I. INTRODUCTION

Non-linear sigma (NLσ) models have been extensively studied in various branches of physics. Specifically, the O(3) NLσ model in two dimensions has been identified as a good toy model of the SU(2) pure Yang-Mills theory in four dimensions, because they share many important features such as conformal invariance, dynamical mass gap and existence of the instanton solution. Moreover, the O(3) NLσ model (with higher differential terms) can be derived as a low energy effective theory for the SU(2) Yang-Mills theory \( F, SU(2) \), Heisenberg model etc. The topological soliton solutions like the instantons and the Hopfions play a crucial role for the non-perturbative aspects in the theories \( F, SU(2) \).

It is of great importance to consider SU(N) generalizations of the O(3) NLσ model, because they share some basic properties with the SU(N) Yang-Mills theory and may possibly be derived from fundamental theories like the SU(N) Yang-Mills theory or the SU(N) Heisenberg model. There may be several variants, and in particular two feasible cases have widely been examined; the \( CP^{N-1} \) and \( F_{N-1} \) NLσ model. The models are NLσ models whose target spaces are the complex projective space \( CP^{N-1} = SU(N)/U(N-1) \) and the flag space \( F_{N-1} = SU(N)/U(1)^{N-1} \), respectively. Note that the \( CP^1(= F_1) \) NLσ model is equivalent to the O(3) NLσ model at least in classical level. Indeed, the \( CP^{N-1} \) models arose as an effective theory from the SU(N) Yang-Mills theory with the minimal case of Maximal Abelian gauge (MAG) \( F, SU(N) \) anti-ferromagnetic SU(N) Heisenberg model \( F, SU(N) \) and so on. On the other hand, the \( F_{N-1} \) model was considered from the SU(N) Yang-Mills theory by means of the maximal case of MAG \( F, SU(N) \). Moreover, the \( F_2 \) model was derived from the SU(3) anti-ferromagnetic Heisenberg model in 1+1 dimensions \( F, SU(3) \) and in 2+1 dimensions \( F, SU(3) \).

In this paper, we study classical solutions of the \( F_2 \) model. There are quite few studies for classical solutions of the \( F_{N-1} \) models, even in the \( N = 3 \) case, though ones of the \( CP^{N-1} \) models have been studied for all \( N \) since the late 70s \( F, SU(N) \). To our knowledge two seminal papers about classical solutions of the \( F_2 \) model are the ones. In \( F, SU(N) \), it was shown that the BPS equation is solved by the \( F_1 \) instantons embedded into the target space \( F_3 \). In \( F, SU(N) \), the author studied classical solutions of the \( F_2 \) model with an additional term called Kalb-Ramond field which is non topological term and then contributes to the Euler-Lagrange equation. In some special choices of the coupling constant, the general static solutions were discussed in context of the integrability \( F, SU(N) \). Note that the Kalb-Ramond field can naturally appear when one derives the \( F_2 \) model from the SU(3) spin chain \( F, SU(N) \). In contrast, from the SU(3) Yang-Mills theory or the SU(3) Heisenberg model on lattice, the term may not be provided.

Main goal of this paper is to find genuine (which means nonembedding) solutions in the \( F_2 \) NLσ model without the Kalb-Ramond term. The model in (3+1)-dimensional Minkowski space-time is defined by the Lagrangian density

\[
\mathcal{L} = - \left( \| Z^a_\mu \partial_\mu Z_b \|^2 + \| Z^b_\mu \partial_\mu Z_c \|^2 + \| Z^c_\mu \partial_\mu Z_a \|^2 \right) .
\]

Note that, the double vertical line norm for (composite) vector fields denotes the Minkowski norm with the metric \( g_{\mu\nu} = (-, +, +, +) \), thus the square of the norm is not positive definite. Here, the fields \( Z_a \) (\( a = A, B, C \)) are three components complex vectors which form a complete basis. Therefore we can construct SU(3) matrix from U(3) matrix \( U = (Z_A, Z_B, Z_C) \) and the fields satisfy the orthonormal condition and the completeness condition i.e.

\[
Z^a_\mu Z^b = \delta_{ab} ,
\]

\[
Z_A \otimes Z^A + Z_B \otimes Z^B + Z_C \otimes Z^C = 1_3
\]

where \( 1_3 \) is the 3 \( \times \) 3 identity matrix. The Lagrangian is invariant under the left global U(3) transformation such as \( U \rightarrow gU, g \in U(3) \). In addition, it is also invariant under the local U(1)\(^3\) transformation i.e. \( g \) three U(1) phase rotations \( Z_a \rightarrow Z^*_a = e^{i \theta_a} Z_a, a = A, B, C \).

