Knot Invariants and New Weight Systems from General 3D TFTs

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Abstract

We introduce and study the Wilson loops in a general 3D topological field theories (TFTs), and show that the expectation value of Wilson loops also gives knot invariants as in Chern-Simons theory. We study the TFTs within the Batalin-Vilkovisky (BV) and Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) framework, and the Ward identities of these theories imply that the expectation value of the Wilson loop is a pairing of two dual constructions of (co)cycles of certain extended graph complex (extended from Kontsevich’s graph complex to accommodate the Wilson loop). We also prove that there is an isomorphism between the same complex and certain extended Chevalley-Eilenberg complex of Hamiltonian vector fields. This isomorphism allows us to generalize the Lie algebra weight system for knots to weight systems associated with any homological vector field and its representations. As an example we construct knot invariants using holomorphic vector bundle over hyperKähler manifolds.
1 Introduction

Knots are embeddings of $S^1$ into some ambient space which is usually $\mathbb{R}^3, S^3$. It is the global feature of the knot, namely, how it is embedded that is of the most interest. One is interested in studying the cohomology of the space of embeddings, which we denote as $\text{Imb}$. This space is disconnected, different components are separated by walls corresponding to singular knots. This leads to the natural definition of knot invariants as $\tilde{H}^0(\text{Imb})$ whose elements are locally constant functions on $\text{Imb}$. The knot polynomials are classes of this
group that behave multiplicatively under the multiplication of knots. For example, the well
known Jones polynomial \[19\] assigns
\[q^{-(n-1)/2}(-q - 1)^{n-1},
q^{-2}(1 - q + q^2 - q^3 + q^4)\]
and
\[q(1 + q^2 - q^3)\]
to the knots in Fig. 1.

Interestingly a simple device called the chord diagram and its extensions appeared repeatedly in the study of knots. A chord diagram by definition consists of a circle and bunch of chords connecting pairs of points of the circle. First of all, if one denotes by \(\Sigma\) the complement of \(\text{Imb}\) in the space of \(C^\infty\) mappings of \(S^1\) into \(S^3\) or \(\mathbb{R}^3\), one can gain knowledge about \(\tilde{H}(\text{Imb})\) by computing \(\tilde{H}(\Sigma)\). The chord diagrams made their appearance here as the labelling of the cells of \(\Sigma\). Vassiliev used this device in his direct computation of the group \(\tilde{H}(\Sigma)\) [38].

The chord diagram and its extension can also be given the structure of a differential complex \((\mathcal{G}, \delta)\), which leads to another independent construction of classes in \(H^0_{\text{dR}}(\text{Imb})\) inspired by Chern-Simons (CS) perturbation theory, due to Bott and Taubes [7] and many others. They studied the de Rham instead of the singular cohomology of \(\text{Imb}\) through integrating certain tautological forms over the configuration space (this step is known as the transfer map). This construction builds 'models' for classes in \(H^0_{\text{dR}}(\text{Imb})\) and the models are labelled by the cycles of the extended graph complex (all of these will be reviewed later).

After carefully blowing up the singular points of the configuration space, which for a physicist means the regularization of short distance singularity, the de Rham differential in \(\Omega(\text{Imb})\) is related through the transfer map to the differential \(\delta\) of the graph complex [9]. More precisely, there is a homomorphism

\[
(\mathcal{G}, \delta) \xrightarrow{\text{transfer}} (\Omega(\text{Imb}), d).
\]

Hence by feeding cycles in the graph complex to the lhs of Eq. [11] one obtains knot invariants from the rhs. At the same time, Kontsevich invented a different configuration space integral including only the chord diagrams; it is believed that two constructions give the same answer [22].

The configuration space integral construction is inspired by the CS perturbation expansion, where the extended graphs are none other than the Feynman diagrams, and the transfer
map is just the Feynman integral while the blowing up of singular points are some sort of Cauchy principle value prescription for regulating the divergences. The CS theory in 3D, like a myriad of other 3D TFT’s can be neatly formulated in the Batalin-Vilkovisky (BV)-Alexandrov-Kontsevich-Schwarz-Zaboronsky (AKSZ) framework [2, 30]: the BV language handles the gauge fixing problem with ease while the AKSZ construction throws the geometrical aspect of the theory into a sharper focus. We shall show in this paper that the graph differential of Eq.1 naturally arises out of the (rather simple) BV yoga and the homomorphism between $\mathcal{G}$ and $\Omega (\text{Imb})$ is one of the manifestations of the Ward identity. The BV machinery is particularly handy in demonstrating the equivalence between the Kontsevich integral and the integrals in Bott Taubes’s construction.

The direct proof of the homomorphism Eq.1 is in fact a one-line proof, but we take a detour of first proving the isomorphism between the said graph complex and certain extension of the Chevalley-Eilenberg (CE) complex, which is a generalization of Kontsevich’s earlier result [22]. Then we prove the homomorphism from the CE complex to the de Rham complex. The first isomorphism gives what is known as the weight system: by constructing the cycles in the CE complex, one finds cycles for the graph complex which can be fed to the transfer map to produce knot invariants. So this detour allows one to interpret the expectation value of the Wilson loop in CS theory as the pairing between two dual constructions of the graph complex. The similar interpretation of the 3D TFT partition function as such a pairing was pointed out in ref.[24] and explained in detail in our earlier work [27]. What is more important, this detour makes it clear what objects can be used as a weight system. We show in this paper that any representation of a homological vector field or a $Q$-structure can be used as a weight system. The $Q$-structure by definition is a deg 1 vector field on a graded manifold (GrMfld) with $Q^2 = 0$. A representation of $Q$ is an extension $Q + Q_R$, which acts on some vector bundle over the GrMfld, such that $(Q + Q_R)^2 = 0$. Our main result is that if $Q$ admits a non-trivial Hamiltonian lift, then we can define a weight system for knot invariants. So far most of the weight systems come from the Lie algebras, which is just one special case of the above general $Q$-structure. Our construction is inspired by the work of Rozansky and Witten [32] and Sawon [33]. The Rozansky-Witten (RW) weight system was also discussed in ref.[29], but the discussion there involves much more sophisticated machinery. The necessity of weight systems other than those from the Lie algebras is called for after the work of Vogel [39], who disproved the conjecture that all weight systems come from semi-simple Lie algebra (the stronger version, which drops the word semi-simple is also believed to be true).

The paper is organized as follows: we first review the construction of knot invariants from Chern-Simons perturbation theory in sec[2], there we demonstrate a recurring theme of this
paper which is the factorization of the partition function or the expectation value of a Wilson loop into the pairing of two dual constructions of graph co(cycle). After giving a ‘picturesque motivation’ for general weight systems, we move on to construct 3D TFT’s whose Feynman rules correspond exactly to these weight systems. For these theories Wilson loops analogous to the CS theory can be constructed and we propose the definition of representation of $Q$-structure in sec.4. And we show that as far as the perturbation expansion is concerned, the calculation proceeds exactly as that of CS theory, and hence non-pathological. In sec.6 we generalize the definition of the graph complex to incorporate the Wilson loops. Even though this was already done by Cattaneo et al [8], we show that the graph differential follows from the Ward identity in the BV formalism and an isomorphism between the graph complex and certain extended Chevalley-Eilenberg complex. Sec.6.3 serves as a tribute to the beautiful work of Bott and Taubes and a concrete example of the our abstract BV manipulation. In sec.7 we perform a low order sample calculation for the weight system of a $Q$-structure and its representation associated with hyperKähler manifolds and holomorphic vector bundles. The knot invariant takes value in $H_\partial$ instead of complex number, which is the main novelty of our result. We point out some loose ends in sec.8 which contains an apologetic review of Vogel’s work as a justification of our considering new weight systems, and some speculations of the nature of these weight systems that are intrinsically different from the conventional ones.

2 Knot Invariants from Chern-Simons Theory

In this section we quickly review the construction of knot invariants and 3-manifold invariants from Chern-Simons theory initiated by Witten [40], for a nice review see [25].

Let $G$ be a simple Lie group and consider the connection $A$ of a principle $G$-bundle over some dim 3 manifold $\Sigma_3$. The Chern-Simons theory is defined by the path integral as

$$Z = \int \mathcal{D}A \exp \left( \frac{ik}{4\pi} \int_{\Sigma_3} \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \right),$$

where one integrates over all the gauge equivalence classes of connections $A$ weighted by the exponential of the Chern-Simons functional. In this expression, $k \in \mathbb{Z}$ is the Chern-Simons level. Since the theory is formulated with only differential forms over $\Sigma_3$ and the partition function $Z$ is expected to be a topological invariant of $\Sigma_3$. In the large $k$ limit, the path integral is done using the stationary phase approximation. The stationary points, which is the solution to the equation of motion, are given by the flat connections. One then breaks up the gauge field into the flat background connection $A_i$ and
the fluctuation $B$: $A = A_i + B$. $Z$ is the sum of $Z_{A_i}$ where $A_i$ range over all gauge equivalence classes of flat connection. $Z_{A_i}$ itself is obtained by integrating over the fluctuation $B$. To integrate over the fluctuations gauge fixing is needed to ensure we are not counting gauge equivalent fluctuations. This is commonly done by imposing the Lorentz gauge (with the help of a metric on $\Sigma_3$). The gauge fixed action around $A_i$ is

$$S_{GF} = S(A_i) + \frac{ik}{4\pi} \mathrm{Tr} \int_{\Sigma_3} B d_i B + \frac{2}{3} B^3 + \phi d_i^3 B + \bar{c} d_i^3 (d_i c + [B, c]) ,$$

where $d_i$ is the gauge covariant derivative with connection $A_i$. Integrating out $\phi$ would put us on the Lorentz gauge, while the ghost anti-ghost $c, \bar{c}$ provides the Fadeev Popov determinant.

To the lowest order in $1/k$, there is the one loop determinant. The norm of the determinant is the Ray-Singer torsion at $A_i$ which is a topological invariant. The phase of the determinant is more subtle: a gravitational Chern-Simons term is needed to remedy the anomalous transformation. We gloss over this point as it is not central to the paper.

Beyond the one loop determinant factor, the higher order perturbation expansion comes from the Feynman diagram calculation. For this, it is rather expedient to assemble the various fields into a super field. Introduce an odd coordinate $\theta^a$ that transforms like 1-form $dx^a$ on $\Sigma_3$. Define a super field

$$A = c + \theta^a B_a + \frac{1}{2} \theta^b \theta^a \tilde{A}_{ab} ,$$

where $\tilde{A} = d_i^3 \bar{c}$. We note that $\bar{c}$ was a 3-form originally, and the change of variable from $\bar{c}$ to $\tilde{A}$ causes a Jacobian which is offset exactly by the Jacobian from integrating out $\phi$. The gauge fixing condition is now neatly summarized by saying that $A$ is co-exact w.r.t $d_i$, and the action condenses to become

$$S_{GF} = S_{A_i} - \frac{ik}{4\pi} \int_{\Sigma_3} d^3 x d^3 \theta \mathrm{Tr} \left( A D_i A + \frac{2}{3} A^3 \right) ,$$

where $D_i A^\alpha = \theta^a (\partial_a \delta^\alpha_\gamma + i f^\alpha_\beta\gamma A_\beta^\gamma) A^\gamma$ is the covariant derivative in the super language and $A = A^\alpha t_\alpha$. The perturbation theory can be carried out using the super Feynman rules [4]. It is also worth mentioning that the form of the gauge fixed action in super language is the motivation of the AKSZ construction of general TFT’s.

The perturbation theory at each order of $1/k$ are also topological invariants of $\Sigma_3$. The invariance can actually be explained by exploring the relation between Feynman integral and certain graph complex, which was first defined by Kontsevich [24]. The proper definition of the graph complex is collected in the sec6.2 but for now suffice it to say that Feynman
diagrams are examples of graphs and that the graph complex is equipped with a differential, which acts on a graph by shrinking in turn each of its propagators.

For the perturbation calculation one first expands the cubic vertex in Eq.3 down the exponential and contract all the fields using the propagators. The resulting Feynman diagrams are of course all tri-valent. Then for a Feynman diagram $\Gamma$, one integrates the propagators (with Lie algebra data stripped off) over the positions of the vertices on $\Sigma_3$; this step assigns each diagram a number $b_\Gamma$. And at the same time, since each cubic vertex carries a structure constant $f^{abc}$ with it, the indices of $f$ will be contracted together according to the given $\Gamma$, resulting in a number $c_\Gamma$. Kontsevich realized that $b_\Gamma, c_\Gamma$ can be used to construct

$$
\sum_\Gamma b_\Gamma \Gamma^* \quad \text{and} \quad \sum_\Gamma c_\Gamma \Gamma.
$$

The former is a cocycle in the graph complex which can be shown using integration by parts; while the latter is a cycle basically due to the Jacobi identity. The partition function $Z = \sum_\Gamma b_\Gamma c_\Gamma$ is just the pairing of the two dual constructions of the graph complex

$$
Z = \langle \sum_\Gamma b_\Gamma \Gamma^*, \sum_\Gamma c_\Gamma \Gamma \rangle = \sum_\Gamma b_\Gamma c_\Gamma.
$$

The topological invariance can now be explained roughly as: the change of metric changes each propagator by an exact form and integration by part of this exact form will cause the differential to hit neighboring propagators which gives delta functions and thereby shrinks the propagators one by one. This manipulation is exactly like the differential of the graph complex, in other words, the change of metric changes $b_\Gamma$ by a coboundary, hence

$$
\delta_g Z = \langle \sum_\Gamma (\delta_g b_\Gamma) \Gamma^*, \sum_\Gamma c_\Gamma \Gamma \rangle = \langle \delta(\cdots), \sum_\Gamma c_\Gamma \Gamma \rangle = \langle \cdots, \partial \sum_\Gamma c_\Gamma \Gamma \rangle = 0,
$$

where we denote by $\partial$ the differential of the graph complex, $\delta$ its dual and $\delta_g$ is the variation w.r.t. the metric.

Before moving on to Wilson loops, we mention in passing that for certain choices of $\Sigma_3$, an exact formula for $Z$ is known. This was done by first looking at $Z_{S^2 \times S^1}$, which is equal to 1, and through performing surgeries relating $Z_{S^2 \times S^1}$ to other $\Sigma_3$’s such as the 3-sphere or the lens space. Major effort has been poured into this arena [18, 31, 13] making it possible to obtain some exact results.

More interesting is the case when there are Wilson loops in the theory. The Wilson loop is given as

$$
W_R = \text{Tr}_R \exp \left( \oint dt A \right),
$$

\footnote{It has been established that $b_\Gamma$ is finite by Axelrod and Singer [1]}

6
where $\mathbb{P}$ means the path ordering and the trace is taken over representation $R$ of the Lie algebra.

Due to the metric independence, the expectation value

$$\langle W_R \rangle = \int DA \, Tr_R \mathbb{P} \exp \left( \oint A \right) e^{S_{CS}}$$

is invariant under continuous deformation of the Wilson loop and so characterizes the topological information of how the Wilson loop is embedded inside $\Sigma_3$, in other words, these are knot invariants by construction.

The fact that the perturbation theory produces knot invariants can be likewise analyzed by taking an excursion to the graph complex, this time slightly generalized. Now one expands both the cubic vertex in the action and the Wilson loop operator into power series. We call vertices from the action the internal ones and those from the Wilson loop peripheral ones, then the propagators will run among all the vertices. It is customary to include an oriented loop into the Feynman diagram to signify the Wilson loop. The resulting Feynman diagrams are tri-valent for the internal vertices and uni-valent for the peripheral ones. These will be an example of the extended graph complex which we define in sec.6.2, for now we just say that there is a properly generalized differential for such graph complex as well.

The Feynman integral prescribes that we should integrate over the entire $\Sigma_3$ the positions of internal vertices, while for the peripheral ones along the Wilson loop and respecting the cyclic order. Similar to the situation earlier, this step assigns a number $b_{\Gamma}$ for every $\Gamma$. The number $c_{\Gamma}$ is also obtained likewise from Lie algebra data. And no surprise that the cochains and chains defined in Eq.4 will remain cocycles and cycles once we properly modify the definition of the differential of the extended complex. This time the deformation of the Wilson loop will change $b_{\Gamma}$ by a coboundary and invariance of the path integral can be analyzed as in Eq.5. Note the $c_{\Gamma}$ for the extended graphs are called weight systems for knots.

The knot polynomials in the introduction are characterized by the so called skein relation. In practice, one calculates the knot invariant by untying the knot until one reaches the trivial knot. In the process, one needs to let two strands of the knot pass each other, and the skein relation dictates the change to the value of the knot polynomial incurred in each such passing. Finally, the value of the knot polynomial is the sum of the cumulative changes. So obtaining the skein relation is absolutely central. In certain situation the skein relation in the Chern-Simons theory can also be obtained exactly, also through the surgery formula. What one does is to gouge out of $\Sigma_3$ a small ball containing two strands of the Wilson loop and glue back the ball but twisted by a diffeomorphism of the boundary. In the few Jones polynomials listed under fig.1 the $q$ is just $\exp \left( 2\pi i/(k + 2) \right)$, where $k$ is the Chern-Simons level.
So far we have seen that we can break apart the partition of Chern-Simons theory with or without Wilson loops into the two independent parts of Eq.\ref{eq:4} making it easier to study/generalize them. In what comes next, we will first generalize the Lie algebra weight system, since it has been an unresolved problem if all weight system comes from Lie algebras. We saw from Eq.\ref{eq:5} that the only requirement for $c_\Gamma$ is that $\sum_{\Gamma} c_\Gamma \Gamma$ should be a cycle in the graph complex. The key quality of the Lie algebra weight system that meets this requirement is the Jacobi identity. Indeed, take three graphs that are identical except in two of the vertices as depicted in fig.\ref{fig:2}. The Feynman rule from the Lie algebra weight system assigns

\[ c_s = f^{ade} f^{bce}, \quad c_t = f^{abe} f^{cde}, \quad c_u = f^{bde} f^{cae} \]

to the $s$-, $t$- and $u$-channel diagram. The graph differential contracts the central propagator and all three channels collapse into the four point vertex. The Jacobi identity then says $c_s + c_t + c_u = 0$ indicating that this is a graph cycle. This graph relation is hieroglyphically denoted as the 'IHX' relation. As a variant to IHX, we can place the lower edge of I and both of the two vertices of H X on a Wilson loop with a given representation. This time IHX relation simply says the representation 'represents' the Lie algebra and is called the STU relation.

