The moduli space of local homogeneous 3-geometries‡

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Abstract: For a canonical formulation of quantum gravity, the superspace of all possible 3-geometries on a Cauchy hypersurface of a 3 + 1-dimensional Lorentzian manifold plays a key role. While in the analogous 2 + 1-dimensional case the superspace of all Riemannian 2-geometries is well known, the structure of the superspace of all Riemannian 3-geometries has not yet been resolved at present.

In this paper, an important subspace of the latter is disentangled: The superspace of all Riemannian 3-geometries has not yet been resolved at present.

In the following, $g \in \text{Geom}(M^d)$ is always understood to denote the geometry as diffeomorphism invariant class of the corresponding metrics. In generic regions, $\text{Geom}(M^d)$ can be equipped with an ultralocal metric, given as a reparametrization invariant form with components

$$G_{\alpha\beta\mu\nu}(x, x') := \frac{\delta^d(x - x')}{\sqrt{\det g}} (a_{\alpha\mu} g_{\beta\nu} + b_{\alpha\nu} g_{\beta\mu} - c_{\alpha\beta} g_{\mu\nu}),$$

(1.6)

In this lecture, we use a Lorentzian path integral instead. However, we should not expect such a metric to exist on all of $\text{Geom}(M^d)$, for the following reason: It is known that the action of $\text{Diff}(M^d)$ on $\text{Met}(M^d)$ is not free in general. So with the natural quotient topology, $\text{Geom}(M^d)$ is not a manifold. At certain symmetrical metrics the quotient has singularities. However, with some techniques from algebraic geometry (in also applied to ADE singularities), a first non-singular resolution of $\text{Geom}(M^d)$ was found in [7], and the minimal non-singular resolution $\text{Geom}_\text{nr}(M^d)$ was finally constructed in [8].

Although the classical action (1.3) is given as the integral over a path of d-geometries $g \in \text{Geom}(M^d)$, for its quantization the superspace $\text{Geom}(M^d)$ is not the appropriate configuration space, because the Euclidean path integral would then contain a divergent integration over the conformal modes of the geometry. In [7] it was suggested to use a Lorentzian path integral instead. However, the superspace $\text{Geom}(M^d)$ could also be regularized, if one succeeds to factor out the conformal modes. Indeed, for $d = 2$ this has been done successfully in [8]. The superspace $\text{Geom}(M^2)$ of all Riemannian 2-geometries $g$ on $M^2$ can be disentangled, into the conformal deformations $\text{Conf}(M^2)$ of any Riemannian geometry $g$, and the moduli space $\mathcal{M}(M^2)$ of all conformal equivalence classes $[g]$:

$$\text{Geom}(M^2) = \text{Conf}(M^2) \times \mathcal{M}(M^2).$$

(1.7)

Since any Riemannian 2-manifold $M^2$ is conformally flat, all local data on $M^2$ is actually contained in $\text{Conf}(M^2)$. However, for $d > 2$, the local data of the manifold $M^d$ is partially also contained in the moduli space

$$\mathcal{M}(M^d) := \frac{\text{Geom}(M^d)}{\text{Conf}(M^d)}$$

(1.8)

Note that mathematically this term applies well to the present case, although some physicists use it with a more limited understanding.

1. Introduction

The fundamental problem of (quantum) gravity uses a topological $d + 1$ decomposition of the $(d + 1)$-dimensional Lorentzian manifold $M^{d+1} = M^d \times M^d$, into a time manifold $M^d$ and a smooth topological $d$-space $M^d$, which for any $t \in M^1$ is a Riemannian manifold $M^d(t)$. The action for pure Einstein gravity on $M^{d+1}$ then becomes

$$S = \int_{M^{d+1}} \sqrt{\det(g_{\mu\nu})} \left(\frac{d+1}{2}\right) R \ d^{d+1}x$$

$$= \int_{M^d} \int_0^{\infty} N \sqrt{\det g(t)} \mathcal{L}(t) \ d^d x \ dt,$$

(1.1)

