THE INVERSE GALOIS PROBLEM FOR CHEREDNIK ALGEBRAS

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Abstract. Given the spherical subalgebra $B$ of a rational Cherednik algebra, we aim to classify all finite groups $\Gamma$ for which there exists a domain $R$ on which $\Gamma$ acts by ring automorphisms, such that $B = R^\Gamma$. We describe such groups in terms of geometry of the center of the reduction of $B$ modulo a large prime.

1. Introduction and main results

Given a simple domain $B$ over $\mathbb{C}$, it is an interesting and natural problem to classify finite groups $\Gamma$ for which there exists a domain $R$ on which $\Gamma$ acts via ring automorphisms such that $B = R^\Gamma$. Given the direct analogy with Galois theory, we refer to this question as the inverse Galois problem for $B$. In [T] we solved this problem for rings of differential operators on smooth affine varieties. Namely, if $D(X) = R^\Gamma$, where $X$ is a smooth affine variety over $\mathbb{C}$ and $\Gamma$ is a finite group of $\mathbb{C}$-automorphisms of a domain $R$, then there exists a smooth affine variety $Y$ such that $R \cong D(Y)$ and $Y \to X$ is a $\Gamma$-Galois etale covering of $X$ [T, Theorem 1]. It was also shown in [T, Theorem 2] that a very generic central quotient of the enveloping algebra of a semi-simple Lie algebra cannot be a nontrivial fixed ring. In this paper we apply the methodology of [T] to the case when $B$ is a (simple) spherical subalgebra of a rational Cherednik algebra defined by Etingof and Ginzburg [EG]. Let us recall their definition.

Let $W$ be a complex reflection group; $\mathfrak{h}$ its reflection representation and $S \subset W$ the set of all complex reflections. Let $(\, , ) : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ be the natural pairing. Given a reflection $s \in S$, let $\alpha_s \in \mathfrak{h}^*$ be an eigenvector of $s$ for eigenvalue 1. Also, let $\alpha_s^\vee \in \mathfrak{h}$ be an eigenvector normalized so that $\alpha_s(\alpha_s^\vee) = 2$. Let $c : S \to \mathbb{C}$ be a function invariant with respect to conjugation by $W$. The rational Cherednik algebra $H_c$ associated to $(W, \mathfrak{h})$ with parameter $c$ is defined as the quotient of $\mathbb{C}[W] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the following relations

$$[x, y] = (y, x) - \sum_{s \in S} c(s)(y, \alpha_s)(\alpha_s^\vee, x), \quad [x, x'] = 0 = [y, y']$$

for all $x, x' \in \mathfrak{h}^*$ and $y, y' \in \mathfrak{h}$. 

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In this note we are concerned with the spherical subalgebra $B_c$ of a Cherednik algebra $H_c$. Recall that

$$B_c = eH_c e, \quad e = \frac{1}{|W|} \sum_{g \in W} g.$$

For $c = 0$, we have that $B_0 = D(\mathfrak{h})^W$. Algebras $B_c$ can be viewed as filtered quantizations of the ring of functions on $(\mathfrak{h} \oplus \mathfrak{h}^*)/W$.

Since $B_c$ is defined over $S = \mathbb{Z}[\frac{1}{n}][c]$, we may define its base change $(B_c)_k = B_c \otimes_S k$ which we denote by $\overline{B}_c$, where $\overline{c}$ is the image of $c$ under the base change map $S \to k$.

The following theorem is the main result of this paper. It relates the inverse Galois problem for $B_c$ to geometry of the center of reduction of $B_c$ modulo a large prime.

**Theorem 1.1.** Let $B_c$ be simple. If $B_c = R^\Gamma$ for a domain $R$ and a finite group $\Gamma$, then there exists a finitely generated ring $S \subset \mathbb{C}$ containing values of $c$ such that the following holds. For any base change $S \to k$ to an algebraically closed field of positive characteristic the group $\Gamma$ is a quotient of the etale fundamental group of the smooth locus of $\text{Spec} \ Z(B_c)$.

We apply Theorem 1.1 to classes of Cherednik algebras for which the center of their reduction modulo $p > 0$ is well-known and (relatively) easy to describe. Namely, we consider two families of spherical subalgebra of the rational Cherednik algebras: one associated to the pair $(S_n, \mathbb{C}^n)$ and a parameter $c \in \mathbb{C}$, the other family of algebras being noncommutative deformations of Kleinian singularities.

