Absolutely continuous invariant measures for random non-uniformly expanding maps

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Abstract  We prove existence of (at most denumerable many) absolutely continuous invariant probability measures for random one-dimensional dynamical systems with asymptotic expansion. If the rate of expansion (Lyapunov exponents) is bounded away from zero, we obtain finitely many ergodic absolutely continuous invariant probability measures, describing the asymptotics of almost every point. We also prove a similar result for higher-dimensional random non-uniformly expanding dynamical systems. The results are consequences of the construction of such measures for skew-products with essentially arbitrary base dynamics and asymptotic expansion along the fibers. In both cases our method deals with either critical or singular points for the random maps.

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1 Introduction

In this work we study the existence of absolutely continuous invariant probability measures for the random iteration of maps of the interval, or of a compact manifold, which have positive Lyapunov exponents but can also have critical points or singularities. We also obtain a decomposition of each absolutely continuous invariant measure into at most denumerably many absolutely continuous ergodic components. This can be seen as an extension of the
results of Pelikan [26], Morita [25] and Buzzi [10] which deal with random iterations of
piecewise expanding maps.

It is well-known that the dynamics of random maps can be modeled by a skew-product
map where the “noise” is driven by the ergodic base transformation. This is the general form
of a Random Dynamical System; see [8, Definition 1.1.1]. Hence our results can also be seen
as a study of the dynamics of skew-product whose maps along the one-dimensional fibers
have critical points or discontinuities, positive Lyapunov exponents and very weak conditions
on the base transformation. We mention the work of Denker and Gordin [15] together with
Heinemann [16] where equilibrium states for random bundle dynamics were studied under
the assumption of expansion along the fibers.

As an example of application of our results, let us consider the map
\[ \phi(\theta, x) = (\alpha(\theta), f(\theta, x)) \]
with \( \alpha : S^1 \to S^1 \) a continuous map with an ergodic \( \alpha \)-invariant probability
measure \( \nu \); and \( f_0(x) = a(\theta) - x^2 \) for \( a(\theta) \) continuous so that \( \phi \) is well-defined, and \( m \)
the Lebesgue measure on the interval \([-2, 2]\). We use the notation \( \phi^n(\theta, x) = (a^n(\theta), f^n_\theta(x)) \)
where we write \( \theta_n = \alpha^n(\theta), n \geq 0 \) and \( f^n_\theta(x) = (f_{\theta_{n-1}} \circ \cdots \circ f_{\theta_0})(x) \), which can be
regarded as the random composition of maps from the family \( f_\theta \) chosen according to the
measure preserving transformation \( \alpha \).

**Corollary 1.1** Assume that there exists \( \lambda > 0 \) such that, for \( \nu \times m\text{-a.e. } (\theta, x) \),
\[ \liminf_{n \to \infty} \frac{1}{n} \log |Df^n_\theta(x)| \geq \lambda \]
(1.1)
Then \( \phi \) admits finitely many ergodic invariant probability measures absolutely continuous
with respect to \( \nu \times m \). Moreover, \( \nu \times m\text{-a.e. } (\theta, x) \) belongs to the basin of one of these
measures.

The weak assumptions of the dynamics of the base map allows us to state our results in
the setting of random dynamical systems; see Corollary D in Sect. 1.1.2 for details. The
assumption (1.3) is natural if we consider random perturbations of certain non-uniformly
expanding maps which are stochastically stable. Namely, if \( f_\theta \) is a non-uniformly expanding
\( C^2 \) local diffeomorphism of a compact manifold \( Y \), \( f_\theta \) is a \( C^2 \) family of maps and \( \alpha : X \to X \)
is the left shift map on the infinite product \( X = [\theta - \epsilon, \theta + \epsilon]^\mathbb{N} \) such that \( f_\theta \) is stochastically
stable, then the skew-product map \( \phi(\theta, x) \) satisfies (1.3); see [4, Theorem B] and compare
with Examples 2 and 3 in Sect. 2 together with the stochastically stable examples from [7].
For other families of non-uniformly expanding maps, even when stochastic stability is known,
non-uniform expansion for random orbits is an interesting open question; see Sect. 2. An
open set of maps satisfying (1.3) is provided by Viana [33], see below. On the other hand, our
work can be useful not only for the study of small random perturbations of a given dynamical
system. In our results, the maps \( f_{\theta_0}, f_{\theta_1}, \ldots \) are not given necessarily by an i.i.d. process
and they can be distant from each other. Our results hold for general random dynamical systems
on the interval which are non-uniformly expanding, see Sects. 1.1.2 and 1.1.1 for the precise
setting of the work.

This work can also be seen as a generalization of the earlier work of Keller [22] which
proves that for maps of the interval with finitely many critical points and non-positive
Schwarzian derivative, existence of absolutely continuous invariant probability is guaranteed
by positive Lyapunov exponents, i.e.,
\[ \limsup_{n \to +\infty} \frac{1}{n} \log |Df^n(x)| > 0 \quad \text{on a positive measure set of points } x. \]
Related results were obtained by Alves et al. [5]. They show that every non-uniformly expanding local diffeomorphism away from a non-degenerate critical/singular set, on any compact manifold, admits a finite number of ergodic absolutely continuous invariant measures describing the asymptotics of almost every point. The notion of non-uniform expansion means that

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < 0$$

Lebesgue almost everywhere. \hfill (1.2)

Some control of recurrence to this critical/singular set must be assumed to construct the absolutely continuous invariant measures. This assumption is usually rather difficult to verify.

The main known example of maps satisfying the conditions of the result of Alves, Bonatti and Viana are the Viana maps. These maps were introduced by Viana [33] and studied by many authors, e.g. [3,4,6,12,29] among others. The maps are skew-products $\varphi : \mathbb{X} \times \mathbb{Y} \to \mathbb{X} \times \mathbb{Y}$, $(\theta, x) \mapsto (\alpha(\theta), f(\theta, x))$, with $\alpha$ being a uniformly expanding circle map and the maps on the fibers being quadratic maps of the interval $f_\theta(x) = a(\theta) - x^2$ for $a(\theta) = a_0 + \beta \sin(2\pi \theta)$, $\beta > 0$ small and $a_0$ a Misiurewicz parameter for $f_{a_0}$. The central direction along $\mathbb{Y}$ is dominated by the strong expansion of the base dynamics along $\mathbb{X}$. For an open class of these maps, Viana [33] proved the positiveness of the Lyapunov exponents and Alves [3] proved the existence of an absolutely continuous invariant measure.

Extensions of the above mentioned results were obtained, among others, by Pinheiro [27], and by one of the authors [31] but, in all cases, either non-uniform expansion (1.2) in all directions, or a weaker form of hyperbolicity (partial hyperbolicity) is demanded. The critical/singular set is also assumed to be non-degenerate. In a remarkable work, Tsujii [32] proves results in this line for generic partially hyperbolic endomorphisms on compact surfaces.

On the other hand, for piecewise expanding maps in higher dimensions, the existence of absolutely continuous invariant measures was obtained by Adl-Zarabi [1], Buzzi [11], Gora and Boyarsky [17], Keller [21] and, among other, Saussol [28]. Again the authors assume uniform expansion with strong expansion rates together with certain boundary conditions on the pieces of the domain where the transformation is not expanding.

Our results demand no partial hyperbolicity or domination conditions and we put no restriction on the dynamics of the base of the skew-product, other than almost everywhere continuity and the existence of an invariant ergodic probability measure. We do not require non-uniform expansion (1.2) in all directions, nor the non-degenerate conditions of the critical set. Along multidimensional fibers (i.e. the dimension of the space $\mathbb{Y}$), we do demand non-uniform expansion and a control of the recurrence to the singular/critical set. Along one-dimensional fibers (i.e., the case where $\mathbb{Y}$ is the interval) with $f_\theta$ having negative Schwarzian, we assume non-uniform expansion only: we do not assume slow recurrence. In particular, the base dynamics can have no absolutely continuous invariant measure with respect to some natural volume form, as we present in some examples. Under these mild conditions we prove the existence of at most denumerable many invariant probability measures absolutely continuous along the fibers. We get finitely many invariant probability measures, instead of denumerable many, if the rate of non-uniform expansion is bounded away from zero. For non-uniformly expanding random dynamical systems on the interval, we get finitely many absolutely continuous measures defined on the interval, describing the asymptotics of almost all random orbits.
1.1 Statements of results

For a topological space $X$ we denote by $\mathcal{B}_X$ the Borel $\sigma$-algebra on $X$. The main setting is the following: let $\mathbb{X}$ and $\mathbb{Y}$ be a separable metrizable and complete (i.e., Polish) topological spaces. Let us consider the skew-product map

$$\varphi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \times \mathbb{Y}$$

$$(\theta, x) \mapsto (\alpha(\theta), f(\theta, x)).$$

We assume that $\varphi$ is at least measurable with respect to the Borel $\sigma$-algebra $\mathcal{B}_X \times \mathcal{B}_Y$ (which equals $\mathcal{B}_{\mathbb{X} \times \mathbb{Y}}$ since both $\mathbb{X}$ and $\mathbb{Y}$ are separable metric spaces; see e.g. [9, Appendix M.10]).

1.1.1 One dimensional fibers

We consider $\mathbb{Y} = I_0$ a compact interval. For $\theta \in \mathbb{X}$, $f_\theta : I_0 \rightarrow I_0, x \rightarrow f(\theta, x)$ is an interval map, possibly with critical points and discontinuities. We denote by $C_\theta$ and $D_\theta$ the set of critical points and discontinuities, respectively, of $f_\theta$, for every $\theta \in \mathbb{X}$. We also use the notations $C = \{ (\theta, x) \in \mathbb{X} \times I_0; x \in C_\theta \}$ and $D = \{ (\theta, x) \in \mathbb{X} \times I_0; x \in D_\theta \}$.

We assume throughout that the discontinuities $D_\theta$ of the interval map $f_\theta$ are in the interior of $I_0$, and that the lateral limits exist at each $x \in D_\theta$; see condition $(H^*_4)$ in what follows.

We assume also that

$$(H_1) \; p := \sup\{ \#(C_\theta \cup D_\theta), \theta \in \mathbb{X} \} < \infty \text{ and } \Gamma := \sup\{ \partial_x f(\theta, x), (\theta, x) \notin D_\theta \} < \infty.$$

The set

$$\mathcal{J} = \{ (\theta, x) \in \mathbb{X} \times I_0; x \in C_\theta \cup D_\theta \}$$

is measurable (i.e. it belongs to $\mathcal{B}_X \times \mathcal{B}_{I_0}$).

$$(H_2) \; \alpha : \mathbb{X} \rightarrow \mathbb{X} \text{ is a measurable map with an ergodic invariant probability measure } \nu \text{ such that } \nu(D_\alpha) = 0, \text{ where } D_\alpha \text{ is the set of discontinuity points of } \alpha.$$

The assumption on the discontinuity set is a natural condition to study the $\varphi$-invariance of weak$^*$ accumulation points of dynamically defined probability measures. Let us consider the map

$$F : \mathbb{X} \rightarrow B(I_0) \quad \theta \mapsto f_\theta : I_0 \rightarrow I_0$$

where $B(I_0)$ is the family of measurable maps from $I_0$ to $I_0$ with the uniform norm:

$$\| F(\tilde{\theta}) - F(\theta) \| = \sup_{x \in I_0} | f_{\tilde{\theta}}(x) - f_\theta(x) |.$$

We write $D_F$ for the set of discontinuities of the map $F$. We further assume some regularity of the map $F$.

$$(H_3) \; \nu(D_F) = 0.$$

We deal with two situations:

$$(H_4) \; \text{the maps } f_\theta \text{ are } C^3, Sf_\theta \leq 0, \text{ for every } \theta \in \mathbb{X} \text{ (here } Sf_\theta \text{ is the Schwarzian derivative of } f_\theta \text{) and the derivatives of } \{ f_\theta \}_{\theta \in \mathbb{X}} \text{ are equicontinuous.}$$

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1 The equicontinuity can be replaced by the following condition: given $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - \xi| < \delta$ then $|f_\theta'(x)| < \epsilon$, for all $\theta \in \mathbb{X}$. This is used in the proof of Theorem 4.1.
Absolutely continuous invariant measures

Let \( \varphi : \mathbb{X} \times I_0 \to \mathbb{X} \times I_0 \) be a skew-product as above satisfying \( (H_1), (H_2), (H_3) \) and \( (H_4) \) (or \( (H_4^*) \)). Assume that \( \varphi \) is non-uniformly expanding along the vertical direction according to \( \nu \times \mu \), on the subset \( Z \subset \mathbb{X} \times I_0 \), if \( (1.3) \) holds for \( \nu \times \mu \)-a.e. \((\theta, x) \in Z\), for some \( \lambda > 0 \).

We recall that for an ergodic \( \varphi \)-invariant probability measure, its \textit{ergodic basin} is the set

\[
B(\mu) = \left\{ \omega = (\theta, x) \in \mathbb{X} \times Y : \lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} g(\varphi^j(\omega)) = \int g \, d\mu \quad \text{for each} \quad g \in C^0(\mathbb{X} \times Y, \mathbb{R}) \right\}.
\]

Our main result in this setting is the following

**Theorem A** Let \( \varphi : \mathbb{X} \times I_0 \to \mathbb{X} \times I_0 \) be a skew-product as above satisfying \( (H_1), (H_2), (H_3) \) and \( (H_4) \) (or \( (H_4^*) \)). Assume that \( \varphi \) is non-uniformly expanding along the vertical direction according to \( \nu \times \mu \), on the subset \( Z \subset \mathbb{X} \times I_0 \). Then \( \varphi \) admits finitely many ergodic invariant probability measures absolutely continuous with respect to \( \nu \times \mu \), whose basins cover \( Z \), up to a \( \nu \times \mu \)-zero measure set.

Note that the existence of an invariant measure for the base dynamics (see condition \( (H_2) \)) is not a restriction in the theorem. Indeed, any \( \varphi \)-invariant measure absolutely continuous (with respect to \( \mu_\mathbb{X} \times \mu \), where \( \mu_\mathbb{X} \) is a measure on \( \mathcal{B}_\mathbb{X} \)) induces an \( \alpha \)-invariant measure which is absolutely continuous [with respect to \( \mu_\mathbb{X} \)].

In the case that the rate of expansion is not bounded away from zero, we have a weaker result.

**Theorem B** Let \( \varphi : \mathbb{X} \times I_0 \to \mathbb{X} \times I_0 \) be a skew-product as above satisfying \( (H_1), (H_2), (H_3) \) and \( (H_4) \) (or \( (H_4^*) \)). Assume that the limit in \( (1.3) \) is greater than 0, for \( \nu \times \mu \)-a.e.
$$(\theta, x) \in \mathbb{Z}. \text{ Then } \varphi \text{ admits an at most denumerable family } \{\mu_i\}_{i \in I} \text{ of ergodic invariant probability measures absolutely continuous with respect to } v \times m. \text{ Moreover } v \times m \text{-a.e. }$$

$$(\theta, x) \in Z \text{ belongs to the basin of some } \mu_i, i \geq 1.\)$$

1.1.2 Random dynamical systems interpretation

Let $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \nu)$ be a probability space and let $\alpha$ be an $\nu$-preserving measurable map on $\mathcal{X}$. A random dynamical system $f$ on the measurable space $(\mathcal{Y}, \mathcal{B}_\mathcal{Y})$ over $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \nu, \alpha)$ is generated by mappings $f_\theta$, $\theta \in \mathcal{X}$, so that (see [8, Definition 1.1.1]):

1. the map $(\theta, x) \rightarrow f_\theta(x)$ is measurable, and
2. it satisfies the cocycle property $f_{\theta}^{n+m} = f_{\theta}^{n} \circ f_{\theta}^{m}$ for all $n, m \in \mathbb{Z}^+$, $\theta \in \mathcal{X}$. The associated random orbits are $x_0, x_1, \ldots$, where $x_0 \in \mathcal{Y}$ and $x_{n+1} = f_{\varphi^n(\theta)}(x_n)$. This random dynamical system (RDS for short) is denoted by $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \nu, \alpha, f)$.

In general there is no common measure invariant for all the maps $f_\theta$, $\theta \in \mathcal{X}$. But one can ask whether there exists a measure (or a finite number of measures) describing the asymptotics of almost all random orbits, in the sense defined to follow. Let us denote by $\delta_x$ the Dirac measure at $x$.

**Definition 1.2** A probability measure $\mu$ on $\mathcal{Y}$ is SRB for the RDS $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \nu, \alpha, f)$ if, for $\nu$-almost every $\theta \in \mathcal{X}$, the set $RB_\theta(\mu)$ of points $x \in \mathcal{Y}$ such that

$$\frac{1}{n} \sum_{k=0}^{n-1} \delta_{f_\theta^{k+1}(\theta) \circ \cdots \circ f_\theta(x)} \rightarrow \mu$$

has positive Lebesgue measure.

We call $RB_\theta(\mu)$ the random basin of $\mu$.

One can associate to the random dynamical system $f$ the skew product $\varphi : \mathcal{X} \times \mathcal{Y} \ni (\alpha(\theta), f_\theta(x))$. Note that, a $\varphi$-invariant measure $\mu$ with marginal $\nu$, that is, such that $\mu(A \times I_0) = \nu(A)$ for every $\nu$-measurable $A \subset \mathcal{X}$, is an invariant measure for the random dynamical system $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \nu, \alpha, f)$; see [8, Definition 1.4.1]. All the $\varphi$-invariant measures obtained in Theorems A, B and E are of this type; see Lemma 3.1 in Sect. 3.

We say that the random map $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \nu, \alpha, f)$ is a

- **random non-uniformly expanding map** on $I_0$ if $\mathcal{X}$ is a Polish space, $\mathcal{Y} = I_0$ and the associated skew-product is non-uniformly expanding along the vertical direction according to $\nu \times m$.

- **admissible random non-uniformly expanding map** on $I_0$ if it is a random non-uniformly expanding map on $I_0$ and the associated skew-product satisfies $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$ (or $(H_4^?)$).

We can state similar results to Theorems A and B in the setting of random non-uniformly expanding maps, since the associated skew-product satisfies the conditions of these results. Moreover, inspired by one result of Buzzi [10, Theorem 0.5], we can state the following probabilistic consequence of our results.

