TRANSFORMATIONS BETWEEN ATTRACTORS OF HYPERBOLIC ITERATED FUNCTION SYSTEMS

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Abstract. This paper is in the form of an essay. It defines fractal tops and code space structures associated with set-attractors of hyperbolic iterated function systems (IFSs). The fractal top of an IFS is associated with a certain shift invariant subspace of code space, whence the entropy of the IFS, and of its set-attractor, may be defined. Given any ordered pair of hyperbolic IFSs, each with the same number of maps, there is a natural transformation, constructed with the aid of fractal tops, whose domain is the attractor \( A_1 \) of the first IFS and whose range is contained in the attractor \( A_2 \) of the second IFS. This transformation is continuous when the code space structure of the IFS is "contained in" the code space structure of the second IFS, and is a homeomorphism between \( A_1 \) and \( A_2 \) when the code space structures are the same. Conversely, if two IFS are homeomorphic then they possess the same code space structure. Hence we obtain that two IFS attractors are homeomorphic then they have the same entropy. Several examples of fractal transformations and fractal homeomorphisms are given.

1. Introduction

In this essay we introduce fractal transformations. The main examples are fascinating mappings between diverse subsets of \( \mathbb{R}^2 \); they can be readily illustrated by using the chaos game. Fractal transformations can be quickly grasped because they rely on basic notions in topology, probability, dynamical systems, and geometry. They may be applied to computer graphics to produce digital content with new look-and-feel \[4\]; they may also be relevant to image compression and biological modelling.

2. Hyperbolic IFS

Definition 1. Let \((X, d_X)\) be a complete metric space. Let \(\{f_1, f_2, \ldots, f_N\}\) be a finite sequence of strictly contractive transformations, \(f_n : X \to X\), for \(n = 1, 2, \ldots, N\). Then

\[ F := \{X; f_1, f_2, \ldots, f_N\} \]

is called a hyperbolic iterated function system or hyperbolic IFS.
A transformation $f_n : X \to X$ is strictly contractive if there exists a number $l_n \in [0, 1)$ such that $d(f_n(x), f_n(y)) \leq l_n d(x, y)$ for all $x, y \in X$. The number $l_n$ is called a contractivity factor for $f_n$ and the number

$$l = \max\{l_1, l_2, ..., l_N\}$$

is called a contractivity factor for $F$.

Let $\Omega$ denote the set of all infinite sequences of symbols $\{\sigma_k\}_{k=1}^{\infty}$ belonging to the alphabet $\{1, ..., N\}$. We write $\sigma = \sigma_1 \sigma_2 \sigma_3 ... \in \Omega$ to denote a typical element of $\Omega$, and we write $\omega_k$ to denote the $k^{th}$ element of $\omega \in \Omega$. Then $(\Omega, d_\Omega)$ is a compact metric space, where the metric $d_\Omega$ is defined by $d_\Omega(\sigma, \omega) = 0$ when $\sigma = \omega$ and $d_\Omega(\sigma, \omega) = 2^{-k}$ when $k$ is the least index for which $\sigma_k \neq \omega_k$. We call $\Omega$ the code space associated with the IFS $F$.

Let $\sigma \in \Omega$ and $x \in X$. Then, using the contractivity of $F$, it is straightforward to prove that

$$\phi_F(\sigma) := \lim_{k \to \infty} f_{\sigma_1} \circ f_{\sigma_2} \circ ... f_{\sigma_k}(x)$$

exists, uniformly for $x$ in any fixed compact subset of $X$, and depends continuously on $\sigma$. See for example [1], Theorem 3. Let

$$A_F = \{\phi_F(\sigma) : \sigma \in \Omega\}.$$ 

Then $A_F \subset X$ is called the attractor of $F$. The continuous function

$$\phi_F : \Omega \to A_F$$

is called the address function of $F$. We call $\phi_F^{-1}(\{x\}) := \{\sigma \in \Omega : \phi_F(\sigma) = x\}$ the set of addresses of the point $x \in A_F$.

Clearly $A_F$ is compact, nonempty, and has the property

$$A_F = f_1(A_F) \cup f_2(A_F) \cup ... \cup f_N(A_F).$$

Indeed, if we define $H(X)$ to be the set of nonempty compact subsets of $X$, and we define $F : H(X) \to H(X)$ by

$$(2.1) \quad F(S) = f_1(S) \cup f_2(S) \cup ... \cup f_N(S),$$

for all $S \in H(X)$, then $A_F$ can be characterized as the unique fixed point of $F$, see [11], section 3.2, and [22].

IFSs may be used to represent diverse subsets of $\mathbb{R}^2$. For example, let $A, B,$ and $C$, denote three noncollinear points in $\mathbb{R}^2$. Let $a$ denote a point on the line segment $AB$, let $b$ denote a point on the line segment $BC$ and let $c$ denote a point on the line segment $CA$, such that $\{a, b, c\} \cap \{A, B, C\} = \emptyset$, see panel (i) of Figure [1]. Let $h_1 : \mathbb{R}^2 \to \mathbb{R}^2$ denote the unique affine transformation such that

$$h_1(ABC) = aBc,$$

by which we mean that $h_1$ maps $A$ to $a$, $B$ to $B$, and $C$ to $c$. Using the same notation, let affine transformations $h_2$, $h_3$, and $h_4$ be uniquely defined by

$$h_2(ABC) = abC, \quad h_3(ABC) = Abc, \quad \text{and} \quad h_4(ABC) = abc.$$ 

Let $F_{a, b, c} = \{\mathbb{R}^2; h_1, h_2, h_3, h_4\}$ where $\alpha = |BC|/|AB|$, $\beta = |CA|/|BC|$ and $\gamma = |Ab|/|CA|$. The attractor of $F_{a, b, c}$ is the filled triangle with vertices at $A$, $B$, and $C$. The attractor of the IFS $\{\mathbb{R}^2; h_1, h_2, h_3\}$ is an affine Sierpinski triangle, as illustrated in (iii) in Figure [1].
FRACTAL TRANSFORMATIONS

Figure 1. (i) The points used to define the affine transformations $h_n : \mathbb{R}^2 \to \mathbb{R}^2$ for $n = 1, 2, 3, 4$; (ii) sketch of the attractor of the IFS \{\(R^2; h_1, h_2, h_3, h_4\)\}; (iii) sketch of the attractor of the IFS \{\(R^2; h_1, h_2, h_3\)\}. Here $\alpha = 0.65$, $\beta = 0.3$, and $\gamma = 0.4$.

| $n$ | $a_n$ | $b_n$ | $c_n$ | $d_n$ | $e_n$ | $l_n$ |
|-----|-------|-------|-------|-------|-------|-------|
| 1   | $-1 + \beta$ | $-\frac{1}{2} + \frac{3}{4} \beta + \frac{1}{4} \alpha$ | $1 - \beta$ | 0 | $\alpha$ | 0 |
| 2   | $\beta + \frac{1}{2} \gamma - \frac{3}{2}$ | $\beta - \frac{3}{2} \gamma + \frac{3}{4} \beta$ | $1 - \beta$ | $1 - \gamma$ | $\frac{1}{4} \gamma - \frac{1}{2}$ | 0 |
| 3   | $\gamma$ | $\frac{1}{4} \gamma$ | $1$ | $-\gamma$ | $-1 + \alpha + \frac{1}{2} \gamma$ | 1 |
| 4   | $\beta + \frac{1}{2} \gamma - \frac{3}{2}$ | $-\frac{3}{2} + \frac{3}{4} \beta + \frac{1}{4} \alpha - \frac{1}{4} \gamma$ | $1 - \beta$ | $1 - \gamma$ | $\alpha - \frac{1}{2} + \frac{1}{2} \gamma$ | 0 |

Table 1.

