ASYMPTOTIC LIMIT OF A NAVIER-STOKES-KORTEWEG SYSTEM WITH DENSITY-DEPENDENT VISCOSITY

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ABSTRACT. In this paper, we study a combined incompressible and vanishing capillarity limit in the barotropic compressible Navier-Stokes-Korteweg equations for weak solutions. For well prepared initial data, the convergence of solutions of the compressible Navier-Stokes-Korteweg equations to the solutions of the incompressible Navier-Stokes equation are justified rigorously by adapting the modulated energy method. Furthermore, the corresponding convergence rates are also obtained.

1. INTRODUCTION

We are interested here in the combined incompressible and vanishing capillarity limit for the Navier-Stokes-Korteweg system which leads to the Navier-Stokes equation. The model we consider originates from the pioneering work by Van der Waals [12] and Korteweg [21] and was derived rigorously by Dunn and Serrin [10] in its modern form by using the second gradient theory. Because of its physical importance, complexity, rich phenomena, and mathematical challenges, the Navier-Stokes-Korteweg system has recently attracted a lot of attentions of many mathematicians. See, for example, [2, 16, 23, 28] and the references cited therein. We consider the dimensionless mass and momentum equations for the particle density \( n(x,t) \) and the mean velocity \( u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \) of a fluid in the three-dimensional torus \( \mathbb{T}^3 \)

\[
\begin{align*}
\partial_t n + \text{div}(nu) &= 0, \ x \in \mathbb{T}^3, t > 0, \\
\partial_t (nu) + \text{div}(nu \otimes u) + \nabla p(n) &= \text{div}(2\mu(n) D(u) + K) - \alpha u,
\end{align*}
\]

with initial conditions

\[
n(x,0) = n_0, u(x,0) = u_0.
\]

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Here, \( p(n) = n^\gamma / \gamma \) with constant \( \gamma > 1 \) denotes the pressure, \( D(u) = (\nabla u + \nabla u^T)/2 \) the strain tensor, \( I \) the unit tensor, and \( K \) the capillarity tensor. \( \mu(n) \) denotes the density-dependent viscosity. In this paper, we assume that \( \mu(n) = \mu n \) for some constant \( \mu > 0 \). The dimensionless parameter \( \sigma > 0 \) is the coefficient of capillarity.

The presence of a positive damping coefficient \( \alpha \) has some importance when dealing with the stability of solutions for small densities. It can guarantee the existence of global weak solutions of the systems (1)-(4) as in [4]. When \( \alpha \) is allowed to be nonnegative, as in [5], the classical definition of weak solutions has to be slightly changed. Apparently, if there has no capillary effects, system (1)-(3) reduces to the Navier-Stokes equations with a damping term. It is easy to observe that

\[
\text{div} K = \sigma \left( \frac{1}{2} \nabla \Delta n^2 - \frac{1}{2} |\nabla n|^2 - \text{div}(\nabla n \otimes \nabla n) \right) = \sigma n \nabla \Delta n, \quad (5)
\]

and that the contribution of capillary effects to energy will be proportional to

\[ \sigma^2 |\nabla n|^2. \]

Our aim in the present paper is to prove rigorously a combined incompressible and vanishing capillarity limit in the framework of the global weak solutions to (1)-(3). To begin with, we scale \( n \) and \( u \) (and thus \( p \)) in the following way:

\[
n(\cdot, t) = n(\cdot, \tau), \quad u(\cdot, t) = \epsilon u(\cdot, \tau),
\]

where a longer time scale \( t = \tau/\epsilon \) (still denote \( \tau \) by \( t \) later for simplicity) is introduced in order to seize the evolution of the fluctuations, and we assume that the viscosity coefficient \( \mu \), and damping coefficient \( \alpha \) are also scaled like

\[ \mu = \epsilon \mu, \quad \alpha = \epsilon \alpha, \]

and furthermore we set \( \sigma = \epsilon^\theta \) for \( 0 < \theta, \epsilon < 1 \). With such scalings, the Navier-Stokes-Korteweg (1)-(3) can be rewritten as (still denote \( \mu \) and \( \alpha \) by \( \mu \) and \( \alpha \) for simplicity, respectively)

\[
\begin{align*}
\partial_t n^\epsilon + \text{div}(n^\epsilon u^\epsilon) &= 0, \quad (6) \\
\partial_t (n^\epsilon u^\epsilon) + \text{div}(n^\epsilon u^\epsilon \otimes u^\epsilon) + \frac{1}{\epsilon^{2\gamma}} \nabla (n^\epsilon)^\gamma \quad &- \frac{1}{\epsilon^{2(1-\theta)}} n^\epsilon \nabla \Delta n^\epsilon = 2\mu \text{div}(n^\epsilon D(u^\epsilon)) - \alpha u^\epsilon, \quad (7)
\end{align*}
\]

with the initial conditions

\[
n^\epsilon(\cdot, 0) = n^\epsilon_0(x), \quad u^\epsilon(\cdot, 0) = u^\epsilon_0(x). \quad (8)
\]

The condition \( 0 < \theta < 1 \) is needed to control the capillary energy (see the energy identity in Lemma 3.2 below). Here we use the superscript to emphasize the dependence on \( \epsilon \) for each variable in (6) and (7).

