Entropy of Extremal Black Hole Solutions of String Theory

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Abstract

For extremal black holes, the thermodynamic entropy is not proportional to the area. The general form allowed by thermodynamics is worked out for three classes of extremal black hole solutions of string theory and shown to be consistent with the entropy calculated from the density of elementary string states. On the other hand, the entanglement entropy does not in general agree with these results.

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1. Introduction

Black hole thermodynamics has been an intriguing subject for many years. The laws of classical black hole physics suggested definitions of temperature and entropy purely by analogy with the laws of thermodynamics, but the scale of these quantities could not be determined that way \cite{1}. It was only with the introduction of quantum theoretical, or more precisely semiclassical, ideas that the scale could be set in terms of Planck’s constant \cite{2}. The temperature defined in this way, related to surface gravity, was later rederived in a euclidean approach where there is a requirement of periodicity on the euclidean time coordinate if conical singularities are to be avoided.

Apart from the obvious question about the origin of a nonzero entropy in this context, the expression for the entropy has itself been a cause for wonder. For ordinary, or what are now called non-extremal black holes, the entropy is proportional to the area of the horizon. Explanations have been sought to be given for this dependence. For instance, it has been pointed out \cite{3} that the ‘entanglement entropy’ for matter outside the black hole is proportional to the area of the horizon. Whether this constitutes an explanation is of course open to debate.

There has been a lot of interest lately in the special case of extremal black holes \cite{4}, \cite{5}, \cite{6}, \cite{7}. The temperature and the entropy behave differently from the case of nonextremal black holes. Thus, when the temperature defined through the surface gravity is zero or infinity, it is found that there is no conical singularity, so that the temperature may really be arbitrary. Again, the thermodynamical entropy fails to be proportional to the area of the horizon. Even the entanglement entropy is not proportional to the area, and further, the two kinds of entropy behave differently.

Another direction which recent research has taken involves black hole solutions of string theory. It has been argued that massive string states may be identified with extremal black hole solutions \cite{8}, \cite{9}, \cite{10}. This presents an opportunity of reaching a better understanding of the entropy of black holes in terms of the underlying theory. The entropy has indeed been calculated \cite{10} from the density of string states. The result is sometimes zero and sometimes nonzero even when the area of the horizon vanishes. By stretching the imagination an area interpretation can be developed for the nonzero entropy, but it appears to fail in the case of the zero entropy. In other words, different approaches have to be used in the two situations.

As mentioned earlier, the thermodynamic entropy of extremal black holes need not be proportional to the area. Instead of seeking an area interpretation, a comparison of the
string result with the correct thermodynamical formula should be made. That is what we do for the entropy of some extremal black hole solutions of string theory in this paper. The calculation on the basis of the string level density is now standard. We shall demonstrate that the expression proportional to the mass that we have advocated earlier \[6\] (see also \[11\]) for the thermodynamic entropy fits very well. We also show that in general the entanglement entropy does not agree with the result.

The extremal black hole solutions considered in this paper are taken from \[12\] and are reviewed in Sec.2, where the expressions for the entropy as given by the density of string states \[9\] are also presented (cf \[10\]). The form of the thermodynamic entropy is derived in Sec. 3. Sec. 4 is devoted to the entropy of scalar matter in the background of these black holes.

2. String based black holes

2.1. The solutions

In four dimensions the massless bosonic fields of heterotic string obtained by toroidal compactification lead to an effective action with an unbroken $U(1)^{28}$ gauge symmetry \[12\]:

$$S = \frac{1}{32\pi} \int d^4 x \sqrt{-G} e^{-\Phi} \left[ R_G + G^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi + \frac{1}{8} G^{\mu\nu} Tr(\partial_\mu \mathcal{M} \partial_\nu \mathcal{M}) \right. \left. - G^{\mu\nu} G^{\rho\sigma} F^{(a)}_{\mu\nu} (\mathcal{M} \nabla)_{ab} F^{(b)}_{\rho\sigma} - \frac{1}{12} G^{\mu\nu} G^{\rho\sigma} G^{\mu\nu} H_{\mu\nu\rho} H_{\mu\nu\rho} \right].$$

Here,

$$\mathcal{L} = \begin{pmatrix} -I_{22} & \ast \\ \ast & I_6 \end{pmatrix},$$

with $I$ representing an identity matrix, $\mathcal{M}$ a symmetric 28 dimensional matrix of scalar fields satisfying

$$\mathcal{M} \nabla \mathcal{M} = \mathcal{L},$$

and there are 28 gauge field tensors

$$F^{(a)}_{\mu\nu} = \partial_\mu A^{(a)}_\nu - \partial_\nu A^{(a)}_\mu, \quad a = 1, \ldots 28$$

and a third rank tensor $H$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + 2 A^{(a)}_{\mu} \mathcal{M}_{ab} F^{(b)}_{\nu\rho} + \text{cyclic permutations of } \mu, \nu, \rho$$

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corresponding to an antisymmetric tensor field \( B \). The canonical metric defined by

\[
g_{\mu\nu} = e^{-\Phi} G_{\mu\nu}
\]

possesses black hole solutions. We shall study some extremal solutions given in [12].

