On the Secrecy Gain of Formally Unimodular Construction $A_4$ Lattices

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Abstract—Lattice coding for the Gaussian wiretap channel is considered, where the goal is to ensure reliable communication between two authorized parties while preventing an eavesdropper from learning the transmitted messages. Recently, a measure called secrecy gain was proposed as a design criterion to quantify the secrecy-goodness of the applied lattice code. In this paper, the theta series of the so-called formally unimodular lattices obtained by Construction $A_4$ from codes over $\mathbb{Z}_4$ is derived, and we provide a universal approach to determine their secrecy gains. Initial results indicate that Construction $A_4$ lattices can achieve a higher secrecy gain than the best-known formally unimodular lattices from the literature. Furthermore, a new code construction of formally self-dual $\mathbb{Z}_4$-linear codes is presented.

I. INTRODUCTION

The study of physical layer security (PLS) has recently received significant attention in the 5G and beyond 5G (B5G) wireless communications [1], [2]. In contrast to cryptographic algorithms, approaches in PLS only utilize the resources at the physical layer of the transmitting parties and provide information-theoretically unbreakable security. It stemmed from Aaron D. Wyner’s landmark paper [3] in 1975, which showed that based on the communication channel characteristics, one can achieve communication that is reliable and at the same time secure against an adversarial eavesdropper.

In the famous wiretap channel (WTC) introduced in [3], a single transmitter (Alice) tries to communicate to a receiver (Bob) while keeping the transmitted messages secure from an unauthorized eavesdropper (Eve). The secure and confidential achievable rate between Alice and Bob for WTC is defined as the secrecy rate. There is a recent focus on designing practical wiretap codes that achieve a high secrecy rate based on lattices over Gaussian WTCs [4]–[6]. Among these works, one of the essential design criteria for good wiretap lattice codes is the secrecy gain [4], [5], which is defined as the maximum attainable secrecy function (the coding gain of a specifically designed lattice $\Lambda_e$ for Eve compared to a regular integer lattice, evaluated in terms of the theta series of lattices. See Section IV-A for an explicit definition.)

Another design criterion for wiretap lattice codes, called the flatness factor, was proposed by Ling et al. [6]. The flatness factor quantifies how much confidential information can leak to Eve in terms of mutual information, while the secrecy gain characterizes Eve’s success probability of correctly guessing the transmitted messages. The secrecy gain and the flatness factor both require small theta series of the designed Eve’s lattice $\Lambda_e$ at a particular point to guarantee secrecy-goodness [6].

In this work, the quality criterion of secrecy gain for lattice coding is considered. Secrecy gains of the so-called unimodular lattices have been studied for well over a decade [4]. In this pioneering work, Belfiore and Solé discovered that there exists a symmetry point in their secrecy functions. Further, they conjectured that for unimodular lattices, the secrecy gain is achieved at the symmetry point of its secrecy function. The conjecture has been further investigated and verified for unimodular (or isodual) lattices in dimensions less than 80 [5], [7]–[9]. The study of secrecy gain was recently also extended to the $\ell$-modular lattices [5], [10], [11], where it is believed that the higher the parameter $\ell$ is, the better secrecy gain we can achieve. Most recently, a new family of lattices, called formally unimodular lattices, or lattices with the same theta series as their dual, was introduced [12]. It was shown that formally unimodular lattices have the same symmetry point as unimodular and isodual lattices, and the Construction $A$ lattices obtained from the formally self-dual codes can achieve a higher secrecy gain than the unimodular lattices. Moreover, for formally unimodular lattices obtained by Construction $A$ from even formally self-dual codes, a sufficient condition to verify Belfiore and Solé’s conjecture on the secrecy gain is also provided. (An even code has all of its codewords with even weights. Otherwise, the code is odd.)

This paper especially focuses on the analysis of the secrecy gain for formally unimodular lattices obtained by Construction $A_4$ from codes over the ring $\mathbb{Z}_4 \triangleq \{0, 1, 2, 3\}$ (also called the quaternary codes) [13]–[15]. Our contributions are three-fold:

i) A code $C$ over $\mathbb{Z}_4$ is formally self-dual if it has the same symmetrized weight enumerator (swe) as its dual. We show that if $C$ is formally self-dual, then its corresponding Construction $A_4$ lattice is formally unimodular.

ii) We provide a novel and universal approach to determine the secrecy gain for Construction $A_4$ lattices obtained from formally self-dual codes over $\mathbb{Z}_4$.

iii) To study secrecy gains of Construction $A_4$ lattices from formally self-dual $\mathbb{Z}_4$-linear codes, we present a new code construction of formally self-dual $\mathbb{Z}_4$-linear codes with respect to swe. There is not much known about this in the literature [16]–[18].

To the best of our knowledge, most of the efforts to solve Belfiore and Solé’s conjecture on the secrecy gain of formally unimodular lattices have only been based on

1 A formally-self dual code has the same weight enumerator as its dual.
the lattices obtained by Construction A from binary codes. The investigation of Construction $A_4$ lattices obtained from formally self-dual codes over $\mathbb{Z}_4$ has not been addressed in the previous literature.

