Exact ground states of a frustrated 2D magnet: deconfined fractional excitations at a first order quantum phase transition

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We introduce a frustrated spin 1/2 Hamiltonian which is an extension of the two dimensional $J_1 - J_3$ Heisenberg model. The ground states of this model are exactly obtained at a first order quantum phase transition between two regions with different valence bond solid order parameters. At this point, the low energy excitations are deconfined spinons and spin-charge separation occurs under doping in the limit of low concentration of holes. In addition, this point is characterized by the proliferation of topological defects that signal the emergence of $Z_2$ gauge symmetry.

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Frustrated magnets are the focus of considerable attention because exotic quantum effects are expected to emerge from the competition between two or more opposite tendencies. While several models in this category are solvable in one dimension, the list is much smaller for higher dimensions. One of the most studied frustrated magnets is the spin 1/2 Heisenberg model with first and second nearest neighbor interactions $J_1$ and $J_2$. In one dimension, this model exhibits a quantum transition as a function of $J_2/J_1$ from a critical state with quasi-long range antiferromagnetic (AF) order to a dimerized phase. Moreover, the exact dimerized ground state has been obtained for the point $J_2/J_1 = 0.5$ by Majumdar and Ghosh [1]. In contrast, two dimensional (2D) frustrated magnets like the $J_1 - J_2$ Heisenberg model on a square lattice still hold many secrets. Different approaches predict a transition between a Néel ordered state and a gapped (non-magnetic) quantum phase for the region $0.4 \leq J_2/J_1 \leq 0.6$. However, the nature of this phase is still debated. More precisely, the question is whether it is a uniform spin liquid [2,3] or a spatially ordered valence bond crystal [4,5,6,7].

The interest in frustration and magnetism is motivated not only by this widely debated question. There are reasons to believe that frustrated magnets may exhibit fractionalized excitations similar to those which appear in the fractional quantum Hall effect. The interest in phases with fractionalized excitations is generated by the increasing number of experimental results showing new physical behaviors in strongly correlated systems. For instance, the normal state of the high temperature superconductors does not exhibit electron-like quasiparticles in its spectrum. Recently, different scenarios were proposed for the realization of points with deconfined fractional excitations. Senthil et al [8] proposed that deconfined critical points may describe quantum phase transitions between Néel ordered states and valence bond solids (VBS) in frustrated magnets. In a second scenario based on the study of models for quantum dimers, the deconfinement point separates two VBS and the spectrum of fractional excitations consists of spinless particles (“photons”) with a quadratic dispersion [9]. Very recently, Tsvelik [10] argued that none of the previous scenarios is realized for a family of frustrated spin Hamiltonians called confederate flag models [11]. As we will see below, his alternative scenario for the deconfinement point at the transition between two VBS has many analogies with the case that we study in the present paper.

In addition, during the last decade attention has focused on the study of inhomogeneous structures that are proposed to emerge from competing interactions. These textures are generated by the proliferation and eventual ordering of one dimensional (1D) topological defects. A prominent example is provided by the stripe phase proposed to exist in the high temperature superconductors. In this case, each stripe is an antiphase boundary for the antiferromagnetic order parameter. In general, it is difficult to prove the existence and understand the origin of these phases due to the complexity of the underlying frustrated model.

In this paper, we will consider the $J_1 - J_2$ Heisenberg Hamiltonian on a square lattice with an additional term that makes the model quasi-exactly solvable for the fully frustrated point $J_2/J_1 = 0.5$. At this point, some of the exact ground states are valence bond crystals with deconfined fractional $S = 1/2$ excitations (spinons). In addition, spin-charge separation occurs if the system is doped with a low concentration of holes. We also show that this particular point has an emergent $Z_2$ gauge symmetry [12] and can be associated with a first order quantum phase transition between two different valence bond orderings. The physical manifestation of the emergent gauge symmetry is a divergent susceptibility for the creation of 1D topological defects that can be identified with twin-boundaries of an underlying orientational ordering. We will see that the common origin of these exotic behaviors is a dynamical decoupling of the 2D magnet into 1D structures.