For reference we note that when $A = (0, 0)$, $B = (0, 1)$, and $C = (0.5, 1)$, the transformations of the IFS $F_{\alpha, \beta, \gamma}$ are given by

$$h_n(x, y) = (a_n x + b_n y + c_n, d_n x + e_n y + l_n)$$

with the parameters specified in Table 1. We will write $\bullet$ to denote the filled triangle $ABC$.

3. Chaos game

When the underlying space is the euclidean plane, one way to sketch the attractor $A_F$ of an IFS $F$ is to plot the set of points

$$\tilde{A}_F = \{f_{\sigma_1} \circ f_{\sigma_2} \circ ... f_{\sigma_K}(x) : \sigma_k \in \{1, 2, ..., N\}, k = 1, 2, ...K\},$$

for some $x \in \mathbb{R}^2$ and some integer $K$. The Hausdorff distance between $A_F$ and $\tilde{A}_F$ is bounded above by $C \cdot l^K$ where the constant $C$ depends only on $F$ and $x$.

A more efficient method is by means of a type of Markov Chain Monte Carlo algorithm which we refer to as the chaos game. Starting from any point $(x_0, y_0) \in \mathbb{R}^2$, a sequence of a million or more points $\{(x_k, y_k)\}_{k=0}^{K}$ is computed recursively; at the $k^{th}$ iteration one of the functions of $F$ is chosen at random, independently of all other choices, and applied to $(x_{k-1}, y_{k-1})$ to produce $(x_k, y_k)$ which is plotted when $k \geq 100$. The result will be usually a sketch of the attractor of the IFS, accurate to within viewing resolution.
The reason that the chaos game yields, almost always, a "picture" of the attractor of an IFS depends on Birkhoff’s ergodic theorem, see for example [9]. The scholarly history of the chaos game is discussed in [12] and [21], and appears to begin in 1935 with the work of Onicescu and Mihok, [17]. Mandelbrot used a version of it to help compute pictures of certain Julia sets, [15] pp.196-199; it was introduced to IFS theory and developed by the author and coworkers, see for example [1], [2], [6], and [8], where the relevant theorems and much discussion can be found. Its applications to fractal geometry were popularized initially by the author and others, see for example [1], [7], [19], and [20].

The sketches in panels (ii) and (iii) of Figure 1 were computed using the chaos game. At each iteration the function \( h_n \) was selected with probability proportional to the area of the triangle \( h_n(ABC) \), for \( n = 1, 2, 3, 4 \).

In section 7 we show how the chaos game may be modified to calculate examples of the fractal transformations that are the subject of this article. Hopefully you will be inspired to try this new application of the chaos game.

4. THE TOPS FUNCTION

We order the elements of \( \Omega \) according to

\[ \sigma < \omega \iff \sigma_k > \omega_k \]

where \( k \) is the least index for which \( \sigma_k \neq \omega_k \). This is a linear ordering, sometimes called the lexicographic ordering.

Notice that all elements of \( \Omega \) are less than or equal to \( \overline{T} = 11111... \) and greater than or equal to \( \underline{T} = NNNNN... \). Also, any pair of distinct elements of \( \Omega \) is such that one member of the pair is strictly greater than the other. In particular, the set of addresses of a point \( x \in A_F \) is both closed and bounded above by \( \overline{T} \). It follows that \( \phi_F^{-1}(\{x\}) \) possesses a unique largest element. We denote this element by \( \tau_F(x) \).

**Definition 2.** Let \( F \) be a hyperbolic IFS with attractor \( A_F \) and address function \( \phi_F : \Omega \rightarrow A_F \). Let

\[ \tau_F(x) = \max \{ \sigma \in \Omega : \phi_F(\sigma) = x \} \text{ for all } x \in A_F. \]

Then

\[ \Omega_F := \{ \tau_F(x) : x \in A_F \} \]

is called the tops code space and

\[ \tau_F : A_F \rightarrow \Omega_F \]

is called the tops function, for the IFS \( F \).

Notice that the tops function \( \tau_F \) is one-to-one. It provides a right-hand inverse to the address function, according to

\[ \phi_F \circ \tau_F = i_{A_F} \]

where \( i_{A_F} \) denotes the identity function on \( A_F \) and \( \circ \) denotes composition of functions. Let

\[ \Phi_F : \Omega_F \rightarrow A_F \]

denote the restriction of \( \phi_F \) to \( \Omega_F \), defined by \( \Phi_F(\sigma) = \phi_F(\sigma) \) for all \( \sigma \in \Omega_F \). Then \( \Phi_F \) is the inverse of \( \tau_F \), namely

\[ \Phi_F = \tau_F^{-1}. \]
We note that although $\Phi_F$ is one-to-one, onto, and continuous, $\tau_F$ may not be continuous. Let $\overline{\Omega}_F$ denote the closure of $\Omega_F$, treated as a subset of the metric space $(\Omega, d_\Omega)$. Let

$$\Phi_F : \overline{\Omega}_F \rightarrow A_F$$

denote the restriction of $\phi_F$ to $\overline{\Omega}_F$. Then $\Phi_F$ is continuous and onto. Notice that the ranges of $\Phi_F$ and $\Phi_F$ are both equal to $A_F$ because $A_F$ is closed.

5. Fractal Transformations

Let $\mathcal{G}$ denote a hyperbolic IFS that consists of $N$ functions. Then $\phi_\mathcal{G} \circ \tau_F : A_F \rightarrow A_\mathcal{G}$ is a mapping from the attractor of $\mathcal{F}$ into the attractor of $\mathcal{G}$. We refer to $\phi_\mathcal{G} \circ \tau_F$ as a fractal transformation.

In order to illustrate transformations between subsets of $\mathbb{R}^2$ we use pictures. We define a picture function to be a function of the form

$$\mathcal{P} : D_\mathcal{P} \subset \mathbb{R}^2 \rightarrow \mathbb{C}$$

where $\mathbb{C}$ is a color space. A picture function $\mathcal{P}$ assigns a unique color to each point in its domain $D_\mathcal{P}$. For example we may have $\mathbb{C} = \{0,1,...255\}^3$ and each point of $\mathbb{C}$ may specify the red, green, and blue components of a color. A picture in the non-mathematical sense may be thought of as a physical representation of the graph of a picture function.

If $T : D_\Omega \subset \mathbb{R}^2 \rightarrow D_\mathcal{P}$ then $\Omega = \mathcal{P} \circ T$ denotes a picture whose domain is $D_\Omega$. We can obtain insights into the nature of $T$ by comparing the picture functions $\mathcal{P}$ and $\mathcal{P} \circ T$, where $\mathcal{P}$ represents a given picture which may be varied. We will use this method to illustrate fractal transformations.

Let

$$\mathcal{F} = \{\mathbb{C}; f_1(z) = sz - 1, f_2(z) = sz + 1, \text{ for all } z \in \mathbb{C} \},$$

$$\mathcal{G} = \{\mathbb{C}; g(z) = s_1 z - 1, g(z) = s_1 z + 1, \text{ for all } z \in \mathbb{C} \},$$

$$\mathcal{H} = \{\mathbb{C}; h(z) = s_2 z - 1, h(z) = s_2 z + 1, \text{ for all } z \in \mathbb{C} \},$$

where $\mathbb{C}$ denotes the complex plane, $s = 0.5(1 + i)$, $s_1 = 0.44(1 + i)$, and $s_2 = 0.535(1 + i)$. We denote the attractors of these IFSs by $A_F$, $A_G$, and $A_H$. In the top row of Figure 2, we illustrate, from left to right, the three picture functions, $\mathcal{P}_\mathcal{G} : A_G \rightarrow \mathbb{C}$, $\mathcal{P}_F : A_F \rightarrow \mathbb{C}$, and $\mathcal{P}_H : A_H \rightarrow \mathbb{C}$. These pictures were obtained by masking a single original digital picture, whose domain we took to be $\{z = x + iy \in \mathbb{C} : -3.5 \leq x \leq 3.5, -3.5 \leq y \leq 3.5\}$, by the complement of each of the sets $A_G$, $A_F$, and $A_H$.

