Equivalence of Intuitionistic Inductive Definitions and
Intuitionistic Cyclic Proofs under Arithmetic

Stefano Berardi (Torino University)
Makoto Tatsuta (National Institute of Informatics)

Abstract

A cyclic proof system gives us another way of representing inductive definitions and efficient proof search. In 2011 Brotherston and Simpson conjectured the equivalence between the provability of the classical cyclic proof system and that of the classical system of Martin-Löf’s inductive definitions. This paper studies the conjecture for intuitionistic logic. This paper first points out that the countermodel of FOSSACS 2017 paper by the same authors shows the conjecture for intuitionistic logic is false in general. Then this paper shows the conjecture for intuitionistic logic is true under arithmetic, namely, the provability of the intuitionistic cyclic proof system is the same as that of the intuitionistic system of Martin-Löf’s inductive definitions when both systems contain Heyting arithmetic HA. For this purpose, this paper also shows that HA proves Podelski-Rybalchenko theorem for induction and Kleene-Brouwer theorem for induction. These results immediately give another proof to the conjecture under arithmetic for classical logic shown in LICS 2017 paper by the same authors.

1 Introduction

An inductive definition is a way to define a predicate by an expression which may contain the predicate itself. The predicate is interpreted by the least fixed point of the defining equation. Inductive definitions are important in computer science, since they can define useful recursive data structures such as lists and trees. Inductive definitions are important also in mathematical logic, since they increase the proof theoretic strength. Martin-Löf’s system of inductive definitions given in [11] is one of the most popular systems of inductive definitions. This system has production rules for an inductive predicate, and the production rule determines the introduction rules and the elimination rules for the predicate.

Brotherston and Simpson [5, 8] proposed an alternative formalization of inductive definitions, called a cyclic proof system. A proof, called a cyclic proof, is defined by proof search, going upwardly in a proof figure. If we encounter the same sequent (called a bud) as some sequent we already passed (called a companion) we can stop. The induction rule is replaced by a case rule, for this purpose. The soundness is guaranteed by some additional condition, called a global trace condition, which can show the case rule decreases some measure of a bud from that of the companion. In general, for proof search, a cyclic proof system can find an induction formula in a more efficient way than Martin-Löf’s system, since a cyclic proof system does not have to choose fixed induction formulas in advance. A cyclic proof system enables us efficient implementation of theorem provers with inductive definitions [4, 6, 7, 9].

Brotherston and Simpson [8] investigated Martin-Löf’s system LKID of inductive definitions in classical logic for the first-order language, and the cyclic proof system CLKIDω for the same language, showed the provability of CLKIDω includes that of LKID, and conjectured the equivalence.

By 2017, the equivalence was left an open question. In general, the conjecture was proved to be false in [2], by showing a counterexample. However, if we restrict both systems to only the natural number inductive predicate and add Peano arithmetic to both systems, the conjecture was proved to be true in [15], by internalizing a cyclic proof in ACA0 and using some results in reverse mathematics. [3] proved that if we add Peano arithmetic to both systems, CLKIDω and LKID are equivalent, namely the conjecture is true under arithmetic, by showing arithmetical Ramsey theorem and Podelski-Rybalchenko theorem for induction.

This paper studies the conjecture for intuitionistic logic, namely, the provability of the intuitionistic cyclic proof system, called CLJIDω, is the same as that of the intuitionistic system of Martin-Löf’s inductive definitions, called LJID. This question is theoretically interesting, and answers will potentially
give new techniques of theorem proving by cyclic proofs to type theories with inductive types and program extraction by constructive proofs.

This paper first points out that the countermodel of [2] also shows the conjecture for intuitionistic logic is false in general. Then this paper shows the conjecture for intuitionistic logic is true under arithmetic, namely, the provability of $CLJID^\omega$ is the same as that of $LJID$, when both systems contain Heyting arithmetical $HA$. Note that a counterexample in [2] does not work for a system that contains $HA$.

We explain main ideas of this paper. The proof transformation given in [3] is intuitionistic, so we can use it for our purpose. Only thing is to intuitionistically show arithmetical Ramsey theorem and Podelski-Rybalchenko theorem for induction, and we cannot use this technique in $HA$. In order to show it we will take the following steps:

1. For each $\pi \in B$, there is $n$ such that $\text{Ind}(>^n\pi)$. $(\cdot)^n$ denotes the $n$-time composition.)
2. Finiteness of path relations $\{>^n\pi \mid \pi \in B\}$.
3. Kleene-Brouwer theorem for induction.
4. Podelski-Rybalchenko theorem for induction.

The global trace condition gives (1). (4) is proved by (3). Combining (2) and (4), we will obtain $\text{Ind}(>^n\pi)$. The places we need arithmetic are the proofs of (3) and (4), since they use sequences of numbers. The claims (1) and (2) can be easily shown in almost the same way as [3]. We will show the claim (3) by refining an ordinary proof of Kleene-Brouwer theorem for orders. We will show the claim (4) by using Erdős trees and (3).

The results of this paper immediately give another proof to the conjecture under arithmetic for classical logic shown in [3] by using the fact $\Gamma \vdash CLKID^\omega + PA \Delta$ implies $E, \Gamma, \neg \Delta \vdash CLJID^\omega + HA$ for some finite set $E$ of excluded middles.

There are not papers that study the conjecture for intuitionistic logic or Kleene-Brouwer theorem for induction in intuitionistic first-order logic. For Podelski-Rybalchenko theorem for induction, [1] intuitionistically showed it but they used second-order logic.

Section 2 describes Brotherston-Simpson conjecture. Section 3 defines $CLJID^\omega + HA$ and $LJID + HA$. Section 4 explains main ideas. Section 5 proves Kleene-Brouwer theorem for induction and Podelski-Rybalchenko theorem for induction. Section 6 discusses proof transformation and the main theorem. We conclude in Section 7.

2 Brotherston-Simpson Conjecture for Intuitionistic Logic

2.1 Intuitionistic Martin-Löf’s Inductive Definition System $LJID$

An intuitionistic Martin-Löf’s inductive definition system, called $LJID$, is defined as the system obtained from classical Martin-Löf’s inductive definition system $LKID$ defined in [8] by restricting every sequent to intuitionistic sequents and replacing $(\rightarrow L)$, $(\lor R)$, and $(\text{Ind } P_j)$ by

\[
\frac{\Gamma \vdash F \quad \Gamma, G \vdash \Delta}{\Gamma, F \rightarrow G \vdash \Delta} \quad (\rightarrow L) \quad \frac{\Gamma \vdash F}{\Gamma \vdash F \lor G} \quad (\lor L) \quad \frac{\Gamma \vdash G}{\Gamma \vdash F \lor G} \quad (\lor R)
\]

\[
\text{minor premises } \quad \Gamma, P_j \vdash F_j \quad (\text{Ind } P_j)
\]

where the minor premises are the same as the minor premises of $(\text{Ind } P_j)$ in $LKID$ (page 9 of [8]). Note that we replace these rules because their formalization in $LKID$ does not work for intuitionistic logic.

2.2 Cyclic Proof System $CLJID^\omega$

An intuitionistic cyclic proof system, called $CLJID^\omega$, is defined as the system obtained from classical cyclic proof system $CLKID^\omega$ defined in [8] by restricting every sequent to intuitionistic sequents and replacing $(\rightarrow L)$ and $(\lor R)$ in the same way as $LJID$. Note that the global trace condition in $CLJID^\omega$ is the same as that in $CLKID^\omega$ (Definition 5.5 of [8]).
2.3 Brotherston-Simpson Conjecture

Brotherston-Simpson conjecture (the conjecture 7.7 in [8]) is that the provability of LKID is the same as that of CLKID. In general, the conjecture was proved to be false in [2], by showing a counterexample. [3] proved that the conjecture is true for any inductive predicates with their stage-number inductive predicates, if we add arithmetic to both systems.

This paper studies an intuitionistic version of the conjecture, namely the equivalence between CLJID and LJID.

The counterexample given in [2] also shows that the equivalence between CLJID and LJID does not hold in general, because the proof of the statement H in [2] is actually in CLJID, and LJID does not prove H since LKID does not prove H. This gives us the following theorem.

Theorem 2.1 There are some signature and some set of production rules such that the provability of CLJID is not the same as that of LJID.

This paper will show that the same results as [3] for intuitionistic logic, namely, the provability of LJID is the same as that of CLJID if we add Heyting arithmetic to both systems. This means that the conjecture is true for intuitionistic logic under arithmetic.

3 Addition of Heyting Arithmetic

In this section, we define systems CLJID + HA and LJID + HA.

Definition 3.1 CLJID + HA and LJID + HA are defined to be obtained from CLJID and LJID by adding Heyting arithmetic. Namely, we add constants and function symbols 0, s, +, ×, the inductive predicate symbol N, the productions for N, and Heyting axioms:

\[
\begin{align*}
&\vdash Nx \rightarrow sx \neq 0, \quad \vdash Nx \land Ny \rightarrow sx = sy \rightarrow x = y, \\
&\vdash Nx \rightarrow x + 0 = x, \quad \vdash Nx \land Ny \rightarrow x + sy = s(x + y), \\
&\vdash Nx \rightarrow x \times 0 = 0, \quad \vdash Nx \land Ny \rightarrow x \times sy = x \times y + x.
\end{align*}
\]

We define \( x < y \) by \( \exists z. x + sz = y \) and \( x \leq y \) by \( x = y \lor x < y \).

We can assume some coding of a sequence of numbers by a number in Heyting arithmetic, because the definitions on pages 115–117 of [14] work also in HA. We write \( \langle t_0, \ldots, t_n \rangle \) for the sequence of \( t_0, \ldots, t_n \). In particular, we write \( \langle \rangle \) for the empty sequence. We define the \( i \)-th element of \( t_0, \ldots, t_n \) as \( t_i \). We also write \( |t| \) and \( (t)_u \) for the length of the sequence \( t \) and the \( u \)-th element of the sequence \( t \) respectively. Note that \( |\langle t_0, \ldots, t_n \rangle| = n + 1 \) and \( (\langle t_0, \ldots, t_n \rangle)_i = t_i \). We write \( \ast \) for the concatenation operation of sequences.

We call an atomic formula an inductive atomic formula when its predicate symbol is an inductive predicate symbol. For a predicate \( P \), we sometimes write \( t \in P \) for \( P(t) \).

4 Main Ideas

In [3], a given proof in CLKID + PA is transformed into a proof in LKID + PA. The main construction of the proof in LKID + PA is summarized by

\[
\begin{align*}
(1) \text{Ineq} &
\quad \langle x_0, x \rangle \succ_{\Pi} \langle y_0, y \rangle \\
&\quad \vdash_{J_1} \text{Ineq} \rightarrow J_1
\end{align*}
\]

\[
\begin{align*}
\text{Ind}(\succ_{\Pi}) &
\quad \vdash_{J_2} \text{Ineq} \rightarrow J_2
\end{align*}
\]

\[
\begin{align*}
(2) Hx_0x &
\quad \vdash_{J} Gx_0x
\end{align*}
\]
One of the key points in this proof construction is the proof of Ind(>Π).

The same construction works for CLJID<sup>+</sup> + HA and LJID + HA except Ind(>Π), since the proof uses only intuitionistic sequents when the goal sequent is an intuitionistic sequent.

In order to intuitionistically show Ind(>Π), we will take the following steps:

1. For each π ∈ B, there is n such that Ind(>Π)<sub>n</sub>.
2. Finiteness of path relations {>π | π ∈ B}.
3. Kleene-Brouwer theorem for induction.
4. Podelski-Rybalchenko theorem for induction.

The global trace condition gives (1). (4) is proved by (3). Combining (2) and (4), we will obtain Ind(>Π). The places we need arithmetic are the proofs of (3) and (4), since they use sequences of numbers.

