Optimal Adaptive Learning in Uncontrolled Restless Bandit Problems

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Abstract

In this paper we consider the problem of learning the optimal policy for uncontrolled restless bandit problems. In an uncontrolled restless bandit problem, there is a finite set of arms, each of which when pulled yields a positive reward. There is a player who sequentially selects one of the arms at each time step. The goal of the player is to maximize its undiscounted reward over a time horizon $T$. The reward process of each arm is a finite state Markov chain, whose transition probabilities are unknown by the player. State transitions of each arm is independent of the selection of the player. We propose a learning algorithm with logarithmic regret uniformly over time with respect to the optimal finite horizon policy. Our results extend the optimal adaptive learning of MDPs to POMDPs.

Index Terms

Online learning, restless bandits, POMDPs, regret, exploration-exploitation tradeoff

I. INTRODUCTION

In an uncontrolled restless bandit problem (URBP) there is a set of arms indexed by $1, 2, \ldots, K$, whose state process is discrete and follows a discrete time Markov rule independent of each other. There is a user/player who chooses one arm at each of the discrete time steps, gets the reward and observes the current state of the selected arm. The control action, i.e., the arm selection, does not affect the state transitions, therefore the system dynamics is uncontrolled. However, it is both used to exploit the instantaneous reward and to decrease the uncertainty about the current state of the system by exploring. Thus, the optimal policy should balance the tradeoff between exploration and exploitation.

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If the structure of the system, i.e., the state transition probabilities and the rewards of the arms are known, then the optimal policy can be found by dynamic programming for any finite horizon. In the case of infinite horizon, stationary optimal policies can be found for the discounted problem by using the contraction properties of the dynamic programming operator. For the infinite horizon average reward problem, stationary optimal policies can be found under some assumptions on the transition probabilities [1], [2]. In this paper, we assume that initially the player has no knowledge about the transition probabilities of the arms. Therefore, the problem we consider is a learning problem rather than an optimization problem given the structure of the system. Our assumption is consistent with the fact that in most of the systems, the player does not have a perfect model for the system at the beginning but learns the model over time. For example, in a cognitive radio network, initially the player (secondary user) may not know how the primary user activity evolves over time, or in a target tracking system, initially the statistics of movement of the target may be unknown. Thus, our goal is to design learning algorithms with fastest convergence rate, i.e., minimum regret where regret of a learning policy at time $t$ is defined as the difference between reward of the optimal policy for the undiscounted $t$-horizon problem with full information about the system model and the undiscounted reward of the learning policy up to time $t$.

In this paper, we show that when the transition probability between any two states of the same arm is always positive, an algorithm with logarithmic regret uniform in time with respect to the optimal policy for the finite time undiscounted problem with known transition probabilities exist. We also claim that logarithmic order is the best achievable order for URBP. To the best of our knowledge this paper is the first attempt to extend the optimal adaptive learning in MDPs to partially observable Markov decision processes (POMDP), in which there are countably many information states.

The organization of the remainder of this paper is as follows. Related work is given in Section II. In Section III, we give the problem formulation, notations and some lemmas that will be used throughout the proofs in the paper. In Section IV, we give sufficient conditions under which the average reward optimality equation has a continuous solution. In Section V, we give an equivalent countable representation of the information state and an assumption under which the regret of a policy can be related to the expected number of times a suboptimal action is taken. In Section VI, an adaptive learning algorithm is given. Then, under an assumption on the structure of the optimal policy, we give an upper bound for the regret of the adaptive learning algorithm in Section VII. In Section VIII, we prove that this upper bound is logarithmic in time. In Section IX, we provide extensions to the adaptive learning algorithm with near-logarithmic regret, and relax the assumption on the structure of the optimal policy. Section X concludes...
the paper.

