Removing Type II singularities off the axis for the 3D axisymmetric Euler equations

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Abstract

In this paper we obtain new local blow-up criterion for smooth axisymmetric solutions to the 3D incompressible Euler equation. If the vorticity satisfies

\[ \int_0^{t^\ast} (t^\ast - t) \| \omega(t) \|_{L^\infty(B(x^\ast, R_0))} dt < +\infty \]

for a ball \( B(x^\ast, R_0) \) away from the axis of symmetry, then there exists no singularity at \( t = t^\ast \) in the torus \( T(x^\ast, R) \) generated by rotation of the ball \( B(x^\ast, R_0) \) around the axis. This implies that possible singularity at \( t = t^\ast \) in the torus \( T(x^\ast, R) \) is excluded if the vorticity satisfies the blow-up rate \( \| \omega(t) \|_{L^\infty(T(x^\ast, R))} = O\left( \frac{1}{(t^\ast - t)^{\gamma}} \right) \) as \( t \to t^\ast \), where \( \gamma < 2 \), and the torus \( T(x^\ast, R) \) does not touch the axis.

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1 Introduction

We consider the Euler equations in the domain \( \mathbb{R}^3 \times (0, +\infty) \)

\[ \begin{aligned}
\partial_t v + (v \cdot \nabla)v &= -\nabla p, \\
\nabla \cdot v &= 0, \\
v(x, 0) &= v_0(x) \quad \forall x \in \mathbb{R}^3
\end{aligned} \]  

(1.1)

where \( v = (v_1(x, t), v_2(x, t), v_3(x, t)) \) is the velocity of the fluid, and \( p = p(x, t) \) represents the pressure. Here, \( v_0 \) stands for initial velocity. Given \( v_0 \in W^{k,q}(\mathbb{R}^3) \), \( k > 3/q+1 \), \( 1 < q < +\infty \), the local in time well-posedness is well-known due to Kato and Ponce\[19\]. The question of spontaneous apparition of singularity in finite time, however, is an
outstanding open problem in the mathematical fluid mechanics (see e.g.\cite{23, 14, 1} for surveys of the problem and the related results). There are also many numerical experiments on the problem (see e.g. \cite{25, 20, 17, 24, 3, 22}).

We say a local in time smooth solution \( v \in C([0, t_\ast); W^{k,q}(\mathbb{R}^3)) \), \( k > \frac{3}{q} + 1 \), \( 1 < q < +\infty \), blows up (or equivalently becomes singular) at \( t = t_\ast \) if

\[
\limsup_{t \to t_\ast} \| v(t) \|_{W^{k,q}(\mathbb{R}^3)} = +\infty \quad \forall k \geq \frac{3}{q} + 1.
\]

The celebrated Beale-Kato-Majda (BKM) criterion \cite{2} says that (1.2) holds if and only if

\[
\int_0^{t_\ast} \| \omega(t) \|_{L^\infty(\mathbb{R}^3)} \, dt = +\infty, \quad \omega = \nabla \times v.
\]

See also \cite{14, 16} for geometric type criteria. Later, Kozono and Taniuchi \cite{21} improved (1.3), replacing \( \| \omega(t) \|_{L^\infty(\mathbb{R}^3)} \) in (1.3) by a weaker norm \( \| \omega(t) \|_{BMO(\mathbb{R}^3)} \). In a very recent paper authors of current paper obtained a localized version of the criterions of \cite{2, 21}, which says that in order to check the blow-up for the solution at particular space-time point \((x_\ast, t_\ast) \in \mathbb{R}^3 \times (0, +\infty)\), it suffices to check if there exists \( r > 0 \) such that

\[
\int_0^{t_\ast} \| \phi(t) \|_{L^\infty(B(x_\ast, r))} \, dt = +\infty,
\]

where and hereafter we use the notation \( B(x, r) = \{ y \in \mathbb{R}^n \mid |x - y| < r \} \), and \( B(r) := B(0, r) \) (In fact, we derived the localized criterion with \( \| \phi(t) \|_{L^\infty(B(x_\ast, r))} \) of (1.4) replaced by a weaker norm \( \| \omega(t) \|_{BMO(B(x_\ast, r))} \)). Note that the local BKM criterion does not rule out the scenario of the blow-up rate

\[
\limsup_{t \to t_\ast} (t_\ast - t)^\gamma \| \phi(t) \|_{L^\infty(B(x_\ast, r))} < +\infty
\]

for some \( x_\ast \in \mathbb{R}^3, r > 0 \), and for \( \gamma \geq 1 \), which includes the case

\[
\| \phi(t) \|_{L^\infty(B(x_\ast, r))} = O \left( \frac{1}{(t_\ast - t)^\gamma} \right) \quad \text{as} \quad t \to t_\ast.
\]

In this paper we study the finite time blow-up problem of system (1.1) under the assumption of axial symmetry. In this case also there are many previous studies from theoretical (e.g. \cite{3, 6, 7, 8, 9, 15}) and computational (see \cite{22, 25} and the references therein) aspects respectively. Our purpose is to obtain a local BKM criterion (1.3) for the axisymmetric Euler equation off the axis region, which implies that we can rule out the singularity having the blow-up rate including (1.5) and (1.6) with \( \gamma < 2 \) at the region. We say the vorticity \( \phi \) has Type I blow-up at \( t_\ast \) in a domain \( \mathcal{D} \subset \mathbb{R}^3 \) if

\[
\limsup_{t \to t_\ast} (t_\ast - t)^\gamma \| \phi(t) \|_{L^\infty(\mathcal{D})} < +\infty.
\]

Otherwise, we say it is of Type II. For comparison we recall the status of the Type I blow-up problem in the axisymmetric Navier-Stokes equations. Due to the the well-known result of Caffarelli-Kohn-Nirenberg [4] for a suitable weak solutions to the 3D
Navier-Stokes equations the one-dimensional Hausdorff measure of the possible singular set must be zero, and therefore all possible singularities should be on the axis of symmetry. On the other hand, according to the works of [26] and [12] independently there exists no Type I singularity on the axis of symmetry. For the case of our axisymmetric Euler equations there exists no available partial regularity type results similar to the Navier-Stokes equations, and therefore we cannot rule out singularity off the axis by an simple argument. Our main result, Theorem 1.1 (and its immediate consequence Theorem 1.2) shows that there exists no Type I (and also part of Type II) singularity off the axis region.

Below we briefly introduce the notion of axisymmetric flow, which is found in standard literature (e.g. [23]). We say \( v \) is an axisymmetric solution of the Euler equations if \( v \) solves (1.1), and can be written as

\[
v = v^r(r, x_3, t)e_r + v^\theta(r, x_3, t)e_\theta + v^3(r, x_3, t)e_3,
\]

where

\[
e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(\frac{x_2}{r}, -\frac{x_1}{r}, 0\right), \quad e_3 = (0, 0, 1), \quad r = \sqrt{x_1^2 + x_2^2}.
\]

are the canonical basis of the cylindrical coordinate system. The Euler equations (1.1) for an axisymmetric solution turn into the following equations.

\[
\begin{align*}
\partial_t v^r + v^r \partial_r v^r + v^3 \partial_3 v^r &= -\partial_r p + \frac{(v^\theta)^2}{r}, \\
\partial_t v^\theta + v^r \partial_r v^\theta + v^3 \partial_3 v^\theta &= -\frac{v^r v^\theta}{r}, \\
\partial_t v^3 + v^r \partial_r v^3 + v^3 \partial_3 v^3 &= -\partial_3 p, \\
\partial_r (rv^r) + \partial_3 (rv^3) &= 0.
\end{align*}
\]

Multiplying (1.9) by \( r \), we see that \( rv^\theta \) satisfies the transport equation

\[
\partial_t (rv^\theta) + v^r \partial_r (rv^\theta) + v^3 \partial_3 (rv^\theta) = 0.
\]

For the vorticity \( \omega = \nabla \times v \) we get

\[
\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3,
\]

where

\[
\omega^r = -\partial_3 v^\theta, \quad \omega^\theta = \partial_3 v^r - \partial_r v^3, \quad \omega^3 = \frac{v^\theta}{r} + \partial_r v^\theta.
\]

Applying \( \partial_3 \) to (1.8) and applying \( \partial_r \) to (1.11), and taking the difference of the two equations, we obtain the following equation for \( \omega^\theta \)

\[
\partial_t \omega^\theta + v^r \partial_r \omega^\theta + v^3 \partial_3 \omega^\theta = \frac{v^r \omega^\theta}{r} + \frac{\partial_3 (v^\theta)^2}{r}.
\]
This leads to the equation
\[
(1.14) \quad \partial_t \left( \frac{\omega^\theta}{r} \right) + v^r \partial_r \left( \frac{\omega^\theta}{r} \right) + v^3 \partial_3 \left( \frac{\omega^\theta}{r} \right) = \frac{\partial_3 (v^\theta)^2}{r^2}.
\]

Our first main theorem is the following improvement of the BKM theorem off the axis region.

**Theorem 1.1.** Let \( v \in C([0, t_*]; \dot{W}^{2, q}(\mathbb{R}^3)) \cap L^\infty(0, t_*; L^2(\mathbb{R}^3)), 3 < q < +\infty \), be an axisymmetric solution to (1.1) in \( \mathbb{R}^3 \times (0, t_*) \). If the following condition is fulfilled
\[
(1.15) \quad \int_0^{t_*} (t_* - t) \| \omega(t) \|_{L^\infty(B(x_*, R_0))} dt < +\infty,
\]
for some ball \( B(x_*, R_0) \subset \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 > 0 \} \), where \( \omega = \nabla \times v \). Then for all \( 0 < R < R_0 \) it holds \( v \in C([0, t_*], W^{2, q}(B(x_*, R))) \). In particular, this implies \( v \in C([0, t_*], W^{2, q}(T(x_*, R))) \). Here, \( T(x_*, R) \) stands for the torus generated by rotation of \( B(x_*, R_0) \) around the axis, i.e.
\[
T(x_*, R) = \{ x \in \mathbb{R}^3 \mid \left( \sqrt{x_1^2 + x_2^2} - \rho_* \right)^2 + (x_3 - x_3^*)^2 < R^2 \},
\]
where \( \rho_* = \sqrt{x_1^{2*} + x_2^{2*}}. \)

As an immediate consequence of this theorem we remove some of Type II as well as Type I singularities in terms of the vorticity blow-up rate off the axis. We have the following:

**Theorem 1.2.** Let \( v \in C([0, t_*]; \dot{W}^{2, q}(\mathbb{R}^3)) \cap L^\infty(0, t_*; L^2(\mathbb{R}^3)), 3 < q < +\infty \), be an axisymmetric solution to (1.1) in \( \mathbb{R}^3 \times (0, t_*) \). Suppose the following vorticity blow-up rate condition holds
\[
(1.16) \quad \sup_{t \in (0, t_*)} (t_* - t)^2 \left[ \log \left( \frac{1}{t_* - t} \right) \right]^\alpha \| \omega(t) \|_{L^\infty(B(x_*, R_0))} < +\infty
\]
for some \( \alpha > 1 \) and some ball \( B(x_*, R_0) \subset \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 > 0 \} \). Then \( v \in C([0, t_*]; W^{2, q}(T(x_*, R))) \) for all \( 0 < R < R_0 \).

**Remark 1.3.** In particular, Theorem 1.2 says that there exists no singularity at \( t = t_* \) in the off-the-axis region if the the vorticity blow-up rate satisfies
\[
(1.17) \quad \| \phi(t) \|_{L^\infty(\mathbb{R}^3)} = O \left( \frac{1}{(t_* - t)^\gamma} \right),
\]
as \( t \to t_* \) if \( 1 \leq \gamma < 2 \). Due to the global BKM criterion, however, the singularity in this case should happen only on the axis. It would be interesting to compare this result with Tao’s construction of a singular solution (see [27], Fig. 3, pp.18) for a modified Euler system, where \( \gamma = 1 \) and the set of singularity is a circle around the axis.
Remark 1.4. In a recent numerical study of the blow-up of the axisymmetric Euler equations by Luo and Hou [22, pp.1766] they computed $\gamma = 2.45$ in (1.17), which is consistent with our rigorous result, since their blow-up region is away from the axis, and near the boundary of the cylinder. As far as the authors know, this is the first explicit computational result with $\gamma \geq 2$ for singularity in the axisymmetric Euler equations.

The key observation in the proof of Theorem 1.1 is that off the axis region, as is well-known, the 3D Euler system behaves like a solution to the 2D Boussinesq system, the scaling property of which is different from the 3D Euler equations, and our careful local analysis takes full advantage of this property. For the proof we introduce a new iteration scheme, the corresponding iteration lemma of which is proved in the appendix B, and also we use a local version of the logarithmic Sobolev inequality proved in [10].

We note also that our local regularity criterion off the axis region, is not an immediate consequence of the corresponding local criterion for the 2D Boussinesq system. We need to show further that $t^* \int_0^{t_*} \|v(t)\|_{L^\infty(B(R,x_0))} dt < +\infty$ is satisfied under our hypothesis for the 3D axisymmetric case, which is accomplished by using the estimate of extra component of vorticity, $\omega^\theta$ together with the local version of the Calderón-Zygmund inequality.

At this moment we do not know whether Theorem 1.1 continues to hold if we impose the condition (1.15) only for two components of vorticity

$$\vec{\omega} := -\partial_3 v^\theta e_r + (\partial_r v^\theta + \frac{v^\theta}{r})e_3.$$ 

However, a similar statement of Theorem 1.1 holds, if we replace the condition on $\omega^\theta$ by a corresponding condition on $v^r$. More precisely, we have the following:

**Theorem 1.5.** Let $v \in C([0,t_*]; W^{2,q}(\mathbb{R}^3)) \cap L^\infty(0,t_*; L^2(\mathbb{R}^3))$, $2 < q < +\infty$, be an axisymmetric solution to (1.1) in $\mathbb{R}^3 \times (0,t_*)$. If the following two conditions are fulfilled

$$\int_0^{t_*} (t_* - t) \|\vec{\omega}(t)\|_{L^\infty(\{r>R\})} dt < +\infty, \quad \int_0^{t_*} \|v^r(t)\|_{L^\infty(\{r>R\})} dt < +\infty$$

for all $0 < R < +\infty$. Then $v \in C([0,t_*]; W^{2,q}_{loc}(\mathbb{R}^3 \setminus \{r = 0\}))$.

**Remark 1.6.** Observing that in view of (1.12) $r v^\theta$ is bounded in $\mathbb{R}^2_+$ due to its conservation along the particle trajectories generated by

$$\vec{v} = v^r e_r + v^3 e_3,$$

it is immediately clear that (1.18) can be replaced by

$$\int_0^{t_*} (t_* - t) \|
abla v^\theta(t)\|_{L^\infty(\{r>R\})} dt < +\infty.$$
The organization of this paper is as follows:

In Section 2 we introduce a generalized 2D Boussinesq system, which includes the 3D axisymmetric Euler system off the axis and the standard 2D Boussinesq system as special cases. For such system we prove a local blow-up criterion, Theorem 2.1, where certain integrability condition of ‘temperature’ gradient together with the velocity integrability leads to a local non blow-up. Establishing this theorem is a major part for the proof of the above main theorems. To prove Theorem 2.1 this we transform the equations from the generalized Boussineq system for \((u, \theta, w)\) into a new system for \((U, \Theta, \Omega)\). In order to deduce \(W^{2,q}_{loc}, q > 3\), estimates for the transformed functions \((U, \Theta)\), we need to handle differential inequality for integrals on different balls in the left and the right hand sides. Appearance of these different balls is originated from the use of cut-off functions, necessary for localizations, which is similar to the case of deriving the Caccioppoli type inequalities in the elliptic equations. In this case we cannot use classical Gronwall’s inequalities to close the differential inequality. To overcome this difficulty we introduce a new of type iteration scheme, and close the differential inequality by iterating a suitable sequence of differential inequalities.

In Section 3, using Theorem 2.1 of Section 2, we prove Theorem 1.1, where we only need to verify the integrability condition of the velocity of Theorem 2.1 is satisfied. Using the fact that the vorticity of the 3D axisymmetric flow has an extra component \(\omega^\theta\) other than \(\tilde{\omega}\), the integrability of the velocity part can be proved. The proof of Theorem 1.2 is an easy consequence of Theorem 1.1.

In Section 4 we prove Theorem 1.5, where the integrability condition of the radial velocity (1.18)_{2 allows us to introduce the functional transform similarly to Section 2. Using this transform, one can show that the integrability condition of \(\nabla v^\theta\) implies the desired estimate of \(\omega^\theta\).

In Appendix A we prove various Gagliardo-Nirenberg type interpolation inequalities involving cut-off functions, which are necessary for the proof in Section 2. These include an improvement to local \(BMO\) norm from \(L^\infty\) norm in the known inequalities.

In Appendix B we prove a Gronwall type iteration lemma, which is crucial to complete our iteration scheme in the proof of Theorem 2.1 in Section 2.

For simplicity of presentation and its proof below we consider the Euler system on the time interval \((-1, 0)\), where \(t = -1\) is the initial time and \(t_* = 0\) is the possible blow-up time.

# 2 Local BKM type criterion for a generalized 2D Boussinesq system

The aim of this section is to prove a local regularity criterion of a generalized 2D Boussinesq equations, which include the 3D axisymmetric system as a special case.
Let $B(1)$ be the unit ball in $\mathbb{R}^2$. We consider the following system on $B(1) \times (-1, 0)$,

\begin{align}
(2.1) \quad & \partial_t \theta + a(y) u \cdot \nabla \theta = 0, \\
(2.2) \quad & \partial_t w + a(y) u \cdot \nabla w = b(y) \cdot \nabla \theta, \\
(2.3) \quad & \nabla \cdot u = 0,
\end{align}

where $u = (u_1(y_1, y_2, t), u_2(y_1, y_2, t))$, $\theta = \theta(y_1, y_2, t)$, and $u$ and $w$ are related by

\begin{equation}
(2.4) \quad \text{curl } u := \partial_2 u_1 - \partial_1 u_2 = d(y) w + e(y) \cdot u.
\end{equation}

Here, $a, b, d$ and $e$ are coefficients in $C^2(B(1))$. Note that the system (1.12), (1.14) and (1.11) reduces to the system (2.1)-(2.4) with identification $(r, x_3) := (y_1 + 2, y_2)$, $u := (r v^r, r v^3)$, $\theta := (r v^\theta)^2$, $w := e^r$ for the coefficient functions $a(y) = 1/(y_1 + 2)$, $b(y) = (0, 1/(y_1 + 2)^4)$, $d(y) = (y_1 + 2)^2$, and $e(x) = (0, -1/(y_1 + 2))$.

