LIFTINGS OF POLYNOMIAL SYSTEMS DECREASING THE MIXED VOLUME

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Abstract. The BKK theorem states that the mixed volume of the Newton polytopes of a system of polynomial equations upper bounds the number of isolated torus solutions of the system. Homotopy continuation solvers make use of this fact to pick efficient start systems. For systems where the mixed volume bound is not attained, such methods are still tracking more paths than necessary. We propose a strategy of improvement by lifting a system to an equivalent system with a strictly lower mixed volume at the expense of more variables. We illustrate this idea providing lifting constructions for arbitrary bivariate systems and certain dense-enough systems.

1. Introduction

The Bernstein–Khovanskii–Kouchnirenko (BKK) theorem [Ber75], [CLO05, Section 7.5] states that the number of isolated solutions of a system of d Laurent polynomials in d variables in the algebraic torus (C*)d is bounded above by the mixed volume of the Newton polytopes of the polynomials. This bound only depends on the monomials occurring in the different polynomials and may or may not be attained by a concrete choice of coefficients. Whether the mixed volume bound is attained depends on the solvability of lower-dimensional systems determined by the convex geometry of the Newton polytopes [Ber75].

Homotopy continuation is an important technique to solve polynomial systems by tracking the solutions of a known start system to the solutions of a system of interest along the paths of a homotopy. The computational complexity of this method depends on the number of paths to be tracked and the cost of tracking each path. It can be difficult to judge how the two costs are related. Ideally one wants to track only as many paths as the system of interest has solutions. Therefore a homotopy continuation method needs a good upper bound on the number of solutions and a suitable start system. In [HS95], Huber and Sturmfels show how to construct efficient start systems using the geometry of the Newton polytopes, introducing mixed subdivisions for sparse homotopy continuation. Solvers based on this approach are the current state of the art and are implemented in PHCpack [Ver11] and as HomotopyContinuation.jl in Julia [BT18]. The number of paths to be tracked in a sparse homotopy is the mixed volume of the Newton polytopes of the target system. In practice, however, the mixed volume bound is often not attained and therefore sparse homotopy continuation still tracks superfluous paths that do not lead to actual torus solutions of the system.

In this paper we propose to study liftings of polynomial systems to equivalent systems with more equations and more variables but fewer paths to be tracked. After laying out the idea, we present methods for liftings that work for certain dense-enough systems of polynomials (Theorem 3.9), system with linear dependency in a facial subsystem (Theorem 4.1), and for bivariate systems (Theorem 5.1).

It is not easy to judge the implications of lifting to the complexity of solving a polynomial system. The celebrated solution of Smale’s 17th problem (see [Lai17] and its references) shows that a path in a homotopy can be tracked in polynomial time. Therefore one solution of a polynomial system is
computable in polynomial time. It is an interesting challenge to investigate the effects of lifting to the computation of individual solutions. First steps with respect to just simple Newton iterations have been undertaken in [AD10]. In the present paper we take the standpoint that solving means to determine all solutions to a system. Our ideas therefore pertain to the complexity of polyhedral homotopy. An estimation of how the number of paths and the cost of tracking an individual path relate to each other is an interesting problem beyond this paper.

We begin by illustrating our lifting approach in a simple example.

**Example 1.1.** Consider the system

\[ \begin{align*}
0 &= (1 - x_1^2)x_2 + 2, \\
0 &= (1 - x_1)^2x_2 + 3.
\end{align*} \tag{1} \]

We aim to solve this system in \((\mathbb{C}^*)^2\). The Newton polytopes of both polynomials agree with \(P = \text{conv}(0, e_2, 2e_1 + e_2)\) (see Figure 1). The mixed volume equals 2. However, in \((\mathbb{C}^*)^2\) the system has a unique solution. According to [Ber75], this happens exactly if a facial subsystem has a root in \((\mathbb{C}^*)^2\) (see Theorem 2.2). In our case this facial subsystem corresponds to the red face in Figure 1:

\[ \begin{align*}
0 &= (1 - x_1^2)x_2, \\
0 &= (1 - x_1)^2x_2, \tag{2}
\end{align*} \]

and has infinitely many solutions \(\{(1, x_2) : x_2 \in \mathbb{C}^*\}\). Now consider the system

\[ \begin{align*}
0 &= y(1 + x_1)x_2 + 2, \\
0 &= y(1 - x_1)x_2 + 3, \\
0 &= y - (1 - x_1). \tag{3}
\end{align*} \]

Solving the last equation for \(y\) and plugging in, recovers the original system. Therefore the lifted system is equivalent to the original. In contrast, the lifted system has mixed volume equal to 1, the true number of torus solutions. What happens is that the original system has a zero at infinity, namely \((1, \infty)\), which can be seen by solving for \(x_2\). This pseudo-solution is taken care of in the lifting. Homotopy continuation for the original system involves tracking a path tending to infinity, while for the lifted system this path is ignored.

In order to illustrate our main lifting strategy and the necessary tools, we discuss further details of Example 1.1. After division by \(x_2\) one observes that the facial subsystem (2) has a solution in \((\mathbb{C}^*)^2\) if and only if the univariate polynomials \((1 - x_1^2)\) and \((1 - x_1)^2\) have a common non-zero root, in our case \(x_1 = 1\). Our approach is to reflect this information in a Newton polytope, so that it contributes to the mixed volume computation. We achieve this by means of a polynomial division in order to substitute the linear factor \((1 - x_1)\) by a new variable \(y\). In the case of a linear factor, the effect of the division on Newton polytopes, the *polytope division*, can be worked out and is a main ingredient in our method (see Definition 3.1). In this way we arrive at an equivalent lifted system (3), Definition 2.3 contains the details of lifting. In the lifted system, the non-genericity that \(y = (1 - x)\) appears as a common factor in both polynomials of (1) is represented also polyhedrally: In this sense we view (1) in a higher-dimensional space of sparse polynomial systems, in which having strictly fewer solutions than the mixed volume is not encoded in the coefficients, but in the geometry of the Newton polytopes. Hence, the mixed volume of the Newton polytopes of the system (3) is strictly less than that of the Newton polytopes of (1). The embedding of information about the number
of solutions in the convex geometry of Newton polytopes can only work if the construction of the lifting depends on the coefficients of the system (1). Theorem 5.1 shows that in the bivariate case there always exist lifts generalizing Example 1.1 as long as the mixed volume bound is not attained.

