Generating collection queries from proofs

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Abstract
Nested relations, built up from atomic types via tupling and set types, form a rich data model. Over the last decades the nested relational calculus, NRC, has emerged as a standard language for defining transformations on nested collections. NRC is a strongly-typed functional language which allows building up queries using products and projections, a singleton-former, and a map operation that lifts queries on tuples to queries on sets. In this work we show that NRC has a strong connection with first-order logic: it contains exactly the transformations that are implicitly definable by a theory $\Sigma$ in first-order logic with quantification suited for nested collections.

We also prove an effective variant of our result, providing a procedure that synthesizes an NRC expression in polynomial time from a proof witnessing that $\Sigma$ provides an implicit definition for one subset of its free variables in terms of another subset of the variables. This synthesis result works off of proofs within an intuitionistic calculus that captures a natural style of reasoning about implicit definability in the context of nested collections.

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1 Introduction

Nested relations are a natural data model for hierarchical data. Nested relations are objects within a type system built up from basic types via tupling and a set-former. In the 1980’s and 90’s, a number of algebraic languages were proposed for defining nested relational queries: transformations on nested collections. Eventually a standard language emerged, the nested relational calculus (NRC). The language is strongly-typed and functional, with queries built up via tuple manipulation operations as well as operators for lifting queries over a type \( T \) to a sets over \( T \), such as singletons constructors and a mapping operator. One common formulation of these uses variables and a “comprehension” operator for forming new complex objects from old ones [9], while an alternative algebraic formalism presents the language as a set of operators that can be freely composed. It was shown that each NRC query can be evaluated in polynomial time in the size of a finite data input, and that when the input and output is “flat” (i.e. only one level of nesting), NRC expresses exactly the queries in the standard relational query language relational algebra. Wong’s thesis [47] summarizes the argument made by this line of work “NRC can be profitably regarded as the ‘right’ core for nested relational languages”. NRC has been the basis for most work on querying nested relations. It is the basis for a number of commercial tools [34], including those embedding nested data transformations in programming languages [33], in addition to having influence in the effective implementation of data transformations in functional programming languages [19, 18].

Although NRC can be applied to other collection types, such as bags and lists, we will focus here on just nested sets. We will show a new connection between NRC and first-order logic. There is a natural logic for describing properties of nested relations, the well-known \( \Delta_0 \) formulas, built up from equalities using quantifications \( \exists x \in \tau, \forall y \in \tau \) where \( \tau \) is a term. For example, a sentence \( \forall x \in c \ x.a \in x.b \) might describe a property of a nested relation \( c \) whose \( a \) attribute is an integer and whose \( b \) attribute is a set of integers. A \( \Delta_0 \) formula \( \Sigma(\vec{R}, Q) \) over variables \( \vec{R} \) and variable \( Q \) thus defines a relationship between \( \vec{R} \) and \( Q \). For such a formula to define a query it must be functional: it must enforce that \( Q \) depends only on the value of \( \vec{R} \). More generally, if we have a formula \( \Sigma(\vec{R}, \vec{A}, Q) \), we say that \( \Sigma \) projectively implicitly defines \( Q \) in terms of \( \vec{R} \) if:

\[
(\ast) \quad \text{For each two bindings } \sigma_1 \text{ and } \sigma_2 \text{ of the variables } \vec{R}, \vec{A}, Q \text{ to nested relations satisfying } \Sigma, \text{ if } \sigma_1 \text{ and } \sigma_2 \text{ agree on each } R_i \text{ they agree on } Q.
\]

That is, \( \varphi \) entails that the value of \( Q \) is a partial function of the value of \( \vec{R} \). We drop the qualifier “projectively” if the set of auxiliary predicates \( \vec{A} \) is empty. Such a sentence \( \varphi \) is total for \( \vec{R} \) if for each valuation for \( \vec{R} \), there is an extension to a valuation for \( \vec{R}, \vec{A}, Q \) such that \( \varphi(\vec{R}, \vec{A}, Q) \). A sentence that is total for \( \vec{R} \) and projectively defines \( Q \) as a function of \( \vec{R} \) defines a query from \( \vec{R} \) to \( Q \) in the obvious way.

A query \( Q \) taking nested relations \( \vec{R} \) to a nested relation \( Q \) is projectively implicitly first-order definable if there is a \( \varphi(\vec{R}, \vec{A}, Q) \) which projectively implicitly defines the predicate \( Q \) in terms of \( \vec{R} \), and for which the corresponding function is \( Q \). Such a query is said to be simply implicitly first-order definable if \( \vec{A} \) is empty.

Note that when we say “for each binding of variables to nested relations” in the definitions above, we include infinite nested relations as well as finite ones. An alternative characterization of \( \varphi \) being a projective implicit definition, which will be more relevant to us in the sequel, is that there is a proof that \( \varphi \) defines a functional relationship. Note that \( (\ast) \) is a first-order entailment: \( \varphi(\vec{R}, \vec{A}, Q) \land \varphi(\vec{R}, \vec{A'}, Q') \models \forall x Q(x) \leftrightarrow Q'(x') \). We refer to a proof of \( (\ast) \) for a given \( \varphi \) and \( \vec{R} \), as a proof of functionality of \( Q \). By the completeness theorem of first-order logic, whenever \( Q \) is projectively implicitly definable according to the semantic definition above, there is a proof that witnesses this. over \( \vec{R} \) assuming \( \varphi \). Such a proof will be in first-order logic with axioms capturing extensionality of sets, the compatibility of the membership relation with the type hierarchy, and
properties of projections and tupling.

**Example 1.** We will consider a formula that describes a transformation over input nested relations \(b_1, b_2, b_3, b_4\) whose types are as follows:

\[
b_1 : \text{Set}(\mathcal{U} \times \mathcal{U}) \quad b_2 : \text{Set}(\mathcal{U} \times \mathcal{U} \times \mathcal{U}) \quad b_3 : \text{Set}(\mathcal{U} \times \mathcal{U} \times \mathcal{U}) \quad b_4 : \text{Set}(\text{Set}(\mathcal{U}))
\]

Above \(\mathcal{U}\) refers to the basic set of elements, the “Ur-elements” in the sequel. \(b_1, b_2, b_3\) are relations in the usual sense, with \(b_1\) binary and the other two ternary. \(b_4\) is a nested relation.

Our formula will relate these to the following “output object” \(q : \text{Set}(\text{Set}(\mathcal{U}))\). That is, \(q\) is a set of sets of Ur-elements.

We let \(\tau_2(i, j, b_2)\) be “the fiber of \(b_2\) over Ur-elements \(i\) and \(j\) = the set of \(x\) such that \((i, j, x)\) is in \(b_2\). We let \(\tau_1(i, b_1)\) be the same but for \(b_1\). We let \(\tau_1(i, b_1)\) be the fiber of Ur-element \(i\) in \(b_1\): the set of \(y\) such that \((i, y)\) \(\in b_1\). Finally, we let \(\pi_i(b_1) : \text{Set}(\mathcal{U})\) be the set of elements \(x\) such that there exists \(y\) with \((x, y) \in b_1\).

Consider the \(\Delta_0\) formula \(\Sigma(b_1, b_2, b_3, b_4, q)\) obtained as the conjunction of

- a formula stating that each set \(s\) in \(q\) is contained in \(b_4\) and has an element \(a\) in \(\pi_1(b_1)\) such that every \(y\) in the \(b_1\)-fiber of \((a, s)\) contains a set sandwiched between the \(b_2\) and \(b_3\)-fibers of \((a, y)\):

\[
\forall s \in q \ (s \subseteq b_4 \land \exists a \in \pi_1(b_1) \forall y \in \tau_1(a, b_1) \exists x \in s \quad \tau_2(a, y, b_2) \subseteq x \subseteq \tau_3(a, y, b_3))
\]

- and another formula stating that for each element of \(\pi_1(b_1)\), there is a set \(s'\) in \(q\) containing \(b_4\), and for every set \(x\) contained in \(s'\), there is some \(y\) in the \(b_1\)-fiber of \(a\) such that \(x'\) is sandwiched between the \(b_3\) and \(b_2\)-fibers of \((a, y)\):

\[
\forall a \in \pi_1(b_1) \exists s' \in q \ (b_4 \subseteq s' \land \forall x' \in s' \exists y \in \tau_1(a, b_1) \quad \tau_3(a, y, b_2) \subseteq x' \subseteq \tau_2(a, y, b_3))
\]

Above we have used quantification bounded by terms built with \(\tau_1, \tau_2, \tau_3\). It is easy Above we have used quantification bounded by terms built with \(\tau_1, \tau_2, \tau_3\). It is easy to see that these quantifications can be turned into quantifications bounded only by expressions built up from variables via projections. We also used atomic formulas with \(\subseteq\), but these can be thought of as abbreviations for \(\Delta_0\) formulas using \(\in\) and equality.

We can prove from \(\Sigma\) that \(q\) is a function of \(b_1, b_2, b_3, b_4\). Thus \(\Sigma\) implicitly defines a query from inputs \(b_1, b_2, b_3, b_4\). We give the argument informally here. Fixing \(b, q\) satisfying \(\Sigma\) and \(\tilde{b}, \tilde{q}\) satisfying \(\Sigma\), we will prove that if \(s \in q\) then \(\tilde{s} \in \tilde{q}\). The proof begins by choosing \(a_s\) for \(s\) as guaranteed by the first conjunct. Now applying the universal quantification in the second conjunct for \(a_s\), we obtain an \(s' \in \tilde{q}\). We will now claim that \(s' = s\). Then using extensionality (sets are determined by their elements), it suffices to argue \(s \subseteq s'\) and \(s' \subseteq s\). The containment \(s \subseteq s'\) follows from the containments \(s \subseteq b_4\) and \(b_4 \subseteq \tilde{b}\). For the remaining containment, we again fix \(x' \in s'\) and argue that \(x' \in s\). Applying the universally-quantified subformula over \(x'\) in the second conjunct to \(a_s\) and \(s'\) we get \(y_0 \in \tau_1(a_s, b_1)\) such that: \(\tau_3(a_s, y_0, b_2) \subseteq x' \subseteq \tau_2(a_s, y_0, b_3)\). Now applying the first conjunct to \(s, a_s, y_0\) we obtain an \(x_0 \in s\) such that \(\tau_2(a_s, y_0, b_2) \subseteq x_0 \subseteq \tau_3(a_s, y_0, b_3)\). Now we see using the containments and extensionality that \(x_0 = x'\). Therefore \(x' \in s\) as required.

This completes the proof that \(Q\) is functional over \(\tilde{b}\).

We first show that the projectively implicitly definable queries are exactly those in a slight variant of NRC. The result is an analog of the well-known Beth definability theorem for first-order logic [8], stating that a property of a first-order structure is defined by a first-order open formula exactly when it is implicitly defined by a first-order sentence. It also extends a result for “ordinary database queries” — those transforming relations to relations. If we add on to Beth’s theorem the restriction that the formulas are “safe” — they depend only on the interpretation of the relation symbols, not
the domain of the structure – then a variant of Beth’s theorem was proven by Segoufin and Vianu [41, 36], identifying the safe projectively implicitly definable queries with those definable in the relational algebra, one of the standard database languages. While the fact that relational algebra is closely-connected with first-order logic is hardly a surprise, but for the nested relational calculus the connection is much less obvious, and the argument much more subtle.

We will then look for an effective variant, stating that from a proof witnessing implicit definability of \( \varphi \) over \( \vec{A} \), we can generate an NRC query with input \( \vec{A} \) that implements \( \varphi \). Such a result will allow us to synthesize NRC queries from implicit definitions. We will give a partial result in this direction, showing that from proofs of functionality in a certain intuitionistic calculus, we can synthesize an NRC query in polynomial time. We argue that this calculus is quite rich, although we can show that it is not complete. We leave open the question whether there is a polynomial time synthesis algorithm for a complete proof calculus.

**Example 2.** Let us return to Example 1. One can verify using \( \Sigma(\vec{b},q) \) that \( q \) is identical to the singleton containing the input \( b_4 \). The transformation from \( \vec{b} \) to the singleton containing \( b_4 \) is easily seen to be definable in NRC.

Our main result is that this is always the case. In fact, we give a procedure that takes as input a proof from \( \Sigma(\vec{b},\ldots,q) \) that \( q \) depends functionally on a subset of its arguments \( \vec{b} \), in the style of the informal argument of Example 1. Our procedure derives an explicit definition of \( q \) from \( \vec{b} \) in NRC. This is an expression \( E \) such that for any \( \vec{b} \), if there is \( q \) such that \( \Sigma(\vec{b},\ldots,q) \) holds, then \( E \) applied to \( \vec{b} \) will produce it.

**Organization.** We overview related work in Section 2 and provide preliminaries in Section 3. Section 4 gives our Beth-style result without effectivity, while Section 5 provides the effective version that synthesizes definitions from proofs. This section contains the details of our proof system, with an example (Figure 3) of how one would use it to prove functionality of an expression. We follow up with an illustration of how our synthesis algorithm would generate an NRC expression from the proof (Example 14).

We close with conclusions in Section 6. In the body of the paper we focus on explaining the results and some proof ideas, with proofs deferred to the appendix.

## 2 Related work

One connection between logic and database query languages is well-known in the context of queries transforming ordinary “flat” relations. Codd’s theorem [12] identifies queries in the query language relational algebra with those in first-order logic that are safe, in that they depend only on the interpretation of the relations. The theorem of Segoufin and Vianu [41] mentioned in the introduction gives a second connection, stating that relational algebra queries are the same as those that satisfy a variant of implicit definability (“determinacy”). The result of [41] makes use of a refinement of Craig’s interpolation theorem due to Otto [37]. The use of interpolation theorems in moving from implicit to explicit is well-established, dating back to Craig [13]. In the database theory context, this connection is motivated by the ability to answer queries defined over one set of “base predicates” using another set of “view predicates”, where the views are defined implicitly by a background theory relating them to the base predicate. There is an extensive literature on this topic in the database literature [21, 29, 4]. The idea that one can use an effective version of Beth’s theorem to synthesize queries from theories first appears in the work of Toman and Weddell [44] and has been developed in a number of directions subsequently [15, 6].

The development of the nested relational model, culminating in the convergence on the query language NRC, has a long history, [1, 3], developing out of work by a number of researchers (e.g. [39, 20]). The thesis of Wong [47] and the related paper of Buneman et al. [9] gave an elegant
presentation of NRC, and summarize the equivalences known between a number of variations on the syntax. More powerful languages than NRC have been proposed for complex objects, including an extension with an operator for forming the powerset of a set. This extension can be captured using the natural logic with membership [2]. Abiteboul and Beeri’s [2] also provides a logical syntax for NRC (“strictly safe calculus”) but this simply encodes the restrictions of NRC within logic.

Quite independently of work on logics for nested relations in databases, researchers in other areas have investigated the relationships between various restricted algebras for manipulating sets. Gandy [16] defines a class of “Basic” functions, and compares them to functions definable by \( \Delta_0 \) formulas. Later languages build on Gandy’s work, particularly for a finer-grained analysis of the constructible sets [26]. An important distinction from the setting of NRC is that these works do not restrict to sets built up from finitely many levels of nesting above the Ur-elements. For instance, Gandy showed that there are basic functions checking whether an input is an ordinal, or is the ordinal \( \omega \); in fact, he showed that there are basic functions that are not primitive recursive. In the setting of [16], the \( \Delta_0 \) functions are strictly more expressive than the basic functions.

Model theorists have looked at generalizing Beth theorem’s to the case where the “implicitly definable structure” has new elements, not just new relations. Hodges and his collaborators [22, 23] explore this in some restricted cases. As we explain later on in this submission, both [22, 23] and the unpublished draft [5] provide model-theoretic tools that can be used to prove our main theorem without any effectivity guarantees.

Lisitsa and Sazonov [30, 31] explore the use of weak set theories in capturing complexity classes of functions over finite sets. The functions that are implicitly definable only over finite instances go beyond first-order logic, and even (under complexity-theoretic hypotheses) beyond polynomial time [28]. All of the Beth-style characterizations of queries provided here thus require us to consider infinite instances.

3 Preliminaries

Nested relations. We deal with schemas that describe objects of various types given by the following grammar.

\[ T, U ::= U | T \times U | \text{Unit} | \text{Set}(T) \]

We have a base type \( U \), which will be assumed to be some infinite collection of Ur-elements, as well as a type Unit containing a single element, and we close the set of types under cartesian products and the Set(\( T \)) type-former for collections of elements of type \( T \).

For simplicity throughout the remainder we will assume exactly one basic type \( \mathcal{U} \), which is infinite, as well as a countable set of constants \( (c_n)_{n \in \mathbb{N}} \) of sort \( \mathcal{U} \) which are assumed to be pairwise distinct in the semantics. We call this set the Ur-elements. From the Ur-elements and a unit type we can build up the set of types via product and the power set operation. We use standard conventions for abbreviating types, with the \( n \)-ary product abbreviating an iteration of binary products.

A nested relational schema consists of declarations of constants of given types.

Example 3. An example of a nested relational schema has objects \( R : \text{Set}(\mathcal{U} \times \mathcal{U}) \) and \( S : \text{Set}(\mathcal{U} \times \text{Set}(\mathcal{U})) \) That is, \( R \) is a set of pairs of Ur-elements: a standard “flat” binary relation. \( S \) is a collection of pairs whose first elements are Ur-elements and whose second elements are sets of Ur-elements. 

The types have a natural interpretation, which we refer to as the universe over \( \mathcal{U} \). The unit type has a unique member and the members of Set(\( T \)) are all sets of members of \( T \). An instance of such a schema is defined in the obvious way, or an \( \mathcal{U} \)-instance if we want to emphasize the set of Ur-elements on which it is based. Notice that nested relational schemas allow one to describe programming language data structures that are built up inductively via the tupling and set constructors, rather than
just sets of tuples. Thus the literature often refers also to the types above as “object types” and to the “complex object data model” [1]. In this work we will sometimes refer to the interpretation of a constant in an instance of a nested relational schema as an object. The subobjects of an object are defined in the obvious way. For example, if \( o \) is an object of type \( \text{Set}(T) \), then it is of the form \( \{t_1, \ldots \} \), where each \( t_i \) is a subobject of \( o \) of type \( T \).

For the schema in Example 3 above, one possible instance has \( R = \{(c_4,c_6),(c_7,c_3)\} \) and \( S = \{(c_4,\{c_6,c_9\})\} \). Note that a relational schema as defined above is a special case of a nested relational schema, in which all the declarations have type \( \text{Set}(U \times \ldots \times U) \).

**Query languages for nested relations.** A nested relational query (over input schema \( \text{SCH}_{\text{in}} \) and output schema \( \text{SCH}_{\text{out}} \) is a function that take as input an instance of \( \text{SCH}_{\text{in}} \), and returns an instance of \( \text{SCH}_{\text{out}} \). For example, suppose our input schema consists of a declaration: \( R : \text{Set}(U \times U) \) Then one possible query would return the nested relation formed by grouping on the first position, which would be written in a standard comprehension syntax as \( Q := \{ \{bv \mid (av,bv) \in R\} \mid \exists bv' (av,bv') \in R \} \).

**Query equivalences.** We say that two queries are equivalent if they agree on all instances (finite and infinite) of a given schema over any set of Ur-elements. It will turn out that for the queries we are interested in, “over any set of Ur-elements” can be freely replaced by “over any infinite set of Ur-elements” or “over some fixed infinite set of Ur-elements”. When we say that a query \( Q \) is expressible in some class of queries \( C \), we mean that there is a query \( Q' \) in \( C \) that is equivalent to \( Q \) in the sense above.

**Nested Relational Calculus.** We review the main language for declaratively transforming nested relations, Nested Relational Calculus (NRC). Queries are associated with an an output type, which are in the type system described above. We let \( \text{Bool} \) denote the type \( \text{Set}(\text{Unit}) \). Then \( \text{Bool} \) has exactly two elements, and will be used to simulate Booleans.

NRC expressions are given by the following grammar:

\[
Q,R ::= x \mid \{ \} \mid (Q,R) \mid \pi_1(Q) \mid \pi_2(Q) \mid \{Q\} \mid \bigcup\{Q \mid x \in R\} \mid \emptyset \mid Q \cup R \mid Q \setminus R \mid c_i
\]

where \( c_i \) are constants and \( x \) is a variable. We leave the typing constraints on expressions (e.g. we can only apply \( \pi_i \) when \( Q \) is of type \( T_1 \times T_2 \), and the result has type \( T_1 \)) implicit. The definition of the free and bound variables of a query are also standard. For example, the union operator \( \bigcup\{Q \mid x \in R\} \) binds \( x \).

The semantics of these expressions should be fairly evident. If \( Q \) has type \( T \), and has input variables \( x_1 \ldots x_n \) of types \( S_1 \ldots S_n \), then the semantics associates with \( Q \) a function that given a binding for the free variables returns a complex object of type \( T \). For example, the expression \( \{ \} \) always returns the empty tuple, while \( \emptyset \) returns the empty set of type \( T \). The expression \( \{e\} \) evaluates to \( \{o\} \), where \( e \) evaluates to \( o \).