The claim (1) can be shown in the same way as Π since that paper did not use classical logic for proving (1). The claim (2) can be easily shown also in intuitionistic logic by using iteration to the least point.

For (3), we will show Kleene-Brouwer theorem for induction, which states that if we have both induction principle for a lifted tree (namely ⟨u⟩*T for some tree T) with respect to the one-step extension relation and induction principle for relations on children, then we have induction principle for the Kleene-Brouwer relation. We can prove it by refining an ordinary proof of Kleene-Brouwer theorem for orders.

For (4), we will show Podelski-Rybalchenko theorem for induction stating that if transition invariant >Π<sub>1</sub> is a finite union of relations >Π<sub>π</sub> such that each Ind(>Π<sub>π</sub>) is provable for some n, and each (>Π<sub>π</sub>) is decidable, then Ind(>Π) is provable.

First each Ind(>Π<sub>π</sub>) is obtained by Ind(>Π<sub>π</sub>). Next by the decidability of each (>Π<sub>π</sub>), we can assume all of (>Π<sub>π</sub>) are disjoint to each other. For simplicity, we explain the idea of our proof for well-foundedness instead of induction principle.

Assume the relation >Π has some decreasing transitive sequence

\[ u_1 >Π u_2 >Π u_3 >Π \ldots \]

in order to show contradiction, where a sequence is called transitive when \( u_i >Π u_j \) for any \( i < j \).

We say the edge from \( u \) to \( v \) is of color \( R_\pi \) when \( u >R_\pi v \). A sequence is called monotonically-colored when for any element there is a color such that the edge from the element to any element after it in the sequence has the same color. Define MS as the set of monotonically-colored finite sequences. It will be shown to be well-founded with the one-step extension relation. A set of sequences beginning with the same element in some tree is called a lifted tree. For a decreasing transitive sequence \( x \) of \( U \), a lifted tree \( T \in U^<\omega \) is called an Erdős tree of \( x \) when the elements of \( x \) are the same as elements of elements of \( T \), every element of \( T \) is monotonically-colored, and the edges from a parent to its children have different colors. Let ET be a function that returns an Erdős tree. Then we consider

\[ ET((u_1)), ET((u_1, u_2)), ET((u_1, u_2, u_3)), \ldots \]

Define MS<sub>(r)</sub> as the set of sequences beginning with \( r \) in MS. Define >KB1,r as the Kleene-Brouwer relation for the lifted tree MS<sub>(r)</sub> and some left-to-right-decreasing relation on children of the lifted tree. Define >KB2,r as the Kleene-Brouwer relation for the lifted tree MS<sub>(r)</sub> and some right-to-left-decreasing relation on children of the lifted tree. By Kleene-Brouwer theorem, (>KB1,r) and (>KB2,r) are well-founded. Define ET2(⟨u₁,...,uₙ⟩) as the (>KB2,u₁)-sorted sequence of elements in ET(⟨u₁,...,uₙ⟩). Then consider

\[ ET2(⟨u₁⟩), ET2(⟨u₁, u₂⟩), ET2(⟨u₁, u₂, u₃⟩), \ldots \]

Define >KB,r as the Kleene-Brouwer relation for >KB1,r and the set of (>KB2,r)-sorted finite sequences of elements in MS<sub>(r)</sub>. This definition is a key idea. By this definition, we have

\[ ET2(⟨u₁⟩) >KB,u₁ ET2(⟨u₁, u₂⟩) >KB,u₂ ET2(⟨u₁, u₂, u₃⟩) >KB,u₃, \ldots \]

Since (>KB,u₁) is well-founded by Kleene-Brouwer theorem, we have contradiction.

In general we need classical logic to derive induction principle from well-foundedness, but the idea we have explained will work well for showing induction principle in intuitionistic logic.
5 HA-Provable Podelski-Rybalchenko Theorem for Induction

This section will prove Podelski-Rybalchenko theorem for induction, inside Heyting arithmetic HA. First we will prove Kleene-Brouwer theorem for induction, inside HA. Next we will show induction for the set MS of monotonically-colored subsequences. Then by applying Kleene-Brouwer theorem to a part of MS and some orders $>_u$,Left and $>_u$,Right, we will obtain two Kleene-Brouwer relations $>_KB^1$,r and $>_KB^2$,r and show their induction principle. Then by applying Kleene-Brouwer theorem to some lifted tree determined by $>_KB^2$,r and the relation $>_KB^1$,r, we will obtain a Kleene-Brouwer relation $>_KB$,r and show its induction principle. Then we will show induction for decreasing transitive sequences is reduced to induction for Erdős trees with the relation $>_KB,r$. Since Erdős trees are in the lifted tree, by combining them, we will prove Podelski-Rybalchenko theorem for induction.

We write $>_R$ or $>$ for a binary relation. We write $<_R$ for the binary relation of the inverse of $>_R$. We write $y<_R x \in X$ for $y<_R x \land y \in X$. We write $x \in \sigma$ when $x$ is an element of the sequence $\sigma$. We write $U^<_\omega$ for the set of finite sequences of elements in $U$. For a set $S$ of sequences, we write $(u) \ast S$ for $\{\langle u \ast \sigma \mid \sigma \in S\}$. For a set $U$ and a binary relation $>_R$ for $U$, the induction principle for $(U,>_R)$ is defined as

$$\text{Ind}(U,>_R, F) \equiv \forall x \in U((\forall y<_R x \in U.Fy) \rightarrow Fx) \rightarrow \forall x \in U.Fx,$$

$$\text{Ind}(U,>_R) \equiv \text{Ind}(U,>_R, F) \text{ (for every formula } F).$$

For a set $U$ a set $T$ is called a tree of $U$ if $T \subseteq U^<_\omega$ and $T$ is nonempty and closed under prefix operations. Note that the empty sequence is a prefix of any sequence. As a graph, the set of nodes is $T$ and the set of edges is $\{(x,y) \in T^2 \mid y = x \ast \langle u \rangle\}$. We call a set $T \subseteq U^<_\omega$ a lifted tree of $U$ when there are a tree $T' \subseteq U^<_\omega$ and $r \in U$ such that $T = \langle r \ast T' \rangle$. We define LiftedTree($T, U$) as a first-order formula that means $T$ is a lifted tree of $U$.

For $x, y \in U^<_\omega$ we define the one-step extension relation $x >_{ext} y$ if $y = x \ast \langle u \rangle$ for some $u$. For a set $T \subseteq U^<_\omega$ and $\sigma \in U^<_\omega$, we define $T_\sigma$ as $\{\rho \in T \mid \rho = \sigma \ast \rho'\}$. Note that $T_\sigma$ is a subset of $T$. For a nonempty sequence $\sigma$, we define $\text{first}(\sigma)$ and $\text{last}(\sigma)$ as the first and the last element of $\sigma$ respectively.

The next lemma shows induction implies $x \neq x$.

**Lemma 5.1** If HA $\vdash \text{Ind}(U,>)$, then HA $\vdash \forall x, y \in U(y < x \rightarrow y \neq x)$.

**Proof.** Fix $x, y \in U$ and assume $y < x$ and $y = x$ in order to show contradiction. Define $Fz$ be $z \neq x$. Then we can show

$$\text{HA} \vdash \forall z \in U((\forall w < z \in U.Fw) \rightarrow Fz)$$

by case analysis for $z \neq x \lor z = x$ as follows. In the first case $z \neq x$, $Fz$. In the second case, by taking $w$ to be $x$ in $\forall w < z \in U.Fw$, we have $Fz$. By $\text{Ind}(U,>)$, $\forall z \in U.Fz$. By taking $z$ to be $x$, we have contradiction. $\square$

**Definition 5.2 (Kleene-Brouwer Relation)** For a set $U$, a lifted tree $T$ of $U$, and the set of binary relations $>_u$ on $U$ for every $u \in U$, we define the Kleene-Brouwer relation $>_KB$ for $T$ and $\{(>_u) \mid u \in U\}$ as follows: for $x, y \in T, x >_{KB} y$ if (1) $x = z \ast \langle u, u_1 \rangle \ast w_1$, $y = z \ast \langle u, u_2 \rangle \ast w_2$, and $u_1 >_u u_2$ for some $z, u, u_1, w_1, u_2, w_2$, or (2) $y = x \ast z$ for some $z \neq \langle \rangle$.

When $>_u$ is some fixed ($>$) for all $u$, for simplicity we call the relation $>_u$ the Kleene-Brouwer relation for $T$ and $>$.

Note that $>_KB$ is a relation on $T$. This Kleene-Brouwer relation is slightly different from ordinary Kleene-Brouwer order for the following points: it creates a relation instead of an order, it uses a set of relations indexed by an element, and it is defined for a lifted tree instead of a tree (in order to use indexed relations).

The next theorem shows Kleene-Brouwer theorem for induction, which states that if we have both induction principle for a lifted tree with respect to the extension relation and induction principle for relations on children, then we have induction principle for the Kleene-Brouwer relation.

**Theorem 5.3 (Kleene-Brouwer Theorem for Induction)** If $\text{HA} \vdash \text{LiftedTree}(T, U)$, $\text{HA} \vdash \text{Ind}(T, >_{ext})$ and $\text{HA} \vdash \forall u \in U.\text{Ind}(U,>_u)$, then $\text{HA} \vdash \text{Ind}(T,>_KB)$. 

5
Proof. By induction on \((T, >_{\text{ext}})\), we will show \(\forall \sigma \in T. \text{Ind}(T_\sigma, >_{\text{KB}})\). After we prove it, we can take \(\sigma\) to be \(\langle \rangle\) to show the theorem, since \(T(\langle \rangle) = T\).

Fix \(\sigma \in T\) in order to show \(\text{Ind}(T_\sigma, >_{\text{KB}})\). Note that we can use induction hypothesis for every \(\sigma \ast (u) \in T\):

\[
\text{Ind}(T_{\sigma \ast (u)}, >_{\text{KB}}). 
\tag{1}
\]

Assume

\[
\forall x \in T_\sigma ((\forall y <_{\text{KB}} x \in T_\sigma.Fy) \rightarrow Fx)
\tag{2}
\]

in order to show \(\forall x \in T_\sigma.Fx\). For simplicity we write \(F(X)\) for \(\forall x \in X.Fx\). Let \(GU \equiv F(T_{\sigma \ast (u)})\). By \(\text{Ind}(U, >_{\text{last}(\sigma)})\) we will show the following claim.

Claim: \(\forall u \in U.Gu\).

Fix \(u \in U\) in order to show \(Gu\).

By IH for \(v\) with \(>_{\text{last}(\sigma)}\) we have

\[
v <_{\text{last}(\sigma)} u \rightarrow F(T_{\sigma \ast (v)}).
\tag{3}
\]

We can show

\[
\forall x \in T_{\sigma \ast (u)}((\forall y <_{\text{KB}} x \in T_{\sigma \ast (u)}.Fy) \rightarrow (\forall y <_{\text{KB}} x \in T_\sigma.Fy))
\tag{4}
\]

as follows. Fix \(x \in T_{\sigma \ast (u)}\), assume

\[
\forall y <_{\text{KB}} x \in T_{\sigma \ast (u)}.Fy
\tag{5}
\]

and assume \(y <_{\text{KB}} x \in T_\sigma\) in order to show \(Fx\). By definition of \(>_{\text{KB}}\), we have \(y \in T_{\sigma \ast (u)}\) for some \(v <_{\text{last}(\sigma)} u\), or \(y \in T_{\sigma \ast (u)}\). In the first case, \(Fx\) by \(3\). In the second case, \(Fx\) by \(\ref{5}\). Hence we have shown \(4\).