II. DEFINITIONS AND PRELIMINARIES

A. Notation

We denote by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ the set of integers, rationals, and reals, respectively. $\{m : n\} \triangleq \{m, m + 1, \ldots, n\}$ for $m, n \in \mathbb{Z}$, $m \leq n$. Vectors are boldfaced, e.g., $x$. Matrices and sets are represented by capital sans serif letters and calligraphic uppercase letters, respectively, e.g., $X$ and $A$. An identity matrix of dimensions $m \times m$ is denoted as $I_m$, and $O_{m \times n}$ represents an all-zero matrix of size $m \times n$. Denote by $d_{\text{Lee}}(x, y)$ the Lee distance between two vectors $x, y$ over binary field $\mathbb{F}_2$ or $\mathbb{Z}_4$. $(x, y)$ denotes the inner product and the $x \circ y$ represents the element-wise (Hadamard/ Schur) product between two vectors over $\mathbb{F}_2$ or $\mathbb{Z}_4$, respectively. We use the code parameters $[n, M]$ or $[n, M, d_{\text{Lee}}]$ to denote a linear code $\mathcal{C}$ of length $n$, $M$ codewords, and minimum Lee distance $d_{\text{Lee}} \triangleq \min_{x, y \in \mathcal{C}} d_{\text{Lee}}(x, y)$. $w_{\text{Ham}}(x)$ denotes the Hamming weight of a vector $x$. $\phi_q : \mathbb{Z}_q \rightarrow \mathbb{Z}$ is defined as the natural embedding, i.e., $\phi_q(x)$ is the remainder of the division of $x$ by $q$. In this work, $q$ can be 2 or 4.

B. Basics on Codes and Lattices

We next recall some definitions of codes over $\mathbb{F}_2$, codes over $\mathbb{Z}_4$, and lattices.

Let $\mathcal{A}$ be an $[n, M]$ binary code. Its weight enumerator is

$$W_{\mathcal{A}}(x, y) = \sum_{c \in \mathcal{A}} x^{n-w(c)} y^{w(c)}.$$ 

Let $\mathcal{C}_1, \mathcal{C}_2$ be two binary linear codes. For $c_1 = (c_{1,1}, \ldots, c_{1,n}) \in \mathcal{C}_1$, $c_2 = (c_{2,1}, \ldots, c_{2,n}) \in \mathcal{C}_2$, we define

$$d_{0,0}(c_1, c_2) \triangleq \{j \in [1 : n] : (c_{1,j}, c_{2,j}) = (0, 0)\}.$$ 

$$d_{0,1}(c_1, c_2) \triangleq \{j \in [1 : n] : (c_{1,j}, c_{2,j}) = (0, 1)\}.$$ 

$$d_{1,0}(c_1, c_2) \triangleq \{j \in [1 : n] : (c_{1,j}, c_{2,j}) = (1, 0)\}.$$ 

$$d_{1,1}(c_1, c_2) \triangleq \{j \in [1 : n] : (c_{1,j}, c_{2,j}) = (1, 1)\}.$$ 

Observe that $d_{0,0}(c_1, c_2) + d_{0,1}(c_1, c_2) + d_{1,0}(c_1, c_2) + d_{1,1}(c_1, c_2) = n$. The joint weight enumerator of $\mathcal{C}_1$ and $\mathcal{C}_2$ is given by

$$jwe_{\mathcal{C}_1, \mathcal{C}_2}(a, b, c, d) \triangleq \sum_{c_1 \in \mathcal{C}_1} \sum_{c_2 \in \mathcal{C}_2} a^{d_{0,0}} b^{d_{0,1}} c^{d_{1,0}} d^{d_{1,1}},$$

where we use the shorthand $d_{ij}(c_1, c_2)$ defined in (1). Detailed properties and MacWilliams identities of the joint weight enumerator can be found in [19, Ch. 5, pp. 147–149].

A $\mathbb{Z}_4$-linear code of length $n$ is an additive subgroup of $\mathbb{Z}_4^n$. If $\mathcal{C}$ is a $\mathbb{Z}_4$-linear code of length $n$, then $\mathcal{C}^\perp \triangleq \{x \in \mathbb{Z}_4^n : (x, y) = 0, \text{ for all } y \in \mathcal{C}\}$ is the dual code of $\mathcal{C}$.

From [15, Prop. 1.1], it is well-known that any $\mathbb{Z}_4$-linear code is permutation equivalent to a code $\mathcal{C}$ with a generator matrix $G$ in standard form

$$G = \begin{pmatrix} I_{k_1} & A \\ O_{k_1 \times k_2} & 2I_{k_2} \end{pmatrix},$$

where $A$ and $C$ are binary matrices, and $B$ is defined over $\mathbb{Z}_4$. Such code $\mathcal{C}$ is said to be a code of type $4^{k_1}2^{k_2}$.

The symmetrized weight enumerator (swe) of a $\mathbb{Z}_4$-linear code $\mathcal{C}$ is defined as

$$\text{swe}_{\mathcal{C}}(a, b, c) = \sum_{c \in \mathcal{C}} a^{n_{1,c}} b^{n_{2,c}} c^{n_{3,c}}.$$ 

where $n_i(c) = \{j \in [1 : n] : c_j = i\}$, $i \in \mathbb{Z}_4$. The corresponding MacWilliams identity for $\mathbb{Z}_4$-linear codes is given by [15, Th. 2.3]

$$\text{swe}_{\mathcal{C}}(a, b, c) = \frac{1}{|\mathcal{C}|} \text{swe}_{\mathcal{C}^\perp}(a + 2b + c, a - c, a - 2b + c).$$

Following the notion of swe, we have the following families of codes over $\mathbb{Z}_4$.

Definition 1 (Self-dual, isodual, formally self-dual codes):

- If $\mathcal{C} = \mathcal{C}^\perp$, $\mathcal{C}$ is a self-dual code.
- If there is a permutation $\pi$ of coordinates such that $\mathcal{C} = \pi(\mathcal{C}^\perp)$, $\mathcal{C}$ is called isodual.
- If $\mathcal{C}$ and $\mathcal{C}^\perp$ have the same symmetrized weight enumerator, i.e., $\text{swe}_{\mathcal{C}}(a, b, c) = \text{swe}_{\mathcal{C}^\perp}(a, b, c)$, $\mathcal{C}$ is a formally self-dual code.

