SK$_1$ OF AZUMAYA ALGEBRAS OVER HENSEL PAIRS

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Abstract. Let $A$ be an Azumaya algebra of constant rank $n^2$ over a Hensel pair $(R, I)$ where $R$ is a semilocal ring with $n$ invertible in $R$. Then the reduced Whitehead group SK$_1(A)$ coincides with its reduction SK$_1(A/IA)$. This generalizes a result of [6] to non-local Henselian rings.

Let $A$ be an Azumaya algebra over a ring $R$ of constant rank $n^2$. Then there is an étale faithfully flat commutative ring $S$ over $R$ which splits $A$, i.e., $A \otimes_R S \cong M_n(S)$. For $a \in A$, considering $a \otimes 1$ as an element of $M_n(S)$, one then defines the reduced characteristic polynomial of $a$ as

$$\text{char}_A(x, a) = \det(x - a \otimes 1) = x^n - \text{Trd}(a)x^{n-1} + \cdots + (-1)^n\text{Nrd}(a).$$

Using descent theory, one can show that $\text{char}_A(x, a)$ is independent of $S$ and the isomorphism above and lies in $R[x]$. Furthermore, the element $a$ is invertible in $A$ if and only if $\text{Nrd}_A(a)$, the reduced norm of $a$, is invertible in $R$ (see [10], III.1.2, and [14], Theorem 4.3). Let $SL(1, A)$ be the set of elements of $A$ with the reduced norm 1. Since the reduced norm map respects the scalar extensions, it defines the smooth group scheme $SL_{1,A} : T \rightarrow SL(1, A_T)$ where $A_T = A \otimes_R T$ for an $R$-algebra $T$. Consider the short exact sequence of smooth group schemes

$$1 \longrightarrow SL_{1,A} \longrightarrow GL_{1,A} \xrightarrow{\text{Nrd}} G_m \longrightarrow 1$$

where $GL_{1,A} : T \rightarrow A_T^*$ and $G_m(T) = T^*$ for an $R$-algebra $T$ where $A_T$ and $T^*$ are invertible elements of $A_T$ and $T$, respectively. This exact sequence induces a long exact sequence

(1) $$1 \longrightarrow SL(1, A) \longrightarrow A^* \xrightarrow{\text{Nrd}} R^* \longrightarrow H^1_{et}(R, SL(1, A)) \longrightarrow H^1_{et}(R, GL(1, A)) \rightarrow \cdots$$

Let $A'$ denote the commutator subgroup of $A^*$. One defines the reduced Whitehead group of $A$ as $SK_1(A) = SL(1, A)/A'$ which is a subgroup of (non-stable) $K_1(A) = A^*/A'$. Let $I$ be an ideal of $R$. Since the reduced norm is compatible with extensions, it induces the map $SK_1(A) \rightarrow SK_1(\overline{A})$, where $\overline{A} = A/IA$. A natural question arises here is, under what circumstances and for what ideals $I$ of $R$, this homomorphism would be injective and/or surjective and thus the reduced Whitehead group of $A$ coincides with its reduction. The following observation shows that even in the case of a split Azumaya algebra, these two groups could differ: consider the split Azumaya algebra $A = M_n(R)$ where $R$ is an arbitrary commutative ring (and $n > 2$). In this case the reduced norm coincides with the ordinary determinant and $SK_1(A) = SL_n(R)/[GL_n(R), GL_n(R)]$. There are examples such that $SK_1(A) \neq 1$, in fact not even torsion. But in this setting, obviously $SK_1(\overline{A}) = 1$ for $\overline{A} = A/mA$ where $m$ is a maximal ideal of $R$ (for some examples see [13], Chapter 2).
If $I$ is contained in the Jacobson radical $J(R)$, then $IA \subset J(A)$ (see, e.g., [4], Lemma 1.4) and (non-stable) $K_1(A) \to K_1(A)$ is surjective, thus its restriction to $SK_1$ is also surjective.

It is observed by Grothendieck ([5], Theorem 11.7) that if $R$ is a local Henselian ring with maximal ideal $I$ and $G$ is an affine, smooth group scheme, then $H^1_{et}(R, G) \to H^1_{et}(R/I, G/I)$ is an isomorphism. This was further extended to Hensel pairs by Strano [15]. Now if further $R$ is a semilocal ring then $H^1_{et}(R, GL(1, A)) = 0$, and thus from the sequence (1) we have the following commutative diagram:

(2)

The aim of this note is to prove that for the Hensel pair $(R, I)$ where $R$ is a semilocal ring, the map $SK_1(A) \to SK_1(A)$ is also an isomorphism. This extends a result of [6] to non-local Henselian rings.

