Web-assisted tunneling in the kicked harmonic oscillator

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We show that heating of harmonically trapped ions by periodic delta kicks is dramatically enhanced at isolated values of the Lamb-Dicke parameter. At these values, quasienergy eigenstates localized on island structures undergo avoided crossings with extended web-states.

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Controlling the state and the time evolution of quantum systems is one of the central themes of current research in experimental and theoretical atomic physics, quantum optics, and mesoscopics. Tailoring wave packets in Rydberg systems \[1\], producing single photons on demand \[2\], creating coherent superpositions of macroscopic persistent-current states \[3\] and, most recently, the controlled production of multiparticle entanglement \[4\], are prominent examples of what is often coined as “quantum state engineering”, such as to stress the almost perfect control we have achieved on matter on the microscopic level. Whilst many of these schemes still rely on an analysis of the quantum dynamics in terms of some unperturbed basis of the quantum system under control, it has become clear during the last decade that generic features of strongly coupled (“complex”) quantum systems allow for novel and often extremely robust strategies of quantum control. In such systems, studied in much detail in the area of quantum chaos, peculiar eigenstates emerge which exhibit unexpected (and unexpectedly robust) localization properties and dynamics. Most prominent examples thereof are nondispersive wave packets in periodically driven quantum systems \[5\], quantum resonances \[6\], and quantum accelerator modes \[7\]. A considerable part of these “strong coupling” quantum control schemes relies on some underlying classical dynamics, which in general is mixed regular-chaotic, and precisely the rich structure of a mixed classical phase space is at their very origin \[8, 9\]. A large class of these systems follows the Kolmogorov-Arnold-Moser (KAM) scenario \[10\], i.e. regular phase space structures are destroyed gradually as the coupling strength is increased. Yet, there is another kind of classically chaotic dynamics, which goes under the name “non-KAM” chaos \[11\], where the phase space flow is fundamentally altered at arbitrarily weak perturbations. Recently, the consequences of such non-KAM transitions have been observed in electron transport in superlattices \[12\], where enhanced transport across the lattice was observed, due to the sudden (non-KAM) appearence and disappearence of unbounded stochastic web-structures in classical phase space, at well-defined values of some control parameter. Hence, enhanced transport was enforced by (controlled) abrupt changes in the underlying classical phase space structure.

Here we expose a different pathway for the controlled enhancement of transport in another non-KAM system, the kicked harmonic oscillator \[13, 14, 15, 16\], at fixed phase space structure, by simply tuning the effective value of \(\hbar\). We will see that, for a suitable choice of the initial oscillator state, dramatic enhancement of the energy absorption by the oscillator from the kicking field can be achieved due to avoided crossings of localized (regular) with extended (web-like) eigenstates of the kicked systems. We argue that our scenario can be easily observed in state-of-the-art experiments on harmonically trapped, kicked cold ions or atoms.

The classical Hamiltonian of the kicked harmonic oscillator \[17\] describes a harmonically trapped (in 1D, with trap frequency \(\nu\)) particle of mass \(m\), subject to a one dimensional, spatially periodic potential (wave vector \(k = 2\pi/\lambda\), modulation depth \(A\)) which is periodically switched on and off at integer multiples of the kicking period \(\tau\):

\[
H = \frac{p^2}{2m} + \frac{mv^2x^2}{2} + A \cos(kx) \sum_n \delta(t - n\tau). \tag{1}
\]

Further inspection of the Hamiltonian reveals that the classical phase space structure is completely determined by the parameters

\[
K = \frac{Ak^2}{m\nu}, \quad \text{and} \quad \alpha = \nu\tau \equiv \frac{2\pi}{q}, \tag{2}
\]

which define the stochasticity parameter and the ratio between kicking and oscillator period, respectively. This is a consequence of the form-invariance of the equations of motion derived from \(\text{11}\), under transformation to the scaled position and momentum coordinates \(v = kx\) and \(u = kp/m\).

A specific property of the kicked harmonic oscillator is that its phase space is unbounded, thus allowing for transport to infinity (which, in the trapped ion scenario, is tantamount to unbounded heating). Furthermore, it exhibits peculiar symmetry properties determined by the ratio \(q = (2\pi/\nu)/\tau\) of oscillator and kicking period: For integer \(q\), in addition to confined regular islands, it displays a stochastic web (reaching out to infinity in phase space) with crystal \((q \in q_c \equiv \{3, 4, 6\})\) or quasi-crystal symmetry \((q \notin q_c)\), along which the system diffuses for...
suitable initial conditions. This web is characterized not only by its symmetry but also by its thickness that broadens (shrinks) as the value of the stochasticity parameter $K$ increases (decreases).

