The unintegrated gluon distribution
from the CCFM equation

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Abstract

The gluon distribution $f(x, k_t^2, \mu^2)$, unintegrated over the transverse momentum $k_t$ of the gluon, satisfies the angular-ordered CCFM equation which interlocks the dependence on the scale $k_t$ with the scale $\mu$ of the probe. We show how, to leading log accuracy, the equation can be simplified to a single scale problem. In particular we demonstrate how to determine the two-scale unintegrated distribution $f(x, k_t^2, \mu^2)$ from knowledge of the integrated gluon obtained from a unified scheme embodying both BFKL ($\log(1/x)$) and DGLAP ($\log \mu^2$) evolution.

1 Introduction

Deep inelastic electron-proton scattering is described in terms of scale dependent parton distributions $q(x, \mu^2)$ and $g(x, \mu^2)$. For less inclusive processes it is however necessary to consider distributions unintegrated over the transverse momentum $k_t$ of the parton, which for the gluon, for example, we denote by $f(x, k_t^2, \mu^2)$. These distributions depend on two hard scales; $k_t$ and the hard scale $\mu$ of the probe. The (conventional) integrated gluon distribution is given by

$$xg(x, \mu^2) = \int_{\mu^2}^{\infty} \frac{dk_t^2}{k_t^2} f(x, k_t^2, \mu^2).$$

Unintegrated distributions are required to describe measurements where transverse momenta are exposed explicitly. For example to describe the $p_T$ spectrum of prompt photons produced in high energy hadron collisions or for dijets or vector mesons produced at HERA.
At very low $x$, that is to leading $\log(1/x)$ accuracy, the unintegrated distribution becomes independent of the hard scale $\mu$, and so from (1) we have

$$f(x, k_t^2, \mu^2) \to \frac{\partial}{\partial \ln \lambda^2} \left( xg(x, \lambda^2) \right) \bigg|_{\lambda^2 = k_t^2}. \quad (2)$$

Clearly (2) cannot remain true as $x$ increases. Indeed we see that it would give negative values for $f$. Moreover even at low $x$, there are significant subleading corrections which, to some level of approximation, modify (2) to the form

$$f(x, k_t^2, \mu^2) \approx \frac{\partial}{\partial \ln \lambda^2} \left( xg(x, \lambda^2) T_g(\lambda, \mu) \right) \bigg|_{\lambda^2 = k_t^2}, \quad (3)$$

where $T_g$ is the Sudakov form factor. In fact (3) is oversimplified. As discussed below, the expression for $f(x, k_t^2, \mu^2)$ is more complicated than (3).

The natural framework for unifying the small and large $x$ domains is the CCFM formalism based on angular ordering [3]–[10], which follows from colour coherence effects [11]. It reduces to the leading order DGLAP formalism at moderate $x$ and it embodies the BFKL formalism at small $x$. The unintegrated gluon distribution $f(x, k_t^2, \mu^2)$ satisfies the CCFM equation [3, 4] which interlocks the two hard scales $(k_t^2, \mu^2)$ in a complicated way. The equation is based on the coherent radiation of gluons, which leads to an angular ordering of the gluon emissions along the chain. The ordering introduces a scale specifying the maximum angle of gluon emission, which turns out to be essentially the hard scale $\mu$ of the probe. At moderate $x$ the angular ordering becomes an ordering in the gluon transverse momenta and the CCFM equation reduces to DGLAP evolution. At very small $x$ the angular ordering does not provide any constraint on the transverse momenta along the chain and, in the leading $\log(1/x)$ approximation, $f(x, k_t^2, \mu^2)$ becomes the $\mu$ independent distribution which satisfies the BFKL equation. On the other hand, although the dependence on the scale $\mu$ only enters at subleading $\log(1/x)$ level, $f$ does depend on $\mu$ through leading $\log \mu^2$ evolution.

The outline of the paper is as follows. The angular-ordered CCFM equation is introduced in Section 2. In Section 3 we simplify this evolution, yet staying within leading log accuracy, to show that the two-scale distribution $f(x, k_t^2, \mu^2)$ can be obtained in terms of the conventional one-scale $g(x, \mu^2)$ distribution. In this way, we are led to a procedure for determining $f(x, k_t^2, \mu^2)$ from a unified DGLAP/BFKL type single-scale evolution equation. This is described in Section 4. Moreover we are able to extend the formalism to incorporate important subleading $\log(1/x)$ effects, which are generated by the so-called consistency condition\footnote{Called the kinematic constraint in [12].} [12, 13] and which subsume the angular ordering constraint at low $x$. We also extend the formalism to include the contributions due to the quark distributions. For comparison, in Section 5 we present the pure DGLAP-type approach to determine $f(x, k_t^2, \mu^2)$, in which the gluon cascade evolves according to evolution strongly ordered in $k_t$. Section 6 contains sample numerical
results for \( f(x, k_t^2, \mu^2) \) obtained from the fully unified approach of Section 4. As expected, we have some diffusion of the gluon transverse momenta into the region \( k_t > \mu \). This is in contrast to the pure DGLAP-type approximation in which the distribution \( f(x, k_t^2, \mu^2) \) is limited to the domain \( k_t < \mu \). Finally, Section 7 contains a summary of the procedure that we have used to determine the unintegrated gluon distribution \( f(x, k_t^2, \mu^2) \).

