Hyperbolic Statics in Space–Time

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Abstract—Based on the concept of a material event as an elementary material source that is concentrated on a metric sphere of zero radius—a light cone of Minkowski space–time, we deduce an analog of Coulomb’s law for a hyperbolic space–time field universally acting between space–time events. The collective field that provides interaction of world lines of a pair of particles at rest contains a standard 3-dimensional Coulomb part and a logarithmic addition. We find that the Coulomb part depends on a fine balance between the causal and geometric space–time characteristics (a concordance of two regularizations).

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1. INTRODUCTION

Special Relativity (SR), formulated at the beginning of the 20th century, has formed a basis for a new understanding of space–time and physical processes taking place in it. One of the key features of SR is its geometric interpretation: in the core of relativistic physics lies the concept of 4D Minkowski space–time $M_{1,3}$ with the pseudo-Euclidean metric

\[ (\eta) = \text{diag}(1, -1, -1, -1). \]

The geometric language of SR makes it possible to explicitly and consistently formulate its essence and main foundations as well as to deduce its various implications and make a consistent transition to General Relativity (GR).

A theoretical basis for this article is provided by the consistent space–time interpretation of elementary objects in 4D Minkowski world. The Newtonian laws were verified previously \cite{1} within the concept of 4D statics of strongly stretched strings. In this paper, we are going to depart from the traditional interpretation of an elementary particle and follow the logic of the Minkowski 4D geometry. It means that we take as an elementary physical object not a particle’s world line, but a true material point of $M_{1,3}$ space, a metric sphere of zero radius. In space–time it corresponds to a light cone with material characteristics concentrated on it. General considerations suggest that these elementary objects–sources correspond to a space–time field. Such field is a 4D hyperbolic analogue of the Coulomb field. According to this approach, the extended structures like world lines or world tubes stretched along a timelike direction should be obtained by alignment (condensation) of elementary event points which can be described in the framework of a certain generalized theory of condensed media in 4D space–time.$^1$ Naturally, there arises a question on the relationship between their collective field and the observable physical fields dealt with in standard physics. This question in its simplest formulation is the main subject of this study.

2. HYPERBOLIC SOLUTION WITH SPHERICAL SYMMETRY

We will use the hyperbolic analogue of Coulomb’s law in Minkowski space as a starting point for our considerations. This analogue can be defined as a spherically symmetric (in the sense of a pseudo-Euclidean sphere) solution of the wave equation

\[ \Box U = 0 \] (1)

in empty space–time that surrounds the center of the hyperbolic sphere. At the same time, we keep in mind that the spherically symmetric solution of the Laplace equation in vacuum

\[ \Delta \phi = 0, \] (2)

is unique (up to a constant), while an immediate inspection shows that it contains all information about its point source. Indeed, the solution of (2) in the

\footnote{This theory would contain timelike forces and interactions that cannot be found in standard relativistic physics.}
form of Coulomb potential \( \phi = q/r \) actually satisfies the equation

\[
\Delta \phi = -4\pi q \delta(x) \delta(y) \delta(z) = -\frac{q \delta(r)}{r^2} \tag{3}
\]
in the whole space. The last equality takes into consideration the transformation of the delta function at a transition to curvilinear coordinates [2].

As in the case of the Coulomb field, we do not put the question of the source structure in Eq. (1): the solution automatically contains the singular characteristics of the source. To obtain this solution, let us choose the hyperbolic spherically symmetric 4D coordinate system with zero in the center:

\[
\begin{align*}
t & = o \cosh \chi, \\
x & = o \sinh \chi \sin \theta \cos \varphi, \\
y & = o \sinh \chi \sin \theta \sin \varphi, \\
z & = o \sinh \chi \cos \theta.
\end{align*} \tag{4}
\]

Here \( o \) is the 4-radius, \( \chi \) is the hyperbolic angle, \( \theta \) and \( \varphi \) are the pair of standard spherical angles. Equations (4) are valid for the domains where \( t^2 - x^2 - y^2 - z^2 > 0 \). The Minkowski metric in this coordinate system can be obtained by using the standard rules for the transformation of an interval. It takes the following form:

\[
d s^2 = d o^2 - o^2 (d \chi^2 + \sinh^2 \chi (d \theta^2 + \sin^2 \theta d \varphi^2)). \tag{5}
\]