From (4), we can conclude that a code in any of these classes has its swe satisfying

$$\text{swe}_{\mathcal{C}}(a, b, c) = \frac{1}{|\mathcal{C}|} \text{swe}_{\mathcal{C}^\perp}(a + 2b + c, a - c, a - 2b + c).$$

A (full rank) lattice $\Lambda \subset \mathbb{R}^n$ is a discrete additive subgroup of $\mathbb{R}^n$, and it can be seen as $\Lambda = \{\Lambda = uL_{n \times n} : u \in \mathbb{Z}^n\}$, where the $n$ rows of $L$ form a lattice basis in $\mathbb{R}^n$. The volume of $\Lambda$ is $\text{vol}(\Lambda) = |\text{det}(L)|$. If a lattice $\Lambda$ has generator matrix $L$, then the lattice $\Lambda^* \subset \mathbb{R}^n$ generated by $(L^{-1})^\dagger$ is called the dual lattice of $\Lambda$. For lattices, the analogue of the weight enumerator of a code is the theta series, defined as follows.

Definition 2 (Theta series): Let $\Lambda$ be a lattice, its theta series is given by

$$\Theta_{\Lambda}(z) = \sum_{\Lambda \in \Lambda} q^{||\Lambda||^2},$$

where $q \triangleq e^{i\pi z}$ and $\text{Im}(z) > 0$.

Analogously, the spirit of the MacWilliams identity can be captured by the Jacobi’s formula [20, eq. (19), Ch. 4]

$$\Theta_{\Lambda}(z) = \text{vol}(\Lambda^*)(\frac{i}{z})^{n/2} \Theta_{\Lambda^*}(\frac{1}{z}).$$
In some particular cases, the theta series of a lattice can be expressed in terms of the \textit{Jacobi theta functions} defined as follows.

\[ \vartheta_2(z) = \sum_{m \in \mathbb{Z}} q^{(m+\frac{1}{2})^2} = \Theta_{\varphi^2} \]
\[ \vartheta_3(z) = \sum_{m \in \mathbb{Z}} q^{m^2} = \Theta_{\varphi^2} \]
\[ \vartheta_4(z) = \sum_{m \in \mathbb{Z}} (-q)^{m^2} \]

Lattices can be classified according to their properties. It is said to be \textit{integral} if the inner product of any two lattice vectors is an integer. An integral lattice such that \( \Lambda = \Lambda^* \) is called a \textit{unimodular} lattice. A lattice \( \Lambda \) is called \textit{isodual} if it can be obtained from its dual \( \Lambda^* \) by (possibly) a rotation or reflection. In [12], a new and broader family was presented, namely the \textit{formally unimodular lattices}, that consists of lattices having the same theta series as their dual, i.e., \( \vartheta_{\Lambda}(z) = \vartheta_{-\Lambda}(z) \).

Lattices can be constructed from binary linear codes through the so-called Constructions A and C [20].

\textbf{Definition 3 (Construction A):} Let \( C \) be a binary \([n, M] \) code, then \( \Lambda_A(C) = \frac{1}{\sqrt{2}}(\varphi_2(C) + 2\mathbb{Z}^n) \) is a lattice.

\textbf{Definition 4 (2-level Construction C):} Let \( C_1, C_2 \) be two linear codes over \( \mathbb{F}_2 \) and \( C_1 \subseteq C_2 \). If the chain \( C_1 \subseteq C_2 \) is closed under the element-wise product, then the packing given by

\[ \Lambda_C(C_1, C_2) = \varphi_2(C_1) + 2\varphi_2(C_2) + 4\mathbb{Z}^n \]  \hspace{1cm} (7)

generates a lattice.

For general choices of \( C_1 \) and \( C_2 \), (7) is a nonlattice packing, and we will denote by \( \Gamma_C(C_1, C_2) \). If \( C_1 \) is the zero code, and \( C_2 \) is linear, then \( \Lambda_C(C_1, C_2) = 2\Lambda_C(C_2) \). If \( C_2 \) is the universe code \( \mathbb{F}_2^n \) and \( C_1 \) is linear, then \( \Lambda_C(C_1, C_2) = \Lambda(C) \).

A packing \( \Gamma \subset \mathbb{R}^n \) is \textit{geometrically uniform} if for any two elements \( e, e' \in \Gamma \) there exists an isometry \( T \) such that \( e' = T(e) \) and \( T(\Gamma) = \Gamma \). It was demonstrated that \( \Gamma_C(C_1, C_2) \) is geometrically uniform [21], for linear codes \( C_1 \) and \( C_2 \).

There is an analogue of Construction A for codes over \( \mathbb{Z}_4 \), which is called \textit{Construction A}_4.

\textbf{Definition 5 (Construction A}_4 \textit{[22, Ch. 12.5.3]):} If \( C \) is a \( \mathbb{Z}_4 \)-linear code, then \( \Lambda_{A_4}(C) = \frac{1}{2}(\varphi_4(C) + 4\mathbb{Z}^n) \) is a lattice.

It is known that \( \Lambda_{A_4}(C) \) is a unimodular lattice if and only if the \( \mathbb{Z}_4 \)-linear code \( C \) is self-dual code [15, Prop. 12.2]. For notational convenience, sometimes the mapping \( \varphi_4 \) is omitted.

The following example illustrates Construction A_4 of the octocode.

