O-MINIMAL METHOD AND GENERALIZED SUM-PRODUCT PHENOMENA

YIFAN JING, SOUKTIK ROY, AND CHIEU-MINH TRAN

Abstract. Using tools from o-minimality, we prove that for two bivariate polynomials $P(x, y)$ and $Q(x, y)$ with coefficients in $\mathbb{R}$ or $\mathbb{C}$ to simultaneously exhibit small expansion, they must exploit the underlying additive or multiplicative structure of the field in nearly identical fashion. This in particular generalizes the main result of Shen [24] and yields an Elekes–Rónyai type structural result for symmetric non-expanders, resolving an issue mentioned by de Zeeuw in [9]. Our result also places sum-product phenomena into a more general picture of model-theoretic interest.

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1. Introduction

Erdős and Szemerédi observed in [12] that there is $\varepsilon \in \mathbb{R}^\times$ such that if $n$ is sufficiently large and $A, B$ are finite subsets of $\mathbb{R}$ with $|A| = |B| = n$, then

$$\max\{|A + B|, |AB|\} > n^{1+\varepsilon};$$

however, $A + B$ or $AB$ might have size comparable to $n$, for example, when $A = B$ is an arithmetic progression or a geometric progression. More recently, Elekes and Rónyai proved in [11] that if $P(x, y) \in \mathbb{R}[x, y]$ is a bivariate polynomial not of the form $f(u(x) + v(y))$ or $f(u(x)v(y))$ where $f, u, v$ are univariate polynomials with real coefficients, then there is $\varepsilon \in \mathbb{R}^\times$ such that for sufficiently large $n$, $A$ and $B$ as in the preceding statement, and $P(A, B) = \{P(a, b) : a \in A, b \in B\}$, we must have

$$|P(A, B)| > n^{1+\varepsilon}.$$ 

These results can be seen as instances of a more general phenomenon: finite subsets of a one-dimensional space have expansion behavior under binary operations unless the situation is “controlled” by a single abelian group. Connections between this and model theory came into the picture after the works of Hrushovski [14] and then of Breuillard, Green, and Tao [3] on the structure of approximate groups; the classification result in the latter is not an instance of the above phenomenon but is very close in spirit; see [15] for a survey on related themes. Recent works suggest that these connections are robust: Elekes–Rónyai type results were recently generalized...
to strongly minimal, o-minimal, and stable settings by Chernikov and Starchenko [6, 8]. In light of all these, one can expect that Erdős-Szemerédi type sum-product results have generalizations to other model-theoretically tame settings. One such generalization was recently given in [2]. This paper obtains a generalization in a different direction, focusing on semi-algebraic/o-minimal settings and with explicit exponent in the bounds.

Throughout the paper, let $K$ be a field, let $P(x, y)$ and $Q(x, y)$ range over $K[x, y]$ (with $x$ and $y$ denoting single variables), and assume that $P(x, y)$ and $Q(x, y)$ have nontrivial dependence on $x$ and $y$. Our main result, which is proven in section 3, is the following:

**Theorem 1.1.** Suppose $K$ is $\mathbb{R}$ or $\mathbb{C}$. Then there is $\alpha = \alpha(\deg P, \deg Q)$ such that exactly one of the following holds:

(i) For all subsets $A$ and $B$ of $K$ with $|A| = |B| = n$

$$\max\{|P(A, B)|, |Q(A, B)|\} > \alpha n^{5/4}.$$  

(ii) $P(x, y) = f(\gamma_1 u(x) + \delta_1 v(y))$ and $Q(x, y) = g(\gamma_2 u(x) + \delta_2 v(y))$ where $f$, $g$, $u$, and $v$ are univariate polynomials over $K$ and $\gamma_1$, $\gamma_2$, $\delta_1$, and $\delta_2$ are in $K$.

(iii) $P(x, y) = f(u^{m_1}(x)v^{n_1}(y))$ and $Q(x, y) = g(u^{m_2}(x)v^{n_2}(y))$ where $f$, $g$, $u$, and $v$ are univariate polynomials over $K$ and $m_1$, $m_2$, $n_1$, and $n_2$ are in $\mathbb{N}^{\geq 1}$.

Taking $P(x, y) = x + y$ and $Q(x, y) = xy$ in Theorem 1.1 recovers the famous sum-product phenomenon with exponent 5/4. Another special case with $P(x, y) = x + y$ is the main result in the paper “Algebraic methods in sum-product phenomena” by Shen [24]. A variant for finite field of Shen’s result was obtained by Bukh and Tsimerman [4]. From Theorem 1.1 we can obtain Theorem 1.2 below; see the proof in Section 4.

**Theorem 1.2.** Suppose $K$ is either $\mathbb{R}$ or $\mathbb{C}$. With the same $\alpha = \alpha(\deg P, \deg Q)$ as in Theorem 1.1, exactly one of the following holds:

(i) For all subsets $A$ of $K$ with $|A| = n$

$$\max\{|P(A, A)|, |Q(A, A)|\} > \alpha n^{5/4}.$$  

(ii) $P(x, y) = f(\gamma_1 u(x) + \delta_1 u(y))$ and $Q(x, y) = g(\gamma_2 u(x) + \delta_2 u(y))$ where $f$, $g$, $u$, and $v$ are univariate polynomials over $K$ and $\gamma_1$, $\gamma_2$, $\delta_1$, and $\delta_2$ are in $K$.

(iii) $P(x, y) = f(u^{m_1}(x)v^{n_1}(y))$ and $Q(x, y) = g(u^{m_2}(x)v^{n_2}(y))$ where $f$, $g$, $u$, and $v$ are univariate polynomials over $K$ and $m_1$, $m_2$, $n_1$, and $n_2$ are in $\mathbb{N}^{\geq 1}$.

The special case of Theorem 1.2 where $P(x, y) = Q(x, y)$ gives us a characterization of a bivariate polynomial which is not a symmetric expander with exponent 5/4; we establish this as Corollary 1.3. This resolves an issue discussed by de Zeeuw in the second paragraph of [9, Section 1.3]. Note that it is a problem which can be stated in
terms of a single polynomial, but considering two polynomials seems necessary for its resolution. There are also analogues of Theorems 1.1 and 1.2 for rational functions; see Remark 4.4 for a more detailed discussion. This can be applied, for example, in showing that any $n$ distinct points on a circle must either give us at least $\beta n^{5/4}$ many distinct distances or $\beta n^{5/4}$ many distinct slopes, where $\beta$ is a constant not depending on $n$.

By standard model-theoretic transfer, the analogues of Theorems 1.1 and 1.2 also hold for all algebraically closed fields and real closed fields. With some basic Galois theory, modified versions where “exactly” is replaced by “at least” in the statement of these two theorems can be obtained for all fields of characteristic 0. Proposition 4.5 covers both these results. This also has some ramifications for large positive characteristic; see Remark 4.6.

We note that the exponent $5/4$ in Theorem 1.1 and Theorem 1.2 should not be optimal. Analogous to the Erdős–Szemerédi sum-product conjecture, it is natural to expect strengthenings of the above two theorems to exponents $r$ arbitrarily close to 2. While this might not be a realistic target at the moment, crossing the $4/3$ threshold in the exponent might be interesting and feasible in view of results by Solymosi [25], Konyagin and Shkredov [17, 18], Rudnev, Shkredov, and Stevens [22], and Shakan [23].

The proof of Theorem 1.1 has three steps. First, we apply the strengthening [21] by Raz–Sharir–Solymosi of Elekes–Rónyai’s result in [11] and reduce to the case where $P(x, y)$ and $Q(x, y)$ are of the form $f(u(x) + v(y))$ or $f(u(x)v(y))$ where $f, u, v$ are univariate polynomials over $K$. Then we use a suitable generalization of the Szemerédi–Trotter Theorem [27] (the result for $\mathbb{R}$ comes from [13] by Fox, Pach, Sheffer, Suk, and Zahl, and that for $\mathbb{C}$ can be deduced from [26] by Solymosi and de Zeeuw) in the same fashion as in Elekes’ classical proof of sum-product phenomena [10]. This allows us to prove that (i) happens except for a special situation. Finally, we analyze the above special situation to show that either (ii) or (iii) must happen. This last step employs ideas from semi-algebraic geometry/o-minimal geometry, in particular, a definable Ramsey-type result.

