Stability of cnoidal waves in the parametrically driven nonlinear Schrödinger equation

I. V. Barashenkov and M. A. Molchan
Department of Mathematics and Applied Mathematics,
University of Cape Town, Private Bag Rondebosch 7701, South Africa and
National Institute for Theoretical Physics (NITheP), Stellenbosch, South Africa

The parametrically driven, damped nonlinear Schrödinger equation has two cn- and two dn-wave solutions. We show that one pair of the cn and dn solutions is unstable for any combination of the driver’s strength, dissipation coefficient and spatial period of the wave; this instability is against periodic perturbations. The second dn-wave solution is shown to be unstable against antiperiodic perturbations — in a certain region of the parameter space. We also consider quasiperiodic perturbations with long modulation wavelength, in the limit where the driving strength is only weakly exceeding the damping coefficient.

PACS numbers: 05.45.Yv

I. INTRODUCTION

Periodic waves arise in media of different physical nature. They find direct applications in plasma physics [1, 2], nonlinear optics [3–5], solid state physics [6, 7], and physics of Bose-Einstein condensates [8]. The nonlinear periodic waves (or cnoidal waves) interpolate between plane waves and solitons.

One of the most important problems associated with periodic waves is their stability in various nonlinear media. To date, stable periodic patterns were discovered in physical settings modelled by self-defocusing nonlinearities [9–13], and in Bose-Einstein condensates confined by periodic potentials [14, 15]. Families of stable cnoidal waves supported by focusing nonlinearities were revealed in quadratic media [16, 17]. All these results pertain to conservative media where self-supported structures exist due to the balance between dispersion and nonlinearity. In the presence of dissipation, an additional balance between gain and loss is needed. In the dissipative case the parameters of the solution (e.g., amplitude and phase) are fixed by the parameters of the governing equation, whereas in conservative settings solutions form continuous families. So far, stable stationary dissipative periodic waves were mostly observed in optical cavities.

In this paper we study stability properties of periodic wave solutions of the parametrically driven, damped nonlinear Schrödinger equation:

\[ i\psi_t + \psi_{xx} - \psi + 2|\psi|^2\psi = h\psi^* - i\gamma\psi, \quad h, \gamma > 0. \]  

(1.1)

Here \( \gamma \) is the damping coefficient and \( h \) the amplitude of the parametric driver. Equation (1.1) is an archetypal equation for small and slowly-varying amplitudes of waves and patterns in spatially-distributed parametrically driven systems. It arises in a wide variety of physical contexts including instabilities in plasma [22, 23], amplitude generation in Josephson junctions [24, 25], and signal amplification effects in fiber optics [26, 27].

At the same time, the knowledge about periodic nonlinear waves of PDNLS is rather scarce. Umeki [29] examined numerically stability properties of the families of cn- and dn-waves of PDNLS considered in the context of water waves in a vertically forced long container (\( h < 0 \)). For a fixed value of the spatial period of the wave, he obtained the corresponding stability diagram and studied the temporal evolution of the perturbation. Umeki’s numerical considerations were restricted to perturbations of the initial wave profile having the same period as the cnoidal wave itself.

In this paper, we examine the linear stability of the cn and dn type solutions of Eq. (1.1) with respect to two broad classes of perturbations. The first class consists of perturbations which are periodic or antiperiodic with the period of the wave. These perturbations have a discrete spectrum and are relatively easy to analyse. On the other hand, the periodic or antiperiodic perturbations turn out to be sufficient to detect the instability of the underlying wave in a wide range of parameter values.

In particular, we prove that out of two cn- and two dn-wave solutions, one pair of cn- and dn-waves is unstable against periodic perturbations — for any combination of the driver’s strength, dissipation coefficient and spatial period of the wave. The other dn wave is shown to be unstable against antiperiodic perturbations — in a certain region of the parameter space.

The second class includes quasiperiodic perturbations which have the form of symmetry eigenvectors modulated by a long wavelength. Here, the small wavenumber of the modulation serves as a perturbation expansion parameter. The perturbation theory can be developed near the lower boundary of the cnoidal-wave existence domain, that is, for driving strengths only slightly exceeding the damping coefficient.

Using this perturbation approach, we reproduce several results obtained by other means. First, we confirm the instability of the cn- and dn-wave solutions of the unperturbed (\( \gamma = h = 0 \)) NLS equation, a result already available in literature. Second, we corroborate our own conclusions on the instability of the two damped-driven dn waves in the \( h \approx \gamma \) case. More importantly, the long-
wavelength-modulation treatment provides useful information on the structure of the spectrum of the cn wave whose stability cannot be classified by restricting to the periodic perturbations.

We complement the perturbation study of the cn wave with a numerical analysis of its linearised spectrum in the $h \approx \gamma$ case.

II. PERIODIC SOLUTIONS AND THEIR LINEARIZATIONS

A. Two pairs of cnoidal waves

Eq. (1.1) has two pairs of nonequivalent periodic solutions. One pair is expressible in terms of the Jacobi cosine function:

$$\Psi_{\pm}^\text{cn} = A_{\pm} q_{\text{cn}}(A_{\pm} x, k) e^{-i\theta_{\pm}},$$

where

$$q_{\text{cn}}(X, k) = \frac{k}{\sqrt{2k^2 - 1}} \text{cn} \left( \frac{X}{\sqrt{2k^2 - 1}}, k \right),$$

$1/\sqrt{2} < k \leq 1$. We will occasionally be referring to these solutions as $\text{cn}^+$ and $\text{cn}^-$, respectively. The other two solutions (referred to as $\text{dn}^+$ and $\text{dn}^-$ in what follows) invoke the Jacobi dn function instead:

$$\Psi_{\pm}^\text{dn} = A_{\pm} q_{\text{dn}}(A_{\pm} x, k) e^{-i\theta_{\pm}},$$

where

$$q_{\text{dn}}(X, k) = \frac{1}{\sqrt{2 - k^2}} \text{dn} \left( \frac{X}{\sqrt{2 - k^2}}, k \right),$$

$0 \leq k \leq 1$. The amplitudes $A_{\pm}$ are given, in both cases, by

$$A_{\pm}^2 = 1 \pm \sqrt{h^2 - \gamma^2},$$

and the phases by

$$\theta_{\pm} = \frac{1}{2} \arcsin \left( \frac{\gamma}{h} \right), \quad \theta_{-} = \pi - \theta_{+}.$$

