpha$ has a zero of order $m$ at $z_0$ if and only if $g - \alpha$ has a zero of order $m$ at $z_0$.

- At points $z_0$ with $\alpha(z_0) = \infty$: The function $\frac{1}{f} - \frac{1}{\alpha}$ has a zero of order $m$ at $z_0$ if and only if $\frac{1}{g} - \frac{1}{\alpha}$ has a zero of order $m$ at $z_0$.

Quite likely this definition has already been discussed and used in the literature. But we couldn’t find any sources. And an exhaustive search is hardly feasible given the sheer mass of publications in this area.

Comparing sharing in the sense of vanishing and sharing in the sense of value, one obviously expects some disagreement at the points where $\alpha(z)$ has a pole.

We already mentioned Example 1, where $f$ and $g$ share $\frac{1}{z}$ in the sense of vanishing but not in the sense of value. The converse also occurs.

**Example 7.** The functions

\[ f = \frac{1}{z} + e^z \quad \text{and} \quad g = \frac{1}{z} + ze^z \]

share $\frac{1}{z}$ IM in the sense of value (not CM!). Note that $\frac{1}{f} - \frac{1}{\alpha} = \frac{-z^2e^z}{1+ze^z}$ and $\frac{1}{g} - \frac{1}{\alpha} = \frac{-z^2e^z}{1+z^2e^z}$. But our main point is that $f$ and $g$ do not share $\frac{1}{z}$ in the sense of vanishing.

Even if $f$ and $g$ share $\alpha$ in the sense of vanishing and in the sense of value, it is not guaranteed that the notions of sharing CM agree.

**Example 8.** Let

\[ f = \frac{1}{z} + e^z, \quad g = \frac{1}{z} + \frac{e^z}{z}, \quad \text{and} \quad \alpha = \frac{1}{z}. \]
Then $f$ and $g$ share $\alpha$ CM in the sense of vanishing, but only IM in the sense of value. Compare Example 4. The discrepancy occurs at $z = 0$.

Here are two reasons why sharing in the sense of value is better.

**Theorem 1.** Sharing in the sense of value (IM or CM) is well-behaved under Möbius transformations. More precisely:

If $f$ and $g$ share $\alpha$ (IM or CM) in the sense of value and if $M$ is a Möbius transformation, then $M(f)$ and $M(g)$ share $M(\alpha)$ (IM resp. CM) in the sense of value.

**Proof.** Obviously, translations and scaling respect any type of sharing we have discussed so far. So it suffices to prove the statement of the theorem for the inversion $z \mapsto \frac{1}{z}$.

Let us first look at points $z_0$ with $\alpha(z_0) \in \mathbb{C}^*$. From $\frac{1}{f} - \frac{1}{\alpha} = \frac{\alpha - f}{\alpha f}$ we then see that $f - \alpha$ has a zero of order $m$ ($> 0$) at $z_0$ if and only if $\frac{1}{f} - \frac{1}{\alpha}$ has a zero of order $m$ at $z_0$. Since the same holds of course for $g$, we obtain that $z \mapsto \frac{1}{z}$ respects the sharing (IM or CM) outside the zeroes and poles of $\alpha$.

At the points with $\alpha(z_0) = 0$ or $\alpha(z_0) = \infty$ this also holds by definition of sharing in the sense of value. □

**Theorem 2.** Let $f$, $g$ and $\alpha$ be meromorphic functions on a domain $D$, such that $\alpha$ is not constant $0$ and also not constant $\infty$. Suppose that on $D$ the functions $f$ and $g$ share the function $\alpha$ CM in the sense of value. Then $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ share the value $1$ CM on $D$.

**Proof.** At points $z_0$ with $\alpha(z_0) \in \mathbb{C}^*$ the functions $f - \alpha$ and $\frac{f}{\alpha} - 1$ obviously have the same order of vanishing.

