On Black-Brane Instability In an Arbitrary Dimension

Barak Kol and Evgeny Sorkin

Racah Institute of Physics
Hebrew University
Jerusalem 91904, Israel
barak.kol, sorkin @phys.huji.ac.il

Abstract: The black-hole black-string system is known to exhibit critical dimensions and therefore it is interesting to vary the spacetime dimension $D$, treating it as a parameter of the system. We derive the large $D$ asymptotics of the critical, i.e. marginally stable, string following an earlier numerical analysis. For a background with an arbitrary compactification manifold we give an expression for the critical mass of a corresponding black brane. This expression is completely explicit for $T^n$, the $n$ dimensional torus of an arbitrary shape. An indication is given that by employing a higher dimensional torus, rather than a single compact dimension, the total critical dimension above which the nature of the black-brane black-hole phase transition changes from sudden to smooth could be as low as $D \leq 11$. 
1. Introduction

In the presence of extra compact dimensions, there exist several phases of black objects depending on the relative size of the object and the relevant length scales in the compactification manifold. More precisely, by “black objects” we mean massive, non-rotating (static) solutions of General Relativity with no extra matter\(^1\).

So far most research on this problem concentrated on the background \(\mathbb{R}^{D-1} \times S^1\), where \(S^1\) is a single compact dimension and often \(D\) has taken the values 5, 6. However, we know of two instances of a critical dimension in this system, where the qualitative behavior changes. Therefore, we consider spacetimes of total dimension \(D\) out of which \(d < D\) dimensions are extended, and we vary both \(D\) and \(d\) as parameters of the system.

The evidence is as follows. In the first case \(^1\) evidence was given that the behavior of the system near the conjectured point of merger between the black hole and black string branches depends on a critical dimension \(D^\ast_{\text{merger}} = 10\), such that for \(D < D^\ast_{\text{merger}}\) there are local tachyonic modes around the tip of the cone (conjectured to be the local geometry close to the thin “waist” of the string) which are absent for \(D > D^\ast_{\text{merger}}\). The second case relates to a different point in the phase diagram – the Gregory-Laflamme point (GL) where the string is marginally tachyonic \(^2\). It was recently found \(^3\) that the order of transition has a critical dimension\(^2\) \(D^\ast_{GL} = \text{“}13.5\text{”}\), such that for \(D < D^\ast_{GL}\) the transition is first order, and otherwise it is second order. Moreover, there is another example, closely

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\(^1\) We assume these are non-singular outside their event horizon.

\(^2\) Of course dimensions are integral, and the notation means only that the change in the order happens between \(D = 13\) and \(D = 14\).
related to the first, of a critical dimension in a different system – the Belinskii-Khalatnikov-Lifshitz (BKL) analysis of the approach to a space-like singularity, where there is a critical dimension $D^{\ast}_{BKL} = 10$, such that for $D > D^{\ast}_{BKL}$ the system becomes non-chaotic (see the review $[4]$ and references therein). This last example is related to our system not only in that it displays a critical dimension, but also in that the equations are similar to those of the merger transition – the BKL equations for the metric components as time approaches the singularity are similar to those for the cone perturbations near its tip.

A description of relevant works follows. Gregory and Laflamme $[2, 5]$ discovered the critical mass for a uniform string, $\mu_{GL}$, and determined it (numerically) for $5 \leq D \leq 10$. Actually, as was realized by Reall $[6]$, in order to determine the critical mass it is enough to know the eigenvalue for the negative mode of the Euclidean Schwarzschild geometry discovered by Gross Perry and Yaffe (GPY) $[7]$ and evaluated there in $4d$. The analysis of $[7]$ was generalized to $d$ dimensions by Prestidge $[8]$ (see also $[9]$ for a related discussion of black holes in a box). Gubser $[10]$ set a method of perturbation analysis around the 5D GL critical string whose first step is to determine $\mu_{GL}$ by a somewhat different way than using $[7]$. That method was perfected by Wiseman $[11, 12]$ who also applied it in 6D. Its generalization to an arbitrary dimension was made in $[3]$, (see appendix A for a summary). In fact, by now there is a considerably growing literature on this black-hole black-string system. See $[12-17]$ which study non-uniform strings and $[18-26]$ which concentrate on black holes in the system.

