ON DYNAMICAL POISSON GROUPOIDS I

Luen Chau Li and Serge Parmentier

September 17, 2002

Abstract. We address the question of duality for the dynamical Poisson groupoids of Etingof and Varchenko over a contractible base. We also give an explicit description for the coboundary case associated with the solutions of (CDYBE) on simple Lie algebras as classified by the same authors. Our approach is based on the study of a class of Poisson structures on trivial Lie groupoids within the category of biequivariant Poisson manifolds. In the former case, it is shown that the dual Poisson groupoid of such a dynamical Poisson groupoid is isomorphic to a Poisson groupoid (with trivial Lie groupoid structure) within this category. In the latter case, we find that the dual Poisson groupoid is also of dynamical type modulo Poisson groupoid isomorphisms. For the coboundary dynamical Poisson groupoids associated with constant $r-$matrices, we give an explicit construction of the corresponding symplectic double groupoids. In this case, the symplectic leaves of the dynamical Poisson groupoid are shown to be the orbits of a Poisson Lie group action.

1. Introduction.

The classical dynamical Yang-Baxter equation (CDYBE) was introduced by Felder in [F] as a consistency condition for the Knizhnik-Zamolodchikov-Bernard equations. The geometric meaning of (CDYBE) was subsequently unraveled by Etingof and Varchenko in the fundamental paper [EV]. While the solutions of the classical Yang-Baxter equation are related to Poisson Lie groups [D], the authors in [EV] showed that an appropriate geometrical setting for (CDYBE) is that of a special class of Poisson groupoids (as defined in [W1]), the so-called coboundary dynamical Poisson groupoids. Given a Lie group $G$, a Lie subgroup $H \subseteq G$, an $Ad^*_H$ invariant open set $U \subset \mathfrak{h}^*$ (here $\mathfrak{h}^*$ is the dual of $\mathfrak{h} = Lie(H)$), and a solution of (CDYBE), Etingof and Varchenko constructed a Poisson bracket on the set $X = U \times G \times U$ compatible with its trivial Lie groupoid structure. This Poisson bracket intertwines a left and a right inclusion of the restricted symplectic cotangent $H \times U$ into $X$ together with a Sklyanin-like term on $G$. In addition, the authors in [EV] identified an appropriate abstract context in which to view these objects as the category of $H$-bi-equivariant Poisson manifolds $\mathcal{C}_U$.

It is classical that the study of Poisson Lie groups relies in an essential way on duality and the construction of doubles [D, STS, LW1]. For Poisson groupoids, the notion of duality was introduced by Weinstein in [W1], and was developed

Key words and phrases. Poisson groupoids, duality, dynamical $r-$ matrices, symplectic double groupoids.
by MacKenzie and Xu in [MX1, MX2]. In the same paper [W1], Weinstein also introduced the notion of symplectic double groupoids (see also [M2]), and described a program for showing that, at least locally, Poisson groupoids in duality arise as the base of a symplectic double groupoid.

In order to state our objectives and results, let us begin by recalling that a symplectic groupoid is a pair \((\Gamma, \Pi)\), consisting of a Lie groupoid \(\Gamma\) together with a non-degenerate Poisson structure \(\Pi\), in such a way that the graph of the multiplication map is a Lagrangian submanifold of \(\Gamma \times \Gamma \times T\Pi\) [W2,K]. It is a classical fact that Poisson structures can be understood at least locally by the notion of symplectic groupoids. On the other hand, double groupoids are intrinsically complicated objects introduced by Ehresmann [E] in the 1960’s and have found usage in category theory [E], homotopy theory [BH], differential geometry [P], and Poisson groups [M3, LW2]. By definition, a double Lie groupoid is a quadruple \((S; \mathcal{H}, \mathcal{V}, B)\) where \(\mathcal{H}\) and \(\mathcal{V}\) are Lie groupoids over \(B\), and \(S\) is equipped with two Lie groupoid structures, a horizontal structure with base \(\mathcal{V}\), and a vertical structure with base \(\mathcal{H}\), such that the structure maps of each groupoid structure on \(S\) are morphisms with respect to the other. Finally, a symplectic double groupoid is a double Lie groupoid \((S; \mathcal{H}, \mathcal{V}, B)\) in which \(S\) is equipped with a symplectic structure such that both \(S \Rightarrow \mathcal{V}\) and \(S \Rightarrow \mathcal{H}\) are symplectic groupoids. Note that for the case of Poisson Lie groups, the program in [W1] which we mentioned above has been carried out globally in [LW2]. Thus a Poisson Lie group and its dual are the bases of a symplectic double groupoid.

This work is the first part of a series to understand the geometry of dynamical Poisson groupoids. Our goal here is three-fold. First of all, for a general dynamical Poisson groupoid \(X = U \times G \times U\) (not necessarily of coboundary type), we would like to characterize certain properties of its (global) dual Poisson groupoid in a simple nontrivial case in which its existence is guaranteed. In this connection, we should point out that in contrast to (finite dimensional) Lie algebras, not all Lie algebroids can be integrated to Lie groupoids [AM]. For (finite dimensional) general Lie algebroids, the necessary and sufficient condition for integrability was only obtained quite recently in [CF]. Thus we work at the outset with the class \(C_\ast\) of Poisson groupoids \(X = U \times G \times U\) (with the trivial Lie groupoid structure) which admits a (base preserving) Poisson groupoid morphism \(I : H \times U \rightarrow X\), where \(U\) is \(Ad_{\Pi}^U\) invariant and contractible. If \(X\) is a dynamical Poisson groupoid in \(C_\ast\), the corresponding Lie algebroid dual \((X)\ast\) must be transitive, i.e., the anchor map is a surjective submersion. Consequently, we can invoke a general theorem of MacKenzie [M1], according to which \((X)\ast \cong TU \oplus (U \times \mathfrak{g'})\), where the latter is the trivial Lie algebroid over \(U\) and \(\mathfrak{g}'\) is a typical fiber of the adjoint bundle of \((X)\ast\). As a result, \((X)\ast\) integrates to a unique global Lie groupoid \(X\ast\) isomorphic to the trivial Lie groupoid \(U \times G' \times U\), where \(G'\) is the connected and simply connected Lie group with \(Lie(G') = \mathfrak{g}'\). Thus the existence of the dual Poisson groupoid is not an issue. Our main result in this direction (Theorem 3.2.4) is the following: if \(X\) is a dynamical Poisson groupoid in \(C_\ast\), then its dual Poisson groupoid \(X\ast\) is isomorphic to a Poisson groupoid \((U \times G' \times U, \{\}, U \times G' \times U)\) in \(C_\ast\). In particular, the Poisson structure \(\{\}, U \times G' \times U\) is uniquely determined by a (unique) Poisson groupoid morphism \(I' : H \times U \rightarrow U \times G' \times U\) and a unique groupoid 1-cocycle \(P'\) on \(U \times G' \times U\). The proof of this theorem consists of two steps: in the first step, we establish the existence of the unique Poisson groupoid morphism \(I'\); while
the second step involves a careful analysis of the form of the Poisson bracket for a Poisson groupoid in $C_G$ (Theorem 2.2.5). As a corollary of Theorem 3.2.4, we obtain via Poisson reduction a reduced duality diagram for the Poisson quotients $G/H \times U$ and $G'/H \times U$ and for the vertex Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$. In the special case when $\mathfrak{h}^* = 0$, this duality diagram is just the well-known diagram of Drinfeld for Poisson Lie groups.

In [EV], extending Belavin and Drinfeld’s classic paper [BD], Etingof and Varchenko obtained a classification of solutions of (CDYBE) for pairs $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras, where $\mathfrak{g}$ is simple and $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra. These solutions of (CDYBE) are parametrized by subsets $S$ of a simple system of roots $\Delta^s$ and closed meromorphic two-forms on $\mathfrak{h}^*$.

Our second objective is to give an explicit study of duality for the coboundary dynamical Poisson groupoids associated with this class of dynamical $r-$ matrices. Note that in this case, the base $U$ (where the $r-$ matrix is analytic) is neither contractible nor simply-connected. We proceed in two steps. To start with, we construct (see Theorem 4.4) an explicit trivialization of the Lie algebroid dual $A(X)^*$ of the (full) coboundary Poisson groupoid $X = U \times G \times U$. This, in particular, establishes the integrability of $A(X)^*$ as a Lie algebroid. Then an argument similar to that of Theorem 3.2.4 applied to any connected and simply connected open subset $U'$ of $U$ shows that the dual Poisson groupoid of $U' \times G \times U'$ is isomorphic to a dynamical Poisson groupoid $U' \times G' \times U'$. Here, the vertex Lie group $G'$ is a semi-direct product $L_S \rtimes I_S$ where $L_S \subset G$ is the Levi factor and $I_S$ is a normal Lie subgroup containing the product $N^+_S \times N^-_S$ of unipotent radicals. More importantly, the Poisson bracket is uniquely determined by the value of a Lie groupoid 1-cocycle $P': U' \times G' \times U' \rightarrow L(\mathfrak{g}'^*, \mathfrak{g}')$ whose partial derivatives are explicitly given in terms of the Lie-Yamaguti data of the reductive pair $(\mathfrak{g}, \mathfrak{h})$.

Our final objective in this paper is to understand how to construct symplectic double groupoids for the coboundary dynamical case in the special instance where the $r-$ matrix is constant. For this class of coboundary dynamical Poisson groupoids, the base is $\mathfrak{h}^*$ and so Theorem 3.2.4 applies. However, from the point of view of constructing the symplectic double groupoids, it is more natural (and considerably simpler) to work directly with the dual Poisson groupoid whose Lie algebroid is $T^*\mathfrak{h}^* \times \mathfrak{g}^*$. Since we have a constant $r-$ matrix, the Lie group $G$ equipped with the Sklyanin bracket is a Poisson Lie group (for simplicity, we assume $G$ is complete) and as it turns out, the dual Poisson groupoid of $X$ is given by $X^* = H \times \mathfrak{h}^* \times G^*$ with appropriate structure maps ($G^*$ is the dual Poisson group of $G$) and the Poisson structure is a product structure (Theorem 5.1.4). The construction of a symplectic double groupoid having $X$ and $X^*$ as side groupoids proceeds via a number of steps (Proposition 5.2.3, Corollary 5.2.6, Corollary 5.2.8, Theorem 5.2.10 and 5.2.13). First of all, we show $X$ and $X^*$ form a matched pair of Lie groupoids. The upshot of this is that $X$ and $X^*$ act on each other via groupoid actions and give rise to a vacant double Lie groupoid $(\mathcal{S}_{vac}, X^*, X, \mathfrak{h}^*)$. However, this is not the correct underlying double Lie groupoid of the sought-for symplectic double groupoid (in contrast to the Poisson group case).

In the second step of the construction, we extend the Lie groupoids $X$ and $X^*$ to the product groupoids $X^*_e = X^* \times H \rightrightarrows H \times \mathfrak{h}^*$ and $X_e = (H \times H) \times X \rightrightarrows H \times \mathfrak{h}^*$ ($H \times H \rightarrow H$ is the coarse groupoid). Then we show that there is a left action of $X^*_e$ on $X$ and a right action of $X_e$ on $X^*$. The corresponding action groupoids
\( S \simeq X_e^* \times X \Rightarrow X \) and \( S \simeq X^* \times X_e \Rightarrow X^* \) then give the horizontal structure and the vertical structure respectively of a nonvacant double Lie groupoid \((S; X^*, X, h^*)\) which has \((S_{vac}; X^*, X, h^*)\) as a double Lie subgroupoid. Finally, we show that the double Lie groupoid \((S; X^*, X, h^*)\) where \( S \) is equipped with an appropriate symplectic structure is a desired symplectic double groupoid. We would like to point out that the actions of the extended Lie groupoids on the unextended ones obey a number of properties (Proposition 5.2.9) which are important in showing that \((S; X^*, X, h^*)\) is a double Lie groupoid. The reader should contrast these properties with actions via ‘twisted automorphisms’ (Proposition 5.2.4, [M3, LW1]). As an application/amplification of this result, we show the existence of a natural Poisson Lie group structure on the set \( H \times H \times G^* \) such that the symplectic leaves of \((X, \{ , \}_X)\) are the orbits of a Poisson action of \( H \times H \times G^* \) on \( X \) (Theorem 5.2.28). Finally, we use this result to describe the symplectic leaves of a natural Poisson quotient associated with \( X \).

The paper is organized as follows. In Section 2, we begin by giving some background material which we recall here for the convenience of the reader. The rest of Section 2 is devoted to the description of all Poisson groupoids \((X, \{ , \})\) which admit a Poisson groupoid morphism \( I : H \times U \longrightarrow X = U \times G \times U \), where \( X \) is the trivial Lie groupoid over a connected base \( U \). Section 3 is concerned with Poisson groupoids in duality with dynamical Poisson groupoids over a contractible base \( U \). It also treats duality diagrams for the Poisson quotients mentioned earlier and for the vertex Lie algebras. In Section 4, we consider the coboundary dynamical Poisson groupoids associated with a class of solutions of (CDYBE) for pairs \((g, h)\) of Lie algebras, where \( g \) is simple, and \( h \) is a Cartan subalgebra of \( g \) [EV]. Here, we obtain a more refined description of the dual Poisson groupoid. Finally, Section 5 treats the coboundary dynamical Poisson groupoids in the constant \( r \)-matrix case in detail. We begin with an explicit description of the dual Poisson groupoid \( X^* \) whose Lie algebroid is \( T^* h^* \times g^* \). Then we move on to the construction of a symplectic double groupoid having \( X \) and \( X^* \) as side groupoids. We conclude the paper by describing the symplectic leaves of \((X, \{ , \})\) as well as a Poisson quotient associated with \( X \).

We shall address the construction of symplectic double groupoids for the general dynamical case, together with its relationship to other works (in particular [LWX]) in a sequel to this paper. On the other hand, the links between duality and the recent work [KW], as well as the relevance of coboundary dynamical Poisson groupoids to integrable systems (see the papers [HM, LX1, LX2] in this connection) will be considered in separate publications.

**Acknowledgements.** L.-C. Li would like to thank the members of Institut G. Desargues for hospitality and CNRS support (UMR 5028) during his visits to Université Lyon 1.

**2. A class of biequivariant Poisson groupoids.**

**2.1. Preliminaries.**

In this preliminary subsection, we recall some of the basic concepts and constructs which we shall use in this paper (other results will be recalled when needed).
Let $\Gamma$ be a Lie groupoid over $B$ (see [DSW,M1] for details), with target and source maps $\alpha, \beta : \Gamma \to B$, and multiplication map $m : \Gamma \times \Gamma \to \Gamma$ defined on the set of composable pairs $\Gamma \times \Gamma := \{(x,y) \mid \beta(x) = \alpha(y)\}$. We shall denote the unit section by $\epsilon : B \to \Gamma$, and the inversion map by $i : \Gamma \to \Gamma$.

**Definition 2.1.1** [W1] (Poisson groupoid.)

A Lie groupoid $\Gamma$ equipped with a Poisson structure $\Pi$ is called a Poisson groupoid if and only if the graph of the multiplication map

$$Gr(m) \subset \Gamma \times \Gamma \times \Gamma$$

is a coisotropic submanifold, i.e. if and only if

$$((\Pi \oplus \Pi \oplus -\Pi)(\omega, \omega') = 0, \quad \forall \omega, \omega' \in (T(Gr(m)))^\perp \subset T^*(\Gamma \times \Gamma \times \Gamma).$$

$\Gamma$ is called a symplectic groupoid if $\Pi$ is non degenerate with $Gr(m)$ a Lagrangian submanifold.

In both cases, we shall say that the Poisson structure and the groupoid structure are compatible.

Let $G$ be a connected Lie group, $H \subset G$ a connected Lie subgroup with respective Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and let $U \subset \mathfrak{h}^*$ be a connected $Ad_H^*$-invariant open subset. In [EV], Etingof and Varchenko introduced the category $\mathcal{C}_U$ of biequivariant Poisson manifolds over $U$ as follows.

An object in $\mathcal{C}_U$ is a Poisson manifold $(X, \{ , \}_X)$ equipped with commuting left Hamiltonian $H-$action $\phi^-$ and right Hamiltonian $H-$action $\phi^+$ with $U$-valued $Ad_H^*$ equivariant momentum maps $j_{\pm} : X \to U$ satisfying the polarity condition

$$\{j_+^* \varphi, j_-^* \psi\}_X = 0, \quad \text{for all } \varphi, \psi \in C^\infty(U).$$

A morphism in $\mathcal{C}_U$ between $(X, \{ , \}_X)$ and $(X', \{ , \}_{X'})$ is an equivariant Poisson map $\sigma : X \to X'$ such that $j_{\pm} \circ \sigma = j_{\pm}$.

**Definition 2.1.2** [EV] (Poisson groupoid in $\mathcal{C}_U$)

We say that $X \in \mathcal{C}_U$ is a Poisson groupoid in $\mathcal{C}_U$ if it is equipped with a compatible groupoid structure over $U$ such that $\alpha = j_-$, $\beta = j_+$.

**Example 2.1.3** (The Hamiltonian unit)

The most basic (but not simplest) symplectic groupoid in $\mathcal{C}_U$ is the (restricted) Hamiltonian unit $H \times U$ equipped with the nondegenerate bracket

$$\{f, g\}(h, p) = -<D'g, \delta f> + <D'f, \delta g> - <p, [\delta f, \delta g]>,$$

$$<D'f, Z> = \frac{d}{dt}|_{t=0} f(e^{tZ}h, p), \quad <\delta f, \lambda> = \frac{d}{dt}|_{t=0} f(h, p+t\lambda), \quad Z \in \mathfrak{h}, \lambda \in \mathfrak{h}^*$$

the $H-$actions

$$\phi^-_{\mathfrak{h}}(h, p) = (kh, p), \quad \phi^+_{\mathfrak{h}}(h, p) = (hk, Ad_{h}^* p),$$

and the action groupoid structure

$$\alpha_0(h, p) = j_- (h, p) = Ad_{h}^* p, \quad \beta_0(h, p) = j_+ (h, p) = p$$

$$(h, j_-(k, q)) \cdot (k, q) = (hk, q), \quad \epsilon(q) = (1, q), \quad i(h, p) = (h^{-1}, Ad_{h}^* p).$$
If $U = \mathfrak{h}^*$, this is clearly isomorphic to the cotangent symplectic groupoid $T^*H$ [W2] under the trivialization map.

We now recall a fundamental construction of [EV] which interprets dynamical $r-$matrices in terms of Poisson groupoids.

Let $\iota : \mathfrak{h} \rightarrow \mathfrak{g}$ be the Lie inclusion. We say that a smooth map $R : U \rightarrow L(g^*, g)$ (here and henceforth we denote by $L(g^*, g)$ the set of linear maps from $g^*$ to $g$) is a classical dynamical $r-$matrix if and only if it is skew symmetric

$$< R(q)(A), B > = - < A, R(q)B >,$$
and satisfies the classical dynamical Yang-Baxter condition

$$dR(q)\iota^* A B - dR(q)\iota^* B A + \iota d < R(q)A, B >$$
$$- [R(q)A, R(q)B] - R(q)ad_{R(q)(A)}^* B + R(q)ad_{R(q)(B)}^* A = \chi(A, B), \quad (2.1.1)$$

where $\chi : g^* \times g^* \rightarrow g$ is $ad_g-$invariant, i.e.

$$ad_X \chi(A, B) + \chi(ad_X^* A, B) + \chi(A, ad_X^* B) = 0,$$

for all $A, B \in g^*, X \in g$, and all $q \in U$.

The dynamical $r-$matrix is said to be $ad_g^*-$equivariant if and only if

$$dR(q)ad_{\iota(Z)}^* q + R(q)ad_{\iota(Z)}^* A + ad_{\iota(Z)} R(q) = 0, \quad (2.1.2)$$

for all $Z \in \mathfrak{h}$, and all $q \in U$.

Note that if $\chi(A, B) = 0$ in Eqn. (2.1.1), the resulting equation is called the classical dynamical Yang-Baxter equation (CDYBE) [F]. On the other hand, if $\chi(A, B) = [T(A), T(B)]$ for some nonzero symmetric map $T : g^* \rightarrow g$ with $ad_X T + Tad_X^* = 0$, the resulting equation is called the modified dynamical Yang-Baxter equation (mDYBE).

Let $X = U \times G \times U$. For $f \in C^\infty(X)$, define its partial derivatives and left/right gradients (w.r.t. $G$) by

$$< \delta_1 f, \lambda > = \frac{d}{dt}_{|0} f(p + t\lambda, x, q), \quad < \delta_2 f, \lambda > = \frac{d}{dt}_{|0} f(p, x, q + t\lambda), \quad \lambda \in \mathfrak{h}^*$$

$$< Df, X > = \frac{d}{dt}_{|0} f(p, e^{tx} x, q), \quad < Df, X > = \frac{d}{dt}_{|0} f(p, xe^{tx}, q), \quad X \in \mathfrak{g}.$$

We shall equip $X$ with the trivial Lie groupoid structure over $U$ with structure maps

$$\alpha(p, x, q) = p, \quad \beta(p, x, q) = q, \quad \epsilon(q) = (q, 1, q), \quad i(p, x, q) = (q, x^{-1}, p)$$
$$m((p, x, q), (q, y, r)) = (p, xy, r). \quad (2.1.3)$$

The following theorem gives the Poisson groupoid analog of coboundary Poisson Lie groups (in the context of trivial Lie groupoids over $U$).

**Theorem 2.1.4**[EV]
(a) The formula
\[
\{ f, g \}_X(p, x, q) = < p, [\delta_1 f, \delta_1 g] > - < q, [\delta_2 f, \delta_2 g] > \\
- < \iota \delta_1 f, Dg > - < \iota \delta_2 f, D' g > \\
+ < \iota \delta_1 g, Df > + < \iota \delta_2 g, D' f > \\
+ < R(p) Df, Dg > - < R(q) D' f, D' g >
\]
defines a Poisson bracket on \( X \) if and only if \( R : U \rightarrow L(g^*, g) \) is an \( \text{ad}_h^* - \)equivariant dynamical \( \tau - \) matrix.

(b) The trivial Lie groupoid \( X \) equipped with the Poisson bracket \( \{ , \}_X \) in (a) and the Hamiltonian \( H - \) actions
\[
\phi_h^-(p, x, q) = (Ad_{h^{-1}}^* p, hx, q), \quad \phi_h^+(p, x, q) = (p, xh, Ad_h^* q),
\]
is a Poisson groupoid in \( C_U \).

We shall call \( (X, \{ , \}_X) \) a coboundary dynamical Poisson groupoid.

Note that the dynamical Poisson groupoid \( X \) of Thm 2.1.5 admits a Poisson groupoid embedding
\[
I : H \times U \rightarrow X : (h, p) \mapsto (Ad_{h^{-1}}^* p, h, p).
\]
where \( H \times U \) is the Hamiltonian unit. As we shall see in later sections, this property turns out to play a crucial role in the study of duality.

We now recall the notion of a Lie algebroid (for more details see [DSW], [M1]).

**Definition 2.1.5** A Lie algebroid is a smooth vector bundle \( q : A \rightarrow B \) equipped with a Lie bracket \( [, ]_A \) on the set \( \Gamma(A) \) of smooth sections of \( A \) and a smooth base preserving bundle map \( a : A \rightarrow TB \), called the anchor map, such that
\[
a[\zeta, \eta]_A = [a(\zeta), a(\eta)]_B \\
[\zeta, f \eta]_A = f [\zeta, \eta]_A + a(\zeta)(f) \eta,
\]
for all \( \zeta, \eta \in \Gamma(A) \) and all \( f \in C^\infty(B) \).

The Lie algebroid of a smooth groupoid \( \Gamma \) over \( B \) is the vector bundle
\[
A(\Gamma) := (\text{Ker}(T\alpha))_{|\epsilon(B)}
\]
over \( B \) with anchor map \( a \) given by the restriction of \( T[\alpha, \beta] \) to \( A(\Gamma) \) (here \([\alpha, \beta](z) = (\alpha(z), \beta(z)), z \in \Gamma) \) and bracket of sections \([X, Y](b) := [X^l, Y^l]_\Gamma(\epsilon(b)) \) where
\[
X^l : \Gamma \rightarrow \text{Ker}(T\alpha)
\]
is the unique left invariant vector field whose restriction to \( \epsilon(B) \) is \( X \).

Let \( V \) be a vector space and let \( \rho : \Gamma \rightarrow \text{Aut}(V) \) be a smooth groupoid morphism where \( \text{Aut}(V) \) is viewed as a groupoid over its unit element \( I_V \).
Definition 2.1.6 A smooth map $\Sigma : \Gamma \rightarrow V$ is called a groupoid $1$–cocycle iff

$$\Sigma(xy) = \Sigma(x) + \rho(x)\Sigma(y)$$

for all $(x, y) \in \Gamma \ast \Gamma$. The induced map $\Sigma_* : A(\Gamma) \rightarrow V$ defined as the restriction of $T\Sigma$ to $A(\Gamma)$ is called the induced Lie algebroid $1$–cocycle.

Finally, we recall the notion of an action of a Lie groupoid $\Gamma \Rightarrow B$ on a manifold $S$ with moment map $f : S \rightarrow B$. (We use the terminology of [MW].)

Let

$$\Gamma *_f S = \{(x, s) \in \Gamma \times S \mid \beta(x) = f(s)\}$$

$$S *_f \Gamma = \{(s, x) \in S \times \Gamma \mid f(s) = \alpha(x)\}$$

Definition 2.1.7 (a) A left action of $\Gamma$ on $S$ with moment $f$ is a smooth map $\phi^l : \Gamma *_f S \rightarrow S : (x, s) \mapsto x \cdot s$ such that

$$f(x \cdot s) = \alpha(x), \quad y \cdot (x \cdot s) = (yx) \cdot s, \quad \epsilon(f(t)) \cdot t = t,$$

for all $(y, x) \in \Gamma \ast \Gamma$, $(x, s) \in \Gamma *_f S, t \in S$.

(b) A right action of $\Gamma$ on $S$ with moment $f$ is a smooth map $\phi^r : S *_f \Gamma \rightarrow S : (s, x) \mapsto s \cdot x$ such that

$$f(s \cdot x) = \beta(x), \quad (s \cdot x) \cdot y = s \cdot (xy), \quad t \cdot \epsilon(f(t)) = t,$$

for all $(x, y) \in \Gamma \ast \Gamma, (s, x) \in S *_f \Gamma, t \in S$.

2.2. Trivial Lie groupoids in $C_U$.

Our purpose in this subsection is to provide an explicit class of Poisson brackets on trivial Lie groupoids which extends the construction of thm 2.1.4 (a), and is essential for our subsequent study of duality.

We assume that the Lie subgroup $H \subset G$ is connected. We begin with a general property.

Proposition 2.2.1 Let $Y$ be a Poisson groupoid over $U$ with source and target maps $\alpha$ and $\beta$ and unit map $\epsilon$. If there exists a (base preserving) Poisson groupoid morphism

$$I : H \times U \rightarrow Y,$$

(here $H \times U$ is the Hamiltonian unit) then $Y$ belongs to $C_U$.

Proof. It follows from a general property of Poisson groupoids [W1] that

$$\{\alpha^*\varphi, \beta^*\psi\}_Y = 0, \forall \varphi, \psi \in C^\infty(U).$$

So it remains to show that $Y$ admits two commuting Hamiltonian $H$– actions $\phi^-, \phi^+$ with equivariant momentum maps $\alpha$ and $\beta$. 
Set, as in Def. 2.1.7,
\[(H \times U) \ast_{\alpha} Y = \{(h, p, y) \mid \beta_0(h, p) = \alpha(y)\}\]
\[Y \ast_{\beta} (H \times U) = \{(y, h, p) \mid \beta(y) = \alpha_0(h, p)\}\]

Here \(\alpha_0, \beta_0\) are as in Example 2.1.3.
The morphism \(I\) induces a left (resp. right) groupoid action of \(H \times U\) on \(Y\) over \(\alpha\) (resp. over \(\beta\)):
\[\phi^{-} : (H \times U) \ast_{\alpha} Y \to Y\]
\[(h, \alpha(y), y) \mapsto I(h, \alpha(y)) \cdot y\]
\[\phi^{+} : Y \ast_{\beta} (H \times U) \to Y\]
\[(y, h, \text{Ad}_{h}^{\beta}(y)) \mapsto y \cdot I(h, \text{Ad}_{h}^{\beta}(y))\]

which, upon the natural identifications
\[(H \times U) \ast_{\alpha} Y \simeq H \times Y, Y \ast_{\beta} (H \times U) \simeq Y \times H,\]
induce a left and a right action of \(H\) on \(Y\) (also denoted \(\phi^{\pm}\)).

We now show that \(\phi^{-}\) is Hamiltonian with momentum map \(\alpha\). (The verification for \(\phi^{+}\) and \(\beta\) is similar.)