which is an integral over all Riemannian hypersurfaces $M^d(t)$, where $\mathcal{L} := R + \text{tr}(K^2) - \text{tr}(K)^2$. On each $M^d(t)$ the Riemannian metric $g(t)$ yields intrinsic Ricci scalar curvature $R(t)$ and the extrinsic curvature form $K(t)$. The latter is the second fundamental form with components

$$K_{\alpha\beta} = -\frac{1}{2N} \left(\frac{\partial g_{\alpha\beta}}{\partial t} - N_{\alpha} g_{\beta} - N_{\beta} g_{\alpha}\right), \quad \alpha, \beta = 1, \ldots, d,$$

(1.2)

where $N$ and $N_\alpha$ are the lapse and shift functions, respectively. Note that, throughout this paper, any Riemannian metric $g$ is per definition sign normalized to $\det g > 0$. Using the conjugate momenta

$$\pi := \frac{1}{\sqrt{\det g}} \frac{\delta S}{\delta g},$$

(1.3)

the Hamiltonian density becomes

$$\mathcal{H} = \pi^{\alpha\beta} g_{\alpha\beta} - \mathcal{L}$$

$$= N \left[ \frac{\text{tr}(\pi^2)}{d-1} - \frac{1}{(d-1)^2} \text{tr}(\pi) - R \right] + 2N\pi^{\alpha\beta}$$

$$= N \mathcal{C} + 2N\pi^{\alpha\beta},$$

(1.4)

where $\mathcal{C}$ and $\mathcal{C}^\alpha$ are the super-Hamiltonian and supermomentum constraints, enforcing reparameterization invariance of $M^1$ and $M^d(t)$, respectively. In particular, the possible reparameterizations of any $M^d$ are given by the connected component $\text{Diff}_0(M^d)$ of its diffeomorphism group $\text{Diff}(M^d)$.

Let $\text{Met}(M^d)$ be the space of all Riemannian metrics on $M^d$. Then the Wheeler superspace of $d$-geometries is given as

$$\text{Geom}(M^d) := \frac{\text{Met}(M^d)}{\text{Diff}(M^d)}$$

(1.5)
of Riemannian $d$-geometries $\text{Geom}(M^d)$ modulo conformal transformations $\text{Conf}(M^d)$. Therefore, for $d > 2$, it makes sense to separate the local data in $\mathcal{M}(M^d)$, neglecting all global properties of $M^d$. When $M^d$ is considered only locally, the mapping class group $\text{Diff}(M^d)/\text{Diff}_0(M^d)$ is negligible, and $\text{Geom}(M^d)$ is really the classical configuration space. For a local homogeneous manifold $M^d$, all geometrical data is given local at just one arbitrarily chosen point $x$ of $M^d$, on a neighbourhood $U(x)$, with infinitesimal limit $\epsilon \to 0$. Therefore we denote $\text{local homogeneous Riemannian } d$-geometries just as $\text{Geom}(d)$. Homogeneous conformal transformations, denoted as $\text{Conf}(d)$, leave the unique flat Riemannian geometry $I^d$ invariant. Therefore, we consider in the following the moduli space of local homogeneous $d$-geometries as defined by

$$\mathcal{M}(d) := \frac{\text{Geom}(d)}{\text{Conf}(d) \setminus \{I^d\}}. \quad (1.9)$$

While for dimension $d \geq 4$ the moduli space $\mathcal{M}(d)$ seems still to be exceedingly complicated, the structure of the moduli space $\mathcal{M}(3)$ of local homogeneous Riemannian 3-geometries is resolved in the present paper.

Sec. 2 resumes the local properties of (Riemannian) homogeneous 3-geometries. In Sec. 3 the moduli space $\mathcal{M}(3)$ is constructed as an algebraic variety, parametrized by scalar geometric invariants. Sec. 4 then shows the consistency of the locally non-Euclidean Hausdorff topology of $\mathcal{M}(3)$ with a topological Morse-like potential on the space $K^3$ of Bianchi Lie isometries of the moduli. Finally, Sec. 5 gives a short discussion of the results.

