TOWARDS TIGHT BOUNDS FOR LOCAL BROADCASTING

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Abstract. We consider the local broadcasting problem in the SINR model, which is a basic primitive for gathering initial information among $n$ wireless nodes. Assuming that nodes can measure received power, we achieve an essentially optimal constant approximate algorithm (with a $\log^2 n$ additive term). This improves upon the previous best $O(\log n)$-approximate algorithm. Without power measurement, our algorithm achieves $O(\log n)$-approximation, matching the previous best result, but with a simpler approach that works under harsher conditions, such as arbitrary node failures. We give complementary lower bounds under reasonable assumptions.

1. Introduction

When a wireless ad-hoc or sensor network starts operating, the nodes must form an infrastructure in a distributed manner without any information about each other. A natural basic primitive is for each node to gather information about all other nodes in its vicinity. The function to achieve this neighborhood learning is called local broadcast.

In the local broadcast problem, each wireless node tries to send (the same) message to all other nodes within a given radius. The objective is to complete the broadcasts within the shortest amount of time. This operation is used as a building block in higher-layer protocols such as routing, synchronization and coordination. The time complexity of those protocols are often dominated by the complexity of the local broadcast operation.

The model we use is the unstructured radio model, which avoids any assumptions of structure or synchronization. Nodes wake up and shut-down asynchronously, meaning nodes can be switched on at arbitrary times, including after other nodes have started operating; they can also shut down at some arbitrary point. There is no global clock to guide the operation of the nodes. The distribution of the nodes can be arbitrary, possibly in worst-case position.

For wireless algorithms, the model of interference is crucial. Most work, both on centralized and distributed algorithms, assumes a graph-based model of interference. The most common graph-based model is the protocol model \cite{7}, where each node has a given transmission radius within which its messages can reach and be decoded by other nodes, and a larger interference radius within which its transmission will disturb (and make it impossible to decode) other messages. More recently, the physical model, or SINR-model, which has been most commonly used in the engineering literature, has received attention in algorithms research. It has been shown to be more faithful to reality, both experimentally and theoretically \cite{16,20}. Here, interference fades slowly with distance, and it adds up. It is neither binary, symmetric, nor local, all of which combine to complicate analysis of SINR algorithms.

1.1. Our contributions. We seek to resolve the exact complexity of the local broadcast problem, both with upper and lower bounds.

We give a randomized distributed algorithm that achieves close to optimal time complexity. For a node $x$, let $N_x$ be the number of nodes that are reachable from $x$ in the case of no interference. The algorithm completes the broadcast for each node $x$ within $O(N_x \log n + \log^2 n)$ slots, with high
probability. We do not need a carrier-sense or collision awareness mechanism for this result (i.e., nodes have no information about the state of activity in the channel except possible reception of a message). This matches the recently achieved results by Yu et al [24]. Our algorithm is, however, simpler. It also can operate under harsher conditions compared to [24] — with asynchronous shutting-down of nodes, the algorithm in [24] may fail since it depends on a network of “leaders” to coordinate transmission decisions.

We then provide an algorithm running in optimal $O(N_x + \log^2 n)$ time. It operates in the same harsh model as the previous one, but assumes that nodes receive acknowledgements for free, i.e., if they manage to broadcast to all nodes in their broadcast region, then an acknowledgment will be returned (in fact, it is enough if this happens with some constant probability). We show that sufficient acknowledgments can be implemented if we relax the restriction of no collision awareness. Namely, if nodes have a “carrier-sense” mechanism that allows them to verify if received signal is above a certain fixed universal threshold, that suffices to deduce that a broadcast was successful. Previously, no better bound was known for the case of a carrier-sense mechanism.

For a lower bound, we show that the term $\log^2 n$ is necessary, under some assumptions. Instead of the SINR model, we prove the lower bound for the protocol model. We also prove the lower bound, not for completely general algorithms, but for “input-determined” algorithms, where the behavior of the algorithm is a (random) function of the messages so far received. Though not completely general, this class intuitively captures most reasonable algorithms possible for this problem.

Regarding the other term, a $\Omega(N_x)$ lower bound is immediate. There is evidence that no algorithm can work in time $o(N_x \log n)$, unless nodes can receive information about the success of their transmissions, but we do not have a formal proof of this.

Our results serve as further indication that the physical model is not significantly more demanding computationally than the protocol mode, at least for problems with uniformly sized neighborhoods like the local broadcast problem.

1.2. Related Work. The local broadcast problem in the SINR model was introduced in [4]. The authors gave two randomized distributed algorithms, both for the asynchronous unstructured radio model. One is a simple Aloha-like protocol that applies in the case of “known competition”, i.e., when each node knows the number of nodes in its proximity. The other, more involved, protocol holds without knowledge of the competition (“unknown competition”). The time complexity of the algorithms is $O(N_x \log n)$ and $O(N_x \log^3 n)$, respectively, where $N_x$ is the maximum number of nodes in any transmission range.

The bounds for unknown competition were improved in [23] to $O(N_x \log^2 n)$, optimizing the algorithm of [4]. Additionally, $O(\log n)$-approximate deterministic algorithms were given for a synchronized model where a carrier-sense primitive was assumed to be available. Finally, an $O(\log n)$-approximate randomized algorithm without a carrier-sense primitive was very recently proposed in [24]. A lower bound of $\Omega(N_x + \log n)$ was also given.