Note that \(\alpha\) is equivariant since \(\alpha(\phi_{\alpha}^{-}(y)) = \alpha(I(k, \alpha(y)) \cdot y) = \alpha(I(k, \alpha(y))) = \alpha_0(k, \alpha(y)) = \text{Ad}_{k}^{\alpha^{-1}} \cdot \alpha(y)\).

Let \(Z^{-}(y) = \frac{d}{dt}\big|_{0} \phi_{\epsilon, x}^{-}(y)\) be the infinitesimal generator of the action corr. to \(Z\).

We want to show \(Z^{-}\) coincides with the Hamiltonian vector field \(\hat{X}_{f_{Z\alpha}}\) where \(f_{Z} \in C^\infty(U)\) is defined by \(f_{Z}(q) = \langle Z, q \rangle, \forall q \in U\).

Since \(I(1, q) = \epsilon(q)\), we have
\[Z^{-}(y) = \frac{d}{dt}\big|_{0} I(e^{tZ}, \alpha(y)) \cdot y = T_{\epsilon, \alpha(y)} T_{1, \alpha(y)} I(Z, 0),\]
which shows that \(Z^{-} \in \ker T_{\beta}.\) On the other hand,
\[Z^{-}(\epsilon \circ \alpha(y)) = \frac{d}{dt}\big|_{0} I(e^{tZ}, \alpha(y)) \cdot I(1, \alpha(y))\]
\[= \frac{d}{dt}\big|_{0} I(e^{tZ}, \alpha(y))\]
\[= T_{(1, \alpha(y))} I(Z, 0),\]
thus \(Z^{-}\) is right invariant. Since \(\hat{X}_{f_{Z\alpha}}\) is also right invariant \([X]\), it suffices to show that both vector fields coincide on \(\epsilon(U)\). But from the Poisson property of \(I\), we have
\[\hat{X}_{f_{Z\alpha}}(\epsilon \circ \alpha(y)) = \Pi_{Y}^{\#} (\epsilon \circ \alpha(y)) d(f_{Z} \circ \alpha)\]
\[= T_{(1, \alpha(y))} I \Pi_{0}^{\#} (1, \alpha(y)) d(f_{Z} \circ \alpha \circ I)\]
\[= T_{(1, \alpha(y))} I \Pi_{0}^{\#} (1, \alpha(y)) d(f_{Z} \circ \alpha_{0})\]
\[= T_{(1, \alpha(y))} I \Pi_{0}^{\#} (1, \alpha(y)) (ad_{Z}^{\alpha}(y), Z)\]
\[= T_{(1, \alpha(y))} I (Z, 0).\]
Now, since $Z^-$ is Hamiltonian, its flow $\phi_{\tau z}$ at $t = 1$ preserves the Poisson bracket of $Y$. Therefore, the connectedness of $\tilde{H}$ implies that $\phi^-$ is Hamiltonian with momentum map $\alpha$. Hence the claim.

For the rest of this subsection, we let $X = U \times G \times U$ be the trivial Lie groupoid of section 2.1 (see eqn (2.1.3)). We shall describe all pairs

$$(X, \{, \}), \quad I : H \times U \to X,$$

where $\{ , \}$ is a Poisson bracket on $X$ compatible with its groupoid structure and $I$ is a morphism of Poisson groupoids, where $H \times U$ is the Hamiltonian unit.

Let $\rho : G \to Aut(V)$ be a representation of $G$ on the vector space $V$. We are going to restrict ourselves to groupoid 1 cocycles $P : X \to V$ which satisfy

$$P(p, xy, q) = P(p, x, r) + I(p)P(r, y, q)$$

for all $p, q, r \in U, x, y \in G$.

**Proposition 2.2.2** $P$ is a 1-cocycle on $X$ iff

$$P(p, x, q) = -l(p) + \pi(x) + \rho(x)l(q),$$

for some smooth map $l : \mathfrak{h}^* \to V$ with $l(q_0) = 0$ for some $q_0 \in U$, and a group 1-cocycle $\pi : G \to V$.

Proof. Clearly any such map is a 1-cocycle. Conversely, if $P$ is a cocycle then $P(p, 1, p) = P(p, 1, q_0) + P(q_0, 1, p) = 0$ and $\pi(x) = P(q_0, x, q_0)$ is a group cocycle. The claim then follows from $P(p, x, q) = P(p, x, q_0) + \rho(x)P(q_0, 1, q) = P(p, 1, q_0) + P(q_0, x, q_0) + \rho(x)P(q_0, 1, q) = -P(q_0, 1, p) + \pi(x) + \rho(x)P(q_0, 1, q)$.

**Proposition 2.2.3** Let $\Pi \in \wedge^2 TX$ be a bivector field. Then the graph of $m : X \times X \to X$ is $\Pi$-coisotropic in $X \times X \times \overline{X}$ iff

$$\Pi^#(p, x, q)(Z_1, B, Z_2) = (-K(p)Z_1 - A^*(p)T^*_1 r_x B, T_1 r_x A(p)Z_1 + T_1 l_x A(q)Z_2 + T_1 r_x P(p, x, q)T^*_1 r_x B, K(q)Z_2 - A^*(q)T^*_1 l_x B),$$

for some smooth maps $K : U \to L(\mathfrak{h}, \mathfrak{h}^*)$, $A : U \to L(\mathfrak{h}, \mathfrak{g})$, and a groupoid 1-cocycle $P : X \to L(\mathfrak{g}^*, \mathfrak{g})$ for the adjoint action. Here, $K$ and $P$ are pointwise skew-symmetric.

Proof. The graph of the multiplication $m$ is

$$Gr(m) = \{((p, x, q), (q, y, r), (p, xy, r))\} \subset X \times X \times \overline{X}.$$

Therefore, $\Omega \in (T_{((p, x, q), (q, y, r), (p, xy, r))}Gr(m))^\perp$ if and only if

$$\Omega = ((Z_1, \omega, Z_2), (-Z_2, T^*_y (r_{y-1} \circ l_x)\omega, Z_3), (-Z_1, -T^*_{xy} r_{y-1} \omega, -Z_3)), $$

for some $Z_1, Z_2, Z_3 \in h$ and $\omega \in (T_2 G)^*$. 


One then verifies (see the appendix for the details) that the $\Pi$–coisotropy of $Gr(m)$:

$$(\Pi \oplus \Pi \oplus -\Pi)(\Omega, \Omega') = 0, \quad \forall \Omega, \Omega' \in (TGr(m))^\perp,$$

is equivalent to our assertion. ■

Now, a map $I : H \times U \to X$ is a (base preserving) groupoid morphism iff

$$I(k, q) = (Ad_k^* q, \chi(k, q), q) \quad (2.2.1)$$

for some smooth map $\chi$ satisfying

$$\chi(hk, q) = \chi(h, Ad_k^* q) \chi(k, q), \quad (2.2.2)$$

in particular $\chi(1, q) = 1$, and if $0 \in U$, the map $H \to G : k \mapsto \chi(k, 0)$ is a group morphism.

Note that Eqn. (2.2.2) says that $\chi$:

$$H \times U \to G$$

is a groupoid morphism when $G$ is viewed as a groupoid over its unit element. Applying the Lie functor to $\chi$ then provides an algebroid morphism

$$A(\chi) : U \times \mathfrak{h} \to \mathfrak{g}$$

$$(q, Z) \mapsto T_{(1, q)} \chi(Z, ad_Z^* q),$$

which we shall henceforth denote as $(q, Z) \mapsto A_\chi(q)Z.$ The morphism property then says that for all $Z, Z' \in \mathfrak{h}, p \in U,$

$$A_\chi(p)[Z, Z'] = dA_\chi(p) \cdot ad_Z^* p \cdot Z' - dA_\chi(p) \cdot ad_{Z'}^* p \cdot Z$$

$$+ [A_\chi(p)Z, A_\chi(p) Z'].$$

Proposition 2.2.4 If $(X, \Pi)$ is a Poisson groupoid with $\Pi^#$ expressed as in Prop. 2.2.3 above, then the map $I$ is a Poisson map iff

(a) $K(p)Z = ad_Z^* p, \quad \forall Z, Z' \in \mathfrak{h}.$

(b) $A_\chi(p) = A(p)$

(c) For all $\alpha, \beta \in \mathfrak{g}^*,$

$$< \alpha, P(I(h, p)) \beta > = < \lambda_\alpha, Z_\beta > = < \lambda_\alpha, Z_\beta > - < \lambda_\beta, Z_\alpha > - < p, [Z_\alpha, Z_\beta] >,$$

where $\lambda_\alpha \in \mathfrak{h}^*$ and $Z_\alpha \in \mathfrak{h}$ are defined by

$$< \lambda_\alpha, Z > = < \alpha, T(h, p)(r(\chi(h, p)))^{-1} \circ \chi)(T_1 l_h Z, 0) >, \quad \forall Z \in \mathfrak{h},$$

$$< Z_\alpha, \lambda > = < \alpha, T(h, p)(r(\chi(h, p)))^{-1} \circ \chi)(0, \lambda) >, \quad \forall \lambda \in \mathfrak{h}^*.$$

Proof. This is a spelled out form of $\{I^* f, I^* g\}_{H \times U} = I^* \{f, g\}_X.$ ■

Assembling the above propositions, we can now formulate the main assertion of this subsection.

Theorem 2.2.5 If $(X, \{ , \})$ is a Poisson groupoid which admits a Poisson groupoid morphism $I : H \times U \to X$ as in Eqn. (2.2.1), then,
(a) The Poisson bracket must be of the form
\[
\{f, g\}(p, x, q) = \langle p, [\delta_1 f, \delta_1 g] \rangle - \langle q, [\delta_2 f, \delta_2 g] \rangle
\]
\[
- \langle A_\chi(p)\delta_1 f, Dg \rangle - \langle A_\chi(q)\delta_2 f, D'g \rangle
\]
\[
+ \langle A_\chi(p)\delta_1 g, Df \rangle + \langle A_\chi(q)\delta_2 g, D'f \rangle
\]
\[
+ \langle Df, P(p, x, q)Dg \rangle,
\]
where the groupoid 1-cocycle $P$ satisfies Prop. 2.2.4 (c).

(b) The Jacobi identity for \{ , \} is equivalent to the condition
\[
\langle \beta, [P\alpha, P\gamma] \rangle - \langle \beta, DP \cdot P\alpha \cdot \gamma \rangle + \langle \beta, \delta_1 P \cdot (A_\chi(p)^* \alpha) \cdot \gamma \rangle
\]
\[
+ \langle \beta, \delta_2 P \cdot (A_\chi(q)^* Ad_\chi^* \alpha) \cdot \gamma \rangle + cp(\alpha, \beta, \gamma) = 0
\]
for all $\alpha, \beta, \gamma \in \mathfrak{g}^*$, where $P$ stands for $P(p, x, q)$ and
\[
\delta_1 P(\lambda) = \frac{d}{dt}|_0 P(p + t\lambda, x, q), \delta_2 P(\lambda) = \frac{d}{dt}|_0 P(p, x, q + t\lambda),
\]
\[
DP \cdot X = \frac{d}{dt}|_0 P(p, e^{tx} x, q).
\]

(c) $(X, \{, \})$ belongs to $\mathcal{C}_U$ with Hamiltonian $H-$ actions
\[
\phi^- : H \times X \rightarrow X
\]
\[
(h, p, x, q) \mapsto (Ad_{h^{-1}}^* p, \chi(h, p)x, q)
\]
\[
\phi^+ : X \times H \rightarrow X
\]
\[
(p, x, q, h) \mapsto (p, x\chi(h, Ad_h^* q), Ad_h^* q).
\]

Proof. (a) This is simply a restatement of Prop. 2.2.3 and Prop. 2.2.4.

(b) We give the main steps (see the appendix for the details of the calculations):

For the Jacobi identity, we use the shorthand notation $J_{ijk}, i, j, k \in \{1, 2, *\}$ to stand for $\{p_i^* \phi, \{p_j^* \phi', p_k^* \psi\}\} + c.p.$, where, as an index, $* = G$, and $p_1, p_G, p_2$ are the projections from $X = U \times G \times U$ onto the first, second, and third factor of $X$ respectively. Thus for example $J_{12*} = \{p_1^* \phi, \{p_2^* \phi', p_G^* \psi\}\} + c.p.$.

Clearly, we have $J_{ijk} = 0$ for $i, j, k \in \{1, 2\}$ and $J_{112} = 0$ and these do not impose any conditions. On the other hand, $J_{*11} = 0 \iff J_{*22} = 0 \iff A_\chi$ satisfies (2.2.3).

Writing $P(p, x, q) = -l(p) + \pi(x) + Ad_x l(q) Ad_x^*$ as in Prop. 2.2.2, we have
\[
J_{*11} = 0 \iff J_{*22} = 0 \iff \langle \alpha, (dl(p)ad_Z^* p + ad_{A_\chi(p)}Z)l(p) + l(p)ad_{A_\chi(p)}Z + d\pi(1)A_\chi(p)Z\beta \rangle > 0
\]
\[
= + \langle dA_\chi(p)(A_\chi(p)^* \alpha)Z, \beta \rangle - \langle dA_\chi(p)(A_\chi(p)^* \beta)Z, \alpha \rangle, \forall \alpha, \beta \in \mathfrak{g}^*.
\]

But the latter follows upon differentiating the identity of Prop. 2.2.4 (c) at $(1, p)$. Indeed, that $\text{lhs}(\ast) = \frac{d}{dt}|_0 < \alpha, P \circ I(e^{tZ}, p)\beta >$ is clear. On the other hand, upon
using
\[ <\lambda, Z_\alpha(1,p)> <\alpha, \frac{d}{dt}|_0 \chi(1,p+t\lambda)\chi(1,p)^{-1}> = 0 \]
\[ <\lambda_\alpha(1,p), Z> <\alpha, \frac{d}{dt}|_0 \chi(e^{tZ},p)\chi(1,p)^{-1}> = <A_\chi(p)\alpha, Z>, \]
we have
\[ \frac{d}{dt}|_0 ( <\lambda_{\alpha}, Z_\beta>(e^{tZ},p) - <\lambda_\beta, Z_\alpha>(e^{tZ},p) - <p, [Z_\alpha(e^{tZ},p), Z_\beta(e^{tZ},p)]> ) \]
\[ = <A_\chi(p)^*\alpha, \frac{d}{dt}|_0 Z_\beta(e^{tZ},p) > - <A_\chi(p)^*\beta, \frac{d}{dt}|_0 Z_\alpha(e^{tZ},p) > \]
\[ = rhs(\ast). \]

Thus, with our assumptions, the bracket \{ , \} satisfies the Jacobi identity if and only if \( J_{***} = 0 \) which is precisely (b).

(c) This is Prop. 2.2.1: The morphism \( I : H \times U \to X \) induces a left groupoid action of \( H \times U \) on \( X \) over \( \alpha_X \)

\[ \phi^- : (H \times U)^{\alpha_X} \to X : ((h,p),(p,x,q)) \mapsto I(h,p) \cdot (p,x,q) \]

and a right groupoid action on \( X \) over \( \beta_X \)

\[ \phi^+ : X^{\beta_X} (H \times U) \to X : ((p,x,q),(k,Ad_k^*q)) \mapsto (p,x,q) \cdot I(k,Ad_k^*q). \]

With the identifications \((H \times U)^{\alpha_X} \simeq H \times X \) and \( X^{\beta_X} (H \times U) \simeq X \times H \),

these actions are the ones given above. ■.

We end this subsection with a definition and some remarks.

**Definition 2.2.6**

Following Etingof and Varchenko, we shall say that the Poisson groupoid \((X, \{ , \})\) of Theorem 2.2.5 is of **dynamical** type iff \( I(k,q) = (Ad_k^{-1}q, k, q) \). In this case, the corresponding Lie algebroid dual \( A(X)^* \) (which is a Lie algebroid) will be called a dynamical Lie algebroid.

**Remarks 2.2.7**

(a) For \( X \) of dynamical type, \( \chi(h,p) = \chi(h,1) = h \), thus \( A_\chi(p)Z = Z \). Therefore, the first six terms of the Poisson bracket coincide with those of the coboundary case. The last term however, which is given by

\[ P(p,x,q) = -l(p) + \pi(x) + A_x l(q)Ad_x^* \]

differs from the coboundary case by the group 1- cocycle \( \pi : G \to L(\mathfrak{g}^*, \mathfrak{g}) \).

As we shall demonstrate in section 4 for a class of solutions of the modified dynamical Yang-Baxter equation on simple Lie algebras, Poisson groupoids of dynamical type with \( \pi \neq 0 \) arise naturally as Poisson groupoid duals (modulo Poisson groupoid isomorphisms) of certain coboundary dynamical groupoids. Note that the situation
here is analogous to that for Poisson Lie groups: Typically, the Poisson Lie group
dual of a Lie group equipped with the Sklyanin bracket is not of coboundary type.
(b) For $X$ of dynamical type, the identity of Prop. 2.2.4 (c) simplifies to
\[ P(Ad^*_{h^{-1}} p, h, p) = 0 \]
In other words $P$ vanishes on the $H-$ orbit of $\epsilon(U) \subset X$. This condition is the
natural extension of the $h-$ equivariance of the dynamical $r-$ matrix which it reduces to when $\pi(x) = 0$.

3. Duality.

The purpose of this section is to characterize the Poisson groupoid dual to a dy-
namical Poisson groupoid $X = U \times G \times U$ where $U \subset h^*$ is an
$Ad^*_{h^{-1}}$ invariant contractible open set, and study some derived duality diag-
rams for Poisson quotients.

3.1. Duality of Poisson groupoids.

Following [W1], [M2], [MX2], we begin by recalling the notion of duality of Poisson
groupoids and the definition of the dual (when it exists) of a Poisson groupoid.
Let $(Y, \{, \}_Y)$ be a Poisson groupoid over $B$ with target and source maps $\alpha, \beta,$ and
unit map $\epsilon$. We use, as above, the sign convention $\{f, g\}_Y = \langle df, \Pi^# dg \rangle$.
Since $Y$ is Poisson, the set of 1- forms $\Omega^1(Y)$ inherits a Lie bracket from $C^\infty(Y)$ [W1], [KSM], given by
\[ [\omega, \omega'] = - L_{\Pi^# \omega} \omega' + L_{\Pi^# \omega'} \omega - d < \omega, \Pi^# \omega' >, \quad (3.1.1) \]
and the map
\[ \Omega^1(Y) \longrightarrow \mathfrak{X}(Y) : \omega \mapsto -\Pi^# \omega \]
is a morphism of Lie algebras, where $\mathfrak{X}(Y)$ is the set of vector fields on $Y$ with
ordinary Lie bracket. Therefore, $T^*Y$ is a Lie algebroid over $Y$.
Now, it follows from a general result in [W1] that the unit submanifold $\epsilon(B)$ of the
Poisson groupoid $Y$ is coisotropic in $Y$, hence its conormal bundle
\[ N^*(\epsilon(B)) := \bigcup_{q \in B} (T_{\epsilon(q)} \epsilon(B))^\perp \subset T^*Y|_{\epsilon(B)} \]
hits a Lie algebroid structure: the bracket of two sections $\theta_1, \theta_2 : B \longrightarrow N^*(\epsilon(B))$ is
\[ [\theta_1, \theta_2]_{N^*(\epsilon)}(q) = [\overline{\theta}_1, \overline{\theta}_2]\epsilon(q) \quad (3.1.2) \]
for arbitrary $\overline{\theta}_1, \overline{\theta}_2 \in Ker \epsilon^*$ subject to $\overline{\theta}_1 \circ \epsilon = \theta_1, \overline{\theta}_2 \circ \epsilon = \theta_2$, while the anchor map $a_* : N^*(\epsilon(B)) \rightarrow TB$ is given by the restriction of $-\Pi^#$ to $N^*(\epsilon(B))$.
Since we have a natural identification $N^*(\epsilon(B)) \simeq A(Y)^*$, we shall therefore always
take $A(Y)^*$ with the induced Lie algebroid structure and the pair $(A(Y), A(Y)^*)$
will be called the tangent Lie bialgebroid of \((Y,\{,\})\). For the precise definition of Lie bialgebroids see [MX1]; note however that \((A,A^*)\) is a Lie bialgebroid if and only if \((A^*,A)\) is.

**Definition 3.1.1** [M2] We shall say that two Poisson groupoids \(Y\) and \(Y'\) over the same base are in duality if and only if the Lie bialgebroids \((A(Y),A(Y)^*)\) and \((A(Y')^*,A(Y')^-)\) are isomorphic. Here, \(A(Y)^-\) is obtained from \(A(Y)\) by changing the sign of both anchor and bracket of sections.

Note that if the Lie algebroid \(A(Y)^*\) is integrable, then there exists (by Lie I) a unique source-simply connected Lie groupoid \(Y^*\) integrating \(A(Y)^*\). In this case, it follows from a general theorem of MacKenzie and Xu [MX2] that the latter may be equipped with a unique Poisson bracket \(\{,\}_{Y^*}\) compatible with its groupoid structure such that \((Y^*,\{,\}_{Y^*})\) has tangent Lie bialgebroid \((A(Y)^*,A(Y))\).

The Poisson groupoids \(Y\) and \(Y^*\) are Poisson groupoids in duality and \((Y^*,\{,\}_{Y^*})\) is called the dual of \((Y,\{,\}_Y)\).

The following example is already in [W1]:

**Example 3.1.2** The Poisson groupoid dual to the Hamiltonian unit \(H \times U\) of Example 2.1.3 is the coarse groupoid \(U \times U\) with Poisson bracket

\[
\{f,g\}_{U \times U}(p,q) = <p,[\delta_1 f,\delta_1 g]> - <q,[\delta_2 f,\delta_2 g]>
\]

Note that \(U \times U\) belongs to \(C_U\) with the Hamiltonian \(H\)– actions

\[
\phi^-_k(p,q) = (Ad^*_k, p, q), \quad \phi^+_k(p,q) = (p, Ad_k^* q).
\]

The associated tangent Lie bialgebroid is given by

\[
(A(H \times U), A(U \times U)) = (U \times h, U \times h^*)
\]

and the respective Lie brackets on smooth sections are as follows

\[
[Z_1, Z_2](q) = [Z_1(q), Z_2(q)]_h + dZ_2(q)ad^*_Z_1(q)dZ_2(q)q - dZ_1(q)ad^*_Z_2(q)q, \quad Z_1, Z_2 : U \to h
\]

\[
[X_1, X_2](q) = dX_2(q)X_1(q) - dX_1(q)X_2(q), \quad X_1, X_2 : U \to h^*.
\]

We now recall a special (and simplest) instance of Lie bialgebroid morphisms. Let \((A,A^*)\) and \((A',(A')^*)\) be two Lie bialgebroids over \(B\) with bundle projections \(q : A \to B, q_* : A^* \to B\), anchors \(a : A \to TB, a_* : A^* \to TB\), and similarly for \((A',(A')^*)\).

**Definition 3.1.3**

A bundle map

\[
\phi : A \to A', \quad q \searrow \searrow q' \nearrow \nearrow
\]

\[
B
\]
is called a Lie bialgebroid morphism if and only if both $\phi$ and $\phi^*$ are Lie algebroid morphisms, i.e.

\[
\begin{align*}
  a' \circ \phi &= a, & \phi[X, Y]_A &= [\phi(X), \phi(Y)]_{A'} \\
  a_* \circ \phi^* &= a'_*, & \phi^*[\alpha', \beta']_{(A')} &= [\phi^*(\alpha'), \phi^*(\beta')]_{A'}.
\end{align*}
\]

for all sections $X, Y : B \to A$, $\alpha', \beta' : B \to (A')^*$.

The following property [MX1] is basic.

**Proposition 3.1.4** Let $Y, Y'$ be Poisson groupoids over $B$ with tangent Lie bialgebroids $(A, A^*)$ and $(A', (A')^*)$. If $\mu : Y \to Y'$ is a base preserving morphism of Poisson groupoids, then

\[
A(\mu) : A \to A'
\]

is a (base preserving) morphism of Lie bialgebroids.

### 3.2. The dual of a dynamical Poisson groupoid

Throughout this subsection, we shall equip $X = U \times G \times U$ with the trivial Lie groupoid structure and we assume that the subgroup $H \subset G$ is connected and simply connected. We begin with a description of the tangent Lie bialgebroid of a Poisson groupoid $(X, \{ , \})$, a special instance of which is described in [BKS].

**Proposition 3.2.1** Let $(X, \{ , \})$ be a Poisson groupoid with Poisson bracket as in Proposition 2.2.3. Then the Lie bialgebroid tangent to $X$ is (isomorphic to) the pair $(U \times h^* \times g, U \times h \times g^*)$ with anchor maps

\[
\begin{align*}
  a & : U \times h^* \times g \to U \times h^* : (q, \lambda, X) \mapsto (q, \lambda) \\
  a_* & : U \times h \times g^* \to U \times h^* : (q, Z, B) \mapsto (q, -K(q)Z + A^*(q)B)
\end{align*}
\]

and Lie brackets of sections $[ , ]$, $[ , ]_s$ given by the following expressions

\[
[(\lambda, X), (\lambda', X')]_s(q) = (d\lambda' \cdot \lambda - d\lambda \cdot \lambda', dX' \cdot \lambda - dX \cdot \lambda' + [X(q), X'(q)]_g)
\]

\[
\begin{align*}
\langle (Z, B), (Z', B') \rangle_s, (\Lambda, Y) & > (q) = \\
\langle -dZ'(K(q)Z - A^*(q)B) + dZ(K(q)Z' - A^*(q)B'), \Lambda \rangle > \\
- \langle Z', dK(q)(\Lambda)Z \rangle > \\
- \langle B, \delta_1 P(\Lambda)B' \rangle > - \langle B', dA(q)(\Lambda)Z \rangle + \langle B, dA(q)(\Lambda)Z' \rangle > \\
+ \langle dB(K(q)Z' - A^*(q)B') - dB'(K(q)Z - A^*(q)B), Y \rangle > \\
+ \langle ad_{A(q)}^*ZB' - ad_{A(q)}^*ZB, Y \rangle > \\
+ \langle B, \partial P(Y)B' \rangle >,
\end{align*}
\]

where $\lambda, \lambda' : U \to h^*$; $X, X' : U \to g$; $Z, Z' : U \to h$; $B, B' : U \to g^*$ are smooth maps, $\Lambda \in h^*$, $Y \in g$, all differentials and sections are evaluated at $q$, and the partial derivatives of the groupoid 1-cocycle $P$ (see Thm. 2.2.5) are evaluated at $(q, 1, q)$. 

Proof: The Lie algebroid $A(X)$ is well known (see, for example, [M1]). Although for the dynamical coboundary case an algebraic description of the Lie algebroid dual $A(X)^*$ was given in [BKS], it was not derived there from the Poisson groupoid using Weinstein’s coisotropic calculus. So we shall briefly indicate the steps of the calculation.

The unit section of $X$ is given by $\epsilon : U \to U \times G \times U : q \mapsto (q,1,q)$. Therefore, $\gamma \in N(\epsilon(U))_q$ if and only if $\gamma = (Z,B,Z)$, for some $Z \in \mathfrak{h}$ and $B \in \mathfrak{g}^*$. Let $\alpha, \alpha' : U \to N(\epsilon(U))$ be two sections written as $\alpha(q) = (Z(q),B(q),Z(q))$ and $\alpha'(q) = (Z'(q),B'(q),Z'(q))$. Set $\omega(p,x,q) = (Z(q),T_x^*l_{Z-B}(q),Z(q))$, and similarly for $\omega'$. By (3.1.2), it suffices to compute

1) to compute $[\alpha, \alpha']_{N(\epsilon(U))}(q) = [\omega, \omega'](q,1,q)$, where the rhs is given by (3.1.1) with Hamiltonian operator

$$\Pi^#(p,x,q)(Z_1,B,Z_2) = (-K(p)Z_1 - A^*(p)T_x^*B, T_1T_x^*A(p)Z_1 + T_1T_x^*A(q)Z_2 + T_1T_x^*P(p,x,q)T_x^*B, K(q)Z_2 - A^*(q)T_x^*B),$$

and

2) to choose an identification of $N(\epsilon(U))$ with $U \times \mathfrak{h} \times \mathfrak{g}^*$.

The computation for (1) is rather standard (although somewhat lengthy) and may be performed with the help of

$$< L_{\Pi^#\omega'}, (\Lambda, X^l, \Lambda') > = d < \omega', (\Lambda, X^l, \Lambda') > \cdot \Pi^#\omega + < \omega', [(\Lambda, X^l, \Lambda'), \Pi^#\omega] >,$$

where $\Lambda, \Lambda' \in \mathfrak{h}^*, X \in \mathfrak{g}$, and $X^l$ is the left invariant vector field on $G$ with $X^l(1) = X$.

