Electrodynamics of Moving Continuous Media with Toroid Polarization

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With regard to the toroid contributions, a modified system of equations of electrodynamics moving continuous media has been obtained. Alternative formalisms to introduce the toroid moment contributions in the equations of electromagnetism has been worked out. The two four-potential formalism has been developed for the electromagnetic continuous media subjected to Lorentz transformations.

Key words: Toroid moments, two-potential formalism

PACS 03.50.De Maxwell theory: general mathematical aspects
PACS 11.10.Ef Lagrangian and Hamiltonian approach

1. INTRODUCTION

The history of electromagnetism is the history of struggle of different rival concepts from the very early days of its existence. Though, after the historical observation by Hertz, all main investigations in electromagnetism were based on Maxwell equations, nevertheless this theory still suffers from some shortcomings inherent to its predecessors. Several attempts were made to remove the internal inconsistencies of the theory. To be short we refer to very few of them. One of the attempts to modify the theory of electromagnetism was connected with the introduction of magnetic charge in Maxwell equation by Dirac [1,2], while keeping the usual definition of E and B in terms of the gauge potentials. A very interesting work in this direction was done by Miller [3]. In this paper he showed the mutual substitution of the sources as follows:

\[ \rho^e \rightarrow \rho^e v = j^e = \nabla \phi^e - \partial \mathbf{A} / \partial t - \text{curl} \mathbf{C} \rightarrow -\text{div} \mathbf{M} = \rho^m \rightarrow \rho^m v = j^m = \nabla \phi^m + \text{curl} \mathbf{P} \rightarrow -\text{div} \mathbf{P} = \rho^e \rightarrow \]

Recently D. Singleton [4,5] gave an alternative formulation of classical electromagnetism with magnetic and electric charges by introducing two four-vector potentials \( A^\mu = (\phi_e, \mathbf{A}) \) and \( C^\mu = (\phi_m, \mathbf{C}) \) and defining E and B fields as

\[
\begin{align*}
\mathbf{E} &= -\nabla \phi^e - \frac{\partial \mathbf{A}}{\partial t} - \text{curl} \mathbf{C} \\
\mathbf{B} &= -\nabla \phi^m - \frac{\partial \mathbf{C}}{\partial t} + \text{curl} \mathbf{A}
\end{align*}
\]

Inserting these newly defined vector potentials into the generalized Maxwell equations [1]

\[
\begin{align*}
\text{curl} \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_e \\
\text{div} \mathbf{E} &= \rho_e \\
-\text{curl} \mathbf{E} &= \frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m \\
\text{div} \mathbf{B} &= \rho_m
\end{align*}
\]

where \( \rho_m \) and \( \mathbf{J}_m \) are the magnetic charge and current, respectively, and imposing the Lorentz gauge condition

\[
\frac{\partial \phi^e}{\partial t} + \text{div} \mathbf{A} = 0, \quad \frac{\partial \phi^m}{\partial t} + \text{div} \mathbf{C} = 0
\]

he arrived at the wave form of Maxwell’s equations with electric and magnetic charges. Note that, a similar theory (two potential formalism) was developed by us few years ago (we will come back to it in Sec. 3). Though this treatment avoids the use of singular, nonlocal variables in electrodynamics with magnetic charge and makes the
Maxwell system more symmetric, since both the charges in this approach are gauge charges, the main defect of this theory in our view is that the existence of magnetic charge still lack of experimental support, hence can be considered as a mathematically convenient one only.

Recently Chubykalo a.o. made an effort to modify the electromagnetic theory by invoking both the transverse and longitudinal (explicitly time independent) fields simultaneously, thus giving an equal footing to both the Maxwell-Hertz and Maxwell-Lorentz equations \[1,2\]. To remove all ambiguities related to the applications of Maxwell’s displacement current they substituted all partial derivatives in Maxwell-Lorenz equations by the total ones, such that

$$\frac{d}{dt} = \frac{\partial}{\partial t} - (\mathbf{V} \cdot \text{grad})$$

(1.4)

where \(\mathbf{V} = d\mathbf{r}_q/dt\) and \(\mathbf{r}_q(t)\) are the velocity and the coordinate of the charge, respectively, at the instant \(t\). Further they separated all field quantities into two independent classes with explicit \(\{\}^*\) and implicit \(\{\}\) time dependence, respectively. Thus, the component \(\mathbf{E}_0\) of the total electric field \(\mathbf{E}\) in every point is understood to depend only on the position of source at a given instant. In other words, \(\mathbf{E}_0\) is rigidly linked with the location of the charge. From this point of view, the partial time derivative in (1.4) must be related only with the explicit time-dependent component \(\mathbf{E}^*\) whereas the convection derivative only with \(\mathbf{E}_0\):

$$\frac{d\mathbf{E}}{dt} = \frac{\partial\mathbf{E}^*}{\partial t} - (\mathbf{V} \cdot \text{grad})\mathbf{E}_0, \quad \mathbf{E} = \mathbf{E}^*(\mathbf{r}, t) + \mathbf{E}_0(\mathbf{R}(t))$$

