A Novel Approach to Elastodynamics: I. The Two-Dimensional Case

A. S. Fokas\textsuperscript{a,*} and D. Yang\textsuperscript{a,b†}

\textsuperscript{a} Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Cambridge CB3 0WA, UK

\textsuperscript{b} Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P. R. China

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Abstract

We introduce a new approach to constructing analytic solutions of the linear PDEs describing elastodynamics. This approach is illustrated for the case of a homogeneous isotropic half-plane body satisfying arbitrary initial conditions and Lamb's boundary conditions. A particular case of this problem, namely the case of homogeneous initial conditions, was first solved by Lamb using the Fourier-Laplace transform. The solution of the general problem can also be expressed in terms of the Fourier transform, but this representation involves transforms of unknown boundary values. This necessitates the formulation and solution of a cumbersome auxiliary problem, which expresses the unknown boundary values in terms of the Laplace transform of the given boundary data. The new approach, which is applicable to arbitrary initial and boundary conditions, bypasses the above auxiliary problem and expresses the solutions directly in terms of the given initial and boundary data.

Keywords: elastodynamics, half space, initial boundary value problem, Lamb's problem, global relation.

1 Introduction

The problem considered in this paper has a long and illustrious history, which begins with the classic works of Sir Horace Lamb in 1904 [1, 2, 3]. In [1], Lamb treated four basic problems, the so called Lamb's problems, including both
the two and the three dimensional cases. These problems have been studied by
several authors, see for example [4, 5, 6, 7, 8, 9, 10, 11, 12]. For the axisymmetric
point-load problem, Pekeris [4] obtained an exact expression for the surface
motion in the case that the point load is given by a Heaviside function. This
result was further extended by Schiel and Protázio [6]. For the three dimensional
cases, Payton, using the Betti-Rayleigh reciprocal theorem [5], obtained the
displacement and the surface motion. Gakenheimer, in his PhD thesis [7], made
a systematic investigation of the travelling normal point load, by using the
Cagniard-deHoop method. Bakkeraq et al. in [9] revisited this problem and
presented a straightforward implementation of the Cagniard-deHoop method.
Several related problems of contact mechanics and wave propagation can be
found in the books of Miklowitz [2], of Johnson [3], and of Payton [13].

The extensive study of Lamb’s problems is perhaps due to the fact that
they arise in a variety of physical situations, as well as to their interesting
mathematical structure.

In this paper, we study two-dimensional problems under the usual assump-
tion of plane strain. We consider arbitrary initial conditions and general stress
boundary conditions, including normal line load, tangential line load, and mixed
line load. We refer to the latter three stress conditions as Lamb’s boundary con-
ditions.

Most studies of Lamb’s problems are based on the Helmholtz decomposition
and on the use of the Laplace transform in time. In particular, Lamb was the
first to use the above approach, as well as a certain superposition which is in
fact equivalent to the use of the Fourier transform. Miklowitz made use of the
Cagniard-deHoop method [2], which itself is based on the Laplace transform.
Payton made use of Green’s functions and of the Laplace transform.

Helmholtz decomposition has the advantage of decomposing the P-waves
and S-waves. However, it has the disadvantage of introducing higher order
derivatives to the boundary conditions. The use of Laplace transform in time,
especially restricts the problem to the case of homogeneous initial conditions.

Our new approach is based on the unified approach for solving linear and
integrable nonlinear PDEs introduced by one of the authors in [14]. A crucial
role in this method is played by the so-called global relations, which are algebraic
relations coupling appropriate transforms of the unknown boundary values with
transforms of the given data. For linear PDEs this method uses three novel steps
[15]-[44]:
1. Derive a representation for the solution in terms of an integral
involving a contour in the complex Fourier plane. This representation is not yet
effective, because in addition to transforms of the given initial and boundary
conditions, it also contains transforms of unknown boundary values.
2. Analyse certain transformations in the complex Fourier plane which leave invariant the
transforms of the unknown boundary values.
3. Eliminate the transforms of the unknown boundary values, by combining the results of the first two steps and
by employing Cauchy’s theorem (or more precisely Jordan’s lemma).

This paper is organised as follows. In section 2, we recall the governing
equations of elastodynamics and derive the global relations. In section 3 we
implement step 1 and in section 4 we implement steps 2 and 3. In section 5 we
use the general representations derived in section 4 in order to analyse Lamb’s problems. These results are further discussed in section 6.

For certain complicated boundary value problems it seems that it is not possible to eliminate from the integral representation of the solution the transforms of the unknown boundary values. However, for some of these problems, by using the global relations, one can derive expressions for the Laplace transforms of the boundary values in terms of the given data [45]. A summary of this less effective approach is presented in the Appendix. It is interesting that for the particular case of zero initial conditions, the formulae presented in the Appendix reduce to the formulae first derived in the classical works of Lamb.

2 Governing Equations and Global Relations

The transient problem for two dimensional elastodynamics in the half plane with Lamb’s boundary conditions is defined as follows: Let $u = u(x, y, t)$, $v = v(x, y, t)$, denote the displacements of a homogeneous isotropic half space body. The governing equations of motion without external body forces, are the Lamé-Navier equations:

\[
\begin{align*}
(\lambda + 2\mu)u_{xx} + (\lambda + \mu)v_{xy} + \mu u_{yy} - \rho u_{tt} &= 0, \\
\mu v_{xx} + (\lambda + \mu)u_{xy} + (\lambda + 2\mu)v_{yy} - \rho v_{tt} &= 0,
\end{align*}
\]

where $\lambda, \mu$ are the Lamé constants and $\rho$ denotes the density of the material, which without loss of generality is normalized to unity, i.e., $\rho \equiv 1$. Let the initial conditions be denoted by

\[
\begin{align*}
u(x, y, 0) &= u_0(x, y), \\
u_t(x, y, 0) &= u_1(x, y),
\end{align*}
\]

(2.2a)

(2.2b)

$-\infty < x < \infty$, $0 < y < \infty$.

Let the stress boundary conditions be denoted by

\[
\begin{align*}
(u_y + v_x)(x, 0, t) &= g_1(x, t), \\
(\lambda + \mu)/2(u_x + \mu v_x)(x, 0, t) &= g_2(x, t),
\end{align*}
\]

(2.3a)

(2.3b)

For a tangential line load, the functions $g_1$ and $g_2$ are given by

\[
g_1(x, t) = \sigma_0 \delta(x) X(t)/\mu, \quad g_2(x, t) = 0;
\]

(2.4)

for a normal line load,

\[
g_1(x, t) = 0, \quad g_2(x, t) = \sigma_0 \delta(x) Y(t)/(\lambda + \mu);
\]

(2.5)

for a moving normal line load with a constant speed $C$,

\[
g_1(x, t) = 0, \quad g_2(x, t) = \sigma_0 \delta(x - Ct)/(\lambda + \mu).
\]

(2.6)

Here $\sigma_0$ is a constant, $\delta(x)$ denotes the Dirac-$\delta$ function, $X(t)$ and $Y(t)$ are functions which depend only on $t$. 

