ON SOME EXAMPLES IN SYMPLECTIC TOPOLOGY

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Abstract.
Article is devoted to the Examples 2 and 3 of the symplectic solvable Lie groups $R$ with some special cohomological properties, which have been constructed by Benson and Gordon. But they are not succeed in constructing corresponding compact forms for symplectic structures on these Lie groups. Recently A. Tralle proved that there is no compact form in the Example 3. But his proof is rather complicated and uses some very special topological methods.

We propose much more simpler (and purely algebraic) method to prove the main result of the Tralle's paper. Moreover we prove that for Example 2 there is no compact form too. But it appears that some modification of the construction of the Example 2 gives some other example of a solvable Lie group $R'$ with the same cohomological properties as $R$, but with a compact form.

Let $(M, \omega)$ be a compact symplectic manifold. This article is devoted to some examples, which have been constructed in [1], where some problems about Kählerian structures on solvmanifolds (i.e. homogeneous spaces $R/\Gamma$ of solvable Lie groups $R$ with discrete stationary subgroups $\Gamma$) are studied. There are Examples 2 and 3 of the symplectic solvable Lie groups $R$ in [1] with some special properties (closely related to the properties of Kählerian Lie groups). But the authors of [1] are not succeed in constructing corresponding compact forms $R/\Gamma$ for this symplectic structures on Lie groups. These examples in [1] have been constructed for the purpose of illustration of the conditions of the main result (see Theorem 2 there about a structure of compact solvmanifolds $R/\Gamma$ which admit a Kähler structure) of this paper. Recently A. Tralle [2] proved that there is no compact form (i.e. a compact solvmanifold $R/\Gamma$ for the Lie group $R$) in the Example 3 from [1]. But the proof in [2] is rather complicated and uses some special topological methods (rational models etc.). We propose much more simpler (and purely algebraic) method to prove the main result from [2]. Moreover we prove that for Example 2 from [1] there is no compact form too. But it appears that some modification of the construction of the Example 2 from [1] gives some other example of a solvable Lie group $R'$ with a compact form $M = R'/\Gamma$.

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Firstly we are going to describe the Lie algebras from the Examples 2 and 3 in [1].

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Example 2. Here a Lie algebra $L(R)$ equals to $\text{Span}(A, B, X_1, X_2, X_3, Z_1, Z_2, Z_3)$, where

\[
\begin{align*}
[X_2, X_3] &= 2Z_1, [X_1, X_3] = Z_2, [X_1, X_2] = -Z_3, \\
[A, X_1] &= -X_1, [A, X_2] = -2X_2, [A, X_3] = 3X_3, \\
[A, Z_1] &= Z_1, [A, Z_2] = 2Z_2, [A, Z_3] = -3Z_3.
\end{align*}
\]

We have $L(R) = A + U$, where

$A = \text{Span}(A, B), U = \text{Span}(X_1, X_2, X_3, Z_1, Z_2, Z_3)$.

Example 3. Here $L(R) = \text{Span}(A, B, X_1, Y_1, X_1, X_2, Y_2, Z_2)$, where

\[
\begin{align*}
[X_1, Y_1] &= Z_1, [X_2, Y_2] = Z_2, \\
[A, X_1] &= X_1, [A, X_2] = -X_2, \\
[A, Y_1] &= -2Y_1, [A, Y_2] = 2Y_2, \\
[A, Z_1] &= -Z_1, [A, Z_2] = Z_2
\end{align*}
\]

Here we have $L(R) = A + U$, where

$A = \text{Span}(A, B), U = \text{Span}(X_1, Y_1, X_2, Y_2, Z_1, Z_2)$.

Now we give some general considerations about Examples 2 and 3 from [1].

