Measure-valued branching processes associated with Neumann nonlinear semiflows

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Abstract. We construct a measure-valued branching Markov process associated with a nonlinear boundary value problem, where the boundary condition has a nonlinear pseudo monotone branching mechanism term \( -\beta \), which includes as a limit case \( \beta(u) = -u^m \), with \( 0 < m < 1 \). The process is then used in the probabilistic representation of the solution of the parabolic problem associated with a nonlinear Neumann boundary value problem. In this way the classical association of the superprocesses to the Dirichlet boundary value problems also holds for the nonlinear Neumann boundary value problems. It turns out that the obtained branching process behaves on the measures carried by the given open set like the linear continuous semiflow, induced by the reflected Brownian motion, while the branching occurs on the measures having non-zero traces on the boundary of the open set, with the behavior of the \( (-\beta) \)-superprocess, having as spatial motion the process on the boundary associated to the reflected Brownian motion.

Mathematics Subject Classification (2010): 60J80, 35J25, 60J45, 60J35, 47D07, 60J50, 31B20.

Key words: Neumann nonlinear semiflow, branching process, nonlinear semigroup, negative definite function, reflected Brownian motion, boundary process.

1 Introduction

Let \( \mathcal{O} \) be a bounded, open subset of \( \mathbb{R}^d \), \( d \geq 1 \), with smooth boundary \( \Gamma \) (for instance, of class \( C^2 \)). Consider the nonlinear parabolic problem

\[
\begin{align*}
\frac{\partial u}{\partial t} - \frac{1}{2} \Delta u + \alpha u &= 0 \quad \text{in } (0, \infty) \times \mathcal{O}, \\
\frac{\partial u}{\partial \nu} + \beta(u) &= g \quad \text{on } \Gamma, \\
u(0, \cdot) &= f \quad \text{in } \mathcal{O},
\end{align*}
\]

(1.1)

where \( \frac{\partial}{\partial \nu} \) is the outward normal derivative to the boundary \( \Gamma \) of \( \mathcal{O} \), \( g \) is a positive continuously differentiable function on \( \Gamma \), \( f \in C(\overline{\mathcal{O}}) \), \( \alpha \in \mathbb{R}_+^* \), and \( \beta : \mathbb{R} \to \mathbb{R}_- \) is the following

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continuous mapping

\[
\beta(u) = \begin{cases} 
\int_0^\infty (e^{-su} - 1)\eta(ds) - bu, & \text{if } u \geq 0, \\
0, & \text{if } u < 0,
\end{cases}
\]

with \(\eta\) a positive measure on \(\mathbb{R}_+\) such that \(\int_{\mathbb{R}_+} s \eta(ds) < \infty\) and \(b \in \mathbb{R}_+\).

We assume that

\[
\int_{\mathbb{R}_+} s \eta(ds) + b \leq \gamma := \inf_{v \in H^1(\Omega)} \frac{1}{2} \|\nabla v\|^2_{L^2(\Omega)} + \alpha \|v\|^2_{L^2(\Gamma)}.
\]

Note that the last inequality is equivalent with the property of the function \(u \mapsto -\beta(u) + \gamma u\) to be nondecreasing. Moreover in this case \(\beta\) is a Lipschitz function.

The function \(-\beta\) is called branching mechanism. An example of \(\beta\) satisfying (1.3) is

\[
\beta_N(u) = \frac{m}{\Gamma(1 - m)} \int_0^N \frac{e^{-su} - 1}{s^{m+1}} ds.
\]

with \(N > 0\) and \(0 < m < 1\). A limit case is therefore \(au^m\) for a convenient number \(a < 0\), since

\[
-u^m = \frac{m}{\Gamma(1 - m)} \int_0^\infty \frac{e^{-su} - 1}{s^{m+1}} ds = \lim_{N \to \infty} \beta_N(u).
\]

If \(\beta = \alpha = 0\) and \(g = 0\) then the solution of the linear problem (1.1) is given by the transition function of the reflected Brownian motion \(B = (B_t)_{t \geq 0}\) on \(\Omega\): \(u(t, \cdot) = \mathbb{E} (f(B_t)), t \geq 0,\) where \(f\) is a bounded, real-valued Borel measurable function on \(\Omega\).

The first aim of this paper is to show that the solution of (1.1) admits a probabilistic interpretation if \(\beta\) is given by (1.2), which is similar with what happens in the linear case (\(\beta = 0\)). More precisely, there exists a branching Markov process \(X = (X_t)_{t \geq 0}\) with state space the set \(M(\overline{\Omega})\) of all positive finite measures on \(\overline{\Omega}\), such that the solution of (1.1) is

\[
u(t, x) = -\ln \mathbb{E}^{x_t}(e_f(X_t)), t \geq 0, x \in \overline{\Omega},
\]

where for a Borel, positive, real-valued function \(f\) on \(\overline{\Omega}\) we considered the exponential mapping \(e_f : M(\overline{\Omega}) \to [0, 1]\), defined as

\[
e_f(\mu) := e^{-\int f d\mu} \text{ for all } \mu \in M(\overline{\Omega}).
\]

The first step of our approach is to prove the existence of the solution of (1.1). We consider the maximal monotone operator \(A\) associated to (1.1) and we show that it is the infinitesimal generator of a nonlinear semigroup of contractions \((V_t)_{t \geq 0}\) on \(L^2(\Omega)\), such that \(u(t, \cdot) := V_tf\) is a solution of (1.1) for each \(f\) from the domain of \(A\). If \(1 \leq d \leq 3\) then \((V_t)_{t \geq 0}\) induces a \(C_0\)-semigroup of (nonlinear) contractions on \(C(\overline{\Omega})\).

The second step is to prove that the map \(f \mapsto V_tf(x), x \in \overline{\Omega}\), is negative definite on \(C_+(\overline{\Omega})\) (:= the set of all positive continuous functions on \(\overline{\Omega}\)). We use essentially an
approximating process in solving (1.1) and a negative definiteness property of the mapping \(-\beta\).

The last step is to follow the so called semigroup approach in order to construct the claimed measure-valued branching process; see [Wat 68], [Fitz 88], [Dyn 02], [LeGa 99], [Li 11], and [Be 11].

We can describe the infinitesimal generator of the branching process \(X\), which shows that it behaves as the linear semiflow \(t \mapsto \mu \circ P_t, t \geq 0, \mu \in M(\mathcal{O}) (:= \text{the set of all positive finite measures on } \mathcal{O})\), where \((P_t)_{t \geq 0}\) is the transition function of the reflected Brownian motion. The branching property of \(X\) holds on the measures with non-zero traces on the boundary \(\Gamma\) of \(\mathcal{O}\); for more details see the final remark of this paper.

Formula (1.4) suggests that we can compare the measure-valued process \(X\) with the \((B, \beta_0)\)-superprocess (in the sense of Dynkin; see [Dyn 02]), where

\[
\beta_0(x, u) = \begin{cases} 
-\beta(u) + g(x), & \text{if } x \in \Gamma, u \geq 0, \\
0, & \text{if } x \in \mathcal{O}, u < 0.
\end{cases}
\]

We may conclude that the classical association of the superprocesses to the Dirichlet boundary value problems also holds for the nonlinear Neumann boundary value problems. However, due to the boundary flux induced by \(g\), this branching process is no longer conservative if \(g \neq 0\).

The second aim of this paper is to prove that the solution of the nonlinear parabolic problem with the dynamic flux on the boundary

\[
\begin{cases} 
\frac{1}{2}\Delta u - \alpha u = 0 & \text{in } \mathcal{O}, \\
\frac{\partial v}{\partial t} + \frac{\partial u}{\partial \nu} + \beta(v) = 0 & \text{on } \Gamma, \\
v = u|_{\Gamma}, & u \in C(\overline{\mathcal{O}}),
\end{cases}
\]

also admits a probabilistic interpretation, which is related to the branching process associate with the equation (1.1) with \(g \equiv 0\). It turns out that the associated measure-valued branching process is precisely the \((Z, \alpha - \beta)\)-superprocess on \(M(\Gamma) (:= \text{the set of all positive finite measures on } \Gamma)\), where \(Z = (Z_t)_{t \geq 0}\) is the boundary process (on \(\Gamma\)), induced by the reflected Brownian motion, or equivalently, by the classical (linear) Neumann boundary value problem. Comparing the infinitesimal operators, one can see that the measure-valued branching process \(X\) behaves on \(M(\Gamma)\) as this superprocess; cf. Final Remark. So, for the process \(X\) on \(M(\overline{\mathcal{O}})\), the role of the ”boundary process” is played by the \((Z, \alpha - \beta)\)-superprocess on \(M(\Gamma)\).

Finally, we thank the anonymous referee for carefully reading the manuscript and for his valuable comments.

