Berry Phase in Atom-Molecule Conversion Systems and Fractional Monopole

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We investigate the geometric phase or Berry phase of adiabatic quantum evolution in an atom-molecule conversion system, and find that the Berry phase in such system consists of two parts: the usual Berry connection term and a novel term from the nonlinearity brought forth by the atom-molecule conversion. The geometric phase can be viewed as the flux of the magnetic field of a monopole through the surface enclosed by a closed path in parameter space. The charge of the monopole, however, is found to be one third of the elementary charge of the usual quantized monopole.

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Berry phase [1], which reveals the gauge structure associated with a phase shift in adiabatic processes in quantum mechanics, has attracted great interest in physics [2]. One classical example of Berry phase is a spin half particle aligned to a magnetic field, and the field is made to rotate adiabatically in a 3-D parameter space (see Fig. 1a). The Berry phase of such system has been interpreted as the flux of a magnetic field of a quantized monopole through the loop enclosed by the loop in parameter space.

On the other hand, association of ultracold atoms into molecules is currently a topic of much experimental and theoretical interest [3] with important applications ranging from the search for the permanent electric dipole moment [4] to BCS-BEC (Bose-Einstein condensate) crossover physics [5]. Through Feshbach resonance [6] or photoassociation [7], a pair of atoms can convert into a bounded molecule. The atom-molecule conversion under mean field treatment is governed by a nonlinear Schrödinger equation, in which the nonlinearity is from the fact that two or more atoms are needed to form one molecule [8]. Since the adiabatic manipulation is an optimal way to yield high conversion efficiency, great efforts and big progress have been made towards the adiabatic condition [9] and adiabaticity [10] of the nonlinear quantum evolution. Nevertheless, the knowledge of the Berry phase for the adiabatic evolution in such nonlinear system is very limited. This system not only lacks superposition principle due to the presence of nonlinearity [11] but also have no $U(1)$-invariance because the chemical potentials of atomic component and molecular component are not identical [12].

In this letter we formulate the adiabatic geometric phase in a general formulism for the atom-molecule conversion systems and derive the explicit expression of the Berry phase analytically. We find strikingly that the circuit integral of Berry connection of the instantaneous eigenstate alone can not account for the geometric phase, while a novel term due to the nonlinearity brought forth by the atom-molecule coupling emerges. Only with the inclusion of this additional contribution, the total geometric phase can be interpreted as a flux of a magnetic field of a monopole through the surface enclosed by the closed path in parameter space. There exists a forbidden cone in Berry sphere (i.e., parameter space, see Fig. 1a') and the Bloch sphere representing projective Hilbert space is deformed dramatically (Fig.1b'). The charge of the monopole is found to be one third of the elementary charge of the usual quantized monopole.

Let us consider an atom-molecule system with energy $\mathcal{H}(\psi, \psi^\dagger; \mathbf{R})$, in which $\mathbf{R}$ denotes all the system paramet-

\[ \begin{align*}
\text{FIG. 1: (color online) The parameter space (a) and Bloch sphere (b) for a spin half particle in magnetic field. (a') and (b') are the parameter space and Bloch sphere for the atom-molecule conversion system, respectively. The parameters change adiabatically along a close path shown as the green circles in the parameter spaces or Berry spheres. Accordingly, the eigenstate will evolve and form a close path schematically plotted as green circles on the Bloch spheres. The gray cone in (a') is the boundary for which $\theta = 2\pi/3$, inside which, i.e., $\theta > 2\pi/3$, no eigenstate exists. See text for details.}
\end{align*} \]
ters that vary in time slowly, \( \hat{\psi} = \{\hat{\psi}_i\} \) and \( \hat{\psi}^\dagger = \{\hat{\psi}^\dagger_i\} \) (with \( i = 1, \cdots, N \)) are the annihilation and creation field operators for atom with \( i = 1, \cdots, M \) and for molecule with \( i = M + 1, \cdots, N \). They obey the commutation relations \[ [\hat{\psi}_i, \hat{\psi}^\dagger_j] = \delta_{ij} \] for bosons. Under the mean field treatment, \( \hat{\psi} \) and \( \hat{\psi}^\dagger \) are replaced by complex numbers \( \psi \) and \( \psi^* \). It is convenient to write \( \mathcal{H}(\psi, \psi^*: \mathbf{R}) = \sum_{i,j} \psi_i^\dagger T_{ij} \psi_j \), then with the help of the above commutation relations, we obtain the following nonlinear Schrödinger equations \( (\hbar = 1) \)

\[
\frac{id\psi_i}{dt} = \sum_k H_{jk}(\psi, \psi^*: \mathbf{R}) \psi_k,
\]

in which the Hamiltonian

\[
H_{jk}(\psi, \psi^*: \mathbf{R}) = T_{jk}(\psi, \psi^*: \mathbf{R}) + \sum_i \psi_i^\dagger \frac{\partial T_{ik}}{\partial \psi^*_j}.
\]