The general situation when a differential acts on a (not necessarily trivalent) graph is drawn in fig.\ref{fig:3}. This figure points out clearly that an $L^\infty$ algebra\footnote{$L^\infty$ structure is most easily formulated in terms of a homological vector field $Q$, $Q^2 = 0$. We use these two terms interchangeably in this paper.} with a hamiltonian

\[ \sum_{l=2}^{n-2} t \left( \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right) \]

Figure 2: Lie algebra weight system

Figure 3: $L^\infty$ algebra weight system
lift will be the natural generalization of Lie algebra weight system in the following way. The $L^\infty$ algebra is a generalization of Lie algebra in the sense that the Jacobi identity fails in a controlled manner. For simplicity, we consider the vector space with an even symplectic structure $(\mathbb{R}^{2n|m}, \Omega_{AB})$, take an odd Hamiltonian function $\Theta$ satisfying $\{\Theta, \Theta\} = 0$. Denote by $\Theta_{A_1\cdots A_n}$ the Taylor coefficient $\partial_{A_1\cdots A_n} \Theta$, and for the time being assume the Taylor coefficient of $\Theta$ has no linear or quadratic term. Then up to quartic order the identity $\{\Theta, \Theta\} = 0$ reads

$$\Theta_{ABE}(\Omega^{-1})^{EF}\Theta_{CDF} + \text{perm} A B C D = 0.$$  

The last relation is the generalization of the Jacobi identity. We can define the weight system by assigning the tensor $\Theta_{A_1\cdots A_l}$ and $\Theta_{A_{l+1}\cdots A_n}$ to the two vertices of fig.3. It is imaginable that after straightening out the sign factors, the rhs of fig.3 exactly equals $\partial_{A_1} \cdots \partial_{A_n} \{\Theta, \Theta\}$, i.e. the rhs of fig.3 is zero and we have graph cycles. The general case when $\Theta$ does have linear and quadratic Taylor coefficient corresponds to the ‘controlled breach’ of Jacobi identity mentioned earlier; and we will deal with this in sec.7.1.

To conclude, we need to invent a TFT such that the various Taylor coefficient of $\Theta$ will appear as interaction vertices. The AKSZ construction for TFT answers this call well.

3 AKSZ Topological Field Theory

We need a TFT that can incorporate the data of a Hamiltonian function satisfying $\{\Theta, \Theta\} = 0$. This is systematically achieved by the AKSZ construction. The reader may see ref.[27] for a more detailed account.

The data for the AKSZ construction is a triple $(\mathcal{M}, \Omega, \Theta)$, where $\mathcal{M}$ is a graded manifold (GrMfld) with deg2 symplectic structure $\Omega$. We use $X^A$ as the coordinate of $\mathcal{M}$. The fields in the theory will be the mappings

$$\text{Maps}(T[1]\Sigma_3, \mathcal{M}) ,$$

where $\Sigma_3$ is some 3D manifold. For this reason, we call $T[1]\Sigma_3$ the source manifold and $\mathcal{M}$ the target manifold. $T[1]\Sigma_3$ is itself a GrMfld, it is the total space of the tangent bundle of $\Sigma_3$. Denote by $(\xi^a, \theta^a)$ the coordinates of the base and fibre; $T[1]$ signifies that we assign $\theta^a$ degree 1. We see $\theta^a$ transforms just like $d\xi^a$ and with the same commutativity property. Any function $F(\xi, \theta)$ on $T[1]\Sigma_3$ can be expanded in power series of $\theta$ as

$$F(\xi, \theta) = F(\xi) + \theta^a F(\xi)_a + \frac{1}{2} \theta^b \theta^a F(\xi)_{ab} + \frac{1}{3!} \theta^c \theta^b \theta^a F(\xi)_{abc} .$$  

(6)
Each term in the expansion is just a differential form over $\Sigma_3$, and we call each component $F_{(0)}$, $F_{(1)}$, $F_{(2)}$, etc. The mapping $\text{Maps}(T[1] \Sigma_3, \mathcal{M})$ will also be described by the superfields $X^A = X^A(\xi, \theta)$ which can be likewise expanded.

The deg 2 symplectic form $\frac{1}{2}\Omega_{AB} dX^A \wedge dX^B$ induces naturally a deg $-1$ symplectic form on the mapping space $\text{Maps}(T[1] \Sigma_3, \mathcal{M})$ according to

$$\omega = \frac{1}{2} \int_{T[1] \Sigma_3} d^6z \left( \Omega_{AB} \delta X^A \delta X^B \right),$$

(7)

where we write $\xi, \theta$ collectively as $z$ and $d^6z \equiv d^3\theta d^3\xi$. And we can define a Laplacian $\Delta$ such that $\Delta^2 = 0$,

$$\Delta \equiv \int_{\Sigma_3} d^3\xi \left( \Omega^{-1} \right)^{AB} (-1)^{AB} \left( \frac{\delta}{\delta X^A(3)} \frac{\delta}{\delta X^B(0)} \frac{\delta}{\delta X^A(1)} \frac{\delta}{\delta X^B(2)} \right).$$

The action is

$$S = S_{\text{kin}} + S_{\text{int}} = \int_{T[1] \Sigma_3} d^6z \left( \frac{1}{2} X^A \Omega_{AB} D X^B - (-1)^{\lambda} \Theta \right), \quad D := \theta^a \partial_a,$$

(8)

where $D$ is just the de Rham differential written in super language. The kinetic term $X^A \Omega_{AB} D X^B$ is a sloppy notation, one should really pick up a Liouville form $\Xi$ such that $d\Xi = \Omega$ and write the kinetic term as $\Xi$. When $\partial \Sigma_3 = \emptyset$ the dependence on the choice of $\Xi$ drops once we expand $\Xi$ into components, see ref. [26]. The sign in front of $\Theta$ is $-(-1)^d$ for theories on $\Sigma_d$ and it is chosen such that the equation of motion reads $D X = \{ \Theta, X \}$.

The action is deceptively simple, the nontrivial part is to define the path integral. We need to first pick a Lagrangian submanifold (LagSubMfld) w.r.t $\omega$ of Eq.7. We recall a submanifold $L$ of a symplectic manifold $(\mathcal{N}, \omega)$ is Lagrangian iff $L$ is maximal submanifold such that $\omega|_L = 0$ (in finite dimensional setting it is middle dimensional submanifold). For our application here, $L \in \text{Maps}(T[1] \Sigma_3, \mathcal{M})$ must be chosen such that the symplectic form Eq.7 vanishes when restricted to it. The choice of $L$ is called gauge fixing since it generalizes the BRST gauge fixing procedure [34]. The path integral integrates the super fields $X$ over $L$,

$$Z_{\text{AKSZ}} = \int_{\mathcal{L}} D X \exp \left( - (S_{\text{kin}} + S_{\text{int}}) \right).$$

The key advantage of this construction is that the complicated gauge fixing issue is encapsulated in the choice of $L$. TFT models like the Rozansky-Witten (RW) model ref. [32] and RW model coupled to CS [21] both emerge from very simple AKSZ actions; these models
only assumed their sophisticated form after some particular choice of $\mathcal{L}$. But most importantly, the various Ward identities that are responsible for the metric independence and other invariance of these models are nothing more than the consequence of the following

$$0 = \int_{\mathcal{L}} \mathcal{D}X \Delta(\cdots).$$

Though it is not always easy to find $\mathcal{L}$, we can nonetheless analyze the general properties of the path integral by using this equation.

The construction may seem all too abstract, but as we have seen familiar theories like the CS fit snugly into this framework. For CS theory, one only need to start from $\mathcal{M} = g[1]$ with $g$ being some Lie algebra whose coordinate we call $A^\alpha$. The symplectic form is taken as the Killing form of the Lie algebra $\Omega_{\alpha\beta} = \eta_{\alpha\beta} = \text{Tr}[t_{\alpha}t_{\beta}]$ and $f_{\alpha\beta\gamma}A^\alpha A^\beta A^\gamma$ is used as $\Theta$. The CS action is the same as Eq.3

$$\int_{T[1]\Sigma_3} d^6z \left[ \eta_{\alpha\beta} A^\alpha D A^\beta + \frac{i}{3} f_{\alpha\beta\gamma} A^\alpha A^\beta A^\gamma \right],$$

except now the superfield $A^\alpha$ is un constrained. And it is not hard to verify that the constraint $A$ being co-exact as in Eq.3 corresponds to a choice of LagSubMfld on which the symplectic form

$$\omega = \frac{1}{2} \int_{T[1]\Sigma_3} d^6z \left( \eta_{\alpha\beta} \delta A^\alpha \delta A^\beta \right)$$

vanishes. The geometrical nature of the AKSZ construction allows one to quickly populate the spectrum of TFT’s [30, 10, 11, 26, 17], much of the theory remain unexplored, though.

We pause to ask, for an arbitrary $\mathcal{M}$, is the AKSZ theory even a sensible quantum theory? As far as the path integral formulation is concerned, there is no difficulty. Since the Gaussian integral will be understood after proper Wick rotations. The determinant $\det \Omega$ is nowhere vanishing, so once we have chosen a sign for $\sqrt{\det \Omega}$ there is no more ambiguity. However to quantize the AKSZ theory is quite a different thing. The quantization of the two special cases: CS or RW theory is worked out in ref.[3] and [32] respectively and is highly non-trivial. Within the scope of this paper, we can only proceed with the assumption that the general AKSZ theory is quantum mechanically non-pathological.

The true innovation of the AKSZ theory comes when one looks at the structure of the partition function. The Feynman rule or the weight system, $c_T$ assigns every vertex a tensor that is the Taylor expansion of $\Theta$. The Jacobi identity for Lie algebra is replaced with the more general relation $\{\Theta, \Theta\} = 0$. This was extensively explored in our previous paper [27],
showing that the path integral is a handy tool to construct characteristic classes related to Θ.

Now we need to include the Wilson loop into the general theory and thereby recruit any such Θ to act as weight systems for knots. The Wilson loop like any other observables in a TFT must be gauge invariant, which in the BV language implies the following

\[ \delta_B \mathcal{O} = \{ S_{\text{kin}} + S_{\text{int}}, \mathcal{O} \} = 0 \text{ and } \Delta \mathcal{O} = 0, \]

where \( \delta_B \) is the BRST transformation in BV language. The second condition is formally fulfilled for all 3D theories. By using

\[ \{ S_{\text{kin}}, f(\mathbf{X}) \} = -Df(\mathbf{X}) ; \quad \{ S_{\text{int}}, f(\mathbf{X}) \} = \{ \Theta, f \}(\mathbf{X}) , \quad f \in C^\infty(\mathcal{M}) \]

the first equation when written in components is

\[
\begin{align*}
0 &= \{ \Theta, \mathcal{O} \}_{(0)} , \\
\delta \mathcal{O}(0) &= \{ \Theta, \mathcal{O} \}_{(1)} , \\
\delta \mathcal{O}(1) &= -\{ \Theta, \mathcal{O} \}_{(2)} , \\
& \quad \vdots
\end{align*}
\]

where the subscript \((p)\) means the \(p\)-form component in the θ-expansion, see Eq.6. One can solve for this equation easily if \( \mathcal{O} \) is a line operator insertion by mimicking the formula for the parallel transport. Pick a curve \( \phi(t), t \in [0, 1] \) embedded in \( \Sigma_3 \), and we define the matrix \( U \) as the path ordered exponentiation

\[ U(t, 0) = \mathbb{P} \exp \left( - \int_0^t dt \theta^i T^i \right) = \mathbb{P} \exp \left( - \int_0^t dt T_t \right), \]

where \( T_t \) is a matrix defined as

\[ (T_t)^A_B = \partial_{\theta^i} T^A_B|_{\theta=0} = \partial_{\mathbf{X}}(X^B_{(0)})X^C_{(1)}. \]

Strictly speaking we should write in the exponential \( \phi^*(T_t) \): the pull back of the 1-form \( T_t \) on \( \Sigma_3 \) to the curve. The pull back \( \phi^* \) will be dropped from now on.

Assuming that

\[ T^A_B = (\Omega^{-1})^{AC} \partial_C \partial_B \Theta , \]

From now on, we adopt the notation that if any function of the target \( \mathcal{M} \) is in boldface, such as \( T \) in Eq.11 it is regarded as a function of the superfield \( \mathbf{X}^A \): \( T = T(\mathbf{X}) \).
we can quickly verify the relations
\[ \partial_t U_A^B(t, 0) = -T_{iC}^A U_B^C(t, 0), \quad \partial_t U_A^B(1, t) = U_A^C(1, t) T_{iB}^C, \]
\[ \sum_B (-1)^B T_A^B T_A^B + (-1)^A \{ \Theta, T_A^A \} = 0. \tag{12} \]

The second line is a consequence of \( \{ \Theta, \Theta \} = 0 \). The BRST transformation (defined in Eq.10) of \( U \) is
\[ \delta_B U_D^C(1, 0) = (-1)^{C+A} \int_0^1 dt U_A^C(1, t) \left( \partial_t (-DT_B^A + \{ \Theta, T_B^A \}) |_{\theta=0} \right) U_D^B(t, 0) \]
\[ = (-1)^{C+A} \int_0^1 dt U_A^C(1, t) \left( - \partial_t A^B + \partial_t \{ \Theta, T_B^A \} |_{\theta=0} \right) U_D^B(t, 0). \]

Integrate by part the \( t \) derivative and use relation Eq.12, we get besides a surface term the following
\[ (-1)^C \int_0^1 dt U_E^C(1, t) \left( (-1)^A T_{iA}^E T_B^A - (-1)^E T_A^E T_B^A + (-1)^E \partial_t \{ \Theta, T_B^E \} |_{\theta=0} \right) U_D^B(t, 0) \]
\[ = (-1)^C \int_0^1 dt U_A^C(1, t) \left( \partial_t (-1)^A T_E^A T_B^A + (-1)^E \{ \Theta, T_B^E \} |_{\theta=0} \right) U_D^B(t, 0) = 0. \]

The surface term is
\[ (-1)^{C+A} U_A^C(1, t) \left( T_B^A(t) \right) U_D^B(t, 0) \bigg|_0^1 = T_{iC}^A(1) U_D^B(1, 0) - (-1)^{C+A} U_A^C(1, 0) T_B^A(0). \]

This shows that if the curve is closed, the trace
\[ W = \sum_A (-1)^A U(1, 0)^A \]
is BRST invariant and hence a valid observable. The sign factor \((-1)^A\) is reminiscent of the interpretation due to Witten that the Wilson loop should be regarded as the partition of a quantum mechanics system attached to the curve.

In the following sections, we will understand the BRST invariance of the Wilson loop from a Lie algebra chain complex point of view, there the BRST invariance will also be related to the closeness of certain graph chain.

### 4 Q-Structures and Their Representations

In CS theory, the gauge fields perched on the Wilson loop can be in representations other than the adjoint one, the same can be achieved for a general TFT given by the data \((\mathcal{M}, \Omega, \Theta)\). We need first the notion of a representation.
For a GrMfld \( M \), a deg 1 vector field \( Q(X) \) satisfying \( Q^2(X) = 0 \) is called a homological vector field or a \( Q \)-structure. The previous nilpotent Hamiltonian function \( \Theta \) on \((M, \Omega)\) with \( \deg \Omega = 2 \) induces a \( Q \)-structure \( Q \cdot = \{ \Theta, \cdot \} \). Of course, any \( Q \)-structure on \( M \) has a Hamiltonian lift on \( T^* [2] M \) since \( T^* [2] M \) is trivially a symplectic GrMfld with a deg 2 symplectic form. But as we shall see later, \( \Theta(X) \) associated with a trivial lift of some \( Q(X) \) tends to give TFT’s with trivial perturbation theory. So it is the ‘genuine’ nilpotent Hamiltonian functions that are of interest to us.

For a \( Q \)-structure over \( M \), consider a vector bundle \( E \to M \). We assume the fibre coordinates of \( E \) are bosonic even though the generalization to graded vector bundle is straightforward (compare with Eq.12). Naming the coordinate of the fibre of \( E \) as \( \xi_a \), we define the representation of a \( Q \)-structure as an extension of \( Q \) into \( Q^R \), with \( Q^R_2 = 0 \),

\[
Q^R(X) = \{ \Theta(X), \cdot \} + R^a_{\ b}(X) \xi_a \frac{\partial}{\partial \xi_b}, \\
0 = Q^R_2 = \{ \Theta, R^a_{\ b} \} \xi_a \partial \xi_b + (R^a_{\ b} R^b_{\ c}) \xi_a \partial \xi_c \Rightarrow \{ \Theta, R \} + R^2 = 0, \tag{13}
\]

Here the matrix \( R \) may itself be a function of \( M \).

The trivial example is of course the representation of a Lie algebra. In this case, \( M \) is \( g[1] \): the linear space of Lie algebra with degree shifted up by 1. We name the coordinate \( A^\alpha \). The bundle \( E \) is the trivial one \( g[1] \times V \) with \( V \) being the representation of the Lie algebra. We extend \( Q \) as

\[
Q = f_{\alpha \beta \gamma} A^\alpha A^\beta \frac{\partial}{\partial A^\gamma} ; \quad Q^R = Q + 2i A^\alpha (t^i_\alpha) \xi_i \frac{\partial}{\partial \xi_j}, \tag{14}
\]

where \( t_\alpha \) is the some matrix satisfying \([t_a, t_b] = i f^c_{ab} t_c\), \( \eta_{\alpha \beta} \) is the Killing metric and \( \eta_{\gamma \delta} f^\delta_{\alpha \beta} = f^\gamma_{\alpha \beta} \) is totally antisymmetric.

For a general representation we can form a BRST invariant line insertion the same way as in Eq.11, it is written as

\[
U(1, 0) = \text{Tr}_{R^P} \exp \left( - \int_0^1 dt \theta^i R \right), \tag{15}
\]

where \( R \) is the matrix \( R^a_{\ b}(X) \). The pull back of 3D superfields (functions on \( T[1] \Sigma_3 \)) to 1D superfields (functions on \( T[1] S^1 \)) is again implicitly understood. The verification of BRST invariance is totally similar and more importantly one can check that under a change of trivialization of \( E \), the Wilson loop changes by a BRST exact piece. In the next subsection, we give some more examples of \( Q \)-structure, for some we can construct interesting representations, for others we have only the “adjoint” representation.