2 Nonlinear \(C_0\)-semigroups generated by the Neumann problem

Everywhere in the following we assume that condition (1.3) holds.
Let $H = L^2(O)$ denote the space of al real-valued square integrable functions on $O$ with the scalar product
\[
\langle u, v \rangle_2 = \int_O u v dx \text{ for all } u, v \in L^2(O),
\]
and the norm $|u|_2 := \langle u, u \rangle_2^{\frac{1}{2}}$. By $H^k(O)$, $k = 1, 2$, we denote the standard Sobolev space in $L^2(O)$ and $C(\overline{O})$ denotes the space of all continuous functions on $\overline{O}$, endowed with the supremum norm $\| \cdot \|_{C(\overline{O})}$. Let further $\sigma$ denote the surface measure on $\Gamma$.

Define the nonlinear operator $A : D(A) \subseteq H \rightarrow H$,
\[
(2.1) \quad \begin{cases} 
D(A) := \{ u \in H^2(O) : \frac{\partial u}{\partial \nu} + \beta(u) = g \text{ on } \Gamma \}, \\
A u := -\frac{1}{2} \Delta u + \alpha u \text{ for all } u \in D(A). 
\end{cases}
\]

Note that $D(A)$ is dense in $H$. We have for all $u, v \in D(A)$
\[
\langle Au - Av, u - v \rangle_2 = \int_O \left( \frac{1}{2} \nabla(u - v)^2 + \alpha(u - v)^2 \right) dx + \frac{1}{2} \int_{\Gamma} (\beta(u) - \beta(v))(u - v) d\sigma 
\geq \int_O \left( \frac{1}{2} |\nabla(u - v)|^2 + \alpha(u - v)^2 \right) dx - \frac{\gamma}{2} \int_{\Gamma} |u - v|^2 d\sigma \geq 0,
\]
because
\[
(2.2) \quad \int_O \left( \frac{1}{2} |\nabla v|^2 + \alpha v^2 \right) dx \geq \gamma \int_{\Gamma} v^2 d\sigma \quad \text{for all } v \in H^1,
\]
where the first inequality holds by the monotonicity of the map $r \mapsto \beta(r) + \gamma r$ and the second one is a consequence of condition (1.3). Hence $A$ is monotone in $H \times H$.

It should be said also that $A$ is a potential operator, $A = \partial \Phi$, where $\Phi : L^2(O) \rightarrow (-\infty, +\infty]$ is the lower semicontinuous, convex function defined as
\[
\Phi(u) := \begin{cases} 
\frac{1}{4} \int_O |\nabla u|^2 dx + \frac{1}{2} \int_{\Gamma} j(u) d\sigma + \frac{\alpha}{2} \int_O u^2 dx \text{ if } u \in H^1(O), \\
+\infty \\
\end{cases}
\text{elsewhere,}
\]
with $j(r) := \int_0^r \beta(s) ds$, $r \in \mathbb{R}$. Moreover, by a fundamental result due to H. Brézis [Brez 72], it follows that $R(I + \lambda A) = H$ for all $\lambda > 0$ (see also [Bar 10], p. 99). In fact, the Brézis regularity result is given for the elliptic problem $u - \lambda \Delta u = f$ in $O$, $\frac{\partial u}{\partial \nu} + \tilde{\beta}(u) = 0$ on $\Gamma$, where $\tilde{\beta}$ is monotone. The extension to the present case (that is $r \mapsto \beta(r) + \gamma r$ is monotone) follows however by the same argument.

By standard existence theory of the Cauchy problem in Banach spaces (see e.g. [Bar 10], p. 163) it follows that $A$ is the infinitesimal generator of a continuous semigroup of contractions $(V_t)_{t \geq 0}$ in $H$, that is
\[
|V_t f - V_t \bar{f}|_2 \leq |f - \bar{f}|_2 \text{ for all } t \geq 0, f, \bar{f} \in H.
\]
Moreover, it turns out that \( u(t) := V_t f, t \geq 0 \), is the unique solution of (1.1) if \( f \in D(A) \) and

\[
\begin{cases}
\frac{d}{dt} V_t f + A V_t f = 0 & \text{for all } t \geq 0, \\
V_0 f = f
\end{cases}
\]

(cf. Theorem 4.12 from \[Bar 10\]).

The next two propositions collect other properties of \((V_t)_{t \geq 0}\).

**Proposition 2.1.** Let \( f \in H \) and define the mapping \( u : [0, \infty) \to H \) as \( u(t) := V_t f, t \geq 0 \). Then the following assertions hold

(i) If \( f \in D(A) \) and \( T > 0 \) then \( u(t) \in D(A) \) for all \( t \geq 0 \) and

\[
u \in C([0, T]; H) \cap L^\infty(0, T; H^2(\O)), \quad \frac{\partial u}{\partial t} \in L^\infty(0, T; H^2(\O)),
\]

\[|A u(t)|_2 \leq |A f|_2 \quad \text{for all } t > 0.\]

(ii) The function \( t \mapsto u(t) \) is right continuous from \((0, \infty)\) to \(H^2(\O)\) and so, if \( 1 \leq d \leq 3 \), then since \(H^2(\O) \hookrightarrow C(\overline{\O})\), we have

\[
\lim_{s \downarrow t} \|u(s) - u(t)\|_{C(\overline{\O})} = 0 \quad \text{for all } t \geq 0.
\]

(iii) If \( f \in H \) then \( u \in C([0, \infty); H) \) and \( \frac{du}{dt} \). \( Au \in L^\infty(\delta, T; H) \) for all \( \delta \in (0, T) \). If \( c \in \mathbb{R}_+ \) and \( c \leq f \) then \( e^{-\alpha c} \leq V_t f \). In particular, if \( f \geq 0 \) then \( V_t f \geq 0 \) for all \( t \geq 0 \). Moreover, if \( f_1, f_2 \in H, f_1 \leq f_2 \) a.e. in \( \O \), then \( V_t f_1 \leq V_t f_2 \) a.e. in \( \O \).

**Proof.** Assertions (i) and (ii) are consequences of the existence theory of the Cauchy problem associated with nonlinear maximal monotone operators in Hilbert spaces (see e.g. \[Bar 10\]). To prove assertion (iii), assume first that \( c \leq f \) and rewrite equation (1.1) as

\[
\begin{cases}
\frac{\partial}{\partial t} (e^{\alpha t} u - c) - \frac{1}{2} \Delta (e^{\alpha t} u - c) = 0 & \text{in } (0, T) \times \O, \\
\frac{\partial}{\partial v} (e^{\alpha t} u - c) + e^{\alpha t} (\beta(u) - g) = 0 & \text{on } (0, T) \times \Gamma,
\end{cases}
\]

multiply by \(e^{\alpha t} u - c\) and integrate on \((0, t) \times \O\), for all \( t \in (0, T)\) to get after some calculation that \(e^{\alpha t} u - c\) is a.e. on \((0, T) \times \O\).

One proves similarly that \((V_t f_2 - V_t f_1)\) is a.e. on \((0, T) \times \O\), involving as above (2.2) and the monotonicity of the function \( u \mapsto \beta(u) + \gamma u \). Indeed, if \( u_i := V_t f_i, i = 1, 2 \), and \( v = u_1 - u_2 \), then

\[
\begin{cases}
\frac{\partial}{\partial t} v - \frac{1}{2} \Delta v + \alpha v = 0 & \text{in } (0, T) \times \O, \\
\frac{\partial}{\partial v} v + \beta(u_1) - \beta(u_2) = 0 & \text{on } \Gamma.
\end{cases}
\]

Multiplying by \( v^+ \) and integrating we get for all \( t \geq 0 \)

\[
\|v^+(t)\|^2_{L^2} + \frac{1}{2} \int_0^t \int_\O |\nabla v^+|^2 dsdx + \alpha \int_0^t \int_\O |v^+|^2 dsdx + \frac{1}{2} \int_\Gamma (\beta(u_1) - \beta(u_2)) v^+ dsd\sigma = 0.
\]
Since $\beta(u_2) - \beta(u_1) \leq \gamma(u_1 - u_2)$ on the set $[v^+ > 0]$, we obtain
\[
\|v^+(t)\|_{L^2}^2 + \frac{1}{2} \int_0^t \int_O |\nabla v^+|^2 dsdx + \alpha \int_0^t \int_O |v^+|^2 dsdx \leq \frac{\gamma}{2} \int_0^t \int_F |v^+|^2 ds\sigma.
\]
By (2.2) we conclude that $\|v^+(t)\|_{L^2}^2 = 0$ for all $t \geq 0$, as claimed. $\Box$

An interesting feature of the family $(V_t)_{t \geq 0}$ is that it is a $C_0$-semigroup on $C(\bar{O})$, namely, we have the following result.