\textbf{Example 1:} The self-dual \( \mathbb{Z}_4 \)-linear code, known as the octocode \( O_8 \), is generated by \( G = (I_4 \ B) \), where

\[ B = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \\ 3 & 3 & 3 & 2 \\ 2 & 3 & 1 & 1 \end{bmatrix} \]

It is of type \( 4^4 \) and its swe [22, Ex. 12.5.13] is given by

\[ \text{swe}_{O_8}(a, b, c) = a^8 + 16a^8 + b^8 + 14a^4c^4 + 112a^3b^3y + 112a^3b^4c^3 \]

A unimodular lattice can be constructed by performing \( \mathbb{E}_8 = \Lambda_{A_4}(O_8) = \frac{1}{2}(O_8 + 42 \mathbb{Z}^8) \). It is equivalent to the well-known Gosset lattice \( \mathbb{E}_8 \) [22, Ex. 12.5.13]. Note that the theta series of the \( \mathbb{E}_8 \) lattice in terms of the Jacobi theta functions is

\[ \vartheta_{\mathbb{E}_8} = \frac{1}{2}(\vartheta_2(z)^8 + \vartheta_3(z)^8 + \vartheta_4(z)^8) \]

The \( \mathbb{E}_8 \) lattice can also be constructed via the binary Construction A, using the [8, 4, 4] extended Hamming code.

\section{Lattices from 2-Level Construction C and Construction A_4}

Some \( \mathbb{Z}_4 \)-linear codes can be obtained from binary linear codes by using the 2-level Construction C as in Definition 4.

\textbf{Proposition 1 ([14, Lemma 2.1]):} Consider two binary linear codes \( C_1, C_2 \), and let \( C = C_1 + 2C_2 \). Then, the code \( C \) over \( \mathbb{Z}_4 \) is linear if and only if \( C_1 \subseteq C_2 \) is closed under the element-wise product.

On one hand, the condition that the chain \( C_1 \subseteq C_2 \) is closed under Schur product guarantees that \( \Lambda_{A_4}(C_1, C_2) \) is a lattice, on the other hand, \( C \) being \( \mathbb{Z}_4 \)-linear assures that \( \Lambda_{A_4}(C) \) is a lattice. Therefore, Proposition 1 standardize the 2-level Construction C and Construction A_4, together with their respective conditions to be a lattice.

Denote by \( C_1 \) an \([n, 2^{k_1}] \) code and \( C_2 \) an \([n, 2^{k_2}] \) code. Once a \( \mathbb{Z}_4 \)-linear code \( C \) can be expressed as \( C_1 + 2C_2 \), and the codes \( C_1 \) and \( C_2 \) are generated, respectively, by

\[ G_1 = \begin{pmatrix} I_{k_1} & X \\ 0_{k_1 \times (k_2 - k_1)} & I_{k_2 - k_1} \end{pmatrix} \]

Then, the generator matrix \( G_2 \) of \( C \) as in (3) becomes [13, Thm. 3]

\[ G = \begin{pmatrix} I_{k_1} & X \\ 0_{k_1 \times (k_2 - k_1)} & 2I_{k_2 - k_1} \end{pmatrix} \]

Up to now, we revisited some results on the construction of \( \mathbb{Z}_4 \)-linear codes from two binary linear codes \( C_1 \) and \( C_2 \), and we notice that the lattice derived from them via Construction A_4 (or analogously 2-level Construction C) can have some properties, such as being unimodular or isodual.

\section{Weight Enumerators and Theta Series}

We now define a few extra notions of weight enumerators and derive an expression for the theta series of the lattice generated via Construction A_4, given the symmetric weight enumerator of the \( \mathbb{Z}_4 \)-linear code \( C \). An analogous relation will be discussed for the joint weight enumerator as well.

From the fact that a 2-level Construction C is geometrically uniform for binary linear codes \( C_1 \) and \( C_2 \) together with the expression of the theta series of periodic packings given in [23], we can state the following result.
**Theorem 1:** Consider a 2-level Construction C lattice given by \( \Gamma_C(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{2}(\mathcal{C}_1 + 2\mathcal{C}_2 + 4\mathbb{Z}^n) \), where \( \mathcal{C}_1, \mathcal{C}_2 \) are binary linear codes. The theta series of \( \Gamma_C(\mathcal{C}_1, \mathcal{C}_2) \) is

\[
\Theta_{\Gamma_C(\mathcal{C}_1, \mathcal{C}_2)}(z) = \sum_{\mathbf{c}_1 \in \mathcal{C}_1} \sum_{\mathbf{c}_2 \in \mathcal{C}_2} q^{\mathbf{d}_0, \mathbf{d}_1}(4z) \left( \frac{\vartheta_2(z)}{2} \right)^{d_{1,0} + d_{1,1}} \vartheta_2^{d_{2,1}}(4z).
\]

**Proof:** Recall that the theta function of periodic packings, not necessarily lattices, can be obtained as follows.

**Proposition 2 ([23]):** Given a periodic constellation \( \Gamma = \bigcup_{k=1}^M (\Lambda + \mathbf{u}_k) \), where \( \Lambda \subset \mathbb{R}^n \) is a lattice and \( \mathbf{u}_1, \ldots, \mathbf{u}_M \in \mathbb{R}^n \) are the \( M \) coset representatives. Then

\[
\Theta_{\Gamma}(z) = \Theta_{\Lambda}(z) + \frac{2}{M} \sum_{k < \lambda \in \Lambda} q^{\parallel \mathbf{u}_k - \mathbf{u}_\lambda \parallel^2}.
\]

For a geometrically uniform packing \( \Gamma \), where the set of distances is preserved for every point, then it reduces to

\[
\Theta_{\Gamma}(z) = \sum_{k=1}^M q^{\parallel \mathbf{u}_k \parallel^2}.
\]

Packing obtained from Construction C are periodic and in particular, a 2-level Construction C, written as \( \mathcal{C}_1 + 2\mathcal{C}_2 + 4\mathbb{Z}^n \), is geometrically uniform, so we can apply Proposition 2, more specifically, (10).