In differential geometry, the wave operator can be invariantly determined by the relation [3]

\[
\Box \equiv \frac{1}{\sqrt{-g}} \frac{\partial}{\partial \xi^\alpha} \left( \sqrt{-g} g^{\alpha\beta} \frac{\partial}{\partial \xi^\beta} \right), \tag{6}
\]

where \( g \) is the metric tensor determinant, \( g^{\alpha\beta} \) are the contravariant components of the metric whose matrix is inverse to \( (g_{\alpha\beta}) \). From (5), it follows that \( g = -o^4 \sinh^4 \chi \sin^2 \theta \), while the inverse metric has the form

\[
(g^{\alpha\beta}) = \text{diag} \left( 1, -\frac{1}{o^2}, -\frac{1}{o^2 \sinh^2 \chi}, -\frac{1}{o^2 \sinh^2 \chi \sin^2 \theta} \right). \tag{7}
\]

Substituting these expressions into the general relation (6), we obtain the expression for the wave operator in the 4D spherical coordinate system:

\[
\Box = \frac{1}{o^3} \frac{\partial}{\partial o} \left( o^3 \frac{\partial}{\partial o} \right) - \frac{1}{o^2} \Delta_{\chi, \theta, \varphi}, \tag{8}
\]

where we have introduced the following notation for the angular part of the wave operator:

\[
\Delta_{\chi, \theta, \varphi} \equiv \frac{1}{\sinh^2 \chi} \left( \frac{\partial}{\partial \chi} \sinh^2 \chi \frac{\partial}{\partial \chi} + \frac{\partial}{\sin \theta \partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta \partial \varphi^2} \right). \tag{9}
\]

Substituting the general form of the spherically symmetric solution \( U = U(o) \) into the operator (8), we obtain the equation

\[
\Box U = \frac{1}{o^3} \frac{\partial}{\partial o} \left( o^3 \frac{\partial U(o)}{\partial o} \right) = 0. \tag{10}
\]

Its general solution has the form

\[
U(o) = \frac{Q}{o^2} + C, \tag{11}
\]

where \( Q \) and \( C \) are integration constants.

Let us consider this solution as an analogue of the fundamental solution for the hyperbolic field whose sources are the \textit{material events}, i.e., a cone with the distributed characteristic \( \Omega \), which we will call the \textit{hyperbolic charge}. A direct inspection shows that our solution satisfies the 4D analogue of the solution\(^3\) of Eq. (3):

\[
\Box U = -\frac{2Q}{o^3} \delta(o). \tag{12}
\]

We have used the term “an analogue of the fundamental solution” because, unlike the classical fundamental solution used in mathematical physics, the singularity of (11) is concentrated on the light cone rather than at a point. We will call the solution (11) the \textit{hyperbolic fundamental solution} of the wave equation, in order to distinguish it from the well-known fundamental solution (the causal Green function) of classical field theory:

\[
G = \theta(t) \frac{\delta(t - r)}{4\pi r}, \tag{13}
\]

where \( \theta(t) \) is Heaviside’s step function.

A simple analysis shows that the hyperbolic fundamental solution of the form (11) satisfies Eq. (12) in the generalized sense in all causal domains.

\(^2\) When the hyperbolic angles are small, one has \( \sinh \chi \approx \chi \), and the expression for \( \Delta_{\chi, \theta, \varphi} \) turns into the Laplace operator in 3D spherical coordinates with \( r = \chi \).