**2. Local properties of homogeneous 3-geometries**

Per definition, a homogeneous geometry admits a transitive action of its isometry group. The Kantowski-Sachs (KS) spaces are the only homogeneous Riemannian 3-spaces not admitting a simply transitive subgroup of their isometry group. The Lie algebra of the latter is $\mathbb{R} \oplus \mathbb{R}$. Any KS space can be obtained as a specific limit of Bianchi IX spaces. (Globally, a hyper-cigar like 3-ellipsoid of topology $S^3$ is stretched to infinite length, becoming a hypercylinder $S^2 \times \mathbb{R}$.) All other homogeneous Riemannian 3-spaces have a simply transitive isometry subgroup, corresponding to one of the Bianchi Lie algebras. Therefore, the following considerations can be restricted to homogeneous 3-geometries of Bianchi type, nevertheless yielding finally a classification of all local homogeneous Riemannian 3-geometries. Let the characteristic isometry group of a local homogeneous 3-geometry be defined to be $\text{IX}$ for KS spaces or its Bianchi type otherwise.

In the case of a transitive Lie isometry of Bianchi type, the Riemannian geometry can be soldered to an orthogonal frame spanned by the Lie algebra generators $e_i$ in the tangent space, i.e.

$$g_{\mu \nu} = e^a_{\mu} e^b_{\nu} g_{a b}, \quad \mu, \nu = 1, 2, 3, \quad (2.1)$$

where $e^a = e^a_\mu dx^\mu = g^{a i} e_i$, $e_i = e^a_\mu \frac{\partial}{\partial x^\mu}$, and $g^{a b} g_{i j} = \delta^a_i \delta^b_j$, with the constant metric

$$(g_{a b}) = \begin{bmatrix}
  e^s & 0 & 0 \\
  0 & e^{s+w-t} & 0 \\
  0 & 0 & e^{s-t} 
\end{bmatrix}. \quad (2.2)$$

Here $s$ fixes the overall scale, while $t$ and $w$ parametrize the anisotropies related respectively to the $e_1$ and $e_2$ direction (preserving isotropy in the respective orthogonal planes).

The data which characterizes the local geometrical structure of a homogeneous Riemannian manifold $M^3$ of characteristic Bianchi isometry can be rendered in form of (i) the local scales of $(2.2)$, (ii) the covariant derivatives

$$D e^k = e^k_{i j} e^i := e^k_{i a} \delta x^a \delta x^b \quad (2.3)$$

of the dual generators $e^k$ in the cotangent frame, and iii) the corresponding Bianchi Lie algebra, represented as

$$[e_i, e_j] = C^k_{i j} e_k. \quad (2.4)$$

The bracket $[\cdot, \cdot]$ defines a Lie algebra, iff the structure constants satisfy the antisymmetry condition

$$C^i_{i j} = 0, \quad (2.5)$$

and the Jacobi condition

$$C^i_{i j} e_k + C^i_{j k} e_j + C^i_{k j} e_i = 0, \quad (2.6)$$

with nondegenerate antisymmetric indices $i, j, k = 1, 2, 3$. With $(2.4)$ a Lie algebra is already completely described by the $2 \times 2$-matrices $C_{i < j, i = 1, 2, 3}$, each with components $C^i_{j k}$, $j, k = 1, 2$. But this description may still carry redundancies: The space of all sets $\{C^i_{j k}\}$ satisfying the Lie algebra conditions $(2.5)$ and $(2.6)$ is a subvariety $W^3 \subset \mathbb{R}^9$. GL(3) basis transformations act on a given set of structure constants as tensor transformations:

$$C^i_{j k} \rightarrow \tilde{C}^i_{j k} := (A^{-1})^i_k C^h_{j g} A_i^f A_j^g A_k^f \forall A \in \text{GL}(3). \quad (2.7)$$

