Markov dilations of semigroups of Fourier multipliers

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Abstract
We describe a Markov dilation for any weak* continuous semigroup \((T_t)_{t \geq 0}\) of selfadjoint unital completely positive Fourier multipliers acting on the group von Neumann algebra \(VN(G)\) of a locally compact group \(G\).

Keywords  Semigroups · Dilations · Fourier multipliers · Completely positive maps · Crossed products

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1 Introduction

Fourier multipliers are the most used operators in (noncommutative) harmonic analysis. In particular, the study of semigroups of these operators is a central issue. In the noncommutative setting, these are connected to a large number of topics, as approximation properties [15, 20] or noncommutative geometry [10].

In this paper, we will focus on weak* continuous semigroup \((T_t)_{t \geq 0}\) of selfadjoint unital completely positive Fourier multipliers acting on the group von Neumann algebra \(VN(G)\) of an arbitrary locally compact group \(G\). We construct a Markov dilation
in the sense of [28, p. 637] for these semigroups, which allows us to state our main result Theorem 3.1.

This construction makes it possible to use martingale theory and obtain results in analysis with probability tools. Indeed, Junge and Mei studied in [29] several BMO-spaces associated to a Markov semigroup on a semifinite von Neumann algebra (i.e. a noncommutative $L^\infty$-space). In particular, the authors obtained interpolation results. Their approach relies on Markov dilations of semigroups. Note that some extensions of these results were generalized to the $\sigma$-finite von Neumann algebras in the paper [17]. Other applications of these dilations include some estimates related to Riesz transforms [29] and connected to the curvature assumption $\Gamma^2 \geq 0$, boundedness of the $H^\infty$ functional calculus of the (negative) generator of the semigroup [27] [19] with applications to maximal inequalities and to ergodic theory.

Observe that Markov dilations are really different from the kind of dilation constructed in the papers [3–5] in the spirit of classical Fendler’s isometric dilation theorem [18] (see also [7, 8] for related works). Furthermore, it is remarkable that the algebra associated to the dilation of this paper is the same that the algebra associated to the dilation constructed in [5]. However, we were unable to find a direct connection between the construction of this paper and the dilation described in [3].

Note that Markov dilations of weak* continuous semigroups of selfadjoint unital completely positive measurable Schur multipliers on $\sigma$-finite measure spaces were explicitly constructed in [6]. In [17], it is described how to obtain Markov dilations of radial semigroups on free Araki-Woods factors. The paper [16] contains a construction of Markov dilations of semigroups of double operator integrals. Finally, an unpublished paper [30] of Junge, Ricard and Shlyakhtenko describes a construction of a Markov dilation for any weak* continuous semigroup of selfadjoint unital completely positive maps on a finite von Neumann algebra (see also the recent [42] for a related construction in the type III case). In the three preceding papers, the construction relies on the use of ultraproduct methods with sharp contrast with the ones of [6] and of this paper. The construction of concrete dilations and Markov dilations remains very important, especially for Ricci curvature bounds, see e.g. [13].

In the discrete setting of a semigroup $(T^n)_{n \in \mathbb{N}}$ associated to an operator $T$ acting on a von Neumann algebra, a notion of Markov dilation was introduced in [2] and thoroughly investigating in [23]. Examples of such dilations are also provided in [38].

Structure of the paper The paper is organized as follows. The next Sect. 2 gives background on probabilities, Fourier multipliers and crossed products. In Sect. 3, we state and prove our dilation result and we describe a reversed result.

2 Preliminaries

Isonormal processes Let $H$ be a real Hilbert space. An $H$-isonormal process on a probability space $(\Omega, \mu)$ [35, Definition 1.1.1] [33, Definition 6.5] is a linear mapping $W : H \to L^0(\Omega)$ from $H$ into the space $L^0(\Omega)$ of measurable functions on $\Omega$ with the following properties:
for any $h \in H$ the random variable $W(h)$ is a centered real Gaussian, \hspace{1cm} (2.1)
for any $h_1, h_2 \in H$ we have $\mathbb{E}(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_H$, \hspace{1cm} (2.2)
the linear span of the products $W(h_1)W(h_2)\cdots W(h_m)$, with $m \geq 0$ and $h_1, \ldots, h_m$ in $H$, is dense in the real Hilbert space $L^2_{\mathbb{R}}(\Omega)$. \hspace{1cm} (2.3)

Here we make the convention that the empty product, corresponding to $m = 0$ in (2.3), is the constant function 1. Moreover, $\mathbb{E}$ is used to denote expected value.

If $(e_i)_{i \in I}$ is an orthonormal basis of $H$ and if $(\gamma_i)_{i \in I}$ is a family of independent standard Gaussian random variables on a probability space $\Omega$ then for any $h \in H$, the family $(\gamma_i(h, e_i)_H)_{i \in I}$ is summable in $L^2(\Omega)$ and

$$W(h) \overset{\text{def}}{=} \sum_{i \in I} \gamma_i(h, e_i)_H, \quad h \in H \hspace{1cm} (2.4)$$
defines an $H$-isonymous process.