The paper is organized as follows. In Sec.II, we briefly review the previous study of an embedding solution of the model. In Sec.III, the basic information for constructing the solutions including the parametrization, the topological charges are given. Sec.IV presents both the static and the time-dependent solutions of the model. Some properties of the energy, for example an energy bound for the static solutions, are devoted in Sec.V. Summary and discussion are in Sec.VI.
II. EMBEDDING SOLUTIONS

As we mentioned, the authors of [21] found a BPS solution which is just an \( F_1 (= CP^1) \) instanton embedded into the target space \( F_2 \). Let us begin with a brief review of their work with introduction of some notations.

The discussion is based on an analogy with the BPS sector in the \( CP^2 \) model. For sake the reader’s understanding, we introduce the covariant derivative

\[
D^{(a)}_i = \partial_i - \left( Z^c_i \partial_i Z_a \right) \quad a = A, B, C. \tag{4}
\]

and write the static energy per unit length such as

\[
E = \frac{1}{2} \int d^2z \sum_{a=A,B,C} \left\| D^{(a)}_i Z_a \right\|^2. \tag{5}
\]

Each term in (5) can be viewed as the static energy of the \( CP^2 \) model linked by the orthonormal condition (2). Thus, by simply applying the BPS bound of the \( CP^2 \) model to the three terms respectively, an energy lower bound may be derived as

\[
E \geq \pi \left( |N_A| + |N_B| + |N_C| \right) \tag{6}
\]

where \( N_a \) is the topological degrees of the map \( Z_a : \mathbb{R}^2 \to S^2 \to CP^2 \) defined as

\[
N_a = \frac{i}{2\pi} \int d^2x \epsilon^{ij} \left( D^{(a)}_i Z_a \right)^j. \tag{7}
\]

Note that there is a constraint among the topological degrees of the form

\[
N_A + N_B + N_C = 0. \tag{8}
\]

The equality in (6) is satisfied if the following (implicitly) coupled first derivative equations are simultaneously satisfied:

\[
D^{(a)}_i Z_a = \pm i e^{ij} D^{(a)}_j Z_a \quad \forall a = A, B, C. \tag{9}
\]

Unfortunately, the solutions of (9) are given by only a very limited class of configurations, i.e. an \( F_1 \) instantons embedded into the target space \( F_2 \), for which one of the vectors \( Z_a \) is fixed as a trivial one. By performing a proper global transformation, the solutions can generally be written as

\[
Z_A = \frac{1}{\sqrt{\Delta}} \left( 1, p(z)/q(z), 0 \right)^T, \\
Z_B = \frac{1}{\sqrt{\Delta}} \left( p(\bar{z})/q(\bar{z}), -1, 0 \right)^T, \\
Z_C = (0, 0, 1)^T,
\]

with \( \Delta \equiv 1 + |p(z)|^2/|q(z)|^2 \). Here, we have introduced the standard complex variables

\[
z = x^1 + ix^2, \quad \bar{z} = x^1 - ix^2, \tag{11}
\]

and the functions \( p(z) \) and \( q(z) \) which are polynomials of \( z \) with no common roots. The solution (10) has apparently \( N_C = 0 \), and therefore, the energy is given by

\[
E = \pi \left( |N_A| + |N_B| \right) = 2\pi N_A = -2\pi N_B \tag{12}
\]

where \( N_A = -N_B \) is given the highest power of \( z \) in either \( p(z) \) or \( q(z) \).

The embedding solutions are unique possibility of solutions in the equation (9). For, (9) requires all the vectors \( Z_a \) are holomorphic or anti-holomorphic, but there is no pair of three components (anti-) holomorphic vectors without constant vectors. If one wants to get more general solution, obviously different strategy has to be implemented. Our approach for the problem is to solve the Euler-Lagrange equation, without touching first-order equations. For the solutions, of course, the energy bound (6) is no longer saturated.

The notable thing is that the nonembedding solutions satisfy a new coupled first-order equations different from (9). As a result, we obtain new energy lower bound which of course is different from (6). Note that the energy is higher than that of embedding solutions and the new energy bound contains a non-topological term. The detailed analysis will be thoroughly discussed in Sec.V. Existence of such solutions will give us deeper understanding of the BPS structure of NLS models whose target space is non-symmetric space.

III. PRELIMINARIES

This section is preliminaries for analyzing the Euler-Lagrange equations and some properties of the energy functional.