(1.5)

where \(\mathbf{r}\) is a fixed distance from the origin of the reference system at rest to the point of observation and \(\mathbf{R}(t) = \mathbf{r} - \mathbf{r}_q(t)\).

Another attempt to modify the equations of electromagnetism is connected with the existence of the third family of multipole moments, namely the toroid one. This theory was developed by us during the recent years. Recently we introduced toroid moments in Maxwell equations exploiting Lagrangian formalism \[10\]. In the Sec. 2 of this paper we give a brief description of this formalism. Moreover, here we develop an alternative method to introduce toroid moments in the equation of electromagnetism. In Sec. 3 we develop two potential formalism suggested by us earlier.

2. INTRODUCTION OF TOROID MOMENTS IN THE EQUATIONS OF ELECTROMAGNETISM

In early fifties, while solving the problem of the multipole radiation of a spatially bounded source, Franz and Wallace \[11,12\] found a contribution to the electric part of radiation at the expense of magnetization. Further Ya. Zel’Dovich \[13\] pointed out the non-correspondence between the existence of two known multipole sets, Coulomb and magnetic, and the number of form-factors for a spin – \(\frac{1}{2}\) charged particles. Following the parity non-conservation law in weak interactions Zel’Dovich suggested a third form-factor in the parametrization of the Dirac spinor particle current. As a classical counterpart of this form-factor he introduced anapole in connection with the global electromagnetic properties of a toroid coil that are impossible to describe within the charge or magnetic dipole moments in spite of explicit axial symmetry of the toroid coil. In 1965 Shirokov and Cheshkov \[14\] constructed the parametrization for relativistic matrix elements of currents of charged and spinning particle, which contain the third set of form-factors. Finally, in 1974 Dubovik and Cheskov \[15\] determined the toroid moment in the framework of classical electrodynamics. Note that anapole and toroid dipole are not the different names of one and the same thing. They are indeed quite different in nature. For example, the anapole cannot radiate at all while the toroid coil and its point-like model, toroid dipole, can. The matter is that the anapole is some composition of electric dipole and actual toroid dipole giving destructive interference of their radiation. Recently a principally new type of magnetism known as aromagnetism was observed in a class of organic substances, suspended either in water or in other liquids \[16\]. Later, it was shown that this phenomena of aromagnetism cannot be explained in a standard way, e.g., by ferromagnetism, since the organic molecules do not possess magnetic moments of either orbital or spin origin. It was also shown that the origin of aromagnetism is the interaction of vortex electric field induced by alternative magnetic one with the axial toroid moments in aromatic substances \[17\].

In a recent work Dubovik and Kuznetsov \[18\] calculated the toroid moment of Majorana neutrino. It was also pointed out that the magnitude of the toroid dipole moment of a Dirac neutrino (\(\nu_d\)) is just the half of that of a majorana one (\(\nu_m\)) and both of them possesses non-trivial torid moments even if \(m_{\nu} = 0\) \[19\].

The latest theoretical and experimental development force the introduction of toroid moments in the framework of conventional classical electrodynamics that in its part inevitably leads to the modification of the equations of electromagnetism and the equations of motion of particles in external electromagnetic field. In the two following subsection we give two alternative schemes of introduction of toroid polarizations in the electromagnetic equations.
To begin with we write the Maxwell equations for electromagnetic fields in vacuum, in the presence of extraneous electric charge \( \rho \) and electric current, that is, charge - in - motion, of density \( j \).

\[
\begin{align*}
\text{curl} B - \frac{1}{c} \frac{\partial E}{\partial t} &= \frac{4\pi}{c} j \quad (2.1a) \\
\text{div} E &= 4\pi \rho \quad (2.1b) \\
\text{curl} E + \frac{1}{c} \frac{\partial B}{\partial t} &= 0 \quad (2.1c) \\
\text{div} B &= 0 \quad (2.1d)
\end{align*}
\]

where \( E \) and \( B \) are the flux densities of electric field and magnetic induction, respectively. Note that, the electric charges and electric currents, being distributed in vacuum, construct the electromagnetic structure of matter \[20\] and are related to the elementary charge \( e \) in the following way

\[
\begin{align*}
\rho(r) &= \sum_n e_n \delta(r - q_n) \quad (2.2a) \\
j(r) &= \sum_n e_n \dot{q}_n \delta(r - q_n) \quad (2.2b)
\end{align*}
\]