Our first algorithm (without carrier-sense) thus matches the result of [24], but has certain advantages. The algorithm from [24] computes a maximal independent set as a set of leaders, which help other nodes to coordinate in an efficient manner. In contrast, our algorithms are simpler. They are variations of the original algorithm of [4], requiring no leader election phase. This approach has advantages in particularly harsh environments. Assuming nodes can shut-down arbitrarily (in which case no guarantee need be made about their success in local broadcasting), a leader based algorithm is undesirable. For example, in [24], once a newly awaken node chooses a leader to attach itself to, it uses that leader for all future contention resolution purposes. This would fail if a leader were to shut down in the meantime. No such problem afflicts our algorithms.
Local broadcasting is related to the radio broadcasting problem in more classical models [1, 14, 2], to initialization and wake-up problems in wireless networks [11, 15] as well as coloring problems on disc graphs [18, 6].

Recently, the SINR model has received considerable attention in the algorithms community, starting with the work of Moscibroda and Wattenhofer [19]. Constant-approximation factors are now known for capacity problems, both with fixed power [3, 9] and power control [12]. See the survey of [5]. Distributed algorithms have been given for dominating sets [21], scheduling [13, 8], coloring [22], and connectivity and capacity [10].

2. Model

The problem is informally as follows. Given is a set \( V \) of \( n \) nodes in the plane. Each node wants to transmit a (single) message to all nodes within its broadcast range in the shortest amount of time. A local broadcast operation is successful if all nodes have performed a successful local broadcast.

We assume that nodes can wake up at any time asynchronously. The nodes are unaware of the network topology, which can be of arbitrary (worst-case) layout. The nodes only have a crude bound on \( n \) (up to a polynomial factor). Without such a bound it is known that no sublinear algorithms are possible [11].

There is no global clock or any synchronization among the nodes. In the analysis, we assume that time is divided into time-slots; this is justified by a standard trick of relating slotted vs. unslotted Aloha (see [4]).

In this paper, all nodes use the same power \( P \), known as the uniform power scheme. We scale values so that \( P = 1 \).

We adopt the SINR model of interference, a non-transmitting node \( v \) will successfully receive a message transmitted by node \( u \) if,

\[
\frac{P}{d(u,v)^\alpha} + \frac{\sum_{w \in S \setminus \{u\}} P}{d(w,v)^\alpha} \geq \beta,
\]

where \( N \) is the ambient noise, \( \beta \) is the required SINR level, \( \alpha > 2 \) is the so-called path loss constant, \( d(u,v) \) is the distance between two points \( u \) and \( v \), and \( S \) is the set of senders transmitting simultaneously.

For any subset \( X \) of the plane, we use the notation \( |X| \) to define the number of nodes in \( X \).

We need the following two definitions:

**Definition 1.** The transmission region \( T_x \) is the ball of some fixed radius \( (R_T) \) around a node \( x \) which \( x \) can reach without any other node transmitting (i.e., \( R_T = \frac{1}{(N\beta)^{1/\alpha}} \)). Clearly, \( N_x = |T_x| \).

Since the signal quality (even without interference) becomes very poor near the boundaries of \( T_x \), to achieve non-trivial results, one needs to define the broadcasting region as somewhat smaller than \( T_x \):

**Definition 2.** The broadcasting region \( B_x \) is a ball of some fixed radius \( (R_B) \) around any node \( x \), containing all nodes to which \( x \) would like to transmit. We set \( R_B = \phi R_T \) for a small constant \( \phi \) (\( \phi = \frac{1}{6} \) suffices).

We will use the notation \( 2B_x \) to mean the ball of radius \( 2R_B \) around \( x \). A probabilistic event is said to happen \( \text{whp} \) (with high probability) if it happens with probability \( 1 - 1/n^c \), for some \( c \geq 1 \).

We assume that a node is not obliged to broadcast to nodes that woke up after itself (or have shut-down before it broadcasts). This is consistent with the algorithm of [4], even if not made explicit. No guarantees are made for nodes that “live” for too short a period of time (i.e., the
time elapsed between wake-up and shut-down is smaller than the claimed running time for the algorithm.

We now define formally the local broadcast operation. A node $x$ is successful in a given time slot if it transmits a message and all nodes within $B_x$ can decode the message, satisfying Eqn. 1. A local broadcast operation is successful when all the nodes have become successful. The time complexity of a node $x$ is measured in terms of the time that elapsed from waking up until the node halts the algorithm, and is evaluated as a function of $N_x = |T_x|$.

3. Results

**Theorem 3.** There exists an algorithm for which the following holds whp: each node $x$ successfully performs a local broadcast within $O(N_x \log n + \log^2 n)$ slots.

We can improve this result to essentially optimal if we assume that the nodes can measure received power:

**Theorem 4.** Assume that in any slot, a node can measure the power received at its receiver (from all other transmitting nodes). Then there exists an algorithm for which the following holds whp: each node $x$ successfully performs a local broadcast within $O(N_x + \log^2 n)$ slots.

Finally, a lower bound:

**Theorem 5.** In the protocol model, there exist instances on $n$ vertices such that

1. There exists a broadcast neighborhood with a constant number of nodes.
2. No input-determined algorithm can complete local broadcast in this region in $o(\log^2 n)$ slots with high probability.

"Input-determined" algorithms are defined in Section 6, where the theorem is proven. Informally, these are algorithms whose behavior in a given slot is a (random) function of the messages received in previous slots.

4. An $O(N_x \log n + \log^2 n)$ time Algorithm

In this section, we will prove Theorem 3. Our algorithm is listed as Algorithm 1 (LocalBroadcast1). The symbols $\gamma, \lambda$ used in the listing are appropriate constants.