The proof of the third step is readily generalizable to many other settings related to o-minimality; see Remark 3.8 for details. We expect that Theorem 1.1 admits suitable generalizations as well, once the corresponding ingredients for the first and second steps are developed. The proof of the third step is quite close in principle to the proof in [24] and the proof in [21], but we use o-minimal techniques instead of algebraic geometry. We believe that algebraic geometry in the proof of [24] can also be substituted with o-minimality, and this will in fact provide the extra flexibility that we need for the aforementioned generalization of Theorem 1.1.
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Notation and conventions. We will keep the notational convention on $K$, $P(x, y)$, and $Q(x, y)$. Moreover, let $m$ and $n$ range over the set $\mathbb{N} = \{0, 1, \ldots \}$ of natural numbers, let $k$ and $l$ range over the set $\mathbb{Z}$ of integers, and let $\mathbb{R}$ and $\mathbb{C}$ be the fields of real and complex numbers, and $(\mathbb{R}, <)$ is the ordered field of real numbers.

2. Preliminaries on logic and o-minimality

To make the paper more accessible to readers without model theory background, we include here a brief introduction to the topics of logic, model theory, and semi-algebraic geometry (through the lenses of o-minimality). We will keep the discussion informal, focus on motivations and the general picture. For more systematic accounts of model theory in general and o-minimality in particular see [19, 29] and [30] respectively.

The logician’s notion of a structure is a generalization of the notion of a field. In this paper, the main structures are $\mathbb{R}$, $\mathbb{C}$, and $(\mathbb{R}, <)$. The method of this paper, modulo some missing ingredients, looks ready for generalization to some expansions of $\mathbb{R}$ and $\mathbb{C}$ (i.e., structures enriching $\mathbb{R}$ and $\mathbb{C}$); examples include the exponential field $(\mathbb{R}, \exp)$ of real numbers and the expansion $\mathbb{C}_{an}$ of $\mathbb{C}$ obtained by adding restriction of analytic functions to bounded closed disks. So the reader may want to keep such structures in mind as well; see Remark 3.8 for further details.

The notion of definable set is the corresponding generalization of the notion of algebraic set (i.e., solution set of a system of polynomials equations in a field). In a field $K$, definable sets include algebraic sets and also sets that can be obtained from these through finitely many applications of taking intersections, unions, complements, and projections to lower dimensions. For instance, the set

$$X = \{(a, b) \in K : \text{there are } c, d \in K \text{ with } c^2 + d^2 = a \text{ and } c^3 + d^3 = b\}$$

is definable in $K$ as it is the projection of the algebraic subset of $K^4$ defined by the system $x = z^2 + t^2, y = z^3 + t^3$ onto the first two coordinates. By the above description, $K^2 \setminus X$ and the projection of $K^2 \setminus X$ onto the first coordinate are also definable. Definable sets are “solution sets” of first order formulas (which, in the case of fields, involve logical symbols like $\land, \lor, \neg, \exists, \forall$ on top of field-theoretic operations). For example, $X$ is the “solution set” of the first-order formula

$$\exists z \exists t (x = z^2 + t^2 \land y = z^3 + t^3).$$
Definable sets in an ordered field $(K, <)$ can be described similarly, but with the role of algebraic sets replaced by that of semi-algebraic sets (i.e., solution sets of systems of polynomial inequalities). In fact, definability in $(\mathbb{R}, <)$ and in $\mathbb{R}$ are equivalent because for $a$ and $b$ in $\mathbb{R}$, we have $a < b$ if and only if there is $c \in \mathbb{R} \setminus \{0\}$ such that $a + c^2 = b$.

The idea of definability extends to more complicated objects like functions, families of sets, groups, etc. In a structure with underlying set $K$, a function $f : X \to K^n$ with $X \subseteq K^m$ is definable if the graph of $f$ is definable; in particular, this implies that $X$ and $\text{Image}(f)$ are definable. Still in the same structure, a family $(X_b)_{b \in Y}$ of subsets of $K^m$ is definable if $Y$ is a definable subset of $K^n$ for some $n$, and the set $X = \{(a, b) \in K^{m+n} : a \in X_b, b \in Y\}$ is definable.

Compared to more restricted notions like algebraic sets, definable sets are very versatile. For example, if $f : X \to \mathbb{R}^n$ is definable in $(\mathbb{R}, <)$, then the set 

$$\{a \in X : f \text{ is differentiable with continuous derivatives at } a\}$$

is definable simply because all the relevant concepts admit epsilon-delta definitions. Unfortunately, this flexibility often comes with the burden that definable sets are often overly complicated and resist geometric understanding. Model theory is, to a certain extent, the study of structures where we do not have these shortcomings. Examples of such structures are $\mathbb{R}$, $\mathbb{C}$, and $(\mathbb{R}, <)$; this is reflected by Fact 2.1 below.

**Fact 2.1.** Let $X$ be definable in $\mathbb{R}$, equivalently, definable in $(\mathbb{R}, <)$, and let $X'$ be definable in $\mathbb{C}$. Then we have the following:

(i) (Tarski–Seidenberg Theorem) $X$ is semi-algebraic \cite[Theorem 3.3.15]{19}.

(ii) (Abstract cell decomposition) $X$ is a disjoint finite union $\sqcup_{i \in I} X_i$, where $X_i$ is an $\mathbb{R}$-definably homeomorphic image of $(0, 1)^{k_i}$ with $k_i \in \{0, \ldots, m\}$; hence, by the Invariance of Domain Theorem, $X_i$ is a connected open set when $k_i = m$ and an open interval when $k_i = m = 1$ for each $i \in I$; moreover, if $f : X \to \mathbb{R}$ is a definable function, we can choose $(X_i)_{i \in I}$ such that $f$ is continuous on $X_i$ for each $i \in I$.

(iii) (Chevalley–Tarski Theorem) $X'$ is constructible (i.e., a boolean combination of algebraic sets in $\mathbb{C}$, or equivalently, a finite union of quasi-affine varieties over $\mathbb{C}$) \cite[Theorem 3.2.2]{19}.

Another advantage in dealing with definable sets is a built-in mechanism for induction, as the image of a definable set under projection to fewer coordinates remains definable. This is particularly powerful when we have good notions of dimension. By Fact 2.1(i) and Fact 2.1(iii), definable sets in $\mathbb{R}$ are essentially real manifolds, and
definable sets in $\mathbb{C}$ are essentially algebraic varieties. Hence, we hope it is believable that there are good notions of dimension $\dim_\mathbb{R}$ and $\dim_\mathbb{C}$ in these cases. Fact 2.2 is about their properties. Items (i) to (iii) of Fact 2.2 are what one would reasonably expect; Fact 2.2(iv) is an easy consequence of Fact 2.1(ii) and Fact 2.2(i)-(iii); Fact 2.2(v) can be seen as an abstract Bezout’s theorem.

**Fact 2.2.** Suppose $K$ is either $\mathbb{R}$ or $\mathbb{C}$, and $X$ and $Y$ are definable in $K$. Then we have the following:

(i) $\dim_K(\emptyset) = -\infty$; $\dim_K X = 0$ if and only if $X$ is finite; $\dim_\mathbb{R}(a, b) = 1$ with $a, b \in \mathbb{R} \cup \{-\infty, +\infty\}$ and $a < b$; $\dim_\mathbb{C} \mathbb{C} = 1$.

(ii) $\dim_K (X \cup Y) = \max\{\dim_K X, \dim_K Y\}$.

(iii) If $(X_b)_{b \in Y}$ is a definable family, and $X = \{(a, b) : a \in X_b\}$, then for each $d \leq \dim X$, we have $Y_d = \{b \in Y : \dim_K (X_b) = d\}$ is definable, and

$$\dim_K X = \max_{d \leq \dim X} (d + \dim_K Y_d);$$

in particular, if $f : X \to Y$ is a definable bijection, then $\dim_K (X) = \dim_K Y$, and $\dim_K (X \times Y) = \dim_K X + \dim_K Y$.