As for 2) the natural identification to make is given by $\iota_- : N(\epsilon(U))_q \to \mathfrak{h} \times \mathfrak{g}^* : (Z,B,Z) \mapsto (Z,B)$. Setting $[(Z,B),(Z',B')]_*(q) = \iota_-[\omega, \omega'](q,1,q) then gives the stated formula.\

Remark 3.2.2 The coboundary dynamical case in [BKS] corresponds to the choice: $\chi(h,q) = h$, and the groupoid 1− cocycle

$$P(p,x,q) = -R(p) + Ad_xR(q)Ad_x^*.$$

Note that $A(q)Z = A_\chi(q)Z = \iota Z$ is constant in this case, while the induced algebroid 1− cocycle defined as

$$P_\chi : U \times \mathfrak{h}^* \times \mathfrak{g} \to L(\mathfrak{g}^*, \mathfrak{g})$$

$$(q,\Lambda,Y) \mapsto (\partial P \cdot Y + \delta_2 P \cdot \Lambda)(q,1,q) = (\partial P \cdot Y - \delta_1 P \cdot \Lambda)(q,1,q)$$

which appears in the bracket of Prop. 3.2.1 is given by

$$P_\chi(q,\Lambda,Y) = dR(q)\Lambda + R(q)ad_\gamma^* + ad_YR(q).$$
The proposition which follows is a special instance of the functorial relationship between Poisson groupoids and Lie bialgebroids.

**Proposition 3.2.3** Let \((X,\{ , \})\) be a Poisson groupoid with Poisson bracket as in Prop. 2.2.3 and let

\[
I : H \times U \longrightarrow U \times G \times U
\]

\[
(h,p) \mapsto (Ad_{h^{-1}}^*p, \chi(h,p), p)
\]

be a groupoid morphism. If \(A(I)\) is a morphism of Lie bialgebroids, then \(I\) is also a Poisson map. Hence \(I\) is a Poisson groupoid morphism.

Proof. We have to show that the three conditions of Prop. 2.2.4 are satisfied. From the definition of \(I\), it is clear that the induced morphism \(A(I) : U \times \mathfrak{h} \longrightarrow U \times \mathfrak{g} \times \mathfrak{h}^*\) is given by

\[
(q,Z) \mapsto (q, A_\chi(q)(Z), ad_Z^*q),
\]

so its dual map \(A(I)^* : U \times \mathfrak{g}^* \times \mathfrak{h} \longrightarrow U \times \mathfrak{h}^*\) is of the form

\[
(q,B,Z) \mapsto (q, A_\chi^*(q)B - ad_Z^*q),
\]

where \(A_\chi(q)(Z) = A(\chi)(q,Z)\). Thus \(A(I)^*\) preserves the anchor maps if and only if

\[
K(q)(Z) = ad_Z^*q \quad \text{and} \quad A^*(q) = A_\chi^*(q),
\]

while a direct calculation shows that

\[
A(I)^*[\{Z,B\},\{Z',B'\}]_* = [A(I)^*(Z,B),A(I)^*(Z',B')]_{U \times \mathfrak{h}^*}
\]

is equivalent to

\[
\langle B, \partial P(A_\chi(Z))B' - \delta_1 P(ad_Z^*q)B' \rangle = \langle B', dA_\chi(A_\chi^*(B))Z > - \langle B, dA_\chi(A_\chi^*(B'))Z \rangle .
\]

Clearly, the anchor conditions are precisely the conditions (a) and (b) of Prop. 2.2.4. Therefore, it remains to verify condition (c). To this end, let \(\rho : G \longrightarrow Aut(L(\mathfrak{g}^*, \mathfrak{g}))\) be the adjoint action and set

\[
\rho_\chi : H \times U \longrightarrow Aut(L(\mathfrak{g}^*, \mathfrak{g}))
\]

\[
(h,p) \mapsto \rho(\chi(h,p)).
\]

Then \(\rho_\chi\) is a groupoid representation. We begin by showing that both sides of condition 2.2.4 (c) are groupoid 1–cocycles for \(\rho_\chi\) (see Def. 2.1.6).

That the left hand side \(P \circ I : H \times U \longrightarrow L(\mathfrak{g}^*, \mathfrak{g})\) is a groupoid 1–cocycle for \(\rho_\chi\) is immediate from the fact that \(P : U \times G \times U \longrightarrow L(\mathfrak{g}^*, \mathfrak{g})\) is such a cocycle for \(\rho\).

As for the right hand side, we have to show that the map \(\Sigma : H \times U \longrightarrow L(\mathfrak{g}^*, \mathfrak{g})\) defined by

\[
\langle \alpha, \Sigma(h,p) \beta \rangle = \langle \lambda_\alpha, Z_\beta \rangle - \langle \lambda_\beta, Z_\alpha \rangle - \langle p, [Z_\alpha, Z_\beta] \rangle
\]

satisfies

\[
\Sigma(hk,p) = \Sigma(h, Ad_{k^{-1}}^*p) + Ad_\chi(h, Ad_{k^{-1}}^*p)\Sigma(k,p)Ad_\chi^*(h, Ad_{k^{-1}}^*p).
\]
But this follows by a direct calculation which makes successive use of the following three identities

\[
< Z_\alpha(hk, p), \lambda > = < Z_\alpha(h, Ad_k^{-1}p), Ad_k^{-1} \lambda > + < Z_{Ad_k^{-1} Ad_k^{-1}p} \lambda > + < Z_{Ad_k^{-1} Ad_k^{-1}p} \lambda > \\
< \lambda_\alpha(hk, p), Z > = < Ad_k \lambda_\alpha(h, Ad_k^{-1}p), Z > \\
< \lambda_\alpha(hk, p), Z > = < Ad_k^{-1} \lambda_\alpha(h, Ad_k^{-1}p), Z > + < p, [Ad_k^{-1} Z_\alpha(h, Ad_k^{-1}p), Z] >
\]

Now, since \( H \) is simply connected, by Prop. 7.3 of [X] the two groupoid 1-coycles \( P \circ I \) and \( \Sigma \) coincide iff their induced algebroid cocycles are the same. But the latter is equivalent to (**) above. This concludes the proof. ■

Recall that a Lie algebroid \( A \) over a (connected) base \( B \) is said to be transitive iff its anchor map \( a: A \to TB \) is a surjective submersion. In this case the Kernel \( Ker a \) of \( a \) is a Lie algebra bundle [M1] called the adjoint bundle of \( A \) whose fibers are called the vertex (or isotropy) Lie algebras \( \mathfrak{k} \) of \( A \). If \( A \) is a transitive Lie algebroid over a contractible base \( B \), it is shown in [M1] that \( A \) isomorphic to the trivial Lie algebroid \( TB \oplus (B \times \mathfrak{k}) \) (Whitney sum), where \( \mathfrak{k} \) is the typical fiber; in particular \( A \) integrates to a global Lie groupoid isomorphic to \( B \times K \times B \) where \( K \) is the connected and simply connected Lie group with \( Lie(K) = \mathfrak{k} \).

With these facts, we immediately obtain a description of the dual of a dynamical Poisson groupoid.

Let \( X \) be a dynamical Poisson groupoid as in Def. 2.2.6 over the contractible base \( U \) with embedding of the Hamiltonian unit given by

\[
I(h, q) = (Ad_h^{-1}q, h, q).
\]

Let \( \iota: \mathfrak{h} \to \mathfrak{g} \) be the inclusion map.

**Theorem 3.2.4 (Duality)**

The dual Poisson groupoid \( X^* \) of \( X \) is isomorphic to the Poisson groupoid \((U \times G' \times U, \{\}, \{\})_{U \times G' \times U}\) where \( G' \) is the connected and simply connected Lie group whose Lie algebra is the vector space \( \mathfrak{k} := \{(Z, A) \in \mathfrak{h} \times \mathfrak{g}^* \ | \ ad^*_Z(q_0) = \iota^* A\} \) for some \( q_0 \in U \), equipped with the Lie bracket

\[
[(Z, A), (Z', A')] = (\ - [Z, Z'] - < A, \delta_1 P(\cdot) A' >, \\
ad^*_Z A' - ad^*_Z A + < A, \partial P(\cdot) A' >),
\]

and the Poisson bracket \( \{\}, \{\} \) is given by Theorem 2.2.5 for a (unique) Poisson groupoid morphism \( I': H \times U \to U \times G' \times U \) and a (unique) groupoid 1-cocycle

\[
P': U \times G' \times U \to L(\mathfrak{k}^*, \mathfrak{k})
\]

for the adjoint action of \( \mathfrak{k} \).

Proof. For the first part, observe that the anchor map of \( A(X)^* \)

\[
a_*(q, Z, A) = (q, -ad^*_Z q + \iota^* A)
\]
is a surjective submersion since $\iota$ is injective. Thus $A(X)^*$ is transitive and therefore, by MacKenzie's theorem, it is isomorphic to the trivial Lie algebroid $A' = \mathfrak{h}^* \times \mathfrak{h}^* \times (Ker a)_q$. Now the fiber $(Ker a)_q$ is the vector space $\mathfrak{h}$ equipped with the Lie bracket given by the restriction of the bracket of sections of $A(X)^*$ of Prop. 3.2.1 (with $K(q)Z = ad_Z q$ and $A(q)Z = \iota Z$). Hence the claim.

For the second part, let
$$\tau : A(X)^* \rightarrow A'$$
be the (base preserving) trivializing isomorphism of MacKenzie's theorem, and denote by
$$T : X^* \rightarrow U \times G' \times U$$
the unique groupoid isomorphism such that $A(T) = \tau$. We may thus transport the Poisson groupoid structure of $X^*$ to $U \times G' \times U$ by setting
$$\{f,g\}_U \times G' \times U = \{f \circ T, g \circ T\}_X \circ T^{-1}.$$

We now show that there is a (base preserving) Poisson groupoid morphism
$$I' : H \times U \rightarrow U \times G' \times U,$$
where $H \times U$ is the Hamiltonian unit.

Consider the Poisson groupoid morphism (this is the anchor map of $X$)
$$J : U \times G \times U \rightarrow U \times U : (p, x, q) \mapsto (p, q)$$
where $U \times U$ is the coarse groupoid of Example 3.1.2. Its induced Lie bialgebroid morphism
$$A(J) : U \times \mathfrak{g} \times \mathfrak{h}^* \rightarrow U \times \mathfrak{h}^*$$
is of course just the anchor map $a$ of $A(X)$. By the lifting property of Lie algebroid morphisms, and Prop. 3.2.3 above, the dual morphism
$$A(J)^* = a^* : U \times \mathfrak{h} \rightarrow U \times \mathfrak{g}^* \times \mathfrak{h}$$
may be lifted uniquely to a (base preserving) Poisson groupoid morphism
$$J^* : H \times U \rightarrow X^*.$$

Thus $I' = T \circ J^*$ is the sought-for Poisson groupoid morphism. The uniqueness of $P'$ now follows from the uniqueness of the Poisson structure of a (suitably simply connected) Poisson groupoid with prescribed tangent Lie bialgebroid [MX2]. Hence the claim. ■

**Caveat**

We shall see in section 5 that, for $\mathfrak{h} \neq 0$, even when $X$ is coboundary with **constant** $r-$ matrix, the vertex group $G'$ is different from the Poisson Lie group dual to $G$ equipped with the Sklyanin bracket $\{ , \}_{(R,-R)}$.

We close this subsection with a description of natural Poisson quotients associated with Thm 3.2.4. We now assume that the contractible set $U$ contains 0.
Let $X$ be as in Thm 2.2.5 with the map $H \to G : h \mapsto \chi(h,0)$ one to one. Consider the restriction of the left Hamiltonian action

$$\phi^- : H \times X \to X : (h, (p, x, q)) \mapsto (Ad_{h^{-1}} p, \chi(h, p) x, q)$$

to $\alpha^{-1}(0) = \{0\} \times G \times U$:

$$\phi^- : H \times G \times U \to G \times U : (h, x, q) \mapsto (\chi(h, 0) x, q).$$

Let $\pi : G \times U \to G/H \times U : (x, p) \mapsto (\pi, p)$ be the canonical projection.

**Proposition 3.2.5** (Hamiltonian reduction)

The Poisson bracket $\{, \}_\text{red.}$ of the reduced space $\alpha^{-1}(0)/H \simeq G/H \times U$ vanishes at $(\overline{1}, 0)$. Its linearization at $(\overline{1}, 0)$ coincides with the vertex Lie algebra of $A(X)^*$ at 0.

Proof. We have to calculate the Poisson bracket $\{f, g\} X(0, x, q)$ of two functions $f, g \in C^\infty(X)$ whose restriction to $\{0\} \times G \times U$ is $H$-invariant.

Since $H$ is connected, the restriction of $f$ to $\{0\} \times G \times U$ is $H$-invariant if and only if

$$A^*_\chi(0) Df(0, x, q) = 0$$

for all $x \in G, q \in U$.

Thus (see Thm 2.2.5)

$$\{f, g\} X(x, 0, x, q) = - <q, [\delta_2 f, \delta_2 g]> - <A^*_\chi(q) \delta_2 f, D'g>$$

$$+ <A^*_\chi(q) \delta_2 g, D'f> + <Df, P(0, x, q) Dg>.$$  

Now, $P(0, 1, 0) = 0$ and $A^*_\chi(0) D'f(0, 1, 0) = A^*_\chi(0) Df(0, 1, 0) = 0$. Therefore,

$$\{f, g\} X(0, 1, 0) = 0.$$

Set $Z = \delta_2 f, Z' = \delta_2 g, A = Df, A' = Dg$, all evaluated at $(0, 1, 0)$. A direct calculation then gives

$$d\{f, g\} X(0, 1, 0)(0, Y, \lambda) = - <\lambda, [Z, Z']> - <dA^*_\chi(0)(\lambda) Z, A'>$$

$$+ <dA^*_\chi(0)(\lambda) Z', A> + <ad_{A^*_\chi(0) Z} A', Y>$$

$$- <ad_{A^*_\chi(0) Z} A, Y> + <A, (\delta P(X) + \delta_2 P(\lambda)) A'>.$$

To conclude, observe that this coincides with the restriction of the Lie bracket of Prop. 3.2.1 (with $K(q)Z = ad_{Z} q$ and $A = A_{\chi}$) to the kernel $\text{Ker} a_{\lambda}$ at 0. $lacksquare$

Prop. 3.2.5 provides in some sense an indirect Poisson integration of the vertex Lie algebra $(\text{Ker} a_{\lambda})_0$ of $A(X)^*$ by the natural quotient space $G/H \times U$. For $X$ dynamical, combining Prop. 3.2.5 with Thm 3.2.4 then gives a reduced vertex diagram reminiscent of the Poisson Lie group duality of Drinfeld.

Let $X = U \times G \times U$ be a dynamical Poisson groupoid as in Def. 2.2.6 with dual Poisson groupoid $X^* \simeq U \times G' \times U$. Assume that the map $H \to G' : h \mapsto \chi'(h, 0)$ is one to one. Denote the units of $G$ and $G'$ by 1 and equip both spaces $G/H \times U, G'/H \times U$ with the Poisson brackets obtained via Poisson reduction.
Let \( h^\perp \subset g^* \) be the annihilator of \( h \).

**Theorem 3.2.6** (Reduced duality diagram)

We have the diagram

\[
\begin{array}{ccc}
X^* & \xrightarrow{\text{red.}} & X \\
\downarrow & \text{lin. at } (T,0) & \downarrow \\
G'/H \times U & \xrightarrow{\text{red.}} & G/H \times U \\
\end{array}
\]

\[A(X)^* \supset (\text{Ker } a_*)_0 = h \times h^\perp \]

In case \( H \) is reduced to its unit, this diagram reduces to that of Drinfeld's duality for Poisson Lie groups.

---

**4. An explicit case study of duality**

In [EV], Etingof and Varchenko obtained, among other things, a classification of solutions of the (CDYBE) for pairs \((g, h)\) of Lie algebras, where \( g \) is simple, and \( h \subset g \) is a Cartan subalgebra. Our purpose in this section is to give an explicit study of duality for the corresponding class of coboundary dynamical Poisson groupoids.

We begin by recalling the general form of these dynamical \( r \)-matrices.

First, let us fix some notation. Let \( g \) be a complex simple Lie algebra with Killing form \((.,.)\), \( h \subset g \) a Cartan subalgebra, and \( g = h \oplus \sum_{\alpha \in \Delta} g_\alpha \) the root space decomposition. We let \( \Delta^* \) be a fixed simple system of roots and denote by \( \Delta^\pm \) the corresponding positive/negative system. For any positive root \( \alpha \in \Delta^+ \), we choose root vectors \( e_\alpha \in g_\alpha \) and \( e_{-\alpha} \in g_{-\alpha} \) which are dual with respect to \((.,.)\) so that \([e_\alpha, e_{-\alpha}] = h_\alpha \). We also fix an orthonormal basis \((x_i)_{1 \leq i \leq \text{rank}(g)}\) of \( h \). Lastly, for a subset of simple roots \( \Gamma \subset \Delta^s \), we shall denote the root span of \( \Gamma \) by \( \langle \Gamma \rangle \subset \Delta \) and set \( \Gamma^\pm = \Delta^\pm \setminus \langle \Gamma \rangle \).

For any subset \( \Gamma \subset \Delta^s \), we give the \( ad_h \)-invariant solutions of \((mDYBE)\) (see eqs. (2.1.1), (2.1.2)) associated with the triple \((g, h; \Gamma)\) as (cf.[EV]):

\[
R(q)B = \sum_{i,j} C_{ij}(q) < x_j, B > x_i + \sum_{\alpha \in \Delta} \phi_\alpha(q) < e_{-\alpha}, B > e_\alpha \quad (4.1)
\]

where

\[
\phi_\alpha(q) = \begin{cases} 
\frac{1}{2} & \text{for } \alpha \in \Gamma^+, \\
-\frac{1}{2} & \text{for } \alpha \in \Gamma^-
\end{cases}
\]

\[
\phi_\alpha(q) = \frac{1}{2} \coth((\alpha, q - \mu)/2) \text{ for } \alpha \in \langle \Gamma \rangle,
\]

and where \( \sum_{i,j} C_{ij} dq^i \otimes dq^j \) is any closed meromorphic 2-form on \( h^* \) and \( \mu \in h^* \) is arbitrary.
We shall denote by $U$ the domain of analyticity of $R$ and let $G$ be the connected and simply-connected Lie group with $\text{Lie}(G) = \mathfrak{g}$. Note that $U$ is trivially $\text{Ad}_H$-invariant as $H$ is abelian, hence we can consider the coboundary dynamical Poisson groupoid $X = U \times G \times U$ associated with $R$. Our immediate goal is to construct an explicit trivialization of the dynamical Lie algebroid $A(X)^* \simeq U \times \mathfrak{h}^* \times \mathfrak{g}^*$. Note that, as $U$ is not contractible, this is not guaranteed by MacKenzie’s theorem. In what follows, we shall make the identification $\mathfrak{g}^* \simeq \mathfrak{g}$ using the Killing form $(\cdot, \cdot)$. Then we have $\mathfrak{h}^\perp \simeq \mathfrak{n} := \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$, and the Lie bracket between the sections of the dynamical Lie algebroid takes the form

$$
[(Z, B), (Z', B')]_*(q)
= (dZ'(q)i^* B(q) - dZ(q)i^* B'(q)) - (dR(q)(\cdot)B(q), B'(q)),
$$

$$
- dB(q)i^* B'(q) + dB(q)i^* B(q)
- [R(q)(B(q)) + Z(q), B'(q)] - [B(q), R(q)(B'(q)) + Z'(q)],
$$

(4.2)

We shall begin our construction with a description of the vertex Lie algebra $\mathcal{V}_q = (\text{Ker} a_*|_q) = \mathfrak{h} \times \mathfrak{h}^\perp \simeq \mathfrak{h} \times \mathfrak{n}$ of $A^*$ at $q \in U$. To do so, let us introduce the following Lie subalgebras of $\mathfrak{g}$ associated with $\Gamma \subset \Delta^*$:

$$
\mathfrak{h}_\Gamma := \langle (h_\gamma)_{\gamma \in \Gamma} \rangle > \mathfrak{c},
$$

$$
\mathfrak{h}^\perp_\Gamma := \text{the orthogonal complement of } \mathfrak{h}_\Gamma \text{ in } \mathfrak{h} \text{ w.r.t. } (\cdot, \cdot)_{|_{\mathfrak{h} \times \mathfrak{h}}},
$$

$$
\mathfrak{l}_\Gamma := \mathfrak{h}_\Gamma \oplus \langle (e_\alpha)_{\alpha \in <\Gamma>} \rangle > \mathfrak{c} \text{ the Levi factor },
$$

$$
\mathfrak{n}^\perp_\Gamma := \langle (e_\alpha)_{\alpha \in \Gamma^\perp} \rangle > \mathfrak{c} \text{ the nilpotent radicals }.
$$

Clearly $[\mathfrak{h}_\Gamma, \mathfrak{n}^\perp_\Gamma] \subset \mathfrak{n}^\perp_\Gamma$ and $[\mathfrak{l}_\Gamma, \mathfrak{n}^\perp_\Gamma] \subset \mathfrak{n}^\perp_\Gamma$.

If $\mathfrak{g}_1, \mathfrak{g}_2$ are two Lie algebras, we denote by $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ equipped with the Lie bracket $[x_1 + x_2, y_1 + y_2] = [x_1, y_1] - [x_2, y_2]$. Let $\mathfrak{J}_\Gamma = \mathfrak{h}^\perp_\Gamma \ltimes (\mathfrak{n}^\perp_\Gamma \ominus \mathfrak{n}^\perp_\Gamma)$ be the semidirect product Lie algebra where $\mathfrak{h}^\perp_\Gamma$ acts on each summand of the anti-direct sum by the adjoint action of $\mathfrak{g}$. Set

$$
\mathfrak{g}' := \mathfrak{l}_\Gamma \ltimes \mathfrak{J}_\Gamma
$$

where $\mathfrak{l}_\Gamma$ also acts on $\mathfrak{J}_\Gamma$ by the adjoint action of $\mathfrak{g}$.

**Proposition 4.1** Let $q \in U$. Then the map $\psi(q) : \mathcal{V}_q \longrightarrow \mathfrak{g}'$ defined by

$$
\psi(q)(0, e_\alpha) = \frac{-1}{2\sinh\left(\frac{(\alpha, q - \mu)}{2}\right)} e_\alpha \quad \text{for } \alpha \in < \Gamma >,
$$

$$
\psi(q)(0, e_\alpha) = -e^{\pm \frac{\pi}{2} (\alpha, q - \mu)} e_\alpha \quad \text{for } \alpha \in \Gamma^\perp,
$$

$$
\psi(q)(Z, 0) = -Z \quad \text{for all } Z \in \mathfrak{h},
$$

is an isomorphism of Lie algebras.

Proof. The Lie bracket of $\mathcal{V}_q = \mathfrak{h} \times \mathfrak{n}$ can be calculated from Eqn. (4.2) and we have

$$
[(Z, n), (Z', n')]_* = (- (dR(q)(\cdot)n, n'), - [R(q)n + Z, n']
- [n, R(q)n' + Z']).
$$
Writing \( n = \sum_{\alpha \in \Delta} n^\alpha e_\alpha \) and similarly for \( n' \), we have

\[
(dR(q) (\Lambda) n, n') = \sum_{\alpha \in <\Gamma>} d\phi_\alpha(q) (\Lambda) n^\alpha n'^\alpha
\]

\[
= \sum_{\alpha \in <\Gamma>} (\frac{1}{4} - \phi_\alpha(q)^2)(\alpha, \Lambda) n^\alpha n'^\alpha.
\]

A direct calculation then gives the following Lie bracket relations:

\[
[(0, e_\alpha), (0, e_\beta)]* = (0, - (\phi_\alpha(q) + \phi_\beta(q))[e_\alpha, e_\beta]) \quad \text{for} \; \alpha \in <\Gamma>, \beta \in \Delta, \; \alpha + \beta \neq 0,
\]

\[
[(0, e_\alpha), (0, e_{-\alpha})]* = (-\frac{1}{4} - \phi_\alpha(q)^2)[e_\alpha, e_{-\alpha}], 0) \quad \text{for} \; \alpha \in <\Gamma>,
\]

\[
[(0, e_\alpha), (0, e_\beta)]* = (0, 0) \quad \text{for} \; \alpha \in \Gamma^+, \beta \in \Gamma^-,
\]

\[
[(0, e_\alpha), (0, e_\beta)]* = (0, -[e_\alpha, e_\beta]) \quad \text{for} \; \alpha, \beta \in \Gamma^+,
\]

\[
[(0, e_\alpha), (0, e_\beta)]* = (0, +[e_\alpha, e_\beta]) \quad \text{for} \; \alpha, \beta \in \Gamma^-,
\]

\[
[(Z, 0), (0, e_\alpha)]* = (0, -\alpha(Z)e_\alpha) \quad \text{for all} \; \alpha \in \Delta, Z \in \mathfrak{h}
\]

\[
[(Z, 0), (Z', 0)]* = (0, 0), \quad Z, Z' \in \mathfrak{h}.
\]

where the bracket \([ , ]\) on the r.h.s. is that of \( \mathfrak{g} \). After rescaling the basis of \( \mathfrak{n} \) by setting

\[
E_\alpha(q) = 2sinh\left(\frac{(\alpha, q - \mu)}{2}\right)e_\alpha, \; \alpha \in <\Gamma> \quad E_\alpha(q) = e^{\pm\frac{1}{2}(\alpha, q - \mu)}e_\alpha, \; \alpha \in \Gamma^\pm,
\]

the above relations yield

\[
[(0, E_\alpha(q)), (0, E_\beta(q))]* = (0, -N_{\alpha, \beta}E_{\alpha + \beta}(q)) \quad \text{for} \; \alpha \in <\Gamma>, \beta \in \Delta, \; \alpha + \beta \neq 0,
\]

\[
[(0, E_\alpha(q)), (0, E_{-\alpha}(q))]* = (-[e_\alpha, -e_{-\alpha}], 0) \quad \text{for} \; \alpha \in <\Gamma>,
\]

\[
[(0, E_\alpha(q)), (0, E_\beta(q))]* = (0, 0) \quad \text{for} \; \alpha \in \Gamma^+, \beta \in \Gamma^-,
\]

\[
[(0, E_\alpha(q)), (0, E_{-\beta}(q))]* = (0, -N_{\alpha, \beta}E_{\alpha + \beta}(q)) \quad \text{for} \; \alpha, \beta \in \Gamma^+,
\]

\[
[(0, E_\alpha(q)), (0, E_{-\beta}(q))]* = (0, +N_{\alpha, \beta}E_{\alpha + \beta}(q)) \quad \text{for} \; \alpha, \beta \in \Gamma^-,
\]

\[
[(Z, 0), (0, E_\alpha(q))]* = (0, -\alpha(Z)E_\alpha(q)) \quad \text{for all} \; \alpha \in \Delta, Z \in \mathfrak{h},
\]

\[
[(Z, 0), (Z', 0)]* = (0, 0), \quad Z, Z' \in \mathfrak{h},
\]

where \( N_{\alpha, \beta} \) are the structure constants of \( \mathfrak{g} \).

We shall check the first Lie bracket above with \( \beta \in \Gamma^- \); the others are similar.

For a root \( \gamma \in \Delta \), set \( x_{\gamma} = (\gamma, q - \mu) \). We have \( \phi_\alpha(q) + \phi_\beta(q) = \frac{1}{2}coth(\frac{x_\alpha}{2}) - \frac{1}{2} = \frac{1}{e^{x_\alpha} - 1} \), \( E_\alpha(q) = e^{\frac{x_\alpha}{2} (1 - e^{-x_\alpha})}e_\alpha \), and \( E_\beta(q) = e^{\frac{-x_\beta}{2} e_{-\beta}} \). Thus,

\[
[(0, E_\alpha(q)), (0, E_\beta(q))]* = (0, -\frac{e^{x_\alpha}}{e^{x_\alpha} - 1} N_{\alpha, \beta}e_{\alpha + \beta}).
\]

Now, if \( \alpha + \beta \) is a root, then it belongs to \( \Gamma^- \); thus \( e_{\alpha + \beta} = e^{\frac{\alpha + \beta}{2}}E_{\alpha + \beta}(q) \), and this immediately gives the assertion.
The Lie bracket relations above show that the structure constants of $V_q$ in the basis $(x_i,0) \ 1 \leq i \leq \text{rank}(g)$; $(0,E_\alpha(q))$, $\alpha \in \Delta$ are opposite to those of $g'$. Therefore the map $\psi : V_q \rightarrow g'$ defined by $(Z,0) \mapsto -Z$ and $(0,E_\alpha(q)) \mapsto -e_\alpha$ is an isomorphism of Lie algebras. Hence the claim.\]

**Corollary 4.2** The map $\tilde{\psi} : U \times g' \rightarrow \text{Ker} \phi_\ast : (q,\xi) \mapsto (q,-\Pi_h \xi,\psi(q)^{-1} (\Pi_n \xi))$ is an isomorphism between the trivial Lie algebra bundle $U \times g'$ and the adjoint bundle $\text{Ker} \phi_\ast$. Here, $\Pi_h$ and $\Pi_n$ are the projections relative to the direct sum decomposition $g = h \oplus n$. ■

Let us briefly comment on the vertex isomorphism of Prop. 4.1. If $\Gamma \subset \Delta^s$ is the empty set, the vertex Lie algebra $V_q, \ q \in U$, of $A(X)^\ast$ is isomorphic to $\text{h}\ltimes (n^+ \oplus n^-)$, which is reminiscent of (although not identical to) the Lie algebra dual of $r$ equipped with the standard constant $r$–matrix (see also example 5.1.8). If $\Gamma = \Delta^s$, we have $V_q \simeq g' = l_\Delta = g$. For a general subset $\Gamma \subset \Delta^s$ the vertex Lie algebra $V_q \simeq g'$ is seen to naturally intertwine the Lie factor $l_r$ with the summand $\mathcal{I}_\Gamma$ which is again reminiscent of the Lie algebra dual of $g$ equipped with the standard constant $r$–matrix.