Certainly, the above system covers the vorticity formulation of the standard Boussinesq system in $\mathbb{R}^2$. Namely, if $(u, p, \theta)$ solves the Boussinesq system in $\mathbb{R}^2$, then $(u, w, \theta)$ with $w = \partial_2 v_1 - \partial_1 v_2$ solves (2.1)-(2.4) with $a = d \equiv 1, b \equiv -e_2$, and $e \equiv 0$. Thus, the following result also applies to the 2D Boussinesq system. Sufficient conditions for global regularity of solutions to the Boussinesq system in whole $\mathbb{R}^2$ has been proved in [11]. Here we present the following local regularity condition for the system (2.1)-(2.4).

**Theorem 2.1.** Let $B(1) \subset \mathbb{R}^2$ be the unit ball, and $a, b, d, e \in C^2(B(1))$ be the coefficients of the system (2.1)-(2.4). Let $2 < q < +\infty$, and

\begin{align}
(u, \theta) & \in C([-1, 0); W^{2, q}(B(1))) \times C([-1, 0); W^{2, q}(B(1))), \\
w & \in C([-1, 0); W^{1, q}(B(1)))
\end{align}

be a solution to (2.1)-(2.4). Suppose that

\begin{equation}
(2.5) \quad u \in C_w([-1, 0]; L^2(B(1)))
\end{equation}

and

\begin{equation}
(2.6) \quad \int_{-1}^{0} \|\nabla \theta(t)\|_{L^\infty(B(1))} dt < +\infty, \quad \int_{-1}^{0} \|u(t)\|_{L^\infty(B(1))} dt < +\infty.
\end{equation}

Then, $u, \theta \in C([-1, 0]; W^{2, q}(B(r)))$ for all $0 < r < 1$.

**Remark 2.2.** As is shown in the next section, the statement of Theorem 1.1 reduces to the statement of Theorem 2.1 by means of a suitable transformation, where the condition (2.5) is satisfied by the global assumption $v \in L^\infty(-1, 0; L^2(\mathbb{R}^3))$ of the solution to the 3D Euler equations (1.1).

For the proof of Theorem 2.1 we will make use of the following Lemma.
Lemma 2.3. Under the assumption of Theorem 2.1 it holds for every $0 < r < 1$

\[(2.7) \quad \int_{-1}^{0} \|w(t)\|_{L^\infty(B(r))} dt < +\infty,\]

\[(2.8) \quad \limsup_{t \to 0^-} (-t)\|w(t)\|_{L^\infty(B(r))} = 0,\]

\[(2.9) \quad \sup_{t \in (-1, 0)} \|\theta(t)\|_{L^\infty(B(r))} < +\infty.\]

\textbf{Proof:} Let $\frac{1}{2} < r < 1$ be fixed. Due to (2.6) we may choose $-1 < t_0 < 0$ such that

\[(2.10) \quad \int_{t_0}^{0} \|u(s)\|_{L^\infty(B(1))} ds \leq \frac{1}{40a_0} \frac{1 - r}{1 + r},\]

where $a_0 = \max_{y \in B(1)} |a(y)|$. Let $r_0 = \frac{1 - r}{2}$. For $(y, t) \in B(r_0) \times (t_0, 0)$ we define the coordinate transformation $(y, t) \mapsto (\tilde{y}(y, t), t)$ by

\[\tilde{y}(y, t) = \varrho(t)y \quad \text{with} \quad \varrho(t) := 1 + 20a_0 \int_{t}^{0} \|u(s)\|_{L^\infty(B(1))} ds.\]

One can check easily that

\[\tilde{y}(y, t) \in B(1) \quad \forall (y, t) \in B(r_0) \times (t_0, 0),\]

\[\partial_t \tilde{y}(y, t) = -20a_0\|u(t)\|_{L^\infty(B(1))} y.\]

Under the above coordinate transform we set

\[U(y, t) = u(\tilde{y}(y, t), t),\]

\[\Theta(y, t) = \theta(\tilde{y}(y, t), t),\]

\[\Omega(y, t) = w(\tilde{y}(y, t), t), \quad (y, t) \in B(r_0) \times (t_0, 0).\]

In addition, define

\[(2.11) \quad W(y, t) = \frac{20a_0\|u(t)\|_{L^\infty(B(1))} y + a(\tilde{y}(y, t))U(y, t)}{\varrho(t)}, \quad (y, t) \in B(r_0) \times (t_0, 0).\]

We claim that

\[(2.12) \quad y \cdot W(y, t) \geq \frac{a_0}{8}\|u(t)\|_{L^\infty(B(1))} \quad \forall (y, t) \in \overline{B(r_0)} \setminus B(r/2) \times (t_0, 0).\]

We now estimate

\[\varrho(t)y \cdot W(y, t) = 20a_0\|u(t)\|_{L^\infty(B(1))} |y|^2 + a(\tilde{y}(y, t))y \cdot U(y, t)\]

\[\geq \frac{5}{4}a_0\|u(t)\|_{L^\infty(B(1))} - a_0|U(y, t)|\]

\[\geq \frac{1}{4}(5a_0\|u(t)\|_{L^\infty(B(1))} - 4a_0\|u(t)\|_{L^\infty(B(1))})\]

\[= \frac{a_0}{4}\|u(t)\|_{L^\infty(B(1))}.\]
From (2.10) we find \( \varrho(t) \leq \frac{t+\varrho}{2+\varrho} \leq \frac{4}{7} \), and therefore (2.12) follows.

Using the chain rule, we see that (2.2) turns into the following equations hold in \( B(r_0) \times (t_0, 0) \)

\[
(2.13) \quad \partial_t \Omega + W \cdot \nabla \Omega = \frac{b(\tilde{\varrho}(y,t))}{\varrho(t)} \cdot \nabla \Theta.
\]

Given \((y, s) \in B(r_0) \times (t_0, t)\), we denote by \( X(\cdot) = X(y, s; \cdot) : [t_0, s] \rightarrow \mathbb{R}^2 \) the trajectory such that

\[
(2.14) \quad \dot{X}(\tau) = W(X(\tau), \tau), \quad X(s) = y.
\]

We claim that \( X(\tau) \in B(r_0) \) for all \( \tau \in [t_0, s] \). Otherwise there exists \( \tau_0 \in [t_0, s] \) such that \( X(\tau_0) \in \partial B(r_0) \times (t_0, 0) \) and \( X(\tau) \in B(r_0) \) for all \( \tau \in (\tau_0, s) \). In particular, the function \( \tau \mapsto |X(\tau)|^2 \) is non increasing at \( \tau = \tau_0 \). This gives

\[
0 \geq \frac{d}{d\tau}|X(\tau)|^2|_{\tau=\tau_0} = 2X(\tau_0) \cdot \dot{X}(\tau_0) = 2X(\tau_0) \cdot W(X(\tau_0), \tau_0),
\]

which contradicts (2.12). On the other hand, by the chain rule (2.13) gives

\[
\frac{d}{d\tau} \Omega(X(\tau), \tau) = \frac{b(\tilde{\varrho}(X(\tau), \tau)}{\varrho(\tau)} \nabla \Theta(X(\tau), \tau).
\]

Recalling that \( X(s) = y \), integration over \((s_0, s), t_0 \leq s_0 \leq t\) yields

\[
\Omega(y, s) = \Omega(X(s_0), s_0) + \int_{s_0}^{s} \frac{b(\tilde{\varrho}(X(\tau), \tau)}{\varrho(\tau)} \nabla \Theta(X(\tau), \tau) d\tau.
\]

Accordingly,

\[
(2.15) \quad \|\Omega(s)\|_{L^\infty(B(r_0))} \leq \|\Omega(s_0)\|_{L^\infty(B(r_0))} + b_0 \int_{s_0}^{s} \|\nabla \Theta(\tau)\|_{L^\infty(B(r_0))} d\tau,
\]

where \( b_0 = \max_{y \in B(1)} |b(y)| \). In (2.15) we insert \( s_0 = t_0 \), integrate both sides over \((t_0, t)\).

This, together with the integration by part gives

\[
\int_{t_0}^{t} \|\Omega(s)\|_{L^\infty(B(r_0))} ds \leq (t-t_0)\|\Omega(t_0)\|_{L^\infty(B(r_0))} - b_0 \int_{t_0}^{t} (-s)' \int_{s_0}^{s} \|\nabla \Theta(\tau)\|_{L^\infty(B(r_0))} d\tau ds
\]

\[
\leq (t-t_0)\|\Omega(t_0)\|_{L^\infty(B(r_0))} + b_0 \int_{t_0}^{t} (-s)\|\nabla \Theta(s)\|_{L^\infty(B(r_0))} ds
\]

\[
\leq (t-t_0)\|w(t_0)\|_{L^\infty(B(1))} + b_0 \int_{t_0}^{t} (-s)\|\nabla \theta(s)\|_{L^\infty(B(1))} ds.
\]
Since \( \| \Omega(s) \|_{B(r_0)} \geq \| w(s) \|_{B(r)} \), this proves (2.7). To verify (2.8), we first note that (2.15) multiplied by \((-s)\) implies
\[
(-s)\| w(s) \|_{L^\infty(B(r))} \leq (-s)\| \Omega(s) \|_{L^\infty(B(r_0))}
\]
\[
\leq (-s)\| w(s_0) \|_{L^\infty(B(r_0))} + b_0 \int_{s_0}^{0} (-\tau) \| \nabla \theta(\tau) \|_{L^\infty(B(r_0))} d\tau.
\]
(2.16)

Applying \( \limsup \) as \( s \to 0^- \) to both sides of (2.16), we are led to
\[
\limsup_{s \to 0^-} (-s)\| w(s) \|_{L^\infty(B(r))}
\leq \limsup_{s \to 0^-} (-s)\| w(s_0) \|_{L^\infty(B(1))} \leq b_0 \int_{s_0}^{0} (-\tau) \| \nabla \theta(\tau) \|_{L^\infty(B(1))} d\tau < +\infty.
\]

On the other hand, the integral on the right-hand side of this inequality tends to 0 as \( s_0 \to 0 \), and we conclude (2.8).

To prove (2.9) we argue as above. In fact, for the same trajectory as in (2.14) we deduce from (2.1) that
\[
\frac{d}{dt} \Theta(X(\tau), \tau) = 0 \quad \text{for all} \quad \tau \in [t_0, s].
\]
(2.17)

Integration of (2.17) over \((t_0, s)\) gives
\[\Theta(X(s), s) = \Theta(X(t_0), t_0) = \theta(\bar{\rho}(X(t_0), t_0), t_0).\]

Accordingly, for all \( r < 1 \) we have
\[\| \theta(s) \|_{L^\infty(B(r))} \leq \| \theta(t_0) \|_{L^\infty(B(1))}.
\]

This completes the proof of the lemma.

**Proof of Theorem 2.1** Applying \( \partial_i, i = 1, 2, \) to (2.13) we get
\[
\partial_i \partial_i \Omega + W \cdot \nabla \partial_i \Omega = -\partial_i W \cdot \nabla \Omega + \frac{b(\bar{\rho}(y, t))}{\varrho(t)} \cdot \partial_i \nabla \theta + \frac{\partial_i b(\bar{\rho}(y, t))}{\varrho(t)} \cdot \nabla \theta.
\]
(2.18)

Let \( \frac{1}{2} < r < 1 \), and choose \(-1 < t_0 < 0 \) to satisfy (2.10). Set \( \rho_* = \frac{r+m}{2} \), and define
\[r_m := \rho_* - (\rho_* - r)^m, \quad m \in \mathbb{N} \cup \{0\}.
\]

Clearly,
\[r_{m+1} - r_m = r \left( 1 - \frac{r}{\rho_*} \right)^m, \quad \text{and} \quad r_m \nearrow \rho_*.\]

Let \( \eta_m \in C^\infty(\mathbb{R}) \) denote a cut off function such that \( 0 \leq \eta_m \leq 1 \) in \( \mathbb{R} \), \( \eta_m \equiv 1 \) on \((-\infty, r_m] \), \( \eta_m \equiv 0 \) in \((r_{m+1}, +\infty) \), and \( 0 \leq -\eta'_m \leq \frac{2}{r_{m+1} - r_m} = 2r^{-1} \left( \frac{\rho_0 + r}{\rho_0 - r} \right)^{m+1} \). Let \( q > \]
2 be fixed. We multiply both sides of (2.18) by \( \partial_t \Omega |\nabla \Omega|^{q-2} \phi_m^{6q} \), where \( \phi_m(y) = \eta_m(|y|) \), integrate the result over \( B(r_{m+1}) \times (t_0, t) \), \( t_0 < t < 0 \), and sum it over \( i = 1, 2 \). Then, applying the integration by parts, we have

\[
\|\nabla \Omega(t) \phi_m^6\|_q^q - 6q \int_{t_0}^t \int_{B(r_{m+1}) \setminus B(r_m)} \frac{W(s) \cdot y}{|y|} |\nabla \Omega(s)| \eta_m \phi_m^{6q-1} \eta_m'(|y|) dyds
\]

\[
= \|\nabla \Omega(t_0) \phi_m^6\|_q^q + I + II + III + IV,
\]

where we set

\[
I + II + III + IV
\]

\[
= \int_{t_0}^t \int_{B(r_{m+1})} \nabla \cdot W(s) |\nabla \Omega(s)|^q \phi_m^{6q} dyds
\]

\[
- q \int_{t_0}^t \int_{B(r_{m+1})} \nabla W(s) : \nabla \Omega(s) \otimes \nabla \Omega(s) |\nabla \Omega(s)|^{q-2} \phi_m^{6q} dyds
\]

\[
+ q \int_{t_0}^t \int_{B(r_{m+1})} \frac{b(\bar{\rho}(y, s))}{\rho(s)} \nabla^2 \Theta(s) \cdot \nabla \Omega(s) |\nabla \Omega(s)|^{q-2} \phi_m^{6q} dyds
\]

\[
+ q \int_{t_0}^t \int_{B(r_{m+1})} \frac{\nabla b(\bar{\rho}(y, s))}{\rho(s)} : \nabla \Theta(s) \otimes \nabla \Omega(s) |\nabla \Omega(s)|^{q-2} \phi_m^{6q} dyds
\]

respectively, where and hereafter we use the notation \( M : N = \sum_{i,j=1}^2 M_{ij}^+ N_{ij}^- \) for the matrices \( M, N \). Since \( B(r_{m+1}) \setminus B(r_m) \subset B(r_0) \setminus B(r/2) \), (2.12) ensures that \( W \cdot x > 0 \) in \( B(r_{m+1}) \setminus B(r_m) \times (t_0, 0) \). Furthermore, recalling that \( \eta_m' \leq 0 \), we see that the sign of the integral on the left-hand side of (2.19) is non-negative. Consequently, it follows that

\[
\|\nabla \Omega(t) \phi_m^6\|_q^q \leq \|\nabla \Omega(t_0) \phi_m^6\|_q^q + I + II + III + IV.
\]

We first focus on the estimate of \( \|\nabla^2 U(t) \phi_m^6\|_q^q \) in terms of \( \|\nabla \Omega(t) \phi_m^6\|_q^q \) in order estimate the left-hand side of (2.20) from below. Recalling the definition of \( U \) and \( \Omega \), by means of the chain rule, the equation (2.4) becomes

\[
\left\{
\begin{array}{l}
\text{curl } U(y, t) = \rho(t) \left\{ d(\bar{\rho}(y, t)) \Omega(y, t) + e(\bar{\rho}(y, t)) \cdot U(y, t) \right\} \\
(y, t) \in B(r_0) \times (t_0, 0).
\end{array}
\right.
\]

In view of (A.17) with \( u = U(t), \psi = \phi_m \) and \( k = 6q \) we estimate