The intuition behind the algorithm is as follows. The “right” probability for $x$ to transmit at is about $\frac{1}{N_x}$ (too high, and collisions are inevitable; too low, nothing happens). The algorithm starts from a low probability, continuously increasing it, but once it starts receiving messages from others, it uses that as an indication that the “right” transmission probability has been reached.

To prove Thm. 3, we will first need the following definition.

**Definition 6.** For any node $x$, the event LowPower occurs at a time slot if the received power at $x$ from other nodes, $P_x \leq \frac{1}{(4(\beta+4)R_\beta)\alpha}$.

The following technical Lemma follows from geometric arguments (see Appendix A for the proof).

**Lemma 7.** If $x$ transmits and LowPower occurs at $x$, all nodes in $2B_x$ receive the message from $x$ (thus a successful local broadcast occurs for $x$).

We will also need the following definition:

**Definition 8.** A FallBack event is said to occur for node $y$ if line 20 is executed for $y$.

We will refer to the transmission probability $p_y$ for a node $y$ at given time slots. This will always refer to the value of $p_y$ in line 9. We first prove a Lemma that bounds the transmission probability in any broadcast region at a given time.
Algorithm 1 LocalBroadcast1 (For any node y)

1: \(tp_y \leftarrow 0\)
2: \(p_y \leftarrow \frac{1}{4n}\)
3: loop
4: \(p_y \leftarrow \max\{\frac{1}{128n}, \frac{p_y}{32}\}\)
5: \(rc_y \leftarrow 0\)
6: loop
7: \(p_y \leftarrow \min\{\frac{1}{16}, 2p_y\}\)
8: for \(j \leftarrow 1, 2, \ldots \delta \log n\) do
9: \(s \leftarrow 1\) with probability \(p_y\)
10: if \(s = 1\) then
11: transmit
12: end if
13: \(tp_y \leftarrow tp_y + p_y\)
14: if \(tp_y > \gamma \log n\) then
15: halt;
16: end if
17: if message received then
18: \(rc_y \leftarrow rc_y + 1\)
19: if \(rc_y > \log n\) then
20: goto line 4
21: end if
22: end if
23: end for
24: end loop
25: end loop

Lemma 9. Consider any node \(x\). Then during any time slot \(t \leq 10n^2\),

\[
\sum_{y \in B_x} p_y \leq \frac{1}{2}
\]

with probability at least \(1 - \frac{1}{n^4}\).

Proof. For contradiction, we will upper bound the probability that Eqn. 2 is violated for the first time at any given time \(t\), after which we will union bound over all \(t \leq 10n^2\).

Let \(T\) be the interval (time period) \(\{t - \delta \log n + 1 \ldots t - 1\}\). Then we claim,

Claim 4.1. In each time slot in the period \(T\),

\[
\frac{1}{2} \geq \sum_{y \in B_x} p_y \geq \frac{1}{4}
\]

Proof. The first inequality is by the assumption that \(t\) is the first slot when Eqn. 2 is violated. The second is because probabilities (at most) double once every \(\delta \log n\) slots (by the description of the algorithm). \(\square\)

We now show that Eqn. 3 is not possible. To that end, we show that in the \(\delta \log n\) interval preceding \(t\), a FALLBACK will occur with high probability:

Claim 4.2. With probability \(1 - \frac{1}{n^8}\), each node \(z \in B_x\) will FALLBACK once in the period \(T\).
Proof. Fix any $z \in B_x$. By the algorithm

(4) \[ p_z \leq \frac{1}{16} \]

Thus, at any time slot,

(5) \[ P(z \text{ does not transmit}) \geq \frac{15}{16} \]

Now, combining Eqn. 4 and Eqn. 3 and defining $B = B_x \setminus \{z\}$,

(6) \[ \sum_{y \in B} p_y \geq \frac{3}{16} \]

For $y \in B_x$ define SUCCESS$_y$ to be the event that $y$ transmits and LOWPOWER occurs for $y$. By Lemma 7, SUCCESS$_y$ implies that $z$ will receive the message from $y$. Thus, the probability of $z$ receiving a message from some node in $B$ in a given round is at least $\frac{15}{16} P(\bigcup_{y \in B} \text{SUCCESS}_y)$.

We claim that for any $y \neq w$ (both in $B$), the events SUCCESS$_y$ and SUCCESS$_w$ are disjoint. This is implicit in Lemma 7, since SUCCESS$_y$ means that $w$ cannot be transmitting and vice-versa. Thus, the probability of $z$ receiving a message from some node in $B$ is at least:

\[ \frac{15}{16} \sum_{y \in B} P(\text{SUCCESS}_y) \geq \frac{15}{16} \sum_{y \in B} \left( \frac{1}{2} \right)^{O(\frac{1}{\phi^2})} \geq \frac{3}{16} \]

where we use Lemma 18 for the first inequality (stated and proved in Appendix A) and Eqn. 6 for the last.

Setting $\delta \geq \frac{10}{128 \left( \frac{1}{4} \right)^{O(\frac{1}{\phi^2})} \frac{1}{16}}$ and using the Chernoff bound, we can show that $z$ will receive $> \log n$ messages in $T$ with probability $1 - \frac{1}{n^2}$, thus triggering the FALLBACK. \hfill \square

Now we show that the above claim implies that Eqn. 3 is not possible.