3
Notations  Hat, “∧”, will denote the two-dimensional Fourier transform with respect to \( x \) and \( y \), whereas tilde, “∼”, will denote the Fourier transform with respect to \( x \). In particular,

\[
\hat{u}(k, l, t) := \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \ e^{-ikx-ily} u(x, y, t), \quad (2.7a)
\]

\[
\hat{v}(k, l, t) := \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \ e^{-ikx-ily} v(x, y, t), \quad (2.7b)
\]

\[
\hat{u}_j(k, l) := \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \ e^{-ikx-ily} u_j(x, y), \quad (2.7c)
\]

\[
\hat{v}_j(k, l) := \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dy \ e^{-ikx-ily} v_j(x, y), \quad (2.7d)
\]

\( k \in \mathbb{R}, \ l \in \mathbb{C}^-, \ t > 0, j = 0, 1, \)

where \( \mathbb{C}^- \) denotes the lower half complex \( l \)-plane.

Furthermore,

\[
\tilde{u}(k, t) := \int_{-\infty}^{\infty} dx \ e^{-ikx} u(x, 0, t), \quad (2.8a)
\]

\[
\tilde{v}(k, t) := \int_{-\infty}^{\infty} dx \ e^{-ikx} v(x, 0, t), \ k \in \mathbb{R}, t > 0. \quad (2.8b)
\]

\[
\tilde{g}_j(k, t) := \int_{-\infty}^{\infty} dx \ e^{-ikx} g_j(x, t), \ k \in \mathbb{R}, t > 0, j = 1, 2. \quad (2.9)
\]

We emphasise that since \( x \in \mathbb{R} \), the \( x \)-Fourier transform is well-defined only for \( k \in \mathbb{R} \); on the other hand, since \( 0 < y < \infty \), the \( y \)-Fourier transform is well-defined for \( l \) in the lower half complex \( l \)-plane.

By applying the two-dimensional Fourier transform to equations (2.1) and by using in the resulting equations the boundary conditions (2.3), we obtain the following equations:

\[
- (\lambda + 2\mu) k^2 \hat{u} + (\lambda + \mu) ik(il\hat{v} - \tilde{v}) + \mu \left[ -l^2 \hat{u} - il\hat{u} - (\tilde{g}_1 - ik\tilde{v}) \right] = \hat{u}_{tt},
\]

\[
- \mu k^2 \hat{v} + (\lambda + \mu) ik(il\hat{u} - \tilde{u}) + (\lambda + 2\mu) \left[ -l^2 \hat{v} - il\hat{v} - \left( \tilde{g}_2 - \frac{\lambda}{\lambda + 2\mu} ik\tilde{v} \right) \right] = \hat{v}_{tt}, \quad k \in \mathbb{R}, \ l \in \mathbb{C}. \quad (2.10a, 2.10b)
\]

Introducing the notations

\[
P(k, l, t) = k\hat{u}(k, l, t) + l\hat{v}(k, l, t), \quad (2.11a)
\]

\[
Q(k, l, t) = l\tilde{u}(k, l, t) - k\tilde{v}(k, l, t), \quad (2.11b)
\]
Solving equations (2.10) for \( \{\omega_1, \omega_2\} \) and the known equations (2.10) become
\[
P_{tt} + (\lambda + 2\mu)(k^2 + l^2)P = F_P, \quad (2.12a)
\]
\[
Q_{tt} + \mu(k^2 + l^2)Q = F_Q, \quad k \in \mathbb{R}, \; l \in \mathbb{C}, \quad (2.12b)
\]
where the functions \( F_P(k, l, t) \) and \( F_Q(k, l, t) \) are defined as follows:
\[
F_P(k, l, t) = -[\mu k \tilde{g}_1(k, t) + i\lambda k^2 \tilde{v}(k, t)] - l[(\lambda + 2\mu)\tilde{g}_2(k, t) + 2\mu k \tilde{v}(k, t)], \quad (2.13a)
\]
\[
F_Q(k, l, t) = [k(\lambda + 2\mu)\tilde{g}_2(k, t) + i\mu k^2 \tilde{u}(k, t)] - l[\mu \tilde{g}_1(k, t) - 2\mu k \tilde{u}(k, t)]. \quad (2.13b)
\]
Solving equations (2.12) for \( \{P, Q\} \), we obtain the following expressions:
\[
k\tilde{v} + l\tilde{u} = \frac{1}{\omega_1} \left( e^{-i\omega_1 t} \int_0^t e^{i\omega_1 s} F_P(k, l, s) ds - e^{i\omega_1 t} \int_0^t e^{-i\omega_1 s} F_P(k, l, s) ds \right) + \left( \frac{1}{2} P_0 + \frac{i}{2} P_1 \right) e^{-i\omega_1 t} + \left( \frac{1}{2} P_0 - \frac{i}{2} P_1 \right) e^{i\omega_1 t}, \quad (2.14a)
\]
\[
l\tilde{u} - k\tilde{v} = \frac{1}{\omega_2} \left( e^{-i\omega_2 t} \int_0^t e^{i\omega_2 s} F_Q(k, l, s) ds - e^{i\omega_2 t} \int_0^t e^{-i\omega_2 s} F_Q(k, l, s) ds \right) + \left( \frac{1}{2} Q_0 + \frac{i}{2} Q_1 \right) e^{-i\omega_2 t} + \left( \frac{1}{2} Q_0 - \frac{i}{2} Q_1 \right) e^{i\omega_2 t}, \quad (2.14b)
\]
where the dispersion relations \( \omega_1 \) and \( \omega_2 \) are given by
\[
\omega_1 = (\lambda + 2\mu)(k^2 + l^2), \quad \omega_2 = \mu(k^2 + l^2), \quad k \in \mathbb{R}, \; l \in \mathbb{C}, \quad (2.15)
\]
and the known functions \( P_0(k, l), P_1(k, l), Q_0(k, l), Q_1(k, l) \) are given in terms of the initial conditions by
\[
P_0 = k\tilde{u}_0 + l\tilde{v}_0, \quad P_1 = k\tilde{u}_1 + l\tilde{v}_1, \quad (2.16a)
\]
\[
Q_0 = l\tilde{u}_0 - k\tilde{v}_0, \quad P_1 = l\tilde{u}_1 - k\tilde{v}_1, \quad (2.16b)
\]
with \( \tilde{u}_j, \tilde{v}_j, \; j = 0, 1 \), defined in equations (2.7).

In what follows, we take
\[
\omega_1 = \sqrt{\lambda + 2\mu(k^2 + l^2)}^{\frac{1}{2}} \quad \text{and} \quad \omega_2 = \sqrt{\mu(k^2 + l^2)}^{\frac{1}{2}}. \quad (2.17)
\]
The function \((k^2 + l^2)^{\frac{1}{2}}\) has the branch points \( \pm ik \) in the complex \( l \)-plane; we connect these two branch points by a branch cut and we fix a branch in the cut plane by the requirement that,
\[
(k^2 + l^2)^{\frac{1}{2}} \sim \text{Re} \left[ \left( \frac{1}{2} \right)^{\frac{1}{2}} \right] \quad \text{as} \quad l \to \infty.
\]
Let \( u^{(j)\pm}, v^{(j)\pm}, U^{(j)}, V^{(j)}, j = 1, 2 \), denote the following unknown functions:

\[
\begin{align*}
    u^{(j)\pm}(k, l, t) &= \int_0^t e^{\pm i\omega_j s} \tilde{u}(k, s) ds, \\
v^{(j)\pm}(k, l, t) &= \int_0^t e^{\pm i\omega_j s} \tilde{v}(k, s) ds, \\
U^{(j)}(k, l, t) &= \frac{1}{2\omega_j} \left( e^{-i\omega_j t} u^{(j)+}(k, l, t) - e^{i\omega_j t} u^{(j)-}(k, l, t) \right), \\
V^{(j)}(k, l, t) &= \frac{1}{2\omega_j} \left( e^{-i\omega_j t} v^{(j)+}(k, l, t) - e^{i\omega_j t} v^{(j)-}(k, l, t) \right),
\end{align*}
\]