We are mainly interested in the formal limit \( \epsilon \to 0 \) of the above system, whereas \( \mu \) and \( \alpha \) remain constant. By letting \( \epsilon \) go to zero in (7), we formally obtain that \( n^\epsilon \) converges to some function \( n(t) > 0 \). If we further assume that the initial datum \( n_0^\epsilon \) is of order \( 1 + o(\epsilon) \) (see (16) below), then one can expect that \( n(t) \equiv 1 \). Thus, the continuity equation (6) yields at the limit \( \text{div} u = 0 \), which is the incompressible condition of a fluid. Hence, we get the following incompressible Navier-Stokes equations (limit system):

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla \pi = 2\mu \text{div}(D(u)) - \alpha u, \quad \text{div} u = 0, \quad (9) \\
u(\cdot, 0) = u_0, \quad (10)
\end{align*}
\]
where the gradient of the hydrostatic pressure \( \pi \) in (9) is the “limit” of
\[
\frac{1}{\epsilon^2} \nabla (n^*)^\gamma - \frac{1}{\epsilon^{2(1-\theta)}} n^* \nabla \Delta n^*.
\]

It is important to mention that the asymptotic limit is a well-known challenging and physically very complex modelling problem for fluid dynamic models. In particular, there exists only partial results for Korteweg type model. In [6], Bresch, Desjardins, and Ducomet studied the quasi-neutral limit of a viscous capillary model of plasma expressed as a so-called Navier-Stokes-Poisson-Korteweg model and proved that the convergence to solutions to the compressible capillary Navier-Stokes equations, in the torus \( \mathbb{T}^3 \), is global in time in energy norm. Motivated by the processes of vanishing capillarity-viscosity limit, Charve, and Hoaspot [8] showed that the global strong solution of the Navier-Stokes-Korteweg system converges to entropic solution of the compressible Euler equations in one dimension. In [17], Jüngel, Lin, and Wu proved a combined incompressible and vanishing capillarity limit in the barotropic compressible Navier-Stokes system whose momentum equation contains a rotational term originating from a Coriolis force, a general Korteweg-type tensor modeling capillary effects, and a density-dependent viscosity.

In the present paper, our aim is to justify the formal convergence of solutions to the Navier-Stokes-Korteweg system (6)-(7) to solutions of the incompressible Navier-Stokes equations (9). We shall apply the method of relative entropy (or the modulated energy) to study the combined incompressible and vanishing capillarity limit in the system (6) and (7). To our knowledge, it is the first work on the combined incompressible and vanishing capillarity limit of weak solutions to Navier-Stokes-Korteweg system.

We should point out that in the present paper we only consider the case of well-prepared initial data. For the general initial data, the fast singular oscillations appear, and some further substantial techniques like the study of wave propagation (see [13,26]) are required to deal with the oscillations in time. It is more difficult to prove the asymptotic limit in this situation, which will be studied in a forthcoming paper. But it is very likely to extend the present convergence results to the case of general initial data. This will be one of the topics of our future work. We also mention that many mathematicians have made contributions to the incompressible limit to the related pure Navier-Stokes model and global existence of smooth or weak solutions to the Korteweg model. See Gamba et al. [14], Jiang-Ou [18], Jüngel [19], Lions-Masmoudi [25], Alazard [1], Hattori-Li [15], Kotschote [22], Chen-Zhao [7], Danchin-Desjardins [9], Li [24], and Tan et al. [29] for more references on that subject.

In this present paper, we denote by \( C \) the generic positive constants independent of \( \epsilon \).

The rest of this paper is organized as follows: in the next section, we state some useful known results, our main result, and the idea of the proof. Section 3 is devoted to the proof of our main result.

2. **Main Result**

Before stating our main results, we first recall the following classical results on the existence of sufficiently regular solutions of the incompressible Navier-Stokes equations (9)-(10) and the global weak solution to the compressible quantum Navier-Stokes equations (6)-(8), respectively.
Proposition 2.1. (Ref. [20, 27]) Assume that \( u_0 \in H^s, s > 3/2 + 3 \) and \( \text{div} u_0 = 0 \). Then there exist \( 0 < T_* < \infty \), the maximal existence time, and a unique smooth solution \((u, \pi)\) of the incompressible Navier-Stokes equations (9)-(10) on \([0, T_*)\) satisfying that \( u(0) = u_0 \). Assume that Proposition 2.1.