**Theorem 1.2.** Let $B_c$ be the spherical subalgebra of a rational Cherednik algebra associated with $(S_n, \mathbb{C}^n)$ with a parameter $c \in \mathbb{C}$. Assume that $B_c$ is simple. If $c$ is irrational then $B_c$ cannot be a fixed ring of a domain under a nontrivial action of a finite group of ring automorphisms. For rational $c$, if $B_c = R^\Gamma$ with finite group $\Gamma$ and domain $R$, then $\Gamma$ must be a quotient of $S_n$.

Next, we consider the case of noncommutative deformations of the Kleinian singularities of type $A_n$ (the spherical subalgebras of Cherednik algebras associated with the pair a cyclic group and its one dimensional representation). These family of algebras is also known as generalized Weyl algebras. Let us recall their definition.

Let $v = \prod_{i=1}^n (h-t_i) \in \mathbb{C}[h]$. Then the algebra $A(v)$ is generated by $x, y, h$ subject to the relations

$$xy = v(h), \quad yx = v(h-1), \quad hx = x(h+1), \quad hy = y(h-1).$$

Recall that if $v = \prod_i (h + \frac{i}{n})$, then $A(v)$ can be identified with the fixed ring of the Weyl algebra $W_1(\mathbb{C})$ under the natural action of the cyclic group of order $n$. On the other hand, when $n = 2$ algebras $A(v)$ correspond to central quotients of $U(\mathfrak{sl}_2)$. It was shown in [S] that a countable family of
primitive quotients of $U(\mathfrak{sl}_2)$ can be realized as $\mathbb{Z}/2\mathbb{Z}$-fixed rings of algebras of differential operators on certain (singular) algebraic curves.

**Theorem 1.3.** Let $A(v)$ be simple. If $A(v) = R^\Gamma$ with $R$ domain and $\Gamma$ a finite group, then $\Gamma$ must be a quotient of $\mathbb{Z}/n\mathbb{Z}$. If in addition $t_i - t_j \notin \mathbb{Q}$ for some $i, j$, then $|\Gamma| < n$.

2. **Proofs**

We start by recalling couple of very basic properties of the spherical subalgebras of rational Cherednik algebras. Namely the PBW property and the Dunkl isomorphism.

The crucial PBW property of $H_c, B_c$, implies that if we equip $H_c, B_c$, with an algebra filtration by putting $\deg(h) = 1, \deg(h^*) = 0, \deg(W) = 0$, then

$$\text{gr}(H_c) = C[W] \ltimes \text{Sym}(h \oplus h^*), \quad \text{gr}(B_c) = \text{Sym}(h \oplus h^*)^W.$$  

Recall that since for any nonzero $f \in \text{Sym}(h^*)$, $\text{ad}(f) = [f, -]$ acts locally nilpotently on $H_c$, we may consider the localization $H_c[f^{-1}]$ (and $B_c[f^{-1}]$ for $f \in C[h]^W$). Then we have the induced filtration on $B_c[f^{-1}]$ and

$$\text{gr}(B_c[f^{-1}]) = \text{Sym}(h \oplus h^*)^{W_f}.$$  

Set $\mathfrak{h}_{reg} = \{ x \in \mathfrak{h}, (x, \alpha) \neq 0, \alpha \in S \}$. Let $\delta \in C[h]^W$ be the defining function of $\mathfrak{h} \setminus \mathfrak{h}_{reg}$. Recall that via the Dunkl embedding we have an isomorphism

$$B_c[\delta^{-1}] \cong D(\mathfrak{h}_{reg}).$$  

**Proof of Theorem [T].** We denote $B_c$ by $B$ throughout the proof. Since $B$ is a simple Noetherian ring and $Z(B) = C$, it follows from the standard facts about fixed rings [M] that $B$ is Morita equivalent to the skew ring $C[\Gamma] \ltimes R$ (see [[T], Lemma 4]). Now, there exists a large enough finitely generated ring $S \subset C$, and models of $B, R$ over $S$, to be denoted by $B_S, R_S$, so that $B_S$ is Morita equivalent to $S[\Gamma] \ltimes R_S$. In particular, $R_S$ is a projective left (and right) $B_S$-module. So for large enough $p \gg 0$ and a base change $S \to k$ to an algebraically closed field of characteristic $p$, we have that $B_k$ is Morita equivalent to $k[\Gamma] \ltimes R_k$.

