Accelerated Zeroth-order Algorithm for Stochastic Distributed Nonconvex Optimization

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Abstract—This paper investigates how to accelerate the convergence of distributed optimization algorithms on nonconvex problems with zeroth-order information available only. We propose a zeroth-order (ZO) distributed primal-dual stochastic coordinates algorithm equipped with “powerball” method to accelerate. We prove that the proposed algorithm has a convergence rate of \( O(\sqrt{p}/\sqrt{nT}) \) for general nonconvex cost functions. We consider solving the generation of adversarial examples from black-box DNNs problem to compare with the existing state-of-the-art approaches. The numerical results demonstrate the faster convergence rate of the proposed algorithm and match the theoretical analysis.

I. INTRODUCTION

In this paper, we focus on solving the following stochastic distributed nonconvex optimization problems

\[
\min_{x \in \mathbb{R}^p} f(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}_{\xi_i}[F_i(x, \xi_i)],
\]

where \( n \) is the total number of agents, \( x \) is the decision variable, \( \xi_i \) is a random variable with dimension \( p \), and \( F_i(\cdot, \xi_i) : \mathbb{R}^p \rightarrow \mathbb{R} \) is the stochastic function. Various algorithms utilizing gradient information aiming to solve such problems in the form of (1) have been proposed and applied to many applications. However, in many realistic problems, it is unable or too expensive to achieve the gradient information [1]–[3]. For example, the simulation based optimization problems [4], the black-box universal attacking of deep neural networks problems [5]–[7], just to name a few. Because of the unavailability of gradient information, we consider that agent \( i \) is only able to access its own stochastic ZO information \( F_i(x, \xi_i) \). For each agent \( i \), the local cost function \( f_i(x) \) is the expectation of the ZO information \( \mathbb{E}_{\xi_i}[F_i(x, \xi_i)] \). Agents communicate with their neighbors via an undirected communication network graph \( \mathcal{G} \).

In recent years, distributed ZO optimization problems obtained more and more attention and have been applied into networked [8], [9]–[12] consider distributed gradient descent methods with ZO information. [13] focus on applying the push-sum technique in distributed ZO optimization in order to handle direct communication between agents. [14] provided distributed ZO algorithm based on primal–dual framework for stochastic distributed nonconvex optimization problems and prove the convergence rate of \( O(\sqrt{p}/\sqrt{nT}) \). To our best knowledge, the proposed algorithm is the first accelerated method in distributed ZO optimization literature.

1) We propose an accelerated ZO algorithm based on primal–dual framework for stochastic distributed nonconvex optimization problems and prove the convergence rate of \( O(\sqrt{p}/\sqrt{nT}) \). To our best knowledge, the proposed algorithm is the first accelerated method in distributed ZO optimization literature.

2) ZODIAC in [20] can be considered as a special case in the proposed algorithm. We theoretically improve the convergence rate of ZODIAC from \( O(\sqrt{p}/\sqrt{p}) \) to \( O(\sqrt{p}/\sqrt{nT}) \).

3) Extensive numerical examples are provided to demonstrate the efficacy of the considered algorithm through benchmark examples on a large-scale agents systems. The rest of this paper is organized as follows. Section II introduces some preliminary concepts. Sections III introduces the proposed algorithm and analyzes its convergence properties. Simulations are presented in Section IV. Finally, concluding remarks are offered in Section V.

Notations: \( \mathbb{N}_0 \) and \( \mathbb{N}_+ \) denote the set of nonnegative and positive integers, respectively. \( [n] \) denotes the set \{1, \ldots, n\} for any \( n \in \mathbb{N}_+ \). \( \|\cdot\| \) represents the Euclidean norm for vectors or the induced 2-norm for matrices. \( \mathbb{R}^p \) and \( S^p \) are the unit ball and sphere centered around the origin in \( \mathbb{R}^p \) under Euclidean norm, respectively. Given a differentiable function \( f, \nabla f \) denotes the gradient of \( f \). \( 1_n \) (\( 0_n \)) denotes the column one (zero) vector of dimen-
sion $n$. col$(z_1, \ldots, z_k)$ is the concatenated column vector of vectors $z_i \in \mathbb{R}^{p_i}$, $i \in [k]$. $I_n$ is the $n$-dimensional identity matrix. Given a vector $[x_1, \ldots, x_n]^T \in \mathbb{R}^n$, diag$([x_1, \ldots, x_n])$ is a diagonal matrix with the $i$-th diagonal element being $x_i$. The notation $A \otimes B$ denotes the Kronecker product of matrices $A$ and $B$. Moreover, we denote $x = \text{col}([x_1, \ldots, x_n])$, $\bar{x} = \frac{1}{n}(1_n \otimes I_p)x$, $x = 1_n \otimes \bar{x}$, $\rho_2(\cdot)$ stands for the spectral radius for matrices and $\rho_2(\cdot)$ indicates the minimum positive eigenvalue for matrices having positive eigenvalues.

II. PRELIMINARIES

The following section discusses some background on graph theory, smooth functions, the gradient estimator, and additional assumptions used in this paper.