Recall that the span of elements $e^{iW(h)}$ where $h \in H$ is weak* dense in $L^\infty(\Omega)$ by [26, Remark 2.15 p. 22]. It is easy to prove that we can replace $H$ by a dense subset of $H$ with Lemma 2.1 below. Using [25, Proposition E.2.2] with $t$ instead of $\xi$ and by observing by (2.2) that the variance $\mathbb{E}(W(h)^2)$ of the Gaussian variable $W(h)$ is equal to $\|h\|_H^2$, we see that

$$\mathbb{E}(e^{itW(h)}) = e^{-\frac{t^2}{2}\|h\|_H^2}, \quad t \in \mathbb{R}, \ h \in H. \hspace{1cm} (2.5)$$

If $u : H \rightarrow H$ is a contraction, we denote by $\Gamma^\infty(u) : L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ the (symmetric) second quantization of $u$ acting on the complex Banach space $L^\infty(\Omega)$. Recall that the map $\Gamma^\infty(u) : L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ preserves the integral.\footnote{That means that for any $f \in L^\infty(\Omega)$ we have $\int_{\Omega} \Gamma^\infty(u)f \, d\mu = \int_{\Omega} f \, d\mu$.} If $u$ is a surjective isometry we have

$$\Gamma^\infty(u)(e^{iW(h)}) = e^{iW(u(h))}, \quad h \in H \hspace{1cm} (2.6)$$

and $\Gamma^\infty(u) : L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ is a *-automorphism of the von Neumann algebra $L^\infty(\Omega)$. If $P : H \rightarrow H$ is an orthogonal projection on a closed subspace $K$, the operator $\Gamma^\infty(u) : L^\infty(\Omega) \rightarrow L^\infty(\Omega)$ is a faithful normal conditional expectation with range $L^\infty(\Omega, \mathcal{F})$ where $\mathcal{F}$ is the $\sigma$-algebra generated by the random variables $W(h)$ for $h \in K$, see [26, Theorem 4.9 p. 46].

Furthermore, the second quantization functor $\Gamma$ satisfies the following elementary result [5, Lemma 2.1]. In the first part, we suppose that the construction\footnote{The existence of a proof of Lemma 2.1 without (2.4) is unclear.} is given by the concrete representation (2.4).

**Lemma 2.1** \hspace{0.5cm} 1. If $L^\infty(\Omega)$ is equipped with the weak* topology then the map $H \rightarrow L^\infty(\Omega), h \mapsto e^{iW(h)}$ is continuous.
2. If \( \pi : G \to B(H) \) is a strongly continuous orthogonal representation of a locally compact group, then \( G \to B(L^\infty(\Omega)) \), \( s \mapsto \Gamma^\infty(\pi_s) \) is a weak* continuous \(^3\) representation on the Banach space \( L^\infty(\Omega) \).

Let \( H \) be a real Hilbert space. Following [34, Definition 2.2] and [33, Definition 6.11], we say that an \( L^2(\mathbb{R}^+, H) \)-isonormal process \( W \) is an \( H \)-cylindrical Brownian motion. In this case, for any \( t \geq 0 \) and any \( h \in H \), we let

\[
W_t(h) \overset{\text{def}}{=} W(1_{[0,t]} \otimes h).
\] (2.7)

We introduce the filtration \( (\mathcal{F}_t)_{t \geq 0} \) defined by

\[
\mathcal{F}_t \overset{\text{def}}{=} \sigma(W_r(h) : r \in [0,t], h \in H),
\] (2.8)

that is the \( \sigma \)-algebra generated by the random variables \( W_r(h) \) for \( r \in [0,t] \) and \( h \in H \).

By [33, p. 77], for any fixed \( h \in H \), the family \( (W_t(h))_{t \geq 0} \) is a Brownian motion. This means by essentially [33, Definition 6.2] that

\[
W_0(h) = 0 \text{ almost surely,}
\] (2.9)

\[
W_t(h) - W_u(h) \text{ is Gaussian with variance } (t-u)\|h\|_H^2 \text{ for any } 0 \leq u \leq t,
\] (2.10)

\[
W_t(h) - W_u(h) \text{ is independent of } \{W_r(h) : r \in [0,u]\} \text{ for any } 0 \leq u \leq t.
\] (2.11)

Indeed by [33, p. 163],

the increment \( W_t(h) - W_u(h) \) is independent of the \( \sigma \)-algebra \( \mathcal{F}_u \). \hspace{1cm} (2.12)

Moreover, by [33, p. 163] the family \( (W_t(h))_{t \geq 0} \) is a martingale with respect to \( (\mathcal{F}_t)_{t \geq 0} \). In particular, the random variable \( W_t(h) \) is \( \mathcal{F}_t \)-measurable. If \( 0 \leq u \leq t \), note that

\[
\|1_{[u,t]} \otimes h\|_{L^2(\mathbb{R}_+)}^2 = \|1_{[u,t]}\|_{L^2(\mathbb{R}_+)}^2 \|h\|_H^2 = (t-u)\|h\|_H^2.
\]

