Efficient Second Order Online Learning by Sketching

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Abstract

We propose Sketched Online Newton (SON), an online second order learning algorithm that enjoys substantially improved regret guarantees for ill-conditioned data. SON is an enhanced version of the Online Newton Step, which, via sketching techniques enjoys a running time linear in the dimension and sketch size. We further develop sparse forms of the sketching methods (such as Oja’s rule), making the computation linear in the sparsity of features. Together, the algorithm eliminates all computational obstacles in previous second order online learning approaches.

1 Introduction

Online learning methods are highly successful at rapidly reducing the test error on large, high-dimensional datasets. First order methods are particularly attractive in such problems as they typically enjoy computational complexity linear in the input size. However, the convergence of these methods crucially depends on the geometry of the data; for instance, running the same algorithm on a rotated set of examples can return vastly inferior results. See Fig. 1 for an illustration.

Second order algorithms such as Online Newton Step \(^{17}\) have the attractive property of being invariant to linear transformations of the data, but typically require space and update time quadratic in the number of dimensions. Furthermore, the dependence on dimension is not improved even if the examples are sparse. These issues lead to the key question in our work: \textit{Can we develop (approximately) second order online learning algorithms with efficient updates?} We show that the answer is “yes” by developing efficient sketched second order methods with regret guarantees. Specifically, the three main contributions of this work are:

1. Invariant learning setting and optimal algorithms (Section \(^2\)). The typical online regret minimization setting evaluates against a benchmark that is bounded in some fixed norm (such as the \(\ell_2\)-norm), implicitly putting the problem in a nice geometry. However, if all the features are scaled down, it is desirable to compare with accordingly larger weights, which is precluded by an apriori fixed norm bound. We study an invariant learning setting similar to the paper \(^{32}\) which compares the learner to a benchmark only constrained to generate bounded predictions on the sequence of examples. We show that a variant of the Online Newton Step \(^{17}\), while quadratic in computation, stays regret-optimal with a nearly matching lower bound in this more general setting.

2. Improved efficiency via sketching (Section \(^3\)). To overcome the quadratic running time, we next develop sketched variants of the Newton update, approximating the second order information using a small number of carefully chosen directions, called a \textit{sketch}. While the idea of data sketching is widely studied \(^{35}\), as far as we know our work is the first one to apply it to a general adversarial online learning setting and provide rigorous regret guarantees. Three different sketching methods are considered: Random Projections \(^1\)\(^\text{18}\), Frequent Directions \(^1\)\(^\text{11}\)\(^\text{22}\), and Oja’s algorithm \(^{27}\)\(^\text{28}\), all of which allow linear running time per round. For the first two methods, we prove regret bounds similar to the full second order update...
whenever the sketch-size is large enough. Our analysis makes it easy to plug in other sketching and online PCA methods (e.g. [10]).

3. Sparse updates (Section 4). For practical implementation, we further develop sparse versions of these updates with a running time linear in the sparsity of the examples. The main challenge here is that even if examples are sparse, the sketch matrix still quickly becomes dense. These are the first known sparse implementations of the Frequent Directions and Oja’s algorithm, and require new sparse eigen computation routines that may be of independent interest.

Empirically, we evaluate our algorithm using the sparse Oja sketch (called Oja-SON) against first order methods such as diagonalized AdaGrad [24] on both ill-conditioned synthetic and a suite of real-world datasets. As Fig. 1 shows for a synthetic problem, we observe substantial performance gains as data conditioning worsens. On the real-world datasets, we find improvements in some instances, while observing no substantial second-order signal in the others.

Related work Our online learning setting is closest to the one proposed in [32], which studies scale-invariant algorithms, a special case of the invariance property considered here (see also [30, Section 5]). Computational efficiency, a main concern in this work, is not a problem there since each coordinate is scaled independently. Orabona and Páli [29] study unrelated notions of invariance. Gao et al. [8] study a specific randomized sketching method for a special online learning setting.

Another line of work focuses on conditioning issues in stochastic optimization, where the examples are drawn i.i.d. from a distribution. The L-BFGS algorithm [23] has recently been studied in the stochastic setting [3][25][26][33][34], but the analysis relies on assumptions of smoothness and strong convexity with pessimistic rates in theory and reliance on the use of large mini-batches empirically. Some recent works [6][14][13][31] consider sketching approaches to stochastic optimization, but do not consider sparse implementation and do not extend in an obvious manner to the online setting.

The Frank-Wolfe algorithm [7][19] is also invariant to linear transformations, but the regret bounds scale as $O(T^{2/3})$ as opposed to the typical $O(\sqrt{T})$ rates [16] where $T$ is the number of examples. Garber and Hazan [9] improve the regret to $O(\sqrt{T})$ but with a much more complicated variant.

Notation Vectors are represented by bold letters (e.g., $x$, $w$, ... ) and matrices by capital letters (e.g., $M$, $A$, ... ). $M_{i,j}$ denotes the $(i,j)$ entry of matrix $M$. $I_d$ represents the $d \times d$ identity matrix, $0_{m \times d}$ represents the $m \times d$ matrix of zeroes, and diag{$x$} represents a diagonal matrix with $x$ on the diagonal. $\lambda_i(A)$ denotes the $i$-th largest eigenvalue of $A$, $\|w\|_A$ denotes $\sqrt{w^\top A w}$, $|A|$ is the determinant of $A$, $\text{tr}(A)$ is the trace of $A$, $(A,B)$ denotes $\sum_{i,j} A_{i,j} B_{i,j}$, and $A \preceq B$ means that $B - A$ is positive semidefinite. The sign function $\text{sgn}(a)$ is 1 if $a \geq 0$ and $-1$ otherwise.

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1 Recent work by [22] also studies sparse updates for a more complicated variant of Frequent Directions which is randomized and incurs extra approximation error.
2 Setup and an Optimal Algorithm

We consider the following setting. On each round $t = 1, 2, \ldots, T$: (1) the adversary first presents an example $x_t \in \mathbb{R}^d$, (2) the learner chooses $w_t \in \mathbb{R}^d$ and predicts $w_t \top x_t$, (3) the adversary reveals a loss function $f_t(w) = \ell_t(w \top x_t)$ for some convex, differentiable $\ell_t : \mathbb{R} \to \mathbb{R}_+$, and (4) the learner suffers loss $f_t(w_t)$ for this round.

The learner’s regret to a comparator $w$ is defined as $R_T(w) = \sum_{t=1}^T f_t(w_t) - \sum_{t=1}^T f_t(w)$. Typical results study $R_T(w)$ against all $w$ with a bounded norm in some geometry. For an invariant update, we relax this requirement and only put bounds on the predictions $w \top x_t$. Specifically, for some pre-chosen constant $C$ we define $K_t \overset{\text{def}}{=} \{ w : |w \top x_t| \leq C \}$. We seek to minimize regret to all comparators that generate bounded predictions on every data point, that is:

$$R_T = \sup_{w \in K} R_T(w) \quad \text{where } K \overset{\text{def}}{=} \bigcap_{t=1}^T K_t = \{ w : \forall t = 1, 2, \ldots, T, \ |w \top x_t| \leq C \}.$$ 

Under this setup, if the data are transformed to $Mx_t$ for all $t$ and some invertible matrix $M \in \mathbb{R}^{d \times d}$, the optimal $w^*$ simply moves to $(M^{-1}) \top w^*$, which still has bounded predictions but might have significantly larger norm. This relaxation is similar to the comparator set considered in [32].

We make two structural assumptions on the loss functions.

Assumption 1. (Scalar Lipschitz) The loss function $\ell_t$ satisfies $|\ell_t'(z)| \leq L$ whenever $|z| \leq C$.

Assumption 2. (Curvature) There exists $\sigma_t \geq 0$ such that for all $u, w \in K$, $f_t(w)$ is lower bounded by $f_t(u) + \nabla f_t(u) \top (w - u) + \frac{\sigma_t}{2} \left( \nabla f_t(u) \top (w - u) \right)^2$.

Note that when $\sigma_t = 0$, Assumption 2 merely imposes convexity. More generally, it is satisfied by squared loss $f_t(w) = (w \top x_t - y_t)^2$ with $\sigma_t = \frac{1}{2L^2}$ whenever $|w \top x_t|$ and $|y_t|$ are bounded by $C$, as well as for all exp-concave functions (see [17, Lemma 3]).

Enlarging the comparator set might result in worse regret. We next show matching upper and lower bounds qualitatively similar to the standard setting, but with stronger invariance properties built in.

Theorem 1. For any online algorithm generating $w_t \in \mathbb{R}^d$ and all $T \geq d$, there exists a sequence of $T$ examples $x_t \in \mathbb{R}^d$ and loss functions $\ell_t$ satisfying Assumptions 1 and 2 (with $\sigma_t = 0$) such that the regret $R_T$ is at least $CL\sqrt{dT/2}$.

We now give an algorithm that matches the lower bound up to logarithmic constants in the worst case but enjoys much smaller regret when $\sigma_t \neq 0$. At round $t + 1$ with some invertible matrix $A_t$ specified later and gradient $g_t = \nabla f_t(w_t)$, the algorithm performs the following update before making the prediction on the example $x_{t+1}$:

$$u_{t+1} = w_t - A_t^{-1} g_t, \quad \text{and} \quad w_{t+1} = \arg\min_{w \in K_{t+1}} \|w - u_{t+1}\|_{A_t}.$$  \hspace{1cm} (1)

The projection onto the set $K_{t+1}$ differs from typical norm-based projections as it only enforces boundedness on $x_{t+1}$ at round $t + 1$. Moreover, this projection step can be performed in closed form.

Lemma 1. For any $x \neq 0, u \in \mathbb{R}^d$ and positive definite matrix $A \in \mathbb{R}^{d \times d}$, we have

$$\arg\min_{w : |w \top x| \leq C} \|w - u\|_A = u - \frac{\tau_C(u \top x)}{x \top x} A^{-1} x, \quad \text{where } \tau_C(y) = \text{sgn}(y) \max\{|y| - C, 0\}.$$ 

If $A_t$ is a diagonal matrix, updates similar to those of Ross et al. [32] are recovered. We study a choice of $A_t$ that is similar to the Online Newton Step (ONS) [17] (though with different projections):

$$A_t = \alpha I_d + \sum_{s=1}^t (\sigma_s + \eta_s) g_s g_s \top$$  \hspace{1cm} (2)

for some parameters $\alpha > 0$ and $\eta_t \geq 0$. The regret guarantee of this algorithm is shown below:
Algorithm 1 Sketched Online Newton (SON)

Input: Parameters $C$, $\alpha$ and $m$.
1. Initialize $u_t = 0_{d \times 1}$.
2. Initialize sketch $(S, H) \leftarrow \text{SketchInit}(\alpha, m)$.
3. for $t = 1$ to $T$ do
   4. Receive example $x_t$.
   5. Projection step: compute $\hat{\alpha} = Sx_t$, $\gamma = \frac{\tau_C(u_t^T u_t)}{x_t^T x_t - \hat{\alpha}^2 H\hat{\alpha}}$ and set $w_t = u_t - \gamma(x_t - S^T H\hat{\alpha})$.
   6. Predict label $y_t = w_t^T x_t$ and suffer loss $\ell_t(y_t)$.
   7. Compute gradient $g_t = \ell'_t(y_t) x_t$ and the to-sketch vector $\hat{g} = \sqrt{\sigma_t + \eta_t g_t}$.
   8. $(S, H) \leftarrow \text{SketchUpdate}(\hat{g})$.
   9. Update weight: $u_{t+1} = w_t - \frac{1}{\alpha}(g_t - S^T Hg_t)$.
10. end for

Theorem 2. Under Assumptions 4 and 2, suppose that $\sigma_t \geq \sigma \geq 0$ for all $t$, and $\eta_t$ is non-increasing. Then using the matrices 2 in the updates 1 yields for all $w \in K$,

$$R_T(w) \leq \frac{\alpha}{2} \|w\|_2^2 + 2(CL)^2 \sum_{t=1}^{T} \eta_t + \frac{d}{2(\sigma + \eta_T)} \ln \left(1 + \frac{(\sigma + \eta_T) \sum_{t=1}^{T} \|g_t\|_2^2}{d} \right).$$

The dependence on $\|w\|_2^2$ implies that the method is not completely invariant to transformations of the data. This is due to the part $\alpha I_d$ in $A_t$. However, this is not critical since $\alpha$ is fixed and small while the other part of the bound grows to eventually become the dominating term. Moreover, we can even set $\alpha = 0$ and replace the inverse with the Moore-Penrose pseudoinverse to obtain a truly invariant algorithm, as discussed in Appendix D. We use $\alpha > 0$ in the remainder for simplicity.

