1. Introduction

In this paper, we prove some vanishing theorems for simple normal crossing pairs, which will play important roles in the study of higher dimensional algebraic varieties. Theorem 1.1 is a generalization of the works of several authors: Kawamata, Viehweg, Kollár, Esnault–Viehweg, Ambro, and others (cf. [Ko1], [KMM], [EV], [Ko2], [A], [F2], [F6], [F7], [F10], and so on).

Theorem 1.1 (cf. Theorem 3.6). Let \((Y, \Delta)\) be a simple normal crossing pair such that \(\Delta\) is a boundary \(\mathbb{R}\)-divisor on \(Y\). Let \(f : Y \to X\) be a proper morphism to an algebraic variety \(X\) and let \(L\) be a Cartier divisor on \(Y\) such that \(L - (K_Y + \Delta)\) is \(f\)-semi-ample.
(i) every associated prime of $R^q f_*\mathcal{O}_Y(L)$ is the generic point of the $f$-image of some stratum of $(Y, \Delta)$.

(ii) let $\pi : X \to V$ be a projective morphism to an algebraic variety $V$ such that

$$L - (K_Y + \Delta) \sim_R f^*H$$

for some $\pi$-ample $R$-Cartier $R$-divisor $H$ on $X$. Then $R^q f_*\mathcal{O}_Y(L)$ is $\pi_*$-acyclic, that is,

$$R^p\pi_*R^q f_*\mathcal{O}_Y(L) = 0$$

for every $p > 0$ and $q \geq 0$.

When $X$ is a divisor on a smooth variety $M$, Theorem 1.1 is contained in [A] and plays crucial roles in the theory of quasi-log varieties. For the details, see [F8] and [F7]. When $X$ is quasi-projective, it is proved in [FF, Section 7]. Here, we need no extra assumptions on $X$. Therefore, Theorem 1.1 is new. The theory of resolution of singularities for reducible varieties has recently been developing (cf. [BM] and [BP]). It refines several vanishing theorems in [F7]. It is one of the main themes of this paper. We will give a proof of Theorem 1.1 in Section 3. Note that we do not treat normal crossing varieties. We only discuss simple normal crossing varieties because the theory of resolution of singularities for reducible varieties works well only for simple normal crossing varieties. We note that the fundamental theorems for the log minimal model program for log canonical pairs can be proved without using the theory of quasi-log varieties (cf. [F9] and [F10]). The case when $Y$ is smooth in Theorem 1.1 is sufficient for [F9] and [F10]. For that case, see [F6] and [F10, Sections 5 and 6]. Our proof of Theorem 1.1 heavily depends on the theory of mixed Hodge structures.

1.2 (Hodge theoretic viewpoint). Let $X$ be a projective simple normal crossing variety with $\dim X = n$. We are mainly interested in $H^\bullet(X, \omega_X)$ or $H^\bullet(X, \omega_X \otimes L)$ for some line bundle $L$ on $X$. By the theory of mixed Hodge structures,

$$\text{Gr}_F^n H^\bullet(X, \mathbb{C}) \simeq H^{*-n}(X, \nu_*\omega_{X^\nu}),$$

where $\nu : X^\nu \to X$ is the normalization, and

$$\text{Gr}_F^0 H^\bullet(X, \mathbb{C}) \simeq H^\bullet(X, \mathcal{O}_X).$$

Note that $F$ is the Hodge filtration on the natural mixed Hodge structure on $H^\bullet(X, \mathbb{Q})$. Let $D$ be a simple normal crossing divisor on $X$. Then we obtain

$$\text{Gr}_F^n H^\bullet(X \setminus D, \mathbb{C}) \simeq H^{*-n}(X, \nu_*\omega_{X^\nu} \otimes \mathcal{O}_X(D)).$$
and
\[ Gr^0_F H^i_c(X \setminus D, \mathbb{C}) \simeq H^i(X, \mathcal{O}_X(-D)). \]
Note that
\[ Gr^0_F H^i_c(X \setminus D, \mathbb{C}) \simeq H^i(X, \mathcal{O}_X) \]
and that
\[ H^{*-n}(X, \nu_* \omega_{X^\nu} \otimes \mathcal{O}_X(D)) \not\cong H^{*-n}(X, \omega_X \otimes \mathcal{O}_X(D)). \]

We also note that \( H^i_c(X \setminus D, \mathbb{Q}) \) need not be the dual vector space of \( H^{2n-i}(X \setminus D, \mathbb{Q}) \) when \( X \) is not smooth. In this setting, we are interested in \( H^i(X, \omega_X(D)) \) or \( H^i(X, \omega_X(D) \otimes L) \). Therefore, we consider the natural mixed Hodge structure on \( H^i_c(X \setminus D, \mathbb{C}) \) and take the dual vector space of \( Gr^0_F H^i_c(X \setminus D, \mathbb{C}) \simeq H^i(X, \mathcal{O}_X(-D)) \)
by the Serre duality. Then we obtain \( H^{n-i}(X, \omega_X(D)) \). We note that if \( L \) is semi-ample then we can reduce the problem to the case when \( L \) is trivial by the usual covering trick. The above observation is crucial for our treatment of the vanishing theorems and the semi-positivity theorems in \([F7]\) and \([FF]\). In this paper, we do not discuss the Hodge theoretic part of vanishing and semi-positivity theorems. We prove Theorem 1.1 by assuming the Hodge theoretic injectivity theorem: Theorem 3.1.

**Remark 1.3** (cf. \([F7]\) Proposition 3.65). We can construct a proper simple normal crossing variety \( X \) with the following property. Let \( f : Y \to X \) be a proper morphism from a simple normal crossing variety \( Y \) such that \( f \) induces an isomorphism \( f|_V : V \simeq U \) where \( V \) (resp. \( U \)) is a dense Zariski open subset of \( Y \) (resp. \( X \)) which contains the generic point of any stratum of \( Y \) (resp. \( X \)). Then \( Y \) is non-projective. Therefore, we can not directly use Chow’s lemma to reduce our main theorem (cf. Theorem 1.1) to the quasi-projective case (cf. \([FF]\) Section 7).

There exists another standard approach to various Kodaira type vanishing theorems. It is an analytic method (see, for example, \([F4]\) and \([F5]\)). At the present time, the relationship between our Hodge theoretic approach and the analytic method is not clear.

We summarize the contents of this paper. In Section 2 we collect some basic definitions and results for the study of simple normal crossing varieties and divisors on them. Section 3 is the main part of this paper. It is devoted to the study of injectivity, torsion-free, and vanishing theorems for simple normal crossing pairs. We note that we do not prove the Hodge theoretic injectivity theorem: Theorem 3.1. We just
quote it from [F7]. Section 4 is an easy application of the vanishing theorem in Section 3. We prove the basic properties of semi divisorial log terminal pairs in the sense of Kollár. In Section 5, we explain our new semi-positivity theorem, which is a generalization of Fujita–Kawamata’s semi-positivity theorem, without proof. It depends on the theory of variations of mixed Hodge structures on compact support cohomology groups and is related to the results obtained in Section 3. Anyway, the vanishing theorem and the semi-positivity theorem discussed in this paper follow from the theory of mixed Hodge structures on compact support cohomology groups.

