Risk minimization and set-valued average value at risk via linear vector optimization

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Abstract In this paper, the market extension of set-valued risk measures for models with proportional transaction costs is linked with set-valued risk minimization problems. As a particular example, the set-valued average value at risk (AV@R) is defined and its market extension and corresponding risk minimization problems are studied. We show that for a finite probability space the calculation of the values of AV@R reduces to linear vector optimization problems which can be solved using known algorithms. The formulation of AV@R as a linear vector optimization problem is an extension of the corresponding scalar result by Rockafellar and Uryasev.

Keywords average value at risk · set-valued risk measures · coherent risk measures · transaction costs · Benson’s algorithm

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1 Introduction

Set-valued risk measures are functions which map a multivariate random variable into a subset of some finite dimensional space, called the space of eligible portfolios. They arise naturally when multivariate random payoffs need to be evaluated and market frictions are present, e.g. in markets with transaction costs. Since set-valued functions are scary objects compared with real-valued ones, one may ask if there are meaningful and non-trivial examples of set-valued risk measures, and, if so, how one can compute and optimize them. The aim of this paper is to introduce and investigate new set-valued versions of the so-called average value at risk and present algorithms which admit its computation for discrete market models.

An important feature of our results concerns the relationship between the "pure" form of the risk measure, called regulator risk measure, and the market model. This relationship is condensed into the
concept of "market-compatibility". For scalar risk measures, this problem does not occur since it is usually assumed (but not always explicitly stated) that the value of a portfolio equals the sum of the values of the positions for each asset where each asset position is evaluated in terms of a "numéraire" (one and the same for all assets since otherwise the operation of taking the sum is not well-defined). In the presence of market frictions, this may lead to a waste of resources and even contradictory results. One may compare [2] and [12] for examples.

This motivates the "set-valuedness" for risk measures when market with frictions are present. In [16], the authors initiated a line of research where the relationships between set-valued risk measures and liquidation mappings are taken into account. In particular, they defined set-valued coherent (sublinear) risk measures for market models with constant proportional transaction costs. Building on this idea, in [11] duality results for set-valued convex risk measures are given. In [12], the duality theory was extended to discrete time market models with proportional transaction costs.

A completely different line of research tries to define real-valued risk measures for multi-variate risks, see for example [4, 25, 5, 6]. In these papers, market models are not discussed, and such risk measures produce a complete order for multi-variate risks which does not always seem appropriate.

In this paper, based on the definition of set-valued risk measures given in [12], we extend (and slightly modify) the concept of market compatibility from the one-period to the multi-period setting. In turns out that the market compatible extension of a set-valued risk measure can be understood as the optimal value of a set-valued minimization problem and, moreover, as a set-valued good deal bound.

The main part of the paper is concerned with set-valued versions of the average value at risk. We introduce its basic version, called regulator AV@R, via a primal representation based on the set-valued extension of the certainty equivalent formulation of the scalar AV@R given in [24]. We define its market extension, and links to set-valued risk minimization and portfolio optimization problems are given. Finally, we present a procedure which admits the computation of values of the set-valued average value at risk and the optimal value of a corresponding risk minimization problem. This procedure relies on an extension of an observation made by Rockafellar and Uryasev in [24]: Within a finite probability space setting, the scalar AV@R can be represented as the optimal value of a linear optimization problem. Our results include the representation of the set-valued AV@R as optimal values of linear vector optimization problems understood in a set-valued sense. Moreover, we will give primal and dual versions and present some examples.

One of the main findings of the computational part of the paper is that a naive scalarization approach (liquidate the terminal payoff and apply a scalar risk measure to the liquidated position) does not necessarily produce points on the efficient frontier of the value of the set-valued risk measure. Compare Remark 4.7 and Example 4.6. We see this in accordance with the discussion in [2].

2 Basic properties of set-valued risk measures

2.1 Eligible portfolios and image spaces

The basic idea behind the concept of risk measures is that one looks for deposits to be given at initial time which make a multivariate random variable $X$ acceptable where $X$ is understood as the payoff vector of some investment, gamble etc. in physical units. Deposits can be made in cash (one or several currencies), bonds or other assets or even combinations of such. The set of all initial portfolios which are potential deposits form a linear space $M$ which is a linear subspace of the space $\mathbb{R}^d$ of all potential initial portfolios consisting of $d$ assets.

A typical example is $M = \mathbb{R}^m \times \{0\}^{d-m}$ for $1 \leq m \leq d$. This means that the first $m$ assets are eligible, i.e. can be used for deposits. Two important particular cases are $m = d$ (all portfolios are eligible, $M = \mathbb{R}^d$) and $m = 1$ (only one, the first asset is eligible – this is the usual scalar case).

Let us denote by $M_+ = M \cap \mathbb{R}_+^d$ the set of eligible portfolios with non-negative holdings in each asset. Throughout this paper, we assume that $M_+$ is non-trivial, i.e. $M_+ \neq \{0\}$. This convex pointed
cone $M_+ \subseteq M$ generates a reflexive, antisymmetric and transitive binary relation (a partial order) $\leq_{M_+}$ on $M$ by means of $x \leq_{M_+} y$ if and only if $y - x \in M_+$ for $x, y \in M$.

Let us denote by $\mathcal{P}(M)$ the set of all subsets of $M$ including $\emptyset$. The order relation $\leq_{M_+}$ on $M$ can be extended to $\mathcal{P}(M)$ by setting

$$A \preceq_{M_+} B \iff B \subseteq A + M_+,$$

for $A, B \in \mathcal{P}(M)$. This relation is reflexive and transitive, but not necessarily antisymmetric. Its asymmetric part coincides with $\preceq$ on the set

$$\mathcal{P}(M_+) = \{D \subseteq M : D = D + M_+\}.$$ 

This means: $A \preceq_{M_+} B$ if, and only if, $A \supseteq B$ whenever $A, B \in \mathcal{P}(M_+)$. Moreover, $(\mathcal{P}(M_+), \preceq)$ is an order complete lattice (every subset has an infimum and a supremum) with

$$\inf_{\mathcal{P}(M_+)} A = \bigcup_{A \in A} A, \quad \sup_{\mathcal{P}(M_+)} A = \bigcap_{A \in A} A,$$

for $A \subseteq \mathcal{P}(M_+)$. This means $A \preceq_{M_+} B$ if, and only if, $A \supseteq B$ whenever $A, B \in \mathcal{P}(M_+)$. Moreover, $(\mathcal{P}(M_+), \preceq)$ is an order complete lattice (every subset has an infimum and a supremum) with

$$\inf_{\mathcal{P}(M_+)} A = \bigcup_{A \in A} A, \quad \sup_{\mathcal{P}(M_+)} A = \bigcap_{A \in A} A,$$

for $A \subseteq \mathcal{P}(M_+)$ while the formula for the supremum is the same as in $\mathcal{P}(M_+)$. 

Since we also will consider functions with closed values, we will work with even more specialized subspaces of $\mathcal{P}(M_+)$, namely

$$\mathcal{F}(M_+) = \{D \subseteq M : D = \text{cl} (D + M_+)\} \subseteq \mathcal{P}(M_+)$$

and

$$\mathcal{G}(M_+) = \{D \subseteq M : D = \text{clco} (D + M_+)\} \subseteq \mathcal{P}(M_+).$$ 

Again, $(\mathcal{F}(M_+), \supseteq)$ and $(\mathcal{G}(M_+), \supseteq)$ are order complete lattices. While the formulas for the supremum are the same as in $\mathcal{P}(M_+)$,

$$\inf_{\mathcal{F}(M_+)} A = \text{cl} \bigcup_{A \in A} A, \quad \inf_{\mathcal{G}(M_+)} A = \text{clco} \bigcup_{A \in A} A,$$

for $A \subseteq \mathcal{F}(M_+)$ and $A \subseteq \mathcal{G}(M_+)$, respectively.

For these facts and more about the canonical extensions of preorders in vector spaces to their power sets, compare [9].

### 2.2 Regulator risk measures

We are given a probability space $(\Omega, \mathcal{F}_T, P)$. A multivariate random variable is a $P$-measurable function $X: \Omega \to \mathbb{R}^d$ for some positive integer $d \geq 2$. If $d = 1$, the random variable is called univariate. Let us denote by $L^0_0 = L^0_0(\Omega, \mathcal{F}_T, P)$ the linear space of the equivalence class of all $\mathbb{R}^d$-valued random variables. As usual, we write

$$(L^0_d)_+ = \left\{X \in L^0_d : X \in \mathbb{R}^d_+ \text{ P-a.s.}\right\}$$

for the closed convex cone of $\mathbb{R}^d$-valued random vectors with $P$-almost surely non-negative components. An element $X \in L^0_d$ has components $X_1, \ldots, X_d$ in $L^0_0 = L^0_1$. The symbol $\mathbb{1}$ denotes the random variable in $L^0_0$ which has $P$-almost surely the value 1.

As usual, the graph of a function $R: L^0_d \to \mathcal{P}(M)$ is the set

$$\text{graph } R = \{(X, u) \in L^0_d \times M : u \in R(X)\} \subseteq L^0_d \times M.$$ 

The basic requirements to risk measures are given in the following definition.
\textbf{Definition 2.1 (\cite{12})} A function \( R: L^0_d \to \mathcal{P}(M_+) \) is called a (regulator) risk measure if it is
\begin{itemize}
\item [(R0)] finite at 0 \( \in L^0_d \); i.e., \( R(0) \neq \emptyset \), \( R(0) \neq M \);
\item [(R1)] \( M \)-translatively:
\begin{equation}
\forall X \in L^0_d, \forall u \in M : R(X + u1_d) = R(X) - u; \tag{2.1}
\end{equation}
\item [(R2)] \((L^0_d)_+\)-monotone:
\[ X^2 - X^1 \in (L^0_d)_+ \implies R(X^2) \supseteq R(X^1). \]
\end{itemize}
A risk measure \( R \) is called convex if for each \( t \in (0, 1) \) and \( X^1, X^2 \in L^0_d \)
\[ R(tX^1 + (1 - t)X^2) \supseteq tR(X^1) + (1 - t)R(X^2), \]
and it is called positively homogeneous if for each \( t > 0 \) and each \( X \in L^0_d \)
\[ R(tX) = tR(X). \]
A convex and homogeneous risk measure is called sublinear. A risk measure \( R \) is called closed if graph \( R \) is closed.

A function \( R: L^0_d \to \mathcal{P}(M_+) \) is convex if and only if graph \( R \) is convex. Note that a convex \( \mathcal{P}(M_+) \)-valued function \( R \) always has convex values. Finally, if \( R: L^0_d \to \mathcal{P}(M_+) \) is closed, then it maps into \( \mathcal{F}(M_+) \), and if it is closed convex, then it maps into \( \mathcal{G}(M_+) \). Note that the image space \( \mathcal{P}(M_+) \) is in accordance with the interpretation of the set \( R(X) \) as the set of all eligible portfolios which compensate for the risk of \( X \): If \( u \in R(X) \) compensates for the risk of \( X \), then \( u + k \) with \( k \in M_+ \) should also compensate for the risk of \( X \).

\textbf{Definition 2.2 (\cite{12})} A set \( A \subseteq L^0_d \) is called an acceptance set if it satisfies \( M1_d \cap A \neq \emptyset \), \( M1_d \cap (L^0_d \setminus A) \neq \emptyset \) and \( A + (L^0_d)_+ \subseteq A \).

