Chiral Symmetry Restoration and $Z_N$ Symmetry

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September 12, 2018

We demonstrate that chiral symmetry restoration in quenched finite temperature QCD depends crucially on the $Z_3$ phase of the Polyakov loop $\mathcal{P}$. This dependence is a general consequence of the coupling of the chiral order parameter to the Polyakov loop. We construct a model for chiral symmetry breaking and restoration which includes the effect of a nontrivial Polyakov loop by calculating the effective potential for the chiral condensate of a Nambu-Jona-Lasinio model in a uniform temperature dependent $A_0$ gauge field background. Above the deconfinement temperature there are three possible phases corresponding to the $Z_3$ symmetric phases of the Polyakov loop in the pure gauge theory. In the phase in which $\text{tr}_c(\mathcal{P})$ is real and positive the first order deconfining transition induces chiral symmetry restoration in agreement with simulation results. In the two phases where $\text{Re}[\text{tr}_c(\mathcal{P})] < 0$ the sign of the leading finite temperature correction to the effective potential is reversed from the normal phase, and chiral symmetry is not restored at the deconfinement transition; this agrees with the recent simulation studies of Chandrasekharan and Christ. In the case of $SU(N)$ a rich set of possibilities emerges. The generality of the mechanism makes it likely to occur in full QCD as well; this will increase the lifetimes of metastable $Z_3$ phases.

PACS numbers: 11.10.Wx 11.30.Rd 11.38.Aw

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I. INTRODUCTION

Restoration of chiral symmetry is observed at high temperatures in simulations of both quenched and full QCD. In interesting recent work Chandrasekharan and Christ have shown through simulation that the chiral properties of quenched finite temperature QCD depend strongly on the $Z_3$ phase of the Polyakov loop $\mathcal{P}$. Only in the phase where $Re[\text{tr}_c(\mathcal{P})] > 0$ and $Im[\text{tr}_c(\mathcal{P})] = 0$ is chiral symmetry restored at the deconfinement transition. In the other two $Z_3$ phases, chiral symmetry is not restored. In this letter we provide a theoretical explanation for these results and make predictions for the general case of $SU(N)$.

The physical mechanism which couples the chiral condensate to the Polyakov loop is simple: finite temperature corrections to the quark determinant involve quark trajectories which are topologically non-trivial in the Euclidean temporal direction. Such trajectories are naturally weighted by powers of the Polyakov loop. When $Re[\text{tr}_c(\mathcal{P})]$ changes sign, the sign of the leading finite temperature correction to the effective potential for the chiral condensate changes as well.

In our efforts to understand the results of Chandrasekharan and Christ, we have studied an effective $U(N_f) \times U(N_c)$ chirally invariant Nambu-Jona-Lasinio (NJL) model coupled to a uniform temperature dependent $A_0$ gauge field. NJL models have been analyzed in great detail, including their finite temperature properties. We extend these models by allowing a dependence on the uniform external gauge field $A_0$. The $A_0$ influence on the phase structure of this model manifests itself solely through the Polyakov loop. The Polyakov loop is taken to be identically zero for temperatures below the deconfinement temperature $T_d$ and jumps to a nonzero value at $T_d$, as in pure $SU(3)$ gauge theory. The deconfining phase transition is known to be first order for $SU(3)$ and believed to be first order for all $SU(N)$ with $N > 3$; $SU(2)$ exhibits a second order phase transition and must be considered separately.

We use the effective potential formalism to study our extended model. Since closed quark loop effects are by definition absent in quenched QCD, it may at first glance seem strange to consider an effective potential which contains a contribution from the quark determinant.
However, consider diagram (a) in Fig. 1. It is one of an entire class of diagrams which presumably contribute to chiral symmetry breaking in the quenched approximation. It has long been argued that such diagrams are treated approximately in NJL models by the interaction shown in diagram (b) \[4\]. If we regard the effective potential as simply a tool to derive the more fundamental gap equation for the chiral condensate, then the potential must include terms such as the one shown in diagram (d) of Fig. 1 since they are obtained from Fierz transformations of quenched QCD terms like the one shown in diagram (c).

