Surface wave scattering at nonuniform fluid interfaces

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Effects of spatially varying interfacial parameters on the propagation of surface waves are studied. These variations can arise from inhomogeneities in coverage of surface active substances such as amphiphilic molecules at the fluid/gas interface. Such variations often occur in phase coexistence regions of Langmuir monolayers. Wave scattering from these surface inhomogeneities are calculated in the limit of small variations in the surface parameters by using the asymptotic form of surface Green’s functions in the first order Born approximation. When viscosity and variations in surface elastic moduli become important, modes other than transverse capillary waves can change the characteristics of propagation. Scattering among these modes provides a mechanism for surface wave attenuation in addition to viscous damping on a homogeneous surfactant covered interface. Experimental detection of waves attenuation and scattering is also discussed.

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I. INTRODUCTION

Investigation of interfacial phenomena has attracted much attention, especially if interfacial properties are modified by a surface active material. Many studies involve amphiphilic molecules at the fluid/air interface with monolayer or less coverage. Small amounts of material at an interface can drastically alter wave propagation and also lead to complex phase behavior. Recent experiments with fluorescence and Brewster angle microscopies have sought to elucidate the structure of monolayer films of fatty acids, cholesterol, and phospholipids. A myriad of phase transitions of the surface film are directly observable as well as theoretically predicted.

These studies often reveal high surfactant concentration domains of various shapes and sizes in the coexistence regions. Figures 1(a) and 1(b) show Brewster angle microscope images of circular and striped domains of triglycerides in the Liquid Condensed/Liquid Expanded and Liquid Expanded/Gaseous “coexistence regions” respectively. The figures span about 1mm. Lighter regions correspond to regions of higher surfactant concentrations. This type of coverage is often associated with a phase coexistence region not unlike that of the gas-liquid phase transition. These first-order like transitions can be described approximately by a thermodynamic Maxwell construction across mechanically unstable regions of the equation of state. External electric fields can also be used to generate local interfacial surfactant concentration gradients. If nonlocal interactions such as dipole forces are important, regular modulated phases appear in the usual two-phase coexistence region.

Experiments have also investigated changes in capillary wave propagation caused by insoluble surfactants at an air-water interface. Wave damping experiments on film covered air/water interfaces show a remarkable increase in the damping rate above that caused by viscous bulk fluid dissipation. Even a single monolayer of surface active substances, such as amphipilic fatty acids, increases the damping rate by as much as five fold by altering the flow trajectories beneath the surface. In addition, at monolayer concentrations, the damping coefficients appear to be sensitive to nonuniform surface coverages. At concentrations where an apparent coexistence of surfactant phases occurs, the damping rate peaked at five or six times that due to viscous dissipation.

Attempts to explain the propagation of water waves covered with surfactant have considered the homogeneous mechanical properties of floating films. Although the theories of Dorrenstein, Levich, and Luccassen all predict a maximum in the damping factor, they are incomplete since they rely on surface parameters which are not well defined in a nonuniform surface. Though a peak in damping rate is experimentally observed, these systems are almost always inhomogeneous. As argued in Refs. (11) and (12), wave propagation should depend on the heterogeneous nature of the interface. The details of this surface inhomogeneity depend on how the film was applied as well as the phase diagrams of these surface materials.

Wave scattering off of surface heterogeneities can serve as an additional damping or attenuation mechanism. Much as light is attenuated in a turbid medium, surface waves can be scattered by inhomogeneities like those in Fig. 1. Gravity wave scattering from a discontinuity in the trough depth has been calculated and extended to multiple scattering and wave localization in a random bottom by Belzons. Single scattering of gravity waves off obstacles or depth discontinuities has been well studied and has been applied to multiple scattering of gravity waves off obstacles protruding from the water surface. Work done in this area has usually been in the context of oceanography and has focussed on gravity waves propagating in the presence spatial variations of the boundaries, particularly the finite depth bottom.
We will consider capillary waves scattering off variations in parameters such as surface elastic constants and surface tension. An exact solution of water wave scattering from discontinuities in boundary condition parameters can be found for very special cases in this paper we will use approximation methods which are valid for small variations in surface tension and interfacial modulus.

In the next Section, we present a model for an inviscid, infinite depth substrate fluid with a spatially varying surface tension. Regions of sharply different surface tensions are stabilized by line tensions at their boundaries. This line tension could be supplied, for example, by a waxed circular thread floating at the interface and separating a surfactant rich monolayer from a clean fluid interface. This model problem leads naturally to a Green’s function describing the response due to a point normal force acting on a uniform surface.

In Section III the asymptotic limit of this Green’s function is used in a Born approximation to describe the scattering of potential flow by a circular domain of different surface tension. The scattering function is used to calculate the attenuation of a plane capillary wave via a scattering cross section.

The effects of bulk viscosity and surface viscoelasticity are included in Section IV. Variations in surface compressibility are the primary source of scattering in this context since the surface tensions of coexisting island phases like those in Fig. 1 are nearly equal. Here the tangential stresses mix both potential and vorticity fields. The lowest order Green’s function matrix is derived. The analysis shows the existence of at least three distinct surface modes: transverse, “in-plane” longitudinal, and “in-plane” shear.

Scattering via the Born approximation for a viscoelastic surface is calculated in Section V using the Green’s functions of Section IV. The dissipation of energy into the various channels is also considered. Wave amplitude and energy attenuation due to these processes is discussed in Section VI, where experimental implications are also considered.

II. MODEL FOR SURFACE TENSION VARIATION ON AN INVISCID FLUID

In this section we investigate capillary wave propagation in the presence of surface tension inhomogeneities with an inviscid Newtonian fluid occupying the half space $z \leq 0$. This first model is somewhat artificial because if one hypothesizes a surface tension discontinuity due to two distinct surfactant phases at an interface, one phase would immediately grow at the expense of the other to reduce the free energy. With short ranged interactions only, the surface tensions of two phases coexisting in equilibrium are necessarily equal. The surface tension changes across modulated phases with large scale structures determined by dipole-dipole interactions are also small, vanishing as the reciprocal island dimension for the configurations shown in Fig. 1. Allowing for fixed surface tension variations is nevertheless instructive and might even be realizable experimentally over a reasonable time period by confining one phase of an insoluble monolayer inside a flexible barrier of negligible stiffness.