**Theorem C** Any admissible random non-uniformly expanding map on $I_0$ admits a finite number of SRB measures. Moreover, the SRB measures are absolutely continuous and, $\nu$-almost surely, the union of their random basins has total Lebesgue measure.
We observe that if $X = \Sigma^{N}$, where $\Sigma$ is an at most countable set, then $X$ is totally disconnected. In addition, setting $f_0 = f_\pi(\theta)$ where $\pi : X \to \Sigma^k$ is a projection on the first $k$-symbols of $\theta \in X$, and $\alpha$ the left shift of $\Sigma^N$ we have both $D_\alpha = \emptyset$ and $D_F = \emptyset$, since $f_0$ depends only on finitely many coordinates of the point $\theta \in X$ (the map $F : X \to B(I_0)$ is locally constant).

Hence we obtain the following as a corollary of Theorem C.

**Corollary D** Let $f_i : I_0 \to I_0$, $i \in \Sigma$ be a countable family of maps of the quadratic family, that is, $f_i(x) = f_\theta(x) = \theta_i - x^2$ with $\theta_i \in [1, 2]$. Let also $X = \Sigma^N$ and $\alpha : X \to X$ be the left shift with some ergodic $\alpha$-invariant probability measure $\nu$.

If $(\Sigma^N, B_{\Sigma^N}, \nu, \alpha, f)$ is a random non-uniformly expanding map on $I_0$, then it admits a finite number of SRB measures. The SRB measures are absolutely continuous and the union of their random basins has total Lebesgue measure $\nu$-a.e.

Similar results holds for families of maps satisfying the non-uniformly expanding conditions with higher-dimensional fibers, as we state in the following Sect. 1.1.3.

### 1.1.3 Higher-dimensional fibers

Assuming a condition of slow recurrence to the set of criticalities and/or discontinuities, which we assume are of a certain non-degenerate type, we can take advantage of the method of proof of Theorems A, B and C to obtain the same conclusion in a setting where the fibers can be higher dimensional manifolds.

Let us assume that $\phi : X \times Y \to X \times Y$ has the same skew-product form as before, but now:

$$(H_5) \quad f : X \times Y \to Y$$ is a Borel measurable map such that $f_\theta : \{\theta\} \times Y \to Y$ is $C^{1+\alpha}$ away from a set of non-degenerate discontinuities $\mathcal{D}_\theta$ and/or criticalities $\mathcal{C}_\theta$ in the compact finite $d$-dimensional manifold $Y$.

We fix a Riemannian metric on $Y$, the corresponding distance function $\text{dist}$ and norm $\| \cdot \|$ to be used in what follows. We also fix a normalized volume form $\text{Leb}$ (Lebesgue measure) on $Y$. The next regularity conditions on the derivatives will be needed.

$$(H_6) \quad f^{'}, f^{'^2} : X \times Y \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d), (\theta, x) \mapsto Df_\theta(x) \text{ and } f^{'^2} : X \times Y \to \mathcal{L}(\mathbb{R}^d, \mathbb{R}^d), (\theta, x) \mapsto Df_\theta(x)^{-1}$$ are Borel measurable maps with respect to the Borel $\sigma$-algebras of $X \times Y$ and $\mathcal{L}(\mathbb{R}^d, \mathbb{R}^d)$. In this last space we consider the topology induced by the usual operator norm $\|L\|_{\theta, x} := \sup(\|L(v)\|_{f_\theta(x)}/\|v\| : 0 \neq v \in T_x Y)$ for a linear map $L : T_x Y \to T_{f_\theta(x)} Y$, $(\theta, x) \in X \times Y$.

We also assume conditions (H1) and (H2) (or $(H_2^*)$) and (H3) on $\mathcal{S}$, $\mathcal{D}_\alpha$ and $\mathcal{D}_F$ as before replacing $I_0$ by $Y$ throughout.

The non-degenerate assumption on the sets $\mathcal{C}_\theta$ and $\mathcal{D}_\theta$ mean that $f_\theta$ behaves like a power of the distance near the set of criticalities/discontinuities. More precisely: there are constants $B > 1$ and $\beta > 0$ for which, writing $\mathcal{S}_\theta$ for $\mathcal{S} \cap \{\theta\} \times \Sigma$.

$$(S1) \quad \frac{1}{B} \text{ dist}(x, \mathcal{S}_\theta)^\beta \leq \frac{\|Df_\theta(x)v\|}{\|v\|} \leq B \text{ dist}(x, \mathcal{S}_\theta)^{-\beta};$$

$$(S2) \quad \left| \log \|Df_\theta(x)^{-1}\| - \log \|Df_\theta(y)^{-1}\| \right| \leq B \frac{\text{ dist}(x, y)}{\text{ dist}(x, \mathcal{S}_\theta)^\beta};$$

$$(S3) \quad \left| \log |\det Df_\theta(x)^{-1}| - \log |\det Df_\theta(y)^{-1}| \right| \leq B \frac{\text{ dist}(x, y)}{\text{ dist}(x, \mathcal{S}_\theta)^\beta};$$

for every $\theta \in X$ and $x, y \in Y \setminus (\mathcal{S}_\theta)$ with $\text{dist}(x, y) < \text{dist}(x, \mathcal{S}_\theta)/2$ and $v \in T_x Y$. 

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Given $\delta > 0$ we define the $\delta$-truncated distance from $x \in \mathcal{Y}$ to $\mathcal{S}_\theta$
\[
\text{dist}_\delta(x, \mathcal{S}_\theta) = \begin{cases} 
1 & \text{if } \text{dist}(x, \mathcal{S}_\theta) \geq \delta, \\
\text{dist}(x, \mathcal{S}_\theta) & \text{otherwise}.
\end{cases}
\]

We say that $\varphi$ is non-uniformly expanding along the fibers according to $\nu \times \text{Leb}$, on $Z \subset \mathcal{X} \times \mathcal{Y}$, if

- $\varphi$ has non-uniform expansion along the vertical direction according to $\nu \times \text{Leb}$ on $Z$: for some $\lambda > 0$,
\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df_{\alpha^j(\theta)}(f_j^j(x))^{-1}\| < -2\lambda, \quad \nu \times \text{Leb} - \text{a.e } (\theta, x) \in Z; \tag{1.4}
\]

- $\varphi$ has slow recurrence to the set of criticalities and discontinuities on the orbit of points of $Z$: for each $\epsilon > 0$ there exists $\delta > 0$ such that
\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log \text{dist}_\delta \left(f_j^j(x), \mathcal{S}_{\alpha^j(\theta)}\right) < \epsilon, \quad \nu \times \text{Leb} - \text{a.e } (\theta, x) \in Z \tag{1.5}
\]

(the reader can recall the definition of $\mathcal{S}$ in the statement of condition $(H_1)$).

Our result in this setting reads as follows.

**Theorem E** Let $\varphi : \mathcal{X} \times \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ be a skew-product as above satisfying $(H_1)$, $(H_2)$ $(H_3)$, $(H_5)$ and $(H_6)$. Assume that $\varphi$ non-uniformly expanding along the fibers according to $\nu \times \text{Leb}$, on the subset $Z \subset \mathcal{X} \times \mathcal{Y}$. Then we obtain the same conclusions as in Theorem A.

In this setting, we also have an analogue of Theorem B: if the limit in (1.4) is smaller than zero, then $\varphi$ admits an at most denumerable family $\{\mu_i\}_{i \in I}$ of ergodic invariant probability measures absolutely continuous with respect to $\nu \times \text{Leb}$, whose basins cover $Z$. The proof is identical to the deduction of the statement of Theorem B from that of Theorem A.

1.2 Strategy of the proof and organization of the text

The basic idea is to define measures on the vertical foliation of the skew-product, depending on the starting vertical leaf $\{\theta\} \times I_0$ or $\{\theta\} \times \mathcal{Y}$; show that these measures depend measurably on $\theta \in \mathcal{X}$ and can be integrated with respect to $\nu$; and then show that weak$^*$ accumulation points of these integrated measures are $\varphi$-invariant.

The assumption of non-uniform expansion along the vertical direction, or along the fibers, enables us to control the densities of these measures along the vertical direction on a certain subset of points which has “positive mass at infinity”. This provides us with an absolutely continuous component for every weak$^*$ accumulation point obtained before. Finally, using the uniqueness of Lebesgue decomposition and the smoothness assumption on $f_\theta$ allows us to obtain an invariant probability measure $\mu$ for the skew-product $\varphi$ which is absolutely continuous with respect to the product measure $\nu \times \text{m}$ of the invariant measure on the base and Lebesgue measure on the interval. In the case of Theorem E, the absolute continuity is respect to $\nu \times \text{Leb}$, where $\text{Leb}$ is the Lebesgue measure on $\mathcal{Y}$. The ergodicity is obtained as a consequence of the fact that the invariant sets, with positive $\nu \times \text{m}$-measure, have $\nu \times \text{m}$-measure bounded away from zero.

In the next Sect. 2 we present some examples of application our main results. In Sect. 3 we construct the basic measures we will use to obtain the invariant probability measures for $\varphi$. In
Sect. 4 we construct an absolutely continuous invariant probability measure for $\varphi$. In Sect. 6, we prove that the invariant sets with positive measure have measure uniformly bounded away from zero. As consequence of this result, we conclude the existence of ergodic absolutely continuous invariant probabilities. From these arguments it also follows the conclusion of Theorems A and B. In Sect. 7 we prove Theorem C, about existence of finitely many SRB continuous invariant probabilities. From these arguments it also follows the conclusion of from zero. As consequence of this result, we conclude the existence of ergodic absolutely we prove that the invariant sets with positive measure have measure uniformly bounded away

2 Some examples and open problems

As mentioned in Sect. 1.1.2, every skew-product map $\varphi(\theta, x) = (\alpha(\theta), f_\theta(x))$ on $\mathbb{X} \times \mathbb{Y}$ presented below can be seen as a RDS $(\mathbb{X}, B_{\mathbb{X}}, \nu, \alpha, f)$ in a standard way; see [8, Definition 1.1.1].

Example 1 Skew-products of quadratic maps have been extensively studied. In [12,33] is proved (1.3), with $\nu$ being Lebesgue measure on $\mathbb{S}^1$, for the maps

$$F : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}, (\theta, x) \mapsto (k \cdot \theta, a_0 - x^2 + a \sin(2\pi \theta))$$

where $k \in \mathbb{Z}^+ \setminus \{1\}$ and $a_0 \in (1, 2]$ is such that 0 is preperiodic for the map $f_{a_0}(x) = a_0 - x^2$. In [29] the same map $F$ as above was studied but with $k$ a real parameter in the interval $(R_0, +\infty)$, where $1 < R_0 < 2$ was shown to exist so that, the map $F$ with $k > R_0$ satisfies (1.3).

In [30] were considered skew-products $G(\theta, x) = (f_{a_1}^k(\theta), f_{a_0}(x) + \alpha s(\theta))$, where $f_{a}(x) := a - x^2$ and $a_0, a_1$ are parameters in the interval $(1, 2]$ such that the critical point is pre-periodic but not periodic, and $s : \mathbb{S}^1 \to [-1, 1]$ is a piecewise $C^1$ map. It was proved that there exist $k_0 \in \mathbb{Z}^+$ and a $C^1$ map $s$ such that, for every small enough $\alpha > 0$ and all integers $k \geq k_0$, the map $G$ satisfies (1.3), with $\mathbb{X} = [f_{a_1}^2(0), f_{a_1}(0)]$ and $\nu$ being Lebesgue measure on the invariant interval $\mathbb{X}$.

Note that the base transformation for the maps in [12,29,33] is (piecewise) expanding. For the maps in [30], it is non-uniformly expanding with critical points.

The existence of absolutely continuous invariant probability measures for all these maps is an immediate consequence of Theorem A, with $\mathbb{X} = \mathbb{S}^1$ and $\varphi = F$ or $\varphi = G$.

Let us mention that the construction of the absolutely continuous invariant probability was obtained in [3] for the maps considered on [12,33]. In [29] this conclusion was only achieved for a full Lebesgue measure subset of $(R_0, +\infty)$. The author in [30] did not obtain absolutely continuous invariant measures. Recently, in [2] was obtained the result for all the maps in [29,30], as a byproduct of the application of inducing to study decay of correlations for the unique absolutely continuous invariant probability measure.

Example 2 We can produce examples where the base dynamics is essentially arbitrary. Let $\mathbb{X}$ be the circle $\mathbb{S}^1$ and $\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ a measurable map preserving an ergodic probability measure $\nu$. Let $\theta \mapsto f_\theta$ be a continuous family of maps of the interval $I_0 = [0, 1]$ such that

- for all $\theta \in \mathbb{S}^1$ the map $f_\theta : I_0 \to I_0$ is 2-to-1, with two branches $f_\theta : [0, 1/2] \to [0, 1/2]$ and $f_\theta : [1/2, 1] \to [1/2, 1]$ both increasing diffeomorphisms;
- on an arc $A$ of $\mathbb{S}^1$ with $\nu(A) \geq 1 - \epsilon$ for some small $\epsilon > 0$ we have
for \( \theta \in A \) the map \( f_{\theta} \) is expanding: there exists \( \sigma > 1 \) such that \( |Df_{\theta}(x)| \geq \sigma \) for all \( x \in I_0 \);

- for \( \theta \in S^1 \setminus A \) the map \( f_{\theta} \) does not contract too much: there exists \( \delta > 0 \) small such that \( |Df_{\theta}(x)| \geq 1 - \delta \) for all \( x \in I_0 \).

In this setting we have that for \( (\nu \times m) \)-a.e. \((\theta, x)\), applying the Ergodic Theorem to the sequence \( (\alpha^j(\theta))_{j \geq 0} \)

\[
\lim \inf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |Df_{\alpha^j(\theta)}(f_0^j(x))| \geq \nu(A) \log \sigma + \nu(S^1 \setminus A) \log(1 - \delta) \\
\geq (1 - \epsilon) \log \sigma - \delta \epsilon > 0,
\]

where \( m \) is the Lebesgue measure on \( I_0 \).

For a concrete expression we may take (see Fig. 1)

\[
f_t(x) = \begin{cases} 
  tx + 2^{\beta}(2 - t)x^1 + \beta & \text{if } x \in [0, \frac{1}{2}) \\
  1 - t(1 - x) - 2^{\beta}(2 - t)(1 - x)^1 + \beta & \text{if } x \in [\frac{1}{2}, 1]
\end{cases}
\]  

(2.1)

with \( \beta \in (0, 1) \) and \( t \in (1/2, 3/2) \). We then take a function \( t : S^1 \to (1/2, 3/2) \) such that, for some small \( a > 0 \), satisfies \( t(A) \subset (1 + a, 3/2) \) and \( t(S^1 \setminus A) \subset (1 - a, 1 + a) \). Finally we define \( \varphi(\theta, x) = (\alpha(\theta), f_t(\theta)(x)) \).

We remark that \( \mathcal{D} = S^1 \times \{1/2\} \) is such that every sequence \( z_k \) converging to \( \mathcal{D} \) on \( S^1 \times I_0 \) is sent to a sequence \( \varphi(z_k) \) whose accumulation points are contained in \( S^1 \times \{0, 1\} \), which is a forward invariant subset of \( \varphi \). This implies the strong non-recurrence condition in \((H_4^+)\).

From Theorem B we have that \( \varphi \) admits an invariant probability measure \( \mu \) absolutely continuous with respect to \( \nu \times m \).

Remark 2.1 We can construct this example with \( \alpha \) a circle diffeomorphism with irrational rotation number and \( \nu \) an ergodic \( \alpha \)-invariant probability which is non-atomic and singular with respect to \( m \); see e.g. [20, Theorem 12.5.1]. We note that in this way we have a base map \( \alpha \) with no average expansion.

Example 3 We can adapt the construction in Example 2 with fibers of arbitrary dimension. We fix \( k > 1 \) in what follows.

Let again \( X \) be the circle \( S^1 \) and \( \alpha : S^1 \to S^1 \) a measurable map preserving an ergodic probability measure \( \nu \). Let now \( \theta \mapsto f_{\theta} \) be a continuous family of maps of the \( k \)-torus \( \mathbb{T}^k \) such that, as before,
on an arc \( A \) of \( \mathbb{S}^1 \) with \( v(A) \geq 1 - \epsilon \) for some small \( \epsilon > 0 \) and some Riemannian norm \( \| \cdot \| \) on \( \mathbb{T}^k \) we have:

- for \( \theta \in A \) the map \( f_\theta \) is expanding: there exists \( \sigma > 1 \) such that \( \| Df_\theta(x)^{-1} \| \leq 1/\sigma \) for all \( x \in \mathbb{T}^k \);

- for \( \theta \in \mathbb{S}^1 \setminus A \) the map \( f_\theta \) does not contract too much: there exists \( \delta > 0 \) small such that \( \| Df_\theta(x)^{-1} \| \leq 1 + \delta \) for all \( x \in \mathbb{T}^k \).

As before, in this setting, we have for \((v \times \text{Leb})\)-a.e. \((\theta, x)\) that, applying the Ergodic Theorem to the sequence \((\alpha^j(\theta))_{j \geq 0}\)

\[
\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Df_{\alpha^j(\theta)}(f_{\alpha^j(\theta)}^j(x))^{-1} \| \leq v(A) \log \sigma + v(\mathbb{S}^1 \setminus A) \log(1 + \delta)
\]

\[
\leq (1 - \epsilon) \log \sigma + \delta \epsilon < 0,
\]

where \( \text{Leb} \) is the some volume from (Lebesgue measure) on \( \mathbb{T}^k \). Since there are no criticalities or discontinuities, this shows that \( \varphi(\theta, x) = (\alpha(\theta), f(\theta, x)) \) is a non-uniformly expanding map along the fibers and we may apply Theorem \( E \) to conclude the existence of a probability measure \( \mu \) absolutely continuous with respect to \( v \times \text{Leb} \).

**Example 4** Now we adapt the previous Example 3 to have a discontinuous family of fiber maps. We repeat the construction, keeping the choice of \( f_\theta \) for \( \theta \in A \) but replacing \( f_\theta \) by the identity map in the torus for \( \theta \in \mathbb{S}^1 \setminus A \).

