Gravitational Waves in Viable $f(R)$ Models

Louis Yang, Chung-Chi Lee, and Chao-Qiang Geng

Department of Physics, National Tsing Hua University, Hsinchu 300, Taiwan
Physics Division, National Center for Theoretical Sciences, Hsinchu 300, Taiwan

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Abstract

We study gravitational waves in viable $f(R)$ theories under a non-zero background curvature. In general, an $f(R)$ theory contains an extra scalar degree of freedom corresponding to a massive scalar mode of gravitational wave. For viable $f(R)$ models, since there always exits a de-Sitter point where the background curvature in vacuum is non-zero, the mass squared of the scalar mode of gravitational wave is about the de-Sitter point curvature $R_d \sim 10^{-66} eV^2$. We illustrate our results in two types of viable $f(R)$ models: the exponential gravity and Starobinsky models. In both cases, the mass will be in the order of $10^{-33} eV$ when it propagates in vacuum. However, in the presence of matter density in galaxy, the scalar mode can be heavy. Explicitly, in the exponential gravity model, the mass becomes almost infinity, implying the disappearance of the scalar mode of gravitational wave, while the Starobinsky model gives the lowest mass around $10^{-24} eV$, corresponding to the lowest frequency of $10^{-9}$ Hz, which may be detected by the current and future gravitational wave probes, such as LISA and ASTROD-GW.

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I. INTRODUCTION

The accelerating expansion of our universe has been supported by various cosmological observations, such as type Ia supernovae [1, 2], large scale structure [3, 4], cosmic microwave background radiation [5, 6] and weak lensing [7]. In order to explain this late time acceleration [8], one can either introduce dark energy, a new form of matter, or modify Einstein's general relativity, i.e., the modification of gravity. One simple way to modify general relativity is to promote the Ricci scalar $R$ in the Einstein-Hilbert action into an $f(R)$ function, which is the so-called $f(R)$ theory [9–13]. A viable model of $f(R)$ can generate a late-time accelerating expansion of our universe, have the radiation-dominated stage followed by the matter-dominated one [14, 15], and be consistent with the solar-system constraint under chameleon mechanism [16–23].

In [24], Chiba showed that an $f(R)$ model will allow a new scalar degree of freedom. This corresponds to a new scalar mode of gravitational wave besides the ordinary tensor one of general relativity. This new scalar mode will be massive and propagate as a longitudinal polarization. Various discussions and predictions about this extra scalar mode of gravitational wave have been given in the literature [25–32]. However, most of them were concentrated on either quadratic or inverse-curvature type of $f(R)$ models, which is highly restricted by the observational results [12, 20, 33–36]. In the article, we will study gravitational wave in viable $f(R)$ models.

The mass of the scalar mode of gravitational wave in viable $f(R)$ models could be quite different from quadratic and inverse-curvature ones. In vacuum, it will be of the order of the Hubble constant because all viable $f(R)$ models need to have de Sitter points which have a non-zero background curvature $R_d$ about square of the Hubble constant. However, when gravitational wave propagates in the galaxy region, the local density of dark matter and baryonic matter will contribute a tremendous background curvature although it is still much smaller then curvature generated by star. This background curvature might make some viable $f(R)$ models return to the ordinary GR very fast. Therefore, the scalar mode will become extremely massive in these cases and, hence, prevent the observable propagation of the fifth force in our galaxy and solar system. On the other hand, viable $f(R)$ theories pass the solar-system constraint by means of chameleon mechanism, which do not require very heavy scalar modes by using the thin-shell argument [18, 21]. In these models, the
scalar modes may still be detectable.

This paper is organized as follows. In Sec. II we review the field equations and linearizations in $f(R)$ theories and demonstrate the modifications of gravitational waves. We apply the analysis on two explicit viable $f(R)$ models: exponential gravity and Starobinsky ones in Sec. III. The results and discussions on the scalar modes of gravitational waves in the inner galaxy region is presented in Sec. IV. Conclusions are given in Sec. V. We use units of $k_B = c = \hbar = 1$ and the gravitational constant $G = M_{Pl}^2$ with the Planck mass of $M_{Pl} = 1.22089 \times 10^{19}$GeV.