The mechanical stress balance at the interface in the presence of a spatially varying surface tension $\sigma(x)$ reads

$$\hat{n} \cdot (\Pi_+ - \Pi_-) = \sigma(\vec{r}) \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \hat{n} - \nabla_\perp \sigma(\vec{r}), \quad (2.1)$$

where $\hat{n}$ is the unit normal vector to the interface, $\nabla_\perp$ is a tangential gradient on the interface, and $R_1$ and $R_2$ are the local principle radii of curvature of the interface. $\Pi_+$ and $\Pi_-$ denote the stress tensor above and below the interface respectively. On an inhomogeneous surface, $\Pi$ and $\sigma$ may depend on the surface coordinates; the projection of these coordinates onto the $x - y$ plane is denoted by $\vec{r}$. For ideal fluids, the stress tensor is proportional to the pressure. In the Monge representation, the linearized normal component of Eq. (2.1) is,

$$\pi(z = 0) = \sigma(\vec{r}) \nabla_\perp^2 \eta(\vec{r}), \quad (2.2)$$

where $\eta$ is the vertical displacement of the interface from its equilibrium position at $z = 0$, and $\nabla_\perp$ is a gradient in the $x - y$ plane. Eq. (2.2) is the usual dynamic boundary condition which determines the dispersion relation for surface waves. Here, $\pi$ is the jump in dynamic pressure across the interface. We assume that the tangential stresses are balanced by surface forces such as line tension.

For an ideal fluid, the motion will be irrotational and the velocity field can be expressed via a potential,

$$v_i = \partial_i \phi(\vec{r}, z) \quad (2.3)$$

satisfying $\nabla^2 \phi(\vec{r}, z) = 0$ in the bulk fluid. At the surface we have,

$$\partial_z \eta(\vec{r}) \simeq \partial_z \phi(\vec{r}, z = 0), \quad (2.4)$$

and the Navier-Stokes equations lead to,

$$\pi(\vec{r}, z) = -\rho \partial_t \phi(\vec{r}, z) - g \eta(\vec{r}), \quad (2.5)$$

where $\pi(\vec{r}, z)$ is the fluid pressure, $\rho$ is the fluid density and $g$ the gravitational constant. Henceforth, gravity will be neglected ($g = 0$). The equation for the velocity potential at the surface reads,

$$-\partial_t^2 \phi + (\sigma_0 + \delta \sigma(\vec{r})) \nabla_\perp^2 \phi(\vec{r}, z) \bigg|_{z=0} = 0 \quad (2.6)$$

We now introduce a harmonic time dependence $\omega$ to all quantities and set $\rho \equiv 1$. Upon adding a unit $z$-component delta function force to Eq. (2.2), we are led to define a surface Green’s function which solves
\[ [\omega^2 + (\sigma_o + \delta\sigma(\vec{r})) \nabla^2 \partial_z] G(\vec{r}, 0) = -i\omega \delta(\vec{r}). \tag{2.7} \]

The deviation of the spatially varying surface tension from the constant value \(\sigma_o\) is denoted by \(\delta\sigma(\vec{r})\). Equation (2.7) can be solved for \(\delta\sigma \equiv 0\) by expanding \(G_0\) in a general Fourier integral,

\[ G_0(\vec{r}, z, \omega) = \int \frac{d^2k}{(2\pi)^2} G_0(k) e^{ik \cdot \vec{r} + ik_iz}, \tag{2.8} \]

The solution to (2.7) can then be formally written as,

\[ G(\vec{k}) = G_0(k) + G_0(k) \int \frac{d^2q}{(2\pi)^2} q^2 |q| \delta\sigma(\vec{k} - \vec{q}) G(\vec{q}) \tag{2.9} \]

with

\[ \sigma_0 G_0(k) = (k^2|k| - p^3)^{-1} \tag{2.10} \]

where \(p \equiv (\omega^2/\sigma_0)^{1/3}\) is real and positive. This transverse mode described by \(G_0(k)\), i.e., capillary waves propagate at \(k = \pm p\).

### III. SCATTERING FROM A SINGLE CIRCULAR DOMAIN

In this section, the scattering of a plane inviscid capillary wave from a single circular domain with a sharp surface tension discontinuity is calculated. The surface tension inside a circular domain of radius \(a\) is \(\sigma_1 = \sigma_0 + \delta\sigma\). A perimeter whose only effect is to confine the domain and maintain the discontinuity is assumed. A schematic of this scattering problem is depicted in Figure 2. The wedge at the left is typically an electrocapillary wavemaker, excited at frequency \(\omega\).

Two approximations will be made. First, a transverse capillary wave impinging on a domain will change its perimeter; we neglect line tension energy of the confining loop in comparison to the areal stretching energy of the surface. Second, we will assume a frozen variation in surface tension due to a fixed domain. This last simplification requires weak scattering, i.e., \(ka \ll 1\), where \(k\) is the wavevector and \(a\) is the domain radius.

For a single domain, a two dimensional analog of the Young-Laplace equation can be defined for circular domain of radius \(a\) with a sharp surface tension discontinuity,

\[ \sigma_0 - \sigma_1 = \frac{\gamma}{2a} \tag{3.1} \]

where \(\sigma_0\) and \(\sigma_1\) are the surface tensions inside and outside the domain respectively and \(\gamma\) is the line tension acting through the circular domain boundary. Equation (3.1) is actually the tangential component of (2.1) when bulk viscosity is neglected. For small \(\delta\sigma\), the scattering is obtained in the first iteration or Born approximation.

In the inviscid case, the elastic scattering mechanism described here will be the only process resulting in attenuation of an incident plane wave in the midst of a collection of scattering centers.

By using Eq. (2.10), we can find the exact surface Green’s function, \(G_0(\vec{r})\), for a homogeneous interface,

\[ G_0(r) = \frac{i\omega}{\sigma_0} \int \frac{d^2k}{(2\pi)^2} e^{-i\vec{k} \cdot \vec{r}} - p^3 \]

\[ = \frac{-\omega}{6\sigma_0} H_0^{(1)}(p\rho) + \frac{i\omega}{\pi^2\sigma_0 p} \int_0^\infty K_0(p\rho)y^4 \, dy \tag{3.2} \]

In the above integrands, \(p_+ \equiv p + i\epsilon\) represents a small damping which removes the ambiguity in the integrals for outgoing Green’s functions. Since \(p\) is real, (3.2) is dominated by the first term at large distances (\(p\rho \gg 1\)) and the asymptotic form becomes,

\[ G_0(r) \sim \frac{\omega}{3\pi^2 p} \frac{e^{ipr + 3\pi i/4}}{\sqrt{2\pi pr}} \tag{3.3} \]

With the help of the Green’s function, \(G_0(\vec{r})\), the solution of Eq. (2.3) can be written in integral form as,

\[ \phi(r, 0) = e^{ipx} + \frac{i}{\omega} \int d^2r' \delta\sigma(r') G_0(|\vec{r} - \vec{r}'|) \nabla^2 \phi(r', 0) \tag{3.4} \]

