MODEL ∞-CATEGORIES I: SOME PLEASANT PROPERTIES OF THE ∞-CATEGORY OF SIMPLICIAL SPACES

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Abstract. Both simplicial sets and simplicial spaces are used pervasively in homotopy theory as presentations of spaces, where in both cases we extract the “underlying space” by taking geometric realization. We have a good handle on the category of simplicial sets in this capacity; this is due to the existence of a suitable model structure thereon, which is particularly convenient to work with since it enjoys the technical properties of being proper and of being cofibrantly generated. This paper is devoted to showing that, if one is willing to work ∞-categorically, then one can manipulate simplicial spaces exactly as one manipulates simplicial sets. Precisely, this takes the form of a proper, cofibrantly generated model structure on the ∞-category of simplicial spaces, the definition of which we also introduce here.

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0. Introduction

0.1. Simplicial spaces and model ∞-categories. A simplicial space can be thought of as a resolution of a space, namely its homotopy colimit; in nice cases this is computed by its geometric realization, and we will henceforth blur the distinction. The operation of taking a homotopy colimit, being homotopy invariant, descends to a functor |−| : sS → S of ∞-categories, from that of simplicial spaces to that of spaces.1 The primary purpose of the present paper is to introduce a new framework for studying this functor. In particular, we give ∞-categorical criteria for determining

- when a map of simplicial spaces becomes an equivalence upon geometric realization, and
- when a homotopy pullback of simplicial spaces remains a homotopy pullback upon geometric realization,

which we have found to be much easier to verify than their existing 1-categorical counterparts. We hope that this encourages homotopy theorists grappling with simplicial spaces to work ∞-categorically: even if a map of simplicial spaces or a homotopy pullback of simplicial spaces began its life 1-categorically, these questions are homotopy invariant and hence inherently ∞-categorical, and thus should be approachable using the framework given here.

To set the stage, let us recall Quillen’s theory of model categories: given a category C equipped with a subcategory W ⊂ C, a model structure on the category C consists of additional data which provide an efficient and computable method of accessing its localization C[W−1]. As a prime example, the Kan–Quillen model structure on the category sSet of simplicial sets provides a combinatorial method of studying the homotopy category sSet[W−1] ∼ Top[W−1] of topological spaces.

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1By “∞-category” we mean “quasicategory”, though in fact we will use a number of different models for ∞-categories throughout this work. We give a rigorous treatment of our foundations in §A.1.
Now, suppose that $\mathcal{C}$ is not merely a category but an $\infty$-category, again equipped with a subcategory $\mathcal{W} \subset \mathcal{C}$. We can analogously localize the $\infty$-category $\mathcal{C}$ at the subcategory $\mathcal{W}$, forming a new $\infty$-category $\mathcal{C}[\mathcal{W}^{-1}]$ equipped with a functor $\mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$, which is initial among those functors from $\mathcal{C}$ which invert all the maps in $\mathcal{W}$. A key example for us is $s\mathcal{S}[\mathcal{W}^{-1}] \simeq \mathcal{S}$, where we denote by $\mathcal{W}_{(-)} \subset s\mathcal{S}$ the subcategory spanned by those maps which become invertible upon geometric realization.

With this background in place, we can now explain the two goals of this paper.

First of all, inspired by Quillen, we introduce the notion of a model $\infty$-category: given an $\infty$-category $\mathcal{C}$ equipped with a subcategory $\mathcal{W} \subset \mathcal{C}$, a model structure on the $\infty$-category $\mathcal{C}$ consists of additional data which allow us to access its $\infty$-categorical localization $\mathcal{C}[\mathcal{W}^{-1}]$. This provides a theory of resolutions which is native to the $\infty$-categorical context. As indicated by the title of this paper, we will continue the development of the theory of model $\infty$-categories in subsequent work.

Then, in precise analogy with the Kan–Quillen model structure on the category of simplicial sets, we endow the $\infty$-category $s\mathcal{S}$ of simplicial spaces with a Kan–Quillen model structure, whose subcategory of weak equivalences is precisely $\mathcal{W}_{(-)} \subset s\mathcal{S}$, which allows us to access its $\infty$-categorical localization $s\mathcal{S}[\mathcal{W}^{-1}] \simeq \mathcal{S}$. This model structure is likewise proper, and is cofibrantly generated by the sets $I_{KQ}, J_{KQ} \subset s\mathcal{S}$ of boundary inclusions $\partial \Delta^n \subset \Delta^n$ and of horn inclusions $\Lambda^i_\nu \subset \Delta^n$, considered as maps of simplicial spaces via the inclusion $s\mathcal{S} \subset s\mathcal{S}$ of simplicial sets as the levelwise-discrete simplicial spaces.

Far from providing just a few assorted tricks, the Kan–Quillen model structure on $s\mathcal{S}$ gives an extensive framework for understanding simplicial spaces vis-à-vis their geometric realizations (which will grow along with the development of the theory of model $\infty$-categories). To illustrate this, we return to the criteria promised above.

- As the Kan–Quillen model structure on $s\mathcal{S}$ is cofibrantly generated, a map is an acyclic fibration if it has the right lifting property against the set $I_{KQ}$ of generating cofibrations. In particular, rlp($I_{KQ}$) $\subset \mathcal{W}_{(-)}$.
  This is comparable to an analogous criterion coming from the Moerdijk model structure of [Moe89] on the category of bisimplicial sets; for an extended comparison, see Remark 7.3.
- As the Kan–Quillen model structure on $s\mathcal{S}$ is right proper, then a pullback in which at least one of the two maps is a fibration (i.e. has rlp($J_{KQ}$)) is a homotopy pullback: that is, it remains a pullback under geometric realization.
  This is comparable to an analogous criterion associated with the $\pi_*$-Kan condition of [BF78]; for an extended comparison, see Remark 6.9. We also contextualize this criterion with respect to the notion of a realization fibration explored in [Rez14] in Remark 6.8.

0.2. Goerss–Hopkins obstruction theory for $\infty$-categories. As the present paper is the first in its series, we spend a moment describing the original motivation for the theory of model $\infty$-categories.

The overarching goal of this project is to generalize Goerss–Hopkins obstruction theory ([GH04, GH]), a powerful tool for obtaining existence and uniqueness results for $E_\infty$ ring spectra via purely algebraic computations, to the equivariant and motivic settings. However, the original obstruction theory is based in a model category of spectra satisfying a number of technical conditions, making it relatively difficult to generalize directly. Relatedly, the foundations for its construction rely on various point-set considerations, which appear for the sake of simplification but play no real mathematical role. Thus, as the obstruction theory ultimately lives on the underlying $\infty$-category of spectra anyways, we instead aim to generalize Goerss–Hopkins obstruction theory to an arbitrary (symmetric monoidal, presentable) $\infty$-category $\mathcal{C}$; this will yield equivariant and motivic obstruction theories simply by specializing to the $\infty$-categories of equivariant and motivic spectra.

Now, Goerss–Hopkins obstruction theory is constructed in the “$E^2$ model structure” on simplicial spectra, originally introduced in [DKS93]. This model structure provides a theory of nonabelian projective resolutions. Correspondingly, suppose we are given a presentable $\infty$-category $\mathcal{C}$, along with a set $\mathcal{S}$ of generators which we assume (without real loss of generality) to be closed under finite coproducts. Then, Goerss–Hopkins obstruction theory for $\mathcal{C}$ will take place in the nonabelian derived $\infty$-category of $\mathcal{C}$, i.e. the $\infty$-category $P_2(\mathcal{C}) = Fun_{\mathcal{C}}(\mathcal{S}^op, \mathcal{S})$ of those presheaves on $\mathcal{S}$ that take finite coproducts in $\mathcal{S}$ to finite products in $\mathcal{S}$, originally introduced in [Lur09a, §5.5.8]. (If $\mathcal{C}$ is the underlying $\infty$-category of an appropriately chosen model category, then $P_2(\mathcal{C})$ will be the underlying $\infty$-category of the $E^2$ model structure on the category of simplicial objects therein.) This admits a natural functor $s\mathcal{C} \to P_2(\mathcal{S})$, given by taking any $Y_\bullet \in s\mathcal{C}$ to the functor $S^\beta \mapsto |\text{hom}_{\mathcal{C}}(S^\beta, Y_\bullet)|$ (for any generator $S^\beta \in \mathcal{S}$, and

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2Note that even if $\mathcal{C}$ is just a 1-category, this will generally differ from the more crude and lossy 1-categorical localization $\mathcal{C}[\mathcal{W}^{-1}]$: the canonical map $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}[\mathcal{W}^{-1}]$ is precisely the projection to the homotopy category $\text{ho}(\mathcal{C}[\mathcal{W}^{-1}]) \simeq \mathcal{C}[\mathcal{W}^{-1}]$.

3For an explanation of this equivalence, see item (8) of §A.

4In fact, to those unfamiliar with the more nuanced techniques of model categories, these considerations might even appear to amount to something like black magic.

5More precisely, it is actually constructed in a certain model category which is monadic over simplicial spectra, whose model structure is lifted along the defining adjunction.
writing “lw” to denote “levelwise”). In fact, this functor is a localization: denoting by $W_{E^2} \subset sC$ the subcategory spanned by those maps which it inverts, it induces an equivalence $sC[W_{E^2}^{-1}] \simeq P_{E}(S)$.

From here, we can explain our motivation for the theory of model $\infty$-categories: the definition of the $\infty$-category $P_{E}(S)$ is extremely efficient, but its abstract universal characterizations alone are insufficient for making the actual computations within this $\infty$-category which are necessary to set up the obstruction theory. Rather, computations therein generally rely on choosing simplicial resolutions of objects, i.e. preimages under the functor $sC \to P_{E}(S)$, and then working in $sC$ to deduce results back down in $P_{E}(S)$. (An important example is the “spiral exact sequence”, which is a key ingredient to setting up the obstruction theory (see e.g. [GH, Lemma 3.1.2(2)]).) Thus, in order to organize these resolutions, we will provide an $E^2$ model structure on the $\infty$-category $sC$, giving an efficient and computable method of accessing the localization $sC[W_{E^2}^{-1}] \simeq P_{E}(S)$.

As a sample application, we will prove the following result in subsequent work, joint with David Gepner. As background, recall that the first application of Goerss–Hopkins obstruction theory was to prove that the Morava $E$-theory spectra admit essentially unique $E_{\infty}$ structures, and moreover that their spaces of $E_{\infty}$ automorphisms are essentially discrete and are given by the corresponding Morava stabilizer groups (see [GH04, §7]). Bootstrapping up their arguments, we use Goerss–Hopkins obstruction theory in the $\infty$-category of motivic spectra to prove

- that the motivic Morava $E$-theory spectra again admit essentially unique $E_{\infty}$ structures,
- that again their spaces of $E_{\infty}$ automorphisms are essentially discrete, but
- that they can admit “exotic” $E_{\infty}$ automorphisms not seen in ordinary topology.

(More precisely, their groups of $E_{\infty}$ automorphisms will generally contain the corresponding Morava stabilizer groups as proper subgroups.)

### 0.3. Outline

We now provide a more detailed outline of the contents of this paper.

- In §1, we define model $\infty$-categories, and we define the notions of Quillen adjunctions and Quillen equivalences between them.
- In §2, we provide some examples of the objects introduced in §1, and we also speculate on the existence of other examples (including some of a foundational nature, some which would provide yet more models for the $\infty$-category of $\infty$-categories, and one related to $E_{\infty}$ deformation theory) whose verifications lie beyond the scope of the current project.
- In §3, we define cofibrantly generated model $\infty$-categories and provide recognition and lifting theorems analogous to the classical ones.
- In §4, we assert the existence of the Kan–Quillen model structure on the $\infty$-category $sS$ of simplicial spaces. The proof (which we only outline, leaving the real substance for §7) relies on the recognition theorem of §3. We also define what it means for a model $\infty$-category to be proper.
- In §5, we collect some auxiliary results regarding spaces and simplicial spaces. In particular, we state a particular result – Lemma 5.4 – which ultimately represents the key piece of not-totally-formal input that makes the entire theory tick (but we defer its proof to §8).
- In §6, we prove some convenient properties enjoyed by the fibrant objects and the fibrations in the Kan–Quillen model structure on $sS$, and we define a “fibrant replacement” endofunctor $Ex_{\infty}$ analogous to the classical one.
- In §7, we prove the main result, that the given data do indeed define a proper, cofibrantly generated model structure on the $\infty$-category $sS$ of simplicial spaces.
- In §8, we prove Lemma 5.4, using the classical theory of model categories and ultimately some rather delicate arguments regarding bisimplicial sets.
- In §9, we carefully lay out the notation, terminology, and conventions that we will adopt in this sequence of papers. The casual reader should feel free to omit this section from a first reading, and to content themselves with the knowledge that we are by and large following those of [Lur09a]. On the other hand, the careful reader may find it useful to peruse this section before proceeding to the main body of the paper.

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1. Model ∞-categories: definitions

In this section, we define model ∞-categories, Quillen adjunctions, and Quillen equivalences. We will provide numerous examples of all of these concepts in §2.

1.1. The definition of a model ∞-category.

**Definition 1.1.** We say that three essentially wide subcategories $W, C, F \subset M$ — called the subcategories of weak equivalences, of cofibrations, and of fibrations, respectively — make an ∞-category $M$ into a model ∞-category if they satisfy the following evident ∞-categorical analogs of the usual axioms for a model category (of which only the lifting axiom takes any thought to generalize).

- $M_{\infty}1$ (limit) $M$ is finitely bicomplete.
- $M_{\infty}2$ (two-out-of-three) $W$ satisfies the two-out-of-three property.
- $M_{\infty}3$ (retract) $W, C,$ and $F$ are all closed under retracts.
- $M_{\infty}4$ (lifting) Let $i \in \text{hom}_M(x, y)$ and $p \in \text{hom}_M(z, w)$, and suppose either that $i \in W \cap C$ and $p \in F$ or that $i \in C$ and $p \in W \cap F$. Then the induced map

$$\text{hom}_M(y, z) \to \lim \left( \begin{array}{c} \text{hom}_M(y, w) \\ \text{hom}_M(x, z) \\ \text{hom}_M(x, w) \end{array} \right)$$

is an effective epimorphism.
- $M_{\infty}5$ (factorization) Every map in $M$ factors via both $F \circ (W \cap C)$ and $(W \cap F) \circ C$.

To indicate that a morphism lies in one of these subcategories, we will use the symbols $\xrightarrow{\sim} , \rightarrow$, and $\to$, respectively. We call $W \cap C \subset M$ the subcategory of trivial or acyclic cofibrations, and we call $W \cap F \subset M$ the subcategory of trivial or acyclic fibrations. Morphisms lying in these subcategories are denoted by the symbols $\approxeq$ and $\eq$, respectively.

**Definition 1.2.** An object of a model ∞-category $M$ is called cofibrant if its essentially unique map from the initial object $\varnothing_M$ is a cofibration, and fibrant if its essentially unique map to the terminal object $pt_M$ is a fibration. We denote the full subcategories of these objects by $M^c, M^f \subset M$, respectively. We write $M^{c,f} = M^c \cap M^f \subset M$ for the full subcategory of bifibrant objects, i.e. objects which are both cofibrant and fibrant.

**Notation 1.3.** For clarity, we will sometimes superscript the data associated to a model ∞-category (for instance, the subcategories $W, C,$ and $F$) with the ambient ∞-category, and we will generally subscript these data with the name of the model category when one exists.

**Remark 1.4.** As in the classical case, one should think of cofibrant objects as being “good for mapping out of”, and of fibrant objects as being “good for mapping into”. Indeed, as we make precise and prove in [MGd], if $x \in M^c$ and $y \in M^f$, then the natural map

$$\text{hom}_M(x, y) \to \text{hom}_{M^{[W^{-1}]}_M}(x, y)$$

is an effective epimorphism, and becomes an equivalence after applying either “∞-categorical equivalence relation” of left homotopy or of right homotopy to the source.

**Remark 1.5.** In view of Remark 1.4, we see that if we are only interested in computing hom-spaces in $M^{[W^{-1}]}_M$, then it suffices to know only which objects of $M$ are co/fibrant (as long as we also have some control over the left and right homotopy relations). However, there are many other constructions that one might perform in an ∞-category besides extracting hom-spaces, and from this point of view we can think of the subcategories $C, F \subset M$ as telling us which objects are relatively co/fibrant, i.e. co/fibrant in some undercategory or overcategory (see Example 2.5 below). Nevertheless, we may view the data of the subcategories $M^c, M^f \subset M$ as a “first approximation” to the data of a model structure. As it is often quite difficult to prove the existence of a model structure, in some examples below we will just content ourselves by producing choices of co/fibrant objects corresponding to a given class of weak equivalences.
Remark 1.6. One of the most important constructions that one might perform in a (1- or ∞-)category is the construction of co/limits. Classically, the theory of homotopy co/limits in model 1-categories gives way of computing co/limits in the ∞-categorical localization (see e.g. [Lur09a, Theorem 4.2.4.1]). (In contrast with the “first approximation” of Remark 1.5, these certainly require the full model structure!) In our situation, a model ∞-category M is meant to give a presentation of its localization M[[W]−1], and thus it is useful to have an analogous theory of “homotopy co/limits” in model ∞-categories, which compute co/limits in their localizations. This will be addressed in [MGg].

Remark 1.7. The ∞-categorical lifting axiom encodes a homotopically coherent version of the usual lifting axiom. It is strictly stronger than the usual lifting axiom for hom-sets in the homotopy category.

Remark 1.8. Factorization systems in ∞-categories (in which fillers are essentially unique) are studied in [Lur09a, §5.2.8]. These are distinct from what we seek here, namely weak factorization systems in ∞-categories.

Remark 1.9. When defining model categories, there are always the questions of whether to require that the factorizations be functorial, and of whether to require bicompleteness or only finite bicompleteness. Of course, in defining model ∞-categories, these questions persist. We have chosen to take the less restrictive definitions. However, see [MGe] for more about the question of functorial factorizations, and see [MGg] for situations in which it will begin to be convenient for our model ∞-categories to be bicomplete.

Remark 1.10. Since the opposite of a model ∞-category is canonically a model ∞-category, many of the statements that we make throughout this series of papers have obvious duals. For conciseness, we will simply make whichever of the pair of dual statements is more convenient, and then we will simply refer to its dual if and when we require it.

Remark 1.11. There are many basic facts about model categories which follow easily and directly from the definitions; these generally remain true for model ∞-categories. For instance, we will use repeatedly that $C = \text{llp}(W \cap F)$, that $W \cap C = \text{rlp}(F)$, that $F = \text{rlp}(W \cap C)$, and that $W \cap F = \text{rlp}(C)$. (The proofs remains the same, see [Hir03, Proposition 7.2.3].)

1.2. The definitions of Quillen adjunctions and Quillen equivalences. Model categories are useful not just in isolation, but in how they relate to one another. Following the classical situation, we make the following definition.

Definition 1.12. Suppose that $M$ and $N$ are two model ∞-categories, and suppose that $F : M \rightleftarrows N : G$ is an adjunction between their underlying ∞-categories. We say that this adjunction is a Quillen adjunction if any of the following equivalent conditions is satisfied:

- $F$ preserves cofibrations and trivial cofibrations;
- $G$ preserves fibrations and trivial fibrations;
- $F$ preserves cofibrations and $G$ preserves fibrations;
- $F$ preserves trivial cofibrations and $G$ preserves trivial fibrations.

(These equivalences are immediate from Remark 1.11.) In this situation, we call $F$ a left Quillen functor and we call $G$ a right Quillen functor.

Remark 1.13. We will prove in [MGd] that a Quillen adjunction $F : M \rightleftarrows N : G$ induces a canonical derived adjunction $LF : M[[W^{-1}N^{-1}]] \rightleftarrows N[[W^{-1}N^{-1}]] : RG$ on localizations.

Of course, we also have the following special case of Definition 1.12.

Definition 1.14. We say that a Quillen adjunction $F : M \rightleftarrows N : G$ is a Quillen equivalence if for all $x \in M^c$ and all $y \in N$, the equivalence

$$\text{hom}_M(x, G(y)) \simeq \text{hom}_N(F(x), y)$$

induces an equivalence of subspaces

$$\text{hom}_{W^c}(x, G(y)) \simeq \text{hom}_{W^c}(F(x), y).$$

(Since $W^c \subset M$ and $W^N \subset N$ are subcategories, this can be checked on path components.)

