Profinite genus of fundamental groups of compact flat manifolds with the cyclic holonomy group of square-free order

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Abstract

In this article we study the extent to which an \(n\)-dimensional compact flat manifold with the cyclic holonomy group of square-free order may be distinguished by the finite quotients of its fundamental group. In particular, we display a formula for the cardinality of profinite genus of the fundamental group of an \(n\)-dimensional compact flat manifold with the cyclic holonomy group of square-free order.

Keywords: Profinite genus; Bieberbach group; compact flat manifold.

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1 Introduction

The study of what structural properties of a manifold can be detected by the set of finite quotients of the fundamental group of a manifold attracted a lot of attention in 21-st century (see [2, 3, 4, 23, 24, 25, 27]).

The \(n\)-dimensional compact flat manifolds were well described by Bieberbach who characterized such manifolds as isometrically covered by a flat torus such that their fundamental groups \(\Gamma\) are torsion-free with a maximal abelian normal subgroup \(M\) (subgroup of all translations) of finite index (see [7]). Consequently, the group \(\Gamma\) is called an \(n\)-dimensional Bieberbach group; the quotient \(G = \Gamma/M\), called the holonomy group, is a finite group acting faithfully on \(M\). Note that \(\Gamma_d\), the full inverse image of a subgroup \(H\) of \(G\) of order \(d\) under the quotient map \(\Gamma \to \Gamma/M\), is an \(n\)-dimensional Bieberbach group with maximal abelian normal subgroup \(M_d\) and holonomy group \(H\).

In this paper we investigate the extent to which a Bieberbach group may be distinguished from each other by its set of finite quotient groups. Since there is no complete classification of Bieberbach groups in all dimensions, it becomes more difficult to investigate this problem in the general case. Thus, it makes sense to consider this problem for some families. For example, in the previous study [15] we give a complete answer to this problem in the case that the holonomy group is cyclic of prime order. In this paper we consider the Bieberbach groups \(\Gamma\) with cyclic holonomy group \(G = C_{p_1} \times \cdots \times C_{p_k}\),

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where $C_{p_i}$ is a cyclic group of prime order $p_i$. Thus, this paper can be considered as a natural continuation of the previous work.

Following [11], we define the genus $g(Γ)$ as the set of isomorphism classes of finitely generated residually finite groups with the profinite completion isomorphic to the profinite completion $\hat{Γ}$ of $Γ$. In this paper we find a formula for the genus of an $n$-dimensional Bieberbach group with the cyclic holonomy group of square-free order.

Fix a faithful $\mathbb{Z}G$-lattice $M$ for a finite group $G$. We define the crystal class $(G, M)$ as the set of all the free-torsion extensions $Γ$ of $G$ by $M$. We say that two crystal classes $(G, M)$ and $(G', M')$ are arithmetically equivalent if $G$ and $G'$ are conjugate subgroups of $\text{GL}(n, \mathbb{Z})$. The resulting equivalence classes are the arithmetic crystal classes. Similar to the definition of $g(Γ)$, let $C(M)$ denote the set of isomorphism classes of $\mathbb{Z}G$-lattices $N$ such that the $\hat{\mathbb{Z}}G$-modules $\hat{N}$ and $\hat{M}$ are isomorphic.

Note that to calculate the cardinality $|C(Γ)|$ of the genus we have to calculate the number of isomorphism classes of Bieberbach groups in the crystal class $(G, M)$ and $|C(M)|$. We first calculate $|C(M)|$. For this purpose, we will divide our study into two cases: special and not special, according to the following

**Definition 1.1.** Let $Γ$ be an $n$-dimensional Bieberbach group with the cyclic holonomy group $G$ of square-free order. Let $D$ be the set of prime divisors of $|G|$ such that for each $p_i \in D$ the $n$-dimensional Bieberbach subgroups $Γ_{p_i}$ of $Γ$ with the cyclic holonomy group $C_{p_i}$ of order $p_i$ have corresponding $\mathbb{Z}C_{p_i}$-modules $M_i$ with all indecomposable summands of $\mathbb{Z}$-rank $p_i - 1$ except one trivial summand of $\mathbb{Z}$-rank 1. We say that $Γ$ is **special**, if $D = \emptyset$.

Let $|G| = \delta = p_1 p_2 \cdots p_k$ be the decomposition of $\delta$ into distinct primes. Denote by $\mathbb{Q}(ζ_\delta)$ the cyclotomic field generated by a primitive $\delta$-th root of unity $ζ_\delta$, by $H(\mathbb{Q}(ζ_\delta))$ its class group and by $\text{Gal}(ζ_\delta)$ its Galois group, the latter acts naturally on $H(\mathbb{Q}(ζ_\delta))$. Using that

$$\text{Gal}(ζ_\delta) \cong \text{Gal}(ζ_{p_1}) \times \cdots \times \text{Gal}(ζ_{p_k})$$

(see [10, Chapter 14, Corollary 27]) and that $\text{Gal}(ζ_{p_i}) \cong C_{p_i - 1}$ contains the unique subgroup $C_2$ or order 2, for a set $D$ of prime divisors of $\delta$ we can define a subgroup $H_D$ of $\text{Gal}(ζ_\delta)$ that has $C_2$ instead of $\text{Gal}(ζ_p)$ as a factor in the direct product in $\text{Gal}(ζ_{p_1}) \times \cdots \times \text{Gal}(ζ_{p_k})$ for each $p_i \in D$.

With these notations, we obtain the following formula for $|C(M)|$:

**Theorem 1.2.** Let $Γ$ be a $n$-dimensional Bieberbach group with maximal abelian normal subgroup $M$ and cyclic holonomy group $G$ of square-free order $\delta$. If $Γ$ is special, then

$$|C(M)| = \left| H_D \cap \prod_{d|\delta} H(\mathbb{Q}(ζ_d)) \right| .$$

Otherwise,

$$|C(M)| = \left| \text{Gal}(ζ_\delta) \setminus \prod_{d|\delta} H(\mathbb{Q}(ζ_d)) \right| .$$

Let $M$ be a $\mathbb{Z}C_{p_i}$-module (for a cyclic group $C_{p_i} = \langle x \rangle$ of prime order $p_i$). Then $M$ can be written in the form $M = M_1 \oplus M_2$, where $M_1$ is the largest direct summand
of $M$ on which $C_p$ acts trivially. For an element $n \in N_{\text{Aut}(M)}(C_p)$ denote by $\bar{n}$ the its natural image in $F_p^* = F_p \setminus \{0\}$ under an identification $\text{Aut}(C_p) \cong F_p^*$.

Now let "bar" denotes the reduction modulo $p$. The normalizer $N_{\text{Aut}(M)}(C_p)$ acts in a natural way on the set of all fixed elements $M^{C_p}$ under the action of $C_p$ on $M$. This induces the action of $N_{\text{Aut}(M)}(C_p)$ on $M^{C_p}$, and hence $N_{\text{Aut}(M)}(C_p)$ acts on $M^{C_p} / \Delta \cdot \bar{M} \cong \bar{M}_1$, where $\Delta = 1 + x + \cdots + x^{p-1}$ (see [7, p. 168] for more details).

Define a new action "$\cdot$" of the normalizer $N_{\text{Aut}(M)}(C_p)$ on $\bar{M}_1$ by

$$n \cdot m := n \cdot \bar{nm}, \quad (m \in \bar{M}_1) \quad (2)$$

where "$\cdot$" denotes the action of $N_{\text{Aut}(M)}(C_p)$ on $\bar{M}_1$ described in the preceding paragraph. Since for each prime $p$ dividing $|G|$ the normalizer $N_{\text{Aut}(M)}(G)$ is a subgroup of $N_{\text{Aut}(M)}(C_p)$, the normalizer $N_{\text{Aut}(M)}(G)$ also acts on $M_1/pM_1$, and hence $N_{\text{Aut}(M)}(G)$ acts on $(M_1/pM_1)^G = (M_1/pM_1)^G$ for each prime $p$ dividing $|G|$.