In this paper we obtain some results on the dimension dependence of the GL effect and in particular of the critical mass. Recall that the dimensionless mass of a black string is defined by $\mu := G_D M/L^{D-3}$ where $G_D$ is the $D$ dimensional Newton constant and $L$ is the asymptotic length of the compact circle. In $[3]$ the critical mass of a black string on $\mathbb{R}^{D-1} \times S^1$ was obtained numerically for a range of $D$’s, up to $D = 50$, and a “phenomenological” exponential behavior $\mu_{GL} \propto \gamma^D$ was observed where $\gamma \approx .68$. In section 2 we derive the large $D$ asymptotics of the critical mass and determine that $\gamma = \sqrt{e/2\pi}$ in agreement with that “phenomenology”. It is interesting to compare this behavior with that of the “equal areas for equal mass” estimator for the critical string$^3$, $\mu_S$: both expressions are exponential in $D$ but with a different base since for $D \gg 1$, $\mu_S \propto \gamma_S^D$ where $\gamma_S = e^{-1/2} \approx 0.606$.

Moreover, the behavior of the critical dimensionless mass turns out to be equivalent to a simple behavior of the critical Schwarzschild radius: $r_0 \simeq (L/2\pi)\sqrt{d}$, showing that for large $d$ the critical string is actually “fat” and as we show in section 4 most probably it cannot decay into a black hole.

In order to derive the asymptotics we use the equation for the negative mode of the $d$ dimensional Euclidean Schwarzschild geometry which is a single, second order, ordinary differential equation (ODE). In the large $d$ limit the equation becomes the radial part of the flat space Laplacian with all the non-trivial information being carried by the boundary condition close to the horizon.

$^3$This is defined to be the mass for which the areas of the uniform black string and a localized, naively undeformed black hole are equal.
In section 3 we proceed to vary the dimension of the compactification manifold and we express the critical mass of a uniform brane in this general background in terms of the lowest eigenvalue of the Laplacian operator on that manifold. In particular, when the compactification manifold is the flat $n$-dimensional torus of an arbitrary shape the expression can be made explicit in terms of the shortest vectors in the reciprocal lattice. We note that there are interesting cases when several modes turn tachyonic simultaneously.

Finally in section 4 we raise the question whether it is possible to see the second order behavior for dimensions smaller than 14 using $T^n$ compactifications. It would be especially interesting if it happened for dimensions $D \leq 11$ where M-theory/string theory is believed to be a consistent theory of quantum gravity. As in [3] we use as an estimator for the critical dimension the intersection between $\muGL(D)$ and $\muS(D)$. We find that the critical dimension can indeed be reduced and for $3 \leq n \leq 6$ it is estimated to be around $D = 10$ (see figure 2).

2. Large dimension asymptotics for $\muGL$

Consider a $D$ dimensional background with $d < D$ extended dimensions, and the other $D - d$ being compact. In this section$^4$ $d = D - 1$ and the compactification manifold is $S^1$ of period $L$.

A perturbation theory around the static uniform black string in $D$ dimensions was recently employed in [3] to construct the non-uniform string branch emerging from the critical GL point. Among other results it was observed that the critical mass follows essentially an exponential law as a function of $D$

$$\muGL \propto \gamma^D,$$

where numeric analysis yields $\gamma \approx 0.686$, the prefactor is approximately 0.47, and our definition of the dimensionless mass is as usual

$$\mu := G_D M / L^{D-3},$$

where $G_D$ is the $D$ dimensional Newton constant$^5$.

This critical mass was compared with $\muS$, the commonly used “equal areas for equal mass” estimator. More precisely it is an estimator for the point of the first order transition, and it is only an estimate since in the absence of exact solutions for caged black holes$^6$ one approximates the mass-area relation with that of a small spherical black hole. One finds

$$\muS = \frac{1}{16 \pi} \frac{\Omega_D^{-3} (D-3)(D-3)(D-3)}{\Omega_{D-1} (D-2)(D-2)(D-1)},$$

where $\Omega_{D-1} := D \pi^{D/2}/(D/2)!$ is the area of the unit $S^{D-1}$ sphere.

$^4$This assumption will be relaxed in later sections.

$^5$In this paper we work in units such that $G_D = 1$.