We will start by considering the following $S = 1/2$ Hamiltonian on a square lattice:

$$ H = J_1 \sum_{\langle i,j \rangle} {S_i} \cdot {S_j} + J_2 \sum_{\langle i,j \rangle} {S_i} \cdot {S_j} + K \sum_{\alpha} (p_{\alpha i}^{\alpha} p_{\alpha j}^{\alpha} + \ h. c. ), $$

where $\langle i,j \rangle$ and $\langle \langle i,j \rangle \rangle$ denote nearest neighbors and second nearest neighbors respectively. The index $\alpha$ denotes the sites of the dual lattice (plaquettes) and $ijkl$ are the four spins of the corresponding plaquette in cyclic order. Note that the plaquette interaction of $H$ is similar to the one introduced by a four cyclic exchange (the only difference is in the sign of the
last term) [13]. In particular, we will consider here the fully frustrated point \( J_2 = J_1/2 \) and \( K = J_1/8 \). For this set of parameters and up to an irrelevant constant the Hamiltonian \( H \) can be rewritten as:

\[
H_p = \frac{3J_1}{2} \sum_\alpha \mathcal{P}_\alpha ,
\]

The operator \( \mathcal{P}_\alpha \) projects the spin state of the plaquette \( \alpha \) onto the subspace with total spin \( S_\alpha^z = 2 \).

**Exact ground states of** \( H_p \). It is clear that any state having at least one singlet dimer per plaquette is a ground state of \( H_p \). This is because \( H_p \) is positive semi-definite and the condition of at least one singlet in the plaquette \( \alpha \) implies that \( S_\alpha^z \leq 1 \) (there is no \( S_\alpha^z = 2 \) component). The same procedure gives rise to the Majumdar-Ghosh [1] and the Affleck-Kennedy-Lieb-Tasaki (AKLT) [14] exact ground states for \( S = 1/2 \) and \( S = 1 \) chains respectively, and to more general ideas for constructing solvable 2D models [15, 16].

We found two different families of states that fulfill the condition of having at least one singlet dimer per plaquette. The first family is generated by the state which is illustrated in Fig.1. These states are simply products of local singlet dimers which are represented by ellipsoids. In other words, the singlet dimers are completely localized and there is emergent \( U(1) \) gauge symmetry associated with this localization [12]. Any array of dimers along a given diagonal direction, \((1,1)\) or \((1,1)\), can be rotated by \( \pi/2 \) as indicated by the arrows of Fig.1. It is important to note that successive rotations have to be done along the same diagonal direction. The degeneracy of this family is \( 2^{N_d+1} \) where \( N_d \propto \sqrt{N_s} \) is the number of diagonal chains and \( N_s \) is the total number of sites. This degeneracy can be associated with a \( Z_2 \) gauge *emergent symmetry* that changes the dimerized order parameter of each individual diagonal zig-zag chain. By \( Z_2 \) gauge symmetry we mean a local symmetry transformation that acts on each individual zig-zag chain mapping one of the two possible dimerized states into the other one. In other words, this family of ground states is formed by configurations with parallel diagonal arrays of vertical or horizontal dimers.

There is an alternative way of viewing the local \( Z_2 \) emergent symmetry. We can imagine that our system has an underlying orientational ordering given by the staggered dimer phase illustrated in Fig.1. The two possible orientations are horizontal and vertical. The energy cost for creating a twin-boundary or inter-phase between the two different orientations is equal to zero. Consequently, the system has a divergent susceptibility for the creation of 1D topological defects, meaning that a weak coupling with another field can easily stabilize a particular array of twin boundaries. If the system is still invariant under translations, the resulting array of topological defects will be periodic. The ordering of 1D topological defects has been proposed in the past as a possible outcome of competing interactions in the high temperature superconductors [12].