Remark 1.15. Continuing along the lines of Remark 1.13, we will moreover prove in [MGd] that the derived adjunction of a Quillen equivalence will induce an equivalence of localizations.
2. Model \(\infty\)-categories: examples

Having discussed the generalities of model \(\infty\)-categories, we now proceed to give a number of examples. In §2.1 we give some examples of model \(\infty\)-categories, in §2.2 we give some examples of Quillen adjunctions and Quillen equivalences between them, and in §2.3 we give some speculative examples whose verifications lie beyond the scope of the current project. This section may be safely omitted from a first reading.

2.1. Examples of model \(\infty\)-categories.

Example 2.1. In the case that \(\mathcal{M}\) is actually a 1-category considered as an \(\infty\)-category with discrete hom-spaces, then Definition 1.1 recovers the classical definition of a model category. Thus, any model 1-category gives an example of a model \(\infty\)-category.

Example 2.2. Any finitely bicomplete \(\infty\)-category \(\mathcal{M}\) has a trivial model structure, in which we set \(\mathbf{W} = \mathcal{M}^{\simeq}\) and \(\mathbf{C} = \mathbf{F} = \mathcal{M}\). We denote this model \(\infty\)-category by \(\mathcal{M}_{\text{triv}}\).

Example 2.3. The main purpose of this paper is to define the Kan–Quillen model structure on \(s\mathcal{S}\), which will be given as Definition 4.5.

Example 2.4. In [MGi], given a suitable model \(\infty\)-category \(\mathcal{M}\), we will define the Reedy and \(E^2\) model structures on the \(\infty\)-category \(s\mathcal{M}\). These model structures will play central roles in the papers [MGa, MGb]. (Conveniently, for the obstruction theory applications we will be able to take \(\mathcal{M}\) to be our \(\infty\)-category of interest (e.g. motivic spectra) equipped with the trivial model structure of Example 2.2.)

Example 2.5. If \(\mathcal{M}\) is a model \(\infty\)-category and \(x \in \mathcal{M}\) is any object, then both the undercategory \(\mathcal{M}_{/x}\) and the overcategory \(\mathcal{M}_{x/}\) inherit the structure of a model \(\infty\)-category, where in both cases the three classes of maps are created by the forgetful functor to \(\mathcal{M}\). Iterating this observation, for any morphism \(x \to y\) in \(\mathcal{M}\) we obtain the structure of a model \(\infty\)-category on \(\mathcal{M}_{x//y}\).

Example 2.6. If \(\mathcal{M}\) is a model \(\infty\)-category and \(\text{ho}(\mathcal{M})\) is finitely bicomplete, then the model structure on \(\mathcal{M}\) descends to a model structure on \(\text{ho}(\mathcal{M})\). In fact, the two-out-of-three, retract, and factorization axioms in \(\mathcal{M}\) are even verifiable in \(\text{ho}(\mathcal{M})\). Moreover, the map

\[
\lim \left( \begin{array}{c} \text{hom}_{\mathcal{M}}(y, w) \\ \text{hom}_{\mathcal{M}}(x, z) \end{array} \right) \rightarrow \lim \left( \begin{array}{c} \text{hom}_{\text{ho}(\mathcal{M})}(y, w) \\ \text{hom}_{\text{ho}(\mathcal{M})}(x, z) \end{array} \right)
\]

is an effective epimorphism, so if the lifting axiom holds in \(\mathcal{M}\) then it must hold in \(\text{ho}(\mathcal{M})\) as well. On the other hand, since in general there is a complicated interplay between co/limits in \(\mathcal{M}\) and co/limits in \(\text{ho}(\mathcal{M})\), it does not appear possible to draw any general conclusions about the relationship between these two model \(\infty\)-categories. (Note that the functor \(\mathcal{M} \to \text{ho}(\mathcal{M})\) is the unit of the left localization \(\text{Cat}_{\infty} \rightleftharpoons \text{Cat}\), but it is not generally itself an adjoint.)

As we will elaborate upon in Remark 2.14, the following example illustrates a particularly “one-sided” sort of model \(\infty\)-category (which is included for its intrinsic interest, not because we necessarily expect this perspective to be computationally helpful in such situations).

Example 2.7. Suppose that \(\mathcal{M}\) is a finitely bicomplete \(\infty\)-category and that \(L : \mathcal{M} \rightleftharpoons LM : i\) is a left localization. This means that we may consider the right adjoint \(i : LM \leftrightarrow \mathcal{M}\) as the inclusion of the reflective subcategory of “local” objects, and that for any \(x \in \mathcal{M}\) and any \(y \in LM \subset \mathcal{M}\), the localization map \(x \to Lx\) induces an equivalence

\[
\text{hom}_{\mathcal{M}}(Lx, y) \rightleftharpoons \text{hom}_{\mathcal{M}}(x, y).
\]

This fits squarely into the “first approximation” rubric of Remark 1.5: we should consider \(\mathcal{M}^L = \mathcal{M}\) and \(\mathcal{M}^i = i(LM) \subset \mathcal{M}\) (with trivial left/right homotopy relations). Hence, it is natural to guess that there might be a model structure \(\mathcal{M}_L\) on \(\mathcal{M}\), in which

- \(\mathbf{W}_L\) is the subcategory of morphisms in \(\mathcal{M}\) that become equivalences in \(LM\),
- \(\mathbf{C}_L = \mathcal{M}\), and
- \(\mathbf{F}_L\) is determined by the lifting condition \(\mathbf{F}_L = \text{rlp}((\mathbf{W} \cap \mathbf{C})_L) = \text{rlp}(\mathbf{W}_L)\).

What remains, then, is to check

- the half of the lifting axiom asserting that \((\mathbf{W} \cap \mathbf{F})_L \subset \text{rlp}(\mathbf{C}_L) = \text{rlp}(\mathcal{M})\), and
the half of the factorization axiom regarding \( F_L \circ (W \cap C)_L = F_L \circ W_L \).

Now, for a map to have \( rlp(M) \) in particular means that it has the right lifting property against the identity map from itself, and this implies that the map is an equivalence (with inverse given by the guaranteed lift). Thus, \( rlp(M) \subset M^\equiv \). On the other hand, clearly \( M^\equiv \subset rlp(M) \), so \( rlp(M) = M^\equiv \), and hence to satisfy the lifting axiom it is both necessary and sufficient to have that \( (W \cap F)_L \subset M^\equiv \). (Of course, the reverse containment follows from the definitions, so it is also necessary and sufficient to have that \( (W \cap F)_L = M^\equiv \).)

The factorization axiom here is more subtle, and there does not appear to be a general criterion for when this might hold. One possibility is that we might try to use the factorization of an arbitrary map \( x \to y \) via the construction

\[
\begin{array}{c}
\text{x} \\
\downarrow \\
Lx \times_{Ly} y \\
\downarrow \\
Ly \\
\end{array}
\]

i.e. via the composite \( x \to Lx \times_{Ly} y \to y \). In the case that the left localization \( L \) is additionally left exact, i.e. in the case that it commutes with finite limits (see [Lur09a, Remark 5.3.2.3]), then this does indeed produce a factorization as \( F_L \circ W_L \):

- the first map is in \( W_L \) by the left exactness of \( L \), and
- the second map is in \( F_L \) since it is a pullback of the map \( Lx \to Ly \), which is directly seen to have \( rlp(W_L) \).

In fact, this does not require the full strength of left exactness, but is only using the weaker notion of \( L \) being a locally cartesian localization (see [GK, 1.2]). Of course, it seems possible that the factorization axiom might hold without \( L \) being a locally cartesian localization.

Anyways, whenever these remaining lifting and factorization conditions hold, we obtain the desired model \( \infty \)-category \( M_L \). Indeed, in this case, the left adjoint is precisely the localization \( M \to M[W_L^{-1}] \cong LM \) (and moreover, the left homotopy relation will be visibly trivial (and hence the right homotopy relation will be trivial as well)).

**Remark 2.8.** There are a number of results in the literature surrounding factorizations in 1-categories which, if suitably generalized to \( \infty \)-categories, would give a much larger class of left localizations in which the factorization axiom holds in the situation of Example 2.7; a notable example is [CHK85, Corollary 3.4]. However, as we are not so deeply interested in this class of model structures anyways, we simply leave this as a remark. (Somewhat relatedly, factorization systems in stable \( \infty \)-categories are explored in [FL].)

**Example 2.9.** A particular case where Example 2.7 does indeed give a model structure is the left localization \( \pi_0 : S \leftarrow \Set : \text{disc} \). Here, we have that \( F_{\pi_0} = rlp(W_{\pi_0}) \) consists of precisely the \( \acute{e}tale \) maps of spaces, i.e. those maps which induce \( \pi_{\geq 1} \)-isomorphisms for every basepoint of the source. (Note that this condition is independent from that of being 0-connected; in particular, such a map may be neither injective nor surjective on \( \pi_0 \).) It follows that \( (W \cap F)_{\pi_0} = S^\equiv \), and so we have the required lifting condition. Moreover, the proposed factorization of Example 2.7 is in this case precisely the standard epi-mono factorization, and this gives the required factorization \( F_{\pi_0} \circ W_{\pi_0} \). As \( S \) is certainly finitely bicomplete, we obtain a model \( \infty \)-category \( S_{\pi_0} \) with \( S \to S[W_{\pi_0}^{-1}] \cong \Set \).

**Example 2.10.** In fact, Example 2.9 generalizes to the left localizations \( \tau_{\leq n} : S \leftarrow S^n : i \). Using the notation of Example 3.11 below, we have that

\[ F_{\tau_{\leq n}} = rlp((I_{\text{triv}}^S)_{\geq n+2}) \cap rlp'(\{S^n \to pt\}) \]

consists of precisely those maps which induce \( \pi_{\geq n+1} \)-isomorphisms for every basepoint in the source. So again \( (W \cap F)_{\tau_{\leq n}} = S^\equiv \), and it is easy to check using the long exact sequence in homotopy for a pullback square that again the suggestion in Example 2.7 yields the factorization \( F_{\tau_{\leq n}} \circ W_{\tau_{\leq n}} \). (Alternatively, an easy cells-and-disks construction yields this factorization as well.) Hence, we obtain a model \( \infty \)-category \( S_{\tau_{\leq n}} \) with \( S \to S[W_{\pi_0}^{-1}] \cong S^n \).

**Remark 2.11.** It is clear that Example 2.10 relies on the fact that there is a good theory of cellular approximation in \( S \) (e.g., the fact that attaching an \((n+2)\)-cell to a space doesn’t change its \(n\)-truncation). Thus, it does not appear to immediately generalize to an arbitrary \( \infty \)-topos: it is important that the generators be suitably compatible with the truncation functors.

**Remark 2.12.** The Kan–Quillen model structure on \( s\Set \) of Definition 4.5 has its weak equivalences created by the left adjoint \( |-| : s\Set \to \Set \), but it is not obtained via Example 2.7. Indeed, this left adjoint is certainly not left exact:
the question of when limits commute with geometric realizations is generally very difficult to answer. (However, in the case of pullbacks, this question is addressed to some extent by Corollary 6.6 below; see also the surrounding Remarks 6.5, 6.8, 6.9, and 6.10. The case of more general limits will also be addressed to some extent in [MGg].)

**Example 2.13.** Given a right localization $i : \mathcal{M} \rightleftarrows \mathcal{M} ; R$ with $\mathcal{M}$ finitely bicomplete, we obtain a dual story to that of Example 2.7. Now we should think of every object as fibrant, and of the left adjoint as the inclusion of the coreflective subcategory of cofibrant objects. In this case, the guess at a model structure $\mathcal{M}_R$ on $\mathcal{M}$ has that

- $W_R$ is the subcategory of morphisms in $\mathcal{M}$ that become equivalences in $RM$,
- $F_R = \mathcal{M}$, and
- $C_R$ is determined by the lifting condition $C_R = llp(W_R)$.

From here, it remains to check

- the lifting condition $(W \cap C)_R \subset llp(F_R) = \mathcal{M}^\tau$, and
- the factorization condition for $(W \cap F)_R \circ C_R = W_R \circ C_R$.

Now, the factorization condition will be implied by the right exactness of the right localization $R$ (or more generally, if the right localization is “locally cocartesian”).

**Remark 2.14.** Example 2.7 reinterprets a left localization as describing a model $\infty$-category in which all objects are cofibrant; dually, Example 2.13 reinterprets a right localization as describing a model $\infty$-category in which all objects are fibrant. Since in an arbitrary model $\infty$-category, not all the objects will be cofibrant and not all the objects will be fibrant, we should think of the notion of a model $\infty$-category as giving a simultaneous generalization of the notions of left and right localizations.

**Example 2.15.** As a simple case of Remark 2.14, we can obtain a “first approximation” (as in Remark 1.5) to a model structure on the $\infty$-category $Sp$ of spectra which would present the $\infty$-category $Sp^{[m,n]}$ of spectra that only have nontrivial homotopy groups in some interval $[m, n] \subset \mathbb{Z}$: the cofibrant objects would be $Sp^{[m,n]} \subset Sp$, while the fibrant objects would be $Sp^{[m,n]} \subset Sp$. This example, though illustrative, is somewhat degenerate, since any weak equivalence between bifibrant objects is already an equivalence. Indeed, the “homotopy relations” would all be trivial, corresponding to the fact that we have an inclusion $Sp^{[m,n]} \subset Sp$ which is a section to the projection $\tau_{\geq m} \circ \tau_{\leq n} \simeq \tau_{\leq n} \circ \tau_{\geq m} : Sp \to Sp^{[m,n]}$.

**Remark 2.16.** Analogously to Example 2.10, we might try to obtain a model structure as in Example 2.13 from the right localization $i : S^{\geq n} \rightleftarrows S^{\geq 1} \coloneqq S^{\geq 1} : \tau_{\geq n}$. However, the existence of such a model structure is much less clear in this case.

We do still have the lifting condition. Indeed, suppose that $x \to y$ is in $(W \cap C)_{\tau_{\geq n}}$. Since $x \to y$ is in $W_{\tau_{\geq n}}$, it induces an isomorphism on $\pi_{\geq n}$. On the other hand, obtaining a lift in the commutative square

$$
\begin{array}{ccc}
x & \xrightarrow{\tau_{\leq n-1} x} & \tau_{\leq n-1} x \\
\downarrow & & \downarrow \\
y & \xrightarrow{\tau_{\leq n-1} y} & \tau_{\leq n-1} y
\end{array}
$$

(in which $\tau_{\geq n}$ applied to the right map yields $id_{pt}$), we see that the map also induces isomorphisms on $\pi_{\leq n-1}$.

But the factorization condition is trickier. Given a map $x \to y$ in $S^{\geq 1}$, the map

$$\tau_{\geq n} y \to \tau_{\geq n} y \sqcup_{\tau_{\geq n} x} x$$

will not necessarily be a $\tau_{\geq n}$-isomorphism. For instance, when $n \geq 2$ and $x$ is a 1-type (so that $\tau_{\geq n} x \simeq pt$), say $x \simeq BG$, then this map will just be the inclusion $\tau_{\geq n} y \to \tau_{\geq n} y \sqcup BG$ of the first factor, which will not generally induce a $\tau_{\geq n}$-isomorphism. (Of course, this observation does not preclude the existence of suitable factorizations.)

### 2.2. Examples of Quillen adjunctions and Quillen equivalences.

**Example 2.17.** If a left localization $L : \mathcal{M} \rightleftarrows LM : i$ induces a model structure $\mathcal{M}_L$ as in Example 2.7, then we obtain a Quillen adjunction $id_{\mathcal{M}} : \mathcal{M}_{triv} \rightleftarrows \mathcal{M}_L : id_{\mathcal{M}}$, whose derived adjunction is precisely the original left localization $L : \mathcal{M} \rightleftarrows LM : i$. This is of course closely related to the theory of left Bousfield localizations of model categories (see e.g. [Hir03, §3.3]). (Dual statements apply to the right localization of Example 2.13.)

**Example 2.18.** As a particular case of Example 2.17, the model $\infty$-category $S_{\leq n}$ of Example 2.10 participates in a Quillen adjunction $id_S : S_{triv} \rightleftarrows S_{\leq n} : id_S$, whose derived adjunction is $\tau_{\leq n} : S \rightleftarrows S^{\leq n} : i$.

**Example 2.19.** Theorem 3.12 will give a set of general criteria for constructing Quillen adjunctions by lifting a cofibrantly generated model structure along a left adjoint.
Example 2.20. As will be immediate from Definitions 4.1 and 4.5, the adjunction \( \pi_0 : sS_{KQ} \rightleftarrows S\text{-Set}_{KQ} : \text{disc} \) is a Quillen equivalence. Note that since the geometric realization functor \( \lvert - \rvert : sS \rightarrow \Delta \) is a left localization, it already follows that it induces an equivalence \( sS[\text{W}_{KQ}^{-1}] \rightleftarrows \Delta \).

Example 2.21. In Remark 6.18, we will see that the “subdivision” and “extension” endofunctors of [Kan57], suitably extended from \( S\text{-Set} \) to \( sS \), define a Quillen equivalence between \( sS_{KQ} \) and itself.

Example 2.22. Recall that the \( \infty \)-category \( CSS \) of complete Segal spaces sits as a left localization \( sS \rightleftarrows CSS \), and moreover admits a natural equivalence \( CSS \simeq \text{Cat}_\infty \) (see e.g. [Rez01, Theorem 7.2] and [JT07, Theorem 4.12]; we will focus on this perspective in [MGc]). As we will see in [MGc], as a particular case of Example 2.19, we can transfer the Kan–Quillen model structure of Definition 4.5 along the adjunction \( sS \rightleftarrows CSS \) to obtain the Thomason model structure on the \( \infty \)-category \( \text{Cat}_{\infty} \) of \( \infty \)-categories. This Quillen adjunction is in fact a Quillen equivalence, and hence Remark 1.15 implies that the Thomason model structure on \( \text{Cat}_{\infty} \) once again presents the \( \infty \)-category \( S \). As explained in [MGc], this model structure resolves some of the less satisfying aspects of the classical Thomason model structure on \( \text{Cat} \) (which also presents \( S \)).

2.3. Speculative examples.

Speculation 2.23. Let us temporarily refer to the model structure of Definition 4.5 as the “strong” Kan–Quillen model structure on \( sS \): its subcategory \( \text{W}_{KQ}^{sS} \subset sS \) of weak equivalences is created by the geometric realization functor \( \lvert - \rvert : sS \rightarrow \Delta \), and it is cofibrantly generated by the sets \( I_{KQ\text{strong}} = \{ \partial \Delta^n \rightarrow \Delta^n \}_{n \geq 0} \) and \( J_{KQ\text{strong}} = \{ \Delta^n \rightarrow \Delta^n \}_{0 \leq i \leq n \geq 1} \). Then, there should exist other Kan–Quillen model structures on \( sS \): these would have the same subcategory of weak equivalences, but would have more cofibrations. For instance, we might define “medium” and “weak” Kan–Quillen model structures by extending the set of generating cofibrations to be given by

\[
I_{KQ\text{medium}} = I_{KQ\text{strong}} \cup \{ S^i \boxtimes \partial \Delta^n \rightarrow S^i \boxtimes \Delta^n \}_{i \geq 1, n \geq 0}
\]

and

\[
I_{KQ\text{weak}} = I_{KQ\text{medium}} \cup \{ S^i \boxtimes \Delta^n \rightarrow \text{pt} S \boxtimes \Delta^n \}_{i \geq 1, n \geq 0},
\]

with sets of generating acyclic cofibrations extended to match. There would then exist Quillen equivalences

\[
sS_{KQ\text{strong}} \rightleftarrows sS_{KQ\text{medium}} \rightleftarrows sS_{KQ\text{weak}}
\]

(in which all underlying functors are \( \text{id}_{sS} \)): moving to the right, more and more maps become cofibrations, while moving to the left, more and more maps become fibrations. This explains the terminology: the geometric realization functor (being a colimit) already plays well with colimits, and hence it does not seem to be particularly useful to identify more maps as cofibrations. On the other hand, these variants would enjoy certain features not shared by \( sS_{KQ\text{strong}} \):

- The model \( \infty \)-category \( sS_{KQ\text{medium}} \) would be obtained by closing up the generating sets under the tensoring, i.e. by performing an enriched small object argument (see Remark 3.8), which would easily provide functorial factorizations.

