QUASI-PERIODIC SOLUTIONS FOR THE TWO-DIMENSIONAL SYSTEMS WITH AN ELLIPTIC-TYPE DEGENERATE EQUILIBRIUM POINT UNDER SMALL PERTURBATIONS

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(Communicated by Rafael Ortega)

ABSTRACT. In this paper, we consider the existence of quasi-periodic solutions for the two-dimensional systems with an elliptic-type degenerate equilibrium point under small perturbations. We prove that under appropriate hypotheses there exist quasi-periodic solutions for perturbed ODEs near the equilibrium point for most parameter values.

1. Introduction. The existence problem of quasi-periodic solutions is a very active research topic of KAM theory for integrable or partially integrable systems under quasi-periodic perturbations. This problem has been studied near non-degenerate elliptic equilibrium points by Jorba and Simo [7], near non-degenerate invariant tori by Friedman [5], Braaksma and Broer [2], Braaksma et al. [3], Broer et al. [4], and Sevryuk [15].

Xu [21] considered the two dimensional quasi-periodic and real analytic system

$$\begin{align*}
\dot{x} &= \Omega y + h_1(x, y, t) + f_1(x, y, t), \\
\dot{y} &= x^3 + h_2(x, y, t) + f_2(x, y, t),
\end{align*}$$

(1.1)

where \( \Omega > 0 \) is a constant, \( h = (h_1, h_2)^T \) \((T\) denotes transpose) is the high-degree term with

\[
h(x, y, t) = \sum_{l_2 \geq 2 \text{ or } l_1 + l_2 \geq 4} h_{l_1l_2}(t)x^{l_1}y^{l_2},
\]

\( f = (f_1, f_2)^T \) is the low-degree small perturbation term with

\[
f(x, y, t) = \sum_{l_2 \leq 1 \text{ and } l_1 + l_2 \leq 3} f_{l_1l_2}(t)x^{l_1}y^{l_2},
\]

and \( h, f \) are quasi-periodic in \( t \) with frequency \( \omega \in \mathbb{R}^{n_0} \). The origin \((0, 0)\) is a hyperbolic-type degenerate equilibrium point of the unperturbed system of (1.1). It is proved that if \((f_1, f_2)^T\) is sufficiently small and the frequency vector \( \omega \) satisfies

2020 Mathematics Subject Classification. Primary: 34J40, 34C27; Secondary: 34E20.

Key words and phrases. KAM theory, quasi-periodic solution, elliptic degenerate equilibrium point.

The second author is supported by the Education Department Project of Hunan Province (No.18C0026) and the NNSF of China (No.11971163).

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the Diophantine condition, system (1.1) has a quasi-periodic solution near the equilibrium point. Since the degenerate equilibrium point of system (1.1) is hyperbolic-type, there is no resonance between the tangent frequency (internal frequency) and the normal frequency (external frequency). So we do not have to overcome the small divisor by digging the ’bad’ parameters. Besides, many scholars also have studied the existence of quasi-periodic solutions for systems with hyperbolic-type degenerate equilibrium points (see [17, 22, 23], for example).

Qiu and Si [14] considered the four-dimensional quasi-periodic system

$$
\begin{align*}
\dot{x}_1 &= x_1 + h_1(x_1, x_2, y_1, y_2, t) + f_1(x_1, x_2, y_1, y_2, t), \\
\dot{x}_2 &= -x_2 + h_2(x_1, x_2, y_1, y_2, t) + f_2(x_1, x_2, y_1, y_2, t), \\
\dot{y}_1 &= y_2 + h_3(x_1, x_2, y_1, y_2, t) + f_3(x_1, x_2, y_1, y_2, t), \\
\dot{y}_2 &= -y_2^3 + h_4(x_1, x_2, y_1, y_2, t) + f_4(x_1, x_2, y_1, y_2, t),
\end{align*}
$$

(1.2)

where $h = (h_1, h_2, h_3, h_4)^T$ and $f = (f_1, f_2, f_3, f_4)^T$ are analytic in $(x_1, x_2, y_1, y_2, t)^T$ and quasi-periodic in $t$ with the frequency $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$. Suppose that $h$ is the high-degree term with

$$
h(x_1, x_2, y_1, y_2, t) = \sum_{\nu \geq 2 \text{ or } l+n+\mu \geq 4} h_{i\mu
u}(t)x_1^l x_2^n y_1^\mu y_2^\nu,$$

and $f$ is the low-degree term with

$$
f(x_1, x_2, y_1, y_2, t) = \sum_{\nu \leq 1 \text{ and } l+n+\mu \leq 3} f_{i\mu
u}(t)x_1^l x_2^n y_1^\mu y_2^\nu.$$

It is proved that system (1.2) has a quasi-periodic solution with frequency $\omega$ near the origin if the perturbation $f$ is sufficiently small and the frequency vector $\omega$ satisfies the small divisor condition. Note that the origin is hyperbolic non-degenerate in the $x$-direction and elliptic degenerate in the $y$-direction. Since the dynamical behavior of the hyperbolic direction cannot be changed under small perturbation, essentially, we only need to analyze the restricted system of (1.2) to the central manifold (i.e., elliptic direction). And resonances occur between the tangent frequency and the drifted normal frequency in the central direction. However, in the process of proving the existence of quasi-periodic solutions for system (1.2), the authors did not discover the resonant phenomenon because of the incorrect expression of the homological equation (10) on page 427 in [14]. In this paper, we will continue studying the system with an elliptic-type degenerate equilibrium point. Before stating our main theorem, we introduce some related work.

Recently, Si and Si [16] have considered the four-dimensional quasi-periodic non-conservative system

$$
\frac{d}{dt}z = J\nabla_z H(z) + G(\omega t, z) + F(\omega t, z, \epsilon), z = (x_1, y_1, x_2, y_2)^T \in \mathbb{R}^4,
$$

(1.3)

where the unforced vector field $X_H := J\nabla_z H(z)$ is a Hamiltonian vector field with symplectic structure $dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and Hamiltonian

$$
H = \frac{1}{2n_1 + 2} x_1^{2n_1+2} + \frac{a}{2m_1 + 2} y_1^{2m_1+2} + \frac{1}{2n_2 + 2} x_2^{2n_2+2} + \frac{b}{2m_2 + 2} y_2^{2m_2+2},
$$

(1.4)

where $n_1$ and $n_2$ are positive integers, $m_1$ and $m_2$ are nonnegative integers, $a, b \in \mathbb{R} \setminus \{0\}$, the terms $G$ and $F$ respectively stand for the high-degree and low-degree quasi-periodic perturbations with $F(\omega t, z, 0) = 0$, which implies that the origin...
\((0,0,0)^T\) is an elliptic-type degenerate equilibrium point of the unperturbed system of (1.3). Compared with system (1.2), it seems to be more general in form. But, to guarantee the existence of quasi-periodic solutions for system (1.3), the authors imposed the following restrictions on terms \(G, F\):

\((\text{H1})\): \(G(\theta, z) (\theta = \omega t)\) is analytic in \((\theta, z)\) and has the following form

\[
G(\theta, z) = \sum_{\eta \in \mathbb{Z}_+^k, ||\eta||_1 \geq \chi} G_\eta(\theta) z^\eta
\]

where

\[
\chi \geq \left\lfloor \frac{\tilde{b}(2(2\beta_2 - \beta_1) + \beta_3)}{(2n_1 + 1)} \right\rfloor + 1
\]

(\(\lfloor \cdot \rfloor\) denotes the integer part), and

\[
\tilde{b} := (2m_1 + 1)(2n_1 + 1)(2m_2 + 1)(2n_2 + 1), \beta_1 := \min\{\beta_{11}, \beta_{12}\},
\]

\[
\beta_2 := \max\{\beta_{11}, \beta_{12}\}, \beta_3 := \max\{\frac{\tilde{b}(n_1 - m_1)}{2n_1(2m_1 + 1)}; \frac{\tilde{b}(n_2 - m_2)}{2n_2(2m_2 + 1)}\}
\]

with \(\beta_{11} := (2m_1 + 1)(2n_1 + 1)(4m_2n_2 + m_2 + n_2)\) and \(\beta_{12} := (2m_2 + 1)(2n_2 + 1)(4m_1n_1 + m_1 + n_1)\).

\((\text{H2})\): \(F(\theta, z, \epsilon) = (F_1, F_2, F_3, F_4)^T\) can be written as

\[
F_j = q_j + \epsilon^{\alpha_j} \tilde{f}_j(\theta, z, \epsilon), \quad j = 1, 2, 3, 4,
\]

where \(q_j, \alpha_j\) are given positive real numbers and \(\tilde{f}_j(\theta, z, \epsilon)\) are analytic in \((\theta, z)\) and \(C^1\)-smooth in \(\epsilon\). Obviously, the perturbations \(f, h\) in systems (1.1) and (1.2) do not satisfy these hypotheses (\(\text{H1}\)) and (\(\text{H2}\)). So the results in [16] cannot cover ours (see Theorem 1.1).

At the same time, Li and Shang [10] considered the real analytic ordinary differential equations

\[
\begin{aligned}
\dot{v}_1 &= \Omega_1(a)v_1^3 + h_1(\omega t, v; a) + \epsilon f_1(\omega t, v; a, \epsilon), \\
\dot{v}_2 &= -\Omega_2(a)v_3 + d_1(a)v_1^2 + h_2(\omega t, v; a) + \epsilon f_2(\omega t, v; a, \epsilon), \\
\dot{v}_3 &= \Omega_2(a)v_2 + d_2(a)v_3^2 + h_3(\omega t, v; a) + \epsilon f_3(\omega t, v; a, \epsilon),
\end{aligned}
\]

where \(v = (v_1, v_2, v_3)^T \in \mathbb{R}^3\), \(a \in \Pi_0 \subset \mathbb{R}\) is a parameter variable, the perturbations \(\epsilon f_j (j = 1, 2, 3)\) are quasi-periodic in \(t\) with frequency \(\omega = (\omega_1, \cdots, \omega_{n_0})\), and

\[
h_j(\omega t, v; a) = \sum_{l_1 + 2l_2 + l_3 \geq 4} h_{j, l_1l_2l_3}(\omega t; a)v_1^{l_1}v_2^{l_2}v_3^{l_3}
\]

are high-degree terms. It is proved that there exist a quasi-periodic solution to system (1.6) for most part parameters of \(\Pi_0\) under appropriate hypotheses. Besides, You [25] studied the existence of quasi-periodic solutions near the hyperbolic-type degenerate lower dimensional tori for Hamiltonian systems and Li and Llave [8] considered delay differential equations with elliptic-hyperbolic equilibrium points. Recently, there are some other results concerning about the invariant tori of degenerate systems, refer to [18, 19].

In this paper we will focus on the following real analytic ordinary differential equation

\[
\begin{aligned}
\dot{x} &= \Omega(a)y + h_1(x, y, \omega t; a) + \epsilon f_1(x, y, \omega t; a), \\
\dot{y} &= -x^3 + h_2(x, y, \omega t; a) + \epsilon f_2(x, y, \omega t; a),
\end{aligned}
\]

(1.7)
where \( a \in \Pi_0 \) is the parameter variable, \( \Omega(a) > 0 \), \( \Pi_0 \subset \mathbb{R} \) is a bounded closed set of positive Lebesgue measure, all functions in (1.7) are continuously differentiable with respect to the parameter \( a \) of positive Lebesgue measure, all functions in (1.7) are continuously differentiable with respect to the parameter \( a \), and \( f_j, h_j \) \((j = 1, 2)\) are quasi-periodic in \( t \) with frequencies \( \omega = (\omega_1, \cdots, \omega_{n_0}) \), real analytic in variables \( x, y, \phi \) with \((x, y) \in \varpi(s_0)\),

\[
\varpi(s_0) = \{(x, y) \in \mathbb{R}^2 : |x|^2 + |y|^2 \leq s_0\},
\]

\( \phi = \omega t \in \mathbb{D}(r_0), \mathbb{D}(r_0) = \{ \phi : |\text{Im} \phi| = \sup_{1 \leq j \leq n_0} |\text{Im} \phi_j| \leq r_0 \} \) and have the following forms

\[ f_j(x, y, \omega t; a) = \sum_{l \in \sum_0^2} f_{j_l} l x^l y^{l_2}, \quad h_j(x, y, \omega t; a) = \sum_{l \in \sum_0^2} h_{j_l} l_x (\omega t; a) x^l y^{l_2} \]

with \( \sum_0^2 = \{ l = (l_1, l_2) \in \mathbb{Z}_+^2 : l_1 + 2l_2 \leq 3 \} \) and \( \sum_G^0 = \{ l = (l_1, l_2) \in \mathbb{Z}_+^2 : l_1 + 2l_2 \geq 4 \} \).