II. GRAVITATIONAL WAVES IN $f(R)$ THEORY

A. $f(R)$ Gravity

We start by considering a general Einstein-Hilbert action

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m(g_{\mu\nu}, \Psi_{\mu\nu}),$$

(2.1)

where $f(R)$ is an arbitrary function of the Ricci scalar $R$, $S_m$ is the action of the matter part and $\kappa^2 \equiv 8\pi G$. In the metric formalism, we vary the action (2.1) with respect to $g_{\mu\nu}$, and the modified Einstein field equation can be obtained as

$$f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu \nabla_\nu) f'(R) = \kappa^2 T_{\mu\nu},$$

(2.2)

where a prime denotes the derivative with respect to $R$, $\nabla_\mu$ is the covariant derivative and $\Box = g^{\mu\nu}\nabla_\mu \nabla_\nu$ is the d’Alembert operator. The trace of the field equation (2.2) gives

$$f'(R)R - 2f(R) + 3\Box f'(R) = \kappa^2 T,$$

(2.3)

where $T = g^{\mu\nu}T_{\mu\nu} = -\rho + 3a^2 P$ is the trace of the matter energy-momentum tensor, and $a$ is the scale factor.

For $f(R)$, the de Sitter stage is a vacuum solution with a positive constant background curvature $R_d$, which is assumed to be homogeneous and static. Consequently, one has

$$\nabla_\mu f'(R_d) = 0 \quad \text{and} \quad f'(R_d)R_d = 2f(R_d).$$

(2.4)

Moreover, from Eq. (2.2), the Ricci tensor satisfies $R_{\mu\nu}|_{R_d} = g_{\mu\nu}R_d/4$. 
B. The Weak-field Approximation

In order to investigate gravitational wave in $f(R)$ theories, we need to study the linearized theory of $f(R)$ gravity. Consider a small perturbation from the FRW metric:

$$g_{\mu\nu} = \mathcal{g}_{\mu\nu} + h_{\mu\nu}, \quad (2.5)$$

where $|h_{\mu\nu}| \ll 1$ is the perturbation and $\mathcal{g}_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2)$ is the FRW background metric. If the evolution of the system is much shorter than Hubble time, we can approximate the background spacetime to be nearly the Minkowski one with $\mathcal{g}_{\mu\nu} \approx \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. We keep the theory to be the first order in $h_{\mu\nu}$ and neglect terms higher than $\mathcal{O}(h^2)$. Thus, the inverse of the metric tensor is given by

$$g^{\mu\nu} = \mathcal{g}^{\mu\nu} - h^{\mu\nu}. \quad (2.6)$$

Note that all indices are raising and lowering by the background metric $\mathcal{g}_{\mu\nu}$. In the metric formalism, the perturbation of connection is

$$\delta \Gamma^\gamma_{\alpha\beta} = \frac{1}{2} \mathcal{g}^{\mu\nu} \left( \partial_\beta h_{\alpha\mu} + \partial_\alpha h_{\mu\beta} - \partial_\mu h_{\alpha\beta} - 2h_{\mu\nu} \bar{\Gamma}^\nu_{\alpha\beta} \right), \quad (2.7)$$

where $\bar{\Gamma}^\nu_{\alpha\beta}$ is the unperturbed connection. The only non-vanishing components of $\bar{\Gamma}^\nu_{\alpha\beta}$ are

$$\bar{\Gamma}^i_{j0} = \bar{\Gamma}^i_{0j} = H \delta^i_j \quad \text{and} \quad \bar{\Gamma}^0_{ij} = a^2 H \delta_{ij}, \quad (2.8)$$

where $H \equiv \dot{a}/a$ is the Hubble constant. The deviation of the Ricci tensor from the background curvature is

$$\delta R_{\alpha\beta} = \partial_\mu \delta \Gamma^\mu_{\alpha\beta} - \partial_\beta \delta \Gamma^\mu_{\alpha\mu} + \bar{\Gamma}^\nu_{\alpha\beta} \delta \Gamma^\mu_{\nu\mu} + \bar{\Gamma}^\mu_{\alpha\beta} \delta \Gamma^\nu_{\nu\mu} - \bar{\Gamma}^\nu_{\alpha\mu} \delta \Gamma^\mu_{\nu\beta} - \bar{\Gamma}^\mu_{\nu\beta} \delta \Gamma^\nu_{\alpha\mu}. \quad (2.9)$$