A. Parametrization

In order to simplify the analysis, we parametrize the vectors \( Z_a \) in terms of complex scalar fields [24]. It can be realized by means of the isomorphism

\[
SU(3)/U(1)^2 \cong SL(3, \mathbb{C})/B_+ \tag{13}
\]

where \( B_+ \) is the Borel subgroup of the upper triangular matrices with determinant being equal to one. In geometrical point of view, the complex scalar fields mean the local coordinates of the manifold \( F_2 \). The coordinates can be introduced through a mapping \( SU(3)/U(1)^2 \to SL(3, \mathbb{C})/B_+ \) which is an extension of the stereographic projection i.e. \( S^2 \to \mathbb{R}^2 \). By a formal result of the map, one can write an element of \( F_2 \) in terms of local coordinates \( u_i, i = 1, 2, 3 \) as the lower triangular matrix

\[
X = \begin{pmatrix} 1 & 0 & 0 \\ u_1 & 1 & 0 \\ u_2 & u_3 & 1 \end{pmatrix}. \tag{14}
\]

Since \( \det X = 1 \), \( X \) is an element of \( SL(3, \mathbb{C}) \). Note that, however, it is not necessary for \( X \) to be an unitary matrix.

Next, in order to obtain the \( SU(3) \) matrix \( U \) or the vectors \( Z_a \) parametrized in terms of the scalar fields, we consider the inverse of the mapping, i.e. \( SL(3, \mathbb{C})/B_+ \to SU(3)/U(1)^2 \). Then, one can construct \( U \) from \( X \) by means of the so-called Iwasawa decomposition: any element of \( SL(3, \mathbb{C}) \) can be factorized into an element of \( SU(3) \) and \( B_+ \) in a unique fashion, up to torus elements of \( U(1)^2 \). The procedure is as following. We regard \( X \) as element of \( SL(3, \mathbb{C}) \) which can be expressed in terms of the column vectors such as

\[
X = (c_1, c_2, c_3) \in SL(3, \mathbb{C}) \tag{15}
\]

with

\[
c_1 = (1, u_1, u_2)^T, \quad c_2 = (0, 1, u_3)^T, \quad c_3 = (0, 0, 1)^T. \tag{16}
\]

In this case, the Iwasawa decomposition can be proved by the Gramm-Schmit orthogonalization process for (16). Then one
obtains mutually orthogonal vectors
\[
\begin{align*}
  e_A &= c_1 , \\
  e_B &= c_2 - \frac{(c_2, e_A)}{(e_A, e_A)} e_A , \\
  e_C &= c_3 - \frac{(c_3, e_B)}{(e_B, e_B)} e_B - \frac{(c_3, e_A)}{(e_A, e_A)} e_A ,
\end{align*}
\]
where the inner product is defined as
\[
(c_i, c_j) = c_j^t c_i .
\]
By normalizing the vectors, one obtains an orthonormal basis. Therefore, we can define \( Z_a = \frac{1}{\sqrt{(e_i, e_j)}} e_a \), \( a = A, B, C \), and they can explicitly be written as
\[
\begin{align*}
  Z_A &= \frac{1}{\sqrt{\Delta_1}} \begin{pmatrix} 1 \\ u_1 \\ u_2 \end{pmatrix} , \\
  Z_B &= \frac{1}{\sqrt{\Delta_1 \Delta_2}} \begin{pmatrix} -u_1^* - u_2 u_3 \\
    1 - u_1 u_2^* u_3 + |u_2|^2 \\
    -u_1^* u_2^* + u_3 |u_2|^2 \end{pmatrix} , \\
  Z_C &= \frac{1}{\sqrt{\Delta_2}} \begin{pmatrix} u_1^* u_3^* - u_2^* \\
    -u_3 \\
    1 \end{pmatrix}
\end{align*}
\]
where
\[
\Delta_1 = 1 + |u_1|^2 + |u_2|^2 , \\
\Delta_2 = 1 + |u_3|^2 + |u_1 u_2 - u_2^*|^2 .
\]
For later convenience, we express the off-diagonal components of the one-form \( Z_i^j dZ_i \) as
\[
\begin{align*}
  Z_B^i dZ_A &= \frac{1}{\Delta_1 \sqrt{\Delta_2}} (P_{11} d u_1 + P_{12} d u_2 + P_{13} d u_3) , \\
  Z_C^i dZ_A &= \frac{1}{\sqrt{\Delta_1 \Delta_2}} (P_{21} d u_1 + P_{22} d u_2 + P_{23} d u_3) , \\
  Z_C^i dZ_B &= \frac{1}{\Delta_2} (P_{31} d u_1 + P_{32} d u_2 + P_{33} d u_3)
\end{align*}
\]
where
\[
\begin{pmatrix} P_{11} & P_{12} & P_{13} \\
  P_{21} & P_{22} & P_{23} \\
  P_{31} & P_{32} & P_{33} \end{pmatrix} = \begin{pmatrix} 1 - u_1^* u_2 u_3^* + |u_2|^2 & -u_1 u_2^* + u_3^* |u_2|^2 & 0 \\
  -u_1^* u_2^* & u_1^* + u_2^* u_3 & -\Delta_1 \\
  -u_3^* (u_1^* + u_2^* u_3) & u_3^* + u_2^* u_3 & 0 \end{pmatrix} .
\]