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*The authors would like to thank Ezra Getzler for pointing this out for us.*
4.1 Examples Q-Structure and Representations

As a cousin of the CS theory, there is the BF theory. The relevant GrMfld for the BF theory is $\mathcal{M} = g[1] \oplus g^*[1]$, where $g^*$ is the dual of the Lie algebra. We name the coordinate of the two summands as $A^\alpha$ and $B_\alpha$. Suppose $V$ is the representation of the Lie algebra, $E = \mathcal{M} \times (V \oplus V)$, with $\xi, \eta$ as the coordinates of the two copies of $V$. The $Q$ structure is the same as in Eq.14, the extension is

$$Q_R = Q + 2i(t_0)^j \left( A^\alpha(\xi_j \frac{\partial}{\partial \xi_j} + \eta_i \frac{\partial}{\partial \eta_j}) + B^\alpha \xi_i \frac{\partial}{\partial \eta_j} \right)$$

defines a representation of $Q$. In this expression, we have used the inverse metric $\eta^{\alpha \beta}$ to raise the index on $B$.

The action can also be written down according to the AKSZ construction

$$S_{BF} = \int_{T[1]\Sigma^3} d^6z B_\alpha D A^\alpha + B_\alpha f^\alpha_{\beta \gamma} A^\beta A^\gamma .$$

One can use this representation to form the Wilson loop in BF theory as in Eq.15. Due to the index structure on $A$ and $B$, any number of $A$’s can be traced together, while the $B$’s must be placed between two $A$’s. So what one gets is the ‘beaded’ Wilson loops. This type of Wilson loops and the knot invariants were studied in ref.\[8\].

The canonical representation for a Lie algebroid. A Lie algebroid is formulated most easily in the graded manifold language. The data required is a deg 1 GrMfld $\mathcal{M}$, such a GrMfld is necessarily of the form $L[1]M$, where $L$ is a vector bundle over some manifold $M$. We denote by $x^\mu, \ell^A$ the even and odd coordinates. Any deg 1 vector field is necessarily of the form

$$Q = 2\ell^A A^\mu_A(x) \frac{\partial}{\partial x^\mu} - f^A_{BC}(x) \ell^B \ell^C \frac{\partial}{\partial \ell^A} .$$

Requiring $Q^2 = 0$ puts constraint on the coefficients

$$A^\mu_{[A} \partial_{\nu} A^\mu_{B]} = A^\mu_C f^C_{AB} ,$$
$$A^\mu_A \partial_\mu f^D_{BC} + f^D_{AX} f^X_{BC} + \text{cyclic in } A_{BC} = 0 ,$$

(16)

$A^\mu_A$ is the called anchor and $f^A_{BC}$ is the structure function. The two conditions constitutes the definition of a Lie algebroid. The AKSZ theory which potentially calculates the characteristic classes of a Lie algebroid can be constructed easily \[11\].

It is interesting to come up with some representation of a Lie algebroid other than the adjoint one. So far we have only the canonical representation, which is abelian. Consider
the line bundle $\wedge^{top} L \otimes \text{vol}$, and a section $s$. $Q_R$ is given by

$$Q_R = Q + \ell^A (\nabla_\mu A_\mu^A + f_{AB}^B) ,$$

where $\nabla_\mu A_\mu^A$ denotes the divergence with respect to $\text{vol}$. The second term is in fact the modular class of the Lie algebroid $\theta = \ell^A (\nabla_\mu A_\mu^A + f_{AB}^B) \in L^\ast$. By construction, $\theta$ is $Q$-closed. Unfortunately, neither the adjoint nor the canonical representation for a general Lie algebroid gives any non-trivial Wilson loops. Note that we are using the term adjoint representation for Lie algebroid in a loose sense, the reader may consult ref.[1] for a more formal definition and more extensive discussion of representations. In general, we expect that some extra structure such as a metric for the Lie algebroid is required to give interesting Wilson loops and knot invariants.

The previous $Q$’s are all based on the Lie algebra-ish constructions, and the TFT constructed from them is Chern-Simons like. We can also have $Q$-structures on purely even GrMfld’s, in which case we need some odd parameters. As an example, for any flat vector bundle $E \to M$ with connection $\Gamma$, $d\Gamma + \Gamma \Gamma = 0$, we can form the following GrMfld $\mathcal{M} = E \oplus T[1]M$. If we name the coordinate of $M$ as $x^\mu$, the fibre of $E$ as $e^i$ and the (odd) fibre of $T[1]M$ as $v^\mu$. The odd coordinate $v^\mu$ transforms the same way as $dx^\mu$ and only plays the role of a parameter. The $Q$-structure is formed as

$$Q = v^\mu \frac{\partial}{\partial x^\mu} ; \quad Q_R = Q + v^\mu \Gamma^i_{\mu j} e^j \frac{\partial}{\partial e^i} .$$

$Q_R^2 = 0$ follows from the flatness condition of $\Gamma$. Once again, to form TFT’s with non-trivial perturbation theory, the fibre of $E$ must have a symplectic structure. In the Rozansky Witten model which is built on a hyperKähler manifold, $E$ will be the holomorphic tangent bundle equipped with the holomorphic symplectic form. The odd coordinate $v^\mu$ may be incorporated as a dynamic variable or be left simply as a parameter of the theory. The precise formulae of Rozansky Witten weight system and its generalization to holomorphic vector bundle will be given in sec.7.1 and it will be clear there that the knot invariant takes value in the Dolbeault-cohomology of the base manifold. Thanks to the work of Beauville and Fujiki [14], there are now two families of hyperkähler manifolds to all dimensions, so the weight system from these families provides substantial generalizations to the Lie algebra weight systems [28, 36].

There is a more tantalizing example related to integrable systems which also reaches to all dimensions. Take $\mathbb{R}^{2n}$ with the standard even symplectic form $1/2 \Omega_{\mu \nu} d\xi^\mu \wedge d\xi^\nu$, and suppose we have $n$ functions $f_i$, satisfying $\{f_i, f_j\} = 0$. Let us pick $n$ odd parameters $t^i$ and
form the nilpotent Hamiltonian function $\Theta = \sum_i t^i f_i$, and the AKSZ TFT

$$S = \int d^6 z \, \xi^\mu \Omega_{\mu \nu} D \xi^\nu + t^i f_i(\xi) \, .$$

We do not have a clear notion of a general representation for such case, but it is likely to be related to the foliation of the phase space $\mathbb{R}^{2n}$ by the conserved charges $f_i$. Nor do we have any clear understanding of the implication of this weight system, except that for a given integrable system, the perturbation theory is in fact a finite series due to the anti-commutativity of $t^i$.

## 5 Recipe for Calculation

Here we spell out the necessary detail for the perturbative calculation, even though all the details were already in ref.[27].

The path integral is defined by the Lagrangian submanifold $\mathcal{L}$ in the mapping space. The choice of $\mathcal{L}$ in general depends on the interaction term $\Theta(X)$. In this section we only focus on the free theory, which means we expand the Wilson loop operator into polynomials of fixed degree and compute its expectation value. Since the theory is free, the equation of motion $D X^A = 0$ says the saddle point corresponds to closed forms on $\Sigma_3$, in particular the 0-form component $X^A_{(0)}$ are constants. So we are expanding around a base point on the target space $M$. An alternative point of view is to take the full interacting theory, but evaluate the path integral using background field method around the constant modes.

From either point of view, the Lorentz gauge used in the CS theory can be applied. To define the Lorentz gauge, we need to pick a metric $h_{ab}$ for $\Sigma_3$, which allows us to define the adjoint operator $d^\dagger$ for de Rham differential $d$. As mentioned previously, the super field can be regarded as poly-forms over $\Sigma_3$, which permits the decomposition into harmonic (h), exact (e) and co-exact (c) part:

$$X^A = (X^A)^h + (X^A)^e + (X^A)^c \, .$$

The symplectic form is decomposed similarly,

$$\omega = \frac{1}{2} \int_{T[1] \Sigma_3} d^6 z \, \Omega_{AB}(\delta(X^A)^h \delta(X^B)^h + 2\delta(X^A)^e \delta(X^B)^c) \, .$$

One prominent feature is that in this symplectic form, the harmonic components decouple from the rest. This sector is the zero modes sector, which can be gauge fixed independently of the rest. Secondly, the choice $(X^A)^e = 0$ corresponds clearly to a valid
choice of the LagSubMfld for the nonzero mode sector. Thirdly, due to the property \(\int_{T[1]\Sigma_3} \delta^6 z \left( X^A \right)^c \left( X^B \right)^c = 0\), the quadratic term in the action drops out, which indirectly excludes all but the tri-valent vertices in the perturbative expansion (see the counting argument in sec.5.2 of [27]).

As an example, let us take \(\Sigma_3\) to be \(S^3\) and for the sake of clarity assume that \(M\) is vector space \(5\) with even symplectic structure \(\Omega\). Take odd nilpotent Hamiltonian \(\Theta\) and the corresponding representation \(R\) defined as automorphism of some vector bundle \(E\) over \(M\) with the property (13). The Taylor coefficients of \(\Theta\) and \(R\) will be used as vertex functions, while \(\Omega^{-1}\) appears in conjunction with the super propagator

\[
\langle X^A(z_1)X^B(z_2) \rangle = (\Omega^{-1})^{AB}G(z_1,z_2).
\]

Take fig.4 as an example, for which the time runs clockwise. The Feynman rules assigns to this figure

\[
(\partial_N \partial_L \partial_F \Theta(x_0)) \Omega^{NB} \Omega^{LC} \Omega^{FD} \Omega^{AS} \text{Tr}_E \left( \partial_A R(x_0) \partial_B R(x_0) \partial_S R(x_0) \partial_C R(x_0) \partial_D R(x_0) \right) \times \\
\int_{S^3} \delta^6 z \int_{0}^{1} d^2 z_0 \int_{0}^{1} d^2 z_1 \cdots \int_{0}^{1} d^2 z_4 G(z,z_1)G(z,z_2)G(z,z_4)G(z_0,z_3)
\]

\[= c_{\Gamma}(x_0) \times b_{\Gamma},\]

where we separated the “algebraic” factor \(c_{\Gamma}\) and the kinematic factor \(b_{\Gamma}\). Several points have to be clarified: The harmonic part of the super field does not participate in the perturbation theory; for \(\Sigma_3 = S^3\) we have only the harmonic 0 and 3-form \((X^A_0)^h, (X^A_3)^h\), we let the LagSubMfld of this sector be given by \((X^A_3)^h = 0\), leaving \((X^A_0)^h\) as a parameter, namely we choose \((X^A_0)^h = x_0\) as the base point for the Taylor expansion of \(\Theta, R\). When the target manifold is a vector space, it may be natural to choose \((X^A_0)^h = 0\) as the base point. Consequently, the first line of the formula is a constant (more generically, the function of

\[\text{In general curved case when dealing with the perturbation theory we will need to apply the exponential map, thus reducing eventually the problem to the linear one.}\]
zero modes); these are the \( c \Gamma \) in the notation of Eq.4. The second line of the formula gives \( b \Gamma \); it is what inspired the construction of Bott and Taubes, we discuss this in sec.6.3.

We have discussed the perturbation calculation only in the Lorentz gauge, but one is free to choose any gauge so long as it is compatible with the separation of zero modes above. In fact, Kontsevich’s integral formula for knot invariants \cite{22} is believed to originate from the same CS theory but with light cone gauge \cite{25}. The light cone gauge sets to zero the cubic vertex in CS action, as a result, there is no internal vertices in the perturbation theory. On the other hand, a general gauge fixing may lead to vertices other than trivalent ones.

The strength of BV-AKSZ construction of TFT is that we can discuss perturbation theory without specifying gauge fixing condition. In the next section, we derive some important Ward identities for the Wilson loop operator under the free theory, which is true for all gauge fixings with the same zero mode sector.

6 Extended Chevalley-Eilenberg and Graph Complex

In earlier section, we have touched upon the interplay of Wilson loops, weight systems and the graph complex. Much of this has been studied earlier, see ref.\cite{8, 5, 22} etc. In this section, we try to understand their relation using the BV formalism. In the BV framework, the CE differential and graph differential arise naturally as a consequence of some standard manipulation, avoiding the untidy analysis used in the discussion of the metric independence and graph relation in sec.2.

6.1 Extended CE Complex

We first introduce a Chevalley-Eilenberg (CE) complex taking value in certain modules. Let \((\mathbb{R}^{2n|m}, \Omega)\) be the super space with the standard even symplectic structure. We define an extended Chevalley-Eilenberg chain complex of the Lie algebra of the Hamiltonian vector fields. The complex is spanned by

\[
\text{CE}_{n,l} = \text{span}(X_{f_0} \wedge \cdots \wedge X_{f_n} \otimes (g_0 \otimes \cdots \otimes g_l)).
\]

(19)

And we use the abbreviation

\[
X_{f_0} \wedge \cdots \wedge X_{f_n} \otimes (g_0 \otimes \cdots \otimes g_l) \overset{\text{abbrev}}{\Rightarrow} (f_0, \cdots, f_n; g_0, \cdots g_l).
\]

(20)

Here \( f, g \) are functions over \( \mathbb{R}^{2n|m} \) with the constraint that \( f \) is at least quadratic and \( g \) is at least linear (the constraint maybe lifted leading to some interesting generalizations). \( X_f \)
is the Hamiltonian vector field induced from $f$. In the second factor cyclic permutation is allowed (in the graded sense): $(g_0 \otimes \cdots \otimes g_l) \sim (-1)^{(g_0+1)(g_1+\cdots+g_l)}(g_1 \otimes \cdots \otimes g_l \otimes g_0)$ and naturally $l + 1 \equiv 0$. Let $c_{n,l} = (f_0, \cdots, f_n; g_0, \cdots, g_l)$, the differential acting on it is given as:

\[
\partial c_{n,l} = \sum_{0 \leq i < j \leq n} (-1)^{s_{ij}}(\{f_i, f_j\}; f_0 \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n; g_0, \cdots, g_l)
- \sum_{0 \leq i \leq n; 0 \leq j \leq l} (-1)^{t_{ij}}(f_0, \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n; g_0 \cdots g_{j-1}, \{f_i, f_j\}; g_{j+1} \cdots g_l)
- \sum_{0 \leq j \leq l} (-1)^{u_{nj}}(f_0, \cdots f_n; g_j g_{j+1}, g_{j+2} \cdots g_l, g_1, \cdots g_{j-1}),
\]

\( (21) \)

\[ s_{ij} = (f_i + 1) \sum_{k=0}^{i-1} (f_k + 1) + (f_j + 1) \sum_{k=0}^{j-1} (f_k + 1) + (f_i + 1)(f_j + 1) + f_i, \]

\[ t_{ij} = (f_i + 1)(\sum_{k=i+1}^{n} (f_k + 1) + \sum_{k=0}^{j-1} (g_k + 1)) + \sum_{k=0}^{n} (f_k + 1) + \sum_{k=0}^{j-1} (g_k + 1), \]

\[ u_{nj} = \sum_{k=0}^{n} (f_k + 1) + (g_j + 1) + \sum_{k=0}^{j-1} (g_k + 1) \sum_{m=j}^{l} (g_m + 1). \]

One certainly recognizes the first line as the conventional differential of the CE complex of Hamiltonian vector fields. The last part resembles the Hoschild differential and is reminiscent of the standard bar-resolution of the Lie group, except that ours has the cyclic property. The second term is simply the action of the Lie algebra on the second factor of the chain.

The definition of the differential is not wanton, it arises out of one single Ward identity in the BV formalism, which was first realized by Schwarz \[35\],

\[
\int \Delta(\cdots) = 0 . \tag{22}
\]

We first list a few useful identities (see \[27\] for more details)

\[
\Delta(\int d^{2d} z f(\vec{X}(z))) = 0 ,
\]

\[
\Delta(\int d^{2d} z_1 f(\vec{X}(z_1)) \int d^{2d} z_2 g(\vec{X}(z_2))) = (-1)^f \int d^{2d} z \{f(\vec{X}(z)), g(\vec{X}(z))\} ,
\]

\[
\{\int d^{2d} z f(\vec{X}(z)), \int d^{2d} z g(\vec{X}(z))\} = (-1)^d \int d^{2d} z \{f(\vec{X}(z)), g(\vec{X}(z))\} ,
\]

\[
\{S_{kin}, f(\vec{X}(z))\} = (-1)^d Df(\vec{X}(z)) , \tag{23}
\]

where $f, g \in C^\infty(\mathcal{M})$ and $d = 3$ is the dimension of the source manifold. One can define a cochain $c^0_L$, which, when evaluated on a chain $(f_0, \cdots, f_n)$, is given by the path integral

\[
c^0_L \circ (f_0, \cdots, f_n) = \int \int d^6 z_0 f(z_0) \cdots d^6 z_n f(z_n) e^{-S_{kin}} .
\]

\[In the expression for $s_{ij}, t_{ij}$ and $u_{nj}$, $f, g$ actually mean $|f|, |g|$, namely the degree of $f, g$. Similar conventions will appear ubiquitously in the paper.\]
Although we study the formal properties of this path integral expression, it makes perfect sense within the perturbative theory. Applying identities Eq.22,23 on $e$ can show that the cochain is in fact a cocycle w.r.t the differential in Eq.21:

$$\partial^n c_L^{n,0} \circ (f_0, \cdots f_{n+1}) = c_L^{n,0} \circ \partial(f_0, \cdots f_{n+1}) = 0, \quad \forall f_i \in C^\infty(M).$$

The cocycle depends on the choice of the Lagrangian submanifold $L$, changing $L$ alters the cocycle by a coboundary

$$(c_L^{n,0}_+ - c_L^{n,0}) \circ (f_0, \cdots f_n) = c^{n-1,0} \circ \partial(f_0, \cdots f_n),$$

for some $\tilde{c}$, the detail is in ref.27. We usually drop the subscript $L$ on $c_L$.

Now we generalize to the case $c^{n,l}$, $l > 0$ by picking a loop in $\Sigma_3$ and strewing some $g_i$’s on it in the prescribed cyclic order.