We choose the scale and asymptotic forms of various backgrounds as in [10] where the gravitational constant is equal to 2:

\[
\langle g_{\mu\nu} \rangle = \eta_{\mu\nu}, \quad \langle e^{-\Phi} \rangle = g^2, \quad \langle M \rangle = I_{28}, \quad \langle B_{\mu\nu} \rangle = 0, \quad \langle A^{(a)}_{\mu} \rangle = 0.
\]

(2.7)

Here \( g \) refers to the string coupling constant.

The dilaton field is nontrivial, though \( H \) still vanishes in the solutions we consider now. The metric \( g_{\mu\nu} \) and the dilaton \( \Phi \) are given by

\[
ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu
\]

\[
= -\frac{r^2 - 2mr}{\Delta^{1/2}} dt^2 + \frac{\Delta^{1/2}}{r^2 - 2mr} dr^2 + \Delta^{1/2} d\Omega^2_{II}
\]

(2.8)

with

\[
\Delta = r^2 \left[ r^2 + 2mr(cosh \alpha cosh \gamma - 1) + m^2 (cosh \alpha - cosh \gamma)^2 \right],
\]

and

\[
e^{\Phi} = \frac{g^2 r^2}{\Delta^{1/2}}.
\]

(2.9)

Here \( \alpha, \gamma \) are real parameters. The time components of the gauge fields are given by

\[
\tilde{A}_t = \begin{cases}
\frac{q_L}{\sqrt{2}} \frac{mr sinh \alpha}{\Delta} \left[ r^2 cosh \gamma + mr (cosh \alpha - cosh \gamma) \right] & L = 1, \ldots 22 \\
\frac{q_R}{\sqrt{2}} \frac{mr sinh \gamma}{\Delta} \left[ r^2 cosh \alpha + mr (cosh \alpha - cosh \gamma) \right] & R = 23, \ldots 28
\end{cases}
\]

(2.10)

with \( \tilde{n}_L, \tilde{n}_R \) denoting respectively 22-component and 6-component unit vectors and

\[
\mathcal{M} = I_{28} + \left( P n_L n_L^T \quad Q n_L n_R^T \right),
\]

(2.11)

where

\[
P = \frac{2m^2 r^2 \ sinh^2 \alpha \ sinh^2 \gamma}{\Delta}
\]

\[
Q = -\frac{2mr}{\Delta} \ sinh \alpha \ sinh \gamma \left[ r^2 + mr (cosh \alpha cosh \gamma - 1) \right].
\]

(2.12)

All other backgrounds vanish for this solution.
The ADM mass of the black hole and its charges are given by

\[ M = \frac{1}{4} m (1 + \cosh \alpha \cosh \gamma) \]
\[ \vec{Q} = \begin{cases} 
\frac{g n_L}{\sqrt{2}} m \sinh \alpha \cosh \gamma & L = 1, \ldots, 22 \\
\frac{g n_R}{\sqrt{2}} m \sinh \gamma \cosh \alpha & R = 23, \ldots, 28 
\end{cases} \] (2.13)

The area of the horizon, which is at \( r = 2m \), is

\[ A_H = 8\pi m^2 (\cosh \alpha + \cosh \gamma), \] (2.14)

and the inverse temperature (as defined in terms of the surface gravity) is given by

\[ \beta_H = 4\pi m (\cosh \alpha + \cosh \gamma). \] (2.15)

Three specific extremal limits were considered in [12].

i) Here,

\[ m \to 0, \quad \alpha = \gamma \to \infty, \quad \text{with} \quad m \cosh^2 \alpha = m_0. \] (2.16)

Then

\[ A_H = 0, \quad T_H = \infty, \] (2.17)

and

\[ M = \frac{m_0}{4}, \quad \vec{Q}_L = \frac{g m_0}{\sqrt{2}} \vec{n}_L, \quad \vec{Q}_R = \frac{g m_0}{\sqrt{2}} \vec{n}_R. \] (2.18)

Consequently,

\[ M^2 = \frac{1}{8g^2} \vec{Q}_L^2 = \frac{1}{8g^2} \vec{Q}_R^2. \] (2.19)