$^6$By a “caged black hole” we mean a black hole in a compactified spacetime.
Performing a series expansion for $D \to \infty$ we obtain
\[
\log(\mu_S) = -\frac{1}{2} D + \text{const} + O\left(\frac{\log(D)}{D}\right).
\] (2.4)

Hence, for large $D$, $\mu_S$ also exhibits an exponential scaling, though with a different exponent, $\gamma_S = 1/\sqrt{e} \approx 0.606$. Since $\gamma_S < \gamma$, $\mu_S$ becomes much smaller than $\mu_{GL}$ as $D$ increases. Moreover, there is a dimension, $D \approx 12.5$ for which $\mu_S$ intersects $\mu_{GL}$. This is an indication that for higher dimensions the black string becomes unstable before the black hole phase can have superior entropy. This, in turn, can be regarded as a hint that above that dimension the unstable GL-string decays to a state different from the black hole, and it is plausible to expect it to decay into a slightly non-uniform string. And indeed, exact calculations in higher orders of perturbation around the GL-point reveal an existence of a critical dimension ($D^* = 13.5$) where the order of the phase transition between the uniform and the non-uniform black strings solutions changes from the first to second order [3].

It is desirable to have an analytic formula for the numerical constant $\gamma$ in (2.1). Unfortunately, the equations that were solved in [3] in order to obtain the scaling (2.1) are two coupled second order, rather complicated, ODEs that required a numerical treatment. In the large $D$ limit these equations simplify, but nevertheless we did not find an analytic solution. Hence, motivated by Reall’s realization [6], we turned to look at the negative mode of the Euclidean Schwarzschild geometry, which yields a single ODE which we were able to solve analytically in the large $D$ limit.

### 2.1 The negative mode of the $d$-dimensional Schwarzschild

We aim to identify the negative modes of perturbed Euclidean Schwarzschild-Tangherlini metric [27] in $d$ (extended) dimensions. The line element for that background is
\[
\begin{align*}
\text{d}s^2 &= +f(r)\text{d}t^2 + f(r)^{-1}\text{d}r^2 + r^2\Omega_{d-2}, \\
f(r) &= 1 - \left(\frac{r_0}{r}\right)^{d-3},
\end{align*}
\] (2.5)
t is periodic with period $4\pi r_0/(d - 3)$, and in this section we work in units where $r_0 = 1$.

The (gauge invariant) spectrum of the perturbations $\phi_{ab}$ in the transverse-traceless gauge is obtained by solving the eigenvalue equations
\[
\triangle_L \phi_{ab} = \lambda \phi_{ab},
\] (2.6)
where in full index notation the Euclidean Lichnerowicz operator is $\triangle_L = -\delta^{cd} \delta^{ab} - 2R^{c d}_{a b}$.

For this spacetime the negative eigenvalue perturbation is static and spherically symmetric [7, 8]. Let us write the ansatz for such a perturbation as
\[
\phi^b_a = \text{diag}\{\psi(r), \chi(r), \kappa(r), \ldots, \kappa(r)\}. 
\] (2.7)

The traceless condition sets
\[
\kappa(r) = -\frac{1}{d-2} [\chi(r) + \psi(r)]
\] (2.8)
while transversality, $\phi^{ab,c} = 0$, implies

$$\psi(r) = \frac{2rf}{rf'-2f} \chi'(r) + \frac{rf'+2(d-1)f}{rf'-2f} \chi(r)$$

(2.9)

Hence, the equations (2.4) reduce to a single linear second order ODE for $\chi(r)$

$$-f \chi''(r) + \left[\frac{2r^2 (f f'' - f'^2) - r (d-2) ff' + 2 d f^2}{r (rf' - 2f)}\right] \chi'(r) + \left[\frac{r f' f'' + r [2(d-1) ff'' - (d+2)f'^2] + 4f f'}{r (rf'-2f)}\right] \chi(r) = -k^2 \chi(r)$$

(2.10)

where for convenience we define $\lambda = -k^2$, such that for the negative eigenvalues $k$ is real. The 4d version of this equation was analyzed in [7], while in [8] (see also [9]) the generalization to arbitrary $d$ was considered.