\(^3\) It should be noted that technically it would be easier to write down the right-hand side in (12) directly in the spherical coordinate system, because this form does not contain the infinite factor \( \Omega_H \), which is an analogue of the factor \( 4\pi \) in (3), which in its turn determines the measure of the set of all directions in \( \mathcal{M}_{1,3} \).
3. STATIC INTERACTION OF PARTICLES

To reveal the intrinsic relationship between the classical field theory and the hyperbolic field, let us first consider the following simple situation: a pair of classical particles (sources) at rest in a certain inertial reference frame. In a 4D coordinate system adjusted to this reference frame, this pair of particles described in \( M_{1,3} \) is a pair of world lines parallel to the time axis and separated by a spatial distance \( r \). These world lines are “weaved” of material events. The superposition principle for the hyperbolic field is valid due to the linearity of the wave equation. It means that the resulting field \( \phi \) of the world line of particle 1, calculated at a certain point on the world line of particle 2, can be obtained by integration:

\[
\phi(t_2, r) = \int_{-T/2}^{T/2} \lambda_1 dt_1 \frac{r^2}{(t_2 - t_1)^2 - r^2}, \quad (14)
\]

where \( \lambda_1 dt_1 = dQ_1, \lambda_1 \) is the linear density of hyperbolic charge 1, \( T \) is the duration of particle history (a regularization parameter). Multiplying \( \phi(t_1, r) \) by the element \( \lambda_2 dt_2 \) of the hyperbolic charge of the second particle world line and integrating along the same line, we obtain

\[
\phi(t_2, r) = \frac{1}{2} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \lambda_1 \lambda_2 \frac{r^2}{(t_2 - t_1)^2 - r^2} dt_1 dt_2. \quad (15)
\]

It is the total energy of hyperbolic interaction between the classical particles (the factor \( 1/2 \) appears because the double integration twice accounts for the same pair of elements on the world lines).

Since the calculation of the integral involves two regularizations, and each of them has a certain physical meaning, a detailed computation is shown below. Substituting the variables \( \xi_1 = t_1/r \), \( \xi_2 = t_2/r \), we arrive at the factorization of dimensional and dimensionless expressions in (15):

\[
\phi(t_2, r) = \frac{\lambda_1 \lambda_2}{2} I(a), \quad (16)
\]

where the dimensionless integral \( I(a) \) depends only on the dimensionless parameter \( a = T/2r \) and is expressed by

\[
I(a) = \int_{-a}^{a} \int_{-a}^{a} \frac{d\xi_1 d\xi_2}{(\xi_2 - \xi_1)^2 - 1}. \quad (17)
\]

In the geometrical sense, it is the integral of an exact 2-form \( (d\xi_1 \wedge d\xi_2)/[(\xi_2 - \xi_1)^2 - 1] \) over the square domain \( \bar{Q}_{2a} \). Let us now choose new coordinates: \( u = \xi_1 - \xi_2, v = \xi_1 + \xi_2 \). The area element \( d\xi_1 \wedge d\xi_2 \) in the integral domain over the square appears because the linearity of the wave equation. It means that the integral (18) in its regularized form now becomes

\[
\bar{Q}_{2a} = \Delta_{\epsilon_1} \cup B_{\epsilon_1} \cup T_{\epsilon_1} \cup T_{\epsilon_2} \cup B_{\epsilon_2} \cup \Delta_{\epsilon_2}, \quad (19)
\]

where the triangular domains \( \Delta_{\epsilon_1}, \Delta_{\epsilon_2} \) are specified by the following inequalities:

\[
\Delta_{\epsilon_1} : -2a - u \leq v \leq 2a + u, \\
-2a \leq u \leq -1 - \epsilon; \\
\Delta_{\epsilon_2} : -2a + u \leq v \leq 2a - u, \\
1 + \epsilon \leq u \leq 2a. \quad (20)
\]

The trapezoidal domains \( T_{\epsilon_1}, T_{\epsilon_2} \) are given by the following inequalities:

\[
T_{\epsilon_1} : -2a - u \leq v \leq 2a + u, \\
-1 + \epsilon \leq u \leq 0; \\
T_{\epsilon_2} : -2a + u \leq v \leq 2a - u, \\
0 \leq u \leq 1 - \epsilon. \quad (21)
\]

The integral (18) in its regularized form now becomes

\[
I(a, \epsilon) = \frac{1}{2} \int_{\bar{Q}_{2a} \setminus (B_{\epsilon_1} \cup B_{\epsilon_2})} \frac{du dv}{u^2 - 1}. \quad (22)
\]

In each of the regularization domains, the integral can be elementarily calculated. After the computing and collecting all data, the result is

\[
I(a, \epsilon) = (2a - 1) \ln(2a - 1) - (2a + 1) \ln(2a + 1) + (2a + 1)(\ln(2 + \epsilon) - \ln(2 - \epsilon)). \quad (23)
\]

