GIAMBELLI COMPATIBLE POINT PROCESSES

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Dedicated to Amitai Regev on the occasion of his 65th birthday

Abstract. We distinguish a class of random point processes which we call Giambelli compatible point processes. Our definition was partly inspired by determinantal identities for averages of products and ratios of characteristic polynomials for random matrices found earlier by Fyodorov and Strahov. It is closely related to the classical Giambelli formula for Schur symmetric functions.

We show that orthogonal polynomial ensembles, z-measures on partitions, and spectral measures of characters of generalized regular representations of the infinite symmetric group generate Giambelli compatible point processes. In particular, we prove determinantal identities for averages of analogs of characteristic polynomials for partitions.

Our approach provides a direct derivation of determinantal formulas for correlation functions.

Introduction

This paper appeared as a result of our attempt to find a connection between the work of Fyodorov and Strahov on evaluating the averages of products and ratios of characteristic polynomials of random matrices, and measures on partitions which exhibit random matrix type behavior.

Among many other things, Fyodorov and Strahov [29, 57] proved the following formula: Let $H$ be a random Hermitian $N \times N$ matrix distributed according to the Gaussian measure

$$P(dH) = \text{const} \cdot \exp(-\text{Tr}(H^2))dH$$

and $D(z) = \det(z - H)$ be its characteristic polynomial. Then for any $d = 1, 2, \ldots$ and $u_1, \ldots, u_d \in \mathbb{C} \setminus \mathbb{R}$, $v_1, \ldots, v_d \in \mathbb{C}$,

$$(0.1) \quad \left\langle \frac{D(v_1) \cdots D(v_d)}{D(u_1) \cdots D(u_d)} \right\rangle = \det \left( \frac{1}{u_i - v_j} \right)^{-1} \cdot \det \left( \frac{1}{u_i - v_j} \frac{D(v_j)}{D(u_i)} \right)$$

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(both determinants have size $d \times d$). By now this result has a number of different proofs and extensions, see [29], [57], [19], [20], [21], [22], [3], [18], [1], [4], [24], [31]. Formulas of this type are of interest in quantum physics and classical number theory, see [2], [28], [23], [33], [38], [39], [40]. Apart from that, (0.1) provides a convenient way to show that the correlation functions of the eigenvalues of $H$ can be written as determinants of a certain kernel. \(^1\)

On the other hand, in recent years there has been a considerable interest in measures on partitions which are in many ways similar to the eigenvalue distributions in Random Matrix Theory. The sources of such measures are quite diverse: they include combinatorics, representation theory, random growth models, random tilings, etc. In this paper we concentrate on the so-called $z$-measures which arise naturally in representation theory of the infinite symmetric group. This 3-parameter family of measures contains a number of other interesting measures on partitions (including the Plancherel measures and measures arising in last passage percolation models) as degenerations, see [14].

One natural question is: What is the analog of formula (0.1) for random partitions? Note that the very existence of such an analog is rather nontrivial: it is not a priori clear what a “characteristic polynomial of a partition” is, and the finite-dimensional averaging in (0.1) should be replaced by essentially an infinite-dimensional one over the space of all partitions.

The main goal of this paper is to provide an analog of (0.1) for the $z$-measures on partitions and their representation theoretic scaling limits, explain a general mechanism of where the identities of type (0.1) come from, and show how these identities imply the determinantal structure of the correlation functions of the underlying point processes. Remarkably, this approach provides the most straightforward derivation of the associated correlation kernels among those known so far.

Let us proceed to a more detailed description of the content of the paper.

a) Giambelli compatible processes. Let us first introduce some notation. Let $\Lambda$ be the algebra of symmetric functions and $\{s_\lambda\}$ be its basis consisting of the Schur functions. The Schur functions are parameterized by partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ which can also be written in the Frobenius notation:

$$\lambda = (p_1, \ldots, p_d \mid q_1, \ldots, q_d)$$

\(^1\)This basic fact lies at the foundation of Random Matrix Theory, see e.g. [48], §5.2.
The Schur functions satisfy the following basic identity called the **Giambelli formula**:

\[
\begin{align*}
    s_{(p_1, \ldots, p_d \mid q_1, \ldots, q_d)} &= \det [s_{(p_i \mid q_j)}]_{i,j=1}^d.
\end{align*}
\]

It turns out that the following remarkable fact holds true: Denote by \(\langle s_\lambda \rangle\) the average of the Schur function \(s_\lambda\) evaluated at the eigenvalues of matrix \(H\) with respect to the Gaussian measure on \(H\) introduced above. Then for any partition \(\lambda = (p_1, \ldots, p_d \mid q_1, \ldots, q_d)\)

\[
\langle s_{(p_1, \ldots, p_d \mid q_1, \ldots, q_d)} \rangle = \det [\langle s_{(p_i \mid q_j)} \rangle]_{i,j=1}^d
\]

or, in other words, the Giambelli formula remains invariant under the averaging.

This fact is closely related to the identity (0.1). More exactly, our first result is the following

**Claim I.** Let \(\langle \cdot \rangle\) be an arbitrary linear map from \(\Lambda\) to \(\mathbb{C}\). Then the following two conditions are equivalent:

(i) For any \(d = 1, 2, \ldots\) and any integers \(p_1 > \cdots > p_d \geq 0, q_1 > \cdots > q_d \geq 0\), the averaged Giambelli formula (0.2) holds.

(ii) For any \(d = 1, 2, \ldots\) the following formal power series identity holds:

\[
\langle H(u_1) \cdots H(u_d) E(v_1) \cdots E(v_d) \rangle = \det \left( \frac{1}{u_i + v_j} \right)^{-1} \det \left( \frac{H(u_i)E(v_j)}{u_i + v_j} \right)
\]

where both determinants are of size \(d \times d\), and \(H(u)\) and \(E(v)\) are the generating functions of the one-row and one-column Schur functions:

\[
H(u) = 1 + \sum_{k=1}^{\infty} \frac{s(k)}{u^k}, \quad E(v) = 1 + \sum_{k=1}^{\infty} \frac{s(1^k)}{v^k}.
\]

If we now evaluate the symmetric functions at \(N\) eigenvalues \(x_1, \ldots, x_N\) of \(H\), then

\[
H(u) = \prod_{i=1}^{N} \frac{1}{1 - x_i u^{-1}}, \quad E(v) = \prod_{i=1}^{N} (1 + x_i v^{-1}),
\]

and averaging over \(H\) turns (0.3) into (0.1).

We also show that in condition (i) above the Schur functions may be replaced by the multiparameter Schur functions (see §3 in [55] or §1.2

\[\text{2A careful reader might object that (0.3) is a formal power series identity while (0.1) is an identity of actual functions in } u_i \text{'s and } v_j \text{'s. It does require some efforts to pass from one to the other and this issue will be addressed in the body of the paper.} \]
below) or by their special case — the Frobenius-Schur functions (see §2 in [55] and §1.3).

The next definition is inspired by Claim I.

**Definition.** A random point process (= a probability measure on point configurations) is called **Giambelli compatible** if there exists a homomorphism of the algebra of symmetric functions $\Lambda$ to a suitable algebra of functions on point configurations such that the linear functional on $\Lambda$ obtained by averaging the images of symmetric functions satisfies the conditions of Claim I.

In this terminology the point process of eigenvalues of random Hermitian matrices with the Gaussian measure is Giambelli compatible.

In this paper we discuss three examples of Giambelli compatible random point processes. Let us describe them one by one.

**b) Orthogonal polynomial ensembles.** Let $\mu$ be an arbitrary measure on $\mathbb{R}$ with finite moments. The $N$-point orthogonal polynomial ensemble on $\mathbb{R}$ associated with $\mu$ is a probability measure on $\mathbb{R}^N$ of the form

$$P_N(dx_1, \ldots, dx_N) = \text{const} \cdot \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{i=1}^N \mu(dx_i).$$

Orthogonal polynomial ensembles are very common in Random Matrix Theory; they are also often called “$\beta = 2$ ensembles”. In particular, for any even degree polynomial $V(x)$ with a nonnegative highest coefficient, the radial part (= projection to eigenvalues) of the unitarily invariant probability measure

$$\text{const} \cdot \exp(-\text{Tr}(V(H))) dH$$

on the Hermitian $N \times N$ matrices is an orthogonal polynomial ensemble with $\mu(dx) = \exp(-V(x)) dx$, see e.g. [26]. Orthogonal polynomial ensembles with discretely supported measures $\mu$ are also quite popular, see e.g. [34], [35], [36], [14], [15], [16], [15], [49].

**Claim II.** Any orthogonal polynomial ensemble defines a Giambelli compatible process with respect to the standard realization of the symmetric functions as functions on $\mathbb{R}^N$.

This fact (more exactly, formula [0.1]) allows one to derive the determinantal formula for the correlation functions of the orthogonal polynomial ensembles, and to express the correlation kernel in terms of

\[3\] The value of $\beta$ refers to the power of the Vandermonde determinant.
the 2-point average $\langle H(u)E(v) \rangle$. This average is in its turn expressible through the orthogonal polynomials associated with $\mu$. See §3 for details.

c) $z$-measures on partitions. These probability measures depend on three (generally speaking, complex) parameters $z, z', \xi$ and assign to a partition $\lambda$ with Frobenius coordinates $(p_1, \ldots, p_d \mid q_1, \ldots, q_d)$ the weight

\begin{equation}
M_{z, z', \xi}(\lambda) = (1 - \xi)^{z' + 1} \left( \frac{1}{(p_i!)^2 (q_i!)^2} \det \left( \frac{1}{p_i + q_j + 1} \right) \right).
\end{equation}

Here $(a)_k = a(a + 1) \cdots (a + k - 1)$ is the Pochhammer symbol.

There are various sets of conditions on $(z, z', \xi)$ that guarantee that these weights are nonnegative and their sum over all partitions is equal to 1; for instance, one can take $z' = \bar{z} \in \mathbb{C}$ and $\xi \in (0, 1)$.

The $z$-measures describe the generalized regular representations of the infinite symmetric group. Briefly, $M_{z, z', \xi}(\lambda)$ are essentially the Fourier coefficients of characters of such representations. We refer to [44], [54] for details.

Apart from that, the $z$-measures degenerate to a variety of measures of rather different origins. When both parameters $z$ and $z'$ are positive integers, the $z$-measures arise in a last passage percolation model, see [34], while when $z$ and $z'$ are integers of different signs, the corresponding measures are directly related to the “digital boiling” growth model, see [32]. In the limit $z' \to \infty$, $\xi \to 0$, and with integral $z \in \mathbb{Z}_+$ the $z$-measures are obtained from pushforwards of the uniform measures on random words built out of an alphabet with $z$ letters under the Robinson-Schensted correspondence, see e.g. [14]. Finally, in the limit when both $z$ and $z'$ tend to infinity and $\xi \to 0$, the $z$-measure becomes the celebrated poissonized Plancherel measure, see e.g. [9].

It is convenient to identify partitions $\lambda = (p_1, \ldots, p_d \mid q_1, \ldots, q_d)$ with finite point configurations on $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ as follows

\begin{equation}
\lambda \leftrightarrow \{ -q_1 - \frac{1}{2}, -q_2 - \frac{1}{2}, \ldots, -q_d - \frac{1}{2}, p_d + \frac{1}{2}, \ldots, p_2 + \frac{1}{2}, p_1 + \frac{1}{2} \} \subset \mathbb{Z}'.
\end{equation}

Then any measure on partitions, in particular, the $z$-measure, defines a random point process on $\mathbb{Z}'$. In order to move on, we need to realize the symmetric functions as functions on partitions. A suitable for us way of doing that was suggested in [13]. Namely, the Newton power sums $p_k \in \Lambda$ (do not
confuse with Frobenius coordinates $p_i$) are specialized as follows: For
\[ \lambda = (p_1, \ldots, p_d \mid q_1, \ldots, q_d) \]
\[ p_k(\lambda) = \sum_{i=1}^{d} (p_i + \frac{1}{2})^k + (-1)^{k-1} (q_i + \frac{1}{2})^k. \]

Then the algebra $\Lambda$ is being mapped to the algebra of polynomial functions on partitions. The images of the generating series $H(u)$ and $E(v)$ under this map have the form
\[ H(u)(\lambda) = \prod_{i \geq 1} \frac{u+i-\frac{1}{2}}{u-\lambda_i+i-\frac{1}{2}} = \prod_{i \geq 1} \frac{u+\lambda_i'-i+\frac{1}{2}}{u-i+\frac{1}{2}} = \prod_{i=1}^{d} \frac{u+q_i+\frac{1}{2}}{u-p_i+\frac{1}{2}}, \]
\[ E(v)(\lambda) = \prod_{i \geq 1} \frac{v+\lambda_i'-i+\frac{1}{2}}{v-i+\frac{1}{2}} = \prod_{i \geq 1} \frac{v+i-\frac{1}{2}}{v-\lambda_i'+i-\frac{1}{2}} = \prod_{i=1}^{d} \frac{v+p_i+\frac{1}{2}}{v-q_i+\frac{1}{2}}. \]

These are the analogs of the characteristic polynomial and its inverse for partitions (here $\lambda'$ denotes the transposed partition).

**Claim III.** The random point process on $\mathbb{Z}'$ corresponding to any $z$-measure or any of its degenerations is Giambelli compatible with respect to the realization of the algebra of symmetric functions on partitions described above.

While for the orthogonal polynomial ensembles the derivation of the determinantal formula for the correlation functions from Giambelli compatibility is of rather limited interest, for the $z$-measures such a derivation provides the simplest known proof of this important fact.

For any finite subset $X$ of $\mathbb{Z}'$ let us denote by $\rho(X)$ the $z$-measure probability that the random point configuration (0.5) contains $X$. Claim III leads to the following result.

**Theorem.** For any finite set $X = \{x_1, \ldots, x_m\} \subset \mathbb{Z}'$ we have
\[ \rho(X) = \det[K(x_i,x_j)]_{i,j=1}^{m}, \]
where
\[ (0.6) \quad K(x,y) = \begin{cases} \Res_{u=y} \frac{\langle H(u)E(-x) \rangle}{x-y}, & x > 0, y > 0, x \neq y, \\ -\Res_{u=x} \Res_{u=y} \frac{\langle H(u)E(-v) \rangle}{x-y}, & x < 0, y > 0, \\ \Res_{v=x} \frac{\langle H(y)E(-x) \rangle}{x-y}, & x > 0, y < 0, \\ -\Res_{v=x} \frac{\langle H(y)E(-v) \rangle}{x-y}, & x < 0, y < 0, x \neq y. \end{cases} \]
Here \( \langle \cdot \rangle \) means averaging over the \( z \)-measure \( M_{z,z',\xi} \), and the indeterminacy arising for \( x = y \) is resolved via the L'Hospital rule.