C. The Scalar Mode $h_f$

The difference between gravitational waves in $f(R)$ and general relativity is that it contains an extra scalar degree of freedom in $f(R)$. This comes from the non-vanishing trace of the field equation. Eq. (2.3) can be viewed as equation of motion for a scalar field $\Phi$. By the identifications

$$\Phi \rightarrow f'(R) \quad \text{and} \quad \frac{dV_{eff}}{d\Phi} \rightarrow \frac{2f(R) - f'(R) R - \kappa^2 \rho}{3}, \quad (2.10)$$
we obtain the Klein-Gordon equation for the scalar field $\Phi$:

$$\Box \Phi = \frac{dV_{\text{eff}}}{d\Phi}. \quad (2.11)$$

In order to have a stable perturbation of spacetime, we must require the background scalar $\Phi_0$ to stay at the stable minimum of the effective potential $V_{\text{eff}}$, i.e.,

$$\frac{dV_{\text{eff}}}{d\Phi} = 0 \quad (2.12)$$

and

$$\frac{d^2V_{\text{eff}}}{d\Phi^2} > 0. \quad (2.13)$$

In vacuum, Eq. (2.12) just gives us the condition for the de-Sitter point curvature (2.4), while Eq. (2.13) requires the mass of the scalar mode to be positive.

Perturbing the trace of the field equation (2.3) with a nonzero constant background curvature $R_0$ which satisfies Eq. (2.12) yields

$$3\Box \delta f' + R_0 \delta f' + f'(R_0) \delta R - 2\delta f = 0. \quad (2.14)$$

Using the relations $\delta f = f''(R_0) \delta R$ and $\delta f' = f''(R_0) \delta R$, we obtain the massive wave equation for the scalar mode [27, 30]

$$\Box h_f = m^2_s h_f, \quad (2.15)$$

where $h_f \equiv \delta f' / f'(R_0)$ is the field of the scalar mode and

$$m^2_s = \frac{1}{3} \left( \frac{f'(R_0)}{f''(R_0) - R_0} \right) \quad (2.16)$$

is the mass squared of it. Note that $m^2_s = V''_{\text{eff}}(\Phi)$ [19]. For any viable $f(R)$ model, the condition $m^2_s > 0$ is needed for the stability of the cosmological perturbation and to prevent the field from being a tachyon [33, 35, 36, 38, 40, 41].

For the FRW metric, Eq. (2.15) should be expressed as

$$\left(-\partial^2_0 + \frac{\partial^2}{a^2} - 3H \partial_0\right) h_f = m^2_s h_f, \quad (2.17)$$

where the term $-3H \partial_0$ gives a damping factor caused by the expansion of the universe. To illustrate the solution of Eq. (2.17), we take the de Sitter universe with a constant $H$. In this case, the solution is a damped plane wave

$$h_f = A(\vec{k}) e^{-\frac{3}{2}Ht} \exp(iq^\mu x_\mu), \quad (2.18)$$
where \( q^\mu \equiv (\omega_m, \vec{k}) \), \( \omega_m = \sqrt{\vec{k}^2 / a^2 + m_s^2 - \frac{9}{4} H^2} \) is the angular frequency and \( A(\vec{k}) \) is the amplitude. For simplicity and without loss of generality, we take \( a = 1 \) and neglect the damping effect as \( \vec{k}^2 / a^2 \gg H^2 \). As a result, Eq. (2.15) leads to a simple plane wave solution

\[
h_f = A(p) \exp(iq^\mu x_\mu),
\]

with \( \omega_m = \sqrt{\vec{k}^2 + m_s^2} \). We can see that \( m_s \) is the cutoff frequency of the scalar mode of gravitational wave. For \( \omega_m < m_s \), the wave vector becomes imaginary. The waveform is an exponential decay in distance, i.e., \( h_f \propto \exp(-\vec{k} \cdot \vec{x}) \). Thus, the scalar will not propagate in space below the cutoff frequency. The massive scalar mode will not propagate at the speed of light with the group-velocity

\[
v_g = \frac{\vec{k}}{\omega_m} = \frac{\sqrt{\omega_m^2 - m_s^2}}{\omega_m}.
\]