Thus the Bogomol’nyi bound is saturated. Although the temperature appears to be infinite here from the surface gravity formula, there is no conical singularity [7] at the horizon in the corresponding Euclidean metric, so that the temperature should really be taken to be arbitrary.

ii) Here,

\[ m \to 0, \quad \gamma \to \infty, \quad \text{with} \quad m \cosh \gamma = m_0, \quad \alpha = \text{finite}. \] (2.20)

Then

\[ A_H = 0, \quad T_H = \frac{1}{4\pi m_0}, \] (2.21)
and
\[ M = \frac{m_0}{4} \cosh \alpha, \quad \vec{Q}_L = \frac{g m_0}{\sqrt{2}} \sinh \alpha \, \vec{n}_L, \quad \vec{Q}_R = \frac{g m_0}{\sqrt{2}} \cosh \alpha \, \vec{n}_R. \quad (2.22) \]

Consequently,
\[ M^2 = \frac{1}{8g^2} \vec{Q}_R^2. \quad (2.23) \]

So in this case the Bogomol'nyi bound is saturated.

iii) Here,
\[ m \to 0, \quad \alpha \to \infty, \quad \text{with} \quad m \cosh \alpha = m_0, \quad \gamma = \text{finite}. \quad (2.24) \]

Then
\[ A_H = 0, \quad T_H = \frac{1}{4\pi m_0} \quad (2.25) \]

and
\[ M = \frac{m_0}{4} \cosh \gamma, \quad \vec{Q}_L = \frac{g m_0}{\sqrt{2}} \cosh \gamma \, \vec{n}_L, \quad \vec{Q}_R = \frac{g m_0}{\sqrt{2}} \sinh \gamma \, \vec{n}_R. \quad (2.26) \]

Consequently,
\[ M^2 = \frac{1}{8g^2} \vec{Q}_L^2. \quad (2.27) \]

Note that the Bogomol'nyi bound is not saturated here:
\[ M^2 \neq \frac{1}{8g^2} \vec{Q}_R^2. \quad (2.28) \]

2.2. Entropy from string level density

The density of states in heterotic string theory is given for a large number \( N \) of oscillators by \([9]\) as
\[ \rho \approx \text{const.}\, N^{-23/2} e^{2a\sqrt{N}}, \quad (2.29) \]

where \( a_L = 2\pi, a_R = \sqrt{2}\pi \). The numbers of oscillators in the left and right sectors are related to the mass and charges of the corresponding states by the usual formula
\[ M^2 = \frac{g^2}{8} \left( \frac{Q_L^2}{g^4} + 2N_L - 2 \right) = \frac{g^2}{8} \left( \frac{Q_R^2}{g^4} + 2N_R - 1 \right). \quad (2.30) \]

To find the level density in terms of the ADM mass of a black hole, one has to combine this formula with the relation between the mass and the charges as applicable for the solution describing that black hole. The three cases have to be discussed separately.

i) In this case, \( N_L = 1 \) and \( N_R = \frac{1}{2} \), so the entropy is zero.
ii) In the second case, $N_R = \frac{1}{2}$ and the entropy arises from large values of $N_L$.

$$S = \log \rho \approx 4\pi \sqrt{N_L} \approx \frac{8\pi}{g} \sqrt{M^2 - \frac{Q^2_1}{8g^2}} = \frac{8\pi}{g \cosh \alpha} M = \frac{2\pi}{g} m_0. \quad (2.31)$$

iii) In this case, $N_L = 1$ and the entropy arises from large values of $N_R$.

$$S = \log \rho \approx 2\sqrt{2\pi} \sqrt{N_R} \approx \frac{4\sqrt{2\pi}}{g} \sqrt{M^2 - \frac{Q^2_R}{8g^2}} = \frac{4\sqrt{2\pi}}{g \cosh \gamma} M = \frac{\sqrt{2\pi}}{g} m_0. \quad (2.32)$$

Thus we see that in all three extremal cases the entropy has a linear dependence on the mass of the black hole, though in one of the two Bogomol’nyi saturated cases the entropy is actually zero. To understand these values, one has to recall our formula \[ S = kM \] for extremal Reissner-Nordstrom black holes with $k$ a constant. A generalization of that result to the case of several charges will be presented in the next section.

