CHAOTIC BEHAVIOR IN THE UNFOLDING OF HOPF-BOGDANOV-TAKENS SINGULARITIES

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On the occasion of Juan J. Nieto 60th birthday

Abstract. A discussion on local bifurcations of codimension one and two is presented for generic unfoldings of Hopf-Bogdanov-Takens singularities of codimension three. Among all identified bifurcations, we focus on Hopf-Zero and Hopf-Hopf bifurcations, since, in certain cases, they can explain the emergence of chaotic dynamics. Moreover, numerical simulations are provided to illustrate that strange attractors appear at least when the second order normal form of the unfolding is considered.

1. Introduction. Let $X$ be a $C^\infty$ four-dimensional vector field. In this paper we consider an equilibrium $p$ of $X$ where $DX(p)$ is conjugated to

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\omega \\
0 & 0 & \omega & 0
\end{pmatrix},
\]

with $\omega \neq 0$. The eigenvalues are a pair of purely imaginary eigenvalues $\pm \omega i$ and a non-semisimple double zero. We refer to $p$ (or to $X$ itself) as a Hopf-Bogdanov-Takens (HBT in what follows) singularity.

Little is known about HBT singularities and their unfoldings. They were studied in the context of weakly coupled nonlinear oscillators with a $Z_2 \oplus Z_2$ symmetry in [23]. Under such symmetry conditions, authors computed a truncated normal form for generic unfoldings and obtained primary bifurcations of codimension one and two. More recently, under identical symmetry conditions, HBT singularities were considered in the context of fluid dynamics (see [29]) where, working again with a truncated normal form and using numerical continuation, Shilnikov homoclinic orbits were detected. A few additional references, including applications, are given in [23].

Particularly remarkable is the role that HBT singularities can play when two (or more) systems are coupled by linear diffusion. A discussion about the relevance of

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certain singularities in that context can be found in [15]. Moreover, the coalescence of Hopf and Bogdanov-Takens bifurcation was observed in a model consisting of two brusselators linearly coupled by diffusion (see [14]). HBT singularities also appear in the unfolding of the four-dimensional nilpotent singularity of codimension four (see [10, 11]).

In this paper we consider truncated normal forms to obtain the complete atlas of Hopf-Zero (HZ in the sequel) and Hopf-Hopf (HH in the sequel) bifurcations which can be exhibited by generic unfoldings of a HBT singularity. A complete description of all classes of HZ and HH bifurcations of codimension two is provided in [18, 22]; we follow the first of these references (degeneracies of higher codimension are considered in [1, 2, 3, 4, 5]). According to Takens’ classification [30], there are six different topological types for HZ singularities, but (see [18]) only the distinction in between four classes of unfoldings is required. Regarding HH singularities, there are twelve distinct unfoldings and nine different topological types for the singularity. Some of these singularities can play the role of organizing centers of chaotic dynamics.

Bearing in mind vector fields, it is very well-known that the simplest configurations which can imply the existence of strange attractors are the Shilnikov homoclinic orbits, that is, homoclinic orbits to a saddle-focus equilibrium point with eigenvalues \( \lambda \) and \(-\rho \pm \omega i\) verifying that \(0 < \rho < \lambda\). Shilnikov [27] proved the existence of infinitely many periodic orbits in any neighborhood of the connection. In fact any neighborhood of the homoclinic orbit contains suspended horseshoes (see [28, 31]) which contain the periodic orbits found in [27]. When the vector field is perturbed to unfold generically the homoclinic connection, horseshoes are destroyed in a process which involves generic unfoldings of homoclinic tangencies between invariant manifolds of periodic orbits. Therefore, according to [24], the appearance of strange attractors also follows (see also [25, 26, 19]).

In spite of the abundance of examples where numerical tools are used to show the existence of the above described homoclinic chaos, analytical proofs are rare. Fortunately, there are results showing the existence of Shilnikov bifurcations in generic unfoldings of certain singularities. Of course, given specific systems, singularities are much more easy to detect than any global configuration, including Shilnikov homoclinic orbits.

Appearance of Shilnikov bifurcations in generic unfoldings of four-dimensional nilpotent singularities of codimension four was argued in [20]. In [21] it was proved that these bifurcations were also generically unfolded by the three-dimensional nilpotent singularity of codimension three. In the case of the unfoldings of HZ singularities, geometrical arguments were provided in [18] to explain how Shilnikov homoclinic orbits could emerge in one of the four possible unfoldings. Broer and Vegter [12] proved that, regarding unfoldings of HZ singularities, the Shilnikov phenomenon is flat in the sense that it can be completely annihilated by a flat perturbation. Finally, an existence theorem valid for any generic unfolding was obtained in [9], putting together previous results in [6, 7, 8, 17]. To the best of our knowledge, there is no similar result for the case of the HH bifurcation where chaos is expected and we can only refer to the geometrical discussion in [18] in order to support the existence of chaotic dynamics in the unfolding of certain HH singularities.

In this paper we study all the cases of HZ and HH singularities which can be unfolded by a HBT singularity. It is remarkable that the HZ case which generically unfolds Shilnikov homoclinic orbits and also the HH case which is likely to unfold chaotic dynamics are included in the catalogue associated to the HBT singularities.
Due to this fact we provide some numerical illustrations of chaos in subfamilies related with both types of codimension-two singularities.

We will consider the normal form for HBT unfoldings obtained in [16] in the general case, without any symmetry assumption. It was used to get a formal classification of the different topological types exhibited by the singularity itself. In this paper we take a second order truncation of the normal form to discuss the catalogue of local bifurcations of equilibria unfolded by HBT singularities. At this primary level, the atlas of bifurcations is not affected for higher order terms in the normal form. A similar discussion can be found in [23], but as far as we know, the dissection between the different cases of HZ and HH singularities has not yet been provided in the literature.