Using (2.5) together with the previous computation, we obtain

\[
\mathbb{E}(e^{iW(1_{[u,t]} \otimes h)}) = e^{-\frac{(t-u)}{2}\|h\|_H^2}, \quad 0 \leq u \leq t, \quad h \in H.
\] (2.13)

**Probabilities** Let \( \Omega \) be a probability space. If \( f \in L^1(\Omega) \) is independent of the sub-\( \sigma \)-algebra \( \mathcal{F} \), then by [24, Proposition 2.6.35] its conditional expectation \( \mathbb{E}_{\mathcal{F}}(f) \) with respect to \( \mathcal{F} \) is given by the constant function:

\[
\mathbb{E}_{\mathcal{F}}(f) = \mathbb{E}(f).
\] (2.14)

\(^3\) That means that \( B(L^\infty(\Omega)) \) is equipped with the point weak* topology.
If \( g \in L^\infty(\Omega) \) is \( \mathcal{F} \)-measurable and \( f \in L^1(\Omega) \), we have by [24, Proposition 2.6.31]

\[
\mathbb{E}_\mathcal{F}(gf) = g \mathbb{E}_\mathcal{F}(f).
\]

**Group von Neumann algebras** Let \( G \) be a locally compact group equipped with a fixed left Haar measure \( \mu_G \). The group von Neumann algebra of \( G \) is the von Neumann algebra generated by the set \( \{ \lambda_s : s \in G \} \) where \( \lambda_s : L^2(G) \to L^2(G), \; f \mapsto (t \mapsto f(s^{-1}t)) \) is the left translation by \( s \).

**Crossed products** We refer to [39] and [40] for more information on crossed products. Let \( M \) be a von Neumann algebra acting on a Hilbert space \( H \). Let \( G \) be a locally compact group equipped with some left Haar measure \( \mu_G \). Let \( \alpha : G \to M \) be a representation of \( G \) on \( M \) which is weak* continuous, i.e., for any \( x \in M \) and any \( y \in M_s \), the map \( G \to M, \; s \mapsto \langle \alpha_s(x), y \rangle_{M,M_s} \) is continuous. For any \( x \in M \), we define the operators \( \pi(x) : L^2(G, H) \to L^2(G, H) \) [39, (2) p. 263] by

\[
(\pi(x)\xi)(s) \overset{\text{def}}{=} \alpha_s^{-1}(x)\xi(s), \quad \xi \in L^2(G, H), \; s \in G. \tag{2.16}
\]

These operators satisfy the following commutation relation [39, (2) p. 292]:

\[
(\lambda_s \otimes \text{Id}_H)\pi(x)(\lambda_t \otimes \text{Id}_H)^* = \pi(\alpha_s(x)), \quad x \in M, \; s \in G. \tag{2.17}
\]

Recall that the crossed product of \( M \) and \( G \) with respect to \( \alpha \) is the von Neumann algebra

\[
M \rtimes_\alpha G \overset{\text{def}}{=} (\pi(M) \cup \{ \lambda_s \otimes \text{Id}_H : s \in G \})''
\]

on the Hilbert space \( L^2(G, H) \) generated by the operators \( \pi(x) \) and \( \lambda_s \otimes \text{Id}_H \) where \( x \in M \) and \( s \in G \). By [39, p. 263], \( \pi \) is a normal injective *-homomorphism from \( M \) into \( M \rtimes_\alpha G \) (hence \( \sigma \)-strong* continuous).

We denote by \( \mathcal{K}(G, M) \) the space of \( \sigma \)-strong* continuous functions \( f : G \to M, \; s \mapsto f_s \) with compact support. If \( f \in \mathcal{K}(G, M) \) then \( f(G) \) is a \( \sigma \)-strong* compact subset of \( M \), hence by [36, Proposition 2.7 d)] a \( \sigma \)-strong* bounded subset of \( M \). Hence it is a strong bounded subset and finally a norm-bounded subset of \( M \) by the principle of uniform boundedness [31, Theorem 1.8.9]. Note that by [39, Proposition p. 186] and [39, p. 41], the bounded function \( G \to M, \; s \mapsto \lambda_s \otimes \text{Id}_H \) is \( \sigma \)-strong* continuous and the norm-bounded function \( s \mapsto \pi(f_s) \) is also \( \sigma \)-strong* continuous. Recall that the product of \( M \) is \( \sigma \)-strong* continuous on bounded subsets by [14, Proposition 2.4.5]. We infer\(^4\) that the function \( G \to M \rtimes_\alpha G, \; s \mapsto \pi(f_s)(\lambda_s \otimes \text{Id}_H) \) is \( \sigma \)-strong* continuous with compact support.

\(^4\) In the book [39], the author considers weak* continuous functions, it is problematic since the product of \( M \) is not weak* continuous even on bounded sets by [31, Exercise 5.7.9] (indeed this latter fact is equivalent to the weak continuity of the product on bounded sets).
So, by [5, Lemma 2.2] and [12, Corollary 2, III p. 38] we can define the element
\[
\int_G f_s \rtimes \lambda_s \, d\mu_G(s)
\]
of the crossed product \( M \rtimes \alpha \) by
\[
\int_G f_s \rtimes \lambda_s \, d\mu_G(s) \overset{\text{def}}{=} \int_G \pi(f_s)(\lambda_s \otimes \text{Id}_H) \, d\mu_G(s).
\] (2.18)

The following is a particular case\(^5\) of [41, Proposition 3.5] and its proof, see also [40, Theorem 1.7 (ii) p. 241]. Note that the von Neumann algebra \( M \) is abelian in the statement. With [12, Proposition 2, III p. 35], the last part is an easy computation left to the reader.