Theorem 3.9 generalizes the above approach to systems of $d$ polynomials in $d$ variables, but the exact lifting strategy does not carry over. In higher dimension the lifting becomes more involved and more conditions are required. The replacement of common linear terms by new variables is implemented by polynomial division and additional assumptions are necessary for the polyhedral geometry to work. A strong but natural additional assumption is that the Newton polytopes of the subsystem for which Bernstein’s criterion fails are saturated in the sense of Definition 3.4. The content of this definition is best seen in Lemma 3.2, which shows that a sparse system behaves polyhedrally well under division by linear univariate terms if and only if it is saturated. Example 3.3 illustrates that this well-behavedness is necessary for the lifting to work. In order to ensure that the mixed volume of the lifting is strictly lower, we additionally need to enforce that the Newton polytopes of the original system are not degenerate in a certain sense (see Lemma 3.7).

Our work describes individual lifting steps, reducing the mixed volume by one in Theorems 3.9 and 4.1 or the degree of a gcd in Theorem 5.1. Of course one would like to be able to make several improvements. Most obviously it would be nice to reduce the mixed volume more by constructing refined lifts. If this is not possible, one would like to lift iteratively, ideally until the mixed volume bound is sharp. At the moment it is an open problem to determine how iterative liftings work.

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2. Preliminaries

We abbreviate $[n] := \{1, \ldots, n\}$. A lattice polytope $P \subseteq \mathbb{R}^d$ is a convex polytope all of whose vertices lie in $\mathbb{Z}^d$. We denote by $\text{Vol}(P)$ the normalized volume of $P$, which equals the standard euclidean volume multiplied by $d!$. The (normalized) mixed volume of $d$-tuples of lattice polytopes living in $\mathbb{R}^d$ is the unique functional satisfying

$$\text{Vol}(\lambda_1 P_1 + \cdots + \lambda_k P_k) = \sum_{i_1=1}^k \cdots \sum_{i_d=1}^k \lambda_{i_1} \cdots \lambda_{i_d} \text{MV}(P_{i_1}, \ldots, P_{i_d}),$$

for all choices of lattice polytopes $P_1, \ldots, P_k \subseteq \mathbb{R}^d$, non-negative scalars $\lambda_1, \ldots, \lambda_k \geq 0$, and $k \in \mathbb{Z}_{\geq 1}$. We refer to of [Sch14, Theorem and Definition 5.1.7] for a detailed treatment of existence and uniqueness of such a functional.

2.1. The BKK theorem. Let $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$. The support of $f$ is

$$\text{supp} f := \{a \in \mathbb{Z}^d : \text{the monomial } x^a \text{ has non-zero coefficient in } f\},$$

and its convex hull $N(f) = \text{conv}(\text{supp}(f))$ is the Newton polytope of $f$. A monomial change of variables is a map transforming a system of Laurent polynomials by sending $x_i \mapsto x^{u_i}$ for all $i \in [d]$ in each of the polynomials, where the $u_i$ form the columns of a unimodular integer matrix $U$, followed by multiplication of each polynomial by some monomial. Any monomial change of variables induces a bijection between the torus solutions of the original system and its transformation.

Theorem 2.1. Let $f_1, \ldots, f_d \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ be Laurent polynomials. Then the number of solutions of the system $f_1 = \cdots = f_d = 0$ in $(\mathbb{C}^*)^d$ is bounded from above by $\text{MV}(N(f_1), \ldots, N(f_d))$.

In order to formulate the conditions under which the above bound is attained, we introduce the notion of a facial system. For any polytope $P \subseteq \mathbb{R}^d$ and any non-zero vector $u \in \mathbb{R}^d$, we denote by $P^u$ the face of $P$ maximizing the functional $(\cdot, u)$. For a polynomial $f \in \mathbb{C}[x_1, \ldots, x_d]$, we denote by $f^u$ the polynomial consisting only of those terms of $f$ that are supported on the face $N(f)^u$. Theorem 2.1. Let $f_1, \ldots, f_d \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ be Laurent polynomials. Then the number of solutions of the system $f_1 = \cdots = f_d = 0$ in $(\mathbb{C}^*)^d$ is bounded from above by $\text{MV}(N(f_1), \ldots, N(f_d))$. In order to formulate the conditions under which the above bound is attained, we introduce the notion of a facial system. For any polytope $P \subseteq \mathbb{R}^d$ and any non-zero vector $u \in \mathbb{R}^d$, we denote by $P^u$ the face of $P$ maximizing the functional $(\cdot, u)$. For a polynomial $f \in \mathbb{C}[x_1, \ldots, x_d]$, we denote by $f^u$ the polynomial consisting only of those terms of $f$ that are supported on the face $N(f)^u$. Theorem 2.1. Let $f_1, \ldots, f_d \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ be Laurent polynomials. Then the number of solutions of the system $f_1 = \cdots = f_d = 0$ in $(\mathbb{C}^*)^d$ is bounded from above by $\text{MV}(N(f_1), \ldots, N(f_d))$.
Theorem 2.2. Let \( f_1, \ldots, f_d \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}] \) be Laurent polynomials. Then the number of solutions of the system \( f_1 = \cdots = f_d = 0 \) in \((\mathbb{C}^*)^d\) equals \( \text{MV}(N(f_1), \ldots, N(f_d)) \) (counting multiplicities) if and only if for any \( u \in \mathbb{R}^d \setminus \{0\} \) the facial system \( f_1^u = \cdots = f_d^u = 0 \) has no solution in \((\mathbb{C}^*)^d\).

Up to an appropriate monomial change of variables, the facial system corresponding to any \( u \in \mathbb{R}^d \setminus \{0\} \) is a system of \( d \) equations in \( d - 1 \) variables (as the corresponding Newton polytopes live in parallel \((n - 1)\)-dimensional hyperplanes orthogonal to \( u \)). In particular, generically, none of these systems has a solution and the BKK bound is attained.

