Equimatchable factor-critical graphs and independence number 2

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Abstract

A graph is equimatchable if each of its matchings is a subset of a maximum matching. It is known that any 2-connected equimatchable graph is either bipartite, or factor-critical, and that these two classes are disjoint. This paper provides a description of \( k \)-connected equimatchable factor-critical graphs with respect to their \( k \)-cuts for \( k \geq 3 \). As our main result we prove that if \( G \) is a \( k \)-connected equimatchable factor-critical graph with at least \( 2k + 3 \) vertices and a \( k \)-cut \( S \), then \( G - S \) has exactly two components and both these components are close to being complete or complete bipartite. If both components of \( G - S \) additionally have at least 3 vertices and \( k \geq 4 \), then the graph has independence number 2. On the other hand, since every 2-connected odd graph with independence number 2 is equimatchable, we get the following result. For any \( k \geq 4 \) let \( G \) be a \( k \)-connected odd graph with at least \( 2k + 3 \) vertices and a \( k \)-cut \( S \) such that \( G - S \) has two components with at least 3 vertices. Then \( G \) has independence number 2 if and only if it is equimatchable and factor-critical. Furthermore, we show that a 2-connected odd graph \( G \) with at least 4 vertices has independence number at most 2 if and only if \( G \) is equimatchable and factor-critical and \( G + e \) is equimatchable for every edge of the complement of \( G \).

Keywords: graph, matching, equimatchable, factor-critical, independence number, cut.

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1 Introduction

A graph is \textit{equimatchable} if each of its maximal matchings is maximum. Equimatchable graphs were introduced in [5], [9], and [11]; in particular Grünbaum [5] asked for a characterisation of all equimatchable graphs. If equimatchable graphs are required to have a perfect matching, the answer turns out to be fairly simple – \( K_{2n} \) and \( K_{n,n} \) for all \( n \) are the only such graphs, see [16]. A general description of all equimatchable graphs in terms of their Gallai-Edmonds decomposition is provided in [8]. Particular consequences of this description are that there is a polynomial-time algorithm recognizing equimatchable graphs, and that every 2-connected equimatchable graph is either bipartite, or factor-critical. On the other hand, if the graph is 2-connected, then the Gallai-Edmonds decomposition provides no additional information about the structure of the graph. Since these early results, a significant attention was given to equimatchable graphs and related concepts of extendability, see [12], [13], and [14] for surveys of the area. Despite considerable effort, the structure of equimatchable graphs is still not very well understood. Particular exceptions are equimatchable factor-critical graphs with cuts of size 1 or 2, which were characterized in [3], and planar and cubic equimatchable graphs, which were characterized in [6]. The aim of this paper is to describe the structure of equimatchable factor-critical graphs with respect to their minimum vertex cuts, extending the results of Favaron [4] to graphs with higher connectivity. We build on a result that for any minimal matching \( M \) isolating a vertex \( v \) of a 2-connected equimatchable factor-critical graph \( G \) the graph \( G - (V(M) \cup \{v\}) \) is connected, which was used in [2] to bound the maximum size of equimatchable factor-critical graphs with a given genus.

Matchings in graphs with independence number 2 were studied during attempts to solve a special case of Hadwiger’s conjecture, see [15] and [1] for details. In particular, it is known that any odd graph with independence number 2 is factor-critical, see for example [15]. We reveal further connections between matchings and graphs with independence number 2.

Our main results can be described as follows. Let \( G \) be a \( k \)-connected equimatchable factor-critical graph with a \( k \)-cut \( S \), where \( k \geq 3 \). If \( G - S \) has at least \( 2k + 3 \) vertices, then \( G - S \) has exactly two components
and both these components are very close to being complete or complete bipartite. If both components of $G - S$ additionally have at least 3 vertices, then both are complete. Furthermore, if we also require $k \geq 4$, then the graph has independence number 2. On the other hand, we show that every 2-connected odd graph with independence number 2 is equimatchable and thus we get the following result. For any $k \geq 4$ let $G$ be a $k$-connected odd graph with at least $2k + 3$ vertices and a $k$-cut $S$ such that $G - S$ has two components with at least 3 vertices. Then $G$ is equimatchable and factor-critical if and only if it has independence number 2. It turns out that independence number is related with equimatchable graphs also in the following way. A 2-connected odd graph $G$ with at least 4 vertices has independence number at most 2 if and only if $G$ is equimatchable and factor-critical and $G + e$ is equimatchable for every edge $e$ of the complement of $G$.

2 Preliminaries

All graphs in this paper are finite, undirected, and simple. All subgraphs are considered to be induced subgraphs unless immediately evident otherwise. If $X$ is a set and $x$ an element of $X$, for brevity we denote the set obtained by removing $x$ from $X$ by $X - x$. If $G$ is a graph and $v$ a vertex of $G$, with a slight abuse of notation we denote by $G - v$ the subgraph of $G$ induced by $V(G) - v$. We say that an edge is between $A$ and $B$ if it has one endpoint in $A$ and the other endpoint in $B$, where $A$ and $B$ are subgraphs, or sets of vertices, of a graph $G$. Similarly, a set of edges or a matching are between $A$ and $B$ if all their edges are between $A$ and $B$. A graph or a component is even if it has even number of vertices, otherwise it is odd. By a cut we always mean a vertex cut. A graph is randomly matchable if it is equimatchable and has a perfect matching; it is known that a graph is connected and randomly matchable if and only if it is isomorphic with $K_{2n}$ or $K_{n,n}$ for some positive integer $n$, see [10]. For a matching $M$ of a graph $G$, by $V(M)$ we denote the vertices of $G$ covered by the edges of $M$. We say that a matching $M$ isolates a vertex $v$ of $G$ if $\{v\}$ is a component of $G - V(M)$. A matching $M$ is a minimal isolating matching of $v$ if $M$ isolates $v$ and no proper subset of $M$ isolates $v$. We repeatedly use the following result.

**Theorem 2.1** (Eiben and Kotrbčík [2]). Let $G$ be a 2-connected equimatchable factor-critical graph. Let $v$ be a vertex of $G$ and $M_v$ a minimal matching isolating $v$. Then $G - (V(M_v) \cup \{v\})$ is connected and randomly matchable.

We assume that the reader is familiar with basic properties of matchings; for more details we refer to [10].

3 Vertex cuts in equimatchable factor-critical graphs

The aim of this section is to describe the structure of equimatchable factor-critical graphs with respect to their minimum vertex cuts. Favaron [4] provided a characterisation of equimatchable factor-critical graphs with connectivity 1 or 2 with respect to their minimum vertex cuts.

**Theorem 3.1** (Favaron [4]). A graph $G$ with vertex-connectivity 1 is equimatchable and factor-critical if and only if all of the following conditions hold:
1. $G$ has exactly one cut-vertex $d$;
2. every connected component $C_i$ of $G - d$ is randomly matchable; and
3. $d$ is adjacent to at least two adjacent vertices of each $C_i$.

While the case of graphs with connectivity 1 is somewhat exceptional, our results for connectivity $k \geq 3$ are in nature very similar to Theorem 3.2 below. In particular, the difficulties with describing the whole larger component in the case when the smaller component is a singleton carry completely to large connectivity, as can be seen from Theorem 3.3.

**Theorem 3.2** (Favaron [4]). Let $G$ be a 2-connected equimatchable factor-critical graph with at least 4 vertices and a 2-cut $S = \{s_1, s_2\}$. Then $G - S$ has precisely two components, one of them even and the other odd. Let $A$ and $B$ denote the even, respectively the odd component of $G - S$, let $a_1$ and $a_2$ be two distinct vertices of $A$ adjacent to $s_1$ and $s_2$, respectively, and, if $|B| > 1$, let $b_1$ and $b_2$ be two distinct vertices of $B$ adjacent to $s_1$ and $s_2$, respectively. Then $G$ has the following structure:
1. $B$ is one of the four graphs $K_{2p+1}$, $K_{2p+1} - \{b_1b_2\}$, $K_{p,p+1}$, $K_{p,p+1} \cup \{b_1b_2\}$ for some nonnegative integer $p$. In the two last cases all neighbours of $S$ in $B$ belong to the larger partite set of $K_{p,p+1}$.
2. $A - \{a_1, a_2\}$ is connected randomly matchable and, if $|B| > 1$, then $A$ is connected randomly matchable.

We extend these results to arbitrary fixed connectivity $k \geq 3$ by showing that if the graph has at least $2k + 3$ vertices, then there are exactly two components and both these components are almost complete or complete
bipartite. Our point of departure is a lemma which allows us to efficiently apply Theorem 2.1 to bound the number of components.

**Lemma 3.3.** Let $G$ be a 2-connected equimatchable factor-critical graph, $M$ a matching of $G$, and $H$ an odd component of $G − V(M)$. Then $G − (H ∪ V(M))$ is connected randomly matchable.

**Proof.** Since $G$ is equimatchable, the matching $M$ can be extended to a maximum matching $M'$ of $G$. The fact that $G$ is factor-critical implies that $M'$ leaves uncovered exactly one vertex $v$ of $G$. Clearly, $M'$ cannot cover all vertices of $H$ and hence $v$ lies in $H$. The matching $M'$ covers all neighbours of $v$ and thus it is an isolating matching of $v$. Consider any minimal matching $M_v$ such that $M_v ⊆ M'$ and $M_v$ isolates $v$. Let $G'$ denote the graph $G − (V(M_v) ∪ \{v\})$. By Theorem 2.1 the graph $G'$ is connected randomly matchable. It is not difficult to see that $M_v$ can contain only edges of $M$ and edges of $H$, and thus $\{v\} ∪ V(M_v) ⊆ H ∪ V(M)$. It follows that $G − (H ∪ V(M)) ⊆ G − (\{v\} ∪ V(M_v)) = G'$ and that the graph $G − (H ∪ V(M))$ can be obtained from $G'$ by removing the vertices covered by the edges of $M − M_v$. It is easy to see that removing any two adjacent vertices of $K_{2n}$ or $K_{n,n}$ leads to $K_{2n−2}$ or $K_{n−1,n−1}$. We conclude that $G − (H ∪ V(M))$ is connected randomly matchable, as claimed.