**Remark 1.** For the general perturbation \( f_j \) which is analytic, we divide it into two parts \( f_j = f_jL + f_jH \) with

\[ f_jL = \sum_{l_1+2l_2 \leq 3} f_{j_l} l x^l y^{l_2} \quad \text{and} \quad f_jH = f_j - f_jL. \]

Then we put the part \( f_jH \) into the high-degree term \( h_j \) in (1.7). Without loss of generality, we assume that the equations in (1.7) take the polynomial perturbation.

Obviously, the origin is an elliptic-type degenerate equilibrium point of the unperturbed system of (1.7). As we know that, when the equilibrium point is elliptic-type, we have to make some assumptions as follow.

(A1) The frequency vector \( \omega = (\omega_1, \cdots, \omega_{n_0}) \) satisfies the Diophantine condition

\[
|\langle k, \omega \rangle| \geq \gamma_0 |k|^{-n_0-1},
\]

where \( k \in \mathbb{Z}^{n_0} \) is nonzero vector, \( |k| = |k_1| + \cdots + |k_{n_0}| \) and \( 0 < \gamma_0 \ll 1 \).

(A2) There is a positive constant \( c_0 \) such that

\[
\min_{a \in \Pi_0} \left\{ \inf_{\omega \in \Pi_0} |\hat{f}_{2,00}(0; a)|, \inf_{\omega \in \Pi_0} |\partial_\omega \hat{f}_{2,00}(0; a)| \right\} \geq c_0
\]

where \( \hat{f}(0; a) \) denotes the zero-th Fourier coefficient of \( f(\phi; a) \).

(A3) There exist positive constants \( c_1, c_2 \) such that

\[
\inf_{a \in \Pi_0} |\Omega(a)| \geq c_1, \quad \inf_{a \in \Pi_0} \left| \frac{d}{da} \Omega(a) \hat{f}_{2,00}(0; a) \right| \geq c_2.
\]

**Remark 2.** If \( \Omega(\neq 0) \) is independent of the parameter \( a \), the condition (A2) leads to the condition (A3).

**Theorem 1.1.** Suppose that for (1.7) the assumptions (A1)-(A3) hold. Then there is a sufficiently small \( \varepsilon^* > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon^* \), there exists a Cantorian-like subset \( \Pi_{\gamma_0} \subset \Pi_0 \) with the Lebesgue measure

\[
\text{Meas} \Pi_{\gamma_0} = \text{Meas} \Pi_0 - O(\gamma_0),
\]

and for any \( a \in \Pi_{\gamma_0} \), the equation (1.7) possesses a quasi-periodic solution \((x(t; a), y(t; a))\) which is real analytic in \( t \), Lipschitz in \( a \in \Pi_{\gamma_0} \) and satisfies

\[
\sup_{x \times \Pi_{\gamma_0}} (|x(t; a)|^2 + |y(t; a)|) = O(\varepsilon^{1/2}).
\]
We remark that on one hand the low-degree and high-degree terms in system (1.7) do not satisfy the hypotheses (H1) and (H2), and it is easy to see that the perturbation in system (1.3) is more restrictive than ours. In fact, if we choose \(m_1 = m_2 = 0, n_1 = n_2 = 1\), the \(\chi \geq 32\) in assumption (H1). Thus, the system (1.3) can not be applied to the following Duffing equation

\[ x'' + x^3 + x^{2n+1} = \varepsilon f(x, \dot{x}, \omega t) \]

with \(2 \leq n \leq 15\). While our result is available. On the other hand, the equilibrium point of (1.6) is elliptic non-degenerate in the \((v_2, v_3)\)-direction and degenerate in the \(v_1\)-direction, while it is elliptic degenerate in the unperturbed system of (1.7). Hence, the estimating in this paper will be more subtle than that in [10].

In order to discuss the existence of quasi-periodic solutions of system (1.7), by some transformations we firstly turn the studied system (1.7) into a normal form (see (3.28)) in which the KAM method can be used directly (see for Section 3). In Section 2, we give a KAM iterative lemma, by which the low-degree terms become smaller and smaller. We iterate it infinitely, then the normal form is transformed into a system (see (3.29)) which clearly possesses the origin as a solution. Thus, we obtain the quasi-periodic solution of system (1.7) via these transformations (see (3.28)). At each iterative step, there are resonances between the tangent frequencies and normal ones. So the small denominators will appear when solving homological equations. With the help of conditions (A1) and (A3), we can deal with them by digging some 'bad' parameter values. Then the term \(\tilde{N}\) in (2.17) which can not be killed will cause the equilibrium point to shift. By the condition (A2), we can find a translation to overcome the shift.

2. The iteration step. Before giving the iterative lemma, we introduce the following recursive parameters and notations. For \(v \geq 1\) and \(0 < \varepsilon_0 \ll 1\), we set:

1. \(\sigma_v = (1^{-2} + \cdots + v^{-2})/(2 \sum_{j=1}^{\infty} j^{-2})\)
2. \(r_v = (1 - \sigma_v)r_0, s_v = (1 - \sigma_v)s_0;\)
3. \(\rho_v = r_v - r_{v+1}, \delta_v = \frac{1}{4}(s_v - s_{v+1});\)
4. \(\epsilon_{v+1} = \epsilon_v^\varepsilon, \text{with } \epsilon_1 = \varepsilon_0^2;\)
5. \(K_v = -\frac{1}{r_v}(v+1)^2(2s+2)\ln \epsilon_1 \text{ and } K_0 = -\frac{8}{r_0}\ln \varepsilon_0;\)
6. \(\gamma_v = \gamma_0/(v+1)^2.\)
7. Let \(D(r_0, s_0) = \varnothing(s_0) \times D(r_0)\) and \(\mathfrak{A}^{\Pi_0}_{r_0, s_0}\) be the set collecting all functions \(f(\phi, z, a)\) defined on \(D(r_0, s_0) \times \Pi_0\) which are analytic in \((\phi, z) \in D(r_0, s_0)\) and continuously differentiable in \(a \in \Pi_0\).
8. We define the norm denoted by \(\| \cdot \|_{r_0, s_0, \Pi_0}\) on \(\mathfrak{A}^{\Pi_0}_{r_0, s_0}\)

\[ \| f \|_{r_0, s_0, \Pi_0} = \max_{s=0,1} \sup_{D(\rho, s_0) \times \Pi_0} | \partial^s_a f |. \]

If \(f\) only depends on variables \(\phi, a\), we denote \(\| f \|_{r_0, s_0, \Pi_0}\) by \(\| f \|_{r_0, \Pi_0}\). If \(f\) only depends on the parameter variable \(a\), we denote \(\| f \|_{r_0, s_0, \Pi_0}\) by \(\| f \|_{r_0, \Pi_0}\).
9. Given \(\varepsilon > 0\), we write \(f = O_{r_0, \Pi_0}(\varepsilon), \text{ if } \| f \|_{r_0, \Pi_0} \leq C\varepsilon, \text{ where } C \text{ is independent of } \varepsilon, a \text{ and } v. \text{ We also write } f < g \text{ (} f \gg g \text{), if there is an absolute constant } C \text{ such that } | f | \leq C| g | \text{ (} | f | \gg C| g |). \text{ Besides, we say } f \sim g, \text{ if there exist positive constants } c \text{ and } C \text{ such that } cg \leq f \leq Cg.\)
(10) For \( f(\phi; a) \in \mathcal{N}_{r, v}^\Pi \) with a Fourier series expansion
\[
f(\phi; a) = \sum_{k \in \mathbb{Z}^n} \hat{f}(k; a)e^{\sqrt{-1}(k, \phi)},
\]
define the truncation operator \( \Gamma_K \) by
\[
\Gamma_K f(\phi; a) = \sum_{|k| \leq K} \hat{f}(k; a)e^{\sqrt{-1}(k, \phi)},
\]
where \( |k| = |k_1| + \cdots + |k_n| \). We have
\[
|||\partial^j_a \hat{f}(k; a)|||_n \leq |||f|||_{r, \Pi} e^{-|k|r}, j = 0, 1,
\]
and
\[
|||\partial^j_a (Id - \Gamma_K)f|||_{r-\rho, \Pi} \leq C_1 \rho^{-n} e^{-\rho K} |||f|||_{r, \Pi}, j = 0, 1, 0 < \rho < r. \tag{2.1}
\]
(11) The constants \( M_0, c_1, c_2, \cdots \) are positive and independent of \( \varepsilon_0, a, v \).

**Lemma 2.1.** For \( v \geq 1 \), suppose that we have obtained the following quasi-periodic system
\[
\begin{cases}
\dot{x} = \varepsilon_0 \left[ \Omega(a)y + \varepsilon_0^2 h^v_1(x, y, \phi; a) + f^v_1(x, y, \phi; a) \right], \\
\dot{y} = N^v_1(x; a) + \varepsilon_0^2 h^v_2(x, y, \phi; a) + f^v_2(x, y, \phi; a),
\end{cases} \tag{2.2}
\]
with the normal form
\[
N^v_1(x, y; a) = \varepsilon_0^3(a)x^3 + \varepsilon_0^2(a)x^2 + \varepsilon_0(a)x + \varepsilon_0^4(a)y + \varepsilon_0^5(a)xy.
\]
We also assume that the system satisfies the following conditions.

(a.1) Frequency condition
\[
|||\varepsilon_j^v(a)|||_{n, v} \sim \varepsilon_0^{-j}, \quad j = 1, 2, 3, \quad |||\varepsilon_4^v(a)|||_{n, v} \leq \varepsilon_0^{-2},
\]
\[
|||\varepsilon_5^v(a)|||_{n, v} \leq \varepsilon_1, \quad \inf_{a \in \Pi_v} \left| \frac{d}{da} (\varepsilon_4^v(a)\Omega^v(a)) \right| \geq \varepsilon_0^{-2}.
\]

(a.2) Smallness condition
For \( j = 1, 2 \), the terms \( f^v_j, h^v_j \) belong to \( \mathcal{N}_{r, v}^\Pi \), where
\[
h^v_j(x, y, \phi; a) = \sum_{l \in \mathbb{Z}^2} h^v_{j, l_1, l_2}(\phi; a)x^{l_1}y^{l_2}
\]
are high-degree terms with
\[
|||h^v_j|||_{r, v, \Pi, v} \leq M_0(1 + \sigma_v) \tag{2.3}
\]
and
\[
f^v_j(x, y, \phi; a) = \sum_{l \in \mathbb{Z}^2} f^v_{j, l_1, l_2}(\phi; a)x^{l_1}y^{l_2}
\]
are the low-degree small perturbation terms with
\[
f^v_{j, 00} = O_{r, \Pi_v} \left( \varepsilon_0^\frac{v}{2} \right), \quad f^v_{j, 10} = O_{r, \Pi_v} \left( \varepsilon_0^\frac{v}{2} \right), \quad f^v_{j, 01} = O_{r, \Pi_v} \left( \varepsilon_0^\frac{v}{2} \right),
\]
\[
f^v_{j, 20} = O_{r, \Pi_v} \left( \varepsilon_0^\frac{v}{2} \right), \quad f^v_{j, 30} = O_{r, \Pi_v} (\varepsilon_v), \quad f^v_{j, 11} = O_{r, \Pi_v} (\varepsilon_v). \tag{2.4}
\]
Then there exist a closed subset \( \Pi_{v+1} \subset \Pi_v \) with
\[
\text{Meas}\Pi_{v+1} \geq \text{Meas}\Pi_v (1 - c_3 \gamma_v). \tag{2.5}
\]
and a quasi-periodic coordinate transformation $F^v : D(r_{v+1}, s_{v+1}) \times \Pi_{v+1} \to \varpi(s_v)$ in the form

$$F^v : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 + \sum_{j=0}^{v} a_j(u,v) \epsilon_j & \sum_{j=0}^{v} b_j(u,v) \epsilon_j \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi(u,v) \\ x_{10}(u) \end{pmatrix},$$

where $\phi \in D(r_{v+1})$, $u \in \Pi_{v+1}$, $(x_1, y_1)^T$ is the new coordinate, $x_{10}(u) \in \mathbb{R}$ with

$$|||x_{10}(u)|||_{\Pi_+} \ll \epsilon_0^{-\frac{3}{2}} \epsilon^\frac{1}{2},$$

and $W_{j}$ is of the same form as $f_j(j = 1, 2)$, taking the following estimates

$$W_{j,00}^v = O_{r_{v+1}, \Pi_{v+1}}(\gamma_0^{-5} \rho_v^{-6n_0} - \epsilon_0^{2-j} \epsilon^\frac{1}{2}),$$

$$W_{j,10}^v = \gamma_0^{-10} \rho_v^{-12n_0} - \epsilon_0^{2-j} \epsilon^\frac{1}{2},$$

$$W_{j,20}^v = \gamma_0^{-18} \rho_v^{-21n_0} - \epsilon_0^{2-j} \epsilon^\frac{1}{2},$$

$$W_{j,30}^v = \gamma_0^{-23} \rho_v^{-27n_0} - \epsilon_0^{2-j} \epsilon^\frac{1}{2},$$

$$W_{j,01}^v = \gamma_0^{-10} \rho_v^{-12n_0} - \epsilon_0^{2-j} \epsilon^\frac{1}{2},$$

$$W_{j,11}^v = \gamma_0^{-18} \rho_v^{-21n_0} - \epsilon_0^{2-j} \epsilon^\frac{1}{2},$$

such that the system (2.2) is transformed into the same form as (2.2) satisfying Conditions (a.1)-(a.2) by replacing $v$ with $v+1$ and $(x, y)$ with $(x_1, y_1)$, respectively.

