An expansion for the sum of a product of an exponential and a Bessel function. II

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Abstract

We examine the sum of a decaying exponential (depending non-linearly on the summation index) and a Bessel function in the form

$$\sum_{n=1}^{\infty} e^{-an^p} \frac{J_\nu(an^p x)}{\left(\frac{1}{2} an^p x\right)^\nu} \quad (x > 0),$$

where $J_\nu(z)$ is the Bessel function of the first kind of real order $\nu$ and $a$ and $p$ are positive parameters. By means of a Mellin transform approach we obtain an asymptotic expansion that enables the evaluation of this sum in the limit $a \to 0$. A similar result is derived for the sum when the Bessel function is replaced by the modified Bessel function $I_\nu(z)$ when $x \in (0, 1)$. The case of even $p$ is of interest since the expansion becomes exponentially small in character. We demonstrate that in the case $p = 2$, a result analogous to the Poisson-Jacobi transformation exists for the above sum.

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1. Introduction

We consider the sum

$$S_{\nu,p}(a, x) = \sum_{n=1}^{\infty} e^{-an^p} \frac{J_\nu(an^p x)}{\left(\frac{1}{2} an^p x\right)^\nu}, \quad p > 0, \ x > 0, \ a > 0, \quad (1.1)$$

where $J_\nu(x)$ is the Bessel function of the first kind of real order $\nu$ and

$$\left(\frac{1}{2}z\right)^{-\nu}J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-z^2/4)^k}{\Gamma(1+\nu+k)k!}.$$

Our interest herein is the asymptotic expansion of $S_{\nu,p}(a, x)$ in the limit $a \to 0$ when convergence of the above sum becomes slow. We employ a Mellin transform approach to express the sum as a Mellin-Barnes integral involving the Riemann zeta function $\zeta(s)$ and the Gauss hypergeometric function $\mathbf{2}_1F_1$. It is found that the resulting (algebraic) asymptotic series converges when $p < 1$ and diverges when $p > 1$; the case $p = 1$ requires the condition $a < 2\pi/\sqrt{1+x^2}$ for convergence.

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In [2] the asymptotic expansion of the sum
\[ \sum_{n=1}^{\infty} e^{-an^2} \frac{J_\nu(bn)}{(\frac{1}{2}bn)^\nu} \]
was examined for \( a \to 0 \) with \( 0 < b < 2\pi \) and \( \nu > -\frac{1}{2} \). The details in this case relied on the use of a double Mellin-Barnes integral involving the zeta function when \( a < b \). When \( a = 0 \), the sum has been considered by Tričković et al. in [4], where approaches using Poisson’s summation formula and Bessel’s integral were employed to derive convergent expansions.

An interesting situation arises when \( p \) is an even integer in the sum (1.1), where the algebraic asymptotic series vanishes to leave an exponentially small contribution. We pay particular attention to the case \( p = 2 \), where it will be demonstrated that a transformation analogous to the well-known Poisson-Jacobi transformation [6, p. 124]
\[ \sum_{n=1}^{\infty} e^{-an^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}} - \frac{1}{2} + \frac{\sqrt{\pi}}{a} \sum_{n=1}^{\infty} e^{-\pi^2n^2/a} \quad (\Re(a) > 0) \quad (1.2) \]
holds for the sum in (1.1).

2. The asymptotic expansion of \( S_{\nu,p}(a, x) \) for \( a \to 0 \)

Let \( a > 0, x > 0, p > 0 \) and \( \nu \) be a real parameter. We consider the asymptotic expansion of the sum
\[ S_{\nu,p}(a, x) = \sum_{n=1}^{\infty} e^{-an^p} \frac{J_\nu(an^p x)}{(\frac{1}{2}an^p x)^\nu} \quad (2.1) \]
for \( a \to 0 \). We adopt a Mellin transform approach as described, for example, in [3, p. 118] and write the above sum as
\[ S_{\nu,p}(a, x) = (\frac{1}{2}x)^{-\nu} \sum_{n=1}^{\infty} h(an^p), \quad h(\tau) := e^{-\tau} \tau^{-\nu} J_\nu(x\tau). \]