The first term of (3.4) represents an incoming plane wave with unit magnitude at the origin of the scatterer generated by a \(z\) component force at \(x \to -\infty\) such that the power law evanescent waves represented by the second term in the last line of (3.4) have long died off. The integral term is the scattered wave due to the “potential” \(\delta\sigma(r')\). In the lowest order approximation for small \(\delta\sigma(r'), \phi(r')\) in the integral is replaced with \(\phi_{inc}(r') = e^{ipx'}\). In the far field, the potential takes the form,

\[ \phi(r) \equiv e^{ipx} + f(\theta) \frac{e^{ipr}}{\sqrt{r}} \tag{3.5} \]

where the scattering factor is the asymptotic limit of the integral in (3.4),

\[ f(\theta) = -i\sqrt{2\pi ip} \frac{a}{3} \frac{\delta\sigma}{\sigma_0} \frac{J_1(p\rho \sin \theta/2)}{\sin \theta/2} \tag{3.6} \]

Validity of this approximation is ensured when the scattered wave just outside the disc is small compared to the incident plane wave, which leads to the requirement,

\[ (pa)^3 \left( \frac{\delta\sigma}{\sigma_0} \right)^2 \ll 1. \tag{3.7} \]

In this asymptotic limit, the total scattering length is,
Upon using the capillary wave dispersion formula, \( \omega = \sqrt{\sigma \rho} \), we see that Eq. (3.8) implies that \( \Sigma_s \sim \omega^2 \), in contrast to the famous Rayleigh result \( \Sigma_s \sim \omega^4 \) for light scattering from the atmosphere.

Nonuniform coverage of air/liquid interfaces however, usually results in a collection of circular scatterers. Scattering from an uncorrelated collection of identical domains (assumed for simplicity here to be of identical size) reduces the forward coherent intensity according to,

\[
|\langle \phi(x) \rangle| \sim e^{-n \Sigma_s(p) x} \tag{3.9}
\]

where \( n \) is the number of scattering domains per area. Beer’s law, equation (3.9), applies in the single scattering regime and is valid when the typical inter-domain distance \( l \gg 2\pi/p \gg a \); our calculation for \( \Sigma_s \ll a \) only describes weak attenuation accurately. An alternative calculation which averages over surface disorder and leads to the same result is given in Appendix B.

### IV. MODEL WITH BULK VISCOSITY AND SURFACE ELASTICITY

In actual experimental systems such as the one depicted in Figure 2, surface layers have additional parameters which are spatially varying. For example, the surface compressibility will be different inside and outside the islands even if the surface tension is uniform. The restoring forces proportional to these surface parameters are coupled to the bulk viscous forces at the interface. Surface elasticity is an important factor in the propagation and attenuation of surface waves and can arise from surfactant molecules at the interface. Fluid motion in the interfacial plane tends to locally distort the film which resists due to its surface compressive or shear elasticity and interfacial 2D viscosity.

The normal component of (2.1) now reads,

\[
(\sigma_0 + \delta \sigma(\vec{r})) \nabla^2_{\perp} v_z - i \omega \pi (\vec{r}, z) + 2 i \omega \nu \partial_z v_z \bigg|_{z=0} = -i \omega f_z(\vec{r}) \tag{4.1}
\]

where \( \sigma_0 \) is the uniform background surface tension, \( \delta \sigma(\vec{r}) \) is the spatial variation in surface tension considered earlier, \( \nu \) is the bulk fluid viscosity, and \( f_z(\vec{r}) \) is proportional to an external pressure in the \( z \)-direction.

Upon considering the intrinsic stresses of the interface, we find that the tangential forces are balanced when,

\[
\nabla_{\perp} \sigma(\vec{r}) + \nabla_{\perp} \cdot \Pi^{(2)} - \nu (\partial_z \vec{v}_{\perp} + \nabla_{\perp} v_z) \bigg|_{z=0} = \vec{f}_{\perp}(\vec{r}), \tag{4.2}
\]

where \( \vec{f}_{\perp}(\vec{r}) \) is proportional to an in-plane stress applied at the interface. The first term on the left side of (4.2) arises from gradients in surface tension and acts like a pressure term in the stress tensor of an ideal two dimensional liquid. The dissipative part of the 2D fluid stress tensor is given by \( \Pi^{(2)}(\vec{r}) \). In the remainder of the paper, we assume \( \nabla_{\perp} \cdot \Pi^{(2)} = 0 \); all of the phenomena can be adequately accounted for by \( \nabla_{\perp} \sigma(\vec{r}) \). Modifications due to dissipative and elastic contributions to \( \Pi^{(2)} \) is considered in Appendix A.

Fluid motion at the surface convects surface molecules and changes the surfactant concentration \( \Gamma(\vec{r}, t) \) locally, so we set

\[
\nabla_{\perp} \sigma(\vec{r}) = \frac{d\sigma}{d\Gamma} \nabla_{\perp} \Gamma(\vec{r}). \tag{4.3}
\]

For an insoluble material, conservation of surfactant at the interface gives

\[
\partial_t \Gamma(\vec{r}) + \nabla_{\perp} \cdot (\vec{v}_{\perp} \Gamma(\vec{r})) \simeq -i \omega \delta \Gamma(\vec{r}) + \Gamma_0 \nabla_{\perp} \cdot \vec{v}_{\perp} = 0, \tag{4.4}
\]

with \( \Gamma(\vec{r}) \equiv \Gamma_0 + \delta \Gamma(\vec{r}) \), where \( \Gamma_0 \) is the equilibrium surfactant concentration inside or outside the domain. Equations (4.3) and (4.4) can be combined to express the surface tension gradient in phase \( j \) in terms of the in-plane velocity field,

\[
\nabla_{\perp} \sigma(\vec{r}) \simeq \frac{i}{\omega} B_j \nabla_{\perp} \cdot (\nabla_{\perp} \cdot \vec{v}_{\perp}) \bigg|_{z=0}, \tag{4.5}
\]

where the bulk modulus is defined by

\[
B_j \equiv -\Gamma_0 \frac{d\sigma}{d\Gamma} \bigg|_{\Gamma=\Gamma_j} \tag{4.6}
\]

with \( j = 0, 1 \) representing \( B \) outside or inside a domain for example. We can now solve equations (4.1) and (4.2). Upon decomposing the velocity into its potential and vorticity components,

\[
v_i = \partial_i \phi + \epsilon_{ijk} \partial_j \psi_k \tag{4.7}
\]

we choose a gauge where \( \psi_z = 0 \), and define \( \beta_j \equiv \epsilon_{ijk} \psi_j \) (the vorticity is \( \Omega = \vec{\nabla} \times \vec{v} = -\vec{z} \times \nabla^2 \vec{\beta} \)) so that,

\[
\vec{v} = \nabla \phi(\vec{r}) + \left( \frac{\partial \beta_j}{\partial x_j - \nabla_{\perp} \cdot \vec{\beta}} \right), \tag{4.8}
\]

where the vector \( \vec{\beta} \) lies in the \( x - y \) plane. The fields \( \phi \) and \( \vec{\beta} \) satisfy,

\[
\nabla^2 \phi(\vec{r}) = 0 \tag{4.9}
\]

and

\[
\partial_t \vec{\beta} = \nu \nabla^2 \vec{\beta} \tag{4.10}
\]

respectively, which are solved by the expansions,
\[
\phi(\vec{r}, z) = \int \frac{d^2 k}{(2\pi)^2} \phi(\vec{k}) e^{i \vec{k} \cdot \vec{r} + |k|z} \tag{4.11}
\]
and
\[
\beta(\vec{r}, z) = \int \frac{d^2 k}{(2\pi)^2} \beta(\vec{k}) e^{i \vec{k} \cdot \vec{r} + lz} \tag{4.12}
\]
where \( l \equiv \sqrt{k^2 - \omega^2 / \nu} \). The positive roots for \( l \) and \(|k|\) are taken to ensure that \( \phi \) and \( \beta \) vanish as \( z \to \infty \).