For various applications of Theorem 1.1 and related topics, see [F7], [FF], and so on.

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We will work over $\mathbb{C}$, the complex number field, throughout this paper. But we note that, by using the Lefschetz principle, all the results in this paper hold over an algebraically closed field $k$ of characteristic zero.

2. Preliminaries

First, we quickly recall basic definitions of divisors. We note that we have to deal with reducible algebraic schemes in this paper. For details, see [M, Lecture 9] and [Fu, Appendix B.4].

2.1. Let $X$ be a noetherian scheme with structure sheaf $\mathcal{O}_X$ and let $\mathcal{K}_X$ be the sheaf of total quotient rings of $\mathcal{O}_X$, that is, for every affine open set $U \subset X$, $\Gamma(U, \mathcal{K}_X)$ is the total quotient ring of $\Gamma(U, \mathcal{O}_X)$. Let $\mathcal{K}_X^*$ denote the (multiplicative) sheaf of invertible elements in $\mathcal{K}_X$, and $\mathcal{O}_X^*$ the sheaf of invertible elements in $\mathcal{O}_X$. We note that $\mathcal{O}_X \subset \mathcal{K}_X$ and $\mathcal{O}_X^* \subset \mathcal{K}_X^*$.

2.2 (Cartier, $\mathbb{Q}$-Cartier, and $\mathbb{R}$-Cartier divisors). A Cartier divisor $D$ on $X$ is a global section of $\mathcal{K}_X^*/\mathcal{O}_X^*$, that is, $D$ is an element of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. A $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-Cartier $\mathbb{R}$-divisor) is an element of $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{Q}$ (resp. $H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{R}$).

2.3 (Linear, $\mathbb{Q}$-linear, and $\mathbb{R}$-linear equivalence). Let $D_1$ and $D_2$ be two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on $X$. Then $D_1$ is linearly (resp. $\mathbb{Q}$-linearly,
or \(\mathbb{R}\)-linearly equivalent to \(D_2\), denoted by \(D_1 \sim D_2\) (resp. \(D_1 \sim_{\mathbb{Q}} D_2\), or \(D_1 \sim_{\mathbb{R}} D_2\)) if
\[
D_1 = D_2 + \sum_{i=1}^{k} r_i(f_i)
\]
such that \(f_i \in \Gamma(X, \mathcal{K}_X^*)\) and \(r_i \in \mathbb{Z}\) (resp. \(r_i \in \mathbb{Q}\), or \(r_i \in \mathbb{R}\)) for every \(i\). We note that \((f_i)\) is a principal Cartier divisor associated to \(f_i\), that is, the image of \(f_i\) by \(\Gamma(X, \mathcal{K}_X^*) \rightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)\). Let \(f : X \rightarrow Y\) be a morphism. If there is an \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(B\) on \(Y\) such that \(D_1 \sim_{\mathbb{R}} D_2 + f^*B\), then \(D_1\) is said to be relatively \(\mathbb{R}\)-linearly equivalent to \(D_2\). It is denoted by \(D_1 \sim_{\mathbb{R},f} D_2\).

2.4 (Supports). Let \(D\) be a Cartier divisor on \(X\). The support of \(D\), denoted by \(\text{Supp} D\), is the subset of \(X\) consisting of points \(x\) such that a local equation for \(D\) is not in \(\mathcal{O}_{X,x}^*\). The support of \(D\) is a closed subset of \(X\).

2.5 (Weil divisors, \(\mathbb{Q}\)-divisors, and \(\mathbb{R}\)-divisors). Let \(X\) be an equi-dimensional reduced separated algebraic scheme. We note that \(X\) is not necessarily regular in codimension one. A (Weil) divisor \(D\) on \(X\) is a finite formal sum
\[
\sum_{i=1}^{n} d_i D_i
\]
where \(D_i\) is an irreducible reduced closed subscheme of \(X\) of pure codimension one and \(d_i\) is an integer for every \(i\) such that \(D_i \neq D_j\) for \(i \neq j\).

If \(d_i \in \mathbb{Q}\) (resp. \(d_i \in \mathbb{R}\)) for every \(i\), then \(D\) is called a \(\mathbb{Q}\)-divisor (resp. \(\mathbb{R}\)-divisor). We define the round-up \(\lceil D \rceil = \sum_{i=1}^{r} \lceil d_i \rceil D_i\) (resp. the round-down \(\lfloor D \rfloor = \sum_{i=1}^{r} \lfloor d_i \rfloor D_i\)), where for every real number \(x\), \(\lceil x \rceil\) (resp. \(\lfloor x \rfloor\)) is the integer defined by \(x \leq \lceil x \rceil < x + 1\) (resp. \(x - 1 < \lfloor x \rfloor \leq x\)). The fractional part \(\{D\}\) of \(D\) denotes \(D - \lfloor D \rfloor\). We define \(D^{<1} = \sum_{d_i < 1} d_i D_i\) and so on. We call \(D\) a boundary \(\mathbb{R}\)-divisor if \(0 \leq d_i \leq 1\) for every \(i\).

Next, we recall the definition of simple normal crossing pairs.

Definition 2.6 (Simple normal crossing pairs). We say that the pair \((X, D)\) is simple normal crossing at a point \(a \in X\) if \(X\) has a Zariski open neighborhood \(U\) of \(a\) that can be embedded in a smooth variety \(Y\), where \(Y\) has regular local coordinates \((x_1, \ldots, x_p, y_1, \ldots, y_r)\) at \(a = 0\) in which \(U\) is defined by a monomial equation
\[
x_1 \cdots x_p = 0
\]
and

\[ D = \sum_{i=1}^{r} \alpha_i(y_i = 0)|_{U}, \quad \alpha_i \in \mathbb{R}. \]

We say that \((X, D)\) is a *simple normal crossing pair* if it is simple normal crossing at every point of \(X\). If \((X, 0)\) is a simple normal crossing pair, then \(X\) is called a *simple normal crossing variety*. If \((X, 0)\) is a simple normal crossing pair, then \(X\) is called a *simple normal crossing variety*. If \(X\) is a simple normal crossing variety, then \(X\) has only Gorenstein singularities. Thus, it has an invertible dualizing sheaf \(\omega_X\). Therefore, we can define the *canonical divisor* \(K_X\) such that \(\omega_X \cong O_X(K_X)\). It is a Cartier divisor on \(X\) and is well-defined up to linear equivalence.

We note that a simple normal crossing pair is called a *semi-snc pair* in \([Ko3, \text{Definition 1.9}]\).

**Definition 2.7** (Strata and permissibility). Let \(X\) be a simple normal crossing variety and let \(X = \bigcup_{i \in I} X_i\) be the irreducible decomposition of \(X\). A *stratum* of \(X\) is an irreducible component of \(X_{i_1} \cap \cdots \cap X_{i_k}\) for some \(\{i_1, \cdots, i_k\} \subset I\). A Cartier divisor \(D\) on \(X\) is *permissible* if \(D\) contains no strata of \(X\) in its support. A finite \(\mathbb{Q}\)-linear (resp. \(\mathbb{R}\)-linear) combination of permissible Cartier divisors is called a *permissible \(\mathbb{Q}\)-divisor* (resp. \(\mathbb{R}\)-divisor) on \(X\).