2.2. Liftings of sparse polynomial systems. For point configurations \( A_1, \ldots, A_d \subseteq \mathbb{Z}^d \), we denote by \( \mathbb{C}[A_1, \ldots, A_d] \) the vector subspace of \( \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]^d \) consisting of all \((f_1, \ldots, f_d)\) with \( \text{supp}(f_i) \subseteq A_i \) for all \( i \in [d] \). For a tuple of lattice polytopes \((P_1, \ldots, P_d)\) we set \( \mathbb{C}[P_1, \ldots, P_d] := \mathbb{C}[P_1 \cap \mathbb{Z}^d, \ldots, P_d \cap \mathbb{Z}^d] \).

Definition 2.3 (Lifting of a system). For a tuple \( F = (f_1, \ldots, f_d) \in (\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}])^d \),
\[
\tilde{F} = (\tilde{f}_1, \ldots, \tilde{f}_d, h_1, \ldots, h_k) \in (\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}, y_1, \ldots, y_k])^{d+k}
\]
is a lifting of the system \( f_1 = \cdots = f_d = 0 \), if a point \((\alpha_1, \ldots, \alpha_d) \in (\mathbb{C}^*)^d\) is a solution of \( F \) if and only if there exists a solution \((\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_k) \in (\mathbb{C}^*)^d \times \mathbb{C}^k\) of \( \tilde{F} \).

Remark 2.4. For our purposes it is important to allow zeroes in the last \( k \) coordinates of the roots of the lifted system. See Remark 3.13 for possible caveats in lifting strategies.

For any tuple \((f_1, \ldots, f_d) \in (\mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}])^d\) we denote by \( S(f_1, \ldots, f_d) \) its number of isolated solutions (counting multiplicities) in the complex torus \((\mathbb{C}^*)^d\).

Definition 2.5 (Solution-preserving map). Let \( A_1, \ldots, A_d \subseteq \mathbb{Z}^d \) and \( B_1, \ldots, B_d \subseteq \mathbb{Z}^d \). For a subset \( U \subseteq \mathbb{C}[A_1, \ldots, A_d] \), we call a map \( \phi: U \to \mathbb{C}[B_1, \ldots, B_d] \) solution-preserving, if for any tuple \((f_1, \ldots, f_d) \in U\), one has \( S((f_1, \ldots, f_d)) = S(\phi(f_1, \ldots, f_d)) \).

The following is a direct consequence of Theorem 2.1.

Proposition 2.6. Let \( A_1, \ldots, A_d \subseteq \mathbb{Z}^d \) and \( B_1, \ldots, B_d \subseteq \mathbb{Z}^d \). Let \( U \subseteq \mathbb{C}[A_1, \ldots, A_d] \) and \( V \subseteq \mathbb{C}[B_1, \ldots, B_d] \) be Zariski-open dense sets. If there exists a solution-preserving map \( \phi_1: U \to \mathbb{C}[B_1, \ldots, B_d] \), then one has \( \text{MV}(A_1, \ldots, A_d) \leq \text{MV}(B_1, \ldots, B_d) \). In particular, if there additionally exists a solution-preserving map \( \phi_2: V \to \mathbb{C}[A_1, \ldots, A_d] \), one has \( \text{MV}(A_1, \ldots, A_d) = \text{MV}(B_1, \ldots, B_d) \).

2.3. Monotonicity of the Mixed Volume.

Definition 2.7. A tuple of lattice polytopes \( P_1, \ldots, P_d \in \mathbb{R}^d \) is essential if one of the following equivalent conditions holds:

(i) \( \text{MV}(P_1, \ldots, P_d) > 0 \),
(ii) there exists a choice of segments \( s_i \subseteq P_i \) for \( i \in [d] \) such that \( s_1, \ldots, s_d \) are linearly independent,
(iii) there is no subset \( \emptyset \neq I \subseteq [d] \) for which the polytopes \( \{P_i: i \in I\} \) can be translated to a common \((|I| - 1)\)-dimensional linear subspace of \( \mathbb{R}^d \),
(iv) there is no subset \( \emptyset \neq I \subseteq [d] \) satisfying
\[
\dim \left( \sum_{i \in I} P_i \right) \leq |I| - 1.
\]

The equivalence of (iii) and (iv) is straightforward to verify and [Sch14, Theorem 5.1.8] contains a proof of the remaining equivalences.

It is well-known that the mixed volume is inclusion-monotonous in each argument. It is, however, not strictly monotonous as one can replace several polytopes in a tuple by strictly smaller ones.
without decreasing the mixed volume. The following theorem of Bihan and Soprunov is a characterization of when the mixed volume of a subtuple is strictly smaller than that of the original tuple. It is a crucial tool in the proofs of both our main results Theorem 3.9 and Theorem 4.1.

**Theorem 2.8.** [BS19, Theorem 3.3] Let $P_1, \ldots, P_d \subseteq \mathbb{R}^d$ and $P_1', \ldots, P_d' \subseteq \mathbb{R}^d$ tuples of polytopes such that $P_i' \subseteq P_i$ for every $i \in [d]$. Given $u \in \mathbb{R}^d$, denote $T_u = \{i \in [d] : P_i' touches P_i\}$. Then $\text{MV}(P_1, \ldots, P_d) > \text{MV}(P_1', \ldots, P_d')$ if and only if there exists $u \in \mathbb{R}^d$ such that the tuple $\{P_i^u : i \in T_u\} \cup \{P_i : i \in [d] \setminus T_u\}$ is essential.