A. Graph Theory

Agents communicate with their neighbors through an underlying network, which is modeled by an undirected graph $G = (V, E)$, where $V = \{1, \ldots, n\}$ is the agent set, $E \subseteq V \times V$ is the edge set, and $(i, j) \in E$ if agents $i$ and $j$ can communicate with each other. For an undirected graph $G = (V, E)$, let $A = (a_{ij})$ be the associated weighted adjacency matrix with $a_{ij} > 0$ if $(i, j) \in E$ if $a_{ij} > 0$ and zero otherwise. It is assumed that $a_{ii} = 0$ for all $i \in [n]$. Let $\deg_i = \sum_{j=1}^{n} a_{ij}$ denotes the weighted degree of vertex $i$. The degree matrix of graph $G$ is $\text{Deg} = \text{diag}([\deg_1, \ldots, \deg_n])$. The Laplacian matrix is $L = (L_{ij}) = \text{Deg} - A$. Additionally, we denote $K_n = I_n - \frac{1}{n}1_n1_n^T$, $L = L \otimes I_p$, $K = K_n \otimes I_p$, $H = \frac{1}{n}(1_n1_n^T \otimes I_p)$. Moreover, from Lemmas 1 and 2 in [21], we know there exists an orthogonal matrix $[r \ R] \in \mathbb{R}^{n \times n}$ with $r = \frac{2}{\sqrt{n}}1_n$, and $R \in \mathbb{R}^{n \times (n-1)}$ such that $R\Lambda_1^{-1}R^T L = LR\Lambda_1^{-1}R^T = K_n$, and $\frac{1}{\rho_2(L)}K_n \preceq R\Lambda_1^{-1}R^T \preceq \frac{1}{\rho_2(L_1)}K_n$, where $\Lambda_1 = \text{diag}([\lambda_2, \ldots, \lambda_n])$ with $0 < \lambda_2 \leq \cdots \leq \lambda_n$ being the eigenvalues of the Laplacian matrix $L$.

B. Smooth Function

Definition 1. A function $f(x) : \mathbb{R}^p \rightarrow \mathbb{R}$ is smooth with constant $L_f > 0$ if it is differentiable and

$$
\|\nabla f(x) - \nabla f(y)\| \leq L_f \|x - y\|, \forall x, y \in \mathbb{R}^p. \tag{2}
$$

C. Gradient Approximation

Denote a random subset of the coordinates $S \subseteq \{1, 2, \ldots, p\}$ where the cardinality of $S$ is $|S| = n_c$. We provide two options of gradient approximation, denoted $\hat{g}_c^i$ and defined by (3) and (4).

$$
\hat{g}_c^i = \frac{p}{n_c} \sum_{i \in S} \left( \frac{F(x + \delta_i e_i, \xi) - F(x, \xi)}{\delta_i} \right) e_i \tag{3}
$$

The coordinates are sampled uniformly, i.e., $Pr(i \in S) = \frac{n_c}{p}$, which guarantees that both (3) and (4) are unbiased estimators of the full coordinate gradient estimator $\sum_{i=1}^{p} (\frac{F(x+\delta_i e_i, \xi) - F(x-\delta_i e_i, \xi)}{2\delta_i}) e_i \tag{22}$

D. Powerball Term

Define the function

$$
\sigma(x, \gamma) = \text{sgn}(x)|x|^\gamma \tag{5}
$$

where $\gamma \in [\frac{1}{2}, 1]$. Note that when $\gamma = 1$, $\sigma(x, 1)$ reduces to $x$. Unlike the “powerball” terms in [23] and [24], under distributed settings, the range of $\gamma$ has to be modified [25].

E. Assumptions

Assumption 1. The undirected graph $G$ is connected.

Assumption 2. The optimal set $\mathbb{K}^*$ is nonempty and the optimal value $f^* > -\infty$.

Assumption 3. For almost all $\xi_i$, the stochastic ZO oracle $F_i(\cdot, \xi_i)$ is smooth with constant $L_f > 0$.

Assumption 4. The stochastic gradient $\nabla_x F_i(x, \xi_i)$ has bounded variance for any jth coordinate of $x$, i.e., there exists $\zeta_i \in \mathbb{R}$ such that $\mathbb{E}_{\xi_i}[(\nabla_x F_i(x, \xi_i) - \nabla f_i(x))^2] \leq \zeta_i^2$, $\forall i \in [n], \forall j \in [p], \forall x \in \mathbb{R}^p$. It also implies that $\mathbb{E}_{\xi_i}[(\nabla_x F_i(x, \xi_i) - \nabla f_i(x))^2] \leq \sigma_2^2$, $\forall i \in [n], \forall x \in \mathbb{R}^p$.

Assumption 5. Local cost functions are similar, i.e., there exists $\sigma_2 \in \mathbb{R}$ such that $\|\nabla f_i(x) - \nabla f_j(x)\|^2 \leq \sigma_2^2$, $\forall i \in [n], \forall x \in \mathbb{R}^p$.

III. ALGORITHM

A. Algorithm Description

We consider the novel distributed primal-dual framework in [20] and apply the “powerball” term described in [5] directly on the estimations of gradient. We summarize the proposed method ZODIAC-PB as Algorithm 1.

$$
\begin{align*}
\text{(a)} \quad & x_{i,k+1} = x_{i,k} - n \left( \alpha \sum_{j=1}^{n} L_{ij} x_{j,k} + \beta \nu_{i,k} + \sigma(\hat{g}_c^i, \gamma) \right), \\
\text{(b)} \quad & \nu_{i,k+1} = \nu_{i,k} + \eta \beta \sum_{j=1}^{n} L_{ij} x_{j,k}, \\
& \forall x_{i,0} \in \mathbb{R}^p, \sum_{j=1}^{n} \nu_{j,0} = 0_p, \forall i \in [n].
\end{align*}
\tag{6}
$$
Algorithm 1 ZODIAC-PB