There is a one-to-one relationship between acceptance sets and risk measures by means of the formulas
\[ A_R = \{ X \in L^0_d : 0 \in R(X) \}, \quad R_A(X) = \{ u \in M : X + u1_d \in A \}, \tag{2.2} \]
compare proposition 2.5 in \cite{12}. Furthermore, \( A = A_{R_A} \) and \( R = R_{A_R} \), see \cite{12}.

The following definition 2.15 from \cite{12} admits to characterize set-valued risk measures with closed values. We use the symbol \( u^k \xrightarrow{M} 0 \) in order to denote a sequence \( \{ u^k \}_{k \in \mathbb{N}} \subseteq M \) with \( \lim_{k \to \infty} u^k = 0 \).

\textbf{Definition 2.3 (\cite{12})} A set \( A \subseteq L^0_d \) is called directionally closed in \( M \) iff \( X \in L^0_d \), \( u^k \xrightarrow{M} 0 \) and \( X + u^k1_d \in A \) for all \( k \in \mathbb{N} \) imply \( X \in A \). For an arbitrary set \( A \subseteq L^0_d \), the set
\[ \text{cl}_MA = \left\{ X \in L^0_d : \exists u^k \xrightarrow{M} 0 : \forall k \in \mathbb{N} : X + u^k1_d \in A \right\} \]
is called the directional closure of \( A \) in \( M \).

Of course, a closed set \( A \) is directionally closed with respect to each linear subspace \( M \subseteq \mathbb{R}^d \). The formulae \cite{12} produce a one-to-one correspondence between directionally closed acceptance sets and risk measures with closed values (proposition 2.16 in \cite{12}). Finally, (topologically) closed acceptance sets are one-to-one with set-valued risk measures with a closed graph, a much stronger condition than closed valuedness (see section 6.1 of \cite{12}).
2.3 Market extensions of risk measures

Scalar risk measures are applied to univariate positions with the underlying assumption that each asset is evaluated in terms of a numéraire and the obtained numbers are added up in order to obtain the value of a portfolio. In markets with frictions, this can lead to undesirable situations: There might not exist an unambiguous value for the portfolio: For example, two dollar/euro positions are (100, 100) and (200, 0). The exchange rate is 1:1 and 1% transaction costs have to be paid. An evaluation of the positions in dollars yields that the second is more worth than the first, while an evaluation in euros produces the opposite result. Thus, the order between values of portfolios is not invariant under a change of numéraire. This makes the idea of working with several numéraires at the same time attractive which is precisely what set-valued risk measure do. In this way, they incorporate trading constraints in a more explicit way.

In this paper, we consider a discrete market: Let $T$ be a time points $\{0, 1, ..., T\}$ be given and a filtered probability space $(\Omega, (\mathcal{F}_t), (\mathbb{P}_t))$ satisfying the usual conditions. A discrete conical market model is a sequence of $(\mathcal{F}_t)$-measurable functions $K_t$, $t = 0, 1, ..., T$ with $\mathbb{R}^d_+ \subseteq K_t \neq \mathbb{R}^d$ such that $K_t(\omega)$ is a closed convex cone for each $\omega \in \Omega$ and all $t \in \{0, ..., T\}$. These cones are called solvency cones and appear, for example, when proportional transaction costs are present. Compare [17],[20],[21],[22].

We denote $K_t^M = K_0 \cap M$ which is non-trivial since $M$ is non-trivial and $\mathbb{R}_+^d \subseteq K_0$. Let $L^0_t(\Omega, \mathcal{F}_t, \mathbb{P})$ be the linear space of the equivalence class of all $\mathbb{R}^d$-valued, $\mathcal{F}_t$-measurable random variables. Further, denote

\[ L^0_t(K_t) = \{X \in L^0_t(\Omega, \mathcal{F}_t, \mathbb{P}) : \{\omega \in \Omega : X(\omega) \in K_t(\omega)\} = 1\}. \]

In the following, we slightly modify and extend the definition of market-compatibility from the one-period setting in [12] to the multi-period setting. Interpretations of this definition, applications, and the relation to existing notions in the literature are given below in remarks 2.5 – 2.7.

**Definition 2.4** A risk measure $R: L^0_t \to \mathcal{P}(M_+)$ is called $K_t$-compatible if $A_R + L^0_t(K_t) \subseteq A_R$ for $t \in \{0, ..., T\}$. A risk measure $R$ is called market-compatible iff it is $K_t$-compatible for all $t \in \{0, ..., T\}$.

**Remark 2.5** Market-compatibility as defined in definition 2.4 can be seen as an extension of definition 2.7 in [12] in the following way. Interpreting the value of a risk measure at $X \in L^0_t$ as the set of initial eligible portfolios which can be hold from $t = 0$ until time $T$ to make the overall position acceptable, one might be interested in two questions.

First, taking trading opportunities at $t = 0$ into account, an agent might be interested in the set of initial eligible portfolios which can be exchanged into an eligible portfolio which makes the payoff $X$ acceptable. This leads to $K_0$-compatibility as defined in [12], which can be expressed as $A_R + K_0^M \mathbb{1} \subseteq A_R$.

Secondly, the agent might be interested to know what risks she could cover with a given available eligible portfolio $u \in M$. Instead of considering only the payoff $X$, she could take trading opportunities into account which start from the zero portfolio at time $t = 0$ and trade self-financingly in the market. These are given by the set

\[ C_T = - \sum_{s=0}^{T} L^0_s(K_s) \subseteq L^0_t, \]

the set of 'freely available' claims. Thus, instead of $X$, one might consider the modified payoffs $X + C_T$. This corresponds to the notion of $K_t$-compatibility or, equivalently, $A_R + L^0_t(K_t) \subseteq A_R$ for $t \in \{0, ..., T\}$. Since $K_t^M \subseteq K_0$, both concepts together again lead to definition 2.4 of market compatibility. This version of market-compatibility includes trades at time $t = 0$ and takes the set of 'freely available' claims into account when evaluating the terminal payoff. Note that even in the one-period case definition 2.4 slightly deviates from [12], since in [12] the modified payoffs under consideration are $X - L^0_t(K_T)$ instead of $X + C_T = X - L^0_t(K_T) - K_0$.

**Remark 2.6** A second interpretation of market-compatibility as given in definition 2.4 is as follows. An agent is interested in the set of initial portfolios $u \in M$ for which there exists a self-financing trading
strategy starting from \( u \) that makes the payoff \( X \) acceptable. This coincides exactly with the notion of so called good deal bounds (GDB)

\[
GDB(X) = \left\{ u \in M : X + u1 \leq \sum_{s=0}^{T} L^0_d(K_s) \in A \right\}
\]

which have been studied in the scalar case e.g. in [19] and are related to risk minimization problems, see e.g. [3]. Below, we will show that also in the set-valued case the market extension of a regulator risk measure \( R: L^0_d \to P(M_+) \) corresponds to a risk minimization problem, and by definition coincides with the notion of a good deal bound as given above. Good deal bounds are used to determine price bounds in incomplete markets by assuming that no good deals should be available in the market. The lower price bound is determined by \(-GDB(-X)\).

**Remark 2.7** Another notion of market compatibility reads as follows (see [7]): A risk measure \( R \) is called market compatible iff \( A_R + L^0_d(K_t) \cap M_t \subseteq A_R \) for all \( t \in \{0, \ldots, T\} \) where \( M_t \subseteq L^0_d(F_t) \) is the space of eligible assets at time \( t \). Thus, trading is only taken into account among the eligible assets.

In some situations, a dynamic risk compensating deposit might be desirable, for example, if deposit requirements in a few currencies (but not all of the \( d \) assets) are allowed, and one wants to update the risk evaluation (like in a margin account, so deposits need to be advanced or can be depleted depending on the new market situation).

If one wants to take the trading opportunities into account before making deposits, a dynamic risk measure \( R_t: L^0_d(F_T) \to P((M_t)_+) \), which is \( L^0_d(K_t) \cap M_t \)-compatible (that is \( A_{R_t} + L^0_d(K_t) \cap M_t \subseteq A_{R_t} \)) for all \( t \in \{0, \ldots, T\} \) should be considered. This is the definition used in [7].

Lemma 4.12 in [7] shows that for dynamic risk measures \( (R_t)_{t=0}^T \) satisfying a certain time consistency property – called multi-portfolio time consistency – \( L^0_d(K_t) \cap M_t \)-compatibility of \( R_t \) for all \( t \in \{0, \ldots, T\} \) coincides with \( A_{R_t} + \sum_{s=t}^{T} L^0_d(K_s) \cap M_s \subseteq A_{R_t} \) for all \( t \in \{0, \ldots, T\} \). For \( t = 0 \) this condition is exactly the definition of market compatibility for the (static) risk measure \( R = R_0 \) proposed above.

Note that if one chooses \( M_t = L^0_d(F_t) \) for all \( t \in \{0, \ldots, T\} \), one recovers definition 2.4.

Recall that the set \( C_T = -\sum_{s=0}^{T} L^0_d(K_s) \subseteq L^0_d \) is the set of all attainable claims with zero initial endowment in a market given by \( (K_t)_{t=0}^T \) (see e.g. section 5.4 in [12] or [20] for details).

**Definition 2.8** The market(-compatible) extension of a regulator risk measure \( R: L^0_d \to P(M_+) \) is given by

\[
\forall X \in L^0_d: \text{R}^{\text{mar}}(X) = R_{(A_R-C_T)}(X).
\]

The closed-valued market extension of \( R \) is given by

\[
\forall X \in L^0_d: \text{cl}_{\text{M}} \text{R}^{\text{mar}}(X) = R_{\text{cl}_{\text{M}}(A_R-C_T)}(X),
\]

and the closed market extension of \( R \) is given by

\[
\forall X \in L^0_d: \text{cl} \text{R}^{\text{mar}}(X) = R_{\text{cl}(A_R-C_T)}(X).
\]

This means \( R^{\text{mar}} \) is defined via its acceptance set \( A_{R^{\text{mar}}} = A_R - C_T \). We will show in the following that \( R^{\text{mar}} \) is a market-compatible risk measure (but might not be finite at zero) with image space \( P(K^M_+) \).

To do so, we will first show that the values of \( R^{\text{mar}} \) are equal to the optimal values of some set-valued risk minimization problems.

**Theorem 2.9** The market extension \( R^{\text{mar}} \) of a regulator risk measure \( R: L^0_d \to P(M_+) \) satisfies

\[
R^{\text{mar}}(X) = \inf_{\mathcal{P}(M_+)} \left\{ R(X + Y) : Y \in C_T \right\}.
\]

The directionally closed market extension \( \text{cl}_{\text{M}} \text{R}^{\text{mar}} \) of a regulator risk measure \( R: L^0_d \to \mathcal{F}(M_+) \) satisfies

\[
\text{cl}_{\text{M}} \text{R}^{\text{mar}}(X) = \inf_{\mathcal{F}(M_+)} \left\{ R(X + Y) : Y \in C_T \right\}.
\]
Proof (Proof of theorem 2.9) Let us only prove the second claim concerning $\text{cl}_M R^{\text{mar}}$ as the result without the directional closure is a simpler version of the same proof. We need to show that the sublevel set $A_S = \{ X \in L^0_d : 0 \in S(X) \}$ of the function

$$S(X) = \text{cl} \bigcup \{ R(X + Y) : Y \in C_T \}$$

satisfies

$$A_S = \text{cl}_M (A_R - C_T).$$

To prove $A_S \supseteq \text{cl}_M (A_R - C_T)$ take $X \in A_R$ and $\hat{Y} \in C_T$. Then

$$0 \in R(X) \subseteq \text{cl} \bigcup \{ R \left( X - \hat{Y} + Y \right) : Y \in C_T \} = S \left( X - \hat{Y} \right),$$

thus $A_S \supseteq A_R - C_T$. The directional closure can be taken since $S$ has by definition closed values which corresponds one-to-one to a directionally closed acceptance set $A_S$ (lemma 6.2 in [12]).

