ASYMPTOTICS OF THE BANANA FEYNMAN AMPLITUDES AT THE LARGE COMPLEX STRUCTURE LIMIT

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ABSTRACT. Recently Bönisch-Fischbach-Klemm-Nega-Osma-Safari \[13\] discovered, via numerical computation, that the leading asymptotics of the $l$-loop Banana Feynman amplitude at the large complex structure limit can be described by the Gamma class of a degree $(1, \ldots, 1)$ Fano hypersurface $F$ in $(\mathbb{P}^1)^{l+1}$. We confirm this observation by using a Gamma-conjecture type result \[14, 9, 11\] for $F$.

1. Introduction

The $l$-loop Banana Feynman amplitude (see \[14, (8.1)-(8.2)], \[3, (2.1)]\) is the integral

$$F(q, t) = \int_{(\mathbb{R}_{>0})^l} \frac{1}{t - \phi_q(y)} \frac{dy_1 \cdots dy_l}{y_1 \cdots y_l}$$

where $\phi_q$ is the Laurent polynomial

$$\phi_q(y) = (q_1 y_1 + \cdots + q_l y_l + q_{l+1})(y_1^{-1} + \cdots + y_l^{-1} + 1).$$

The parameters $q_i$, denoted by $\xi^2$ in \[3\], are the squares of the internal masses and $t$ is the square of the external momentum. When $t$ is a large positive number, the integrand has a pole along the hypersurface $(\phi_q(y) = t)$ and the integral diverges. We regularize the integral by means of analytic continuation: we know that the integral converges for $t < 0$, and it can then be analytically continued to the complex plane (of $t$) minus the branch cut $[T, \infty)$, where $T := (\sum_{i=1}^{l+1} \sqrt{q_i})^2 = \min\{\phi_q(y) : y \in (\mathbb{R}_{>0})^l\}.$

The Feynman amplitude can be regarded as a relative period of the mixed Hodge structure of the pair $(\mathbb{P}_\Delta \setminus M_{q,t}, \partial \mathbb{P}_\Delta \setminus M_{q,t})$, where $\mathbb{P}_\Delta$ is an $l$-dimensional toric variety such that $\phi_q^{-1}(t) \subset (\mathbb{C}^*)^l$ is compactified to an anticanonical hypersurface $M_{q,t} \subset \mathbb{P}_\Delta$ (intersecting every toric stratum properly) and $\partial \mathbb{P}_\Delta = \mathbb{P}_\Delta \setminus (\mathbb{C}^*)^l$ is the toric boundary. As such, it satisfies \textit{inhomogeneous} Picard-Fuchs differential equations with respect to the parameters $q$ and $t$, which extend the Picard-Fuchs equations for $M_{q,t} = \phi_q^{-1}(t)$. We refer the reader to \[14, 11\] and references therein for differential equations, Hodge-theoretic and arithmetic aspects of the Feynman amplitudes.

In the present notes, we study the asymptotics of the Banana Feynman amplitude $F(q, t)$ near the large complex structure limit $t = \infty$ (or equivalently $q_1 = \cdots = q_{l+1} = 0$) of $M_{q,t}$.

**Theorem 1.** Let $F$ be a degree $(1, \ldots, 1)$ Fano hypersurface in $(\mathbb{P}^1)^{l+1}$ and let $p_1, \ldots, p_{l+1} \in H^2(F)$ denote the hyperplane classes pulled back from $(\mathbb{P}^1)^{l+1}$. For $q_1, \ldots, q_{l+1}, t > 0$, we have

$$F(q, t + 10) \sim \frac{1}{t} \int_F e^{-p \log(q/t)} \cup \widehat{\Gamma}_F \cdot e^{\pm \pi i \gamma_1(F)} \Gamma(1 - c_1(F)) \quad \text{as } t \to \infty$$

where $p \log(q/t) = \sum_{i=1}^{l+1} p_i \log(q_i/t)$ and the sign depends on whether we perform the analytic continuation anti-clockwise or clockwise from a negative real $t$.

This follows from the power series expansion of $F(q, t)$ we give in Theorem 8 below. The class $\widehat{\Gamma}_F \in H^4(F)$ in the theorem is the Gamma class \[14, 3, 11\] of the tangent bundle.
$TF$; it is explicitly given as

$$
\widehat{\mathcal{F}}_F = \frac{\Gamma(1 + p_1)^2 \cdots \Gamma(1 + p_{l+1})^2}{\Gamma(1 + p_1 + \cdots + p_{l+1})} = \frac{e^{-2\gamma c_1(F)}}{\Gamma(1 + c_1(F))}
$$

where $\Gamma(1 + z) = \int_0^\infty e^{-t^2} dt$ is the Euler $\Gamma$-function (when evaluating it at a cohomology class, we take its Taylor expansion) and $\gamma = 0.577 \cdots$ is the Euler constant. We also note that $c_1(F) = p_1 + \cdots + p_{l+1}$.