D. The Tensor Mode \( h_T^{\mu \nu} \)

Perturbing the field equation (2.2) under the de-Sitter curvature \( R_d \) leads to

\[
f'(R_d)\delta R_{\mu \nu} + R_{\mu \nu}|_{R_d} \delta f' - \frac{1}{2} \vec{\nabla}_{\mu} \delta f - \frac{1}{2} h_{\mu \nu} f(R_d) + (\vec{\nabla}_{\mu} \Box - \nabla_\mu \nabla_\nu) \delta f' = 0.
\]

Using \( \delta f = \frac{f'}{f} \delta f' \), \( R_{\mu \nu}|_{R_d} = \vec{\nabla}_{\mu} R_d / 4 \) and the condition for the de-Sitter stage curvature (2.4), Eq. (2.21) becomes

\[
\delta R_{\mu \nu} + \frac{1}{4} \vec{\nabla}_{\mu} R_d h_f - \frac{1}{2} \vec{\nabla}_{\mu} f' h_f - \frac{1}{4} R_d h_{\mu \nu} + (\vec{\nabla}_{\mu} \Box - \nabla_\mu \nabla_\nu) h_f = 0.
\]

The total perturbation of metric can be decomposed into the tensor part \( h_T^{\mu \nu} \) and scalar part \( h_S^{\mu \nu} \) in the way that

\[
h_{\mu \nu} = h_T^{\mu \nu} + h_S^{\mu \nu},
\]

where \( h_T^{\mu \nu} = \vec{\nabla}_{\mu} h_f \) and \( b \) is an unknown factor which will be determined later. Note that we require the tensor mode \( h_T^{\mu \nu} \) to be traceless because we do not want it to give any contribution to the perturbation of Ricci scalar \( \delta R \) and couple to the scalar mode \( h_f \). Since the theory is linearized, the perturbation of Ricci curvature tensor can also be separated into two parts

\[
\delta R_{\mu \nu} = \delta R_T^{\mu \nu} + \delta R_S^{\mu \nu},
\]
where $\delta R_{\mu\nu}^T$ and $\delta R_{\mu\nu}^S$ represent the perturbations contributed from the tensor mode $h_{\mu\nu}^T$, and scalar mode $h_{\mu\nu}^S$, respectively. In the de Sitter universe, we have $\partial_0 H = 0$ and $R_d = 12H^2$. Thus, $\delta R_{\mu\nu}^S$ can be written as

$$
\delta R_{\mu\nu}^S = -b \left( \frac{1}{2} g_{\mu\nu} \Box + \nabla_\mu \nabla_\nu \right) h_f. \quad (2.25)
$$

Inserting this into Eq. (2.22), we obtain

$$
\delta R_{\mu\nu}^T - \frac{1}{4} R_d h_{\mu\nu}^T + \left[ \left( 1 - \frac{b}{2} \right) g_{\mu\nu} \Box - (1 + b) \nabla_\mu \nabla_\nu + \frac{1}{4} (1 - b) g_{\mu\nu} R_d - \frac{1}{2} g_{\mu\nu} f' \right] h_f = 0. \quad (2.26)
$$

Since the wave equation for the scalar mode (2.15) does not involve any off-diagonal term, and we do not want the coupling between scalar and tensor modes, the term $\nabla_\mu \nabla_\nu h_f$ should not appear here. To cancel the $\nabla_\mu \nabla_\nu$ term, we must pick $b = -1$. This gives

$$
\delta R_{\mu\nu}^T - \frac{1}{4} R_d h_{\mu\nu}^T + \left( 3 \Box + R_d - \frac{f'}{f} \right) h_f = 0, \quad (2.27)
$$

where the terms in parentheses vanish by using the wave equation for the scalar mode (2.15). It is clear that the extra scalar degree of freedom in $f(R)$ can totally decouple from the ordinary tensor mode:

$$
\delta R_{\mu\nu}^T - \frac{1}{4} R_d h_{\mu\nu}^T = 0. \quad (2.28)
$$

Similar to gravitational wave in GR, we assume the tensor mode to be divergenceless, which means that it has to satisfy the Lorenz gauge condition, i.e., $\partial^\mu h_{\mu\nu}^T = 0$, and be transverse, $h_{\mu\nu}^T h_{\mu\nu} = 0$. The perturbation of Ricci tensor for the tensor mode can be simplified in this case as

$$
\delta R_{\mu\nu}^T = -\frac{1}{2} \left( -\partial_0^2 + \frac{\partial_1^2}{a^2} + H \partial_0 - 4H^2 \right) h_{\mu\nu}^T. \quad (2.29)
$$