The implication of this regret bound is the following: in the worst case where $\sigma = 0$, we set $\eta_t = \frac{1}{t\sqrt{T}}$, and the bound simplifies to

$$R_T(w) \leq \frac{\alpha}{2} \|w\|_2^2 + \frac{CL}{2} \sqrt{Td} \ln \left(1 + \frac{\sum_{t=1}^{T} \|g_t\|_2^2}{\alpha CL \sqrt{Td}} \right) + 4CL\sqrt{d},$$

essentially only losing a logarithmic factor compared to the lower bound in Theorem 1. On the other hand, if $\sigma_t \geq \sigma > 0$ for all $t$, then we set $\eta_t = 0$ and the regret simplifies to

$$R_T(w) \leq \frac{\alpha}{2} \|w\|_2^2 + \frac{d}{2\sigma} \ln \left(1 + \frac{\sigma \sum_{t=1}^{T} \|g_t\|_2^2}{d} \right),$$

extending the $O(d \ln T)$ results in [17] to the weaker Assumption 2 and a larger comparator set $K$.

3 Efficiency via Sketching

Our algorithm so far requires $\Omega(d^2)$ time and space just as ONS. In this section we show how to achieve regret guarantees nearly as good as the above bounds, while keeping computation within a constant factor of first order methods.

Let $G_t \in \mathbb{R}^{d \times d}$ be a matrix such that the $t$-th row is $\hat{g}_t^T$, where we define $\hat{g}_t = \sqrt{\sigma_t + \eta_t g_t}$ to be the to-sketch vector. Our previous choice of $A_t$ (Eq. 2) can be written as $\alpha I_d + G_t^T G_t$. The idea of sketching is to maintain an approximation of $G_t$, denoted by $S_t \in \mathbb{R}^{m \times d}$ where $m \ll d$ is a small constant called the sketch size. If $m$ is chosen so that $S_t^T S_t$ approximates $G_t^T G_t$ well, we can redefine $A_t$ as $\alpha I_d + S_t^T S_t$ for the algorithm. To see why this admits an efficient algorithm, notice that by the Woodbury formula
independent random Gaussian variable drawn from \( N \). We consider Gaussian Random Projection sketch: Random Projection (RP).

Random projections are classical methods for sketching [18, 1, 20]. Here is set to (\( \Sigma H \) be realized by two rank-one updates:

\[
\gamma \text{ and summarize it in Algorithm 1.}
\]

When \( k \) is the last row of \( S \) maintains the invariant that the last row of \( S \) is always \( 0 \). On each round, the vector \( \tilde{g}_t \) is inserted into the last row of \( S_{t-1} \), then the covariance of the resulting matrix is eigendecomposed into \( V_t^\top \Sigma_t V_t \) and \( S_t \) is set to \( (\Sigma_t - \rho_t I_m)^{\frac{1}{2}} V_t \) where \( \rho_t \) is the smallest eigenvalue. Since the rows of \( S_t \) are orthogonal to each other, \( H_t \) is a diagonal matrix and can be maintained efficiently (see Algorithm [2]). The sketch update works in \( O(md) \) time (see [11] and Appendix G.2) so the total running time is \( O(md) \) per round. We call this combination FD-SON and prove the following regret bound with notation \( \Omega_k = \sum_{i=k+1}^{d} \lambda_i (G^\top_t G_t) \) for any \( k = 0, \ldots, m - 1 \).

\textbf{Theorem 3.} Under Assumptions [7] and [3] suppose that \( \sigma_t \geq \sigma \geq 0 \) for all \( t \) and \( \eta_t \) is non-increasing.
FD-SON ensures that for any \( w \in \mathcal{K} \) and \( k = 0, \ldots, m-1 \), we have

\[
R_T(w) \leq \frac{\alpha}{2} \|w\|_2^2 + 2(CL)^2 \sum_{t=1}^{T} \eta_t + \frac{m}{2(\sigma + \eta t)} \ln \left(1 + \frac{\text{TR}(S_t^T S_t)}{m\alpha}\right) + \frac{m \Omega_k}{2(m-k)(\sigma + \eta t)\alpha}.
\]

Instead of the rank, the bound depends on the spectral decay \( \Omega_k \), which essentially is the only extra term compared to the bound in Theorem 2. Similarly to previous discussion, if \( \sigma_t \geq \sigma > 0 \), setting \( \eta_t = 0 \) gives the bound \( \frac{\alpha}{2} \|w\|_2^2 + \frac{m}{2\sigma} \ln \left(1 + \frac{\text{TR}(S_t^T S_t)}{m\alpha}\right) + \frac{m \Omega_k}{2(m-k)\sigma \alpha} \). With \( \alpha \) tuned well, we pay logarithmic regret for the top \( m \) eigenvectors, but a square root regret \( O(\sqrt{\Omega_k}) \) for remaining directions not controlled by our sketch. This is expected for deterministic sketching which focuses on the dominant part of the spectrum. When \( \alpha \) is not tuned well, we still get sublinear regret as long as \( \Omega_k \) is sublinear. If the rank of \( G_T \) is smaller than \( m \), then \( \Omega_{m-1} = 0 \) and thus the sketch does not bring any extra regret. The discussion for the case \( \sigma = 0 \) is similar.

Oja’s Algorithm. Oja’s algorithm \cite{1982, 1999} is not usually considered as a sketching algorithm but seems very natural here. This algorithm uses online gradient descent to find eigenvectors and eigenvalues of data in a streaming fashion, with the to-sketch vector \( \hat{g}_t \)'s as the input. Specifically, let \( V_t \in \mathbb{R}^{m \times d} \) denote the estimated eigenvectors and the diagonal matrix \( \Lambda_t \in \mathbb{R}^{m \times m} \) contain the estimated eigenvalues at the end of round \( t \). Oja’s algorithm updates as:

\[
\Lambda_t = (I_m - \Gamma_t) \Lambda_{t-1} + \Gamma_t \operatorname{diag}\{V_{t-1} \hat{g}_t\}^2, \quad V_t \leftarrow \text{orth} \quad V_{t-1} + \Gamma_t V_{t-1} \hat{g}_t \hat{g}_t^T
\]

where \( \Gamma_t \in \mathbb{R}^{m \times m} \) is a diagonal matrix with (possibly different) learning rates of order \( \Theta(1/t) \) on the diagonal, and the “\( \text{orth} \)” operator represents an orthonormalizing step\(^2\). The sketch is then \( S_t = (t \Lambda_t)^{1/2} V_t \). The rows of \( S_t \) are orthogonal and thus \( H_t \) is an efficiently maintainable diagonal matrix (see Algorithm 5). We call this combination Oja-SON.

The time complexity of Oja’s algorithm is \( O(m^2d) \) per round due to the orthonormalizing step. To improve the running time to \( O(md) \), one can only update the sketch every \( m \) rounds (similar to the block power method \cite{1990, 2020}). The regret guarantee of this algorithm is unclear since existing analysis for Oja’s algorithm is only for the stochastic setting (see e.g. \cite{2019, 2021}). However, Oja-SON provides good performance experimentally.

4 Sparse Implementation

In many applications, examples (and hence gradients) are sparse in the sense that \( \|x_t\|_0 \leq s \) for all \( t \) and some small constant \( s \ll d \). Most online first order methods enjoy a per-example running time depending on \( s \) instead of \( d \) in such settings. Achieving the same for second order methods is more difficult since \( \Lambda_t^{-1} g_t \) (or sketched versions) are typically dense even if \( g_t \) is sparse.

We show how to implement our algorithms in sparsity-dependent time, specifically, in \( O(m^2 + ms) \) for RP-SON and FD-SON and in \( O(m^3 + ms) \) for Oja-SON. We emphasize that since the sketch would still quickly become a dense matrix even if the examples are sparse, achieving purely sparsity-dependent time is highly non-trivial (especially for FD-SON and Oja-SON), and may be of independent interest. Due to space limit, below we only briefly mention how to do it for Oja-SON. Similar discussion for the other two sketches can be found in Appendix G. Note that mathematically these updates are equivalent to the non-sparse counterparts and regret guarantees are thus unchanged.

There are two ingredients to doing this for Oja-SON: (1) The eigenvectors \( V_t \) are represented as \( V_t = F_t Z_t \), where \( Z_t \in \mathbb{R}^{m \times d} \) is a sparsely updatable direction (Step 3 in Algorithm 5) and \( F_t \in \mathbb{R}^{m \times m} \) is a matrix such that \( F_t Z_t \) is orthonormal. (2) The weights \( w_t \) are split as \( \hat{w}_t + Z_{t-1}^T b_t \), where \( b_t \in \mathbb{R}^m \) maintains the weights

\(^2\)For simplicity, we assume that \( V_{t-1} + \Gamma_t V_{t-1} \hat{g}_t \hat{g}_t^T \) is always of full rank so that the orthonormalizing step does not reduce the dimension of \( V_t \).
Since the first sparse implementation of online eigenvector computation to the best of our knowledge.

Again, it is clear that all the computations scale with $s$ and not $d$, so both $\bar{w}_{t+1}$ and $b_{t+1}$ require only $O(m^2 + ms)$ time to maintain. Furthermore, the prediction $w_{t+1}^T x_t = \bar{w}_{t+1}^T x_t + b_{t+1}^T z_{t-1} x_t$ can also be computed in $O(ms)$ time. The $O(m^2)$ in the overall complexity comes from a Gram-Schmidt step in maintaining $F_t$ (details in Appendix H).

The pseudocode is presented in Algorithms 4 and 5 with some details deferred to Appendix H. This is the first sparse implementation of online eigenvector computation to the best of our knowledge.

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Algorithm 4 Sparse Sketched Online Newton with Oja’s Algorithm

**Input:** Parameters $C$, $\alpha$ and $m$.

1. Initialize $\bar{u} = 0_{d \times 1}$ and $b = 0_{m \times 1}$.
2. $(\Lambda, F, Z, H) \leftarrow$ SketchInit($\alpha, m$) (Algorithm 5).
3. for $t = 1$ to $T$ do
   4. Receive example $x_t$.
   5. **Projection step:** compute $\hat{x} = FZ x_t$ and $\gamma = \tau C (\hat{u}^T x_t + b^T Z x_t)$.
      