Recall that the pair $(R, I)$ where $R$ is a commutative ring and $I$ an ideal of $R$ is called a Hensel pair if for any polynomial $f(x) \in R[x]$, and $b \in R/I$ such that $\overline{f}(b) = 0$ and $\overline{f}'(b)$ is invertible in $R/I$, then there is $a \in R$ such that $\overline{a} = b$ and $f(a) = 0$ (for other equivalent conditions, see Raynaud [12], Chap. XI).

In order to prove the statement, we use a result of Vaserstein [17] which establishes the (Dieudonné) determinant in the setting of semilocal rings. The crucial part is to prove a version of Platonov’s congruence theorem [11] in the setting of an Azumaya algebra over a Hensel pair. The approach to do this was motivated by Suslin in [16]. We also need to use the following facts established by Greco in [3, 4].

**Proposition 1** ([4], Prop. 1.6). Let $R$ be a commutative ring, $A$ be an $R$-algebra, integral over $R$ and finite over its center. Let $B$ be a commutative $R$-subalgebra of $A$ and $I$ an ideal of $R$. Then $IA \cap B \subseteq \sqrt{IB}$.

**Corollary 2** ([3], Cor. 4.2). Let $(R, I)$ be a Hensel pair and let $J \subseteq \sqrt{I}$ be an ideal of $R$. Then $(R, J)$ is a Hensel pair.

**Theorem 3** ([3], Th. 4.6). Let $(R, I)$ be a Hensel pair and let $B$ be a commutative $R$-algebra integral over $R$. Then $(B, IB)$ is a Hensel pair.

We are in a position to prove the main theorem of this note.

**Theorem 4.** Let $A$ be an Azumaya algebra of constant rank $n^2$ over a Hensel pair $(R, I)$ where $R$ is a semilocal ring with $n$ invertible in $R$. Then $SK_1(A) \cong SK_1(A)$ where $A = A/IA$. 
Proof. Since for any \( a \in A \), \( \text{Nrd}_A(a) = \sqrt{\text{Nrd}_A(a)} \), it follows that there is a homomorphism \( \phi : \text{SL}(1, A) \to \text{SL}(1, \overline{A}) \). We first show that \( \ker \phi \subseteq A' \), the commutator subgroup of \( A' \). In the setting of valued division algebras, this is the Platonov congruence theorem \([11]\). We shall prove this in several steps. Clearly \( \ker \phi = \text{SL}(1, A) \cap 1 + IA \). Note that \( A \) is a free \( R \)-module (see \([1] \), II, §5.3, Prop. 5).

(i) The group \( 1 + I \) is uniquely \( n \)-divisible and \( 1 + IA \) is \( n \)-divisible.

Let \( a \in 1 + I \). Consider \( f(x) = x^n - a \in R[x] \). Since \( n \) is invertible in \( R \), \( \overline{f}(x) = x^n - 1 \in \overline{R}[x] \) has a simple root. Now this root lifts to a root of \( f(x) \) as \( (R, I) \) is a Hensel pair. This shows that \( 1 + I \) is \( n \)-divisible. Now if \( (1 + a)^n = 1 \) where \( a \in I \), then \( a(a^{n-1} + na^{n-2} + \cdots + n) = 0 \). Since the second factor is invertible, \( a = 0 \), and it follows that \( 1 + I \) is uniquely \( n \)-divisible.

Now let \( a \in 1 + IA \). Consider the commutative ring \( B = R[a] \subseteq A \). By Theorem \([4]\) \( (B, IB) \) is a Hensel pair. On the other hand by Prop. \([1]\) \( IA \cap B \subseteq \sqrt{IB} \). Thus by Cor. \([2]\) \( (B, IA \cap B) \) is also a Hensel pair. But \( a \in 1 + IA \cap B \). Applying the Hensel lemma as in the above, it follows that \( a \) has a \( n \)-th root and thus \( 1 + IA \) is \( n \)-divisible.

(ii) \( \text{Nrd}_A(1 + IA) = 1 + I \).