This equation has three singular points in the region outside the horizon $r \geq r_0$: at the horizon $r = r_0$ itself, at infinity and at $r = r_s$ which is the solution to $rf' - 2f = 0$. $r_s$ is given by

$$r_s^{d-3} = \frac{d-1}{2}$$

(2.11)

and we note that it is exactly the critical radius for trapping light, namely, a light ray originating from infinity will fall into the black hole if (and only if) it crosses $r_s$. Regularity of $\chi(r)$ at the horizon (a regular singular point) and at infinity (an irregular singular point) determines the solution. At the horizon, where $f(r_0) = 0$ the characteristic exponents are $-1, 0$ namely $\chi \sim (r-r_0)^\sigma$ with $\sigma = -1, 0$, and so regularity of (2.10) imposes

$$\chi'(r_0)/\chi(r_0) = -\left[d - \frac{k^2}{2(d-3)}\right].$$

(2.12)

At $r = r_s$, on the other hand, the characteristic exponents are $\sigma = 0, 3$ and both solutions are finite (though with possible log's). Looking at a generic solution with $\chi \sim (r-r_s)^\sigma$ we find

$$\chi'(r_s)/\chi(r_s) = -\frac{1}{2rf} \left[rf' + 2(d-1)f\right]_{r=r_s} = -\frac{d}{r_s}.$$  

(2.13)

It is important to realize that this relation is not a boundary condition, but rather a consequence of the equation itself. We observe that since the characteristic exponents are $0, 3$, regularity at $r_s$ can be obtained by extending the equation (2.10) to become fourth order (such that the exponents $\sigma = 1, 2$ are solutions as well), which is what happens in Gubser’s gauge (see the Appendix).

### 2.2 Large $d$ limit

In general, the solution to the equation (2.10) must be obtained numerically [7-9]. However, let us consider this equation in the limit $d \to \infty$. In this limit $f(r) \to 1$ and $f'(r), f''(r) \to 0$ for $r > r_s > 1$ and hence the eigenvalues equation (2.10) simplifies to

$$\chi''(r) + \frac{d}{r} \chi'(r) - k^2 \chi(r) = 0,$$

(2.14)
which is the flat space Laplacian (in a $d + 2$ dimensional spacetime). In this approximation $r_s$ is not singular anymore and hence (2.13) does not hold automatically. Instead we impose it as a boundary condition

$$\chi'(r_s)/\chi(r_s) = -\frac{d}{r_s}.$$  

(2.15)

A more rigorous treatment would derive this effective boundary condition by solving the original equation in the vicinity of $r = 1, r_s$ and matching it in the large $d$ limit with the solutions of (2.14).

We shall now show that the solution to (2.14) subject to the boundary condition (2.15) and regularity at infinity can be found analytically.

Changing variables in (2.14) by $\chi(r) = u(r) r^{-\nu}$ with $\nu = (d - 1)/2$ we obtain for $u(r)$

$$r^2 u''(r) + ru(r) - (\nu^2 + k^2 r^2) u(r) = 0$$  

(2.16)

which is just a modified Bessel equation whose solution is $u(r) = C_1 K_\nu(k r) + C_2 I_\nu(k r)$ (2.17)

where $I_\nu$ and $K_\nu$ are the modified Bessel functions of the first and the second kind respectively [28]. The asymptotic boundary condition forbids a growing exponential solution. Hence $C_2 = 0$ and the selected solution is $K_\nu$, whose asymptotic behavior is $K_\nu(z) \sim e^{-z}/\sqrt{2\pi z}$. The boundary condition (2.15) dictates

$$x K'_\nu(x) + (d - \nu) K_\nu(x) = 0$$

$$x := k r_s.$$  

(2.18)

Using the recurrence relations for $K_\nu$ one arrives to a simpler algebraic equation

$$K_\nu(x) - x K_{\nu-1}(x) = 0,$$  

(2.19)

that determines the eigenvalue $k$ as a function of $\nu$. We note that in order to arrive at this equation one relies on the cancellation $d - 2\nu = 1$, which is a delicate feature of this process. The leading behavior of $k = k(d)$ can be gotten by making the self-consistent assumption $k \ll d$, in which case one may use the leading behavior $K_\nu(z) \simeq \Gamma(\nu) (x/2)^{-\nu}/2$ valid for $x \ll \nu$. Substituting into (2.19) we get

$$k = \frac{1}{r_s} \sqrt{d - 3} \simeq \sqrt{d}, \quad d \gg 1,$$  

(2.20)

confirming the assumption $k \ll d$, and at the same time showing that the critical string is also surprisingly “fat” $r_0/L = k/(2\pi) \gg 1$ at large $d$.