**Proof.** In the proof of the lemma we drop the index $v$ and write $'+$ for $'+1'$ to simplify notation. Thus, $f_j = f_j$, $f_j^{v+1} = f_j^+$, and so on. It will be divided into three steps to prove the lemma.

**Step 1. Homology Equations**

Firstly, we set $z = (x, y)^T$, $F(\phi, z; a) = (f_1(\phi, x, y; a), f_2(\phi, x, y; a))^T$, $J_{z_0} = diag(\epsilon_0, 1)$, $N(z; a) = (\Omega(a)y, N_1(z; a))^T$, $H(\phi, z; a) = (\epsilon_0^5 h_1(\phi, x, y; a), \epsilon_0^5 h_2(\phi, x, y; a))^T$. Then the system (2.2) can be written as

$$J_{z_0}^{-\frac{1}{2}} = N(z; a) + H(\phi, z; a) + F(\phi, z; a).$$

Next we define a quasi-periodic transformation

$$F_1 : z = u + U(\phi, u; a),$$

where $u = (x_1, y_1)^T$ and $U = (U_1, U_2)^T$ with

$$U_j(\phi, u; a) = \sum_{l \in \Sigma^l} U_{j,l_1l_2}(\phi; a)x_1^{l_1}y_1^{l_2}.$$  

Substituting the transformation $F_1$ into system (2.2), we obtain

$$J_{z_0}^{-1}(E_2 + DU)(\dot{u} - J_{z_0}N(u; a))$$

$$= -J_{z_0}^{-1}W_{\partial z}U + DN(u)U - J_{z_0}^{-1}DUJ_{z_0}N(u; a) + \Gamma\mathcal{K} F(\phi, u; a) + R_1$$

$$+ \int_0^1 DF(\phi, u + \xi U)Ud\xi + H \circ F_1 + (Id - \Gamma\mathcal{K}) F(\phi, u; a),$$

where $DU$, $DN$ and $DF$ are the Jacobian matrices of $U$, $N$ and $F$ with respect to $u$ or $z$, $\Gamma\mathcal{K}$ is the truncation operator, $\partial z U$ denotes the partial derivative of $U$ with respect to $\phi$, and $R_1 = (0, R_{12})^T$ with

$$R_{12} = c_3(a)(3x_1U_1^2 + U_1^3) + c_2(a)U_1^2 + c_5(a)U_1U_2.$$
Note that the second line in (2.14) consists of high-degree terms and smaller low-degree terms. Therefore, the transformation \( F_1 \) for which we want to look satisfies
\[ -J_{x_0}^{-1} \partial_y U + DNU - J_{x_0}^{-1} DU J_{x_0} N(u; a) + \Gamma K F(\phi, u; a) = \tilde{N}(u; a) + R_2, \tag{2.16} \]
where
\[ \tilde{N}(u; a) = \left( 0, \sum_{i=0}^{j} N_{i,j} t_2(a) x_1^i y_1^j \right) \tag{2.17} \]
is the drift from \( \Gamma K F \) which will be determined later and \( R_2 = (R_{21}, R_{22})^T \) consists of high-degree terms with
\[ R_{21} = -\Omega(a)(3U_{1,30} x_1^2 + U_{1,11} y_1) y_1 - \varepsilon_0^{-1} \varepsilon_3(a) U_{1,11} x_1 - \varepsilon_0^{-1} \varepsilon_5(a) U_{1,11} x_1^2 y_1 \]
and
\[ R_{22} = (3 \varepsilon_3(a) x_1^2 + \varepsilon_5(a) y_1) \sum_{2 \leq t_1 + 2t_2 \leq 3} U_{1,11} t_2(x_1^i y_1^j) + \varepsilon_5(a) U_{2,30} x_1^4 + 2 \varepsilon_2(a) (U_{1,30} x_1^4 \]
\[ +U_{1,11} x_1^2 y_1) - \varepsilon_0 \Omega(a)(3U_{2,30} x_1^2 + U_{2,11} y_1) y_1 - \varepsilon_3(a) U_{2,11} x_1^2. \tag{2.18} \]
By comparing the coefficients in front of \( x^i (l_1 = 0, 1, 2, 3) \), \( y \), \( xy \) in (2.16), ones get that
\[
\begin{align*}
(2.19)
\end{align*}
\]
\[
\begin{align*}
(2.20)
\end{align*}
\]
To solve these homological equations, we expand all functions in (2.19) and (2.20) with respect to \( \phi \) into Fourier series and obtain the following algebraic equations.
\[ i(k, \omega) \tilde{U}_{1,00} (k; a) - \varepsilon_0 \Omega(a) \tilde{U}_{2,00} (k; a) = \varepsilon_0 \tilde{f}_{1,00} (k; a). \tag{2.21} \]
\[ i(k, \omega) \tilde{U}_{1,10} (k; a) + \varepsilon_1(a) \tilde{U}_{1,01} (k; a) - \varepsilon_0 \Omega(a) \tilde{U}_{2,10} (k; a) = \varepsilon_0 \tilde{f}_{1,10} (k; a), \tag{2.22} \]
\[ i(k, \omega) \tilde{U}_{1,20} (k; a) + \varepsilon_2(a) \tilde{U}_{1,01} (k; a) + \varepsilon_1(a) \tilde{U}_{1,11} (k; a) - \varepsilon_0 \Omega(a) \tilde{U}_{2,20} (k; a) = \varepsilon_0 \tilde{f}_{1,20} (k; a), \tag{2.23} \]
(2.24)
\[ i(k, \omega)\dot{U}_{1,30}(k; a) + e_3(a)\dot{U}_{1,01}(k; a) + e_2(a)\dot{U}_{1,11}(k; a) \\
- \varepsilon_0 \Omega(a)\dot{U}_{2,30}(k; a) = \varepsilon_0 \hat{f}_{1,30}(k; a), \]
(2.27)
\[ i(k, \omega)\dot{U}_{2,00}(k; a) - e_1(a)\dot{U}_{1,00}(k; a) - e_4(a)\dot{U}_{2,00}(k; a) \\
= \hat{f}_{2,00}(k; a) - N_{2,00}(a)\delta_k^0, \]
(2.29)
\[ i(k, \omega)\dot{U}_{2,10}(k; a) + e_1(a)\dot{U}_{2,01}(k; a) - 2e_2(a)\dot{U}_{1,00}(k; a) - e_1(a)\dot{U}_{1,10}(k; a) \\
- e_4(a)\dot{U}_{2,10}(k; a) - e_5(a)\dot{U}_{2,00} = \hat{f}_{2,10}(k; a) - N_{2,10}(a)\delta_k^0, \]
where \( k \in \mathbb{Z}^n \) with \(|k| \leq K \) and \( \delta_k^0 \) is the Dirac function. At first, we consider the case \( k = 0 \), let
\[
\begin{align*}
\dot{U}_{1,l,12}(0; a) &= 0, \quad \dot{U}_{2,l,12}(0; a) = -\Omega^{-1}(a)\hat{f}_{1,l,12}(0; a), \quad l \in \sum_0^0 D, \\
N_{2,00}(a) &= \hat{f}_{2,00}(0; a) - \Omega^{-1}(a)e_4(a)\hat{f}_{1,00}(0; a), \\
N_{2,l,0}(a) &= \hat{f}_{2,l,0}(0; a) - \Omega^{-1}(a)(e_5(a)\hat{f}_{1,l,0}(0; a) \\
&\quad + e_4(a)\hat{f}_{1,l,0}(0; a) - e_1(a)\dot{f}_{1,01}(0; a) - e_{1,1}(a)\dot{f}_{1,11}(0; a)), l = 1, 2, 3, \\
N_{2,l,1}(a) &= \hat{f}_{2,l,1}(0; a) + (l_1 + 1)\varepsilon_0\hat{f}_{1,l,1}(0; a), l_1 = 0, 1, \end{align*}
\]
with \( \varepsilon_0(a) \equiv 0 \), then the equations (2.21)-(2.32) hold true for \( k = 0 \). From (2.33) and (2.4), it is obvious that
\[
\begin{align*}
|||N_{2,00}(a)||_1 &\ll \varepsilon^2, |||N_{2,10}(a)||_1 \ll \varepsilon^2, |||N_{2,01}(a)||_1 \ll \varepsilon^2, \\
|||N_{2,20}(a)||_1 &\ll \varepsilon, |||N_{2,30}(a)||_1 \ll \varepsilon, \quad \text{and} \quad |||N_{2,11}(a)||_1 \ll \varepsilon. \end{align*}
\]
If \( k \neq 0 \), firstly, we give a rough outline of how to solve the equations. By some observations, one can find that the equations (2.21) and (2.27) are independent of others. Thus, for \( j = 1, 2 \), it is easy to get the corresponding solutions \( \dot{U}_{j,00}(k; a) \) which will be regarded as known functions later. We also find that the equations (2.22), (2.25), (2.28) and (2.31) are independent of the rest and obtain the solutions...
We take the subset $\Pi$ which are denoted by $U_{00}$ and $X$ throughout. So the small divisors are $\hat{\Pi}$ in form. For $I_{d_1 \pm}$ to be the roots of the quadratic equation $\hat{X} - e_0(a) = 0$. Thus, $d_1(X - X_1)$ and $d_1 - X - X_2$. Then we have to dig some parameters from $\Pi$ so that small divisors satisfy Melnikov conditions and that the Fourier series of functions $U_{00}(\phi; a)$ are convergent. Since the small divisors are differential with respect to $\alpha$, in view of the Frequency condition, one gets

$$|\partial_{\alpha} d_1 \pm| = |\partial_{\alpha} X_j| \sim |X_j| \sim \varepsilon_0^\gamma.$$  

We take the subset $\Pi^+_{\gamma}$ of $\Pi$ as follows

$$\Pi^+_{\gamma} = \left\{ a \in \Pi : |d_{1+}| \geq \frac{\varepsilon_0^\gamma a}{|k|^{n_0 + 1}}, |d_{1-}| \geq \frac{\varepsilon_0^\gamma a}{|k|^{n_0 + 1}}, 0 < |k| \leq K \right\},$$

where $k \in \mathbb{Z}^{n_0}$. Now, we give the estimates of $U_{00}(\phi; a)$ for $j = 1, 2$. By (2.36), it follows that

$$U_{00}(\phi; a) = \sum_{k \neq 0, |k| \leq K} \varepsilon_0 \left( \frac{X - e_4(a)}{(X - X_1)(X - X_2)} \hat{f}_{1,00}(k; a) + \Omega(a) \hat{f}_{2,00}(k; a) e^{ik\phi} \right)$$

$$= \sum_{k \neq 0, |k| \leq K} \left( \varepsilon_0 \hat{f}_{1,00}(k; a) + \varepsilon_0 \frac{X_1 - e_4(a)}{(X - X_1)(X - X_2)} \hat{f}_{1,00}(k; a) \right) e^{ik\phi} \hat{f}_{2,00}(k; a) e^{ik\phi} = I_1 + I_2 + I_3$$

in form. For $I_1$, we have

$$\sup_{\mathbb{D}(r/4) \times \Pi^+_{\gamma}} |I_1| = \sup_{\mathbb{D}(r/4) \times \Pi^+_{\gamma}} \left| \sum_{k \neq 0, |k| \leq K} \varepsilon_0 \frac{X - e_4(a)}{(X - X_2)} \hat{f}_{1,00}(k; a) e^{ik\phi} \right|$$

$$\leq \varepsilon_0 \sup_{\mathbb{D}(r/4) \times \Pi^+_{\gamma}} \left| \sum_{k \neq 0, |k| \leq K} \left( \frac{1}{X} + \frac{X_2}{X(X - X_2)} \right) \hat{f}_{1,00}(k; a) e^{ik\phi} \right|$$

$$\leq \varepsilon_0 \sum_{k \neq 0, |k| \leq K} \left( \left| k \right|^{n_0 + 1} e^{-\frac{|k|}{\gamma}} + \left| k \right|^{2n_0 + 2} e^{-\frac{|k|}{\gamma^2}} \right) ||f_{1,00}||_{r, \Pi}$$

$$\leq \varepsilon_0 \gamma^{-2} \rho^{-3n_0 - 2} ||f_{1,00}||_{r, \Pi},$$

where $\rho$ is a constant.
where in the third line we have used the facts 
\[ |f_{1,00}(k; a)| \leq C ||f_{1,00}||_{r, \Pi} e^{-\rho |k|} \]
and 
\[ |X_2| \sim \varepsilon_0^5. \]

