A Note on Echelon-Ferrers Construction

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Abstract

Echelon-Ferrers is one of the important techniques to help researchers to improve lower bounds for constant-dimension codes. Fagang Li [4] combined the linkage construction and echelon-Ferrers to obtain some new lower bounds of constant-dimension codes. This is a note on the proof of the construction.

**keywords:** Echelon-Ferrers, Constant-Dimension Codes, Projective Space, Reduced Echelon Form, Linkage Construction

1 Introduction

Subspace coding was proposed by R. Koetter and F. R. Kschischang in [3] to correct errors and erasures in random network coding. The *projective space* of order \( n \) over the finite field \( \mathbb{F}_q \), denoted \( \mathcal{P}_q(n) \), is the set of all subspaces of the vector space \( \mathbb{F}_q^n \). The set of all \( k \)-dimensional subspaces of an \( \mathbb{F}_q \)-vector space \( V \) will be denoted by \( G_q(k, n) \). A widely used distance measure for subspace codes is the *subspace distance*

\[
d_S(U, W) := \dim(U + W) - \dim(U \cap W) = 2 \cdot \dim(U + W) - \dim(U) - \dim(W),
\]

where \( U \) and \( W \) are subspaces of \( \mathbb{F}_q^n \). A set \( \mathcal{C} \) of subspaces of \( V \) is called a *subspace code*. The *minimum distance* of \( \mathcal{C} \) is given by

\[d = \min\{d_S(U, W) \mid U, W \in \mathcal{C}, U \neq W\} .\]

If the dimension of the codewords is fixed as \( k \), we use the notation \((n, \#\mathcal{C}, d, k)_q\) and call \( \mathcal{C} \) a constant dimension code (CDC for short).

For fixed ambient parameters \( q, n, k \) and \( d \), the main problem of subspace coding asks for the determination of the maximum possible size \( A_q(n, d, k) := M \) of an \((n, M, \geq d, k)_q\) subspace code.

A plethora of results on the construction of CDCs are invented in the literature, see e.g. [2]. The report [2] describes the underlying theoretical base of an on-line database, which can be found at [http://subspacecodes.uni-bayreuth.de](http://subspacecodes.uni-bayreuth.de) that tries to collect up-to-date information on the best lower and upper bounds for subspace codes.

Recently, Fagang Li combined the two methods of linkage construction and echelon-Ferrers to obtain some new lower bounds of CDCs. The method in [4] is based on the following lemma.
Lemma 1. \((\text{[I]}\)) Let \(X, Y \in \mathcal{P}_q(n)\), then \(d_S(X, Y) \geq d_H(v(X), v(Y))\), where \(d_H\) is the Hamming metric \((\text{see [3]})\).

The construction is correct, but using the lemma \([\text{I}]\) in the proof seems wrong. In this following texts, we are going to give a counterexample to the employing of the lemma.

2 Construction

Let \(X\) be a \(k\)-dimensional subspace of \(G_q(k, n)\). We represent \(X\) by the matrix in reduced row echelon form \(E(X)\), whose \(k\) rows form a basis for \(X\). The identifying vector of \(X\), denoted by \(v(X)\), is a binary vector of length \(n\) and weight \(k\), where the \(k\) ones of \(v(X)\) are exactly the pivots of \(E(X)\).

Remove the zeroes from each row of \(E(X)\) to the left of the pivot, and after that remove the columns which contain the pivots. All the remaining entries are shifted to the right. Then we obtain the Ferrers tableaux form of a subspace \(X\), denoted by \(F(X)\). The Ferrers diagram of \(X\), denoted by \(F(X)\), is obtained from \(F(X)\) by replacing the entries of \(F(X)\) with dots.

Let \(F\) be a Ferrers diagram with \(m\) dots in the rightmost column and \(\ell\) dots in the top row. A linear rank-metric code \(C_F\) of \(F_q^{m \times \ell}\) is called a Ferrers diagram rank-metric (FDRM) code, if for any codeword \(M\) of \(C_F\), all entries of \(M\) not in \(F\) are zeroes. An FDRM code \(C_F\) is denoted an \([F, d, \delta]\) FDRM code, if \(\text{rank}(A) \geq \delta\) for any nonzero codeword \(A\), and \(\text{dim}(C_F) = d\).

Given a matrix \(M \in F_q^{k \times m}\), the row space of \(M\) over \(F_q\) is denoted by \(\text{im}(M)\).

The combing construction is based on the following theorem:

Theorem 1. Let \(n_1 \geq k, n_2 \geq k, k \geq d\). For \(i = 1, 2\), let \(\mathcal{M}_i \subseteq F_q^{k \times n_i}\) be SC-representing sets with cardinality \(N_i\), and \(d_S(\mathcal{M}_i) = d\). Suppose that \(C_R \subseteq F_q^{k \times n_2}\) is a linear rank-metric code with \(d_R(C_R) = \frac{d}{2}\) and \(|C_R| = N_R\).

Let the identifying vector \(v_j\) with length \(n := n_1 + n_2\) and weight \(k\) satisfy the following properties for \(j = 1, 2, \ldots\):

(a) For each \(v_j\), the number of ones in the first \(n_1\) positions and last \(n_2\) positions are both greater than or equal to \(\frac{d}{2}\).