\[
\|\nabla^2 U(t) \phi_m^6\|_q^q \leq c \|\nabla \text{curl } U(t) \phi_m^6\|_q^q + c \|\nabla \phi_m\|^{3q-2} \|U(t) \phi_m^2\|_2.
\]
From (2.21) deducing
\[
\nabla \text{curl} \, U = \varrho(t) d(\bar{\rho}(\cdot)) \nabla \Omega + e(\bar{\rho}(\cdot)) \cdot \nabla U \\
+ \varrho(t)^2 \nabla d(\bar{\rho}(\cdot)) \Omega + \varrho(t)^2 \nabla e(\cdot) U,
\]
we estimate in \(B(r_0) \times (t_0, 0)\)
\[
|\nabla \text{curl} \, U| \leq c(|\nabla \Omega| + |\nabla U| + |U|),
\]
with a constant \(c > 0\) depending only on \(\|d\|_\infty, \|\nabla d\|_\infty, \|e\|_\infty\) and \(\|\nabla d\|_\infty\). Accordingly,
\[
(2.23) \quad \|\nabla \text{curl} \, U(t) \phi_m^6\|_q \leq c \|\nabla \Omega(t) \phi_m^6\|_q + c \|\nabla U(t) \phi_m^6\|_q + c \|U(t) \phi_m^6\|_q.
\]
By means of (A.16) with \(m = k = 6\) we find
\[
(2.24) \quad \|\nabla U(t) \phi_m^6\|_q \leq c \|U(t) \phi_m^6\|_q \frac{1}{q} \|\nabla \text{curl} \, U(t) \phi_m^6\|_q + c \|\nabla \phi_m\|_\infty \|U(t) \phi_m^5\|_q.
\]
Estimating the second term on the right-hand side of (2.23) using (2.24) along with Young’s inequality, it follows that
\[
(2.25) \quad \|\nabla \text{curl} \, U(t) \phi_m^6\|_q \leq c \|\nabla \Omega(t) \phi_m^6\|_q + c(1 + \|\nabla \phi_m\|_\infty) \|U(t) \phi_m^5\|_q.
\]
Furthermore, employing (A.2) with \(m = 6\) and \(k = 1\), we get
\[
\|U(t) \phi_m^5\|_q \leq c \|U(t) \phi_m^4\|_q^\frac{q'}{q} \|\nabla U(t) \phi_m^6\|_q^\frac{1}{q} \|\nabla \phi_m\|_\infty \|U(t) \phi_m^4\|_2,
\]
where \(q' := \frac{q}{q-1}\). This inequality combined with (2.24) yields
\[
(2.26) \quad \|U(t) \phi_m^5\|_q \leq c \|U(t) \phi_m^4\|_q^\frac{q'}{q} \|\nabla \text{curl} \, U(t) \phi_m^6\|_q^\frac{1}{q} \|\nabla \phi_m\|_\infty \|U(t) \phi_m^4\|_2^\frac{q-2}{q} + c \|\nabla \phi_m\|_\infty \|U(t) \phi_m^4\|_2^\frac{q-2}{q}.
\]
Applying Young’s inequality, we infer from (2.26)
\[
(2.27) \quad \|U(t) \phi_m^5\|_q \leq c \|\nabla \text{curl} \, U(t) \phi_m^6\|_q^\frac{2-q'}{q} \|U(t) \phi_m^4\|_2^\frac{2}{2-q} \|\nabla \phi_m\|_\infty \|U(t) \phi_m^4\|_2 + c \|\nabla \phi_m\|_\infty \|U(t) \phi_m^4\|_2.
\]
Finally, inserting (2.27) into the right-hand side of (2.25) and using Young’s inequality, we obtain
\[
(2.28) \quad \|\nabla \text{curl} \, U(t) \phi_m^6\|_q \leq c \|\nabla \Omega(t) \phi_m^6\|_q + c(1 + \|\nabla \phi_m\|_\infty) \|U(t) \phi_m^4\|_2.
\]
Combining (2.22) and (2.28), noting that
\[
(1 + \|\nabla \phi_m\|_\infty) \leq c(r_{m-1} - r_m)^{-1}, \quad \|U(t)\|_{L^2(B(r_{m+1}))} \leq \|u\|_{L^\infty(-1,0; L^2(B(1)))} < +\infty
\]
by the assumption of the theorem, we get
(2.29) \[ \|\nabla^2 U(t)\phi_m^6\|_q^q \leq c\|\nabla\Omega(t)\phi_m^6\|_q^q + c(r_{m-1} - r_m)^{-3q+2} \]
with a constant \( c > 0 \) independent of \( t \in (t_0, 0) \) and \( m \in \mathbb{N} \).
We continue to estimate the right-hand side of (2.20) from above. Calculating
\[
\nabla \cdot W(y, t) = \frac{40a_0\|u(t)\|_{L^\infty(B(1))}}{\varrho(t)} + \nabla a(\tilde{\varrho}(y, t)) \cdot U(y, t),
\]
\[
\nabla W(y, t) = \frac{20a_0\|u(t)\|_{L^\infty(B(1))}}{\varrho(t)} I + \nabla a(\tilde{\varrho}(y, t)) \otimes U(y, t) + \frac{a(\tilde{\varrho}(y, t))}{\varrho(t)} \nabla U(y, t),
\]
where \( I \) denotes the \( 2 \times 2 \) unit matrix, we easily get
\[
I + II = (2 - q) \int_{t_0}^{t} \int_{B(r_{m+1})} \frac{20a_0\|u\|_{L^\infty(B(1))}}{\varrho} |\nabla \Omega|^q \phi_m^6 dy ds
\]
\[
+ \int_{t_0}^{t} \int_{B(r_{m+1})} \nabla a(\tilde{\varrho}(\cdot)) \cdot U |\nabla \Omega|^q \phi_m^6 dy ds
\]
\[
- q \int_{t_0}^{t} \int_{B(r_{m+1})} \nabla a(\tilde{\varrho}(\cdot)) \cdot \nabla \Omega U \cdot \nabla \Omega |\nabla \Omega|^{q-2} \phi_m^6 dy ds
\]
\[
+ \int_{t_0}^{t} \int_{B(r_{m+1})} \frac{a(\tilde{\varrho}(\cdot))}{\varrho} \nabla U : \nabla \Omega \otimes \nabla \Omega |\nabla \Omega|^{q-2} \phi_m^6 dy ds.
\]
Since the first term of the right-hand side is non-positive, we find
\[
I + II \leq c \int_{t_0}^{t} \int_{B(r_{m+1})} (|U| + |\nabla U|) |\nabla \Omega|^q \phi_m^6 dy ds,
\]
where \( c > 0 \) depends on \( q, \|a\|_\infty \) and \( \|\nabla a\|_\infty \). Furthermore, it is readily seen that
\[
III + IV \leq c \int_{t_0}^{t} \int_{B(r_{m+1})} (|\nabla \Theta| + |\nabla^2 \Theta|) |\nabla \Omega|^{q-1} \phi_m^6 dy ds,
\]
where \( c > 0 \) depends on \( q, \|b\|_\infty \) and \( \|\nabla b\|_\infty \). Inserting the above estimates of \( I, II, III, \) and \( IV \) into (2.20), we are led to
(2.30) \[
\|\nabla \Omega(t)\phi_m^6\|_q^q \leq \|\nabla \Omega(t_0)\phi_m^6\|_q^q + c \int_{t_0}^{t} \int_{B(r_{m+1})} (|U| + |\nabla U|) |\nabla \Omega|^q \phi_m^6 dy ds
\]
\[
+ c \int_{t_0}^{t} \int_{B(r_{m+1})} (|\nabla \Theta| + |\nabla^2 \Theta|) |\nabla \Omega|^{q-1} \phi_m^6 dy ds.
\]
For $0 < \varepsilon < 1$ and $t \in (t_0, 0)$ we set
\[ Z_m(t) = Z^U_m(t) + Z^\Theta_m(t) \]
with
\[ Z^U_m(t) := \| \nabla^2 U(t) \phi^6_m \|^q \quad \text{and} \quad Z^\Theta_m(t) := \varepsilon^{-q}(-t)^q \| \nabla^2 \Theta(t) \phi^6_m \|^q. \]

By Young’s inequality we find
\[ |\nabla^2 \Theta(y, s)||\nabla \Omega(y, s)|^{q-1} \leq \varepsilon (-s)^{-1} \left\{ \varepsilon^{-q}(-s)^q |\nabla^2 \Theta(y, s)|^q + |\nabla \Omega(y, s)|^q \right\}. \]

Therefore,
\[ \int_{t_0}^{t} \int_{B(r_{m+1})} |\nabla^2 \Theta||\nabla \Omega|^{q-1} \phi^6_m dyds \leq \varepsilon \int_{t_0}^{t} (-s)^{-1} Z_m(s) ds, \]
and
\[ \int_{t_0}^{t} \int_{B(r_{m+1})} |\nabla \Theta||\nabla \Omega|^{q-1} \phi^6_m dyds = \int_{t_0}^{t} \int_{B(r_{m+1})} |\nabla \Theta| \phi^6_m |\nabla \Omega|^{q-1} \phi^6_m dyds \]
\[ \leq \int_{t_0}^{t} \| \nabla \Theta(s) \phi^6_m \|_q \| \nabla \Omega(s) \phi^6_m \|_q^{q-1} ds. \]

Combining (2.30) with (2.31) and (2.32), we get
\[ \| \nabla \Omega(t) \phi^6_m \|^q \leq c Z^U_m(t_0) + c \int_{t_0}^{t} \left[ \| U(s) \|_{L^\infty(B(r_{m+1}))} + \| \nabla U(s) \|_{L^\infty(B(r_{m+1}))} \right] Z^U_m(s) ds \]
\[ \quad + \kappa \varepsilon \int_{t_0}^{t} (-s)^{-1} Z_m(s) ds + c \int_{t_0}^{t} \| \nabla \Theta(s) \phi^6_m \|_q Z^U_m(s)^{q-1} s ds \]
with a constant $\kappa > 0$ independent of $\varepsilon$.

Now we turn to the estimation of second gradient of $\Theta$. First, by the chain rule we derive from (2.2) the following equation for $\Theta$ in $B(r_0) \times (t_0, 0)$
\[ \partial_t \Theta + W \cdot \nabla \Theta = 0. \]
We apply, $\partial_i, \partial_j, i, j = 1, 2$ to (2.35), to obtain
\[ \partial_i \partial_j \Theta + W \cdot \nabla \partial_i \partial_j \Theta = -\partial_i W \cdot \nabla \partial_j \Theta - \partial_j W \cdot \nabla \partial_i \Theta - \partial_i \partial_j W \cdot \nabla \Theta. \]
We now multiply both sides of (2.35) by \( \varepsilon^{-q}(-s)^q \partial_t \Theta \mid \nabla^2 \Theta \mid^{q-2} \phi_m^6 \), integrate the result over \( B(r_{m+1}) \times (t_0, t) \), \( t_0 < t < 0 \), and sum it over \( i = 1, 2 \). Then, applying the integration by parts, we have

\[
\varepsilon^{-q}(-t)^q \| \nabla^2 \Theta(t) \phi_m^6 \|_q^q \\
- 6q \int_{t_0}^t \int_{B(r_{m+1}) \setminus B(r_m)} (-s)^q \frac{W \cdot y}{|y|} |\nabla^2 \Theta|^{q-2} \phi_m^{6q-1} \eta'_m(|y|) \varepsilon^{-q}(-s) dy ds
\]

\[
= \varepsilon^{-q}(-t_0)^q \| \nabla^2 \Theta(t_0) \phi_m^6 \|_q^q + \int_{t_0}^t \int_{B(r_{m+1})} \nabla \cdot W |\nabla^2 \Theta|^{q-2} \phi_m^{6q} \varepsilon^{-q}(-s) dy ds
\]

\[
- 2q \int_{t_0}^t \int_{B(r_{m+1})} \nabla W : \nabla^2 \Theta \otimes \nabla^2 \Theta |\nabla^2 \Theta|^{q-2} \phi_m^{6q} \varepsilon^{-q}(-s) dy ds
\]

\[
- q \int_{t_0}^t \int_{B(r_{m+1})} \nabla^2 W : \nabla \Theta : \nabla^2 \Theta |\nabla^2 \Theta|^{q-2} \phi_m^{6q} \varepsilon^{-q}(-s) dy ds
\]

(2.36)

\[
= \varepsilon^{-q}(-t_0)^q \| \nabla^2 \Theta(t_0) \phi_m^6 \|_q^q + I + II + III.
\]

Once more, using the fact that \( y \cdot W(y, s) > 0 \) for all \((y, s) \in B(r_0) \times (t_0, 0)\), we get from (2.36)

(2.37)

\[
Z_m^O(t) \leq Z_m^O(t_0) + I + II + III.
\]

Arguing similarly to the above, we estimate

\[
I + II \leq c \int_{t_0}^t \int_{B(r_{m+1})} (|U(s)| + |\nabla U(s)|) |\nabla^2 \Theta|^{q-2} \phi_m^{6q} \varepsilon^{-q}(-s) dy ds
\]

(2.38)

\[
\leq c \int_{t_0}^t \left[ \|U(s)\|_{L^\infty(B(r_{m+1}))} + \|\nabla U(s)\|_{L^\infty(B(r_{m+1}))} \right] Z_m^O(s) ds.
\]

Furthermore, computing from (2.31)

\[
\nabla^2 W = \frac{1}{q} \left\{ \nabla a(\tilde{\psi}(\cdot)) \otimes \nabla U + \nabla U \otimes \nabla a(\tilde{\psi}(\cdot)) + \nabla^2 a(\tilde{\psi}(\cdot)) U + a(\tilde{\psi}(\cdot)) \nabla^2 U \right\},
\]

we immediately get

\[
III \leq c \int_{t_0}^t \int_{B(r_{m+1})} (|U| + |\nabla U| + |\nabla^2 U|) |\nabla \Theta| |\nabla^2 \Theta|^{q-2} \phi_m^{6q} \varepsilon^{-q}(-s) dy ds
\]
with a constant $c > 0$, depending on $q, \|a\|_{\infty}, \|a\|_{\infty}$ and $\|\nabla^2 a\|_{\infty}$. Using the fact
\[
|\nabla^2 U(y, s)||\nabla^2 \Theta(y, s)|^{q-1} = \varepsilon^{-1}(-s)\varepsilon^{-1}(-s)^{-q-1} |\nabla^2 U y, (s)||\nabla^2 \Theta(y, s)|^{q-1}
\]
(2.39) we find
\[
\int_{\mathcal{B}(r_{m+1})} \int_{t_0}^t |\nabla^2 U||\nabla \Theta||\nabla^2 \Theta|^{q-1} \phi_m^6 \varepsilon^{-q} (-s)^q dy ds \\
\leq \varepsilon^{-1} \int_{t_0}^t (-s) \|\nabla \Theta(s)\|_{L^\infty(B(r_{m+1}))} Z_m(s) ds.
\]
Similarly, using Hölder’s inequality for the other terms of $III$, we estimate
\[
III \leq c\varepsilon^{-1} \int_{t_0}^t (-s) \|\nabla \Theta(s)\|_{L^\infty(B(r_{m+1}))} Z_m(s) ds \\
+ c\varepsilon^{-1} \int_{t_0}^t \|U(s)\|_{L^\infty(B(r_{m+1}))} (-s) \|\nabla \Theta(s)\|_q \phi_m^6 ||Z_m^\Theta(s)\|_q^{q-1} ds \\
+ c\varepsilon^{-1} \int_{t_0}^t \|\nabla U(s)\|_{L^q(B(r_{m+1}))} (-s) \|\nabla \Theta(s)\|_\infty \phi_m^3 ||Z_m^\Theta(s)\|_\infty^{q-1} ds.
\]
(2.40)
Inserting the estimates (2.38) and (2.40) into the right-hand side of (2.37), we arrive at
\[
Z_m^\Theta(t) \leq Z_m^\Theta(t_0) \\
+ c \int_{t_0}^t \left[ \|U(s)\|_{L^\infty(B(r_{m+1}))} + \|\nabla U(s)\|_{L^\infty(B(r_{m+1}))} + (-s) \|\nabla \Theta(s)\|_{L^\infty(B(r_{m+1}))} \right] Z_m(s) ds \\
+ c \int_{t_0}^t \|U(s)\|_{L^\infty(B(r_{m+1}))} (-s) \|\nabla \Theta(s)\|_q \phi_m^6 ||Z_m^\Theta(s)\|_q^{q-1} ds \\
+ c \int_{t_0}^t \|\nabla U(s)\|_{L^q(B(r_{m+1}))} (-s) \|\nabla \Theta(s)\|_\infty \phi_m^3 ||Z_m^\Theta(s)\|_\infty^{q-1} ds
\]
(2.41)
with a constant $c$ depending on $\varepsilon$. Taking the sum of (2.38) and (2.41), and taking
into account (2.29), we obtain

\[ Z_m(t) \leq c Z_m(t_0) + c(r_m + 1 - r_m)^{3q + 2} + c \int_{t_0}^{t} \left[ \| U(s) \|_{L^\infty(\mathbb{R}^{n+1})} + \| \nabla U(s) \|_{L^\infty(\mathbb{R}^{n+1})} + (s) \| \nabla \Theta(s) \|_{L^\infty(\mathbb{R}^{n+1})} \right] Z_m(s) \, ds \]

\[ + \kappa \varepsilon \int_{t_0}^{t} (s)^{-1} Z_m(s) \, ds + J_1 + J_2 + J_3, \]

(2.42)

where we set

\[ J_1 = \int_{t_0}^{t} \| U(s) \|_{L^\infty(\mathbb{R}^{n+1})} (s) \| \nabla \Theta(s) \|_{L^q(\mathbb{R}^{n+1})} Z_m^{q-1}(s) \, ds, \]

\[ J_2 = \int_{t_0}^{t} \| \nabla \Theta(s) \|_{L^q(\mathbb{R}^{n+1})} Z_m^{q-1}(s) \, ds, \]

and

\[ J_3 = \int_{t_0}^{t} \| \nabla U(s) \|_{L^q(\mathbb{R}^{n+1})} (s) \| \nabla \Theta(s) \|_{L^q(\mathbb{R}^{n+1})} Z_m^{q-1}(s) \, ds. \]

As we shall show below, the integrals \( J_1, J_2 \) and \( J_3 \) are lower order terms dominated by the other terms in the right-hand side of (2.42). To see this we first estimate the term \( \| \nabla \Theta(s) \|_{L^q(\mathbb{R}^{n+1})} \) as follows. Observing (2.9) (cf. Lemma 2.3), we easily verify that

\[ \Theta_{\infty} := \sup_{t \in (t_0, 0)} \| \Theta(t) \|_{L^\infty(\mathbb{R}^{n+1})} < +\infty. \]

Integrating by parts, we calculate

\[ \| \nabla \Theta(s) \|_{L^q(\mathbb{R}^{n+1})} \]

\[ = - \int_{\mathbb{R}^{n+1}} \Theta(y, s) \Delta \Theta(y, s) \| \nabla \Theta(y, s) \|^{q-2} \phi_m^{q-3}(y) \, dy \]

\[ - (q - 2) \sum_{i=1}^{2} \int_{B(r_m+1)} \Theta(y, s) \partial_i \Theta(y, s) \partial_i \nabla \Theta(y, s) \phi_m(y) \cdot \nabla \Theta(y, s) \| \nabla \Theta(y, s) \|^{q-4} \phi_m^{q-3}(y) \, dy \]

\[ - 6q \int_{B(r_m+1)} \Theta(y, s) \nabla \Theta(y, s) \| \nabla \Theta(y, s) \|^{q-2} \phi_m^{q-1}(y) \, dy := A + B + C. \]

(2.44)

Thanks to (2.43) along with Hölder’s inequality and Young’s inequality, we immediately
get

\[ A + B \leq c \Theta_\infty \int_{B(r_{m+1})} |\nabla^2 \Theta(y, s)| \phi_m^6 |\nabla \Theta(y, s)|^{q-2} \phi_m^{3(q-2)}(y) dy \]

\[ \leq c \Theta_\infty \| \nabla^2 \Theta(s) \phi_m^6 \|_q \| \nabla \Theta(s) \phi_m^3 \|_{q-2} \]

\[ \leq c \Theta_\infty \frac{2}{q} \| \nabla^2 \Theta(s) \phi_m^6 \|_q^2 + \frac{1}{4} \| \nabla \Theta(s) \phi_m^3 \|_q^q. \]

Similarly, by means of Hölder’s inequality and Young’s inequality, we infer

\[ C \leq 6q \int_{B(r_{m+1})} |\Theta(y, s)| \| \nabla \phi_m(y) \| \| \nabla \Theta(y, s) \|_{q-1}^{q-1} \phi_m^3(y) dy \]

\[ \leq c \Theta_\infty \| \nabla \phi_m \|_\infty \| \nabla \Theta(s) \phi_m^3 \|_{q-1} \leq c \Theta_\infty (r_{m+1} - r_m)^{-1} \| \nabla \Theta(s) \phi_m^3 \|_{q-1} \]

\[ \leq c \Theta_\infty (r_{m+1} - r_m)^{-q} + \frac{1}{4} \| \nabla \Theta(s) \phi_m^3 \|_q^q. \]

Inserting the above estimates of \(A, B\) and \(C\) into the right-hand side of (2.44), and absorbing \(\frac{1}{2} \| \nabla \Theta(s) \phi_m^3 \|_q^q\) into the left-hand side, we deduce that

\[ \| \nabla \Theta(s) \phi_m^3 \|_q \leq c \Theta_\infty \frac{1}{2} \| \nabla^2 \Theta(s) \phi_m^6 \|_q^2 + c \Theta_\infty (r_{m+1} - r_m)^{-1} \]

\[ = c \Theta_\infty \varepsilon - \frac{3}{2} (s) - \frac{1}{2} Z_m^e(s) 1 \frac{1}{2} + c \Theta_\infty (r_{m+1} - r_m)^{-1} \]

\[ \leq c (s) - \frac{1}{2} Z_m^e(s) 1 \frac{1}{2} + c (r_{m+1} - r_m)^{-1}. \]

(2.45)