\( k \in \mathbb{R}, l \in \mathbb{C}, t \geq 0, j = 1, 2. \)

Similarly, let \( g^{(j)\pm}, f^{(j)\pm}, G^{(j)}, F^{(j)}, j = 1, 2 \), denote the following known functions:

\[
\begin{align*}
    g^{(j)\pm}(k, l, t) &= \int_0^t e^{\pm i\omega_j s} \tilde{g}_1(k, s) ds, \\
f^{(j)\pm}(k, l, t) &= \int_0^t e^{\pm i\omega_j s} \tilde{g}_2(k, s) ds, \\
G^{(j)}(k, l, t) &= \frac{1}{2\omega_j} \left( e^{-i\omega_j t} g^{(j)+}(k, l, t) - e^{i\omega_j t} g^{(j)-}(k, l, t) \right), \\
F^{(j)}(k, l, t) &= \frac{1}{2\omega_j} \left( e^{-i\omega_j t} f^{(j)+}(k, l, t) - e^{i\omega_j t} f^{(j)-}(k, l, t) \right),
\end{align*}
\]

\( k \in \mathbb{R}, l \in \mathbb{C}, t \geq 0, j = 1, 2. \)

Using the above notations, equations (2.14) become

\[
\begin{align*}
k\tilde{u} + l\tilde{v} &= (\lambda k^2 + l^2(\lambda + 2\mu)) V^{(1)} + 2\mu kl U^{(1)} + N_P, \\
l\tilde{u} - k\tilde{v} &= -\mu(k^2 - l^2) U^{(2)} - 2\mu kl V^{(2)} + N_Q,
\end{align*}
\]

where the known functions \( N_P(k, l, t) \) and \( N_Q(k, l, t) \) are defined as follows:

\[
\begin{align*}
    N_P(k, l, t) &= -i(\lambda + 2\mu) F^{(1)}(k, l, t) - i\mu k G^{(1)}(k, l, t) \\
    &\quad + \left( \frac{1}{2} P_0(k, l) + \frac{i}{2} \frac{P_1(k, l)}{\omega_1} \right) e^{-i\omega_1 t} + \left( \frac{1}{2} P_0(k, l) - \frac{i}{2} \frac{P_1(k, l)}{\omega_1} \right) e^{i\omega_1 t}, \tag{2.21a}
\end{align*}
\]

\[
\begin{align*}
    N_Q(k, l, t) &= i k(\lambda + 2\mu) F^{(2)}(k, l, t) - i\mu l G^{(2)}(k, l, t) \\
    &\quad + \left( \frac{1}{2} Q_0(k, l) + \frac{i}{2} \frac{Q_1(k, l)}{\omega_2} \right) e^{-i\omega_2 t} + \left( \frac{1}{2} Q_0(k, l) - \frac{i}{2} \frac{Q_1(k, l)}{\omega_2} \right) e^{i\omega_2 t}. \tag{2.21b}
\end{align*}
\]

We will refer to equations (2.20) as the global relations. The global relations express the two-dimensional Fourier transforms of the solution \((u, v)\) in terms
of the given initial and boundary data, as well as in terms of the transforms $U^{(1)}, U^{(2)}, V^{(1)}, V^{(2)}$ of the unknown boundary values. The important observation is that equations (2.20) are valid for all values of $l$ in the lower half complex $l-$plane. It turns out that using this fact it will be possible to eliminate the unknown transforms.

### 3 An Integral Representation Involving the Unknown Transforms

We first observe that for any fixed $k \in \mathbb{R}$ and any fixed $t$, $0 \leq t < T$, $T > 0$, the functions $U^{(j)}, V^{(j)}, G^{(j)}, F^{(j)}, j \in \{1, 2\}$, are analytic in the complex $l-$plane. Indeed, these functions are all single-valued, thus it only remains to establish the analyticity in the neighbourhood $l = \pm ik$. The definition of $U^{(1)}$, i.e. equation (2.18c), implies that this function possesses the following expansion for $l$ near $\pm ik$:

$$U^{(1)}(k, l, t) = \sum_{n=1}^{\infty} \frac{\omega_{n}^{2n-2}}{(2n-1)!} \int_{0}^{t} (s-t)^{2n-2} \tilde{u}(k, s) ds. \quad (3.1)$$

Similar expansions are also valid for $U^{(2)}, V^{(1)}, V^{(2)}, G^{(1)}, G^{(2)}, F^{(1)}, F^{(2)}$.

Solving equations (2.20) we find

$$\hat{u} = \frac{1}{k^2 + l^2} \left\{ (\lambda k^3 + kl^2(\lambda + 2\mu))V^{(1)} + 2\mu k^2 U^{(1)} - \mu(k^2l - l^3)U^{(2)} - 2\mu kl^2 V^{(2)} + kN_P + lN_Q \right\}, \quad (3.2a)$$

$$\hat{v} = \frac{1}{k^2 + l^2} \left\{ (\lambda k^2l + l^3(\lambda + 2\mu))V^{(1)} + 2\mu kl^2 U^{(1)} + \mu(k^3 - kl^2)U^{(2)} + 2\mu k^2 V^{(2)} + lN_P - kN_Q \right\}. \quad (3.2b)$$

We observe that the following important transformation is valid in the complex $l-$plane:

$$l \rightarrow -l \text{ maps } \omega_1 \text{ to } -\omega_1, \ \omega_2 \text{ to } -\omega_2, \text{ and leaves } U^{(j)}, V^{(j)}, j = 1, 2 \text{ invariant.} \quad (3.3)$$

Employing the transformation $l \rightarrow -l$ in equations (3.2a) and (3.2b) and then adding the resulting equations to (3.2a) and (3.2b), we obtain the following
Applying the inverse Fourier transform formula to these equations, we obtain

\[ \hat{u}(k, l, t) + \hat{u}(k, -l, t) = \frac{1}{k^2 + l^2} \left\{ 2(\lambda k^3 + kl^2(\lambda + 2\mu))V^{(1)}(k, l, t) - 4\mu k^2V^{(2)}(k, l, t) + k(N_P(k, l, t) + N_P(k, -l, t)) + l(N_Q(k, l, t) - N_Q(k, -l, t)) \right\}, \tag{3.4a} \]

\[ \hat{v}(k, l, t) + \hat{v}(k, -l, t) = \frac{1}{k^2 + l^2} \left\{ 4\mu k^2U^{(1)}(k, l, t) + 2\mu(k^3 - kl^2)U^{(2)}(k, l, t) + l(N_P(k, l, t) - N_P(k, -l, t)) - k(N_Q(k, l, t) + N_Q(k, -l, t)) \right\}. \tag{3.4b} \]