Combining \ref{1} with \ref{2}, we have

\[
\forall x \in T_{\sigma \ast (u)}((\forall y <_{\text{KB}} x \in T_{\sigma \ast (u)}.Fy) \rightarrow F(x)).
\tag{6}
\]

By IH \ref{1} for \(\sigma \ast (u)\), we have \(\text{Ind}(T_{\sigma \ast (u)}, >_{\text{KB}})\), namely,

\[
\forall x \in T_{\sigma \ast (u)}((\forall y <_{\text{KB}} x \in T_{\sigma \ast (u)}.Fy) \rightarrow Fx) \rightarrow \forall x \in T_{\sigma \ast (u)}.Fx.
\tag{7}
\]

By \ref{1} \ref{6}, \(F(T_{\sigma \ast (u)})\). Hence we have shown the claim.

If \(y <_{\text{KB}} \sigma \in T_\sigma\), we have \(y \in T_{\sigma \ast (u)}\) for some \(u\), since \(y <_{\text{KB}} \sigma\) implies \(y \neq \sigma\) by definition of KB and Lemma \ref{5} for \(\sigma\). By the claim, \(Fx\). Hence

\[
\forall y <_{\text{KB}} \sigma \in T_\sigma.Fy.
\tag{8}
\]

By letting \(x := \sigma\) in \ref{2}, we have \((\forall y <_{\text{KB}} \sigma \in T_\sigma.Fx) \rightarrow F\sigma\). By \ref{8}, \(F\sigma\). Combining it with the claim, \(\forall x \in T_\sigma.Fx\). □

**Definition 5.4** For a set \(U\) and a relation \(>\) for \(U\), we define the set \(DS(U, >)\) of decreasing sequences as \(\{\langle x_0, \ldots, x_{n-1}\rangle \mid n \geq 0, x_i \in U, \forall i < n - 1.(x_i > x_{i+1})\}\).

We define the set \(DT(U, >)\) of decreasing transitive sequences by \(\{\langle x_0, \ldots, x_{n-1}\rangle \mid n \geq 0, x_i \in U, \forall i \leq n - 1.(i < j \rightarrow x_i < x_j)\}\).

We define \(>_{R_1, \ldots, R_k}\) as the union of \(>_{R_i}\) for all \(1 \leq i \leq k\). We define \(>_{R_1 + \ldots + R_k}\) as the disjoint union of \(>_{R_i}\) for all \(1 \leq i \leq k\). (Whenever we use it, we implicitly assume the disjointness is provable in \(\text{HA}\).)

We define Monosed\(_{R_1, \ldots, R_k}(x)\) to hold when \(x = \langle x_0, \ldots, x_{n-1}\rangle \in DT(U, >_{R_1 + \ldots + R_k})\) and \(\forall i < n - 1.(i < j \rightarrow \bigwedge_{1 \leq j \leq k} (x_i >_{R_j} x_{i+1} \rightarrow x_i >_{R_j} x_j))\). Note that \(n\) may be 0.

We define MS as \(\{x \in DT(U, >_{R_1 + \ldots + R_k}) \mid \text{Monosed}_{R_1, \ldots, R_k}(x)\}\).

MS is the set of monotonically-colored finite sequences. Note that \(MS_{(r)}\) is a subset of MS and a lifted tree of \(U\) for any \(r \in U\).
Definition 5.5 For a relation $>_{R_i}$ on $U_i$ for $1 \leq i \leq k$, we define a relation $>_{R_1 \times \ldots \times R_k}$ on $U_1 \times \ldots \times U_k$ by: $(x_1, \ldots, x_k) >_{R_1 \times \ldots \times R_k} (y_1, \ldots, y_k)$ if there is some $i$ such that $x_i >_{R_i} y_i$ and $x_j = y_j$ for all $j \neq i$.

The next lemma shows induction for cartesian product.

Lemma 5.6 If $HA \vdash \text{Ind}(U_i, >_{R_i})$ for $1 \leq i \leq k$, then $HA \vdash \text{Ind}(U_1 \times \ldots \times U_k, >_{R_1 \times \ldots \times R_k})$.

Proof. First we will show the case $k = 2$.
For simplicity, we write $U$ for $U_1 \times U_2$ and $>_{\times}$ for $>_{R_1 \times R_2}$.
Assume
\[ \forall x \in U((\forall y <_\times x \in U.Fy) \rightarrow Fx) \] in order to show $\forall x \in U.Fx$.
Define $Gx_1 \equiv \forall x_2 \in U_2.F(x_1, x_2)$. We will show
\[ \forall x_1 \in U_1((\forall y_1 <_{R_1} x_1 \in U_1.Gy_1) \rightarrow Gx_1). \] Fix $x_1 \in U_1$ and assume
\[ \forall y_1 <_{R_1} x_1 \in U_1.Gy_1 \] in order to show $Gx_1$. We will show
\[ \forall x_2 \in U_2((\forall y_2 <_{R_2} x_2.F(x_1, y_2)) \rightarrow F(x_1, x_2)). \] Fix $x_2 \in U_2$ and assume
\[ \forall y_2 <_{R_2} x_2.F(x_1, y_2), \] in order to show $F(x_1, x_2)$. We will show
\[ \forall y <_\times (x_1, x_2) \in U.Fy. \] Fix $y <_\times (x_1, x_2) \in U$ in order to show $Fy$. Let $(y_1, y_2)$ be $y$. Consider cases by $y <_\times (x_1, x_2)$.
Case 1. $y_1 = x_1$ and $y_2 <_{R_2} x_2$.
By taking $y_2$ to be $y_2$ in [13], $F(x_1, y_2)$. Hence $Fy$.
Case 2. $y_1 <_{R_1} x_1$ and $y_2 = x_2$.
By taking $y_1$ to be $y_1$ in [11], $Gy_1$. Hence $\forall x_2 \in U_2.F(y_1, x_2)$. By taking $x_2$ to be $x_2$ in it, $F(y_1, x_2)$. Hence $Fy$.
In both cases, $Fy$. Hence we have shown [14]. By taking $x$ to be $(x_1, x_2)$ in [9], we have $F(x_1, x_2)$. Hence we have shown [12]. By $\text{Ind}(U_2, >_{R_2})$ for $\lambda x_2.F(x_1, x_2)$, we have $\forall x_2 \in U_2.F(x_1, x_2)$. Hence $Gx_1$. Hence we have shown [10]. By $\text{Ind}(U_1, >_{R_1})$ for $G$, we have $\forall x_1 \in U_1.Gx_1$. Hence $\forall x \in U.Fx$.
We have shown the case $k = 2$.
Next we will show the case $k > 2$. We use induction on $k$ to show the claim. By IH, we have $\text{Ind}(U_1 \times \ldots \times U_{k-1}, >_{R_1 \times \ldots \times R_{k-1}})$. By using the case $k = 2$ for it and $\text{Ind}(U_k, >_{R_k})$, we have $\text{Ind}(U_1 \times \ldots \times U_k, >_{(R_1 \times \ldots \times R_{k-1}) \times R_k})$. Since $(>_{(R_1 \times \ldots \times R_{k-1}) \times R_k})$ is $(>_{R_1 \times \ldots \times R_k})$, we have the claim. \[ \square \]

The next lemma shows that induction principle for each relation implies induction principle for monotonically-colored sequences. This lemma can be proved by refining Lemma 6.4 (1) of [1] from second-order logic to first-order logic.

Lemma 5.7 If $HA \vdash \text{Ind}(DT(U, >_{R_i}), >_{\text{ext}})$ for all $1 \leq i \leq k$, then $HA \vdash \forall r \in U.\text{Ind}(\text{MS}_{(r)}, >_{\text{ext}})$.

Proof. Fix $r \in U$ in order to show $\text{Ind}(\text{MS}_{(r)}, >_{\text{ext}})$.
Assume
\[ \forall \sigma \in \text{MS}_{(r)}((\forall \rho <_{\text{ext}} \sigma \in \text{MS}_{(r)}.F\rho) \rightarrow F\sigma) \] in order to show $\forall \sigma \in \text{MS}_{(r)}.F\sigma$.
For $1 \leq i \leq k$, define
\[ \text{Seq}_i(\sigma) = (x_{n_1}, \ldots, x_{n_m}) \]
where $\sigma = \langle x_1, \ldots, x_n \rangle$, $\{x_{n_1}, \ldots, x_{n_m}\} = \{x_j \mid x_j >_{R_{i}} x_{j+1}\}$, and $n_1 <_{N} \ldots <_{N} n_m$ for the natural number order $>_{N}$. Formally $\text{Seq}_I(\sigma) = \rho$ is an abbreviation for some HA-formula $F(\sigma, \rho)$. Note that $\text{Seq}_I(\sigma)$ may be $\langle \rangle$.

For simplicity we write $\text{DT}_k$ for $\text{DT}(U, >_{R_{1}}) \times \ldots \times \text{DT}(U, >_{R_{k}})$.

We define $\text{ext}^k$ for $\text{DT}_k$ by

$$(x_1, \ldots, x_k) >_{\text{ext}^k} (y_1, \ldots, y_k)$$

where for some $1 \leq i \leq k$, $x_i >_{\text{ext}} y_i$ and $x_j = y_j$ for all $j \neq i$. Note that the set of elements of $\sigma$ is the union of the sets of $\text{Seq}_I(\sigma) (1 \leq i \leq k)$ and $\{\text{last}(\sigma)\}$.

Define

$G(x_1, \ldots, x_k) \equiv \forall \sigma \in \text{MS}_{\langle \rangle}((1 \leq i \leq k. \text{Seq}_I(\sigma) = x_i) \rightarrow F \sigma).$

We write $\overrightarrow{x}$ for $(x_1, \ldots, x_k)$. We will show

$$\forall \overrightarrow{y} \in \text{DT}_k((\forall \overrightarrow{y} <_{\text{ext}^k} \overrightarrow{x} \in \text{DT}_k.G(\overrightarrow{y})) \rightarrow G(\overrightarrow{x})).$$

(16)

Fix $\overrightarrow{x}$ and assume

$$\forall \overrightarrow{y} <_{\text{ext}^k} \overrightarrow{x} \in \text{DT}_k.G(\overrightarrow{y})$$

in order to show $G(\overrightarrow{x})$. Fix $\sigma \in \text{MS}_{\langle \rangle}$ and assume $\text{Seq}_I(\sigma) = x_i$ for all $1 \leq i \leq k$ in order to show $F \sigma$.

We can show $\forall \rho <_{\text{ext}} \sigma \in \text{MS}_{\langle \rangle}. F \sigma$ as follows. Assume $\rho <_{\text{ext}} \sigma$. Let $\rho = \sigma * (u)$. Then last$(\sigma) >_{R_{i}} u$ for some $1 \leq i \leq k$. Then

$$\text{Seq}_I(\rho) = \text{Seq}_I(\sigma)* \langle \text{last}(\sigma) \rangle, \quad \text{Seq}_J(\rho) = \text{Seq}_J(\sigma) \quad (\forall j \neq i).$$

Hence $(\text{Seq}_I(\rho), \ldots, \text{Seq}_k(\rho)) <_{\text{ext}^k} (\text{Seq}_I(\sigma), \ldots, \text{Seq}_k(\sigma))$, namely, $\text{Seq}_I(\rho), \ldots, \text{Seq}_k(\rho)) <_{\text{ext}^k} \overrightarrow{x}$. By (17), $G(\text{Seq}_I(\rho), \ldots, \text{Seq}_k(\rho))$. Hence $F \rho$. Hence we have shown $\forall \rho <_{\text{ext}} \sigma \in \text{MS}_{\langle \rangle}. F \sigma$.

By (16), $F \sigma$. Hence we have shown $G(\overrightarrow{x})$. Hence we have shown (16).