For the converse implication, consider $\hat{S} \left( X \right) = \bigcup \{ R \left( X + Y \right) : Y \in C_T \}$. It holds $A_{\hat{S}} \subseteq A_R - C_T$ since by definition of $\hat{S}$ whenever $X \in A_{\hat{S}}$ there exists a $\hat{Y} \in C_T$ with $0 \in R \left( X + \hat{Y} \right)$. Since $S(X) = \text{cl}_M \hat{S}(X)$ it holds $A_S = \text{cl}_M A_{\hat{S}} = \text{cl}_M (A_R - C_T)$.

To summarize, the market extension of $R$ at $X$ can be seen as the optimal value of the set-valued optimization problem "minimize the regulator risk measure over all random variables which are the sum of $X$ and an attainable claim with zero initial endowment." Here, "minimization" is understood as looking for the infimum in $\mathcal{P}(M_+)$ and $\mathcal{F}(M_+)$, respectively, with respect to the relation $\supseteq$.

The question arises if the market extension has a closed graph (not only closed values) given that the regulator risk measure is closed. This is the case if the market satisfies the robust no-arbitrage property (which implies that the set $C_T$ is closed, see [20]) and the sum of the two closed sets $A_R$ and $C_T$ is closed. This is in general not true, but under convexity assumptions, conditions could be given in the spirit of Dieudonne’s theorem. We leave this questions to further research since it is not of primary interest for the present paper.

**Proposition 2.10** The market extension $R^{\text{mar}}$ of a regulator risk measure $R: L^0_d \to \mathcal{P}(M_+)$ is $M$-translative, $(L^0_d)_+$-monotone, market-compatible and maps into

$$\mathcal{P}(K^M_0) = \{ D \subseteq M : D = D + K^M_0 \}.$$ 

Moreover, if $R$ is convex (positively homogeneous, sublinear), then so is $R^{\text{mar}}$.

Finally, all of the above are also true for $\text{cl}_M R^{\text{mar}}$ and $\text{cl} R^{\text{mar}}$, and these two functions map into

$$\mathcal{F}(K^M_0) = \{ D \subseteq M : D = \text{cl} \left( D + K^M_0 \right) \}.$$ 

**Proof** $R^{\text{mar}}$ is $M$-translative by definition (proposition 2.4 in [12]). The function $R^{\text{mar}}$ maps into $\mathcal{P}(K^M_0)$ if and only if $A_{R^{\text{mar}}} + K^M_0 \subseteq A_{R^{\text{mar}}}$ (proposition 2.8 in [12]). To prove this, take $X \in A_{R^{\text{mar}}}$ and $k \in K^M_0$. Then, by theorem 2.9

$$R^{\text{mar}} (X + k) = \bigcup \{ R \left( X + k \mathbb{I} + Y \right) : Y \in C_T \}$$

$$= \bigcup \{ R \left( X + Y \right) : Y \in k \mathbb{I} + C_T \}$$

$$\supseteq \bigcup \{ R \left( X + Y \right) : Y \in C_T \} = R^{\text{mar}} (X) \ni 0.$$ 

The inclusion follows from $C_T + k \mathbb{I} \supseteq C_T$. The extension $R^{\text{mar}}$ is $K_t$-compatible for all $t \in \{ 0, \ldots, T \}$ if, and only if, $A_{R^{\text{mar}}} - C_T \subseteq A_{R^{\text{mar}}}$. This can be proven analogously replacing $k \mathbb{I} \subseteq K^M_0 \mathbb{I}$ by $\hat{Y} \in -C_T$ in the above argument. In particular, $R^{\text{mar}}$ is $K_t$-compatible for each $t \in \{ 0, \ldots, T \}$. Since $K_t$-compatibility is equivalent to $L^0_d \left( K_t \right)$-monotonicity (see proposition 6.5 of [12], for example), $R^{\text{mar}}$ is $L^0_d \left( K_T \right)$-monotone and hence $(L^0_d)_+$-monotone.

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Moreover, \( R_{\text{mar}} \) is convex (positively homogeneous, sublinear) if \( R \) is convex (positively homogeneous, sublinear) since if \( A_R \) is convex (a cone, a convex cone), then \( A_{R_{\text{mar}}} = A_R - C_T \) is convex (a cone, a convex cone).

All of the properties remain true if the closed-valued or closed version of \( R_{\text{mar}} \) is taken since this corresponds to the directional closure and the (topological) closure of the acceptance set of \( R_{\text{mar}} \), respectively.

**Remark 2.11** An investor starts with initial portfolio vector \( x \in \mathbb{R}^d \) and invests self-financially in the market until time \( T \). The set of possible terminal portfolios is given by \( x \mathbb{1} + C_T \). She wants to pick a terminal portfolio which minimizes the risk with respect to a given risk measure \( R: L^0_d \to \mathcal{P}(M_+) \). The corresponding portfolio optimization problem reads as follows:

\[
\inf_{\mathcal{P}(M_+)} \{ R(x \mathbb{1} + Y) : Y \in C_T \} = \bigcup_{X \in x \mathbb{1} + C_T} R(X) = R_{\text{mar}}(x \mathbb{1}).
\]

(2.3)

Therefore, the portfolio optimization problem (2.3) is a special case of the risk minimization problem considered in theorem 2.9. The right hand side of this formula can be understood as an optimal value function depending on the initial endowment; such a function features prominently in (scalar) utility maximization theory. It shares many properties with set-valued risk measures, but is a function on \( \mathbb{R}^d \) rather than \( L^0_d \). This also means that one can solve the portfolio optimization problem as soon as one can compute the market extension of the regulator risk measure.

### 3 Set-valued average value at risk

In this section, we turn to set-valued versions of a sublinear (coherent) risk measure which has been baptized – for good reasons – ”Average Value at Risk” by H. Föllmer and A. Schied [8]. The scalar AV\@R for univariate random variables has several reformulations, see [8][11][23]. Rockafellar and Uryasev [24] discovered that the computation and optimization of the scalar AV\@R for univariate random variables over a finite probability spaces reduces to a linear programming problem. We will extend this method to a set-valued setting which allows to deal with proportional bid-ask spreads. The decisive tool is the set-valued duality theory developed in [9][12] whereas the particulars of the discretization and the finite-dimensional duality are due to [28].

Set-valued extensions of AV\@R for multivariate positions can already be found in [11][10] for the special case of a constant solvency cone.

Let \( L^0_{d} = L^0_{d}(\Omega, \mathcal{F}_T, P) \) denote the linear space of all \( X \in L^0_d \) with \( \int_{\Omega} |X(\omega)| \, dP < +\infty \), where \(|\cdot|\) stands for an arbitrary, but fixed norm on \( \mathbb{R}^d \), and the usual identification of functions differing only on sets of \( P \)-measure zero is assumed. And let \( L^\infty_{d} = L^\infty_{d}(\Omega, \mathcal{F}_T, P) \) the linear space of equivalence classes of random vectors \( X \in L^0_d \) with \( \text{ess.sup}_{\omega \in \Omega} |X(\omega)| < \infty \). We write \( (L^p_d)_{+} = \{ X \in L^p_d : X \in \mathbb{R}^d_{+}, P \text{-a.s.} \} \), \( p \in \{0, 1, \infty \} \) for the closed convex cone of \( \mathbb{R}^d \)-valued \( \mathcal{F}_T \)-measurable random vectors in \( L^p_{d} \) with non-negative components.

The following definition gives a primal representation for the set-valued average value at risk that extends the certainty equivalent representation of the scalar AV\@R as in [24].

**Definition 3.1** Let \( \alpha \in (0, 1]^d \) and \( X \in L^1_{d} \). The average value at risk of \( X \) is defined as

\[
AV@R^\alpha_{\text{eq}}(X) = \left\{ \text{diag} (\alpha)^{-1} E[Z] - z : Z \in (L^1_{d})_{+}, X + Z - z \mathbb{1} \in (L^1_{d})_{+}, z \in \mathbb{R}^d \right\} \cap M.
\]

(3.1)

**Remark 3.2** If \( m = d = 1 \), the two conditions \( Z \in (L^1_{d})_{+} \) and \( X + Z - z \mathbb{1} \in (L^1_{d})_{+} \) are equivalent to \( Z \geq (-X + z \mathbb{1})^+ \) with \( X^+ = \max \{0, X\} \text{ (P-a.s.)} \). We obtain \( AV@R^\alpha_{\text{eq}}(X) = AV@R^\alpha_{\text{eq}}(X) + \mathbb{R}_+ \) with

\[
AV@R^\alpha_{\text{eq}}(X) = \inf_{z \in \mathbb{R}} \left\{ \frac{1}{\alpha} E\left[\left(-X + z \mathbb{1}\right)^+\right] - z \right\}
\]

(3.2)
which is the optimized certainty equivalent representation of the AV@R found by Rockafellar and Uryasev [24]. Compare also Föllmer and Schied [8], formula 4.42.

**Remark 3.3** The following observations will lead to an interpretation of formula (6.31). Let $M = \mathbb{R}^m \times \{0\}^{d-m}$ (hence $M_+ = \mathbb{R}^m_+ \times \{0\}^{d-m}$) with $1 \leq m \leq d$, i.e., the first $m$ assets are eligible. In this case, the objective and the constraints can be considered component-wise, and we obtain for $i = 1, \ldots, m$

\[
\left\{ \frac{1}{\alpha_i} \mathbb{E}[Z_i] - z_i : Z_i \in L^1_+, X_i + Z_i - z_i \mathbb{1} \in L^1_+, z_i \in \mathbb{R} \right\}
\]

\[
= \left\{ \frac{1}{\alpha_i} \mathbb{E}[Z_i] - z_i : Z_i \geq \max \{0, z_i - X_i\} = (z_i - X_i)^+, z_i \in \mathbb{R} \right\}
\]

\[
= \inf \left\{ \frac{1}{\alpha_i} \mathbb{E} \left[ (z_i - X_i)^+ \right] - z_i : z_i \in \mathbb{R} \right\} + \mathbb{R}_+ = AV@R_{\alpha_i}^\text{comp} (X_i) + \mathbb{R}_+.
\]

For $i = m+1, \ldots, d$, there must exist $Z_i \in L^1_+$, $z_i \in \mathbb{R}^d$ such that $0 = \frac{1}{\alpha_i} \mathbb{E}[Z_i] - z_i$ and $X_i + Z_i - z_i \mathbb{1} \in L^1_+$. This means $AV@R_{\alpha_i} (X_i) \leq 0$. Altogether, the set-valued regulator AV@R produces the component-wise scalar AV@R for the first $m$ components plus the cone $\mathbb{R}^m_+$ under the constraint that the scalar AV@R for the last $d-m$ components is at most zero. If the latter is not the case, $AV@R_{\alpha_i}^\text{comp} (X) = \emptyset$. So, the set-valued regulator AV@R is either "point plus cone" or the empty set. Of course, this may change if $M$ has a different structure.

**Proposition 3.4** The function $X \mapsto AV@R_{\alpha_i}^\text{comp} (X)$ is a sublinear risk measure on $L^1_+$ with image space $\mathcal{C}(M_+)$. In particular, $AV@R_{\alpha_i}^\text{comp} (0) \subseteq M$ is a convex cone.

