Abstract

Plane electromagnetic wave propagating in perfect electromagnetic conductor (PEMC) is considered. Its wave number has no connection with the frequency. An interface is introduced between an ordinary isotropic medium and PEMC. The wave in PEMC is matched to plane electromagnetic wave incident normally on the interface from the ordinary medium and reflected from it. Then the plane-parallel slab made of PEMC is considered and a plane wave is found in it.

1 Introduction

Recently Lindell and Sihvola [1] generalized the notions of perfect electric conductor and perfect magnetic conductor to perfect electromagnetic conductor (PEMC) for which the constitutive relation between the electromagnetic fields is exotic in comparison with the ordinary media like vacuum or air. In differential-form representation of the fields this relation reads \( G = \alpha F \) where \( G = D - H \wedge dt \) and \( F = B + E \wedge dt \). The pseudoscalar parameter \( \alpha \) was called admittance in [2] and axion field in [3]. Its zero limit yields the perfect magnetic conductor, its infinity limit yields the perfect electric conductor. For the three-dimensional fields \( E, B, H, D \) the above constitutive relation means \( D = \alpha B, H = -\alpha E \).

It is worth to consider the Maxwell energy-momentum tensor in PEMC. The best appearance of this quantity for our purposes is the energy-momentum 3-form [see [4], eq. (50)]:

\[
\Sigma_i = \frac{1}{2}(F \wedge e_i|G - G \wedge e_i|F),
\]

where \( e_i \) is the 4-dimensional vector basis of the tangent space. If one substitutes \( G = \alpha F \) with pseudoscalar \( \alpha \) into this expression, it yields zero. This means that for any electromagnetic field in PEMC the energy density, energy flux and the stress is zero. The same must be true for the electromagnetic wave in such a medium.

In [1], among others, a problem of linearly polarized plane electromagnetic wave incident normally on a PEMC boundary was considered by the method of duality.
transformation. It was found that the reflected wave contains the cross-polarized term, that is, the component of the electric field perpendicular to that of the incident field. A similar problem was considered in [3], namely the propagation of plane electromagnetic wave in the ordinary medium with additional piecewise constant axion field. It was shown that the reflection of the wave occurs at an interface between two media with different axion values.

We present the explicit formula for a linearly polarized plane electromagnetic wave in PEMC medium which—as it was mentioned above—does not contain energy nor transmits energy. We consider also the plane electromagnetic wave incident normally from the vacuum on a boundary of PEMC and use the standard boundary conditions to match it with the wave in PEMC. It turns out that the reflected wave must be present on the side of the vacuum, and it contains the cross-polarized term. It is an open question whether the wave present in PEMC could be called transmitted wave, if it does not transmit any energy.

Afterwards, we consider a plane-parallel slab made of PEMC and the plane electromagnetic wave in it such that on the left-hand side of the slab, the same incident and reflected waves are present, and on the right-hand side, no wave is present.

2 Plane electromagnetic wave

In the differential-form formulation of electrodynamics the electromagnetic fields are the following objects: $B$ is two-form, $D$ is twisted two-form, $E$ is one-form, $H$ is twisted one-form. The Maxwell equations are general, i.e., independent on the metric of space and properties of a medium (we write them in a region devoid of charges and currents):

\begin{align}
\mathbf{d} \wedge B &= 0, \\
\mathbf{d} \wedge E + \frac{\partial B}{\partial t} &= 0.
\end{align}

(2)

\begin{align}
\mathbf{d} \wedge D &= 0, \\
\mathbf{d} \wedge H - \frac{\partial D}{\partial t} &= 0,
\end{align}

(3)

The boundary conditions on a flat interface (a plane) $S$ without surface charges and currents have the form

\begin{align}
(D_1 - D_2)_S &= 0, \\
(B_1 - B_2)_S &= 0, \\
(E_1 - E_2)_S &= 0, \\
(H_1 - H_2)_S &= 0,
\end{align}

(4)

(5)

where the subscript $S$ denotes the restriction of a given form to the plane $S$. When the external form is parallel to $S$, its restriction to $S$ is zero. For the explanation of direction of an external form see [5]. For instance, in the Cartesian coordinates $x, y, z$ in flat space the one-form $dx$ is parallel to the $(Y, Z)$-plane and so on with the cyclic change of variables, whereas the two-form $dx \wedge dy$ is parallel to the $Z$-axis and so on.

\[1\] Restriction of a differential form to $S$ means taking its values only on vectors and bivectors parallel to $S$. 