In both Examples the Lie group $R$ (simply connected) has a form $R \times (R \times \phi N)$, where $R' = R \times \phi N$ is a semi-direct product, corresponding to a homomorphism $\phi : R \to \text{Aut}(N)$. In both this Examples $N$ are some six-dimensional 2-nilpotent Lie groups, $L(N) = U$. For the Example 2 the Lie algebra for Lie group $N$ is the free 2-nilpotent Lie algebra with three generators, for the Example 3 we have $N = N_3(R) \times N_3(R)$, where $N_3(R)$ is the unique three-dimensional non-Abelian simply connected nilpotent Lie group ($N_3(R)$ is isomorphic to the group of all real unipotent 3-matrices). In both cases the homomorphism $\phi$ has only real characteristic numbers and therefore Lie groups $R$ are triangular (or, in other terms, completely solvable); about these groups see, for example,[3]. For the triangular Lie groups there is some very important property: the algebra of cohomologies $H^*(L(R), R)$ for the corresponding Lie algebra $L(R)$ is isomorphic to $H^*(R/\Gamma, R)$ (where $\Gamma$ is a lattice in $R$, i.e. the discrete subgroup in $R$ with a compact factor-space $R/\Gamma$) [5].

Also in both Examples we have $[R, R] = N$. We need some general result about the lattices in triangular Lie groups. This result can be found in [6], but we prefer to give here some explicit (and more easier than in [6]) proof.

**Proposition 1.** Let $R$ be a triangular Lie group and $\Gamma$ be a lattice in $R$. Then the commutator $[\Gamma, \Gamma]$ of $\Gamma$ is a lattice in $[R, R]$. In particular, $\Gamma \cap [R, R]$ is a lattice in $R$.

**Proof.** We may suppose that $R$ is simply connected. Consider $N = [R, R], N$ is a simply connected nilpotent Lie group, therefore we have a natural structure of an algebraic group on $N$ (see, for example [7,4]). Let us consider the algebraic closure $N_1$ of a subgroup $[\Gamma, \Gamma]$ in $N$, this subgroup is a lattice in $N_1$. It is clear that $\Gamma$ normalizes $N_1$ (because $\Gamma$ normalizes $[\Gamma, \Gamma]$). We have the following simple Lemma (see [8]):
Lemma 1 [8]. Let $R$ be a triangular connected Lie group and $F$ be a connected Lie subgroup in $R$. Then the normalizer $N_R(F)$ of $F$ in $R$ is connected.

By the way, this Lemma shows that the condition 1) in the main result of [9] (Theorem 1 - proof of the Benson-Gordon conjecture in some very special case) may be excluded. But the second condition in this theorem are extremely strong and therefore there are a only few situations where this theorem may be applied.

Due to this Lemma 1 we see that the normalizer $N_R(N_1)$ of $N_1$ in $R$ is connected ($N_1$ is connected as an algebraic subgroup in the connected nilpotent algebraic group $N$). But $N_R(N_1)$ contains $\Gamma$ and it is easy to understand that $N_R(N_1)$ is necessarily equals to $R$.

Now we consider $R_1 = R/N_1, \Gamma_1 = \Gamma/[\Gamma, \Gamma]$, where $\Gamma_1$ is an Abelian lattice in the triangular Lie group $R_1$. It is easy to understand (due to triangularity of $R_1$) that $R_1$ must be Abelian too. Consequently $N_1 \supset [R, R]$, therefore $N_1 = [R, R]$. We get that $[\Gamma, \Gamma]$ is a lattice in $R$. The intersection $\Gamma \cap [R, R]$ is a discrete subgroup in $[R, R]$ and contains $[\Gamma, \Gamma]$, therefore $\Gamma \cap [R, R]$ is the lattice in $[R, R]$ too.

We continue our consideration of Examples 2 and 3 from [1]. For the nilpotent Lie group $N$, mentioned above, we have due to Proposition 1 that $\Gamma \cap N$ is a lattice in $N$ (in both Examples).

It is well known (see, for example [7]) that if a nilpotent Lie group $N$ has a lattice, then the corresponding Lie algebra has a rational structure (i.e., it’s structure constants in an appropriate basis are rational). If we consider a natural structure of an algebraic group on $N$ (we suppose that $N$ is simply connected) then in this case $N$ will be defined over $\mathbb{Q}$ and the lattice in $N$ is commensurable with a subgroup $N_\mathbb{Z}$ of integer points of $N$.