In Section 2 we recall the normal form obtained in [16] and fix the truncated unfolding that it is considered along the paper. Local bifurcations are studied in Section 3, where we also state and prove our main results. Only attending to the linear part at the equilibrium points, we describe a basic bifurcation diagram of saddle-node, Hopf, Hopf-Zero and Hopf-Hopf bifurcations. Later, as already mentioned, we focus on discussing the different cases of HZ and HH bifurcations of saddle-node, Hopf, Hopf-Zero and Hopf-Hopf bifurcations. Later, as already mentioned, we focus on discussing the different cases of HZ and HH bifurcations which can appear depending on the values of certain coefficients in the normal form of the HBT singularity. In Section 4, numerical simulations showing the emergence of chaotic behaviors are provided. These simulations were done bearing in mind the HZ and HH cases which are related with the homoclinic chaos.

2. Normal form. As argued in [16], any \( C^\infty \) family \( X_\nu \), with \( \nu \in \mathbb{R}^3 \) and such that \( X_0 \) is a HBT singularity, can be written as

\[
\begin{align*}
x' &= y + R_1(x, y, u, v, \nu) \\
y' &= a_{0,0}(\nu) + a_{1,0}(\nu)x + b_{0,0}(\nu)y + A_{2,0}(\nu)x^2 + B_{1,0}(\nu)xy + A_{0,1}(\nu)(u^2 + v^2) + R_2(x, y, u, v, \nu) \\
u' &= c_{0,1}(\nu)u - d_{0,1}(\nu)v + (C_{1,1}(\nu)u - D_{1,1}(\nu)v)x + R_3(x, y, u, v, \nu) \\
v' &= d_{0,1}(\nu)u + c_{0,1}(\nu)v + (D_{1,1}(\nu)u + C_{1,1}(\nu)v)x + R_4(x, y, u, v, \nu),
\end{align*}
\]

with \( R_i(x, y, u, v, \nu) = O(\|(x, y, u, v)\|^3) \) for \( i = 1, \ldots, 4 \), \( a_{0,0}(0) = a_{1,0}(0) = b_{0,0}(0) = c_{0,1}(0) = 0 \) and \( d_{0,1}(0) = 1 \). Moreover, assuming the generic conditions \( A_{2,0}(0) \neq 0, \quad B_{1,0}(0) \neq 0, \quad A_{0,1}(0) \neq 0, \quad C_{1,1}(0) \neq 0 \) it is proved that, rescaling variables and time, and after the appropriate \( \nu \)-dependent translation in the first variable, family (2) can be transformed into

\[
\begin{align*}
x' &= y + \tilde{R}_1(x, y, u, v, \nu) \\
y' &= \tilde{a}_{0,0}(\nu) + \tilde{x}^2 + \tilde{b}_{0,0}(\nu)y + \kappa(u^2 + v^2) + \tilde{R}_2(x, y, u, v, \nu) \\
u' &= \tilde{c}_{0,1}(\nu)u - \tilde{d}_{0,1}(\nu)v + (\tilde{C}_{1,1}(\nu)u - \tilde{D}_{1,1}(\nu)v)x + \tilde{R}_3(x, y, u, v, \nu) \\
v' &= \tilde{d}_{0,1}(\nu)u + \tilde{c}_{0,1}(\nu)v + (\tilde{D}_{1,1}(\nu)u + \tilde{C}_{1,1}(\nu)v)x + \tilde{R}_4(x, y, u, v, \nu),
\end{align*}
\]

where \( \tilde{R}_i(x, y, u, v, \nu) = O(\|(x, y, u, v)\|^3) \) for \( i = 1, \ldots, 4 \), \( \tilde{a}_{0,0}(0) = \tilde{b}_{0,0}(0) = \tilde{c}_{0,1}(0) = 0, \tilde{d}_{0,1}(0) \neq 0, \tilde{C}_{1,1}(0) \neq 0 \) and \( \kappa = \text{sign}(A_{2,0}(0)A_{0,1}(0)) \).

The transversality condition

\[
\det(DP(0)) \neq 0,
\]
with $P(\nu) = (a_{0,0}(\nu), b_{0,0}(\nu), c_{0,1}(\nu))$, is also assumed. Hence, we can introduce new parameters $(\lambda_1, \lambda_2, \mu) = (a_{0,0}(\nu), b_{0,0}(\nu), c_{0,1}(\nu))$ to get

$$
\begin{cases}
x' &= y + \tilde{R}_1(x, y, u, v, \lambda_1, \lambda_2, \mu) \\
y' &= \lambda_1 + x^2 + \lambda_2 y + xy + \kappa(u^2 + v^2) + \tilde{R}_2(x, y, u, v, \lambda_1, \lambda_2, \mu) \\
u' &= \mu u - \tilde{a}_{0,1}(\lambda_1, \lambda_2, \mu)v + (\tilde{C}_{1,1}(\lambda_1, \lambda_2, \mu)u - \tilde{D}_{1,1}(\lambda_1, \lambda_2, \mu)v)x + \tilde{R}_3(x, y, u, v, \lambda_1, \lambda_2, \mu) \\
v' &= \tilde{a}_{0,1}(\lambda_1, \lambda_2, \mu)u + \mu v + (\tilde{D}_{1,1}(\lambda_1, \lambda_2, \mu)u + \tilde{C}_{1,1}(\lambda_1, \lambda_2, \mu)v)x + \tilde{R}_4(x, y, u, v, \lambda_1, \lambda_2, \mu),
\end{cases}
$$

with $\tilde{R}_i(x, y, u, v, \lambda_1, \lambda_2, \mu) = O(||(x, y, u, v)||^3)$ for $i = 1, \ldots, 4$, $\tilde{a}_{0,1}(0, 0, 0) \neq 0$ and $\tilde{C}_{1,1}(0, 0, 0) \neq 0$.