(iv) If $K = \mathbb{R}$ and $X \subseteq \mathbb{R}^m$, then $\dim X = m$ if and only if a subset of $X$ is open.

(v) (algebraic boundedness) If $(X_b)_{b \in Y}$ is a definable family in $K$, then there is $N \in \mathbb{N}^{\geq 1}$ such that either $|X_b| < N$ or $\dim_K (X_b) \geq 1$.

An expansion $(\mathbb{R}, \ldots)$ of $\mathbb{R}$ is **o-minimal** if every $X \subseteq \mathbb{R}$ definable in $(\mathbb{R}, \ldots)$ is a finite union of points and open intervals. The analogues of Fact 2.1(ii) and Fact 2.2 hold in all such $(\mathbb{R}, \ldots)$. In fact, for expansion of the field $\mathbb{R}$, one can take Fact 2.1(ii) as an alternative definition for o-minimality. An important example of an o-minimal expansion of $\mathbb{R}$ is $(\mathbb{R}, \exp)$ where $\exp$ is the exponential map; this allows us to include transcendental functions while keeping algebraic features (e.g., analogue of Fact 2.2(v)).

### 3. Proof of the main theorem

In this section, we will prove Theorem [1.1]. For convenience, we will use the terms $\mathbb{R}$-definable and $\mathbb{C}$-definable as short-hands for definable in $\mathbb{R}$ and definable in $\mathbb{C}$ respectively.

For $r \in \mathbb{R}$ with $1 \leq r < 2$ and $\alpha \in \mathbb{R}^{>0}$, $P(x, y)$ is **$(r, \alpha)$-expanding over $K$** if for all subsets $A$ and $B$ of $K$ with $|A| = |B| = n$ we have

$$|P(A, B)| > \alpha n^{5/4}.$$ We say that $P(x, y)$ is **additive over $K$** if it has the form $f(u(x) + v(y))$ where $f$, $u$, $v$ are univariate polynomials with coefficients in $K$. We define **multiplicative over $K$** for $P(x, y)$ likewise, replacing $f(u(x) + v(y))$ with $f(u(x)v(y))$. 

We will reduce Theorem 1.1 to a special case using Fact 3.1, which consists of Elekes–Rónyai type structural results for non-expander polynomials. The case where \( k = \mathbb{R} \) of Fact 3.1 is a recent Theorem by Raz, Sharir, and in Solymosi [21]; this refines Elekes–Rónyai’s original result in [11]. The case where \( K = \mathbb{C} \) is folklore as communicated privately to us by de Zeeuw. This follows from the proof in [21] together with Solymosi–de Zeeuw’s incidence bound in [26], but some steps in [21] must be modified to work over \( \mathbb{C} \).

**Fact 3.1.** Suppose \( K \) is either \( \mathbb{R} \) or \( \mathbb{C} \). Then there is \( \alpha = \alpha(\deg P) \) such that if \( P(x, y) \in K[x, y] \) is not \((4/3, \alpha)\)-expanding over \( K \), then \( P(x, y) \) is either additive or multiplicative.

**Corollary 3.2.** If Theorem 1.1 holds in the special case where \( P(x, y) \) and \( Q(x, y) \) are each either additive or multiplicative, then Theorem 1.1 holds in general.

For \( r \in \mathbb{R} \) with \( 1 \leq r < 2 \) and \( \alpha \in \mathbb{R}^>0 \), we say that \( P(x, y) \) and \( Q(x, y) \) form an \((r, \alpha)\)-expanding pair over \( K \) if for all subsets \( A \) and \( B \) of \( K \) with \(|A| = |B| = n\)

\[
\max\{|P(A, B)|, |Q(A, B)|\} > \alpha n^r.
\]

We say that \( P(x, y) \) and \( Q(x, y) \) form an additive pair over \( K \) if

\[
P(x, y) = f(\gamma_1 u(x) + \delta_1 v(y)) \quad \text{and} \quad Q(x, y) = g(\gamma_2 u(x) + \delta_2 v(y))
\]

where \( f, g, u \) and \( v \) are univariate polynomials over \( K \) and \( \gamma_1, \gamma_2, \delta_1, \delta_2 \) are in \( K \).

Finally, we say that \( P(x, y) \) and \( Q(x, y) \) form a multiplicative pair over \( K \) if

\[
P(x, y) = f(u^{m_1}(x)v^{n_1}(y)) \quad \text{and} \quad Q(x, y) = g(u^{m_2}(x)v^{n_2}(y))
\]

where \( f, g, u \) and \( v \) are univariate polynomials over \( K \) and \( m_1, m_2, n_1, n_2 \) are in \( \mathbb{N}^>1 \).

Note that these three definitions correspond to the three cases of Theorem 1.1. It is easy to see that if \( P(x, y) \) and a copy of itself form an \((r, \alpha)\)-expanding pair over \( K \), then \( P(x, y) \) is \((r, \alpha)\)-expanding over \( K \). Similar observations hold when \( P(x, y) \) and a copy of itself form an additive pair or a multiplicative pair over \( K \).

For \((b_1, b_2) \in K^2\), denote by \( C_{b_1, b_2} \) the curve

\[
\{(P(a, b_1), Q(a, b_2)) : a \in K \text{ for } (b_1, b_2) \in S^2\}.
\]

For a \( K \)-definable \( S \subseteq K \), set \( \mathcal{C}_{P, Q}(S) \) to be the definable family \((C_{b_1, b_2})_{(b_1, b_2) \in S^2} \). We say that \( \mathcal{C}_{P, Q}(S) \) is scattered if for all \((b_1, b_2) \in S^2\),

\[
\{(b_1', b_2') \in S^2 : \dim_K(C_{b_1, b_2} \cap C_{b_1', b_2'}) = 1\}
\]

is finite.

As we will see, the feature that distinguishes case (i) from cases (ii) and (iii) of Theorem 1.1 when \( P(x, y) \) and \( Q(x, y) \) are each either additive or multiplicative is essentially the existence of a cofinite subset \( S \) of \( K \) such that \( \mathcal{C}_{P, Q}(S) \) is scattered.
We need a suitable version of Szemerédi–Trotter theorem. The case where $K = \mathbb{R}$ of Fact 3.3 is a very special case of the main result in [13]. The case where $K = \mathbb{C}$ is essentially known in the field as privately communicated to us by de Zeeuw. This can be deduced from a result in [26]. To be more precise, Fact 3.3 with the $K_{k,k}$-free condition replaced by a $K_{2,k}$-free condition can be derived quite immediately. To get the actual version of Fact 3.3, we can employ the trick in the proof of Corollary 4.2 of [20] by Mojarrad, Pham, Valculescu, and de Zeeuw.

**Fact 3.3.** Assume $K$ is either $\mathbb{R}$ or $\mathbb{C}$, $\phi(x_1, x_2, y_1, y_2)$ is a formula in the language of fields (possibly with parameters from $K$), and $G = (X, Y, E)$ is a bipartite graph with the following properties:

(i) $X = X_1 \times X_2$ where $X_1$ and $X_2$ are finite subsets of $K$;
(ii) $Y$ is a finite subset of $K^2$;
(iii) for $c \in X$ and $d \in Y$, $(c, d)$ is in $E$ if and only if $\phi(c, d)$.

If $G$ is $K_{k,k}$-free, then

$$|E| \leq \beta((|X||Y|)^{2/3} + |X| + |Y|),$$

where $\beta$ depends only on $k$ and the complexity of $\phi$ as described in [13].

The generalization of Fact 3.3 where $X$ is just assumed to be finite subset of $K^2$ but not necessary a Cartesian product as in (i) also recently becomes available. For $\mathbb{R}$, this is still a special case of the main theorem of [13]. The result for $\mathbb{C}$ can be obtained from a recent preprint of Walsh [31] by using the same trick as above.