- The model \( \infty \)-category \( sS_{KQ\text{weak}} \) would have all objects cofibrant, just as \( S\text{-Set}_{KQ} \). However, it seems that the primary importance of this fact is that it implies left properness, so this may not be much of an advantage, since \( sS_{KQ\text{strong}} \) is already (practically trivially) left proper.

The existence of these alternate Kan–Quillen model structures almost follows easily from Theorem 3.12 (the recognition theorem for cofibrantly generated model \( \infty \)-categories) and our proof of Theorem 4.4 (the main theorem, which asserts the existence of \( sS_{KQ\text{strong}} \)): more precisely, using the results presented here, it is straightforward to verify all the conditions given in Theorem 3.12 except for condition (3). On the other hand, it seems eminently plausible that Smith’s recognition theorem for combinatorial model categories (see [Lur09a, Proposition A.2.6.8]), especially its simpler special case given by Lurie (see [Lur09a, Proposition A.2.6.13]), would admit a straightforward generalization to the \( \infty \)-categorical setting. From here, a version of [Lur09a, Proposition A.2.6.13] would guarantee that any set of maps \( I \subset sS \) containing \( I_{KQ\text{strong}} \) would constitute a set of generating cofibrations for a model structure on \( sS \) with subcategory of weak equivalences given by \( \text{W}_{KQ}^{sS} \subset sS \). (Condition (1) would be satisfied by a combination of variants of [Lur09a, Example A.2.6.11 and Corollary A.2.6.12], condition (2) would be true (even without the assumption that \( I \supset I_{KQ\text{strong}} \)) because geometric realization (being a colimit) commutes with pushouts, and condition (3) would follow from the fact that \( rlp(I) \subset rlp(I_{KQ\text{strong}}) \subset \text{W}_{KQ}^{sS} \)).
Remark 2.24. It seems that $sS_{KQ}^\text{weak}$ would be closely related to the Moerdijk model structure on $ssSet$ described in Remark 7.3, and a putative set of generating acyclic cofibrations
\[ J_{KQ_{\text{strong}}} \cup \{ S^j \boxtimes A^p \to S^j \boxtimes A^n \}_{1 \leq i \leq n, 1 \leq j \leq 1} \]
specifies closely related to our comparison with the $\pi_\ast$-Kan condition in Remark 6.9.

Speculation 2.25. There should exist a Kan–Quillen model structure on the $\infty$-category $sSp$ of simplicial spectra. However, the “levelwise infinite loopspace” functor $\Omega^\infty : sSp \to SS_\ast$ isn’t conservative, and so it wouldn’t make sense to lift this from a Kan–Quillen model structure on $sS_\ast$. (By contrast, the usual model structure on simplicial abelian groups is lifted from $sS_{KQ}$.) This should give rise to model structures e.g. on simplicial module spectra over a (simplicial) ring spectrum, and in other stable contexts. Alternatively, such applications might all be handled sufficiently by the $E^n$ model structure of $[MG]$.

Speculation 2.26. There should exist a model structure on simplicial objects in algebras over an operad which accounts for free resolutions. For example, this should recover as a special case the model structure on simplicial commutative rings alluded to in [Qui, §2] (and laid out explicitly in [Sch97, §3.1]), and should provide a framework organizing the “prove it for a free simplicial resolution, then prove that it commutes with (sifted) colimits” computations with “$B$-structured $n$-disk algebras” (e.g. $E_n$ algebras) that appear throughout [AF].

Speculation 2.27. There should exist a Joyal model structure on $sS$, whose fibrant objects are the “homotopical quasicategories”, namely those $Y \in sS$ such that for all $n \geq 0$ and all $0 < i < n$, the inner horn inclusion $\Lambda^i_n \to \Delta^n$ induces an effective epimorphism $Y(\Delta^n) \to Y(\Lambda^i_n)$ (identifying $Y \in sS$ with its image under the left Kan extension $sS = \text{Fun}(\Delta^{op}, S) \to \text{Fun}(sSet^{op}, S)$ along the Yoneda embedding). This should moreover participate in a left Bousfield localization $sS_{\text{Joyal}} \rightleftarrows sS_{KQ}^\text{weak}$ with the “weak” Kan–Quillen model structure of Speculation 2.23.

The proof of the Joyal model structure on $sS$ would presumably follow that of the one on $sSet$ fairly closely; a short and streamlined exposition of the latter is given in [DS11, Appendix C] (in contrast with the one given in [Lur09a], which proceeds by using the model category (Cat$_\infty Set$)$_{\text{Bergner}}$.)

There should similarly exist other model $\infty$-categories which present Cat$_\infty$ based on model 1-categories that do, e.g. a Bergner model structure on “$sS$-enriched categories” and a Barwick–Kan model structure on relative $\infty$-categories.

Speculation 2.28. The central theorem regarding formal moduli problems for $E_n$ algebras, described in [Lur10] (and made precise in [Lur11]), posits an equivalence between the $\infty$-categories of formal $E_n$ moduli problems and of augmented $E_n$ algebras (see [Lur10, Theorem 6.20]). This equivalence takes an augmented $E_n$ algebra to its associated Maurer–Cartan functor.

The inverse equivalence is somewhat trickier to describe. When the formal $E_n$ moduli problem is affine or pro-affine (i.e. corepresented by a small $E_n$ algebra or by a pro-object in such), then this inverse equivalence is implemented by Koszul duality (see [Lur10, Example 8.5 and Remark 8.9]). However, more generally one must take a resolution of the given formal $E_n$ moduli problem by a smooth hypercovering consisting of pro-affine ones, apply Koszul duality levelwise to this simplicial object, and then take the colimit. Thus, in general this inverse equivalence is given by the derived functor of Koszul duality (in analogy with e.g. the statement that the cotangent complex is the derived functor of derivations).

Hence, there should then exist a model structure on the $\infty$-category of simplicial objects in formal $E_n$ moduli problems (presumably related to the model structure of Speculation 2.26): appropriate levelwise pro-affine objects would be cofibrant, smooth hypercoverings would be acyclic fibrations, and then e.g. for an arbitrary formal $E_n$ moduli problem, [Lur10, Proposition 8.19] would provide a cofibrant replacement by an acyclic fibration.

3. Cofibrantly generated model $\infty$-categories

In this section we discuss cofibrantly generated model $\infty$-categories. As in the classical situation, these are model structures which are determined by a relatively small amount of data, namely by a set of generating cofibrations and a set of generating acyclic cofibrations, which simultaneously

- generate the subcategories $C$ and $W \cap C$, respectively, in a suitable sense (as their names suggest),
- detect the subcategories $W \cap F$ and $F$, respectively, in accordance with Remark 1.11, and
- are suited for obtaining the factorizations required by Definition 1.1.

We will use this setup to define the Kan–Quillen model structure on the $\infty$-category of simplicial spaces in §4.

We begin with a sequence of definitions. They are all direct generalizations of their 1-categorical counterparts (after replacing a set of maps with a set of homotopy classes of maps), and it is routine to verify that they enjoy completely analogous properties (see e.g. [Hir03, §10.4-5]). We also point out once and for all that these definitions do not depend on choices of representatives for the elements of the given set $I$ of homotopy classes of maps.
Definition 3.1. Given a set $I$ of homotopy classes of maps in $\mathcal{C}$, the subcategory $I$-inj $\subset \mathcal{C}$ of $I$-injectives is the subcategory of maps with rlp($I$).

Definition 3.2. Given a set $I$ of homotopy classes of maps in $\mathcal{C}$, the subcategory $I$-cof $\subset \mathcal{C}$ of $I$-cofibrations is the subcategory of maps with lfp(rlp($I$)).

Definition 3.3. Assume that $\mathcal{C}$ admits pushouts and sequential colimits. Given a set $I$ of homotopy classes of maps in $\mathcal{C}$, the subcategory $I$-cell $\subset \mathcal{C}$ of relative $I$-cell complexes is the subcategory of maps that can be constructed as transfinite compositions of pushouts of elements of $I$. An object is called an $I$-cell complex if its essentially unique map from $\emptyset_{\mathcal{C}}$ is a relative $I$-cell complex. Note that ($I$-cell)-inj $= I$-inj and that $I$-cell $\subset I$-cof.

Definition 3.4. Given a cardinal $\kappa$, an object $x \in \mathcal{C}$ is called $\kappa$-small relative to $I$ if for every regular cardinal $\lambda \geq \kappa$ and every $\lambda$-sequence $\{y_\beta\}_{\beta < \lambda}$ of relative $I$-cell complexes, $\text{colim}_{\beta < \lambda} \hom_{\mathcal{C}}(x, y_\beta) \sim \hom_{\mathcal{C}}(x, \text{colim}_{\beta < \lambda} y_\beta)$. An object of $\mathcal{C}$ is called small relative to $I$ if it is $\kappa$-small relative to $I$ for some cardinal $\kappa$. An object of $\mathcal{C}$ is called $\kappa$-small if it is $\kappa$-small relative to $\mathcal{C}$, and is called small if it is $\kappa$-small for some cardinal $\kappa$.

Definition 3.5. We say that a set $I$ of homotopy classes of maps in $\mathcal{C}$ permits the small object argument if the sources of its elements are small relative to $I$.

Proposition 3.6. Suppose that $\mathcal{C}$ is an $\infty$-category that admits pushouts and sequential colimits, and suppose that $I$ is a set of homotopy classes of maps in $\mathcal{C}$ which has all coproducts and permits the small object argument. Then every map in $\mathcal{C}$ admits a factorization into a relative $I$-cell complex followed by an $I$-injective.

Proof. The proof runs identically to that of [Hir03, Proposition 10.5.16], except that we take a coproduct over homotopy classes of commutative squares and we choose arbitrary representatives for these classes when forming the pushout. (See also [Lur11, Proposition 1.4.7].)

Definition 3.7. We refer to Proposition 3.6 as the small object argument.

Remark 3.8. Although the above definitions are analogous to the classical ones, there is one wrinkle that appears in the $\infty$-categorical case. Namely, the classical small object argument is visibly functorial: one simply takes a coproduct over the set of commutative squares (and no choices of representatives of homotopy classes is necessary). In an $\infty$-category, however, we instead have a space of commutative squares. Thus, it would be more natural in some respects for us to instead carry out an “$\mathbb{S}$-enriched small object argument”, which would then be similarly functorial. However, this has two drawbacks for us.

First of all, to do so would shrink the right class of the associated weak factorization system. (Indeed, in enriched category theory, passing from an unenriched lifting condition to an enriched lifting condition is equivalent to closing up the given set of maps under tensors with the enriching category.) We will be making much use of fibrations and trivial fibrations (for instance in the model $\infty$-category of Definition 4.5), and so it is in our best interest to keep these classes as large as possible: given that the stronger statement holds (i.e. that in our cases of interest, it suffices to check an unenriched lifting condition), it seems reasonable to incorporate it into the theory.

More crucially, however, a key feature of the $E^2$ model structure of [MGi] is that, given a presentable $\infty$-category $\mathcal{C}$ with chosen set of generators $\mathcal{G}$, we will obtain a cofibrant replacement of an object $X \in \mathcal{C}$ by an object $Y_* \in s\mathcal{C}$ which is given in each level as a coproduct of elements of $\mathcal{G}$ (see [DKS93, 3.3 and 5.5] and [Lur09a, Lemma 5.5.8.13 and Proposition 5.5.8.10(5)]), as opposed to some more elaborate construction involving tensorings with spaces. This is useful because, denoting by $F : \mathcal{C} \to \mathcal{A}$ our topology-to-algebra functor of interest (for instance, a homology theory $E_*$), it affords us an approximation of $F(X) \in \mathcal{A}$ by $F^{\text{bw}}(Y_*) \in s\mathcal{A}$, which is only useful inasmuch as it is algebraically accessible (for instance, a levelwise-projective simplicial $E_*$-module). If we were to use an $\mathbb{S}$-enriched small object argument to construct the cofibrantly generated $E^2$ model structure, we would cease to have any control whatsoever over the value of the functor $F^{\text{bw}}$ on these cofibrant replacements.

However, we will eventually want to have a suitably functorial unenriched small object argument at our disposal. As we have indicated, this is a rather subtle issue; thus, we defer its treatment to [MGe], and for now we settle for a non-functorial unenriched small object argument.

We now come to the main definition of this section.

Definition 3.9. A cofibrantly generated model $\infty$-category is a model $\infty$-category $\mathcal{M}$ such that there exist sets of homotopy classes of maps $I$ and $J$, respectively called the generating cofibrations and the generating acyclic cofibrations, both permitting the small object argument, such that $\mathcal{W} \cap \mathcal{F} = \text{rlp}(I)$ and $\mathcal{F} = \text{rlp}(J)$. (It follows from Remark 1.11 that also $\mathcal{C} = I$-cof and $\mathcal{W} \cap \mathcal{C} = J$-cof.)
Example 3.10. Let $S_{\text{triv}}$ denote the trivial model structure of Example 2.2 on the $\infty$-category $S$ of spaces. Combined with a few basic observations coming from the theory of CW complexes, Lemma 5.1 shows that $S_{\text{triv}}$ is cofibrantly generated by the sets $I^S_{\text{triv}} = \{S^{n-1} \to pt_S\}_{n \geq 0}$ and $J^S_{\text{triv}} = \{\text{id}_S\}$.

Example 3.11. Recall the model structure $S_{\pi_0}$ on the $\infty$-category $S$ of Example 2.9: $W_{\pi_0}$ is created by $\pi_0 : S \to \text{Set}$, $C_{\pi_0} = S$, and $F_{\pi_0} = \text{rlp}(W_{\pi_0})$. This model structure is not cofibrantly generated (or at least, not obviously so), but we nevertheless have that $F_{\pi_0} = \text{rlp}(\{S^0 \to pt_S\})$.

Of course, this observation generalizes to the model structure $S_{\pi_{\leq n}}$ of Example 2.10, as indicated there.

We have the following recognition theorem for cofibrantly generated model $\infty$-categories.

Theorem 3.12. Let $M$ be an $\infty$-category which is both cocomplete and finitely complete, and let $W \subset M$ be a subcategory which is closed under retracts and satisfies the two-out-of-three property. Suppose $I$ and $J$ are sets of homotopy classes of maps in $M$, both permitting the small object argument, such that

1. $J$-cof $\subset (I$-cof $\cap W)$,
2. $I$-inj $\subset (J$-inj $\cap W)$, and
3. either
   a. $(I$-cof $\cap W) \subset J$-cof or
   b. $(J$-inj $\cap W) \subset I$-inj.

Then the sets $I$ and $J$ define a cofibrantly generated model structure on $M$ whose weak equivalences are $W$.

Proof. With the small object argument in hand, the proof runs identically to that of [Hir03, Theorem 11.3.1].

We also have the following lifting theorem.

Theorem 3.13. Let $M$ be a cofibrantly generated model $\infty$-category with generating cofibrations $I$ and generating acyclic cofibrations $J$, and let $F : M \simeq N : G$ be an adjunction with $N$ finitely bicomplete. If $FI$ and $FJ$ both permit the small object argument and $G$ takes relative $FJ$-cell complexes into $W^M$, then $FI$ and $FJ$ define a cofibrantly generated model structure on $N$ in which $W^N$ is created by $G$. Moreover, with respect to this lifted model structure, the adjunction $F \dashv G$ becomes a Quillen adjunction.

Proof. The proof runs identically to that of [Hir03, Theorem 11.3.2].

4. THE DEFINITION OF THE KAN–QUILLEN MODEL STRUCTURE

We are now in a position to state the main result of this paper (Theorem 4.4), which gives a systematic way of manipulating simplicial spaces in their capacity as “presentations of spaces” via the geometric realization functor. This sits in precise analogy with the 1-category of simplicial sets, and so we begin by recalling the following.

Definition 4.1. The Kan–Quillen model structure on $s\text{Set}$, denoted $s\text{Set}_{\text{KQ}}$, is the proper model structure which is cofibrantly generated by the sets $I^K_{\text{KQ}} = \{\partial \Delta^n \to \Delta^n\}_{n \geq 0}$ and $J^K_{\text{KQ}} = \{\Lambda_i^n \to \Delta^n\}_{0 \leq i \leq n \geq 1}$ (see e.g. [Hir03, Example 11.1.6 and Theorem 13.1.13]).

In order to be precise regarding model $\infty$-categories (and in case the reader has forgotten the classical definition), we also give the following.

Definition 4.2. A model $\infty$-category is called left proper if its weak equivalences are preserved under pushout along cofibrations, and dually is called right proper if its weak equivalences are preserved under pullback along fibrations. A model $\infty$-category is called proper if it is both left proper and right proper.

The categories of simplicial sets and simplicial spaces are related in the following way.

Notation 4.3. Recall the adjunction $\pi_0 : S \rightleftarrows \text{Set} : \text{disc}$. Applying $\text{Fun}(\Delta^{op}, -)$, this induces an adjunction which we again denote by $\pi_0 : sS \rightleftarrows s\text{Set} : \text{disc}$. We will generally omit this right adjoint from the notation unless we mean to emphasize it.
Using this terminology and notation, we can now state the main result of this paper.

**Theorem 4.4.** The sets $I^S_{KQ} = \text{disc}(I^{s\text{Set}}_{KQ})$ and $J^S_{KQ} = \text{disc}(J^{s\text{Set}}_{KQ})$ define a proper, cofibrantly generated model structure on $sS$, in which the weak equivalences are created by the geometric realization functor $|-| : sS \to s$. 

**Proof.** We appeal to Theorem 3.12, verifying
- condition (1) in Proposition 7.1,
- condition (2) in Proposition 7.2,
- condition (3)(b) in Proposition 7.8, and
- that $I^S_{KQ}$ and $J^S_{KQ}$ both permit the small object argument in Corollary 5.3.

The weak equivalences are closed under retracts and satisfy the two-out-of-three property because they are pulled back from a class of equivalences. Lastly, left properness is immediate since the weak equivalences are created by a colimit functor (which commutes with pushouts), and right properness is proved as Corollary 6.7. □

**Definition 4.5.** In analogy with Definition 4.1, we also refer to the model structure on $sS$ defined by Theorem 4.4 as the Kan–Quillen model structure, denoted $sS_{KQ}$.

**Remark 4.6.** In the theory of model 1-categories, one of the most useful consequences of right properness is that a pullback in which just one of the maps is already a homotopy pullback (and dually for left properness). This remains true in the theory of model ∞-categories; once the theory of “homotopy co/limits” has been set up in [MGg], the proof runs identically (see e.g. [Hir03, §13.3]). However, for the sake of self-containment, we will also directly prove this consequence of the right properness of $sS_{KQ}$ as Corollary 6.6. (In fact, we use this as an input to the proof of right properness in Corollary 6.7.) The dual corollary of left properness is as trivial to verify as the left properness itself.

**Remark 4.7.** In the theory of model 1-categories, there are many other adjectives that one might attach to a model structure: combinatorial, cellular, tractable, etc. The model ∞-category $sS_{KQ}$ enjoys completely analogous properties to those enjoyed by $s\text{Set}_{KQ}$. However, we won’t need these observations for now, so we just leave them here as a remark. (The model ∞-category $sS_{KQ}$ will also be simplicio-spatial, the correct analog of a simplicial model 1-category; see [MGf].)

**Remark 4.8.** Note that if we apply the small object argument for $I^S_{KQ}$ or $J^S_{KQ}$ to a map in $sS$ whose source is in $s\text{Set}$, then the intermediate object will also be in $s\text{Set}$. In particular, by the usual transfinite induction arguments, $\text{rlp}(I^S_{KQ}) = \text{rlp}(\text{disc}((C^s_{KQ})$) and $\text{rlp}(J^S_{KQ}) = \text{rlp}(\text{disc}((W \cap C)_{KQ}^{s\text{Set}}))$. We will use these facts without further comment.