With the Mellin transform given by \( H(s) = \int_0^\infty \tau^{s-1} h(\tau) d\tau \), we have [5, p. 385(2)]
\[ H(s) = \int_0^\infty \tau^{s-\nu-1} e^{-\tau} J_\nu(x\tau) d\tau = \frac{(\frac{1}{2}x)^\nu}{\Gamma(1 + \nu)} \Gamma(s) \quad 2F_1\left(\frac{1}{2}s, \frac{1}{2}s + \frac{1}{2}; 1 + \nu; -x^2\right), \]
where \( \Gamma(s) \) is the Gauss hypergeometric function [1, p. 384]. Then upon use of the Mellin inversion theorem [3, p. 118] we obtain the integral representation in the form
\[ S_{\nu,p}(a, x) = (\frac{1}{2}x)^{-\nu} \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} H(s)(an^p)^{-s} ds = \frac{(\frac{1}{2}x)^{-\nu}}{2\pi i} \int_{c-i\infty}^{c+i\infty} H(s) \zeta(sp) a^{-s} ds \]
\[ = \frac{1}{\Gamma(1 + \nu)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \zeta(sp) 2F_1\left(\frac{1}{2}s, \frac{1}{2}s + \frac{1}{2}; 1 + \nu; -x^2\right) a^{-s} ds, \quad (2.2) \]
where \( \zeta(s) \) is the Riemann zeta function and the integration path is such that \( c > 1/p \).

The integrand in (2.2) has simple poles at \( s = 1/p, s = 0 \) and, in general, at \( s = -k (k = 1, 2, \ldots) \). However, if \( p \) is an odd integer, the poles in \( \Re(s) < 0 \) are at \( s = -1, -3, \ldots \) on account of the trivial zeros of \( \zeta(s) \) at \( s = -2, -4, \ldots \) And if \( p \) is an even integer there are no poles in \( \Re(s) < 0 \).

Displacement of the integration path to the left over the poles (when \( p \neq 2, 4, \ldots \)) then formally produces
\[ S_{\nu,p}(a, x) = \frac{1}{\Gamma(1 + \nu)} \left\{ \frac{1}{p} a^{-1/p} \Gamma(1/p) 2F_1\left(\frac{1}{2p}, \frac{1}{2p} + \frac{1}{2}; 1 + \nu; -x^2\right) - \frac{1}{2} + \Upsilon(a) \right\} \quad (2.3) \]
with
\[ \Upsilon(a) = \sum_{k=1}^{\infty} \frac{(-a)^k}{k!} \zeta(-kp) _2F_1(-\frac{1}{2}, k; 1 + \nu; -x^2), \]
where we have used the fact that \( \zeta(0) = -\frac{1}{2} \). From the functional relation [1, p. 603]
\[ \zeta(s) = 2^s \pi^{s-1} \zeta(1-s) \Gamma(1-s) \sin \frac{1}{2} \pi s, \] we find
\[ \zeta(-kp) = -\left(\frac{2\pi}{p}\right)^{-kp} \zeta(1+kp) \Gamma(1+kp) \sin \left(\frac{1}{2} \pi kp\right) \]
so that the residue sum \( \Upsilon(a) \) can be written alternatively as
\[ \Upsilon(a) = -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-a)^k}{k!} \left(\frac{a}{(2\pi)^p}\right)^k \zeta(1+kp) \Gamma(1+kp) \sin \left(\frac{1}{2} \pi kp\right) \]
\[ \times _2F_1(-\frac{1}{2}k, -\frac{1}{2}k + \frac{1}{2}; 1 + \nu; -x^2). \] (2.5)

Throughout the paper we define the acute angle \( \phi := \arctan x \). To discuss the convergence of the series (2.5) we require the large-\( k \) behaviour of the above hypergeometric function. From (A.2), this is given by
\[
\frac{1}{\Gamma(1+\nu)} _2F_1(-\frac{1}{2}k, -\frac{1}{2}k + \frac{1}{2}; 1 + \nu; -x^2) 
\sim \frac{(1+x^2)^{k+\nu/2+3/4}}{\sqrt{\pi(xk)^{\nu+1/2}}} \sin((k+\nu+\frac{3}{2})\phi - \frac{1}{2} \pi \nu + \frac{1}{4} \pi) \quad (k \to \infty).
\]
Hence we see that the late terms in \( \Upsilon(a) \) are controlled in absolute value by
\[
\frac{(1+x^2)^{k/2}}{k^{\nu+1/2}} \frac{\Gamma(1+kp)}{k!} \left(\frac{a \sqrt{1+x^2}}{(2\pi)^p}\right)^k
\]
since \( \zeta(1+kp) \approx 1 \) for \( k \to \infty \). Consequently the residue sum \( \Upsilon(a) \) converges absolutely when \( p < 1 \) and diverges when \( p > 1 \). When \( p = 1 \), the sum (where only odd values of \( k \) contribute) is absolutely convergent provided \( a < 2\pi/\sqrt{1+x^2} \) and divergent otherwise. This result may be summarised in the following theorem:

**Theorem 1.** For \( p > 0 \) (when \( p \neq 2, 4, \ldots \)), \( a > 0 \), \( x > 0 \), and \( \nu \) real the following expansion holds:
\[ S_{\nu,p}(a, x) = \frac{1}{\Gamma(1+\nu)} \left\{ \frac{1}{p} a^{-1/p} \Gamma(1/p) _2F_1(\frac{1}{2p}, \frac{p+1}{2p}; 1 + \nu; -x^2) - \frac{1}{2} \right\} \]
\[ -\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-a)^k}{k!} \left(\frac{a}{(2\pi)^p}\right)^k \zeta(1+kp) \Gamma(1+kp) \sin \left(\frac{1}{2} \pi kp\right) _2F_1(-\frac{1}{2}k, -\frac{1}{2}k + \frac{1}{2}; 1 + \nu; -x^2). \] (2.6)

When \( p < 1 \), the expansion (2.6) is an equality, but is asymptotic when \( p > 1 \). When \( p = 1 \), absolute convergence of the infinite series holds when \( a < 2\pi/\sqrt{1+ x^2} \), otherwise it is divergent.

In the following subsection we analyse the case \( p = 1 \) more carefully.

### 2.1 The case \( p = 1 \)

When \( p = 1 \), the poles in \( \mathcal{R}(s) < 0 \) are situated at \( s = -1, -3, \ldots \). We displace the integration path in (2.2) to the left to coincide with the path \( s = -2N + it \), \( t \in (-\infty, \infty) \), where \( N \) is a positive integer. We find
\[ S_{\nu,1}(a, x) = \frac{1}{\Gamma(1+\nu)} \left\{ \frac{1}{a} _2F_1(\frac{1}{2}, 1 + \nu; -x^2) - \frac{1}{2} \right\} \]
\[+ \frac{1}{\pi} \sum_{k=0}^{N-1} (-k) \zeta(2k+2) \frac{2F_1(-k, -k - \frac{1}{2}; 1 + \nu; -x^2)}{a^{2k+1}} + R_N(a),\]

where the remainder \(R_N(a)\) is

\[
R_N(a) = \frac{1}{2\pi i} \int_{c-2N+i\infty}^{c-2N+i\infty} \Gamma(s) \zeta(s) \frac{2F_1(\frac{1}{2}s, \frac{1}{2}s + \frac{1}{2}; 1 + \nu; -x^2)}{a^{-s}} ds
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(-2N + it) \zeta(-2N + it) \frac{2F_1(-N + \frac{1}{2}it, -N + \frac{1}{2} + \frac{1}{2}it; 1 + \nu; -x^2)}{a^{2N-it}} dt.
\]

Upon observing from (2.4) that

\[
\Gamma(-s) \zeta(-s) = -\frac{(2\pi)^s}{2\cos \frac{1}{2}\pi s},
\]

we find that

\[
|R_N(a)| < \frac{1}{4\pi} \frac{a^{2N}}{2\pi} \zeta(1 + 2N) \int_0^\infty \frac{|2F_1(-N + \frac{1}{2}it, -N + \frac{1}{2} + \frac{1}{2}it; 1 + \nu; -x^2)|}{\cosh \frac{1}{2}\pi t} dt.
\]

From (A.6) the modulus of the hypergeometric function satisfies the bound \(K(1 + x^2)^N e^{-\phi t}/(N^2 + \frac{a^2}{4}t^2)^{\nu/2+1/4}\), where \(K\) is a positive constant and \(0 < \phi < \frac{1}{2}\pi\). Hence

\[
|R_N(a)| < \frac{K}{2\pi} \frac{a\sqrt{1 + x^2}}{2\pi} \zeta(1 + 2N) \int_0^\infty \frac{\cos \phi t}{\cosh \frac{1}{2}\pi t} \frac{dt}{(N^2 + \frac{a^2}{4}t^2)^{\nu/2+1/4}} = O\left(\frac{(a\sqrt{1 + x^2})^{2N}}{2\pi}\right).
\]

Hence, as \(N \to \infty\) the remainder \(R_N(a) \to 0\) when \(a < 2\pi/\sqrt{1 + x^2}\). If this last condition is not met the series is asymptotic in character. Then we have the exact result

\[
S_{\nu,1}(a, x) = \frac{1}{\Gamma(1 + \nu)} \left\{ \frac{1}{a} \frac{2F_1(\frac{1}{2}, 1; 1 + \nu; -x^2)^2}{2\pi} - \frac{1}{2} + \frac{1}{\pi} \sum_{k=0}^{\infty} (-k) \zeta(2k+2) \frac{2F_1(-k, -k - \frac{1}{2}; 1 + \nu; -x^2)^2}{a^{2k+1}} \right\} (a < 2\pi/\sqrt{1 + x^2}).
\]