Eq. (2.19) may be solved as a series in $1/d$ although it is probably not justified since it represents only the leading $d$ limit of the original equations where various $1/d$ corrections were already neglected. Still we record that if one uses more orders in the Taylor series of $K_\nu(x)$ the solution of (2.19) can be expanded as $x = \sqrt{d - 1}/\sqrt{d} + O(1/d^{3/2})$ for $d \gg 1$. This formula yields less than 0.2% discrepancy with the numerical solution of (2.19) for $d = 20$ and the discrepancy is even smaller for larger $d$. Moreover, this estimate differs by less than 4% from the full solution of eq. (2.10) for $d = 50$ and by 2.5% for $d = 100$ (see Table 1 in appendix A). This is an additional confirmation of the relationship between the GL tachyon and the GPY negative mode.
2.3 The critical mass

We can get now some insight into the mass scaling (2.1) of the marginally stable black strings. From the definition of $\mu$ (2.2) with $L = 2\pi/k_{GL}$ we get the critical mass for the GL-instability

$$\mu_{GL} = \frac{(d-2)\Omega_{d-2}}{16\pi} \frac{G_D}{G_d} \frac{r_0 k_{GL}}{2\pi}^{d-3},$$

(2.21)

(recall that $G_D/G_d$ is the volume of the internal manifold which is $L$ in this section). Expanding $\log(\mu_{GL})$ in series for $d \to \infty$ and keeping the leading terms we get

$$\log(\mu_{GL}(k_{GL})) = d \log\left(\sqrt{\frac{e}{2\pi}}\frac{k_{GL}}{\sqrt{d}}\right) + \text{const} + O\left[\frac{1}{d} \log\left(\frac{k_{GL}}{\sqrt{d}}\right) + \frac{\log(d)}{d}\right].$$

(2.22)

Since according to (2.20) $k_{GL} \to \sqrt{d}$ in this limit, we obtain from the above expansion

$$\log(\mu_{GL}) = d \log\left(\sqrt{\frac{e}{2\pi}}\right) + \text{const}$$

(2.23)

Hence we learn that for large $D = d + 1$ the critical mass should scale as $\mu \propto \tilde{\gamma}^D$ with

$$\tilde{\gamma} = \sqrt{\frac{e}{2\pi}} \approx 0.658$$

(2.24)

The numerical value $\gamma = 0.686$ differs from this $\tilde{\gamma}$ by about 4%. Note, however, that the numerical estimate was obtained for relatively small $D$'s [3], which should account for this deviation.

3. Torus compactification

In this section we determine the critical mass for GL instability of black branes on the $D$-dimensional background $\mathbb{R}^d \times T^n$, namely $d$ extended dimensions and $n$ compactified on a torus, and $D = d + n$. The torus is completely general, so its period vectors may be of any size and with arbitrary angles between each other.

The relevant black branes are the uniform $n$-branes $B_n = \text{Schw} \times T^n$, namely those which do not depend on the internal coordinates $\vec{z} \equiv z^i$, $i = 1, \ldots, n$. Their metric is given by a sum of the $d$ dimensional Schwarzschild metric (2.3) and the flat torus metric $ds_{T^n}^2 = dz^i dz^i$. In addition, the torus satisfies the periodicity boundary conditions: it is defined by a lattice which may be given by $n$ period vectors $\vec{e}_i$ such that any function $Y(\vec{z})$ must satisfy $Y(\vec{z} + \vec{e}) = Y(\vec{z})$ where $\vec{e}$ belongs to the lattice: $\vec{e} = \sum_{i=1}^n m^i \vec{e}_i$ with $m^i$ being arbitrary integers.

The critical mass is defined such that a perturbation mode $h_{\mu\nu}(r, \vec{z})$ turns marginally tachyonic

$$\triangle_L h_{\mu\nu}(r, \vec{z}) = 0,$$

(3.1)

where $\triangle_L$ is the Lichnerowicz operator in the total space $B_n$. Separating the compact variables from the extended ones, and restricting to perturbations which are scalar with respect to $T^n$ we have

$$h_{\mu\nu}(r, \vec{z}) = h_{\mu\nu}(r) Y(\vec{z}).$$

(3.2)
The Lichnerowicz operator in the total space decomposes into

\[
\triangle_L h_{\mu\nu}(r, \vec{z}) = Y(\vec{z}) \triangle_L h_{\mu\nu}(r) + h_{\mu\nu}(r) (-\triangle Y(\vec{z})) ,
\]

(3.3)

where \(\triangle\) is the Laplacian operator in the compact space \(T^n\) and \(\triangle_L h_{\mu\nu}(r)\) describes the Lichnerowicz operator on the \(d\) dimensional Schwarzschild geometry. Denoting the eigenvalues by \(\lambda_{\text{Schw}}\), \(\lambda_{T^n}\)

\[
\triangle_L h_{\mu\nu}(r) = \lambda_{\text{Schw}} h_{\mu\nu}(r)
-\triangle Y(\vec{z}) = \lambda_{T^n} Y(\vec{z}) ,
\]