We still have non-uniform expansion and we note that the discontinuities of the map \( F \) are on the boundary \( \partial A \) of the arc \( A \) of the circle, which is formed by a two points on the circle. Hence condition \((H_3)\) is satisfied. We apply Theorem \( E \) to obtain a \( \varphi \)-invariant probability \( \eta \) absolutely continuous with respect to \( v \times \text{Leb} \).

**Example 5** We present an example of a \( C^\infty \) map \( T \) away from a denumerable singular set, which is non-uniformly expanding and has infinitely many ergodic absolutely continuous invariant probability measures.

On the one hand, considering \( \alpha = T \) as the base map and a constant fiber map \( f(x) = 4x(1 - x) \) of the interval which has positive Lyapunov exponents for Lebesgue almost all point, a unique critical point and negative Schwarzian derivative, we obtain a direct product \( \varphi = \alpha \times f \). The map \( f \) admits a unique ergodic absolutely continuous invariant probability measure \( \mu \). Thus we can apply our arguments to each ergodic absolutely continuous invariant probability measure \( v_k \) for \( \varphi \) to obtain \( v_k \times \mu \) as an ergodic absolutely continuous invariant probability measure for \( \varphi \). In this way \( \varphi \) has a countable set of distinct absolutely continuous invariant probability measures.

On the other hand, considering the direct product \( \varphi = \alpha \times T \) of any map \( \alpha \) of a metric space with an ergodic probability measure \( v \), with \( T \) on the fibers, we obtain an example with infinitely many ergodic absolutely continuous invariant measures \( v \times v_k \) with the same marginal \( v \).

The map \( T \) is easily described as the standard doubling map

\[
f : x \in [0, 1] \mapsto \begin{cases} 2x & \text{if } 0 \leq x < 1/2 \\ 2x - 1 & \text{if } 1/2 \leq x \leq 1 \end{cases}
\]

conveniently rescaled on the unit interval infinitely many times, as follows, see Fig. 2:

\[
T(x) := \sum_{n \geq 1} \begin{cases} \frac{1}{2^n} + \frac{1}{2^n} f\left(2^n(x - 2^{-n})\right) & \text{if } x \in \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \\ 0 & \text{otherwise} \end{cases}
\]

\( \varepsilon \) Springer
It is clear that $DT \equiv 2$ and $DT^2 \equiv 0$ outside the compact set $\mathcal{S} := \{0\} \cup \{2^{-n}, 2^{-n} + 2^{-(n+1)} : n \in \mathbb{Z}^+\}$. It is easy to see that Lebesgue measure $m$ on $[0, 1]$ is invariant and each interval $[2^{-n}, 2^{-n+1}]$ supports an ergodic component of $m$ given by the normalized restriction of $m$ to this interval.

Moreover it is straightforward to check that the set $\mathcal{S}$ satisfies conditions $(S1)$ through $(S3)$ with constants $B = \beta = 1$, so $\mathcal{S}$ is a non-degenerate singular set for $T$. In addition, conditions $(H_2)$, $(H_3)$ and $(H_4^*)$ are also easily checked.

However the slow recurrence condition is not satisfied: for each given $\delta > 0$ and $N > 1$ there exists $k > N$ such that $2^{-k+1} < \delta$ and we have

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \text{dist}_\delta(T^j(x), \mathcal{S}) \geq k > N \quad \text{for all} \quad x \in (2^{-k}, 2^{-k+1}).$$

But this condition fails in a small set: for each $N > 1$ the points for which the above inequality holds are contained in $[0, 2^{-(\log_2 N)+1}]$, where $[x]$ denotes the integer part of $x$.

2.1 Problems

We list below some open problems related with our setting of random non-uniformly expanding maps.

1. Consider the family $f_\theta(x) = a_0 + \theta - x^2$ of quadratic maps of $Y = \mathbb{R}$ as in Example 1, set $X = [-\epsilon, \epsilon]^N$ for some fixed $\epsilon > 0$ and let $\alpha : X \to X$ be the left shift map on $X$ endowed with the ergodic invariant measure $\nu = \lambda_\epsilon^N$, where $\lambda_\epsilon$ is Lebesgue measure on $[-\epsilon, \epsilon]$. Is $\varphi(\theta, x) = (\alpha(\theta), f_{\theta_1}(x))$ non-uniformly expanding for random orbits for some parameter $a_0 \in \mathbb{R}$? (Or, equivalently, is $\varphi$ non-uniformly expanding along the vertical direction?)

2. Consider $X$, $\alpha$, $\nu$ as in the previous item (1). Let $f_\theta(x) = f_1(x) + \theta \mod 1$ be a family of local diffeomorphisms of the circle, where $f_1$ is given in Example 2. Is $\varphi(\theta, x) = (\alpha(\theta), f_{\theta_1}(x))$ non-uniformly expanding for random orbits for some exponent $\beta \in (0, 1)$?

We note that according to [7] the answer is affirmative if $f_1$ is defined with exponents $\beta \geq 1$.

3. Consider $f_\theta$ as in the previous item (2). Let $\alpha(\theta) = \theta + \omega \mod 1$ (for some fixed $\omega$) be an irrational rotation with uniquely ergodic measure $\nu = \text{Lebesgue measure}$. Is $\varphi(\theta, x) = (\alpha(\theta), f_{\theta}(x))$ non-uniformly expanding along the vertical direction? What if
we consider $\alpha$ a circle diffeomorphism with irrational rotation number and $\nu$ an ergodic $\alpha$-invariant probability which is non-atomic and singular with respect to Lebesgue measure.

## 3 Basic invariant measures

We assume from now on that the skew-product map satisfies $(H_1)$, $(H_2^*)$, $(H_3)$ and $(H_4)$ (or $(H_4^*)$) in the case $\mathbb{Y} = I_0$, or it satisfies $(H_1)$, $(H_2^*)$, $(H_3)$, $(H_5)$ and $(H_6)$ in the case $\mathbb{Y}$ is other compact manifold. The condition $(H_2^*)$ is as follows

$$(H_2^*) \, \alpha : \mathbb{X} \to \mathbb{X} \text{ is a bimeasurable bijection with an ergodic invariant probability measure } \nu \text{ such that } \nu(D_\alpha) = 0 \text{ (we recall that } D_\alpha \text{ is the set of discontinuity points of } \alpha).$$

In Sect. 5 we show how to replace condition $(H_2^*)$ by $(H_2)$.

We recall that $m$ is the normalized Lebesgue measure on $I_0$. Since $\alpha$ is invertible, the functions $f_{\alpha^{-j}(\theta)}^j$ are well defined and they send $[\alpha^{-j}(\theta)] \times I_0$ on $[\theta] \times I_0$, for $\theta \in \mathbb{X}$, $j \geq 1$.

Thus, we can define the following measures on $I_0$, for every $\theta \in \mathbb{X}$ and every $n \in \mathbb{N}$,

$$
\eta_n(\theta) = \frac{1}{n} \sum_{j=1}^{n} \left( f_{\alpha^{-j}(\theta)}^j \right)_* m
$$

and using them, for every $n \in \mathbb{N}$ we define the following measures on $\mathbb{X} \times I_0$,

$$
\eta_n = \int \eta_n(\theta) \, d\nu(\theta).
$$

The integral above means that for any continuous function $g : \mathbb{X} \times I_0 \to \mathbb{R}$ we have

$$
\eta_n(g) = \int g \, d\eta_n = \int \left( \int g(\theta, x) \, d\eta_n(\theta)(x) \right) \, d\nu(\theta).
$$

We recall from Sect. 1.1 that $\mathcal{B}_\mathbb{X}$ is the Borel $\sigma$-algebra on $\mathbb{X}$. To be able to define the measure $\eta_n$, we need that for every continuous function $h : I_0 \to \mathbb{R}$ the map

$$
\theta \mapsto \eta_n(\theta)(h) = \int h \, d\eta_n(\theta)
$$

is measurable. This is proved in “Appendix”.

Assuming that these measures are all well-defined, we can easily prove some key properties of the accumulation points of $(\eta_n)_{n \geq 1}$.

**Lemma 3.1** For every probability measure $\eta$ which is a weak* limit of $(\eta_n)_{n \geq 1}$ we have that $\eta_n(A \times I_0) = \nu(A)$ for each $n \geq 1$ and $\eta(A \times I_0) = \nu(A)$, for all $A \in \mathcal{B}_\mathbb{X}$.

**Proof** We fix $A$ and $\eta$ as in the statement. Then we have for all $n \in \mathbb{Z}^+$ by definition $\eta_n(A \times I_0) = \int_A \eta_n(\theta)(I_0) \, d\nu(\theta) = \nu(A)$. If we take $A \in \mathcal{B}_\mathbb{X}$ such that $\eta(\delta(A \times I_0)) = \eta((\delta A) \times I_0) = 0$, then using $\eta_n \xrightarrow{k \to +\infty} \eta$ we get $\eta(A \times I_0) = \nu(A)$. Since the family of these sets generates $\mathcal{B}_\mathbb{X}$ modulo $\eta$-null sets, we are done. \qed

**Lemma 3.2** For every probability measure $\eta$ which is a weak* limit of $(\eta_n)_{n \geq 1}$ we have that $\eta(D) = 0$, where $D$ is the set of discontinuity points of $\varphi$.

**Proof** We consider the following cases.
Case 1 The maps $f_\theta$ are $C^3$ for all $\theta \in \mathbb{X}$, that is, there are no discontinuities along the vertical direction: $\mathcal{D}_\theta = \emptyset$ for all $\theta \in \mathbb{X}$. Thus, it holds that $\mathcal{D} \subset (\mathcal{D}_a \times I_0) \cup (\mathcal{D}_F \times I_0)$. Then we have, by Lemma 3.1, that $\eta(\mathcal{D}) \leq \eta(\mathcal{D}_a \times I_0) + \eta(\mathcal{D}_F \times I_0) \leq \nu(\mathcal{D}_a) + \nu(\mathcal{D}_F) = 0$ by $(H^3)$ and $(H_3)$.

Case 2 We have discontinuities $\mathcal{D}_\theta \neq \emptyset$ for some $\theta \in \mathbb{X}$. But we assume that there is no recurrence to the set $\mathcal{D} = \{(\theta, x) : x \in \mathcal{D}_\theta, \theta \in \mathbb{X}\}$; Sect. 1.1. see condition $(H^3)$. Hence for every given $n \in \mathbb{Z}^+$ we can find an open neighborhood $V = V_\ell$ of $\mathcal{D}$ in $\mathbb{X} \times I_0$ such that $\varphi^k(V) \cap V = \emptyset$ for all $k = 1, \ldots, \ell$. This implies that for any $z \in \mathbb{X} \times I_0$ we have $\sum_{j=1}^n \chi_{V_l}(\varphi^j(z)) > (n/\ell) + 1$. Thus, since $\eta(V) \leq \liminf_{n \to +\infty} \eta_n(V)$ (see e.g. [9, Theorem 2.1]), it is enough to estimate for every big enough $n \in \mathbb{Z}^+$, using that $v$ is $\alpha$-invariant and that $\alpha$ is invertible

$$\eta_n(V) = \int \int \frac{1}{n} \sum_{j=1}^n \chi_{V}(\varphi^j(\alpha^{-j}(\theta), x)) \, d\nu(\theta) \, d\nu(\theta)$$

$$= \int \int \frac{1}{n} \sum_{j=1}^n \chi_{V}(\varphi^j(\alpha^{-j}(\theta), x)) \, d\nu(\theta)$$

$$= \int \int \frac{1}{n} \sum_{j=1}^n \chi_{V}(\varphi^j(\theta, x)) \, d\nu(\theta) \leq \frac{2}{\ell}.$$ 

So for every $\ell > 1$ we can find and open neighborhood $V$ of $\mathcal{D}$ such that $\eta(V) \leq 2/\ell$. Finally, since $\mathcal{D} \subset (\mathcal{D}_a \times I_0) \cup (\mathcal{D}_F \times I_0) \cup \mathcal{D}$ we obtain from the above together with Lemma 3.1

$$\eta(\mathcal{D}) \leq \eta(\mathcal{D}_a \times I_0) + \eta(\mathcal{D}_F \times I_0) + \eta(\mathcal{D}) = \nu(\mathcal{D}_a) + \nu(\mathcal{D}_F) = 0$$

as stated.

Case 3 In the higher dimensional setting, we have slow recurrence to the set of discontinuities $\mathcal{D} \subset \mathcal{F}$ of $\varphi$ in the vertical direction. Arguing by contradiction, let us assume that $\eta(\mathcal{D}) > 0$. Then there exists $\alpha > 0$ such that $\eta(B(\mathcal{D}, \varphi)) > \alpha$ for all $\varphi > 0$. We fix $0 < \varepsilon < a$ and then find $\delta > 0$ given by the slow recurrence condition (1.5). After that we fix $0 < \varepsilon < \delta$ so that

$$\inf\{ - \log \text{dist}((\theta, x), \mathcal{D}) : (\theta, x) \in B(\mathcal{D}, \varphi) \} > 1 \quad \text{and} \quad \eta(\partial B(\mathcal{D}, \varphi)) = 0, \quad n \geq 1$$

and also $\eta(\partial B(\mathcal{D}, \varphi)) = 0$. Then we note that, for each $n \geq 1$, since $\nu$ is $\alpha$-invariant

$$a < \eta_n(B(\mathcal{D}, \varphi)) = \int \int \frac{1}{n} \sum_{j=1}^n \chi_{B(\mathcal{D}, \varphi)}(\varphi^j(\alpha^{-j}(\theta), x)) \, d\text{Leb}(\theta) \, d\nu(\theta)$$

$$= \int \int \frac{1}{n} \sum_{j=1}^n \chi_{B(\mathcal{D}, \varphi)}(\varphi^j(\alpha^{-j}(\theta), x)) \, d\text{Leb}(\theta) \, d\nu(\theta)$$

$$= \int \int \frac{1}{n} \sum_{j=1}^n \chi_{B(\mathcal{D}, \varphi)}(\varphi^j(\theta, x)) \, d\text{Leb}(\theta) \, d\nu(\theta)$$

$$\leq \int \int \frac{1}{n} \sum_{j=1}^n - \log \text{dist}_\delta \left( f^j_{\theta}(x), c_n \right) \, d\text{Leb}(\theta) \, d\nu(\theta).$$
Moreover, for big enough $n$ we get $\varepsilon > \eta_n(B(\mathcal{D}, \varrho)) \geq a$ thus $a < \varepsilon$. This contradiction concludes the proof, since $D \subseteq (\mathcal{D}_a \times I_0) \cup (\mathcal{D}_F \times I_0) \cup \mathcal{D}$ as in Case 2. \hfill $\Box$

**Lemma 3.3** Every weak$^*$ limit of $(\eta_n)_{n \geq 1}$ is a $\varphi$-invariant probability measure.

*Proof* Let us suppose, without loss of generality, that the sequence converges in the weak$^*$ topology to some probability measure, i.e., $\eta_n \to \eta$ when $n \to \infty$. See Lemma 4.6 and Remark 4.7.

Let $g : X \times I_0 \to \mathbb{R}$ be a continuous and bounded function. We note that $\eta_n(g \circ \varphi)$ can be rewritten as

$$\int \int g(\varphi(\theta, x)) \, d\eta_n(\theta)(x) \, d\nu(\theta) = \int \left( \frac{1}{n} \sum_{j=1}^{n} (f_{\alpha^{-1}(\theta)} \circ \cdots \circ f_{\alpha^{-j}(\theta)}) \ast m \right) g(\circ \varphi) \, d\nu(\theta)$$

$$= \frac{1}{n} \sum_{j=1}^{n} \int \int g(\alpha(\theta), f_{\theta}(f_{\alpha^{-1}(\theta)} \circ \cdots \circ f_{\alpha^{-j}(\theta)}(x))) \, d m(x) \, d\nu(\theta).$$

But the last integral equals

$$\int \int \frac{1}{n} \left( \int \sum_{j=1}^{n+1} g(\alpha(\theta), (f_{\theta} \circ f_{\alpha^{-1}(\theta)} \circ \cdots \circ f_{\alpha^{-j+1}(\theta)})(x)) \, d m(x) \right) \, d\nu(\theta)$$

that is $\int \left( \frac{n+1}{n} \eta_{n+1}(\alpha(\theta))(g) - \frac{1}{n} ((f_{\theta}) \ast m)(g(\alpha(\theta), \cdot)) \right) \, d\nu(\theta).$ We note that the last integral is bounded by $\sup |g|$, which is finite.

Now since $\eta(D) = 0$ by Lemma 3.2, we then arrive at (see e.g. [9, Theorem 2.1])

$$(\varphi \ast \eta)g = \lim_{n \to \infty} \eta_n(g \circ \varphi) = \lim_{n \to \infty} \int \frac{n+1}{n} \eta_{n+1}(\alpha(\theta))(g) \, d\nu(\theta).$$

But $\nu$ is $\alpha$-invariant and the function $\theta \mapsto \eta_{n+1}(\alpha(\theta))(g)$ is measurable, hence the last expression equals

$$\lim_{n \to \infty} \int \frac{n+1}{n} \eta_{n+1}(\theta)(g) \, d\nu(\theta) = \lim_{n \to \infty} \frac{n+1}{n} \eta_{n+1}(g) = \eta(g).$$

This concludes the proof. \hfill $\Box$

### 4 Absolutely continuous invariant measures

Now we are going to define measures which are absolutely continuous along the vertical fibers. For this, we will use the notion of hyperbolic-like times used in [31].

#### 4.1 Notations and main technical result

We state a result for sequences of one dimensional maps. This result is used to analyze the dynamics of the skew-product restricted to the vertical leaves. Since we have to consider
skew-products in the different settings \((H_4)\) and \((H^+_4)\), we also need to state the result for sequences of one dimensional maps with conditions given by these two settings. For \(k \geq 0\), let us denote by \(\mathcal{C}_k\) and \(\mathcal{D}_k\) the set of critical points and the set of discontinuities, respectively, of \(f_k : I_0 \rightarrow I_0\).

We say that:

- a sequence of one dimensional maps \(\{f_k\}\) satisfies \(\widetilde{H}_4\) if: \(f_k\) are \(C^1\) maps, \(p := \sup\{\#C_k, k \in \mathbb{N}\} < \infty\) and \(\Gamma := \sup\{|f'_k(x)|, k \in \mathbb{N}, x \in I_0\} < \infty\) and the sequence \(\{f'_k\}\) is equicontinuous.