Similarly, following the above calculations (2.43)-(2.45), we also have

\[ \| \nabla U(s) \phi_m^3 \|_q \leq c \| U(s) \|_{L^\infty(B(r_{m+1}))} \| \nabla^2 U(s) \phi_m^6 \|_q^2 + c \| U(s) \|_{L^\infty(B(r_{m+1}))} (r_{m+1} - r_m)^{-1} \]

(2.46)

\[ = c \| U(s) \|_{L^\infty(B(r_{m+1}))} \| \nabla^2 U(s) \phi_m^6 \|_q^2 + c \| U(s) \|_{L^\infty(B(r_{m+1}))} \| \nabla \Theta(s) \phi_m^3 \|_q^q. \]

Obviously,

\[ \| \nabla \Theta(s) \phi_m^6 \|_q \leq \| \nabla \Theta(s) \phi_m^3 \|_q, \quad \text{and} \quad \| \nabla U(s) \phi_m^6 \|_q \leq \| \nabla U(s) \phi_m^3 \|_q. \]

Therefore, using (2.45), and then applying Young’s inequality, we find

\[ J_1 \leq c \int_{t_0}^t \| U(s) \|_{L^\infty(B(r_{m+1}))} \left[ (-s)^{\frac{1}{2}} (Z_m^e(s))^{\frac{2q-1}{2q}} + (r_{m+1} - r_m)^{-1} (Z_m^e(s))^{\frac{2q-1}{2q}} \right] ds \]

\[ \leq c \int_{t_0}^t \| U(s) \|_{L^\infty(B(r_{m+1}))} Z_m^e(s) ds + c (r_{m+1} - r_m)^{-q} \int_{t_0}^t \| U(s) \|_{L^\infty(B(r_m))} ds \]

\[ + c \int_{t_0}^t \| U(s) \|_{L^\infty(B(r_{m+1}))} (-s)^q ds. \]
Noting that $\|U(s)\|_{L^\infty(B(r_{m+1}))} \leq \|u(s)\|_{L^\infty(B(1))}$, and observing (2.6)2, we find

$$J_1 \leq c \int_{t_0}^{t} \|u(s)\|_{L^\infty(B(1))} Z_m(s) ds + c(r_{m+1} - r_m)^{-q} \int_{t_0}^{t} \|u(s)\|_{L^\infty(B(1))} ds$$

$$+ c \int_{t_0}^{t} \|u(s)\|_{L^\infty(B(1))} ds$$

$$\leq c \int_{t_0}^{t} \|u(s)\|_{L^\infty(B(1))} Z_m(s) ds + c(r_{m+1} - r_m)^{-q}.$$

Using (2.45) together with Young’s inequality, we easily get

$$J_2 \leq c \int_{t_0}^{t} \left[ (-s)^{\frac{1}{2}} (Z_m^\Theta(s))^{\frac{2}{2q}} (Z_m^U(s))^{\frac{2-1}{q}} + (r_{m+1} - r_m)^{-1}(Z_m^U(s))^{\frac{2-1}{q}} \right] ds$$

$$\leq c \int_{t_0}^{t} \left[ (-s)^{\frac{1}{2}} (Z_m(s))^{\frac{2q-4}{2q}} + c(r_{m+1} - r_m)^{-1}Z_m(s)^{\frac{q-1}{q}} \right] ds$$

$$\leq c \int_{t_0}^{t} Z_m(s) ds + c \int_{t_0}^{t} (-s)^q ds + c(r_{m+1} - r_m)^{-q}$$

$$\leq c \int_{t_0}^{t} Z_m(s) ds + c(r_{m+1} - r_m)^{-q}.$
Using (2.45), (2.46) and Young’s inequality, we easily estimate

\[
J_3 \leq c \int_{t_0}^{t} (-s)^{\frac{1}{2}} \|U(s)\|_{L^\infty(B(r_{m+1})))}^{\frac{1}{2}} (Z_m^u(s))^{\frac{1}{2}} (Z_m^\theta(s))^{\frac{2q-1}{2q}} ds
\]

\[
+ c \int_{t_0}^{t} (-s)^{\frac{1}{2}} \|U(s)\|_{L^\infty(B(r_{m+1}))}(Z_m^\theta(s))^{\frac{2q-1}{2q}} (r_{m+1} - r_m)^{-1} ds
\]

\[
+ c \int_{t_0}^{t} (-s)^{\frac{1}{2}} \|U(s)\|_{L^\infty(B(r_{m+1}))}(Z_m^\theta(s))^{\frac{q-1}{q}} (r_{m+1} - r_m)^{-2} ds
\]

\[
\leq c \int_{t_0}^{t} \|U(s)\|_{L^\infty(B(r_{m+1}))} Z_m(s) ds + c(r_{m+1} - r_m)^{-2}
\]

\[
\leq c \int_{t_0}^{t} \|u(s)\|_{L^\infty(B(1))} Z_m(s) ds + c(r_{m+1} - r_m)^{-2}.
\]

Inserting the estimates of \(J_1, J_2\) and \(J_3\) into the right-hand side of (2.42), noting that \(2q < 3q - 2\) for \(q > 2\), and observing (2.6)_1, we arrive at

\[
Z_m(t) \leq c Z_m(t_0)
\]

\[
+ c \int_{t_0}^{t} \left[ \|\nabla U(s)\|_{L^\infty(B(r_{m+1}))} + (-s)\|\nabla \Theta(s)\|_{L^\infty(B(r_{m+1}))} + \|u(s)\|_{L^\infty(B(1))} \right] Z_m(s) ds
\]

\[
+ \kappa \varepsilon \int_{t_0}^{t} (-s)^{-1} Z_m(s) ds + c(r_{m+1} - r_m)^{-3q+2},
\]

where \(c\) stands for a positive constants, depending on \(q, \varepsilon, \alpha, C_0, u, \theta\) but not on \(t\) and \(m\), while the constant \(\kappa\) depends on \(q, \alpha, C_0, u, \theta\) but not on \(\varepsilon, t\) and \(m\).

By the similar argument to [10, cf. (5.14)], observing (2.21), we get

\[
\|\nabla U(s)\|_{L^\infty(B(r_{m+1}))}
\]

\[
\leq c \left\{ 1 + |\text{curl} U(s)|_{BMO(B(\rho_0))} \right\} \log(e + Z_{m+1}(s))
\]

\[
\leq c \left\{ 1 + \|U(s)\|_{L^\infty(B(\rho_0))} + \|\Omega(s)\|_{L^\infty(B(\rho_0))} \right\} \log(e + Z_{m+1}(s)).
\]

Combining (2.47) and (2.48), and noting that \((r_{m+1} - r_m)^{-1} \leq c \left( \frac{\rho_0 + \rho}{\rho_0 - \rho} \right)^m\), we are led to

\[
e + Z_m(t) \leq c_0 + d^m + \int_{t_0}^{t} (\alpha(s) \log(e + Z_{m+1}(s)) + f(s)) Z_m(s) ds,
\]

20
\[
\alpha(s) = c \left( 1 + \|u(s)\|_{L^\infty(B(1))} + \|\Omega(s)\|_{L^\infty(B(\rho))} \right), \\
f(s) = c \left[ (s) \|\nabla \theta(s)\|_{L^\infty(B(1))} + \|u(s)\|_{L^\infty(B(1))} \right] + \kappa \varepsilon(s)^{-1}, \\
c_0 = c(Z_m(t_0) + e) \leq c \left( 1 + \|\nabla^2 u(t_0)\|^q_{L^q(B(1))} + c\|\nabla^2 \theta(t_0)\|^q_{L^q(B(1))} \right), \\
d = c \left( \frac{r_0 + r}{r_0 - r} \right)^{3q-2}.
\]

We now define
\[
\begin{align*}
Y_m(t) &:= c_0 + d^m + \int_{t_0}^{t} \{ \alpha(s) \log(e + Z_{m+1}(s)) + f(s) \} Z_m(s) \, ds, \\
t &\in (t_0, 0), \quad m \in \mathbb{N}.
\end{align*}
\]

Thus, (2.49) gives
\[
Z_m(t) \leq Y_m(t), \quad \text{and therefore}
\]
\[
Y'_m(t) = \left\{ \alpha(t) \log(e + Z_{m+1}(s)) + f(s) \right\} Z_m(s) \leq \left\{ \alpha(t) \log(e + Y_{m+1}(s)) + f(s) \right\} Y_m(s).
\]

Dividing both sides of (2.50) by \(e + Y_m(s)\) and setting
\[
\beta_m(s) := \log(e + Y_m(s)), \quad s \in [t_0, 0),
\]
we deduce from (2.50) the following recursive differential inequality
\[
\beta'_m(s) \leq \alpha(t) \beta_{m+1}(s) + f(s).
\]

We now fix \(t \in (t_0, 0)\), and integrate (2.51) over \((t_0, t)\). This yields
\[
\beta_m(t) \leq X_m(t_0) + \int_{t_0}^{t} \alpha(s)\beta_{m+1}(s) \, ds + \int_{t_0}^{t} f(s) \, ds
\]
\[
= \log(e + Y_m(t_0)) + \int_{t_0}^{t} \alpha(s)\beta_{m+1}(s) \, ds + \int_{t_0}^{t} f(s) \, ds
\]
\[
= \log(e + c_0 + d^m) + \int_{t_0}^{t} \alpha(s)\beta_{m+1}(s) \, ds + \int_{t_0}^{t} f(s) \, ds
\]
\[
\leq m \log d + g(t) + \int_{t_0}^{t} \alpha(s)\beta_{m+1}(s) \, ds,
\]
where we set
\[
g(\tau) = 1 + \log(e + c_0) + \int_{t_0}^{\tau} f(s) \, ds, \quad \tau \in [t_0, 0).
\]
In order to apply Lemma B.1 we still need to check if (B.2) is fulfilled. By the assumption of the theorem we have

\[ \max_{\tau \in [t_0, t]} Z_m(\tau) \leq \| \nabla^2 u \|_{L^\infty(-t, t; L^q(B(1)))} + \varepsilon^{-1}(t) \| \nabla^2 \theta \|_{L^\infty(-t, t; L^q(B(1)))} =: c_1(t) < +\infty \]

for all \( m \in \mathbb{N} \). Accordingly, for \( \tau \in [t_0, t] \)

\[ Y_m(\tau) \leq c_0 + d^m + \int_{t_0}^{\tau} (\alpha(s) \log(e + c_1) + f(s))c_1 ds \leq c_2 + d^m, \]

for some constant \( c_2 > 0 \) depending on \( t \) but independent on \( \tau \). Thus, for all \( \tau \in [t_0, t] \)

\[ \beta_m(\tau) = \log(e + Y_m(\tau)) \leq \log(e + c_2 + d^m) \leq 1 + \log(e + c_2) + m \log d. \]

Clearly, the condition (B.2) of Lemma B.1 is satisfied. Indeed,

\[ \max_{\tau \in [t_0, t]} \beta_m(\tau) \leq 1 + \log(e + c_2) + m \log d \leq K^m \quad \forall m \in \mathbb{N}. \]

with \( K = 1 + \log(e + c_2) + \log d \). We are now in a position to apply Lemma B.1 with \( C = \log d \), which shows that

\begin{equation}
\beta_0(t) \leq g(t) + \log d \int_{t_0}^{t} \alpha(s) ds^\alpha + \int_{t_0}^{t} \alpha(s) \int_{s}^{t} \alpha(\tau) d\tau ds.
\end{equation}

\begin{equation}
\beta_0(t) \leq g(t) + \log d \int_{t_0}^{t} \alpha(s) ds^\alpha + \int_{t_0}^{t} f(s) e^{\int_{s}^{t} \alpha(\tau) d\tau} ds + c e^{\int_{t_0}^{t} \alpha(\tau) d\tau}.
\end{equation}

Setting

\[ r_1 = \frac{3r + 1}{4}, \]

owing to (2.10), we see that \( g(y, t) \in B(r_1) \) for all \( t \in (t_0, 0) \), and thanks to Lemma 2.3 we get

\[ \int_{t_0}^{0} \| \Omega(t) \|_{L^\infty(B(r_1))} dt \leq \int_{t_0}^{0} \| w(t) \|_{L^\infty(B(r_1))} dt < +\infty. \]

This shows that

\[ a_0 := \int_{t_0}^{0} \alpha(t) dt \]

\[ = c(-t_0) + \int_{t_0}^{0} [\| u(t) \|_{L^\infty(B(1))} + \| \Omega(t) \|_{L^\infty(B(\rho_1))}] dt < +\infty. \]
Therefore, choosing $|t_0|$ small enough, we may assume that $a_0 \leq 1$, and recalling the definition of $g$, we obtain from (2.54)

$$
\beta_0(t) \leq c + (1 + e) \int_{t_0}^{t} f(s) ds + (\log d) e + ce
$$

$$
\leq c + (1 + e) \int_{t_0}^{t} f(s) ds.
$$

From the definition of $f$, observing (2.6), we get a constant $c > 0$ such that

$$
\beta_0(t) \leq c - (1 + e) \kappa \varepsilon (\log(-t) - \log(-t_0)) \leq c + \log(-t)^{(1+e)\kappa \varepsilon}.
$$

Consequently,

$$
e + Y_0(t) \leq c(-t)^{(1+e)\kappa \varepsilon} \quad \forall t \in [t_0, 0).
$$

Since $Z_0 \leq Y_0$, it follows that

$$
\int_{-1}^{0} \left( \|\nabla^2 u(t)\|_{L^q(B(r))}^q + (-t)^{-q} \|\nabla^2 \vartheta(t)\|_{L^q(B(r))}^q \right) dt < +\infty.
$$

Choosing $\varepsilon > 0$ so that $(1 + e)\kappa \varepsilon \leq \frac{1}{2}$, we get from (2.55)

$$
\int_{-1}^{0} \left( \|\nabla^2 u(t)\|_{L^2(B(r))}^2 + (1-e)\kappa \varepsilon \cdot \|\nabla \vartheta(t)\|_{L^2(B(r))}^2 \right) dt < +\infty.
$$

By Sobolev’s embedding theorem we get from (2.56) and (2.6) together with (2.9) for all $0 < r < 1$

$$
\int_{-1}^{0} \|\nabla u(t)\|_{L^\infty(B(r))} dt < +\infty.
$$

To complete the proof of the theorem, we first show that $\nabla \vartheta \in L^\infty(B(r) \times (-1, 0))$ for all $0 < r < 1$. We apply, $\partial_i$, $i \in \{1, 2\}$, to (2.34), which gives

$$
\partial_t \partial_i \Theta + W \cdot \nabla \partial_i \Theta = -\partial_i W \cdot \nabla \Theta \quad \text{in} \quad B(r_0) \times (t_0, 0).
$$

Multiplying both sides of (2.58) by $\partial_i \Theta$, and taking the sum from $i = 1$ to 2, we arrive at

$$
\partial_t |\nabla \Theta|^2 + W \cdot \nabla |\nabla \Theta|^2 = -2 \nabla W : \nabla \Theta \otimes \nabla \Theta.
$$

Let $(y, s) \in B(r_0) \times (t_0, t)$. We denote by $X(\cdot) = X(y, s; \cdot) : [t_0, s] \to \mathbb{R}^2$ the trajectory such that

$$
\dot{X}(\tau) = W(X(\tau), \tau), \quad X(s) = y.
$$
As we have seen in the proof of Lemma 2.3, \(X(\tau) \in B(r_0)\) for all \(\tau \in [t_0, s]\). In addition from (2.59) together with (2.60) we deduce that

\[
\begin{align*}
\frac{d}{d\tau} |\nabla \Theta(X(\tau), \tau)|^2 &= -2\nabla W(X(\tau), \tau) : \nabla \Theta(X(\tau), \tau) \otimes \nabla \Theta(X(\tau), \tau).
\end{align*}
\]  

Integrating both sides of (2.61) over \((t, s)\) for some \(t \in (t_0, s)\), we obtain

\[
|\nabla \Theta(X(t), t)|^2 = |\nabla \Theta(X(t_0), t_0)|^2 - 2 \int_{t_0}^{t} \nabla W(X(\tau), \tau) : \nabla \Theta(X(\tau), \tau) \otimes \nabla \Theta(X(\tau), \tau) d\tau
\]

\[
\leq |\nabla \Theta(X(t_0), t_0)|^2 + 2 \int_{t_0}^{t} \|\nabla W(\tau)\|_{L^\infty(B(r_0))} |\nabla \Theta(X(\tau), \tau)|^2 d\tau.
\]

Noting that thanks to (2.57) and (2.60),

\[
\begin{align*}
\int_{t_0}^{0} \|\nabla W(\tau)\|_{L^\infty(B(r_0))} d\tau &< +\infty,
\end{align*}
\]

and we are in a position to apply Gronwall’s Lemma, which shows that

\[
|\nabla \Theta(y, s)| \leq |\nabla \Theta(X(t_0), t_0)| e^{0} \int_{0}^{s} \|\nabla W(\tau)\|_{L^\infty(B(r_0))} d\tau
\]

\[
\leq \|\nabla \Theta(t_0)\|_{L^\infty(B(r_0))} e^{0} \int_{0}^{s} \|\nabla W(\tau)\|_{L^\infty(B(r_0))} d\tau < +\infty.
\]

Whence,

\[
\|\nabla \theta\|_{L^\infty(B(r) \times (-1, 0))} < +\infty.
\]

Next, multiplying both sides of (2.18) by \(\partial_i \Omega |\nabla \Omega|^{q-2}, q > 3\), and taking the sum over \(i = 1, 2\), we get

\[
\begin{align*}
\frac{1}{q} \partial_i |\nabla \Omega|^q + \frac{1}{q} W \cdot \nabla |\nabla \Omega|^q &= -\nabla W \cdot \nabla \Omega \otimes \nabla \Omega |\nabla \Omega|^{q-2} + \frac{b(\tilde{\theta}(y, t))}{\tilde{\theta}(t)} \otimes \nabla \Theta \cdot \nabla |\nabla \Omega|^{q-2}
\end{align*}
\]

\[
\begin{align*}
+ \frac{\nabla b(\tilde{\theta}(y, t))}{\tilde{\theta}(t)} \cdot \nabla \Theta \otimes \nabla \Omega |\nabla \Omega|^{q-2}
\end{align*}
\]

\[
\leq |\nabla W||\nabla \Omega|^q + c_1 |\nabla^2 \Theta||\nabla \Omega|^{q-1} + c_2 |\nabla \Theta||\nabla \Omega|^{q-1},
\]

where \(c_1, c_2\) are constants, depending on \(b\). Integrating both sides of (2.64) over \(B(r_0)\) and applying integration by parts, and Hölder’s inequality, we are led to

\[
\begin{align*}
\frac{1}{q} \partial_i |\nabla \Omega(t)|_{L^q(B(r_0))}^{q} + \frac{1}{q} \int_{\partial B(r_0)} \frac{y}{r_0} \cdot W(y, t) |\nabla \Omega(y, t)|^q dS
\end{align*}
\]

\[
\leq \left(\frac{1}{q} + 1\right) |\nabla W(t)||L^\infty(B(r_0))| |\nabla \Omega(t)||_{L^q(B(r_0))}^{q} + c_1 |\nabla^2 \Theta(t)||_{L^0(B(r_0))} |\nabla \Omega(t)||_{L^q(B(r_0))}^{q-1}
\]

\[
+ c_2 |\nabla \Theta(t)||_{L^q(B(r_0))} |\nabla \Omega(t)||_{L^q(B(r_0))}^{q-1}.
\]
First note that, according to (2.12), the second term on the left-hand side is nonnegative. Furthermore, thanks to (2.56) and (2.63) the second and the third term on the right-hand side belong to \( L^1(t_0, 0) \), and the function \( \| \nabla W(\cdot) \|_{L^\infty(B(0))} \) belongs to \( L^1(t_0, 0)(\text{see (2.62)}) \), we are in a position to apply Gronwall’s Lemma, which shows that

\[
\sup_{t \in (t_0, 0)} \| \nabla \Omega(t) \|_{L^q(B(0))} < +\infty.
\]