Assume that the Wilson loop $\mathcal{K}$ is embedded in $\Sigma_3$ by the function $\phi^a(t)$, $t \in [0, 1]$, $a = 1, 2, 3$. We want to describe the integration of any form over the Wilson loop (a 1-cycle) in the super language. Take a 1-form $\psi_a d\xi^a$ on $\Sigma_3$ and rewrite it as a function of $T[1] \Sigma_3$: $\psi(\xi, \theta) = \psi_a \theta^a$, then tautologically we have

$$\int_{\mathcal{K}} \phi^* \psi = \int_{\mathcal{K}} dt \phi^a \partial \theta^a \psi = \int_{T[1] \mathcal{K}} dt d\theta^a \psi.$$ 

In the last step we have used the somewhat imprecise notation $d\theta^a$ to denote $\dot{\phi}^a \partial \theta^a$. Obviously there is well-defined operation of pulling back superfields from $T[1] \Sigma_3$ to $T[1] S^1$. We place $l + 1$ insertions $g_i$ on the loop with prescribed cyclic ordering; this may be written as

$$W^l(g_0, g_1, \cdots g_l) = \int_0^1 dt_0 d\theta_0^a g_0(z_0) \int_{t_0-1}^{t_0} dt_1 d\theta_1^a g_1(z_1) \int_{t_0-1}^{t_1} dt_2 d\theta_2^a g_2(z_2) \cdots \int_{t_0-1}^{t_{l-1}} dt_l d\theta_l^a g_l(z_l),$$

(24)

where $z$ denotes both $t, \theta^a$. The lower limit of the integrals is perhaps not what one is used to having in a Wilson loop. But these integration limits indeed describes $l + 1$ insertions that are distributed on the loop with fixed cyclic ordering. When all the insertions are the same, which is the case of a true Wilson loop, we can rewrite the lower limit in the conventional way

$$W^l(g, g, \cdots g) = (l + 1) \int_0^1 dt_0 d\theta_0^a g(z_0) \int_0^{t_0} dt_1 d\theta_1^a g(z_1) \cdots \int_0^{t_{l-1}} dt_l d\theta_l^a g(z_l).$$

We define the cochain $c^{n,l}$ by the path integral with both bulk insertions and Wilson loop

$$c^{n,l} \circ (f_0, \cdots, f_n; g_0, \cdots, g_l) = \int_{\mathcal{L}} \int d^6 z_0 f_0 \cdots d^6 z_n f_n \cdot W^l(g_0, \cdots g_l) e^{-S_{kin}}.$$  

(25)

\footnote{Strictly speaking, the line insertion we are dealing with here is different from that of Eq.[11] but we use the term Wilson loop nonetheless.}
And we investigate the Ward identity

\[
0 = \int_{\mathcal{L}} \Delta (\cdots e^{-S_{\text{kin}}}) = \int_{\mathcal{L}} \Delta \left( \int d^6 z_0 f_0 \cdots \int d^6 z_n f_n \right) \cdot W^t(g_0, \cdots, g_l) e^{-S_{\text{kin}}} \\
+ (-1)^{\sum_{k=0}^{n} (f_k + 3)} \int_{\mathcal{L}} \{ \int d^6 z_0 f_0 \cdots \int d^6 z_n f_n, W^t(g_0, \cdots, g_l) \} e^{-S_{\text{kin}}} \\
- (-1)^{\sum_{k=0}^{n} (f_k + 3)} \int_{\mathcal{L}} \int d^6 z_0 f_0 \cdots \int d^6 z_n f_n \cdot \{ S_{\text{kin}}, W^t(g_0, \cdots, g_l) \} e^{-S_{\text{kin}}} ,
\]

where we dropped terms like \( S_{\text{kin}}, \int d^6 z f \) = \( \int d^6 z D f = 0 \). \( \Delta \) acting on the first part gives a series of terms resembling the first line of Eq.\(^{21}\) which is the usual Lie algebra differential \( [27]. \) We denote three terms in Eq.\(^{26}\) as \( \partial_t, \partial_V, \partial_H \) after ref.\(^{7}. \) The internal \( \partial_t \) collapses two bulk insertions, and the vertical \( \partial_V \) collapses a bulk insertion with one on the Wilson loop because the effect of the Poisson bracket is that it removes one \( f_i \) and replace one \( g_j \) with \( \{ f_i, g_j \}. \) To see the effect of \( \partial_H, \) we need to calculate the bracket \( \{ S_{\text{kin}}, W \}. \)

The bracket formally gives total derivatives, as

\[
\{ S_{\text{kin}}, \int_a^b dt d\theta^t g(z) \} = + \int_a^b dt d\theta^t D g(z) = \int_a^b dt \partial_t g ,
\]

but due to the upper and lower limits of the \( t \) integral in \( W, \) one gets some important surface terms (we set \( t_{-1} := 1 \) in the following)

\[
\{ S_{\text{kin}}, W \} = \sum_{k=2}^{l} (-1)^{\sum_{k=0}^{n} (g_m + 1)} \int_{t_0}^{1} dt_0 d\theta^t_0 g_0(z_0) \int_{t_{-1}}^{t_0} \cdots \\
\int_{t_{-1}}^{t_{-1}} dt_{k-1} d\theta^t_{k-1} g_{k-1} g_k(z_{k-1}) \int_{t_{-1}}^{t_{-1}} \cdots \int_{t_{-1}}^{t_{-1}} dt_l d\theta^t_l g_l(z_l) \\
+ (-1)^{(g_l + 1)} \sum_{l}^{1} \int_{t_0}^{1} dt_0 d\theta^t_0 g_l g_0(z_0) \int_{t_{-1}}^{t_0} \cdots \int_{t_{-1}}^{t_{-1}} dt_l d\theta^t_l g_l(z_l) .
\]

Written concisely,

\[
\{ S_{\text{kin}}, W \} = (-1)^{g_0 + 1} W(g_0 g_1, \cdots, g_l) + \cdots + (-1)^{\sum_{m=0}^{n} (g_m + 1)} W(g_0, \cdots, g_{l-1} g_l) \\
+ (-1)^{(g_l + 1)} \sum_{l}^{1} W(g_l g_0, g_1, \cdots, g_{l-1}) ,
\]

in other words, the horizontal \( \partial_H \) collapses two neighboring points on the Wilson loop.
Collecting these results, the Ward identity is summarized as

\[
0 = (\delta c_{n,l}^n) \circ (f_0, \cdots f_n; g_0, \cdots g_l)
\]

\[
= \sum_{0 \leq i < j \leq n} (-1)^{s_{ij}} c_{n-1,l}^{n-1} \circ (\{f_i, f_j\}; f_0 \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n; g_0, \cdots g_l)
\]

\[
- \sum_{0 \leq i \leq n; 0 \leq j \leq l} (-1)^{t_{ij}} c_{n,l}^{n-1} \circ (f_0, \cdots, \hat{f}_i, \cdots f_n; g_0, \cdots, g_{j-1}, \{f_i, g_j\}; g_{j+1}, \cdots g_l)
\]

\[
- \sum_{0 \leq j \leq l} (-1)^{u_{nj}} c_{n,l}^{n,l-1} \circ (f_0, \cdots f_n; g_j g_{j+1}, g_{j+2} \cdots g_l, g_0, \cdots g_{j-1})
\]

with the sign factors none other than those of Eq.21. This tells us that the path integral with both bulk and Wilson line insertions is a cocycle of the extended CE complex

\[
c_{n,l}^n \circ \partial((f_0, \cdots, f_n; g_0, \cdots g_l)) = (\delta c_{n,l}^n) \circ (f_0, \cdots, f_n; g_0, \cdots g_l) = 0 .
\]

Similarly, \(c_{n,l}^n\) depends on \(L\) according to \(c_{n,l}^n \delta L = \delta \tilde{c}\) for some \(\tilde{c}\). Hence the cohomology class \([c_{n,l}^n]L\) is independent of the continuous deformation of \(L\).

### 6.2 Extended Graph Complex

The extended CE complex of the last section is intimately related to the extended graph complex to be introduced presently.

First recall the definition of an ordinary graph complex. A graph is a 1-dimensional CW complex (whose vertices need not be trivalent). We mostly consider closed graphs (without external legs), the open graphs appear briefly in our review of Vogel’s construction. An important ingredient of the graph is its orientation. The orientation is given by the ordering of all the vertices and orienting of all the edges. We remark here that there are other orientation schemes that are equivalent to the current one. For example, the orienting of all the edges can be replaced by the ordering of all the legs from all vertices. Another convenient scheme is to order the incident legs for each vertex and order all the even valent vertices. We refer the reader to sec.2.3.1 of ref.\[12\] for a full discussion of the orientation.

We say the two orientations are the same (resp. opposite) if the sum of the number of permutations of the vertices plus the number of flips of edges required to take one orientation to another is even (resp.odd). Consider the linear space spanned by all the graphs with all the orientations, then the graph complex is the quotient of this space by the relation \(\sim\): \((\Gamma, -or) = -(\Gamma, or)\). Denote by \([\Gamma_{or}]\) the corresponding element of \((\Gamma, or)\) in the quotient space. A direct consequence of this quotient is the absence of edges starting and ending on the same vertex.
The graph complex is equipped with a differential $\partial_I$, which acts on the graph by collapsing one edge in turn. To be precise, if an edge runs from the $i$th vertex to the $j$th with $i < j$, then one collapses the two vertices making a new vertex labelled by $i$ inheriting all the legs from vertex $i$ and $j$ except the collapsed one. The labelling of vertices after $j$ move forward one notch. Finally one includes a sign $(-1)^{j+i+1} (-1)^j$ if the collapsed edge runs from $j$ to $i$ for the resulting graph.

There is an important isomorphism between this graph complex and the CE chain complex. Let $\text{Ham}_{2n|m}$ be the Lie algebra of formal Hamiltonian vector fields on $\mathbb{R}^{2n|m}$, and as usual we will think of this in terms of the Hamiltonian functions that generate these vector fields. Let $\text{Ham}^0_{2n|m} \subset \text{Ham}_{2n|m}$ be those Hamiltonian functions having zero constant and linear term in their Taylor expansion and $\text{osp}_{2n|m} \subset \text{Ham}_{2n|m}$ be the ones with only quadratic term. The chain complex $c.(\text{Ham}^0_{2n|m}, \text{osp}_{2n|m})$ is the $\text{osp}_{2n|m}$ co-invariants of usual CE complex of $\text{Ham}^0_{2n|m}$ (the orbits of the $\text{osp}$ action). There is the following isomorphism

$$\mathcal{G} \sim \lim_{n \to \infty} CE (\text{Ham}^0_{2n|m}, \text{osp}_{2n|m}) .$$

(27)

The necessity of the direct limit $n \to \infty$ will become clear shortly.

The extended graph also includes an oriented circle in addition to the ingredients of the graph complex introduced above. We distinguish between the internal vertices and vertices on the circle (peripheral vertices). The edges of the extended graph may now run amongst both types of vertices. The orientation of such graphs is determined by the ordering of the internal vertices, the orientation of the edges and the ordering of the vertices on the circle.

The extended graph also has a differential, which consists of, apart from $\partial_I$, the collapsing of a pair of adjacent vertices on the circle $\partial_H$ and the collapsing of one internal vertex with one on the circle ($\partial_V$). Fig.5 is an illustration of $\partial_I, \partial_H, \partial_V$. The sign factors are given in fig.6. We prove in the appendix that this extended graph complex is isomorphic to the extended CE complex in a purely algebraic way. The importance of this isomorphism is: all the properties the extended CE complex has are enjoyed by the extended graph complex. Our proof in
Figure 6: Sign factor for differential, here $I$ is the number of internal vertices. In the third picture, if $j$ is the $l^{th}$–the last–vertex, then we rename the new vertex 0 with sign factor $(-1)^{l+1}$.

sec\textsuperscript{6.1} of the Feynman integral as a cocycle of the extended CE complex can be grafted over and we have the key result: the Feynman integral is a cocycle in the graph complex.

In fact thanks to this isomorphism, we can prove some stronger results about the relation between the extended graph complex and the de Rham complex $\Omega_{\cdot}(\text{Imb})$ of the space of embeddings: $\mathcal{K} \hookrightarrow \Sigma_3$. Our result is stronger in the sense that it is valid for any metric of $\Sigma_3$.

In sec\textsuperscript{6.1} we took the embedding of the Wilson loop into $\Sigma_3$ as $\phi^a(t)$, but now we think of it as a function

$$\text{Imb} \times S^1 \xrightarrow{\phi(t)} \Sigma_3,$$

and write $\phi^a_\Sigma(t)$ when necessary to emphasize its dependence on Imb. Thus the Feynman integral, being a cocycle, evaluates any graph and returns a number, which actually is a 0-form of Imb. How do we differentiate this function? In general, to deform a Wilson loop infinitesimally, we pick a vector field $v$ defined on a neighborhood of the Wilson loop, and deform the Wilson loop along $v$: $\phi^a \rightarrow \phi^a + v^a$. Let $\psi$ be a 1-form on $\Sigma_3$ that is integrated along the Wilson loop, we have the following standard manipulation

$$\delta_v \int_a^b dt \dot{\phi}^a \psi_a (\phi(t)) = \int_a^b dt (\dot{v}^a \psi_a + \dot{\phi}^a (\partial_b \psi_a) v^b) = \int_a^b dt (\dot{v}^a \psi_a + \dot{\phi}^b (\partial_a \psi_b) v^a) ,$$

$$= \int_a^b \phi^a ( d_t v \psi + t_v d\psi ) = \int_a^b \phi^a L_v \psi .$$

Looking at the definition of Wilson loop insertions Eq\textsuperscript{24} where the $g$’s correspond to the 1-form $\psi$ here, we see that $\psi$ depends on $\phi$ implicitly through its dependence on the fields $X$. To deform the Wilson loop in this case, we need only find operator that deforms $X$ to $X + L_v X$. 

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This is done easily in the BV formalism. Define
\[ \Psi_v = \frac{1}{2} \sum_{A,B} \int d^6 z \, X^A \Omega_{AB}(L_v X^B)(-1)^B , \]
where \( L_v \) is the Lie derivative along \( v \). The Cartan formula gives \( L_v = \{ d, \iota_v \} \), which may be written in super language as \( \{ D, v^a \partial_{\theta^a} \} \). \( \Psi_v \) obviously has the desired property
\[ \{ \Psi_v, \cdot \} = \frac{1}{(3 - p)!} \sum_{p=0}^{3} \sum_{C} \frac{1}{(p-1)!} (L_v X^C) \frac{\partial}{\partial X^C} = \sum_{C} (L_v X^C) \frac{\partial}{\partial X^C} . \]
\( \Psi_v \) has zero bracket with any bulk insertions \( \{ \Psi_v, \int d^6 z \, f \} = \int d^6 z L_v f = \int d^6 z D \iota_v f = 0 \) and its bracket with the line insertions effectively deforms the Wilson line. But we stress that \( \{ \Psi_v, \mathcal{O}(X) \} = L_v \mathcal{O}(X) \) only when \( \mathcal{O} \) depends on \( \Sigma_3 \) only through \( X \), and for the same reason we have the counter intuitive relation,
\[ \{ \Psi_v, \Psi_v \} = \Psi_{[v,u]} = (1) \Psi_{[u,v]} . \]

We apply the Ward identity again,
\[ 0 = \int \mathcal{L} \left( \Psi_v \int d^6 z f_0 \cdots \int d^6 z f_n \cdot W^l(g_0, \cdots g_l) \, e^{-S_{kin}} \right) \Rightarrow \]
\[ \int \mathcal{L} \{ \Psi_v, \int d^6 z f_0 \cdots \int d^6 z f_n \cdot W^l(g_0, \cdots g_l) e^{-S_{kin}} \} = - \int \mathcal{L} \Psi_v \Delta \left( \int d^6 z f_0 \cdots \int d^6 z f_n \cdot W^l(g_0, \cdots g_l) e^{-S_{kin}} \right) . \]

This relation can be rewritten as
\[ \delta_v (e^{n^l} \circ (f, \cdots; g, \cdots)) = \tilde{c} \circ \partial (f, \cdots; g, \cdots) , \quad \text{for} \quad \tilde{c} \circ (f, \cdots; g, \cdots) = - \int \mathcal{L} \Psi_v (\int f \cdots W(g, \cdots)) e^{-S_{kin}} . \quad (28) \]

From Eq 28 we have proved that a Feynman integral will produce constant functions on Imb if we feed into it cycles in the extended CE complex. Furthermore by the isomorphism between graph complex and CE complex, we conclude that the Feynman integral sends graph cycles to \( H^0(\text{Imb}) \). This is all we need as far as knot invariant is concerned. But we can forge on and tap more into the Ward identity.

Let us now regard the vector field \( v \) not only as a vector field on \( \Sigma_3 \), but also as a vector field on Imb and think of \( \tilde{c} \) as the contraction of \( v \) with a 1-form \( \Omega^1(\text{Imb}) \). Continuing in this track, we can define \( q \)-forms \( c^{n^l}_{(q)} \circ (f_0, \cdots f_n; g_0, \cdots g_l) \in \Omega^q(\text{Imb}) \). This form, when evaluated on \( v_1, \cdots v_q \) is defined as
\[ \iota_{v_1} \cdots \iota_{v_q} (c^{n^l}_{(q)} \circ (f, \cdots; g, \cdots)) = (-1)^{q+1} \int \mathcal{L} \Psi_{v_1} \cdots \Psi_{v_q} (\int f \cdots W(g, \cdots)) e^{-S_{kin}} . \quad (29) \]
We claim that \( c(q) \) maps the extended CE complex homomorphically to \( \Omega(\text{Imb}) \). To see this, take \( q = 2 \) as an example, from the Ward identity

\[
0 = \int \Delta \left( \Psi_{v_1} \Psi_{v_2} (\int f \cdots W(g \cdots)) e^{-S_{\text{kin}}} \right),
\]

we get

\[
\int \Psi_{v_1} \Psi_{v_2} \Delta \left( (\int f \cdots W(g \cdots)) e^{-S_{\text{kin}}} \right) + \int \Psi_{v_1} \{ \Psi_{v_2}, (\int f \cdots W(g \cdots)) e^{-S_{\text{kin}}} \} - \int \Psi_{v_2} \{ \Psi_{v_1}, (\int f \cdots W(g \cdots)) e^{-S_{\text{kin}}} \} = 0.
\]

To identify the second and third term, we observe

\[
L_u \int \Psi_{v_1} O(X) = \int \{ \Psi_{v_1}, \Psi_{v} O(X) \} + \int \Psi_{\{u,v\}} O(X) = \int \{ \Psi_{v_1}, \Psi_{v_2} O(X) \} + \int \Psi_{\{u,v\}} O(X) = \int \Psi_{\{v_2, v_1\}} O(X) = \int \Psi_{\{v_1, v_2\}} O(X).
\]

So the Ward identity can be rewritten as

\[
( - \iota_{v_1} \iota_{v_2} \delta c(2) + L_{v_2} \iota_{v_1} c(1) - L_{v_1} \iota_{v_2} c(1) - \iota_{[v_1, v_2]} c(1)) \circ (f, \cdots ; g, \cdots) = 0.
\]

Note the first \( \delta \) in the left bracket is the CE differential, while the last three terms in this bracket is just the definition of the exterior differentiation. The verification for general \( q \) proceeds similarly.