The next lemma guarantees the existence of a large number of independent edges between any subset of a cut and a component separated by the cut.

**Lemma 3.4.** Let $G$ be a k-connected graph with a k-cut $S$, where $k ≥ 0$. Let $H$ be a component of $G − S$. Then for arbitrary set of vertices $X ⊆ S$ the graph $G$ contains at least $\min(|H|, |X|)$ independent edges between $H$ and $X$.

**Proof.** We prove the lemma by contradiction. Let $l$ be the maximum number of independent edges of $G$ between $H$ and $X$ and suppose that $l < \min(|H|, |X|)$. Since any set of independent edges between $H$ and $X$ is a matching between the vertices of $H$ and $X$, any maximum matching between $H$ and $X$ has size $l$. By König’s theorem [7] the maximum size of a matching between $H$ and $X$ equals the minimum cardinality of a vertex cover of all edges between $H$ and $X$. Hence there is a vertex set $Y ⊆ (H ∪ X)$ such that $|Y| = l$ and $Y$ cover all edges between $H$ and $X$. Since $|Y| < |H|$, the set $H − Y$ contains at least one vertex and $(S − X) ∪ Y$ is a vertex cut of $G$. Using $|Y| < |X|$ we get that the size of $(S − X) ∪ Y$ satisfies $(|S| − |X|) + |Y| = k − |X| + |Y| < k$, which contradicts the fact that $G$ is k-connected.

We are now ready to prove that in the case where there is a component with at least $k$ vertices and a component with precisely one vertex there are exactly two components and the larger component, except the vertices matched with the cut, is complete or complete bipartite. However, as stated earlier, a description of the structure of the graph induced on $V(M)$ and of the edges between $V(M)$ and $C$ seems to be quite difficult and remains to be an open problem.

**Theorem 3.5.** Let $G$ be a k-connected equimatchable factor-critical graph with a k-cut $S$ such that $G − S$ has a component with a single vertex and a component with at least $k$ vertices, where $k ≥ 2$. Then $G − S$ has exactly two components and there is a matching $M$ between $S$ and $C$ covering all vertices of $S$. Furthermore, $C − V(M)$ is connected randomly matchable.

**Proof.** Existence of a matching $M$ between $S$ and $C$ covering all vertices of $S$ is a consequence of Lemma 3.1. Let $v$ be the vertex of the single-vertex component of $G − S$. Lemma 3.1 implies that $v$ is adjacent to every vertex of $S$ and thus $M$ is a minimal isolating matching of $v$. By Theorem 2.1 the graph $G − (V(M) ∪ \{v\})$ is connected and randomly matchable, which completes the proof.

The next lemma implies that if the graph has at least $2k + 3$ vertices, then removing any minimum cut yields precisely two components.

**Lemma 3.6.** Let $G$ be a k-connected equimatchable factor-critical graph with a k-cut $S$, where $k ≥ 2$. If $G$ has at least $2k + 3$ vertices, then $G − S$ has precisely two components.

**Proof.** For a contradiction suppose that $H_1, \ldots, H_l$ are the components of $G − S$ for some $l ≥ 3$. Let $M$ be a matching between $S$ and $H_1 ∪ H_2$ covering as many vertices of $S$ as possible while leaving uncovered odd number of vertices of both $H_1$ and $H_2$. Observe that such a matching always exists since $k ≥ 2$ and, by Lemma 3.1 every vertex of $S$ is adjacent to every component of $G − S$. First we prove that if $M$ leaves uncovered at least 2 vertices of $S$, then it leaves uncovered precisely one vertex in both $H_1$ and $H_2$. Indeed, suppose for the contrary that $M$ leaves uncovered at least two vertices $s_1$ and $s_2$ of $S$ and more than one vertex in, say, $H_1$. Note that in this case $M$ leaves uncovered at least 3 vertices of $H_1$. Denote by $M_1$ the edges of $M$ incident with
Let $X = \{s_1, s_2\} \cup (S \cap V(M_1))$. Applying Lemma 3.3 to $H_1$ and $X$ yields that there is a matching $M'$ between $H_1$ and $X$ covering all vertices of $X$. It can be easily seen that $M'' = M' \cup (M - M_1)$ is a matching between $S$ and $H_1 \cup H_2$ which leaves uncovered odd number of vertices in both $H_1$ and $H_2$, and that $M''$ is larger than $M$, which contradicts the maximality of $M$.

We proceed to extend $M$ to a matching $N$ between $S$ and $G - S$ such that $N$ covers all vertices of $S$ and leaving uncovered odd number of vertices in both $H_1$ and $H_2$. If $M$ covers all vertices of $S$, then let $N = M$. If $M$ leaves uncovered precisely one vertex $s$ of $S$, then let $N = M \cup \{e\}$, where $e$ is any edge joining $s$ with $H_3$, note that such an edge always exists by Lemma 3.3. Finally, if $M$ leaves uncovered at least 2 vertices of $S$, then it leaves uncovered exactly one vertex in both $H_1$ and $H_2$ as shown above, and $|V(G)| \geq 2k + 3$ implies that $H_3 \cup \cdots \cup H_l$ contains more vertices than $S - V(M)$. Therefore, by Lemma 3.3 there is a matching $N''$ between $S - V(M)$ and $H_3 \cup \cdots \cup H_l$ covering all vertices of $S - V(M)$. Now $N = M \cup N''$ is the desired matching covering all vertices of $S$ and leaving uncovered odd number of vertices in both $H_1$ and $H_2$.

To complete the proof it suffices to show that $N$ cannot be extended to a maximum matching of $G$, contradicting the fact that $G$ is equimatchable. Indeed, $N$ leaves uncovered odd number of vertices in both $H_1$ and $H_2$ and separates $H_1$ and $H_2$ from the rest of the graph and thus any maximal matching $N''' \supseteq N$ leaves uncovered at least one vertex in both $H_1$ and $H_2$. Since $G$ is equimatchable and factor-critical, any maximum matching of $G$ leaves uncovered precisely one vertex of $G$ and hence $N'''$ cannot be a maximum matching. The proof is now complete.

To deal with the cases where the smaller component of $G - S$ has at least two vertices we will need the following lemma.

**Lemma 3.7.** Let $G$ be a $k$-connected equimatchable factor-critical graph with a $k$-cut $S$, where $k \geq 2$. Assume that $G - S$ has a component $C$ with at least $k$ vertices and $G - (S \cup C)$ has a component with exactly two vertices. Then $G - S$ has exactly two components and there is a matching $M$ between $S$ and $C$ covering all vertices of $S$. Furthermore, for any matching $M'$ between $S$ and $C$ covering all vertices of $S$ and for each vertex $x$ of $C \cap V(M')$, the subgraph of $G$ induced by $(C - V(M')) \cup \{x\}$ is connected randomly matchable.

**Proof.** Existence of a matching $M$ between $S$ and $C$ covering all vertices of $S$ is a consequence of Lemma 3.3. Let $M$ be any matching between $S$ and $C$ covering all vertices of $S$ and let $D$ be a component of $G - (S \cup C)$ with exactly two vertices. Let $x$ be any vertex of $C$ covered by $M$ and let $s$ be the vertex of $S$ matched by $M$ with $x$. Lemma 3.3 implies that there is a vertex of $D$, say $d$, adjacent to $s$. Let $d'$ be the vertex of $D$ different from $d$. Consider the set $M' = (M - \{sx\}) \cup \{ds\}$; clearly $M'$ is a matching and $\{d'\}$ is an odd component of $G - V(M')$. Thus by Lemma 3.3 the graph $G - (V(M') \cup \{d'\}) = (C - V(M)) \cup \{x\}$ is connected randomly matchable, which completes the proof.

The following theorem provides a characterisation of $k$-connected equimatchable factor-critical graphs with a $k$-cut $S$ such that $G - S$ contains a component with at least $k$ vertices and a component with precisely 2 vertices. We indicate the end of a proof of a claim by ■.

**Theorem 3.8.** Let $G$ be a $k$-connected equimatchable factor-critical graph with a $k$-cut $S$, where $k \geq 3$. Assume that $G - S$ has a component $C$ with at least $k$ vertices and $G - (S \cup C)$ has a component with exactly two vertices. Then $G - S$ has exactly two components. Furthermore, if $S$ contains an edge, then $C$ is a complete graph. If $S$ does not contain an edge, then there is a nonnegative integer $m$ and sets $\{x_1, \ldots, x_m\}$ of vertices of $C$ and $\{y_1, \ldots, y_m\}$ of vertices of $S$ such that $x_iy_i$ is not an edge of $G$ for every $i \in \{1, \ldots, m\}$ and $C \cup S \cup \{x_1y_1, \ldots, x_my_m\}$ is isomorphic with $K_{n,n+1}$ for some $n$.

**Proof.** Let $S = \{s_1, \ldots, s_k\}$ and let $D = \{d_1, d_2\}$ be a component of $G - (C \cup S)$ with exactly two vertices. Note that since $G$ is odd, it has at least $2k + 3$ vertices and hence, by Lemma 3.3, $D$ is the only component of $G - (C \cup S)$. By Lemma 3.3 there is a matching $M$ between $S$ and $C$ which covers all vertices of $S$. Denote by $X$ the set $C \cap V(M)$ and let $C' = C - X$. The fact that $G - S$ has exactly two components follows from Lemma 3.3. The rest of the proof is split into two cases.

**Case A** There is an edge in $S$.

**Claim A1.** If $rs$ is an edge in $S$ and $u$ and $v$ are the two vertices of $C$ matched by $M$ with $r$ and $s$, respectively, then $\{u, v, w\}$ is a triangle for any vertex $w$ of $X - \{u, v\}$.