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We seek the solutions to the Maxwell equations in the form of plane wave propagating along the $X$-axis, i.e. when the fields are functions of a single scalar variable $\eta = \omega t - kx$, called the phase:

$$E(x, t) = E_y(\eta)\, dy + E_z(\eta)\, dz,$$
$$B(x, t) = B_{xy}(\eta)\, dx \wedge dy + B_{xz}(\eta)\, dx \wedge dz.$$  

(6) (7)

Usually $E_y, E_z, B_{xy}, B_{xz}$ are taken as combinations of sine and cosine functions of $\eta$ which is tantamount to assume that the wave is time-harmonic. We present our reasoning without this assumption.

The Maxwell equation (2) is automatically satisfied, because $dB_{xy}$ and $dB_{xz}$ are one-forms parallel to $dx$. We calculate $d \wedge E = -kE'_y(\eta)\, dx \wedge dy - kE'_z(\eta)\, dx \wedge dz$ and substitute into (2):

$$-kE'_y(\eta)\, dx \wedge dy - kE'_z(\eta)\, dx \wedge dz + \omega B'_{xy}(\eta)\, dx \wedge dy + \omega B'_{xz}(\eta)\, dx \wedge dz = 0,$$

$$\omega B'_{xy} - kE'_y = 0 \quad \text{and} \quad \omega B'_{xz} - kE'_z = 0.$$  

Since the basic two-forms are linearly independent, we obtain

$$\omega B'_{xy} - kE'_y = 0 \quad \text{and} \quad \omega B'_{xz} - kE'_z = 0.$$  

Left-hand sides are functions of single variable $\eta$, hence their integration yields

$$\omega B_{xy} - kE_y = \text{const}, \quad \omega B_{xz} - kE_z = \text{const}.$$  

It is natural to omit constant fields, so the following proportionality is obtained

$$B_{xy} = \frac{k}{\omega} E_y \quad \text{and} \quad B_{xz} = \frac{k}{\omega} E_z,$$

for the functions of the scalar variable $\eta$. It follows from this that

$$B = \frac{k}{\omega} dx \wedge E.$$  

(8)

We similarly obtain for the two other fields

$$H(x, t) = H_x(\eta)\, dy + H_y(\eta)\, dz,$$

(9) and

$$D = -\frac{k}{\omega} dx \wedge H.$$  

(10)

2.1 Plane wave in the conventional isotropic medium

The constitutive relations for the isotropic medium with the electric permittivity $\varepsilon$ and magnetic permeability $\mu$ have the form

$$D = \varepsilon \ast E, \quad B = \mu \ast H,$$  

(11)

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where $*$ denotes the Hodge star which maps the basic one-forms into basic two-forms as follows:

\[
* \, dx = dy \wedge dz, \quad * \, dy = dz \wedge dx, \quad * \, dz = dx \wedge dy. \tag{12}
\]

We treat here $D$ and $H$ as ordinary, non twisted forms, because this does not influence the reasoning.

The relations (11) allow us to show that $\frac{k}{\omega} = \sqrt{\frac{\varepsilon}{\mu}}$ and to derive from (6) and (8) two other electromagnetic fields for the plane wave. We omit the standard reasoning in the derivation. We now summarize the results and add the subscript $+$ to denote the fact that the planes of constant phase propagate in the positive $X$-direction:

\[
E_+(x, t) = E_y(\eta) \, dy + E_z(\eta) \, dz, \tag{13}
\]

\[
B_+(x, t) = \sqrt{\frac{\varepsilon}{\mu}} \, dx \wedge [E_y(\eta) \, dy + E_z(\eta) \, dz], \tag{14}
\]