As an example, the special case \(\nu = -\frac{1}{2}\), where \(J_{-1/2}(anx)/(\frac{1}{2}anx)^{-1/2} = \cos(anx)/\sqrt{\pi}\) and

\[2F_1(-k, -k - \frac{1}{2}; \frac{1}{2}; -x^2) = (1 + x^2)^{k+1/2} \cos(2(k + \frac{1}{2})\phi),\]

yields

\[
S_{-\frac{1}{2},1}(a, x) = \frac{1}{\sqrt{\pi}} \left\{ \frac{1}{a(1 + x^2)} - \frac{1}{2} + \frac{1}{\pi} \sum_{k=0}^{\infty} (-k) \zeta(2k+2) \cos(2(k + \frac{1}{2})\phi) X^{2k+1} \right\},
\]

where \(X := a\sqrt{1 + x^2}/(2\pi)\). Evaluation of the sum then produces

\[
S_{-\frac{1}{2},1}(a, x) = \frac{1}{\sqrt{\pi}} \left\{ \frac{1}{4} \left( \coth(\pi X e^{i\phi}) + \coth(\pi X e^{-i\phi}) \right) - \frac{1}{2} \right\} = \frac{1}{2\sqrt{\pi}} \left\{ \frac{\sinh a}{\sinh a - \cos ax} - 1 \right\}
\]

\[
= \frac{1}{\sqrt{\pi}} \left\{ \frac{e^a \cos ax - 1}{1 - 2e^a \cos ax + e^{2a}} \right\}.
\]

This last result is readily verified to be the case by some straightforward algebra applied to the sum \(\pi^{-1/2} \sum_{n\geq1} e^{-an} \cos(anx)\).
3. The case $p = 2$

In the case $p = 2$, we have from (2.2) the result

$$S_{\nu,2}(a, x) = \frac{1}{\Gamma(1 + \nu)} \left\{ \frac{1}{2} \sqrt{\frac{\pi}{a}} \text{$_2$F}_1 \left( \frac{1}{4}, \frac{3}{4}; 1 + \nu; -x^2 \right) - \frac{1}{2} \right\}$$

$$+ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(-s)\zeta(-2s) \text{$_2$F}_1 \left( -\frac{1}{2}s, -\frac{1}{2}s + \frac{1}{2}; 1 + \nu; -x^2 \right) a^s ds \right\}, \quad (3.1)$$

where in the integral we have put $s \to -s$ and $c > 0$. Since there are no poles of the integrand in $\Re(s) > 0$ (due to the trivial zeros of $\zeta(-2s)$) the expansion $Y(a)$ vanishes. This indicates that this contribution is exponentially small in the limit $a \to 0$.

Noting from (2.4) that

$$\Gamma(-s)\zeta(-2s) = \pi^{-2s-1/2}\zeta(1 + 2s)\Gamma(s + \frac{1}{2}),$$

we express the integral (with the further change of variable $s \to u - \frac{1}{2}$) as

$$\sqrt{\frac{\pi}{a}} \int_{c-i\infty}^{c+i\infty} \left( \frac{\pi^2}{a^2} \right)^{-u} Y(u)\zeta(2u) \text{$_2$F}_1 \left( -\frac{1}{2}u + \frac{1}{4}, -\frac{1}{2}u + \frac{3}{4}; 1 + \nu; -x^2 \right) du \quad (c > \frac{1}{2})$$

$$= \sqrt{\frac{\pi}{a}} \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \frac{\pi^2}{a^2} \right)^{-u} Y(u) \text{$_2$F}_1 \left( -\frac{1}{2}u + \frac{1}{4}, -\frac{1}{2}u + \frac{3}{4}; 1 + \nu; -x^2 \right) du. \quad (3.2)$$

From the form\(^1\) of the integrand in (3.2) we may expect a generalised Poisson-Jacobi transformation to hold for $S_{\nu,2}(a, x)$.