On $W^3$ this yields a natural equivalence relation $C \sim \tilde{C}$, defined by

$$C^k_{i j} \sim \tilde{C}^k_{i j} : \Leftrightarrow \exists A \in \text{GL}(3) : \tilde{C}^k_{i j} = (A^{-1})^k_i C^h_{j g} A_i^f A_j^g. \quad (2.8)$$

The associated projection

$$\pi : \left\{ \begin{array}{c}
  W^3 \\
  C
\end{array} \right\} \rightarrow K^3 := W^3/\text{GL}(3) \quad (2.9)$$

yields just the space $K^3$ of all Bianchi Lie algebras as quotient space.

The present choice of the real parameters $s, t, w$ for datum (i) simplifies the following calculations in a specific triad basis for data (ii) and (iii), chosen in consistency with the representations of $\mathbb{R}^3$. The basis is given by matrices $(e_\alpha^a)$, with anholonomic rows $a = 1, 2, 3$ indexing the generators of the algebra, and holonomic coordinate columns $\alpha = 1, 2, 3$. The coordinates will also be denoted as $x := x^t$, $y := x^s$, $z := x^w$. For the different Bianchi types the matrices $(e_\alpha^a)$ take the following form:

**Bianchi I:**

$$(e_\alpha^a) = \begin{bmatrix}
  1 \\
  1 \\
  1
\end{bmatrix}. \quad (2.10)$$

**Bianchi II:**

$$(e_\alpha^a) = \begin{bmatrix}
  1 & -z \\
  0 & 1 \\
  1 & 0
\end{bmatrix}. \quad (2.11)$$

**Bianchi IV:**

$$(e_\alpha^a) = \begin{bmatrix}
  1 & e^x & 0 \\
  x e^x & e^x & e^z
\end{bmatrix}. \quad (2.12)$$

**Bianchi V:**

$$(e_\alpha^a) = \begin{bmatrix}
  1 & e^x \\
  e^z & e^x
\end{bmatrix}. \quad (2.13)$$
Bianchi VI, $h = A^2$:

$$
(e^a_α) = \begin{bmatrix}
1 & e^{Ax} \cosh x & -e^{Ax} \sinh x \\
-e^{Ax} \sinh x & e^{Ax} \cosh x & 0 \\
-\sinh y & e^{Ax} \sinh x & e^{Ax} \cosh x \\
\end{bmatrix},
$$

(2.14)

Bianchi VII, $h = A^2$:

$$
(e^a_α) = \begin{bmatrix}
1 & e^{Ax} \cos x & -e^{Ax} \sin x \\
e^{Ax} \sin x & e^{Ax} \cos x & 0 \\
0 & 0 & 1 \\
\end{bmatrix},
$$

(2.15)

Bianchi VIII:

$$
(e^a_α) = \begin{bmatrix}
cosh y \cos z & -\sin z & 0 \\
cosh y \sin z & \cos z & 0 \\
\sinh y & 0 & 1 \\
\end{bmatrix},
$$

(2.16)

Bianchi IX:

$$
(e^a_α) = \begin{bmatrix}
cos y \cos z & -\sin z & 0 \\
cos y \sin z & \cos z & 0 \\
-\sin y & 0 & 1 \\
\end{bmatrix},
$$

(2.17)

For each of the lines $\{\mathrm{VI}_h\}$ and $\{\mathrm{VII}_h\}$ the parameter range is given by $\sqrt{h} = A \in [0, \infty]$. Let us also remind that, $\mathrm{III} := \mathrm{VII}_1$.

