Conformal Bootstrap Analysis for Localization: Symplectic Case

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Abstract

The localization phenomena due to the random potential scattering is widely discussed in the electron and photon systems, where the theoretical approach is the nonlinear σ model with the replica method or with the supersymmetry. In this article, we discuss the application of the conformal bootstrap method to the localization by the small determinants. The possible correspondence to the symplectic Anderson localization is discussed.
1 Introduction

The conformal bootstrap has a powerful algorithm for obtaining very precise critical exponents in three dimensional Ising model. It has been applied to other interesting cases. The references may be found in a recent review article [1], which includes the basic explanation of the conformal bootstrap method. The conformal bootstrap method by the use of small determinants has been applied for the non-unitary model such as Yang-Lee edge singularity [2, 3]. The polymer and the branched polymer, which are also non-unitary, are considered by the conformal bootstrap method [4]. The random systems are treated by the $N \to 0$ limit of $O(N)$ invariant models due to the replica method. The random magnetic field Ising model (RFIM) is studied also by the conformal bootstrap method [5].

In this article, we apply this conformal bootstrap method to the localization problems. The Anderson localization has passed more than 50 years from the discovery, and has been applied to electron systems and photon systems [6]. The renormalization group method for the localization is based on the field theoretical model, known as the nonlinear $\sigma$ model with the limit $N \to 0$ of Grassmann manifold $O(2N)/O(N) \times O(N)$ or its non-compact version [7]. The isomorphic relation $O(\infty)/O(N)$ converts the orthogonal to the symplectic Grassmannian manifold. The symplectic localization corresponds to the limit of $p = N = 0$ in $Sp(N)/Sp(p) \times Sp(N-p) = O(-2N)/O(-2p) \times O(-2N+2p)$, and we take the coupling constant $t$ of this nonlinear sigma model to $t \to -t$ from the reason of the non-compactness [8, 9]. In two dimensions, $\beta$-function is not asymptotic free, and similar to the Ising model, and it suggests the existence of the phase transition in two dimensions. There are many types of the nonlinear $\sigma$ models, defined on the symmetric spaces. The polymer case has critical phenomenon , in the replica limit $N \to 0$, in two dimensions. The symplectic localization case, in the replica limit $N \to 0$, has a phase transition. Although we assume the present analysis for the symplectic localization by $Sp(2N)/Sp(N) \times Sp(N)$, there remains a possibility that the present bootstrap analysis may describes the localization of $Sp(N)/U(N)$ non-linear $\sigma$ model in the replica limit $N \to 0$ [10].

Among three different types of the Anderson localization, the symplectic case, where the time reversal symmetry is preserved but the symmetry of space inversion is broken, leads to the existence of the phase transition in two dimensions. The strong spin-orbit case belongs to the symplectic case [8]. The numerical analysis of the localization has precise predictions of the critical exponents due to the finite scaling [11], and the re-considering of the results of $\beta$-function has been proposed to obtain the exponents [12]. We consider the model, which satisfies the conditions of (3) in below, and we call it as a localization model, in which the density of state does not show the singular behavior.

Our study is a continuation of the previous works [3, 4, 5] about the degeneracy of the primary operators in the conformal bootstrap. In the $N \to 0$ limit in $O(N)$ invariant system, it is important to notice that the crossover exponent of $O(N)$ [13], $\hat{\phi}$ becomes one in the $N \to 0$ limit, and hence two prime operators $\Delta_\epsilon = D - \frac{1}{2}$ and $\Delta_T = D - \frac{1}{2}$ has a degeneracy. We have used a blow up technique which provides the degenerated fixed point in a blow up plane [3, 5]. The blow up means the small difference between $\Delta_\epsilon$ and $\Delta_T$. The blow up is essential for the resolution of the singularity in the algebraic geometry [14, 15, 16].