**Remark 4.9.** The adjunction $\pi_0 : sS \rightleftharpoons s\text{Set} : \text{disc}$ could be used to lift the Kan–Quillen model structure on $sS$ to the one on $s\text{Set}$ via Theorem 3.13, except of course that that’s totally circular – the construction of $sS_{KQ}$ certainly depends on that of $s\text{Set}_{KQ}$. As we mentioned in Example 2.20, this adjunction is even a Quillen equivalence, whose derived adjunction is the identity adjunction on their common localization $S$.

**Notation 4.10.** In the course of proving Theorem 4.4, it will be important to take care in distinguishing which facts have been proved and which facts have not; otherwise, our arguments might end up being circular, for example as discussed in Remark 7.7. Thus, in order to be totally clear about this distinction, we take the following conventions regarding maps of simplicial sets and of simplicial spaces.

- We will only decorate our arrows if the corresponding property is relevant to the argument.
- We will only use the decoration $\Rightarrow$ if we’ve actually proved that the map becomes a weak equivalence upon geometric realization.
- When working in $s\text{Set}_{KQ} \subset sS_{KQ}$, we will use all standard results and notation.
- By the nature of the arguments, the only cofibrations in $sS_{KQ}$ that appear will actually lie in the subcategory $s\text{Set}_{KQ}$. On the other hand, rather than use the decorations $\to$ or $\Rightarrow$ for maps in $sS_{KQ}$, we will instead label the arrows with their lifting properties (so $\text{rlp}(J^S_{KQ})$ or $\text{rlp}(I^S_{KQ})$, respectively).
- For convenience, we will still write $sS^f_{KQ}$ for those objects whose terminal map has $\text{rlp}(J^S_{KQ})$. Note that we will have $sS^c_{KQ} = s\text{Set} \subset sS$.

### 5. Auxiliary results on spaces and simplicial spaces

Before proving Theorem 4.4, we first lay out some auxiliary results regarding spaces and simplicial spaces. We begin with the following result, which gives a criterion for a map of spaces to be an equivalence.
Lemma 5.1. A map $Y \xrightarrow{\varphi} Z$ in $S$ is an equivalence iff it has $\operatorname{rlp} \{ S^{n-1} \rightarrow \operatorname{pt}_S \}_{n \geq 0}$. More generally, a map in $S$ is $n$-connected iff it has $\operatorname{rlp} \{ S^{i-1} \rightarrow \operatorname{pt}_S \}_{0 \leq i \leq n}$.

Proof. First, note that if $Y \rightarrow Z$ has $\operatorname{rlp} \{ S^{i-1} \rightarrow \operatorname{pt}_S \}$, then the map $[S^{i-1}, Y]_S \rightarrow [S^{i-1}, Z]_S$ is an inclusion. Since a map off of $S^{i-1}$ is basedly nullhomotopic iff it’s freely so, this means that for any basepoint $y \in Y$, the map $\pi_{i-1}(Y, y) \rightarrow \pi_{i-1}(Z, \varphi(y))$ is an inclusion.

On the other hand, considering the map $S^{i-1} \rightarrow \operatorname{pt}_S$ as the standard inclusion $S^{i-1} \rightarrow D^i$, we see that if we begin with the constant map $S^{i-1} \rightarrow Y$ at some point $y \in Y$, then an extension of the composite $S^{i-1} \rightarrow Y \rightarrow Z$ over $S^{i-1} \rightarrow D^i$ really just selects a map $S^i \rightarrow Z$ which is based at $\varphi(y)$. So, if $Y \rightarrow Z$ has $\operatorname{rlp} \{ S^{i-1} \rightarrow \operatorname{pt}_S \}$, then also $\pi_i(Y, y) \rightarrow \pi_i(Z, \varphi(y))$ is a surjection for any basepoint $y \in Y$.

Combining these two consequences of $Y \rightarrow Z$ having $\operatorname{rlp} \{ S^{i-1} \rightarrow \operatorname{pt}_S \}$ proves both claims. \hfill \Box

We now turn to simplicial spaces. We begin with the necessary smallness results.

Lemma 5.2. An object of $\mathbf{sSet}$ with finitely many nondegenerate simplices is $\omega$-small as an object of $\mathbf{sSet}$.

Proof. This follows from the fact that finite sets are small in $S$ and from the inductive definition of a map of simplicial objects. \hfill \Box

Corollary 5.3. The sets $I_{KQ}^S$ and $J_{KQ}^S$ permit the small object argument.

Proof. This follows from Lemma 5.2. \hfill \Box

Many of the remaining results of this paper will rely critically on the following one. Its proof is rather involved, and so we postpone it to $\S 8$. Roughly speaking, the result concerns the inductive construction of representations of maps in $S$ by maps in $\mathbf{sSet}$ in which the source is in $\mathbf{sSet} \subset \mathbf{sSet}$.

Lemma 5.4. Suppose we are given any $W \in \mathbf{sSet}$, any $K \in \mathbf{sSet}$, and any pushout

$$
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & K \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & L
\end{array}
$$

in $\mathbf{sSet}$. Suppose further that we are given any point of the pullback

$$
\lim \left( \begin{array}{c}
\operatorname{hom}_S(K, W) \\
\downarrow \\
\operatorname{hom}_S([L], [W]) \rightarrow \operatorname{hom}_S([K], [W])
\end{array} \right).$

Then there exists some $i \geq 0$ such that if the front square in the cube

$$
\begin{array}{ccc}
\partial \Delta^n & \rightarrow & K \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & L
\end{array}
$$

is also a pushout in $\mathbf{sSet}$, then $L' \xrightarrow{\varphi} L$ in $\mathbf{sSet}_{KQ}$ and the map

$$
\begin{array}{ccc}
\operatorname{hom}_S(L', W) & \rightarrow & \operatorname{hom}_S(K, W) \\
\downarrow \\
\operatorname{hom}_S([L'], [W]) \leftarrow \operatorname{hom}_S([L], [W]) \rightarrow \operatorname{hom}_S([K], [W])
\end{array}
$$

is essentially surjective onto the chosen point.
Remark 5.5. Lemma 5.4 is not as strong as one might hope. First of all, it would be nice if the last map in its statement were actually an effective epimorphism, but to deduce this we would need to be able to bound the number $i$ as we run through the path components of the pullback, and this does not appear to be possible. But even more seriously, if instead we have a cofibration $K \hookrightarrow L$ in $s\text{Set}_{KQ}$ which can only be obtained through multiple pushouts of maps in $I_{KQ}^{\text{Set}}$, then Lemma 5.4 cannot be made to guarantee the existence of an extension $L' \to W$ in $s\mathcal{S}$ (for some $L' \sim L$ in $s\text{Set}_{KQ}$) modeling the chosen extension $|L| \to |W|$ in $\mathcal{S}$.

For instance, suppose that $L = \Delta^1 \times \Delta^1$, and that the map $K \to L$ is the inclusion of its boundary (so that $K$ is a simplicial square, and the map $K \to L$ in $s\text{Set}_{KQ}$ presents the map $S^1 \to pt_\mathcal{S}$ in $\mathcal{S}$). The minimal way to present this as a composition of pushouts of maps in $I_{KQ}^{\text{Set}}$ is as

$$K \to M \to N \to L,$$

where $M$ is the 1-skeleton of $L$ and the latter two maps are each obtained by attaching a 2-simplex. However, when we attempt to extend our given map $K \to W$ along $K \to M$ using Lemma 5.4, we may need to subdivide the 1-simplex that we’re attaching, and so we only obtain an extension $M' \to W$ in $s\mathcal{S}$ for some factorization $K \to M' \to M$ in $s\text{Set}_{KQ}$. If any such subdivisions are required, then the two remaining holes to be filled in $M'$ will now have at least four edges each, and so we are no closer to “filling the hole” than when we started.

On the other hand, in the case that the map $K \to L$ is $\emptyset \to \Delta^0$, then no subdivisions are required (indeed, subdivision preserves both $\emptyset_{\text{Set}}$ and $\Delta^0$, or alternatively we can see this from the fact that the map $W_0 \to |W|$ is an effective epimorphism). Thus, if we are given any $W \in s\mathcal{S}$ and any map $S^n \to |W|$ in $\mathcal{S}$, we can first present the composite $pt_\mathcal{S} \to S^n \to |W|$ in $\mathcal{S}$ by some map $\Delta^0 \to W$ in $s\mathcal{S}$, and then taking $K = \Delta^0$ and forming the pushout

$$\partial \Delta^n \to \Delta^0 \to \Delta^n \to L$$

in $s\mathcal{S}$, we are guaranteed a factorization $\Delta^0 \to L' \to L$ in $s\text{Set}_{KQ}$ and a map $L' \to W$ in $s\mathcal{S}$ presenting the chosen map $|L| \simeq S^n \to |W|$ in $\mathcal{S}$. Since so many arguments in $\mathcal{S}$ go by considering arbitrary maps into a space from a sphere, the existence of these smallest models in $s\text{Set}_{KQ}$ for the spheres in $\mathcal{S}$ seems like a real stroke of luck.

In any case, we do not expect Lemma 5.4 to be particularly useful in the long run. Once the model $\infty$-category $s\mathcal{S}_{KQ}$ and the fundamental theorem of [MGd] have been set up, we expect that most arguments will simply run using those stronger and more satisfying results. In particular, see Corollary 6.2 (and Remark 6.3) below.

6. Fibrancy, fibrations, and the $\text{Ex}^\infty$ functor

We now undertake a study of fibrant objects and fibrations in $s\mathcal{S}_{KQ}$, which behave quite analogously to their counterparts in $s\text{Set}_{KQ}$.

We begin by studying fibrant objects. First of all, Lemma 5.4 admits a much cleaner analog when $W \in s\mathcal{S}_{KQ}$ is fibrant.

Lemma 6.1. In Lemma 5.4, if $W \in s\mathcal{S}_{KQ}^f$ then we may take $i = 0$. Hence, the map

$$\text{hom}_\mathcal{S}(L,W) \to \text{hom}_\mathcal{S}(K,W) \to \text{hom}_\mathcal{S}(|L|,|W|) \to \text{hom}_\mathcal{S}(|K|,|W|)$$

is an effective epimorphism.

Proof. We will argue using the diagram in $s\text{Set}_{KQ}$ shown in Figure 1, in which some of the objects and morphisms have yet to be constructed. First of all, recall that the top square of the cube is a pushout by definition of $L$. Also, observe that we can also build $L'$ via the iterated pushout in the front two squares, where we have $\text{sd}^i(\Delta^n) \sim M$ since $s\text{Set}_{KQ}$ is left proper. Next, choose any acyclic object $M' \in s\text{Set}_{KQ}$ admitting a cofibration from $M \coprod_{\partial \Delta^n} \Delta^n$, and use it to form the left face of the cube. (The maps from $M$ and $\Delta^n$ to this pushout are cofibrations, which is why the maps from $M$ and $\Delta^n$ to $M'$ are also cofibrations.) Then, define $L''$ by declaring that the bottom square of the cube is a pushout; by an easy diagram chase, the back square of the cube is therefore a pushout as well.
Figure 1. The diagram in $s\text{Set}_{KQ}$ used in the proof of Lemma 6.1.

Now, since $W \rightarrow \text{pt}_{sS}$ has $\text{rlp}(J_{KQ}^{s})$, we are guaranteed an extension

\[
K \rightarrow W
\]

in $sS$, and the composite map $L \rightarrow L'' \rightarrow W$ satisfies the same hypotheses as were required of the map $L' \rightarrow W$. Thus, we may take $i = 0$, as claimed.

Carrying out this same argument for all path components of the pullback implies that the indicated map is indeed an effective epimorphism. □

Corollary 6.2. If $W \in s\text{Set}_{KQ}$, then for any $K \rightarrow L$ in $s\text{Set}_{KQ}$, the map

\[
\text{hom}_{sS}(L, W) \rightarrow \lim \begin{pmatrix}
\text{hom}_{sS}(K, W) \\
\text{hom}_{S}(|L|, |W|) \\
\text{hom}_{S}(|K|, |W|)
\end{pmatrix}
\]

is an effective epimorphism.

Proof. First, we present the map $K \rightarrow L$ as a transfinite composition of pushouts of maps in $J_{KQ}^{s\text{Set}}$. Then, the result follows by transfinite induction, applying Lemma 6.1 at each successor ordinal and using the universal property of the colimit (which here is a colimit both in $s\text{Set}$ and in $sS$) at each limit ordinal. □

Remark 6.3. In the special case that $K = \varnothing_{s\text{Set}}$, Corollary 6.2 reduces to the statement that for any $L \in s\text{Set} = s\text{Set}_{KQ}$ and any $W \in s\text{Set}_{KQ}$, the map $\text{hom}_{sS}(L, W) \rightarrow \text{hom}_{S}(|L|, |W|)$ is an effective epimorphism. This is a hint of the fundamental theorem of model $\infty$-categories as applied to $s\text{Set}_{KQ}$; recall Remark 1.4.

We now turn from fibrant objects to fibrations. The next result is crucial, and provides a basis for many of the convenient properties enjoyed by the model $\infty$-category $s\text{Set}_{KQ}$. Its proof is relatively straightforward, though somewhat long (although not as long as it looks, since it contains a fair number of diagrams); afterwards, we state some of its immediate consequences, and we give some discussion regarding comparisons with previously known results.

Proposition 6.4. Suppose the map $Y \rightarrow Z$ in $sS$ has $\text{rlp}(J_{KQ}^{s})$, and suppose we are given any point $\text{pt}_{sS} \rightarrow Z$. Let $F_{z} \in sS$ be the fiber of $Y \rightarrow Z$ over $z$, and let $F|_{z} \in S$ be the fiber of $|Y| \rightarrow |Z|$ over $|z|$. Then the natural map $|F_{z}| \rightarrow F|_{z}$ is an equivalence in $S$. 
Proof. We use the criterion of Lemma 5.1. So, suppose that

\[
\begin{array}{ccc}
S^{n-1} & \longrightarrow & |F_z| \\
\downarrow & & \downarrow \\
pt_S & \longrightarrow & F_{|z|}
\end{array}
\]

is any commutative square in \(S\), for any \(n \geq 0\). Since \(F_z \to pt_S\) also has \(rlp(J_{KQ}^S)\) as this property is closed under pullbacks, by Corollary 6.2 we may present the upper map in the above diagram as a map \(\partial \Delta^n \to F_z\) in \(sS\).

From here, our argument will play back and forth between the diagrams shown in Figures 2 and 3. The former takes place in \(sS_{KQ}\), while the latter takes place in \(S\); in both, many of the objects (and all of the dotted arrows) have yet to be constructed. For clarity, we proceed in steps.

(1) Given the composite map \(\partial \Delta^n \to F_z \to Y\) in \(sS\) and its chosen extension

\[
\begin{array}{ccc}
|\partial \Delta^n| & \longrightarrow & |F_z| \\
\downarrow & & \downarrow \\
|\Delta^n| & \longrightarrow & F_{|z|}
\end{array}
\]

in \(S\), by Lemma 5.4 there exists a factorization \(\partial \Delta^n \to (\Delta^n)' \xrightarrow{\sim} \Delta^n\) in \(sS\) and a dotted arrow \((\Delta^n)' \to Y\) as in Figure 2 which models this extension in \(S\).

(2) For expository convenience, we consider the object \(\Delta^0 \in sS\) with its essentially unique map \(\Delta^0 \xrightarrow{\sim} pt_S\) as selecting a composite map \(|\Delta^0| \xrightarrow{\sim} |pt_S| \xrightarrow{|z|} |Z|\).

(3) Choose any vertex of \((\Delta^n)'\), and use this to define \((\Delta^n)'\) in \(sS\) by the pushout diagram

\[
\begin{array}{ccc}
\partial \Delta^1 & \longrightarrow & (\Delta^n)' \sqcup \Delta^0 \\
\downarrow & & \downarrow \\
\Delta^1 & \longrightarrow & (\Delta^n)''.
\end{array}
\]

Observe that both maps \(\Delta^0 \to (\Delta^n)''\) and \((\Delta^n)' \to (\Delta^n)''\) are in \(W_{KQ}^S\).

(4) Now, we have the solid commutative diagram

\[
\begin{array}{ccc}
|(\Delta^n)'| & \longrightarrow & |\Delta^n| \\
\downarrow & & \downarrow \\
|\Delta^0| & \longrightarrow & |pt_S| \\
\end{array}
\]

in \(S\), and hence we can obtain a dotted arrow \(|(\Delta^n)''| \to |Z|\) making the entire diagram commute, as in Figure 3.

(5) Thus, we have a map \((\Delta^n)' \sqcup \Delta^0 \to Z\) in \(sS\) and a chosen extension

\[
\begin{array}{ccc}
|(\Delta^n)' \sqcup \Delta^0| & \longrightarrow & |Z| \\
\downarrow & \downarrow \\
|(\Delta^n)''|
\end{array}
\]
in $\mathcal{S}$. (Note that geometric realization commutes with colimits, and in particular with coproducts.) So by Lemma 5.4, there is a factorization $(\Delta^n)' \sqcup \Delta^0 \to (\Delta^n)'' \xrightarrow{\sim} (\Delta^n)''$ in $s\mathcal{S}_{KQ}$ and a dotted arrow $(\Delta^n)'' \to Z$ as in Figure 2 which models this extension in $\mathcal{S}$.

(6) It is easy to see that that in fact, we have $(\Delta^n)' \xrightarrow{\sim} (\Delta^n)''$ in $s\mathcal{S}_{KQ}$ (for instance because $s\mathcal{S}_{KQ}$ is left proper, and so the defining map to $(\Delta^n)''$ from some subdivision of $\Delta^1$ is in $\mathcal{W}_{KQ}^{s\mathcal{S}}$). Since the map $Y \to Z$ in $\mathcal{S}$ has $\text{rlp}(J_{KQ}^{nS})$, it follows that there exists a lift $(\Delta^n)'' \to Y$ as in Figure 2.
(7) We now have the solid commutative diagram

```
∂Δ^n → Fz → Y
\downarrow \downarrow \downarrow
|\Delta^n| → |Fz| → |Y|
```

in sS, and so by the universal property of the pullback we can obtain a dotted arrow Δ^0 → Fz making the entire diagram commute, as in Figure 2.

(8) Now, taking geometric realization gives the entire diagram in S of Figure 3 (including the dotted arrows). In particular, we obtain the desired lift

```
|\partial Δ^n| → |Fz| → |Y|
```

in S. (It is straightforward to see that this does indeed commute, as Fz| is defined as a pullback.)

\[ \Box \]

Remark 6.5. In either sSet or sS, one can always take the fiber of a map over a given point in its target. However, inasmuch as we are interested in simplicial sets and simplicial spaces as presenting spaces via geometric realization, we can view Proposition 6.4 as saying that in either case, this fiber is only “homotopically meaningful” – that is, it only computes the fiber in S – if the original map is a fibration in the corresponding Kan–Quillen model structure.

While Proposition 6.4 only addresses the question of when taking fibers commutes with geometric realization, we can use it to address the same question regarding more general pullbacks.

Corollary 6.6. Suppose the map Y → Z in sS has rlp(J^{sS}_{KQ}). Then for any map W → Z in sS, the natural map

```
|W × Z Y| → |W| × |Z| |Y|
```

is an equivalence in S, i.e. the pullback of Y → Z along any map commutes with geometric realization.

Proof. It suffices to show that in the diagram

```
|W × Z Y| → |Y|
\downarrow \downarrow \downarrow
|W| → |Z|
```

in S, for every point of |W| the upper map induces an equivalence on induced fibers.

To begin, note that by Lemma 5.4 (or since the map W_0 → |W| is an effective epimorphism), every point pt_S → |W| in S is represented by a point pt_{sS} ≃ Δ^0 → W in sS. For such a point pt_{sS} → W, denote by F_w ∈ sS the fiber of the map W ×_Z Y → W over w.

Now, as rlp(J^{sS}_{KQ}) is closed under pullbacks, then the map W ×_Z Y → W also has rlp(J^{sS}_{KQ}). So in the diagram

```
F_w → W ×_Z Y → Y
\downarrow \downarrow \downarrow
pt_{sS} → W → Z
```

in sS_{KQ}, both the left square and the large rectangle are pullbacks (since pullbacks are stable under composition), and by Proposition 6.4 these both remain pullbacks when we apply |−| : sS → S. This proves the indicated sufficient condition.

\[ \Box \]
As stated in Remark 4.6, Corollary 6.6 already spells out one of the most useful consequences of the right properness of $s\mathcal{K}_Q$. However, for the sake of completeness, since we have not actually proved that this model structure is right proper, we do so now.