The above system is not invariant under usual \( U(1) \) transformation, instead, it is invariant under the following co-diagonal \( U(1) \) transformation,

\[
U(\eta) = e^{i\Theta(\eta)}, \quad \Theta(\eta) = \left( \begin{array}{cc}
\eta I^M & \mathbf{0} \\
\mathbf{0} & \kappa \eta I^{N-M}
\end{array} \right),
\]

in which \( I^K \) is the \( K \) rank unit matrix and \( \kappa \) is an integer related with molecule structure, e.g. for diatomic molecule \( \kappa = 2 \). Obviously, \( \kappa = 1 \) the system reduces to an atomic system and \( U(\eta) \) is just the ordinary \( U(1) \) transformation.

The eigenvalue equation of the above system is

\[
\sum_k H_{jk}(\bar{\psi}(\mathbf{R}), \bar{\psi}^*(\mathbf{R}); \mathbf{R}) \bar{\psi}_k(\mathbf{R}) = \mu_j(\mathbf{R}) \bar{\psi}_j(\mathbf{R}),
\]

where \( \mu_j(\mathbf{R}) = \mu(\mathbf{R}) \) for \( j \leq M \) (atom) and \( \mu_j(\mathbf{R}) = \kappa \mu(\mathbf{R}) \) for \( j > M \) (molecule). The above eigenvalue equation defines the eigenfunction \( \bar{\psi} \) and the eigenvalue (or chemical potential) \( \mu \) that are functions of the adiabatic parameter \( \mathbf{R} \).

The equation \( (1) \) and its conjugate construct a canonical structure of classical dynamics with the energy \( \mathcal{H}(\psi, \psi^*: \mathbf{R}) \) as classical Hamiltonian and \( (\psi, i\psi^*) \) as a canonical variable pair. The gauge symmetry of \( \mathcal{H} \) given by \( (2) \) implies that the total atom number is conserved and the dynamics of the overall phase can be separated from the rest of the degrees of freedom. For simplicity and without losing generality we denote \( \lambda = \arg \psi_1 \) and set total phase as \( \lambda \) for atomic components and \( \kappa \lambda \) for molecular components, respectively. The normalization condition is \( \sum_{i=1}^M |\psi_i|^2 + \kappa \sum_{j=M+1}^N |\psi_j|^2 = 1 \). The other variables form a close set of Hamiltonian dynamics. We choose a new set of canonical variables \( (q, p) \) as \( q = (q_1, \cdots, q_i, \cdots, q_{N-1}) \) with \( q_i = -\arg(\psi_{i+1}) + \kappa_i \arg(\psi_i) \), \( p = (p_1, \cdots, p_i, \cdots, p_{N-1}) \) with \( p_i = |\psi_{i+1}|^2 \), in which \( \kappa_i = 1 \) for \( i \leq M \) and \( \kappa_i = \kappa \) for \( i > M \). From \( (1) \) and its canonical structure and using the normalization condition, we obtain the dynamical equations for the overall phase and other variables,

\[
\frac{d\lambda}{dt} = \frac{\partial}{\partial q} - \mathcal{H}(p, q) - \Lambda(p, q),
\]

\[
\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} - \dot{\lambda} - \frac{\partial \mathcal{H}}{\partial p}.
\]

in which \( \Lambda(p, q) = \frac{\partial}{\partial q} - (\sum_{i,j,k} \psi_i^\dagger \psi_j \frac{\partial T_{ik}}{\partial \psi^*_j} \psi_k) = \frac{\partial}{\partial q} - (\sum_{i,j,k} \psi_i^\dagger \psi_j T_{ik} \psi_k) = \frac{\partial}{\partial q} - (\sum_{i,j,k} \psi_i^\dagger \psi_j T_{ik} \psi_k) \)