Eq. (2.5) carries the entire information on the domain of existence of the four periodic solutions on the $(h, \gamma)$-parameter plane. For the given $\gamma$, the cnoidal waves $\Psi_{\pm}^\text{cn}$ and $\Psi_{\pm}^\text{dn}$ exist for all $h > \gamma$ whereas the domain of the solutions $\Psi_{\text{cn}}^\gamma$ and $\Psi_{\text{dn}}^\gamma$ is bounded on both sides: $\gamma < h < \sqrt{1 + \gamma^2}$. The two cn solutions have the spatial periods $L_{\text{cn}}/A_{\pm}$, where

$$L_{\text{cn}} = L_{\text{cn}}(k) = 2K(k) \sqrt{2k^2 - 1},$$

and the dn solutions are periodic with the periods $L_{\text{dn}}/A_{\pm}$, where

$$L_{\text{dn}} = L_{\text{dn}}(k) = 2K(k) \sqrt{2 - k^2}.$$ 

In Eqs. (2.6) and (2.7), $K$ is the complete elliptic integral of the first kind. As $k$ varies from $1/\sqrt{2}$ to 1, $L_{\text{cn}}$ grows, monotonically, from 0 to infinity. Less obvious fact is that $L_{\text{dn}}$ is also a monotonically growing function – growing from $\sqrt{2\pi}$ to infinity as $k$ varies from 0 to 1. (See the Appendix.) Therefore, for the given $h$ and $\gamma$ the period of the cnoidal wave can be used as its third parameter, in lieu of the elliptic modulus $k$.

As $k \to 1$, the periodic solutions $\Psi_{\pm}^\text{cn}$ and $\Psi_{\pm}^\text{dn}$ approach the $\psi_\pm$ soliton, while the $\Psi_{\pm}^\gamma$ and $\Psi_{\pm}^\delta$ tend to the soliton $\psi_\pm$ [21].

B. Linearised problem

To examine the stability of cnoidal waves (2.1)–(2.3), we write $\psi(x, t) = \Psi_\pm(x) + \delta \psi(x, t)$, where $\Psi_\pm(x)$ is the stationary solution in question, and linearize in $\delta \psi$. Letting

$$\delta \psi(x, t) = e^{-i\theta_{\pm}} \left[ U(x, t) + iV(x, t) \right]$$

yields

$$U_T = L_0 V, \quad -V_T - 2\gamma V = L_1 U,$$

where the operators $L_{0,1}$ are given by

$$L_0 = -\frac{d^2}{dX^2} + 1 \mp \mathcal{E} - 2q^2$$

and

$$L_1 = -\frac{d^2}{dX^2} + 1 - 6q^2.$$ 

In (2.8), $T = A_+^2 t$ is the scaled time variable, and in $A_+ x$ is the scaled spatial coordinate. In (2.8) we have scaled the damping coefficient and introduced a parameter $\mathcal{E}$ which measures the driving-damping difference:

$$\tilde{\gamma} = \gamma/A_+^2, \quad \mathcal{E} = 2\sqrt{h^2 - \gamma^2}/A_+^2.$$ 

The notation $q$ stands for $q_{\text{cn}}$ or $q_{\text{dn}}$ [Eq. (2.2)] or Eq. (2.1) depending on whether we linearize about a cn or dn solution. The top sign in front of $\mathcal{E}$ in Eq. (2.9a) corresponds to the $\Psi_{\text{cn}}^\gamma$ and $\Psi_{\text{dn}}^\gamma$ solutions; the bottom sign selects the $\Psi_{\text{cn}}^\delta$ and $\Psi_{\text{dn}}^\delta$ cnoidal waves. The cnoidal wave is deemed unstable provided Eqs. (2.8) have solutions growing faster than $\exp(\tilde{\gamma} T)$ in time.

We do not impose any periodicity requirements on $U$ and $V$. All we require is that $U(X, T)$ and $V(X, T)$ be bounded on the whole line $-\infty < X < +\infty$.

We call $\Lambda$ a point of spectrum of an operator $\mathcal{L}$ if the equation $\mathcal{L}y = \Lambda y$ has a solution $y(X)$ bounded for all $X$, $-\infty < X < \infty$. The spectrum of the operators (2.9a) and (2.9b) can be found exactly. Consider first the cn solutions, that is, assume that $q(X)$ in Eqs. (2.9a)–(2.9b)
is given by Eq. (2.2). Then, defining $\xi = X/\sqrt{2k^2-1}$, we obtain:

$$L_0 = \frac{\xi^{(1)} - 1}{2k^2 - 1} \mp \mathcal{E},$$  \hspace{1cm} (2.11)

and

$$L_1 = \frac{\xi^{(2)} - 1 - 2k^2}{2k^2 - 1},$$  \hspace{1cm} (2.12)

where

$$\xi^{(\ell)} = -\frac{d^2}{dx^2} + \ell(\ell + 1)k^2\mathrm{sn}^2(\xi, k)$$

is the $\ell$-gap Lamé operator \cite{30}. Therefore the spectra of $L_0$ and $L_1$ result from the spectra of $\xi^{(1)}$ and $\xi^{(2)}$ (given in \cite{30}) by shift and scaling.

Thus, the spectrum of $L_0$ consists of a finite band

$$\Lambda \in \left[\frac{k^2 - 1}{2k^2 - 1} \mp \mathcal{E}, \ + \mathcal{E}\right],$$  \hspace{1cm} (2.13)

and a semi-infinite band,

$$\Lambda \in \left[\frac{k^2}{2k^2 - 1} \mp \mathcal{E}, \ - \infty\right).$$

The spectrum of $L_1$ comprises two finite bands

$$\left[-1 - \frac{2\sqrt{1 - k^2 + k^4}}{2k^2 - 1}, \ - \frac{2k^2}{2k^2 - 1}\right],$$

$$\left[0, \ \frac{2(1 - k^2)}{2k^2 - 1}\right],$$

and a semi-infinite band

$$\left[-1 + \frac{\sqrt{1 - k^2 + k^4}}{2k^2 - 1}, \ - \infty\right).$$

In the case of the dn solutions \cite{28} the spectrum structures are similar. Namely, assuming that $q(x)$ is given by Eq. (2.4) and defining $\xi = X/\sqrt{2 - k^2}$, the operators $L_0$ and $L_1$ can be expressed as

$$L_0 = \frac{\xi^{(1)} - k^2}{2 - k^2} \mp \mathcal{E},$$  \hspace{1cm} (2.14)

and

$$L_1 = \frac{\xi^{(2)} - 2 - k^2}{2 - k^2},$$  \hspace{1cm} (2.15)

respectively. Accordingly, the spectrum of $L_0$ comprises two bands,

$$\left[\mp \mathcal{E}, \ \frac{1 - k^2}{2 - k^2} \mp \mathcal{E}\right],$$

$$\left[\frac{1}{2 - k^2} \mp \mathcal{E}, \ - \infty\right),$$  \hspace{1cm} (2.16)

while the spectrum of $L_1$ consists of three:

$$\left[-1 - \frac{2\sqrt{1 - k^2 + k^4}}{2 - k^2}, \ - \frac{2(1 - k^2)}{2 - k^2}\right],$$

$$\left[-\frac{2k^2}{2 - k^2}, \ 0\right],$$

$$\left[-1 + \frac{\sqrt{1 - k^2 + k^4}}{2 - k^2}, \ - \infty\right).$$  \hspace{1cm} (2.17)

In Eqs. (2.13)–(2.17) the top sign in front of $\mathcal{E}$ pertains to the $\Psi_+$ solutions and the bottom sign to $\Psi_-$. The spectrum of the $\psi_+$ and $\psi_-$ solutions \cite{21} is recovered be sending $k \to 1$. In this case, each finite band collapses into a discrete eigenvalue. As a result, the spectrum of $L_0$ consists of a single discrete eigenvalue $\Lambda_0 = \mp \mathcal{E}$ (and a continuum of values $\Lambda \geq 1 \mp \mathcal{E}$) whereas the spectrum of $L_1$ includes two discrete eigenvalues, $\Lambda_0 = -3$ and $\Lambda_1 = 0$ (and a continuum $\Lambda \geq 1$).