The attractor $A_F$ is a so-called twin-dragon fractal. It is an example of a just-touching attractor: that is, $f_1(A_F) \cap f_2(A_F)$ is non-empty and equals $f_1(\partial A_F) \cap f_2(\partial A_F)$ where $\partial A_F$ denotes the boundary of $A_F$. This contrasts with $A_G$ which is totally disconnected, perfect, and in fact homeomorphic to the classical cantor set. This also contrasts with $A_H$ which is such that there exists a disk in $\mathbb{R}^2$, of non-zero radius, which is contained in $h_1(A_H) \cap h_2(A_H)$.

The bottom row of Figure 2 illustrates the pictures, from left to right, $\mathcal{P}_\mathcal{G} \circ \phi_\mathcal{G} \circ \tau_F$, $\mathcal{P}_F \circ \phi_\mathcal{F} \circ \tau_F$, and $\mathcal{P}_H \circ \phi_\mathcal{H} \circ \tau_F$. They were computed by a variant of the chaos game as explained in section 7. The domain of each of these pictures is $A_F$. We notice that $\mathcal{P}_F \circ \phi_\mathcal{F} \circ \tau_F = \mathcal{P}_F$, which is true regardless of the choice of IFS $\mathcal{F}$ since $\phi_\mathcal{F} \circ \tau_F$ is the identity on $A_F$. We notice that both $\mathcal{P}_G \circ \phi_\mathcal{G} \circ \tau_F$ and $\mathcal{P}_H \circ \phi_\mathcal{H} \circ \tau_F$ have features in common with the underlying digital picture; for example, $\mathcal{P}_G \circ \phi_\mathcal{G} \circ \tau_F$ displays something like the texture of the hat, near the middle of the bottom-left image.
Examples of fractal transformations are illustrated using the three picture functions $\Psi_G$, $\Psi_F$, and $\Psi_H$ shown in the top row, from left to right. The domains of these functions are the attractors $A_G$, $A_F$, $A_H$ of three IFSs $F$, $G$, and $H$, defined in the text. The bottom row illustrates the pictures $\Psi_G \circ \phi_G \circ \tau_F$, $\Psi_F \circ \phi_F \circ \tau_F$, and $\Psi_H \circ \phi_H \circ \tau_F$.

The bottom right image shows parts of the hat, repeated several times, and some clearly delineated small twin-dragon tiles.

We are led to consider the following questions. Under what conditions on general IFSs $F$ and $G$ is the fractal transformation $\phi_G \circ \tau_F$ continuous? When does it provide a homeomorphism between $A_F$ and $A_G$?

**Definition 3.** The address structure of $F$ is defined to be the set of sets

$$C_F = \{ \phi_F^{-1}(\{x\}) \cap \overline{\Omega}_F : x \in A_F \}.$$

The address structure of an IFS is a certain partition of $\overline{\Omega}_F$. Let $C_G$ denote the address structure of $G$. Let us write $C_F \prec C_G$ to mean that for each $S \in C_F$ there is $T \in C_G$ such that $S \subset T$. Notice that if $C_F = C_G$ then $\Omega_F = \Omega_G$. Some examples of address structures are given in section 6.

**Theorem 1.** Let $F$ and $G$ be two hyperbolic IFSs such that $C_F \prec C_G$. Then the fractal transformation $\phi_G \circ \tau_F : A_F \to A_G$ is continuous. If $C_F = C_G$ then $\phi_G \circ \tau_F$ is a homeomorphism.

The proof relies on a standard result in topology, Lemma 2 below, which we present in the context of metric spaces.

**Lemma 1.** (cf. [16], bottom of p. 194.) Let $F : X \to Y$ be a continuous mapping from a compact metric space $X$ onto a metric space $Y$. Then $S \subset Y$ is open if and only if $F^{-1}(S) \subset X$ is open.
Proof. If \( S \subset Y \) is open then \( F^{-1}(S) \subset X \) is open because \( F : X \to Y \) is continuous. Suppose that \( F^{-1}(S) \) is open. Then \( X \setminus F^{-1}(S) \) is closed. But a closed subset of a compact metric space is compact. The continuity of \( F \) now implies that \( F(X \setminus F^{-1}(S)) \) is compact and hence closed. But \( F(X \setminus F^{-1}(S)) = Y \setminus S \). Hence \( S \) is open.

**Lemma 2.** (cf. [16], Proposition 7.4 on p.195) Let \( F : X \to Y \) be a continuous mapping from a compact metric space \( X \) onto a metric space \( Y \). Let \( H : Y \to Z \) where \( Z \) is a metric space. Let \( H \circ F : X \to Z \) be continuous. Then \( H : Y \to Z \) is continuous.

**Proof.** Let \( O \subset Z \) be open. Then \((H \circ F)^{-1}(O) = F^{-1}(H^{-1}(O))\) is open. But then by Lemma 1 \( H^{-1}(O) \) is open. Hence \( H : Y \to Z \) is continuous. \( \square \)

**Proof of Theorem 2.** In Lemma 2 we set \( X = \overline{\Omega}_F \), \( Y = A_F \), and \( Z = A_G \). We choose \( F : X \to Y \) to be \( \overline{\Omega}_F : \overline{\Omega}_F \to A_F \). Then \( F : X \to Y \) is a continuous mapping from a compact metric space \( X \) onto a metric space \( Y \). We also choose \( H : Y \to Z \) to be

\[
H = \phi_G \circ \tau_F : A_F \to A_G.
\]

Now look at the function

\[
G := H \circ F = \phi_G \circ \tau_F \circ \overline{\Omega}_F : \overline{\Omega}_F \to A_G.
\]

If \( \sigma \in \overline{\Omega}_F \), then both \( \sigma \) and \( (\tau_F \circ \overline{\Omega}_F)(\sigma) \) belong to the same set in the \( C_F \). Since \( C_F \prec C_G \) it follows that both \( \sigma \) and \( (\tau_F \circ \overline{\Omega}_F)(\sigma) \) belong to the same set in \( C_G \). It follows that

\[
(\phi_G \circ \tau_F \circ \overline{\Omega}_F)(\sigma) = \phi_G(\sigma) \text{ for all } \sigma \in \overline{\Omega}_F.
\]

But \( \phi_G : \Omega \to A_G \) is continuous. Hence \( G \) is continuous.

We have shown that the conditions in Lemma 2 hold. It follows that \( H = \phi_G \circ \tau_F \) is continuous.