2 The CCFM equation

The unintegrated gluon distribution satisfies the CCFM evolution equation [3, 4] based on angular ordering. In unfolded form the equation is

\[
f(x, k_t^2, \mu^2) = f_0(x, k_t^2) + \frac{\alpha_s}{2\pi} \int_0^1 dz \int \frac{d^2 q}{\pi q^2} \Theta(1 - z - q_0/q) \left[ \Theta(z - x) \Theta(\mu - qz) \Theta(k_t^2) \right.
\]
\[
\left. \left( \frac{k_t^2}{k_t^2} P(z) f \left( \frac{x}{z}, k_t^2, q^2 \right) - z P(z) \Theta(\mu - q) f(x, k_t^2, q^2) \right) \right]
\]
\[
- \frac{\alpha_s}{2\pi} 2 N_C \int_0^1 \frac{dz}{z} \int \frac{d^2 q}{\pi q^2} \Theta(q - q_0) \Theta(k_t^2 - q^2) f \left( \frac{x}{z}, k_t^2, q^2 \right).
\]

where \( k_t' \equiv |k_t + (1 - z)q| \). For simplicity, \( \alpha_s \) is taken outside the integrals, but the scale will be specified carefully in the final equations (in particular see (23)). The driving term \( f_0 \) is of non-perturbative origin and is assumed to contribute only for \( k_t^2 < q_0^2 \). The remaining terms contribute in the \( k_t^2 > q_0^2 \) domain. The driving term thus gives the non-perturbative starting gluon distribution

\[
xg(x, q_0^2) = \int q_0^2 \frac{dk_t^2}{k_t^2} f_0(x, k_t^2).
\]

This angular-ordered equation for \( f \), which embodies both DGLAP and BFKL evolution, is shown schematically in Fig. 1. The first term in the square brackets in (4) describes real gluon emission with angular ordering imposed. The term containing \( f(x, k_t^2, q^2) \) is related to the virtual corrections corresponding to the unfolded Sudakov form factor, whilst the last term in (4) represents the virtual corrections which, when resummed, give rise to gluon reggeization. The latter correspond to the BFKL part of the (unfolded) non- Sudakov form factor. Angular ordering along the chain, a portion of which is shown in Fig. 2, requires

\[
z_{n-1} q_{n-1} < q_n \quad \text{where} \quad q_n \equiv q_{n}/(1 - z_n)
\]

and \( z_n = x_n/x_{n-1} \). The angular ordering continues up to a maximum angle with the limit expressed in terms of the hard scale via \( \Theta(\mu - qz) \). When \( z \) is away from the \( z \sim 0 \) and \( z \sim 1 \)

\footnote{The folded form (which actually is the CCFM equation [3, 4]) contains Sudakov and non-Sudakov form factors, which arise from the resummation of virtual corrections and screen the singularities as \( z \to 1 \) and \( z \to 0 \) respectively.}
\[ f(x, k_t^2, \mu^2) = f(x, k_t^2, \mu^2) + v_{irtual\ contributions} \]

domains, angular ordering is equivalent to the strong \( k_t \) ordering of pure DGLAP LO evolution. At first sight there appear to be three types of large logarithms in (4). First the usual DGLAP logarithms coming from the \(|k'_t| \equiv |k_t + (1-z)q| \ll q\) domain. Second there are the BFKL-type \( \log(1/x) \) contributions originating from the \( 1/z \) part of the real emission term in (4) and the gluon reggeization contribution. These two terms can be combined together in the function

\[
F(x, k_t^2) = \frac{\alpha_s}{2\pi 2N_C} \int_0^1 \frac{dz}{z} \int d^2q \frac{k_t^2}{q^2} \left[ f \left( \frac{x}{z}, |k_t + q|^2, q^2 \right) F \left( \frac{x}{z}, \mu, q^2 \right) \right] - F \left( \frac{k_t^2}{q^2}, q^2 \right) \]

where we have assumed \( z \ll 1 \), in \( k'_t \) of (7). Finally there is a danger that in (4) we have a large logarithm from the region \( q^2 \ll k_t^2 \). However, we see from (4) that the function \( f \left( x, k_t^2, q^2 \right) \) only extends into the region \( k_t^2 > q^2 \) as a result of the BFKL small \( x \) effects, which are subleading at finite \( x \). That is, to leading log accuracy, it can be shown, in the so-called “single-loop” approximation, that if the last term in (4) is neglected and \( \Theta(\mu - qz) \) is replaced by \( \Theta(\mu - q) \), then the function \( f \left( x, k_t^2, q^2 \right) \) vanishes for \( k_t^2 > q^2 \). Thus we limit the integration regions in (4) to the strongly-ordered domain \( k_t^2 \ll q^2 \) for those contributions in which the unintegrated gluon is multiplied by the part \( \bar{P}(z) \) of the splitting function \( P(z) \) which is non-singular at low \( z \). That is

\[
\bar{P}(z) = P(z) - \frac{2N_C}{z}. \]

It should be noted that the BFKL part, (8), of (4) (for which this approximation is not justified) is free from singularity as \( q \to 0 \), since the potential singularity of the real emission term is cancelled by the virtual contribution.
Figure 2: A portion of the evolution chain. Angular ordering requires $z_{n-1}q_{n-1} < q_n$, where $q_n \equiv q_{tn}/(1 - z_n)$ and $z_n = x_n/x_{n-1}$.