Applying the inverse Fourier transform formula to these equations, we obtain

\[
\begin{align*}
\hat{u}(x, y, t) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \ e^{ikx + ily} \hat{u}(k, l, t) \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \ e^{ikx + ily} \left\{ 2(\lambda k^3 + kl^2(\lambda + 2\mu))V^{(1)}(k, l, t) - 4\mu k^2V^{(2)}(k, l, t) + k(N_P(k, l, t) + N_P(k, -l, t)) + l(N_Q(k, l, t) - N_Q(k, -l, t)) \right\}, \tag{3.5a} \\
\hat{v}(x, y, t) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \ e^{ikx + ily} \hat{v}(k, l, t) \\
&= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \ e^{ikx + ily} \left\{ 4\mu k^2U^{(1)}(k, l, t) + 2(\mu k^3 - kl^2)U^{(2)}(k, l, t) + l(N_P(k, l, t) - N_P(k, -l, t)) - k(N_Q(k, l, t) + N_Q(k, -l, t)) \right\}.
\end{align*}
\tag{3.5b} \]

Denote by \( H_1(k, l, t) \) and \( H_2(k, l, t) \) the following functions appearing in equations \((3.5)\):

\[
\begin{align*}
H_1(k, l, t) &= \frac{1}{k^2 + l^2} \left\{ 2(\lambda k^3 + kl^2(\lambda + 2\mu))V^{(1)}(k, l, t) - 4\mu k^2V^{(2)}(k, l, t) \right\}, \tag{3.6a} \\
H_2(k, l, t) &= \frac{1}{k^2 + l^2} \left\{ 4\mu k^2U^{(1)}(k, l, t) + 2(\mu k^3 - kl^2)U^{(2)}(k, l, t) \right\}. \tag{3.6b} 
\end{align*}
\]

We observe that for any fixed \( k \in \mathbb{R} \) and fixed \( t, 0 \leq t < T, T > 0 \), the above functions are analytic in the entire complex \( l \)-plane. Indeed, equation \((3.1)\) and the analogous equation for \( U^{(2)} \), imply that in the neighbourhood of \( l = \pm i k \), the following expansion is valid:

\[ H_2(k, l, t) = 2i\mu k \int_0^t \hat{u}(k, s) ds + o(\omega_1^2). \tag{3.7} \]
Similarly,
\[ H_1(k, l, t) = 2i\lambda k \int_0^t \tilde{v}(k, s)ds + o(\omega_1^2). \] (3.8)

The restriction \( y > 0 \), as well as the analyticity of the functions \( H_1(k, l, t) \) and \( H_2(k, l, t) \), allow us to deform the contour of integration from the real axis to a contour \( \gamma_k \) in the upper half \( l - \)plane (the particular choice of \( \gamma_k \) will be determined in the next section):

\[
\begin{align*}
    u(x, y, t) &= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \frac{e^{ikx+ily}}{k^2 + l^2} \left\{ 2(\lambda k^3 + kl^2(\lambda + 2\mu))V^{(1)}(k, l, t) - 4\mu kl^2V^{(2)}(k, l, t) \right\} + \\
    &\quad \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \frac{e^{ikx+ily}}{k^2 + l^2} \left\{ k[N_P(k, l, t) + N_P(k, -l, t)] + l[N_Q(k, l, t) - N_Q(k, -l, t)] \right\},
\end{align*}
\]

(3.9a)

\[
\begin{align*}
    v(x, y, t) &= \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \frac{e^{ikx+ily}}{k^2 + l^2} \left\{ 2\mu kl^2U^{(1)}(k, l, t) + (\mu k^3 - \mu kl^2)U^{(2)}(k, l, t) \right\} + \\
    &\quad \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \frac{e^{ikx+ily}}{k^2 + l^2} \left\{ l[N_P(k, l, t) - N_P(k, -l, t)] - k[N_Q(k, l, t) + N_Q(k, -l, t)] \right\}.
\end{align*}
\]

(3.9b)

4 The Elimination of the Transforms of the Unknown Boundary Values

Let
\[
    l_{21} = -l\left(\frac{\lambda + 2\mu}{\lambda + 2\mu} + \frac{\lambda + \mu}{\mu} \frac{k^2}{l^2}\right)^{\frac{1}{2}}
\] (4.1)

and
\[
    l_{12} = -l\left(\frac{\mu}{\lambda + 2\mu} - \frac{\lambda + \mu}{\lambda + 2\mu} \frac{k^2}{l^2}\right)^{\frac{1}{2}}.
\] (4.2)

The function \( l_{12} \) has two branch points \( \pm k\sqrt{\frac{\lambda + \mu}{\lambda + 2\mu}} \), which we connect by a horizontal branch cut; the function \( l_{21} \) has two branch points \( \pm ik\sqrt{\frac{\lambda + \mu}{\lambda + 2\mu}} \), which we connect by a vertical branch cut, see Fig.1. We fix the branches by the requirements that,

\[
    l_{21} \sim -l\sqrt{\frac{\lambda + 2\mu}{\mu}}, \quad l_{12} \sim -l\sqrt{\frac{\mu}{\lambda + 2\mu}}, \quad \text{as } l \to \infty.
\]
The following transformations are valid in the cut $l$-plane:

\[ l \rightarrow l_{21} \text{ maps } \omega_2 \text{ to } -\omega_1, \quad U^{(2)} \text{ to } U^{(1)}, \quad V^{(2)} \text{ to } V^{(1)}; \quad (4.3a) \]

\[ l \rightarrow l_{12} \text{ maps } \omega_1 \text{ to } -\omega_2, \quad U^{(1)} \text{ to } U^{(2)}, \quad V^{(1)} \text{ to } V^{(2)}. \quad (4.3b) \]

Using in equations (2.20a) and (2.20b) the transformations $l \rightarrow l_{12}$ and $l \rightarrow -l$ respectively, and then combining the two resulting equations, we obtain the following equations:

\[ k\hat{u}(k, l_{12}, t) + l_{12}\hat{v}(k, l_{12}, t) = (\lambda k^2 + I_{12}^2(\lambda + 2\mu))V^{(2)}(k, l, t) \]
\[ + 2\mu kl_{12}U^{(2)}(k, l, t) + N_P(k, l_{12}, t), \]
\[ -l\hat{u}(k, -l, t) - k\hat{v}(k, -l, t) = -\mu(k^2 - l^2)U^{(3)}(k, l, t) + 2\mu klV^{(2)}(k, l, t) + N_Q(k, -l, t). \quad (4.4b) \]

Similarly, using in equations (2.20a) and (2.20b) the transformations $l \rightarrow -l$ and $l \rightarrow l_{21}$ respectively, and then combining the two resulting equations, we obtain the following equations:

\[ k\hat{u}(k, -l, t) - l\hat{v}(k, -l, t) = (\lambda k^2 + I^2(\lambda + 2\mu))V^{(1)}(k, l, t) \]
\[ - 2\mu klU^{(1)}(k, l, t) + N_P(k, -l, t), \]
\[ l_{21}\hat{u}(k, l_{21}, t) - k\hat{v}(k, l_{21}, t) = -\mu(k^2 - l_{21}^2)U^{(1)}(k, l, t) \]
\[ - 2\mu kl_{21}V^{(1)}(k, l, t) + N_Q(k, l_{21}, t). \quad (4.5b) \]
Let
\begin{align}
C_1(k,l) &= \lambda k^2 + l_1^2(\lambda + 2\mu), \quad C_2(k,l) = 2\mu kl_{12}, \quad (4.6a) \\
C_3(k,l) &= 2\mu kl, \quad C_4(k,l) = -\mu(k^2 - l^2), \quad (4.6b) \\
D_1(k,l) &= \lambda k^2 + l^2(\lambda + 2\mu), \quad D_2(k,l) = -2\mu kl, \quad (4.6c) \\
D_3(k,l) &= -2\mu kl_{21}, \quad D_4(k,l) = -\mu(k^2 - l_{21}^2), \quad (4.6d) \\
\Delta_1(k,l) &= C_1(k,l)C_4(k,l) - C_2(k,l)C_3(k,l), \quad (4.6e) \\
\Delta_2(k,l) &= D_1(k,l)D_4(k,l) - D_2(k,l)D_3(k,l). \quad (4.6f)
\end{align}