**2.8.** Let \(X\) be a simple normal crossing variety. Let \(\text{PerDiv}(X)\) be the abelian group generated by permissible Cartier divisors on \(X\) and let \(\text{Weil}(X)\) be the abelian group generated by Weil divisors on \(X\). Then we can define natural injective homomorphisms of abelian groups

\[ \psi : \text{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} \to \text{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K} \]

for \(\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \text{and } \mathbb{R}\). Let \(\nu : \tilde{X} \to X\) be the normalization. Then we have the following commutative diagram.

\[
\begin{array}{ccc}
\text{Div}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} & \overset{\sim}{\longrightarrow} & \text{Weil}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{K} \\
\nu^* \downarrow & & \downarrow \nu^* \\
\text{PerDiv}(X) \otimes_{\mathbb{Z}} \mathbb{K} & \underset{\psi}{\longrightarrow} & \text{Weil}(X) \otimes_{\mathbb{Z}} \mathbb{K}
\end{array}
\]

Note that \(\text{Div}(\tilde{X})\) is the abelian group generated by Cartier divisors on \(\tilde{X}\) and that \(\psi\) is an isomorphism since \(\tilde{X}\) is smooth.

By \(\psi\), every permissible Cartier (resp. \(\mathbb{Q}\)-Cartier or \(\mathbb{R}\)-Cartier) divisor can be considered as a Weil divisor (resp. \(\mathbb{Q}\)-divisor or \(\mathbb{R}\)-divisor). Therefore, various operations, for example, \(\cup D, D^{<1}\), and so on, make sense for a permissible \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(D\) on \(X\).

We note the following easy example.
Example 2.9. Let $X$ be a simple normal crossing variety in $\mathbb{C}^3 = \text{Spec}\mathbb{C}[x, y, z]$ defined by $xy = 0$. We put $D_1 = (x+z) \cap X$ and $D_2 = (x-z) \cap X$. Then $D = \frac{1}{2} D_1 + \frac{1}{2} D_2$ is a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. In this case, $\lceil D \rceil = (x = z = 0)$ on $X$. Therefore, $\lceil D \rceil$ is not a Cartier divisor on $X$.

Definition 2.10 (Simple normal crossing divisors). Let $X$ be a simple normal crossing variety and let $D$ be a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$. If $D$ is reduced and $(X, D)$ is a simple normal crossing pair, then $D$ is called a simple normal crossing divisor on $X$.

Remark 2.11. Let $X$ be a simple normal crossing variety and let $D$ be a permissible $\mathbb{K}$-Cartier $\mathbb{K}$-divisor on $X$ where $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. If $\text{Supp}D$ is a simple normal crossing divisor on $X$, then $\lceil D \rceil$ and $\lceil D \rceil^\prec$ (resp. $\{D\}$, $D <^1$, and so on) are Cartier (resp. $\mathbb{K}$-Cartier) divisors on $X$.

The following lemma is easy but important.

Lemma 2.12. Let $X$ be a simple normal crossing variety and let $B$ be a permissible $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on $X$ such that $\lceil B \rceil = 0$. Let $A$ be a Cartier divisor on $X$. Assume that $A \sim_\mathbb{R} B$. Then there exists a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $C$ on $X$ such that $A \sim_\mathbb{Q} C$, $\lceil C \rceil = 0$, and $\text{Supp}C = \text{Supp}B$.

Proof. We can write $B = A + \sum_{i=1}^{k} r_i (f_i)$, where $f_i \in \Gamma(X, \mathcal{K}_X^*)$ and $r_i \in \mathbb{R}$ for every $i$. Here, $\mathcal{K}_X$ is the sheaf of total quotient rings of $\mathcal{O}_X$. Let $P \in X$ be a scheme theoretic point corresponding to some stratum of $X$. We consider the following affine map

$$
\mathbb{R}^k \to H^0(X_P, \mathcal{K}_{X_P}^* / \mathcal{O}_{X_P}) \otimes_{\mathbb{Z}} \mathbb{K}
$$

given by $(a_1, \cdots, a_k) \mapsto A + \sum_{i=1}^{k} a_i (f_i)$, where $X_P = \text{Spec}\mathcal{O}_{X,P}$ and $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. Then we can check that

$$
\mathcal{P} = \{(a_1, \cdots, a_k) \in \mathbb{R}^k \mid A + \sum a_i (f_i) \text{ is permissible} \} \subset \mathbb{R}^k
$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Therefore, we see that

$$
\mathcal{S} = \{(a_1, \cdots, a_k) \in \mathcal{P} \mid \text{Supp}(A + \sum a_i (f_i)) \subset \text{Supp}B \} \subset \mathcal{P}
$$

is an affine subspace of $\mathbb{R}^k$ defined over $\mathbb{Q}$. Since $(r_1, \cdots, r_k) \in \mathcal{S}$, we know that $\mathcal{S} \neq \emptyset$. We take a point $(s_1, \cdots, s_k) \in \mathcal{S} \cap \mathbb{Q}^k$ which is general in $\mathcal{S}$ and sufficiently close to $(r_1, \cdots, r_k)$ and put $C = A + \sum_{i=1}^{k} s_i (f_i)$. By construction, $C$ is a permissible $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $C \sim_\mathbb{Q} A$, $\lceil C \rceil = 0$, and $\text{Supp}C = \text{Supp}B$. \qed
We close this section with an important definition.

**Definition 2.13** (Strata and permissibility for pairs). Let \((X, D)\) be a simple normal crossing pair such that \(D\) is a boundary \(\mathbb{R}\)-divisor on \(X\). Let \(\nu : X^\nu \to X\) be the normalization. We define \(\Theta\) by the formula

\[
K_{X^\nu} + \Theta = \nu^*(K_X + D).
\]

Then a *stratum* of \((X, D)\) is an irreducible component of \(X\) or the \(\nu\)-image of a log canonical center of \((X^\nu, \Theta)\). We note that \((X^\nu, \Theta)\) is log canonical. When \(D = 0\), this definition is compatible with Definition 2.7. A permissible \(\mathbb{R}\)-Cartier \(\mathbb{R}\)-divisor \(B\) on \(X\) is permissible with respect to \((X, D)\) if \(B\) contains no strata of \((X, D)\) in its support.

### 3. Vanishing theorems

Let us start with the following injectivity theorem (cf. [F7, Proposition 2.23]). The proof of Theorem 3.1 in [F7] is purely Hodge theoretic. It uses the theory of mixed Hodge structures on compact support cohomology groups (cf. 1.2). For the details, see [F7, Chapter 2].

**Theorem 3.1** (Hodge theoretic injectivity theorem). Let \((X, S + B)\) be a simple normal crossing pair such that \(X\) is proper, \(S + B\) is a boundary \(\mathbb{R}\)-divisor, \(S\) is reduced, and \(\Delta = 0\). Let \(L\) be a Cartier divisor on \(X\) and let \(D\) be an effective Cartier divisor whose support is contained in \(\text{Supp} B\). Assume that \(L \sim_{\mathbb{R}} K_X + S + B\). Then the natural homomorphisms

\[
H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)),
\]

which are induced by the inclusion \(\mathcal{O}_X \to \mathcal{O}_X(D)\), are injective for all \(q\).

