Nonpower Expansions for QCD Observables at Low Energies

Dmitry Shirkov

Bogoliubov Lab. of Theor. Physics, JINR, Dubna, 141980, Russia

Abstract

A comprehensive review is presented of the progress made in further developing the ghost-free Analytic Approach to low-energy QCD since “QCD-97” meeting. It is now formulated as a logically closed “Analytic Perturbation Theory” algorithm. Its most essential feature is nonpower functional expansions for QCD observables.

1. Intro

The claim is that current practice to use expansions for QCD observables in powers of the same effective coupling function $\bar{\alpha}_s$, i.e., $[\bar{\alpha}_s(Q^2)]^k$, $[\bar{\alpha}_s(s)]^k$ or $[\bar{\alpha}_s(r)]^k$, in different pictures ($Q^2$, $s$, $r$ ...) as, e.g., it is implicitly recommended by PDG\cite{1}, is neither based theoretically, nor adequate practically to low-energy QCD.

Instead, one should use non-power functional expansions with non-power sets of functions $\{A_k(Q^2)\}$, $\{A_k(s)\}$ and $\{\aleph_k\}$, related by some suitable integral transformations.

To comment this, consider a class of homogeneous Linear Integral Transformations (LITs)

$$ f_E \rightarrow f_i(x_i) = \int K_i \left( \frac{x_i}{Q^2} \right) f_E(Q^2) \frac{dQ^2}{Q^2}, \quad (1) $$

that describe transitions from the Euclidean picture to Minkowskian and Distance ones:

$$ f_E(Q^2) \rightarrow f_M(s); \quad f_E(Q^2) \rightarrow f_D(1/r^2). $$

Being applied to renormalisation-invariant functions they could relate, in particular, effective couplings properly defined in these pictures

$$ \alpha_E(Q^2) \rightarrow \alpha_M(s); \quad \alpha_E(Q^2) \rightarrow \alpha_D(r). \quad (2) $$

Meanwhile, generally, LITs (1) (and reverse transformations) can be used only within a class of functions $f_i$ that are free of nonintegrable singularities. For example, a common expression for QCD effective coupling $\bar{\alpha}_s$ with powers of $L = \ln(Q^2/\Lambda^2)$ in denominators (like eq.(9.5) in ref.[1]) is not suitable here. Due to this, in our further discussion we use some adequate models for $\bar{\alpha}_s$, free of unphysical singularities.

Moreover, LITs change the form of observable expansion; being applied to powers of some $\bar{\alpha}(Q^2)$, they yield expressions that are not powers of the $\bar{\alpha}(Q^2)$ image. They transform expansions in coupling powers $\bar{\alpha}^k$ into expansions over non-power sets $\{A_k(Q^2)\}$, $\{A_k(s)\}$ and $\{\aleph_k(r^{-2})\}$.

We give explicit examples of these sets and construct non-power expansions for observables that demonstrate quicker convergence and reduced renorm-scheme sensitivity.
2. Different pictures in HEP

2.1. Various HEP representations

An essential property of any successive theoretical scheme for physical phenomena is the possibility to use various pictures for their description. In HEP, these are space-time, energy–momentum, eikonal and some others. Below, we discuss the momentum-transfer (Euclidean part of the energy–momentum 4-space defined by \(Q^2 = q^2 - q_0^2 \geq 0\)), the center-of-mass-energy squared (Minkowskian part of the energy–momentum 4-space with \(s = q_0^2 - q^2 \geq 0\)) as well as the distance (space-like part of space-time \(r^2 = x^2 - x_0^2 \geq 0\)) pictures.

Remind, first, that Euclidean momentum-transfer picture was, historically, a natural launch-platform for LITs (1).

