Vacuum Rabi Oscillation of an Atom without Rotating-Wave Approximation

WANG Fa-Qiang(王发强)**, LIU Wei-Ci(刘伟慈), LIANG Rui-Sheng(梁瑞生)

Lab of Photonic Information Technology, School of Information and Photoelectronic Science and Engineering, South China Normal University, Guangzhou 510006

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We have investigated vacuum Rabi oscillation of an atom coupled with single-mode cavity field exactly, and compared the results with that of the Jaynes–Cummings (J-C) model. The results show that for resonant case, there is no Rabi oscillation for an atom. For small detuning and weak coupling case, the probability for the atom in excited state oscillates against time with different frequencies and amplitudes from that of the J-C model. It exhibits that the counter-rotating wave interaction could significantly affect the dynamic behaviour of the atom, even under the condition in which the RWA is considered to be justified.

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A two-level atom interacting with a single cavity mode, described by the Jaynes–Cummings (J-C) model, which neglects counter-rotating terms, is widely used in quantum optics and has the potential to constitute the basic building block of quantum computers.[1–4] Making a rotating-wave approximation (RWA) in the Hamiltonian strongly simplifies the mathematical treatment of the problem and usually gives exact solution of the approximate Hamiltonian. In spite of the simplicity of the J-C model, the dynamics has turned out to be various and complex, describing many physical phenomena, such as Rabi oscillations, collapse–revivals, squeezing, and atom–field entanglement.[5]

Generally, the RWA, neglecting counter-rotating, is considered to be justified for small detuning and small ratio of the atom–field coupling divided by the atomic transition frequency. In atom–field cavity systems, this ratio is typically of the order $10^{-7} \sim 10^{-6}$. Recently, cavity systems with very strong couplings have been discussed.[6] The ratio might become order of magnitudes larger in solid state systems and the counter-rotating wave terms on the decay behaviour of an atoms coupled with one-mode cavity, without rotating-wave approximation. For simplicity, we will investigate vacuum Rabi oscillation of the system.

Now we restrict our attention to a two-level atom coupled with a perfect one-mode cavity field, of which the Hamiltonian is[2]

$$H = H_a + H_f + H_{af},$$

where

$$H_a = \frac{\omega_0 \sigma_z}{2},$$

$$H_f = \omega a^\dagger a,$$

$$H_{af} = g(\sigma_+ + \sigma_-)(a^\dagger + a),$$

where $\omega_0$ is the atomic transition frequency between the ground state $|0\rangle$ and excited state $|1\rangle$, $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$, and $\sigma_+ = |1\rangle\langle 0| + |0\rangle\langle 1|$ are pseudospin operators of atom; $a^\dagger$ and $a$ are the creation and annihilation operators of the cavity field mode corresponding frequency $\omega$; and $g$ is the coupling constant between the transition $|1\rangle - |0\rangle$ and the field mode.

The non-perturbative reduced master equation of the atom could be derived by path integrals[8]

$$\frac{\partial}{\partial t} \rho_a = -i \mathcal L_a \rho_a - \int_0^t ds \langle \mathcal L_{af} e^{-i\mathcal L_0(t-s)} \mathcal L_{af} e^{-i\mathcal L_0(s-t)} \rangle_f \rho_a,\quad (5)$$

where $\mathcal L_0$, $\mathcal L_a$ and $\mathcal L_{af}$ are the Liouvillian operators defined as

$$\mathcal L_0 \rho \equiv [H_a + H_f, \rho]$$

and

$$\mathcal L_a \rho \equiv [H_a, \rho]$$

and $\mathcal L_{af}$ stands for partial trace of cavity mode $\text{Tr}_f \{\cdots \rho_f\}$, $\rho_f$ is the initial density matrix of the cavity field.[8] By the formula

$$L_1 \rho = [H_1, \rho], \quad L_0 \rho = [H_0, \rho],$$

$$e^{L_0 t} L_1 e^{-iL_0 t} \rho = [e^{H_0 t} H_1 e^{-iH_0 t}, \rho],$$

the non-perturbative reduced master equation of the atom could be obtained if the cavity field is initially in vacuum state

$$\frac{\partial}{\partial t} \rho_a = [\varepsilon_0 J_0 + \varepsilon_+ J_+ + \varepsilon_- J_-] \rho_a - g^2 [\alpha R + f] \rho_a$$