When \( C_F = C_G \) it is readily verified that \( \phi_G \circ \tau_F : A_F \to A_G \) is one-to-one and onto and that its inverse is \( \phi_F \circ \tau_G \). Also \( C_F \prec C_G \) implies \( C_G \prec C_F \) and so, by the first part of the theorem, \( \phi_F \circ \tau_G \) is continuous. Hence \( \phi_G \circ \tau_F : A_F \to A_G \) is a homeomorphism. \( \square \)

6. **Examples of address structures**

6.1. **Backwards orbits.** Let \( x \in A_F \). Let \( \sigma \in \Omega \) be such that \( \phi_F(\sigma) = x \). Assume that the \( f_n \)'s are one-to-one. Then define \( x_0 = x \) and \( x_k = f^{-1}_{\sigma_k}(x_{k-1}) \) for \( k = 1, 2, 3, ... \). Notice that \( x_k = \phi_F(\sigma_k \sigma_{k+1} \sigma_{k+2} \ldots) \). We call \( \{x_k\}_{k=0}^\infty \) a backwards orbit of \( x \) (under the IFS \( F \)).

The set of all addresses of \( x \) can be calculated by following all possible backwards orbits of \( x \). Define a sequence of points \( \{\overline{x}_k\}_{k=0}^\infty \) in \( A_F \) and an address \( \overline{\sigma} = \overline{\sigma}_1 \overline{\sigma}_2 \overline{\sigma}_3 \ldots \in \Omega \), as follows. Let \( \overline{x}_0 = x \). For each \( k = 1, 2, 3, ... \) first choose

\[
\overline{\sigma}_k \in \{n \in \{1, 2, ..., N\} : \overline{x}_{k-1} \in f_n(A_F)\}
\]

and then define

\[
\overline{x}_k = f^{-1}_{\overline{\sigma}_k}(\overline{x}_{k-1}).
\]

Then \( \overline{\sigma} \in \phi_F^{-1}(\{x\}) \) and all \( \overline{\sigma} \in \phi_F^{-1}(\{x\}) \) can be obtained in this manner.
6.2. Some notation. We use the notation \((PQ) = PQ\{P,Q\}\) to denote the straight line segment which connects the two points \(P\) and \(Q\) in \(\mathbb{R}^2\), without its endpoints. We write \(\Omega'\) to denote the set of all finite length strings of symbols from the alphabet \(\{1,2,...,N\}\), including the empty string "\(\emptyset\)". We write \(|\sigma|\) to denote the length of \(\sigma \in \Omega'\). We define \(\omega \sigma = \omega_1 \omega_2 ... \omega_{|\omega|}\sigma_1 \sigma_2 ... \sigma_{|\sigma|}\) for all \(\omega, \sigma \in \Omega'\). Similarly we define \(\omega \sigma = \omega_1 \omega_2 ... \omega_{|\omega|}\sigma_1 \sigma_2 ... \sigma_{|\sigma|}\) for all \(\omega \in \Omega', \sigma \in \Omega\). We write \(S : \Omega' \to \Omega'\) to denote the shift operator defined by \(S(\sigma) = \sigma_2 \sigma_3 ... \sigma_{|\sigma|}\) when \(|\sigma| \geq 1\) and \(S(\emptyset) = \emptyset\). We write \(f_\sigma = f_{\sigma_1} \circ f_{\sigma_2} \circ ... \circ f_{\sigma_{|\sigma|}}\) for all \(\sigma \in \Omega'\) with \(|\sigma| \geq 1\) and \(f_{\emptyset}\) denotes the identity function.

6.3. Example 1. An interesting example of address structures is provided by the IFS \(\mathcal{F} = \mathcal{F}_{\alpha,\beta,\gamma} = \{\mathbb{R}^2; h_1, h_2, h_3, h_4\}\), introduced at the end of section 2. Here we prove that

\begin{equation}
\mathcal{C}_{\mathcal{F}_{\alpha,\beta,\gamma}} = \mathcal{C}_{\mathcal{F}_{\alpha',\beta',\gamma'}}
\end{equation}

for all \(\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in (0, 1)\) by calculating the address structure \(\mathcal{C}_{\mathcal{F}_{\alpha,\beta,\gamma}}\).

In this case there is only one backwards orbit \(\{x_k\}_{k=0}^\infty\) of each \(x \in \mathcal{A}_\mathcal{F}\). This is because of the form of \(\mathcal{F}\), and because the \(h_n\)'s are affine and so preserve ratios of distances between points which lie on any given straight line: for example if \(x \in ab = h_2(\bullet) \cap h_4(\bullet)\) then \(h_2^{-1}(x) = h_4^{-1}(x)\). Indeed, the mapping \(T : \bullet \to \bullet\) defined as in equations [8.1] is continuous and the backwards orbit of \(x\) is the same as the orbit of \(x\) under \(T\) treated as a dynamical system.

Any point \(x_K \in \{x_k\}_{k=0}^\infty\) on the backwards orbit of \(x\), such that more than one map \(h_n^{-1}\) may be applied, is such that \(x_K\) belongs to the set

\[\bigcup_{i \neq j} h_i(\bullet) \cap h_j(\bullet) = (ab) \cup (bc) \cup (ca) \cup \{a, b, c\}.\]

If \(x_K \in (ab)\) then \(\sigma_K \in \{2, 4\}\), if \(x_K = a\) then \(\sigma_K \in \{1, 2, 4\}\), and so on. For example, if \(x = \phi_F(\omega_1 \omega_2 \omega_3 ... )\) and the only point on the backwards orbit of \(x\) which lies in \((ab) \cup (bc) \cup (ca) \cup \{a, b, c\}\) is \(x_K \in (ab)\) then \(\phi_F^{-1}\{x\} = \{\sigma \in \Omega : \sigma_k = \omega_k\} \text{ for all } k \neq K\), and \(\sigma_K \in \{2, 4\}\).
Let \( \triangle \) denote the boundary of \( \bullet \) as a subset of \( \mathbb{R}^2 \), \( \nabla = (ab) \cup (bc) \cup (ca) \), and

\[
\Xi = \triangle \cup \nabla \cup \left( \bigcup_{|\sigma| \geq 1} h_\sigma(\nabla) \right).
\]

Then

(i) each of the sets \( \triangle, \nabla, h_1(\nabla), h_2(\nabla), h_3(\nabla), h_4(\nabla), h_{11}(\nabla), h_{12}(\nabla), \ldots \) is disjoint;

(ii) \( T(\Xi) = \mathcal{F}(\Xi) = \Xi \);

(iii) \( T \) is one-to-one on \( h_\sigma(\nabla) \) and \( T(h_\sigma(\nabla)) = h_{S(\sigma)}(\nabla) \) for all \( \sigma \in \Omega' \) with \( |\sigma| \geq 1 \);

(iv) \( T \) is one-to-one on \( \nabla \) and \( T(\nabla) = \triangle \setminus \{A, B, C\} \);

(v) \( T \) is two-to-one on \( \triangle \) and \( T(\triangle \setminus \{A, B, C\}) = T(\triangle) = \triangle \).

The transformation \( T \) maps \( \triangle \) continuously onto itself. If \( x \) goes around \( \triangle \) clockwise once, then \( T(x) \) goes around \( \triangle \) anticlockwise twice. It does so in such a way that \( T(\{A\}) = \{A\} \), \( T(\{Ac\}) = \{Ab\} \cup \{b\} \cup \{bC\} \), \( T(\{c\}) = \{C\} \) and so on.

This information provides us with the directed graph, with labelled edges, shown in Figure 4. We denote this graph by \( G \). It is such that that there is a bijective correspondence between the points of \( \triangle \) and the set of all paths in \( G \). A path in \( G \) is obtained by starting at any node and successively following edges in the directions of the arrows, yielding an infinite sequences of edges. The set of addresses of the point represented by a path in \( G \) consists of all sequences of the numbers \( \{1, 2, 3, 4\} \) which can be read off successively from the path, with one symbol from each edge. For example, the only possible address for \( \{A\} \) is \( 3 \in \mathbb{Z} = 3333... \), and the set of addresses of a point in \( \{Ab\} \) may be \( 3\{1, 2, 3\}222..., \) or \( 33\{2, 3, 4\}111..., \) or one or more which begin \( 32222\).