2.2. Pictures related by integral transformations

Transition from an Euclidean function \(f_E(Q^2)\) to the Distance one \(f_D(r^2)\) can be performed with an appropriate Fourier transformation. For example, in the three-dimensional case one has

\[
f_E(Q^2) \rightarrow \mathbb{F}[f_E](r^2) = f_D(r^2);
\]

\[
\mathbb{F}[f_E](r^2) \equiv r \int_0^\infty \text{d}Q \sin(Qr) f_E(Q^2).
\]

At the same time, in current QCD analysis, people use a relation between the Euclidean and Minkowskian pictures based on definition of the Adler function

\[
D(Q^2) = Q^2 \int_0^\infty \frac{ds}{(s + Q^2)^2} R(s).
\]

We treat this relation as integral transformation

\[
R(s) \rightarrow D(Q^2) \equiv \mathbb{D}[R](Q^2),
\]

that transforms a real function \(R(s)\) of the Minkowskian domain (real \(s \geq 0\)) to a real function \(D(Q^2)\) of the Euclidean one (real \(Q^2 \geq 0\)). The reverse transformation \(R = D^{-1}\) has the form of a contour integral.

Both the transformations (3) and (4) are consistent with RG–invariance and are non–consistent with existence of unphysical pole at \(\Lambda^2\).

3. Ghost-free APT construction

3.1. Ghost-free QCD coupling in Minkowskian

In the early 80s, by summing the \(\pi^2\)–terms arising in the course of logs branching

\[\ell \equiv \ln \left( \frac{Q^2}{\Lambda^2} \right) \rightarrow \ln \left( \frac{s}{\Lambda^2} \right) - i\pi \equiv L - i\pi\]

under the transition \(Q^2 \rightarrow -s\) Radsinskij, Krasnikov and A. Pivovarov obtained remarkable expressions for Minkowskian QCD coupling and its “effective powers”. This operation \(\sum_{\pi^2}\), in the one-loop case for \(\bar{\alpha}_s^{(1)} = 1/\beta_0 \ell\), results in the explicit form[5]:

\[
\bar{\alpha}_s^{(1)} \rightarrow \bar{\alpha}_M^{(1)}(s) = \frac{1}{\beta_0 \pi} \arccos \frac{L}{\sqrt{L^2 + \pi^2}}; \quad (5)
\]

\[
\left[ \bar{\alpha}_M^{(1)}(s) = \mathcal{A}_1^{(1)}(s) \right]_{L>0} = \frac{1}{\beta_0 \pi} \arctan \frac{\pi}{L}
\]

and for square and cube of \(\bar{\alpha}_s^{(1)}\) in expressions[6]

\[
\left( \frac{1}{\beta_0^2 f^2} \right) \rightarrow \mathcal{A}_2^{(1)}(s) = \frac{1}{\beta_0^2} \frac{L}{[L^2 + \pi^2]^2}; \quad (6)
\]

\[
\left( \bar{\alpha}_s^{(1)} \right)^3 \rightarrow \mathcal{A}_3^{(1)}(s) = \frac{L}{\beta_0^3 [L^2 + \pi^2]^2},
\]

which are not powers of \(\bar{\alpha}_M^{(1)}(s)\). As it was noticed later[7], they are ghost-free.

\[\text{See also Ref.[8].}\]
3.2. Ghost-free QCD coupling at Euclidean

In the Euclidean momentum-transfer domain analogous ghost-free construction for QCD coupling and observables was proposed\cite{9} in the mid-90s. This Analytic approach was based on imperative of analyticity in the cut $Q^2$–plane. Its algorithm, first realized \cite{10} in QED, combines RG invariance the Källen–Lehmann spectral representation with— see, Sections 53 and 54 in \cite{3} and the first of eqs.\cite{9} below.