In summary, we defined a graph cochain \( c_{n,l}^{(q)} \) taking values in \( \Omega^q(\text{Imb}) \),

\[
c_{n,l}^{(q)} : G_{n,l} \to \Omega^q(\text{Imb}),
\]

where \( G_{n,l} \) is the graph with \( n + 1 \) internal and \( l + 1 \) peripheral vertices. \( c_{n,l}^{(q)} \) has the property

\[
\delta c_{(0)}^{p} = 0, \quad dc_{(0)}^{p} = \delta c_{(1)}^{p-1}, \quad \cdots \quad dc_{(q+1)}^{p} = \delta c_{(q+1)}^{p-1},
\]

where we define \( c_{(q)}^{p} = \sum_{n+l=p} c_{n,l}^{(q)} \), \( \delta \) is the graph cochain differential and \( d \) de Rham differential on \( \text{Imb} \). The first of these properties implies that the knot invariant constructed by evaluating \( c_{(0)}^{p} \) on a graph cycle depends only on the class of this cycle in \( H_c(\mathcal{G}) \). The second one of the identities shows that \( c_{(0)}^{p} \) when evaluated at cycles of the CE complex is invariant under the deformation of the knot.

---

8One can say that this mapping gives a homomorphism \((G_{n,l,s}, \delta) \to (\Omega^{3n+l+2s}(\text{Imb}), d)\), with \( s \) being the number of edges (propagators). This is essentially Eq.1. Here \( G_{n,l,s} \) is the graph with \( n + 1 \) internal, \( l + 1 \) peripheral vertices and \( s \)-edges. \( G_{n,l,s}^{\ast} \) is its dual object.
To give a fair assessment of the proof given above, the BV path integral proof, compact as it maybe, is subject to the many weaknesses inherent in the path integral approach. Some of the weakness maybe improved, others not. Just to point out a few, the statement \[ \Delta \int d^6z f = 0 \] requires regularization. In odd dimension, the heat kernel regulation of \( \Delta \) was used in ref. [27] to cure this. Secondly, any time we use the fact \( \int L \cdot \cdots (\int d^6z Df) \cdot \cdots = 0 \), we need to check the validity of integration by part, since \( f \) may collide with other insertions and the singularity invalidates integration by parts. The Ward identity \( \int L \Delta (\cdots) = 0 \) is rigorous for finite dimension integral, but how is it faring in an infinite dimensional setup is beyond our grip. Despite all these problems the BV formal considerations capture amazingly well the intricate properties of the perturbation theory. The purpose of the next section is to review Bott Taubes construction. This construction is just a special case of the more general discussion of the current section, with a flat metric on \( \Sigma_3 \), and a specific choice of \( L \).

6.3 Configuration Space Integral

In this section, we try to make the previous abstract discussion more concrete, and most importantly, to investigate when does our path integral proof break down. We do so by reviewing the Bott-Taubes construction [7] and fit their formulae into our framework.

Bott and Taubes exclusively worked with source manifold \( S^3 \), which is represented as \( \mathbb{R}^3 \) with an added point \( \{ \infty \} \) and a flat metric. They tried to construct ansatz for closed forms on \( \Omega (\text{Imb}) \) by mimicking the Feynman integral. Given a knot parameterized by \( \phi^a(t) \), \( \phi(0) = \phi(1) \), consider the following trivial bundle structure over \( \text{Imb} \times C^0_{n,l} \subset \text{Imb} \times (S^3)^n \times (S^1)^l \), where \( S^1 \) is identified with its image in \( S^3 \) under \( \phi \) and the superscript 0 means none of the points on \( (S^3)^n \times (S^1)^l \) are allowed to coincide and no point in \( \mathbb{R}^3 \) can be \( \infty \). \( C^0_{n,l} \) is called the configuration space, the first \( n \) copies of \( S^3 \) label the position of the internal vertex and the \( l \) copies of \( S^1 \) labels the position of the peripheral vertices. Take any graph in our extended complex, one can write down a form on \( \text{Imb} \times C^0_{n,l} \) as follows: suppose \( C^0_{n,l} \) is parameterized as \( (x_1, \cdots, x_n, t_1, \cdots t_l) \), then for each edge in the graph running from vertex \( i \) to \( j \), include a propagator

\[ \omega_{ij} = \frac{1}{4\pi|x_i - x_j|^3} \epsilon_{abc}(x_i - x_j)^a \wedge d(x_i - x_j)^b \wedge d(x_i - x_j)^c, \]

where \( x_i = \phi(t_{i-n}) \) if \( i > n \). The form on \( \text{Imb} \times C^0_{n,l} \) we want to construct is the product of all such \( \omega_{ij} \)'s; it is called the tautological form in ref. [7]. We stress that \( \omega_{ij} \) is a form on
Imb × $C^0_{n,l}$, and it can have 'legs' in Imb if $i$ or $j > n$, because one should write
\[ d\phi^a(t) = \dot{\phi}(t) dt + \delta\phi^a(t), \]
and the variation $\delta\phi$ is a 1-form with leg in Imb.

There is a convenient way of viewing the propagator, let
\[ G_{ij} := \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|}, \quad (31) \]
be a map from $C^0_{2,0}, C^0_{1,1}, C^0_{0,2}$ to $S^2$ depending whether $i, j$ both less than $n$, one less one greater than $n$ or both greater than $n$. $\omega_{ij}$ can be viewed as the pull back of the volume form $\mu$ on $S^2$
\[ \omega_{ij} = G_{ij}^* \mu, \quad (32) \]

\[ \mu = \frac{1}{4\pi r^3} \epsilon_{abc} r^a dr^b \wedge dr^c; \quad \int_{S^2} \mu = 1. \]

We pause to make the remark that this propagator is exactly the super propagator in our AKSZ TFT. In the gauge choice scheme of ref. [27], $\mathcal{L}$ corresponds to setting $X^e$, the exact part of $X$, to zero. The super propagator is
\[ \langle X(u, \theta_1), X(v, \theta_2) \rangle = \frac{1}{2} \theta_1^b \theta_1^a \langle X_{ab}(u), X(v) \rangle - \theta_1^a \theta_2^b \langle X_{a}(u), X_b(v) \rangle + \frac{1}{2} \theta_1^a \theta_2^b \langle X(u), X_{ab}(v) \rangle \]
\[ = \frac{1}{2} (\theta_1^a - \theta_2^a)(\theta_1^b - \theta_2^b) \epsilon_{ab} \partial_0 G(u, v), \]
where $G$ is the same as in Eq. (31). This is exactly the propagator given in Eq. (32).

The importance of this point is that: since the volume form of $S^2$ is closed, the propagators will be closed forms in Imb × $C^0_{n,l}$ since they are the pull back of closed forms. Furthermore as $\mu^2 = 0$, graphs with two edges running between the same pair of vertices will be set to zero. Thirdly, the volume form of $S^2$ is always well-defined; so long as we can extend the mapping $G_{ij}$ from $C^0_{2,0}, C^0_{1,1}, C^0_{0,2}$ smoothly to some compactified configuration space $C_{2,0}, C_{1,1}, C_{0,2}$, the propagator will be well defined. $G_{ij}$ is most easily extended to $C_{0,2}$: when $t_i = t_j = t$ one simply defines $G_{ij} = \dot{\phi} / |\vec{\phi}|$. Since $\phi$ is an embedding, $\dot{\phi} \neq 0$. The extension to $C_{2,0}$ or $C_{1,1}$ will need some resolution of singularity. In general, one needs to extend all the $G_{ij}$ defined on $C^0_{n,l}$ smoothly to some compactified $C_{n,l}$.

After the compactification, the trivial bundle structure becomes Imb × $C^0_{n,l}$ with a compact fibre. And the tautological form is closed on the total space of the bundle. The instinctive step to take will be to integrate this form fibrewise. This step is called the transfer map in ref. [7] and from the way the tautological form is constructed, the transfer is seen to be a
map from the graph complex to $\Omega (\text{Imb})$. Since $C_{n,l}$ is compact, this integration is \textit{finite} to start with. One asks is the image of the transfer map a closed form on $\text{Imb}$. If the answer is yes, then it will be a knot invariant.

Since the tautological forms are closed, we can apply the Stokes theorem as follows

$$0 = \int_{C_{n,l}} d \text{taut forms} = \delta \int_{C_{n,l}} \text{taut forms} + \int_{\partial C_{n,l}} \text{taut forms} .$$

(33)

The integral of tautological forms over configuration space is nothing but $b_T$ in Eq.18. Had the fibre $C_{n,l}$ been a closed manifold, we could have concluded that the transfer map does give a closed form on $\text{Imb}$. But $C_{n,l}$ has complicated boundaries in general. We remind the reader that amongst the summands of tautological forms, there are terms with one leg in $C_{n,l}$ and one leg in $\text{Imb}$, which has the correct degree to be integrated over the $\partial C_{n,l}$ leaving a 1-form on $\text{Imb}$.

The boundary comes from various sources. The obvious one is when two insertions on the Wilson loop collide, which is certainly of codimension 1 in the configuration space and we have seen how to compactify these already. The configuration when two internal points collide appears to be of codimension 3, but since the limit $\lim_{y \to x} G_{ij}(x, y)$ depends on from which direction $y$ approaches $x$, one needs to replace the colliding point with an $S^2$ to keep track of how the limit is taken. To be more specific, let $\vec{r} \in S^2$ be the direction of the collision, and $z$ be the colliding point. Substitute $x = z - \vec{r} \epsilon, y = z + \vec{r} \epsilon$. Define the limiting value of $G_{ij}(x, y)|_{x=\pm \epsilon} G_{ij}(x, y) = \vec{r}/|\vec{r}| \in S^2$. Due to this $S^2$, the codimension of this singular configuration is $1 = 3 - 2$. This is essentially the blow up. Similarly, the configuration when an internal point collides with a peripheral point is also a codim 1 boundary. The three singular configurations corresponds neatly to three types of boundary operations of fig.5. These three are called the \textit{principle faces} in ref.7. It is interesting that, even though out of the three principal faces, only one comes naturally, the BV manipulation of the previous section can detect all three types without any strain.

But how about when $n \geq 3$ points colliding? In general, the stratum structure of the compactified configuration space of $n$ points is labelled by a \textit{grove}. For example $\{\{1, 2, 3\}\}$ means points 1,2,3 are colliding with a different terminal speed, and this case also gives a codim 1 boundary. While $\{\{1, 2, 3, 4\}, \{1, 2, 3\}\}$ means that points 1 ~ 4 are colliding, but to order $\epsilon$, 1,2,3 are inseparable. Once we zoom in, we find that 1,2,3 can be resolved by looking up to order $\epsilon^2$. In this case, the grove looks like fig.7. In general each successive blowup implies a closer look at the singularity and the grove can have many layers. The appendix of ref.7 and Thurston’s thesis 37 give a very down-to-the-earth review of this.

The codimension of a boundary labelled by a grove is the number of subsets, so any
number of points colliding at a different speed is also of codim 1 after the blow up. But thanks to the work of Bott and Taubes, most of these hidden faces do not contribute. The only worrisome case is when all the points in a graph collide, which has to be examined case by case. Any how, excluding this last subtlety, we have demonstrated (up to sign) that the Stokes theorem Eq.33 can be read as follows (where $\delta_K$ is the deformation of knot)

$$\delta_K b_{\Gamma} = b_{\partial\Gamma}$$

and hence the transfer map is a homomorphism between the graph complex and $\Omega(\text{Imb})$. The same homomorphism for knots embedded in $\mathbb{R}^n$, $n > 3$ can also be proved [8].

7 Knot Invariant from $Q$-structure and its Representation

So far we have managed to convert the problem of knot invariants to the seeking of cycles of the extended graph complex, and the seeking of weight systems to the cycles in the extended CE complex. From now on, we solely discuss the problem at the level of CE complex and forget about the path integral or Chern-Simons theory.

Out of the three parts the CE differential is made of, $\partial_I$ is a differential only involving the first part of the CE chain, and is nilpotent by itself. We know that cycles w.r.t to $\partial_I$ can be constructed from the $Q$-structures as in ref. [27]. Namely (see the notation of Eq.20)

$$c_{n,0} = (\Theta, \cdots \Theta)$$

is $\partial_I$-closed and simply serves as a weight system for the 3-manifold invariants. The idea behind is roughly the construction for secondary characteristic classes, for the evaluation of a cocycle on the above cycle is analogous to plugging in the flat connection of a $G$-bundle to a cocycle of the CE complex of the Lie algebra of $G$ (see sec.3 of [27]).

The shape of $\partial_V$ suggests that it described some action of $Q$ on the line insertions $W(g_0, \cdots g_l)$ compatible with the action of $\partial_H$ which multiplies two adjacent $g$’s. It is fairly clear that, the object we are trying to sniff out is some analogue of representation of the
the path integral to distract us.

The representation of a \(Q\)-structure defined in Eq.13, which we repeat here for convenience

\[
Q_R = \{\Theta, \cdot\} + R^a_b \xi_a \frac{\partial}{\partial \xi_b},
\]
\[
0 = Q^2_R = \{\Theta, R\} + R^2 = 0.
\]

We can immediately form the cycle (in the notation of Eq.20)

\[
c = \sum_{n+l=N} \frac{1}{n!l!} (\Theta, \ldots, \Theta; R^a_{a_2}, R^a_{a_3}, \ldots, R^a_{a_l}).
\]  

(34)

We mention the following to ward off some likely confusions. The representation matrix \(R^a_b\) is merely the coefficient of the chain rather than being a part of the definition of the chain. To be precise, let us denote by \(\{A\}, \{B\}\) the collective indices \(\{A_1, \ldots, A_m\}, \{B_1, \ldots, B_n\}\), and choose the monomials \(x^{(A)} = x^{A_1} \ldots x^{A_m}\) as the basis for the Hamiltonian functions on \(M\). We Taylor expand the \(R\)'s and write the second part of \(c\) as

\[
(R^a_{a_2}, R^a_{a_3}, \ldots, R^a_{a_l}) = \pm(\partial_{\{C_1\}} R^a_{b_1})(\partial_{\{C_2\}} R^a_{b_2}) \cdots (\partial_{\{C_1\}} R^a_{b_1}) (x^{\{C_1\}}, x^{\{C_2\}}, \ldots, x^{\{C_1\}})
\]

\[
= \pm \text{Tr} (\partial_{\{C_1\}} R \cdot \partial_{\{C_2\}} R \cdots \partial_{\{C_1\}} R) (x^{\{C_1\}}, x^{\{C_2\}}, \ldots, x^{\{C_1\}}),
\]

where the \(\pm\) sign come from pulling the matrices out of the chain. So only the second factor is the CE chain, while the trace factor are merely the coefficients. Omitting the indices on \(R\), the differential acting on \(c\) gives

\[
\partial c = \sum_{n+l=N} \frac{1}{n!l!} \left(\left(-\frac{n(n-1)}{2}\right) \left(\{\Theta, \Theta, \ldots, \Theta; R, \ldots, R\}_{n-2}^{n-l} \right) - n l (\Theta, \ldots, \Theta; \{\Theta, R, \ldots, R\}^{n-1}_{n-1}) - l (\Theta, \ldots, \Theta; \{R, R, \ldots, R\}^{l-1}_{l-1}) \right)
\]

\[
= \sum_{n+l=N} \left(-\frac{1}{2(n-2)!} \left(\{\Theta, \Theta, \ldots, \Theta; R, \ldots, R\}^{n-2}_{n-2} \right) - \frac{1}{(n-1)!} \left(\Theta, \ldots, \Theta; \{\Theta, R, \ldots, R\} + \{R, R, \ldots, R\} \right) \right) = 0.
\]

In general one can allow \(\xi\) above to have different statistics, for example when one uses the “adjoint” representation. We recall in that case

\[
R^A_B = (\Omega^{-1})^{AC} \partial_C \partial_B \Theta; \quad \sum_C R^A_C R^C_B (-1)^C + \{\Theta, R^A_B\} (-1)^A = 0.
\]
The cycle is formed in just the same manner, but we warn the reader to be extra careful with the sign factors in Eq.21 when checking the cycle condition.

7.1 Weight System Valued in $H_Q$

In the discussion of the isomorphism between the graph complex and the CE complex, we required $\Theta$ to be at least quadratic and $R$ linear. This restriction of course does not make sense on a curved manifold, and even when the manifold is flat such as $\mathbb{R}^{2n|m}$, the restriction is still too awkward.

The solution in fact brings about a pleasant generalization. We pick a base point $x_0$ on $\mathcal{M}$, identify the neighborhood of $x_0$ with the tangent bundle of $\mathcal{M}$ at $x_0$. This is usually done with the exponential map, but here we only do the simple minded Taylor expansion: the neighboring points of $x_0$ are parameterized as $x_0 + \xi$ and Taylor expand $\Theta(x_0 + \xi), R(x_0 + \xi)$ into formal power series. The $\xi$ space is equipped with the symplectic form $\Omega(x_0)$. Denote

$$\Theta' = \sum_{n=2}^{\infty} \frac{1}{n!} \partial C_1 \cdots \partial C_n \Theta(x_0) \xi^{C_1} \cdots \xi^{C_n},$$

$$R' = \sum_{n=1}^{\infty} \frac{1}{n!} \partial C_1 \cdots \partial C_n \mathcal{R}(x_0) \xi^{C_1} \cdots \xi^{C_n},$$

where the summation starts from 2 and 1 respectively. By Taylor expanding the equation $\{\Theta, \Theta\} = 0$ and $\{\Theta, R\} + RR = 0$ into power series in $\xi$ one clearly sees that

$$\{\Theta', \Theta'\}_\xi = -2Q^A(x_0) \frac{\partial}{\partial x^A_0} \Theta',$$

$$\{\Theta', R'\}_\xi + \{R', R'\}_\xi = -Q^A(x_0) \frac{\partial}{\partial x^A_0} R' - RR' - R'R,$$  \hspace{1cm} (35)

where $Q^A(x_0)$ is $-(\Omega^{-1})^{AB} \partial_B \Theta$ evaluated at $x_0$. In these equations we have written $\{., .\}_\xi$ to stress the bracket is taken in $\xi$ space. These two equations imply $\Theta', R'$ satisfy the same relation as Eq.13 up to $Q(x_0)$-exact term.