In the sequel, we thus assume that every NRC expression is implicitly associated with a list of input types \( T \) such that each free variable corresponds to a \( T_i \) and an output type \( S \). We may write \( Q : T_1, \ldots, T_n \to S \) to mean that \( Q \) is such an NRC expression, and refer to \( S \) as the output type of \( Q \). We often abuse notation by identifying an NRC expression with the associated query. For example, if \( Q \) is an NRC expression and \( i \) is an object of the input type of \( Q \), we will write \( Q(i) \) for the output of \( Q \) on \( i \).

As explained in [47], the following NRC queries are definable with their expected semantics.

- For every type \( T \) there is an NRC query \( =_T \) of type \( \text{Bool} \) representing equality of elements of type \( T \). In particular, there is an expression \( =_{\text{Ur}} \) representing equality between Ur-elements.
- For every type \( T \) there is an NRC query \( e_T \) of type \( \text{Bool} \) representing membership between an element of type \( T \) in an element of type \( \text{Set}(T) \).
Further, if $E$ is a NRC expression with free variable $x$ of type $T$ and $F$ is an expression of type $T$, then the NRC expression
\[ \bigcup \{ \{E\} \mid x \in \{F\} \} \]
represents the substitution of $x$ for $F$ in $E$. Combining this with the first observations above, we can see that for expressions $E_1$ and $E_2$ of type $T$, we have an expression representing $E_1 \equiv_T E_2$ of type Bool. Using this, we will often treat $\equiv_T$ and $\in_T$ as additional constructors of the language.

Boolean operations $\land, \lor, \neg$ can also be represented as NRC queries with output type Bool. For example $\neg x$ is just $\{\{\}\}\setminus x$. Using the observation about substitution as above, we see that given $E$ of type Bool we can obtain an expression $\neg E$ of type Bool, and as with $\equiv_T$ and $\in_T$ we will treat the Boolean operations as primitives.

Arbitrary arity tupling and projection operations $(Q_1, \ldots, Q_n)$, $\pi_j(Q)$ for $j \geq 2$ can be seen as abbreviations for a composition of binary operations. Further

$\equiv$ If $B$ is an expression of type Bool and $Q_1, Q_2$ expressions of type $T$, then there is an expression case($B, Q_1, Q_2$) of type $T$ that implements “if $B$ then $Q_1$ else $Q_2$”.  

$\equiv$ If $Q_1$ and $Q_2$ are expressions of type Set($T$), then there are expressions $Q_1 \cap Q_2$ and $Q_1 \setminus Q_2$ of type Set($T$).

The derivations of these are not difficult. For example, the conditional required by the first item is given by:
\[ \bigcup \{Q_1 \mid x \in B\} \cup \bigcup \{Q_2 \mid x \in (\neg B)\} \]

**Example 4.** Consider an input schema including a binary relation $R : \text{Set}((\mathcal{U} \times \mathcal{U}))$. The query $Q_{\text{Proj}}$ with free variable $R$ returning the projection of $R$ on the first component can be expressed in NRC as $\bigcup \{\pi_1(t) \mid r \in R\}$. The query $Q_{\text{Filter}}$ with free variable $R$ and also $v$ of type $\mathcal{U}$ that filters $R$ down to those pairs which agree with $v$ on the first component can be expressed in NRC as $\bigcup \{\text{case}((\pi_1(t) =_\mathcal{U} v), \{r, \emptyset\} \mid r \in R\}$. Finally consider the query $Q_{\text{Group}}$ that groups $R$ on the first component, returning an object of type Set($\mathcal{U} \times \text{Set}(\mathcal{U})$). This can be expressed in NRC as $\bigcup \{(v, \bigcup \{\pi_2(t) \mid r \in Q_{\text{Filter}}\}) \mid v \in Q_{\text{Proj}}\}$.

The language NRC can not define certain natural queries whose output type is $\mathcal{U}$. To get a canonical language for such queries, let NRC[\text{Get}] denote the extension of NRC with the family of operations Get$_T : \text{Set}(T) \rightarrow T$ that extracts the unique element from a singleton. Get$_T$ was considered in [47], with connection to parallel evaluation explored in [43]. The semantics are: if $E$ returns a singleton set $\{x\}$, then Get$_T(E)$ returns $x$; otherwise it returns some default object of the appropriate type. The semantics of Get$_T(x)$ on non-singleton $x$ is not particularly important; to fix ideas, we can define for each type $T$ a default element $d_T$ that will be the output of Get$_T(x)$ when $x$ is not a singleton; take $d_{\mathtt{unit}} = c_0$, $d_{\text{Set}(T)} = \emptyset$, $d_{\text{unit}} = ()$ and $d_T \times d_S = (d_T, d_S)$.

In [43], it is shown that Get is not expressible in NRC at sort $\mathcal{U}$. For $T \neq \mathcal{U}$, Get$_T$ is actually definable from other NRC constructs; the important case is $T = \mathcal{U}$.

$\Delta_0$ formulas. We need a logic appropriate for talking about nested relations, which can be implemented effectively on finite inputs. A natural and well-known class are the $\Delta_0$ formulas. They are built up from equality of Ur-elements via the Boolean operators $\land, \neg$ and relativized existential quantification. All terms involving tupling and projections are allowed. We assume an infinite family of (semantically) distinct constants $(c_n)_{n \in \mathbb{N}}$ of sort $\mathcal{U}$. Formally, we deal with multi-sorted first-order logic, with sorts corresponding to each of our types. We use the following syntax for $\Delta_0$ formulas and terms. Terms are built using tupling, projections and constants $c_T$ of type $\mathcal{U}$. All formulas and terms are assumed to be well-typed in the obvious way, with the expected sort of $t$ and $u$ to be $\mathcal{U}$ in $t =_\mathcal{U} u$ and $t \neq_\mathcal{U} u$, while in $t \in_\mathcal{U} u$ the sort of $t$ is $T$ and the sort of $u$ is Set($T$).
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\[ t, u ::= x \left| (t, u) \right| \pi_1(t) \mid \pi_2(t) \mid c \]

\[ \varphi, \psi ::= \top \mid \bot \mid \varphi \lor \psi \mid \varphi \land \psi \mid \forall x \in T \; \varphi(x) \mid \exists x \in T \; \varphi(x) \]

Note that there is no primitive negation or equalities for sorts other than \( \mathcal{U} \). This does not limit expressiveness of formulas with respect to classical semantics. Negation \( \neg \varphi \) may be defined by induction on \( \varphi \) by dualizing every connective; we write \( \varphi \Rightarrow \psi \) for \( \neg \varphi \lor \psi \) in the sequel. Equality, inclusion and membership predicates may be defined as notations by induction on the involved types.

\[ t \in_T u ::= \exists x' \in u \; t =_T x' \quad t \subseteq_T u ::= \forall z \in_T t \; z \in_T u \]

\[ t =_{\text{Set}(T)} u ::= t \subseteq_T u \land u \subseteq_T t \quad t =_{\text{Unit}} u ::= \top \quad (\text{since all elements of this type are equal}) \]

\[ t =_{T_1 \times T_2} u ::= \pi_1(t) =_{T_1} \pi_1(u) \land \pi_2(t) =_{T_2} \pi_2(u) \]

The advantage of not having \( e \) at higher types as an atomic predicate, but rather as a derived predicate, is that set-extensionality

\[(\forall z \in_T x \; z \in_T y) \land (\forall z \in_T y \; z \in_T x) \Rightarrow x =_{\text{Set}(T)} y\]

holds in first-order logic without having to assume any additional axiom. In fact, the only critical axioms needed to reason about \( \Delta_0 \) formulas are that the equality \( =_{\mathcal{U}} \) is a congruence for terms and relations, and \( \pi_i(t_1, t_2) = t_i \) holds for every \( t_1, t_2 \) and \( i \in \{1, 2\} \). This is going to be helpful when considering proof systems.

**NRC and \( \Delta_0 \) formulas.** Since we have a type of boolean in NRC, one may ask about the expressiveness of queries \( T_1, \ldots, T_n \rightarrow \text{Bool} \). It turns out that they are equivalent to \( \Delta_0 \) formulas. This gives one justification for focusing on \( \Delta_0 \) formulas.

◨ **Proposition 5.** There is a polynomial time function taking a \( \Delta_0 \) formula \( \varphi(\vec{x}) \) and producing an NRC query \( \text{Verify}_{\varphi}(\vec{x}) \), where the query takes as input \( \vec{x} \) and returns true if and only if \( \varphi \) holds.

This useful result is proved by an easy induction over \( \varphi \).

## 4 Beth-type theorem for Nested Relations

We consider an input schema \( SC_{in} \) with one input object \( o_{in} \) and an output schema with one output object \( o_{out} \). Using product objects, we can easily model any nested relational query in this way. We consider a \( \Delta_0 \) formula \( \varphi(o_{in}, o_{out}, \vec{a}) \) with variables for \( o_{in}, o_{out} \) and possibly others. A formula is functional in \( o_{in} \) if for each nested relation \( o_{in} \) there is exactly one \( o_{out} \) such that \( \varphi(o_{in}, o_{out}, \vec{a}) \) holds for some \( \vec{a} \). A formula \( \varphi(o_{in}, o_{out}, \vec{a}) \) projectively implicitly defines a query \( Q \) from \( o_{in} \) to \( o_{out} \) if for each \( o_{in} \), \( \varphi(o_{in}, o_{out}, \vec{a}) \) holds for some \( \vec{a} \) if and only if \( Q(o_{in}) = o_{out} \). Note that when this happens, \( \varphi \) must be functional in \( o_{in} \). We drop “projectively” if \( \vec{a} \) is empty.

**Example 6.** Consider the query \( Q_{\text{Group}} \) from Example 4. It has a simple implicit \( \Delta_0 \) definition: first, define the auxiliary formula \( \chi(x, p, R) \) stating that \( \pi_1(p) \) is \( x \) and \( \pi_2(p) \) is the fiber of \( R \) above \( x \)

\[ \chi(x, p, R) ::= \pi_1(p) = x \land \left( \forall t' \in R \; [ \pi_1(t') = x \Rightarrow \pi_2(t') \in \pi_2(p) ] \right) \land \forall z \in \pi_2(p) \; (x, z) \in R \]

Then \( Q_{\text{Group}} \) is implicitly defined by \( \forall t \in R \; \exists p \in q \; \chi(\pi_1(t), p, R) \land \forall p \in q \; \chi(\pi_1(p), p, R) \).

We will show the following:

◨ **Theorem 7.** The following are equivalent for a query \( Q \):

\[ Q \text{ is projectively implicitly definable by a } \Delta_0 \text{ formula} \]

\[ Q \text{ is implicitly definable by a } \Delta_0 \text{ formula} \]
The following is the “easy direction” of Theorem 7, which can be shown via a direct construction:

**Proposition 8.** For every NRC[Get] expression $Q$ we can obtain (in polynomial time) a $\Delta_0$ formula that implicitly defines $Q$.

The other direction is our first main result:

**Theorem 9.** For any $\Delta_0$ formula $\varphi(o_{in}, o_{out}, \vec{a})$ which is functional, there is an NRC query $q$ such that whenever $\varphi(o_{in}, o_{out}, \vec{a})$ holds, then $q(o_{in}) = o_{out}$.

In particular, if in addition for each $o_{in}$ there is some $o_{out}$ and $\vec{a}$ such that $\varphi(o_{in}, o_{out}, \vec{a})$ holds, then $q$ and $\varphi$ define the same query.

**Finite instances versus all instances.** In Theorem 9 we emphasize that our theorem concerns the class $\text{Fun}_{\text{All}}$ of queries $Q$ such that there is a $\Delta_0$ formula $\varphi$ which defines a functional relationship between $o_{in}$ and $o_{out}$ on all instances, finite and infinite, and where the function agrees with $Q$. We can consider $\text{Fun}_{\text{All}}$ as a class of transformations on all instances or of finite instances, but the class is defined by references to all instances. Expressed semantically

$$\varphi(o_{in}, o_{out}, \vec{a}) \land \varphi(o_{in}, o_{out}', \vec{a}) \models o_{out}' = o_{out}$$

Our results say that $\text{Fun}_{\text{All}}$ is identical with the set of queries given by NRC expressions. Recalling the comment in the introduction, an equivalent characterization of $\text{Fun}_{\text{All}}$ is proof-theoretic: these are the queries such that there is a classical proof of functionality in a complete first-order proof system using some basic axioms about Ur-elements, products and projection functions, and the extensionality axiom for the membership relation.

Whether one thinks of $\text{Fun}_{\text{All}}$ semantically or proof-theoretically, our results say that $\text{Fun}_{\text{All}}$ is identical with the set of queries given by NRC expressions. But the proof-theoretic perspective is crucial for the effective version we will present later.

It is natural to ask about the analogous class of queries $\text{Fun}_{\text{Fin}}$ of transformations over finite instances for which there is a $\Delta_0$ $\varphi$ which is functional for all finite inputs and where the corresponding function agrees with $Q$. It is well-known that $\text{Fun}_{\text{Fin}}$ is not identical to NRC and is not so well-behaved.

The query returning the powerset of a given relation is in $\text{Fun}_{\text{Fin}}$: this is a variant of a well-known example of Gurevich, see e.g.

From this we can see that $\text{Fun}_{\text{Fin}}$ contains transformations of high data complexity. Indeed, even when considering transformations from flat relations to flat relations, $\text{Fun}_{\text{Fin}}$ contains transformations whose membership in polynomial time would imply that $\text{UP} \cap \text{coUP}$, the class of problems such that both the problem and its complement can be solved by an unambiguous non-deterministic polynomial time machine, is identical to $\text{PTIME}$. More importantly for our goals, membership in $\text{Fun}_{\text{Fin}}$ is not witnessed by proofs in any effective proof system, since this set is not computably enumerable.

**Sketch of the proof of Theorem 9.** In the body of the paper, we will only sketch the proof of Theorem 9. Although it is quite lengthy, the main ideas are not difficult.

The first step is an interpretation characterization, showing that the NRC queries are expressively equivalent to the queries defined by $\Delta_0$ interpretations. A $\Delta_0$ interpretation describes each relationship in the output of a query via a $\Delta_0$ formula over the input. The definition of interpretation is quite technical, and thus we defer the details to the appendix. The correspondence of NRC with interpretations co-incides with the idea that NRC queries can be “simulated” by ordinary queries on flat relations. This is well-known in both the theory [46] and practice [11] of NRC, and similar statements have been proven for related query languages such as XQuery [7]. The proofs are direct, via inductively-defined transformations in each direction.
The next step is to show that every projectively implicitly definable query can be expressed as an interpretation. This can be recovered from a model-theoretic result outside of the setting of nested relations, an analog of Beth’s theorem for multi-sorted logic. Let $\Sigma G$ be any multi-sorted signature, and let $\text{Sorts}_0$ and $\text{Sorts}_1$ be a collection of sorts such that $\text{Sorts}_0 \subset \text{Sorts}_1$. We say that a relation $R$ is over $\text{Sorts}_0$ if all of its arguments are in $\text{Sorts}_0$. Let $\Gamma$ be a set of sentences in $\Sigma G$. Given a model $M$ for $\Sigma G$, let $\text{Sorts}_0(M)$ be the union of the domains of relations over $\text{Sorts}_0$, and let $\text{Sorts}_1(M)$ be defined similarly. We say that $\text{Sorts}_1$ is implicitly interpretable over $\text{Sorts}_0$ relative to $\Sigma$ if:

For any models $M_1$ and $M_2$ of $\Sigma$, if there is a mapping $m$ from $\text{Sorts}_0(M_1)$ to $\text{Sorts}_0(M_2)$ that preserves all relations over $\text{Sorts}_0$, then $m$ extends to a unique mapping from $\text{Sorts}_1(M_1)$ to $\text{Sorts}_1(M_2)$ which preserves all relations over $\text{Sorts}_1$.

The variant of Beth’s theorem for multi-sorted logic will say:

if $\text{Sorts}_1$ is implicitly interpretable over $\text{Sorts}_0$ relative to $\Sigma$, then $\text{Sorts}_1$ is interpretable in the usual sense over $\text{Sorts}_0$ relative to $\Sigma$.

In our application $\text{Sorts}_0$ will be the set-theoretic universe lying below the input constants, while $\text{Sorts}_1$ will correspond to the set-theoretic universe lying below the outputs. Projective implicit definability of the output from the input in our sense will easily apply the unique mapping property above, while interpretability of $\text{Sorts}_1$ in $\text{Sorts}_0$ will imply that our query is given by an interpretation, which using the interpretation characterization above will imply that our query is expressible in NRC (with Get in the case that the output type is $U$).

It seems that this “multi-sorted Beth theorem” has never appeared explicitly in the published literature. But similar results appear in Hodge’s book [22] and unpublished work from the logic group in Budapest [5]. We defer the details of the multi-sorted Beth result, as well as its application to Theorem 9, to the appendix as well. The main step is to move from implicit interpretability to the fact that in every model, every component in an output is definable with parameters from the input. This is proven by contraposition, with the lack of a definition leading to the construction of a model violating implicit interpretability.

The equivalence of NRC with interpretations holds also when only finite nested relations are considered, while the equivalence of interpretations with implicit definability requires that implicit definability holds considering both finite and infinite nested relations.

## 5 The effective Beth result

We will now present an effective variant of Theorem 9.

**Restricted proof system.** To state our effective Beth result, we will need to present a proof system that can generate proofs of functionality. We present a special-purpose sequent calculus in Figure 1 deriving judgments $\Theta; \Gamma \vdash \Phi$ where $\Gamma$ is a multi-set of $\Delta_0$ formulas, $\Theta$ a multi-set of membership formulas $t \in u$, and $\Phi$ is a $\Delta_0$ formula with one of the following shapes: $t \in_T u, t =_T u$ or $t \subseteq_T u$. A multi-set of formulas will also be called a context, and above we write $C, C'$ for the concatenation of contexts $C$ and $C'$. Informally, a judgment $\Theta; \Gamma \vdash \Phi$ is meant to be read as “If all the containments in $\Theta$ and formulas in $\Gamma$ hold, then $\Phi$ does”. In the figure, we use $\text{FV}$ to denote the free variables of a context, and use $\Phi[t/x]$ as usual denotes the result of substituting $t$ for $x$ in $\Phi$.

The main essential restriction on the proof system is that it is intuitionistic. There is no obvious way to deduce $\Theta; \Gamma \vdash \neg \Phi$ from $\Theta; \Gamma, \neg \Phi \vdash \bot$ in general. Informally, this means that we forbid reasoning by contradiction. In particular, this means that some sequents are classically valid but not derivable in our calculus. For instance, consider $w \in r, \forall x \in l \in r, \forall y \in w \in l \in r \vdash l \in r$. This is seen to be classically valid by considering separately the following three cases: $l$ non-empty, $w$ non-empty and $l = w = \emptyset$. However, it is also easy to check that this cannot be derived intuitionistically.
Figure 1 Our intuitionistic sequent calculus for proofs of implicit definability
Generating collection queries from proofs

A proof of functionality of Example 11.

Informally as follows (putting references to proof steps in Figure 3 in parentheses).

We note that many natural proof rules are admissible in our system; they are conservative in terms of the set of proofs that they enable. We collect the most useful cases in Figure 2. Showing that they are admissible is done by rather elementary inductions, and it can be noted that eliminating those additional proof rules can be done in polynomial time in the size of proof trees and the types of the involved formulas. This list is not meant to be exhaustive, as it can be shown that the derivable sequents in our system are exactly those derivable in a more standard sequent calculus for multi-sorted intuitionistic logic. We offer a more detailed discussion in the appendix.

**Provable implicit definitions.** By a proof of functionality of \( o_{\text{out}} \) over \( o_{\text{in}} \) assuming \( \varphi(o_{\text{in}}, o_{\text{out}}, \vec{a}) \) we mean a formal derivation of a sequent \( \varphi(o_{\text{in}}, o_{\text{out}}, \vec{a}) \vdash o_{\text{out}} =_{\text{T}} o'_{\text{out}} \) in our proof system. Such a proof witnesses that \( \varphi(o_{\text{in}}, o_{\text{out}}, \vec{a}) \) projectively implicitly defines \( o_{\text{out}} \) over \( o_{\text{in}} \).

This notion allows to state our effective Beth result:

**Theorem 10.** There is a PTIME procedure which takes as input a proof of functionality of a formula \( \varphi(o_{\text{in}}, o_{\text{out}}, \vec{a}) \) over vocabulary \( o_{\text{in}} \), and returns an NRC query \( q \) such that whenever \( \varphi(o_{\text{in}}, o_{\text{out}}, \vec{a}) \) holds, then \( q(o_{\text{in}}) = o_{\text{out}} \).

Let us provide a detailed example to illustrate Theorem 10.

**Example 11.** Given a set of sets of Ur-elements \( X \in \text{Set}((\mathcal{U})) \), say that an Ur-element \( a \) distinguishes a set \( x \in X \) if \( x \) is the unique element of \( X \) containing \( a \). Consider the query taking as input such an \( X \) and returning the set of Ur-elements that distinguish some element of \( X \). This is implicitly definable by a \( \Delta_0 \) formula \( \varphi(X, q) \) stating that every \( a \) in \( q \) distinguishes some element of \( X \) and conversely. Writing this in our restricted syntax for \( \Delta_0 \) formulas, in which membership of higher-order objects must be expressed using bounded quantification and equality, we obtain an implicit definition

\[
\varphi(X, q) := (\forall a \in q \exists x \in X \psi(X, x, a)) \land (\forall x \in X \forall a \in x [\chi(X, x, a) \Rightarrow a \in U q])
\]

where

\[
\chi(X, x, a) := \forall y \in X (a \in U y \Rightarrow x =_{\text{Set}(\mathcal{U})} y) \quad \text{and} \quad \psi(X, x, a) := a \in U x \land \chi(X, x, a)
\]

Above \( a \in U x \), \( x =_{\text{Set}(\mathcal{U})} y \), and \( a \in U q \) are actually abbreviations for more complex formulas built out through bounded quantification and \( \Rightarrow \) is a derived connective.