**Claim 4.3.** There exists a time slot in $T$ such that

\[ \sum_{y \in B_x} p_y < \frac{1}{4}. \]

**Proof.** For any $y \in B_x$, let $p_y^1$ be the value of $p_y$ in the first slot of $T$. Let $p_y^f$ be the value of $p_y$ in the slot when FALLBACK happened for $y$. Since probabilities can at most double during $T$,

(7) \[ \sum_{y \in B_x} p_y^f \leq 2 \sum_{y \in B_x} p_y^1 \leq 1, \]

the last inequality using the fact that $\sum_{y \in B_x} p_y^1 \leq \frac{1}{2}$ (Eqn. 3).

Now by lines 4 and 7 of the algorithm, in the slot after FALLBACK, $p_y = \max\left\{ \frac{1}{128n}, \frac{p_y^f}{32} \right\} \leq \frac{1}{128n} + \frac{p_y^f}{32}$. Since probabilities at most double during $T$, the value of $p_y$ at the final slot of $T$ is at most $\frac{1}{64n} + \frac{p_y^f}{16}$. Summing over all $y$, during the final slot of $T$,

\[ \sum_{y \in B_x} p_y \leq \frac{n}{32n} + \sum_{y \in B_x} \frac{p_y^f}{8} \leq \frac{1}{32} + \frac{1}{8} < \frac{1}{4} \]

contradicting Eqn. 3. We used Eqn. 7 in the second inequality. \hfill \square
The proof of the Lemma is completed by union bounding over time slots $t \leq 10n^2$. □

Now we prove that nodes stop running the algorithm by a certain time.

**Lemma 10.** Each node $x$ stops executing within $O(N_x \log n + \log^2 n)$ slots, whp.

**Proof.** Fix $x$. We derive four claims that together imply the lemma.

First, by the halting condition of line 14.

**Claim 4.4.** The number of slots for which $p_x \geq \frac{1}{32}$ is $O(\log n)$.

Assume that $x$ experienced $k$ FALLBACKs. Consider the times $t_x(1), t_x(2) \ldots t_x(k)$ when a FALLBACK happened for $x$. Now,

**Claim 4.5.** $t_x(1) = O(\log^2 n)$. Also, there are $O(\log^2 n)$ slots after $t_x(k)$.

**Proof.** The two claims are very similar. Let us prove the latter one. Since FALLBACK does not occur after $t_x(k)$, the probability doubles every $\delta \log n$ slots. Since the minimum probability is $\Omega(\frac{1}{n})$, by $O(\log^2 n)$ slots, the probability will reach $\frac{1}{32}$. Once this happens, the algorithm terminates in $O(\log n)$ additional slots, by Claim 4.4 □

Given the above claim it suffices to bound $t_x(k) - t_x(1)$. By Claim 4.4 we can also restrict ourselves to slots for which $p_x < \frac{1}{32}$. For these slots, line 7 does not need the min clause, i.e., $p_y \leftarrow 2p_y$ each time line 7 is executed.

Define $b_i$ such that $p_x = \frac{1}{2^i}$ at time $t_x(i)$. Note that if $n$ is a power of 2, $b_i$ is always an integer (the case of other values of $n$ can be easily managed).

We can characterize the running time between two FALLBACKs as follows.

**Claim 4.6.** $t_x(i + 1) - t_x(i) \leq (b_i - b_{i+1} + 5)\delta \log n$, for all $i = 1, 2 \ldots k - 1$.

**Proof.** During slots in $[t_x(i), t_x(i + 1)]$, $p_x$ doubles every $\delta \log n$ slots (by the description of the algorithm and the fact that $p_x < \frac{1}{32}$). Let $b$ be such that $p_x = \frac{1}{2^b}$ at time $t_x(i + 1) - 1$. Then,

$$\frac{1}{2^b} = 2^{\left\lfloor \frac{t_x(i + 1) - t_x(i)}{\delta \log n} \right\rfloor} \Rightarrow b - b_i = \left\lfloor \frac{t_x(i + 1) - t_x(i)}{\delta \log n} \right\rfloor \frac{2^{b_i}}{2^b}$$

By lines 7 and 4 of the algorithm, $b_{i+1} \leq b + 4$, and thus,

$$b_i - b_{i+1} + 4 \geq \left\lfloor \frac{t_x(i + 1) - t_x(i)}{\delta \log n} \right\rfloor \Rightarrow b_i - b_{i+1} + 5 \geq \frac{t_x(i + 1) - t_x(i)}{\delta \log n} \right\rfloor, \right. \right.$$

completing the proof of the Lemma. □

Thus, the running time $t_x(k) - t_x(1)$ can be bounded by:

$$t_x(k) - t_x(1) = (t_x(k) - t_x(k - 1)) + (t_x(k - 1) - t_x(k - 2)) \ldots + (t_x(2) - t_x(1))$$

$$\leq ((b_{k-1} - b_k + 5) + (b_{k-2} - b_{k-1} + 5) \ldots + (b_1 - b_2 + 5))\delta \log n$$

$$= (b_1 - b_k + 5k)\delta \log n$$

(8) $= O(\log^2 n + k \log n).$
where we use Claim 4.6, the non-negativity of $b_k$ and the fact that $b_i = O(\log n)$ (as $p_x = \Omega(\frac{1}{n})$).

To complete the proof of the Lemma, we need a bound on $k$:

**Claim 4.7.** $k = O(N_x)$.

**Proof.** The total number of possible transmissions that $x$ could possibly hear is $O(N_x \log n)$, whp. This is because each node transmits $O(\log n)$ times, whp (by Lemma 19 in Appendix A) and a node can only hear messages from nodes in $T_x$ (by the definition of $T_x$). But nodes only FALLBACK once for every $\log n$ messages received (by the condition immediately preceding line 20). The claim is proven.