The next lemma is an easy generalization of the vanishing theorem of Reid–Fukuda type for simple normal crossing pairs, which is a very special case of Theorem 3.6 (i). However, we need Lemma 3.2 for our proof of Theorem 3.6.

**Lemma 3.2** (Relative vanishing lemma). Let \(f : Y \to X\) be a proper morphism from a simple normal crossing pair \((Y, \Delta)\) to an algebraic variety \(X\) such that \(\Delta\) is a boundary \(\mathbb{R}\)-divisor on \(Y\). We assume that \(f\) is an isomorphism at the generic point of any stratum of the pair \((Y, \Delta)\). Let \(L\) be a Cartier divisor on \(Y\) such that \(L \sim_{\mathbb{R}, f} K_Y + \Delta\). Then \(R^q f_* \mathcal{O}_Y(L) = 0\) for every \(q > 0\).

**Proof.** By shrinking \(X\), we may assume that \(L \sim_{\mathbb{R}} K_Y + \Delta\). By applying Lemma 2.12 to \(\{\Delta\}\), we may further assume that \(\Delta\) is a \(\mathbb{Q}\)-divisor and \(L \sim_{\mathbb{Q}} K_Y + \Delta\).
Step 1. We assume that $Y$ is irreducible. In this case, $L - (K_Y + \Delta)$ is nef and log big over $X$ with respect to the pair $(Y, \Delta)$. Therefore, $R^q f_* \mathcal{O}_Y(L) = 0$ for every $q > 0$ by the vanishing theorem of Reid–Fukuda type (see, for example, [F7, Lemma 4.11]).

Step 2. Let $Y_1$ be an irreducible component of $Y$ and let $Y_2$ be the union of the other irreducible components of $Y$. Then we have a short exact sequence

$$0 \to \mathcal{O}_{Y_1}(-Y_2|_{Y_1}) \to \mathcal{O}_Y \to \mathcal{O}_{Y_2} \to 0.$$ 

We put $L' = L|_{Y_1} - Y_2|_{Y_1}$. Then we have a short exact sequence

$$0 \to \mathcal{O}_{Y_1}(L') \to \mathcal{O}_Y(L) \to \mathcal{O}_{Y_2}(L|_{Y_2}) \to 0$$

and $L' \sim Q K_{Y_1} + \Delta|_{Y_1}$. On the other hand, we can check that

$$L|_{Y_2} \sim Q K_{Y_2} + Y_1|_{Y_2} + \Delta|_{Y_2}.$$ 

We have already known that $R^q f_* \mathcal{O}_{Y_1}(L') = 0$ for every $q > 0$ by Step 1. By the induction on the number of the irreducible components of $Y$, we have $R^q f_* \mathcal{O}_{Y_2}(L|_{Y_2}) = 0$ for every $q > 0$. Therefore, $R^q f_* \mathcal{O}_Y(L) = 0$ for every $q > 0$ by the exact sequence:

$$\cdots \to R^q f_* \mathcal{O}_{Y_1}(L') \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_{Y_2}(L|_{Y_2}) \to \cdots.$$ 

So, we finish the proof of Lemma 3.2. \hfill \Box

Although Lemma 3.2 is a very easy generalization of the relative Kawamata–Viehweg vanishing theorem, it is sufficiently powerful for the study of reducible varieties once we combine it with the recent results in [BM] and [BP]. In Section 4, we will see an application of Lemma 3.2 for the study of semi divisorial log terminal pairs.

It is the time to state the main injectivity theorem for simple normal crossing pairs. It is a direct application of Theorem 3.1. Our formulation of Theorem 3.3 is indispensable for the proof of our main theorem: Theorem 3.6.

**Theorem 3.3** (Injectivity theorem for simple normal crossing pairs). Let $(X, \Delta)$ be a simple normal crossing pair such that $X$ is proper and that $\Delta$ is a boundary $\mathbb{R}$-divisor on $X$. Let $L$ be a Cartier divisor on $X$ and let $D$ be an effective Cartier divisor that is permissible with respect to $(X, \Delta)$. Assume the following conditions.

(i) $L \sim \mathbb{R} K_X + \Delta + H$,

(ii) $H$ is a semi-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor, and

(iii) $tH \sim \mathbb{R} D + D'$ for some positive real number $t$, where $D'$ is an effective $\mathbb{R}$-Cartier $\mathbb{R}$-divisor that is permissible with respect to $(X, \Delta)$. 


Then the homomorphisms
\[ H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)), \]
which are induced by the natural inclusion \( \mathcal{O}_X \to \mathcal{O}_X(D) \), are injective for all \( q \).

**Remark 3.4.** For the definition and the basic properties of semi-ample \( \mathbb{R} \)-Cartier \( \mathbb{R} \)-divisors, see \([F10]\) Definition 4.11, Lemma 4.13, and Lemma 4.14.

**Proof of Theorem 3.3.** We put \( S = \triangle \cup \Delta \) and \( B = \{\Delta\} \) throughout this proof. We obtain a projective birational morphism \( f : Y \to X \) from a simple normal crossing variety \( Y \) such that \( f \) is an isomorphism outside \( \text{Supp}(D + D' + B) \), and that the union of the support of \( f^*(S + B + D + D') \) and the exceptional locus of \( f \) has a simple normal crossing support on \( Y \) (cf. \([BP]\) Theorem 1.5). Let \( B' \) be the strict transform of \( B \) on \( Y \). We may assume that \( \text{Supp}B' \) is disjoint from any strata of \( Y \) that are not irreducible components of \( Y \) by taking blowing-ups. We write
\[ K_Y + S' + B' = f^*(K_X + S + B) + E, \]
where \( S' \) is the strict transform of \( S \) and \( E \) is \( f \)-exceptional. By the construction of \( f : Y \to X \), \( S' \) is Cartier and \( B' \) is \( \mathbb{R} \)-Cartier. Therefore, \( E \) is also \( \mathbb{R} \)-Cartier. It is easy to see that \( E_+ = \lceil E \rceil \geq 0 \). We put \( L' = f^*L + E_+ \) and \( E_- = E_+ - E \geq 0 \). We note that \( E_+ \) is Cartier and \( E_- \) is \( \mathbb{R} \)-Cartier because \( \text{Supp}E \) is simple normal crossing on \( Y \).