**Proposition 2.2** Let \( M \) be an abelian von Neumann algebra acting on a Hilbert space \( H \) equipped with a weak* continuous action \( \alpha \) of a locally compact group \( G \). Suppose that there exists a strongly continuous function \( u : G \to \mathcal{U}(M) \) such that
\[
u(sr) = u(s)\alpha_s(u(r)), \quad s, r \in G.
\] (2.19)

Then \( V : L^2(G, H) \to L^2(G, H), \xi \mapsto (s \mapsto u(s^{-1})\xi(s)) \) is a unitary and we have a *-isomorphism \( U : M \rtimes \alpha \to M \rtimes \alpha \), \( x \mapsto VxV^* \) such that
\[
U(\lambda_s \otimes \text{Id}_H) = \pi(u(s)^*)(\lambda_s \otimes \text{Id}_H) \quad \text{and} \quad U(\pi(x)) = \pi(x), \quad s \in G, x \in M.
\]
Moreover, for any \( f \in \mathcal{K}(G, M) \), we have
\[
U\left(\int_G f_s \rtimes \lambda_s \, d\mu_G(s)\right) = \int_G u(s)^* f_s \rtimes \lambda_s \, d\mu_G(s).
\] (2.20)

Now, we suppose that the von Neumann algebra \( M \) is finite and equipped with a normal finite faithful trace \( \tau \). By [21, Lemma 3.3] [39, Theorem p. 301] [40, Theorem 1.17 p. 249], there exists a unique normal semifinite faithful weight \( \varphi_{\ltimes} \) on the crossed product \( M \rtimes \alpha \) which satisfies for any \( f, g \in \mathcal{K}(G, M) \) the fundamental “noncommutative Plancherel formula”
\[
\varphi_{\ltimes} \left( \left( \int_G f_s \rtimes \lambda_s \, d\mu_G(s) \right)^* \left( \int_G g_s \rtimes \lambda_s \, d\mu_G(s) \right) \right) = \int_G \tau(f_s^* g_s) \, d\mu_G(s)
\] (2.21)
and the relations
\[
\sigma_t^{\varphi_{\ltimes}}(\pi(x)) = \pi(x) \quad \text{where} \quad x \in M \quad \text{and} \quad t \in \mathbb{R}
\]
and
\[
\sigma_t^{\varphi_{\ltimes}}(\lambda_s \otimes \text{Id}_H) = \Delta^\mu_G(s)(\lambda_s \otimes \text{Id}_H) \pi([D(\tau \circ \alpha_s) : \text{D}]) \quad \text{for} \quad s \in G, x \in M.
\]

If \( M = \mathbb{C} \), we recover the Plancherel weight \( \varphi_G \) on the group von Neumann algebra \( \text{VN}(G) \) [40, p. 67]. If each \( \alpha_s : M \to M \) is trace preserving, we obtain in particular
\[
\sigma_t^{\varphi_{\ltimes}}(\lambda_s \otimes \text{Id}_H) = \Delta^\mu_G(s)(\lambda_s \otimes \text{Id}_H), \quad s \in G, t \in \mathbb{R}.
\]

\(^5\) The function \( u : G \to \mathcal{U}(M) \) is a \( \alpha \)-1-cocycle.
Using [12, Proposition 2, III p. 35], we deduce that

\[
\sigma_t^\phi (\int_G f_s \rtimes \lambda_s \, d\mu_G(s)) = \int_G \Delta_t^\phi G(s) f_s \rtimes \lambda_s \, d\mu_G(s), \quad f \in \mathcal{K}(G, M), \quad t \in \mathbb{R}.
\]

(2.22)

By [22, Theorem 4.1], we have the following result. Note that the proof of [22, Theorem 4.1] does not use the fact that \( G \) is abelian. The second part is an obvious observation left to the reader.

**Lemma 2.3** Let \( G \) be a locally compact group and \( \alpha : G \to \text{Aut}(M) \) be a weak* continuous action on a von Neumann algebra \( M \) equipped with a normal semifinite faithful weight. Let \( \mathbb{E} : M \to M \) be a weight preserving faithful normal conditional expectation such that \( \mathbb{E}\alpha_s = \alpha_s \mathbb{E} \) for any \( s \in G \).