It is well-known that $B_k$ is finite over its center, more specifically [BFG, Theorem 9.1.1]

$$\text{gr } Z(B_k) = \text{gr}(B_k)^p.$$  

Let $f \in Z(B_k)$ be a nonzero element that vanishes on the singular locus of Spec($Z(B_k)$). As the smooth and the Azumaya loci of Spec($Z(B_k)$) coincide [[BC, Theorem ]], we get that $(B_k)_f$ is an Azumaya algebra over $Z(B_k)_f$ and $(B_k)_f$ is Morita equivalent to $k[\Gamma] \ltimes (R_k)_f$. Then just as in [[T], Proposition
1], we can conclude that Spec \( Z(R_k)_f \rightarrow \text{Spec } Z(B_k)_f \) is a \( \Gamma \)-Galois etale covering and

\[
(R_k)_f = B_k \otimes_{Z(B_k)} Z(R)_f.
\]

Therefore, if \( U \) denotes the smooth locus of Spec\( Z(B_k) \), and \( Y \) denotes the preimage of \( U \) under the projection Spec\( Z(R_k) \rightarrow \text{Spec } Z(B_k) \), then \( Y \rightarrow U \) is \( \Gamma \)-Galois covering. In particular, for any \( g \in k[\hbar] \), \( \text{ad}(g) \) acts locally nilpotently on \( (R_k)_f \). Which implies that \( \text{ad}(g) \) acts locally nilpotently on \( R_k \) as \( R_k \) is \( Z(B_k) \)-torsion free (since \( R_k \) is projective over \( B_k \)). It follows that if \( R_k \) is a domain, then \( \Gamma \) is a quotient of the etale fundamental group of the smooth locus of Spec\( Z(B_k) \) = \( U \). Thus, all it remains to show is that \( R_k \) is a domain.

Next we argue that \( \text{ad}(\delta) \) acts locally nilpotently on \( R \). Indeed, put \( B' = B \otimes B^{op} \) and \( f = \delta \otimes 1 - 1 \otimes \delta \). So, \( B' \) is a spherical subalgebra of a Cherednik algebra associated to \( (W \times W, \hbar \otimes \hbar) \). We can view \( R \) as a left \( B' \)-module. Recall that we have the filtration on \( B'_S[f^{-1}] \) so that \( \text{gr}(B'_S[f^{-1}]) \) is a finitely generated commutative \( S \)-algebra. Equipping \( R_S[f^{-1}] \) with a compatible filtration gives that \( \text{gr}(R_S[f^{-1}]) \) is a finitely generated \( \text{gr}(B'_S[f^{-1}]) \)-module, so by the generic flatness theorem there is a localization \( S' \) of \( S \) so that \( \text{gr}(R'_S[f^{-1}]) \) and hence \( R'_S[f^{-1}] \) is a free \( S' \)-module. On the other hand, since \( R_k[f^{-1}] = 0 \) for all base changes \( S' \rightarrow k \) for \( \text{char}(k) \gg 0 \), we conclude that \( R[f^{-1}] = 0 \). Therefore the action of \( \text{ad}(\delta) \) on \( R \) is locally nilpotent.

Let \( \tilde{R}' = R[\delta^{-1}] \). Since \( R'^{\text{et}} = D(\hbar^{c\theta}) \), it follows from [1], Theorem 1] that \( R' \cong D(Y) \) for some smooth affine variety \( Y \). Hence \( R'_k \) is a domain for \( \text{char}(k) \gg 0 \), as desired.

\(\square\)

To use Theorem 1.1, we need to know the \( p' \)-part of the etale fundamental group of the smooth locus of Spec\( Z(B_c) \). For this purpose we utilize the following.

**Remark 2.1.** Let \( X \) be a complete smooth variety over an algebraically closed field \( k \) of characteristic \( p \), and \( U \subset X \) be an open subset such that \( X \setminus U \) is a divisor with normal crossings in \( X \). Let \( \tilde{X} \) be a complete smooth lift of \( X \) over \( W(k) \) (\( W(k) \) is the ring of Witt vectors over \( k \)), \( \tilde{U} \subset \tilde{X} \) be an open subset lifting \( U \), such that \( \tilde{X} \setminus \tilde{U} \) is a divisor with normal crossings over \( W(k) \). Then any \( p' \)-degree Galois covering of \( U \) admits a lift to a Galois covering of \( \tilde{U} \) [LO, Corollary A.12], which yields that any \( p' \)-quotient of the etale fundamental group of \( U \) must be a quotient of the fundamental group of \( U_C \).