So, to describe the system as a whole the equations (2.1) and (2.2) should be supplemented by the equation of motion of micro-particle, i.e.,

\[
m_n \ddot{q}_n = e_n E + e_n c \dot{q}_n \times B \quad (2.3)
\]

In this process there occurs a vast number of problems connected with the different areas of this vast field:

Write the classical and quantum analogies of the equations of motion of a point-like particle possessing toroid dipole (with usual properties);

Solve the boundary-value problems for the model with all the of vector polarizations;

Formulate the electrodynamics of continuous infinite media for the latter case.

Since the electromagnetic field in media generates bound charges and bound currents, the source parts in (2.1a) and (2.1b) should be supplemented as follows \[21\]

\[
\begin{align*}
\rho_{\text{total}} &= \rho + \rho_{\text{bound}} = \rho - \text{div}P \quad (2.4a) \\
j_{\text{total}} &= J = j + \frac{\partial P}{\partial t} + c \text{curl} M \quad (2.4b)
\end{align*}
\]

Then we may write the Maxwell equations for electromagnetic fields in media as

\[
\begin{align*}
\text{curl} H - \frac{1}{c} \frac{\partial D}{\partial t} &= \frac{4\pi}{c} j \quad (2.5a) \\
\text{div} D &= 4\pi \rho \quad (2.5b) \\
\text{curl} E + \frac{1}{c} \frac{\partial B}{\partial t} &= 0 \quad (2.5c) \\
\text{div} B &= 0 \quad (2.5d)
\end{align*}
\]

where we denote \( D = E + 4\pi P \) and \( H = B - 4\pi M \) are the flux densities of electric displacement vector and magnetic field, respectively.

Note that \( \partial P/\partial t \), known as bound-charge current density is independent of the details of the model and is a conduction current. The difference between an “ordinary” conduction current density and the current density \( \partial P/\partial t \) is that the first involves free charge \( \rho \) (foreign charge over which we have some control, i.e., charge that can be added to or remove from an object) in motion, when the second bound charge (the integral parts of atoms or molecules of the dielectric) in motion. Another obvious practical distinction is that it is impossible to get a steady bound charge current that goes forever unchanged. The bound currents \( c \text{curl} M \) are associated with molecular or atomic magnetic moments, including the intrinsic magnetic moment of particles with spin, whereas free currents \( j \) are “ordinary” conduction currents flowing on macroscopic paths and can be started or stopped with a switch and measured with an ammeter.

The foregoing equations can be obtained from a Lagrangian describing the interacting system of electromagnetic field and non-relativistic test charged particle \[22\]
\[ L = \frac{1}{2} m \dot{\mathbf{q}}^2 + \frac{1}{8\pi} \int \left( \frac{\dot{\mathbf{A}}^2}{c^2} - (\text{curl}\mathbf{A})^2 \right) d\mathbf{r} + \frac{1}{c} \int J(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) d\mathbf{r} \]  

(2.6)

Here the total current of the system \( \mathbf{J} \) coincides with the current \( \mathbf{j} = e\dot{\mathbf{q}} \), generated by the free charges (charged particles) in motion since there is no polarization current.

It can be easily verified that variation of the Lagrangian (2.6) with respect to the particle coordinates gives the second law of Newton with the Lorentz force, i.e., the equation (2.3)

\[ m \dot{\mathbf{q}} = e\mathbf{E}(\mathbf{q}, t) + \frac{e}{c} \dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}, t) \]

whereas the variation with respect to field variables gives (2.1a), i.e.,

\[ \text{curl}\mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = -\frac{4\pi}{c} \mathbf{j} \]  

(2.7)

where we denote \( \mathbf{B} = \text{curl}\mathbf{A} \) and \( \mathbf{E} = -c^{-1}(\partial\mathbf{A}/\partial t) \). It should be emphasized that \( \mathbf{E} \) in (2.3) and (2.7) is the transverse part of the total electric field. The longitudinal electric field in question is entirely electrostatic. The Hamiltonian, corresponding to the Lagrangian (2.6) reads

\[ H[\Pi, \mathbf{A}; p, q] = p \cdot \dot{q} + \int \Pi \cdot \mathbf{A} d\mathbf{r} - L \]

(2.8)

where the corresponding conjugate momenta are \( p = m\dot{q} + (e/c)\mathbf{A}(\mathbf{q}, t) \), \( \Pi(\mathbf{r}) = (4\pi c^2)^{-1} \dot{\mathbf{A}} \).

