1. Introduction

For a graph $G$, the graph $R(G)$ of a graph $G$ is the graph obtained by adding a new vertex for each edge of $G$ and joining each new vertex to both end vertices of the corresponding edge. Let $I(G)$ be the set of newly added vertices, i.e., $I(G) = V(R(G)) \setminus V(G)$. The generalized $R$-vertex corona of $G$ and $H_i$, for $i = 1, 2, ..., n$, denoted by $R(G) \boxtimes_{n} H_i$, is the graph obtained from $R(G)$ and $H_i$ by joining the $i$-th vertex of $V(G)$ to every vertex in $H_i$. The generalized $R$-edge corona of $G$ and $H_i$ for $i = 1, 2, ..., m$, denoted by $R(G) \boxtimes_{m} H_i$, is the graph obtained from $R(G)$ and $H_i$ by joining the $i$-th vertex of $I(G)$ to every vertex in $H_i$. In this paper, we derive closed-form formulas for resistance distance and Kirchhoff index of $R(G) \boxtimes_{n} H_i$ and $R(G) \boxtimes_{m} H_i$ whenever $G$ and $H_i$ are arbitrary graphs. These results generalize the existing results.

1. Introduction

All graphs considered in this paper are simple and undirected. The resistance distance between vertices $u$ and $v$ of $G$ was defined by Klein and Randić [9] to be the effective resistance between nodes $u$ and $v$ as computed with Ohm’s law when all the edges of $G$ are considered to be unit resistors. The Kirchhoff index $Kf(G)$ was defined in [9] as $Kf(G) = \sum_{uv \in E(G)} r_{uv}(G)$, where $r_{uv}(G)$ denotes the resistance distance between $u$ and $v$ in $G$. These novel parameters are in fact intrinsic to the graph theory and has some nice properties and applications in chemistry. For the study of resistance distance and Kirchhoff index, one may be referred to the recent works ([2], [4], [5], [7]-[25]) and the references therein.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $d_i$ be the degree of vertex $i$ in $G$ and $D_G = \text{diag}(d_1, d_2, \ldots, d_{|V(G)|})$ the diagonal matrix with all vertex degrees of $G$ as its diagonal entries. For a graph $G$, let $A_G$ and $B_G$ denote the adjacency matrix and vertex-edge incidence matrix of $G$, respectively. The matrix $L_G = D_G - A_G$ is called the Laplacian matrix of $G$, where $D_G$ is the diagonal matrix of vertex degrees of $G$. We use $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{|G|}(G) = 0$ to denote the eigenvalues of $L_G$. For other undefined notations and terminology from graph theory, the readers may refer to [6] and the references therein.

In [14], Lu et.al generalized the corona operation and defined the generalized $R$-vertex corona. For a graph $G$, the graph $R(G)$ of a graph $G$ is the graph obtained by adding a new vertex for each edge of $G$ and
joining each new vertex to both end vertices of the corresponding edge. Let \( I(G) \) be the set of newly added vertices, i.e \( I(G) = V(R(G)) \setminus V(G) \).

**Definition 1.1** ([14]) The generalized \( R \)-vertex corona of \( G \) and \( H_i \) for \( i = 1, 2, \ldots, n \), denoted by \( R(G) \oplus \wedge^n_{i=1} H_i \), is the graph obtained from \( R(G) \) and \( H_i \) by joining the \( i \)th vertex of \( V(G) \) to every vertex in \( H_i \).

**Definition 1.2** The generalized \( R \)-edge corona of \( G \) and \( H_i \) for \( i = 1, 2, \ldots, m \), denoted by \( R(G) \oplus \wedge^m_{i=1} H_i \), is the graph obtained from \( R(G) \) and \( H_i \) by joining the \( i \)th vertex of \( I(G) \) to every vertex in \( H_i \).

Bu et al. investigated resistance distance in subdivision-vertex join and subdivision-edge join of graphs [2]. Liu et al. [12] gave the resistance distance and Kirchhoff index of \( R \)-vertex join and \( R \)-edge join of two graphs. In [11], the resistance distance of subdivision-vertex and subdivision-edge corona are obtained. Motivated by the results, in this paper we consider the generalization of the \( R \)-vertex corona and the \( R \)-edge corona to the case of \( n(m) \) different graphs and we obtain the resistances distance and the Kirchhoff index in terms of the corresponding parameters of the factors. These results generalize the existing results in [13].

2. Preliminaries

The \( [1] \)-inverse of \( M \) is a matrix \( X \) such that \( MXM = M \). If \( M \) is singular, then it has infinite \([1]\)-inverse [1]. For a square matrix \( M \), the group inverse of \( M \), denoted by \( M^g \), is the unique matrix \( X \) such that \( MXM = M, XMX = X \) and \( MX = XM \). It is known that \( M^g \) exists if and only if \( \text{rank}(M) = \text{rank}(M^2) \) ([1],[3]). If \( M \) is real symmetric, then \( M^g \) exists and \( M^g \) is a symmetric \([1]\)-inverse of \( M \). Actually, \( M^g \) is equal to the Moore-Penrose inverse of \( M \) since \( M \) is symmetric [3].

It is known that resistance distances in a connected graph \( G \) can be obtained from any \([1]\)-inverse of \( G \) ([4]). We use \( M^{[1]} \) to denote any \([1]\)-inverse of a matrix \( M \), and let \( (M)_{uv} \) denote the \((u,v)\)-entry of \( M \).