**Proof** $AV@R_{\alpha_i}^\text{comp}$ maps into $\mathcal{P}(M_+)$ since for $k \in M_+$ we obtain

\[
AV@R_{\alpha_i}^\text{comp} (X) + k
\]

\[
= \left\{ \text{diag} (\alpha)^{-1} (E[Z] - z + k) : Z \in (L^1_+)_+, X + Z - z \mathbb{1} \in (L^1_+)_+, z \in \mathbb{R}^d \right\} \cap M
\]

\[
= \left\{ \text{diag} (\alpha)^{-1} (E[Z] - z) : Z \in (L^1_+)_+, X + Z - z \mathbb{1} \in (L^1_+)_+, z \in \mathbb{R}^d \right\} \cap M
\]

\[
\subseteq AV@R_{\alpha_i}^\text{comp} (X).
\]

With a similar arguments, one easily checks that $AV@R_{\alpha_i}^\text{comp}$ is $M$-translative.

$AV@R_{\alpha_i}^\text{comp}$ is finite at zero. Indeed, while obviously $0 \in AV@R_{\alpha_i}^\text{comp} (0) \neq \emptyset$, take $k \in M_+ \setminus \{0\}$ (non-empty by assumption) and $t > 0$. Assume $-tk \in AV@R_{\alpha_i}^\text{comp} (0)$. Then, there is $Z \in (L^1_+)_+$, $z \in \mathbb{R}^d$ such that $Z - z \mathbb{1} \in (L^1_+)_+$ and $-tk = \text{diag} (\alpha)^{-1} E[Z] - z$. These conditions imply $Z \in (L^1_+)_+$ and

\[
Z - tk = -\text{diag} (\alpha)^{-1} E[Z] - z \in (L^1_+)_+.
\]

At least one component of $k$ is positive, say $k_i > 0$. Multiplying the above inclusion with $Y_j \in (L^\infty_+)_{+}$ defined by $Y_j = e^j \mathbb{1}$ and $Y_j \equiv 0$ for $j \neq i$ and taking the expected value we obtain

\[
\mathbb{E}[Z_i] \left( 1 - \frac{1}{\alpha_i} \right) - tk_i \geq 0.
\]

Since $\alpha_i \leq 1$ and $\mathbb{E}[Z_i] \geq 0$, this clearly contradicts $tk_i > 0$. Therefore, $-tk \notin AV@R_{\alpha_i}^\text{comp} (0)$ for $t > 0$ and $AV@R_{\alpha_i}^\text{comp} (0) \neq M$. 
AV@\(R^{\alpha}_{\alpha}^g\) is \((L^1_d)_+\)-monotone since for \(U \in (L^1_d)_+\) we obtain
\[
AV@\(R^{\alpha}_{\alpha}^g\) (X - U) = \left\{ \text{diag} (\alpha)^{-1} E [\hat{Z}] - z : Z \in (L^\infty_d)_+, X - U + Z - z \mathbb{1} \in (L^1_d)_+, z \in \mathbb{R}^d \right\} \cap M
\]
\[
= \left\{ \text{diag} (\alpha)^{-1} E [\hat{Z}] - z : Z \in (L^\infty_d)_+, X + Z - z \mathbb{1} \in U + (L^1_d)_+ \subseteq (L^1_d)_+, z \in \mathbb{R}^d \right\} \cap M
\]
\[
\subseteq AV@\(R^{\alpha}_{\alpha}^g\) (X).
\]
Finally, \(AV@\(R^{\alpha}_{\alpha}^g\)\) is sublinear. While positive homogeneity is easy to check, we outline the proof for subadditivity. Taking \(X_1, X_2 \in L^1_d\) we obtain
\[
AV@\(R^{\alpha}_{\alpha}^g\) (X_1) + AV@\(R^{\alpha}_{\alpha}^g\) (X_2)
\]
\[
= \left\{ \text{diag} (\alpha)^{-1} E [\hat{Z}^1] - z^1 + \text{diag} (\alpha)^{-1} E [\hat{Z}^2] - z^2 : Z^1 \in (L^\infty_d)_+, z^1 \in \mathbb{R}^d,
X_1 + Z^1 - z^1 \mathbb{1} \in (L^1_d)_+, Z^2 \in (L^\infty_d)_+, X_2 + Z^2 - z^2 \mathbb{1} \in (L^1_d)_+, z^2 \in \mathbb{R}^d \right\} \cap M
\]
\[
\subseteq \left\{ \text{diag} (\alpha)^{-1} E [\hat{Z}^1 + Z^2] - z^1 - z^2 : Z^1 + Z^2 \in (L^\infty_d)_+, X_1 + X_2 + Z^1 - z^1 \mathbb{1} - z^2 \mathbb{1} \in (L^1_d)_+, z^1 + z^2 \in \mathbb{R}^d \right\} \cap M
\]
\[
= \left\{ \text{diag} (\alpha)^{-1} E [\hat{Z}'] - z' : Z' \in (L^\infty_d)_+, X_1 + X_2 + Z' - z' \mathbb{1} \in (L^1_d)_+, z' \in \mathbb{R}^d \right\} \cap M
\]

Since \(AV@\(R^{\alpha}_{\alpha}^g\)\) is sublinear it is convex and has convex values, so it maps into \(C(M+)\). One easily checks that \(AV@\(R^{\alpha}_{\alpha}^g\)(0)\) is a convex cone.

Note that in the above proof we even showed \(AV@\(R^{\alpha}_{\alpha}^g\)(0) \cap -M+ \setminus \{0\} = \emptyset\) which is stronger than being finite at zero. Compare the notion of normalization in \([11]\), definition 2.1 and the discussion about normalization in \([7]\).

In the following, \(L^1_d(K_t) = L^1_d \cap L^0_d(K_t)\) for \(t \in \{0, 1, \ldots, T\}\).

**Proposition 3.5** The market extension of \(AV@\(R^{\alpha}_{\alpha}^g\)\) is given by
\[
AV@\(R^{\alpha}_{\alpha}^n_{mar}\) (X) = \left\{ \text{diag} (\alpha)^{-1} \mathbb{1} E [\hat{Z}] - z : Z \in (L^1_d)_+, X + Z - z \mathbb{1} \in \sum_{s=0}^T L^1_d(K_s), z \in \mathbb{R}^d \right\} \cap M.
\]

and \(AV@\(R^{\alpha}_{\alpha}^n_{mar}\) maps \(L^1_d\) into \(C(K^M_0)\).

**Proof** By theorem [2.9] the market extension of \(AV@\(R^{\alpha}_{\alpha}^g\)\) is given by
\[
AV@\(R^{\alpha}_{\alpha}^n_{mar}\) (X) = \bigcup \left\{ AV@\(R^{\alpha}_{\alpha}^g\) (X - Y) : Y \in \sum_{s=0}^T L^1_d(K_s) \right\}
\]
\[
= \left\{ \text{diag} (\alpha)^{-1} \mathbb{1} E [\hat{Z}] - z : Z \in (L^\infty_d)_+, X + Z - z \mathbb{1} \in \sum_{s=0}^T L^1_d(K_s), z \in \mathbb{R}^d \right\} \cap M
\]
which already is the desired formula.

**Remark 3.6** Note that in the setting of remark [3.3] \((M = \mathbb{R}^m \times \{0\}^{d-m})\), the market extension \(AV@\(R^{\alpha}_{\alpha}^n_{mar}\)\) is less restrictive than \(AV@\(R^{\alpha}_{\alpha}^g\)\) since instead of imposing the strict condition \(AV@\(R^{\alpha}_{\alpha} (X)_i \leq 0\) for all components of \(X\) in the non-eligible assets \(i = m+1, \ldots, d\), one first trades \(X\) and then evaluates the risk in terms of \(AV@\(R^{\alpha}_{\alpha}^g\) of the resulting payoff.
4 Set-valued AV@R over finite probability spaces

In the rest of the paper, we impose the following assumptions.

**Notations and standing assumptions.**

(H1) $|\Omega| = N, \mathcal{F}_T = 2^\Omega$, where the probability measure $P$ is given by $N$ numbers $p_1, p_2, \ldots, p_N > 0$ with $\sum_{n=1}^N p_n = 1$ and $P(\{\omega_n\}) = p_n$, $n = 1, \ldots, N$.

(H2) The vectors $b^1, \ldots, b^m \in \mathbb{R}^d$ form a basis of the space $M$ of eligible portfolios, the vectors $b^{m+1}, \ldots, b^d \in \mathbb{R}^d$ form a basis of $M^\perp$. Of course, $1 \leq m \leq d$, and $b^1, \ldots, b^d$ form a basis of $\mathbb{R}^d$.

(H3) The cone $K_0$ is spanned by $h^1, \ldots, h^t \in \mathbb{R}^d$, thus it is a finitely generated (hence closed) convex cone.

(H4) The polyhedral closed convex cone $K_0^M = K_0 \cap M$ is generated by $g^1, \ldots, g^L \in \mathbb{R}^d$. Note that this collection can be entirely different from $h^1, \ldots, h^t$.

(H5) For each $\omega \in \Omega$, the cone $K_T(\omega)$ is spanned by $k^1(\omega), \ldots, k^L(\omega)(\omega)$, thus it is a finitely generated (hence closed) convex cone.

Note that assumptions (H3) and (H5) are always satisfied in markets with proportional transaction costs, where the solvency cones are generated by the bid and ask exchange rates between any two of the $d$ assets as for example considered in [17][26][18].

4.1 The regulator case

4.1.1 The discrete version of $AV\alpha R_c^{reg}$

In the following, we shall reformulate $AV\alpha R_c^{reg}$ given in (3.1) of definition 3.1 in a linear programming language. We use the representation of the random variables $X, Z: \Omega \rightarrow \mathbb{R}^d$ by $x_m = X_i(\omega_n)$ and $z_{in} = Z_i(\omega_n)$ $i = 1, \ldots, d$, $n = 1, \ldots, N$ and set

$$\hat{x} = (x_{11}, \ldots, x_{1d}, x_{21}, \ldots, x_{2dN})^T \in \mathbb{R}^{dN}$$
$$\hat{z} = (z_{11}, \ldots, z_{1d}, z_{21}, \ldots, z_{2dN})^T \in \mathbb{R}^{dN}.$$  

First, the condition $Z \in (L_d^\infty)_+$ is equivalent to $\hat{z} \in \mathbb{R}^{dN}_+$. Next, using

$$\hat{I}_d = \begin{pmatrix} I_d \\ \vdots \\ I_d \end{pmatrix} \in \mathbb{R}^{dN \times d},$$

where $I_d$ is the quadratic $d \times d$-unit matrix and $\hat{I} \in \mathbb{R}^{dN}$, we can write the condition $Z + X - z I \in (L_d^1)^+$ as

$$\hat{z} + \hat{x} - \hat{I}_d z = \hat{I}.$$  

The objective function $(Z, z) \mapsto \text{diag}(\alpha)^{-1} E[Z] - z$ can be given a matrix form as follows. If we define

$$P_{(n)} = \begin{pmatrix} p_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_n \end{pmatrix} \in \mathbb{R}^{d \times d}, \text{ } n = 1, \ldots, N,$$

then $\hat{P} = (P_{(1)}, P_{(2)}, \ldots, P_{(N)})$ is a $d \times dN$-matrix and

$$\text{diag}(\alpha)^{-1} E[Z] - z = \text{diag}(\alpha)^{-1} \hat{P} \hat{z} - z.$$
Finally, the constraint \( \text{diag}(\alpha)^{-1}E[Z] - z \in M \) can be written as follows. Denote by
\[
B_{(d-m)}(\alpha^{-1}, \alpha^{m+1}) = \begin{pmatrix}
\alpha_1^{m+1} & \cdots & \alpha_d^{m+1} \\
\vdots & \ddots & \vdots \\
\alpha_1^d & \cdots & \alpha_d^d
\end{pmatrix} \in \mathbb{R}^{(d-m) \times d}
\]
the matrix containing the generating vectors of \( M^\perp \) as rows. Then, the condition \( \text{diag}(\alpha)^{-1}E[Z] - z \in M \) is equivalent to \( B_{(d-m)}(\alpha^{-1}, \alpha^{m+1}) P \hat{\varepsilon} - z = 0 \).