By Lemma 5.6 for Ind$(\text{DT}(U, >_{R_{i}}), >_{\text{ext}^k})$ for all $1 \leq i \leq k$, we have Ind$(\text{DT}_k, >_{\text{ext}^k})$. By it and (10), we have

$$\forall \overrightarrow{x} \in \text{DT}_k.G(\overrightarrow{x}).$$

For every $\sigma \in \text{MS}_{\langle \rangle}$, by letting $x_i = \text{Seq}_I(\sigma)$ for all $1 \leq i \leq k$, we have $G(\overrightarrow{x})$, and hence we have $F \sigma$. □

Next we create Kleene-Brouwer relations $>_\text{KB}_{1,r}$ and $>_\text{KB}_{2,r}$ for monotonically-colored sequences beginning with $r$. Then we consider the set of $(>_\text{KB}_{2,r})$-sorted finite sequences of monotonically-colored finite sequences beginning with $r$. It is a lifted tree. Then, by induction principle for MS, the lifted tree is well-founded with the one-step extension relation. The Kleene-Brouwer relation for the lifted tree and $>_\text{KB}_{1,r}$ gives us $>_\text{KB}_{r}$ for the lifted tree. Since an Erdős tree is in the lifted tree, this will later show induction principle for Erdős trees.

**Definition 5.8** For $u \in U$, we define $>_\text{u,Left}$ for $U$ by: $u_1 >_{u,\text{Left}} u_2$ if $u >_{R_{i}} u_1, u >_{R_{i}} u_2$, and $j < l$ for some $j, l$.

We define $>_\text{KB}_{1,r}$ for $\text{MS}_{\langle \rangle}$ as the KB relation for $\text{MS}_{\langle \rangle} \subseteq U^\omega$ and $(>_\text{u,Left}) \subseteq U^2$ for all $u \in U$.

For $u \in U$, we define $>_\text{u,Right}$ for $U$ by: $u_1 >_{u,\text{Right}} u_2$ if $u_1 <_{\text{u,Left}} u_2$.

We define $>_\text{KB}_{2,r}$ for $\text{MS}_{\langle \rangle}$ as the KB relation for $\text{MS}_{\langle \rangle} \subseteq U^\omega$ and $(>_\text{u,Right}) \subseteq U^2$ for all $u \in U$.

We define $>_\text{KB}_{r}$ for $\text{DS}(\text{MS}_{\langle \rangle}, >_{\text{KB}_{2,r}})\langle (\langle r \rangle) \rangle$ as the KB relation for $\text{DS}(\text{MS}_{\langle \rangle}, >_{\text{KB}_{2,r}})\langle (\langle r \rangle) \rangle \subseteq \text{MS}_{\langle \rangle}^\omega$ and $>_\text{KB}_{1,r}$.

$>_\text{u,Left}$ is the left-to-right-decreasing order of children of $u$ in some ordered tree of $U$ in which the edge label $R_{i}$ is put to an edge $(x, y)$ such that $x >_{R_{i}} y$, each parent has at most one child of the same edge label, and children are ordered by their edge labels with $R_{1} < \ldots < R_{k}$. Similarly $>_\text{u,Right}$ is the right-to-left-decreasing order of children of $u$ in the ordered tree.

**Definition 5.9** For $u \in U \subseteq N$, finite $T \subseteq MS$ such that $\forall \rho \in T. \forall v \in \rho. (v >_{R_{1}} \ldots >_{R_{k}} u)$, and for $\sigma \in T$, we define the function insert by:

$$\text{insert}(u, T, \sigma) =$$

$$\text{insert}(u, T, \sigma * \langle v \rangle) \text{ if last}(\sigma) >_{R_{i}} u, v = \mu v. (\sigma * \langle v \rangle) \in T \land \text{last}(\sigma) >_{R_{i}} v),$$

$$T \cup \{\sigma * \langle u \rangle\} \text{ otherwise},$$

8
where $\mu v.F(v)$ denotes the least element $v$ with the natural number order such that $F(v)$. Formally $\text{insert}(u, T, \sigma) = T'$ is an abbreviation of some HA-formula $G(u, T, \sigma, T')$. It is the same for ET below.

For $x \in DT(U, >_{R_1+\ldots+R_k}) - \{()\}$, we define $ET(x) \subseteq MS$ by

$\text{ET}((u)) = \{ (u) \}$,
$\text{ET}(x \cdot (u)) = \text{insert}(u, ET(x), \langle \text{first}(x) \rangle)$ if $x \neq ()$.

Note that $\text{insert}(u, T, \sigma)$ adds a new element $u$ to the set $T$ at some position below $\sigma$ to obtain a new set. $ET(x)$ is an Erdős tree obtained from the decreasing transitive sequence $x$.

The next lemma (1) states a new element is inserted at a leaf. It is proved by induction on the number of nodes in the Erdős tree $ET(x)$.

**Lemma 5.10** (1) For $u \in U$, $T \subseteq MS$, and $\sigma \in T$, if $u \notin \rho$ for all $\rho \in T$, $\sigma = \langle x_0, \ldots, x_{n-1} \rangle$, $x_i >_{R_j} x_{i+1}$ implies $x_i >_{R_j} u$ for all $i < n - 1$, and $\text{insert}(u, T, \sigma) = T'$, then there is some $\rho \in T_\sigma$ such that $\rho \cdot \langle u \rangle$ is a maximal sequence in $T'$.

**Lemma 5.10** (2) If $\sigma \cdot \langle u, u_1 \rangle \ast \rho_1, \sigma \cdot \langle u, u_2 \rangle \ast \rho_2 \in ET(x), u >_{R_1} u_1$, and $u >_{R_2} u_2$, then $u_1 = u_2$.

**Definition 5.11** For $x \in DT(U, >_{R_1+\ldots+R_k}) - \{()\}$, we define

$\text{ET}(x) \equiv \langle x_0, \ldots, x_{n-1} \rangle$

where $\{x_0, \ldots, x_{n-1}\} = \text{ET}(x)$ and $\forall i < n - 1. (x_i >_{KB_2, \text{first}(x)} x_{i+1})$.

Note that $>_{KB_2, \text{first}(x)}$ is a total order on $ET(x)$ by Lemma 5.10 (2). $ET(x)$ is the decreasing sequence of all nodes in the Erdős tree $ET(x)$ ordered by $>_{KB_2, \text{first}(x)}$.

The next lemma shows $ET2$ is monotone. It is the key property of reduction in Lemma 5.13.

**Lemma 5.12** HA $\vdash \forall r \in U, \forall x, y \in DT(U, >_{R_1+\ldots+R_k})(r). (x >_{\text{ext}} y \to ET(2)(x) >_{KB, r} ET(2)(y))$.

**Proof.** Fix $r \in U$ and $x, y \in DT(U, >_{R_1+\ldots+R_k})(r)$ and assume $x >_{\text{ext}} y$. Let $y = x \cdot \langle u \rangle$. Then $ET(y) = \text{insert}(u, ET(x), \langle r \rangle)$. By Lemma 5.10 (1), we have $\sigma$ such that $ET(y) = ET(x) + \{ \sigma \cdot \langle u \rangle \}$. Then we have two cases:

Case 1. $\text{last}(ET2(x)) >_{KB_2, r} \sigma \ast \langle u \rangle$.

Then $ET2(x) = ET(x) \ast (\sigma \ast \langle u \rangle)$. By definition, $ET2(x) >_{KB, r} ET2(x)$.

Case 2. $\sigma \ast \langle u \rangle >_{KB_2, r} x \ast r$ for some $\tau \in ET2(x)$.

Let $\rho$ be the next element of $\sigma \ast \langle u \rangle$ in $ET2(y)$. Then $ET2(x) = \alpha \ast \langle \rho \rangle \ast \beta$ and $ET2(y) = \alpha \ast \langle \sigma \ast \langle u \rangle, \rho \rangle \ast \beta$. By definition of $ET2$, $\sigma \ast \langle u \rangle >_{KB_2, r} \rho$. Since $\sigma \ast \langle u \rangle$ is maximal in $ET(y)$ by Lemma 5.10 (1), there is no $\alpha \neq ()$ such that $\sigma \ast \langle u \rangle >_{KB_1, r} \rho$. Hence $ET2(x) >_{KB_1, r} ET2(y)$.

The next lemma shows that induction for decreasing transitive sequences is reduced to induction for Erdős trees with $>_{KB_1, r}$.

**Lemma 5.13** HA $\vdash \forall r \in U. \text{Ind}(ET2(DT(U, >_{R_1+\ldots+R_k})(r)), >_{KB, r})$ implies HA $\vdash \text{Ind}(DT(U, >_{R_1+\ldots+R_k}), >_{\text{ext}})$.

**Proof.** In this proof, for simplicity, we write DT for $DT(U, >_{R_1+\ldots+R_k})$ and we also write $ET_r$ for $ET2(DT(U, >_{R_1+\ldots+R_k})(r))$.

Assume HA $\vdash \forall r \in U. \text{Ind}(ET_r, >_{KB, r})$. Assume

$\forall x \in DT((\forall y <_{\text{ext}} x \in DT.Fy) \to Fx)$.

in order to show

$\forall x \in DT.Fx$.

Define $Gy \equiv \forall z \in DT(z \neq () \to ET(2)(z) = y \to Fz)$. We will show

$\forall r \in U. \forall x \in ET_r((\forall y <_{KB, r} x \in ET_r.Gy) \to Gx)$.

Fix $r \in U$ and $x \in ET_r$ and assume

$\forall y <_{KB, r} x \in ET_r.Gy$
in order to show \( Gx \). Fix \( x_0 \in DT \) and assume \( x_0 \neq \langle \rangle \) and \( ET2(x_0) = x \) in order to show \( Fx_0 \).

We can show

\[
\forall y_0 <_{ext} x_0 \in DT. Fy_0
\]
as follows. Let \( r \) be \( first(x_0) \). Fix \( y_0 <_{ext} x_0 \in DT \). Then \( x_0, y_0 \in DT_{(r)} \). Let \( y = ET2(y_0) \). By Lemma 5.12 \( y <_{KB,r} x \). By (21) and \( y \in ET_r, Gy \). Hence \( Fy_0 \).

By taking \( x \) to be \( x_0 \) in (18), \( Fx_0 \). Hence \( \forall x_0 \in DT(x_0 \neq \langle \rangle \rightarrow x = ET2(x_0) \rightarrow Fx_0) \), namely, \( Gx \).

We have shown (20).

By (20) and \( \forall r \in U.Ind(ET_r, >_{KB,r}) \), we have \( \forall r \in U. \forall x \in ET_r. Gx \).

For any \( x \in DT \) such that \( x \neq \langle \rangle \), by taking \( r \) to be \( first(x) \) and \( x \) to be \( ET2(x) \) in it we have \( G(ET2(x)) \). Hence \( Fx \). For \( x = \langle \rangle \), by taking \( x \) to be \( \langle \rangle \) in (18), \( F(\langle \rangle) \). Combining them, we have (19).

\( \square \)

The next lemma shows induction holds when we restrict the universe.

**Lemma 5.14** \( HA \vdash Ind(U, >) \) and \( HA \vdash V \subseteq U \) imply \( HA \vdash Ind(V, >) \).

**Proof.** We will show \( Ind(V, >) \) for \( F \), namely,

\[
\forall x \in V((\forall y < x \in V. Fy) \rightarrow Fx) \rightarrow \forall x \in V. Fx.
\]

(22)

Let \( Gx \) be \( x \in V \rightarrow Fx \). By \( Ind(U, >) \) for \( G \), we have

\[
\forall x \in U((\forall y < x \in U. Gy) \rightarrow Gx) \rightarrow \forall x \in U. Gx.
\]

By predicate logic, it is equivalent to (22). \( \square \)

The next lemma shows induction is implied from induction for decreasing sequences.

**Lemma 5.15** \( HA \vdash Ind(DS(U, >), >_{ext}) \) implies \( HA \vdash Ind(U, >) \).

**Proof.** In this proof, for simplicity, we write \( DS \) for \( DS(U, >) \).

Assume

\[
\forall x \in U((\forall y < x \in U. Fy) \rightarrow Fx)
\]

(23)
in order to show \( \forall x \in U. Fx \).