      Obtain $\bar{w} = \bar{u} - \gamma x_t$ and $b \leftarrow b + \gamma (t - 1) F^T \Lambda H \hat{x}$ (Equation 6).
6. Predict label $y_t = \bar{w}^T x_t + b^T Z x_t$ and suffer loss $\ell_t(y_t)$.
7. Compute gradient $g_t = \ell_t'(y_t) x_t$ and the to-sketch vector $\hat{g} = \sqrt{\sigma_t + \eta_t} g_t$.
8. $(\Lambda, F, Z, H, \delta) \leftarrow$ SketchUpdate($\hat{g}$) (Algorithm 5).
9. Update weight: $u = w - \frac{\alpha}{\gamma} g_t - (\delta^T b) \hat{g}$ and $b \leftarrow b + \frac{\alpha}{\gamma} t F^T \Lambda H F Z g_t$ (Equation 5).
10. end for

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Algorithm 5 Sparse Oja’s Sketch

**Internal State:** $t, \Lambda, F, Z, H$ and $K$.

**SketchInit($\alpha, m$)**

1. Set $t = 0, \Lambda = 0_{m \times m}, F = K = \alpha H = I_m$ and $Z$ to any $m \times d$ matrix with orthonormal rows.
2. Return $(\Lambda, F, Z, H)$.

**SketchUpdate($\hat{g}$)**

1. Update $t \leftarrow t + 1$. Pick a diagonal stepsize matrix $\Gamma_t$ to update $\Lambda \leftarrow (I - \Gamma_t) \Lambda + \Gamma_t \text{diag}(FZ\hat{g})^2$.
2. Set $\delta = A^{-1} \Gamma_t FZ\hat{g}$ and update $K \leftarrow K + \delta g^T Z^T + Zg\delta^T + (g^T \hat{g}) \delta \delta^T$.
3. Update $Z \leftarrow Z + \delta g^T$.
4. $(L, Q) \leftarrow$ Decompose($F, K$) (Algorithm 13), so that $LQZ = FZ$ and $QZ$ is orthogonal. Set $F = Q$.
5. Set $H \leftarrow \text{diag}\{\frac{1}{\alpha + t\Lambda}, \ldots, \frac{1}{\alpha + t\Lambda_{m,m}}\}$.
6. Return $(\Lambda, F, Z, H, \delta)$.

on the subspace captured by $V_{t-1}$ (same as $Z_{t-1}$), and $\bar{w}_t$ captures the weights on the complementary subspace which are again updated sparsely.

We describe the sparse updates for $\bar{w}_t$ and $b_t$ below with the details for $F_t$ and $Z_t$ deferred to Appendix H. Since $S_t = (t \Lambda_t)^{1/2} V_t = (t \Lambda_t)^{1/2} F_t Z_t$ and $w_t = \bar{w}_t + Z_{t-1} b_t$, we know $u_{t+1}$ is equal to

$$w_t - (I_d - S_t^T H_t S_t) \frac{\partial}{\partial w_t} = w_t - \frac{\partial}{\partial w_t} (Z_t - Z_{t-1}) b_t + Z_t^T (b_t + \frac{1}{\alpha} F_t^T (t \Lambda_t H_t) F_t Z_t g_t) \overset{\text{def}}{=} u_{t+1}.$$

(5)

Since $Z_t - Z_{t-1}$ is sparse by construction and the matrix operations defining $b_{t+1}'$ scale with $m$, overall the update can be done in $O(m^2 + ms)$. Using the update for $w_{t+1}$ in terms of $u_{t+1}, w_{t+1}$ is equal to

$$u_{t+1} - \gamma_t (I_d - S_t^T H_t S_t) x_{t+1} = u_{t+1} + \gamma_t x_{t+1} + Z_t^T (b_{t+1}' + \gamma F_t^T (t \Lambda_t H_t) F_t Z_t x_{t+1}) \overset{\text{def}}{=} u_{t+1}.$$

(6)

Again, it is clear that all the computations scale with $s$ and not $d$, so both $\bar{w}_{t+1}$ and $b_{t+1}$ require only $O(m^2 + ms)$ time to maintain. Furthermore, the prediction $w_{t+1}^T x_t = \bar{w}_{t+1}^T x_t + b_{t+1}^T z_{t-1} x_t$ can also be computed in $O(ms)$ time. The $O(m^2)$ in the overall complexity comes from a Gram-Schmidt step in maintaining $F_t$ (details in Appendix H).

The pseudocode is presented in Algorithms 4 and 5 with some details deferred to Appendix H. This is the first sparse implementation of online eigenvector computation to the best of our knowledge.
Figure 2: Oja’s algorithm’s eigenvalue recovery error.

Figure 3: (a) Comparison of two sketch sizes on real data, and (b) Comparison against AdaGrad on real data.

5 Experiments

Preliminary experiments revealed that out of our three sketching options, Oja’s sketch has the best empirical performance. For more thorough evaluation, we implemented the sparse version of Oja-SON in the open source toolkit Vowpal Wabbit. In this section, we compare it with AdaGrad [5, 24] on both synthetic and real-world datasets. Each algorithm takes a stepsize parameter: $\alpha$ serves as a stepsize for Oja-SON and a scaling constant on the gradient matrix for AdaGrad. We try both the methods with the parameters set to $2^j$ for $j = -3, -2, \ldots, 6$ and report the best results. We keep the stepsize matrix for eigenvalues in Oja-SON fixed as $\Gamma_t = \frac{1}{t} I_m$ throughout. All methods make one online pass over data.

5.1 Synthetic Datasets

To investigate Oja-SON’s performance in the setting it is really designed for, we generated a range of synthetic ill-conditioned datasets as follows. We picked a random Gaussian matrix $Z \sim \mathbb{R}^{T \times d}$ ($T = 10,000$ and $d = 100$) and a random orthonormal basis $V \in \mathbb{R}^{d \times d}$. We chose a specific spectrum $\lambda \in \mathbb{R}^d$ where the first $d - 10$ coordinates are 1 and the rest increase linearly to some fixed condition number parameter $\kappa$.

We let $X = Z \text{diag}\{\lambda\}^{\frac{1}{2}} V^\top$ be our example matrix, and created a binary classification problem with labels $y = \text{sign}(\theta^\top x)$, where $\theta \in \mathbb{R}^d$ is a random vector. We generated 20 such datasets with the same $Z, V$ and labels $y$ but different values of $\kappa \in \{10, 20, \ldots, 200\}$. Note that if the algorithm is truly invariant, it would have the same behavior on these 20 datasets.

Fig. 1 (in Section 1) shows the final progressive error rate (that is, fraction of misclassified examples after one pass over data) for AdaGrad and Oja-SON (with sketch size $m = 0, 5, 10$) as the condition number increases. As expected, the plot confirms the performance of first order methods such as AdaGrad degrades when the data is ill-conditioned. The plot also shows that as we increase the sketch size, Oja-SON becomes more accurate: when $m = 0$ (no sketch at all), Oja-SON is vanilla gradient descent and is worse than AdaGrad as expected; when $m = 5$, the accuracy greatly improves; and finally when $m = 10$, the accuracy of Oja-SON is substantially better and hardly worsens with $\kappa$.

To further explain the effectiveness of Oja’s algorithm in identifying top eigenvalues and eigenvectors, the plot in Fig. 2 shows the largest relative difference between the true and estimated top 10 eigenvalues as Oja’s algorithm sees more data. This gap drops quickly after seeing just 500 examples.

5.2 Real-world Datasets

Next we evaluated Oja-SON on 23 benchmark datasets from the UCI and LIBSVM repository (see Appendix I for description of these datasets). Note that some datasets are very high dimensional but very sparse (e.g.
for 20news, \(d \approx 102,000\) and \(s \approx 94\), and consequently methods with running time quadratic (such as ONS) or even linear in dimension rather than sparsity are prohibitive.

In Fig. 3(a), we show the effect of using sketched second order information, by comparing sketch size \(m = 0\) and \(m = 10\) for Oja-SON (concrete error rates in Appendix I). We observe significant improvements in 5 datasets (acoustic, census, heart, ionosphere, letter), demonstrating the advantage of using second order information.

However, we found that Oja-SON was outperformed by AdaGrad on most datasets, mostly because the diagonal adaptation of AdaGrad greatly reduces the condition number on these datasets. Moreover, one disadvantage of SON is that for the directions not in the sketch, it is essentially doing vanilla gradient descent. We expect better results using diagonal adaptation as in AdaGrad in off-sketch directions.

To incorporate this high level idea, we performed a simple modification to Oja-SON: upon seeing example \(x_t\), we feed \(D_t^{-\frac{1}{2}} x_t\) to our algorithm instead of \(x_t\), where \(D_t \in \mathbb{R}^{d \times d}\) is the diagonal part of the matrix \(\sum_{\tau=1}^{t-1} g_\tau g_\tau^\top\). The intuition is that this diagonal rescaling first homogenizes the scales of all dimensions. Any remaining ill-conditioning is further addressed by the sketching to some degree, while the complementary subspace is no worse-off than with AdaGrad. We believe this flexibility in picking the right vectors to sketch is an attractive aspect of our sketching-based approach.

With this modification, Oja-SON outperforms AdaGrad on most of the datasets even for \(m = 0\), as shown in Fig. 3(b) (concrete error rates in Appendix I). The improvement on AdaGrad at \(m = 0\) is surprising but not impossible as the updates are not identical—our update is scale invariant like Ross et al. [32]. However, the diagonal adaptation already greatly reduces the condition number on all datasets except splice (see Fig. 4 in Appendix I for detailed results on this dataset), so little improvement is seen for sketch size \(m = 10\) over \(m = 0\). For several datasets, we verified the accuracy of Oja’s method in computing the top-few eigenvalues (Appendix I), so the lack of difference between sketch sizes is due to the lack of second order information after the diagonal correction.

The average running time of our algorithm when \(m = 10\) is about 11 times slower than AdaGrad, matching expectations. Overall, SON can significantly outperform baselines on ill-conditioned data, while maintaining a practical computational complexity.

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4 \(D_1\) is defined as \(0.1 \times I_d\) to avoid division by zero.
References

[1] Dimitris Achlioptas. Database-friendly random projections: Johnson-lindenstrauss with binary coins. *Journal of computer and System Sciences*, 66(4):671–687, 2003.

[2] Akshay Balsubramani, Sanjoy Dasgupta, and Yoav Freund. The fast convergence of incremental pca. In *Advances in Neural Information Processing Systems 26*, 2013.

[3] Richard H Byrd, SL Hansen, Jorge Nocedal, and Yoram Singer. A stochastic quasi-newton method for large-scale optimization. *arXiv preprint arXiv:1401.7020*, 2014.

[4] Nicolo Cesa-Bianchi, Alex Conconi, and Claudio Gentile. A second-order perceptron algorithm. *SIAM Journal on Computing*, 34(3):640–668, 2005.

[5] John Duchi, Elad Hazan, and Yoram Singer. Adaptive subgradient methods for online learning and stochastic optimization. *The Journal of Machine Learning Research*, 12:2121–2159, 2011.

[6] Murat A Erdogdu and Andrea Montanari. Convergence rates of sub-sampled newton methods. In *Advances in Neural Information Processing Systems 28*, pages 3034–3042, 2015.

[7] Marguerite Frank and Philip Wolfe. An algorithm for quadratic programming. *Naval research logistics quarterly*, 3(1-2):95–110, 1956.

[8] Wei Gao, Rong Jin, Shenghuo Zhu, and Zhi-Hua Zhou. One-pass auc optimization. In *Proceedings of the 30th International Conference on Machine Learning*, 2013.

[9] Dan Garber and Elad Hazan. A linearly convergent conditional gradient algorithm with applications to online and stochastic optimization. *arXiv preprint arXiv:1301.4666*, 2013.