The second family (Figs.1, c and d) appears when we force the states to have interfaces between vertical and horizontal configurations along the two diagonal directions \((1,1)\) and \((1,1)\). Under this condition the state is forced to create a defect at the intersection between the two diagonal interfaces. In particular, it is possible to have an \( S = 1/2 \) defect or localized *spinon* as is illustrated in Fig.1. It is easy to prove that there is no ground state with more than one localized spinon. Since the position of the defect is arbitrary, the degeneracy of each of these configurations is proportional to \( N_s \).

![FIG. 1: Different ground states of \( H_p \). The ellipsoids represent singlet dimers.](image)

**Deconfined fractional excitations.** What are the low energy excitations of the ground states of Fig.1? Let us first consider the family of solutions illustrated in Fig.1. As we can see in Fig.1, if we excite one singlet dimer into a triplet state, the two parallel \( S = 1/2 \) spins can propagate along the diagonals without an energy cost proportional to the separation between them. Consequently, the two spinon excitations are *deconfined*. Note that when the spinons propagate along one diagonal zig-zag chain they do not “see” the other chains because the two possible dimerized configurations (horizontal and vertical) have exactly the same energy. In other words, the effective dimensionality of the low energy spectrum is dynamically reduced from \( D=2 \) to \( D=1 \). The emergent \( Z_2 \) gauge symmetry that we mentioned above is the mathematical manifestation of this dynamical decoupling. The \( Z_2 \) dimerized order parameter of each diagonal zig-zag chain is decoupled from the corresponding order parameter of the other chains. The most notorious physical consequence is the emergence of fractional excitations which are characteristic of 1D systems.

Note that the previous analysis is only valid when the spinons are moving along the two diagonal directions. If, for instance, they propagate horizontally, they will “feel” the confining interaction characteristic of 2D systems because the horizontal chains are not dynamically decoupled. Consequently, although our system is 2D, its low energy spinon excitations are free to move only along 1D paths. The same type of deconfined spinon excitations are obtained for the second family of ground states.
A similar situation occurs when one hole is introduced (Fig. 3b). The charge and the spin of the added hole get deconfined because there is no energy cost for the string generated in between the two excitations. In other words, we can expect a non-Fermi liquid behavior when a magnetic system described by $H_y$ is doped away from half-filling. All of these “exotic” behaviors are just a consequence of a dynamical decoupling which is signaled by an emergent $Z_2$ gauge symmetry and that only occurs at the point under consideration: $J_2 = J_1/2$ and $K = J_1/8$. To understand what can be the physical role of this point we need to move away from it.

![Image](318x271 to 559x554)

**Fig. 3:** a) Staggered dimer phase (SD). b) Zig-zag dimer phase (ZD). c) The two order parameters change discontinuously at $g = 0$ indicating a first order quantum phase transition between the phases illustrated in a) and b).

**First order quantum phase transition.** The special point described by $H_y$ can be easily converted into a first order quantum phase transition point. Note that the two configurations of Fig. 3 are the periodic ground states with the shortest periods within the set shown in Fig. 4 Therefore, they are the leading candidates to remain as ground states when we depart from the $H_y$ point. For instance, we can add a term to the Hamiltonian which favors the staggered dimer ordering shown in Fig. 3 when the coupling constant $g$ is negative or the zig-zag dimer configuration of Fig. 3 when $g$ is positive. There are different realizations of such a term and we will not focus in any particular one. The two different dimer phases shown in Fig. 3 break simultaneously the translation and the rotation symmetry of the square lattice. The first dimer configuration (Fig. 3a) is four-fold degenerate while the second one (Fig. 3b) has an eight-fold degeneracy. The symmetry order parameters for each of these phases are:

$$D^s = \frac{1}{N_s} \sum_j (S_j \cdot S_{j+\eta}) e^{iQ \cdot r_j},$$

for the staggered dimer ordering of Fig. 3a and

$$D^z_x = \frac{1}{N_s} \sum_j (S_j \cdot S_{j+\hat{x}} + S_{j+2\hat{x}} \cdot S_{j+2\hat{x}+\hat{y}}) e^{i\hat{Q} \cdot r_j},$$