Figure 3 contains a formal derivation of functionality for \( \varphi(X, q) \). We may render this proof informally as follows (putting references to proof steps in Figure 3 in parentheses).

**Proof of functionality of Example 11.** Assume \( \varphi(X, q) \) and \( \varphi(X, q') \). To show \( q = q' \), we need to show that \( q \subseteq q' \) and \( q' \subseteq q \). Since the roles of \( q \) and \( q' \) are symmetric, without loss of generality, it suffices to give the proof that \( q \subseteq q' \) (1). So fix \( z \in q \) (2). Since \( \varphi(X, q) \) holds, according to its first conjunct, we have in particular that there exists some \( x \in X \) such that \( \psi(X, x, z) \) holds
Verify $\phi$ to $\Delta$. Proposition 12. Calculus admits efficient interpolation.

For each entailment between formulas $\chi$, $\psi$, $\phi$, $a$, $x$, $z$, $q$, $\gamma$, $\Theta$, $\Gamma$, $\Delta$, $\Omega$, and $\sigma$, let $\mathcal{L}$ and $\mathcal{R}$ be contexts and $\psi$ a formula. Let $\tilde{I}$ be the collection of variables shared between $\Theta_L$, $\Gamma_L$, and $\Theta_R$, $\Gamma_R$, $\psi$. Then for every derivation

$$
\Theta_L; \Theta_R; \Gamma_L; \Gamma_R \vdash \psi
$$

there exists a $\Delta_0$ formula $\theta$ with free variables $\tilde{I}$ such that the following holds

$$
\Theta_L; \Gamma_L \models \theta \quad \text{and} \quad \Theta_R; \Gamma_R, \theta \models \psi
$$

Further the interpolant $\theta$ can be found in polynomial time from the derivation.

The interpolation result should be thought of as giving us the result we want for Boolean queries. From it we can derive that a formula whose truth value is implicitly defined by a set of input variables must be given as a $\Delta_0$ formula over those inputs. By Proposition 5, these formulas can be converted to NRC.

The main lemma. Our main result is deduced from the following lemma, which can be thought of as a strengthening of interpolation for our restricted proof system.
Lemma 13. Let \( L, R \) be sets of variables, \( \Theta = \Theta_I, \Theta_L, \Theta_R \) be a \( \in \)-context and \( \Gamma = \Gamma_L, \Gamma_R \) be a context such that
\[
\text{FV}(\Theta_I) \subseteq L \cap R \quad \text{FV}(\Theta_L) \subseteq L \setminus R \quad \text{FV}(\Theta_R) \subseteq R \setminus L \quad \text{FV}(\Gamma_L) \subseteq L \quad \text{FV}(\Gamma_R) \subseteq R
\]
The following holds, assuming that \( t \) and \( u \) are terms of suitable types such that \( x \) which is semantically equivalent to \( \bigcup \).\( \bigcup \) which is equivalent to the union \( \Delta \) (Proposition 12) and the conversion of \( \text{NRC} \) and the final \( \text{R} \) Lemma 13.

Further the desired queries can be constructed in time polynomial in the proof.

Proof of Theorem 10. A proof that \( \Sigma(\vec{t}, \vec{a}, q) \) defines \( q \) as a function of \( \vec{t} \) is exactly a proof that \( \Sigma(\vec{t}, \vec{a}, q) \), \( \Sigma(\vec{t}, \vec{a}, q') \vdash \theta \) where \( q' \) is a new constant and \( \Sigma(\vec{t}, \vec{a}, q') \) denotes a copy of \( \Sigma \) substituting \( q' \) for \( q \). Applying Lemma 13 with \( \Theta \), \( \Gamma_L = \Sigma, \Gamma_R = \Sigma', L \) the free variables of \( \Gamma_L \) and \( R \) the free variables of \( \Gamma_R \) gives Theorem 10.

Lemma 13 is proven by induction on the derivation, which requires examining every proof rule in Figure 1. The more interesting cases are the left-hand side rules for first-order connectives (\( \land \)-L, \( \lor \)-L, \( \forall \)-L and \( \exists \)-L) and the rules for the right-hand side formulas \( \in \text{Set} \)-R and \( = \text{U} \)-R. Regarding the left-hand side rules, since the right-hand side formula of both the premise and conclusion is of the shape \( t \in \Gamma u \), the inductive invariant requires us to output a \( \text{NRC} \) query bounding the term \( t \). To prove the inductive step, we use the binary union operator \( Q_1 \cup Q_2 \) of \( \text{NRC} \) for the rule \( \lor \)-L and the big union operator \( \bigcup \{ Q \mid x \in y \} \) for the rule \( \exists \)-L. On the other hand, the inductive steps for the rules \( \land \)-L and \( \forall \)-L do not require modifying the query obtained as part of the induction hypothesis. To treat the inductive steps corresponding to the rules \( \subseteq \)-R and \( = \text{U} \)-R, we use a combination of interpolation (Proposition 12) and the conversion of \( \Delta_0 \) formulas to Boolean queries in \( \text{NRC} \) (Proposition 5).

Example 14. Let us illustrate the algorithm provided by Lemma 13 on the proof tree in Figure 3 by providing the corresponding intermediate \( \text{NRC} \) expressions that are synthesized, starting from top to bottom: from step (7) to (5), the \( \text{NRC} \) expression is the singleton \( \{ z' \} \). After the conclusion of the subsequent \( \exists \)-L rule, the expression becomes
\[
\bigcup \{ \{ z' \} \mid z' \in x \}
\]
which is semantically equivalent to \( x \). After the next \( \exists \)-L rule at step (3), we obtain
\[
\bigcup \bigcup \{ \{ z' \} \mid z' \in x \} \mid x \in X
\]
which is equivalent to the union \( \bigcup X \). The final expression is then obtained right after step (2), by first computing an interpolant \( \theta(X, z) \) such that \( z \in q \land \phi(X, q) \vdash \theta(X, z) \) and \( \theta(X, z) \land \phi(X, q') \vdash z \in q' \).

Computing according to the procedure underlying Proposition 12 yields \( \theta(X, a) = \exists x \in X \phi(X, x, a) \) and the final \( \text{NRC} \) expression
\[
\bigcup \{ \text{case(Verify}_a(X, a), \{ a \}, \theta) \mid a \in \bigcup \{ \{ z' \} \mid z' \in x \} \mid x \in X \}
\]

We now detail two cases of the inductive argument required to prove Lemma 13, the other cases being relegated to the appendix. We also omit the routine complexity analysis of the underlying algorithm.
Rule $\exists$-L. If the last proof rule used is
\[ \Theta, x \in \mathcal{T} y; \Gamma, \varphi \vdash t \in \mathcal{T} v \quad x \notin \text{FV}(\Theta, \Gamma, y, t, v) \]
then we have two cases: either \( y \notin \text{FV}(\Theta_l) \), in which case the induction hypothesis gives a query with free variables ranging over \( \text{FV}(\Theta_l) \) and we may use that to conclude. Otherwise, \( x \) is also among the free variables of \( Q'(\vec{i}, x) \) given by the induction hypothesis, as well as \( y \). We may conclude by considering
\[ Q = \bigcup \{ Q'(\vec{i}, x) \mid x \in y \} \]

Rule $\subseteq$-R. If the last proof rule used is
\[ \Theta, z \in \mathcal{T} t; \Gamma \vdash z \in \mathcal{T} u \quad z \notin \text{FV}(\Theta, \Gamma, t, u) \]
then the inductive hypothesis gives us a query \( Q'(\vec{i}) \) such that \( \Theta; \Gamma \vdash z \in Q' \). Apply interpolation to the premise so as to obtain a \( \Delta_0 \) formula \( \theta(\vec{i}, z) \) such that
\[ \Theta_l, t; \Theta_L; \Gamma_L, z \in t \vdash \theta(\vec{i}, z) \quad \text{and} \quad \Theta_R; \Gamma_R, \theta(\vec{i}, z) \models z \in u \]
In this case, we take \( Q(\vec{i}) = \{ z \in Q'(\vec{i}) \mid \theta(\vec{i}, z) \} \), which is NRC-definable as
\[ \bigcup \{ \text{case}(\text{ Verify}_\theta(\vec{i}, z), \{ z \}, \emptyset) \mid z \in Q'(\vec{i}) \} \]

Now, let us assume that \( \Gamma \) is valid and show that \( t \subseteq Q \) and \( Q \subseteq u \).

- Suppose that \( z \in q \). By the induction hypothesis, we know that \( z \in Q' \). But we also know that \( \Gamma_L \) is valid, so that \( \theta(\vec{i}, z) \) holds. By definition, we thus have \( z \in Q \).
- Now suppose that \( z \in Q' \), that is, that \( z \in Q \) and \( \theta(\vec{i}, z) \) holds. The latter directly implies that \( z \in q' \) since \( \Gamma_R \) is valid.

6 Conclusion

We have proven that for queries other that do not yield Ur-element type, the language NRC expresses exactly the queries that can be implicitly defined by \( \Delta_0 \) formulas, with a similar statement holding when we include queries at Ur-element type and replace NRC with NRC[Get]. This connection between the NRC language studied in data management and programming languages and logic is, to our knowledge, surprising and non-trivial. Our other main result is a PTIME procedure that synthesizes an NRC query from a proof of functionality in a natural intuitionistic proof system for these queries.

One application of our synthesis result is to querying with respect to view definitions, a central topic in databases [29, 44, 21, 36, 4]. Our first Beth-style result, Theorem 9, already implies that in looking to implement an NRC query over a base schema using NRC views, it suffices to look for a rewriting of the query in NRC over the view schema. This is a nested relational analog of results for relational algebra by Segoufin and Vianu [41]. Still, we think the synthesis procedure for collections can be of interest to a number of communities that look at transformation of collections.

We have to stress that our effective result is only partial. We have shown that our intuitionistic calculus is, unsurprisingly, not complete with respect to classical logic. In ongoing work we are investigating whether there is a complete calculus that still allows for PTIME generation of NRC queries from proofs of functionality.
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A  Proofs for Section 3

Proof of Proposition 5: NRC queries that verify $\Delta_0$ formulas

Recall the statement:

There is a polynomial time function taking a $\Delta_0$ formula $\phi(\bar{x})$ and producing an NRC query $\text{Verify}_\phi(\bar{x})$, where the query takes as input $\bar{x}$ and returns true if and only if $\phi$ holds.

Proof. First, one should note that every term in the logic can be translated to a suitable NRC query of the same sort. We can then proceed by induction over the formula $\phi(\bar{x})$.

- If $\phi(\bar{x})$ is an equality $t = t'$ or a membership $t \in t'$, this is obvious.
- If $\phi(\bar{x})$ is a disjunction $\phi_1(\bar{x}) \lor \phi_2(\bar{x})$, we take $\text{Verify}_\phi(\bar{x}) = \text{Verify}_{\phi_1} \cup \text{Verify}_{\phi_2}$. We proceed similarly for disjunction thanks to $\cap$.
- If $\phi(\bar{x})$ is a negation, we use the definability of negation in NRC.
- If $\phi(\bar{x})$ begins with a bounded existential quantification $\exists z \in y \psi(\bar{x}, y, z)$, we simply set $\text{Verify}_\phi(\bar{x}, y) = \bigcup \{ \text{Verify}_{\psi(\bar{x}, y, z)} \mid z \in y \}$. Universal quantification is then treated similarly by using negation in NRC.

Note that the converse (without the polynomial time bound) also holds; this will follow from Lemma 23 that is proven later in the appendix.

B  Proofs for Section 4, I: the easy direction of Theorem 9

Recall the statement:

For every NRC[Get] expression $Q$ we can obtain a $\Delta_0$ formula that implicitly defines $Q$.

This can be done by induction on the structure of $Q$. For example, consider the case of the singleton constructor $E = \{F\}$. Inductively we have $\phi_F(\bar{x}, q_2)$ defining $F$, and from there we can define $E$ by:

$$(\exists q_2 \in q_1 \top) \land (\forall q_2 \in q_1 \phi_F(\bar{x}, q_2))$$

We discuss briefly the inductive case of the union operator. One approach, is to break this operator down into a simpler union operator where the variable can only iterate over another variable. The full union operator can be recovered if we also allow a composition operation. The simpler operator is easy to handle inductively. Composition can be handled without a blow-up if we allow projective implicit definitions, because projective implicit definitions are closed under composition. From our prior results, we know that projective implicit definitions are no more expressive than implicit ones.

An alternative is to rely on the NRC[Get] normalization result mentioned at the end of Lemma 24: we can pre-process NRC[Get] expressions to be composition-free: in unions we do not iterate over complex queries. For these normalized expressions, the creation of implicit definitions can be done in PTIME.

C  Proofs for Section 4, II: nested relational queries and interpretations

Interpretations and nested relations

We mentioned in the body that the proof of our first Beth-style theorem, Theorem 9 has two components: an equivalence of NRC with interpretations, and an equivalence of interpretations with
This means the following

We first review the notion of an interpretation. Let \( \text{SCH}_{in} \) and \( \text{SCH}_{out} \) be multi-sorted vocabularies, where \( \text{SCH}_{out} \) contains only relation symbols. A first-order interpretation for \( \text{SCH}_{in} \) to \( \text{SCH}_{out} \) consists of:

- for each output sort \( S_{out} \), a sequence of input sorts \( \tau(S_{out}) = \vec{S} \),
- a formula \( \varphi_{S'}^S(\vec{x}_1, \vec{x}_2) \) for each output sort \( S' \) in \( \text{SCH}_{out} \),
- a formula \( \varphi_{\text{Domain}}^S(\vec{x}_1) \) for each output sort \( S' \) in \( \text{SCH}_{out} \),
- a formula \( \varphi_R(\vec{x}_1, \ldots, \vec{x}_n) \) for every relation \( R \) of arity \( n \) in \( \text{SCH}_{out} \),

subject to the following constraints

- \( \varphi_{\text{Domain}}^S(\vec{x}) \) should define a partial equivalence relation, i.e. be symmetric and transitive,
- \( \varphi_{\text{Domain}}^S(\vec{x}) \) should be equivalent to \( \varphi_{\text{Domain}}^S(\vec{x}, \vec{x}) \),
- \( \varphi_R(\vec{x}_1, \ldots, \vec{x}_n) \) and \( \varphi_{\text{Domain}}^S(\vec{x}_1, \vec{y}_i) \) for \( 1 \leq n \leq n \) (where \( S_i \) is the output sort associated with position \( i \) of the relation \( R \)) should jointly imply \( \varphi_R(\vec{y}_1, \ldots, \vec{y}_n) \).

In \( \varphi_{\text{Domain}}^S \) and \( \varphi_{\text{Domain}}^S \), each \( \vec{x}_i \) is a tuple containing variables of sorts agreeing with the prescribed sequence of input sorts for \( S' \). Given a structure \( M \) for the input sorts, we call a binding of these variables to input elements of the appropriate input sorts an \( M, S' \) input match. If in relation \( R \) position \( i \) is of output sort \( S_i \), then in \( \varphi_R(\vec{t}_1, \ldots, \vec{t}_n) \) \( \vec{t}_i \) is a tuple of variables variables of sorts agreeing with the prescribed sequence of input sorts for \( S' \). Each of the above formulas is over the vocabulary of \( \text{SCH}_{in} \).

An interpretation \( I \) defines a function from structures over vocabulary \( \text{SCH}_{in} \) to structures over vocabulary \( \text{SCH}_{out} \) as follows:

- The domain of sort \( S' \) is the set of equivalence classes of the partial equivalence relation defined by \( \varphi_{\text{Domain}}^S \) over the \( M, S' \) input matches.
- A relation \( R \) is interpreted by the set of those tuples \( \vec{a} \) such that \( \varphi_R(\vec{t}_1, \ldots, \vec{t}_n) \) holds for some \( \vec{t}_1 \ldots \vec{t}_n \) with each \( \vec{t}_i \) a representative of \( a_i \).

**Interpretations defining nested relational queries.** We now consider how to define a nested relational query via an interpretation. The input sorts will be simply the types over the input schema. This is the standard multi-sorted representation of an input schema that we referred to in discussing \( \Delta_0 \) formulas. In addition to the element relation, the ur-element constants \( c_i \) and tuple-manipulation functions on these sorts, we will have a constant symbol for the input object \( o_{in} \), whose sort is the sort of \( o_{out} \). Note that a closed \( \Delta_0 \) formula in such a language must bound outermost quantifications by \( o_{in} \).

Our output will consist of a distinguished top object \( o_{out} \), whose type is the type of the output schema, and representatives for the sets of subobjects of \( o_{out} \). Concretely speaking, define the notion of subtype of a type \( T \) as follows.

- Every type \( T \) is a subtype of itself.
- If \( T \) is a subtype of \( T' \), then \( T \) is a subtype of \( \text{Set}(T') \), \( T' \times T'' \) and \( T'' \times T' \) for arbitrary \( T'' \).

A \( \Delta_0 \) interpretation for a nested relational input schema \( \text{SCH}_{in} \) of type \( T_{in} \) and output schema \( \text{SCH}_{out} \) of type \( T_{out} \) is an interpretation whose input sorts are \( U \) and the subtypes of \( T_{in} \) and output sorts are subtypes of \( T_{out} \). We call this multi-sorted signature the **multi-sorted encoding** of a given nested relational schema.

We also bake into our interpretations the requirement that equality is a congruence for all of our terms, and that the axioms pertaining to the extensionality of sets and cartesian structure are still valid. This means the following

- for every output function symbol \( f(\vec{x}) \) represented by terms \( \vec{t}(\vec{x}) \), we have

\[
\forall \vec{x} \vec{y} \left( \bigwedge_i \varphi_{\text{Domain}}^S(\vec{x}_i, \vec{y}_i) \rightarrow \varphi_{S'}^S(f(\vec{x}), \vec{t}(\vec{y})) \right)
\]
where \( S' \) is the sort of the output of \( f \) and the \( S_i \) correspond to the arities.

Similarly, if \( \pi_i \) is represented by \( \vec{\alpha}_i \) and Tuple by Tuple, we require that

\[
\forall \vec{x} \vec{y} \varphi^T_i (\vec{\alpha}_i (\text{Tuple}(\vec{x}, \vec{y})), \vec{x}) \quad \forall \vec{x} \vec{y} \varphi^T_i (\vec{\alpha}_i (\text{Tuple}(\vec{x}, \vec{y})), \vec{y})
\]

\[
\forall \vec{z} \varphi^T_{i \times T_i} (\text{Tuple}(\vec{\alpha}_i (\vec{z}), \vec{\alpha}_i (\vec{z})), \vec{z})
\]

Similarly, for the unit type we have

\[
\forall \vec{x} \vec{y} (\varphi^{\text{Unit}}_{\text{Domain}} (\vec{x}) \land \varphi^{\text{Unit}}_{\text{Domain}} (\vec{y})) \rightarrow \varphi^{\text{Unit}} (\vec{x}, \vec{y})
\]

The extensionality axiom for sets should hold

\[
\forall \vec{x} \vec{y} \left( \varphi^T_{\text{Domain}} (\vec{x}) \land \varphi^T_{\text{Domain}} (\vec{y}) \land \forall \vec{z} (\varphi^T_{\text{Set}} (\vec{z}, \vec{x}) \leftrightarrow \varphi^T_{\text{Set}} (\vec{z}, \vec{y})) \right) \rightarrow \varphi^T_{\text{Set}} (\vec{x}, \vec{y})
\]

Furthermore we require that the interpretation preserves equality of Ur-elements.

\[
\forall x y \varphi^I (x, y) \rightarrow x =_\mathcal{M} y
\]

Note that this is not necessarily an equivalence, as some Ur-elements may not belong to the support of the output.

The output of an interpretation is a structure in the multi-sorted encoding of the output nested relational schema, but it is not technically a nested relational instance as required by our semantics for nested relational queries. We can convert the output to a semantically appropriate entity via a modification of the well-known Mostowski collapse [35]. We define Collapse\((e, M)\) on elements \(e\) of the domain of a structure \(M\) for the multi-sorted encoding of a schema, by structural induction on the type of \(e\):

- If \(e\) has sort \(T_1 \times T_2\) then we set
  \[
  \text{Collapse}(e, I) = \text{Collapse}(\pi_1(e), I), \text{Collapse}(\pi_2(e), I)
  \]
  \[
  \text{Collapse}(e, I) = \{ \text{Collapse}(t, I) \mid t \in e \}
  \]

Collapsing \(I\) for a model \(M\) with constants \(c_1\ldots c_n\) is the instance interpreting each \(c_i\) by \(\text{Collapse}(c_i, I)\).

**Definition 15.** We say that a nested relational query from \(Q\) from \(\text{SCH}_{in}\) to \(\text{SCH}_{out}\) is defined by a \(\Delta_0\) interpretation \(I\) if, for every instance \(A\) of \(\text{SCH}_{in}\) regarded as a structure is mapped to a structure \(A'\) with top output sort \(S_{\text{out}}\), such that \(S_{\text{out}}\) in \(\text{Collapse}(A')\) is a singleton whose unique element is the output of \(Q\) on \(A\).