3. Liftings via polynomial division

**Definition 3.1.** Let $P \subseteq \mathbb{R}^d$ be a lattice polytope. The quotient $Q_i(P)$ and remainder $R_i(P)$ of $P$ are

$$Q_i(P) = \text{conv}(x - e_i, x - (x, e_i)e_i : x \in P \cap \mathbb{Z}^d, (x, e_i) > 0),$$

$$R_i(P) = \text{conv}(x - (x, e_i)e_i : x \in P \cap \mathbb{Z}^d).$$

**Lemma 3.2.** Let $f, q, r \in \mathbb{C}[x_1, \ldots, x_d]$ such that $q$ and $r$ are quotient and remainder, respectively, of the polynomial division of $f$ by $(x_i - \alpha)$ for some $\alpha \in \mathbb{C}^*$. Then $N(q) \subseteq Q_i(N(f))$ and $N(r) \subseteq R_i(N(f))$.

**Proof.** For any monomial $x_1^{a_1} \cdots x_i^{a_i} \cdots x_d^{a_d}$ with $a_i > 0$, polynomial division by $(x_i - \alpha)$ gives

$$x_1^{a_1} \cdots x_i^{a_i} \cdots x_d^{a_d} = (x_1^{a_1} \cdots x_i^{a_i-1} \cdots x_d^{a_d})(x_i - \alpha) + \alpha(x_1^{a_1} \cdots x_i^{a_i-1} \cdots x_d^{a_d}).$$

Therefore, for every monomial $x_1^{a_1} \cdots x_i^{a_i} \cdots x_d^{a_d}$ occurring in $f$, the division algorithm can only produce monomials of the form $x_1^{a_1} \cdots x_i^{a_i} \cdots x_d^{a_d}$ for $0 \leq b_i < a_i$ in $q$, showing $N(q) \subseteq Q_i(N(f))$.

The only monomials which cannot be divided by $(x_i - \alpha)$ are those in which $x_i$ does not occur. So the only monomials which can occur in the remainder $r$ are of the form $x_1^{a_1} \cdots x_i^{a_i} \cdots x_d^{a_d}$ where $x_1^{a_1} \cdots x_i^{a_i} \cdots x_d^{a_d}$ occurs in $f$ for some $a_i \geq 0$. This shows $N(r) \subseteq R_i(N(f))$. \hfill \Box

For a generic choice of coefficients of $f$ in $\mathbb{C}[\text{supp}(f)]$, it can be shown that equality holds in both cases of Lemma 3.2. The proof consists of analyzing in each step of polynomial division which conditions on the coefficients accumulate.

**Example 3.3.** Consider a polynomial of the form

$$f(x_1, x_2) = a + bx_1^2 + cx_1^2x_2^2.$$ Division of $f$ by a term $(x_1 - \alpha)$ results in a decomposition

$$f(x_1, x_2) = (cx_1x_2^2 + ax_2^2 + bx_1 + ab)(x_1 - \alpha) + \alpha^2cx_2^2 + \alpha^2b + a = Q(x_1, x_2)(x_1 - \alpha) + R(x_2),$$

where one has $N(Q) = Q_1(N(f))$ and $N(R) = R_1(N(f))$. For different polynomials $Q'(x_1, x_2) \in \mathbb{C}[Q_1(N(f))]$ and $R'(x_1) \in \mathbb{C}[R_1(N(f))]$ with the same Newton polytopes as $Q$ and $R$, respectively, the polynomial

$$f'(x_1, x_2) := Q'(x_1, x_2)(x_1 - \alpha) + R'(x_1),$$

generally has a Newton polytope strictly larger than $N(f)$. To see this let $\beta_1$ be the coefficient of the monomial $x_2^2$ in $Q'$ and $\beta_2$ the coefficient of the same monomial in $R'$. Then the coefficient of $x_2^2$ in $f'$ equals $-\alpha \beta_1 + \beta_2$, while $f$ does not contain the monomial $x_2^2$ at all (see Figure 2).

**Definition 3.4.** A lattice polytope $P \subseteq \mathbb{R}^d$ is $i$-saturated if either $\langle x, e_i \rangle = 0$, for all $x \in P \cap \mathbb{Z}^d$, or if one has

$$P = \text{conv}((Q_i(P) + [0, e_i]) \cup R_i(P)).$$

The polytope $P$ is $(1, \ldots, k)$-saturated if

(i) $P$ is 1-saturated, and
Figure 2. Newton polytopes $N(f) \not\ni (0, 2)$ and $N(f') \ni (0, 2)$ of Example 3.3 for generic $Q'$ and $R'$.

Figure 3. Example of a 1-saturated polygon $P$.

Figure 4. Example of a polygon $P$ that is not 1-saturated, but is 2-saturated.

(ii) $\mathcal{R}_{(1, \ldots, i)}(P) := \mathcal{R}_i(\mathcal{R}_{i-1}(\ldots (\mathcal{R}_1(P)))$ is $(i + 1)$-saturated for all $i \in [k - 1]$.

Finally, $P$ is saturated if it is $(1, \ldots, d)$-saturated. All definitions here extend to polynomials considering their Newton polytopes.

See Figures 3 and 4 for an example.

Remark 3.5. The inclusion $P \subseteq \text{conv}(\{Q_i(P) + [0, e_i]\} \cup \mathcal{R}_i(P)$ always holds.

Remark 3.6. The polytope $P$ is $i$-saturated if and only if $\mathcal{R}_i(P) = P \cap \{\langle e_i, \cdot \rangle = 0\}$.

The polytopes $P_j$ in the following lemma are those arising from lifting by division and consist of $N(f_j - f_u)$, the remainder and several different polytopes arising from substitutions.

Lemma 3.7. Set $u := - e_d$ and let $f_1, \ldots, f_d \in \mathbb{C}[x_1, \ldots, x_d]$ such that the polynomials $f_1^u, \ldots, f_d^u$ are $(1, \ldots, k)$-saturated for some $k \in [d - 1]$. For $j \in [d]$ let

$$P_j := \text{conv} \left( N(f_j - f_u^j) \cup \mathcal{R}_{(1, \ldots, k)}(N(f_u^j)) \cup \bigcup_{i=0}^{k-1} (Q_{i+1}(\mathcal{R}_{(1, \ldots, i)}(N(f_u^j))) + e_{d+1+i}) \right),$$

where $\mathcal{R}_{(1, \ldots, 0)}$ is the identity. Then

$$\text{MV}(P_1, \ldots, P_d, [0, e_1], \ldots, [0, e_k]) = \text{MV}(N_1, \ldots, N_d)$$

with

$$N_j := \text{conv}(\pi(N(f_j)), 0, e_{d+1}: l \in [k], x_l \text{ occurs in } f_u^j),$$

where $\pi: \mathbb{R}^{d+k} \to \mathbb{R}^d$ is the projection forgetting the first $k$ coordinates.