In the simplest case of 1-loop QCD coupling, the analyticization operation $\mathcal{A}$ yields analytic Euclidean QCD coupling

$$\mathcal{A} \left[ \frac{1}{\ell} \right] \Rightarrow \beta_0 \alpha_E^{(1)}(Q^2) = \frac{1}{\ell} - \frac{\Lambda^2}{Q^2 - \Lambda^2} ; \ell = \ln \frac{Q^2}{\Lambda^2} .$$

Analogous expressions with “subtracted” high-order pole singularities are valid for powers of $\tilde{\alpha}_s^{(1)}$; e.g., for coupling squared

$$\frac{1}{\ell^2} \Rightarrow \frac{1}{\ell^2} + \frac{Q^2 \Lambda^2}{(Q^2 - \Lambda^2)^2} = \beta_0^2 \mathcal{A}_2^{(1)}(Q^2) \neq \left( \beta_0 \alpha_E^{(1)} \right)^2 .$$

By construction, new analytic functions $\{ \mathcal{A}_k \} = \mathcal{A}_1(= \alpha_E), \mathcal{A}_2, \ldots : \star$ are free of unphysical singularities; $\star$ are finite in the IR limit with

$$\alpha_E(0) = 1/\beta_0, \quad \mathcal{A}_{k \geq 2}(0) = 0 ; \quad (7)$$

$\star$ include non-perturbative structures; $\star$ in the weak-coupling UV limit $\rightarrow \tilde{\alpha}_s^{(k)} \sim [\ell]^{-k} .$$

These properties remain valid for higher-loop cases. In particular, IR limiting values \cite{7} provide the basis for remarkable stability of the $\mathcal{A}_k$ functions behavior in the low-energy domain. On Fig.1, one can see that while the 2-loop curve\footnote{For analytic expressions and numerical tables of 2-loop functions see papers \cite{9,11}.} for $\alpha_E(Q^2)$ differs from the 1-loop one in the few GeV region by 10-20\%\%, the 3-loop approximation practically coincides with the 2-loop curve, the difference being of the order of 2-3 per cent. The same is true for Minkowskian coupling. This, in turn, yields low sensitivity with respect to change of renormalization scheme (RS).

$\star$ Fig. 1. Space-like and time-like global analytic couplings in a few GeV domain with $\Lambda^{(s=3)} = 350$ MeV. Here, $\alpha_{E,M}$ in 1-loop case are drown by dash-and-dotted lines and for 2– and 3– loop cases by solid curves.

3.3. Minkowskian vs Euclidean; the APT

As it was noticed later \cite{12}, both the sets of Euclidean and Minkowskian functions are connected by LITs $\mathcal{A}_k(Q^2) = \mathbb{D} \left[ \tilde{\mathcal{A}}_k \right] , \tilde{\mathcal{A}}_k(s) = \mathbb{R} \left[ \mathcal{A}_k \right] . \quad$ In the 1-loop case \cite{7}

$$\mathbb{D} \left[ \frac{1}{\pi} \arccos \frac{L}{\sqrt{L^2 + \pi^2}} \right] = \frac{1}{\ln(Q^2/\Lambda^2)} - \frac{\Lambda^2}{Q^2 - \Lambda^2} ,$$

$$\mathbb{R} \left[ \frac{1}{\ln^2(Q^2/\Lambda^2)} + \frac{Q^2 \Lambda^2}{(Q^2 - \Lambda^2)^2} \right] = \frac{1}{L^2 + \pi^2} .$$

Accordingly, for an Euclidean RG-invariant function $I_E(Q^2)$ — initially expandable in powers of the coupling constant $\alpha_s$ — there appears a relation with its Minkowskian image $I_M(s) = \mathbb{R}[I_E]$ \cite{7}

$$\mathbb{D} \left[ I_M = \sum_k i_k \mathcal{A}_k \right] = I_E(Q^2) = \sum_k i_k \mathcal{A}_k(Q^2) .$$

Thus, starting with common RG-improved perturbation expansion for an Euclidean function $I_{pt}(Q^2, \alpha_s) , \quad$ after applying operation of analyticization $\mathcal{A}$, and of $\pi^2$–summation $\sum_{\pi^2}$ we arrive at non-power functional expansions related by (8) with numerical coefficients $i_k$ . The whole construction has been named \cite{7,12,13} the An-
alytic Perturbation Theory (APT). It was duly elaborated\[12\] for real QCD with various numbers $f$ of active quarks. The logic of its structure is presented in Fig.2.

\[
\alpha_s(Q^2), \alpha_k^R \]

\[\text{RG + PT}\]

\[\sum_{\sigma^2} \rho(\sigma) \]

\[\alpha_M(s) \]

\[\alpha_E(Q^2) \]

\[\alpha_0(r) \]

\[\rho(\sigma) \]

\[\text{Minkow} \]

\[\text{Euclid} \]

\[\text{Distance} \]

Fig.2 Logic of APT scheme.