Since
\[
\frac{1}{(X - X_1)(X - X_2)} = \frac{1}{X(X - X_1)} \frac{X_1}{(X - X_1)(X - X_2)},
\]
one can check that
\[
sup_{\mathbb{D}(r - \rho/4) \times \Pi^+_{\varepsilon}} |I_2| \lesssim \varepsilon_0 \gamma^{-3} \rho^{-4n_0 - 3} ||f_{1,00}||_{r, \Pi}.
\]

By the same method, we conclude that
\[
sup_{\mathbb{D}(r - \rho/4) \times \Pi^+_{\varepsilon}} |I_3| \lesssim \varepsilon_0 \gamma^{-4} \rho^{-5n_0 - 4} ||f_{2,00}||_{r, \Pi}.
\]

Therefore, it deduces that
\[
sup_{\mathbb{D}(r - \rho/4) \times \Pi^+_{\varepsilon}} |U_{1,00}| \lesssim \varepsilon_0 \gamma^{-4} \rho^{-5n_0 - 4} (||f_{1,00}||_{r, \Pi} + ||f_{2,00}||_{r, \Pi}).
\] (2.39)

Then we estimate \( \partial_a U_{1,00} \). Firstly, the expression of \( \partial_a I_1 \) is displayed below.

\[
\partial_a I_1 = \varepsilon_0 \sum_{k \neq 0, |k| \leq K} \left( \frac{\partial_a f_{1,00}(k; a)}{X - X_2} + \frac{(X_2)\partial_a f_{1,00}(k; a)}{(X - X_2)^2} \right) e^{ik \cdot \phi}
\]

\[
= \varepsilon_0 \sum_{k \neq 0, |k| \leq K} \left( \frac{\partial_a f_{1,00}}{X} + \frac{X_2 \partial_a f_{1,00}}{X(X - X_2)} + \frac{(X_2)\partial_a f_{1,00}}{X(X - X_2)^2} + \frac{X_2 (X_2)\partial_a f_{1,00}}{(X - X_2)^2} \right) e^{ik \cdot \phi}.
\]

Due to (2.38), it follows that
\[
sup_{\mathbb{D}(r - \rho/4) \times \Pi^+_{\varepsilon}} |\partial_a I_{1,00}| \lesssim \varepsilon_0 \gamma^{-3} \rho^{-4n_0 - 3} ||f_{1,00}||_{r, \Pi}. \tag{2.40}
\]

As to the terms \( \partial_a I_2 \) and \( \partial_a I_3 \), we adopt the same method and show that
\[
sup_{\mathbb{D}(r - \rho/4) \times \Pi^+_{\varepsilon}} |\partial_a I_{2,00}| \lesssim \varepsilon_0 \gamma^{-4} \rho^{-5n_0 - 4} ||f_{1,00}||_{r, \Pi} \tag{2.41}
\]

and
\[
sup_{\mathbb{D}(r - \rho/4) \times \Pi^+_{\varepsilon}} |\partial_a I_{3,00}| \lesssim \varepsilon_0 \gamma^{-5} \rho^{-6n_0 - 5} ||f_{2,00}||_{r, \Pi}. \tag{2.42}
\]

Based on (2.39)-(2.42), it is clear that
\[
||U_{1,00}||_{r - \rho, 4, \Pi^+_{\varepsilon}} \lesssim \varepsilon_0 \gamma^{-5} \rho^{-6n_0 - 5} (||f_{1,00}||_{r, \Pi} + ||f_{2,00}||_{r, \Pi}). \tag{2.43}
\]

Similarly, recall that \( \tilde{U}_{2,00}(0; a) = -\Omega^{-1}(a) f_{1,00}(0; a) \) (see (2.33)) and (2.37),
\[
||U_{2,00}||_{r - \rho, 4, \Pi^+_{\varepsilon}} \lesssim \gamma^{-5} \rho^{-6n_0 - 5} (||f_{1,00}||_{r, \Pi} + ||f_{2,00}||_{r, \Pi}). \tag{2.44}
\]

Hence, for \( j = 1, 2 \),
\[
||U_{j,00}||_{r - \rho, 4, \Pi^+_{\varepsilon}} \lesssim \gamma^{-5} \rho^{-6n_0 - 5} \varepsilon_0^2 \varepsilon_0^5. \tag{2.45}
\]

Next, we consider the linear equations (2.22), (2.25), (2.28) and (2.31) and find the following small divisor
\[
X^2 (X^2 - \xi_1^2(a) - 4\varepsilon_0 \Omega(a) e_1(a)),
\]
that is,
\[ X \pm i\sqrt{-(e_4(a))^2 + 4\varepsilon_0 e_1(a)\Omega(a)}} \]
which are denoted by \( d_{2\pm} \) respectively. Again, by Frequency condition, we show that \( |\partial_a d_{2\pm}| \sim \varepsilon_0^g \). Then we define the the subset \( \Pi^2_+ \) of \( \Pi^1_+ \) as follows
\[
\Pi^2_+ = \left\{ a \in \Pi^1_+ : |d_{2+}| \geq \frac{\varepsilon_0^g \gamma}{|k|^{n_0+1}}, |d_{2-}| \geq \frac{\varepsilon_0^g \gamma}{|k|^{n_0+1}}, 0 < |k| \leq K \right\}.
\]
Analogously, we can obtain the expressions and estimates of \( U_{j,10}(\phi; a) \) and \( U_{j,01}(\phi; a) \), for \( j = 1, 2 \). Since the computation is too tedious to be lost in, we omit it and only give the final results as follows. For \( j = 1, 2 \),
\[
|\|U_{j,10}\|_{r-\rho/2, \Pi^2_+} \| \lesssim \gamma^{-10} \rho^{-12n_0-10} \varepsilon_0^{-2-j} \varepsilon \|\|U_{j,01}\|_{r-\rho/2, \Pi^2_+} \| \lesssim \gamma^{-10} \rho^{-12n_0-10} \varepsilon_0^{-2-j} \varepsilon .
\]
We continue studying the linear equations \((2.23), (2.26), (2.29)\) and \((2.32)\) and find
\[
d_{3\pm \pm} = : i(k, u) \pm i\sqrt{2}[(e_4(a))^2 + 6\varepsilon_0 e_1(a)\Omega(a)] + \sqrt{\Delta}/2
\]
with \( \Delta = [(e_4(a))^2 + 6\varepsilon_0 e_1(a)\Omega(a)]^2 - 4\varepsilon_0^2(e_1(a)\Omega(a))^2 \) as the small divisors. Similarly, we can prove that \( |\partial_a d_{3\pm \pm}| \sim \varepsilon_0^g \) and take the subset \( \Pi^3_+ \) of \( \Pi^2_+ \) as follows.
\[
\Pi^3_+ = \left\{ a \in \Pi^2_+ : |d_{3++}| \geq \frac{\varepsilon_0^g \gamma}{|k|^{n_0+1}}, |d_{3--}| \geq \frac{\varepsilon_0^g \gamma}{|k|^{n_0+1}}, 0 < |k| \leq K \right\},
\]
where \( k \in \mathbb{Z}^{n_0} \) and \( 0 < |k| \leq K \). Moreover, we list the estimates of \( U_{j,20}, U_{j,11} \) with \( j = 1, 2 \) below.
\[
|\|U_{j,20}\|_{r-3\rho/4, \Pi^3_+} \| \lesssim \gamma^{-18} \rho^{-21n_0-18} \varepsilon_0^{-\frac{11}{2}+\epsilon},
\]
\[
|\|U_{j,11}\|_{r-3\rho/4, \Pi^3_+} \| \lesssim \gamma^{-18} \rho^{-21n_0-18} \varepsilon_0^{-2-j} \varepsilon .
\]
Finally, we investigate the last two equations and discover the same divisors as that in equations \((2.21)\) and \((2.27)\), namely \( d_{1\pm} \). Thus, there is no parameter to be removed. Let \( \Pi_+ = \Pi^3_+ \), one can verify that
\[
\text{Meas}\Pi_+ \geq \text{Meas}\Pi - 8 \sum_{0 < |k| \leq K} \frac{\gamma}{|k|^{n_0+1}},
\]
which implies that measure estimate \((2.5)\) holds true. Furthermore, we have
\[
|\|U_{j,30}\|_{r, \Pi_+} \| \lesssim \gamma^{-23} \rho^{-27n_0-23} \varepsilon_0^{-2-j} \varepsilon .
\]
Obviously, \( F_1: \varpi(s-2\delta) \rightarrow \varpi(s-\delta) \) if \( \varepsilon_0 \) is sufficiently small. Moreover, \( U_j \in \mathcal{A}_{r_+, s-2\delta}^{\Pi_+} \).
Step 2. Translation
Let
\[
\tilde{N}(u) = N(u) + \tilde{N}(u).
\]
We hope that \( \tilde{N} \) and \( N \) have the same form, so we only need to translate the first variable \( x_1 \). Consider the equilibrium point equation
\[
(e_3+N_{2,30}(0; a)x_1^2 + (e_2+N_{2,20}(0; a))x_1^2 + (e_1+N_{2,10}(0; a))x_1 + N_{2,00}(0; a))x_1 = 0.
\]
With the help of Lemma 4 in [10] and (2.34), (2.4), the equation (2.50) has a real root $x_{10}(a)$ satisfying

$$||x_{10}(a)||_{\Pi_+} \leq \epsilon_0^{-\frac{3}{2}} e^k. \quad (2.51)$$

Hence, (2.6) holds true. Take the translation

$$F_2: x_1 = x_{10} + x_{11}, \ y_1 = y_{11} \quad (2.52)$$

and

$$N^+(z_+; a) = \tilde{N} \circ F_2(z_+) := \left( \begin{array}{c} \Omega(a)y_{11} \\ N_1^+(z_+; a) \end{array} \right). \quad (2.53)$$

Then, for sufficiently small $\epsilon_0$ it is clear that $F_2: \varpi(s-3\delta) \to \varpi(s-\delta)$ and

$$N_1^+(z_+; a) = e_1^+(a)x_{11}^3 + e_2^+(a)x_{11}^2 + e_3^+(a)x_{11} + e_4^+(a)y_{11} + e_5^+(a)x_{11}y_{11} + 1$$

with

$$e_1^+(a) - e_1(a) = 3(e_3(a) + 2e_0(a))x_{10}^2 + 2(e_2(a) + 2e_0(a))x_{10} + N(a);$$

$$e_2^+(a) - e_2(a) = 3(e_3(a) + 2e_0(a))x_{11} + N(a);$$

$$e_3^+(a) - e_3(a) = N(a);$$

$$e_4^+(a) - e_4(a) = N(a);$$

$$e_5^+(a) - e_5(a) = N(a).$$

According to (2.51) and (2.34), the frequency condition still hold true when all superscripts $v$ are replaced by $v+1$.