**Proposition 2.2.** Assume that $1 \leq d \leq 3$. Then the family $(V_t)_{t \geq 0}$ induces a $C_0$-semigroup of $\alpha$-quasi-contractions on $C(\bar{O})$, i.e., for each $f, \bar{f} \in C(\bar{O})$ we have

\[
V_t f \in C(\bar{O}) \text{ and } \|V_t f - V_t \bar{f}\|_* \leq e^{\alpha t} \|f - \bar{f}\|_* \text{ for all } t \geq 0,
\]
\[
V_t V_s f = V_{t+s} f \text{ for all } t, s \geq 0,
\]
\[
\lim_{t \to 0} \|V_t f - f\|_* = 0.
\]

Here $\| \cdot \|_*$ is a norm on $C(\bar{O})$ which is equivalent with the supremum norm, to be defined below. In addition we have $V_t(S) \subseteq S$ for all $t \geq 0$, where $S := C_+(\bar{O})$.

**Proof.** Since $1 \leq d \leq 3$ we have $H^2(O) \subseteq C(\bar{O})$ and so, by Proposition 2.1 $V_t f \in C(\bar{O})$ for all $f \in H$ and $t \geq 0$.

We show now that the operator $A_0$ defined as
\[
A_0 u := -\frac{1}{2} \Delta u + \alpha u \quad \text{for } u \in D(A_0) := \{u \in H^2(O) : \Delta u \in C(\bar{O}), \frac{\partial u}{\partial \nu} + \beta(u) = g \text{ on } \Gamma\}
\]
is quasi-$m$-accretive in $C(\bar{O})$, i.e., for some $\lambda_0 > 0$ and any $0 < \lambda < \lambda_0$ we have
\[
\|(1 + \lambda \alpha)I + \lambda A_0\|^{-1} v - ((1 + \lambda \alpha)I + \lambda A_0)^{-1} \bar{v}\|_* < \|v - \bar{v}\|_* \text{ for all } v, \bar{v} \in C(\bar{O}).
\]

Indeed, if $u := ((1 + \lambda \alpha)I + \lambda A_0)^{-1} v$ and $\bar{u} := ((1 + \lambda \alpha)I + \lambda A_0)^{-1} \bar{v}$, then
\[
(1 + \lambda \alpha)(u - \bar{u}) - \frac{\lambda}{2} \Delta (u - \bar{u}) + \lambda \alpha (u - \bar{u}) = v - \bar{v} \text{ in } O \text{ and } \frac{\partial u}{\partial \nu}(u - \bar{u}) + \beta(u) - \beta(\bar{u}) = 0 \text{ on } \Gamma.
\]

We choose now $\varphi \in C^2(\bar{O})$ such that
\[
\varphi \geq \rho > 0 \text{ and } 2\alpha - \varphi \Delta \frac{1}{\varphi} \geq 0 \text{ in } \bar{O}, \quad \frac{1}{\varphi} \frac{\partial \varphi}{\partial \nu} \geq L \text{ on } \Gamma, \text{ with } L := \text{Lip}(\beta).
\]

An example of such a function $\varphi$ is $\varphi(x) := \exp(\delta \psi(x))$, where $\delta > 0$ is sufficiently small and $\Delta \psi = K$ in $O$, $\frac{\partial \psi}{\partial \nu} = \frac{K}{\delta}$ on $\Gamma$; $K$, $\delta$ are such that $Km(O) = \frac{\delta}{\delta} \sigma(\Gamma)$, where $m$ is the Lebesgue measure.

We set
\[
\|z\|_* := \sup\{|z(x)\varphi(x)| : x \in \bar{O}\} \text{ for all } z \in C(\bar{O}) \text{ and } c := \|v - \bar{v}\|_*.
\]

Let $y := u \varphi$ and $\bar{y} := \bar{u} \varphi$. We have
\[
y - \bar{y} - \frac{\lambda}{2} \Delta (y - \bar{y}) + \lambda \alpha (y - \bar{y}) \frac{\lambda}{2} \varphi \Delta \frac{1}{\varphi} (y - \bar{y}) - \frac{\lambda}{2} \Delta (y - \bar{y}) = \varphi (v - \bar{v}) \quad \text{in } O,
\]
\[
\frac{\partial}{\partial \nu}(y - \bar{y}) - \frac{1}{\varphi} \frac{\partial \varphi}{\partial \nu}(y - \bar{y}) + \beta(\frac{1}{\varphi}y) - \beta(\frac{1}{\varphi}\bar{y}) = 0 \text{ on } \Gamma.
\]

This yields
\[
(y - \bar{y} - c) - \frac{\lambda}{2} \Delta(y - \bar{y} - c) - \lambda \varphi \nabla(\frac{1}{\varphi}) \nabla(y - \bar{y} - c) + (\lambda \alpha - \frac{\lambda}{2} \varphi \Delta(\frac{1}{\varphi}))(y - \bar{y} - c) = \varphi(v - \bar{v}) - (\lambda \alpha - \frac{\lambda}{2} \varphi \Delta(\frac{1}{\varphi}))c - c \leq 0 \text{ in } \mathcal{O},
\]
\[
\frac{\partial}{\partial \nu}(y - \bar{y} - c) = \frac{1}{\varphi} \frac{\partial \varphi}{\partial \nu}(y - \bar{y}) - \beta(\frac{1}{\varphi}) - \beta(\frac{1}{\varphi}\bar{y}) \text{ on } \Gamma.
\]

Multiplying by \((y - \bar{y} - c)^{+}\) and integrating on \(\mathcal{O}\), we get via Green’s formula
\[
\int_{\mathcal{O}} ((y - \bar{y} - c)^{+})^2 dx + \frac{\lambda}{2} \int_{\mathcal{O}} |\nabla(y - \bar{y} - c)^{+}|^2 dx 
\]
\[
\leq \frac{1}{2} \int_{\mathcal{O}} |(y - \bar{y} - c)^{+}|^2 dx + \frac{\lambda^2}{2} \int_{\mathcal{O}} |\varphi \nabla(\frac{1}{\varphi})|^2 (|y - \bar{y} - c)^{+}|^2 dx 
\]
\[
+ \int_{\Gamma} (\beta(\frac{1}{\varphi}y) - \beta(\frac{1}{\varphi}\bar{y}))(y - \bar{y} - c)^{+} d\sigma - \int_{\Gamma} \frac{1}{\varphi} \frac{\partial \varphi}{\partial \nu}(y - \bar{y})(y - \bar{y} - c)^{+} d\sigma.
\]

Taking into account that by (2.4)
\[
(\beta(\frac{1}{\varphi}y) - \beta(\frac{1}{\varphi}\bar{y}))(y - \bar{y} - c)^{+} - \frac{1}{\varphi} \frac{\partial \varphi}{\partial \nu}(y - \bar{y})(y - \bar{y} - c)^{+} \leq 0 \text{ on } \Gamma,
\]

it follows that for \(\lambda \in (0, \lambda_0)\), \(\lambda_0\) sufficiently small, we have \(\int_{\mathcal{O}} |(y - \bar{y} - c)^{+}|^2 = 0\). Hence \(\varphi(u - \bar{u}) \leq c\) in \(\mathcal{O}\) and similarly it follows that \(\varphi(u - \bar{u}) \geq -c\) in \(\mathcal{O}\), \(\|u - \bar{u}\|_{s} \leq \|v - \bar{v}\|_{s}\). We conclude that \(\mathcal{A}_0\) is quasi-\(m\)-accretive in \(C(\overline{\mathcal{O}})\) as claimed.

By Crandall-Liggett theorem (see [Bar 10], p. 131) \(\mathcal{A}_0\) generates a \(C_{0}\)-semigroup of \(\alpha\)-quasi-contractions \((\bar{V}_t)_{t \geq 0}\) on \(C(\overline{\mathcal{O}})\) (endowed with the norm \(\| \cdot \|_{s}\)), given by the exponential formula
\[
\bar{V}_tf = \lim_{n \to \infty} \left( I + \frac{t}{n} \mathcal{A}_0 \right)^{-n} f \text{ in } C(\overline{\mathcal{O}}) \text{ for all } f \in C(\overline{\mathcal{O}}) \text{ uniformly in } t \text{ on compact intervals}.
\]

Since \((I + \frac{t}{n} \mathcal{A}_0)^{-1} f = (I + \frac{t}{n} \mathcal{A})^{-1} f\) for all \(t > 0\) and \(f \in C(\overline{\mathcal{O}})\), and \(V_t f = \lim_{n \to \infty} (I + \frac{t}{n} \mathcal{A})^{-n} f \) in \(L^2(\mathcal{O})\) for \(f \in L^2(\mathcal{O})\), we infer that
\[
V_t f = \bar{V}_t f \text{ for all } f \in C(\overline{\mathcal{O}}) \text{ and } t \geq 0.
\]

Therefore for all \(f, \bar{f} \in C(\overline{\mathcal{O}})\)
\[
\|V_t f - V_t \bar{f}\|_{s} \leq e^{\alpha t} \|f - \bar{f}\|_{s}, \quad t \geq 0, \text{ and } \lim_{t \to 0} \|V_t f - f\|_{s} = 0
\]
as claimed. The fact that \(V_t(S) \subseteq S\) is a direct consequence of assertion (iii) in Proposition 2.1.

\[\square\]

Remark. The results from Section 2 are valid for more general functions \(\beta\) which satisfies the following condition: \(\beta\) is continuous and \(u \mapsto \beta(u) + \gamma u\) is monotonically nondecreasing on \([0, \infty)\), where \(\gamma\) is defined by (1.3).
3 Negative definite properties of the nonlinear evolution equation

Let \( M := M(\mathcal{O}) \) be the set of all finite positive measures on \( \mathcal{O} \), endowed with the weak topology. It is a locally compact topological space with a countable base. In order to construct a semigroup of (linear) kernels on \( M \), induced by \( (V_t)_{t \geq 0} \), we start with preliminaries on the harmonic analysis of the semigroup \( S := C_+ (\mathcal{O}) \); for more details see [BeChRe 84], [Fitz 88], and [Dyn 02].