For the localization problem, the system becomes localized due to the ran-
dom potential or random scattering. The diffusion coefficient becomes null when the localization occurs. The diffusion propagator is two-body Green function. The one body Green function, which imaginary part is the density of state, has no singularity. This means the critical exponent $\beta$ is vanishing for the localization problem \[7\]. The exponent $\beta$ is given by the scaling law,

$$\beta = \nu(D - 2 + \eta)$$

Since the scale dimension $\Delta_\phi$ is

$$\Delta_\phi = \frac{D - 2 + \eta}{2}$$

we have $\Delta_\phi = 0$ for the localization in any space dimensions $D$.

Thus we have a constraint for the primary operators in the localization as

$$\Delta_\phi = 0, \quad \Delta_T = \Delta_\epsilon$$ (3)

Some investigation has been done in the Yang-Lee edge singularity for the critical dimensions $D_c$, in which $\Delta_\phi = \Delta_\epsilon = \Delta_T = 0$ is realized \[3\]. The situation is analogous.

The conformal bootstrap does not assume the Lagrangian, and instead only the symmetrical constraints determine the scale dimensions. For the localization, the usual method is starting from the nonlinear $\sigma$ model, and the perturbational expansion gives the $\beta$-function successively. There is a different method, which uses the scattering amplitude like Virasoro-Shapiro amplitude of the string theory, and it determines the $\beta$ function \[17\]. The conformal bootstrap method is generally related to the four point scattering amplitude from the point of AdS/CFT correspondence \[18\]. The relation of the scattering amplitude to the localization will be discussed in a different article. We have not a clear method to treat the symplectic or orthogonal localization except the nonlinear $\sigma$ model or the generalized effective Lagrangian written by the Riemann tensors, which is based upon the tree scattering amplitudes \[17\]. In this paper, we use the appropriate values of the 4-spin operator, denoted by $Q$, to find the symplectic localization fixed point. In the perturbed analysis of the nonlinear $\sigma$ model, the spin operators such as $Q$, is essential.

## 2 Conformal bootstrap

The determinant method \[2\] is used for the evaluation of $\Delta_\phi$ under the condition of $\Delta_\phi = 0$ as \[3\].

The briefly explanation of the determinant method for the conformal bootstrap theory is following. The conformal bootstrap theory is based on the conformal group $O(D, 2)$, and the conformal block $G_{\Delta, L}$ is the eigenfunction of Casimir differential operator $\tilde{D}_2$. The eigenvalue of this Casimir operator is $C_2$,

$$\tilde{D}_2 G_{\Delta, L} = C_2 G_{\Delta, L},$$

$$C_2 = \frac{1}{2}\Delta(\Delta - D) + L(L + D - 2).$$

The solutions of the Casimir equation have been studied \[19\, 20\, 21\]. The conformal block $G_{\Delta, L}(u, v)$ has two variables $u$ and $v$, which denote the cross
determination of the values of $\Delta$. The matrix elements of minors are expressed
numbers of the truncated variables $\Delta$, we need to consider the minors for the
taken about ratios, $u = (x_{12}x_{34}/x_{13}x_{24})^2$, $v = (x_{14}x_{23}/x_{13}x_{24})^2$, where $x_{ij} = x_i - x_j$
($x_i$ is two dimensional coordinate). They are expressed as $u = z\bar{z}$ and $v = (1 - z)(1 - \bar{z})$. For the particular point $z = \bar{z} = 1/2$, the conformal block $G_{\Delta,L}(u,v)$
for spin zero ($L=0$) case has a simple expression,

$$G_{\Delta,0}(u,v)|_{z=\bar{z}} = \left(\frac{2}{1-z}\right)^{\Delta/2}{\genfrac{[}{]}{0pt}{}{3}{2}}_3F_2[\frac{\Delta}{2}, \frac{\Delta}{2}, D/2 + 1; \frac{D+1}{2}, \frac{D-1}{2} + 1; \frac{z^2}{4(z-1)}]$$