In [10] all nilpotent Lie algebras of dimension no more than 6 are classified (over an arbitrary field of characteristic 0). In particular we get a classification of the rational Lie algebras up to dimension 6. This classification may be considered as a classification of the lattices (up to commensurability) in real nilpotent simply connected Lie groups $N$. In is interesting to mention that in some such $N$ there are an infinite series of non commensurable lattices (it is true, for example, for $N_3 \times N_3$).

Let us consider now the Example 3 from [1] in details. Here $N = N_3 \times N_3$. We have $L(R) = R \oplus L'$, where $R = \text{Span}(B), L' = R + U$ (here $R = \text{Span}(A), U = L(N)$. An action of $A$ on $U$ is defined by the matrix $\Phi = \text{diag}(1, -2, -1, -1, 2, 1)$. Let us suppose that there is some lattice $\Gamma$ in $R$. Then by virtue of Proposition 1 there is a lattice in $N$. Therefore on $N$ (and on the corresponding Lie algebra $L(N)$) it must be some rational structure. It is easy to understand that there is a lattice $\Gamma' = Z + \phi(\Gamma \cap N)$ in $R'$. Let $\gamma$ be a generator of $Z$ in this decomposition. Then the action of $\gamma$ on $L(N)$ is equal to the action of $C \cdot J$, where $C = \text{exp}(\Phi)t_0$ for some $t_0 \in R$ and $J$ is some unipotent matrix (in fact $J$ is one of the elements of the adjoint Lie group for $N$). The characteristic numbers of $C$ are $z, 1/z, 1/z, z^2, 1/z^2$, where $z = \text{exp}(t_0), t_0 \neq 0$.

An action of $C$ on $L(N)$ induces an action on $L_{ab} = L(N)/[L(N), L(N)]$ - abelianization of $L = L(N)$. For the lattice $D = \Gamma \cap N$ its intersection $D \cap [N, N]$ is a lattice in $[N, N]$ (see [7]), consequently $D/D \cap [N, N]$ is a lattice in Abelian Lie group $N/[N, N] = R^3$. The action of $C$ on $[N, N]$ preserves a lattice (which is isomorphic to $Z^2$. The characteristic numbers of this action are $z, 1/z$, therefore $z + 1/z = n$ (as the trace of matrix) for some $n \in N$. 

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Let us consider the rational structure (i.e. structure of rational Lie algebra) on \( L(N) \), which is corresponding to the lattice \( \Gamma \cap N \).

**Lemma 2.** Let \( L \) be a Lie algebra over \( \mathbb{Q} \), for which \( L \otimes \mathbb{R} = n_3(\mathbb{R}) \times n_3(\mathbb{R}) \). Then \( L \) is isomorphic (over \( \mathbb{Q} \)) to some Lie algebra \( L(p_1, p_2, \ldots p_k) \), where \( p_i \) are the some pairwise different primes or zero, which defined by such commutator relations:

\[
[X_1, X_2] = X_5, [X_2, X_4] = X_5, [X_1, X_4] = X_6, [X_2, X_3] = q \cdot X_6
\]

, where \( q = p_1 \cdot p_2 \cdot \ldots \cdot p_k \).

**Proof.** We use the classification of six-dimensional Lie nilpotent algebras over field \( K = \mathbb{Q} \) (see [10]). Our lattice \( D \) is 2-nilpotent and the ranga of its center and commutator equal to 2 (is follows from the corresponding properties of \( N \) - see above). There are only a few Lie algebras with such properties in the list from [10]. They are (we use the notation from [4]):

\( g_{3,1} \oplus g_{3,1} \) (where \( g_{3,1} = n_3 \), \( g_{5,1} \oplus \mathbb{Q}, g_{6,4}, g_{6,5} \)).