Finally, upon using (4.11), (4.12), (4.5), (2.3) in (1.1), and taking a time derivative of (4.12), the normal and tangential stress balances at the surface \( z = 0 \) after Fourier transforming in the in-plane \( \vec{r} \) coordinate take the form,

\[
\mathbf{L}(q) \psi(\vec{q}) = \mathbf{F}(\vec{q}) + \int \frac{d^2 k}{(2\pi)^2} \mathbf{M}(\vec{k}, \vec{q}) \psi(\vec{k}) \tag{4.13}
\]

where \( \mathbf{F}(\vec{q}) \) is the \( q^\text{th} \) Fourier component of \((f_x, f_y, f_z)\). In (4.13), \( \psi(\vec{r}) \equiv (\phi, \beta_x, \beta_y) \) and \( \vec{k} \) and \( \vec{q} \) are wavevectors in the \( x-y \) plane. Spatial variations in the surface tension \( \sigma(\vec{r}) \) and interfacial bulk modulus \( B \) are included in the matrix

\[
\mathbf{M}(\vec{k}, \vec{q}) \equiv \frac{i}{\omega} \begin{bmatrix} \delta \sigma(\vec{q} - \vec{k}) k^2 |k| & \delta \sigma(\vec{q} - \vec{k}) k^2 k_x & \delta \sigma(\vec{q} - \vec{k}) k^2 k_y \\ \delta B(\vec{q} - \vec{k}) i k^2 k_x & \delta B(\vec{q} - \vec{k}) k^2 l & \delta B(\vec{q} - \vec{k}) k_x k_y \\ \delta B(\vec{q} - \vec{k}) i k^2 k_y & \delta B(\vec{q} - \vec{k}) k_x k_y & \delta B(\vec{q} - \vec{k}) k^2 l \end{bmatrix}. \tag{4.14}
\]

When the film is homogeneous, \( \mathbf{M} = 0 \).

The coefficients \( \sigma \) and \( B \) in \( \mathbf{L}(\vec{q}) \) are constants, e.g. the values \( \sigma_0 \) and \( B_0 \) outside an isolated circular domain, or mean values appropriate for a heterogeneous collection of many domains. In equilibrium, for large domains, the surface tension contrast will be small as discussed above. Hence, in the following analysis (unlike that for the nonviscous fluid), we make the simplifying assumption that the surface tension is constant in monolayers in coexistence and near equilibrium. Scattering will be solely due to the contrast in \( B \). The compressional modulus for phase \( j = (0,1), B_j \), can be measured by the slopes of the isotherms just outside the coexistence plateau as shown by the dotted lines in Fig. 3. These values will be different in the two phases even for monolayers in equilibrium.

The matrix \( \mathbf{L}(\vec{q}) \) corresponding to an interface with \( B = B_0 \) is,

\[
\mathbf{L}(\vec{q}) \equiv \begin{bmatrix} i (\omega - \frac{\sigma_0 q^2}{\omega} + 2i \nu q^2) & q_x \left[ 2i \nu l - \frac{\sigma_0 q^2}{\omega} \right] & q_y \left[ 2i \nu l - \frac{\sigma_0 q^2}{\omega} \right] \\ -iq_x \left[ \frac{4B_0}{\omega} q^2 + 2
u (l^2 + q^2) \right] + 4B_0 q^2 l + \nu (l^2 + q^2) & -q_x q_y \left[ \frac{4B_0}{\omega} q^2 l + \nu \right] & -q_x q_y \left[ \frac{4B_0}{\omega} q^2 l + \nu \right] \end{bmatrix} \tag{4.15}
\]

Upon inverting (4.13), the equation for \( \psi \) takes the form

\[
\psi(\vec{q}) = \mathbf{G}(\vec{q}) \mathbf{F}(\vec{q}) + \mathbf{G}(\vec{q}) \int \frac{d^2 k}{(2\pi)^2} \mathbf{M}(\vec{k}, \vec{q}) \psi(\vec{k}), \tag{4.16}
\]

with a matrix Green’s function \( \mathbf{G}(\vec{q}) \equiv \mathbf{L}^{-1}(\vec{q}) \). For fixed \( \omega \), the poles of the Green’s function give the wavevectors of the possible modes. For the region of parameter space most accessible to experiments, there are two poles corresponding to outgoing capillary waves. Appendix A shows that if \( \nabla \cdot \mathbf{\Pi}''(2) \neq 0 \), additional modes corresponding to in-plane motion also exist.

When viscosity is small, the “transverse” capillary wave pole is near the inviscid pole; it is shifted from \( k = p \) by a small real part and develops a small imaginary part due to viscous damping. When \( x \equiv \nu (\omega / \sigma_0)^{1/3} \ll 1 \),

\[
k_t \simeq p + \Delta = p + \Delta' + i \Delta'' \tag{4.17}
\]

The functions \( \Delta' \) and \( \Delta'' \) are plotted in Figure 4(a) as a function of \( y \equiv B_0 / \sigma_0 \).

\[
\Delta'' \simeq \frac{1}{3} \left( \frac{\omega^2}{\sigma_0} \right)^{1/3} \frac{y^2 \sqrt{x/2} - yx}{y^2 - y \sqrt{2x} + x} \tag{4.19}
\]

are plotted in Figure 4(a) as a function of \( y \equiv B_0 / \sigma_0 \).

For \( x \ll 2B / \nu^3 \) the longitudinal pole is approximately,

\[
k_l \simeq \nu^{1/4} (l^2)^{3/4} e^{\pi i / 8} + \ldots \tag{4.20}
\]

The relative dissipation between the transverse and longitudinal modes, \( \text{Im} \{k_t\} / \text{Im} \{k_l\} \), are plotted in Figure 4(b). The “shear” mode found in the Appendix A can usually be omitted since it has a large imaginary part and is highly damped. The asymptotic form of the Green’s function including the transverse and longitudinal modes is explicitly displayed in Appendix C.