To demonstrate that this is the case in a particular example we consider $\nu = -\frac{1}{2}$, where

$$\text{$_2$F}_1 \left( -\frac{1}{2}u + \frac{1}{4}, -\frac{1}{2}u + \frac{3}{4}; \frac{1}{2}; -x^2 \right) = (1 + x^2)^{-u/2-1/4} \cos(u - \frac{1}{2})\phi.$$

Then the integral in (3.2) becomes

$$\frac{1}{(1 + x^2)^{1/4}} \int_{c-i\infty}^{c+i\infty} Y^{-u} \Gamma(u) \cos((u - \frac{1}{2})\phi) du, \quad Y := \frac{\pi^2n^2}{a\sqrt{1 + x^2}}$$

$$= \frac{1}{(1 + x^2)^{1/4}} \sum_{k=0}^{\infty} \frac{(-Y)^k}{k!} \cos((k + \frac{1}{2})\phi) = \frac{1}{(1 + x^2)^{1/4}} e^{-Y \cos \phi} \cos(Y \sin \phi - \frac{1}{2}) \phi.$$

Since

$$\text{$_2$F}_1 \left( \frac{1}{4}, \frac{3}{4}, \frac{1}{2}; -x^2 \right) = \frac{\cos \frac{1}{2} \phi}{(1 + x^2)^{1/4}},$$

we finally have the expansion

$$S_{-1/2,2}(a, x) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} e^{-an^2} \cos(an^2 x) = \frac{1}{\sqrt{\pi}} \left\{ \frac{1}{2} \sqrt{\frac{\pi}{a}} \cos \frac{1}{2} \phi \right\}$$

$$+ \sqrt{\frac{\pi}{a}} \left\{ \frac{1}{(1 + x^2)^{1/4}} \sum_{n=1}^{\infty} \exp \left[ -\frac{\pi^2n^2}{a(1 + x^2)} \right] \cos \left( \frac{\pi^2n^2x}{a(1 + x^2)} - \frac{1}{2} \phi \right) \right\}, \quad (3.3)$$

which is a generalised Poisson-Jacobi transformation. This result, however, is easily verified upon use of (1.2) applied to the sum $\pi^{-1/2} \sum_{n \geq 1} e^{-an^2} \cos(an^2 x)$.

\(^1\)In the absence of the $\text{$_2$F}_1$ function, the integral in (3.2) can be evaluated by the Cahen-Mellin integral as $\exp(-\pi^2n^2/a)$; see, for example, [3, p. 80].
3.1 The general case

We study the general case of the integral in (3.2), viz.

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \frac{\pi^2 n^2}{a} \right)^{-u} \Gamma(u) {}_2F_1 \left( -\frac{1}{2} u + \frac{1}{4}, -\frac{1}{2} u + \frac{3}{4}; 1 + \nu; -x^2 \right) du \quad (c > 0)
\]

\[
= \sum_{k=0}^{\infty} \frac{(-\chi)^k}{k!} {}_2F_1 \left( \frac{1}{2} k + \frac{1}{4}, \frac{1}{2} k + \frac{3}{4}; 1 + \nu; -x^2 \right), \quad \chi := \frac{\pi^2 n^2}{a}
\]

upon displacement of the integration path to the left over the poles of \( \Gamma(u) \). When \( x^2 < 1 \), we can series expand the hypergeometric function to obtain

\[
\sum_{k=0}^{\infty} \frac{(-\chi)^k}{k!} \sum_{r=0}^{\infty} \frac{(k + \frac{1}{2})_{2r}}{(1 + \nu)_r r!} (-x^2/4)^r = \frac{1}{\sqrt{\pi}} \sum_{r=0}^{\infty} \Gamma(2r + \frac{1}{2}) \Gamma(2r + 1) (x^2/4)^r \sum_{k=0}^{\infty} \frac{(-\chi)^k (2r + \frac{1}{2})_k}{k!(\frac{1}{2})_k}.
\]

The inner sum can be expressed as a confluent hypergeometric function in the form

\[
{}_1F_1 (2r + \frac{1}{2}; \frac{1}{2}; -\chi) = e^{-\chi} {}_1F_1 (-2r; \frac{1}{2}; \chi) = e^{-\chi} \frac{(2r)!}{(4r)!} H_{4r}(\sqrt{\chi})
\]

by application of Kummer’s theorem and \( H_n(x) \) is the Hermite polynomial [1, p. 328]. Thus the integral in (3.4) can be evaluated as

\[
\sqrt{\pi a} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 / a} P_\nu \left( x, \frac{\pi^2 n^2}{a} \right),
\]

where

\[
P_\nu \left( x, \frac{\pi^2 n^2}{a} \right) := \sum_{r=0}^{\infty} H_{4r}(\pi n/\sqrt{a}) (1 + \nu)_r r! (-x^2/64)^r.
\]

The form (3.5) (with \( x^2 < 1 \)) demonstrates the Poisson-Jacobi-type structure for \( S_{\nu,2}(a,x) \) in (3.1). However, we have been unable to express \( P_\nu (x, \pi^2 n^2 / a) \) in a simpler, more recognisable form.