So finally we have just the large logarithms coming from either $k_t' \ll k_t$ or from $z \ll 1$. Our aim is to develop an approximate treatment of the CCFM equation which incorporates both types of large logarithms.

3 Simplification of the CCFM equation

To simplify the angular-ordered equation, (4), we rearrange the equation and retain only terms which generate large logarithms. To achieve this it is convenient to add and subtract the term

$$\frac{\alpha_s}{2\pi} \int_0^1 dz \int \frac{d^2q}{\pi q^2} \Theta \left(1 - z - \frac{q_0}{q}\right) \Theta(\mu - q) \Theta(k_t^2 - k_t'^2) z P(z) \frac{k_t^2}{k_t'^2} f(x, k_t'^2, q^2)$$

from the right-hand-side of (4), and to group together contributions containing the singular $2N_C/z$ part of the splitting function

$$P(z) \equiv \bar{P}(z) + \frac{2N_C}{z}. \quad (11)$$

In this way we obtain the approximate form

$$f(x, k_t^2, \mu^2) = f_0(x, k_t^2) + \frac{\alpha_s}{2\pi} \int_0^1 dz \Theta(k_t - q_0) \Theta \left(\mu - \frac{k_t}{1 - z}\right)$$

$$\times \left[ \bar{P}(z) \Theta(z - x) \frac{x}{z} g \left(\frac{x}{z}, \left(\frac{k_t}{1 - z}\right)^2\right) - z P(z) x g \left(x, \left(\frac{k_t}{1 - z}\right)^2\right) \right]$$

$$+ \frac{\alpha_s}{2\pi} \int_0^1 dz \int \frac{dq^2}{q^2} \Theta \left(1 - z - \frac{q_0}{q}\right) \Theta(\mu - q) z P(z)$$

$$\times \left[ \int \frac{dk_t'^2}{k_t'^2} \Theta(k_t^2 - k_t'^2) f(x, k_t'^2, q^2) \frac{q}{2} \delta \left(q - \frac{k_t}{1 - z}\right) - f(x, k_t^2, q^2) \Theta \left(q - \frac{k_t}{1 - z}\right) \right] + F(x, k_t^2), \quad (12)$$
where the BFKL-type $\log(1/x)$ contribution $F(x, k_t^2)$ is given by (8). The second term on the right-hand-side is the pure DGLAP contribution in the large $x$ limit. It comes from the strongly-ordered configuration

$$k_t^2 \equiv |\mathbf{k}_t + (1-z)\mathbf{q}|^2 \ll q^2$$

in the second term on the right-hand-side of (4). In this configuration the variable $q$ becomes $k_t/(1-z)$. We have also made the large $z$ approximation such that

$$\bar{P}(z) \Theta(\mu - qz) \approx \bar{P}(z) \Theta(\mu - q) \approx \bar{P}(z) \Theta \left( \mu - \frac{k_t}{1-z} \right).$$

With these approximations we may rewrite the second term using

$$\int_{k_t^2/(1-z)^2}^{k_t^2} \frac{dk_t^2}{k_t^2} \left[ \Theta(z-x) \bar{P}(z) f \left( x, \frac{k_t^2}{1-z} \right) - zP(z) f \left( x, k_t^2, \frac{k_t}{1-z} \right) \right]$$

$$= \Theta(z-x) \bar{P}(z) \frac{x}{z} g \left( x, \frac{k_t^2}{1-z} \right) - zP(z) xg \left( x, \frac{k_t}{1-z} \right)$$

(15)

where the upper limit $k_t^2$ of the $dk_t^2$ integration has, to leading log $\mu^2$ accuracy, been replaced by $(k_t/(1-z))^2$.

Finally, the third term on the right-hand-side of (12) corresponds to the difference between (10) and the virtual Sudakov (DGLAP-type) contribution given by the second term in the square brackets in (3), that is to the contribution

$$\frac{\alpha_s}{2\pi} \int_0^1 dz \int \frac{dz}{z} \theta \left( 1-z - \frac{q_0}{q} \right) \Theta \left( \mu - q \right) zP(z) \left[ \frac{k_t^2}{k_t^2} f(x, k_t^2, q^2) - f(x, k_t^2, q^2) \right].$$

(16)

The first integral is evaluated using the strongly-ordered configuration $(k_t^2 \ll k_t^2, q \approx k_t/(1-z))$, while the second integration is restricted to the region $q(1-z) > k_t$. It should be noted that for $q(1-z) \ll k_t$ the two terms in the integrand of (16) cancel, since then $k_t' \approx k_t$.