A real-valued function \( \varphi : S \rightarrow \mathbb{R} \) is called negative definite if for each \( n \geq 2 \), \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) with \( \alpha_1 + \cdots + \alpha_n = 0 \), and \( u_1, \ldots, u_n \in S \) we have \( \sum_{i,j} \alpha_i \alpha_j \varphi(u_i + u_j) \leq 0 \).

A function \( \varphi : S \rightarrow \mathbb{R} \) is named positive definite provided that for each \( n \geq 1 \), \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \), and \( u_1, \ldots, u_n \in S \) we have \( \sum_{i,j} \alpha_i \alpha_j \varphi(u_i + u_j) \geq 0 \).

Basic properties of the negative and positive definite functions are the following:

1) If \( \varphi : S \rightarrow \mathbb{R} \) is linear or constant then it is negative definite. The linear combination with positive coefficients of negative (resp. positive) definite functions is also negative (resp. positive) definite. The pointwise limit of a sequence of negative (resp. positive) definite functions is negative (resp. positive) definite.

2) Let \( \varphi : S \rightarrow \mathbb{R} \). Then \( \varphi \) is negative definite if and only if \( e^{-t\varphi} \) is positive definite for all \( t > 0 \).

We present now two results which are essentially used in the semigroup approach to the construction of the superprocesses (see e.g., [Fitz 88] and [Be 11]).

**Lemma 3.1.** If \( a \in \mathbb{R}_+ \), \( b \in \mathbb{R} \) and \( \varphi : S \rightarrow \mathbb{R} \) is negative definite, then the function \( -\beta \circ \varphi + a \varphi + b \) is also negative definite.

**Proof.** By (3.2) the function \( e^{-s\varphi} \) is positive definite for all \( s \geq 0 \). So, by (3.1) the function \( (1 - e^{-s\varphi}) \) is negative definite and therefore \( \int_0^\infty (1 - e^{-s\varphi}) \eta(ds) \) has the same property. Again by (3.1) we conclude that \( -\beta \circ \varphi + a \varphi + b \) is negative definite. \( \square \)

It is easy to see that the exponential mapping \( e_f : M \rightarrow [0, 1] \) is a Borel measurable function on \( M \) and that \( S \ni f \mapsto e_f(\mu) \) is positive definite on \( S \) for each \( \mu \in M \). Further we denote by \( \mathcal{B}(M) \) the Borel \( \sigma \)-algebra on \( M \).

**Proposition 3.2.** ([Fitz 88], [Be 11]). The following assertions hold.

(i) Let \( \xi \) be a finite, positive measure on \((M, \mathcal{B}(M))\) and consider the map \( \varphi_\xi : S \rightarrow \mathbb{R}_+ \) defined as

\[
\varphi_\xi(f) = \int_M e_f(\mu)\xi(d\mu), \quad f \in S.
\]

Then \( \varphi_\xi \) is positive definite and if \( (f_n) \subset S \), \( \lim_{n \to \infty} \|f_n\rVert_{C(\mathcal{O})} = 0 \), then \( \lim_{n \to \infty} \varphi_\xi(f_n) = \varphi_\xi(0) \).

(ii) Let \( \varphi : S \rightarrow [0, 1] \) be a positive definite map such that \( \lim_{n \to \infty} \varphi(\frac{1}{n}) = \varphi(0) \).

Then there exists a unique finite positive measure \( \xi \) on \((M, \mathcal{B}(M))\) such that \( \varphi = \varphi_\xi \), that is

\[
\varphi(f) = \int_M e_f \, d\xi \quad \text{for all } f \in S.
\]
We can state now the main result of this section.

**Proposition 3.3.** The following assertions hold.

(i) For each \( t \geq 0 \) and \( x \in \mathcal{O} \) the function \( S \ni f \mapsto V_t f(x) \) is negative definite.

(ii) For each \( t \geq 0 \) there exists a unique sub-Markovian kernel \( Q_t \) on \((M, \mathcal{B}(M))\) such that

\[
Q_t(e_f) = e_{V_t f} \text{ for all } f \in S.
\]

(iii) The family \((Q_t)_{t \geq 0}\) induces a Feller semigroup on \( C_0(M) \) \(:= \text{the space of real-valued continuous functions on } M, \text{vanishing at infinity}\).

**Proof.** (i) We consider the iteration process:

\[
\begin{aligned}
\frac{\partial u_{n+1}}{\partial t} - \frac{1}{2} \Delta u_{n+1} + \alpha u_{n+1} &= 0 \quad \text{in } (0, T) \times \mathcal{O}, \\
\frac{\partial u_{n+1}}{\partial \nu} + \beta(u_n) &= g \quad \text{on } (0, T) \times \Gamma, \\
\end{aligned}
\]

\[
\begin{aligned}
u_{n+1}(0) &= f \quad \text{in } \mathcal{O}, \quad f \in S.
\end{aligned}
\]

We claim that for \( n \to \infty, \; u_n \to u \) in \( L^2(0,T; H^1(\mathcal{O})) \) and \( \beta(u_n) \to \beta(u) \) in \( L^2((0,T) \times \Gamma) \), where \( u \) is the solution of (1.1). Indeed, for each \( v \in L^2(\Gamma) \) we denote by \( F(v) \) the trace \( \tau(u) \) of \( u \), solution to

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \frac{1}{2} \Delta u + \alpha u &= 0 \quad \text{in } [0, T) \times \mathcal{O}, \\
\frac{\partial u}{\partial \nu} + \beta(v) &= g \quad \text{on } [0, T) \times \Gamma, \\
u(0, \cdot) &= f \quad \text{in } \mathcal{O}.
\end{aligned}
\]

Here we take the norm of \( H^1(\mathcal{O}) \) as \( |||u|||_{H^1(\mathcal{O})} := (\int_{\mathcal{O}} |\nabla u|^2 + 2\alpha u^2 dx)^{1/2} \). We prove first that for \( |||\beta|||_{\text{Lip}} \) sufficiently small, \( F \) is a contraction on \( L^2((0,T) \times \Gamma) \). If \( v, \tilde{v} \in L^2(\Gamma) \) and \( u, \tilde{u} \) are related to \( v \) and respectively \( \tilde{v} \) as before, then

\[
\begin{aligned}
\frac{\partial}{\partial t} (u - \tilde{u}) - \frac{1}{2} \Delta (u - \tilde{u}) + \alpha (u - \tilde{u}) &= 0 \quad \text{in } [0, T) \times \mathcal{O}, \\
\frac{\partial}{\partial \nu} (u - \tilde{u}) + \beta(v) - \beta(\tilde{v}) &= g \quad \text{on } [0, T) \times \Gamma, \\
(u - \tilde{u})(0, \cdot) &= 0 \quad \text{in } \mathcal{O}.
\end{aligned}
\]

This yields

\[
(3.5) \quad \frac{1}{2} ||u(t) - \tilde{u}(t)||^2_{L^2(\mathcal{O})} + \frac{1}{2} \int_0^T \int_{\mathcal{O}} |\nabla (u - \tilde{u})|^2 dx ds + \alpha \int_0^T \int_{\mathcal{O}} |u - \tilde{u}|^2 dx ds \leq \int_0^T \int_{\Gamma} |\beta(v) - \beta(\tilde{v})||u - \tilde{u}| ds d\sigma \leq
\]

\[
|||\beta|||_{\text{Lip}} \int_0^T ||v - \tilde{v}||_{L^2(\Gamma)} ||u - \tilde{u}||_{L^2(\Gamma)} ds \leq \gamma^{-1} |||\beta|||_{\text{Lip}} ||v - \tilde{v}||_{L^2((0,T) \times \Gamma)} ||u - \tilde{u}||_{L^2(0,T;H^1(\mathcal{O}))},
\]

where \( \gamma \geq 1 \) is a constant depending only on \( S \).
where $\gamma$ is given by (1.3). Hence $\|u - \bar{u}\|_{L^2(0,T;H^1(\mathcal{O}))} \leq 2\gamma^{-\frac{1}{2}}\|\beta\|_{\text{Lip}}\|v - \bar{v}\|_{L^2((0,T) \times \Gamma)}$ and this yields by the trace theorem $\|u - \bar{u}\|_{L^2((0,T) \times \Gamma)} \leq 2\gamma^{-\frac{1}{2}}\|\beta\|_{\text{Lip}}\|v - \bar{v}\|_{L^2((0,T) \times \Gamma)}$. Consequently $F$ is a contraction for

$$(3.6) \quad \|\beta\|_{\text{Lip}} < \frac{\gamma}{2}$$

and so, the sequence $(u_n)_{n \geq 0}$ which is defined by $\tau(u_{n+1}) = F(\tau(u_n))$, is strongly convergent in $L^2(\Gamma)$ to $\tau(u)$. This implies also by (3.5) that $(u_n)_{n \geq 0}$ is convergent in $L^2(0,T;H^1(\mathcal{O})) \cap C([-T,T];L^2(\mathcal{O}))$. Now we can get rid of condition (3.6) by rescaling equation (1.1) via the transformation $t \rightarrow \lambda s$.