Applying the above claim to Eqn. 8, $t_x(k) - t_x(1) \leq O(\log^2 n + k \log n) = O(N_x \log n + \log^2 n)$, completing the argument.

The final piece of the puzzle is to show that for each node, a successful local broadcast happens whp during one of its $\Theta(\gamma \log n)$ transmissions.

**Lemma 11.** By the time a node halts, it has successfully locally broadcast a message, whp.

**Proof.** The expected number of transmission made by a node is $\gamma \log n$ (by the algorithm). By Lemmas 18 and 7 during each such transmission, local broadcast succeeds with probability $\frac{1}{2} \left( \frac{1}{4} \right)^{O(\frac{1}{\sigma^2})}$, at least. Thus, the expected number of successful local broadcasts is $\frac{1}{2} \left( \frac{1}{4} \right)^{O(\frac{1}{\sigma^2})} \gamma \log n$. Setting $\gamma$ to a high enough constant, and using Chernoff bounds, with high probability, a successful local broadcast happens at least once.

Lemmas 10 and 11 together imply Thm. 3.

5. Improved Algorithm with Received Power Measurement

We will assume for this section that nodes can measure total received power from other nodes (even when transmitting). In hardware implementations, the received power is usually available as RSSI (Received signal strength indicator). Additionally, filtering out one’s signal (thus being able to measure received power even when transmitting) is also possible in many hardware implementations.

With this primitive we are able to design an algorithm that completes local broadcasting in time $O(N_x + \log^2 n)$, with high probability (thus proving Thm. 4).

Our new algorithm (Algorithm 2) is identical to the previous one, except for an extra halting condition in line 12—A node halts if LOWPOWER happens, which it can clearly measure with the received power measurement primitive discussed above.

To show why this leads to the improved bound, recall the proof of Lemma 10. In proving that Lemma, we showed in Claim 4.7 that $k = O(N_x)$ (where $k$ is the number of FALLBACKs for $x$). We will show instead that for Algorithm 2

**Lemma 12.** $k = O\left( \frac{N_x + \log n}{\log n} \right)$

**Proof.** As before, since we FALLBACK once for every $\log n$ received messages, it suffices to show that whp, the number of transmissions from $T_x$ that a node will hear is $O(N_x + \log n)$.

By Lemma 18, for any node $x$ transmitting, LOWPOWER occurs with a constant probability. Thus, for any given transmission, the number of unhalted nodes in $T_x$ reduces by 1 with some constant probability $c$. For contradiction, assume nodes in $T_x$ transmit more than $101^c \cdot N_x + 10 \log n$ times. Using Chernoff bounds, it is easy to show that, whp, LOWPOWER will occur for $> N_x$ transmitting nodes, which is a contradiction (since nodes halt once they complete a LOWPOWER and there are only $N_x$ nodes in $T_x$).
Algorithm 2 LocalBroadcast2 (for any node $y$)

1: $tp_y \leftarrow 1$
2: $p_y \leftarrow \frac{1}{4n}$
3: loop
4: $p_y \leftarrow \max\{\frac{1}{128n}, \frac{p_y}{32}\}$
5: $rc_y \leftarrow 0$
6: loop
7: $p_y \leftarrow \min\{\frac{1}{16}, 2p_y\}$
8: for $j \leftarrow 1, 2, \ldots \delta \log n$ do
9: $s \leftarrow 1$ with probability $p_y$
10: if $s = 1$ then
11: transmit
12: if LOWPOWER occurs then
13: halt;
14: end if
15: end if
16: $tp_y \leftarrow tp_y + p_y$
17: if $tp > \gamma \log n$ then
18: halt;
19: end if
20: if message received then
21: $rc_y \leftarrow rc_y + 1$
22: if $rc_y > \log n$ then
23: goto line 4
24: end if
25: end if
26: end for
27: end loop
28: end loop

6. Lower Bound

In this section, we prove Thm. 3, thus showing that the $O(\log^2 n)$ in the running time may be necessary. As indicated, we prove the bound in the protocol model of interference [7]. This is a widely used and simpler model of wireless interference. In the protocol model, there is a transmission range $R_T$ and interference range $R_I$. A transmission from $x$ to $y$ succeeds if $d(x, y) \leq R_T$ and $d(y, z) > R_I$ for all other transmitting nodes $z$.

The algorithmic result of Section 4 applies to the protocol model as well, i.e., local broadcasting is possible in $O(N_x \log n + \log^2 n)$ time. Here $N_x$ is the number of nodes in the ball of radius $R_T$ around $x$ (thus, the transmission and broadcast regions are identical). The analysis for the SINR model can be applied naturally to the protocol model.

Though our lower bound result does not apply directly to the SINR model, it does apply to the type of algorithm employed in the paper. The fact that we measure success by the event LOWPOWER is essentially equivalent to establishing an interference perimeter around nodes. Thus, our lower bound indicates that getting rid of the $O(\log^2 n)$ factor in the SINR model, if at all possible, would have to use different techniques.

We need the following two assumptions:
Nodes do not have any “carrier sense ability”, thus only external information they can get is a message reception.

The algorithm is “input-determined”, i.e., the action of the algorithm is a function of messages it has received thus far and its own random bits. We define this precisely in Definition 13.

Note that Algorithm 1 is clearly input-determined. Though it is possible to conjure up algorithms that are not, it is difficult to imagine how such an algorithm would help. Closing this gap remains an intriguing open problem.