Our next step is to construct a flat connection $\theta_\ast : TU \simeq U \times h^\ast \rightarrow U \times h \times g^\ast$ satisfying the condition $[\theta_\ast(\lambda), \tilde{\psi}(\xi)]_\ast = \tilde{\psi}(d\xi \cdot \lambda)$ for $\lambda : U \rightarrow h^\ast$ and $\xi : U \rightarrow g'$. To simplify notation, we shall identify the elements $(Z,n) \in h \times n \simeq V_q$ of the vertex Lie algebra with $Z + n \in g$ from now onwards.

Let $C^\# : U \rightarrow L(h^\ast,h)$ be the map defined by $C^\#(q)\lambda = \sum_{i,j} C_{ij}(q) \lambda(x_j) x_i$. We shall seek $\theta_\ast$ in the form $\theta_\ast(q,\lambda) = (q,f(q)\lambda,\lambda)$, where $f : U \rightarrow L(h^\ast,h)$. By definition, $\theta_\ast$ is a flat connection if and only if $\theta_\ast[\lambda,\lambda'] = [\theta_\ast(\lambda),\theta_\ast(\lambda')]'$ for $\lambda,\lambda' : U \rightarrow h^\ast$. By using Eqn. (4.2), a straightforward calculation shows that this is equivalent to the following two conditions:

1. $df(q)(\lambda'(q)) \lambda(q) - df(q)(\lambda(q)) \lambda'(q) = -(dR(q)(. \lambda(q),\lambda'(q))$  
2. $[R(q)(\lambda(q)),\lambda'(q)] + [\lambda(q),R(q)(\lambda'(q))] = 0$

for $q \in U$

On the other hand, the condition $[\theta_\ast(\lambda), \tilde{\psi}(\xi)]_\ast = \tilde{\psi}(d\xi \cdot \lambda)$ is equivalent to

3. $(dR(q)(. \lambda(q),n) = 0$  
4. $d\psi^{-1}(q)\lambda(q)(n) - [f(q)\lambda(q),\psi^{-1}(q)(n)]$  
   $- ([R(q)(\lambda(q)),\psi^{-1}(q)(n)] + [\lambda(q),R(q)(\psi^{-1}(q)(n))]) = 0$,  

for $n \in n$ and $q \in U$.

From the properties of $R$, (2) and (3) are immediately seen to hold. We now examine condition (4). Set $\psi(q)(e_\alpha) = \psi_\alpha(q)e_\alpha$, for all $\alpha \in \Delta$, then $d\psi^{-1}(q)\lambda(q)(e_\alpha) = \phi_\alpha(q)\psi_\alpha(q)^{-1}(q)(\lambda(q),\alpha)e_\alpha$. Meanwhile, it is easy to check that

$[f(q)\lambda(q),\psi^{-1}(q)(e_\alpha)] = \psi_\alpha(q)^{-1}(q)\alpha(q)(e_\alpha)$  
$([R(q)(\lambda(q)),\psi^{-1}(q)(e_\alpha)] + [\lambda(q),R(q)(\psi^{-1}(q)(e_\alpha))])$  
$
= \phi_\alpha(q)^{-1}(q)(\alpha((C^\#(q)(\lambda(q))) + (\alpha(\lambda(q)\phi_\alpha(q))e_\alpha.$
Therefore, condition (4) is equivalent to
\[ \alpha (f(\lambda)(q) + C^\#(q)(\lambda(q))) = 0, \text{ for all } \alpha \in \Delta, \]
that is to \( f = -C^\# \). Finally, inserting \( f = -C^\# \) into condition (1) shows that it is trivially satisfied as it is equivalent to the closedness of the 2-form \( \sum_{i,j} C_{ij} dq^i \otimes dq^j \). Hence we have

**Proposition 4.3** The map
\[
\theta_\ast : TU \simeq U \times \mathfrak{h}^* \longrightarrow U \times \mathfrak{h} \times \mathfrak{g}^*
\]
\[(q, \lambda) \mapsto (q, -C^\#(q)\lambda, \lambda)\]
is a flat connection on \( U \times \mathfrak{h} \times \mathfrak{g}^* \) satisfying \([\theta_\ast(\lambda), \tilde{\psi}(\xi)]_\ast = \tilde{\psi}(d\xi \cdot \lambda)\) for \( \lambda : U \longrightarrow \mathfrak{h}^* \) and \( \xi : U \longrightarrow \mathfrak{g}' \).

**Theorem 4.4** (Trivialization) Let \( A' := U \times \mathfrak{h}^* \times \mathfrak{g}' \) be the trivial Lie algebroid over \( U \) (see Prop. 3.2.1), then the (bijective) bundle map
\[
\sigma : A' \longrightarrow U \times \mathfrak{h} \times \mathfrak{g}^*
\]
\[(q, \lambda, \xi) \mapsto \theta_\ast(q, \lambda) + \tilde{\psi}(q, \xi)\]
is an isomorphism of Lie algebroids. Its inverse is given by \( \tau(q, Z, \lambda + n) = (q, \lambda, -C^\#(q)\lambda - Z + \psi(q)n) \).

Proof. This is clear from the properties of \( \theta_\ast \) and the fact that \( \tilde{\psi} \) is an isomorphism of Lie algebra bundles. \( \blacksquare \)

Note that the theorem implies, in particular, that the dynamical Lie algebroid \( A(X)^* \simeq U \times \mathfrak{h} \times \mathfrak{g}^* \) is integrable. In what follows, we let \( U' \) be a connected and simply-connected open subset of \( U \) and we consider the coboundary Poisson groupoid \( X(U') = U' \times G \times U' \) associated with \( R \). We also let \( G' \) be the connected and simply connected Lie group with \( Lie(G') = \mathfrak{g}' \) and denote by
\[
T : X(U')^* \longrightarrow X' = U' \times G' \times U'
\]
the unique (base preserving) Lie groupoid isomorphism such that \( A(T) = \tau_{U' \times h \times g^*} \).

If we define the Poisson bracket on \( X' \) by
\[
\{f, g\}_{X'} = \{f \circ T, g \circ T\}_{X(U')^*} \circ T^{-1},
\]
then \( (X', \{\cdot, \cdot\}_{X'}) \) and \( (X(U'), \{\cdot, \cdot\}_{X(U')}) \) are Poisson groupoids in duality (see Def 3.1.1) and \( T \) is an isomorphism of Poisson groupoids.

The following theorem characterizes the Poisson groupoid \( (X', \{\cdot, \cdot\}_{X'}) \).

Let \( j : \mathfrak{h} \longrightarrow \mathfrak{g}' : Z \mapsto Z \) be the inclusion.

**Theorem 4.5** The Poisson groupoid \( (X', \{\cdot, \cdot\}_{X'}) \) is of dynamical type with Poisson bracket
\[
\{f, g\}_{X'}(p, u, q) = \langle p, [\delta_1 f, \delta_1 g] \rangle - \langle q, [\delta_2 f, \delta_2 g] \rangle
\]
\[- \langle j\delta_1 f, Dg \rangle - \langle j\delta_2 f, D'g \rangle
\]
\[+ \langle j\delta_1 g, Df \rangle + \langle j\delta_2 g, D'f \rangle
\]
\[+ \langle Df, P'(p, u, q)Dg \rangle,
\]
where $P' : U' \times G' \times U' \to L(g^*, \mathfrak{g}^*)$ is the unique skew symmetric groupoid cocycle whose tangent cocycle $P'_*(q, \Lambda, Z + n) := -\delta_1 P'(\Lambda) + \partial P'(Z + n)$ is given by

$$(n, \delta_1 P'(\Lambda) n') = (\Pi_h [\psi^* n, \psi^* n'], \Lambda)$$

$$\lambda, \delta_1 P'(\Lambda) \lambda') = (dC^#(\lambda') \lambda - dC^#(\Lambda) \lambda', \Lambda)$$

$$(n, \partial P'(\mathfrak{n}) n') = -(\Pi_n [\psi^* n, \psi^* n'], \psi^{-1} n)$$

$$\lambda, \partial P'(n) \lambda') = -(d(\psi^{-1})(\lambda) \psi^* n', n) - ([C^#(\lambda), n], n)$$

$$\partial P'(\mathfrak{Z}) = 0.$$ 

Here, $\Pi_h$ and $\Pi_n$ are the projections relative to the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}, n, n', n \in \mathfrak{n}, \mathfrak{Z} \in \mathfrak{h}$, and $\lambda, \lambda', \Lambda \in \mathfrak{h}^* \simeq \mathfrak{h}$, and the differentials of $P'$ are taken at $(q, 1, q)$.

Proof. For the sake of clarity, we shall begin by repeating the argument of Theorem 3.2.4 here. Consider the Poisson groupoid morphism $J : X \to U' \times U' : (p, h, q) \mapsto (p, q),$ with induced Lie bialgebroid morphism $A(J)$. Applying Prop. 3.2.3 to the dual morphism 

$A(J)^* : U' \times \mathfrak{h} \to U' \times \mathfrak{h} \times \mathfrak{g}^*$

we infer the existence of a (base preserving) Poisson groupoid morphism $J^* : H \times U' \to X^*$ and so of a morphism

$I' = T \circ J^* : H \times U' \to U' \times G' \times U'$.

Now, (see Eqn (2.2.1)) $I'$ is necessarily of the form

$$(h, p) \mapsto (p, \chi'(h, p), p),$$

for some groupoid morphism $\chi' : H \times U' \to G'$ with tangent map $A(\chi') : U' \times \mathfrak{h} \to \mathfrak{g}'$. Therefore, the Poisson bracket $\{., \}_{X'}$ is given by Thm 2.2.5 for $A_{X'}$ and some groupoid 1–cocycle $P' : X' \to L(g^*, \mathfrak{g}^*)$.

Denote the Lie algebroid of $U' \times G' \times U'$ by $A'$ and let $(A', [., .], a; A'^*, [., .]'$, $a'_*)$ be the Lie bialgebroid structure of Prop. 3.2.1. For $A'^*$, we have $K = 0$ since $\mathfrak{h}$ is abelian, and $A = A_{X'}$. The Poisson bracket $\{., \}_{X'}$ is now uniquely determined by the duality requirement (see Def. 3.1.1) that the trivialization map $\tau$ of Thm. 4.4 be a Lie bialgebroid isomorphism from $(A(X)^*, A(X)^{-})$ to $(A', A'^*)$, that is, by the condition that the map

$$\tau^* : A'^* = U' \times \mathfrak{h} \times g'^* \to A(X)^{-} = U' \times \mathfrak{h}^* \times \mathfrak{g}$$

$$(q, Z, \lambda + n) \mapsto (q, -\lambda, C^#(\lambda) + Z + \psi^*(n))$$
satisfies
\[-a_{A(X)} \tau^* = a'_{\delta}, \quad \tau^*[(Z, \lambda + n), (Z', \lambda' + n')] = -[\tau^*(Z, \lambda + n), \tau^*(Z', \lambda' + n')]_{A(X)}\].

Now (see Prop. 3.2.1) the anchor condition is equivalent to \(A_{\chi'}(q)(Z) = j(Z) = Z\) so the groupoid morphism \(\chi' : H \times U' \to G'\) is just the inclusion of \(H\) into \(G'\). Therefore (see Def. 2.2.6) \(X'\) is of dynamical type. On the other hand, a direct calculation shows that the bracket condition holds if and only if \(\delta_1 P'\) and \(\partial P'\) satisfy the equations given above. Hence the claim. ■

**Remarks 4.6**

(a) The relationship between \(P'\) and \(P'_s\) is as follows. Fix \(q_0 \in U'\) and write \(P'\) as
\[P'(p, u, q) = -l(p) + \pi(u) + Ad_u l(q) Ad^*_u\]
for some map \(l : U' \to L(g'', g')\) with \(l(q_0) = 0\) and some group cocycle \(\pi : G' \to L(g'', g')\). We have
\[P'_s(q, A, X') = d\pi(1) X' + ad_X l(q) + l(q) ad_{X'} + dl(q) A\]
Therefore (this is a special case of a result of [X])
\[dl(q)(\Lambda) = P'_s(q, \Lambda, 0), \quad d\pi(u) T_1 l_u X' = Ad_u d\pi(1) X' Ad^*_u = Ad_u P'_s(q_0, 0, X') Ad^*_u\].

(b) Writing out the equations of Thm 4.5 for \(\delta_1 P'\) using the basis \((e_\beta)\) of \(n\) and integrating yields
\[l(q)(e_\beta) = (\phi_\beta(q) - \phi_\beta(q_0)) e_\beta, \text{ if } \beta \in < \Gamma >\]
\[l(q)(e_\beta) = e^{(\beta, \mu)}(e^{-(\beta, q_0)} - e^{-(\beta, q)}) e_\beta, \text{ if } \beta \in \Gamma^+\]
\[l(q)(e_\beta) = e^{-(\beta, \mu)}(e^{(\beta, q)} - e^{(\beta, q_0)}) e_\beta, \text{ if } \beta \in \Gamma^-\]
\[l(q)(\lambda) = (C^#(q) - C^#(q_0))(\lambda) \text{ for all } \lambda \in h^*\]

On the other hand, the remaining equations evaluated at \(q = q_0\) give
\[< \lambda, d\pi(1)(n) e_\beta >= - (\phi_\beta(q_0)(\lambda, \beta) + (C^#(q_0)\lambda, \beta)) < e_\beta, n >\]
\[< e_\alpha, d\pi(1)(n) e_\beta >= - \frac{\psi_{-\alpha}(q_0)\psi_{-\beta}(q_0)}{\psi_{-(\alpha+\beta)}(q_0)} < [e_\alpha, e_\beta], n >\]
\[< \lambda, d\pi(1)(n) \lambda' >= 0, \quad d\pi(1) Z = 0\]
for all \(\alpha, \beta \in \Delta, \lambda, \lambda' \in h^*, Z + n \in g'\), where we have set \(\psi(e_\alpha) = \psi_\alpha(q)e_\alpha\).

The latter equations allow, in principle, for an explicit expression of \(\pi(u)\) but we shall postpone this integration to a future publication as it will not be needed in the rest of this paper.

5. Coboundary dynamical Poisson groupoids - the constant \(r-\) matrix case.
The purpose of this section is two-fold. In Section 5.1, we give a construction of the dual $X^*$ of the coboundary dynamical Poisson groupoid $X = \mathfrak{h}^* \times G \times \mathfrak{h}^*$ (of Theorem 2.1.4) for the constant $r-$ matrix case, i.e., for the case where $R$ is a constant map from $\mathfrak{h}^*$ to $L(\mathfrak{g}^*, \mathfrak{g})$. As the reader will see, the construction involves the use of Poisson Lie group theory. More specifically, the Poisson Lie group $G$ equipped with the Sklyanin bracket admits an extension to a bigger Poisson Lie group whose dual is critical in the construction. In Section 5.2, we construct a symplectic double groupoid which has $X$ and $X^*$ as its side groupoids. This leads, in particular, to a description of the symplectic leaves of $X$ as orbits of a Poisson Lie group action. We shall discuss the non-constant $r-$ matrix case in a forthcoming publication.

5.1. The dual Poisson groupoid.

Let $\iota : \mathfrak{h} \rightarrow \mathfrak{g}$ be the inclusion map. We assume here that the Lie groups $G$ and $H$ are connected and simply connected. Let $R : \mathfrak{g}^* \rightarrow \mathfrak{g}$ be a skew-symmetric constant $r-$ matrix which satisfies Eqn. (2.1.1) and Eqn.(2.1.2). Recall that the group $G$ equipped with the Sklyanin bracket

$$\{f, g\}_G(x) = < R(Df), Dg > - < R(D)f, D'g >$$

(5.1.1)

is a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}, [\cdot, \cdot]; \mathfrak{g}^*, [\cdot, \cdot])$ where

$$[A, B]_s = ad^*_{R(A)}B - ad^*_{R(B)}A.$$  

(5.1.2)

Lemma 5.1.1 $H$ is a trivial Poisson Lie subgroup of $G$.

Proof. Since $R$ is $\mathfrak{h}-$ equivariant, we have

$$< [A, B]_s, \iota Z > = < ad^*_{R(A)}B, \iota Z > - < ad^*_{R(B)}A, \iota Z >$$

$$= < B, [R(A), \iota Z] > + < A, [\iota Z, R(B)] >$$

$$= < B, [R(A), \iota Z] > - < A, R(ad^*_{\iota Z}B) > = 0,$$

therefore $[\mathfrak{g}^*, \mathfrak{g}^*]_s \subset \mathfrak{h}^\perp$. In particular, $\mathfrak{h}^\perp$ is an ideal in $\mathfrak{g}^*$ and the connected Lie subgroup $H \subset G$ is a Poisson Lie subgroup with tangent Lie bialgebra $(\mathfrak{h}, \mathfrak{h}^*)$ defined by

$$[\iota^* A, \iota^* B]_{\mathfrak{h}^*} := \iota^* [A, B]_s.$$  

Hence the Lie bracket of $\mathfrak{h}^*$ is identically zero and $H \subset G$ is a trivial Poisson Lie subgroup. ■

Let $(G^*, \{\cdot, \cdot\}_s)$ be Drinfeld’s Poisson Lie group dual to $(G, \{\cdot, \cdot\}_G)$, and let ([STS], [LW])

$$\varphi^+ : G^* \times \overline{G} \rightarrow G^*, \quad \varphi^- : \overline{G^*} \times G \rightarrow G$$

(5.1.3)

be the right and left dressing actions. Recall that $\varphi^+$ and $\varphi^-$ are Poisson Lie group actions and that $G$ and $G^*$ act on each other by twisted automorphisms

$$\varphi^x(uv) = \varphi^x(v) \varphi^x(u), \quad \varphi^- u(x) = \varphi^- u(x) \varphi^- u(y).$$

(5.1.4)
Lemma 5.1.2

Proof. For \( G \) and \( G^* \), the Poisson brackets of \( G \) and \( G^* \) by the formulae

\[
\begin{align*}
\{\phi, \psi\}_*(u) &= -d\phi(u)\lambda^+(T_1l_u^*d\psi(u))(u) \\
\{f, g\}_G(x) &= df(x)\lambda^-(T_1r_x^*dg(x))(x),
\end{align*}
\]

where \( \lambda^+(X)(u), \lambda^-(A)(x) \) are the infinitesimal generators of \( \varphi^+ \) and \( \varphi^- \).

It follows from Lemma 5.1.1 that the restriction of the right dressing action to \( H \) induces a left Hamiltonian action

\[ \phi^l : H \times G^* \to G^* : (h, u) \mapsto \varphi^+_h(u). \] (5.1.6)

Moreover, since \( G^* \) acts trivially on \( H \subset G \) (i.e. \( \varphi^-_h(h) = h, u \in G^*, h \in H \)), by (5.1.4) for each \( h \in H \), \( \phi^l_h \) is an automorphism of \( G \).

Caveat Note that our conventions differ from those of [LW1]. Indeed \( \varphi^+ \) is their left dressing action made right, while \( \varphi^- \) is their right dressing action made left.

Recall also that the dressing vector fields \( \lambda^+ \) and \( \lambda^- \) may fail to globally integrate to define \( \varphi^+ \) and \( \varphi^- \). In this subsection, we need only assume that \( \phi^l \) is globally defined.

By construction, the map \( t^* : g^* \to \mathfrak{h}^* \) is a morphism of Lie algebras. Let

\[ I^* : G^* \to \mathfrak{h}^* \]

be the (unique) morphism of Lie groups integrating \( t^* \).

Lemma 5.1.2 \( I^* \) is an \( Ad_{t^*}^|- \) equivariant momentum map for the action \( \phi^l \).

Proof. For \( Z \in \mathfrak{h} \), let \( j_Z \in C^\infty(G^*, \mathfrak{h}^*) \) be defined by \( j_Z(u) = < I^*(u), Z > \). We have to show that the Hamiltonian vector field \( \tilde{X}_{j_Z} \) coincides with the infinitesimal generator \( -\lambda^+(tZ) \) of the action \( \phi^l \). But

\[
\frac{d}{dt} j_Z(ue^{tA}) = \frac{d}{dt} I^*(ue^{tA}), Z >
\]

\[
= \frac{d}{dt} < I^*(u) + I^*(e^{tA}), Z > = < t^*A, Z > .
\]

Therefore

\[ \tilde{X}_{j_Z}(u) = -\lambda^+(T_1l_u^*dj_Z(u))(u) = -\lambda^+(T_1l_u^*T_u^*l_u-1tZ)(u) = -\lambda^+(tZ)(u). \]

It remains to show \( Ad_{t^*}^{I^*}I^*(u) = I^*(\varphi^+_h(u)) \). Since both sides are group morphisms from \( G^* \) to \( \mathfrak{h}^* \) and \( G^* \) is connected, it is enough to check that the induced Lie morphisms are equal. Now

\[
\frac{d}{dt} I^*(\varphi^+_h(e^{tA})) = \frac{d}{dt} Ad_{t^*}^{I^*}I^*(e^{tA})
\]

reads as

\[ t^*T_1\varphi^+_hA = Ad_{t^*}^{I^*}t^*A. \]

But this equality follows from \( T_1\varphi^+_h = Ad_{t^*}^{I^*} \) and the \( Ad_{t^*}^{I^*} \) equivariance of \( t^* \).

Hence the claim.
Proposition 5.1.3

(a) The set $G \times \mathfrak{h}^*$ equipped with the multiplication $(x, p)(y, q) = (xy, p + q)$ and the Poisson bracket

$$\{f, g\}(x, p) = \langle p, [\delta f, \delta g] \rangle + \langle R(Df), Dg \rangle - \langle R(D'f), D'g \rangle + \langle (Df - D'f), \iota(\delta g) \rangle - \langle (Dg - D'g), \iota(\delta f) \rangle$$

is a Poisson Lie group.

(b) The Drinfeld Poisson Lie group dual of $(G \times \mathfrak{h}^*, \{, \})$ is the set $H \times G^*$ equipped with the semi-direct multiplication

$$(h, u)(k, v) = (hk, u \varphi^+_{h-1}(v))$$

and the Poisson bracket

$$\{\phi, \psi\}_*(h, u) = -\partial_\tau \phi \lambda^+ (T^*_1 \varphi \partial_\tau \psi)(u),$$

where $\partial_\tau \phi$ is the partial derivative w.r.t. $G^*$.

Proof. (a) This may be checked by a standard calculation which makes use of Lemma 5.1.1, so we shall leave out the details of the verification.

(b) The tangent Lie bialgebra of $G \times \mathfrak{h}^*$ is given by $(g \oplus \mathfrak{h}^*, [ , ]_{\mathfrak{g}} \kappa \mathfrak{h}^*, [ , ]')$ where

$$[Z + A, Z' + A']' = [Z, Z'] - ad_{\iota(Z)} A' + ad_{\iota(Z)} A + [A, A']_s.$$ 

Therefore the dual group is $H \times G^*$ as stated, while the multiplicativity of the Poisson bracket $\{ , \}_*$ follows from the Hamiltonian property (5.1.6) and the multiplicativity of the Poisson bracket of $G^*$. Hence the assertion.

Note that $\mathfrak{h} \times \mathfrak{h}^+ \subset \mathfrak{h} \times \mathfrak{g}^*$ is a Lie subalgebra which is isomorphic to the vertex Lie algebra of Thm. 3.2.4.

Let $X = \mathfrak{h}^* \times G \times \mathfrak{h}^*$ be the dynamical Poisson groupoid of Thm 2.1.4 with constant $r-$ matrix taken to be $-R$. By the proof of Thm. 3.2.4 and Prop. 2.2.1, the dual groupoid $X^*$ belongs to $\mathcal{C}_{\mathfrak{h}^*}$. In the theorem below we shall give the explicit $\mathcal{C}_{\mathfrak{h}^*}$-structure of $X^*$.

If $f \in C^\infty(H \times \mathfrak{h}^* \times \mathfrak{h}^*)$ we define $\delta f$ and $D'f$ by the formulae

$$\langle \delta f, \lambda \rangle = \frac{d}{dt}\bigg|_0 f(h, p + t\lambda, u), \langle D'f, Z \rangle = \frac{d}{dt}\bigg|_0 f(h e^{tZ}, p, u),$$

and we denote by $\partial_\tau f$ the partial derivative w.r.t. $G^*$.

Theorem 5.1.4 (Dual Poisson groupoid (second form))

Let $X$ be as above.

(a) The set $\Gamma := H \times \mathfrak{h}^* \times G^*$ together with the product Poisson bracket

$$\{f, g\}_\Gamma(h, p, u) = -\langle D'g, \delta_1 f \rangle + \langle D'f, \delta_1 g \rangle - \langle p, [\delta_1 f, \delta_1 g] \rangle - \partial_\tau f \lambda^+ (T^*_1 \varphi \partial_\tau g)(u),$$
the commuting Hamiltonian actions of $H$

$$\phi_k^-(h, p, u) = (kh, p, \varphi_k h^{-1} u), \quad \phi_k^+(h, p, u) = (hk, Ad_k^* p, u)$$

with equivariant momentum maps

$$j_-(h, p, u) = Ad_{h^{-1}}^* p + I^*(u), \quad j_+(h, p, u) = p$$

($I^*$ is as in Lemma 5.1.2), and the groupoid structure

$$\alpha = j_-, \quad \beta = j_+, \quad \epsilon(q) = (1, q, 1)$$

$$(h, j_-(k, q, v), u) \cdot (k, q, v) = (hk, q, u\varphi_{k^{-1}}^+(v))$$

$$i(h, p, u) = (h^{-1}, j_-(h, p, u), \varphi_{h^{-1}}^+(u^{-1}))$$

is a Poisson groupoid in $C_{h^*}$.

(b) The Poisson groupoid $\Gamma$ of (a) is the Poisson groupoid dual $X^*$ of $X$.

Proof. (a) That the actions $\phi^\pm$ are Hamiltonian with equivariant momentum maps $j_\pm$ follows from Example 2.1.3, the Hamiltonian property of the action $\phi^l$ (Eqn. (5.1.6)), and Lemma 5.1.2. On the other hand, an easy verification, using $\varphi_{h^{-1}}^+(uv) = \varphi_{h^{-1}}^+(u)\varphi_{h^{-1}}^+(v)$ and the $Ad_{h^*}^*$ -equivariance of $I^*$, shows that the groupoid axioms (for these axioms see e.g. [W1]) are satisfied. The lengthy check that the graph of the multiplication

$$Gr(m) \subset \Gamma \times \Gamma \times \Gamma$$

is a coisotropic submanifold is postponed to the appendix.

(b) We have to show that the Lie bialgebroid tangent to $\Gamma$ is isomorphic to the Lie bialgebroid $(A(X)^*, A(X)) \simeq (\mathfrak{h}^* \times \mathfrak{h} \times \mathfrak{g}^*, \mathfrak{h}^* \times \mathfrak{h}^* \times \mathfrak{g})$ of Prop. 3.2.1 (with constant $r-$ matrix $-R$). We shall only sketch the main steps.