1: Input: positive number $\alpha$, $\beta$, $\eta$, and positive sequences $\{\delta_i\}$, $\gamma$.
2: Initialize: $x_i, 0 \in \mathbb{R}^p$ and $v_i, 0 = 0, \forall i \in [n]$.
3: for $k = 0, 1, \ldots$ do
4: for $i = 1, \ldots, n$ in parallel do
5: Broadcast $x_i, k$ to $\mathcal{N}_i$ and receive $x_j, k$ from $j \in \mathcal{N}_i$.
6: Select coordinates independently and uniformly;
7: Select $\xi_i, k$ independently;
8: Option 1: sample $F_i(x_i, k + \delta_i, e_i, k, \xi_i)$, and $F_i(x_i, k, \xi_i)$;
9: Update $x_i, k+1$ by (6a) with (3);
10: Option 2: sample $F_i(x_i, k + \delta_i, e_i, k, \xi_i)$, and $F_i(x_i, k - \delta_i, e_i, k, \xi_i)$;
11: Update $x_i, k+1$ by (6a) with (4);
12: Update $v_i, k+1$ by (6b);
13: end for
14: end for
15: Output: $\{x_k\}$.

B. Convergence Analysis

Theorem 1. Suppose Assumptions 1-5 hold. For any given $T > n^3/p$, let $\{x_k, k = 0, \ldots, T\}$ be the output generated by Algorithm 7 with
\[
\alpha = \kappa_1 \beta, \quad \beta = \frac{\kappa_2 \sqrt{pT}}{\sqrt{n}}, \quad \eta = \frac{\kappa_2}{\beta},
\]
\[
\delta_i, k \leq \frac{\kappa_8}{p^2 n^3(k + 1)^2}, \quad \forall k \leq T, \tag{7}
\]
where $\kappa_1, \beta \in \left(0, \min\left\{\frac{(\kappa_1 - 1)\rho(L) - 1}{\rho(L)} + 1, \frac{1}{\beta}\right\}\right)$, and $\kappa_8 > 0$, then we have,
\[
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \left[\|f(\bar{x}_k)\|^2\right] = O\left(\frac{\sqrt{p}}{\sqrt{n}T}\right) + O\left(\frac{n}{T}\right), \tag{8a}
\]
\[
\mathbb{E} [f(\bar{\theta}_k)] - f^* = O(1), \tag{8b}
\]
\[
\frac{1}{T} \sum_{k=0}^{T-1} \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^{n} \|x_i, k - \bar{x}_k\|^2\right] = O\left(\frac{n}{T}\right). \tag{8c}
\]

In order to prove Theorem 1 we introduce the following lemmas.

Lemma 1. (Lemma 2 in [20]) Consider $f(x) = \mathbb{E}_F[F(x, \xi)]$, we have the following relationship,
\[
\mathbb{E} \left[\|g_0^0\|^2\right] \leq 2(p - 1) \|\nabla f(x)\|^2 + 2p \sigma_2^2 + \frac{3p^2}{n_e} \left(\zeta^2 + \frac{L_2^2 \sigma_k^2}{2}\right)
\]

where $\delta_k = \max\left\{\delta_i, i \in [p]\right\}$.

Lemma 2. By using the powerball term in (5) and when $\gamma \in \left[\frac{1}{2}, 1\right)$, we have $\|g_0^0\|_{p_\gamma} \leq \|g_0^0\|_{p_1}$.

Proof. The proof follows the proof of Lemma 1 in [25] directly.

Lemma 3. Suppose Assumptions 1-5 hold. Let $\{x_k\}$ be the sequence generated by Algorithm 7, $g_k^0 = \text{col}(g_{1, k}^0, \ldots, g_{n, k}^0)$, $g_k^0 = n \nabla f(\bar{x}_k)$, $g_k^0 = Hg_k^0 = 1_n \otimes \nabla f(\bar{x}_k)$, then
\[
\mathbb{E} \left[\|g_k^0\|_{p_1}^2\right] \leq 6(p - 1) \|g_k^0\|_{p_1}^2 + 6(p - 1) L_2^2 \|x_k\|_K^2 + 6n(p - 1) \sigma_2^2 + 3n p^2 \left(\zeta^2 + \frac{L_2^2 \sigma_k^2}{2}\right)
\]

Lemma 4. Suppose Assumptions 1-5 hold, and we have fixed parameters $\alpha = \kappa_1 \beta$, $\beta$, and $\eta = \frac{\kappa_2}{\beta}$, where $\beta$ is large enough, $\kappa_1 > \frac{1}{\rho(L)} + 1$ and $\kappa_2 \in (0, \min\left\{\frac{(\kappa_1 - 1)\rho(L) - 1}{\rho(L)} + 1, \frac{1}{\beta}\right\})$ are constants. Let $\{x_k\}$ be the sequence generated by Algorithm 7 then
\[
\mathbb{E} [W_{k+1}] \leq W_k - \kappa_4 \|x_k\|_K^2 - \frac{1}{2} \left(\kappa_2 - 5 \kappa_3\right) \left\|v_k + \frac{1}{\beta} g_k^0\right\|_K^2 - \frac{1}{8} \eta \|g_k^0\|_{1+\gamma}^2 + O(n p \eta^2) + O(n p^2 \eta \delta_k^2), \tag{11a}
\]
\[
\mathbb{E} [W_{k+1}] \leq W_k + 2 \kappa_2 \|x_k\|_K^2 - \frac{1}{8} \eta \|g_k^0\|_2^2 + O(p) \eta^2 + O(n p \eta \delta_k^2). \tag{11b}
\]

Proof. We provide the proof of Lemma 4 in the appendix.