Passing on to physical notations \( 2a = T/r \) in (23), we obtain:

\[
I(a, \epsilon) = \left( \frac{T}{r} - 1 \right) \ln \left( \frac{T}{r} - 1 \right) - \left( \frac{T}{r} + 1 \right) \ln \left( \frac{T}{r} + 1 \right) + \left( \frac{T}{r} + 1 \right)(\ln(2 + \epsilon) - \ln(2 - \epsilon)). \quad (24)
\]
If we now consider the exact limits $T \rightarrow \infty$, $\epsilon \rightarrow 0$ in this expression, it will diverge independently of the order in which the limiting processes are done. Let us consider a world in which the parameters $T$ and $\epsilon$ differ from their perfect limiting values. These parameters have different physical meanings: the value of $T$ reflects “the duration of history” of the source particles, while the value of $\epsilon$ is responsible for causality. If $\epsilon = 0$, the interaction due to the hyperbolic field propagates strictly at the speed of light along the cones. Small deviations of $\epsilon$ from zero correspond to the picture in which the cones are slightly “blurred”. Thus the parameter $\epsilon$ acquires the meaning of an additional “fundamental variable” which could be formally described as

$$\epsilon = \delta c/c, \quad (25)$$

where $\delta c$ is the absolute uncertainty of the speed of light $c$ (a “fundamental constant”). To describe such a world which has more general properties than has Minkowski space–time in SR, it would be natural to start by considering not the limit of the expression (24) where $T \rightarrow \infty$, $\epsilon \rightarrow 0$, but rather its asymptotic form under these conditions. Restricting ourselves to a couple of first terms of the corresponding expansions (and cutting inessential additive constants), we obtain:

$$I(a, \epsilon) \overset{\text{as}}{=} \frac{\epsilon T}{r} + 2 \ln r + O(\epsilon^3) + O((T/r)^5). \quad (26)$$

The expression (26) can be divided into the logarithmic part and the Coulomb part. The latter is retained due to the finite value of the product $\epsilon T$, which is responsible for a specific balance between the history duration and causality.

With (16), the asymptotic approximation of the final expression for the interaction energy of the pair of particles–sources at rest is

$$\phi_{12}(r) \overset{\text{as}}{=} \frac{\alpha_1 \alpha_2}{r} + \lambda_1 \lambda_2 \ln r, \quad (27)$$

where $\alpha_3 = \lambda_1 \sqrt{\epsilon T}/2$ are the Coulomb charges, $\lambda_i$ are the logarithmic charges which coincide with the linear density of the original hyperbolic charge. The function (27) is plotted in the figure.

The Coulomb part of the potential (27) dominates at small distances, the logarithmic one dominates at large distances. According to the asymptotic theory described here, the “small distances” are defined by the natural condition $r \ll \epsilon T$. To make a rough agreement with observations, let us set $\epsilon \lesssim 10^{-10}$ (the present accuracy of speed of light measurements), $T \gtrsim 10^{24}$ s (the today’s notion for the time scale of the Universe existence). Then the Coulomb domain can be determined by the inequality $r \ll 10^{14}$ m, which confidently covers the Solar system scale. On the other hand, the logarithmic part inevitably dominates on the cosmological scale. As can be easily demonstrated, the logarithmic potential naturally ensures the flat nature of rotation curves related to a common massive center, making it unnecessary to refer to the dark matter concept. Indeed, Newton’s second law for the attractive force in rotary motion is $\sim 1/r$, and we obtain:

$$\frac{v^2}{r} \sim \frac{A}{r} \Rightarrow v \sim \text{const}. \quad (28)$$

We would like to stress that our approach eliminates the need for the dark matter concept in principle, as is also the case with theories like MOND (Modified Newton Dynamics) [4] or AGD (Anisotropic Geometrodynamics) [5, 6]. Contrary to MOND, we do not modify the form of Newton’s second law, and unlike AGD, we do not rely mainly on geometry. Our approach deals with a principally new definition of an elementary physical object and with a modified fundamental law of interaction between such objects.

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