Truncating at second order in $(x, y, u, v)$, we get

$$X_{(\lambda_1, \lambda_2, \mu)}^2 := \begin{cases}
x' &= y \\
y' &= \lambda_1 + x^2 + \lambda_2 y + xy + \kappa(u^2 + v^2) \\
u' &= \mu u - \tilde{a}_{0,1}v + (\tilde{C}_{1,1}u - \tilde{D}_{1,1}v)x \\
v' &= \tilde{a}_{0,1}u + \mu v + (\tilde{D}_{1,1}u + \tilde{C}_{1,1}v)x
\end{cases} \tag{3}$$

where we use the notation $X_{(\lambda_1, \lambda_2, \mu)}^2$ to refer to the vector field whose components are given by the left hand side equations in (3). Note that, to make the notation more manageable, in the sequel we do not make explicit that $\tilde{a}_{0,1}$, $\tilde{C}_{1,1}$ and $\tilde{D}_{1,1}$ are functions depending on $(\lambda_1, \lambda_2, \mu)$.

In the next section, we obtain the whole catalogue of local bifurcations which are exhibited by family (3) in a neighborhood of $(\lambda_1, \lambda_2, \mu) = (0, 0, 0)$. Starting the analysis with a truncation is common when one works with singularities with normal forms which are invariant by rotations. The reader can compare with the analysis of Hopf-Zero and Hopf-Hopf bifurcations in [18, 22].

**Remark 1.** Note that our interest is the unfolding of the singularity exhibited at $(x, y, u, v) = (0, 0, 0, 0)$ when $(\lambda_1, \lambda_2, \mu) = (0, 0, 0)$. Hence, regarding local bifurcations, we focus on equilibrium points of $X_{(\lambda_1, \lambda_2, \mu)}^2$ arising in a neighborhood of the origin in the phase space, when parameters vary in a neighborhood of $(\lambda_1, \lambda_2, \mu) = (0, 0, 0)$.

**Remark 2.** It is worth noting that the truncated family (3) exhibits some non-generic features. Namely, the plane $u = v = 0$ is invariant by the flow. This property restricts the possibilities for the dynamical behavior in family (3). Indeed, arguments to prove or to explain the appearance of chaotic dynamics in the unfolding of certain HZ and HH singularities are based in how heteroclinic cycles involving equilibrium points break to create Shilnikov homoclinic orbits. However, the formation of these connections is not compatible with the existence of invariant planes. In Section 4, we show numerical simulations of chaotic behavior in family (3), but it must be noticed that this emerging chaos is linked to bifurcations of the periodic orbits arising in the family. Of course, generically, the invariant plane does
not persist when (not normalized) higher order terms are considered. Nevertheless, the truncated family may show heteroclinic cycles and therefore studying their existence becomes essential. As it happens in the case of HZ and HH bifurcations, the non-robustness of the cycles when higher order terms are considered could lead to interesting dynamical behaviors, including homoclinic and heteroclinic orbits.

3. Bifurcations in the truncated family. In this section we prove the results below.

**Theorem 3.1.** When $\lambda_1 = \mu = 0$ and $\lambda_2 \neq 0$, family (3) exhibits a Hopf-Zero singularity. According to the classification of unfoldings provided in [18], four cases can be distinguished

- **Case I:** If $\kappa = -1$ and $\lambda_2 \hat{C}_{1,1} > 0$,
- **Case II:** If $\kappa = -1$ and $\lambda_2 \hat{C}_{1,1} < 0$,
- **Case III:** If $\kappa = 1$ and $\lambda_2 \hat{C}_{1,1} > 0$,
- **Case IV:** If $\kappa = 1$ and $\lambda_2 \hat{C}_{1,1} < 0$.

**Theorem 3.2.** When $\lambda_2 = \sqrt{-\lambda_1}$ and $\mu - \hat{C}_{1,1} \sqrt{-\lambda_1} = 0$, family (3) exhibits a Hopf-Hopf singularity. According to the classification of unfoldings provided in [18], eight cases can be distinguished:

- **Case IVb:** If $\kappa = -1$ and $C_{1,1} < 0$,
- **Case VIa:** If $\kappa = -1$ and $0 < C_{1,1} < \frac{1}{4}$,
- **Case V:** If $\kappa = -1$ and $\frac{1}{4} < C_{1,1} < \frac{1}{2}$,
- **Case VIb:** If $\kappa = 1$ and $\frac{1}{2} < C_{1,1} < 0$,
- **Case Ib:** If $\kappa = 1$ and $0 < C_{1,1} < \frac{1}{4}$,
- **Case Ia:** If $\kappa = 1$ and $\frac{1}{4} < C_{1,1} < \frac{1}{2}$,
- **Case III:** If $\kappa = 1$ and $\frac{1}{2} < C_{1,1}$.

**Remark 3.** It is not difficult to prove that the distinction between the different cases is not affected for higher order terms in the normal form.

**Remark 4.** Case III of Hopf-Zero bifurcation and case VIa of Hopf-Hopf bifurcation are the cases where chaotic behaviors are likely to emerge.

We first study the elementary bifurcations. Let $G : \mathbb{R}^4 \times \mathbb{R}^3 \rightarrow \mathbb{R}^4$ given by the right-hand side of the system (3). It can be easily checked, bearing $\hat{a}_{0,1} \neq 0$ in mind, that the Implicit Function Theorem can be applied to the equation $G(x, y, u, v, \lambda_1, \lambda_2, \mu) = (0, 0, 0, 0)$ in a neighborhood of the origin in $\mathbb{R}^4 \times \mathbb{R}^3$. It follows that there exist neighborhoods $B_1$ and $B_2$ of $(0, 0, 0)$ and $(0, 0, 0, 0)$, respectively, and there exist unique functions $\xi_i(x, \lambda_2, \mu)$, with $i = 1, 2, 3, 4$, defined from $B_1$ to $B_2$ such that

$$G(x, \xi_1(x, \lambda_2, \mu), \xi_2(x, \lambda_2, \mu), \xi_3(x, \lambda_2, \mu), \xi_4(x, \lambda_2, \mu), \lambda_2, \mu) = (0, 0, 0, 0)$$

for all $(x, \lambda_2, \mu) \in B_1$. Moreover, points on the graph of $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)$ are the only solutions of the equation in $B_1 \times B_2$. On the other hand, it is straightforward to check that

$$\xi_1(x, \lambda_2, \mu) = 0, \quad \xi_2(x, \lambda_2, \mu) = 0, \quad \xi_3(x, \lambda_2, \mu) = 0 \quad \text{and} \quad \xi_4(x, \lambda_2, \mu) = -x^2.$$ 