We denote $\bar{M}_1 = M_1 - \{0\}$ and state the following

**Theorem 1.3.** Let $\Gamma$ be an $n$-dimensional Bieberbach group with maximal abelian normal subgroup $M$ and cyclic holonomy group $G$ of square-free order $\delta$. Then,

$$|\mathfrak{g}(\Gamma)| = \sum_{M \in T} \prod_{p | \delta} |N_{\text{Aut}(M)}(G) \backslash (\bar{M}_1)^G|,$$

where $T$ is a set of representatives for the isomorphism classes of $\mathbb{Z}G$-lattices in $C(M)$ and $\bar{M}_{1,p}$ is the largest direct summand of $M$ on which $C_p$ acts trivially.

**Corollary 1.4.**

$$|\mathfrak{g}(\Gamma)| \leq |H(\mathbb{Q}(\zeta_\delta))|^a \left( \max \{|(\bar{M}_{1,p})^G| : p | \delta \} \right)^b,$$

where $a$ is the number of divisors of $\delta$ and $b$ is the number of prime divisors of $\delta$.

**Corollary 1.5.** If $\Gamma$ is a special Bieberbach group, then

$$|\mathfrak{g}(\Gamma)| = \sum_{M \in T} \prod_{p | D} |N_{\text{Aut}(M)}(G) \backslash \bar{M}_1| \prod_{q | \delta} |N_{\text{Aut}(M)}(G) \backslash (\bar{M}_{1,q})^G|.$$

We also deduce as a corollary the main result of [15].

**Corollary 1.6.** Let $\Gamma$ be an $n$-dimensional Bieberbach group with maximal abelian normal subgroup $M$ and cyclic holonomy group $G$ of square-free order. If $|G|$ is a prime number, then $|\mathfrak{g}(\Gamma)| = |C(M)|$.

The outline of this paper is as follows. In Section 2, we summarize without proofs the Oppenheim’s classification of integral representations of a cyclic group of square-free order – important result used in the proof of Theorem 1.2 – and presents some preliminaries. Section 3 contains a special case of Theorem 1.3 which generalizes the results of [15] (see Theorem 3.9) and the proofs of our main results. We also give an example of use of the formula of Theorem 1.3.
Notation

$\delta =$ square-free positive integer.

$a | b = a$ divides $b$ (for $a, b \in \mathbb{Z}$).

$\zeta_d =$ primitive $d$-th root of unity (for $d \in \mathbb{Z}$).

$\Phi_d(x) =$ $d$-th cyclotomic polynomial (for $d \in \mathbb{Z}$).

$\mathbb{Q}(\zeta_d) =$ cyclotomic field generated by $\zeta_d$; $\mathbb{Z}[\zeta_d] =$ ring of integers of $\mathbb{Q}(\zeta_d)$ (for $d \in \mathbb{Z}$).

$\text{Gal}(\zeta_d) =$ Galois group of $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$.

$G =$ cyclic group of order $\delta$.

$\hat{\Gamma} =$ profinite completion of $\Gamma$ (see [18]).

$H \backslash X =$ set of orbits of the action of the group $H$ on a set $X$.

$|X| =$ cardinality of the set $X$.

2 Preliminaries

2.1 Modules over cyclic groups of square-free order

For the convenience of the reader we repeat the relevant results of the Oppenheim’s classification of integral representations of a cyclic group of square-free order [17] without proofs.

Let $D_0$ denote the set of all $n \in \mathbb{Z}$ that divides $\delta$ such that there is only an even number of distinct primes in its decomposition. Similarly, let $D_1$ denote the set of all $n \in \mathbb{Z}$ that divides $\delta$ such that there is only an odd number of distinct primes in its decomposition.

In $\mathbb{Z}[x]$ set

$$s_0(x) = \prod_{d \in D_0} \Phi_d(x), \quad s_1(x) = \prod_{d \in D_1} \Phi_d(x)$$

and let $s_i = s_i(g), \ i = 0, 1$.

Definition 2.1. A $\mathbb{Z}G$-lattice is a $\mathbb{Z}G$-module which is finitely generated and free as a $\mathbb{Z}$-module.

Lemma 2.2 ([17], Lemma 3.4). Let $M$ be a $\mathbb{Z}G$-lattice. Then,

$M_0 = \{m \in M : s_0m = 0\}$

and $M_1 = M/M_0$ are $\mathbb{Z}G$-lattices.

Note that the module $M$ is an extension of $M_1$ by $M_0$, i.e., an exact sequence

$$1 \to M_0 \to M \to M_1 \to 1$$

of $\mathbb{Z}G$-modules. It is well known that there is a bijection between the set of equivalence classes of extensions of $M_1$ by $M_0$ and $\text{Ext}_{\mathbb{Z}G}^1(M_1, M_0)$ (see [19] for details). Thus, to characterize a $\mathbb{Z}G$-lattice $M$ we must determine not only the structure of $M_0$ and $M_1$ but also the group $\text{Ext}_{\mathbb{Z}G}^1(M_1, M_0)$.

Proposition 2.3 ([17], p. 17). Let $M_0$ and $M_1$ be as in Lemma 2.2. Then,

$$M_i \cong \bigoplus_{d \in D_i} \mathcal{M}_d$$

where $\mathcal{M}_d := t_dM_i$ with $t_d := s_i/\Phi_d(g), \ i = 0, 1$. 

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Note that $s_0M_0 = 0 = s_1M_1$. Thus, $\Phi_d(g)\mathcal{M}_d = 0$ for all $d \mid \delta$. Hence, $\mathcal{M}_d$ is a $\mathbb{ZG}/\Phi_d(g)\mathbb{ZG}$-module.

Let $\zeta_d$ be a primitive $d$-th root of unity. We have a ring isomorphism

$$\frac{\mathbb{ZG}}{\Phi_d(g)\mathbb{ZG}} \cong \mathbb{Z}[\zeta_d]$$

given by $g \mapsto \zeta_d$. Thus, we may then turn $\mathcal{M}_d$ into a $\mathbb{Z}[\zeta_d]$-module by setting $\zeta_d^m := gm$, $m \in \mathcal{M}_d$. Since $\mathbb{Z}[\zeta_d]$ is a Dedekind domain, it follows that we may write

$$\mathcal{M}_d \cong I_{1,d} \oplus I_{2,d} \oplus \cdots \oplus I_{r(d),d}$$

where the $\{I_{j,d}\}$ are ideals in $\mathbb{Z}[\zeta_d]$ (see [9, Theorem 4.13]). Moreover, the isomorphism invariants of $\mathcal{M}_d$ are its rank $r(d)$ and the ideal class of the product $I_{1,d}I_{2,d} \cdots I_{r(d),d}$.

**Proposition 2.4** ([17], Theorem 4.1 and Corollary 4.8). Let $M_0$ and $M_1$ be as in Lemma 2.2. Then,

$$\text{Ext}^1_{\mathbb{ZG}}(M_1, M_0) \cong \bigoplus_{(s,t) \in D_1 \times D_0} \Lambda(s,t)$$

where $\Lambda(s,t)$ is the $\mathbb{Z}[\zeta_s]/\Phi_d(\zeta_s)\mathbb{Z}[\zeta_s]$-module of $r(s) \times r(t)$ matrices with entries in $\mathbb{Z}[\zeta_s]/\Phi_d(\zeta_s)\mathbb{Z}[\zeta_s]$.

Let $L(s,t)$ denote $\mathbb{Z}[\zeta_s]/\Phi_d(\zeta_s)\mathbb{Z}[\zeta_s]$. Suppose now that $M$ is an extension of $M_1$ by $M_0$ corresponding to an element $\lambda \in \text{Ext}^1_{\mathbb{ZG}}(M_1, M_0)$. According to Proposition 2.4 we can write

$$\lambda = (\lambda(s_1, t_1), \ldots, \lambda(s_i, t_i))$$

where $(s_i, t_i) \in D_1 \times D_0$ and $\lambda(s_i, t_i)$ is an $r(s_i) \times r(t_i)$ matrix with entries in $L(s_i, t_i)$, $i = 1, \ldots, l$.

**Lemma 2.5** ([17], Lemma 4.4). Let $s \mid \delta$, $t \mid \delta$ and $t > s$. Then, $L(s,t)$ is a trivial ring unless there is a prime $p$ such that $t = ps$. If $t = ps$, then $L(s,t) = F_{p,1} \oplus \cdots \oplus F_{p,v}$, where $F_{p,i}$ is a field of characteristic $p$, $i = 1, \cdots, v$.