Simplifying the expressions for \(\Delta_1\) and \(\Delta_2\) we find
\begin{align}
\Delta_1(k,l) &= \mu^2(k^2 - l^2)^2 - 4\mu^2 k^2 l_{12}, \quad (4.7a) \\
\Delta_2(k,l) &= (\lambda k^2 + l^2(\lambda + 2\mu))^2 - 4\mu^2 k^2 l_{21}. \quad (4.7b)
\end{align}

Equations (4.4) imply
\begin{align}
V^{(2)}(k,l,t) &= \frac{1}{\Delta_1(k,l)}[C_4(k,l)(k\hat{u}(k,l_{12}, t) + l_{12}\hat{v}(k,l_{12}, t) - N_P(k,l_{12}, t)) \\
&\quad - C_2(k,l)(-\hat{l}\hat{u}(k,-l,t) - \hat{k}\hat{v}(k,-l,t) - N_Q(-l))], \quad (4.8a) \\
U^{(2)}(k,l,t) &= \frac{1}{\Delta_1(k,l)}[-C_3(k,l)(k\hat{u}(k,l_{12}, t) + l_{12}\hat{v}(k,l_{12}, t) - N_P(k,l_{12}, t)) \\
&\quad + C_1(k,l)(-\hat{l}\hat{u}(k,-l,t) - \hat{k}\hat{v}(k,-l,t) - N_Q(k,-l,t))]; \quad (4.8b)
\end{align}

Equations (4.5) imply
\begin{align}
V^{(1)}(k,l,t) &= \frac{1}{\Delta_2(k,l)}[D_4(k,l)(k\hat{u}(k,-l,t) - \hat{l}\hat{v}(k,-l,t) - N_P(k,-l,t)) \\
&\quad - D_2(k,l)(l_{21}\hat{u}(k,l_{21}, t) - \hat{k}\hat{v}(k,l_{21}, t) - N_Q(l_{21}))], \quad (4.9a) \\
U^{(1)}(k,l,t) &= \frac{1}{\Delta_2(k,l)}[-D_3(k,l)(k\hat{u}(k,-l,t) - \hat{l}\hat{v}(k,-l,t) - N_P(k,-l,t)) \\
&\quad + D_1(k,l)(l_{21}\hat{u}(k,l_{21}, t) - \hat{k}\hat{v}(k,l_{21}, t) - N_Q(k,l_{21}, t))]. \quad (4.9b)
\end{align}

We fix the choice of the contour \(\gamma_k\) by requiring that every term in the RHS of (4.8) and (4.9) does not have a pole or a branch point above this contour, see Fig[2].

Regarding the zeros of \(\Delta_j, j = 1, 2\), we note that they are of the form \(l = \alpha k\), for some constant \(\alpha \in \mathbb{C}\). For example, if \(\lambda = 2\mu\), we find that the zeros of \(\Delta_1\) are
\begin{align*}
l &\approx (-1.624 \pm 0.126i)k, \quad l \approx \pm 0.357i k, \quad l \approx \pm 1.056i k, \quad l \approx (1.624 \pm 0.126i)k;
\end{align*}
whereas the zeros of $\Delta_2$ are

\[ l \approx (-0.295 \pm 0.442i)k, \ l \approx \pm 0.885ik, \ l \approx \pm ik, \ l \approx (0.295 \pm 0.442i)k. \]

Substituting equations (4.8) and (4.9) in equations (3.9), and using Jordan's lemma in the complex $l-$plane above the contour $\gamma_k$, it follows that $\hat{u}(k,-l,t)$, $\hat{u}(k,l_{12},t)$, $\hat{v}(k,-l,t)$, $\hat{v}(k,l_{12},t)$, $\hat{v}(k,l_{21},t)$, yield a zero contribution. Hence equations (3.9) become the following equations:

\[
\begin{align*}
\hat{u}(x,y,t) &= \int_{-\infty}^{\infty} dk \int_{\gamma_k} dl \frac{e^{ikx+ily}}{(k^2+l^2)\Delta_1(k,l)} \left\{ 2(\lambda^3 + kl^2(\lambda + 2\mu))[-D_4(k,l)N_P(k,-l,t) + D_2(k,l)N_Q(k,l_{21},t)] - 4\mu kl^2[-C_4(k,l)N_P(k,l_{12},t) + C_2(k,l)N_Q(k,-l,t)] \right\} + \\
&\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \left\{ k[N_P(k,l,t) + N_P(k,-l,t)] + l[N_Q(k,l,t) - N_Q(k,-l,t)] \right\},
\end{align*}
\]

\[
\begin{align*}
\hat{v}(x,y,t) &= \int_{-\infty}^{\infty} dk \int_{\gamma_k} dl \frac{e^{ikx+ily}}{(k^2+l^2)\Delta_2(k,l)} \left\{ 2\mu kl^2[D_3(k,l)N_P(k,-l,t) - D_1(k,l)N_Q(k,l_{21},t)] + (\mu k^3 - \mu kl^2)[C_3(k,l)N_P(k,l_{12},t) - C_1(k,l)N_Q(k,-l,t)] \right\} + \\
&\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \left\{ k[N_P(k,l,t) - N_P(k,-l,t)] - l[N_Q(k,l,t) - N_Q(k,-l,t)] \right\},
\end{align*}
\]

where \( \{C_j, D_j\}_1^4 \) and \( \{\Delta_j\}_1^2 \) are defined in (4.6) and (4.7), and the known functions $N_P$ and $N_Q$ are defined in (2.21).
We summarize the above result in the following proposition:

**Proposition 4.1** Let \((u, v)\) satisfy the Lamé-Navier equations (2.1) in the half plane 

\[-\infty < x < \infty, \ 0 < y < \infty, \ t > 0,\]

with the initial conditions (2.2) and the stress boundary conditions (2.3). Assume that the given functions

\[\{u_j(x, y), v_j(x, y)\}_{j=0,1}, \ \{g_j(x, t)\}_{j=1,2}\]

have sufficient smoothness and decay. A solution of the above initial-boundary value problem, which decays for large \((x, y)\), is given by equations (4.10), where:

a) The known functions \((N_P, N_Q)\) are defined in (2.21) in terms of the transforms of the initial and boundary data (see equations (2.15)-(2.19)).

b) \([C_j, D_j]^1_4\) and \([\Delta_j]^2_1\) are given by equations (4.6) and (4.7).

c) The contours \(\gamma_k\) depicted in Fig.2, are deformations of the real axis and determined by the requirement that the zeros of \(\Delta_j, j = 1, 2\) and the branch points of \(l_{21}\) and \(l_{12}\) are all below \(\gamma_k\).

