Non-local amplification of intense vorticity in turbulent flows

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(Dated: October 27, 2021)

The nonlinear and nonlocal coupling of vorticity and strain-rate constitutes a major hindrance in understanding the self-amplification of velocity gradients in turbulent fluid flows. Utilizing high-resolution direct numerical simulations of isotropic turbulence in periodic domains of up to $12288^3$ grid points, and Taylor-scale Reynolds number $R_t$ in the range 140 – 1300, we investigate this nonlocality by decomposing the strain-rate tensor into local and non-local contributions obtained through Biot-Savart integration of vorticity in a sphere of radius $R$. We find that vorticity is predominantly amplified by the non-local strain coming beyond a characteristic scale size, which varies as a simple power-law of vorticity magnitude. The underlying dynamics preferentially align vorticity with the most extensive eigenvector of non-local strain. The remaining local strain aligns vorticity with the intermediate eigenvector and does not contribute significantly to amplification; instead it surprisingly attenuates intense vorticity, leading to breakdown of the observed power-law and ultimately also the scale-invariance of vorticity amplification, with important implications for prevailing intermittency theories.

Complex non-linear physical systems are often characterized by formation of extreme events, which strongly deviate from Gaussianity, necessitating anomalous corrections to mean-field descriptions \cite{1,2}. Fluid turbulence, described by the three-dimensional incompressible Navier-Stokes equations (INSE), is an emblematic example of such a system, where extreme events are associated with intermittent formation of large velocity gradients, organized into thin filaments of intense vortices \cite{3–5}. The amplification of such intense gradients is readily described by the vortex-stretching mechanism, which expresses the non-linear stretching of vorticity $\omega$ by the strain-rate tensor $S_{ij}$ in the INSE (written as the vorticity equation):

\begin{equation}
\frac{D\omega_i}{Dt} = \omega_j S_{ij} + \nu \nabla^2 \omega_i ,
\end{equation}

where $\nu$ is the kinematic viscosity.

The canonical description based on angular momentum conservation dictates that as vortical filaments are stretched by strain, they become thinner and spin faster, enabling gradient amplification, and simultaneously driving the energy cascade from large to small-scales \cite{8,9}. Though Eq. (1) is valid pointwise, this multiscale description can be analyzed by realizing that vorticity and strain are related non-locally via Biot-Savart integral over the entire flow domain:

\begin{equation}
S_{ij}(x) = \frac{PV}{8\pi} \int_S \left( \epsilon_{kli} r_{lj} + \epsilon_{jkl} r_{ki} \right) \omega_l(x') \frac{r_k}{r^3} d^3x' ,
\end{equation}

where $r = x - x'$, $r = |r|$ and $\epsilon_{ijk}$ is the Levi-Civita symbol. This integral essentially couples all the scales, providing a direct means to understand the non-locality of gradient amplification, without involving additional complexities such as the pressure field \cite{11,12}. However, the integral in Eq. (2) is analytically intractable, leading to outstanding challenges in turbulence theory and also in establishing the regularity of INSE \cite{12}. In this Letter, we investigate the nonlocality of vorticity self-amplification by tackling the Biot-Savart integral in Eq. (2) via direct numerical simulations (DNS) of INSE \cite{3}.

To analyze the nonlocality w.r.t. a scale size $R$, the integration domain in Eq. (2) is separated into a spherical neighborhood of radius $r \leq R$, and the remaining domain \cite{12}:

\begin{equation}
S_{ij}(x) = \int_{r > R} \left[ \cdots \right] d^3x' + \int_{r \leq R} \left[ \cdots \right] d^3x' ,
\end{equation}

where $S_{ij}^{NL}$ represents the non-local or background strain acting on the vorticity to stretch it, and $S_{ij}^L$ is the local strain induced in response to stretching. We utilize DNS to compute $S_{ij}^{L,NL}$ and investigate their interaction with vorticity for various $R$, allowing us to quantify the degree of nonlocality of vortex-stretching, and thereafter relate it to vortical structures in the flow.

While computing $S_{ij}^{L,NL}$ through numerical integration is possible in DNS \cite{13}, it is prohibitively expensive at high Reynolds numbers \cite{16}. Instead, as derived in our recent work \cite{15}, non-local (and local) strain can be efficiently computed for any $R$ by applying a transfer function to the total strain in Fourier space: $\tilde{S}_{ij}^{NL}(k, R) = f(kR) \tilde{S}_{ij}(k)$, with $f(kR) = 3 \sin(kR) - kR \cos(kR))/((kR)^3$, thus bypassing the direct evaluation of the Biot-Savart integral. This novel approach is used to analyze a large DNS database, generated using the canonical setup of forced stationary
isotropic turbulence in a periodic domain, utilizing the highly-accurate Fourier pseudo-spectral methods. Special attention is given to maintain a grid-resolution of smaller than the Kolmogorov length scale η to resolve the extreme events accurately. The database corresponds to Taylor-scale Reynolds number $R_\lambda$ in the range 140–1300, on up to grids of $12288^3$ (for additional details see [13, 18–20]).