**Representation of $u_{n+1}$ as the solution of an integral equation.** Define the linear continuous operator $A : H^1(\mathcal{O}) \rightarrow (H^1(\mathcal{O}))'$ as

$$\langle (H^1(\mathcal{O})), (Au, \psi) \rangle_{H^1(\mathcal{O})} := \frac{1}{2} \int_{\mathcal{O}} (\nabla u \cdot \nabla \psi + 2\alpha u \psi) \, dx \quad \text{for all } u, \psi \in H^1(\mathcal{O}),$$

and also the mapping $\bar{\beta} : H^1(\mathcal{O}) \rightarrow (H^1(\mathcal{O}))'$ as

$$\langle (H^1(\mathcal{O})), (\bar{\beta}(u), \psi) \rangle_{H^1(\mathcal{O})} := -\frac{1}{2} \int_{\Gamma} (\beta(u) - g) \tau(\psi) \, d\sigma \quad \text{for all } u, \psi \in H^1(\mathcal{O}),$$

where $\tau(\psi) \in L^2(\Gamma)$ is the trace of $\psi$ on $\Gamma$. Then (3.4) is equivalent with

$$\begin{align*}
\left\{ \begin{array}{ll}
\frac{\partial u_{n+1}}{\partial t} = -Au_{n+1} + \bar{\beta}(u_n) & \text{in } (0, T) \times \mathcal{O}, \\
u_{n+1}(0) = f & \text{in } \mathcal{O},
\end{array} \right.
\end{align*}$$

which may be rewritten as (d’Alembert formula)

$$(3.7) \quad \langle \psi, u_{n+1}(t) \rangle_2 = \langle e^{At} \psi, f \rangle_2 - \frac{1}{2} \int_0^t \int_{\Gamma} \tau(e^{A(t-s)} \psi) \left( \beta(u_n(s)) - g \right) \, d\sigma \, ds$$

for all $\psi \in H^1(\mathcal{O}), \ 0 < t < T, \ n \in \mathbb{N}$. By Lemma 3.1 and (3.7) it follows inductively that the function $f \mapsto \langle \psi, u_{n+1}(t) \rangle_2$ is negative definite for all $\psi \in H^1(\mathcal{O}), \ \psi \geq 0$. Passing to the limit and using (3.1) we get that $u := \lim_{n \rightarrow \infty} u_n$ has the same property and consequently assertion (i) holds.

(ii) From (i) and (3.2) it follows that for each $\mu \in M$ the map

$$f \mapsto e_{V_i f}(\mu), \ f \in S,$$

is positive definite and by Proposition 2.2 we have $\lim_{n \rightarrow \infty} e_{V_i f}(\mu) = e_{V_{i,0}}(\mu)$. Proposition 3.2 (ii) implies the existence of a finite measure $Q_{t,\mu}$ on $M$ such that

$$Q_{t,\mu}(e_f) = e_{V_i f}(\mu) \text{ for all } f \in S.$$

We have $Q_{t,\mu}(1) = Q_{t,\mu}(e_0) = e_{V_{i,0}}(\mu) \leq 1$ since $V_{i,0} \geq 0$, so $Q_{t,\mu}$ is a sub-probability on $M$. The map $\mu \mapsto Q_{t,\mu}(e_f)$ is $\mathcal{B}(M)$-measurable for all $f \in S$ and since the family $\{e_f : f \in S\}$ generates the $\sigma$-algebra $\mathcal{B}(M)$, we deduce by a monotone class argument.
that \( \mu \mapsto Q_{t,\mu}(F) \) is also \( \mathcal{B}(M) \)-measurable for all positive \( \mathcal{B}(M) \)-measurable function \( F \) on \( M \). Consequently

\[
F \mapsto Q_{t}F(\mu) := Q_{t,\mu}(F), \; \mu \in M,
\]
defines a sub-Markovian kernel on \( M \).

(iii) Since the family \((V_t)_{t \geq 0}\) is a (nonlinear) semigroup it follows that \((Q_t)_{t \geq 0}\) is semi-group of sub-Markovian kernels on \( M \). Let \( S' := \{ f \in S : f > 0 \} \), \( \mathcal{S}' := \{ e_f : f \in S' \} \) and \( \mathcal{S} \) the linear space spanned by \( \mathcal{S}' \). Since by Proposition 2.1 (iii) \( V_t(S') \subseteq S' \) for all \( t \geq 0 \), it follows that \( Q_t(\mathcal{S}') \subseteq \mathcal{S}' \) and therefore \( Q_t(\mathcal{S}) \subseteq \mathcal{S} \). Because \( \mathcal{S} \) is an algebra, separating the points of \( M \) (by Stone-Weierstrass Theorem), it is a dense subset of \( C_0(M) \) and therefore \( Q_t(C_0(M)) \subseteq C_0(M) \) for all \( t \geq 0 \). By Proposition 2.2 we have

\[
\lim_{t \searrow 0} \| V_t f - f \|_{C(\mathcal{S})} = 0
\]
and thus \( \lim_{t \searrow 0} Q_t(e_f)(\mu) = e_f(\mu) \) for all \( \mu \in M \), hence \( Q_t(F) \) converges to \( F \) for every \( F \in \mathcal{S} \), pointwise on \( M \), as \( t \) decreases to zero. We conclude that the above pointwise convergence holds for every \( F \in C_0(M) \) and therefore \((Q_t)_{t \geq 0}\) is a \( C_0 \)-semigroup on \( C_0(M) \).

\[\Box\]

4 Measure-valued branching processes

4.1 Branching processes on the closure of the open set

In this subsection we associate to the equation (1.1) a branching process with state space the set \( M = M(\mathcal{O}) \) of all finite measures on the closure of the open set \( \mathcal{O} \).

Recall that if \( p_1, p_2 \) are two finite measures on \( M \), then their convolution \( p_1 \ast p_2 \) is the finite measure on \( M \) defined for every \( h \in \mathcal{B}_+(M) := \{ \text{the set of all positive } \mathcal{B}(M) \text{-measurable functions on } M \} \) by

\[
\int_{M} p_1 \ast p_2(\text{d}\nu)h(\nu) := \int_{M} p_1(\text{d}\nu_1) \int_{M} p_2(\text{d}\nu_2)h(\nu_1 + \nu_2).
\]

A bounded kernel \( Q \) on \((M, \mathcal{B}(M))\) is called branching kernel provided that

\[
Q_{\mu + \nu} = Q_{\mu} \ast Q_{\nu} \quad \text{for all } \mu, \nu \in M,
\]

where \( Q_{\mu} \) denotes the measure on \( M \) such that \( \int \text{d}hQ_{\mu} = Qh(\mu) \) for all \( h \in \mathcal{B}_+(M) \).

A right (Markov) process with state space \( M \) is called branching process provided that its transition function is formed by branching kernels. The probabilistic interpretation of this analytic branching property of a process is as follows: if we take two independent versions \( X \) and \( X' \) of the process, starting respectively from two measures \( \mu \) and \( \mu' \), then \( X + X' \) and the process starting from \( \mu + \mu' \) are equal in distribution.

Example 4.1. Let \((P_t)_{t \geq 0}\) be the transition function of the reflected Brownian motion on \( \mathcal{O} \).