**Corollary 3.4.** Assume $K$ is either $\mathbb{R}$ or $\mathbb{C}$, $N$ is in $\mathbb{N}^{>0}$, and there is a cofinite $S \subseteq K$ such that $|K \setminus S| \leq N$ and $\mathcal{E}_{P,Q}(S)$ is scattered. Then $P(x, y)$ and $Q(x, y)$ form a $(5/4, \alpha)$-expanding pair with $\alpha = \alpha(\deg P, \deg Q, N)$.

**Proof.** Recall that $\mathcal{E}_{P,Q}(S)$ is $(C_{b_1, b_2})(b_1, b_2) \in S^2$. The family $(C_{b_1, b_2} \cap C_{b'_1, b'_2})(b_1, b_2; b'_1, b'_2) \in S^4$ is also $K$-definable. For $(b_1, b_2) \in S^2$, set

$$Y_{b_1, b_2} = \{(b'_1, b'_2) \in S^2 : \dim_K(C_{b_1, b_2} \cap C_{b'_1, b'_2}) = 1\}.$$ 

Then the family $(Y_{b_1, b_2})(b_1, b_2) \in S^2$ is $K$-definable by Fact 2.2(iii). Using Fact 2.2(v), we obtain $k > 0$ such that for all $(b_1, b_2) \in S^2$, the following holds:

1. For all $(b'_1, b'_2) \in K^2$, if $|C_{b_1, b_2} \cap C_{b'_1, b'_2}| \geq k$, then $\dim_K(C_{b_1, b_2} \cap C_{b'_1, b'_2}) = 1$.
2. If $|Y_{b_1, b_2}| \geq k$, then $\dim_K Y_{b_1, b_2} \geq 1$.

Let $\phi(x_1, x_2, y_1, y_2)$ be the formula in the language of fields such that $\phi(c_1, c_2, d_1, d_2)$ holds for $(c_1, c_2) \in K^2$ and $(d_1, d_2) \in S^2$ if and only if

$$(c_1, c_2) \in C_{d_1, d_2}.$$ 

For subsets $A$ and $B$ of $K$ with $|A| = |B| = n$, define $G = (X, Y, E)$ as in Fact 3.3 with $X_1 = P(A, B \cap S)$, $X_2 = Q(A, B \cap S)$, $Y = (B \cap S) \times (B \cap S)$, and $\phi(c_1, c_2, d_1, d_2)$
as above. By the assumption that $\mathcal{C}_{P,Q}(S)$ is scattered and our choice of $k$, the graph $G$ is $K_{k,k}$-free. Applying Fact 3.3, we get a constant $\beta$ depending only on $\phi$ and $k$ such that

$$|E| \leq \beta\left(|P(A, B \cap S)|^{3/2} |Q(A, B \cap S)|^{3/2} + |P(A, B \cap S)| + Q(A, B \cap S)\right).$$

Let $d$ be maximum degree of $P(x, y)$ and $Q(x, y)$. Note that the curve $C_{b_1,b_2}$ passes through at least $n/d$ points in $X$ for each $(b_1, b_2) \in Y$, namely, the points of the form $(P(a, b_1), Q(a, b_2))$ for $a \in A$. In particular, $|E| \geq n(n - N)^2/2d$ when $n$ is sufficiently large. Hence, we get $\max\{|P(A, B)|, |Q(A, B)|\} \geq \alpha n^{5/4}$ for a constant $\alpha$ depending only on $k$, the complexity of $\phi$, and $N$ which in turn depends only on $\deg P$, $\deg Q$, and $N$. \hfill \Box

In light of Corollary 3.2 and Corollary 3.4, Theorem 1.1 reduces essentially to the special case where $P(x, y)$ and $Q(x, y)$ are each either additive or multiplicative. We make use of the following easy lemma which is a restatement of [1, Lemma 9]. This can be viewed as either a “definable Ramsey” theorem or a dimensional version of the strong Erdős-Hajnal property described in [7].

**Fact 3.5.** Suppose $X \subseteq \mathbb{R}^m$ is $\mathbb{R}$-definable, and $(X_b)_{b \in Y}$ is an $\mathbb{R}$-definable family of subsets of $X$ with $\dim_{\mathbb{R}}(X_b) = \dim_{\mathbb{R}} X$ for every $b \in Y$. Then there are $\mathbb{R}$-definable $X' \subseteq X$ and $\mathbb{R}$-definable $Y' \subseteq Y$ such that $\dim_{\mathbb{R}}(X') = \dim_{\mathbb{R}} X$, $\dim_{\mathbb{R}}(Y') = \dim_{\mathbb{R}} Y$, and $X'$ is a subset of $X_b$ for all $b \in Y'$.

We identify the underlying set of $\mathbb{C}$ with $\mathbb{R}^2$ in the standard way. Then every $\mathbb{C}$-definable set can be viewed as an $\mathbb{R}$-definable set. Note that if $X$ is $\mathbb{C}$-definable and $\dim_{\mathbb{C}} X = n$, then $\dim_{\mathbb{R}} X = 2n$ by Fact 2.1(iii), Noether normalization lemma, and Fact 2.2(i-iii).

Assume $K$ is either $\mathbb{R}$ or $\mathbb{C}$ and $S$ is a subset of $K$. A **decomposition** of $\mathcal{C}_{P,Q}(S)$ consists of a finite set $I$, and an $\mathbb{R}$-definable family $\mathcal{C}_{P,Q}^i(S) = (C_{b_1,b_2}^i)_{(b_1,b_2) \in S^2}$ for each $i \in I$ such that for all $(b_1, b_2) \in S^2$, we have

$$\dim_{\mathbb{R}} \left( C_{b_1,b_2} \setminus \bigcup_{i \in I} C_{b_1,b_2}^i \right) < \dim_{\mathbb{R}} K.$$

We say that $\mathcal{C}_{P,Q}^i(S)$ for $i \in I$ as above is **scattered** if for all $(b_1, b_2) \in S$ such that

$$\dim_{\mathbb{R}} \{ (b_1', b_2') \in S^2 : \dim_{\mathbb{R}}(C_{b_1,b_2}^i \cap C_{b_1',b_2'}^{i'}) = \dim_{\mathbb{R}} K \} < \dim_{\mathbb{R}} K.$$

Note that $\mathcal{C}_{P,Q}(S)$ forms a decomposition of itself. It is easy to see that this notion of being scattered coincides with the previous one because of the relationship between $\dim_{\mathbb{C}} X$ and $\dim_{\mathbb{R}} X$ for a $\mathbb{C}$-definable set $X$. 
Corollary 3.6. Assume $K$ is either $\mathbb{R}$ or $\mathbb{C}$, $S$ is a subset of $K$, and $I$ together with $\mathcal{C}_{P,Q}(S)$ for $i \in I$ forms a decomposition of $\mathcal{C}_{P,Q}(S)$. If $\mathcal{C}_{P,Q}(S)$ is not scattered, then for some $i \in I$, $\mathcal{C}_{i,P,Q}(S)$ is not scattered.

Proof. Recall that $\mathcal{C}_{P,Q}(S) = (C_{b_1,b_2})(b_1,b_2) \in S^2$, and $\mathcal{C}_{P,Q}(S) = (C_{b_1,b_2})(b_1,b_2) \in S^2$ for $i \in I$. For $(b_1,b_2) \in S^2$, let $Y_{b_1,b_2} = \{(b_1', b_2') \in S^2 : \dim_\mathbb{R}(C_{b_1,b_2} \cap C_{b_1',b_2'}) = \dim_\mathbb{R} K\}$ and let

$$Y_{b_1,b_2}^i = \{(b_1', b_2') \in S^2 : \dim_\mathbb{R}(C_{b_1,b_2} \cap C_{b_1',b_2'}) = \dim_\mathbb{R} K\}.$$ 

Hence, $Y_{b_1,b_2}$ and $Y_{b_1,b_2}^i$ are $\mathbb{R}$-definable for all $i \in I$ and $(b_1,b_2) \in K^2$ by Fact 2.2(iii).