- a sequence of one dimensional maps \(\{f_k\}\) satisfies \(\widetilde{H}^+_4\) if: \(f_k\) is a map such that restricted to each connected component of \(I_0 \setminus \mathcal{D}_k\), is a \(C^1\) diffeomorphism onto its image, \(p := \sup\{|f'_k(x)|, k \in \mathbb{N}, x \notin \mathcal{D}_k\} < \infty\).

Finally we assume that for every \(\ell \in \mathbb{Z}^+\), there exist \(\epsilon > 0\) and neighborhoods \(V_\epsilon \mathcal{D}_k\) of \(\mathcal{D}_k\) (for all \(k \geq 0\)) such that

\[
\left| f^{j}_i\right| (V_\varepsilon \mathcal{D}_i) \cap V_\varepsilon \mathcal{D}_{i+j} = \emptyset \quad \text{for } i \geq 0, 1 \leq j \leq \ell. \tag{4.1}
\]

where \(f^{j}_i = f_{i+j-1} \circ \cdots \circ f_{i+1} \circ f_{i}\).

Let us recall some additional definitions (see [31] for more details). For every \(x \in I_0\), \(i \in \mathbb{N}\), we denote

\[
f^i(x) := f_{i-1} \circ \cdots \circ f_1 \circ f_0(x),
\]

and we write \(T_i(\{f_k\}, x)\) for the maximal interval \(T \subset I_0\), containing \(x\) such that \(f^{i}_i(T)\) is a \(C^3\) diffeomorphism, and \(r_i(\{f_k\}, x)\) for the minimum between the lengths of the connected components of \(f^i(T_i(\{f_k\}, x) \setminus \{x\})\).

The following is a central technical result in our arguments. For the proof see Sect. 4.3.

**Theorem 4.1** Let \(\{f_k\}\) be a sequence of maps \(f_k : I_0 \rightarrow I_0\) which satisfies \(\widetilde{H}_4\) (or \(\widetilde{H}^+_4\)). Assume that for some \(\lambda > 0\),

\[
\liminf_{n \rightarrow \infty} \frac{1}{n} \log |Df^n(x)| > 2\lambda
\]

for every \(x \in E \subset I_0\). Then, there exists \(\varsigma > 0\) such that

\[
\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} r_i(\{f_k\}, x) \geq 3\varsigma \tag{4.2}
\]

Lebesgue almost every \(x \in E\). Moreover, in the case of \(\{f_k\}\) satisfy condition \(\widetilde{H}_4\), \(\varsigma\) depends only on \(\lambda\), the modulus of continuity and the uniform bound for the derivatives of the sequence \(\{f_k\}\), and in the uniform bound \(p\) for the number of critical points. In the case of \(\{f_k\}\) satisfy condition \(H^+_4\), \(\varsigma\) depends only on \(\lambda\), the uniform bound for the derivatives of the sequence \(\{f_k\}\) (outside of discontinuities), the uniform bound \(p\) for the number of discontinuity points and the uniformity of \(\epsilon\) on condition (4.1).

For our purposes the following sets are very useful:

\[
\mathcal{H}_i(\{f_k\}, \sigma) = \{ x \in I_0; r_i(\{f_k\}, x) > \sigma \};
\]

\[
\mathcal{H}_i(\{f_k\}, \sigma) = \{ x \in I_0; r_i(\{f_k\}, x) > \sigma \text{ and } |f^i(T_i(\{f_k\}, x))| > 3\sigma \}.
\]

We will prove below that every connected component of \(\mathcal{H}_i(\{f_k\}, \sigma)\) is sent diffeomorphically by \(f^i\) onto its image with bounded distortion and the Lebesgue measure of the image
is bounded away from zero. We are interested in applying the last theorem to every sequence \( \{f_{a_i(\theta)}\}_{j \in \mathbb{Z}^+} \), for each \( \theta \in \mathbb{X} \). For simplicity, from now on, for \( i \in \mathbb{N}, r_i(\theta, x) \) denotes the set \( r_i(\{f_k\}, x) \), where \( f_k = f_{a^k(\theta)} \) for every \( k \geq 0, \theta \in \mathbb{X} \). Analogously for the sets \( T_i(\theta, x), \mathcal{H}_i(\theta, \sigma) \) and \( H_i(\theta, \sigma) \).

We need the following result showing that \( (\tilde{H}_4^*) \) is a consequence of \( (H_4^*) \).

**Lemma 4.2** The above condition (4.1) is a consequence of the assumption \( (H_4^*) \).

**Proof** We fix \( \ell \in \mathbb{Z}^+ \) and \( V \) given by \( (H_4^*) \). Consider also \( i \geq 0 \) and \( 1 \leq j \leq \ell \). We note that, by the skew-product nature of \( \varphi \)

\[
\varphi^j(V \cap (\{\alpha^j(\theta)\} \times I_0)) \subseteq (\{\alpha^{i+j}(\theta)\} \times I_0) \cap \varphi^j(V).
\]

We now observe that the intersection in (4.1) equals

\[
\pi_2\left(\varphi^j(V \cap (\{\alpha^j(\theta)\} \times I_0)) \cap (\{\alpha^{i+j}(\theta)\} \times I_0)\right) \subseteq \pi_2((\{\alpha^{i+j}(\theta)\} \times I_0) \cap \varphi^j(V \cap V) = \emptyset,
\]

where \( \pi_2 : \mathbb{X} \times I_0 \to I_0 \) is the projection on the second coordinate. So we can use the neighborhoods \( V \) given by \( (H_4^*) \) to obtain the neighborhoods \( V_e \mathcal{D}_i \) in (4.1).

**Remark 4.3** The fact that \( \epsilon > 0 \) does not depend on the sequence of maps chosen relies on the choice in \( (H_4^*) \) of the neighborhood \( V \) of the closure \( \mathcal{D} \) of the set of discontinuities in \( \mathbb{X} \times I_0 \).

We need the following result in the rest of the arguments.

**Lemma 4.4** [Pliss] Given \( A \geq c_2 > c_1 > 0 \), let \( \zeta = (c_2 - c_1)/(A - c_1) \). Then, given any real numbers \( a_1, \ldots, a_N \) such that

\[
\sum_{j=1}^N a_j \geq c_2 N \quad \text{and} \quad a_j \leq A \text{ for every } 1 \leq j \leq N,
\]

there are \( l > \zeta N \) and \( 1 < n_1 < \cdots < n_l \leq N \) so that

\[
\sum_{j=n_l+1}^{n_i} a_j \geq c_1(n_i - n) \text{ for every } 0 \leq n < n_i \text{ and } i = 1, \ldots, l.
\]

**Proof** See [24, Lemma 11.8].

Thus, by the last theorem and using the Lemma of Pliss, we have the following.

**Corollary 4.5** Let \( \varphi : \mathbb{X} \times I_0 \to \mathbb{X} \times I_0 \) be a skew-product as above satisfying \( (H_1) \) and \( (H_4) \) (or \( (H_1) \) and \( (H_4^*) \)). Assume that there exists a set \( E \subseteq \mathbb{X} \times I_0 \) and \( \lambda > 0 \) such that

\[
\liminf_{n \to \infty} \frac{1}{n} \log |Df_\theta^n(x)| > 2\lambda
\]

for every \( (\theta, x) \in E \) and let us denote by \( E(\theta) \) the \( \theta \)-section of the set \( E \), that is, \( E(\theta) = \{x \in I_0 : (\theta, x) \in E\} \). Then there exist \( \zeta > 0 \) and \( \xi > 0 \) such that for \( n \) big enough do not depend on \( \theta \)

\[
\int \frac{1}{n} \sum_{i=1}^{n} m(\mathcal{H}_i(\theta, \zeta) \cap E(\theta)) \, d\nu(\theta) \geq \frac{\zeta}{2} (v \times m)(E).
\]
Proof Let us fix $\theta \in \mathbb{X}$ and consider the sequence $\{f_{\alpha^j(\theta)}\}_{j \in \mathbb{Z}^+}$. Let $\zeta > 0$ be the constant found on Theorem 4.1. We consider the measure $\pi_n \in \{1, 2, \ldots, n\}$ defined by $\pi_n(B) = \#(B)/n$, for every subset $B$. Using Fubini's theorem, we have

$$
\frac{1}{n} \sum_{i=1}^{n} m(H_i(\theta), \zeta) \cap E(\theta)) = \int \int_{I_0} \chi(x, i) d m(x) d \pi_n(i) = \int \int_{I_0} \chi(x, i) d \pi_n(i) d m(x)
$$

where $\chi(x, i) = 1$ if $x \in H_i(\theta, \zeta) \cap E(\theta)$ and $\chi(x, i) = 0$ otherwise. Applying Pliss Lemma 4.4, we conclude the existence of $\zeta > 0$ such that $\int \chi(x, i) d \pi_n(i) \geq \zeta$ if $x$ is such that $x \in E(\theta)$ and $\sum_{i=1}^{n} r_i(\theta, x) \geq 2\zeta n$. Hence

$$
\frac{1}{n} \sum_{i=1}^{n} m(H_i(\theta, \zeta) \cap E(\theta)) \geq \zeta \left\{ \left\{ x \in E(\theta) : \sum_{i=1}^{n} r_i(\theta, x) \geq 2\zeta n \right\} \right\}
$$

By Theorem 4.1, we have that

$$
m\left( \left\{ x \in E(\theta) : \sum_{i=1}^{n} r_i(\theta, x) \geq 2\zeta n, \text{ for all } n \geq N \right\} \right) \rightarrow m(E(\theta))
$$

when $N \rightarrow \infty$. Since the constant $\zeta$ is the same for any sequence $\{f_{\alpha^j(\theta)}\}_{j \in \mathbb{Z}^+}$, varying $\theta \in \mathbb{X}$, the result follows using the Dominated Convergence Theorem. \hfill $\square$

For any $\sigma > 0$, if $r_i(\{f_k\}, x) > 2\sigma$ then $|f^i(T_i(\{f_k\}, x))| > 4\sigma$. Thus, $H_i(\{f_k\}, 2\sigma) \subset H_i(\{f_k\}, \sigma) \subset H_i(\{f_k\}, \sigma)$. Therefore, we get a similar result to the last corollary for $H_i$ instead of $H_i$.

4.2 Construction of absolutely continuous invariant probability measures

Assume that we are in the conditions of Theorem B. Clearly, the set $Z$ in the statement of Theorems B and A may be taken positively invariant under $\varphi$. Given any $\lambda > 0$, let $Z(\lambda)$ be the set of points in $Z$ for which the inequality (1.3) holds. Then $Z(\lambda)$ is positively invariant. As usual, $Z(\theta, \lambda)$ denotes the $\theta$-section of the set $Z(\lambda)$.

Let us fix a constant $\lambda > 0$. Let $\zeta > 0$ be the constant found on Theorem 4.1. Thus, we define the following measures on $I_0$, for every $n \in \mathbb{N}$ and $\theta \in \mathbb{X}$

$$
\mu_n(\theta) = \frac{1}{n} \sum_{j=1}^{n} (f_{\alpha^{-j}(\theta)})_n \left( m \mid Z(\alpha^{-j}(\theta), \lambda) \cap H_j(\alpha^{-j}(\theta), \zeta) \right). \tag{4.3}
$$

Using these measures, for every $n \in \mathbb{N}$ we define the following on $\mathbb{X} \times I_0$,

$$
\mu_n = \int \mu_n(\theta) d v(\theta) \tag{4.4}
$$

Again we need to show that for every continuous function $h : I_0 \rightarrow \mathbb{R}$ the map

$$
\theta \mapsto \mu_n(\theta)(h) = \int h d \mu_n(\theta)
$$

is measurable. This is proved in “Appendix”.

Lemma 4.6 For all $n \geq 1$ and $A \in B_{\mathbb{X}}$ we have $\mu_n(A \times I_0) \leq v(A)$. Moreover, this conditions ensures that the sequence $(\mu_n)_{n \geq 1}$ of measures is tight in $\mathbb{X} \times I_0$; thus it is relatively compact in the weak* topology of measures in $\mathbb{X} \times I_0$.
Proof We just observe that \( \mu_n(A \times I_0) = \int_A \mu_n(\theta)(I_0) \, dv(\theta) \) by definition, and also \( \mu_n(\theta)(I_0) \leq 1 \) for each \( \theta \). In addition, from this property and the assumption that \( \mu_n \) are Borel measures on \( X \) which is a separable metrizable and complete topological space, given \( \epsilon > 0 \) we can fix a compact subset \( X_0 \subset X \) such that \( v(X \setminus X_0) < \epsilon \) and we obtain
\[
\mu_n(X \times I_0 \setminus (X_0 \times I_0)) = \mu_n((X \setminus X_0) \times I_0) \leq v(X \setminus X_0) < \epsilon
\]
uniformly in \( n \geq 1 \), as required for tightness of the family \( (\mu_n)_{n \geq 1} \). We can now apply Prokhorov’s Theorem to obtain the final conclusion of the statement of the lemma; see [9, Chapter 1, Section 5].

Remark 4.7 Lemma 3.1 together with the previous arguments also shows that \( (\eta_n)_{n \geq 1} \) is a tight sequence of probability measures in \( X \times I_0 \).

Now we obtain the absolute continuity of \( \mu_n(\theta) \) with respect to \( m \).

Lemma 4.8 There exists \( K > 0 \) such that for any measurable subset \( A \subset I_0 \),
\[
\mu_n(\theta)(A) \leq K \, m(A)
\]
for every \( \theta \in X, n \in \mathbb{N} \). Moreover, \( K \) depends only on the constant \( \zeta \) in the definition of \( H \).\( \Box \)

Proof Let \( J \) be a connected component of \( H(\alpha^{-j}(\theta), \zeta) \). Let us consider the maximal interval \( T \), containing \( J \), such that \( f^j_{\alpha^{-j}(\theta)} \) restricted to \( T \) is a \( C^3 \) diffeomorphism. There exists \( \tau \), depending only on \( \zeta \), such that \( f^j_{\alpha^{-j}(\theta)}(T) \) contains a \( \tau \)-scaled neighborhood of \( f^j_{\alpha^{-j}(\theta)}(J) \) (i.e., both connected components of \( f^j_{\alpha^{-j}(\theta)}(T) \backslash f^j_{\alpha^{-j}(\theta)}(J) \) have length \( \geq \tau |f^j_{\alpha^{-j}(\theta)}(J)| \)). By Koebe Principle (see [14, Theorem IV.1.2]), \( f^j_{\alpha^{-j}(\theta)} \) restricted to \( J \) has bounded distortion (by a constant \( K' \) depending only on \( \zeta \)). On the other hand, \( |f^j_{\alpha^{-j}(\theta)}(J)| \) is bounded away from zero. Thus, we conclude that \( (f^j_{\alpha^{-j}(\theta)})_*(m | Z(\alpha^{-j}(\theta), \lambda) \cap H_j(\alpha^{-j}(\theta), \zeta))(A) \leq K \, m(A) \) for any measurable set \( A \subset I_0 \).\( \Box \)

From the previous lemma we deduce the absolute continuity of \( \mu_n \) with respect to \( v \times m \).

Lemma 4.9 Let \( K > 0 \) be as in Lemma 4.8. Then for any \( W \in \mathcal{B}_X \times \mathcal{B}_{I_0} \) we have \( \mu_n(W) \leq K \cdot (v \times m)(W) \) for all \( n \in \mathbb{N} \).

Proof The set \( \mathcal{A} = \{W \in \mathcal{B}_X \times \mathcal{B}_{I_0}; \mu_n(W) \leq K \cdot (v \times m)(W)\} \) is a \( \sigma \)-algebra. On the other hand, if \( W = F \times A \) for some \( F \in \mathcal{B}_X, A \in \mathcal{B}_{I_0} \), we conclude, from the definition of \( \mu_n \) and the last claim, that \( W \in \mathcal{A} \). This is enough to conclude the proof.\( \Box \)

Now we extend the results of the previous lemmas to the cluster points of the sequence \( \mu_n \) in the weak* topology.

Lemma 4.10 Let \( K > 0 \) be as in Lemma 4.8. Then for any weak* limit \( \mu \) of \( \{\mu_n\} \), we have \( \mu(W) \leq K \cdot (v \times m)(W) \) for any \( W \in \mathcal{B}_X \times \mathcal{B}_{I_0} \).

Proof The set \( \mathcal{A} = \{W \in \mathcal{B}_X \times \mathcal{B}_{I_0}; \mu(W) \leq K \cdot (v \times m)(W)\} \) is a \( \sigma \)-algebra. Since \( \mu_{n_k} \) converges in the weak* topology to \( \mu \), \( \mu(W) \leq \liminf \mu_{n_k}(W) \) for any open set \( W \). Also note that from Lemma 4.9, for open sets \( W \in \mathcal{B}_X \times \mathcal{B}_{I_0} \), \( \mu_n(W) \leq K \cdot (v \times m)(W) \) for all \( n \in \mathbb{N} \).

As these sets generate \( \mathcal{B}_X \times \mathcal{B}_{I_0} \), the claim follows.\( \Box \)
By definition $\mu_n$ is a part of the measure $\eta_n$, for all $n \in \mathbb{N}$. Let $\xi_n$ be a measure such that
\begin{equation}
\eta_n = \mu_n + \xi_n \tag{4.5}
\end{equation}
for all $n \in \mathbb{N}$. From Lemma 4.6 and Remark 4.7 we assume, without loss of generality, that there exist some subsequence $\{\eta_k\}_k$ and measures $\eta$, $\mu$, $\xi$ such that $\eta_n, \mu_{n_k}, \xi_{n_k}$ converge to $\eta$, $\mu$, $\xi$, when $k \to \infty$, respectively. We then have
\begin{equation}
\eta = \mu + \xi. \tag{4.6}
\end{equation}

Let $\beta_1$ and $\beta_2$ be measures on the same measurable space. As usually, if $\beta_1$ is absolutely continuous with respect to $\beta_2$, we write $\beta_1 \ll \beta_2$; and if $\beta_1$ is singular with respect to $\beta_2$, we write $\beta_1 \perp \beta_2$.

Next we show that the Lebesgue decomposition of an invariant measure with respect to any finite measure $\sigma$, for a non-singular transformation, is formed by invariant measures.