We start with some definitions. For any node \( x \), and any time \( t \), we define a binary function:

\[
I(x,t) = \begin{cases} 
1 & \text{if } x \text{ successfully decoded a message,} \\
0 & \text{otherwise.} 
\end{cases}
\]

Assume (without loss of generality) that nodes cannot decode messages in slots where they are transmitting.

Define \( I_r(x) \) to be the string containing all bits \( I(x,t) \) for \( t = 1 \ldots r \). Let \( T(x,t) \) define whether or not node \( x \) transmits at time \( t \).

Now we define precisely what we mean by input-determined:

**Definition 13.** An algorithm is said to be input-determined if for any node \( x \) and any time slot \( t \),

\[
\mathbb{P}(T(x,t) = 1|I_{t-1}(x) = B, E_x) = \mathbb{P}(T(x,t) = 1|I_{t-1}(x) = B)
\]

for any binary string \( B \in \{0,1\}^{t-1} \) and any event \( E_x \) that is a function of \((\cup_{y \in [n], r \in [1,t]} T(y,r)) \setminus \{T(x,t)\}\). This is to say: once the reception history \( I_{t-1}(x) \) is known, the behavior of the algorithm does not depend on \( E_x \). This implies,

\[
\mathbb{P}(T(x,t) = 1|E_x) = \sum_{B \in \{0,1\}^{t-1}} \mathbb{P}(T(x,t) = 1|I_{t-1}(x) = B) \mathbb{P}(I_{t-1}(x) = B|E_x)
\]

Define the string \( 0_t \) to be the string of \( t \) zeroes. Now, consider the behavior of any algorithm. For any \( t \), define \( p_t \) by

\[
p_t = \mathbb{P}(T(x,t) = 1|N_{t-1})
\]

for any node \( x \) (\( x \) is arbitrary as the nodes are indistinguishable), and where \( N_{t-1} \) is the event that in each slot up to \( t - 1 \), there were either 0 or more than 2 nodes transmitting in the system.

Assume a construction where each node is in every other node’s interference range (but not necessarily in its transmission range). Then the event \( N_{t-1} \) implies that \( I_{t-1}(x) = 0_{t-1} \) for all \( x \).

Our lower bound will present such a construction, and then show that \( N_t \) occurs with significant probability for \( t = o(\log^2 n) \). Clearly, this means that no messages were decoded, thus local broadcast has not happened for any node. We now claim that the probabilities in a single slot are independent across nodes, or,

**Lemma 14.** Consider an input-determined algorithm and a node \( x \). Let \( \hat{T}(x,t) \) define any arbitrary collection of transmissions at time \( t \) by some other nodes in the system. Then,

\[
\mathbb{P}(T(x,t)|N_{t-1}, \hat{T}(x,t)) = p_t
\]

**Proof.** First note that the event \( N_{t-1} \cup \hat{T}(x,t) \) meets the conditions of \( E_x \) as laid down in Defn. 13, since it is a function of the past events, plus, events in the present excluding \( T(x,t) \).
Since the algorithm is input-determined,

\[ P(T(x, t) | N_{t-1}, \tilde{T}(x, t)) = \sum_{B \in \{0, 1\}^{t-1}} P(T(x, t) = 1 | I_{t-1}(x) = B) \cdot P(I_{t-1}(x) = B | N_{t-1}, \tilde{T}(x, t)) \]

Note that clearly \( P(I_{t-1}(x) = B | N_{t-1}, \tilde{T}(x, t)) = 1 \) for \( B = 0 \) and \( P(I_{t-1}(x) = B | N_{t-1}, \tilde{T}(x, t)) = 0 \) for all other \( B \). Thus \( P(T(x, t) | N_{t-1}, \tilde{T}(x, t)) = P(T(x, t) = 1 | I_{t-1}(x) = 0_{t-1}) \). A similar argument shows that \( p_t = P(T(x, t) = 1 | I_{t-1}(x) = 0_{t-1}) \), completing the proof of the Lemma. \( \square \)

We now provide a construction of nodes leading to the lower bound. Consider two transmission regions that are non-overlapping, yet close enough that they are included in each other’s interference region. One will have a constant number of nodes, the other will have \( \Delta \) nodes, a value which will be set later.

\textbf{Figure 1.} The two transmission regions that are in each others interference regions.
The top one has a lot of nodes, the bottom one only a few.

Partition the range \([\frac{1}{n^2}, 1] = \bigcup_{j=0}^{r} R_j \) where \( R_0 = [-\infty, n^2] \) and for \( j > 0, R_j = [\frac{1}{n^2}16^j, \frac{1}{n^2}16^j+1) \) and \( r = \Theta(\log n) \). Consider for any \( t \), the sequence \( p_1, \ldots p_t \). Define, for a range \( R_i \) a weight function \( w_i = \frac{|R_i \cap \{p_1, \ldots p_t\}|}{t} \).

Fix any \( j \). For any \( i \), define the function \( f_i^j = \left( \frac{2}{e} \right)^{|i-j|+1} \).

Now we claim that (proof in Appendix A):

\textbf{Lemma 15.} There must be range \( R_j \) such that \( j \leq \frac{\log n}{4} \) and

\[ \sum_i f_i^j w_i = O\left( \frac{1}{\log n} \right) \]

Consider the \( j \) found in the above Lemma. Set \( \Delta = \frac{1}{P_j} \) where \( P_j = \frac{4}{n^2}16^j \).