In (10), we identify \( \lambda \in 4\mathbb{Z}^n, M = |\mathcal{C}_1| |\mathcal{C}_2|, \) and \( \mathbf{u}_1 = (0, \ldots, 0) \), since \( \mathcal{C}_1, \mathcal{C}_2 \) are linear codes and thus contain the zero codeword. Notice that in our context, \( \mathbf{u}_k \in \mathcal{C}_1 + 2\mathcal{C}_2 \) and initially, let us fix \( k \) and set \( \mathbf{u} = \mathbf{u}_k \) to simplify.

As \( \mathbf{u} \in \mathcal{C}_1 + 2\mathcal{C}_2 \), there exist \( \mathbf{c}_1 \in \mathcal{C}_1 \) and \( \mathbf{c}_2 \in \mathcal{C}_2 \) such that \( \mathbf{u} = \mathbf{c}_1 + 2\mathbf{c}_2 \). The coordinates of \( \mathbf{u} \) can be 0, 1, 2 or 3 and their frequency are given respectively by \( d_{0,0}(\mathbf{c}_1, \mathbf{c}_2), d_{1,0}(\mathbf{c}_1, \mathbf{c}_2), d_{0,1}(\mathbf{c}_1, \mathbf{c}_2), \) and \( d_{1,1}(\mathbf{c}_1, \mathbf{c}_2) \), as in (1).

By fixing the \( i \)-th coordinate of \( \mathbf{u} \), we have as possible exponents of \( q \) in (10)

\[
4z_i + u_i = \begin{cases} 
4z_i, & \text{if } u_i = 0 \\
4(z_i + \frac{1}{2}), & \text{if } u_i = 1 \\
4(z_i + \frac{1}{2}), & \text{if } u_i = 2 \\
4(z_i + \frac{3}{2}), & \text{if } u_i = 3
\end{cases}
\]

The corresponding theta series associated to each one of the previous cases are

\[
\Theta_{4z}(z) = \vartheta_3(16z), \quad \Theta_{4(z_i + \frac{1}{2})}(z) = \vartheta_2(16z),
\]

\[
\Theta_{4(z_i + \frac{3}{2})}(z) = \Theta_{4(z_i + \frac{1}{2})}(z) = \frac{\vartheta_3(4z)}{2},
\]

By incorporating such results into the fixed \( n \)-dimensional vector \( \mathbf{u} \), we have that

\[
\sum_{z \in \mathbb{Z}^n} q^{\parallel z + u \parallel^2} = \vartheta_3^{d_{0,0}}(16z) \left( \frac{\vartheta_3(4z)}{2} \right)^{d_{1,0} + d_{1,1}} \vartheta_2^{d_{2,1}}(16z).
\]

Finally, running through all \( k \) vectors \( \mathbf{u}_k \) and considering the scaled version \( \Gamma_C(\mathcal{C}_1, \mathcal{C}_2) = \frac{1}{2}(\mathcal{C}_1 + 2\mathcal{C}_2 + 4\mathbb{Z}^n) \), we get

\[
\Theta_{\Gamma}(z) = \sum_{k=1}^M \sum_{z \in \mathbb{Z}^n} q^{\parallel 4(z + u_k) \parallel^2} = \sum_{c_{1k} \in \mathcal{C}_1, c_{2k} \in \mathcal{C}_2} q^{d_{0,0}}(4z) \left( \frac{\vartheta_3(z)}{2} \right)^{d_{1,0} + d_{1,1}} \vartheta_2^{d_{2,1}}(4z),
\]

(12)

where \( \mathbf{u}_k = c_{1k} + 2c_{2k} \), for \( c_{1k} \in \mathcal{C}_1 \) and \( c_{2k} \in \mathcal{C}_2 \).

Theorem 1 is general and can be applied to nonlattice packings. We relate now the theta series of \( \Gamma_C(\mathcal{C}_1, \mathcal{C}_2) \) to \( \text{jwe} \).

**Corollary 1:** The theta series of a 2-level Construction C lattice, in terms of the \( \text{jwe} \) of two codes, is

\[
\Theta_{\Gamma(\mathcal{C}_1, \mathcal{C}_2)}(z) = \text{jwe}_{\mathcal{C}_1, \mathcal{C}_2}(\vartheta_3(4z), \vartheta_2(4z), \vartheta_3(z)/2, \vartheta_2(z)/2).
\]

If we consider the \( \mathbb{Z}_4 \)-linear code \( \mathcal{C} \), the theta series of a Construction \( \Lambda_4 \) lattice can be expressed as follows.

**Corollary 2:** Let \( \mathcal{C} \) be a \( \mathbb{Z}_4 \)-linear code with \( \text{swe}_\mathcal{C}(a, b, c) \), then the theta series of \( \Lambda_4(\mathcal{C}) \) is

\[
\Theta_{\Lambda_4(\mathcal{C})}(z) = \text{swe}_\mathcal{C}(\vartheta_3(4z), \vartheta_2(z)/2, \vartheta_2(4z)).
\]

**Proof:** If the \( \mathbb{Z}_4 \)-linear code \( \mathcal{C} \) is such that \( \mathcal{C} = \mathcal{C}_1 + 2\mathcal{C}_2 \), the result comes immediately from Corollary 1, since \( \text{swe}_\mathcal{C}(a, b, c) = \text{jwe}_{\mathcal{C}_1, \mathcal{C}_2}(a, c, b, b) \). For a general \( \mathbb{Z}_4 \)-linear code \( \mathcal{C} \), the same proof of Theorem 1 can be applied, since \( \Lambda_4(\mathcal{C}) = \mathcal{C} + 4\mathbb{Z}^n \) is also a periodic packing and the coordinates of an element in \( \Lambda_4(\mathcal{C}) \) are also described as in (11). The only difference is that the exponents in (12) are replaced by \( n_0(c), n_1(c) + n_3(c), \) and \( n_2(c) \) respectively, and the result follows.