**Proof of Claim A1.** Choose an arbitrary vertex $w$ from $X - \{u, v\}$ and let $t$ be the vertex of $S$ matched by $M$ to $w$. By Lemma 3.3 there is an edge between $s$ and $t$ and $D$ is a vertex of $D$ by $d'$. Clearly, $M'$ is a matching of $G$ which isolates $d'$. Let $x$ be any vertex from $\{u, v, w\}$. Applying Lemma 3.3 to $C' \cup \{x\}$ and $M$ yields that $C' \cup \{x\}$ is randomly matchable and thus it
has a perfect matching $M_x$. Observe that the set $M_x \cup M'$ is a matching of $G$ which leaves uncovered precisely the vertices in $\{d', u, v, w\} \setminus \{x\}$. Because $\{u, v, w\} \subseteq C$ and $S$ is a cut separating $C$ and $D$, there is no edge between $d'$ and $\{u, v, w\}$. The fact that $G$ is equimatchable and factor-critical implies that the two vertices in $\{u, v, w\} \setminus \{x\}$ are joined by an edge. Since $x$ was arbitrary vertex from $\{u, v, w\}$, the claim follows.

If $k = 3$, then the result follows from Claim A1. Therefore, from now on we assume $k \geq 4$.

**Claim A2.** The subgraph of $G$ induced by $X$ is a complete graph.

**Proof of Claim A2.** Let $rs$ be an edge of $S$. Our aim is to show that there is an edge between arbitrary two vertices $y$ and $z$ of $X$. Denote by $x_r$ and $x_s$ the two vertices of $X$ joined by $M$ to $r$ and $s$, respectively. If $y$ or $z$ belongs to $\{x_r, x_s\}$, then $y$ and $z$ are joined by an edge by Claim A1. Hence we can assume that $\{y, z\} \cap \{x_r, x_s\} = \emptyset$. Claim A1 applied to $\{x_r, x_s, y\}$ shows that $x_r y$ is an edge of $G$. Applying Lemma 5.7 to $C' \cup \{z\}$ and $M$ yields that $C' \cup \{z\}$ is randomly matchable and thus it has a perfect matching $M'$. Let $s_y$ and $s_z$ be the vertices of $S$ joined by $M$ to $y$ and $z$, respectively. Consider the set $M'' = (M - \{rx_r, ysz_z\}) \cup M' \cup \{ys, e\}$, where $e$ is the edge in $D$. It is not difficult to see that $M''$ is a matching which leaves uncovered exactly the vertices $r, s_y,$ and $s_z$. Hence $\{r, s_y, s_z\}$ contains an edge $e$ and the result follows by using Claim A1 on $e$ and $\{x_r, y, z\}$.

**Claim A3.** The subgraph of $G$ induced by $C'$ is a complete graph.

**Proof of Claim A3.** Assume that the edge in $S$ is $rs$. Let $x_r$ and $x_s$ be the vertices of $X$ joined by $M$ to $r$ and $s$, respectively, and let $y$ be an arbitrary vertex of $X - \{x_r, x_s\}$. By Claim A1 applied to $rs$ the subgraph of $G$ induced by $\{y, x_r, x_s\}$ is a triangle. Let $s_y$ be the vertex of $S$ joined to $y$ by $M$. By Lemma 5.3 there is a vertex of $D$, say $d$, adjacent to $s_y$. Let $d'$ be the vertex of $D$ different from $d$. Consider the set $M' = M - \{rx_r, sxs_y, ysz_y\} \cup \{rs, ds_y\}$; clearly $M'$ is a matching isolating $d'$. By Theorem 2.1 the graph $C' \cup \{y, x_r, x_s\}$ is either $K_{2m}$ or $K_{2,m_n}$. Since $\{y, x_r, x_s\}$ induces a triangle and is contained in $C' \cup \{y, x_r, x_s\}$, the graph $C' \cup \{y, x_r, x_s\}$ is complete. In particular $C'$ is a complete graph, as claimed.

**Claim A4.** The subgraph of $G$ induced by $C'$ is a complete graph.

**Proof of Claim A4.** By Claim A2 the set $X$ induces a complete graph and by Claim A3 the set $C'$ induces a complete graph. Lemma 5.7 implies that for each vertex $x$ of $X$ the graph $C' \cup \{x\}$ is connected randomly matchable. It is not difficult to see that if $C'$ is a complete graph, then also $C' \cup \{x\}$ is a complete graph for each $x$ of $X$. It follows that each vertex of $X$ is adjacent to every vertex of $C'$ and thus $C'$ is a complete graph, as claimed.

The preceding claim completes the case where there is an edge in $S$ and the first part of the proof.

**Case B1.** The set $S$ is independent.

**Claim B1.** The set $X$ is independent.

**Proof of Claim B1.** For a contradiction suppose that $x_1 x_2$ is an edge in $X$ and let $x_3$ be an arbitrary vertex of $X - \{x_1, x_2\}$. Let $M'$ be an arbitrary perfect matching of $C' \cup \{x_1\}$, due to Lemma 5.7 such a matching exists. Furthermore, let $s_i$ be the vertex matched by $M$ with $x_i$ for $i = 1, 2, 3$ and let $e$ be the edge in $D$. The matching $M'' = (M - \{s_1 x_1, s_2 x_2, s_3 x_3\}) \cup M' \cup \{x_1 x_2, e\}$ leaves uncovered only the vertices $s_1, s_2,$ and $s_3$. By the assumption of Case B) the matching $M''$ is maximal, contradicting the fact that $G$ is equimatchable and factor-critical.

**Claim B2.** The subgraph of $G$ induced by $C'$ is isomorphic with $K_{n+1,n}$ for some $n \geq 0$.

**Proof of Claim B2.** If $C'$ contains only one vertex, then the claim holds. Since $C'$ is odd, we can assume $|V(C')| \geq 3$. Lemma 5.7 implies that for each $x \in X$ the graph $C' \cup \{x\}$ is connected randomly matchable. If $C' \cup \{x\}$ is $K_{m,m}$ for some $x \in X$, then $C'$ is clearly $K_{m,m-1}$ and the claim holds.

For a contradiction suppose that $C' \cup \{x\}$ is $K_{2m}$ for each $x \in X$. Let $x$ be an arbitrary vertex of $X$, let $M'$ be a perfect matching of $C' \cup \{x\}$, and $bc$ an edge of $M'$ not incident with $x$. Observe that both $b$ and $c$ are adjacent with each vertex $x'$ of $X$ since $C' \cup \{x'\}$ is $K_{2m}$. Let $x_b$ and $x_c$ be two vertices of $X - x$. Finally, let $s_x, s_b,$ and $s_c$ be the vertex matched by $M$ with $x, x_b,$ respectively $x_c$. It follows that the set $M''$ defined by $M'' = (M - \{s_x x, s_b x_b, s_c x_c\}) \cup (M' - \{bc\}) \cup \{xy, x_c y, d_1 d_2\}$ is a matching which covers all vertices of $G$ except $s_x, s_b,$ and $s_c$. Because $S$ is independent, $M''$ is a maximal matching leaving uncovered 3 vertices, which contradicts the fact that $G$ is equimatchable factor-critical and completes the proof of the claim.

Denote by $U$ the smaller and by $W$ the larger partite set of $C'$.

**Claim B3.** There is no edge between $X$ and $U$. 

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Proof of Claim B3. We proceed by contradiction. Suppose that \( u \) is a vertex of \( U \) adjacent to a vertex \( x \) of \( X \). Let \( s \) be the vertex of \( S \) matched with \( x \) by \( M \) and let \( d \) be any vertex of \( D \) adjacent to \( s \); such a vertex \( d \) exists by Lemma 3.4. Clearly, the set \( M' = (M - \{sx\}) \cup \{ds,xu\} \) is a matching of \( G \). It is not difficult to see that any maximal matching containing \( M' \) leaves unmatched at least two vertices of \( W \), which contradicts the fact that \( G \) is equimatchable and factor-critical.

Claim B4. There is no edge between \( S \) and \( W \).

Proof of Claim B4. For a contradiction suppose that there is a vertex \( w \) of \( W \) adjacent to some vertex \( s \) of \( S \). Let \( t \) be any vertex of \( S - s \) and let \( d \) be a vertex of \( D \) adjacent to \( t \), such a vertex \( d \) exists by Lemma 3.4. Furthermore, let \( x_s \) and \( x_t \) be the vertices of \( X \) matched by \( M \) with \( s \) and \( t \) respectively, and let \( N = (M - \{sx_s,tx_t\}) \cup \{sw,td\} \). Clearly, \( N \) is a matching of \( G \). Note that \( N \) leaves uncovered \( D - d \) and \( (C' \cup \{x_s,x_t\}) \setminus w \). Since \( S \) is a cut, the vertex in \( D - d \) is not adjacent with any vertex in \( (C' \cup \{x_s,x_t\}) \setminus w \). Claim 3.4 and the choice of \( w \) imply that \( C' - w \) is \( K_{n,n} \) for some \( n \). Furthermore, by Claim B3 there is no edge between \( \{x_s,x_t\} \) and \( U \). It follows that \( (C' \cup \{x_s,x_t\}) \setminus w \) is a subgraph of \( K_{n+2,n} \) and thus any maximal matching of \( G \) containing \( N \) leaves uncovered \( d \) and at least two vertices of \( (C' \cup \{x_s,x_t\}) \setminus w \), contradicting the fact that \( G \) is equimatchable factor-critical.

Claim B5. Each vertex of \( X \) is adjacent to every vertex of \( W \).

Proof of Claim B5. Let \( x \) be a vertex of \( X \) and \( w \) a vertex of \( W \). By Claim 3.4 the graph \( C' \) is \( K_{n,n+1} \) and by the definition of \( W \) the vertex \( w \) lies in the larger partite set of \( C' \). It follows that there is a perfect matching \( M' \) of \( C' - \{w\} \). Let \( s \) be a vertex of \( S \) matched with \( x \) by \( M \). By Lemma 3.4 there is an edge \( e \) between \( s \) and \( D \). Let \( d \) be the vertex of \( D \) not covered by \( e \). Let \( M'' = (M - \{xs\}) \cup M' \cup \{e\} \). Clearly, \( M'' \) is a matching which covers all vertices of \( G \) except \( d \) and \( w \). Since \( C \) and \( D \) are different components of \( G \) and \( x \) and \( w \) lie in \( C \), the vertex \( d \) is adjacent with neither \( x \), nor \( w \). Using the fact that \( G \) is factor-critical and equimatchable we get that \( x \) and \( w \) are adjacent, which completes the proof.