**Corollary 6.7.** $W^{s\mathcal{F}}_{\mathcal{K}_Q}$ is preserved by pullback along maps that have rlp($J^{s\mathcal{F}}_{\mathcal{K}_Q}$).

**Proof.** Suppose that we have a diagram $W \xrightarrow{\sim} Z \to Y$ in $s\mathcal{K}_Q$. By Corollary 6.6, the induced diagram

$$
\begin{array}{ccc}
|W \times Z Y| & \longrightarrow & |Y| \\
\downarrow & & \downarrow \\
|W| & \xrightarrow{\sim} & |Z|
\end{array}
$$

in $\mathcal{S}$ is a pullback square. But this implies that the upper map is an equivalence in $\mathcal{S}$, i.e. that the map $W \times Z Y \to Y$ is in $W^{s\mathcal{F}}_{\mathcal{K}_Q}$.

We now give some comparison of Corollary 6.6 with various results in the literature.

**Remark 6.8.** In an unpublished note, Rezk defines a realization fibration to be a map $Y \to Z$ in $s\mathcal{S}$ such that all pullbacks commute with geometric realization, and he gives a complete characterization (see [Rez14, Definition 1.1 and Proposition 5.10]). In this language, we can restate Corollary 6.6 as asserting that all maps in $F_{\mathcal{K}_Q}^{s\mathcal{F}}$ are realization fibrations. On the other hand, as geometric realization (being a sifted colimit) commutes with finite products, every terminal map $Y \to \text{pt}_{s\mathcal{S}}$ is a realization fibration, but it clearly need not be in $F_{\mathcal{K}_Q}^{s\mathcal{F}}$ in general.

**Remark 6.9.** The most directly comparable result in the literature to Corollary 6.6 is the $\pi_*-\text{Kan condition}$ of Bousfield–Friedlander (originally introduced in [BF78, Appendix B], but see also [GJ99, Chapter IV, §4]). This condition is stated in terms of bisimplicial sets, but with preferred “geometric” and “simplicial” coordinates; given these distinctions it becomes visibly invariant under weak equivalence in the model category $\mathcal{S}(s\text{Set}_{\mathcal{K}_Q})_{\text{Reedy}}$, i.e. it descends to the underlying $\infty$-category $s\mathcal{S}$. The precise definition ([BF78, B.3]) is too convoluted to give here, but roughly speaking it requires of an object $W \in s\mathcal{S}$ that the levelwise homotopy groups $\pi^l_{\text{lw}}(W)$ satisfy the Kan extension condition in the simplicial direction, for all possible basepoints and for all $t \geq 1$. Bousfield–Friedlander give three conditions under which the $\pi_*-\text{Kan condition}$ is satisfied:

- when the underlying object $W \in s\mathcal{S}$ is levelwise connected,
- when the bisimplicial set is a bisimplicial object (so presents a simplicial object in grouplike $E_1$ algebras in $\mathcal{S}$), and
- more generally, when the underlying object $W \in s\mathcal{S}$ has that each $W_n \in \mathcal{S}$ is simple and for all $t \geq 1$ the map

$$
\text{hom}^\text{lw}_{\text{ht}(\mathcal{S})}(S^t, W) \to \text{hom}^\text{lw}_{\text{ht}(\mathcal{S})}(\text{pt}, W) \cong \pi^l_{\text{lw}}(W)
$$

is in $F_{s\text{Set}}^{s\mathcal{F}}$.

Aside from these conditions, the $\pi_*-\text{Kan condition}$ is extremely difficult to check, and requires knowing a good deal about the simplicial space in question. In any case, in the language of [Rez14], Bousfield–Friedlander’s main result concerning the $\pi_*-\text{Kan condition}$ ([BF78, Theorem B.4]) is that if the map $Y \to Z$ in $s\mathcal{S}$ has that

- both $Y$ and $Z$ satisfy the $\pi_*-\text{Kan condition}$, and
- the map $\pi^l_{\text{lw}}(Y) \to \pi^l_{\text{lw}}(Z)$ is in $F_{\mathcal{K}_Q}^{s\text{Set}}$,

then the map is a realization fibration.

Thus, it might seem that the criterion of Corollary 6.6 for a map to be a realization fibration is far simpler than Bousfield–Friedlander’s $\pi_*-\text{Kan condition}$; one might even term it a “homotopy-coherent $\pi_0$-Kan condition”. However, we should not make light of the “homotopy-coherent” part of this phrase, i.e. the fact that the lifting condition takes places in an $\infty$-category. Indeed, given an object $W \in s\mathcal{S}$, attempting to check that $W \in s\mathcal{K}_Q^{s\mathcal{F}}$ by working with a particular representative in $s(s\text{Set}_{\mathcal{K}_Q})_{\text{Reedy}}$ ends up feeling a lot like checking that that representative has the $\pi_*-\text{Kan condition}$. On the other hand, we expect that it should be strictly easier to be in $s\mathcal{K}_Q^{s\mathcal{F}}$ than to satisfy the $\pi_*-\text{Kan condition}$: the former only asks for lifts once we have chosen coherent homotopies between various maps from spheres, whereas the latter asks for a lift whenever there exist not-necessarily-coherent homotopies between such maps. To be slightly more precise, attempting to check that $W \in s\mathcal{K}_Q^{s\mathcal{F}}$ using a representative in $s(s\text{Set}_{\mathcal{K}_Q})_{\text{Reedy}}$ ends up feeling a lot like checking the extension condition against maps of the form $S^t \boxtimes \Delta^n \to S^t \boxtimes \Delta^n$. However, we have not verified that this is the case.

(Of course, this is all reminiscent of the distinction between enriched and unenriched lifting conditions which arose in Remark 3.8. One might venture to conjecture that, at least in the absolute case of a map $W \to \text{pt}_{s\mathcal{S}}$ in $s\mathcal{S}$,
having $\text{rlp}(\mathcal{J}_{\mathcal{K}Q}^S)$ is actually equivalent to having $\text{rlp}(\mathcal{S} \Box J_{\mathcal{K}Q}^S)$: in other words, that the fibrant objects would remain the same whether we define the model structure on $\mathcal{S}^\flat$ using the enriched or the unenriched lifting condition.)

In any case, we nevertheless view Corollary 6.6 as an improvement of [BF78, Theorem B.4] for a few different reasons.

First of all, it is certainly conceptually simpler. There is not much sense to be made of the $\pi_n$-Kan condition (at least as far as we can see), but we find Remark 6.5 quite satisfying. This is in addition to the above observation regarding the difference between having chosen coherent homotopies and merely asserting the existence of not-necessarily-coherent homotopies.

Second of all, if it is indeed true that if $W \in \mathcal{S}$ satisfies the $\pi_n$-Kan condition then $W \in \mathcal{S}^\flat_{\mathcal{K}Q}$, then it seems plausible that the condition in [BF78, Theorem B.4] is actually secretly asking for the map $Y \to Z$ to be a fibration between fibrant objects in $\mathcal{S}^\flat_{\mathcal{K}Q}$. In comparison, Corollary 6.6 provides a relative version, which only requires that the map be a fibration but does not require that the target (and hence the source) be fibrant.

Finally, we have found Corollary 6.6 to be much more usable in the $\infty$-categorical context than [BF78, Theorem B.4]. Strong evidence is provided by the proof of the fundamental theorem of model categories in [MGd] (which in fact was the motivation for the development of $\mathcal{S}^\flat_{\mathcal{K}Q}$ in the first place). In that proof, it is easy to see that where the former result is used, the latter result would not suffice at all.

**Remark 6.10.** It is well known that for a simplicial space which is levelwise connected, taking loopspaces (with respect to any compatible choices of basepoints) commutes with geometric realization. (This follows from the result [BF78, Theorem B.4] discussed in Remark 6.9, see e.g. [GJ99, Chapter IV, Corollary 4.11].) In fact, any pullback in $\mathcal{S}$ in which the common target in the cospan is levelwise connected commutes with geometric realization (see [Lur14, Lemma 5.5.6.17]). Of course, there are many interesting simplicial spaces which are not levelwise connected – for instance, any nontrivial simplicial set – and so this result is of somewhat limited use when manipulating simplicial spaces and their geometric realizations.

We now return to the general theory. In the proof of Proposition 7.2 we will need to have a version of the $\text{Ex}^\infty$ functor for simplicial spaces, so we take a moment to develop that now. For simplicial sets, this was originally defined and explored in [Kan57, §3-4]; it is developed in more modern terminology in [GJ99, Chapter III, §4].

**Definition 6.11.** Recall that any $\Delta^n \in \text{sSet}$ admits a subdivision, denoted $\text{sd}(\Delta^n) \in \text{sSet}$; this is the nerve of its poset of nondegenerate simplices. Recall further that this admits a map $\text{sd}(\Delta^n) \to \Delta^n$, called the last vertex map, induced by the map of posets given by taking a simplex to its last vertex. Recall still further that we can extend this definition to any $K \in \text{sSet}$ by defining

$$\text{sd}(K) = \text{colim}(\Delta^n \to K) \in \Delta \times _{\text{sSet}} \text{sSet} / K \text{ sd}(\Delta^n),$$

and that we obtain an induced last vertex map $\text{sd}(K) \to K$. We now extend this even further to any $Y \in \mathcal{S}$ by defining

$$\text{sd}(Y) = \text{colim}(\Delta^n \to Y) \in \Delta \times _{\text{sd}(\mathcal{S})} \text{sd}(\Delta^n);$$

in the same way, this also admits a last vertex map $\text{sd}(Y) \to Y$. Note that this does indeed extend the functor $\text{sd} : \text{sSet} \to \text{sSet}$, as $\text{sSet} \subset \mathcal{S}$ is a full subcategory (so that for any $K \in \text{sSet} \subset \mathcal{S}$, we have an equivalence

$$\Delta \times _{\text{sSet}} \text{sSet} / K \xrightarrow{\sim} \Delta \times _{\text{sSet}} \mathcal{S} / K$$

of $\infty$-categories). This clearly defines a functor $\text{sd} : \mathcal{S} \to \mathcal{S}$.

**Definition 6.12.** We define the extension of $Y \in \mathcal{S}$ to be the object $\text{Ex}(Y) \in \mathcal{S}$ defined by $\text{Ex}(Y)_n = \text{hom}_{\mathcal{S}}(\text{sd}(\Delta^n), Y)$, with simplicial structure maps cocomputed by the cosimplicial structure maps of $\text{sd}(\Delta^n) \in c(\text{sSet})$. This defines a functor $\text{Ex} : \mathcal{S} \to \mathcal{S}$, which extends the usual functor $\text{Ex} : \text{sSet} \to \text{sSet}$ (again since $\text{sSet} \subset \mathcal{S}$ is a full subcategory).

**Notation 6.13.** We write $\text{sd}^i = \text{sd}^{[i]}$ and $\text{Ex}^i = \text{Ex}^{[i]}$ for any $i \geq 0$.

**Lemma 6.14.** The functors $\text{sd}$ and $\text{Ex}$ define an adjunction $\text{sd} : \mathcal{S} \rightleftarrows \mathcal{S} : \text{Ex}$, and hence the functors $\text{sd}^i$ and $\text{Ex}^i$ define an adjunction $\text{sd}^i : \mathcal{S} \rightleftarrows \mathcal{S} : \text{Ex}^i$ for any $i \geq 0$.

**Proof.** The first statement follows directly from the definitions and the fact that, as in any presheaf category, any $Y \in \mathcal{S}$ is recoverable as

$$Y \simeq \text{colim}(\Delta^n \to Y) \in \Delta \times _{\text{sSet}} \mathcal{S} / Y \Delta^n.$$ 

The second statement is obtained by composing the adjunction $i$ times. \qed
Notation 6.15. By Lemma 6.14, the last vertex map \( \text{sd}(Y) \to Y \) is adjoint to a map \( Y \to \text{Ex}(Y) \). We write
\[
\text{Ex}^\infty(Y) = \text{colim}(Y \to \text{Ex}(Y) \to \text{Ex}^2(Y) \to \cdots).
\]

Remark 6.16. Using the classical theory of \( s\text{Set}_{KQ} \), we can see that \( \text{Ex}^\infty \) cannot be a right adjoint. For instance, it does not commute with the countably infinite product of copies of the “simplicial infinite line”, i.e. the nerve of the poset \((\mathbb{Z}, \leq)\). (This product is not acyclic, but the countably infinite product of any acyclic Kan complexes is again acyclic.)

We now give the result which we will need in the proof of Proposition 7.2.

Proposition 6.17. For any \( Y \in s\mathcal{S} \), \( \text{Ex}^\infty(Y) \in s\mathcal{S}^f_{KQ} \).

Proof. The proof of [GJ99, Chapter III, Lemma 4.7], given as it is by a universal computation involving the map \( \Lambda^n_\ast \to \Delta^n_\ast \), works equally well in our setting to show that for any \( Y \in s\mathcal{S} \) and for any map \( \Lambda^n_\ast \to \text{Ex}(Y) \), there exists an extension
\[
\begin{array}{ccc}
\Lambda^n & \longrightarrow & \text{Ex}(Y) \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & \text{Ex}^2(Y).
\end{array}
\]

By Corollary 5.3, it follows that \( \text{Ex}^\infty(Y) \to \text{pt}_{s\mathcal{S}} \) is in \( \text{rlp}(J_{KQ}^{s\mathcal{S}}) \), i.e. that \( \text{Ex}^\infty(Y) \in s\mathcal{S}^f_{KQ} \). \( \square \)

Remark 6.18. Many of the usual results about the functors \( \text{sd} : s\mathcal{S} \to s\mathcal{S} \) and \( \text{Ex} : s\mathcal{S} \to s\mathcal{S} \) extend to our setting.

For instance, the functor \( \text{Ex}^\infty : s\mathcal{S} \to s\mathcal{S} \) (with its canonical map from \( \text{id}_{s\mathcal{S}} \)) is a fibrant replacement functor in \( s\mathcal{S}_{KQ} \). To see this, in light of Proposition 6.17 it suffices to show that the map \( Y \to \text{Ex}^\infty(Y) \) is in \( W_{KQ}^{s\mathcal{S}} \). Since \( W_{KQ}^{s\mathcal{S}} \) is closed under transfinite composition, it suffices to show that \( Y \to \text{Ex}(Y) \) is in \( W_{KQ}^{s\mathcal{S}} \). For this, we first use the small object argument to produce a map \( Y' \to Y \) in \( s\mathcal{S} \) that has \( \text{rlp}(J_{KQ}^{s\mathcal{S}}) \) with \( Y' \in s\mathcal{S} \subset s\mathcal{S} \). Then, we see that the map \( \text{Ex}(Y') \to \text{Ex}(Y) \) also has \( \text{rlp}(J_{KQ}^{s\mathcal{S}}) \) since a commutative square
\[
\begin{array}{ccc}
\partial\Delta^n & \longrightarrow & \text{Ex}(Y') \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & \text{Ex}(Y)
\end{array}
\]
is adjoint to a commutative square
\[
\begin{array}{ccc}
\text{sd}(\partial\Delta^n) & \longrightarrow & Y' \\
\downarrow & & \downarrow \text{rlp}(J_{KQ}^{s\mathcal{S}}) \\
\text{sd}(\Delta^n) & \longrightarrow & Y,
\end{array}
\]
and there is always a lift in the latter square (which is equivalent to a lift in the former square) since \( \text{rlp}(J_{KQ}^{s\mathcal{S}}) = \text{rlp}(\text{disc}(C_{KQ}^{s\mathcal{S}})) \). It follows from Proposition 7.2 below that we have both \( Y' \to \text{Ex}(Y) \) and \( \text{Ex}(Y') \to \text{Ex}(Y) \), and hence from the diagram
\[
\begin{array}{ccc}
Y' & \longrightarrow & \text{Ex}(Y') \\
\downarrow & & \downarrow \\
Y & \longrightarrow & \text{Ex}(Y)
\end{array}
\]
we deduce that also \( Y \to \text{Ex}(Y) \) since \( W_{KQ}^{s\mathcal{S}} \) satisfies the two-out-of-three property. On the other hand, our choice to use an unenriched lifting condition (and hence to be relatively restrictive about which maps are cofibrations) means that the map \( Y \to \text{Ex}(Y) \) is not generally in \( C_{KQ}^{s\mathcal{S}} \).

From here, it is not hard to see that in fact we have a Quillen equivalence \( \text{sd} : s\mathcal{S}_{KQ} \rightleftarrows s\mathcal{S}_{KQ} : \text{Ex} \). Indeed, it is straightforward to check that \( \text{sd} : s\mathcal{S} \to s\mathcal{S} \) preserves both \( I_{KQ}^{s\mathcal{S}} \)-cell and \( J_{KQ}^{s\mathcal{S}} \)-cell, so that this adjunction is a Quillen adjunction. Then, the condition for being a Quillen equivalence follows from the facts that \( \text{sd}(K) \to K \) for any \( K \in s\mathcal{S}_{KQ} = s\mathcal{S}_{KQ} \), that \( Y \to \text{Ex}(Y) \) for any \( Y \in s\mathcal{S} \) (as shown above), and that \( W_{KQ}^{s\mathcal{S}} \) satisfies the two-out-of-three property.
One can similarly verify using standard arguments that $\text{Ex}^\infty$ preserves $F_{KQ}^S$, finite limits, filtered colimits, and $0^\text{th}$ spaces.

7. The Proof of the Kan–Quillen Model Structure

We now turn to the components of the proof of Theorem 4.4. Recall that this appeals to the recognition result for cofibrantly generated model $\infty$-categories given by Theorem 3.12; we verify the various criteria in turn, as itemized in the proof of Theorem 4.4 above.

**Proposition 7.1.** $J^S_{KQ}^{\text{cof}} \subset (I^{\text{cof}} \cap W)^S_{KQ}$.

**Proof.** First, since $J^S_{KQ} \subset I^S_{KQ}$-cell, then $J^S_{KQ} \text{-inj} \supset (I^S_{KQ}^{\text{cell}}) \text{-inj} = I^S_{KQ} \text{-inj}$, so $J^S_{KQ}^{\text{cof}} \subset I^S_{KQ}^{\text{cof}}$. So it remains to show that $J^S_{KQ}^{\text{cof}} \subset W^S_{KQ}$.

To show that $J^S_{KQ}^{\text{cof}} \subset W^S_{KQ}$, we claim that it suffices to show that $J^S_{KQ}^{\text{cell}} \subset W^S_{KQ}$, more precisely, we claim that any map in $J^S_{KQ}^{\text{cof}}$ is a retract of a map in $J^S_{KQ}^{\text{cell}}$, and so the result follows from the fact that $W^S_{KQ}$ is closed under retracts. Indeed, by Corollary 5.3, we can apply the small object argument for $J^S_{KQ}$ to any map in $J^S_{KQ}^{\text{cof}}$ to factor it as a map in $J^S_{KQ}^{\text{cell}}$ followed by a map in $J^S_{KQ}^{\text{inj}}$. Then, by the retract argument (in the form of [Hir03, Proposition 7.2.2(1)], whose proof for 1-categories carries over verbatim to $\infty$-categories), it follows that the original map in $J^S_{KQ}^{\text{cof}}$ is a retract of the map in $J^S_{KQ}^{\text{cell}}$.

Finally, to see that $J^S_{KQ}^{\text{cell}} \subset W^S_{KQ}$, since a sequential colimit of equivalences is an equivalence, by transfinite induction it suffices to show that if we have a pushout square

$$
\begin{array}{ccc}
\Lambda^n & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & Z
\end{array}
$$

in $S$, then $|Y| \xrightarrow{\sim} |Z|$ in $S$. But this follows from the fact that geometric realization (being a colimit) commutes with pushouts, so the induced square

$$
\begin{array}{ccc}
|\Lambda^n| & \longrightarrow & |Y| \\
\downarrow & & \downarrow \\
|\Delta^n| & \longrightarrow & |Z|
\end{array}
$$

is a pushout in $S$. \qed

**Proposition 7.2.** $I^S_{KQ}^{\text{inj}} \subset (J^{\text{inj}} \cap W)^S_{KQ}$.

**Proof.** First, since $J^S_{KQ} \subset I^S_{KQ}$-cell, then $I^S_{KQ}^{\text{inj}} = (I^S_{KQ}^{\text{cell}}) \text{-inj} \subset J^S_{KQ}^{\text{inj}}$. So it remains to show that $I^S_{KQ}^{\text{inj}} \subset W^S_{KQ}$.