\[
H_+(x, t) = \sqrt{\frac{\varepsilon}{\mu}} \, dz \wedge [E_x(\eta) \, dx + E_y(\eta) \, dy] = \sqrt{\frac{\varepsilon}{\mu}} \, dz \wedge [E_x(\eta) \, dx + E_y(\eta) \, dy], \tag{15}
\]

\[
D_+(x, t) = \varepsilon \, dx \wedge [E_z(\eta) \, dy - E_y(\eta) \, dz], \tag{16}
\]

where $\eta = \omega t - kx$. We write also similar plane wave with the phase propagating in the negative $X$-direction:

\[
E_-(x, t) = E_y(\eta) \, dy + E_z(\eta) \, dz, \tag{17}
\]

\[
B_-(x, t) = -\sqrt{\frac{\varepsilon}{\mu}} \, dx \wedge [E_y(\eta) \, dy + E_z(\eta) \, dz], \tag{18}
\]

\[
H_-(x, t) = \sqrt{\frac{\varepsilon}{\mu}} \, dz \wedge [E_x(\eta) \, dx + E_y(\eta) \, dy] = \sqrt{\frac{\varepsilon}{\mu}} \, dz \wedge [E_x(\eta) \, dx + E_y(\eta) \, dy], \tag{19}
\]

\[
D_+(x, t) = \varepsilon \, dx \wedge [E_z(\eta) \, dy - E_y(\eta) \, dz], \tag{20}
\]

where $\eta = \omega t + kx$.

### 2.2 Plane wave in PEMC medium

We now write the constitutive relations for the PEMC medium

\[
H = -\alpha E, \tag{21}
\]

\[
D = \alpha B. \tag{22}
\]

They allow us to write immediately the expressions for the fields $H$ and $D$ when (6) and (8) are given. Thus we summarize the formulas for the electromagnetic fields constituting the plane electromagnetic wave propagating in the positive direction of the $X$-axis:

\[
\tilde{E}_+(x, t) = f_1(\xi) \, dy + f_2(\xi) \, dz, \tag{23}
\]

\[
\tilde{B}_+(x, t) = \frac{k}{\omega} \, dx \wedge [f_1(\xi) \, dy + f_2(\xi) \, dz], \tag{24}
\]

\[
\tilde{H}_+(x, t) = -\alpha [f_1(\xi) \, dy + f_2(\xi) \, dz], \tag{25}
\]
\[ \tilde{D}_+(x, t) = \frac{\alpha \tilde{k}}{\omega} \, dx \wedge \left[ f_1(\xi_+) \, dy + f_2(\xi_+) \, dz \right], \]

where \( \xi_+ = \omega t - \tilde{k} x \). There is no condition imposed on the quotient \( \frac{\tilde{k}}{\omega} \); this fact is different from the situation known from the conventional medium.

The three fields \( \tilde{B}_+, \tilde{H}_+, \tilde{D}_+ \) are parallel to \( \tilde{E}_+ \) because the following relations are satisfied:

\[ \tilde{B}_+ = \frac{\tilde{k}}{\omega} \, dx \wedge \tilde{E}_+, \quad \tilde{H}_+ = -\alpha \tilde{E}_+, \quad \tilde{D}_+ = \frac{\alpha \tilde{k}}{\omega} \, dx \wedge \tilde{E}_+. \]

For this reason the energy density \( w \) and the energy flux density \( S \) are zero

\[ w = \frac{1}{2} (\tilde{E} \wedge \tilde{D} + \tilde{H} \wedge \tilde{B}) = 0, \quad S = \tilde{E} \wedge \tilde{H} = 0, \]

as expected from the considerations in the Introduction.

