Does the coframe geometry can serve as a unification scheme?

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The coframe field model is known as a viable model for gravity. The principle problem is an interpretation of six additional degrees of freedom. We construct a general family of connections which includes the connections of Levi-Civita and Weitzenböck as the limiting cases. We show that for a special choice of parameters, a subfamily of connections is invariant when the infinitesimal field of transformations (antisymmetric tensor) satisfies the pair of vacuum Maxwell equations — one for torsion and one for non-metricity. Moreover, the vacuum Maxwell equations turn to be the necessary and sufficient conditions for invariance of the viable coframe action (alternative to GR).

Consequently, for the viable models, the coframe field is proved to have the Maxwell-type behavior in addition to the known gravity sector.

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I. INTRODUCTION. ABSOLUTE FRAME VARIABLE

Absolute (teleparallel, fermparallel, ...) frame variable $e_α$ was introduced in physics by Einstein in 1928 [1]. This object is rather widely used in standard GR as well as in its various alternations, see [2] and the reference given therein. The frame $e_α$ and its dual, the coframe $ϑ^α$, have a well defined geometrical sense. In particular, they may be used as a reference basis. For a fixed absolute basis $\{e_α, ϑ^α\}$, this construction gives an invariant meaning to the components of a tensor, thus it emerges violation of Lorentz invariance. However, allowing global Lorentz transformations of the absolute frame field, the frame components of a tensor are merely transformed by the Lorentz transformation law. Thus, some interrelation emerges between Lorentz invariant field theories and diffeomorphism invariant gravity.

In local coordinates [3], the frame field, $e_α = e_α^a \partial_a$, and its dual, the coframe field, $ϑ^α = ϑ^α_{\beta} dx^\beta$, are expressed by $4 \times 4$ matrices which are reciprocal to each other

\[
e_α^a \partial^3_a = \delta^3_a, \quad e_α^a \partial^3_b = \delta^3_b.
\]

Thus we have another intrigued property of the frame field variable: 16 = 10 + 6. It is most desirable to have a separation of sixteen independent variables of the frame field to ten variables for gravity plus six variables for electromagnetic field. Unfortunately, this idea does not work in such a simple form. Ten gravity variables are easily extracted from the coframe by use of the metric tensor ($η_αβ = \text{diag}(-1, 1, 1, 1)$)

\[
g = η_αβ \partial^α \otimes \partial^β, \quad g_{ab} = η_αβ \partial^α_a \partial^β_b.
\]

As for the six remaining components of the coframe, a corresponded Lorentz invariant algebraic combination fails to exist. Consequently, the problem is:

What physical interpretation can be given to these six remained degrees of freedom? More pretentiously, is it possible to extract six electromagnetic field strengths from the frame/coframe variables?

II. COFRAME FIELD MODEL

Let us start with a brief account of the coframe field model, see [7] for an exterior form representation. Let a manifold be endowed with two smooth basis fields $ϑ^α(x)$ and $e_α(x)$, which are assumed to be fixed up to global (rigid) Lorentz transformations.

The action is required to be quadratic in the first order derivatives $ϑ^α_m,n$. Although the second order derivative terms in the form of total divergence are also admissible, we neglect such additions. A global Lorentz and diffeomorphism invariant action functional may be easily constructed from the exterior derivative components $ϑ^α_{[a,b]}$.

Since it is preferable to deal with a quantity who’s indices are all of the same nature, we introduce two tensors

\[
T_{ab}^c = e_α^a \partial^3_{[a,b]}, \quad C_{αβγ} = e_α^a e_β^b \partial^γ_{[a,b]}.
\]

A general “quadratic” action functional may be written now as

\[
A = \frac{κ}{2} \int C_{αβγ} F^{αβγ} \sqrt{-g} dx^4 + (m) A.
\]

Here $κ$ is a coupling constant, $(m) A$ is an action for a matter field, and $F^{αβγ}$ is a tensor which is assumed to be linear in the components of $C_{αβγ}$:

\[
F^{αβγ} = λ^{αβγμρ} C_{μρ},
\]

The tensor $λ^{αβγμρ}$ carries only the frame indices, so its general expression may be written as

\[
λ^{αβγμρ} = μ_1 η^{αμ} η^{βρ} + μ_2 η^{αμ} η^{βρ} η^{γμ} + μ_3 η^{αγ} η^{δρ} η^{βν}.
\]