(i) If \( t \geq 0 \) and \( \alpha \geq 0 \) then the kernel \( Q_{t}^{0} \) on \( M \) defined as

\[
Q_{t}^{0}F(\mu) := F(\mu \circ e^{-\alpha t}P_{t}), \quad F \in \mathcal{B}_+(M), \quad \mu \in M,
\]
is a branching kernel. (ii) The linear semiflow on $M$, $\mu \mapsto \mu \circ e^{-at}P_t$, is a continuous path branching (deterministic) process with transition function $(Q_t^0)_{t \geq 0}$. If $f \in C^2(M)$, $F := e_f$, and $\mu \in M$, then there exists

$$\lim_{t \searrow 0} \frac{Q_t^0 F(\mu) - F(\mu)}{t} =: L^0 F(\mu)$$

and we have $L^0 F(\mu) = \int M(\mu(dx)) \frac{1}{2} \Delta F'(\mu, x) - \alpha F'(\mu, x)]$, where recall that the variational derivative of a function $F : M \to \mathbb{R}$ is

$$F'(\mu, x) := \lim_{t \searrow 0} \frac{1}{t} (F(\mu + t\delta_x) - F(\mu)), \; \mu \in M, \; x \in \mathcal{O}.$$ 

Now, we present the main result of this paper. Let $(Q_t)_{t \geq 0}$ be the Feller semigroup given by assertion (iii) of Proposition 3.3, induced by the solution of (1.1).

**Theorem 4.2.** There exists a branching Markov process $X = (X_t)_{t \geq 0}$ with state space $M$, such that the following assertions hold.

(i) $X$ is a Hunt process with transition function $(Q_t)_{t \geq 0}$, or equivalently, for every $f \in S$, the solution $(V_t f)_{t \geq 0}$ of the nonlinear parabolic problem (1.1) has the representation

$$(4.1) \quad V_t f(x) = -\ln \mathbb{E}^x(e_f(X_t); t < \zeta), \; t \geq 0, \; x \in \mathcal{O},$$

where $\zeta$ denotes the life time of $X$.

(ii) Let $(L, D(L))$ be the infinitesimal generator of $X$, that is $Q_t = e^{tL}$, $t \geq 0$, i.e. $(L, D(L))$ is the generator of the $C_0$-semigroup $(Q_t)_{t \geq 0}$ on $C_0(M)$. If $f \in D(A)$, $F := e_f$, and $\mu \in H^1(\mathcal{O})$ with $\Delta \mu \in H$, then there exists

$$\lim_{t \searrow 0} \frac{Q_t F(\mu) - F(\mu)}{t} =: LF(\mu) \quad \text{and}$$

$$(4.2) \quad LF(\mu) = \frac{1}{2} \Delta \mu - \alpha \mu, \; F'(\mu) \rangle_2 - \frac{1}{2} \int \mu \frac{\partial}{\partial \nu} F'(\mu) \, d\sigma + \int \mu|_\Gamma (dy) \left( \int_0^\infty \left[ F(\mu + s\delta_y) - F(\mu) \right] \eta(ds) - \frac{1}{2} g(y) F(\mu) - b F'(\mu, y) \right).$$

**Proof.** (i) By assertion (ii) of Proposition 3.3 we have $Q_t(e_f) = e_{V_t f}$ for all $t \geq 0$ and $f \in S$. Using also a monotone class argument it follows that $Q_t$ is a branching kernel on $M$ for all $t \geq 0$; for details see [Wat 68], [Li 11], [Fitz 88], and [Be 11]. From the basic existence theorem for Hunt processes (cf. Theorem 9.4 in [BlGe 68]), since $(Q_t)_{t \geq 0}$ is a Feller semigroup according to Proposition 3.3 (iii), there exists the claimed Hunt process with transition function $(Q_t)_{t \geq 0}$. Hence $Q_t F(\mu) = \mathbb{E}^\mu(F(X_t); t < \zeta)$ for all $\mu \in M$ and $F \in B_+(M)$. The equality (4.1) follows now by (3.3).

(ii) Since $f$ belongs to the domain of $A$, given by (2.1), using (2.3) we get

$$LF(\mu) = \frac{d(e_{V_t f}(\mu))}{dt} = -e_f(\mu) \left\langle \mu, \frac{d}{dt} V_t f(0) \right\rangle_2$$

$$= F(\mu) \langle \mu, Af \rangle_2 = -F(\mu) \left[ \int \mu \frac{1}{2} \Delta f - \alpha \int \mu f \right].$$
By the Green formula and again by (2.1)

\[ \int_{\Omega} \mu \Delta f = \int_{\Omega} f \Delta \mu + \int_{\Gamma} \frac{\partial f}{\partial \nu} \mu d\sigma - \int_{\Gamma} f \frac{\partial \mu}{\partial \nu} d\sigma = \int_{\Omega} f \Delta \mu + \int_{\Gamma} [(g - \beta(f))\mu - f \frac{\partial \mu}{\partial \nu}] d\sigma. \]

We have also

\[ F(\mu) \int_{\Gamma} \beta(f)\mu d\sigma = F(\mu) \int_{\Gamma} \mu(y) \left[ \int_{0}^{\infty} (e^{-sf(y)} - 1) \eta(ds) - bf(y) \right] \sigma(dy) \]

\[ = \int_{0}^{\infty} \eta(ds) \int_{\Gamma} \mu(y) F(\mu)[F(s\delta_y) - 1] \sigma(dy) - bF(\mu) \int_{\Gamma} \mu f d\sigma \]

\[ = \int_{0}^{\infty} \eta(ds) \int_{\Gamma} \mu(y) [F(\mu + s\delta_y) - F(\mu)] \sigma(dy) - bF(\mu) \int_{\Gamma} \mu f d\sigma \]

\[ = \int_{\Gamma} \mu|\Gamma(dy) \left( \int_{0}^{\infty} [F(\mu + s\delta_y) - F(\mu)] \eta(ds) - bF(\mu)f(y) \right). \]

Since \( F'(\mu, \cdot) = -fF(\mu) \) we conclude that (4.2) holds. \( \square \)

4.2 Branching processes on the boundary

Define the nonlinear operator \( \Lambda : D(\Lambda) \subseteq L^2(\Gamma) \longrightarrow L^2(\Gamma) \) as

\[ D(\Lambda) := \{ \varphi \in H^{1,2}(\Gamma) : \text{there exists } u \in H^1(\Omega) \text{ s.t. } \frac{1}{2} \Delta u - \alpha u = 0 \text{ in } \Omega, \]

\[ u|_{\Gamma} = \varphi, \frac{\partial u}{\partial \nu} \in L^2(\Gamma) \}, \]

\[ \Lambda \varphi := -\frac{\partial u}{\partial \nu} - \beta(\varphi) \text{ for all } \varphi \in D(\Lambda), \]

where \( u \in H^1(\Omega) \) is s.t. \( \frac{1}{2} \Delta u - \alpha u = 0 \text{ in } \Omega \) and \( u|_{\Gamma} = \varphi. \)

The exact meaning of (4.3) is the following. \( D(\Lambda) \) is the space of all \( \varphi \in H^{1,2}(\Gamma) \) with the property that there are \( u \in H^1(\Omega) \) and \( \eta \in L^2(\Gamma) \) such that

\[ \int_{\Omega} u(\Delta \psi - 2\alpha \psi)dx = \int_{\Gamma} \varphi \frac{\partial \psi}{\partial \nu} d\sigma - \int_{\Gamma} \eta \psi d\sigma \text{ for all } \psi \in H^2(\Omega). \] (4.4)

(In fact \( \eta = \frac{\partial u}{\partial \nu} \) and \( \varphi = \tau(u). \)) The operator \( \Lambda : D(\Lambda) \longrightarrow L^2(\Gamma) \) is defined by

\[ \int_{\Gamma} \Lambda \varphi \xi d\sigma = -\int_{\Gamma} (\eta + \beta(\varphi)) \xi d\sigma \text{ for all } \xi \in L^2(\Gamma), \varphi \in D(\Lambda), \] (4.5)

where \( \eta = \frac{\partial u}{\partial \nu} \) is defined by (4.4).