B. Topological properties

We consider static configurations in two space dimensions \((x^1, x^2)\) and also with wave components traveling along \( x^3 \) axis with the speed of light. The vectors \( \mathbf{e} \) provide a mapping from \( \mathbb{R}^2 \) (or \( \mathbb{R}^2 \times \mathbb{R}^2 \)) into \( F_2 \). However, for finiteness of the energy, the vectors should go to a constant at infinity on the \( x^1 x^2 \) plane. Thus, the plane can be identified as \( S^2 \). Since both the static and time-dependent configurations localize only on the plane, they are topologically classified into homotopy classes of a mapping \( S^2 \to F_2 \), i.e., \( \pi_2(F_2) \).

In addition, there exists a useful theorem (see e.g. [11]):
\[
\pi_2(G/H) = \pi_1(H) \text{ where } \pi_1(H) \subseteq \pi_1(G)
\]
formed by closed paths in \( H \) which can be contracted to a point in \( G \). Therefore, these configurations are characterized by the second homotopy group
\[
\pi_2(F_2) = \pi_2(SU(3)/U(1)^2) = \pi_1(U(1)^2) = \mathbb{Z} + \mathbb{Z}
\]
which guarantees that the solutions are classified by two independent integers. Note that there are three topological degrees defined in \[10\], but independent degrees are just two due to the identity \[8\].

We can also define a topological invariant by means of Kähler structure of the flag space \( F_2 \), though \( F_2 \) is not Kähler manifold. Note that the non-symmetric manifolds, unlike symmetric manifolds, possesses a family of invariant metrics including both Kähler and non-Kähler metrics. In order to simplify the discussion, we consider the Kähler case. Then the corresponding Kähler potential can be defined as
\[
K = K_1 + K_2
\]
where \( K_j = m_j \log \Delta_j, \ j = 1, 2, \) with \( m_j \) being a positive constant. Since the manifold is now the Kähler one, it possesses the closed two-form \( \Omega \) i.e. \( d \Omega = 0 \). Here, \( d = \partial + \bar{\partial} = d u_1 \frac{\partial}{\partial u_1} + d u_2 \frac{\partial}{\partial u_2} \) denote the exterior derivative, while the operator \( \partial \) and \( \bar{\partial} \) are called the Dolbeault operators. The condition \( d \Omega = 0 \) is equivalent to
\[
\Omega = i \partial \bar{\partial} K .
\]
If one divides the two-form into two parts \( \Omega_j = i \partial \bar{\partial} K_j \), they are given by
\[
\begin{align*}
  \Omega_1 &= \frac{im_1}{\Delta_1} \{ (1 + |u_2|^2) d u_1 \wedge d u_1^* - u_1 u_2^* d u_3 \wedge d u_2^* \\
  &\quad - u_1^* u_2 d u_2 \wedge d u_1 + (1 + |u_1|^2) d u_2 \wedge d u_2^* \} , \\
  \Omega_2 &= \frac{im_2}{\Delta_2} \{ (1 + |u_3|^2) d (u_1 u_3 - u_2) \wedge d (u_1^* u_3 - u_2^*) \\
  &\quad - u_3 (u_1^* u_3^* - u_2^*) d (u_1 u_3 - u_2) \wedge d u_3^* \\
  &\quad - u_3^* (u_1 u_3 - u_2) d (u_1^* u_3^* - u_2^*) + (1 + |u_1 u_3 - u_2|^2) d u_3 \wedge d u_3^* \} .
\end{align*}
\]
The topological invariant \( Q \) is defined in terms of the integral of the Kähler two-form
\[
Q \equiv Q_1 + Q_2, \quad Q_j = \int \Omega_j .
\]
The relation of the topological invariant and the winding numbers is directly calculable by the integration
\[
\begin{align*}
  Q_1 &= -i m_1 \int d^2 x \varepsilon^{ij} (D_i^A Z_\Lambda) \Lambda \wedge (D_j^A Z_\Lambda) \wedge \\
  &= 2 \pi m_1 N_A , \\
  Q_2 &= i m_2 \int d^2 x \varepsilon^{ij} (D_i^C Z_\Sigma) \Lambda \wedge (D_j^C Z_\Sigma) \wedge \\
  &= -2 \pi m_2 N_C .
\end{align*}
\]
Consequently, after an appropriate normalization, i.e. \( m_1 = m_2 = \frac{1}{2\pi} \), the topological invariant is characterized just by two integers \( N_A \) and \( N_C \) as
\[
Q = N_A - N_C
\]
TABLE I. The values of \((N_A, N_C)\) for the winding numbers \(n_1, n_2, n_3\). As was shown in Sec III-B, the topological charge is given by \(Q = N_A - N_C\).