Claim B6. Each vertex of \( S \) is adjacent to either all, or all but one vertices of \( X \cup U \).

Proof of Claim B6. Suppose to the contrary that there is a vertex \( s \) of \( S \) and two vertices \( v_1 \) and \( v_2 \) from \( X \cup U \) such that \( s \) is adjacent neither to \( v_1 \), nor to \( v_2 \). Let \( x \) be the vertex of \( X \) matched with \( s \) by \( M \) and note that \( x \) is different from both \( v_1 \) and \( v_2 \). If \( v_1 \in X \), then let \( y_1 = v_1 \), otherwise let \( y_1 \) be an arbitrary vertex of \( X - \{x,v_2\} \). Similarly, if \( v_2 \in X \), then let \( y_2 = v_2 \), otherwise let \( y_2 \) be an arbitrary vertex of \( X - \{x,y_1\} \). Let \( t_1 \) and \( t_2 \) be the two vertices of \( S \) matched by \( M \) with \( y_1 \), respectively \( y_2 \). Let \( M' \) be a set of two independent edges between \( D \) and \( \{t_1,t_2\} \); such two edges exist by Lemma 3.4. Recall that the graph \( C' \) is isomorphic with \( K_{n,n+1} \) by Claim 3.4 and that each vertex of \( X \) is adjacent with every vertex of \( W \) by Claim 3.4. Using the last two observations it is not difficult to prove that \( C' \cup \{x_1,y_2\} \) has a matching \( N ' \) which covers all vertices of \( C' \cup \{x_1,y_2\} \) except \( v_1 \) and \( v_2 \): a straightforward case analysis on \( \{y_1,y_2\} \cap \{v_1,v_2\} \) is left to the reader. Consider the set \( N' = (M - \{sx,t_1y_1,t_2y_2\}) \cup M' \cup N \). It is easy to see that \( N' \) is a matching which covers all vertices of \( G \) except \( v_1 \) and \( v_2 \). Since \( v_1 \) and \( v_2 \) belong to \( X \) and the other to \( U \), then they are not adjacent by Claim B3. If both \( v_1 \) and \( v_2 \) are from \( X \), then they are not adjacent by Claim 3.4. If both \( v_1 \) and \( v_2 \) are from \( U \), then they are not adjacent by the definition of \( U \). Since by our assumption \( s \) is adjacent with neither \( v_1 \), nor \( v_2 \), we get a contradiction with the fact that \( G \) is equimatchable and factor-critical.

Claim B7. Each vertex of \( X \) is adjacent to either all, or all but one vertices of \( S \).

Proof of Claim B7. Suppose for the contrary that there is a vertex \( x \) of \( X \) and two vertices \( t_1 \) and \( t_2 \) of \( S \) such that \( x \) is adjacent to neither \( t_1 \), nor \( t_2 \). Let \( s \) be the vertex of \( S \) matched with \( x \) by \( M \) and let \( y_1 \) and \( y_2 \) be the two vertices matched by \( M \) with \( t_1 \) and \( t_2 \), respectively. By Claim 3.4 the vertex \( s \) is adjacent with at least one of \( y_1 \) and \( y_2 \); without loss of generality we assume that \( s \) is adjacent to \( y_1 \). By Lemma 3.4 the graph \( C' \cup \{y_2\} \) is randomly matchable and hence it has a perfect matching \( M' \). Let \( M'' = (M - \{sx,t_1y_1,t_2y_2\}) \cup M' \cup \{e,xy_1\} \), where \( e \) is the edge in \( D \). It is not difficult to see that \( M'' \) is a matching which covers all vertices of \( G \) except \( x,t_1 \), and \( t_2 \). By the assumption of Case B) the vertices \( t_1 \) and \( t_2 \) are not adjacent and by our assumption \( x \) is adjacent to neither \( t_1 \), nor \( t_2 \). Therefore, \( M'' \) is a maximal matching leaving uncovered 3 vertices, contradicting the fact that \( G \) is equimatchable and factor-critical.

Claim B8. Each vertex of \( U \) is adjacent to either all, or all but one vertices of \( S \).
For each triple of pairwise distinct vertices \( u, v, w \) of \( G \) the graph \( G' \) is isomorphic with \( K_{n,n+1} \) and by Claim A5 both vertices \( u, v, w \) are adjacent to every vertex from the larger partite set of \( G' \). Therefore, there exists a perfect matching of \( G' \), which covers all vertices of \( G' \). It is not difficult to see that \( M'' \) is a matching which covers all vertices of \( G' \) except \( w, v, u \). By the assumption of Case B) the vertices \( v, w \) are adjacent, and by our assumption \( u, v, w \) is adjacent to neither \( t_1, t_2 \). It follows that \( M'' \) is a maximal matching leaving uncovered 3 vertices, which contradicts the fact that \( G' \) is equimatchable and factor-critical.

The proof of Claim A5 is now complete.

Denote by \( H \) the subgraph of \( G \) induced by \( U \cup S \). Claims \( C_5, H_5 \) imply that \( U \cup W \cup X \cup S = H \) is a bipartite graph with partite sets \( X \cup U \) and \( S \cup W \). Claim A2 and the definition of \( U \) and \( W \) yield that each vertex of \( U \) is adjacent to every vertex of \( W \). By Claim A5 each vertex of \( X \) is adjacent to every vertex of \( W \).

From Claims \( C_5, H_5 \) and \( H_5 \) we get that there is a nonnegative integer \( m \) and sets of vertices \( \{t_1, \ldots, t_m\} \subseteq S \) and \( \{y_1, \ldots, y_m\} \subseteq X \cup U \) such that \( t_iy_i \notin E(G) \) for all \( i \in \{1, \ldots, m\} \) and that \( H \cup \{t_1y_1, \ldots, t_my_m\} \) is a complete bipartite graph. The proof is now complete.

The following observation may be easily verified.

**Observation 3.9.** Let \( G \) be isomorphic with \( K_{n,n} \) for some \( n \geq 1 \) and let \( u, v \) and \( w \) be two vertices of \( G \). If \( G - \{u, v\} \) is randomly matchable, then \( u \) and \( v \) are adjacent.

**Theorem 3.10.** Let \( G \) be a \( k \)-connected equimatchable factor-critical graph with at least \( 2k + 3 \) vertices and a \( k \)-cut \( S \) such that \( G - S \) has two components with at least \( 3 \) vertices, where \( k \geq 3 \). Then \( G - S \) has exactly two components and both are complete graphs.

**Proof.** By Lemma 3.6 the graph \( G - S \) has precisely two components, denote these components by \( C \) and \( D \), respectively. First we deal with the case where both \( C \) and \( D \) are strictly smaller than \( k \); this case is much simpler. Take any two vertices \( c, c' \) of a component of \( G - S \), say of \( C \). Let \( l = |V(C)| \). Since \( |V(C)| < k \), there are \( l \) independent edges between \( S \) and \( C \) by Lemma 3.4. Therefore, we can choose a set \( M_C \) of \( l - 2 \) independent edges between \( S \) and \( C - \{c, c'\} \). Since \( |V(G)| \geq 2k + 3 \), by Lemma 3.4 there is a set \( M_D \) of \( k - l + 2 \) independent edges between \( D \) and \( S - V(M_C) \). Let \( M = M_C \cup M_D \) and observe that \( M \) is a matching of \( G \). It is not difficult to see that the vertex \( c \) can be in \( G - V(M) \) adjacent only to \( c' \), and similarly \( c' \) can be adjacent only to \( c \). Since \( G \) is equimatchable and factor-critical, the matching \( M \) can be extended to a maximum matching of \( G \), which leaves unmatched precisely one vertex of \( G \). Clearly, this is possible only if \( c \) and \( c' \) are adjacent. Since the choice of \( c \) and \( c' \) was arbitrary, it follows that both components of \( G - S \) are complete, as claimed.

From now on we assume that at least one component of \( G - S \), say \( C \), has at least \( k \) vertices. By Lemma 3.4 there is a set \( M \) of \( k \) independent edges between \( C \) and \( S \) covering all vertices of \( S \). Denote by \( X \) the set of vertices \( C \cap V(M) \) and let \( C' = C - X \). We distinguish two cases.

**Case A)** \( D \) is even. First observe that in this case \( C' \) is odd and denote by \( H \) an odd component of \( C' \). Clearly, \( G, M, H \) satisfy the assumptions of Lemma 3.6 which implies that \( G - (H \cup V(M)) \) is connected randomly matchable. Since \( D \) is a component of \( G - (H \cup V(M)) \), it follows that \( H \) is the only component of \( C' \) and thus \( H \cup V(M) = S \cup C \). Consequently, \( D = G - (H \cup V(M)) \) and hence \( D \) is connected randomly matchable. To prove that \( D \) is complete we proceed by contradiction and suppose that \( D \) is \( K_{n,n} \) for some \( n \geq 2 \). Since \( k \geq 3 \), by Lemma 3.3 there are at least three independent edges between \( D \) and \( S \) and at least two of these edges, say \( sd \) and \( s'd' \), have their endvertices in the same partite set of \( D \), where \( d \) and \( d' \) are vertices of \( D \). Let \( x \) and \( x' \) be the vertices of \( X \) matched by \( M \) with \( s \) and \( s' \), respectively. Let \( M'' = (M - \{sx, s'x'\}) \cup \{sd, s'd'\} \) and let \( H' \) be an odd component of \( C' \cup \{x, x'\} \). By Lemma 3.6 the graph \( G - (H' \cup V(M'')) \) is randomly matchable. On the other hand, \( G - (H' \cup V(M'')) = D - \{d, d'\} \) and thus \( d \) and \( d' \) are adjacent by Observation 5.6 contradicting the fact that \( d \) and \( d' \) lie in the same partite set. Therefore, we conclude that \( D \) is isomorphic with \( K_{2n} \).