This together with \( u \in C([-1, t_0]; W^{2,q}(B(1))) \), which is an assumption of the theorem, and (2.4) shows that

\[
curl u \in L^\infty(-1, 0; W^{1,q}(B(r))) \quad \forall 0 < r < 1.
\]

Employing Lemma A.7, together with the assumption \( u \in C_w([-1, 0]; L^2(B(1))) \), we get

\[
(2.65) \quad u \in L^\infty(-1, 0; W^{2,q}(B(r))) \quad \forall 0 < r < 1.
\]

Next, we claim

\[
(2.66) \quad u \in C(B(r) \times [-1, 0)) \quad \forall 0 < r < 1.
\]

To see this, let \( \{(x_k, t_k)\} \) be a sequence in \( B(r) \times (-1, 0), \) \( 0 < r < 1 \), which converges to \( (x_0, 0) \in B(r) \times \{0\} \) as \( k \to \infty \). Since \( W^{1,q}(B(r)) \) is compactly embedded into \( C(B(r)) \), we get from (2.65) and the assumption \( u \in C_w([-1, 0]; L^2(B(1))) \) that \( u(t_k) \to u(0) \) uniformly in \( B(r) \) as \( k \to +\infty \). On the other hand, by virtue of Sobolev’s embedding theorem we see that \( W^{1,q}(B(r)) \) is continuously embedded into \( C^\gamma(B(r)) \), for \( \gamma = 1 - \frac{2}{q} \). Hence, using triangle inequality, we find

\[
|u(x_k, t_k) - u(x_0, 0)| \leq |u(x_k, t_k) - u(x_0, t_k)| + |u(x_0, t_k) - u(x_0, 0)| \\
\leq c|x_k - x_0|^\gamma u(t_k)\|_{W^{1,q}(B(r))} + \|u(t_k) - u(0)\|_{L^\infty(B(r))}.
\]

Clearly, as \( k \to +\infty \), the second term on the right-hand side tends to zero, while the first term tends to zero thanks (2.65). Whence, (2.66).

To estimate the second gradient of \( \Theta \), we argue similarly as the above. Multiplying both sides of (2.35) by \( \partial_i \partial_j \Theta |\nabla^2 \Theta|^{q-2} \), and taking the sum over \( i, j = 1, 2 \), we find

\[
\frac{1}{q} \partial_i |\nabla^2 \Theta|^{q} + \frac{1}{q} \nabla \cdot \nabla |\nabla^2 \Theta|^{q} \\
= - \sum_{i,j=1}^2 \partial_i W \cdot \nabla \partial_j \Theta \partial_i \partial_j \Theta |\nabla^2 \Theta|^{q-2} - \sum_{i,j=1}^2 \partial_j W \cdot \partial_i \nabla \Theta \partial_i \partial_j \Theta |\nabla^2 \Theta|^{q-2} \\
- \sum_{i,j=1}^2 \partial_i \partial_j W \cdot \partial_i \partial_j \Theta \cdot \nabla \Theta |\nabla^2 \Theta|^{q-2} \\
(2.67) \quad \leq 2 \|\nabla W\| |\nabla^2 \Theta|^{q} + |\nabla^2 W\| \|\nabla \Theta\| |\nabla^2 \Theta|^{q-1}.
\]
Integrating both sides of (2.69) over $B(r_0)$, and applying integration by parts, using Hölder’s inequality, and observing (2.12), we infer

$$
\frac{1}{q} \partial_t \| \nabla^2 \Theta(t) \|^q_{L^q(B(r_0))} \leq 2 \| \nabla W(t) \|_{L^\infty(B(r_0))} \| \nabla^2 \Theta(t) \|^q_{L^q(B(r_0))}
+ \| \nabla \Theta(t) \|_{L^\infty(B(r_0))} \| \nabla^2 W(t) \|_{L^q(B(r_0))} \| \nabla^2 \Theta(t) \|^q_{L^q(B(r_0))}
\leq (2 \| \nabla W(t) \|_{L^\infty(B(r_0))} + 1) \| \nabla^2 \Theta(t) \|^q_{L^q(B(r_0))}
+ c \| \nabla \Theta(t) \|^q_{L^q(B(r_0))} \| \nabla^2 W(t) \|^q_{L^q(B(r_0))},
$$

(2.68)

where we used Young’s inequality in the second inequality. From (2.63) and (2.65) combined with (2.71) we find the term of the last line of (2.68) belongs to $L^\infty(t_0, 0)$. Since the function $\| \nabla W(\cdot) \|_{L^\infty(B(r_0))}$ belongs to $L^\infty(t_0, 0)$, once more using Gronwall’s lemma, as above we see that $\nabla^2 \Theta \in L^1(t_0, 0; L^q(B(r_0)))$. Again observing (2.68), we get

$$
\partial_t \| \nabla^2 \Theta(t) \|^q_{L^q(B(r_0))} \in L^1(t_0, 0), \quad \Theta \in L^\infty(t_0, 0; W^{2, q}(B(r_0))).
$$

(2.69)

Let $\phi \in C^\infty_c(B(1))$ denote a cut off function with $\phi \equiv 1$ on $B(r)$, $0 < r < 1$. We apply $\partial_i \partial_j, i, j \in \{1, 2\}$ to (2.1) and multiply the resultant equation by $\phi$. This gives

$$
\partial_t (\partial_i \partial_j \phi) + au \cdot \nabla (\partial_i \partial_j \phi) = f,
$$

where

$$
f = -\partial_i (au) \cdot \nabla \partial_j \phi - \partial_j (au) \cdot \nabla \partial_i \phi - \partial_i \partial_j (au) \nabla \phi + au \cdot \nabla \phi \partial_i \partial_j \phi.
$$

Thanks to (2.63), (2.66) and (2.69) we see that $f, \partial_i \partial_j \phi, \partial_i \partial_j \phi \in L^q(\mathbb{R}^2 \times (t_0, 0))$. Furthermore, in view of (2.65) and Sobolev’s embedding theorem we get $au \in L^\infty(t_0, 0; W^{1, \infty}(B(1)))$. Thus, we are in a position to apply Lemma C.1 with $h = \partial_i \partial_j \phi$ and $au$ in place of $u$. Accordingly, $\partial_i \partial_j \phi \in C([t_0, 0]; L^q(\mathbb{R}^2))$, which implies

$$
\theta \in C([t_0, 0]; W^{2, q}(B(r))) \quad \forall 0 < r < 1.
$$

(2.70)

This together with $\theta \in C([-1, t_0]; W^{2, q}(B(1)))$ yields $\theta \in C([-1, 0]; W^{2, q}(B(r)))$ for all $0 < r < 1$. By a similar reasoning we infer from (2.13) that $w \in C([-1, 0]; W^{1, q}(B(r)))$ for all $0 < r < 1$. This together with (2.1) and (2.66) implies $u \in C([-1, 0]; W^{2, q}(B(r)))$ for all $0 < r < 1$. Whence, the proof of Theorem 1.1 is complete.

## 3 Proof of Theorem 1.1 and 1.2

In the proof of Theorem 1.1 below we make use of the following

**Lemma 3.1.** Let $B(\xi_0, \rho)$ be a ball in $\mathbb{R}^2$. Furthermore, let $\phi \in C^\infty_c(B(\xi_0, \rho))$ denote a cut off function such that $0 \leq \phi \leq 1$, and $|\nabla \phi| \leq c \rho^{-1}$. Then for every $u \in L^2(B(\xi_0, \rho))$ with $\text{curl } u \in L^\infty(B(\xi_0, \rho))$ it holds

$$
\| u \phi^5 \|_\infty \leq c \| u \phi^4 \|^\frac{3}{2} \| \text{curl } u \phi^4 \|^\frac{1}{2} + c \rho^{-1} \| u \phi^2 \|_2.
$$

(3.1)
Proof: By virtue of Lemma A.9 with $n = 2$ and Lemma A.10 we get

$$
\|\nu^5\|_\infty \leq C\|\nu^5\|_2^{1/2}\|\nabla (\nu^5)\|_{\text{BMO}}^{1/2}
$$

$$
\leq C\|\nu^5\|_2^{1/2}\left(\|\text{curl}(\nu^5)\|_{\text{BMO}} + \|\nabla \cdot (\nu^5)\|_{\text{BMO}}\right)^{1/2}
$$

$$
\leq C\|\nu^5\|_2^{1/2}\left(\|\text{curl} \nu^5\|_{\text{BMO}} + |u \cdot \nabla \phi|_{\text{BMO}}\right)^{1/2}
$$

(3.2)

Using Calderón Zygmund’s inequality, we estimate

$$
\|\nabla \nu^4\|_4 \leq C\|\text{curl} \nu^4\|_4 + \|\nabla \cdot \nu^4\|_4 + c \rho^{-1}\|\nu^3\|_4
$$

$$
\leq C\|\text{curl} \nu^4\|_4 + c \rho^{-1}\|\nu^3\|_4
$$

(3.3)

Noting that $\|\nu^3\|_4 \leq \|\nu^2\|_2^{1/2}\|\nu^4\|_\infty^{1/2}$, we infer from (3.3)

$$
\|\nabla \nu^4\|_4 \leq c \rho^{1/2}\|\nu^4\|_\infty + c \rho^{-1}\|\nu^3\|_2\|\nu^4\|_\infty
$$

(3.4)

On the other hand, by Gagliardo-Nirenberg’s inequality we find

$$
\|\nu^4\|_\infty \leq C\|\nu^4\|_2^{1/2}\|\nabla \nu^4\|_4^{1/2} + c \rho^{-1/2}\|\nu^3\|_2^{1/2}\|\nu^4\|_\infty^{1/2}
$$

$$
\leq C\|\nu^4\|_2^{1/2}\|\nabla \nu^4\|_4^{1/2} + c \rho^{-1}\|\nu^2\|_2\|\nu^4\|_\infty^{1/2}
$$

Applying Young’s inequality, we obtain

(3.5)  \[ \|\nu^4\|_\infty \leq c\|\nu^4\|_2^{1/2}\|\nabla \nu^4\|_4^{1/2} + c \rho^{-1}\|\nu^2\|_2. \]

Combining (3.4) and (3.5), and applying Young’s inequality, we deduce that

(3.6)  \[ \|\nu^4\|_\infty \leq c \rho^{1/2}\|\nu^4\|_2^{1/2}\|\text{curl} \nu^4\|_\infty^{1/2} + c \rho^{-1}\|\nu^2\|_2. \]

We estimate the second term on the right-hand side of (3.2) by using (3.6) and then apply Young’s inequality. This leads to the estimate (3.1). \[ \blacksquare \]

Proof of Theorem 1.1 Let $v \in C([-1,0]; W^{2,q}(\mathbb{R}^3))$, $3 < q < +\infty$, be a solution to the Euler equation (1.1) satisfying (1.15) for some ball $B(x_*, R_0)$.

In our discussion below let $\xi_* := (\sqrt{x_{1,*}^2 + x_{2,*}^2}, x_{*,3}) \in \mathbb{R}^2$. In order to apply Theorem 2.1 our first aim will be to check that for all $0 < R < R_0$ the following conditions holds

(3.7)  \[ \int_{-1}^{0} \|\tilde{w}(t)\|_{L^\infty(B(\xi_*, R))} dt < +\infty, \]
where $\vec{v} = (v^r, v^3)$, and $B(\xi, R) = \{y \in \mathbb{R}^2 | |y - \xi| < R\}.$

**Proof of (3.7):** Observing (1.14), we see that

$$\partial_t \omega^\theta + \vec{v} \cdot \vec{\nabla} \omega^\theta = \frac{v^r}{r} \omega^\theta + 2\frac{v^\theta \partial_3 v^\theta}{r} \quad \text{in } \mathbb{R}^2_+ \times (-1, 0),$$

where $\vec{\nabla} = (\partial_r, \partial_3)$.

Let $0 < R < R_0$ be arbitrarily chosen, but fixed. Set $R_1 = \frac{R_0 + R}{2}$. Let $\phi \in C^\infty_c(B(\xi, R_0))$ such that $0 \leq \phi \leq 1$, $\phi \equiv 1$ on $B(\xi, R_1)$ and $|\nabla \phi| \leq c(R_0 - R)^{-1}$. We multiply both sides of (3.8) by $\phi \text{sign}(\omega^\theta)|\omega^\theta|^{-\frac{1}{2}}$, and set $V = \phi^2|\omega^\theta|^{\frac{1}{2}}$. This gives

\begin{equation}
\partial_t V + \vec{v} \cdot \vec{\nabla} V = 6\nabla \phi \cdot \vec{v} \phi^5 |\omega^\theta|^{\frac{1}{2}} + \frac{v^r}{r} V + 2\frac{\phi^2 v^\theta \partial_3 v^\theta}{r \text{sign}(\omega^\theta)V} \quad \text{in } \mathbb{R}^2_+ \times (-1, 0).
\end{equation}

Let $(\rho, z, s) \in B(\xi, R_1) \times (-1, 0)$ be arbitrarily chosen. By $X(\tau) : (-1, s) \to \mathbb{R}^2_+$ we denote the trajectory such that

$$\dot{X}(\tau) = \vec{v}(X(\tau), \tau), \quad X(s) = \xi_0 = (\rho, z).$$

We claim that there exists a constant $c > 0$ independent of $(\rho, z, s)$ such that

$$|V(\rho, z, s)| \leq \max\{(-s)^{-\frac{1}{2}}, \|\omega^\theta(1)\|_\infty\}$$

\begin{equation}
+ c \int_{t_0}^s \left(\|\omega^\theta(\tau)\|_\infty + \|\partial_3 v^\theta(\tau)\|_\infty\right) d\tau + c.
\end{equation}

In case $|V(\rho, z, s)| \leq (-s)^{-\frac{1}{2}}$, (3.10) is obvious. Thus, without the loss of generality we may assume that $V(\rho, z, s) > (-s)^{-\frac{1}{2}}$. There are two cases, either $V(X(\tau), \tau) > (-\tau)^{-\frac{1}{2}}$ for all $\tau \in (-1, s)$, then we set $t_0 = -1$, or there exists $-1 \leq t_0 < s$ such that

$$V(X(t_0), t_0) = (-t_0)^{-\frac{1}{2}}, \quad V(X(\tau), \tau) \geq (-\tau)^{-\frac{1}{2}} \quad \forall \tau \in (t_0, s).$$

The sign of $\omega^\theta$ does not change in $[t_0, s]$ (since $|\omega^\theta|$ does not touch zero), we may assume that $\text{sign}(\omega^\theta) = 1$ in $[t_0, s]$. Using the chain rule, we derive from (3.9)

\begin{equation}
\frac{d}{d\tau} V(X(\tau), \tau) = F(\tau) + G(\tau),
\end{equation}

where we set

$$F(\tau) = 6r(X(\tau))^{-1} \nabla \phi(X(\tau)) \cdot \vec{v}(X(\tau), \tau) \phi(X(\tau))^5 \omega^\theta(X(\tau), \tau)^{\frac{1}{2}}$$

$$+ r(X(\tau))^{-1} V(X(\tau), \tau) v^r(X(\tau), \tau),$$

$$G(\tau) = 2\frac{\phi(X(\tau))^2 v^\theta(X(\tau), \tau) \partial_3 v^\theta(X(\tau), \tau)}{r(X(\tau)) V(X(\tau), \tau)}, \quad \tau \in (t_0, s).$$
Integrating both sides of (3.11) over \((t_0, s)\), we are led to

\[
V(R, z, s) \leq \max\{(-t_0)^{-\frac{1}{2}}, V(X(-1))\} + \int_{t_0}^{s} (F(\tau) + G(\tau))d\tau
\]

(3.12)

\[
\leq \max\{(-s)^{-\frac{1}{2}}, \|\omega^{\theta}(-1)\|_{L^{\infty}(\mathbb{R}^2)}^{\frac{1}{2}}\} + \int_{t_0}^{s} (F(\tau) + G(\tau))d\tau.
\]

Firstly, it is readily seen that

\[
F(\tau) \leq c\|\bar{v}(\tau)\phi^5\|_{\infty}\|\omega^{\theta}(\tau)\|_{\frac{1}{2}}^s.
\]

Thanks to (3.1) with \(u = \bar{v}(t)\), using Young’s inequality, and recalling that \(v \in L^{\infty}(-1, 0; L^2(\mathbb{R}^3))\), we get

\[
F(\tau) \leq c\|\bar{v}(\tau)\phi^4\|_{\frac{1}{2}}\|\omega^{\theta}(\tau)\|_{\infty} + c\|\bar{v}(\tau)\phi^2\|_2\|\omega^{\theta}(\tau)\|_{\frac{1}{2}}^s
\]

\[
\leq c\|r^{\frac{1}{2}}\bar{v}(\tau)\|_{\frac{1}{2}}\|\omega^{\theta}(\tau)\|_{\infty} + c\|r^{\frac{1}{2}}\bar{v}(\tau)\|_2\|\omega^{\theta}(\tau)\|_{\frac{1}{2}}^s
\]

(3.13)

\[
\leq c\|\omega^{\theta}(\tau)\|_{\infty} + c,
\]

where \(c > 0\), depending only on \((R_0 - R)^{-1}\).

Secondly,

\[
|G(\tau)| \leq c(-\tau)^{\frac{1}{4}}\|rv^{\theta}\|_{L^{\infty}(B(\xi, R_0))}\|\partial_3 v^{\theta}(\tau)\|_{L^{\infty}(B(\xi, R_0))} \leq c\|\partial_3 v^{\theta}(\tau)\|_{L^{\infty}(B(\xi, R_0))}
\]

(3.14)

where the constant \(c\) depends only on \((R_0 - R)^{-1}\).