[10] Dan Garber, Elad Hazan, and Tengyu Ma. Online learning of eigenvectors. In *Proceedings of the 32nd International Conference on Machine Learning*, pages 560–568, 2015.

[11] Mina Ghashami, Edo Liberty, Jeff M Phillips, and David P Woodruff. Frequent directions: Simple and deterministic matrix sketching. *arXiv preprint arXiv:1501.01711*, 2015.

[12] Mina Ghashami, Edo Liberty, and Jeff M. Phillips. Efficient frequent directions algorithm for sparse matrices. *arXiv preprint arXiv:1602.00412*, 2016.

[13] Alon Gonen and Shai Shalev-Shwartz. Faster sgd using sketched conditioning. *arXiv preprint arXiv:1506.02649*, 2015.

[14] Alon Gonen, Francesco Orabona, and Shai Shalev-Shwartz. Solving ridge regression using sketched preconditioned svrg. In *Proceedings of the 33rd International Conference on Machine Learning*, 2016.

[15] Moritz Hardt and Eric Price. The noisy power method: A meta algorithm with applications. In *Advances in Neural Information Processing Systems 27*, pages 2861–2869, 2014.

[16] Elad Hazan and Satyen Kale. Projection-free online learning. In *Proceedings of the 29th International Conference on Machine Learning*, 2012.

[17] Elad Hazan, Amit Agarwal, and Satyen Kale. Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2-3):169–192, 2007.

[18] Piotr Indyk and Rajeev Motwani. Approximate nearest neighbors: towards removing the curse of dimensionality. In *Proceedings of the thirtieth annual ACM symposium on Theory of computing*, pages 604–613. ACM, 1998.

[19] Martin Jaggi. Revisiting frank-wolfe: Projection-free sparse convex optimization. In *Proceedings of the 30th International Conference on Machine Learning*, pages 427–435, 2013.

[20] Daniel M Kane and Jelani Nelson. Sparser johnson-lindenstrauss transforms. *Journal of the ACM (JACM)*, 61(1):4, 2014.
[21] Chun-Liang Li, Hsuan-Tien Lin, and Chi-Jen Lu. Rivalry of two families of algorithms for memory-restricted streaming pca. arXiv preprint arXiv:1506.01490, 2015.

[22] Edo Liberty. Simple and deterministic matrix sketching. In Proceedings of the 19th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 581–588. ACM, 2013.

[23] Dong C Liu and Jorge Nocedal. On the limited memory bfgs method for large scale optimization. Mathematical programming, 45(1-3):503–528, 1989.

[24] H. Brendan McMahan and Matthew Streeter. Adaptive bound optimization for online convex optimization. In Proceedings of the 23rd Annual Conference on Learning Theory (COLT), 2010.

[25] Aryan Mokhtari and Alejandro Ribeiro. Global convergence of online limited memory bfgs. arXiv preprint arXiv:1409.2045, 2014.

[26] Philipp Moritz, Robert Nishihara, and Michael I Jordan. A linearly-convergent stochastic l-bfgs algorithm. arXiv preprint arXiv:1508.02087, 2015.

[27] Erkki Oja. Simplified neuron model as a principal component analyzer. Journal of mathematical biology, 15(3):267–273, 1982.

[28] Erkki Oja and Juha Karhunen. On stochastic approximation of the eigenvectors and eigenvalues of the expectation of a random matrix. Journal of mathematical analysis and applications, 106(1):69–84, 1985.

[29] Francesco Orabona and Dávid Pál. Scale-free algorithms for online linear optimization. In The 26th International Conference on Algorithmic Learning Theory (ALT), 2015.

[30] Francesco Orabona, Koby Crammer, and Nicolo Cesa-Bianchi. A generalized online mirror descent with applications to classification and regression. Machine Learning, 99(3):411–435, 2015.

[31] Mert Pilanci and Martin J Wainwright. Newton sketch: A linear-time optimization algorithm with linear-quadratic convergence. arXiv preprint arXiv:1505.02250, 2015.

[32] Stéphane Ross, Paul Mineiro, and John Langford. Normalized online learning. In Proceedings of the 29th Conference on Uncertainty in Artificial Intelligence (UAI), 2013.

[33] Nicol N Schraudolph, Jin Yu, and Simon Günter. A stochastic quasi-newton method for online convex optimization. In International Conference on Artificial Intelligence and Statistics, pages 436–443, 2007.

[34] Jascha Sohl-Dickstein, Ben Poole, and Surya Ganguli. Fast large-scale optimization by unifying stochastic gradient and quasi-newton methods. In Proceedings of the 31st International Conference on Machine Learning, 2014.

[35] David P Woodruff. Sketching as a tool for numerical linear algebra. Foundations and Trends in Machine Learning, 10(1-2):1–157, 2014. ISSN 1551-305X. doi: 10.1561/0400000060.
A Proof of Theorem 1

Proof. Assuming $T$ is a multiple of $d$ without loss of generality, we pick $x_t$ from the basis vectors $\{e_1, \ldots, e_d\}$ so that each $e_i$ appears $T/d$ times (in an arbitrary order). Note that now $K$ is just a hypercube:

$$K = \{ w : |w^\top x_i| \leq C, \ \forall i \} = \{ w : \|w\|_\infty \leq C \}.$$  

Let $\sigma_1, \ldots, \sigma_T$ be independent Rademacher random variables such that $\Pr(\sigma_t = +1) = \Pr(\sigma_t = -1) = \frac{1}{2}$. Define loss function $\ell_i(p) = (\sigma_t L)p$, so that Assumptions 1 and 2 are clearly satisfied with $\sigma_t = 0$. We show that, for any online algorithm, 

$$\sqrt{\frac{T}{d}} \leq C L \sqrt{d},$$

which implies the statement of the theorem.

First of all, note that $\mathbb{E}[\ell_i(w_t^\top x_t)]|_{\sigma_1, \ldots, \sigma_{t-1}} = 0$ for any $w_t$. Hence we have

$$\mathbb{E}
\left[
\sum_{i=1}^{T} \ell_i(w_t^\top x_t) - \inf_{w \in K} \sum_{i=1}^{T} \ell_i(w^\top x_t)
\right] = \mathbb{E}
\left[
\sup_{w \in K} \sum_{i=1}^{T} -\ell_i(w^\top x_t)
\right] = L \mathbb{E}
\left[
\sup_{w \in K} \sum_{i=1}^{T} \sigma_i x_t
\right],$$

which, by the construction of $x_t$, is

$$CLE \left[ \left\| \sum_{i=1}^{T} \sigma_i x_t \right\| \right] = CLd \mathbb{E}
\left[
\sum_{i=1}^{T/d} \sigma_i
\right] \geq CLd \sqrt{\frac{T}{2d}} = CL \sqrt{\frac{dT}{2}},$$

where the inequality is due to the Khintchine inequality. This concludes the proof.

B Projection

We prove a more general version of Lemma 1 which does not require invertibility of the matrix $A$ here.

Lemma 2. For any $x \neq 0, u \in \mathbb{R}^{d \times 1}$ and positive semidefinite matrix $A \in \mathbb{R}^{d \times d}$, we have

$$w^* = \arg\min_{w : \|w - u\|_A \leq C} \|w - u\|_A = \left\{ \begin{array}{cl}
  u - \frac{\tau_C(x^\top x)}{x^\top A x} A^\dagger x & \text{if } x \in \text{range}(A) \\
  u - \frac{\tau_C(w^\top x)}{x^\top (I - A^\dagger A)x} (I - A^\dagger A) x & \text{if } x \notin \text{range}(A) 
\end{array} \right.$$  

where $\tau_C(y) = \text{sgn}(y) \max\{|y| - C, 0\}$ and $A^\dagger$ is the Moore-Penrose pseudoinverse of $A$. (Note that when $A$ is rank deficient, this is one of the many possible solutions.)

Proof. First consider the case when $x \in \text{range}(A)$. If $|x^\top u| \leq C$, then it is trivial that $w^* = u$. We thus assume $x^\top u \geq C$ below (the last case $x^\top u \leq -C$ is similar). The Lagrangian of the problem is

$$L(w, \lambda_1, \lambda_2) = \frac{1}{2} (w - u)^\top A (w - u) + \lambda_1 (w^\top x - C) + \lambda_2 (w^\top x + C)$$

where $\lambda_1 \geq 0$ and $\lambda_2 \leq 0$ are Lagrangian multipliers. Since $w^\top x$ cannot be $C$ and $-C$ at the same time, the complementary slackness condition implies that either $\lambda_1 = 0$ or $\lambda_2 = 0$. Suppose the latter case is true, then setting the derivative with respect to $w$ to 0, we get $w^* = u - \lambda_1 A^\dagger x + (I - A^\dagger A) z$ where $z \in \mathbb{R}^{d \times 1}$ can be arbitrary. However, since $A(I - A^\dagger A) = 0$, this part does not affect the objective value at all and

5By adding a suitable constant, these losses can always be made nonnegative while leaving the regret unchanged.
we can simply pick $z = 0$ so that $w^*$ has a consistent form regardless of whether $A$ is full rank or not. Now plugging $w^*$ back, we have

$$L(w^*, \lambda_1, 0) = -\frac{\lambda_1^2}{2} x^\top A^\top x + \lambda_1 (u^\top x - C)$$

which is maximized when $\lambda_1 = \frac{u^\top x - C}{x^\top A^\top x} \geq 0$. Plugging this optimal $\lambda_1$ into $w^*$ gives the stated solution. On the other hand, if $\lambda_1 = 0$ instead, we can proceed similarly and verify that it gives a smaller dual value (0 in fact), proving the previous statement is indeed optimal.

We now move on to the case when $x \notin \text{range}(A)$. First of all the stated solution is well defined since $x^\top (I - A^\top A)x$ is nonzero in this case. Moreover, direct calculation shows that $w^*$ is in the valid space: $|w^\top x| = |u^\top x - \tau C(u^\top x)| \leq C$, and also it gives the minimal possible distance value $\|w^* - u\|_A = 0$, proving the lemma.

### C Proof of Theorem 2

We first prove a general regret bound that holds for any choice of $A_t$ in update 1:

$$u_{t+1} = u_t - A_t^{-1} g_t,$$

$$w_{t+1} = \arg\min_{w \in K_{t+1}} \|w - u_{t+1}\|_{A_t}.$$

This bound will also be useful in proving regret guarantees for the sketched versions.

**Proposition 1.** For any sequence of positive definite matrices $A_t$ and sequence of losses satisfying Assumptions 1 and 2, the regret of updates 1 against any comparator $w \in K$ satisfies

$$2R_T(w) \leq \|w\|_{A_0}^2 + \sum_{t=1}^T g_t^\top A_t^{-1} g_t + \sum_{t=1}^T (w_t - w)^\top (A_t - A_{t-1} - \sigma_t g_t^\top g_t^\top)(w_t - w)$$

where $R_G$ and $R_D$ represent the "Gradient Bound" and "Diameter Bound" respectively.