(i) The isomorphism $A(\Gamma) \simeq A(X)^*$. We have to compute (see the end of section 2.1) the value on $\epsilon(\mathfrak{h}^*)$ of the Lie bracket of two left invariant sections

$$X^l, X'^l : \Gamma \longrightarrow Ker T \alpha \subset T\Gamma.$$

We have

$$A(\Gamma)_q := Ker T_{(l, 1, 1)}\alpha = \{(Z, ad_Z^* q - \iota^* A, A) \mid Z \in \mathfrak{h}, A \in \mathfrak{g}^* \} \simeq \mathfrak{h} \times \mathfrak{g}^*,$$

where the identification $\simeq$ is by dropping the middle term. Now the left invariant vector field $X^l$ whose restriction to $\epsilon(\mathfrak{h}^*)$ is

$$X : \mathfrak{h}^* \longrightarrow A(\Gamma) : p \mapsto (q, Z(q), A(q))$$

is given by

$$X^l(h, q, u) = (T_1 l_h Z(q), ad_{Z(q)}^* q - \iota^* A(q), T_1 l_u Ad_{h^{-1}} A(q)).$$
A lengthy calculation then shows that

\([X, X'](q) := [X^t, X'^t]|_{\Gamma}(1, q, 1)\)

is given, after the identification \(\simeq\), by

\[
[X, X'](q) = \left( dZ'(ad^*_{dZ}q - \iota^*A) - dZ(ad^*_{dZ}q - \iota^*A') + [Z(q), Z'(q)],
\right.
\]

\[
+ dA'(ad^*_{dZ}q - \iota^*A) - dA(ad^*_{dZ}q - \iota^*A')
\]

\[- ad^*_{dZ}A' + ad^*_{dZ}A + [A(q), A'(q)]\)

where \((\mathfrak{g}^*, [, ]_*)\) is as in Eqn.(5.1.2) and all maps and differentials are evaluated at \(q \in \mathfrak{h}^*\). Thus the bracket indeed coincides, up to sign, with the one given in Prop. 3.2.1 for \(A(q) = \iota, K(q)Z = ad^*_{dZ}q\), and \(P\) as in Remark 3.2.2 with \(r-\) matrix \(-R\).

Now, applying the Lie functor to the morphism

\[
[\alpha, \beta] : H \times \mathfrak{h}^* \times G^* \to \mathfrak{h}^* \times \mathfrak{h}^* : (h, q, u) \mapsto (\alpha(h, q, u), \beta(h, q, u))
\]

shows that the anchor \(a_*(q, Z, A) = (q, ad^*_{dZ}q - \iota^*A)\).

(ii) The isomorphism \(A(H \times \mathfrak{h}^* \times G^*)^* \simeq A(X)\). The unit section is \(\epsilon : \mathfrak{h}^* \to \Gamma : q \mapsto (1, q, 1)\). Therefore \(\gamma \in N^*(\epsilon(\mathfrak{h}^*))q\) if and only if \(\gamma = (\lambda, 0, X)\) for some \(\lambda \in \mathfrak{h}^*\) and \(X \in \mathfrak{g}\).

Let \(\theta, \theta' : \mathfrak{h}^* \to N^*(\epsilon(\mathfrak{h}^*)) \simeq \mathfrak{h}^* \times \mathfrak{h}^* \times \mathfrak{g}\) be two sections expressed as \(\theta(q) = (\lambda(q), 0, X(q))\) and \(\theta'(q) = (\lambda'(q), 0, X'(q))\). We set

\[
\overline{\theta}(h, q, u) = (T^\iota_l h, 1, \lambda(q), 0, T^\iota_l u, X(q)),
\]

and similarly for \(\overline{\theta}'\). By Eqn. (3.1.2), it suffices to calculate

\[
[\overline{\theta}, \overline{\theta}'](q) = [\overline{\theta}, \overline{\theta}](1, q, 1),
\]

where the rhs is given by Eqn. (3.1.1) with Hamiltonian operator

\[
\Pi^\#_1(h, q, u)(\overline{\theta}) = (0, -\lambda(q), -\lambda^+(X(q))(u)).
\]

A calculation making use of standard properties of the Lie derivative and of the dressing field \(\lambda^+\) shows that \([\lambda(q), X(q), 0]\) coincides with that of the trivial Lie algebroid of Prop. 3.2.1. Finally observe that the anchor map, which is the restriction of \(-\Pi^\#\) to \(N^*(\epsilon(\mathfrak{h}^*))\), is given by \(a(q, X, \lambda) = (q, \lambda)\). (Note that the dressing field \(\lambda^+\) vanishes at \(u = 1\).) This concludes the proof of the theorem. ■

In the special case when \(R = 0\), we have \(G^* = \mathfrak{g}^*\) equipped with the Lie-Poisson structure. In this case \(I^* = \iota^* : \mathfrak{g}^* \to \mathfrak{h}^*\) and \(\varphi^+_h(A) = Ad^*_h(A)\). Specializing Thm 5.1.4 to this situation, we have

**Corollary 5.1.5** (Dual Poisson groupoid for \(R = 0\))

Let \(X = \mathfrak{h}^* \times G \times \mathfrak{h}^*\) be the coboundary dynamical Poisson groupoid of Thm 2.1.4 with \(R = 0\). Then the Poisson groupoid dual \(X^*\) of \(X\) is the set \(H \times \mathfrak{h}^* \times \mathfrak{g}^*\) equipped with the Poisson bracket

\[
\{f, g\}(h, p, A) = - < D'g, \delta_1 f > + < D'f, \delta_1 g >
\]

\[- < p, [\delta_1 f, \delta_1 g] > + < A, [\delta f, \delta g] >,
\]
\[ \langle \delta f, B \rangle = \frac{d}{dt} \big|_{t=0} f(h, p, A + tB) \]

and the groupoid structure

\[ \alpha(h, p, A) = Ad_h^* p + i^* A, \quad \beta(h, p, A) = p, \quad \epsilon(p) = (1, p, 0) \]

\[ (h, \alpha(k, q, B), A) \cdot (k, q, B) = (hk, q, A + Ad_h^* B) \]

\[ i(h, p, A) = (h^{-1}, \alpha(h, p, A), -Ad_h^* A). \]

We now describe the trivialization of the Lie groupoid \( \Gamma = H \times \mathfrak{h}^* \times G^* \).

Let \( H^\perp \) be the connected and simply connected Lie subgroup of \( G^* \) with \( \text{Lie}(H^\perp) = (\mathfrak{h}^\perp, [\cdot, \cdot]_s) \) and

\[ H \ltimes H^\perp \subset H \ltimes G^* \]

be the Lie subgroup with Lie algebra \( \mathfrak{h} \ltimes \mathfrak{h}^\perp \subset \mathfrak{h} \ltimes \mathfrak{g}^* \) (see the note in the proof of Prop. 5.1.3).

**Proposition 5.1.6** (Trivialization)

*Equip \( \mathfrak{h}^* \times (H \ltimes H^\perp) \times \mathfrak{h}^* \) with the trivial Lie groupoid structure over \( \mathfrak{h}^* \), and let \( \Gamma \) be the Lie groupoid in Theorem 5.1.4. If \( s \) is an arbitrary linear section \( s : \mathfrak{h}^* \to \mathfrak{g}^* \) of \( i^* \), the map

\[ \Sigma : \mathfrak{h}^* \times (H \ltimes H^\perp) \times \mathfrak{h}^* \to \Gamma \]

\[ (p, (k, u), q) \mapsto (k, q, \exp(s(p)) u \varphi_{k^{-1}}^+(\exp(-s(q)))) \]

is a Lie groupoid isomorphism.*

Proof. We use an elementary device (see [M1]) according to which if \( \sigma : \mathfrak{h}^* \to \Gamma \) is a global smooth section of the restriction of \( \alpha \) to the fiber \( \beta^{-1}(0) \) and \( G' \subset G \) is the isotropy subgroup at 0, then the map

\[ \Sigma : \mathfrak{h}^* \times G' \times \mathfrak{h}^* \to \Gamma \]

\[ (p, x', q) \mapsto \sigma(p) \cdot x' \cdot i(\sigma(q)) \]

is a Lie groupoid isomorphism.

Now \( G' = \beta^{-1}(0) \cap \alpha^{-1}(0) = H \ltimes H^\perp \), while \( \alpha|_{\beta^{-1}(0)} : \beta^{-1}(0) \to \mathfrak{h}^* : (h, 0, u) \mapsto I^*(u) \). Observe that for any linear section \( s : \mathfrak{h}^* \to \mathfrak{g}^* \), the map

\[ \sigma : \mathfrak{h}^* \to \Gamma : p \mapsto (1, 0, \exp s(p)) \]

is a smooth section of \( \alpha|_{\beta^{-1}(0)} \) since \( I^*(\sigma(p)) = I^*(\exp(s(p))) = \exp_{h^*} i^*(s(p)) = p \). Calculating \( \sigma(p) \cdot (h, 0, u) \cdot i(\sigma(p)) \) in \( \Gamma \) immediately yields the claim. \( \blacksquare \)

**Caveat** Note that if \( \mathfrak{h} \neq 0 \) the group \( G' \) is not in general isomorphic to the the Poisson Lie group dual \( \mathfrak{g}^* \). For example, if \( R = 0 \), \( G^* = \mathfrak{g}^* \) but \( G' = H \ltimes \mathfrak{h}^\perp \). So \( G' \) and \( G^* \) may differ even topologically.

**Remark 5.1.7** By Thm. 3.2.4, the Poisson bracket \( \{ \cdot, \cdot \}_* \) on \( \mathfrak{h}^* \times (H \ltimes H^\perp) \times \mathfrak{h}^* \) defined by \( \{ \Sigma^*, \Sigma^* g \}_* := \Sigma^* \{ f, g \}_\Gamma \) has the form given by Thm. 2.2.5. However,
even for standard Poisson Lie groups, the explicit bracket transport turns out to be rather cumbersome.

We close this subsection with the following

**Example 5.1.8** Let \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_- \) be the root space decomposition of a complex simple Lie algebra \( \mathfrak{g} \), as in section 4. Let \( R = \Pi_{\mathfrak{n}_+} - \Pi_{\mathfrak{n}_-} \) be the standard \( r- \)matrix. In what follows, we shall scale the Poisson bracket by \( 1/2 \) to match with standard conventions. Let \( N_{\pm} \) be the (connected and simply connected) unipotent subgroups of \( G \) with Lie algebra \( \mathfrak{n}_{\pm} \). Note that \( H = \mathfrak{h} \cong \mathfrak{h}^* \).

The dual group \( G^* \) is the set \( \mathfrak{h} \times \mathfrak{h} \times N_+ \times N_- \) with semi-direct group law

\[
(Z, A, B) \cdot (Z', A', B') = (Z + Z', A e^{\frac{Z}{2}} A' e^{-\frac{Z}{2}}, e^{-\frac{Z}{2}} B' e^{\frac{Z}{2}}).
\]

The dressing action of \( \mathfrak{h} \) on \( G^* \) is given by

\[
\phi_+^Y(Z, A, B) = (Z, e^Y A e^{-Y}, e^Y B e^{-Y}).
\]

The Poisson Lie group \( H \times G^* \) of Prop. 5.1.3 (b) is the set \( \mathfrak{h} \times \mathfrak{h} \times N_+ \times N_- \) with group law

\[
(Y, Z, A, B) \cdot (Y', Z', A', B') = (Y + Y', Z + Z', A e^{\frac{Z}{2}} A' e^{-\frac{Z}{2}}, e^{-\frac{Z}{2}} e^Y B' e^{-Y} e^{\frac{Z}{2}} B),
\]

and hence the vertex subgroup \( G' \) of the groupoid \( \Gamma \) is the set \( \mathfrak{h} \times N_+ \times N_- \) with group law

\[
(Y, A, B) \cdot (Y', A', B') = (Y + Y', A e^Y A' e^{-Y}, e^Y B' e^{-Y} B).
\]

Finally, the map

\[
\mathfrak{h}^* \times G' \times \mathfrak{h}^* \rightarrow \Gamma = \mathfrak{h}^* \times \mathfrak{h} \times N_+ \times N_- \times \mathfrak{h}^*
\]

\[
(p, Y, A, B, q) \mapsto (q, Y, e^{\frac{p}{2}} A e^{-\frac{p}{2}}, e^{-\frac{p}{2}} B e^{\frac{p}{2}}, p - q)
\]

gives an explicit trivialization of the Lie groupoid \( \Gamma \).

### 5.2. Construction of the associated symplectic double groupoid.

Our goal in this subsection is to construct, for the constant \( r- \)matrix case (taken to be \(-R\)), a symplectic double groupoid having \( \mathfrak{h}^* \times G \times \mathfrak{h}^* \) and \( \mathfrak{h} \times \mathfrak{h}^* \times G^* \) as its side Poisson groupoids.

We begin by recalling the notion of double Lie groupoids [E], [M3], and symplectic double groupoids [W1], [LW2], [M2].

**Definition 5.2.1**

(a) A double Lie groupoid consists of a quadruple \((S; \mathcal{H}, \mathcal{V}, B)\) where \( \mathcal{H} \) and \( \mathcal{V} \) are Lie groupoids over \( B \), and \( S \) is equipped with two Lie groupoid structures, a horizontal structure with base \( \mathcal{V} \), and a vertical structure with base \( \mathcal{H} \), such that the structure maps (source, target, multiplication, unit section and inversion) of each
groupoid structure on \( S \) are morphisms with respect to the other. We call \( \mathcal{H} \) and \( \mathcal{V} \) the side groupoids of \( S \), and \( B \) the double base. \((S; \mathcal{H}, \mathcal{V}, B)\) is displayed as in Fig. 5.2.1 below.

(b) A double Lie groupoid \((S; \mathcal{H}, \mathcal{V}, B)\) is called symplectic if \( S \) is equipped with a symplectic structure such that both \( S \Rightarrow \mathcal{V} \) and \( S \Rightarrow \mathcal{V} \) are symplectic groupoids.

\[
\begin{array}{ccc}
S & \xrightarrow{\tilde{\alpha}_S, \tilde{\beta}_S} & \mathcal{V} \\
\tilde{\alpha}_V, \tilde{\beta}_V & \downarrow & \downarrow \alpha_V, \beta_V \\
\mathcal{H} & \xrightarrow{\alpha_H, \beta_H} & B
\end{array}
\]

Fig. 5.2.1

We shall consider the case where the Poisson Lie group \( G \) is complete. In this case, the Drinfeld double \( D \) can be identified with \( G \times G^* \) [STS], [LW1] with multiplication

\[
(g_1, u_1) \cdot (g_2, u_2) = ((\varphi_{g_1}^-(g_2^{-1}))^{-1} g_2, u_1 (\varphi_{g_1}^+(u_2^{-1}))^{-1}).
\]

(5.2.2)

As a first step in the construction, we show that \( X = \mathfrak{h}^* \times G \times \mathfrak{h}^* \) and \( X^* = H \times \mathfrak{h}^* \times G^* \) form a matched pair of Lie groupoids in the sense of the following

**Definition 5.2.2** [M3] Two Lie groupoids \( \mathcal{V} \) and \( \mathcal{H} \) over the same base \( B \) are said to form a matched pair of Lie groupoids iff the manifold

\[
\mathcal{V} \ast \mathcal{H} = \{(v, h) \in \mathcal{V} \times \mathcal{H} \mid \beta_\mathcal{V}(v) = \alpha_\mathcal{H}(h)\}
\]

admits a Lie groupoid structure over \( B \) such that

(a) the maps \( h \mapsto \overline{h} = (\epsilon_\mathcal{V}(\alpha_\mathcal{H}(h)), h) \) and \( v \mapsto \overline{v} = (v, \epsilon_\mathcal{H}(\beta_\mathcal{V}(v))) \) are morphism of Lie groupoids from \( \mathcal{H} \) and \( \mathcal{V} \) to \( \mathcal{V} \ast \mathcal{H} \) respectively,

(b) the map \( \mathcal{V} \ast \mathcal{H} \rightarrow \mathcal{V} \ast \mathcal{H} : (v, h) \mapsto \overline{v\overline{h}} \) is a diffeomorphism.

In this case, the groupoid \( \mathcal{V} \ast \mathcal{H} \) is called the matched product of \( \mathcal{V} \) and \( \mathcal{H} \).

**Proposition 5.2.3** The Lie groupoids \( X \) and \( X^* \) form a matched pair with matched product given by the trivial groupoid \( \mathfrak{h}^* \times M \times \mathfrak{h}^* \) where the vertex group \( M = H \times (G \times G^*) \) is the direct product of \( H \) with the Drinfeld double (see (5.2.2)).

Proof. Clearly, \( X \ast X^* \) may be identified with the manifold \( \mathfrak{h}^* \times M \times \mathfrak{h}^* \). Equip the latter with the trivial groupoid structure. The groupoids \( X \) and \( X^* \) are embedded as wide subgroupoids of \( \mathfrak{h}^* \times M \times \mathfrak{h}^* \) through the morphisms \((p, g, q) \mapsto (p, 1, g, 1, q), (h, p, u) \mapsto (Ad_{h^{-1}}^p + I^*(u), h, h, u), p)\). Finally, for \((p, h, g, u, q) \in \mathfrak{h}^* \times M \times \mathfrak{h}^*\), we have the unique factorization

\[
(p, h, g, u, q) = (p, 1, \varphi^-_{gh^{-1}}(g^{-1}), 1, Ad_{h^{-1}}^p + I^*(\varphi_{gh}^+(u)))
\]

\[
\cdot (Ad_{h^{-1}}^p + I^*(\varphi_{gh}^+(u)), h, h, \varphi_{gh}^+(u), q).
\]

Hence it follows that \((X, X^*)\) is a matched pair. \( \blacksquare \)
that these extended objects act on the unextended ones through groupoid actions.

We therefore obtain two maps

\[ (\psi^- : (H \times h^* \times G^*) \ast_{\alpha_X} (h^* \times G \times h^*) \longrightarrow h^* \times G \times h^*) \] (5.2.4.a)

\[ ((h, p, u), (p, g, q)) \mapsto (Ad_{h}^{-1}p + I^*(u), \varphi_{u}(hgh^{-1}), Ad_{h}^{-1}q + I^*(\varphi_{hgh^{-1}}(u))) \}, \]

and

\[ (\psi^+ : (H \times h^* \times G^*) \ast_{\beta_X} (h^* \times G \times h^*) \longrightarrow H \times h^* \times G^*) \] (5.2.4.b)

\[ ((h, p, u), (p, g, q)) \mapsto (h, q, \varphi_{hgh^{-1}}(u)). \]

Proposition 5.2.4 \( \psi^- \) is a left groupoid action of \( X^* \) on \( X \) with moment map \( \alpha_X \) and \( \psi^+ \) is a right groupoid action of \( X \) on \( X^* \) with moment map \( \beta_X \). Furthermore, the following conditions are satisfied

(a) \( \beta_X(\psi^-_{g^-}(g_+)) = \alpha_X(\psi^+_{g^+}(g_-)), \)

for all \( g_+ \in X, g_- \in X^* \) with \( \beta_X(g_-) = \alpha_X(g_+) \),

(b) \( \psi^-_{g^-}(g_+g'_+) = \psi^-_{g^-}(g_+}\psi^-_{g^-+g'_-}(g'_+) \),

for all \( g_- \in X^*, g_+, g'_+ \in X \) with \( \beta_X(g_-) = \alpha_X(g_+), \beta_X(g_+) = \alpha_X(g'_+) \),

(c) \( \psi^+_{g^+}(g_+g'_-) = \psi^+_{g^+}(g_-)\psi^+_{g^+g'_-}(g'_-) \),

for all \( g_+ \in X, g_-, g'_- \in X^* \) with \( \beta_X(g_-) = \alpha_X(g'_-), \beta_X(g'_-) = \alpha_X(g_+) \).

Proof. The proof consists of direct checking and we shall omit the details. See, however, Prop. 5.2.9 below.

From standard consideration [M3], the upshot of the above proposition is that one can construct a vacant double Lie groupoid \( (S_{vac}; X^*, X, h^*) \) (vacant means that the double source map \( \beta_+ : S_{vac} \rightarrow X^* \ast_{\beta} X \) is a diffeomorphism) having \( X \) and \( X^* \) as its side groupoids. Indeed the horizontal structure of the vacant double is given by the left action groupoid \( X^* \ltimes X \) corresponding to \( \psi^- \), while the vertical structure is given by the right action groupoid \( X^* \rtimes X \) associated with \( \psi^+ \). However, \( S_{vac} \) is not the correct underlying double Lie groupoid of the symplectic double groupoid which we are looking for, as is clear from dimension considerations. Nevertheless, as we shall show in what follows, the sought for double Lie groupoid can be constructed by extending the objects \( X \) and \( X^* \). It turns out that these extended objects act on the unextended ones through groupoid actions.
which restrict to $\psi^\pm$, and the corresponding left/right action groupoids then give the desired horizontal/vertical structures. Before we carry out the details of this construction, let us make an important remark. As we know from Thm 5.1.4, the Poisson structure on $X^*$ is the product of the standard symplectic structure on $H \times h^*$ and the multiplicative structure on $G^*$. Since $H \times h^*$ is symplectic, the coarse groupoid $(H \times h^*) \times (H \times h^*)^\sim \Rightarrow H \times h^*$ is a symplectic groupoid. On the other hand, there is a symplectic groupoid $G \times G^* \Rightarrow G^*$ [Lu] with structure maps given as follows:

$$
\alpha(g, u) = \varphi_{g^{-1}}^+(u^{-1})^{-1}, \quad \beta(g, u) = u
$$

(5.2.5)

Therefore the product groupoid

$$(H \times h^*) \times (H \times h^*) \Rightarrow (G \times G^*) \Rightarrow H \times h^* \times G^*
$$

(5.2.6)

is a symplectic groupoid over $X^*$. It turns out that this product groupoid is isomorphic to the right action groupoid alluded to above.

We now introduce the extensions of $X$ and $X^*$. Since $H$ is a groupoid over a point, we have the product groupoid

$$X_e^* = X^* \times H \Rightarrow h^*
$$

(5.2.7).

Let $J_+ = \alpha_X$ and define $\Psi^- : X_e^* \times J_+ \times X \rightarrow X$ by

$$
\Psi^-_{(h, p, u, k)}(p, g, q) = (Ad_{h^{-1}}^* p \cdot \alpha(u), \varphi_u^-(h g k^{-1}), Ad_k^* q \cdot \beta^+_{h g k^{-1}}(u)).
$$

(5.2.8)

**Proposition 5.2.5** $\Psi^-$ is a left groupoid action of $X_e^*$ on $X$ with moment map $J_+$ such that $\Psi^-_{(h, p, u, k)}(p, g, q) = \psi^-_{(h, p, u)}(p, g, q)$ for all $(h, p, u) \in X^*$, $(p, g, q) \in X$.

Proof. Clearly, $(\tilde{x}, x) = ((\tilde{h}, Ad_{h^{-1}}^* p + \alpha(u), \tilde{u}, \tilde{k}), (h, p, u, k))$ is a composable pair in $X_e^*$ and we have

$$\tilde{x} x = (\tilde{h} h, p, \tilde{u} \varphi_{h^{-1}}^+(u), \tilde{k} k).
$$

Therefore,

$$
\Psi^-_{\tilde{x} x}(p, g, q) = (Ad_{(h h)^{-1}}^* p \cdot \alpha(u), \varphi_{(h h)^{-1}}^+(u), \varphi_{(h h)^{-1}}^+ (u), \tilde{h} h g (\tilde{k} k)^{-1},
$$

$$Ad_{(h h)^{-1}}^* q \cdot \beta^+_{h h g (\tilde{k} k)^{-1}}(u)).
$$

On the other hand,

$$
\Psi^-_{x \tilde{x}}(p, g, q) = (Ad_{(h h)^{-1}}^* p + Ad_{h^{-1}}^* \alpha(u) + \alpha(u), \varphi_{h h g (\tilde{k} k)^{-1}}^+(u), \tilde{h} h g (\tilde{k} k)^{-1},
$$

$$Ad_{(h h)^{-1}}^* q + Ad_k^* \beta^+_{h h g (\tilde{k} k)^{-1}}(u) + I^* (\varphi_{h h g (\tilde{k} k)^{-1}}^+ (u))).
$$

Since $I^*$ is an $H-$ equivariant homomorphism, the equality of the first components is clear. Now using the same property of $I^*$, we have

$$
Ad_{k}^* I^* (\varphi_{h h g (\tilde{k} k)^{-1}}^+ (u)) + I^* (\varphi_{h h g (\tilde{k} k)^{-1}}^+ (u))
$$

$$= I^* (\varphi_{h h g (\tilde{k} k)^{-1}}^+ (u)) \varphi_{h h g (\tilde{k} k)^{-1}}^+ (u)).
$$
But from Eqn. (5.1.4) and the fact that $H$ is a trivial Poisson Lie subgroup of $G$, we find

$$\varphi^+_{hgh(\tilde{k}k-1)}(\tilde{u}\varphi^+_{h-1}(u)) = \varphi^+_{\varphi_{h^{-1}}^{-1}(\varphi_{h^{-1}}(\tilde{u}))} \varphi^+_{hgh(\tilde{k}k-1)}(\varphi^+_{h^{-1}}(u))$$

$$= \varphi^+_{h\varphi^+_{h^{-1}}^{-1}(\varphi_{h^{-1}}(u))} \varphi^+_{hgh(\tilde{k}k-1)}(\varphi^+_{h^{-1}}(u))$$

$$= \varphi^+_{h\varphi^+_{h^{-1}}^{-1}(\varphi_{h^{-1}}(u))} \varphi^+_{hgh(\tilde{k}k-1)}(\varphi^+_{h^{-1}}(u)).$$

Hence we have equality of the third components. Finally it follows from the above calculation that

$$\varphi^-_{\tilde{u}}(\tilde{h}\varphi^-_{u}(h\tilde{k}k^{-1})\tilde{k}^{-1}) = \varphi^-_{\tilde{u}}\varphi^-_{\varphi_{h^{-1}}^{-1}(\tilde{u})}(hgh(\tilde{k}k^{-1}))$$

$$= \varphi^-_{\tilde{u}\varphi_{h^{-1}}^{-1}(\tilde{u})}(hgh(\tilde{k}k^{-1})).$$

This completes the proof that $\Psi^-_{\tilde{x}x} = \Psi^-_{\tilde{x}x}$. The assertion on the relationship between $\Psi^-$ and $\psi^-$ is clear. □

The following corollary is a direct consequence of the definition of an action groupoid.

**Corollary 5.2.6** The left action groupoid $X^*_e \times X \rightarrow X$ corresponding to $\Psi^-$ has structure maps given by

$$\tilde{\alpha}_H : X^*_e \times X \rightarrow X : ((h, p, u, k), (p, g, q)) \mapsto \Psi^-_{(h, p, u, k)}(p, g, q)$$

$$\tilde{\beta}_H : X^*_e \times X \rightarrow X : ((h, p, u, k), (p, g, q)) \mapsto (p, g, q)$$

$$\tilde{m}_H : (X^*_e \times X) \ast (X^*_e \times X) \rightarrow X^*_e \times X$$

$$= ((h_1 p_1, u_1, k_1, \Psi^-_{(h_2, p_2, u_2, k_2)}(p_2, g_2, q_2)) \cdot ((h_2, p_2, u_2, k_2), (p_2, g_2, q_2))$$

$$= ((h_1 h_2, p_2, u_1 \varphi^+_{h_{1^{-1}}}(u_2), k_1 k_2), (p_2, g_2, q_2))$$

$$\tilde{\epsilon}_H : X^*_e \times X \rightarrow X^*_e \times X : ((p, g, q)) \mapsto ((1, p, 1, 1), (p, g, q))$$

$$\tilde{\iota}_H : X^*_e \times X \rightarrow X^*_e \times X : ((h, p, u, k), (p, g, q)) \mapsto$$

$$(h^{-1}, \text{Ad}^*_h p + \iota^*(u), \varphi^+_h(u^{-1}), k^{-1}),$$

$$\text{Ad}^*_h p + \iota^*(u), \varphi^-_{h\text{Ad}^*_h p + \iota^*(u)}(h\text{Ad}^*_h p + \iota^*(u)) \mapsto \Psi^-_{h\text{Ad}^*_h p + \iota^*(u)}(h\text{Ad}^*_h p + \iota^*(u))).$$

For the extension of $X$, we consider the coarse groupoid $H \times H \Rightarrow H$ and let

$$X_e = (H \times H) \times X \Rightarrow H \times \mathfrak{h}^*$$

be the product groupoid. Introduce the map

$$J_- : X^* \rightarrow H \times \mathfrak{h}^* : (h, p, u) \mapsto (h, p)$$

and define

$$\Psi^+ : X^*_e \ast J_- X_e \rightarrow X^* : \Psi^+_{(h, k, p, g, q)}(h, p, u) = (k, q, \varphi^+_{h\text{Ad}^*_h p + \iota^*(u)}(h\text{Ad}^*_h p + \iota^*(u)))).$$

(5.2.9)

(5.2.10)

(5.2.11)
Proposition 5.2.7 \( \Psi^+ \) is a right groupoid action of \( X_e \) on \( X^* \) with moment map \( J_- \) and we have \( \Psi^+_{(h,h,p,g,q)}(h,p,u) = \Psi^+_{(p,g,q)}(h,p,u) \) for all \( (h,p,u) \in X^*, (p,g,q) \in X \).