C. Stability eigenvalues and symplectic eigenvalue problem

We assume that the linear system \cite{28} has separable solutions of the form

$$U(X, T) = \text{Re} \left[e^{\eta T} \tilde{u}(X)\right]; \quad V(X, T) = \text{Re} \left[e^{\eta T} \tilde{v}(X)\right],$$

where $\tilde{u}$, $\tilde{v}$ and $\eta$ are complex. Here $\eta$ and $\tilde{u}$, $\tilde{v}$ are eigenvalues and eigenfunctions in the eigenvalue problem

$$L_0 \tilde{v} = \eta \tilde{u}, \quad L_1 \tilde{u} = -(\eta + 2\gamma)\tilde{v}. \hspace{1cm} (2.18)$$

We will be referring to this problem as the “linearised eigenvalue problem”, whereas $\eta$ will be called “stability eigenvalues” below.

Making a substitution $(\tilde{u}, \tilde{v}) \rightarrow (u, v)$ where $\lambda \tilde{v} = \eta \tilde{u}$, $\tilde{u} = u$, and

$$\lambda^2 = \eta(\eta + 2\gamma),$$  \hspace{1cm} (2.19)

we transform (2.18) to

$$\mathcal{H}\vec{y} = \lambda J\vec{y}, \hspace{1cm} (2.20)$$

where

$$\mathcal{H} = \begin{pmatrix} L_1 & 0 \\ 0 & L_0 \end{pmatrix},$$  \hspace{1cm} (2.21)

and

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} u \\ v \end{pmatrix}.$$  \hspace{1cm} (2.22)

One advantage of this formulation is that eigenvalues $\lambda$ depend on $h$ and $\gamma$ only in combination $\mathcal{E}$ (that is, only as $h^2 - \gamma^2$); thus a two-parameter eigenvalue problem is reduced to a one-parameter problem. Another merit is that the operator $J^{-1}\mathcal{H}$ is symplectic, that is, generates a Hamiltonian flow. The spectrum of symplectic operators consists of pairs of opposite pure-imaginary values, real pairs and complex quadruplets. If $\lambda$ is a real or pure imaginary point of spectrum, then $-\lambda$ is another one; if a complex $\lambda$ is in the spectrum, then so are $-\lambda$, $\lambda^*$, and $-\lambda^*$. We will be referring to the eigenvalue problem (2.20) as the “symplectic eigenvalue problem”, and $\lambda$’s as the “symplectic eigenvalues”. Having found a symplectic eigenvalue $\lambda$, we can readily recover the corresponding growth rate $\text{Re} \eta$ from (2.19):

$$\text{Re} \eta = -\gamma + \text{Re} \sqrt{\gamma^2 + \lambda^2}. \hspace{1cm} (2.22)$$
(Here we have kept the largest of the two growth rates.)

Before proceeding to the analysis of the symplectic spectrum, three remarks are in order. First of all, we note that if the symplectic eigenvalues ±λ are real, the corresponding stability eigenvalues are real as well:

$$\eta = -\dot{\gamma} \pm \sqrt{\dot{\gamma}^2 + \lambda^2}.$$  

Thus the occurrence of a real symplectic eigenvalue immediately implies instability of the underlying cnoidal wave. This instability has monotonic growth.

Second, since the potentials in the operators \( L_0 \) and \( L_1 \) are even, it is sufficient to consider only even and odd eigenfunctions \( \tilde{y}(X) \).

Finally, we note that the potentials of the operators \( L_{0,1} \) are periodic with the period \( L_{cn} \) in the case of the \( cn^\pm \) solutions and \( L_{dn} \) in the case of the \( dn^\pm \) cnoidal waves. Therefore, the subspace of periodic functions with period \( L_{cn} \) respectively \( L_{dn} \) is invariant under the action of the operators \( L_{0,1} \) in the case of \( cn^\pm \) respectively \( dn^\pm \) solutions. This implies that the eigenvalue problem (2.20) is well posed on the subspace of periodic functions with the corresponding period. The subspace of antiperiodic functions, that is, functions satisfying \( \tilde{y}(X + L_{cn, dn}) = -\tilde{y}(X) \), is also invariant. Accordingly, the eigenvalue problem (2.20) is well posed on the subspace of antiperiodic functions.

III. INSTABILITY TO PERIODIC AND ANTIPERIODIC PERTURBATIONS

To establish instability of a solution it is sufficient to demonstrate its instability to a particular class of perturbations. In this section we will show that the cnoidal waves \( cn^- \) and \( dn^- \) are unstable w.r.t. periodic perturbations, for any combination of the parameters \( h, \gamma \) and \( k \). We will also show that the wave \( dn^+ \) is unstable to antiperiodic perturbations — in some region of the parameter space.

A. Instability of \( cn^- \)

In the case of the cnoidal wave \( cn^- \), we will show that the symplectic eigenvalue problem (2.20) has real eigenvalues \( \lambda \) associated with the eigenfunctions satisfying boundary conditions of the third kind,

$$u_X(0) = u \left( \frac{L}{2} \right) = 0, \quad v_X(0) = v \left( \frac{L}{2} \right) = 0. \quad (3.1)$$

Here \( L = L_{cn} \) is the half-period of the cnoidal wave, defined by Eq. (2.10). Due to the boundary condition at the origin, the eigenfunction \( \tilde{y}(X) \) has to be even.