**Claim A1.** The graph \( C' \cup \{x\} \) is connected randomly matchable for each \( x \in X \).

**Proof of Claim A1.** Let \( s \) be the vertex of \( S \) matched by \( M \) with \( x \). Since \( S \) is a minimum cut, there is a vertex \( d \) of \( D \) adjacent with \( s \). Let \( H \) be an odd component of \( D - d \) and let \( M' = (M \cup \{ds\}) - sx \). Lemma 3.6 applied to \( H \) and \( M' \) implies that \( G - (H \cup V(M')) \) is connected randomly matchable and that \( H = D - d \). Therefore, \( C' \cup \{x\} = G - (H \cup V(M')) \) and thus \( C' \cup \{x\} \) is connected randomly matchable, as claimed.

**Claim A2.** For each triple of pairwise distinct vertices \( x, y, z \) of \( X \) the graph \( C' \cup \{x, y, z\} \) is isomorphic with \( K_{2n} \) for some \( n \).
Proof of Claim B2. Let \( s_x, s_y \) and \( s_z \) be the vertices matched by \( M \) with \( x, y, \) and \( z, \) respectively. By Lemma 3.4 there are three pairwise distinct vertices \( d_x, d_y, \) and \( d_z \) of \( D \) adjacent to \( s_x, s_y, \) and \( s_z, \) respectively. Since \( D \) is even and \(|D| \geq 4, \) the graph \( D = \{d_x, d_y, d_z\} \) is odd and thus contains an odd component \( H. \) Using Lemma 3.3 on \( H \) and \((M - \{x, y, z, s_x, s_y, z, s_z\}) \cup \{s_x, s_y, z, s_z\} \) we get that \( C' \cup \{x, y, z\} \) is connected randomly matchable. Since \( C' \cup \{v\} \) is connected randomly matchable for each \( v \in \{x, y, z\} \) by Claim A1, Observation 3.9 used on all pairs from \( \{x, y, z\} \) implies that \( \{x, y, z\} \) induces a triangle. The last observation implies that \( C' \cup \{x, y, z\} \) is complete and concludes the proof of the claim.

Using Claim A2 on all triples of vertices of \( X \) implies that the graph induced by \( C \) is complete, as claimed.

Case B) \( D \) is odd.

Let \( l = \min(|D|, k) \) and note that \( l \geq 3. \) Our first aim is to show that \( C \) is complete.

Claim B1. The subgraph of \( G \) induced by \( C' \) is connected randomly matchable.

Proof of Claim B1. Lemma 3.3 applied to \( D \) and \( M \) implies that \( G - (D \cup V(M)) \) is connected randomly matchable. The claim follows from the fact that \( C' = G - (D \cup V(M)) \).

Claim B2. For each two vertices \( x \) and \( x' \) of \( X \) the graph \( C' \cup \{x, x'\} \) is connected randomly matchable. Furthermore, the vertices \( x \) and \( x' \) are adjacent.

Proof of Claim B2. Let \( s \) and \( s' \) be the vertices of \( S \) matched by \( M \) with \( x \) and \( x' \), respectively. By Lemma 3.4 there are two independent edges \( ss' \) and \( s's' \), where \( d \) and \( d' \) are vertices of \( D. \) Since \( D \) is odd and \(|D| \geq 3, \) there is an odd component \( H \) of \( D - \{d, d'\}. \) Let \( M' = (M \cup \{sd, s'd'\}) - \{xs, x's'\}. \) Clearly, \( M' \) is a matching and thus Lemma 3.3 applied to \( H \) and \( M' \) implies that \( C' \cup \{x, x'\} \) is connected randomly matchable. If \( C' \cup \{x, x'\} \) is a complete graph, then \( c \) and \( c' \) are adjacent and there is nothing left to prove. If \( C' \cup \{x, x'\} \) is \( K_{n,n} \) for some \( n \geq 2, \) then we get that \( x \) and \( x' \) are adjacent by Claim 11 and Observation 3.9.

Using Claim B2 on all pairs of vertices \( x \) and \( x' \) of \( X \) implies that the subgraph of \( G \) induced by \( C \) is complete. Since \( C' \) is even, we can thus assume that \(|C'| \geq 2. \)

Claim B3. For each two vertices \( x \) and \( y \) of \( X \) the graph \( C' \cup \{x, y\} \) is \( K_{2n} \) for some \( n. \)

Proof of Claim B3. Let \( z \) be a vertex of \( X - \{x, y\}. \) By Claim 12 the graph \( C' \cup \{x, y\} \) is connected randomly matchable. Suppose for a contradiction that \( C' \cup \{x, y\} \) is \( K_{n,n} \) for some \( n \geq 2. \) Since \( C' \cup \{x, y\} \) is \( K_{n,n} \) and \( n \geq 2, \) Claim 12 implies that \( C' \) is \( K_{n-1,n-1}. \) Let \( A \) and \( B \) denote the partite sets of \( C'. \) Since \( x \) and \( y \) are adjacent, without loss of generality we may assume that the partite sets of \( C' \cup \{x, y\} \) are \( A \cup \{x\} \) and \( B \cup \{y\}. \) In particular, \( x \) is not adjacent to any vertex of \( A \) and thus \( C' \cup \{x, z\} \) is not a complete graph. Similarly, \( C' \cup \{y, z\} \) is also not a complete graph. Therefore, Claim 12 used on \( x \) and \( z \) imply that \( z \) is adjacent with at least one of \( y \) and \( z \) is adjacent to all vertices of \( B, \) which is a contradiction.

We conclude that \( C' \) is complete by using Claim 13 on all pairs of vertices \( x \) and \( x' \) of \( X. \)

Now we prove that \( D \) is complete. By Lemma 3.4 there is a set of \( l \) independent edges \( \{s_1d_1, \ldots, s_l d_l\} \) between \( S \) and \( D, \) where \( d_1, \ldots, d_l \) are vertices of \( D. \) For each \( i \in \{1, \ldots, l\} \) denote the graph \( D - d_i \) by \( D_i \) and let \( x_i \) be the vertex of \( X \) matched by \( s_i. \)

Claim B4. For each \( i \in \{1, \ldots, l\} \) the graph \( D_i \) is connected randomly matchable.

Proof of Claim B4. Let \( M_i = (M - \{s_i, x_i\}) \cup \{d_i, s_i\} \) and let \( H_i \) be an odd component of \( C' \cup \{x_i\}. \) Lemma 3.3 applied to \( G, M_i, \) and \( H_i \) yields that \( G - (H_i \cup V(M_i)) \) is connected randomly matchable and thus \( H_i \) is the only component of \( C' \cup \{x_i\}. \) Consequently, \( D_i = G - (H_i \cup V(M_i)) \) and thus \( D_i \) is connected randomly matchable, as claimed.

Since \( l \geq 3, \) it is easy to see that if \( D \) contains only three vertices, then Claim 14 for \( i = 1, 2, \) and \( 3 \) implies that \( D \) is complete. Therefore, we can assume \(|V(D)| \geq 5. \)

Claim B5. If \( D_i \) is a complete graph for some \( i \in \{1, \ldots, l\}, \) then \( D \) is a complete graph.

Proof of Claim B5. Assume that \( D_i \) is a complete graph for some \( i \in \{1, \ldots, l\}. \) It is easy to see that for any \( j \in \{1, \ldots, l\} \) the graph \( D_i - d_j \) contains a triangle. Since \( D_i - d_j \) is contained in \( D_j, \) we get that \( D_j \) is a complete graph for each \( j \in \{1, \ldots, l\} \) by Claim 14. The proof of the claim is concluded by observing that for each pair of vertices \( d \) and \( d' \) of \( D \) there is some \( m \in \{1, \ldots, l\} \) such that both \( d \) and \( d' \) are contained in \( D_m, \)

Claim B6. If there is a pair of integers \( i \) and \( j \) from \( \{1, \ldots, l\} \) such that both \( D_i \) and \( D_j \) are isomorphic with \( K_{n,n} \) for some \( n, \) then \( d_i \) and \( d_j \) are not adjacent.
Proof of Claim B6. Let \( m \) be an integer from \( \{1, \ldots, l\} \setminus \{i, j\} \) and note that \( D_m \) is connected randomly matchable by Claim H4. Observe that \( D_m \) is \( K_{n,n} \), since otherwise \( D_m - d_i \subseteq D_i \) would contain a triangle. Since \( D_j \) is \( K_{n,n} \), the graph \( D_j - d_i \) is \( K_{n,n-1} \). Let \( A \) denote the set of vertices of \( D \) lying in the larger partite set of \( D_j \). By comparing \( D_j \) and \( D_j - d_i \), it is easy to see that \( d_i \) is adjacent to all vertices of \( A \). Furthermore, \( D_i = (D_j - d_i) \cup \{d_j\} \) and thus also \( d_j \) is adjacent to all vertices of \( A \). It follows that both \( d_i \) and \( d_j \) are in \( D_m \) adjacent to all vertices of \( A \cap D_m \). The fact that \( |V(D)| \geq 5 \) implies \( n \geq 2 \) and thus \( A \cap D_m \) contains a vertex \( d \). The proof is concluded by observing that \( d_i \) and \( d_j \) are not adjacent, since otherwise \( D_m \) would contain the triangle \( \{d, d_i, d_j\} \).

Recall that \( l \geq 3 \) and observe that if one of \( D_1, D_2, \) and \( D_3 \) is a complete graph, then we are done by Claim H4. Therefore, we can assume that \( D_1, D_2, \) and \( D_3 \) are \( K_{n,n} \) for some integer \( n \). Let \( M' = (M - \{s_1, s_2, s_3\}) \cup \{s_1d_1, s_2d_2, s_3d_3\} \), and let \( H' \) be an odd component of \( C' \cup \{x_1, x_2, x_3\} \). Clearly, \( G, M' \), and \( H' \) satisfy the assumptions of Lemma \[\text{B.3}\] which in turn implies that \( G - (H' \cup V(M')) = D - \{d_1, d_2, d_3\} \) is connected randomly matchable. Since \( D - d_1 \) is \( K_{n,n} \) for some \( n \) by our assumption, Observation \[\text{B.9}\] implies that \( d_2 \) and \( d_3 \) are adjacent. On the other hand, Claim H4 yields that \( d_2 \) and \( d_3 \) are not adjacent, which is a contradiction. The proof is now complete. \( \blacksquare \)

We conclude this section by showing that the requirement on the number of vertices in Lemma \[\text{B.6}\] cannot be relaxed. More precisely, for every \( k \geq 3 \) we construct a \( k \)-connected equimatchable factor-critical graphs with \( 2k + 1 \) vertices and a \( k \)-cut \( S \) such that \( G - S \) has \( k \) components and show that this bound is tight.

