Central Extensions of Gauge Groups Revisited

Andrei Losev \(^1\), Gregory Moore \(^2\), Nikita Nekrasov \(^3\), and Samson Shatashvili \(^4\)*

\(^{1,3}\) Institute of Theoretical and Experimental Physics, 117259, Moscow, Russia
\(^3\) Department of Physics, Princeton University, Princeton NJ 08544
\(^{1,2,4}\) Dept. of Physics, Yale University, New Haven, CT 06520, Box 208120

losev@genesis5.physics.yale.edu
moore@castalia.physics.yale.edu
nikita@puhep1.princeton.edu
samson@euler.physics.yale.edu

We present an explicit construction for the central extension of the group Map\((X, G)\) where \(X\) is a compact manifold and \(G\) is a Lie group. If \(X\) is a complex curve we obtain a simple construction of the extension by the Picard variety Pic\((X)\). The construction is easily adapted to the extension of Aut\((E)\), the gauge group of automorphisms of a nontrivial vector bundle \(E\).

November 26, 1995; Revised December 19, 1995

* On leave of absence from St. Petersburg Steklov Mathematical Institute, St. Petersburg, Russia.
1. Statement of the problem

This paper continues an investigation into the application and generalization of two-dimensional field theory to higher dimensional theories begun in [1]. Here we explain how the higher dimensional analog of the central charge of a current algebra is “integrated” to the central extension of the corresponding group.

Let $X$ be an $n$-dimensional compact manifold, $G$ a Lie group and $\mathcal{G} = \text{Map}(X, G)$ the space of differentiable maps. It is well-known that when $G$ is simple and simply connected the covering group of $G$ has a universal central extension by

$$\hat{G} = \Omega^1(X; \mathbb{R})/Z^1_\mathbb{Z}(X)$$

(1.1)

where $\Omega^j(X)$ is the space of all differentiable $j$-forms on $X$, $Z^j(X)$ is the space of closed $j$-forms, and $Z^j_\mathbb{Z}$ is the space of closed forms with integral periods [2]. Correspondingly, the Lie algebra is extended by the space:

$$J = \Omega^1(X)/d\Omega^0(X) \cong Z^{n-1}(X)^\vee$$

(1.2)

by means of the cocycle:

$$\langle c(X, Y), \alpha \rangle = \frac{1}{8\pi^2} \int_X \alpha \wedge \text{Tr}(XdY)$$

(1.3)

for $\alpha \in Z^{n-1}(X)$ and $X, Y \in \Omega^0(X; g)$.

It has been emphasized in [3] that it would be desirable to make the abstract construction of the universal central extension $\hat{G}$ more explicit. In this note we give such a construction. It is similar to Mickelsson’s approach [4] for the case $X = S^1$ and follows the ideas of section (4.4) of [2]. A different solution to this problem, for the case when $X$ is a Riemann surface, was recently proposed in [3].

A slight generalization of the above problem replaces the group $\text{Map}(X, G)$ by the group $\text{Aut}(E)$ of gauge transformations of a principal $G$-bundle $E$ over $X$. In order to write down the Lie algebra cocycle one fixes a connection $\nabla$ in the adjoint bundle $ad(E)$ and defines:

$$c_\nabla(X, Y) = \frac{1}{8\pi^2} \text{Tr}(X\nabla Y)$$

(1.4)

Our construction generalizes to give the universal central extension of $\text{Aut}(E)$.

---

1 Tr is normalized so that, if $\tilde{G}$ is the simply connected cover, an integral generator of $H^4(B\tilde{G}; \mathbb{Z})$ is defined by $\frac{1}{8\pi^2} \text{Tr} F^2$.

2 Under a change of connection by an $ad(E)$-valued one form $A$ the cocycle changes by a coboundary: $c_{\nabla + A} - c_\nabla = \delta \epsilon_A$, where $\epsilon_A(X) = \frac{1}{8\pi^2} \text{Tr}(XA)$. 