Define \( Gx \equiv F(last(x)) \). We will show

\[
\forall x \in DS((\forall z <_{ext} x \in DS. Gz) \rightarrow Gx).
\]

(24)

Fix \( x \in DS \) and assume

\[
\forall z <_{ext} x \in DS. Gz
\]

(25)
in order to show \( Gx \).

We can show

\[
\forall y < last(x) \in U. Fy
\]

(26)
as follows. Assume \( y < last(x) \) in order to show \( Fy \). Then \( x * \langle y \rangle \in DS \) and \( x >_{ext} x * \langle y \rangle \). By taking \( z \) to be \( x * \langle y \rangle \) in (25), we have \( G(x * \langle y \rangle) \). By definition of \( G \), \( F(last(x * \langle y \rangle)) \). Hence \( Fy \). We have shown (26).

By taking \( x \) to be \( last(x) \) in (23), we have \( F(last(x)) \). Hence \( Gx \). Hence we have shown (24).

By \( Ind(DS, >_{ext}) \) with (24), we have \( \forall x \in DS. Gx \). By taking \( x \) to be \( \langle x \rangle \) in it, we have \( G(\langle x \rangle) \). By definition of \( G \), we have \( Fx \). \( \square \)

The next lemma shows induction for a power of a relation implies induction for the relation.

**Lemma 5.16** \( HA \vdash Ind(U, >^n) \) implies \( HA \vdash Ind(U, >) \).
Proof. We can assume $n > 1$.
Assume
\[
\forall x \in U((\forall y < x \in U.Fy) \rightarrow Fx)
\]
in order to show $\forall x \in U.Fx$.
We will show
\[
\forall x \in U((\forall y < x \in U.Fy) \rightarrow Fx).
\]
Fix $x \in U$ and assume
\[
\forall y < x \in U.F y
\]
in order to show $Fx$.
By induction on $m$, we will show
\[
\forall m \leq n. \forall w < n^{-m} x \in U.Fw
\]
Case 1. $m = 0$.
Assume $w < x \in U$. By taking $y$ to be $w$ in (29), $Fw$.
Case 2. $m > 0$.
Assume $w < n^{-m} x \in U$ in order to show $Fw$. We can show $\forall y < w \in U.Fy$ as follows. Assume $y < w$. Then $y < (n^{-m} - 1) x$. By IH for $m - 1$, $Fy$.
By taking $x$ to be $w$ in (27), $Fw$.
We have shown (30). By taking $m$ to be $n$ and $w$ to be $x$ in it, $Fx$. Hence we have shown (28). By Ind$(U, \succ n)$, $\forall x \in U.Fx$. □

Define
\[
\text{Trans}(U, \succ R) \equiv \forall xyz \in U(x \succ y \land y \succ z \rightarrow x \succ R z),
\]
\[
\text{Decide}(U, \succ R) \equiv \forall xyz \in U(x \succ y \lor y \succ x \lor \neg(x \succ y)).
\]

The next theorem states that if some powers of relations $\succ R_i$ have induction principle, $\succ R_i$, are decidable and their union is transitive, then the union has induction principle. This theorem is the same as Theorem 6.1 in [3] except HA and the decidability condition Decide$(U, \succ R_i)$.

**Theorem 5.17 (Podelski-Rybalchenko Theorem for Induction)** If HA $\vdash$ Ind$(U, \succ n R_i)$, HA $\vdash$ Decide$(U, \succ R_i)$, ..., HA $\vdash$ Ind$(U, \succ n R_k)$, HA $\vdash$ Decide$(U, \succ R_k)$, and HA $\vdash$ Trans$(U, \succ R_1 + \ldots + R_k)$, then Ind$(U, \succ R_1 + \ldots + R_k)$.

Proof. We will discuss in HA.
By Lemma 5.16, Ind$(U, \succ R_i)$. Define $\succ R'_1$ as $\succ R_i$ and $\succ R'_{i+1}$ as $(\succ R_{i+1}) - (\succ R_i) - \ldots - (\succ R_1)$. Then $(\succ R'_1), \ldots, (\succ R'_{k})$ are disjoint and $\forall xy \in U(x \succ R_1 + \ldots + R_{k} y \rightarrow x \succ R'_1 + \ldots + R'_{k} y)$ by Decide$(U, \succ R_i)$ for $1 \leq i \leq k$. Since $(\succ R'_i) \subseteq (\succ R_i)$, Ind$(U, \succ R'_i)$. For simplicity, from now on we write $\succ R_i$ for $\succ R'_i$ in this proof. We will show Ind$(U, \succ R_1 + \ldots + R_k)$.

From Ind$(U, \succ R_i)$, we have Ind$(\text{DT}(U, \succ R_i), \succ \text{ext})$ for $1 \leq i \leq k$. By Lemma 5.7, we have $\forall r \in U.\text{Ind}(U, \succ u, \text{Left})$. By taking $U$ to be $U$, $T$ to be $\text{MS}(r)$, and $>u$ to be $>u, \text{Left}$ in Theorem 5.3, for $>\text{KB}_1, r$, we have $\forall r \in U.\text{Ind}(\text{MS}(r), >\text{KB}_1, r)$. By Theorem 5.3, for $>\text{KB}_2, r$, we have $\forall r \in U.\text{Ind}(\text{MS}(r), >\text{KB}_2, r)$ similarly. Hence $\forall r \in U.\text{Ind}(\text{DS}(\text{MS}(r), >\text{KB}_2, r), >\text{ext})$. From Lemma 5.14, we have $\forall r \in U.\text{Ind}(\text{DT}(U, >R_1 + \ldots + R_k), >\text{ext})$. By taking $T$ to be $\text{DS}(\text{MS}(r), >\text{KB}_2, r)$, $U$ to be $\text{MS}(r)$, and $>u$ to be $>\text{KB}_1, r$, we have $\forall r \in U.\text{Ind}(\text{MS}(r), >\text{KB}_1, r)$. By Theorem 5.3, for $>\text{KB}_2, r$, we have $\forall r \in U.\text{Ind}(\text{DT}(U, >R_1 + \ldots + R_k), >\text{ext})$. By Lemma 5.14, we have $\forall r \in U.\text{Ind}(\text{DT}(U, >R_1 + \ldots + R_k), >\text{KB}_2, r)$. By Lemma 5.14, we have $\forall r \in U.\text{Ind}(\text{DT}(U, >R_1 + \ldots + R_k), >\text{ext})$.

6 Proof Transformation
This section defines main proof transformation from CLJID$^\omega$ + HA to LJID + HA. The proof is the same as [3] except Theorem 5.17 and Lemma 6.6. First we will define stage numbers and path relations, and then define proof transformation using them.

For notational convenience, we assume a cyclic proof Π in this section. Let the buds in Π be $J_{1i}$ ($i \in I$) and the companions be $J_{2j}$ ($j \in K$). Assume $f : I \rightarrow K$ such that the companion of a bud $J_{1i}$ is $J_{2f(i)}$. □
\[
\begin{array}{c}
P t\\
Q_1 \overline{v}_1 \ldots Q_n \overline{v}_n P_1 t_1 \ldots P_m t_m
\end{array}
\]

\[
\begin{array}{c}
P t\\
Q_1 \overline{v}_1 \ldots Q_n \overline{v}_n \quad v > v_1 P_1' t_1 v_1 \ldots v > v_m P_m' t_m v_m \quad N v
\end{array}
\]

Figure 1: Production Rules

6.1 Stage Numbers for Inductive Definitions

In this subsection, we define and discuss stage transformation.

We introduce a stage number to each inductive atomic formula so that the argument of the formula comes into the inductive predicate at the stage of the stage number. This stage number will decrease by a progressing trace. A proof in \(\text{LJID} + \text{HA}\) will be constructed by using the induction on stage numbers.

First we give stage transformation of an inductive atomic formula. We assume a fresh inductive predicate symbol \(P'\) for each inductive predicate symbol \(P\) and we call it a stage-number inductive predicate symbol. \(P'(\overline{t}, v)\) means that the element \(\overline{t}\) comes into \(P\) at the stage \(v\). We transform \(P(\overline{t})\) into \(\exists v P'(\overline{t}, v)\). We call a variable \(v\) a stage number of \(\overline{t}\) when \(P'(\overline{t}, v)\). \(P(\overline{t})\) and \(\exists v P'(\overline{t}, v)\) will become equivalent by inference rules introduced by the transformation of production rules. We call \(P'(\overline{t}, v)\) a stage-number inductive atomic formula.

Secondly we give stage transformation of a production rule. We transform the production of \(P\) (the first rule) in Figure 1 into the production of \(P'\) (the second rule) in Figure 1 where \(v, v_1, \ldots, v_m\) are fresh variables.

Next we give stage transformation of a sequent. For given fresh variables \(\overline{v}\), we transform a sequent \(J\) into \(J'\overline{t}\) defined as follows. We define \(\Gamma^*\) as the sequent obtained from \(\Gamma\) by replacing \(P(\overline{t})\) by \(\exists v P'(\overline{t}, v)\). For fresh variables \(\overline{v}\), we define \((\Gamma)^*_{\overline{v}}\) as the sequent obtained from \(\Gamma^*\) by replacing the \(i\)-th element of the form \(\exists v P'(\overline{t}, v)\) in the sequent \(\Gamma^*\) by \(P'(\overline{t}, v_i)\). We define \((\Gamma \vdash \Delta)^*\) by \(\Gamma^* \vdash \Delta^*\), and define \(\Delta^*\) by \(\Delta\). We write \((a_i)_{i \in I}\) for the sequence of elements \(a_i\) where \(i\) varies in \(I\). We extend the notion of proofs by allowing open assumptions. We write \(\Delta \vdash_{\text{CLJID}^* + \text{HA}} \Pi\) with assumptions \((J_i)_{i \in I}\) when there is a proof with assumptions \((J_i)_{i \in I}\) and the conclusion \(\Pi \vdash \Delta\) in \(\text{CLJID}^* + \text{HA}\).

Definition 6.1 In a path \(\pi\) in a proof, we define \(\text{Ineq}(\pi)\) as the set of the forms \(v > v'\) and \(v = v'\) for any stage numbers \(v, v'\) eliminated by every case distinction in \(\pi\).

The next lemma shows a stage number is a number.

Lemma 6.2 (1) \(P'^* \overline{t} v \vdash_{\text{CLJID}^* + \text{HA}} N v\).
(2) \(P'^* \overline{t} v \vdash_{\text{LJID} + \text{HA}} N v\).

Proof. (1) and (2) are proved by (Case \(P'\)) and (Ind \(P'\)) respectively. □

An example for \(\text{Ineq}(\pi)\) is as follows. When the path \(\pi\) has the case distinction

\[
\Gamma^*, u = t(y), v = v, Q_1 u_1(y), \ldots, Q_n u_n(y),
\]

\[
v > v_1, P'_1 t_1(y) v_1, \ldots, v > v_m, P'_m t_m(y) v_m,
\]

\[
N v \vdash \Delta^*
\]

we take \(\hat{v} = v, v > v_1, \ldots, v > v_m\) for \(\text{Ineq}(\pi)\).

The proof of the next proposition gives stage transformation of a proof into a proof of the stage transformation of the conclusion of the original proof. We write \(\Pi^*\) for the stage transformation of \(\Pi\).