For a linear quantum case, both \( H_{ij} \) and \( T_{ij} \) are the functions of the parameter \( \mathbf{R} \) only, so that the last term in Eq.\( (5) \) vanishes, i.e., \( \Lambda(p, q) = 0 \). The second term in the right-hand of Eq.\( (5) \) is the energy, whose time integral gives so called dynamical phase. The time integral of the first term is the Aharonov-Anandan phase for a cyclic quantum evolution. The above observation is readily extended to the adiabatic evolution of a quantum eigenstate, because the adiabatic theorem of quantum mechanics dictates that an initial nondegenerate eigenstate remains to be an instantaneous eigenstate and the evolution will be cyclic when the parameters move slowly in a circuit. In this case, the second term is the eigenenergy and the first term is just the Berry connection, i.e. \( i < \phi(\mathbf{R})|\nabla|\phi(\mathbf{R}) > \). Then the Berry phase equals to the circuit integral of the Berry connection.

However, for our nonlinear system, the contribution of the last term in \( (5) \) should be taken into account. Notice that the chemical potential is usually not identical to the energy while the dynamic phase should be the time integral of the chemical potential, we need to evaluate following quantity in adiabatic limit,

\[
\Xi(p, q; \mathbf{R}) = \mathcal{H}(p, q) + \Lambda(p, q) - \mu(\mathbf{R}),
\]

We denote \( p = \mathcal{P}(\mathbf{R}) + \delta p \) and \( q = \mathcal{Q}(\mathbf{R}) + \delta q \). Here \( \mathcal{P}(\mathbf{R}) \) and \( \mathcal{Q}(\mathbf{R}) \) are the fixed points of Eq.\( (6) \) that are local energy minima of system and therefore correspond to the eigenstates defined by \( (4) \). The vector \( (\delta p, \delta q) \) represents the correction to the adiabatic eigenstates in the order of \( \frac{d\mathbf{R}}{dt} \). As will be shown, \( (\delta p, \delta q) \) contains some secular terms in addition to the rapid oscillations. These secular terms will be accumulated in the nonlinear adiabatic evolution and contribute to the geometric phase. We expand the quantity \( \Xi(p, q; \mathbf{R}) \) around the fixed point,

\[
\Xi(p, q; \mathbf{R}) = \frac{\partial \Xi}{\partial \mathbf{R}}|_{(\mathbf{R}, \mathbf{Q})} \delta \mathbf{R} + \frac{\partial \Xi}{\partial (\mathbf{Q}, \mathbf{P})} |_{(\mathbf{R}, \mathbf{Q})} \delta \mathbf{Q} + o(\delta p^2, \delta q^2). \]

Here we use the relations: \( \mathcal{H}(\mathbf{R}, \mathbf{P}) + \Lambda(\mathbf{R}, \mathbf{P}) - \mu(\mathbf{R}) \) and \( \partial \mathcal{H}(\mathbf{R}, \mathbf{P})/\partial \mathbf{P} |_{\mathbf{R}, \mathbf{Q}} = \partial \mathcal{H}(p, q)/\partial q |_{\mathbf{R}, \mathbf{Q}} = 0 \). On the other hand, the \( (\delta p, \delta q) \) can be evaluated from
following Hamiltonian equations,
\[\dot{q} = \frac{\partial^2 H}{\partial p \partial p}|_{(\hat{p}, \hat{q})} \delta p + \frac{\partial^2 H}{\partial q \partial q}|_{(\hat{p}, \hat{q})} \delta q + o(\delta q^2, \delta p^2), \tag{9}\]
\[\dot{p} = -\frac{\partial^2 H}{\partial q \partial p}|_{(\hat{p}, \hat{q})} \delta p - \frac{\partial^2 H}{\partial q \partial q}|_{(\hat{p}, \hat{q})} \delta q + o(\delta q^2, \delta p^2). \tag{10}\]
Omitting the higher order terms, keeping the secular terms by average over the fast oscillations, we obtain
\[(\langle \delta p \rangle, \langle \delta q \rangle)^T = \Omega^{-1} \left(\frac{\partial \tilde{T}}{\partial R}, \frac{\partial \tilde{T}}{\partial R^*} \right)^T, \tag{11}\]
where the matrix \(\Omega = \left(\begin{array}{cc} -\frac{\partial^2 H(p, \theta)}{\partial q \partial p} & -\frac{\partial^2 H(p, \theta)}{\partial q \partial q} \\ -\frac{\partial^2 \tilde{H}(p, \theta)}{\partial q \partial p} & -\frac{\partial^2 \tilde{H}(p, \theta)}{\partial q \partial q} \end{array} \right) \big|_{(\hat{p}, \hat{q})}\) is the Hessian matrix of the classical Hamiltonian.