The contribution (16) represents the virtual corrections which have to be resummed. The resummation is performed in the Appendix. We obtain

$$f(x, k_t^2, \mu^2) = \frac{\partial}{\partial \ln k_t^2} \left[ T_g(k_t, \mu) xg(x, k_t^2) \right]$$

$$+ \int_{k_t^2}^{\mu^2} \frac{dq^2}{q^2} T_g(q, \mu) \int_{q_0^2}^{k_t^2} \frac{dk_t^2}{k_t^2} \frac{\partial L(x, k_t^2, q^2)}{\partial \ln k_t^2},$$

(17)

where

$$L(x, k_t^2, \mu^2) = \frac{\alpha_s}{2\pi} \theta(k_t - q_0) \int_0^1 dz \Theta \left( \mu - \frac{k_t}{1-z} \right) \left[ \bar{P}(z) \Theta(z-x) \frac{x}{z} g \left( x, \frac{k_t}{1-z} \right) \right]$$

$$- zP(z) xg \left( x, \frac{k_t}{1-z} \right),$$

(18)
and where the Sudakov form factor
\[ T_g(q, \mu) = \exp\left(- \int_{q^2}^{\mu^2} \frac{dp^2}{p^2} \frac{\alpha_S(p^2)}{2\pi} \int_0^{1-k_t/p} dz' \, z' P(z') \right). \] (19)

The cut-off \( z' < 1 - k_t/p \) enters on account of the kinematic structure of the real emission term, where the upper limit of the \( p \) integration is given by \( k_t/(1 - z') \). Note that \( T_g(q, \mu) \) therefore implicitly depends on \( k_t \). The integration limits defining the Sudakov form factor should be understood as arising from the \( \Theta \) function constraints
\[ \Theta(1 - k_t/p) \Theta(\mu - p) \Theta(p - q). \] (20)

This implies \( T_g(q, \mu) = 1 \) if these constraints are not satisfied. In particular \( T_g = 1 \) for \( k_t > \mu \) or \( q > \mu \).

A nice feature of the result (17) is that the unintegrated gluon \( f(x, k_t^2, \mu^2) \) is entirely specified in terms of the integrated gluon \( xg \). The next step is to introduce a single scale unified equation, which embodies both BFKL and DGLAP type effects, to determine \( xg \).

4 Strategy for determining the unintegrated gluon

We have emphasized that the angular-ordered equation (12) is a ‘two-scale’ evolution equation for \( f(x, k_t^2, \mu^2) \). That is the scales \( k_t^2 \) and \( \mu^2 \) are intertwined by angular ordering. In the previous section we have shown how the two-scale unintegrated gluon \( f(x, k_t^2, \mu^2) \) can be determined once we know the integrated gluon \( xg \). Here we describe the procedure to obtain \( xg \) from a unified evolution equation for a single scale auxiliary distribution
\[ h(x, \mu^2) = \left( \frac{\mu^2}{\mu^2} \right)^2 \left( \frac{\mu^2}{k_t^2} \right)^2 f(x, k_t^2, \mu^2) \]. (21)

If we integrate both sides of equation (12) over \( k_t^2 \) up to \( \mu^2 \), differentiate with respect to \( \log \mu^2 \), then we find that \( h(x, \mu^2) \) satisfies the evolution equation
\[ h(x, \mu^2) = \frac{\alpha_S}{2\pi} \int_0^{1-q_0/\mu} dz \left\{ \Theta(z - x) \, \tilde{P}(z) \, \frac{x}{z} \, g \left( \frac{x}{z}, \mu^2 \right) \right\} + F(x, k_t^2 = \mu^2), \] (22)

where \( F(x, k_t^2) \) is defined by (8). Note that the integral in (22) has no singularity close to \( z = 1 \). We can now derive a relation expressing the two-scale unintegrated gluon distribution \( f(x, k_t^2, \mu^2) \) in terms of the one-scale distribution \( xg \) (or \( h \)). From (17) we get the following expression for \( f(x, k_t^2, \mu) \):
\[ f(x, k_t^2, \mu^2) = T_0(k_t, \mu) h(x, k_t^2) + T_g(k_t, \mu) \int_0^{1-k_t/\mu} dz \, z P(z) \alpha_S k_t^2 (1 - z^2) xg(x, k_t^2) \].
\[-\frac{\alpha_s}{2\pi} T_g(k_t, \mu) \int_0^{1-q_0/k_t} dz \left[ \Theta(z-x) \bar{P}(z) \frac{x}{z} g \left( \frac{x}{z}, k_t^2 \right) - P(z) x g \left( x, k_t^2 \right) \right] \]
\[+ \Theta(\mu-k_t) \int_0^{1-k_t/\mu} dz \frac{\alpha_s}{2\pi} T_g \left( \frac{k_t}{1-z}, \mu \right) \left[ \Theta(z-x) \bar{P}(z) \frac{x}{z} g \left( \frac{x}{z}, \frac{k_t}{1-z} \right)^2 \right] - P(z) x g \left( x, \frac{k_t}{1-z} \right)^2 \right], \tag{23}\]

where the scale of $\alpha_s$ is taken to be the scale of the appropriate gluon, except for the second term on the right-hand-side. We may safely set the $1-z$ cut-off $q_0/k_t$ to zero, but not the cut-offs $k_t/\mu$.

Note that in the leading $\ln(1/x)$ approximation we may set $T_g = 1$ and neglect all the integral terms in (23), since they do not generate $\ln(1/x)$ contributions. In this approximation the unintegrated gluon is simply

\[f(x, k_t^2, \mu^2) = h(x, k_t^2),\]

with no dependence on the scale $\mu$.