Proposition 6.3 (Stage Transformation) For any fresh variables \(\overline{v}\), \(\Gamma \vdash_{\text{CLJID}^* + \text{HA}} \Delta\) with assumptions \((\Gamma_i \vdash \Delta_i)_{i \in I}\) without any buds implies \((\Gamma^*_{\overline{v}} \vdash_{\text{CLJID}^* + \text{HA}} \Delta^*)_{i \in I}\) with assumptions \((\text{Ineq}(\pi_i), (\Gamma_i)^*_{\overline{v}} \vdash \Delta^*)_{i \in I}\) without any buds for some fresh variables \((\overline{v}_i)_{i \in I}\), where \(\pi_i\) is the path from the conclusion to the assumption \((\Gamma_i)^*_{\overline{v}} \vdash \Delta^*\).
Proof. By induction on the proof. We will transform the proof $\Pi$ of $\Gamma \vdash_{CLJID+HA} \Delta$ with assumptions $(J_i)_{i \in I}$ into the proof of $\Pi^* \vdash_{CLJID+HA} \Delta^*$ with assumptions $(\text{Ineq}(\pi_i), (J_i)^*_{\overrightarrow{v}})_{i \in I}$ by transforming each rule as follows.

Case 1. The rule is not $(P R), (\text{Case}), (\text{Cut}), (\text{Axiom})$ with a common inductive atomic formula, or logical rules with some of the main formula and the auxiliary formulas being an inductive atomic formula in the antecedent.

Since the main formula and the auxiliary formulas are not inductive atomic formulas in the antecedent, we can just replace each sequent $J_i$ by $(J_i)^*_{\overrightarrow{v}}$. If the rule is an assumption, since $I = \{1\}$, we take $\overrightarrow{v}_1$ to be $\overrightarrow{v}$.

For example,

$$\frac{\Gamma \vdash P_1 t_1}{\Gamma \vdash P_1 t_1 \lor P_2 t_2} (\lor R_1)$$

is transformed into

$$\frac{(\Gamma)^*_{\overrightarrow{v}} \vdash \exists v P_1' t_1 v_1}{(\Gamma)^*_{\overrightarrow{v}} \vdash \exists v P_1' t_1 v_1 \lor \exists v P_2' t_2 v_2} (\lor R_1)$$

Case 2. The rule is $(\text{Axiom})$ with a common inductive atomic formula.

We transform

$$\frac{\Gamma, P(\overrightarrow{t}) \vdash P(\overrightarrow{t})}{(\text{Axiom})}$$

into

$$\frac{\Gamma^*_{\overrightarrow{v}}, P'(\overrightarrow{t}, v) \vdash P'(\overrightarrow{t}, v)}{(\text{Axiom})}$$

$$\frac{\Gamma^*_{\overrightarrow{v}}, P'(\overrightarrow{t}, v) \vdash \exists v P'(\overrightarrow{t}, v)}{(\exists R)}$$

Case 3. The rule is $(\text{Cut})$, or logical rules with some of the main formula and the auxiliary formulas being an inductive atomic formula in the antecedent.

We replace each sequent $J_i$ by $(J_i)^*_{\overrightarrow{v}}$ and use

$$\frac{\Gamma^*_{\overrightarrow{v}}, P'(\overrightarrow{t}, v) \vdash \Delta^*}{(\exists L)}$$

Note that by IH we can assume each $P'(\overrightarrow{t}, v)$ has a fresh $v$, so we can use $(\exists L)$. For example, we transform

$$\frac{\Gamma, Pt \vdash \neg Pt}{(\neg R)}$$

into

$$\frac{\Gamma^*_{\overrightarrow{v}}, P' tv \vdash \exists v P' tv}{(\exists L)}$$

$$\frac{\Gamma^*_{\overrightarrow{v}}, \exists v P' tv \vdash \neg \exists v P' tv}{(\neg R)}$$

Case 4. The rule is $(P R)$. Assume the production rule of $P$ and its stage transformation in Figure 1.

Let $\Sigma_0$ be the set of all assumptions of this production rule of $P'$ and $t_0$ be $v_1 + \ldots + v_n + 1$. By $(P' R)$ for this production rule,

$$\Sigma_0 \vdash P' \overrightarrow{t} v.$$

Since $P_1 \overrightarrow{t}_i v_i + N v_i$, $(\Sigma_1)^*_{\overrightarrow{v}} \vdash N v_i$ where $\Sigma_1$ is $Q_1 \overrightarrow{v}_1, \ldots, Q_m \overrightarrow{v}_m, P_1 \overrightarrow{t}_1, \ldots, P_n \overrightarrow{t}_n$ and $\overrightarrow{v}$ is $v_1 \ldots v_n$.

By (Subst) with $v := t_0$, and (Cut) with $t_0 > v_i$ and $(\Sigma_1)^*_{\overrightarrow{v}} \vdash N t_0$,

$(\Sigma_1)^*_{\overrightarrow{v}} \vdash P' \overrightarrow{t} t_0$. 

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By (∃R), then by (∃L), (∑1)∗ ⊢ ∃vP′ t v.

By IH,

\[ \Gamma^o \vdash Q_j \overline{u}_j, \]
\[ \Gamma^o \vdash \exists v P' t_i v_i. \]

By (Cut), \( \Gamma^o \vdash \exists v P' t v. \)

Case 5. The rule is (Case \( P \)). Assume the production rule of \( P \) and its stage transformation in Figure 1.

Let \( \overline{v} \) be \( \overline{v}' \). Let \( v_1, \ldots, v_m \) be fresh variables. Let the rule be

\[ \text{case distinctions} \]
\[ \Gamma, \overline{v} \vdash \Delta \]

with the case distinctions

\[ \Gamma, \overline{u} = \overline{v}(\overline{y}), Q_1 \overline{u}_1(\overline{y}), \ldots, Q_n \overline{u}_n(\overline{y}), \]
\[ P_1 \overline{t}_1(\overline{y}), \ldots, P_m \overline{t}_m(\overline{y}) \vdash \Delta. \]

Let

\[ \Sigma \equiv (\overline{u} = \overline{v}(\overline{y}), Q_1 \overline{u}_1(\overline{y}), \ldots, Q_n \overline{u}_n(\overline{y}), \]
\[ P_1' \overline{t}_1(\overline{y}) v_1, \ldots, P_m' \overline{t}_m(\overline{y}) v_m), \]
\[ \Sigma_1 \equiv (\overline{v} = v, v > v_1, \ldots, v > v_m). \]

By IH with \( \overline{v}' v_1 \ldots v_m \) for the case distinction we obtain a proof of

\[ \Gamma^o, \Sigma \vdash \Delta^∗ \]

with assumptions (Ineq(\( \pi'_i \)), (\( J_i \)\( _{\overline{v}_i}^o \))\( _{i \in I} \)) for some \( (\overline{v}_i)_{i \in I} \) where \( \pi'_i \) is the path from the transformation of the case distinction to \( (\overline{v}_i)_{\overline{v}_i}^o \). Add \( \Sigma_1 \) to the antecedent of every sequent in the path \( \pi'_i \), by renaming fresh variables to keep freshness of all fresh variables in the path. Then we have a proof of

\[ \Sigma_1, \Gamma^o, \Sigma \vdash \Delta^∗ \]

with assumptions \( (\Sigma_1, \text{Ineq}(\pi'_i), (J_i)_{\overline{v}_i}^o)_{i \in I} \). We have

\[ \text{case distinctions} \]
\[ \Gamma^o, P' \overline{v} \vdash \Delta^∗ \]

(Case \( P' \))

with the case distinctions

\[ \Gamma^o, \Sigma_1, \Sigma, Nv \vdash \Delta^∗ \]

By (Wk) with \( Nv \) we obtain the case distinction. By (Case \( P' \)),

\[ \Gamma^o, P' \overline{v} \vdash \Delta^∗. \]

Let \( \pi_i \) be the path from \( \Gamma^o, P' \overline{v} \vdash \Delta^∗ \) to \( (J_i)_{\overline{v}_i}^o \). Then

\[ \text{Ineq}(\pi_i) = (\Sigma_1, \text{Ineq}(\pi'_i))). \]

Hence we have a proof of

\[ \Gamma^o, P' \overline{v} \vdash \Delta^∗ \]

with assumptions \( (\text{Ineq}(\pi_i), (J_i)_{\overline{v}_i}^o)_{i \in I} \). \( \square \)
6.2 Path Relation

In this section, we will introduce path relations and discuss them.

We assume a subproof $\Pi_j$ of $\Pi$ such that it does not have buds, its conclusion is $J_{2j}$ and its assumptions are $J_{1i}$, $(i \in I_j)$.

For $J$ in $\Pi_j^*$, we define $\bar{J}$ as $\langle v_1, \ldots, v_k \rangle$ where $J$ is $\Gamma_{v_1 \ldots v_k}^* \vdash \Delta^*$. For a path $\pi$ from the conclusion to an assumption in $\Pi_j^*$, we write $\bar{\pi}$ for the corresponding path in $\Pi$.

We extend this notation to a finite composition of $\pi$'s. By the correspondence ('), a stage-number inductive atomic formula in $\Pi_j^*$ corresponds to an inductive atomic formula in $\Pi$, and a path, a trace, and a progressing trace in $\Pi_j^*$ correspond to the same kind of objects in $\Pi$.

**Definition 6.4** For a finite composition $\pi$ of paths in $\{\Pi_j^* \mid j \in K\}$ such that $\bar{\pi}$ is a path in the infinite unfolding of $\Pi$, we define the path relation $\triangleright_{\pi}$ by

$$x \triangleright_{\pi} y \equiv |x| = |J_2| \land |y| = |J_1| \land \bigwedge_{F(q_1, q_2)} (x)_{q_2} > (y)_{q_4} \land \bigwedge_{G(q_1, q_2)} (x)_{q_2} = (y)_{q_4},$$

where $J_1$ and $J_2$ are the top and bottom sequents of $\pi$ respectively, $J_1$ and $J_2$ are those of the path $\bar{\pi}$, $F(q_1, q_2)$ is that there is some progressing trace from the $q_2$-th atomic formula in $J_2$ to the $q_1$-th atomic formula in $J_1$, $G(q_1, q_2)$ is that there is some non-progressing trace from the $q_2$-th atomic formula in $J_2$ to the $q_1$-th atomic formula in $J_1$.

We define $B_1$ as the set of paths from conclusions to assumptions in $\Pi_j^*$ ($j \in K$). We define $B$ as the set of finite compositions of elements in $B_1$ such that if $\pi \in B$ then $\bar{\pi}$ is a path in the infinite unfolding of $\Pi$.

**Definition 6.5** For $\pi \in B$, define $x \triangleright_{\pi} y$ by

$$x \triangleright_{\pi} y \equiv (x)_0 = j \land (y)_0 = f(i) \land (x)_1 \triangleright_{\pi}(y)_1,$$

where $J_{1i}$ is the top sequent of $\bar{\pi}$, and $J_{2j}$ is the bottom sequent of $\bar{\pi}$.

Note that $(\ )_0$ and $(\ )_1$ are operations for a number that represents a sequence of numbers defined in Section 3. The first element is a companion number.

**Lemma 6.6** $\{\triangleright_{\pi} \mid \pi \in B\}$ is finite.

*Proof.* Define $C_n$ as $\{\triangleright_{\pi_1 \ldots \pi_n} \mid m \leq n, \pi_i \in B_1\}$. Since $\triangleright_{\pi}$ is a relation on $N \times N^{\leq p}$ where $p$ is the maximum number of inductive atomic formulas in the antecedents of $\Pi$, there is $L$ such that $|C_n| \leq L$ for all $n$. Then we have the least $n$ such that $C_{n+1} = C_n$. Then $|\{\triangleright_{\pi} \mid \pi \in B\}| = |C_n|$. □

The next proposition shows a sequent is implied from its stage-number transformation.

**Proposition 6.7** For any fresh variables $\bar{v}$, $\Gamma_{\bar{v}}^* \vdash_{\text{LJD+HA}} \Delta^*$ implies $\Gamma \vdash_{\text{LJD+HA}} \Delta$.