The conformal bootstrap determines $\Delta$ by the condition of the crossing symmetry $x_1 \leftrightarrow x_3$. For the practical calculations, the point $z = \bar{z} = 1/2$ is chosen.
The conformal bootstrap analysis by a small size of matrix has been investigated
for Yang-Lee edge singularity with accurate results of the scale dimensions by
Gliozzi [2]. In this paper, we emphasize the importance of the structure of minors
along the Plücker relations as shown in [3]. In this $3 \times 3$ minors, if the value of the space dimension $D$ is given, the value of $\Delta_\phi = \Delta_T$ is determined
from the intersection point. However, if one goes to larger minors, for instance
$4 \times 4$ or $5 \times 5$ minors, one need the values of additional scale dimensions of
operator product expansion (OPE), like spin 4 and spin 6 so on. In localization
problem, the value of $\Delta_\phi$ is fixed as zero, so there is advantage to use the $3 \times 3$
determinants, which provides $\Delta_\phi = \Delta_T$ for each dimension $D$.

The bootstrap method is comprised of the crossing symmetry of the four point function. The four point correlation function for the scalar field $\phi(x)$ is given by

$$< \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)> = \frac{g(u,v)}{|x_{12}|^{2\Delta_\phi}|x_{34}|^{2\Delta_\phi}}.$$  

The notation of $x_{ij} = x_i - x_j$ ($x_i$ is two dimensional coordinate) and the scaling
dimension $\Delta_\phi$ for the field $\phi(x)$ are used. The amplitude $g(u,v)$ is expanded as the sum of conformal blocks $G_{\Delta,L}$ ($L$ is a spin),

$$g(u,v) = 1 + \sum_{\Delta,L} p_{\Delta,L}G_{\Delta,L}(u,v)$$  

where $u$ and $v$ are cross ratios, defined by $u = (x_{12}x_{34}/x_{13}x_{24})^2$ and $v = (x_{14}x_{23}/x_{13}x_{24})^2$.

The crossing symmetry of the exchange $x_1 \leftrightarrow x_3$ implies

$$\sum_{\Delta,L} p_{\Delta,L} \frac{v^{\Delta_\phi}G_{\Delta,L}(u,v) - u^{\Delta_\phi}G_{\Delta,L}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}} = 1.$$  

A Minor method is consist of the derivatives at the symmetric point $z = \bar{z} = 1/2$. By the change of variables $z = (a + \sqrt{b})/2$, $\bar{z} = (a - \sqrt{b})/2$, derivatives are taken about $a$ and $b$. Since the numbers of equations become larger than the numbers of the truncated variables $\Delta$, we need to consider the minors for the determination of the values of $\Delta$. The matrix elements of minors are expressed by,

$$f_{\Delta,L}^{(m,n)} = (\partial_a^m \partial_b^n) \frac{v^{\Delta_\phi}G_{\Delta,L}(u,v) - u^{\Delta_\phi}G_{\Delta,L}(v,u)}{u^{\Delta_\phi} - v^{\Delta_\phi}}|_{a=1,b=0}$$  

and the minors of $3 \times 3$ $d_{ijk}$ are the determinants such as

$$d_{ijk} = \det(f_{\Delta,L}^{(m,n)})$$
where \(i, j, k\) are numbers chosen differently from \((1,\ldots,6)\), following the dictionary correspondence to \((m, n)\) as 1 \(\rightarrow\) \((2,0)\), 2 \(\rightarrow\) \((4,0)\), 3 \(\rightarrow\) \((0,1)\), 4 \(\rightarrow\) \((0,2)\), 5 \(\rightarrow\) \((2,1)\), 7 \(\rightarrow\) \((6,0)\) and 8 \(\rightarrow\) \((4,1)\). Since we consider the scalar operator \(\Delta_T\), three components \(f_{\Delta, L}\) of the \(3 \times 3\) determinant are \(\Delta = \Delta_\epsilon\), \(\Delta = \Delta_T\) and \((\Delta = D, L = 2)\). The basic one is \(d_{123}\), which involves the second, fourth derivatives of \(a\) and the first derivative of \(b\). We consider \(d_{123}, d_{124}, d_{125}, d_{234}\) and \(d_{134}\). The \(3 \times 3\) minor, for instance \(d_{123}\), is

\[
\begin{vmatrix}
\frac{\partial^2 a}{\partial x^2} & \frac{\partial^2 a}{\partial x \partial y} & \frac{\partial^2 a}{\partial y^2} \\
\frac{\partial^2 b}{\partial x^2} & \frac{\partial^2 b}{\partial x \partial y} & \frac{\partial^2 b}{\partial y^2}
\end{vmatrix}
\]