5 **The Normal Line Load with the Homogeneous Initial Conditions**

Consider a normal line load suddenly applied to an isotropic elastic half plane body. In this case,

\[u_0 = u_1 = v_0 = v_1 = 0; \ g_1 = 0, \ g_2 = \sigma_0 \delta(x)h(t)/(\lambda + \mu),\]

where \(\sigma_0\) is a constant and \(h(t)\) is the Heaviside function defined by 

\[h(t) = 0, \ t \leq 0; \ h(t) = 1, \ t > 0.\]

In what follows, we compute the functions needed in equations (4.10). Equations (2.9) and (2.19) imply

\[\tilde{g}_1(k, t) = 0, \ \tilde{g}_2(k, t) = \frac{\sigma_0}{\lambda + \mu} h(t),\]

(5.2a)

\[g^{(j)\pm}(k, l, t) = 0, \ G^{(j)}(k, l, t) = 0,\]

(5.2b)

\[f^{(j)\pm}(k, l, t) = \frac{\sigma_0}{\lambda + \mu} \frac{1}{\pm i\omega_j} (e^{\pm i\omega_j t} - 1),\]

(5.2c)

\[F^{(j)}(k, l, t) = -i\sigma_0 \frac{1}{\lambda + \mu} \left( \frac{1}{\omega_j} - \frac{\cos(\omega_j t)}{\omega_j^2} \right), \ j = 1, 2.\]

(5.2d)

Thus, the known functions \(N_P\) and \(N_Q\) defined in (2.21), are given by

\[N_P(k, l, t) = -l\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_l^2} - \frac{\cos(\omega_1 t)}{\omega_1^2} \right),\]

(5.3a)

\[N_Q(k, l, t) = k\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_2^2} - \frac{\cos(\omega_2 t)}{\omega_2^2} \right),\]

(5.3b)
The above equations imply
\[
N_P(k, l, t) + N_P(k, -l, t) = 0, \quad (5.4a)
\]
\[
N_Q(k, l, t) + N_Q(k, -l, t) = 2k\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_2^2} - \frac{\cos(\omega_2 t)}{\omega_2^2} \right), \quad (5.4b)
\]
\[
N_P(k, l, t) - N_P(k, -l, t) = -2l\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \left( \frac{1}{\omega_1^2} - \frac{\cos(\omega_1 t)}{\omega_1^2} \right), \quad (5.4c)
\]
\[
N_Q(k, l, t) - N_Q(k, -l, t) = 0, \quad (5.4d)
\]
\[
N_P(k, l_{12}, t) = -\frac{l_{12}}{k} N_Q(k, l, t), \quad (5.4e)
\]
\[
N_Q(k, l_{21}, t) = -\frac{k}{l} N_P(k, l, t). \quad (5.4f)
\]

Substituting the above relations and equations (4.6) and (4.7) into equations (4.10), we find the following result:

**Proposition 5.1** Let \((u, v)\) satisfy the Lamé-Navier equations (2.1) in the half plane \(-\infty < x < \infty,\ 0 < y < \infty,\ t > 0\), with homogeneous initial condition and the normal line load boundary condition (5.1). The solution of this initial-boundary value problem, which decays for large \((x, y)\), is given by

\[
u(x, y, t) = -\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \frac{e^{ikx + ily}}{(k^2 + l^2)[(k^2 + l^2)(\lambda + 2\mu)^2 - 4\mu^2 k^2 l^2 l_{12}]} \left\{2(\lambda k^3 + kl^3)(\lambda + 2\mu)(1 - \cos(\omega_1 t)) + 4\mu kl^2 l_{12}(1 - \cos(\omega_2 t))\right\},
\]

\[
u(x, y, t) = -\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \frac{e^{ikx + ily}}{(k^2 + l^2)[(k^2 + l^2)(\lambda + 2\mu)^2 - 4\mu^2 k^2 l^2 l_{21}]} \left\{2\mu^2 l^2 [2\mu ll_{21} + (\lambda k^2 + l^2)(\lambda + 2\mu)] \frac{1 - \cos(\omega_1 t)}{\omega_1^2} \right. \\
- \left. 2\mu^2 (k^4 - k^2 l^2)(k^2 - l^2 - 2ll_{12}) \frac{1 - \cos(\omega_2 t)}{\omega_2^2} \right\},
\]

\[
u(x, y, t) = -2\sigma_0 \frac{\lambda + 2\mu}{\lambda + \mu} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dl \frac{e^{ikx + ily}}{k^2 + l^2} \left\{l^2 \frac{1 - \cos(\omega_1 t)}{\omega_1^2} + k^2 \frac{1 - \cos(\omega_2 t)}{\omega_2^2} \right\},
\]

where \((\omega_1, \omega_2)\) are defined by equations (2.17) and \(l_{12}, l_{21}\) are defined by equations (4.2), (4.1) respectively.

Similar results can be obtained for the other Lamb’s problems.
6 Conclusions

The main result of this paper is the derivation of equations (4.10). These equations express the displacements \( u(x, y, t) \) and \( v(x, y, t) \) in terms of an integral along the real line and integrals along contours \( \gamma_k \) of the complex \( l \)-plane; these integrals involve transforms of the given initial and boundary data. Indeed, equations (4.10) involve the functions \( N_P(k, l, t) \) and \( N_Q(k, l, t) \), which are defined in equations (2.21) in terms of the Fourier transforms \( P_0(k, l, t), Q_0(k, l, t), P_1(k, l, t), Q_1(k, l, t) \) of the initial data (see equations (2.16)), as well as in terms of certain known transforms \( G^{(1)}(k, l, t), G^{(2)}(k, l, t), F^{(1)}(k, l, t), F^{(2)}(k, l, t) \) of the boundary data (see equations (2.19)).

The starting point of the derivations of equations (4.10) is the derivation of the global relations (2.20). These equations are the direct consequence of the application of the two-dimensional Fourier transform and of the substitution in the resulting equations of the given initial and boundary conditions. Equations (2.20) involve the known functions \( N_P \) and \( N_Q \), as well as the unknown functions \( U^{(1)}, U^{(2)}, V^{(1)}, V^{(2)} \) (these functions involve certain transforms of the unknown boundary values \( u(x, 0, t) \) and \( v(x, 0, t) \), see equations (2.8) and (2.18)). The elimination of the above unknown functions is achieved by the following steps:

1. By employing the transformation \( l \to -l \), which leave the unknown functions \( U^{(1)}, U^{(2)}, V^{(1)}, V^{(2)} \) invariant, and by utilising the analyticity properties of these functions, we obtain the integral representations (3.9). These representations involve an integral along the real \( k \)-axis and an integral along the contour \( \gamma_k \) of the complex \( l \)-plane.

2. Using the transformations \( l \to l_{12} \) and \( l \to l_{21} \) which map \( U^{(2)} \) to \( U^{(1)}, V^{(2)} \) to \( V^{(1)} \) and \( U^{(1)} \) to \( U^{(2)}, V^{(1)} \) to \( V^{(2)} \) respectively, we express the unknown functions \( (U^{(1)}, U^{(2)}, V^{(1)}, V^{(2)}) \) in terms of known functions as well as in terms of certain unknown functions which however are analytic in a certain domain of the complex \( l \)-plane, see equations (4.8) and (4.9).