**Lemma 4.11** Let us assume that a measurable transformation $T : (X, \mathcal{X}) \to (X, \mathcal{X})$ satisfies $T_*\sigma \ll \sigma$ for some finite measure $\sigma$ in $(X, \mathcal{X})$ (that is, $T$ is non-singular with respect to $\sigma$). We assume also that a $T$-invariant probability measure $\eta$ is given with Lebesgue decomposition $\eta = \mu + \xi$, with $\mu \ll \sigma$ and $\xi \perp \sigma$. Then both $\mu$ and $\xi$ are $T$-invariant measures.

**Proof** Since $\xi \perp \sigma$, we may find $E \in \mathcal{X}$ such that $\sigma(E) = 0$ and $\xi(X \setminus E) = 0$. In particular, $\xi(A) = \xi(A \cap E)$ for all $A \in \mathcal{X}$. Because $\sigma(E) = 0 = \sigma(T^{-1}(E))$ we get
\begin{equation}
\xi(T^{-1}(E)) = \xi(T^{-1}E) + \xi(T^{-1}(E)) = \eta(T^{-1}(E)) = \eta(E) = \mu(E) + \xi(E) = \xi(E)
\end{equation}
and $E$ is $T$-invariant $\xi \mod 0$, i.e., $\xi(E \bigtriangleup T^{-1}(E)) = 0$. Hence $\xi(X \setminus T^{-1}(E)) = 0$ and $\sigma(T^{-1}(E)) = 0$. Thus for $A \in \mathcal{X}$
\begin{equation}
\xi(T^{-1}(A)) = \xi(T^{-1}(A) \cap T^{-1}(E)) = \xi(T^{-1}(A \cap E)) = \xi(A \cap E) = \xi(A)
\end{equation}
since $\xi(T^{-1}(A \cap E)) = \eta(T^{-1}(A \cap E)) = \eta(A \cap E) = \xi(A \cap E)$. We have proved that $\xi$ is $T$-invariant.

Therefore
\begin{equation}
\mu + \xi = \eta = T_*\eta = T_*\mu + T_*\xi = T_*\mu + \xi
\end{equation}
shows that $T_*\mu = \mu$ and $\mu$ is also $T$-invariant \hfill $\square$

4.2.1 Existence of absolutely continuous invariant probability measure

Now we use the previous results to complete the proof of existence of an absolutely continuous invariant probability measure for $\varphi$. The ergodicity is proved in Sect. 6.

**Proposition 4.12** Let $\varphi : \mathbb{X} \times I_0 \to \mathbb{X} \times I_0$ be a skew-product as above satisfying $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$ (or $(H_4^*)$). Assume that $\nu \times \mathfrak{m}(Z(\lambda)) > 0$. Then there exists an absolutely continuous invariant probability measure which gives positive mass to $Z(\lambda)$.

**Proof** Let us consider the measures $\mu_n$ given by (4.4). We recall that by (4.5) and (4.6) we have $\eta = \mu + \xi$ with $\mu \ll \nu \times \mathfrak{m}$. By the Lebesgue Decomposition Theorem, there exist (unique) measures $\xi^{ac}$ and $\xi^{s}$ such that $\xi^{ac} \ll \nu \times \mathfrak{m}$, $\xi^{s} \perp \nu \times \mathfrak{m}$ and $\xi = \xi^{ac} + \xi^{s}$. Then we have a decomposition of $\eta = (\mu + \xi^{ac}) + \xi^{s}$ as a sum of one absolutely continuous measure and a singular one (both with respect to $\nu \times \mathfrak{m}$). On the other hand, notice that
Absolutely continuous invariant measures

ϕ∗(ν × m) ≪ ν × m (it follows from the invariance of ν and by the non-singularity of f(θ, ·), for every θ ∈ X).

The previous Lemma 4.11 ensures that μ + ξac is an absolutely continuous ϕ-invariant measure.

We claim that μ + ξac is a non-zero finite measure. It suffices to prove that there exists γ > 0 such that, μn(X × I0) > γ for all n big enough. Using that α−1 is invariant by ν and defining the family sj(θ) := m(Hj(θ, ς) ∩ Z(θ, λ)) of measurable functions for j ≥ 1, we have for all n ∈ Z+,

μn(X × I0) = \int X \left( \frac{1}{n} \sum_{j=1}^{n} m(H_j(\alpha^{-j}(\theta), \varsigma) \cap Z(\alpha^{-j}(\theta), \lambda)) \right) d\nu(\theta)

= \frac{1}{n} \sum_{j=1}^{n} \int X s_j(\theta) d\nu(\theta) = \frac{1}{n} \sum_{j=1}^{n} \int X sj(\theta) d\nu(\theta)

= \frac{1}{n} \sum_{j=1}^{n} m(H_j(\theta, \varsigma) \cap Z(\theta, \lambda)) d\nu(\theta).

By Corollary 4.5 this last integral is bounded away from zero, as long as the set Z(λ) has positive ν × m-measure. More precisely, we have μn(Z(λ)) ≥ ζ_2(ν × m)(Z(λ)) for all big enough n. Hence the normalization of μ + ξac satisfies the conditions of the statement. □

4.3 Proof of the technical result

Here we present a proof of Theorem 4.1. The proof in the setting (H_4^n) is similar to the proof on the setting (H_4). The result on the setting (H_4^n) corresponds to Theorem B in [31], but here we do not assume the equicontinuity of the sequence \{f_k\}. For completeness, we prove the result on the setting (H_4^n) and we remark the modifications for the proof on the setting (H_4).

4.3.1 Definitions and fundamental lemmas

In order to simplify the notation we say that \(f^j(x) \in V_\epsilon \varnothing\) if \(f^j(x) \in V_\epsilon \varnothing_j\) for \(j \in \mathbb{N}\). By the recurrence property on the setting (H_4^n) [see Eq. 4.1] we have that

Lemma 4.13 Given γ > 0, there exists ε > 0 such that for n big enough,

\[
\frac{1}{n} \sum_{j=0}^{n-1} \chi_{V_\epsilon \varnothing}(f^j(x)) < \gamma
\] (4.7)

for any \(x \in I_0\).

Remark 4.14 Note that the lemma also holds on setting (H_4). In this case, (4.7) holds for any x such that \(\log |Df^n(x)| > \lambda n\) and ε depends on \(\lambda\). We use the equicontinuity of the sequence \(f'_k\) instead of condition (4.1).

On the other hand, since the derivative of the maps of the sequence \(f_k\) is bounded from above outside of the set of discontinuities, it holds the following result.
Lemma 4.15 Given $\epsilon > 0$ and $l \in \mathbb{N}$, there exists $\delta > 0$ such that for any subinterval $J \subset I_0$,

$$\text{if } |J| \leq 2\delta \text{ and } f_i^j(J) \cap \mathcal{D}_{i+j} = \emptyset \text{ for } 0 \leq j < k \text{ then } |f_i^k(J)| < \epsilon$$

for all $i \geq 0$ and $0 < k \leq l$.

Proof Let us consider $\Gamma = \sup \{|Df_k(x)|; k \in \mathbb{N}, x \notin \mathcal{D}_k\}$. The lemma follows from the next claim: for all $i, k \geq 0$, for any interval $J \subset I_0$, if $f_i^j(J) \cap \mathcal{D}_{i+j} = \emptyset$ for $0 \leq j < k$, then $|f_i^k(J)| \leq \Gamma^k |J|$.

Remark 4.16 Notice that this lemma also holds on the setting $(\tilde{\mathcal{H}}_k)$, replacing the set $\mathcal{D}$ by $\mathcal{C}$.

The main part in the proof of Theorem 4.1 is the control of the Lebesgue measure of the sets $Y_n(\lambda) \cap A_n ([f_k], \delta)$, where $Y_n(\lambda) = \{x, \log |Df^n(x)| > \lambda n\}$ and

$$A_n(\delta) = A_n ([f_k], \delta) := \left\{ x \in I_0 : \frac{1}{n} \sum_{i=1}^{n} r_i([f_k], x) < \delta^2, \quad r_n([f_k], x) > 0 \right\},$$

for $n \in \mathbb{N}$ and $\delta > 0$ (and $r_i$ as was defined just before the statement of Theorem 4.1). For simplicity, we denote by $A_n(\delta)$ the set $A_n ([f_k], \delta)$ and by $r_i(x)$ the number $r_i([f_k], x)$.

We introduce the following sets.

For $\delta > 0, a_i \in \{0, 1\}$ for $i = 1, 2, \ldots, n$,

$$C_\delta(a_1, a_2, \ldots, a_n) := \{ x \in I_0 : r_i(x) \geq \delta \text{ if } a_i = 1, \quad 0 < r_i(x) < \delta \text{ if } a_i = 0 \}$$

Note that for every $x \in A_n(\delta)$, there exist $a_1, \ldots, a_n$ (with $a_i \in \{0, 1\}$ for $i = 1, \ldots, n$) and a component $C_\delta(a_1, a_2, \ldots, a_n)$ such that $x \in J$.

The key lemma in the proof of Theorem 4.1 is the following. Let $\#X$ denotes the number of connected components of $X$.

Lemma 4.17 Given $\lambda > 0$, there exists $\delta > 0$ such that $\sum \#C_\delta(a_1, \ldots, a_n) \leq \exp(n\lambda/2)$, where the sum is over all $a_1, \ldots, a_n$ such that $a_1 + a_2 + \cdots + a_n < \delta n$. Moreover, the dependence of $\delta$ is as $\zeta$ on the statement of Theorem 4.1.

We need to decompose the interval $I_0$ set in a convenient way. Given $\epsilon > 0, m \leq n$, $\{t_1, \ldots, t_m\} \subset \{0, 1, \ldots, n - 1\}$, we define

$$K_{n,\epsilon}(t_1, \ldots, t_m) = \{ x \in I_0 : f^j(x) \in V_\epsilon \mathcal{D} \text{ if and only if } j \in \{t_1, \ldots, t_m\} \}$$

By Lemma 4.13 we conclude that given $\gamma > 0$, there exists $\epsilon > 0$ such that for $n$ big enough,

$$I_0 = \bigcup_{m=0}^{\gamma n} \bigcup_{t_1, \ldots, t_m} K_{n,\epsilon}(t_1, \ldots, t_m)$$

where the second union is over all subsets $\{t_1, \ldots, t_m\} \subset \{0, 1, \ldots, n - 1\}$.

Let us denote by $\#\{I \subset C_\delta(a_1, \ldots, a_n) ; I \cap K_{n,\epsilon}(t_1, \ldots, t_m) \neq \emptyset\}$ the number of connected components of $C_\delta(a_1, \ldots, a_n)$ whose intersection with $K_{n,\epsilon}(t_1, \ldots, t_m)$ is non empty.

From the last equation we conclude that

$$\sum_{a_1, \ldots, a_n} \#C_\delta(a_1, \ldots, a_n) \leq \sum_{a_1, \ldots, a_n} \sum_{t_1, \ldots, t_m} \#\{I \subset C_\delta(a_1, \ldots, a_n) ; I \cap K_{n,\epsilon}(t_1, \ldots, t_m) \neq \emptyset\}$$

(4.8)

where the first sum is over all $a_1, \ldots, a_n$ such that $a_1 + \cdots + a_n < \delta n$ and the second sum is over all subsets $\{t_1, \ldots, t_m\} \subset \{0, 1, \ldots, n - 1\}$ with $m < \gamma n$. \(\square\) Springer
Remark 4.18 In the setting $(\tilde{H}_4)$, we count the number of components of $C_\delta(a_1, \ldots, a_n)$ whose intersection with $Y_n(\lambda)$ is non-empty. In order to do it, instead of the sets $K_{n,e}(t_1, \ldots, t_m)$, we use the sets $Y_n, e(t_1, \ldots, t_m) := Y_n(\lambda) \cap K_{n,e}(t_1, \ldots, t_m)$.

4.3.2 Components of $C_\delta(a_1, \ldots, a_s)$

We state some claims related to the number of connected components of the sets $C_\delta(a_1, \ldots, a_n)$. Recall that $p$ is the maximum number of elements in any $\mathcal{D}_k$ (for $k \geq 0$). Given $I \subseteq I_0$ and $s \in \mathbb{N}$, we say $f^s(I) \cap \mathcal{D} = \emptyset$ (resp. $\neq \emptyset$) if $f^s(I) \cap \mathcal{D}_i = \emptyset$ (resp. $\neq \emptyset$).

Claim 4.19 For any $a_1, a_2, \ldots, a_s$ with $a_j \in \{0, 1\}$ for all $j$,

$$\#C_\delta(a_1, \ldots, a_s, 0) + \#C_\delta(a_1, \ldots, a_s, 1) \leq 3(p + 1)\#C_\delta(a_1, \ldots, a_s)$$

Proof Let $I$ be a component of $C_\delta(a_1, \ldots, a_s)$. If $f^s(I) \cap \mathcal{D} = \emptyset$ and $I' \subset I$ is a component of $C_\delta(a_1, \ldots, a_s, 0)$, it can not exist one component of $C_\delta(a_1, \ldots, a_s, 1)$ at each side of $I'$. So, there exist at most two components of $C_\delta(a_1, \ldots, a_s, 0)$ in $I$. Hence, $I$ is divided at most in 3 components of $C_\delta(a_1, \ldots, a_s, 0) \cup C_\delta(a_1, \ldots, a_s, 1)$.

If $f^s(I) \cap \mathcal{D} \neq \emptyset$, $I$ is divided at most in $p + 1$ components. Each one of these components have a boundary which goes by $f^s$ to $\mathcal{D}$ and is divided (as for the last case) at most in 3 components of $C_\delta(a_1, \ldots, a_s, 0) \cup C_\delta(a_1, \ldots, a_s, 1)$.

Claim 4.20 Let $s, n \in \mathbb{N}$ and $J$ be a component of $C_\delta(a_1, \ldots, a_s, 0)$. If $f^{s+i}(J) \cap \mathcal{D} = \emptyset$ for $1 \leq i \leq n$, then

$$\#\left\{ I \subseteq C_\delta(a_1, \ldots, a_s, 0^{i+1}) : I \subseteq J \right\} \leq i + 1.$$ 

for $1 \leq i \leq n$, where $0^{i+1}$ means that the last $i + 1$ terms are equal to 0.

Proof For $i = 1$ the proof is contained in the proof of Claim 4.19. Now, note that every connected component of $C_\delta(a_1, \ldots, a_s, 0^i)$ gives rise to one or two components of $C_\delta(a_1, \ldots, a_s, 0^{i+1})$. The proof of Claim 4.20 follows by induction on $i$, showing that at most one component of $C_\delta(a_1, \ldots, a_s, 0^i)$ gives rise to two components of $C_\delta(a_1, \ldots, a_s, 0^{i+1})$.

To bound the number of connected components whose intersection with $K_{n,e}(t_1, \ldots, t_m)$ is non-empty, we have the following claim.

Claim 4.21 Let $l \in \mathbb{N}$ and $\epsilon > 0$ be constants and let $\delta = \delta(l)$ be the number given by Lemma 4.15. For any $a_1, \ldots, a_s$ with $a_j \in \{0, 1\}$, $\{t_1, \ldots, t_m\} \subset \{0, 1, \ldots, n - 1\}$.

If $\{s + 1, \ldots, s + i\} \cap \{t_1, \ldots, t_m\} = \emptyset$ and $i \leq l$, then

$$\#\left\{ I \subseteq C_\delta(a_1, \ldots, a_s, 0^{i+1}) : I \cap K_{n,e}(t_1, \ldots, t_m) \neq \emptyset \right\} \leq (i + 1)\#\left\{ I \subseteq C_\delta(a_1, \ldots, a_s, 0) : I \cap K_{n,e}(t_1, \ldots, t_m) \neq \emptyset \right\}.$$ 

Proof Let $I$ be a component of $C_\delta(a_1, \ldots, a_s, 0)$. Then $|f^{s+1}(I)| \leq 2\delta$. Let $i_0 \in \{1, 2, \ldots, i\}$ the first number such that $f^{s+i_0}(I) \cap \mathcal{D} \neq \emptyset$. Since $f^{s+j}(I) \cap \mathcal{D} \neq \emptyset$ for $0 \leq j < i_0$, by Lemma 4.15, $|f^{s+i_0}(I)| < \epsilon$. Then, for all $x \in I$, $f^{s+i_0}(x) \in V_\epsilon \mathcal{D}$. Since $\{s + 1, \ldots, s + i\} \cap \{t_1, \ldots, t_m\} = \emptyset$, then $I \cap K_{n,e}(t_1, \ldots, t_m) = \emptyset$. Hence, if $I \cap K_{n,e}(t_1, \ldots, t_m) \neq \emptyset$ and $\{s + 1, \ldots, s + i\} \cap \{t_1, \ldots, t_m\} = \emptyset$, then $f^{s+j}(I) \cap \mathcal{D} = \emptyset$ for all $0 \leq j \leq i$, with $i \leq l$. Thus, claim follows using Claim 4.20.
4.3.3 Proof of Lemma 4.17

Given \( m < n, \delta > 0 \) and \( \epsilon > 0 \), let us consider \( a_1, \ldots, a_n \) with \( a_i \in \{0, 1\} \) (such that \( a_1 + a_2 + \cdots + a_n < \delta n \)) and \( \{t_1, \ldots, t_m\} \subset \{0, \ldots, n - 1\} \). We can decompose the sequence \( a_1 \ldots a_n \) in maximal blocks of 0’s and 1’s. We write the symbol \( \xi \) in the \( j \)th position if \( a_j = 1 \) or, \( a_j = 0 \) and \( j = t_k \) for some \( k \in \{1, \ldots, m\} \). In this way we have,

\[
a_1a_2 \ldots a_n = \xi^{i_1}0^{i_2}\xi^{i_3}0^{i_4} \ldots \xi^{i_h}0^{i_h} \tag{4.9}
\]

with \( 0 \leq i_k, j_k \leq n \) for \( k = 1, \ldots, h \), \( \sum_{k=1}^{h} i_k + j_k = n \) and \( \sum_{k=1}^{h} i_k < m + \delta n \).