Hence, the manifold of equilibria in $B_1 \times B_2$ is given by:

$$y = 0, \quad u = 0, \quad v = 0, \quad \lambda_1 = -x^2.$$
Primary bifurcations in the unfolding of a HBT singularity. A sphere $\lambda_1^2 + \lambda_2^2 + \mu^2 = \delta$ is fixed and the front view corresponds to $\lambda_1 < 0$. Hence, points (respectively lines) correspond to bifurcation curves (respectively surfaces). We assume that $\hat{C}_{1,1} < 0$.

It follows that, close to $(0,0,0,0)$ and for $\| (\lambda_1, \lambda_2, \mu) \|$ small enough, family $X^2_{(\lambda_1, \lambda_2, \mu)}$ given in (3) has no equilibrium points if $\lambda_1 > 0$, a unique one at $P^0 = (0,0,0,0)$ if $\lambda_1 = 0$ and two equilibria $P^\pm = (\pm \sqrt{-\lambda_1}, 0,0,0)$ when $\lambda_1 < 0$. On the one hand, assuming $\lambda_1 < 0$ we obtain

$$DX^2_{(\lambda_1, \lambda_2, \mu)}(P^\pm) =$$

$$ \begin{pmatrix}
  0 & 1 & 0 & 0 \\
  \pm 2\sqrt{-\lambda_1} & \lambda_2 + \sqrt{-\lambda_1} & 0 & 0 \\
  0 & 0 & \mu + \hat{C}_{1,1}\sqrt{-\lambda_1} & -\hat{d}_{0,1} \mp \hat{D}_{1,1}\sqrt{-\lambda_1} \\
  0 & 0 & \hat{d}_{0,1} \pm \hat{D}_{1,1}\sqrt{-\lambda_1} & \mu \pm \hat{C}_{1,1}\sqrt{-\lambda_1}
\end{pmatrix}.$$ 

We find four cases of non-hyperbolicity at $P^\pm$:

- A Hopf bifurcation at $P^-$ when $\lambda_2 = \sqrt{-\lambda_1}$ and $\mu - \hat{C}_{1,1}\sqrt{-\lambda_1} \neq 0$ (see the curve $H_1^-$ in Figure 1).
- A Hopf bifurcation at $P^+$ when $\lambda_2 \neq \sqrt{-\lambda_1}$ and $\mu - \hat{C}_{1,1}\sqrt{-\lambda_1} = 0$ (see the curve $H_2^-$ in Figure 1).
- A Hopf bifurcation at $P^+$ when $\mu + \hat{C}_{1,1}\sqrt{-\lambda_1} = 0$ (see the curve $H_1^+$ in Figure 1).
- A Hopf-Hopf bifurcation at $P^-$ when $\lambda_2 = \sqrt{-\lambda_1}$ and $\mu - \hat{C}_{1,1}\sqrt{-\lambda_1} = 0$ (see the point $HH^-$ in Figure 1).
On the other hand, assuming $\lambda_1 = 0$ we get

$$DX^2_{(\lambda_1, \lambda_2, \mu)}(P^0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \mu & -\hat{d}_{0,1} \\ 0 & 0 & \hat{d}_{0,1} & \mu \end{pmatrix}$$

and we distinguish four non-hyperbolic cases:

- A saddle-node bifurcation at $(0, 0, 0, 0)$ when $\lambda_2 \neq 0$ and $\mu \neq 0$ (see the curve $SN_0$ in Figure 1).
- A Bogdanov-Takens bifurcation at $(0, 0, 0, 0)$ when $\lambda_2 = 0$ and $\mu \neq 0$ (see the points $BT^0_{sp}$ and $BT^0_{down}$ in Figure 1).
- A Hopf-Zero bifurcation at $(0, 0, 0, 0)$ when $\lambda_2 \neq 0$ and $\mu = 0$ (see the points $HZ^0_{left}$ and $HZ^0_{right}$ in Figure 1).
- A Hopf-Bogdanov-Takens bifurcation at $(0, 0, 0, 0)$ when $\lambda_2 = 0$ and $\mu = 0$.

All the different elementary bifurcations are shown in Figure 1. To make the picture simple, we consider a sphere $\lambda^2_1 + \lambda^2_2 + \mu^2 = \delta$ with $\delta$ small and on it we plot the bifurcations. Due to our focus is on HZ and HH bifurcations, we do not include the study of the genericity for saddle-node, Hopf and Bogdanov-Takens bifurcations.

**Remark 5.** As already mentioned in Remark 1, we are only interested in bifurcations exhibited by the system in a neighborhood of $(\lambda_1, \lambda_2, \mu) = (0, 0, 0)$. As expected, the curves and surfaces that we have found in the study of the truncated family show a bifurcation diagram that extends far from the origin, but we are still only interested in the dynamics that we can observe for small values of the parameters.

**Remark 6.** Note that we are assuming that $\hat{C}_{1,1} < 0$ in Figure 1. Otherwise, $H^+$ and $H_2^-$ should switch their positions and also the $HH^-$ point should move up.