The efficacy of vortex-stretching is controlled by the alignment between vorticity and strain-rate, and is commonly studied in the eigenframe of strain tensor – given by the eigenvalues $\lambda_i$ ($\lambda_1 \geq \lambda_2 \geq \lambda_3$) and the corresponding eigenvectors $\mathbf{e}_i$. Incompressibility imposes $\lambda_1 + \lambda_2 + \lambda_3 = 0$, giving $\lambda_1 > 0$ and $\lambda_2 < 0$. (The corresponding quantities for local/non-local strain are defined with superscripts L/NL). It is well-known that $\lambda_2$ is positive on average and vorticity preferentially aligns with the intermediate (second) eigenvector of the total strain rate $[19, 21–23]$. This alignment is often regarded as anomalous, since an analogy with stretching of material-lines suggests that vorticity should align with the first eigenvector of total strain, corresponding to the largest eigenvalue $[24]$.

The earlier work of [14], based on direct evaluation of the Biot-Savart integral for a single value of $R = 12\eta$ at very low Reynolds number $R_\lambda = 100$, provides some evidence that vorticity preferentially aligns with the first eigenvector of the non-local strain (similar to stretching of material-lines), whereas the anomalous alignment results from local dynamics. In the following, we provide a comprehensive investigation of the alignment properties, as a function of $R$ and over a drastically larger $R_\lambda$-range. In addition, we also condition on the enstrophy, $\Omega = \omega_i \omega_i$, to analyze generation of intense vorticity. To this end, we extract the second-moment of directional cosines: $\langle (e_{NL}^i \cdot \hat{\omega})^2 \rangle$, whose averages are individually bounded between 0 and 1 (with 1/3 corresponding to a uniform distribution), and additionally also add up to unity, i.e., $\sum_{i=1}^3 (e_{NL}^i \cdot \hat{\omega})^2 = 1$ [19].

The directional cosines are shown as a function of scale-size $R/\eta$ in Fig. 1 and conditioned on $\Omega/\langle \Omega \rangle$ to separate the extreme events. The alignments for $S^L$ are explored first in Fig. 1a-c, corresponding to $\Omega/\langle \Omega \rangle = 1, 100, 1000$. We observe that for all $R/\eta$, vorticity preferentially aligns with second eigenvector of $S^L$, with a tendency to be orthogonal to first and third eigenvectors. The alignment properties become more pronounced as $\Omega$ increases.

Overall, this result conforms to the picture of axisymmetric vortex tubes, where the velocity field is approximately two-dimensional, resulting in preferential alignment of vorticity with the second eigenvector of $S^L$ [13, 23, 26]. Interestingly, vorticity is more orthogonal to the first eigenvector compared to the third for small $R$ ($\lesssim 10\eta$), with the difference becoming more pronounced for large $\Omega$ in panel c (we return to this behavior later). At large $R$, this trend is reversed, approaching the well known result corresponding to total strain ($\text{as } S^L = S$ for $R \rightarrow \infty$) [19, 21].

The alignment of vorticity with $S^{NL}$ is shown next in Fig. 1d-f. The known alignment between vorticity and the intermediate eigenvector of $S$ is recovered at $R = 0$ (where $S^{NL} = S$). However, as $R$ increases, a switch occurs and $\omega$ preferentially aligns with the first eigenvector of $S^{NL}$, more strongly as $\Omega$ increases (while vorticity is al-

FIG. 1. Conditional second moment of the alignment cosines between vorticity and eigenvectors of the local (L) and non-local (NL) strain tensors at $R_\lambda = 1300$ (solid lines) and 650 (dashed lines), and various conditioning values of enstrophy $\Omega$. The dotted line at 1/3 in each panel corresponds to a uniform distribution of the cosines. Note, $S^L_{ij} = 0$ at $R = 0$, with the alignments being undefined.
As \( \Omega \) increases, the normalized magnitude of \( S^L \) approaches unity at a smaller \( R \), whereas that of \( S^{NL} \) falls of towards zero in a similar fashion. This critical distance, say \( R_c(\Omega) \), at which their relative magnitudes are equal, steadily decreases with \( \Omega \), qualitatively consistent with the switching of alignment in Fig. 1f.