Now let $\alpha(z_0) = 0$. By assumption, $f - \alpha$ and $g - \alpha$ either both have a zero of the same order at $z_0$ or both do not vanish at $z_0$. So the functions $\frac{f - \alpha}{\alpha} = \frac{f}{\alpha} - 1$ and $\frac{g - \alpha}{\alpha} = \frac{g}{\alpha} - 1$ either both have a zero of the same order at $z_0$ or both do not vanish at $z_0$. This is what we wanted.

At points $z_0$ with $\alpha(z_0) = \infty$ we use that by Theorem 1 the functions $\frac{1}{f}$ and $\frac{1}{g}$ share $\frac{1}{\alpha}$ CM in the sense of value. Then by the previous argument $\frac{\alpha}{f}$ and $\frac{\alpha}{g}$ share the value $1$ CM. Another application of $z \mapsto \frac{1}{z}$ finishes the proof. □

However, this does not mean that one can get rid of all problems concerning $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ sharing $1$ CM by simply changing the definition of sharing a small function from sharing in the sense of vanishing to sharing in the sense of value. This is a change that comes at a certain price and could create new problems somewhere else. For example, if $f$ and $g$ share $\alpha$ in the sense of vanishing, every zero of $f - \alpha$ also is a zero of $g - \alpha$ and of $f - g$. But from sharing in the sense of value we only get...
that every zero of $f - \alpha$ that is not a pole of $\alpha$ also is a zero of $g - \alpha$ and of $f - g$.  

Another claim that can be found in the literature is that if two meromorphic functions $f$ and $g$ share the small function $\alpha (\neq 0, \neq \infty)$ IM, then $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ share the value 1 IM. See for example the proofs of [3, Theorems 1.1 and 1.2], [5, Theorems 1 and 2], and [9, Theorem 1].

Again the claim is not true in general. Just see Example 5 (= Example 8), where $f$ and $g$ share $\alpha$ in the sense of vanishing and in the sense of value, but $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ do not share the value 1.

But one can give a more striking counter-example in which all functions involved are entire.

**Example 9.** Let 

$$f = (\sin z) + (\sin z)e^{z^2}, \quad g = (\sin z) + (\sin z)^2 e^{z^2}, \quad \text{and} \quad \alpha = \sin z.$$  

Then $f$ and $g$ share $\alpha$ IM in the sense of vanishing and in the sense of value, but $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ do not share the value 1.

In this case the problems are coming from the zeroes of $\alpha$. This example also shows that there can hardly be a reasonable way to define the problem away.

We do of course not claim that our Examples 5, 6 and 9 are direct counterexamples to the specified theorems in the papers we mentioned. The functions in these papers satisfy many more conditions, for example $g$ being a derivative of $f$ or a differential polynomial of $f$, or there are additional sharing properties. But without additional arguments the status of those theorems is questionable.

Since $\alpha$ is small, the counting function of the poles and zeroes of $\alpha$ is small, even when they are counted with their multiplicities. (This is not necessarily true when they are counted with their multiplicities as poles or zeroes of $f$, as those multiplicities might grow rapidly.) So if $f$ and $g$ share $\alpha$ IM (resp. CM), one can still say that the truncated counting function of the points where $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ do not share the value 1 IM (resp. not share the value 1 CM) is small.

But it makes a difference whether one can apply a well-known theorem on functions that share the value 1 CM (compare for example [4, Theorem A]), or whether one would need a theorem on functions that share the value 1 outside a small set of arguments.

Many papers argue correctly that $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ share the value 1 outside the zeroes and poles of $\alpha$. Some other papers avoid the whole problem by imposing the extra condition that $\alpha(z)$ has no common poles and no common zeroes with $f(z)$ or $g(z)$.

Before we continue our discussion of this, we have to recall the definition of weighted sharing, which has proved to be very useful in the theory of value sharing. Weighted sharing (of values) was introduced by Lahiri in [4] to have some finer degrees of division between sharing CM and sharing IM.