We are now ready to prove Theorem 1.

Proof. Denote $\bar{V}_k = \|x_k\|_K^2 + \left\|v_k + \frac{1}{\beta} g_k^0\right\|_K^2 + n(f(\bar{x}_k) - f^*)$. We have $W_k$...
= \frac{1}{2}\|x_k\|^2 + \frac{1}{2}\|v_k + \frac{1}{\beta_k}g_k^0\|^2_{Q + \kappa_k K} \\
+ x_k^T K\left(v_k + \frac{1}{\beta_k}g_k^0\right)\mathcal{N}(f(x_k) - f^*) \\
\geq \frac{1}{2}\|x_k\|^2 + \frac{1}{2}\left(\frac{1}{\rho(L)} + \kappa_1\right)\|v_k + \frac{1}{\beta_k}g_k^0\|^2_{K} \\
- \frac{1}{2\kappa_1}\|x_k\|^2 - \frac{1}{\kappa_1}\|v_k + \frac{1}{\beta_k}g_k^0\|^2_{K} + n(f(x_k) - f^*) \\
\geq \min\left\{\frac{1}{2\rho(L)}, \frac{\kappa_1 - 1}{2\kappa_1}\right\} \hat{V}_k \geq 0, \quad (12)

Additionally, we can get $W_k \leq \left(\frac{\kappa_1 + 1}{2} + \frac{1}{2\rho(L)}\right)\hat{V}_k$.

Consider that Lemma $\ref{lemma:1}$ is satisfied. So (11a) and (11b) hold. Summing (11a) over $k \in [0, T]$ and applying (12), we have

$$
\frac{1}{T + 1}\sum_{k=0}^{T} \mathbb{E}\left[\|x_{i,k} - \bar{x}_k\|^2\right] \\
\leq \frac{1}{\kappa_4}\left(\frac{W_0}{n(T + 1)} + \frac{O(n)\delta^2_k}{T} + \frac{O(n/p)\eta^2_k\kappa_k}{\sqrt{T(T + 1)}}\right) \\
= O\left(\frac{n}{T}\right), \quad (13)
$$

where $W_0 = O(n)$, $W_0 = O\left(\frac{1}{p}\right)$, $nO(n^2)/\delta_k^2 = O\left(\frac{n}{T}\right)$, and $O(n/p)\eta^2_k\kappa_k = O\left(\frac{n}{pT}\right)$, which gives (8c).

From (11b), (7), and (12), summing (11b) over $k \in [0, T]$ similar to the way to get (8c), we have

$$
\frac{1}{T + 1}\sum_{k=0}^{T} \mathbb{E}\left[\|\nabla f(\bar{x}_k)\|^2_{\gamma+1}\right] = \frac{1}{n(T + 1)}\sum_{k=0}^{T} \mathbb{E}\left[\|g_k^0\|^2_{1+\gamma}\right] \\
\leq 8\left(\frac{W_{4,0}}{n(T + 1)\eta} + \frac{2\ell_f L^2}{n(T + 1)}\sum_{k=0}^{T} \mathbb{E}[\|x_k\|^2_{K}] + \frac{O(p)}{n} \\
+ \frac{O(\sqrt{np})}{n\sqrt{T(T + 1)}}\right). \quad (14)
$$

Noting that $\eta = \sqrt{n}/\sqrt{T}$, due to $T > n^3/p$, from (14) and (13), we have

$$
\frac{1}{T}\sum_{k=0}^{T-1} \mathbb{E}[\|\nabla f(\bar{x}_k)\|^2_{\gamma+1}] = O(\frac{\sqrt{n}}{\sqrt{T}}) + O\left(\frac{n}{T}\right),
$$

which gives (8a).

Summing (11b) over $k \in [0, T]$, and using (7) yield

$$
\mathbb{E}[f(\bar{x}_{T+1}) - f^*) = \mathbb{E}[W_{4,T+1}] \\
\leq W_{4,0} + \frac{2\sqrt{n}}{\sqrt{pT}}\sum_{k=0}^{T} \mathbb{E}[\|x_k\|^2_{K}] + nO(p)\eta^2\frac{T + 1}{T} \\
+ O(n/p)\eta^2\sqrt{\frac{T + 1}{T}}. \quad (15)
$$

Noting that $W_{4,0} = O(n)$ and $\sqrt{n}/\sqrt{pT} < 1$ due to $T > n^3/p$, from (13) and (15), we have $\mathbb{E}[f(\bar{x}_{T+1}) - f^*) = O(1)$, which gives (8b).

\end{proof}

Remark 1. Theorem $\ref{theorem:1}$ indicates that we improve the convergence rate of the algorithm in $[20]$ from $O\left(\frac{\sqrt{n}}{T}\right)$ to $O\left(\frac{\sqrt{n}}{\sqrt{T}}\right)$, which is the same convergence rate of the algorithm proposed in $[19]$.

IV. NUMERICAL EXPERIMENTS

A. Black-box binary classification

We consider a non-linear least square problem [6], [26], [27], i.e., problem with $f_i(x) = (y_i - \phi(x; a_i))^2$ for $i \in [n]$, where $\phi(x; a_i) = \frac{1}{1+e^{-a_i^T x}}$. For preparing the synthetic dataset, we randomly draw samples $a_i$ from $\mathcal{N}(0, I)$, and we set a optimal vector $x_{opt} = 1$, the label is $y_i = 1$ if $\phi(x_{opt}; a_i) \geq 0.5$ and 0 otherwise. The training set has 200 samples per agent and test set has 10,000 samples. We set the dimension $d$ of $a_i$ as 100, batchsize is 1, and the total iteration number as 500. As suggested in the work [27], the smooth parameter $\delta = \frac{10}{T}$.