**Lemma 2.1** ([3]) Let \( G \) be a connected graph. Then
\[
r_{uv}(G) = (L_G^{[1]})_{uv} + (L_G^{[1]})_{vu} - (L_G^{[1]})_{uw} - (L_G^{[1]})_{vw} = (L_G^s)_{uv} + (L_G^s)_{vu} - 2(L_G^s)_{uv}.
\]

Let \( 1_n \) denotes the column vector of dimension \( n \) with all the entries equal one. We will often use \( 1 \) to denote an all-ones column vector if the dimension can be read from the context.

**Lemma 2.2** ([2]) For any graph \( G \), we have \( L^s_G 1 = 0 \).

**Lemma 2.3** ([24]) Let
\[
M = \begin{pmatrix}
A & B \\
C & D \\
\end{pmatrix}
\]
be a nonsingular matrix. If \( A \) and \( D \) are nonsingular, then
\[
M^{-1} = \begin{pmatrix}
A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\
-S^{-1}CA^{-1} & S^{-1} \\
\end{pmatrix}
\]
\[
= \begin{pmatrix}
(A - BD^{-1}C)^{-1} & -A^{-1}BS^{-1} \\
-S^{-1}CA^{-1} & S^{-1} \\
\end{pmatrix}
\]
where \( S = D - CA^{-1}B \).

For a square matrix \( M \), let \( \text{tr}(M) \) denote the trace of \( M \).

**Lemma 2.4** ([15]) Let \( G \) be a connected graph on \( n \) vertices. Then
\[
Kf(G) = n\text{tr}(L_G^{[1]}) - 1^T L_G^{[1]} 1 = ntr(L_G^s).
\]

**Lemma 2.5**([10]) Let \( G \) be a connected graph of order \( n \) with edge set \( E \). Then
\[
\sum_{uv \in E} r_{uv}(G) = n - 1.
\]
For a vertex \( i \) of a graph \( G \), let \( T(i) \) denote the set of all neighbors of \( i \) in \( G \).
Lemma 2.6([2]) Let $G$ be a connected graph. For any $i, j \in V(G)$,

$$r_{ij}(G) = d_i^{-1}(1 + \sum_{k \in T(i)} r_{ki}(G) - d_i^{-1} \sum_{k \in T(i)} r_{kl}(G)).$$

Lemma 2.7 ([12]) Let $G$ be a graph of order $n$. For any $a, b > 0$ satisfying $b \neq a$, we have

$$(L_G + aI_n - \frac{a}{b}J_{n \times n})^{-1} = (L_G + aI_n)^{-1} + \frac{1}{a(b - n)}J_{n \times n},$$

where $J_{n \times n}$ denotes the $n \times n$ matrix with all entries equal to one.

Lemma 2.8 ([13]) Let

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

be a symmetric block matrix. If $D$ is nonsingular, then

$$X = \begin{pmatrix} H^* & -H^*BD^{-1} \\ -D^{-1}B^TH^* & D^{-1} + D^{-1}B^TH^*BD^{-1} \end{pmatrix}$$

is a symmetric $[1]$-inverse of $L$, where $H = A - BD^{-1}B^T$.

Lemma 2.9 ([9]) Let $k$ be a cut-vertex of a graph, and let $i$ and $j$ be vertices occurring in different components which arise upon deletion of $k$. Then $r_{ij} = r_{ik} + r_{kj}$.

3. The resistance distance and Kirchhoff index of $R(G) \boxplus \bigwedge_{i=1}^n H_i$

In this section, we focus on determining the resistance distance and Kirchhoff index of generalized $R$-vertex corona $R(G) \boxplus \bigwedge_{i=1}^n H_i$ whenever $G$ and $H_i(i = 1, \ldots, n)$ be an arbitrary graph.

Theorem 3.1 Let $G$ be a connected graph with $n$ vertices and $m$ edges and let $H_i$ be a graph with $t_i$ vertices for $i = 1, 2, \ldots, n$. Then $R(G) \boxplus \bigwedge_{i=1}^n H_i$ have the resistance distance and Kirchhoff index as follows:

(i) For any $i, j \in V(G)$, we have

$$r_{ij}(R(G) \boxplus \bigwedge_{i=1}^n H_i) = \frac{2}{3}(L_G^*h_i + \frac{2}{3}L_G^*i_j - \frac{2}{3}L_G^*i_j) = r_{ij}(G).$$

(ii) For any $i, j \in V(H_k)(k = 1, 2, \ldots, n)$, we have

$$r_{ij}(R(G) \boxplus \bigwedge_{i=1}^n H_i) = ((L_{H_k} + I_{h_k})^{-1}_{ii} + ((L_{H_k} + I_{h_k})^{-1}_{ij} - 2((L_{H_k} + I_{h_k})^{-1}_{ij}).$$

(iii) For any $i, j \in V(G)$, we have

$$r_{ij}(R(G) \boxplus \bigwedge_{i=1}^n H_i) = \frac{2}{3}r_{ij}(G).$$

(iv) For any $i \in V(G)$, $j \in V(H_k)(k = 1, 2, \ldots, n)$, we have

$$r_{ij}(R(G) \boxplus \bigwedge_{i=1}^n H_i) = r_{ij}(R(G)) + r_{ij}(F_k),$$

where $F_k = H_k \lor [v]$, i.e, $F_k$ is the graph obtained by adding new edges from an isolated vertex $v$ to every vertex of $H_k$.

(v) For any $i \in V(H_k)$, $j \in V(H_k)$, we have

$$r_{ij}(R(G) \boxplus \bigwedge_{i=1}^n H_i) = r_{ij}(R(G)) + r_{ij}(F_k) + r_{ij}(F_k),$$

where $F_k = H_k \lor [v]$, i.e, $F_k$ is the graph obtained by adding new edges from an isolated vertex $v$ to every vertex of $H_k$. 