Note that $\dim_\mathbb{R}(C_{b_1,b_2} \cap C_{b_1',b_2'}) = \dim_\mathbb{R} K$ if and only if $\dim_K(C_{b_1,b_2} \cap C_{b_1',b_2'}) = 1$ for all $(b_1,b_2)$ and $(b_1',b_2')$ in $S^2$, so $Y_{b_1,b_2}$ is also $K$-definable by Fact 2.2(iii). From the assumption that $\mathcal{C}_{P,Q}(S)$ is not scattered, we obtain $(c_1,c_2) \in K^2$ with $\dim_K(Y_{c_1,c_2}) \geq 1$, or equivalently,

$$\dim_\mathbb{R}(Y_{c_1,c_2}) \geq \dim_\mathbb{R} K.$$ 

For $(b_1', b_2') \in S^2$, we have $\dim_\mathbb{R}(C_{b_1',b_2'} \setminus \bigcup_{i \in I} C_{b_1',b_2'}) < \dim_\mathbb{R} K$, so Fact 2.2(ii) implies there is $i \in I$ such that

$$\dim_\mathbb{R}(C_{c_1,c_2} \cap C_{b_1',b_2'}) \geq \dim_\mathbb{R} K.$$ 

It follows that $Y_{c_1,c_2} = \bigcup_{i \in I} Y_{c_1,c_2}^i$. Using Fact 2.2(ii), we get $i \in I$ such that $\dim_\mathbb{R} Y_{c_1,c_2}^i \geq \dim_\mathbb{R} K$. Fix such $i$. Note that $C_{c_1,c_2} \cap C_{b_1',b_2'} \subseteq C_{c_1,c_2}$ and

$$\dim_\mathbb{R}(C_{c_1,c_2} \cap C_{b_1',b_2'}) = \dim_\mathbb{R} C_{c_1,c_2} = \dim_\mathbb{R} K \quad \text{for} \quad (b_1', b_2') \in Y_{c_1,c_2}^i.$$ 

So we can apply Fact 3.5 to get $X' \subseteq K^2$ with $\dim_\mathbb{R} X' = \dim_\mathbb{R} K$ and $Y' \subseteq K^2$ with $\dim_\mathbb{R} Y' \geq \dim_\mathbb{R} K$ such that

$$X' \subseteq C_{c_1,c_2} \cap C_{b_1',b_2'} \quad \text{for all} \quad (b_1', b_2') \in Y'.$$ 

Hence, we have $\dim(C_{b_1,b_2} \cap C_{b_1',b_2'}) \geq \dim_\mathbb{R} K$ for all $(b_1,b_2)$ and $(b_1',b_2')$ in $Y'$. The desired conclusion follows.

Proposition 3.7. If $K$ is either $\mathbb{R}$ or $\mathbb{C}$, $P(x,y)$ and $Q(x,y)$ are individually either additive or multiplicative, and the family $\mathcal{C}_{P,Q}(S)$ is not scattered for all cofinite $S \subseteq K$, then exactly one of the following possibilities holds:

(i) $P(x,y)$ and $Q(x,y)$ form an additive pair over $K$;

(ii) $P(x,y)$ and $Q(x,y)$ form a multiplicative pair over $K$.

Moreover, there is $N = N(\deg P, \deg Q)$ such that the same conclusion holds with the weaker assumption that $\mathcal{C}_{P,Q}(S)$ is not scattered for all cofinite $S \subseteq K$ with $|K \setminus S| < N$. \qed
Proof. Toward proving the first statement, we will show under the given assumption that we cannot have one of \( P(x, y) \) and \( Q(x, y) \) additive and the other multiplicative over \( K \). In particular, this gives us that the two possibilities in the proposition are mutually exclusive. Suppose to the contrary that \( P(x, y) = f(u_1(x) + v_1(y)) \) and \( Q(x, y) = g(u_2(x)v_2(y)) \) where \( f, g, u_1, v_1, u_2, v_2 \) are univariate polynomials with coefficient in \( K \). Assume temporarily that we have shown the contradiction in the special case where

\[
P(x, y) = u_1(x) + v_1(y) \quad \text{and} \quad Q(x, y) = u_2(x)v_2(y).
\]

Note that \( f, g, u_1, v_1, u_2, v_2 \) are nonconstant as \( P(x, y) \) and \( Q(x, y) \) have nontrivial dependence on \( x \) and \( y \). This allows us to obtain a finite \( \mathbb{R} \)-definable family \( (U_i)_{i \in I} \) of Euclidean open subsets of \( K \) where \( K \setminus \bigcup_{i \in I} U_i \) is finite such that \( f \) and \( g \) are injective on \( U_i \) for each \( i \in I \). Set

\[
D_{b_1, b_2} = \{(u_1(a) + v_1(b_1), u_2(a)v_2(b_2)) : a \in K\}.
\]

To deduce the contradiction from the above temporary assumption, we need to show for an arbitrary cofinite \( S \subseteq K \), that \( \mathcal{D}_{P, Q}(S) = (D_{b_1, b_2}(b_1, b_2)) \in S^2 \) is not scattered. Shrinking \( S \) if necessary, we can assume that \( u_1(x) + v_1(b_1) \) and \( u_2(x)v_2(b_2) \) are nonconstant polynomials for all \( (b_1, b_2) \) in \( S^2 \). The family \( \mathcal{C}_{P, Q}(S) = (C_{b_1, b_2}(b_1, b_2)) \in S^2 \) has \( C_{b_1, b_2} = f \times g(D_{b_1, b_2}) \). For \( (i_1, i_2) \in I^2 \) and \( (b_1, b_2) \in K^2 \), set

\[
D_{b_1, b_2}^{i_1, i_2} = D_{b_1, b_2} \cap (U_{i_1} \times U_{i_2}) \quad \text{and} \quad C_{b_1, b_2}^{i_1, i_2} = (f \times g)D_{b_1, b_2}^{i_1, i_2}.
\]

As \( K \setminus \bigcup_{i \in I} U_i \) is finite and \( u_1(x) + v_1(b_1) \) and \( u_2(x)v_2(b_2) \) are nonconstant polynomials for all \( (b_1, b_2) \) in \( S^2 \), we get \( \dim_{\mathbb{R}}(D_{b_1, b_2} \setminus \bigcup_{(i_1, i_2) \in I^2} D_{b_1, b_2}^{i_1, i_2}) < \dim_{\mathbb{R}} K \) for all \((b_1, b_2)\) in \( S^2 \). It follows that

\[
\dim_{\mathbb{R}} \left( C_{b_1, b_2} \setminus \bigcup_{(i_1, i_2) \in I^2} C_{b_1, b_2}^{i_1, i_2} \right) < \dim_{\mathbb{R}} K \quad \text{for all} \quad (b_1, b_2) \in S^2.
\]

As \( \mathcal{C}_{P, Q}(S) \) is not scattered, we can apply Corollary 3.6 to get \((i_1, i_2) \in I^2 \) and \((b_1, b_2) \in S^2 \) such that

\[
\dim_{\mathbb{R}} \left\{ (b_1', b_2') \in S^2 : \dim_{\mathbb{R}}(C_{b_1, b_2}^{i_1, i_2} \cap C_{b_1', b_2'}^{i_1, i_2}) = \dim_{\mathbb{R}} K \right\} \geq \dim_{\mathbb{R}} K.
\]

It is easy to see that \( f \times g \) is an injective and definable map on \( U_i \times U_j \). By Fact 2.2(iii), \( \dim_{\mathbb{R}}(D_{b_1, b_2}^{i_1, i_2} \cap D_{b_1, b_2}^{i_1, i_2}) = \dim_{\mathbb{R}} K \) whenever \( \dim_{\mathbb{R}}(C_{b_1, b_2}^{i_1, i_2} \cap C_{b_1', b_2'}^{i_1, i_2}) = \dim_{\mathbb{R}} K \) for \((b_1, b_2)\) and \((b_1', b_2')\) in \( S^2 \). Hence, \( \mathcal{D}_{P, Q}(S) \) is not scattered, as desired.