Proposition 3.11. Let \( G \) be a \( k \)-connected equimatchable factor-critical graph with a \( k \)-cut \( S \) for any \( k \geq 3 \). Then \( G - S \) has at most \( k \) components and this bound is tight for every \( k \geq 3 \).

Proof. If \( |V(G)| \geq 2k + 3 \), then \( G - S \) has exactly \( 2 \leq k \) components by Lemma \[\text{B.6}\]. Therefore, we can assume that \( |V(G)| \leq 2k + 1 \). Clearly, the number of components of \( G - S \) is at most \( |V(G - S)| \leq k + 1 \), with equality if and only if \( G - S \) consists from \( k + 1 \) singletons. However, it is easy to see that if \( G - S \) consists from \( k + 1 \) singletons, then for arbitrary vertex \( s \) of \( S \) the graph \( G - s \) cannot have a perfect matching, which contradicts factor-criticality of \( G \). Therefore, the number of components of \( G - S \) is at most \( k \).

To show that this bound is tight, for each \( k \geq 3 \) we construct a \( k \)-connected equimatchable factor-critical graph \( G_k \) with \( 2k + 1 \) vertices and a \( k \)-cut \( S \) such that \( G - S \) has exactly \( k \) components. Let \( V(G_k) = C \cup D \cup S \), where \( C \) and \( S \) form independent sets of \( G_k \) with sizes \( k - 1 \) and \( k \), respectively, and \( D \) is a copy of \( K_2 \). The edges of \( G_k \) are precisely all the edges between \( S \) and \( G_k - S \) and the edge in \( D \). Clearly, the graph \( G_k \) is \( k \)-connected for every \( k \geq 3 \). To show that \( G_k - v \) has a perfect matching for each vertex \( v \), we distinguish whether \( v \) belongs to \( C \), \( S \), or \( D \). If \( v \) is a vertex of \( S \), then a perfect matching of \( G_k - v \) can be constructed by taking the edge from \( D \) and a matching between \( S - v \) and \( C \) covering all vertices of \( (C \cup S) - v \). If \( v \) is a vertex of \( C \) or \( D \), then \( G_k - v \) contains \( K_{k,k} \) as a subgraph and hence also admits a perfect matching.

To prove that any matching \( M \) of \( G_k \) can be extended to a maximum matching, we distinguish two cases according to whether \( M \) contains the edge of \( C \) or not. If \( M \) contains the edge \( c_1c_2 \) of \( C \), then \( M - \{c_1c_2\} \) is a matching in \( G_k - \{c_1, c_2\} \), which in turn is isomorphic to \( K_{k,k-1} \). Since \( K_{k,k-1} \) is equimatchable, the matching \( M - \{c_1c_2\} \) can be extended to a matching \( M \) of \( G_k - \{c_1, c_2\} \) covering all but one vertex. It follows that \( M' \cup \{c_1c_2\} \) is the desired maximum matching of \( G_k \) containing \( M \). If \( M \) does not contain the edge of \( C \), then \( M \) contains only edges from \( G_k - E(C) \), which is isomorphic to an equimatchable graph \( K_{k+1,k} \). It follows that \( M \) can be extended to a matching of \( G_k \) covering all but one vertex, which completes the proof. \( \blacksquare \)

We note that Theorem 3.10 and 3.8 cannot be extended to graphs with connectivity 2. More precisely, for graphs with connectivity 2 neither the fact that \( G - S \) has two components with at least three vertices implies that the components are complete, nor presence, respectively absence, of an edge in \( S \) forces the structure described in Theorem 3.8.

4 Graphs with independence number 2

In this section we investigate the relationship between equimatchability and independence number. We focus on odd \( k \)-connected graphs with \( k \geq 4 \), at least \( 2k + 3 \) vertices, and a \( k \)-cut which separates at least two components with at least 3 vertices and show that such graphs are equimatchable factor-critical if and only if their independence number equals 2. In one direction, we show that if a graph with independence number 2 is odd, then it is equimatchable, and if it is even, then it is very close to being equimatchable. In the reverse direction, we use the characterisation of \( k \)-connected equimatchable factor-critical graphs with at least \( 2k + 3 \) vertices and a \( k \)-cut separating at least two components with at least three vertices from Theorem 3.10 to show that if \( k \geq 4 \), then all such graphs have independence number 2. Finally, we provide examples showing that it is
not possible to extend these results to graphs in which every minimum cut separates a component with at most 2 vertices – even if such graphs are equimatchable factor-critical, they can have arbitrarily large independence number. Note that Proposition 3.11 from the previous section shows that these result can neither be extended to graphs with at most 2\(k+1\) vertices, since in such graphs \(G-S\) can have \(k\) components and hence also independence number at least \(k\).

We start with two propositions showing close relationship between equimatchable and almost-equimatchable graphs, and graphs with independence number 2. In the following proof we assume that the reader is familiar with the concept of Gallai-Edmonds decomposition, see [10] for details. We use notation consistent with [10], more precisely, \(D\) is the set of vertices of \(G\) uncovered by at least one maximum matching of \(G\). Furthermore, \(A\) is the set of vertices of \(G-D\) adjacent to at least one vertex of \(D\) and \(C\) is the set \(V(G) - (A \cup D)\). For a discussion concerning how Gallai-Edmonds decomposition relates to equimatchable graphs see [8].

**Proposition 4.1.** Let \(G\) be a graph with independence number 2. If \(G\) is odd, then \(G\) is equimatchable. If \(G\) is even, then either \(G\) is randomly matchable, or \(G\) is not equimatchable, has a perfect matching, and every maximal matching of \(G\) leaves uncovered at most two vertices.

*Proof.* For any maximal matching \(M\) the set of vertices not covered by \(M\) induces an independent set. Hence any maximal matching of a graph with independence number 2 leaves uncovered at most 2 vertices. Since the parity of the number of vertices not covered by a matching is the same as the parity of the number of vertices of the graph, if \(G\) is odd, then any maximal matching of \(G\) leaves uncovered exactly one vertex. Consequently, all maximal matchings of \(G\) have the same size and \(G\) is equimatchable. If \(G\) is even, then every maximal matching of \(G\) leaves uncovered 0 or 2 vertices. We distinguish two cases: either \(G\) is equimatchable, or not. If \(G\) is equimatchable with a perfect matching, then it is isomorphic with \(K_{2n}\) or \(K_{n,n}\) for some nonnegative integer \(n\) by [10]. Suppose that \(G\) is equimatchable and every maximal matching of \(G\) leaves uncovered exactly 2 vertices, our aim is to show that there are no such graphs. Since \(G\) is even, it cannot be factor-critical. Furthermore, \(G\) does not have a perfect matching and thus it has a nontrivial Gallai-Edmonds decomposition. It is well known that the number of vertices uncovered by any maximum matching equals the difference between the number of components in \(D\) and the number of vertices in \(A\), see for example [10]. It follows that there are at least 3 components in \(D\), which contradicts the fact that the independence number of \(G\) is 2. The only remaining possibility is that \(G\) is not equimatchable, in which case \(G\) has a perfect matching, every its maximal matching leaves uncovered at most 2 vertices, and it has a maximal matching which leaves uncovered precisely 2 vertices.

Odd graphs with independence number 2 are described by the following proposition.

**Proposition 4.2.** Let \(G\) be a connected odd graph with independence number 2. Then \(G\) is either factor-critical, or an union of two complete graphs, one even and one odd, joined by a set of pairwise incident edges.

*Proof.* If \(G\) is 2-connected, then by [8] it is either bipartite, or factor-critical. If \(G\) is factor-critical, then there is nothing to prove. Therefore, we can assume that \(G\) is bipartite. Since each partite set of a bipartite graph form an independent set, each partite sets of \(G\) has size at most 2. From the fact that \(G\) is odd follows that \(G = P_3\) and thus it has a cutvertex. Therefore, there is no 2-connected odd bipartite graph with independence number 2, which completes the proof of the first case.

If \(G\) has a cutvertex \(v\), then \(G - v\) has exactly two components, otherwise the independence number of \(G\) would be at least 3. Moreover, both components of \(G - v\) are complete, since otherwise there would be an independence set with size 3 consisting from two nonadjacent vertices of one component and any vertex of the second component. If there are two vertices \(u\) and \(w\) from different components of \(G - v\) such that \(v\) is not adjacent to neither of them, then again \(\{u, v, w\}\) is an independent set of size 3. Hence \(v\) is adjacent with every vertex of at least one component of \(G - v\) and \(G\) is an union of two complete graphs, one even and one odd, joined by a set of pairwise incident edges.

We now turn our attention to the independence number of equimatchable factor-critical graphs.

**Lemma 4.3.** Let \(G\) be a \(k\)-connected equimatchable factor-critical graph with a \(k\)-cut \(S\) for some \(k \geq 3\). Assume that \(G-S\) has precisely two components \(C\) and \(D\), both of them complete. Then for each vertex \(s\) of \(S\) there is a matching \(M\) containing only edges from \(S-s\) such that \(|S-V(M)| = 2\) if \(k\) is even and \(|S-V(M)| = 3\) if \(k\) is odd.