From compatibility of the reduced norm, it follows that \( \text{Nrd}_A(1 + IA) \subseteq 1 + I \). Now using the fact that \( 1 + I \) is \( n \)-divisible, the equality follows.

(iii) \( \text{SK}_1(A) \) is \( n^2 \)-torsion.

We first establish that \( N_{A/R}(a) = \text{Nrd}_A(a)^n \). One way to see this is as follows. Since \( A \) is an Azumaya algebra of constant rank \( n^2 \), \( i : A \otimes A^{op} \cong \text{End}_R(A) \cong M_{n^2}(R) \) and there is an étale faithfully flat \( S \) algebra such that \( j : A \otimes S \cong M_n(S) \). Consider the following diagram

\[
\begin{align*}
A \otimes A^{op} \otimes S & \xrightarrow{i \otimes 1} \text{End}_R(A) \otimes S \xrightarrow{\cong} \text{End}_S(A \otimes S) \xrightarrow{\cong} M_{n^2}(S) \\
A^{op} \otimes A \otimes S & \xrightarrow{1 \otimes j} A^{op} \otimes M_n(S) \xrightarrow{\cong} M_n(A^{op} \otimes S) \xrightarrow{\cong} M_{n^2}(S)
\end{align*}
\]

where the automorphism \( \psi \) is the compositions of isomorphisms in the diagram. By a theorem of Artin (see, e.g., \([10]\), §III, Lemma 1.2.1), one can find an étale faithfully flat \( S \) algebra \( T \) such that \( \psi \otimes 1 : M_{n^2}(T) \to M_{n^2}(T) \) is an inner automorphism. Now the determinant of the element \( a \otimes 1 \otimes 1 \) in the first row is \( N_{A/R}(a) \) and in the second row is \( \text{Nrd}_A(a)^n \) and since \( \psi \otimes 1 \) is inner, thus they coincide.

Therefore if \( a \in \text{SL}(1, A) \), then \( N_{A/R}(a) = 1 \). We will show that \( a^{n^2} \in A' \). Consider the sequence of \( R \)-algebra homomorphism

\[
f : A \to A \otimes A^{op} \to \text{End}_R(A) \cong M_{n^2}(R) \to M_{n^2}(A)
\]
and the \( R \)-algebra homomorphism \( i : A \to M_{n^2}(A) \) where \( a \) maps to \( aI_{n^2} \), where \( I_{n^2} \) is the identity matrix of \( M_{n^2}(A) \). Since \( R \) is a semilocal ring, the Skolem-Noether theorem is present in this setting (see \([10]\), Prop. 5.2.3) and thus there is \( g \in \text{GL}_{n^2}(A) \) such that \( f(a) = gi(a)g^{-1} \). Also, since \( A \) is a finite algebra over \( R \), \( A \) is a semilocal ring. Since \( n \) is invertible in \( R \), by Vaserstein’s result \([17]\), the Dieudonné determinant extends to the setting of \( M_{n^2}(A) \).
Taking the determinant from \(f(a)\) and \(gi(a)g^{-1}\), it follows that \(1 = N_{A/R}(a) = a^{n^2}c_a\) where \(c_a \in A'\). This shows that \(SK_1(A)\) is \(n^2\)-torsion.

**(iv).** Platonov’s Congruence Theorem: \(SL(1, A) \cap (1 + IA) \subseteq A'\).

Let \(a \in SL(1, A) \cap (1 + IA)\). By part (i), there is \(b \in 1 + IA\) such that \(b^{n^2} = a\). Then \(Nrd_A(a) = Nrd_A(b)^{n^2} = 1\). By part (ii), \(Nrd_A(b) \in 1 + I\) and since \(1 + I\) is uniquely \(n\)-divisible, \(Nrd_A(b) = 1\), so \(b \in SL(1, A)\). By part (iii), \(b^{n^2} \in A'\), so \(a \in A'\). Thus \(ker \phi \subseteq A'\) where \(\phi : SL(1, A) \rightarrow SL(1, A')\).

It is easy to see that \(\phi\) is surjective. In fact, if \(\alpha \in SL(1, A)\) then \(1 = Nrd_A(\alpha) = Nrd_A(b)\) thus, \(Nrd_A(a) \in 1 + I\). By part (i), there is \(r \in 1 + I\) such that \(Nrd_A(ar^{-1}) = 1\) and \(ar^{-1} = \alpha\). Thus \(\phi\) is an epimorphism. Consider the induced map \(\overline{\phi} : SL(1, A) \rightarrow SL(1, A)/A'\). Since \(I \subseteq J(R)\), and by part (iii), \(ker \phi \subseteq A'\) it follows that \(ker \overline{\phi} = A'\) and thus \(\overline{\phi} : SK_1(A) \cong SK_1(A)\).