Formally multiplying (7) by \( u^\epsilon \), integrating by parts, and making use of the continuity equation (6) yield the energy inequality

\[
\frac{d}{dt} E^\epsilon(t) + D^\epsilon(t) dx \leq 0,
\]

where the energy \( E^\epsilon(t) \) and energy dissipation \( D^\epsilon(t) \) are defined by, respectively,

\[
E^\epsilon(t) = \int_{\mathbb{T}^3} \left( \frac{1}{2} n^\epsilon |u^\epsilon|^2 + \frac{1}{\epsilon^2 \gamma (\gamma - 1)} (n^\epsilon)^\gamma + \frac{1}{2 \epsilon^{2(1-\sigma)}} |\nabla n^\epsilon|^2 \right) dx,
\]

\[
D^\epsilon(t) = \int_{\mathbb{T}^3} (\alpha |u^\epsilon|^2 + 2 \mu n^\epsilon |D(u^\epsilon)|^2) dx.
\]

In fact, as long as the initial data satisfy

\[
E^\epsilon(0) = \int_{\mathbb{T}^3} \left( \frac{1}{2} n_0^\epsilon |u_0^\epsilon|^2 + \frac{1}{\epsilon^2 \gamma (\gamma - 1)} (n_0^\epsilon)^\gamma + \frac{1}{2 \epsilon^{2(1-\sigma)}} |\nabla n_0^\epsilon|^2 \right) dx < +\infty,
\]

one has the global estimate \( \sup_{t \geq 0} E^\epsilon(t) \leq E^\epsilon(0) \), which provides a global priori bounds for solutions of the above system.

The global existence of weak solutions for the system (6)-(8) without damping \((\alpha = 0)\) was obtained in [4]. But the “weak solutions” is in the nonclassical sense of weak solutions (multiplying the momentum equation by the density). When the damping \( \alpha \) is positive, the velocity \( u^\epsilon \) makes sense by itself independently of the density \( n^\epsilon \) since \( u^\epsilon \) belongs to \( L^2([0, T]; L^2(\mathbb{T}^3)) \). Thus the existence of global weak solutions to system (6)-(8) can be obtained in the classical sense of weak solutions by a straightforward modification of the arguments in [4]. Here, we omit the details.

Proposition 2.2. Let \( \alpha > 0, \mu > 0, \epsilon > 0, T > 0, \gamma \geq 2 \). Suppose that the initial data \((n_0^\epsilon, U_0^\epsilon)\) and \( E^\epsilon(0) \) are finite. Then the Navier-Stokes-Korteweg system (6)-(7) with (8) admits at least one global weak solution \((n^\epsilon, u^\epsilon)\).

1. \( n^\epsilon \) belongs to \( L^\infty([0, T]; L^\gamma(\mathbb{T}^3)) \); \( \sqrt{n^\epsilon} u^\epsilon \) and \( \nabla n^\epsilon \) belong to \( L^\infty([0, T]; L^2(\mathbb{T}^3)) \); finally, \( u^\epsilon \) and \( \sqrt{n^\epsilon} D(u^\epsilon) \) belong to \( L^2([0, T]; L^2(\mathbb{T}^3)) \).

2. The continuity equation (6) is satisfied in the sense of distributions.

3. For all \( v \in C^\infty(\mathbb{T}^3 \times [0, T]), \) compactly supported in \( \mathbb{T}^3 \times [0, T] \), one has

\[
\int_{\mathbb{T}^3} (n^\epsilon u^\epsilon \cdot v)(t = 0) dx + \int_0^T \int_{\mathbb{T}^3} \left( n^\epsilon u^\epsilon \cdot \partial_t v + n^\epsilon u^\epsilon \otimes u^\epsilon : D(v) \right. \\
+ p(n^\epsilon) \text{div} v - \frac{1}{2 \epsilon^{2(1-\sigma)}} (n^\epsilon)^2 \Delta \text{div} v \\
+ \frac{1}{2 \epsilon^{2(1-\sigma)}} |\nabla n^\epsilon|^2 \text{div} v + \frac{1}{\epsilon^{2(1-\sigma)}} (\nabla n^\epsilon \otimes \nabla n^\epsilon) : D(v) \\
- 2 \mu n^\epsilon D(u^\epsilon) : D(v) - \alpha u^\epsilon \cdot v \left. \right) dxdt = 0.
\]
We denote by
\[ h(n^\epsilon) = \frac{1}{\gamma(\gamma - 1)} \left( (n^\epsilon)^\gamma - 1 - \gamma(n^\epsilon - 1) \right). \]
In fact, \( h(n^\epsilon) \) stands for the free energy per unit volume.

The main result of this paper can be stated as follows.