We next deal with the special case assumed in the preceding paragraph. An easy degree argument gives us that \( u_2'(x) \) is not a constant polynomial, and \( \frac{u_1'(x)}{u_2'(x)} \) is a nonconstant rational function. Hence, we get a finite \( \mathbb{R} \)-definable family \( (U_j)_{j \in J} \) of Euclidean open subsets of \( K \) with \( \dim_{\mathbb{R}}(K \setminus \bigcup_{j \in J} U_j) < \dim_{\mathbb{R}} K \) and such that for
As we have arranged that \( j, f, g, u \) are also homeomorphism from \( U_j \) to the respective images. For \( j \in J \) and \((b_1, b_2) \in K^2, \) set

\[
D^j_{b_1, b_2} = \{(u_1(a) + v_1(b_1), u_2(a)v_2(b_2)) : a \in U_j \}.
\]

Let \( S \) be a cofinite subset of \( K \) such that \( u_1(x) + v_1(b_1) \) and \( u_2(x)v_2(b_2) \) are nonconstant functions for all \((b_1, b_2) \in S^2 \). Using an argument involving Corollary 3.3 similar to the one in the preceding paragraph, we obtain \( j \in J \) and \((b_1, b_2) \in S^2 \) such that

\[
\dim_k \{(b_1', b_2') \in S^2 : \dim_k (D^j_{b_1, b_2} \cap D^{j}_{b_1', b_2'}) = \dim_k K \} \geq \dim_k K.
\]

Suppose \((b_1', b_2') \in S^2 \) is such that \( \dim_k (D^j_{b_1, b_2} \cap D^{j}_{b_1', b_2'}) = \dim_k K \). Note that

\[
a \mapsto (u_1(a) + v_1(b_1), u_2(a)v_2(b_2)) \quad \text{and} \quad a \mapsto (u_1(a) + v_1(b_1'), u_2(a)v_2(b_2'))
\]

are also homeomorphism from \( U_j \) onto the respective images. So using Fact 2.2(iv), we obtain a definable open subset \( U' \) of \( U_j \) and \( \mathbb{R} \)-definable differentiable function \( \lambda : U' \to U_j \) such that

\[
(u_1(x) + v_1(b_1), u_2(x)v_2(b_2)) = (u_1(\lambda(x)) + v_1(b_1'), u_2(\lambda(x))v_2(b_2')) \quad \text{on } U'.
\]

Differentiating \( u_1(x) + v_1(b_1) = u_1(\lambda(x)) + v_1(b_1') \), we get \( u_1'(x) = u_1'(\lambda(x))\lambda'(x) \) on \( U' \). Taking logarithmic derivative of both sides of \( u_2(x)v_2(b_2) = u_2(\lambda(x))v_2(b_2) \), we get

\[
\frac{u_2'(x)}{u_2(x)} = \frac{u_2'(\lambda(x))}{u_2(\lambda(x))}\lambda'(x) \quad \text{on } U'.
\]

Dividing the two equations involving \( \lambda'(x) \), we get

\[
\frac{u_1'u_2}{u_2'}(x) = \frac{u_1'u_2}{u_2'}(\lambda(x)) \quad \text{on } U'.
\]

As we have arranged that \( \frac{v_2u_1'}{v_2'} \) is a homeomorphism on \( U_j, \lambda(x) = x \) on \( U' \). It follows that \( v_1(b_1) = v_1(b_1') \) and \( v_2(b_2) = v_2(b_2') \). This is a contradiction, as it implies that the set \( \{(b_1', b_2') \in S^2 : \dim_k (D^j_{b_1, b_2} \cap D^{j}_{b_1', b_2'}) = \dim_k K \} \) is finite.

Now suppose \( P(x, y) \) and \( Q(x, y) \) are each multiplicative but do not form a multiplicative pair over \( K \). Assume that

\[
P(x, y) = f(u_1(x)v_1(y)) \quad \text{and} \quad Q(x, y) = g(u_2(x)v_2(y)).
\]

where \( f, g, u_1, v_1, u_2, v_2 \) are nonconstant univariate polynomial. If \( u_1^m(x) = eu_2^n(x) \) and \( v_1^{m'}(y) = e'v_2^{n'}(y) \) for \( m, n, m', n' \in \mathbb{N}^2 \) and \( e, e' \in \mathbb{C} \), then \( P(x, y) \) and \( Q(x, y) \) forms a multiplicative pair. So we can assume that

\[
u_1^m(x) \neq eu_2^n(x) \quad \text{for all } m, n \in \mathbb{N}^2 \text{ and } e \in \mathbb{C}.
\]
If $\frac{u_1'u_2}{u_2'u_1} = c$ with $c \in \mathbb{C}$, then by considering behavior when $|x|$ is large, we see that $c$ is a rational number $m/n$ and $u_1'(x) = eu_2'(x)$ and $e \in \mathbb{C}$. Hence, $\frac{u_1'u_2}{u_2'u_1}$ is not a constant function. We then deduce the contradiction in the same fashion in the preceding two paragraph substituting the role of $\frac{u_1'u_2}{u_2'u_1}$ with that of $\frac{u_1'u_2}{u_2'u_1}$. Similarly, we can rule out the case where $P(x, y)$ and $Q(x, y)$ are each additive but do not form a additive pairs using $\frac{u_1'u_2}{u_2'u_1}$ in the place of $\frac{u_1'u_2}{u_2'u_1}(x)$. We are left with the desired possibilities that $P(x, y)$ and $Q(x, y)$ either form an additive pair or a multiplicative pair.

Finally, we explain why the same conclusion can be reached under the weaker assumption of the second statement. Note that in the second paragraph, we only need the assumption that $\mathcal{D}_{P,Q}(S)$ is not scattered for one single set $S$ such that $u_1(x) + v_1(b_1)$ and $u_2(x)v_2(b_2)$ are nonconstant for all $(b_1, b_2) \in S^2$. This happens to also be the requirement on $S$ appearing in the first paragraph. There are at most $\deg P \cdot \deg Q$ many $(b_1, b_2) \in K$ such that either $u_1(x) + v_1(b_1)$ or $u_2(x)v_2(b_2)$ is a constant polynomial. Similarly considering the other cases in the third paragraph, we see that $N(\deg P, \deg Q) = \deg P \deg Q + 1$ is sufficient for our purpose.

Proof of Theorem [7.7] Let $N = N(\deg P, \deg Q)$ be as in Proposition [3.7] and set $\alpha = \alpha(\deg P, \deg Q, N)$ be as in Corollary [3.4] So $\alpha$ depends only on $\deg P$ and $\deg Q$, and we can write $\alpha = \alpha(\deg P, \deg Q)$. Suppose $P(x, y)$ and $Q(x, y)$ do not form a $(5/4, \alpha)$-expanding pair. Using Corollary [3.2] we can arrange that $P(x, y)$ and $Q(x, y)$ are each either additive or multiplicative over $K$. By Corollary [3.3] the family $\mathcal{C}_{P,Q}(S)$ is not scattered for all cofinite $S \subseteq K$ with $|K \setminus S| < N$. Applying Proposition [3.7] yields that at least one of the three cases in the statement of the theorem holds, and further the additive and multiplicative cases are mutually exclusive.