V. FIRST ORDER SCATTERING FROM VISCOELASTIC DOMAINS

To consider the effects of surface elasticity, the inclusion of a nonzero bulk viscosity is necessary to satisfy the tangential force balance equation. Thus, there is inelastic wave attenuation even without surface inhomogeneities and elastic wave scattering. In this Section, we
calculate the scattering of surface waves from domains of different modulus $B$ only. The variation in the modulus, $\delta B = B_1 - B_0$, of domains near equilibrium can be estimated from the isothermal construction in Fig. 3. Since the periods of the surface waves are probably smaller than the molecular diffusion times, the relevant quantity is the isentropic modulus rather than the isothermal modulus found from Fig. 3. We assume equality of surface tensions inside and outside the domains.

Incident waves with various amounts of transverse and longitudinal (and possibly shear) mode components can be generated with surface forces at various angles from the normal. The incoming wave we consider will be that generated by a force which acts solely in the $z$ direction at the surface. The relative amounts of transverse and longitudinal components is fixed by the coefficients of the relevant Green’s functions given in Appendix C.

In the large $r$ limit, Eq. (4.16) yields the form,

$$
\psi(\vec{r}) \simeq \psi_0(x) + \sum_{a,b=t,l} \tilde{g}_{a,b}(\theta)e^{ik_a x} \frac{e^{ik_b r}}{r} \tag{5.1}
$$

where $\psi_0(x)$ is a plane normalized combination of transverse and longitudinal waves generated by a force per length $f_z$ with infinite extent in the $y$ direction. The wave induced by the wavemaker will initially have transverse and longitudinal modes with both $\phi$ and $\beta_2$ components. By integrating the asymptotic forms of the Green’s function from Appendix C with respect to the direction along the wavemaker blade, we can determine $\psi_0$ to leading order in $\nu$. The vertical interfacial displacement generated by a plane wavemaker,

$$
\eta(x) \simeq \left[ -\frac{i}{3\sigma_0 p} e^{ik_1 x} + \frac{e^{5\pi i/8} \nu^{5/4}}{2B_0^{3/2}\omega^{1/4}} e^{ik_2 x} \right] f_z, \tag{5.2}
$$

determines the nature of the waves incident on the scattering domains. If $Im\{k_i\} \ll Im\{k_1\}$, incident waves are mainly transverse capillary waves; if $Im\{k_i\} \gg Im\{k_1\}$, the incident waves are predominantly longitudinal waves. The ratio of the imaginary parts of the transverse and longitudinal modes are plotted in Fig. 4(b) for various values of $B$.

The form factor in Eq. (5.1) describes a complicated scattering process. The $\phi$ and $\beta_2$ components of $\psi$ scatter among themselves; as suggested by the indexes $a,b = t,l$ on the scattering amplitudes $g_{a,b}(\theta)$. The components of $\tilde{g}_{a,b}(\theta)$ governing the scattering of $\phi$ has four terms describing scattering between transverse and longitudinal modes. To lowest order in $\omega \nu^{3/2}/\sigma_0^2$ and $\omega \nu^{3}/B_0^2$ these are,

$$
g_{t,t}(\theta) \simeq e^{5\pi i/8} a^{2} \sqrt{\pi p} p^{7/4} \left( \frac{2B_0 - \sigma_0}{\sigma_0} \right)^{5/4} \left( \frac{\delta B}{B_0} \right) \frac{J_1(qa)}{qa} \cos \theta, \tag{5.3}
$$

$$
g_{t,l}(\theta) \simeq e^{\pi i/4} \sqrt{2\pi k_1 (pa)} \frac{3\sigma_0 (2B_0 - \sigma_0) \nu^2}{4B_0^3} \left( \frac{\delta B}{B_0} \right) \frac{J_1(qa)}{qa} \cos \theta \tag{5.4}
$$

and,

$$
g_{t,l}(\theta) \simeq a^2 p^{9/4} \sqrt{\pi} \frac{\sigma_0^{5/4} \nu^{3/2}}{2B_0} \left( \frac{\delta B}{B_0} \right) \frac{J_1(qa)}{qa} \cos \theta. \tag{5.5}
$$

In the above equations, $q \equiv \sqrt{k_1^2 + k_2^2 - 2k_1 k_2 \cos \theta}$. Each of these scattering functions correspond to a particular intermode scattering. For example, $g_{t,t}(\theta)$ represents the scattering of an impinging transverse wave into a longitudinal wave. The two limiting cases depicted in Figures 5(a) and 5(b) will be of interest.

In Fig. 5(a), the longitudinal mode is more dissipative than the transverse mode. For modes described by Eqs. (4.17), (4.18), (4.19), and (4.20), this first case corresponds to $Im\{k_1\} \gg t^{-1} \gg \Delta t$. In this case, we may assume that only transverse capillary waves reach the scattering centers. Some of the transverse amplitude, however, is scattered into the longitudinal mode which is then quickly viscously dissipated leading to enhanced inelastic attenuation of the velocity potential. For an areal density of identical island scatterers $n$, the transmitted potential decays according to

$$
\langle |\phi(x)| \rangle \sim \exp\left[-\frac{n}{2} (\Sigma_{t,t} + \Sigma_{t,l}) x - \alpha x \right] \tag{5.7}
$$

where the scattering and absorption lengths, $\Sigma_{t,t}$ and $\Sigma_{t,l}$ are given by angular integrals over $g_{t,t}(\theta)$ and $g_{t,l}(\theta)$,

$$
\Sigma_{t,t} \simeq \frac{4(pa)^{1/2} \nu}{\sqrt{\sigma_0 p}} \left( \frac{\delta B}{B_0} \right)^2 \tag{5.8}
$$

and,

$$
\Sigma_{t,l} \simeq \frac{(pa)^{1/2} \nu^{5/2} p^{3/4}}{4B_0^3 \sigma_0^{1/4}} (2B_0 - \sigma_0)^2 \left( \frac{\delta B}{B_0} \right)^2. \tag{5.9}
$$

To make these and subsequent expressions dimensionally meaningful, let $\sigma_0, B_0 \rightarrow \sigma_0/\rho, B_0/\rho$ where $\rho = 1$ g/cm$^2$ for the interface between air and water. Since $\omega \sim p^{3/2}$ for capillary waves, these scattering lengths depend on frequency and viscosity according to $\Sigma_{t,t} \sim \nu \omega^{-3/2}$, and $\Sigma_{t,l} \sim \nu \omega^{-11/6}$. Note that $\Sigma_{t,l}$ describes an elastic scattering process, which could be detected by placing a wave detector at an angle to the incident wavevector, and measuring the scattering amplitude $g_{t,t}(\theta)$ directly. $\Sigma_{t,t}$ acts
as an enhanced viscous dissipation under the conditions described above. In our approximations, $\Sigma_{l,t} \gg \Sigma_{t,l}$; the major scattering process is elastic. Only when $\sigma_0/B_0$ is very large can $\Sigma_{t,l}$ be comparable to $\Sigma_{l,t}$. The damping from viscous dissipation is described by $\alpha = \Delta''$ displayed in Eq. (4.18).