Let us write down another plane wave propagating along the same axis with the opposite phase velocity. The fields of this wave are expressed by scalar functions \( g_1, g_2 \) of another scalar variable \( \xi_- = \omega t + \tilde{q} x \):

\[ \tilde{E}_-(x, t) = g_1(\xi_-) \, dy + g_2(\xi_-) \, dz, \]

\[ \tilde{B}_-(x, t) = -\frac{\tilde{q}}{\omega} \, dx \wedge \left[ g_1(\xi_-) \, dy + g_2(\xi_-) \, dz \right], \]

\[ \tilde{H}_-(x, t) = -\alpha \left[ g_1(\xi_-) \, dy + g_2(\xi_-) \, dz \right], \]

\[ \tilde{D}_-(x, t) = -\frac{\alpha \tilde{q}}{\omega} \, dx \wedge \left[ g_1(\xi_-) \, dy + g_2(\xi_-) \, dz \right], \]

Again no condition is imposed on the quotient \( \frac{\tilde{q}}{\omega} \). We deliberately have chosen \( \tilde{q} \) different from \( \tilde{k} \). The obvious assumption is that both are positive scalars.

### 3 Reflection from PEMC boundary

Let the space be divided on two parts by the plane \( x = 0 \). For the left half-space \( x < 0 \) we assume homogeneous medium characterized by \( \alpha = 0 \) and constant values \( \varepsilon, \mu \). For the right half-space \( x > 0 \) we assume \( \varepsilon = \mu = 0 \) and constant value \( \alpha \) which means that it is PEMC medium. Consider a plane linearly polarized electromagnetic wave travelling along the \( X \)-axis in the left half-space. Such a normally incident wave will partially penetrate the PEMC medium and partially be reflected from the interface. Therefore we assume that in the left medium the electromagnetic field will be a superposition of plane waves, right- and left-travelling along the \( X \)-axis.

We assume that the right-travelling wave has the linear polarization parallel to \( Y \)-axis; such a field is expressed as follows

\[ E_+(x, t) = E_{y+}(\eta_+) \, dy, \]

\[ B_+(x, t) = \sqrt{\varepsilon \mu} \, dx \wedge E_{y+}(\eta_+) \, dy, \]

\[ H_+(x, t) = \sqrt{\frac{\varepsilon}{\mu}} E_{y+}(\eta_+) \, dz, \]
\[ D_+(x,t) = -\varepsilon dx \land E_{y+}(\eta_+) dz. \] (35)

One cannot expect that the reflected wave will have the same linear polarization — in fact, it has to contain a component of \( E \) parallel to \( dz \), because this gives rise to a component of \( H \) parallel to \( dy \) which by [21] must be present in the right medium. Thence we admit that the reflected wave in the left medium has the general form [17–20]. Thus the full electromagnetic field in the left medium is the following superposition

\[ E(x,t) = [E_{y+}(\eta_+) + E_{y-}(\eta_-)] dy + E_{z-}(\eta_-) dz, \] (36)
\[ B(x,t) = \sqrt{\varepsilon \mu} dx \land \{[E_{y+}(\eta_+) - E_{y-}(\eta_-)] dy - E_{z-}(\eta_-) dz\}, \] (37)
\[ H(x,t) = \sqrt{\frac{\varepsilon}{\mu}} \{E_{z-}(\eta_-) dy + [E_{y+}(\eta_+) - E_{y-}(\eta_-)] dz\}, \] (38)
\[ D(x,t) = \varepsilon dx \land \{E_{z-}(\eta_-) dy - [E_{y+}(\eta_+) + E_{y-}(\eta_-)] dz\}. \] (39)

In order to not restrict generality we assume that in the right medium the electromagnetic field is also a superposition of two opposite travelling plane waves, hence we write the sums of expressions [23–26] with the corresponding expressions [28–31]

\[ \tilde{E}(x,t) = [f_1(\xi_+) + g_1(\xi_-)] dy + [f_2(\xi_+) + g_2(\xi_-)] dz, \] (40)
\[ \tilde{B}(x,t) = \frac{1}{\omega} dx \land \{[\tilde{k}f_1(\xi_+) - \tilde{q}g_1(\xi_-)] dy + [\tilde{k}f_2(\xi_+) - \tilde{q}g_2(\xi_-)] dz\}, \] (41)
\[ \tilde{H}(x,t) = -\alpha \{[f_1(\xi_+) + g_1(\xi_-)] dy + [f_2(\xi_+) + g_2(\xi_-)] dz\}, \] (42)
\[ \tilde{D}(x,t) = \frac{\alpha}{\omega} dx \land \{[\tilde{k}f_1(\xi_+) - \tilde{q}g_1(\xi_-)] dy + [\tilde{k}f_2(\xi_+) - \tilde{q}g_2(\xi_-)] dz\}. \] (43)