The following change of parameters

\[
μ_1 = ρ_1 + ρ_2 + ρ_3, \quad μ_2 = 2ρ_2, \quad μ_3 = −2ρ_3
\]

appears to be useful for the exterior form representation [7]. With (6), the action functional (4) may be rewritten in a compact exterior differential form

\[
A = \frac{κ}{4} \int dϑ^α ∧ * F_{α} + (m) A.
\]
Here, $\ast$ denotes the Hodge dual while the 2-form $F^\alpha$ is

$$F^\alpha = \frac{1}{2} F^{\alpha \beta \gamma} \, \theta^\beta \wedge \theta^\gamma = \frac{1}{2} F^{\alpha \beta \gamma} \, m^\beta \, m^\gamma \, dx^m \wedge dx^n.$$  

(9)

Variation of (8) yields the coframe field equation

$$d \ast F^\alpha = \ast (c) T^\alpha + \ast (m) T^\alpha .$$  

(10)

Here $(m)T^\alpha = \delta((m)A)/\delta \vartheta^\alpha$ is the 3-form of the matter energy-momentum current. Such an object is related to the energy-momentum tensor as $T^\alpha = T^\alpha \ast \vartheta^\alpha$. The additional 3-form $(c)T^\alpha$ comes from variation of the Hodge dual operator which itself depends on the coframe. Being related to the absolute frame, the corresponded energy-momentum tensor takes the regular Yang-Mills form,

$$(c)T^\nu = - C^{\alpha \beta \gamma} F_{\alpha \nu \gamma} + \frac{1}{4} \delta^\nu \, C^{\alpha \beta \gamma} F_{\alpha \beta \gamma} .$$  

(11)

Since the right hand side of (10) is conserved, it has to be identified as the total energy-momentum current for the system of the coframe and matter fields. Consequently, $(c)T^\alpha$ accepts the meaning of the coframe energy-momentum current. This Hilbert type consideration is supplemented with the Noether procedure, which yields the same expression (11), see [7].

For a generic choice of the parameters $\rho_\alpha$, (10) is a well posed system of 16 independent equations for 16 independent coframe components. Being a vector-valued 3-form expression, the field equation may be covariantly reduced to a system of two tensorial equation, the symmetric and the antisymmetric one. We write them symbolically as

$$\mathcal{E}Q_{(ab)} = (m)T_{(ab)} , \quad \mathcal{E}Q^{[ab]} = (m)T^{[ab]} .$$  

(12)

The left hand side of the antisymmetric equation vanishes identically if and only if

$$\rho_1 = 0 , \quad \rho_2 = -1/2 , \quad \rho_3 = 1 .$$  

(13)

In this case, we remain with a system of only 10 independent equation. So only 10 combinations of the coframe are determined. Although, it is enough to determine the metric tensor (2).

The coframe energy-momentum tensor (11) is well defined for a generic set of parameters $\rho_\alpha$. This object is invariant under global Lorentz transformations and covariant under smooth changes of coordinates. Moreover, the tensor (11) is traceless, in accordance to the scale invariance of the coframe Lagrangian.

### III. Gravity Sector

Let us briefly recall how the coframe field model works in the gravity sector. Observe that every polynomial constructed from the derivatives of the metric is expressed, by (2), in the derivatives of the coframe. Consequently, the standard Einstein-Hilbert action has to appear as a special case of the general coframe action. Indeed, the action (8) with the parameters (13) is equivalent to the Einstein-Hilbert action (up to a total derivative). For this set of parameters, the action (4) and the field equation (10) have a hidden symmetry: They are invariant under local Lorentz transformations.

$$\vartheta^\alpha \mapsto L^\alpha \beta \vartheta^\beta , \quad L^\alpha \beta (x) \in SO(1,3).$$  

(14)

For the parameters (13), the equation (10) must have a coframe solution corresponded by (2) to the Schwarzschild metric. It is natural to look for possible spherical-symmetric solutions of (10) with an arbitrary set of parameters. The answer is as following [6]:

(i) For the set of the parameters

$$\rho_1 = 0 , \quad \rho_2 = \text{arbitrary} , \quad \rho_3 = 1 ,$$  

(15)

the field equation (10) has a unique static spherical-symmetric solution of a ”diagonal form”:

$$\vartheta^0 = 1 - \frac{m}{2r} \, dx^0 , \quad \vartheta^i = \left(1 + \frac{m}{2r}\right)^2 \, dx^i ,$$  

(16)

where $i = 1, 2, 3$. This coframe corresponds to the Schwarzschild metric in the isotropic coordinates.