Proof. This is clear. \( \blacksquare \)

Corollary 5.2.8 The right action groupoid \( X^* \rtimes X_e \rightrightarrows X^* \) corresponding to \( \Psi^+ \) has structure maps given by

\[
\tilde{\alpha}_\Psi : X^* \times X_e \to X^* : ((h,p,u),(h,k,p,g,q)) \mapsto (h,p,u)
\]

\[
\tilde{\beta}_\Psi : X^* \rtimes X \to X^* : ((h,p,u),(h,k,p,g,q)) \mapsto \Psi^+_{(h,k,p,g,q)}(h,p,u)
\]

\[
\tilde{m}_\Psi : (X^* \rtimes X_e) \ast (X^* \rtimes X_e) \to X^* \rtimes X_e :
\]

\[
((h_1,p_1,u_1),(h_1,k_1,p_1,g_1,q_1)) \cdot (\Psi^+_{(h_1,k_1,p_1,g_1,q_1)}(h_1,p_1,u_1),(k_1,k_2,q_1,g_2,q_2)) = ((h_1,p_1,u_1),(h_1,k_2,p_1,g_1,q_2))
\]

\[
\tilde{e}_\Psi : X^* \to X^* \rtimes X_e : (h,p,u) \mapsto ((h,p,u),(h,h,1,p))
\]

\[
\tilde{i}_\Psi : X^* \rtimes X_e \to X^* \rtimes X_e : ((h,p,u),(h,k,p,g,q)) \mapsto ((k,q,\varphi^+_{(h,k,q,g)}(u),(h,h,q,g^{-1},p)). \blacksquare
\]

Let \( Pr_1 : X^* \to H \) be the projection onto the first factor of \( X^* = H \rtimes \mathfrak{h}^* \rtimes G^* \).

Proposition 5.2.9 The groupoid actions \( \Psi^\pm \) satisfy the following properties:

(a) \( \beta_X(\Psi^-(g_-,k))(g_+)) = \alpha_X(\Psi^+_{(Pr_1(g_-),k,g_+)}(g_-)) \)

for all \( g_- \in X^*, g_+ \in X \) with \( \beta_X(g_-) = \alpha_X(g_+) \) and all \( k \in H \).

(b) \( \Psi^-_{(g_-,k')} (g_++g'_+) = \Psi^-_{(g_-,k)} (g_+) \Psi^-_{(Pr_1(g_-),k,g_+)} (g_-) \Psi^+_{(Pr_1(g_-),k',g_+)} (g'_+) \)

for all \( g_- \in X^*, g_+, g'_+ \in X \) with \( \beta_X(g_-) = \alpha_X(g_+), \beta_X(g_+) = \alpha_X(g'_+) \) and for all \( k, k' \in H \).

(c)

\[
\Psi^+_{(Pr_1(g_-)Pr_1(g'_-),k,k',g_+)}(g_-g'_-) = \Psi^+_{(Pr_1(g_-),k,\Psi^-_{(g'_-,k')} (g_+))} (g_-) \Psi^+_{(Pr_1(g_-),k',g_+)} (g'_-)
\]

for all \( g_+ \in X, g_-, g'_- \in X^* \) with \( \beta_X(g_-) = \alpha_X(g'_-), \beta_X(g'_-) = \alpha_X(g_+) \) and for all \( k, k' \in H \).

Proof. We shall check (b) and (c).

(b) Let \( g_- = (h,p,u), g_+ = (p,g_1,q), g'_+ = (q,g_2,r) \). Then

\[
\Psi^-(g_+,g'_+) = (Ad_{h^{-1}} p + I^*(u), \varphi^{-h g_1 g_2 k'^{-1}} (u), Ad_{k'^{-1}} r + I^*(\varphi^+_{h g_1 g_2 k'^{-1}} (u)));
\]
Proof. We have to show that the structure maps of the horizontal (resp. vertical) structure on \( S \) is a groupoid morphism. The rest of the proof will be left to the interested reader.

(c) Let \( g_- = (h, p, u), g'_- = (h', p', u') \) satisfy \( \beta(g_-) = \alpha(g'_-), \) i.e. \( p = Ad_{h'}^{-1}p' + I^*(u') \), and let \( g_+ = (p', g, q) \) so that \( \beta(g'_- - (g'_-)) = \alpha(g_-) \). We have

\[
\Psi_{(g_-, k)}^{-} (\Psi_{(Pr_1(g_-), k, g_+)}^{*}(g'_-)) = (\text{Ad}_{h^{-1}}^* p + I^*(u), \\
\varphi_{h^{-1}}^{-}(h g_1 k^{-1}) \varphi_{h^{-1}}^{-}(k g_2 k')^{-1}, \text{Ad}_{h^{-1}}^* r + I^*(\varphi_{h^{-1}}^{-}(h g_2 k^{-1})))
\]

Hence the assertion follows from (5.1.4).

On the other hand,

\[
\Psi_{(Pr_1(g_-), k, g_+)}^{*}(g_- g'_-) \Psi_{(Pr_1(g_-), k', g_+)}^{*}(g'_-)
\]

\[
= (k k', q, \varphi_{h^{-1}}^+ (k k')^{-1}(h g_1 k^{-1})(g_2 k'^{-1})(u) \varphi_{h^{-1}}^+ (k k') X).
\]

The assertion then follows from the calculation in the proof of Prop. 5.2.5. ■

Let \( S \) be a double Lie groupoid. Then \( S \) supports both the left action groupoid structure \( X_e^* \times X \) and the right action groupoid structure \( X^* \times X_e \).

**Theorem 5.2.10** If the horizontal structure on \( S \) is \( X_e^* \times X \) and the vertical structure on \( S \) is \( X^* \times X_e \), then \( (S, X^*, X, e^*) \) is a double Lie groupoid.

Proof. We have to show that the structure maps of the horizontal (resp. vertical) structure on \( S \) are groupoid morphisms with respect to the vertical (resp. horizontal) structure. We shall illustrate the role played by the properties in Prop. 5.2.9 by checking that \( \tilde{\alpha}_H \) and \( \tilde{\eta}_V \) are groupoid morphisms. The rest of the proof will be left to the interested reader.

(i) \( \tilde{\alpha}_H \) is a groupoid morphism.

\[
\tilde{\alpha}_H((g_-, k, g_+) : (\Psi_{(Pr_1(g_-), k, g_+)}^{*}(g'_-), k', g'_+)) \]

\[
= \tilde{\alpha}_H(g_-, k', g_+ g'_+)
= \Psi_{g_+ g'_+}^{-}(g_+ g'_+)
= \Psi_{g_+ g'_+}^{-}(g_+ g'_+) \Psi_{(Pr_1(g_-), k, g_+)}^{*}(g'_-) \Psi_{(Pr_1(g_-), k, g_+)}^{*}(g'_-) \]

\[
= \tilde{\alpha}_H(g_-, k, g_+) \tilde{\alpha}_H(\Psi_{(Pr_1(g_-), k, g_+)}^{*}(g'_-), k', g'_+).
\]

(ii) \( \tilde{\eta}_V \) is a groupoid morphism.
\[\tilde{i}_V((g_-, k, \Psi^-_{(g'_-, k')}(g'_+))(g'_-, k', g_+))\]
\[= \tilde{i}_V(g_-, k, k', g'_+)(g'_-, g'_+)\]
\[= (\Psi^-_{(Pr_1(g_-), k, k')}(g_-, Pr_1(g'_-, g'_+))(g'_-, g'_+), Pr_1(g_-)Pr_1(g'_-, g'_+), g'_+), g'_+), g'_+))\]
\[= (\Psi^-_{(Pr_1(g_-), k, k')}(g_-, Pr_1(g'_-, g'_+))(g'_-, g'_+), Pr_1(g_-)Pr_1(g'_-, g'_+), g'_+), g'_+))\]
\[= (\Psi^-_{(Pr_1(g_-), k, k')}(g_-, Pr_1(g'_-, g'_+)g'_+), Pr_1(g_-)Pr_1(g'_-, g'_+), g'_+), g'_+))\]
\[= \tilde{i}_V(g_-, k, \Psi^-_{(g'_-, k')}(g'_+))(g'_-, k', g_+). \blacksquare\]

To clarify the relation between the vacant double Lie groupoid \(\mathcal{S}_{vac; X^*, X, h^*}\) and the double Lie groupoid \(\mathcal{S}; X^*, X, h^*\) in the above theorem, we introduce the following definition

**Definition 5.2.11** Let \((S_1; \mathcal{H}_1, \mathcal{V}_1, P_1)\) be a double Lie groupoid. A double Lie subgroupoid of \((S_1; \mathcal{H}_1, \mathcal{V}_1, P_1)\) is a double Lie groupoid \((S_2; \mathcal{H}_2, \mathcal{V}_2, P_2)\) such that the Lie groupoids \(S_2 \supseteq \mathcal{H}_2, S_2 \supseteq \mathcal{V}_2, \mathcal{H}_2 \supseteq P_2, \mathcal{V}_2 \supseteq P_2\) are respectively Lie subgroupoids of \(S_1 \supseteq \mathcal{H}_1, S_1 \supseteq \mathcal{V}_1, \mathcal{H}_1 \supseteq P_1, \mathcal{V}_1 \supseteq P_1\).

**Corollary 5.2.12** The vacant double Lie groupoid \(\mathcal{S}_{vac; X^*, X, h^*}\) associated with the matched pair \((X, X^*)\) is a double Lie subgroupoid of the double Lie groupoid \(\mathcal{S}; X^*, X, h^*\) in Thm 5.2.10.

Proof. As the side groupoids of the two double groupoids Lie groupoids are identical, it suffices to show that the Lie groupoids \(S_{vac} \supseteq X, S_{vac} \supseteq X^*\) are respectively Lie subgroupoids of \(S \supseteq X, S \supseteq X^*\). For the horizontal structures, it suffices to observe that \(X^* \times X \to X^* \times X : (g_-, g_+)) \mapsto (g_-, Pr_1(g_-), g_+)\) is an injective immersion. The other case is similar. \(\blacksquare\)

We now turn to the description of the symplectic properties of the double Lie groupoid \(\mathcal{S}; X^*, X, h^*\) of Thm. 5.2.10. (Recall that \(X = h^* \times G \times h^*\) is the dynamical groupoid for the constant \(r\)- matrix \(-R\).)

To begin with, a simple computation (using Eqn.(5.1.4)) shows that the map

\[\rho : (H \times h^*) \times (H \times h^*) \times G \times G^* \to X^* \times X_e\]
\[(h, p, k, q, g, u) \mapsto ((h, p, \varphi_g^+, (u^{-1})^{-1}), (h, k, p, h^{-1} \varphi_{g^{-1}}(g^{-1})^{-1}k, q))\] \hspace{1cm} (5.2.12)

is an isomorphism of groupoids; here the domain is the product groupoid of Eqn (5.2.6) and the range is the right action groupoid of Cor. 5.2.8.
Recall that the Poisson bracket of the symplectic groupoid $S' = (H \times \mathfrak{h}^*) \times (H \times \mathfrak{h}^*) \rightrightarrows X^*$ of Eqn. (2.5.6) is explicitly given by

$$\{F, F'\}_{S'}(h, p, k, q, g, u)$$

$$= - < D'_1 F', \delta_1 F > + < D'_1 F, \delta_1 F' > - < p, [\delta_1 F, \delta_1 F'] >$$

$$+ < D'_2 F', \delta_2 F > - < D'_2 F, \delta_2 F' > + < q, [\delta_2 F, \delta_2 F'] >$$

$$- < \partial F, \lambda_-(DF')(g) > + < \partial F, \lambda_+(DF')(u) >$$

$$- < D'F', D_* F > + < D'F, D_* F' >,$$

(5.2.13)

where the indices 1 and 2 indicate partial derivatives and left/right gradients w.r.t. the appropriate factor in the first and second copies of $H \times \mathfrak{h}^*$ and the index $*$ indicates partial derivative w.r.t. $G^*$. Using the bijection $\rho$ we may transport this Poisson bracket to $S$ by setting

$$\{F_1, F_2\}_S \circ \rho = \{F_1 \circ \rho, F_2 \circ \rho\}_{S'}.$$

Since $\rho$ is a Lie groupoid isomorphism, $(X^* \ltimes X, \{, \}_S)$ is a symplectic groupoid. We now come to the main result of this section.

**Theorem 5.2.13** The double Lie groupoid $(S; X^*, X, \mathfrak{h}^*)$ where $S$ is equipped with the Poisson bracket $\{, \}_S$ is a symplectic double groupoid.

In order to prove the theorem, it remains to show that $(X^*_e \ltimes X, \{, \}_S)$ is a symplectic groupoid. For this purpose, we shall use the isomorphic image $S' \rightrightarrows X$ of $X^*_e \ltimes X \rightrightarrows X$ under the map $\rho$ and the bracket $\{, \}_{S'}$. By direct computation, $S' \rightrightarrows X$ has target and source maps

$$\alpha(h, p, k, q, g, u)$$

$$= (Ad_{h^{-1}}^* p + I^*(\varphi_{g^{-1}}^*(u^{-1})^{-1}), g, Ad_{k^{-1}}^* q + I^*(u)),$$

$$\beta(h, p, k, q, g, u)$$

$$= (p, h^{-1} \varphi_{u^{-1}}^*(g^{-1})^{-1} k, q),$$

(5.2.14.a)

multiplication map

$$m((h_1, p_1, k_1, q_1, u_1), (h_2, p_2, k_2, q_2, g_2, u_2))$$

$$= (h_1 h_2, p_2, k_1 k_2, g_2, g_1, u_1 \varphi_{k_1^{-1}}^+(u_2)),$$

(5.2.14.b)

where

$$(p_1, h_1^{-1} \varphi_{u_1^{-1}}^*(g_1^{-1})^{-1} k_1, q_1) = (Ad_{h_2^{-1}}^* p_2 + I^*(\varphi_{g_2^{-1}}^*(u_2^{-1})^{-1}), g_2, Ad_{k_2^{-1}}^* q_2 + I^*(u_2)),$$

and unit section

$$\epsilon(p, g, q) = (1, p, 1, q, g, 1).$$

(5.2.14.c)

We shall verify the conditions of the following proposition of Libermann in [L], and show that the unique Poisson structure induced on the base $X$ indeed coincides with that of Thm 2.1.5.
Moreover, the canonical projection $S \to P$ be a Lie groupoid equipped with a symplectic form $\Omega$. If $\Gamma$ is $\beta-$ connected, the $\alpha-$ foliations and the $\beta-$ foliations are symplectically orthogonal, and $\epsilon(P) \subset \Gamma$ is Lagrangian, then $\Gamma$ is a symplectic groupoid over $P$.

In our case, that the $\beta-$ fibers of $S' \Rightarrow X$ are connected and $\epsilon(h^*)$ is Lagrangian are easy to check and we shall leave the details to the reader. In order to establish the other condition, we begin with two Propositions which allow us to identify the target and source maps of (5.2.14.a) with canonical projections of natural group actions.

Let $H \times (H \ltimes G^*)$ be the product of $H$ with the Lie group $H \ltimes G^*$ of Prop. 5.1.3 (b).

**Proposition 5.2.15** The left action of $S' \Rightarrow X$ on itself induces a left action of the group $H \times (H \ltimes G^*)$ on $S'$ given by

$$(h', k', u') \cdot (h, p, k, q, g, u) = (h'h, p, k'k, q, h'\varphi^{-}(u')^{(g^{-1})k'k^{-1}}, u'\varphi_{k'k^{-1}}^{+}(u)).$$

Moreover, the canonical projection $S' \to H \times (H \ltimes G^*) \backslash S' \simeq X$ coincides with $\beta$.

Proof. Clearly,

$$(h', k', u') \cdot (h, p, k, q, g, u) = m((h', p', k', q', g', u'), (h, p, k, q, g, u))$$

for unique $(p', q', g') \in h^* \ltimes h^* \times G$ and it is easy to show that this defines a left $H \times (H \ltimes G^*)$ action on $S'$. Now, if we identify each $H \times (H \ltimes G^*)$--orbit with its unique intersection with $\{1\} \times h^* \times \{1\} \times h^* \times G \times \{1\}$, then $H \times (H \ltimes G^*) \backslash S' \simeq X$ and an easy calculation shows the projection map coincides with $\beta$. ■

In a similar way, we have

**Proposition 5.2.16** The right action of $S' \Rightarrow X$ on itself induces a right action of the group $H \times (H \ltimes G^*)$ on $S'$ given by

$$(h, p, k, q, g, u) \cdot (h', k', u') = (hh', Ad_{h'}p + I^*(\varphi_{k'}^{-}(g^{-1})_{hh_{h}}(u'^{-1}))), kk',$$

$$Ad_{k'}q + I^*(\varphi_{k'}^{+}(u'^{-1})), g, u\varphi_{k'k^{-1}}^{+}(u')).$$

Moreover, the canonical projection $S' \to S'/H \times (H \ltimes G^*) \simeq X$ coincides with $\alpha$. ■

If $\phi \in C^\infty(X)$, then it follows from the above propositions that $\alpha^*\phi$ is right $H \times (H \ltimes G^*)$--invariant and $\beta^*\phi$ is left $H \times (H \ltimes G^*)$--invariant. Conversely, it is clear that right/left $H \times (H \ltimes G^*)$ invariant functions on $S'$ are of the above form.
Lemma 5.2.17  For $\phi \in C^\infty(X)$, we have

(a.1) $D^*_1(\alpha^* \phi) = ad^*_1(\alpha^* \phi) p$

(a.2) $D^*_2(\alpha^* \phi) = ad^*_2(\alpha^* \phi) q$

(a.3) $D^*_r(\alpha^* \phi) = Ad_{\varphi^{-1}_{u-1}(g^{-1})} \delta_1(\alpha^* \phi) + Ad_k \delta_2(\alpha^* \phi)$,

(b.1) $D^*_1(\beta^* \phi) + \iota^* D(\beta^* \phi) = 0$

(b.2) $D^*_2(\beta^* \phi) - \iota^* D'(\beta^* \phi) + \iota^* T_u l_{u-1} \lambda^+ \left(D^*_r(\beta^* \phi)\right)(u) = 0$

(b.3) $D^*_s(\beta^* \phi) - T_g l_{g-1} \lambda^- \left(D(\beta^* \phi)\right)(g) = 0$

Here all partial derivatives and left/right gradients are evaluated at $(h, p, k, q, g, u) \in S'$. Proof. These are the infinitesimal versions of the invariance properties of $\alpha^* \phi$ and $\beta^* \phi$ which can be obtained by using the following basic formulae:

\[
\begin{align*}
\frac{d}{dt}|_0 \varphi^+_{uX}(u) &= \lambda^+(X)(u), \\
\frac{d}{dt}|_0 \varphi^-_{uX}(u) &= \lambda^-(X)(u), \\
\frac{d}{dt}|_0 \varphi^+_{\alpha}(t\omega) &= Ad^*_g \alpha, \\
\frac{d}{dt}|_0 \varphi^-_{\alpha}(t\omega) &= Ad^*_g \alpha.
\end{align*}
\]

Lemma 5.2.18  

(a) For all $g \in G, u \in G^*$ and $Z \in h$, we have

\[
Ad^*_u Ad^*_{\varphi^{-1}_{u-1}(g^{-1})} Z = Ad_g Z + T_g l_{g-1} \lambda^- \left(Tr_{\varphi^+_{\varphi^{-1}_{u-1}(u^{-1})}}(Z)(\varphi^+_{\varphi^{-1}_{u-1}(g^{-1})})(u)\right)(g).
\]

(b) $\varphi^+_{g^{-1}} \circ r_{u^{-1}} \circ \varphi^+_{\varphi^{-1}_{u-1}(g^{-1})} = r_{\varphi^+_{g^{-1}}(u^{-1})}$

for all $g \in G, u \in G^*$.

Proof.

(a)  

\[
\begin{align*}
Ad^*_u Ad^*_{\varphi^{-1}_{u-1}(g^{-1})} Z &= \frac{d}{dt}|_0 \varphi^+_{uX}(e^{tAd^*_{\varphi^{-1}_{u-1}(g^{-1})} Z}) \\
&= \frac{d}{dt}|_0 g^{-1} e^{tZ} \varphi^-_{\pi Z (\varphi^+_{\varphi^{-1}_{u-1}(g^{-1})}(u)) \varphi^+_{g-1}(u^{-1})}(g),
\end{align*}
\]

by repeated application of Eqn (5.1.4) and the triviality of the action of $G^*$ on $H$

\[
Ad_g Z + T_g l_{g-1} \lambda^- \left(Tr_{\varphi^+_{\varphi^{-1}_{u-1}(u^{-1})}}(Z)(\varphi^+_{\varphi^{-1}_{u-1}(g^{-1})})(u)\right)(g)
\]

where we have used the formulae in the proof of Lemma 5.2.17.

(b)  

\[
\begin{align*}
\varphi^+_{g^{-1}} \circ r_{u^{-1}} \circ \varphi^+_{\varphi^{-1}_{u-1}(g^{-1})} &= \varphi^+_{g^{-1}}(\varphi^+_{\varphi^{-1}_{u-1}(g^{-1})} (u') u^{-1}) \\
&= \varphi^+_{\varphi^{-1}_{u-1}(g^{-1})} (\varphi^+_{\varphi^{-1}_{u-1}(g^{-1})} (u')) \varphi^+_{g^{-1}}(u^{-1})
\end{align*}
\]

by Eqn (5.1.4)

\[
u^' \varphi^+_{g^{-1}}(u^{-1}).
\]
Hence the assertion. ■

**Proposition 5.2.19 (Polarity condition)** For all \( \phi, \psi \in C^\infty(X) \), we have

\[
\{ \alpha^* \phi, \beta^* \psi \}_S = 0.
\]

Proof. Let \( \hat{\phi} = \alpha^* \phi, \hat{\psi} = \beta^* \psi \). By invoking the identities (a.1), (a.2) and (b.3) of Lemma 5.2.17, we have

\[
\{ \hat{\phi}, \hat{\psi} \}_S(h, p, k, q, g, u) = - < D_1 \hat{\psi}, Ad_h \delta_1 \hat{\phi} > + < D_2 \hat{\psi}, Ad_k \delta_2 \hat{\phi} > + < d_* \hat{\phi}, \lambda_+ (D_* \hat{\psi})(u) > - < D^\prime \hat{\psi}, D_* \hat{\phi} >.
\]

Next, using (b.1), (b.2) and (b.3) of the same lemma successively gives

\[
\{ \hat{\phi}, \hat{\psi} \}_S(h, p, k, q, g, u) = - < D^\prime \hat{\psi}, Ad_{g^{-1}h} \delta_1 \hat{\phi} > - Ad_{u^{-1}} Ad_\varphi^{-1} (g^{-1}) h \delta_1 \hat{\phi} + D_* \hat{\phi}.
\]

Now using (a.3) together with \( Ad_{u^{-1}} Z = Z, \forall Z \in \mathfrak{h} \), we obtain

\[
\{ \hat{\phi}, \hat{\psi} \}_S(h, p, k, q, g, u) = - < D^\prime \hat{\psi}, Ad_{g^{-1}h} \delta_1 \hat{\phi} - Ad_{u^{-1}} Ad_\varphi^{-1} (g^{-1}) h \delta_1 \hat{\phi} > + < D^\prime \hat{\psi}, T_g l_{g^{-1}} \lambda_+ (Ad_{g^{-1}h} T_a r_{u^{-1}} \lambda_+ (Ad_\varphi^{-1} (g^{-1}) h \delta_1 \hat{\phi}))(u))(g) >
\]

where in the last equality we have used Lemma 5.2.18 (a).

But

\[
\lambda_+ (Ad_\varphi^{-1} (g^{-1}) h \delta_1 \hat{\phi})(u) = T \varphi_{\varphi^{-1} (g^{-1})} (\varphi_{(g^{-1})} (u)) = T \varphi_{\varphi^{-1} (g^{-1})} (u^{-1})^{-1},
\]

hence the assertion that \( \{ \hat{\phi}, \hat{\psi} \}_S = 0 \) now follows from Lemma 5.18 (b). ■

**Lemma 5.2.20** For \( \phi \in C^\infty(X) \), we have

(a) \( \delta_1 (\alpha^* \phi) = Ad_{h^{-1}} \delta_1 \phi (\alpha(s)) \),

(b) \( \delta_2 (\alpha^* \phi) = Ad_{k^{-1}} \delta_2 \phi (\alpha(s)) \),

(c) \( D(\alpha^* \phi) = D\phi (\alpha(s)) - Tl_{\varphi_{\varphi^{-1} (g^{-1})} (\delta_1 \phi (\alpha(s))) (\varphi_{(g^{-1})} (u^{-1}))} \),

(d) \( D_* (\alpha^* \phi) = \delta_2 \phi (\alpha(s)) + Ad_{u^{-1}} Ad_{\varphi^{-1} (g^{-1})} \delta_1 \phi (\alpha(s)) \).
\[
= \delta_2 \phi(\alpha(s)) + Ad_g^{-1} \delta_1 \phi(\alpha(s)) \\
- T_g l_{g^{-1}}^{-1} \lambda^-(T_l \varphi_{g^{-1}(u^{-1})}^+(\varphi_{g^{-1}(u^{-1})}^+(\delta_1 \phi(\alpha(s))))(g)) \quad \text{(second form)}.
\]

where all the partial derivatives and right gradients of \(\alpha^* \phi\) are evaluated at \(s = (h, p, k, q, g, u) \in S'\).

Proof. We shall establish the formulae in (d), leaving the other parts to the interested reader. For \(\gamma \in g^*\), we have

\[
< D_s'(\alpha^* \phi), \gamma > \\
= \frac{d}{dt}\bigg|_0 \phi\left(Ad_k^* p - I^*(\alpha^*(\gamma) + g, Ad_k^* q + I^*(ue^\gamma))\right) \\
= < \delta_1 \phi(\alpha(s)), -T I^* \frac{d}{dt}\big|_0 \varphi_{g^{-1}(e^{-t \gamma} u^{-1})} > + < \delta_2 \phi, \iota^* \gamma > .
\]

Since

\[
\frac{d}{dt}\bigg|_0 \varphi_{g^{-1}(e^{-t \gamma} u^{-1})} = -T_1 r_{\varphi_{g^{-1}(u^{-1})}^+} Ad_{\varphi_{g^{-1}(u^{-1})}^+ g^{-1} \gamma},
\]

and

\[
T_1 (I^* \circ r_{\varphi_{g^{-1}(u^{-1})}^+}) = T_1 I^* = \iota^*,
\]

it follows that

\[
D_s'(\alpha^* \phi) = \delta_2 \phi(\alpha(s)) + Ad_{\varphi_{g^{-1}(u^{-1})}^+ g^{-1} \delta_1 \phi(\alpha(s)),
\]

and this gives the first form of \(D_s(\alpha^* \phi)\). To obtain the second form of \(D_s(\alpha^* \phi)\) from the first one, simply apply Lemma 5.2.18 (a) and the fact that

\[
\lambda^+(Z)(\varphi_{g^{-1}(u^{-1})}^+ (u^{-1})^{-1}) \\
= -T(l_{\varphi_{g^{-1}(u^{-1})}^+ (u^{-1})^{-1}} \varphi_{g^{-1}(u^{-1})}^+ (u^{-1}) = Z \in h.
\]

**Lemma 5.2.21** For all \(g \in G, u \in G^*, \) and \(Z \in h, \) we have

(a) \(T_u I^* \lambda^+(Z)(u) = ad_Z I^*(u),\)

(b) \(Ad_g^* T_l \varphi_{g^{-1}(u^{-1})}^+ \lambda^+(Z)(\varphi_{g^{-1}(u^{-1})}^+ (u^{-1})) \)

\[
= -T_1 r_{\varphi_{g^{-1}(u^{-1})}^+} Ad_{\varphi_{g^{-1}(u^{-1})}^+ g^{-1} Z}(u),
\]

(c) \(T_l \varphi_{g^{-1}(u^{-1})}^+ \lambda^+(Ad_g Z)(\varphi_{g^{-1}(u^{-1})}^+ (u^{-1})) \)

\[
= -Ad_{\varphi_{g^{-1}(u^{-1})}^+ g^{-1}} Ad_{\varphi_{g^{-1}(u^{-1})}^+ g^{-1}} T_1 l_{\varphi_{g^{-1}(u^{-1})}^+} \lambda^+(Z)(u).
\]

Proof. (a) This is the infinitesimal version of the \(H-\) equivariance of the map \(I^*.\)
(b) 
\[ \lambda^+(Ad_{\varphi_u^{-1}(g^{-1})}Z)(u) \]
\[ = \frac{d}{dt}{\big|}_{t=0} \varphi_{e^{tAd_{\varphi_u^{-1}(g^{-1})}Z}}(u) \]
\[ = T\varphi_{\varphi_u^{-1}(g^{-1})-1}^+(\varphi_{g^{-1}}^+(u^{-1})^{-1}) \]
\[ = -T\varphi_{\varphi_u^{-1}(g^{-1})-1}^+Tr_{\varphi_{g^{-1}}^+(u^{-1})^{-1}}Tl_{\varphi_{g^{-1}}^+(u^{-1})^{-1}}^+\lambda^+(Z)(\varphi_{g^{-1}}^+(u^{-1})) \]
(where we have used \( Ad_u^*Z = Z \) for \( u \in G^*, Z \in \mathfrak{h} \)).