Let us assume that \( a_1, \ldots, a_n \) are as in (4.9). Let \( l, \epsilon \) and \( \delta \) be as in Lemma 4.15. Using claims 4.19 and 4.21 we have,

\[
\#\{I \subset C_\delta(a_0, \ldots, a_n), I \cap K_{n, \epsilon}(t_1, \ldots, t_m) \neq \emptyset\} \leq \frac{(3(p + 1))(l - 1)^{\frac{1}{l}} + 1 (3(p + 1))^{i_1})}{(3(p + 1)) \sum_{k=1}^{h} i_k (3(p + 1))^{i_k} (l + 1)^{\frac{1}{l} + 1} (3(p + 1))^{i_1}} \leq \frac{(3(p + 1))^{\frac{1}{l} + 1} (3(p + 1))^{i_1}}{(3(p + 1))^{m + \delta n + h} (l + 1)^{\frac{1}{l} + 1} (3(p + 1))^{i_1}} \leq \frac{\exp(n \psi_0(l, \gamma, \delta))}{(3(p + 1))^2 + 2 \delta + \gamma} \tag{4.10}
\]

where \( \psi_0(l, \gamma, \delta) = 3(\delta + \gamma) \log(3(p + 1)) + 2(\delta + \gamma + \frac{1}{l}) \log(2l) \).

Using (4.10) and Stirling’s formula in Eq. (4.8), we conclude that

\[
\sum_{a_1, \ldots, a_n} \#C_\delta(a_1, \ldots, a_n) \leq \exp(n \psi_3(l, \gamma, \delta))
\]

where \( \psi_3(l, \gamma, \delta) = \psi_0(l, \gamma, \delta) + \psi_1(\gamma) + \psi_2(\delta), \psi_1(\gamma) \rightarrow 0 \) and \( \psi_2(\delta) \rightarrow 0 \) when \( \gamma \rightarrow 0 \) and \( \delta \rightarrow 0 \), respectively.

Hence, we have to choose \( l \) such that

\[
\frac{2}{l} \log(2l) < \frac{\lambda}{14} \tag{4.11}
\]

and, let \( \gamma > 0 \) be such that

\[
2\gamma \log(2l) < \frac{\lambda}{14}, \quad 3\gamma \log(3(p + 1)) < \frac{\lambda}{14}, \quad \text{and} \quad \psi_1(\gamma) < \frac{\lambda}{14} \tag{4.12}
\]

Next, we find \( \epsilon > 0 \), using Lemma 4.13. Finally, given \( \epsilon \) and \( l \), let \( \delta > 0 \) be the constant given by Lemma 4.15 and satisfying

\[
2\delta \log(2l) < \frac{\lambda}{14}, \quad 3\delta \log(3(p + 1)) < \frac{\lambda}{14}, \quad \text{and} \quad \psi_3(\delta) < \frac{\lambda}{14} \tag{4.13}
\]

With this choice, \( \psi_3(l, \gamma, \delta) \leq \frac{\lambda}{14} \). Hence the first part of Lemma 4.17 is proved. On the other hand, observe that the choice of \( \delta \) is given fundamentally by Lemmas 4.13 and 4.15. Namely, \( \delta \) depends on: the constant \( \lambda \) in the definition of \( Y_n(\lambda) \); the uniformity of \( \epsilon \) (given \( \ell \in \mathbb{N} \)) on the Eq. (4.1); the uniform boundedness of \( |Df_k| \) on the proof of Lemma 4.15; and the uniform boundedness of the number of discontinuity points for \( f_k \), where \( k \geq 0 \). This concludes the proof of Lemma 4.17. \( \square \)
4.3.4 Proof of Theorem 4.1

Note that for every \( N \in \mathbb{N} \) it holds

\[
E \cap \left( \bigcap_{n \geq N} Y_n(\lambda) \right) \cap \mathcal{C} \left( \bigcup_{n \geq N} A_n(\delta) \cap Y_n(\lambda) \right) \subseteq E \cap \left( \bigcap_{n \geq N} \mathcal{C}A_n(\delta) \cap Y_n(\lambda) \right).
\]

where \( \mathcal{C}B \) denotes the complement set of \( B \) and \( |B| \) denotes the Lebesgue measure of \( B \). By the hypotheses of theorem, \( |E \cap (\bigcap_{n \geq N} Y_n(\lambda))| \) converges to the Lebesgue measure of \( E \).

On the other hand, note that if \( J \) is a component of \( C_\delta(a_1, \ldots, a_n) \) (with \( a_1 + \cdots + a_n < \delta n \)) then \( |J \cap Y_n(\lambda) \cap A_n(\delta)| \leq |I_0| \exp(-n\lambda) \). Then, using Lemma 4.17 we conclude that \( |\bigcup_{n \geq N} A_n(\delta) \cap Y_n(\lambda)| \) converges to zero when \( N \to \infty \). Therefore, \( |\bigcap_{n \geq N} (\mathcal{C}A_n(\delta) \cap Y_n(\lambda)) \cap E| \) converges to \( |E| \) when \( N \to \infty \). Thus, we conclude that (4.2) holds considering \( 3\zeta = \delta^2 \).

\( \Box \)

5 Non-invertible base transformation

Let \( \varphi : X \times I_0 \to X \times I_0 \) or \( \varphi : X \times Y \to X \times Y \) be a skew-product satisfying \((H_2)\) and the remaining conditions of Theorems A or B. We define a natural extension \( \hat{\varphi} \) of this map and we prove that it satisfies \((H_2^*)\) and also the remaining conditions of the statement of the Main Theorems.

5.1 Inverse limit construction

We use a standard construction which allows to define, for an endomorphism of a measure space, an induced invertible bimeasurable map of a new measure space. For more details, see for instance [13, Chapter 10.4]. We perform the construction with the map \( \alpha : X \to X \).

First consider the (inverse limit) space \( \hat{X} \) which is formed by points

\[
\hat{\theta} = (\theta_0, \theta_{-1}, \theta_{-2}, \ldots),
\]

where \( \theta_{-i} \in X \) for \( i \geq 0 \) and \( \alpha(\theta_{-i}) = \theta_{-i+1} \) for \( i \geq 1 \). Then we have

1. \( X^\mathbb{N} \) with the product topology is a metrizable space (see \([19, \text{Lemma } 111.15]\));
2. \( X^\mathbb{N} \) is separable (see \([19, \text{Theorems } 111.14 \text{ and } 58.7]\));
3. as a topological space (in fact, a metrizable space), \( X^\mathbb{N} \) admits the Borel \( \sigma \)-algebra \( \mathcal{B}_{X^\mathbb{N}} \), which is the \( \sigma \)-algebra generated by the open sets of the product topology on \( X^\mathbb{N} \);
4. the product \( \sigma \)-algebra \( \prod_{i \in \mathbb{N}} \mathcal{B}_{X_i} \) on \( X^\mathbb{N} \) coincides with \( \mathcal{B}_{X^\mathbb{N}} \). \( (X_i = X \text{ for all } i \in \mathbb{N}) \);
5. as subset of \( X^\mathbb{N} \), \( \hat{X} \) is endowed with the product topology, and therefore has a Borel \( \sigma \)-algebra \( \mathcal{B}_{\hat{X}} \);
6. \( \mathcal{B}_{\hat{X}} \) coincides with the \( \sigma \)-algebra obtained by intersecting \( \prod_{i \in \mathbb{N}} \mathcal{B}_{X_i} \) with \( \hat{X} \).

Now, \( \hat{X} \) with the \( \sigma \)-algebra \( (\prod_{i \in \mathbb{N}} \mathcal{B}_{X_i}) \cap \hat{X} \) is a measurable space. For the sets of the form

\[
(A)_n = \{ \hat{\theta} = (\theta_0, \theta_{-1}, \theta_{-2}, \ldots) \in \hat{X}; \ \theta_{-n} \in A \}
\]

where \( A \in \mathcal{B}_X \) and \( n \geq 0 \), we define \( \hat{\nu}((A)_n) = \nu(A) \). Since these sets generate the \( \sigma \)-algebra and the conditions of compatibility of Kolmogorov’s Theorem are satisfied, we have a measure \( \hat{\nu} \) defined on the \( \sigma \)-algebra.
We can consider the map $\hat{\alpha} : \hat{X} \to \hat{X}$ given by

$$\hat{\alpha}((\theta_0, \theta_{-1}, \theta_{-2}, \ldots)) = (\alpha(\theta_0), \alpha(\theta_{-1}), \alpha(\theta_{-2}), \ldots) = (\alpha(\theta_0), \theta_0, \theta_{-1}, \theta_{-2}, \ldots).$$

This map is invertible $\hat{\alpha}^{-1}((\theta_0, \theta_{-1}, \theta_{-2}, \ldots)) = (\theta_{-1}, \theta_{-2}, \theta_{-3}, \ldots)$. The measure $\hat{\nu}$ is invariant with respect to $\hat{\alpha}$.

Therefore we have constructed an invertible map $\hat{\alpha}$, bimeasurable (with the Borel $\sigma$-algebra $B_{\hat{X}}$) on a metric space $\hat{X}$, such that $\pi_0 \circ \hat{\alpha}(\hat{\theta}) = \alpha \circ \pi_0(\hat{\theta})$ for every $\hat{\theta} \in \hat{X}$, where $\pi_0(\hat{\theta}) = \theta_0$. It is also useful to define the natural projection map $P : \hat{X} \times I_0 \to \hat{X} \times I_0$, by $P(\hat{\theta}, x) = (\pi_0(\hat{\theta}), x) = (\theta_0, x)$.

5.2 Non-invertible base

Let us define the map $\hat{\varphi} : \hat{X} \times I_0 \to \hat{X} \times I_0$, $\hat{\varphi}(\hat{\theta}, x) = (\hat{\alpha}(\hat{\theta}), \hat{f}(\hat{\theta}, x))$, where $\hat{f}(\hat{\theta}, x) = f(\theta_0, x)$. Since $\hat{\alpha}$, $P$ and $f$ are measurable then $\hat{\varphi}$ is measurable, i.e, $\hat{\varphi}^{-1}(B_{\hat{X}} \times B_{I_0}) \subset B_{\hat{X}} \times B_{I_0}$.

Note that the set of critical and discontinuity points for $\hat{f}_0$ projects onto the corresponding set for $f_0$. Hence the measurability of the set

$$\hat{\mathcal{F}} = \{(\hat{\theta}, x) \in \hat{X} \times I_0 : x \in C_{\theta_0} \cup D_{\theta_0}\} = P^{-1}(\mathcal{F})$$

follows from the measurability of the set $\mathcal{F}$ and of the map $P$. Thus, $\hat{\varphi}$ satisfies condition ($H_1$).

We note that the set of discontinuity points $\mathcal{D}_a$ of $\hat{\alpha}$ coincides with the set

$$(\mathcal{D}_a)_0 = \{\hat{\theta} \in \hat{X}; \theta_0 \in \mathcal{D}_a\}$$

and so $\hat{\nu}(\mathcal{D}_a) = \nu(\mathcal{D}_a) = 0$. On the other hand, for the map $\hat{F} : \hat{X} \to B(I_0)$, $\hat{\theta} \mapsto \hat{f}_0 = f_0$ we have that $\mathcal{D}_F \subset (\mathcal{D}_F)_0$.

Hence the map $\hat{\varphi}$ satisfies conditions ($H_1^*$) and ($H_3$). The map $\hat{\varphi}$ clearly satisfies condition ($H_4$) (resp. ($H_4^*$)) if the map $\varphi$ satisfies the condition ($H_4$) (resp. ($H_4^*$)).

Moreover, if $\varphi$ is non-uniformly expanding along the vertical direction according to $v \times m$, on the subset $Z$, then $\hat{\varphi}$ is non-uniformly expanding along the vertical direction according to $\hat{\nu} \times m$, on the subset $P^{-1}Z$. It also holds that $\hat{\nu} \times m(P^{-1}Z) = \nu \times m(Z)$.

Thus, $\hat{\varphi}$ is a skew-product in the conditions of Theorems A and B.

We remark that in order to prove the relative compactness of the sequences of measures $\{\eta_n\}$ and $\{\mu_n\}$ (see Lemma 4.6 and Remark 4.7) we use the fact that $X$ is a separable metrizable and complete topological space. The space $\hat{X}$ can fail to be complete. To solve this problem, we can consider $\hat{\nu}$ as a measure defined on $X^N$ (stating that $\hat{\nu}(X^N \setminus \hat{X}) = 0$). Thus, we can find a closed set $X_1 \subset \hat{X}$ such that $\hat{\nu}(\hat{X} \setminus X_1) < \epsilon$. On the other hand, since $X^N$ is a separable metrizable and complete topological space, we can find a compact set $X_2 \subset X^N$, such that $\hat{\nu}(X^N \setminus X_2) < \epsilon$. Hence, for the compact set $X_1 \cap X_2$, we have that $\hat{\nu}(\hat{X} \setminus (X_1 \cap X_2)) < 2\epsilon$. Therefore, considering $X_0 = X_1 \cap X_2$ in Lemma 4.6 and Remark 4.7, the relative compactness of the sequences of measures $\{\eta_n\}$ and $\{\mu_n\}$ follows.

Hence we may repeat the same sequence of steps in the arguments in Sect. 4 assuming Theorem 4.1 to conclude the result in Proposition 4.12: there exists an invariant probability measure $\hat{\mu}$ which is absolutely continuous with respect to $\hat{\nu} \times m$, with $\hat{\mu}(P^{-1}(Z(\lambda))) > 0$.

Now we push this measure for the original space $X \times I_0$.

**Lemma 5.1** $P_\ast \hat{\mu}$ is an $\varphi$-invariant probability measure which is absolutely continuous with respect to $\nu \times m$, and $P_\ast \hat{\mu}(Z(\lambda)) > 0$. 

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Lemma 6.2
Given subsets, their number in a finite measure space must be finite. Has \( \nu \) and then \( P \) of \( f_i \) since the measure \( \nu \) and continuity of \( \hat{\mu} \), we have that

\[
P_* \hat{\mu} \left( \varphi^{-1}(A) \right) = \hat{\mu} \left( P^{-1} \varphi^{-1}(A) \right) = \hat{\mu} \left( (P \circ \hat{\varphi})^{-1}(A) \right) = \hat{\mu} \left( \hat{\varphi}^{-1}(P^{-1}A) \right) = P_* \hat{\mu}(A)
\]

and then \( P_* \hat{\mu} \) is invariant with respect to \( \varphi \).

On the other hand, if \((\nu \times m)(A) = 0\), then \((\nu \times m)(P^{-1}A) = 0\). Using the absolute continuity of \( \hat{\mu} \), we conclude that \( P_* \hat{\mu}(A) = 0 \).

\[\square\]

6 Finitely many ergodic basins

Here we conclude the proofs of Theorems A and B, proving that the invariant sets with positive \( \nu \times m \)-measure, have mass bounded away from zero.

Given \( \lambda > 0 \), let \( Z(\lambda) \subset \mathbb{X} \times I_0 \) the set of points with vertical Lyapunov exponents greater than \( 2\lambda \), i.e., points in \( Z \) for which the inequality (1.3) holds.

Proposition 6.1
Given \( \lambda > 0 \), there exists \( b > 0 \) such that every \( \varphi \)-invariant subset \( G \subset Z(\lambda) \) with positive \( \nu \times m \)-measure satisfies \((\nu \times m)(G) > b\).

This ensures that the ergodic basins \( B_\zeta = B(\mu_i) \) of the measures provided by Theorem A has \( \nu \times m \)-measure uniformly bounded away from zero. Since these are pairwise disjoint subsets, their number in a finite measure space must be finite.

For the proof of Proposition, we need the following result.

Lemma 6.2
Given \( \zeta > 0 \), there exists \( K_1 > 0 \) such that, for any \( i \in \mathbb{N} \) and any \((\theta, x) \in \mathbb{X} \times I_0 \), if \( r_i(\theta, x) > \zeta \), there exists \( J_i(x) \subset I_0 \) such that \( f_\theta^i(J_i(x)) = B(f_\theta^i(x), \zeta/2) \), \( f_\theta^i \) restricted to \( J_i(x) \) is a \( C^3 \) diffeomorphism and

\[
\frac{1}{K_1} \leq \frac{|Df_\theta^i(y)|}{|Df_\theta^i(z)|} \leq K_1 \quad \text{for all } y, z \in J_i(x).
\]  

Proof
Let \((\theta, x) \in \mathbb{X} \times I_0 \) such that \( r_i(\theta, x) > \zeta \). By definition of \( r_i \), there exists \( T_i \subset I_0 \) such that \( x \in T_i \), \( f_\theta^i \) restricted to \( T_i \) is a \( C^3 \) diffeomorphism and the connected components of \( f_\theta^i(T_i) \setminus \{f_\theta^i(x)\} \) have length \( > \zeta \). Let us choose \( J_i(x) \subset T_i \) such that \( f_\theta^i(J_i(x)) = B(f_\theta^i(x), \zeta/2) \). Note that \( f_\theta^i(T_i) \) contains an \( 1/2 \)-scaled neighborhood of \( f_\theta^i(J_i(x)) \). It means that both connected components of \( f_\theta^i(T_i) \setminus \{f_\theta^i(J_i(x))\} \) have length \( > |f_\theta^i(J_i(x))|/2 \). By Koebe Principle (see [14, Theorem IV.1.2]), there exists \( K_1 \) such that (6.1) holds. The distortion \( K_1 > 0 \) does not depend on the point \((\theta, x)\), nor the iterate \( i \).