Proof. Assume $\text{MV}(P_1, \ldots, P_d, [0, e_1], \ldots, [0, e_k]) = 0$. By the projection formula for mixed volumes (see e.g. [Sch14, Theorem 5.3.1]) this is equivalent to $\text{MV}(\pi(P_1), \ldots, \pi(P_d)) = 0$, and we set the Minkowski sum of any set with the empty set to be empty. By construction, one therefore has

$$\pi(P_j) \cap \{\langle e_{k+1}, \cdot \rangle = \cdots = \langle e_d, \cdot \rangle = 0\} = \text{conv}(0, e_{d+1}: x_l \text{ occurs in } f_u^j),$$

finishing the proof upon noting that $\mathcal{R}_{(1, \ldots, k)}(N(f_u^j)) = \pi(N(f_u^j))$.

□
Remark 3.8. One has $\text{MV}(N_1, \ldots, N_d) > 0$ if $k = d-1$, the $N(f_i), \ldots, N(f_d)$ are all full-dimensional and the $f_i^n, \ldots, f_d^n$ are saturated.

Our main theorem follows now. It makes several special choices, some of which can be assumed without loss of generality. A discussion follows the proof in Remark 3.12. For example, we only consider facial subsystems in direction of the last coordinate.

Theorem 3.9. Let $k \in [d-1]$ and $u := -e_d$. Assume $f_1, \ldots, f_d \in \mathbb{C}[x_1, \ldots, x_d]$ satisfy

(i) $f_i$ is not divisible by $x_d$ for any $i \in [d]$,
(ii) $f_1^n, \ldots, f_d^n$ are saturated,
(iii) $\text{MV}(N_1, \ldots, N_d) > 0$, where the $N_j$ are as in Lemma 3.7 and
(iv) the system $f_1 = \cdots = f_d = 0$ has a solution in $(\mathbb{C}^*)^k \times \{0\}^{d-1-k}$ for some $k \in [d-1]$.

Then there exists a lifting of $(f_1, \ldots, f_d)$ to $\tilde{f}_1, \ldots, \tilde{f}_d, h_1, \ldots, h_k \in \mathbb{C}[x_1, \ldots, x_d, y_1, \ldots, y_k]$, explicitly given in the proof, satisfying

$$\text{MV}(N(\tilde{f}_1), \ldots, N(\tilde{f}_d), N(h_1), \ldots, N(h_k)) < \text{MV}(N(f_1), \ldots, N(f_d)).$$

Proof. The polynomials $f_1^n, \ldots, f_d^n$ are a system in the variables $x_1, \ldots, x_{d-1}$ and have a common root $\alpha \in (\mathbb{C}^*)^k \times \{0\}^{d-1-k}$. We define a map $\phi(f_1, \ldots, f_d)$ that sends any tuple $(g_1, \ldots, g_d) \in \mathbb{C}[N(f_1), \ldots, N(f_d)]$ to the tuple

$$\tilde{g}_1(x_1, \ldots, x_d, y_1, \ldots, y_k) = y_1 q_1^1 + \cdots + y_k q_k^1 + r_1 + (g_1 - g_1^n),$$

$$\vdots$$

$$\tilde{g}_d(x_1, \ldots, x_d, y_1, \ldots, y_k) = y_1 q_1^d + \cdots + y_k q_k^d + r_d + (g_d - g_d^n),$$

$$\tilde{h}_1(x_1, \ldots, x_d, y_1, \ldots, y_k) = y_1 - (x_1 - \alpha_1),$$

$$\vdots$$

$$\tilde{h}_k(x_1, \ldots, x_d, y_1, \ldots, y_k) = y_k - (x_k - \alpha_k).$$

where $q_1^j, \ldots, q_k^j$ are the quotients from polynomial division of $g_i^n$ by $(x_1 - \alpha_1), \ldots, (x_k - \alpha_k)$ in this order (and with respect to an arbitrary monomial order). The polynomial $r_i$ is the remainder and therefore a polynomial only in the variables $x_{d+1}, \ldots, x_{d-1}$. It is straightforward to see that $\phi(f_1, \ldots, f_d)$ maps a system to a lifting in the sense of Definition 2.3. For $i \in [d]$ define $P_i$ as in Lemma 3.7 and for $j \in [k]$ let

$$\Delta_j := \text{conv}(0, e_j, e_{d+j}) = N(h_j).$$

By Lemma 3.2, the image of $\phi(f_1, \ldots, f_d)$ lies in the vector space $\mathbb{C}[P_1, \ldots, P_d, \Delta_1, \ldots, \Delta_k]$. Let $U \subset \mathbb{C}[P_1, \ldots, P_d, \Delta_1, \ldots, \Delta_k]$ be the Zariski-open subset for which the coefficients of all $y_i$ in $h_j$ are non-zero. Consider the map which solves the $h_j$, giving $y_i = x_i - \alpha_i$, and substitutes this in the first $d$ equations. Since the facial subsystem $f_1^n, \ldots, f_d^n$ is saturated, this is a well-defined map $\tilde{\phi}(f_1, \ldots, f_d): U \to \mathbb{C}[N(f_1), \ldots, N(f_d)]$. It is also solution-preserving. Proposition 2.6 then yields the equality $\text{MV}(P_1, \ldots, P_d, \Delta_1, \ldots, \Delta_k) = \text{MV}(N(f_1), \ldots, N(f_d))$.