We now consider the boundary conditions [1, 3] on the plane \( x = 0 \). The conditions [1] are satisfied trivially, because the two-forms \[37, 39, 41\] and \[43\] are parallel to the interface (they contain the factor \( dx \)), hence their restrictions to it are zero. The one-forms \[36, 38, 40, 42\] are perpendicular to the interface, hence their restrictions to it are equal to themselves. Thus the boundary conditions [4] reduce to

\[ E(0,t) = \tilde{E}(0,t), \quad H(0,t) = \tilde{H}(0,t), \]

and, after equating independent components, yield four equalities

\[ E_{y+}(\omega t) + E_{y-}(\omega t) = f_1(\omega t) + g_1(\omega t), \] (44)
\[ E_{z-}(\omega t) = f_2(\omega t) + g_2(\omega t), \] (45)
\[ \sqrt{\frac{\varepsilon}{\mu}} E_{z-}(\omega t) = -\alpha [f_1(\omega t) + g_1(\omega t)], \] (46)
\[ \sqrt{\frac{\varepsilon}{\mu}} [E_{y+}(\omega t) - E_{y-}(\omega t)] = -\alpha [f_2(\omega t) + g_2(\omega t)]. \] (47)

It is interesting to notice that \( f_i \) and \( g_i \) appear only in sums \( f_i(\omega t) + g_i(\omega t) \), hence the field present in PEMC medium can be chosen in arbitrary combination of the
right- and left-travelling waves. By eliminating \( f_i + g_i \) from the above equations we arrive at the following two linear equations

\[
\sqrt{\frac{\varepsilon}{\mu}} E_{z-} = -\alpha (E_{y+} + E_{y-}),
\]

\[
\sqrt{\frac{\varepsilon}{\mu}} (E_{y+} - E_{y-}) = -\alpha E_{z-},
\]

which allow us to express the components \( E_{y-} \) and \( E_{z-} \) of the reflected wave by the component \( E_{y+} \) of the incident wave:

\[
E_{y-} = \frac{\varepsilon - \alpha^2 \mu}{\varepsilon + \alpha^2 \mu} E_{y+}, \quad E_{z-} = -\frac{2\alpha \sqrt{\varepsilon \mu}}{\varepsilon + \alpha^2 \mu} E_{y+}.
\]

(48)

We insert this into (17) and (19) to obtain the field strengths of the reflected wave:

\[
E_-(x, t) = \frac{1}{\varepsilon + \alpha^2 \mu} E_{y+}(\eta_-) \left[ (\varepsilon - \alpha^2 \mu) \, dy - 2\alpha \sqrt{\varepsilon \mu} \, dz \right],
\]

(49)

\[
H_-(x, t) = -\frac{\sqrt{\varepsilon / \mu}}{\varepsilon + \alpha^2 \mu} E_{y+}(\eta_-) \left[ 2\alpha \sqrt{\varepsilon \mu} \, dy + (\varepsilon - \alpha^2 \mu) \, dz \right].
\]

(50)

Formula (49) coincides (after appropriate change of notation) with the formula (41) in [1]. The electric field (32) of the incident wave is parallel to \( dy \). The electric field (49) of the reflected wave contains the extra component parallel to \( dz \). Lindell and Sihvola in [1] call it “cross-polarized component”.

The Poynting two-form for the reflected wave fields (49) and (50) reads

\[
S_-(x, t) = E_-(x, t) \wedge H_-(x, t) = -\frac{\sqrt{\varepsilon / \mu}}{(\varepsilon + \alpha^2 \mu) \varepsilon} \left[ (\varepsilon - \alpha^2 \mu) + 4\alpha^2 \varepsilon \mu \right] E_{y+}^2(\eta_-) \, dy \wedge dz
\]

\[
= -\sqrt{\varepsilon / \mu} E_{y+}^2(\eta_-) \, dy \wedge dz.
\]

This ought to be compared with the Poynting two-form of the incident wave (32) and (34)

\[
S_+(x, t) = E_+(x, t) \wedge H_+(x, t) = \sqrt{\varepsilon / \mu} E_{y+}^2(\eta_-) \, dy \wedge dz
\]

We see that \( S_- \) is opposite to \( S_+ \) on the interface \( x = 0 \). This implies that the reflection coefficient \( T = |S_-|/|S_+| \) is precisely one, what is to be expected because no energy can be transmitted into PEMC medium.