Let \(R\) be a semilocal ring and \((R, J(R))\) a Hensel pair. Let \(A\) be an Azumaya algebra over \(R\) of constant rank \(n\) and \(n\) invertible in \(R\). Then by Theorem 4, \(SK_1(A) \cong SK_1(\overline{A})\) where \(\overline{A} = A/J(R)A\). But \(J(A) = J(R)A\), so \(\overline{A} = M_{k_1}(D_1) \times \cdots \times M_{k_r}(D_r)\) where \(D_i\) are division algebras. Thus \(SK_1(A) \cong SK_1(\overline{A}) = SK_1(D_1) \cdots \times SK_1(D_r)\).

Using a result of Goldman [2], one can remove the condition of Azumaya algebra having a constant rank from the Theorem.

**Corollary 5.** Let \(A\) be an Azumaya algebra over a Hensel pair \((R, I)\) where \(R\) is semilocal and the least common multiple of local ranks of \(A\) over \(R\) is invertible in \(R\). Then \(SK_1(A) \cong SK_1(\overline{A})\) where \(\overline{A} = A/IA\).

**Proof.** One can decompose \(R\) uniquely as \(R_1 \oplus \cdots \oplus R_t\) such that \(A_i = R_i \otimes_R A\) have constant ranks over \(R_i\) which coincide with local ranks of \(A\) over \(R\) (see [2], §2 and Theorem 3.1). Since \((R_i, IR_i)\) are Hensel pairs, the result follows by using Theorem 4. \(\square\)

**Remarks 6.** Let \(D\) be a tame unramified division algebra over a Henselian field \(F\), i.e., the value group of \(D\) coincides with value group of \(F\) and \(char(F)\) does not divide the index of \(D\) (see [18] for a nice survey on valued division algebras). Let \(V_D\) be the valuation ring of \(D\) and \(U_D = V_D^*\). Jacob and Wadsworth observed that \(V_D\) is an Azumaya algebra over its center \(V_F\) (Theorem 3.2 in [18] and Example 2.4 in [8]). Since \(D^* = F^*U_D\) and \(V_D \otimes_{V_F} F \simeq D\), it can be seen that \(SK_1(D) = SK_1(V_D)\). On the other hand our main Theorem states that \(SK_1(V_D) \simeq SK_1(D)\). Comparing these, we conclude the stability of \(SK_1\) under reduction, namely \(SK_1(D) \simeq SK_1(D)\) (compare this with the original proof, Corollary 3.13 in [31]).

Now consider the group \(CK_1(A) = A^*/R^*A'\) for the Azumaya algebra \(A\) over the Hensel pair \((R, I)\). A proof similar to Theorem 3.10 in [3], shows that \(CK_1(A) \cong CK_1(\overline{A})\). Thus in the case of tame unramified division algebra \(D\), one can observe that \(CK_1(D) \cong CK_1(D)\).

For an Azumaya algebra \(A\) over a semilocal ring \(R\), by [1] one has

\[R^*/\text{Nrd}_A(A^*) \cong H^1_{et}(R, SL(1, A)).\]
If \((R, I)\) is also a Hensel pair, then by the Grothendieck-Strano result,
\[ R^*/\text{Nrd}_A(A^*) \cong H^1_{\text{ét}}(R, \text{SL}(1, A)) \cong H^1_{\text{ét}}(\overline{R}, \text{SL}(1, \overline{A})) \cong \overline{R}/\text{Nrd}_{\overline{A}}(\overline{A}). \]

However specializing to a tame unramified division algebra \(D\), the stability does not follow in this case. In fact for a tame and unramified division algebra \(D\) over a Henselian field \(F\) with the valued group \(\Gamma_F\) and index \(n\) one has the following exact sequence (see [7], Theorem 1):
\[ 1 \rightarrow H^1(F, \text{SL}(1, D)) \rightarrow H^1(F, \text{SL}(1, D)) \rightarrow \Gamma_F/n\Gamma_F \rightarrow 1. \]

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