$$+ [\nu_0 K_0 + \nu_+ K_+ + \nu_- K_-] \rho_a, \quad (6)$$

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**Email: fqwang@scnu.edu.cn

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where \( \varepsilon_0 = -2i(\omega_0 - g^2 \alpha^I + g^2 f^I) \), \( \varepsilon_+ = g^2(\alpha + f^*) \), \( \varepsilon_- = g^2(\alpha^* + f) \). \( \nu_0 = 2g^2(\alpha R - f R) \), \( \nu_+ = 2g^2 \alpha R \), \( \nu_- = 2g^2 f R \). \( J_0 \), \( J_+ \), \( J_- \), \( K_0 \), \( K_+ \) and \( K_- \) are super-operators defined as

\[
J_0 \rho_0 \equiv \left[ \frac{\rho_0}{4}, \rho_0 \right], \quad J_+ \rho_- \equiv \frac{\rho_0}{2} \rho_0 + \rho_0 \sigma_+ + J_- \rho_0 \equiv \rho_0 \rho_0 - \sigma_-, \\
K_0 \rho_0 \equiv \left( \rho_0 \rho_0 + \rho_0 \sigma_+ - J_+ \rho_0 \right)/2, \\
K_+ \rho_- \equiv \rho_0 \rho_0 - J_- \rho_0 \equiv \rho_0 \rho_0 - \sigma_+, \\
\alpha = \frac{1 - \exp(-i \Delta t)}{i \Delta}, \\
f = \frac{-\exp(i \delta t) + 1}{i \delta},
\]

where \( \Delta = \omega + \omega_0, \delta = \omega_0 - \omega; \alpha R, \alpha^I, \alpha^* \) and \( f R, f^I, f^* \) are the real part, image part and conjugate of \( \alpha \) and of \( f \) respectively; \( \alpha \) comes from the counter-rotating wave interaction and \( f \) comes from the rotating-wave interaction.

Using the algebraic approach, the formal solution of Eq. (7) is obtained\,[2,9]

\[
\rho_a(t) = e^{-T_k} T e^{j f t} dt (e^{j (0 + \epsilon, J_+ + \epsilon), J_-}) \\
\times T e^{j f t} dt (e^{j (0 + K_0 + \epsilon, K_+, K_- - K_0)} \rho_0(0),
\]

where \( \Gamma_k = g^2(\alpha R + FR) \) and

\[
\tilde{\alpha} = \int_0^t \alpha dt = \frac{1 - \exp(-i \Delta t) - i \Delta t}{\Delta^2} \equiv \tilde{\alpha} R + i \tilde{\alpha} I, \\
F = \int_0^t f dt = \frac{1 + i \delta t - \exp(i \delta t)}{\delta^2} \equiv FR + i F I,
\]

where \( \tilde{\alpha} R, \tilde{\alpha} I, \tilde{\alpha}^* \) and \( FR, FI, F^* \) are real part, image part and conjugate of \( \tilde{\alpha} \) and of \( F \), respectively. \( T \) is the time-ordering operator.

Using the decomposition of SU(2) operator, the time-ordered exponential operators could be disentangled.\,[2] The exact solution of master equation Eq. (6) is obtained

\[
\rho_a(t) = e^{-T_k} \tilde{\rho}(t),
\]

\[
\tilde{\rho}(t) = \left( \begin{array}{cc}
lp_0^{10}(0) + m \rho_{00}^{00}(0) & x \rho_0^{10}(0) + y \rho_0^{01}(0) \\
q \rho_0^{01}(0) + r \rho_0^{10}(0) & n \rho_{00}(0) + p \rho_{00}(0)
\end{array} \right),
\]

\[
l = e^{k_b/2} + e^{-k_b/2} k_b k_b, \\
n = e^{-k_b/2}, \quad p = e^{-k_b/2} k_b, \quad m = e^{-k_b/2} k_b,
\]

\[
x = e^{j n/2} + e^{-j n/2} j_+ j_-, \quad y = e^{-j n/2} j_+, \quad z = e^{j n/2} j_+
\]

where \( j_+, j_- \) and \( k_+, k_- \) satisfy the following differential equations\,[2]

\[
\dot{X}_+ = \mu_+ - X_+^2 + p \rho_0 X_+, \\
\dot{X}_0 = \mu_0 - 2 \mu X_+, \\
\dot{X}_- = \mu_- \exp(X_0),
\]

\[
\mu = \varepsilon \quad \text{for} \quad X = j \quad \text{and} \quad \mu = \nu \quad \text{for} \quad X = k. \quad \text{Generally, the Riccati equation could not be solved analytically. Next, we will investigate it numerically.}
\]