which of course is consistent with the formal argument \(22\).

There are exact expressions of the topological invariants corresponding to \(32\). However, the field configurations in this model are characterized by two integers and it is not so straightforward to find proper form of the topological charge interpreted as the net number of solitons. The definition \(31\) seems suitable for the net number of solitons for the parametrization \(19\).

As we shall see in Sec.V, the static solutions of the equations are proportional to \(11\) like standard BPS type solutions and then we call \(31\) as the topological charge.

\[ \Delta_1 \Delta_2 (P_{11} \partial^\mu \partial_\mu u_1 + P_{12} \partial^\mu \partial_\mu u_2) - 2 \Delta_2 (P_{11} \partial_\mu u_1 + P_{12} \partial_\mu u_2) (u_1^{\dagger} \partial^\mu u_1 + u_2^{\dagger} \partial^\mu u_2) \]

\[ - \Delta_1 [(u_1 + u_2 u_3) (u_3^{\dagger} \partial^\mu u_1 - \partial^\mu u_2) + \Delta_1 \partial^\mu u^*_{12} (u_3 \partial^\mu u_1 - \partial^\mu u_2)] = 0, \]

\[ \Delta_1 \Delta_2 (P_{21} \partial^\mu \partial_\mu u_1 + P_{22} \partial^\mu \partial_\mu u_2 - \partial^\mu u_3 \partial_\mu u_1) \]

\[ - (P_{21} \partial_\mu u_1 + P_{22} \partial_\mu u_2) |\Delta_2 [u_1^{\dagger} \partial^\mu u_1 + u_2^{\dagger} \partial^\mu u_2] + \Delta_1 [(u_1^{\dagger} u_2^{\dagger} - u_2^{\dagger} u_1^{\dagger}) \partial^\mu (u_1 u_3 - u_2) + u_3^{\dagger} \partial^\mu u_3^*] = 0, \]

\[ \Delta_1 \Delta_2 \left( P_{31} \partial^\mu \partial_\mu u_1 + P_{32} \partial^\mu \partial_\mu u_2 + P_{33} \partial^\mu \partial_\mu u_3 - 2 (u_1 + u_2 u_3) \partial_\mu u_1 \partial^\mu u_3 \right) \]

\[ - 2 \Delta_1 (P_{31} \partial_\mu u_1 + P_{32} \partial_\mu u_2 + P_{33} \partial_\mu u_3) \left( (u_1 u_2^* - u_2 u_1^*) \partial^\mu (u_1 u_3 - u_2) + u_3^{\dagger} \partial^\mu u_3 \right) \]

\[ - \Delta_2 \left[ (1 - u_1 u_2 u_3 + |u_2|^2) \partial^\mu u_1^{\dagger} + (-u_1^* u_2 + u_3 + u_3 |u_1|^2) \partial_\mu u_2 \right] (u_3 \partial^\mu u_1 - \partial^\mu u_2) = 0, \]

where the explicit form of \(\Delta_1, \Delta_2\) and \(P_{ij}\) \(i, j = 1, 2, 3\) are given in \(20\) and \(24\), respectively.

Since the equations are quite complicated and highly nonlinear, for the moment we introduce the simplified ansatz: all the complex scalars \(u_i\) depend only on \(z\) and \(y_\pm\) \((y_\pm\) is defined as \(y_\pm = x^0 \pm x^3\), i.e.

\[ u_i = u_i(z, y_\pm) \quad \text{and} \quad u_i^* = u_i^*(z, y_\pm). \]

It is easy to see that the ansatz satisfies

\[ \partial^\mu \partial_\mu u_i = 0, \quad \partial^\mu u_i \partial_\mu u_j = 0. \]

One can directly check that the second equation \(38\) is automatically satisfied by the ansatz \(10\). \(37\) and \(39\) are certainly simplified

\[ [P_{12} (u_3 \partial^\mu u_1^{\dagger} - \partial^\mu u_2) + \Delta_1 \partial^\mu u_1] (u_3 \partial^\mu u_1 - \partial^\mu u_2) = 0, \]

\[ [P_{11} \partial^\mu u_1 + P_{12} \partial^\mu u_2] (u_3 \partial^\mu u_1 - \partial^\mu u_2) = 0. \]

Therefore, if we find the field configurations satisfying

\[ u_3 \partial_\mu u_1 - \partial_\mu u_2 = 0, \]

they exactly solve \(37\). \(38\) and \(39\).