It is easy to verify that for these algebras tensored by \( \mathbb{R} \) only \( g_{3,1} \oplus g_{3,1} \) and \( g_{6,5} \) are isomorphic to \( n_3(\mathbb{R}) \times n_3(\mathbb{R}) \).

For \( g_{6,5} \) the commutator relations are:

\[
[X_1, X_2] = X_5, [X_2, X_4] = X_5, [X_1, X_4] = X_6, [X_2, X_3] = q \cdot X_6
\]

, where \( q \geq 0 \) belongs to \( \mathbb{Q}/\mathbb{Q}^2 \) (for \( q = 0 \) we get \( g_{3,1} \oplus g_{3,1} \)). Therefore we may suppose that \( q = p_1 \cdot p_2 \cdot \ldots \cdot p_k \).

For \( q = 2 \) some construction of the corresponding rational Lie algebra can be found in [10].

Now we consider Lie algebras \( g_{6,5} \) for \( q > 0 \). It is easy to calculate the group of their automorphisms. There are two special one-dimensional groups of these automorphisms - \( A_\alpha, B_\beta \):

\[
A_\alpha : X_1 \to X_1, X_2 \to X_2, X_3 \to \alpha X_3, X_4 \to \alpha X_4, X_5 \to \alpha X_5, X_6 \to \alpha X_6,
\]

\[
B_\beta : X_1 \to \beta X_1, X_2 \to \beta X_2, X_3 \to X_3, X_4 \to X_4, X_5 \to \beta X_5, X_6 \to \beta X_6,
\]

where \( \alpha, \beta \in \mathbb{R} \setminus \{0\} \).

It is easy to proof that \( \text{Aut}(g_{6,5}) = (A \cdot B) \cdot U \), here \( F = A \cdot B \) is the reductive part (an Abelian two-dimensional group) of \( \text{Aut}(g_{6,5}) \) and \( U \) - its unipotent radical \( (\dim U = 7) \). A matrix realization of \( F \) can be written in a such form:

\[
F = \{\text{diag}(\beta, \beta, \alpha, \alpha, \alpha \beta, \alpha \beta)\},
\]

the latter two elements \( \alpha \beta \) are corresponding to the action on the center of our Lie algebra.

We have a rational structure on \( L(N) \) due to lattice in \( N \) (see above). In virtue of Lemma 2 this structure is isomorphic to \( g_{6,5} \) for some \( q \geq 0 \). Our first case will be \( q > 0 \). The lattice in \( R' \) gives as an automorphism of \( L(N) \) with the characteristic numbers \( z, 1/z, 1/z, z^2, 1/z^2, \) where \( z = \exp(t_0) \). We have \( z > 0 \).
The only case when the set of these numbers \(z, 1/z, 1/2, z^2, 1/2^z\) equals to a set \((\beta, \beta, \alpha, \alpha, \alpha\beta, \alpha\beta)\) is when \(z = 1\). But \(z = \exp(t_0)\) and \(t_0 \neq 0\). We have a contradiction.

Now we consider the case when \(q = 0\), here \(L = L(N) = n_3(Q) \oplus n_3(Q)\). In this case the group of automorphisms has a decomposition \(GL_2(Q) \times GL_2(Q) \cdot U\), here \(U\) is the nilradical (dimU = 8), two \(GL_2(Q)\) are corresponding to the groups of linear transformations of \(n_3/[n_3, n_3]\). Let us consider \(L_{ab} = L/[L, L] = Q^4\) and the induced action of \(C\) on this vector space. The characteristic numbers of this action are \(z, 1/z, z^2, 1/z^2\). We know that \(z\) is real and positive. If \(z \neq 1\) then all the numbers mentioned above are distinct. Therefore this case there are only three decompositions of \(L_{ab}\) into a direct sum of two two-dimensional subspaces. If \(e_1, e_2, e_3, e_4\) are proper vectors, corresponding to \(z, 1/z, z^2, 1/z^2\), then these decompositions are \(\text{Span}(e_1, e_2) \oplus \text{Span}(e_3, e_4), \text{Span}(e_1, e_3) \oplus \text{Span}(e_2, e_4), \text{Span}(e_1, e_4) \oplus \text{Span}(e_2, e_3)\). We are going to show that the first and the last decompositions over \(Q\) are impossible.