When obtaining (23) from (17), we have chosen to neglect a contribution coming from the derivative of the Sudakov form factor, $T_g(q, \mu)$, with respect to $k_t$, which arises from the $k_t$ dependence of the regulator, see (19). For this reason the unintegrated gluon of (23) does not precisely integrate to $x g(x, \mu^2)$, although the corrections are subleading in $\log \mu^2$. The discrepancy is indeed negligible at low $x$, but can become of the order of 20% or so for large values of $x > 0.1$. Rather than complicating (23) by including the derivative, we eliminate the discrepancy by changing the regulator in the form factor (19) from $k_t/p$ to $q/p$, that is we take the Sudakov form factor

\[T_g(q, \mu) = \exp \left( - \int_{q^2}^{\mu^2} \frac{dp^2}{p^2} \frac{\alpha_s(p^2)}{2\pi} \int_0^{1-q/p} dz' P(z') \right). \tag{24}\]

This approximation is justified since in our case either $q = k_t$ or $q \sim k_t$. Within this approximation it is evident that the unintegrated gluon (23) integrates exactly to $x g(x, \mu^2)$; the sum of the first two terms on the right-hand-side of (23) form the total derivative

\[k_t^2 \frac{\partial}{\partial k_t} \left[ T_g(k_t, \mu) x g(x, k_t^2) \right], \tag{25}\]

and the integrals of the third and fourth terms cancel each other.

From (21) we see that the integrated gluon distribution $g$ can be expressed in terms of $h$, namely

\[x g(x, k_t^2) = x g(x, q_0^2) + \int_{q_0^2}^{k_t^2} \frac{d\mu^2}{\mu^2} \bar{h}(x, \mu^2). \tag{26}\]
Equations (22) and (23), together with (8) and (26), form a system of coupled equations. If we substitute \( f \) of (23) into the \( F \) term on the right-hand-side of (22), and take account of (26), then we obtain an integral equation for \( h \). We may solve this equation for the single-scale auxiliary distribution \( h(x, \mu^2) \), and then compute the two-scale unintegrated gluon \( f(x, k_t^2, \mu^2) \) from (23).

It is convenient to simplify the integral equation (22) for \( h(x, \mu^2) \) using approximations which are valid to leading log accuracy. To be precise we simplify the computation of \( F(x, k_t^2) \) of (8). First instead of allowing the scale \( q^2 \) of \( f \) to vary, we note that in (8) the dominant values of \( q^2 \) are such that \( q^2 \approx k_t^2 \). Moreover we notice that in the strongly ordered domain, \( k_t^2(\equiv |k_t + q|) \ll k_t^2 \), that the first term on the right-hand-side of (8) can be simplified using

\[
\int \frac{d^2 q}{\pi q^2} k_t^2 f \left( \frac{x}{z}, |k_t + q|^2, q^2 \right) \approx \int k_t^2 \frac{d k_t^2}{k_t^2} f \left( \frac{x}{z}, k_t^2, k_t^2 \right) \equiv \frac{x}{z} g \left( \frac{x}{z}, k_t^2 \right),
\]

see (10). In the remaining contributions to \( F(x, k_t^2) \) of (8) we can use (23) to approximate \( f \) by the first term,

\[
f(x, k_t^2, q^2) \approx h(x, k_t^2),
\]

noting that \( q^2 \approx k_t^2 \) and \( T_g(k_t, k_t) = 1 \). The other terms in (23) give only subleading \( \log(1/x) \) contributions to the BFKL kernel. Since this contribution to \( F(x, k_t^2) \) goes beyond the strongly-ordered part of the kernel, it is also subleading in \( \log \mu^2 \). After these approximations, eq. (22) for \( h(x, \mu^2) \) for \( \mu^2 > q_0^2 \) may be written\(^3\)

\[
h(x, \mu^2) = h_0(x, \mu^2) + \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 dz \int_{q_0^2}^{\mu^2} \frac{dq^2}{q^2} \left\{ \Theta(z - x) \tilde{P}(z) h \left( \frac{x}{z}, q^2 \right) - z P(z) h(x, q^2) \right\}
\]

\[
+ \frac{\alpha_s(\mu^2)}{2\pi} 2N_C \int_x^1 \frac{dz}{z} \int \frac{dq^2}{q^2} \Theta(k_t^2 - q_0^2) \left\{ \frac{\mu^2}{k_t^2} h \left( \frac{x}{z}, k_t^2 \right) - \Theta(\mu^2 - q^2) h \left( \frac{x}{z}, \mu^2 \right) \right\}
\]

where \( k_t^2 = |k_t + q|^2 \) with \( k_t^2 = \mu^2 \). The driving term, which arises from the substitution of (26) for \( x g \) in (22) and (27), is given by

\[
h_0(x, \mu^2) = \frac{\alpha_s(\mu^2)}{2\pi} \int_0^1 dz \left\{ \Theta(z - x) P(z) \frac{x}{z} g \left( \frac{x}{z}, q_0^2 \right) - z P(z) x g(x, q_0^2) \right\}.
\]

Notice that the strongly-ordered contribution of \( F(x, k_t^2 = \mu^2) \) has combined with the residual DGLAP contribution in (22) with the effect that \( \tilde{P}(z) \rightarrow P(z) \).