Altogether, we end up with
\[
AV@R_{\alpha}^{reg}(X) = \left\{ \text{diag}(\alpha)^{-1} \hat{P} \hat{\varepsilon} - z : B_{(d-m)}(\alpha^{-1}, \alpha^{m+1}) P \hat{\varepsilon} - z = 0, \right. \\
\text{\vspace{.1cm}}
\left. \hat{\varepsilon} + \hat{\tau} - \hat{I} \hat{d} = \hat{\tau}, \hat{\varepsilon}, \hat{\tau} \in \mathbb{R}^{dN}, \hat{\tau} \in \mathbb{R}_+^{dN}, z \in \mathbb{R}^d \right\}
\]  

This shows that \( AV@R_{\alpha}^{reg}(X) \) is the image of a polyhedral set under a linear function mapping \( \mathbb{R}^{2dN+d} \) into \( M \). In particular, under the standing assumptions \( AV@R_{\alpha}^{reg} \) has a closed graph and closed images. Since the infimum in \( F(M_+) \) is the closure of the union, the representation above can be understood as a set-valued optimization problem. This point of view will turn out to be particularly useful when it comes to duality results. See below and compare [10][20].

4.1.2 The dual of the discrete version of \( AV@R_{\alpha}^{reg} \)

We shall construct the set-valued dual of \( AV@R_{\alpha}^{reg} \).

Let us denote by \( (M_+)^+ \) the positive dual cone of \( M_+ \) in \( M \), that is \( (M_+)^+ = \{ u \in M : \forall v \in M_+ : v^T u \geq 0 \} \).

**Proposition 4.1** Under assumptions (H1) and (H2), the set-valued dual of \( AV@R_{\alpha}^{reg} \) can equivalently be written

1. in matrix form as a linear vector optimization problem given by
\[
AV@R_{\alpha}^{reg}(X) = \left\{ \text{diag}(\alpha)^{-1} \hat{P} \hat{\varepsilon} - z : B_{(d-m)}(\alpha^{-1}, \alpha^{m+1}) P \hat{\varepsilon} - z = 0, \right. \\
\text{\vspace{.1cm}}
\left. \eta_1 \geq 0, \hat{I}_\beta \eta_1 + (B_{(d-m)}(\alpha^{-1}, \alpha^{m+1}) P \hat{\varepsilon} - z = 0, \eta_2 = v, v \in (M_+)^+ \setminus \{0\} \right\},
\]
where \( S_{(\eta_1, v)}(\hat{\varepsilon}) = \{ z \in M : \eta^T \hat{\varepsilon} \leq v^T z \} \) for \( \eta, \hat{\varepsilon} \in \mathbb{R}^{dN}, v \in (M_+)^+ \).

2. in linear form using CPS-like dual variables
\[
AV@R_{\alpha}^{reg}(X) = \left\{ \text{diag}(\alpha)^{-1} \hat{P} \hat{\varepsilon} - z : v \in (E[Y] + M^\perp) \cap (M_+)^+ \setminus \{0\}, \right. \\
\text{\vspace{.1cm}}
\left. Y \in (L_d^\infty)_+, \text{diag}(\alpha)^{-1} E[Y] - Y \in (L_d^\infty)_+ \right\},
\]
where \( F_{(\hat{\varepsilon}, v)}[-X] = \{ u \in M : E[-X^TY] \leq v^T u \} \) and CPS stands for consistent pricing system;

3. with respect to EMM-like vector probability measures
\[
AV@R_{\alpha}^{reg}(X) = \left\{ (Q, w) \in W^\alpha : (E^Q[-X] + G(w)) \cap M, \right. \\
\text{\vspace{.1cm}}
\left. G(w) = \{ x \in \mathbb{R}^d : 0 \leq w^T x \} \right\}
\]

where \( G(w) = \{ x \in \mathbb{R}^d : 0 \leq w^T x \} \) and
\[
W^\alpha = \{ (Q, w) \in W : \text{diag}(w) \left( \text{diag}(\alpha)^{-1} e \mathbb{I} - \frac{dQ}{dP} \right) \in (L_d^\infty)_+ \},
\]

\[
W = \{ (Q, w) \in M^P_{1,d} \times \mathbb{R}^d : \text{diag}(w) \frac{dQ}{dP} \in (L_d^\infty)_+ \}.
\]

Here, \( e = (1, \ldots, 1)^T \in \mathbb{R}^d \), and \( M^P_{1,d} = M^P_{1,d}(\Omega, \mathcal{F}_T) \) denotes the set of all vector probability measures with components being absolutely continuous with respect to \( P \), i.e. \( Q : \mathcal{F}_T \to [0, 1] \) is a probability measure on \( (\Omega, \mathcal{F}_T) \) such that \( \frac{dQ}{dP} \in L^1 \) for \( i = 1, \ldots, d \). EMM stands for equivalent martingale measure.
Proof (1) Defining the matrices
\[ C_1 = \left( \text{diag}(\alpha)^{-1} \hat{P} \ 0 \right), \quad C_2 = -I_d, \quad A_1 = \left( \begin{array}{cc} I_{dN} \\ B_{(d-m)} \text{diag}(\alpha)^{-1} \hat{P} \\ -I_{dN} \end{array} \right), \]
\[ A_2 = \left( \begin{array}{c} -\hat{I}_d \\ -B_{(d-m)} \end{array} \right), \quad b = \left( \begin{array}{c} -\hat{x} \\ 0 \end{array} \right) \]
and the new variables \( x^1 = \left( \begin{array}{c} \hat{x} \\ \hat{t} \end{array} \right), \) \( x^2 = z \) we may write
\[ AV\hat{R}_n^{reg} (X) = \left\{ C_1 x^1 + C_2 x^2 : A_1 x^1 + A_2 x^2 = b, \ x^1 \in \mathbb{R}^{2dN} \right\}. \]
This problem has precisely the form of a linear vector optimization problem with constraints in the form of equations and inequalities and ordering cone \( M_+ \). Thus, by Corollary 3 in [10], its set-valued dual is given by
\[ \bigcap \left\{ S_{(\eta,-\nu)} (b) : A_{\eta}^T v \leq C_1^T v, \ A_{\nu}^T v = C_2^T v, \ v \in (M_+)^\perp \setminus \{0\} \right\}. \]
Plugging in the matrices defined above, using \( \eta = \left( \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right) \) with \( \eta_1 \in \mathbb{R}^{dN}, \eta_2 \in \mathbb{R}^{d-m} \) and observing that \( S_{(\eta,-\nu)} (b) = S_{(\eta_1,-\nu)} (-\hat{x}) \), we end up with (4.2).
(2) Defining the matrix
\[ \hat{D} = \left( \begin{array}{ccc} P_{(1)} & 0 & \cdots \\ 0 & P_{(2)} & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots & 0 \end{array} \right) \in \mathbb{R}^{dN \times dN}, \]
we may see that \( \hat{D} \) is invertible. Thus, we can substitute \( \eta_1 = \hat{D} \hat{y}, \ \hat{y} \in \mathbb{R}^{dN} \) and obtain the following equivalent reformulation of the dual problem
\[ AV\hat{R}_n^{reg} (X) = \bigcap \left\{ S_{(\hat{D}\hat{y},-\nu)} (-\hat{x}) : \hat{D} \hat{y} \leq \left( \text{diag}(\alpha)^{-1} \hat{P} \right)^T \left( v - (B_{(d-m)})^T \eta_2 \right), \right. \\
\left. 0 \leq \hat{D} \hat{y}, \ M_+^\perp \hat{D} \hat{y} + (B_{(d-m)})^T \eta_2 = v, \ v \in (M_+)^\perp \setminus \{0\} \right\}. \]
If the components of the vector \( \hat{y} \in \mathbb{R}^{dN} \) written as
\[ \hat{y} = (y_{11}, \ldots, y_{d1}, y_{12}, \ldots, y_{dN})^T \in \mathbb{R}^{dN} \]
are interpreted as the values of a random vector \( Y : \Omega \to \mathbb{R}^d \) with \( y_{im} = Y_i (\omega_n), \ i \in \{1, \ldots, d\}, \ n \in \{1, \ldots, N\} \), then
\[ S_{(\hat{D}\hat{y},-\nu)} (-\hat{x}) = \left\{ u \in M : E \left[ -X^T Y \right] \leq u^T \hat{u} \right\}, \]
the right hand side of this expression has been denoted as \( F_{(M,Y)}^{(M)} [-X] \) in [11][12]. Furthermore,
\[ I_d^T \hat{D} \hat{y} = E [Y] \]
and \( (B_{(d-m)})^T \eta_2 \in M_+^\perp \). Thus, the condition \( I_d^T \hat{D} \hat{y} + (B_{(d-m)})^T \eta_2 = v \) means
\[ v - (B_{(d-m)})^T \eta_2 = E [Y] \]
and \( v \in E [Y] + M_+^\perp \). Moreover, \( 0 \leq \hat{D} \hat{y} \) if and only if \( Y (\omega_n) \in \mathbb{R}^d_+ \) for all \( v \in \mathbb{R}^d_+ \). The condition \( \hat{D} \hat{y} \leq \left( \text{diag}(\alpha)^{-1} \hat{P} \right)^T (v - (B_{(d-m)})^T \eta_2) \) is equivalent to
\[ \forall n \in \{1, \ldots, N\} : \ \text{diag}(\alpha)^{-1} E [Y] - Y (\omega_n) \in \mathbb{R}^d_+. \]
The overall result is the dual representation of \( AV@R^{v^{cg}} \) under (H1), (H2) as given in (1.3).

(3) A one-to-one relationship between elements \( (Y, v) \) satisfying \( Y \in (L^{\infty}_d)_+ \), \( v \in (E[Y] + M^+) \cap (M_+)^+ \setminus \{0\} \) and elements \( (Q, w) \in W \) such that

\[
F^{M}_{(Y, v)} [-X] = (E^Q [-X] + G(w)) \cap M
\]

follows from lemma 4.1 in [11] by setting \( K = \mathbb{R}^{\hat{d}}_+ \) in [11]. This provides the desired result as the additional property \( \text{diag}(\alpha)^{-1} E[Y] - Y \in (L^\infty_d)_+ \) for \( (Y, v) \in (L^\infty_d)_+ \times v \in (E[Y] + M^+) \cap (M_+)^+ \setminus \{0\} \) implies the corresponding property \( \text{diag}(w) \left( \text{diag}(\alpha)^{-1} e \mathbb{I} - \frac{dN}{\partial p} \right) \in (L^\infty_d)_+ \) for \( (Q, w) \in W \) and vice versa.