The results in Figs. 1d-f allow us to identify characteristic length scales, which demarcate the relative importance of local and non-local dynamics, and its dependence on \( \Omega \). The analysis of Burgers vortices presented in [13], establishes that that \( R^L(\Omega) \sim R(\Omega) \), and they physically identify the radii of vortex tubes in the flow [14]. A simple method to obtain the radius of a vortex tube is from a balance between viscosity \( \nu \) and some effective strain \( S \), giving \( R(\langle \Omega \rangle) \sim (\nu/S)^{1/2} \) [28]. Utilizing strain corresponding to mean-field, i.e. \( S \sim \langle \epsilon \rangle/\nu \), where \( \langle \epsilon \rangle \) is the mean-dissipation rate, results in the well-known expression for the Kolmogorov length scale \( \eta = (\nu^3/\langle \epsilon \rangle)^{1/4} \). However, strain acting on intense vorticity grows with vorticity, given by the power-law [7, 19]:

\[
\langle |S|^2 \rangle / \Omega^n, \quad 0 < \gamma < 1
\]

where the exponent \( \gamma \) weakly increases with \( R_\lambda \), ostensibly approaching unity at \( R_\lambda \rightarrow \infty \) [28]. Utilizing Eq. (5), and \( \langle \epsilon \rangle = \nu\langle \Omega \rangle \) from statistical homogeneity, the radius of tubes \( R^* \) can be written as a function of \( \Omega \):

\[
R^* / \eta = (\langle \Omega \rangle / \langle \Omega \rangle)^{-\gamma/4} .
\]

To test the result in Eq. (5), Fig. 3 shows the curves for \( R_c(\Omega) / \eta \) (dashed lines) and \( R_c(\Omega) / \eta \) (solid lines) extracted from Fig. 1d-f and Fig. 2 respectively. Firstly, we observe that both \( R_c(\Omega) \) and \( R^c(\Omega) \) are always comparable and follow the same trend for moderately intense vorticity, consistent with the power-law predicted by Eq. (5) (represented by the black dashed line). For very intense events (\( \langle \Omega \rangle / \langle \Omega \rangle \gtrsim 100 \)), \( R_c(\Omega) \) is still consistent with the power-law, but \( R^c(\Omega) \) starts deviating. However, these deviations occur at slightly increasing values of \( \Omega \) when \( R^c_3 \) increases. We note that over the range of \( R_3 \) (from 390 to 1300), the exponent \( \gamma/4 \) only varies from 0.17 to 0.19 (respectively), and this small change in slope is also faintly visible for the curves corresponding to \( R_c \). It is worth noting that such a dependence of vortex radius on \( \Omega \) was not possible to detect in earlier studies at significantly lower \( R_\lambda \) [14].

To analyze deviations of \( R^c_3 \) at large \( \Omega \), we consider the enstrophy production term, \( P_\Omega = \omega_i\omega_jS_{ij} \), which also represents the effective strain acting to amplify vorticity by factoring in the alignments. Similar to Eq. (3), we can also decompose \( P_\Omega \) as \( P_\Omega = P^L_\Omega + P^{NL}_\Omega \), where \( P^{NL}_\Omega = \omega_i\omega_jS_{ij}^{NL} \). The conditional expectation of the non-local production \( \langle P^{NL}_\Omega \rangle \), normalized by the total conditional production \( \langle P_\Omega \rangle \), is shown in Fig. 4.

For regions of moderately strong vorticity (\( \Omega \lesssim 10(\langle \Omega \rangle) \)), the normalized production term \( P^{NL}_\Omega \) behaves qualitatively similar as non-local strain in Fig. 2 – it starts at
reiterate that vorticity is pre-
sents at increasing Ω values with $R_\lambda$, in agreement with the deviations of $R_c^\Omega(\Omega)$ in Fig. 3.

A breakdown of individual contributions from each eigenvalue for both $P^{NL}_\Omega$, normalized by the total production, is shown next in Fig. 5. Fig. 5-b shows that the first eigenvalue of non-local strain provides most of the production, with the contributions from the second and third eigenvalues largely canceling each other; except at small $R$, where the second eigenvalue provides a small but significant contribution. The contributions to the local production in Fig. 5-d shows a very weak role of the intermediate eigenvalue for small $R$, despite the very strong alignment observed in Fig. 1-c. Rather, the contributions from first and third eigenvalues are more prominent, with the third eigenvalue ultimately leading to overall negative local production at large $\Omega$ and small $R$ (which can also be traced to the slightly better alignment of vorticity with the third eigenvector instead of the first, also observed in Fig. 1-c). These results highlight the non-trivial role of nonlinearity, going beyond a simple kinematic alignment switching as hypothesized earlier [13, 25].