We use $\Theta'$ and $R'$ to form the following chain

$$c(x_0) = \sum_{n+l=N} \frac{1}{n!l!} (\Theta', \cdots, \Theta'; R_{a_1}^{(n_1)}, R_{a_2}^{(n_2)}, \cdots, R_{a_l}^{(n_l)}),$$

and it depends now on $x_0$. Suppressing the indices on $\mathcal{R}$ henceforth and using Eq.35 plus the graded cyclicity we have

$$\partial c(x_0) = Q(x_0) \cdot \left( \sum_{n+l=N} \frac{1}{(n-1)!l!} (\Theta', \cdots, \Theta'; R', \cdots R') \right).$$

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Seeing \( c(x_0) \) is closed up to \( Q \)-exact terms, the proper generalization is clearly staked out: we should talk about cycles in the CE complex (or graph complex) with coefficients in the \( Q \)-cohomology group \( H_Q \). This change is easily incorporated in the path integral. We split any field into \( X^A = x^A_0 + \xi^A \), where \( x^A_0 \) is the zero mode, and the path integral is only over the non-zero modes. The details are in ref. [27]. Thus in practice \( c(x_0) \) can be calculated by using the perturbation theory, as for example using the Lorentz gauge as in Section 5.

This concept of weight system valued in \( H_Q \) was already in ref. [32] and further developed in ref. [33, 20]. The construction in ref. [32] is that on a hyperKähler manifold with a holomorphic symplectic form \( \Omega_{ij} \), one can define power series in \( \xi^i \)

\[ \Theta_i = \sum_{n=0}^{\infty} \frac{1}{(n+3)!} \xi^{i_1} \cdots \xi^{i_{n+3}} (\nabla_{i_1} \cdots \nabla_{i_n} R_{i_{n+1} j}^{\ j \ i_{n+3}}) \Omega_{j i_{n+2}} \]

where \( R_{i j}^k \) is the curvature tensor and \( \Theta, R, \Omega \) are also implicitly functions of \( x_0^i \), which plays the role of the base point above. \( \Theta \) satisfies

\[ \bar{\partial}_{\bar{j}} \Theta_{\bar{i}} = -\frac{1}{2} \{ \Theta_{\bar{j}}, \Theta_{\bar{i}} \} \xi \]

Introducing a formal \( \deg 1 \) parameter \( v^i \), which behaves like \( dx_0^i \), we can define a \( Q \)-structure

\[ Q_{rw} = v^i \partial_i + \{ v^i \Theta_i, \cdot \} \xi \]

Since \( \Theta_i \) has no linear term in \( \xi \), the CE chain (without the Wilson loop part) thus formed

\[ (v^i \Theta_i, \cdots v^i \Theta_i) \]

is a cycle up to \( \bar{\partial} \)-exact terms. In ref. [32], this is used as a weight system for 3-manifold invariant valued in \( H_\bar{\partial} \). One can wedge it with proper powers of \( \Omega \) and integrate it over the hyperKähler manifold to get a complex valued 3-manifold invariant.

If one includes the Wilson loops and take the “adjoint” representation for \( Q_{rw} \), a cycle in the extended CE complex is constructed as

\[ \sum_{n+l=N} \frac{1}{n!l!} \left( v\Theta_i \cdots v\Theta_i; v\Theta_i \cdots v\Theta_i \right) \]

The path integral evaluates this cycle producing a knot invariant valued in \( H_\bar{\partial} \).

Then it was point out in ref. [33] that from any holomorphic vector bundle \( E \) over \( M \) with curvature \( K \), we can construct a representation for \( Q_{rw} \). First denote the (even) coordinate
of the fibre of $E$ as $e_I$ and define the following

$$(R^p_{i_1\cdots i_{p+1}})^i_j = \nabla_{i_1\cdots i_p} R^p_{i_{p+1} j}, \quad (K^p_{i_1\cdots i_{p+1}})^I_J = \nabla_{i_1\cdots i_p} K^p_{i_{p+1} j},$$

$$\Theta^i = \sum_{p=0}^{\infty} \frac{1}{(p+3)!} (R^p_{i_1\cdots i_{p+1}})^i_j \xi^{i_1} \cdots \xi^{i_{p+3}},$$

$$(K^p_{i_1\cdots i_{p+1}})^I_J = \sum_{p=0}^{\infty} \frac{1}{(p+1)!} (K^p_{i_1\cdots i_{p+1}})^I_J \xi^{i_1} \cdots \xi^{i_{p+1}},$$

$\Theta$ and $K$ satisfy a neat relation

$$\bar{\partial}_i [\Theta_j] = -\{\Theta_i, \Theta_j\},$$

$$\bar{\nabla}_i [K_j] = K_i [K_j] - \{\Theta_i, K_j\}. \quad (36)$$

So we can define the representation

$$\hat{Q}_{rw} = v^i \bar{\nabla}_i + v^i (\{\Theta_i, \cdot\} - (K_i)^I_J e_I \frac{\partial}{\partial e_J}),$$

where

$$\bar{\nabla}_i = \partial_i + (A_i)^I_J e_I \frac{\partial}{\partial e_J}$$

is the covariant derivative of $E$, then $\hat{Q}_{rw}^2 = 0$. This formula is proved in appendix. This agrees with our definition of the representation for a graded manifold equipped with homological vector field.

We can use this construction to form weight system for knots, we write down the necessary formulae. To avoid clutter, we absorb $v$ and write $K = v^i K_i$ and $\Theta = v^i \Theta_i$. We also raise the holomorphic indices from the left by $\Omega^{-1}$: $\Theta^i = (\Omega^{-1})^{ij} \Theta_j = (\Omega^{-1})^{ij} \partial \xi^j \Theta_i$, etc.

According to the discussion above, we form a cycle $c$,

$$c = \frac{1}{4} \langle ; K, K, K, K \rangle - \frac{1}{3} (\Theta; K, K, K) + \frac{1}{4} (\Theta; \Theta; K, K) - \frac{1}{6} (\Theta, \Theta, \Theta; K).$$

We defined a mapping $\beta$ from the extended CE chain to graph chain in the paragraph around Eq.42 of the appendix. Applying $\beta$ to the cycle $c$ we obtain (the last term in $c$ drops)

$$\beta_c = \frac{1}{2} \text{Tr}(K_i, K^i, K^j, K_j) [\Gamma_1] + \frac{1}{4} \text{Tr}(K_i, K_j, K^i, K^j) [\Gamma_2] - \frac{1}{3} \Theta^{ijk} \text{Tr}(K_i, K_j, K_k) [\Gamma_3] + \frac{1}{4} \Theta^{ijkl} \text{Tr}(K_i, K_j) [\Gamma_4]. \quad (37)$$

The $[\Gamma_{1,2,3,4}]$ refers to the graphs in fig 16. Of course at this low order, we can not expect anything new from the RW weight system, because up to degree 10, Bar-Natan has explicitly
computed the dimension of knot invariants and shown that all are covered by the Lie algebra weight systems. Still, we need to clean up the messy expression above. First, by using Eq.36 and dropping total $\bar{\partial}$ derivatives it is straightforward to regroup the four terms as

$$\beta c = -\frac{1}{4} \text{Tr}(K_i, K_i, K_j, K_j)(2[\Gamma_1] + [\Gamma_2]) + \frac{1}{4} \Theta_{kl} \Theta^{kl} \text{Tr}(K_i, K_j) \left( -\frac{2}{3} [\Gamma_3] - \frac{1}{2} [\Gamma_2] + [\Gamma_4] \right),$$

and this is again a linear combination of two graph cycles as in the case of Lie algebra weight system Eq.44, except the coefficient now takes value in $H^\cdot(M)$.

In general this is all we can say for the RW weight system, but when the graph in question are made of wheels, the coefficients can be further shuffled to be expressed in terms of characteristic classes. The wheels are most easily defined by pictures fig.8. In fig.8 the circle or the rim may either be the boundary circle of the extended graph or be made of loop of internal edges. The rim basically provides a ‘trace’ of the relevant indices. It is easy to see that when the rim of fig.8 is the boundary circle, its weight will be given by $\text{Tr}[K_{i_1} K_{i_2} \cdots K_{i_n}]$, where $K_i = \partial_\xi K|_{\xi=0}$ and $i_1, i_2, \cdots$ are the indices of the ’spokes’. While if the rim is made of internal edges, the weight is given by

$$v^{i_1} \cdots v^{i_n} R_{i_{1}i_{1}}^{t_1} R_{i_{2}i_{2}}^{t_2} \cdots R_{i_{n}i_{n}}^{t_{n-1}} = v^{i_1} \cdots v^{i_n} \text{Tr}[R_{i_{1}i_{1}} R_{i_{2}i_{2}} \cdots R_{i_{n}i_{n}}].$$

In both cases, if the indices $i_1, \cdots i_n$ were anti-symmetrized, then the weight is none other than the $n^{th}$ Chern character of the bundle $E$ and the holomorphic tangent bundle of $M$. In general, all the Chern classes are expressible as weights of properly chosen graphs but not vice versa (see Sawon’s discussion sec.2.2 of ref.[33]).

With less than 4 vertices, all graphs can be formed by sewing together the spokes of wheels. For example, $\Gamma_{1,2}$ are made by sewing the 4 spokes of a 4-wheel while $\Gamma_4$ is made by sewing together 2 2-wheels. So it is possible to reexpress these weights as polynomials of Chern classes, (in fact Chern character is more convenient here). To do this, we need a

![Figure 8: A wheel is made of the rim and its spokes](image-url)
simple but useful formula,

\[(\omega^{-1})^{a_1 b_1} \cdots (\omega^{-1})^{a_p b_p} O_{a_1 b_1 \cdots a_p b_p} = e^{a_1 b_1 \cdots a_n b_n} \omega_{a_1 b_1} \cdots \omega_{a_n b_n} O_{a_{n-p} b_{n-p} \cdots a_{n-p} b_{n-p}} \frac{4^p (-1)^p p!}{(n-p)!(2p)!|\phi\omega|} \]

where \(\omega_{ab}\) is a symplectic form on a dim \(2n\) manifold, \(O\) is a \(2p\)-form and \(e^{a_1 \cdots a_{2n}}\) is the Levi-Civita symbol. This formula allows us to finesse away \(\omega^{-1}\) form lhs. To derive this formula, one can consider the following fermion integral

\[I = \int d^{2n} \psi \, e^{\psi^a \omega_{ab} \psi^b + 2 J_a \psi^a} \]

and evaluate it in two ways, 1. brute force, which does not involve \(\omega^{-1}\) since \(\psi\) are fermions, 2. complete the square which does involve \(\omega^{-1}\). By differentiating both sides w.r.t \(J\) one obtains Eq.39

Back to our manipulation of the weights. Let us first give some short-hands to lighten the formulae. Denote the cumbersome number on the rhs of Eq.39 by \(I_\Omega^2\). And for any form \(\tau \in \Omega^{q,q}(M)\) we use \(\tau_{i_1 \cdots i_q}\) to denote \(1/q! v^{i_1} \cdots v^{i_q} \tau_{i_1 \cdots i_q}\), since the anti-holomorphic part of the forms only go along for the ride. Finally define \(\langle \tau \rangle = e^{i_1 \cdots i_{2n} \tau_{i_1 \cdots i_{2n}}}\) for any \(\tau \in \Omega^{2n,2n}(M)\). By applying Eq.39 we get

\[\Theta_{ikl} \Theta_{jkl} \text{Tr}[K^i K^j] = -16\pi^4 ch_2(M)_{ij} ch_2(E)^{ij} \]

\[= -8\pi^4 \left( (I_\Omega^2)^2 \langle \Omega^{n-1} ch_2(M) \rangle \langle \Omega^{n-1} ch_2(E) \rangle - I_\Omega^2 \langle \Omega^{n-2} ch_2(M) ch_2(E) \rangle \right) \]

and for the 4 \(K\) term

\[(\Omega^{-1})^{ij} (\Omega^{-1})^{kl} \text{Tr}[K_i K_j K_k K_l] = 4! 16\pi^4 (\Omega^{-1})^{ij} (\Omega^{-1})^{kl} ch_4(E)^{ijkl} = 4! 16\pi^4 I_\Omega^2 \langle \Omega^{n-2} ch_4(E) \rangle \]

Thus Eq.38 becomes

\[\beta_c = -4\pi^4 I_\Omega^2 \langle \Omega^{n-2} ch_4(E) \rangle \cdot c_1 , \]

\[-2\pi^4 \left( (I_\Omega^2)^2 \langle \Omega^{n-1} ch_2(M) \rangle \langle \Omega^{n-1} ch_2(E) \rangle - I_\Omega^2 \langle \Omega^{n-2} ch_2(M) ch_2(E) \rangle \right) \cdot \left( \frac{1}{6} c_1 + c_2 \right) , \]

\[c_1 = 2[\Gamma_1] + [\Gamma_2] , \]

\[c_2 = -\frac{2}{3} [\Gamma_3] - \frac{1}{2} [\Gamma_2] + [\Gamma_4] . \]

This is a slight strengthening of Sawon’s result; in his treatment he required the dimension of the hyperKähler manifold to be twice the number of vertices and the spokes of the wheels.
totally anti-symmetrized to convert the weights into *Chern numbers*. The dimension requirement is too restrictive, for we will then not be able to use high dimension hyperKähler manifolds as weights for low degree graphs. Here by using the cohomology group $H_\partial$ as coefficients, we can circumvent the restriction.

However, in his thesis [33], Sawon was able to plumb much deeper into the relation between wheels and Chern classes. By using the so called wheeling theorem [6], he managed to identify subclasses of wheels and their weight as polynomials of Chern classes of $TM$ and $E$: $ch(M)Td^{1/2}(E)$.

8 Unresolved Problems

In this last section, we discuss some loose ends and unresolved problems.

8.1 Vogel’s Construction

This subsection serves as an explanation why do we try to generalize the Lie algebra weight system.

The graph complex serves as the middle man between $\Omega(\text{Imb})$ and the extended CE complex. On one side, the mapping $c^{n,l}_{(q)} : G_{n,l} \to \Omega^q(\text{Imb})$ is a homomorphism and thus it models $H^q(\text{Imb})$ on $H^q(G)$. Even though it is not clear to us whether this mapping is into or onto, it is shown in ref. [8] that this mapping does produce infinitely many nontrivial classes in $H^q(\text{Imb})$ for the case $S^1 \hookrightarrow \mathbb{R}^n, n > 3$.

On the other side, every Lie algebra weight system produces cycles in $H(G)$ through the construction Eq.34 with $\Theta$ and $R$ given by Eq.14. If this construction exhausted all of $H(G)$, then by feeding these cycles into $c^{n,l}_{(q)}$, we can reach the entire image of $c^{n,l}_{(q)}$.

A conjecture by Bar-Natan is that all the weight systems come from semi-simple Lie (super)algebra with an invariant bilinear form and finite dimensional representation. But this conjecture was negated by Vogel, which was in fact Bar-Natan’s wish. This result calls for the need of new weight systems. We try to review Vogel’s construction, for the paper [39] does not provide the most pleasant bed time reading.

Let us specialize to the Lorentz gauge, in which the graph cochains $c^{(0)}$ only ‘respond’ to diagrams with tri-valent internal vertices and uni-valent peripheral vertices (uni-trivalent graphs). Such cochains are cocycles automatically, since the differential $\delta$ acts on a graph by splitting a vertex into two, each of which is at least trivalent or univalent depending on whether the initial vertex is internal or peripheral, but $c^{(0)}$ has only trivalent or univalent vertices to start with. To descend to graph cohomology, we need only mod out coboundaries,
this amounts to modding out the graphs I H X (the lhs of fig.2). Furthermore, the remark about orientation in sec.6.2 says it is enough to orient the graph by ordering the three legs at each vertex. The flipping of the cyclic ordering at a vertex flips the sign of the graph; this is called the AS relation. Thus all the diagrams with tri-valent internal and uni-valent peripheral vertices, mod out by the IHX, AS relation is nothing but a very specific representative of the cohomology group $H^*({\mathcal G})$. This is usually denoted as $CD_*$ in the literature.

Vogel defined a module structure on $CD_*$, the ring for this module is denoted $\Lambda$, which consists of trivalent graphs with 3 external legs, satisfying the anti-symmetry under permutation of the three legs and the relation in fig.9. The ring multiplication is done by picking

![Diagram](image)

Figure 9: Members of $\Lambda$ are drawn as a blob, this relation says the insertion of this blob into any vertex does not depend on which vertex one chooses

any vertex in a uni-trivalent graph $\Gamma$, and insert a member of $\Lambda$ into it. The relation fig.9 says one can make insertions into an arbitrary vertex and the result does not change. This action is also compatible with AS and IHX relations so it is a cohomology operation.