The structure constants can be reobtained from $ds^2$ and the triad by

$$
C_{ijk} = ds^2([e_i, e_j], e_k), \quad C^k_{ij} = C_{ijk} g^{jk}.
$$

(2.18)

The metrical connection coefficients are determined as

$$
\Gamma^k_{ij} = \frac{1}{2} g^{kr} (C_{ijr} + C_{jri} + C_{rji}).
$$

(2.19)

W.r.t. the triad basis, the curvature operator is defined as

$$
\mathcal{R}_{ij} := \nabla_{[e_i, e_j]} - (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i}),
$$

(2.20)

the Riemann tensor components are

$$
R_{hijk} := \langle e_k, \mathcal{R}_{ij} e_h \rangle >, \quad \mathcal{R}_{ij} := R^k_{ij} g^{jk}.
$$

(2.21)

and the Ricci tensor is

$$
R_{ij} := R^k_{ij} = \Gamma^f_{ij} \Gamma^e_{ef} - \Gamma^f_{ie} \Gamma^e_{jf} + \Gamma^e_{if} C^f_{ej}.
$$

(2.22)

From (2.22) we may form the following scalar invariants of the geometry: The Ricci curvature scalar

$$
R := R^i_{ii};
$$

(2.23)

the sum of the squared eigenvalues

$$
N := R^i_{ij} R^{ij},
$$

(2.24)

the trace-free scalar

$$
S := S^i_{jh} S^{jh} = R^i_{jk} R^{jk} - RN + \frac{2}{9} R^3,
$$

(2.25)

where $S^i_{jh} := R^i_{jk} - \frac{1}{2} \delta^i_{jh} R$, and, related to the York tensor,

$$
Y := R_{ijk} g^{il} g^{jm} g^{kn} R_{lmnm}, \quad \text{with}
$$

(2.26)

$$
R_{ij;k} := e^α_{βγ} e^β_{αm} e^γ_{αn} R_{αβγ},
$$

yielding the following normalized invariants, which depend only on the homogeneously conformal class of the geometry:

$$
\tilde{N} := 1, \quad \tilde{R} := R/\sqrt{\tilde{N}}, \quad \tilde{S} := S/N^{3/2}, \quad \tilde{Y} := Y/N^{3/2}.
$$

(3.1)

For a non-flat Riemannian 3-space, the invariant $\tilde{Y}$ vanishes, iff the 3-geometry is conformally flat. A general conformal transformation is not necessarily homogeneous. Hence there may exist homogeneous spaces, which are in the same conformal class, but in different homogeneously conformal classes. Note that a rescaling (3.1) does not change the Bianchi or KS type of isometry.

The moduli space $\mathcal{M}(3)$ is given by classifying space $\text{Geom}(3)$ of non-flat local homogeneous Riemannian 3-geometries modulo homogeneous conformal transformations $\text{Conf}(3)$ (3.1). Since the flat Bianchi I geometry, as $\text{Conf}(3)$-invariant center of the projection, has been excluded, the moduli space $\mathcal{M}(3)$ can be parametrized by the invariants $\tilde{R}$, $\tilde{S}$ and $\tilde{Y}$, given for each fixed Bianchi type as a function of the anisotropy parameters $t$ and $w$. For the non-flat Bianchi Lie algebras these invariants are listed explicitly in [11]. $\mathcal{M}(3)$ can also be embedded into a minimal cube, spanned by $\tilde{R}/\sqrt{\tilde{N}}, \sqrt{\tilde{S} t^2}, \tilde{Y} \in [0, \tilde{N}]$.

Fig. 4 describes points of the moduli space which are of Bianchi type VI and VII or lower. Fig. 2 shows likewise points for Bianchi type VIII and IX.

A projection of $\mathcal{M}(3)$ to the $\tilde{R}$-$\tilde{S}$-plane was already considered in [14]. For a homogeneous space with 2 equal Ricci eigenvalues the corresponding point in the $\tilde{R}$-$\tilde{S}$-plane lies on a double line $L_2$, which has a range defined by $|\tilde{R}| \leq \sqrt{3}$ and satisfies the algebraic equation

$$
162 S^2 = (3 - \tilde{R}^2)^3.
$$

(3.3)