We need the following corollary of the Chebotarev density theorem. It contains slightly more than [WWW, Theorem 1.1]. We present a short proof for a reader’s convenience.

**Lemma 2.1.** Let \( S \) be a finitely generated domain containing \( \mathbb{Z} \) and \( c \in S \). Then there are infinitely many primes \( p \) and ring homomorphisms \( \phi_p : S \rightarrow \)
If \( c \notin \mathbb{Q} \) then there exist infinitely many primes \( p \) and homomorphisms \( \phi_p : S \to F_q \), so that \( \phi_p(c) \notin F_p \) and \( q \) is a power of \( p \).

**Proof.** By the Noether normalization theorem, there exists \( l \in \mathbb{N} \) and algebraically independent \( x_1, \ldots, x_n \in S_l \) so that \( S_l \) is integral over \( S_l[x_1, \ldots, x_n] \). Let \( I \) be a prime ideal in \( S_l \) lying over \( (x_1, \ldots, x_n) \) (such ideal exists since \( \text{Spec}(S) \to \text{Spec}(S_l[x_1, \ldots, x_n]) \) is surjective by the going-up theorem). So, \( S_l/I = R \) is an integral domain finite over \( \mathbb{Z} \). Let \( S'/l \) be the integral closure of \( \mathbb{Z} \) in \( R \). Then \( R = S'/l \). Thus suffices to show that there exists a homomorphism \( \phi_p : S' \to F_p \) for infinitely many \( p \). This is a consequence of the Chebotarev density theorem.

We have that the image of the map \( \text{Spec}(S) \to \text{Spec}\mathbb{Z}[c_1] \) contains a nonempty open subset. If \( c_1 \) is algebraic, then all but finitely many prime ideals in \( \mathbb{Z}[c_1] \) lift to \( S \). By the Chebotarev density theorem there are infinitely many primes \( I \subset \mathbb{Z}[c_1] \) such that the image of \( c_1 \) in the quotient \( \mathbb{Z}[c_1]/I \cong F_q \) does not belong to \( F_p \). Let \( I' \in \text{Spec}(S) \) be a lift of \( I \). Now any homomorphism \( S/I' \to F_p \) lifting \( \mathbb{Z}[c_1]/I \to F_q \) will do. Finally, let \( c_1 \) be transcendental. Let \( f \in \mathbb{Z}[c_1] \) be such that \( \text{Spec}(\mathbb{Z}[c_1]/f) \) lifts to \( \text{Spec}(S) \). Thus it suffices to show that there are infinitely many primes \( p \) for which there exists \( t \in \mathbb{Z}[c_1] \) such that \( f \notin (p, t) \) and \( \mathbb{Z}[c_1]/(p, t) = F_q \) for \( q > p \). For this purpose we can take any \( p \) that does not divide \( f \), then take a nonlinear irreducible \( t \in F_p[c_1] \) that does not divide \( f \) mod \( p \). Then let \( t \) be any lift of \( t \).

**Remark 2.2.** Given a Cherednik algebra \( H_c \) associated with an arbitrary pair \((W, \mathfrak{h})\), we expect that there is a base change to a characteristic \( p \) field \( k \) for infinitely many values of \( p \), such that

\[
\text{Spec}(\mathbb{Z}(H_c)) = \text{Spec}(\mathbb{Z}(B_c)) = (\mathfrak{h}_k \oplus \mathfrak{h}_k^*)/W.
\]

This in view of Theorem 1.1 would imply that if \( B_c = R^F \), where \( B_c \) is simple and \( R \) is a domain, then \( \Gamma \) must be a quotient of \( W \).

For the proof of Theorem 1.2 we need to recall the definition of the \( n \)-th Calogero-Moser space. Consider the following subscheme of pairs of \( n \)-by-\( n \) matrices over \( \mathbb{C} \)

\[
X = \{(A, B) | \text{rank}([A, B] + \text{Id}_n) = 1\}.
\]

It is known that \( PGL_n(\mathbb{C}) \) acts freely on \( X \) by conjugation, and the \( n \)-th Calogero-Moser space, denoted by \( \text{CM}_n \), is defined as the quotient

\[
X/PGL_n(\mathbb{C}) = \text{CM}_n.
\]