Proof of Proposition 6.1
Let \( G \subset Z(\lambda) \) be a forward \( \varphi \)-invariant set, such that \((\nu \times m)(G) > 0\). Given \( \lambda > 0 \), let us consider the constant \( \zeta > 0 \) given by Theorem 4.1. Let \( K_1 > 0 \) the constant found on Lemma 6.2. Denoting by \( G(\theta) \) the \( \theta \)-section of \( G \), i.e., \( G(\theta) := \{x \in I_0; (\theta, x) \in G\} \), let us define the measurable set

\[
B_\zeta^G := \left\{ \theta \in \mathbb{X}, \ m \left( G(\theta) \right) \geq \frac{\zeta}{4K_1} \right\}
\]

Since the measure \( \nu \) is ergodic for the map \( \alpha \), then

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_{B_\zeta^G}(\alpha^i(\theta)) = \int \chi_{B_\zeta^G} \nu
\]  

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for all \( \theta \) in a \( \nu \)-full measure set. Let \( \theta_0 \in X \) be a point such that \( m(G(\theta_0)) > 0 \) and (6.2) holds for \( \theta = \theta_0 \). By Theorem 4.1 applied to the set \( E = G(\theta_0) \), we can find a point \( x_0 \in G(\theta_0) \) such that \( \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} r_i(\theta_0, x_0) \geq 3 \zeta \). We can assume that \( x_0 \) is a density point of \( G(\theta_0) \). Thus, there exists \( \epsilon_0 > 0 \) such that for all \( \epsilon < \epsilon_0 \),

\[
\frac{m(G(\theta_0) \cap B(x_0, \epsilon))}{m(B(x_0, \epsilon))} \geq \frac{1}{2}
\]

On the other hand, we can find \( N \in \mathbb{N} \) such that \( \sum_{i=1}^{n} r_i(\theta_0, x_0) \geq 2 \zeta n \) and log \( |Df_{\theta_0}^n(x_0)| \) \( \geq \lambda n \) for all \( n \geq N \). Then, if \( r_i(\theta_0, x_0) \geq \zeta \), for \( i \geq N \), the interval \( J_i(x_0) \) found on Lemma 6.2 is such that \( |J_i(x_0)| \leq K_1 \zeta e^{-\lambda i} \). Therefore, there exists \( n_0 \geq N \) such that \( J_i(x_0) \subset B(x_0, \epsilon_0) \), provided \( i \geq n_0 \) (and obviously only when \( r_i(\theta_0, x_0) \geq \zeta \)).

**Claim 6.3** If \( r_i(\theta_0, x_0) \geq \zeta \) and \( i \geq n_0 \) then \( \alpha^i(\theta_0) \in B_G^\zeta \).

**Proof** Let \( J^i(x_0) \) the maximal ball centered at \( x_0 \) contained in \( J_i(x_0) \). Using Lemma 6.2,

\[
\frac{m(f_{\theta_0}^i(J^i(x_0) \cap G(\theta_0)))}{m(f_{\theta_0}^i(J^i(x_0)))} \geq \frac{1}{K_1} \frac{m(J^i(x_0) \cap G(\theta_0))}{m(J^i(x_0))} \geq \frac{1}{2K_1}
\]

for \( i \geq n_0 \). Then we have that \( m(f_{\theta_0}^i(J^i(x_0) \cap G(\theta_0))) \geq \zeta / 4K_1 \). Since \( f_{\theta_0}^i(G(\theta_0)) \subset G(\alpha^i(\theta_0)) \) (by the forward \( \varphi \)-invariance of \( G \)), the claim follows.

An immediate consequence of the claim is that for all \( n \geq n_0 \),

\[
\sum_{i=n_0}^{n} \chi_{B_G^\zeta(\alpha^i(\theta_0))} \geq \# \{ n_0 \leq i \leq n; r_i(\theta_0, x_0) \geq \zeta \}
\]

Now, using Pliss Lemma (see Lemma 4.4), there exists \( \zeta = \zeta(\zeta) > 0 \) such that for \( n \geq n_0 \),

\[
\frac{\# \{ 1 \leq i \leq n; r_i(\theta_0, x_0) \geq \zeta \}}{n} \geq \zeta
\]

since \( \sum_{i=1}^{n} r_i(\theta_0, x_0) \geq 2 \zeta n \), for \( n \geq n_0 \). Hence the limit in (6.2) for \( \theta = \theta_0 \) is greater than \( \zeta \). It means that \( \nu(B_G^\zeta) \geq \zeta \). Thus, Proposition 6.1 follows considering \( b = \zeta \zeta / 4K_1 \).

Finally we can conclude the proof of Theorems A and B.

**Proof of Theorem A** Assume that \( (\nu \times m)(Z) > 0 \) (otherwise, there is nothing to prove). By assumption, there exists \( \lambda > 0 \) such that \( Z \setminus Z(\lambda) \) has zero \( (\nu \times m) \)-measure. Let \( \mu_0 \) be the \( \varphi \)-invariant probability measure absolutely continuous with respect to \( \nu \times m \) given by Proposition 4.12 and Lemma 5.1. Considering the normalized restriction to the forward invariant set \( Z(\lambda) \), we can assume that \( \mu_0(Z(\lambda)) = 1 \). Since every invariant set, with positive \( \nu \times m \)-measure, has \( \nu \times m \)-measure greater than \( b \) (by Proposition 6.1), we can decompose \( \mu_0 \) in a finite number of ergodic components. Then \( \mu_0 = \sum_{i=1}^{s} a_i \mu_i \), where \( a_i > 0 \), \( \sum_{i=1}^{s} a_i = 1 \) and \( \mu_i \) are ergodic \( \varphi \)-invariant absolutely continuous probability measures.

If \( Z_1 = Z(\lambda) \setminus \bigcup_{i=1}^{s} B(\mu_i) \) still has positive \( \nu \times m \)-measure, then we can repeat the arguments of Sect. 4 for the set \( Z_1 \subset Z(\lambda) \) instead of \( Z(\lambda) \). Repeating this argument, we obtain the ergodic components as in the statement of Theorem A such that \( \nu \times m \)-a.e. point in \( Z(\lambda) \) is in the basin of one of these measures. The number of such measures is finite, since the basin of each of them is a collection of pairwise disjoint invariant sets with \( \nu \times m \)-positive measure, and Proposition 6.1 holds. \( \square \)
Absolutely continuous invariant measures

Proof of Theorem B  Since $Z = \bigcup_{n \in \mathbb{N}} Z(1/n)$, the previous argument applied to each $Z(1/n)$ provides finitely many ergodic probability measures whose basins cover $Z(1/n)$, for each $n \geq 1$. This concludes the proof of Theorem B. □

7 SRB measures for random non-uniformly expanding maps

Let $(\mathcal{X}, \mathcal{B}_\mathcal{X}, \nu, \alpha, f)$ be an admissible random non-uniformly expanding map on $I_0$. Let us consider the associated skew-product $\varphi$ defined on $\mathcal{X} \times I_0$. By Theorem A, there exist $\mu_1, \ldots, \mu_t$, $\varphi$-invariant ergodic probabilities, such that $(\nu \times m)$-a.e. $(\theta, x)$ is in the basin of one of these measures. Denote by $B_i$ the ergodic basin $B(\mu_i)$ of the measure $\mu_i$, for $1 \leq i \leq t$. As usual, $B_i(\theta)$ denotes the $\theta$-section of the set $B_i$.

Proof of Theorem C  Define $p_i$ as the projection on $I_0$ of $\mu_i$. By a straightforward calculation, we can prove that $RB(\theta) \supseteq B_i(\theta)$. As $\mu_i$ is absolutely continuous with respect to $\nu \times m$, then $\nu \times m(B_i(\theta)) > 0$. Since $B_i$ is $\varphi$-invariant and $\nu$ is $\alpha$-ergodic, then $m(B_i(\theta)) > 0$ for $\nu$-almost every $\theta \in \mathcal{X}$. It implies that $p_i$ is a SRB probability for the random dynamical system.

Since $m \left(I_0 \setminus \bigcup_{i=1}^t B_i(\theta)\right) = 0$, for $\nu$-almost every $\theta \in \mathcal{X}$, then $\nu$-almost surely, the union of the random basins of $p_1, \ldots, p_t$ has total Lebesgue measure. Clearly, these measures are absolutely continuous with respect to Lebesgue measure. □

8 Higher dimensional fibers

Here we outline the arguments in the higher-dimensional fiber case. The strategy is the same as the one presented for one-dimensional fibers.

We start by considering the sequences $\eta_n(\theta)$ and $\eta_n$ as in Sect. 3. Then we use the notion of hyperbolic times from [5] to redefine $\mu_n(\theta)$ replacing $H_n(\theta, \varsigma)$ by $H_n(\theta)$. Finally we just have to obtain the analogous results to Corollary 4.5 and Lemmas 4.8, 4.9 and 4.10.

After this the argument follows the proof of Theorem A through Lemmas 4.10 and 4.11.

In what follows, since hyperbolic times have been extensively investigated recently, we cite most of the results from other published works.

8.1 Hyperbolic times and their properties

Given $0 < \sigma < 1$ and $b, \delta > 0$, we say that the positive integer $n$ is a $(\sigma, \delta, b)$-hyperbolic time for $(\theta, x) \in \mathcal{X} \times \mathcal{Y}$ if

\[
\prod_{j=n-k}^{n-1} \left\| Df_{\alpha^j(\theta)}(f_{\theta}^j(x))^{-1} \right\| \leq \sigma^k \quad \text{and} \quad \text{dist}_\delta (f_{\theta}^k(x), \mathcal{Y} \cap (\{\alpha^k(\theta)\} \times \mathcal{Y})) \geq e^{-bk}
\]

for $k = 0, \ldots, n - 1$. (8.1)

We now outline the properties of these special times. For detailed proofs see [5, Lemma 5.2, Corollary 5.3] and [4, Proposition 2.6, Corollary 2.7, Proposition 5.2].

Proposition 8.1  There are constants $C_1, \delta_1 > 0$ depending on $(\sigma, \delta, b)$ and $\varphi$ only such that, if $n$ is $(\sigma, \delta, b)$-hyperbolic time for $(\theta, x)$, then there are neighborhoods $V_n(\theta, x)$ of $(\theta, x)$ on $\{\theta\} \times \mathcal{Y}$, such that

\[ \text{(8.1)} \]
(1) $f^n_\theta \mid V_n(\theta, x)$ maps $V_n(\theta, x)$ diffeomorphically to the ball of radius $\delta_1$ around $f^n_\theta(x)$ inside $\{\alpha^n(\theta)\} \times \mathcal{Y}$;

(2) $\text{dist} \left( f^n_\theta \cdot f^{-k}_\theta(y), f^n_\theta \cdot f^{-k}_\theta(z) \right) \leq \sigma^{k/2} \cdot \text{dist} \left( f^n_\theta(y), f^n_\theta(z) \right)$ for every $0 \leq k \leq n$ and $y, z \in V_n(\theta, x)$;

(3) for $y, z \in V_n(\theta, x)$

$$\frac{1}{C_1} \leq \frac{\left| \det Df^n_\theta(y) \right|}{\left| \det Df^n_\theta(z) \right|} \leq C_1.$$

The following ensures existence of infinitely many hyperbolic times for Lebesgue almost every point for non-uniformly expanding maps with slow recurrence to the singular set. A complete proof can be found in [5, Section 5].

**Theorem 8.2** Let $\varphi : \mathbb{X} \times \mathcal{Y} \to \mathbb{X} \times \mathcal{Y}$ be as in the statement of Theorem E, i.e., non-uniformly expanding along the fibers according to $\nu \times \text{Leb}$, on a subset $Z$ of $\mathbb{X} \times \mathcal{Y}$.

Then there are $\sigma \in (0, 1)$, $\delta, b > 0$ and there exists $\rho = \rho(\sigma, \delta, b) > 0$ such that $\nu \times \text{Leb}$-a.e. $(\theta, x) \in Z$ has infinitely many $(\sigma, \delta, b)$-hyperbolic times. Moreover if we write $0 < n_1 < n_2 < n_2 < \ldots$ for the hyperbolic times of $(\theta, x) \in Z$, then their asymptotic frequency satisfies

$$\liminf_{N \to \infty} \frac{\#\{k \geq 1 : n_k \leq N\}}{N} \geq \rho \quad \text{for} \quad \nu \times \text{Leb}$ - a.e. $(\theta, x) \in Z.$$

Now we define, in this setting

$$\mathcal{H}_n(\sigma, \delta, b) := \{ (\theta, x) \in \mathbb{X} \times \mathcal{Y} : n \text{ is a}(\sigma, \delta, b) \text{-hyperbolic time for (} \theta, x \text{)} \}$$

and, having fixed $\sigma, \delta, b$ according to Theorem 8.2, we set

$$H_n(\theta) := \{ (\theta) \times \mathcal{Y} \} \cap \mathcal{H}_n(\sigma, \delta, b).$$

### 8.2 Hyperbolic times on fibers

Now we are able to state and prove the analogous result to Corollary 4.5 with the same arguments.

**Lemma 8.3** Given $\lambda > 0$, let $Z(\lambda) \subset Z \subset \mathbb{X} \times \mathcal{Y}$ be such that $(\theta, x) \in Z(\lambda)$ satisfies

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \| Df^j_\theta(\alpha^j(\theta))(f^j_\alpha(\theta))^{-1} \| < -2\lambda < 0.$$

If $n$ is big enough we have $\int \frac{1}{n} \sum_{i=1}^{n} \text{Leb} \left( H_i(\theta) \right) \ d\nu(\theta) \geq \frac{\rho}{2} (\nu \times \text{Leb})(Z(\lambda))$, where $\rho > 0$ is given by Theorem 8.2.

We assume the measurability of $H_n(\theta)$ in what follows. This will be proved in “Appendix”.

We define the measures $\mu_n(\theta)$ on $\mathcal{Y}$, for every $\theta \in \mathbb{X}$ and every $n \in \mathbb{N}$, adapting (4.3) as follows

$$\mu_n(\theta) = \frac{1}{n} \sum_{j=1}^{n} \left( f^j_{\alpha^{-j}(\theta)} \right)_*(\text{Leb} | Z(\alpha^{-j}(\theta), \lambda) \cap H_j(\alpha^{-j}(\theta))).$$

and then we define the measures $\mu_n$ on $\mathbb{X} \times \mathcal{Y}$ as in (4.4).

We need to show that these measures are well-defined and again this is proved in “Appendix”.
Lemma 8.4 There exists $K > 0$ such that $\mu_n(\theta)(A) \leq K \cdot \text{Leb}(A)$ for any measurable subset $A \subset Y$ and every $\theta \in \mathcal{X}$, $n \in \mathbb{N}$.

The proof of this result follows [4, Proposition 5.2].

Proof of Lemma 8.4 Take $\delta_1 > 0$ given by Proposition 8.1. It is sufficient to prove that there is some uniform constant $\tilde{K} > 0$ such that if $A$ is a Borel set in $\{\theta\} \times Y$ with diameter smaller than $\delta_1/2$ then

$$\text{Leb} \left( \left( f_{\alpha^{-n}(\theta)}^{n} \right)^{-1}(A) \cap Z(\alpha^{-n}(\theta), \lambda) \cap H_n(\alpha^{-n}(\theta)) \right) \leq \tilde{K} \cdot \text{Leb}(A).$$

Let $A$ be a Borel set in $\{\theta\} \times Y$ with diameter smaller than $\delta_1/2$ and $B$ an open ball of radius $\delta_1/2$ containing $A$. We may write

$$(f_{\alpha^{-n}(\theta)}^{n})^{-1}(B) = \bigcup_{k \geq 1} B_k,$$

where $(B_k)_{k \geq 1}$ is a (possibly finite) family of two-by-two disjoint open sets in $\{\alpha^{-n}(\theta)\} \times Y$. Discarding those $B_k$ that do not intersect $Z(\alpha^{-n}(\theta), \lambda) \cap H_n(\alpha^{-n}(\theta))$, we choose for each $k \geq 1$ a point $x_k \in Z(\alpha^{-n}(\theta), \lambda) \cap H_n(\alpha^{-n}(\theta)) \cap B_k$.

For $k \geq 1$ let $V_n(\alpha^{-n}(\theta), x_k)$ be the neighborhood of $x_k$ in $\{\alpha^{-n}(\theta)\} \times Y$ given by Proposition 8.1. Since $B$ is contained in $B(f_{\alpha^{-n}(\theta)}^{n}(x_k), \delta_1)$, the ball of radius $\delta_1$ around $f_{\alpha^{-n}(\theta)}^{n}(x_k)$ in $\{\theta\} \times Y$, and $f_{\alpha^{-n}(\theta)}^{n}$ is a diffeomorphism from $V_n(\alpha^{-n}(\theta), x_k)$ onto $B(f_{\alpha^{-n}(\theta)}^{n}(x_k), \delta_1)$, we must have $B_k \subset V_n(\alpha^{-n}(\theta), x_k)$ (recall that by our choice of $B_k$ we have $f_{\alpha^{-n}(\theta)}^{n}(B_k) \subset B$).

As a consequence of this and item (3) of Proposition 8.1, we have for every $k$ that the map $f_{\alpha^{-n}(\theta)}^{n} | B_k : B_k \to B$ is a diffeomorphism with bounded distortion:

$$\frac{1}{C_1} \leq \frac{| \det Df_{\alpha^{-n}(\theta)}^{n}(y) |}{| \det Df_{\alpha^{-n}(\theta)}^{n}(z) |} \leq C_1 \quad \text{for all } y, z \in B_k.$$

This finally gives that $\text{Leb} \left( f_{\alpha^{-n}(\theta)}^{n} \right)(A) \cap Z(\alpha^{-n}(\theta), \lambda) \cap H_n(\alpha^{-n}(\theta))$ is bounded from above by

$$\sum_{k} \text{Leb} \left( f_{\alpha^{-n}(\theta)}^{n}(A \cap B) \cap B_k \right) \leq \sum_{k} \frac{\text{Leb}(A \cap B)}{\text{Leb}(B)} \cdot \text{Leb}(B_k) \leq \tilde{K} \cdot \text{Leb}(A),$$

where $\tilde{K} > 0$ is a constant only depending on $C_1$, on the volume of the ball $B$ of radius $\delta_1/2$, and on the volume of $Y$.  

The analogous statements to Lemmas 4.9 and 4.10 are proved in the exact same way. At this point, we have the analogous results to Corollary 4.5 and Lemmas 4.8, 4.9 and 4.10. The rest of the argument proving the existence of absolutely continuous invariant measures is entirely analogous. We also obtain a similar statement to Proposition 4.12.

For the ergodic decomposition, the arguments are the same as in Sect. 6, including a result analogous to Proposition 6.1 whose proof is standard and follows [5, Lemma 5.6] using the bounded distortion property provided by item (3) of Proposition 8.1.

8.3 Non-invertible base map with higher-dimensional fibers

With the notation introduced in Sect. 5, we define the map $\hat{\phi} : \hat{\mathcal{X}} \times Y \to \hat{\mathcal{X}} \times Y$, $\hat{\phi}(\hat{\theta}, x) = (\hat{\alpha}(\hat{\theta}), \hat{f}(\hat{\theta}, x))$, where $\hat{f}(\hat{\theta}, x) = f(\theta_0, x)$. In the exact same manner as in Sect. 5, we
deduce that this map satisfies conditions \((H_1), (H_2^*)\) and \((H_3)\), if \(\varphi\) satisfies conditions \((H_1)\) through \((H_3)\).