All other algebraically possible points of the $\tilde{R}$-$\tilde{S}$-plane lie inside the region surrounded by the line $L_2$. At the branch points $\tilde{R} = \pm \sqrt{3}$ of $L_2$ all Ricci eigenvalues are equal. These homogeneous spaces possess a 4-dimensional isometry group. Homogeneous spaces possessing a 4-dimensional isometry group are represented by points on $L_2$. If one Ricci eigenvalue equals $R$, i.e. if there exists a pair $(a, -a)$ of Ricci eigenvalues, the corresponding point in the $\tilde{R}$-$\tilde{S}$-plane lies on a line $L_0$, defined by the range $|\tilde{R}| \leq \sqrt{2}$ and the algebraic equation

$$
\tilde{S} = \frac{(2 - \tilde{S}^2)}{\tilde{R}^2}.
$$

(3.4)

In the case that one eigenvalue of the Ricci tensor is zero, the corresponding point in the $\tilde{R}$-$\tilde{S}$-plane lies on a line $L_0$, defined by the range $|\tilde{R}| \leq \sqrt{2}$ and the algebraic equation

$$
\tilde{S} = 11 \tilde{R}^2 - \tilde{R}.
$$

(3.5)
At the branch points of the curve $L_9$ the Ricci tensor has a triple eigenvalue, which is negative for geometries of Bianchi type $V$, and positive for type $IV$ geometries with metrics $(t, w) = (0, 0)$. These constant curvature geometries are all conformally flat with $\hat{Y} = 0$. Besides the flat Bianchi I geometry, the remaining conformally flat spaces with $\hat{Y}$ are the KS space $(\hat{R}, \hat{S}, \hat{Y}) = (\sqrt{2}, -\sqrt{2}, 0)$ and, point reflected, the Bianchi type IIIa, corresponding to the initial point of a Bianchi III line segment ending at the Bianchi II point in Fig. 1.

The point $(1, 0, 0)$ of Fig. 1 admits both types, Bianchi V and VIIb with $h > 0$. Nevertheless, this point corresponds only to one homogeneous space, namely the space of constant negative curvature. This is possible, because this space has a 6-dimensional Lie group, which contains the Bianchi V and VIIb subgroups. Note that in the flat limit $V \to I$, the additional Bianchi groups VIIb change with $h \to 0$.

Similarly, the Bianchi III points of Fig. 1 lie on the curve $L_2$. However, these points are also of Bianchi type $VIII$. In fact, each of them correspond to one homogeneous geometry only. However, the latter admits a 4-dimensional isometry group, which has two 3-dimensional subgroups, namely Bianchi III and VIIb, both containing the same 2-dimensional non-Abelian subgroup.

Altogether, the moduli space $M(3)$ of local homogeneous Riemannian 3-spaces is a $T_3$ (Hausdorff) space. But it is not a topological manifold: The line of VIIb moduli is a common boundary of 3 different 2-faces, namely that of the IX moduli, that of the VIII moduli, and with $h \to 0$ that of all moduli of type VIIb with $h > 0$. With $M(3)$ also Geom(3) is not locally Euclidean; rather both are stratifiable varieties. Geom(3) is a projective cone with sections $M(3)$ and the flat geometry $I^3$ as singularity. In the following Section it is shown, that the topology of $M(3)$ is consistent with the natural topology on the space $K^3$ of Bianchi Lie algebras, which may provide a Morse-like isometry potential on $M(3)$.

4. A potential on the isometry components of $M(3)$

Let us now examine the 3-dimensional characteristic isometries of the moduli in more detail. The space $K^3$ of Bianchi Lie algebras is naturally equipped with the quotient topology $\kappa^3$, generated by the projection $\pi$ from the subspace topology on $W^3 \subset R^9$. Below it will become clear that $\kappa^3$ does not satisfy the separation axiom $T_1$, i.e. $K^3$ contains non-closed points, or in other words, there is a non-trivial transition from $A$ to some $B \neq A$ in the closure $c(A)$ of $A$. Non-trivial ($A \neq B$) transitions are special limits, which exist only due to the non-$T_1$ property of $\kappa^3$. This property implies that $\kappa^3$ is not Hausdorffian (i.e. the separation axiom $T_2$ fails with $T_1$). Here a transitions from $A$ to $B$ is defined by

$$A \geq B : \iff B \in c(A).$$

By this definition, transitions are transitive and yield a natural partial order. A transition $A \geq B$ is non-trivial, iff $A > B$.