Thus, the entries $a_{ij} \in L(s,t)$ of the matrix $\lambda(s,t)$ can be written as

$$a_{ij} = (\alpha_{ij}^1, \ldots, \alpha_{ij}^v)$$

where $\alpha_{ij}^k \in F_{p,k}$. Then to $\lambda(s,t)$ corresponds the $v$-tuple of matrices $((\alpha_{ij}^1), \ldots, (\alpha_{ij}^v))$. Set $\rho_k(\lambda(s,t)) = \text{rank}(\alpha_{ij}^k)$.

We need a little more notation to state the classification of the $\mathbb{ZG}$-lattices. Let

$$D^* = \{(s,t) : s \mid \delta, t \mid \delta \text{ and } s/t \text{ is a prime power}\}$$

and let $D^*_1 = \{(s,t) : (s,t) \in D^*, s \in D_1\}$.

**Proposition 2.6** ([17], Theorem 4.13). Let $M$ be a $\mathbb{ZG}$-lattice. A full set of isomorphism invariants of $M$ consists of:

(i) The $\mathbb{ZG}$-rank of $\mathcal{M}_d$, $r(d)$, for each $\mathcal{M}_d$ and $d \mid \delta$.

(ii) The ideal class of product $I_{1,d}I_{2,d} \cdots I_{r(d),d}$ associated with $\mathcal{M}_d$, for each $d \mid \delta$. 

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(iii) \( \{ \rho_k(\lambda(s, t)) : (\lambda(s, t)) \in \text{Ext}^1_{\mathbb{Z}G}(M_1, M_0), (s, t) \in D^*_1, k = 1, \ldots, v \} \).

In what follows, \( r(d, M) \) and \( \lambda(s, t, M) \) denote \( r(d) \) and \( \lambda(s, t) \) on the module \( M \), respectively. Moreover, to shorten notation, let \( [I_{M_d}] \) denote the ideal class of product \( I_{1,d}I_{2,d} \cdots I_{r(d),d} \) associated with \( M_d \), for each \( d \mid \delta \).

By [17, Theorem 7.3] and [9, Proposition 31.15] we have the following profinite version of Proposition 2.6.

**Proposition 2.7.** Let \( M \) and \( N \) be \( \mathbb{Z}G \)-lattices. Then, \( \hat{M} \cong \hat{N} \) as \( \hat{\mathbb{Z}}G \)-modules if and only if

(i) \( r(d, M) = r(d, N) \) for each \( d \mid \delta \).

(ii) \( \rho_k(\lambda(s, t, M)) = \rho_k(\lambda(s, t, N)) \) for each \( k = 1, \ldots, v \) and \( (s, t) \in D^*_1 \).

The next result will be useful to us, later on.

**Proposition 2.8** ([26], Proposition 5.1). Let \( C_n \) be a cyclic group of order \( n \) (\( n \) square-free) and let \( M, N, \) and \( L \) be \( \mathbb{Z}C_n \)-lattices. Then, \( M \oplus L \cong N \oplus L \) implies \( M \cong N \) if and only if \( n = 6, 10, 14, \) or \( n \) is prime.

The next result tells us that the direct-sum cancellation holds for \( \hat{\mathbb{Z}}G \)-modules.

**Proposition 2.9.** Let \( M_1, M_2, N_1 \) and \( N_2 \) be \( \mathbb{Z}G \)-lattices. Then,

\[ \hat{M}_1 \oplus \hat{M}_2 \cong \hat{N}_1 \oplus \hat{N}_2 \]

if and only if

\[ r(d, M_1) + r(d, M_2) = r(d, N_1) + r(d, N_2) \]

and

\[ \rho_k(\lambda(s, t, M_1)) + \rho_k(\lambda(s, t, M_2)) = \rho_k(\lambda(s, t, N_1)) + \rho_k(\lambda(s, t, N_2)) \]

for each \( d \mid \delta \) and \( k = 1, 2, \ldots, v \).

**Proof.** The result follows from [17, Theorem 7.4] and [9, Proposition 31.15].

\( \square \)

### 2.2 Galois groups acting on ideal class groups

Let \( \zeta_m \) be a primitive \( m \)-th root of unity. It is well known that the Galois group \( \text{Gal}(\zeta_m) \) of the cyclotomic field \( \mathbb{Q}(\zeta_m) \) is isomorphic to \( (\mathbb{Z}/m\mathbb{Z})^\times \) (the multiplicative group of units of the ring \( \mathbb{Z}/m\mathbb{Z} \)).

**Proposition 2.10** ([10], Chapter 14, Corollary 27). Let \( m = p_1^{a_1}p_2^{a_2} \cdots p_k^{a_k} \) be the decomposition of the positive integer \( m \) into distinct prime powers. Then,

\[ \text{Gal}(\zeta_m) \cong \text{Gal}(\zeta_{p_1^{a_1}}) \times \cdots \times \text{Gal}(\zeta_{p_k^{a_k}}). \]

The next lemma shows that there is an action of the Galois group \( \text{Gal}(\zeta_m) \) on the ideal class group \( H(\mathbb{Q}(\zeta_m)) \).

**Lemma 2.11** ([7], Chapter IV, Exercise 6.2). Let \( A \) and \( B \) be ideals of \( \mathbb{Z}[\zeta] \). If \( A \) and \( B \) are in the same ideal class, then \( \sigma(A) \) and \( \sigma(B) \) are in the same ideal class for any \( \sigma \in \text{Gal}(\zeta_m) \).
Remark 2.12. (i) Given any positive square-free integer $\delta$ suppose that $d \mid \delta$. Then, there is $u \in \mathbb{Z}$ such that $\delta = du$. Since $\zeta^\ast$ is a primitive $d$-th root of unity, the field $\mathbb{Q}(\zeta_d)$ is a subfield of $\mathbb{Q}(\zeta_\delta)$. Thus, for any $\sigma$ in $\text{Gal}(\mathbb{Q}(\zeta_\delta))$ we have $\sigma|_{\mathbb{Q}(\zeta_d)} \in \text{Gal}(\mathbb{Q}(\zeta_d))$. Therefore, $\text{Gal}(\mathbb{Q}(\zeta_\delta))$ acts via automorphisms on $H(\mathbb{Q}(\zeta_d))$ for each $d \mid \delta$.

(ii) From (i) it follows that $\text{Gal}(\mathbb{Q}(\zeta_\delta))$ acts via automorphisms on the direct product $\prod_{d \mid \delta} H(\mathbb{Q}(\zeta_d))$. We let $\text{Gal}(\mathbb{Q}(\zeta_\delta)) \setminus \prod_{d \mid \delta} H(\mathbb{Q}(\zeta_d))$ be the orbit set.

Definition 2.13. Let $M$ and $N$ be $\mathbb{Z}[G]$-modules. A semi-linear homomorphism from $M$ to $N$ is a pair $(f, \varphi)$ where $f : M \to N$ is an abelian group homomorphism and $\varphi$ is an automorphism of $G$ such that

$$f(x \cdot m) = \varphi(x) \cdot f(m)$$

for $x \in G$ and $m \in M$.

Let $\varphi$ be an automorphism of a finite group $G$ and let $M$ be a $\mathbb{Z}[G]$-module. Let $(M)^\varphi$ denote the $\mathbb{Z}[G]$-module $M$ given by the action $g \ast m := \varphi(g) \cdot m$ for $g \in G$ and $m \in M$.

Proposition 2.14. Let $M$ and $N$ be $\mathbb{Z}[G]$-lattices. Then, $M$ will be semi-linearly isomorphic to $N$ if and only if

(i) $r(d, M) = r(d, N)$ for each $d \mid \delta$.

(ii) $\rho_k(\lambda(s, t, M)) = \rho_k(\lambda(s, t, N))$ for each $k = 1, \ldots, v$ and $(s, t) \in D_1^\ast$.

(iii) $\sigma \cdot [I_{M_d}] = [J_{N_d}]$ for each $d \mid \delta$ and for some $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_\delta))$.