It is well known that in classical dynamics the addition of a total time derivative to a Lagrangian leads to a new Lagrangian with the equations of motion unaltered. Lagrangians obtained in this manner are said to be equivalent. In general, the Hamiltonians following from the equivalent Lagrangians are different. Even the relationship between the conjugate and the kinetic momenta may be changed [23]. Let us now construct an equivalent Lagrangian to that of (2.6) in the following way

\[ L_1 = L - \frac{1}{c} \frac{d}{dt} \int \mathbf{P}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) d\mathbf{r} - \int \text{div}(\mathbf{M} \times \mathbf{A}) d\mathbf{r} \]  

(2.9)

The new Lagrangian is a function of the variables \( q, \dot{q} \) and a functional of the field variables \( \mathbf{A}, \dot{\mathbf{A}} \), and the equations of motion follow from the variational principle. We have

\[ \frac{\partial L_1}{\partial \mathbf{A}_i} = \frac{1}{c} (\mathbf{J} - \dot{\mathbf{P}} - c\text{curl}\mathbf{M})_i = \dot{\mathbf{j}}_i \]

\[ \sum_j \frac{\partial}{\partial x_j} \frac{\partial L_1}{\partial (\mathbf{A}_i/\partial x_j)} = \frac{1}{4\pi} \left[ \text{curl}\mathbf{B} - 4\pi c \text{curl}\mathbf{M} \right]_i = \frac{1}{4\pi} \left[ \text{curl}\mathbf{H} \right]_i \]

\[ \frac{\partial}{\partial t} \frac{\partial L_1}{\partial (\mathbf{A}_i/\partial t)} = -\frac{1}{4\pi c} \frac{\partial}{\partial t} \left[ \mathbf{E} + 4\pi \mathbf{P} \right]_i = -\frac{1}{4\pi c} \frac{\partial \mathbf{D}_i}{\partial t} \]

Applying the Euler-Lagrange equations of motion

\[ \frac{\partial}{\partial t} \frac{\partial L_1}{\partial \mathbf{A}_i(\mathbf{r})} + \sum_j \frac{\partial}{\partial x_j} \frac{\partial L_1}{\partial (\mathbf{A}_i/\partial x_j)} - \frac{\partial L_1}{\partial \mathbf{A}_i} = 0 \]

we obtain (2.5a), i.e.,

\[ \text{curl}\mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} \mathbf{j} \]

The corresponding Hamiltonian now can be written as

\[ H_1[\Pi, \mathbf{A}; p, q] = \frac{1}{2m} [p - \frac{e}{c} \mathbf{A}(\mathbf{q}, t)]^2 + \frac{1}{8\pi} \int [\mathbf{D}^2 + \mathbf{B}^2] d\mathbf{r} \]

\[ - \int \mathbf{D} \cdot \mathbf{P} d\mathbf{r} + 2\pi \int \mathbf{P}^2 d\mathbf{r} - \int \mathbf{M} \cdot \mathbf{B} d\mathbf{r} \]  

(2.10)
In the same way we can introduce the toroid polarizations in the system. If we take into account that the total current now has the form \[ \mathbf{j}_{\text{total}} = \mathbf{j} + \frac{\partial \mathbf{P}}{\partial t} + \frac{\partial \text{curl} \mathbf{T}^e}{\partial t} + c \text{curl} \mathbf{M} + c \text{curl} \text{curl} \mathbf{T}^m \] (2.11)

we again obtain (2.5a) with \( \mathbf{D} = \mathbf{E} + 4\pi(\mathbf{P} + \text{curl} \mathbf{T}^e) \) and \( \mathbf{H} = \mathbf{B} - 4\pi(\mathbf{M} + \text{curl} \mathbf{T}^m) \). Here \( \mathbf{T}^e \) and \( \mathbf{T}^m \) are the toroid polarizations of electric and magnetic type respectively. In terms of \( \mathbf{D} \) and \( \mathbf{B} \) the system (2.5) can be written as follows

\[
\begin{align*}
\text{curl} \mathbf{B} &= \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + 4\pi \{ \text{curl} \mathbf{M} + \text{curl} \text{curl} \mathbf{T}^m \} + \frac{4\pi}{c} \mathbf{j} \\
\text{div} \mathbf{D} &= 4\pi \rho \\
\text{curl} \mathbf{D} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + 4\pi \{ \text{curl} \mathbf{P} + \text{curl} \text{curl} \mathbf{T}^e \} \\
\text{div} \mathbf{B} &= 0
\end{align*}
\] (2.12a)