Let \( L_0 \) be defined by Eq. (2.3a) with \( q = q_{cn} \) and consider an eigenvalue problem

$$L_0 y = \lambda y \quad (3.2)$$

with the mixed boundary conditions

$$y_X(0) = 0; \quad y \left( \frac{L}{2} \right) = 0. \quad (3.3)$$

One eigenvalue of \( L_0 \) is \( \Lambda_1 = \mp \mathcal{E} \); it is associated with the eigenfunction

$$y_1(X) = cn \left( \frac{X}{\sqrt{2k^2 - 1}}, k \right). \quad (3.4)$$

The eigenfunction does not have zeros inside the interval \((0, L/2)\); therefore \( \Lambda_1 \) is the lowest eigenvalue of the regular Sturm-Liouville problem (3.2)+(3.3). If we are considering the linearization about the solution \( cn^- \), the eigenvalue \( \Lambda_1 = \mp \mathcal{E} \) is strictly positive. Hence the operator \( L_0 \), defined by the differential expression (2.9a) and boundary conditions (3.3), is positive definite. On the subspace of functions satisfying (3.1), the system (2.20) can be written in the form

$$L_1 u = -\lambda^2 L_0^{-1} u, \quad (3.5)$$

where \( L_1 \) is symmetric and \( L_0^{-1} \) symmetric and positive definite. The smallest eigenvalue \( (\mp \lambda^2)_0 \) of the generalised eigenvalue problem (3.5) is given by the minimum of the Rayleigh quotient

$$(-\lambda^2)_0 = \min \frac{\langle u | L_1 | u \rangle}{\langle u | L_0^{-1} | u \rangle}, \quad (3.6)$$

where the minimum is evaluated over all functions \( u(X) \) satisfying the boundary conditions (3.1). Here the scalar product is defined by

$$\langle u | v \rangle = \int_0^{L/2} u(X)v(X)dX.$$

Turning to the operator \( L_1 \), we note that it has a negative eigenvalue \( \mu = -2k^2/(2k^2-1) \) with the eigenfunction

$$z_1(X) = cn \left( \frac{X}{\sqrt{2k^2-1}}, k \right) dn \left( \frac{X}{\sqrt{2k^2-1}}, k \right) \quad (3.7)$$

which satisfies the boundary condition (3.3). Therefore, the quadratic form \( \langle u | L_1 | u \rangle \) attains negative values on the space of functions with the boundary condition (3.3). Eq. (3.6) implies then that the smallest eigenvalue \( -\lambda^2 \) of the eigenvalue problem (3.5) is negative and hence the vector eigenvalue problem (2.20) has a real eigenvalue \( \lambda \). By (2.22) we conclude, eventually, that there are perturbations with positive growth rates Re \( \eta \).

Since the eigenfunction \( \tilde{y}(X) \) is even, \( \tilde{y}(L/2) = 0 \) implies \( \tilde{y}(-L/2) = 0 \). On the other hand, the derivative \( \tilde{y}_X(X) \) is odd; thus we have

$$\tilde{y} \left( -\frac{L}{2} \right) = -\tilde{y} \left( \frac{L}{2} \right), \quad \tilde{y}_X \left( -\frac{L}{2} \right) = -\tilde{y}_X \left( \frac{L}{2} \right). \quad (3.8)$$
We have thus established instability of the \( dn^- \) solution against perturbations periodic with the period \( L_{dn} \), the period of the \( dn^- \) cnoidal wave. The \( dn^- \) wave is unstable for any choice of the parameters \( h \) and \( \gamma \), and any \( k \).

C. Instability of \( dn^+ \): antiperiodic perturbations

So far we have demonstrated the instability of the \( cn^- \) and \( dn^- \) waves, with the unstable perturbations exhibiting a monotonic growth. Another solution that turns out to be prone to the instability of a similar type is the \( dn^- \) wave; however this time our proof will only be valid in a part of the \( (h, \gamma, k) \) parameter space.

In the case of the \( dn^+ \) wave, the operator \( \mathcal{L}_0 \) has an eigenvalue \( \Lambda_1 = 1/(2 - k^2) - \varepsilon \) with an eigenfunction

\[
\tilde{y}_1(X) = \text{sn} \left( \frac{X}{\sqrt{2 - k^2}}, k \right).
\]

The eigenfunction \( \tilde{y}_1 \) satisfies mixed boundary conditions

\[
y(0) = 0, \quad y \left( \frac{L}{2} \right) = 0 \tag{3.12}
\]

and does not have zeros inside the interval \( (0, L/2) \). Hence the eigenvalue \( \Lambda_1 \) is the smallest eigenvalue of \( \mathcal{L}_0 \) under the boundary conditions \( (3.12) \).

The eigenvalue \( \Lambda_1 \) is positive and the operator \( \mathcal{L}_0 \) is positive definite in two adjacent parameter regions. One parameter region is

\[
h < \sqrt{\frac{1}{9} + \gamma^2}, \tag{3.13}
\]

with \( k \) taking any values between 0 and 1. The second region is

\[
\sqrt{\frac{1}{9} + \gamma^2} < h < \sqrt{1 + \gamma^2}, \tag{3.14a}
\]

with the elliptic modulus being bounded from below:

\[
k^2 > \frac{3}{2} - \frac{1}{2} \sqrt{h^2 - \gamma^2}. \tag{3.14b}
\]

We now assume that the parameter vector \( (h, \gamma, k) \) lies in one of the above two regions.

The operator \( \mathcal{L}_1 \) has a negative eigenvalue \( 2(k^2 - 1)/(2 - k^2) \) with an eigenfunction satisfying the boundary conditions \( (5.12) \):

\[
\tilde{z}_1(X) = \text{sn} \left( \frac{X}{\sqrt{2 - k^2}}, k \right) \text{dn} \left( \frac{X}{\sqrt{2 - k^2}}, k \right).
\]

Therefore the minimum of the Rayleigh quotient \( (3.6) \) on the space of functions satisfying \( (5.12) \) is negative and the
symplectic eigenvalue problem \( (2.20) \) has a real eigenvalue \( \lambda \). This means that the \( \text{dn}^+ \) wave is unstable to perturbations satisfying

\[
u (0) = u_X \left( \frac{L}{2} \right) = 0, \quad \nu (0) = v_X \left( \frac{L}{2} \right) = 0.
\]

The boundary condition \( \bar{\nu}(0) = 0 \) singles out the odd eigenfunction; hence it satisfies the antiperiodicity conditions on the interval \((-L/2, L/2)\):

\[
\bar{\nu} \left( -\frac{L}{2} \right) = -\bar{\nu} \left( \frac{L}{2} \right), \quad \bar{\nu} X \left( -\frac{L}{2} \right) = -\bar{\nu} X \left( \frac{L}{2} \right).
\]

We conclude that when \( h, \gamma \) and \( k \) belong to the region \((3.13) + (3.14)\), the cnoidal wave \( \text{dn}^+ \) is unstable under perturbations of period twice the period of the wave.

**IV. THE UNPERTURBED NONLINEAR SCHRÖDINGER: STABILITY OF THE \( \text{cn} \) WAVE**

When \( h = \gamma \), Eq. \((2.10)\) gives \( \mathcal{E} = 0 \) and the spectrum of symplectic eigenvalues \((2.20)\) coincides with the spectrum of stability eigenvalues of the cnoidal wave of the undamped undriven NLS.

**A. Small eigenvalues: general setting**

In this subsection we will obtain small symplectic eigenvalues.