*Proof.* First we can show $\Gamma^* \vdash_{\text{LJD+HA}} \Delta^*$ by

$$\exists v P' \bar{v}, \Gamma^* \vdash \Delta^* \quad \exists L$$

Secondly by

$$\vdash_{\text{LJD+HA}} P' \bar{v} \iff \exists v P'(\bar{v}, v)$$

we have

$$\Gamma \vdash_{\text{LJD+HA}} \Delta.$$

□

The next lemma shows $\triangleright_{\pi}$ is an abstraction of $\text{Ineq}(\pi)$.

**Lemma 6.8** For a proof $\Pi$ without any buds, if $\pi$ is a path from $(J_2)^{2x}_{\bar{x}}$ to $(J_1)^{2y}_{\bar{y}}$ in $\Pi^0$,

$$\text{Ineq}(\pi) \vdash_{\text{HA}} \langle \bar{x} \rangle \triangleright_{\pi} \langle \bar{y} \rangle.$$
Proof. By induction on $|\pi|$.

Case 1. $|\pi| = 0$. $J_1 = J_2$. $x >_\pi y$ is $|x| = m \wedge |y| = m \wedge (x)_0 = (y)_0 \wedge \ldots \wedge (x)_{m-1} = (y)_{m-1}$ where $m$ is the number of stage-number inductive atomic formulas in $\langle J_1 \rangle$. Hence $\vdash_{HA} x >_\pi y$.

Case 2. $|\pi| > 0$. Consider cases according to the last rule of $\pi$. Let $\pi = \pi_1 \pi_2$ such that $|\pi_1| = 1$. Let the top sequent of $\pi_1$ be $\langle J_1 \rangle$. Let $x,y,z$ be $\langle \bar{\nu} \rangle$, $\langle \bar{\nu} \rangle$, $\langle \bar{\nu} \rangle$ respectively.

By IH, $\text{Ineq}(\pi_1) \vdash_{HA} x >_{\pi_1} y$. We will show $\text{Ineq}(\pi_1) \vdash_{HA} x >_{\pi_1} z$. Since the rule that changes the stage number is only (Case), we will show only the case (Case). Assume the production rule of $P$ and its stage transformation in Figure 1

Let the path for the rule (Case $P'$) be

$$\frac{\Gamma \vdash_{\pi_2}, \Sigma_1, \Sigma, Nu \vdash \Delta^*}{\Gamma \vdash_{\pi_2}, P^\prime \bar{\nu} \vdash \Delta^*}$$

where

$$\Sigma = (\bar{t} = t_0(\bar{y}), Q_1 \bar{w}_1(\bar{y}), \ldots, Q_n \bar{w}_n(\bar{y}), P_1 \bar{r}_1(\bar{y})v_1, \ldots, P_n \bar{r}_m(\bar{y})v_m),$$

$$\Sigma_1 = (\bar{v} = v, v > v_1, \ldots, v > v_m).$$

Then $x = \langle \bar{t}, \bar{v} \rangle$ and $z = \langle \bar{t}, v_1, \ldots, v_m \rangle$, and $$x >_{\pi_1} z \iff v_1 < \bar{v} \wedge \ldots \wedge v_m < \bar{v}.$$ Since $\text{Ineq}(\pi_1) \equiv \Sigma_1$, $\text{Ineq}(\pi_1) \vdash_{HA} x >_{\pi_1} z$. Since $\text{Ineq}(\pi) = (\text{Ineq}(\pi_1), \text{Ineq}(\pi_2))$, $\text{Ineq}(\pi) \vdash_{HA} x >_{\pi_1} z >_{\pi_2} y$. Since $$(>_{\pi_1}) \circ (>_{\pi_2}) \subseteq (>_{\pi_1 \pi_2}),$$ $\text{Ineq}(\pi) \vdash_{HA} x >_{\pi_1 \pi_2} y$. □

The next lemma is the only lemma that uses the global trace condition.

**Lemma 6.9** For all $\pi \in B$, there is $n > 0$ such that $\vdash_{HA} \text{Ind}(U, >^n_{\pi})$.

**Proof.** Let the bottom sequent of $\pi$ be $J_{2g}$ and the top sequent be $J_{1i}$. Let the companion of $J_{1i}$ be $J_{2k}$.

Case 1. $j \neq k$.

Assume

$$H \equiv \forall x \in U.(\forall y <_{\pi} x \in U.Fy) \rightarrow Fx$$

and fix $x \in U$.

Assume $y <_{\pi} x \in U$. By taking $x$ to be $y$ in $H$,

$$(\forall z <_{\pi} y \in U.Fz) \rightarrow Fy.$$

By $\neg(z <_{\pi} y)$ from $y <_{\pi} x$ and $j \neq k$, we have $Fy$. Hence $H \vdash_{HA} \forall y <_{\pi} x \in U.Fy$.

By taking $x$ to be $x$ in $H$, we have $Fx$. Hence $H \vdash_{HA} \forall x \in U.Fx$. Hence $\vdash_{HA} \text{Ind}(U, >_{\pi}, F)$. We can take $n$ to be 1.

Case 2. $j = k$.

By applying the global trace condition to the infinite path $\bar{x}_\omega$, there is a progressing trace in the path. Hence there are $n, m, q$ such that the trace passes the $q$-th stage-number inductive atomic formula in the top sequent of $\pi^m$ and the $q$-th stage-number inductive atomic formula in the top sequent of $\pi^m+n$.

Define $x <_q y$ by $((x)_1)_q < ((y)_1)_q$. By mathematical induction we can easily show

$$(\forall x \in U.(\forall y <_q x \in U.Fy) \rightarrow Fx) \rightarrow \forall x \in U.Fx.$$

If $y <_q x$, then $y <_{\pi^n} x$, and hence $((y)_1)_q < ((x)_1)_q$, which implies $y <_q x$. Therefore

$$(\forall x \in U.(\forall y <_{\pi^n} x \in U.Fy) \rightarrow Fx) \rightarrow \forall x \in U.Fx,$$

which is $\text{Ind}(U, >_{\pi^n}, F)$. □

We define $>_n$ as $\{ >_{\pi} \mid \pi \in B \}$. Note that $>_n$ is transitive, since the top sequent of $\pi_1$ is the bottom sequent of $\pi_2$ by the first element, and $((>_{\pi_1}) \circ (>_{\pi_2})) \subseteq (>_{\pi_1 \pi_2})$. 

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6.3 Proof Transformation

This section gives main proof transformation.

The next lemma shows we can replace (Case) rules of CLJIDω + HA by (Ind) rules of LJID + HA.

**Lemma 6.10** If there is a proof with some assumptions and without any buds in CLJIDω + HA, then there is a proof of the same conclusions with the same assumptions in LJID + HA.

*Proof.* It is sufficient to replace the rule (Case) by the rule (Ind). This is straightforward and has been proved by Lemma 4.1.4 in [5]. We give only a proof idea here.

Assume the production rule of $P$ in Figure [1]. We can replace

\[
\begin{array}{c}
\vdash \Pi_i \\
\Gamma, \bar{\gamma} = \bar{t} \langle \bar{\gamma} \rangle, \Sigma \vdash \Delta \\
\hline
\Gamma, P\bar{\gamma} \vdash \Delta
\end{array}
\]  

(Case)

by

\[
\begin{array}{c}
\vdash \Pi'_i \\
\Sigma' \vdash F\bar{\gamma} \langle \bar{\gamma} \rangle (\forall i \in I) \\
\hline
\Gamma, P\bar{\gamma} \vdash \Delta
\end{array}
\]  

(Ind $P$)

where

\[
F\bar{\gamma} \equiv \forall \bar{\gamma} (\bar{\gamma} = \bar{u} \rightarrow \Gamma \rightarrow \Delta),
\]

\[
F'\bar{\gamma} \equiv P\bar{\gamma} \land F\bar{\gamma}.
\]

\[\square\]

The next is the main lemma and shows each bud in a cyclic proof is provable in LJID + HA.

**Lemma 6.11** For every bud $J$ of a proof in CLJIDω + HA and fresh variables $\bar{v}$, $(J)_{\bar{v}}$ is provable in LJID + HA.

*Proof.* For $j \in K$, define

\[
G_j x \equiv \forall \bar{v} (x = \langle \bar{v} \rangle \rightarrow (\Gamma_{2j})_{\bar{v}} \rightarrow \Delta_{2j}^*)
\]

where $\bar{v}$ is FV(($\Gamma_{2j}$)$_{\bar{v}}$, $\Delta_{2j}^*$).

Define

\[
G_{x_0} x \equiv \bigwedge_{j \in K} (x_0 = j \rightarrow G_j x),
\]

\[
H_{x_0} x \equiv \forall y_0. \langle y_0, y \rangle < \Pi (x_0, x) \rightarrow G y_0 y.
\]

We will show that for every companion $J$ in $\Pi$ and fresh variables $\bar{v}$, $(J)_{\bar{v}}$ is provable in LJID + HA. Fix a companion in $\Pi$ and fresh variables $\bar{v}$. Let the companion be $J_{2j}$. We will construct a proof of $(J_{2j})_{\bar{v}}$ in LJID + HA. We have a subproof $\Pi_j$ of $\Pi$ such that it does not have buds, its conclusion is $J_{2j}$ and its assumptions are $J_{1i}$ ($i \in I_j$). By Proposition [6.3]

\[
(\text{Ineq}(\pi_{j}), (\Gamma_{1i})_{\bar{v}} \vdash \Delta_{1i}', i \in I_j, \\
\vdash \Pi_j^\circ, \\
(\Gamma_{2j})_{\bar{v}} \vdash \Delta_{2j}^*)
\]

for some fresh variables $\langle \bar{v} \rangle_i \in I_j$, where $\pi_{j}$ is the path from $J_{2j}$ to $J_{1i}$ in $\Pi_j$. Next we transform $\Pi_j^\circ$ into the (LJID + HA)-proof $\Pi_j'$ with the same conclusion and the same assumptions by Lemma [6.10]

Next we add

\[
x_0 = j, x = \langle \bar{v} \rangle, H_{x_0} x
\]
to every antecedent in $\Pi_j'$ to obtain

\[
(x_0 = j, x = \langle \overline{v} \rangle, Hx_0x, \text{Ineq}(\pi_{ji}), (\Gamma_{1i})^0_\overline{v}, \vdash \Delta_{1i}^j)_{i \in I_j}
\]

\[
\vdash \text{Ind}(\rangle_\Pi, G) \quad \vdash Hx_0x \vdash x_0 = j \to G_jx
\]

Let $\Pi_j$ be the proof in Figure 2. We have a proof $\Pi''$ of $\forall x_0x. Gx_0x$ with the assumption $\text{Ind}(\rangle_\Pi, G)$ in Figure 3.

By applying Theorem 5.17 to $U = N \times N^\leq_\pi$ and $\{ \rangle_\pi \mid \pi \in B \}$,

\[
\text{HA} \vdash \text{Ind}(U, \rangle_\pi^B) \quad (\forall \pi \in B)
\]

implies

\[
\text{HA} \vdash \text{Ind}(U, \rangle_\Pi).
\]

By Lemma 6.9 we have

\[
\vdash_{\text{HA}} \text{Ind}(U, \rangle_\pi^B) \quad (\forall \pi \in B).
\]

By the definition of $\rangle_\pi$,

\[
\vdash_{\text{HA}} \text{Decide}(U, \rangle_\pi)
\]

for all $\pi \in B$.