Let us suppose that the first decomposition is defined over \(Q\) (for the rational structure on \(L = L(N)\)). Then \(z + z^2 = k, 1/z + 1/z^2 = l\) for some natural \(k, l \in \mathbb{N}\). From this relations it follows that \(z = (kl - 1)/l\), i.e. \(z\) must be rational. But we have also \(z + 1/z \in \mathbb{N}\). It is easy to understand that it is possible only if \(z = 1\). We know that \(z \neq 1\), therefore this decomposition into the direct sum is impossible. Analogously we can prove that the third decomposition is impossible too.

As we consider the case \(q = 0\), when the rational structure on \(L(N)\) is isomorphic to \(n_3 \oplus n_3\), it is necessary to have some decomposition for \(n_{ab}\) (which is corresponding to the decomposition of \(L\) into a direct sum due to its definition). Moreover, this decomposition of \(L_{ab}\) must generate some decomposition of \(L\) into a direct sum. As it is proved above, we have only one variant of decomposition: \(L_{ab} = \text{Span}(e_1, e_3) \oplus \text{Span}(e_2, e_4)\). But \([e_1, e_2] = [e_2, e_4] = 0\) (due to commutator relations), therefore this decomposition can not correspond to the decomposition of \(L\) into a direct sum of ideals \((e_1, e_3)\) cannot generate such ideal, also for \((e_2, e_4)\). Once again we get a contradiction.

In all subcases we have the contradictions. Therefore there is no lattices in the Lie group \(R\) from the Example 3 in [2].

Now we are proceed to the Example 2 from [1]. Here we are going to prove the nonexistence of any lattice in the corresponding Lie group \(R\) too. But for this example we’ll give below some modification \(R^*\) of \(R\) with the same geometrical and cohomological properties as \(R\) and with some lattice.

The proof of nonexistence of a lattice for \(R\) from the Example 2 is analogous but much more shorter that in the case of the Example 3. Here we have corresponding Lie group \(R’ = R \cdot \phi N\). We suppose that there is a lattice in \(R\), then we have a lattice \(\Gamma = \mathbb{Z} \cdot D\) in \(R’\), where \(D = \Gamma \cap N\) is a lattice in \(N\). The Lie algebra of \(N\) is the free 2-nilpotent Lie algebra with 3 generators, it is \(g_{0, 3}\) over \(Q\) in [10]).

We have \(L = V + \Lambda^2 V, \text{dim}V = 3, Z(L) = \Lambda^2 V\). The action on \(L\) of the generator \(B \in R\) in the decomposition of \(L(R)\) has the characteristic numbers \(-1, -2, 3, 1, 2, -3\) (it follows from the definition of \(L\)). Therefore the action of the generator \(\gamma\) of \(Z\) in the decomposition of \(\Gamma\) has the characteristic numbers \(1/z, 1/z^2, z^3, z, z^2, 1/z^3\), where \(z = \exp(Bt_0), t_0 \neq 0\). As the commutator is invariant under this action of \(\gamma\), for the corresponding rational structure on \(L\) we have two actions, defined over \(Q\) - on \(V\) and \(Z(n) = \Lambda^2\) (three-dimensional vector spaces). The corresponding characteristic numbers for the action on \(V\) are \(1/z, 1/z^2, z^3, \ldots\).
therefore, in particular, for their sum (=trace) we have $1/z + 1/z^2 + z^3 = m$ for some $m \in N$. analogously for the action on $\Lambda^2V$ the characteristic numbers are $z, z^2, 1/z^3$, hence $z + z^2 + 1/z^3 = n$ for some $n \in N$. We get two equations, they may be rewritten in a form of a polynomial system of equations:

- $x_5 - mx^2 + x + 1 = 0$,
- $x_5 + x^4 - nx + 3 + 1 = 0$.