(ii) If the parameters differ from (15), any exact solution of a ”diagonal form” does not have the Newtonian behavior at infinity.

Thus, the condition $\rho_1 = 0$ identifies a family of viable models with Schwarzschild solution. Another justification of this condition comes from consideration of the first order approximation to the coframe field model [9]. In this case, the coframe variable is reduced to a sum of symmetric and antisymmetric matrices. It means that, in linear approximation, we can treat the coframe field as a system of two independent fields. It is natural to require all the field-theoretic constructions, i.e. the action, the field equation and the energy-momentum tensor, to accept the same reduction to two independent expression. It is remarkably that such separation appears if and only if $\rho_1 = 0$.

Thus, instead of a unique gravity model based on the Riemannian metric, for the coframe variable, we have a whole family (15) of viable gravity models parameterized by the parameter $\rho_2$.

For $\rho_2 = -1/2$, the coframe model is equivalent to the standard GR. The equivalence proved on the level of the action, of the field equation and of the exact solutions. The energy-momentum expression (11), however, is not invariant under local transformations of the coframe. Actually, it is no more than a type of a pseudo-tensor. The local invariance also decreases the number of degrees of freedom to ten metric components.

For $\rho_2 \neq -1/2$, we have an alternative model of sixteen independent degrees of freedom. The action is only global Lorentz invariant so it is not correct to require more symmetries for the energy-momentum tensor. In
particular, the field equation and the energy-momentum tensor of the coframe field are well defined. So the energy-momentum problem of GR is solved in this alternative context. The price is a set of new problems: (i) What interpretation can be given for the additional six degrees of freedom? (ii) Which value of the parameter \( \rho_2 \) must be chosen? (iii) What geometries can be related to different values of the parameters?

IV. “GEOMETRIZATION” OF THE COFRAME MODEL

Although the coframe variable itself has a well defined geometrical sense, the action (4) and the field equation (10) are not related yet to any specific geometry.

We accept the Cartan viewpoint which treats a geometrical structure as a pair \( \{ g_{ab}, \Gamma_{ab}^c \} \) of two independent objects: the metric field \( g_{ab} \) and the connection field \( \Gamma_{ab}^c \). Moreover, we require \( g_{ab} \) and \( \Gamma_{ab}^c \) to be explicitly constructed from the coframe components.

The metric tensor (2) is already constructed from the coframe. Due to the index content, this construction is unique (up to a scalar factor).

As for the field of connection \( \Gamma_{ab}^c \), we require it to be linear in the first order derivatives of the coframe components \( \theta^a_{m,n} \). The coefficients in this linear combination are polynomial in the coframe components. Observe that the similar requirement, being accepted in the Riemannian geometry when the connection is linear in the first order derivatives of the metric, gives a unique Levi-Civita connection. In the coframe background the situation is rather different: In fact, we have here a whole family of connections. Recall the properties: (i) The connection is a set of 4\(^3 \) components which change by a specific inhomogeneous linear law; (ii) The difference of two connections is a tensor of type (1,2).

On a manifold endowed with a coframe field an absolute (curve independent) sense can be given to the parallelism of distance vectors. Namely, two vectors may be declared parallel one to another when they have the proportional components being referred to the local absolute frames. It means that the covariant derivatives of the absolute coframe components are zero relative to some special connection \( \hat{\Gamma}_{ab}^c \), which is referred to as the Weitzenböck connection. From \( \theta^a_{a,b} = 0 \), we have, by (1), the Weitzenböck connection as

\[
\hat{\Gamma}_{ab}^c = \epsilon_a^c \theta^a_{a,b} .
\]

Under a transform of coordinates, this expression changes by the proper inhomogeneous linear law.