Proof. Let $M$ and $N$ be $\mathbb{Z}[G]$-lattices. Note that $M$ and $N$ are semi-linearly isomorphic if and only if $M \cong (N)^\varphi$ as $\mathbb{Z}[G]$-modules for some $\varphi \in \text{Aut}(G)$. As $G$ is a cyclic group of square-free order $\delta$, by Proposition 2.6 we have

(i) $r(d, M) = r(d, N)$ for each $d \mid \delta$.

(ii) $\rho_k(\lambda(s, t, M)) = \rho_k(\lambda(s, t, N))$ for each $k = 1, \ldots, v$ and $(s, t) \in D_1^\ast$.

(iii) $I_{M_d} \cong (J_{N_d})^\varphi$ as $\mathbb{Z}[\zeta_d]$-modules for each $d \mid \delta$ and for some $\varphi \in \text{Aut}(G)$ (see [8, Lemma 22.2]).

For each $d \mid \delta$, we will denote by $C_d$ the subgroup of $G$ of order $d$. Recall that $\text{Gal}(\mathbb{Q}(\zeta_\delta)) \cong (\mathbb{Z}/\delta\mathbb{Z})^\ast \cong \text{Aut}(G)$. So to finish the proof it suffices to show that $(J_{N_d})^\varphi \cong \varphi^{-1}.J_{N_d}$ as $\mathbb{Z}[G]$-modules. Suppose $\varphi(c) = c^l$ where $c \in C_d$ and $l$ is a positive integer such that $(l, \delta) = 1$. Consider the map $\eta : (J_{N_d})^\varphi \to \varphi^{-1}.J_{N_d}$ defined by $u \mapsto \varphi^{-1}(u)$. It is clear that $\eta$ is a group isomorphism, so we have to show that $\eta$ is a $C_d$-homomorphism. Note that

$$\eta(c \ast u) = \eta(\varphi(c) \cdot u) = \varphi^{-1}(c^l \cdot u) = \varphi^{-1}(c^l) \varphi^{-1}(u) = \zeta_d^l \varphi^{-1}(u) = c \cdot \eta(u)$$

where $c \in C_d$. This finishes the proof of Proposition 2.14. □
2.3 Bieberbach groups and Cohomology of Groups

If $\Gamma$ is any Bieberbach group, then $\Gamma$ satisfies an exact sequence

$$1 \to M \to \Gamma \to H \to 1,$$

where $M$ is a maximal abelian subgroup (subgroup of all translations) of $\Gamma$ and $H$ is a finite group.

**Proposition 2.15** ([7], Chapter I, Proposition 4.1). Let $\Gamma$ be an $n$-dimensional Bieberbach group. Then, the translation subgroup $M$ is the unique normal, maximal abelian subgroup of $\Gamma$.

Note that the exact sequence (3) gives a natural structure of $ZH$-module on $M$. Since $M$ is maximal abelian, it follows that $M$ is a faithful $ZH$-module. Recall that a crystal class $(H, M)$ is the set all $n$-dimensional Bieberbach groups $\Gamma$ that appears as an extension of $H$ by $M$ and that two crystal classes $(H, M)$ and $(H', M')$ are arithmetically equivalent if there exist isomorphisms $\varphi : H \to H'$ and $f : M \to M'$ such that $f(h \cdot m) = \varphi(h) \cdot f(m)$ for all $h \in H$ and $m \in M$, i.e., if $H$ and $H'$ are conjugate subgroups of $\text{GL}(n, \mathbb{Z})$. The resulting equivalence classes are the arithmetic crystal classes.

**Lemma 2.16.** Isomorphic Bieberbach groups determine the same arithmetic crystal class.

**Proof.** Let $\Gamma_1$ and $\Gamma_2$ be $n$-dimensional Bieberbach groups with the holonomy groups $H_1, H_2$ and maximal abelian normal subgroups $M_1, M_2$, respectively. Suppose that $F : \Gamma_1 \to \Gamma_2$ is an isomorphism. By Proposition 2.15, $F(M_1) = M_2$. Hence, $F$ induces an isomorphism $\varphi : H_1 \to H_2$. Let $\pi_1 : \Gamma_1 \to H_1$ and $\pi_2 : \Gamma_2 \to H_2$ be the natural projections. Let $\gamma_1 \in \Gamma_1$ and suppose $\pi_1(\gamma_1) = h_1 \in H_1$. By the definition of $\varphi$, we have $\pi_2(F(\gamma_1)) = \varphi(h_1)$. Hence, if $f$ denotes the restriction of $F$ to $M$, then

$$f(h_1 \cdot m) = F(\gamma_1 m \gamma_1^{-1}) = F(\gamma_1)F(m)F(\gamma_1)^{-1} = \varphi(h_1) \cdot f(m),$$

where $m \in M_1$. Therefore, $\Gamma_1$ and $\Gamma_2$ determine the same arithmetic crystal class. \(\square\)

**Lemma 2.17.** Let $(H, M)$ and $(H', M')$ be belong to the same arithmetic crystal class. Then, for each Bieberbach group $\Gamma$ in $(H, M)$ there is an isomorphic group $\Gamma'$ in $(H', M')$.

**Proof.** Let $\Gamma$ be a Bieberbach group of the crystal class $(H, M)$. Let $\xi : H \times H \to M$ be a 2-cocycle corresponding to the extension

$$1 \to M \to \Gamma \to H \to 1,$$

and consider $\Gamma$ to be $M \times H$ with the multiplication

$$(m_1, h_1)(m_2, h_2) = (m_1 + h_1 m_2 + \xi(h_1, h_2), h_1 h_2),$$

where $m_1, m_2 \in M$ and $h_1, h_2 \in H$. Since $(H, M)$ and $(H', M')$ are in the same arithmetic crystal class, there exist isomorphisms $\varphi : H \to H'$ and $f : M \to M'$ such that

$$f(h \cdot m) = \varphi(h) \cdot f(m)$$
for all \( h \in H \) and \( m \in M \). Now define a map \( F : M \times H \rightarrow M' \times H' \) by

\[
F(m, h) = (f(m), \varphi(h))
\]

for \( m \in M \) and \( h \in H \). Clearly, \( F \) is a bijective and

\[
F[(m_1, h_1)(m_2, h_2)] = F(m_1 + h_1 m_2 + \xi(h_1, h_2), h_1 h_2)
= (f(m_1) + \varphi(h_1)f(m_2) + f(\xi(h_1, h_2)), \varphi(h_1)\varphi(h_2))
= (f(m_1) + \varphi(h_1)f(m_2) + f(\xi(f^{-1}f_1 f^{-1}f, f^{-1}f_2 f^{-1}f)), \varphi(h_1)\varphi(h_2))
= (f(m_1) + \varphi(h_1)f(m_2) + f(\xi(f^{-1}\varphi(h_1)f, f^{-1}\varphi(h_2)f)), \varphi(h_1)\varphi(h_2))
= (f(m_1) + \varphi(h_1)f(m_2) + \xi'(\varphi(h_1), \varphi(h_2)), \varphi(h_1)\varphi(h_2))
= (f(m_1), \varphi(h_1))(f(m_2), \varphi(h_2))
\]

where \( \xi'(\varphi(h_1), \varphi(h_2)) = f(\xi(f^{-1}\varphi(h_1)f, f^{-1}\varphi(h_2)f)) \) is a 2-cocycle from \( H' \times H' \) to \( M' \). Therefore, \( F(\Gamma) = \Gamma' \) is a Bieberbach group of the crystal class \((H', M')\). \( \square \)

**Remark 2.18.** It follows from Lemmas 2.16 and 2.17 that to find all possible Bieberbach groups (up to isomorphism) of an arithmetic crystal class it is sufficient to find all possible Bieberbach groups of only one representative of the class.

Recall that the groups \( \Gamma \) that satisfies an exact sequence (3) can be classified by elements of the second cohomology group \( H^2(H, M) \) of \( H \) with coefficients in \( M \). It is known that if \( H \) is a finite group and \( M \) is a finitely generated \( \mathbb{Z}H \)-module, then \( H^2(H, M) \) is a finite group (see [19, Corollary 9.41]).

**Proposition 2.19 ([7], Chapter III, Theorem 2.1).** Let \( \Gamma \) be the extension corresponding to \( \alpha \in H^2(H, M) \). Then, \( \Gamma \) is torsion-free if and only if, for any cyclic subgroup \( C_p \) of \( H \) of prime order \( p \), the image of the restriction homomorphism \( \text{res}^2(\alpha) \in H^2(C_p, M) \) is not zero.