**Proof.** Since $u_{t+1}$ is the projection of $u_{t+1}$ onto $K_{t+1}$, by the property of projections (see for example [10 Lemma 8]), the algorithm ensures

$$\|w_{t+1} - w\|_{A_t}^2 \leq \|u_{t+1} - w\|_{A_t}^2 = \|w_t - w\|_{A_t}^2 + g_t^\top A_t^{-1} g_t - 2g_t^\top (w_t - w)$$

for all $w \in K \subseteq K_{t+1}$. By the curvature property in Assumption 1, we then have that

$$2R_T(w) \leq \sum_{t=1}^T 2g_t^\top (w_t - w) - \sigma_t (g_t^\top (w_t - w))^2$$

$$\leq \sum_{t=1}^T g_t^\top A_t^{-1} g_t + \|w_t - w\|_{A_t}^2 - \|w_{t+1} - w\|_{A_t}^2 - \sigma_t (g_t^\top (w_t - w))^2$$

$$= \|w\|_{A_0}^2 + \sum_{t=1}^T g_t^\top A_t^{-1} g_t + (w_t - w)^\top (A_t - A_{t-1} - \sigma_t g_t^\top g_t^\top)(w_t - w),$$

which completes the proof.

**Proof of Theorem 2.** We apply Proposition 1 with the choice: $A_0 = \alpha I_d$ and $A_t = A_{t-1} + (\sigma_t + \eta_t) g_t^\top g_t^\top$, which gives $\|w\|_{A_0}^2 = \alpha \|w\|_2^2$ and

$$R_D = \sum_{t=1}^T \eta_t (w_t - w)^\top g_t^\top g_t^\top (w_t - w) \leq 4(C L)^2 \sum_{t=1}^T \eta_t,$$
where the last equality uses the Lipschitz property in Assumption 1 and the boundedness of $w^T x_t$ and $w^T x_t$.

For the term $R_G$, define $\hat{A}_t = \frac{\alpha}{\sigma + \eta T} I_d + \sum_{s=1}^T g_s g_s^T$. Since $\sigma_t \geq \sigma$ and $\eta_t$ is non-increasing, we have $\hat{A}_t \preceq \frac{1}{\sigma + \eta T} A_t$, and therefore:

$$R_G \leq \frac{1}{\sigma + \eta T} \sum_{t=1}^T g_t^T \hat{A}_t^{-1} g_t = \frac{1}{\sigma + \eta T} \sum_{t=1}^T \left( \hat{A}_t - \hat{A}_{t-1} \right)$$

$$\leq \frac{1}{\sigma + \eta T} \sum_{t=1}^T \ln \left( \frac{\hat{A}_t}{\hat{A}_{t-1}} \right) = \frac{1}{\sigma + \eta T} \ln \left( \frac{\hat{A}_T}{\hat{A}_0} \right)$$

$$= \frac{1}{\sigma + \eta T} \sum_{i=1}^d \ln \left( 1 + \frac{(\sigma + \eta T) \lambda_i \left( \sum_{t=1}^T g_t^T g_t \right)}{\alpha} \right)$$

$$\leq \frac{d}{\sigma + \eta T} \ln \left( 1 + \frac{(\sigma + \eta T) \sum_{t=1}^T \|g_t^2\|}{\alpha} \right)$$

where the second inequality is by the concavity of the function $\ln |X|$ (see [17, Lemma 12] for an alternative proof), and the last one is by Jensen’s inequality. This concludes the proof.

\[\square\]

D A Truly Invariant Algorithm

In this section we discuss how to make our adaptive online Newton algorithm truly invariant to invertible linear transformations. To achieve this, we set $\alpha = 0$ and replace $A_t^{-1}$ with the Moore-Penrose pseudoinverse $A_t^\dagger$:

$$\hat{A}_t = \frac{\alpha}{\sigma + \eta T} I_d + \sum_{s=1}^T g_s g_s^T$$

$$u_{t+1} = w_t - A_t^\dagger g_t,$n

$$w_{t+1} = \arg\min_{w \in K_{t+1}} \| w - u_{t+1} \|_{A_t}.$$  

(7)

When written in this form, it is not immediately clear that the algorithm has the invariant property. However, one can rewrite the algorithm in a mirror descent form:

$$w_{t+1} = \arg\min_{w \in K_{t+1}} \| w - w_t + A_t^\dagger g_t \|_{A_t}^2$$

$$= \arg\min_{w \in K_{t+1}} \| w - w_t \|_{A_t}^2 + 2(w - w_t)^T A_t A_t^\dagger g_t$$

$$= \arg\min_{w \in K_{t+1}} \| w - w_t \|_{A_t}^2 + 2w^T g_t$$

where we use the fact that $g_t$ is in the range of $A_t$ in the last step. Now suppose all the data $x_t$ are transformed to $M x_t$ for some unknown and invertible matrix $M$, then one can verify that all the weights will be transformed to $M^{-T} w_t$ accordingly, ensuring the prediction to remain the same.

Moreover, the regret bound of this algorithm can be bounded as below. First notice that even when $A_t$ is rank deficient, the projection step still ensures the following: $\| w_{t+1} - w_t \|_{A_t}^2 \leq \| u_{t+1} - u_t \|_{A_t}^2$, which is

\[\text{See Appendix B for the closed form of the projection step.}\]
Let \( \text{Theorem 4.} \) case.

proceed similarly to the proof of \([4, \text{Theorem 4.2}]\) to show that this term is of order \( O \) nonzero eigenvalues of \( t \) time step \( \lambda \). We have

\[
\sum_{t=1}^{T} g_t^\top \hat{A}_t^i g_t \leq r + \frac{(1 + r) r}{2} \ln \left( 1 + \frac{2 \sum_{t=1}^{T} \|g_t\|^2}{(1 + r) r \lambda^*} \right).
\]

The key now is to bound the term \( \sum_{t=1}^{T} g_t^\top \hat{A}_t^i g_t \) where we define \( \hat{A}_t = \sum_{s=1}^{t} g_s g_s^\top \). In order to do this, we proceed similarly to the proof of \([3, \text{Theorem 4.2}]\) to show that this term is of order \( O(d^2 \ln T) \) in the worst case.

**Theorem 4.** Let \( \lambda^* \) be the minimum among the smallest nonzero eigenvalues of \( \hat{A}_t \) \( (t = 1, \ldots, T) \) and \( r \) be the rank of \( \hat{A}_T \). We have

\[
\sum_{t=1}^{T} g_t^\top \hat{A}_t^i g_t \leq r + \frac{(1 + r) r}{2} \ln \left( 1 + \frac{2 \sum_{t=1}^{T} \|g_t\|^2}{(1 + r) r \lambda^*} \right).
\]

Proof. First by Cesa-Bianchi et al. \([4, \text{Lemma D.1}]\), we have

\[
g_t^\top \hat{A}_t^i g_t = \begin{cases} 
\frac{1}{\det_+(\hat{A}_{t-1})} & \text{if } g_t \notin \text{range}(\hat{A}_{t-1}) \\
1 - \frac{\det_+(\hat{A}_{t-1})}{\det_+(\hat{A}_t)} & \text{if } g_t \in \text{range}(\hat{A}_{t-1})
\end{cases}
\]

where \( \det_+(M) \) denotes the product of the nonzero eigenvalues of matrix \( M \). We thus separate the steps \( t \) such that \( g_t \in \text{range}(\hat{A}_{t-1}) \) from those where \( g_t \notin \text{range}(\hat{A}_{t-1}) \). For each \( k = 1, \ldots, r \) let \( T_k \) be the first time step \( t \) in which the rank of \( A_t \) is \( k \) (so that \( T_1 = 1 \)). Also let \( T_{r+1} = T + 1 \) for convenience. With this notation, we have

\[
\sum_{t=1}^{T} g_t^\top \hat{A}_t^i g_t = \sum_{k=1}^{r} \left( g_{T_k}^\top \hat{A}_{T_k}^i g_{T_k} + \sum_{t=T_k+1}^{T_{k+1}-1} g_t^\top \hat{A}_t^i g_t \right)
\]

\[
= \sum_{k=1}^{r} \left( 1 + \sum_{t=T_k+1}^{T_{k+1}-1} \left( 1 - \frac{\det_+(\hat{A}_{t-1})}{\det_+(\hat{A}_t)} \right) \right)
\]

\[
= r + \sum_{k=1}^{r} \sum_{t=T_k+1}^{T_{k+1}-1} \ln \frac{\det_+(\hat{A}_t)}{\det_+(\hat{A}_{t-1})}
\]

Fix any \( k \) and let \( \lambda_{k,1}, \ldots, \lambda_{k,k} \) be the nonzero eigenvalues of \( \hat{A}_{T_k} \) and \( \lambda_{k,1} + \mu_{k,1}, \ldots, \lambda_{k,k} + \mu_{k,k} \) be the nonzero eigenvalues of \( \hat{A}_{T_{k+1}-1} \). Then

\[
\ln \frac{\det_+(\hat{A}_{T_{k+1}-1})}{\det_+(\hat{A}_{T_k})} = \ln \prod_{i=1}^{k} \frac{\lambda_{k,i} + \mu_{k,i}}{\lambda_{k,i}} = \sum_{i=1}^{k} \ln \left( 1 + \frac{\mu_{k,i}}{\lambda_{k,i}} \right)
\]

Hence, we arrive at

\[
\sum_{t=1}^{T} g_t^\top \hat{A}_t^i g_t \leq r + \sum_{k=1}^{r} \sum_{i=1}^{k} \ln \left( 1 + \frac{\mu_{k,i}}{\lambda_{k,i}} \right).
\]
Algorithm 6 Random Projection Sketch for RP-SON

Internal State: \( S \) and \( H \).

SketchInit(\( \alpha, m \))
1. Set \( S = 0_{m \times d} \) and \( H = \frac{1}{\alpha} I_m \).
2. Return \((S, H)\)

SketchUpdate(\( \hat{g} \))
1. Draw \( r \sim N(0, 1/\sqrt{m}) \) and update \( S \leftarrow S + r \hat{g}^\top r \).
2. Compute \( q = S \hat{g}^\top \hat{g} - \hat{g}^\top \) and update \( H \leftarrow H - \frac{H q r^\top H}{1 + q^\top H r} \) and \( H \leftarrow H - \frac{H r^\top H}{1 + q^\top H r} \).
3. Return \((S, H)\).

To further bound the latter quantity, we use \( \lambda^* \leq \lambda_{k,i} \) and Jensen’s inequality:

\[
\sum_{k=1}^r \sum_{i=1}^k \ln \left( 1 + \frac{\mu_{k,i}}{\lambda_{k,i}} \right) \leq \sum_{k=1}^r \sum_{i=1}^k \ln \left( 1 + \frac{\mu_{k,i}}{\lambda^*} \right) = \frac{(1 + r)r}{2} \ln \left( 1 + \frac{2 \sum_{k=1}^r \sum_{i=1}^k \mu_{k,i}}{(1 + r)r \lambda^*} \right)
\]

Finally noticing that

\[
\sum_{i=1}^k \mu_{k,i} = \text{Trace}(\hat{A}_{T_{k+1}-1}) - \text{Trace}(\hat{A}_{T_k}) = \sum_{t=T_{k+1}}^{T_{k+1}-1} \text{Trace}(g_t g_t^\top) = \sum_{t=T_{k+1}}^{T_{k+1}-1} \|g_t\|_2^2
\]

completes the proof.

Taken together, Eq. (8) and Theorem 4 lead to the following regret bounds (recall the definitions of \( \lambda^* \) and \( r \) from Theorem 4).