It is now immediate to explicitly evaluate (using formula (0.4)) the 2-point average \( \langle H(u)E(v) \rangle \) and the whole correlation kernel \( K(x,y) \) in terms of the Gauss hypergeometric function. We do this simple computation in the body of the paper and thus rederive the hypergeometric kernel of [12].\(^4\) Details on \( z \)-measures on partitions are presented in \( \S 4 \).

d) Spectral \( z \)-measures. These measures describe the (spectral) decomposition of the generalized regular representations of the infinite symmetric group on irreducibles. The spectral \( z \)-measures have continual infinite-dimensional support and they are not easy to describe in simple terms.

One way to obtain the spectral \( z \)-measures is to take a certain scaling limit of the \( z \)-measures on partitions described above as \( \xi \to 1 \). Another, more direct approach is to represent them as a unique solution of an infinite-dimensional moment problem.

More exactly, the probability measures that we are interested in live on the space of pairs of nonincreasing sequences \( (\alpha, \beta) \) of nonnegative real numbers whose total sum is finite:

\[
\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \quad \beta_1 \geq \beta_2 \geq \cdots \geq 0, \quad \sum_i \alpha_i + \sum_j \beta_j < \infty.
\]

There is a standard way of realizing the algebra of symmetric functions \( \Lambda \) by functions on such pairs of sequences. Namely, the images of the Newton power sums take the form (cf. (0.5))

\[
(0.7) \quad p_k(\alpha, \beta) = \sum_i \alpha_i^k + (-1)^{k-1} \sum_j \beta_j^k,
\]

see e.g. [17], Ex. I.3.23. The role of moments is played by the averages of images of the Schur functions under this map. The representation theoretic definition of the spectral \( z \)-measures implies that these averages can be explicitly computed:

\[
(0.8) \quad \langle s_\lambda \rangle = (zz')^d \prod_{i=1}^d \frac{(z+1)_{p_i}(z'+1)_{p_i}(-z+1)_{q_i}(-z'+1)_{q_i}}{p_i!q_i!} \det \left( \frac{1}{p_i + q_j + 1} \right)
\]

for any partition \( \lambda = (p_1, \ldots, p_d \mid q_1, \ldots, q_d) \), cf. (0.4).

\(^4\)It should be noted that the proof in [12] was not a derivation but a verification. Known derivations of the hypergeometric kernel are somewhat indirect: they use an \( sl(2) \)-action on the infinite wedge space [50], more general Schur measures on partitions [51], [8], [37], [56], or nontrivial analytic continuation arguments [17].
Let us view pairs of sequences \((\alpha, \beta)\) as point configurations
\((-\beta_1, -\beta_2, \ldots, \alpha_2, \alpha_1)\)
in \(\mathbb{R}^* = \mathbb{R} \setminus \{0\}\). Then any spectral \(z\)-measure defines a point process on \(\mathbb{R}^*\). Formula (0.7) defines a map of \(\Lambda\) to functions on such point configurations. It is not hard to see that (0.8) implies

**Claim IV.** The random point processes on \(\mathbb{R}^*\) associated with the spectral \(z\)-measures are Giambelli compatible.

This fact and the product formulas
\[
H(u)(\alpha, \beta) = \prod_{i=1}^{\infty} \frac{1 + \beta_i u^{-1}}{1 - \alpha_i u^{-1}}, \quad E(v)(\alpha, \beta) = \prod_{i=1}^{\infty} \frac{1 + \alpha_i v^{-1}}{1 - \beta_i v^{-1}},
\]
which hold with probability 1, allow us to obtain the determinant formula for the correlation functions of the point processes on \(\mathbb{R}^*\) and to express the correlation kernel in terms of the 2-point average \(\langle H(u)E(v) \rangle\) by a formula similar to (1.6) with residues replaced by jumps across the real axis. A straightforward computation leads to explicit expressions for \(\langle H(u)E(v) \rangle\) and, thus, for the correlation kernel in terms of the confluent hypergeometric functions or, equivalently, the Whittaker functions.

This argument yields a relatively short derivation of the Whittaker kernel obtained earlier by much heavier machinery in [6], see also [10], [12]. Since the Whittaker kernel essentially provides a complete solution to a problem of harmonic analysis on the infinite symmetric group, we see that the formalism of Giambelli compatible processes delivers adequate tools for a direct solution of this problem. Details and references on spectral \(z\)-measures can be found in §5.

### 1. Preliminaries on Schur functions, multiparameter Schur functions and Frobenius-Schur functions

In this section our main references are Macdonald’s book [47] (symmetric functions in general) and Olshanski, Regev and Vershik [55] (multiparameter Schur functions and Frobenius-Schur functions).

#### 1.1. Schur functions

Let \(\Lambda\) denote the algebra of symmetric functions. The algebra \(\Lambda\) can be considered as the algebra of polynomials \(\mathbb{C}[p_1, p_2, \ldots]\) in power sums \(p_1, p_2, \ldots\). Then it can be realized, in different ways, as an algebra of functions, depending on a specialization of the generators \(p_k\). The elements \(h_k\) and \(e_k\) (the complete homogeneous
symmetric functions and the elementary symmetric functions) can be introduced through the generating series:

\[ 1 + \sum_{k=1}^{\infty} h_k t^k = \exp \left( \sum_{k=1}^{\infty} \frac{p_k}{k} t^k \right) = \left( 1 + \sum_{k=1}^{\infty} e_k (-t)^k \right)^{-1}. \]

The Schur function \( s_\mu \) indexed by a Young diagram \( \mu \) can then be introduced through the Jacobi–Trudi formula:

\[ s_\mu = \det [h_{\mu_i - i+j}], \]

where, by convention, \( h_0 = 1, h_{-1} = h_{-2} = \ldots = 0 \), and the order of the determinant is any number greater or equal to \( l(\mu) \) (the number of nonzero row lengths of \( \mu \)).

Define the generating series for \( \{h_k\} \) and \( \{e_k\} \) as formal series in \( u \) by

\[ H(u) = 1 + \sum_{k=1}^{\infty} \frac{h_k}{u^k}, \quad E(u) = 1 + \sum_{k=1}^{\infty} \frac{e_k}{u^k}. \]

For \( p, q = 0, 1, \ldots \), let \( (p|q) \) denote the hook Young diagram \( (p+1,q) \), and let \( s_{(p|q)} \) be the “hook” Schur function associated with this diagram. The following formula holds ([47], Ex. I.3.14):

\[ (1.1) \quad H(u) E(v) = 1 + (u + v) \sum_{p, q = 0}^{\infty} \frac{s_{(p|q)}}{u^{p+1} v^{q+1}}. \]

In the Frobenius notation, a Young diagram is written as

\[ \mu = (p_1, \ldots, p_d|q_1, \ldots, q_d), \]

where \( d \) is the number of diagonal boxes, and

\[ p_i = \mu_i - i, \quad q_i = \mu'_i - i, \]

where \( \mu' \) is the transposed diagram.

In what follows we exploit the expression of the general Schur functions through the hook Schur functions given by the Giambelli formula ([47], Ex. I.3.9)

\[ s_\mu = \det [s_{(p_i|q_j)}]_{i,j=1}^d. \]

1.2. Multiparameter Schur functions. Let \( a = (a_i)_{i \in \mathbb{Z}} \) be an arbitrary sequence of complex numbers. The multiparameter analogs \( h_{k,a} \) of the complete homogeneous functions are introduced by the expression

\[ 1 + \sum_{k=1}^{\infty} \frac{h_{k,a}}{(u-a_1) \ldots (u-a_k)} = H(u) = 1 + \sum_{k=1}^{\infty} \frac{h_k}{u^k}. \]
Since \( h_{k,a} = h_k + \text{lower terms} \),
\[
\{h_{k,a}\}_{k=1,2,...} \text{ is a system of algebraically independent generators of } \Lambda.
\]
We agree that 
\[
h_{0,a} = 1, \quad h_{-1,a} = h_{-2,a} = \ldots = 0.
\]
For \( r \in \mathbb{Z} \), let \( \tau^r \cdot a \) be the result of shifting \( a \) by \( r \) digits to the left,
\[
(\tau^r a)_i = a_{i+r}.
\]
The multiparameter Schur function \( s_{\mu,a} \) indexed by an arbitrary Young diagram \( \mu \) is defined by
\[
s_{\mu,a} = \text{det} [h_{\mu,-i+j;\tau^1-j} a]_i^d,
\]
where the order of the determinant is any number greater or equal to \( l(\mu) \). From the above definition and from the result of Macdonald [47], Example I.3.21, it is clear that the multiparameter Schur functions \( s_{\mu,a} \) satisfy the Giambelli formula
\[
(1.2) \quad s_{\mu,a} = \text{det} [s_{(p_1|q_j);a}]_{i,j=1}^d,
\]
where the determinant has order \( d = d(\mu) \) and \( p_1, \ldots, p_d, q_1, \ldots, q_d \) denote the Frobenius coordinates of \( \mu \).

As shown in [55], formula (1.1) can be generalized as follows:
\[
(1.3) \quad H(u)E(v) = 1 + (u + v) \sum_{p,q=0}^{\infty} \frac{s_{(p|q);a}}{(u|a)^p(v|\hat{a})^{q+1}},
\]
where \( \hat{a} \) stands for the "dual" sequence attached to \( a \), i. e.
\[
\hat{a}_i = -a_{-i+1},
\]
and
\[
(x|a)^m = \begin{cases} 
(x - a_1) \ldots (x - a_m), & m \geq 1, \\
1 & m = 0.
\end{cases}
\]

1.3. Frobenius–Schur functions. The Frobenius–Schur functions are a special case of the multiparameter Schur functions:
\[
F s_\mu = s_{\mu,a}, \quad a_i = i - 1/2.
\]
By (1.2), the Frobenius-Schur functions satisfy the Giambelli formula
\[
F s_\mu = \text{det} [F s_{(p|q)}]_{i,j=1}^d,
\]
and the next formula is a particular case of (1.3):

\[ H(u)E(v) = 1 + (u+v) \sum_{p,q=0}^{\infty} \left( u - \frac{1}{2} \right) \ldots \left( u - \frac{2p+1}{2} \right) \left( v - \frac{1}{2} \right) \ldots \left( v - \frac{2q+1}{2} \right). \]

1.4. The Young graph. Let \( Y \) denote the set of all Young diagrams including the empty diagram \( \emptyset \). We regard \( Y \) as the set of the vertices of a graph, called the Young graph and denoted also by \( Y \). The edges of the graph \( Y \) are couples of diagrams \((\mu, \lambda)\) such that \( \lambda \) is obtained from \( \mu \) by adding a box (we denote this relation as \( \mu \leftarrow \lambda \)).

Let \( \dim(\mu, \lambda) \) be the number of all paths going from a vertex \( \mu \) to a vertex \( \lambda \) with \( |\lambda| > |\mu| \). We agree that \( \dim(\mu, \mu) = 1 \), and \( \dim \lambda = \dim(\emptyset, \lambda) \). Clearly, if \( \mu \subset \lambda \) then \( \dim(\mu, \lambda) \) is equal to the number of the standard Young tableaux of skew shape \( \lambda/\mu \), and if \( \mu \) is not contained in \( \lambda \) then \( \dim(\mu, \lambda) = 0 \).

1.5. Polynomial functions on \( Y \). Given \( \lambda \in Y \), let \( a_1, \ldots, a_d, b_1, \ldots, b_d \) stand for its modified Frobenius coordinates:

\[ a_i = \lambda_i - i + \frac{1}{2}, \quad b_i = \lambda'_i - i + \frac{1}{2}, \quad i = 1, \ldots, d. \]

Following [43] (see also [55]) we realize \( \Lambda \) as an algebra of functions on \( Y \) using the following specialization of the Newton power sums

\[ p_k(\lambda) = \sum_{i=1}^{d} a_i^k + (-1)^{k-1} \sum_{i=1}^{d} b_i^k, \]

where \( \lambda \) ranges over \( Y \). Then each \( f \in \Lambda \) becomes a function \( f(\lambda) \) on \( Y \). Such functions were called in [43] the polynomial functions on \( Y \).

In particular, the generating series for \( \{h_k\} \) and \( \{e_k\} \) take the form (see e.g. Macdonald [47], Ex. 1.3.23, and Olshanski, Regev and Vershik [55]):

\[ H(u)(\lambda) = \prod_{i=1}^{d} \frac{u + b_i}{u - a_i}, \quad E(v)(\lambda) = \prod_{i=1}^{d} \frac{v + a_i}{v - b_i}. \]

Note that these expressions are rational functions in \( u \) or \( v \) for any fixed \( \lambda \).