3.4. Sketch of the global APT algorithm

The most elegant form of the APT formalism uses a spectral density $\rho(\sigma)$ taken from perturbative input. Then all involved functions in the above-mentioned pictures look like

\[
A_k = \frac{1}{\pi} \int_0^\infty \frac{\rho_k(\sigma) d\sigma}{\sigma + Q^2}, \quad \mathfrak{A}_k = \frac{1}{\pi} \int_0^\infty \frac{d\sigma}{\sigma} \rho_k(\sigma),
\]

\[
\mathcal{N}_k \left( \frac{1}{r^2} \right) = \int_0^\infty \frac{\rho_k(\sigma) d\sigma}{\sigma} \left( 1 - e^{-r\sqrt{\sigma}} \right).
\]

In the 1-loop case,

\[
\rho_1^{(1)} = \frac{1}{\beta_0 \left[ L_0^2 + \pi^2 \right]}, \quad L_0 = \ln \left( \frac{\sigma}{\Lambda^2} \right)
\]

with higher spectral functions expressed via $\rho_1^{(1)}$ by simple iterative relation

\[
k \beta_0 \rho_{k+1}^{(1)}(\sigma) = -\frac{d \rho_k^{(1)}(\sigma)}{d L_0},
\]

that implies analogous iterative relations for $A_k$ and $\mathfrak{A}_k$. As it has been noted above, eqs.(10) and (11) were generalized for a higher-loops case with transitions across heavy quark thresholds and were successively used [13] for fitting various data.

Let us emphasize an important ansatz built-in the presented APT construction: spectral functions, like (10), are defined on the basis of RG-improved perturbation theory. Due to this, they contain no additional parameters and non-perturbative elements. Hence, this construction can be considered as a “minimal APT scheme”.

Here, it can be added that some detail of the APT construction in the distance picture was recently considered in our paper Ref.[14].

4. Non-power expansions for observables

To summarize APT for observables, we state that instead of usual expansions in powers of the same universal QCD effective $\alpha_s$ function

\[
\{ \hat{\alpha}_k^R(Q^2, \alpha_s) \}, \{ \hat{\alpha}_k^E(s, \alpha) \}, \{ \hat{\alpha}_k^D(1/r^2, \alpha_s) \}
\]

one should use “perturbatively motivated” expansions in terms of non-power sets

\[
\{ A_k(Q^2, \alpha_s) \}, \{ \mathfrak{A}_k(s, \alpha_s) = R[A_k](s) \},
\]

\[
\{ \mathcal{N}_k(1/r^2, \alpha_s) = F[A_k](1/r^2) \}
\]

with some of them being non-analytic at $\alpha_s = 0$.

4.1. Properties of non-power functions

Review now the main properties of non-power sets $\{ A_k \}, \{ \mathfrak{A}_k \}, \{ \mathcal{N}_k \}$. Qualitatively, all three are similar:

I. They consist of ghost-free functions

II. First functions, new couplings, $\alpha_E, \alpha_M, \alpha_D$

\[\triangleright\] are monotonic;

\[\triangleright\] In the IR limit, they are finite $= 1/\beta_0 \approx 1.4$;

\[\triangleright\] In the UV limit $\sim 1/\ln x \sim \hat{\alpha}_s(x)$.

III. All other functions ($k \geq 2$):

\[\triangleright\] start from zero $A_k(0), \mathfrak{A}_k(0), \mathcal{N}_k(0) = 0$;

\[\triangleright\] in the UV limit $\sim 1/(\ln x)^k \sim \hat{\alpha}_s^k(x)$.

\[\triangleright\] 2nd ones, $A_2, \mathfrak{A}_2, \mathcal{N}_2$ obey max at $\sim \Lambda^2$.

\[\triangleright\] Higher ones $k \geq 3$ oscillate at $\sim \Lambda^2$ and obey

\[k - 1\] zeroes².