We will often identify a \(\Delta_0\) interpretation with the corresponding query, speaking of its input and output as a nested relation (rather than the corresponding structure).

**Example 16.** Consider an input schema including a binary relation \(R : \text{Set}(U \times \text{Set}(U))\), so an input object is a set of pairs, with each pair consisting of an Ur-element and a set of Ur-elements.

The multi-sorted encoding for such a nested relation would be a signature with

- a sort corresponding to subobjects of type \(\text{Set}(U \times \text{Set}(U))\), abbreviated below as \(R_{\text{top}}\). This will store the top-level object
- a sort corresponding to subobjects of the input of type \(\text{Set}(U \times \text{Set}(U))\), abbreviated \(R_{\text{e}}\) below, containing the set of members of the top-level
a sort corresponding to subobjects of the input of type \( \text{Set}(\mathcal{U}) \), abbreviated as \( R_{\in,2} \), and storing the second component of a pair in the top-level object

the \( \mathcal{U} \) sort, storing all the Ur-elements that lie inside the input nested relation.

We have projection functions and element relations going between these sorts in the obvious way, and a constant \( R \) of sort \( R_{\text{top}} \).

If we consider the following instance of the nested relational schema

\[
R_0 = \{ (a, \{a, b\}), (a, \{a, c\}), (b, \{a, c\}) \}
\]

Then the corresponding encoded structure could consist of:

- \( R_{\text{top}} \) containing only the constant \( R_0 \)
- \( R_{\in} \) consisting of the elements of \( R_0 \)
- \( \mathcal{U} \) consisting of \( \{a, b, c\} \), item \( R_{\in,2} \) consisting of the sets \( \{a, b\}, \{a, c\} \)
- the projections and element relations interpreted in the natural way

Notice that all of the values in the encoded structure lie “inside the input object”, and thus a \( \Delta_0 \) formula evaluated over such an encoding can only return such values.

Consider the query that groups on the first component, returning an output object of type \( O = \text{Set}(\mathcal{U} \times \text{Set}(\text{Set}(\mathcal{U}))) \). This is a variation of the query in Example 4. On the example input \( R_0 \) the query would return

\[
O_0 = \{ (a, \{\{a, b\}, \{a, c\}\}), (b, \{\{a, c\}\}) \}
\]

The output would be represented by a structure having sorts \( O_{\text{top}}, O_{\text{top}, \in}, \ldots \). It is easy to capture this query with a \( \Delta_0 \) interpretation. For example, the interpretation could represent the element \( O_{\text{top}, \in} \) by members of \( R_{\in,1} \), thus the domain formula for the output sort \( O_{\text{top}, \in} \) could be \( \varphi_{\text{Domain}}(x) = x \), where \( x \) is of sort \( R_{\in,1} \).

We will often make use of the following observation about interpretations:

\begin{itemize}
  \item \textbf{Proposition 17.} Nested relational interpretations can be composed, and their composition correspond to the underlying composition of queries.
\end{itemize}

The composition of nested relational interpretations amount to the usual composition of FO-interpretations (see e.g. [7]) and an easy check that the additional requirements we impose on nested relational interpretations still are satisfied up to logical equivalence.

We can now state the equivalence of \( \text{NRC} \) and interpretations formally:

\begin{itemize}
  \item \textbf{Theorem 18.} Every query in \( \text{NRC}[\text{Get}] \) can be translated effectively to an equivalent \( \Delta_0 \) interpretation. Conversely, for every \( \Delta_0 \) interpretation, one can effectively form an equivalent \( \text{NRC}[\text{Get}] \) query.
\end{itemize}

As mentioned in the body of the paper, this characterization holds when equivalence is over finite nested relational inputs and also when arbitrary nested relation are allowed as inputs to the queries.

\section*{Coding with monadic schemas}

Before beginning the proof of Theorem 18 we first note that it is possible to reduce questions about definability within \( \text{NRC} \) to the case of a very simple kind of schema.

A \textit{monadic type} is a type built only using the atomic type \( \mathcal{U} \) and the type constructor \( \text{Set} \). Monadic types are in one-to-one correspondence with natural numbers by setting \( \mathcal{U}_0 := \mathcal{U} \) and \( \mathcal{U}_{n+1} := \text{Set}(\mathcal{U}_n) \).

A monadic type is thus a \( \mathcal{U}_n \) for some \( n \in \mathbb{N} \). A nested relational schema is monadic if it contains only monadic types, and a \( \Delta_0 \) formula is said to be monadic if it all of its variables have monadic types.

Restricting to monadic formulas is not essential for our arguments, but it simplifies the type system significantly and thus helps streamline certain arguments by induction. It turns out that by the
Generating collection queries from proofs

usual “Kuratowski encoding” of pairs by sets, we can reduce all of our questions about definability to the case of monadic schemas \(^1\). The following proposition implies that we can derive all of our main results for arbitrary schemas from their restriction to monadic formulas. We will thus restrict to monadic formulas for the remainder of the argument.

\[\text{Proposition 19. For any nested relational schema } SCH, \text{ there is a monadic nested relational schema } SCH', \text{ an injection } \text{Convert} \text{ from instances of } SCH \text{ to instances of } SCH' \text{ that is definable in NRC, and an NRC[Get] query } \text{Convert}^{-1} \text{ such that } \text{Convert}^{-1} \circ \text{Convert} \text{ is the identity query } SCH \rightarrow SCH.\]

Furthermore, there is a \(\Delta_0\) formula \(\text{Im}_{\text{Convert}}\) from \(SCH'\) to \(\text{Bool}\) such that \(\text{Im}_{\text{Convert}}(i')\) holds if and only if \(i' = \text{Convert}(i)\) for some instance \(i\) of \(SCH\).

To prove the proposition, we give an encoding of general nested relational schemas into monadic nested relational schemas that will allow us to reduce the equivalence between NRC queries, interpretations and implicit definitions to the case where input and outputs are monadic. To do so, it is crucial to check that this encoding may be defined either through NRC queries or interpretations. We first give the definition in terms of NRC queries.

The first step toward defining these encodings is actually to emulate in a sound way the cartesian product structure. For every \(n \in \mathbb{N}\), we have that, for this encoding to make sense in the typed monadic setting, the types of \(a\) and \(NRC\) projections that satisfy the usual equations associated with cartesian product structure.

\[\text{Proposition 20. For every } n_1, n_2 \in \mathbb{N}, \text{ there are NRC queries } \hat{\text{Pair}}(x, y) : U_{n_1, n_2} \rightarrow U_{\max(n_1, n_2) + 2} \text{ and NRC[Get] queries } \hat{\pi}_i(x) : U_{n_1, n_2} \rightarrow U_{\max(n_1, n_2) + 2} \text{ for } i \in \{1, 2\} \text{ such that the following equations hold}\]

\[\hat{\pi}_1(\hat{\text{Pair}}(a_1, a_2)) = a_1 \quad \hat{\pi}_2(\hat{\text{Pair}}(a_1, a_2)) = a_2\]

Furthermore, there is a \(\Delta_0\) formula \(\text{Im}_{\hat{\text{Pair}}} (x)\) such that \(\text{Im}_{\hat{\text{Pair}}} (a)\) holds if and only if there exists \(a_1, a_2\) such that \(\hat{\text{Pair}}(a_1, a_2) = a\). In such a case, the following also holds

\[\hat{\text{Pair}}(\hat{\pi}_1(a), \hat{\pi}_2(a)) = a\]

\[\text{Proof. We adapt the Kuratowski encoding of pairs } (a, b) \mapsto \{\{a\}, \{a, b\}\}. \text{ The notable thing here is that, for this encoding to make sense in the typed monadic setting, the types of } a \text{ and } b \text{ need to be the same. This will not be an issue because we have NRC-definable embeddings}\]

\[\uparrow_{m, n} : U_n \rightarrow U_m\]

for \(n \leq m\) defined as the \(m - n\)-fold composition of the singleton query \(x \mapsto \{x\}\). This will be sufficient to define the analogues of pairing for monadic types and thus to define \(\text{Convert}_T\) by induction over \(T\). On the other hand, \(\text{Convert}_T^{-1}\) will require a suitable encoding of projections. This means that to decode an encoding of a pair, we need to make use of a query inverse to the singleton construct \(\uparrow\).

But we have this thanks to the Get construct. We let

\[\downarrow_{m, n} : U_m \rightarrow U_n\]

the query inverse to \(\uparrow_{m, n}\), defined as the \(m - n\)-fold composition of Get.

Firstly, we define the family of queries \(\hat{\text{Pair}}_{n, m}(x_1, x_2), \text{ where } x_i \text{ is an input of type } U_n \text{ for } i \in \{1, 2\}\) and the output is of type \(U_{\max(n_1, n_2) + 2}\), as follows

\[\hat{\text{Pair}}_{n_1, n_2}(x_1, x_2) := \{\{\uparrow x_1\}, \{\uparrow x_1, \uparrow x_2\}\}\]

\[\text{1 This simplification was suggested by Szymon Toruńczyk}\]
The associated projections $\pi_i^{\mathcal{N}}(x)$ where $x$ has type $\mathcal{U}^{\max(n_1,n_2)+2}$ and the output is of type $\mathcal{U}_i$ are a bit more challenging to construct. The basic idea is that there is first a case distinction to be made for encodings $\text{Pair}_{n,m}(x_1,x_2)$: depending on whether $\uparrow x_1 = \uparrow x_2$ or not. This can be actually tested by a NRC query. Once this case distinction is made, one may informally compute the projections as follows:

- if $\uparrow x_1 = \uparrow x_2$, both projections can be computed as a suitable downcasting $\downarrow$ (the depth of the downcasting is determined by the output type, which is not necessarily the same for both projections).
- otherwise, one needs to single out the singleton $\{\uparrow x_1\}$ and the two-element set $\{\uparrow x_1, \uparrow x_2\}$ in NRC. Then, one may compute the first projection by downcasting the singleton, and the second projection by first computing $\{\uparrow x_2\}$ as a set difference and then downcasting with $\downarrow$.

We now give the formal encoding for projections, making a similar case distinction. To this end, we first define a generic NRC query

$$\text{AllPairs}_T(x) : \text{Set}(T) \rightarrow \text{Set}(T \times T)$$

computing all the pairs of distinct elements of its input $x$

$$\text{AllPairs}_T(x) = \bigcup \{ \{(y,z) \} \mid y \in x \setminus \{ z \} \mid z \in x \}$$

Note in particular that $\text{AllPairs}(\emptyset) = \emptyset$ if and only if $i$ is a singleton or the empty set. The projections can thus be defined as

$$\check{\pi}_1(x) := \begin{cases} \text{case} (\text{AllPairs}(x) = \emptyset, \downarrow x, \downarrow \bigcup \{ \pi_1(z) \cap \pi_2(z) \mid z \in \text{AllPairs}(x) \} ) \\ \text{case} (\text{AllPairs}(x) = \emptyset, \downarrow x, \downarrow (x \setminus \check{\pi}_1(x))) \end{cases}$$

$$\check{\pi}_2(x) := \begin{cases} \text{case} (\text{AllPairs}(x) = \emptyset, \downarrow x, \downarrow \bigcup \{ \pi_1(z) \cap \pi_2(z) \mid z \in \text{AllPairs}(x) \} ) \\ \text{case} (\text{AllPairs}(x) = \emptyset, \downarrow x, \downarrow (x \setminus \check{\pi}_2(x))) \end{cases}$$

These definitions crucially ensure that, for every object $a_i$ with $i \in \{1,2\}$, we have

$$\check{\pi}_i (\overline{\text{Pair}}(a_1,a_2)) = a_i$$

Now all remains to be done is to define $\text{Im}_{\text{Pair}}$. Before that, it is helpful to define a formula $\text{Im}_{\text{Pair}}(x)$ which holds if and only if $x$ is in the image of $\text{Im}_{\text{Pair}}$. As a preliminary step, define generic $\Delta_0$ formulas $\text{IsSing}(x)$ and $\text{IsTwo}(x)$ taking an object of type $\text{Set}(T)$ and returning a Boolean indicating whether the object is a singleton or a two-element set. Defining $\text{Im}_{\text{Pair}}$ is straightforward using $\text{IsSing}$ and Boolean connectives. Then $\text{Im}_{\text{Pair}_{n,m}}(x)$ can be defined as follows for each $n \in \mathbb{N}$

$$\text{Im}_{\text{Pair}_{n,m}}(x) := \left( \text{IsSing}(x) \land \text{Im}_{\text{Pair}_{n,m}}(x) \right) \lor \left( \text{IsTwo}(x) \land \text{Im}_{\text{Pair}_{n,m}}(x) \right)$$

$$\text{Im}_{\text{IsSing}}(x) := \exists z \in x \text{ IsSing}(z)$$

$$\text{Im}_{\text{IsTwo}}(x) := \exists z' \in x \left( \text{IsTwo}(z) \land \text{IsSing}(z') \land \forall y \in z \ y \in z' \right)$$

Then, the more general $\text{Im}_{\text{Pair}_{n_1,n_2}}$ can be defined using $\text{Im}_{\text{Pair}}$ where $m = \max(n_1,n_2)$.

$$\text{Im}_{\text{Pair}_{n_1,n_2}}(x) := \text{Im}_{\text{Pair}_{n,m}}(x) \cap \text{Im}_{\text{Pair}_{n_1,m}}(\check{\pi}_1(x)) \cap \text{Im}_{\text{Pair}_{n_2,m}}(\check{\pi}_2(x))$$

One can then easily check that $\text{Im}_{\text{Pair}}$ does have the advertised property: if $\text{Im}_{\text{Pair}}(a)$ holds for some object $a$, then there are $a_1$ and $a_2$ such that $\overline{\text{Pair}}(a_1,a_2) = a$ and we have

$$\overline{\text{Pair}}(\check{\pi}_1(a),\check{\pi}_2(a)) = a$$
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We are now ready to give the proof of Proposition 19.

**Proof.** Convert$_T$, Convert$^{-1}_T$ and Im$_{\text{Convert}_T}$ are defined by induction over $T$. Beforehand, define the map $d$ taking a type $T$ to a natural number $d(T)$ so that Convert maps instances of type $T$ to monadic types $U_d(T)$:

\[
\begin{align*}
    d(U) &= 0 \\
    d(T_1 \times T_2) &= 2 + \max(d(T_1), d(T_2))
\end{align*}
\]

Convert$_T$, Convert$^{-1}_T$ and Im$_{\text{Convert}_T}$ are then defined by the following clauses, where we write $\text{Map}(z \mapsto Q)(x)$ for the NRC expression $\bigcup \{ \{Q \} \mid z \in x \}$.

\[
\begin{align*}
    \text{Convert}_U(x) &:= x \\
    \text{Convert}_{\text{Set}(T)}(x) &:= \text{Map}(z \mapsto \text{Convert}_T(z))(x) \\
    \text{Convert}_{T_1 \times T_2}(x) &:= \langle \text{Pair}(\text{Convert}_{T_1}(\pi_1(x)), \text{Convert}_{T_2}(\pi_2(x))) \rangle \\
    \text{Convert}^{-1}_U(x) &:= x \\
    \text{Convert}^{-1}_{\text{Set}(T)}(x) &:= \text{Map}(z \mapsto \text{Convert}^{-1}_T(z))(x) \\
    \text{Convert}^{-1}_{T_1 \times T_2}(x) &:= \langle \langle \text{Convert}^{-1}_{T_1}(\hat{\pi}_1(x)), \text{Convert}^{-1}_{T_2}(\hat{\pi}_2(x)) \rangle \rangle \\
    \text{Im}_{\text{Convert}_U}(x) &:= \text{True} \\
    \text{Im}_{\text{Convert}_{\text{Set}(T)}}(x) &:= \forall z \in x \text{ Im}_{\text{Convert}_T}(z) \\
    \text{Im}_{\text{Convert}_{T_1 \times T_2}}(x) &:= \text{Im}_{\text{Pair}(\text{Convert}_{T_1}(\pi_1(x)), \text{Convert}_{T_2}(\pi_2(x)))} \\
\end{align*}
\]

It is easy to check, by induction over $T$, that for every object $a$ of type $T$

\[
\text{Convert}^{-1}(\text{Convert}(a)) = a
\]

and that for every object $b$ of type $U_d(T)$, if Im$_{\text{Convert}_T}(b) = \text{True}$, then it lies in the image of Convert$_T$ and Convert$(\text{Convert}^{-1}(b)) = b$. ▫

We have seen so far that it is possible to reduce questions about definability within NRC to the case of monadic schema; this was Proposition 19. Now we turn to the analogous statement for interpretations.

**Proposition 21.** For any object schema $\text{SCH}$, there is a monadic nested relational schema $\text{SCH}'$, a $\Delta_0$ interpretation $\mathcal{I}_{\text{Convert}}$ from instances of $\text{SCH}$ to instances of $\text{SCH}'$, and another interpretation $\mathcal{I}_{\text{Convert}^{-1}}$ from instances of $\text{SCH}'$ to instances of $\text{SCH}$ compatible with Convert and Convert$^{-1}$ as defined in Proposition 21 in the following sense: for every instance $I$ of $\text{SCH}$ and for every instance $J$ of $\text{SCH}'$ in the codomain of Convert, we have

\[
\text{Convert}^{-1}(J) = \text{Collapse}(\mathcal{I}_{\text{Convert}^{-1}}(J)) \quad \text{Convert}(I) = \text{Collapse}(\mathcal{I}_{\text{Convert}}(I))
\]

Before proving Proposition 21, it is helpful to check that a number of basic NRC connectives may be defined at the level of interpretations.

**Proposition 22.** The following $\Delta_0$-interpretations are definable:

- $\mathcal{I}_{\text{Sing}}$ defining the query $x \mapsto \{x\}$.
- $\mathcal{I}_{\cup}$ defining the query $x, y \mapsto x \cup y$. 
Furthermore, assuming that $\mathcal{I}$ is a $\Delta_0$-interpretation defining a query $Q$ and $\mathcal{I}'$ is a $\Delta_0$-interpretation defining a query $R$, the following $\Delta_0$-interpretations are also definable:

- $\text{Map}(\mathcal{I})$ defining the query $x \mapsto \{Q(y) \mid x \in y\}$.
- $(\mathcal{I}, \mathcal{I}')$ defining the query $x, y \mapsto (Q(x), R(y))$.

**Proof.** For the singleton construction $\{e\}$ with $e$ of type $T$, we take the interpretation $\mathcal{I}_e$ for $e$, where $e$ itself is interpreted by a constant $c$ and we add an extra level represented by an input constant $c'$. Then $\varphi_{\text{Domain}}^{\mathcal{I}}(x) = x = c'$ where $\varphi_{\text{Domain}}^{\mathcal{I}}(x, y) = x = c \land y = c'$.

- The empty set $\{\}$ at type $\text{Set}(T)$ is given by the trivial interpretation where $\varphi_{\text{Domain}}^{\mathcal{I}}(x) = x = c$ for some constant $c$ and $\varphi_{\text{Domain}}^{\mathcal{I}}$ is set to false for $T'$ a subtype of $T$, as well as all the $\varphi_{\mathcal{I}}^{T'}$.

- For the binary union $\cup : \text{Set}(T), \text{Set}(T) \to \text{Set}(T)$, the interpretation is easy: $T$ is interpreted as itself. The difference between input and output is that $\text{Set}(T) \times \text{Set}(T)$ is not an output sort and that $\text{Set}(T)$ is interpreted as a single element, the constant $c_0$ of $\mathcal{U}$.

$$\begin{align*} 
\varphi_{\text{Domain}}^{\mathcal{I}}(x) &:= x = c_0 \\
\varphi_{\text{Domain}}^{\mathcal{I}}(x, y) &:= z \in \pi_1(o_m) \lor z \in \pi_2(o_m)
\end{align*}$$

We now discuss the Map operator. Assume that we have an interpretation $\mathcal{I}$ defining a query $S \to T$ that we want to lift to an interpretation $\text{Map}(\mathcal{I}) : \text{Set}(S) \to \text{Set}(T)$. Let us write $\varphi_{\text{Domain}}^{T}$, $\psi_{\mathcal{I}}^{T}$ and $\psi_{\mathcal{I}}^{T'}$ for the formulas making up $\mathcal{I}$ and reserve the $\varphi$ formulas for $\text{Map}(\mathcal{I})$. At the level of sort, let us write $\tau^T$ and $\tau^{\text{Map}(\mathcal{I})}$ to distinguish the two.

For every $T' \leq T$ such that $T'$ is not $\mathcal{U}$, Unit or a cartesian product, we set $\tau^{\text{Map}(\mathcal{I})}(T') = S$, $\tau^T$. This means that objects of sort $T'$ are interpreted as in $\mathcal{I}$ with an additional tag of sort $S$. We interpret the output object $\text{Set}(T)$ as the constant $c_0$ of $\mathcal{U}$.