What remains is to show that when $P$ and $Q$ form either an additive pair or a multiplicative pair, then there exist $n$ and sets $A, B$ with $|A| = |B| = n$ such that $\max\{|P(A, B)|, |Q(A, B)|\} \leq \alpha n^{5/4}$. Consider the case that

$$P(x, y) = f(\gamma_1u(x) + \delta_1v(y)) \text{ and } Q(x, y) = g(\gamma_2u(x) + \delta_2v(y)).$$

If $K = \mathbb{R}$, with $x$ and $y$ suitably replaced by $\pm x + c$ and $\pm y + d$ where $c$ and $d$ are constants in $\mathbb{R}$, we can assume that $\mathbb{R}^0$ is in the range of both $u$ and $v$. If $K = \mathbb{C}$, no modification is necessary. For $k > 0$, let

$$A(k) = \{u^{-1}(\gamma_1\gamma_2l + \delta_1\delta_2l') : l, l' \in \mathbb{N}^k\}, \quad B(k) = \{v^{-1}(\gamma_1\delta_2l + \delta_1\gamma_2l') : l, l' \in \mathbb{N}^k\}.$$ 

For any given $n$, we suitably choose $k \in \mathbb{N}^0$ such that $n \leq |A(k)| \leq n \deg u$ and $n \leq |B(k)| \leq n \deg v$, and we take $A \subseteq A(k)$ and $B \subseteq B(k)$ of size exactly $n$ each. Since $\alpha$ is fixed, these clearly satisfy $|P(A, B)|, |Q(A, B)| \leq \alpha n^{5/4}$ for all sufficiently large $n$. The multiplicative case is very similar but simpler. With $P(x, y) = f(u^{m_1}(x)v^{n_1}(y)),$
and \( Q(x, y) = g(u^{m_2}(x)v^{n_2}(y)) \), the only difference is to let \( A(k) = \{u^{-1}(2^l) : l \in \mathbb{N} \leq k \} \) and \( B(k) = \{v^{-1}(2^l) : l \in \mathbb{N} \leq k \} \).

**Remark 3.8.** The same strategy with obvious modifications will also allow us to prove analogues of Proposition 3.7 in more general settings where \( P(x, y) \) and \( Q(x, y) \) are replaced by definable binary functions in o-minimal expansions of \( \mathbb{R} \) or binary analytic functions restricted to bounded open sets of \( \mathbb{C} \). Functions of the latter type can be interpreted in an o-minimal expansion of \( \mathbb{R} \), and we expect them to be useful when we consider generalized sum-product phenomena for complex analytic functions, but the corresponding \( A \) and \( B \) are finite subset of a fixed bounded open subset of \( \mathbb{C} \).

As mentioned in the introduction, the analogue of the main theorem should hold in much more general settings. The missing ingredients include appropriate analogue of Szemerédi-Trotter Theorem (and Corollary 3.4) and of the Elekes–Rónyai Theorem in these settings. For o-minimal expansion of \( \mathbb{R} \), the analogue of the Szemerédi-Trotter Theorem is known [1].

For the case where \( K = \mathbb{C} \), there is also a proof using algebraic geometry, essentially a suitable translation of the above strategy. So in combination with material in Section 4, we can recover the main result of this paper using only algebraic geometry. We leave the details to the interested reader.

### 4. Applications

**Lemma 4.1.** Suppose \( K \) has \( \text{char} \ K = 0 \), \( f, \hat{f}, u, \hat{u}, v, \hat{v} \) are nonconstant univariate polynomials with coefficients in \( K \) such that \( f \) and \( \hat{f} \) are monic, \( u, \hat{u}, v, \hat{v} \) each have constant coefficient equal to 0, and

\[
f(u(x) + v(y)) = \hat{f}(\hat{u}(x) + \hat{v}(y)).
\]

Then we must have \( f = \hat{f}, u = \hat{u} \) and \( v = \hat{v} \).

**Proof.** Taking partial derivatives with respect to \( x \) and \( y \) and manipulating the equations, we get

\[
\frac{u'(x)}{\hat{u}'(x)} = \frac{v'(y)}{\hat{v}'(y)}.
\]

Hence, both sides of the equation must be equal to a constant \( c \in K \). Hence, \( u(x) = cu(x) + d \) and \( v(y) = cv(y) + e \). By the assumptions on \( u, \hat{u}, v, \hat{v} \) we see that \( d = e = 0 \). Further, \( c = 1 \) since \( f \) and \( \hat{f} \) are monic. The desired conclusion follows.

**Lemma 4.2.** Suppose \( K \) has \( \text{char} \ K = 0 \), \( f, \hat{f}, u, \hat{u}, v, \hat{v} \) are nonconstant univariate polynomials with coefficients in \( K \) such that \( u, \hat{u}, v, \hat{v} \) are monic, neither \( u(x)v(y) \)
nor \( \hat{u}(x)\hat{v}(y) \) can be written as \( \tilde{u}^n(x)\tilde{v}^n(y) \) where \( \tilde{u} \) and \( \tilde{v} \) are univariate polynomials over \( K \), and \( n \geq 2 \). If

\[
 f(u(x)v(y)) = \hat{f}(\hat{u}(x)\hat{v}(y)),
\]

then we must have \( f = \hat{f}, u = \hat{u} \) and \( v = \hat{v} \).

**Proof.** Using model-theoretic transfer principle, we can reduce to the case where \( K = \mathbb{C} \). Taking partial derivatives with respect to \( x \) and \( y \), and manipulating the equations we get

\[
 \frac{u'(x)\hat{u}(x)}{u'(x)u(x)} = \frac{v'(y)\hat{v}(y)}{v'(y)v(y)}. \]

Hence, both sides of the equation must be equal to a constant \( c \in \mathbb{C} \). Letting \( |x| \) go to infinity, we get \( c = \deg u/\deg \hat{u} = m/n \) with \( m/n \) a rational number in lowest terms. So \( nud'x)/u(x) = m\hat{u}'(x)/\hat{u}(x) \). Integrating and taking exponential, we get \( u^m = d\hat{u}^m \) with \( d \in \mathbb{C} \). By the assumption that \( u \) and \( \hat{u} \) are monic, \( d = 1 \) and \( u^n = \hat{u}^n \). As \( K[x] \) is a unique factorization domain, we obtain a univariate polynomial \( \tilde{u} \) with coefficients in \( K \) such that \( u(x) = \tilde{u}^n(x) \) and \( \hat{u}(x) = \tilde{u}^n(x) \). Likewise, we get a univariate polynomials \( \tilde{v} \) with coefficients in \( K \) such that \( v(y) = \tilde{v}^n(y) \) and \( \hat{v}(y) = \tilde{v}^n(y) \). Then \( u(x)v(y) = \tilde{u}^m(x)\tilde{v}^m(y) \) and \( \hat{u}(x)\hat{v}(y) = \tilde{u}^n(x)\tilde{v}^n(y) \). By the assumption on \( u, \hat{u}, v, \) and \( \hat{v} \), we must have \( m = n = 1 \). The desired conclusion follows.

We are now ready to deduce Theorem 1.2.

**Proof of Theorem 1.2.** Suppose we are not in case (i) of Theorem 1.2. Then, applying Theorem 1.1 on the pair of polynomials \( (P(x, y), Q(x, y)) \), we find ourselves in case (ii) or (iii) of Theorem 1.1. Suppose we are in case (ii) of Theorem 1.1. Then we get

\[
 P(x, y) = f(\gamma_1u(x) + \delta_1v(y)) \quad \text{and} \quad Q(x, y) = g(\gamma_2u(x) + \delta_2v(y)),
\]

where we can further assume that \( f \) and \( g \) are monic and \( u \) and \( v \) have constant coefficient 0. Applying Theorem 1.1 again, now on the pair \( (\hat{P}(x, y), Q(x, y)) \) with \( \hat{P}(x, y) = P(y, x) \), we end up in case (ii) of Theorem 1.1 again due to mutual exclusivity of the cases of Theorem 1.1. Combined with an application of Lemma 4.1 on \( Q(x, y) \), we get

\[
 \hat{P}(x, y) = \hat{f}(\hat{\gamma}_1u(x) + \hat{\delta}_1v(y)) \quad \text{and} \quad Q(x, y) = g(\gamma_2u(x) + \delta_2v(y)),
\]

where \( \hat{f} \) is also monic. Hence \( P(x, y) = \hat{f}(\delta_1v(x) + \hat{\gamma}_1u(y)) \). Now we apply Lemma 4.1 to \( P(x, y) \) to conclude the proof for this special case. We treat the situation where we are in case (iii) of Theorem 1.1 similarly, replacing the role of Lemma 4.1 by that of Lemma 4.2. The exactness part of Theorem 1.2 follows easily from exactness part of Theorem 1.1. \( \square \)

Applying Theorem 1.2 when \( P(x, y) = Q(x, y) \) gives us the following corollary.
Corollary 4.3. Suppose K is either $\mathbb{R}$ or $\mathbb{C}$. Then there is $\alpha = \alpha(\deg P)$ such that exactly one of the following possibilities hold:

(i) For all $n$ and all subsets $A$ of $K$ with $|A| = n$, we have $|P(A, A)| > \alpha n^{5/4}$.