**Lemma 4.3.** The operator \( \Lambda \) is maximal monotone in \( L^2(\Gamma) \).
Proof. It suffices to show that \( R(I + \Lambda) = L^2(\Gamma) \) and \( \|(I + \Lambda)^{-1}\|_{\text{Lip}(L^2(\Gamma))} \leq 1 \). On the other hand, for each \( f \in L^2(\Gamma) \) the equation \( \varphi + \Lambda \varphi = f \) reduces to

\[
\begin{cases}
\frac{1}{2} \Delta u - \alpha u = 0 & \text{in } \mathcal{O}, \\
u + \frac{\partial u}{\partial \nu} + \beta(u) = f & \text{on } \Gamma, \\
u|_{\Gamma} = \varphi.
\end{cases}
\]

(4.6)

Of course (4.6) should be taken in the sense of (4.4)-(4.5), that is, for all \( \psi \in H^2(\mathcal{O}) \)

\[
\int_{\mathcal{O}} u(\Delta \psi - 2\alpha \psi) dx = \int_{\Gamma} \varphi \frac{\partial \psi}{\partial \nu} d\sigma - \int_{\Gamma} (f - \beta(\varphi) - \varphi) \psi d\sigma.
\]

Equation (4.6) (without the Dirichlet boundary condition \( u|_{\Gamma} = \varphi \)) has a unique weak solution \( u \in H^1(\mathcal{O}) \), that is

\[
\int_{\mathcal{O}} (\nabla u \cdot \nabla \psi + 2\alpha u \psi) dx + \int_{\Gamma} (u - f + \beta(u) - \varphi) \psi d\sigma = 0 \quad \text{for all } \psi \in H^1(\mathcal{O}).
\]

(4.6')

Here is the argument. We can rewrite (4.6') as

\[
L_1 u + L_2 v = f,
\]

where \( L_1 v := \frac{\partial u}{\partial \nu} - \gamma v \) and \( L_2 v := \beta(v) + \gamma v \), with

\[
D(L_1) := \{ v \in L^2(\mathcal{O}) : v = u|_{\Gamma}, \frac{1}{2} \Delta u - \alpha v = 0 \text{ in } \mathcal{O}, \frac{\partial v}{\partial \nu} \in L^2(\Gamma) \}.
\]

By the Lax-Milgram Lemma it follows that \( R(I + L_1) = L^2(\Gamma) \) and so, \( L_1 \) is \( m \)-accretive (or equivalently, it is maximal monotone). It is clear by the monotonicity of the function \( u \mapsto \beta(u) + \gamma u \) that also \( R(I + L_2) = L^2(\Gamma) \). Then by Rockafellar’s perturbation result (see [Bar 10], page 44), it follows that \( \Lambda = L_1 + L_2 \) is maximal monotone (\( m \)-accretive) and so \( f \in R(I + \Lambda) \), as claimed.

This clearly implies via Green’s formula that \( (u, \varphi = u|_{\Gamma}) \) satisfy (4.7). Hence \( R(I + \Lambda) = L^2(\Gamma) \). By (4.4) we have for all \( f, \bar{f} \in L^2(\Gamma) \) and \( u, \bar{u} \) the corresponding solutions of (3.2)

\[
\begin{cases}
\frac{1}{2} \Delta (u - \bar{u}) - \alpha (u - \bar{u}) = 0 & \text{in } \mathcal{O}, \\
u - \bar{u} + \frac{\partial}{\partial \nu} (u - \bar{u}) + \beta(u) - \beta(\bar{u}) = f - \bar{f} & \text{on } \Gamma.
\end{cases}
\]

(4.8)

This yields, again via the Green’s formula,

\[
\int_{\mathcal{O}} (|\nabla (u - \bar{u})|^2 + 2\alpha(u - \bar{u})^2) dx + \int_{\Gamma} [(u - \bar{u})^2 + \beta(u) - \beta(\bar{u})(u - \bar{u})] d\sigma = \int_{\Gamma} (f - \bar{f})(u - \bar{u}) d\sigma.
\]

As in the proof of the monotonicity of the operator \( \mathcal{A} \) in Section 2, using the monotonicity of the map \( r \mapsto \beta(r) + \gamma r \) and condition (1.3), we get

\[
\|f - \bar{f}\|_{L^2(\Gamma)} \cdot \|u - \bar{u}\|_{L^2(\Gamma)} \\
\geq \|u - \bar{u}\|^2_{L^2(\Gamma)} + \|\nabla (u - \bar{u})\|^2_{L^2(\mathcal{O})} + \|u - \bar{u}\|^2_{L^2(\mathcal{O})} - \gamma \|u - \bar{u}\|^2_{L^2(\Gamma)} \\
\geq \|u - \bar{u}\|^2_{L^2(\Gamma)}
\]

and therefore

\[
\|u - \bar{u}\|_{L^2(\Gamma)} \leq \|f - \bar{f}\|_{L^2(\Gamma)} \quad \text{for all } f, \bar{f} \in L^2(\Gamma),
\]

as claimed. \( \square \)
By the generation theory of $C_0$-semigroups of contractions (as in Section 2, see, e.g., [Bar 10]), we infer that the equation

$$\begin{cases}
\frac{dv(t)}{dt} + \Lambda v(t) = 0 \quad \text{for all } t \geq 0, \\
v(0) = f
\end{cases}$$

has for each $f \in L^2(\Gamma)$ a unique solution $v(t) = W_t f$ for all $T > 0$, given by the exponential formula

$$W_t f = \lim_{n \to \infty} (I + \frac{t}{n})^{-n} f \quad \text{for all } t \geq 0, \text{ uniformly on compacts of } \mathbb{R}_+.$$

We have also for $f \in D(\Lambda)$

$$\frac{d^+ W_t f}{dt} + \Lambda W_t f = 0 \quad \text{for all } t > 0.$$

**Remark** The above result remains true for any continuous and sub-linear function $\beta$ such that $r \mapsto \beta(r) + \gamma r$ is monotonically increasing.

Let us define the linear operator $\Lambda_1 : D(\Lambda_1) \subseteq L^2(\Gamma) \rightarrow L^2(\Gamma)$ as

$$D(\Lambda_1) = \{ \psi \in H^2(\Gamma), \frac{1}{2} \Delta u - \alpha u = 0, \quad u|_{\Gamma} = \varphi, \quad \frac{\partial u}{\partial \nu} \in L^2(\Gamma) \},$$

$$\Lambda_1 \varphi := -\frac{\partial u}{\partial \nu}.$$ 

Since $\beta$ is Lipschitzian, we may define also the Lipschitz operator $\Lambda_2 : L^2(\Gamma) \rightarrow L^2(\Gamma)$, $\Lambda_2(\varphi) := -\beta(\varphi)$. We have $D(\Lambda_1) = D(\Lambda)$, $\Lambda = \Lambda_1 + \Lambda_2$ and so, by (4.9) we have

$$W_t f = e^{-t \Lambda_1} f - \int_0^t e^{-(t-s) \Lambda_1} \Lambda_2 W_s f ds \quad \text{in } \Gamma \text{ for all } t \geq 0,$$

where $e^{-t \Lambda_1}$ is the $C_0$-semigroup of quasi-contractions generated by $-\Lambda_1$ on $L^2(\Gamma)$.

**Theorem 4.4.** There exists a branching Markov process $Z^\Gamma = (Z^\Gamma_t)_{t \geq 0}$ with state space the set $M(\Gamma)$ ($:= \text{the set of positive finite measures on } \Gamma$) such that

$$W_t f = -\ln \mathbb{E}^\delta (e_f(Z^\Gamma_t)) \quad \text{for all } t \geq 0 \text{ and } f \in C(\Gamma),$$

and the following assertions hold.

(i) The process $Z^\Gamma = (Z^\Gamma_t)_{t \geq 0}$ is precisely the $(Z, \alpha - \beta)$-superprocess on $M(\Gamma)$, where $Z = (Z_t)_{t \geq 0}$ is the process on the boundary, induced by the reflected Brownian motion.

(ii) Let $(L^\Gamma, D(L^\Gamma))$ be the generator on $C_0(M(\Gamma))$ of $Z^\Gamma = (Z^\Gamma_t)_{t \geq 0}$. If $f \in C^1(\Gamma)$ then $F := e_f$ belongs to $D(L^\Gamma)$ and for every $\mu \in M(\Gamma)$ we have

$$L^\Gamma F(\mu) = \int_{\Gamma} \mu(dy) \left( F(\mu) \frac{\partial u}{\partial \nu}(y) + \int_0^\infty [F(\mu + s\delta_y) - F(\mu)] \eta(ds) - b F(\mu, y) \right),$$
where \( u \in H^1(\Omega) \) is such that \( \frac{1}{2} \Delta u - \alpha u = 0 \) and \( u|_\Gamma = f \). In particular, if \( \mu \in H^1(\Omega) \) with \( \Delta \mu \in H \), then we have

\[
L^\Gamma F(\mu|_\Gamma) = \left( \frac{1}{2} \Delta u - \alpha \mu, F'(\mu|_\Gamma) \right)_2 - \int_\Gamma \frac{\partial \mu}{\partial \nu} F'(\mu|_\Gamma) d\sigma
\]

\[
+ \int_\Gamma \mu|_\Gamma(dy) \left( \int_0^\infty [F(\mu|_\Gamma + s\delta_y) - F(\mu|_\Gamma)] \eta(ds) - bF'(\mu|_\Gamma, y) \right),
\]

where \( F \) is the extension of \( F \) from \( M(\Gamma) \) to \( M(\overline{\Omega}) \), defined as \( F := e_u \).