Remark 2. Kerr [12, Example 9.10] outlined another way to evaluate the asymptotics of the $l$-loop Banana Feynman integral.

Remark 3. Theorem 5.7 confirms the numerical computation by Bönisch-Fischbach-Klemm-Nega-Safari [3, §3]. By taking the imaginary and the real parts of Theorem 5.7, we get the following asymptotics as $t \to \infty$:

$$
\Im \mathcal{F}(q, t - i0) \sim \frac{1}{t} \int_F e^{-p \log(q/t)} \bigcup \frac{\prod_{j=1}^{l+1} \Gamma(1 + p_j)^2}{\Gamma(1 + c_1(F))^2} \pi c_1(F)
$$

(4)

$$
\Re \mathcal{F}(q, t - i0) \sim \frac{1}{t} \int_F e^{-p \log(q/t)} \bigcup \widehat{\mathcal{F}}_W
$$

(5)

where $W \subset F$ is an anticanonical hypersurface, i.e. the intersection of two degree $(1, \ldots, 1)$ hypersurfaces in $(\mathbb{P}^1)^{l+1}$; this is a mirror of $M_{q,t}$. These asymptotics coincide with [3, (3.18), (3.36), (3.37)].

Remark 6. By the reflection principle, the imaginary part of $\mathcal{F}(q, t - i0)$ with $t > 0$ can be understood as the difference $\frac{1}{i2\pi} (\mathcal{F}(q, t - i0) - \mathcal{F}(q, t + i0))$ of two analytic continuations. This can then be equated with the residue integral

$$
\pi \int_{\phi_q^{-1}(t) \cap (\mathbb{R}_{>0})^l} \operatorname{Res} \left( \frac{1}{t - \phi_q(y)} \frac{dy_1 \cdots dy_l}{y_1 \cdots y_l} \right)
$$

over the vanishing cycle $\phi_q^{-1}(t) \cap (\mathbb{R}_{>0})^l \subset M_{q,t}$. The Calabi-Yau Gamma conjecture [8, 10] predicts that the asymptotics of such vanishing periods should be given by the Gamma class of the mirror partner $W$ of $M_{q,t}$, as in [3]; in the case at hand this has been proved in [10, Theorem 5.7]. On the other hand, the asymptotics [3] of the real part of $\mathcal{F}(q, t)$ discovered in [3] is slightly beyond the scope of the Calabi-Yau Gamma conjecture; it is related to (a degeneration of) the mixed Hodge structure (see also the recent work [4]). In this paper, we will derive this from the Fano Gamma conjecture [3, 11, 3, 10, 3].

Remark 7. We can interpret $\widehat{\mathcal{F}}_F \cdot \Gamma(1 - c_1(F))$ as the Gamma class of the total space $K_F$ of the canonical bundle of $F$. See [2] for the relation to local mirror symmetry.

2. Proof of the Asymptotics

Theorem 6 follows immediately from the following result (compare [14, Proposition 5.1]).

Theorem 8. Let $q_1, \ldots, q_{l+1}$ be positive real numbers. For $t \ll 0$, we have

$$
\mathcal{F}(q, t) = \frac{1}{t} \int_F I_W(q/(-t), -1) \cup \widehat{\mathcal{F}}_F \cdot \Gamma(1 - c_1(F))
$$

1The factor $e^{-2\gamma c_1(F)}$ is missing in the second expression of [14, (3.37)].
where $I_W(q, z)$ is the cohomology-valued hypergeometric series

$$I_W(q, z) = e^{p \log q / z} \sum_{d = (d_1, \ldots, d_{l+1}) \in \mathbb{N}^{l+1}} \frac{\prod_{i=1}^{d_1+\cdots+d_{l+1}} (p_1 + \cdots + p_{l+1} + k z)^2}{\prod_{i=1}^{d_1+1} \prod_{k=1}^{d_i} (p_i + k z)^2} q^d$$

with $p \log q = \sum_{i=1}^{l+1} p_i \log q_i$ and $q^d = q_1^{d_1} \cdots q_{l+1}^{d_{l+1}}$.