First we consider the static solutions within the ansatz \(10\) which satisfy \(31\). Note that if one scalar field is set to a constant, there are only trivial and embedding solutions. In
the intriguing case where all the scalars aren’t constants, the solutions can generally be written as

\[ u_1 = \frac{p_1(z)}{q_1(z)}, \quad u_2 = \frac{p_2(z)}{q_2(z)}, \quad u_3 = \frac{\partial_z u_2}{\partial_z u_1} \quad (45) \]

where \( p_i(z) \) and \( q_i(z) \) are irreducible polynomials of \( z \) and \( u_1 \) isn’t proportional to \( u_2 \). The winding number of the scalar field \( u_i \) is equal to number of the poles of the scalar field including those at infinity and thereby

\[ n_i = \max\{\deg(p_i), \deg(q_i)\}, \quad i = 1, 2 \]
\[ n_3 = n_2 - n_1. \quad (46) \]

According to the derivation of the topological charge in the \( CP^N \) model \[19\], the topological degrees \( N_A \) and \( N_C \) are given by pair of the winding numbers \( n_i \) as

\[ N_A = \max(0, n_1, n_2) - \min(0, n_1, n_2), \quad (47) \]
\[ N_C = \max(-n_2, -n_3, 0) - \min(-n_2, -n_3, 0). \quad (48) \]

As we summarize in Table \[I\] the topological degrees \( (N_A, N_C) \) are thus defined by pair of the winding numbers with opposite signs.

The most simple choice of the static solution \[15\] seems

\[ u_1 = c_1 z^{n_1}, \quad u_2 = c_2 z^{n_2}, \quad u_3 = \frac{c_2 p_2}{c_1 q_1} z^{n_2 - n_1} \quad (49) \]

where \( c_i \) are non-zero complex constants. In Fig \[1\] we plot the topological charge density of \[49\]. In the case of \((n_1, n_2) = (1, 2)\), it is lump shaped of which the peak locates at the origin, while for higher winding numbers, the solutions always exhibit the crater like structure. Anisotropic configuration can also be examined. The solution

\[ u_1 = c_1 z, \quad u_2 = z^2 - 1, \quad u_3 = \frac{2}{c_1} z \quad (50) \]
are not allowed for the case that one or more of the functions \( \alpha, \beta, A \) of the energy per unit length is given as

\[
\frac{1}{a_3} - z^3, \quad \frac{1}{a_2} - z^4, \quad \frac{1}{a_3} - z.
\]

From left to right \( a_3 = 0.5, a_2 = 0.5, a_1 = 0.5 \)

V. PROPERTY OF THE ENERGY

In this section, we discuss the energy of the static solutions \((51)\) and also of the wave solution \((53)\). We show that the static solutions saturate to an energy lower bound different from \((6)\). It is proved that the solutions \((51)\) are obtained through coupled first derivative equations which correspond to the saturation condition. For the traveling wave solution \((53)\), the energy can be written by the topological charge plus Noether charges which are relevant to some \(U(1)\) symmetries.

Using the completeness condition \((3)\), the static energy per unit length \((5)\) can be written as

\[
E = \int d^2x \left( \| D_i^{(A)} Z_A \|^2 + \| D_i^{(C)} Z_C \|^2 - \| Z_C \partial_i Z_A \|^2 \right).
\]

We again emphasize that the first and second term in \((54)\) are the static energy of the \(CP^2\) model. Applying the BPS bound of the \(CP^2\) model to the two terms, we obtain the energy bound

\[
E \leq \frac{1}{2} \int d^2x \left\| D_i^{(A)} Z_A \mp i \varepsilon i^j D_j^{(A)} Z_A \right\|^2 + \frac{1}{2} \int d^2x \left\| D_i^{(C)} Z_C \mp i \varepsilon i^j D_j^{(C)} Z_C \right\|^2
\]

\[
\geq 2\pi (|N_A| + |N_C|) - \int d^2x \left\| Z_C \partial_i Z_A \right\|^2.
\]