We compare the proposed algorithms with ZODIAC only since ZODIAC achieves better result than other state-of-the-art algorithms. The communication topology of $N = 500$ agents is generated randomly following the Erdős - Rényi model with probability of $1.01 \log(N)/N$ in Figure $\ref{fig:1}$ The training loss and testing accuracy are shown in Figure $\ref{fig:2}$ and Figure $\ref{fig:3}$ respectively. We can easily see that the proposed algorithm converges faster than ZODIAC and returns a better result in terms of testing accuracy, shown in Table $\ref{table:1}$.

B. Generation of adversarial examples from black-box DNNs

We consider the benchmark example of generation of adversarial examples from black-box DNNs in ZO optimization literature [6], [27], [28]. In image classification
tasks, convolutional neural networks are vulnerable to adversarial examples [5] even under small perturbations, which leads to misclassifications. Considering the setting of zeroth-order attacks [29], the model is hidden and no gradient information is available. We treat this task of generating adversarial examples as a zeroth-order optimization problem.

Formally, the loss function is given as in (16)

\[
\begin{align*}
    f_i(x) &= c \cdot \max \{F_i(0.5 \cdot \tanh(\tanh^{-1} 2a_i + x)) \\
    &\quad - \max_{j \neq i} F_j(0.5 \cdot \tanh(\tanh^{-1} 2a_i + x)), 0\} \\
    &\quad + \|0.5 \cdot \tanh(\tanh^{-1} 2a_i + x) - a_i\|_2^2
\end{align*}
\]

(16)

where \((a_i, y_i)\) denotes the pair of the \(i\)th natural image \(a_i\) and its original class label \(y_i\). The output of function 

\[ F(z) = [F_1(z), \ldots, F_N(z)] \] 

is the well-trained model prediction of the input \(z\) in all \(N\) image classes. The well-trained DNN model [4] on MNIST handwritten has 99.4% test accuracy on natural examples [6]. The purpose of this experiment is to generate false examples to attack the DNN model in order to have a wrong prediction, i.e. if feeding an original image with label 1, the DNN predicts it as 1, however after generating the false example based on the original 1, the DNN should make a wrong prediction. We conduct two experiments on 10 agents and 50 agents scenarios.

1) 10 agents: We compare the proposed algorithm with several existing algorithms, namely ZODIAC [20], ZODPDA [19], ZO-GDA [15], and ZONE-M [17] on a communication topology with 10 agents following the Erdős - Rényi model with probability of 0.4. The digit we consider to attack is 4. Additionally, we compare with centralized ZO algorithms, namely ZO-SCD [30], and ZO-SGD [31] as baselines. The training loss is shown in Figure 4 and the distortion of the generated examples is shown in Table II, we can conclude that the ZODIAC-PB outperformed among all the algorithms compared in the literature.

\[\text{Fig. 2: Training Loss.}\]

\[\text{Fig. 3: Testing Accuracy.}\]

\[\text{Fig. 4: Performance comparison of training loss for 10 agents.}\]

\[\text{TABLE I: Accuracy}\]

| Algorithm   | Accuracy(%) |
|-------------|-------------|
| ZODIAC-PB   | 94.15       |
| ZODIAC      | 92.56       |

\[\text{TABLE II: Distortion (10 agents)}\]

| Algorithm   | \(l_2\) Distortion |
|-------------|-------------------|
| ZODIAC-PB   | 4.92              |
| ZODIAC      | 7.18              |
| ZODPDA      | 6.44              |
| ZO-GDA      | 7.23              |
| ZONE-M      | 9.96              |
| ZO-SGD      | 5.69              |
| ZO-SCD      | 5.14              |

[https://github.com/carlini/nn_robust_attacks]
2) **50 agents**: In this case, we only compare the proposed algorithms with ZODIAC to attack digit 0 since ZODIAC achieves better result than other state-of-the-art algorithms. We tested them on 50 agents respectively, the topology are shown in Figure. The graphs are generated randomly following the Erdős - Rényi model with probability of 0.4.

The distortion of the generated examples is shown in Table III and the generated examples and prediction results are shown in Table IV. In this experiment, we can conclude that the proposed algorithm accelerate the convergence in Figure 6. Moreover, the distortion generated from ZODIAC-PB is 5.67, which is around 34.7% improvement.

![Communication topology of 50 agents.](image1)

**Fig. 5:** Communication topology of 50 agents.

![Performance comparison of training loss for 50 agents.](image2)

**Fig. 6:** Performance comparison of training loss for 50 agents.

| Algorithm   | $l_2$ Distortion |
|-------------|------------------|
| ZODIAC-PB   | **5.67**         |
| ZODIAC      | 8.68             |

**TABLE III:** Distortion (50 agents)

V. **Conclusions**

In this paper, we investigated the acceleration of ZO stochastic distributed nonconvex optimization problems and proposed ZODIAC-PB based on the primal–dual framework. We demonstrated that the proposed algorithm achieves the convergence rate of $O(\frac{1}{\sqrt{p}}\sqrt{nT})$ for general nonconvex cost functions. Additionally, we illustrated the efficacy of ZODIAC-PB through benchmark examples on a large-scale multi-agent topology in comparison with the existing state-of-the-art centralized and distributed ZO algorithms.