---
2. General construction

2.1. Extension of the universal covering of the group $\text{Map}_0(X, G)$

We begin by constructing the extension of the universal covering $U\mathcal{G}$ of the component of the identity $\mathcal{G}_0 = \text{Map}(X, G)_0$ of $\mathcal{G} = \text{Map}(X, G)$. If $B \subset A$ and $D \subset C$ we let $\text{Map}((A, B); (C, D))$ denote the space of smooth maps $f$ of $A$ to $C$, such that $f(B) \subset D$. Introducing $I = [0, 1]$ we define:

\[
\begin{align*}
\mathcal{P}\mathcal{G} & \equiv \text{Map}((X \times I, X \times \{1\}); (G, 1)) \\
\Omega\mathcal{G} & \equiv \text{Map}((X \times I, X \times \{0, 1\}); (G, 1))
\end{align*}
\]

and let $\Omega_0\mathcal{G} \subset \Omega\mathcal{G}$ be the component of the identity. The construction is summarized by the diagram:

\[
\begin{array}{cccc}
1 & \to & J & \to & \hat{U}\mathcal{G} & \to & U\mathcal{G} & \to & 1 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
1 & \to & \mathcal{N} & \to & \hat{\mathcal{P}}\mathcal{G} & \to & \mathcal{P}\mathcal{G} & \to & 1
\end{array}
\]

(2.2)

In the rightmost column of (2.2) we represent the group $U\mathcal{G}$ as a quotient. To obtain the middle line we first construct a topologically trivial extension of $\mathcal{P}\mathcal{G}$ by the space of two-forms $\Omega^2(X \times I)$ using the group law:

\[
(g_1, e_1) \cdot (g_2, e_2) = (g_1 g_2, e_1 + e_2 + C(g_1, g_2))
\]

where $C$ is a cocycle given by:

\[
C(g_1, g_2) = \frac{1}{8\pi^2} \text{Tr}(g_1^{-1}dg_1 \wedge dg_2 g_2^{-1})
\]

(2.3)

We would like to construct an embedding $\psi$ of $\Omega_0\mathcal{G}$ as a normal subgroup of $\hat{\mathcal{P}}\mathcal{G}$ using the 3-form $\omega_3 = \text{Tr}(g^{-1}dg)^3$ on the group $G$. Accordingly, we choose an extension $\tilde{h}(x, t, \bar{t})$ of $h \in \Omega_0\mathcal{G}$ to $X \times I \times \bar{I}$ such that $\tilde{h}(x, t, \bar{t} = 1) = 1$ and write:

\[
\psi(h) \equiv \left(h, \frac{1}{24\pi^2} \int_{\bar{I}} \tilde{h}^* \omega_3 \right)
\]

(2.5)
The second entry of (2.3) is an element of $\Omega^2(X \times I)$ which depends on $\tilde{h}$. Two choices of extension lead to a difference by the closed two-form: $\frac{1}{24\pi^2} \int_{S^1} \tilde{h}^* \omega_3$. This two-form has integral periods since for any cycle $\gamma$ in $Z_2(X \times I, X \times \{0,1\})$ we have the corresponding period:

$$\frac{1}{24\pi^2} \int_{\gamma \times S^1} \tilde{h}^* \omega_3 \in \mathbb{Z}. \quad (2.6)$$

Thus the difference for two choices of extension is an element of $Z_2^2(X \times I, X \times \{0,1\})$, the space of closed two-forms vanishing on $X \times \{0,1\}$ and having integral periods. Hence we must extend $\mathcal{P}\mathcal{G}$ in (2.2) by the quotient space $\mathcal{N} \equiv \Omega^2(X \times I)/Z_2^2(X \times I, X \times \{0,1\})$. Using the Polyakov-Wiegmann formula

$$(g_1 g_2)^* \omega_3 = g_1^* \omega_3 + g_2^* \omega_3 - d(3\text{Tr}(g_1^{-1} dg_1 \wedge dg_2 g_2^{-1})) \quad (2.7)$$

one easily checks that $\psi$ is a group homomorphism and that the image is a normal subgroup.

The projection in the middle column of (2.2) is defined by restriction of $g$ to the boundary and gives $\hat{U}\mathcal{G} \cong \hat{\mathcal{P}G}/\psi(\Omega_0 \mathcal{G})$. Correspondingly, we have a map of the centers $\mathcal{N} \to \mathcal{J}$ by integration along $I$.