A general coframe connection may be represented now as the Weitzenböck connection plus a tensor of type (1,2)

\[
\Gamma_{ab}^c = \hat{\Gamma}_{ab}^c + K_{ab}^c ,
\]

where \( K_{bc}^a \) is linear in the first order derivatives

\[
K_{ab}^c = \chi_{abm} \theta^a_{n,p} \chi^m_{(n,p)} = \chi_{abm} \theta^a_{(n,p)} .
\]

The general form of the “coupling tensor” is

\[
\chi_{abm} = \alpha_1 \delta_{m}^{\alpha} \delta_{n}^{\beta} + \alpha_2 \delta_{m}^{\alpha} \delta_{n}^{\beta} + \alpha_3 \delta_{m}^{\alpha} \delta_{n}^{\beta} + \beta_1 g_{ab} \chi^a_{n,m} + \beta_2 g_{ab} \chi^m_{n,m} + \beta_3 g_{ab} \chi^m_{n,m} .
\]

Consequently the additional (“contortion”) tensor is

\[
K_{ab}^c = \alpha_1 \hat{T}_{ab}^c + \alpha_2 \delta_{m}^{\alpha} \hat{T}_{mb}^m + \alpha_3 \delta_{m}^{\alpha} \hat{T}_{ma}^m + g^m \left( \beta_1 g_{ab} \hat{T}_{nm}^m + \beta_2 g_{am} \hat{T}_{nb}^m + \beta_3 g_{bm} \hat{T}_{na}^m \right) .
\]

Hence, in contrast to the metric geometry, we have a 6-parametric connection constructed from the coframe components. Every connection \( \Gamma_{bc}^a \) is characterized by two tensors. The torsion, \( T_{ab}^c = -T_{ba}^c \), is defined as

\[
T_{ab}^c = \Gamma_{[ab]}^c .
\]

The non-metricity tensor, \( Q_{ab} = Q_{cba} \), is defined as

\[
Q_{ab} = -\nabla_c g_{ab} = -g_{ab,c} + \Gamma_{ac} + \Gamma_{bc} .
\]

where \( \Gamma_{ac} = \Gamma_{ac}^{m} g_{mb} \).

For the Weitzenböck connection, the torsion is given by the \( \hat{T}_{bc}^a \) while the non-metricity tensor is zero. The Levi-Civita connection is defined by setting both tensors to zero.

For the general connection (18,21), the torsion is

\[
T_{ab}^c = (1 + \alpha_1) \hat{T}_{ab}^c + \frac{1}{2} \left( \alpha_2 - \alpha_3 \right) \left( \hat{T}_{ab}^c - \hat{T}_{nc}^c \hat{T}_{mb}^n - \hat{T}_{ma}^n \right) + \frac{1}{2} \left( \beta_2 - \beta_3 \right) g^{cn} \left( g_{am} \hat{T}_{nb}^m - g_{mb} \hat{T}_{na}^m \right) .
\]

Consequently, the connection (18) is identically torsion-free if and only if

\[
\alpha_1 = -1 , \quad \alpha_3 = \alpha_2 , \quad \beta_2 = \beta_3 .
\]

The non-metricity tensor of the general connection (18,21) is

\[
Q_{ab} = \left( \alpha_1 + \beta_2 \right) \left( \hat{T}_{abc} + \hat{T}_{bca} \right) + 2 \alpha_2 g_{ab} \hat{T}_{mc}^m + \left( \alpha_3 - \beta_1 \right) \left( g_{ac} \hat{T}_{mb}^m - g_{bc} \hat{T}_{ma}^m \right) .
\]

Hence, the connection (18) is metric-compatible if

\[
\alpha_1 = -\beta_2 , \quad \alpha_2 = 0 , \quad \alpha_3 = \beta_1 .
\]

Thus, the unique torsion-free and metric-compatible coframe connection is given by the set of parameters

\[
-\alpha_1 = \beta_2 = \beta_3 = 1 , \quad \alpha_2 = \alpha_3 = \beta_1 = 0 .
\]
Certainly, it is not more than the ordinary Levi-Civita connection (31), which we can express now by the Weitzenböck connection
\[ \Gamma_{abc} = \Gamma_{(ab)c} + \Gamma_{[ca]b} + \Gamma_{[cb]a} . \]  

Two tensors, the torsion and the non-metricity, characterize the coframe connection uniquely. Indeed, let a metric \( g \) be fixed and two tensors \( T_{ab} = -T_{ba} \) and \( Q_{abc} = Q_{cba} \) be given. The corresponding unique connection is \([8]\).