Assuming that $T \neq \mathcal{U}$, Unit, $\text{Map}(\mathcal{I})$ is determined by setting the following

$$\begin{align*} 
\varphi_{\text{Domain}}^{\mathcal{I}}(a) &:= \exists s \in o_m, \psi_{\text{Domain}}^{T}(a)[s/o_m] \\
\varphi_{\text{Domain}}^{\mathcal{I}}(a, s, x) &:= \psi_{\mathcal{I}}^{T}(a, x)[s/o_m] \\
\varphi_{\text{Domain}}^{T}(a, s, x) &:= \psi_{\text{Domain}}^{T}(a)[s/o_m] \\
\varphi_{\mathcal{I}}^{T}(a, s, x) &:= \varphi_{\text{Domain}}^{T}(a)[x/o_m] \\
\varphi_{\mathcal{I}}^{T'}(a, s, x) &:= \exists x', \psi_{\mathcal{I}}^{T'}(x'[s/o_m]) \land \varphi_{\mathcal{I}}^{T'}(a, s, x', x') \\
\varphi_{\mathcal{I}}^{T'}(x, s, x') &:= s \in o_m \\
\varphi_{\mathcal{I}}^{T'}(x, s, x') &:= \varphi_{\text{Domain}}^{T'}(x)[s/o_m]
\end{align*}$$

where $[x/o_m]$ means that we replace occurrences of the constant $o_m$ by the variable $x$ and sorts $T'$ and $T' \times T''$ are subtypes of $T$. Note that this definition is technically by induction over the type, as we use $\varphi_{\mathcal{I}}^{T'}$ to define $\varphi_{\mathcal{I}}^{T'}$. In case $T$ is $\mathcal{U}$ or Unit, the last two formulas $\varphi_{\text{Domain}}^{T}$ and $\varphi_{\mathcal{I}}^{T}$ need to change. If $T = \mathcal{U}$, then we set

$$\varphi_{\text{Domain}}^{\mathcal{U}}(c_0) := \varphi_{\mathcal{I}}^{\mathcal{U}}(c_0, c_0) := \exists s \in o_m \top$$

and if $T = \mathcal{U}$, we set

$$\varphi_{\text{Domain}}^{\mathcal{U}}(a) := \varphi_{\mathcal{I}}^{\mathcal{U}}(a) := \exists s \in o_m \psi_{\text{Domain}}^{T}(a)[s/o_m]$$

Finally we need to discuss the pairing of two queries $(\mathcal{I}_1, \mathcal{I}_2) : S \to T_1 \times T_2$. Similarly as for map we reserve $\varphi_{\text{Domain}}^{T}$, $\psi_{\mathcal{I}}^{T}$ and $\varphi_{\mathcal{I}}^{T}$ formulas for the interpretation $(\mathcal{I}_1, \mathcal{I}_2)$. We write $\varphi_{\text{Domain}}^{\mathcal{I}}$, $\psi_{\mathcal{I}}^{\mathcal{I}}$ and $\psi_{\mathcal{I}}^{\mathcal{I}}$ for components of $\mathcal{I}$ and $\theta_{\text{Domain}}^{\mathcal{I}}$, $\theta_{\mathcal{I}}^{\mathcal{I}}$ and $\theta_{\mathcal{I}}^{\mathcal{I}}$ for components of $\mathcal{I}'$. Without loss of generality, we may assume that $\mathcal{I}_1$ and $\mathcal{I}_2$ have the same set of output sorts; if they do not, extend them by
interpreting extra sorts as empty sets. For instance, if $T$ is not an output sort of $\mathcal{I}_1$, set $\tau^T(T)$ to be the empty tuple and $\psi^T_{\text{Domain}} := \text{False}$.

Now, the basic idea is to interpret output sorts of $\langle \mathcal{I}_1, \mathcal{I}_2 \rangle$ as tagged unions of elements that either come from $\mathcal{I}_1$ or $\mathcal{I}_2$. Concretely, for every $T$ subtype of either $T_1$ or $T_2$, we set

$$
\tau^T(\mathcal{I}_1, \mathcal{I}_2)(T) := U, \tau^T_1(T), \tau^T_2(T)
$$

$$
\psi^T_{\text{Domain}}(u, \vec{x}, \vec{y}) := (u = c_0 \land \psi^T_{\text{Domain}}(\vec{x})) \lor (u = c_1 \land \theta^T_{\text{Domain}}(\vec{y}))
$$

$$
\phi^T_{\text{Domain}}(u, \vec{x}, \vec{y}) := (u = u' = c_0 \land \psi^T_{\text{Domain}}(\vec{x}, \vec{y})) \lor (u = u' = c_1 \land \theta^T_{\text{Domain}}(\vec{x}, \vec{y}))
$$

To complete the interpretation, it is sufficient to set $\tau^T_2(\mathcal{I}_1, \mathcal{I}_2)(T_1 \times T_2) := U, \tau^T_1(T), U, \tau^T_2(T_2)$, use the canonical projection and pairing terms and to set

$$
\psi^{T_1 \times T_2}_{\text{Domain}}(u, \vec{x}, v, \vec{y}) := u = c_0 \land \psi^{T_1}_{\text{Domain}}(\vec{x}) \land v = c_1 \land \psi^{T_2}_{\text{Domain}}(\vec{y})
$$

Proof of Proposition 21. Similarly as with Proposition 19, we define auxiliary interpretations $\mathcal{I}_1, \mathcal{I}_1, \mathcal{I}_{\text{Pair}}, \mathcal{I}_{\text{Def}}$, and $\mathcal{I}_{\text{Def}}$, mimicking the relevant constructs of Proposition 19. Then we will dispense with giving the recursive definitions of $\mathcal{I}_{\text{Convert}}$ and $\mathcal{I}_{\text{Convert}^{-1}}$, as they will be obvious from inspecting the clauses given in the proof of Proposition 19 and replicating them using Proposition 22 together with closure under composition of interpretations (Proposition 17).

$\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_{\text{Pair}}$ are easy to define through Proposition 22, so we focus on the projections $\mathcal{I}_{\mathcal{K}_1,n_2}^n$ and $\mathcal{I}_{\mathcal{K}_2,n_2}^n$, defining queries from $\mathcal{U}_m$ to $\mathcal{U}_n$ for $i \in \{1, 2\}$ where $m := \max(n_1, n_2)$. Note that in both cases, the output sort is part of the input sorts. Thus an output sort will be interpreted by itself in the input, and the formulas will be trivial for every sort lying strictly below the output sort: we take

$$
\psi^{\mathcal{K}_1_{n_2}}_{\text{Domain}}(x, y) := x \in y \land \psi^{\mathcal{K}_1_{n_2-1}}_{\text{Domain}}(y) \quad \psi^{\mathcal{K}_2_{n_2}}_{\text{Domain}}(x, y) := x = y \quad \psi^{\mathcal{K}_2_{n_2}}_{\text{Domain}}(x) := \top
$$

for every $k < n_i$ (i according to which projection we are defining). The only remaining important data that we need to provide are the formulas $\psi^{\mathcal{K}_1_{n_2}}_{\text{Domain}}$, which, of course, differ for both projections. We provide those below, calling $o_m$ the designated input object. For both cases, we use an auxiliary predicate $x \in^k y$ standing for $\exists y_1 \in y \ldots \exists y_{k-1} \in y_{k-2} \ldots x \in y_{k-1}$ for $k > 1$; for $k = 0$, we take $x \in^1 y$ to be $x \in y$ and $x \in^0 y$ for $x = y$.

For $\mathcal{I}_{\mathcal{K}_1,n_2}$, we set

$$
\psi^{\mathcal{K}_1_{n_2}}_{\text{Domain}}(x) := \forall z \in o_m \exists z' \in z \ x \in^m - n_1 z'
$$

The basic idea is that the outermost $\forall \exists$ ensures that we compute the intersection of the two sets contained in the encoding of the pair.

For $\mathcal{I}_{\mathcal{K}_2,n_2}$, first note that there are obvious $\Delta_0$-predicates $\text{IsSing}(x)$ and $\text{IsTwo}(x)$ classifying singletons and two element sets. This allows us to write the following $\Delta_0$ formula

$$
\psi^{\mathcal{K}_2_{n_2}}_{\text{Domain}}(x) := \bigvee \begin{cases} 
\text{IsSing}(x) \land \forall z \in o_m \exists z' \in z \ x \in^m - n_2 z' \\
\text{IsTwo}(x) \land \exists z' \in o_m \exists y \in z' \ (y \notin z \land x \in^m - n_2 z')
\end{cases}
$$

It is then easy to check that, regarded as queries, those interpretation also implement the projections for Kuratowski pairs.

From NRC queries to interpretations. One direction of Theorem 18 is given in the following lemma:
Lemma 23. There is an EXPTIME computable function taking an NRC\[Get\] query $Q$ to an equivalent FO interpretation $I_Q$.

Note that very similar results occur in the literature, so we only sketch the argument. The paper [46] proves this lemma, but without formalizing the output as an interpretation. In the context of the XML query language XQuery, [7] proves a transformation to first-order interpretations over trees. As noted in [27], there is a very close relationship between XQuery and NRC, and the translation to interpretations in [7] can be easily lifted to NRC. There is also similarity to results from the 1960’s of Gandy [16]. Gandy defines a class of set functions that are similar to NRC, and shows that they are “substitutable”. This is the core of the argument for translating NRC to interpretations.

Proof. We can assume that the input and output schemas are monadic, using Proposition 21. Indeed, if we solve the problem for monadic queries, we can reduce the problem of finding an interpretation for an arbitrary query $Q(\vec{x})$ as follows: give a $\Delta_0$ interpretation $I$ for the query $\text{Convert}(Q(\text{Convert}^{-1}(x)))$ (where Convert and Convert$^{-1}$ are taken as in Proposition 19) and then, using closure under composition of interpretations (see e.g. [7]), one can then leverage Proposition 21 to produce the composition of $I_{\text{Convert}}^{-1} \circ I$ and $I_{\text{Convert}}$ which is equivalent to the original query $Q$.

The argument proceeds by induction on the structure of $Q$:

For the set difference, since interpretations are closed under composition, it suffices to prove that we can code the query

$$(x, y) \mapsto x \setminus y$$

at every sort $\text{Set}(U_n)$. Each sort gets interpreted by itself. We thus set

$$\varphi_{\text{Domain}}^{U_n}(z) := \begin{cases} \pi_1(o_m) \land z \notin \pi_1(o_m) & \text{if } x \\ \exists z' \pi_1(o_m) \land z \in n^{-k} z' & \text{if } y \\ \exists z \pi_1(o_m) \land \varphi_{\text{Domain}}^{U_k}(z) \land \varphi_{\text{Domain}}^{U_{k+1}}(z') & \text{if } z \\
\end{cases}$$

To get $\text{Get}$ queries, it suffices the interpretation $\text{Set}(U) \to U$ which follows

$$\varphi_{\text{Domain}}^{\text{Set}}(a) := (\exists z \pi_1(o_m) z = a) \lor (\neg(\exists z \pi_1(o_m) \land a = c_0))$$

For the binding operator

$$\bigcup \{Q_1 \mid x \in Q_2\}$$

we exploit the classical decomposition

$$\bigcup \circ \text{Map}(Q_1) \circ Q_2$$

As interpretations are closed under composition and the mapping of queries through $\text{Set}$ was handled in Proposition 22, it suffices to give an interpretation for the query $\bigcup: \text{Set}((\text{Set}(T))) \to \text{Set}(T)$ for every sort $T$. This is straightforward: each sort gets interpreted as itself, except for $\text{Set}(T)$ itself which gets interpreted as the singleton \{c_0\}. The only non-trivial clause are the following

$$\varphi^T_{\text{Domain}}(x, y) := \varphi_{\text{Domain}}^T : \exists y' \pi_1(o_m) \land x \in y'$$

From interpretations to NRC\[Get\] queries. The other direction of Theorem 18 is provided by:
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Lemma 24. There is a polynomial time function taking a $\Delta_0$ interpretation to an equivalent NRC[Get] query.

This direction will not be used in our Beth-style result, but it is of interest in showing that NRC[Get] and $\Delta_0$ interpretations are equally expressive.

Proof. (of Lemma 24) Using Proposition 19, it suffices to show this for queries that have monadic input schemas as input and output.

Fix a $\Delta_0$ interpretation $I$ with input $U_n$ and output $U_m$.

Before we proceed, first note that for every $d \leq m$, there is an NRC query

$$E_d : U_n \to \text{Set}(U_d)$$

collecting all of the subobjects of its input of sort $U_d$. It is formally defined by the induction over $n - d$.

$$E_m(x) := \{x\} \quad E_d(x) = \bigcup E_{d-1}(x)$$

Write $E_{d_1, ..., d_k}(x)$ for $(E_{d_1}, ..., E_{d_k})(x)$ for every tuple of integers $d_1, ..., d_k$.

For $d \leq m$, let $d_1, ..., d_k$ be the tuple such that the output sort $U_d$ is interpreted by the list of input sorts $U_{d_1}, ..., U_{d_k}$. By induction over $d$, we build a NRC queries $Q_d : U_m, U_{d_1}, ..., U_{d_k} \to U_d$ such that, provided that $\varphi^{U_d}_{\text{Domain}}(\vec{a})$ and $\varphi^{U_{d+1}}_{\text{Domain}}(\vec{b})$ hold, we have

$$\varphi^d_i(\vec{a}, \vec{b}) \quad \text{if and only if} \quad Q_d(\vec{a}) \in Q_{d+1}(\vec{b})$$

For $Q_0 : U_m, U \to U$, we simply take the second projection. Now assume that $Q_d$ is defined and that we are looking to define $Q_{d+1}$. We want to set

$$Q_{d+1}(x_m, \vec{y}) := \{Q_d(x_m, \vec{x}) \mid \vec{x} \in E_{d_1, ..., d_k}(x_m) \land \text{Verify}_{\varphi}^d(x_m, \vec{x}, y_m, \vec{y})\}$$

which is NRC-definable as follows

$$\bigcup \left\{ \text{case} \left( \text{Verify}_{\varphi}^d(x_m, \vec{x}, y_m, \vec{y}), \{Q_d(x_m), \{\}\right\} \mid \vec{x} \in E_{d_1, ..., d_k}(x_m) \right\}$$

where Verify is given as in Proposition 5 and $\{Q(x, \vec{y}) \mid \vec{x} \in Q(\vec{y})\}$ is a notation for $\bigcup \{Q(x, \vec{y}) \mid x_1 \in \pi_1(Q(\vec{y})) \land \ldots \land x_k \in \pi_k(Q(\vec{y}))\}$. It is easy to check that the inductive invariant holds.

Now, consider the query $Q_m : U_n, U_{m_1}, ..., U_{m_k} \to U_m$. Consider the query $R := \{Q_m(x_m, \vec{y}) \mid \vec{y} \in E_{m_1, ..., m_k}(x_m) \land \varphi^{U_m}_{\text{Domain}}(\vec{y})\}$ which is also NRC-definable using Verify. Since the inductive invariant holds at level $m$, $R$ returns the singleton containing the output of $I$. Therefore Get($R$) : $U_n \to U_m$ is the desired NRC[Get] query equivalent to the interpretation $I$. \hfill $\blacktriangleleft$

Note that the argument can be easily modified to produce an NRC query that is composition-free: in union expressions $\bigcup \{Q_1 \mid x \in Q_2\}$, the variable range $Q_2$ is always another variable. In composition-free queries, we allow as a native construct case($B, Q_1, Q_2$) where $B$ is a Boolean combination of atomic Boolean queries, since we can not use composition to derive the conditional from the other operations.

Thus every NRC query can be converted to one that is composition-free, and similarly for NRC[Get]. The analogous statements have been observed before for related languages like XQuery [7].
D Proofs for Section 4, III: model theoretic proof of Theorem 9

This subsection will be devoted to the model-theoretic proof of Theorem 9.

Given the equivalence with interpretations given by Theorem 18, it suffices to show the following:

◮ Theorem 25. For any \( \Delta_0 \) formula \( \varphi(o_{in}, o_{out}, \vec{a}) \) which implies that \( o_{out} \) is a function of \( o_{in} \), there is a \( \Delta_0 \) interpretation \( I \) such that whenever \( \varphi(o_{in}, o_{out}, \vec{a}) \) holds, then \( I(o_{in}, o_{out}) \) holds.

We first argue that we can convert implicit projective \( \Delta_0 \)-definitions to implicit \( \Delta_0 \)-definitions (i.e., suppress auxiliary parameters \( \vec{a} \)).

◮ Lemma 26. For any \( \Delta_0 \) formula \( \varphi(o_{in}, o_{out}, \vec{a}) \) such that \( o_{out} \) is a (partial) function of \( o_{in} \), there is another \( \Delta_0 \) formula \( \psi(o_{in}, o_{out}) \) which is also functional such that \( \varphi(o_{in}, o_{out}, \vec{a}) \Rightarrow \psi(o_{in}, o_{out}) \).

Proof. Having \( \varphi \) functional means that we have an entailment

\[
\varphi(o_{in}, o_{out}, \vec{a}) \models \varphi(o_{in}, o_{out}', \vec{a}') \Rightarrow o_{out} = o_{out}'
\]

Applying \( \Delta_0 \) interpolation (see e.g. Proposition 12), we may obtain a formula \( \theta(o_{in}, o_{out}) \) such that

\[
\varphi(o_{in}, o_{out}, \vec{a}) \models \theta(o_{in}, o_{out}) \quad \text{and} \quad \theta(o_{in}, o_{out}) \land \varphi(o_{in}, o_{out}', \vec{a}') \models o_{out} = o_{out}'
\]

In particular, it means that we may derive the following entailment

\[
\varphi(o_{in}, o_{out}, \vec{a}) \models [\theta(o_{in}, o_{out}') \land \theta(o_{in}, o_{out}'')] \Rightarrow o_{out} = o_{out}''
\]

to which we may apply interpolation again to obtain a formula \( D(o_{in}) \) such that

\[
\varphi(o_{in}, o_{out}, \vec{a}) \models D(o_{in}) \quad \text{and} \quad D(o_{in}) \land \theta(o_{in}, o_{out}') \land \theta(o_{in}, o_{out}'') \models o_{out} = o_{out}''
\]

We now claim that \( \psi(o_{in}, o_{out}) := D(o_{in}) \land \theta(o_{in}, o_{out}) \) is an implicit definition extending \( \varphi \). Functionality of \( \psi \) is a consequence of the second entailment witnessing that \( D \) is an interpolant. Finally, the implication \( \exists \vec{a} \varphi(o_{in}, o_{out}, \vec{a}) \models \psi(o_{in}, o_{out}) \) is given by the combination of the first entailments witnessing that \( \theta \) and \( D \) are interpolants.

From this point on, we thus suppose that we do not have auxiliary parameters \( \vec{a} \) in our implicit definitions.

We further argue that, given Proposition 19 and Proposition 21, it suffices to consider only monadic nested relational schemas.

Given a \( \Delta_0 \) projective implicit definition \( \varphi(o_{in}, o_{out}) \) we can form a new definition that computes the composition of \( \text{Convert}_{SC^{\bot}H_{out}} \), a projection onto the first component, the query, and \( \text{Convert}_{SC^{\bot}H_{out}} \), capturing it by a new formula \( \varphi'(o_{in}', o_{out}') \) that is functional over a monadic schema. Assuming that we have proven the theorem in the monadic case, we would get an NRC query \( Q' \) from \( SC^{\bot}H_{out} \) to \( SC^{\bot}H_{out}' \) agreeing with this formula on its domain. Now we can compose \( \text{Convert}_{SC^{\bot}H_{in}} \cdot Q' \cdot \text{Convert}_{SC^{\bot}H_{out}}^{-1} \), and the projection to get an NRC query agreeing with the partial function defined by \( \varphi(o_{in}, o_{out}) \) on its domain, as required.

As explained in the body, in order to prove Theorem 25, we will show a more general result concerning multi-sorted logic.

Let \( S \Sigma G \) be any multi-sorted signature, \( \text{Sorts}_1 \) be its sorts and \( S_0 \) be a subset of \( \text{Sorts}_1 \). We say that a relation \( R \) is \textit{over} \( \text{Sorts}_0 \) if all of its arguments are in \( \text{Sorts}_0 \). Let \( \Sigma \) be a set of sentences in \( S \Sigma G \). Given a model \( M \) for \( S \Sigma G \), let \( \text{Sorts}_1(M) \) be the union of the domains of relations over \( \text{Sorts}_0 \), and let \( \text{Sorts}_1(M) \) be defined similarly.

We say that \( \text{Sorts}_1 \) is \textit{implicitly interpretable} over \( \text{Sorts}_0 \) relative to \( \Sigma \) if:
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For any models $M_1$ and $M_2$ of $\Sigma$, if there is a mapping $m$ from $\text{Sorts}_0(M_1)$ to $\text{Sorts}_0(M_2)$ that preserves all relations over $\text{Sorts}_0$, then $m$ extends to a unique mapping from $\text{Sorts}_1(M_1)$ to $\text{Sorts}_1(M_2)$ which preserves all relations over $\text{Sorts}_1$.

We say that $\text{Sorts}_1$ is explicitly interpretable over $\text{Sorts}_0$ relative to $\Sigma$ if for all $S$ in $\text{Sorts}_1$ there is a formula $\psi_S(\bar{x}, y)$ where $\bar{x}$ are variables with sorts in $\text{Sorts}_0$, $y$ a variable of sort $S$, such that:

- In any model $M$ of $\Sigma$, $\psi_S$ defines a partial function $F_S$ mapping $\text{Sorts}_0$ tuples on to $S$.
- For every relation $R$ of arity $n$ over $\text{Sorts}_1$, there is a formula $\psi_R(\bar{x}, \ldots \bar{x}_n)$ using only relations of $\text{Sorts}_0$ and only quantification over $\text{Sorts}_0$ such that in any model $M$ of $\Sigma$, the pre-image of $R$ under the mappings $F_S$ for the different arguments of $R$ is defined by $\psi_R(\bar{x}_1, \ldots \bar{x}_n)$.