Now suppose \( x \in h_\sigma(\nabla) \) for some \( \sigma \in \Omega' \) with \( |\sigma| \geq 1 \). Then by repeated application of (iii) above we find that the first \( |\sigma| \) terms in any address of \( x \) are precisely \( \sigma \), and \( T^{|\sigma|}(x) \in \nabla \). Since \( T^{|\sigma|}(\nabla) \) maps \( h_\sigma(\nabla) \) one-to-one onto \( \nabla \), it follows that the set of all sets of addresses of all points in \( h_\sigma(\nabla) \) is the same as the set of all sets of addresses of all points in \( \nabla \) after \( \sigma \) has been appended to the front of each of the latter addresses. So what is the set of all sets of addresses of all points in \( \nabla \)?

We have \( \nabla = (ab) \cup (bc) \cup (ca) \). Let us deal with \( (ab) \). The transformation \( T \) maps \( (ab) \) one-to-one onto \( (AB) = (Ac) \cup \{c\} \cup (cB) \). It follows that the set of all sets of addresses of all points in \( (ab) \), which we denote by \( C(ab) \), is determined by the set of sets of addresses of all points in \( (Ac) \cup \{c\} \cup (cB) \), which we denote by \( C((Ac) \cup \{c\} \cup (cB)) \). Specifically,

\[
C(ab) = \{ \{\eta \sigma : \eta \in \{2, 4\}, \sigma \in \pi \} : \pi \in C((Ac) \cup \{c\} \cup (cB)) \}
\]

The set of sets of addresses \( C((Ac) \cup \{c\} \cup (cB)) \) corresponds to the set of paths in \( G \) which start at a node labelled \( (Ac) \), \( \{c\} \), or \( (cB) \). We can similarly describe the address structures of \( (bc) \), and \( (ca) \). We are thus able, in principle, to write down the set of addresses of each \( x \in \Xi \); in particular, the set of all sets thus obtained does not depend on \( \alpha, \beta, \) or \( \gamma \).

Next we deal with \( \Xi^C = \triangle \setminus \Xi \). Since all points with multiple addresses lie in \( \Xi \) and the backwards orbit of each point in \( \Xi^C \) lies in \( \Xi^C \) it follows that each point in \( \Xi^C \) has a unique address, and in particular \( \phi_{\mathcal{F}}^{-1}(\Xi^C) = \Omega \setminus \phi_{\mathcal{F}}^{-1}(\Xi) \) does not depend on \( \alpha, \beta, \) or \( \gamma \).
Finally we note that $\phi_F^{-1}(\Xi) = \Omega$. Hence $\Omega_F = \Omega$. Hence $\Omega_F = \phi_F^{-1}(\Xi) \cup \phi_F^{-1}(\Xi^c)$. Since the equivalence class structures of both $\phi_F^{-1}(\Xi)$ and $\phi_F^{-1}(\Xi^c)$ do not depend on $\alpha, \beta,$ or $\gamma$, it follows that equation (6.1) is true.

So, for example, let $F = F_{0.5,0.5,0.5} = \{\triangle; h_1, h_2, h_3, h_4\}$ and $G = F_{0.65,0.3,0.4} = \{\triangle; g_1, g_2, g_3, g_4\}$. Then $C_F = C_G$ and, by Theorem 1, the fractal transformation $\phi_G \circ \tau_F : \triangle \rightarrow \triangle$ is a homeomorphism. Figure 6 illustrates the action of this homeomorphism. The figure on the left shows the set $S$, defined to be the union of the attractors of the two IFSs $\{\triangle; h_1, h_3, h_4\}$ and $\{\triangle; h_2, h_3, h_4\}$. The image on the right shows the set $\tilde{S}$, defined to be the union of the attractors of the IFSs $\{\triangle; g_1, g_3, g_4\}$ and $\{\triangle; g_2, g_3, g_4\}$. The two sets are related by $\phi_G \circ \tau_F(S) = \tilde{S}$. Figure 7 illustrates two other homeomorphisms associated with the family $F_{\alpha,\beta,\gamma}$.

These examples were computed as described in section 7.

6.4. Example 2. An example of address structures $C_F$ and $C_G$ such that $C_F \prec C_G$ and $C_F \neq C_G$ is provided by taking $F = \{\square; f_1, f_2, f_3, f_4\}$ and $G = \{\square; g_1, g_2, g_3, g_4\}$ to be the IFSs of affine maps specified in Tables 2 and 3 respectively. Here $\square \subset \mathbb{R}^2$. 
denotes the filled square with vertices at $I = (1, 1)$, $J = (1, 0)$, $K = (0, 0)$, $J = (0, 1)$. The attractor $A_F$ of $F$ is represented by the fern image in Figure 5. The attractor $A_G$ of $G$ is $\square$.

The transformations of $F$ are such that

\begin{equation}
(6.2) \quad f_1(i) = m, f_1(k) = k, f_2(i) = i, f_2(k) = m, \\
f_3(i) = m, f_3(k) = l, f_4(i) = m, f_4(k) = j,
\end{equation}

where the points $i, j, k, l, m \in A_F$ are approximately as labelled in Figure 5. Furthermore $f_p(A_F) \cap f_q(A_F) = m$ whenever $p, q \in \{1, 2, 3, 4\}$ with $p \neq q$. It is readily deduced that $k = \phi_F(\Omega), i = \phi_F(\overline{\Omega}), m = \phi_F(1\overline{\Omega}) = \phi_F(2\overline{\Omega}) = \phi_F(3\overline{\Omega}) = \phi_F(4\overline{\Omega})$, that

$$\Omega_F = \{ \sigma \in \Omega : S^{cn}(\sigma) \notin \{1\overline{\Omega}, 2\overline{\Omega}, 3\overline{\Omega}, 4\overline{\Omega}\} \text{ for all } n \in \{0, 1, 2, \ldots\} \},$$

that $\overline{\Omega}_F = \Omega$ and that the address structure of $F$ is

$$C_F = C_F^{(1)} \cup C_F^{(2)}$$

where

$$C_F^{(1)} = \{ \sigma : \sigma \in \Omega, S^{cn}(\sigma) \notin \{1\overline{\Omega}, 2\overline{\Omega}, 3\overline{\Omega}, 4\overline{\Omega}\} \text{ for all } n \in \{0, 1, 2, \ldots\} \},$$

$$C_F^{(2)} = \{ \sigma'1\overline{\Omega}, \sigma'2\overline{\Omega}, \sigma'3\overline{\Omega}, \sigma'4\overline{\Omega} : \sigma' \in \Omega' \}.$$

To determine the address structure of $G$, we note that $\square$ is the union of four rectangular tiles $g_n(\square)$ which share portions of their boundaries. The transformations of $G$ are such that

\begin{equation}
(6.3) \quad g_1(I) = M, g_1(K) = K, g_2(I) = I, g_2(K) = M, \\
g_3(I) = M, g_3(K) = L, g_4(I) = M, g_4(K) = J,
\end{equation}

where the points $I, J, K, L, M \in A_G$ are approximately as labelled in Figure 5.