So, suppose that the map $Y \to Z$ is in $I^S_{KQ}^{\text{inj}}$, i.e. that it has rlp($I^S_{KQ}$). By Lemma 5.1, it suffices to show that any diagram

$$
\begin{array}{ccc}
S^{n-1} & \longrightarrow & |Y| \\
\downarrow & & \downarrow \\
pt & \longrightarrow & |Z|
\end{array}
$$

in $S$ admits a lift, for any $n \geq 0$. Using the observation in Remark 5.5, by Lemma 5.4 there exists a factorization $\varnothing_{\text{Set}} \to K \xrightarrow{\delta} (\Delta^{n-1}/\partial \Delta^{n-1})$ in $\text{Set}_{KQ}$ and a map $K \to Y$ in $S$ presenting the upper map in this diagram, where $K$ has only finitely many nondegenerate simplices.

Now, the above diagram gives us a nullhomotopy of the composite $S^{n-1} \simeq |K| \to |Y| \to |Z|$ in $S$, and we would like to extend the composite $K \to Y \to Z$ over a cofibration from $K$ into an acyclic object of $\text{Set}_{KQ}$ in a way which presents this nullhomotopy. To do this, write $M = K \times \Delta^{1}/K \times \Delta^{11}$ (with its natural inclusion $K \cong K \times \Delta^{10} \to M$ in $\text{Set}_{KQ}$), and then by Proposition 6.17 and Corollary 6.2 we conclude that there must exist
an extension

\[
\begin{array}{ccc}
K & \longrightarrow & Z \\
\downarrow & & \downarrow \\
M & \longrightarrow & \Ex^\infty(Z)
\end{array}
\]

in \(s\Set_{KQ}\) modeling the above nullhomotopy. However, since \(M\) also has only finitely many nondegenerate simplices, then by Lemma 5.2 there must exist a factorization

\[
\begin{array}{ccc}
K & \longrightarrow & L \\
\downarrow & & \downarrow \\
M & \longrightarrow & \Ex^i(Z) \longrightarrow \Ex^\infty(Z)
\end{array}
\]

for some \(i < \infty\). Via the adjunction \(sd^i : s\Set \rightleftarrows \Ex^i : \sd^i\) of Lemma 6.14, the above extension yields the extension

\[
\begin{array}{ccc}
\sd^i(K) & \overset{\sim}{\longrightarrow} & K \\
\downarrow & & \downarrow \\
\sd^i(M) & \longrightarrow & Z
\end{array}
\]

in \(s\Set_{KQ}\), and plugging this back into the original diagram gives us the diagram

\[
\begin{array}{ccc}
\sd^i(K) & \overset{\sim}{\longrightarrow} & K \\
\downarrow & & \downarrow \\
\sd^i(M) & \longrightarrow & Z
\end{array}
\]

in \(s\Set_{KQ}\). Now, since by assumption \(Y \rightarrow Z\) has \(rlp(I_{Moer}^{s\Set}) = rlp(\disc(C^{s\Set}_{KQ}))\), then there must exist a lift

\[
\begin{array}{ccc}
\sd^i(K) & \overset{\sim}{\longrightarrow} & K \\
\downarrow & & \downarrow \\
\sd^i(M) & \longrightarrow & Z
\end{array}
\]

in \(s\Set_{KQ}\), and upon extracting the outer rectangle and taking geometric realizations, this yields the desired lift

\[
S^{n-1} \longrightarrow |Y| \\
\downarrow & & \downarrow \\
\pt_S & \longrightarrow |Z|
\]

in \(S\).

\[\square\]

**Remark 7.3.** In the same spirit as Remark 6.9, the criterion of Proposition 7.2 is comparable to the lifting criterion coming from the generating cofibrations in the *Moerdijk model structure* on \(ss\Set\) (originally introduced in [Moe89, §1], but see also [GJ99, Chapter IV, §3.3]). However, to actually make a direct comparison requires a bit of care, and so we explain this in some detail.

First of all, the diagonal functor \(\diag : \Delta^{op} \rightarrow \Delta^{op} \times \Delta^{op}\) induces an adjunction \(\diag : s\Set \rightleftarrows ss\Set : \diag^*\), and the Moerdijk model structure on \(ss\Set\) is induced by the standard lifting theorem (on which Theorem 3.13 is based) applied to the model structure \(s\Set_{KQ}\); in fact, this yields a Quillen equivalence \(\diag : s\Set_{KQ} \rightleftarrows ss\Set_{Moer} : \diag^*\). Denoting the external product by \(-\boxtimes- : s\Set \times s\Set \rightarrow ss\Set\), the generating cofibrations for this model structure are thus given by

\[I_{Moer}^{ss\Set} = \{\diag(\partial \Delta^n \rightarrow \Delta^n)\}_{n \geq 0} = \{\partial \Delta^n \boxtimes \partial \Delta^n \rightarrow \Delta^n \boxtimes \Delta^n\}_{n \geq 0},\]

and we have that \(rlp(I_{Moer}^{ss\Set}) \subset W_{Moer}^{ss\Set}\).

Next, recall that we can present the \(\infty\)-category \(s\Set\) using the model category \(s(s\Set_{KQ})_{\text{Reedy}}\). Thus, any map \(Y \rightarrow Z\) in \(s\Set\) can be presented as a map \(Y' \rightarrow Z'\) in \(ss\Set\). If the latter map happens to have \(rlp(I_{ss\Set}^{s\Set})\), then we will have that the induced map \(\diag^*(Y') \rightarrow \diag^*(Z')\) will be in \(W_{KQ}^{s\Set}\); since in \(s(s\Set_{KQ})_{\text{Reedy}}\) the diagonal always computes the homotopy colimit (see for instance [Lur09a, Example A.2.9.31]), then this map in \(s\Set_{KQ}\) presents the map \(|Y| \rightarrow |Z|\) in \(S\), which is therefore an equivalence.
Now, observe that the maps in $I^s_{\text{Moer}}$ are cofibrations (between cofibrant objects) when considered in $s(s\text{Set}_{\text{KQ}})_{\text{Reedy}}$. Moreover, note that if

- the maps $A \rightarrow B$ and $Y \xrightarrow{\varphi} Z$ in $sS$ are respectively presented by the maps $A' \xrightarrow{f'} B'$ and $Y' \xrightarrow{\varphi'} Z'$ in $s(s\text{Set}_{\text{KQ}})_{\text{Reedy}}$,
- $Z' \in s(s\text{Set}_{\text{KQ}})_{\text{Reedy}}$, and
- $\varphi \in \text{rlp}(\{1\})$ in $sS$,

then $\varphi' \in \text{rlp}(\{1'\})$ in $s(s\text{Set})$. (This follows from the fact that under these hypotheses, $(s(s\text{Set}_{\text{KQ}})_{\text{Reedy}})_{A'/\varphi'/Z'}$ presents $sS_{A'/\varphi'/Z'}$.) Together, these imply that if the map $Y \rightarrow Z$ in $sS$ has the right lifting property against the set

$$I^s_{\text{Moer}} = \{S^{n-1} \boxtimes \partial \Delta^n \rightarrow pt_S \boxtimes \Delta^n\}_{n \geq 0} = \{S^{n-1} \boxtimes \partial \Delta^n \rightarrow \Delta^n\}_{n \geq 0},$$

then it can be presented by a map $Y' \rightarrow Z'$ in the model category $s(s\text{Set}_{\text{KQ}})_{\text{Reedy}}$ which has $\text{rlp}(I^s_{\text{Moer}})$, and therefore $[Y] \simto [Z]$ in $S$. Hence, we obtain that $\text{rlp}(I^s_{\text{Moer}}) \subset W^s_{\text{KQ}}$. This, finally, allows us to make a direct comparison.

Of course, what we come to see now is that it appears much easier to have $\text{rlp}(I^s_{\text{KQ}})$ than to have $\text{rlp}(I^s_{\text{Moer}})$. The sets $I^s_{\text{Moer}}$ and $I^s_{\text{KQ}}$ of homotopy classes of maps in $sS$ are illustrated in Figure 4, in which the various shapes and their positions are meant to vaguely indicate the different simplicial levels at which these spaces live as well as the simplicial structure maps between them. The maps in $I^s_{\text{Moer}}$ are obtained by simultaneously coning off a $\partial \Delta^n$ worth of $(n-1)$-spheres and adding a new nondegenerate point in the $n$th simplicial level, while the maps in $I^s_{\text{KQ}}$ are simply maps of levelwise-discrete simplicial spaces. In particular, maps in $I^s_{\text{Moer}}$ have real homotopical content, and thus checking that a map has $\text{rlp}(I^s_{\text{Moer}})$ is indeed much more difficult than checking that it has $\text{rlp}(I^s_{\text{KQ}})$. From here, essentially identical remarks to those made in Remark 6.9 apply; in particular, the result that $\text{rlp}(I^s_{\text{Moer}}) \subset W^s_{\text{KQ}}$ is far too weak to be useful in the proof of the fundamental theorem of model $\infty$-categories given in [MGd].

**Remark 7.4.** There is also the “$\mathcal{W}$ model structure” on $ss\text{Set}$ of [CR07], which admits a left Quillen equivalence to $ss\text{Set}_{\text{Moer}}$ (see [CR07, Theorem 9]). However, this is of course also inherently 1-categorical, and hence any $\infty$-categorical lifting criteria that come of it will likewise necessarily contain far more geometric content than the maps in $I^s_{\text{KQ}}$ (compare with Remarks 7.3 and 7.5).

**Remark 7.5.** If one thinks of the 1-category of topological spaces or of simplicial sets in place of the $\infty$-category $S$, then Proposition 7.2 may seem somewhat implausible. For instance, the functor $\text{disc} : s\text{Set} \rightarrow sS$ of $\infty$-categories is modeled by an evident functor $\text{const} : s\text{Set}_{\text{triv}} \rightarrow s(s\text{Set}_{\text{KQ}})_{\text{Reedy}}$. On underlying 1-categories, this functor participates in an adjunction $\text{const} : s\text{Set} \rightleftarrows ss\text{Set} : ((-)_0)^{\text{lw}}$ with the “levelwise 0-simplices” functor, which takes each constituent simplicial set to its set of 0-simplices. This right adjoint clearly does not know anything about the higher homotopical information in the bisimplicial set, and in particular cannot recover its geometric realization. Hence, one might deduce that asking for the right lifting property in $sS$ against maps in the image of $\text{disc} : s\text{Set} \rightarrow sS$ could not possibly tell us about the functor $[-] : s\text{Set} \rightarrow S$. However, asking for the 0-simplices of a simplicial set isn’t a homotopical operation in $s\text{Set}_{\text{KQ}}$, and so we cannot expect this maneuver to tell us anything $\infty$-categorical. Indeed, the above right adjoint is certainly not a relative functor, nor is it even a right Quillen functor (with respect to the indicated model structures), corresponding to the fact that the functor $\text{disc} : s\text{Set} \rightarrow sS$ isn’t a left adjoint.

**Example 7.6.** For a concrete nonexample of Proposition 7.2, we show that the map $\text{const}(S^1) \rightarrow \text{const}(pt_S)$ in $sS$ (which is not in $W^s_{\text{KQ}}$) does not have $\text{rlp}(\{\partial \Delta^2 \rightarrow \Delta^2\})$. This will illustrate the capability of “simplicial” spheres to detect “geometric” spheres in $sS_{\text{KQ}}$. 

![Figure 4](image-url)
The quickest way to proceed is to use the adjunction $|\cdot| = \colim : sS \rightleftarrows S : \const$. This gives a canonical commutative square

$$
\begin{array}{ccc}
\partial \Delta^2 & \longrightarrow & \const(S^1) \\
\downarrow & & \downarrow \\
\Delta^2 & \longrightarrow & \const(\pt_S)
\end{array}
$$

in $sS$ which corresponds to the evident commutative square

$$
\begin{array}{ccc}
|\partial \Delta^2| & \sim & S^1 \\
\downarrow & & \downarrow \\
|\Delta^2| & \sim & \pt_S
\end{array}
$$

in $S$. Moreover, a lift in either square yields a lift in the other, but a lift in the latter diagram would imply that its vertical maps are also equivalences, which is clearly false.

But we can also describe the above commutative square in $sS$ more explicitly. Namely, we can define the upper map $\partial \Delta^2 \to \const(S^1)$ in $sS$ by giving a weak natural transformation of simplicial topological spaces, as illustrated in Figure 5 (using the same schematics as were employed in Figure 4). Let us parametrize the circle as the group-theoretic quotient $\mathbb{R}/\mathbb{Z}$. Then, we begin at level 0 by sending $\Delta^{(0)}$ to 0, $\Delta^{(1)}$ to $\frac{1}{3}$, and $\Delta^{(2)}$ to $\frac{2}{3}$. Since $\partial \Delta^2$ is 1-skeletal, it remains to fill in the commutative diagram

$$
\begin{array}{ccc}
L_1(\partial \Delta^2) & \longrightarrow & (\partial \Delta^2)_1 & \longrightarrow & M_1(\partial \Delta^2) \\
\downarrow & & \downarrow & & \downarrow \\
L_1(\const(S^1)) & \longrightarrow & \const(S^1)_1 & \longrightarrow & M_1(\const(S^1))
\end{array}
$$

in $S$. To do this, we map the degenerate elements of $(\partial \Delta^2)_1$ so that the left square commutes on the nose. Then, we map the nondegenerate elements of $(\partial \Delta^2)_1$ to $\const(S^1)_1 = S^1$ by sending $\Delta^{(01)}$ to $\frac{1}{6}$, sending $\Delta^{(12)}$ to $\frac{1}{2}$, and sending $\Delta^{(02)}$ to $\frac{5}{6}$. To select a homotopy witnessing the homotopy commutativity of the right square, for $i \neq j$ we choose the evident paths of length $\frac{1}{6}$ from the image of each $\Delta^{(ij)}$ to the images of $\Delta^{(i)}$ and $\Delta^{(j)}$ (as indicated by the squiggly arrows in Figure 5). Again, it is clear that this map cannot be extended over $\Delta^2$. (This can also be realized as an actual natural transformation of simplicial topological spaces if we’re willing to use a fatter model for the object $\partial \Delta^2 \in sS$, despite the fact that the simplicial topological space which is constant at the circle isn’t actually fibrant in the corresponding Reedy model structure.)

Remark 7.7. In the proof of Proposition 7.2 above, one might be tempted to apply the small object argument for $I_{KQ}^S$ to the map $K \to Z$ to obtain a factorization $K \to L \to Z$ with the map $K \to L$ in $I_{KQ}^S$-cell (so that $L \in sS \subset sS$) and with the map $L \to Z$ in $I_{KQ}^S$-inj; then, we could proceed with the proof using standard techniques in $sS \inj_{KQ}$, using $L \in sS \inj_{KQ}$ as a replacement for $Z \in sS_{KQ}$. If this worked, it would allow us to sidestep the extension of the functor $\Ex^\infty$ from $sS \setto set$ to $sS$. However, such an argument would be circular: we can certainly obtain such a factorization $K \to L \to Z$, but to conclude that the map $L \to Z$ is in $W_{KQ}^S$ because it is in $I_{KQ}^S$-inj uses precisely the result that we are trying to prove.
In contrast with Remark 7.7, now that we have Proposition 7.2 in hand, we can use this technique of reducing to $s\text{Set}_{KQ}$. We employ it in proving the following result, the last of this section.

**Proposition 7.8.** $(J \text{-} inj \cap W)_{KQ}^{sS} \subset J_{KQ}^{sS} \text{-} inj$.

**Proof.** Suppose that the map $Y \to Z$ in $sS$ has rlp($J_{KQ}^{sS}$) and geometrically realizes to an equivalence in $S$. We must show that $Y \to Z$ also has rlp($I_{KQ}^{sS}$), i.e. that any commutative square

$$
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\Delta^n & \longrightarrow & Z
\end{array}
$$

in $sS$ admits a lift, for any $n \geq 0$. We argue by constructing the diagram (and in particular the dotted arrow, which solves the above lifting problem) in $sS_{KQ}$ given in Figure 6, beginning with only the outermost square. For clarity,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram}
\caption{The diagram in $sS_{KQ}$ used in the proof of Proposition 7.8.}
\end{figure}

we proceed in steps.

1. We use the small object argument for $I_{KQ}^{sS}$ to obtain the the factorization $\partial \Delta^n \to Y' \to Y$ in $sS_{KQ}$, where $Y' \in s\text{Set} \subset sS$ and the latter map has rlp($I_{KQ}^{sS}$); by Proposition 7.2, this latter map is also in $W_{KQ}^{sS}$.

2. We use the small object argument for $J_{KQ}^{sS}$ to obtain the factorization $Y' \to Z' \to Z$ in $sS_{KQ}$ of the composite map $Y' \to Y \to Z$, where $Z' \in s\text{Set} \subset sS$ and the latter map has rlp($I_{KQ}^{sS}$); again by Proposition 7.2, this latter map is also in $W_{KQ}^{sS}$.

3. Since the map $Z' \to Z$ has rlp($I_{KQ}^{sS}$), we are guaranteed a lift in the square

$$
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & Y' \\
\downarrow & & \downarrow \text{rlp($I_{KQ}^{sS}$)} \\
\Delta^n & \longrightarrow & Z
\end{array}
$$

4. We use the small object argument for $J_{KQ}^{sS}$ to obtain the factorization $Y' \cong Y'' \to Z'$ in $s\text{Set}_{KQ} \subset sS_{KQ}$ of the map $Y' \to Z'$. 
(5) Since the map $Y \to Z$ has $\text{rlp}(J_{KQ}^s)$, we are guaranteed a lift in the square

$$
\begin{array}{ccc}
Y' & \cong & Y \\
\downarrow & & \downarrow \\
Y'' & \to & Z \\
\end{array}
$$

Since $W_{KQ}^s$ has the two-out-of-three property, then this lift is in $W_{KQ}^s$.

(6) Again since $W_{KQ}^s$ has the two-out-of-three property, the map $Y'' \to Z'$ must be in $W_{KQ}^s$. But its objects are both in $s\text{Set}$, so in fact it is in $W_{KQ}^{s\text{Set}}$.

(7) In $s\text{Set}_{KQ}$ we have that $(W \cap F)^{s\text{Set}} = \text{rlp}(I_{KQ}^s)$, so we must have a lift in the square

$$
\begin{array}{ccc}
\partial \Delta^n & \to & Y' \\
\downarrow & & \downarrow \\
\Delta^n & \to & Z' \\
\end{array}
$$

which gives us the dotted arrow in the diagram in Figure 6.

\[\square\]

8. The proof of Lemma 5.4

We now give the proof of Lemma 5.4, completing the proof of Theorem 4.4.

**Proof of Lemma 5.4.** We begin by observing that $L' \cong L$ in $s\text{Set}_{KQ}$ (for any choice of $i \geq 0$) because this model structure is left proper, so these are both homotopy pushouts.

To prove the rest of the statement, we proceed in steps for clarity. To fix notation, suppose that the chosen point of the pullback selects a pair of maps $(\varphi, \psi) \in \text{hom}_s(K, W) \times \text{hom}_s(|L|, |W|)$.

We present the $\infty$-category $s\mathcal{S}$ via the model category $s(s\text{Set}_{KQ})\text{Reedy}$, and we denote by $\text{const}: s\text{Set}_{\text{triv}} \to s(s\text{Set}_{KQ})\text{Reedy}$ the evident right Quillen functor modeling the right adjoint disc $s\text{Set} \to s\mathcal{S}$ (though we will continue to suppress the latter).

(1) We choose an arbitrary fibrant representative $W' \in s(s\text{Set}_{KQ})^{\text{f}}_{\text{Reedy}}$ for the object $W \in s\mathcal{S}$, and then we choose an arbitrary map $\text{const}(K) \xrightarrow{\varphi'} W'$ in $s(s\text{Set}_{KQ})_{\text{Reedy}}$ which presents the map $K \xrightarrow{\varphi} W$ in $s\mathcal{S}$.