It is left to show that the Newton polytopes of the system

$$\phi(f_1, \ldots, f_d)(f_1, \ldots, f_d) = (\tilde{f}_1, \ldots, \tilde{f}_d, h_1, \ldots, h_k)$$

have strictly lower mixed volume than $(P_1, \ldots, P_d, \Delta_1, \ldots, \Delta_k)$. Set $u := -(e_{d+1} + \cdots + e_d + \cdots + e_{d+k}) \in \mathbb{R}^{d+k}$. Since $\alpha$ is a solution of $(f_1^n, \ldots, f_d^n)$, we have $r_1(\alpha) = \cdots = r_d(\alpha) = 0$. Hence, the polynomials $\tilde{f}_1, \ldots, \tilde{f}_d$ do not have a constant term and their Newton polytopes do not contain the origin. Therefore, in the notation of Theorem 2.8, one has $T_u = \{d+1, \ldots, d+k\}$. Moreover, $\Delta_j^\alpha = [0, e_j]$ for $j \in [k]$, and so the tuple $(P_1, \ldots, P_d, \Delta_1^\alpha, \ldots, \Delta_k^\alpha)$ is essential by Lemma 3.7. This implies $\text{MV}(N(f_1), \ldots, N(f_d)) < \text{MV}(N(f_1), \ldots, N(f_d))$ by Theorem 2.8. \hfill \qed

Remark 3.10. The assumption of saturatedness of the lifted facial subsystem is necessary, as otherwise the image of $\tilde{\phi}(f_1, \ldots, f_d)$ generically does not lie in $\mathbb{C}[N(f_1), \ldots, N(f_d)]$. 

Remark 3.11. Analyzing the constraints on the coefficients imposed in the polynomial division, it can be seen that the lifting in Theorem 3.9 decreases the mixed volume by exactly one. However, this theorem should not be the final word. Theorem 5.1 contains a quantitative version in the bivariate case and lifting with more general polynomials could be more efficient.

Remark 3.12. Several assumptions of Theorem 3.9 seem restrictive but do not sacrifice generality. Transforming a system using monomial changes of variables allows to apply Theorem 3.9 for a considerably larger class of systems. Whenever any facial subsystem has a solution, one can transform the system so that this subsystem is obtained in direction $u = -e_d$ and contains a constant term. Moreover, this transformation can be chosen so that the entire system consists of polynomials with non-negative exponents. Furthermore, saturatedness of the Newton polytopes can sometimes be attained by applying further transformations. It is an interesting open problem to give an explicit characterization of tuples of lattice polytopes that can be transformed to a saturated tuple using affine unimodular transformations.

Remark 3.13. Definition 2.3 allows lifted solutions to lie in $(\mathbb{C}^*)^d \times \mathbb{C}^k$. In particular, certain torus-solutions of the original system may correspond to non-torus solutions of the lifting. This can cause problems in applying the lifting strategy of Theorem 3.9 to solve a concrete non-generic system. Assume that, in the setting of the above theorem, the facial subsystem in direction $u$ has a solution at $x_1 = \alpha_1, \ldots, x_k = \alpha_k$ and there exists a torus solution $s$ of the full system with $s_i = \alpha_i$ for some $i \in [k]$. Then $s$ corresponds to a non-torus solution in the lifting because one of the $g_i$ vanishes. However, these non-torus solutions are not polyhedral in the sense that a small perturbation of coefficients turns them into torus solutions.

Remark 3.14. One drawback of the lifting in Theorem 3.9 is that it adds $k$ polynomials for a mixed volume reduction of one. In general, it would be interesting to explore the liftings of a given system that only add a single polynomial. A first insight is that there is no point in substituting a monomial. Indeed, consider a lifting of the form

$$
\begin{align*}
\tilde{g}_1(x_1, \ldots, x_d, y) &= yq_1 + r_1 \\
\tilde{g}_d(x_1, \ldots, x_d, y) &= yq_d + r_d \\
h(x_1, \ldots, x_d, y) &= y - x_1^{a_1} \cdots x_d^{a_d}
\end{align*}
$$

where $q_i$ and $r_i$ are the quotients and remainders from polynomial division of $g_i$ by $h$, i.e. replacing $x_1^{a_1} \cdots x_d^{a_d}$ by $y$ where possible. By the projection formula for mixed volumes (see e.g. [Sch14, Theorem 5.3.1]) such a lifting does not alter the mixed volume. Theorem 3.9 substitutes binomials, which struck us as the next logical step. Buying into some complications on the polyhedral side, more complicated liftings seem feasible. For example, it would be interesting to investigate liftings in which the new polynomial is of the form $p_1 y + p_2$ with $p_1, p_2 \in \mathbb{C}[x_1, \ldots, x_d]$ where $p_1$ is not constant.

4. LIFTING USING DEPENDENCY IN THE FACIAL SUBSYSTEM

In this section we present a different type of lifting that does not assume the facial subsystem to be saturated. It does not use polynomial division as in Theorem 3.9, but exploits dependencies in the facial subsystem. The lifting replaces two linearly dependent facial polynomials by a new variable. For simplicity, we formulate it for $f_i^n = \lambda f_j^n$ but more complicated dependencies of the $f_i^n$ can be also exploited under more restrictive conditions. See Remark 4.2.

Theorem 4.1. Let $f_1, \ldots, f_d \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_d^{\pm 1}]$ and $u \in \mathbb{R}^d \setminus \{0\}$ such that furthermore:

(i) $f_i^n = \lambda f_j^n$,
(ii) $\text{MV}(N(f_j^n), N(f_k^n), \ldots, N(f_d^n)) > 0$,
(iii) $f_1 \neq f_2^n$. 

\begin{itemize}
\item[(i)]
\item[(ii)]
\item[(iii)]
\end{itemize}
Then there exists a lift of the system with polynomials \( \tilde{f}_1, \tilde{f}_2, f_3, \ldots, f_d, h \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_{d+1}^{\pm 1}] \), explicitly given in the proof, satisfying
\[
\text{MV}(N(\tilde{f}_1), N(f_2), N(f_3), \ldots, N(f_d), N(g)) < \text{MV}(N(f_1), \ldots, N(f_d)).
\]