$$D^z_y = \frac{1}{N_s} \sum_j (S_j \cdot S_{j+\hat{y}} + S_{j+2\hat{y}} \cdot S_{j+2\hat{y}+\hat{x}}) e^{i\hat{Q} \cdot r_j},$$

where $Q = (\pi, \pi)$, $\bar{Q} = (\pi, -\pi)$, and $\eta = \{\hat{x}, \hat{y}\}$. Note that $(D^s_x, D^s_y) = (\pm 1, 0)$ or $(0, \pm 1)$ for the four equivalent ground states of the staggered dimer phase. For the zig-zag phase, the non-zero component of $(D^z_x, D^z_y)$ takes the four possible values $\{1, -1, -1, 1\}$ that are necessary to identify the eight equivalent configurations. The remaining symmetry group of the zig-zag phase, $G^{zz}$, is a subgroup of $G^{sst}$, the symmetry group of the staggered dimer phase. The level crossing that occurs at $g = 0$ between the staggered and the zig-zag dimer states gives rise to a first order quantum phase transition. In other words, at this point there is a discontinuous change of the order parameters $D^s$ and $D^{zz}$ (see Fig. 3). As we mentioned above, this level crossing is accompanied by the softening of the twin-boundary defects of the staggered dimer phase. Hence, we can think of the zig-zag dimer phase as a “condensation” of these twin-boundaries.
two chains has been reduced to zero and a $Z_2$ gauge symmetry emerges at the transition point. In general, we can say that these are quantum phase transitions between two broken symmetry states whose order parameters have something in common: given a particular decomposition of our 2D system into 1D chains described by some macroscopic variable, both order parameters characterize different inter-chain orderings.

For real systems, we do not expect the dynamical decoupling into 1D systems to be perfect. In general, there is always some residual interaction that makes this coupling weak but non-zero. A similar situation occurs in the materials that provide an experimental realization of one dimensional systems. The structure of these materials contains weakly coupled chains and the coupling becomes relevant when temperature is lower than some characteristic value $T_0$. However, the system still behaves as an array of decoupled chains for $T > T_0$. We expect the same behavior for 2D systems that get dynamically deconfined into 1D structures.

These ideas can be extended to other lattices. For instance, for a honeycomb lattice we can consider a Hamiltonian that is dynamically deconfined into 1D structures.

In summary, we introduced a simple extension of the $J_1-J_2$ Heisenberg model and we obtained different exact ground states for the fully frustrated point $J_2/J_1 = 0.5, K = J_1/8$. Both the ground states and their low energy excitations exhibit exotic behaviors like the softening of 1D topological defects and the emergence of deconfined fractional excitations. When some of the spins are replaced by holes, the phenomenon of spin-charge separation occurs in the limit of low concentration of holes. In addition, we showed that the point under consideration can be easily identified with a first order quantum phase transition.

The emergence of deconfined fractional excitations was recently proposed to occur at a critical point by different authors [8, 9, 10]. In particular, our model is an isotropic version of the anisotropic Confederate Flag model studied by Tsvelik [10]. By analyzing the four chain model, he finds that there are two VBS separated by an approximately (1+1) Lorentz invariant quantum critical point. However, the exponent that he obtains for the average dimerization is quite small, indicating that in the limit of infinite number of chains the transition could become first order as in the isotropic model discussed in this paper. Our results then show that this phenomenon is not restricted to quantum critical points. First order quantum phase transition points can also produce strong deviations from the normal properties of 2D systems. In our case, the common origin of these deviations is a dynamical decoupling of the 2D magnet into 1D systems. Such a dynamical decoupling was proposed for 2D strongly correlated models in the context of the high temperature superconductors [18]. As is well known, this effective reduction in the dimensionality of the system gives rise to completely different properties: the usual Fermi liquid is replaced by a Luttinger liquid and the magnetic spectrum is dominated by $S = 1/2$ spinon excitations.

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