We substitute now \( E_{y-} \) and \( E_{z-} \) known from (48) into (44), (45) in order to express \( f_i + g_i \) by the component of the incident wave:

\[
f_1(\omega t) + g_1(\omega t) = \frac{2\varepsilon}{\varepsilon + \alpha^2 \mu} E_{y+}(\omega t),
\]

(51)

\[
f_2(\omega t) + g_2(\omega t) = -\frac{2\alpha \sqrt{\varepsilon \mu}}{\varepsilon + \alpha^2 \mu} E_{y+}(\omega t).
\]

(52)
We insert $g_1$, $g_2$ calculated from these equation into (40, 42)

$$\tilde{E}(x, t) = \left[ f_1(\xi_+) + \frac{2\varepsilon}{\varepsilon + \alpha^2 \mu} E_{y+}(\xi_-) - f_1(\xi_-) \right] dy$$

$$+ \left[ f_2(\xi_+) - \frac{2\alpha \sqrt{\varepsilon \mu}}{\varepsilon + \alpha^2 \mu} E_{y+}(\xi_-) - f_2(\xi_-) \right] dz, \quad (53)$$

$$\tilde{B}(x, t) = \frac{1}{\omega} dx \wedge \left\{ \left[ \tilde{k} f_1(\xi_+) - \frac{2\varepsilon \tilde{q}}{\varepsilon + \alpha^2 \mu} E_{y+}(\xi_-) + \tilde{q} f_1(\xi_-) \right] dy$$

$$+ \left[ \tilde{k} f_2(\xi_+) + \frac{2\alpha \sqrt{\varepsilon \mu} \tilde{q}}{\varepsilon + \alpha^2 \mu} E_{y+}(\xi_-) + \tilde{q} f_2(\xi_-) \right] dz \right\} \quad (54)$$

The other fields are obtained through $\tilde{H} = -\alpha \tilde{E}$, $\tilde{D} = \alpha \tilde{B}$. The arbitrary functions $f_1, f_2$ are still present in these formulas, therefore the electromagnetic wave in PEMC, after fulfilling the boundary conditions, remains arbitrary to high degree.

If one chooses $f_1 = f_2 = 0$, the formulas (53) reduce to

$$\tilde{E}(x, t) = \frac{2E_{y+}(\xi_-)}{\varepsilon + \alpha^2 \mu} (\varepsilon dy - \alpha \sqrt{\varepsilon \mu} dz),$$

$$\tilde{B}(x, t) = -\frac{2\tilde{q} E_{y+}(\xi_-)}{\varepsilon + \alpha^2 \mu} dx \wedge (\varepsilon dy - \alpha \sqrt{\varepsilon \mu} dz),$$

which show that only the left-travelling wave in present in PEMC. On the other hand, if one chooses

$$f_1(\omega t) = \frac{2\varepsilon}{\varepsilon + \alpha^2 \mu} E_{y+}(\omega t), \quad f_2(\omega t) = -\frac{2\alpha \sqrt{\varepsilon \mu} \tilde{q}}{\varepsilon + \alpha^2 \mu} E_{y+}(\omega t),$$

the formulas (53) assume the form

$$\tilde{E}(x, t) = \frac{2E_{y+}(\xi_+)}{\varepsilon + \alpha^2 \mu} (\varepsilon dy - \alpha \sqrt{\varepsilon \mu} dz),$$

$$\tilde{B}(x, t) = -\frac{2\tilde{k} E_{y+}(\xi_+)}{\varepsilon + \alpha^2 \mu} dx \wedge (\varepsilon dy - \alpha \sqrt{\varepsilon \mu} dz),$$

and in this case only the right-travelling wave is present in PEMC.