1. There exists a weight preserving faithful normal conditional expectation \( \mathbb{E} \rtimes \text{Id}_{\mathcal{VN}(G)} : M \rtimes_\alpha G \to M \rtimes_\alpha G \) such that for any \( s \in G \) and any \( x \in M \)

\[
(\mathbb{E} \rtimes \text{Id}_{\mathcal{VN}(G)})(\pi(x)) = \pi(\mathbb{E}(x)), \quad (\mathbb{E} \rtimes \text{Id}_{\mathcal{VN}(G)})(\lambda_s \otimes \text{Id}_H) = \lambda_s \otimes \text{Id}_H.
\]

2. For any function \( f \in \mathcal{K}(G, M) \), we have

\[
(\mathbb{E} \rtimes \text{Id}_{\mathcal{VN}(G)})(\int_G f_s \rtimes \lambda_s \, d\mu_G(s)) = \int_G \mathbb{E}(f_s) \rtimes \lambda_s \, d\mu_G(s).
\]

(2.23)

**Weights** We will use the following result which is a particular case of [39, Theorem 6.2 p. 83]. Recall that a normal semifinite weight \( \psi \) commutes with a normal semifinite faithful weight \( \varphi \) if \( \psi \circ \sigma_t^\varphi = \psi \) for any \( t \in \mathbb{R} \), see [39, p. 68] and that \( n_\varphi \) is a weak* dense *-subalgebra of \( M \) such that \( A \subseteq n_\varphi \) which is \( \sigma^\psi \)-invariant such that

\[
\psi(x^*x) = \varphi(x^*x), \quad x \in A.
\]

Then \( \varphi = \psi \).

**Fourier multipliers** Let \( G \) be a locally compact group. We say that a weak* continuous operator \( T : \mathcal{VN}(G) \to \mathcal{VN}(G) \) is a Fourier multiplier if there exists a continuous function \( \phi : G \to \mathbb{C} \) such that for any \( s \in G \) we have \( T(\lambda_s) = \phi(s) \lambda_s \). In this case, \( \phi \) is bounded and for any function \( f \in C_c(G) \) the element \( \int_G \phi(s) f(s) \lambda_s \, d\mu_G(s) \) belongs to the von Neumann algebra \( \mathcal{VN}(G) \) and

\[
T\left( \int_G f(s) \lambda_s \, d\mu_G(s) \right) = \int_G \phi(s) f(s) \lambda_s \, d\mu_G(s).
\]

(2.24)
In this case, we let \( M_{\phi} \overset{\text{def}}{=} T \) and we say that \( \phi \) is the symbol of \( T \). We refer to the books [32] and [9] and references therein for more information.

**Semigroups of Fourier multipliers** Consider a locally compact group \( G \) with identity element \( e \). Let \((T_t)_{t \geq 0}\) be a weak* continuous semigroup of selfadjoint unital completely positive Fourier multipliers. There exists a (unique) continuous real-valued conditionally negative definite function \( \psi : G \to \mathbb{R} \) satisfying \( \psi(e) = 0 \) such that
\[
T_t(\lambda_s) = e^{-t\psi(s)}\lambda_s, \quad t \geq 0, \ s \in G.
\]

In this case, there exists a real Hilbert space \( H \) together with a mapping \( b_{\psi} : G \to H \) and a homomorphism \( \pi : G \to O(H) \) such that the 1-cocycle law holds
\[
\pi_s(b_{\psi}(r)) = b_{\psi}(sr) - b_{\psi}(s), \quad \text{i.e.} \quad b_{\psi}(sr) = b_{\psi}(s) + \pi_s(b_{\psi}(r)) \tag{2.25}
\]
for any \( s, r \in G \) and such that
\[
\psi(s) = \|b_{\psi}(s)\|_H^2, \quad s \in G. \tag{2.26}
\]

We refer to the book [11] for more information on affine isometric actions of groups and 1-cocycles.

### 3 Markov dilations of semigroups of Fourier multipliers

Our main result is the following theorem which gives a standard Markov dilation. Here, we equip the von Neumann algebra \( \text{VN}(G) \) with the Plancherel weight.

**Theorem 3.1** Let \( G \) be a locally compact group. Consider a weak* continuous semigroup \((T_t)_{t \geq 0}\) of selfadjoint unital completely positive Fourier multipliers on \( \text{VN}(G) \) defined by (2.26). Then there exists a von Neumann algebra \( M \) equipped with a normal semifinite faithful weight \( \phi_M \), an increasing filtration \((M_t)_{t \geq 0}\) of the algebra \( M \) with associated weight preserving normal faithful conditional expectations \( \mathbb{E}_t : M \to M_t \) and weight preserving unital normal injective \(*\)-homomorphisms \( \pi_t : \text{VN}(G) \to M_t \) such that
\[
\mathbb{E}_u \pi_t = \pi_{uT_{t-u}}, \quad 0 \leq u \leq t. \tag{3.1}
\]

Moreover, we have the following properties.

1. If \( G \) is discrete then the weight \( \phi_M \) is a normal finite faithful trace.
2. If \( G \) is unimodular then the weight \( \phi_M \) is a normal semifinite faithful trace.
3. If \( G \) is amenable then the von Neumann algebra \( M \) is injective.