Moreover the argument about relative compactness and the proofs of Lemmas 4.6 and 5.1 need no change. We are left to show that if \(\varphi\) is non-uniformly expanding along the fibers, then \(\hat{\varphi}\) is likewise. But this follows from

- the easy observation that \(\hat{\varphi}^k(\hat{\theta}, x) = (\sigma^k(\hat{\theta}), f_{\theta_0}^k(x))\);
- together with the fact that the full \(v \times \text{Leb}\)-measure subset \(W\) of \(\mathbb{X} \times \mathbb{Y}\) satisfying the conditions \((1.4)\) and \((1.5)\) of non-uniform expansion and slow recurrence provides the set \(\hat{W} = \pi^{-1}(W)\) which also has full \(\hat{v} \times \text{Leb}\)-measure on \(\hat{\mathbb{X}} \times \mathbb{Y}\).

So the points \((\hat{\theta}, x) \in \hat{W}\) will satisfy conditions \((1.4)\) and \((1.5)\). Hence \(\hat{\varphi}\) is non-uniformly expanding along the fibers, with a bijection \(\hat{\alpha}\) as the base transformation.

We can now apply the same arguments of Sects. 3 and 4 to \(\hat{\varphi}\). So our main results also hold if we replace condition \((H_2^*)\) by condition \((H_2)\).

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9 Appendix: Measurability

Here we prove that the measures \(\eta_n\) defined on Sect. 3 together with the measures \(\mu_n\) defined on Sect. 4 are well-defined. We consider separately the case with one dimensional fibers and the case with higher dimensional fibers.

9.1 The measures \(\eta_n\) are well defined

By the Hahn Extension Theorem, it is enough to define the measures on rectangles \(A \times J\) with \(A \in B_\mathbb{X}\) and \(J \in B_{I_0}\). It easily follows from

**Proposition 9.1** Let \(J \subset I_0\) be a Borel set. For every \(n \in \mathbb{N}\), the function \(\mathbb{X} \ni \theta \mapsto \eta_n(\theta)(J)\) is measurable.

**Proof of Lemma 8.4** Let us fix a set \(J \in B_{I_0}\). To prove the measurability of \(\theta \mapsto \eta_n^i(\theta)(J)\) it suffices to prove the measurability of the functions \(\theta \mapsto \eta_n^i(\theta) := (f^i_{\theta^{-1}(\theta)})^* m(J)\), for \(i \in \mathbb{N}\). Let us define the following functions

\[
\begin{align*}
\alpha^{-1} \times i d : \mathbb{X} \times I_0 &\to \mathbb{X} \times I_0 \\
(\theta, x) &\mapsto (\alpha^{-1}(\theta), x) \\
\pi_\mathbb{X} : \mathbb{X} \times I_0 &\to \mathbb{X} \\
(\theta, x) &\mapsto \theta \\
\pi_{I_0} : \mathbb{X} \times I_0 &\to I_0 \\
(\theta, x) &\mapsto x
\end{align*}
\]

and \(\chi_J\) is the characteristic function of \(J\). The projection maps are clearly measurable, considering on \(\mathbb{X} \times I_0\) the \(\sigma\)-algebra \(B_\mathbb{X} \times B_{I_0}\). Since compositions of measurable maps are measurable maps, \(\alpha^{-1} \times i d(\theta, x) = (\alpha^{-1} \circ \pi_\mathbb{X}(\theta, x), \pi_{I_0}(\theta, x))\) is also measurable.

With these notations, we have that \(\eta_n^i(\theta) = \int_{I_0} \phi_i(\theta, x) \, d m(x)\), where \(\phi_i : \mathbb{X} \times I_0 \to \mathbb{R}\) is defined by

\[
(\theta, x) \mapsto \phi_i(\theta, x) := \chi_J \circ \pi_{I_0} \circ \varphi^i \circ (\alpha^{-1} \times i d)^i(\theta, x).
\]

Using Fubini’s Theorem, the measurability of \(\phi_i\) (considering the \(\sigma\)-algebra \(B_\mathbb{X} \times B_{I_0}\)) implies the measurability of \(\theta \mapsto \eta_n^i(\theta)\) (considering the \(\sigma\)-algebra \(B_\mathbb{X}\)). \(\square\)
9.2 The measures $\mu_n$ are well defined

We assume the skew-product satisfies the property $(H_4)$. The proof for the case of $(H_4^*)$ is entirely analogous. It is enough to substitute $\mathcal{C}$ by $\mathcal{D}$.

As in the case of $\eta_n$, the well-definition of the measures $\mu_n$ follows from Hahn Extension Theorem and the following result which implies that these measures are defined on the algebra of the rectangles.

**Proposition 9.2** Let $J \subset I_0$ be a borelian set. For every $n \in \mathbb{N}$, the function $X \ni \theta \mapsto \mu_n(\theta)(J)$ is measurable.

In the definition of the measures $\mu_n$ appear the sets $H_j(\theta, \zeta)$ $(j \in \mathbb{N}, \theta \in X)$. These sets depend on the maps $r_j(\theta, x)$ and $I^+_j(\theta, x) := |f^j_\theta(T^j(\theta, x))|$. We study first the measurability of these functions.

Let us recall the definition of the function $r_i$ (given in Sect. 4). Given $i \in \mathbb{N}$ and a point $(\theta, x) \in X \times I_0$, we denote by $T_i(\theta, x)$ the maximal interval such that $f^j_\theta(T_i(\theta, x)) \cap \mathcal{C}_{\mathcal{D}_j}(\theta) = \emptyset$ for all $j < i$. Thus $r_i(\theta, x)$ denotes the minimum of the lengths of the connected components of $f^j_\theta(T_i(\theta, x) \setminus \{x\})$.

**Lemma 9.3** The maps $r_i : X \times I_0 \to \mathbb{R}$ are measurable, for all $i \in \mathbb{N}$.

**Proof of Lemma 8.4** For fixed $\theta \in X$, $x \mapsto r_i(\theta, x)$ is a continuous function, since $f^j_\theta$ (for $\theta \in X$, $i \in \mathbb{N}$) are piecewise continuous $C^3$ maps. Hence, by [23, Lemma 9.2], we conclude $r_i$ is measurable, if for fixed $x \in I_0$ the function $\theta \mapsto r_i(\theta, x)$ is measurable. We claim that this last condition is true. To prove it, we write $r_i(\cdot, x)$ as a composition of measurable maps.

For $i \in \mathbb{N}$, let us define the set

$$\mathcal{C}^i = \bigcup_{j=0}^{i-1} \varphi^{-j} \mathcal{C} \cup (X \times \partial I_0)$$

Given $(\theta, x) \in X \times I_0$, the interval $T_i(\theta, x) = (a_i(\theta, x), b_i(\theta, x))$ can be defined in the following way

$$a_i(\theta, x) = \sup(E^{x^-})_{\theta} := \sup\{y \in I_0; (\theta, y) \in E^{x^-}\}$$

$$b_i(\theta, x) = \inf(E^{x^+})_{\theta} := \inf\{y \in I_0; (\theta, y) \in E^{x^+}\}$$

where $E^{x^-} = (X \times (-\infty, x] \cap I_0) \cap \mathcal{C}^i$ and $E^{x^+} = (X \times [x, +\infty) \cap I_0) \cap \mathcal{C}^i$. The sets $E^{x^-}$ and $E^{x^+}$ are measurable, since by hypotheses $(H_1)$, $\mathcal{C}$ is measurable. Then, for fixed $x \in I_0$, the measurability of the functions $\theta \mapsto a_i(\theta, x)$ and $\theta \mapsto b_i(\theta, x)$ follows from the next result.

**Claim 9.4** Let $E$ be a set in $B_X \times B_0$ and let $S : X \to I_0, s : X \to I_0$ be functions defined by $S(\theta) = \sup E_0 = \sup\{y \in I_0; (\theta, y) \in E\}, s(\theta) = \inf E_0 = \inf\{y \in I_0; (\theta, y) \in E\}$. Then $S$ and $s$ are measurable maps.

**Proof of Lemma 8.4** We prove first for the map $S$. Let $b \in \mathbb{R}$ be a constant. We want to prove that $S^{-1}((b, +\infty)) \in B_X$. First, let us suppose that $E$ is an open set on $X \times I_0$. Let $\theta_0$ be any point in $S^{-1}((b, +\infty))$. Then there exists $y_0 \in I_0$ such that $y_0 > b$ and $(\theta_0, y_0) \in E$. The openness of $E$ shows the existence of open sets $A \subset X$ and $B \subset I_0$ such that $(\theta_0, y_0) \in A \times B \subset E$. Thus $A \subset S^{-1}((b, +\infty))$ and it shows that $S^{-1}((b, +\infty))$ is an open set.
In the general case, given any measurable set $E$, let us consider the sets

$$B\left(E, \frac{1}{n}\right) = \left\{ z \in \mathbb{K} \times I_0; \text{ dist}(z, w) < \frac{1}{n} \text{ for some } w \in E \right\}.$$ 

for $n \in \mathbb{N}$. We consider the functions $S_n(\theta) = \sup\{y \in I_0; (\theta, y) \in B(E, 1/n)\}$. These functions are measurable by what we have proved. Since $S = \inf_{n \in \mathbb{N}} S_n$, the measurability of $S$ follows. \hfill \Box

Using the measurability of $a_i(\theta, x)$ and $b_i(\theta, x)$ we conclude the measurability of $\theta \mapsto r_j(\theta, x)$ (all for fixed $x \in I_0$). It finishes the proof of Lemma 9.3.

Now, we want to prove the measurability of the maps $l^n_j$. Let us consider a sequence of measurable partitions $\cdots \subset \mathcal{P}_{n+1} \subset \mathcal{P}_n \subset \cdots \subset \mathcal{P}_1$ of $I_0$ such that the norm of $\mathcal{P}_n$ is less than $1/n$. Choose a point $x^n_i$ in each $P^n_i$ element of $\mathcal{P}_n$ and define the functions

$$l^n_j(\theta, x) := |f^j_\theta(T^j(\theta, x^n_i))| \text{ for all } x \in P^n_i.$$ 

We also consider the map $l_j := \lim_{n \to \infty} l^n_j$.

**Lemma 9.5** The maps $l_j : \mathbb{K} \times I_0 \to \mathbb{R}$ are measurable for all $j \in \mathbb{N}$.

**Proof of Lemma 8.4** For fixed $x \in I_0$, the maps $\theta \mapsto |f^j_\theta(T^j(\theta, x))|$ are measurable, since

$$|f^j_\theta(T^j(\theta, x))| = |\pi_{I_0} \circ \varphi^j \circ (id, a_i(\cdot, x))(\theta) - \pi_{I_0} \circ \varphi^j \circ (id, b_i(\cdot, x))(\theta)|.$$ 

Therefore the maps $l^n_j$ are measurable. Obviously it implies the measurability of maps $l_j$. \hfill \Box

**Proof of Proposition 9.2** By Lemma 9.5, the map $l_j$ is measurable and $l_j(\theta, x) = l^n_j(\theta, x)$ if $r_j(\theta, x) > 0$. By Lemma 9.3, the sets $\mathcal{H}(\sigma):=\{z \in \mathbb{K} \times I_0; r_j(z) > \sigma\}$ are measurable, for any $\sigma > 0$. These facts imply that $H_j(\sigma) := \mathcal{H}(\sigma) \cap (l^n_j)^{-1}(3\sigma, \infty)$ is a measurable set.

Let us fix a set $J \in \mathcal{B}_{I_0}$. As on Proposition 9.1, to prove the measurability of $\theta \mapsto \mu_n(\theta)(J)$ it suffices to prove the measurability of the functions $\theta \mapsto \mu^i_j(\theta) := (f^i_\alpha J(\theta, x)) \ast (m |H_{\alpha^{-1}}(\zeta, \cdot) \cap Z(\alpha^{-1}(\theta), \lambda))(J)$, for $i \in \mathbb{N}$. Now, we have that $\mu^i_j(\theta) = \int_{I_0} \phi_i(\theta, x) \psi_i(\theta, x) d m(x)$, where $\phi_i, \psi_i : \mathbb{K} \times I_0 \to \mathbb{R}, \phi_i$ are respectively defined in (9.1) and

$$(\theta, x) \mapsto \psi_i(\theta, x) := \chi_{H_i}(\zeta) \circ (\alpha^{-1} \times id)^i(\theta, x) \cdot \chi_{Z(\alpha)} \circ (\alpha^{-1} \times id)^i(\theta, x).$$

Once again, using Fubini’s Theorem, the measurability of $(\theta, x) \mapsto \phi_i(\theta, x) \psi_i(\theta, x)$ implies the measurability of $\theta \mapsto \mu^i_j(\theta)$. \hfill \Box

9.3 Higher-dimensional fibers

9.3.1 The measures $\eta_n$ are well defined

This case is precisely the same as the case with one-dimensional fibers, so we have nothing to add.

9.3.2 The measures $\mu_n$ are well defined

From the definition of $\mu_n$ in the higher dimensional case, we see that it is enough to show that for every $n \in \mathbb{N}$ and Borel set $S \subset \mathbb{K}$ the function $\mathbb{K} \ni \theta \mapsto \mu_n(\theta)(S)$ is measurable. For this it is enough to prove the following.

\begin{equation}
\mathbb{C} \text{ Springer}
\end{equation}
Lemma 9.6  The function $\mathbb{X} \ni \theta \mapsto \text{Leb} \left( \mathcal{H}_j (\alpha^{-j} (\theta)) \cap (f^j_{\alpha^j (\theta)})^{-1} (S) \right)$ is measurable for each fixed $j \in \mathbb{N}$ and measurable $S \subset \mathbb{Y}$.

Analogously to the previous subsection, we consider the maps

$$\alpha^{-1} \times id : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \times \mathbb{Y} \quad \pi_\mathbb{X} : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \quad \pi_\mathbb{Y} : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}$$

$$(\theta, x) \mapsto (\alpha^{-1} (\theta), x) \quad (\theta, x) \mapsto \theta \quad (\theta, x) \mapsto x$$

and $\chi_S$ the characteristic function of $S$. These functions are all measurable with respect to the corresponding Borel $\sigma$-algebras. We consider also $\chi_{\mathcal{H}_n}$ the characteristic function of $\mathcal{H}_n(\sigma, \delta, b)$.

Lemma 9.7 The set $\mathcal{H}_n(\sigma, \delta, b)$ is a Borel subset of $\mathbb{X} \times \mathbb{Y}$.

Proof of Lemma 8.4  According to the definition of $(\sigma, \delta, b)$-hyperbolic time

$$\mathcal{H}_n(\sigma, \delta, b) = \{ (\theta, x) \in \mathbb{X} \times \mathbb{Y} : (8.1) \text{ is true for } (\theta, x) \}$$

is an intersection of at most finitely many sets of the form $\{ (\theta, x) \in \mathbb{X} \times \mathbb{Y} : g(\theta, x) > c \}$ for a measurable function $g : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{R}$ and some constant $c \in \mathbb{R}$. Indeed, if we define for $k = 0, \ldots, n - 1$

$$g_k(\theta, x) := \prod_{j=-n}^{-k+1} \| D f_{\alpha^{-j} (\theta)} (f^j_{\theta} (x))^{-1} \| \quad \text{and} \quad d_k(\theta, x) := \text{dist}_\mathbb{X} \left( f^k_{\theta} (x), \mathscr{S} \cap (s_k(\theta)) \times \mathbb{Y} \right) ,$$

then we can write

$$\mathcal{H}_n(\sigma, \delta, b) = \{ (\theta, x) \in \mathbb{X} \times \mathbb{Y} : g_k(\theta, x) < \sigma^k \quad \text{and} \quad d_k(\theta, x) > e^{-b k}, k = 0, \ldots, n - 1 \} .$$

Thus $\mathcal{H}_n(\sigma, \delta, b)$ is a Borel subset of $\mathbb{X} \times \mathbb{Y}$ as soon as we show that $g_k, d_k$ are measurable functions for each $k \geq 0$.

Clearly $g_k$ is measurable from condition $(H_6)$. For the functions $d_k : \mathbb{X} \times \mathbb{Y} \rightarrow [0, +\infty)$ we clearly have

$$d_k(\theta, x) = D(\alpha^k(\theta), f^k_{\theta} (x)) \quad \text{where} \quad D(\theta, x) = \inf \xi_{(\theta, x)}$$

and we define

$$\xi(\theta, x, y) = \xi_{(\theta, x)}(y) := \text{dist}_\mathbb{X} (x, y) \cdot \chi_{\mathscr{S}} (\theta, y) + \delta \cdot (1 - \chi_{\mathscr{S}} (\theta, y)) .$$

Clearly $\xi : \mathbb{X} \times \mathbb{Y} \times \mathbb{Y} \rightarrow [0, \delta]$ is measurable, so $D : \mathbb{X} \times \mathbb{Y} \rightarrow [0, \delta]$ is also measurable and $d_k$ is a composition of $D$ with other measurable maps from condition $(H_5)$. This completes the argument showing that $\mathcal{H}_n(\sigma, \delta, b)$ is a Borel subset of $\mathbb{X} \times \mathbb{Y}$.  

Now we are ready to prove the first lemma.

Proof of Proposition 9.6  We note that we can write

$$\text{Leb} \left( \mathcal{H}_j (\alpha^{-j} (\theta)) \cap (f^j_{\alpha^j (\theta)})^{-1} (S) \right) = \int \phi_j(\theta, x) \psi_j(\theta, x) \, d \text{Leb}(x), \quad (9.2)$$

where

$$\phi_j(\theta, x) := \chi_S \circ \pi_\mathbb{Y} \circ \varphi^j \circ (\alpha^{-j} \times id)^j (\theta, x) \quad \text{and} \quad \psi_j(\theta, x) := \chi_{\mathcal{H}_n} \circ (\alpha^{-j} \times id)^j (\theta, x) .$$

Since both $\phi_j$ and $\psi_j$ are Borel measurable from $\mathbb{X} \times \mathbb{Y}$ to $\mathbb{R}$, Fubini’s Theorem ensures that $(9.2)$ is a measurable function of $\theta \in \mathbb{X}$, as we need. This concludes the proof.  

With Lemma 9.6 we complete the proof of the measurability of all functions used in the previous sections.
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