\[ \Gamma_{abc} = \Gamma_{abc} + (T_{abc} + T_{cabc} - T_{bca}) + \frac{1}{2}(Q_{abc} - Q_{cab} + Q_{bca}) , \]  

where \( \Gamma_{abc} \) is used for the Levi-Civita connection:
\[ \Gamma_{abc} = \frac{1}{2}(g_{ac,b} + g_{bc,a} - g_{ab,c}) \]  

V. MAXWELL EQUATIONS

The connection (18) is invariant under global (rigid) transformation of the absolute coframe. We will now examine how it changes under local linear transformations \( \vartheta^a \mapsto L^a_{\beta}(x) \vartheta^\beta \). In a coordinate basis, the coframe components change as
\[ \vartheta^a \mapsto L^a_{\beta}(x) \vartheta^\beta , \quad e_a^\alpha \mapsto (L^{-1})^\beta_\alpha e^a_{\beta} . \]  
The infinitesimal version of this transformation with \( L^a_{\beta} = \delta^a_{\beta} + X^a_{\beta} \) is
\[ \vartheta^a \mapsto \vartheta^a + X^a_{\beta} \vartheta^\beta , \quad e^a_{\alpha} \mapsto e^a_{\alpha} - X^a_{\alpha} e^a_{\beta} . \]  
Let us examine now, under what conditions the geometrical structure is invariant under these transformations. Since the coframe field appears in the geometrical structure, \( \{g_{ab}(\vartheta^a), \Gamma_{abc}(\vartheta^a) \} \), only implicitly, (32) is a type of a gauge transformation. Invariance of the metric tensor restricts \( L^a_{\beta} \) to a pseudo-orthonormal matrix. In the infinitesimal version, it means that the matrix \( X^a_{\beta} = X^a_{\beta} \eta_{\alpha a} \) is antisymmetric.

The Levi-Civita connection \( \Gamma_{bc}^a \) is invariant under the transformations (32) with an arbitrary matrix \( X_{ab} \). As for the Weitzenböck connection, it changes as
\[ \Delta \Gamma_{abc} = \vartheta^a_{\beta} \vartheta^\beta_{a} X_{\alpha \beta,b} . \]  

For a generic set of parameters \( \alpha_i, \beta_i \), the left hand sides of (35) do not vanish identically. Instead, we have here two first order partial differential equations for the antisymmetric tensor field \( X_{ab} \). Instead of \( X_{ab} \) we define an antisymmetric matrix with coordinate indices
\[ F_{ab} = X_{\mu\nu} \eta^\mu_{\alpha} \vartheta^\nu_{\beta} . \]  
Moreover let us restrict to the case when the derivatives of the coframe field are small comparing to the derivatives of the matrix \( X_{\mu\nu} \). Under this condition, (34) may be rewritten as
\[ \Delta \Gamma_{abc} = F_{ca,b} . \]  

Under the transformations (33), the torsion (24) changes as
\[ \Delta T_{abc} = \frac{1}{2}(1 + \alpha_1)(F_{ca,b} - F_{cb,a}) - \frac{1}{4}(\alpha_2 - \alpha_3)(g_{ac,F^m_{b,m} - g_{bc,F^m_{a,m}}}) + \frac{1}{4}(\beta_2 - \beta_3)(F_{ac,b} - 2F_{ab,c} - F_{bc,a}) . \]  

For a special family of connections with parameters
\[ \alpha_2 = \alpha_3 , \quad (\beta_2 - \beta_3) + 2(1 + \alpha_1) = 0 , \]  
we obtain
\[ \Delta T_{abc} = (1 + \alpha_1)(F_{ab,c} + F_{bc,a} + F_{ca,b}) . \]  
Under the transformations (33), the non-metricity (26) changes as
\[ \Delta Q_{abc} = -\frac{1}{2}(\alpha_1 + \beta_2)(F_{bc,a} + F_{ac,b}) - \alpha_2 g_{ab} F^m_{c,m} \]  
\[ -\frac{1}{2}(\alpha_3 - \beta_1)(g_{ac} F^m_{b,m} + g_{bc} F^m_{a,m}) . \]  
For a family of connections with parameters
\[ \alpha_1 + \beta_2 = 0 , \quad \alpha_3 - \beta_1 = 0 , \]  
we have
\[ \Delta Q_{cab} = -\alpha_2 g_{ab} F^m_{c,m} \]  
Consequently, we have derived a nonempty family of connections that are invariant under the transformation (33) provided the antisymmetric tensor \( F_{ab} \) satisfies the Maxwell field equations
\[ F_{ab,c} + F_{bc,a} + F_{ca,b} = 0 , \quad F^m_{a,m} = 0 . \]  
Let us look now how these “Maxwell transformations” are connected to the coframe Lagrangian. Recall that we are looking for a symmetry which distinguishes the viable models with Schwarzschild solutions. Since the
second order term is involved only as a total derivative, we can write the viable Lagrangian as

