Rheology of a dense granular material

Takahiro Hatano
Earthquake Research Institute, University of Tokyo, 1-1-1 Yayoi, Bunkyo, Tokyo 113-0032, Japan
E-mail: hatano@eri.u-tokyo.ac.jp

Abstract. In order to understand the nature of friction in dense granular materials, a discrete element simulation on granular layers subjected to isobaric plain shear is performed. It is found that the friction coefficient increases as the power of the shear rate, the exponent of which does not depend on the details of the computational model. Using a nondimensional parameter that is known as the inertial number, this power law can be cast in a generalized form so that the friction coefficients at different confining pressures collapse on the same curve. We also show that the shear rate dependence of the volume fraction obeys a power law.

1. Introduction
Friction is one of the oldest problems in science because it dominates various phenomena in our daily life. In particular, flow dynamics of granular materials, which is ubiquitous in earth science and engineering, is governed by laws that describe behaviors of friction coefficient (ratio of shear stress to normal pressure). Such examples are avalanche, landslide, debris flow, silo flow, etc.

In addition, the nature of friction on faults, which plays a key role in earthquake mechanics [1, 2], is also attributed to that of granular rock because fault zone consists of layers of fine rock granules that are ground-up by the fault motion of the past. To find suitable laws of friction in granular materials under specific conditions is thus an essential problem in natural science and engineering.

Although the frictional properties of granular materials are so important, our understanding is still limited. In the context of earthquake mechanics, sliding velocity (or shear rate) dependence of friction coefficient, which is equivalent to rheology under constant pressure condition, is a matter of focus because it determines dynamics of slip [2]. In experiments on thin granular layers subjected to relatively low sliding velocities ranging from nm/s to mm/s, the behavior of friction coefficient can be described by a phenomenological law in which friction coefficient logarithmically depends on slip velocity. This is known as the rate and state dependent friction (RSF) law [3]. Although the RSF law applies to low speed friction at interfaces between two solids as well as that in granular layers, it is not applicable to high speed friction. For example, several experiments indicated nonlogarithmic increase of friction coefficient in granular layers at higher slip velocities [4, 5, 6]. The same tendency was also observed in experiments on friction between two sheets of paper [7, 8]. Thus anomalous strengthening of friction at high velocities seems universal. However, at this point, we do not know any friction law that is valid at higher slip velocities.

In this paper, we perform a computer simulation on sheared granular layers under constant pressure in order to understand the nature of friction outside the region where the RSF law is
valid. A new law is reported in which friction coefficient increases as the power of shear rate. This law describes friction in granular layers under high shear rates and high pressure (on the order of tens of MPa) that may be relevant to seismic slip on faults.

2. The computational model
The grains are assumed to be spheres, and their diameters range uniformly from 0.7d to 1.0d. The interaction force follows the discrete element method (DEM) [9]. Consider a grain i of radius \( R_i \) located at \( \mathbf{r}_i \) with the translational velocity \( \mathbf{v}_i \) and the angular velocity \( \mathbf{\Omega}_i \). This grain interacts with another grain j whenever overlapped; i.e. \( |\mathbf{r}_{ij}| < R_i + R_j \), where \( \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \).

The interaction consists of two kinds of forces, each of which is normal and transverse to \( \mathbf{r}_{ij} \), respectively. Introducing the unit normal vector \( \mathbf{n}_{ij} = \mathbf{r}_{ij}/|\mathbf{r}_{ij}| \), the normal force acting on i, which is denoted by \( \mathbf{F}_{ij}^{(n)} \), is given by \([f(\epsilon_{ij}) + \zeta|\mathbf{n}_{ij} \cdot \dot{\mathbf{r}}_{ij}|]|\mathbf{n}_{ij}\), where \( \epsilon_{ij} = 1 - |\mathbf{r}_{ij}|/(R_i + R_j) \). A function \( f(\epsilon) \) describes elastic repulsion between grains. Here we test two models: \( f(\epsilon) = k\epsilon \) (the linear force) and \( f(\epsilon) = k\epsilon^{3/2} \) (the Hertzian force) [10]. Note that the constant \( k/d^2 \) is on the order of the Young’s modulus of the grains. In order to define the transverse force, we utilize the relative tangential velocity \( \mathbf{v}_{ij}^{(t)} \) defined by \( (\dot{\mathbf{r}}_{ij} - \mathbf{n}_{ij} \cdot \dot{\mathbf{r}}_{ij}) + (R_i\mathbf{\Omega}_i + R_j\mathbf{\Omega}_j)/(R_i + R_j) \times \mathbf{r}_{ij} \) and introduce the relative tangential displacement vector \( \Delta_{ij}^{(t)} = \int_{\Omega_{ij}} \mathbf{v}_{ij}^{(t)} \cdot \mathbf{d}t \). The subscript in the integral indicates that the integral is performed when the contact is rolling, i.e., \( |\Delta_{ij}^{(t)}| < k_t \) or \( \Delta_{ij}^{(t)} \cdot \mathbf{v}_{ij}^{(t)} < 0 \). Then the tangential force acting on the particle i is written as