2.2. Central extension for non-simply-connected $\mathcal{G}$

When $\mathcal{G}$ is not simply connected we take a composition of the above extension of the universal covering with the extension of the group $\mathcal{G}$ by its fundamental group:

$$1 \to \pi_1(\mathcal{G}) \to U\mathcal{G} \to \mathcal{G} \to 1 \quad (2.8)$$

to get the universal central extension of $\mathcal{G}$:

$$1 \to \hat{\mathcal{J}} \to \hat{\mathcal{G}} \to \mathcal{G} \to 1$$

$$\hat{\mathcal{J}} \equiv \hat{\Omega}\mathcal{G}/\psi(\Omega_0 \mathcal{G}) \quad (2.9)$$

Here $\hat{\Omega}\mathcal{G}$ is the restriction of the extension $\hat{\mathcal{P}G}$ of the group $\mathcal{P}G$ to its subgroup $\Omega\mathcal{G}$. In order to show that $\hat{\mathcal{J}}$ is in the center of $\hat{U}\mathcal{G}$ one must use (2.7) and the result that the fundamental group of any Lie group is abelian. In general $\hat{\mathcal{J}}$ is itself an extension

$$1 \to \mathcal{J} \to \hat{\mathcal{J}} \to \pi_1(\mathcal{G}) \to 1 \quad (2.10)$$

If $\pi_1(\mathcal{G})$ has no torsion then $\hat{\mathcal{J}} \cong \pi_1(\mathcal{G}) \oplus \mathcal{J}$ since the projection of an abelian group to $\mathbb{Z}^n$ splits.
2.3. Extension of Aut(E)

The above construction generalizes to the gauge group of a nontrivial bundle \( E \) by making the following replacements. Let \( \pi : X \times I \to X \) be a projection. The group \( \mathcal{P}G \) is replaced by the subgroup of Aut(\( \pi^*(\mathcal{E}) \)) of automorphisms which are trivial at \( t = 1 \). \( \Omega \hat{G} \) is replaced by the subgroup of automorphisms which are trivial at \( t = 0, 1 \).

Generalizing the extension in (2.3) requires a choice of connection on Aut(\( \mathcal{E} \)) and is defined by the cocycle:

\[
\psi_{\nabla}(h) = \left( h, \frac{1}{24\pi^2} \int_I \left[ \text{Tr}(h^{-1}\tilde{\nabla}h)^3 + 3\text{Tr} \ F_\tilde{\nabla}[h^{-1}\tilde{\nabla}h + (\tilde{\nabla}h)h^{-1}] \right] \right) .
\]

(2.11)

Here we use the 3-form originally discovered in [6] in connection with multi-dimensional solitons (for a recent application see [7]). The extra terms in the second entry in (2.11) are required in order for \( \psi_{\nabla} \) to define a group homomorphism, or, equivalently, in order to satisfy the PW formula:

\[
\psi_{\nabla}(g_1 g_2) = \psi_{\nabla}(g_1) + \psi_{\nabla}(g_2) + c_{\nabla}(g_1, g_2).
\]

Again, two choices of extension of \( \tilde{h} \) lead to an ambiguity in (2.11) by an element of \( \mathbb{Z}_2(X \times I, X \times \{0, 1\}) \) (the periods are related to characteristic numbers of vector bundles constructed from \( E \)). So, the group Aut(\( E \)) is extended by the same space \( \hat{J} \).

2.4. Extension when Aut(E) is not connected

Now suppose that \( \pi_0(\text{Aut}(E)) \) is not trivial so Aut(\( E \)) = \( \amalg \alpha \text{Aut}(E)_\alpha \). The construction above generalizes by letting \( \mathcal{P}G = \amalg \mathcal{P}G_\alpha \) where, for each component \( \alpha \) we choose a standard element \( g_\alpha^0 \) (with \( g_\alpha^0 = 1 \)) and let \( \mathcal{P}G_\alpha = \{ g \in \text{Aut}(E) : g(x, t = 1) = g_\alpha^0(x) \} \).

The group \( \Omega_0G \) remains unchanged as does the construction.

2.5. Explicit formulas for the central extension

In order to make the construction more explicit we must make two choices. First, for each \( \gamma \in \pi_1(\mathcal{G}) \), we choose a representative \( L(\gamma) \in \Omega \mathcal{G} \). Second, we choose a continuation \( \Phi(g) \) of \( g \), i.e. a map of sets \( \mathcal{G} \to \mathcal{P}G \) that inverts the restriction of \( \mathcal{P}G \) to \( X \). In

\footnote{We continue to assume that Aut(\( E \)) is connected.}
general neither \(L\) nor \(\Phi\) is a homomorphism. Indeed, if \(G\) is simple then \(\Phi\) cannot be a homomorphism. Elements of the centrally extended group \(\hat{G}\) are left-cosets

\[(g, \Phi(g), 0)(1, L(\gamma), \lambda)\psi(\Omega_0 G) \subset \hat{\mathcal{P}G}\]  

(2.12)

where \(\lambda \in J\). Elements of the center \(\hat{J} = \hat{\Omega G}/\psi(\Omega_0 G)\) of the extension (2.9) are also cosets: \((1, L(\gamma), \lambda)\psi(\Omega_0 G) \subset \hat{\Omega G}\).