Inserting (3.13) and (3.14) into (3.12), we arrive at

\[
|\omega^{\theta}(\rho, z, s)|^{\frac{1}{2}} = V(\rho, z, s)
\]

\[
\leq \max\{(-s)^{-\frac{1}{2}}, \|\omega^{\theta}(-1)\|_{\infty}\}
\]

\[
+ c\int_{t_0}^{s} \|\omega^{\theta}(\tau)\|_{L^{\infty}(B(\xi, R_0))} + \|\partial_3 v^{\theta}(\tau)\|_{L^{\infty}(B(\xi, R_0))}d\tau + c
\]

with a constant \(c > 0\) depending on \((R_0 - R)^{-1}\) but independent of \((\rho, z, s)\). This completes the proof of (3.10).

Accordingly,

\[
\|\omega^{\theta}(s)\|_{L^{\infty}(B(\xi, R_1))} \leq cs^{-\frac{1}{2}} + c\int_{t_0}^{s} \|\omega(\tau)\|_{L^{\infty}(B(\xi, R_0))}d\tau,
\]

(3.15)

with a constant \(c > 0\) depending on \((R_0 - R)^{-1}\), but not on \(z\) and \(s\).

On the other hand, for given \((\rho, z, s) \in B(\xi, R)\), we may choose a cut-off function \(\phi \in C_c^{\infty}(B((\rho, z), \sigma))\), with \(\sigma = \frac{R_0 - R}{4}\) such that \(\phi(\rho, z) = 1\) and \(|\nabla \phi| \leq c\sigma^{-1}\). Then we
apply (3.1) with \( u = \tilde{v}(s) \), which shows that

\[
|\tilde{v}(\rho, z, s)| \leq c\|\tilde{v}(s)\phi^4\|_2^{\frac{1}{2}}\|\omega^\theta(s)\|_{L^\infty(B(\xi, R_1))}^\frac{1}{2} + c\sigma^{-1}\|\tilde{v}(s)\phi^4\|_2
\]

\[
\leq c\varepsilon_1\|r\frac{1}{2}\tilde{v}(s)\|_2^{\frac{1}{2}}\|\omega^\theta(s)\|_{L^\infty(B(\xi, R_1))}^\frac{1}{2} + c\sigma^{-1}x_1^\frac{1}{2}\|r\frac{1}{2}\tilde{v}(s)\|_2
\]

Thus, combining (3.16) and (3.15), we get

\[
(3.16)
\]

\[
|\tilde{v}(s)|_{L^\infty(B(\xi, R))} \leq c(1 + \|\omega^\theta(s)\|_{L^\infty(B(\xi, R_1))}^\frac{1}{2}) \leq c\varepsilon_1\|\omega^\theta(s)\|_{L^\infty(B(\xi, R_1))}^\frac{1}{2} + c \int_{t_0}^s \|\omega(\tau)\|_{L^\infty(B(x_*, R_0))}d\tau.
\]

Since the right-hand side of (3.17) belongs to \( L^1(-1, 0) \), we have (3.7).

Let \( 0 < R < \rho_* \), where \( \rho_* = \sqrt{x_1^2 + x_2^2} \) is given in Theorem 1.1. Setting

\[
(r, x_3) := \left(\rho_* + R y_1, x_3, x_2 + R y_2\right), \quad (y_1, y_2) \in B(1),
\]

we define

\[
u(y_1, y_2, t) = r(v^r(r, x_3, t), v^3(r, x_3, t))
\]

\[
\theta(y_1, y_2, t) = (r v^\theta(r, x_3, t))^2
\]

\[
w(y_1, y_2, t) = \frac{\omega^\theta(r, x_3, t)}{r}
\]

we see that \((u, \theta, \omega)\) solves the system (2.1), (2.2), (2.3) in \( B(1) \times (-1, 0) \) with

\[
a(y) = \frac{1}{R(\rho_* + R y_1)}, \quad b_1(y) = 0, \quad b_2(y) = \frac{1}{R(\rho_* + R y_1)},
\]

while the equation \( \omega^\theta = \partial_3 v^r - \partial_r v^3 \) turns into

\[
\text{curl } u := \partial_1 u_2 - \partial_2 u_1 = d(y) w + e(y) \cdot u
\]

in \( B(1) \times (-1, 0) \), where

\[
d(y) = R(\rho_* + R y_1)^2, \quad e_1(y) = 0, \quad e_2(y) = \frac{R}{\rho_* + R y_1}.
\]

Obviously, \( a, b, d, e \in C^\infty(B(1)) \). Furthermore, by our assumption (1.15) we get (2.6). Indeed, recalling the relation \( \partial_3 v^\theta = \omega^r \) and \( \partial_3 v^3 = \omega^3 - \frac{v^\theta}{r} \), we see that

\[
\partial_1 \theta = 2 R v^\theta \omega^3(r, x_3), \quad \partial_2 \theta = -2 R v^\theta \omega^3(r, x_3)
\]

Taking into account of the fact that \( v^\theta \) preserved along the particle trajectories (cf. (1.12)), (2.6) follows from (1.15).
Thus $(u, \theta, w)$ solves (2.3), (2.1), (2.3), (2.4), and (2.6) holds. In addition thanks to (3.7) the condition in (2.6) also holds. In order to apply Theorem 2.1 it only remains to verify that $u \in C_w([-1, 0]; L^2(B(1)))$. Indeed, recalling that $v \in L^\infty(-1, 0; L^2(\mathbb{R}^3))$, and noting that the following identity

$$
\int^t_{-1} \int_{\mathbb{R}^3} v(s) \cdot \partial_t \varphi(s) + v(s) \otimes v(s) : \nabla \varphi(s) dxds = \int_{\mathbb{R}^3} v(t) \cdot \varphi(t) dx
$$

holds true for all $t \in (-1, 0)$ and for all $\varphi \in C^\infty_c(\mathbb{R}^3 \times (-1, 0])$ with $\nabla \cdot \varphi = 0$. Therefore there exists a unique $v(0) \in L^2(\mathbb{R}^3)$ such that $v(t) \rightharpoonup v(0)$ weakly in $L^2(\mathbb{R}^3)$ as $t \nearrow 0$. By this definition of $v$ we get $v \in C_w([-1, 0]; L^2(\mathbb{R}^3))$. Thus, by virtue of the definition of $u$ we get $u \in C_w([-1, 0]; L^2(B(1)))$. Accordingly, we are in a position to apply Theorem 2.1. This completes the proof of Theorem 1.1. \hfill ■

**Proof of Theorem 1.2.** Since (1.16) implies (1.15), the assertion of Theorem 1.2 is an immediate consequence of Theorem 1.1. \hfill ■

### 4  Proof of Theorem 1.5

Given $R > 0$, we denote below $H_R = \{ x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 > R^2 \}$. Thanks to Theorem 1.1 the statement of Theorem 1.5 will be an immediate consequence of the following.

**Lemma 4.1.** Let $v \in C([-1, 0); W^{1,\infty}(\mathbb{R}^3)) \cap L^\infty(-1, 0; L^2(\mathbb{R}^3))$, $2 < q < +\infty$, be an axisymmetric solution to (1.1) in $\mathbb{R}^3 \times (-1, 0)$. If the for $0 < R < +\infty$ the following two conditions are fulfilled

$$
(4.1) \quad \int^0_{-1} (-t) \| \nabla v^\theta(t) \|_{L^\infty(H_R)} dt < +\infty, \quad \int^0_{-1} \| v^\tau(t) \|_{L^\infty(H_R)} dt < +\infty.
$$

Then

$$
(4.2) \quad \int^0_{-1} \| \omega^\theta(t) \|_{L^\infty(H_R)} dt < +\infty,
$$

$$
(4.3) \quad \limsup_{t \to 0^-} (-t) \| \omega^\theta(t) \|_{L^\infty(H_R)} = 0.
$$

**Proof:** Let $0 < R < +\infty$ be fixed. According to (4.1) there exists $t_0 \in (-1, 0)$, such that

$$
(4.4) \quad \int^0_{t_0} \| v^\tau(s) \|_{L^\infty(H_R)} ds \leq \frac{R}{16}.
$$

We now set $\varrho(r, t) = r + 8R^{-1}(r - R) \int^0_t \| v^\tau(s) \|_{L^\infty(H_R)} ds$. Owing to (4.4) we see that for all $t \in (t_0, 0)$

$$
(4.5) \quad \varrho(r, t) \geq r + \frac{r - R}{2} = \frac{3r - R}{2} \geq \frac{R}{4} \quad \forall \frac{R}{2} \leq r < +\infty.
$$

31
Clearly,
\[
\partial_t \varrho(r, t) = -8 R^{-1} \| v^r(s) \|_{L^\infty(H_R^{\frac{3}{4}})} (r - R),
\]
\[
\partial_r \varrho(r, t) = 1 + 8 R^{-1} \int_0^t \| v^r(s) \|_{L^\infty(H_R^{\frac{3}{4}})} ds.
\]

We define
\[
V(r, x_3, t) = v(\varrho(r, t), x_3, t),
\]
\[
\Theta(r, z, t) = \vartheta(\varrho(r, t), x_3, t),
\]
\[
\Omega(r, x_3, t) = \varrho(r, t)^{-1} \omega^\vartheta(\varrho(r, t), z, t), \quad (r, z, t) \in H_R^{\frac{3}{2}} \times (t_0, 0).
\]

In addition, define
\[
\begin{cases}
W^r(r, x_3, t) = \frac{-\partial}_t \varrho(r, t) + V^r(r, x_3, t), \\
W^3(r, x_3, t) = V^3(r, x_3, t),
\end{cases}
\]
\[
(r, x_3, t) \in H_R^{\frac{3}{2}} \times (t_0, 0).
\]

Note that, (4.5) implies
\[
\varrho(r, t) \geq \frac{r}{2} \quad \forall \quad \frac{R}{2} < r < +\infty, t_0 < t \leq 0.
\]

Therefore the functions \(V, \Theta\) and \(\Omega\) are well defined on \(H_R^{\frac{3}{2}} \times (t_0, 0)\).

To proceed, we verify
\[
W^r(r, x_3, t) \leq -\frac{4}{3} \| v^r(t) \|_{L^\infty(H_R^{\frac{3}{4}})} \quad \forall \quad (r, x_3, t) \in (\overline{H_R^{\frac{3}{2}}} \setminus H_R^{\frac{3}{4}}) \times (t_0, 0).
\]

In fact, according to (1.13), we estimate
\[
\partial_r \varrho(r, t) W^r(r, x_3, t) = -\partial_t \varrho(r, t) + V^r(r, x_3, t)
\]
\[
= 8 R^{-1} (r - R) \| v^r(t) \|_{L^\infty(H_R^{\frac{3}{4}})} + V^r(r, x_3, t)
\]
\[
\leq -2 \| v^r(t) \|_{L^\infty(H_R^{\frac{3}{4}})} + \| V^r(t) \|_{L^\infty(H_R^{\frac{3}{4}})}
\]
\[
\leq -2 \| v^r(t) \|_{L^\infty(H_R^{\frac{3}{4}})} + \| v^r(t) \|_{L^\infty(H_R^{\frac{3}{4}})}.
\]

Since \(1 \leq \partial_r \varrho(r, t) \leq \frac{3}{2} \) by (4.4) it follows (4.7).

By the chain rule we see that (1.13) turns into the following equations in \(H_R^{\frac{3}{2}} \times (t_0, 0)\)
\[
\partial_t \Omega + W \cdot \nabla \Omega = 2 \frac{\Theta \partial_3 \Theta}{\varrho(r, t)^2}.
\]

For \((r, x_3, s) \in H_R^{\frac{3}{2}} \times (t_0, t)\) by \(X = (X^r, X^3) = X(r, x_3, s; \cdot) : [t_0, s] \to \mathbb{R}^2\) we denote the particle trajectory such that
\[
\dot{X}(\tau) = W(X(\tau), \tau), \quad X(s) = (r, x_3).
\]
Let \((r, x_3, s) \in H_{\frac{R}{2}} \times (t_0, 0)\).

We claim that \(X(\tau) \in H_{\frac{R}{2}}\) for all \(\tau \in [t_0, s]\). Otherwise, there exists \(\tau_0 \in [t_0, s]\) such that \(X(\tau_0) \in \partial H_{\frac{R}{2}} = \{r = \frac{R}{2}\}\) and \(X(\tau) \in H_{\frac{R}{2}}\) for all \(\tau \in (\tau_0, s]\). This gives

\[
\dot{X}(\tau_0) = W^r(X(\tau_0), \tau_0) \geq 0,
\]

which contradicts (4.7), since \(\|v^r(t)\|_{L^\infty(H_{R/2})}\) must be strictly positive. Thus, the claim is proved. On the other hand, by the chain rule, (4.8) gives

\[
\frac{d}{d\tau}\Omega(X(\tau), \tau) = 2\frac{\Theta(X(\tau), \tau)\partial_3\Theta(X(\tau), \tau)}{\varrho(X^r(\tau), \tau)^2}.
\]

Recalling that \(X(s) = (r, x_3)\), integration over \((s_0, s)\), \(t_0 \leq s_0 \leq t\), yields

\[
\Omega(r, x_3, s) = \Omega(X(s_0), s_0) + 2\int_{s_0}^{s} \frac{\Theta(X(\tau), \tau)\partial_3\Theta(X(\tau), \tau)}{\varrho(X^r(\tau), \tau)^2}d\tau.
\]

Accordingly, using (4.6), and noting that \(|\varrho(r, t)\Theta(r, x_3, t)| = |\varrho(r, t)v^\theta(\varrho(r, t), x_3, t)| \leq \|rv^\theta\|_\infty \leq c < +\infty\), we get

\[
\|\Omega(s)\|_{L^\infty(H_{R/2})} \leq \|\Omega(s_0)\|_{L^\infty(H_{R/2})} + cR^{-3}\int_{s_0}^{s} \|\partial_3\Theta(\tau)\|_{L^\infty(H_{R/2})}d\tau.
\]

\[(4.9)\]

\[
\leq \|\Omega(s_0)\|_{L^\infty(H_{R/2})} + cR^{-3}\int_{s_0}^{s} \|\partial_3v^\theta(\tau)\|_{L^\infty(H_{R/2})}d\tau.
\]

In (4.9) we take \(s_0 = t_0\), and integrate both sides over \((t_0, t)\). This leads to

\[
\int_{t_0}^{t} \|\Omega(s)\|_{L^\infty(H_{R/2})}ds \leq (t - t_0)\|\Omega(s_0)\|_{L^\infty(H_{R/2})} - cR^{-3}\int_{t_0}^{t} (-s)\int_{s_0}^{s} \|\partial_3v^\theta(\tau)\|_{L^\infty(H_{R/2})}d\tau ds
\]

\[
\leq (t - t_0)\|\Omega(t_0)\|_{L^\infty(H_{R/2})} + cR^{-3}\int_{t_0}^{t} (-s)\|\partial_3v^\theta(s)\|_{L^\infty(H_{R/2})}ds.
\]

Observing (4.1), this proves that (4.2). To verify (4.3), we first note that (4.9) multiplied by \((-s)\) implies

\[
(4.10)\quad (-s)\|\omega(s)\|_{L^\infty(H_R)} \leq (-s)\|\omega(s_0)\|_{L^\infty(H_{R/2})} + cR^{-3}\int_{s_0}^{0} (-\tau)\|\partial_3v^\theta(\tau)\|_{L^\infty(H_{R/2})}d\tau.
\]

33
Applying \( \limsup \) as \( s \to 0^- \) to both sides of (4.10), we are led to

\[
\limsup_{s \to 0^-} (-s) \| \omega(s) \|_{L^\infty(H_R)} = \limsup_{s \to 0^-} (-s) \| \omega(t_0) \|_{L^\infty(H_{\frac{3}{4}})} + cR^{-3} \int_{s_0}^{0} (-\tau) \| \partial_3 v^\theta(\tau) \|_{L^\infty(H_{\frac{3}{4}})} d\tau 
\]

\[
\leq cR^{-3} \int_{s_0}^{0} (-\tau) \| \partial_3 v^\theta(\tau) \|_{L^\infty(H_{\frac{3}{4}})} d\tau.
\]

On the other hand, the integral on the right-hand side of this inequality tends to 0 as \( s_0 \to 0 \) we conclude (4.13). \( \blacksquare \)

### A Gagliardo-Nirenberg inequalities with cut-off

Below we denote the Hölder conjugate of \( q \) by \( q' = \frac{q}{q-1} \). Following similar argument to the proof of [10, Lemma 2.3], we get

**Lemma A.1.** Let \( \psi \in C_c^\infty(B(r)) \), \( 0 < r < +\infty \), such that \( 0 \leq \psi \leq 1 \) in \( B(r) \). For all \( u \in W^{1,q}(B(r)) \), \( 2 < q < +\infty \) with \( \nabla \cdot u = 0 \) a.e. in \( B(r) \) and for all \( m \geq 2 \) it holds

\[\| \nabla u\psi^m \|_q \leq c \| \text{curl} u\psi^m \|_q + c \| \nabla \psi \|_{L^\infty}^2 \| u\psi^{m-a} \|_2, \tag{A.1}\]

\[\| u\psi^{m-k} \|_q \leq c \| u\psi^{m-\frac{2k}{q}} \|_{L^{\frac{q'}{2}}} 2 \| \text{curl} u\psi^m \|_q^{1-\frac{q'}{2}} + c \| \nabla \psi \|_{L^\infty}^{\frac{q-2}{q}} \| u\psi^{m-ka} \|_2. \tag{A.2}\]

**Lemma A.2.** Let \( \psi \in C_c^\infty(B(r)) \), \( 0 < r < +\infty \), such that \( 0 \leq \psi \leq 1 \) in \( B(r) \). For all \( u \in W^{1,q}(B(r)) \), \( 2 < q < +\infty \) for all \( m \geq \frac{2q-2}{q} \) it holds

\[\| u\psi^m \|_\infty \leq c \| u\psi^{m-\frac{q}{2}} \|_2^{1-\frac{q'}{2}} \| \nabla u\psi^m \|_q^{\frac{q'}{2}} + c \| \nabla \psi \|_{L^\infty} \| u\psi^{m-\frac{q}{2}} \|_2. \tag{A.3}\]

For further discussion below we recall the notion of the local BMO space. For \( 0 < r < +\infty \) we say \( u \in BMO(B(r)) \) if

\[|u|_{BMO(B(r))} := \sup_{\substack{x \in B(r) \\ 0 < \rho \leq r}} r^{-2} \int_{B(x,\rho) \cap B(r)} |u(x) - u_{B(x,\rho) \cap B(r)}| dx < +\infty, \]

Here we have used the following notation for the mean for a given set \( \Omega \subset \mathbb{R}^2 \) and \( v \in L^1(\Omega) \)

\[v_\Omega = \int_{\Omega} v \, dx = \frac{1}{m(\Omega)} \int_{\Omega} v \, dx,
\]

where \( m \) stands for the two dimensional Lebesgue measure.
Finally, combining (A.5) and (A.7), and applying Young’s inequality, we obtain (A.4).

\[ \| \nabla \psi^6 \|_{BMO} \leq c \| (\text{curl } u) \psi^5 \|_{BMO} + c \left\{ r \| \nabla \psi \|_\infty^3 + \| \nabla \psi \|_\infty^2 \right\} \| u \psi \|_2. \]

**Proof:** Using Calderón-Zygmund’s inequality, we find that

\[ \| (\nabla u) \psi^6 \|_{BMO} \leq c \| \text{curl}(\psi^6 u) \|_{BMO} + c \| u \cdot \nabla \psi^6 \|_{BMO} \]

\[ \leq c \| (\text{curl } u) \psi^6 \|_{BMO} + c \| \nabla \psi \|_\infty \| u \psi \|_5. \]

On the other hand, in view of (A.3) with \( q = 4 \) and \( m = 5 \) we get

\[ \| u \psi^5 \|_\infty \leq c \| u \psi \|_2^3 \| \nabla u \psi^5 \|_4^2 + c \| \nabla \psi \|_\infty \| u \psi \|_2. \]

We estimate the first term on the right-hand side of (A.6) by (A.1) with \( q = 4 \) and \( m = 5 \). This together with [10, Lemma B.3] gives

\[ \| u \psi^5 \|_\infty \leq c \| u \psi \|_2^3 \| (\text{curl } u) \psi^5 \|_4^2 + c \| \nabla \psi \|_\infty \| u \psi \|_2. \]

\[ \leq cr^\frac{5}{4} \| u \psi \|_2^\frac{5}{2} \| (\text{curl } u) \psi^5 \|_BMO(B(r)) + c \| \nabla \psi \|_\infty \| u \psi \|_2. \]

Finally, combining (A.5) and (A.7), and applying Young’s inequality, we obtain (A.4).