(11)

We have Plücker relation \[3\] for \(3 \times 6\) minors such as

\[
[146][235] + [124][356] - [134][256] + [126][345] - [136][245] + [123][456] = 0 \quad (12)
\]

where \([ijk] = d_{ijk}\). Also we have

\[
[146][235] = -3[123][456] - [125][346] + [135][246] \quad (13)
\]

### 3 \(3 \times 3\) minors

We take \(\Delta_\phi = 0\) for the localization. The loci of the minors \(d_{ijk} = 0\) are shown in \((\Delta_\epsilon, \Delta_T)\) plane.

**Table 1. Scale dimension \(\Delta_\epsilon\) for localization from \(3 \times 3\) determinant \(d_{123}\).**

The value of \(\Delta_\epsilon = \Delta_T\) is obtained from the intersection point of the loci of \(d_{123} = 0\) and the line of \(\Delta_\epsilon = \Delta_T\). For \(D = 1.245\), there is no solution as indicated by (*). The value of \(\nu\) becomes smaller than the mean field value 0.5 for \(D > 6\), which means the fixed point may be unstable in this \(3 \times 3\) determinant method.

| \(D\) | \(\Delta_T = \Delta_\epsilon\) | \(\frac{1}{\nu} = D - \Delta_\epsilon\) | \(\nu\) |
|------|----------------|----------------|---|
| 1.245 | * | * | * |
| 1.247 | 1.12 | 0.13 | 7.69 |
| 1.5 | 1.30 | 0.20 | 5.0 |
| 2 | 1.52 | 0.48 | 2.08 |
| 3 | 2.01 | 0.99 | 1.01 |
| 4 | 2.56 | 1.44 | 0.694 |
| 5 | 3.15 | 1.85 | 0.541 |
| 6 | 3.77 | 2.23 | 0.448 |
| 7 | 4.40 | 2.60 | 0.385 |
| 8 | 5.05 | 2.95 | 0.339 |
| 10 | 6.38 | 3.62 | 0.276 |

For \(D = 2.0\), the zero loci of \(d_{123}, d_{124}, d_{134}, d_{125}\) are shown in Fig.1. The value \(\Delta_\epsilon = 1.52\) obtained from \(d_{123}\) is different from the numerical analysis of the finite size scaling, which provides \(\Delta_\epsilon = 1.634\) \[11\].
Figure 1: Localization in D=2: The fixed point $\Delta_\epsilon = \Delta_T = 1.52$ is obtained. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.

Figure 2: Localization in D=3: The fixed point $\Delta_\epsilon = \Delta_T = 2.01$ is obtained from the zero loci of $d_{123}$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$. 

5
Figure 3: Localization in D=1.5: The fixed point $\Delta_\epsilon = \Delta_T = 1.30$ is obtained. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.

For $D = 3$, the zero loci of $d_{123}, d_{124}, d_{134}, d_{125}$ are shown in Fig.2.

For $D = 1.5$, the zero loci of $d_{123}, d_{124}, d_{134}, d_{125}$ are shown in Fig.3.
Figure 4: Localization in D=1.247 from 3 × 3 determinant $d_{123}$: The fixed point $\Delta_\epsilon = \Delta_T = 1.12$ is obtained from the central circle, which disappears at $D = 1.245$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.

4 4 × 4 determinants

It is easy to increase the numbers of operators. For instance 4 × 4 or 5 × 5 determinants, the spin 4 operator $L=4$, and spin 6 operator can be included, also including the operator for the correction to scaling.

The problem is that we don’t know the values of scale dimension of spin 4 or spin 6. These values are treated as parameters for the higher rank determinants.