Just two last properties play an essential role is the RS-insensitivity and in better convergence of expansions in the low-energy region.

² On mathematical nature of these sets see papers [15,16].
It is curious that the mechanism of liberation of singularities is quite different: for $A_k$, in the space-like domain, it involves subtraction of nonperturbative, power in $Q^2$, structures. Meanwhile, in the time-like region, it is based only on resummation of “$\pi^2$–terms”. Graphically, nonperturbative analyticization in the $Q^2$–channel looks like a slightly distorted reflection (as $Q^2 \rightarrow s = -Q^2$) of the perfectly perturbative “$\pi^2$–summation” result in the $s$–channel. The effect of “distorting mirror” (first discussed by Milton-Solovtsov [7]) is illustrated in Figs. 1 and 3.

4.2. Better convergence and RS insensitivity

Due to a quite different from the usual oscillation behavior of higher (with $k \geq 3$) functions, the APT nonpower expansions

$$d_{m}(Q^2) = d_1 \alpha_E(Q^2) + d_2 A_2(Q^2) + d_3 \mathcal{A}_3 + ...$$

$$r_s(s) = d_1 \alpha_M(s) + d_2 \mathcal{A}_2(s) + d_3 \mathcal{A}_3(s) + ...$$

obey better convergence in the hadronization region. We demonstrate this by a Table from paper [13]. All results are given in the $\overline{\text{MS}}$ scheme.

As it follows from the comparison between the 3rd columns for usual (PT) case and for the APT one, the relative contribution of the 3rd ($\sim \bar{\alpha}_3^2$) term drastically diminishes; it is not any more of the same order of magnitude as the 2nd term.

| Process                  | PT          | APT          |
|--------------------------|-------------|--------------|
| GLS Sum Rules            | 65 24       | 11           |
| Bjorken Sum Rules        | 55 26       | 19           |
| Incl. V. $\tau$-decay    | 55 29       | 16           |
| $10\text{GeV } e^+e^- \rightarrow$ hadrons | 96 8 | -4 |
| $Z_o \rightarrow$ hadrons | 98.6 3.7   | -2.3         |

In turn, this leads to essential reducing of dependence of the results on change of the renormalisation-scheme prescription. For further numerical examples of reduced RS sensitivity see papers [17,18,19]. One more fresh illustration is provided by recent analysis of the pion form factor papers [17,18,19].

4.3. Issue of the APT non-uniqueness

The described minimal APT scheme obeys one specific feature: it does not contain any new parameters. Like in usual QCD case, there is only one adjustable parameter; it can be chosen as $\Lambda_{(f=5)}$ or $\bar{\alpha}_s(m_{Z_o})$. Aesthetically, this feature looks very nice. In particular, as a thorough numerical analysis reveals, within the APT framework with its two different coupling functions $\alpha_M$ and $\alpha_E$, it turns out to be possible to correlate a great bulk of QCD data more successively[13] (with smaller $\chi^2$ value) than by a standard algorithm with the only $\bar{\alpha}_s$ – see the paper [21].

Meanwhile, there exist nonperturbative evidence that in the 1 GeV region QCD coupling could reach values close to 0.6-0.9.[22–c] (see also our review [23]) which is impossible for the minimal APT effective couplings at realistic $\Lambda$ values.

In APT, there are several possibilities for insert-
ing adjustable “low-energy parameters”. The simplest – from physical point of view – one, is to attribute some effective low-energy mass to partons. This idea is not a new one — see, e.g., refs.

In the APT context this trick was successively used by Baldicchi and Prosperi [25] in calculating mass spectrum of light mesons.

Another way is to insert some non-perturbative structure into spectral density \( \rho(\sigma) \) in eqs.(9). An example of this kind can be found in Refs.[26].

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