\[-\min(\mu |\mathbf{r}_{ij}|/|\mathbf{r}_{ij}|, k_t |\Delta_{ij}^{(t)}|)|\mathbf{F}_{ij}^{(n)}|\].

In the case that \( \mu = 0 \), the tangential force vanishes and the rotation of particles does not affect the translational motion. The parameter values adopted in the present simulation are given in Table 1.

The configuration of the system mimics a typical experiment on granular layers subjected to simple shear. (See the inset of Fig. 1.). Note that there is no gravity in the system. The system spans \( L_x \times L_y \times L_z \) volume, and is periodic in the \( x \) and the \( y \) directions. We prepare two systems of different aspect ratio, each of which contains approximately 10,000 particles: \( 25d \times 25d \times 8d \), and \( 15d \times 15d \times 25d \). As we shall discussed later, the difference of the aspect ratio does not affect the rheology. In the \( z \) direction, there exist two rough walls that consist of the same kind of particles as those in the bulk. The particles that consist walls are randomly placed on the boundary and their relative positions are fixed. The walls are parallel to each other and displaced antiparallel along the \( y \) axis at constant velocities \( \pm V/2 \), while they are prohibited to move along the \( x \) axis. One of the walls is allowed to move along the \( z \) axis so that the pressure is kept constant at \( P \). Using the mass of the wall \( M_w \) that is defined as the sum of the masses of the constituent particles, the \( z \) coordinate of the wall \( Z_w \) is described by the following equation of motion; \( M_w \dot{Z}_w = F_z - PS \), where \( F \) denotes the sum of the forces between the wall particles and the bulk particles, and \( S \) denotes the area of the wall. Then the \( z \) component of the velocity of the wall particles is given by \( \dot{Z}_w \). Note that the friction coefficient of the system is defined by \( F_y/PS \), which is denoted by \( M \) in the following.

| polydispersity | \( \zeta \sqrt{d/km} \) | \( k_t d \) | \( \mu \) | \( Pd^2/k \) |
|----------------|-----------------|-----------|------|------------|
| 30 \%          | 1               | 0.005     | 0 - 0.6 | \( 3.8 \times 10^{-5} - 1.1 \times 10^{-2} \) |
Figure 1. Temporal evolution of the friction coefficient (the black line) and that of the $z$ coordinate of the upper wall (the red line). The system is initially at rest and reaches a steady state after certain amount of slip. The system generally dilates upon sliding. Here the dilatation is approximately 4%. The layer thickness is $26d$ here. The inset shows the schematic of the simulation system, which consists of approximately 10,000 grains. The red and blue particles constitute the walls, through which the shear rate and the pressure are controlled.

3. Results

3.1. Transient
The system reaches a steady state after a certain amount of displacement of the walls. We judge that the system reaches a steady state if each of the following quantities does not show apparent trends and seems to fluctuate around a certain value: the friction coefficient, the $z$ coordinate of the wall (i.e., the density), and the granular temperature. In addition, snapshots of the velocity profile are observed to ensure the realization of uniform shear flow. A typical temporal behavior of the friction coefficient and the $z$ coordinate of the wall are shown in Fig. 1. We confirm that these transient behaviors are quite similar to those observed in experiments. Here we do not investigate such transients and restrict ourselves to steady-state friction.

3.2. Realization of uniform shear flow
Because uniform shear flow is unstable in a certain class of granular systems, we must check the internal velocity profiles at steady states. There is a strict tendency that shear flow is localized near the walls in the case that the confining pressure is weak and/or the sliding velocity of the walls is large. This kind of spatial inhomogeneity is rather ubiquitous in granular flow, and is extensively investigated by Nott et al. [11]. Note that uniform shear flow is realized in the present simulation at lower slip velocities and higher confining pressures. Hereafter we discuss exclusively the case in which uniform shear is realized.