The 4 × 4 minor is written as, for instance $d_{1237}$,

$$d_{1237} = \det \begin{pmatrix} f_{\Delta, L=0}^{(2,0)} & f_{\Delta, L=0}^{(2,0)} & f_{\Delta_T, L=0}^{(2,0)} & f_Q^{(2,0)} \\ f_{(0,1)}^{(2,0)} & f_{(0,1)}^{(2,0)} & f_{(0,1)}^{(2,0)} & f_Q^{(0,1)} \\ f_{(6,0)}^{(2,0)} & f_{(6,0)}^{(2,0)} & f_{(6,0)}^{(2,0)} & f_Q^{(0,0)} \\ f_{\Delta, L=0}^{(2,0)} & f_{\Delta, L=0}^{(2,0)} & f_{\Delta_T, L=0}^{(2,0)} & f_Q^{(2,0)} \end{pmatrix}$$

(14)

The indices of $d_{ijkl}$ correspond to the dictionary rule: 1 → (2, 0), 2 → (4, 0), 3 → (0, 1), 4 → (0, 2), 5 → (2, 1), 7 → (6, 0) and 8 → (4, 1).

For $D = 2.0$, from $d_{1235} = 0$, we obtain the intersection points $\Delta_\epsilon = 1.635$, by putting the value of $Q$ as $Q = 1.55$ as Fig. 5. The value $\Delta_\epsilon = 1.635$ leads to $\nu = 2.74$. In Fig. 5, the zero loci of four minors, $d_{1235}, d_{1237}, d_{1357}$ and $d_{2347}$ are shown. The two lines of $d_{1235}$ and $d_{1357}$ are overlapping (degenerate) for $Q = 1.55$. For $Q = 1.7$, these four lines become separated as in Fig. 6, and the degeneracy between $d_{1235}$ and $d_{1357}$ is resolved for $Q = 1.7$.

The obtained value $\Delta_\epsilon = 1.635$, which provides $\nu = 2.74$ is very closed the estimate by the finite scaling analysis, which gives $\nu = 2.73 \pm 0.02$ for the symplectic localization in two dimensions [11].

For $D = 2$, if we take $Q = 2.5$, there appears an elliptic circle by $d_{1357}$ which intersects the straight line of $\Delta_\epsilon = \Delta_T$ at $\Delta_\epsilon = 1.6$. This leads to $\nu = 2.5$. Other loci do not show a circle shape. The point at which $d_{1357}$ does not move
Figure 5: Localization in $D=2.0$ by $4 \times 4$ determinants of $d_{1235}, d_{1237}, d_{2347}$: The fixed point $\Delta_\epsilon = \Delta_T = 1.6$ are obtained with the value $Q = 2.5$ from $d_{1357}$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.

so much when $Q$ is changed. For instance if we take $Q = 1.55$, we obtain the estimation $\Delta_\epsilon = 1.635$, which leads to $\nu = 2.74$.

For $D = 2.5$, $\Delta_\epsilon = \Delta_T = 1.68$ is obtained for $Q = 2.55$.

For $D = 3$, the value of $\Delta_\epsilon$ is estimated as 2.05 by the degeneracy of $d_{1235}, d_{1357}$ and $d_{2347}$ as shown in Fig. 7, where $Q = 3.2$ is taken. This leads to $\nu = 1.05$, which is a bit less than the numerical estimate of $\nu = 1.3$. If the value $Q$ decreases, the value of $\nu$ increases from the intersection of $d_{1235}$.

For $D = 4$, as shown in Fig.9, the four lines of zero loci of the determinants $d_{1235}, d_{1237}, d_{1357}$ and $d_{2347}$ are close to the line of $\Delta_\epsilon = \Delta_T$ at $\Delta_\epsilon^2 = 2.93$, which leads to the value of $\nu = 0.935$.

In Table 2, the values of fixed points for other dimensions are shown.