Then

\[
\text{Lemma 6.9} \vdash \text{Decide}(U, \rangle_\pi) \quad \text{Theorem 5.17}
\]

\[
\vdash \text{Ind}(U, \rangle_\Pi, G)
\]

\[
\vdash_{\text{HA}} \text{Decide}(U, \rangle_\pi)
\]

We have shown that for every companion $J$ in $\Pi$ and fresh variables $\overline{v}$, $(J)^0_\overline{v}$ is provable in $\text{LJID + HA}$.  

\[
\end{proof}
\]
Fix a bud be $J_{lk}$ in $\Pi$ and fresh variables $\overline{v}$. Let $J_{2j}$ be the companion of the bud. Since $(J_{2j})_{\overline{v}'}$ is provable, $(J_{lk})_{\overline{v}'}$ is provable in $\text{LJID} + \text{HA}$. □

We write $\text{LJID} + \text{HA} + (\Sigma, \Phi)$ for the system $\text{LJID} + \text{HA}$ with the signature $\Sigma$ and the set $\Phi$ of production rules. Similarly we write $\text{CLJID}^{\omega} + \text{HA} + (\Sigma, \Phi)$ for simplicity, in $\Phi$ we write only $P$ for the set of production rules for $P$. We define $\Sigma_N = \{0, s, +, \times, <, N\}$ and $\Phi_N = \{N\}$. We write $P''$ for $(P')'$.

The next is the main proposition stating a cyclic proof is transformed into an $(\text{LJID} + \text{HA})$-proof with stage-number inductive predicates.

**Proposition 6.12** If a sequent $J$ is provable in $\text{CLJID}^{\omega} + \text{HA} + (\Sigma_N \cup \{\overline{P}\}, \Phi_N \cup \{\overline{P}'\})$, then $J$ is provable in $\text{LJID} + \text{HA} + (\Sigma_N \cup \{N', \overline{P}, \overline{P}'\}, \Phi_N \cup \{\overline{P}, \overline{P}'\})$ where $N', \overline{P}'$ are the stage-number inductive predicates of $N, \overline{P}$.

**Proof.** Let $\Pi_1$ be the cyclic proof of $\Gamma \vdash_{\text{CLJID}^{\omega} + \text{HA}} \Delta$. Let $(J_{1i})_{i \in I}$ be all the buds in $\Pi_1$. Define $\Pi_2$ be a proof obtained from $\Pi_1$ by removing all bud-companion relations. Then $\Pi_2$ is a proof of $\Gamma \vdash \Delta$ with assumptions $(J_{1i})_{i \in I}$ and without buds in $\text{CLJID}^{\omega} + \text{HA}$. Choose fresh variables $\overline{v}$. By Proposition 6.3 there is $(\overline{v}, i)_{i \in I}$ such that $\Gamma_{\overline{v}, i} \vdash \Delta^*$ with assumptions $(\text{Ineq}(\pi_i), (J_{1i})_{\overline{v}, i})_{i \in I}$ and without buds in $\text{CLJID}^{\omega} + \text{HA}$. By Lemma 6.11 $\Gamma_{\overline{v}, i} \vdash \Delta^*$ with assumptions $(\text{Ineq}(\pi_i), (J_{1i})_{\overline{v}, i})_{i \in I}$ and without buds is provable in $\text{LJID} + \text{HA}$. By (Wk), there is a proof $\Pi_3$ of $\Gamma_{\overline{v}, i} \vdash \Delta^*$ with assumptions $(\overline{J}_{1i})_{\overline{v}, i} \vdash \Delta^*$ in $\text{LJID} + \text{HA}$. Let $\Pi_3$ of $\Gamma_{\overline{v}, i} \vdash \Delta^*$ with assumptions $(\overline{J}_{1i})_{\overline{v}, i} \vdash \Delta^*$ in $\text{LJID} + \text{HA}$. Combining them with $\Pi_3$, $\Gamma_{\overline{v}, i} \vdash \Delta^*$ is provable in $\text{LJID} + \text{HA}$. By Proposition 6.11 $\Gamma \vdash \Delta$ is provable in $\text{LJID} + \text{HA}$. □

The next shows conservativity for stage-number inductive predicates.

**Proposition 6.13 (Conservativity of $N'$ and $P''$)** Let $\Sigma = \Sigma_N \cup \{\overline{Q}, \overline{P}, \overline{P}'\}$, $\Phi = \Phi_N \cup \{\overline{P}, \overline{P}'\}$, $\Sigma' = \Sigma \cup \{N', \overline{P}'\}$, and $\Phi' = \Phi \cup \{N', \overline{P}'\}$. Then $\text{LJID} + \text{HA} + (\Sigma', \Phi')$ is conservative over $\text{LJID} + \text{HA} + (\Sigma, \Phi)$.

**Proof.** Define $(\overline{v})$ by replacing $N'$ by $\lambda x y. N x \land N y \land x \leq y$ and replacing $P_j''$ by $\lambda \overline{x}. P_j' \overline{x} y \land N z \land y < z$ for all $j$.

By induction on a proof, we will show that $\text{LJID} + \text{HA} + (\Sigma', \Phi') \vdash J$ implies $\text{LJID} + \text{HA} + (\Sigma, \Phi) \vdash J$.

Use case analysis by the last rule. If the last rule is not (Ind $N'$) or (Ind $P_j''$), the claim follows immediately from IH.

Case 1. The last rule is (Ind $N'$).

Let the rule be

$$
\frac{\Gamma, N v \vdash F_0 v \quad \Gamma, F x v_1, v_1 < v, N v \vdash F s v x}{\Gamma, N a b \vdash F a b} \quad (\text{Ind } N')
$$

By IH for these two premises, we have

$$
\overline{\Gamma}, N v \vdash \overline{F}_0 v
$$

(31)

and

$$
\overline{\Gamma}, F x v_1, v_1 < v, N v \vdash \overline{F}_s v x.
$$

(32)

Define $F' a$ be $\forall z \in N, (a \leq z \rightarrow \overline{F} a z)$. By (Ind $N$) we will show $\overline{\Gamma}, Na \vdash F' a$.

We can show the first premise $\Gamma \vdash F'0$, since it immediately follow from (31).

We can show the second premise $\Gamma, N x, F' x \vdash F' s x z$ as follows. Assume $\overline{\Gamma}, N x, F' x$ and fix $z \in N$ and assume $sz \leq z$ in order to show $\overline{F} s x z$. Then there is $s' \in N$ such that $z = ss'$. Then $x \leq s'$. By $F' x$, $Fr z'$. By taking $v_1$ to be $sz'$ and $v$ to be $z$ in (32), $\overline{F} s x z$.

By (Ind $N$), we have $\overline{\Gamma}, Na \vdash F' a$. Hence $\overline{\Gamma}, Na, Nb, a \leq b \vdash \overline{F} a b$. Hence $(\overline{\Gamma}, N' a b) \vdash \overline{F} a b$.

Case 2. The last rule is (Ind $P_j''$).

Let the rule be

$$
\frac{\forall i \in I \quad J_i}{\Gamma, P_j'' \vdash F_j' \overline{a} b \overline{c}} \quad (\text{Ind } P_j'')
$$

(33)
Define $F^*_i \overline{a} b$ for $N b \land \forall z \in N (b < z \rightarrow \overline{F}_k \overline{a} b z)$ for all $k$.

We will show $\Gamma, P^*_j \overline{a} b \vdash F^*_j \overline{a} b$ by

$$J'_i \vdash (\forall i \in I) \quad \text{(Ind } P'_j)$$

We will show each $J'_i$. Let $J'_i$ be

$\Gamma, Q_1 \overline{u}_1, \ldots, Q_n \overline{u}_n, F^*_i \overline{t}_1 v_1, v_1 < v, \ldots, F^*_m \overline{t}_m v_m, v_m < v, N v \vdash F^*_i \overline{t}_v v$.

and the production rules be

$$P \overline{t}$$

$$Q_1 \overline{u}_1 \ldots Q_n \overline{u}_n \quad P_1 \overline{t}_1 \ldots P_m \overline{t}_m$$

$$\quad \vdash v_1 < v \quad \ldots \quad P'_m \overline{t}_m v_m \quad v_m < v \quad N v$$

$$P' \overline{t}_v v$$

$$Q_1 \overline{u}_1 \ldots Q_n \overline{u}_n \quad P'_{i \in I} v_1 < v \quad \ldots$$

$$P'_{m \in M} \overline{t}_m v_m \quad v_m < v \quad N v$$

$$P'_{m \in M} v w$$

We can assume the production rule for $J'_i$ is the stage-number transformation of the production rule for $J_i$. Then $J'_i$ is

$$\Gamma, Q_1 \overline{u}_1, \ldots, Q_n \overline{u}_n, F^*_i \overline{t}_1 v_1, v_1 < w, v_1 < v, \ldots, F^*_m \overline{t}_m v_m, v_m < w, v_m < v, N v w_0, w_0 < w, N w \vdash F^*_i v w$$

By IH for $J_i$ we have $\tilde{J}_i, namely

$$F^*_i \overline{t}_1 v_1, v_1 < w, v_1 < v, \ldots, F^*_m \overline{t}_m v_m, v_m < w, v_m < v,$$

$$N v, N w_0, v \leq w_0, w_0 < w, N w \vdash F^*_i v w.$$ (33)

We can show $J'_i$ as follows. Assume the antecedent, fix $z$, and assume $N z$ and $v < z$ in order to show $\overline{F} v z$. Take $w$ to be $z, w_0$ to be $s v_0$, and $w_0$ to be $v$ in [33]. Then $v_1 < w$ by $v_1 < v$ and $v < z$. We also have $w_0 < w$ by $v < z$. Moreover $\overline{F}_i \overline{t}_v w_0$ by $\overline{F}_i \overline{t}_v v_1$. Hence $\overline{F}_i \overline{t}_v z$ by [33]. Hence we have shown $J'_i$.

By (Ind $P'$), we have $\Gamma, P'_j \overline{a} b \vdash F'_j \overline{a} b$. Hence $\tilde{\Gamma}, P'_j \overline{a} b, N c, b < c \vdash \tilde{F}_j \overline{a} b c$. Hence we have

$$(\Gamma, P'_j \overline{a} b c) \vdash F'_j \overline{a} b c.$$ \hfill \square

The next main theorem shows Brotherston-Simpson conjecture under arithmetic for intuitionistic logic.

**Theorem 6.14 (Equivalence of LJID + HA and CLJID + HA)** Let $\Sigma = \Sigma_N \cup \{ \overline{Q}, \overline{P}, \overline{P}' \}$ and $\Phi = \Phi_N \cup \{ \overline{P}, \overline{P}' \}$. Then the provability of $CLJID^w + HA + (\Sigma, \Phi)$ is the same as that of $LJID + HA + (\Sigma, \Phi)$.

**Proof.** (1) $LJID + HA + (\Sigma, \Phi)$ to $CLJID^w + HA + (\Sigma, \Phi)$.

For this claim, we can obtain a proof from the proof of Lemma 7.5 in [8] by restricting every sequent into intuitionistic sequents and replacing $LKID + (\Sigma, \Phi)$ and $CLKID^w + (\Sigma, \Phi)$ by $LJID + (\Sigma, \Phi)$ and $CLJID^w + (\Sigma, \Phi)$ respectively.

(2) $CLJID^w + HA + (\Sigma, \Phi)$ to $LJID + HA + (\Sigma, \Phi)$.

Let $\Sigma' = \Sigma \cup \{ N', \overline{P}' \}$ and $\Phi' = \Phi \cup \{ N', \overline{P}' \}$. Assume $J$ is provable in $CLJID^w + HA + (\Sigma, \Phi)$. By Proposition 6.13 $J$ is provable in $LJID + HA + (\Sigma, \Phi)$. By Proposition 6.13 $J$ is provable in $LJID + HA + (\Sigma, \Phi)$. \hfill \square
7 Conclusion

We have studied Brotherston-Simpson conjecture for intuitionistic logic. We have pointed out that the countermodel of [2] shows the Brotherston-Simpson conjecture for intuitionistic logic is false in general. We have shown HA-provability of Kleene-Brouwer theorem for induction and Podelski-Rybalchenko theorem for induction. By using them, we have shown the conjecture for intuitionistic logic is true under arithmetic, namely, the provability of the intuitionistic cyclic proof system is the same as that of the intuitionistic system of Martin-Lof’s inductive definitions when both systems contain Heyting arithmetic.

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