*Proof.* Let \(s\) be a given vertex of \(S\) and first assume that \(k\) is even. One of the components of \(G-S\), say \(C\), is odd and the other is even. Let \(c\) be a vertex of \(C\) adjacent to \(s\) and denote by \(M_D\) and \(M_C\) a perfect matching of \(D\) and of \(C-c\), respectively. Clearly, the set \(N\) defined by \(N = M_D \cup M_C \cup \{sc\}\) is a matching of \(G\) and
hence it can be extended to a matching $N'$ leaving only one vertex of $G$ uncovered. The only vertex not covered by $N'$ lies in $S$, therefore $N' - N$ is the desired matching.

In the rest of the proof we assume that $k$ is odd, which implies that $|C|$ and $|D|$ have the same parity. First we consider the case where both $|C|$ and $|D|$ are even. Let $M_C$ and $M_D$ be perfect matchings of $C$ and $D$, respectively. Since $G$ is equimatchable and factor-critical, the matching $M_C \cup M_D$ can be extended to a matching $N$ leaving only one vertex $s'$ of $G$ uncovered. Note that necessarily $s'$ lies in $S$. Let $M = N - M_C \cup M_D$. If $s = s'$, then let $e$ be an arbitrary edge of $M$, otherwise let $e$ be the edge of $M$ incident with $s$. It is easy to see that $M - e$ is the desired matching.

Finally we consider the case where both $|C|$ and $|D|$ are odd. Let $s'$ be a vertex of $S$ different from $s$ and let $c$ be a vertex of $C$ and $d$ a vertex of $D$ adjacent to $s$ and $s'$, respectively. Furthermore, let $M_C$ and $M_D$ be perfect matchings of $C - c$ and $D - d$, respectively. Since $G$ is equimatchable and factor-critical, the matching $N$ defined by $N = M_C \cup M_D \cup \{sc, s'd\}$ can be extended to a matching $N'$ leaving uncovered only one vertex of $G$. As in the previous cases, it is easy to see that the vertex uncovered by $N'$ lies in $S$. Therefore, $N' - N$ is the desired matching, which completes the proof.

Lemma 4.4. Let $G$ be a $k$-connected equimatchable factor-critical graph with at least $2k + 3$ vertices and a $k$-cut $S$, where $k \geq 4$. Assume that $G - S$ has two components $C$ and $D$, each with at least 3 vertices. Then for any vertices $s \in S$, $c \in C$, and $d \in D$ the subgraph of $G$ induced by $\{c, d, s\}$ contains at least one edge.

Proof. Theorem 3.10 implies that both $C$ and $D$ are complete and that $G - S$ does not have any other components. For the rest of the proof let $c, d, s$ be arbitrary, but fixed, vertices of $G$ such that $c \in C$, $d \in D$, and $s \in S$. We will need the following two claims.

Claim 1. If there is a matching $M$ covering all vertices of $S - s$ such that $V(M) \cap \{c, d, s\} = \emptyset$ and both $C - V(M)$ and $D - V(M)$ are odd, then the subgraph of $G$ induced by $\{c, d, s\}$ contains at least one edge.

Proof of Claim 1. Since both $C$ and $D$ are complete and both $C - V(M)$ and $D - V(M)$ are odd, the subgraphs of $G$ induced by $C - V(M)$ and $D - V(M)$ are odd complete graphs. Therefore, there are matchings $M_C$ and $M_D$ of $C - V(M)$ and $D - V(M)$ covering all vertices of $C - (V(M) \cup \{c\})$ and $D - (V(M) \cup \{d\})$, respectively. It follows that $M' = M \cup M_C \cup M_D$ is a matching of $G$ covering all vertices of $G$ except $c$, $d$, and $s$. Since $G$ is equimatchable and factor-critical, $M'$ can be extended to a maximum matching of $G$, that is, a matching covering all but one vertices of $G$. Consequently, the subgraph of $G$ induced by $\{c, d, s\}$ contains at least one edge, as claimed.

Claim 2. Let $s$ be a vertex from $S$. If $s$ is adjacent to only one vertex of some component of $G - S$, then $s$ is adjacent to all vertices of the other component of $G - S$.

Proof of Claim 2. Assume that $s$ is adjacent to a single vertex of $D$, say $d$. Let $R = (S \cup \{d\}) - s$ and note that $R$ is a $k$-cut of $G$ such that $G - R$ has two components, namely $C \cup \{s\}$ and $D - d$. If $|D| \geq 4$, then both components of $G - R$ have at least 3 vertices and hence both are complete by Theorem 3.10. In particular, $s$ is adjacent to every vertex of $C$, as claimed. If $|D| = 3$, then the components of $G - R$ have size 2 and at least $k + 1$, respectively. By Theorem 3.8, either $C \cup \{s\}$ is a complete graph, or $R \cup C \cup \{s\}$ is a complete bipartite graph minus a matching. Since $C$ is a complete graph containing a triangle, the graph $R \cup C \cup \{s\}$ cannot be bipartite and the claim follows.

By Lemma 4.3 there is a matching $M$ containing only edges from $S - s$ such that $|S - V(M)| \leq 3$. Let $C' = C - c$, $D' = D - d$, and $S' = S - (V(M) \cup \{s\})$. We distinguish three cases depending on the parity of $C$ and $D$.

First let $k$ be even. Since $k$ is even, one of the components of $G - S$ is even, say $C$, and the other is odd. By Lemma 4.3 the set $S'$ contains only one vertex, denote it by $s_1$. If $s_1$ is adjacent to some vertex in $C'$, then we are done by Claim 1. Otherwise, $c$ is the only neighbour of $s$ in $C$ and by Claim 2 the vertex $s_1$ is adjacent to all vertices of $D$. Since $k \geq 4$, $M$ contains at least one edge, say $s_2s_3$. By Lemma 4.3 there are two independent edges between $\{s_2, s_3\}$ and $D$. At least one of the edges, say the one incident with $s_2$, does not have $d$ as an endvertex. Denote this edge by $e$. Using Lemma 4.3, again yields that there are two independent edges between $\{s_1, s_3\}$ and $C$. Clearly, one of these edges is $s_1c$ and hence there is an edge $f$ between $s_3$ and $C'$. Since $|D| \geq 3$ and $s_1$ is adjacent to all vertices of $D$, there is an edge $g$ between $D'$ and $s_1$ such that $e$ and $g$ are independent. Applying Claim 1 to the matching $(M - \{s_2s_3\}) \cup \{e, f, g\}$ finishes the proof.

Assume that $k$ is odd and that both components of $G - S$ are even. Since $k$ is odd, by Lemma 4.3 the set $S'$ contains precisely two vertices, denote them by $s_1$ and $s_2$. By Lemma 4.3 there are two independent edges between $S'$ and $C$ and thus at least one of them, say $s_1c'$, is not incident with $c$. If there exists an edge $s_2d'$, where $d \in D'$, then the matching $M \cup \{s_1c', s_2d'\}$ satisfies the assumptions of the Claim 1 and we are done. If there is no edge between $s_2$ and $D'$, then $d$ is the only vertex of $D$ adjacent to $s_2$. By Lemma 5.4 there are
two independent edges between $D$ and $\{s_1, s_2\}$, one of them is necessarily $s_2d$ and thus the other is $s_1d''$, where $d'' \neq d$. Furthermore, by Claim 2 the vertex $s_2$ is adjacent to all vertices $C$ and thus there is an edge $s_2e''$, where $e'' \neq c$. Applying Claim 1 to the matching $M \cup \{s_1d'', s_2e''\}$ completes the proof of this case.

Finally, if $k$ is odd and also both components of $G - S$ are odd, then again $S'$ has two vertices by Lemma 4.19. First observe that if there is a matching $M'$ between $S'$ and $C'$ or between $S'$ and $D'$, which covers both vertices of $S'$, then applying Claim 1 to $M \cup M'$ yields the desired result. We proceed to show that it is always possible to construct such a matching. Let $s_1$ and $s_2$ be the vertices in $S'$. Since $k$ is odd, the matching $M$ contains at least one edge of $S$, say $s_3s_4$. By Lemma 3.4 there is a set of two independent edges $M_C$ between $S'$ and $C$ and a set of two independent edges between $S'$ and $D$. If $M_C$ does not cover either $c$ or $d$, then $M \cup M_C$ is the desired matching and we are done, similarly for $M_D$. Therefore, we can assume that all matchings between $S'$ and $C$ covering $S'$ cover also $c$ and analogously all matchings between $S'$ and $D$ covering $S'$ cover also $d$. In the rest of the proof we distinguish two cases.

First assume that one of the vertices of $S'$, say $s_1$, is in $C$ adjacent only to $c$. By Claim 2 the vertex $s_1$ is adjacent to all vertices of $D$. It follows that $s_2$ is in $D$ adjacent only to $d$, since otherwise there would be a set of two independent edges between $S'$ and $D'$. By Lemma 3.4 there are three independent edges $s_1c_1, s_2c_2, s_3c'$ between $C$ and $\{s_1, s_2, s_3\}$. Since $s_1$ is in $C$ adjacent only with $c$, we necessarily have $c_1 = c$. Similarly, by Lemma 3.4 there are three independent edges $s_1d_1, s_2d_2, s_3d'$ between $D$ and $\{s_1, s_2, s_4\}$. Again, since $s_2$ is in $D$ adjacent only to $d$, we have $d_2 = d$. It follows that $M' = (M - s_3s_4) \cup \{s_1d_1, s_2c_1, s_3c', s_3d'\}$ satisfy the assumptions of Claim 1 which completes the proof of this case.

Second assume that both vertices of $S'$ are adjacent to at least two vertices of both $C$ and $D$. It follows that there is a vertex $c'$ of $C'$ such that there is no edge between $C' - c'$ and $S'$, since otherwise there would be a set of two independent edges between $S'$ and $C'$. Similarly, there is a vertex $d'$ of $D'$ such that there is no edge between $D' - d'$ and $S'$. By Lemma 3.4 there are three independent edges between $\{s_1, s_2, s_3\}$ and $C$. One of these edges is $s_3c'$, where $c'$ is different from both $c$ and $c'$, since $s_1$ and $s_2$ are adjacent to precisely two vertices of $C$. Without loss of generality we can assume that the other two are $s_1c$ and $s_2c'$. Similarly, by Lemma 3.4 there are three independent edges between $\{s_1, s_2, s_4\}$ and $D$. Again, one of these edges is $s_4d'$. Since both $s_1$ and $s_2$ are adjacent to precisely two vertices of $C$, each of them is adjacent to both $d$ and $d'$ and thus $s_1d'$ is an edge of $G$. To conclude the proof of it suffices to observe that the matching $(M - s_3s_4) \cup \{s_1d', s_2c', s_3c', s_4d'\}$ satisfy the assumptions of Claim 1.