(vi) $Kf(R(G) \Box \Lambda_{i=1}^n H_i)$

\[
= (n + 2m + \sum_{i=1}^{n} t_i) \left( \frac{2}{3n} Kf(G) + \frac{m}{2} + \frac{1}{2} \text{tr}(D_G L_G^\delta) - \frac{n-1}{4} + \sum_{i=1}^{n} \frac{1}{\mu_i(H_i) + 1} \right) + 2\text{tr}(Q^T L_G^\delta Q) - \left( \frac{m}{2} + \frac{1}{4} \pi^T L_G^\delta \pi + \pi^T L_G^\delta \delta + \sum_{i=1}^{n} t_i + \delta^T L_G^\delta \delta \right),
\]

where $Q$ equals (1), $\pi^T = (d_1, d_2, ..., d_n)$, $\delta^T = (t_1, t_2, ..., t_n)$.

**Proof** Let $R(G)$ and $D_G$ be the incidence matrix and degree matrix of $G$. With a suitable labeling for vertices of $R(G) \Box \Lambda_{i=1}^n H_i$, the Laplacian matrix of $R(G) \Box \Lambda_{i=1}^n H_i$ can be written as follows:

\[
L_{R(G) \Box \Lambda_{i=1}^n H_i} = \begin{pmatrix}
  P + L_G & -R(G) & -Q \\
  -R^T(G) & 2I_m & 0 \\
  -Q^T & 0 & T
\end{pmatrix},
\]

where

\[
P = \begin{pmatrix}
  d_1 + t_1 & 0 & 0 & ... & 0 \\
  0 & d_2 + t_2 & 0 & ... & 0 \\
  0 & 0 & ... & ... & 0 \\
  0 & 0 & 0 & ... & d_n + t_n
\end{pmatrix}, \quad Q = \begin{pmatrix}
  1^T & 0 & 0 & ... & 0 \\
  0 & 1^T & 0 & ... & 0 \\
  0 & 0 & ... & ... & 0 \\
  0 & 0 & 0 & ... & 1^T
\end{pmatrix}, \quad T = \begin{pmatrix}
  L_{H_1} + I_{t_1} & 0 & 0 & ... & 0 \\
  0 & L_{H_2} + I_{t_2} & 0 & ... & 0 \\
  0 & 0 & ... & ... & 0 \\
  0 & 0 & 0 & ... & L_{H_n} + I_{t_n}
\end{pmatrix}.
\]

First we begin with the computation of (1)-inverse of $R(G) \Box \Lambda_{i=1}^n H_i$.

By Lemma 2.8, we have

\[
H = L_G + P - \left( -R(G) - Q \right) \begin{pmatrix}
  \frac{1}{2} I_m & 0 \\
  0 & T^{-1}
\end{pmatrix} \begin{pmatrix}
  -R^T(G) \\
  -Q^T
\end{pmatrix}
\]

\[
= L_G + P - \left( -\frac{1}{2} R(G) - QT^{-1} \right) \begin{pmatrix}
  -R^T(G) \\
  -Q^T
\end{pmatrix}
\]

\[
= L_G + D_G + \left( \begin{array}{cccc}
  t_1 & 0 & 0 & ... & 0 \\
  0 & t_2 & 0 & ... & 0 \\
  0 & 0 & ... & ... & 0 \\
  0 & 0 & 0 & ... & t_n
\end{array} \right) \left( \begin{array}{cccc}
  1 & 0 & 0 & ... & 0 \\
  0 & 1 & 0 & ... & 0 \\
  0 & 0 & ... & ... & 0 \\
  0 & 0 & 0 & ... & 1
\end{array} \right)
\]

\[
= \frac{3}{2} L_G,
\]

so $H^* = \frac{3}{2} L_G^\delta$.

According to Lemma 2.8, we calculate $-H^* B D^{-1}$ and $-D^{-1} B^T H^*$.

\[
-H^* B D^{-1} = -\frac{3}{2} L_G^\delta \left( -R(G) - Q \right) \begin{pmatrix}
  \frac{1}{2} I_m & 0 \\
  0 & T^{-1}
\end{pmatrix}
\]

\[
= -\frac{3}{2} L_G^\delta \left( -\frac{1}{2} R(G) - QT^{-1} \right) \begin{pmatrix}
  -\frac{1}{2} R(G) & \frac{3}{2} L_G R(G) & \frac{3}{2} L_G Q
\end{pmatrix}
\]

and

\[
-D^{-1} B^T H^* = -(H^* B D^{-1})^T = \begin{pmatrix}
  \frac{3}{2} R^T(G) L_G^\delta \\
  \frac{3}{2} Q^T L_G^\delta
\end{pmatrix}.
\]
We are ready to compute the $D^{-1}B^TH^*BD^{-1}$.

$$D^{-1}B^TH^*BD^{-1} = \left( \frac{1}{2}I_m 0 \ T^{-1} \right) \left( \begin{array}{cc} -R^T(G) & -Q^T \\ -Q & -Q \end{array} \right) \left( \frac{1}{2}I_m 0 \ T^{-1} \right)$$

$$= \left( \begin{array}{c} \frac{1}{2}R^T(G)L^*_G R(G) \\ \frac{1}{2}Q^T L^*_G R(G) \end{array} \right) \left( \begin{array}{c} \frac{1}{2}R^T(G)L^*_G R(G) \\ \frac{1}{2}Q^T L^*_G R(G) \end{array} \right).$$