(3.4)

we have that the total eigenvalue (which should vanish by (3.1)) can be written as the sum

\[
0 = \lambda_{\text{Schw}} + \lambda_{T^n} .
\]

(3.5)

In order to find a zero eigenvalue one of the eigenvalues should be negative. Actually there is only one such mode – it is the GPY mode of the Schwarzschild geometry \([7]\). For this reason it is enough to consider modes which are tensor on the Schwarzschild geometry and scalar on the torus. So we take

\[
\lambda_{\text{Schw}} = -\frac{k_d^2}{r_0^2}
\]

(3.6)

where \(k_d^2\) are the eigenvalues defined in (2.10), computed numerically in \([8]\) and whose large \(d\) asymptotics is (2.21). Substituting back into (3.5) we find that the critical \(r_0\) below which the black brane destabilizes is

\[
r_0^2 = \frac{k_d^2}{\lambda_{\text{min}}}
\]

(3.7)

where \(\lambda_{\text{min}}\) is the minimal (non-zero) eigenvalue of the Laplacian on \(T^n\). Actually the argument is more general and applies to any stable compactifying manifold. In the case of \(T^n\) this eigenvalue may be described explicitly

\[
\lambda_{T^n} = |\vec{k}|^2
\]

(3.8)

where \(\vec{k}\) is in the reciprocal lattice, namely \(\vec{k} \cdot \vec{e} = 2\pi j, j \in \mathbb{Z}\) for any lattice vector \(\vec{e}\). Finding the minimal eigenvalue of the Laplacian reduces to finding the shortest vector in the reciprocal lattice of \(T^n\).

The qualitative options

There are interesting (even if degenerate) cases where there are several vectors in the reciprocal lattice which lie closest to the origin, for example, a cubic lattice and a triangular one. Clearly that would mean that several modes would turn marginally tachyonic at the same brane mass

\[
\# \text{ (marginal tachyons)} =
 = \# \text{ (reciprocal lattice vector closest to the origin)} := \# \{\vec{k} : |\vec{k}|^2 = |\vec{k}_{\text{min}}|^2 / 2\} ,
\]

(3.9)
Two tachyon instability

\[ \theta = \pi/3 \]

\[ \theta = 2\pi/3 \]

Figure 1: The shaded area designate the modular domain of \( T^2 \) (the sides \( I \) and \( I' \) are identified). The two tachyon instability develops for torii inhabiting the arc \( \tau = \exp(i\theta) \) with \( \pi/3 \leq \theta \leq 2\pi/3 \).

where in the last expression \( \vec k \) is in the reciprocal lattice and the division by 2 accounts for the trivial degeneracy of \( \vec k \) and \( -\vec k \).

An example: \( T^2 \)

For the example of \( T^2 \) we can find the locus of these degenerate torii explicitly. They are precisely the torii such that the basis vectors \( \vec e_1, \vec e_2 \) satisfy \( |\vec e_1| = |\vec e_2| \) and the angle between them, \( \theta \), is in the range \( \pi/3 \leq \theta \leq \pi/2 \). In terms of the modular parameter \( \tau \) that means that it lies on the boundary of the modular domain on the arc \( \tau = \exp(i\theta) \) with \( \pi/3 \leq \theta \leq 2\pi/3 \) see figure (1).

4. Indication for critical dimensions with \( T^n \) compactification

In the black-string case there is a critical dimension \( D^* = 13.5 \) for the phase transition between the uniform and the non-uniform states. Namely, for a compactification on \( T^1 \), the transition is of first order below \( D^* \), while above \( D^* \) it is continuous. In \([3]\) it was shown that \( D^* \) is well estimated by \( \bar{D}^* \) defined by the intersection \( \mu_{GL}(\bar{D}^*) = \mu_S(\bar{D}^*) \). Let us generalize and estimate such a critical dimension for the \( T^n \) compacification. At the end of this section we mention other estimates which inquire whether at given \( \mu \) a black-hole can “fit into” the compact dimension (as stressed recently in \([29]\)).