4. Two generalisations

An extension of the sum in (1.1) is given by

\[
S_{\nu,p}^\mu (a,x) = \sum_{n=1}^{\infty} e^{-an^p \nu} J_\nu (an^p x) \frac{x^\nu}{(2in^p x)^\nu-\mu},
\]

where \( \mu \) is real. The same procedure employed in Section 2 yields the integral representation

\[
S_{\nu,p}^\mu (a,x) = \frac{(\frac{1}{2})^\mu}{\Gamma(1 + \nu)} \frac{1}{2\pi i} \Gamma(s + \mu) \zeta(sp) \int_{c-i\infty}^{c+i\infty} \Gamma(s + \mu - \nu, 1 + \nu; -x^2) a^{-s} ds,
\]

where \( c > \max\{1/p, -\mu\} \). Poles of the integrand are situated at \( s = 1/p \) and \( s = -\mu - k \), although some poles can be deleted on account of the trivial zeros of \( \zeta(sp) \) and it is possible to have a double pole at \( s = 1/p \) when \( \mu = -1/p, -1/p - 1, \ldots \). As an example, we display the particular case \( p = 2 \) and \( \mu \neq -\frac{1}{2}, -\frac{3}{2}, \ldots \), to obtain

\[
S_{\nu,p}^\mu (a,x) = \frac{(\frac{1}{2})^\mu}{\Gamma(1 + \nu)} \Gamma(\mu + \frac{1}{2}) \frac{1}{2^{\nu / 2}} {}_2F_1 \left( \frac{\nu + 1}{2}, \frac{1}{2} \mu + \frac{3}{4}; 1 + \nu; -x^2 \right)
\]
\[-\sin \frac{\pi \mu}{\pi} \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{a}{(2\pi)^2} \right)^{n+k} \zeta(1+2\mu+2k)\Gamma(1+2\mu+2k)_{2}F_{1}(-\frac{1}{2}k; -\frac{1}{2}k + \frac{1}{2}; 1 + \nu; -x^2) \}, \]

(4.2)

where we have made use of the functional relation in (2.4).

When \( \mu = 1 \) it is seen that the infinite sum of residues in (4.2) vanishes, which provides us with another example of a generalised Poisson-Jacobi transformation. When \( \nu = \frac{1}{2} \), we obtain

\[ S_{1/2,2}(a, x) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} e^{-an^2} \sin(an^2x) = \frac{x}{\sqrt{\pi}} \left\{ \frac{1}{4} \sqrt{\pi} a \left( \frac{1}{2} \right)_{2}F_{1} \left( \frac{1}{4}, \frac{3}{4}; \frac{1}{2}; -x^2 \right) + R(a) \right\} \]

\[ = \frac{1}{\sqrt{\pi}} \left\{ \frac{1}{2} \sqrt{\frac{\pi}{a}} \sin \frac{\pi}{2} \phi + xR(a) \right\}. \]

(4.3)

Here

\[ R(a) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(1-s)\zeta(-2s)_{2}F_{1} \left( \frac{1-s}{2}, \frac{2-s}{2}; \frac{3}{2}; -x^2 \right) a^s ds \quad (c > 0) \]

\[ = \sqrt{\pi} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \frac{\pi^2}{a} \right)^{-u} \Gamma(u)\zeta(2u)_{2}F_{1} \left( \frac{1}{2}, \frac{3}{4}; \frac{1}{2}; -x^2 \right) du, \]

since from (2.4)

\[ \Gamma(1-s)\zeta(-2s) = -\frac{(u - \frac{1}{2})}{\pi^s a^{1/2}} \Gamma(u)\zeta(2u) \quad (s \to u - \frac{1}{2}). \]

Making use of the identity

\[ _{2}F_{1} \left( -\frac{1}{2}u + \frac{3}{4}, -\frac{1}{2}u + \frac{5}{4}; \frac{3}{2}; -x^2 \right) = (1 + x^2)^{u/2 - 1/4} \frac{\sin((u - \frac{1}{2})\phi)}{x(u - \frac{1}{2})}, \]

we then obtain

\[ xR(a) = -\sqrt{\pi} \frac{1}{a} \left( 1 + x^2 \right)^{1/4} \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( \frac{\pi^2 n^2}{a \sqrt{1 + x^2}} \right)^{-u} \Gamma(u) \sin((u - \frac{1}{2})\phi) du \]

\[ = \sqrt{\pi} \frac{1}{a} \left( 1 + x^2 \right)^{1/4} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{(-Y)^k}{k!} \sin((k + \frac{1}{2})\phi), \quad Y := \frac{\pi^2 n^2}{a \sqrt{1 + x^2}}. \]