Note that for \( m = d, M = \mathbb{R}^d, M_+ = \mathbb{R}^d_+ \) the random variable \( Y \) in (1.3) is a consistent price system (see [15][26] for the one-period market \( (K_0, K_T = K_T(\omega)) \)). This motivates the name "CPS-like". Compare the discussion in [12].

4.1.3 Examples

We will use Benson’s algorithm in the variant described in [13] to calculate the optimal value \( AV@R^{v^{cg}} \) of the linear vector optimization problem (1.1). Benson’s algorithm considers linear vector optimization problems with inequality constraints in the following form.

\[
\text{minimize } P(x) \text{ with respect to } \leq_{M_+} \text{ subject to } Bx \geq b.
\]

For the calculation of \( AV@R^{v^{cg}} \), this input parameters are given by the ordering cone \( M_+ \) (which is finitely generated since it is the intersection of the linear subspace \( M \subseteq \mathbb{R}^d \) with the finitely generated cone \( \mathbb{R}^d_+ \)) and

\[
P = \left( \text{diag}(\alpha)^{-1} \hat{P} - I_d \right), \quad B = \begin{pmatrix}
I_d N & -\hat{I}_d \\
0 & I_d N \\
B_{(d-m)} \text{diag}(\alpha)^{-1} \hat{P} - B_{(d-m)} & 0 & -\hat{I}_d \\
0 & -B_{(d-m)} \text{diag}(\alpha)^{-1} \hat{P} & B_{(d-m)}
\end{pmatrix}, \quad b = \begin{pmatrix}
-\hat{x} \\
0 \\
0
\end{pmatrix}
\]

with the variable \( x = \begin{pmatrix}
z \\
\hat{z}
\end{pmatrix} \in \mathbb{R}^{dN+d} \).

**Example 4.2** Consider an example with \( d = 2 \) assets. Let \( M = \mathbb{R}^d \) and \( N = 2 \) (binomial model) with the following payoff

\[
X(\omega_1) = \begin{pmatrix} 12 \\ -20 \end{pmatrix}, \quad X(\omega_2) = \begin{pmatrix} 4 \\ -6 \end{pmatrix}.
\]

The significance level is \( \alpha = (0.01, 0.02)^T, p = (0.4, 0.6)^T \). \( AV@R^{v^{cg}} = PP + R^2 \) with one vertex \( PP = (-4, 20)^T \). Thus, the minimal risk compensating portfolio for a risk manager/regulator is \( -4 \) units of the first asset and \( 20 \) units of the second asset. \( AV@R^{v^{cg}} \) coincides with the worst case risk measure in this particular example since \( \alpha_i < p_n \) for all \( i, n = 1, 2 \).

**Example 4.3** Consider \( d = 2 \) assets. Let \( M = \mathbb{R}^d \) and \( N = 5 \) with the following payoff

\[
X(\omega_1) = \begin{pmatrix} 6 \\ 3 \end{pmatrix}, \quad X(\omega_2) = \begin{pmatrix} -8 \\ -6 \end{pmatrix}, \quad X(\omega_3) = \begin{pmatrix} -4 \\ 2 \end{pmatrix},
\]

\[
X(\omega_4) = \begin{pmatrix} -90 \\ -6 \end{pmatrix}, \quad X(\omega_5) = \begin{pmatrix} -80 \\ -60 \end{pmatrix}.
\]

The significance level is \( \alpha = (0.05, 0.05)^T, p = (0.25, 0.4, 0.3, 0.02, 0.03)^T \). \( AV@R^{v^{cg}} = PP + R^2 \) with the vertex \( PP = (84, 38.4)^T \). In contrast to the previous example, the significance levels do have an impact here on the risk measure.

More complex examples with \( d = 5 \) assets and \( M = \mathbb{R}^2 \times \{0\}^3 \) and a random variable \( X \) that is the payoff of an outperformance option can be found in examples [4.8 - 4.10] at the end of the paper where \( AV@R^{v^{cg}} \) as well as \( AV@R^{max} \) are calculated.
4.2 The market extension and minimization of AV@R

In this section, we will restrict ourselves to one-period models. The reason is that market extensions or risk minimization problems including trading at times $t = 0, ..., T$ typically involve path dependent strategies in markets with transaction costs and should rather be formulated in a recursive way. Also time consistency issues will play a role.

For the special case of superhedging in markets with transaction costs (which is an example of a set-valued multi-portfolio time consistent coherent risk measure, see [12], [7]) a recursive multi-period algorithm had been presented in [21], which leads to a sequence of linear vector optimization problems. For more general risk measures, a set-valued Bellman’s principle should be explored, but we will leave this to further research.

Let us assume from now on that trading is possible at time $t = 0$ and $t = T$. Then, we can formulate $AV@R_{\alpha \text{mar}}$ as a linear vector optimization problem and use Benson’s algorithm to solve it.

4.2.1 The discrete version of $AV@R_{\alpha \text{mar}}$

We now want to reformulate the description of $AV@R_{\alpha \text{mar}}$ given in (3.3) in the language of linear vector optimization in order to calculate the values of $AV@R_{\alpha \text{mar}}$ with help of Benson’s algorithm. All conditions in (3.3) are the same as in $AV@R_{\alpha \text{reg}}$, except the condition $Z + X - z \mathbb{I} \in L^1_d(K_T) + K_0 \mathbb{I}$.

Let

$$H = \begin{pmatrix} h^1 \cdot \cdot \cdot h^I \\ \vdots \cdot \cdot \cdot \vdots \\ h^I \cdot \cdot \cdot h^I \end{pmatrix} \in \mathbb{R}^{d \times I}$$

be the matrix of generating vectors of $K_0$, see assumption (H3).

Under assumption (H5), we have that for each $\omega \in \Omega$ the cone $K_T(\omega)$ is spanned by $k^1(\omega), ..., k^J(\omega)(\omega)$. Denote

$$\hat{J} = \sum_{n=1}^{N} J(\omega_n).$$

Let

$$A_{(n)} = \begin{pmatrix} k^1_{1n} \cdot \cdot \cdot k^I_{1n} \\ \vdots \cdot \cdot \cdot \vdots \\ k^I_{dn} \cdot \cdot \cdot k^I_{dn} \end{pmatrix} \in \mathbb{R}^{d \times J_n}, n = 1, ..., N$$

be the matrices containing the generating vectors of $K_T(\omega)$ as columns and

$$\hat{A} = \begin{pmatrix} A_{(1)} & 0 & \ldots & 0 \\ 0 & A_{(2)} & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & A_{(N)} \end{pmatrix} \in \mathbb{R}^{dN \times J}.$$

a diagonal matrix, where 0 stands for blocks of zeros of appropriate dimensions. Consider the vector

$$s = (s_{11}, ..., s_{1J_1}, s_{21}, ..., s_{2J_2}, ..., s_{NJ_N}) \in \mathbb{R}^{J}.$$

Then the condition $Z + X - z \mathbb{I} \in L^1_d(K_T) + K_0 \mathbb{I}$ can equivalently be written as

$$\hat{z} + \hat{x} - \hat{I}_d z = \hat{A}s + \hat{I}_d H t$$

for $s \in \mathbb{R}^J_+$ and $t \in \mathbb{R}^I_+$. 
Using this and the results of section 4.1 for the other conditions we obtain
\[
AV@R_{\alpha}^{mar}(X) = \{ \text{diag}(\alpha)^{-1} \hat{P} \hat{z} - z : B_{(d-m)}(\text{diag}(\alpha)^{-1} \hat{P} \hat{z} - z) = 0, \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (4.6) \}
\]
\[
\hat{z} + \hat{\epsilon} - \hat{I}_d \hat{z} = \hat{A}_s + \hat{I}_d H t, \ \hat{z} \in \mathbb{R}^{dN}, \ \ s \in \mathbb{R}^{J}, \ t \in \mathbb{R}^{I}, \ z \in \mathbb{R}^d. \]

Therefore, as in the regulator case, \( AV@R_{\alpha}^{mar}(X) \) is the image of a polyhedral set under a linear function mapping \( \mathbb{R}^{(N+1)d+I+J} \) into \( M \subseteq \mathbb{R}^d \), and this can be seen as the optimal value of an \( C(K_0^M) \)-valued minimization problem.

4.2.2 The dual of the discrete version of \( AV@R_{\alpha}^{mar} \)

By \( K_0^+ \) and \( (K_0^M)^+ \) we denote the positive dual cones of the cones \( K_0 \) in \( \mathbb{R}^d \) and \( K_0^M \) in \( M \), respectively. Thus,
\[
(K_0^M)^+ = \{ v \in M : \forall u \in K_0^M : v^T u \geq 0 \} \subseteq M.
\]

It holds \( (K_0^M)^+ = (K_0^+ + M^+) \cap M \) with \( M^+ = \{ v \in \mathbb{R}^d : \forall u \in M : v^T u = 0 \} \) since \( K_0^+ + M^+ \) is the dual cone of \( K_0^M \) in \( \mathbb{R}^d \). The reader should be aware that both \( (K_0^M)^+ \) and \( K_0^+ + M^+ \) are dual cones of \( K_0^+ \), the first one in \( \mathbb{R}^d \), the second one in \( \mathbb{R}^d \).

By \( K_T^+ \) we denote the set-valued mapping \( \omega \mapsto [K_T(\omega)]^+ \), and \( \{ L_1^0(K_T) \}^+ \subseteq L_\alpha^\infty \) denotes the dual cone of \( L_1^0(K_T) \). It holds \( [L_1^0(K_T)]^+ = L_\alpha^\infty \left(K_T^+ \right) \), see section 6.3 in [12].

**Proposition 4.4** Under assumptions (H1)-(H5), the set-valued dual of \( AV@R_{\alpha}^{mar} \) can equivalently be written

1. in matrix form as a linear vector optimization problem given by
\[
AV@R_{\alpha}^{mar}(X) = \bigcap \{ S_{(\eta,-v)}(\hat{\epsilon}) : \eta \leq \left( \text{diag}(\alpha)^{-1} \hat{P} \right)^T (v - (B_{(d-m)})^T \eta_2), \ 0 \leq \hat{A}^T \eta_1, \ 0 \leq H^T \hat{I}^T \eta_1, \ \hat{I}^T \eta_1 + (B_{(d-m)})^T \eta_2 = v, \ v \in (K_0^M)^+ \setminus \{0\} \}; \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (4.7)
\]

2. in linear form using CPS-like variables
\[
AV@R_{\alpha}^{mar}(X) = \bigcap \{ F_{(Y,\omega)}([-X]) : v \in (E[Y] + M^+) \cap (K_0^M)^+ \setminus \{0\}, \ \ Y \in L_\alpha^\infty \left(K_T^+ \right), \ E[Y] \in K_0^+, \ \text{diag}(\alpha)^{-1} E[Y] - Y \in (L_\alpha^\infty)^+ \}; \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ (4.8)
\]

where \( F_{(Y,\omega)}([-X]) = \{ u \in M : E \left([-X^T Y] \leq v^T u \right) \}; \)

3. with respect to EMM-like vector probability measures
\[
AV@R_{\alpha}^{mar}(X) = \bigcap_{(Q, w) \in W} \{ E^Q \left([-X] + G(w) \right) \cap M, \ \ \ \ \ \ \ (4.9) \}
\]

where
\[
W_\alpha = \left\{ (Q, w) \in W : \text{diag}(w) \left( \text{diag}(\alpha)^{-1} e^I - \frac{dQ}{dP} \right) \in (L_\alpha^\infty)^+ \right\},
\]
\[
\bar{W} = \left\{ (Q, w) \in M_{1,d}^F \times (K_0^+) \setminus M^+ : \text{diag}(w) \frac{dQ}{dP} \in L_\alpha^\infty \left(K_T^+ \right) \right\}.
\]

**Remark 4.5** Note that the set of dual variables differ slightly from the ones in [12], lemma 3.4 and theorem 4.2 for market compatible risk measures in a one-period framework. This is due to the fact that the notion of market compatibility used in this paper differs slightly from the definition used in definition 2.7 in [12] in the sense that we use \( K_0 \) instead of \( K_0^M \) in the condition for \( K_0^M \)-compatibility. This is motivated by the dynamic framework and explained in details in remark 2.5. If we used market
compatibility as defined in [12], the condition $0 \leq H^T I_d^T \eta_1$ in (4.7) would be replaced by $0 \leq G^T I_d^T \eta_1$ where $G$ is the matrix of generating vectors of $K_0^M$. In the formulation with random variables the condition $E[Y] \in K_0^+$ in (4.8) would then read as

$$E[Y] \in K_0^+ + M^\perp = \left\{ v \in \mathbb{R}^d : \forall u \in K_0^M : v^T u \geq 0 \right\}.$$ 

$K_0^+ + M^\perp$ is the dual cone of $K_0^M$ in $\mathbb{R}^d$. In the vector probability measure formulation the condition $w \in (K_0)^+ \setminus M^\perp$ in (4.9) would be replaced by $(K_0)^+ \setminus M^\perp + M^\perp$, which is exactly the set appearing in the general duality result in theorem 4.2 in [12].