Based on Lemma 2.8, the following matrix

$$N = \left( \begin{array}{ccc} \frac{2}{3}L^*_G & \frac{1}{2}L^*_G R(G) & \frac{2}{3}L^*_G \\ \frac{1}{2}L^*_G R(G) & \frac{1}{2}R^T(G)L^*_G R(G) & \frac{1}{2}R^T(G)L^*_G Q \\ \frac{1}{2}Q^T L^*_G R(G) & \frac{1}{2}Q^T L^*_G R(G) & T^{-1} + Q^T L^*_G Q \end{array} \right)$$

is a symmetric [1]- inverse of $L_{R(G)\triangle^* H}$. For any $i, j \in V(G)$, by Lemma 2.1 and the Equation (2), we have

$$r_{ij}(R(G) \triangle^* H) = 2 \left( \frac{1}{3}L^*_G \right)_{ij} + \frac{1}{3}L^*_G (L^*_G)_{ij} - \frac{1}{3}L^*_G (L^*_G)_{ij} = \frac{2}{3}r_{ij}(G),$$

as stated in (i).

For any $i, j \in V(H_k)(k = 1, 2, ..., n)$, by Lemma 2.1 and the Equation (2), we have

$$r_{ij}(R(G) \triangle^* H) = ((L_{H_k} + I_k)^{-1})_{ij} + ((L_{H_k} + I_k)^{-1})_{ij} - 2((L_{H_k} + I_k)^{-1})_{ij},$$

as stated in (ii).

From the left side of above equation, we can obviously have

$$r_{ij}(F_k) = ((L_{H_k} + I_k)^{-1})_{ij} + ((L_{H_k} + I_k)^{-1})_{ij} - 2((L_{H_k} + I_k)^{-1})_{ij},$$

where $F_k = H_k \cup \{v\}$, i.e., $F_k$ is the graph obtained by adding new edges from an isolated vertex $v$ to every vertex of $H_k$.

For any $i, j \in R(G)$, by Lemma 2.1 and the Equation (2), we have

$$r_{ij}(R(G) \triangle^* H) = r_{ij}(R(G)).$$

By Lemma 3.1 in [7], $r_{ij}(R(G)) = \frac{2}{3}r_{ij}(G)$, so $r_{ij}(R(G) \triangle^* H) = \frac{2}{3}r_{ij}(G)$, as stated in (iii).

For any $i \in V(G), j \in V(H_k)(k = 1, 2, ..., n)$, since $i$ and $j$ belong to different components, then by Lemma 2.9, we have

$$r_{ij}(R(G) \triangle^* H) = r_{i\alpha}(R(G)) + r_{\beta j}(F_k),$$

as stated in (iv).

For any $i \in V(H_k), j \in V(H_j)$, by Lemma 2.9, we have

$$r_{ij}(R(G) \triangle^* H) = r_{i\alpha}(R(G)) + r_{\beta j}(F_k) + r_{\beta j}(F_k),$$

as stated in (v).

By Lemma 2.4, we have

$$Kf(R(G) \triangle^* H) = (n + m + \sum_{i=1}^{n} t_i)tr(N) - 1^T N 1^T$$

$$= (n + m + \sum_{i=1}^{n} t_i) \left( \frac{2}{3}tr(L^*_G) + tr \left( \frac{1}{2}I_m + \frac{1}{4}R^T(G)L^*_G R(G) \right) + 
+ tr \left( T^{-1} + Q^T L^*_G Q \right) \right) - 1^T N 1^T$$

$$= (n + m + \sum_{i=1}^{n} t_i) \left( \frac{2}{3}Kf(G) + \frac{m}{2} + \frac{1}{4} \sum_{i=1,j=1}^{n} [(L^*_G)_{ii} + (L^*_G)_{jj} + 2(L^*_G)]_{ij} + 
+ tr \left( T^{-1} + Q^T L^*_G Q \right) \right) - 1^T N 1^T.$$
By Lemma 2.4, we get
\[
Kf(R(G) \square \wedge_{i=1}^n H_i) = (n + m + \sum_{i=1}^n l_i) \left( \frac{2}{3n} Kf(G) + \frac{m}{2} + \frac{1}{4} \sum_{i,j \neq j \in E(G)} [2(L^g_G)_{ii} + 2(L^q_G)_{jj}] \right)
- N_1 + tr(T^{-1} + QT^T L^q_G) - 1^T N_1^T
\]
\[
= (n + m + \sum_{i=1}^n l_i) \left( \frac{2}{3n} Kf(G) + \frac{m}{2} + \frac{1}{4} tr(DG) - \frac{n - 1}{4} \right)
+ tr(T^{-1} + QT^T L^q_G) - 1^T N_1^T.
\]

Note that the eigenvalues of \((L(H_i) + l_i) (i = 1, 2, ..., n)\) are \(\mu_1(H_i) + 1, \mu_2(H_i) + 1, ..., \mu_n(H_i) + 1\). Then
\[
tr(T^{-1}) = \sum_{i=1}^n \sum_{j=1}^{l_i} \frac{1}{\mu_j(H_i)} + 1.
\] (3)

By Lemma 2.2, \(L^q_G \mathbf{1} = 0\) and \((1^T (R(T(G) L^q_G) Q T(G)) 1)^T = 1^T (Q^T L^q_G R(G)) 1\), then
\[
1^T N_1 = \frac{m}{2} + \frac{1}{4} 1^T (R(T(G) L^q_G R(G)) \mathbf{1} + 1^T (R(T(G) L^q_G Q T(G) 1)
+ 1^T T^{-1} 1 + 1^T (Q^T L^q_G Q 1).
\]