We can conclude that the packing obtained from Construction \( \Lambda_4 \) via \( \mathcal{C}_1 + 2\mathcal{C}_2 \) over \( \mathbb{Z}_4 \) has exactly the same theta series as the packing constructed by the 2-level Construction C via \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \). Despite this equivalence, both results have their own importance, as Corollary 1 can be applied to any choices of \( \mathcal{C}_1, \mathcal{C}_2 \) and Corollary 2 is restricted to lattices.

In the end of this section, we highlight the following result.

**Corollary 3:** If \( \mathcal{C} \) is a formally self-dual \( \mathbb{Z}_4 \)-linear code, then \( \Lambda_4(\mathcal{C}) \) is formally unimodular.

This is a direct consequence of Corollary 2.

**IV. Secrecy Gain of Formally Unimodular Lattices**

**A. The Secrecy Function of a Lattice**

We start by the definition of secrecy gain [5].

**Definition 6 (Secrecy function and secrecy gain [5, Defs. 1 and 2]):** Let \( \Lambda \) be a lattice with volume \( \text{vol}(\Lambda) = \nu^n \). The secrecy function of \( \Lambda \) is defined by

\[
\Xi_\Lambda(\tau) \triangleq \frac{\Theta_{\nu \Xi_\Lambda}(i\tau)}{\Theta_{\Lambda}(i\tau)},
\]

for \( \tau \triangleq -iz > 0 \). The (strong) secrecy gain of a lattice is given by \( \xi_\Lambda \triangleq \sup_{\tau > 0} \Xi_\Lambda(\tau) \).
It was shown in [5] that the higher the secrecy gain of a lattice, the more security of the lattice wiretap code is. Hence, the objective here is to design a good lattice $\Lambda$ to achieve a high secrecy gain.

Under the design criterion of secrecy function, we summarize the following three important observations for the formally unimodular lattices [12].

1) The secrecy function of a formally unimodular lattice $\Lambda$ has exactly the same symmetry point at $\tau = 1$ as a unimodular or an isodual lattice, i.e., $\Xi_\Lambda(\tau) = \Xi_\Lambda(\frac{1}{\tau})$.

2) Similar to Belfiore and Solé's conjecture from [24], it is also conjectured that the secrecy function of a formally unimodular lattice $\Lambda$ achieves its maximum at $\tau = 1$, i.e., $\xi_\Lambda = \Xi_\Lambda(1)$.

3) It was demonstrated that formally unimodular lattices can outperform the secrecy gain of unimodular lattices. In particular, the unimodular and formally unimodular lattices constructed via Construction A are compared, and it indicates that formally unimodular lattices obtained from formally self-dual codes via Construction A always achieve better secrecy gains than the Construction A unimodular lattices obtained from self-dual codes (see [12, Tab. I] for details).

Using these observations, we next explore the secrecy gain of formally unimodular lattices obtained by Construction A4 from formally self-dual codes over $\mathbb{Z}_4$.

B. Secrecy Gain of Construction A4 Lattices obtained from Formally Self-Dual Codes over $\mathbb{Z}_4$

In this subsection, we derive a closed-form expression of the theta series of a Construction A4 lattice obtained from a formally self-dual $\mathbb{Z}_4$-linear code. Let $C$ be a $\mathbb{Z}_4$-linear code. From (5) and the following identities from [20, Eq. (23), Ch. 4], and [20, Eq. (31), Ch. 4], respectively,

$$\vartheta_{3}(z) + \vartheta_{4}(z) = 2\vartheta_{3}(4z), \vartheta_{3}(z) - \vartheta_{4}(z) = 2\vartheta_{2}(2z), (13)$$

$$\vartheta_{41}(z) + \vartheta_{41}(z) = \vartheta_{31}(z),$$

we obtain

$$\Theta_{\Lambda_{44}}(\vartheta)(z) = \text{swe}_{\vartheta}(\vartheta_{3}(4z), \vartheta_{3}(z)/2, \vartheta_{2}(2z)),$$

where (15) holds since if $C$ is formally self-dual, $|C^\perp| = 4^{n/2}$. Now we are able to state the following main theorem.

**Theorem 2:** Let $C$ be a formally self-dual code over $\mathbb{Z}_4$. Then

$$\left[\Xi_{\Lambda_{44}}(\vartheta)(\tau)\right]^{-1} = \frac{\text{swe}_{\vartheta}(1 + t, \sqrt{1 - t^2}, 1 - t)}{2^n},$$

where $0 < t(\tau) \neq \vartheta_{4}(\tau)/\vartheta_{3}(\tau) < 1$. Moreover, define $h_{\vartheta}(t) \triangleq \text{swe}_{\vartheta}(1 + t, \sqrt{1 - t^2}, 1 - t)$ for $0 < t < 1$. Then, maximizing the secrecy function $\Xi_{\Lambda_{44}}(\vartheta)(\tau)$ is equivalent to determining the minimum of $h_{\vartheta}(t)$ on $t \in (0, 1)$.