**Corollary 1.** If \( \sigma_t = 0 \) for all \( t \) and \( \eta_t \) is set to be \( \frac{1}{CL} \sqrt{\frac{d}{T}} \), then the regret of the algorithm defined by Eq. (7) is at most

\[
\frac{CL}{2} \sqrt{\frac{T}{d}} \left( r + \frac{(1 + r)r}{2} \ln \left( 1 + \frac{2 \sum_{t=1}^T \|g_t\|_2^2}{(1 + r)r \lambda^*} \right) \right) + 4CL \sqrt{Td}.
\]

On the other hand, if \( \sigma_t \geq \sigma > 0 \) for all \( t \) and \( \eta_t \) is set to be 0, then the regret is at most

\[
\frac{1}{2\sigma} \left( r + \frac{(1 + r)r}{2} \ln \left( 1 + \frac{2 \sum_{t=1}^T \|g_t\|_2^2}{(1 + r)r \lambda^*} \right) \right).
\]

### E Regret Bound for RP-SON

The pseudocode of the RP sketch is presented in Algorithm 6. Recall the notation \( R_G \) and \( R_D \) in Proposition 4 and let \( r \) be the rank of \( G_T \), we prove the following regret bound:

**Theorem 5.** Under Assumptions 1 and 2 if the sketch size \( m = \Omega((r + \ln(T/\delta))\epsilon^{-2}) \), then RP-SON ensures

1. \( \mathbb{E}[R_D] \leq 4(CL)^2 \sum_{t=1}^T \eta_t \), and
2. \( R_G \leq \frac{1}{T} \sum_{t=1}^T \|g_t\|_2^2 (\alpha I_d + G_t G_t^\top)^{-1} g_t \) with probability at least \( 1 - \delta \).
Proof. We apply the property of the random projection method (see for example [35 Theorem 2.3]): as long as \( m = \Omega((r + \ln(T/\delta))\epsilon^{-2}) \), with probability at least \( 1 - \delta \),

\[
(1 - \epsilon)G_t^T G_t \preceq S_t^T S_t \preceq (1 + \epsilon)G_t^T G_t \quad \text{for all} \ t = 1, \ldots, T
\]

which implies \( A_t^{-1} \preceq \frac{1}{1 - \epsilon}(\alpha I_d + G_t^T G_t)^{-1} \) and thus \( R_G \leq \frac{1}{1 - \epsilon} \sum_{t=1}^{T} g_t^T (\alpha I_d + G_t^T G_t)^{-1} g_t \). For \( R_D \), first fix all the randomness before drawing \( r_t \) and let \( E_t \) be the corresponding conditional expectation, then we have

\[
E_t[A_t - A_{t-1}] = E_t\left[ S_{t-1}^T r_t g_t^T + g_t r_t^T S_{t-1} + \| r_t \|^2 \hat{g}_t \hat{g}_t^T \right] = (\sigma_t + \eta_t) g_t g_t^T.
\]

Since \( w_t, w \) and \( g_t \) are fixed, we continue with

\[
E_t \left[ (w_t - w)^T (A_t - A_{t-1} - \sigma_t g_t g_t^T) (w_t - w) \right] = \eta_t (w_t - w)^T g_t g_t^T (w_t - w) \leq 4(CL)^2 \eta_t.
\]

Therefore, taking the overall expectation gives \( E[R_D] \leq 4(CL)^2 \sum_{t=1}^{T} \eta_t \).

This theorem implies that the bound on \( R_D \) is the same as the one without using sketch, and the term \( R_G \) is only constant larger.

F Proof of Theorem 3

Proof. We again first apply Proposition 1 (recall the notation \( R_G \) and \( R_D \) stated in the proposition). By the construction of the sketch, we have \( A_t - A_{t-1} = S_{t-1}^T S_t - S_{t-1}^T S_{t-1} = \tilde{g}_t \tilde{g}_t^T - \rho_t V_t V_t^T \leq \rho_t V_t V_t^T \). It follows immediately that \( R_D \) is again at most \( 4(CL)^2 \sum_{t=1}^{T} \eta_t \). For the term \( R_G \), we will apply the following guarantee of Frequent Directions (see the proof of Theorem 1.1 of [11]): \( \sum_{t=1}^{T} \rho_t \leq \frac{\Omega_k}{m - k} \). Specifically, since \( \text{TR}(V_t A_t^{-1} V_t^T) \leq \frac{1}{\alpha} \text{TR}(V_t V_t^T) = \frac{m}{\alpha} \) we have

\[
R_G = \sum_{t=1}^{T} \frac{1}{\sigma_t + \eta_t} \langle A_t^{-1}, A_t - A_{t-1} + \rho_t V_t V_t^T \rangle
\]

\[
\leq \frac{1}{\sigma + \eta T} \sum_{t=1}^{T} \left( \langle A_t^{-1}, A_t - A_{t-1} + \rho_t V_t V_t^T \rangle \right)
\]

\[
= \frac{1}{\sigma + \eta T} \sum_{t=1}^{T} \left( \langle A_t^{-1}, A_t - A_{t-1} \rangle + \rho_t \text{TR}(V_t A_t^{-1} V_t^T) \right)
\]

\[
\leq \frac{1}{(\sigma + \eta T)} \sum_{t=1}^{T} \langle A_t^{-1}, A_t - A_{t-1} \rangle + \frac{m\Omega_k}{(m - k)(\sigma + \eta T)\alpha}.
\]

Finally for the term \( \sum_{t=1}^{T} \langle A_t^{-1}, A_t - A_{t-1} \rangle \), we proceed similarly to the proof of Theorem 2

\[
\sum_{t=1}^{T} \langle A_t^{-1}, A_t - A_{t-1} \rangle \leq \sum_{t=1}^{T} \ln \frac{|A_t|}{|A_{t-1}|} = \ln \frac{|A_T|}{|A_0|} = \sum_{i=1}^{d} \ln \left( 1 + \frac{\lambda_i(S_i^T S_i)}{\alpha} \right)
\]

\[
= m \sum_{i=1}^{d} \ln \left( 1 + \frac{\lambda_i(S_i^T S_i)}{\alpha} \right) \leq m \ln \left( 1 + \frac{\text{TR}(S_i^T S_i)}{m\alpha} \right)
\]

where the first inequality is by the concavity of the function \( \ln |X| \), the second one is by Jensen’s inequality, and the last equality is by the fact that \( S_i^T S_i \) is of rank \( m \) and thus \( \lambda_i(S_i^T S_i) = 0 \) for any \( i > m \). This concludes the proof. \( \square \)
Algorithm 7 Sparse Sketched Online Newton with Random Projection

Input: Parameters $C$, $\alpha$ and $m$.
1: Initialize $\tilde{u} = 0_{d \times 1}$, $b = 0_{m \times 1}$ and $(S, H) \leftarrow$ SketchInit($\alpha, m$) (Algorithm 6).
2: for $t = 1$ to $T$ do
3:   Receive example $x_t$.
4:   Projection step: compute $\hat{x} = S x_t$, $\gamma = \frac{\gamma t (\tilde{u}^\top x_t + b^\top \hat{x})}{\hat{x}^\top H x}$, $\tilde{w} = \tilde{u} - \gamma x_t$ and $b \leftarrow b + c H \hat{x}$.
5:   Predict label $y_t = \tilde{w}^\top x_t + b^\top \hat{x}$ and suffer loss $\ell_t(y_t)$.
6:   Compute gradient $g_t = \ell_t'(y_t)x_t$ and the to-sketch vector $\hat{g} = \sqrt{\sigma_t + \eta_t} g_t$.
7:   $(S, H) \leftarrow$ SketchUpdate($\hat{g}$) (Algorithm 7).
8:   Update $\tilde{u} = \tilde{w} - (r^\top b) \hat{g} - \frac{1}{\alpha} g_t$ and $b \leftarrow b + \frac{1}{\alpha} H S g_t$.
9: end for

G Sparse updates for RP and FD sketches

G.1 Random Projection

We recall the updates of RP sketch. Since $\hat{g}_t$ is sparse, $S_t = S_{t-1} + r \hat{g}_t^\top$ is easily updated in $O(ms)$ time. $H_t$ can also be updated in $O(m^2 + ms)$ time clearly. However, since the sketch $S_t$ is getting denser and denser, direct update of the weight vector is a dense operation too. The solution is to represent and store $w_t$ in the form of $\tilde{w}_t + S_{t-1} b_t$ for some $\tilde{w}_t \in \mathbb{R}^d$ and $b_t \in \mathbb{R}^m$. Note that now computing the prediction $w_t^\top x_t$ needs $O(ms)$ time. Rewriting the update rules we have

$$u_{t+1} = w_t - \frac{1}{\alpha} g_t + \frac{1}{\alpha} S_t^\top H_t S_t g_t = \tilde{w}_t + S_{t-1} b_t - \frac{1}{\alpha} g_t + \frac{1}{\alpha} S_t^\top H_t S_t g_t$$

$$= \tilde{w}_t - \tilde{g}_t r_t^\top b_t - \frac{1}{\alpha} g_t + S_t^\top (b_t + \frac{1}{\alpha} H_t S_t g_t) \overset{\text{def}}{=} u_{t+1} \quad \overset{\text{def}}{=} b_{t+1}$$

Since $g_t$ and $\tilde{g}_t$ are sparse, computing $\tilde{u}_{t+1}$ and $b_{t+1}$ needs $O(m^2 + ms)$ time. Finally, for the projection step, $c_t$ can clearly be computed in $O(m^2 + ms)$ time, and the update rule of $\tilde{w}_{t+1}$ and $b_{t+1}$ is thus derived as follows:

$$w_{t+1} = u_{t+1} - \gamma_t (x_{t+1} - S_t^\top H_t S_t x_{t+1})$$

$$= \tilde{u}_{t+1} + S_{t-1} b_{t+1} - \gamma_t (x_{t+1} - S_t^\top H_t S_t x_{t+1})$$

$$= \tilde{u}_{t+1} - \tilde{g}_t x_{t+1} + S_t^\top (b_{t+1} + \gamma_t H_t S_t x_{t+1}) \overset{\text{def}}{=} w_{t+1} \quad \overset{\text{def}}{=} b_{t+1}$$

which again takes $O(m^2 + ms)$ time. Taken together, the total time complexity per round is $O(m^2 + ms)$.

The pseudocode for this version of the algorithm is presented in Algorithm 7.

G.2 Frequent Directions

The sparse version of our algorithm with the Frequent Directions option is much more involved. We begin by taking a detour and introducing a fast and epoch-based variant of the Frequent Directions algorithm proposed in [11]. The idea is the following: instead of doing an eigendecomposition immediately after inserting a new $\tilde{g}$ every round, we double the size of the sketch (to $2m$), keep up to $m$ recent $\tilde{g}$’s, do the decomposition only at the end of every $m$ rounds and finally keep the top $m$ eigenvectors with shrunk eigenvalues. The advantage of this variant is that it can be implemented straightforwardly in $O(md)$ time on average without doing a complicated rank-one SVD update, while still ensuring the exact same guarantee with the only price of doubling the sketch size.
Algorithm 8 Frequent Direction Sketch (epoch version)

**Internal State:** $\tau, D, V, G$ and $H$.

**SketchInit**($\alpha, m$)
1. Set $\tau = 1, D = 0_{m \times m}, G = 0_{m \times d}, H = \frac{1}{d}I_{2m}$ and let $V$ be any $m \times d$ matrix whose rows are orthonormal.
2. Return $((0_{2m \times d}, H)$.