Equation (29) is the single-scale unified BFKL/DGLAP equation for the gluon that was proposed in Ref. [14]. There it was shown that it is straightforward to incorporate a major

\(^3\)In order to be consistent with (22), the upper limit of the \( z \) integration in the second term on the right-hand-side of (29) should be \( 1 - q_0/q \) rather than 1. The integrals are, of course, regular at \( z = 1 \) and so the discrepancy is subleading in \( \ln \mu^2 \).
part of the subleading order \(\log(1/x)\) (or BFKL) effects by imposing a consistency condition to ensure that the virtuality of the exchanged gluon is dominated by its transverse momentum squared. This is achieved by including the theta function \(\Theta(\mu^2 - qz^2)\) in the real emission contribution in the last term of (23). Other important subleading terms arising from using the complete DGLAP splitting function and from the running of \(\alpha_S\) are automatically included in our framework [14]. This formalism was used to fit to deep inelastic scattering data was made and the auxiliary function \(h(x, \mu^2)\) was determined [14]. It was checked that the corresponding integrated gluon \(xg(x, k_t^2)\) computed from (22) was compatible with the gluons obtained in the MRS, CTEQ global parton analyses [15].

In [14] the contribution of the quark distributions was included in (29) for \(h(x, \mu^2)\). To incorporate the quarks in the present analysis we must also include the contribution of the singlet quark distribution \(\Sigma\) in (23) for \(f(x, k_t^2, \mu^2)\). That is we make the replacement

\[
\bar{P}(z) \frac{x}{z} g \rightarrow \bar{P}(z) \frac{x}{z} g \left( \frac{x}{z}, \mu'^2 \right) + P_{gq}(z) \frac{x}{z} \sum \left( \frac{x}{z}, \mu'^2 \right)
\]

in the real emission part of the third and fourth terms on the right-hand-side of (23), where \(\mu'\) is the appropriate scale. Recall that \(P(z) \equiv P_{gg}(z)\). In the second term, and in the virtual part of the third and fourth terms, we make the replacement

\[
zP(z) xg \rightarrow (zP(z) + 2n_f zP_{gq}(z)) xg(x, \mu^2),
\]

where \(n_f\) is the number of active flavours, and the scale \(\mu^2 = k_t^2\) or \(k_t^2/(1-z)^2\) as appropriate. Finally we have to modify the Sudakov form factor so that (24) becomes

\[
T_g(q, \mu) = \exp \left( - \int_{q^2}^{\mu^2} \frac{\alpha_S(p^2)}{2\pi} d\mu^2 \int_0^{1-q/p} dz' \left[ P(z') + \sum_q P_{gq}(z') \right] \right).
\]

The above procedure allows the determination of the approximate solution \(f(x, k_t^2, \mu^2)\) of the CCFM equation, which incorporates both a full (or so-called “all-loop” [5, 6]) resummation of leading \(\ln(1/x)\) contributions, as well as the resummation of leading \(\ln \mu^2\) contribution, and the inclusion of dominant subleading \(\ln(1/x)\) terms.

5 The pure DGLAP limit

It is informative to compare the predictions above for the unintegrated gluon \(f(x, k_t^2, \mu^2)\) with those obtained in the DGLAP (or so-called “single-loop” [5, 6]) approximation, in which \(\Theta(\mu - qz)\) in (4) is replaced by \(\Theta(\mu - q)\), and the last term in (4) is neglected. After making these modifications we repeat the procedures of Sections 3 and 4 and obtain the DGLAP form

\[
f(x, k_t, \mu^2) = \int_x^{1-k_t/\mu} dz \frac{\alpha_s(k_t^2/(1-z)^2)}{2\pi} T_g \left( \frac{k_t}{1-z}, \mu \right) P(z) \frac{x}{z} g \left( \frac{x}{z}, \left( \frac{k_t}{1-z} \right)^2 \right)
\]
\[ + \int_0^{1-k_t/\mu} dz z P(z) \frac{\alpha_S(k_t^2/(1-z)^2)}{2\pi} \left[ T_g(k_t, \mu) xg(x, k_t^2) - T_g \left( \frac{k_t}{1-z}, \mu \right) xg \left( x, \left( \frac{k_t}{1-z} \right)^2 \right) \right], \]

with \( f = 0 \) if \( k_t > \mu \). Apart from the last term, this is the equation for the unintegrated gluon introduced in Ref. [3]. The last term, which is only non-zero on account of different scales, introduces subleading corrections. Its inclusion improves the accuracy of the integration of \( f(x, k_t^2, \mu^2) \) to reproduce \( xg(x, \mu^2) \), see (34). Note that the DGLAP or ‘single-loop’ unintegrated gluon vanishes for \( k_t \geq \mu \), as indeed it must [5, 6]. In the results presented below we include the quark contributions as described in (31)–(33).