Combining Eq.(8) and (11) and with the help of Eq.(5), we find that, except for the dynamical phase (i.e., the time integral of the chemical potential), the eigenstate of the nonlinear atom-molecule conversion system acquires following additional phase during the adiabatic cyclic evolution \[\tag{12}\]
\[\gamma_g = \int \frac{\partial \tilde{T}}{\partial R} \cdot dR - \int (\frac{\partial \tilde{T}}{\partial p}, \frac{\partial \tilde{T}}{\partial q})|_{(\hat{p}, \hat{q})} \cdot \Omega^{-1} \left(\frac{\partial \tilde{T}}{\partial R}, \frac{\partial \tilde{T}}{\partial R^*} \right)^T \cdot dR.\]
In contrast to previous works \[\tag{17}\], the adiabatic geometric phase in the atom-molecule system is dramatically modified. The first term is the usual expression of the Berry phase that can be rewritten as the circuit integral of the Berry connection \[\oint \tilde{\phi}(R) \cdot d\tilde{\phi}(R) = \langle \tilde{\phi}(R) | \nabla \tilde{\phi}(R) \rangle - \langle \nabla \tilde{\phi}(R) | \tilde{\phi}(R) \rangle \]. The novel second term indicates that, the high-order correction to adiabatic approximate solution that is negligible in linear case, could be accumulated in the nonlinear adiabatic evolution with an infinite time duration in adiabatic limit and contributes a finite phase with geometric nature.

As an illustration, we consider following atom-molecule conversion model whose energy takes the form,
\[\mathcal{H} = \frac{R \cos \theta}{2} \left( \psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2 \right) + \sqrt{\frac{3}{8}} \frac{R \sin \theta}{2} \left( e^{-i \phi} \psi_1^\dagger T_{11} \psi_1 + h.c. \right), \tag{13}\]
where \(\psi = (\psi_1, \psi_2)\) and \(\psi^\dagger = \{\psi_1^\dagger, \psi_2^\dagger\}\) are the annihilation and creation operators for atom and molecule respectively, the terms \(\psi_1^\dagger T_{11} \psi_1 + h.c.\) describe coupling between atom pairs and diatomic molecules, and \(R = (R, \theta, \varphi)\) are parameters. Obviously, the system is invariant under the transformation \(U(\eta) = e^{i \Theta(\eta)}\).

Let us rewrite \(\mathcal{H}(\psi, \psi^*; R) = \sum_{i,j} \psi_i^T_{ij} T_{ij}(\psi, \psi^*; R) \psi_j\), where the matrix elements \(T_{11} = -T_{22} = \frac{R \cos \theta}{2}, T_{12} = T_{21} = \frac{\sqrt{3}}{2} R \sin \theta e^{-i \phi} \psi_1^\dagger T_{11} \psi_1 + h.c.\) and the nonlinear Schrödinger equation takes the form of Eq.(1) with
\[H(\psi, \psi^*; R) = \left(\begin{array}{c} \frac{R \cos \theta}{2} \psi_1 + \sqrt{\frac{3}{8}} e^{-i \phi} R \sin \theta \psi_1 \psi_1^\dagger/2 \end{array} \right), \tag{14}\]
where \(\psi_1^\dagger \psi_2^\dagger \psi_1 \psi_2\) are complex amplitudes for atom and molecule respectively. The projective Hilbert space is spanned by the vector \(\mathbf{n}_a = (2\sqrt{2} R \sin \theta \psi_1^\dagger \psi_2^\dagger \psi_1, 2\sqrt{2} R \sin \theta \psi_1^\dagger \psi_2^\dagger \psi_2, \psi_1^2 - 2\psi_2^2)\). Obviously, every point in this space corresponds to a class of quantum states among which the states are only different in co-diagonal total phases (see Eq. \[\tag{3}\]). With the normalization condition \(\psi_1^2 + 2\psi_2^2 = 1\), we plot the projection space in Fig. 2(b), which is a "tear-drop" shaped surface \[\tag{18}\].