**SketchUpdate**($\hat{g}$)
1. Insert $\hat{g}$ into the $\tau$-th row of $G$.
2. if $\tau < m$ then
3. Let $e$ be the $2m \times 1$ basic vector whose $(m + \tau)$-th entry is 1 and $q = S\hat{g} - \frac{\hat{g}^\top\hat{g}}{2} e$.
4. Update $H \leftarrow H - \frac{\hat{g}eq^\top H}{1 + q^\top q}$ and $H \leftarrow H - \frac{\hat{g}eq^\top H}{1 + q^\top q}$.
5. Update $\tau \leftarrow \tau + 1$.
6. else
7. $(V, \Sigma) \leftarrow \text{ComputeEigenSystem} \left( \begin{pmatrix} DV \\ G \end{pmatrix} \right)$ (Algorithm 9).
8. Set $D$ to be a diagonal matrix with $D_{i,i} = \sqrt{\Sigma_{i,i} - \Sigma_{m,m}}$, $\forall i \in [m]$.
9. Set $H \leftarrow \text{diag} \left\{ \frac{1}{\alpha + D_{1,1}^2}, \ldots, \frac{1}{\alpha + D_{m,m}^2}, \frac{1}{\alpha}, \ldots, \frac{1}{\alpha} \right\}$.
10. Set $G = 0_{m \times d}$.
11. Set $\tau = 1$.
12. end if
13. Return $\left( \begin{pmatrix} DV \\ G \end{pmatrix}, H \right)$.

Although we can use any available algorithm that runs in $O(m^2d)$ time to do the eigendecomposition (Line 7 in Algorithm 8), we explicitly write down the procedure of reducing this problem to eigendecomposing a small square matrix in Algorithm 9 which will be important for deriving the sparse version of the algorithm. Lemma 3 proves that Algorithm 9 works correctly for finding the top $m$ eigenvector and eigenvalues.

Algorithm 9 ComputeEigenSystem($S$)

**Input:** $S = \begin{pmatrix} DV \\ G \end{pmatrix}$.

**Output:** Output $V' \in \mathbb{R}^{m \times d}$ and diagonal matrix $\Sigma \in \mathbb{R}^{m \times m}$ such that the $i$-th row of $V'$ and the $i$-th entry of the diagonal of $\Sigma$ are the $i$-th eigenvector and eigenvalue of $S^\top S$ respectively.
1. Compute $M = GV^\top$.
2. Decompose $G - MV$ into the form $LQ$ where $L \in \mathbb{R}^{m \times r}$, $Q$ is a $r \times d$ matrix whose rows are orthonormal and $r$ is the rank of $G - MV$ (e.g. by a Gram-Schmidt process).
3. Compute the top $m$ eigenvectors ($U \in \mathbb{R}^{m \times (m + r)}$) and eigenvalues ($\Sigma \in \mathbb{R}^{m \times m}$) of the matrix
   $$ \begin{pmatrix} D^2 & 0_{m \times r} \\ 0_{r \times m} & 0_{r \times r} \end{pmatrix} + \begin{pmatrix} M^\top \\ L^\top \end{pmatrix} \begin{pmatrix} M & L \end{pmatrix} $$.
4. Return $(V', \Sigma)$ where $V' = U \begin{pmatrix} V \\ Q \end{pmatrix}$.

**Lemma 3.** The outputs of Algorithm 9 are such that the $i$-th row of $V'$ and the $i$-th entry of the diagonal of $\Sigma$ are the $i$-th eigenvector and eigenvalue of $S^\top S$ respectively.
Proof. Let $W^T \in \mathbb{R}^{d \times (d-m-r)}$ be an orthonormal basis of the null space of $\left( \begin{array}{c} V \\ Q \end{array} \right)$. By Line 2 we know that $GW^T = 0$ and $E = (V^T Q^T W^T)$ forms an orthonormal basis of $\mathbb{R}^d$. Therefore, we have

$$S^T S = V^T D^2 V + G^T G$$

$$= E \left( \begin{array}{ccc} D^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{c} E^T + EE^T G^T GEE^T \\ V^T \end{array} \right)$$

$$= E \left( \begin{array}{ccc} D^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \left( \begin{array}{ccc} V^T G^T \\ QG^T \\ WG^T \end{array} \right) \left( \begin{array}{ccc} (GV^T)QG^T \\ GW^T \end{array} \right) \left( \begin{array}{c} E^T \\ V^T \end{array} \right)$$

$$= (V^T Q^T) \left( \begin{array}{ccc} D^2 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) + \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{ccc} M_{\cdot}^T \\ L_{\cdot}^T \end{array} \right) \left( \begin{array}{c} V \\ Q \end{array} \right)$$

where in the last step we use the fact $GQ^T = (MV + LQ)Q^T = L$. Now it is clear that the eigenvalue of $C$ will be the eigenvalue of $S^T S$ and the eigenvector of $C$ will be the eigenvector of $S^T S$ after left multiplied by matrix $(V^T Q^T)$, completing the proof.

We are now ready to present the sparse version of SON with Frequent Direction sketch (Algorithm 10). The key point is that we represent $V_t$ as $F_t Z_t$ for some $F_t \in \mathbb{R}^{m \times m}$ and $Z_t \in \mathbb{R}^{m \times d}$, and the weight vector $w_t$ as $\bar{w}_t + Z_{t-1} b_{t}$ and ensure that the update of $Z_t$ and $\bar{w}_t$ will always be sparse. To see this, denote the sketch $S_t$ by $\left( \begin{array}{c} D_t F_t Z_t \\ G_t \end{array} \right)$ and let $H_{t,1}$ and $H_{t,2}$ be the top and bottom half of $H_t$. Now the update rule of $u_{t+1}$ can be rewritten as

$$u_{t+1} = w_t - (I_d - S_t^T H_t S_t) g_t$$

$$= \bar{w}_t + Z_{t-1} b_t - \frac{1}{\alpha} g_t + \frac{1}{\alpha} (Z_t^T F_t^T D_t, G_t^T) \left( \begin{array}{c} H_{t,1} S_t g_t \\ H_{t,2} S_t g_t \end{array} \right)$$

$$= \bar{w}_t + \frac{1}{\alpha} (G_t^T H_{t,2} S_t g_t - g_t) - (Z_t - Z_{t-1})^T b_t + Z_t^T (b_t + \frac{1}{\alpha} F_t^T D_t H_{t,1} S_t g_t)$$

We will show that $Z_t - Z_{t-1} = \Delta_t G_t$ for some $\Delta_t \in \mathbb{R}^{m \times m}$ shortly, and thus the above update is efficient due to the fact that the rows of $G_t$ are collections of previous sparse vectors $\bar{g}$.

Similarly, the update of $w_{t+1}$ can be written as

$$w_{t+1} = u_{t+1} - \gamma_t (x_{t+1} - S_t^T H_t S_t x_{t+1})$$

$$= \bar{w}_{t+1} + Z_{t+1} b_{t+1} - \gamma_t (x_{t+1} + \gamma_t (Z_t^T F_t^T D_t, G_t^T) \left( \begin{array}{c} H_{t,1} S_t x_{t+1} \\ H_{t,2} S_t x_{t+1} \end{array} \right)$$

$$= \bar{w}_{t+1} + \gamma_t (G_t^T H_{t,2} S_t x_{t+1} - x_{t+1}) + Z_t^T (b_{t+1} + \gamma_t F_t^T D_t H_{t,1} S_t x_{t+1})$$

It is clear that $\gamma_t$ can be computed efficiently, and thus the update of $w_{t+1}$ is also efficient. These updates correspond to Line 9 and 10 of Algorithm 10.

It remains to perform the sketch update efficiently. Algorithm 11 is the sparse version of Algorithm 8. The challenging part is to compute eigenvectors and eigenvalues efficiently. Fortunately, in light of Algorithm 9 using the new representation $V = FZ$ one can directly translate the process to Algorithm 12 and find that the eigenvectors can be expressed in the form $N_1 Z + N_2 G$. The only tricky part is the decomposing step (Line 2 of Algorithm 12). Essentially, this step requires finding $L$ and $Q$ such that $LQ \bar{R} = \bar{P} \bar{R}$ and the
Algorithm 10 Sparse Sketched Online Newton with Frequent Directions

\textbf{Input:} Parameters $C, \alpha$ and $m$.

1. Initialize $\bar{u} = 0_{d \times 1}$, $b = 0_{m \times 1}$ and $(D, F, Z, G, H) \leftarrow \text{SketchInit} (\alpha, m)$ (Algorithm 11).
2. Let $S$ denote the matrix $\begin{pmatrix} DFZ \\ G \end{pmatrix}$ throughout the algorithm (without actually computing it).
3. Let $H_1$ and $H_2$ denote the upper and lower half of $H$, i.e. $H = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}$.
4. for $t = 1$ to $T$
   5. Receive example $x_t$.
   6. Projection step: compute $\hat{x} = Sx_t$ and $\gamma = \frac{\tau c(\bar{u}^\top x_t + b^\top Zx_t)}{\bar{u}^\top x_t - \bar{x}^\top H \bar{x}}$.
      Obtain $\bar{w} = \bar{u} + \gamma (G^\top H_2 \hat{x} - x_t)$ and $b \leftarrow b + \gamma F^\top DH_1 \hat{x}$.
   7. Compute gradient $y_t = \bar{w}^\top x_t + b^\top Zx_t$ and suffer loss $\ell_t(y_t)$.
   8. Compute gradient $g_t = \ell'_t(y_t)x_t$ and the to-sketch vector $\hat{g} = \sqrt{\sigma_t + m} g_t$.
   9. $(D, F, Z, G, H, \Delta) \leftarrow \text{SketchUpdate} (\hat{g})$ (Algorithm 11).
10. Update $\bar{u} = \bar{w} + \frac{1}{\alpha} (G^\top H_2 Sg - g) - G^\top \Delta^\top b$ and $b \leftarrow b - \frac{1}{\alpha} F^\top DH_1 Sg$.
11. end for

The average running time of Algorithm 10 is $O(m^2 + ms)$ per round, and the regret bound is exactly the same as the one stated in Theorem 5.

\section{Details for sparse Oja's algorithm}

We finally summarize the results of this section in the following theorem.

\textbf{Theorem 6.} The average running time of Algorithm 10 is $O(m^2 + ms)$ per round, and the regret bound is exactly the same as the one stated in Theorem 5.

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\textbf{Theorem 6.} The average running time of Algorithm 10 is $O(m^2 + ms)$ per round, and the regret bound is exactly the same as the one stated in Theorem 5.

\section{Details for sparse Oja’s algorithm}

We finally provide the missing details for the sparse version of the Oja’s algorithm. Since we already discussed the updates for $\bar{w},$ and $b$, in Section 4, we just need to describe how the updates for $F_t$ and $Z_t$ work. Recall that the dense Oja’s updates can be written in terms of $F$ and $Z$ as

\begin{equation}
\begin{aligned}
\Lambda_t &= (I_m - \Gamma_t) \Lambda_{t-1} + \Gamma_t \text{diag} \{ F_{t-1} Z_{t-1}^{-1} \hat{g}_t \}^2 \\
F_t Z_t &\leftarrow \text{orth} \ F_{t-1} Z_{t-1} + \Gamma_t F_{t-1} Z_{t-1} \hat{g}_t \hat{g}_t^\top = F_{t-1} (Z_{t-1} + \Gamma_t F_{t-1} Z_{t-1} \hat{g}_t \hat{g}_t^\top).
\end{aligned}
\end{equation}

Here, the update for the eigenvalues is straightforward. For the update of eigenvectors, first we let $Z_t = Z_{t-1} + \delta_t \hat{g}_t^\top$, where $\delta_t = F_{t-1}^{-1} \Gamma_t F_{t-1} Z_{t-1} \hat{g}_t$ (note that under the assumption of Footnote 2, $F_t$ is always invertible). Now it is clear that $Z_t - Z_{t-1}$ is a sparse rank-one matrix and the update of $\bar{u}_{t+1}$ is efficient. Finally it remains to update $F_t$ so that $F_t Z_t$ is the same as orthonormalizing $F_{t-1} Z_{t-1}$, which can in fact be achieved by applying the Gram-Schmidt algorithm to $F_{t-1}$ in a Banach space where inner product is defined as $\langle a, b \rangle = a^\top K_t b$ where $K_t$ is the Gram matrix $Z_t Z_t^\top$ (see Algorithm 13). Since we can maintain $K_t$ efficiently based on the update of $Z_t$:

\begin{equation}
K_t = K_{t-1} + \delta_t \hat{g}_t^\top Z_{t-1} + Z_{t-1} \hat{g}_t \delta_t^\top + (\hat{g}_t^\top \hat{g}_t) \delta_t \delta_t^\top,
\end{equation}

the update of $F_t$ can therefore be implemented in $O(m^3)$ time.
Algorithm 11 Sparse Frequent Direction Sketch

**Internal State:** $\tau, D, F, Z, G, H$ and $K$.

**SketchInit($\alpha, m$)**
1. Set $\tau = 1, D = 0_{m \times m}, F = K = I_m, H = \frac{1}{\alpha}I_{2m}, G = 0_{m \times d}$, and let $Z$ be any $m \times d$ matrix whose rows are orthonormal.
2. Return $(D, F, Z, G, H)$.