**Proof (Proof of proposition 4.4)** (1) Defining the matrices

$$C_1 = \left( \text{diag}(\alpha)^{-1} \hat{P} 0 0 \right), \quad C_2 = -I_d, \quad A_1 = \left( \begin{array}{cc} I_{dN} & -\hat{A} - \hat{I}_d \hat{H} \\ B(d-m) \text{diag}(\alpha)^{-1} \hat{P} & 0 & 0 \end{array} \right),$$

$$A_2 = \left( \begin{array}{c} -\hat{I}_d \\ -B(d-m) \end{array} \right), \quad b = \left( \begin{array}{c} -\hat{\bar{\epsilon}} \\ 0 \end{array} \right)$$

and the new variables $x^1 = \left( \begin{array}{c} \hat{\bar{\epsilon}} \\ s \\ t \end{array} \right) \in \mathbb{R}^{4N+J+I}$, $x^2 = z$ we may write

$$AV@R^{mar}(X) = \left\{ C_1 x^1 + C_2 x^2 : A_1 x^1 + A_2 x^2 = b, \ x^1 \in \mathbb{R}^{4N+J+I} \right\}.$$ 

Again, this is the image set of a polyhedral set under a linear function and can be understood as a "linear" set-valued optimization problem with constraints in the form of equations and inequalities and ordering cone $K_0^M$. Its set-valued dual is by corollary 3 in [10] given by

$$\bigcap \left\{ S_{(\eta,-v)}(b) : A_1^T \eta \leq C_1^T v, \ A_2^T \eta = C_2^T v, \ v \in (K_0^M)^+ \setminus \{0\} \right\}.$$ 

Plugging in the matrices defined above, using $\eta = \left( \begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right)$ with $\eta_1 \in \mathbb{R}^{dN}$, $\eta_2 \in \mathbb{R}^{d-m}$, and observing that $S_{(\eta,-v)}(b) = S_{(\eta,-v)}(-\hat{\bar{\epsilon}})$, we end up with

$$\bigcap \left\{ S_{(\eta,-v)}(-\hat{\bar{\epsilon}}) : I_{dN} \eta_1 + \left( B(d-m) \text{diag}(\alpha)^{-1} \hat{P} \right)^T \eta_2 \leq \left( \text{diag}(\alpha)^{-1} \hat{P} \right)^T v, \ -\hat{A}^T \eta_1 \leq 0, \ -\hat{I}_d^T \eta_1 + (B(d-m))^T \eta_2 = I_d v, \ v \in (K_0^M)^+ \setminus \{0\} \right\}.$$ 

and thus with (4.7).

(2) Recalling the definition of the matrix $\hat{D}$ as in (4.3), let us substitute $\eta_1 = \hat{D} \hat{y}$, $\hat{y} \in \mathbb{R}^{dN}$. The components of the vector $\hat{y} \in \mathbb{R}^{dN}$ written as

$$\hat{y} = (y_1, \ldots, y_{d_1}, y_{d_2}, \ldots, y_{dN})^T \in \mathbb{R}^{dN}$$

are interpreted as the values of a random vector $Y : \Omega \to \mathbb{R}^d$ with $y_i = \mathcal{Y}_t(\omega_n)$, $i \in \{1, \ldots, d\}$, $n \in \{1, \ldots, N\}$. Then,

$$S_{(\hat{D}\hat{y},-v)}(-\hat{\bar{\epsilon}}) = \left\{ u \in M : E[-X^T Y] \leq u^T u \right\} = F_{(Y,v)}^M \left[ -X \right],$$

using the notation of [11][12]. Furthermore,

$$\hat{I}_d^T \hat{D} \hat{y} = E[Y]$$

and $(B(d-m))^T \eta_2 \in M^\perp$. Thus, the condition $\hat{I}_d^T \hat{D} \hat{y} + (B(d-m))^T \eta_2 = v$ means

$$v - (B(d-m))^T \eta_2 = E[Y]$$
and \( v \in E[Y] + M^\perp \). Moreover, \( 0 \leq \hat{A}^T \hat{D} \hat{g} \) if and only if \( Y(\omega_n) \in K_T (\omega_n)^+ \) for all \( n \in N \) and \( 0 \leq \left( \hat{I}_d H \right)^T \hat{D} \hat{g} \) if and only if \( E[Y] \in K_0^+ \). The condition \( \hat{D} \hat{g} \leq \left( \text{diag}(\alpha)^{-1} \hat{P} \right)^T (v - (B_{(d-m)})^T \eta_2) \) is equivalent to

\[
\forall n \in \{1, \ldots, N\} : \text{diag}(\alpha)^{-1} E[Y] - Y(\omega_n) \in \mathbb{R}^d_+.
\]

The overall result is the dual representation of \( AV@R^\text{mar} \) under (H1)-(H5) as given in (4.8).

(3) A one-to-one relationship between elements \((Y, v)\) satisfying \( Y \in L_d^\infty (K_T^+) \), \( v \in (E[Y] + M^\perp) \cap (K_0^M)^+ \setminus \{0\} \) and elements \((Q, w) \in \tilde{W}\) such that

\[
F_{M,v}^q[-X] = (E^Q[-X] + G(w)) \cap M
\]

follows by lemma 3.4 in [12]. Note that the property \( E[Y] \in K_0^+ \) in (4.8) ensures that \( w \in (K_0)^+ \setminus M^\perp \) instead of the larger set \((K_0)^+ \setminus M^\perp + M^\perp \) obtained by lemma 3.4 in [12]. This provides the desired result as the additional property \( \text{diag}(\alpha)^{-1} E[Y] - Y \in (L_d^\infty)_+ \) for \((Y, v) \in L_d^\infty (K_T^+) \times (E[Y] + M^\perp) \cap (K_0^M)^+ \setminus \{0\} \) implies the corresponding property \( \text{diag}(w)(\text{diag}(\alpha)^{-1}eI - \frac{dQ}{dP}) \in (L_d^\infty)_+ \) for the element \((Q, w) \in \tilde{W}\) and vice versa.

### 4.2.3 Examples

As before we use Benson’s algorithm in the variant described in [13] to calculate the optimal value \( AV@R^\text{mar} \) of the linear vector optimization problem given in (4.6). Recall that Benson’s algorithm considers linear vector optimization problems with inequality constraints in the following form.

\[
\text{minimize } P(x) \text{ with respect to } \leq_{K_0^M} x \text{ subject to } Bx \geq b.
\]

For the calculation of \( AV@R^\text{mar} \) this input parameters are given by the ordering cone \( K_0^M \) (generated by \( g^1, \ldots, g^r \)) and

\[
P = \left( \text{diag}(\alpha)^{-1} \hat{P} \right) 0 0 -I_d), \quad b = \begin{pmatrix} -\hat{x} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad x = \begin{pmatrix} z \\ s \\ t \end{pmatrix} \in \mathbb{R}^{dN+J+I+d},
\]

\[
B = \begin{pmatrix} I_{dN} & -\hat{A} & -\hat{I}_d & -\hat{I}_d \\ B_{(d-m)} \text{diag}(\alpha)^{-1} \hat{P} & 0 & 0 & -B_{(d-m)} \end{pmatrix}.
\]

Here, we used the fact that the equality \( \hat{z} + \hat{x} - \hat{I}_dz = \hat{A}s + \hat{I}_dHt \) is equivalent to the inequality \( \hat{z} + \hat{x} - \hat{I}_dz \geq \hat{A}s + \hat{I}_dHt \) since \( L_{d}^1(K_T) + K_0 I + (L_d^\infty)_+ = L_d^1(K_T) + K_0 I \).

**Example 4.6** Consider as in example 12 \( d = 2 \) assets, a USD cash account and one risky stock. Let \( M = \mathbb{R}^d \) and \( N = 2 \) (binomial model) with the following payoff

\[
X(\omega_1) = \begin{pmatrix} 12 \\ -20 \end{pmatrix}, \quad X(\omega_2) = \begin{pmatrix} 4 \\ -6 \end{pmatrix}.
\]

The significance level is \( \alpha = (0.01, 0.02)^T \), \( p = (0.4, 0.6)^T \). Thus, the \( AV@R \) is a type of worst case risk measure in this case. Let the bid-ask prices of the risky stock at \( t = 0 \) be given by \( S_{0,b} = 0.72 \), \( S_{0,a} = 1 \) and at \( t = T \) by \( S_{T,b} (\omega_1) = 0.75 \), \( S_{T,a} (\omega_1) = 1.11 \) and \( S_{T,b} (\omega_2) = 0.7 \), \( S_{T,a} (\omega_2) = 0.9 \). Then \( AV@R^\text{mar}(X) \) has two vertices given by \((-12, 20)\) and \((-39, 56)\) and a recession cone given by \( K_0 \).

Optimal strategies that correspond to the market extension of the risk measure are calculated using the solution concept for linear vector optimization problems as described in [20], see also section 6 in [21].
for a short introduction to the topic. Benson’s algorithm (in the variant described in [13]) provides also a solution. In our example a solution is given by the set of efficient points \( \{x^1, x^2\} \subseteq \mathbb{R}^{dN+J+I+d} \) with

\[
x^1 = (z^1, s^1, t^1, z^1)^T = (0, 0, 0, 0, 0, 0, 11.4286, 0, 0, 12, -20)^T
\]

\[
x^2 = (z^2, s^2, t^2, z^2)^T = (0, 0, 0, 0, 36, 0, 50, 0, 0, 39, -56)^T.
\]

In this example the set of efficient directions is empty. By the definition of a solution it holds

\[\co \{Px^1, Px^2\} + K_0 = AV@R_{\alpha}^{mar}(X).\]

The values of \( s^i, t^i \) in \( x^i \), \( i = 1, 2 \) yield two different optimal strategies (that correspond to the two vertices). The number of optimal strategies in this example is infinite.