Since \( f^*H \) is an \( \mathbb{R}_{>0} \)-linear combination of semi-ample Cartier divisors, we can write \( f^*H \sim_{\mathbb{R}} \sum_i a_iH_i \), where \( 0 < a_i < 1 \) and \( H_i \) is a general Cartier divisor on \( Y \) for every \( i \). We put
\[ B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D + D') + (1 - \varepsilon) \sum_i a_iH_i \]
for some \( 0 < \varepsilon \ll 1 \). Then \( L' \sim_{\mathbb{R}} K_Y + S' + B'' \). By the construction, \( \cup B'' = 0 \), the support of \( S' + B'' \) is simple normal crossing on \( Y \), and \( \text{Supp}B'' \supset \text{Supp}f^*D \). So, Theorem 3.1 implies that the homomorphisms
\[ H^q(Y, \mathcal{O}_Y(L')) \to H^q(Y, \mathcal{O}_Y(L' + f^*D)) \]
are injective for all \( q \). By Lemma 3.2, \( R^q f_*\mathcal{O}_Y(L') = 0 \) for every \( q > 0 \) and it is easy to see that \( f_*\mathcal{O}_Y(L') \simeq \mathcal{O}_X(L) \). By the Leray spectral sequence, the homomorphisms
\[ H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L + D)) \]
are injective for all \( q \). \( \square \)
Lemma 3.5. Let $f : Z \to X$ be a proper morphism from a simple normal crossing pair $(Z,B)$ to an algebraic variety $X$. Let $\overline{X}$ be a projective variety such that $\overline{X}$ contains $X$ as a Zariski dense open subset. Then there exist a proper simple normal crossing pair $(\overline{Z},B)$ that is a compactification of $(Z,B)$ and a proper morphism $\overline{f} : \overline{Z} \to \overline{X}$ that compactifies $f : Z \to X$. Moreover, $\overline{Z} \setminus Z$ is a divisor on $\overline{Z}$, $\text{Supp} B \cup \text{Supp}(\overline{Z} \setminus Z)$ is a simple normal crossing divisor on $\overline{Z}$, and $\overline{Z} \setminus Z$ has no common irreducible components with $B$. We note that we can make $B$ a $K$-Cartier $K$-divisor on $\overline{Z}$ when so is $B$ on $Z$, where $K$ is $\mathbb{Z}$, $\mathbb{Q}$, or $\mathbb{R}$. When $f$ is projective, we can make $\overline{Z}$ projective.

Proof. Let $\overline{B} \subset \overline{Z}$ be any compactification of $B \subset Z$. By blowing up $\overline{Z}$ inside $\overline{Z} \setminus Z$, we may assume that $f : Z \to X$ extends to $\overline{f} : \overline{Z} \to \overline{X}$, $\overline{Z}$ is a simple normal crossing variety, and $\overline{Z} \setminus Z$ is of pure codimension one (see [BM, Theorem 1.5]). By [BP, Theorem 1.2], we can construct a desired compactification. Note that we can make $\overline{B}$ a $K$-Cartier $K$-divisor by the argument in [BP, Section 8].

Theorem 3.6 below is our main theorem of this paper, which is a generalization of Kollár’s torsion-free and vanishing theorem (cf. [Ko1, Theorem 2.1]). The reader find various applications of Theorem 3.6 in [F7]. We note that Theorem 3.6 for embedded normal crossing pairs was first formulated by Florin Ambro for his theory of quasi-log varieties (cf. [A]). For the details of the theory of quasi-log varieties, see [F7, Chapter 3].

Theorem 3.6. Let $(Y,\Delta)$ be a simple normal crossing pair such that $\Delta$ is a boundary $\mathbb{R}$-divisor on $Y$. Let $f : Y \to X$ be a proper morphism to an algebraic variety $X$ and let $L$ be a Cartier divisor on $Y$ such that $L - (K_Y + \Delta)$ is $f$-semi-ample.

(i) every associated prime of $R^q f_* O_Y(L)$ is the generic point of the $f$-image of some stratum of $(Y,\Delta)$.

(ii) let $\pi : X \to V$ be a projective morphism to an algebraic variety $V$ such that

$$L - (K_Y + \Delta) \sim_{\mathbb{R}} f^* H$$

for some $\pi$-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $H$ on $X$. Then $R^q f_* O_Y(L)$ is $\pi_*$-acyclic, that is,

$$R^p \pi_* R^q f_* O_Y(L) = 0$$

for every $p > 0$ and $q \geq 0$.

Proof. We put $S = \cup \Delta$, $B = \{\Delta\}$, and $H' \sim_{\mathbb{R}} L - (K_Y + \Delta)$ throughout this proof. Let us start with the proof of (i).
Step 1. First, we assume that $X$ is projective. We may assume that $H'$ is semi-ample by replacing $L$ (resp. $H'$) with $L + f^*A'$ (resp. $H' + f^*A'$), where $A'$ is a very ample Cartier divisor on $X$. Suppose that $R^q f_* \mathcal{O}_Y(L)$ has a local section whose support does not contain the $f$-images of any strata of $(Y, S + B)$. More precisely, let $U$ be a non-empty Zariski open set and let $s \in \Gamma(U, R^q f_* \mathcal{O}_Y(L))$ be a non-zero section of $R^q f_* \mathcal{O}_Y(L)$ on $U$ whose support $V \subset U$ does not contain the $f$-images of any strata of $(Y, S + B)$. Let $\nabla$ be the closure of $V$ in $X$. We note that $\nabla \setminus V$ may contain the $f$-image of some stratum of $(Y, S + B)$. Let $Y_2$ be the union of the irreducible components of $Y$ that are mapped into $\nabla \setminus V$ and let $Y_1$ be the union of the other irreducible components of $Y$. We put

$$K_{Y_1} + S_1 + B_1 = (K_Y + S + B)|_{Y_1}$$

such that $S_1$ is reduced and that $\nabla B_1 = 0$. By replacing $Y$, $S$, $B$, $L$, and $H'$ with $Y_1$, $S_1$, $B_1$, $L|_{Y_1}$, and $H'|_{Y_1}$, we may assume that no irreducible components of $Y$ are mapped into $\nabla \setminus V$. Let $C$ be a stratum of $(Y, S + B)$ that is mapped into $\nabla \setminus V$. Let $\sigma : Y' \to Y$ be the blowing-up along $C$. We put $E = \sigma^{-1}(C)$. We can write

$$K_{Y'} + S' + B' = \sigma^*(K_Y + S + B)$$

such that $S'$ is reduced and $\nabla B' = 0$. Thus,

$$\sigma^* H' \sim_{\mathbb{R}} \sigma^* L - (K_{Y'} + S' + B')$$

and

$$\sigma^* H' \sim_{\mathbb{R}} \sigma^* L - E - (K_{Y'} + (S' - E) + B').$$

We note that $S' - E$ is effective. We replace $Y$, $H'$, $L$, $S$, and $B$ with $Y'$, $\sigma^* H'$, $\sigma^* L - E$, $S' - E$, and $B'$. By repeating this process finitely many times, we may assume that $\nabla$ does not contain the $f$-images of any strata of $(Y, S + B)$. Then we can find a very ample Cartier divisor $A$ on $X$ with the following properties.

(a) $f^* A$ is permissible with respect to $(Y, S + B)$, and

(b) $R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)$ is not injective.

We may assume that $H' - f^* A$ is semi-ample by replacing $L$ (resp. $H'$) with $L + f^* A$ (resp. $H' + f^* A$). If necessary, we replace $L$ (resp. $H'$) with $L + f^* A''$ (resp. $H' + f^* A''$), where $A''$ is a very ample Cartier divisor. Then, we have

$$H^0(X, R^q f_* \mathcal{O}_Y(L)) \simeq H^q(Y, \mathcal{O}_Y(L))$$

and

$$H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \simeq H^q(Y, \mathcal{O}_Y(L + f^* A)).$$
We obtain that
\[ H^0(X, R^q f_* \mathcal{O}_Y(L)) \to H^0(X, R^q f_* \mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \]
is not injective by (b) if $A''$ is sufficiently ample. So,
\[ H^q(Y, \mathcal{O}_Y(L)) \to H^q(Y, \mathcal{O}_Y(L + f^*A)) \]
is not injective. It contradicts Theorem 3.3. Therefore, every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains the $f$-image of some stratum of $(Y, \Delta)$ in its support. We finish the proof when $X$ is projective.