**Example 2:** Consider two formally self-dual codes over $\mathbb{Z}_4$ in dimension 8. The first one is the octacode $O_8$, and its swe $\vartheta_{O_8}$ is presented in [15, Ex. 14.3]. Also, consider the formally self-dual code $\vartheta_{S_8}$ from [17, pp. 83–84], one can obtain

$$\vartheta_{\vartheta_{S_8}}(a, b, c) = e^8 + 64b^8 + 12a^8b^4 + 64a^4b^8 + 16a^2c^8 + 40a^3b^2c^3 + 30a^4c^4 + 12a^5b^2c + 16a^6c^2 + d^8.$$

We have that $h_{\vartheta_{S_8}}(t) = 256(1 - t^4 + t^8)$ and $h_{\vartheta_{O_8}}(t) = 256(1 - 4t^4 + 8t^8)$. Then we solve $h_{\vartheta_{S_8}}'(t) = 0$, we get as unique solution in the interval $t \in (0, 1), t = \sqrt[4]{\frac{1}{3}}$. For $t \in (0, 1/\sqrt[4]{3})$, $h_{\vartheta_{S_8}}(t) > 0$ and for $t \in (1/\sqrt[4]{3}, 1)$, $h_{\vartheta_{S_8}}(t) > 0$, meaning that $t = 1/\sqrt[4]{3}$ is a minimum, as we wanted. Therefore, $\xi_{\Lambda_{44}}(\vartheta_{S_8}) \approx 1.333$, which coincides with the best known secrecy gain up to now in this dimension.

Proceeding analogously for $C$, we have $h_{\vartheta_{C}}(t) = 64(2t^8 + t^6 + (\sqrt{1 - t^2} + 2)t^4 - 2(\sqrt{1 - t^2} - 1)t^2 + 2(\sqrt{1 - t^2} + 1))$ and $h_{\vartheta_{C}}(t) = 0$ for $t = 1/\sqrt[4]{3}$, which is also a minimum. For this code, $\xi_{\Lambda_{44}}(\vartheta_{C}) \approx 1.282 < \xi_{\Lambda_{44}}(\vartheta_{S_8})$.

**Example 3:** Gulliver and Harada presented in [16] optimal formally self-dual codes over $\mathbb{Z}_4$ in dimensions 6, 8, 10 and 14, together with their swes. Each $h_{\vartheta_{C}}(t), t = 6, 8, 10, 14$ achieves its minimum at $t = 1/\sqrt[4]{3}$. Therefore, we have $\xi_{\Lambda_{44}}(\vartheta_{C}) \approx 1.172, \xi_{\Lambda_{44}}(\vartheta_{C}_{10}) \approx 1.333, \xi_{\Lambda_{44}}(\vartheta_{C}_{14}) \approx 1.379$, and $\xi_{\Lambda_{44}}(\vartheta_{C}_{16}) \approx 1.871$, which coincide or are very close to best secrecy gains from [12, Tab. I].

**Example 4:** In this example we consider the swe of the isodual code $D_{4.22}$ presented in [25, p. 230, Prop. 4.2]. Using Theorem 2, we get $\xi_{\Lambda_{44}}(\vartheta_{D_{4.22}}) \approx 3.403$. We remark that the best known secrecy gain for formally unimodular lattice in this dimension is 3.34, which is presented in [12, Tab. I].

C. Secrecy Gain of Construction A4 Lattices obtained from $\mathcal{C}_1 + 2\mathcal{C}_2$ $\mathbb{Z}_4$-Linear Codes

We review an important class of binary Reed-Muller codes in coding theory.

**Definition 7 (Reed-Muller codes [19, Ch. 13]):** For a given $m \in \mathbb{N}$, the $r$-th order binary Reed-Muller code $\mathcal{R}(r, m)$ is a linear $[n = 2^m, k = \sum_{t=0}^{r} (\binom{m}{t})]$ code for $r \in [0 : m]$, constructed as the vector space spanned by the set of all $m$-variable Boolean monomials of degree at most $r$. 
Reed-Muller codes have interesting properties, such as being nested. In order to get Z₄-linear codes from pairs of Reed-Muller binary codes, we still need to guarantee that the chain is closed under the element-wise product, which is true for the chains described next.

A result connecting the construction of Z₄-linear codes and Reed-Muller chains is the following.

**Proposition 3** ([15, Ex. 12.8]): The Z₄-linear code \( \overline{\mathcal{C}}_{2m} \) induces a unimodular lattice \( \Lambda_{\mathcal{C}_{2m}}(\overline{\mathcal{C}}_{2m}) = \frac{1}{2} (\overline{\mathcal{C}}_{2m} + 4Z^{2m}) \).

**Example 5:** Proposition 3 gives an even unimodular lattice in dimension 16 obtained via \( \Lambda_{\mathcal{C}_{16}}(\overline{\mathcal{C}}_{16}) \), where \( \overline{\mathcal{C}}_{16} = \mathcal{C}(1, 4) + 2 \mathcal{C}(2, 4) \) is isomorphic to \( Z_8 \times Z_8 \). For such lattice, \( \xi_{\Lambda_{\mathcal{C}_{16}}(\overline{\mathcal{C}}_{16})} \approx 1.778 < 2.141 \), see Table I. If one considers \( \overline{\mathcal{C}}_{32} = \mathcal{C}(1, 5) + 2 \mathcal{C}(3, 5) \), then BW \( \overline{\mathcal{C}}_{32} = \sqrt{2} \Lambda_{\mathcal{C}_{32}}(\overline{\mathcal{C}}_{32}) = \frac{\sqrt{2}}{2} (\overline{\mathcal{C}}_{32} + 4Z^{32}) \) is an unimodular lattice in dimension 32.