Note that equations (6.3) are the same as equations (6.2) upon substitution of $f_1, f_2, f_3, f_4, i, j, k, l,$ and $m$, by $g_1, g_2, g_3, g_4, I, J, K, L,$ and $M$ respectively. It is readily deduced that $K = \phi_G(\overline{\Omega}), I = \phi_G(\overline{\Omega}), M = \phi_G(1\overline{\Omega}) = \phi_G(2\overline{\Omega}) = \phi_G(3\overline{\Omega}) = \phi_G(4\overline{\Omega})$, and that $\overline{\Omega}_G = \Omega$. As a consequence $C_F < C_G$: if $s \in C_F$ then either $s \in C_F^{(1)}$ or $s \in C_F^{(2)}$; if $s \in C_F^{(1)}$ then $s$ is a singleton and, since $C_G$ is a partition of $\Omega$, there must be $t \in C_G$ such that $s \subset t$; if $s \in C_F^{(2)}$ then $s = \{ \sigma'1\overline{\Omega}, \sigma'2\overline{\Omega}, \sigma'3\overline{\Omega}, \sigma'4\overline{\Omega} \}$ for some $\sigma' \in \Omega'$, and since $M = \phi_G(1\overline{\Omega}) = \phi_G(2\overline{\Omega}) = \phi_G(3\overline{\Omega}) = \phi_G(4\overline{\Omega})$, it follows that $C_G$ contains a set that contains $s$. Hence $C_F < C_G$ and, by Theorem 1, the fractal transformation $\phi_G \circ \tau_F$ from the fern-shaped set onto $\square$ is continuous. This transformation is illustrated in Figure $\square$ as described at the start of section $\square$. Note however that in this case $C_F \neq C_G$ because there is a set in $C_G$ which consist of a pair of distinct addresses, whereas all sets in $C_F$ contain either one or four distinct addresses.

If, in this example, we change $G$ to $\tilde{G}$ specified in Table $\square$ then the attractor is still the filled square, that is $A_{\tilde{G}} = \square$, but Equation (6.3) no longer holds and we can show that the fractal transformation $\phi_{\tilde{G}} \circ \tau_F$ from the fern-shaped set onto $\square$ is not continuous. This lack of continuity is illustrated in Figure $\tilde{\square}$ as described in section $\tilde{\square}$.
Figure 5. (i) Shows the points $i, j, k, l, m$ and (ii) shows the points $I, J, K, L, M$.

| $n$ | $a_n$ | $b_n$ | $c_n$ | $d_n$ | $e_n$ | $l_n$ |
|-----|-------|-------|-------|-------|-------|-------|
| 1   | 0.85  | -0.05 | 0.125 | 0.05  | 0.85  | -0.039|
| 2   | 0.06  | 0.02  | 0.45  | 0.0   | 0.165 | 0.835 |
| 3   | 0.17  | 0.22  | 0.195 | -0.22 | 0.17  | 0.776 |
| 4   | -0.17 | -0.22 | 0.805 | -0.22 | 0.17  | 0.776 |

Table 2.

| $n$ | $a_n$ | $b_n$ | $c_n$ | $d_n$ | $e_n$ | $l_n$ |
|-----|-------|-------|-------|-------|-------|-------|
| 1   | 0.8   | 0.0   | 0.0   | 0.0   | 0.8   | 0.0   |
| 2   | 0.2   | 0.0   | 0.8   | 0.0   | 0.8   | 0.2   |
| 3   | -0.2  | 0.0   | 1.0   | 0.0   | 0.8   | 0.0   |
| 4   | 0.8   | 0.0   | 0.0   | 0.0   | -0.2  | 1.0   |

Table 3.

| $n$ | $a_n$ | $b_n$ | $c_n$ | $d_n$ | $e_n$ | $l_n$ |
|-----|-------|-------|-------|-------|-------|-------|
| 1   | -0.8  | 0.0   | 0.8   | 0.0   | -0.8  | 0.8   |
| 2   | -0.2  | 0.0   | 1.0   | 0.0   | -0.2  | 1.0   |
| 3   | 0.8   | 0.0   | 0.0   | 0.0   | 0.2   | 0.8   |
| 4   | 0.2   | 0.0   | 0.8   | 0.0   | 0.8   | 0.0   |

Table 4.

7. Pictures of tops functions

When the underlying space is $\mathbb{R}^2$ we can use the chaos game to compute illustrations of fractal transformations.

Let two hyperbolic IFSs

$\mathcal{F} := \{ \square, f_1, ..., f_N \}$ and $\mathcal{G} := \{ \square, g_1, ..., g_N \}$
and a picture function
\[ \mathcal{P} : \square \to \mathbb{C} \]
be given, where
\[ \square := \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}. \]
Let \( \mathcal{P}_G \) denote \( \mathcal{P} \) restricted to \( A_G \), that is \( \mathcal{P}_G = \mathcal{P}|_{A_G} \). Then we define a new picture
\[ \mathcal{P}_F : A_F \to \mathbb{C} \]
by
\[ \mathcal{P}_F = \mathcal{P}_G \circ \phi_G \circ \tau_F. \]
We say that \( \mathcal{P}_F \) is defined by tops plus color-stealing.

In order to make a physical picture of \( \mathcal{P}_F \) and thus illustrate the tops function \( \phi_G \circ \tau_F \) we use a variant of the chaos game. To work at finite precision we partition the set \( \square \subset \mathbb{R}^2 \) into a finite set of small rectangles, say ten thousand of them, which we refer to as pixels. Each point \((x, y) \in \square\) belongs to exactly one pixel, which we denote by \( p(x, y) \).

Start from an arbitrary pair of points \((x_0^F, y_0^F) \in \square\) and \((x_0^G, y_0^G) \in \square\). Let \( K \) be a large number such as ten million. For \( k = 1, 2, ..., K \) let \( \sigma_k \) denote an element of \( \{1, 2, ..., N\} \) chosen at random, independently of all other choices. Let
\[ (x_k^F, y_k^F) = f_{\sigma_k}(x_{k-1}^F, y_{k-1}^F) \quad \text{and} \quad (x_k^G, y_k^G) = g_{\sigma_k}(x_{k-1}^G, y_{k-1}^G). \]
For each iterative step \( k > 100 \), if the color of the pixel \( p((x_k^F, y_k^F)) \) was not assigned at an earlier step \( l < k \) such that \( \sigma_l \sigma_{l-1}...\sigma_1 > \sigma_k \), then plot the pixel \( p((x_k^F, y_k^F)) \) in the color \( \mathcal{P}_G(x_k^G, y_k^G) \).

The reason this algorithm converges in practice to produce a stable physical picture that approximates \( \mathcal{P}_G \circ \phi_G \circ \tau_F \) is described in Chapter 4 of [5]. Again, it depends on Birkhoff’s ergodic theorem. Intuitively, ergodicity of the shift transformation ensures that, almost always, the sequences \( \{(x_k^F, y_k^F)\} \) and \( \{(x_k^G, y_k^G)\} \) repeatedly visit all of the points that represent the points of \( A_F \) and \( A_G \) respectively. Let \( \sigma^{(k)} = \sigma_k \sigma_{k-1}...\sigma_1 \). Then the point \((x_k^F, y_k^F)\) is very close to \( \phi_F(\sigma^{(k)}) \) when \( k \) is sufficiently large; indeed
\[ d_{R^2}(\phi_F(\sigma^{(k)}), (x_k^F, y_k^F)) \leq l^k d_{R^2}(\phi_F(\sigma^{(1)}), (x_0^F, y_0^F)). \]
Similarly \((x_k^G, y_k^G)\) is very close to \( \phi_G(\sigma^{(k)}) \) when \( k \) is sufficiently large. Hence, to a good approximation, the color of the pixel \( p(\phi_F(\sigma^{(k)})) \) is updated to become the color of the pixel \( p(\phi_G(\sigma^{(k)})) \) except when \( \sigma^{(l)} > \sigma^{(k)} \) for some \( l < k \) for which \( p(\phi_F(\sigma^{(l)})) = p(\phi_F(\sigma^{(k)})) \).