**Step 2.** Next, we assume that $X$ is not projective. Note that the problem is local. So, we shrink $X$ and may assume that $X$ is affine. We can write $H' \sim R H'_1 + H'_2$, where $H'_1$ (resp. $H'_2$) is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (resp. a semi-ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor) on $Y$. We can write $H'_2 \sim \sum_a a_i A_i$, where $0 < a_i < 1$ and $A_i$ is a general effective Cartier divisor on $Y$ for every $i$. Replacing $B$ (resp. $H'_1$) with $B + \sum a_i A_i$ (resp. $H'_1$), we may assume that $H'$ is a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Without loss of generality, we may further assume that $(Y, B + S + H')$ is a simple normal crossing pair. We compactify $X$ and apply Lemma 3.5. Then we obtain a compactification $\tilde{f} : \tilde{Y} \to X$ of $f : Y \to X$. Let $\overline{H'}$ be the closure of $H'$ on $\overline{Y}$. If $\overline{H'}$ is not a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then we take blowing-ups of $\overline{Y}$ inside $\overline{Y} \setminus Y$ and obtain a semi-ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\tilde{H}'$ on $\overline{Y}$ such that $\tilde{H}'|_Y = H'$. Let $\overline{B}$ (resp. $\overline{S}$) be the closure of $B$ (resp. $S$) on $\overline{Y}$. We may assume that there is a Cartier divisor $\overline{L}$ on $\overline{Y}$ such that $\overline{L}|_Y = L$ by modifying $\overline{Y}$ suitably. More precisely, we construct a coherent sheaf $\mathcal{F}$ on $\overline{Y}$ which is an extension of $\mathcal{O}_Y(L)$, flatten $\mathcal{F}$, take more blowing-ups outside $Y$, and obtain a new compactification $\overline{Y}$ which satisfies all the desired properties. Note that a flattening of $\mathcal{F}$ can be constructed, for example, by using Grothendieck’s Quot scheme $\text{Quot}^1_{\mathcal{O}_Y^\mathbb{Q}}$ (cf. [N, Theorem 5.14] and [AK, Section 2]). In this situation, $\overline{H}' \sim \overline{L} - (K_{\overline{Y}} + S + B)$ does not necessarily hold. We can write
\[ H' + \sum_i b_i(f_i) = L - (K_Y + S + B), \]
where $b_i$ is a real number and $f_i \in \Gamma(Y, K_Y^*)$ for every $i$. We put
\[ E = \overline{H}' + \sum_i b_i(f_i) - (\overline{L} - (K_{\overline{Y}} + S + B)). \]
We note that we can see $f_i \in \Gamma(\overline{Y}, K_{\overline{Y}}^*)$ for every $i$. We replace $\overline{L}$ (resp. $\overline{B}$) with $\overline{L} + \Gamma E^\dagger$ (resp. $\overline{B} + \{ -E \}$). Then we obtain the desired
property of $R^q f_* \mathcal{O}_{\mathcal{Y}}(L)$ since $\mathcal{Y}$ is projective. We note that $\text{Supp}E$ is in $\mathcal{Y} \setminus Y$. So, we finish the whole proof of (i).

From now on, we prove (ii).

**Step 1.** We assume that $\dim V = 0$. In this case, we can write $H \sim_{\mathbb{R}} H_1 + H_2$, where $H_1$ (resp. $H_2$) is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor (resp. an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor) on $X$. So, we can write $H_2 \sim_{\mathbb{R}} \sum a_i A_i$, where $0 < a_i < 1$ and $A_i$ is a general very ample Cartier divisor on $X$ for every $i$. Replacing $B$ (resp. $H$) with $B + \sum a_i f^* A_i$ (resp. $H_1$), we may assume that $H$ is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. We take a general member $A \in |mH|$, where $m$ is a sufficiently large and divisible integer, such that $A' = f^* A$ and $R^q f_* \mathcal{O}_Y(L + A')$ is $\pi_*$-acyclic, that is, $\Gamma$-acyclic, for all $q$. By (i), we have the following short exact sequences,

$$0 \to R^q f_* \mathcal{O}_Y(L) \to R^q f_* \mathcal{O}_Y(L + A') \to R^q f_* \mathcal{O}_{A'}(L + A') \to 0$$

for every $q$. Note that $R^q f_* \mathcal{O}_A(L + A')$ is $\pi_*$-acyclic by induction on $\dim X$ and $R^q f_* \mathcal{O}_Y(L + A')$ is also $\pi_*$-acyclic by the above assumption. Thus, $E_2^{pq} = 0$ for $p \geq 2$ in the following commutative diagram of spectral sequences.

$$
\begin{array}{ccc}
E_2^{pq} = H^p(X, R^q f_* \mathcal{O}_Y(L)) & \xrightarrow{\varphi^{pq}} & H^{p+q}(Y, \mathcal{O}_Y(L)) \\
\downarrow{\varphi^{pq}} & & \downarrow{\varphi^{p+q}} \\
\overline{E}_2^{pq} = H^p(X, R^q f_* \mathcal{O}_Y(L + A')) & \xrightarrow{\varphi^{p+q}} & H^{p+q}(Y, \mathcal{O}_Y(L + A'))
\end{array}
$$

We note that $\varphi^{p+q}$ is injective by Theorem 3.3. We have that $E_2^{1q} \xrightarrow{\alpha} H^{1+q}(Y, \mathcal{O}_Y(L))$ is injective by the fact that $E_2^{pq} = 0$ for $p \geq 2$. We also have that $\overline{E}_2^{pq} = 0$ by the above assumption. Therefore, we obtain $E_2^{1q} = 0$ since the injection

$$E_2^{1q} \xrightarrow{\alpha} H^{1+q}(Y, \mathcal{O}_Y(L)) \xrightarrow{\varphi^{1+q}} H^{1+q}(Y, \mathcal{O}_Y(L + A'))$$

factors through $\overline{E}_2^{1q} = 0$. This implies that $H^p(X, R^q f_* \mathcal{O}_Y(L)) = 0$ for every $p > 0$.