**Definition 2.20.** An element \( \alpha \in H^2(H, M) \) is special, if it defines a Bieberbach group.

Let \( X(H, M) \) denote the subset of \( H^2(H, M) \) formed by all the special elements.

We need a little more notation to characterize all possible Bieberbach groups in the crystal class. The normalizer \( \mathcal{N}_{\text{Aut}(M)}(H) \) of \( H \) in \( \text{Aut}(M) \) acts on \( H^2(H, M) \) by the formula

\[
[\phi \circ c](h_1, h_2) = \phi(c(\phi^{-1}h_1\phi, \phi^{-1}h_2\phi)), \quad (4)
\]

where \( c : H \times H \rightarrow M \) is a 2-cocycle, \( \phi \in \mathcal{N}_{\text{Aut}(M)}(H) \), and \( h_1, h_2 \in H \) (see [7, p. 168]).

**Remark 2.21.** Let \( M \) be a \( \mathbb{Z}C_p \)-module for a cyclic group \( C_p = \langle x \rangle \) of prime order \( p \). Then, \( M \) can be written in the form \( M = M_1 \oplus M_2 \), where \( M_1 \) is the largest direct summand of \( M \) on which \( C_p \) acts trivially. Thus, \( H^2(C_p, M) \cong M_1 \) (see [7, Chap. IV, Theorem 5.1]). Therefore, we can define an action of \( \mathcal{N}_{\text{Aut}(M)}(C_p) \) on \( H^2(C_p, M) \) by the formula (2). We claim that this action is equal to the action \( "\circ " \) defined in (4). Indeed, let \( \bar{\phi} \in \mathcal{N}_{\text{Aut}(M)}(C_p) \) and define \( \tilde{\phi} \in \text{Aut}(C_p) \) by \( \tilde{\phi}(x) = \phi^{-1}x\phi \). Note that if \( c : C_p \times C_p \rightarrow M \) is a 2-cocycle, then

\[
[\tilde{\phi} \circ c](x, y) = c(\bar{\phi}(x), \tilde{\phi}(y))
\]
and
\[
[\varphi_*(c)](x, y) = \varphi(c(x, y)),
\]
for \( x, y \in C_p \), define homomorphisms \( \tilde{\varphi}_* : H^2(C_p, M) \to H^2(C_p, M) \) and \( \varphi_* : H^2(C_p, M) \to H^2(C_p, M) \), respectively. Thus, the action \( \otimes \) defined in (4) can be rewritten as
\[
\phi \otimes \alpha = \varphi_*(\tilde{\varphi}_*(\alpha)),
\]
where \( \alpha \in H^2(C_p, M) \). On the other hand, since \( H^2(C_p, M) \cong \tilde{M}_1 \), the action \( \bullet \) defined in (2) can be rewritten as
\[
\phi \bullet m = \phi \cdot \tilde{m} = \varphi_*(\tilde{\varphi}_*(m)),
\]
where \( m \in \tilde{M}_1 \). Therefore, from (5) and (6) we see that the action \( \otimes \) defined in (4) is equal to the action \( \bullet \) defined in (2) in this case.

**Proposition 2.22** ([13], Theorem 5.2). There exists a one-to-one correspondence between the isomorphism classes of Bieberbach groups in the crystal class \((H, M)\) and the orbits of the action of \( \mathcal{N}_{\text{Aut}(M)}(H) \) on \( X(H, M) \).

**Lemma 2.23.** Let \( C_p \) be a cyclic group of prime order \( p \). Then, \( \mathcal{N}_{\text{Aut}(M)}(C_p) \) acts transitively on \( H^2(C_p, M)^* \).

To prove this, we will need the following definition:

**Definition 2.24.** Let \( C_p \) be a cyclic group of prime order \( p \). A finitely generated \( \mathbb{Z}C_p \)-module \( M \) is exceptional if all indecomposable summands of \( M \) have \( \mathbb{Z} \)-rank \( p - 1 \) except one trivial summand of \( \mathbb{Z} \)-rank 1.

**Proof of Lemma 2.23.** Suppose that \( M \) is an exceptional \( \mathbb{Z}C_p \)-module, i.e., \( M = \bigoplus_{i=1}^r A_i \oplus \mathbb{Z} \) where \( A_i \) are ideals of \( \mathbb{Z}[\zeta_p] \). For each \( \varphi \in \text{Gal}(\zeta_p) \), consider the map \( \phi : \bigoplus_{i=1}^r A_i \oplus \mathbb{Z} \to \bigoplus_{i=1}^r \varphi(A_i) \oplus \mathbb{Z} \) defined by
\[
a_1 \oplus \cdots \oplus a_r \oplus u \mapsto \varphi(a_1) \oplus \cdots \oplus \varphi(a_r) \oplus u,
\]
for \( a_i \in A_i \) and \( u \in \mathbb{Z} \). Recall that \( \text{Gal}(\zeta_p) \cong \text{Aut}(C_p) \); every \( \varphi \in \text{Aut}(C_p) \) is of the form \( \varphi(x) = x^k \) where \( x \) is a generator of \( C_p \) and \( k \) is some integer between 0 and \( p \). Thus,
\[
\varphi(x \cdot a_i) = \varphi(\zeta_p a_i) = \varphi(\zeta_p) \varphi(a_i) = \varphi(x) \cdot \varphi(a_i),
\]
for \( i = 1, \ldots, r \). Since \( \mathbb{Z} \) is a trivial \( \mathbb{Z}C_p \)-module, we have \( \phi(x \cdot m) = \varphi(x) \cdot \phi(m) \) for each \( m \in M \). Hence, \( \phi \in \mathcal{N}_{\text{Aut}(M)}(C_p) \). As \( H^2(C_p, M) = H^2(C_p, \mathbb{Z}) \cong \tilde{\mathbb{Z}} \), we have
\[
\phi \bullet m = \tilde{\phi} m = km,
\]
where \( m \in \tilde{\mathbb{Z}} \) and \( \tilde{\phi} = k \). Therefore, by construction we have \( \mathcal{N}_{\text{Aut}(M)}(C_p) \) acts transitively on \( H^2(C_p, M)^* \) in this case.

Now, if \( M \) is a non-exceptional \( \mathbb{Z}C_p \)-module, then \( \mathcal{N}_{\text{Aut}(M)}(C_p) \) also acts transitively on \( H^2(C_p, M)^* \) by [7, Theorem 6.1, p. 140].

Let \( p \) be a prime number. Let \( H_p \) denote the \( p \)-primary component of the group \( H \).
Proposition 2.25 ([6], p. 84). Let \( H \) be a finite group and \( P \) a normal \( p\)-Sylow subgroup of \( H \). Then,

(i) \( H^n(H, M) = \bigoplus_{p|\mid H} H^n(H, M)_p \) for all \( n > 0 \).

(ii) \( H^n(H, M)_p \cong H^n(P, M)^{G/P} \) for all \( n > 0 \).

In particular, if \( G \) is a cyclic group of square-free order, then \( X(G, M) = \prod_{p\mid\mid G} (\overline{M}_{1,p}^* G/C_p) = \prod_{p\mid\mid G} (\overline{M}_{1,p}^* G) \) where \( M_{1,p} \) is the largest direct summand of \( M \) on which \( C_p \) acts trivially (see Proposition 2.19).

Lemma 2.26. Let \( M \) be a \( \mathbb{Z}G \)-lattice for a cyclic group \( G \) of square-free order \( \delta \). Then,

\[ |N_{\text{Aut}(M)}(G) \setminus X(G, M)| = \prod_{p\mid\mid G} |N_{\text{Aut}(M)}(G) \setminus (\overline{M}_{1,p}^* G)|, \tag{7} \]

where \( M_{1,p} \) is the largest direct summand of \( M \) on which \( C_p \) acts trivially.

Proof. Since \( N_{\text{Aut}(M)}(G) \) is a subgroup of \( N_{\text{Aut}(M)}(C_p) \) for each \( p \mid \delta \), the result follows immediately from Proposition 2.25.