**Definition 4.** Let $f(z)$, $g(z)$ and $\alpha(z)$ be meromorphic functions on a domain
D. We say that the functions $f$ and $g$ share the function $\alpha$ with weight $m$ in the sense of vanishing if for each $k = 1, 2, \ldots, m$ the $k$-fold zeroes of $f - \alpha$ coincide with the $k$-fold zeroes of $g - \alpha$, and the zeroes of $f - \alpha$ of multiplicity bigger than $m$ coincide with the zeroes of $g - \alpha$ of multiplicity bigger than $m$.

In the latter case the multiplicities are not necessarily the same. So sharing with weight $\infty$ is sharing CM, and sharing with weight $0$ is sharing IM.

Analoguously we could refine Definition 3 into a definition of sharing $\alpha$ with weight $m$ in the sense of value. By Theorem 1 and its proof at points $z_0$ with $\alpha(z_0) \in \mathbb{C}^*$ a $k$-fold zero of $f - \alpha$ at $z_0$ corresponds to a $k$-fold zero of $\frac{1}{f} - \frac{1}{\alpha}$ at $z_0$. So we can build this into the formulation and give the following equivalent definition.

**Definition 5.** Let $f(z)$, $g(z)$ and $\alpha(z)$ be meromorphic functions on a domain $D$. We say that the functions $f$ and $g$ share the function $\alpha$ with weight $m$ in the sense of value if outside the poles of $\alpha$ the functions $f$ and $g$ share $\alpha$ with weight $m$ in the sense of vanishing, and outside the zeroes of $\alpha$ the functions $\frac{1}{f}$ and $\frac{1}{g}$ share $\frac{1}{\alpha}$ with weight $m$ in the sense of vanishing.

This definition contains everything we need as special cases, namely sharing IM in the sense of value (weight $0$), sharing CM in the sense of value (weight $\infty$), and sharing a value with weight. Also, when $\alpha$ has no poles, sharing $\alpha$ (IM, CM, with weight $m$) in the sense of value coincides with the corresponding notion of sharing $\alpha$ in the sense of vanishing.

As we already pointed out, Definition 5 is well-behaved under Möbius transformations.

The functions $f$ and $g$ in Example 7 share $\frac{1}{z}$ with weight 1 in the sense of value.

The proof of [6, Theorem 1.1] contains a statement of the form that if $f$ and $g$ share $\alpha$ with weight $m$, then $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ share the value 1 with weight $m$.

Unfortunately, this claim is also not true in general, not even if all functions involved are entire. To see that we generalize Example 9.

**Example 10.** Let $m$ be a nonnegative integer and

$$f = (\sin z)^{m+1} + (\sin z)^{m+1}e^{z^2}, \quad g = (\sin z)^{m+1} + (\sin z)^{m+2}e^{z^2}, \quad \text{and} \quad \alpha = (\sin z)^{m+1}.$$ 

Then $f$ and $g$ share $\alpha$ with weight $m$ in the sense of vanishing and in the sense of value, but $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ do not share the value 1, not even IM.

In this case the problems are coming from the zeroes of $\alpha$. And again we see that there is probably no hope of completely getting rid of the problem.

In contrast, the claim in the proof of [1, Theorem 1.1] that if $f$ and $g$ weakly share $\alpha$ with weight 2, then $\frac{f}{\alpha}$ and $\frac{g}{\alpha}$ share the value 1 weakly with weight 2 is correct, because it is already built into the definition of sharing weakly [1, Definition 1.3] that there might be a small set where the sharing doesn’t hold.
Finally we point out that the converse of the claims discussed above is also not true, not even if most of the functions involved are entire.

**Example 11.** Let

\[ f = 1 + e^{z^2} \quad \text{and} \quad g = 1 + \frac{e^{z^2}}{\sin z}. \]

Then \( f \) and \( g \) share the value 1 CM, but \( \alpha f \) and \( \alpha g \) do **not** share the small function \( \alpha = \sin z \), neither in the sense of vanishing nor in the sense of value.

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