In order to find the topology $\kappa^3$ and all its transitions in $K^3$, an explicit description of the Bianchi Lie algebras is useful. In fact, it can be given in terms of the nonvanishing matrices $C_{i,j,k}$, i.e. $K^3$ contains non-closed points, or in other words, there is a non-trivial transition from $A$ to some $B \neq A$ in the closure $c(A)$ of $A$. Non-trivial ($A \neq B$) transitions are special limits, which exist only due to the non-$T_1$ property of $\kappa^3$. Furthermore, this representation can be normalized modulo an overall scale of the basis $e_1, e_2, e_3$, and $C_5$ can be chosen in some normal form (use the Jordan normal form).

In the semisimple representation category, there are only the simple Lie algebras $VII \equiv so(1, 2) = su(1, 1)$ and $IX \equiv so(3) = su(2)$, given by

$$C_{<3>} (VII) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C_{<3>} (IX) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

All other algebras are in the solvable representation category. They all have an Abelian ideal span $\{e_1, e_2\}$. Hence, with vanishing $C_{<3>} (VIII) = C_{<3>} (IX) = 0$, they are described by $C_{<3>} (IX)$ only. The "inhomogeneous" algebras $VII \equiv \{e(1,1) = (1,1)\}$ (local geometry of a Minkowski plane) and $VII = e(2) = (2)\) (local isometry of an Euclidean plane) are determined by

$$C_{<3>} (VII) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_{<3>} (VII) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By geometric specialization of the algebra $IV$ the geometric multiplicity of its eigenvalue is increases to 2, yielding the pure vector type algebra $V$, given by

$$C_{<3>} (IV) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$
correspond to rigid Lie algebras \( C \), i.e. those which cannot be deformed to any \( A \in \mathbb{R}^3 \) with \( A \not\subset C \) and index \( J(A) \geq J(C) \). In this sense, the isolated open points are the isolated locally maximal elements.

Fig. 3 shows on each horizontal level \( L(i) \) the algebras of equal index \( i \), which are the sources for the level \( L(i+1) \) below, and possible targets for the level \( L(i+1) \) above.

Let us now consider, for any \( A \in L(i) \subset \mathbb{R}^3 \), the component \( \mathcal{M}(A) \subset \mathcal{M}(3) \), which is given for \( A \neq I \) by all local homogeneous moduli of 3-geometries with characteristic isometry \( A \), and for \( A = I \) by \( \mathcal{M}(I) = \emptyset \). With \( \dim \emptyset := -1 \) and components \( \text{Geom}(A) \subset \text{Geom}(3) \) of characteristic isometry of type \( A \in \mathbb{R}^3 \), the relation

\[
J(A) = \dim \text{Geom}(A) + 1 + \dim \mathcal{M}(A)
\]

is satisfied for all \( A \in \mathbb{R}^3 \). By Eq. (4.9) the Morse-like potential \( J \) on \( \mathbb{R}^3 \), which was constructed purely topologically from \( \kappa^3 \), provides also a potential for the isometry components of \( \mathcal{M}(3) \) in terms of their dimension.

The Morse-like isometry potential for components \( \mathcal{M}(A) \subset \mathcal{M}(3) \) of the moduli space might be used to determine an evolution of a 3-geometry's isometry towards the minimum, corresponding to the flat geometry \( I \). However, a 3-geometry in the interior of its characteristic isometry component \( \text{Geom}(A) \) with \( J(A) = i > 0 \), contained in some nonminimal potential level \( L(i) \), might be considered as metastabilized against a transition: Since all isometry components of lower potential level can be reached only on the boundary of the isometry component \( \text{Geom}(A) \), for that given geometry, a transition to lower level is inhibited by the geometry's distance to that boundary.