According to the discussion in the \(CP^2\) model, the equality in \((55)\) is satisfied when the scalar fields are functions of only \(z\) (or \(\bar{z}\)), i.e. \( u_i = u_i(z) \). Thus the static solutions \((51)\) obviously saturate the new bound \((55)\). Unlike standard BPS bound, \((55)\) has non-topological term (the last term in \((55)\)).
The equations (58) are equivalent to the Cauchy-Riemann solutions (45): following coupled first-order equations corresponding to the bound (55) and the lower inequality in (56), we obtain the harmonic map solutions in the $\text{CP}^2$ instanton-anti instanton bound state which described by the interpretation that the nonembedding solutions include long to the same topological sector. This is consistent with the maximum of the non-topological term (without minus sign) gives the equivalence between the two bounds (6) and (56), the value of the term runs

$$0 \leq \int d^2x \| Z_\alpha^A \partial_i Z_\alpha \|^2 \leq \pi (|N_A| - |N_B| + |N_C|).$$

(56)

One can find from (21) that $Z_\alpha^A \partial_i Z_\alpha$ is proportional to $u_3 \partial_i u_1 - \partial_i u_2$. Therefore, for the solutions of (44), this non-topological term identically vanishes (then takes the lower bound of the inequality of (56)). Consequently, the static energy of the nonembedding solutions (44) is given by

$$E = 2\pi (|N_A| + |N_C|) = 2\pi Q.$$

(57)

On the other hand, for the embedding solutions which $u_1$ and $u_3$ are constants, the term satisfies the lower bound of the inequality (56) and then it corresponds to the energy bound (6) with $N_B = 0$. Thus, we find that the static energy is twice greater than that of the embedding solution which belongs to the same topological sector. This is consistent with the interpretation that the nonembedding solutions include instanton-anti instanton bound state which described by the harmonic map solutions in the $\text{CP}^2$ model.

Note that, from the saturation conditions of the energy bound (56) and the lower inequality in (56), we obtain the following coupled first-order equations corresponding to the solutions (44):

$$D_\alpha^{(A)} Z_\lambda = i e^{ij} D_j^{(A)} Z_\lambda,$$

$$D_\alpha^{(C)} Z_C = -i e^{ij} D_j^{(C)} Z_C,$$

(58)

and

$$Z_\lambda^i \partial_i Z_\lambda = 0.$$

(59)

The equations (58) are equivalent to the Cauchy-Riemann equations for the complex scalar fields $u_1$ and (59) exactly is (44). This clearly means that the solutions (44) of the Euler-Lagrange equations (50) can be obtained by the first-order equations (58) and (59). Note that even when the saturation of the energy bound (55) is attained, the energy does not take a stationary point due to the presence of the non topological term. According to the above discussion, however, if both (58) and (59) are satisfied, the energy is stationary, i.e. maximal energy in the bound (55). Therefore we conclude that the solutions (59) which satisfy both the saturation conditions, are saddle-point.

The solutions (56) possess basic properties as BPS solutions, such that they saturate an energy lower bound and are the solutions of first order equations. They seem to be unstable, because they are not of the global minima in a given topological sector, which is fulfilled by the embedding solutions. It is worth to check behavior of the energy for changing the parameters of the configuration and confirm the stability nature of our solutions. In Fig.4 we plot the energy per unit length for the configuration

$$u_1 = \frac{1 - a_1}{a_1} z, u_2 = \frac{1 - a_2}{a_2} z^4, u_3 = \frac{1 - a_3}{a_3} z^7,$$

(60)

with the parameters $a_i \in [0, 1]$. The nonembedding solutions locates on the ridge line of the surface, while the embedding solutions are on the left front corner. There is a zero mode along the ridge, but they essentially be unstable. A natural question occurs where the nonembedding solutions might eventually decay into the embedding ones. In order to see this in detail, we plot the cuts of the energy surface at several values of $a_1$ for fixed $a_2 = 0.5$ in Fig.4. For the configuration (60), unless $u_1 u_3 = u_2$ or $u_2 = 0$, at infinities of the $x^1 x^2$ plane the fields $Z_\lambda$ approach the value