We now give explicit formulae for the multiplication of the cosets. \(\hat{J}\) is itself a central extension (2.10). Thus, as a set it is the space of pairs \((\gamma, \lambda)\), but the multiplication involves a cocycle: \((\gamma_1, \lambda_1)(\gamma_2, \lambda_2) = (\gamma_1 + \gamma_2, \lambda_1 + \lambda_2 + CL(\gamma_1, \gamma_2))\). The cocycle \(CL\) is defined by choosing a homotopy \(h\) between the loops \(L(\gamma_1 \ast \gamma_2)\) and \(L(\gamma_1) \ast L(\gamma_2)\) and writing:

\[CL(\gamma_1, \gamma_2) = \frac{1}{24\pi^2} \int_{I \times I} h^* \omega_3\]  

(2.13)

Multiplication of two cosets (2.12) in \(\hat{P}G\) leads to the central element \((\gamma(g_1, g_2), \lambda(g_1, g_2))\) given by:

\[\gamma(g_1, g_2) = [\varphi_{1,2}] \in \pi_1(\mathcal{G})\]

\[\lambda(g_1, g_2) = \int_I C(\Phi(g_1), \Phi(g_2)) + \frac{1}{24\pi^2} \int_{I \times I} \tilde{h}^* \omega_3,\]  

(2.14)

The loop \(\varphi_{1,2} \in \Omega \mathcal{G}\) is obtained by “glueing together” the paths \(\Phi(g_1) \cdot \Phi(g_2)\) and \(\Phi(g_1 \cdot g_2)\). More precisely, denoting by \((\cdot)^{inv}\) the operation of taking the inverse in the semigroup of paths, \(\varphi_{1,2} = (\Phi(g_1) \cdot \Phi(g_2))^{inv} (\Phi(g_1 \cdot g_2))^{inv}\). \(\tilde{h}\) is a homotopy in \(\mathcal{G}\) between the loop \(\varphi_{1,2}\) and the representative of its homotopy class \(L([\varphi_{1,2}])\).

2.6. Relation to the descent procedure

In light of the above results it is instructive to reconsider the descent procedure for constructing gauge group cocycles with values in functionals of gauge connections [8]. Recall that one introduces three operations \(d, d^{-1}, \delta\), on differential-form valued functionals of group elements and gauge connections such that \(\delta^2 = d^2 = 0\). \(\delta\) is a group cochain differential [8]; \(d^{-1}\) is defined in [9] and is essentially the operation \(\int_I\) used in equation (2.5) above. Starting from a \(2m\)-form \(\Omega_{2m} = \text{Tr} F^m\), which satisfies \(\delta \Omega_{2m} = d \Omega_{2m} = 0\), one applies the operation \(d^{-1}\delta\) a total of \(k\) times to get \(\Xi = \int_X d^{-1}\delta \cdots d^{-1}\delta d^{-1}\Omega_{2m}\), which satisfies \(\delta \Xi = 0\) and hence is a cocycle of degree \(k\) in dimension \(2m - k - 1\).

---

\(^4\) * stands for the path multiplication.
Comparing with (1.3) it is apparent that one might have started with an \( n + 3 \)-form \( \Omega_{n+3}' = \alpha \wedge \text{Tr}F^2 \) where \( \alpha \) is a closed \( n - 1 \)-form with integer periods. Repeating the descent procedure with the appropriate definition of \( d^{-1} \), one obtains a 2-cocycle taking values in \( J \). For example, if \( H^3(G) \) is trivial, the form \( \text{Tr}(g^{-1}dg)^3 \) on the group \( G \) is exact: \( \text{Tr}(g^{-1}dg)^3 = db_2 \), where \( b_2 \) is a 2-form on a group \( G \). The cocycle \( C_d \) obtained using the descent procedure is:

\[
C_d(g_1, g_2) = \lambda(g_1, g_2) + \int_I (\Phi(g_1)^*b_2 + \Phi(g_2)^*b_2 - \Phi(g_1g_2)^*b_2),
\]

(2.15)

and differs from the one presented in this paper by a coboundary. One can show that the cocycle \( C_d \) is independent of the choice of the section \( \Phi \), while under a change of \( b_2 \) it changes by a coboundary.