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**APPENDIX**

**Proof of Lemma 4.** Consider the following Lyapunov candidate function

\[
W_k = \frac{1}{2} \|x_k\|_K^2 + \frac{1}{2} \|v_k + \frac{1}{\beta} g_{k}^{\alpha}\|_{Q_{\gamma}}^2
\]

\[+ \frac{1}{2} \|v_k + \frac{1}{\beta} g_{k}^{\alpha}\|_{Q_{\gamma}}^2 + n(f(x_k) - f^*) \tag{17}
\]

where \(Q = RA_1^{-1}R^T \otimes I_p\). Additionally, we denote \(g_{i,k}^{\alpha} = \nabla f_i(x_k + \delta_{i,k}e_1)\), \(g_{i,k}^{\alpha} = \text{col}(g_{1,k}^{\alpha}, \ldots, g_{n,k}^{\alpha})\), \(g_{k}^{\alpha} = Hg_{k}^{\alpha}\), \(\bar{g}_{k}^{\alpha} = \frac{1}{n} \otimes I_p g_{k}^{\alpha}\), and \(\bar{g}_{k}^{\alpha} = 1_n \otimes \bar{g}_{k}^{\alpha} = Hg_{k}^{\alpha}\).

(i) We have

\[
\mathbb{E}[W_{k+1}] = \mathbb{E}\left[\frac{1}{2} \|x_{k+1}\|_K^2\right]
\]

\[
= \mathbb{E}\left[\frac{1}{2} \|x_k - \eta(\alpha L x_k + \beta v_k + \sigma(g_{k}^{\alpha}, \gamma))\|_K^2\right]
\]

\[
= \mathbb{E}\left[\frac{1}{2} \|x_k\|_K^2 - \eta \alpha \|x_k\|^2 L \right. + \frac{1}{2} \eta^2 \alpha^2 \|x_k\|^2 L^2
\]

\[
- \eta \beta \|x_k\|_K^2 - \eta \beta (I_{np} - \eta \alpha L)K(v_k + \frac{1}{\beta} g_{k}^{\alpha}, \gamma)
\]

\[
+ \frac{1}{2} \eta^2 \beta^2 \left\|v_k + \frac{1}{\beta} g_{k}^{\alpha}(\gamma)\|_K^2\right\|
\]

**TABLE IV:** Comparison of generated adversarial examples from a black-box DNN on MNIST: digit class “0”.

| Image ID | Original | Classified as |
|----------|----------|---------------|
| 3        | ![Image](image1.png) | 0             |
| 10       | ![Image](image2.png) | 0             |
| 13       | ![Image](image3.png) | 0             |
| 25       | ![Image](image4.png) | 0             |
| 28       | ![Image](image5.png) | 0             |
| 55       | ![Image](image6.png) | 0             |
| 69       | ![Image](image7.png) | 0             |
| 71       | ![Image](image8.png) | 0             |
| 101      | ![Image](image9.png) | 0             |
| 126      | ![Image](image10.png) | 0             |

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where (a) holds due to Lemma 1 and 2 in [21]; (b) holds due to the Cauchy–Schwarz inequality; (g) holds due to Lemma 1 and 2 in [21]; (h) holds due to \( \rho(Q + \kappa_1 K) \leq \rho(Q) + \kappa_1 \rho(K) \), and \( \rho(K) = 1 \); (i) holds due to \( \|g^0_{k+1} - g^0_k\|^2 \leq \eta^2 L_f^2 \|\hat{g}_k\|^2 \leq \eta^2 L_f^2 \|g_k\|^2 \).

Moreover, we have the following two inequalities hold:

\[
\|g^0_{k+1}\|_{Q+\kappa_1 K} \leq \left( \frac{1}{\rho_2(L)} + \kappa_1 \right) \|g^0_k\|_K. \tag{21}
\]

\[
\|v_k + \frac{1}{\beta_k} g^0_k\|_{Q+\kappa_1 K} \leq \left( \frac{1}{\rho_2(L)} + \kappa_1 \right) \|v_k + \frac{1}{\beta_k} g^0_k\|_K. \tag{22}
\]

Then, from (19)–(22), we have

\[
W_{2,k+1} \leq W_{2,k} + \eta \beta \lambda x^T_k (K + \kappa_1 L) \left( g^0_k + \frac{1}{\beta} L \nu \right) + \frac{1}{2} \eta \left( \frac{1}{\rho_2(L) + \kappa_1} \right) \|v_k + \frac{1}{\beta_k} g^0_k\|^2_K + \frac{1}{\beta} \lambda \|x_k\|^2_{Q+\kappa_1 K} + \frac{\eta}{2} \left( \frac{1}{\rho_2(L) + \kappa_1} \right) \|x_k\|^2_{Q+\kappa_1 K}.
\]