II. EFFECTIVE POTENTIAL

We begin with a Euclidean action of the form

\[
\mathcal{L}_E = \bar{\Psi}(\not{\partial} + m)\Psi - G \sum_{a=0}^{N_f^2-1} \left[ (\bar{\Psi}\lambda^a\Psi)^2 + (\bar{\Psi}i\gamma_5\lambda^a\Psi)^2 \right]
\]  \hspace{1cm} (2.1)

where the quark field \(\Psi\) carries flavor and color \[3\]. The \(\lambda^a\) are the generators of \(U(N_f)\). The covariant derivative reflects the presence of a uniform background temporal \(SU(N_c)\) gauge field:

\[
D_\mu = \partial_\mu + igA_\mu(x) \quad \text{with} \quad A_\mu(x) = \delta_{0\mu}A_0.
\]  \hspace{1cm} (2.2)

Let \(S_0\) denote the propagator for a free quark with mass \(m\), and let \(S\) denote the quark propagator of the interacting theory. Following Cornwall, Jackiw, and Tomboulis \[3\], an effective action \(\Gamma(S)\) for the theory in Eq. (2.1) is given by

\[
\Gamma(S) = \text{Tr} \left[ \ln \left( S_0^{-1}S \right) \right] - \text{Tr} \left( S_0^{-1}S - 1 \right) - G \sum \text{(diagrams)},
\]  \hspace{1cm} (2.3)

where the diagrams are shown in Fig. 1 with each internal fermion line being associated with the propagator \(S\). The two rightmost diagrams sum to zero if \(S(x, x)\) is a scalar in spinor space. In the presence of a uniform background \(A_0\) gauge field this scalar property does not hold at finite temperature. For simplicity, however, we consider only the first diagram. This is the Hartree approximation and the leading term in a \(1/N_c\) expansion. It follows that
\[ \Gamma(S) = \text{Tr} \left[ \ln \left( S_0^{-1}S \right) \right] - \text{Tr} \left( S_0^{-1}S - 1 \right) - 2G \int_0^\beta dt \int d^3x \sum_f \{ \text{tr}_{cd} [S_f(x,x)] \}^2 \]  

(2.4)

where the sum is over all flavors and the subscripted trace is over Dirac and color indices alone. \( \beta \) is the inverse temperature. The equation for the propagator is

\[ S_f^{-1}(x,y) = S_0f^{-1}(x,y) + 4G \text{tr}_{cd} [S_f(x,x)] \delta(x,y). \]  

(2.5)

Letting \( \sigma_f = \text{tr}_{cd} [S_f(x,x)] \), the quark mass of a single flavor in the interacting theory is given by \( M_f = m_f + 4G \sigma_f \). Conventionally, \( m_f \) and \( M_f \) are considered to be the current and constituent quark masses, respectively.

The phase structure of the model is conveniently studied using the effective potential

\[ V(\sigma, A_0, T) = \sum_f \left\{ 2G\sigma_f^2 - 2 \sum k_0 \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \text{tr}_c \ln \left( [k_0 + gA_0]^2 + \omega_{f\vec{k}}^2 \right) \right\}, \]  

(2.6)

where \( \omega_{f\vec{k}} = \sqrt{k^2 + (m_f + 4G\sigma_f)^2} \), and the sum over \( k_0 \) is the mode sum over Matsubara frequencies. Evaluating this mode sum and discarding an irrelevant constant,

\[ V(\sigma, A_0, T) = \sum_f \left[ 2G\sigma_f^2 - 2N_c \int \frac{d^3k}{(2\pi)^3} \omega_{f\vec{k}} \right. \]

\[ - \frac{2}{\beta} \sum_{n=1}^\infty \frac{(-1)^n}{n} \text{tr}_c \left( P^n + P^{\dagger n} \right) \int \frac{d^3k}{(2\pi)^3} e^{-n\beta\omega_{f\vec{k}}}, \]  

(2.7)

where \( P = \exp(i\beta gA_0) \) is the Polyakov loop associated with the background field \( A_0 \). The second term in Eq. (2.7) is an integral representation of the zero-temperature quark determinant which sums all possible zero-point energies. The infinite series of integrals in the third term represents the finite temperature corrections. Intuitively, each order in \( n \) is associated with quark paths that wind around space-time in the temporal direction a net number of times \( n \). This is essentially an image expansion of the finite temperature quark determinant \[6,7\]. The standard NJL model is recovered by setting \( A_0 \) to zero.