3 Profinite genus

3.1 Bieberbach groups with the cyclic holonomy group of square-free order

In this section we shall determine a full set of isomorphism invariants for the profinite completion of an \( n \)-dimensional Bieberbach group with the cyclic holonomy group of square-free order and also for an arithmetic crystal class.

Lemma 3.1. Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group with cyclic holonomy group \( G \) of square-free order \( \delta \) and \( M \) its maximal abelian normal subgroup. Then, \( \Gamma = M_{n-1} \oplus \mathbb{Z} \) admits a \( \mathbb{Z}G \)-decomposition, where \( \mathbb{Z} \) is trivial module generated by the \( \delta \)-th power of some element \( c \) of \( \Gamma \) and \( \Gamma = M_{n-1} \rtimes C \) with \( C = \langle c \rangle \).

Proof. By [20, Theorem 2] we have the following exact sequence

\[ 1 \to M_{n-1} \to \Gamma \to \mathbb{Z} \to 1. \]

Since \( \mathbb{Z} \) is free, this sequence splits as a semidirect product \( \Gamma = M_{n-1} \rtimes C \).

Lemma 3.2. Let \( \Gamma_1 \) and \( \Gamma_2 \) be \( n \)-dimensional Bieberbach groups with the cyclic holonomy group \( G \) of square-free order \( \delta \) and maximal abelian normal subgroups \( M \) and \( N \), respectively. Then, \( \Gamma_1 \) and \( \Gamma_2 \) are in the same arithmetic crystal class if and only if

(i) \( r(d, M) = r(d, N) \) for each \( d \mid \delta \).

(ii) \( \rho_k(\lambda(s, t, M)) = \rho_k(\lambda(s, t, N)) \) for each \( k = 1, \ldots, v \) and \( (s, t) \in D_1^* \).

(iii) \( \sigma \cdot [I_{M_d}] = [J_{N_d}] \) for each \( d \mid \delta \) and for some \( \sigma \in \text{Gal}(\zeta_\delta) \).

Proof. This is a consequence of Proposition 2.14.
Proposition 3.3. Let $\Gamma_1$ and $\Gamma_2$ be $n$-dimensional Bieberbach groups with the cyclic holonomy group $G$ of order $\delta = 6, 10, 14$, or $\delta$ is prime and maximal abelian normal subgroups $M$ and $N$, respectively. Then, $\Gamma_1 \cong \Gamma_2$ if and only if

(i) $r(d, M) = r(d, N)$ for each $d \mid \delta$.

(ii) $\rho_k(\lambda(s, t, M)) = \rho_k(\lambda(s, t, N))$ for each $k = 1, \cdots, v$ and $(s, t) \in D_1^*$.

(iii) $\sigma \cdot [I_{M_d}] = [I_{N_d}]$ for each $d \mid \delta$ and for some $\sigma \in \text{Gal}(\zeta_\delta)$.

Proof. If $\Gamma_1 \cong \Gamma_2$, then $M$ is semi-linearly isomorphic to $N$ by Lemma 2.16. Therefore, it follows from Proposition 2.14 that the conditions (i)–(iii) are fulfilled.

Conversely, let $M$ and $N$ be $\Z G$-lattices satisfying the conditions (i)–(iii). By Lemma 3.1, $M = M_{n-1} \oplus \Z$ and $N = N_{n-1} \oplus \Z'$ such that $\Gamma_1 = M_{n-1} \rtimes C_1$ and $\Gamma_2 = N_{n-1} \rtimes C_2$ where $C_1, C_2$ contain $\Z$ and $\Z'$ as subgroups of index $\delta$ and acts on $M_{n-1}, N_{n-1}$ as $G$. Since the conditions (i)–(iii) are satisfied for $M_{n-1} \oplus \Z$ and $N_{n-1} \oplus \Z'$, it follows from Proposition 2.14 that $M_{n-1} \oplus \Z$ is semi-linearly isomorphic to $N_{n-1} \oplus \Z'$. Hence, there is a semi-linearly isomorphism $(f, \varphi)$ from $M_{n-1}$ to $N_{n-1}$ by Proposition 2.8. Let $\Theta : C_1 \to C_2$ be the isomorphism induced by $\varphi : G \to G$. Now, consider the map $F : M_{n-1} \rtimes C_1 \to N_{n-1} \rtimes C_2$ given by

$$F(m, c) = (f(m), \Theta(c)),$$

for $m \in M_{n-1}$ and $c \in C_1$. We have $F$ is a group homomorphism, because $(f, \varphi)$ is a semi-linearly homomorphism from $M_{n-1}$ to $N_{n-1}$. Since $F$ is clearly bijective, it is an isomorphism, as we wanted. \hfill $\Box$

Lemma 3.4 ([15], Lemma 3.2). Let $\Gamma_1$ and $\Gamma_2$ be $n$-dimensional Bieberbach groups with maximal abelian normal subgroups $M_1$ and $M_2$ and holonomy groups $G_1$ and $G_2$, respectively. If $\psi : \widehat{\Gamma}_1 \to \widehat{\Gamma}_2$ is an isomorphism, then there are isomorphisms $\phi : \widehat{M}_1 \to \widehat{M}_2$ and $\varphi : G_1 \to G_2$ such that the following diagram commutes:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{M}_1 & \longrightarrow & \widehat{\Gamma}_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\
& & \downarrow{\phi} & & \downarrow{\psi} & & \downarrow{\varphi} & & \\
1 & \longrightarrow & \widehat{M}_2 & \longrightarrow & \widehat{\Gamma}_2 & \longrightarrow & G_2 & \longrightarrow & 1.
\end{array}
$$

Proposition 3.5. Let $\Gamma_1$ and $\Gamma_2$ be $n$-dimensional Bieberbach groups with the cyclic holonomy group $G$ of square-free order $\delta$ and maximal abelian normal subgroups $M$ and $N$, respectively. Then, $\Gamma_1 \cong \Gamma_2$ if and only if

(i) $r(d, M) = r(d, N)$ for each $d \mid \delta$.

(ii) $\rho_k(\lambda(s, t, M)) = \rho_k(\lambda(s, t, N))$ for each $k = 1, \cdots, v$ and $(s, t) \in D_1^*$.

Proof. Suppose $\Gamma_1 \cong \Gamma_2$. By Lemma 3.4, there are isomorphisms $\phi : \widehat{M}_1 \to \widehat{M}_2$ and $\varphi : G \to G$ such that the following diagram is commutative

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \widehat{M}_1 & \longrightarrow & \widehat{\Gamma}_1 & \longrightarrow & G & \longrightarrow & 1 \\
& & \downarrow{\phi} & & \downarrow{\psi} & & \downarrow{\varphi} & & \\
1 & \longrightarrow & \widehat{M}_2 & \longrightarrow & \widehat{\Gamma}_2 & \longrightarrow & G & \longrightarrow & 1.
\end{array}
$$
Then, \( \phi : \hat{M}_1 \to \hat{M}_2 \) is a \( \hat{Z}G \)-module isomorphism. Now, the statements (i) and (ii) follow immediately from Proposition 2.7.

Conversely assume that \( M \) and \( N \) are maximal abelian normal subgroups of \( \Gamma_1 \) and \( \Gamma_2 \), respectively, satisfying the conditions (i) and (ii). Since \( G \) is a cyclic group of square-free order \( \delta \), it follows from Lemma 3.1 that \( M = M_{n-1} \oplus \mathbb{Z} \) and \( N = N_{n-1} \oplus \mathbb{Z}' \) such that \( \Gamma_1 = M_{n-1} \rtimes C_1 \) and \( \Gamma_2 = N_{n-1} \rtimes C_2 \) where \( C_1, C_2 \) contain \( \mathbb{Z} \) and \( \mathbb{Z}' \) as subgroups of index \( \delta \) and acts on \( M_{n-1}, N_{n-1} \) as \( G \). Hence, \( \hat{M} = \hat{M}_{n-1} \oplus \hat{\mathbb{Z}}, \hat{N} = \hat{N}_{n-1} \oplus \hat{\mathbb{Z}}' \) and \( \hat{\Gamma}_1 = \hat{\Gamma}_{n-1} \rtimes \hat{C}_1, \hat{\Gamma}_2 = \hat{\Gamma}_{n-1} \rtimes \hat{C}_2 \), where \( \hat{C}_1 \) and \( \hat{C}_2 \) act on \( \hat{M}_{n-1} \) and \( \hat{N}_{n-1} \) as \( G \). By Proposition 2.7,

