The Ground State of the $D = 11$ Supermembrane: the External Dirichlet Problem

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Abstract. We analyse the outer Dirichlet problem for the Hamiltonian of the $SU(N)$ regularized $D = 11$ Supermembrane. We show the existence and uniqueness of the solution for this problem. The solution belongs to a suitable Hilbert space, introduced in the present work, which is defined in terms of the supersymmetric charges. The existence and uniqueness of the ground state of the $D = 11$ supermembrane is directly related to the patching of solutions of the inner and outer Dirichlet problems.

1. Introduction

The quantization of the Supermembrane theory has been a challenge of fundamental importance since the introduction of the theory in [1] thirty years ago. Many aspects related to this quantization remain open. One in particular is the existence of the ground state, which to these days remains to be confirmed. If this ground state exists, it should be related to the $D = 11$ supergravity multiplet, which is the eigenfunction associated to zero eigenvalue for the Hamiltonian of the zero modes of the Supermembrane theory [5]. In order to confirm the existence of the ground state of the theory, one needs to show the existence and uniqueness of the ground state for the Hamiltonian of the non zero modes which should be invariant under $SO(9)$. This Hamiltonian was first derived from a $(0 + 1)$ reduction of the $D = 10$ Super Yang Mills Theory in a completely different context [2, 3, 4]. The $SU(N)$ regularized Hamiltonian for the nonzero modes of the Supermembrane [5] coincides with the Hamiltonian of the BFSS matrix model [6]. Hence the existence of the ground state for this matrix model, which is still undetermined, turns out to be exactly the missing step in the proof of existence of the ground state for the $D = 11$ Supermembrane.

Due to the complexity of this task, the problem of finding a nontrivial solution has been split into different regions of the the space. This has been considered asymptotically in several works [7, 8, 9, 10, 11, 14]. Under the presumption of existence, the structure of the possible ground state near the origin has been examined in [14]. In the works [15, 16, 17] we proposed a strategy to tackle the ground state problem involving the decomposition of $\mathbb{R}^D$ into an inner and outer region, and confirmed existence and uniqueness in the inner region. The present work is devoted to the complementary outer Dirichlet problem. Once the existence and uniqueness of this outer
solution has been obtained, then it has to be patched in an appropriate sense with the inner solution. If the patch is smooth enough, then the overall state is indeed the ground state of the Hamiltonian of the non-zero modes of the $D = 11$ Supermembrane.

2. Existence of a solution for the outer region

Below we follow the notation set in [17]. Let $\Omega$ be a ball of radius $R$ in $\mathbb{R}^D$ and let $\Omega_C = \mathbb{R}^D / \Omega$. Let $S_\mu$ denote the bilinear form

$$
S_\mu(u, v) := (Qu, Qv)_{L^2(\Omega_C)} + (Q^1 u, Q^1 v)_{L^2(\Omega_C)} + \mu(u, v)_{L^2(\Omega_C)}
$$

(1)

where $\mu$ is a real, strictly positive parameter. This bilinear form gives rise to an inner product in the linear subspace $C_0^\infty(\Omega_C)$, the functions with compact support in $\Omega_C$. That this is the case is a consequence of the condition (K) introduced in [17]. This condition is satisfied by the supersymmetric charges also in $\Omega_C$. The completion of $C_0^\infty(\Omega_C)$ with respect to the metric associated to this inner product defines the Hilbert space, $S_\mu(\Omega_C)$. From the definition (1) it follows that $S_\mu(\Omega_C) \subset L^2(\Omega_C)$.

We introduce the operator $(-\Delta + V + \mu)$ with domain

$$
\text{Dom}(-\Delta + V + \mu) \subset S_\mu(\Omega_C) \cap H^2_{loc}(\Omega_C)
$$

(2)

where $V$ is the potential of the Supermembrane Hamiltonian. See [17] for details.

From the results in [17] it follows that given $f \in C_0^\infty(\Omega_C)$, there exists $\Psi \in \text{Dom}(-\Delta + V + \mu)$ such that the outer boundary value problem:

$$
\begin{cases}
(-\Delta + V + \mu)\Psi = f & \text{in } \Omega_C \\
\Psi = 0 & \text{on } \partial \Omega
\end{cases}
$$

(3)

has a unique solution. Consequently the image of $(-\Delta + V + \mu)$ is dense in $L^2(\Omega_C)$. Also the operator $(-\Delta + V + \mu)$ is bounded from below, since $-\Delta + V$ is a positive operator and $\mu > 0$. Hence $(-\Delta + V + \mu)^{-1}$ is a bounded operator. From both these properties it follows that the domain of $(-\Delta + V + \mu)^{-1}$ is $L^2(\Omega_C)$ and the solution $\Psi$ of the boundary value problem exists and is unique for all $f \in L^2(\Omega_C)$.