4 Plane wave in PEMC slab

The reasoning of previous section is based on the assumption that PEMC medium fills the whole half-space $x > 0$. This assumption is nonphysical, it is natural to assume, rather, that the PEMC medium forms a plane-parallel slab defined by the condition $0 < x < b$. What conditions should be imposed on the electromagnetic fields on the other interface $x = b$? Since no energy is transmitted through PEMC, we suppose that the same occurs for $x > b$. The zero energy flux cannot be accomplished by a superposition of left- and right-travelling waves giving total energy flux
equal to zero, because there is no physical reason for a presence of the electromagnetic wave incoming from the right infinity. In this manner we arrive to conclusion that the electromagnetic fields should vanish for \( x > b \). The boundary conditions \( 4 \) are satisfied trivially because the two-forms \( 41, 43 \) are parallel to \( d \). The other condition \( 5 \) assumes the form

\[
\tilde{E}(b, t) = 0, \quad \tilde{H}(b, t) = 0.
\]

It is sufficient to consider the first equality, because \( \tilde{H} = -\alpha \tilde{E} \), hence we assume that the two square brackets in \( 53 \) vanish

\[
f_1(\omega t - \tilde{k}b) - f_1(\omega t + \tilde{q}b) + \frac{2\varepsilon}{\varepsilon + \alpha^2 \mu} E_{y+}(\omega t + \tilde{q}b) = 0, \tag{55}
\]

\[
f_2(\omega t - \tilde{k}b) - f_2(\omega t + \tilde{q}b) - \frac{2\alpha\sqrt{\varepsilon \mu}}{\varepsilon + \alpha^2 \mu} E_{y+}(\omega t + \tilde{q}b) = 0. \tag{56}
\]

Equations \( 55, 56 \) can be rewritten with the following change of notation: \( \omega t + \tilde{q}b = u \), \( \tilde{k}b + \tilde{q}b = a \), \( \omega t - \tilde{k}b = u - a \):

\[
f_1(u - a) - f_1(u) + \frac{2\varepsilon}{\varepsilon + \alpha^2 \mu} E_{y+}(u) = 0, \tag{57}
\]

\[
f_2(u - a) - f_2(u) - \frac{2\alpha\sqrt{\varepsilon \mu}}{\varepsilon + \alpha^2 \mu} E_{y+}(u) = 0 \tag{58}
\]

These equations ought to be satisfied for all \( u \in \mathbb{R} \) and fixed \( a \). They are functional equations which I do not know how to solve.

Let us assume now that the incident wave is time-harmonic, i.e.

\[
E_{y+}(u) = A \cos(u - \delta), \tag{59}
\]

with a given constant \( A \). Then we look for the unknown function \( f_1 \) in the form

\[
f_1(u) = C \cos u. \tag{60}
\]

The constants \( C \) and \( \delta \) are to be found.

We rewrite equation \( 57 \) with the use of \( 59, 60 \):

\[
C \cos(u - a) - C \cos u + \frac{2\varepsilon A}{\varepsilon + \alpha^2 \mu} \cos(u - \delta) = 0.
\]

A simple trigonometry yields

\[
\left( C \cos a - C + \frac{2\varepsilon A}{\varepsilon + \alpha^2 \mu} \cos \delta \right) \cos u + \left( C \sin a + \frac{2\varepsilon A}{\varepsilon + \alpha \mu} \sin \delta \right) \sin u = 0.
\]

Sine and cosine are linearly independent functions, thence

\[
C \cos a - C + \frac{2\varepsilon A}{\varepsilon + \alpha^2 \mu} \cos \delta = 0.
\]

\[
C \sin a + \frac{2\varepsilon A}{\varepsilon + \alpha \mu} \sin \delta = 0.
\]
This system of two equations has two solutions:

\[ C_I = \frac{\varepsilon A}{(\varepsilon + \alpha \mu) \sin(a/2)}, \quad \delta_I = \frac{a - \pi}{2}, \]
\[ C_{II} = -\frac{\varepsilon A}{(\varepsilon + \alpha \mu) \sin(a/2)}, \quad \delta_{II} = \frac{a + \pi}{2}. \]

Then, according to (57, 58), we have two possibilities. The first one reads

\[ E_{y+}(u) = A \cos \left( u - \frac{a}{2} + \frac{\pi}{2} \right) = -A \sin \left( u - \frac{a}{2} \right), \quad (61) \]
\[ f_1(u) = \frac{\varepsilon A}{(\varepsilon + \alpha \mu) \sin(a/2)} \cos u. \quad (62) \]

The other possibility is not essentially different – it gives only opposite sign in front of \( A \).