\[
\hat{M}_{n-1} \oplus \hat{\mathbb{Z}} \cong \hat{N}_{n-1} \oplus \hat{\mathbb{Z}}
\]
as \( \hat{Z}G \)-modules and hence

\[
r(d, M_{n-1}) + r(d, \mathbb{Z}) = r(d, N_{n-1}) + r(d, \mathbb{Z}')
\]
and

\[
\rho_k(\lambda(s, t, M_{n-1})) + \rho_k(\lambda(s, t, \mathbb{Z})) = \rho_k(\lambda(s, t, N_{n-1})) + \rho_k(\lambda(s, t, \mathbb{Z}'))
\]
for each \( d \mid \delta \) and \( k = 1, 2, \cdots, v \) by Proposition 2.9. As \( \mathbb{Z} \cong \mathbb{Z}' \) as \( ZG \)-modules, by Proposition 2.6 we have

\[
r(d, \mathbb{Z}) = r(d, \mathbb{Z}') \quad \text{and} \quad \rho_k(\lambda(s, t, \mathbb{Z})) = \rho_k(\lambda(s, t, \mathbb{Z}'))
\]
for each \( d \mid \delta \) and \( k = 1, 2, \cdots, v \). Hence,

\[
r(d, M_{n-1}) = r(d, N_{n-1}) \quad \text{and} \quad \rho_k(\lambda(s, t, M_{n-1})) = \rho_k(\lambda(s, t, N_{n-1}))
\]
for each \( k = 1, 2, \cdots, v \) and \( d \mid \delta \), so that by Proposition 2.7, \( \hat{M}_{n-1} \cong \hat{N}_{n-1} \) as \( \hat{Z}G \)-modules. Hence, \( \hat{\Gamma}_1 \cong \hat{\Gamma}_2 \), and the proposition is proved.

We finish this section with the Charlap’s classification of Bieberbach groups with the holonomy group of prime order.

**Definition 3.6.** A Bieberbach group \( \Gamma \) with prime order holonomy group \( C_p \) is exceptional if its maximal abelian normal subgroup \( M \) is an exceptional \( ZC_p \)-module.

**Proposition 3.7 ([7], Chapter IV, Theorem 6.3).** There is a one-to-one correspondence between the isomorphism classes of non-exceptional Bieberbach groups whose holonomy group has prime order \( p \) and 4-tuples \((a, b, c; \theta)\) where \( a, b, c \in \mathbb{Z} \) with \( a > 0 \), \( b \geq 0 \), \( c \geq 0 \), \((a, c) \neq (1, 0)\), \((b, c) \neq (0, 0)\) and \( \theta \in \text{Gal}(\zeta_p)\backslash H(\mathbb{Q}(\zeta_p)) \).

Note that \( \text{Gal}(\zeta_p) \) is cyclic of even order if \( p > 2 \); hence \( \text{Gal}(\zeta_p) \) has a subgroup \( C_2 \) of order 2 if \( p > 2 \).

**Proposition 3.8 ([7], Chapter IV, Theorem 6.4).** There is a one-to-one correspondence between the isomorphism classes of exceptional Bieberbach groups whose holonomy group has prime order \( p \) and pairs \((b, \theta)\) where \( b > 0 \), and \( \theta \in C_2 \backslash H(\mathbb{Q}(\zeta_p)) \).
3.2 Proofs of main results

Proof of Theorem 1.2. Let $\Gamma_1$ and $\Gamma_2$ be $n$-dimensional Bieberbach groups with the cyclic holonomy group of square-free order and maximal abelian normal subgroups $M$ and $N$, respectively. Suppose $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$. By Lemma 3.4, we can assume that $\Gamma_1$ and $\Gamma_2$ have the same cyclic holonomy group $G$ of square-free order $\delta$.

For each $d \mid \delta$, let $C_d$ denote the subgroup of $G$ of order $d$ and let $\Gamma_{i,d}$ denote the $n$-dimensional Bieberbach subgroup of $\Gamma_i$ with the holonomy group $C_d$, for $i = 1, 2$. Combining Propositions 3.5 and 2.7, we have

$$\hat{\Gamma}_1 \cong \hat{\Gamma}_2 \iff \hat{M} \cong \hat{N} \text{ as } \hat{\mathbb{Z}}G\text{-modules}$$

$$\iff \text{for all } d \mid \delta, \hat{M} \cong \hat{N} \text{ as } \hat{\mathbb{Z}}C_d\text{-modules} \quad (8)$$

$$\iff \text{for all } d \mid \delta, \hat{\Gamma}_{1,d} \cong \hat{\Gamma}_{2,d}.$$ 

Suppose that there are prime numbers $p$ and $q$ dividing $\delta$ such that the Bieberbach groups $\Gamma_{1,p}$ and $\Gamma_{2,q}$ are exceptional. Since $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$, it follows from (8) that $\hat{\Gamma}_{1,d} \cong \hat{\Gamma}_{2,d}$ for each $d \mid \delta$. In particular, $\hat{\Gamma}_{1,p} \cong \hat{\Gamma}_{2,p}$ and $\hat{\Gamma}_{1,q} \cong \hat{\Gamma}_{2,q}$. This implies that, if $p \neq q$, then the Bieberbach groups $\Gamma_{1,p}$ and $\Gamma_{2,q}$ are exceptional, for $i = 1, 2$. Thus, as $\delta = p_1 p_2 \cdots p_k$ where $p_l$ ($l = 1, \ldots, k$) are distinct prime numbers, we can assume that there is a positive integer $r$ with $1 \leq r \leq k$ such that $D = \{q_1, \ldots, q_r\}$ is a subset of $\{p_1, \ldots, p_k\}$ with $r$ distinct elements, so that the Bieberbach groups $\Gamma_{i,q}$ are exceptional, for each $q \in D$ and $i = 1, 2$.

We conclude from Proposition 2.10 that the Galois group $\text{Gal}(\zeta_d)$ has a subgroup $\mathcal{H}_D$ that has $\mathbb{Z}/2\mathbb{Z}$ instead of $\text{Gal}(\zeta_q)$ as a factor in the direct product for all $q \in D$. Since $\text{Gal}(\zeta_d) \cong (\mathbb{Z}/\delta \mathbb{Z})^\times \cong \text{Aut}(G)$, we have $\text{Aut}(G)$ contains an isomorphic copy of $\mathcal{H}_D$, which we will also denote by $\mathcal{H}_D$.

Now, if $[(G, M)]$ denotes the arithmetic crystal class of $\Gamma_1$, then, by Lemma 3.2 and Proposition 2.14, we have

$$\Gamma_1, \Gamma_2 \in [(G, M)] \iff M \cong (N)^\varphi \text{ as } \mathbb{Z}G\text{-modules for some } \varphi \in \mathcal{H}_D$$

$$\iff \text{for all } d \mid \delta, M \cong (N)^\varphi \text{ as } \mathbb{Z}C_d\text{-modules for some } \varphi \in \mathcal{H}_D$$

$$\iff \text{for all } d \mid \delta, \Gamma_{1,d}, \Gamma_{2,d} \in [(C_d, M)].$$

Since the Bieberbach groups $\Gamma_{i,q}$ ($q \in D, i = 1, 2$) are exceptional, it follows from Propositions 3.5 and 3.8 together with Lemma 3.2 that

$$|\mathcal{C}(M)| = \left|\mathcal{H}_D \setminus \prod_{d \mid \delta} H(\mathbb{Q}(\zeta_d))\right|.$$ 

On the contrary, if for each prime $p$ dividing $\delta$ the Bieberbach groups $\Gamma_{i,p}$ ($i = 1, 2$) are not exceptional, then by Propositions 3.5 and 3.7 together with Lemma 3.2, we have

$$|\mathcal{C}(M)| = \left|\text{Gal}(\zeta_d) \setminus \prod_{d \mid \delta} H(\mathbb{Q}(\zeta_d))\right|,$$

and therefore the Theorem 1.2 is proved. □
In particular, using Proposition 3.3 and similar arguments as in the proof of Theorem 1.2, we get the following generalization of Theorem 1.1 of [15].