Now \( Ad_g^* = T_1\varphi_g^+ \) and by a straightforward calculation using Eqn.(5.1.4), we find
\[ \varphi_g^+ \circ r_{\varphi_u^{-1}(u^{-1})} \circ \varphi_{\varphi_u^{-1}(g^{-1})}^+ = r_u^{-1}. \]

Hence the assertion follows.

(c) We have
\[ \lambda^+(Ad_gZ)(\varphi_{g^{-1}}^+(u^{-1})) \]
\[ = \frac{d}{dt}{\big|}_{t=0} \varphi_{e^{tAd_gZ}}(\varphi_{g^{-1}}^+(u^{-1})) \]
\[ = \frac{d}{dt}{\big|}_{t=0} \varphi_{g^{-1}}^+((\varphi_{r_u^{-1}(e^tZ)(u)}^{-1})^{-1}) \]
\[ = \frac{d}{dt}{\big|}_{t=0} \varphi_{g^{-1}}^+(\varphi_{e^tZ}(u))^{-1} \]
by the triviality of the action of \( G^* \) on \( H \)
\[ = -T_{u-1}\varphi_{g^{-1}}^+T_1r_{u-1}Tu^{-1}\lambda^+(Z)(u). \]

On the other hand, for \( \bar{u} \in G^* \), we find
\[ l_{\varphi_{g^{-1}}^+(u^{-1})^{-1}} \circ \varphi_{g^{-1}}^+ \circ r_u^{-1}(\bar{u}) = \varphi_{g^{-1}}^+(u^{-1})^{-1}\varphi_{\varphi_u^{-1}(g^{-1})}(\bar{u})\varphi_{g^{-1}}^+(u^{-1}) \]
from which we deduce that
\[ T_1(l_{\varphi_{g^{-1}}^+(u^{-1})^{-1}} \circ \varphi_{g^{-1}}^+ \circ r_u^{-1}) = Ad_{\varphi_{g^{-1}}^+(u^{-1})^{-1}}Ad_{\varphi_u^{-1}(g^{-1})}. \]

The assertion is now clear. ■

**Proposition 5.2.22** The map \( \alpha : (S', \{ , \}_S) \longrightarrow (X, \{ , \}_X) \) is a Poisson map.

Proof. We want to show
\[ \{ \alpha^* \phi, \alpha^* \psi \}_S = \alpha^* \{ \phi, \psi \}_X \]
for all \( \phi, \psi \in C^\infty(X) \). We shall evaluate \( \{ \alpha^* \phi, \alpha^* \psi \}_S \) at \( s = (h, p, k, q, g, u) \in S' \) and set \( x = \alpha(s) \).
By using Lemma 5.2.17 (a.1), (a.2) and Lemma 5.2.20 (a), (b), we have

\[
\{\alpha^* \phi, \alpha^* \psi\}_{\mathcal{S}(s)} = < Ad_{k-1}^* q, [\delta_1 \phi(x), \delta_1 \psi(x)] > - < Ad_{k-1}^* q, [\delta_2 \phi(x), \delta_2 \psi(x)] > \\
- < D'(\alpha^* \phi), T_g l_{g-1} \lambda^- (D(\alpha^* \psi))(g) > - D_s(\alpha^* \psi) > \\
+ < D_s(\alpha^* \phi), T_u r_{u-1} \lambda^+ (D'_{s}(\alpha^* \psi))(u) > - D'(\alpha^* \psi)>
\]

Now, it follows from Lemma 5.2.20 (c) and (d) (second form) that

\[
T_g l_{g-1} \lambda^- (D(\alpha^* \psi))(g) - D_s(\alpha^* \psi) \\
= T_g l_{g-1} \lambda^- (D(\psi)(x))(g) - \delta_2 \psi(x) - Ad_{g-1} \delta_1 \psi(x).
\]

On the other hand, by using Lemma 5.2.20 (c), (d) (first form) and Lemma 5.2.21 (b), we obtain

\[
T_u r_{u-1} \lambda^+ (D'_{s}(\alpha^* \psi))(u) - D'(\alpha^* \psi) \\
= T_u r_{u-1} \lambda^+ (\delta_2 \psi(x))(u) - D'(\psi(x)).
\]

Consequently,

\[
- < D'(\alpha^* \phi), T_g l_{g-1} \lambda^- (D(\alpha^* \psi))(g) > - D_s(\alpha^* \psi) > \\
+ < D_s(\alpha^* \phi), T_u r_{u-1} \lambda^+ (D'_{s}(\alpha^* \psi))(u) > - D'(\alpha^* \psi) > \\
= < D'(\phi(x)), \delta_2 \psi(x) > + < D\phi(x), \delta_1 \psi(x) > \\
- < D'(\psi(x)), \delta_2 \phi(x) > - < D(\psi(x)), \delta_1 \phi(x) > \\
- < D'(\phi(x)), T_g l_{g-1} \lambda^- (D(\psi(x))(g) > \\
+ T_1 + T_2 + T_3 + T_4,
\]

where

\[
T_1 = < Ad_{g-1}^* T_l_{\varphi_{g-1}^+(u^{-1})-1} \lambda^+ (\delta_1 \phi(x))(\varphi_{g-1}^+(u^{-1})), T_g l_{g-1} \lambda^- (D(\psi))(g) > \\
+ < T_g l_{g-1} \lambda^- (T_l_{\varphi_{g-1}^+(u^{-1})-1} \lambda^+ (\delta_1 \phi(x))(\varphi_{g-1}^+(u^{-1}))(g), D'(\psi) >,
\]

\[
T_2 = - < \iota^* T_l_{\varphi_{g-1}^+(u^{-1})-1} \lambda^+ (\delta_1 \phi(x))(\varphi_{g-1}^+(u^{-1})), \delta_1 \psi(x) >,
\]

\[
T_3 = < \delta_1 \phi(x), \iota^* T_u r_{u-1} \lambda^+ (\delta_2 \psi(x))(u) >,
\]

\[
T_4 = < Ad_{u-1}^* Ad_{\varphi_{u-1}^+(g^{-1})-1} \delta_1 \phi(x), T_u r_{u-1} \lambda^+ (\delta_2 \psi(x))(u) > \\
- < Ad_{g-1}^* T_l_{\varphi_{g-1}^+(u^{-1})-1} \lambda^+ (\delta_1 \phi(x))(\varphi_{g-1}^+(u^{-1})), \delta_2 \psi(x) >.
\]

Using the relation

\[
< \lambda^- (\gamma)(g), \nu > = - < \gamma, T_g r_{g-1} \lambda^- (T_1^* r_g \nu)(g) >,
\]
it is immediate that $T_1 = 0$. For the term $T_2$, note that

$$t^* T_2 \varphi_{u^{-1}(u^{-1})} = T_2 \varphi_{u^{-1}(u^{-1})} I^s.$$ 

Hence it follows from Lemma 5.2.21 (a) that

$$T_2 = - \langle \text{ad}_{\delta_1(x)}^\ast I^s(\varphi_{g^{-1}(u^{-1})}), \delta_1(x) \rangle = - \langle I^s(\varphi_{g^{-1}(u^{-1})}), [\delta_1(x), \delta_1(x)] \rangle.$$ 

Similarly we have

$$T_3 = - \langle I^s(u), [\delta_2(x), \delta_2(x)] \rangle.$$ 

Assembling the calculations, we find

$$\{ \alpha^* \phi, \alpha^* \psi \}_S(s) = \{ \phi, \psi \}_X(x) + T_4.$$ 

Hence it remains to show that $T_4 = 0$. To do so, we invoke the relation

$$\langle \lambda^+(X)(u), \omega \rangle = - \langle X, T_4 l_u \lambda^+(T_4 l_u \omega)(u) \rangle$$

and Lemma 5.2.21 (c) to rewrite $T_4$ as

$$T_4 = - \langle \delta_1(x), -t^* Ad_{\varphi_{u^{-1}(u^{-1})}}^\ast Ad_{g^{-1}}^\ast (u^{-1}) T_4 l_u \lambda^+(\delta_2(x))(u) \rangle + t^* Ad_{\varphi_{g^{-1}(u^{-1})}}^\ast (u^{-1}) T_4 l_u \lambda^+(\delta_2(x))(u).$$

But from the triviality of the $Ad_G^\ast$ action on $h$, it follows that

$$t^* Ad_{\varphi_{g^{-1}(u^{-1})}}^\ast (u^{-1}) = t^* Ad_{\varphi_{u^{-1}(u^{-1})}}^\ast Ad_{g^{-1}}^\ast (u^{-1}).$$

Therefore, $T_4 = 0$. ■

Combining the above Proposition with Proposition 5.2.16, we have

**Corollary 5.2.23** The right action of $H \times (H \ltimes G^\ast)$ on $S'$ in Proposition 5.2.16 is admissible, i.e., functions in $C^\infty(S')$ invariant under the action form a Lie subalgebra of $C^\infty(S')$. Furthermore, the quotient Poisson structure on $X \simeq S'/H \ltimes (H \ltimes G^\ast)$ coincides with $\{ , \}_X$. ■

We shall skip the proof of the next two lemmas.

**Lemma 5.2.24** For $\phi \in C^\infty(X)$, we have

1. $\delta_1(\beta^\ast \phi) = \delta_1(\phi(\beta(s)))$,
2. $\delta_2(\beta^\ast \phi) = \delta_2(\phi(\beta(s)))$,
3. $D_1^\ast (\beta^\ast \phi) = -t^* D(\phi(\beta(s)))$,
4. $D_2^\ast (\beta^\ast \phi) = t^* D(\phi(\beta(s)))$. 
\[(e) \, D(\beta^* \phi) = Ad_{\varphi_g^{-1}(u^{-1})-1} Ad_{h^{-1}} D\phi(\beta(s)), \]

\[(f) \, D'_s(\beta^* \phi) = -Tr_{\varphi_{u^{-1}(g^{-1})-1}} \lambda^- (Ad_{k^{-1}} D'\phi(\beta(s)))(\varphi_{u^{-1}}^{-1}(g^{-1})), \]

where all the partial derivatives and left/right gradients of $\beta^* \phi$ are evaluated at $s = (h, p, k, q, g, u) \in S'$. \hfill \blacksquare

**Lemma 5.2.25** For all $g \in G, u \in G^*$, and $\gamma \in \mathfrak{g}^*$, we have

\[ Ad_g \, Ad_{\varphi_g^{-1}(u^{-1})-1} = Ad_u \, Ad_{\varphi_{u^{-1}(g^{-1})-1}} \gamma \]

\[- T_1 r_{u^{-1}} \lambda^+ (Tr_{\varphi_{u^{-1}(g^{-1})-1}} \lambda^- (Ad_{\varphi_{u^{-1}(g^{-1})-1}} \gamma))(\varphi_{u^{-1}}^{-1}(g^{-1}))(u). \hfill \blacksquare\]

**Proposition 5.2.26** The map $\beta : (S', \{ , \}_{S'}) \rightarrow (X, \{ , \}_X)$ is an anti-Poisson map.

Proof. Let $\phi, \psi \in C^\infty(X)$, $s = (h, p, k, q, g, u) \in S'$ and denote $\beta(s)$ by $x$. From Lemma 5.2.24 (a) - (d) and Lemma 5.2.17 (b.1), we immediately have

\[ \{\beta^* \phi, \beta^* \psi\}_{S'}(s) \]

\[ \leq <\iota^* D\psi(x), \delta_1 \phi(x) > - <\iota^* D\phi(x), \delta_1 \psi(x) > - < p, [\delta_1 \phi(x), \delta_1 \psi(x)] > \]

\[ + <\iota^* D'\psi(x), \delta_2 \phi(x) > - <\iota^* D'\phi(x), \delta_2 \psi(x) > + < q, [\delta_2 \phi(x), \delta_2 \psi(x)] > \]

\[ + < D'_s(\beta^* \phi), T_u l_{\varphi_{u^{-1}}}(D'_s(\beta^* \psi))(u) - Ad_{u^{-1}} D'(\beta^* \psi) > . \]

Hence we have to show that

\[ < D'_s(\beta^* \phi), T_u l_{\varphi_{u^{-1}}}(D'_s(\beta^* \psi))(u) - Ad_{u^{-1}} D'(\beta^* \psi) > \]

\[ = < d_G \phi(x), \Pi_G^# (h^{-1} \varphi_{u^{-1}}^{-1}(g^{-1})^{-1} k) d_G \psi(x) > . \]

To do so, we apply Lemma 5.2.24 (e), (f) and Lemma 5.2.25, this yields

\[ T_u l_{\varphi_{u^{-1}}}(D'_s(\beta^* \psi))(u) - Ad_{u^{-1}} D'(\beta^* \psi) \]

\[ = - Ad_{\varphi_{u^{-1}(g^{-1})-1}} Ad_{h^{-1}} D\psi(x). \]

Consequently,

\[ < D'_s(\beta^* \phi), T_u l_{\varphi_{u^{-1}}}(D'_s(\beta^* \psi))(u) - Ad_{u^{-1}} D'(\beta^* \psi) > \]

\[ = < Tr_{\varphi_{u^{-1}}^{-1}(g^{-1})} \lambda^- (Ad_{k^{-1}} D'\phi(x))(\varphi_{u^{-1}}^{-1}(g^{-1})), Ad_{\varphi_{u^{-1}(g^{-1})-1}} Ad_{h^{-1}} D\psi(x) > \]

\[ = < d_G \psi(x), T_{\varphi_{u^{-1}(g^{-1})}}(h^{-1} \varphi_{u^{-1}}^{-1} \circ r_{\varphi_{u^{-1}}^{-1}(g^{-1})-1} k) \Pi_G^# (\varphi_{u^{-1}}^{-1}(g^{-1})) \]

\[ = - < d_G \psi(x), \Pi_G^# (h^{-1} \varphi_{u^{-1}}^{-1}(g^{-1})^{-1} k) d_G \phi(x) > , \]

where in the last step, we have used the fact that $\Pi_G$ is multiplicative and that $\Pi_G$ vanishes on $H$. This completes the proof. \hfill \blacksquare
As a consequence of the Prop. 5.2.26 and Prop. 5.2.15, we obtain

**Corollary 5.2.27** The left action of $H \times (H \ltimes G^*)$ on $S'$ in Proposition 5.2.15 is admissible (see Cor. 5.2.23). Furthermore, the quotient Poisson structure on $X \simeq H \times (H \ltimes G^*) \setminus S'$ coincides with $-\{ , \}_X$. ■

This completes the proof that $S' \cong X$ is a symplectic groupoid.

We now turn to the description of the symplectic foliation of $X$.

Equip $H \times (H \ltimes G^*)$ with the product of the trivial Poisson Lie group structure on $H$ and the Poisson Lie group structure of Prop. 5.1.3 on $H \ltimes G^*$. It is easy to see that the groupoid action $\Psi^-$ of Eqn. (5.2.8) restricts to a group action

$$\tilde{\Psi}^- : H \times (H \ltimes G^*) \times X \to X,$$

given by

$$( (k, (h, u)), (p, g, q)) \mapsto \Psi^-_{(h, p, u, k)}(p, g, q).$$

**Theorem 5.2.28**

(a) $\tilde{\Psi}^-$ is a left Poisson Lie group action.

(b) The symplectic leaf $L_{(p, g, q)}$ in $(X, \{ , \}_X)$ passing through the point $(p, g, q)$ is the orbit of $(p, g, q)$ under the action $\tilde{\Psi}^-$ i.e.

$$L_{(p, g, q)} = \{ (Ad^r_{h^{-1}} p + I^*(u), \varphi^ -_{u}(hgk^{-1}), Ad^r_{k^{-1}} q + I^*(\varphi^+_{hgk^{-1}}(u))) \mid (k, (h, u)) \in H \times (H \ltimes G^*) \}.$$

Proof. (a) Equip $X$ with the Poisson bracket of Thm. 2.1.5 and $H \times (H \ltimes G^*)$ with the Poisson Lie bracket

$$\{ \phi, \psi \}(k, (h, u)) = \partial_\phi \phi(u) \Pi_\star(u) \partial_\psi \psi(u)$$

go of Prop. 5.1.3. We have to show that the action

$$\tilde{\Psi}^- : H \times (H \ltimes G^*) \times X \to X$$

$$( (k, (h, u)), (p, g, q)) \mapsto (Ad^r_{h^{-1}} p + I^*(u), \varphi^ -_{u}(hgk^{-1}), Ad^r_{k^{-1}} q + I^*(\varphi^+_{hgk^{-1}}(u)))$$

satisfies

$$\{ f \circ \tilde{\Psi}^-, f' \circ \tilde{\Psi}^- \}_{H \times (H \ltimes G^*) \times X} = \{ f, f' \}_X \circ \tilde{\Psi}^-.$$ (P)

To begin with, a direct calculation making use of the equivariance of $I^*$, Eqn. (5.1.4), and the triviality of the action of $G^*$ on $H$ yields the following expressions for the partial derivatives of $f \circ \tilde{\Psi}^-$ at $( (k, (h, u)), p, g, q) \in H \times (H \ltimes G^*) \times X$:

$$\partial_\star(f \circ \tilde{\Psi}^-) = T^*_u r_{u^{-1}}(t\delta_1 f - \Pi^*(\varphi^ -_{u}(hgk^{-1}))Df + Ad^r_{\varphi^ -_{u}(hgk^{-1})}t\delta_2 f),$$

$$\delta_1(f \circ \tilde{\Psi}^-) = Ad_{h^{-1}} \delta_1 f,$$

$$\delta_2(f \circ \tilde{\Psi}^-) = Ad_{k^{-1}} \delta_2 f,$$

$$\partial(f \circ \tilde{\Psi}^-) = T^*_{g^{-1}} + \partial_\star(f \circ \tilde{\Psi}^-) + Ad^r_{\varphi^ -_{h g}(u)}D'f + \Pi^*_{\varphi^+_{h g}(u)}t Ad_{h^{-1}} \delta_2 f,$$
where $\Pi^*$ (resp. $\Pi^i$) stands for the Poisson tensor of $G$ (resp. $G^*$) in the right (resp. left) invariant frame.

We shall restrict ourselves to an outline of the main steps of the calculation of $\text{lhs}(P) - \text{rhs}(P)$. We use the shorthand notation $d_{ij} = (\text{lhs})_{ij}(P) - (\text{rhs})_{ij}(P)$, $i, j \in \{1, 2, \star\}$, where 1 (resp. 2) stands for $f = p^*_1 \phi$ (resp. $f = p^*_2 \phi$), $\phi \in C^\infty(\mathfrak{h}^*)$, and $\star$ stands for $f = p^*_G \psi$, $\psi \in C^\infty(G)$. Thus for example

$$d_{11} = \{p^*_1 \phi \circ \tilde{\Psi}^-, p^*_G \psi \circ \tilde{\Psi}^\star\} \circ \tilde{\Psi}^- - \{p^*_1 \phi, p^*_G \psi\} \circ \tilde{\Psi}^-.$$ 

We now compute $d_{ij}$ for the various cases.

(1) $d_{11} = -<\iota \phi, \Pi^*_1(u)\iota \delta \psi > - < I^*(u), [\delta \phi, \delta \psi] > = 0$ by Lemma 5.2.21 (a).

(2) $d_{22} = -2 <\iota Ad_{\varphi^{-1}_{h^2}(u)} \delta \phi, \Pi^i_1(\varphi^+_{hg^{-1}}(u))\iota Ad_{\varphi^{-1}_{h^2}(u)} \delta \psi > + <\iota \delta \phi, \Pi^i_1(\varphi^+_{hg^{-1}}(u))\iota \delta \psi >$

$$+ < Ad_{\varphi^{-1}_{h^2}(u)} \delta \phi, \Pi^i_1(\varphi^+_{hg^{-1}}(u)) Ad_{\varphi^{-1}_{h^2}(u)} \iota \delta \psi >$$

$$+ <\iota Ad_{\varphi^{-1}_{h^2}(u)} \delta \phi, \Pi^i_1(\varphi^+_{hg^{-1}}(u)) \Pi^i_1(\varphi^+_{hg^{-1}}(u))\iota Ad_{\varphi^{-1}_{h^2}(u)} \delta \psi >.$$

Using the fact that the action $H \times G^* \rightarrow G^*$ : $(k, v) \mapsto \varphi^+_{k^{-1}}(v)$ is Hamiltonian, we have

$$\Pi^i_1(\varphi^+_{hg^{-1}}(u)) = \Pi^i_1(\varphi^+_{k^{-1}}(\varphi^+_{hg^{-1}}(u))) = Ad_{k^{-1}} \Pi^i_1(\varphi^+_{hg^{-1}}(u)) Ad_{k^{-1}} \quad (E.1)$$

while, since $\Pi$ is multiplicative and vanishes on $H \subset G$, we obtain

$$\Pi^i_1(hg^{-1}) = Ad_k \Pi^i_1(g) Ad_{k^{-1}} \quad (E.2)$$

Therefore,

$$d_{22} = -<\iota \delta \phi, \Pi^i_1(\varphi^+_{hg^{-1}}(u)) \iota \delta \psi >$$

$$+ < Ad_{\varphi^{-1}_{h^2}(u)} \delta \phi, \Pi^i_1(\varphi^+_{hg^{-1}}(u)) Ad_{\varphi^{-1}_{h^2}(u)} \iota \delta \psi >$$

$$+ <\iota \delta \phi, \Pi^i_1(\varphi^+_{hg^{-1}}(u)) \Pi^i_1(hg^{-1}) \Pi^i_1(\varphi^+_{hg^{-1}}(u)) \iota \delta \psi >.$$

Next, observe that the Poisson property of $\varphi^+ : G^* \times G \rightarrow G^*$ may be written as

$$Ad^e_{\varphi^+_x}(x) \Pi^i_1(u) Ad^e_{\varphi^+_x}(x)$$

$$= Ad_{\varphi^+_x}(u) \left( - \Pi^i_1(\varphi^+_x(u)) \Pi^i_1(\varphi^+_x(u)) + \Pi^i_1(\varphi^+_x(u)) \right) Ad^e_{\varphi^+_x}(u).$$

Thus, $d_{22} = 0$ follows from $Ad^e_{\varphi^+_x}(x) Z = \iota Z$, $Z \in \mathfrak{h}$, $v \in G^*$. 

(3) $d_{12} = <\iota \delta \phi, (\Pi^i_1(u) Ad_{\varphi^{-1}_{h^2}(u)} - Ad_{g^{-1}} \Pi^i_1(\varphi^+_{hg^{-1}}(u)) Ad_{k^{-1}}) \iota \delta \psi >$

$$= <\iota \delta \phi, (\Pi^i_1(u) Ad_{\varphi^{-1}_{h^2}(u)} - Ad_{(hg^{-1})^{-1}} \Pi^i_1(\varphi^+_{hg^{-1}}(u)) \iota \delta \psi >.$$

But it follows from Lemma 5.2.21 (b) that

$$Ad^e_{\varphi^{-1}_{x^{-1}}} \Pi^i_1(\varphi^+_{x^{-1}}(u)) \iota Z = Ad_{u^{-1}} \Pi^i_1(u) Ad_{\varphi^{-1}_{x^{-1}}} \iota Z.$$
Thus, $d_{12} = 0$ follows from $Ad_u^* tZ = tZ$.

(4)

$$d_{1*} = - < i\delta \phi, \Pi^u \Pi^r (\varphi^-_u (h g k^{-1}) D \psi) >$$

$$- < i\delta \phi, Ad^{*}_{g-1 h^-1} Ad_{\varphi^-_{h g^{-1} (u)}} Ad^{*}_{\varphi^-_{h g^{-1}}} D \psi > + < i\delta \phi, D \psi >.$$  

which, upon using $\varphi^-_u (h g k^{-1}) = \varphi^-_u (h g) k^{-1}$ and the multiplicativity

$$\Pi^r (\varphi^-_u (h g) k^{-1}) = \Pi^r (\varphi^-_u (h g)),$$

becomes

$$d_{1*} = < i\delta \phi, (- \Pi^u (u) \Pi^r (\varphi^-_u (h g)) - Ad^{*}_{h g} - Ad_{\varphi^-_{h g^-1} (u)} Ad^{*}_{\varphi^-_{h g^-1}} + Id) D \psi >.$$  

But by taking the derivative at $t = 0$ in the identity

$$\varphi^-_u (e^{tX} x) = \varphi^-_u (e^{tX}) \varphi^-_{\varphi^-_{h g^{-1} (u)}} (x)$$

and dualizing immediately yields

$$Id = Ad_u Ad_{x^{-1}} Ad_{\varphi^-_{h g^-1} (u)} Ad^{*}_{\varphi^-_{h g^-1} (x)} + \Pi^u (u) \Pi^r (\varphi^-_u (x)). \quad (E.3)$$

Thus $d_{1*} = 0$ again follows from the triviality of $Ad^{*}_{G^*}$ on $\mathfrak{g}$.

(5)

$$d_{2*} = < i\delta \phi, (- Ad^{*}_{\varphi^-_u (h g k^{-1})} \Pi^u (u) \Pi^r (\varphi^-_u (h g k^{-1})) Ad^{*}_{\varphi^-_u (h g k^{-1})}^{-1}$$

$$+ Ad^{*}_{k^{-1}} \Pi^l (\varphi^l_{h g^{-1} (u)}) \Pi^l (g) Ad_{\varphi^l_{h g^{-1} (u)}} Ad^{*}_{\varphi^l_{h g^{-1} (u)^{-1}}} D' \psi >.$$  

Now, by taking the derivative at $t = 0$ in the identity

$$k \varphi^+ (e^{tX} k^{-1}) = \varphi^+_{\varphi^+_{h g^{-1} (u)}} (k e^{tX} k^{-1})$$

and dualizing, we find

$$Ad_{x^{-1}} Ad^{*}_{k} = Ad^{*}_{(\varphi^+_{h g^{-1} (u)})^{-1}}. \quad (E.4)$$

Therefore,

$$d_{2*} = < i\delta \phi, (- Ad^{*}_{\varphi^-_u (h g k^{-1})} \Pi^u (u) \Pi^r (\varphi^-_u (h g k^{-1})) Ad^{*}_{\varphi^-_u (h g k^{-1})}^{-1}$$

$$+ \Pi^l (\varphi^l_{h g^{-1} (u)}) \Pi^l (g) Ad_{\varphi^l_{h g^{-1} (u)}} Ad^{*}_{\varphi^l_{h g^{-1} (u)^{-1}}} D' \psi >,$$

where we have also used the identities (E.1) and (E.2) above.