The characterizing property of the Frobenius-Schur functions that we are going to exploit is expressed by the relation

\[ F_{s_{\mu}}(\lambda) = \frac{\dim(\mu, \lambda) n_{qm}^{i,m}}{\dim \lambda}, \]
where \( m = |\mu|, \ n = |\lambda|, \) and
\[
n^m_i = \begin{cases} \frac{n(n-1)\ldots(n-m+1)}{n}, & n \geq m, \\
0, & n < m. \end{cases}
\]

2. GIAMBELLI COMPATIBILITY AND POINT PROCESSES

**Definition 2.1.** Assume \( f \mapsto \langle f \rangle \) is a linear functional on the algebra \( \Lambda \) of symmetric functions, such that \( \langle 1 \rangle = 1 \). Let us say that \( \langle \cdot \rangle \) is Giambelli compatible if for any Young diagram \( \lambda = (p_1, \ldots, p_d|q_1, \ldots, q_d) \)

\[
\langle s_\lambda \rangle = \det \left( \langle s_{(p_i|q_j)} \rangle \right)_{i,j=1}^d.
\]

**Proposition 2.2.** A linear functional \( \langle \cdot \rangle \) is Giambelli compatible in the sense of the above definition if and only if for all \( d = 1, 2, \ldots \)

\[
\left( \det \left( \frac{H(u_i)E(v_j)}{u_i + v_j} \right)_{i,j=1}^d \right) = \det \left( \langle H(u_i)E(v_j) \rangle \right)_{i,j=1}^d.
\]

Here we regard \( H(u_i)E(v_j)/(u_i + v_j) \) as elements of the algebra

\( \Lambda[[u_1^{-1}, \ldots, u_d^{-1}, v_1^{-1}, \ldots, v_d^{-1}]]_{\text{loc}}, \)

where the subscript “loc” means localization with respect to \( \prod (u_i^{-1} + v_j^{-1}) \), which makes it possible to deal with

\[
\frac{1}{u_i + v_j} = \frac{u_i^{-1}v_j^{-1}}{u_i^{-1} + v_j^{-1}}.
\]

**Proof.** Let us show that (2.1) implies (2.2). Indeed, by (1.1)

\[
\det \left( \frac{H(u_i)E(v_j) - 1}{u_i + v_j} \right)_{i,j=1}^d = \det \left( \sum_{p_i, q_j=0}^{\infty} \frac{s_{(p_i|q_j)}}{u_i^{p_i+1}v_j^{q_j+1}} \right)_{i,j=1}^d = \sum_{p_1, \ldots, p_d=0}^{\infty} \sum_{q_1, \ldots, q_d=0}^{\infty} \frac{\det \left( \langle s_{(p_i|q_j)} \rangle \right)_{i,j=1}^d}{u_1^{p_1+1} \ldots u_d^{p_d+1}v_1^{q_1+1} \ldots v_d^{q_d+1}}.
\]

Applying \( \langle \cdot \rangle \) to the both sides we obtain

\[
\left( \det \left( \frac{H(u_i)E(v_j) - 1}{u_i + v_j} \right)_{i,j=1}^d \right) = \sum_{p_1, \ldots, p_d=0}^{\infty} \sum_{q_1, \ldots, q_d=0}^{\infty} \frac{\langle \det \left( \langle s_{(p_i|q_j)} \rangle \right)_{i,j=1}^d \rangle}{u_1^{p_1+1} \ldots u_d^{p_d+1}v_1^{q_1+1} \ldots v_d^{q_d+1}}.
\]
\[= \sum_{p_1, \ldots, p_d=0}^{\infty} \frac{\det \left( \langle s_{(p_i|q_j)} \rangle \right)_{i,j=1}}{u_i^{p_i+1} \cdots u_d^{p_d+1} \cdot v_1^{q_i+1} \cdots v_d^{q_d+1}} = \det \left( \sum_{p_i, q_j=0}^{\infty} \langle s_{(p_i|q_j)} \rangle \right)_{i,j=1}^{d} \]

where in the second equality we have used the Giambelli compatibility assumption.

Now we aim to remove the \(-1\)'s. Let \(A\) and \(B\) denote the \(d \times d\) matrices with entries

\[A(i, j) = \frac{H(u_i)E(v_j) - 1}{u_i + v_j}, \quad B(i, j) = \frac{1}{u_i + v_j}, \quad i, j = 1, \ldots, d.\]

Next, let \(A_{IJ}\) and \(B_{IJ}\) denote their submatrices corresponding to subsets \(I, J \subset \{1, \ldots, d\}\) with \(|I| = |J|\). The above argument shows that

\[\langle \det A_{IJ} \rangle = \det \langle A_{IJ} \rangle.\]

Since \(B\) has numerical entries, we have

\[\langle \det(A + B) \rangle = \det \langle A + B \rangle,\]

as follows from the expansion

\[\det(A + B) = \sum_{I, J} \pm \det A_{IJ} \det B_{IJ},\]

where \(\bar{I}\) stands for the complement to \(I\) in \(\{1, \ldots, d\}\).

Thus, we have proved that (2.1) implies (2.2). Finally, the whole argument above can be inverted, which proves the inverse implication.

\[\Box\]

**Proposition 2.3.** The Giambelli compatibility property (2.1) remains intact if we replace in (2.1) the Schur functions \(s_\lambda\) by the multiparameter Schur functions \(s_{\lambda; a}\). That is if we require

\[(2.3) \quad \langle s_{\lambda; a} \rangle = \det \left( \langle s_{(p_i|q_j); a} \rangle \right)_{i,j=1}^{d} \]

for any Young diagram \(\lambda = (p_1, \ldots, p_d | q_1, \ldots, q_d)\).

**Proof.** Indeed, the transition formulas between multiparameter Schur functions with different parameters (see [55], Theorem 7.3) imply that conditions (2.1) and (2.3) are equivalent. Another way to see this is to observe that the proof of Proposition 2.2 used only relations which hold equally well for the multiparameter Schur functions. \[\Box\]
Assume $S$ is a Borel space equipped with a probability Borel measure $P$. Let $\mathcal{A}(S, P)$ be the set of Borel functions $f$ on $S$ such that $|f|, |f|^2, |f|^3, \ldots$ belong to $L^1(S, P)$. Clearly, $\mathcal{A}(S, P)$ is an algebra.

Let $\langle \cdot \rangle_P$ denote the expectation on $\mathcal{A}(S, P)$: the linear functional determined by integration with respect to measure $P$.

**Definition 2.4.** Assume we are given an algebra morphism $\phi : \Lambda \to \mathcal{A}(S, P)$. Let us say that the triple $(S, P, \phi)$ is Giambelli compatible if the pullback of $\langle \cdot \rangle_P$ on $\Lambda$ is a Giambelli compatible functional in the sense of Definition 2.1.

Finally, recall some basic definitions related to random point processes; for more detailed information, see Daley and Vere–Jones [25] and Lenard [46].

Let $\mathfrak{X}$ be a locally compact space. By a point configuration in $\mathfrak{X}$ we mean a finite or countably infinite collection of points of the space $\mathfrak{X}$ with no accumulation points. The set of all point configurations in $\mathfrak{X}$ will be denoted by $\text{Conf}(\mathfrak{X})$; it admits a natural Borel structure. By definition, a random point process on $\mathfrak{X}$ is defined by specifying a Borel map $S \to \text{Conf}(\mathfrak{X})$, where $(S, P)$ is a Borel space with a probability measure. Then the pushforward $\mathcal{P}$ of $P$ is a probability measure on $\text{Conf}(\mathfrak{X})$, hence one can speak about random point configurations on $\mathfrak{X}$. Since only the resulting measure $\mathcal{P}$ is actually relevant, a point process is often viewed simply as a couple $(\text{Conf}(\mathfrak{X}), \mathcal{P})$, the “source” probability space $(S, P)$ being unnecessary or playing only an auxiliary role. However, in the concrete examples we deal with in sections 4 and 5, the situation is somewhat different: we are primarily interested in describing a measure $P$ on a space $S$ while the point process generated by $(S, P)$ is used rather as a tool.

The $m$th correlation measure $\rho_m$ ($m = 1, 2, \ldots$) of a random point process is a symmetric measure on $\mathfrak{X}^m = \mathfrak{X} \times \cdots \times \mathfrak{X}$ ($m$ times) determined by

$$\left\langle \sum_{y_1, \ldots, y_m \in \mathfrak{X}} F(y_1, \ldots, y_m) \right\rangle_{\mathcal{P}} = \int F(x_1, \ldots, x_m) \rho_m(dx_1 \ldots dx_m),$$

where the sum is taken over all ordered $m$-tuples of pairwise distinct points taken from the random point configuration $X$ and $F$ is a test function on $\mathfrak{X}^m$.

The space $\mathfrak{X}$ usually comes with a natural reference measure $\nu(dx)$ such that $\rho_m$ is absolutely continuous with respect to $\nu^\otimes m$ for all $m$. In such a case one can consider the density of $\rho_m$ with respect to $\nu^\otimes m$, which is called the $m$th correlation function of the process. We will
denote this function as \( \rho_m(x_1, \ldots, x_m) \). The process is called *determinantal* if there exists a function \( K(x, y) \) on \( \mathcal{X} \times \mathcal{X} \) such that for any \( m = 1, 2, \ldots \)

\[
\rho_m(x_1, \ldots, x_m) = \det (K(x_i, x_j))_{i,j=1,\ldots,m}.
\]

In our concrete examples, the point processes turn out to be determinantal ones, and we will show how determinantal identity (2.2), which holds for Giambelli compatible triples \((S, P, \phi)\), leads to determinantal identity (2.4).

3. The unitary ensemble of Random Matrix Theory

3.1. Basic notation. Fix an arbitrary measure \( \alpha \) on \( \mathbb{R} \) with finite moments and also fix \( N = 1, 2, \ldots \). In this section we take \( \mathcal{X} = \mathbb{R} \) and consider the subset \( \text{Conf}_N(\mathbb{R}) \subset \text{Conf}(\mathbb{R}) \) consisting of \( N \)-point configurations \( X = (x_1, \ldots, x_N) \). We also regard \( \text{Conf}_N(\mathbb{R}) \) as the “source” space \( S \). On this space we define a probability measure \( P_{\alpha,N} \), as follows:

\[
P_{\alpha,N}(dX) = \text{const}_N V^2(x) \alpha^\otimes N(dX),
\]

where \( \alpha^\otimes N(dX) = \prod_{i=1}^N \alpha(dx_i) \), \( V(X) = \prod_{1 \leq i < j \leq N} (x_i - x_j) \) is the Vandermonde determinant, and \( \text{const}_N \) is the normalization constant.

For a symmetric function \( f(x_1, \ldots, x_N) \) of the \( x_i \)'s, we denote by \( \langle f \rangle_{\alpha,N} \) its average with respect to \( P_{\alpha,N} \).

If we interpret the points \( x_1, x_2, \ldots, x_N \) of the random point configuration \( X \) as eigenvalues of a random \( N \times N \) Hermitian matrix, then measure \( P_{\alpha,N} \) determines a unitary invariant \((\beta = 2)\) ensemble of Random Matrix Theory (see Mehta [48], Deift [26] for details).

3.2. Giambelli compatibility. To any \( f \in \Lambda \) we assign a function \( \phi(f) \) on configurations \( X \) in a natural way

\[
(\phi(f))(X) = f(x_1, \ldots, x_N, 0, 0, \ldots), \quad X = (x_1, \ldots, x_N).
\]

Since, by assumption, all moments of \( \alpha \) are finite, \( (\phi(f))(X) \) belongs to \( \mathcal{A}(\text{Conf}_N(\mathbb{R}), P_{\alpha,N}) \). To simplify the notation we will write \( f(X) \) instead of \( (\phi(f))(X) \). Note that

\[
s_\lambda(X) = \frac{\det \left( x_i^{\lambda_j+N-j} \right)_{i,j=1}^N}{V(x_1, \ldots, x_N)} \quad \ell(\lambda) \leq N; \quad s_\lambda(X) = 0, \quad \ell(\lambda) > N,
\]

and

\[
H(u)(X) = \prod_{i=1}^N \frac{u}{u - x_i}, \quad E(v)(X) = \prod_{i=1}^N \frac{v + x_i}{v}.
\]
Theorem 3.1. The triple \((\text{Conf}_N(\mathbb{R}), P_{\alpha,N}, \phi)\) is Giambelli compatible in the sense of Definitions 2.1 and 2.4.

Proof. Let \(A_n\) denote the \(n\)th moment of \(\alpha\),
\[
A_n = \int_{\mathbb{R}} x^n \alpha(dx), \quad n = 0, 1, \ldots.
\]
Assume first \(\ell(\lambda) \leq N\). The above expression for \(s_{\lambda}(X)\) together with the definition of \(P_{\alpha,N}\) imply
\[
\langle s_{\lambda} \rangle_{\alpha,N} = \text{const} \int \ldots \int \det(x_i^{\lambda_i+N-j}) \det(x_i^{-j}) \alpha(dx_1) \ldots \alpha(dx_N) = \text{const} \det(A_{\lambda_i+N-i+N-j}).
\]
Here the second equality is obtained by a well-known trick, see, e.g., [58]. All determinants above are of order \(N\).

Hence we obtain
\[
\langle s_{\lambda} \rangle_{\alpha,N} = \begin{cases} 
\text{const} \det(A_{\lambda_i+N-i+N-j})_{i,j=1}^N, & \ell(\lambda) \leq N, \\
0, & \text{otherwise}.
\end{cases}
\]

Now, our claim becomes a particular case of a general theorem due to Macdonald (see [47, Example I.3.21]) which says:

Let \(\{h_{rs}\}_{r \in \mathbb{Z}, s \in \mathbb{Z}_+}\) be any collection of commuting indeterminates such that
\[
h_{0s} = 1, \quad h_{-1,s} = h_{-2,s} = \cdots = 0 \quad \forall s \in \mathbb{Z}_+,
\]
and set
\[
\tilde{s}_\lambda = \text{det}(h_{\lambda_i-i+j,j-1})_{i,j=1}^k
\]
where \(k\) is any number \(\geq \ell(\lambda)\). Then we have
\[
\tilde{s}_\lambda = \text{det}(\tilde{s}_{(p_i|q_i)})_{i,j=1}^d, \quad \lambda = (p_1, \ldots, p_d | q_1, \ldots, q_d).
\]

To apply Macdonald’s theorem consider the matrix \(g = (g_{kl})\) of format \(\infty \times N\) with entries \(g_{kl} = A_{k+l}\), where \(k \in \mathbb{Z}_+, l = 0, \ldots, N-1\). Next, multiply \(g\) on the right by a suitable nondegenerate matrix \(N \times N\) in such a way that the resulting matrix \(g' = (g'_{kl})\) be strictly lower triangular:
\[
g'_{kl} = \delta_{kl}, \quad 0 \leq k \leq l \leq N-1.
\]

Then
\[
\langle s_{\lambda} \rangle_{\alpha,N} = \text{det}(g'_{\lambda_i+N-i,N-j})_{i,j=1}^N, \quad \ell(\lambda) \leq N.
\]

Setting
\[
h_{rs} = \begin{cases} 
g'_r, & s = 0, 1, \ldots, N-1, r \geq 0 \\
g'_r, & s \geq N, r \geq 0 \\
\delta_{r0}, & s \geq 0, r < 0,
\end{cases}
\]
it is readily seen that \( \langle s_\lambda \rangle_{\alpha,N} \) coincides with \( \det(h_{\lambda, i+j, j-1}) \) both for \( \ell(\lambda) \leq N \) and for \( \ell(\lambda) > N \) (in the latter case the determinant vanishes).

\[ \square \]

3.3. The correlation kernel. It is well known (see, e.g., [48]) that the process \((\text{Conf}_N(\mathbb{R}), P_{\alpha,N})\) is determinantal and its correlation kernel \(K(x, y)\) is essentially the kernel of the projection operator in \(L^2(\mathbb{R}, \alpha)\) whose range is the space of polynomials of degree \(\leq N - 1\). The kernel can be written explicitly in terms of orthogonal polynomials \(\pi_0, \pi_1, \ldots\) corresponding to the weight \(\alpha\).

Here we present a different expression for \(K(x, y)\) which does not involve orthogonal polynomials; instead of them we are dealing with averages \(\langle H(u)E(v) \rangle_{\alpha,N}\).