The eigenequations take the forms of
\[H(\tilde{\phi}, \tilde{\phi}^*; R) \left(\begin{array}{c} \tilde{\phi}_1 \tilde{\phi}_2 \end{array} \right) = \left(\begin{array}{c} \muicted \varphi/2 \sqrt{1 - 2 \tilde{\phi}_2^2} \end{array} \right) \tag{15}\]
with the eigenvalue (or chemical potential) \(\mu \pm \frac{R \cos \theta}{2} \pm 1\).
Following our general formalism, we choose the total phase as \(\lambda = \arg \psi_1 \) and define the canonical pair as \(q = -\arg \psi_2 + 2 \arg \psi_1 \) and \(p = -\psi_2^2\). Hence, we have \(\mathcal{H} = \frac{1}{2} \psi_2^2 \) and \(\mathcal{H} \neq 0\). Substituting the concrete expressions of matrix \(\{T_{ij}\}\) to the definition of the quantity \(\Lambda\), we get \(\Lambda(p, q) = \sqrt{\frac{3}{8}} R \sin \theta (1 - 2p) \sqrt{p} \cos(q - \varphi)\).
On the other hand, from the system energy \(\mathcal{H}(p, q) = \frac{R \cos \theta}{2} (1 - 3p) + \frac{8}{3} R \sin \theta (1 - 2p) \sqrt{p} \cos(q - \varphi)\), we get
\[\Omega^{-1} = \sqrt{\frac{3}{8} R \sin \theta} \left(\begin{array}{cc} 0 & -\frac{2R \cos \theta}{3} \\ \frac{1 - 2p}{1 - 2p} & 0 \end{array} \right) \tag{16}\]. After lengthy calculation, we obtain the Berry phase according to formula \[\tag{12}\],
\[\gamma_g = \int \mathcal{F} \mathrm{d} \varphi + \int \frac{1}{9} \frac{R \cos \theta}{2} \mathrm{d} \varphi \tag{16}\]
\[\gamma_g = \frac{1}{6} \int (1 + \cos \theta) \mathrm{d} \varphi, \tag{17}\]
The above theoretical formulation on the Berry phase has been verified numerically by directly integrating the Schrödinger equation. On the other hand, we recognize that the above system although admits the quantal equations of motion, appears formally to have classical structure if we regard the total phase and total particle number as a pair of canonical conjugate variables. We thus could exploit this particular feature to construct a canonical transformation to action-angle variables. With including the canonical motions in projective Hilbert space.
represented by Eq.(6), we have derived the Hannay’s angle of a geometric nature associated with the adiabatic evolution. The Hannay’s angle is found to exactly equal to minus Berry phase of Eq.(16) \[19\]. This fact indicates a novel connection between Berry phase and Hannay’s angle in contrast to the usual derivative form \[20\], and supports our new expression of Berry phase from the other aspect.

For the linear systems, such as the spin-half system, i.e., \( H = -\frac{i}{\hbar} \mathbf{R} \cdot \mathbf{σ} \) where \( \mathbf{σ} \) are pauli matrices and \( \mathbf{R} = (R \sin \theta \cos \phi, R \sin \theta \sin \phi, R \cos \theta) \) is a vector in the 3-D parameter space. The Berry phase equals to the circuit integral of the Berry connection and is interpreted as the flux of the magnetic field of a virtual quantized monopole through the surface enclosed by the loop in parameter space, i.e., \( B_m = g_0 \frac{R}{g} \) with the elementary charge \( g_0 = \frac{e}{2} \). In general, the degeneracies of the spectrum in parameter space play an important role in connection with the geometric phase. Each degeneracy can be seen as a charge distribution located at the contact point between energy surfaces. Because the eigenstates are smooth and single valued outside the degeneracies, the total charge of the distribution, i.e., the monopole charge is necessarily an integer multiple of the elementary charge \( g_0 = 1/2 \). The non-elementary monopoles with integer multiples of \( g_0 \) have been found in case of light propagating and in condensed matter physics \[21\], \[22\]. The mechanism for the production of monopole charges larger than the elementary \( g_0 \) is due to constraints that act on the system \[23\].