**Proof** Here, we suppose that \( H, \pi \) and \( b_{\psi} \) are defined as in (2.25). Let \( W : L^2_{\mathbb{R}^+}(\mathbb{R}^+, H) \to L^0(\Omega) \) be an \( H \)-cylindrical Brownian motion on a probability space \((\Omega, \mu)\), see Sect. 2. For any \( s \in G \), we will use the second quantization \( \alpha_s \overset{\text{def}}{=} \Gamma_\infty(\text{Id}_{L^2_{\mathbb{R}^+}(\mathbb{R}^+)}) \otimes \mathbb{E}_s \).
$\pi_s) : L^\infty(\Omega) \to L^\infty(\Omega)$ which is integral preserving. In particular, if $r, s \in G$ and if $t \geq 0$, we have

$$
\alpha_s(e^{-\sqrt{2iW_t(b_\psi(r))}}) = \Gamma^\infty(Id_{L^2(\mathbb{R}^+)} \otimes \pi_s)(e^{-\sqrt{2iW_t(b_\psi(r))}}) \overset{\text{(2.6)}}{=} e^{-\sqrt{2iW_t}(\pi_s(b_\psi(r)))}.
$$

(3.2)

Since the orthogonal representation $\pi$ is strongly continuous, we obtain by Lemma 2.1 a continuous action $\alpha : G \to \text{Aut}(L^\infty(\Omega))$. So we can consider the crossed product $M \overset{\text{def}}{=} L^\infty(\Omega) \rtimes_\alpha G$ equipped with its canonical normal semifinite faithful weight $\varphi_M \overset{\text{def}}{=} \varphi_\pi$. We denote by $J : \text{VN}(G) \to L^\infty(\Omega) \rtimes_\alpha G$ the canonical unital normal injective $*$-homomorphism. Using [12, Proposition 2, III p. 35], for any $f \in C_c(G)$, we see that

$$
J\left(\int_G f(s)\lambda_s \ d\mu_G(s)\right) = \int_G f(s) 1 \rtimes \lambda_s \ d\mu_G(s).
$$

(3.3)

The same proof as the one of [6, Lemma 3.2], shows that the map $J$ is weight preserving. For any $t \in \mathbb{R}$, we consider the function $u_t : G \to U(L^\infty(\Omega)), s \mapsto e^{-\sqrt{2iW_t}(b_\psi(s))}$. The map $b_\psi : G \to H$ is continuous. By the first point of Lemma 2.1, the map $L^2_\mathbb{R}(\mathbb{R}^+, H) \to L^\infty(\Omega), g \mapsto e^{iW(g)}$ is continuous if $L^\infty(\Omega)$ is equipped with the weak* topology, hence with the weak operator topology when we consider that the von Neumann algebra $L^\infty(\Omega)$ acts on $L^2(\Omega)$. Recall that by [31, Exercice 5.7.5] or [39, p. 41] the weak operator topology and the strong operator topology coincide on the unitary group $U(L^\infty(\Omega))$. So by composition, the function $u_t$ is continuous if $U(L^\infty(\Omega))$ is equipped with the strong operator topology. For any $t \geq 0$ and any $r, s \in G$, note that

$$
u_t(sr) = e^{-\sqrt{2iW_t}(b_\psi(sr))} \overset{\text{(2.25)}}{=} e^{-\sqrt{2iW_t}(b_\psi(s))} e^{-\sqrt{2iW_t}(\pi_s(b_\psi(r)))} \overset{\text{(3.2)}}{=} u_t(s)\alpha_s(e^{-\sqrt{2iW_t}(b_\psi(r))}) = u_t(s)\alpha_s(u_r(r)).
$$

Hence (2.19) is satisfied. By Proposition 2.2, for any $t \geq 0$, we have a unitary $V_t : L^2(G, L^2(\Omega)) \to L^2(G, L^2(\Omega)), \xi \mapsto (s \mapsto u_t(s^{-1})(\xi(s)))$ and a $*$-isomorphism

$$
U_t : L^\infty(\Omega) \rtimes_\alpha G \to L^\infty(\Omega) \rtimes_\alpha G
$$

such that for any function $f \in \mathcal{K}(G, L^\infty(\Omega))$

$$
U_t\left(\int_G f_s \rtimes \lambda_s \ d\mu_G(s)\right) = \int_G e^{\sqrt{2iW_t}(b_\psi(s))} f_s \rtimes \lambda_s \ d\mu_G(s), \ t \geq 0.
$$

(3.4)

\[\square\]
Lemma 3.2 For any $t \geq 0$, the map $U_t$ is weight preserving.

**Proof** We will use Lemma 2.4 with the weights $\varphi_\infty$ and $\varphi_\times U_t$ on $L^\infty(\Omega) \rtimes_\alpha G$. Note that the space of elements $\int_G f_s \rtimes \lambda_s d\mu_G(s)$ for $f \in K(G, L^\infty(\Omega))$ is a *-subalgebra which is $\sigma^{\varphi_\times}$-invariant by (2.22), weak* dense in $L^\infty(\Omega) \rtimes_\alpha G$ and included in $n_{\varphi_\infty}$. The formulas (2.22) and (3.4) show that each $U_t$ and $\sigma_t^{\varphi_\times}$ commute. So, we have

$$\varphi_\times \circ U_t \circ \sigma_t^{\varphi_\times} = \varphi_\times \circ \sigma_t^{\varphi_\times} \circ U_t = \varphi_\times \circ U_t.$$ 