Note that $\phi_S$, the mapping between the elements $y$ in $S$ and the tuples in $\text{Sorts}_0$ that interpret them, can use arbitrary relations. The key property is that when we pull a relation $R$ over $\text{Sorts}_1$ back using the mappings $\psi_S$, then we obtain something definable using $\text{Sorts}_0$.

**Theorem 27.** For any $\Sigma, \text{Sorts}_0, \text{Sorts}_1$, $\text{Sorts}_1$ is explicitly interpretable over $\text{Sorts}_0$ if and only if it is implicitly interpretable over $\text{Sorts}_0$.

Before proving the theorem, we explain how it implies Theorem 25. In this explanation we assume a monadic schema for both input and output. Thus every element $e$ in an instance has sort $U_n$ for some $n \in \mathbb{N}$.

Consider a $\Delta_0$ formula $\phi(\alpha_{\text{in}}, \alpha_{\text{out}})$ that is functional in $\alpha_{\text{in}}$. $\phi$ can be considered as a multi-sorted first-order formula with sorts for every subtype occurrence of the input as well as distinct sorts for every subtype occurrence of the output other than $U$. Because we are dealing with monadic input and output schema, every sort other than $U$ will be of the form $\text{Set}(T)$, and these sorts have only the element relations $\in_T$ connecting them. We refer to these as input sorts and output sorts. We modify $\phi$ by asserting that all elements of the input sorts lie underneath the constant symbol for $\alpha_{\text{in}}$, and all elements of the output sorts lie underneath the constant symbol for $\alpha_{\text{out}}$. Since $\phi$ was $\Delta_0$, this does not change the semantics. We also conjoin to $\phi$ the sanity axioms for the schema, including extensionality axiom at the sorts corresponding to each object type. Let $\phi^*$ be the resulting formula.

Given models $M$ and $M'$ of $\phi^*$, we define relations $\equiv_i$ connecting elements of $M$ of depth $i$ with elements of $M'$ of depth $i$. For $i = 0$, $\equiv_i$ is the identity: that is, it connects elements of $U$ if and only if they are identical. For $i = j + 1$, $\equiv_i(x, x')$ holds exactly when for every $y \in x$ there is $y' \in x'$ such that $y \equiv_j y'$, and vice versa.

The fact that $\phi$ is functional tells us that:

Suppose $M \models \phi^*$, $M' \models \phi^*$ and $M$ and $M'$ are identical on the input sorts. Then the mapping taking a $y \in M$ of depth $i$ to a $y' \in M'$ such that $y' \equiv_i y$ is an isomorphism of the output sorts that is the identity on $U$. Further, any isomorphism of $\text{Sorts}_1(M)$ on to $\text{Sorts}_1(M')$ that is the identity on $U$ must be equal to $m$: one can show this by induction on the depth $i$ using the fact that $\phi^*$ includes the extensionality axiom.

From this, we see that the output sorts are implicitly interpretable in the input sorts relative to $\phi^*$. Using Theorem 27, we conclude that the output sorts are explicitly interpretable in the input sorts relative to $\phi^*$. Applying the conclusion to the formula $x = x$, where $x$ is a variable of a sort corresponding to object type $T$ of the output, we obtain a first-order formula $\phi^*_{\text{Domain}}(\bar{x})$ over the input sorts. Applying the conclusion to the formula $x = y$ for $x, y$ variables corresponding to the object type $T$ we get a formula $\phi^*_{\equiv_T}(\bar{x}, \bar{x}')$ over the input sorts. Finally, applying the conclusion to the element relation $\varepsilon_T$ at every level of the output, we get a first-order formula $\phi^*_{\varepsilon_T}(\bar{x}, \bar{x}')$ over the input sorts. Because $\phi^*$ asserts that each element of the input sorts lies beneath a constant for $\alpha_{\text{in}}$, we can convert all quantifiers to bind only beneath $\alpha_{\text{in}}$, giving us $\Delta_0$ formulas. It is easy to verify that
these formulas give us the desired interpretation. This completes the proof of Theorem 25, assuming Theorem 27.

We now begin the proof of Theorem 27. Although the theorem appears to be new, each of the components is a variant of arguments that already appear in the model theory literature, and in the course of the proof we will point the reader to some of these analogs.

We first note that it suffices to consider complete theories.

**Proposition 28.** Theorem 27 follows from its restriction to a complete theory.

Proof. Fix a \( \Sigma \) satisfying the hypothesis, but not the conclusion, and let \( \rho \) be a sentence in the vocabulary of \( \Sigma \). We claim that one of \( \rho, \neg \rho \) can be added to \( \Sigma \) in such a way that the conclusion of the theorem still fails. This would suffice, since then we can inductively complete \( \Sigma \).

The hypothesis of the theorem, implicit interpretability of \( \text{Sorts}_1 \) over \( \text{Sorts}_0 \) relative to \( \Sigma \), is preserved under extending \( \Sigma \). Suppose \( \text{Sorts}_1 \) is explicitly interpretable over \( \text{Sorts}_0 \) via \( \Theta_1 \) relative to \( \Sigma \cup \{ \rho \} \), and also that \( \text{Sorts}_1 \) is explicitly interpretable over \( \text{Sorts}_0 \) via \( \Theta_2 \) relative to \( \Sigma \cup \{ \neg \rho \} \). Consider the sentence \( \Sigma_1 \) stating that \( \Sigma \) holds and if \( \rho \) holds then \( \text{Sorts}_1 \) is interpreted via \( \Theta_1 \) applied to \( \text{Sorts}_0 \). Then \( \Sigma_1 \) is implicitly definable over \( \text{Sorts}_0 \), and thus by the standard Beth Definability theorem, there is a sentence \( \Sigma'_1 \) over \( \text{Sorts}_0 \) that holds of models \( M \) that extend to a \( \Sigma_1 \) structure. Similarly we get a sentence \( \Sigma'_2 \) over \( \text{Sorts}_0 \) that holds of a \( \text{Sorts}_0 \) structure \( M \) whenever \( M \) has an expansion that either satisfies \( \rho \) or agrees with \( \Theta_2 \). We can form an interpretation that acts as \( \Theta_1 \) when \( \Sigma'_1 \) holds and as \( \Theta_2 \) when \( \Sigma'_2 \) holds, and this gives the desired conclusion.

In tracking the remaining results, at certain points we will not require the uniqueness claim in implicit interpretability. We say that \( \Sigma_1 \) is weakly implicitly interpretable in this case.

Given a model \( M \) of \( \Sigma \) and \( x_0 \in \text{Sorts}_1 \) within \( M \), the type of \( x_0 \) with parameters from \( \text{Sorts}_0 \) is the set of all formulas satisfied by \( x_0 \), using any sorts and relations but only constants from \( \text{Sorts}_0 \).

A type \( p \) is isolated over \( \text{Sorts}_0 \) if there is a formula \( \varphi(x, \bar{a}) \) with parameters \( \bar{a} \) from \( \text{Sorts}_0 \) such that \( M \models \varphi(x, \bar{a}) \rightarrow \gamma(x) \) for each \( \gamma \in p \). The following is a variation of Lemma 12.5.3 in [22]:

**Lemma 29.** Suppose \( \text{Sorts}_1 \) is weakly implicitly interpretable over \( \text{Sorts}_0 \) with respect to \( \Sigma \). Then in any model \( M \) of \( \Sigma \) the type of any \( \bar{b} \) over \( \text{Sorts}_1 \) with parameters from \( \text{Sorts}_0 \) is isolated over \( \text{Sorts}_0 \).

Proof. Fix a counterexample \( \bar{b} \), and let \( \Phi \) be the set of formulas in \( \text{Sorts}_1 \) with parameters from \( \text{Sorts}_0 \) satisfied by \( \bar{b} \) in \( M \). We claim that there is a model \( M' \) with \( \text{Sorts}_0(M') \) identical to \( \text{Sorts}_0(M) \) where there is no tuple satisfying \( \Phi \). This follows from the failure of isolation and the omitting types theorem (recall that \( \Sigma \) is complete).

Now we have a contradiction of weak implicit interpretability, since the identity mapping on \( \text{Sorts}_0 \) can not extend to an isomorphism of relations over \( \text{Sorts}_1 \) from \( M \) to \( M' \).

The next step is to argue that every element of \( \text{Sorts}_1 \) is definable by a formula using parameters from \( \text{Sorts}_0 \). This property, coupled with the “reduction property” defined further below, is referred to as *coordinatisation* [23, 22], since it states that every element in \( \text{Sorts}_1 \) is identified by “co-ordinates” from \( \text{Sorts}_0 \).

**Lemma 30.** Assume implicit interpretability of \( \text{Sorts}_1 \) over \( \text{Sorts}_0 \) relative to \( \Sigma \). In any model \( M \) of \( \Sigma \), for every element \( e \) of a sort \( S_1 \) in \( \text{Sorts}_1 \), there is a first-order formula \( \psi_e(y, x) \) with variable \( y \) having sort in \( \text{Sorts}_0 \) and \( x \) a variable of sort \( S_1 \), along with a tuple \( \bar{a} \) in \( \text{Sorts}_0(M) \) such that \( \psi_e(\bar{a}, x) \) is satisfied only by \( e \) in \( M \).

In a single-sorted setting, this can be found in [22] Theorem 12.5.8 where it is referred to as “Gaifman’s coordinatisation theorem”, credited independently to unpublished work of Haim Gaifman and Dale Myers. The multi-sorted version is also a variant of Remark 1.2, part 4 in [24], which points
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to a proof in the appendix of [10]; the remark assumes that $\text{Sorts}_1$ is the set of all sorts. Another variation is Theorem 3.3.4 of [5].

Proof. We can assume $M$ is countable. By Lemma 29, the type of every $e$ is isolated by a formula $\varphi(\vec{x}, \vec{a})$ with parameters from $\text{Sorts}_0$ and relations from $\text{Sorts}_1$. We claim that $\varphi$ defines $e$: that is, $e$ is the only satisfier. If not, then there is $e' \neq e$ that satisfies $\varphi$. Consider the relation $\vec{e} \equiv \vec{e}'$ holding if $\vec{e}$ and $\vec{e}'$ satisfy all the same formulas using relations and variables from $\text{Sorts}_1$ and parameters from $\text{Sorts}_0$. Isolation implies that $e \equiv e'$. Further, isolation of types shows that $\equiv$ has the back-and-forth property: given $\vec{d} \equiv \vec{d}'$, and consider $\vec{e}$. We have $\gamma(\vec{x}, \vec{y}, \vec{a})$ isolating the type of $\vec{d}, \vec{e}$, and further $\vec{d}$ satisfies $\exists \vec{y} \gamma(\vec{x}, \vec{y}, \vec{a})$ and thus so does $\vec{d}'$ with witness $\vec{e}'$. But then using $\vec{d} \equiv \vec{d}'$ again we see that $\vec{d}, \vec{e} \equiv \vec{d}', \vec{e}'$. Using countability of $M$ we can thus create a mapping on $M$ fixing $\text{Sorts}_0$ pointwise, preserving all relation in $\text{Sorts}_1$, and taking $\vec{b}$ to $\vec{b}'$. But this contradicts implicit interpretability. ▲

Lemma 31. The formula in Lemma 30 can be taken to depend only on the sort $S$.

Proof. Consider the partial type over the single variable $x$ in $S$ consisting of the formulas $\neg \delta_\varphi(x)$ where $\delta_\varphi(x)$ is the following formula

$$\exists \vec{b} \left[ \varphi(\vec{b}, x) \land \forall x' \left( \varphi(\vec{b}, x') \Rightarrow x' = x \right) \right]$$

where the $\vec{b}$ tuple range over $\text{Sorts}_0$. By Lemma 30, this type cannot be satisfied in a model of $\Sigma$. Since it is unsatisfiable, by compactness, there are finitely many formulas $\varphi_1(\vec{b}, x), \ldots, \varphi_n(\vec{b}, x)$ such that $\forall x \ \bigwedge_{i=1}^n \delta_\varphi(x)$ is satisfied. Therefore, each $\varphi_i(\vec{b}, x)$ defines a partial function from tuples of $S_0$ to $S$ and every element of $S$ is covered by one of the $\varphi_i$. Assuming that $\Sigma$ enforces that $\text{Sorts}_0$ has at least two elements (if not, the main theorem is obvious), we can combine the $\varphi_i(\vec{b}, x)$ into a single formula $\psi(\vec{b}, \vec{e}, x)$ defining a surjective partial function form $S_0$ to $S$ where $\vec{e}$ is an additional parameter in $\text{Sorts}_0$ selecting some $i \leq n$. ▲

Consider the formulas $\psi_\Sigma$ produced by Lemma 31. For a relation $R$ of arity $n$ over $\text{Sorts}_1$, where the $i^{th}$ argument has sort $S_i$, consider the formula

$$\psi_R(\vec{x}_1 \ldots \vec{x}_n) = \exists \vec{y}_1 \ldots \vec{y}_n \ R(\vec{y}_1 \ldots \vec{y}_n) \land \bigwedge_i \psi_{S_i}(\vec{x}_i, \vec{y}_i)$$

where $\vec{x}_i$ is a tuple of variables of sorts in $\text{Sorts}_0$. The formulas $\psi_\Sigma$ for each sort $S$ and the formulas $\psi_R$ for each relation $R$ are as required by the definition of explicitly interpretable, except that they may use quantified variables and relations of $\text{Sorts}_1$, while we only want to use variables and relations from $\text{Sorts}_0$. We take care of this in the following lemma:

Lemma 32. If $\psi(\vec{x})$ is a formula with free variables over sorts in $\text{Sorts}_0$ and relations from $\text{Sorts}_1$. If $\text{Sorts}_1$ is implicitly interpretable over $\text{Sorts}_0$ relative to $\Sigma$, then there is $\psi'(\vec{x})$ such that $\Sigma \models \forall \vec{x} \ \psi(\vec{x}) \leftrightarrow \psi'(\vec{x})$ and $\psi'$ uses only relations and variables over $\text{Sorts}_0$.

Lemma 32 can be proven using a standard Beth definability argument: $\psi$ can be considered (via a straightforward rewriting of $\psi$ and $\Sigma$) as a single-sorted formula over the union of all sorts with unary predicates for each sort. We claim that $\psi$ is implicitly definable relative to $\Sigma$ over the predicates corresponding to $\text{Sorts}_0$. Given two structures that agree on all predicates corresponding to $\text{Sorts}_0$, we can translate them back to multi-sorted structures that agree on $\text{Sorts}_0$, and now implicit interpretability gives an isomorphism on all sorts that preserves $\text{Sorts}_0$, and thus preserves the arguments $\vec{x}$. Thus $\psi$ is preserved. The classical projective Beth definability theorem implies that $\psi$ can be rewritten to a formula that only uses the structure on $\text{Sorts}_0$.

Alternatively we can argue directly that formulas over $\text{Sorts}_1$ do not allow us to define any more subsets of $\text{Sorts}_0$ than we can with formulas over $\text{Sorts}_0$. In the application to nested relational
algebra, this is reminiscent of the “conservativity theorem” [47], saying that using objects built up in NRC does not allow us to define any new queries over flat relations. Following [22], we say that \( \text{Sorts}_1 \) has the uniform reduction property over \( \text{Sorts}_0 \) if for every formula \( \varphi(\vec{x}) \) over \( \text{Sorts}_1 \) with \( \vec{x} \) variables of sort in \( \text{Sorts}_0 \) there is a formula \( \varphi^*(\vec{x}) \) over \( \text{Sorts}_0 \) such that for every model \( M \) of \( \Sigma \),
\[
M \models \forall \vec{x} \varphi(\vec{x}) \leftrightarrow \varphi^*(\vec{x})
\]

\( \blacktriangleright \text{Lemma 33.} \) Assuming weak implicit interpretability, \( \text{Sorts}_1 \) has the uniform reduction property over \( \text{Sorts}_0 \).

The lemma essentially dates to Pillay and Shelah’s [38], where it is stated in a single-sorted context. We present a proof as argued in Lemma 12.5.1 of [22] and Theorem 3.3.3 of [5]:

Proof. Assume not, with \( \varphi \) as a counterexample. By the compactness theorem, we know that there is a countable model \( M \) of \( \Sigma \) containing \( \vec{c}, \vec{c}' \) that agree on all formulas in \( \text{Sorts}_0 \) but that disagree on \( \varphi \). As in Lemma 30, we can obtain a mapping on \( M \) preserving \( \text{Sorts}_0 \) but sending \( \vec{c} \) to \( \vec{c}' \). This contradicts weak implicit interpretability, since the mapping cannot be extended. \( \blacktriangleleft \)

Lemma 33 immediately gives an alternative proof of Lemma 32. Above we obtained the formulas \( \psi_R \) for each relation symbol \( R \) needed for an explicit interpretation. We can obtain formulas defining the necessary equivalence relations \( \psi = \) and \( \psi_{\text{Domain}} \) easily from these.

Putting Lemmas 30, 31, and 32 together yields a proof of Theorem 27.

\[ \blacktriangleleft \text{Proofs for Section 5} \]

\textbf{Strength of the proof system}

Although we do not know whether our proof system derives every classically valid \( \Delta_0 \) sequent, we can show that it derives all sequents of the shape we consider that are constructively derivable in the sense of intuitionistic logic.

Let us first recall the syntax of multi-sorted first-order logic, with equality at every sort and a predicate \( \in_T \) for every sort \( T \) representing membership.

\[
\varphi, \psi ::= t \in_T u \mid t =_T u \mid \top \mid \bot \mid \varphi \land \psi \mid \varphi \lor \psi \mid \varphi \Rightarrow \psi \mid \forall x^T \varphi \mid \exists x^T \varphi
\]

We will deal with the case where the terms are built up using Ur-element constants, the unit constant, the pairing function and the projection functions. The intuitionistic sequent calculus we adopt for first-order logic with equality, projection, and pairing is shown in Figure 4, with the structural rules (weakening and contraction) omitted. It is a straightforward extension of the textbook definition of the sequent calculus LJ for intuitionistic first-order logic (see e.g. [42, Sections 7.2 and 9.3] and [45, Chapter 3]) due to Gentzen [17] to accommodate our typing discipline and additional rules concerning equalities, projection and pairing. The main technical distinction between LJ and the sequent calculus for classical logic LK is that there is a single conclusion formula on the right, rather than a list of formulas. This prevents one from deriving the law of excluded middle \( \vdash \varphi \lor \neg \varphi \) for arbitrary \( \varphi \) in LJ. Note that this does not imply that the calculus is incomplete for (translations of) the restricted sequents that we deal with in our calculus.

The extensions of LJ to accommodate typed terms, equality, and the projection and pairing functions are straightforward. Although we are not aware of a source describing exactly the proof system above, [25, Chapter 4] describes an equivalent system based on natural deduction and [45, Section 4.7] extends LJ with rules for equality without types.
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\[
\begin{align*}
\text{AX} & \quad \text{Γ, } \varphi \vdash \varphi \\
\land\text{-L} & \quad \text{Γ, } \varphi_1, \varphi_2 \vdash \psi \quad \text{Γ, } \varphi_1 \land \varphi_2 \vdash \psi \\
\land\text{-R} & \quad \text{ Γ } \vdash \varphi \quad \text{ Γ } \vdash \varphi \land \psi \\
\top\text{-R} & \quad \text{ Γ } \vdash \top \\
\bot\text{-L} & \quad \text{ Γ, } \bot \vdash \varphi \\
\lor\text{-L} & \quad \text{ Γ, } \varphi \vdash \psi \quad \text{ Γ, } \varphi_1 \lor \varphi_2 \vdash \psi \\
\lor\text{-R} & \quad \text{ Γ } \vdash \varphi_1 \quad i \in \{1,2\} \\
\Rightarrow\text{-L} & \quad \text{ Γ } \vdash \varphi \quad \text{ Γ, } \psi \vdash \theta \\
\Rightarrow\text{-R} & \quad \text{ Γ } \vdash \varphi \Rightarrow \psi \\
\forall\text{-L} & \quad \text{ Γ, } \forall y \varphi \vdash \psi \\
\forall\text{-R} & \quad \text{ Γ } \vdash \varphi_1 \quad i \in \{1,2\} \\
\exists\text{-L} & \quad \text{ Γ, } \varphi \vdash \psi \quad x \notin \text{FV}(\Gamma, \psi) \\
\exists\text{-R} & \quad \text{ Γ } \vdash \varphi[t/x] \\
\equiv\text{-L} & \quad \text{ Γ } \vdash \varphi[s/x,t/y] = \varphi[t,s/y] \\
\equiv\text{-R} & \quad \text{ Γ } \vdash t = t \\
\times\eta & \quad \text{ Γ }[(x_1,x_2)/x] \vdash \varphi[(x_1,x_2)/x] \\
\times\beta & \quad \text{ Γ }[\pi_1(t_1,t_2)/x] \vdash \varphi[\pi_1(t_1,t_2)/x] \\
\text{Unit}_{\eta} & \quad \text{ Γ }[\mathcal{I}/x] \vdash \varphi[\mathcal{I}/x] \\
\end{align*}
\]

**Figure 4** The intuitionistic sequent calculus (LJ) for multi-sorted first-order logic with equality and pairs

In this section, we define a translation of the sequents \( \Theta; \Gamma \vdash \varphi \) of our restricted proof system into sequents \( \overline{\Theta}; \overline{\Gamma} \vdash \overline{\varphi} \) of the calculus displayed in Figure 4, which we refer to as LJ from now on.