One can apply the Lie algebra weight system before and after the action of $\Lambda$ and see what happens. To do this, Let $u \in \Lambda$, $b \in H^*({\mathcal G})$ and $c$ is a graph cycle formed using the recipe of Eq.34.14 with Lie algebra $g$, it turns out that

$$(u \circ b)(c) = \chi_g(u) \times b(c) ,$$

where $u \circ b$ is the action of $u$ on $b$ and $\chi_g(u)$ is a number called character by Vogel which only depends on the Lie algebra and $u$. The rough reason for this simple relation is that, the application of Lie algebra weight system to $u$ turns it into an anti-symmetric rank 3 tensor in the Lie algebra. Letting $e^a$ be the basis of the Lie algebra $g$, the relation fig.9 is now written

$$[e^c, u(e^a, e^b)] - u([e^a, e^b], e^c) = 0 .$$

The semi-simpleness of the Lie algebra says the adjoint representation has no non-trivial ideal, then the above relation implies $u(e^a, e^b)$ is proportional to $[e^a, e^b]$ (Thm.6.1 [39]). So the effect of inserting $u$ into a trivalent vertex is like computing vertex correction, which corrects the tree level vertex by a factor $\chi_g(u)$—the charge renormalization in physics.
The hard part of the work of Vogel is to show that there is a certain $u$ such that $\chi_g(u) = 0$ for all semi-simple Lie (super)algebra with an invariant bilinear form and finite dimensional representation. For example, to construct $u$ such that $\chi_{su(m)}(u) = 0$, it suffices to consider $sl(m, \mathbb{C})$ since the latter is the complexification of the former. Apply the $sl(m)$ weight system to $u$ will result in products like

$$\text{Tr}[\cdots t^d \cdots t_a \cdots t_b \cdots t_c \cdots t_d \cdots],$$

where $t$’s are the traceless $m \times m$ matrices, the defining representation of $sl(m, \mathbb{C})$. Such matrices have a very convenient double-line notation, and hence their products can be represented as ribbon graphs. In the case of $sl(m, \mathbb{C})$, the ribbon graphs correspond to oriented open surfaces. This object is fact a polynomial algebra generated by the disc $t$ and torus $\beta$. In particular, the product above will be an oriented open surface with 3 marked points on its boundary corresponding to $a, b, c$. It can be generated starting from a disc with three marked points on the boundary, which is the tree level vertex, by applying $t$ and $\beta$. This view point is important because then for a fixed $u$, the coefficient of the polynomial of $t, \beta$ representing $u$ does not depend on $m$, allowing us to define one $u$ that annihilates all $sl(m, \mathbb{C})$. The effect of the disc is to create an extra index loop in the surface, multiplying $\chi_{sl(m)}$ by $m$; while gluing the torus does not affect $\chi_{sl(m)}$. It turns out that $x_n$ as shown in fig.10 for $n = 2p + 1$ has character (Prop 5.4, Thm 6.4, 7.1 of ref. [39])

![Figure 10: Generating element of $\Lambda$](image)

$$\chi_{sl(m)}(x_n) = t^n + t(4\beta)^p + 2^n t \beta \frac{t^{2p} - \beta^p}{t^2 - \beta}$$

and $3tx_5 - 6t^3x_3 - x_3^2 + 4t^6$ kills all $sl(m, \mathbb{C})$. The character $\chi_{su(m)}$, however complicated, is an algebraic function of the entries of the matrix $t^n$. The uniqueness of analytic continuation tells us that $\chi_{su(m)}$ is the same function of entries of $t$ as $\chi_{sl(m)}$ and hence $\chi_{su(m)}(u) = 0$. One finds a killer $u$ for each of the semi-simple Lie algebra and multiply them together, the product then annihilates all.

By acting $\Lambda$ on the ‘Mercedes-Benz’ diagram gives a map from $\Lambda$ to $H \cdot (G)$. This map is shown to be injective. In particular, we have $0 \neq u \circ (\text{Mercedes})$, but this cocycle will annihilate any graph cycle constructed from Lie (super)algebras satisfying the conditions.
above, since $\chi_g(u) = 0$. Hence $u \circ \text{(Mercedes)}$ is a graph cocycle that eludes all the Lie algebra weight systems.

We hope this sketchy review will explain why do we bother about general weight system in this paper. But as shown by Vogel, the member of $\Lambda$ of lowest degree that annihilates all Lie algebra weight system has 23 vertices, so one has to come up with constructive ways of testing the potency of Rozansky-Witten weight system up that order, we certainly will not be able to tie up this loose thread in this paper.

8.2 Quantization, Surgery and Skein Relation

In this paper we considered the properties of perturbative theory for a general BV-AKSZ 3D theory. However we failed to bring to light how to quantize non-perturbatively any one of the TFT constructed in the BV-AKSZ framework. But the quantization is crucial in the computation of the expectation value of Wilson loop as a whole and thus it is crucial to the understanding of the new knot invariants arising through these theories. As we briefly mentioned in the sec[2] one pins down the knot invariants by investigating the skein relations, which is done in CS theory through surgeries. What one does is to take a ball $B^3$ enclosing the locus where one strand goes over another strand. Gouge out this ball, glue it back again after a nontrivial diffeomorphism of the boundary of $B^3$. If the diffeomorphism is chosen shrewdly, one finds that the previous over crossing becomes an under crossing and the value of the knot invariant changes by a quantity determined by the surgery. The ability of getting exact formula for surgery in CS theory is due to the ability to quantize this theory and to relate it to the conformal field theory. Then, one can obtain the formula for surgery by computing the modular transformation matrix in the corresponding CFT. For the quantization of other AKSZ TFT’s, only the case of RW theory is known [32]. The surgery formula of RW theory is worked out when there is no Wilson loops, so one may certainly try to generalize the result of Rozansky and Witten to the case of holomorphic vector bundles over hyperKähler manifolds.

Without the complete knowledge of quantization, or equivalently, the structure of the Hilbert space, one may use the path integral to obtain (at least perturbatively) the skein relation. The procedure will probably involve performing path integral over $B^3$, but with arbitrary (BRST invariant) boundary conditions on $S^2 = \partial B^3$. The computation with general boundary condition allows one to obtain surgery formula and therewith the skein relation, regardless of what is happening outside of the ball. This might be the point where TFT’s with odd couplings, e.g. the parameter $\bar{v}^3$ in RW theory, become interesting.

Finally let us conclude with one intriguing comment. If we believe that our BV-AKSZ
theories are quantum-mechanically consistent even non-perturbatively then we should be able to construct the corresponding knot polynomials. In particular the theories with “odd coupling constants” such as RW or odd Chern-Simons for integrable model would give rise to knot polynomials which depend on the odd couplings and thus have just finite number of terms. This idea is most intriguing for us and indeed is the main motivation behind our study.

9 Summary

We summarize the major results in this paper. We have proved that there is an isomorphism between the extended Chevalley-Eilenberg (CE) complex of the Lie algebra of Hamiltonian vector fields with the extended graph complex. This is an extension of the classic result due to Kontsevich and many other authors. We also used BV machinery to show that the path integral in a 3D TFT gives a cocycle in the extended CE complex. Putting these two results together, we concluded that the path integral is a cocycle of the extended graph complex, this result generalizes that of ref. In the BV framework, we can easily prove that there is a homomorphism between the graph complex and de Rham complex of the space of embeddings $S^1 \to \Sigma_3$. This so called transfer map is basically the Feynman integral.

Next, we applied these new results to bear upon the study of knot invariants. We can form a 3D TFT associated with the Hamiltonian lift $\Theta$ of a $Q$-structure using the AKSZ construction. The perturbation expansion uses the Taylor coefficients of $\Theta$ as vertex functions. This is a weight system for the extended graph and the IHX relation is guaranteed by $\{\Theta, \Theta\} = 0$, this is a novelty compared to the Lie algebra weight system in the sense $\{\Theta, \Theta\} = 0$ takes the place of the Jacobi identity. The representation of the Lie algebra is replaced with extensions of the $Q$-structure. We also showed how to construct cycles in the extended graph complex from $Q$-structures and their representations by applying the isomorphism of the previous paragraph.

In these TFT’s, we showed how to construct Wilson loops in general. And the partition function or the expectation value of the Wilson loop can be interpreted as the pairing between two dual constructions of the extended graph complex, one from the $Q$-structure weight system, another from the Feynman integral. Finally, we worked out the necessary formulae for the knot invariants with the weight system associated with a holomorphic vector bundle on a hyperKähler manifold, which is a generalization of the Rozansky-Witten weight system.

Acknowledgement:
It is our pleasure to thank Francesco Bonechi, Alberto Cattaneo, Ezra Getzler and Alexei Morozov for many illuminating discussions. The research of M.Z. was supported by VR-grant 621-2008-4273.

A The $\mathcal{Q}$-Structure from Holomorphic Vector Bundle

Let us take a GrMfld $\mathcal{M} = T^{0,1}[1]M$, where $M$ is Kähler with coordinate $x^i, x^{\bar{i}}$ and $v^{\bar{i}}$ is the odd fibre coordinate. There is a $\mathcal{Q}$-structure corresponding to the Dolbeault differential $v^{\bar{i}} \partial_{\bar{i}}$, we can find a representation of the $\mathcal{Q}$-structure from a holomorphic vector bundle $E \to M$. Denote by $\nabla$ the full covariant derivative $d + \Gamma + A$ with $\Gamma$ the Levi-Civita connection and $A$ the connection of $E$. The holomorphicity implies we can choose the curvature $K_{\mu\nu}^I$ of $E$ satisfy $K_{02}^0 = (\bar{\partial} + A_{01})^2 = 0$. Then the Bianchi identity implies for example $\nabla_{[\bar{i}} K^I_{\bar{j}]} J = 0$.

By applying $\nabla_{l_1} \cdots \nabla_{l_n}$ to the Bianchi identity, we have

$$0 = \frac{1}{(n+1)!} \nabla_{l_1} \cdots \nabla_{l_n} \nabla_{[\bar{j}} K_{l_{n+1}}^I J_{l_{n+1}]} K + \text{perm in } l = \sum_{k=0}^{n-1} \frac{1}{(n+1)!} \nabla_{l_1} \cdots \nabla_{l_k} \left[ K_{l_{k+1}}^I J_{l_{k+2}} M \nabla_{l_{k+2}} \cdots \nabla_{l_n} K_{l_{n+1}}^M K - R_{l_{k+1}}^m_{l_{k+2}} \nabla_{l_{k+3}} \cdots \nabla_{l_n} K_{l_{n+1}}^J M \right]$$

$$- K_{l_{k+1}}^M K \nabla_{l_{k+2}} \cdots \nabla_{l_n} K_{l_{n+1}}^I M \right]$$

$$+ \frac{1}{(n+1)!} \nabla_{\bar{j}} \nabla_{l_1} \cdots \nabla_{l_n} K_{l_{n+1}}^I K + \text{perm in } l, \text{ anti in } [\bar{i}, \bar{j}].$$

Define some short hands

$$(R_{l_{i_1} \cdots l_{i_p+1}}^p)_{[i} = \nabla_{l_{i_1}} \cdots \nabla_{l_{i_p+1}} R_{l_{i_p+1}}^{i_p+1} ; (K_{l_{i_1} \cdots l_{i_p+1}}^p)_{[i} = \nabla_{l_{i_1}} \cdots \nabla_{l_{i_p+1}} K_{l_{i_p+1}}^{i_p+1}.$$

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note that all $l$’s are symmetric due to the Kähler property. Continue on

\[
\sum_{k=0}^{n-1} \sum_{p=0}^{k} \frac{1}{(n+1)!} C_{k}^{p} \left[ - [K_{j_{1} \ldots l_{p+1}}^{p}, K_{l_{p+2} \ldots l_{n+1}}^{n-p-1}]_{K}^J + (R_{j_{1} \ldots l_{p+1}}^{p}, K_{l_{p+2} \ldots l_{n+1}}^{n-p-1})_{K}^J (n-k) \right] \\
+ \frac{1}{(n+1)!} \nabla_{j}(K_{l_{1} \ldots l_{n+1}}^{n})_{K}^J + \text{perm in } l, \text{ anti in } [i, j] \\
= \sum_{p=0}^{p=n-1} \frac{1}{2(n-p+1)!} \left[ - \frac{1}{(n+1)!} C_{k}^{p} \left[ - [K_{j_{1} \ldots l_{p+1}}^{p}, K_{l_{p+2} \ldots l_{n+1}}^{n-p-1}]_{K}^J \\
+ \frac{1}{(n+1)!} \nabla_{j}(K_{l_{1} \ldots l_{n+1}}^{n})_{K}^J + \text{perm in } l, \text{ anti in } [i, j] \right] \\
\right]
\]

So if we define

\[
R_{i} = \sum_{p=0}^{\infty} \frac{1}{(p+2)!} (R_{j_{1} \ldots l_{p+1}}^{p})^i_j \xi^{l_{1}} \ldots \xi^{l_{p+1}} \xi^{j} \partial_{\xi}^i , \\
(K_{i})^j_j = \sum_{p=0}^{\infty} \frac{1}{(p+1)!} (K_{j_{1} \ldots l_{p+1}}^{p})^i_j \xi^{l_{1}} \ldots \xi^{l_{p+1}} ,
\]

Then the above relation can be written concisely

\[
\nabla_{j}[i] K_{j} = K_{[j} K_{i]} - R_{[i} K_{j]} .
\]

So one can place $K$’s on the Wilson loop by tracing the $I, J$ index. And the above relation says that the diagrams I+H+X (see fig. 2 the lower edge of figure I and the vertices of H X are now on the Wilson loop) is something $\partial$-exact.

If furthermore, $M$ is also hyperKähler, with symplectic form $\Omega_{ij}$ we can define

\[
\Theta_{i} = \sum_{p=0}^{\infty} \frac{1}{(p+3)!} (R_{j_{1} \ldots l_{p+1}}^{p})^i_j \xi^{l_{1}} \ldots \xi^{l_{p+3}} \Omega_{l_{p+2}} \xi^{l_{p+3}} ,
\]

We obtain

\[
\nabla_{j}[i] K_{j} = K_{[j} K_{i]} - \{ \Theta_{[i}, K_{j]} \} .
\]

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If $E$ is the tangent bundle of $M$, then we recover our previous result

$$\nabla_i \Theta_j = \partial_i \Theta_j = -\frac{1}{2} \{\Theta_i, \Theta_j\}.$$  

## B Isomorphism between Extended Graph and CE Complex

We prove that this graph complex is isomorphic to the following generalized CE complex. The complex is spanned by

$$c_{n,l} = \mathbb{X}_{f_0} \wedge \cdots \wedge \mathbb{X}_{f_n} \otimes (g_0 \otimes \cdots \otimes g_l) \Rightarrow (f_0, \cdots, f_n; g_0, \cdots, g_l).$$

We allow cyclic permutations in the second factor, with $l + 1 \equiv 0$, etc. The differential is defined as in Eq.21. The differential commutes with the $osp$ action, so it makes sense to consider the $osp$ co-invariants of the CE complex, that is the orbits of the $osp$ action.

Before proceeding to the proof, we temporarily change the labelling of chains from $0 \cdots n$, $0 \cdots l$ to $1 \cdots n$, $1 \cdots l$ and offer our apology for this confusion.

We first re-prove the known result Eq.27 to set the stage. Once this is clear, the generalization is straightforward. Most of the labor comes from keeping track of signs, so we use the following technical contraptions. Let $x^p$ denote the coordinate of $\mathbb{R}^{2n|m}$, with the (trivial) symplectic structure $\Omega = 1/2 \Omega_{pq} dx^p \wedge dx^q$. Enlarge the set of coordinates into the \{x^p, i = 0, 1 \cdots \}, $x^p_i$ for different $i$ are the same as $x^p$, but the label $i$ allows us to treat them as formally independent variables. Introduce some formal degree 1 variables \{t_i, i = 0, 1 \cdots \}. This way, we are able to define a Laplacian which induces the Poisson bracket, even though strictly speaking, even brackets are not induced by Laplacians. We can write

$$(f_1, \cdots f_n) = \mathbb{X}_{f_1} \wedge \cdots \wedge \mathbb{X}_{f_n} \Rightarrow t_1 f_1(x_1) \cdot t_2 f_2(x_2) \cdots t_n f_n(x_n),$$

where the odd parameter $t$ takes care of the grading shift due to $\deg \mathbb{X}_f = \deg f + 1$.

The Laplacian is defined somewhat awkwardly

$$\Delta := - \sum_{i<j} R_{ij} \frac{\partial}{\partial t_j} (\Omega^{-1})_{pq} \frac{\partial}{\partial x^p_i} \frac{\partial}{\partial x^q_j},$$

\[A version of the proof in the \mathbb{R}^{2n}$ case may be found in ref.\[12, and ref.\[15 contains the \mathbb{R}^{2n|m}$ case but is rather sketchy in showing the homomorphism\]
where \( R_{i,j} \) renames \( x_j, t_j \) as \( x_i, t_i \). It is easy to check the following

\[
\Delta t_m f(x_m) t_n g(x_n) = -(-1)^{x^f x^p} R_m n \frac{\partial}{\partial n} (\Omega^{-1})^{pq} (t_m \partial_p f(x_m)) (t_n \partial_q g(x_n)) \\
= -(-1)^{x^f x^p + 1 + x^p + f + x^p} t_m \{ f(x_m), g(x_m) \} = (-1)^{f} t_m \{ f(x_m), g(x_m) \}.
\]

This relation mimics the second one of Eq. (23). We need to check \( \Delta^2 = 0 \) to ensure \( \Delta \) induces a differential,

\[
\Delta^2 = \sum_{i<j; k<l} R_{k,i} \frac{\partial}{\partial t_i} (\Omega^{-1})^{rs} \frac{\partial}{\partial x^r} \frac{\partial}{\partial x^s} R_{i,j} \frac{\partial}{\partial t_j} (\Omega^{-1})^{pq} \frac{\partial}{\partial x^p} \frac{\partial}{\partial x^q}.
\]

When \( k, l, i, j \) are not the same, this certainly vanishes, due to the antisymmetry of \( t \). However we may also have the following possibilities

\[ l = i ; \quad \Delta^2 \Rightarrow R_{k,i} \frac{\partial}{\partial t_i} (\Omega^{-1})^{rs} \left[ \frac{\partial}{\partial x^r} + \frac{\partial}{\partial x^s} \right] (\Omega^{-1})^{pq} \frac{\partial}{\partial x^p} \frac{\partial}{\partial x^q}, \]

\[ k = i ; \quad \Delta^2 \Rightarrow R_{i,j} \frac{\partial}{\partial t_i} (\Omega^{-1})^{rs} \left[ \frac{\partial}{\partial x^r} + \frac{\partial}{\partial x^s} \right] (\Omega^{-1})^{pq} \frac{\partial}{\partial x^p} \frac{\partial}{\partial x^q}. \]

The third term is zero by itself. The fourth one depending whether \( j < l \) or \( j > l \) can be reshuffled and cancel the first and second term respectively.