Take $F = F_1 \circ F_2$. Then $F: \varpi(s-3\delta) \to \varpi(s-\delta)$ and the transformation $F$ is defined by

$$x = x_{11} + x_{10} + U_1(\phi, x_{11} + x_{10}, y_{11}; a) \triangleq x_{11} + x_{10} + W_1^+(\phi, x_{11} + y_{11}; a),$$

$$y = y_{11} + U_2(\phi, x_{11} + x_{10}, y_{11}; a) \triangleq y_{11} + W_2^+(\phi, x_{11} + y_{11}; a).$$

In view of $W_j^+ = U_j \circ F_2$ and the estimates (2.45)-(2.49), (2.51), we have that $W_j^+ \in \mathcal{Q}_{r, s-3\delta} \cap \mathcal{Q}_{r, s-3\delta}$ satisfying (2.7)-(2.12), for $j = 1, 2$, moreover,

$$||F - Id||_{r, s-3\delta, \Pi_+} \leq \gamma^{-23} \rho^{-27n_0} - 23 \epsilon. \quad (2.53)$$

and (by Cauchy estimate)

$$||DF||_{r, s-3\delta, \Pi_+} \leq 1 + \delta^{-1} \gamma^{-23} \rho^{-27n_0} - 23 \epsilon \leq 1 + \frac{3}{2} \epsilon. \quad (2.54)$$

Step 3. Estimates of remainder terms

Inserting the translation $F_2$ into the equation (2.14), we have

$$\dot{z}_+ = J_{e_0}N_+^+(z_+) + (E_2 + DU \circ F_2)^{-1}J_{e_0}(P^0 + P^1 + P^2 + P^3 + P^4 + P^5), \quad (2.55)$$

where

$$P^0 = -(J_{e_0}^{-1}DU \cdot J_{e_0}N) \circ F_2, \quad P^1 = R_1 \circ F_2, \quad (2.56)$$

$$P^2 = R_2 \circ F_2, \quad P^3 = \int_0^1 (DF(\phi, u + \xi U) \cdot U) \circ F_2 d\xi, \quad (2.57)$$

$$P^4 = H \circ F, \quad P^5 = (Id - \Gamma_K)F \circ F_2.$$

Now, our aim is to rewrite the system (2.55) in the form of (2.2) and prove the condition (a.2) holds for (2.55) replacing $v$ by $v+1$. Let $P_L^l$ and $P_H^l$ ($l = 0, 1, \cdots, 5$) be the low-degree terms and high-degree terms of $P^l$, respectively. We assume that $P_L^l = (P_L^{1l}, P_L^{2l})^T$ and $P_H^{l} = P_H^{l, 0} + P_H^{l, 10} + P_H^{l, 20} + P_H^{l, 30} + P_H^{l, 01} + P_H^{l, 11}(j = 1, 2)$ are the coefficients in front of $x^0, x, x^2, x^3, y, xy$ in the expression of $P^l$. Since $P^l$ ($l = 0, 1, 2, 3$) are polynomials in $z_+$, the term $P_H^l$ satisfy

$$||P_H^l||_{r, s, \Pi_+} \leq \gamma^{-23} \rho^{-27n_0} - 23 \epsilon. \quad (2.57)$$
To estimate the coefficients $P^l_{j,l_1l_2}$ ($l = 0, 1, 2, 3$), the authors in [10] use the Cauchy inequality. However, this method is not accurate enough and fails in our model. In this paper, we will write the precise expression, at least the main part, of the coefficients one by one. We start with the term $P^0_{jL}$. According to (2.13), (2.17), (2.52) and (2.56), by some computations, we show that

$$P^0_{j,00} = -e_0^{-j-2}(U_{j,01} + U_{j,11}x_{10}) \sum_{l_1=0}^{3} N_{2,l_10}(0)x_{10}^{l_1},$$

$$P^0_{j,10} = -e_0^{-j-2}[(U_{j,01} + U_{j,11}x_{10}) \sum_{l_1=1}^{3} l_1 N_{2,l_10}(0; a)x_{10}^{l_1-1} - U_{j,11} \sum_{l_1=0}^{3} N_{2,l_10}(0; a)x_{10}^{l_1}],$$

$$P^0_{j,20} = -e_0^{-j-2}(U_{j,01} + U_{j,11}x_{10})(N_{2,20}(0; a) + 3N_{2,30}x_{10})$$

$$-e_0^{-j-2}U_{j,11} \sum_{l_1=1}^{3} l_1 N_{2,l_10}(0; a)x_{10}^{l_1-1},$$

$$P^0_{j,30} = -e_0^{-j-2}(U_{j,01} + U_{j,11}x_{10})N_{2,30}(0; a) - e_0^{-j-2}U_{j,11}(N_{2,20}(0; a) + 3N_{2,30}(0; a)x_{10}),$$

$$P^0_{j,01} = -e_0^{-j-2}(U_{j,01} + U_{j,11}x_{10})(N_{2,01}(0; a) + N_{2,11}(0; a)x_{10}),$$

$$P^0_{j,11} = -e_0^{-j-2}(U_{j,01} + U_{j,11}x_{10})N_{2,11}(0; a) - e_0^{-j-2}U_{j,11}(N_{2,01}(0; a) + N_{2,11}(0; a)x_{10}).$$

Then, if $e_0$ is small enough, the estimates (2.34), (2.45)-(2.49) and (2.51) lead to

$$\| \| P^0_{j,00} \| \|_{r_+, \Pi^+} < e_0^{-j-2} \| \| U_{j,01} \| \|_{r_+, \Pi^+} \| \| N_{2,00} \| \|_{r_+, \Pi^+} < \gamma^{-10} \rho^{-(12n_0+10)} \epsilon_+ < e_+^5,$$

$$\| \| P^0_{j,10} \| \|_{r_+, \Pi^+} < e_0^{-j-2} \| \| U_{j,11} \| \|_{r_+, \Pi^+} \| \| N_{2,00} \| \|_{r_+, \Pi^+} < \gamma^{-18} \rho^{-(21n_0+18)} \epsilon_+ < e_+^5,$$

$$\| \| P^0_{j,20} \| \|_{r_+, \Pi^+} < e_0^{-j-2} \| \| U_{j,11} \| \|_{r_+, \Pi^+} \| \| N_{2,10} \| \|_{r_+, \Pi^+} < \gamma^{-18} \rho^{-(21n_0+18)} \epsilon_+ < e_+^5,$$

$$\| \| P^0_{j,30} \| \|_{r_+, \Pi^+} < e_0^{-j-2} \| \| U_{j,11} \| \|_{r_+, \Pi^+} \| \| N_{2,20} \| \|_{r_+, \Pi^+} < \gamma^{-18} \rho^{-(21n_0+18)} \epsilon_+ < e_+^5,$$

$$\| \| P^0_{j,01} \| \|_{r_+, \Pi^+} < e_0^{-j-2} \| \| U_{j,01} \| \|_{r_+, \Pi^+} \| \| N_{2,01} \| \|_{r_+, \Pi^+} < \gamma^{-10} \rho^{-(12n_0+10)} \epsilon_+ < e_+^5,$$

$$\| \| P^0_{j,11} \| \|_{r_+, \Pi^+} < e_0^{-j-2} \| \| U_{j,11} \| \|_{r_+, \Pi^+} \| \| N_{2,11} \| \|_{r_+, \Pi^+} < \gamma^{-18} \rho^{-(21n_0+18)} \epsilon_+ < e_+^5, \quad (2.58)$$

where $\epsilon_+ = e_+^5$. Next, by the definition of $R_1$ (refer to (2.15)), the Frequency condition and estimates (2.45)-(2.49), we know that $e_2(a)U^2_{jL} \circ F_2$ is the largest part of term $P^1_{jL}$. Similarly, it follows that

$$\| \| P^1_{j,00} \| \|_{r_+, \Pi^+} < e_0^{-j-2} \| \| W_{1,00} \| \|_{r_+, \Pi^+} \| \| W_{1,10} \| \|_{r_+, \Pi^+} < \epsilon_+^5,$$

$$\| \| P^1_{j,10} \| \|_{r_+, \Pi^+} < e_0^{-j-2} \| \| W_{1,00} \| \|_{r_+, \Pi^+} \| \| W_{1,10} \| \|_{r_+, \Pi^+} < \epsilon_+, \quad (2.59)$$

Since the $R_2$ consists only of high-degree terms, $P^2_{jL}$ can be absorbed by $P^1_{jL}$. For $P^3_{jL}$, we rewrite it as $f_j(\phi, u + U) \circ F_2 - f_j(\phi, u) \circ F_2$, whose main part is

$$M P^3_{jL} \triangleq (f_{1,10} + 2f_{1,20}x_{11} + 3f_{1,30}x_{11}^2 + f_{1,11}y_{11})W_1 + (f_{1,01} + f_{1,11}x_{11})W_2.$$
Like the $P_3^0$, we calculate the coefficients of $MP_j^3$ and obtain the estimates in the following

$\begin{align*}
\|P_{3,00}\|_{r_+,\Pi_+} &\leq \|f_{1,01}\|_{r_+,\Pi_+} \|W_{2,00}\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{3,10}\|_{r_+,\Pi_+} &\leq \|f_{1,11}\|_{r_+,\Pi_+} \|W_{2,00}\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{3,20}\|_{r_+,\Pi_+} &\leq \|f_{1,11}\|_{r_+,\Pi_+} \|W_{2,10}\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{3,30}\|_{r_+,\Pi_+} &\leq \|f_{1,11}\|_{r_+,\Pi_+} \|W_{2,20}\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{3,01}\|_{r_+,\Pi_+} &\leq \|f_{1,11}\|_{r_+,\Pi_+} \|W_{1,00}\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{3,11}\|_{r_+,\Pi_+} &\leq \|f_{1,11}\|_{r_+,\Pi_+} \|W_{2,01}\|_{r_+,\Pi_+} < \epsilon_+. 
\end{align*}$

(2.60)

To estimate $P^4$, we divide $H$ into two parts $H = H' + H'' = \epsilon_0^2 (h_1' + h_2')^T$ with

$$h_j'(\phi, x, y, a) = \sum_{l_1+2l_2=4} h_{j,l_1l_2} x^{l_1} y^{l_2}, \quad h_j'' = h_j - h_j', \quad j = 1, 2.$$ 

By the Cauchy inequality and (2.3), we have

$$||h_{j,l_1l_2}||_{r,\Pi} \leq M_0 \frac{(1 + \sigma)}{s^{l_1+2l_2}} \quad \text{for } l_1 + 2l_2 = 4, \quad \text{and } ||h_j''||_{r,s,\Pi} < 4M_0(1 + \sigma).$$

(2.61)

Let

$$P^4(\phi, z_+) = H'(\phi, \mathcal{F}(z_+)), \quad P^4''(\phi, \mathcal{F}(z_+)).$$

Then $P^4 = P^4 + P^4''$. Since $P^4$ is a polynomial in $z_+$, the coefficients $P_{j,l_1l_2}$ can be calculated directly. And using the estimates (2.45)-(2.49) and (2.51), for $j = 1, 2$, we conclude that

$\begin{align*}
\|P_{j,00}'\|_{r_+,\Pi_+} &\leq \|f_{0,0}'\|_{r_+,\Pi_+} < \epsilon_0^7 \epsilon_+^{14} \epsilon_+^{20} < \epsilon_+^7, \\
\|P_{j,10}'\|_{r_+,\Pi_+} &\leq \|f_{0,1}'\|_{r_+,\Pi_+} < \epsilon_0^7 \epsilon_+^{14} \epsilon_+^{20} < \epsilon_+^7, \\
\|P_{j,20}'\|_{r_+,\Pi_+} &\leq \|f_{0,2}'\|_{r_+,\Pi_+} < \epsilon_0^7 \epsilon_+^{14} \epsilon_+^{20} < \epsilon_+^7, \\
\|P_{j,30}'\|_{r_+,\Pi_+} &\leq \|f_{0,3}'\|_{r_+,\Pi_+} < \epsilon_0^7 \epsilon_+^{14} \epsilon_+^{20} < \epsilon_+^7, \\
\|P_{j,01}'\|_{r_+,\Pi_+} &\leq \|f_{0,1}'\|_{r_+,\Pi_+} < \epsilon_0^7 \epsilon_+^{14} \epsilon_+^{20} < \epsilon_+^7, \\
\|P_{j,11}'\|_{r_+,\Pi_+} &\leq \|f_{0,0}'\|_{r_+,\Pi_+} < \epsilon_0^7 \epsilon_+^{14} \epsilon_+^{20} < \epsilon_+^7. 
\end{align*}$

(2.62)

Here, we use the fact that $s \geq s_0/2$ and $s_0$ is a positive constant. Due to (1.8), (2.61) and the estimates (2.7)-(2.12), if $\epsilon_0$ is small enough, one gets

$$\|P_{j,00}''\|_{r_+,\Pi_+} \leq \|h_j''\|_{r_+,\Pi_+} < \epsilon_+^{24} \epsilon_+^{20} \epsilon_+^{20} < \epsilon_+^7.$$

Hence, by the Cauchy inequality again, it is clear that

$$\|P_{j,l_1l_2}''\|_{r_+,\Pi_+} < \epsilon_+^{24} \epsilon_+^{20} \epsilon_+^{20} < \epsilon_+^7 \quad 0 \leq l_1 + 2l_2 < 3.$$ 

(2.63)

Based on the estimates (2.62) and (2.63), we have

$\begin{align*}
\|P_{j,00}'\|_{r_+,\Pi_+} &\leq \|f_{0,0}'\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{j,10}'\|_{r_+,\Pi_+} &\leq \|f_{0,1}'\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{j,20}'\|_{r_+,\Pi_+} &\leq \|f_{0,2}'\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{j,30}'\|_{r_+,\Pi_+} &\leq \|f_{0,3}'\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{j,01}'\|_{r_+,\Pi_+} &\leq \|f_{0,1}'\|_{r_+,\Pi_+} < \epsilon_+^7, \\
\|P_{j,11}'\|_{r_+,\Pi_+} &\leq \|f_{0,0}'\|_{r_+,\Pi_+} < \epsilon_+^7. 
\end{align*}$