### 3.1. Hopf-Zero bifurcations: Proof of Theorem 3.1.

If we assume that $\lambda_1 = \mu = 0$ and $\lambda_2 \neq 0$, family (3) reduces to

$$X^2_{(0, \lambda_2, 0)} := \begin{cases} 
    x' = y \\
    y' = x^2 + \lambda_2 y + xy + \kappa(u^2 + v^2) \\
    u' = -\hat{d}_{0,1} v + (\hat{C}_{1,1} u - \hat{D}_{1,1} v) x \\
    v' = \hat{d}_{0,1} u + (\hat{D}_{1,1} u + \hat{C}_{1,1} v) x.
\end{cases}$$

Introducing new variables $(X, Y, U, V)$ such that $(x, y, u, v)^t = J(X, Y, U, V)^t$, with

$$J = \begin{pmatrix} 1 & \frac{1}{\lambda_2} & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix}.$$
the linear part is transformed into Jordan canonical form. Namely, we obtain
\[
\begin{align*}
X' & = -\frac{1}{\lambda_2} X^2 - \frac{2+\lambda_1}{\lambda_2^2} XY - \frac{1+\lambda_2}{\lambda_2^2} Y^2 - \frac{\kappa}{\lambda_2^2} (U^2 + V^2) \\
Y' & = \lambda_2 Y + X^2 + \frac{2+\lambda_1}{\lambda_2^2} XY + \frac{1+\lambda_2}{\lambda_2^2} Y^2 + \kappa (U^2 + V^2) \\
U' & = -\hat{d}_{0,1} V + (\hat{C}_{1,1} U - \hat{D}_{1,1} V)(X + \frac{1}{\lambda_2} Y) \\
V' & = \hat{d}_{0,1} U + (\hat{D}_{1,1} U + \hat{C}_{1,1} V)(X + \frac{1}{\lambda_2} Y).
\end{align*}
\]

As usual, to make the computations easier, we introduce complex coordinates \(Z = U + iV\) and \(Z = U - iV\) to get
\[
\begin{align*}
X' & = -\frac{1}{\lambda_2} X^2 - \frac{2+\lambda_1}{\lambda_2^2} XY - \frac{1+\lambda_2}{\lambda_2^2} Y^2 - \frac{\kappa}{\lambda_2^2} Z\overline{Z} \\
Y' & = \lambda_2 Y + X^2 + \frac{2+\lambda_1}{\lambda_2^2} XY + \frac{1+\lambda_2}{\lambda_2^2} Y^2 + \kappa Z\overline{Z} \\
Z' & = \hat{d}_{0,1} iZ + (\hat{C}_{1,1} + i\hat{D}_{1,1}) XZ + \frac{1}{\lambda_2} (\hat{C}_{1,1} + i\hat{D}_{1,1}) YZ \\
\overline{Z}' & = -\hat{d}_{0,1} i\overline{Z} + (\hat{C}_{1,1} - i\hat{D}_{1,1}) X\overline{Z} + \frac{1}{\lambda_2} (\hat{C}_{1,1} - i\hat{D}_{1,1}) Y\overline{Z}.
\end{align*}
\]

To obtain the normal form up to second order we need to compute the action of the Lie bracket \([L, \bullet]\), where \(L\) is the linear vector field
\[L = \lambda_2 Y \frac{\partial}{\partial Y} + \hat{d}_{0,1} iZ \frac{\partial}{\partial Z} - \hat{d}_{0,1} iZ \frac{\partial}{\partial \overline{Z}}\]
over generic monomials. We obtain
\[
\begin{align*}
\left[ L, X^k Y^l Z^m \overline{Z}^n \right] \frac{\partial}{\partial X} & = -\left( \lambda_2 l + \hat{d}_{0,1} (m-n)i \right) X^k Y^l Z^m \overline{Z}^n \frac{\partial}{\partial X} \\
\left[ L, X^k Y^l Z^m \overline{Z}^n \right] \frac{\partial}{\partial Y} & = \left( \lambda_2 (1-l) - \hat{d}_{0,1} (m-n)i \right) X^k Y^l Z^m \overline{Z}^n \frac{\partial}{\partial Y} \\
\left[ L, X^k Y^l Z^m \overline{Z}^n \right] \frac{\partial}{\partial Z} & = -\left( \lambda_2 l + \hat{d}_{0,1} (m-n-1)i \right) X^k Y^l Z^m \overline{Z}^n \frac{\partial}{\partial X}.
\end{align*}
\]

Recall that the advantage of introducing complex coordinates is that the linear part corresponds to a diagonal matrix. Hence, as we can easily check with the above computations, each monomial is an eigenvector of the Lie bracket. All monomials associated to a non-zero eigenvalue are removable by a local change of coordinates. It easily follows that the normal form up to second order is given by
\[
\begin{align*}
X' & = -\frac{1}{\lambda_2} X^2 - \frac{\kappa}{\lambda_2^2} Z\overline{Z} \\
Y' & = \lambda_2 Y + \frac{2+\lambda_1}{\lambda_2^2} XY \\
Z' & = \hat{d}_{0,1} iZ + (\hat{C}_{1,1} + i\hat{D}_{1,1}) XZ \\
\overline{Z}' & = -\hat{d}_{0,1} i\overline{Z} + (\hat{C}_{1,1} - i\hat{D}_{1,1}) X\overline{Z}.
\end{align*}
\]

Note that \(Y = 0\) is an invariant manifold. In fact, taking \(Y = 0\) we obtain the reduction to the center manifold up to second order. Recovering variables \((X, U, V)\) we obtain
\[
\begin{align*}
X' & = -\frac{1}{\lambda_2} X^2 - \frac{\kappa}{\lambda_2^2} (U^2 + V^2) \\
U' & = -\hat{d}_{0,1} iV + \hat{C}_{1,1} XU - \hat{D}_{1,1} XV \\
V' & = \hat{d}_{0,1} iU + \hat{D}_{1,1} XU + \hat{C}_{1,1} XV.
\end{align*}
\]
Finally, introducing polar coordinates \((R, \Theta)\) such that \(U = R \cos \Theta\) and \(V = R \sin \Theta\) we get
\[
\begin{align*}
X' &= -\frac{1}{\lambda^2}X^2 - \frac{\kappa}{\lambda^2}R^2 \\
R' &= \tilde{C}_{1,1}XR \\
\Theta' &= \tilde{d}_{0,1} + \tilde{D}_{1,1}X.
\end{align*}
\]
Because \(\Theta' \neq 0\) for \(X\) small enough, we consider the decoupled systems for \(X'\) and \(R'\). After rescaling by \(\tilde{X} = X/\lambda^2\) and \(\tilde{R} = R/|\lambda|\), we obtain
\[
\begin{align*}
\tilde{X}' &= -\tilde{X}^2 - \kappa\tilde{R}^2 \\
\tilde{R}' &= \lambda^2\tilde{C}_{1,1}\tilde{X}\tilde{R}.
\end{align*}
\]