Assume for simplicity that \(\alpha\) is a pure atomic measure. Then we may speak about probability \(\text{Prob}(X)\) of each individual configuration \(X\). We may assume \(X \subset \mathcal{X}\), where \(\mathcal{X}\) is a discrete subset of \(\mathbb{R}\), the support of \(\alpha\).

By definition, the \(m\)-point correlation function \(\rho_m(Y)\), where \(Y = (y_1, \ldots, y_m)\) is a subset of \(\mathcal{X}\), is given by

\[ (3.1) \quad \rho_m(Y) = \sum_{X \supseteq Y} \text{Prob}(X). \]

(To pass from correlation measures to correlation functions we use the counting measure on \(\mathcal{X}\) as the reference measure.)

**Proposition 3.2.** Let \(\alpha\) be a pure atomic measure supported by a discrete subset \(\mathcal{X} \subset \mathbb{R}\). Then correlation functions \((3.1)\) are given by a determinantal formula

\[ \rho_m(Y) = \det(K(y_i, y_j)) \]

with the correlation kernel

\[ K(x, y) = \text{Res}_{u=y} \left( \frac{\langle E(-x)H(u) \rangle_{\alpha,N}}{x - y} \right) \]

(for \(x = y\) the value of the kernel can be found using the L’Hospital rule).

A proof of this result is given in [18], §2.8. To make a connection with the notation of [18], note that

\[ (E(-v)H(u))(X) = \frac{u^N}{v^N} \prod_{i=1}^{N} \frac{v - x_i}{u - x_i}. \]
The argument of [18] relies on identity (2.2) (see formula 2.8.4 in [18]). We can now obtain this identity as a direct corollary of Theorem 3.1 and Proposition 2.2.

As shown in [18], §2.8, the result of Proposition 3.2 implies the classical expression of the kernel in terms of orthogonal polynomials.

A similar approach is presented in detail in the next section for the more difficult case of \( z \)-measures.

Finally, it is worth noting that the assumption that \( \alpha \) is pure atomic can be removed. Then instead of residues of functions with isolated singularities one has to deal with jumps on a contour of functions which are holomorphic outside this contour (in our case, the contour is the support of \( \alpha \)).

4. **Z-measures as Giambelli compatible point processes**

In this section, the “source” space \( S \) is the set \( \mathcal{Y} \) of Young diagrams and as \( P \) we take the so-called \( z \)-measures. The related point processes live on the discrete space \( \mathbb{Z}' = \mathbb{Z} + \frac{1}{2} \), the lattice of semi–integers. We give only necessary definitions and refer to ([12], [54]) for motivation and details. We show that the \( z \)-measures are Giambelli compatible. Using this fact we then prove that the related lattice point processes are determinantal, and we derive a formula for their correlation kernel.

4.1. **The \( z \)-measures.** Let

\[
(a)_k = a(a + 1) \ldots (a + k - 1), \quad (a)_0 = 1,
\]

denote the Pochhammer symbol. We fix two complex parameters \( z, z' \) such that the numbers \((z)_k(z')_k\) and \((-z)_k(-z')_k\) are real and strictly positive for any \( k = 1, 2, \ldots \). These assumptions on \( z, z' \) are satisfied if and only if one of the following two conditions holds:

- either \( z' = \overline{z} \) and \( z \in \mathbb{C} \setminus \mathbb{Z} \)
- or \( z, z' \in \mathbb{R} \) and there exists \( m \in \mathbb{Z} \) such that \( m < z, z' < m+1 \).

Let \( \mathcal{Y}_n \) denote the finite set of Young diagrams with \( n \) boxes \((n = 1, 2, \ldots)\). The \( z \)-measure on \( \mathcal{Y}_n \) with parameters \( z, z' \) is defined by

\[
M_{z,z'}^{(n)}(\lambda) = \frac{\prod_{(i,j) \in \lambda} (z + j - i)(z' + j - i)}{(zz')_n^n} \frac{(\dim \lambda)^2}{n!}, \quad \lambda \in \mathcal{Y}_n,
\]

where “\((i, j) \in \lambda\)” stands for the box of diagram \( \lambda \) with row coordinate \( i \) and column coordinate \( j \), and \( \dim \lambda \) denotes the number of the standard Young tableaux of shape \( \lambda \). This is a probability measure.
Further, let $\xi \in (0, 1)$ be an additional parameter. The mixed measure with parameters $z, z', \xi$ is a probability measure on the set $\mathcal{Y}$ of all Young diagrams defined by

$$M_{z, z', \xi}^{(n)}(\lambda) = M_{z, z'}^{(n)}(\lambda) \cdot (1 - \xi) z^z (z')^{n} \xi^n, \quad n = |\lambda|,$$

where $M_{z, z'}^{(0)}(\emptyset) := 1$.

### 4.2. Giambelli compatibility

Recall that $\Lambda$ can be viewed as the algebra of “polynomial functions” on $\mathcal{Y}$. We write $\langle f \rangle_{M_{z, z', \xi}}$ for the expectation of a function $f$ with respect to the probability measure $M_{z, z', \xi}$. It turns out that the quantities $\langle f \rangle_{M_{z, z', \xi}}$ are readily computed for the Frobenius–Schur functions $F_{s_{\mu}}$.

**Proposition 4.1.** For any $\mu \in \mathcal{Y}$,

$$\langle F_{s_{\mu}} \rangle_{M_{z, z', \xi}} = (1 - \xi)^{z^z} \sum_{n=m}^{\infty} \frac{(z')^n \xi^n n^m}{n!} \sum_{\lambda \in \mathcal{Y}_n} \dim(\mu, \lambda) \frac{M_{z, z'}^{(n)}(\lambda)}{\dim \lambda}.$$

**Proof.** The computation relies on formula (1.5). First of all, note that this formula implies that $F_{s_{\mu}}(\lambda) \geq 0$ for any $\lambda$, which justifies transformations of infinite sums below.

From (1.5) we obtain

$$\langle F_{s_{\mu}} \rangle_{M_{z, z', \xi}} = (1 - \xi)^{z^z} \sum_{n=m}^{\infty} \frac{(z')^n \xi^n n^m}{n!} \sum_{\lambda \in \mathcal{Y}_n} \dim(\mu, \lambda) \frac{M_{z, z'}^{(n)}(\lambda)}{\dim \lambda},$$

where $m = |\mu|$. Now we use the fact that the function

$$\varphi_{z, z'}(\lambda) := \frac{M_{z, z'}^{(\lambda)}(\lambda)}{\dim \lambda}$$

is harmonic on the Young graph in the sense of Vershik and Kerov [59]. That is,

$$\varphi_{z, z'}(\mu) = \sum_{\lambda: \mu \nrightarrow \lambda} \varphi_{z, z'}(\lambda) \quad \forall \mu \in \mathcal{Y}_m,$$

see [53, 44, 41, 50] for different proofs. Iterating this relation we obtain

$$\varphi_{z, z'}(\mu) = \sum_{\lambda \in \mathcal{Y}_n} \dim(\mu, \lambda) \varphi_{z, z'}(\lambda) \quad \forall n \geq m.$$
Plugging this into (4.3) gives

\[
\langle F s_{\mu} \rangle_{M_{z,z',\xi}} = \frac{M_{z,z'}^{(m)}(\mu)}{\dim \mu} \left(1 - \xi\right)^{zz'} \sum_{n \geq m} \frac{(zz')_{n} \xi^{n} n^{1+m}}{n!} \sum_{n \geq m} \frac{(zz' + m)_{n-m} \xi^{n-m}}{(n-m)!}.
\]

Finally, observe that the latter sum equals \((1 - \xi)^{-zz'-m}\), and use the explicit expression for \(M_{z,z'}^{(m)}(\mu)\) (see (4.1)). This gives (4.2).

The first consequence of Proposition 4.1 is that all functions \(f(\lambda)\), where \(f \in \Lambda\), are summable with respect to \(M_{z,z',\xi}\). Therefore, the map \(f \mapsto f(\cdot)\) determines a morphism \(\phi : \Lambda \rightarrow A(\mathbb{Y}, M_{z,z',\xi})\).

**Proposition 4.2.** The triple \((\mathbb{Y}, M_{z,z',\xi}, \phi)\) is Giambelli compatible.

**Proof.** Let us show that

\[
\langle F s_{\mu} \rangle_{M_{z,z',\xi}} = \det \left[ \langle F s_{(p_i|q_j)} \rangle_{M_{z,z',\xi}} \right]_{i,j=1}^{d}
\]

Indeed, we can rewrite (4.2) in terms of Frobenius coordinates: for the product over the boxes this is easy, and for the dimension we use the formula

\[
\frac{\dim \mu}{|\mu|!} = \prod_{i<j} (p_i - p_j)(q_i - q_j) \prod_{i} p_i! q_i! \prod_{i,j} (p_i + q_j + 1) = \det \left[ \frac{1}{p_i! q_j!(p_i + q_j + 1)} \right]
\]

(see, e.g., [53]). Then we obtain (4.4) which in turn implies the claim, by virtue of Proposition 2.3.

**4.3. Computation of \(\langle H(u)E(v) \rangle_{M_{z,z',\xi}}\).** Below \(u\) and \(v\) are assumed to be complex variables (rather than formal parameters, as in §2). To ensure the existence \(\langle H(u)E(v) \rangle_{M_{z,z',\xi}}\) we need an estimate of \(\langle H(u)E(v) \rangle(\lambda)\). It is provided by the next lemma, which is stated in a slightly greater generality, because we will need a similar estimate in §5.

**Lemma 4.3.** Assume that \(u\) is a complex variable subject to constraints

\[
\varepsilon < |\arg(u)| < \pi - \varepsilon
\]

with a certain \(\varepsilon > 0\). Take \(\delta > 0\) and two infinite sequences \(\alpha_1 \geq \alpha_2 \geq \cdots \geq 0\), \(\beta_1 \geq \beta_2 \geq \cdots \geq 0\) such that

\[
\sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq \delta.
\]
Then
\[ \prod_{i=1}^{\infty} \frac{1 + \beta_i u^{-1}}{1 - \alpha_i u^{-1}} \leq e^{C \delta |u^{-1}|}, \]
where \( C = C(\varepsilon) > 0 \) is a constant depending only on \( \varepsilon \).

**Proof.** The numerator admits a trivial estimate,
\[ |1 + \beta_i u^{-1}| \leq 1 + |\beta_i u^{-1}| \leq e^{\beta_i |u^{-1}|}, \]
which implies
\[ \prod_{i=1}^{\infty} (1 + \beta_i u^{-1}) \leq e^{\delta |u^{-1}|}. \]

As for the denominator, we will estimate it separately for \( i \in I \) and for \( i \notin I \), where
\[ I = \{ i \mid \alpha_i |u^{-1}| \geq \frac{1}{2} \}. \]
Assume first \( i \notin I \). Then \( \alpha_i |u^{-1}| \leq \frac{1}{2} \), whence
\[ \left| \frac{1}{1 - \alpha_i u^{-1}} \right| \leq \frac{1}{1 - \alpha_i |u^{-1}|} \leq 1 + 2\alpha_i |u^{-1}|, \]
because
\[ \frac{1}{1 - x} \leq 1 + 2x, \quad 0 \leq x \leq \frac{1}{2}. \]

Therefore,
\[ \left| \prod_{i \notin I} \frac{1}{1 - \alpha_i u^{-1}} \right| \leq \prod_{i \notin I} (1 + 2\alpha_i |u^{-1}|) \leq e^{2\delta |u^{-1}|}. \]

Now assume that \( i \in I \). Then \( \alpha_i |u^{-1}| \geq \frac{1}{2} \) and \( \alpha_i \geq \frac{1}{2} |u| \). Since the sum of all \( \alpha_i \) does not exceed \( \delta \), we obtain that \( |I| \) (the cardinality of \( I \)) does not exceed \( 2\delta |u^{-1}| \). Next, the constraints on \( \arg(u) \) imply that \( |\text{Im} u^{-1}| \geq C_1 |u^{-1}| \) with a certain constant \( C_1 \) depending only on \( \varepsilon \). Therefore
\[ |\alpha_i| \text{Im} u^{-1}| \geq \alpha_i C_1 |u^{-1}|, \]
whence
\[ \left| \frac{1}{1 - \alpha_i u^{-1}} \right| \leq \frac{1}{|\alpha_i| \text{Im} u^{-1}|} \leq \frac{2}{C_1} \]
and
\[ \left| \prod_{i \in I} \frac{1}{1 - \alpha_i u^{-1}} \right| \leq \left( \frac{2}{C_1} \right)^{|I|} \leq \left( \frac{2}{C_1} \right)^{2\delta |u^{-1}|} \leq e^{C_2 \delta |u^{-1}|}. \]

Combining all these estimates we obtain the desired inequality with \( C = 3 + C_2 \). \( \Box \)
Corollary 4.4. Fix \( \varepsilon > 0 \) and let \( u_1, \ldots, u_m, v_1, \ldots, v_m \) be complex variables subject to constraints

\[ \varepsilon < |\arg(u_i)| < \pi - \varepsilon, \quad \varepsilon < |\arg(v_i)| < \pi - \varepsilon, \quad i = 1, \ldots, m, \]

and such that \( |u_i|, |v_i| \) are large enough (greater than a constant depending on \( \varepsilon \) and \( \xi \)). Then

\[
\sum_{\lambda \in \mathbb{Y}} |(H(u_1)E(-v_1) \ldots H(u_m)E(-v_m))(\lambda)| M_{z,z',\xi}(\lambda) < \infty.
\]

Proof. Recall that

\[
H(u)(\lambda) = \prod_{j=1}^{d} \frac{1 + b_j u^{-1}}{1 - a_j u^{-1}}, \quad E(-v)(\lambda) = \prod_{j=1}^{d} \frac{1 - a_j v^{-1}}{1 + b_j v^{-1}}
\]

and note that \( \sum_j (a_j + b_j) = n := |\lambda| \). By virtue of Lemma 4.3,

\[
|\langle \sum \rangle| \leq e^{C \sum_{i} (|u_i|^{-1} + |v_i|^{-1})}
\]

with the same constant \( C = C(\varepsilon) \) as in the lemma. Then it follows from the definition of \( M_{z,z',\xi} \) that the sum in question is finite provided that \( u_i \) and \( v_i \) are so large that

\[
e^{C \sum_{i} (|u_i|^{-1} + |v_i|^{-1})} \xi < 1.
\]

Below we use the standard notation \( F(a, b; c; \zeta) \) for the Gauss hypergeometric function with parameters \( a, b, c \) and argument \( \zeta \). Recall that \( F(a, b; c; \zeta) \) is well defined for \( \zeta \in \mathbb{C} \setminus [1, +\infty) \). Moreover, \( F(a, b; c; \zeta) / \Gamma(c) \) is an entire function of parameters \( (a, b, c) \in \mathbb{C}^3 \). In particular, \( F(a, b; c; \zeta) \) is a meromorphic function in \( c \) with poles at \( c = 0, -1, -2, \ldots \).