Next, by taking the derivative at $t = 0$ of the expression

$$\varphi^+_{x} (ue^{tA}) = \varphi^+_{\varphi^+_{x (x)}} (u) \varphi^+_{x} (e^{tA}),$$
we obtain
\[ Id = Ad_{\varphi^+_{\{u\}}} Ad_{\varphi^-_{\{x\}}} Ad_u Ad_{x} + \Pi^l_{\{u\}}(\varphi^l_{\{u\}}) \Pi^l_{\{x\}}. \]  
\( (E.5)\)

Combining (E.3) with (E.5) then yields
\[ Ad_{\varphi^-_{\{x\}}} \Pi^l_{\{u\}}(\varphi^-_{\{u\}}) Ad_{\varphi^+_{\{x\}}} = Ad_{\varphi^+_{\{u\}}} \Pi^l_{\{u\}}(\varphi^+_{\{u\}}) \Pi^l_{\{x\}} Ad_{\varphi^+_{\{u\}}}^{-1}. \]
Thus \( Ad_u Z = Z \) implies \( d_{2*} = 0. \)

\( (6) \)

\[ d_{**} = - < D'\phi, Ad_{\varphi^+_{\{h g k\}}}^{-1} \Pi^l_{\{\varphi^-_{\{h g k\}}} Ad_{\varphi^-_{\{h g k\}}}^{-1}, D'\psi > \]
\[ + < D'\phi, - Ad_k Ad_{\varphi^+_{\{r g k\}}}^{-1} \Pi^l_{\{g\}} Ad_{\varphi^+_{\{r g k\}}}^{-1} Ad_k + \Pi^l_{\{\varphi^-_{\{h g k\}}} ) D'\psi >. \]

Here we use the Poisson property of \( \varphi^- : \overline{G^*} \times G \rightarrow G \) which may be expressed in the form
\[ Ad_{\varphi^-_{\{x\}}}^{-1} \Pi^l_{\{\varphi^-_{\{u\}} \Pi^l_{\{\varphi^-_{\{x\}}} Ad_{\varphi^-_{\{x\}}}^{-1} \]
\[ = - Ad_{\varphi^+_{\{u\}}}^{-1} \Pi^l_{\{x\}} Ad_{\varphi^+_{\{u\}}}^{-1} + \Pi^l_{\{\varphi^-_{\{x\}}}. \]

Therefore,
\[ d_{**} = < D'\phi, Ad_{\varphi^+_{\{h g k\}}}^{-1} \Pi^l_{\{h g k\}} Ad_{\varphi^+_{\{h g k\}}}^{-1}, D'\psi > \]
\[ - < D'\phi, Ad_k Ad_{\varphi^-_{\{r g k\}}}^{-1} \Pi^l_{\{g\}} Ad_{\varphi^-_{\{r g k\}}}^{-1} Ad_k D'\psi >. \]

Thus, \( d_{**} = 0 \) follows from the identities (E.2) and (E.4).
This concludes the verification that \( \text{lhs}(P) - \text{rhs}(P) = 0. \)

(b) This follows from a general result of Weinstein according to which the symplectic leaf passing through \( (p, g, q) \) of the base \( X \) in the full symplectic realization

\[ S \]
\[ \tilde{\beta}_{\mathcal{H}} \vee \gamma \tilde{\alpha}_{\mathcal{H}} \]
\[ X \]
\[ X \]
of Theorem 5.2.13 is given by \( \tilde{\alpha}_{\mathcal{H}}(\tilde{\beta}_{\mathcal{H}}^{-1}(p, g, q)). \)  

To conclude the paper we give two corollaries of Thm 5.2.28.
First, for the special case when \( R = 0 \), we have \( G^* = g^*, I^* = i^* : g^* \rightarrow h^*, \)
\( \varphi^+_g = Ad_g^* \), and \( \varphi^-_u = Id. \) Therefore, the Poisson Lie group structure on \( H \times (H \times g^*) \) is given by
\[ (k, (h, A)) : (k', (h', A')) = (k k', (h h', A + Ad^*_{h-1} A')) \]
\[ \{ f, g \}_{H \times (H \times g^*)}(k, (h, A)) = < A, [\delta f, \delta g] >. \]
and the group action becomes
\[
\tilde{\psi}_0^*: H \times (H \ltimes g^*) \times X \to X
\]
\[
((k, (h, A), (p, g, q)) \mapsto (Ad^*_h - 1 A + \iota^*(A), h g k^{-1}, Ad^*_k - 1 q + \iota^*(Ad^*_h g_k - 1 A)).
\]

**Corollary 5.2.29** (Symplectic leaves for \( R = 0 \))

(a) \( \tilde{\psi}_0^* \) is a left Poisson Lie group action

(b) The symplectic leaf \( L_{(p,g,q)} \) in \((X, \{ \cdot, \cdot \}_X)\) passing through the point \((p,g,q)\) is the orbit of \((p,g,q)\) under the action \( \tilde{\psi}_0^* \). ■

Next, we consider the symplectic foliation of a Poisson quotient which we now introduce. Recall from Theorem 2.1.4 (b) that \( X \) has a pair of Hamiltonian \( H \)–
actions with \( \alpha \) and \( \beta \) as momentum maps respectively. Combining the two actions, we obtain the action
\[
h \cdot (p, g, q) = (Ad^*_h - 1 p, h g h^{-1}, Ad^*_h - 1 q)
\]
which is also Hamiltonian and its equivariant momentum map is given by \( J = \alpha - \beta \).
Now, \( 0 \in h^* \) is clearly a regular value of \( J \) and the corresponding isotropy subgroup is \( H \). Hence it follows from Poisson reduction \([MR] \) that \( J^{-1}(0)/H \) inherits a Poisson structure \( \{ \cdot, \cdot \}_{J^{-1}(0)/H} \) satisfying
\[
\{ f_1, f_2 \}_{J^{-1}(0)/H} \circ \pi = \{ \tilde{f}_1, \tilde{f}_2 \}_X \circ i.
\]
Here, \( i: J^{-1}(0) \to X \) is the inclusion map, \( \pi: J^{-1}(0) \to J^{-1}(0)/H \) is the canonical projection, \( f_1, f_2 \in C^\infty(J^{-1}(0)/H) \), and \( \tilde{f}_1, \tilde{f}_2 \) are (locally defined) smooth extensions of \( \pi^* f_1, \pi^* f_2 \) with differentials vanishing on the tangent spaces of the \( H \)–
orbits.

**Corollary 5.2.30** The symplectic leaves of \((J^{-1}(0)/H, \{ \cdot, \cdot \}_{J^{-1}(0)/H})\) are given by the connected components of \( L_{(p,g,q)} \bigcap J^{-1}(0)/H, (p,g,q) \in X \).

Proof. This is a consequence of the theorem and a result in \([MR] \), as the triple \((X, J^{-1}(0), E)\) is Poisson reducible, where \( E \) is the tangent space to the \( H \)–
orbits of the action in (5.2.15). ■

Clearly, the symplectic leaves of the quotient \( G/H \times U \) in Proposition 3.2.5 can also be obtained in a similar way.
Appendix.

A1. Proof of Proposition 2.2.3.

The most general bivector field on \( X \) is of the form

\[
\Pi(df, dg)(p, x, q) = K_1(\delta_1 f, \delta_1 g) + K_2(\delta_2 f, \delta_2 g) + R(\delta_1 f, \delta_2 g) \\
- R(\delta_1 g, \delta_2 f) + P_1(\delta_1 f, \partial g) - P_1(\delta_1 g, \partial f) \\
+ P_2(\delta_2 f, \partial g) - P_2(\delta_2 g, \partial f) + P_G(\partial f, \partial g),
\]

where \( K, R, P_1, P_G \) are evaluated at \((p, x, q)\).

Set

\[
\Omega_{(\omega, Z_1, Z_2, Z_3)} = ((Z_1, \omega, Z_2), (-Z_2, T_y^\ast (r_y^{-1} \circ l_x)\omega, Z_3), (-Z_1, -T_{xy}^\ast r_y^{-1}\omega, -Z_3)),
\]

and denote \((\Pi \oplus \Pi \ominus \Pi)((p, x, q), (q, y, r), (p, xy, r))\) by \( \Pi_m \). Fix a reference point \( q_0 \in U \). We have

\[
\begin{align*}
\Pi_m(\Omega_{(0, Z, 0, 0)}, \Omega_{(0, 0, Z, 0)}) &= 0 \iff R = 0 \\
\Pi_m(\Omega_{(0, Z, 0, 0)}, \Omega_{(0, 0, Z', 0)}) &= 0 \iff K_1(p, x, q) = K_1(p, 1, q_0) =: K(p) \\
\Pi_m(\Omega_{(0, 0, Z, 0)}, \Omega_{(0, 0, Z', 0)}) &= 0 \iff K_2(p, x, q) = -K(q).
\end{align*}
\]

Now,

\[
\begin{align*}
\Pi_m(\Omega_{(0, 0, 0, 0)}, \Omega_{(0, Z', 0, 0)}) &= 0 \iff P_1(p, x, q)(Z', \omega) = P_1(p, xy, r)(Z', T_{xy}^\ast r_y^{-1}\omega).
\end{align*}
\]

Setting successively \( y = 1, r = q_0 \), and \( x = 1, \omega = T^\ast_{xy} r_y \omega' \) in the latter equality yields

\[
P_1(p, y, r)(Z', \omega') = P_1(p, 1, q_0)(Z', T_{xy}^\ast r_y \omega') =: A_1(p)Z', T_{xy}^\ast r_y \omega' > .
\]

Similarly

\[
\begin{align*}
\Pi_m(\Omega_{(0, 0, 0, 0)}, \Omega_{(0, 0, 0, Z')}) &= 0 \iff \\
P_2(p, x, r)(Z', \omega) &= P_2(q_0, 1, r)(Z', T^\ast_{1x} l_x \omega) =: A_2(r)Z', T_{1x}^\ast l_x \omega > .
\end{align*}
\]

Moreover,

\[
\Pi_m(\Omega_{(0, 0, 0, 0)}, \Omega_{(0, 0, Z', 0)}) = 0 \iff A_1(p) = A_2(p).
\]

It only remains to demand that \( \Pi_m(\Omega_{(0, 0, 0, 0)}, \Omega_{(0, 0, 0, 0)}) = 0 \). But working in the right invariant frame \( P_G(\omega, \omega') = < T_{xy}^\ast \omega, P(T_{xy}^\ast \omega') > , \) the latter condition is equivalent to the cocycle property

\[
P(p, xy, r) = P(p, x, q) + Ad_x P(q, y, r) Ad_x^*.
\]

Hence the assertion. ■

A2. Proof of Thm 2.2.5 (b).
We have to check the Jacobi identity for the bracket
\[
\{ f, g \} \chi(p, x, q) = \langle p, [\delta_1 f, \delta_1 g] \rangle - \langle q, [\delta_2 f, \delta_2 g] \rangle
\]
\[
- \langle A_\chi(p)\delta_1 f, Dg \rangle - \langle A_\chi(q)\delta_2 f, D'g \rangle
\]
\[
+ \langle A_\chi(p)\delta_1 g, Df \rangle + \langle A_\chi(q)\delta_2 g, D'f \rangle
\]
\[
+ \langle Df, P(p(x, q)Dg) \rangle.
\]
We shall use (up to sign) the same notation \( J_{ijk} \) as in the text. If \( a \in \mathfrak{h} \), we define (as in [EV]) the functions \( a_1, a_2 \in C^\infty(U \times G \times U) \) by \( a_1(p, h, q) = \langle p, a \rangle \) and \( a_2(p, h, q) = \langle q, a \rangle \). Finally, for \( Y \in \mathfrak{g} \), the left (resp. right) invariant vector field on \( G \) whose value at 1 is \( Y \) will be denoted by \( Y^l \) (resp. \( Y^r \)).

We now compute \( J_{ijk} \) for the various cases.

First of all, it is clear that \( J_{ijk} = 0 \), \( i, j, k \in \{1, 2\} \).

On the other hand, we have
\[
J_{12} = \{ \{ p_G f, a_1 \}, b_2 \} + \{ \{ a_1, b_2 \}, p_G f \} + \{ \{ b_2, p_G^* f \}, a_1 \}
\]
\[
= (A_\chi(q)(b))^l (A_\chi(p)(a))^r (f)(x) - (A_\chi(p)(a))^r (A_\chi(q)(b))^l (f)(x) = 0.
\]

while
\[
J_{11} = \{ \{ p_G f, a_1 \}, b_1 \} + \{ \{ a_1, b_1 \}, p_G f \} + \{ \{ b_1, p_G^* f \}, a_1 \}
\]
\[
= - \langle dA_\chi(p) \cdot ad_{a_1} p \cdot a, Df(x) \rangle + (A_\chi(p)b)^l (A_\chi(p)a)^l (f)(x)
\]
\[
- \langle A_\chi(p)[a, b], Df(x) \rangle + \langle dA_\chi(p) \cdot ad_{a_1} p \cdot b, Df(x) \rangle
\]
\[
- (A_\chi(p)(a))^l (A_\chi(p)b)^l (f)(x)
\]
\[
= \langle dA_\chi(p) \cdot ad_{a_1} p \cdot b - dA_\chi(p) \cdot ad_{b_1} p \cdot a
\]
\[
+ [A_\chi(p)a, A_\chi(p)b] - A_\chi(p)((a, b)), Df(x) \rangle.
\]

Similarly,
\[
J_{22} = - \langle dA_\chi(p) \cdot ad^*_{a_1} p \cdot b + dA_\chi(p) \cdot ad^*_{b_1} p \cdot a
\]
\[
- [A_\chi(p)a, A_\chi(p)b] + A_\chi(p)((a, b)), D'f(x) \rangle.
\]

So \( J_{ij} = 0 \), \( i, j \in \{1, 2\} \) \( \Leftrightarrow A_\chi : \mathfrak{h}^* \times \mathfrak{h} \to \mathfrak{g} \) is a morphism of Lie algebroids.

Now,
\[
J_{1*} = \{ a_1, \{ p_G^* f, p_G^* g \} \} + \{ p_G^* f, \{ p_G^* g, a_1 \} \} + \{ p_G^* g, \{ a_1, p_G^* f \} \}
\]
\[
= \langle Df, \delta_1 P \cdot ad_{a_1} p \cdot Dg \rangle - (A_\chi(p)a)^r (PDg)^r (f)
\]
\[
- \langle Df, DP \cdot A_\chi(p)(a) \cdot Dg \rangle + (A_\chi(p)a)^r (PDf)^r (g)
\]
\[
+ \langle dA_\chi(p) \cdot (A_\chi(p)^* Df) \cdot a, Dg \rangle - (PDf)^r (A_\chi(p)a)^r (g)
\]
\[
- \langle dA_\chi(p) \cdot (A_\chi(p)^* Dg) \cdot a, Df \rangle + (PDg)^r (A_\chi(p)a)^r (f)
\]
\[
= \langle Df, \delta_1 P \cdot ad^*_{a_1} p \cdot Dg + ad_{A_\chi(p)a}(P(Dg)) \rangle
\]
\[
+ \langle Df, P(ad_{A_\chi(p)a}Dg) - DP \cdot (A_\chi(p)a) \cdot Dg \rangle
\]
\[
+ \langle dA_\chi(p) \cdot (A_\chi(p)^* Df) \cdot a, Dg \rangle - \langle dA_\chi(p) \cdot (A_\chi(p)^* Dg) \cdot a, Df \rangle.
\]
Similarly,

\[ J_{2^*} = \langle Df, -\delta_2 P \cdot ad_\alpha^* q \cdot Dg - D'P \cdot (A_\chi(q)(a)) \cdot Dg \rangle \]
\[ + \langle dA_\chi(q) \cdot (A_\chi(q)^* D'f) \cdot a, D'g \rangle - \langle dA_\chi(q) \cdot (A_\chi(q)^* D'q) \cdot a, D'f \rangle. \]

Writing the groupoid 1-cocycle as

\[ P(p, x, q) = -l(p) + \pi(x) + Ad_x l(q) Ad_x^*, \]

we have

\[ \delta_1 P \cdot \Lambda = -dl(p) \cdot \Lambda \]
\[ DP \cdot X = d\pi(1)(X) + ad_\chi \pi(x) + \pi(x) ad_\chi^* \]
\[ + ad_\chi Ad_x l(q) Ad_x^* + Ad_x l(q) Ad_x^* ad_\chi \]
\[ \delta_2 P \cdot \Lambda = Ad_x dl(q) \cdot \Lambda Ad_x^* \]
\[ D'P \cdot X = Ad_x d\pi(1) \cdot X Ad_x^* + Ad_x ad_\chi l(q) Ad_x^* \]
\[ + Ad_x l(q) ad_\chi^* Ad_x^* \]

Inserting the latter into \( J_{1^*} \) and \( J_{2^*} \) yields

\[ J_{1^*} = 0 \iff J_{2^*} = 0 \iff \]
\[ \langle \alpha, (dl(p)ad_\alpha^* p + ad_\chi_{x}(p)l(p) + l(p)ad_\chi^* p) \alpha + d\pi(1)A_\chi(p)a \beta \rangle \]
\[ = + \langle dA_\chi(p) \cdot (A_\chi(p)^* \alpha) \cdot a, \beta \rangle - \langle dA_\chi(p) \cdot (A_\chi(p)^* \beta) \cdot a, \alpha \rangle, \]
\[ \forall \alpha, \beta \in g^*, a \in h. \]

Finally,

\[ J_{m^*} = \{ p^*_C f, \{ p^*_C g, p^*_C h \} \} + c.p. (f, g, h) \]
\[ = \langle Dg, \delta_2 P \cdot (A_\chi(p)^* Df) \cdot Dh + \delta_2 P \cdot (A_\chi(p)^* D'f) \cdot Dh \rangle \]
\[ - (PDf)^*(PDh)^*(g) + (PDf)^*(PDg)^*(h) \]
\[ - \langle Dg, DP \cdot (PDf) \cdot Dh > + c.p. (f, g, h), \]

which may easily be brought to the form stated in Thm 2.2.5 (b).

A3. Proof of Theorem 5.1.4.

We have to show that the graph of the multiplication

\[ Gr(m) \subset \Gamma \times \Gamma \times \Gamma \]

is a coisotropic submanifold. We use (as in (5.1.6)) the notation \( \phi^*_h(v) = \phi^*_v(h) = \varphi^*_{h^{-1}}(v), h \in H, v \in G^*. \) We have

\[ Gr(m) = \]
\[ \{ (h, Ad_{k^{-1}}^* q + I^*(v), u), (k, q, v), (hk, q, u\phi^*_h(v))) \mid h, k \in H, u, v \in G^*, q \in h^* \}, \]
since for some $60 \text{ LUEN CHAU LI AND SERGE PARMENTIER}$

$T_*(Gr(m)) = \{(Z_1, -Ad_{k-1}^* ad_{T_k l_{k-1} Z_2}) q + Ad_{k-1}^* \lambda + T_v (I^*) V, U, (Z_2, \lambda, V), (T_k h, Z_2 + T_h r_k Z_1, \lambda, T_u r_{\phi_h^l(v)} U + T_{\phi_h^l(v)} l_u (T_h \phi_h^l Z_1 + T_v \phi_h^l V)) \mid Z_1 \in T_h H, Z_2 \in T_h, \lambda \in h^*, U \in T_u G^*, V \in T_v G^* \}$.

Hence $\Omega \in (T_*(Gr(m))^\perp \text{ if and only if}$

$$\Omega = ((-T_h^* r_k \mu - T_h^* (l_u \circ \phi_h^l) A, z_1, -T_u^* r_{\phi_h^l(v)} A),$$

$$(-T_k^* l_h \mu - T_k^* l_{k-1} ad_{(A_{k-1} - z_1)} q, z_2, -T_v^* I^* z_1 - T_v^* (l_u \circ \phi_h^l) A),$$

$$(\mu, -Ad_{k-1}^* z_1 - z_2, A)),$$

for some $\mu \in T_h^* H, z_1, z_2 \in h, A \in T_{u \phi_h^l(v)} G^* \text{. For } \Omega, \Omega' \in (T_*(Gr(m))^\perp, \text{ we have}$

$$(\Pi \oplus \Pi \oplus -\Pi)(\Omega, \Omega') = < T_h^* r_k \mu' + T_h^* (l_u \circ \phi_h^l) A', T_h^* l_h z_1 >$$

$$- < T_h^* r_k \mu + T_h^* (l_u \circ \phi_h^l) A, T_1^* l_h z_1' > - < Ad_{k-1}^* q + I^*(v), [z_1, z_1'] >$$

$$- < T_v^* \lambda^+(T^*_1 l_u (T_v^* r_{\phi_h^l(v)} A'))(u) >$$

$$+ < T_k^* l_h \mu' + T_k^* l_{k-1} ad_{(A_{k-1} - z_1)} q, T_1^* l_h z_2 >$$

$$- < T_k^* l_h \mu + T_k^* l_{k-1} ad_{(A_{k-1} - z_1)} q, T_1^* l_h z_2' > - < q, [z_1, z_1'] >$$

$$- < T_v^* I^* z_1 + T_v^* (l_u \circ \phi_h^l) A, \lambda^+(T^*_1 l_u (T_v^* r_{\phi_h^l(v)} A'))(v) >$$

$$- < \mu', T_1^* l_h (Ad_{k-1} z_1 + z_2) > + < \mu, T_1^* l_h (Ad_{k-1} z_1' + z_2') >$$

$$+ < q, [Ad_{k-1} z_1 + z_2, Ad_{k-1} z_1' + z_2'] > + < A, \lambda^+(T^*_1 l_u (u \phi_h^l(v)) (u \phi_h^l(v))) > .$$

We now treat separately the three types of terms which do not obviously cancel out:

$(1) := < I^*(v), [z_1, z_1'] > - < T_v^* I^* z_1, \lambda^+(T^*_1 l_u (T_v^* r_{\phi_h^l(v)} A'))(v) > .$

Since $I^*$ is a morphism of groups, $I^* l_v(w) = I^*(w) + I^*(w)$ therefore $T_1^* (I^* l_v) = \iota^*$, and hence

$$T_v I^* \lambda^+(T^*_1 (I^* l_u) z_1')(v) = T_v I^* \lambda^+(\iota(z_1'))(v)$$

$$= \frac{d}{dt}|_0 I^*(\phi^l_{e^+}(z_1))(v)$$

$$= \frac{d}{dt}|_0 Ad^*_{e^+}(I^*(v)) = ad^*_{z_1} I^*(v).$$

Thus $(1) = 0$.

$(2) := < T_u^* r_{\phi_h^l(v)} A, \lambda^+(T^*_1 l_u (T_v^* r_{\phi_h^l(v)} A'))(u) >$

$$- < T_v^* (l_u \circ \phi_h^l) A, \lambda^+(T^*_1 l_u (T_v^* (l_u \circ \phi_h^l) A'))(v) >$$

$$+ < A, \lambda^+(T^*_1 l_u (u \phi_h^l(v)) (u \phi_h^l(v)) > .$$
Let $\Pi_*$ be the Poisson tensor of $G^*$. We have

$$
(2) = \Pi_*(u)(T_u^* r_{\phi_h^1(v)} A, T_u^* r_{\phi_h^1(v)} A')
+ \Pi_*(v)(T_v^* (l_u \circ \phi_h^0) A, T_v^* (l_u \circ \phi_h^0) A') - \Pi_*(u \phi_h^1(v))(A, A')
= +\Pi_*(v)(T_v^* (l_u \circ \phi_h^0) A, T_v^* (l_u \circ \phi_h^0) A') - \Pi_*(\phi_h^1(v))(T_{\phi_h^1(v)}^* l_u A, T_{\phi_h^1(v)}^* l_u A'),
$$

where in the last equality we have used the multiplicativity of $\Pi_*$. Thus $(2) = 0$ follows from the Hamiltonian property of $\phi_h^1 : G^* \to G^*$.

$$(3) = <T_h^* (l_u \circ \phi_h^0) A', T_1 l_h z_1> - <T_v^* l_v \iota z_1, \lambda \iota ((T_v^* (l_u \circ \phi_h^0) A'))(v)>.$$ We have

$$<T_h^* (l_u \circ \phi_h^0) A', T_1 l_h z_1> = <A', T_1 (l_u \circ \phi_h^0 \circ l_h) z_1>
= <A', T_1 (l_u \circ \phi_h^0 \circ l_h) z_1>
= <T_v^* (l_u \circ \phi_h^0) A', T_1 \phi_h^0 z_1>,
$$

and $T_v^* l_v^* = T_v^* l_{v-1} \circ \iota$. Therefore

$$
(3) = <T_v^* (l_u \circ \phi_h^0) A', T_1 \phi_h^0 (z_1)> + <T_v^* l_{v-1} \iota (z_1), \Pi_*(v)(T_v^* (l_u \circ \phi_h^0) A')>
= <T_v^* (l_u \circ \phi_h^0) A', T_1 \phi_h^0 (z_1)} - \Pi_*(v)(T_v^* l_{v-1} \iota (z_1)) \geq 0.
$$

Hence the proof. $lacksquare$

**References**

[AM] Almeida, R. and Molino, P., *Suites d’Atiyah et feuilletages transversalement complets*, C. R. Acad. Sci. Paris, Série I, t. 300 (1985), 13-15.

[BD] Belavin, A.A. and Drinfel’d, V., *Triangle equations for simple Lie algebras*, Mathematical Physics Reviews (Ed. Novikov et al.) Harwood, New York (1984), 93-165.

[BH] Brown, R. and Higgins, P. J., *On the connection between the second relative homotopy groups of some related spaces*, Proc. London Math. Soc. 36 (1978), 193-212.

[BKS] Bangoura, M. and Kosmann-Schwarzbach, Y., *Equations de Yang-Baxter dynamique classique et algébroides de Lie*, C. R. Acad. Sc. Paris, Série I 327 (1998), 541-546.

[CDW] Coste, A., Dazord, P. and Weinstein, A., *Groupoides symplectiques*, Publications du Département de Mathématiques de l’Université de Lyon 2/A (1987), 1-65.

[CF] Crainic, M. and Fernandes, R., *Integrability of Lie brackets*, LANL e-print archive math.DG/0105033 [http://xxx.lanl.gov].

[D] Drinfeld, V.G., *Hamiltonian structures on Lie groups, Lie bialgebras, and the geometric meaning of the classical Yang-Baxter equations*, Soviet Math. Dokl. 27 (1983), 667-671.

[DSW] Cannas da Silva, A. and Weinstein, A., *Geometric models for noncommutative algebras*, Berkeley Mathematics Lecture Notes 10. Amer. Math. Soc., Providence, RI (1999).

[E] Ehresmann, C., *Catégories structurées*, Ann. Sci. Ecole Norm. Sup. 80 (1963), 349-426.

[EV] Etingof, P. and Varchenko, A., *Geometry and classification of solutions of the classical dynamical Yang-Baxter equation*, Commun. Math. Phys. 192 (1998), 77-120.

[F] Felder, G., *Conformal field theory and integrable systems associated to elliptic curves*, Proc. ICM Zurich, Birkhäuser, Basel (1994), 1247–1255.

[HM] Hurtubise, J., Markman, E., *Elliptic Sklyanin integrable systems for arbitrary reductive groups*, LANL e-print Archive math.AG/0203031.

[K] Karasev, M., *Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets*, Math. USSR Izvestiya 28 (1987), 497-527.

[KW] Kinyon,M. and Weinstein, A., *Leibniz algebras, Courant algebroids, and multiplications on reductive homogeneous spaces*, Amer. J. math. 123 (2001), 525-550.

[L] Libermann, P., *On symplectic and contact groupoids*, Differential geometry and its applications (Opava 1992), Math. Publ.1, Silesian Univ. Opava, Opava (1993), 29-45.
[Lu] Lu, J.H., Ph.D. Thesis, Berkeley, 1990.
[LW1] Lu, J.-H., Weinstein, A., Poisson Lie groups, dressing transformations, and Bruhat decompositions., J. Diff. Geom. 31 (1990), 501–526.
[LW2] Lu, J.-H., Weinstein, A., Groupoides symplectiques doubles des groupes de Lie-Poisson, C. R. Acad. Sci. Paris, Ser. I, t. 309 (1989), 951-954.
[LWX] Liu, Z.-J., Weinstein, A., and Xu, P., Manin triples for bialgebroids, J. Diff. Geom. 45 (1997), 547-574.
[WX] Liu, Z.-J., Weinstein, A., and Xu, P., A class of integrable spin Calogero-Moser systems, to appear in Commun. Math. Phys.
[M1] MacKenzie, K., Lie groupoids and Lie algebroids in differential geometry. LMS Lecture Notes Series 124, Cambridge University Press, 1987.
[M2] , On symplectic double groupoids and the duality of Poisson groupoids, Internat. J. Math. 10 (1999), 435-456.
[M3] , Double Lie algebroids and second-order geometry I., Adv. Math. 94 (1992), 180-239.
[MR] Marsden J.E. and Ratiu T., Reduction of Poisson manifolds, Letters in Math. Phys. 11 (1986), 161-169.
[MW] Mikami K. and Weinstein A., Moments and reduction for symplectic groupoids, Publ. RIMS, Kyoto University 24 (1988), 121-140.
[MX1] MacKenzie, K. and Xu, P., Lie bialgebroids and Poisson groupoids, Duke Math. J. 73 (1994), 415–452.
[MX2] , Integration of Lie bialgebroids, Topology 39 (2000), 445-467.
[P] Pradines, J., Géométrie différentielle au-dessus d’un groupoïde, C. R. Acad. Sc. Paris, Ser. I, t. 266 (1968), 1194-1196.
[STS] Semenov-Tian-Shansky, M., Dressing transformations and Poisson Lie group actions, Publ. RIMS, Kyoto University 21 (1985), 1237-1260.
[W1] Weinstein, A., Coisotropic calculus and Poisson groupoids, J. Math. Soc. Japan 4 no. 40 (1988), 705–727.
[W2] Weinstein, A., Symplectic groupoids and Poisson manifolds, Bull. Amer. Math. Soc. 16 (1987), 101-104.
[X] Xu, P., On Poisson groupoids, Internat. J. Math. 6 (1995), 101-124.

L.-C. Li, Department of Mathematics, Pennsylvania State University, University Park, PA 16802. USA
E-mail address: luenli@math.psu.edu

S. Parmentier, Institut Girard Desargues (UMR 5028 du CNRS), Université Lyon 1, 43 Blvd du 11 Novembre 1918, 69622 Villeurbanne cedex, France
E-mail address: serge@desargues.univ-lyon1.fr