First, considering a single tachyon instability, when one of the torus dimensions is much larger than all the rest, then the situation is exactly analogous to that of the black string transition. Hence, in this case the critical dimension is still 13.5.
Figure 2: The dimension $\tilde{D}^*$ for which $\mu_{GL}(\tilde{D}^*, n) = \mu_S(\tilde{D}^*, n)$ as a function of the dimension $n$ of the square torus $T^n$. The closest integer dimension above $\tilde{D}^*$ estimates a critical dimension for a change of order in the black brane phase transition. For $n = 1$ this estimate is known to be short by about 1 from the actual $D^*$: $D^*_{(n=1)} \approx 13.5 \approx \tilde{D}^*_{(n=1)} + 1$. For $3 \leq n \leq 6$ the estimate is around $10D$, making it plausible that the actual critical dimension may be as low as $D^* \leq 11$, where a consistent theory of quantum gravity is believed to exist.

Another important example is that of a square $n$-torus where, as we found in previous section, $n$ tachyons appear simultaneously below the critical mass. In this case one expects the unstable modes to deform the horizon along all internal directions. We would like to know whether the transition to the non-uniform state is smooth or not. Therefore, generalizing equation (2.3) from the string case, we define an “equal area for equal mass” estimator

$$\mu_S = \frac{1}{16\pi} \left[ \frac{\Omega_{d-2}^{D-3}}{\Omega_{D-2}^{d-3}} \right] \frac{(d-2)(D-3)(d-2)}{D(d-2)(d-3)(d-2)} \frac{1}{n},$$

(4.1)

The estimate for a critical dimension is obtained by such a $\tilde{D}^*$ that $\mu_S(\tilde{D}^*, n) = \mu_{GL}(\tilde{D}^*, n) = \mu_{GL}(\tilde{D}^* - n)$. In figure 2 we plot this $\tilde{D}^*$ as a function of the dimension of the internal torus space, $n$. Above the curve the black hole state is not entropically favorable at the critical mass. This indicates probably a smooth transition to a non-uniform brane emerging from the GL-point. Interestingly, the value of this critical dimension decreases as more internal dimensions are considered. We learn from figure 2 that for $3 \leq n \leq 6$ the total dimension of the spacetime should be at least $D = 10$ to allow a continuous decay of an unstable black brane to the non-uniform state. This estimate is somewhat marginal since as we noted above in the $T^1$ case the actual critical dimension is 13.5 while the estimate is around 12.5, namely one less. This indicates that even though the estimate for $D^*$ is closer to 10 than in the $T^1$ case, the actual calculations should be performed in order to establish whether indeed $D^* = 10$ or 11.

In addition, one should exercise care using the approximate mass (4.1) since it turns out that above certain $D$ and $n$ the Schwarzschild black hole “does not fit” into the torus,
namely its radius becomes “too large”, 2ρ₀/L > 1. In such a case the equation for μₛ (4.1) is outside its domain of validity. Explicitly, one obtains

\[ \frac{2\rho_0}{L} = 2 \left( \frac{\Omega_{d-2}}{\Omega_{D-2}} \right)^{\frac{1}{n}} \left( \frac{d-2}{D-2} \right)^{\frac{d-2}{n}} . \]  

(4.2)

Hence, for the D = 10 example, the Schwarzschild black hole would not fit into the torus for n = 5, 6 being slightly larger, by 3 and 8 percents respectively, than L (see also [29]). Actually, there is a little more space in the compact dimension than appears first due to the “Archimedes law for caged black holes” [20, 25] which tells us that a black hole “repels” an amount of space around it that is proportional to its own size. The effect vanishes in the large d limit, but may cause an appreciable correction for 5 ≤ D ≤ 10. It would be interesting to generalize its computation in [25] from T¹ compactification to Tⁿ.

Another interesting estimator can be gotten by considering the Schwarzschild radius ρ₀ for a would be black-hole with mass μGL. One finds

\[ \rho_0^{D-3} = \frac{G_D}{G_d} \frac{(d-2)\Omega_{d-2}}{(D-2)\Omega_{D-2}} r_0^{d-3} , \]

(4.3)

where r₀ is the Schwarzschild radius of the critical brane. For square torus G_D/G_d = Lⁿ, and taking the large d limit where k/GL = r₀(2π/L) ∼ √d one gets

\[ (D-3) \log \left( \frac{2\rho_0}{L} \right) = \frac{D}{2} \log(D) + O(D,d) , \]

(4.4)

from which we see that for large D (and independently of d) the black hole cannot fit in the compact dimension at the GL point, and therefore the black brane must decay into a different end-state, presumably a non-uniform black brane. This estimator can be improved a little by replacing the Schwarzschild radius with the radius in conformal coordinates ρ₀ → ρₖ = ρ₀/2^(2/(d-3)) (see [13]), and by incorporating the “Archimedes effect”, but these improvements would not change the large D result above.