Proof. Without loss of generality, potentially after a monomial change of variables, we assume \( u = -e_d \) and \( \max_{p \in N(f_i)} (u, p) = 0 \) for all \( i \in [d] \). Then the polynomials \( f_1^u, \ldots, f_d^u \) are a system in the variables \( x_1, \ldots, x_{d-1} \). Let \( \phi(f_1, \ldots, f_d) \) send any system \( (g_1, \ldots, g_d) \in \mathbb{C}[N(f_1), \ldots, N(f_d)] \) to the system \( (\tilde{g}_1, \tilde{g}_2, g_3, \ldots, g_d, h) \), where:
\[
\tilde{g}_1(x_1, \ldots, x_d, y) = y + g_1^u - f_1^u + (g_1 - g_1^u),
\]
\[
\tilde{g}_2(x_1, \ldots, x_d, y) = \lambda y + g_2^u - f_2^u + (g_2 - g_2^u),
\]
\[
h(x_1, \ldots, x_d, y) = y - f_1^u.
\]
One verifies \( \phi(f_1, \ldots, f_d)(g_1, \ldots, g_d) \) is a lift in the sense of Definition 2.3. For \( i = 1, 2 \), let
\[
P_i := \text{conv}\left( \{ e_{d+1} \} \cup (N(f_i) \times \{ 0 \}) \right),
\]
and furthermore
\[
\Delta = \text{conv}\left( \{ e_{d+1} \} \cup (N(f_i^u) \times \{ 0 \}) \right).
\]
Then the image of \( \phi(f_1, \ldots, f_d) \) lies in the vector space \( \mathbb{C}[P_1, P_2, N(f_3), \ldots, N(f_d), \Delta] \). Let \( U \subset \mathbb{C}[P_1, P_2, N(f_3), \ldots, N(f_d), \Delta] \) be the Zarsiki-open subset of systems for which the coefficient of \( y \) in \( h \) does not vanish. Then the map \( \phi(f_1, \ldots, f_d) : U \rightarrow \mathbb{C}[N(f_1), \ldots, N(f_d)] \), solving \( h = 0 \) and substituting \( y \) into the first two polynomials is solution preserving. Proposition 2.6 therefore implies
\[
\text{MV}(P_1, P_2, N(f_3), \ldots, N(f_d), \Delta) = \text{MV}(N(f_1), \ldots, N(f_d)).
\]

It is left to show that the Newton polytopes of the system
\[
\phi(f_1, \ldots, f_d)(f_1, \ldots, f_d) = (\tilde{f}_1, \tilde{f}_2, f_3, \ldots, f_d, h),
\]
have strictly lower mixed volume than \( (P_1, P_2, N(f_3), \ldots, N(f_d), \Delta) \). Set \( v = -e_d - e_{d+1} \). By construction, \( N(\tilde{f}_1) \) and \( N(\tilde{f}_2) \) only contain monomials in which \( x_d \) occurs. All other monomials are subtracted in the construction of the lifting. Then, in the notation of Theorem 2.8, one has \( T_v = \{ 3, \ldots, d+1 \} \). Moreover one has
\[
(P_1, P_2, N(f_3)^v, \ldots, N(f_d)^v, \Delta^v) = (P_1, P_2, N(f_3)^u, \ldots, N(f_d)^u, N(f_2^u)),
\]
which is essential. To see this we use assumption (ii) to produce \( d-1 \) linearly independent segments in \( N(f_3^u), \ldots, N(f_d^u), N(f_2^u) \). Additionally there are one segment in \( P_1 \) between a point in \( N(f_1^u) \) and a point in \( N(f_1) \) (by (iii)), and a segment in \( P_2 \) between a point in \( N(f_2^u) \) and the point \( e_{d+1} \). The whole collection of segments is linearly independent. Theorem 2.8 implies
\[
\text{MV}(N(\tilde{f}_1), N(\tilde{f}_2), N(f_3), \ldots, N(f_d), \Delta) < \text{MV}(P_1, P_2, N(f_3), \ldots, N(f_d), \Delta).
\]

Remark 4.2. Theorem 4.1 can be generalized to other linear dependencies among the \( f_i^u \). If for some \( i \in [d] \) one has \( f_i^u = \sum_{j \neq i} \lambda_j f_j^u \) and \( N(f_i^u) \supset N(f_j^u) \) for all \( j \neq i \), then the resubstitution using \( \tilde{\phi} \) has the correct codomain. Making the formulation more technical, also more general linear dependencies could be used for lifting.

Example 4.3. Consider the following polynomial system in three variables:
\[
f_1(x_1, x_2, x_3) = 1 + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 + x_1 x_3 + x_2 x_3,
\]
\[
f_2(x_1, x_2, x_3) = 1 + x_1^2 x_2^2 + x_1^2 x_3^2 + 2x_2^2 + x_1 x_3 + x_2 x_3,
\]
\[
f_3(x_1, x_2, x_3) = 2 + x_1 x_2 + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 + x_1 x_3 + x_2 x_3.
\]
The mixed volume of the Newton polytopes of the \( f_i \) is 16. Denote by \( f_i^u(x_1, x_2) \) the bracketed subsystems which are the facial subsystems with respect to \( u = (0, 0, -1) \). They have the common solutions \( (\pm \sqrt{2}, \pm i\sqrt{2}) \). It also holds that \( f_1^u = f_2^u \), so the first condition of Theorem 4.1 is satisfied.
The second condition is satisfied as well because the three Newton polytopes of the $f_i^n$ agree, so the mixed volume is just the lattice volume of some full-dimensional polytope, hence positive. The substitution $y = f_1^n(x_1, x_2) = f_2^n(x_1, x_2)$ gives the following lifted system in 4 variables:

\[
\begin{align*}
\tilde{f}_1(x_1, x_2, x_3, y) &= y + x_1^2 + x_1 x_3 + x_2 x_3, \\
\tilde{f}_2(x_1, x_2, x_3, y) &= y + 2x_3^2 + x_1 x_3 + x_2 x_3, \\
\tilde{f}_3(x_1, x_2, x_3, y) &= \tilde{f}_3(x_1, x_2, x_3), \\
h(x_1, x_2, y) &= y - (1 + x_1^2 x_3^2 + x_2^2 x_3^2).
\end{align*}
\]

This system has mixed volume $12 < 16$. Still, this is not best possible as the number of torus solutions is smaller than 12.