Note that \(R(G) \mathbf{1} = \pi\), where \(\pi^T = (d_1, d_2, ..., d_n)\), then \(1^T (R(T(G) L^q_G R(G)) \mathbf{1} = \pi^T L^q_G \pi\), so
\[
1^T N_1 = \frac{m}{2} + \frac{1}{4} \pi^T L^q_G \pi + \pi^T L^q_G Q \mathbf{1} + 1^T T^{-1} 1 + 1^T (Q^T L^q_G Q 1).
\] (4)

Let \(R_i = L(H_i) + l_i (i = 1, 2, ..., n)\), then
\[
1^T T^{-1} 1^T = \left( \begin{array}{cccc} 1^T & 1^T & \cdots & 1^T \\ 1^T & 1^T & \cdots & 1^T \\ \vdots & \vdots & \ddots & \vdots \\ 1^T & 1^T & \cdots & 1^T \\ \end{array} \right) \left( \begin{array}{cccc} R^{-1} & 0 & 0 & \cdots & 0 \\ 0 & R^{-1} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & R^{-1} \\ \end{array} \right) \left( \begin{array}{c} 1_{t_1} \\ 1_{t_2} \\ \vdots \\ 1_{t_n} \\ \end{array} \right).
\]
\[
= \sum_{i=1}^n 1^T (L(H_i) + l_i) \mathbf{1} = \sum_{i=1}^n l_i,
\] (5)

and
\[
1^T Q^T = \left( \begin{array}{cccc} 1^T & 1^T & \cdots & 1^T \\ 1^T & 1^T & \cdots & 1^T \\ \vdots & \vdots & \ddots & \vdots \\ 1^T & 1^T & \cdots & 1^T \\ \end{array} \right) \left( \begin{array}{cccc} 1_{t_1} & 0 & 0 & \cdots & 0 \\ 0 & 1_{t_2} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1_{t_n} \\ \end{array} \right).
\]
\[
= (t_1, t_2, ..., t_n) = \delta^T.
\] (6)

Plugging (3), (4), (5) and (6) into \(Kf(R(G) \square \wedge_{i=1}^n H_i)\), we obtain the required result in (vi).
4. The resistance distance and Kirchhoff index of \( R(G) \otimes \Lambda_{i=1}^m H_i \)

In this section, we focus on determining the resistance distance and Kirchhoff index of generalized R-edge corona \( R(G) \otimes \Lambda_{i=1}^m H_i \) whenever \( G \) and \( H_i \) for \( i = 1, 2, \ldots, n \) be an arbitrary graph.

**Theorem 4.1** Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges. Let \( H_i \) be a graph with \( t_i \) vertices for \( i = 1, 2, \ldots, m \). Then \( R(G) \otimes \Lambda_{i=1}^m H_i \) have the resistance distance and Kirchhoff index as follows:

(i) For any \( i, j \in V(G) \), we have

\[
\begin{align*}
    r_{ij}(R(G) \otimes \Lambda_{i=1}^m H_i) &= 2 \left( L^*_G \right)_{ij} + 2 \left( L^*_G \right)_{ij} - \frac{2}{3} \left( L^*_G \right)_{ij} = 2 \frac{2}{3} r_{ij}(G).
\end{align*}
\]

(ii) For any \( i, j \in V(H_k)(k = 1, 2, \ldots, m) \), we have

\[
\begin{align*}
    r_{ij}(R(G) \otimes \Lambda_{i=1}^m H_i) &= (L_{H_k} + I_{k}) - \left( \frac{1}{2} + t_k \right) j_{k}^{-1} + \left( L_{H_k} + I_{k} - \frac{1}{2} + t_k \right) \left( j_{k}^{-1} \right) - 2(L_{H_k} + I_{k} - \frac{1}{2} + t_k) j_{k}^{-1}.
\end{align*}
\]

(iii) For any \( i, j \in V(G) \), we have

\[
\begin{align*}
    r_{ij}(R(G) \otimes \Lambda_{i=1}^m H_i) &= \frac{2}{3} r_{ij}(G).
\end{align*}
\]

(iv) For any \( i \in V(G), j \in V(H_k)(k = 1, 2, \ldots, n) \), we have

\[
\begin{align*}
    r_{ij}(R(G) \otimes \Lambda_{i=1}^m H_i) &= r_{ij}(R(G)) + r_{ij}(F_k),
\end{align*}
\]

where \( F_k = H_k \cup \{v\} \), i.e., \( F_k \) is the graph obtained by adding new edges from an isolated vertex \( v \) to every vertex of \( H_k \).

(v) For any \( i \in V(H_k), j \in V(H_i) \), we have

\[
\begin{align*}
    r_{ij}(R(G) \otimes \Lambda_{i=1}^m H_i) &= r_{ij}(R(G)) + r_{ij}(F_k) + r_{ij}(F_i),
\end{align*}
\]

where \( F_k = H_k \cup \{v\} \), i.e., \( F_k \) is the graph obtained by adding new edges from an isolated vertex \( v \) to every vertex of \( H_k \).

(vi) \( Kf(R(G) \otimes \Lambda_{i=1}^m H_i) \)

\[
\begin{align*}
    Kf(R(G) \otimes \Lambda_{i=1}^m H_i) &= (n + 2m) + \sum_{i=1}^{n} t_i \left( \frac{2}{3n} Kf(G) + \frac{m}{2} + \frac{1}{3} \text{tr}(D_G L^*_G) - \frac{n - 1}{2} + \sum_{i=1}^{m} \frac{1}{\mu_i(H_i)} + 1 \right) \\
    &+ 2 \left( F^T R^T(G) L^*_G R(G) F \right) - \left( \frac{m}{2} + \frac{1}{3} \pi^T L^*_G \pi + \sum_{i=1}^{m} t_i + \frac{2}{3} \pi^T(G) L^*_G R(G) \delta + \pi^T L^*_G \delta \right) \\
    &+ \frac{1}{2} \sum_{i=1}^{m} t_i (2 + t_i) + \frac{2}{3} \delta^T R^T(G) L^*_G R(G) \delta,
\end{align*}
\]

where \( F \) equals (7), \( \pi^T = (d_1, d_2, \ldots, d_m) \), \( \delta^T = (t_1, 0, \ldots, 0, t_2, 0, \ldots, 0, \ldots, t_m) \).