3. Specializations

Let us now assume that \( \pi_1(G) \) has no torsion. We have constructed the universal central extension of \( G \) by \( J \oplus \pi_1(G) \). All other central extensions are formed by taking quotients \( \widetilde{G}/\widetilde{J}_1 \), where \( \widetilde{J}_1 \subset J \oplus \pi_1(G) \). Here we list some interesting examples.

First, to get a one-dimensional central extension we note that the group of characters of \( J \oplus \pi_1(G) \) is \( \mathbb{Z}^{n-1} \oplus \mathbb{Z} \mathbb{Z}(X) \). Given an element \((\alpha, \chi)\) of this space we can take \( \widetilde{J}_1 \) to be \( \ker(\alpha \oplus \chi) \). The corresponding group extension is given by \( \exp[2\pi i \int_X \alpha \wedge \lambda(g_1, g_2) + 2\pi i \chi(\gamma(g_1, g_2))] \). Second, if \( X \) is equipped with a metric then we can take \( \widetilde{J}_1 = d^*\Omega^2 + Z_\mathbb{Z}^1 \). In this case the connected component of the center is the torus \( H^1(X; \mathbb{R})/H^1(X; \mathbb{Z}) \).

Third, if \( X \) is also equipped with a complex structure then we can work over the complex numbers and choose \( \widetilde{J}_1 = [\bar{\partial}^j\Omega^{0,2} \ominus \Omega^{1,0}] + Z_\mathbb{Z}^1 \). In this case the connected component of the center is the connected component of the Picard variety

\[
\text{Pic}_0(X) \equiv H^{1,0} \setminus H^1(X; \mathbb{C})/H^1(X; \mathbb{Z}) .
\]

(3.1)

Finally, if \( X \) is a complex curve we do not need a metric to construct the central extension by the Jacobian of the curve. In this way we obtain a solution of the problem posed in \[3\] and solved independently using a very different and beautiful technique of holomorphic geometry in \[3\]. Moreover, since in this case \( \pi_1(G) = \mathbb{Z} \), the center is the full Picard variety.
Acknowledgements

We would like to thank I. Frenkel for emphasizing to us the importance of this problem. We would also like to thank L. Faddeev and B. Khesin for useful remarks and discussions. The research of A. Losev was partially supported by RFFI grant 95-01-01101. The research of G. Moore is supported by DOE grant DE-FG02-92ER40704, and by a Presidential Young Investigator Award; that of S. Shatashvili, by DOE grant DE-FG02-92ER40704, by NSF CAREER award and by OJI award from DOE.
References

[1] A. Losev, G. Moore, N. Nekrasov, and S. Shatashvili, “Four-Dimensional Avatars of Two-Dimensional RCFT,” hep-th/9509151.

[2] A. Pressley and G. Segal, ”Loop Groups”, Oxford Clarendon Press, 1986

[3] P.I. Etingof and I.B. Frenkel, “Central Extensions of Current Groups in Two Dimensions,” Commun. Math. Phys. 165(1994) 429

[4] J. Mickelsson, “Kac-Moody groups, topology of the Dirac determinant bundle and fermionization,” Commun. Math. Phys. 110(1987)173.

[5] I. Frenkel and B. Khesin, “Four dimensional realization of two dimensional current groups,” Yale preprint, July 1995.

[6] L. D. Faddeev, “Some Comments on Many Dimensional Solitons”, Lett. Math. Phys., 1 (1976) 289-293.

[7] A. Gerasimov, “Localization in GWZW and Verlinde formula,” hep-th/9305090

[8] L. Faddeev and S. Shatashvili, Theor. Math. Fiz., 60 (1984) 206, L. Faddeev, Phys. Lett. B145 (1984) 81, J. Mickelsson, CMP, 97 (1985) 361, A. Reiman, M. Semenov-Tian-Shansky, L. Faddeev, Funct. Anal. and Appl.,vol.18,No.4(1984)319

[9] S. Novikov,”The Hamiltonian formalism and many-valued analogue of Morse theory”, Russian Math.Surveys 37:5(1982),1-56; E. Witten, “Global Aspects of Current Algebra,” Nucl.Phys.B223:422,1983