where we forced to redefine \( \mathbf{B} \) and \( \mathbf{D} \) as \( \mathbf{B} \) and \( \mathbf{D} \) respectively. Indeed, due to the introduction of toroid polarizations, having independent origin in terms of atomic and molecular current and charge distributions, the quantities \( \mathbf{B} \) and \( \mathbf{D} \) as well as \( \mathbf{E} \) and \( \mathbf{H} \) lost their initial meaning. The existence of the vorticities \( \mathbf{T}^e \) and \( \mathbf{T}^m \), generally speaking, can be imputed to the one and the same physical volume. So in what follows, we substitute \( \mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H} \) by \( \mathcal{E}, \mathcal{B}, \mathcal{D}, \mathcal{H} \).

The equation of motion (2.3) should also be rewritten as follows

\[
m\ddot{\mathbf{q}} = e\mathcal{E} + \frac{e}{c}\dot{\mathbf{q}} \times \mathcal{B}
\] (2.13)

The equation (2.12a) can be derived as earlier by constructing a Lagrangian equivalent to \( L_1 \) such that

\[
L_2 = L_1 - \frac{1}{c} \frac{d}{dt} \int \text{curl} \mathbf{T}^e(r) \cdot \mathbf{A}(r) \, dr - \int \text{div} \{ \text{curl} \mathbf{T}^m \times \mathbf{A} \} \, dr
\] (2.14)

The corresponding Hamiltonian reads

\[
H_2[\Pi, \mathbf{A}; p, q] = \frac{1}{2m} \left[ \mathbf{p} - \frac{e}{c} \mathbf{A}(q, t) \right]^2 + \frac{1}{8\pi} \int [\mathbf{D}^2 + \mathbf{B}^2] \, dr \\
- \int \mathbf{D} \cdot \mathbf{[P + \text{curl} \mathbf{T}^e]} \, dr + 2\pi \int [\mathbf{P + \text{curl} \mathbf{T}^e}]^2 \, dr - \int [\mathbf{M + \text{curl} \mathbf{T}^m}] \cdot \mathbf{B} \, dr
\] (2.15)

Note that in our previous work we impose the following additional condition

\[
\text{curl} \mathbf{T}^{m,e} = \pm \frac{1}{c} \mathbf{T}^{e,m}
\] (2.16)

The relation (2.16) demands some comments. Both \( \mathbf{T}^e \) and \( \mathbf{T}^m \) represent the closed isolated lines of electric and magnetic fields. So they have to obey the usual differential relations similar to the free Maxwell equations (2.26). However, remark that signs here are opposite to the corresponding one in Maxwell equations because the direction of electric dipole is accepted to be chosen opposite to its inner electric field (2.7). It should be emphasized that the relation (2.16) is a local one and it is not necessary to demand the condition (2.16) to be held to introduce toroid polarizations to the electromagnetic equations. We would also like to remark that the equations (2.12a) and (2.12c) can be derived strait from (2.5a) and (2.5c) making the following substitutions in them:

\[
\mathbf{P} \Rightarrow \mathbf{P} + \text{curl} \mathbf{T}^e, \quad \mathbf{M} \Rightarrow \mathbf{M} + \text{curl} \mathbf{T}^m
\]

and redefining the vectors \( \mathbf{B} \) and \( \mathbf{D} \) as earlier.

3. TWO POTENTIAL FORMALISM

It is generally inferred that the divergence equations of the Maxwell system are ”redundant” since they are the consequences of curl equations under the condition of continuity [33]. Recently Krivsky a.o. [32] claimed that to describe the free electromagnetic field it is sufficient to consider the curl-subsystem of Maxwell equations since the
equalities $\text{div} \mathbf{E} = 0$ and $\text{div} \mathbf{B} = 0$ are fulfilled identically. Contrary to this statement, it has been proved that the divergence equations are not redundant and that neglecting these equations is at the origin of spurious solutions in computational electromagnetics [34,35]. Here we construct generalized formulation of Maxwell equations including both curl and divergence subsystems. In this section we develop two potential formalism (a similar formalism was developed by us earlier with the curl-subsystem taken into account only). Note that in the ordinary one potential formalism ($\mathbf{A}, \phi$) the second set of Maxwell equations are fulfilled identically. So that all the four Maxwell equations bring their contribution individually; in our view, one has to rewrite the Maxwell equation in terms of two vector and two scalar potentials. To this end we introduce so-called double potential [36,37] to the system (2.1), i.e.,

$$\text{curl} \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c} j_{\text{free}} + 4\pi \{ \text{curl} \mathbf{M} + \text{curl} \text{curl} \mathbf{T}^m \}$$