It is informative to see how the full equation, (23), for \( f(x, k_t^2, \mu^2) \) reduces to the DGLAP limit (34). A crucial observation is that in the DGLAP domain \( (k_t < \mu) \) it is possible, within the leading \( \ln(1/x) \) and \( \ln(\mu^2) \) accuracy, to replace \( \bar{P}(z) \) by \( P(z) \) in (23). Thus, provided \( k_t < \mu \), we obtain the following more symmetric formula

\[
\begin{align*}
    f(x, k_t^2, \mu^2) &= T_g(k_t, \mu) h(x, k_t^2) + T_g(k_t, \mu) \int_0^{1-k_t/\mu} dz z P(z) \frac{\alpha_S(k_t^2/(1-z)^2)}{2\pi} xg(x, k_t^2) \\
    &- \frac{\alpha_S(k_t^2)}{2\pi} T_g(k_t, \mu) \int_0^{1-q_0/k_t} dz P(z) \left[ \Theta(z - x) \frac{x}{z} g \left( \frac{x}{z}, k_t^2 \right) - z xg \left( x, k_t^2 \right) \right] \\
    &+ \Theta(\mu - k_t) \int_0^{1-k_t/\mu} dz \frac{\alpha_S}{2\pi} T_g \left( \frac{k_t}{1-z}, \mu \right) P(z) \left[ \Theta(z - x) \frac{x}{z} g \left( \frac{x}{z}, \left( \frac{k_t}{1-z} \right)^2 \right) \right. \\
    &\left. - \frac{z xg \left( x, \left( \frac{k_t}{1-z} \right)^2 \right)}{2\pi} \right].
\end{align*}
\]

(35)

In the DGLAP limit the first and third terms on the right-hand-side of (35) exactly cancel\(^4\); they simply represent the DGLAP equation for \( h(x, k_t^2) \) of (21). Thus (35) reduces to (34).

6 Numerical evaluation of the unintegrated gluon

In Fig. 3 we show the \( k_t \) distributions of three different unintegrated gluons at each of four different values of \( x \) at a hard scale \( \mu^2 = 100 \text{ GeV}^2 \). Briefly

(i) the continuous curves are the gluons \( f(x, k_t^2, \mu^2) \) obtained from (23), with the quark terms included, using the auxiliary function \( h(x, k_t^2) \) of Ref. [14], which was itself obtained from a fit to deep inelastic scattering data using a unified BFKL/DGLAP equation. (The curves have been smoothed in the transition region \( k_t \sim \mu \).

\(^4\)Strictly speaking the cancellation is only exact when \( q_0 \to 0 \) in (35).
(ii) the dot-dashed curves show $h(x, k_t^2)$ itself \[14\], which is independent of $\mu^2$,

(iii) the dashed curves show $f(x, k_t^2, \mu^2)$ calculated from the pure DGLAP equation, \[33\], using in this case the auxiliary function $h(x, k_t^2)$ obtained in \[14\] from pure DGLAP evolution from exactly the same starting distributions $(xg(x, q_0^2)$ etc) as those found in the unified fit.

Figure 3: The continuous curves show the $k_t$ dependence of the unintegrated gluon distribution $f(x, k_t^2, \mu^2)$ for $\mu^2 = 100$ GeV$^2$. For comparison we also show the input auxiliary function $h(x, k_t^2)$ (dot-dashed curves) \[14\] and the $k_t$ dependence coming from pure DGLAP evolution (dashed curves).

Note that the third set of gluons are shown solely to illustrate the difference between two types of evolution. The gluons of the third set have not been constrained by a fit to the data, so should not be regarded as realistic.
In the pure DGLAP case, (iii), we see that the distributions are confined to the domain \( k_t < \mu \), as anticipated from strong ordering. On the other hand the distributions \( f(x, k_t^2, \mu^2) \) obtained in the unified BFKL/DGLAP framework develop a more and more extensive \( k_t > \mu \) tail as \( x \) decreases. At small \( k_t \) and low \( x \) the magnitude of the unintegrated gluon calculated from the unified scheme is about a factor of two less than that of the gluon coming from the pure DGLAP approach of case (iii). This is due to the imposition of the consistency constraint in case (i) which suppresses the magnitude of the gluon. If this constraint were absent the distributions of cases (i) and (iii) would not be that different. We note that the auxiliary function \( h \) of case (ii) remains different from the unintegrated gluon \( f \) of case (i) down to very small values of \( x \).

For \( k_t > \mu \) we see that \( f \) is greater than \( h \), whereas the DGLAP-driven unintegrated gluon vanishes, as it must. In this domain inspection of (22) and (23) shows that \( f \) comes purely from the BFKL contribution,

\[
f(x, k_t^2, \mu^2) = F(x, k_t^2).
\]  

On the other hand \( h \) is smaller than \( f \) due to the negative contribution of the integral term in (22). The latter is a DGLAP contribution which is ruled out when angular ordering is imposed. It is negative because the \( 2N_C/z \) contribution has been subtracted from the real emission contribution, but not from the virtual term. We see that for \( x < 0.01 \) that there is about a factor of 2 discrepancy between \( h \) and the true unintegrated gluon \( f \).