Let
\[ 100 < k_1 < k_2 < k_3 < ... < k_M \leq K \]
denote the sequence of successive values of \( k \) at which such updates occur. Then \( \{\sigma^{(k_i)}\}_{i=1}^{M} \) is an increasing sequence of addresses, each associated with a point in the pixel \( p(\phi_F(\sigma^{(k_i)})) \). Hence, again invoking ergodicity, \( \{\sigma^{(k_i)}\}_{i=1}^{M} \) approaches the highest address of all points in the pixel \( p(\phi_F(\sigma^{(k_i)})) \). The address \( \sigma^{(k_M)} \) is our approximation to \( \sup\{\tau_F(\sigma) : \sigma \in \tau_F(\phi_F(\sigma^{(k_i)}))\} \). In general we expect it to become increasingly accurate with increasing \( K \). According to this approximation, the pixel \( p(\phi_F(\sigma^{(k_i)})) \) is assigned the color of the pixel \( p(\phi_G(\tau_F(\sigma^{(k_M)}))) \). Thus we obtain a sensible pixel-based approximation to \( \mathcal{P}_G \circ \phi_G \circ \tau_F \).

In Figure 6 we illustrate two different fractal transformations from a fern-like fractal to a filled square, computed using this algorithm. For the picture on the left
Figure 6. The ferns on the left and right are both obtained by fractal transformations. The one on the left is continuous image of the central image.

$\mathcal{F}$ and $\mathcal{G}$ are as discussed in section 6.3 with $\mathcal{C}_F \prec \mathcal{C}_G$, $\mathcal{C}_F \neq \mathcal{C}_G$, so that $\phi_G \circ \tau_F$ is continuous. The picture $\Psi_G$ is represented in the center of Figure 6. It has been chosen to have apparently continuously varying intensity so that the continuity of $\phi_G \circ \tau_F$ is illustrated by the smooth variation of intensity in the left-hand fern image, which represents a close-up on $\Psi_F = \Psi_G(\phi_G \circ \tau_F)$. To produce the picture on the right the IFS $\mathcal{G}$ has been switched, from the one in Table 3 to the one in Table 4, so that $\phi_G \circ \tau_F$ is not continuous and $\Psi_F(\phi_G \circ \tau_F)$ is no longer smoothly varying.

In Figure 7 we illustrate two examples, computed using the modified chaos game described here, in each of which the fractal transformation $\phi_G \circ \tau_F : A_F \rightarrow A_G$ is a homeomorphism. The homeomorphisms are constructed using IFSs of the form $F_{a,b,c}$ discussed in sections 2 and 6.3. In both examples $A_F = A_G = △$, the filled triangle with vertices at $A = (0,0)$, $B = (1,0)$, and $C = (0.5,1)$. Also in both cases, $\Psi_G : △ \rightarrow \mathcal{C}$ corresponds to the grayscale picture of a caged bird in the top triangle in Figure 7. The image at bottom left shows $\Psi_G \circ \phi_G \circ \tau_F$ when $\mathcal{F} = F_{0.525,0.525,0.525}$ and $\mathcal{G} = F_{0.475,0.475,0.475}$. In this case the corresponding subtriangles have the same areas at all levels with the consequence that the fractal transformation $\phi_G \circ \tau_F$ is area-preserving. To produce the image at the bottom right we used $\mathcal{F} = F_{0.4,0.4,0.475}$ and $\mathcal{G} = F_{0.5,0.5,0.5}$.

8. The tops dynamical system

In general, to determine the nature of the fractal transformation $\phi_G \circ \tau_F : A_F \rightarrow A_G$ we need to know the tops code space $\Omega_F$. Here we prove that $\Omega_F$ is shift invariant. Consequently it may be described in terms of the orbits of an associated dynamical system $T_F : A_F \rightarrow A_F$.

Throughout this section we assume that the transformations of the IFS $\mathcal{F}$ are one-to-one. Let $\mathcal{S}_F : \Omega_F \rightarrow \Omega$ denote the shift transformation, defined by

$$\mathcal{S}_F(\sigma_1\sigma_2\sigma_3...) = \sigma_2\sigma_3\sigma_4...$$

for all $\sigma_1\sigma_2\sigma_3... \in \Omega_F$. Let

$$G_F := \{(x, \tau_F(x)) : x \in A_F\}$$

denote the graph of the tops function $\tau_F$. 
Figure 7. Two examples of fractal homeomorphisms applied to the picture at the top. The transformations from the top image to the one at bottom left is area-preserving.

Lemma 3. Let \((x, \sigma) \in G_F\). Then \((f_{\sigma_1}^{-1}(x), S_F(\sigma)) \in G_F\).

Proof. \((x, \sigma) \in G_F\) implies \(x \in A_F\), \(\sigma \in \Omega_F\) and \(\tau_F(x) = \sigma\). In particular, \(\phi_F(\sigma) = x\) for any \(z \in X\),

\[
\lim_{k \to \infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \ldots \circ f_{\sigma_k}(z) = x.
\]

Using the continuity and invertibility of \(f_{\sigma_1}\) it follows that

\[
\lim_{k \to \infty} f_{\sigma_2} \circ f_{\sigma_3} \circ \ldots \circ f_{\sigma_k}(z) = f_{\sigma_1}^{-1}(x).
\]

This says that \(\phi_F(S_F(\sigma)) = f_{\sigma_1}^{-1}(x)\) which tells us that \(S_F(\sigma) \in \phi_F^{-1}(\{f_{\sigma_1}^{-1}(x)\})\).

Now suppose that there is \(\omega \in \phi_F^{-1}(\{f_{\sigma_1}^{-1}(x)\})\) with \(\omega > S_F(\sigma)\). Then \(\phi_F(\omega) = f_{\sigma_1}^{-1}(x)\) which implies \(f_{\sigma_1}(\phi_F(\omega)) = \phi_F(\sigma_1 \omega) = x\). Let \(\bar{\sigma} = \sigma_1 \omega\). Then \(\bar{\sigma} > \sigma\) and \(\phi_F(\bar{\sigma}) = x\) which contradicts the assertion that \(\sigma\) is the largest element of \(\Omega\) such that \(\phi_F(\sigma) = x\). Hence \(S_F(\sigma) \in \Omega_F\) and \(\tau_F(f_{\sigma_1}^{-1}(x)) = S_F(\sigma)\).

Lemma 4. Let \((x, \sigma) \in G_F\). Then \((f_1(x), 1\sigma) \in G_F\).

Proof. \((x, \sigma) \in G_F\) implies \(\tau_F(x) = \sigma\). Hence \(x = \phi_F(\sigma)\) and so \(f_1(x) = \phi_F(1\sigma)\).

Now suppose that \((f_1(x), 1\sigma) \notin G_F\). Then there is \(\omega > 1\sigma\) such that \(\phi_F(\omega) = f_1(x)\). But then \(\omega = 1\bar{\sigma}\) where \(\bar{\sigma} > \sigma\) and \(\phi_F(1\bar{\sigma}) = f_1(x)\). This implies \(\phi_F(\bar{\sigma}) = x\) with \(\bar{\sigma} > \sigma\) which implies \(\tau_F(x) > \sigma\) which is a contradiction. Hence \((f_1(x), 1\sigma) \in G_F\).

It follows from Lemmas 3 and 4 that the mapping \(\hat{T}_F : G_F \to G_F\) specified by

\[
\hat{T}_F(x, \sigma) = (f_{\sigma_1}^{-1}(x), S_F(\sigma))\]

for all \((x, \sigma) \in G_F\)

is well-defined and onto.