We further define $h_{ij}^T = a^2 \mathcal{H}_{ij}$ to absorb the effect of expansion of the universe $[42]$, and choose the tensor mode to propagate in the z direction with

$$
\mathcal{H}_{ij} = \begin{pmatrix}
  h_+ & h_x & 0 \\
  h_x & -h_+ & 0 \\
  0 & 0 & 0
\end{pmatrix}, \quad (2.30)
$$

where $h_+$ and $h_x$ are the plus and cross polarizations, respectively. The first and second time derivatives of $h_{ij}^T$ can be expressed as

$$
\partial_0 h_{ij}^T = a^2 (\partial_0 + 2H) \mathcal{H}_{ij} \quad \text{and} \quad \partial_0^2 h_{ij}^T = a^2 (\partial_0^2 + 4H \partial_0 + 4H^2) \mathcal{H}_{ij}. \quad (2.31)
$$
Inserting these and Eq. (2.29) into (2.28), we obtain the wave equation for the tensor mode

$$
-\partial_0^2 + \frac{\partial^2}{a^2} - 3H\partial_0 \right) \mathcal{H}_{ij} = 0
$$

(2.32)
or

$$
\Box h_\alpha = 0, \quad \alpha = +, \times.
$$

(2.33)

Note that Eq. (2.32) is equivalent to Eq. (5.1.53) in Ref. [37] and Eq. (5.61) in Ref. [42]. Clearly, the tensor mode is exactly the same as that in GR when a traceless gravitational wave propagates in a non-zero de-Sitter curvature $R_d$ background. In what follows, we will concentrate on the scalar mode of gravitational wave.

### III. GRAVITATIONAL WAVES IN VAILABLE $f(R)$ MODELS

The conditions for a cosmological viable $f(R)$ model include (i) the positivity of the effective gravitational coupling, (ii) the stability of cosmological perturbations [36, 40, 41, 43], (iii) the stability of the late-time de-Sitter point [14, 44–46], (iv) the asymptotic behavior to $\Lambda$CDM at the high curvature regime, (v) the solar system constraint, and (vi) the constraint from the violation of the equivalence principle [18, 19, 21]. The typical examples of the viable $f(R)$ models are Hu-Sawicki [19], Starobinsky [36], Tsuikawa [22] and exponential gravity models [47]. To illustrate our results, we will concentrate on the exponential and Starobinsky models. Our study can be easily extended to other viable models.

Usually a viable $f(R)$ model can use chameleon mechanism to pass the constraints from the solar-system and equivalence principle [18, 19, 21]. In this case, a light-mass scalar is allowed by introducing the thin shell condition. As a result, the mass of the scalar mode does not have to be very heavy [18].

#### A. $\Lambda$CDM

We can take the cosmological constant model or the $\Lambda$CDM model as a special case of $f(R)$ with

$$
f(R) = R - 2\Lambda,
$$

(3.1)

where $\Lambda$ is the cosmological constant.
In this case, the mass of the scalar mode is infinite, i.e., $m_s^2 = \infty$, which requires infinite large energy to excite the scalar mode. Clearly, there is no scalar mode in the $\Lambda$CDM model.

Although the de Sitter curvature $R_d$ is not zero, i.e.,

$$R_d = 4 \Lambda \quad (\Lambda CDM), \quad (3.2)$$

the contribution from $R_d$ is negligible because $\Lambda \approx H_0^2 \approx (10^{-33} eV)^2$, where $H_0$ is the present Hubble parameter.

B. Exponential Gravity

The exponential gravity has been studied intensively in the literature [22, 47–54]. The form of $f(R)$ in the exponential gravity model is given by

$$f(R) = R - \beta R_S \left(1 - e^{-R/R_S}\right), \quad (3.3)$$

where $R_S$ is the characteristic curvature scale and $\beta$ is a model parameter. The viable conditions are satisfied when $\beta > 1$ and $R_S > 0$ [47, 52]. The feature is that it is free from the fine tuning problem and it has only one parameter more than the $\Lambda$CDM model.