(ii) $P(x, y) = f(\gamma u(x) + \delta u(y))$ where $f$, $g$, and $u$ are univariate polynomials over $K$, and $\gamma$ and $\delta$ are in $K$.

(iii) $P(x, y) = f(u^m(x)u^n(y))$ where $f$, $g$, and $u$ are univariate polynomials over $K$, and $m$ and $n$ are in $\mathbb{N}^{\geq 1}$.

Remark 4.4. As mentioned earlier in the introduction, the analogues of Theorem 1.1 and 1.2 also hold for rational functions. For the proof, we need to replace Fact 3.1 by the corresponding statement for rational functions. The latter can be deduced from a recent result by [16]; note that the first case of their main theorem is of additive form and the last two cases are of multiplicative form.

We now discuss application mentioned in the introduction. Suppose $a_1, \ldots, a_n$ are $n$ distinct points on a circle $T \subseteq \mathbb{R}^2$. Note that $T$ minus a point can be naturally parametrized by $\mathbb{R}$ using a combination of affine transformations and inversions. Hence, the distances and slopes obtained by a pair of points $a$ and $a'$ chosen among $a_1, \ldots, a_n$ can be expressed as $f(a, a')$ and $g(a, a')$, where $f$ and $g$ are rational functions with real coefficients not depending on the choice of $a_1, \ldots, a_n$. Applying the analogue of Theorem 1.2, we deduce that either $a_1, \ldots, a_n$ gives us at least $\beta n^{5/4}$ many distinct distances or $\beta n^{5/4}$ many distinct slopes, where $\beta$ is a constant not depending on $n$ or $T$.

Proposition 4.5. Suppose $K$ is a field with $\text{char } K = 0$. We obtain weakened analogues of Theorem 1.1, Theorem 1.2, and Corollary 4.3, with “exactly” replaced by “at least” in the respective statements. Moreover, when $K$ is algebraically closed or real closed, then the full analogues of these results hold.

Proof. We will only prove the proposition for the analogue of Theorem 1.1; the deduction of the analogues of the other two statement from this is similar to what we have done earlier in this section. For a given pair $(d_1, d_2)$ and $n \in \mathbb{N}^{\geq 1}$, Theorem 1.1 implies the following when $K$ is $\mathbb{R}$ or $\mathbb{C}$: For all tuples $(c_{1,k,l})_{0 \leq k+l \leq d_1}$ and $(c_{2,k,l})_{0 \leq k+l \leq d_2}$ of elements in $K$, and all subsets $A$ and $B$ of $K$ with $|A| = |B| = n$, with $P(x, y) = \sum_{0 \leq k+l \leq d_1} c_{1,k,l} x^k y^l$ and $Q(x, y) = \sum_{0 \leq k+l \leq d_2} c_{2,k,l} x^k y^l$ the inequality

$$\max\{|P(A, B)|, |Q(A, B)|\} \leq \alpha(\deg P, \deg Q)n^{5/4}$$

implies that $P(x, y)$ and $Q(x, y)$ form either an additive pair or a multiplicative pair. For each such $(d_1, d_2, n)$, the preceding statement admits a first-order expression in the language of fields. Hence, it also holds in all algebraically closed and real closed fields, as the respective theories are complete. Therefore, in all algebraically closed and real closed fields, the negation of (i) implies (ii) or (iii) in the corresponding
analogues of Theorem 1.1. The "exactness" part for algebraically closed and real closed fields are simpler and can be achieved similarly.

Now fix a field $K$ with char $K = 0$, and suppose there is $n \in \mathbb{N} \geq 1$, and finite subsets $A$ and $B$ of $K$ with $|A| = |B| = n$ such that

$$\max\{|P(A, B)|, |Q(A, B)|\} \geq \alpha (\deg P, \deg Q) n^{5/4}.$$ 

Let $K^a$ be the algebraic closure of $K$. Viewing $P(x, y)$ and $Q(x, y)$ as elements of $K^a[x, y]$ and applying the analogue of Theorem 1.1 we get that either

$$P = f(\gamma_1 u(x) + \delta_1 v(y)) \quad \text{and} \quad Q = g(\gamma_2 u(x) + \delta_2 v(y))$$ 

or

$$P = f(u^{m_1}(x)v^{n_1}(y)) \quad \text{and} \quad Q = g(u^{m_2}(x)v^{n_2}(y)),$$

where $f$, $g$, $u$, and $v$ are univariate polynomials with coefficient in $K^a$, $\gamma_1$, $\gamma_2$, $\delta_1$, and $\delta_2$ are in $K^a$, and $m_1$, $m_2$, $n_1$, and $n_2$ are in $\mathbb{N} \geq 1$. In the additive case, we can moreover arrange that $f$ and $g$ are monic, and $u$ and $v$ are monic and have zero constant coefficients; in the multiplicative case, we can arrange that $u$ and $v$ are monic, $\gcd(m_1, n_1) = \gcd(m_2, n_2) = 1$ and $u(x)v(y)$ can not be written $\tilde{u}^n(x)\tilde{v}^n(y)$ where $\tilde{u}$ and $\tilde{v}$ are univariate polynomials over $K$, and $n \geq 2$. We will show that $f$, $g$, $u$, and $v$ have coefficients in $K$, and $\gamma_1$, $\gamma_2$, $\delta_1$, $\delta_2$ are in $K$.

Let $G = \text{Aut}(K^a/K)$ be the absolute Galois group of $K$. It suffices to show that the natural actions of $G$ on $K^a$ and the ring of univariate polynomials with coefficients in $K^a$ fix $f$, $g$, $u$, $v$, $\gamma_1$, $\gamma_2$, $\delta_1$, and $\delta_2$. We treat $P(x, y)$ in the additive case. Let $\sigma$ be in $G$. As $P(x, y)$ is in $K[x, y]$, it is fixed by $\sigma$. Hence

$$f(\gamma_1 u(x) + \delta_1 v(y)) = f^\sigma(\gamma_1^\sigma u^\sigma(x) + \delta_1^\sigma v^\sigma(y))$$

where we use exponential notation for group actions. Applying Lemma 4.1, we get $f = f^\sigma$, $u = u^\sigma$, $v = v^\sigma$, $\gamma_1 = \gamma_1^\sigma$, and $\delta_1 = \delta_1^\sigma$. We can deal with $Q(x, y)$ identically. The multiplicative case is similar, but using Lemma 4.2 instead of Lemma 4.1.

**Remark 4.6.** There is evidence that the full analogues of Theorem 1.1, Theorem 1.2, and Corollary 1.3 do not hold in all fields of characteristic 0. For example, $x^2 + y^2$ is not 1-expanding over $\mathbb{C}$, but 1-expanding over $\mathbb{Q}$ due to a result by Chang [5].

Using model-theoretic transfer principle, we can show that when $K$ has positive characteristic $p$, and $|A| = |B| = n$ is very small compared to $p$, if

$$\max\{|P(A, B)|, |Q(A, B)|\} \geq \alpha (\deg P, \deg Q) n^{5/4},$$

then $P(x, y)$ and $Q(x, y)$ must form either an additive or a multiplicative pair. There are more involved results due to Tao along similar lines when $n$ is relatively large compared to the size of the field [28]. That suggests that similar results about arbitrary pairs of polynomials should also be true for intermediate values of $n$ in finite fields, but no proof is currently known.
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA IL, USA
E-mail address: yifanjing17@gmail.com

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA IL, USA
E-mail address: souktik2@illinois.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME IN, USA
E-mail address: mtran6@nd.edu