We are going to solve this system. For this we use the method of finding the Groebner Basis for the systems of polynomial equations [11]. With aid of the computer program Maple V we get such set of polynomials (they generate the same polynomial ideal as the equations from our system) as the Groebner Basis for our system:

- $-7m^2 + m^3 - m^4 - m^5 - 13mn - m^2n + 5m^3n - 3m^4n - 7n^2 - mn^2 + 10m^2n^2 + n^3 + 5mn^3 + m^3n^3 - n^4 - 3mn^4 - n^5$,
- $-28m + 4m^2 - 4m^3 - 4m^4 - 20n - 16mn + 29m^2n - 8m^3n - m^4n - 8n^2 + 8mn^2 + 17m^2n^2 - m^2n^2 - 5mn^3 + 12m^2n^3 + 2m^3n^3 + m^4n^3 - 10n^4 - 12mn^4 - 2m^2n^4 + 3m^3n^4 - 6n^5 - 12mn^5 + m^2n^5 - 6n^6 - m^2n^6 + 2n^7 + mn^7 + 12nx + 8n^2x + 23n^3x + 9n^4x + 12n^5x + n^5x - n^6x$,
- $-52m + 6m^2 + 4m^3 - 6m^4 - 20n - 68mn + 63m^2n - 14m^3n - m^4n - 52n^2 + 26mn^2 + 24m^2n^2 + 3m^3n^2 + 3m^4n^2 + 16m^3n^3 - 13mn^3 - 4m^2n^3 + 9n^3n^3 - 18n^4 - 39mn^4 + 4m^3n^4 - 20n^5 - mn^5 - 3m^2n^5 + 3n^6 + 3mn^6 + 40mx + 68nx + 18n^2x + 33n^3x + 22n^4x + 2n^5x + 4n^6x - 3n^7x$,
- $-20 - 26m - 7m^2 - 8m^3 - 3m^4 - 20m + 36mn + 9m^2n - 7m^3n - 2m^4n + 24n^2 + 33mn^2 - 18m^2n^2 + 4m^3n^2 - m^4n^2 + 18n^3 - 14mn^3 + 8n^2n^3 - 3m^3n^3 - 14n^4 + 8m^4 - 3m^2n^4 + 10n^5 + 2mn^5 + m^2n^5 - 2n^6 - mn^6 - 20x + 4nx - 31n^2x + 9n^3x - 14x^2 - n^3x + n^5x - 10n^2x^2$,
- $40 - 52m + 6m^2 + 4m^3 - 6m^4 - 20n - 68mn + 63m^2n - 14m^3n - m^4n - 52n^2 + 26mn^2 + 24m^2n^2 + 3m^3n^2 + 3m^4n^2 + 16m^3n^3 - 13mn^3 - 4m^2n^3 + +9n^3n^3 - 18n^4 - 39mn^4 + 4m^3n^4 - 20n^5 - mn^5 - 3m^2n^5 + 3n^6 + 3mn^6 + 40mx + 68nx + 18n^2x + 33n^3x + 22n^4x + 2n^5x + 4n^6x - 3n^7x + 40nx^2 - 40x^3$.

Let us consider the corresponding polynomial equations. First equation is the resultant of our system. Next two equations are linear in $x$. If in one of these two equations the coefficient for $x$ is nonzero, we get that $x$ is rational. But a rational root of our initial equation $x^5 - mx^2 + x + 1 = 0$ have to divide 1, therefore this root equals to 1 (we remind that $x > 0$). For the second equation the coefficient for $x$ equals to $12n + 8n^2 + 23n^3 + 9n^4 + 12n^5 + n^7 - n^8$, the only root in $\mathbb{N}$ of this polynomial is $n = 3$. For the coefficient from the second equation we get $-120 + 40m = 0$ (using $n = 3$), therefore $m=3$. Our initial system for $m = n = 3$ has only one solution $x = 1$. Therefore we get $x = 1$, but it is a contradiction and there is no lattices in $R$ from Example 2 in [1].