**Table 2. Estimation of $\nu$ by $4 \times 4$ determinants for various dimensions**

| $D$ | $Q$ | $\Delta_T = \Delta_\epsilon$ | $\nu$ |
|-----|-----|-------------------------------|-------|
| 2   | 2.5 | 1.6                           | 2.5   |
| 2.5 | 2.55| 1.68                          | 1.22  |
| 3   | 3.2 | 2.05                          | 1.05  |
| 4   | 4.2 | 2.93                          | 0.93  |
| 5   | 5.25| 3.75                          | 0.80  |
| 6   | 6.35| 4.5                           | 0.66  |
| 8   | 8.45| 6.1                           | 0.52  |
Figure 6: Localization in $D=2.0$ by $4 \times 4$ determinant $d_{1235}, d_{1237}, d_{1357}$ and $d_{2347}$: The fixed point $\Delta_\epsilon = \Delta_T = 1.635$ are obtained from $d_{1234}$ and $d_{1357}$ with the value $Q = 1.55$. This value leads to $\nu = 2.74$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.

Figure 7: Localization in $D=2.5$ by $4 \times 4$ determinants of $d_{1235}, d_{1237}, d_{1357}, d_{2347}$: The fixed point $\Delta_\epsilon = \Delta_T = 1.68$ are obtained with the value $Q = 2.55$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.
Figure 8: Localization in D=3.0 by $4 \times 4$ determinants of $d_{1235}, d_{1237}, d_{2347}$: The fixed point $\Delta_\epsilon = \Delta_T = 2.05$ are obtained with the value $Q = 3.2$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.

Figure 9: Localization in D=4.0 by $4 \times 4$ determinants $d_{1235}, d_{1237}, d_{1357}$ and $d_{2347}$: The fixed point $\Delta_\epsilon = \Delta_T = 2.93$ are obtained with the value $Q = 4.2$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$. 

Figure 10: Localization in $D = 5.0$ by $4 \times 4$ determinants $d_{1235}, d_{1237}, d_{1357}$ and $d_{2347}$: The fixed point $\Delta_\epsilon = \Delta_T = 3.75$ are obtained with the value $Q = 5.25$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.

Figure 11: Localization in $D = 6.0$ by $4 \times 4$ determinants $d_{1235}, d_{1237}, d_{1357}$ and $d_{2347}$: The fixed point $\Delta_\epsilon = \Delta_T = 4.5$ are obtained with the value $Q = 6.35$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.
Figure 12: Localization in $D=8.0$ by $4 \times 4$ determinants $d_{1235}$, $d_{1237}$, $d_{1357}$ and $d_{2347}$: The fixed point $\Delta_\epsilon = \Delta_T = 6.1$ are obtained with the value $Q = 8.45$. The axis is $(x, y) = (\Delta_\epsilon, \Delta_T)$.

5 Discussion

In this paper, we examine the conformal bootstrap method for the localization by putting $\Delta_\phi = 0$ condition in the blow up plane where $\Delta_\epsilon$ differs slightly from $\Delta_T$. Similar analysis has been done for the replica $N \to 0$ limit of polymer, branched polymer and for the random field Ising model [3, 4, 5].

There is a fixed point as shown in Table 2 in the region of $2 < D < 8$, and the value of the critical exponent $\nu$ is similar to the values by the finite size scaling for $D = 2$ and $D = 3$ [11]. This case is symplectic localization class, where the phase transition exists in two dimensions. We are not yet conclusive about the upper critical dimension. Around $D = 8$, we have no clear fixed point. Usually the upper critical point becomes a free field, in which almost all zero loci of minors intersect at a single point, similar to polymer and branched polymers [4]. At $D = 8$, we have no such phenomena, and $D = 8$ may not be the upper critical point of the localization. The upper critical dimension $D = \infty$ has been suggested for the localization [12, 22], and our present work does not exclude this possibility.

The extension of this article is possible for including the spin 6 operators and also including the scalar operator for the correction of scaling. Some calculations of these extensions do not change the present results. The other localization of the orthogonal and unitary classes are remained in a future work.

The result obtained here by the simple conformal bootstrap suggests the relevance to the symplectic Anderson localization of the strong spin orbit coupling. There are possibilities of the present analysis which correspond to other universality class, such as quantum Hall effect or as sated in the introduction, such as $Sp(N)/U(N)$ case. This will remain an interesting study.

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