(3.1a)

$$\text{div} \mathbf{D} = 4\pi \rho$$

(3.1b)

$$\text{curl} \mathbf{D} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + 4\pi \{ \text{curl} \mathbf{P} + \text{curl} \text{curl} \mathbf{T}^e \}$$

(3.1c)

$$\text{div} \mathbf{B} = 0$$

(3.1d)

Before developing the two potential formalism we first rewrite system (2.1) in terms of vector and scalar potentials $\mathbf{A}, \phi$ such that $\mathbf{B} = \text{curl} \mathbf{A}$, $\mathbf{E} = -\nabla \phi - (1/c)(\partial \mathbf{A}/\partial t)$. Following any text book we can write system (2.1) as

$$\Box \mathbf{A} = -\frac{4\pi}{c} j_{\text{tot}} = -\frac{4\pi}{c} j_{\text{free}} + \frac{\partial \mathbf{P}}{\partial t} + c \text{curl} \mathbf{M}$$

(3.2a)

$$\Box \phi = -4\pi [\rho - \text{div} \mathbf{P}]$$

(3.2b)

under Lorentz gauge, i.e., $\text{div} \mathbf{A} + (1/c)(\partial \phi/\partial t) = 0$ and

$$\Box \mathbf{A} = \frac{4\pi}{c} j_{\text{tot}} - \frac{\partial \mathbf{P}}{\partial t} + c \text{curl} \mathbf{M}$$

(3.3a)

$$\nabla^2 \phi = -4\pi [\rho - \text{div} \mathbf{P}]$$

(3.3b)

under Coulomb gauge, i.e., $\text{div} \mathbf{A} = 0$. Here $\Box = \nabla^2 - (1/c^2)(\partial^2/\partial t^2)$. Note that to obtain (3.2) or (3.3) it is sufficient to consider (2.1a) and (2.1b) only since the two others are fulfilled identically.

Let us now develop two potential formalism. Two potential formalism was first introduced in [36] and further developed in [37,10]. In both papers we introduce only two vector potentials $\alpha^m, \alpha^e$ and use only the curl-subsystem of the Maxwell equations with the additional condition $\text{div} \alpha^{m,e} = 0$. Thus, in our view our previous version of two potential formalism lack of completeness. In the present paper together with the vector potentials $\alpha^m, \alpha^e$ we introduce two scalar potentials $\varphi^m$ and $\varphi^e$ such that

$$\mathbf{B} = \text{curl} \alpha^m + \frac{1}{c} \frac{\partial \alpha^e}{\partial t} + \nabla \varphi^m$$

(3.4a)

$$\mathbf{D} = \text{curl} \alpha^e - \frac{1}{c} \frac{\partial \alpha^m}{\partial t} - \nabla \varphi^e$$

(3.4b)

It can be easily verified that system of equations (3.1) are invariant under this transformation and take the form

$$\Box \alpha^m = -\frac{4\pi}{c} [j + c \text{curl} \mathbf{M} + c \text{curl} \text{curl} \mathbf{T}^m]$$

(3.5a)

$$\Box \varphi^m = 0$$

(3.5b)

$$\Box \alpha^e = -\frac{4\pi}{c} [\text{curl} \mathbf{P} + \text{curl} \text{curl} \mathbf{T}^e]$$

(3.5c)

$$\Box \varphi^e = -4\pi \rho$$

(3.5d)

under $\text{div} \alpha^{m,e} + (1/c)(\partial \varphi^{m,e}/\partial t) = 0$ and

$$\Box \alpha^m = -\frac{4\pi}{c} [j + c \text{curl} \mathbf{M} + c \text{curl} \text{curl} \mathbf{T}^m - \frac{1}{4\pi} \nabla \frac{\partial \varphi^e}{\partial t}]$$

(3.6a)

$$\nabla^2 \varphi^m = 0$$

(3.6b)

$$\Box \alpha^e = -\frac{4\pi}{c} [\text{curl} \mathbf{P} + \text{curl} \text{curl} \mathbf{T}^e - \frac{1}{4\pi} \frac{\partial \varphi^m}{\partial t}]$$