The classification of HZ unfoldings in [18] is based on the signs of the coefficients
\[-\kappa\quad\text{and}\quad\lambda^2\tilde{C}_{1,1}.
\]
The different cases are given in Table 1.

| \(\lambda^2\tilde{C}_{1,1}\) | \(\lambda^2\tilde{C}_{1,1} < 0\) | \(\lambda^2\tilde{C}_{1,1} > 0\) |
|---|---|---|
| \(\kappa = -1\) | Case II | Case I |
| \(\kappa = 1\) | Case IV | Case III |

**Table 1.** Cases of HZ bifurcation unfolded by a HBT singularity. With the appropriate choice for coefficients in the normal form of the HBT singularity, any case can be obtained. In Case III, it can be expected the emergence of chaotic behavior when higher order terms are considered.

**Remark 7.** In cases II and III the determinacy of the bifurcation diagram involves a generic condition stated in terms of third order terms in the normal form. Because it does not affect to our main result, computations of third order terms in the normal form are not included in this paper.

### 3.2. Hopf-Hopf bifurcations: Proof of Theorem 3.2.
First we translate the equilibrium point \(P^-\) to the origin introducing \(\hat{x} = x + \sqrt{-\lambda_1}\) and fix parameters on the Hopf-Hopf bifurcation curve, that is, we take \(\lambda_2 = -1\) and \(\mu = -\tilde{C}_{1,1}\). Hence family (3) reduces to
\[
\begin{align*}
\hat{x}' &= y \\
y' &= -2\lambda_2\hat{x} + \hat{x}^2 + \hat{x}y + k(u^2 + v^2) \\
u' &= -\left(\tilde{d}_{0,1} - \lambda_2\tilde{D}_{1,1}\right)v + \left(\tilde{C}_{1,1}u - \tilde{D}_{1,1}v\right)\hat{x} \\
v' &= \left(\tilde{d}_{0,1} - \lambda_2\tilde{D}_{1,1}\right)u + \left(\tilde{D}_{1,1}u + \tilde{C}_{1,1}v\right)\hat{x}
\end{align*}
\]
and introducing $\hat{y} = -\frac{1}{\sqrt{2\lambda_2}} y$ we obtain

\[
\begin{cases}
\dot{x}' = -\sqrt{2\lambda_2} \hat{y} \\
\dot{y}' = \sqrt{2\lambda_2} \hat{x} - \frac{1}{\sqrt{2\lambda_2}} \dot{x}^2 + \dot{x} \hat{y} - \frac{1}{\sqrt{2\lambda_2}} (u^2 + x^2) \\
u' = -(\hat{d}_{0,1} - \lambda_2 \hat{D}_{1,1}) v + (\hat{C}_{1,1} u - \hat{D}_{1,1} v) \dot{x}
\end{cases}
\]

As expected for a Lie bracket $[L, \bullet]$, all monomials are eigenvectors of the bracket and hence those associated to a non-zero eigenvalue can be removed by reduction to normal form. It easily follows that all second order terms can be removed and there only remain the third order monomials listed below.

\[ \{ Z^2 W \frac{\partial}{\partial Z}, Z W W \frac{\partial}{\partial Z}, Z W W \frac{\partial}{\partial W}, Z W W \frac{\partial}{\partial W}, W^2 W \frac{\partial}{\partial W}, Z Z W \frac{\partial}{\partial W}, Z Z W \frac{\partial}{\partial W}, \} \]
Introducing new complex coordinates $P = X + iY$ and $Q = U + iV$ with
\[
\begin{align*}
P &= Z - \frac{a_{2,0,0}}{\wp_{2,0,0}} Z^2 - \frac{a_{1,1,0}}{\wp_{1,1,0}} ZZ - \frac{a_{0,2,0}}{\wp_{0,2,0}} Z^2 - \frac{a_{0,0,1}}{\wp_{0,0,1}} W W \\
Q &= W - \frac{c_{0,1,1}}{\wp_{0,1,1}} Z W
\end{align*}
\]
we obtain the following normal form (truncated at order three)
\[
\begin{align*}
P' &= \alpha P + A_{2,1,0,0} P^2 P + A_{1,0,1,1} P Q Q \\
\bar{P}' &= -\alpha \bar{P} + A_{2,1,0,0} P \bar{P}^2 + A_{1,0,1,1} \bar{P} \bar{Q} Q \\
Q' &= \beta i Q + C_{0,0,2,1} Q^2 Q + C_{1,1,1,0} P \bar{P} Q \\
\bar{Q}' &= -\beta i \bar{Q} + C_{0,0,2,1} Q \bar{Q}^2 + C_{1,1,1,0} P \bar{P} \bar{Q}
\end{align*}
\]
with
\[
\begin{align*}
A_{2,1,0,0} &= \frac{1}{16\lambda_2} - \frac{5 + \lambda_2}{24\lambda_2^2} i \\
A_{1,0,1,1} &= \frac{k(1 - 2\tilde{C}_{1,1,1})}{4\lambda_2} - \frac{\kappa}{2\lambda_2^2} \frac{1}{i}
\end{align*}
\]
and
\[
\begin{align*}
C_{0,0,2,1} &= \frac{\kappa \tilde{C}_{1,1,1}}{2\lambda_2} + \frac{\kappa \bar{D}_{1,1,1}}{2\lambda_2} i \\
C_{1,1,1,0} &= \frac{\tilde{C}_{1,1,1}}{4\lambda_2} + \frac{\bar{D}_{1,1,1}}{4\lambda_2} i.
\end{align*}
\]

**Remark 8.** A number of calculations has been intentionally omitted. To get the third order normal form, it is necessary to compute the inverse of the change of variables (4) up third order. This is a tedious task that we have done using Matlab. Of course, the code is available from the authors on request.