**Step 2.** We assume that $V$ is projective. By replacing $H$ (resp. $L$) with $H + \pi^* G$ (resp. $L + (\pi \circ f)^* G$), where $G$ is a very ample Cartier divisor on $V$, we may assume that $H$ is an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. If $G$ is a sufficiently ample Cartier divisor on $V$, then we have

$$H^k(V, R^q \pi_* R^q f_* \mathcal{O}_Y(L) \otimes G) = 0$$
for every \( k \geq 1, \)
\[
H^0(V,R^p\pi_*R^qf_*O_Y(L) \otimes O_Y(G)) \simeq H^p(X,R^qf_*O_Y(L) \otimes O_X(\pi^*G)) \\
\simeq H^p(X,R^qf_*O_Y(L + f^*\pi^*G))
\]
for every \( p \) and \( q, \) and \( R^p\pi_*R^qf_*O_Y(L) \otimes O_Y(G) \) is generated by its global sections for every \( p \) and \( q. \) Since
\[
L + f^*\pi^*G - (K_Y + \Delta) \sim_{\mathbb{R}} f^*(H + \pi^*G),
\]
and \( H + \pi^*G \) is ample, we can apply Step 1 and obtain
\[
H^p(X,R^qf_*O_Y(L + f^*\pi^*G)) = 0
\]
for every \( p > 0. \) Thus, \( R^p\pi_*R^qf_*O_Y(L) = 0 \) for every \( p > 0 \) by the above arguments.

**Step 3.** When \( V \) is not projective, we shrink \( V \) and may assume that \( V \) is affine. By the same argument as in Step 1, we may assume that \( H \) is \( \mathbb{Q} \)-Cartier. Let \( \overline{\pi}: \overline{X} \to \overline{V} \) be a compactification of \( \pi: X \to V \) such that \( \overline{X} \) and \( \overline{V} \) are projective. We may assume that there exists a \( \overline{\pi} \)-ample \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( \overline{H} \) on \( \overline{X} \) such that \( \overline{H}|_X = H. \) By Lemma 3.3, we can compactify \( f: (Y,S+B) \to X \) and obtain \( \overline{f}: (Y,\overline{S}+\overline{B}) \to \overline{X}. \) We note that \( \overline{f}^*\overline{H} \sim_{\mathbb{R}} \overline{L} - (K_{\overline{Y}} + \overline{S} + \overline{B}) \) does not necessarily hold, where \( \overline{L} \) is a Cartier divisor on \( \overline{Y} \) constructed as in Step 2 in the proof of (i). By the same argument as in Step 2 in the proof of (i), we obtain that \( R^p\pi_*R^qf_*O_Y(L) = 0 \) for every \( p > 0 \) and \( q \geq 0. \)

We finish the whole proof of (ii). \( \square \)

### 4. Semi divisorial log terminal pairs

Let us start with the definition of *semi divisorial log terminal pairs* in the sense of Kollár. For details of singularities which appear in the minimal model program, see [F3] and [Ko3].

**Definition 4.1** (Semi divisorial log terminal pairs). Let \( X \) be a pure-dimensional reduced \( S_2 \) scheme which is simple normal crossing in codimension one. Let \( \Delta = \sum_i a_i\Delta_i \) be an \( \mathbb{R} \)-Weil divisor on \( X \) such that \( 0 < a_i \leq 1 \) for every \( i \) and that \( \Delta_i \) is not contained in the singular locus of \( X, \) where \( \Delta_i \) is a prime divisor on \( X \) for every \( i \) and \( \Delta_i \neq \Delta_j \) for \( i \neq j. \) Assume that \( K_X + \Delta \) is \( \mathbb{R} \)-Cartier. The pair \((X,\Delta)\) is *semi divisorial log terminal* (sdlt, for short) if \( a(E,X,\Delta) > -1 \) for every exceptional divisor \( E \) over \( X \) such that \( (X,\Delta) \) is not a simple normal crossing pair at the generic point of \( c_X(E), \) where \( c_X(E) \) is the center of \( E \) on \( X. \)
We note that if \((X, \Delta)\) is sldt and \(X\) is irreducible then \((X, \Delta)\) is a divisorial log terminal pair (dlt, for short). The following theorem is a direct generalization of \([F7, \text{Theorem 4.14}]\) (cf. \([F11, \text{Proposition 2.4}]\)). It is an easy application of Lemma 3.2.

**Theorem 4.2** (cf. \([F11, \text{Theorem 5.2}]\)). Let \((X, D)\) be a semi divisorial log terminal pair. Let \(X = \bigcup_{i \in I} X_i\) be the irreducible decomposition. We put
\[
Y = \bigcup_{i \in J} X_i \subset X
\]
for \(J \subset I\). Then \(Y\) is Cohen–Macaulay, semi-normal, and has only Du Bois singularities. In particular, each irreducible component of \(X\) is normal and \(X\) itself is Cohen–Macaulay.

We note that an irreducible component of a semi-normal scheme need not be semi-normal (cf. \([Ko3, \text{Example 9.8}]\)). We also note that an irreducible component of a Cohen–Macaulay scheme need not be Cohen–Macaulay. We learned the following example from Shunsuke Takagi.

**Example 4.3.** We put
\[
R = \mathbb{C}[x, y, z, w]/(yz - xw, xz^2 - y^2w).
\]
Then \(X = \text{Spec}R\) is a reduced reducible two-dimensional Cohen–Macaulay scheme. An irreducible component
\[
Y = \text{Spec}R/(y^3 - x^2z, z^3 - yw^2)
\]
of \(X\) is not Cohen–Macaulay. It is because
\[
R/(y^3 - x^2z, z^3 - yw^2) \simeq \mathbb{C}[s^4, s^3t, st^3, t^4].
\]

The Cohen–Macaulayness of \(X\) is very important for various duality theorems. We use it in the proof of Theorem 5.1 in \([FF]\).

Let us start the proof of Theorem 4.2.

**Proof of Theorem 4.2.** By \([BP, \text{Theorem 1.2}]\), there is a morphism \(f : Z \to X\) given by a composite of blowing-ups with smooth centers such that \((Z, f^{-1}_*D + \text{Exc}(f))\) is a simple normal crossing pair and that \(f\) is an isomorphism over \(U\), where \(U\) is the largest Zariski open subset of \(X\) such that \((U, D|_U)\) is a simple normal crossing pair. Then we can write
\[
K_Z + D' = f^*(K_X + D) + E,
\]
where \(D'\) and \(E\) are effective and have no common irreducible components. By construction, \(E\) is \(f\)-exceptional and \(\text{Supp}(E + D')\) is a
simple normal crossing divisor on $Z$. Since $X$ is $S_2$ and simple normal crossing in codimension one, $X$ is semi-normal. Then we obtain $f_*\mathcal{O}_Z \cong \mathcal{O}_X$. Let $Z = \bigcup_{i \in I} Z_i$ be the irreducible decomposition. We consider the short exact sequence

$$0 \to \mathcal{O}_V(-W|_V) \to \mathcal{O}_Z \to \mathcal{O}_W \to 0,$$

where $W = \bigcup_{i \in J} Z_i$ and $V = \bigcup_{i \in I \setminus J} Z_i$. Therefore,

$$0 \to \mathcal{O}_V(\mathcal{E} - W|_V) \to \mathcal{O}_Z(\mathcal{E}) \to \mathcal{O}_W(\mathcal{E}) \to 0$$

is exact. By Lemma 3.2, $R^i f_* \mathcal{O}_Z(\mathcal{E}) = 0$ for every $i > 0$. We note that

$$\mathcal{E} \sim_{R,f} K_Z + D' + \{-E\}.$$  

Since $\mathcal{E} - W|_V \sim_{R,f} K_V + (D' + \{-E\})|_V$, $R^i f_* \mathcal{O}_V(\mathcal{E} - W|_V) = 0$ for every $i > 0$ by Lemma 3.2 again. Therefore, we obtain that