\[ (v) L = R + \frac{1}{2}(2 \rho_2 + 1) T^{abc}_{\ o}(T^o_{abc} + 2 T^o_{cab}) + 2(\rho_3 - 1) T^m_{an} T^n_{am} \]

Under the transformations (33), this Lagrangian is transformed as

\[ \Delta (v) L = \frac{1}{2}(2 \rho_2 + 1) T^{abc}_{\ o}(F_{ab,c} + F_{bc,a} + F_{ca,k}) + 2(\rho_3 - 1) T^m_{an} F^n_{a,n} \]

Thus, for the non-Einstein models, the Lagrangian is invariant if and only if the local Lorentz transformations satisfy the vacuum Maxwell equations (44).

VI. DISCUSSION

We discuss briefly and somewhat speculatively how this construction can work.

1. Variables. The set of coframe fields is separated to equivalence classes, while the equivalence relation is given by (infinitesimal) Lorentz transformations. The fields from the same equivalence class generate the same metric which is a representation of the gravity field. The antisymmetric tensor of infinitesimal Lorentz transformations represents the electromagnetic field.

2. Action. An action for a system of a metric field and a field of connection is known from the metric-affine gravity field. The motion of singularities is described by the geodesic equation, which does not involve the torsion part of the connection. Thus, in order to have the Lorentz force in an addition to the Newtonian one, the connection has to contain the non-metricity ingredient.

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[1] The classical references are given in the recent historical reviews H. F. M. Goenner, Living Rev. Rel. 7, 2 (2004); T. Sauer, arXiv:physics/0405142.

[2] K. Hayashi and T. Shirafuji, Phys. Rev. D 19, 3524 (1979) J. Nitsch and F. W. Hehl, Phys. Lett. B 90, 98 (1980); F. Mueller-Hoissen and J. Nitsch, Phys. Rev. D 28, 718 (1983); E. W. Mielke, Annals Phys. 219, 78 (1992); U. Muench, F. Gronwald and F. W. Hehl, Gen. Rel. Grav. 30, 933 (1998); R. S. Tung and J. M. Nester, Phys. Rev. D 60, 021501 (1999); M. Blagojevic and M. Vasilic, Class. Quant. Grav. 17, 3785 (2000); I. L. Shapiro, Phys. Rept. 357, 113 (2000) R. T. Hammond, Rept. Prog. Phys. 65, 599 (2002); Y. N. Obukhov and J. G. Pereira, Phys. Rev. D 67, 044016 (2003)

[3] The Roman index refers to a specific coordinate, while the Greek index is a label for the absolute basis element involved. These indices are principally different, in particular, they cannot be contracted in \( \theta^a_{\ a} \) or \( e_a^a \).

[4] F. W. Hehl, J. D. McCrea, E. W. Mielke and Y. Neeman, Phys. Rept. 258, 1 (1995); F. W. Hehl and A. Macias, Int. J. Mod. Phys. D 8, 399 (1999)

[5] Y. Itin and S. Kaniel, J. Math. Phys. 41, 6318 (2000)

[6] Y. Itin, Int. J. Mod. Phys. D 10, 547 (2001)

[7] Y. Itin, Class. Quant. Grav. 19, 173 (2002); Gen. Rel. Grav. 34, 1819 (2002); J. Phys. A 36, 8867 (2003)

[8] J.A. Schouten, Ricci-Calculus, An Introduction to Tensor Analysis and its Geometrical Applications (2nd ed., Springer-Verlag, New York, 1954).

[9] Y. Itin, J. Math. Phys. in press, [arXiv:gr-qc/0409021].