**Theorem 2.3.** Let \( \theta \in (0, 1) \), \( \gamma \geq 2 \) and \( s \in \mathbb{N} \) with \( s > \frac{3}{2} + 3 \). Let \( u_0 \in H^s(\mathbb{T}^3) \) in (10) be a divergence-free vector field on \( \mathbb{T}^3 \). Let \( (u, \pi) \) be the smooth solution to incompressible Navier-Stokes equations (9)-(10) on \( \mathbb{T}^3 \times [0, T_*) \), satisfying the condition (11). Let \( (n_0^\epsilon, u_0^\epsilon) \) be a sequence of initial data such that
\[
\frac{1}{\epsilon^2} \int_{\mathbb{T}^3} h(n_0^\epsilon) dx \leq C\epsilon, \tag{16}
\]
\[
\| \sqrt{n_0^\epsilon} u_0^\epsilon - u_0 \|^2_{L^2(\mathbb{T}^3)} \leq C\epsilon, \tag{17}
\]
\[
\frac{1}{\epsilon^{2(1-\theta)}} \| \nabla n_0^\epsilon \|^2_{L^2(\mathbb{T}^3)} \leq C\epsilon. \tag{18}
\]
Then there exists a sequence \( (n^\epsilon, u^\epsilon) \) of weak solutions to the compressible Navier-Stokes-Korteweg equations (6)-(7) with initial data \( (n_0^\epsilon, u_0^\epsilon) \) in the periodic domain \( \mathbb{T}^3 \). Moreover, for any \( T < T_* \), we have that
\[
\frac{1}{\epsilon^2} \| n^\epsilon - 1 \|^2_{L^\infty([0, T]; L^\gamma(\mathbb{T}^3))} \leq C\epsilon^\beta, \tag{19}
\]
\[
\| \sqrt{n^\epsilon} u^\epsilon - u \|^2_{L^\infty([0, T]; L^2(\mathbb{T}^3))} \leq C\epsilon^\beta, \tag{20}
\]
\[
\| n^\epsilon u^\epsilon - u \|^2_{L^\infty([0, T]; L^{2(1+\gamma)}(\mathbb{T}^3))} \leq C\epsilon^\beta. \tag{21}
\]
where \( \beta = \min\{1 - \theta, \frac{2}{\gamma}\} \).

**Remark 2.1.** We explain that \( \gamma \geq 2 \) imposed in the above results is a (technical) restriction. The energy inequality (24) (see below) only provides a \( L^2([0, T]; L^2(\mathbb{T}^3)) \) bound for \( u^\epsilon \), so this hypothesis is needed to infer an estimate (see (38) below) for \( \int_0^T \int_{\mathbb{T}^3} (n^\epsilon - 1) u^\epsilon \cdot u dx dt \) in \( L^1([0, T]; L^1(\mathbb{T}^3)) \).

In the next section, we are going to prove Theorem 2.3. The proof of our result is based on the modulated energy method, first introduced by Brenier [3] and later extended to different directions, for instance [11, 17]. The idea of modulated energy method is to modulate the energy of the given system by test functions, and to obtain a stability inequality when these test functions are the solution to the limiting system. To this end, we introduce the following form of the modulated energy:
\[
\mathcal{H}^\epsilon(t) = \int_{\mathbb{T}^3} \left\{ \frac{1}{2} n^\epsilon |u^\epsilon|^2 + \frac{1}{\epsilon^2} h(n^\epsilon) + \frac{1}{2\epsilon^{2(1-\theta)}} |\nabla n^\epsilon|^2 \right\} dx, \tag{22}
\]
where \( u \) is the smooth solution of the incompressible Navier-Stokes equations (9)-(10). We shall employ the evolution equations and elaborated computations to prove the inequality
\[
\mathcal{H}^\epsilon(t) \leq C \int_0^t \mathcal{H}^\epsilon(s) ds + \epsilon^\eta \tag{23}
\]
for some positive constant \( \eta > 0 \). The Gronwall lemma then implies the result.
3. Proof of Theorem 2.3

From the energy inequality (12) and the conservation of mass, we have for almost all \( t \in [0, T], \)
\[
\int_{T^3} \left\{ \frac{1}{2} n^\epsilon |u^\epsilon|^2 + \frac{1}{\epsilon^2} h(n^\epsilon) + \frac{1}{2\epsilon^2(1-\theta)} \| \nabla n^\epsilon \|^2 \right\} \, dx \\
+ \int_0^t \int_{T^3} (\alpha |u^\epsilon|^2 + 2\mu |D(u^\epsilon)|^2) \, dx \, ds \\
\leq \int_{T^3} \left( \frac{1}{2} n_0^\epsilon |u_0^\epsilon|^2 + \frac{1}{\epsilon^2} h(n_0^\epsilon) + \frac{1}{2\epsilon^2(1-\theta)} \| \nabla n_0^\epsilon \|^2 \right) \, dx \\
\leq C. \tag{24}
\]
Therefore, we have the following properties:
\[
\sqrt{n^\epsilon u^\epsilon} \text{ is bounded in } L^\infty([0, T]; L^2(T^3)), \tag{25}
\]
\[
\frac{1}{\epsilon^2} h(n^\epsilon) \text{ is bounded in } L^\infty([0, T]; L^1(T^3)), \tag{26}
\]
\[
\frac{1}{\epsilon^2(1-\theta)} \| \nabla n^\epsilon \|^2 \text{ is bounded in } L^\infty([0, T]; L^1(T^3)), \tag{27}
\]
\[
u^\epsilon \text{ is bounded in } L^2([0, T]; L^2(T^3)). \tag{28}
\]