(2.64)

Moreover,

$$\|P_{j,01}'\|_{r_+,\Pi_+} < \epsilon_+^7, \quad \|P_{j,11}'\|_{r_+,\Pi_+} < \epsilon_+^7, \quad \|P_{j,00}''\|_{r_+,\Pi_+} < \epsilon_+^7, \quad \|P_{j,10}''\|_{r_+,\Pi_+} < \epsilon_+^7, \quad \|P_{j,20}''\|_{r_+,\Pi_+} < \epsilon_+^7.$$ 

(2.65)
Therefore, it follows that
\[
||| P_H^5|||_{r_+, s_+, \Pi_+} \leq ||| P^4 - H|||_{r_+, s_+, \Pi_+} + ||| H|||_{r_+, s_+} + ||| P_h^2|||_{r_+, s_+, \Pi_+} \\
\leq ||| H \circ \mathcal{F} - H|||_{r_+, s_+, \Pi_+} + M_0(1 + \sigma) + \epsilon_+ \\
\leq \delta^{-1}||| H|||_{r_+, s_+} ||| \mathcal{F} - Id|||_{r_+, s_+} + M_0(1 + \sigma) + \epsilon_+ \\
\leq \delta^{-1} \gamma^{-23} \rho^{-2(27n_0 + 23)} \epsilon + M_0(1 + \sigma) + \epsilon_+ \\
\leq M_0(1 + \sigma)(1 + \epsilon_+^2) \tag{2.66}
\]
where we have used (2.53) in the fourth line.
Note that the $P^5$ consists only of low-degree terms, so $P_h^5 = 0$. Since $P^5$ is also a polynomial in $z_+$, one can calculate its coefficients as follow.

\[
P_{j,00}^5 = (Id - \Gamma_K) \sum_{l=0}^{3} f_{j,l,10}(\phi; a)x_{10}^{l}, \quad P_{j,10}^5 = (Id - \Gamma_K) \sum_{l=1}^{3} l_1 f_{j,l,10}(\phi; a)x_{10}^{l-1}, \\
P_{j,20}^5 = (Id - \Gamma_K)(f_{j,20}(\phi; a) + 3f_{j,30}(\phi; a)x_{10}), \quad P_{j,30}^5 = (Id - \Gamma_K)f_{j,30}(\phi; a), \\
P_{j,01}^5 = (Id - \Gamma_K)(f_{j,01}(\phi; a) + f_{j,11}(\phi; a)x_{10}), \quad P_{j,11}^5 = (Id - \Gamma_K)f_{j,11}(\phi; a),
\]
Recalling the definition of the truncation operator $\Gamma_K$ and $K_\nu$ (using (2.1)) and (2.51), we get

\[
||| P_{j,00}^5|||_{r_+, \Pi_+} \leq \rho^{-n_0} \epsilon^5 \epsilon^5 < \epsilon_+^5, \quad ||| P_{j,10}^5|||_{r_+, \Pi_+} \leq \rho^{-n_0} \epsilon^5 \epsilon^5 < \epsilon_+^5, \\
||| P_{j,20}^5|||_{r_+, \Pi_+} \leq \rho^{-n_0} \epsilon^5 \epsilon^5 < \epsilon_+^5, \quad ||| P_{j,30}^5|||_{r_+, \Pi_+} \leq \rho^{-n_0} \epsilon^5 \epsilon^5 < \epsilon_+^5, \\
||| P_{j,01}^5|||_{r_+, \Pi_+} \leq \rho^{-n_0} \epsilon^5 \epsilon^5 < \epsilon_+^5, \quad ||| P_{j,11}^5|||_{r_+, \Pi_+} \leq \rho^{-n_0} \epsilon^5 \epsilon^5 < \epsilon_+^5. \tag{2.67}
\]

Let $P = P_0 + \cdots + P^5 = (P_1, P_2)^T$. Inequalities (2.58)-(2.60), (2.64) and (2.67) imply

\[
||| P_{j,00}^4|||_{r_+, \Pi_+} < \epsilon_+^5, \quad ||| P_{j,10}^4|||_{r_+, \Pi_+} < \epsilon_+^5, \quad ||| P_{j,20}^4|||_{r_+, \Pi_+} < \epsilon_+^5, \quad ||| P_{j,30}^4|||_{r_+, \Pi_+} < \epsilon_+^5, \quad ||| P_{j,01}^4|||_{r_+, \Pi_+} < \epsilon_+^5, \quad ||| P_{j,11}^4|||_{r_+, \Pi_+} < \epsilon_+^5, \quad ||| P_{j,11}^4|||_{r_+, \Pi_+} < \epsilon_+^5. \tag{2.68}
\]
while the estimates (2.57) and (2.66) lead to

\[
||| P_H|||_{r_+, s_+, \Pi_+} \leq M_0(1 + \sigma)(1 + \epsilon_+^2) \leq M_0(1 + \sigma_+) \tag{2.69}
\]
for sufficiently small $\epsilon_0$.

Set
\[
\mathcal{P} = J_\epsilon^{-1}(E_2 + DU \circ \mathcal{F}_2)^{-1}J_\epsilon \left[ P^0 + P^1 + P^2 + P^3 + P^4 + P^5 \right].
\]
Then
\[
F^+ = \mathcal{P}_l, \quad H^+ = \mathcal{P}_h,
\]
where $\mathcal{P}_l$ and $\mathcal{P}_h$ are the low-degree terms and high-degree terms of $\mathcal{P}$, respectively.

In view of the fact that $J_\epsilon^{-1}(E_2 + DU \circ \mathcal{F}_2)^{-1}J_\epsilon = E_2 + O(\epsilon_+^2)$ and the estimates (2.68), (2.69), one can check that $F^+$ and $H^+$ satisfy the condition (a.2) with $v+1$.

The proof of the lemma 2.1 is complete. $\square$
3. Proof of Theorem 1.1. At the beginning of the proof, we scale the system (1.7) by
\[ x \to \varepsilon_0 x, \ y \to \varepsilon_0^2 y, \ \varepsilon = \varepsilon_0^4, \] (3.1)
and obtain the new system
\[
\begin{aligned}
\dot{x} &= \varepsilon_0 [\Omega(a) y + \varepsilon_0^2 h_1^0(x, y, \omega t; a) + \varepsilon_0^2 f_1^0(x, y, \omega t; a)], \\
\dot{y} &= -\varepsilon_0 x^3 + \varepsilon_0^2 h_2^0(x, y, \omega t; a) + \varepsilon_0^2 f_2^0(x, y, \omega t; a), 
\end{aligned}
\] (3.2)
where \( h_j^0 = \varepsilon_0^{-4} h_j(\varepsilon x, \varepsilon^2 y, \omega t; a) \) and \( f_j^0 = f_j(\varepsilon x, \varepsilon^2 y, \omega t; a) \), \( j = 1, 2 \). According to the definition of terms \( h_j \), it implies that \( ||h_j^0||_{r_0, s_0, l_0} \leq M_0 \).

To prove the Theorem 1.1 by Lemma 2.1, ones have to turn the system (3.2) into the form of the system (2.2). For the purpose, we firstly introduce the change of coordinates
\[ \mathcal{F}_0: z = u + U_0^0(u, \phi; a), \]
where \( z = (x, y)^T, \ u = (x_1, y_1)^T \) and \( U_0^0 = (U_1^0, U_2^0)^T \) with \( U_j^0(\phi, u; a) = \sum_{l_i \in \Sigma_{D_j}} U_{j, l_1, l_2}^0 (\phi; a) x_1^{l_1} y_1^{l_2} \). Substituting the change into the system (3.2), we obtain
\[
\begin{aligned}
J_{\varepsilon_0}^{-1}(E_0 + DU_0^0)(u - J_{\varepsilon_0}N_0^0(u)) \\
= - J_{\varepsilon_0}^{-1} \omega_\partial aU_0^0 - DN_0^0(u)U_0^0 - J_{\varepsilon_0}^{-1} DU_0^0 J_{\varepsilon_0}N_0^0(u) + \Gamma_{K_0} F_0^0(u, \phi; a) \\
+ R_1^0 + \int_0^1 DF_0^0(u + \xi U_0^0, \phi; a) U_0^0 d\xi + H_0 \circ \mathcal{F}_0^0 + (Id - \Gamma_{K_0}) F_0^0(u, \phi; a) \quad (3.3)
\end{aligned}
\]
where \( N_0^0(u) = (\Omega(a) y_1, -\varepsilon_0 x_1^3)^T \) is the normal form,
\[
\begin{aligned}
F_0^0 = (\varepsilon_0^2 f_1^0(x, y, \omega t; a), \varepsilon_0^2 f_2^0(x, y, \omega t; a))^T, \\
H_0 = (\varepsilon_0^2 h_2^0(x, y, \omega t; a), \varepsilon_0^2 h_2^0(x, y, \omega t; a))^T,
\end{aligned}
\]
and
\[ R_1^0 = \left( 0, -\varepsilon_0 \left[ \left( 3x_1 (U_1^0)^2 + (U_1^0)^3 \right) \right] \right)^T. \]

Analogously, we hope that the transformation \( \mathcal{F}_0^0 \) satisfies
\[
-J_{\varepsilon_0}^{-1} \omega_\partial aU_0^0 + DN_0^0(u)U_0^0 - J_{\varepsilon_0}^{-1} DU_0^0 J_{\varepsilon_0}N_0^0(u) + \Gamma_{K_0} F_0^0(\phi, u; a) = \hat{N}_0^0(u) + R_2^0, \quad (3.4)
\]
where
\[ \hat{N}_0^0 = \left( 0, \sum_{l_i \in \Sigma_{D_j}} N_{2, l_1, l_2}^0(a) x_1^{l_1} y_1^{l_2} \right)^T \quad (3.5) \]
is the drift from \( \Gamma_{K_0} F_0^0 \) which will be determined later (see (3.17)) and \( R_2^0 = (R_{21}^0, R_{22}^0)^T \) consists of high-degree terms with
\[
\begin{aligned}
R_{21}^0 &= U_{1, 1, 1}^0 x_1^4 - \Omega(a) (3U_{1, 3, 0} x_1^2 + U_{1, 1, 1} y_1), \\
R_{22}^0 &= -3\varepsilon_0 x_1^2 \sum_{2 \leq l_1, 2 \leq l_2 \leq 3} U_{l_1, l_2}^0 (\phi; a) x_1^{l_1} y_1^{l_2} - \varepsilon_0 \Omega(a) (3U_{2, 3, 0} x_1^2 + U_{2, 1, 1} y_1) + \varepsilon_0 U_{2, 1, 1} x_1^4.
\end{aligned}
\]
Comparing the coefficients of lower-degree terms in (3.4), we obtain
\[
\begin{aligned}
\omega_\partial a U_{l_1, 0}^0 - \varepsilon_0 \Omega(a) U_{2, l_0}^0 &= \varepsilon_0 \delta^{l_1} \Gamma_{K_0} f_{1, l_1, 0}(\phi; a), \quad l_1 = 0, 1, 2, \\
\omega_\partial a U_{1, 0}^0 - \varepsilon_0 U_{2, 0}^0 &= \varepsilon_0 \Gamma_{K_0} f_{1, 3, 0}(\phi; a), \\
\omega_\partial a U_{l_0}^0 + \varepsilon_0 \Omega(a) U_{1, l_0}^0 &= \varepsilon_0 \delta^{l_0} \Gamma_{K_0} f_{1, 0, 1}(\phi; a), \\
\omega_\partial a U_{1, l_0}^0 + 2\varepsilon_0 \Omega(a) U_{1, 0}^0 &= \varepsilon_0 \Gamma_{K_0} F_{1, 1, 1}(\phi; a), \\
\end{aligned}
\] (3.6)
\[
\begin{align*}
\omega \partial_y U^0_{2,1,0} &= \varepsilon_0^{2+l_1} \Gamma_{k,0} f_{2,1,0}(\phi; a) - N^0_{2,1,0}(a), \quad l_1 = 0, 1, \\
\omega \partial_y U^0_{2,2,0} + 3\varepsilon_0 U^0_{1,0} &= \varepsilon_0^4 \Gamma_{k,0} f_{2,2,0}(\phi; a) - N^0_{2,2,0}(a), \\
\omega \partial_y U^0_{2,3,0} - \varepsilon_0 U^0_{2,1} + 3\varepsilon_0 U^0_{1,0} &= \varepsilon_0^5 \Gamma_{k,0} f_{2,3,0}(\phi; a) - N^0_{2,3,0}(a), \\
\omega \partial_y U^0_{2,0,1} + \varepsilon_0 \Omega(a) U^0_{2,10} &= \varepsilon_0^6 \Gamma_{k,0} f_{2,0,1}(\phi; a) - N^0_{2,0,1}(a), \\
\omega \partial_y U^0_{2,11} + 2\varepsilon_0 \Omega(a) U^0_{2,20} &= \varepsilon_0^7 \Gamma_{k,0} f_{2,11}(\phi; a) - N^0_{2,11}(a),
\end{align*}
\]

where we have used the fact \( f_j^0 = f_j(\varepsilon x, \varepsilon^2 y, \omega t; a), \quad j = 1, 2 \). These homology equations are different from those in (2.19) and (2.20). Analogously, by making Fourier expansion with respect to \( \phi \), ones turn them into the algebraic equations as follows.