The existence and uniqueness of $\Psi$ also follows from the Riesz representation theorem. In fact, given $f \in L^2(\Omega_C)$, there always exists $\Psi \in S_\mu(\Omega_C)$ such that

$$
S_\mu(u, \Psi) = (u, f)_{L^2(\Omega_C)}
$$

(4)

for all $u \in S_\mu(\Omega_C)$. This again implies that $\Psi$ is the solution of the boundary value problem for any $L^2(\Omega_C)$. The regularity properties of the elliptic operator ensures that $\Psi \in L^2(\Omega_C) \cap H^2_{loc}(\Omega_C)$.

Now consider the bilinear form $S$ defined on the subspace $C_0^\infty(\Omega_C)$:

$$
S(u, v) := (Qu, Qv)_{L^2(\Omega_C)} + (Q^1 u, Q^1 v)_{L^2(\Omega_C)}.
$$

(5)

From property (K) in [17] which is also satisfied by the corresponding charges on $\Omega_C$, this quadratic form also defines an inner product in $C_0^\infty(\Omega_C)$. Its completion with respect to the associated norm defines the Hilbert space $S(\Omega_C)$.

Consider the operator $(-\Delta + V)$ with domain: $\text{Dom}(-\Delta + V) \subset S(\Omega_C) \cap H^2_{loc}(\Omega_C)$. According to the Riesz representation theorem, for any $f \in C_0^\infty(\Omega_C)$ there always exists a unique solution in $\varphi \in S(\Omega_2) \cap H^2_{loc}(\Omega_C)$ such that

$$
\begin{cases}
(-\Delta + V)\varphi = f & \text{in } \Omega_C \\
\varphi = 0 & \text{on } \partial \Omega
\end{cases}
$$

(6)
We notice that the solution \( \varphi \) has not been shown to lie in \( L^2(\Omega_C) \). However, we can map it to the solution \( \Psi \) of problem (3) which indeed belongs to \( L^2(\Omega_C) \). In fact,

\[
\Psi = (-\Delta + V + \mu)^{-1}(-\Delta + V)\varphi = \varphi - \mu(\Delta + V + \mu)^{-1}\varphi
\]

(7)
is well defined because \((-\Delta + V)\varphi \in C_0^\infty(\Omega_C)\), and it satisfies

\[
S_\mu(\Psi, u) = S(\varphi, u) = (f, u)_{L^2(\Omega_C)} \text{ with } \mu \in S_\mu(\Omega_C),
\]

(8)
for any \( f \in C_0^\infty(\Omega_C) \). Hence \( \Psi \) is the solution to problem (3), \( \Psi \in S_\mu(\Omega_C) \cap H^2_{loc}(\Omega_C) \).

The outer Dirichlet problem for the Hamiltonian of the \( D = 11 \) Supermembrane may be formulated as follows: given \( g \in \text{Dom}(-\Delta + V) \), find \( \Phi \in S_\mu(\Omega_C) \cap H^2_{loc}(\Omega_C) \) such that

\[
\begin{cases}
(-\Delta + V + \mu)\Phi = 0 & \text{in } \Omega_C \\
\Phi = g & \text{on } \partial \Omega
\end{cases}
\]

(9)

where \( \Phi \) is restricted by the \( SU(N) \) constraint. That is, the reduction under the \( SU(N) \) regularization procedure of the area preserving constraint of the \( D = 11 \) Supermembrane.

Following the previous analysis we can prove the existence and uniqueness of \( \Phi \in S(\Omega_C) \cap H^2_{loc}(\Omega_C) \) which solves the outer Dirichlet problem. It remains to show that \( \Phi \) also belongs to \( L^2(\Omega_C) \). This result surely depends on the dimension of the target space of the supermembrane. For the \( D = 11 \) Supermembrane we expect it to be true.

3. Conclusions
We argued on the existence and uniqueness of the outer Dirichlet problem associated to the Hamiltonian of the \( SU(N) \) regularized \( D = 11 \) Supermembrane. The argument we have presented shows that the solution belongs to the Hilbert space \( S(\Omega_C) \) and locally to \( H^2_{loc}(\Omega_C) \). In order to construct the external ground state we require a in \( L^2(\Omega_C) \). The missing step for the latter surely depends on the structure of the supersymmetric charges. However the improvement we have obtained compared to the results currently available in the literature is that we have shown the existence of a solution and based on it, the analysis of the asymptotic behavior is largely simplified.

4. Acknowledgements
MPGM is supported by Mecesup ANT1655, Universidad de Antofagasta, (Chile). A.R. and MPGM are partially supported by Projects Fondecyt 1161192 (Chile).

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