The evaluation of the quark determinant at finite temperature is slightly subtle. After regularization, the zero-temperature portion of the determinant contains an \( M^4 \log(M) \) term. This logarithmic term is cancelled by a similar term from the finite temperature contribution; every order in the summation over \( n \) contributes to this cancellation. On the
other hand, the integrals associated with each order in $n$ can be performed analytically, with no need for the imposition of a cutoff, resulting in modified Bessel functions $K_n$. It is tempting to approximate the finite temperature correction by simply keeping the first few terms in the series. However, this approximation fails to reproduce the correct behavior of the effective potential for small constituent masses and can lead to unphysical behavior.

The image expansion nevertheless remains an effective tool for understanding the role of $Z_N$ phases in critical behavior. The basic physics for the change in critical behavior associated with the Polyakov loop is given by the $n = 1$ term in Eq. (2.7). Terms higher order in $n$ are suppressed by powers of $\exp(-M/T)$, making this a reliable approximation when $T << M$. Keeping only the $n = 1$ term, the effective potential can be written as

$$V(\sigma, A_0, T) = \sum_f \left\{ 2G\sigma_f^2 - 2N_c \int \frac{d^3 \vec{k}}{(2\pi)^3} \omega_f \frac{2N_c}{\pi} \int_0^\Lambda dk k^2 \omega_{f k} \right\}.$$  

The crucial point is that the sign of the $n = 1$ finite temperature correction to $V(\sigma, A_0, T)$ changes with the sign of $\text{tr}_c(P + P^\dagger)$. If the effect of finite temperature corrections is to raise the normal minimum relative to $M = 0$, phases with $\text{Re}\{\text{tr}_c(P)\} < 0$ will instead lower this minimum. This suppresses chiral symmetry restoration in these phases.

A quantitative demonstration is best performed with the full effective potential. It is convenient to introduce a further approximation to powers of the Polyakov loop:

$$\text{tr}_c(P^n) \approx N_c \left( \frac{\text{tr}_c P}{N_c} \right)^n.$$  

This approximation neglects the formation of triality zero states in the gluon plasma. That is, each quark moves independently in its own fixed background. We also regulate $V(\sigma, A_0, T)$ by introducing a non-covariant cut-off, $\Lambda$. Denoting the trace of the Polyakov loop by $\phi$, Eq. (2.7) now can be resummed as

$$V(\sigma, \phi, T) = \sum_f \left\{ 2G\sigma_f^2 - \frac{N_c}{\pi^2} \int_0^\Lambda dk k^2 \omega_{f k} \right\}.$$  

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+ \frac{N_c}{\pi^2 \beta} \int_0^\Lambda dk k^2 \ln \left[ 1 + e^{-\beta \omega_f k} \left( \frac{\phi + \phi^*}{N_c} \right) + e^{-2\beta \omega_f k} \left( \frac{\phi^* \phi}{N_c^2} \right) \right]. \tag{2.10}

In this approximation, if $\phi$ is zero, all finite temperature effects are absent from the potential.

Other approximations are also defensible. Consider

$$V(\sigma, \phi, T) = \sum_f \left\{ 2G\sigma_f^2 - \frac{N_c}{\pi^2} \int_0^\Lambda dk k^2 \omega_f k \right.$$ 

$$+ \frac{N_c}{\pi^2 \beta} \int_0^\Lambda dk k^2 \ln \left[ 1 + e^{-\beta \omega_f k} \left( \frac{\phi + \phi^*}{N_c} \right) + e^{-2\beta \omega_f k} \right] \right\}, \tag{2.11}$$

which differs from the previous expression only in the last term, where the Polyakov loop for the quark is cancelled against its inverse for the antiquark. In this approximation, some finite temperature effects of quark-antiquark pairs are taken into account. In fact, even when $\phi = 0$, there is a chiral symmetry restoration transition at exactly twice the critical temperature of the ordinary NJL model.

### III. DISCUSSION AND NUMERICAL RESULTS

Following Ref. [8], we determine the noncovariant cut-off $\Lambda$ and the constituent quark mass $M$ by fixing the chiral condensate and the pion decay constant at zero temperature: $\sigma = (246.7 \text{ MeV})^3$ and $f_\pi^2 = (93 \text{ MeV})^2$. For the remainder of this letter we will use the numerical solution $M = 335 \text{ MeV}$, $\Lambda = 631 \text{ MeV}$, and $G\Lambda^2 = 2.2$, which assumes a current quark mass $m = 4 \text{ MeV}$ and $N_f = 2$ [8, 11].