It is well-known that \( \text{CM}_n \) is a smooth, affine variety over \( \mathbb{C} \). In the following proof we also need that the Calogero-Moser spaces are simply connected. This follows from the fact that the \( n \)-th Calogero-Moser space is homeomorphic to the Hilbert scheme of \( n \)-points on the plane which is known to be simply connected based on its cell decomposition.
Proof of Theorem 1.2. If \( c \) is rational then after a base change to a field \( k \) of characteristic \( p \), we have that \[ \text{Spec}(\mathbb{Z}(B_c)) = (\mathfrak{h} \oplus \mathfrak{h}^*)/S_n. \]

Hence using Remark 2.1 the \( p' \)-etale fundamental group of the smooth locus of \( \text{Spec}(\mathbb{Z}(B_c)) \) is \( S_n \). Let \( c \) be irrational. By Lemma 2.1 for any finitely generate subring \( S \subset \mathbb{C} \), there are infinitely many primes \( p \) and algebraically close fields \( k \) of characteristic \( p \) with a base change \( S \to k \), such that \( \bar{c} \not\in F_p \). Then as explained in [BFG], we have \[ \text{Spec} \mathbb{Z}(B_c) \cong (\text{CM}_n)_k. \]

Since \( (\text{CM}_n)_k \) admits a smooth simply connected lift to characteristic 0 (namely \( \text{CM}_n \)), the desired assertion follows.

Proof of Theorem 1.3. Let \( S \subset \mathbb{C} \) be as in the conclusion of Theorem 1.1. Denote by \( \bar{t}_1, \ldots, \bar{t}_n \) images of \( t_1, \ldots, t_n \) after a base change \( S \to k \) to an algebraically closed field of characteristic \( p \). The center of \( A(\bar{v}) = A(v) \otimes_S k \) is known to be generated by \( x_p = x^p, y_p = y^p, h_p = h^p - h \) subject to the following relation \[ x_p y_p = \prod_{i=1}^n (h_p - (\bar{t}_i^p - \bar{t}_i)). \]

After reordering if necessary, let \( \bar{t}_1, \ldots, \bar{t}_k \) be representatives of all distinct cosets of \( \bar{t}_i + F_p \) with multiplicities \( a_1, \ldots, a_k \). So \[ x_p y_p = \prod_{i=1}^k (h^p - h - (\bar{t}_i^p - \bar{t}_i))^{a_i}. \]

Hence the singular locus of \( \text{Spec} Z(A(\bar{v})) \) is \[ \{ x_p = 0 = y_p, h_p = \bar{t}_i^p - \bar{t}_i, a_i > 1 \}. \]

Let \( \alpha_i, 1 \leq i \leq k \) be lifts of \( \bar{t}_i^p - \bar{t}_i \) in \( W(k) \). Let \( U \) be the smooth locus of \[ \text{Spec} W(k)[a,b,c]/(ab - \prod_{i \leq k} (c - \alpha_i)^{a_i}). \]

Then \( U \) is a lift of the smooth locus of \( \text{Spec} Z(A(\bar{v})) \) over \( W(k) \) and the fundamental group of \( U_C \) is \( \mathbb{Z}/\gcd(a_1, \ldots, a_k)\mathbb{Z} \). Using Lemma 2.1, there are infinitely many \( p \) for which there exists a base change such that \( \bar{t}_i \in F_p \) for all \( i \). Hence, \( \Gamma \) must be a quotient of \( \mathbb{Z}/n\mathbb{Z} \). Let \( t_1 - t_2 \notin \mathbb{Q} \). Then again using Lemma 2.1 there exists a base change so that \( t_1^p - t_1 \neq t_2^p - t_2 \). Therefore, in the corresponding partition \( (a_1, \ldots, a_k) \) all numbers are less than \( n \). Hence, \( \gcd(a_1, \ldots, a_k) = d < n \). Thus \( \Gamma \) must be a quotient of \( \mathbb{Z}/d\mathbb{Z} \). \( \square \)
Remark 2.3. It seems natural to expect that Theorem 1.1 should hold for general filtered quantizations. Namely, given a simple $\mathbb{C}$-domain $A$ that can be equipped with an ascending filtration such that the corresponding associated graded algebra is a finitely generated commutative $\mathbb{C}$-domain. In this setting, if $A = R^G$, then it seems reasonable to expect that $G$ must appear as a quotient of the étale fundamental group of the Azumaya locus of $\text{Spec}(Z(A_k))$ for $\text{char}(k) \gg 0$. The proof of Theorem 1.1 can easily be adapted to prove this provided that $R_k$ is a domain. It also follows from the proof that $R$ must be a Harish-Chandra bimodule over $A$.

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