**Theorem 3.9.** Let \( \Gamma, \Gamma_d \) and \( \mathcal{H}_D \) be as in Theorem 1.2. If \( \delta = 6, 10, 14 \), or \( \delta \) is prime, then

\[
|g(\Gamma)| = \left| \mathcal{H}_D \setminus \prod_{d|\delta} H(\mathbb{Q}(\zeta_d)) \right|
\]

for the special case. Otherwise,

\[
|g(\Gamma)| = \left| \text{Gal}(\zeta_\delta) \setminus \prod_{d|\delta} H(\mathbb{Q}(\zeta_d)) \right|.
\]

Since the action of the Galois group on ideal class group is transitive if and only if the class group is trivial, we have

**Corollary 3.10.** Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group with the cyclic holonomy group of order \( \delta = 6, 10, 14 \), or \( \delta \) is prime. Then, \( |g(\Gamma)| = 1 \) if and only if for each integer \( d \) dividing \( \delta \) the class group \( H(\mathbb{Q}(\zeta_d)) \) is trivial.

We can now prove our main theorem.

**Proof of Theorem 1.3.** Let \( \Gamma \) be an \( n \)-dimensional Bieberbach group with maximal abelian normal subgroup \( M \) and cyclic holonomy group \( G \) of square-free order \( \delta \). It follows from Remark 2.18 that to find all the isomorphism classes of Bieberbach groups of the arithmetic crystal classes that corresponds to the \( \mathbb{Z}G \)-lattices in \( \mathcal{C}(M) \), it is sufficient to consider a set \( T \) of representatives for the isomorphism classes of \( \mathbb{Z}G \)-lattices in \( \mathcal{C}(M) \). Theorem 1.2 gives us a formula for the cardinality of \( T \). Now, applying first Proposition 2.22 and then Lemma 2.26, we have

\[
|g(\Gamma)| = \sum_{M \in T} |N_{\text{Aut}(M)}(G(X(G,M)) |
\]

where \( M_{1,p} \) is the largest direct summand of \( M \) on which \( C_p \) acts trivially.

\[\square\]

**Proof of Corollary 1.4.** To simplify notation, we let \( \max\{|(\bar{M}_{1,p})^* G|\} \) stand for \( \max\{|(\bar{M}_{1,p})^* G| : p | \delta \} \). By Theorem 1.3,

\[
|g(\Gamma)| = \sum_{M \in T} \left( \prod_{p|\delta} |N_{\text{Aut}(M)}(G)((\bar{M}_{1,p})^* G)| \right)
\]

\[
\leq \sum_{M \in T} \left( \prod_{p|\delta} \max\{|(\bar{M}_{1,p})^* G|\} \right)
\]

\[
= \sum_{M \in T} \left( \max\{|(\bar{M}_{1,p})^* G|\} \right)^b
\]

\[
= |\mathcal{C}(M)| \left( \max\{|(\bar{M}_{1,p})^* G|\} \right)^b
\]

\[
\leq (H(\mathbb{Q}(\zeta_\delta)))^a \left( \max\{|(\bar{M}_{1,p})^* G|\} \right)^b, \quad \text{(by Theorem 1.2)}
\]

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where \( a \) is the number of divisors of \( \delta \) and \( b \) is the number of prime divisors of \( \delta \).

**Proof of Corollary 1.5.** This follows from Theorem 1.3 together with Lemma 3.1.

**Proof of Corollary 1.6.** This is a consequence of Theorem 1.3 together with Lemma 2.23.

### 3.3 Examples

Suppose that \( G \) is a cyclic group of square-free order \( \delta > 1 \). Let \( \langle a, b \mid R \rangle \) be a presentation of \( G \), i.e., \( G \cong F_2/\tilde{R} \) where \( F_2 \) is the free group on \( \{a, b\} \) and \( \tilde{R} \) is the normal closure of \( R \) in \( F_2 \). This defines an exact sequence

\[
1 \to \tilde{R} \to F_2 \to G \to 1.
\]

Hence, if \([\tilde{R}, \tilde{R}]\) denotes the commutator subgroup of \( \tilde{R} \), then (9) induces the exact sequence

\[
1 \to M \to \Gamma \to G \to 1,
\]

where \( M = \tilde{R}/[\tilde{R}, \tilde{R}] \) and \( \Gamma = F_2/[\tilde{R}, \tilde{R}] \). Note that \( M \) is a free abelian group whose rank is given by the Schrier’s formula \( \text{rank}(F_2) - 1 = \delta + 1 = \delta + 1 \) (see [14, Chapter 6, Proposition 2]). Moreover, we claim that \( M \) is maximal abelian in \( \Gamma \). Suppose the claim were false. Then we could find \( x \in F_2 \) such that the image of \( x \) in \( G \) is non-trivial, so that \([x, r] \in [\tilde{R}, \tilde{R}] \) for all \( r \in \tilde{R} \). Thus, since \( G \) is a cyclic group, we have \( M \) is in the center of \( \Gamma \). Hence, \( \Gamma \) is abelian and \([F_2, F_2] = [\tilde{R}, \tilde{R}] \). As \( M \) has finite index in \( \Gamma \), we have \( \text{rank}(\tilde{R}) = \text{rank}(F_2) \), and hence \(|G| = 1 \), a contradiction. Then, \( \Gamma \) is an \((\delta + 1)\)-dimensional Bieberbach group whose holonomy group is \( G \). By [21, Corollary 1], \( M = \mathbb{Z}G \oplus \mathbb{Z} \) where \( \mathbb{Z} \) is a trivial \( \mathbb{Z}G \)-module. Thus, by Theorem 1.3,

\[
|g(\Gamma)| = \sum_{M \in \mathcal{F}} \left( \prod_{p\mid \delta} |\mathcal{N}_{\text{Aut}(M)}(G)\backslash \mathbb{F}_p^*| \right),
\]

since \( \mathbb{Z} \) is the largest direct summand of \( M \) on which \( C_p \) acts trivially for each prime \( p \mid \delta \). Since \( \mathbb{Z}G \) is a permutation \( \mathbb{Z}G \)-module (i.e., the \( \mathbb{Z} \)-basis of \( \mathbb{Z}G \) is fixed under the action of \( G \)), we have \( G \leq \text{Sym}(G) \leq \text{GL}(\delta + 1, \mathbb{Z}) \cong \text{Aut}(M) \), where \( \text{Sym}(G) \) is the group of all permutations of the elements of \( G \). Hence, \( \mathcal{N}_{\text{Aut}(M)}(G) \) contains the holomorph of \( G \) by [12, Theorem 6.3.2], i.e., the group \( \text{Hol}(G) = G \rtimes \text{Aut}(G) \). As \( \text{Aut}(C_p) \leq \text{Aut}(G) \) for each prime \( p \mid \delta \), we have \( \mathcal{N}_{\text{Aut}(M)}(G) \) acts transitively on \( \mathbb{F}_p^* \) for each prime \( p \mid \delta \). Thus,

\[
|g(\Gamma)| = \left| \text{Gal}(\zeta_\delta) \backslash \prod_{d\mid \delta} H(\mathbb{Q}(\zeta_d)) \right|,
\]

by Theorem 1.2.

In particular, if \( \delta = 6, 10, 14, 15, 21 \) or \( \delta \) is prime \( \leq 19 \), for example, then \(|g(\Gamma)| = 1\) because for each \( d \mid \delta \) the class group \( H(\mathbb{Q}(\zeta_d)) \) is trivial (see [1, Table 10]).

Now, if \( \delta = 46, 55, 105 \) or \( \delta \) is prime \( > 19 \), for example, then \(|g(\Gamma)| > 1\) because \( 23 \mid 46 \) and \( |H(\mathbb{Q}(\zeta_{46})| \geq 3; |H(\mathbb{Q}(\zeta_{55})| \geq 10; |H(\mathbb{Q}(\zeta_{105})| \geq 13; \text{ and } |H(\mathbb{Q}(\zeta_p)| > 1 \text{ for all prime } p > 19 \text{ (see [22, Theorem 11.1 and p. 353]).}} \)
Remark 3.11. In general, calculating the normalizer of a finite subgroup in $GL(n, \mathbb{Z})$ is not easy. We refer the interested reader to [5], which presents a method for calculating such normalizers.

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