\[
i(k, \omega) \tilde{U}^0_{2,1,0}(k; a) - \varepsilon_0 \Omega(a) \tilde{U}^0_{2,1,0}(k; a) = \varepsilon_0^{3+l_1} \hat{f}_{1,1,0}(k; a), \quad l_1 = 0, 1, 2,
\]

\[i(k, \omega) \tilde{U}^0_{2,0,1}(k; a) - \varepsilon_0 \Omega(a) \tilde{U}^0_{2,0,1}(k; a) = \varepsilon_0^{5+l_1} \hat{f}_{1,0,1}(k; a), \quad l_1 = 0, 1, 2,
\]

\[i(k, \omega) \tilde{U}^0_{2,1,1}(k; a) - 2\varepsilon_0 \Omega(a) \tilde{U}^0_{2,2,0}(k; a) - \varepsilon_0 \Omega(a) \tilde{U}^0_{2,1,1}(k; a) = \varepsilon_0^{5+l_1} \hat{f}_{1,1,1}(k; a),
\]

\[i(k, \omega) \tilde{U}^0_{2,2,1}(k; a) = \varepsilon_0^{5+l_1} \hat{f}_{2,1,0}(k; a) - N^0_{2,2,1}(a) \delta^0_k, \quad l_1 = 0, 1,
\]

\[i(k, \omega) \tilde{U}^0_{2,2,0}(k; a) + 3\varepsilon_0 \tilde{U}^0_{2,0,0}(k; a) = \varepsilon_0^4 \hat{f}_{2,2,0}(k; a) - N^0_{2,2,0}(a) \delta^0_k,
\]

\[i(k, \omega) \tilde{U}^0_{2,3,0}(k; a) + 3\varepsilon_0 \tilde{U}^0_{2,1,0}(k; a) - \varepsilon_0 \Omega(a) \tilde{U}^0_{2,0,1}(k; a) = \varepsilon_0^5 \hat{f}_{2,3,0}(k; a) - N^0_{2,3,0}(a) \delta^0_k,
\]

\[i(k, \omega) \tilde{U}^0_{2,1,1}(k; a) + 2\varepsilon_0 \Omega(a) \tilde{U}^0_{2,2,0}(k; a) = \varepsilon_0^4 \hat{f}_{2,1,1}(k; a) - N^0_{2,1,1}(a) \delta^0_k,
\]

where \( k \in \mathbb{Z}^n \) with \( |k| \leq K_0 \) and \( \delta^0_k \) is the Dirac function. At first, we consider the case \( k = 0 \), let

\[
\begin{align*}
\hat{U}^0_{1,1,l_1}(0; a) &= 0, \quad \hat{U}^0_{2,1,l_2}(0; a) = -\varepsilon_0^{2+l_1+2l_2} \Omega^{-1}(a) \hat{f}_{1,1,0}(0; a), \quad l_1 \in \sum_0^0, \\
N^0_{2,1,l_1}(a) &= \varepsilon_0^{2+l_1} \hat{f}_{2,1,0}(0; a), \quad l_1 = 0, 1, 2, \\
N^0_{2,3,0}(a) &= \varepsilon_0^5 \hat{f}_{2,3,0}(0; a) - \Omega^{-1}(a) \hat{f}_{1,0,1}(0; a), \\
N^0_{2,1,1}(a) &= \varepsilon_0^{4+l_1} \hat{f}_{2,1,1}(0; a) + (l_1 + 1) \hat{f}_{1,1,0}(0; a), \quad l_1 = 0, 1,
\end{align*}
\]

then the equations (3.8)-(3.16) hold true for \( k = 0 \). For the case \( k \neq 0 \), we can solve them one by one in the following order.

\[(3.12) \rightarrow (3.8)(l_1=0,1) \rightarrow (3.13) \rightarrow (3.8)(l_1=2) \rightarrow (3.15) \rightarrow (3.16) \rightarrow (3.10) \rightarrow (3.11) \rightarrow (3.14) \rightarrow (3.9).\]

In above process, the small divisors are always \( i(k, \omega) \). According to the assumption \((A1)\), there is no resonance. Thus, the parameter set \( \Pi_1 \) is the same as \( \Pi_0 \). And we can also obtain the estimates of the change \( \mathcal{F}^0_1 \) as below.

\[|||U^0_{1,l_1,l_2}|||_{r_1, \Pi_1} \leq \varepsilon_0^{3+l_1+2l_2} \gamma^5_0 \rho_0^{-(6n_0+5)}, \quad |||U^0_{2,l_1,l_2}|||_{r_1, \Pi_1} \leq \varepsilon_0^{2+l_1+2l_2} \gamma^5_0 \rho_0^{-(6n_0+5)},\]

for \( l_1 + 2l_2 \leq 3 \). From (3.17), it is easy to check that the drifted terms satisfy

\[|||N^0_{2,l_1,l_2}(a)|||_{r_1, \Pi_1} \leq \varepsilon_0^{2+l_1+2l_2}, \quad \text{for} \ l_1 + 2l_2 \leq 3. \]

Next, we reduce \( N^0(u) + \tilde{N}^0(u) \) to a normal form as in (2.2) by the translation \( \mathcal{F}^0_2 : x_1 = x_{10} + x_{11}, \ y_1 = y_{1+} \).

\[x_1 \]
where the $x_{10}$ is a real root of the algebraical equation
\[-(\varepsilon_0 - N_{2,30}^0(a))x_{10}^3 + N_{2,20}^0(a)x_{10}^2 + N_{2,10}^0(a)x_{10} + N_{2,00}^0(a) = 0.\]

By (3.17) and the Lemma 4 in [10], we get
\[x_{10} = \varepsilon_0^\frac{1}{3} f_{2,00}^0(0; a) + O(\varepsilon_0^\frac{2}{3}).\]

According to the assumptions (A2), it implies
\[|||x_{10}|||_{1} \sim \varepsilon_0^\frac{1}{3}.\] (3.19)

Thus, the change $F^0 = F_1^0 \circ F_2^0 : \varpi(s_1) \to \varpi(s_0 - \delta_0)$, transforms the system (3.2), denoting the new variable $x_{1+}$, $y_{1+}$ by $x$, $y$, into the form
\[
\begin{align*}
\dot{x} &= \varepsilon_0 \left[\Omega(a)y + \varepsilon_0^h h_1(x, y; \phi; a) + f_1^1(x, y; \phi; a)\right], \\
\dot{y} &= N_1^1(x; a) + \varepsilon_0^h h_2(x, y; \phi; a) + f_2^1(x, y; \phi; a),
\end{align*}
\] (3.20)

with the normal form
\[N^1(z) = \left( e_1^1(a)x^3 + e_4^1(a)x^2 + e_1^2(a)x + e_5^1(a)y + e_5^2(a)xy \right), \]

where
\[
\begin{align*}
e_1^1(a) &= -3(\varepsilon_0 - N_{2,30}^0(a))x_{10}^2 + 2N_{2,20}^0(a)x_{10} + N_{2,10}^0(a), \\
e_1^2(a) &= -3(\varepsilon_0 - N_{2,30}^0(a))x_{10} + N_{2,20}^0(a), \\
e_1^3(a) &= -\varepsilon_0 + N_{2,30}^0(a), \\
e_2^1(a) &= N_{2,01}^0(a) + N_{2,11}^0(a)x_{10}, \\
e_3^1(a) &= N_{2,11}^0(a),
\end{align*}
\] (3.21)-(3.23)

$h_j^1$ and $f_j^1$ are the high-degree and low-degree terms for $j = 1, 2$, respectively. Due to (3.18), (3.19) and (3.21)-(3.23), it is clear that
\[|||e_j^1(a)|||_{1} \sim \varepsilon_0^{-\frac{6-j}{3}} (j = 1, 2, 3), \quad |||e_4^1(a)|||_{1} \sim \varepsilon_0^\frac{1}{3}, \quad |||e_5^1(a)|||_{1} \sim \varepsilon_0^\frac{2}{3}.\] (3.24)

By some analysis, we know that the largest part of terms $(f_1^1, f_2^1)^T$ comes from $H^0 \circ F^0$. Therefore, the corresponding estimates are
\[
\begin{align*}
|||f_{1,00}^1|||_{1 \times 1} &\sim \varepsilon_0^\frac{3}{4}, \\
|||f_{1,10}^1|||_{1 \times 1} &\sim \varepsilon_0^\frac{3}{4}, \\
|||f_{2,20}^1|||_{1 \times 1} &\sim \varepsilon_0^\frac{5}{4}, \\
|||f_{1,01}^1|||_{1 \times 1} &\sim \varepsilon_0^\frac{3}{4}, \\
|||f_{1,11}^1|||_{1 \times 1} &\sim \varepsilon_0^\frac{5}{4}, \\
|||f_{1,12}^1|||_{1 \times 1} &\sim \varepsilon_0^\frac{5}{4},
\end{align*}
\] (3.25)

where $f_{j,i_1,i_2}^1$ is the coefficient of $x^{i_1}y^{i_2}$ in low-degree term $f_{j}^1$ for $j = 1, 2$. As to the estimates of high-degree terms $h_j^1$, if $\varepsilon_0$ is small enough, then $|||h_j^1|||_{1 \times 1} \leq M_0(1 + \sigma_1)$.

Set $\varepsilon_1 = \varepsilon_0^\frac{2}{3}$. Due to (3.24) and (3.25), we can check that the system (3.20) satisfies the Conditions (a.1) and (a.2) in lemma 2.1 with $v = 1$. So, by lemma 2.1, we obtain a sequence of quasi-periodic coordinate transformations $F^v : D(r_{v+1}, s_{v+1}) \times \Pi_{v+1} \to \varpi(s_v)$ in the form
\[
F^v : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_{1+} \\ y_{1+} \end{pmatrix} + \begin{pmatrix} x_{10}^v(a) \\ 0 \end{pmatrix} + \begin{pmatrix} W_{1}^v(\phi, x_{1+}, y_{1+}; a) \\ W_{2}^v(\phi, x_{1+}, y_{1+}; a) \end{pmatrix},
\]

where $x_{10}^v, W_{1}^v$ and $W_{2}^v$ are analytic in $(\phi, x_{+}, y_{+}) \in D(r_{v+1}, s_{v+1})$ and continuously differentiable in $a \in \Pi_{v+1}$, $v = 1, 2, \cdots$. Let
\[F_v = F_1 \circ \cdots \circ F^v, \quad D_\infty = D(\frac{r_0}{2}, \frac{s_0}{2}), \quad \Pi_{v_0} = \bigcap_{v=0}^{\infty} \Pi_v.\]
By (2.5) and \( \Pi_{v+1} \subset \Pi_v \), we obtain

\[
\text{Meas}(\Pi_0 - \Pi_{\gamma_0}) \leq \sum_{v=0}^{\infty} \text{Meas}(\Pi_v - \Pi_{v+1}) \leq c \text{Meas} \Pi_0 \sum_{v=0}^{\infty} \frac{\gamma_0}{(v+1)^2},
\]

that is

\[
\text{Meas}\Pi_{\gamma_0} = \text{Meas}\Pi_0 - O(\gamma_0).
\]

And by the transformation \( \mathcal{F}_v \), the system (3.20) becomes

\[
\begin{cases}
\dot{x} = \varepsilon \Omega(a)y + \varepsilon^2 h_1^{v+1}(x, y, \phi; a) + f_1^{v+1}(x, y, \phi; a), \\
\dot{y} = N_1^{v+1}(x; a) + \varepsilon^2 h_2^{v+1}(x, y, \phi; a) + f_2^{v+1}(x, y, \phi; a).
\end{cases}
\]

(3.27)

From (2.54) and (2.6)-(2.12), it follows that

\[
\|\mathcal{F}_v - \mathcal{F}_{v+1}\|_{r_{\ast}, s_{\ast}, \Pi_{\ast}} \leq \prod_{\mu=1}^{v-1} \|D\mathcal{F}_\mu\|_{r_{\ast}, s_{\ast}, \Pi_{\ast}} \|W_0^0 + W^v\|_{r_{\ast}, s_{\ast}, \Pi_{\ast}} \leq 2\varepsilon_0^\frac{\delta}{2},
\]

where \( W_0^0 = (x_{10}^v, 0)^T \) and \( W^v = (W_1^v, W_2^v)^T \). Thus, \( \{\mathcal{F}_v\} \) is convergent under the norm \( \| \cdot \|_{\frac{\delta}{2}, \frac{\delta}{2}, \frac{\delta}{2}} \). Assume that \( \mathcal{F}_\infty \) is the limit of \( \mathcal{F}_v \). Obviously, \( \mathcal{F}_\infty \) is of the form

\[
\mathcal{F}_\infty : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_{1+} \\ y_{1+} \\ \phi_{1+} \\ 0 \end{pmatrix} + \begin{pmatrix} W_1^\infty(\phi, x_{1+}, y_{1+}; a) \\ W_2^\infty(\phi, x_{1+}, y_{1+}; a) \end{pmatrix},
\]

(3.28)

where \( x_{10}^v, W_1^\infty \) and \( W_2^\infty \) are analytic in \( (\phi, x_{\ast}, y_{\ast}) \in \mathcal{D}_\infty \), and Lipschitz in \( a \in \Pi_{\gamma_0} \).