(iii) We have

\[
W_{3,k+1} = x^T_{k+1} K \left( v_{k+1} + \frac{1}{\beta} g^0_{k+1} \right).
\]

\[
\text{E} \left[ W_{3,k+1} \right] \leq \text{E} \left[ x^T_k - \eta (\alpha L x_k + \beta v_k + g^0_k + \sigma(g_k, \gamma) - g^0_k) \right]^T K \left( v_k + \frac{1}{\beta} g^0_k + \eta \beta L x_k + \frac{1}{\beta} (g^0_{k+1} - g^0_k) \right)
\]

\[
= \left( x^T_k - \eta (\alpha L x_k + \beta v_k + g^0_k + \sigma(g_k, \gamma) - g^0_k) \right)^T K \left( v_k + \frac{1}{\beta} g^0_k + \eta \beta L x_k + \frac{1}{\beta} (g^0_{k+1} - g^0_k) \right)
\]
+ \frac{1}{\beta} x_k^\top (K - \eta \alpha L) E[g_k^{n+1} - g_k^n] - \eta \beta \|v_k + \frac{1}{\beta} g_k^n\|_K^2 \\
- \eta \left( v_k + \frac{1}{\beta} g_k^n \right)^\top K E[g_k^{n+1} - g_k^n] \\
- \eta \langle g_k^* - g_k^n \rangle^\top K \left( v_k + \frac{1}{\beta} g_k^n + \eta \beta L x_k \right) \\
- \frac{1}{\beta} \mathbb{E} \left[ \sigma(g_k^*, \gamma) - g_k^n \right]^\top K (g_k^{n+1} - g_k^n)

\leq x_k^\top (K - \eta \alpha L) (v_k + \frac{1}{\beta} g_k^n) + \frac{1}{2} \eta^2 \beta^2 \|L x_k\|^2 \\
+ \frac{1}{2} \eta^2 \beta^2 \left\| v_k + \frac{1}{\beta} g_k^n \right\|_K^2 + \frac{1}{2} \eta^2 \beta^2 \mathbb{E} \left[ \|g_k^{n+1} - g_k^n\|^2 \right] \\
+ \frac{1}{2} \eta \|x_k\|_K^2 + \frac{1}{2} \eta^2 \beta^2 \mathbb{E} \left[ \|g_k^{n+1} - g_k^n\|^2 \right] \\
+ \frac{1}{2} \eta^2 \beta^2 \|L x_k\|^2 + \frac{1}{2} \eta \beta \|v_k + \frac{1}{\beta} g_k^n\|_K^2 \\
- \eta \beta \left\| v_k + \frac{1}{\beta} g_k^n \right\|_K^2 \\
+ \frac{1}{2} \eta^2 \beta^2 \left\| v_k + \frac{1}{\beta} g_k^n \right\|_K^2 + \frac{1}{2} \eta^2 \beta^2 \mathbb{E} \left[ \|g_k^{n+1} - g_k^n\|^2 \right] \\
+ \frac{1}{2} \eta \|x_k\|_K^2 + \frac{1}{2} \eta^2 \beta^2 \mathbb{E} \left[ \|g_k^{n+1} - g_k^n\|^2 \right] \\
+ \frac{1}{2} \eta^2 \mathbb{E} \left[ \sigma(g_k^*, \gamma) - g_k^n \right]^2 + \frac{1}{2} \eta \beta \|v_k + \frac{1}{\beta} g_k^n\|_K^2 \\
- \|v_k + \frac{1}{\beta} g_k^n\|^2_{\beta^2(\beta L + 1/2) + \eta \beta^2 \alpha L^2 + \eta \beta^2 L^2 K}

(\text{j})$ holds since $K_i = L K_{i} = L, \mathbb{E}[g_k^n] = g_k^*$, and that $x_{i,k}$ and $v_{i,k}$ are independent; (k) holds due to the Cauchy–Schwarz inequality, the Jensen’s inequality, and $\rho(K) = 1$; (l) holds due to $\|g_k^* - g_k^n\|^2 \leq 2L^2 \|x_k\|^2_K + \frac{1}{\beta} L^2 \|g_k^n\|^2_K + \frac{4}{\beta^2} \mathbb{E} \|\sigma(g_k^*, \gamma)^2\|_K^2 \leq 4L^2 \|x_k\|^2_K + 4nL^2 \|g_k^n\|^2_K + 2\mathbb{E} \|\sigma(g_k^*, \gamma)^2\|_K^2$.

(iv) We have

$$
\begin{align*}
\mathbb{E}[W_{4,k+1}] &= \mathbb{E}[n(f(\bar{x}_{k+1}) - f^*)] = \mathbb{E}[\hat{f}(\bar{x}_{k+1}) - n f^*] \\
&= \mathbb{E}[f(\bar{x}_{k+1}) - n f^* + f(\bar{x}_{k+1}) - \hat{f}(\bar{x}_{k+1})] \\
&\leq \mathbb{E}[\hat{f}(\bar{x}_{k+1}) - n f^* - \eta \langle g_k^* \rangle^\top g_k^0 + \frac{1}{\beta} \eta^2 L f\|g_k^0\|^2] \\
&\leq W_{4,k} - \eta \langle g_k^* \rangle^\top g_k^0 + \frac{1}{\beta} \eta^2 L f\|g_k^0\|^2 \\
&\leq W_{4,k} - \frac{1}{\beta} \eta \|g_k^0\|^2 + \frac{1}{\beta} \eta^2 L f\|g_k^0\|^2 + \frac{1}{\beta} \eta^2 L f\|g_k^0\|^2
\end{align*}
$$

(\text{m}) holds due to $f$ is smooth; (n) holds due to $\mathbb{E}[g_k^n] = g_k^*$, $x_{i,k}$ and $v_{i,k}$ are independent; (o) holds due to $\|g_k^*\|^2 = (g_k^*)^\top H g_k^* = (g_k^*)^\top H H g_k^* = (g_k^*)^\top g_k^n$; (p) holds due to the Cauchy–Schwarz inequality; and (q) holds due to $\|g_k^* - g_k^n\|^2 \leq 2L f^2 \|x_k\|^2_K + \frac{np}{4} \|g_k^n\|^2$.