Using the well known John-Nirenberg inequality, we can get the following

**Lemma A.4.** Let \( u \in BMO(B(r)) \). Then \( u \in \cap_{1 \leq q < \infty} L^q(B(r)) \), and it holds

\[ \| u \|_{L^q(B(r))} \leq cr^\frac{n}{q} \| u \|_{BMO(B(r))} + cr^{\frac{n}{q} - n} \| u \|_{L^1(B(r))} = cr^\frac{n}{q} \| u \|_{BMO(B(r))}. \]

For an elementary proof see [10, Lemma B.3].

Arguing as in [10] we shall show the following.

**Lemma A.5.** Let \( u \in W^{1,1}(B(r)) \) with \( \text{curl } u \in BMO(B(r)) \). Then for all \( \psi \in C^\infty_c(B(r)) \) with \( 0 \leq \psi \leq 1 \) we get

\[ \| (\text{curl } u) \psi^5 \|_{BMO} \leq c \left\{ 1 + r^3 \| \nabla \psi \|_\infty^3 \right\} (\| \text{curl } u \|_{BMO(B(r))} + cr^{-2} \| u \|_{L^2(B(r))}). \]

**Proof:** Assume \( r = 1 \). Let \( \eta \in C^\infty_c(B(1)) \) such that \( |\nabla \eta| \leq c \) and \( \int_{B(1)} \eta dx \geq c \), where \( c > 0 \) stands for a constant depending only on \( n \). For \( f \in L^1(B(1)) \) we define the mean

\[ \tilde{f}_{B(1)} = \frac{1}{B(1)} \int_{B(1)} \eta dx \int_{B(1)} f \eta dx. \]
First we see that
\[ \| \text{curl } u \|_{L^1(B(1))} = \| \text{curl } u - \text{curl } u_{B(1)} \|_{L^1(B(1))} + |\text{curl } u_{B(1)}| \leq c \| \text{curl } u \|_{BMO(B(1))} + c \| u \|_{L^1(B(1))}. \tag{A.10} \]
Using (A.10), we estimate for \( \rho \geq \frac{1}{2} \) and \( x_0 \in \mathbb{R}^3 \)
\[
\int_{B(x_0, \rho)} |(\text{curl } u) \psi^5 - (\text{curl } u \psi^5)_{B(x_0, \rho)}| \, dx \\
\leq c \| \text{curl } u \|_{L^1(B(1))} \\
\leq c \| \text{curl } u \|_{BMO(B(1))} + c \| u \|_{L^1(B(1))}.
\]
In case \( \rho \leq \frac{1}{2} \) and \( B(x_0, \rho) \cap B(1) \neq \emptyset \) there exists \( y_0 \in B(1) \) such that \( B(x_0, \rho) \subset B(y_0, 2\rho) \) and
\[
\int_{B(x_0, \rho)} |\text{curl } u \psi^5 - (\text{curl } u \psi^5)_{B(x_0, \rho)}| \, dx \\
\leq c \int_{B(y_0, 2\rho)} \int_{B(y_0, 2\rho)} |\text{curl } u(x) \psi^5(x) - |\text{curl } u(y) \psi^5(y)| \, dx \, dy \\
\leq c \| \text{curl } u \|_{BMO(B(1))} + c \int_{B(y_0, 2\rho)} \int_{B(y_0, 2\rho)} |\text{curl } u(x)\| \psi^5(x) - \psi^5(y)\| \, dx \, dy.
\]
By the fundamental theorem of differentiation and integration we calculate
\[
\psi^5(y) - \psi^5(x) = 5\psi^4(\xi_1) \nabla \psi(\xi_1) \cdot (y - x) \\
= 5\psi^4(x) \nabla \psi(\xi_1) \cdot (y - x) + 5(\psi^4(\xi_1) - \psi^4(x)) \nabla \psi(\xi_1) \cdot (y - x) \\
= 5\psi^4(x) \nabla \psi(\xi_1) \cdot (y - x) \\
+ \psi^2(\xi_2) \prod_{i=1}^{2} (6 - i) \nabla \psi(\xi_i) \cdot (\xi_{i-1} - x).
\]
For some \( \xi_i \in [x, y], i = 1, 2 \). This along with (A.10) yields
\[
\int_{B(y_0, 2\rho)} \int_{B(y_0, 2\rho)} |\text{curl } u(x)| \psi^5(x) - \psi^5(y) \| \, dx \, dy \\
\leq c \| \nabla \psi \|_{L^\infty(B(1))} \int_{B(y_0, 2\rho)} |\text{curl } u(x)| \psi^4(x) \, dx \\
+ c \| \nabla \psi \|_{L^\infty}^2 \| \text{curl } u \|_{L^1(B(1))} \\
\leq c \| \nabla \psi \|_{L^\infty} \| \text{curl } u \|_{L^1(B(1))} + c \| \nabla \psi \|_{L^\infty}^2 \| \text{curl } u \|_{L^1(B(1))}.
\]
By using Hölder’s inequality, we find that
\[
\| (\text{curl } u) \psi^4 \|_2 \leq \| \text{curl } u \|_{L^1(B(1))}^{\frac{1}{2}} \| (\text{curl } u) \psi^6 \|_4^{\frac{2}{3}} \\
\leq \| \text{curl } u \|_{L^1(B(1))}^{\frac{1}{2}} \| (\text{curl } u) \psi^5 \|_4^{\frac{2}{3}}.
\]
Applying the embedding $L^6(B(1)) \hookrightarrow \text{BMO}(B(1))$ (cf. Lemma A.4), we get

$$
\|(\text{curl } u) \psi^4\|_2 \leq c \| \text{curl } u \|_{L^1(B(1))}^{\frac{3}{2}} \|(\text{curl } u) \psi^5\|_{\text{BMO}}^{\frac{3}{2}}.
$$

Combining the above inequalities, and applying Young’s inequality together with (A.10), we arrive at

$$
\|(\text{curl } u) \psi^5\|_{\text{BMO}} \leq c \| \text{curl } u \|_{\text{BMO}(B(1))} + c \| \nabla \psi \|_{L^\infty} \| \text{curl } u \|_{L^1(B(1))}
$$

(A.11)

$$
\leq c(1 + \| \nabla \psi \|_{L^\infty}^3) \left( \| \text{curl } u \|_{\text{BMO}(B(1))} + c \| u \|_{L^1(B(1))} \right).
$$

Whence, (A.9) follows immediately from (A.11) by standard scaling argument.  ■

Combining Lemma A.5 and Lemma A.3 we get

**Corollary A.6.** Let $\psi \in C_c^\infty(B(r))$, $0 < r < +\infty$, with $0 \leq \psi \leq 1$. For every $u \in W^{1,1}(B(r))$ such that $\nabla \cdot u = 0$ and $\text{curl } u \in \text{BMO}(B(r))$ it holds

$$
\| \nabla u \psi^6 \|_{\text{BMO}(B(r))} \leq c \left( 1 + r^3 \| \nabla \psi \|_{L^\infty}^3 \right) \| \text{curl } u \|_{\text{BMO}(B(r))}
$$

(A.12)

$$
+ c \left( r^{-2} + r \| \nabla \psi \|_{L^\infty}^3 \right) \| u \|_{L^2(B(r))}.
$$

**Proof:** Combining (A.4) and (A.9) along with Young’s inequality, we infer

$$
\| \nabla u \psi^6 \|_{\text{BMO}(B(r))}
$$

$$
\leq c \left( 1 + r^3 \| \nabla \psi \|_{L^\infty}^3 \right) \left( \| \text{curl } u \|_{\text{BMO}(B(r))} + cr^{-2} \| u \|_{L^2(B(r))} \right)
$$

$$
+ c \left( r \| \nabla \psi \|_{L^\infty}^3 + \| \nabla \psi \|_{L^\infty}^2 \right) \| u \|_{L^2(B(r))}
$$

(A.13)

$$
\leq c \left( 1 + r^3 \| \nabla \psi \|_{L^\infty}^3 \right) \| \text{curl } u \|_{\text{BMO}(B(r))} + c \left( r^{-2} + r \| \nabla \psi \|_{L^\infty}^3 \right) \| u \|_{L^2(B(r))}.
$$

Whence, (A.12).  ■

**Lemma A.7.** Let $u \in W^{2,q}(B(r))$, $2 \leq q < +\infty$. Let $m, k \in \mathbb{R}$ such that $2 \leq m < +\infty$ and $0 < k \leq 2m$. Then for every $\psi \in C_c^\infty(B(r))$ it holds

$$
\| \nabla u \psi^m \|_q \leq c \| \nabla u \psi^{2m-k} \|_q^\frac{1}{2} \| \nabla^2 u \psi^k \|_q^\frac{1}{2} + c \| \nabla \psi \|_{L^\infty} \| u \psi^{m-1} \|_q.
$$

(A.14)

If in addition, if $\nabla \cdot u = 0$ almost everywhere in $B(r)$, then for $2 \leq m < +\infty$ and $m + 1 < k \leq 2m$ it holds

$$
\| \nabla^2 u \psi^k \|_q \leq c \| (\text{curl } u) \psi^k \|_q + c \| \nabla \psi \|_{L^\infty} \| u \psi^{k-2} \|_q,
$$

(A.15)

$$
\| \nabla u \psi^m \|_q \leq c \| \nabla u \psi^{2m-k} \|_q^\frac{1}{2} \| (\text{curl } u) \psi^k \|_q^\frac{1}{2} + c \| \nabla \psi \|_{L^\infty} \| u \psi^{m-1} \|_q.
$$

(A.16)
Proof: Applying integration by parts, and using Hölder’s inequality, we get
\[
\|\nabla u \psi^m\|_q^2 = - \int_{B(\rho)} u \nabla u \cdot \nabla |\nabla u|^{q-2} \psi^m dx - \int_{B(\rho)} u \nabla u |\nabla u|^{q-2} \cdot \nabla \psi^m dx
\]
\[
\leq c \|u \psi^{2m-k}\|_q \|\nabla^2 u \psi^k\|_q \|\nabla u \psi^m\|_{q-2}
\]
\[
+ c \|\nabla \psi\|_\infty \|u \psi^{m-1}\|_{L^q(B(\rho))} \|\nabla u \psi^m\|_{q-1},
\]
and Young’s inequality gives (A.14).

Suppose \( \nabla \cdot u = 0 \) almost everywhere in \( B(r) \). We first apply (A.1) with \( \nabla u \) in place of \( u \) and \( m = k \), and then use (A.14) with \( m = k - 1 \). This gives
\[
\|\nabla^2 u \psi^k\|_q \leq c \|\nabla \psi\|_\infty \|u \psi^{k-1}\|_q
\]
\[
\leq c \|\nabla \psi\|_\infty \|u \psi^{k-2}\|_q \|\nabla^2 u \psi^k\|_q
\]
Applying Young’s inequality, we obtain (A.15). The estimate (A.16) is now an immediate consequence of (A.14) and (A.15).

Combining Lemma A.1 and Lemma A.7, we get the following

Corollary A.8. For all \( u \in W^{2,q}(B(r)) \), for all \( \psi \in C^\infty_c(B(r)) \) with \( 0 \leq \psi \leq 1 \) and for all \( k > 5 \) we get
\[
(A.17) \quad \|\nabla^2 u \psi^k\|_q \leq c \|\nabla \psi\|_\infty \|u \psi^{k-2}\|_q \leq \frac{c}{\sqrt{q}} \|u \psi^{k-4}\|_2.
\]

Proof: Let \( k > q \). The estimate (A.2) with \( m = k \) and \( k = 2 \) reads
\[
\|u \psi^{k-2}\|_q \leq c \|u \psi^{k-2}\|_q \|\nabla \psi\|_\infty \|u \psi^{k-4}\|_2
\]
Combining this inequality with (A.15), and applying Young’s inequality, we obtain (A.17).

Next, we shall establish a Gagliardo-Nirenberg inequality involving the \( BMO \) norm of the gradient, which improves the case with corresponding \( L^\infty \) norm.

Lemma A.9. For \( u \in L^2(\mathbb{R}^n) \cap W^{1,1}_{loc}(\mathbb{R}^n) \) such that \( \nabla u \in BMO \) it holds
\[
(A.18) \quad \|u\|_\infty \leq c \|u\|_2 \|\nabla u\|_{BMO}^m.
\]

Proof: First let us show that for every \( u \in L^2(\mathbb{R}^n) \cap W^{1,1}_{loc}(\mathbb{R}^n) \) such that \( \nabla u \in BMO \) it holds
\[
(A.19) \quad \|u\|_\infty \leq c (\|u\|_2 + |\nabla u|_{BMO})
\]
for some constant \( c > 0 \) depending only on \( n \). To prove (A.19) let \( x_0 \in \mathbb{R}^n \) be fixed. We take a cut off function \( \zeta \in C^\infty_c(Q(x_0,1)) \) such that \( 0 \leq \zeta \leq 1 \), \( |\nabla \zeta| \leq 3 \), and
\[
\int_{Q(x_0,1)} \eta dx \geq 2^{-n},
\]
and define the generalized mean
\[
\tilde{u}_{B(x_0,1)} = \frac{1}{Q(x_0,1)} \int_{Q(x_0,1)} u \zeta dx.
\]

38
By virtue of Sobolev embedding theorem, and Jensen’s inequality we find
\[
\|u - \nabla u_Q(x_0,1)(x - x_0)\|_{L^\infty(Q(x_0,1))} \\
\leq c\|u - \nabla u_Q(x_0,1)(x - x_0)\|_{L^2(Q(x_0,1))} + c\|\nabla u - \nabla u_Q(x_0,1)\|_{L^{2n}(Q(x_0,1))} \\
\leq c\|u\|_{L^2(Q(x_0,1))} + c|\nabla u_Q(x_0,1)| + c\|\nabla u - \nabla u_Q(x_0,1)\|_{L^{2n}(Q(x_0,1))}.
\]

Here, \(Q(x_0, 1)\) stands for the usual cube. Using integration by parts, we see that
\[
|\nabla u_Q(x_0,1)| \leq 2^n \left| \int_{Q(x_0,1)} u \nabla \zeta dx \right| \leq c\|u\|_{L^2(Q(x_0,1))}.
\]

Furthermore, employing John-Nirenberg’s inequality \([18]\), we get
\[
\|\nabla u - \nabla u_Q(x_0,1)\|_{L^{2n}(Q(x_0,1))} \leq c|\nabla u|_{BMO(Q(x_0,1))} \leq c|\nabla u|_{BMO}.
\]

Combining the last two inequalities, we obtain the desired estimate (A.19).