3.3. Shear rate dependence of the friction coefficient
We investigate behaviors of the friction coefficient of the system, which is denoted by $M$. First we discuss the dependence of $M$ on the shear rate, $\gamma$. Because the shear rate is proportional
to the slip velocity in uniform shear flow, we can identify the shear rate dependence as the slip velocity dependence. Here we use a nondimensional slip velocity \( U = V/\sqrt{kd/m} \) for that purpose and a nondimensional shear rate is expressed as \( Ud/L_z \). In order to grasp the main point of our result, it is convenient to begin with the models in which friction between individual particles and the rotation of particles are not incorporated.

Recall that we test two models, each of which has a different type of interaction. These two models show essentially the same shear-rate dependence that is described as

\[
M = M_0 + \left( \frac{U}{U^*} \right)^\phi,
\]

where \( M_0 \) denotes the friction coefficient for \( U \to 0 \). Note that the values of \( \phi \) and \( M_0 \) do not depend on the interaction model; i.e. \( \phi = 0.3 \pm 0.05 \) and \( M_0 = 0.06 \pm 0.005 \). Here we test several values of the confining pressure and find that the intercept \( M_0 \) and the exponent \( \phi \) does not depend on the confining pressure, while the rate constant \( U^* \) does. Note that the layer thickness does not affect the friction law.

Then we wish to discuss the effect of the tangential force and the rotation of particles. That is, we adopt nonzero \( \mu \) in the present simulation. As shown in Fig. 4, in the Hertzian force model, the friction coefficient obeys Eq. (1) with \( M_0 \) being 0.27. It should be stressed that the exponent \( \phi \) is again approximately 0.3. This behavior is essentially the same as that of the linear force model. Indeed, in the case that \( \mu = 0.2 \), the Hertzian model and the linear force model bear almost the same frictional strength. Again, the confining pressure does not affect \( M_0 \) and \( \phi \) and the pressure dependence is expressed only through the constant \( U^* \).
Note that the friction coefficient of these models range from 0.3 to 0.4, which are not significantly discrepant from those obtained in an experiment on spherical glass beads [12].

3.4. The dynamic yield strength

Although the constant $M_0$ in Eq. (1) looks like the static friction coefficient, note that $M_0$ is defined in the $U \rightarrow 0$ limit and different from the static friction coefficient above which systems at rest begin to flow. In order to distinguish the two concepts, $M_0$ is referred to as dynamic yield strength. The difference is important when we consider the stability of slip, as will be discussed in the last paragraph of this paper. The value of the dynamic yield strength depends on, but not equals to, the friction coefficient between individual grains, $\mu$. In the linear force model, $M_0 \simeq 0.05$ for $\mu = 0$, while $M_0 \simeq 0.26$ for $\mu = 0.2$, and $M_0 \simeq 0.36$ for $\mu = 0.6$. Similar dependence was also observed in a two dimensional DEM simulation [13].

3.5. The inertial number and a master curve

Then we discuss the universality of the friction law, Eq. (1), and recast it in a more general form. Here we use a nondimensional parameter known as the inertial number [14]:

$$I = \frac{V}{L_z} \sqrt{\frac{m}{Pd}}.$$  

(2)

Indeed the inertial number is much more useful than the sliding velocity in describing rheology of granular media [13, 15]. As a remarkable result, which is shown in Fig. 5, we can see that the friction coefficient at different pressures collapse on the same curve. This master curve is described as

$$M = M_0 + s_1 I^\phi,$$  

(3)

where $s_1$ is a numerical factor that does not depend on the pressure. This equation is the main point of this paper. We stress that the model dependence, such as the value of $\mu$ or the type of a repulsive force, is expressed by $M_0$, while the exponent $\phi$ does not change. In particular, the dependence of $M$ (the "macroscopic" friction coefficient) on $\mu$ (the "microscopic" friction coefficient) is fully expressed by the dynamic yield strength, $M_0$. The numerical factor $s_1$ is insensitive to $\mu$, but somewhat depends on the degree of inelasticity. We discuss the behavior of $s_1$ in the next section.
3.6. The effect of inelasticity