By Lemma 4.3, the average \( \langle H(u)E(v) \rangle_{M_{z,z',\xi}} \) is well defined and is an analytic function in \( (u, v) \), provided that \( (u, v) \) range over a suitable domain in \( \mathbb{C}^2 \).

Proposition 4.5. The average \( \langle H(u)E(v) \rangle_{M_{z,z',\xi}} \) is given by the formula:

\[
\langle H(u)E(v) \rangle_{M_{z,z',\xi}} = F(z, z'; -u + \frac{1}{2}; \frac{\xi}{\xi - 1}) F(-z, -z'; -v + \frac{1}{2}; \frac{\xi}{\xi - 1})
\]

\[
+ \frac{zz'\xi}{(1 - \xi)(u - \frac{1}{2})(v - \frac{1}{2})} F(z + 1, z' + 1; -u + \frac{3}{2}; \frac{\xi}{\xi - 1}) F(-z + 1, -z' + 1, -v + \frac{3}{2}; \frac{\xi}{\xi - 1}).
\]
Proof. By (1.3),

\[(4.6) \quad (H(u)E(v))_\lambda = 1 + (u + v) \sum_{p, q \geq 0} \frac{F_{s(p|q)}(\lambda)}{(u - \frac{1}{2})^{p+1}(v - \frac{1}{2})^{q+1}}.\]

Since we know \(\langle F_{s(p|q)} \rangle_{M_{s',\xi}}\), it is tempting to average this relation over \(\lambda\)'s. However, we have to be careful at this point, because now we are dealing with actual functions in \((u, v)\) (not with formal series in \(u^{-1}, v^{-1}\), as in §2). Moreover, we cannot even expect that the resulting expression would possess an expansion at \((u, v) = (\infty, \infty)\), because, for a fixed \(\xi\), the right-hand side of (4.5) is not a meromorphic function near \((u, v) = (\infty, \infty)\) and hence does not admit such an expansion.

This difficulty can be overcome using the following trick: we will regard \(\xi\) not as a numeric parameter but as a formal indeterminate. Observe that both sides of (4.5) are analytic functions in \(\xi\) near \(\xi = 0\) such that the coefficients of the Taylor expansion at \(\xi = 0\) are rational functions in \((u, v)\) admitting an expansion at \((u, v) = (\infty, \infty)\) (in more detail, these rational functions are finite sums \(\sum f_i(u)g_i(v)\), where \(f_i\) and \(g_i\) are rational functions in one variable). Thus, we may prove (4.5) as an identity in the algebra of formal power series in \(u^{-1}, v^{-1},\) and \(\xi\).

This provides a justification for the formal computation below.

By (4.2), for the hook diagram \(\mu = (p + 1, 1^q)\) we have

\[\langle F_{s(p|q)} \rangle_{M_{s',\xi}} = \left(\frac{\xi}{\xi - 1}\right)^{p+q+1} \frac{z' \cdot (z + 1)_p(z' + 1)_p(-z + 1)_q(-z' + 1)_q}{p!q!(p + q + 1)}.\]

Denote the expression in the right-hand side by \(A(p, q)\). Then we obtain from (4.6)

\[\langle H(u)E(v) \rangle_{M_{s',\xi}} = 1 + (u + v) \sum_{p, q \geq 0} \frac{A(p, q)}{(u - \frac{1}{2})^{p+1}(v - \frac{1}{2})^{q+1}}.\]

Write

\[u + v = (u - p - \frac{1}{2}) + (v - q - \frac{1}{2}) + (p + q + 1),\]

and plug this expression into the sum. Then we obtain

\[\langle H(u)E(v) \rangle_{M_{s',\xi}} = 1 + \sum_{p, q \geq 0} \frac{A(p, q)}{(u - \frac{1}{2})^{p+1}(v - \frac{1}{2})^{q+1}} + \sum_{p, q \geq 0} \frac{A(p, q)}{(u - \frac{1}{2})^{p+1}(v - \frac{1}{2})^{q}} + \sum_{p, q \geq 0} \frac{(p + q + 1)A(p, q)}{(u - \frac{1}{2})^{p+1}(v - \frac{1}{2})^{q+1}}.\]

Decompose the first sum into two parts in such a way that the first part corresponds to summation over index \(q\) with \(p\) being equal to zero, while the second part is summation over \(p \geq 1\) and \(q \geq 0\). Replace index
by $p - 1$ in the second part, then $A(p, q)$ is replaced by $A(p + 1, q)$. Decompose the second sum in the same way, and obtain

$$
\langle H(u)E(v) \rangle_{M_{z,z'}} = 1 + \sum_{q \geq 0} \frac{A(0, q)}{(v - \frac{1}{2})^{q+1}} + \sum_{p \geq 0} \frac{A(p, 0)}{(u - \frac{1}{2})^{p+1}}
$$

$$
+ \sum_{p, q \geq 0} \frac{(p + q + 1)A(p, q) + A(p, q + 1) + A(p + 1, q)}{(u - \frac{1}{2})^{p+1}(v - \frac{1}{2})^{q+1}}.
$$

The first two sums can be immediately rewritten in terms of hypergeometric functions. To compute the last sum we observe that

$$
A(p + 1, q) + A(p, q + 1) + (p + q + 1)A(p, q)
$$

$$
= \left( \frac{\xi}{1 - \xi} \right)^{p+q+2} \frac{(z)pq+1(z')pq+1(-z)q+1(-z')q+1}{(p+1)!(q+1)!}
$$

$$
+ \frac{1}{1 - \xi} \left( \frac{\xi}{1 - \xi} \right)^{p+q+1} \frac{zz'(z+1)p(z'+1)p(-z+1)q(-z'+1)q}{p!q!}.
$$

It follows that the last sum can be decomposed into two sums and rewritten in terms of hypergeometric functions. With these preparations we find

$$
\langle H(u)E(v) \rangle_{M_{z,z'}, \xi} = 1 + 
\left( F(-z, -z'; -v + \frac{1}{2}; \frac{\xi}{1 - \xi}) - 1 \right)
$$

$$
+ \left( F(z, z'; -u + \frac{1}{2}; \frac{\xi}{1 - \xi}) - 1 \right)
$$

$$
+ \left( F(-z, -z'; -v + \frac{1}{2}; \frac{\xi}{1 - \xi}) - 1 \right)
$$

$$
+ \frac{\xi zz'}{(1 - \xi)^2(u - \frac{1}{2})(v - \frac{1}{2})} \frac{F(z + 1, z' + 1; -u + \frac{3}{2}; \frac{\xi}{1 - \xi})}{F(-z + 1, -z' + 1; -v + \frac{3}{2}; \frac{\xi}{1 - \xi})}.
$$

After simplifications we obtain desired formula (4.5). \qed

### 4.4. Correlation measures and controlling measures.

Set

$$
\mathbb{Z}' = \mathbb{Z} + \frac{1}{2} = \left\{ \ldots, -\frac{3}{2}, -1, \frac{1}{2}, \frac{3}{2}, \ldots \right\}.
$$

As in §1.5, for any $\lambda \in \mathbb{Y}$ we define the modified Frobenius coordinates of $\lambda$ as

$$
a_i = a_i(\lambda) = p_i + \frac{1}{2} = \lambda_i - i + \frac{1}{2}, \quad b_i = b_i(\lambda) = q_i + \frac{1}{2} = \lambda'_i - i + \frac{1}{2},
$$
where \( i = 1, \ldots, d \) and \( d = d(\lambda) \) denotes the number of diagonal boxes in \( \lambda \). Note that

\[
a_1 > \ldots > a_d > 0, \quad b_1 > b_2 > \ldots > b_d > 0, \quad \sum_{i=1}^{d} (a_i + b_i) = |\lambda|.
\]

Using this notation we assign to an arbitrary Young diagram a point configuration \( X = X(\lambda) \in \text{Conf}(Z') \), as follows

\[
X(\lambda) = \{-b_1, \ldots, -b_d, a_d, \ldots, a_1\}.
\]

Note that a point configuration \( X \) on \( Z' \) comes from a Young diagram \( \lambda \) if and only if \( X \) is finite and balanced in the sense that it has equally many points to the left and to the right of zero.

Thus, the correspondence \( \lambda \mapsto X(\lambda) \) defines a bijection between Young diagrams \( \lambda \) and balanced configurations \( X \), and we will often identify \( \lambda \) and \( X(\lambda) \). Assume we are given a probability measure \( M \) on \( \mathcal{Y} \). Then we obtain a point process on \( Z' \) with “source space” \( (\mathcal{Y}, M) \).

Let \( \rho_m \) stand for the \( m \)th correlation measure of this process; \( \rho_m \) is supported by the subset

\[
(Z')^m_0 = \{(x_1, \ldots, x_m) \in (Z')^m \mid x_i \neq x_j, \ i \neq j\}.
\]

Since \( Z' \) is a discrete space, there is no essential difference between correlation measures and correlation functions (see the end of section 2; here we take the counting measure on \( Z' \) as the reference measure). Note that \( \rho_m(x_1, \ldots, x_m) \) is the probability that the random configuration contains \( \{x_1, \ldots, x_m\} \).

We will introduce one more concept, that of controlling measures \[53\]. The definition is as follows. First, to an arbitrary \( \lambda \in \mathcal{Y} \) we assign a measure on \( Z' \):

\[
\sigma_\lambda = \sum_{i=1}^{d} (a_i \delta_{a_i} + b_i \delta_{-b_i}),
\]

where \( a_i, b_i \) are the modified Frobenius coordinates of \( \lambda \) and \( \delta_x \) stands for the delta measure at \( x \in Z' \). Second, for any \( m = 1, 2, \ldots \) we take the \( m \)th power \( (\sigma_\lambda)^{\otimes m} \), which is a measure on \( (Z')^m \), and then average it with respect to our initial probability measure \( M \):

\[
\sigma_m = \langle (\sigma_\lambda)^{\otimes m} \rangle_M = \sum_{\lambda \in \mathcal{Y}} (\sigma_\lambda)^{\otimes m} M(\lambda).
\]

**Lemma 4.6.** We have

\[
\sigma_m = |x_1 \ldots x_m|(\rho_m + \ldots),
\]
where the dots denote a measure supported by \((Z')^m \setminus (Z')_0^m\). In particular, on \((Z')_0^m\), the measure \(\sigma_m\) coincides with the measure \(\rho_m\) multiplied by the function \(|x_1 \ldots x_m|\).

**Proof.** Assume first that \(M\) is the delta measure at a point \(\lambda \in \mathcal{Y}\). Then we have

\[
\sigma_m = \sum_{x_1, \ldots, x_m \in X(\lambda)} |x_1 \ldots x_m| \delta_{x_1} \otimes \cdots \otimes \delta_{x_m},
\]

\[
\rho_m = \sum_{x_1, \ldots, x_m \in X(\lambda) \text{ pairwise distinct}} \delta_{x_1} \otimes \cdots \otimes \delta_{x_m}.
\]

Clearly, this implies the desired equality in the special case \(M = \delta_\lambda\). In the general case, both \(\sigma_m\) and \(\rho_m\) are obtained from these expressions by averaging with respect to \(M\), which completes the proof.

A detailed description of the “rest measure” supported by \((Z')^m \setminus (Z')_0^m\) is given in [53]. □

Recall that the Cauchy transform of a measure \(\nu\) on \(\mathbb{R}^m\) is given by

\[
\hat{\nu}(u_1, \ldots, u_m) = \int_{\mathbb{R}^m} \frac{\nu(dx)}{(u_1 - x_1) \ldots (u_m - x_m)}, \quad (u_1, \ldots, u_m) \in (\mathbb{C} \setminus \mathbb{R})^m.
\]

It is well defined if \(\nu\) satisfies the growth condition

\[
\int_{\mathbb{R}^m} \frac{\nu(dx)}{(1 + |x_1|) \ldots (1 + |x_m|)} < \infty.
\]

Note that the initial measure \(\nu\) can be reconstructed from its transform \(\hat{\nu}\).

In particular, if \(\nu\) is a pure atomic measure whose support has no accumulation points, then \(\hat{\nu}\) is a meromorphic function in each variable \(u_i\), and for any point \((x_1, \ldots, x_m)\) in the support of \(\nu\), we have

\[
\nu(x_1, \ldots, x_m) = \text{Res}_{u_1 = x_1} \ldots \text{Res}_{u_m = x_m} \hat{\nu}(u_1, \ldots, u_m).
\]

**Lemma 4.7.** Let \(M\) be a probability measure on \(\mathcal{Y}\) and \(\sigma_m\) be the corresponding \(m\)-th controlling measure on \((Z')^m\). Assume that \(\sigma_m\) satisfies the growth condition ensuring the existence of the Cauchy transform \(\hat{\sigma}_m(u_1, \ldots, u_m)\). Further, assume that the average

\[
\langle H(u_1)E(-v_1) \ldots H(u_m)E(-v_m) \rangle_M
\]

is well defined for \((u_1, \ldots, u_m)\) and \((v_1, \ldots, v_m)\) ranging over a domain \(\mathcal{D} \subset (\mathbb{C} \setminus \mathbb{R})^m\).
Then for any \((u_1, \ldots, u_m) \in D\) we have

\[
\hat{\sigma}_m(u_1, \ldots, u_m) = u_1 \ldots u_m 
\times \left\{ \frac{\partial^m}{\partial v_1 \ldots \partial v_m} (H(u_1)E(-v_1) \ldots H(u_m)E(-v_m))_M \right\}_{v_1 = u_1 \atop \vdots \atop v_m = u_m}.
\]

**Proof.** Assume first that \(M\) is the delta measure at \(\lambda \in \mathbb{Y}\). Then

\[
\sigma_m = \sigma_{\lambda}^\otimes = \left( \sum_{x \in X(\lambda)} |x| \delta_x \right)^\otimes \subseteq \mathbb{C}^m,
\]

whence

\[
\hat{\sigma}_m(u_1, \ldots, u_m) = \sum_{x_1, \ldots, x_m \in X(\lambda)} \frac{|x_1 \ldots x_m|}{(u_1 - x_1) \ldots (u_m - x_m)}.
\]

On the other hand,

\[
\langle H(u_1)E(-v_1) \ldots H(u_m)E(-v_m) \rangle_M 
= (H(u_1)E(-v_1) \ldots H(u_m)E(-v_m))(\lambda) 
= \prod_{i=1}^m \prod_{j=1}^d \frac{(1 + b_j u_i^{-1})(1 - a_j v_i^{-1})}{(1 - a_j u_i^{-1})(1 + b_j v_i^{-1})}.
\]

Differentiating over \(v_1, \ldots, v_m\) and then specializing \(v_i = u_i\) for all \(i = 1, \ldots, m\) gives

\[
\prod_{i=1}^m \left( \frac{1}{u_i} \sum_{j=1}^d \left( \frac{a_j}{u_i - a_j} + \frac{b_j}{u_i + b_j} \right) \right),
\]

which leads to the same result after multiplication by \(u_1 \ldots u_m\).