Next, given \(\lambda > 0\), we define \(u_\lambda(x) = u(\lambda x), x \in \mathbb{R}^n\). Applying the chain rule together with the transformation formula of the Lebesgue integral we find for any ball \(B(x, r) \subset \mathbb{R}^n\)
\[
\frac{1}{|B(x, r)|} \int_{B(x, r)} |\nabla u_\lambda - (\nabla u_\lambda)_{B(x, r)}| dy \\
= \frac{\lambda}{|B(\lambda x, \lambda r)|} \int_{B(\lambda x, \lambda r)} |\nabla u(y) - (\nabla u)_{B(\lambda x, \lambda r)}| dy.
\]

From this identity we easily deduce
\[(A.20) \quad |\nabla u_\lambda|_{BMO} = \lambda|\nabla u|_{BMO}.
\]

We now assume that \(|\nabla u|_{BMO} > 0\). Otherwise, since \(\nabla u\) is harmonic and \(u \in L^2(\mathbb{R}^n)\) would get \(u \equiv 0\). Thus, choosing
\[
\lambda = \|u\|_2^{\frac{2}{n+2}}|\nabla u|^{-\frac{2}{n+2}}_{BMO},
\]

by the aid of \((A.19)\) and \((A.20)\) we find
\[
\|u\|_\infty = \|u_\lambda\|_\infty \leq c(\|u_\lambda\|_2 + |\nabla u_\lambda|_{BMO}) = c(\lambda^{-\frac{n}{2}}\|u\|_2 + \lambda|\nabla u|_{BMO}) \\
= 2c\|u\|_2^{\frac{2}{n+2}}|\nabla u|^{\frac{n}{n+2}}_{BMO}.
\]

Whence, \((A.18)\). ■

In order to estimate the right-hand side of \((A.18)\) we use the Calderón-Zygmund inequality as follows
**Lemma A.10.** For every \( u \in L^2(\mathbb{R}^2) \cap W^{1,1}_{\text{loc}}(\mathbb{R}^2) \) with \( \nabla u \in BMO \) the following estimate holds true

\[
|\nabla u|_{BMO} \leq c \left( |\text{curl} \, u|_{BMO} + |\nabla \cdot u|_{BMO} \right).
\]

**Proof:** Denoting the Helmholtz projection by \( \mathbb{P} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \), we may write \( u = \mathbb{P} u + (u - \mathbb{P} u) \). Clearly, there exists potentials \( \Phi, \Psi \in C^1(\mathbb{R}^2) \) with \( \nabla \Phi, \nabla \Psi \in L^2(\mathbb{R}^2) \), such that

\[
u = \nabla \Phi + \nabla \perp \Psi.
\]

Having \( \nabla \cdot u = -\Delta \Psi \), by the Calderón-Zygmund inequality, we deduce that

\[
|\nabla^2 \Psi|_{BMO} \leq c |\nabla \cdot u|_{BMO}.
\]

Similarly, observing that \( -\Delta \Psi = \text{curl} \, u \), once more applying Calderón-Zygmund’s inequality, we find

\[
|\nabla^2 \Phi|_{BMO} \leq c |\text{curl} \, u|_{BMO}.
\]

From the last two estimates we obtain (A.21). \( \blacksquare \)

As an immediate consequence of Lemma A.9 and Lemma A.10, we get

**Corollary A.11.** For every \( u \in L^2(\mathbb{R}^2) \cap W^{1,1}_{\text{loc}}(\mathbb{R}^2) \) with \( \nabla u \in BMO \) and \( |\nabla \cdot u| \leq c |u| \) in \( \mathbb{R}^2 \) it holds

\[
\|u\|_{\infty} \leq c \left( 1 + \|u\|_{BMO}^{1/2} \|\text{curl} \, u\|_{BMO}^{1/2} \right).
\]

**B Gronwall type iteration lemma**

**Lemma B.1 (Iteration lemma).** Let \( \beta_m : [t_0, t_1] \to \mathbb{R}, \ m \in \mathbb{N} \cup \{0\} \) be a sequences of continuous functions. Furthermore let \( \alpha, g \in L^1(t_0, t_1) \) with \( \alpha \geq 0 \). We assume that the following recursive of integral inequality holds true for a constant \( C > 0 \)

\[
\beta_m(t) \leq Cm + g(t) + \int_{t_0}^{t} \alpha(s) \beta_{m+1}(s) ds, \quad m \in \mathbb{N} \cup \{0\}.
\]

Furthermore, suppose that there exists a constant \( K > 0 \) such that

\[
\max_{t \in [t_0, t_1]} |\beta_m(t)| \leq K^m \quad \forall \ m \in \mathbb{N}.
\]

Then the following inequality holds true for all \( t \in [t_0, t_1] \)

\[
\beta_0(t) \leq g(t) + C \int_{t_0}^{t} \alpha(s) \int_{t_0}^{t} \alpha(\tau) ds e^{s - \tau} \, d\tau + \int_{t_0}^{t} \alpha(s) g(s) e^{s - \tau} \int_{t_0}^{s} \alpha(\tau) d\tau ds.
\]
Proof: Iterating (B.1) \(m\)-times, we see that

\[
\beta_0(t) \leq g(t) + \int_{t_0}^t \alpha(s_1) \beta_1(s_1) ds_1
\]

\[
\leq g(t) + \int_{t_0}^t \alpha(s_1)(C + g(s_1)) ds_1 + \int_{t_0}^t \int_{t_0}^{s_1} \alpha(s_1) \alpha(s_2)(2C + g(s_2)) ds_2 ds_1
\]

\[
+ \ldots + \int_{t_0}^t \int_{t_0}^{s_1} \int_{t_0}^{s_1} \ldots \int_{t_0}^{s_1} \alpha(s_1) \alpha(s_2) \ldots \alpha(s_m)(Cm + g(s_m)) ds_m \ldots ds_2 ds_1
\]

\[
+ \int_{t_0}^t \int_{t_0}^{s_1} \int_{t_0}^{s_1} \ldots \int_{t_0}^{s_1} \alpha(s_1) \alpha(s_2) \ldots \alpha(s_{m+1}) \beta_{m+1}(s_{m+1}) ds_{m+1} \ldots ds_2 ds_1
\]

\[
(B.4)
\]

\[
= g(t) + \int_{t_0}^t \alpha(s)(C + g(s)) ds + \sum_{k=2}^m I_k + J_m,
\]

where

\[
I_k = \int_{t_0}^t \int_{s_k}^{s_{k-1}} \ldots \int_{t_0}^{s_k} \alpha(s_1) \alpha(s_2) \ldots \alpha(s_{k-1})(Ck + g(s_k)) ds_{k-1} \ldots ds_2 ds_1, \quad k = 2, \ldots, m.
\]

With the help of Fubini’s theorem we compute

\[
I_k = \int_{t_0}^t \int_{s_k}^{s_{k-2}} \ldots \int_{t_0}^{s_k} \alpha(s_1) \alpha(s_2) \ldots \alpha(s_{k-1}) ds_{k-1} \ldots ds_2 ds_1 \alpha(s_k)(Ck + g(s_k)) ds_k
\]

Using the following identity

\[
(B.5)
\]

\[
\int_{s}^{t} \ldots \int_{s}^{s_{k-2}} \prod_{j=1}^{k-1} \alpha(s_j) ds_{k-1} \ldots ds_1 = \frac{1}{(k-1)!} \left( \int_{s}^{t} \alpha(\tau) d\tau \right)^{k-1},
\]

we obtain

\[
I_k = \int_{t_0}^t \alpha(s)(Ck + g(s)) \frac{1}{(k-1)!} \left( \int_{s}^{t} \alpha(\tau) d\tau \right)^{k-1} ds
\]

\[
= -\frac{C}{(k-1)!} \int_{t_0}^t \frac{d}{ds} \left( \int_{s}^{t} \alpha(\tau) d\tau \right)^{k} ds + \int_{t_0}^t \alpha(s) g(s) \frac{1}{(k-1)!} \left( \int_{s}^{t} \alpha(\tau) d\tau \right)^{k-1} ds
\]

\[
= \frac{C}{(k-1)!} \left( \int_{t_0}^t \alpha(\tau) d\tau \right)^{k} + \int_{t_0}^t \alpha(s) g(s) \frac{1}{(k-1)!} \left( \int_{s}^{t} \alpha(\tau) d\tau \right)^{k-1} ds.
\]
Furthermore, by our assumption on \( \{ \beta_m \} \) along with (B.3) with \( k = m + 1 \) we see that

\[
|J_m| = \left| \int_0^t \beta_{m+1}(s) \frac{1}{m!} \left( \int_s^t \alpha(\tau) d\tau \right)^m ds \right|
\]

\[
\leq K(t_1 - t_0) \frac{1}{m!} \left( K \int_{t_0}^{t_1} \alpha(\tau) d\tau \right)^m \to 0 \quad \text{as} \quad m \to +\infty.
\]

Therefore, letting \( m \to \infty \) in the right-hand side of (B.4), we arrive at

\[
\beta_0(t) \leq g(t) + C \int_{t_0}^t \alpha(s) ds \int_{s_0}^s \alpha(\tau)d\tau + \int_{t_0}^t \alpha(s) g(s) \int_{s_0}^s \alpha(\tau)d\tau ds.
\]

Whence, (B.3) follows. \( \square \)

\section{Continuity for solutions to the transport equation}

\textbf{Lemma C.1.} Given \( f \in L^q(\mathbb{R}^n \times (t_0,0)) \), \( 1 < q < +\infty \), and \( u \in L^\infty(t_0,0; W^{1,\infty}(\mathbb{R}^n)) \cap C(\mathbb{R}^n \times [t_0,0]) \), let \( h \in L^q(\mathbb{R}^n \times (t_0,0)) \) be a weak solution to

\begin{equation}
\partial_t h + u \cdot \nabla h = f \quad \text{in} \quad \mathbb{R}^n \times (t_0,0).
\end{equation}

Then, \( h \in C([t_0,0]; L^q(\mathbb{R}^n)) \), and it holds for all \( t \in [t_0,0] \)

\begin{equation}
\| h(t) \|^q_q = \| h(t_0) \|^q_q + \int_{t_0}^t \int_{\mathbb{R}^n} \nabla \cdot u |h|^q dx ds + g \int_{t_0}^t \int_{\mathbb{R}^n} fh |h|^{q-2} dx ds.
\end{equation}

\textbf{Proof:} By \( \eta \in C^\infty(\mathbb{R}(1)) \) we denote the usual mollifying kernel. We set \( \eta_\varepsilon(x) = \varepsilon^{-n} \eta(\varepsilon^{-1} x) \), \( 0 < \varepsilon < +\infty \), and define for \( v \in L^1_{\text{loc}}(\mathbb{R}^n) \)

\[
v_\varepsilon(x) = \int_{\mathbb{R}^n} v(x-y) \eta_\varepsilon(y) dy = \int_{\mathbb{R}^n} v(y) \eta_\varepsilon(x-y) dy, \quad x \in \mathbb{R}^n.
\]

Testing (C.1) with \( \eta_\varepsilon(x - \cdot) \), we see that \( h_\varepsilon \) solves

\begin{equation}
\partial_t h_\varepsilon + \nabla \cdot (uh)_\varepsilon = f_\varepsilon + (\nabla \cdot uh)_\varepsilon.
\end{equation}

Clearly, \( h_\varepsilon \in L^q(t_0,0; L^q(\mathbb{R}^n)) \) with \( \partial_t h_\varepsilon \in L^q(t_0,0; L^q(\mathbb{R}^n)) \). Eventually, redefining \( h_\varepsilon(t) \) on a set of measure zero in \([t_0,0]\), we may assume that \( h_\varepsilon \in C([t_0,0]; L^q(\mathbb{R}^n)) \).
We estimate for \((x, t) \in \mathbb{R}^n \times [t_0, 0]\)
\[
\nabla \cdot (u h_\varepsilon(x, t) - (u \cdot \nabla h_\varepsilon)(x, t)
= \int_{\mathbb{R}^n} u(x - y, t) h(x - y, t) \cdot \nabla \eta_\varepsilon(y) - u(x, t) h(x - y, t) \cdot \nabla \eta_\varepsilon(y) dy
= \int_{\mathbb{R}^n} (u(x - y, t) - u(x, t)) (h(x - y, t) - h(x, t)) \cdot \nabla \eta_\varepsilon(y) dy.
\]

Accordingly,
\[
(C.4) \quad |\nabla \cdot (u h_\varepsilon(x, t) - (u \cdot \nabla h_\varepsilon)(x, t)| \leq c \|\nabla u(t)\|_\infty \int_{B(\varepsilon)} |h(x - y, t) - h(x, t)| dy
\]

Multiplying this inequality by \(|h_\varepsilon(x, t)|^{q-1}\), integrating over \(\mathbb{R}^n\) with respect to \(x\), and applying Hölder’s inequality, we find
\[
\int_{\mathbb{R}^n} |\nabla \cdot (u h_\varepsilon(x, t) - (u \cdot \nabla h_\varepsilon)(x, t)| |h_\varepsilon(x, t)|^{q-1} dx
\leq c \|\nabla u(t)\|_\infty \int_{B(\varepsilon)} \int_{\mathbb{R}^n} |h(x - y, t) - h(x, t)| |h_\varepsilon(x, t)|^{q-1} dy dx
\leq c \|\nabla u(t)\|_\infty \int_{B(\varepsilon)} \int_{\mathbb{R}^n} |h(x - y, t) - h(x, t)| |h_\varepsilon(x, t)|^{q-1} dx dy
\leq c \|\nabla u(t)\|_\infty \int_{B(\varepsilon)} \|h(x - y, t) - h(x, t)\|_q dy \|h_\varepsilon(x, t)\|_q^{q-1} \leq c \|\nabla u(t)\|_\infty \|h_\varepsilon(x, t)\|_q^q.
\]

Multiplying \((C.3)\) by \(q h_\varepsilon |h_\varepsilon|^{q-2}\), and applying integration by parts, together with \((C.5)\) we see that for almost every \(t \in (t_0, 0)\)
\[
\|h_\varepsilon(t)\|_q^q = \|h_\varepsilon(t_0)\|_q^q - q \int_{t_0}^t \left( \nabla \cdot (u h_\varepsilon - u \cdot \nabla h_\varepsilon) h_\varepsilon \right) |h_\varepsilon|^{q-2} dx ds
+ q \int_{t_0}^t \left( \nabla \cdot (u h_\varepsilon) h_\varepsilon \right) |h_\varepsilon|^{q-2} dx ds + \int_{t_0}^t \int_{\mathbb{R}^n} |\nabla \cdot (u h_\varepsilon)|^{q-2} dx ds
+ \int_{t_0}^t \int_{\mathbb{R}^n} f_\varepsilon h_\varepsilon |h_\varepsilon|^{q-2} dx ds
\leq \|h(t_0)\|_q^q + c(1 + \|\nabla u\|_\infty) \|h_\varepsilon\|_q^q + \|f\|_q^q.
\]

This yields, \(h \in L^\infty(t_0, 0; L^q(\mathbb{R}^n))\). On the other hand, from \((C.1)\) we deduce that \(\partial_t h \in L^q(t_0, 0; W^{-1,q}(\mathbb{R}^n))\). Thus, eventually redefining \(h(t)\) on a set of measure zero, we get \(h \in C_0([t_0, 0]; L^q(\mathbb{R}^n))\).
Let $t \in (t_0, 0]$. Multiplying $[C.3]$ by $qh_\varepsilon|h_\varepsilon|^{q-2}$ and integrating over $\mathbb{R}^n \times (t_0, t)$, we find

$$
\|h_\varepsilon(t)\|_q^q = \|h_\varepsilon(t_0)\|_q^q - q \int_{t_0}^{t} \int_{\mathbb{R}^n} (\nabla \cdot (uh)_\varepsilon - (u \cdot \nabla h_\varepsilon)) h_\varepsilon|h_\varepsilon|^{q-2} dx ds
$$

$$
+ q \int_{t_0}^{t} \int_{\mathbb{R}^n} (\nabla \cdot uh)_\varepsilon h_\varepsilon|h_\varepsilon|^{q-2} dx ds + q \int_{t_0}^{t} \int_{\mathbb{R}^n} \nabla \cdot u|h_\varepsilon|^q dx ds
$$

(C.6)

+ q \int_{t_0}^{t} \int_{\mathbb{R}^n} f_\varepsilon h_\varepsilon|h_\varepsilon|^{q-2} dx ds.

We claim that

$$
\begin{aligned}
\left\{ - \int_{t_0}^{t} \int_{\mathbb{R}^n} (\nabla \cdot (uh)_\varepsilon - (u \cdot \nabla h_\varepsilon)) h_\varepsilon|h_\varepsilon|^{q-2} dx ds \\
+ \int_{t_0}^{t} \int_{\mathbb{R}^n} (\nabla \cdot uh)_\varepsilon h_\varepsilon|h_\varepsilon|^{q-2} dx ds \rightarrow 0 & \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned}
$$

(C.7)

To see this, we first verify that

$$
- \nabla \cdot (uh)_\varepsilon + u \cdot \nabla h_\varepsilon + (\nabla \cdot uh)_\varepsilon := \sigma_\varepsilon \rightarrow 0 \quad \text{weakly in } \quad L^q(\mathbb{R}^n \times (t_0, 0))
$$

as $\varepsilon \rightarrow 0$. Indeed, thanks to $[C.4]$ we see that $\sigma_\varepsilon \in L^q(\mathbb{R}^n \times (t_0, 0))$. Let $\{\varepsilon_k\}$ be a sequence of positive numbers with $\varepsilon_k \rightarrow 0$ as $k \rightarrow +\infty$. Eventually passing to a subsequence, we get $\sigma \in L^q(\mathbb{R}^n \times (t_0, 0))$ such that

$$
\sigma_{\varepsilon_k} \rightarrow \sigma \quad \text{weakly in } \quad L^q(\mathbb{R}^n \times (t_0, 0)) \quad \text{as } \quad k \rightarrow +\infty.
$$

On the other hand, we get for all $\varphi \in C^\infty_c(\mathbb{R}^n \times (t_0, 0))$

$$
\int_{t_0}^{0} \int_{\mathbb{R}^n} \left( - \nabla \cdot (uh)_{\varepsilon_k} + u \cdot \nabla h_{\varepsilon_k} + (\nabla \cdot uh)_{\varepsilon_k} \right) \varphi dx ds
$$

$$
= \int_{t_0}^{0} \int_{\mathbb{R}^n} \left( (uh)_{\varepsilon_k} - (uh)_{\varepsilon_k} \right) \cdot \nabla \varphi dx ds + \int_{t_0}^{0} \int_{\mathbb{R}^n} \left( - \nabla \cdot uh_{\varepsilon_k} + (\nabla \cdot uh)_{\varepsilon_k} \right) \varphi dx ds
$$

$$
\rightarrow 0 \quad \text{as } \quad k \rightarrow \infty.
$$

This provides us with

$$
\int_{t_0}^{0} \int_{\mathbb{R}^n} \sigma \varphi dx ds = 0 \quad \forall \varphi \in C^\infty_c(\mathbb{R}^n \times (t_0, 0)).
$$
Accordingly, $\sigma = 0$. Whence, (C.8).

Next, by means of Lebesgue’s theorem of dominated convergence, we see that

$$(C.9) \quad h_\varepsilon |h_\varepsilon|^{q-2} - h|h|^{q-2} \to 0 \quad \text{in} \quad L^q(\mathbb{R}^n \times (t_0, 0)) \quad \text{as} \quad \varepsilon \to 0.$$ 

Clearly,

$$-\int_t^{t_0} \int_{\mathbb{R}^n} \left( \nabla \cdot (uh)_\varepsilon - \nabla \cdot (uh_\varepsilon) \right) h_\varepsilon |h_\varepsilon|^{q-2} dxds + \int_t^{t_0} \int_{\mathbb{R}^n} (\nabla \cdot uh)_\varepsilon h_\varepsilon |h_\varepsilon|^{q-2} dxds$$

$$= \int_0^t \int_{\mathbb{R}^n} \left( - \nabla \cdot (uh)_\varepsilon + u \cdot \nabla h_\varepsilon + (\nabla \cdot uh)_\varepsilon \right) \left( h_\varepsilon |h_\varepsilon|^{q-2} - h|h|^{q-2} \right) dxds$$

$$+ \int_0^t \int_{\mathbb{R}^n} \left( - \nabla \cdot (uh)_\varepsilon + u \cdot \nabla h_\varepsilon + (\nabla \cdot uh)_\varepsilon \right) h|h|^{q-2} dxds.$$ 

The first integral on the right-hand side of the above identity converges to 0 in view of (C.9), while the second integral converges to 0 according to (C.8). Whence, (C.7). Thus, thanks to (C.9), (C.6) and the fact that $h_\varepsilon(t) \to h$ in $L^q(\mathbb{R}^n)$ for all $t \in [t_0, 0]$ and $f_\varepsilon \to f$ in $L^q(\mathbb{R}^n \times (t_0, 0))$ as $\varepsilon \to 0$, we are in a position to pass $\varepsilon \to 0$ in (C.6). This yields the following identity

$$(C.10) \quad \|h(t)\|_q^q = \|h(t_0)\|_q^q + \int_0^t \int_{\mathbb{R}^n} \nabla \cdot u|h|^{q-2} dxds + q \int_0^t \int_{\mathbb{R}^n} fh|h|^{q-2} dxds.$$ 

In particular, $\|h(t)\|_q^q \in C([t_0, 0])$, which together with $h \in C_w([t_0, 0; L^q(\mathbb{R}^n)])$ yields $h \in C([t_0, 0]; L^q(\mathbb{R}^n))$. This completes the proof of the lemma. 

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