5. Liftings for Bivariate Systems

In this section we generalize the lifting construction from Theorem 3.9 for bivariate systems, so that it yields a mixed volume reduction of more than one. In the bivariate case the facial subsystems are univariate and thus saturativeness is not an issue. For the proof of the following theorem we need another formula for the mixed volume which can be found, e.g., in [Sch14] and [BS19, Remark 2.3]. We only state it in the case of three lattice polytopes $P_1, P_2, P_3 \subseteq \mathbb{R}^3$. For any $v \in \mathbb{R}^3 \setminus \{0\}$ we denote by $\text{ht}_{P_i}(v)$ the maximum of $(v, p)$ for all $p \in P_i$. Then

\[
\text{MV}(P_1, P_2, P_3) = \sum_{v \in \mathbb{Z}^3 \setminus \{0\} \text{ primitive}} \text{ht}_{P_i}(v) \text{MV}_{v^+}(P_2^v, P_3^v),
\]

where $\text{MV}_{v^+}$ denotes the mixed volume with respect to the linear subspace $v^+$ with its induced lattice, and $P_2^v$ and $P_3^v$ denote the faces of $P_2$ and $P_3$ maximizing $v$, viewed (after a lattice translation) as subsets of $v^+$. In contrast to the proofs of Theorems 3.9 and 4.1, the proof of the following theorem does not rely on the strict monotonicity criterion of [BS19].

**Theorem 5.1.** Let $f_1, f_2 \in \mathbb{C}[x_1, x_2]$ and $u := -e_2$. Assume that $f_1^n$ and $f_2^n$ have a non-zero constant term and that $m = \deg \gcd(f_1^n, f_2^n) > 0$. Then there exists a lift $\tilde{f}_1, \tilde{f}_2, g \in \mathbb{C}[x_1, x_2, y]$, explicitly given in the proof, satisfying

\[
\text{MV}(N(\tilde{f}_1), N(f_2), N(g)) \leq \text{MV}(N(f_1), N(f_2)) - m.
\]

**Proof.** We adapt the proof of Theorem 3.9. Here $\phi_{(f_1, f_2)}$ maps $g_1, g_2$ to

\[
\begin{align*}
\tilde{g}_1(x_1, x_2, y) &= yq_1 + r_1 + (g_1 - g_1^n), \\
\tilde{g}_2(x_1, x_2, y) &= yq_2 + r_2 + (g_2 - g_2^n), \\
h(x_1, x_2, y) &= y - \gcd(f_1^n, f_2^n),
\end{align*}
\]

where now $q_j$ and $r_j$ are quotient and remainder of the division of $f_j^n$ by $\gcd(f_1^n, f_2^n)$. Again, $\phi_{(f_1, f_2)}$ lifts $(g_1, g_2)$ in the sense of Definition 2.3. The system $\phi_{(f_1, f_2)}(g_1, g_2)$ lies in $\mathbb{C}[P_1, P_2, \Delta]$, where

\[
\begin{align*}
P_1 &:= \text{conv}\left(\{0\} \cup \langle e_3 + [0, (\deg(f_1^n) - m)e_1] \cup N(f_1 - f_1^n)\right), \\
P_2 &:= \text{conv}\left(\{0\} \cup \langle e_3 + [0, (\deg(f_2^n) - m)e_1] \cup N(f_2 - f_2^n)\right), \\
\Delta &:= \text{conv}\left(\{e_3\} \cup [0, me_1]\right).
\end{align*}
\]

Let $U \subset \mathbb{C}[P_1, P_2, P_3]$ be the Zariski-open subset for which the coefficient of $y$ in $h$ is non-zero. The generic resubstitution $\phi_{(f_1, f_2)}: U \to \mathbb{C}[N(f_1), N(f_2)]$ which solves $h = 0$ for $y$ and plugs the result into $\tilde{g}_1$ and $\tilde{g}_2$ is well defined. Therefore, Proposition 2.6 yields the equality $\text{MV}(P_1, P_2, \Delta) = \text{MV}(N(f_1), N(f_2))$. We now show that the mixed volume decreases by $m$ upon lifting. The remainder $r_i$ in $\tilde{f}_i$ vanishes for $i = 1, 2$. Without loss of generality one can assume that $x_2$ occurs in $f_2$ as it occurs in $f_1$ or $f_2$ or we had a univariate system to start with. Let $k \geq 1$ be minimal such that
In order to determine a lifting as in Theorem 5.1, it suffices to compute the gcd of two univariate polynomials via the Euclidean algorithm. In particular, it is not necessary to compute the common roots of the facial subsystem as in Theorem 3.9.

References

[AD10] Jan Albersmeyer and Moritz Diehl. The lifted Newton method and its application in optimization. SIAM Journal on Optimization, 20(3):1655–1684, 2010.
[Ber75] David N. Bernstein. The number of roots of a system of equations.

[BS19] Frédéric Bihan and Ivan Soprunov. Criteria for strict monotonicity of the mixed volume of convex polytopes. Advances in Geometry, 19(4):527–540, 2019.
[BT18] Paul Breiding and Sascha Timme. HomotopyContinuation.jl: A package for homotopy continuation in Julia.

[CLO05] David A. Cox, John Little, and Donal O’Shea. Using algebraic geometry, volume 185 of Graduate Texts in Mathematics. Springer, New York, second edition, 2005.

[HS95] Birkett Huber and Bernd Sturmfels. A polyhedral method for solving sparse polynomial systems. Mathematics of computation, 64(212):1541–1555, 1995.

[Lai17] Pierre Lairez. A deterministic algorithm to compute approximate roots of polynomial systems in polynomial average time. Foundations of computational mathematics, 17(5):1265–1292, 2017.

[Sch14] Rolf Schneider. Convex bodies: the Brunn-Minkowski theory, volume 151 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, expanded edition, 2014.

[Ver11] Jan Verschelde. Polynomial homotopy continuation with PHCpack. ACM Communications in Computer Algebra, 44(3/4):217–220, 2011.