$$0 \to f_* \mathcal{O}_V(\mathcal{E} - W|_V) \to f_* \mathcal{O}_Z(\mathcal{E}) \cong \mathcal{O}_X \to f_* \mathcal{O}_W(\mathcal{E}) \to 0$$

is exact and that $R^i f_* \mathcal{O}_W(\mathcal{E}) = 0$ for every $i > 0$. Since $\mathcal{E}|_W$ is effective and $\mathcal{O}_X \to f_* \mathcal{O}_W(\mathcal{E})$ factors through $\mathcal{O}_Y$, we have $\mathcal{O}_Y \cong f_* \mathcal{O}_W \cong f_* \mathcal{O}_W(\mathcal{E})$. Therefore, $Y$ is semi-normal because so is $W$. In the derived category of coherent sheaves on $Y$, the composition

$$\mathcal{O}_Y \to Rf_* \mathcal{O}_W \to Rf_* \mathcal{O}_W(\mathcal{E}) \cong \mathcal{O}_Y$$

is a quasi-isomorphism. Therefore, $Y$ has only Du Bois singularities because $W$ is a simple normal crossing variety. On the other hand, $R^i f_* \omega_W = 0$ for every $i > 0$ by Lemma 3.2. By applying the Grothendieck duality to (1):

$$\mathcal{O}_Y \to Rf_* \mathcal{O}_W \to Rf_* \mathcal{O}_W(\mathcal{E}) \cong \mathcal{O}_Y,$$

we obtain

$$\omega_Y^\bullet \xrightarrow{a} Rf_* \omega_W^\bullet \xrightarrow{b} \omega_Y^\bullet,$$

where $\omega_Y^\bullet$ (resp. $\omega_W^\bullet$) is the dualizing complex of $Y$ (resp. $W$). Note that $b \circ a$ is a quasi-isomorphism. Thus we have

$$h^i(\omega_Y^\bullet) \subset R^i f_* \omega_W^\bullet = R^{i+d} f_* \omega_W$$

where $d = \dim Y = \dim W$. This implies that $h^i(\omega_Y^\bullet) = 0$ for every $i > -\dim Y$. Thus, $Y$ is Cohen–Macaulay and $\omega_Y^\bullet \cong \omega_Y[d]$. □

As a byproduct of the proof of Theorem 4.2, we obtain the following useful vanishing theorem. Roughly speaking, Proposition 4.4 says that $Y$ has only semi-rational singularities.
Proposition 4.4. In the notation of the proof of Theorem 4.2, \( f_\ast \mathcal{O}_W \cong \mathcal{O}_Y \) and \( R^i f_\ast \mathcal{O}_W = 0 \) for every \( i > 0 \).

Proof. By (2) in the proof of Theorem 4.2, we obtain
\[
\omega_Y \xrightarrow{\alpha} f_\ast \omega_W \xrightarrow{\beta} \omega_Y
\]
where \( \beta \circ \alpha \) is an isomorphism. Since \( \omega_W \) is locally free and \( f \) is an isomorphism over \( U \), \( f_\ast \omega_W \) is a pure sheaf of dimension \( d \). Thus \( f_\ast \omega_W \cong \omega_Y \) because they are isomorphic over \( U \). Then we obtain \( Rf_\ast \omega_\bullet_W \cong \omega_\bullet_Y \) in the derived category of coherent sheaves on \( Y \). By the Grothendieck duality, \( Rf_\ast \mathcal{O}_W \cong R\mathcal{H}om(Rf_\ast \omega_\bullet_W, \omega_\bullet_Y) \cong \mathcal{O}_Y \) in the derived category of coherent sheaves on \( Y \). Therefore, \( f_\ast \mathcal{O}_W \cong \mathcal{O}_Y \) and \( R^i f_\ast \mathcal{O}_W = 0 \) for every \( i > 0 \). \( \square \)

As an easy application of Theorem 4.2, we have an adjunction formula for sldt pairs.

Corollary 4.5 (Adjunction for sdlt pairs). In the notation of Theorem 4.2, we define \( D_Y \) by
\[
(K_X + D)|_Y = K_Y + D_Y.
\]
Then the pair \((Y, D_Y)\) is semi divisorial log terminal.

Proof. By Theorem 4.2, \( Y \) is Cohen–Macaulay. In particular, \( Y \) satisfies Serre’s \( S_2 \) condition. Then it is easy to see that the pair \((Y, D_Y)\) is semi divisorial log terminal. \( \square \)

We close this section with an important remark.

Remark 4.6. Let \((X, D)\) be a semi divisorial log terminal pair in the sense of Kollár (see Definition 4.1). Then it is a semi divisorial log terminal pair in the sense of [F1, Definition 1.1]. A key point is that any irreducible component of \( X \) is normal (see Theorem 4.2). When the author defined semi divisorial log terminal pairs in [F1, Definition 1.1], the theory of resolution of singularities for reducible varieties (cf. [BM] and [BP]) was not available.

5. Semi-positivity theorem

In [F7, Chapter 2], we discuss mixed Hodge structures on compact support cohomology groups for the proof of Theorem 3.1. In [FF], we investigate variations of mixed Hodge structures on compact support cohomology groups. By the Hodge theoretic semi-positivity theorem obtained in [FF, Section 6], we can prove the following theorem as an application of Theorem 3.6. For related topics, see [Ko2].
Theorem 5.1 (Semi-positivity theorem). Let \((X, D)\) be a simple normal crossing pair such that \(D\) is reduced and let \(f : X \to Y\) be a projective surjective morphism onto a smooth complete algebraic variety \(Y\). Assume that every stratum of \((X, D)\) is dominant onto \(Y\). Let \(\Sigma\) be a simple normal crossing divisor on \(Y\) such that every stratum of \((X, D)\) is smooth over \(Y_0 = Y \setminus \Sigma\). Then \(R^i f_* \omega_{X/Y}(D)\) is locally free for every \(i\). We put \(X_0 = f^{-1}(Y_0)\), \(D_0 = D|_{X_0}\), and \(d = \dim X - \dim Y\). We further assume that all the local monodromies on \(R^{d-i}(f|_{X_0 \setminus D_0})_* \mathbb{Q}_{X_0 \setminus D_0}\) around \(\Sigma\) are unipotent. Then we obtain that \(R^i f_* \omega_{X/Y}(D)\) is a semi-positive (in the sense of Fujita–Kawamata) locally free sheaf on \(Y\).

We note that Theorem 5.1 is a generalization of Fujita–Kawamata’s semi-positivity theorem (cf. [Ka1]). We also note that Theorem 5.1 contains the main theorem of [F2]. In [F2], we use variations of mixed Hodge structures on cohomology groups of smooth quasi-projective varieties. However, our formulation in [FF] based on mixed Hodge structures on compact support cohomology groups is more suitable for reducible varieties than the formulation in [F2] (cf. 1.2). Theorem 3.6 and Theorem 5.1 show that the theory of mixed Hodge structures on compact support cohomology groups is useful for the study of higher dimensional algebraic varieties. For details, see [F7, Chapter 2] and [FF].

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