3. Using (4.8) and (4.9) in equations (3.9) and employing Jordan’s lemma we obtain equations (4.10).

The main advantages of the new approach are the following:

1. The new method provides an analytic solution of Lamb’s problem with arbitrary initial and boundary conditions.

2. This solution is expressed in terms of the given initial and boundary data. The relevant representation is novel even for the particular case of homogeneous initial conditions (this case has been analysed by several authors). An alternative approach using the Laplace transform is briefly discussed in the Appendix. By comparing equations (4.10) and equation (6.1), the advantage of the new formulae becomes clear. Actually, taking into consideration that the initial-boundary value problems of the equations of elastodynamics are well posed for any finite \( t \), the use of the Laplace transform, which requires \( t \to \infty \), is clearly inappropriate.

3. The new method can be employed for the solution of several related initial-boundary value problems, including the problem of the isotropic half space.
in the axisymmetric case. Furthermore, it can be extended to problems in three dimensions \[40\].

**Appendix**

It is possible to analyse Lamb’s problem by using only the transformations (3.3) instead of using the transformations (3.3) and the transformations (4.3). However, in this case one cannot eliminate directly all unknown boundary values. Instead, one can derive a complicated expression for the unknown boundary values in terms of the given initial and boundary data. (This approach is similar with the one used in [45] for solving Crighton’s problem). Indeed, let the contour \(\gamma_{k2}\) be a simple curve in the lower half \(l-\)plane, determined by the requirement that it does not cross the branch cut associated with \(\omega_1\) and \(\omega_2\), see Fig.3. Let \(K\) be the integral operator defined by

\[
(K[f])(k,t) = \frac{1}{2\pi} \int_{\gamma_{k2}} \frac{f(k,l,t) \, dl}{(k^2 + l^2)^{1/2}},
\]

for any function \(f(k,l,t)\) with appropriate smoothness and decay. Integrating the global relations (2.20) along \(\gamma_{k2}\) we find that the functions

\[
\tilde{h}(k,t) = \begin{pmatrix} \tilde{u}(k,t) \\ \tilde{v}(k,t) \end{pmatrix}, \quad \tilde{g}(k,t) = \begin{pmatrix} \tilde{g}_1(k,t) \\ \tilde{g}_2(k,t) \end{pmatrix},
\]

satisfy a system of Volterra integral equations of the second kind:

\[
\tilde{h}(k,t) = (K[N])(k,t) * \tilde{g}(k,t) + (K[M])(k,t) * \tilde{h}(k,t) + (K[H])(k,t),
\]

(6.1)

\(k \in \mathbb{R}, 0 \leq t < T\),

where \(T > 0\), \(*\) denotes the convolution operation with respect to \(t\),

\[
N(k,l,t) = \begin{pmatrix} i\sqrt{\mu} e^{i\omega_2 t} & -i\sqrt{\mu + 2\mu} e^{i\omega_2 t} \\ i\sqrt{\lambda + 2\mu} e^{i\omega_1 t} & i\sqrt{\lambda + 2\mu} e^{i\omega_1 t} \end{pmatrix},
\]

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\[ M(k, l, t) = \left( \begin{array}{cc} \sqrt{\mu k^2} e^{i\omega_2 t} & 2\sqrt{\mu k} e^{i\omega_2 t} \\ \frac{2\mu}{\sqrt{\lambda + 2\mu}} k e^{i\omega_1 t} & \frac{2\mu}{\sqrt{\lambda + 2\mu}} k^2 e^{i\omega_1 t} \end{array} \right), \]

and the known function \( H(k, l, t) \) is defined by

\[
H(k, l, t) = \left( i e^{i\omega_1 t} (i\omega_1 P_0(k, l) + P_1(k, l)), i e^{i\omega_2 t} (i\omega_2 Q_0(k, l) + Q_1(k, l)) \right)^T.
\]

The solution of the integral equations (6.1) yields the unknown transforms appearing in (2.20).

Equations (6.1) can be solved in closed form by using the Laplace transform with respect to \( t \) and let

\[
w_1 = (p^2 + (\lambda + 2\mu) k^2)^{\frac{1}{2}}, \quad w_2 = (p^2 + \mu k^2)^{\frac{1}{2}},
\]

\[
\Delta = \left( \frac{p^2 + 2\mu k^2}{w_1^2 w_2^2} - 4 \sqrt{\frac{\mu}{\lambda + 2\mu} \frac{\mu k^2}{w_1^2}} \right).
\]

The solution of (6.1) is given by

\[
\hat{h} = \left( \begin{array}{c} \frac{\sqrt{\mu k^2}}{w_1^2} \frac{2\mu}{\sqrt{\lambda + 2\mu}} k \left( \begin{array}{c} \frac{(\lambda + 2\mu)(p^2 + 2\mu k^2) - 2\sqrt{\mu(\lambda + 2\mu)w_1^2}}{(p^2 + 2\mu k^2)^2 - 4\sqrt{\mu(\lambda + 2\mu)w_1^2}} \\
\frac{\sqrt{\mu k^2}}{w_1^2} \left( \begin{array}{c} \frac{1}{\sqrt{\frac{\mu k^2}{w_1^2}}} \frac{2\mu}{\sqrt{\lambda + 2\mu}} k \left( \begin{array}{c} \frac{i}{2\pi} \int_{\gamma_{k_2}} \frac{i\omega_2 Q_0 + Q_1}{\mu(p - i\omega_2)} \frac{dl}{l(k^2 + l^2)^2} \\
\frac{i}{2\pi} \int_{\gamma_{k_2}} \frac{i\omega_2 Q_0 + Q_1}{\mu(p - i\omega_2)} \frac{dl}{l(k^2 + l^2)^2} \end{array} \right) \right) \\
\right) \right) \left( \begin{array}{c}
\end{array} \right)
\]

The zeros of \( \Delta \) coincide with the zeros of Rayleigh’s function. When \( \mu/\lambda > 0.906 \), the known transforms \( \hat{g}(k, p) \), for each fixed \( k \), do not have poles with positive real parts.

In the particular case of the problem of homogeneous initial condition and the normal line load boundary condition, equation (6.4) reduces to the classic Lamb’s solution [1][2].

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References

[1] H. Lamb, *On the propagation of tremors over the surface of an Elastic Solid*, Phil. Trans. Royal. Soc. London, Series A, Vol. 203, pp. 1-42, 1904.

[2] J. Miklowitz, *The theory of elastic waves and waveguides*, North-Holland Publishing Company, 1978.

[3] K. L. Johnson, *Contact mechanics*, Cambridge University Press, Cambridge, 1985.

[4] C. L. Pekeris, *The seismic surface pulse*, Proc. Nat. Acad. Sci., vol. 41, 1955.

[5] R. G. Payton, *An application of the dynamic Betti-Rayleigh reciprocal theorem to moving-point loads in elastic media*, Quart. Appl. Math. Vol. 21, pp. 299-313, 1964.

[6] K. Schiel and S. J. Protázio, *Transient-wave solution for Lamb’s problem at the free surface*, Bulletin of the Seismological Society of America, Vol. 79, No. 6, pp. 1956-1971, Dec., 1989.