Factorising the eigenfunction \( \bar{\nu} \) into a periodic function \( \bar{\nu} \) and an exponential, \( \bar{\nu} = \bar{\nu} e^{i n X} \), the symplectic eigenvalue problem \((2.20)\) becomes

\[
\mathcal{H} \bar{\nu} = 2i \kappa \bar{\nu} X - \kappa^2 \bar{\nu} + \lambda J \bar{\nu}.
\]

Without loss of generality, the period of the function \( \bar{\nu} \) could be taken equal to the period of the potential of the operator \( \mathcal{H} \): \( L_{\text{cn}} \) in the case of the \( \text{cn} \)-wave, and \( L_{\text{dn}} \) in the case of the \( \text{dn} \)-wave. This will indeed be our choice in the \( \text{dn} \) situation. However, in the case of the \( \text{cn} \) wave, it is convenient to regard \( \bar{\nu} \) as a \( 2L_{\text{cn}} \)-periodic function — that is, choose the period of \( \bar{\nu} \) to coincide with the period of the cnoidal wave. This convention is equivalent to the previous one, and is equally general.

The symplectic spectrum includes a four-fold zero eigenvalue; associated with these are two periodic eigenvectors and two generalised eigenvectors. When \( \kappa \) is small, we expand the eigenvalues and eigenfunctions in powers of \( \kappa \):

\[
\bar{\nu} = \bar{\nu}_0 + \kappa \bar{\nu}_1 + \kappa^2 \bar{\nu}_2 + ..., \quad \lambda = \lambda_1 \kappa + \lambda_2 \kappa^2 + ...
\]

Substituting in \((4.1)\) we equate coefficients of like powers of \( \kappa \).

The coefficient of \( \kappa^0 \) gives \( \mathcal{H} \bar{\nu}_0 = 0 \). The general solution is of the form

\[
\bar{\nu}_0 = \begin{pmatrix} Aq_X \\ Bq \end{pmatrix},
\]

where \( A \) and \( B \) are two arbitrary constants. Next, the order \( \kappa^1 \) produces

\[
\mathcal{H} \bar{\nu}_1 = 2i \partial_X \bar{\nu}_0 + \lambda_1 J \bar{\nu}_0,
\]

or, componentwise,

\[
\mathcal{L}_1 u_1 = 2i A \partial_X^2 q - \lambda_1 B q, \quad \mathcal{L}_0 v_1 = (2i B + \lambda_1 A) q_X,
\]

where \( u_1 \) and \( v_1 \) are the top and bottom components of the vector \( \bar{\nu}_1 \); \( \bar{\nu}_1 = (u_1, v_1)^T \). We note that

\[
\mathcal{L}_1 \partial_X (X q) = -2 q, \quad \mathcal{L}_1 (X q_X) = -2 \partial_X^2 q, \quad \mathcal{L}_0 (X q) = -2 X q;
\]

therefore one solution to Eq. \((4.3)\) is

\[
\tilde{u}_1 = -i A X q_X + \frac{\lambda_1}{2} B (X q)_X,
\]

and one solution to \((4.6)\) is

\[
\tilde{v}_1 = -\frac{1}{2} (\lambda_1 A + 2i B) X q.
\]

However neither of these solutions is periodic.

**B. \( \text{cn} \) wave**

From this point on, our analysis depends on which periodic solution we consider. We start with the \( \text{cn} \) wave. In this case both components of the nonzero-order approximation \( \bar{\nu}_0 \) are \( 2L \)-periodic, and we will attempt to construct its perturbation \( \bar{\nu} \) with the same period. Here \( L \) is our short-hand notation for \( L_{\text{cn}} \).

In order to obtain a periodic \( u_1 \), we can add to \( \tilde{u}_1 \) a multiple of \( \partial_X q \), the nonperiodic homogeneous solution of equation \((4.5)\):

\[
\partial_X q(x) = \text{sn}(\xi, \kappa) d\text{sn}(\xi, k) \times \left[ E(\text{am} \xi, k) + \frac{X}{(1 - k^2)(2k^2 - 1) + (2k^2 - 1)^2} - \frac{\text{cn}(\xi, k)}{2k^2 - 1} + \frac{k^2}{2k^2 - 1 - k^2 \text{sn}^2(\xi, k)} \right].
\]

In \((4.9)\), \( \xi = X / \sqrt{2k^2 - 1} \) and \( E(\varphi, k) \) is the incomplete elliptic integral of the second kind:

\[
E(\varphi, k) = \int_0^\varphi \sqrt{1 - k^2 \sin^2 \theta} d\theta.
\]
The resulting nonhomogeneous solution

\[ u_1 = -i AX q_X + \frac{\lambda_1}{2} B(Xq)_X + C \partial_k q \]  

(4.10)
is even and so the periodicity condition \( u_1(L) = u_1(-L) \) is satisfied automatically, for any \( C \); therefore, one only needs to satisfy \( \partial_X u_1(L) = \partial_X u_1(-L) \). For the odd \( \partial_X u_1(X) \), this reduces to

\[ \partial_X u_1(L) = 0. \]  

(4.11)

Substituting (4.10) in (4.11) we find

\[ C = (\lambda_1 B - iA) \frac{L}{L_k}, \]

where \( L_k = \partial_k L \).

In a similar way, we take

\[ v_1 = - \left( \frac{1}{2} \lambda_1 A + iB \right) Xq + Dz, \]  

(4.12)

where \( z \) is the nonperiodic homogeneous solution of (4.10),

\[ z(X) = \xi \text{cn}(\xi, k) \]

\[ + \frac{1}{1-k^2} [\text{sn}(\xi, k) \text{dn}(\xi, k) - 2\text{cn}(\xi, k) E(\text{am}\xi, k)]. \]

Eq. (4.12) is odd and its derivative \( \partial_X v_1 \) is even; hence the periodicity condition \( \partial_X v_1(L) = \partial_X v_1(-L) \) is in place and we only need to make sure that \( v_1(L) = v_1(-L) \). This amounts to

\[ v_1(L) = 0. \]  

(4.13)

Substituting Eq. (4.12) in (4.13), the constant \( D \) is identified:

\[ D = - (\lambda_1 A + iB) \frac{K}{K_k}. \]

where \( (u_2, v_2)^T = \mathbf{J}_2 \). The solvability conditions for
where the elements of the matrix \( M \) are given by
\[
M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix},
\]
and
\[
\lambda_1 \lambda_1^* + n_11 = \left\langle qX | qX \right\rangle, \quad m_{11} = \frac{1}{2} \left\langle qX | w \right\rangle,
\]
where \( \lambda \) and \( \lambda^* \) are the roots of the equation
\[
\lambda^2 - \left( \lambda + \rho \right) \lambda + \rho \phi = 0.
\]
Here \( \rho = \left\langle qX | qX \right\rangle - \left\langle qX | v_1 \right\rangle \) and \( \phi = \left\langle qX | qX \right\rangle - \left\langle qX | v_1 \right\rangle \).