(b) The Hamming distance of two distinct identifying vectors is greater than or equal to \(d\).

Let \(C_{\mathcal{F}_1} \subseteq F_q^{k \times (n-k)}\) be an FDRM code and \(d_R(C_{\mathcal{F}_1}) = \frac{d}{2}\), where \(C_{\mathcal{F}_1}\) is a Ferrers diagram corresponding to the identifying vector \(v_j\).

Define the subspace code \(C\) of length \(n = n_1 + n_2\) as \(C = C_1 \cup C_2 \cup C_3\), where \(C_1 = \{\text{im}(U|M) \mid U \in \mathcal{M}_1, M \in C_R\}\); \(C_2 = \{\text{im}(U_{k \times n_1} | U) \mid U \in \mathcal{M}_2\}\); \(C_3 = \cup_j C_{\mathcal{F}_j}\).

Then \(C\) is an \((n, N, d, k)_q\) CDC with \(N = N_2 + N_1 \cdot N_R + \sum_j |C_{\mathcal{F}_j}|\).

Proof. We note that \(C_1, C_2, C_3\) are pairwise disjoint, therefore, the cardinality of the code is \(N_2 + N_1 \cdot N_R + \sum_j |C_{\mathcal{F}_j}|\).
According to the definition, \( C_{\mathcal{F}_1} \) is a CDC with \( d_s(C_{\mathcal{F}_1}) \geq d \). Hence, it is sufficient to prove that for any \( w_1 \in C_1, w_2 \in C_2, d_s(c_1, c_2) \geq d \).

For any identifying vector \( v_j \ (j = 1, 2, \cdots) \), we note that this vector has \( \frac{d}{2} \) ones in the last \( n_2 \) positions, and can be illustrated in reduced row echelon form as follows:

\[
\begin{pmatrix}
M_1 & M_3 \\
M_2 & M_4
\end{pmatrix} := \begin{pmatrix}
\ast_{(k-\frac{d}{2}) \times n_1} & \ast_{(k-\frac{d}{2}) \times n_2} \\
0_{\frac{d}{2} \times n_1} & \ast_{\frac{d}{2} \times n_2}
\end{pmatrix}_{k \times n}
\]

The first \( n_1 \) columns of the last \( \frac{d}{2} \) rows of the matrix are all 0, that is, \( M_2 \) is a zero matrix with the size of \( \frac{d}{2} \times n_1 \).

Denote

\[
M := \begin{pmatrix}
A_q(n_1, d, k) & Q_q(n_2, \frac{d}{2}, k) \\
M_1 & M_3 \\
0 & M_4
\end{pmatrix}_{2k \times n}
\]

The rank of \( A_q(n_1, d, k) \) is \( k \), hence, the rank of the matrix \( M \) is at least \( k + \frac{d}{2} \). This completes the proof.

\[\square\]

### 3 A Counterexample

In this section, we present a counterexample to Lemma \[\text{Lemma 1}\]. In the proof of Theorem \[\text{Theorem 1}\] Lemma \[\text{Lemma 1}\] was utilized repeatedly.

Follow the construction steps to construct \( A_2(8, 4, 4) \):

1) Lift MRD code: \( C_1 = [E_4, Q_2(4, 4, 2)] \), where \( Q_2(4, 4, 2) \) is a maximum rank distance code with the parameters \( 4 \times 4, d = 2, q = 2 \);

2) \( C_2 = [SQ_2(4, 4, 2), E_4] \), where \( SQ_2(4, 4, 2) \) is the codeword which is in \( Q_2(4, 4, 2) \) and has a rank at most 2.

3) We apply the combing method to add the following Ferrers diagram rank-metric codes with the corresponding identifying vector \((00110011)\). Notice that the Hamming distances between these identifying vectors are at least 4. Then the following four codewords could be added to \( C_3 \):

\[
C_3 = \left\{ \begin{pmatrix}
0 & 0 & 1 & 0 & x_1 & x_2 & 0 & 0 \\
0 & 0 & 0 & 1 & x_3 & x_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \mid x_i \in \mathbb{F}_2, 1 \leq i \leq 4, \text{rank} \begin{pmatrix}
x_1 & x_2 \\
x_3 & x_4
\end{pmatrix} \geq 2 \right\}
\]

Define

\[
a_1 := \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad a_2 := \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
However, one can easily to check that the codewords $a_1$ and $a_2$ satisfy

$$\begin{cases} 
    a_1 \in C_2, \\
    a_2 \in C_3, \\
    d_H(a_1, a_2) < 4.
\end{cases}$$

The last inequality contradicts that any two codewords should have a distance greater than or equal to 4.

Remark: Starting from the original intention in the paper [1], the true meaning of Lemma I is: If all subset of codewords are composed of Ferrers diagram rank-metric (FDRM) codes, then this lemma can be used directly. But if it is applied to other construction methods, this lemma can’t be used directly. That is, if only the condition $d_H(v_x, v_y) \geq d$ is satisfied, it cannot be guaranteed that it meets the distance requirements. The echelon-Ferrers construction has its own geometric meaning rather than simple applying Lemma I.

References

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