The standard NJL model is recovered by setting $\phi = N_c$ in Eq. (2.10). Chiral symmetry is restored by a second-order phase transition; with the $\Lambda$ and $G$ given above, the critical temperature is $T_c = 194.6 \text{ MeV}$. Our model will behave as the zero-temperature solution of the standard NJL model until the Polyakov loop develops an expectation value; in particular, chiral symmetry will remain broken while $\phi = 0$. Thus, in our model the critical temperature for chiral symmetry restoration $T_c$ will always be greater than or equal to the temperature for deconfinement $T_d$. We have shown recently in Ref. [11] that the unquenched version of this model has $T_c$ strictly greater than $T_d$ for certain ranges of phenomenological parameters.
However, in lattice simulations it appears that $T_c = T_d$. Requiring this behavior places a mild constraint on this model as shown in Fig. 3. At this time, the constraint is not a strong test of the model, because $\phi$ is not known: the Polyakov loop is multiplicatively renormalized in a way that depends on the regularization scheme. In particular, lattice simulations do not directly measure $\phi$.

If the Polyakov loop is in one of the $Z_3$ phases of QCD in which $\phi + \phi^* < 0$, the potential $V(\sigma_f, \phi, T)$ will not have a minimum at $\sigma_f = 0$, unless the temperature is significantly greater than the $T_c$ of the corresponding real phase. This is precisely the behavior observed in quenched QCD simulations [1]. In Fig. 3 we compare the effective potentials of the real and complex $Z_3$ phases for $\phi = 2.0$ and $T = 252.1$ MeV to the zero-temperature effective potential where $\phi = 0$. The relative depth of the well in the complex phase has increased as expected given the change in sign of $\phi + \phi^*$.

The general case of a theory with $SU(N)$ gauge fields can also be considered. The mechanism will be the same whether the full theory or a phenomenological, low energy approximation is studied. The sign of the most important finite temperature correction to the effective potential depends on the sign of the real part of $\phi$. In general, the temperature dependence of the chiral condensate will be different in different $Z_N$ phases. Only phases in which the Polyakov loops are related by complex conjugation will have the same behavior. Thus, for both $SU(2N)$ and $SU(2N + 1)$, there will be $N + 1$ distinct behaviors.

The finite temperature behavior when $\phi_d$ is purely imaginary is determined by higher order terms in the image expansion [see Eq. (2.7)]. In $SU(4)$, scaling the temperature by a factor of two yields a potential identical to the original NJL model, except the remaining exponential in Eq. (2.10) is multiplied by a factor of $|\phi_d|^2/N^2_c$. Naively, this would imply that the critical temperature is more than double that of the phase where $\phi$ is real and positive.

In the high temperature phase of full QCD, the $Z_N$ phases are not equivalent; quark effects lift the $Z_N$ degeneracy and only the phase in which the Polyakov loop is real and positive is stable. The other phases are metastable, at least in the standard Euclidean
formalism. The action of the bounce solution has been calculated under the assumption that chiral symmetry has been restored in all phases \[12\]. However, it seems likely that the same mechanism discussed here for the quenched approximation is also operative in the full theory. Because the strength of the $Z_N$ symmetry breaking decreases with increasing mass, the absence of chiral symmetry breaking in those phases will tend to decrease the action of the bounce solution, and therefore increase the lifetime of the metastable states. This will also modify the thermodynamic properties of these metastable phases.

ACKNOWLEDGEMENTS

We wish to thank the U.S. Department of Energy for financial support under grant number DE-FG02-91-ER40628.
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FIGURES

FIG. 1. Feynman diagrams contributing to the constituent quark mass in quenched QCD (a) and the NJL model (b) and related diagrams contributing to the effective potential (c and d, respectively).

FIG. 2. Feynman diagrams contributing to the effective action.

FIG. 3. Boundary in the $\phi, T$-plane between parameter values which yield one phase transition and parameter values which yield two. Only points above the curve are compatible with simulation results.

FIG. 4. Comparison of the effective potentials of the real and complex $Z_3$ phases for $\phi = 2.0$ and $T = T_\chi(\phi) = 252.1$ MeV to the zero-temperature effective potential where $\phi = 0$. 
$\phi_d = 2.0$

$T = 252.1 \text{ MeV, Complex Phase}$

$T = 0.0 \text{ MeV}$

$T = 252.1 \text{ MeV, Real Phase}$