It is clear that \( x_{10}^v(a) = \sum_{v=1}^{\infty} x_{10}^v(a) \approx \varepsilon_0^{-\frac{\delta}{2}} \varepsilon_1^\frac{\delta}{2} \varepsilon_0^\frac{\delta}{2} \). By lemma 2.1 and letting \( v \to \infty \) in (3.27), the \( \mathcal{F}_\infty \) transforms (3.20) into

\[
\begin{cases}
\dot{x}_{1+} = \varepsilon \Omega(\phi)x_{1+} + \varepsilon^2 h_1^\infty(x_{1+}, y_{1+}; a), \\
\dot{y}_{1+} = N_1^\infty(x_{1+}; a) + \varepsilon^2 h_2^\infty(x_{1+}, y_{1+}; a),
\end{cases}
\]

(3.29)

where \( h_j^\infty \) are the high-degree terms for \( j = 1, 2 \). Obviously, \( (0, 0)^T \) is a solution of (3.29). Hence \( (x(t), y(t)) = (x_{10}^\infty(a) + W_1^\infty(\omega t, 0; a), W_2^\infty(\omega t, 0; a))^T \) is a solution of (3.20). With the help of the transformation \( \mathcal{F}_0 \) and (3.1), we obtain the quasi-periodic solution of (1.7) as follows

\[
\begin{cases}
x(t) = \varepsilon_0(x_{10}(a) + x_{10}^\infty(a) + W_1^\infty(\omega t, 0; a) + U_1^0(\omega t, W_1^\infty(\omega t, 0; a), W_2^\infty(\omega t, 0; a); a)), \\
y(t) = \varepsilon_0^2(W_2^\infty(\omega t, 0; a) + U_2^0(\omega t, W_1^\infty(\omega t, 0; a), W_2^\infty(\omega t, 0; a); a)),
\end{cases}
\]

(3.30)

which are analytic in \( \omega t \), Lipschitz in \( a \in \Pi_{\gamma_0} \). Furthermore,

\[
\sup_{\mathbb{R} \times \Pi_{\gamma_0}} (|x(t; a)|^2 + |y(t; a)|) = O(\varepsilon_0^2) = O(\varepsilon^2).
\]

The proof of Theorem 1.1 is complete.

4. Applications. In this section, we will apply the results above to study the nonlinear Hill equations with quasi-periodic forcing terms and the Stoker’s problem.

1. Nonlinear Hill equations. Hill equations, as an important class of mathematical physics equations, have been used to study the periodic motion of the moon. Then the scholars pay attentions to searching for the periodic solutions of Hill equations and introduce many different methods such as upper and lower solutions, degree theory or the fixed point theorems. As to the quasi-periodic solutions, ones usually
make use of the KAM theory. In what follows, we shall consider the nonlinear Hill equations with quasi-periodic forcing terms

\[ \ddot{x} + \lambda x^{2n+1} + (a_1 + ea(t))x = f(t), \]

where \( n \in \mathbb{N}^+, \) \( a_1 \geq 0, \) \( \lambda \in [1, 2] \) is the parameter, \( a(t) \) and \( f(t) \) are real analytic and quasi-periodic in \( t \) with frequencies \( \tilde{\omega} \in \mathbb{R}^{n_1} \) and \( \tilde{\omega} \in \mathbb{R}^{n_2} \) respectively, and \( 0 < \epsilon \ll 1. \) Let \( y = \dot{x} \), then equation (4.1) turns into

\[
\begin{aligned}
\dot{x} &= y, \\
\dot{y} &= -a_1 x - \lambda x^{2n+1} - ea(t)x + f(t).
\end{aligned}
\]

If \( a_1 > 0, \) the origin is a non-degenerate elliptic equilibrium point of the unperturbed system of (4.2). According to the Theorem 2.5 in [7], the system (4.2) possesses lots of quasi-periodic solutions. When \( a_1 = 0, \) the origin is degenerate elliptic, and we will apply our main Theorem 1.1 to the equations (4.2) with \( n = 1. \) For this, we firstly make some assumptions. (B1) The frequency vector \( \omega := (\tilde{\omega}, \tilde{\omega}) \) satisfies the Diophantine condition

\[ |(k, \omega)| \geq \gamma_0 |k|^{-n_1-n_2-1}, \]

where \( k \in \mathbb{Z}^{n_1+n_2} \) is nonzero vector, \( |k| = |k_1| + \cdots + |k_{n_1+n_2}| \) and \( 0 < \gamma_0 \ll 1. \) (B2) The zero-th Fourier coefficient of \( f(t), \) denoted by \( f(0), \) does not vanish. (B3) The forcing term \( f(t) \) is small, moreover, \( \sup_t |f(t)| < \epsilon. \)

Then we scale the system (4.2) by

\[ x \to \lambda^{-1} x, \ y \to \lambda^{-2} y, \]

and obtain the new system

\[
\begin{aligned}
\dot{x} &= \lambda^{-1} y, \\
\dot{y} &= -x^3 - \epsilon \lambda a(t)x + \lambda^2 f(t),
\end{aligned}
\]

where \( \lambda \in [1, 2] \) is the parameter. Observe that \( \Omega(\lambda) = \lambda^{-1}, h_1 = h_2 = f_1 = 0 \) and \( f_2 = -\lambda a(t)x + \epsilon^{-1} \lambda^2 f(t), \) it is not difficult to check that the system (4.3) satisfies the conditions (A1)-(A3) of Theorem 1.1. It follows that

**Theorem 4.1.** Suppose that for (4.1) the assumptions (B1)-(B3) hold. Then there is a sufficiently small \( \epsilon^* > 0 \) such that for \( 0 < \epsilon \leq \epsilon^* \), there exists a Cantorian-like subset \( \Pi_{\gamma_0} \subset [1, 2] \) with the Lebesgue measure

\[ \text{Meas} \Pi_{\gamma_0} = 1 - O(\gamma_0), \]

and for any \( \lambda \in \Pi_{\gamma_0}, \) the equation (4.1) possesses a solution \( x(t) \) which is real analytic and quasi-periodic with frequency \( \omega := (\tilde{\omega}, \tilde{\omega}) \). Moreover, it satisfies

\[ \sup_{\mathbb{R}} (|x(t)|^2 + |\dot{x}(t)|) = O(\epsilon^2). \]

Note that the term \(-\epsilon a(t)x + f(t)\) in (4.2) does not satisfy the hypothesis (H2) (refer to (1.5)), thus the results in [16] can not be applied to the system (4.2).

2. The Stoker’s problem. Consider a nonlinear oscillator with damping and quasi-periodic forcing, of the form

\[ \ddot{x} + c \dot{x} + a^2 x + x^3 = \epsilon f(t), \]  

where \( a, c \) are parameters and \( f \) is real analytic and quasi-periodic in \( t \) with the fixed rationally independent frequency \( \omega \in \mathbb{R}^{n_0}. \) One will expect that the above system has a quasi-periodic solution with the same frequency as the forcing \( f, \) so-called response solution. At first, J.J. Stoker [20] show the existence of such
solution when given \(a, c > 0\) and \(\varepsilon\) is sufficiently small by a contraction argument. For fixed \(\varepsilon\) and \(c\) close to 0, whether response solutions exist, which is nowadays called “Stoker’s problem”. There are two pioneering works in this subject: J. Moser [11] solved Stoker’s problem without damping (i.e. \(c = 0\)) via the KAM method, M. Friedman [5] considered the Stoker’s problem with small damping. It requires that \(a \neq 0\) for the results mentioned above. When \(a = 0\), the origin of system (4.4) is degenerate elliptic and the dynamical behavior will be more complicated. In [6], the author obtained the response solutions for the case \(a = 0\) and \(c \gg 1\). In the following part, we will consider the case \(a = 0\) and \(c\) is small enough.

Set \(y = \dot{x}\), then equation (4.4) with \(a = 0\) can be written as

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -cy - x^3 + \varepsilon f(t).
\end{align*}
\]

(4.5)

Comparing with (4.2), we find that equation (4.5) lacks the parameter \(\lambda\). For this, we introduce the internal parameter \(\lambda\) by \(x \rightarrow x + \varepsilon_0^2 \lambda\), where \(\lambda \in [1, 2]\) and \(0 < \varepsilon_0 \ll 1\). Then the equations (4.5) are transformed into

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -cy - (x + \varepsilon_0^2 \lambda)^3 + \varepsilon f(t).
\end{align*}
\]

(4.6)

By the scale (3.1), one gets the following system.

\[
\begin{align*}
\dot{\varepsilon}_0 y &= \varepsilon_0 x^3 - 3\varepsilon_0^5 \lambda x^2 - 3\varepsilon_0^5 \lambda^2 x - cy - \varepsilon_0^2 (\lambda^3 - f(t)).
\end{align*}
\]

(4.7)

Now we will apply the Lemma 2.1 instead of the main Theorem 1.1 to the equations (4.7) which can be treated as the system (2.2) with \(\Omega(\lambda) = 1\), \(e_0^5 = 0\), \(e_4^0 = -c\), \(e_0^0(\lambda) = -\varepsilon_0\), \(e_0^0(\lambda) = -3\varepsilon_0^2 \lambda\), \(e_0^0(\lambda) = -3\varepsilon_0^2 \lambda^2\), \(h_1^0 = h_2^0 = f_1^0 = 0\) and \(f_2^0 = -\varepsilon_0^2 (\lambda^3 - f(t))\). Recall that \(e_1 = \varepsilon_0^2 \), unfortunately, \(f_2^0\) does not satisfy condition (2.4) with \(v = 1\). Before using the Lemma 2.1, we shall make the term \(f_2^0\) smaller by some transformation denoted by \(\mathcal{F}\) so that the condition (2.4) holds true. For this purpose, we make some assumptions on the equation (4.4).

(C1) The frequency vector \(\omega\) satisfies the Diophantine condition

\[|(k, \omega)| \geq \gamma_0 |k|^{-n_0-1},\]

where \(k \in \mathbb{Z}^{n_0}\) is nonzero vector, \(|k| = |k_1| + \cdots + |k_{n_0}|\) and \(0 < \gamma_0 \ll 1\).

(C2) The zero-th Fourier coefficient of \(f(t)\), denoted by \(\hat{f}(0)\), satisfies that \(|\hat{f}(0)| < 1\).

(C3) The constant \(c\) related to the damping is small and \(c < \varepsilon_1^2 = \varepsilon_1^{21}\).

By almost the same proof of Lemma 2.1, we obtain \(\mathcal{F}\) and the new system to which the Lemma 2.1 can be applied. Then we follow the process of verifying the Theorem 1.1 and have the results below.

**Theorem 4.2.** Suppose that for (4.4) the assumptions (C1)-(C3) hold. Then there is a sufficiently small \(\varepsilon^* > 0\) such that for \(0 < \varepsilon \leq \varepsilon^*\), the equation (4.4) possesses a solution \(x(t)\) which is real analytic and quasi-periodic with frequency \(\omega\). Moreover, it satisfies \(\sup_{\mathbb{R}}(|x(t)|^2 + |\dot{x}(t)|) = O(\varepsilon^2)\).
Acknowledgments. The authors would like to thank the referees for their valuable comments and suggestions. Besides, the authors are grateful to Professor Li Xuemei for her invaluable encouragement.

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Received November 2019; revised May 2020.

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