Now we will first investigate the de-Sitter curvature $R_d$ in the model. The first and second derivatives of $f(R)$ with respect to $R$ are

$$f'(R) = 1 - \beta e^{-R/R_S} \quad \text{and} \quad f''(R) = \frac{\beta}{R_S} e^{-R/R_S}. \quad (3.4)$$

According to the condition for the de-Sitter curvature (2.4), $R_d$ satisfies

$$(1 - \beta e^{-R_d/R_S}) R_d = 2R_d - 2\beta R_S \left(1 - e^{-R_d/R_S}\right). \quad (3.5)$$

Defining $x \equiv R_d/R_S$, Eq. (3.5) becomes

$$x = 2\beta - \beta e^{-x} (x + 2). \quad (3.6)$$

The numerical solutions for this equation are shown in Fig. The factor $e^{-x} (x + 2)$ decreases very fast when $\beta > 1$, which is generally required by the viable condition for the exponential gravity. Therefore, we can obtain the asymptotic solution of $x$ for a large $\beta$:

$$x = 2\beta \quad \text{for} \quad \beta \gg 1. \quad (3.7)$$
Figure 1. Numerical solution of $x = R_d/R_S$ versus the model parameter $\beta$ in the exponential gravity model, where the solid line indicates the exact solution of Eq. (3.6) and the dashed line presents the approximated solution $x = 2\beta$.

Figure 2. $m_s^2/R_S$ versus $\beta$ in the region of $\beta = 0$ to 2 (left panel) and $\beta = 1$ to 3.5 (right panel) in the exponential gravity model, where the solid lines indicate the numerical solution of Eq. (3.8), $m_s^2/R_S$ is essentially zero when $\beta = 1$, and the dashed lines show the approximated solution $x = 2\beta$.

From Eq. (2.16), we derive the mass squared of the scalar mode in the exponential gravity as

$$m_s^2 = \frac{1}{3} R_S \left( \frac{e^{R_d/R_S} - \beta}{\beta} - \frac{R_d}{R_S} \right) = \frac{1}{3} R_S \left( \frac{1}{\beta} e^x - 1 - x \right).$$

The numerical solutions for $m_s^2/R_S$ are presented in Fig. 2. Since in the large curvature regime $R/R_S \gg 1$, the theory will recover the cosmological constant model, $R_S$ is roughly inverse proportional to $\beta$ in the way that

$$\beta R_S \cong 2\Lambda = 9.94 \times 10^{-66} eV^2$$

(3.9)
Figure 3. $m_s^2$ versus $\beta$ in the region of $\beta = 1$ to 2 (left panel) and $\beta = 1$ to 5 in log scale (right panel) in the exponential gravity model, where the solid lines present the approximated numerical solution of $m_s^2$ obtained by Eq. (3.10) and the dots in the right panel show the exact value of $m_s^2$ presented in Table I.

Table I. Numerical results of the scalar mode mass $m_s$ in vacuum with respect to different $\beta$ in the exponential gravity model.

| $\beta$ | $h$   | $y_H^{ini}$ | $\Omega_m^0$ | $R_S \ (10^{-66} eV^2)$ | $m_s \ (10^{-33} eV)$ |
|----------|-------|-------------|--------------|------------------------|----------------------|
| 4        | 0.7050| 2.618       | 0.2761       | 2.452                  | 24.36                |
| 3        | 0.7059| 2.609       | 0.2758       | 3.263                  | 11.39                |
| 2        | 0.7103| 2.558       | 0.2738       | 4.824                  | 5.069                |
| 1.27     | 0.7194| 2.45        | 0.2701       | 7.39                   | 1.86                 |

with the value of $\Lambda$ obtained from WMAP 7, SDSS 7 and SCP Union2 observations. Eq. (3.8) then can be approximated as

$$m_s^2 \approx \frac{2\Lambda}{3\beta} \left( \frac{1}{\beta} e^{x} - x - 1 \right).$$

(3.10)

In Fig. 3 we depict the result of the mass squared $m_s^2$ versus $\beta$ by using Eq. (3.10). We also calculate the exact $m_s$ without any approximation, and the results of $\beta = 1.27, 2, 3$ and 4 are shown in Table I where we have used the values of $R_S$ obtained from our previous result in Ref. [53] under the constraints of WMAP 7, SDSS 7 and SCP Union 2 measurements.