So the weights $\varphi_\times \circ U_t$ and $\varphi_\times$ commutes by [39, pp. 67-68]. It is easy to check that the weight $\varphi_\times \circ U_t$ is normal and semifinite. If $f \in K(G, L^\infty(\Omega))$, we have

$$\varphi_\times \circ U_t \left( \left( \int_G f_s \rtimes \lambda_s d\mu_G(s) \right) \right)^* \left( \int_G f_s \rtimes \lambda_s d\mu_G(s) \right) = \varphi_\times \left( \left( U_t \left( \int_G f_s \rtimes \lambda_s d\mu_G(s) \right) \right)^* \left( U_t \left( \int_G f_s \rtimes \lambda_s d\mu_G(s) \right) \right) \right) \tag{3.4}$$

$$= \varphi_\times \left( \left( \int_G e^{\sqrt{2i}W_t(b_\psi(s))} f_s \rtimes \lambda_s d\mu_G(s) \right) \right)^* \left( \int_G e^{\sqrt{2i}W_t(b_\psi(s))} f_s \rtimes \lambda_s d\mu_G(s) \right) \tag{2.21}
\int_G \int_{\Omega} e^{-\sqrt{2i}W_t(b_\psi(s))} f_s^* e^{\sqrt{2i}W_t(b_\psi(s))} f_s d\mu_G(s) = \int_G \int_{\Omega} f_s^* f_s d\mu_G(s) \tag{2.21}
= \varphi_\times \left( \left( \int_G f_s \rtimes \lambda_s d\mu_G(s) \right) \right)^* \left( \int_G f_s \rtimes \lambda_s d\mu_G(s) \right).$$

We conclude with Lemma 2.4 that $\varphi_\times \circ U_t = \varphi_\times$ for any $t \geq 0$. \hfill \Box

For any $t \geq 0$, we define the unital normal injective *-homomorphism

$$\pi_t \overset{\text{def}}{=} U_t J : VN(G) \to L^\infty(\Omega) \rtimes_\alpha G. \tag{3.5}$$

Each $\pi_t$ is weight preserving by composition. For any $t \geq 0$, we also define the canonical normal conditional expectations $E_{\mathcal{F}_t} : L^\infty(\Omega) \to L^\infty(\Omega)$ on $L^\infty(\Omega, \mathcal{F}_t)$ where the $\sigma$-algebra $\mathcal{F}_t$ is defined in (2.8). Recall that $(W_t(h))_{t \geq 0}$ is a Brownian motion for any fixed $h \in H$. Hence for any $0 \leq u \leq t$ and any $s \in G$ the random variable

$$W(1_{[u,t]} \otimes b_\psi(s))
= W(1_{[0,t]} \otimes b_\psi(s)) - W(1_{[0,u]} \otimes b_\psi(s)) \overset{(2.7)}{=} W_t(b_\psi(s)) - W_u(b_\psi(s)) \tag{3.6}$$

is independent by (2.12) from the $\sigma$-algebra $\mathcal{F}_u \overset{(2.8)}{=} \sigma(W_r(h) : r \in [0, u], h \in H)$. Consequently, the random variable $e^{\sqrt{2i}W(1_{[u,t]} \otimes b_\psi(s))}$ is also independent from the $\sigma$-algebra $\mathcal{F}_u$.

**Lemma 3.3** The $\sigma$-algebra $\mathcal{F}_u$ is equal to the $\sigma$-algebra $\mathcal{G}$ generated by the random variables $W(g)$ where $g \in L^2_{\mathcal{F}}([0, u], H)$. \hfill \qed
Proof} It suffices to show that the space $L^\infty(\Omega, \mathcal{F}_u)$ is weak* dense in $L^\infty(\Omega, \mathcal{G})$. Note that by [24, Remark 1.2.20 p. 24] the subspace $E$ of elements $\sum_{k=1}^{m} 1_{(c_k, d_k]} \otimes h_k$, where $0 \leq c_1 < d_1 < c_2 < d_2 < \cdots < c_m < d_m \leq u$ and $h_1, \ldots, h_m \in H$, is dense in the real Hilbert space $L^2_{\mathbb{R}}([0, u], H)$. Hence by the discussion after (2.4) the span of elements $e^{iW(f)}$ where $f \in E$ is weak* dense in the space $\Gamma_1(L^2_{\mathbb{R}}([0, u], H))$, which identifies to $L^\infty(\Omega, \mathcal{G})$. We can conclude since $L^\infty(\Omega, \mathcal{F}_u)$ contains these elements.