Recovering variables $(X, Y, U, V)$, we obtain
\[
\begin{align*}
X' &= -\alpha Y + (\Re(A_{2,1,0,0}) X - \Im(A_{2,1,0,0}) Y) (X^2 + Y^2) \\
&\quad + (\Re(A_{1,0,1,1}) X - \Im(A_{1,0,1,1}) Y) (U^2 + V^2) \\
Y' &= \alpha X + (\Im(A_{2,1,0,0}) X + \Re(A_{2,1,0,0}) Y) (X^2 + Y^2) \\
&\quad + (\Im(A_{1,0,1,1}) X + \Re(A_{1,0,1,1}) Y) (U^2 + V^2) \\
U' &= -\beta V + (\Re(C_{0,0,2,1}) U - \Im(C_{0,0,2,1}) V) (U^2 + V^2) \\
&\quad + (\Re(C_{1,1,1,0}) U - \Im(C_{1,1,1,0}) V) (X^2 + Y^2) \\
V' &= -\beta V + (\Im(C_{0,0,2,1}) U + \Re(C_{0,0,2,1}) V) (U^2 + V^2) \\
&\quad + (\Im(C_{1,1,1,0}) U + \Re(C_{1,1,1,0}) V) (X^2 + Y^2).
\end{align*}
\]

Finally, introducing polar coordinates $(R_1, \Theta_1)$ and $(R_2, \Theta_2)$ with $X = R_1 \cos \Theta_1$, $Y = R_1 \sin \Theta_1$, $U = R_2 \cos \Theta_2$ and $V = R_2 \sin \Theta_2$, we get
\[
\begin{align*}
R_1' &= \Re(A_{2,1,0,0}) R_1^3 + \Re(A_{1,0,1,1}) R_1 R_2^2 \\
R_2' &= \Re(C_{1,1,1,0}) R_1^2 R_2 + \Re(C_{0,0,2,1}) R_2^3 \\
\Theta_1' &= \alpha + \Im(A_{2,1,0,0}) R_1^2 + \Im(A_{1,0,1,1}) R_2^2 \\
\Theta_2' &= \beta + \Im(C_{1,1,1,0}) R_1 R_2 + \Im(C_{0,0,2,1}) R_2^2.
\end{align*}
\]
Because $\Theta_1' > 0$ and $\Theta_2' > 0$ for $R_1$ and $R_2$ small enough, following [18], we ignore the variation of $\Theta_1$ and $\Theta_2$ and consider the decoupled systems for $R_1'$ and $R_2'$. We
Table 2. Cases of HH bifurcation unfolded by a HBT singularity. In Case VIa, it can be expected the emergence of chaotic behavior when higher order terms are considered.

| $\kappa = -1$ | Case IVb | Case VIIa | Case VIIb | Case V |
|--------------|----------|-----------|-----------|-------|
| $\kappa = 1$ | Case VIa | Case Ib   | Case Ia   | Case III |

Using the expressions in (5) and (6), we obtain

\[
\begin{align*}
    b &= \Re(A_{1,0,1,1}) / \Re(C_{0,0,2,1}), \\
    c &= \Re(C_{1,1,1,0}) / \Re(C_{0,2,0,1}), \\
    d &= \sign(\Re(C_{0,0,2,1})).
\end{align*}
\]

Using the expressions in (5) and (6), we obtain

\[
\begin{align*}
    b &= \kappa(1 - 2\hat{C}_{1,1}) / 2|\hat{C}_{1,1}|, \\
    c &= 4\hat{C}_{1,1}, \\
    d &= \sign(\kappa\hat{C}_{1,1}).
\end{align*}
\]

For the classification, we also need to pay attention to the following value

\[
\Delta = d - bc = \sign(\kappa\hat{C}_{1,1})(4\hat{C}_{1,1} - 1).
\]

Attending to the classification in [18, Table 7.5.2], which depend on the signs of $b$, $c$, $d$ and $\Delta$, we obtain eight of the twelve possible cases for the codimension two Hopf-Hopf bifurcation: see Table 2. As argued in [18], HH unfoldings in case VIa may exhibit chaotic dynamics.

4. Numerical simulations: Chaotic behavior. Introducing polar coordinates $(r, \theta)$ such that $u = r \cos \theta$ and $v = r \sin \theta$, we obtain from (3)

\[
\begin{align*}
    x' &= y, \\
    y' &= \lambda_1 + x^2 + \lambda_2 y + xy + kr^2, \\
    r' &= \mu r + \hat{C}_{1,1} x r, \\
    \theta' &= \hat{d}_{0,1} + \hat{D}_{1,1} x.
\end{align*}
\]

and, because $\hat{d}_{0,1} \neq 0$ when $(\lambda_1, \lambda_2, \mu)$ is close enough to $(0, 0, 0)$, the angular component can be decoupled to obtain the following reduced family of three-dimensional vector fields

\[
\begin{align*}
    x' &= y, \\
    y' &= \lambda_1 + x^2 + \lambda_2 y + xy + kr^2, \\
    r' &= \mu r + \hat{C}_{1,1} x r.
\end{align*}
\] (7)

In this section, we provide illustrations of the chaotic behavior exhibited by family (7) when parameters are chosen close to type III Hopf-Zero and type VIa Hopf-Hopf bifurcations. Although numerical simulations are shown for family (7), bifurcations diagrams correspond to family (3). Attractors in the four-dimensional phase-space
corresponding to family (3) are obtained from those exhibited in Figures 3 and 5 by rotating around \( u = v = 0 \).