When the atom is initially in excited state \( \rho_a(0) = 1 \), the probability \( P_e \) for the atom in excited state at time \( t \) is \( P_e = l \) (Eq. (13)). The counterpart result for the Jaynes–Cummings model is

\[
P_{e JC} = 1 - \left[ \frac{2g \sin(\Omega/2)}{\Omega} \right]^2,
\]

where \( \Omega = \sqrt{\delta^2 + 4g^2} \).

Generally, the RWA is considered valid for small detuning and small ratio of the atom–field coupling divided by the atomic transition frequency. In order to compare our results with that of RWA, we focus the discussion for the case of weak coupling and small detuning. We refer to our exact solution as exact model in the following discussion. In all the figures, the lines for the J-C model are calculated from Eq. (20) and the lines for the exact model are calculated from Eq. (13).

(A) For resonant and weak coupling case, Fig. 1 reveals that the probability \( P_e \), for the J-C model, oscillates against time, while it decays to zero in finite time for exact model. Equations (11) and (20) show that there is an attenuation factor \( \exp(-g^2 t^2 / 2) \) for exact model when \( \omega_0 \gg 1 \), while \( P_{e JC} = 1 - \sin^2(g t) \) for the J-C model.

\[
\text{Fig. 1.} \quad P_e \text{ as a function of } gt \text{ for the initially excited atom for } \omega_0 = 50g \text{ and } \delta = 0: \text{ dotted line for the J-C model, solid line for the exact model.}
\]
Form Eqs. (6)–(8), we could find that the contribution of energy-conserving process, corresponding to rotating-wave terms $\sigma_- a^\dagger$ and $\sigma_+ a$ in Hamiltonian, varies with frequency of $\omega_0 - \omega$. The one of energy-non-conserving process, corresponding to counter-rotating wave terms $\sigma_+ a^\dagger$ and $\sigma_- a$ in Hamiltonian, varies with frequency of $\omega_0 + \omega$. The attenuation factor in Eq. (11) comes from the destructive superposition of the different frequency contribution associated with rotating-wave terms and counter-rotating wave terms.

From Fig. 2, we could find, with the decreasing rotation associated with rotating-wave terms and counter-rotating wave terms. The exact model, while that is $\Omega$ for the J-C model. For the oscillation frequency, when the coupling strength $g$ decreases to zero, the oscillating frequency for $P_e$ for the J-C model, $\Omega = \sqrt{\delta^2 + 4g^2}$, will decrease to $\delta$, which is the oscillating frequency for $P_e$ for exact model.

From the above discussion, we find that $P_e$ will decay to zero for the resonant case and there is Rabi oscillation for detuning case. That is because spontaneous emission is inhibited if there is detuning between atomic frequency and cavity mode, and enhanced if the cavity is resonant. [10,11]

In summary, we have investigated vacuum Rabi oscillation of an atom coupled with single-mode cavity field exactly, and compared the results with that of the J-C model.

Firstly, the results show that, for resonant case, there is no Rabi oscillation for an atom, while there is Rabi oscillation for an atom in the J-C model.

Secondly, for the small detuning and weak coupling case, the probability for the atom in excited state will oscillate against time with different frequencies and amplitudes from that of the J-C model. Thirdly, the results also reveal that the Rabi oscillation frequency is only dependent on the detuning value, while it is dependent on both the coupling strength and the detuning value for J-C model.

On the whole, it exhibits that there is a significant effect on the dynamic behaviours of the atom due to the counter-rotating wave interaction even under the condition in which the RWA is considered to be justified, which is coincident with the result of Ref. [12].

![Fig. 2. $P_e$ as a function of $\delta t$ for initially excited atom with $\delta = 0.1\omega_0$. J-C model: dotted line for $\omega_0 = 20g$ and dashed line for $\omega_0 = 40g$; exact model: solid line for $\omega_0 = 20g$ and solid line plus x for $\omega_0 = 40g$.](image)

![Fig. 3. Oscillating amplitude difference of $P_e$ for the J-C model and the exact model as a function of $g/\omega_0$ for the initially excited atom with $\delta = 0.1\omega_0$.](image)

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