For $\beta \gg 1$, the mass squared $m_s^2$ becomes

$$m_s^2 \approx \frac{2\Lambda}{3} \left( \frac{1}{\beta^2} e^{2\beta} - 2 - \frac{1}{\beta} \right).$$

(3.11)
When $\beta$ is $O(1)$, $m_s$ is around $10^{-33}eV$. However, the cosmological observations do not give any significant upper bound on $\beta$. Thus, $m_s$ could be arbitrary large in this case. As $\beta \to \infty$, corresponding to the $\Lambda$CDM model with $m_s \to \infty$, the scalar mode of gravitational wave vanishes.

C. Starobinsky Model

In Ref. [36], Starobinsky proposed the following $f(R)$ form:

$$f(R) = R - \lambda R_c \left(1 - \left(1 + \frac{R^2}{R_c^2}\right)^{-n}\right),$$

(3.12)

where $R_c$ is roughly the present cosmological density and $\lambda$ and $n$ are positive model parameters. From the solar system constraint and the bound on the violation of the equivalence principle, one gets $n > 0.9$ [21]. In Fig. 4 we present the vacuum curvature $R_d$ at the de Sitter stages obtained by Eq. (2.14). We can see that $R_d/R_c \simeq 2\lambda$ when $\lambda \gg 1$. Since for $R \gg R_c$, the model will restore the $\Lambda$CDM model, we have $\lambda R_c \simeq 2\Lambda$ and $R_d \simeq 4\Lambda = 1.99 \times 10^{-65}eV^2$ when $\lambda \gg 1$.

In Fig. 5 we depict the numerical solutions of the mass squared of the scalar mode $m_s^2$ derived from Eq. (2.16).
IV. GRAVITATIONAL WAVE IN INNER GALAXY

A scalar mode of gravitational wave with effective zero or non-zero mass in \( f(R) \) is a very different prediction from the ordinary GR. However, we will show that in the presence of matter density, the scalar mode might not be able to exist in these viable \( f(R) \) models. Consider a scalar mode of gravitational wave propagating within our Galaxy halo. The local homogeneous density of dark matter and baryonic matter is roughly \( \rho \approx 10^{-24} \text{g/cm}^3 \). If we take this matter density into our analysis, it will give a large contribution to the background curvature compared to the vacuum de Sitter curvature. (The ratio of the matter density \( \rho \) to de Sitter curvature \( R_d \) is about \( \kappa^2 \rho / R_d \approx \kappa^2 \rho / 4 \Lambda \approx 10^5 \).) In this case, the condition for the background curvature \( R_0 \) should be modified as

\[
 f'(R_0)R_0 = 2f(R_0) - \kappa^2 \rho, \tag{4.1}
\]

where \( R_0 \) is the background curvature with matter. Note that for viable \( f(R) \) models, the solutions to Eq. (4.1) can be approximated as \( R_0 \approx \kappa^2 \rho \) at the high curvature regime.

In the case of the exponential gravity (3.3), Eq. (4.1) gives

\[
 x = 2 \beta + r - \beta e^{-x} (x + 2), \tag{4.2}
\]

where \( x \equiv R_0 / R_S \) and \( r \equiv \kappa^2 \rho / R_S \) are the ratios of the background curvature and matter density to \( R_S \), respectively. Since \( \beta R_S \approx 2 \Lambda \) from (3.9), we find that the solution of Eq. (4.2) is extremely large,

\[
 x \simeq r \simeq \frac{\kappa^2 \rho}{R_d/2\beta} \simeq 2 \times 10^5 \beta, \tag{4.3}
\]
which just leads to $R_0 \simeq \kappa^2 \rho$. Thus, in the exponential gravity, the mass of the scalar mode will become an extreme in the galaxy region:

$$m_s \approx \sqrt{\frac{2\Lambda}{3\beta^2} e^{2\times10^5 \beta}} \approx \infty.$$  \hfill (4.4)

The corresponding cutoff frequency $\omega_m$ is also infinite. As a result, it is almost impossible to detect this scalar mode within our Galaxy under the exponential gravity scenario. Moreover, for any source that is massive enough to generate gravitational waves, we expect them to lay in the region with density higher than the baryonic/dark matter density $10^{-24} g/cm^3$. Therefore, the scalar mode will not have the chance to propagate from the source in the exponential gravity.