The first strategy consists of no trade at \( t = 0 \left( k_0^1 = HT^1 = (0, 0)^T \in K_0 \right) \), no trade at \( t = T \) if \( \omega_1 \) occurs \( \left( k_T^1(\omega_1) = A_1(s_{11}^1, ..., s_{1J}^1)^T \in (0, 0)^T \in K_T(\omega_1) \right) \) and to sell 11.4286 units of stock (at price \( S_{T,b}(\omega_2) = 0.7 \) which yields 8$) if \( \omega_2 \) occurs \( \left( k_T^2(\omega_2) = A_2(s_{21}^1, ..., s_{2J}^1)^T = (-8, 11.4286)^T \in K_T(\omega_2) \right) \). Let us denote by \( Y^1 \in C_T = -K_0 \mathbb{I} - L_2(1) \) the outcome of this optimal strategy starting from zero capital at \( t = 0 \). Note that \( AV@R_{\alpha}^{reg} (X + Y^1) = (-12, 20)^T + R_{\alpha}^2 \), which corresponds to the first vertex of \( AV@R_{\alpha}^{mar}(X) \).

The second strategy consists of no trade at \( t = 0 \left( k_0^2 = HT^2 = (0, 0)^T \in K_0 \right) \), to sell 36 units of stock (at price \( S_{T,b}(\omega_1) = 0.75 \) which yields 27$) if \( \omega_1 \) occurs \( \left( k_T^3(\omega_1) = A_3(s_{31}^3, ..., s_{3J}^3)^T = (-27, 36)^T \in K_T(\omega_1) \right) \) and to sell 50 units of stock (at price \( S_{T,b}(\omega_2) = 0.7 \) which yields 35$) if \( \omega_2 \) occurs \( \left( k_T^2(\omega_2) = A_2(s_{21}^2, ..., s_{2J}^2)^T = (-35, 50)^T \in K_T(\omega_2) \right) \). Let us denote by \( Y^2 \) the outcome of this optimal strategy starting from zero capital at \( t = 0 \). Note that \( AV@R_{\alpha}^{reg} (X + Y^2) = (-39, 56)^T + R_{\alpha}^2 \), which corresponds to the second vertex of \( AV@R_{\alpha}^{mar}(X) \).

Both strategies lead to terminal wealth \( X + Y^i, i = 1, 2 \) which is risk minimal for the risk measure \( AV@R_{\alpha}^{reg} \) among all possible terminal wealths \( X + C_T \) as described in theorem 2.9. It holds

\[AV@R_{\alpha}^{mar}(X) = \co \left( AV@R_{\alpha}^{reg} (X + Y^1) \cup AV@R_{\alpha}^{reg} (X + Y^2) \right) + K_0.\]

**Remark 4.7** Let us compare the set-valued approach to risk measurement to the scalar approach in which it is often assumed that a multivariate random variable \( X \) is first liquidated into a given numéraire asset and then a scalar risk measure is applied to the liquidated value of \( X \) (see e.g. [2], example 2.5, or the standard assumption of liquidation in the scalar approach to utility maximization in markets with transaction costs, see [14]).

For example 4.6 the results are as follows. The scalar \( AV@R \) is calculated from the liquidated payoff, where we use the liquidation functions according to the bid-ask prices at time \( T \)

\[
l_1(X)(\omega) = X_1(\omega) + X_2(\omega) S_{T,b}(\omega) I_{x_2(\omega) \geq 0} + X_2(\omega) S_{T,a}(\omega) I_{x_2(\omega) < 0},
\]

\[
l_2(X)(\omega) = X_2(\omega) + \frac{X_1(\omega)}{S_{T,a}(\omega)} I_{x_1(\omega) \geq 0} + \frac{X_1(\omega)}{S_{T,b}(\omega)} I_{x_1(\omega) < 0}.\]

In example 4.8 the liquidated payoff is \( l_1(X) = (-10.2, -1.4) \) if \( X \) is liquidated into the first asset, and \( l_2(X) = (-9.19, -1.6) \) if \( X \) is liquidated into the second asset. One obtains \( AV@R_{\alpha_1}(l_1(X)) = 10.2 \) USD and \( AV@R_{\alpha_2}(l_2(X)) = 9.19 \) units of stock. Note that the set \( AV@R_{\alpha}^{mar}(X) \) from example 4.6 intersects the \( x \) and \( y \) axis at \((8, 0)\) and \((0, 8)\). Clearly, the scalar \( AV@R \)‘s are not on the boundary of the set \( AV@R_{\alpha}^{mar}(X) \), but in the interior of \( AV@R_{\alpha}^{mar}(X) \).

**Example 4.8** Consider an example with \( d = 5 \) assets. The first asset \( S^1 \) is a USD cash account (zero interest rate), the second is another currency (e.g. EUR), denoted in USD, and the other assets are risky stocks denoted in USD. As the space of eligible assets we choose the space spanned by the first and the second asset (the currencies), i.e. \( M = \mathbb{R}^2 \times \{0\}^d \). We will use a multi-dimensional one-period tree that approximates a \( d - 1 \)-dimensional Black Scholes model for \( d - 1 \) correlated risky assets, where the stock price dynamics under the real world probability measure \( P \) are given by

\[
dS_t^i = S_t^i(\mu d t + \sigma dW_t^i), \quad i = 2, ..., d
\]
for Brownian motions \( W^i \) and \( W^j \) with correlation \( \rho_{i,j} \in [-1,1] \) for \( i \neq j \). The input parameters are as follows. The initial prices of the 5 assets in USD are given by \( S_0 = (1,1.3,50,6,25)^T \), the covariance matrix of the 4 risky assets is
\[
\begin{pmatrix}
0.010 & 0.004 & 0.002 & 0.018 \\
0.004 & 0.040 & 0.012 & 0.006 \\
0.002 & 0.012 & 0.0225 & 0.012 \\
0.018 & 0.006 & 0.012 & 0.040 \\
\end{pmatrix},
\]
and \( \mu = (0.03,0.1,0.06,0.12)^T \). Let the length of the one-period model under consideration be one year.

We will follow the method in [19] to set up a tree for the correlated risky assets by transforming the stock price process \( S \) into a process with independent components using the decoupling with the Cholesky decomposition, see [19] and also section 4.2 in [21] for more details in a setting similar to ours in markets with transaction costs. We adapt the method to obtain a tree under the real world probability measure. The one-period tree will have \( 2^{d-1} = 16 \) branches, i.e. \( N = 16 \) in this example and the probabilities of each path are given by \( 2^{-d} \). Now, let us assume that the proportional transaction costs for the risky assets are given by \( \lambda = (\lambda^1,\ldots,\lambda^d)^T = (0.07,0.05,0.01,0.01)^T \) and that the bid and ask prices at \( t \in \{0,T\} \) are given by
\[
(S^i_0)^t = S^i_t(1 - \lambda^i), \quad (S^i_T)^t = S^i_t(1 + \lambda^i), \quad i = 2,\ldots, d.
\]

Furthermore, let us assume an exchange between any two risky assets can not be made directly, only via cash in USD by selling one asset and buying the other. Since all risky assets are denoted in USD, the solvency cone \( K_t \) for \( t \in \{0,T\} \) is generated by the columns of the following matrix (see e.g. [22])
\[
\begin{pmatrix}
(S^1_T)^2 - (S^1_T)^2 & (S^1_T)^3 & (S^1_T)^4 - (S^1_T)^4 \quad (S^2_T)^5 - (S^2_T)^5 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
\end{pmatrix}.
\]

As the payoff \( X \) we consider an outperformance option with strike \( K = (1 + \lambda_1)S^1_0 = 1.378 \) and physical delivery. A vector \( c \) is defined as \( (S^i_T)^2 = c_i(S^i_T)^2 \) for \( i \in \{2,\ldots,5\} \). Let the payoff \( X \) be \(-K\) in the USD account, \( c_i \) units of asset \( i \) for the smallest \( i \) satisfying \( c_i(S^i_T)^2 = \max_{j \in \{2,\ldots,5\}} (c_j(S^j_T)^2) \geq K \) and zero in the other assets. If \( \max_{j \in \{2,\ldots,5\}} (c_j(S^j_T)^2) < K \) the payoff is the zero vector. The maturity \( T \) is chosen as one year.

Let us calculate \( AV@R^{reg} (X) \) and \( AV@R^{mar} (X) \) with significance levels \( \alpha = (0.1,0.08,0.09,0.1,0.05)^T \). The vector optimization problem to calculate \( AV@R^{reg} (X) \) has 85 variables, 166 constraints and 2 objectives. The corresponding vector optimization problem for \( AV@R^{mar} (X) \) has 221 variables, 302 constraints and 2 objectives. \( AV@R^{reg} (X) \) has one vertex at \((1.3910,0\),\) which is the smallest cash deposit in the first two assets necessary to compensate the risk of \( X \) without involving trading. The recession cone of \( AV@R^{reg} (X) \) is \( R_2^2 \).

The set \( AV@R^{mar} (X) \) has two vertices given by \((0.8858,-0.7160)^T \) and \((0.4200,-0.3771)^T \) and a recession cone equal to \( K^M_0 \).

**Example 4.9** Let us consider the same model as in example 4.8, now with an annual interest rates of 5% for the riskless asset denoted USD. Let \( (S_0)^{1} = (1 + r)^{-1} = 0.9524 \) and \( (S_T)^{1} = 1 \). All the other input parameters are as before. The solvency cones change in the sense that in the matrix [4.1] all the values \( \pm 1 \) are replaced by \( \pm (S_0)^{1} \), see [22]. \( AV@R^{reg} (X) \) has one vertex at \((1.391,0\),\) which is the smallest deposit in the first two assets (USD bond and EUR) necessary to compensate the risk of \( X \) without involving trading. The recession cone of \( AV@R^{reg} (X) \) is \( R_2^2 \). To calculate the smallest deposit in cash (USD and EUR), one just needs to multiply the number of USD bonds with the initial bond price \( (S_0)^{1} = 0.9524 \). The set \( AV@R^{mar} (X) \) has four vertices given by \((4.7411, -3.3883)^T \), \((4.1666, -2.9941)^T \), \((3.9181, -2.8235)^T \) and \((0.8716, -0.7160)^T \) (in USD bonds and EUR) and a recession cone equal to \( K^M_0 \).
Example 4.10 Let us consider the same model as in example 4.9 now with transaction costs \( \lambda_0 = 0.03 \) for the riskless asset. That means \((S_0^1) = 1 - \lambda_0(1+r)^{-1}, (S_0^2) = 1 + \lambda_0(1+r)^{-1}\) and \((S_0^3) = 1 + \lambda_0\). All the other input parameters are as before. The solvency cones have now 20 generating vectors instead of 8, for details see e.g. [22].

The set \( AV \cap R^{reg} (X) \) has one vertex at \((1.391, 0)^T\), which is the smallest deposit in the first two assets (USD bond and EUR) necessary to compensate the risk of \( X \) without involving trading. The recession cone of \( AV \cap R^{reg} (X) \) is \( R^2_0 \). The set \( AV \cap R^{mar} (X) \) has seven vertices (in USD bonds and EUR) given by the column of the following matrix

\[
\begin{pmatrix}
6.9490 & 5.0535 & 4.2335 & 1.8394 & 1.3989 & 1.1107 & 1.0965 \\
-4.7770 & -3.5156 & -2.9696 & -1.3630 & -1.0652 & -0.8621 & -0.8520
\end{pmatrix}
\]

and a recession cone equal to \( K^M \). The linear vector optimization problem for \( AV \cap R^{mar} (X) \) has 425 variables, 506 constraints and 2 objectives.

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