The homomorphism from a CE chain to the graph chain is defined as: fix \( n \) vertices labelled \( 1 \cdots n \) to correspond to \( n \) dummy variables \( x_i \); every way of connecting \( n \) vertices with edges gives a graph \( \Gamma_{1\cdots n} \). Here the subscript denotes the dummy variable \( x_1 \cdots x_n \). Since the vertices of the graph are already ordered, we need only choose an orientation of all the edges to fix the orientation of graph. For an edge that runs from vertex \( i \) to \( j \), form the operator \( (\Omega^{-1})^{pq} \partial_{x^p} \partial_{x^q} \). Doing the same for every edge in \( \Gamma_{1\cdots k} \) gives the operator \( \beta_{\Gamma_{1\cdots k}} \). A graph like the following will form the operator

![Figure 11: Box](image)

\[
\beta_{\Gamma} = ((\Omega^{-1})^{p_{1}q_{3}} \frac{\partial}{\partial x_{3}^{p_{3}}} \frac{\partial}{\partial x_{1}^{q_{1}}})((\Omega^{-1})^{p_{1}q_{3}} \frac{\partial}{\partial x_{1}^{p_{1}}} \frac{\partial}{\partial x_{3}^{q_{3}}})((\Omega^{-1})^{p_{2}q_{2}} \frac{\partial}{\partial x_{3}^{p_{2}}} \frac{\partial}{\partial x_{4}^{q_{2}}})
\]

\[
((\Omega^{-1})^{r_{2}s_{1}} \frac{\partial}{\partial x_{2}^{r_{2}}} \frac{\partial}{\partial x_{3}^{s_{1}}})((\Omega^{-1})^{r_{3}s_{2}} \frac{\partial}{\partial x_{4}^{r_{3}}} \frac{\partial}{\partial x_{3}^{s_{2}}})((\Omega^{-1})^{r_{3}s_{1}} \frac{\partial}{\partial x_{4}^{r_{3}}} \frac{\partial}{\partial x_{4}^{s_{1}}}).
\]

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graphs with loops are zero. Next we define homomorphism taking a CE chain \( c \) to the following
\[
\beta c = \sum_{\Gamma_{1\ldots k}} [\Gamma_{1\ldots k}] \int dt_k \cdots dt_1 \cdot \beta_{\Gamma_{1\ldots k}} c \bigg|_{x=0} ,
\]
where the sum is over all possible ways of connecting \( n \) vertices. The way this operator acts on the CE chain formally resembles the Wick formula for Gaussian integral. If the reader had been wondering about the seeming loss of the graded commutativity in Eq.11 the sum of all ways of connecting is the remedy. As an example, take \( c = t_1 f_2(x_1) t_2 f_1(x_2) \cdots = (-1)^{(f_1+1)(f_2+1)} t_2 f_1(x_2) t_1 f_2(x_1) \cdots \). Since the \( t_{1,2}, x_{1,2} \) are dummy variables, we can rename them in \( c \) and in \( \beta \) simultaneously. The resulting graph will be \((-1)^{(f_1+1)(f_2+1)} \beta (t_1 f_1(x_1) t_2 f_2(x_2) \cdots )\), as it should be.

Next we show that this is a homomorphism: \( \beta \partial c = \partial \beta c \). We investigate \( \partial_t \) first, since this one also appears independently in the un-extended CE/graph complex. Take \( c = t_1 f_1(x_1) \cdots t_{n+1} f_{n+1}(x_{n+1}) \),
\[
\partial_t c = \sum_{1 \leq i < j \leq n+1} c_{ij},
\]
\[
c_{ij} = (-1)^{s_{ij}} t_0 \{ f_i, f_j \}(x_0) t_1 f_1(x_1) \cdots \hat{t} \cdots \hat{j} \cdots t_{n+1} f_{n+1}(x_{n+1})
\]
\[
\beta c_{ij} = \sum_{\Gamma} [\Gamma_{0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n+1}] \int dt_{n+1} \cdots \hat{t}_j \cdots \hat{t}_i \cdots dt_0 \beta_{\Gamma_{0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n+1}} c_{ij} \bigg|_{x=0} ,
\]
where \( \Gamma_{0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n+1} \) means the \( n \) vertices in this graph correspond to the dummy variable \( x_0, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots x_n \).

Variable \( x_0 \) will appear among the differentials of \( \beta_{\Gamma_{0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n+1}} \), but if we agree to understand the following
\[
(\Omega^{-1})^{pq} \frac{\partial}{\partial x_0^p} \frac{\partial}{\partial x_k^q} \to (\Omega^{-1})^{pq} \left( \frac{\partial}{\partial x_0^p} + \frac{\partial}{\partial x_j^p} + \frac{\partial}{\partial x_k^q} \right),
\]
for every \( x_0 \) in \( \beta \), we can rewrite \( \beta c_{ij} \) as
\[
\beta c_{ij} = \sum_{\Gamma} [\Gamma_{0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n+1}] \int dt_{n+1} \cdots \hat{t}_j \cdots \hat{t}_i \cdots dt_0 \beta_{\Gamma_{0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n+1}}
\]
\[
R_{0,ij} (-1) \frac{\partial}{\partial t_j} (\Omega^{-1})^{pq} \frac{\partial}{\partial x_i^p} \frac{\partial}{\partial x_j^q} (t_1 f_1(x_1) \cdots t_{n+1} f_{n+1}(x_{n+1}))
\]
\[
= \sum_{\Gamma,\hat{\Gamma}} [\Gamma_{0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n+1}] \int dt_{n+1} \cdots dt_1 \beta_{\Gamma_{0,\ldots,\hat{i},\ldots,\hat{j},\ldots,n+1}} (-1)^{i+j-1} (t_1 f_1(x_1) \cdots t_{n+1} f_{n+1}(x_{n+1})) \bigg|_{x=0} ,
\]
\[47\]
where \( \hat{\Gamma} \) is obtained from \( \Gamma \) by breaking up the vertex 0 in \( \Gamma \) into vertex \( i, j \) \((i < j)\) in all possible ways as in fig.12. Since we assumed that all \( f \) are at least cubic, the uni- or bi-valent vertex will not occur in the breaking. The sign factor \((-1)^{i+j+1}\) is taken to be the

\[
\begin{align*}
\text{Figure 12: Breaking into two internal vertices}
\end{align*}
\]

definition of the incidence number between \([\Gamma_{0,\ldots,i,\ldots,n+1}]\) and \([\hat{\Gamma}_{1,\ldots,n+1}]\). This means the last line is exactly \( \partial_I \beta c \) and we have re-produced the proof of ref.\[12, 15\]. The discrepancy of the incidence number here with that of fig.6 is due to two reasons, there we combined vertices \( i, j \) and named it \( i \) instead of 0 and also the labelling of the vertices there stars from 0 instead of 1. These two factors causes \((-1)^{i+j+1} \rightarrow (-1)^{i+j+1+(i-1)} \rightarrow (-1)^{(j+1)+1+(i-1)} = (-1)^{j+1}\).

To proceed to the extended graph, we use extra formal odd parameters \( t_{n+1} \cdots t_{n+l} \); this way we can write the entire chain as one function \((f_1, \ldots, f_n; g_1, \ldots, g_l) = t_1f_1(x_1) \cdots t_nf_n(x_n)t_{n+1}g_1(x_{n+1}) \cdots t_{n+l}g_l(x_{n+l})\).

One need not worry about seeming loss of cyclicity here either.

To construct the operator, let \( \Gamma_{1,\ldots,n;1,\ldots,l} \) denote an extended graph. We form \( \beta \) like before, but also for every edge that runs from a peripheral vertex to an internal one, we include an operator

\[
(\Omega^{-1})^{pq} \frac{\partial}{\partial x_{n+j}^p} \frac{\partial}{\partial x_n^q}; \quad 1 \leq j \leq l, 1 \leq i \leq n,
\]

and for every edge that runs from vertex \( i \) to \( j \) on the Wilson loop, we include

\[
(\Omega^{-1})^{pq} \frac{\partial}{\partial x_{n+i}^p} \frac{\partial}{\partial x_{n+j}^q}; \quad 1 \leq i, j \leq l.
\]

To investigate \( \partial_{\nu} \), let \( c = (f_1, \ldots, f_n; g_1, \ldots, g_l) \),

\[
\partial_{\nu} c = \sum_{1 \leq i \leq n; 1 \leq j \leq l} c_{ij},
\]

\[
c_{ij} = -(-1)^{i+j}(t_1f_1(x_1)) \cdots (t_if_i(x_i)) \cdots (t_nf_n(x_n)) \cdots (t_{n+1}g_1(x_{n+1}) \cdots t_{n+j}f_i(x_{n+j}) \cdots t_{n+l}g_l(x_{n+l})).
\]

\[
\beta c_{ij} = \sum_{\Gamma}[\Gamma_{1,\ldots,i,\ldots,n;1,\ldots,l}] \int dt_{n+1} \cdots dt_{n+l} dt_{n} \cdots dt_{i} \cdots dt_{1} \beta \Gamma_{1,\ldots,i,\ldots,n;1,\ldots,l} c_{ij} \big|_{x=0}.
\]
The only tricky terms in $\beta_\Gamma$ are of the type 

$$(\Omega^{-1})^{pq} \frac{\partial}{\partial x^i_{n+j}} \frac{\partial}{\partial x^p_k} , \ k \neq i .$$

But if we agree to replace

$$\frac{\partial}{\partial x^i_{n+j}} \Rightarrow \frac{\partial}{\partial x^i_{n+j}} + \frac{\partial}{\partial x^j}$$

within $\beta_\Gamma$, then $\beta_{ci}$ can be written as

$$\beta_{ci} = \sum_{\Gamma} [\Gamma_1, \ldots, \hat{i}, \ldots, l] \int \frac{dt_{n+l}}{t_n} \cdots \frac{dt_i}{t_1}$$

$$R_{n+j,i} \beta_{\Gamma_1, \ldots, \hat{i}, \ldots, l}( - \frac{\partial}{\partial t_i} (\Omega^{-1})^{rs} \frac{\partial}{\partial x^r_{n+j}} \frac{\partial}{\partial x^s_i} f_1, \cdots f_n; g_1, \cdots g_l)$$

$$= \sum_{\hat{\Gamma}_{j,l}} [\hat{\Gamma}_{1}, \ldots, \hat{i}, \ldots, l] \int \frac{dt_{n+l}}{t_n} \cdots \frac{dt_1}{t_1} \beta_{\hat{\Gamma}_{1}, \ldots, \hat{i}, \ldots, l} f_1, \cdots \frac{f}{t_1} g_1, \cdots g_l |_{x=0} ,$$

where $\hat{\Gamma}$ is obtained from $\Gamma$ by splitting the $j$'th peripheral vertex into an internal one and an peripheral one as in fig.13. And $(-1)^i$ is defined to be the incident number between

![Figure 13: Breaking into an internal and a peripheral vertex](image)

$[\Gamma_{1, \ldots, \hat{i}, \ldots, n, \ldots, l}]$ and $[\hat{\Gamma}_{1, \ldots, n, \ldots, l}]$. So we have again $\beta \partial_H c = \partial_V \beta c$.

Now look at $\partial_H c$,

$$\partial_H c = \sum_{1 \leq j < l} c_j + c_l ,$$

$$c_j = -(-1)^{\sum j (f_{k+1}) + \sum j (g_{k+1})} (t_1 f_1(x_1)) \cdots (t_n f_n(x_n))$$

$$t_{n+1} g_{1}(x_{n+1}) \cdots t_{n+j} g_{j} g_{j+1}(x_{n+j}) t_{n+j+2} g_{j+2}(x_{n+j+2}) \cdots t_{n+l} g_{l}(x_{n+l}) , \quad 1 \leq j < l ,$$

$$c_l = -(-1)^{\sum l (f_{k+1}) + (g_{k+1}) \sum l (g_{k+1})} (t_1 f_1(x_1)) \cdots (t_n f_n(x_n))$$

$$t_{n+1} g_1(x_{n+1}) \cdots t_{n+l-1} g_{l-1}(x_{n+l-1}) ,$$

$$\beta c_j = \sum_{\Gamma} [\Gamma_{1, \ldots, \hat{j+1}, \ldots, l}] \int \frac{dt_{n+l}}{t_n} \cdots \frac{dt_{n+j+1}}{t_{n+j+1}} \cdots \frac{dt_1}{t_1} R_{n+j,n+j+1} \beta_{\hat{\Gamma}_{1, \ldots, \hat{j+1}, \ldots, l}} c_j |_{x=0} ,$$

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For this term, we only need to worry about operators like
\[(\Omega^{-1})^{pq} \frac{\partial}{\partial x_{n+j}} \frac{\partial}{\partial x_k}, \text{ or } (\Omega^{-1})^{pq} \frac{\partial}{\partial x_{n+j}} \frac{\partial}{\partial x_{n+i}}, \quad i \neq j + 1\]

But if we agree to replace
\[\frac{\partial}{\partial x_{n+j}} \Rightarrow \frac{\partial}{\partial x_{n+j}} + \frac{\partial}{\partial x_{n+j+1}},\]
within \(\beta\), then we can write \(\beta c_j, \quad j < l\)
as
\[
\beta c_j = -\sum \Gamma_{1, \ldots, n, 1, \ldots, j, \ldots, l} [\int dt_{n+l} \cdots \hat{t}_{n+j+1} \cdots dt_1 
R_{n+j, n+j+1} \beta_{\Gamma_{1, \ldots, n, \ldots, j, \ldots, l}} \frac{\partial}{\partial t_{n+j+1}} (f_1, \cdots f_n; g_1, \cdots g_l) \big|_{x=0} \n= \sum \Gamma_{1, \ldots, n, 1, \ldots, j, \ldots, l} [\int dt_{n+l} \cdots dt_1 (-1)^{n+j} \beta_{\Gamma_{1, \ldots, n, \ldots, j, \ldots, l}} (f_1, \cdots f_n; g_1, \cdots g_l) \big|_{x=0},
\]

where \(\hat{\Gamma}\) is obtained by breaking vertex \(j\) in \(\Gamma\) into vertex \(j\) and \(j + 1\) in all possible ways as in fig.14. Naturally, we take \((-1)^{n+j+1}\) as the incidence number between \([\Gamma_{1, \ldots, n, \ldots, j, \ldots, l}]\) and \([\hat{\Gamma}_{1, \ldots, n, \ldots, j, \ldots, l}]\). One special case which we do not cover is when \(j = l\), but the reader can easily verify that \(\partial_H\) combines vertex \(l\) and \(1\) and names the new vertex 1 with incident number \((-1)^{n+l-1}\). The final conclusion is that \(\beta \partial_H c = \partial_H \beta c\) and we completed the proof that \(\beta\) induces a chain map between the extended CE complex and the extended graph complex.

Next we construct the inverse of \(\beta\). Take a super vector space \(\mathbb{R}^{2n|m}\), \(n, m\) big enough (bigger than the number of edges in a graph). Let \(p_i, q_i, \quad i = 1 \sim n\) be the bosonic coordinates with the bracket \(\{p_i, q_j\} = -\{q_j, p_i\} = \delta_{ij}\). Let us number the edges from 1 through \(k\), for an edge \(i\), we associate \(p_i\) to the vertex from which it issues forth and \(q_i\) to the vertex on which it ends. This way, we form a polynomial for each vertex. For example, the fig.15 gives
\[
c = (q_1 p_5 p_4; q_5 p_2 q_3; p_1, q_2, p_3, q_4). \quad (43)
\]
Figure 15: Example of the inverse map, the vertices are labelled as: 1 and 2 for the left and right internal vertex, 1 for the lower left peripheral vertex and numbering increases counterclockwise.

Clearly $\beta$ acting on $c$ only produces one nonzero term which is exactly the graph we want. It is perhaps queer that only the bosonic part of $\mathbb{R}^{2n|m}$ is used in this construction, but one must remember that $c$ is to be regarded as a representative of the orbit under the $osp_{2n|m}$ action. One can also use the odd part of $\mathbb{R}^{2n|m}$ to form the same inverse. But this time, it is more convenient to use the alternative orientation scheme for the graph complex, one orders all the even valent vertices and orders all incident legs of every vertex \footnote{See the remark in the second paragraph of sec.6.2}. Let $\xi^i$ be the odd coordinates, even though we can no longer choose $p$ or $q$ make distinction between income or outgoing legs, we can place the $\xi$’s in one vertex in the order conforming to the ordering of the incident legs for that vertex.

Finally as an exercise, let us see an example of graph cycles constructed from the CE cycle. Let $A^\alpha$ be the odd coordinate of $su(n)[1]$, $so(n)[1]$, $sp(2n)[1]$, $\eta_{\alpha\beta} = Tr[T^\alpha T^\beta]$ be the killing metric and also let

$$c = \frac{1}{4} ( ; A, A, A, A ) + \frac{1}{3} ( \frac{1}{3} Tr[A^3]; A, A, A )$$

$$+ \frac{1}{2!} ( \frac{1}{3} Tr[A^3], \frac{1}{3} Tr[A^3]; A, A ) + \frac{1}{3!} ( \frac{1}{3} Tr[A^3], \frac{1}{3} Tr[A^3], \frac{1}{3} Tr[A^3]; A ) ,$$

where $A = A^\alpha T^\alpha$. One can check directly that $c$ is closed. Applying $\beta$, we get the four graphs in fig.\[16\] with coefficients (where $T^\alpha = \eta^{\alpha\beta} T^\beta$)

Figure 16: Lowest order cycles
\[
\beta c = -\frac{1}{2} \text{Tr}[T^\alpha T^\beta T^\beta][\Gamma_1] - \frac{1}{4} \text{Tr}[T^\alpha T^\beta T^\beta T^\beta][\Gamma_2] - \frac{i}{3} f_{\alpha\beta\gamma} \text{Tr}[T^\alpha T^\beta T^\gamma][\Gamma_3] - \frac{1}{4} f_{\alpha\beta\gamma} f^{\alpha\beta\gamma}[\Gamma_4]
\]
\[
= \frac{d_r C_2(G)}{2} \left( \frac{1}{4}[\Gamma_2] + \frac{1}{3}[\Gamma_3] - \frac{1}{2}[\Gamma_4] \right) - \frac{d_r C_2^2(r)}{4} \left( [\Gamma_2] + 2[\Gamma_1] \right),
\]
where \(d_r, G\) is the dimension of representation \(r\) and adjoint representation, \(C_2\) is the second Casimir. So the result is the linear combination of two cycles in the two brackets above. Just for a record, the coefficient for the first cycle is \(n^2(n^2 - 1), n(n - 1)(n - 2), n(2n + 1)(n + 1)\) and \(- (n^2 - 1)^2/(4n), -n(n - 1)^2, -n(2n + 1)/8\) for the second.

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