**Proof** Let \( R(G) \) and \( D_G \) be the incidence matrix and degree matrix of \( G \). With a suitable labeling for vertices of \( R(G) \otimes \Lambda_{i=1}^m H_i \), the Laplacian matrix of \( R(G) \otimes \Lambda_{i=1}^m H_i \) can be written as follows:

\[
\begin{align*}
    L_{R(G) \otimes \Lambda_{i=1}^m H_i} = \begin{pmatrix}
        L_G + D_G & -R(G) & 0 & 0 \\
        -R^T(G) & P & -M & 0 \\
        0 & -M^T & Q
    \end{pmatrix},
\end{align*}
\]
where
\[ P = \begin{pmatrix}
2 + t_1 & 0 & 0 & \ldots & 0 \\
0 & 2 + t_2 & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 2 + t_m \\
\end{pmatrix}_{m \times m}, \quad M = \begin{pmatrix}
1^T_{t_1} & 0 & 0 & \ldots & 0 \\
0 & 1^T_{t_2} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 1^T_{t_m} \\
\end{pmatrix}_{m \times (t_1 + t_2 + \ldots + t_m)} \]
\[ Q = \begin{pmatrix}
L_{H_1} + I_{t_1} & 0 & 0 & \ldots & 0 \\
0 & L_{H_2} + I_{t_2} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & L_{H_{m}} + I_{t_m} \\
\end{pmatrix}. \]

Let \( A = L_G + D_G, B = \left( -R(G) \quad 0 \right), B^T = \begin{pmatrix}
-R^T(G) \\
0
\end{pmatrix} \) and \( D = \begin{pmatrix}
P \\
-M^T \\
Q
\end{pmatrix}. \)

First, we will compute \( D^{-1} \). By Lemma 2.3, we have
\[ S = \begin{pmatrix}
L_{H_1} + I_{t_1} & 0 & 0 & \ldots & 0 \\
0 & L_{H_2} + I_{t_2} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 2 + t_m \\
\end{pmatrix}^{-1} + \begin{pmatrix}
1^T_{t_1} & 0 & 0 & \ldots & 0 \\
0 & 1^T_{t_2} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 1^T_{t_m} \\
\end{pmatrix}
\]
\[ S^{-1} \begin{pmatrix}
(L_{H_1} + I_{t_1} - \frac{1}{\Delta t_1} I_{t_1})^{-1} & 0 & 0 & \ldots & 0 \\
0 & (L_{H_2} + I_{t_2} - \frac{1}{\Delta t_2} I_{t_2})^{-1} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & (L_{H_{m}} + I_{t_m} - \frac{1}{\Delta t_{m}} I_{t_m})^{-1}
\end{pmatrix} = \begin{pmatrix}
P - MQ^{-1}M^T
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}I_{m^2}
\end{pmatrix}.
\]

According to Lemma 2.3, we have
\[ P - MQ^{-1}M^T = \begin{pmatrix}
2 + t_1 & 0 & 0 & \ldots & 0 \\
0 & 2 + t_2 & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 2 + t_m \\
\end{pmatrix} - \begin{pmatrix}
1^T_{t_1} & 0 & 0 & \ldots & 0 \\
0 & 1^T_{t_2} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 1^T_{t_m} \\
\end{pmatrix}
\]
\[ \begin{pmatrix}
(L_{H_1} + I_{t_1})^{-1} & 0 & 0 & \ldots & 0 \\
0 & (L_{H_2} + I_{t_2})^{-1} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & (L_{H_{m}} + I_{t_m})^{-1}
\end{pmatrix} = 2I_{m^2}.
\]

so \( (P - MQ^{-1}M^T)^{-1} = \frac{1}{2}I_{m^2} \).

By Lemma 2.3, we have
\[ -P^{-1}MS^{-1} = \begin{pmatrix}
\frac{1}{\Delta t_1} & 0 & 0 & \ldots & 0 \\
0 & \frac{1}{\Delta t_2} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & \frac{1}{\Delta t_{m}}
\end{pmatrix} \begin{pmatrix}
1^T_{t_1} & 0 & 0 & \ldots & 0 \\
0 & 1^T_{t_2} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & 1^T_{t_m}
\end{pmatrix}
\]
\[ \begin{pmatrix}
(L_{H_1} + I_{t_1} - \frac{1}{\Delta t_1} I_{t_1})^{-1} & 0 & 0 & \ldots & 0 \\
0 & (L_{H_2} + I_{t_2} - \frac{1}{\Delta t_2} I_{t_2})^{-1} & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & \ldots & (L_{H_{m}} + I_{t_m} - \frac{1}{\Delta t_{m}} I_{t_m})^{-1}
\end{pmatrix} = 2I_{m^2}.
\]
\[
F = \left( \begin{array}{cccc}
\frac{1}{2}I_{t_1}^T & 0 & 0 & \ldots & 0 \\
0 & \frac{1}{2}I_{t_1}^T & 0 & \ldots & 0 \\
0 & 0 & \ldots & \ldots & 0 \\
0 & 0 & 0 & \ldots & \frac{1}{2}I_{t_m}^T
\end{array} \right)
\]

Similarly, \(-S^{-1}MT^{-1} = N^T\), so \(D^{-1} = \left( \begin{array}{cc}
\frac{1}{2}I_m \\
F \end{array} \right)
\).