In particular, the projection of \(\hat{T}_F\) on \(\Omega_F\) yields the symbolic dynamical system \(S_F : \Omega_F \to \Omega_F\), because from Lemma 4 we have

\[
S_F(\Omega_F) = \Omega_F.
\]
The projection of $\tilde{T}_F : G_F \to G_F$ onto $A_F$ yields what we call the tops dynamical system

$$T_F : A_F \to A_F$$

where

$$T_F(x) = \begin{cases} f_1^{-1}(x) & \text{if } x \in D_1 := f_1(A_F), \\ f_2^{-1}(x) & \text{if } x \in D_2 := f_2(A_F) \setminus f_1(A_F), \\ \vdots & \text{for all } x \in A_F. \end{cases}$$

(8.1)

for all $x \in A_F$. Lemma [4] implies

$$T_F(A_F) = A_F.$$

Theorem 2. The tops dynamical systems $T_F : A_F \to A_F$ is related to the symbolic dynamical system $S_F : \Omega_F \to \Omega_F$ by the tops function $\tau_F : A_F \to \Omega_F$, according to

$$S_F = \tau_F \circ T_F \circ \tau_F^{-1}.$$ 

If $\Omega_F \subset \Omega_G$ then

$$(\phi_G \circ \tau_F \circ T_F)(x) = (T_G \circ \phi_G \circ \tau_F)(x) \text{ for all } x \in A_F.$$ 

If $C_F = C_G$ then the tops dynamical systems $T_F : A_F \to A_F$ and $T_G : A_G \to A_G$ are topologically conjugate.

Proof. Let $\Phi_F = \tau_F^{-1}$ be as discussed at the end of section [4]. Then we claim that

$$T_F \circ \Phi_F = \Phi_F \circ S_F.$$ 

Since $S_F$ maps $\Omega_F$ onto itself and $\Phi_F$ maps $\Omega_F$ onto $A_F$ it follows that the mapping $\Phi_F \circ S_F$ takes $\Omega_F$ onto $A_F$. (Similarly, $T_F \circ \Phi_F$ maps $\Omega_F$ onto $A_F$.)

Let $\sigma = \sigma_1 \sigma_2 \sigma_3 \ldots \in \Omega_F$. Then $S_F(\sigma) = \sigma_2 \sigma_3 \ldots \in \Omega_F$ and

$$(\Phi_F \circ S_F)(\sigma) = \lim_{k \to \infty} (f_{\sigma_1} \circ f_{\sigma_2} \circ \ldots \circ f_{\sigma_k})(z).$$

On the other hand

$$\Phi_F(\sigma) = \lim_{k \to \infty} (f_{\sigma_1} \circ f_{\sigma_2} \circ \ldots \circ f_{\sigma_k})(z)$$

$$= f_{\sigma_1} \left( \lim_{k \to \infty} (f_{\sigma_2} \circ f_{\sigma_3} \circ \ldots \circ f_{\sigma_k})(z) \right) = f_{\sigma_1}(\Phi_F(\sigma_2 \sigma_3 \ldots)),$$

belongs to $A_F$ and lies in the range of $f_{\sigma_1}$ and so must belong to $D_{\sigma_1}$ as defined in Equation [8.1]. Hence

$$(T_F \circ \Phi_F)(\sigma) = \Phi_F(\sigma_2 \sigma_3 \ldots) = (\Phi_F \circ S_F)(\sigma)$$

for all $\sigma \in \Omega_F$.

We now apply $\tau_F$ to both sides of this last equation to complete the proof of the first assertion in the theorem.

Now assume that $\Omega_F \subset \Omega_G$. Then, since

$$S_F(\sigma) = S_G(\sigma)$$

for all $\sigma \in \Omega_F$, it follows from the first part of the theorem that

$$(\tau_F \circ T_F \circ \Phi_F)(\sigma) = (\tau_G \circ T_G \circ \Phi_G)(\sigma)$$

for all $\sigma \in \Omega_F$. It follows that

$$(\tau_F \circ T_F \circ \Phi_F \circ \tau_F)(x) = (\tau_G \circ T_G \circ \Phi_G \circ \tau_F)(x)$$
for all $x \in A_F$. But $\Phi_F \circ \tau_F = i_{A_F}$ and
\[(\Phi_G \circ \tau_F)(x) = (\phi_G \circ \tau_F)(x)\]
for all $x \in A_F$. Hence
\[(\tau_F \circ T_F)(x) = (\tau_G \circ T_G \circ \phi_G \circ \tau_F)(x)\]
for all $x \in A_F$. Applying $\phi_G$ to both sides we obtain
\[(\phi_G \circ \tau_F \circ T_F)(x) = (\phi_G \circ \tau_G \circ T_G \circ \phi_G \circ \tau_F)(x)\]
for all $x \in A_F$. But $\phi_G \circ \tau_G = i_{A_G}$. This completes the proof of the second assertion in the theorem.

Finally, let us suppose that $C_F = C_G$. Then Theorem 1 implies that $\phi_G \circ \tau_F$ is a homeomorphism from $A_F$ onto $A_G$. Also $C_F = C_G$ implies $\Omega_F = \Omega_G$ which implies, via the previously proven part of this theorem,
\[T_F(x) = (\phi_G \circ \tau_F)^{-1} \circ T_G \circ \phi_G \circ \tau_F)(x)\]
for all $x \in A_F$. $\square$

If the domains $\{D_n : n = 1, 2, ..., N\}$ are known then it is easy to compute the tops function. Just follow the orbit of $x$ under the tops dynamical system and keep track of the sequence of indices $\sigma_1 \sigma_2 \sigma_3 \sigma_4 \ldots$ visited by the orbit.

In the special case where the IFS is totally disconnected and the $f_n$s are one-to-one then $T : A \to A$ is defined by $T(x) = f_n^{-1}(x)$ where $n$ is the unique index such that $x \in f_n(A)$. This dynamical system has been considered elsewhere, for example in [11] and [14]. In this case $\phi_F : \Omega \to A_F$ is a homeomorphism, $\tau_F = \phi_F^{-1}$, and $T_F$ it is conjugate to the shift transformation according to $T_F = \phi_F \circ S \circ \phi_F^{-1}$.

Theorem 2 says in particular that $T_F : A_F \to A_F$ is a factor of $S \circ \Omega_F \to \Omega_F$, and as defined for example in [13] p. 68, because $\Phi_F \circ S \circ T_F = T_F \circ \Phi_F$ where $\Phi_F = \tau_F^{-1}$ is continuous; this tells us that the topological entropy of $T_F$ is less than or equal to the topological entropy of $S_F$. [13] Proposition 3.1.6, p. 111. If $\tau_F$ is continuous then Theorem 2 says that the two dynamical systems $T_F : A_F \to A_F$ and $S_F : \Omega_F \to \Omega_F$ are topologically conjugate, see [13] p. 60, and it follows that the two systems must have the same topological entropy.

This suggests that we may compare the complexity of some subsets of $\mathbb{R}^2$ by assigning to them the topological entropy of a corresponding shift dynamical system. Let $\mathcal{M}$ denote the set of all attractors of hyperbolic IFSs in $\mathbb{R}^2$, whose transformations are all affine and invertible, such that the associated tops function is continuous. Then we can define the topological entropy of each $A_F$ to be the infimum of the entropies of the set of corresponding shift dynamical systems. In this way we arrive at a geometry-based definition of the topological entropy of some subsets of $\mathbb{R}^2$. Is it useful?

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