In the case of the Starobinsky model \cite{3.12}, the situation is quite different. The scalar mode of gravitational wave can have a light mass in the galaxy region. The minimum bound of the scalar mode mass is $m_s \gtrsim 10^{-24} eV$ when $\rho = 10^{-24} g/cm^3$. The corresponding cutoff frequency is quite small $f_m \gtrsim 10^{-9} \text{ Hz}$. This feature will allow the propagation of the scalar mode inside the galaxy. Hence, detecting the scalar mode in the Starobinsky model will be possible. In Fig. 6 we depict the mass squared versus the model parameter $n$ with different fixed values of $\lambda$, where we have used $\lambda R_c \cong 2\Lambda$. The scalar mode can still be very heavy when the index $n$ goes to a large value, but the mass dependence on the parameter $\lambda$ is not quite significant.

Figure 6. $m_s^2$ versus $n$ in the Starobinsky model with matter density $\rho = 10^{-24} g/cm^3$ and $\lambda R_c \cong 2\Lambda$. 

\[ \begin{array}{cccc}
\lambda = 1 & \lambda = 2 & \lambda = 3 & \lambda = 4 \\
\end{array} \]
V. CONCLUSIONS

We have discussed gravitational waves in viable $f(R)$ theories. Using the weak field approximation on the field equation, we have confirmed that $f(R)$ will give an extra massive scalar mode besides the ordinary tensor mode in the standard GR. We have explicitly investigated the situations of the extra scalar mode of gravitational wave in the exponential gravity and Starobinsky models of the viable $f(R)$ gravity theories. In vacuum, we have shown that the typical mass squared of the scalar mode is in the order of the de-Sitter curvature $m_s^2 \sim R_d \approx 10^{-66} eV^2$ in both models.

However, in the galaxy region, the situations will be different if we consider the background curvature by the presence of the galactic density. The small matter density $\rho = 10^{-24} g/cm^3$ is still $10^5$ larger than the de-Sitter curvature $R_d$ in both models. In the exponential gravity, since the mass of the scalar mode in galaxy is undetectable large, it will be unable to measure a gravitational wave. On the other hand, in the Starobinsky model, the mass can be much smaller with its lower bound in galaxy being about $10^{-24} eV$ (or $10^{-9}$ Hz). Therefore, it is possible to observe the scalar mode of gravitational wave in the Starobinsky scenario if there is an astrophysical source which generates this scalar mode. The extreme difference of the scalar mode masses also makes the distinction of $f(R)$ theories from $\Lambda$CDM possible by probing this scalar mode of gravitational wave.

Recently, there is an underway space-based gravitational wave probing experiment, the Laser Interferometer Space Antenna (LISA) [58], which is a proposed joint mission of the European Space Agency (ESA) and NASA. It will measure the low-frequency band ($10^{-5}$ to 1 Hz) of gravitational waves with high signal-to-noise ratio. The gravitational sources within this band [59] include supermassive black holes, intermediate-mass black holes, extreme-mass-ratio black hole inspirals, galactic compact binaries and some primordial gravitational wave sources. It is possible that these sources also generate the scalar mode of gravitational wave. Since LISA will be located far from the Earth and other gravitational sources, the background curvature of it is very low compared to the ground-based experiments, which allows the propagation of the scalar mode in some viable $f(R)$. As a result, LISA has a great chance to direct detect not only the ordinary gravitational wave but also the clues of the deviation from Einstein’s GR by analyzing the scalar mode behavior of gravitational wave.

1 Laser Interferometer Space Antenna, http://sci.esa.int/lisa
Moreover, if the scalar mode becomes observable, many interesting features in the viable $f(R)$ gravity models\cite{12, 60} will appear. We note that other gravitational wave probes, such as ASTROD-GW\cite{61}, with the sensitivity in the $10^{-7} - 10^{-1}$ Hz band, may also detect the extra scalar mode. It is clear that an observation of the scalar mode with a frequency larger than $10^{-8}$ Hz by LISA or ASTROD-GW would be associated with the Starobinsky model and is rules out the exponential one. Finally, we remark that the scalar mode in a viable $f(R)$ cannot be observed by the ground gravitational searches due to the large background curvature.

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