(3.6c)

$$\nabla^2 \varphi^e = -4\pi \rho$$

(3.6d)
under \( \text{div } \alpha^{m,e} = 0 \). The solutions to the systems (3.3) and (3.6) can be written as follows (see for example [10,38]): The solutions to the d’Alembert equation

\[
\Box F(r, t) = f(r, t) \tag{3.7}
\]

look

\[
F(r, t) = -\frac{1}{4\pi} \int_{\text{all space}} \frac{f(r', t') \, dr'}{|r - r'|} \bigg|_{t' = t - |r - r'|/c} \tag{3.8}
\]

whereas the solutions to the Poisson equation

\[
\nabla^2 F(r) = f(r) \tag{3.9}
\]

read

\[
F(r) = -\frac{1}{4\pi} \int \frac{f(r') \, dr'}{|r - r'|} \tag{3.10}
\]

Let us rewrite the equations (3.5) and (3.6) for the fields subject to Lorentz transformation. If the fields in stationary frame (unprimed) are connected with those in moving one (primed) in the following way

\[
\begin{align*}
B &= \gamma (B' + \frac{1}{c} \beta \times E') \tag{3.11a} \\
E &= \gamma (E' - c\beta \times B') \tag{3.11b} \\
P &= \gamma (P' + \frac{1}{c} \beta \times M') \tag{3.11c} \\
M &= \gamma (M' - c\beta \times P') \tag{3.11d} \\
T^c &= \gamma (T'^c + \frac{1}{c} \beta \times T'^{m'}) \tag{3.11e} \\
T^m &= \gamma (T'^m - c\beta \times T'^c) \tag{3.11f} \\
\alpha^m &= \gamma (\alpha'^m + \beta \phi'') \tag{3.11g} \\
\phi^m &= \gamma (\phi'^m - \beta \alpha'^e) \tag{3.11h} \\
\alpha^e &= \gamma (\alpha'^e - \beta \phi'^m) \tag{3.11i} \\
\phi^e &= \gamma (\phi'^e + \beta \alpha'^m) \tag{3.11j} \\
\rho &= \gamma (\rho' + \frac{1}{c} \beta J') \tag{3.11k} \\
J &= \gamma (J' + c\beta \rho') \tag{3.11l}
\end{align*}
\]

The equations (3.5) now can be written as

\[
\begin{align*}
\Box \alpha^{m'} &= -\frac{4\pi \gamma^2}{c} \left[ \gamma^{-1} j' + c \, \text{curl } (M' + \text{curl } T'^{m'}) - \text{curl } (\beta \times P' + \text{curl } (\beta \times T'^e)) \right], \tag{3.12a} \\
\Box \phi'^m &= \beta \Box \alpha'^e \tag{3.12b} \\
\Box \alpha'^e &= -\frac{4\pi \gamma^2}{c} \left[ \text{curl } (P' + \text{curl } T'^e) + \frac{1}{c} \, \text{curl } (\beta \times M' + \text{curl } (\beta \times T'^m)) \right], \tag{3.12c} \\
\Box \phi'^e &= -\beta \Box \alpha'^m - 4\pi (\rho' + \beta J'/c) \tag{3.12d}
\end{align*}
\]

It is necessary to emphasize that the potential descriptions electrotoroidic and magnetotoroidic media are completely separated. The properties of the magnetic and electric potentials \( \alpha^m \) and \( \alpha^e \) under the temporal and spatial inversions are opposite [25]. The potential \( \alpha^e \) (\( \alpha^m \)) is related to the toroidness of the medium \( T^c \) (\( T^m \)) as \( B \) (\( D \)) to \( M \) (\( P \)).
4. CONCLUSION

The modified equations of electrodynamics has been obtained in account of toroid moment contributions. The two-potential formalism has been further developed for the equations obtained. Note that introduction of free magnetic current $j^m_{\text{free}}$ and magnetic charge $\rho^m$ in the equations (3.1c) and (3.1d) respectively leads to the equations obtained by Singleton [5].