The roots have nonzero real parts; this implies that the cn-solution of the unperturbed nonlinear Schrödinger equation is unstable for all \( k \). This fact is known in literature.

The eigenvalues \( \lambda = \lambda_k + O(\kappa^2) \), with small \( \kappa \), lie on four rays emanating out of the origin on the complex-\( \lambda \) plane. The full spectrum of symplectic eigenvalues, obtained numerically, is displayed in the top row of Fig. 1.

The result \[4.21\] for the dominant behaviour of the eigenvalues near the origin is shown by the dashed line. The analytical result is seen to accurately reproduce the numerically computed eigenvalues. The four rays bend and join, pairwise, forming an eight-shaped curve centred at the origin (Fig. 1(a-c)). In addition, the spectrum fills the imaginary axis of \( \lambda \) (with a gap).

C. The unperturbed nonlinear Schrödinger: Stability of the \( \text{dn} \) wave

In the case of the \( \text{dn} \) wave, the eigenvalue problem \[4.1\] and expansion \[4.2\] remain in place. The null eigenfunction \( \Psi_0 \) is periodic with the period of the \( \text{dn} \) wave: \( \Psi_0(X+L) = \Psi_0(X) \), where \( L = L_{dn} \), and we will assume that the eikonal function \( \Psi \) satisfies the same boundary conditions.

The \( \epsilon \) corrections \( u \) and \( v \) are still given by equations \[4.10\] and \[4.12\], respectively, where the homogeneous solutions are, this time,
\[
\partial_k u = \frac{k \text{sn}(\xi, k) \text{cn}(\xi, k)}{(2-k^2)^{1/2}} \left[ \frac{E(\text{am} \xi) - X}{1-k^2} (2-k^2) \right] + \frac{k \text{dn}(\xi, k)}{(1-k^2)(2-k^2)} \left[ (2-k^2) \text{cn}^2(\xi, k) - 1 \right]
\]
and
\[
z(\xi) = k^2 \text{sn}(\xi, k) \text{sn}(\xi, k) - 2 \text{dn}(\xi, k) E(\text{am} \xi, k),
\]
where
\[
\xi = \frac{X}{\sqrt{2-k^2}}.
\]

Since the function
\[
u_1(X) = -i AX q_X + \frac{\lambda_1}{2} B(X q_X) + C q_k
\]
is even, the only periodicity condition that needs to be verified, is
\[
\partial_X \nu_1 \left( \frac{L}{2} \right) = 0.
\]
Substituting \[4.22\] in \[4.23\], the constant \( C \) is evaluated to be
\[
C = \left( \lambda_1 B - i A \right) \frac{L}{L_k},
\]
where \( L_k = \partial_k L \). On the other hand, the function

\[
v_1 = -\left( \frac{1}{2} A\lambda_1 + iB \right) X q + D z
\]

is odd; hence the periodicity condition reduces to

\[
v_1 \left( \frac{L}{2} \right) = 0.
\]

This gives

\[
D = - (\lambda_1 A + iB) \frac{K}{E}.
\]

At the order \( \kappa^2 \) we obtain equations \([4.14]-[4.15]\), with the solvability conditions \([4.16]-[4.17]\), where, this time, the scalar product is defined as

\[
\langle f | g \rangle = \int_{-L/2}^{L/2} f(X) g(X) dX.
\]

The elements of the matrix \([4.18]\) are given by Eqs. \([4.19]\) where, this time, the functions \( p(X) \) and \( w(X) \) are given by

\[
p = 2X q_X + \frac{L}{L_k} q_k(X), \quad w = X q + \frac{K}{E} z(X).
\]

The conclusion is that the \( dn \)-wave solution of the unperturbed NLS is unstable, for any \( k \). This fact is known to workers in the field. Furthermore, the real positive symplectic eigenvalues \( \lambda \) translate into real positive growth rates \( \eta \). This implies that the \( dn \) wave is also unstable as a solution of the damped-driven NLS with \( h = \gamma \).

V. THE DAMPED-DRIVEN \( \text{cn} \) WAVES WITH \( h = \gamma \)

That the \( cn \) wave of the unperturbed NLS is unstable for any \( k \) does not mean, however, that the damped-driven wave with \( h = \gamma \) is necessarily unstable. In the damped-driven situation, the stability is determined by the linearised eigenvalues \( \eta \), not symplectic eigenvalues \( \lambda \). Transforming \( \lambda \) to \( \eta \) by the rule \([2.19]\), the symplectic spectrum shown in the top row of Fig. 1 is mapped to the spectrum shown in the bottom row of the same figure.

Substituting for \( q \) we get

\[
m_{11} = -\frac{k(1-k^2)}{4(2-k^2)^2} \frac{(LN)_k}{N}, \quad n_{11} = \frac{k^2}{(2-k^2)^2} \frac{L^2}{L_k}, \quad m_{12} = m_{21} = -\frac{k^2}{L^2} \frac{L}{(2-k^2)^2 N L_k} \times \left( \sqrt{1-k^2}N \right)_k \left( \frac{L}{\sqrt{1-k^2}} \right)_k,
\]

\[
m_{22} = \frac{1}{2} \frac{(LN)_k}{L_k}, \quad n_{22} = -\frac{1-k^2}{2} \frac{L^2}{(2-k^2)^3 N},
\]

where \( L \) is given by \([2.7]\) and

\[
N = \int_{L/2}^{L/2} q^2(X) dX = \frac{2}{\sqrt{2-k^2} E}.
\]

Hence we find the coefficients of the biquadratic equation \([4.20]\):

\[
a_0 = [(LN)_k]^2, \quad c_0 = \frac{8k^2}{(2-k^2)^3} L^4,
\]

and

\[
b_0 = -\frac{4}{k(2-k^2)} LN^2 \left( \frac{L}{N} \right)_k.
\]

The discriminant of the equation is

\[
D = \frac{4L^4}{k(2-k^2)^2} (k^2 L_k N - N_k L_k) (L_k N - k^2 N_k L).
\]

Since \( L_k > 0 \) and \( N_k < 0 \), this is positive and so Eq. \([4.20]\) has two positive roots, \( (\lambda_1^2)_a > 0 \) and \( (\lambda_2^2)_b > 0 \).

A. Stability to long-wavelength perturbations

The symplectic spectrum includes a double zero eigenvalue resulting from the phase invariance of the unperturbed NLS equation (invariance w.r.t. constant and constant-velocity phase rotations) and another double zero resulting from its translation and Galilian invariances. The map \([2.19]\) leaves only two eigenvalues at the origin: one resulting from the translation symmetry and the other one corresponding to the symmetry w.r.t. the velocity boosts.