**Lemma 4.5.** Let $G$ be a $k$-connected equimatchable factor-critical graph with at least $2k + 3$ vertices and a $k$-cut $S$ such that $G - S$ has two components with at least 3 vertices, where $k \geq 4$. Then the independence number of $G$ is 2.

**Proof.** By Theorem 3.10 both $C$ and $D$ are complete and $G - S$ does not have any other components. Since both $C$ and $D$ are complete, no independent set of $G$ can contain more than one vertex from any of them. Observe that $G$ cannot have an independent set $\{c, d, s\}$ where $c \in C$, $d \in D$, and $s \in S$ by Lemma 3.4.

There are two remaining possible types of independent sets of size 3 in $G$. More precisely, either $G$ has an independent set consisting of 3 vertices of $S$, or $G$ has an independent set consisting of 2 vertices of $S$ and a vertex of $C \cup D$. For a contradiction suppose that $I$ is such an independent set of size 3 in $G$. Let $T = I \cap S$ and let $C' = C - I$. If $C'$ is odd, let $F$ be the set containing an arbitrary edge between $C'$ and $S - T$, otherwise let $F = \emptyset$. Furthermore, if $C'$ is odd, let $T' = T \cup \{s\}$, where $s$ is the vertex of $S$ incident with the edge in $F$, otherwise let $T' = T$. It is not difficult to see that if $D$ is odd, then there is a vertex $s'$ in $S - T'$. Therefore, if $D$ is odd, then by Lemma 3.4 there is a vertex $d$ of $D$ adjacent to $s'$ and we set $F' = F \cup \{ds'\}$. If $D$ is even, we set $F' = F$. Let $U$ be the set of vertices covered by the edges in $F'$. Since $|C \cup D| \geq k - 1$ by the assumptions, Lemma 3.4 implies that there is a matching $M$ between $S - (T \cup U)$ and $(C' \cup D) - U$ such that both $C' - V(M \cup F')$ and $D - V(M \cup F')$ are even and $M$ covers all vertices of $S - (T \cup U)$. Finally, let $M_C$ and $M_D$ be perfect matchings of $C' - (U \cup V(M))$ and $D - (U \cup V(M))$, respectively. It is easy to see that the matching $M'$ defined by $M' = M \cup M_C \cup M_D \cup F'$ covers all vertices of $G$ except $I$. Since $G$ is equimatchable and factor-critical, $M'$ can be extended to a matching leaving uncovered exactly one vertex. It follows that $I$ contains an edge, contradicting the fact that it is an independent set. The proof is now complete.

The following theorem is the main result of this section.

**Theorem 4.6.** Let $G$ be a $k$-connected odd graph with at least $2k + 3$ vertices and a $k$-cut $S$ such that $G - S$ has two components with at least 3 vertices, where $k \geq 4$. Then $G$ has independence number at most 2 if and only if it is equimatchable and factor-critical.

**Proof.** If $G$ is equimatchable and factor-critical, then its independence number is 2 by Lemma 4.5. In the reverse direction, assume that $G$ has independence number 2. Then $G$ is equimatchable by Proposition 4.14. Furthermore, Proposition 4.2 implies that $G$ is factor-critical, which completes the proof.
The following theorem reveals further connection between independence number 2 and equimatchable graphs.

**Theorem 4.7.** A 2-connected odd graph $G$ with at least 4 vertices has independence number at most 2 if and only if $G$ is equimatchable and factor-critical and $G \cup \{e\}$ is equimatchable for each edge of the complement $\overline{G}$ of $G$.

**Proof.** If $G$ has independence number 2, then it is equimatchable and factor-critical by Proposition 4.1 and 4.2. Clearly, for any edge $e$ of the complement $\overline{G}$ of $G$ the graph $G \cup \{e\}$ has again independence number 2 and thus it is equimatchable by Proposition 4.4. In the reverse direction, assume that $G$ is equimatchable factor-critical and that $G \cup \{e\}$ is equimatchable for every edge of the complement $\overline{G}$ of $G$. For a contradiction suppose that $G$ has an independent set $\{x, y, z\}$ of size 3. Since $G$ is factor-critical, $G - z$ has a perfect matching $M$. Let $x'$ and $y'$ be the vertices matched with $x$, respectively $y$, by $M$. Note that since $\{x, y, z\}$ is an independent set we have $x \neq x'$ and $y \neq y'$. It is easy to see that $M' = (M - \{xx', yy'\}) \cup \{x'y'\}$ is a maximal matching of $G \cup \{x'y'\}$ which leaves uncovered precisely three vertices. On the other hand, $M$ is a matching of $G \cup \{x'y'\}$ leaving uncovered precisely one vertex. It follows that $M'$ is a maximal matching of $G \cup \{x'y'\}$, which is not maximum, contradicting equimatchability of $G \cup \{x'y'\}$. We conclude that the independence number of $G$ is at most 2, which completes the proof.

Equimatchable graphs $G$ such that $G \cup \{e\}$ is equimatchable for every edge $e$ of the complement $\overline{G}$ of $G$ are further investigated in [3], together with other extremal classes of equimatchable graphs.

Our final two results show that Lemma 4.5, and thus also Theorem 4.6, can be extended neither to equimatchable graphs without two components with at least 3 vertices, nor to the case of graphs with connectivity 3.

**Proposition 4.8.** For every triple of integers $n, k, \text{ and } m$ such that $k \geq 3$ and $m \in \{1, 2\}$ there is a $k$-connected equimatchable factor-critical graph $G$ with an independent set of size at least $n$ and a $k$-cut $S$ such that $G - S$ has a component of size $m$.

**Proof.** First assume that $m = 1$. Let $l = \max\{n, k\}$ and denote by $H$ a copy of $K_{l, l}$. Choose a set $S$ of $k$ vertices of $H$ in such a way that $S$ contains at least one vertex from each partite set of $H$. The desired graph $G$ is constructed by taking a new vertex $v$ and joining it with every vertex in $S$. Clearly, $G$ is $k$-connected and $S$ is a $k$-cut of $G$. Since $\{v\}$ is a component of $G - S$ and $m = 1$, the graph $G - S$ has a component with $m$ vertices. Furthermore, it is easy to directly verify that $G$ is factor-critical and equimatchable. The proof of this case is concluded by observing that each partite set of $H$ forms an independent set of $G$ with size $l \geq n$.

Now we assume that $m = 2$. Let $l = \max\{n, k\}$ and denote by $H_1$ a copy of $K_{l, l+1}$ and by $H_2$ a copy of $K_2$. Denote by $S$ a set of $k$ vertices from the larger partite set of $H_1$. The desired graph $G$ is constructed by joining both vertices of $H_2$ with all vertices of $S$. It can be easily verified that the resulting graph is $k$-connected, equimatchable, and factor-critical. Clearly $S$ is a $k$-cut of $G$ such that $G - S$ has a component with $m$ vertices. Finally, $G$ contains an independent set with $l + 1 \geq n$ vertices, which completes the proof.

**Proposition 4.9.** For every pair of odd integers $m$ and $n$ such that $m + n \geq 4$ there is a 3-connected equimatchable factor-critical graph $G$ with independence number 3 and a 3-cut $S$ such that $G - S$ has two components with sizes $m$ and $n$, respectively.

**Proof.** For any given pair of odd integers $m$ and $n$ we construct a graph $G(m, n)$ with the required properties as follows. Let $C$, respectively $D$, be a copy of the complete graph on $m$ and $n$ vertices, respectively and let $S$ be an independent set on 3 vertices. To obtain $G(m, n)$ we join every vertex of $C \cup D$ with every vertex of $S$. Since $m + n \geq 4$, the graph $G(m, n)$ is 3-connected.

To prove that $G(m, n)$ is factor-critical, first let $v \in (C \cup D)$ and let $G' = G(m, n) - v$. It is easy to see that there is a set $M$ of 3 independent edges between $S$ and $(C \cup D) - v$ such that $G' - V(M)$ consists of two even complete graphs. Therefore, $M$ can be extended to a perfect matching of $G'$. If $v \in S$, then there is an edge $sc$ between $S - v$ and $C$ and an edge $s'd'$, independent from $sc$, between $S - v$ and $D$. Since removing $\{s, s', c, d\}$ from $G - v$ yields two even complete components, $\{sc, s'd'\}$ can be extended to a perfect matching of $G - v$, which in turn is factor-critical, as claimed.

In the rest of the proof we show that $G(m, n)$ is equimatchable. Let $M$ be a maximal matching of $G(m, n)$. Since $C$ and $D$ are complete, $M$ leaves uncovered at most one vertex of each $C$ and $D$. Assume that $M$ leaves uncovered a vertex $c$ from $C$. Since $M$ is maximal, then clearly $M$ must cover all vertices of $S$. Therefore, $D - V(M)$ is an even complete graph and thus $M$ covers all vertices of $D$ by its maximality. We conclude that $c$ is the only vertex of $G(m, n)$ uncovered by $M$ and hence $M$ is a maximum matching. Analogous argument also shows that if $M$ leaves uncovered a vertex from $D$, then $M$ is maximum.

Therefore, we can assume that $M$ covers all vertices from $C \cup D$. Since both $C$ and $D$ are odd, to cover all vertices of $C \cup D$ the matching $M$ has to cover precisely two vertices of $S$. Consequently, $M$ leaves uncovered
exactly one vertex of $S$ and $M$ is maximum, as required. Since both $C$ and $D$ are odd, if $M$ does not leave uncovered exactly one vertex of $S$ leaves uncovered also at least one vertex of $C \cup D$, and hence $M$ cannot be a maximal matching of $G(m,n)$.

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