[7] D. C. Gakenheimer, *Transient excitation of an elastic-half space by a point load travelling on the surface*, PhD Thesis, California Institute of Technology, 1969.

[8] J. R. Barber, *Surface displacements due to a steadily moving point force*, J. Appl. Mech., Vol. 63, Jun., 1976.

[9] M. C. M. Bakker, M. D. Verweij, B. J. Kooij, H. A. Dietermana, *The travelling point load revisited*, Wave Motion 29, 1999.

[10] R. G. Payton, *Transient motion of an elastic half-space due to a moving surface line loads*, Int. J. Engng Sci. Vol. 5, pp. 49-79, 1967.

[11] J. Miklowitz, W. R. Garrott, *Lamb’s problem for an impulsive line load on a Laminated Composite*, Modern problems in elastic wave propagation, IUTAM, Symposium at Northwestern University, Illinois, USA, 1977.

[12] H. G. Georgiadis and J. R. Barber, *Steady-state transonic motion of a line load over an elastic half-space. The corrected Cole/Huth solution*, Journal of Applied Mechanics, Vol. 60, No. 3, 772-774, 1993.

[13] R. G. Payton, *Elastic wave propagation in transversely isotropic media*, Martinus Nijhoff Publishers, USA, 1983.

[14] A. S. Fokas, *A unified transform method for solving linear and certain nonlinear PDEs*, Proc. Math. Phys. Eng. Sci. 453, pp. 1411-1443, 1997.

[15] A. S. Fokas, *A unified approach to boundary value problems*, SIAM, 2008.
[16] D. G. Crowdy, A. S. Fokas, *Explicit integral solutions for the plane elasto-static semi-strip*, Proc. Roy. Soc. London Ser. A 460, pp. 1289-1309, 2004.

[17] D. ben-Avraham, A. S. Fokas, *The solution of the modified Helmholtz equation in a wedge and an application to diffusion-limited coalescence*, Phys. Lett. A 263, pp. 355-359, 1999.

[18] D. ben-Avraham, A. S. Fokas, *The modified Helmholtz equation in a triangular domain and an application to diffusion-limited coalescence*, Phys. Rev. E 64, 2001.

[19] A. S. Fokas, A. A. Kapaev, *A Riemann-Hilbert approach to the Laplace equation*, J. Math. Anal. and Appl. 251, pp. 770-804, 2000.

[20] A. S. Fokas, *On the integrability of linear and nonlinear PDEs*, J. Math. Phys. 41, pp. 4188-4237, 2000.

[21] A. S. Fokas, *Two dimensional linear PDE’s in a convex polygon*, Proc. R. Soc. Lond. A 457, pp. 371-393, 2001.

[22] A. S. Fokas, M. Zyskin, *The fundamental differential form and boundary value problems*, Quart. J. Mech. Appl. Math. 55, pp. 457-479, 2002.

[23] A. S. Fokas, *A new transform method for evolution PDEs*, IMA J. Appl. Math. 67, pp. 1-32, 2002.

[24] A. S. Fokas, *Boundary-value problems for linear PDEs with variable coefficients*, Proc. R. Soc. London A 460, pp. 1131-1151, 2004.

[25] P. A. Treharne, A. S. Fokas, *Boundary-value problems for systems of evolution equations*, IMA J. Appl. Math 69, 2004.

[26] A. S. Fokas, B. Pelloni, *Boundary value problems for Boussinesq type systems*, Math. Phys. Anal. Geom. 8, pp. 59-96, 2005.

[27] A. S. Fokas, B. Pelloni, *A transform method for evolution PDEs on the interval*, IMA J. Appl. Maths 75, pp. 564-587, 2005.

[28] A. S. Fokas, D. T. Papageorgiou, *Absolute and convective instability for evolution PDEs on the half-line*, Studies in Appl. Maths. 114, pp. 95-114, 2005.

[29] A. S. Fokas, D. A. Pinotsis, *The Dbar formalism for certain linear non-homogeneous elliptic PDEs in two dimensions*, Eur. J. Appl. Math. 17, I. 3, pp. 323-346, 2006.

[30] A. S. Fokas, D. A. Pinotsis, *Quaternions, evaluation of integrals and boundary value problems*, Comp. Meth. Funct. Th. 7, pp. 443-476, 2007.

[31] P. A. Treharne, A. S. Fokas, *Initial-boundary value problems for linear PDEs with variable coefficients*, Camb. Phil. Soc. 143, pp. 221-242, 2007.
[32] G. Dassios, A.S. Fokas, The basic elliptic equations in an equilateral triangle, Proc R Soc A, 461, pp. 2721-2748, 2005.

[33] G. Dassios, A.S. Fokas, Methods for Solving Elliptic PDEs in Spherical Coordinates, SIAM J. APPL. MATH. Vol. 68, No. 4, pp. 1080-1096, 2008.

[34] E. A. Spence, A. S. Fokas, A new transform method I: domain-dependent fundamental solutions and integral representations, Proc R Soc A, 466, pp. 2259-2281, 2010.

[35] E. A. Spence, A. S. Fokas, A new transform method II: the global relation, and boundary value problems in polar co-ordinates, Proc R Soc A, 466, pp. 2283-2307, 2010.

[36] B. Pelloni, The spectral representation of two-point boundary value problems for linear evolution equations, Proc R Soc London Ser A, 461, pp. 2965-2984, 2005.

[37] Y. Antipov, A. S. Fokas, A transform method for the modified Helmholtz equation on the semi-strip, Math. Proc. Cambridge Philos. Soc. 137, pp. 339-365, 2004.

[38] A. S. Fokas, A. A. Kapaev, On a transform method for the Laplace equation in a polygon, IMAI. Appl. Math. 68, pp. 355-408, 2003.

[39] S. Fulton, A. S. Fokas, C. Xenophontos, An analytical method for linear elliptic PDEs and its numerical implementation, J. Comput. Appl. Math. 167, pp. 465-483, 2004.

[40] A. G. Sifalakis, A. S. Fokas, S. R. Fulton, Y. G. Saridakis, The generalised Dirichlet-Neumann map for linear elliptic PDE’s and its numerical implementation, J. Comput. Appl. Math. 219, pp. 9-34, 2008.

[41] N. Flyer, A. S. Fokas, A hybrid analytical numerical method for solving evolution partial differential equations. I. The half-line, Proc. R. Soc. 464, pp. 1823-1849, 2008.

[42] N. Flyer, A. S. Fokas, S. A. Smitheman, E. A. Spence, A semi-analytical numerical method for solving evolution and elliptic partial differential equations, J. Comp. Appl. Math. 227, pp. 59-74, 2009.

[43] K. Kalimeris, A. S. Fokas, The heat equation in the interior of an equilateral triangle, Stud. Appl. Math. 124, pp. 283-305, 2010.

[44] S. A. Smitheman, E. A. Spence, A. S. Fokas, A spectral collocation method for the Laplace and modified Helmholtz equations in a convex polygon, IMA J. Num. Anal. (in press), 2010.

[45] A. C. L. Ashton, A. S. Fokas, A novel method of solution for the fluid-loaded plate, Proc. R. Soc. A 465, pp. 3667-3685, 2009.
[46] A. S. Fokas, D. Yang, *A novel approach to elastodynamics: II. The three dimensional case*, (preprint), 2010.