**SketchUpdate($\hat{g}$)**
1. Insert $\hat{g}$ into the $\tau$-th row of $G$.
2. if $\tau < m$ then
   3. Let $e$ be the $2m \times 1$ basic vector whose $(m + \tau)$-th entry is 1 and compute $q = S\hat{g} - \frac{\hat{g}^\top \hat{g}}{\tau} e$.
   4. Update $H \leftarrow H - \frac{Hqe^\top H}{1 + e^\top Hq}$ and $H \leftarrow H - \frac{He^\top H}{1 + q^\top He}$.
   5. Set $\Delta = 0_{m \times m}$.
   6. Set $\tau \leftarrow \tau + 1$.
3. else
   4. $(N_1, N_2, \Sigma) \leftarrow \text{ComputeSparseEigenSystem} \left( \begin{pmatrix} DFZ \\ G \end{pmatrix}, K \right)$ (Algorithm 12).
   5. Compute $\Delta = N_1^{-1}N_2$.
   6. Update Gram matrix $K \leftarrow K + \Delta GZ^\top + ZG^\top \Delta^\top + \Delta GG^\top \Delta^\top$.
   7. Update $F = N_1, Z \leftarrow Z + \Delta G$, and let $D$ be such that $D_{i,i} = \sqrt{\Sigma_{i,i} - \Sigma_{m,m}}, \forall i \in [m]$.
   8. Set $H \leftarrow \text{diag}\left\{ \frac{1}{\alpha + D_{1,1}^2}, \cdots, \frac{1}{\alpha + D_{m,m}^2}, \frac{1}{\alpha}, \cdots, \frac{1}{\alpha} \right\}$.
   9. Set $G = 0_{m \times d}$.
   10. Set $\tau = 1$.
4. end if
5. Return $(D, F, Z, G, H, \Delta)$.

I Experiment Details

This section reports some detailed experimental results omitted from Section 5.2. Table 1 includes the description of benchmark datasets, Table 2 reports concrete error rates, and Table 3 shows that Oja’s algorithm estimates the eigenvalues accurately.

As mentioned in Section 5.2, we see substantial improvement for the splice dataset when using Oja’s sketch even after the diagonal adaptation. We verify that the condition number for this dataset before and after the diagonal adaptation are very close (682 and 668 respectively), explaining why a large improvement is seen using Oja’s sketch. Fig. 4 shows the decrease of error rates as Oja-SON with different sketch sizes sees more examples. One can see that even with $m = 1$ Oja-SON already performs very well. This also matches our expectation since there is a huge gap between the top and second eigenvalues of this dataset (50.7 and 0.4 respectively).
Algorithm 12 ComputeSparseEigenSystem(S, K)

Input: $S = \begin{pmatrix} DFZ \\ G \end{pmatrix}$ and Gram matrix $K = ZZ^\top$.

Output: Output $N_1, N_2 \in \mathbb{R}^{m \times m}$ and diagonal matrix $\Sigma \in \mathbb{R}^{m \times m}$ such that the $i$-th row of $N_1Z + N_2G$ and the $i$-th entry of the diagonal of $\Sigma$ are the $i$-th eigenvector and eigenvalue of the matrix $S^\top S$.

1: Compute $M = GZ^\top F^\top$.
2: $(L, Q) \leftarrow$ Decompose $\left( \begin{pmatrix} -MF & I_m \\ GZ^\top & GG^\top \end{pmatrix}, \begin{pmatrix} K \\ GZ^\top \\ GG^\top \end{pmatrix} \right)$ (Algorithm 13).
3: Let $r$ be the number of columns of $L$. Compute the top $m$ eigenvectors $(U \in \mathbb{R}^{m \times (m+r)})$ and eigenvalues $(\Sigma \in \mathbb{R}^{m \times m})$ of the matrix $\begin{pmatrix} D^2 & 0_{m \times r} \\ 0_{r \times m} & 0_{r \times r} \end{pmatrix} + \begin{pmatrix} M^\top \\ L^\top \end{pmatrix} \begin{pmatrix} M \\ L \end{pmatrix}$.
4: Set $N_1 = U_1F + U_2Q_1$ and $N_2 = U_3Q_2$ where $U_1$ and $U_2$ are the first $m$ and last $r$ columns of $U$ respectively, and $Q_1$ and $Q_2$ are the left and right half of $Q$ respectively.
5: Return $(N_1, N_2, \Sigma)$.

Algorithm 13 Decompose(P, K)

Input: $P \in \mathbb{R}^{m \times n}$, $K \in \mathbb{R}^{m \times m}$ such that $K$ is the Gram matrix $K = RR^\top$ for some matrix $R \in \mathbb{R}^{n \times d}$ where $n \geq m, d \geq m$.

Output: Output $L \in \mathbb{R}^{m \times r}$ and $Q \in \mathbb{R}^{r \times n}$ such that $LQR = PR$ where $r$ is the rank of $PR$ and the rows of $QR$ are orthonormal.

1: Initialize $L = 0_{m \times m}$ and $Q = 0_{m \times n}$.
2: for $i = 1$ to $m$ do
3: Let $p^\top$ be the $i$-th row of $P$.
4: Compute $\alpha = QKp, \beta = p - Q^\top \alpha$ and $c = \sqrt{\beta^\top K \beta}$.
5: if $c \neq 0$ then
6: Insert $\frac{1}{c} \beta^\top$ to the $i$-th row of $Q$.
7: end if
8: Set the $i$-th entry of $\alpha$ to be $c$ and insert $\alpha$ to the $i$-th row of $L$.
9: end for
10: Delete the all-zero columns of $L$ and all-zero rows of $Q$.
11: Return $(L, Q)$. 

23
Table 1: Datasets used in experiments

| Dataset      | #examples | avg. sparsity | #features |
|--------------|-----------|---------------|-----------|
| 20news       | 18845     | 93.89         | 101631    |
| a9a          | 48841     | 13.87         | 123       |
| acoustic     | 78823     | 50.00         | 50        |
| adult        | 48842     | 12.00         | 105       |
| australian   | 690       | 11.19         | 14        |
| breast-cancer| 683       | 10.00         | 10        |
| census       | 299284    | 32.01         | 401       |
| cod-rna      | 271617    | 8.00          | 8         |
| covtype      | 581011    | 11.88         | 54        |
| diabetes     | 768       | 7.01          | 8         |
| gisette      | 1000      | 4971.00       | 5000      |
| heart        | 270       | 9.76          | 13        |
| ijcnn1       | 91701     | 13.00         | 22        |
| ionosphere   | 351       | 30.06         | 34        |
| letter       | 20000     | 15.58         | 16        |
| magic04      | 19020     | 9.99          | 10        |
| mnist        | 11791     | 142.43        | 780       |
| mushrooms    | 8124      | 21.00         | 112       |
| rcv1         | 781265    | 75.72         | 43001     |
| real-sim     | 72309     | 51.30         | 20958     |
| splice       | 1000      | 60.00         | 60        |
| w1a          | 2477      | 11.47         | 300       |
| w8a          | 49749     | 11.65         | 300       |

Figure 4: Error rates for Oja-SON with different sketch sizes on splice dataset
Table 2: Error rates for different algorithms

| Dataset     | Oja-SON |     |     |     | ADA-GRAD |
|-------------|---------|-----|-----|-----|---------|
|             | Without Diagonal Adaptation | With Diagonal Adaptation |
| m = 0       | m = 10  | m = 0 | m = 10 |
| 20news      | 0.121338 | 0.121338 | 0.049590 | 0.049590 | 0.068020 |
| a9a         | 0.204447 | 0.195203 | 0.155953 | 0.155953 | 0.156414 |
| acoustic    | 0.305824 | 0.260241 | 0.257894 | 0.257894 | 0.259493 |
| adult       | 0.199763 | 0.199803 | 0.150830 | 0.150830 | 0.181582 |
| australian  | 0.366667 | 0.366667 | 0.162319 | 0.157971 | 0.289855 |
| breast-cancer | 0.374817 | 0.374817 | 0.036603 | 0.036603 | 0.358712 |
| census      | 0.093610 | 0.062038 | 0.051479 | 0.051439 | 0.069629 |
| cod-rna     | 0.175107 | 0.175107 | 0.049710 | 0.049643 | 0.081066 |
| covtype     | 0.042304 | 0.042312 | 0.050827 | 0.050818 | 0.045507 |
| diabetes    | 0.433594 | 0.433594 | 0.329427 | 0.328125 | 0.391927 |
| gisette     | 0.208000 | 0.208000 | 0.152000 | 0.152000 | 0.154000 |
| heart       | 0.477778 | 0.388889 | 0.244444 | 0.244444 | 0.362963 |
| ijcnn1      | 0.046826 | 0.046826 | 0.034536 | 0.034645 | 0.036913 |
| ionosphere  | 0.188034 | 0.148148 | 0.182336 | 0.182336 | 0.190883 |
| letter      | 0.306650 | 0.232300 | 0.232350 | 0.232450 | 0.237350 |
| magic04     | 0.000263 | 0.000263 | 0.000158 | 0.000158 | 0.000210 |
| mnist       | 0.062336 | 0.062336 | 0.040031 | 0.039182 | 0.046561 |
| mushrooms   | 0.003323 | 0.002339 | 0.002462 | 0.002462 | 0.001969 |
| rcv1        | 0.055976 | 0.052694 | 0.052764 | 0.052766 | 0.059938 |
| real-sim    | 0.045140 | 0.043577 | 0.029498 | 0.029498 | 0.031670 |
| splice      | 0.343000 | 0.343000 | 0.294000 | 0.294000 | 0.301000 |
| w1a         | 0.000161 | 0.000161 | 0.000485 | 0.000485 | 0.003633 |
| w8a         | 0.000101 | 0.000101 | 0.000422 | 0.000422 | 0.000221 |

Table 3: Largest relative error between true and estimated top 10 eigenvalues using Oja’s rule.

| Dataset     | Relative eigenvalue difference |
|-------------|-------------------------------|
| a9a         | 0.90                          |
| australian  | 0.85                          |
| breast-cancer | 5.38                        |
| diabetes    | 5.13                          |
| heart       | 4.36                          |
| ijcnn1      | 0.57                          |
| magic04     | 11.48                         |
| mushrooms   | 0.91                          |
| splice      | 8.23                          |
| w8a         | 0.95                          |