As is customary for two-sided sequent calculi, rules introducing logical connectives can be split into left-hand side and right-hand side rules. We make this distinction in our naming conventions, using L and R in rule names to indicate left and right rules. Informally speaking, a rule is left if the right-hand side formula stays the same in the premises and the conclusion and the corresponding connective occurs in the left-hand side of the conclusion. Right rules can be similarly characterized. Some rules are neither right nor left. For LJ, these would be the axiom rule AX and the rules \( \times\eta \), \( \times\beta \) and Unit\( \eta \).

**Translation to LJ sequents.** We will need to perform some translations from the membership contexts and \( \Delta_0 \) formulas used in our context to the multi-sorted first-order formulas used in LJ. \( \Delta_0 \) formulas \( \varphi \) as defined in Section 3 can be regarded as a particular case of general formulas with an abbreviated syntax. Formally, for each \( \Delta_0 \) formula \( \varphi \) we have a corresponding first-order formula \( \overline{\varphi} \).
defined in the usual way

\[
\begin{align*}
(t =_U u)^* & := t =_U u \quad (t \not= _U u)^* := t =_U u \Rightarrow \bot \\
\top^* & := \top \\
\Gamma \land \psi^* & := \psi^* \land \psi^* \\
\forall x \in T \; \phi^* & := \forall x^T \; (x \in T \Rightarrow \phi^*) \quad (\exists x \in T \; \phi)^* := \exists x^T \; (x \in T \land \phi^*)
\end{align*}
\]

Recall that sequents in our restricted system are of the shape \( \Theta; \Gamma \vdash \psi \) where \( \Theta \) is a multiset of pairs of formulas \( t \not\in U, \Gamma \) a list of \( \Delta_0 \) formulas and \( \psi \) a special right-hand side formula of shape either \( t \in_U u, t \subseteq_U u \) or \( t =_U u \). Given such contexts, we write \( \tilde{\Theta} \) for the multiset of formulas \( \{ \phi^* \mid \phi \in \Gamma \} \) and \( \tilde{\Theta} \) for the multiset \( \{ t \in_U u \mid (t \not\in U, \Gamma) \in \Theta \} \). As for right-hand side formulas \( \psi \), we define the notation \( \tilde{\psi} \) by recursion on the type of the main connective of \( \psi \) as follows:

\[
\begin{align*}
t \in_T u & := \exists z' (z' \in u \land t \equiv_T z') \\
t \equiv_{\text{set}}(T) u & := t \subseteq_T u \land t \equiv_T u \\
t \equiv_{\text{Unit}} u & := \top \\
t \equiv_{\pi_1 \times \pi_2} u & := \pi_1(t) \equiv_{\pi_1} \pi_1(u) \land \pi_2(t) \equiv_{\pi_2} \pi_2(u) \\
t \equiv_U u & := t =_U u
\end{align*}
\]

Translating proofs to LJ. We are now ready to state the first direction concerning the equivalence between LJ and our proof system.

Lemma 34. If \( \Theta; \Gamma \vdash \psi \) is derivable in our restricted system, then LJ derives \( \tilde{\Theta}, \tilde{\Gamma} \vdash \tilde{\psi} \).

Towards a proof of Lemma 34, first notice that for every rule

\[
\frac{\Theta; \Gamma \vdash \psi \ldots}{\Theta'; \Gamma' \vdash \psi'}
\]

of our restricted system, the rule

\[
\frac{\tilde{\Theta}, \tilde{\Gamma} \vdash \tilde{\psi} \ldots}{\tilde{\Theta}', \tilde{\Gamma}' \vdash \tilde{\psi}'}
\]

is easily seen to be admissible in LJ, save for one:

\[
\frac{\Theta, t \in_U z; \Gamma \vdash u \in_U z \quad z \not\in \text{FV}(\Theta, \Gamma, t, u)}{\Gamma \vdash t =_U u}
\]

It is helpful to treat the sequents of the type \( \Theta, t \in_U z; \Gamma \vdash u \in_U z \) with \( z \not\in \text{FV}(\Theta, \Gamma, t, u) \) as a special case.

Proposition 35. For every contexts \( \Theta, \Gamma \) and terms \( t \) and \( u \) of type \( U \) whose free variables do not include \( z \), if the sequent \( \Theta, t \in_U z; \Gamma \vdash u \in_U z \) is derivable in the restricted system, then LJ derives \( \Theta, \Gamma \vdash t =_U u \).

Proof. The proof goes by induction on the proof in the restricted system. For most cases, the induction hypothesis is used in a very simple way. We focus on the two representative cases: the rule for universal quantification and the axiom

- If the last rule applied is a \( \forall \) rule, with \( \Gamma = \Gamma', \forall x \in_T y \; \phi \)

\[
\frac{\Theta, t \in_U z; \Gamma', \phi[v/x] \vdash u \in_U z}{\Theta, t \in_U z; \Gamma', \forall x \in_T y \; \phi \vdash u \in_U z}
\]

then we must have \( v \in_T y \) occurring in \( \Theta, t \in_U z \). By assumption, \( z \) does not occur freely in \( \Theta \), so we have necessarily that \( v \) does not have \( z \) as a free variable. Therefore \( z \) does not occur free in
either $\Theta$, $\Gamma'$ or $\varphi[v/x]$, so we can conclude by applying the inductive hypothesis and using the rule $\forall$-L of LJ.

Induction hypothesis

\[
\begin{align*}
\Theta, t \in U z, \Gamma', \varphi^*[v/x] &\vdash t =_U u \\
\Theta, t \in U z, \Gamma', v \in y &\vdash \varphi^*[v/x] \vdash t =_U u \\
\Theta, t \in U z, \Gamma', (\forall x \in y \varphi)^* &\vdash t =_U u
\end{align*}
\]

- If the last rule applied is an axiom

\[
\begin{align*}
\Theta, t \in_U z; \Gamma, t =_U t_1, \ldots, t_{k-1} =_U u \vdash u \in_U z
\end{align*}
\]

then one may derive $\Theta, t \in_U z; \Gamma, t =_U t_1, \ldots, t_{k-1} =_U u \vdash t =_U u$ in LJ by applying the $\vdash$-L rule to all equalities in succession.

\[
\begin{align*}
t =_U u \vdash t =_U u \\
\vdots \\
t =_U t_2, \ldots, t_{k-1} =_U u \vdash t =_U u \\
t =_U t_1, \ldots, t_{k-1} =_U u \vdash t =_U u \\
\Theta, t \in_U z; \Gamma, t =_U t_1, \ldots, t_{k-1} =_U u \vdash t =_U u
\end{align*}
\]

Proof of Lemma 34. The proof goes by induction over the proof of $\Theta; \Gamma \vdash \psi$ in the restricted system. Now that we have proven Proposition 35, all the cases are straightforward. We only outline a few.

- If the last rule applied is $=_U$-R

\[
\begin{align*}
\Theta, t \in_U z; \Gamma \vdash u \in_U z &\quad z \notin \text{FV}(\Theta, \Gamma, t, u) \\
\Gamma \vdash t =_U u
\end{align*}
\]

then we may use the induction hypothesis together with Proposition 35.

- If the last rule applied is an axiom

\[
\begin{align*}
\Theta, t \in_U v; \Gamma, t =_U t_1, \ldots, t_{k-1} =_U u \vdash u \in_U v
\end{align*}
\]

we may derive in the same way as in LJ by applying the $\vdash$-L rule $k$ times.

\[
\begin{align*}
t \in_U v \vdash t \in_U v \\
t \in_U v, t =_U u \vdash u \in_U v \\
\vdots \\
t \in_U v, t =_U t_2, \ldots, t_{k-1} =_U u \vdash u \in_U v \\
t \in_U v, t =_U t_1, \ldots, t_{k-1} =_U u \vdash u \in_U v \\
\Theta, t \in_U v; \Gamma, t =_U t_1, \ldots, t_{k-1} =_U u \vdash u \in_U v
\end{align*}
\]

- If the last rule applied is $\in_{\text{Set}}$-R

\[
\begin{align*}
\Theta, t \in_T u; \Gamma \vdash t =_T t' \\
\Theta, t \in_T u; \Gamma \vdash t' =_T u
\end{align*}
\]
Lemma 36. If the sequent $\Theta, \Gamma \vdash \psi$ is derivable in LJ, then $\Theta; \Gamma \vdash \psi$ is derivable in the restricted system.

This direction is harder to prove than Lemma 34, so we will decompose this result in multiple steps:

1. First, we note that we have the subformula property for LJ: any formula $\phi$ occurring in a LJ-proof tree is necessarily a subformula of some formula occurring at the root, up to substitution of terms. This allows us to distinguish a special class of formulas which we call sub$LJ_0$ formulas and consider LJ sequents containing only such formulas.

2. For sequents containing only sub$LJ_0$ formulas, we note that if we replace the rules $\exists$-L, $\forall$-L, $\exists$-R and $\forall$-R by the bounded variants

\[
\begin{align*}
\forall$-LBV &: & \Gamma, t \in y, \phi[t/x] & \vdash \psi & & \forall$-RBV &: & \Gamma, z \in y \vdash \phi & \quad z \notin FV(\Gamma) & \\
\exists$-LBV &: & \Gamma, x \in y, \phi \vdash \psi & \quad x \notin FV(\Gamma, \psi, y) & & \exists$-RBV &: & \Gamma, t \in y \vdash \phi[t/x] & \quad y \notin FV(\Gamma, \psi, y, \Delta)
\end{align*}
\]

while deriving the same sequents as LJ, while retaining the constraint that the right-hand side formula be neither a conjunction, universal quantification or implication when left-hand side rules are applied. We will call the corresponding system LJBoundVarQ.

3. Then, we note that LJBoundVarQ is equivalent to its restriction where left rules cannot be applied if the right-hand side formula under consideration is a conjunction, an implication or a universal quantification.

4. Finally, the translation can go by induction on such restricted proofs.

We now go through these steps in more detail.

Step 1. That LJ has the subformula property is obvious from inspection of the proof rules. We identify the set of subformulas of (translation of) $\Delta_0$ formulas, that we call sub$LJ_0$ formulas.
generating collection queries from proofs

we want to show that \( LJBoundedConn \).

To this end, we make a case analysis according to the last \( t \in \Gamma \) and the corresponding proof systems:

Let us first focus on the admissibility of \( \Rightarrow \) and \( \land \) and the following instances of \( \land \):

\[
\begin{align*}
\Rightarrow \text{-LB} & \quad \Gamma, t \in u, \varphi[t/x] \vdash \psi \\
\land \text{-RB} & \quad \Gamma, t \in u, t \in u \land \varphi
\end{align*}
\]

\[
\begin{align*}
\forall \text{-LB} & \quad \Gamma, t \in u, \varphi[t/x] \vdash \psi \\
\forall \text{-RB} & \quad \Gamma, z \in u \vdash \varphi \\
\exists \text{-LB} & \quad \Gamma, x \in u, \varphi \vdash \psi \\
\exists \text{-RB} & \quad \Gamma, t \in y \vdash \exists (x \in u \land \varphi)
\end{align*}
\]

\[
\begin{align*}
\forall \text{-LBV} & \quad \Gamma, t \in u, \forall x (x \in u \Rightarrow \varphi) \vdash \psi \\
\forall \text{-RBV} & \quad \Gamma, z \in u \vdash \forall z (\varphi) \\
\exists \text{-LBV} & \quad \Gamma, x \in u, \forall x (x \in u \Rightarrow \varphi) \vdash \psi \\
\exists \text{-RBV} & \quad \Gamma, t \in y \vdash \exists (x \in u \land \varphi)
\end{align*}
\]

and the corresponding proof systems:

- We call \( LJBoundedConn \) the system \( LJD_0 \) with the addition of the rules \( \Rightarrow \) and \( \land \) but omitting the rules \( \Rightarrow \text{-L} \) and the following instances of \( \land \):

\[
\begin{align*}
\Gamma & \vdash t \in u \\
\Gamma & \vdash \varphi \\
\Gamma & \vdash t \in u \land \varphi
\end{align*}
\]

- We call \( LJBoundedQ \) the system \( LJBoundedConn \) with the addition of the rules \( \forall \text{-LB} \), \( \forall \text{-RB} \), \( \exists \text{-LB} \) and \( \exists \text{-RB} \), but omitting the rules \( \forall \text{-L} \), \( \forall \text{-R} \), \( \exists \text{-L} \) and \( \exists \text{-R} \).

- We call \( LJBoundedVarQ \) the system \( LJ2 \) with the addition of the rules \( \forall \text{-LBV} \), \( \forall \text{-RBV} \), \( \exists \text{-LBV} \) and \( \exists \text{-RBV} \), but omitting the rules \( \forall \text{-LB} \), \( \forall \text{-RB} \), \( \exists \text{-LB} \) and \( \exists \text{-RB} \).

We can now show that all those systems derive the same sequents thanks to a series of lemmas stating that when moving from \( LJD_0 \) to \( LJBoundedConn \) to \( LJBoundedQ \) to \( LJBoundedVarQ \), in each step the rules we have removed remain admissible using the rules we have added. The admissibility of each individual rule mentioned in the lemmas can be shown by a lengthy induction.

**Lemma 38.** The rules \( \Rightarrow \text{-L} \) and \( \land \text{-R} \) are admissible in \( LJBoundedConn \).

**Proof.** Let us first focus on the admissibility of \( \land \text{-R} \). By induction on the depth of a \( LJBoundedConn \) proof of

\[
\begin{align*}
\Gamma & \vdash t \in \Gamma, u \\
\Gamma & \vdash \varphi
\end{align*}
\]

we want to show that \( \Gamma \vdash t \in \Gamma, u \land \varphi \) is derivable in \( LJBoundedConn \). Note that if the first conjunct is not a formula of the shape \( t \in \Gamma, u \), we may conclude using an instance of \( \land \text{-R} \) of \( LJBoundedConn \). To this end, we make a case analysis according to the last \( LJBoundedConn \) rule applied to derive \( \Gamma \vdash t \in \Gamma, u \). As they are many cases, we only outline a few representative ones. Most cases are easy because it cannot be the case that a right-hand side rule of \( LJBoundedConn \) may be applied, since \( t \in \Gamma, u \) is an atomic formula.
If the last rule applied was an axiom, this means that $t \in T_u$ was part of $\Gamma$. In this case

$$\Gamma \vdash \psi \quad \frac{}{\Gamma \vdash t \in T_u \land \psi}$$

is an instance of $\land$-RB, the designated replacement of $\land$-R.

If the last rule applied was $\land$-L, assuming that $\Gamma = \Gamma'$, $\phi_1 \land \phi_2$ $\vdash t \in T_u$ then the induction hypothesis gives us a proof of $\Gamma', \phi_1, \phi_2 \vdash t \in T_u \land \psi$, so we may build the tree

Induction hypothesis

$$\frac{\Gamma', \phi_1, \phi_2 \vdash t \in T_u \land \psi}{\Gamma \vdash \psi}$$

by applying the rule $\land$-L.

The admissibility of $\Rightarrow$-L is handled similarly, noticing that, since we are dealing with sub-$\Delta_0$ formulas, the antecedent of an implication in such a rule is also an atomic formula $t \in T_u$.

Corollary 39. $LJ\Delta_0$ and $LJBoundedConn$ derive the same sequents.

Proof. Thanks to Lemma 38, it is then obvious that all the rules of $LJ\Delta_0$ are admissible in $LJBoundedConn$, so every sequent derivable in $LJ\Delta_0$ is derivable in $LJBoundedConn$. The converse is obvious.

Lemma 40. The rules $\forall$-L, $\forall$-R, $\exists$-L and $\exists$-R are admissible in $LJBoundedQ$.

Proof. Let us focus on $\forall$-L. We assume that we have a $LJBoundedQ$ derivation of

$$\Gamma, t \in T_u \Rightarrow \phi[t/x] \vdash \psi$$

and we show, by induction on its depth, that we way obtain a $LJBoundedQ$ derivation of $\Gamma, \forall x (x \in T_u \Rightarrow \phi[t/x] \vdash \psi)$. As usual, one should proceed by case analysis on the last rule applied to get $\Gamma, t \in T_u \Rightarrow \phi[t/x] \vdash \psi$. In all but one case, the main formula under consideration is not $t \in T_u \Rightarrow \phi[t/x]$ and it is easy to use the induction hypothesis. The only interesting case thus occurs when the last rule applied was the $\Rightarrow$-LB rule

$$\Gamma, \phi \vdash \psi \quad \frac{}{\Gamma, t \in T_u \Rightarrow \phi[t/x] \vdash \psi}$$

In such a case, we know that $t \in T_u$ is a formula occurring in $\Gamma$, so we replace the application of this rule with the new rule $\forall$-LB of $LJBoundedQ$ to conclude.

$$\Gamma, \phi \vdash \psi \quad \frac{}{\Gamma, \forall x (x \in T_u \Rightarrow \phi) \vdash \psi}$$

The reasoning for the other rule $\exists$-R is extremely similar, where the only interesting case occurs upon applying a rule $\land$-RB. The last two rules are also handled similarly, the interesting case for the admissibility of $\forall$-R (respectively $\exists$-L) being $\Rightarrow$-R (respectively $\land$-L).

Corollary 41. $LJBoundedConn$ and $LJBoundedQ$ derive the same sequents.

Lemma 42. The rules $\forall$-LB, $\forall$-RB, $\exists$-LB and $\exists$-RB are admissible in $LJBoundVarQ$. 

Proof. All four cases are proven in a similar manner. Exceptionally, the induction this time is not over the size of the proofs, but rather on a quantity computed from the bounding term occurring in the main quantifier of the rule. For instance, this would be $t$ in the following instance of $\exists$-LB:

$$
\exists\text{-LB} \quad \Gamma, x \in t, \varphi \vdash \psi \quad \frac{}{\Gamma, \exists x (x \in t \land \varphi) \vdash \psi}
$$

The “size” of such a term $t$ is the pair $(v_t, r_t)$ computed as follows:

There is an intuitive notion of size for types defined by induction:

$$s(U) = 1$$
$$s(\text{Unit}) = 1$$
$$s(T_1 \times T_2) = 1 + s(T_1) + s(T_2)$$
$$s(\text{Set}(T)) = 1 + s(T)$$
$$s(T_1 \times T_2) = 1 + s(T_1) + s(T_2)$$

From this we can define the “variable size” of a term $t$, denoted $v_t$, to be the sum of the size of the free variables of $t$.

$$v_t = \sum_{x \in \text{FV}(t)} s(T)$$

$r_t$ is the intuitive notion of size for terms, computed by induction over $t$:

$$r_{c_i} = 1$$
$$r_{(t, u)} = 1 + r_t + r_u$$
$$r_{(i)} = 1$$
$$r_{\pi_i(t)} = 1 + r_t$$

Then we can use the fact that the lexicographic product of $\mathbb{N}$ with itself is well-founded to run induction over the pair $(v_t, r_t)$. Let us do so for the rule $\exists$-LB. To this end, suppose that $t$ is a term such that the rule

$$\Gamma, x \in u, \varphi \vdash \psi \quad \frac{}{\Gamma, \exists x (x \in u \land \varphi) \vdash \psi}$$

is admissible in LJBoundVarQ for every $u$ such that either $v_u < v_t$ or $v_u = v_t$ and $r_u < r_t$. We proceed with a case analysis to show that the same rule with $t$ instead of $u$ is admissible.

If $t$ is a variable, then this is an instance of the rule $\exists$-LBV of LJBoundVarQ.

Otherwise, if $t$ has a free variable $z$ of type $T_1 \times T_2$, one may apply the rule $\times$η

$$
\Gamma, [(z_1, z_2)/z]. \exists x \in t[(z_1, z_2)/z]. \varphi[(z_1, z_2)/z] \vdash \psi[(z_1, z_2)/z] \quad \Gamma, \exists x \in t \varphi \vdash \psi
$$

and conclude using our induction hypothesis since $v_t[(z_1, z_2)/z] < v_z$.

Otherwise, if $t$ has no such free variable, but is itself not a free variable, then it is necessarily of the shape $\pi_i(t_1, t_2)$ for some $i \in \{1, 2\}$, so we may apply the rule $\times$β

$$
\Gamma, \exists x \in t_1 \varphi \vdash \psi \quad \frac{}{\Gamma, \exists x \in t \varphi \vdash \psi}
$$

and conclude using our induction hypothesis as we have $v_{t_i} \leq v_t$ and $r_{t_i} < r_t$. 

$\blacktriangle$
Corollary 43. LJ\textsc{BoundedQ} and LJ\textsc{BoundVarQ} derive the same sequents.

Lemma 44. LJ\textsc{Δ0} and LJ\textsc{BoundVarQ} derive the same sequents.

Proof. Combine Corollaries 39, 41 and 43.