Then since

\[ \sum_{k=0}^{\infty} \frac{(-Y)^k}{k!} \sin((k + \frac{1}{2})\phi) = -e^{-Y} \cos \phi \sin(Y \sin \phi - \frac{1}{2} \phi), \]

we finally obtain

\[ xR(a) = -\sqrt{\pi} \frac{1}{a} \left( 1 + x^2 \right)^{1/4} \sum_{n=1}^{\infty} \exp \left[ -\frac{\pi^2 n^2}{a(1 + x^2)} \right] \sin \left( \frac{\pi^2 n^2 x}{a(1 + x^2)} - \frac{1}{2} \phi \right). \]

(4.4)

Combination of (4.3) and (4.4) then gives the expansion of \( S_{1/2,2}(a, x) \) for \( a > 0 \) and clearly has the form of a generalised Poisson-Jacobi transformation. This result is easily verified upon use of (1.2) applied to the sum \( \pi^{-1/2} \sum_{n \geq 1} e^{-an^2} \sin(an^2x) \).

Another extension is the sum

\[ T_{\nu,p}(a, x) = \sum_{n=1}^{\infty} e^{-an^p} \frac{I_{\nu}(an^p x)}{(\frac{2}{a} n^p x)^\nu}, \quad x \in (0, 1), \]

\[ = \frac{1}{\Gamma(1+\nu)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \zeta(s)_{2}F_{1} \left( \frac{1}{4}, \frac{3}{4}; \frac{1}{2} + \frac{1}{2}; \frac{1}{2} + \nu; x^2 \right) a^{-s} ds \quad (c > 1/p), \]

(4.5)
where $I_0(z)$ is the modified Bessel function of the first kind. Then we obtain the same expansion given in (2.3) and (2.5) with the argument of the hypergeometric function replaced by $x^2$. When $0 < p < 1$ the infinite series of residues is convergent but divergent (asymptotic) when $p > 1$.

The special case $p = 1$ yields

$$T_{\nu,1}(a, x) = \frac{1}{\Gamma(1 + \nu)} \left\{ \frac{1}{a} {2F_1}(\frac{1}{2}, 1; 1 + \nu; x^2) - \frac{1}{2} \right\}$$

$$+ \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \left( \frac{a}{2\pi} \right)^{2k+1} \zeta(2k + 2) {2F_1}(-k, -k + \frac{1}{2}; 1 + \nu; x^2) \right\}. \quad (4.6)$$

From (A.5), the leading large-$k$ behaviour of the above hypergeometric function is given by

$$2F_1(-k, -k + \frac{1}{2}; 1 + \nu; x^2) \sim \frac{\Gamma(1 + \nu)}{2\sqrt{\pi}} \frac{(1 + x)^{2k+\nu+3/2}}{(kx)^{\nu+1/2}} \quad (k \to \infty)$$

so that the infinite sum in (4.6) converges when $a < 2\pi/(1 + x)$. If $a(1 + x)/(2\pi) = 1$, we have convergence of the sum for $\nu > -\frac{1}{2}$, since it is easily seen that the hypergeometric series is positive for $k \geq 1$.

### 5. Concluding remarks

In Theorem 1 we have presented the expansion of $S_{\nu,p}(a, x)$ for $p > 0$ ($p \neq 2, 4, \ldots$) with the parameters $a > 0$ and $x > 0$. This expansion is convergent when $p < 1$ but is asymptotic when $p > 1$. In the case $p = 1$ convergence requires the condition $a < 2\pi/\sqrt{1 + x^2}$. When $p$ is an even integer the character of the expansion changes to become exponentially small. In two cases when $p = 2$ it was possible to display a generalised Poisson-Jacobi transformation, although both follow in a straightforward manner from (1.2). An attempt at the general case managed to establish the standard infinite sum of exponentials of the form $\exp[-\pi^2n^2/a]$ in (3.5), although the factor $P_{\nu}(x, \pi^2n^2/a)$ could not be simplified.

The situation when $p = 4$, or higher, is more complicated and preliminary investigation suggests a composite of expansions each containing the exponential factor $\exp[-\pi^2n^2/a]$. We do not consider this situation further here, nor how the expansion might change as $p \to 2$.