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A. Determination of the critical mass μGL(D)

In this appendix we briefly review the first order calculations of a Lorentzian perturbation theory around the marginally static GL-string, B₁, in D = d + 1 dimensions. In fact, the first order is precisely what one needs to determine the onset of instability. An appropriate perturbation theory was first developed in 5D by Gubser [10], perfected by Wiseman [11, 12] in 6D and generalized in [3] to an arbitrary dimension.

The most general ansatz for static black string solutions can be written as

\[ ds^2 = -e^{2A} f dt^2 + e^{2B} (f^{-1} dr^2 + dz^2) + e^{2C} r^2 d\Omega_{D-3}^2 , \]

\[ f = 1 - (r_0/r)^{D-4} , \]

(A.1)
where $A, B$ and $C$ depend on $r, z$ only. When these functions vanish the metric becomes that of a static uniform black string. We will work here in units such that the horizon located at $r_0 = 1$.

Let us expand the metric functions around the marginally static uniform solution in powers of some small parameter\footnote{A good candidate for such a parameter is the dilatonic charge or the tension along the $z$-direction\cite{19, 21}, though for our purposes $\hat{\lambda}$ even does not have to be specified at this stage.} $\hat{\lambda}$ such that in the limit $\hat{\lambda} \to 0$ the perturbed string joins the GL-point

$$ A = a(r)\hat{\lambda}\cos(kz) + O(\hat{\lambda}^2), \quad B = b(r)\hat{\lambda}\cos(kz) + O(\hat{\lambda}^2), \quad C = c(r)\hat{\lambda}\cos(kz) + O(\hat{\lambda}^2). \quad (A.2) $$

The wavenumber $k$ is related to the asymptotic length of the compact circle by $k = \frac{2\pi}{L}$.

Upon substituting this expansion into the Einstein equations one obtains a set of ODEs for $a, b$ and $c$. It turns out that in our case $b$ can obtained algebraically from

$$ b = \frac{rf'}{2 (D-3) f + rf'} a + \frac{2 (D-3)}{2} \left( a' + (D-3) c' \right), \quad (A.3) $$

while for $a$ and $c$ there is a shooting problem to solve the set of two coupled linear second order ODEs

$$ -fa'' - \frac{2 (D-3) f + 3 r f'}{2 r} a' - \frac{(D-3) f'}{2} c' + k^2 a = 0, \quad (A.4) $$

subject to the regularity boundary conditions at the horizon, $r = r_0$,

$$ a' = \frac{2}{3(D-4)} k^2 a + \frac{D-3}{3} \left( -2 a + 2 c - \frac{k^2}{D-4} c \right), \quad (A.6) $$

$$ c' = 2 a - 2 c + \frac{k^2}{D-4} c, $$

and at the infinity, $r \to \infty$, where $a$ and $c$ must vanish.

These equations do not have singular points other than horizon and the spatial infinity unlike the Euclidian perturbation equation (2.10). However, there is a price to pay – instead of a single second order equation, as in a Euclidean theory, one has to deal with two coupled equations.

In \cite{3} the equations (A.4-A.6) were solved numerically for various $D$’s. The wavenumber $k$ obtained in this way is precisely that of the marginally tachyonic mode. (Indeed, the static method that is employed here converges only for this mode. For $k$ other than $k_{GL}$ there is no perturbed static solution since above $k_{GL}$ the string is non static being GL-unstable, while below $k_{GL}$ there is no static perturbed solution either, since perturbations must decay.) The calculated values of the critical wavenumber, $k_{GL}$, are listed in Table 1.
Table 1: Numerically computed static mode wavenumbers $k_{GL}$ in units of $r_0^{-1}$.

| D  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 |
|----|----|----|----|----|----|----|----|----|
| $k_{GL}$ | 0.876 | 1.27 | 1.58 | 1.85 | 2.09 | 2.30 | 2.50 | 2.69 |
| D  | 13 | 14 | 15 | 16 | 20 | 30 | 50 | 100 |
| $k_{GL}$ | 2.87 | 3.03 | 3.19 | 3.34 | 3.89 | 5.06 | 6.72 | 9.75 |

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