Next, for small \( k \ll \tilde{\gamma} \), the small-\( \eta \) branch of the map is simply \( \eta = (2\tilde{\gamma})^{-1} \lambda^2 \). Hence any ray \( \lambda = e^{i\phi}r \) emanating out of the origin on the \( \lambda \)-plane and making the angle \( \phi \) to the real axis, is mapped to a ray making double that angle to the real axis on the \( \eta \)-plane: \( \eta = e^{2i\phi}r' \). The branches of the spectral curve with \( \text{Re} \lambda > 0 \) emanate out of the origin as two rays making the angles \( \pm \phi \) to horizontal. Eq. \([4.21]\) implies that the value of \( \cot \phi \) is
smaller than 1 for all $k > \frac{1}{\sqrt{2}}$ (see the Appendix) and so $\phi$ lies between $\frac{\pi}{4}$ and $\frac{\pi}{2}$. These two rays (and their negatives) are mapped to rays with $\Re \eta < 0$. Therefore, no instability of the damped-driven cnoidal waves with $h = \gamma$ is associated with the neighbourhood of $\eta = 0$, no matter what is the value of $k$.

This conclusion is valid for all $\gamma$, including very small ones. A natural question, therefore, is how the stable $\eta$-spectrum becomes unstable when $\gamma$ reaches zero; that is, how can a diagram from the bottom row in Fig. 1 (two rays at the angle $2\phi$ to horizontal) evolve into the corresponding diagram in the top row (two rays at the angle $\phi$ to horizontal). The answer is that for small $\tilde{\gamma}$, $\tilde{\gamma} \ll \kappa$, the map (2.19) reduces to $\eta = \lambda$. On a relatively large scale, namely on the scale $|\eta| \gg \tilde{\gamma}$, the spectrum of $\eta$ looks indistinguishable from the corresponding spectrum of $\lambda$; it includes an eight-shaped curve centred at the origin. However if we zoom in on very small $\eta$ ($|\eta| \sim \tilde{\gamma}$), we will observe that the four rays do not reach the origin on the $\eta$-plane, but form a shape shown in the inset to Fig. 3. As $\gamma \to 0$, the box shown in the inset shrinks to the origin.

B. The cn waves with $h = \gamma$: arbitrary perturbations

The periodic wave is unstable if the linearised-spectrum curve crosses into the $\Re \eta > 0$ half-plane. For the crossing points $\Re \eta = 0$, Eq. (2.19) gives

$$\Im \lambda = \pm \frac{\Re \lambda}{\sqrt{1 - (\Re \lambda/\tilde{\gamma})^2}}.$$ (5.1)

This is an equation for two curves on the $\lambda$-plane. These two curves are straight lines at the origin, making $45^\circ$ to the horizontal. As $\Im \lambda \to \pm \infty$, the curves are asymptotic to the vertical straight lines $\Re \lambda = \pm \tilde{\gamma}$ (see Fig. 4).
of the eight-shaped spectral curve make the angle greater than 45° of the eight only when \( \gamma > \gamma_c(k) \) and unstable otherwise.

Note that when \( h = \gamma \), the original and scaled damping coefficients coincide: \( \gamma = \gamma_c \). Therefore, on the \((\gamma, h)\)-plane, the stability region is given by the ray \( h = \gamma \) with \( \gamma > \gamma_c(k) \). The inverse function \( k_c(\gamma) \) [where \( k_c(\gamma_c(k)) = k \)] defines the boundary of the stability region on the \((\gamma, k)\)-plane (Fig. 5).

VI. SOLUTIONS \( cn^+ \) AND \( cn^- \): \( h > \gamma \)

In this section, we calculate the deformation of the linear eigenvalue spectrum associated with the cnoidal waves \( cn^+ \) and \( cn^- \) as \( h \) grows from the value \( h = \gamma \).

It is not difficult to evaluate the change of the symplectic spectrum as \( h \) deviates from \( \gamma \) and hence the parameter \( \mathcal{E} \) deviates from zero. The corresponding spectrum coincides with the spectrum of the undamped NLS with nonzero parametric driving.

In addition to expanding \( \tilde{\mathcal{Y}} \) and \( \lambda \) as in (4.12), we let \( \mathcal{E} = \mathcal{E}_0 k^2 \) in the operator (2.9a) [which, in turn, is a matrix element of the operator (2.21)]. Substituting these expansions in equation (4.11), the orders \( \kappa^0 \) and \( \kappa^1 \) produce the same equations as in section IV with the same set of solutions. The first difference from the case \( \mathcal{E} = 0 \) arises at the order \( \kappa^2 \) where the equation (4.15) is replaced with

\[
\mathcal{L}_0 v_2 = 2i\partial_X v_1 - (1 + \mathcal{E}_0) v_0 + \lambda_2 w_0 + \lambda_1 v_1,
\]

and the corresponding solvability condition (4.17) with

\[
2i(q|\partial_X v_1) - B(1 + \mathcal{E}_0) q | q \rangle + \lambda_1 (q | w_1) = 0.
\]

[On the other hand, the solvability condition (4.16) is not changed.] As a result, the only matrix element of the

For the given \( h = \gamma \), the cnoidal wave is unstable if the corresponding pair of curves (5.1) intersects the locus of the symplectic spectrum on the \( \lambda \)-plane. Since the rays of the eight-shaped spectral curve make the angle greater than 45° to horizontal at the origin, the curves (5.1) do not have to intersect the spectral curve. They will cross through the “eight” only when \( \gamma \) is large enough (Fig. 4). For some critical value \( \gamma = \gamma_c(k) \), the curves will just touch the “eight”. This critical value defines the range of stability of the cn wave for the given \( k \): the wave is stable when \( \gamma > \gamma_c(k) \) and unstable otherwise.

For some critical value \( \gamma = \gamma_c(k) \), the curves (5.1) intersect the locus of the symplectic spectrum on the \( \lambda \)-plane, 

\[
\text{FIG. 3. Spectrum of linearised eigenvalues of the cn solution with very small } \gamma. \text{ (In this plot, } \gamma = 0.005 \text{ and } k^2 = 0.8. \text{) On a large scale, the spectrum of } \eta \text{ looks like the spectrum of } \lambda; \text{ however zooming in on the neighbourhood of the origin (dotted box, enlarged in the inset) the shape typical for } \eta-\text{curves emerges.}
\]

\[
\text{FIG. 4. The symplectic spectrum of the cn wave (solid) and the curves (5.1) (dashed). For small } \gamma, \text{ the curves intersect the spectrum. The “internal” section of the spectrum corresponds to linearised eigenvalues with } \Re \eta < 0 \text{ whereas the part which lies outside the dashed curves is “unstable”: } \Re \eta > 0. \text{ For large } \gamma, \text{ the entire symplectic spectrum is stable. In this plot, } k^2 = 0.65.
\]