In Fig. 5(a), the opposite situation obtains, $\Delta'' > Im\{k_i\}$. The predominant mechanism is scattering of longitudinal waves. Here, the relevant scattering lengths are,

$$\Sigma_{l,t} \approx 9 \pi^2 (pa)^4 \frac{\sigma_0^{19/8} \rho^{9/8} \nu^{17/4} (2B_0 - \sigma_0)^2}{32B_0^{13/2}} \left( \frac{\delta B}{B_0} \right)^2$$

and,

$$\Sigma_{t,l} \approx \pi^2 (pa)^4 \frac{\sigma_0^{5/2} B_0^3}{8B_0^4} \left( \frac{\delta B}{B_0} \right)^2,$$

which behave as $\Sigma_{l,t} \sim \nu^{17/4} \omega^{41/12}$ and $\Sigma_{t,l} \sim (\nu \omega)^3$. The approximations $\omega \nu^3/B_0^2 \ll 1$ and $\omega \nu^3/\sigma_0^2 \ll 1$ give $\Sigma_{l,t} \gg \Sigma_{t,l}$; in this case, the predominant scattering is that of longitudinal waves scattering into transverse waves which are then quickly dissipated by the bulk viscosity. Again, only when $\sigma_0/B_0 \gg 1$ is the elastic longitudinal $\to$ longitudinal scattering comparable. Here, simple viscous attenuation of longitudinal waves behaves as,

$$\alpha = Im\{k_i\} = \frac{\sqrt{2} - \sqrt{2} \nu^{1/4} \omega^{3/4}}{2 \sqrt{B_0}}.$$ (5.12)

VI. IMPLICATIONS FOR EXPERIMENTS

Surface wave damping is expected to have experimental signatures with the techniques used in recent studies. Induced surface waves can be probed by light scattering and detecting the beam deflection or measuring the power spectrum. Alternatively, direct mechanical measurements due to the force from impinging waves are also used.

Our calculations for one scatterer can be tested by scattering an induced wave off a single isolated domain. The asymptotic radial distribution of scattered potential should follow those given in equations (3.3) and (5.1) for the elastic (inviscid) and inelastic (viscous) cases respectively. Because a single surfactant domain is rarely seen, an isolated domain may have to be constructed by placing a thin confining thread on the surface of a homogeneous monolayer and aspirating the material outside of this loop. Provided the solubility into the bulk liquid is small, this arrangement, although metastable, may be quite long-lived. In practice, the scattering from the thread loop alone may have to be considered. Electric fields have also been used to manipulate surfactant concentrations and orientations at the interface. Single circular domains of surfactant rich or depleted phases have been produced with this method.

Damping of induced height fluctuations propagating through an array of scattering domains can be directly measured by probing the amplitude as a function of distance from the source. Experimentally, the nucleation of surfactants results in domains of various sizes and shapes. Our analysis describes a monodisperse collection of scatterers but can be extended to include polydispersity by averaging over domain size distributions.

Using Eqs. (5.8) and (5.9) as guides, an estimate of the scattering effect can made. For a membrane which has only a surface tension variation, scattering attenuation above that of viscous damping will be noticeable when,

$$n \left( \frac{\delta \sigma}{\sigma_0} \right)^2 \geq \frac{\nu \omega}{a (pa)^3 \sigma_0},$$ (6.1)

The value $\delta \sigma$ is difficult to measure directly. The surface tension difference inside and outside a domain of radius $a$ is related through Eq. (3.3); however, these quantities are difficult to measure but would be expected to be small for systems near equilibrium. The plateaus in the isotherms are rarely perfectly flat due to long-ranged dipolar interactions between surface molecules and the slow relaxation to equilibrium. An experimental estimate gives line tensions on the order of $\gamma \approx 10^{-6}$ erg/cm. For domains of radius $\approx 0.1$ mm, this tension gives $\delta \sigma \approx 10^{-3}$ dyne/cm. For a fluid with $\sigma_0 \approx 70$ dyne/cm driven at a frequency $\omega = 2\pi \times 1000$ s$^{-1}$ and covered with $\approx 10^3$ scatterers per cm$^2$, Eq. (6.1) requires unphysical bulk viscosities $\nu \lesssim 10^{-13}$ cm$^2$/s before attenuation due to scattering will be noticeable.

When viscosity is explicitly included, the transverse wave attenuation given by (5.7) includes a damping term $-\Delta'' \sigma$ which has a maximum (Fig. 4(a)) as predicted by homogeneous theories of Dorrenstein and others. For the coexistence of fatty acids in the low coverage regime, the modulus $B$ is expected to be much smaller than the surface tension; thus it is not clear whether the maximum seen in experiments can be described by this peak. However, for wavelengths much larger than the typical interdomain distance, a definition of an effective $B$ may be consistent with observations. Appendix A shows that if only surface viscosities $\eta^{(2)}$ and $\zeta^{(2)}$ are important, $\Delta''$ has no predicted peak.

The attenuation effects of elastic and inelastic scattering are measured by the scattering lengths (5.8), (5.9), (5.10), and (5.11), as well as the viscous dissipation coefficients $\Delta''$ and $Im\{k_i\}$. Detection of interfacial scattering via damping is feasible when the relevant $n \Sigma$ is comparable to $\Delta''$ or $Im\{k_i\}$. In the transverse $\to$ transverse wave scattering, this condition leads to,

$$n \left( \frac{\delta B}{B_0} \right)^2 \geq \frac{\nu^{1/4} \sigma_0^{1/4}}{\pi^2 (pa)^4 \sqrt{\nu}}.$$ (6.2)
For $\omega = 2\pi \times 1000 \text{ s}^{-1}$, $\sigma_0 \sim 70 \text{ dyne/cm}$, $\nu = 0.01 \text{ cm}^2/\text{s}$, and $a \sim 1 \text{ mm}$, Eq. (1.2) yields $n(\delta B/B_0)^2 \geq 1.4$. Note that this experimentally realizable criterion for detecting scattering through wave attenuation depends strongly on $pa$, elastic scattering having a larger effect when the wavelength is smaller than the domain radius $a$. For the process involving longitudinal wave scattering pictured in Fig. 5(b), the attenuation is proportional to $n\Sigma_{1,1}$. The criterion for observing inelastic scattering assisted damping is,

$$n \left( \frac{\delta B}{B_0} \right)^2 \geq \frac{p^{5/8} B_0^{7/2}}{\nu^{11/4} \sigma_0^{17/8} \pi^2 (pa)^{3/2}}.$$  \hspace{1cm} (6.3)

Though it is possible to find parameters such that Eqs. (1.2) and (6.3) are satisfied, the accuracy of the Born approximation is questionable when $pa$ or $(\delta B/B_0)$ are too large. However, experiments do show enhanced damping when the transverse capillary wavelength is small compared to the typical domain size, $(pa > 1)$, and no observable damping above viscous dissipation for large wavelengths.