Next we begin with the computation of \([1]\)-inverse of \(L_{(G)\in\Lambda_{m}^nM_{i}}\).

By Lemma 2.8, we have

\[
H = L_G + D_G - \left( \begin{array}{cc}
R(G) & 0 \\
\frac{1}{2}R(G) & R(G)F
\end{array} \right) \left( \begin{array}{c}
R^T(G) \\
0
\end{array} \right)
\]

so \(H^* = \frac{3}{2}L_G^*\).

According to Lemma 2.8, we calculate \(-H^*BD^{-1}\) and \(-D^{-1}B^T H^*\).

\[
-H^*BD^{-1} = \frac{3}{2}L_G^* \left( \begin{array}{cc}
-R(G) & 0 \\
\frac{1}{2}R(G) & R(G)F
\end{array} \right) \left( \begin{array}{c}
\frac{1}{2}I_m \\
F \end{array} \right) S^{-1}
\]

and

\[
-D^{-1}B^T H^* = -(H^*BD^{-1})^T = \left( \begin{array}{c}
\frac{1}{2}R^T(G)L_G^* \\
\frac{3}{2}F^T R^T(G)L_G^* \\
\frac{3}{2}F^T R^T(G)L_G^* R(G)F
\end{array} \right)
\]

We are ready to compute the \(D^{-1}B^T H^*BD^{-1}\).

\[
D^{-1}B^T H^*BD^{-1} = \frac{2}{3} \left( \begin{array}{cc}
\frac{1}{2}I_m \\
F \end{array} \right) S^{-1} \left( \begin{array}{c}
-\frac{3}{2}R^T(G) \\
0
\end{array} \right) L_G^* \left( \begin{array}{c}
-R(G) \\
0
\end{array} \right) \left( \begin{array}{c}
\frac{1}{2}I_m \\
F \end{array} \right) S^{-1}
\]

Based on Lemma 2.3 and 2.8, the following matrix

\[
N = \left( \begin{array}{cccc}
\frac{2}{3}L_G^* R(G) & \frac{1}{2}R^T(G)L_G^* R(G) & \frac{3}{2}L_G^* R(G)F \\
\frac{1}{2}R^T(G)L_G^* & \frac{1}{2}I_m + \frac{1}{2}R^T(G)L_G^* R(G) & F + \frac{1}{2}R^T(G)L_G^* R(G)F \\
\frac{1}{2}F^T R^T(G)L_G^* & \frac{1}{2}F^T R^T(G)L_G^* R(G) & S^{-1} + \frac{3}{2}F^T R^T(G)L_G^* R(G)F
\end{array} \right)
\]

is a symmetric \([1]\)-inverse of \(L_{(G)\in\Lambda_{m}^nM_{i}}\).

For any \(i, j \in V(G)\), by Lemma 2.1 and the Equation (8), we have

\[
r_{ij}(R(G) \ominus \Lambda_{i=1}^n H_i) = \frac{2}{3}(L_G^*)_{ij} + \frac{1}{3}(L_G^*)_{ij} - \frac{4}{3}(l_G^*)_{ij} = \frac{2}{3}r_{ij}(G),
\]

as stated in (i).

For any \(i, j \in V(H_k)(k = 1, 2, \ldots, m)\), by Lemma 2.1 and the Equation (8), we have

\[
r_{ij}(R(G) \ominus \Lambda_{i=1}^n H_i) = \left( L_{H_k} + I_k - \frac{1}{2 + t_k} j_{ij} \right)^{-1} + \left( L_{H_k} + I_k - \frac{1}{2 + t_k} j_{ij} \right)^{-1}
\]

as stated in (i).
as stated in (ii).

From the left side of above equation, we can obviously have
\[ r_{ij}(F_k) = ((L_{Rh} + I_h)^{-1})_{ii} + ((L_{Rh} + I_h)^{-1})_{jj} - 2((L_{Rh} + I_h)^{-1})_{ij}, \]
where \( F_k = H_k \cup \{v\} \), i.e, \( F_k \) is the graph obtained by adding new edges from an isolated vertex \( v \) to every vertex of \( H_k \).

For any \( i, j \in R(G) \), by Lemma 2.1 and the Equation (8), we have
\[ r_{ij}(R(G) \ominus \bigwedge_{i=1}^m H_i) = r_{ij}(R(G)). \]

By Lemma 3.1 in [7], \( r_{ij}(R(G)) = \frac{2}{3} r_{ij}(G) \), so \( r_{ij}(R(G) \ominus \bigwedge_{i=1}^m H_i) = \frac{2}{3} r_{ij}(G) \), as stated in (iii).

For any \( i \in V(G) \), \( j \in V(H_k)(k = 1, 2, ..., m) \), since \( i \) and \( j \) belong to different components, then by Lemma 2.9, we have
\[ r_{ij}(R(G) \ominus \bigwedge_{i=1}^m H_i) = r_{ik}(R(G)) + r_{jl}(F_k), \]
as stated in (iv).

For any \( i \in V(H_k) \), \( j \in V(H_l) \), by Lemma 2.9, we have
\[ r_{ij}(R(G) \ominus \bigwedge_{i=1}^m H_i) = r_{ik}(R(G)) + r_{jl}(F_l), \]
as stated in (v).