A similar reasoning applied to equation (58) leads to the result

\[ f_2(u) = -\frac{\alpha \sqrt{\varepsilon \mu} A}{(\varepsilon + \alpha \mu) \sin(a/2)} \cos u. \quad (63) \]

We substitute (61, 62, 63) into (53, 54) and obtain the following time-harmonic plane electromagnetic wave in the plane-parallel slab \( 0 < x < b \) of PEMC medium

\[ \tilde{\mathbf{E}}(x, t) = \frac{A}{\varepsilon + \alpha^2 \mu} \left[ \frac{\cos(\omega t - \tilde{k}x)}{\sin a/2} - 2\sin(\omega t + \tilde{q}x - a/2) \right. \]
\[ \left. -\frac{\cos(\omega t + \tilde{q}x)}{\sin a/2} \right] (\varepsilon \, dy - \alpha \sqrt{\varepsilon \mu} \, dz), \quad (64) \]
\[ \tilde{\mathbf{B}}(x, t) = \frac{A}{(\varepsilon + \alpha^2 \mu)\omega} \left[ \tilde{k} \cos(\omega t - \tilde{k}x) \right. \]
\[ \left. -\frac{\cos(\omega t + \tilde{q}x)}{\sin a/2} \right] \left[ \tilde{q} \cos(\omega t + \tilde{q}x) \right. \]
\[ \left. -\frac{\cos(\omega t - \tilde{k}x)}{\sin a/2} \right] dx \wedge (\varepsilon \, dy - \alpha \sqrt{\varepsilon \mu} \, dz), \quad (65) \]

where \( a = \tilde{k}b + \tilde{q}b \). As is visible from (61, 62), the values \( a = 2n\pi \) for integer \( n \) are not allowable for the time-harmonic wave. This fact can be interpreted as follows. As mentioned earlier, the wave numbers \( \tilde{k}, \tilde{q} \) are independent of the wave number \( k \) in the left ordinary medium, i.e. in the half-space \( x < 0 \). The values of \( \tilde{k} \) and \( \tilde{q} \) which yield \( \tilde{k}b + \tilde{q}b = 2n\pi \) can not be present in the solution (61, 62) valid for the PEMC slab.

5 Conclusion

A plane electromagnetic wave propagating in PEMC has been presented. The normal reflection of plane electromagnetic wave from the PEMC boundary has been considered with the use of boundary conditions. It turned out that the field strengths of the reflected wave are the same as in \( \Pi \).
An interesting result is that the wave present in PEMC may contain two arbitrary functions $f_1, f_2$ which are present in the right- and left-travelling waves. By and appropriate choice of them one can assure the presence of the right-travelling wave only, or the left-travelling one only. This is possible because no energy is transported by these waves. In a sense they are virtual waves. Moreover, the two oppositely travelling waves in PEMC may contain wave numbers $\tilde{k}$ and $\tilde{q}$ in their phases, which may be different from each other and from the wave number $k$ in the ordinary medium.

The above mentioned observations are true for the PEMC medium extending from $x = 0$ to infinity. Since this is nonphysical situation, we have assumed that PEMC extends from $x = 0$ to $x = b$, i.e. it forms a plane-parallel slab. On one side of it, there are incident and reflected waves, whereas on the other side of it, there is no electromagnetic fields at all. We have shown that the functions $f_1, f_2$ must have unique shapes for the time-harmonic wave incident from the left.

Acknowledgement

A principal part of the work has been done during my stay at Cologne University with the financial support of the European Union. I am deeply grateful to Friedrich Hehl and Yuri Obukhov for stimulating suggestions and discussions which make this paper possible to emerge.

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