Let us consider some variant of the Example 2 from [1]. Let $L$ be a Lie algebra with basis

$$A, B, X_1, X_2, X_3, Z_1, Z_2, Z_3$$

and commutator relations

$$[X_1, X_2] = -Z_3, [X_1, X_3] = Z_2, [X_2, X_3] = 2X_1$$

$$[A, X_1] = \lambda_1 X_1, [A, X_2] = \lambda_2 X_2, [A, X_3] = \lambda_3 X_3,$$
\[ [A, Z_1] = (\lambda_2 + \lambda_3)Z_1, [A, Z_2] = (\lambda_1 + \lambda_3)Z_2, [A, Z_3] = (\lambda_1 + \lambda_2)Z_3. \]

for some \( \lambda_i \in \mathbb{R} \). Following [1] we set

\[ \omega = \alpha \wedge \beta + \mu_1 \wedge \zeta_1 + \mu_2 \wedge \zeta_2 + \mu_3 \wedge \zeta_3 \]

where \( \alpha, \beta, \mu_1, \mu_2, \mu_3, \zeta_1, \zeta_2, \zeta_3 \) - the dual basis in \( L \).

It is easy to check that all the properties of \( L \) and \( \omega \) from [1] are true for our more general Lie algebra \( L \) if \( \lambda_i \neq 0, i = 1, 2, 3 \) and \( \lambda_1 + \lambda_2 + \lambda_3 = 0 \). Therefore as in [1] we get the Hard Lefschetz property for our \( L \). Also the cohomology ring for \( L \) is isomorphic to the cohomology ring for the Kähler manifold \( T^2 \times \mathbb{CP}^3 \). We are going to prove that for some triples \( \lambda_1, \lambda_2, \lambda_2 \) we can construct a lattice in a simply connected triangular Lie group \( R \) corresponding to our Lie algebra \( L \).

Let \( x^3 - px^2 + qx - 1 \) be a polynomial with integer coefficients and three real roots. For example, the equation \( x^3 - 5x^2 + 6x - 1 = 0 \) has the roots \( x_1 \approx 0.198, x_2 \approx 1.555, x_3 \approx 3.247 \). We take \( \lambda_i = \ln(x_i) \), in our example \( \lambda_1 \approx -1.619, \lambda_2 \approx 0.441, \lambda_3 \approx 1.777 \).

For these \( \lambda_i \) the action of the matrix \( C = \text{diag}(exp(\lambda_1, \lambda_2, \lambda_3)) \) on \( V \) is conjugate to some action from \( \text{GL}(3, \mathbb{Z}) \). The corresponding action on \( A^2V \) is integer-valued too. Therefore there is a lattice \( D \) in \( N \) which is invariant under the action of some element \( \gamma \) from the group of automorphisms. We set \( \Gamma_1 = \mathbb{Z} \times D \), it is a lattice in \( R_1 \). The group \( \Gamma = \Gamma_1 \times \mathbb{Z} \) is a lattice in \( R = R_1 \times R \). Therefore we get a lattice in our Lie group \( R \) (which is slightly different from \( R \) in Example 2 [1]). The compact symplectic manifold \( M = R/\Gamma \) gives us an example which failed to be found in [1]. For this \( M \) all the cohomological properties are the same as for Kähler manifold. Moreover, the (nilpotent) minimal model for \( M \) is the same that for the Kähler manifold \( T^2 \times \mathbb{CP}^3 \) (as in [1]). It is not known if there is a complex structure on this \( M \). Is it is really no complex structure on this \( M \), therefore there is no Kähler structure on it. In this case we’ll find that there is no cohomological invariants which can characterize Kähler solvmanifolds. Some other examples of such kind are, as indicated in [1], six-dimensional manifolds \( \text{Sol} \times T^2 \) and \( \text{Sol} \times \text{Sol} \), where \( \text{Sol} \) is constructed in [1], Example 1.

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