[1] P.A.M. Dirac, Proceedings of Royal Society, A 133, 60-72 (1931)
[2] P.A.M. Dirac, Physical Review 74, 817-830 (1948)
[3] M.A. Miller, Izvestia Vysshikh Uchebnykh Ustavov, Radiofizika 29, No 9, 991 (1986)
[4] D. Singleton, International Journal of Theoretical Physics 34(1), 37-46 (1995)
[5] D. Singleton, American Journal of Physics 64(4), 452-458 (1996)
[6] J.D. Jackson, Classical Electrodynamics (Wiley, New York, 1975)
[7] E. Andrew, A.E. Chubykalo and R. Smirnov-Rueda, Physical Review E, 53(5) 5373-5381 (1996)
[8] A.E. Chubykalo and R. Smirnov-Rueda, Modern Physics Letters A, 12(1) 1-24 (1997)
[9] A.E. Chubykalo, M.W. Evans and R. Smirnov-Rueda, Foundations of Physics Letters, 10(1) 93-98 (1997)
[10] V.M. Dubovik, B. Saha and M.A. Martseyuk, ICTP Internal Reports IC/IR/96/9, 23 p. (1996)
[11] W. Franz, Zs. f. Phys., 127, 363 (1950)
[12] P.R. Wallace, Phys. Rev., 81, 493 (1951); 82, 297 (1951)
[13] Ya.B. Zel’dovich, Zhurnal Eksperimentalnoi i Thereticheskoi Fiziki, 6 1184 (1958)
[14] A.A. Cheshkov and Yu.M. Shirokov, Zhurnal Eksperimentalnoi i Thereticheskoi Fiziki, 44 1982 (1965)
[15] V.M. Dubovik and A.A. Cheshkov, Fizika Elementarnikh Chastits i Atomnogo Yadra, 5 318 (1974)
[16] A.A. Spartakov, and N.A. Tolstoi, Psi’ma v Zhurnal Eksperimentalnoi i Thereticheskoi Fiziki, 52(3) 796 (1990)
[17] M.A. Martseyuk, and N.M. Martseyuk, Psi’ma v Zhurnal Eksperimentalnoi i Thereticheskoi Fiziki, 53(5) 229 (1991)
[18] V.M. Dubovik, and V.E. Kuznetsov, JINR Preprint E2-96-53 (1996)
[19] E.N. Bukina, V.M. Dubovik, and V.E. Kuznetsov, Yadernaya Fizika, (in press) (1998)
[20] E.A. Turov, The Material Equations of Electrodynamics, (Moscow, Nauka, 1983)
[21] E.M. Purcell, Electricity and Magnetism, (McGraw-Hill, 1960)
[22] W.P. Healy, Non-Relativistic Quantum Electrodynamics, (Academic Press, 1982)
[23] E.A. Power, and T. Thirunamachandran, American Journal Physics, 46(4), 370 (1978)
[24] V.M. Dubovik, and S.V. Shabanov, in Lakhtakia, A. (ed.), Essays on the Formal Aspects of Electromagnetic Theory, (World Scientific, Singapore), 399 (1993)
[25] V.M. Dubovik, and A.M. Kurbatov, in Barut, A.O., Feranchuk, I.D., Shmir, Ya.M. and Tomil’chik, L.M. (eds), Quantum Systems: New Trends and Methods, (World Scientific, Singapore), 117 (1994).
[26] M.A. Miller, Uspekhi Fizicheskikh Nauk, 142(1) 147 (1984)
[27] V.L. Ginzburg, Works of Lebedev Physics Institute, 176, 3 (1986).
[28] I.S. Zheludev, Izvestia Academi Nauk USSR, Physics, 33 204 (1969)
[29] V.M. Dubovik, M.A. Martseyuk, and N.M. Martseyuk, Phys. Part. Nucl., 24(4) 453 (1993).
[30] V.M. Dubovik, I.V. Lunegov, and M.A. Martseyuk, Fizika Elementarnikh Chastits i Atomnogo Yadra, 26 72 (1995)
[31] V.M. Dubovik, and V.V. Tugushev, Physics Reports, 187(4), 145 (1990)
[32] I.Yu. Krivsky, V.M. Simulik, and Z.Z. Torich, Preprint: National Academy of Science, Ukraine, KINR -96 - 3, 20p.
[33] J.A. Stratton, Electromagnetic Theory, (Mc-Graw Hill, New York, 1941)
[34] Bo-nan Jiang, Jie Wu, and L.A. Povinelli, J. Comp. Phys., 125, 104 (1996)
[35] P. Hillion, J. Comp. Phys., 132, 154 (1997)
[36] V.M. Dubovik and E.N. Magar, J. Mosc. Phys. Soc., 3, 1 (1994).
[37] V.M. Dubovik, B. Saha and M.A. Martseyuk, S. Jeffers et al. (eds.) The Present Status of the Quantum Theory of Light, (Kluwer Acad. Publs.), 141-150. (1997)
[38] O.D. Jefimenko, Electricity and Magnetism, (Electret Scientific Company, 1989)
[39] S. Shanmugadhasan, Proceedings of Cambridge Philosophical Society, 59,743 (1963).