### 3. Thermodynamic Entropy

For nonextremal black holes, the laws of black hole physics suggest that the entropy is proportional to the area of the horizon. When the scale is fixed by comparing the temperature thus suggested with that given by the semiclassical calculations of \[ S = kM \], the entropy turns out to be a quarter of the area. If one is interested in an extremal black hole, one may be tempted to regard it as a special limiting case of a sequence of nonextremal black holes and thus infer that the same formula should hold for the entropy. It was pointed out in the context of Reissner-Nordstrom black holes \[ S = kM \] that the extremal and nonextremal cases of the euclidean version are topologically different, so that continuity need not hold. It was also argued that the entropy in the extremal case should vanish. Subsequently it was shown \[ S = kM \] that if the derivation of an expression for the thermodynamic entropy is attempted afresh for the extremal case, one obtains a result proportional to the mass of the black hole with an undetermined scale. We shall now see how the arguments of \[ S = kM \] can be adapted to the stringy black holes. The three cases have to be treated separately.

i) This is the simplest case. The first law of thermodynamics takes the form

$$\tilde{T}dS = dM - \Phi \cdot d\bar{Q}, \quad (3.1)$$
where $\bar{\Phi}$ represents the chemical potential corresponding to the charge $\vec{Q}$ and the temperature has been written as $\tilde{T}$ to indicate the possibility of its being different from the infinite temperature $T_H$. We can make use of the $O(22) \times O(6)$ symmetry to write

$$\bar{\Phi} = \begin{cases} \frac{\sqrt{2} f_L \vec{n}_L}{4g} & L = 1, \ldots, 22 \\ \frac{\sqrt{2} f_R \vec{n}_R}{4g} & R = 23, \ldots, 28 \end{cases}$$  

(3.2)

where $f_L, f_R$ are unknown functions of $m_0$. There are standard expressions for the chemical potential in nonextremal cases, but we cannot use them for two reasons: first, extremal black holes are not continuously connected to nonextremal black holes, and secondly, the standard expressions are calculated by differentiating the mass with respect to charges at constant area in the anticipation that constant area and constant entropy are synonymous!

By using (2.18) and (3.1) we can now write

$$\tilde{T} dS = (1 - f_L - f_R) dm_0 / 4.$$  

(3.3)

Here it is understood that only such thermodynamic processes are allowed which leave the black hole in the class being considered, i.e., all variations are in the parameters $m_0$ and the unit vectors $\vec{n}_L, \vec{n}_R$. Now the partition function can be written as

$$Z = \exp(-W / \tilde{T}),$$  

(3.4)

where the grand canonical thermodynamic potential is

$$W = M - \tilde{T} S - \bar{\Phi} \cdot \vec{Q}.$$  

(3.5)

Moreover, in the leading semiclassical approximation, $Z$ can be taken to be the exponential of the negative classical action, which vanishes in this case as the area vanishes. Hence $W$ vanishes too and

$$\tilde{T} S = M - \bar{\Phi} \cdot \vec{Q} = (1 - f_L - f_R) m_0 / 4.$$  

(3.6)

Comparison of (3.3) and (3.6) yields

$$\frac{dS}{dm_0} = S / m_0,$$  

(3.7)

i.e.,

$$S = km_0$$  

(3.8)
with some undetermined constant $k$. As $k$ may be taken to vanish, the vanishing string answer is consistent with the thermodynamical expression for the entropy.

ii) This case is slightly more complicated because of the existence of an extra parameter $\alpha$. We can still introduce the chemical potential by (3.2), but the $f$-s are now unknown functions of both $m_0$ and $\alpha$. Because of the vanishing area and classical action, we have an analogue of (3.6):

$$T_H S = M - \Phi \cdot \vec{Q} = \frac{(\cosh \alpha - f_L \sinh \alpha - f_R \cosh \alpha)m_0}{4}. \quad (3.9)$$

Note that we have not written $\tilde{T}$ here: this is because the temperature is not arbitrary in this case but has to be $T_H$. Using (2.21), we then have

$$S = \pi m_0^2 (\cosh \alpha - f_L \sinh \alpha - f_R \cosh \alpha). \quad (3.10)$$

Further, the first law of thermodynamics yields

$$T_H \frac{\partial S}{\partial m_0} = \frac{\partial M}{\partial m_0} - \Phi \cdot \frac{\partial \vec{Q}}{\partial m_0} = \left(\cosh \alpha - f_L \sinh \alpha - f_R \cosh \alpha\right) \frac{m_0}{4}. \quad (3.11)$$

This can be written in view of (3.10) as

$$\frac{\partial S}{\partial m_0} = \frac{S}{m_0}, \quad (3.12)$$

whence

$$S = k(\alpha)m_0, \quad (3.13)$$

with $k(\alpha)$ now an undetermined function of $\alpha$. This function cannot be fixed by considering the analogue of (3.11) where the $m_0$-derivatives are replaced by $\alpha$-derivatives; what happens is that $f_L, f_R$ get expressed in terms of $k$. The string answer for the entropy is indeed of the form (3.13), with $k(\alpha)$ actually taking the constant value $\frac{2\pi}{g}$.