**Lemma 3.1.** Let \((n^\epsilon, u^\epsilon)\) be the weak solution to Navier-Stokes-Korteweg equations (6)-(8) on \([0, T].\) Then there exists a constant \( C > 0 \) such that for all \( \epsilon \in (0, 1) \) and \( \gamma \geq 2, \)
\[
\| n^\epsilon - 1 \|_{L^\infty([0, T]; L^\gamma(T^3))} \leq C \epsilon^{\frac{2}{\gamma}}. \tag{29}
\]
**Proof.** If \( \gamma \geq 2, \) we claim that
\[
|n^\epsilon - 1|^\gamma \leq h(n^\epsilon) \tag{30}
\]
for \( n^\epsilon \geq 0. \) Then the result follows immediately from the energy inequality (24) or property (26). \( \square \)

**Lemma 3.2.** Let \( T > 0, \gamma \geq 2, \) and \( 0 < \theta < 1. \) Then
\[
\mathcal{H}^\epsilon(t) \leq C \epsilon^\beta \tag{31}
\]
uniformly in \([0, T],\) where \( \beta = \min\{1 - \theta, \frac{2}{\gamma}\}. \)
**Proof.** The energy and the energy dissipation of incompressible Navier-Stokes equations (9)-(10) are
\[
E_{NS}(t) = \frac{1}{2} \int_{T^3} |u|^2 \, dx, \tag{32}
\]
\[
D_{NS}(t) = \int_{T^3} (\alpha |u|^2 + 2\mu |D(u)|^2) \, dx. \tag{33}
\]
We find the energy identity of incompressible Navier-Stokes equations (9)-(10):
\[
\frac{d}{dt} E_{NS}(t) + D_{NS}(t) = 0, \quad \forall \ t \in [0, T],
\]
which implies that for any \( t \in [0, T], \)
\[
E_{NS}(t) + \int_0^t D_{NS}(s) \, ds = E_{NS}(0) = \frac{1}{2} \int_{T^3} |u_0|^2 \, dx. \tag{34}
\]
To derive the integration inequality for $\mathcal{H}'(t)$, we use $u$ as a test function in the weak formulation of momentum equation (7) to yield the following equality for almost all $t$:

\[
\begin{align*}
&\int_{T^3} n^e u^e \cdot u dx = \int_{T^3} (n^e u^e \cdot u)(t = 0) dx + \int_0^t \int_{T^3} n^e u^e \cdot \partial_s u dx ds \\
&+ \int_0^t \int_{T^3} (n^e u^e \otimes u^e) : D(u) dx ds + \frac{1}{\varepsilon^{2(1-v)}} \int_0^t \int_{T^3} (\nabla n^e \otimes \nabla n^e) : D(u) dx ds \\
&- 2\mu \int_0^t \int_{T^3} n^e D(u^e) : D(u) dx ds - \alpha \int_0^t \int_{T^3} u^e \cdot u dx ds,
\end{align*}
\]

(35)

where we have used the facts that $\text{div} \ u = 0$ and

\[
 n^e \nabla \Delta n^e = \frac{1}{2} \nabla \Delta (n^e)^2 - \frac{1}{2} \nabla (n^e)^2 - \text{div} (\nabla n^e \otimes \nabla n^e).
\]

Using the definitions of the energy and energy dissipation as well as (35), by integration by parts, we calculate $\mathcal{H}'(t)$ as follows:

\[
\begin{align*}
\mathcal{H}'(t) + 2\mu \int_0^t \int_{T^3} n^e D(u^e) - D(u)|^2 dx ds + \alpha \int_0^t \int_{T^3} |u^e - u|^2 dx ds \\
= (E^e + E_{NS})(t) + \int_0^t (D^e + D_{NS})(s) ds + \frac{1}{2} \int_{T^3} (n^e - 1)|u|^2 dx \\
- \int_{T^3} n^e u^e \cdot u dx + 2\mu \int_0^t \int_{T^3} (n^e - 1)|D(u)|^2 dx ds \\
- 4\mu \int_0^t \int_{T^3} n^e D(u^e) : D(u) dx ds - 2\alpha \int_0^t \int_{T^3} u^e \cdot u dx ds \\
\leq (E^e + E_{NS})(0) - \int_{T^3} (n^e u^e \cdot u)(t = 0) dx + \frac{1}{2} \int_{T^3} (n^e - 1)|u|^2 dx \\
- \int_{T^3} n^e u^e \cdot \partial_s u dx ds - \int_{T^3} (n^e u^e \otimes u^e) : \nabla u dx ds \\
- \frac{1}{\varepsilon^{2(1-v)}} \int_0^t \int_{T^3} (\nabla n^e \otimes \nabla n^e) : \nabla u ds + 2\mu \int_0^t \int_{T^3} (n^e - 1)|D(u)|^2 dx ds \\
- 2\mu \int_0^t \int_{T^3} n^e D(u^e) : D(u) dx ds - \alpha \int_0^t \int_{T^3} u^e \cdot u dx ds \\
= \mathcal{H}'(0) - \frac{1}{2} \int_{T^3} (n_0^e - 1)|u_0|^2 dx + \frac{1}{2} \int_{T^3} (n^e - 1)|u|^2 dx \\
- \int_{T^3} n^e u^e \cdot \partial_s u dx ds - \alpha \int_{T^3} u^e \cdot u dx ds \\
- \int_{T^3} (n^e u^e \otimes u^e) : \nabla u dx ds - \frac{1}{\varepsilon^{2(1-v)}} \int_0^t \int_{T^3} (\nabla n^e \otimes \nabla n^e) : \nabla u ds + 2\mu \int_0^t \int_{T^3} n^e D(u^e) : D(u) dx ds \\
+ 2\mu \int_0^t \int_{T^3} (n^e - 1)|D(u)|^2 dx ds - 2\mu \int_0^t \int_{T^3} n^e D(u^e) : D(u) dx ds \\
= \mathcal{H}'(0) - \frac{1}{2} \int_{T^3} (n_0^e - 1)|u_0|^2 dx + \frac{1}{2} \int_{T^3} (n^e - 1)|u|^2 dx + \sum_{k=1}^5 I_k,
\end{align*}
\]

(36)
where

\[
I_1 = -\int_0^t \int_{T^3} n' u' \cdot \partial_s u dx ds - \alpha \int_0^t \int_{T^3} u' \cdot u dx ds,
\]

\[
I_2 = -\int_0^t \int_{T^3} (n' u' \otimes u') : \nabla u dx ds,
\]

\[
I_3 = -\frac{1}{\epsilon^{2(1-\nu)}} \int_0^t \int_{T^3} (\nabla n' \otimes \nabla n') : u dx ds,
\]

\[
I_4 = 2\mu \int_0^t \int_{T^3} (n' - 1) |D(u)|^2 dx ds,
\]

\[
I_5 = -2\mu \int_0^t \int_{T^3} n' D(u') : D(u) dx ds.
\]

Now, we begin to treat the integrals \(I_k\) (with \(k = 1, 2, 3, 4, 5\)) term by term. From (9), we have

\[
I_1 = \int_0^t \int_{T^3} n' u' \cdot ((u \cdot \nabla) u) dx ds - 2\mu \int_0^t \int_{T^3} n' u' \cdot \text{div}(D(u)) dx ds
\]

\[
+ \int_0^t \int_{T^3} n' u' \cdot \nabla \pi dx ds + \alpha \int_0^t \int_{T^3} (n' - 1) u' \cdot u dx ds.
\]

In fact, we only need to consider the last two terms on the right hand side of the above equation since the first two terms will be canceled later (see (41) and (44)). Using Lemma 3.1, the Young inequality, continuity equation (6), property (28), and integrating by parts, we get

\[
\int_0^t \int_{T^3} n' u' \cdot \nabla \pi dx ds = -\int_0^t \int_{T^3} \text{div}(n' u') \pi dx ds = \int_0^t \int_{T^3} \partial_s n' \pi dx ds
\]

\[
= \int_{T^3} (n' - 1) \pi dx - \int_{T^3} (n'_0 - 1) \partial_s \pi dx - \int_0^t \int_{T^3} \partial_s (n' - 1) \pi dx ds
\]

\[
\leq C \|n' - 1\|_{L^\infty([0,T]\times(T^3))} - \int_{T^3} (n'_0 - 1) \partial_s \pi dx
\]

\[
\leq C \epsilon^{\frac{2}{p}} + \frac{1}{\epsilon^2} \|n'_0 - 1\|^2_{L^\gamma(T^3)} + C \epsilon^{\frac{2}{1-\nu}}
\]

\[
\leq C \epsilon^{\frac{2}{p}},
\]

and

\[
\alpha \int_0^t \int_{T^3} (n' - 1) u' \cdot u dx ds \leq \alpha \|u\|_{L^\infty([0,T]\times(T^3))} \int_0^t \int_{T^3} n' - 1 \|u'\| dx ds
\]

\[
\leq C \|n' - 1\|_{L^2([0,T];L^2(T^3))} \|u'\|_{L^2([0,T];L^2(T^3))}
\]

\[
\leq C \|n' - 1\|_{L^\infty([0,T];L^\gamma(T^3))}
\]