Thus, we have verified the desired relation for \(M = \delta_\lambda\). In the general case, we average over \(\lambda\)'s with respect to measure \(M\). To justify the interchange of the operation “differentiation over \(v_i\)’s followed by specialization \(v_i = u_i\)” with the averaging operation, we observe that the former operation can be written as a multiple contour Cauchy-type integral.

Let us abbreviate

\[
F_m(u_1, \ldots, u_m) = \left\{ \frac{\partial^m}{\partial v_1 \ldots \partial v_m} (H(u_1)E(-v_1) \ldots H(u_m)E(-v_m))_M \right\}_{v_1 = u_1 \atop \vdots \atop v_m = u_m}
\]

By the above lemma, this function, which is initially defined in a domain \(D \subset (\mathbb{C} \setminus \mathbb{R})^m\), can be extended to the whole \(\mathbb{C}^m\) as a function
which is meromorphic in each variable $u_i$ with possible poles at points of the lattice $\mathbb{Z}'$.

**Corollary 4.8.** Under the hypotheses of the Lemma 4.7, for any $m$-tuple of pairwise distinct numbers $x_1, \ldots, x_m \in \mathbb{Z}'$,

$$\rho_m(x_1, \ldots, x_m) = \text{sgn}(x_1) \ldots \text{sgn}(x_m) \operatorname{Res} \ldots \operatorname{Res} F_m(u_1, \ldots, u_m),$$

where $\text{sgn}(x) = 1$ for $x > 0$ and $\text{sgn}(x) = -1$ for $x < 0$.

**Proof.** Indeed, by (4.8),

$$\sigma_m(x_1, \ldots, x_m) = \operatorname{Res} \ldots \operatorname{Res} \tilde{\sigma}_m(u_1, \ldots, u_m).$$

Then we apply Lemma 4.7. $\square$

4.5. **Computation of the correlation functions.** Here we apply the above result to computing the correlation functions for the point process determined by $M = M_{z, z', \xi}$.

First of all, it should be noted that the two assumptions on $M$ made in Lemma 4.7 are satisfied for $M = M_{z, z', \xi}$.

In more detail, one of the assumptions was the growth condition on $\sigma_m$. We claim that for $M = M_{z, z', \xi}$, the measure $\sigma_m$ actually satisfies a stronger condition: it is a finite measure. To see this, we observe that $\sigma_{\lambda}$ has mass $\sum (a_i + b_i) = |\lambda|$, hence $\sigma_{\lambda}^\otimes m$ has mass $|\lambda|^m$. Averaging over $\lambda$’s and recalling the definition of $M_{z, z', \xi}$ we obtain that the total mass of $\sigma_m$ equals

$$(1 - \xi)^{zz'} \sum_{n=0}^{\infty} n^m \frac{(zz')_n \xi^n}{n!} < \infty.$$

Another assumption was that the average of

$$(H(u_1)E(-v_1) \ldots H(u_m)E(-v_m))(\lambda)$$

with respect to $M$ exists provided that $(u_1, \ldots, u_m)$ and $(v_1, \ldots, v_m)$ range in a suitable domain $\mathcal{D}$. For $M = M_{z, z', \xi}$ this is indeed true due to the estimate established in Lemma 4.3. As $\mathcal{D}$ one can take any domain of the form

$$\varepsilon < | \arg(u_i) | < \pi - \varepsilon, \quad |u_i| \gg 0, \quad i = 1, \ldots, m.$$
Theorem 4.9. The point process on $\mathbb{Z}'$ corresponding to the measure $M_{z,z',\xi}$ is determinantal and its correlation kernel can be written as

\[
K(x, y) = \begin{cases} 
\text{Res}_{u=y} \langle E(-x)H(u) \rangle, & x > 0, y > 0, x \neq y, \\
-\text{Res}_{u=x} \text{Res}_{u'=y} \langle E(-u')H(u) \rangle, & x < 0, y > 0, \\
\langle E(-x)H(y) \rangle, & x > 0, y < 0, \\
-\text{Res}_{u=x} \langle E(-u)H(y) \rangle, & x < 0, y < 0, x \neq y,
\end{cases}
\]

(4.9)

where $\langle \cdot \rangle$ means $\langle \cdot \rangle_{M_{z,z',\xi}}$, and the indeterminancy arising for $x = y$ is resolved via the L'Hospital rule.

The statement of the theorem needs a few comments:

1) By Proposition 4.5, the quantity $\langle E(-v)H(u) \rangle$, which is initially defined (as a function in $(u,v)$) in a domain of $\mathbb{C}^2$, actually can be extended to a meromorphic function on the whole $\mathbb{C}^2$. In the above formula for the kernel we use this meromorphic extension.

2) Note that $\langle E(-v)H(u) \rangle$ has poles at $u \in \mathbb{Z}'_- = \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\}$ and at $v \in \mathbb{Z}'_+ = \{\frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots\}$. This is readily seen from the formula of Proposition 4.5.

3) Let us explain what we mean by application of the L'Hospital rule. In the proof below we actually show that the diagonal entries of the kernel are given by

$$K(x, x) = \text{Res}_{u=x} \langle G(u) \rangle, \quad x \in \mathbb{Z'},$$

where, by definition,

$$G(u) = G(u, \lambda) = \left( \frac{\partial}{\partial v} E(-v)H(u) \right)_{v=u}.$$ 

On the other hand, the above expression for $K(x, y)$ makes sense not only when $x, y$ are (distinct) points on the lattice $\mathbb{Z}'$ but also if $y \in \mathbb{Z}'_+$ and $x$ is a complex number with $\text{Re} x > 0$, or if $x \in \mathbb{Z}'_-$ and $y$ is a complex number with $\text{Re} y < 0$. (Indeed, this follows from the preceding comment.) Then we can apply the L'Hospital rule to examine the limit values of this extended kernel on the diagonal, and it is readily seen that

$$\langle G(y) \rangle = \lim_{x \to y} K(x, y), \quad y \in \mathbb{Z}'_-; \quad \langle G(x) \rangle = \lim_{y \to x} K(x, y), \quad x \in \mathbb{Z}'_+.$$
Proof. Fix \( m = 1, 2, \ldots \) and assume that \( u_1, \ldots, u_m, v_1, \ldots, v_m \) are complex variables subject to appropriate constraints on the argument and the modulus, as in Corollary 4.4. This will ensure existence of the necessary averages.

We start with the determinantal identity of Proposition 2.2, which we rewrite as

\[
\left\langle \det \begin{pmatrix}
\frac{E(-v_1)H(u_1)}{v_1-u_1} & \frac{E(-v_1)H(u_m)}{v_1-u_m} \\
\vdots & \vdots \\
\frac{E(-v_m)H(u_1)}{v_m-u_1} & \frac{E(-v_m)H(u_m)}{v_m-u_m}
\end{pmatrix}\right\rangle = \det \begin{pmatrix}
\frac{E(-v_1)H(u_1)}{v_1-u_1} & \frac{E(-v_1)H(u_m)}{v_1-u_m} \\
\vdots & \vdots \\
\frac{E(-v_m)H(u_1)}{v_m-u_1} & \frac{E(-v_m)H(u_m)}{v_m-u_m}
\end{pmatrix}
\]

To justify the passage from formal series to actual functions we use the same trick as in the proof of Proposition 4.5.

We multiply both sides of this identity by the product \( \prod_{i=1}^{m} (v_i - u_i) \)
(in more detail, we multiply the \( i \)th row of the matrix in the left–hand side or in the right–hand side by \( (v_i - u_i) \)), then we differentiate with respect to \( v_1, \ldots, v_m \), and finally we specialize \( v_1 = u_1, \ldots, v_m = u_m \).

The multiplication of the \( i \)th row by \( (v_i - u_i) \) has the following consequences: First, the same factor in the denominator of the diagonal entry is cancelled. Second, when we apply \( \partial/\partial v_i \) to an off–diagonal entry, we only have to differentiate this factor \( (v_i - u_i) \), because \( v_i - u_i \) vanishes after specialization \( v_i = u_i \).

Using these observations and the notation \( G(u) \) introduce above we obtain the identity

\[
\left\langle \det \begin{pmatrix}
\frac{G(u_1)}{E(-u_2)H(u_1)} & \frac{E(-u_1)H(u_2)}{u_1-u_2} & \frac{E(-u_1)H(u_m)}{u_1-u_m} \\
\frac{E(-u_2)H(u_1)}{u_2-u_1} & G(u_2) & \frac{E(-u_2)H(u_m)}{u_2-u_m} \\
\frac{E(-u_m)H(u_1)}{u_m-u_1} & \frac{E(-u_m)H(u_2)}{u_m-u_2} & G(u_m)
\end{pmatrix}\right\rangle = \det \begin{pmatrix}
\langle G(u_1) \rangle & \frac{E(-u_1)H(u_2)}{u_1-u_2} & \frac{E(-u_1)H(u_m)}{u_1-u_m} \\
\frac{E(-u_2)H(u_1)}{u_2-u_1} & \langle G(u_2) \rangle & \frac{E(-u_2)H(u_m)}{u_2-u_m} \\
\frac{E(-u_m)H(u_1)}{u_m-u_1} & \frac{E(-u_m)H(u_2)}{u_m-u_2} & \langle G(u_m) \rangle
\end{pmatrix}
\]
Recall that $E(-u)H(u) \equiv 1$. Using this fact we can simplify the determinant in the left-hand side, which gives

\[
(4.10) \quad \left\langle \det \begin{pmatrix}
\frac{G(u_1)}{u_1 - u_2} & \frac{1}{u_2 - u_1} & \cdots & \frac{1}{u_1 - u_m} \\
\frac{G(u_2)}{u_2 - u_1} & G(u_2) & \cdots & \frac{1}{u_1 - u_m} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{u_m - u_1} & \frac{1}{u_m - u_2} & \cdots & G(u_m)
\end{pmatrix} \right\rangle
= \det \begin{pmatrix}
\langle G(u_1) \rangle & \langle E(-u_1)H(u_2) \rangle & \cdots & \langle E(-u_1)H(u_m) \rangle \\
\langle E(-u_2)H(u_1) \rangle & \langle G(u_2) \rangle & \cdots & \langle E(-u_2)H(u_m) \rangle \\
\cdots & \cdots & \cdots & \cdots \\
\langle E(-u_m)H(u_1) \rangle & \langle E(-u_m)H(u_2) \rangle & \cdots & \langle G(u_m) \rangle
\end{pmatrix}
\]

Next, we remark that we may remove the constraints on variables $u_i, v_i$ and regard the above formulas as an identity of meromorphic functions. Indeed, the off–diagonal entries in right–hand side are meromorphic functions. Indeed, the off–diagonal entries in right–hand side are meromorphic by virtue of Proposition 4.5, and the same property for the diagonal entries $\langle G(u_i) \rangle$ is verified using the remark preceding Corollary 4.8. As for the left–hand side, we expand the determinant, apply averaging term–wise, and then use the same remark to conclude that any quantity of the form $\langle G(u_{i_1}) \cdots G(u_{i_k}) \rangle$ with $i_1 < \cdots < i_k$ is meromorphic.

Now take the residues of both sides of (4.10) at $u_i = x_i$, where $i = 1, \ldots, m$ and the $x_i$’s are pairwise distinct points of $\mathbb{Z}$. In the left–hand side, only the product of diagonal entries gives a nontrivial contribution. Comparing with the formula of Proposition 4.8, we conclude that the result in the left–hand side of (4.10) is equal to

\[
\text{Res}_{u_1 = x_1} \cdots \text{Res}_{u_m = x_m} F_m(u_1, \ldots, u_m) = \text{sgn}(x_1) \cdots \text{sgn}(x_m) \rho_m(x_1, \ldots, x_m).
\]

To handle the right–hand side of (4.10) we may assume, without loss of generality, that among the $x_i$’s, the first $k$ numbers are positive while the last $l = m - k$ numbers are negative. Then it is convenient to write the matrix in the right–hand side as a $2 \times 2$ block matrix, according to partition $m = k + l$. Taking into account the location of poles of $\langle E(-u_i)H(u_j) \rangle$ (see comment 2 after the statement of the theorem) we can take the residues inside the matrix in an appropriate way. Namely, the matrix entries in block $(1, 1)$ are equipped with symbol $\text{Res}_{u_j = x_j}$; those in block $(1, 2)$ are equipped with symbols $\text{Res}_{u_i = x_i} \text{Res}_{u_j = x_j}$; in block $(2, 1)$ there are no residues at all; and in block $(2, 2)$ we use $\text{Res}_{u_i = x_i}$.
In our present notation, the sign in (4.11) is equal to \((-1)^{l}\). Using this fact and comment 3 to the statement of the theorem we finally obtained the desired determinantal expression

\[
\rho_m(x_1, \ldots, x_m) = \det[K(x_i, x_j)],
\]

where the kernel is given by (4.9).

\[\square\]

4.6. The discrete hypergeometric kernel. Let us introduce some notation. Let \(h(x)\) be the function on \(\mathbb{Z}' = \mathbb{Z}'_+ \sqcup \mathbb{Z}'_-\) given by

\[
h(x) = \begin{cases} \frac{(zz')^{1/4}x^{z-x}}{\Gamma(x + \frac{1}{2})} \frac{F(x, 1-x, -z, -1; u + \frac{3}{2}; \xi)}{\Gamma(-x + \frac{1}{2})}, & x \in \mathbb{Z}'_+, \\ \frac{(zz')^{1/4}x^{-z/2}(1-\xi)^{1/2}}{\sqrt{2}} \frac{F(1+z, 1+z', -u+\frac{3}{2}, \xi)}{\xi}, & x \in \mathbb{Z}'_- \end{cases}
\]

and \(m(u)\) be the \(2 \times 2\) matrix–valued function given by

\[
m(u) = \begin{pmatrix} m_{11}(u) & m_{12}(u) \\ m_{21}(u) & m_{22}(u) \end{pmatrix} = \begin{pmatrix} \sqrt{zz'}\xi^3 F(-z, -z', u + \frac{1}{2}; \xi) \\ -\sqrt{zz'}\xi^3 F(1-z, 1-z', u + \frac{3}{2}; \xi) \end{pmatrix} \begin{pmatrix} \frac{F(1+z, 1+z', -u+\frac{3}{2}, \xi)}{1-\xi} \\ \frac{F(z, z', -u+\frac{1}{2}; \xi)}{-u+\frac{3}{2}} \end{pmatrix}
\]

We also write the kernel \(K(x, y)\) in matrix form

\[
K(x, y) = \begin{pmatrix} K_{11}(x, y) & K_{12}(x, y) \\ K_{21}(x, y) & K_{22}(x, y) \end{pmatrix}
\]

where \(x > 0, y > 0\) in \(K_{11}; x > 0, y < 0\) in \(K_{12}; x < 0, y > 0\) in \(K_{21}; x < 0, y < 0\) in \(K_{22}.