(2) Recall that the diagonal functor $\text{diag}: \Delta^{op} \to \Delta^{op} \times \Delta^{op}$ induces an adjunction

$$
\text{diag}_{!}: s\text{Set} \rightleftarrows s(s\mathcal{S}): \text{diag}^*.
$$

Applying its right adjoint to $\varphi'$ yields a map

$$
K \equiv \text{diag}^*(\text{const}(K)) \xrightarrow{\text{diag}^*(\varphi')} \text{diag}^*(W')
$$

in $s\mathcal{S}$. Since this right adjoint $\text{diag}^*: s(s\text{Set}_{KQ})_{\text{Reedy}} \to s\text{Set}_{KQ}$ is a relative functor (see [GJ99, Chapter IV, Proposition 1.9]) and considered in $\text{RelCat}_{BK}$ models the functor $|\cdot|: s\mathcal{S} \to \mathcal{S}$ (see [GJ99, Chapter IV, Exercise 1.6]), then the map $\text{diag}^*(\varphi')$ in $s\text{Set}_{KQ}$ models the map $|K| \xrightarrow{\varphi} |W|$ in $\mathcal{S}$.

(3) By assumption, we have an extension

$$
\begin{array}{ccc}
|K| & \xrightarrow{|\varphi|} & |W| \\
\downarrow & & \downarrow \\
|L| & \xrightarrow{\psi} & |W|
\end{array}
$$
in $S$. Since the $\infty$-category $S_{|K|/}$ is presented by the model category $(sSet_{K/})_{KQ}$, then this can be modeled as an extension

$$
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & K \\
\downarrow & & \downarrow g \\
\Delta^n & \longrightarrow & L \\
\end{array}
\xrightarrow{\psi} \Ex^\infty(\diag^*(W'))
$$

in $sSet_{KQ}$. To simplify our diagrams we will henceforth omit $L$, since it's defined as a pushout anyways.

(4) Recall that the map $\diag^*(W') \xrightarrow{\simeq} \Ex^\infty(\diag^*(W'))$ is defined as a transfinite composition

$$
\diag^*(W') \xrightarrow{\simeq} \Ex(\diag^*(W')) \xrightarrow{\simeq} \Ex^2(\diag^*(W')) \xrightarrow{\simeq} \cdots
$$

in $sSet_{KQ}$. Since $\Delta^n$ is small as an object of $sSet_{\partial \Delta^n/}$, there must be a factorization

$$
\begin{array}{ccc}
\partial \Delta^n & \longrightarrow & K \\
\downarrow & & \downarrow g \\
\Delta^n & \longrightarrow & \Ex^i(\diag^*(W')) \\
\end{array}
\xrightarrow{\psi} \Ex^\infty(\diag^*(W'))
$$

in $sSet_{KQ}$ for some $i < \infty$.

(5) Via the adjunction $sd^i : sSet \rightleftarrows sSet : \Ex^i$, the extension in step (4) is equivalent to the extension

$$
\begin{array}{ccc}
\sd^i(\partial \Delta^n) & \longrightarrow & \partial \Delta^n \\
\downarrow & & \downarrow \\
\sd^i(\Delta^n) & \longrightarrow & \Delta^n \\
\end{array}
\xrightarrow{\psi} \K \xrightarrow{\diag^*(\varphi')} \diag^*(W').
$$

i.e. an extension

$$
\begin{array}{ccc}
K & \xrightarrow{\diag^*(\varphi')} & \diag^*(W') \\
\downarrow & & \downarrow \\
L' & & \\
\end{array}
$$

(6) Via the adjunction $\diag_! : sSet \rightleftarrows ssSet : \diag^*$, the extension in step (5) is equivalent to the extension

$$
\begin{array}{ccc}
\diag_!(\sd^i(\partial \Delta^n)) & \longrightarrow & \diag_!(\partial \Delta^n) \\
\downarrow & & \downarrow \\
\diag_!(\sd^i(\Delta^n)) & \longrightarrow & \diag_!(\Delta^n) \\
\end{array}
\xrightarrow{\psi^f} W'.
$$
(7) The diagram in step (6) extends to a diagram

\[ \text{diag}(\text{sd}^i(\partial \Delta^n)) \rightarrow \text{diag}(\partial \Delta^n) \rightarrow \text{diag}(K) \rightarrow W' \]

\[ \text{const}(\text{sd}^i(\partial \Delta^n)) \rightarrow \text{const}(\partial \Delta^n) \rightarrow \text{const}(K) \rightarrow \varphi' \]

\[ \text{diag}(\text{sd}^i(\Delta^n)) \rightarrow \text{diag}(\Delta^n) \rightarrow \text{const}(\text{sd}^i(\Delta^n)) \rightarrow \text{const}(\Delta^n) \rightarrow \phi' \]

in \text{s(sSet}_{\text{KQ}})_{\text{Reedy}}, in which all unlabeled oblique arrows are components of the natural transformation \text{diag} \cong \text{diag}_i \text{diag}^* \text{const} \rightarrow \text{const}

in \text{Fun}(\text{sSet}, \text{ssSet}) induced by the counit of the adjunction \text{diag}_i \dashv \text{diag}^*; indeed, the counit is initial among maps of the form \text{diag}^*(\_)^\sharp.

(8) The map \text{diag}(\text{sd}^i(\Delta^n)) \rightarrow \text{const}(\text{sd}^i(\Delta^n)) in the diagram of step (7) is a weak equivalence in \text{s(sSet}_{\text{KQ}})_{\text{Reedy}} by Lemma 8.1 below. Hence, upon applying the localization \text{s(sSet}_{\text{KQ}})_{\text{Reedy}} \rightarrow \text{sS} to that diagram, we obtain the desired extension

\[ \text{sd}^i(\partial \Delta^n) \rightarrow \partial \Delta^n \rightarrow K \rightarrow W \]

in \text{sS}. (Working back through the proof, it is clear that this does indeed model the extension

\[ |K| \rightarrow |W| \]

\[ |L'| \rightarrow |L| \rightarrow \psi \]

in \text{sS}.)

\[ \Box \]

**Lemma 8.1.** For any acyclic \( M \in \text{sSet}_{\text{KQ}}, \) the component

\[ \text{diag}_i(M) \cong \text{diag}_i(\text{diag}^*(\text{const}(M))) \rightarrow \text{const}(M) \]

of the counit of the adjunction \( \text{diag}_i : \text{sSet} \rightleftarrows \text{ssSet} : \text{diag}^* \) is a weak equivalence in \text{s(sSet}_{\text{KQ}})_{\text{Reedy}}.

**Proof.** We begin by choosing a presentation of \( M \) as a transfinite composition of pushouts of the generating acyclic cofibrations \( J_{\text{KQ}}^{\text{Set}} = \{ \Lambda^n_i \rightarrow \Delta^n \}_{0 \leq i \leq n \geq 1}. \) Note that \( \text{diag}_i \) is a left adjoint, so it commutes with pushouts; thus, this also gives us a presentation of \( \text{diag}_i(M) \) as a transfinite composition of pushouts of maps in \( \text{diag}_i(J_{\text{KQ}}^{\text{Set}}) = \{ \text{diag}_i(\Lambda^n_i) \rightarrow \text{diag}_i(\Delta^n) \}_{0 \leq i \leq n \geq 1}. \)

We now argue by transfinite induction. Clearly the result holds for \( M = \Delta^0. \) To obtain the inductive step at any successor ordinal, we will show below that we have a commutative square

\[ \text{diag}_i(\Lambda^n_i) \xrightarrow{\cong} \text{const}(\Lambda^n_i) \]

\[ \text{diag}_i(\Delta^n) \xrightarrow{\cong} \text{const}(\Delta^n) \]
in \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\). Then, since \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\) is left proper (for instance because all its objects are cofibrant), the induced map between the pushouts of the front and back faces in the diagram

\[
\begin{array}{ccc}
\text{const}(\Lambda^n_0) & \xrightarrow{\approx} & \text{const}(M) \\
\downarrow & & \downarrow \\
\text{diag}(\Lambda^n_0) & \xrightarrow{\approx} & \text{diag}(M) \\
\text{const}(\Delta^n) & \xrightarrow{\approx} & \text{diag}(\Delta^n) \\
\end{array}
\]

in \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\) will again be a weak equivalence. To obtain the inductive step at any limit ordinal, we observe that both colimits and weak equivalences in \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\) are defined levelwise, and that weak equivalences in \(s\mathcal{SSet}_{KQ}\) are closed under transfinite composition (for instance by arguments in the style of steps (3)-(5) in the proof of Lemma 5.4 above).

So, it only remains to show that we have a commutative square in \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\) as claimed above. We verify the illustrated assertions in turn.

- Both vertical maps are monomorphisms and hence are cofibrations in \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\).
- We have an isomorphism \(\text{diag}(\Delta^n) \cong \Delta^n \times \Delta^n\), under which identification the lower map is given by \(\Delta^n \times \Delta^n \to \Delta^n \times \Delta^0\). At level \(j\) this is just the map \(\coprod (\Delta^n), \Delta^n \to \coprod (\Delta^n), \Delta^0\), which is a weak equivalence in \(s\mathcal{SSet}_{KQ}\). So the lower map is indeed a weak equivalence in \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\).
- To see that the upper map is also a weak equivalence, we recall the explicit description of \(\text{diag}(\Lambda^n_0)\) (given both in the proof of [Moe89, Lemma 1.3] and in the text leading up to [GJ99, Chapter IV, Lemma 3.10]), that

\[
\text{diag}(\Lambda^n_0)_j \cong \left\{ (\alpha, \beta) \in \text{hom}_\Delta([j],[n]) \times \text{hom}_\Delta([k],[n]) : \right. \left. \text{there exists some } l \in [n] \text{ with } l \neq i \right\}.
\]

For any fixed \(j \geq 0\), we must show that the map \(\text{diag}(\Lambda^n_0)_j \to \text{const}(\Lambda^n_0)_j\) is a weak equivalence in \(s\mathcal{SSet}_{KQ}\). The latter object is discrete (i.e. its only nondegenerate simplices are 0-simplices), and so this is equivalent to showing that the preimage of each such 0-simplex is acyclic in \(s\mathcal{SSet}_{KQ}\). Such a 0-simplex is precisely the datum of a map \(\alpha \in \text{hom}_\Delta([j],[n]) \cong \text{hom}_{\mathcal{SSet}}(\Delta^j, \Delta^n)\) whose image on 0-simplices does not cover \((\Delta^n)_0 \setminus \{\Delta^i\}\). Define the subset \(T^c \subset (\Delta^n)_0\) to be the union of \(\{\Delta^i\}\) and the image of \(\alpha\), and let \(T^c \subset (\Delta^n)_0\) be its complement. Then, the preimage of the 0-simplex of \(\text{const}(\Lambda^n_0)_j\), corresponding to \(\alpha\) is the subobject of \(\Delta^n \in s\mathcal{SSet}\) consisting of those simplices whose 0-simplices do not contain all of \(T^c \subset (\Delta^n)_0\), which is indeed acyclic in \(s\mathcal{SSet}_{KQ}\) since \(T^c\) is nonempty.

\[\square\]

**Remark 8.2.** Lemma 8.1 fails drastically if we do not assume that \(M \in s\mathcal{SSet}_{KQ}\) is acyclic. In fact, in Remark 7.3, the stark difference between the sets of homotopy classes of maps \(j_{\text{Moer}}^n\) and \(j_{\text{KQ}}^n\) in \(s\mathcal{S}\) is precisely due to the difference between (the weak equivalences classes of) the objects \(\text{diag}(\partial \Delta^n) \cong \partial \Delta^n \times \partial \Delta^n\) and \(\text{const}(\partial \Delta^n)\) in \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\).

**Remark 8.3.** Using the arguments of Remark 7.3, one can use Lemma 8.1 to give an alternative proof of Corollary 6.6. The key point is that we have a diagram

\[
\begin{array}{ccc}
\text{diag}(\Lambda^n_0) & \xrightarrow{\approx} & \text{const}(\Lambda^n_0) \\
\downarrow & & \downarrow \\
\text{diag}(\Delta^n) & \xrightarrow{\approx} & \text{const}(\Delta^n) \\
\end{array}
\]

in \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\). Hence, if the map \(Y \to Z\) in \(s\mathcal{S}\) has \(\text{rlp}(j_{\text{KQ}}^n)\), then it can be presented by a fibration \(Y' \to Z'\) in \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\) that additionally has \(\text{rlp}(\text{diag}(j_{\text{KQ}}^n))\), i.e. it is also in \(\mathcal{F}_{\text{Reedy}}^{s\mathcal{SSet}_{\text{Moer}}}\). Since both \(s(\mathcal{SSet}_{KQ})_{\text{Reedy}}\) and \(s\mathcal{S}_{\text{Moer}}\) are right proper, it follows that all pullbacks of the map \(Y' \to Z'\) in \(s\mathcal{S}_{\text{Moer}}\) simultaneously compute homotopy pullbacks in both model structures.
APPENDIX A. NOTATION, TERMINOLOGY, AND CONVENTIONS

As this is the first paper in its series, we take the opportunity to spell out once and for all the precise foundations on which the project is built.

A.1. We begin with our philosophy surrounding the semantics of the signifier “∞-category”.

(1) For definiteness, we ground ourselves in the theory of quasicategories: an ∞-category is a quasicategory. We will refer to these as “quasicategories” only when we mean to make specific reference to their properties or manipulation as such, which we will avoid doing to the largest extent possible. We use [Lur09a] as our primary reference, but we note that many of the ideas given there have their origins in [Joyc, Joyb, Joya].

In order to proceed with the enumeration of our foundations, we must immediately lay out the following basic conventions.

- As we have already indicated, we will be ignoring all set-theoretic issues. They are irrelevant to our aims, and in any case can be dispensed with by appealing to the usual device of Grothendieck universes (see [Lur09a, §1.2.15]).

- If an ∞-category C has an initial (resp. terminal) object, we will write ΩC (resp. ptC) for any such object, or we will simply write Ω (resp. pt) if the ambient ∞-category C is clear from the context. For ∞-categories of co/pointed objects, we will make the abbreviations CΩ = C/Ω and Cpt = Cpt/Ω.

- We write S for the ∞-category of spaces. By definition, up to equivalence we can take this to either be Top[[W]] or sSet[[W]], where in either case the symbol W denotes the weak homotopy equivalences. In particular, by “space” we will mean an object of S; when we mean to refer to an object of Top, we will instead use the term “topological space”. The ∞-category S of spaces plays the same fundamental role in the theory of ∞-categories that the category Set of sets plays in the theory of categories: whereas categories are naturally enriched in sets, ∞-categories are naturally enriched in spaces (see item (5) below).

We adopt the following conventions regarding S.

- A map in S is called
  - a monomorphism if it induces a π₀-monomorphism and a π₁-surjection of the source;
  - étale if it induces a π₁-surjection of every basepoint of the source;
  - an effective epimorphism if it induces a π₀-surjection.

- There is an evident adjunction π₀ : S ⇄ Set : disc, and we call a space essentially discrete (or sometimes simply discrete) if it is in the essential image of the right adjoint. We will only include this right adjoint in the notation if we mean to emphasize it.

- More generally, for any n ≥ 0 we have a truncation adjunction τ≤n : S ⇄ S≤n : i and a cotruncation adjunction i : S≥n ⇄ S≥n−1 : τ≥n. (In the special case that n = 0, the truncation adjunction reduces to the adjunction π₀ : S ⇄ Set : disc given above.)

- We will refer to spaces as “∞-groupoids” when we mean to emphasize the fact that they are just particular examples of ∞-categories.

- We write Cat∞ for the ∞-category of ∞-categories. We adopt the following conventions regarding Cat∞.

  - To say that an ∞-category C is a subcategory of some other ∞-category C′ means, in the most invariant possible language, that we have a chosen functor C → C′ which is (homotopically) faithful: that is, for all x, y ∈ C, the induced map home(x, y) → home(F(x), F(y)) is a monomorphism of spaces. We will call the functor F the inclusion of a subcategory, but we will suppress it from the notation and simply write C ⊂ C′ as shorthand. A subcategory C ⊂ C′ is uniquely specified by the resulting subcategory ho(C) ⊆ ho(C′) of its homotopy category.

---

6Of course, by Top we mean to denote any “convenient” category of topological spaces.

7Note that these are not quite the monomorphisms in Cat∞. Rather, the monomorphisms are precisely the pseudomononic functors, i.e. the inclusions of subcategories which are full on equivalences. This is perhaps most easily seen by appealing to the equivalence
More generally, if $I$ is a class of maps in $S$, then a functor in $\mathsf{Cat}_\infty$ is called a local $I$ if all the induced maps on hom-spaces are in $I$.

For an $\infty$-category $\mathcal{C} \in \mathsf{Cat}_\infty$, we denote by $\mathcal{C}^\approx \in S$ its maximal subgroupoid. This construction is right adjoint to the inclusion $S \subset \mathsf{Cat}_\infty$ of the subcategory of $\infty$-groupoids into the $\infty$-category of $\infty$-categories.

For an $\infty$-category $\mathcal{C} \in \mathsf{Cat}_\infty$, we denote by $\mathcal{C}^{\text{end}} \in S$ its $(\infty,1)$-groupoid completion. This construction is left adjoint to the inclusion $S \subset \mathsf{Cat}_\infty$.

(2) Despite our grounding declared in item (1), our notion of “$\infty$-category” is nevertheless a rather flexible one: over the course of these papers, we will interchange fluidly between a number of distinct but essentially equivalent notions thereof. In accordance with current best practices, the ones that we will employ all appear naturally as objects in various model categories. For the reader’s convenience, we itemize these notions and their ambient model categories here, and we give some indication of the roles that they will play in this series of papers.

(a) The notion of a quasicategory plays a distinguished role in these papers, as indicated in item (1). These are precisely the bifibrant objects in $\mathsf{sSet}_{\text{Joyal}}$, the category of simplicial sets equipped with the Joyal model structure of [Lur09a, Theorem 2.2.5.1]. We view these as the most convenient of the notions to employ as an ambient framework, which advantage is surely in large part due to the abundance of theory that has been built up around them.

(b) The notion which most closely adheres to the intuition of a “category enriched in spaces” is that of a category enriched in simplicial sets, simply called a $\mathsf{sSet}$-enriched category for short. These organize into the model category $(\mathsf{Cat}_{\mathsf{sSet}})_{\text{Bergner}}$ under the Bergner model structure of [Ber07, Theorem 1.1] (or see [Lur09a, Proposition A.3.2.4] for a generalization). Just as when one uses simplicial sets to present spaces one should generally be working with Kan complexes, when considering a $\mathsf{sSet}$-enriched category as an $\infty$-category one should generally be working with a category which is in fact enriched in Kan complexes: indeed, these are precisely the fibrant objects of $(\mathsf{Cat}_{\mathsf{sSet}})_{\text{Bergner}}$. We write $\mathcal{C} : \mathsf{sSet}_{\text{Joyal}} \rightleftarrows (\mathsf{Cat}_{\mathsf{sSet}})_{\text{Bergner}} : \mathsf{N}^{\text{hc}}$ for the Quillen equivalence of [Lur09a, Theorem 2.2.5.1].

Categories enriched in simplicial sets will make an appearance in [MGe], in which we will want to easily associate to an $\infty$-category a functor to it from a 1-category which is both essentially surjective and a local effective epimorphism. This is achieved by the canonical map to any $\mathsf{sSet}$-enriched category from its $0$th level (i.e. from its underlying category). They also provide an explicit bridge from $\mathsf{RelCat}_{\mathsf{BK}}$ to $\mathsf{sSet}_{\text{Joyal}}$ (see item (d) below).

(c) The notion which is most “homotopy invariant” is that of a complete Segal space. These are actually bisimplicial sets, thought of as simplicial spaces via choices of distinguished “simplicial” and “geometric” directions. They are precisely the bifibrant objects in $\mathsf{ssSet}_{\text{Rezk}}$, the category of bisimplicial sets equipped with the Rezk model structure of [Rez01, Theorem 7.2] (there called the “complete Segal space” model structure).