Remark 9. The hypergeometric series $I_W(q, z)$ is the Givental $I$-function for the anticanonical hypersurface $W \subset F$, which is mirror to $M_{q,t}$. Here we regard it as a function taking values in $H^*(F)$, rather than in $H^*(W)$.

A crucial observation [17, p.41] is the fact that the Laurent polynomial $\phi_q(y)$ is a mirror of the Fano manifold $F$. The Givental mirror [4, p.150, equation (**) of the $(1, \ldots, 1)$-hypersurface $F \subset (\mathbb{P}^1)^{l+1}$ is given by the oscillatory integral

$$\int_{C \subset \{u_1 + \cdots + u_{l+1} = 1\}} e^{-\left(\frac{u_1}{y_1} + \cdots + \frac{u_{l+1}}{y_{l+1}}\right)} \frac{d u_1 \cdots d u_{l+1}}{d(u_1 + \cdots + u_{l+1})}.$$ 

By the Przyjalkowsky change of variables [13]

$$u_1 = \frac{y_1}{1 + y_1 + \cdots + y_l}, \quad \cdots \quad u_l = \frac{y_l}{1 + y_1 + \cdots + y_l}, \quad u_{l+1} = \frac{1}{1 + y_1 + \cdots + y_l},$$

the above oscillatory integral can be rewritten as

$$\int_{C'} e^{-\left(1 + y_1 + \cdots + y_l\right)} \frac{d y_1 \cdots d y_l}{y_1 \cdots y_l}.$$ 

The phase function equals the Laurent polynomial $-\phi_q(y)$ after the change of variables $y_i \to y_i^{-1}$. Therefore, the Gamma-conjecture type result [10, Theorem 5.7] implies that we have

$$\int_{(\mathbb{R}_{>0})^l} e^{-\phi_q(y)} \frac{d y_1 \cdots d y_l}{y_1 \cdots y_l} = \int_F I_F(q, -1) \cup \hat{F}$$

for $q_1, \ldots, q_{l+1} > 0$, where $I_F$ is the Givental $I$-function [4] for $F$

$$I_F(q, z) = e^{p \log q / z} \sum_{d = (d_1, \ldots, d_{l+1}) \in \mathbb{N}^{l+1}} \frac{\prod_{i=1}^{d_1+\cdots+d_{l+1}} (p_1 + \cdots + p_{l+1} + k z)^2}{\prod_{i=1}^{d_1+1} \prod_{k=1}^{d_i} (p_i + k z)^2} q^d.$$ 

We substitute $r q_i$ for $q_i$ in the equation (10) and perform the Laplace transformation with respect to $r$. We find

$$\int_0^\infty e^{r t} dr \int_{(\mathbb{R}_{>0})^l} e^{-\phi_{r q}(y)} \frac{d y_1 \cdots d y_l}{y_1 \cdots y_l} = \int_0^\infty e^{r t} dr \int_F I_F(r q, -1) \cup \hat{F}$$

for $t < 0$. A similar computation appeared in [10, Section 5.1]. Using $\phi_{r q}(y) = r \phi_q(y)$ and performing the integration in $r$ first\footnote{This is legitimate, because the integrand is a positive continuous function.} the left-hand side yields the Feynman amplitude

$$\int_{(\mathbb{R}_{>0})^l} \left(\int_0^\infty e^{(t-\phi_q(y)) r} dr\right) \frac{d y_1 \cdots d y_l}{y_1 \cdots y_l} = - \int_{(\mathbb{R}_{>0})^l} \frac{1}{t-\phi_q(y)} \frac{d y_1 \cdots d y_l}{y_1 \cdots y_l} = - F(q, t).$$

The right-hand side can be computed termwise, using

$$\int_0^\infty e^{r t} \prod_{i=1}^{l+1} (r q_i)^{d_i-p_i} dr = \Gamma(1 + \sum_{i=1}^{l+1} (d_i-p_i)) \frac{q^{d-p}}{(-t)^{1+\sum_{i=1}^{l+1} (d_i-p_i)}}.$$ 

Note that the coefficient of $q^d$ in the series $I_F(q, -1)$ has the norm bounded by $C^{1+|d|/|d|!}$ for some $C > 1$, where $|d| = d_1 + \cdots + d_{l+1}$. From this it follows that, for a sufficiently negative $t \ll 0$, we can interchange the sum over $d$ and the integral and that the right-hand side of (11) converges; in particular the left-hand side also does. This proves Theorem 8.
Acknowledgements. I thank Albrecht Klemm for inviting me to think about the Banana Feynman amplitudes and Matt Kerr for helpful comments. This research is supported by JSPS Kakenhi Grant Number 16H06335, 16H06337, 20K03582.

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