Corollary 4.10. With the notation introduced above, the correlation kernel for the point process on \(\mathbb{Z}'\) corresponding to the measure \(M_{z, z', \xi}\) can be written in the form

\[
(4.13) \quad \begin{pmatrix} K_{11}(x, y) & K_{12}(x, y) \\ K_{21}(x, y) & K_{22}(x, y) \end{pmatrix} = h(x)h(y) \times \begin{pmatrix} -m_{11}(x)m_{21}(y) + m_{21}(x)m_{11}(y) & m_{11}(x)m_{22}(y) - m_{21}(x)m_{12}(y) \\ m_{22}(x)m_{11}(y) - m_{21}(x)m_{12}(y) & -m_{22}(x)m_{12}(y) + m_{12}(x)m_{22}(y) \end{pmatrix}
\]

\[
\frac{x-y}{x-y} \quad \frac{x-y}{x-y}
\]

\[
\frac{x-y}{x-y} \quad \frac{x-y}{x-y}
\]
where the indeterminacies of type 0/0 on the diagonal are removed by the L'Hospital rule.

Proof. First of all, it is worth noting that the kernel written above differs from that of Theorem 4.9 by the transformation $K(x, y) \mapsto h(x)K(x, y)(h(y))^{-1}$, which does not affect the correlation functions.

The claim of the corollary is obtained by direct computation of the kernel of Theorem 4.9 using the explicit expression of Proposition 4.5 and the knowledge of the residues of $F(a, b; c; \zeta)$ (here $\zeta = \frac{\xi}{1-\xi}$) at points $c = 0, -1, \ldots$:

$$\text{Res}_{c=-n} F(a, b; c; \zeta) = (-1)^n \zeta^{n+1} \frac{(a)^{n+1}(b)^{n+1}}{n!(n+1)!} F(a+n+1, b+n+1; n+2; \zeta),$$

see [27], 2.8 (19).

Note that the result of Corollary 4.10 agrees with the result obtained in [12]. The kernel (4.13) is called the discrete hypergeometric kernel.

Remark 4.11. As was pointed out in Borodin [5], Section 8, the matrix $m$ appears in a discrete Riemann–Hilbert problem. Namely, set

$$w(x) = \begin{pmatrix} 0 & -h^2(x) \\ 0 & 0 \end{pmatrix}, \quad x \in \mathbb{Z}'; \quad w(x) = \begin{pmatrix} 0 & 0 \\ -h^2(x) & 0 \end{pmatrix}, \quad x \in \mathbb{Z}'_-. $$

We are looking for a $2 \times 2$ matrix–valued function $m = m(u)$ with simple poles such that

1. $m(u)$ is analytic in $\mathbb{C} \setminus \mathbb{Z}'$.
2. $\text{Res}_{u=x} m(u) = \lim_{u \to x} (m(u)w(x))$, $x \in \mathbb{Z}'$.
3. $m(u) \to 1$ as $u \to \infty$.

One can show that this problem has a unique solution, which is the matrix (4.12).

It is worth noting that the formula of Proposition 4.5 can also be written in terms of $m(u)$:

$$(4.14) \quad \langle E(-v)H(u) \rangle_{M_{z',x';\xi}} = m_{11}(v)m_{22}(u) - m_{21}(v)m_{12}(u).$$

The “jump” condition (2) allows one to quickly derive (4.13) from (4.14).

Remark 4.12. One should not think that the $z$-measures and their degenerations exhaust all known examples of Giambelli compatible measures on partitions. There exists a wider class of (generally speaking, complex) Giambelli compatible measures, which are constructed as follows.
Take any algebra homomorphism \( \pi : \Lambda \to \mathbb{C} \) (where \( p_1 \) is the first power sum), and for any \( \xi \in \mathbb{C} \) with \( |\xi| < 1 \) set
\[
M_{\pi, \xi}(\lambda) = (1 - \xi)^t \cdot \frac{\pi(F s\lambda) \dim \lambda}{|\lambda|!} (-\xi)^{|\lambda|}, \quad \lambda \in \mathbb{Y}.
\]
Then \( \sum_{\lambda \in \mathbb{Y}} M_{\pi, \xi}(\lambda) = 1 \), see [13]. In fact, the \( z \)-measures are special cases of measures \( M_{\pi, \xi} \), see [13], §2. One can prove that any \( M_{\pi, \xi} \) (or the corresponding point process) is Giambelli compatible. Some examples of positive measures \( M_{\pi, \xi} \), other than the \( z \)-measures, can be found in [13], §§6.1–6.2.

5. The Whittaker kernel.

In this section we discuss some Giambelli compatible point processes on a continuous space, the punctured line \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \). These processes provide a solution to a problem of harmonic analysis on the infinite symmetric group (see [53], [10], [12], [54]); they are determined (in a certain precise sense) by the measures \( M^{(n)}_{z, z', \xi} \). The correlation functions of these processes were first found in [6] by rather heavy computations. Then a simpler derivation was obtained in [12]; it relies on a scaling limit transition from lattice processes corresponding to measures \( M^{(n)}_{z, z', \xi} \), as \( \xi \) approaches the critical value \( \xi = 1 \). Here we aim to demonstrate that using the Giambelli compatibility property makes it possible to substantially simplify and clarify the initial approach of [6]. Since two detailed proofs have already been published, we only sketch the main steps of the argument (note that it is quite similar to that of §4). Some omitted technical details can be recovered with the help of [7], [6].

5.1. The spaces \( \Omega \) and \( \tilde{\Omega} \). Let \( \mathbb{R}^\infty \) denote the direct product of countably many copies of \( \mathbb{R} \) equipped with the product topology. By \( \tilde{\Omega} \) we denote the subspace of triples \( \omega = (\alpha, \beta, \delta) \in \mathbb{R}^\infty \times \mathbb{R}^\infty \times \mathbb{R} \) such that
\[
\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0), \quad \sum_{i=1}^{\infty} \alpha_i + \sum_{j=1}^{\infty} \beta_j \leq \delta.
\]
The space \( \tilde{\Omega} \) is locally compact in the induced topology. We will use it as a “source” space \( S \). By definition, the morphism \( \phi \) of algebra \( \Lambda \) into the algebra of functions on \( \tilde{\Omega} \) is determined on the generators \( p_k \in \Lambda \) as follows
\[
(5.1) \quad \phi(p_1)(\omega) \equiv \delta, \quad \phi(p_k)(\omega) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} \beta_j^k, \quad k = 2, 3, \ldots.
\]
The map \( \phi \) is an embedding. To simplify the notation, given \( f \in \Lambda \), we will abbreviate \( f(\omega) = \phi(f)(\omega) \). One can prove that the functions \( f(\omega) \) are continuous on \( \tilde{\Omega} \).

Finally, we will also need the subspace \( \Omega := \{ \omega \in \tilde{\Omega} | \delta = 1 \} \), which is called the Thoma simplex. Note that \( \Omega \) is compact.

5.2. The measures \( P_{z,z'} \) and \( \tilde{P}_{z,z'} \). Fix parameters \( z, z' \) as in section 4. It is known that there exists a unique probability measure \( P_{z,z'} \) on the Thoma simplex \( \tilde{\Omega} \), such that

\[
(5.2) \quad \int_{\tilde{\Omega}} s_{\lambda}(\omega) P_{z,z'}(d\omega) = \frac{M_{z,z'}^{(\Lambda)}(\lambda)}{\dim \lambda}, \quad \forall \lambda \in \mathbb{Y}.
\]

Uniqueness of \( P_{z,z'} \) follows from the fact that the image of \( \Lambda \) is dense in the space of continuous functions on the compact space \( \tilde{\Omega} \). Existence is a more deep claim; it follows from a general theory developed in [42] (see also [54]). Note that (5.2) can be viewed as an infinite-dimensional moment problem: an unknown measure is characterized by its “moments”, which are indexed by \( \lambda \)'s. The measures \( P_{z,z'} \) are interesting because they govern the decomposition of certain natural representations of the infinite symmetric group, see [44], [54], and references therein.

For certain reasons explained in [7], [6], [11] we prefer to deal with a modification of \( P_{z,z'} \). Consider the gamma distribution on the positive half-line \( \mathbb{R}_{>0} \) with parameter \( zz' \):

\[
\text{GAMMA}_{zz'}(dr) = \frac{1}{\Gamma(zz')} r^{zz'-1} e^{-r} dr, \quad r > 0.
\]

The modified measure, denoted as \( \tilde{P}_{z,z'} \), lives on \( \tilde{\Omega} \) and is defined as the pushforward of \( P_{z,z'} \otimes \text{GAMMA}_{zz'} \) under the map

\[
\Omega \times \mathbb{R}_{>0} \rightarrow \tilde{\Omega}, \quad ((\alpha, \beta), r) \mapsto (r \cdot \alpha, r \cdot \beta, r).
\]

Clearly, \( \tilde{P}_{z,z'} \) is again a probability measure.

In a certain precise sense, \( P_{z,z'} \) is the limit of measures \( M_{z,z'}^{(n)} \) as \( n \rightarrow \infty \) while \( \tilde{P}_{z,z'} \) is the limit of measures \( M_{z,z',\xi} \) as \( \xi \rightarrow 1 \).

5.3. Giambelli compatibility. It is readily verified that all functions \( f(\omega) \) on \( \tilde{\Omega} \) coming from elements \( f \in \Lambda \) are integrable with respect to \( \tilde{P}_{z,z'} \). Hence, the map \( \phi \) as defined in §5.1 sends \( \Lambda \) to \( \mathcal{A}(\tilde{\Omega}, \tilde{P}_{z,z'}) \).

Proposition 5.1. The triple \( (\tilde{\Omega}, \tilde{P}_{z,z'}, \phi) \) is Giambelli compatible.
Proof. Set $n = |\lambda|$ and observe that $s_\lambda(r \cdot \omega) = r^n \cdot s_\lambda(\omega)$. It follows that

$$
\int_\Omega s_\lambda(\omega) \tilde{P}_{z,z'}(d\omega) = (zz')_n \int_\Omega s_\lambda(\omega) P_{z,z'}(d\omega) = (zz')_n \frac{M^{(n)}_{z,z'}(\lambda)}{\dim \lambda}.
$$

Denoting integration with respect to $\tilde{P}_{z,z'}$ as $\langle \cdot \rangle_{\tilde{P}_{z,z'}}$ we thus get

$$
\langle s_\lambda \rangle_{\tilde{P}_{z,z'}} = (zz')_n \frac{M^{(n)}_{z,z'}(\lambda)}{\dim \lambda}.
$$

Then we use formula (4.1) and the expression of $\dim \lambda$ in terms of Frobenius coordinates, as in the proof of Proposition 4.2.

5.4. Computation of $\langle E(v)H(u) \rangle_{P_{z,z'}}$ and $\langle E(v)H(u) \rangle_{\tilde{P}_{z,z'}}$. The definition of $\phi$ (see (5.1)) implies that

$$
H(u)(\omega) = e^{\gamma u - 1} \prod_{i=1}^{\infty} \frac{1 + \beta_i u^{-1}}{1 - \alpha_i u^{-1}}, \quad E(v)(\omega) = e^{\gamma v - 1} \prod_{i=1}^{\infty} \frac{1 + \alpha_i v^{-1}}{1 - \beta_i v^{-1}},
$$

where we are using the notation

$$
\gamma = \delta - \sum_i \alpha_i - \sum_j \beta_j.
$$

Actually, $\gamma = 0$ almost surely (with respect to probability measure $\tilde{P}_{z,z'}$), see Theorem 6.1 in [54]. Hence, the exponential prefactors could be omitted. However, a priori we cannot use this fact because it appears as a consequence of the computation of the correlation functions.

We will regard $u$ and $v$ as complex variables. Note that infinite products are well defined provided that $u, v \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 4.3 implies that $\langle H(u)E(v) \rangle_{\tilde{P}_{z,z'}}$ makes sense when $(u, v)$ ranges over a suitable domain in $(\mathbb{C} \setminus \mathbb{R})^2$. As in section 4, the possibility of computing this quantity is based on the knowledge of $\langle s_\lambda \rangle_{\tilde{P}_{z,z'}}$. Our computation goes in two steps. First we evaluate the average over $(\Omega, P_{z,z'})$ and then we pass to $\tilde{\Omega}$ using the ray integral transform with respect to measure $\text{GAMMA}_{z,z'}$. The reason is that on the Thoma simplex $\Omega$ we can use the formula

$$
(5.3) \quad \langle H(u)E(v) \rangle = 1 + (u + v) \sum_{p,q \geq 0} \frac{\langle s(p/q) \rangle}{u^{p+1}v^{q+1}}
$$

whereas on $\tilde{\Omega}$ such a series diverges.
Let \( F_3(a, a', b, b'; c; x, y) \) denote the hypergeometric function in two variables \( x \) and \( y \), defined by the series

\[
F_3(a, a', b, b'; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}m!n!} x^m y^n.
\]

It possesses an Euler–type integral representation

\[
F_3(a, a', b, b'; c; x, y) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \int_{s \geq 0, t \geq 0, s+t \leq 1} \frac{s^{b-1}t^{b'-1}(1-s-t)^{c-b-b'-1}dsdt}{(1-sx)^a(1-ty)^a'}
\]

(see Erdelyi [27], §5.7–5.8). The series converges in the polydisc \(|x| < 1, |y| < 1\) and can be analytically continued to a larger domain using the integral representation. Note that using the generalized function \( s_+^a/\Gamma(a) \) supported by the half–line \( s \geq 0 \) (see Gelfand and Shilov [30]), the integral representation can be rewritten as

\[
(5.4) \quad F_3(a, a', b, b'; c; x, y) = \Gamma(c) \int \frac{s^{b-1}t^{b'-1}(1-s-t)^{c-b-b'-1}dsdt}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')} \frac{ds dt}{(1-sx)^a(1-ty)^a'}.
\]

**Lemma 5.2.** For \( u, v \in \mathbb{C} \setminus [0,1] \)

\[
\langle E(v)H(u) \rangle_{p_{z,z'}} = F_3(z, -z, z'; z; z; u^{-1}, v^{-1})
\]

\[
+ \frac{1}{uv(zz' + 1)} F_3(z + 1, -z + 1, z' + 1, -z' + 1; z; z + 2; u^{-1}, v^{-1}).
\]

**Proof.** For \( u, v \in \mathbb{C} \setminus [0,1] \), \( H(u)E(v)(\omega) \) is uniformly bounded on \( \omega \in \Omega \), hence the quantity \( \langle E(v)H(u) \rangle_{p_{z,z'}} \) is well defined and is a holomorphic function in \( u, v \). Assume that \(|u^{-1}| < 1, |v^{-1}| < 1\) first. Then we may apply formula (5.3), where \( \langle \cdot \rangle \) means \( \langle \cdot \rangle_{p_{z,z'}} \). Using the explicit expression

\[
\langle s(p|q) \rangle_{p_{z,z'}} = \frac{M_{p,z}(p|q)}{\dim(p|q)} = \frac{(z+1)_p(z'+1)_p(-z+1)_q(-z'+1)_q}{(zz'+1)p+q!q!(p+q+1)}
\]

one can verify the desired formula directly. Then we use analytic continuation.