With the harmonic oscillator annihilation operator $\hat{a} = (\hat{v} + \hat{u})/2\eta$ and its hermitean conjugate $\hat{a}^\dagger$ derived from the scaled center of mass coordinates of the trapped particle, the quantum version of (1) reads

$$\hat{H} = \hbar \hat{a}^\dagger \hat{a} + \hbar \hat{K} \sum_{n=0}^{\infty} \delta(t - n\tau),$$

with $\hat{K} = K/2\eta^2$, and the Lamb-Dicke parameter $\eta = k\sqrt{\hbar/2mv}$. Note that the latter measures the ratio of the width of the harmonic oscillator ground state in units of the wave length of the kicking potential, and that its squared value plays the role of an effective Planck constant – which can be tuned easily in ion trap experiments.

Since (3) is periodic in time, the time evolution of any initial state $|\psi_0\rangle$ is given by the action of the one-cycle Floquet propagator

$$\hat{U}\langle \varepsilon_j | = e^{-i\alpha \hat{a}^\dagger \hat{a}} e^{-iK \cos[\eta(\hat{a} + \hat{a}^\dagger)]} \langle \varepsilon_j | = e^{i\phi_j} |\varepsilon_j\rangle,$$

with $\phi_j$ the quasi-energies and $|\varepsilon_j\rangle$ the associated Floquet eigenstates. Due to the $\delta$-term in (3), the Floquet propagator factorizes into a free evolution and a kicking part, and inspection of the former shows that the above classical condition for the emergence of a web structure in classical phase space is also quantum mechanically distinguished, since integer $q$ is equivalent to a (fractional) revival condition 18 for the free evolution right upon the subsequent kick. One therefore expects quantum signatures of web-sustained transport in classical phase space – which we shall now establish by combining dynamical and spectral information.

Let us start with Fig. 1 where we plot the mean energy of the kicked (quantum) particle, initially prepared in the harmonic oscillator ground state, as a function of the number of kicks imparted on it by the external field, for slightly different values of the Lamb-Dicke parameter $\eta$, and $q = 6$. We see that a change of $\eta$ by merely one percent dramatically affects the energy absorption by the particle from the field, eventually leading to quantum transport which is much more efficient than the classical excitation process. Note that the latter is unaffected by such tiny changes of $\eta$, since the results displayed in the figure are obtained for fixed $K$ and $q$, hence for fixed phase space structure!

Whilst this latter observation appears puzzling on a first glance, it is readily resolved by inspection of the evolution of the quasienergy spectrum of (3) as a function of $\eta$. A global view thereof is provided by Fig. 2 for quasicrystal ($q = 5$) and crystal ($q = 6$) symmetries, respectively, and otherwise the same parameters as in Fig. 1. Only those eigenvalues associated with eigenstates which have an overlap larger than $10^{-3}$ with $|\psi_0\rangle$ are shown. For both symmetries, two distinct initial states $|\psi_0\rangle$ are considered: $|\psi_0\rangle = |0\rangle$ (circles), and $|0\rangle$ displaced to the points $(1.2, 2.0)$ and $(1.3, 3.0)$, located in the chaotic region of phase space, for $q = 6$ and $q = 5$ (dots), respectively.

We see that, for the vacuum initial state, the level dynamics are rather similar in the quasicrystal and in the crystal case, what is just the quantum signature of the locally similar phase space structures in the vicinity of the origin. In contrast, they are quite distinct for displaced initial conditions, since, under these premises, the quantum dynamics probes the rather distinct web structures of the underlying classical phase space (illustrated by the insets in the figure, which show how a single trajectory explores phase space in the long time limit – note the different scales!). Independently of symmetry and initial condition, an abundance of avoided crossings is apparent from both plots – a typical signature of the strong coupling between the driving field and the center of mass motion of the trapped particle. Also note that, in the crystal case with $|\psi_0\rangle$ in the chaotic domain, some avoided crossings are organized in a regular manner, what strongly suggests a semiclassical origin – which, however, is not relevant for our present purpose.

For the vacuum initial state, both symmetries exhibit isolated or overlapping avoided crossings (of variable size) at well defined values of $\eta$, which turn out to cause the enhanced quantum transport observed in Fig. 1. The inset in Fig. 1 zooms into the level structure around the
FIG. 2: Spectrum of the Floquet operator $\hat{U}$ as a function of the Lamb-Dicke parameter $\eta$, for $K = 2.0$, $q = 5$ (top) and $q = 6$ (bottom). Only eigenstates with an overlap larger than $10^{-3}$ with the initial state $|\psi_0\rangle$ are represented. Filled circles represent $|\psi_0\rangle = |0\rangle$, while dots refer to a displaced vacuum state centered at $(1.3, 3.0)$ (top) and $(1.2, 2.0)$ (bottom). The insets show the classical phase space explored by a single trajectory after 40000 kicks, when launched at a point near the origin, for the same $K$ and $q$. The arrow in the bottom plot refers to the inset of Fig. 1.