However, it is fruitful to consider a theory of complete Segal spaces internally to the world of $\infty$-categories, i.e. to define them as a subcategory $\mathcal{CSS} \subset \mathsf{S}$ of the $\infty$-category of simplicial spaces.\(^8\)\(^9\)

From this viewpoint, a complete Segal space can be thought of as a “homotopically correct” version of the nerve of a category: the equivalence $CSS(-) : \mathsf{Cat}_\infty \to \mathcal{CSS}$ takes an $\infty$-category $\mathcal{C}$ to the simplicial space defined by

$$CSS(\mathcal{C}) \bullet = \hom_{\mathsf{Cat}_\infty}^\text{lw}(\ast, \mathcal{C}),$$

i.e. the levelwise hom-space from the standard cosimplicial category $\Delta \to \mathsf{Cat}$ (so that for any $n \geq 0$, $CSS(\mathcal{C})_n = \hom_{\mathsf{Cat}_\infty}([n], \mathcal{C})$).\(^10\)\(^11\) The inverse equivalence takes a complete Segal space $\mathcal{C} \bullet \in \mathcal{CSS}$ to $\mathcal{C} \in \mathsf{CSS}$ of item (2)(c) below: as the inclusion $\mathcal{CSS} \subset \mathsf{S}$ preserves limits (being a right adjoint), a map in $\mathcal{CSS}$ is a monomorphism precisely if it is a monomorphism when considered in $\mathsf{S}$.

\(^8\)This perspective is explored in detail (and in greater generality) in [Lur09b, §1].

\(^9\)In a sense to be made precise in item (3), this $\infty$-category coincides with the “underlying $\infty$-category” of the model category $\mathsf{ssSet}_{\text{Rezk}}$ of complete Segal spaces.

\(^10\)Indeed, a simplicial set is the nerve of a category precisely if it satisfies the Segal condition.

\(^11\)Note that the complete Segal space of a category does not generally coincide with its nerve.
the $\infty$-category
\[
\int_{[n] \in \Delta} C_n \times [n],
\]
where we consider $C_n \in \mathcal{S} \subset \mathcal{Cat}_\infty$ by considering spaces as $\infty$-groupoids. In fact, there is a left adjoint to the inclusion $\mathcal{CSS} \subset \mathcal{S}$, and we should think of the adjunction $\mathcal{S} \leftarrow \mathcal{CSS} \simeq \mathcal{Cat}_\infty$ as being a homotopical analog of the usual “nerve/homotopy category” adjunction $\mathcal{S} \leftarrow \mathcal{Cat}$.

Given a relative category, one can define its resk nerve (see [Rez01, 3.3], where it is called the “classification diagram”), and this often yields its corresponding complete Segal space (see [LMG]). One can perform an analogous construction for a relative $\infty$-category, and in [MGc] we will prove that for the underlying relative $\infty$-category $(M, W)$ of a model $\infty$-category,

- the resulting simplicial space is in fact a complete Segal space (i.e. an object of the subcategory $\mathcal{CSS} \subset \mathcal{S}$), and moreover
- via the equivalence $\mathcal{CSS} \simeq \mathcal{Cat}_\infty$, it corresponds to the localization $M[\mathbb{W}^{-1}]$.

Finally, the simplest notion is that of a relative category, considered as an $\infty$-category via its hammock localization (see [DK80a, 2.1]), which we can think of as its “$\infty$-categorical localization”. These organize into the model category $\RelCat_{BK}$ of [BK12b, Theorem 6.1]. We write $\mathcal{Z}^H : \RelCat_{BK} \rightarrow (\Cat sSet)_{Bergner}$ for the hammock localization functor, which is a relative functor (see [BK12a, Theorem 1.8]); in fact, it is even a weak equivalence in $\RelCat_{BK}$ (see [BK12a, Theorem 1.7] and item (3) below).

We mainly use relative categories as a technical device that allows us to make rigorous sense of the underlying $\infty$-category of a relative category, and hence in particular of a model category. In this situation, we will say that the model category gives a presentation of its underlying $\infty$-category. It follows from [DK80b, Proposition 4.4] that we can extract hom-spaces (presented as objects of $sSet_{KQ}$) in the underlying $\infty$-category of a model category using co/simplicial resolutions (see [DK80b, 4.3]). See §A.3 below for details of our usage of model categories as presentations of $\infty$-categories.

The assertion made in item (2) that these various notions of $\infty$-categories are all “essentially equivalent” is rather multifaceted. We therefore give a careful account of this assertion. Our perspective is espoused in a number of relatively recent papers, notably [BSP] (from the introduction of which this item is more or less directly lifted), and seems to represent the emerging consensus among practitioners of higher category theory.

First of all, these four model categories are all connected by a diagram of Quillen equivalences (see [BSP, Figure 1] and the references cited therein). Thus, any homotopically meaningful manipulations that we might make using one of these notions can equally well be made using any other notion.

However, there is still cause for potential concern: the diagram of [BSP, Figure 1] does not commute, even up to natural isomorphism. However, a moment’s reflection should reassure us that this is a stronger request than we should really be making: after all, we would generally like to consider objects of a model category up to weak equivalence, not up to isomorphism. Thus, it is helpful to reinterpret this diagram within one of the given model categories. Rather than choose a particular one, we will simply refer to objects of this model category as ‘$\infty$-categories’ (with scare-quotes) for the remainder of the item; as explained in item (11) below, Quillen equivalences between model categories induce weak equivalences of underlying ‘$\infty$-categories’.

This conceptual leap leads us to the alternative point of view that what we are looking at is a not-necessarily-commutative diagram of weak equivalences of ‘$\infty$-categories’. This may not seem like an improvement in and of itself, but in fact we are saved by the following remarkable result ([Toën05, Théorème 6.3], reproved as [Lur09b, Theorem 4.4.1] and generalized as [BSP, Theorem 8.2]), originally stated within the model category $sSet_{\Rezk}$ of complete Segal spaces.

- The ‘$\infty$-category’ of complete Segal spaces (i.e. the ‘$\infty$-category’ corresponding to $sSet_{\Rezk}$) – and hence any ‘$\infty$-category’ weakly equivalent to it – has an essentially discrete derived automorphism space, which is equivalent as a group to $\mathbb{Z}/2$.

---

12This formula follows from [Lur09b, Corollary 4.3.15], but note that this is ultimately just an instance of the “generalized nerve/realization” Quillen equivalence first proved as [DK84, Theorem 3.1].

13In fact, the Rezk nerve functor $N^{\Rezk} : \RelCat_{BK} \rightarrow sSet_{\Rezk}$ also creates weak equivalences, but be warned that the objects in its image are not generally complete Segal spaces, even up to weak equivalence in $s(sSet_{KQ})_{Reedy}$ (again see [LMG]).
• Furthermore, the essentially unique nontrivial derived automorphism of this ‘∞-category’ is given by the involution of taking opposites, and is therefore detected by considering its restriction to the full subcategory generated by the objects $[0], [1] \in \mathcal{C}at$ (considered as objects of each of these various model categories).

It now follows readily that the diagram of [BSP, Figure 1] commutes as a diagram in an ∞-category; more precisely, as a diagram internal to the quasicategory corresponding to the ambient model category of ‘∞-categories’.

A.2. We now establish some conventions surrounding our usage of ∞-categories.

(4) As a rule, the statements we make will generally be invariant under equivalence of ∞-categories. In fact, when we make statements about ∞-categories, we will generally mean to be working in the ∞-category $\mathcal{C}at_\infty$ of ∞-categories. However, this is merely a matter of aesthetics: the sufficiently motivated reader should readily be able to turn our invariant arguments about ∞-categories into simplex-by-simplex arguments about quasicategories.

On the other hand, as a matter of completeness and rigor, whenever a result that we cite regarding quasicategories is not a priori an invariant one, we show explicitly that it descends from $s\mathbf{Set}_{\text{Joyal}}$ to $\mathcal{C}at_\infty$. In the pursuit of this aim, we will occasionally need to refer to a particular quasicategory; when we do so we will use “typewriter text” to denote the quasicategory and align the notation accordingly, so that for instance $\mathcal{C}$ would denote a quasicategory presenting the ∞-category $\mathcal{C}$ (see item (9)).

(5) Many of our arguments will implicitly rely on the existence of a hom-bifunctor

$$\hom(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{S}$$

for an arbitrary ∞-category $\mathcal{C}$. This is achieved by the twisted arrow ∞-category construction (see [Lur14, Proposition 5.2.1.11]).

(6) A cocomplete ∞-category $\mathcal{C}$ is canonically tensored over $\mathcal{S}$; we denote this tensoring by $- \otimes - : \mathcal{S} \times \mathcal{C} \to \mathcal{C}$. (This is constructed objectwise in [Lur09a, Corollary 4.4.4.9]. In the case that $\mathcal{C}$ is presentable, the symmetric monoidal structure on the ∞-category of presentable ∞-categories, for which $\mathcal{S}$ is the unit object; see [Lur14, Proposition 4.8.1.14 and Example 4.8.1.19]. But in any case, one can use the former result to give an ad hoc construction of almost any tensoring which one needs in practice.)

(7) When attempting to define a functor of categories, if the source is a Reedy category and the target is bicomplete, then we can define our functor inductively using latching and matching objects. In fact, this statement continues to hold if the target is a bicomplete ∞-category $\mathcal{C}$ (see item (9)).

(8) Given a relative ∞-category $(\mathcal{M}, \mathbf{W}) \in \mathcal{R}el\mathcal{C}at_\infty$, its localization $\mathcal{M}[\mathbf{W}^{-1}] \in \mathcal{C}at_\infty$ might be more carefully termed its free localization. This construction is left adjoint to the functor $\mathcal{C}at_\infty \to \mathcal{R}el\mathcal{C}at_\infty$ taking an ∞-category $\mathcal{C}$ to its corresponding minimal relative ∞-category $(\mathcal{C}, \mathcal{C}^{\sim})$.

We warn the reader that this notion does not generally agree with the definition of “localization” studied in [Lur09a, §5.2.7] (see [Lur09a, Warning 5.2.7.3]), namely a functor admitting a fully faithful right adjoint. When we discuss it, we will refer to this latter notion as a left localization; its right adjoint may then be referred to as the inclusion of a reflective subcategory. These are actually a special case of free localizations (see [Lur09a, Proposition 5.2.7.12]).

Of course, there is the dual notion of a right localization (into a coreflective subcategory), although due to the overall handedness of mathematics (boiling down to the fact that we’re generally more comfortable thinking about $\mathcal{S}et$ than about $\mathcal{S}et^{op}$), this arises less frequently in practice and in particular does not appear anywhere in [Lur09a] (hence the unambiguity of the terminology “localization” used there).

Note that a (free) localization which is neither a left nor a right localization can nevertheless admit a section; see for instance Example 2.15.

A.3. Note that we are considering model categories as objects of study in their own right: they are nothing more than model ∞-categories whose hom-spaces are essentially discrete. However, we will also be using model

\footnote{Somewhat confusingly, accessible left localizations of presentable ∞-categories additionally satisfy a universal property among left adjoint functors (see [Lur09a, Proposition 5.5.4.20]).}
categories as presentations of their underlying $\infty$-categories (as indicated in item (2)). Thus, we must also establish our conventions regarding their manipulation in this capacity.

For historical context, we will make some attempt to reference the primary sources for results concerning model categories. However, the body of literature is vast, and so as catch-all resources we will generally refer to [Hir03] and [GJ99], especially for the more classical results.\(^{15}\)

(9) In keeping with our general desire for our language to remain independent of any noncanonical choices, when we choose a representative in a model category of an object or a map in its underlying $\infty$-category, we will only mean a representative up to equivalence in the underlying $\infty$-category.

(10) Given a simplicial model category $M_\bullet$ (with underlying model category $M$), another notion of “underlying $\infty$-category” is given by the full simplicial subcategory $M^c_\bullet \in \mathsf{Cat}_{s\mathsf{Set}}$ on the bifibrant objects. By [DK80b, Proposition 4.8], this is weakly equivalent to $L^H(M,W)$ in $(\mathsf{Cat}_{s\mathsf{Set}})_{\mathsf{Bergner}}$.\(^{16}\) In making connections between model categories and $\infty$-categories, the results of [Lur09a] generally assume that the given model categories are simplicial. As a result, some of the connections that we make will carry this same caveat.

(11) Suppose that $F : M \rightleftarrows N : G$ is a Quillen adjunction. Note that the functors $F$ and $G$ do not define functors of underlying relative categories: they do not generally preserve weak equivalences. Nevertheless, we prove in [MGh] that a Quillen adjunction between model categories induces an associated adjunction of quasicategories. By Kenny Brown’s lemma (or rather its immediate consequence [Hir03, Corollary 7.7.2]), the composites

$$M^c \hookrightarrow M \xrightarrow{F} N$$

and

$$M \xleftarrow{G} N \leftarrow N'$$

do preserve weak equivalences, and these respectively present the left and right adjoint functors.

We collect a few related remarks.

- As a particular case, we immediately obtain that left Bousfield localizations present left localizations (and dually).
- In the case of a Quillen adjunction of simplicial model categories, this result is proved as [Lur09a, Proposition 5.2.4.6].
- In fact, [DK80b, Proposition 5.4] already almost produces an adjunction of $\mathsf{ho}(s\mathsf{Set}_{KQ})$-enriched categories from a Quillen adjunction (though this is a weaker notion than that of an adjunction of quasicategories). However, it only gives weak equivalences of corresponding hom-objects: it doesn’t actually provide candidate left and right adjoint functors! This might be thought of as something like a categorification of the situation involving the Whitehead theorem: a list of abstract isomorphisms between the homotopy groups of two spaces does not suffice to show that they are weakly equivalent; rather, it is necessary to have a map inducing those isomorphisms. On the other hand, this result does relatively directly imply the much easier result that a Quillen equivalence induces an equivalence of underlying $\infty$-categories.

(12) If $M$ is a model category and $f \in \mathsf{hom}_M(x,y)$, it is easy to check that the induced adjunction $M_{x/} \rightleftarrows M_{y/}$ is always a Quillen adjunction. If $f$ is additionally a weak equivalence, we might hope that this is then a Quillen equivalence. For this to hold, however, we need for every pushout of $f$ along a cofibration to be a weak equivalence. This will be true either

- if $f$ is an acyclic cofibration, or
- if $M$ is left proper.

This observation allows us to partially address the question of when a slice of a model category presents the corresponding slice $\infty$-category: we have only established the connection for simplicial model categories, though this will suffice for our purposes. Namely, let $M_\bullet$ be a simplicial model category.

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\(^{15}\)The former requires that its model categories have functorial factorizations, whereas we do not. We will never use general results on model categories that depend on functorial factorizations.

\(^{16}\)In the diagram in the statement of [DK80b, Proposition 4.8], the right arrow should also be labeled as a weak equivalence (in $(\mathsf{Cat}_{s\mathsf{Set}})_{\mathsf{Bergner}}$), as indicated by its proof.
• Suppose that \( x \in M^c \). If we choose any factorization \( x \xrightarrow{\sim} x' \to \text{pt}_M \), then we obtain a Quillen equivalence \((M_{x'})_* \stackrel{\simeq}{\to} (M_{x'})_*\) with \( x' \in M^{cc} \). Since Quillen equivalences induce equivalences of underlying \( \infty \)-categories, the dual result to [Lur09a, Lemma 6.1.3.13] implies that \((M_{x'})_*\) (and hence also the underlying model category \(M_{x'})\) presents the undercategory of the object of the underlying \( \infty \)-category of \(M_*\) corresponding to \( x \).

• On the other hand, if \( M_* \) is left proper, then this statement holds for any \( x \in M \). Indeed, if we choose any factorization \( \partial M \to x'' \xrightarrow{\sim} x \), then we obtain a Quillen equivalence \((M_{x''/})_* \stackrel{\simeq}{\to} (M_{x/})_*\), which reduces us to the previous case.

(13) If \( M_* \) is a combinatorial simplicial model category, then it follows from [Lur09a, Remark A.3.3.11, Proposition A.3.3.12, Remark A.3.3.13, and Theorem 4.2.4.1] that homotopy co/limits in \( M_* \) are equivalent to co/limits in its underlying \( \infty \)-category. (See those results for a precise statement.)

(14) In the proof of Lemma 5.4 in the present paper, we make computations in the \( \infty \)-category \( sS \) using the model category \( s(sSet_{KQ})_{\text{Reedy}} \). That the latter presents the former follows from [Lur09a, Proposition 4.2.4.4 and Remark A.2.9.21]. Moreover, the Reedy and injective model structures coincide, and hence the Reedy cofibrations are precisely the levelwise cofibrations (and hence every object of \( s(sSet_{KQ})_{\text{Reedy}} \) is cofibrant).

A.4. Finally, we end §A by laying out a few other miscellaneous conventions.

(15) Following standard conventions, in our diagrams that involve adjunctions, we will always keep left adjoints above and/or to the left of their right adjoints to whatever extent possible. For added clarity, we will use the “turnstile”, which sits on the right adjoint and points towards the left adjoint. Even in the absence of an ambient diagram, we’ll write \( F \dashv G \) to denote that \( F \) is left adjoint to \( G \). Meanwhile, in-line adjunctions will always have their left adjoints on top.

(16) For reference, we list here the notations for all the “named” model structures which appear in this sequence of papers (including those mentioned in item (2)) – first those on 1-categories, then those on \( \infty \)-categories.

| name | underlying (1- or \( \infty \)-category) | notation | localization |
|------|----------------------------------|----------|-------------|
| Kan-Quillen | simplicial sets | \( sSet_{KQ} \) | \( \$ \) |
| Joyal | simplicial sets | \( sSet_{Joyal} \) | \( \mathcal{C}at_{\infty} \) |
| Bergner | \( sSet \)-enriched categories | \( (\mathcal{C}at_{sSet})_{Bergner} \) | \( \mathcal{C}at_{\infty} \) |
| Rezk | bisimplicial sets | \( sSet_{Rezk} \) | \( \mathcal{C}at_{\infty} \) |
| Barwick-Kan | relative categories | \( \mathcal{C}at_{BK} \) | \( \mathcal{C}at_{\infty} \) |
| Moerdijk | bisimplicial sets | \( sSet_{Moerdijk} \) | \( \$ \) |
| Reedy-(Kan-Quillen) | bisimplicial sets | \( s(sSet_{KQ})_{Reedy} \) | \( sS \) |
| Thomason | categories | \( \mathcal{C}at_{Thomason} \) | \( \$ \) |
| Kan-Quillen | simplicial spaces | \( sSet_{KQ} \) | \( \$ \) |
| Thomason | \( \infty \)-categories | \( (\mathcal{C}at_{\infty})_{Thomason} \) | \( \$ \) |

Whenever we draw a diagram which takes place in a model (\( \infty \))-category, we explicitly mention the ambient model structure for emphasis. However, we will only decorate those aspects of the diagram which are relevant to the argument.

(17) Given a set \( I \) of homotopy classes of maps in a (1- or \( \infty \))-category, we write llp(\( I \)) and rlp(\( I \)) for the sets of homotopy classes of maps that have the left or right lifting property with respect to \( I \), respectively.\(^{17}\) To be explicit, note that a commutative square in an \( \infty \)-category is presented by a map from \( \Delta^1 \times \Delta^1 \) to a quasicategory. To obtain a lift through that commutative square is then to obtain an extension over the map \( \Delta^1 \times \Delta^1 \cong \Delta^{[1,1]} \coprod_{\Delta^{[0,0]}} \Delta^{[1,0]} \rightarrow \Delta^3 \) in \( sSet_{Joyal} \).

\(^{17}\)A lifting property with respect to a subcategory by definition means a lifting property with respect to the homotopy classes of maps contained in that subcategory.
There are certain decorations which are sometimes useful to include for emphasis or clarity but are at other times useful to exclude for simplicity. For instance, we may write \((-)^\bullet\) to emphasize that an object is cosimplicial, but we may omit this decoration if we are considering the entire cosimplicial object at once and have no plans to extract its constituents. We list these here.

| decoration | meaning               |
|------------|-----------------------|
| \((-)^\bullet\) | cosimplicial object    |
| \((-)\)    | simplicial object      |
| \((-)^\text{conv}\) | functor being taken levelwise |

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