\[
\leq C \epsilon^{\frac{2}{p}}.
\]
Therefore, we obtain that
\[
I_1 = \int_0^t \int_{T^3} n' u' \cdot ((u \cdot \nabla) u) dx ds + C \epsilon^{\frac{2}{3}} \\
- 2\mu \int_0^t \int_{T^3} n' u' \cdot \text{div}(D(u)) dx ds.
\]
In order to estimate \( I_2 \), we express it as follows:
\[
I_2 = - \int_0^t \int_{T^3} n'(u' - u) \otimes (u' - u) : \nabla u dx ds - \int_0^t \int_{T^3} (n' u \otimes u') : \nabla u dx ds \\
+ \int_0^t \int_{T^3} (n' u \otimes u) : \nabla u dx ds - \int_0^t \int_{T^3} (n' u' \otimes u) : \nabla u dx ds \\
\leq C \int_0^t \int_{T^3} n'|u' - u|^2 dx ds - \int_0^t \int_{T^3} (n' u' \cdot ((u \cdot \nabla) u)) dx ds + I_{21} + I_{22},
\]
where
\[
I_{21} = \int_0^t \int_{T^3} (n' u \otimes u) : \nabla u dx ds,
\]
\[
I_{22} = - \int_0^t \int_{T^3} (n' u' \otimes u) : \nabla u dx ds.
\]
Notice that the second term on the right hand side of the above inequality will be canceled by the first term on the right hand of (40). Similar to the estimate of \( I_1 \), using the Lemma 3.1 and the inequality (11), we get
\[
I_{21} = \int_0^t \int_{T^3} n'(u \cdot \nabla) u \cdot u dx ds \\
= \int_0^t \int_{T^3} (n' - 1)(u \cdot \nabla) u \cdot u dx ds \\
\leq C \epsilon^{\frac{2}{3}},
\]
where we have used the equality
\[
\int_0^t \int_{T^3} (u \cdot \nabla) u \cdot u dx = \frac{1}{2} \int_0^t \int_{T^3} u \cdot \nabla |u|^2 dx ds = 0.
\]
From the continuity equation (6) and the inequality (11), we get, by integration by parts, that
\[
I_{22} = - \frac{1}{2} \int_{T^3} (n' - 1)|u|^2 dx + \frac{1}{2} \int_{T^3} (n'_0 - 1)|u_0|^2 dx \\
+ \frac{1}{2} \int_0^t \int_{T^3} (n' - 1)\partial_\alpha |u|^2 dx ds \\
\leq \frac{1}{2} \int_{T^3} (n' - 1)|u|^2 dx + \frac{1}{2} \int_{T^3} (n'_0 - 1)|u_0|^2 dx \\
+ C\|n' - 1\|_{L^\infty((0,T];L^2(T^3))}\|\partial_\alpha |u|^2\|_{L^\infty((0,T];L^{\gamma}(T^3))} \\
\leq \frac{1}{2} \int_{T^3} (n' - 1)|u|^2 dx + \frac{1}{2} \int_{T^3} (n'_0 - 1)|u_0|^2 dx + C \epsilon^{\frac{2}{3}}.
\]
To justify the calculation in the above inequality, we need to use the definition of the weak solution to the compressible Navier-Stokes-Korteweg system (6)-(8) as in Proposition 2.2. Therefore, we obtain that

\[
I_2 \leq C \int_0^t \int_{\mathbb{T}^3} n^\mu |u^\epsilon - u|^2 \, dx \, ds - \int_0^t \int_{\mathbb{T}^3} n^\mu u^\epsilon \cdot ((u \cdot \nabla)u) \, dx \, ds
- \frac{1}{2} \int_{\mathbb{T}^3} (n^\epsilon - 1)|u|^2 \, dx + \frac{1}{2} \int_{\mathbb{T}^3} (n_0^\epsilon - 1)u_0^2 \, dx + C\epsilon^\frac{3}{2}.
\]  

(41)

For \( I_4 \), one gets that

\[
I_4 \leq C\|n^\epsilon - 1\|_\infty \|D(u)\|^2_{L^2([0,T];L^2(\mathbb{T}^3))} \leq C\epsilon^{\frac{3}{2}}.
\]

(43)

Using (11), Lemma 3.1, and the Hölder inequality, we have that

\[
I_4 \leq C\|n^\epsilon - 1\|_\infty \|D(u)\|^2_{L^2([0,T];L^2(\mathbb{T}^3))} \leq C\epsilon^{\frac{3}{2}}.
\]