\[
\text{FIG. 5. The critical value of the elliptic modulus as a function of } \gamma. \text{ Shaded is the stability domain; for } k < k_c(\gamma) \text{ the wave is unstable.}
\]
matrix $\mathcal{M}$ that is altered by allowing a nonzero $\mathcal{E}$, is $n_{22}$ which becomes
\[ n_{22} = \frac{4k}{\sqrt{2k^2-1}} K^2 \pm \mathcal{E}_0 N. \]
The quartic equation (4.20) is replaced with
\[ a_0 \lambda_1^4 + (b_0 \mp \mathcal{E}_0 b_1) \lambda_1^2 + (c_0 \mp \mathcal{E}_0 c_1) = 0, \]
where
\[ b_1 = -2(LN)kL_k, \quad c_1 = -\frac{4LN^2}{(1-k^2)(2k^2-1)} \]
are two negative constants.
Defining $\mu$ such that $\lambda_1 = \mathcal{E}_0^{1/2} \mu$, the quartic equation becomes
\[ a_0 \mu^4 + \left( b_0 \mp \frac{\mathcal{E}_0}{\mathcal{E}_0} b_1 \right) \mu^2 + \frac{c_0 \mp \mathcal{E}_0 c_1}{\mathcal{E}_0} = 0. \quad (6.1) \]
Here $\mathcal{E}_0$ is a positive parameter.
Consider the top sign in (6.1), that is, consider eigenvalues pertaining to the cnoidal wave $cn^+$. For $\mathcal{E}_0 = \infty$, two roots of equation (6.1) are equal to zero, $\mu = 0$, while the other two roots take opposite pure imaginary values:
\[ \mu = \pm i(-b_1/a_0)^{1/2}. \]
As $\mathcal{E}_0$ decreases from large values, two pairs of opposite roots move along the imaginary axis, collide pairwise, and appear into the complex plane, forming a quadruplet $\pm \mu, \pm \mu^*$. As $\mathcal{E}_0 \to 0$, the four complex roots diverge from the origin along the straight lines making the angles $\pm \phi$ to the horizontal, with $\phi$ as in (4.21). The trajectories of the roots of the equation (6.1) are related to the manifestly positive integral
\[ \mathcal{E}_0 \to 0, \text{ and (b) that the derivative of this expression is positive:} \]
\[ \frac{d}{dk} [E^2 - (1 - k^2) K^2] = \frac{2}{k} (E - K)^2. \]
3. Since $(LN)k > 0$ while $N_k < 0$, one is led to conclude that $L_k > 0$, the period of the $dn$ wave is a monotonically growing function of $k$. An independent way to see this is to note that the derivative
\[ \frac{dL_{dn}}{dk} = \frac{2\sqrt{2 - k^2}}{k} \left( \frac{E}{1 - k^2} - \frac{2K}{2 - k^2} \right) \]
can be related to the manifestly positive integral
\[ \int_0^K \frac{\sin^2(\xi, k) \csc^2(\xi, k)}{dn^2(\xi, k)} d\xi = \frac{(2 - k^2)E - 2(1 - k^2)K}{3k^4}. \]
4. Finally, we prove the inequality
\[ \frac{k'}{k} \frac{K(k) - E(k)}{E(k)} < 1 \quad (A2) \]
for all $\frac{1}{2} < k^2 < 1$. Here $k' = \sqrt{1 - k^2}$ is the complementary modulus of the elliptic integrals.
We start by writing (A2) as
\[ g(k) < 1, \quad (A3) \]
where we have defined
\[ g(k) = -\frac{k'}{E(k)} \frac{dE}{dk}. \]
At the point \( k_0 = 1/\sqrt{2} \), the integral \( K(k) \) has the property \( dK/dk = K \) which allows to relate \( K \) and \( E \),

\[
K = \frac{1}{k^2(1 + k)} E,
\]

and subsequently evaluate \( g(k_0) \):

\[
g(k_0) = \frac{\sqrt{2} - 1}{\sqrt{2} + 1}.
\]

Therefore, at the point \( k_0 \) the value of the function \( g(k) \) is smaller than \( 1 \) and will remain smaller than \( 1 \) in some neighbourhood of this point. We will now show that this neighbourhood extends all the way to \( k = 1 \).

Assume the contrary, that is, assume that there is a point \( k_*, k_0 < k_* < 1 \), such that

\[
g(k_*) = 1. \tag{A4}
\]

Using the hypergeometric equation satisfied by \( E(k) \),

\[
kk'^2 \frac{d^2 E}{dk'^2} + k'^2 \frac{dE}{dk'} + kE = 0,
\]

we obtain a Riccati equation

\[
\frac{df}{dk} = \frac{f^2}{k} + \frac{k}{k'} \tag{A5}
\]

for the function

\[
f(k) = \frac{k}{k'} g(k).
\]

Integrating both sides of (A5) from \( k_0 \) to \( k_* \) we obtain

\[
f(k_*) - f(k_0) = \int_{k_0}^{k_*} \left[ \frac{f^2}{k} + \frac{k}{k'} \right] dk. \tag{A6}
\]

Since \( f(k) < k/k' \) in the interval \( (k_0, k_*) \), the integral admits a simple bound:

\[
\int_{k_0}^{k_*} \left[ \frac{f^2}{k} + \frac{k}{k'} \right] dk < \int_{k_0}^{k_*} \left[ \frac{1}{k} \left( \frac{k}{k'} \right)^2 + \frac{k}{k'^2} \right] dk. \tag{A7}
\]

The integral in the right-hand side of (A7) is tabular and hence (A6) gives

\[
f(k_*) < 1 + \ln \frac{k_0^2}{1 - k_*^2}, \tag{A8}
\]

where we have also used \( f(k_0) < 1 \).

On the other hand, one can readily show that the function

\[
F_1(k) = 1 + \ln \frac{k_0^2}{1 - k_*^2}
\]

is smaller than \( F_2(k) = k/k' \) for all \( k > k_0 \). [Indeed, we have \( F_1(k_0) = F_2(k_0) = 1 \) but \( dF_1/dk < dF_2/dk \) for all \( 0 < k < 1 \).] In particular, \( F_1(k_*) \) is smaller than \( F_2(k_*) \), that is,

\[
1 + \ln \frac{k_0^2}{1 - k_*^2} < \frac{k_*}{k_*'} \tag{A9}
\]

where \( k_*' = \sqrt{1 - k_*^2} \). Using (A9), the inequality (A8) becomes \( f(k_*) < k_*/k_*' \) or, equivalently,

\[
g(k_*) < 1 \tag{A10}
\]

which contradicts (A4).

The contradiction proves that no point \( k_* < 1 \) satisfying (A4) can exist. Therefore the inequality (A3) (and thus, (A2)) remain valid for all \( k \) between \( k_0 \) and 1.

**ACKNOWLEDGEMENTS**

M.M. was supported by the National Research Foundation (NRF) of South Africa and the National Institute for Theoretical Physics (NITheP).

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