### Appendix: The large-$k$ behaviour of some hypergeometric functions

In this appendix we determine the large-$k$ asymptotic behaviour of the hypergeometric functions appearing in the main body of the paper. We first consider the function

$$F \equiv 2F_1(-k, -k + \frac{1}{2}; 1 + \nu; -x^2) \quad (k \to +\infty), \quad (A.1)$$

where $x^2 > 0$. From [1, pp. 388, 390] we have

$$F = (1 + x^2)^{2k+\nu+3/2} 2F_1(k + \nu + 1; k + \nu + \frac{1}{2}; 1 + \nu; -x^2)$$

$$= (1 + x^2)^{2k+\nu+3/2} \frac{\Gamma(1 + \nu)\Gamma(k + 1)}{2\pi i \Gamma(k + \nu + 1)} \int_0^{(1+)} \frac{t^{k+\nu}(t - 1)^{-k-1}}{(1 + x^2 t)^{k+\nu+3/2}} dt, \quad (A.2)$$

where the integration path is a closed contour starting and finishing at $t = 0$ that encircles $t = 1$ in the positive sense. The above integral can be written as

$$\frac{1}{2\pi i} \int_0^{(1+)} e^{\psi(t)} f(t) dt,$$
Then we obtain the leading large-\( k \) behaviour given by

\[
2F_1(-k, -k - \frac{1}{2}; 1 + \nu; -x^2) \sim \frac{\Gamma(1 + \nu)}{2\sqrt{\pi}} \frac{(1 + x^2)^{k+\nu/2+3/4}}{(xk)^{\nu+1/2}} \sin\left((2k + \nu + \frac{3}{2})\phi - \frac{1}{2}\pi\nu + \frac{1}{4}\pi\right) \quad (k \to \infty). \tag{A.4}
\]

In the special case \( \nu = -\frac{1}{2} \), (A.4) yields

\[
2F_1(-k, -k - \frac{1}{2}; 1 + \nu; -x^2) \sim (1 + x^2)^{k+1/2} \cos((2k + 1)\phi),
\]

which is the exact result.

For the function with positive argument

\[
2F_1(-k, -k - \frac{1}{2}; 1 + \nu; x^2) \quad x \in (0, 1),
\]

the procedure is similar, where now \( \psi(t) = \log[(t/((t - 1)(1 - x^2)t)] \) with saddles at \( \pm 1/x \). The integration path can be deformed to pass over the saddle at \( t_s = 1/x \), where the direction of integration at the saddle is \( \frac{1}{2}\pi - \frac{1}{4}\arg \psi''(t_s) = \frac{1}{4}\pi + \phi \); the steepest descent path through the saddle at \( -i/x \) is the reflection of the path in the upper half-plane. The integration path can then be expanded to infinity around an infinite arc to pass over both saddles, which contribute equally to the integral.

Application of the saddle-point method [1, p. 47] shows that the contribution to the integral from the saddle \( t_s = i/x \) is

\[
-\sqrt{\frac{2\pi}{k(-\psi''(t_s))}} f(t_s)e^{k\psi(t_s)} = -\sqrt{\frac{\pi}{k}} \frac{(1 + x^2)^{-k-\nu/2-3/4}}{x^{\nu+1/2}} \exp\left[-i(2k + \nu + \frac{3}{2})\phi + \frac{1}{2}\pi i(\nu - \frac{1}{2})\right]
\]

with the conjugate expression from the saddle \( t_s = -i/x \). Hence, noting that the ratio of gamma functions multiplying the integral in (A.2) is \( \Gamma(1 + \nu)k^{-\nu} \) for large \( k \), we obtain the final result

\[
2F_1(-k, -k - \frac{1}{2}; 1 + \nu; -x^2) \sim \frac{\Gamma(1 + \nu)}{\sqrt{\pi}} \frac{(1 + x^2)^{k+\nu/2+3/4}}{(xk)^{\nu+1/2}} \sin\left((2k + \nu + \frac{3}{2})\phi - \frac{1}{2}\pi\nu + \frac{1}{4}\pi\right) \quad (k \to \infty).
\]

Finally, we require an estimate for

\[
|2F_1(-N + \frac{1}{2}it, -N + \frac{1}{2} + \frac{1}{2}it; 1 + \nu; -x^2)|, \quad t \in (-\infty, \infty)
\]
that appears in the estimation of the remainder term $R_N(a)$ in (2.7). The procedure follows that employed in the estimation of (A.1) with $k$ replaced by $N - \frac{1}{2}it$. The modulus of the exponential factor in (A.3) becomes $e^{-\phi t}/(1 + x^2)^N$, with the result that

$$|_{2F1}(-N + \frac{1}{2}it, -N + \frac{1}{2} + \frac{1}{2}it; 1 + \nu; -x^2)| \leq \frac{K(1 + x^2)^N e^{-\phi t}}{(N^2 + \frac{1}{4}t^2)^{\nu/2 + 1/4}}, \quad (A.6)$$

where $K$ is a positive constant.

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