(v) Define $W_{k+1} = \sum_{i=1}^4 W_{i,k+1}$ and then we have the following inequality.

$$
\begin{align*}
\mathbb{E}[W_{k+1}] &\leq W_k - \|x_k\|^2_{\eta \alpha L - \frac{1}{4} \eta \gamma L K - \frac{1}{4} \eta^2 \alpha^2 L^2 - \eta (1 + 3n) L^2 K} \\
&\quad + \|v_k + \frac{1}{\beta} g_k^n\|^2_{\frac{1}{4} \eta^2 \gamma^2 \beta^2} \\
&\quad + n L^2 \|f\|^2_{\frac{1}{4} + \left\| L f(\gamma) \right\|^2_{\frac{1}{4}}} \\
&\quad + \frac{1}{\beta} \mathbb{E} \left[ \sigma(g_k^*, \gamma) \right]^2 \\
&\quad + \|x_k\|^2_{\frac{\eta \beta^2}{\beta^2} \left( 1 + \frac{3}{\beta^2} \right) L^2} \\
&\quad + n L^2 \|f\|^2_{\frac{1}{4} + \left\| L f(\gamma) \right\|^2_{\frac{1}{4}}} \\
&\quad + \|x_k\|^2_{\left( 1 + \frac{1}{\beta^2} \right) L^2} \\
&\quad + \eta \left( \frac{1}{\beta^2} \right) \left( 1 + \frac{3}{\beta^2} \right) L^2 \mathbb{E} \left[ \|g_k^0\|^2 \right] \\
&\quad + \eta \left[ \left( 1 + \frac{1}{\beta^2} \right) L^2 \mathbb{E} \left[ \|\sigma(g_k^*, \gamma)\|^2 \right] \\
&\quad + n L^2 \|f\|^2_{\frac{1}{4} + \left\| L f(\gamma) \right\|^2_{\frac{1}{4}}} \\
&\quad + \|x_k\|^2_{\frac{\eta \beta^2}{\beta^2} \left( 1 + \frac{3}{\beta^2} \right) L^2} \\
&\quad + \eta \left( \frac{1}{\beta^2} \right) \left( 1 + \frac{3}{\beta^2} \right) L^2 \mathbb{E} \left[ \|g_k^0\|^2 \right] \\
&\quad + \|x_k\|^2_{\frac{\eta \beta^2}{\beta^2} \left( 1 + \frac{3}{\beta^2} \right) L^2} \\
&\quad + \eta \left( \frac{1}{\beta^2} \right) \left( 1 + \frac{3}{\beta^2} \right) L^2 \mathbb{E} \left[ \|g_k^0\|^2 \right]
\end{align*}
$$

(27)

where (r) holds due to (10a), (10b), $\alpha = \kappa \beta$, $\eta = \frac{\kappa^2}{\beta}$.
and

\[ M_1 = (\alpha - \beta)L - \left(1 + 3L_f^2 + \frac{6L_f^4\kappa_1}{\beta^2} (p - 1)\right)K, \]
\[ M_2 = \beta^2 L + (2\alpha^2 + \beta^2)L^2 + 8L_f^2K + 6(p - 1)\left(3 + \frac{1}{2}L_f + \frac{2L_f^2}{\beta^2}\kappa_1 + \frac{L_f^2}{2\beta^2}\right)K, \]
\[ \kappa_3 = \frac{1}{\rho_2(L)} + \kappa_1 + 1, \]
\[ b_0^2 = \frac{1}{2}\eta(2\beta - \kappa_3) - 2.5\kappa_2^2, \]
\[ b_1 = 6\rho\kappa_3 L_f^4 \frac{\eta}{\beta^2} + 12\rho(\kappa_3 + 1)L_f^4 \frac{\eta^2}{\beta^2}, \]
\[ c_1 = \left(3 + \frac{1}{2}L_f + \frac{2L_f^2}{\beta^2}\kappa_1 + \frac{L_f^2}{2\beta^2}\right)n\eta^2 + \frac{L_f^2\kappa_1}{\beta^2}n\eta, \]
\[ c_2 = \frac{3}{4}pn + \eta n\left(\frac{p}{2} + 6\right) + \frac{p^2 L_f^2}{2\beta^2} \kappa_1, \]
\[ c_3 = c_2 + \left(c_1 - \frac{L_f^2\kappa_1}{\beta^2}n\eta\right)p^2\eta. \]

Consider \( p \geq 1, \alpha = \kappa_3 \beta, \kappa_1 > 1, \beta \) is large enough, and \( \eta = \frac{\kappa_2}{\beta} \), we have

\[ \eta M_1 \geq [(\kappa_1 - 1)\rho_2(L) - 1]\kappa_2 K. \quad (28) \]
\[ \eta^2 M_2 \leq [\rho(L) + (2\kappa_1^2 + 1)\rho(L^2) + 1]\kappa_2^2 K. \quad (29) \]
\[ b_0^2 \geq \frac{1}{2}(\kappa_2 - 5\kappa_2^2). \quad (30) \]

From (27)–(30), let \( \kappa_4 = [(\kappa_1 - 1)\rho_2(L) - 1]\kappa_2 - [\rho(L) + (2\kappa_1^2 + 1)\rho(L^2) + 1]\kappa_2^2 \) we know that (11a) holds. Similar to the way to get (11a), we have (11b).