**Proof.** (i) Let \((S_t)_{t \geq 0}\) be the transition function of the the process on the boundary \( Z = (Z_t)_{t \geq 0} \), induced by the reflected Brownian motion. \((S_t)_{t \geq 0}\) induces a \( C_0\)-semigroup on \( C(\Gamma) \) (see, e.g., [SaUe 65]) and its generator is \((\Lambda_1 + \alpha, D(\Lambda_1))\): \( e^{-\Lambda t} = e^{-\alpha t} S_t, \ t \geq 0 \); see e.g., [BeVo 14]. By the classical result on the existence of the continuous state branching processes (see Theorem 2.1 and Theorem 2.3 from [Wat 68]; see also [Fitz 88], [Li 11], [Be 11], and [BeLuOp 12]) there exists a branching process \( Z^\Gamma &= (Z^\Gamma_t)_{t \geq 0} \) on \( M(\Gamma) \), such that its transition function \((Q^\Gamma_t)_{t \geq 0}\) is a \( C_0\)-semigroup on \( C_0(M(\Gamma)) \), and we have

\[
Q^\Gamma_t(e_f) = e_{W^\Gamma_t}, \ f \in C_+(\Gamma), t \geq 0,
\]

where \((W^\Gamma_t f)_{t \geq 0}\) is the solution of the integral equation

\[
W^\Gamma_t f = S_t f + \int_0^t S_{t-s} (\alpha - \beta) (W^\Gamma_s f) \, ds, \ t \geq 0.
\]

The process process \( Z^\Gamma \) is the \((Z, \alpha - \beta)\)-superprocess on \( M(\Gamma) \). By Proposition 3.1 from [Be 11] it follows that \((W^\Gamma_t f)_{t \geq 0}\) is a solution of (4.11) too, hence \( W^\Gamma_t f = W^\Gamma_s f \) for all \( t \geq 0 \).

Since \( Q^\Gamma_t(e_f) = \mathbb{E}(e_f(Z^\Gamma_t)) \), the equality (4.12) is a consequence of (4.15).

(ii) The fact that \( F = e_f \in D(L^\Gamma) \) follows from Theorem 2.3 in [Wat 68] since \( f \in D(\Lambda_1) \) if \( f \in C^1(\Gamma) \). Arguing as in the proof of (4.2) we have

\[
L^\Gamma F(\mu) = \frac{d(e_{W^\Gamma_t f}(\mu))}{dt} = -e_f(\mu) \mu(\frac{d^+ W^\Gamma_t f(0)}{dt})(0) = F(\mu) \int_\Gamma \mu(dy) [\Lambda_1 f(y) + \beta(f(y))]
\]

and we deduce further that (4.13) and (4.14) hold.

**Final Remark.** (i) Comparing the equalities (4.2) and (4.14), one can see that

\[
L^\Gamma F(\mu|_\Gamma) = L^\Gamma F(\mu|_\Gamma),
\]

with the notations and under the conditions of assertion (ii) of Proposition 4.4, where recall that \( F|_\Gamma = F \). Observe also that if the measure \( \mu \) has compact support in \( \Omega \), then

\[
L^\Gamma F(\mu) = L^0 F(\mu),
\]

under the conditions of assertion (ii) of Theorem 4.2, where \( L^0 \) is defined in Example 4.1.

(ii) The above assertion (i) justifies the probabilistic interpretation presented in the Introduction. By (4.17) the measure-valued branching process \( X \) on \( M(\overline{\Omega}) \) behaves on measures carried by \( \Omega \) like the linear continuous semiflow, induced by the reflected Brownian motion, presented in Example 4.1. The non-local part of the generator \( L \) of the branching
process $X$, as it is described in (4.2) (the "Lévy measure" part, see e.g. [Sha 88]), indicates that the jumps of $X$ occur only on the set of measures, having non-zero traces on the boundary $\Gamma$. In particular, by (4.16) the process $X$ behaves as the $(Z, \alpha - \beta)$-superprocess on measures carried by $\Gamma$.

(iii) Recall that for the Neumann problem, the boundary process $Z$ on $\Gamma$ is given by the time moments when the Brownian motion on $O$ is reflected at the boundary. Analogously, the $(Z, \alpha - \beta)$-superprocess on $M(\Gamma)$ describes the branching moments of the measure-valued process $X$, associated to the problem (1.1).

(iv) By (4.14) the infinitesimal operator of the $(Z, \alpha - \beta)$-superprocess, the measure-valued "boundary process", has no second order (differential) term, as it happens in the classical case of the infinitesimal operator of the boundary process $Z$, associated to the (linear) Neumann problem; cf. [SaUe 65], p. 570.

(v) Probabilistic representations for solutions of nonlinear Neumann boundary value problems are related to the "catalytic super-Brownian motion". Such a representation formula, similar to (1.4), is given in [DeVo 05] for the nonnegative solution of a mixed Dirichlet nonlinear Neumann boundary value problem; we thank P. J. Fitzsimmons for suggesting us this connection.

(vi) Recall that the nonlinear part $-\beta$ in the Neumann boundary condition of the parabolic problem (1.1) is classically the branching mechanism of a superprocesses associated to the Dirichlet boundary value problem. It is possible to construct branching processes starting with other types of branching mechanisms (for relevant examples see [BeLu 16] and also [BeDeLu 15]) and note that the state space of these "non-local branching" processes is the set of all finite configurations of $O$ (= the set of all finite sums of Dirac measures concentrated in points of $O$). It is known to solve the corresponding Dirichlet boundary value problem; see e.g. [BeOp 11] and [BeOp 14]. However, it is a challenge to investigate the appropriate nonlinear parabolic problem, similar to (1.1).

Note added in proof. This work is a version of the article with the same title [J. Math. Anal. Appl. 441 (2016), 167-182], which details the proofs of a few technical results mentioned without proof in that work. Also, some inaccuracies were eliminated.

References

[Bar 10] Barbu, V., Nonlinear Differential Equations of Monotone Type in Banach Spaces, Springer, 2010.

[BeChRe 84] Berg, C., Christensen, J. P. R., and Ressel, P., Harmonic Analysis on Semigroups, Springer, 1984.

[Be 11] Beznea, L., Potential theoretical methods in the construction of measure-valued branching processes, J. European Math. Soc. 13 (2011), 685–707.

[BeDeLu 15] Beznea, L., Deaconu, M., and Lupaşcu, O., Branching processes for the fragmentation equation. Stochastic Processes and their Applications 125 (2015), 1861–1885.
[BeLu 16] Beznea, L. and Lupaşcu, O., Measure-valued discrete branching Markov processes. Trans. Amer. Math. Soc. 368 (2016), 5153–5176.

[BeLuOp 12] Beznea, L., Lupaşcu, O., and Oprina, A.-G., A unifying construction for measure-valued continuous and discrete branching processes. In Complex Analysis and Potential Theory, CRM Proceedings and Lecture Notes, vol. 55, Amer. Math. Soc., Providence, RI, 2012, 47–59.

[BeOp 11] Beznea L., and Oprina, A.-G., Nonlinear PDEs and measure-valued branching type processes. J. Math. Anal. Appl. 384 (2011), 16–32.

[BeOp 14] Beznea L., and Oprina, A.-G., Bounded and $L^p$-weak solutions for nonlinear equations of measure-valued branching processes. Nonlinear Analysis 107 (2014), 34–46.

[BeVl 14] Beznea, L. and Vlăduţ, S., Markov processes on the Lipschitz boundary for the Neumann and Robin problems, J. Math. Anal. Appl. 455 (2017), 292–311.

[BlGe 68] Blumenthal, R. M. and Getoor, R. K., Markov Processes and Potential Theory Academic Press, New York, 1968.

[Brez 72] Brézis, H., Problemes unilateraux. J. Math. pures et appl. 51 (1972), 1–168.

[DeVo 05] Delmas, J.-F. and Vogt, P., Non-linear Neumann’s condition for the heat equation: a probabilistic representation using catalytic super-Brownian motion. Ann. I. H. Poincaré PR 41 (2005), 817–849

[Dyn 02] Dynkin, E. B., Diffusions, Superdiffusions and Partial Differential Equations, Amer. Math. Soc. Colloq. Publ. 50, Amer. Math. Soc., 2002.

[Fitz 88] Fitzsimmons, P. J., Construction and regularity of measure-valued Markov branching processes. Israel J. Math. 64 (1988), 337–361.

[LeGa 99] Le Gall, J.-F., Spatial Branching Processes, Random Snakes and Partial Differential Equations (Lectures in Mathematics ETH Zürich) Birkhäuser, 1999.

[Li 11] Li, Z. H., Measure-Valued Branching Markov Processes, Probab. Appl., Springer, 2011.

[SaUe 65] Sato, K. and Ueno, T., Multi-dimensional diffusion and the Markov process on the boundary. J. Math. Kyoto Univ. 4 (1965), 529–605.

[Sha 88] Sharpe, M., General Theory of Markov Processes, Academic Press, Boston, 1988.

[Wat 68] Watanabe, S., A limit theorem of branching processes and continuous state branching processes, J. Math. Kyoto Univ. 8 (1968), 141–167.