5. Discussion

With the flat center of projection \( \mathbb{R}^3 \) excluded, the moduli space \( \mathcal{M}(3) \), of local homogeneous Riemannian 3-geometries \( \text{Geom}(3) \) modulo homogeneous conformal transformations \( \text{Conf}(3) \), has been constructed as an algebraic variety. The explicit parametrization of \( \mathcal{M}(3) \) by \( R, S, Y \), and of \( \text{Conf}(3) \) by \( N \), yields invariant measures on both, and on \( \text{Geom}(3) \), just given by the scalar invariants of geometry. Although the moduli space \( \mathcal{M}(3) \) is not locally Euclidean, it is a Hausdorff space. Nevertheless, Eq. (4.9) shows that its topology is also compatible with the non-Hausdorffian topology \( \kappa^3 \) of the space \( \mathbb{R}^3 \) of all Bianchi-Lie algebras, which characterize the moduli already up to differences in their anisotropic scales. Independently from regularity requirements in the construction of \( \mathcal{M}(3) \), Eq. (4.9), and its possible interpretation related to an isometry potential, show that the definition (1.9) is, at least for \( d = 3 \), the right one.

With the (perhaps a little artificial) notion of characteristic isometry, assigning the Bianchi Lie isometry \( \mathcal{I}X = \text{so}(3) \) to the KS spaces, the problem of occasionally missing simply transitive isometries could be circumvented for the considered dimension \( d = 3 \). For \( d > 3 \) this problem might be much worse. There, also the parametrization of \( \text{Geom}(d) \) by scalar invariants is essentially more difficult, due to the additional presence of the Weyl tensor.

The analogous construction for local homogeneous Lorentzian 3-geometries is complicated by the null cone structure, as additional datum of the geometry. In this case, partial results were obtained in [13]. The non-uniqueness of the flat Lorentzian 3-geometry implies further singularities, which have to be taken into account in a moduli construction.

For for canonical quantization, in the local homogeneous case considered here, the factorization of non-flat 3-geometries, into conformal modes \( \text{Conf}(3) \) and moduli \( \mathcal{M}(3) \), provides both, a possible regularization of an Euclidean path integral over 3-geometries, and the foundation for a conformal mode quantization with fixed moduli. The latter technique is extensively used in minisuperspace constructions of multidimensional geometries (see e.g. [13,14,15] and references therein).

Although global properties of the homogeneous 3-geometries have not been considered here, it should be clear that the global properties are partially dependent on the local ones: Any global homogeneous 3-geometry of a given Thurston type, admits only very specific local geometries (cf. [15]), corresponding to characteristic points in the moduli space \( \mathcal{M}(3) \).

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Fig. 1: Riemannian Bianchi geometries II, IV, V, VI\(_h\)(w = 0), VI\(_h\) (\(\sqrt{\pi} = 0, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, 1, 2\)), VII\(_h\) (\(\sqrt{\pi} = 0, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, 1\)); w.r.t. the common origin, the axes of the 3 planar diagrams, are: \(R/\sqrt{3}\) to the right, \(\sqrt{6}\) up, and 2 tanh \(\hat{Y}\) both, left and down.
Fig. 2: Riemannian Bianchi geometries II, V, V₁, V₁₁, VII₁, VIII(\(t, w\)) (\(t = -5, -1, 0, 1, 5\)), IX(\(t, w\)) (\(t = 0, \frac{1}{2}, 1, 2, 5\));

w.r.t. the common origin, the axes of the 3 planar diagrams, are: 
\(\hat{R}/\sqrt{3}\) to the right, \(\sqrt{6}\hat{S}\) up, and \(2\tanh\hat{Y}\) both, left and down.
Fig. 3: The topological space $K^3$ (right and left images have to be identified for the algebras IV and V; the locally maximal algebras IV, VI$_h$ and VII$_h$, $0 \leq h < \infty$, form a 1-parameter set of sources of arrows) as isometry potential.
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