Alternatively, the scattered waves may be directly detected. The incident waves vectors are scattered in magnitude (in the inelastic case) and direction. Using light scattering, this experiment may be realized by rotating the light scattering plane an angle $\theta$ away from the direction of the incident wavevector as shown in Fig. 6. For a small number of elastic scatterers $(\nu = 0)$, the signal would be proportional to $|f(\theta)|^2$ where the scattering function is given by Eq. (1.5). At higher densities, the random scattering is expected to effectively contribute to the thermal ripplons near the source. A difference in amplitude of the thermal spectra peak measured at $\theta = \pi/2$ is expected depending on whether the wave maker is on.

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APPENDIX A: INTRINSIC SURFACE STRESS EFFECTS

When surface viscosities are included in the dissipative part of the two dimensional stress tensor $\Pi^{(2)}$, the behavior of the surface modes can change. For a two dimensional Newtonian liquid,

$$\Pi^{(2)}_{ij} = \eta^{(2)}(\partial_i v_j + \partial_j v_i - \delta_{ij} \partial_k v_k) + \zeta^{(2)} \delta_{ij} \partial_k v_k, \quad \hspace{1cm} (A1)$$

where $\eta^{(2)}$ and $\zeta^{(2)}$ are the two dimensional shear and dilatational viscosities respectively. Using (A1) in (1.2), the tangential stress balance becomes,

$$\nabla_\perp \sigma(\vec{r}) + \eta^{(2)} \nabla_\perp^2 \vec{v}_\perp + \zeta^{(2)} \nabla_\perp \cdot (\nabla_\perp \vec{v}_\perp) = \nu(\partial_t \vec{v}_\perp + \nabla_\perp v_z) = f_\perp(\vec{r}). \quad \hspace{1cm} (A2)$$

In Section IV, we neglected surface viscosities. If however, $B/\omega \ll \eta^{(2)}, \zeta^{(2)}$, then the transverse pole is given by (1.17) with

$$\Delta' \simeq \frac{\sqrt{2}}{6} \left( \frac{\omega^2}{\sigma_0} \right)^{1/3} \frac{x^{3/2} y (y + 4)}{y^2 x + y \sqrt{2} x + 1} \quad \hspace{1cm} (A3)$$

and,

$$\Delta'' \simeq \frac{1}{3} \left( \frac{\nu \omega}{\sigma_0} \right) \frac{y^2 x^2 y + 4}{y^2 x + y \sqrt{2} x + 1} \quad \hspace{1cm} (A4)$$

In the above expression, $x \ll 1$, and $y \equiv (\eta^{(2)} + \zeta^{(2)}) \omega^{2/3} / \nu \sigma_1^{1/3}$. Both $\Delta'$ and $\Delta''$ are monotonic in $y$. The longitudinal mode for $\omega \gg \nu^3 / (\eta^{(2)} + \zeta^{(2)})^2$ becomes,

$$k_l \simeq \frac{(\nu \omega)^{1/4} e^{\pi i / 8}}{\sqrt{\eta^{(2)} + \zeta^{(2)}}} + \ldots \quad \hspace{1cm} (A5)$$

and a “shear” mode appears at,

$$k_s = \frac{\nu}{\sqrt{2 \nu \eta^{(2)}}} \sqrt{1 - \frac{2 i \omega \eta^{(2)} \nu^2}{\nu^3}} \approx \frac{\nu}{\nu \nu} \left[ 1 - \frac{i \omega \eta^{(2)} \nu^2}{2 \nu^3} + \ldots \right] \quad \hspace{1cm} (A6)$$

where the last approximation assumes $\omega \ll \nu^3 / (\eta^{(2)})^2$.

We can also consider the case of a purely elastic interface; a relevant example may be a polymerized membrane on the air/water interface. For an isotropic elastic membrane, $\nabla_\perp \cdot \Pi^{(2)}$ is replaced with $-\delta F / \delta \vec{u}$ with the free energy of the elastic sheet,

$$F = \frac{1}{2} \int d^2 r \left[ \lambda(r) u_{\perp i}^2 + 2 \mu(r) u_{\perp i}^2 \right] \quad \hspace{1cm} (A7)$$

where $u_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is a two-dimensional strain tensor and $u_i$ are the in-plane displacements from equilibrium. $\lambda$ and $\mu$ are spatially varying Lamé coefficients which may be $\lambda_0$, $\mu_0$ outside the domains and $\lambda_1$, $\mu_1$ inside. With this model, the transverse and longitudinal wavevectors are given by (4.17), (4.18), (4.19), and (4.20) respectively, but with an effective interfacial modulus given by

$$B_0 = 2 \mu_0 + \lambda_0 - \Gamma_0 \frac{d \sigma}{d \Gamma} \bigg|_{\Gamma = \Gamma_0}. \quad \hspace{1cm} (A8)$$

For $\mu \neq 0$, an elastic shear mode also occurs at,
where the last approximation is valid when $\omega \gg \mu^2/\nu^2$. The modes given by (4.17), (4.18), (4.19), (4.20), and (4.21) describe a solid with vanishing shear modulus or a liquid membrane with small surface viscosities. Surface viscoelasticity of the interface can be incorporated by allowing $B$ to be complex: $B \rightarrow B + i\omega C$.

APPENDIX B: DISORDER AVERAGED GREEN’S FUNCTION $\langle G \rangle$

In this section, we develop another means of calculating the attenuation of the coherent velocity potential in the viscid model. In the limit of small variations in the surface tension and sparse uncorrelated scatterers, the result is identical with that obtained from the first Born iteration. Equation (2.9) can be represented diagrammatically as in Figure 7(a). The result of ensemble averaging over uncorrelated domains, $\langle \delta \sigma(\vec{r})\delta \sigma(\vec{r}') \rangle \simeq \epsilon \delta(\vec{r} - \vec{r}')$, is shown in Figure 7(b).