Note that by the choice of \( j \), \( n^2 \geq \Delta \geq n \). Now we bound \( P(N_t | N_{t-1}) \) for \( t > 1 \) (the claim also applies to \( P(N_1) \)).

\textbf{Lemma 16.} Assume \( t \) is such that \( p_t \in R_i \) for any \( i \). Then \( P(N_t | N_{t-1}) \geq 1 - f_i^j \).

\textit{Proof.} Follows from Lemmas 20, 21 and 22 (in Appendix A). \( \square \)

Now,

\textbf{Lemma 17.} If \( t = o(\log^2 n) \), then, \( P(N_t) = 1 - \frac{1}{n^{o(1)}} \)
Proof. By Lemma 16,
\[ P(N_t) = P(N_1)P(N_2|N_1)P(N_3|N_2) \ldots P(N_t|N_{t-1}) \]
\[ = \prod_i (1 - f_i^j)^{w_i^t} \]
(10)
The following claim can be proven using basic calculus:

**Claim 6.1.** If \( x \leq \frac{2}{7} \), then \( 1 - x \geq 1 \frac{1}{16^t} \)

Continuing with Eqn. (10) using the above claim:
\[ P(N_t) = \prod_i (1 - f_i^j)^{w_i^t} \geq \prod_i \frac{1}{16^t f_i^j} = \frac{1}{16^t \sum_i f_i^j} \]
\[ \geq \frac{1}{16^t O(1/\log n)} = \frac{1}{16^t o(\log n)} = \frac{1}{n^{o(1)}} \]
completing the proof. The second inequality is from Eqn. 9. The equality right after that uses the assumption \( t = o(\log^2 n) \). \( \Box \)

This Lemma shows that if \( t = o(\log^2 n) \), then with not too small probability \( \frac{1}{n^{o(1)}} \), the event \( N_t \) occurs. Recall that \( N_t \) implies that none of the nodes received any messages by time \( t \), thus local broadcasting has not completed (even in the “bottom” broadcasting range that has only a constant number of nodes). This completes the proof of Thm. 5.

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Proof. [of Lemma 7] Consider any \( y \in 2B_x \). By definition of \( 2B_x \), \( d(x, y) \leq 2R_B \). Now consider any other transmitting node \( z \). We will show that,

Claim A.1. \( d(z, x) \leq 3(\beta + 2) d(z, y) \)

Proof. By the signal propagation model, \( \frac{1}{d(z, x)^\alpha} \) is the power received at \( x \) from \( z \). Since LowPower occurred,

\[
\frac{1}{d(z, x)^\alpha} \leq \frac{1}{((4\beta + 4)R_B)^\alpha} \Rightarrow d(z, x) \geq 4(\beta + 4)R_B
\]

By the triangle inequality, \( d(z, y) \geq d(z, x) - d(x, y) > 4(\beta + 4)R_B - 2R_B \geq 3(\beta + 4)R_B \), proving the claim. \( \square \)

This implies, by basic computation and summing over all transmitting \( z \), that

\[
(11) \quad P_y \leq \left( \frac{4}{3} \right) \alpha P_x
\]

Now, the SINR at node \( y \) (in relation to the message sent by \( x \)) is

\[
\frac{\frac{1}{2^\alpha R_T^\alpha}}{P_y + N} \geq \frac{\frac{1}{2^\alpha R_T^\alpha}}{\left( \frac{4}{3} \right) \alpha P_x + N} \geq \frac{\frac{1}{2^\alpha R_T^\alpha}}{\left( \frac{4}{3} \right) \alpha \frac{1}{((4(\beta + 4))R_T)^\alpha} + \frac{2}{R_B^\alpha \beta}} \geq \beta
\]

Explanation of numbered (in)equalities:

(1) By Eqn. [11]
(2) Plugging in the bound of \( P_x \) (since LowPower occurs at \( x \)) and noting that \( N = \frac{1}{3R_T^\alpha} = \frac{\phi^\alpha}{\beta R_B} \) from the definitions of \( R_T \) and \( R_B \).
(3) Follows from simple computation once \( \phi \) is set to a small enough constant (\( \phi = \frac{1}{6} \) suffices).

Thus the SINR condition is fulfilled, and \( y \) receives the message from \( x \). \( \square \)

Lemma 18. Consider any slot \( t \) and any node \( z \). Assume that in that slot, for all broadcast regions \( B_x \), \( \sum_{y \in B_x} \frac{1}{2} \leq \frac{1}{2} \). Then, LowPower occurs for \( z \) with probability at least \( \frac{1}{2} \left( \frac{1}{4} \right)^{\frac{1}{2} O \left( \frac{1}{n^\alpha} \right)} \).

Proof. Let \( B = B_x \setminus \{ x \} \). We first prove that there is a substantial probability that no node in \( B \) transmits. Assuming this probability is \( P_n \)
Lemma 20. If is \( \gamma \) by the description of the algorithm, when the node stops, its total transmission probability

Proof. Consider the following Chernoff-type bound: Let \( \{X_i\} \) be independent Poisson trials such that \( X = \sum_i X_i \) and \( \mu = E(X) \). Then, \( P(X \leq (1-\delta)\mu) \leq e^{-\mu\delta^2/2} \). See Thm. 4.5 of [17] for a reference.

Now let \( X \) be the number of transmissions in the slot and we would like to lower bound the probability of there being at most 2 of them. Note that Chernoff bound needs independence between the variables, but we have shown that in Lemma 14.

\[ P_n \geq \prod_{w \in B} (1 - p_w) \geq \prod_{w \in B_x} (1 - p_w) \geq \left( \frac{1}{4} \right)^{\sum_w p_w} \geq \left( \frac{1}{4} \right)^{\frac{1}{2}} \]

The third inequality is from Fact 3.1 [4], and the last from the bound \( \sum_w p_w \leq \frac{1}{2} \).