Now we deal with the last term \( I_5 \). To estimate it, we rewrite it as follows:

\[
I_5 = \mu \int_0^t \int_{\mathbb{T}^3} n^\mu u^\epsilon \cdot (\Delta u + \nabla \text{div} u) \, dx \, ds + \mu \int_0^t \int_{\mathbb{T}^3} (\nabla n^\epsilon \otimes u^\epsilon + u^\epsilon \otimes \nabla n^\epsilon) : \nabla u \, dx \, ds
= 2\mu \int_0^t \int_{\mathbb{T}^3} n^\mu u^\epsilon \cdot \text{div}(D(u)) \, dx \, ds + \mu \int_0^t \int_{\mathbb{T}^3} (\nabla n^\epsilon \otimes u^\epsilon + u^\epsilon \otimes \nabla n^\epsilon) : \nabla u \, dx \, ds,
\]

where we have used the equality \( 2\text{div}(D(u)) = \Delta u \) with \( \text{div} u = 0 \). Applying the Cauchy-Schwarz inequality and properties (27) and (28), the last integral is bounded by

\[
C\|\nabla n^\epsilon\|_{L^2([0,T];L^2(\mathbb{T}^3))} \|u^\epsilon\|_{L^2([0,T];L^2(\mathbb{T}^3))} \leq C\epsilon^{1-\theta}.
\]

Therefore, we obtain that

\[
I_5 \leq 2\mu \int_0^t \int_{\mathbb{T}^3} n^\mu u^\epsilon \cdot \text{div}(D(u)) \, dx \, ds + C\epsilon^{1-\theta}.
\]

(44)

Inserting (39)-(44) into (36), we get

\[
\mathcal{H}^\epsilon(t) \leq \mathcal{H}^\epsilon(t = 0) + C \int_0^t \mathcal{H}^\epsilon(s) \, ds + C\epsilon^\beta, \quad \beta = \min\{1-\theta, \frac{2}{7}\}.
\]

Using the initial conditions (16)-(18), we have that \( \mathcal{H}^\epsilon(0) \leq C\epsilon \) since

\[
\int_{\mathbb{T}^3} n_0^\epsilon |u_0^\epsilon - u_0|^2 \, dx \leq 2 \int_{\mathbb{T}^3} |\sqrt{n_0^\epsilon} u_0^\epsilon - u_0|^2 \, dx + 2 \int_{\mathbb{T}^3} |(1 - \sqrt{n_0^\epsilon})u_0|^2 \, dx
\leq 2 \int_{\mathbb{T}^3} |\sqrt{n_0^\epsilon} u_0^\epsilon - u_0|^2 \, dx + C \int_{\mathbb{T}^3} |1 - \sqrt{n_0^\epsilon}|^2 \, dx
\leq 2 \int_{\mathbb{T}^3} |\sqrt{n_0^\epsilon} u_0^\epsilon - u_0|^2 \, dx + C \int_{\mathbb{T}^3} |n_0^\epsilon - 1|^2 \, dx
\leq C\epsilon.
\]

Here, we have used the following elementary inequality:

\[
|1 - \sqrt{x}|^2 \leq C|1 - x|^k, \quad \forall \ k \geq 1,
\]

(45)

for some positive constant \( C \) and any \( x \geq 0 \). Thus the proof of Lemma 3.2 is completed. □
We are now in a position to prove Theorem 2.3. From Lemma 3.2 and inequality (30), we claim that estimate (19) holds. Using Lemma 3.2, inequality (45), and the Hölder inequality, we have that
\[
\|\sqrt{n^\epsilon u^\epsilon - u}\|_{L^2(T^3)}^2 \leq 2 \|\sqrt{n^\epsilon (u^\epsilon - u)}\|_{L^2(T^3)}^2 + 2 \|(n^\epsilon - 1)u\|_{L^2(T^3)}^2
\]
\[
\leq C \epsilon^\beta + C \int_{T^3} |n^\epsilon - 1|^\gamma dx
\]
\[
\leq C \epsilon^\beta + C \epsilon^2 \int_{T^3} h(n^\epsilon) dx
\]
\[
\leq C \epsilon^\beta,
\]
for any \( t \in [0,T] \). Therefore, we conclude that (20) holds. Using the Hölder inequality and the fact that \( 1 < \frac{2\gamma}{\gamma + 1} < \gamma \) as \( \epsilon \to 0 \),
\[
\|n^\epsilon u^\epsilon - u\|_{L^{\frac{2\gamma}{\gamma + 1}}(T^3)}^2 \leq 2 \|n^\epsilon (u^\epsilon - u)\|_{L^{\frac{2\gamma}{\gamma + 1}}(T^3)}^2 + 2 \|(n^\epsilon - 1)u\|_{L^{\frac{2\gamma}{\gamma + 1}}(T^3)}^2
\]
\[
\leq 2 \|\sqrt{n^\epsilon}\|_{L^{2}(T^3)}^2 \|\sqrt{n^\epsilon (u^\epsilon - u)}\|_{L^{2}(T^3)}^2 + 2 \|n^\epsilon - 1\|_{L^{2}(T^3)}^2 \|u\|_{L^{2}(T^3)}^2
\]
\[
\leq C \epsilon^\beta + C \epsilon^2
\]
\[
\leq C \epsilon^\beta.
\]
Then, we conclude that (21) holds. Thus the proof of Theorem 2.3 is finished. \( \square \)

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