By Lemma 2.4, we have
\[
Kf(R(G) \ominus \bigwedge_{i=1}^m H_i) = (n + m + \sum_{i=1}^m t_i)tr(N) - 1^T N 1
\]
\[ = (n + m + \sum_{i=1}^m t_i) \left( \frac{2}{3} \sum_{i,j \in E(G)} (L_{G}^*)_{ij} + \frac{1}{2} \sum_{i,j \in E(G)} \left[ (L_{G}^*)_{ii} + (L_{G}^*)_{jj} - 2(L_{G}^*)_{ij} \right] \right) + \frac{1}{6} \sum_{i,j \in E(G)} (L_{G}^*)_{ij} + 1 \]
\[ = (n + m + \sum_{i=1}^m t_i) \left( \frac{2}{3} Kf(G) + \frac{m}{2} + \frac{1}{6} \sum_{i,j \in E(G)} (L_{G}^*)_{ij} + 1 \right) \]
\[ + \sum_{i,j \in E(G)} (L_{G}^*)_{ij} - 1^T N 1. \]

By Lemma 2.5, we get
\[
Kf(R(G) \ominus \bigwedge_{i=1}^m H_i) = (n + m + \sum_{i=1}^m t_i) \left( \frac{2}{3} Kf(G) + \frac{m}{2} + \frac{1}{6} \sum_{i,j \in E(G)} (L_{G}^*)_{ij} \right)
\[ - r_{ij}(G) + \left( S^{-1} + \frac{2}{3} F^T R(G)^T R(G) F \right) - 1^T N 1 \]
\[ = (n + m + \sum_{i=1}^m t_i) \left( \frac{2}{3} Kf(G) + \frac{m}{2} + \frac{1}{6} \sum_{i,j \in E(G)} (L_{G}^*)_{ij} \right) \]
\[ + \sum_{i,j \in E(G)} (L_{G}^*)_{ij} - 1^T N 1. \]

Note that the eigenvalues of \((L_{Rh} + I_h, -\frac{1}{2} I_h) \) \((i = 1, 2, ..., m)\) are \( \mu_1(H_i) + 1, \mu_2(H_i) + 1, ..., \mu_h(H_i) + 1 \). Then
\[ tr(S^{-1}) = \sum_{i=1}^m \sum_{j=1}^m \frac{1}{\mu_i(H_i)} + 1. \]  

(9)
By Lemma 2.2, $L^*_{ii} \mathbf{1} = 0$ and $(\mathbf{1}^T \left( R^T(G)L^*_{ii} Q \right) \mathbf{1})^T = \mathbf{1}^T \left( Q^T L^*_{ii} R(G) \right) \mathbf{1}$, then

$$1^T \mathbf{N} \mathbf{1} = \frac{m}{2} + \frac{1}{6} \mathbf{1}^T \left( R^T(G)L^*_{ii} R(G) \right) \mathbf{1} + \mathbf{1}^T \mathbf{F} \mathbf{1} + \mathbf{1}^T \mathbf{F}^T \mathbf{1} + \frac{2}{3} \mathbf{1}^T R^T(G)L^*_{ii} R(G) \mathbf{F} \mathbf{1} + \mathbf{1}^T S^{-1} \mathbf{1} + \frac{2}{3} \mathbf{1}^T \left( F^T R^T(G)L^*_{ii} R(G) F \right) \mathbf{1}.$$ 

Note that $R(G) \mathbf{1} = \pi$, where $\pi^T = (d_1, d_2, ..., d_n)$, then $1^T \left( R^T(G)L^*_{ii} R(G) \right) \mathbf{1} = \pi^T L^*_{ii} \pi$, so

$$1^T \mathbf{N} \mathbf{1} = \frac{m}{2} + \frac{1}{6} \pi^T L^*_{ii} \pi + \pi^T L^*_{ii} Q(G) \mathbf{1} + \mathbf{1}^T S^{-1} \mathbf{1} + \mathbf{1}^T \left( Q^T L^*_{ii} Q \right) \mathbf{1}. \tag{10}$$

Let $R_i = L_{H_i} + I_i - \frac{1}{\alpha+1} j_i$, then

$$1^T S^{-1} \mathbf{1} = \left( \begin{array}{ccc} 1^T \mathbf{t}_1 & 1^T \mathbf{t}_2 & \cdots & 1^T \mathbf{t}_m \end{array} \right) \left( \begin{array}{cccc} R_1^{-1} & 0 & 0 & \cdots \ 0 & R_2^{-1} & 0 & \cdots \ 0 & 0 & \cdots & 0 \ 0 & 0 & \cdots & R_m^{-1} \end{array} \right) \left( \begin{array}{c} \mathbf{1}_t \ \mathbf{1}_t \ \cdots \ \mathbf{1}_t \end{array} \right). \tag{11}$$

and

$$1^T \mathbf{F}^T = \frac{1}{2} \left( \begin{array}{ccc} 1^T \mathbf{t}_1 & 1^T \mathbf{t}_2 & \cdots & 1^T \mathbf{t}_m \end{array} \right) \left( \begin{array}{cccc} \mathbf{1}_t & 0 & 0 & \cdots \ 0 & \mathbf{1}_t & 0 & \cdots \ 0 & 0 & \cdots & 0 \ 0 & 0 & \cdots & \mathbf{1}_t \end{array} \right). \tag{12}$$

Plugging $(9), (10), (11)$ and $(12)$ into $Kf(R(G) \ominus \bigwedge_{i=1}^m H_i)$, we obtain the required result in $(vi)$.

5. Conclusion

In this paper, using the Laplacian generalized inverse approach, we obtained the resistance distance and Kirchhoff indices of $R(G) \ominus \bigwedge_{i=1}^m H_i$ and $R(G) \ominus \bigwedge_{i=1}^m H_i$ whenever $G$ and $H_i$ are arbitrary graph. These results generalize the existing results in [13].

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