This completes the first step. The second step, the passage to average over \( \hat{\Omega} \), is based on the relation
\[ \langle H(u)E(v) \rangle_{\tilde{P}_{z,z'}} = \frac{1}{\Gamma(z')} \int_0^\infty \langle H(ur^{-1})E(vr^{-1}) \rangle_{P_{z,z'}} r^{z'-1} e^{-r} dr. \]

It turns out that the result is expressed through the classical Whittaker function \( W_{\kappa,\mu}(x) \) (see \[27\], §6, for the definition). This function possesses the integral representation (see \[27\], 6.11 (18))

\[
W_{\kappa,\mu}(x) = e^{-x/2}x^{\mu+1/2} \int_0^\infty t^{\kappa+\mu-1/2} (1 + t)^{\kappa+\mu+1/2} e^{-xt} dt.
\]

The integral converges for \( \text{Re} \ x > 0 \) and admits an analytic continuation to the larger domain \( \mathbb{C} \setminus (-\infty, 0] \).

In the next proposition we assume that \( u, v \in \mathbb{C} \setminus \mathbb{R} \) are such that \( (H(u)E(v))(\omega) \) is integrable with respect to measure \( \tilde{P}_{z,z'} \) on \( \tilde{\Omega} \). By Lemma 4.3, this holds at least for for large \( |u| \) and \( |v| \).

**Proposition 5.3.** Under these assumptions we have

\[
\langle H(u)E(v) \rangle_{\tilde{P}_{z,z'}} = e^{-v+u^2/2} (z-z')^{-v} \cdot \frac{2F_0(z+1, z'; u^{-1})}{u} + \frac{zz'}{w} 2F_0(z+1, z'+1; u^{-1}) \cdot \frac{2F_0(-z+1, -z'+1; v^{-1})}{w}.
\]

**Proof.** Direct computation using Lemma 5.2 and integral representations (5.4) and (5.5). \( \Box \)

**Remark 5.4.** On a heuristical level, this result can be obtained directly from (5.3) with \( \langle \cdot \rangle \) understood as \( \langle \cdot \rangle_{\tilde{P}_{z,z'}} \). The formal summation leads to

\[
2F_0(z, z'; u^{-1}) 2F_0(-z, -z'; v^{-1}) + \frac{zz'}{w} 2F_0(z+1, z'+1; u^{-1}) 2F_0(-z+1, -z'+1; v^{-1}).
\]

Here \( 2F_0(a, b; x) \) is a divergent hypergeometric series, which, however, can be interpreted as an asymptotic series for the Whittaker function (see \[27\], 6.9 (5)).

**5.5. Correlation measures and controlling measures.** The contents of the present subsection is similar to that of §4.4.

Instead of the lattice \( \mathbb{Z}' \) we are dealing with punctured line \( \mathbb{R}^* = \mathbb{R} \setminus \{0\} \). To an arbitrary point \( \omega = (\alpha, \beta, \delta) \in \tilde{\Omega} \) we assign a point
configuration \( X(\omega) \subset \mathbb{R}^* \) as follows: we remove the possible 0’s from the sequences \( \alpha \) and \( \beta \) and then set

\[
X(\omega) = \{-\beta_1, -\beta_2, \ldots, \alpha_2, \alpha_1\}.
\]

For instance, in the special case when both \( \alpha \) and \( \beta \) are zero sequences, the configuration \( X(\omega) \) is empty. Note that the correspondence \( \omega \mapsto X(\omega) \) is not injective, because we cannot restore \( \delta \) from \( X(\lambda) \). However, the restriction to the subset of \( \omega \)'s with \( \gamma = 0 \) is injective.

Assume we are given a probability measure \( P \) on \( \tilde{\Omega} \). Then we obtain a point process on \( \mathbb{R}^* \) with “source space” \( (\tilde{\Omega}, P) \). Let \( \rho_m \) stand for the \( m \)th correlation measure of this process. As a reference measure on \( \mathbb{R}^* \) we take Lebesgue measure. If \( \rho_m \) is absolutely continuous with respect to Lebesgue measure, we can pass to the correlation function, which we will denote as \( \rho_m(x_1, \ldots, x_m) \). (Even if \( \rho_m \) is not absolutely continuous, \( \rho_m(x_1, \ldots, x_m) \) makes sense as a generalized function.) Informally, \( \rho_m(x_1, \ldots, x_m) \) is the density of the probability that the random configuration intersects each of the infinitesimal intervals \([x_i, x_i + dx_i]\), \( i = 1, \ldots, m \).

Next, we assign to any \( \omega \in \tilde{\Omega} \) a measure on \( \mathbb{R} \),

\[
\sigma_\omega = \sum_{i=1}^{\infty} \left( \alpha_i \delta_{\alpha_i} + \beta_i \delta_{-\beta_i} \right) + \gamma \delta_0,
\]

of total mass \( \delta \), and then we define the \( m \)th controlling measure \( \sigma_m \) on \( \mathbb{R}^m \) (\( m = 1, 2, \ldots \)) as follows:

\[
\sigma_m = \int_{\tilde{\Omega}} \sigma_\omega \otimes^m P(d\omega).
\]

The controlling measures contain all the information about the correlation measures, see [53]. In particular, there is a simple correspondence between the restrictions of \( \sigma_m \) and \( \rho_m \) to the subset \( (\mathbb{R}^*)^0 \subset (\mathbb{R}^*)^m \) of vectors \( (x_1, \ldots, x_m) \) with distinct coordinates:

\[
\sigma_m = \left| x_1 \ldots x_m \right| \rho_m \quad \text{on} \quad (\mathbb{R}^*)^0.
\]

Assuming that \( \sigma_m \) satisfies the growth condition (4.7) we can introduce its Cauchy transform \( \tilde{\sigma}_m \). It is well known (and readily verified) that \( \sigma_m \) can be restored from \( \tilde{\sigma}_m \) as follows

\[
\sigma_m(x) = \text{Jump}_{u=x} \tilde{\sigma}_m(u) := \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} (\tilde{\sigma}_m(x-i\varepsilon) - \tilde{\sigma}_m(x+i\varepsilon)), \quad x \in \mathbb{R}^m,
\]

where the limit means weak limit of generalized functions.
Arguing as in Lemma 4.7 we have
\[ \hat{\sigma}_m(u_1, \ldots, u_m) = \left\{ \frac{\partial^m}{\partial v_1 \cdots \partial v_m} (H(u_1)E(-v_1) \cdots H(u_m)E(-v_m)) \right\}_{v_1 = u_1}^{v_m = u_m} \]

As in §4, set
\[ F_m(u_1, \ldots, u_m) = \left\{ \frac{\partial^m}{\partial v_1 \cdots \partial v_m} (H(u_1)E(-v_1) \cdots H(u_m)E(-v_m)) \right\}_{v_1 = u_1}^{v_m = u_m} \]

Then we have on \((\mathbb{R}^*)^m_0\)
\[ \rho_m(x_1, \ldots, x_m) = \text{sgn}(x_1) \cdots \text{sgn}(x_m) \text{ Jump } \cdots \text{ Jump } [F_m(u_1, \ldots, u_m)]. \]

5.6. Computation of the correlation functions. We set \(P = \hat{P}_{z,z}^{*}\).

One can verify that the corresponding controlling measures \(\sigma_m\) are finite measures, so that their Cauchy transforms \(\hat{\sigma}_m\) are well defined.

**Theorem 5.5.** The point process on \(\mathbb{R}^*\) corresponding to the measure \(P_{z,z'}\) is determinantal and its correlation kernel can be written as
\[ K(x, y) = \begin{cases} \text{Jump } \frac{\langle E(-x)H(u) \rangle}{x - y}, & x > 0, y > 0, x \neq y, \\ -\text{Jump } \frac{\langle E(-u)H(u) \rangle}{x - y}, & x < 0, y > 0, \\ \frac{\langle E(-x)H(y) \rangle}{x - y}, & x > 0, y < 0, \\ -\text{Jump } \frac{\langle E(-u)H(y) \rangle}{x - y}, & x < 0, y < 0, x \neq y, \end{cases} \]

where \(\langle \cdot \rangle\) means \(\langle \cdot \rangle_{\hat{P}_{z,z'}}\), and the indeterminacy arising for \(x = y\) is resolved via the L’Hospital rule.

**Idea of proof.** We compute \(\rho_m\) on the subset \((\mathbb{R}^*)^m_0\) of \((\mathbb{R}^*)^m\) (one can check that this subset has full measure with respect to \(\rho_m\), see Theorem 2.5.1 in [7]). The scheme of the argument is similar to that of the proof of Theorem 4.9: we use the formula of Proposition 5.3 and and the determinantal identity (2.2). Let us briefly describe how to justify this identity. Here we cannot apply the trick of Proposition 4.5; instead of this we rearrange the proof of Proposition 2.2 using the two–step
procedure of §5.4. Namely, we start with integration over the Thoma simplex:

\[
\left\langle \det \left( \frac{H(u_i) E(v_j) - 1}{u_i + v_j} \right) \right\rangle_{i,j=1}^{d} \quad p_{z,z'} = \sum_{p_1, \ldots, p_d = 0}^{\infty} \sum_{q_1, \ldots, q_d = 0}^{\infty} \left\langle \det \left( s(p_i|q_j) \right) \right\rangle_{i,j=1}^{d} p_{z,z'}
\]

Then, using the explicit expression of Proposition 5.3 we write the sum in terms of an integral:

\[
\Gamma(z') (z')^d \frac{1}{u_1 \ldots u_d v_1 \ldots v_d} \int \int \sum_{\tau \in S_d} \sgn(\tau) \prod_{i=1}^{d} \frac{s^z}{\Gamma(z' + 1)} \frac{t^{z'}_i}{\Gamma(-z' + 1)} \left( 1 - \frac{u_i s_i}{u_i} \right)^{-z-1} \left( 1 - \frac{w_{\tau(i)} t_i}{v_i} \right)^{z-1} \frac{(1 - \sum_{i=1}^{d} (s_i + t_i))^{zz'-d-1}}{\Gamma(zz' - d)} d w_i d s_i d t_i.
\]

where \( S_d \) is the symmetric group of degree \( d \) and \( \sgn(\tau) \) stands for the signature of a permutation \( \tau \in S_d \). To see the equivalence it suffices to expand the factors \((1 - \ldots)^{z - 1} \) and use the Dirichlet integral. The integration over \( w_i \)’s can be explicitly performed, see proof of Lemma 2.2.4 in [7], which simplifies the formula. In particular, the sum over \( \tau \) can be turned into a determinant inside the integral.

This formula splits into a \( d \times d \) determinant of “2-point” averages \( \langle \cdot \rangle_{\tilde{P}_{z,z'}} \) under the ray transform, which follows from the one-dimensional integration formula

\[
\int_{0}^{\infty} \frac{1 - r^{-1} \sum_{i=1}^{d} (s_i + t_i))^{zz'-d-1}}{\Gamma(zz' - d)} r^{zz'-d-1} e^{-r} dr = e^{-\sum_i (s_i + t_i)}.
\]

\[\square\]

5.7. The Whittaker kernel. Write \( \mathbb{R}^* \) as \( \mathbb{R}_+ \sqcup \mathbb{R}_- \) (strictly positive and strictly negative reals) and define a function \( h \) on \( \mathbb{R}^* \) by

\[
h(x) = \begin{cases} \frac{(zz')^{1/4}}{\sqrt{\Gamma(z + 1) \Gamma(z' + 1)}} x^{(z+z')/2} e^{-x/2}, & x > 0 \\ \frac{(zz')^{1/4}}{\sqrt{\Gamma(-z + 1) \Gamma(-z' + 1)}} (-x)^{-(z+z')/2} e^{x/2}, & x < 0. \end{cases}
\]
Let \( m(u) \) be the following \( 2 \times 2 \) matrix–valued function on \( \mathbb{C} \setminus \mathbb{R} \):

\[
\begin{pmatrix}
    u - \frac{z + z'}{2} e^{z' \frac{u + u'}{2}} W_{\frac{z + z'}{2}, \frac{z - z'}{2}}(u) & \sqrt{z' z} (u) \frac{z - z'}{2} e^{\frac{z' u}{2}} W_{\frac{z + z'}{2}, \frac{z - z'}{2}}(u)
    -\sqrt{z' z} u \frac{z - z'}{2} e^{\frac{z' u}{2}} W_{\frac{z + z'}{2}, \frac{z - z'}{2}}(u) & (u) \frac{z - z'}{2} e^{-\frac{z' u}{2}} W_{\frac{z + z'}{2}, \frac{z - z'}{2}}(u)
\end{pmatrix}
\]

**Corollary 5.6.** After the transformation \( K(x, y) \to h(x) K(x, y)(h(y))^{-1} \) the kernel of Theorem 5.5 can be written in the same form as in Corollary 4.10, with \( h \) and \( m \) as defined above.

This is exactly the Whittaker kernel of [6], [12]. About this kernel, see also [8], [52].

As in §4, \( m(u) \) turns out to be a solution to a Riemann–Hilbert problem, see [5].

Finally, note that the expression of Proposition 5.3 can be written in terms of \( m(u) \), just as in (4.14).

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