For our nonlinear system, when the parameters \( \mathbf{R} = (R, \theta, \varphi) \) are considered as spherical coordinates of a vector in a 3-D space, and then from \[17\] we get the vector potential, \( \mathbf{A} = \frac{i}{\hbar} \mathbf{R} \left( \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta \right) \). Here, for convenience we only consider the branch of \( \mu_+ \). Hence, the Berry phase of atom-molecule conversion system can also be interpreted as the flux of a magnetic field of a virtual monopole through the surface enclosed by the closed path in parameter space (see in fig. 2(a’)), that is, \( \mathbf{B} = \nabla \times \mathbf{A}_\text{spin} = \frac{g \mathbf{R}}{2} \). Strikingly, the monopole charge \( g = \frac{1}{2} g_0 \), one third of the elementary charge. We attribute the fractional charge to the symmetry breaking of the parameter space by the boundary. From \[15\], we see \( |\psi_2|^2 \) increases with \( \theta \) monotonously. When \( \theta = 2\pi/3 \), \( |\psi_2|^2 = 1/\sqrt{2} \) reaching its extreme value (since \( |\psi_1|^2 + 2 |\psi_2|^2 = 1 \)). It implies that there is no eigenstate in the regime \( \theta > 2\pi/3 \), i.e., the Berry sphere of this system is a defected sphere with a forbidden cone bounded by \( \theta = 2\pi/3 \). This curious structure has been illustrated in fig. 1(a’).

In summary, we have investigated the adiabatic geometric phase in the atom-molecule conversion systems. A novel formula of geometric phase is derived and an exotic monopole with fractional elementary charge is found. The above phase and monopole are expected to be observed in the future ultracold atom experiments.

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\[1\] M.V. Berry, Proc. R. Soc. Lond. A 392, 45 (1984).
\[2\] Geometric Phase in Physics, edited by A. Shapere and F. Wilczek (World Scientific, 1989); The Geometric Phase in Quantum Systems, A. Bohm, A. Mostafazadeh, H.Koizumi, Q.Niu, J.Zwanziger, (Springer, 2003).
\[3\] T. Köhler, K. Góral, and P.S. Julienne, Rev. Mod. Phys. 78, 1311 (2006).
\[4\] J. J. Hudson, B. E. Sauer, M. R. Tarbutt, and E. A. Hinds, Phys. Rev. Lett. 89, 023003 (2002)
\[5\] C. A. Regal, M. Greiner, and D. S. Jin, Phys. Rev. Lett. 92, 040403 (2004).
\[6\] E. A. Donley, et al, Nature London 417, 529 (2002); J. L. Roberts, et al, Phys. Rev. Lett. 86, 4211 (2001).
\[7\] R. Wynar et al., Science 287, 1016 (2000); M. Mackie, R. Kowalski, and J. Javanainen, Phys. Rev. Lett. 84, 3803 (2000).
\[8\] A. Vardi, V.A. Yurovsky, and J.R. Anglin, Phys. Rev. A 64, 063611 (2001); H.Y. Ling, H. Pu, and B. Seanman, Phys. Rev. Lett. 93, 250403 (2004); H. Jing, J. Cheng, and P. Meystre, Phys. Rev. Lett. 99, 133002 (2007).
\[9\] Han Pu, et al, Phys. Rev. Lett. 98, 050406 (2007).
\[10\] A. P. Itin and S. Watanebe, Phys. Rev. Lett. 99, 223903 (2007)
\[11\] Y. S. Kivshar and B. A. Malomed, Rev. Mod. Phys. 61, 763 (1989).
\[12\] Shaoying Meng, Li-Bin Fu, and Jie Liu, Phys. Rev. A Phys. Rev. A 73, 053410 (2008).
\[13\] Jie Liu, Biao Wu, and Qian Niu, Phys. Rev. Lett. 90, 170404 (2003).
\[14\] Y. Aharonov and J.S. Anandan, Phys. Rev. Lett. 58, 1593(1987).
\[15\] Hong Y. Ling, et al, Physical Review A 75 033615 (2007)
\[16\] This is for the atom components. For the molecule components, the phase should time the factor \( \kappa \).
\[17\] J. C. Garrison, R.Y. Chiao, Phys. Rev. Lett. 60, 165 (1988); Biao Wu, Jie Liu, and Qian Niu, Phys. Rev. Lett. 94, 140402 (2005).
\[18\] I. Tikhonenkov, et al, Phys. Rev. A 73, 043605 (2006)
\[19\] Detailed calculation on the Hannay’s angle will be presented elsewhere.
\[20\] J. H. Hannay, J. Phys. A 18, 221 (1985); M. V. Berry, J. Phys. A 18, 15 (1985).
\[21\] Berry, M V, 1986, 'The Aharonov-Bohm effect is real physics not ideal physics’ in Fundamental aspects of quantum theory, eds. V Gorini and A Frigerio, Plenum, NATO ASI series vol. 144, 319-320.
\[22\] B. Simon, Phys. Rev. Lett. 51, 2167 (1983).
\[23\] P. Leboeuf and A. Mouchet, J. Phys. A 36, 2847 (2003).