The results in Fig. 1-5 reiterate that vorticity is predominantly amplified non-locally, analogous to linear dynamics of material-line-stretching; whereas the nonlinear effects are local and restricted to small distances, but still playing an important role. Since as vorticity is amplified beyond a threshold, the local effects directly counteract further amplification, reflecting a fundamental change in the nature of extreme events. It marks a breakdown of scale-invariance (self-similarity) of vorticity amplification at small-scales, also explaining why the power-law derived in Eq. 10 fails to capture the behavior of $R_c^\Omega(\Omega)$ (in Fig. 3) for large $\Omega$. In contrast, for Burgers vortices, for which $R_c^\Omega(\Omega) = R_c(\Omega)$, the stretching produced by local strain is always zero [13], i.e., the self-attenuation mechanism is always absent.

The breakdown of scale-invariance can further be shown by considering the critical scale $R_c^P = R_c^\Omega(\Omega)$, defined by the condition that non-local enstrophy production recovers most of the total production (as shown in Fig. 6). Remarkably, we find that $R_c^P$ seemingly becomes constant at large $\Omega$, marking a critical scale below which the non-local effects do not penetrate and local dynamics dominate. A comparison with Fig. 3 shows that the value of $R_c^P$ and range of $\Omega$ where its constant are consistent with where $R_c^\Omega$ deviates from $R_c$ – once again consistent with the onset of self-attenuation mechanism [13].

The breakdown of scale-invariance (self-similarity) of vortex-stretching leads to some important consequences for turbulence theory and modeling. Prevalent intermittency theories postulate that gradient amplification and the resulting energy-cascade is self-similar across scales, until regularized by viscosity. In fact, such an assumption

unity for $R = 0$ and monotonically decreases to zero at $R \to \infty$. However, when conditioned on extreme values of $\Omega (\gtrsim 100\Omega)$, the normalized $P^{NL}_\Omega$ overshoots unity at small $R$, before decreasing more sharply at larger $R$. Since $P^{NL}_\Omega / P_\Omega = 1 - P^{NL}_\Omega / P_\Omega$, this observation implies that local production is negative for small $R$, and thus counteracts vorticity amplification for large $\Omega$. This is in fact a manifestation of the self-attenuation mechanism recently identified in [13], which provides an inviscid mechanism to arrest vorticity growth and supports regularity of Navier-Stokes equations. Note, viscosity plays an implicit role, since stationarity imposes a conditional balance between net inviscid production and viscous de-

FIG. 5. The individual contributions from each eigenvalue to the non-local (NL) and local (L) enstrophy production terms, normalized by the production based on total strain, at $R_\lambda = 1300$. 

FIG. 4. Conditional expectation of the enstrophy production based on non-local strain, $\langle \omega_i \omega_j S^\Omega_{ij}(\Omega) \rangle$, normalized by the corresponding enstrophy production for total strain, as a function of $R/\eta$, at $R_\lambda = 1300$ (solid lines) and 650 (dashed lines).
is directly built into celebrated Kolmogorov’s hypotheses and also multifractal and shell models [31]. However, current results point to an intricate role of nonlinearity, which acts in conjunction with viscosity to attenuate the most extreme events. This casts serious doubts on the dimensional estimate of the scale where viscous effects become prevalent, as used by phenomenological models. In fact, there is mounting evidence that such models are inadequate at characterizing extreme events, even at large Reynolds numbers [14][17][18]. A similar situation also applies to large-eddy simulation, where local dynamics are unresolved (by definition). The current results call for development of new models which can, for instance, appropriately capture the self-attenuation mechanism.

In conclusion, using state-of-the-art DNS, we have analyzed non-locality of vorticity-amplification by directly tackling the global Biot-Savart integral. We show that vorticity is predominantly amplified by non-local strain, the characteristic scale of nonlocality, which varies as a simple power-law of vorticity magnitude. The nonlinear effects are captured by the remaining local strain, revealing that the nature of extreme events is fundamentally different due to the self-attenuation mechanism [13], ultimately leading to a breakdown of the observed power-law and scale-invariance of vortex-stretching mechanism. Further investigations are ongoing and are expected to provide essential ingredients for improved intermittency theories and turbulence models.

Acknowledgements: We gratefully acknowledge the Gauss Centre for Supercomputing e.V. for providing computing time on the supercomputers JUQUEEN and JUWELS at Jülich Supercomputing Centre (JSC), where the simulations reported in this paper were performed.

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