Inelasticity of the grains is an important ingredient that potentially affects the frictional strength. In our model, inelasticity is modeled by the viscous coefficient $\zeta$, which appears in the expression of the normal repulsive force. We perform several runs of the simulation with different values of $\zeta$, and find that the decrease of $\zeta$ reduces the friction coefficient in the region where $I \geq 0.01$, while the frictional strength is independent of $\zeta$ for the smaller $I$ region. This behavior is consistent with those obtained in Refs. [13, 16]. Nevertheless, Eq. (1) still holds with $s_1$ being a smaller value. For example, when $\zeta = 0.05$ the friction coefficient is described by Eq. (3) with $s_1 \simeq 0.27$ while $s_1 \simeq 0.45$ when $\zeta = 20$. However, the functional form of $s_1(\zeta)$ is not clear at this point. We wish to stress that the degree of inelasticity affects only $s_1$; i.e., the exponent $\phi$ is independent.

4. Discussions

From the discussions so far, we can conclude that the details of the present model do not affect the validity of Eq. (3), which is the main result of this paper. Importantly, the exponent $\phi$ seems to be universal; it is approximately 0.3 regardless of the details of the model and the control parameters. The velocity-strengthening nature of this friction law does not contradict experiments [4, 5, 6, 7, 8]. In addition, it illustrates universality of power laws in rheological properties of random media [17, 18, 19], such as glassy materials [20, 21, 22], foams [23], and human neutrophils [24]. In the following we discuss four important points that are peripherally related to the main result.

4.1. The dilatation law

First, we discuss the dependence of the volume fraction to the inertial number. Surprisingly, decrease of the volume fraction caused by shear flow is also described by a power law [25].

$$\nu_0 - \nu = s_2 I^\delta,$$

where $\nu_0$ is the volume fraction in the $\gamma \to 0$ limit. Note that the constants $s_2$ and $\delta$ do not depend on the details of the model, where $s_2 \simeq 0.11$ and $\delta = 0.57 \pm 0.03$. This dilatation law also illustrates the ubiquity of power laws in granular matter. This fact leads to a speculation that the friction coefficient does not explicitly depend on the shear rate in the slow shear limit [16] so that the friction law reads $\mu(\nu) \simeq \mu_0 + (\nu_0 - \nu)^{\phi/\delta}$. Although it remains a speculation at this point, we remark that the similar behavior was observed in a two dimensional DEM simulation [26].
4.2. The effect of the boundary condition
The second point we wish to discuss is the relation between the present result and power-law rheology in systems at constant volume. In particular, Xu and O’Hern [27] found a power-law relation between the shear stress and the shear rate in a two dimensional granular material consisting of frictionless particles. They estimated the exponent to be 0.65. However, at constant volume condition, the pressure also depends on the shear rate so that the behavior of friction coefficient is generally different from that of shear stress. Therefore, power-law rheology in systems under constant volume condition are not directly related to the present result.

4.3. The effect of dimensionality
The next point we wish to discuss is the effect of dimensionality. Da Cruz et al. simulated a two dimensional granular system that is sheared under constant pressure [13]. In contrast to the present study, they obtained a linear rheology expressed in terms of the inertial number; i.e., $\phi \simeq 1$ in Eq. (2). The difference is attributed to the dimensionality of the systems, which affect the nature of contact between particles. In particular, the distribution of tangential force is strongly anisotropic in their two dimensional system, while such anisotropy is not observed in our three dimensional system. Accordingly, in the case that tangential force was switched off, their system exhibited a friction law that is quite similar to ours.

4.4. Relevance to earthquake machanics
As the last point of interest, we discuss relevance of our result to earthquake mechanics by comparing it to a friction law recently proposed by Jop et al. [28], which seems to be validated in experiments on inclined plane flow. We stress that such flow is characterized by relatively large inertial number (typically $10^{-1} \leq I \leq 10^{0}$), while our simulation involves much smaller values ($10^{-5} \leq I$) as shown in Fig. 5. Such small inertial numbers correspond to configurations that are considered to be typical to seismic motion of faults. For example, in the case that $d = 1$ mm, $V = 1$ m/s, $L_z = 4$ cm, and $P = 100$ MPa, the corresponding inertial number is $10^{-4}$.

Because the friction law found here is velocity-strengthening, one may wonder that it cannot explain stick-slip motion. Recall that we discuss exclusively stationary-state dynamic friction. Taking static friction into account, unstable slip is inevitable because static friction is always stronger than dynamic friction, which is mainly due to dilatation. Therefore power-law friction in stationary states does not contradict unstable slip on faults.

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