4.1. Chaotic behavior close to the Hopf-Zero bifurcation point in Case IV. We consider family (7) with \( \hat{C}_{1,1} = -1 \) and \( \kappa = 1 \). Moreover, we fix \( \lambda_2 = -0.1 \). It follows from Theorem 3.1 that, with such assumptions and when \( \lambda_1 = \mu = 0 \), family (3) exhibits a HZ singularity corresponding to Case III.

![Figure 2](image)

**Figure 2.** Green line corresponds to the segment of parameters where simulations in Figure 3 have been done. Note that we start after the birth of an invariant torus when parameters cross the secondary Hopf bifurcation. This bifurcation curve has been obtained using Matcont [13].

Figure 2 provides a partial bifurcation diagram (fixing \( \lambda_2 \)) of family (3) close to the HZ point. There is a saddle node bifurcation along the line \( \lambda_1 = 0 \). When \( \lambda_1 < 0 \), we find the equilibrium points \( P_{\pm} = (\pm\sqrt{-\lambda_1}, 0, 0, 0) \). There is a Hopf bifurcation at \( P_- \) along the curve \( H_2 \) and a limit cycle appears. Such limit cycle undergoes through a secondary Hopf bifurcation (or Neimark-Sacker bifurcation) producing an invariant torus. Our numerical simulations start on the left-hand side of this bifurcation curve (see Figure 2).

The description below follows the plots in Figure 3 from top to bottom and from left to right. Note that we have fixed \( \mu = -0.3 \) and \( \lambda_1 \) varies in the interval \([-0.326, -0.291]\). When \( \lambda_1 = -0.291 \), a limit cycle arises (it corresponds to an invariant torus when we consider family (3)). Orbits for \( \lambda_1 = -0.310, \lambda_1 = -0.313 \) and \( \lambda_1 = -0.315 \) show a sequence of period-doubling bifurcations, and a strange attractor is detected when \( \lambda_1 = -0.319 \) (the maximal Lyapunov exponent is 0.042). After a transition through chaotic behavior, we detect a window of periodicity: a periodic orbit of period 5 is found when \( \lambda_1 = -0.321 \). After that, chaos emerges again: the maximal Lyapunov exponents are 0.048 and 0.052 when \( \lambda_1 = -0.322 \) and \( \lambda_1 = -0.326 \), respectively.
Figure 3. Examples of period-doubling and chaotic dynamics close to a Hopf-Zero bifurcation. In all cases, we have $\lambda_2 = -0.1$, $\mu = -0.3$ and $\tilde{C}_{1,1} = -1$. Initial conditions are $x = y = u = v = 0.04$. 

$\lambda_1 = -0.291$  
$\lambda_1 = -0.310$

$\lambda_1 = -0.313$  
$\lambda_1 = -0.315$

$\lambda_1 = -0.319$  
$\lambda_1 = -0.321$

$\lambda_1 = -0.322$  
$\lambda_1 = -0.326$
4.2. Chaotic behavior close to the Hopf-Hopf bifurcation point in Case IV. We consider family (7) with $C_{1,1} = -1$ and $\kappa = 1$. Moreover, we fix $\lambda_2 = 0.21$. It follows from Theorem 3.2 that, with such assumptions and when $\lambda_1 = -\lambda_2^2$ and $\mu = \sqrt{-\lambda_1}$, family (3) exhibits a HH singularity corresponding to Case VIa.

![Numerical Simulation](image)

**Figure 4.** Red line corresponds to the segment of parameters where simulations in Figure 5 have been done. Note that we start after the birth of an invariant torus when parameters cross the secondary Hopf bifurcation curve. This bifurcation curve has been obtained using Matcont [13].

Figure 4 provides a partial bifurcation diagram (fixing $\lambda_2$) of family (3) close to HZ point. There is a Hopf bifurcation at the point $P^- \lambda_1 (\lambda_1 = -\lambda_2^2)$. There is also another Hopf bifurcation at $P^- \mu = \sqrt{-\lambda_1}$. Moreover, we observe a secondary Hopf bifurcation (or Neimark-Sacker bifurcation) which produces an attracting invariant torus. Numerical simulations start from the left-hand side of this bifurcation curve (see Figure 4).

The description below follows the plots in Figure 5 from top to bottom and from left to right. Note that we have fixed $\mu = -0.3$ and $\lambda_1$ varies in the interval $[-0.2129794796, -0.2129725000]$. When $\lambda_1 = -0.2129725000$, a limit cycle is shown (it corresponds to an invariant torus when we consider family (3)). The orbit shown for $\lambda_1 = -0.2129725798$ has already undergone through several period doubling bifurcations. After the cascade of period doubling, we detect strange attractors. The maximal Lyapunov exponents for the orbits shown when $\lambda_1 = -0.212972959$ and $\lambda_1 = -0.2129794796$ are positive (0.011 and 0.012, respectively).

5. **Conclusions.** In this paper, the first steps of what will be a long term task to get a deep understanding of the bifurcation diagram of a HBT singularity are given. The possibility of displaying singularities of codimension two which are the germ of chaotic behaviors and also the numerical simulations presented here guarantee that the unfoldings of HBT singularities are going to exhibit an extraordinary richness.
Before tackling an analytical study, it would be good to use numerical continuation techniques to explore in more detail the bifurcation diagram of the truncated family. Particularly, it is crucial to know how all the bifurcation surfaces arising at the bifurcation curves of codimension two glue together.

Also, as usual when working with unfoldings of singularities from a theoretical point of view, it will be necessary to consider appropriate rescalings in the general normal form. The goal is to express any arbitrary generic unfolding as the perturbation of a limit family for which, hopefully, it is possible to understand its dynamics and, in addition, such knowledge allows to draw conclusions about the perturbed system.

![Diagram](image)

**Figure 5.** Examples of period-doubling and chaotic dynamics close to a Hopf-Hopf bifurcation. In all cases, we have $\lambda_2 = 0.21$, $\mu = -0.3$ and $\hat{C}_{1,1} = -1$. Initial conditions are $x = y = u = v = 0.04$.

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