The code \( \overline{\mathcal{C}}_{32} \) as in (16) is self-dual. Hence, Theorem 2 can be applied and we get \( \xi_{\Lambda_{\mathcal{C}_{32}}(\overline{\mathcal{C}}_{32})} \approx 7.11 \), which is the best known secrecy gain up to now [5, p. 5698].

**V. FORMALLY SELF-DUAL Z₄-LINEAR CODES FROM NESTED BINARY CODES**

In this section, we present now a novel construction of formally self-dual codes over Z₄ and investigate the corresponding lattice properties via Construction A₄.

First, we are going to investigate how to generate formally self-dual codes over Z₄, which by itself is an interesting research topic [16], [17].

The dual of a Z₄-linear code \( \mathcal{C} = \mathcal{C}_1 + 2 \mathcal{C}_2 \) is as follows.

**Lemma 1:** Let \( \mathcal{C} = \mathcal{C}_1 + 2 \mathcal{C}_2 \) be a Z₄-linear code. Then, \( \mathcal{C}^\perp = \mathcal{C}_2^\perp + 2 \mathcal{C}_1^\perp \).

**Proof:** First, we notice that \( \mathcal{C}_2^\perp \subseteq \mathcal{C}_1 \), which comes from the fact that \( \mathcal{C} = \mathcal{C}_1 + 2 \mathcal{C}_2 \) is linear over Z₄, therefore \( \mathcal{C}_1 \subseteq \mathcal{C}_2 \) is closed under element-wise product.

We will demonstrate that \( \mathcal{C}_2^\perp + 2 \mathcal{C}_1^\perp \subseteq \mathcal{C}^\perp \). Consider an element \( \mathcal{C}_2 + 2 \mathcal{C}_1 + 2 \mathcal{C}_2 \in \mathcal{C}^\perp \), which by itself is an interesting fact: optimal \( d_{\text{Lee}}(\mathcal{C}) \) does not imply higher secrecy gain: For length 12, a Lee-optimal code has \( d_{\text{Lee}} = 6 \). One such code has secrecy gain \( \xi_{\Lambda_{\mathcal{C}_{12}}(\mathcal{C}_{12})} \approx 1.456 < 1.6 \), achieved by a \( d_{\text{Lee}} = 4 \) code.

As a result, we summarize the secrecy gains of some Construction A₄ lattices obtained from formally self-dual codes over Z₄ in Table I.
VI. CONCLUSION

In this work, we studied the secrecy gains of the Construction $A_4$ lattices from formally self-dual $Z_4$-linear codes. We showed that these Construction $A_4$ lattices are formally unimodular and presented a universal approach to determine their secrecy gains. We found that it is possible to obtain a better secrecy gain from Construction $A_4$ formally unimodular lattices than that from Construction $A$ formally unimodular lattices. Furthermore, a novel code construction of formally self-dual $Z_4$-linear codes is given.

APPENDIX A

PROOF OF THEOREM 3

We start the proof by using the following useful identities [19, Ch. 5, pp. 448]:

\[
W_{e_1}(x, y)W_{e_2}(z, t) = jw_{e_1, e_2}(xz, xt, yz, yt), \quad (18)
\]

\[
jwe_{e_1, e_2}(a, b, c, d) = jw_{e_2, e_1}(a, c, b, d). \quad (19)
\]

Observe that

\[
jwe_{e_1, e_2}(xz, xt, yz, yt) = W_{e_1}(x, y)W_{e_2}(z, t)
\]

\[
\overset{(i)}{=} \frac{1}{|e_1^+|} W_{e_1^+}(x + y, x - y) \frac{1}{|e_2^+|} W_{e_2^+}(z + t, z - t) \]

\[
\overset{(ii)}{=} \frac{1}{|e_1^+|} W_{e_2}(x + y, x - y) \frac{1}{|e_2^+|} W_{e_1}(z + t, z - t) \]

\[
\overset{(18)}{=} \frac{1}{|e_1^+| |e_2^+|} jwe_{e_1, e_2}(xz + xt + yz + yt, xz - xt + yz - yt, xz + xt - yz - yt, xz - xt - yz + yt) \]

\[
\overset{(19)}{=} \frac{1}{|e_1^+| |e_2^+|} jwe_{e_1, e_2}(xz + xt + yz + xt, xz + xt - yz - yt, xz - xt + yz - yt, xz - xt - yz + yt),
\]

where $(i)$ follows by the MacWilliams identity and $(ii)$ holds because $W_{e_2}(x, y) = W_{e_1^+}(x, y)$ and $W_{e_1}(z, t) = W_{e_2^+}(z, t)$. Thus, we have

\[
jwe_{e_1, e_2}(a, b, c, d)
\]

\[
= \frac{1}{|e_1^+| |e_2^+|} jwe_{e_1, e_2}(a + b + c + d, a + b - c - d, a - b + c - d, a - b - c + d). \quad (20)
\]

Now, from Proposition 1, Lemma 1, and the fact that $\text{swe}_{e_1}(a, b, c) = \text{swe}_{e_1^+ + 2e_2}(a, b, c) = jwe_{e_1, e_2}(a, c, b, c)$, we can further get

\[
\overset{(i)}{=} \frac{1}{|e_1^+| |e_2^+|} \text{swe}_{e_1^+ + 2e_2}(a + c + 2b, a - c, a + c - 2b)
\]

\[
= \frac{1}{|e_1^+| |e_2^+|} \text{swe}_{e_1^+ + 2e_2}(a + c + 2b, a - c, a + c - 2b). \quad (22)
\]

Therefore, by comparing (21) with (22), we obtain $\text{swe}_{e_1} = \text{swe}_{e_1^+}$. This completes the proof.

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