### 7 Summary

Here we have addressed the issue of obtaining a reliable determination of the (scale dependent) gluon distribution \( f(x, k_t^2, \mu^2) \), unintegrated over the gluon transverse momentum \( k_t \), where \( \mu \) denotes the hard scale of the probe. In the leading \( \log(1/x) \) approximation the distribution is given simply by the derivative of the unintegrated gluon with respect to its scale \( \lambda = k_t \), see (2), and satisfies the BFKL equation. We correct this simple relation by going beyond the leading \( \log(1/x) \) approximation to include both subleading contributions and DGLAP effects. The final result for \( f(x, k_t^2, \mu^2) \) is given in (23). To obtain this result we use the appropriate gluon cascade formalism based on angular ordering, which leads to the CCFM equation embodying both BFKL and DGLAP evolution. It is important to note that the CCFM equation gives a well-defined framework to calculate the very quantity that we seek: the unintegrated gluon distribution \( f(x, k_t^2, \mu^2) \). Using this formalism we devise a procedure to determine \( f(x, k_t^2, \mu^2) \) from the integrated gluon distribution \( xg(x, q^2) \), its derivative \( h(x, q^2) \) and the Sudakov form factor \( T_g(q, \mu) \), cf. (23). An important ingredient is the solution of a (single scale) evolution equation for \( h(x, q^2) \) which embodies both BFKL and DGLAP effects. From the low \( x \) viewpoint it includes subleading effects from (i) the consistency constraint which limits the available phase space to the region in which the virtuality of the exchanged gluon is dominated by its transverse momentum squared, (ii) DGLAP effects generated by that part of the splitting function \( P_{gg}(z) \).
which is not singular in the limit \( z \rightarrow 0 \), (iii) the inclusion of the quark contribution and, (iv) allowing the coupling \( \alpha_s \) to run and depend on the local scale(s) characteristic of the vertices of the cascade.

We presented sample results to show that the structure of the \( k_t \) distribution of the gluon \( f(x, k_t^2, \mu^2) \) can be significantly different from that of \( h(x, k_t^2) \), down to very small values of \( x \). There are important consequences for the description of hadron-initiated hard processes in which the \( k_t \) of the gluon is probed locally.

**Appendix**

Eq. (17) for the unintegrated gluon \( f(x, k_t^2, \mu^2) \) is obtained from (12) by resumming the virtual corrections given in the third term on the right-hand-side of (12). Here we show how the resummation is performed. First we note that this virtual correction term can be written as a derivative, that is

\[
\frac{\alpha_s}{2\pi} \int_0^1 dz \int k_t^2 z P(z) \left[ \int \frac{dk_t^2}{k_t^2} \Theta(k_t^2 - k_t^2) f(x, k_t^2, q^2) \delta \left( \frac{q - k_t}{1 - z} \right) - f(x, k_t^2, q^2) \Theta \left( \frac{q - k_t}{1 - z} \right) \right]
\]

\[
= -k_t^2 \frac{\partial}{\partial k_t^2} \left[ A(k_t, q) R(x, k_t^2, q^2) \right] \tag{37}
\]

where

\[
R(x, k_t^2, q^2) = \int_{k_t^2}^{k_t^2} \frac{dk_t^2}{k_t^2} f(x, k_t^2, q^2), \tag{38}
\]

\[
A(k_t, q) = \frac{\alpha_s}{2\pi} \int_0^1 dz \int P(z) \Theta(1 - z - k_t/q). \tag{39}
\]

The function \( R(x, k_t^2, q^2) \) has a simple physical meaning. It is the gluon distribution for fixed impact parameter \( b \sim 1/k_t \) at scale \( q \). Note that if \( b = 1/q \) then the distribution \( R \) reduces to the integrated gluon \( x g(x, q^2) \). Using (37) we see that (12) can be expressed as an integro-differential equation for \( R \)

\[
f(x, k_t^2, \mu^2) \equiv k_t^2 \frac{\partial R(x, k_t^2, \mu^2)}{\partial k_t^2}
\]

\[
= L(x, k_t^2, \mu^2) + F(x, k_t^2) - k_t^2 \frac{\partial}{\partial k_t^2} \int_{k_t^2}^{\mu^2} \frac{dq^2}{q^2} A(k_t, q) R(x, k_t^2, q^2), \tag{40}
\]

where \( L \) and \( F \) are defined by (18) and (8) respectively.

In order to solve (18) for \( R \) we integrate both sides over \( dk_t^2/k_t^2 \) up to \( k_t^2 \) and obtain the following integral equation

\[
R(x, k_t^2, \mu^2) = x g_0(x) + \int_{q_0}^{k_t^2} \frac{dk_t^2}{k_t^2} \left[ L(x, k_t^2, \mu^2) + F(x, k_t^2) \right]
\]

\[
- \int_{k_t^2}^{\mu^2} \frac{dq^2}{q^2} A(k_t, q) R(x, k_t^2, q^2). \tag{41}
\]
From (21) and (22) we see, if $\mu = k_t$, that the first two terms on the right-hand-side of (41) are just $xg(x, k^2_t)$. The solution of (41) may be therefore written

$$R(x, k^2_t, \mu^2) = T_g(k_t, \mu) xg(x, k^2_t)$$

$$+ \int_{k^2_t}^{\mu^2} \frac{dq^2}{q^2} T_g(q, \mu) q^2 \frac{\partial}{\partial q^2} \left( \int_{q_0}^{k^2_t} \frac{dk^2_{t}}{k^2_{t}} L(x, k^2_{t}, q^2) \right), \quad (42)$$

where the Sudakov form factor

$$T_g(q, \mu) = \exp \left( - \int_{q^2}^{\mu^2} \frac{dp^2}{p^2} A(k_t, p) \right) \quad (43)$$

is in agreement with (13). Eq. (23) then follows from (12) after differentiation by $\partial / \partial \ln k^2_t$.

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