Step 3. Recall that a right-hand side rule is one that changes the right-hand side formula. Among the rules of LJ\textsc{BoundVarQ}, these are the rules \(-R, \land-R, \land-RB, \rightarrow-R, \lor-RBV\) and \(3-RBV\). We call a proof tree right-focused if every occurrence of sequent \(Γ ⊢ ψ\) in the tree such that the top-level connective of \(ψ\) is either \(\lor\), \(→\) or \(\land\) is necessarily the conclusion of a right-hand side rule.

The rationale behind this choice is that the rules \(\land-R, \land-RB, \rightarrow-R\) and \(\lor-RBV\) are invertible (if their conclusion is true, so are all the premises), so they may be safely applied eagerly.

Lemma 45. If \(Γ ⊢ φ\) is derivable in LJ\textsc{BoundVarQ}, then there is a right-focused LJ\textsc{BoundVarQ} proof tree of deriving \(Γ ⊢ φ\).

Proof. The result is proven by induction over the depth of the proof-tree, and is straightforward. We sketch one of the case: if the last rule applied is \(\lor-L\) and the right-hand side formula is an implication \(Γ, φ₁, ψ \Rightarrow θ \quad Γ, φ₂, ψ \Rightarrow θ\)

by the induction hypotheses, we have right-focused proofs π₁ with conclusion \(Γ, φᵣ, ψ \Rightarrow θ\) for \(i ∈\{1, 2\}\). We may then build the tree

\[
\frac{π₁}{Γ, φ₁, ψ \Rightarrow θ} \quad \frac{π₂}{Γ, φ₂, ψ \Rightarrow θ} \quad \frac{Γ, φ₁ \lor φ₂, ψ \Rightarrow θ}{Γ, φ₁ \lor φ₂, ψ \Rightarrow θ}
\]

which is right-focused.

Step 4. First, we observe that LJ\textsc{BoundVarQ} has a stronger variant of the subformula property: if all formulas in the conclusion sequent \(Γ ⊢ ψ\) is the translation of some \(Δ₀\) formula, then all formulas occurring in a proof tree are actually \(Δ₀\) formulas.

Lemma 46. If \(Θ; Γ ⊢ ψ\) has a right-focused proof tree in LJ\textsc{BoundVarQ}, then there is a proof of \(Θ; Γ ⊢ ψ\) in our restricted system.

The proof goes by induction over the right-focused LJ\textsc{BoundVarQ} proof tree. All cases are immediate, except for the case of the congruence rule

\[
Γ[s/x,t/y] ⊢ ψ[s/x,t/y]
\]

This particular case can be treated by showing that the obvious counterpart to this rule is admissible in the restricted system before embarking on the proof of Lemma 46.

Proposition 47. The following rule is admissible in our restricted proof system

\[
Θ[s/x,t/y]; Γ[s/x,t/y] ⊢ ψ[s/x,t/y]
\]

Proposition 47 is proved by induction on proofs of \(Θ[t/x,t/y]; Γ[t/x,t/y] ⊢ ψ[t/x,t/y]\) and presents no surprise. Then, similarly to Lemma 34, Lemma 46 is proven by a routine induction on the proof of the desired sequent in LJ\textsc{BoundVarQ}, which allows us to complete the proof of Lemma 36.

Proof of Lemma 36. Assume \(Θ; Γ ⊢ ψ\) is derivable in LJ. Because of the subformula property, it is also derivable in LJ\textsc{Δ0} and thus, by Lemma 44, it is also derivable in LJ\textsc{BoundVarQ}. Then, Lemma 45 shows that it can be done using a right-focused proof, and then Lemma 46 allows us to conclude that \(Θ; Γ ⊢ ψ\) is derivable in the restricted system.
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Proof of Proposition 12, interpolation for the intuitionistic proof system

We restate the result, abusing notation by eliding the difference between membership contexts and Λ0 formulas, since it will not be crucial in this argument:

Let ΛL and ΛR be multi-sets each consisting possibly of formulas and membership contexts and ψ a formula. Let ̅t be the collection of variables that occur in ΛL and which also occur in ΛR, ψ. Then for every derivation

$$\Lambda_L, \Lambda_R \vdash \psi$$

there exists a Λ0 formula θ with free variables ̅t such that the following holds

$$\Lambda_L \models \theta \quad \text{and} \quad \Lambda_R, \theta \models \psi$$

Further, there is a polynomial-time algorithm which outputs Θ when given as input a formal derivation of ΛL, ΛR \vdash \psi.

We use induction on the complexity of the proofs, following the template presented in Fitting’s textbook [14], see also the expositions of this method in [6, 44]. We present here further representative cases of the rules, omitting many cases that are either trivial or similar to rules that are already covered below.

The base case consists of rules with no hypotheses.
Consider first the case of a proof consisting only of an application of the rule:

$$\Lambda, t \not\models t \vdash u \in T \nu$$

Note that t \not\models t is a Λ0 formula representing False, just as t =_{\mathcal{U}} t represents True.
If t \not\models t is in ΛL, we generate t \not\models t, while if it is in ΛR we generate t =_{\mathcal{U}} t.
For the hypothesis-free rule:

θ, t_0 \in_{\mathcal{U}} u, \Gamma, t_0 =_{\mathcal{U}} t_1, \ldots, t_{k-1} =_{\mathcal{U}} t_k \vdash t_k \in_{\mathcal{U}} u

we will generate t \in_{\mathcal{U}} u if t \in_{\mathcal{U}} u is in ΛL, and otherwise \neg(t \in_{\mathcal{U}} u).

We now consider the case where final application is the rule:

$$\Lambda, t \in_{\text{Set}(T)} \nu \vdash t =_{\text{Set}(T)} u$$

First consider the subcase where t \in_{\text{Set}(T)} \nu is in ΛL within the bottom sequent. Thus our goal is to find an interpolant θ’ which contains only variables common to ΛL, t \in_{\text{Set}(T)} \nu and ΛR, u \in_{\text{Set}(T)} \nu.
We apply the induction hypothesis with the same decomposition of the left side into L and R. It gives us a θ such that ΛL, t \in_{\text{Set}(T)} \nu \models \theta and ΛR, \theta \models x =_{\text{Set}(T)} u, and θ includes only variables that are common to ΛL, t \in_{\text{Set}(T)} \nu and ΛR, x =_{\text{Set}(T)} u. Thus all the variables in θ meet the criteria for θ’ except possibly for t.
We set θ’ = \exists \nu v. θ. The free variables in θ’ are those of θ other than t, and also v, and thus they meet the desired criteria.
It is easy to see using the properties of θ’ that ΛL, t \in_{\text{Set}(T)} \nu \models \theta’ and ΛR, \theta’ \models u \in_{\text{Set}(T)} \nu as required.
In the other subcase, where t \in_{\text{Set}(T)} \nu is in ΛR, we can apply the induction hypothesis as above and set θ’ = θ.
We now turn to the case where the last proof rule is:

$$\Lambda, z \in_T, t \models z \in_T u \quad z \not\in \text{FV}(\Lambda, t, u)$$

$$\Lambda \models t \subseteq_T u$$
We call the induction hypothesis on the top sequent, splitting the formulas the same way but putting $z \in \tau t$ in $\Lambda_R$. We can use the inductively formed interpolant directly.

Let us turn to the case where the last rule applied is:

\[
\Lambda, t \in \tau z, \varphi[t/y] \vdash v \in \tau w
\]

\[
\Lambda, t \in \tau z, \forall y \in \tau z \varphi \vdash v \in \tau w
\]

We first consider the subcase where $\forall y \in \tau z \varphi$ is in $\Lambda_R$ in the bottom. We can apply the induction hypothesis to the top sequent with the partition of formulas being the one induced from the partition on the bottom. The induction gives us a $\theta$ that may use the variable $t$, which may not occur in any formula within $\Lambda_R$ in the bottom sequent, and hence is not allowed in our interpolant for the bottom. If this happens, then this implies that $t \in \tau z$ is in $\Lambda_L$ on the bottom. In this case we set $\theta' = \exists y \in \tau z \varphi$. It is clear that $\Lambda_L, t \in \tau z \models \theta'$. Since $t$ does not occur in $\Lambda_L$ and $\varphi[t/y]$, $\theta \models v \in \tau w$ by induction, we conclude that $\Lambda_L, \varphi, \theta' \models v \in \tau w$ as required.

Now consider the subcase where $\forall y \in \tau z \varphi$ is in $\Lambda_L$ in the bottom sequent. We apply induction in the same way, to obtain $\theta$ as above. The only difficult case is when $t$ only occurs in formulas within $\Lambda_R$ on the bottom. In this case we can check that $\theta' = \forall y \in \tau \varphi$ can be used as the desired interpolant.

### F Proof of Lemma 13: the inductive invariant for the query synthesis algorithm

Recall the statement of Lemma 13, which gives the inductive invariant used in query synthesis:

Let $L, R$ be sets of variables, $\Theta = \Theta_l, \Theta_e, \Theta_r$ be a $\in$-context and $\Gamma = \Gamma_l, \Gamma_r$ be a context such that

\[
FV(\Theta_l) \subseteq L \cap R \quad FV(\Theta_e) \subseteq L \setminus R \quad FV(\Theta_r) \subseteq R \setminus L \quad FV(\Gamma_l) \subseteq L \quad FV(\Gamma_r) \subseteq R
\]

The following holds, assuming that $t$ and $u$ are terms of suitable types such that $FV(t) \in L \setminus R$ and $FV(u) \in R \setminus L$, and that $\bar{t}$ ranges over $L \cap R$:

- If $\Theta; \Gamma \vdash t =_T u$ is derivable, there is a NRC query $Q$ with free variables $\bar{t}$ of type $T$ such that $\Theta; \Gamma \models t = Q = u$.
- If $\Theta; \Gamma \vdash t \subseteq_T u$ is derivable, there is a NRC query $Q$ with free variables $\bar{t}$ of type $Set(T)$ such that $\Theta; \Gamma \models t \subseteq Q \subseteq u$.
- If $\Theta; \Gamma \vdash t \in_T u$ is derivable, there is a NRC query $Q$ with free variables $\bar{t}$ of type $Set(T)$ such that $\Theta; \Gamma \models t \in Q$.

Further the desired queries can be constructed in time polynomial in the size of the proof (e.g. measured in terms of the number of steps and the maximal size of a sequent in each step).

Note that the invariants imply that the sets $\Theta_A$ are pairwise disjoint, and that the following “downward closure property” holds.

For all variables $q$, if $q \in L \setminus R$ and $x \in q \in \Theta$, then $x \in q$ must be in $\Theta_l$ and hence $x$ must be in $L \setminus R$; and similarly when the roles of $R$ and $L$ are swapped.

**Proof.** We proceed by induction over the proof tree, calling $Q$ the desired output that we want to create in the inductive step.

- If the last proof rule used is a structural rule, we directly use the induction hypothesis.
- If the last proof rule used is $=_T-R$

\[
\Theta; \Gamma \vdash t \subseteq u \quad \Theta; \Gamma \vdash u \subseteq t
\]

\[
\Theta; \Gamma \vdash t = u
\]
then one has a query $Q'$ such that $\Gamma \vdash t \subseteq Q' \subseteq u$ by applying the induction hypothesis on the first subproof. Since the system is sound, we do have $\Gamma \vdash t = u$, so $\Gamma \vdash t = Q' = u$. We can thus take $Q = Q'$.

- If the last proof rule used is $=_{\text{R}}$

\[
\Theta; \; \Gamma \vdash \pi_1(t) =_{T_1} \pi_1(u) \quad \Theta; \; \Gamma \vdash \pi_2(t) =_{T_2} \pi_2(u) \\
\Theta; \; \Gamma \vdash t =_{T_1 \times T_2} u
\]

The induction hypothesis yields queries $Q_1$ and $Q_2$ such that

\[
\Theta; \; \Gamma \vdash \pi_1(t) = Q_1 = \pi_1(u) \quad \text{and} \quad \Theta; \; \Gamma \vdash \pi_2(t) = Q_2 = \pi_2(u)
\]

It suffices to take $Q = (Q_1, Q_2)$.

- If the last proof rule used is $=_{\text{Unit-R}}$

\[
\Theta; \; \Gamma \vdash t =_{\text{Unit}} u
\]

Then the constant query returning the unique element of Unit works.

- If the last proof used is $=_{\text{L-R}}$

\[
\Theta, \; t \in_U z; \; \Gamma \vdash u \in_U z \quad z \notin \text{FV}(\Theta, \Gamma, t, u) \\
\Theta; \; \Gamma \vdash t =_{U} u
\]

The induction hypothesis gives us a query $Q'$ of type $\text{Set}(U)$ such that

\[
\Theta; \; \Gamma, \; t \in_U z \vdash u \in Q'
\]

Furthermore, by using interpolation, there is a formula $\theta(\vec{i}, z)$ such that

\[
\Theta_I, \Theta_L; \; \Gamma_L, \; t \in_U z \vdash \theta(\vec{i}, z) \quad \text{and} \quad \Theta_R; \; \Gamma_R, \; \theta(\vec{i}, z) \vdash u \in_U z
\]

Since $z$ is fresh, this means that we have

\[
\Theta; \; \Gamma, \; \theta(\vec{i}, z) \vdash z = \{t\} = \{u\} \quad \text{and} \quad \Theta; \; \Gamma \vdash u \in Q'
\]

So we may take $Q$ to be the unique element of $\{x \in Q' \mid \theta(\vec{i}, \{x\})\}$, which can be formally defined in NRC as

\[
Q = \text{Get} \left( \bigcup \{\text{case}(\text{Verify}_\theta(\vec{i}, \{x\}), \{x\}, \theta \mid x \in Q') \} \right)
\]

- If the last proof rule used is $\subseteq_{\text{R}}$

\[
\Theta, \; z \in_T t; \; \Gamma \vdash z \in_T u \quad z \notin \text{FV}(\Theta, \Gamma, t, u) \\
\Theta; \; \Gamma \vdash t \subseteq_T u
\]

then the inductive hypothesis gives us a query $Q'(\vec{i})$ such that

\[
\Theta; \; \Gamma \vdash z \in Q'
\]

Apply interpolation to the premise so as to obtain a $\Delta_0$ formula $\theta(\vec{i}, z)$ with

\[
\Theta_I, \Theta_L; \; \Gamma_L, \; z \in t \vdash \theta(\vec{i}, z) \quad \text{and} \quad \Theta_R; \; \Gamma_R, \; \theta(\vec{i}, z) \vdash z \in u
\]

In this case, we take

\[
Q(\vec{i}) = \{z \in Q'(\vec{i}) \mid \theta(\vec{i}, z)\}
\]

which is NRC-definable as

\[
\bigcup \{\text{case}(\text{Verify}_\theta(\vec{i}, z), \{z\}, \theta) \mid z \in Q'(\vec{i})\}
\]

Now, let us assume that $\Gamma$ is valid and show that $t \subseteq Q$ and $Q \subseteq u$. 

Suppose that \( z \in q \). By the induction hypothesis, we know that \( z \in Q' \). But we also know that \( \Gamma_L \) is valid, so that \( \theta(t, z) \) holds. By definition, we thus have \( z \in Q \).

Now suppose that \( z \in Q' \), that is, \( z \in Q \) and \( \theta(t, z) \) holds. The latter directly imply that \( z \in Q' \) since \( \Gamma_R \) is valid.

If the last proof rule used is \( \in_{\text{Set}} \cdot \Gamma \)

\[
\frac{\Theta, t \in_{\text{Set}(T)} v; \Gamma \vdash u \in_{\text{Set}(T)} u}{\Theta, t \in_{\text{Set}(T)} v; \Gamma \vdash u \in_{\text{Set}(T)} v}
\]

then, by using the induction hypothesis on the premise, we get a query \( Q' \) which is equal to \( u \) assuming \( \Theta, t \in_{\text{set}(T)} v; \Gamma \). So we may take \( Q = \{Q'\} \).

If the last proof rule used is \( \in_{\text{U} \cdot \Gamma} \)

\[
\frac{\Theta, t_0 \in_{\text{U}} u; \Gamma, t_0 =_{\text{U}} t_1, \ldots, t_{k-1} =_{\text{U}} t_k \vdash \Theta \in_{\text{U}} v}{\Theta, t_0 \in_{\text{U}} u; \Gamma, t_0 =_{\text{U}} t_1, \ldots, t_{k-1} =_{\text{U}} t_k \vdash \Theta \in_{\text{U}} v}
\]

then, we make a case analysis:

If \( \text{FV}(u) \setminus L \) is non-empty, then \( \text{FV}(t_0) \subseteq R \setminus L \). This means that there is necessarily some \( t_i \) such that \( t_i =_{\text{U}} t_{i+1} \) is a left formula and, either \( i = 0 \) or \( t_{i-1} =_{\text{U}} t_i \). In any case, this enforces that \( \text{FV}(t_i) \subseteq L \cap R \). One can then regard \( t_i \) as a \( \text{NRC} \) query and pick \( Q = \{t_i\} \) as output.

Otherwise \( \text{FV}(u) \subseteq L \cap R \), in which case \( u \) can be regarded as a \( \text{NRC} \) query; we then take \( Q = u \).

If the last rule used is \( \neq_{\text{U} \cdot \Gamma} \)

\[
\frac{\Theta, \Gamma; t_0 =_{\text{U}} t_1, \ldots, t_{k-1} =_{\text{U}} t_k \neq_{\text{U}} t_k \vdash \Theta \in_{\text{U}} v}{\Theta, \Gamma; t_0 =_{\text{U}} t_1, \ldots, t_{k-1} =_{\text{U}} t_k \neq_{\text{U}} t_k \vdash \Theta \in_{\text{U}} v}
\]

then any query can be used, since the premise is a contradiction.

If the last proof rule used is \( \times \)

\[
\frac{\Theta[t_i/y]; \Gamma[t_i/y] \vdash (t \in_{T} u)[t_i/y]}{\Theta; \Gamma \vdash (t \in_{T} u)[t_i/y] \text{ for } i \in \{1, 2\}} \quad \frac{\Theta; \Gamma, u =_{\text{U}} t \vdash v \in_{T} w}{\Theta; \Gamma, u =_{\text{U}} t \vdash v \in_{T} w}
\]

the query obtained using the induction hypothesis allows to directly conclude.

If the last proof rule used is \( \times \eta \)

\[
\frac{\Theta[(x_1, x_2)/x]; \Gamma[(x_1, x_2)/x] \vdash (t \in_{T} u)[(x_1, x_2)/x]}{\Theta; \Gamma \vdash (t \in_{T} u)\text{ for } x_1, x_2 \notin \text{FV}(\Theta; \Gamma, t, u)\text{ }}
\]

then the induction hypothesis yields a query \( Q' \). If \( x \notin L \cap R \), then we also have the same for \( x_1 \) and \( x_2 \) in the premise so that \( Q' \) has exactly the right variables. Otherwise, \( x_1 \) and \( x_2 \) are among the free variables of \( Q' \) and \( x \in L \cap R \). Writing \( Q'(\vec{z}, x_1, x_2) \) to display the free variables, it suffices to set

\[
Q(\vec{z}, x) = Q'(\vec{z}, \pi_1(x), \pi_2(x))
\]

If the last proof rule is \( \bot \)

\[
\frac{\Theta; \Gamma, \bot \vdash t \in_{T} u}{\Theta; \Gamma, \bot \vdash t \in_{T} u}
\]

then, as above, any query can be used since the premise is contradictory.

If the last proof rule is \( \wedge \)

\[
\frac{\Theta; \Gamma, \phi \vdash t \in_{T} u}{\Theta; \Gamma, \phi \wedge \psi \vdash t \in_{T} u}
\]

one may directly take the query given by the induction hypothesis.
Generating collection queries from proofs

If the last proof rule used is $\lor$-L

\[
\frac{\Theta; \Gamma, \phi \vdash t \in T \quad \Theta; \Gamma, \psi \vdash t \in T}{\Theta; \Gamma, \phi \lor \psi \vdash t \in T}
\]

the induction hypothesis yields queries $Q_1$ and $Q_2$ of sort $\text{Set}(T)$ such that

\[
\Theta; \Gamma, \phi \models t \in Q_1 \quad \text{and} \quad \Theta; \Gamma, \psi \models t \in Q_2
\]

So we may take $Q = Q_1 \cup Q_2$.

If the last proof rule used is $\forall$-L

\[
\frac{\Theta, t \in T \; ; \; \Gamma, \phi[t/x] \vdash v \in T' \; w}{\Theta, t \in T \; ; \; \Gamma, \forall x \in T \; \phi \vdash v \in T' \; w}
\]

then we have the guarantee that either $\text{FV}(\phi[t/x]) \subseteq L$ or $\text{FV}(\phi[t/x]) \subseteq R$ according to whether $\forall y \in T \; \phi$ belongs to $\Gamma_L$ or $\Gamma_R$. In any case, one may directly use the query obtained via the induction hypothesis.

If the last proof rule used is $\exists$-L

\[
\frac{\Theta, x \in T \; y; \; \Gamma, \phi \vdash t \in T \; v \quad x \notin \text{FV}((\Gamma, y, t, v))}{\Theta, \exists x \in T \; y \; \phi \vdash t \in T \; v}
\]

then we have two cases: either $y \notin L \cap R$, in which case the induction hypothesis gives a query which has exactly the right set of variable and we may use that to conclude. Otherwise, $x$ is among the free variables of the query $Q'(\vec{i}, x)$ given by the induction hypothesis, as well as $y$. We may thus set

\[
Q = \bigcup \{Q'(\vec{i}, x) \mid x \in y\}
\]

to conclude.