FIG. 3: (Color online) Husimi representations for the eigenstates associated with the points A, B, C and D of the avoided crossing in the inset of Fig. 1. Phase space is spanned by the coordinates $v/2\eta$ and $u/2\eta$, respectively. Clearly, the eigenstates localized in the vicinity of the hyperbolic point at the origin (B and C) dominate the dynamics in the asymptotic part ($\eta = 0.459$ and $\eta = 0.469$) of the anticrossing in the inset of Fig. 1 whilst extended states localized on the stochastic web (A and D) exhibit equal weight at the center of the crossing parameter window for enhanced quantum transport.

avoided crossing at ($\eta = 0.464; \phi = 1.35$) indicated by an arrow in the lower panel of Fig. 2. Here, two quasienergies (as a matter of fact, two quasienergy bands) undergo an isolated avoided crossing, with maximal tunneling coupling between the associated eigenstates at its center at $\eta = 0.464$. Note from Fig. 1 that enhanced quantum transport is observed precisely at this value. In contrast, for only slightly smaller or larger values of the Lamb-Dicke parameter, i.e. slightly shifted with respect to the center of the avoided crossing, the trapped particle absorbs energy with a much smaller rate, rather similar to the classical energy absorption. Hence, the width of isolated avoided crossings induces the sharp response of the system to tiny changes in $\eta$, opening a narrow parameter window in which the system can efficiently tunnel from its initial state $|\psi_0\rangle$ into another eigenstate which mediates rapid transport.

But which eigenstates of the kicked harmonic oscillator promote enhanced diffusion? Let us inspect the phase space projection of the anticrossing states in the inset of Fig. 1 “on the left” (labels A and C) and “on the right” (labels B and D) of the crossing’s center, which are represented by their Husimi functions in Fig. 3. As we approach the crossing from small values of $\eta$, the lower branch of the anticrossing in the inset of Fig. 1 has the largest overlap with $|\psi_0\rangle$. The corresponding eigenstate for $\eta = 0.459$ is plotted in the lower left panel of Fig. 3 and is clearly localized on the hyperbolic fixed point at the origin, and on its first replica on the crystal lattice. It anticrosses with the state (A) shown in the upper left panel of the figure. Clearly, this latter state is extended (note that it has large densities far away from the origin), localized on the stochastic web, reaching out to the effective boundary of the phase space which is defined by the finite size of the basis which we use for the diagonalization of $\hat{U}$. Since such web states − by their very nature − can never be converged numerically, we carefully checked that the associated eigenvalues saturate at
space structure, with $K = 2.0$ and $q = 6$. The signal is strongly enhanced at those values of $\eta$ which define avoided crossings between localized and web-like eigenstates.

a finite basis size, and that their localization properties within a finite phase space domain remain structurally invariant when increasing the numerical basis. On the other side of the avoided crossing, at $\eta = 0.469$, the anticrossing states have exchanged their localization properties, as illustrated in the right panels of Fig. 3 and as to be expected from the usual anticrossing scenario: Now the upper branch of the crossing provides the strongest support for $|\psi_0\rangle$, and the associated eigenstate (B) is once again localized on the hyperbolic fixed point at the origin, whereas the lower branches’ eigenstate (D) is localized on the web.

Consequently, the web eigenstates of the kicked harmonic oscillator lend support for enhanced quantum transport as observed in Fig. 1 at sharply defined values of the effective Planck constant (parametrized by the Lamb-Dicke parameter), where near-resonant tunneling from the hyperbolic fixed point onto the web becomes possible. Since the Lamb-Dicke parameter can be tuned by variation of the trap frequency $\nu$ or of the kicking lattice’s wavelength $2\pi/k$, such web-enhanced tunneling from the harmonic oscillator ground state will have a marked signature on the experimentally observed heating rate vs. $\eta$, as illustrated in Fig. 4. Note that each of the peaks in the mean energy signal (after a fixed number of kicks) plotted vs. $\eta$ in this plot corresponds to anticrossings (or, for the broader resonance structures, to overlapping avoided crossings) between localized and extended states alike the one identified in Figs. 1 and 3 at fixed phase space structure.

Tuning the Lamb Dicke parameter (at easily variable interaction time!) thus allows for sensitive probing of the local density of states by an experimentally robust observable, and, vice versa, for efficient control of the diffusion properties of the complex quantum system under study. Also note that Fig. 4 is reminiscent of conductance fluctuations which occur in solids [20], mesoscopic systems [21] and strongly driven atoms [22] – which bear experimentally accessible information on the spectral structure that supports the detected probability current.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4.png}
\caption{Mean energy (left y-axis) after 600 kicks, as a function of the Lamb-Dicke parameter $\eta$, and for fixed classical phase space structure, with $K = 2.0$ and $q = 6$. The signal is strongly enhanced at those values of $\eta$ which define avoided crossings between localized and web-like eigenstates.}
\end{figure}

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