$$Z_\lambda \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad Z_B \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad Z_C \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$
up to phase factors. The configuration \( U \) with \( u_1 u_3 = u_2 \) or \( u_2 = 0 \) do not support the boundary condition \( \text{tilde}(1) \) and then, they belong to a different topological sector from the nonembedding and embedding solutions. It clearly indicates that the solutions cannot continuously deform into the configuration which satisfy \( u_1 u_3 = u_2 \) or \( u_2 = 0 \) with keeping finiteness of the energy. One can find the topological barrier associated with \( u_1 u_3 = u_2 \) lies between the nonembedding and the embedding solutions. It is a common feature among all topological classification in Table \( I \) that such a barrier lies between nonembedding and embedding solutions. For configurations corresponding to the domain \( i \), or \( ii. \), the above example, the barrier exists on the line corresponding to \( u_1 u_3 = u_2 \) which do not obey the boundary condition of the solutions at infinity. For the domain \( iii. \) or \( vi. \), the condition \( u_1 u_3 = u_2 \) does not satisfy the boundary condition of solutions at the origin. In this case, one can also easily prove the condition \( u_1 u_3 = u_2 \) lies between nonembedding and embedding solutions. The situation for the domain \( iii. \) or \( vi. \) are little more complicate than for the others. Nonembedding solutions in the domain exist in region \( c_1 c_2 c_3 < 0 \) and, on the other hand, embedding solutions are derived as \( c_1, c_2 \rightarrow \infty \) with keeping the condition \( c_1 = c_2 c_3 \), which gives \( c_1 c_2 c_3 > 0 \). Therefore during deformation from a nonembedding solution to an embedding solution, at least one of the coefficient have to change the sign. At the point \( c_1 = 0 \), \( i = 1, 2, 3 \), the boundary condition of the solutions are broken and therefore the barrier appears between solutions. This observation indicates the nonembedding solution cannot decay into the embeddings in classical level, at least, through a deformation described by variations of the parameter in \( 60 \). However, quantum mechanically the tunneling between these solutions may occurs and then the decay process may be able to achieve. Some readers may expect the other possibility. One can find lower energy region behind the ridges in Fig. 4, to achieve. Some readers may expects and then the decay process may be able to achieve. It is worth to comment on scaling property like Derrick's theorem for the traveling wave solutions. Because the traveling wave solutions localize only on the \( x_1 x_2 \) plane, we consider the rescaling \( x_i \rightarrow \lambda x_i, i = 1, 2 \). Then, the energy contribution from the \( x_3 \)-derivative term is scaled like a potential term. On the other hand, as discussed in the \( Q \)-lump case \( 20 \), the contribution from the time-derivative term behaves like a quadratic term in \( x_i \)-derivatives. The scaling property of the terms are opposite, but when \( \lambda = 1 \), the contribution from the \( x_3 \)-derivative term and the time-derivative term are equal. Therefore, the traveling wave solutions satisfy the Derrick type condition.

### VI. SUMMARY AND DISCUSSION

In this paper, we have constructed both static and time-dependent solutions of codimension two in the F\(_2\) NL\( \sigma \) model by analytically solving the second order Euler-Lagrange equation. The static solutions possess novel features; the solutions saturate an energy lower bound, and satisfy coupled first order differential equations corresponding the saturation condition, then in this sense they are BPS solutions. On the other hand, caused by the fact that a non-topological term is contained in the energy bound, they are saddle point solutions, i.e., sphalerons. As far as we know, our solutions are a rare example of "BPS sphalerons". The time-dependent solutions are given by a product of localizing component on the \( x^2 \) plane and traveling wave being parallel to \( x^2 \) with speed of light. The energy of the wave solutions are given
FIG. 5. Combination of winding numbers corresponding to the solutions in the form \([53]\), where the infinite energy (per unit length) solutions are represented by red squares and black dots correspond to finite energy solutions. Points with no genuine solutions are shown by cross marks.

Since we have found several analytical solutions of the model, it is certainly worth to discuss such existence in the context of integrable theories in higher dimensions based on discussion of the integrable submodels possessing an infinite number of local conservation laws \([31, 32]\) as a generalization of the previous work \([24, 25]\), which study integrability of two dimensional NL\(\sigma\) models on homogeneous spaces.

Existence and stability of the solutions which we obtained implies that instanton-anti instanton creation/annihilation may occur, through the quantum tunneling effect, in condensed matters described by the SU\((3)\) antiferromagnetic Heisenberg model such as cold atoms in an optical lattice \([24]\), and rotating dense quark matter \([25]\). In addition, we hope this work gives a hint for finding finite energy nonembedding solutions of the SU\((3)\) pure Yang-Mills theory on the four dimensional Euclidean space.

Apart from the BPS and also the analytical study, it is also important to discuss solutions of a model which breaks the scale invariance by a higher derivative order term and a potential term. Such baby-skyrmion solutions may be more easy to find in several physical applications. We will report results of these issues in forthcoming article.

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