Consider the projection $P_u : L^2_{\mathbb{R}}(\mathbb{R}^+, H) \to L^2_{\mathbb{R}}(\mathbb{R}^+, H)$ on the closed subspace $L^2_{\mathbb{R}}([0, u], H)$. By the previous lemma and by an observation following (2.6), we see that $\mathbb{E}_{\mathcal{F}_u} = \Gamma^\infty(P_u \otimes \text{Id}_H)$ for any $u \geq 0$. Consequently, for any $s \in G$ and any $u \geq 0$, we obtain that

$$\alpha_s \mathbb{E}_{\mathcal{F}_u} = \Gamma^\infty(\text{Id}_{L^2_{\mathbb{R}}(\mathbb{R}^+)} \otimes \pi_s) \Gamma^\infty(P_u \otimes \text{Id}_H) = \Gamma^\infty(P_u \otimes \pi_s) = \mathbb{E}_{\mathcal{F}_u} \alpha_s.$$ 

So by Lemma 2.3, we can consider the map $\mathbb{E}_t \overset{\text{def}}{=} \mathbb{E}_{\mathcal{F}_t} \times \text{Id}_{\mathcal{VN}(G)} : L^\infty(\Omega) \times_{\alpha} G \to L^\infty(\Omega, \mathcal{F}_t) \times_{\alpha} G$. We introduce the von Neumann algebra $N_t \overset{\text{def}}{=} L^\infty(\Omega, \mathcal{F}_t) \times_{\alpha} G$. Moreover, we have

\begin{equation}
\mathbb{E}_{\mathcal{F}_u}(e^{\sqrt{2iW_i(b_\psi(s))}}) \overset{(2.15)}{=} e^{\sqrt{2iW_u(b_\psi(s))}} \mathbb{E}_{\mathcal{F}_u}(e^{\sqrt{2iW(1_{[u,t]} \otimes b_\psi(s))}}) \\
\overset{(2.14)}{=} e^{\sqrt{2iW_u(b_\psi(s))}} \mathbb{E}(e^{\sqrt{2iW(1_{[u,t]} \otimes b_\psi(s))}}) \overset{(2.13)}{=} e^{-(t-u)\|b_\psi(s)\|^2} e^{\sqrt{2iW_u(b_\psi(s))}}.
\end{equation}

(3.7)

For any function $f \in C_c(G)$ and any $t \geq 0$, we have

$$\pi_t \left( \int_G f(s) \lambda_s \, d\mu_G(s) \right) \overset{(3.5)}{=} \pi_t J \left( \int_G f(s) \lambda_s \, d\mu_G(s) \right) = \pi_t \left( \int_G f(s) \lambda_s \, d\mu_G(s) \right) \overset{(3.4)}{=} \int_G f(s) e^{\sqrt{2iW_i(b_\psi(s))}} \times \lambda_s \, d\mu_G(s).$$

(3.8)

Similarly, for any $0 \leq u \leq t$, we have

$$\pi_u T_{t-u} \left( \int_G f(s) \lambda_s \, d\mu_G(s) \right) \overset{(2.24)}{=} \pi_u \left( \int_G e^{-(t-u)\|b_\psi(s)\|^2} f(s) \lambda_s \, d\mu_G(s) \right) \overset{(3.8)}{=} \int_G f(s) e^{-(t-u)\|b_\psi(s)\|^2} e^{\sqrt{2iW_u(b_\psi(s))}} \times \lambda_s \, d\mu_G(s).$$

(3.9)

$\square$ Springer
We finally obtain for any $0 \leq u \leq t$ and any function $f \in C_c(G)$

$$
\mathbb{E}_u \pi_t \left( \int_G f(s) \lambda_s \, d\mu_G(s) \right) \overset{(3.8)}{=} \mathbb{E}_u \left( \int_G f(s)e^{\sqrt{2}iW_t(b_{\psi}(s))} \lambda_s \, d\mu_G(s) \right)
$$

$$
\overset{(2.23)}{=} \int_G f(s)\mathbb{E}_s \left( e^{\sqrt{2}iW_t(b_{\psi}(s))} \right) \lambda_s \, d\mu_G(s)
$$

$$
\overset{(3.7)}{=} \int_G f(s)e^{-(t-u)\|b_{\psi}(s)\|^2}e^{\sqrt{2}iW_u(b_{\psi}(s))} \lambda_s \, d\mu_G(s)
$$

$$
\overset{(3.9)}{=} \pi_u T_{t-u} \left( \int_G f(s) \lambda_s \, d\mu_G(s) \right).
$$

By weak* density, the proof is complete.

Now, we prove the last assertions. Note each $\alpha_s: L^\infty(\Omega) \to L^\infty(\Omega)$ preserves the integral. The first is well-known, e. g. [37, Corollary 7.11.8]. The second is folklore. The third is [1, Proposition p. 301].

Similarly, we can prove the following reversed Markov dilation.

**Theorem 3.4** Let $G$ be a locally compact group. Consider a weak* continuous semigroup $(T_t)_t \geq 0$ of selfadjoint unital completely positive Fourier multipliers on $\text{VN}(G)$ defined by (2.26). There exists a von Neumann algebra $M$ equipped with a normal semifinite faithful weight, a decreasing filtration $(M_t)_t \geq 0$ of $M$ with associated weight preserving normal faithful conditional expectations $\mathbb{E}_t: M \to M_t$ and weight preserving unital normal injective *-homomorphisms $\pi_t: \text{VN}(G) \to M_t$ such that

$$
\mathbb{E}_u \pi_t = \pi_u T_{t-u}, \quad 0 \leq t \leq u.
$$

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