In this summation, the cross terms representing correlated scattering are neglected. Inverting the series and defining an ensemble averaged Green’s function,

$$
\langle G(k) \rangle \simeq G_0(k) \sum_{n=0}^{\infty} \epsilon G_0(q)k^3 \int_0^{\Lambda} \frac{d^2 q}{(2\pi)^2} q^2 |q| G_0(q) \\
= G_0(k) \sum_{n=0}^{\infty} \frac{\epsilon k^3}{n!} \int_0^{\Lambda} \frac{d^2 q}{(2\pi)^2} q^2 |q| G_0(q) \\
= (G_0(k))^{-1} - \frac{i \sigma_R - i \sigma I}{\sigma_0} |k| k^2 - p^3
$$

(B1)

The cut-off $\Lambda^{-1}$ gives a measure of the typical spacing between domains of different surface tension. This equation is an expansion in the small quantity $\epsilon$. After performing the integral over $G_0(q)$ and formally summing the series, the propagator can be written in the form,

$$
\langle G(k) \rangle^{-1} = \frac{\sigma_R - i \sigma I}{\sigma_0} |k| k^2 - p^3
$$

(B2)

The form of $\langle G(k) \rangle^{-1}$ is that of $G_0(k)^{-1}$, but with $\sigma_0$ replaced by an effective complex surface tension where $\sigma_I = \epsilon p^2/6\sigma_0$ and $\sigma_R$ is a $\Lambda$ dependent real part proportional to $\epsilon$. The Fourier transform of $G(k)$ yields,

$$
\langle G(k) \rangle \simeq \frac{-ie^{2i\theta}}{6p [\sigma_0(\sigma_R^2 + \sigma_I^2)]^{1/3}} H^{(1)}_0 (k_0 r) + \ldots
$$

(B3)

where $\theta = \frac{1}{3} \tan^{-1}(\sigma_I/\sigma_R)$ and,

$$
\left[ (\omega^2 - \sigma)q^2 + 2i\nu q^2 \right] \left[ \frac{iB}{\omega} q^2l + 2\nu q^2 - i\omega \right] + q^2(\sigma q^2 - 2i\nu q^2) \left[ \frac{iB}{\omega} (q^2 + 2\nu q^2) \right] \left[ 1 + \frac{i\nu q^2}{\omega} \right]
$$

(B4)

The attenuation of a one dimensional plane wave is found by integrating the $y$ coordinate in $B_3$,

$$
\int_{-\infty}^{\infty} \langle G(r) \rangle dy \propto e^{i\kappa x} \sim e^{-\alpha x}.
$$

(B5)

where the damping factor $\alpha$ is,

$$
\alpha = p \left( \frac{\sigma_0}{\sqrt{\sigma_R^2 + \sigma_I^2}} \right)^{1/3} \left[ \frac{1}{3} \tan^{-1}(\sigma_I/\sigma_R) \right] \approx \frac{\epsilon p^3}{18\sigma_0^2} + O(\epsilon^2)
$$

(B6)

The correlation $\epsilon$ for randomly distributed, noninteracting domains can be found by considering the averages,

$$
\langle \sigma(\vec{r}) \rangle = \sigma_0 c + \sigma_1 (1 - c)
$$

(B7)

and,

$$
\langle \sigma(\vec{r})^2 \rangle = \sigma_0^2 c + \sigma_1^2 (1 - c) - \langle \sigma(\vec{r}) \rangle^2,
$$

(B8)

where $c$ is the area fraction occupied by the domains. Hence,

$$
\langle \delta \sigma(\vec{r})^2 \rangle \equiv \langle ([\sigma(\vec{r}) - \langle \sigma(\vec{r}) \rangle]^2 \rangle = c(1-c)(\sigma_0 - \sigma_1)^2
$$

(B9)

If we define the correlation on a lattice with spacing of the domain size $\pi a^2$ as $\langle \delta \sigma(\vec{r}) \delta \sigma(\vec{r}') \rangle \equiv \epsilon \delta(\vec{r} - \vec{r}') \simeq \epsilon \delta_j/\pi a^2$, we find,

$$
\epsilon \simeq \pi a^2 c(1 - c)(\sigma_0 - \sigma_1)^2.
$$

(B10)

Using this result in (B6), we find that the attenuation $\alpha$ for small area fractions $c$ compares with that found in Section III, equations (3.8) and (3.9).

APPENDIX C: ASYMPTOTIC FORM OF THE UNIFORM SURFACE GREEN’S FUNCTION

We briefly outline the calculation of the asymptotic forms of the Green’s function matrix for a nonviscous compressible two dimensional fluid at the interface of an incompressible viscous substrate fluid. The zeroes of the determinant of the matrix $\mathbf{L}(\vec{q})$,
are shown in Fig. 8 and define the modes of the system. Note that these eigenmodes are not orthogonal and are hence coupled to each other. The Green’s function is found by Fourier transforming each element of $L^{-1}(\mathbf{q})$ via the contours shown in Figure 3. Contributions from the shear mode near the branch point at $q = \sqrt{i\omega/\nu}$ and the branch cut at are neglected as well as the integrations on the imaginary axes. The leading behavior of the Green’s function is

$$G_{11}(r) \simeq \frac{-\omega}{6\sigma_0} H_0^{(1)}(k_1 r) + \frac{3\omega\nu^2}{8B_0} H_0^{(1)}(k_r)$$

(3.2)

$$G_{11}(r) \simeq \frac{\sqrt{\nu\omega} e^{3\pi i/4}}{6\sigma_0} \left( \frac{r_i}{r} \right) H_1^{(1)}(k_1 r) + \frac{e^{i\pi/8} \nu^{5/4} \omega^{1/4}}{4B_0^{5/2}} \left( \frac{r_i}{r} \right) H_1^{(1)}(k_1 r)$$

(3.3)

and,

$$G_{11}(r) \simeq \frac{e^{3\pi i/4} \sqrt{\nu\omega}}{6B_0} \left( \frac{r_i}{r} \right) H_1^{(1)}(k_1 r) - \frac{e^{i\pi/8} (2B_0 - \sigma_0) \nu^{5/4} \omega^{1/4}}{4B_0^{5/2}} \left( \frac{r_i}{r} \right) H_1^{(1)}(k_1 r)$$

(3.4)

where the coefficients of the Hankel functions are expansions in small viscosity of the residues at the poles $k_1$ and $k_l$. Though not needed in this study, similar expressions can be found for $G_{23}(\mathbf{r})$ and $G_{jj}(\mathbf{r})$.

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FIG. 5. (a) Limit when longitudinal mode is damped near each scatterer. The transverse waves are weakly damped on interdomain length scales $l$. (b) Longitudinal waves are long-ranged, transverse waves are strongly damped; $\text{Im}\{k_d\} \ll l^{-1}$.

FIG. 6. Schematic of a proposed experiment to detect scattering from interfacial inhomogeneities at the air/water interface.

FIG. 7. (a) Diagram representing the sum in (A1). (b) Green’s function averaged over frozen disorder.

FIG. 8. Approximate location of poles in the complex $q$-plane. The shear mode in our analyses has a large imaginary component. The contours are the integrals used to evaluate the Green’s functions.