Let \( P_T \) be the probability that no other node transmits in \( T_x \). Since \( R_B = \phi R_T \), \( T_x \) can be covered by \( O(\frac{1}{\sigma^2}) \) broadcast regions (this can be shown using basic geometric arguments). Thus,

\[ P_T \geq P_n \left( \frac{1}{\sigma^2} \right) \geq \left( \frac{1}{4} \right)^{\frac{1}{2}} O\left( \frac{1}{\sigma^2} \right) \]

(12)

Since no other node in \( T_x \) is transmitting, we only need to bound the signal received from outside \( T_x \).

To this end, we need the following Claim (which is a restatement of Lemma 4.1 of [4] and can be proven by standard techniques):

Claim A.2. Assume that for all broadcast regions \( B_x \), \( \sum_{y \in B_x} p_y \leq \frac{1}{2} \). Consider a node \( x \). Then the expected power received at node \( x \) from nodes not in \( T_x \) can be upper bounded by

\[ \frac{1}{8} \alpha - \frac{1}{8} 3^2 \frac{\phi^2}{R_B^2} \leq \frac{1}{2(4(\beta + 4)R_B)^\alpha} \]

for appropriately small \( \phi \).

Then by Markov’s inequality, with probability at least \( \frac{1}{2} \), the power received from nodes outside of \( T_x \) is at most \( \frac{1}{(4(\beta + 4)R_B)^\alpha} \).

Thus, with probability \( \frac{1}{2} P_T \), LOWPOWER occurs at \( x \), proving the Lemma.

Lemma 19. With high probability, each node transmits at least \( \frac{1}{2} \gamma \log n \) times, and at most \( 2 \gamma \log n \) times.

Proof. By the description of the algorithm, when the node stops, its total transmission probability is \( \gamma \log n \). By the standard Chernoff bound, the actual number of transmissions is very close to this number, whp.

Lemma 20. If \( p_i \in R_i \) and \( i < j \), then \( P(N_i|N_{i-1}) > 1 - \left( \frac{3}{5} \right)^{j-i+1} \).

Proof. Essentially, by Markov’s inequality. Let the number of nodes transmitting at time \( t \) be \( T \). Then, \( E(T) = p_i \Delta \leq \frac{1}{n^2} 16^{i+1} \frac{\phi^2}{4^j 16^i} \) (by the choice of \( p_i \)). Now, if \( N_i \) is the event of \( N_t \) not occurring, then

\[ P(N_i|N_{i-1}) \leq P(T \geq 1|N_{i-1}) \leq E(T|N_{i-1}) = \frac{1}{4} \cdot 16^{j-i-1} \]

This implies the Lemma after some elementary manipulations.

Lemma 21. Let \( p_i \in R_i \) and \( i > j \), then \( P(N_i|N_{i-1}) > 1 - \left( \frac{2}{5} \right)^{i-j+1} \).

Proof. Consider the following Chernoff-type bound: Let \( \{X_i\} \) be independent Poisson trials such that \( X = \sum_i X_i \) and \( \mu = E(X) \). Then, \( P(X \leq (1-\delta)\mu) \leq e^{-\mu\delta^2/2} \). See Thm. 4.5 of [17] for a reference.

Now let \( X \) be the number of transmissions in the slot and we would like to lower bound the probability of there being at most 2 of them. Note that Chernoff bound needs independence between the variables, but we have shown that in Lemma 14.
Set $\delta = 1 - 2/\mu$, which means that $(1 - \delta)\mu = 2$. It is easy to verify that $\mu \geq 4$. Thus,
\[
\mathbb{P}(N_t|N_{t-1}) \leq \mathbb{P}(X \geq 2|N_{t-1}) \leq e^{-(1-\frac{2}{\mu})^2\mu/2} \\
\leq e^{-(\mu/8)} = \exp(-\frac{16^i}{16^j})
\]
which implies the Lemma after some calculations.

\begin{proof}
\end{proof}

**Lemma 22.** If $p_t \in R_j$, $\mathbb{P}(N_t|N_{t-1}) \geq 1 - \frac{2}{e}$

\begin{proof}
To see this, note that $N_t$ occurs iff the number of nodes transmitting in the slot is not 1. So we need to upper bound the probability of exactly 1 node transmitting. Note that by Lemma 14, the transmission probabilities at time $t$ are iid. Let this iid probability be $p$. The probability of exactly one transmission is $\Delta p(1 - p)^{\Delta - 1}$ which can be seen to be upper bounded by $\frac{2}{e}$ for large enough $\Delta$ using calculus.

\end{proof}

**Proof.** [of Lemma 15] By the definition of $w_i$,
\begin{equation}
\sum_{i=0}^{r} w_i = 1
\end{equation}

Now it is elementary (using bounds for geometric series) to check that for any $i$,
\begin{equation}
\sum_{j} f_{ij} = \Theta(1)
\end{equation}

We can see that,
\[
\sum_{j} \sum_{i} f_{ij} w_i = \sum_{i} w_i \left( \sum_{j} f_{ij} \right) = \Theta \left( \sum_{i} w_i \right) = \Theta(1)
\]
The first equality is rearrangement, the rest follows from Eqns. 14 and 13. The Lemma now follows from noting that the sum over $j$ has $\Theta(\log n)$ terms.