iii) This case is similar to the previous one and clearly leads to

$$S = k(\gamma)m_0, \quad (3.14)$$

where $k(\gamma)$ is now the constant $\frac{\sqrt{2\pi}}{g}$. 8
4. Entanglement entropy

The entropy of scalar matter outside a black hole was calculated in [3] for a Schwarzschild black hole. It can be easily generalized for other black holes and written as

\[ S = \frac{8\pi^3}{45\beta^3} \int_{r_h+\epsilon}^{L} dr \, g_{rr}^{1/2} (-g_{tt})^{-3/2} g_{\theta\theta}, \]  

(4.1)

where \( r_h \) is the location of the horizon and the integration runs from the ‘brick wall’, which is a distance \( \epsilon \) outside the horizon, to a large value \( L \). These can be thought of as the ultraviolet and infrared cutoffs respectively. The \( L \)-dependent part has to be subtracted because it arises even in the absence of the black hole. \( \beta \) is the inverse of the temperature, which is taken to be the Hawking temperature when it is finite. We shall apply this formula to the three extremal black holes under consideration.

i) Here, one finds

\[ S = \frac{8\pi^3}{45\beta^3} \int_{\epsilon}^{m_0} dr \, (2m_0)^{3/2}r^{1/2} = -\left( \frac{16\pi^3}{135\beta^3} \right) (2m_0\epsilon)^{3/2}. \]  

(4.2)

Unlike the usual situation, this expression remains finite in the limit of vanishing \( \epsilon \), i.e., there is no ultraviolet divergence. The entire entropy can be absorbed in the long distance part, so it is natural to take the entanglement entropy to be zero. This is consistent with the string result as well as the form derived for the thermodynamic entropy. Thus all answers, including the prediction of [5], agree.

ii) Here, the expressions given above lead to

\[ S = \frac{8\pi^3}{45\beta^3} \int_{\epsilon}^{m_0} dr \, \frac{m_0^3}{r} = \frac{1}{360} \log \frac{1}{\epsilon}. \]  

(4.3)

It is customary to replace the cutoff \( \epsilon \) by the proper distance of the brick wall at \( r_h + \epsilon \) from the horizon, i.e.,

\[ \tilde{\epsilon} = \int_{0}^{\epsilon} dr \, \sqrt{g_{rr}} \approx 2\sqrt{m_0\epsilon}. \]  

(4.4)

This means

\[ S = \frac{1}{360} \log \frac{1}{\tilde{\epsilon}^2}. \]  

(4.5)

This logarithmic dependence is known from [4] and is reminiscent of \((1 + 1)\) dimensional black holes. It does not agree with the expression given by the string level density.

iii) Here too the same expression follows.
5. Discussion

Although we demonstrated in \cite{3} and \cite{11} that some extremal black holes do not satisfy the area formula and have thermodynamic entropies proportional to the mass, it may not have been clear whether our arguments apply to stringy black holes.

We have therefore considered here three classes of extremal stringy black holes. Our treatment of the thermodynamics does lead to expressions for the entropy proportional to the mass. In the last two cases, the entropy calculated from the density of string states is indeed proportional to the mass. In the first case, the same approach leads to a vanishing entropy, which is also consistent with the general form derived by us on the basis of thermodynamics.

It may appear somewhat disappointing that the thermodynamic approach gives an expression for the entropy in terms of an undetermined constant or function $k$. The same thing happened in the case of the Reissner-Nordstrom black hole \cite{3}. As we argued there, this is bound to happen in the case of zero or infinite $T_H$ where the actual temperature is arbitrary and does not introduce a scale as is done for nonextremal black holes by $T_H$. In the cases considered here with finite $T_H$, a scale is of course involved. But there are many different ways of embedding a black hole in string theory \cite{8} and in general the string level density depends on the embedding. As the thermodynamically derived expression has to accommodate all these possible values, $k$ has to remain undetermined without further specification.

As the entropy of matter in the background of a black hole is often studied in the context of the entropy of a black hole, we have also compared this kind of entropy with the other kinds. In the last two cases, the answer is not of the form derived from thermodynamics and hence inconsistent with the value given by the density of string states. In the first case, the matter entropy can be taken to be zero, and hence made consistent with the string result as well as thermodynamics. In general, the matter entropy must be said to be of a different form, and so the thermodynamical entropy cannot easily be explained in terms of this kind of entropy.
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