II. RELATED WORK

Related work in optimal adaptive learning started with the paper of Lai and Robbins [3], where asymptotically optimal adaptive policies for the multi-armed bandit problem with i.i.d. reward process for each arm were constructed. These are index policies and it is shown that they achieve the optimal regret both in terms of the constant and the order. Later Agrawal [4] considered the i.i.d. problem and provided sample mean based index policies which are easier to compute, order optimal but not optimal in terms of the constant in general. Anantharam et. al. [5], [6] proposed asymptotically optimal policies with multiple plays at each time for i.i.d. and Markovian arms respectively. However, all the above work assumed parametrized distributions for the reward process of the arms. Auer et. al. [7] considered the i.i.d. multi-armed bandit problem and proposed sample mean based index policies with logarithmic regret when reward processes have a bounded support. Their upper bound holds uniformly over time rather than asymptotically but this bound is not asymptotically optimal. Following this approach, Tekin and Liu [8], [9] provided policies with uniformly logarithmic regret bounds with respect to the best single arm policy for restless and rested multi-armed bandit problems and extended the results to single player multiple plays and decentralized multi-player models in [10]. Decentralized multi-player versions of the i.i.d. multi-armed bandit problem under different collision models were considered in [11], [12], [13], [14]. Other research on adaptive learning focused on Markov Decision Processes (MDP) with finite state and action space. Burnetas and Katehakis [15] proposed index policies with asymptotically logarithmic regret, where the indices are the inflations of right-hand sides of the estimated average reward optimality equations based on Kullback Leibler (KL) divergence, and showed that these are asymptotically optimal both in terms of the order and the constant. However, they assumed that the support of the transition probabilities are known. Tewari and Bartlett [16] proposed a learning algorithm that uses $l_1$ distance instead of KL divergence with the same order of regret but a larger constant. Their proof is simpler than the proof in [15] and does not require the support of the transition probabilities to be known. Auer and Ortner [17] proposed another algorithm with logarithmic regret and reduced computation for the MDP problem, which solves the average reward optimality equations only when a confidence interval is halved. In all the above work the MDPs are assumed to be irreducible.

Contrary to the related work in Markovian restless bandits [10], which compare the performance of the learning algorithm with the best static policy, in this paper we compare our learning algorithm with the best dynamic policy.
III. PROBLEM FORMULATION AND ASSUMPTIONS

In this paper we study strong regret algorithms for uncontrolled restless Markovian bandit problems (URBP). Consider $K$ mutually independent uncontrolled restless Markovian arms, indexed by the set $\mathcal{K} = \{1, 2, \ldots, K\}$ whose states evolve in discrete time steps $t = 1, 2, \ldots$ according to finite state Markov chain with unknown transition probabilities.

Let $S^k$ be the state space of arm $k$. For simplicity of presentation, without loss of generality, assume that for state $x \in S^k$, $r^k_x = x$, i.e., the state of an arm also represents the reward from that arm. Then, the state space of the system is the Cartesian product of the state spaces of individual arms which is denoted by $S = S^1 \times \ldots \times S^K$. Let $p^k_{ij}$ denote the transition probability from state $i$ to $j$ of arm $k$. Then the transition probability matrix of arm $k$ is $P^k$ whose $ij$th element is $p^k_{ij}$. The set of transition probability matrices is denoted by $P = (P^1, \ldots, P^K)$. We assume that $P^k$s are such that the induced Markov chains are ergodic. Therefore for each arm there exists a unique stationary distribution which is given by $\pi^k = (\pi^k_x)_{x \in S^k}$. At each time step, the state of the system is a $K$-dimensional vector of states of arms which is given by $x = (x^1, \ldots, x^K) \in S$.

Next, we define notations that will frequently be used in the following sections. Let $e^k_x$ represent the unit vector with dimension $|S^k|$, whose $x$th element is 1, and all other elements are 0, $\mathbb{N} = \{1, 2, \ldots\}$ represent the set of natural numbers, $\mathbb{Z}_+ = \{0, 1, \ldots\}$ represent the set of non-negative integers, $(v \bullet w)$ represent the standard inner product of vectors $v, w$, $||v||_1, ||v||_\infty$ represent respectively the $l_1$ and $l_\infty$ norm of vector $v$, and $||P||_1$ represent the induced maximum row sum norm for matrices. For a vector $v$, $(v_{-u}, v')$ represents the vector whose $u$th element is $v'$, while all other elements are the same as the elements of $v$. For a vector of matrices $P$, $(P_{-u}, P')$ represents the vector of matrices whose $u$th matrix is $P'$, while all other matrices are the same as the matrices of $P$. The transpose of a vector $v$ or matrix $P$ is denoted by $v^T$ and $P^T$ respectively. The list below gives some of the quantities that frequently appear in the results in this paper.

Notation 1:  
- $\beta = \sum_{t=1}^\infty 1/t^2$, $\pi^k_{\min} = \min_{x \in S^k} \pi^k_x$
- $\pi_{\min} = \min_{k \in \mathcal{K}} \pi^k_{\min}$
- $r^{\max} = \max_{x \in S^k, k \in \mathcal{K}} r^k_x$
- $S^{\max} = \max_{k \in \mathcal{K}} |S^k|$

There is a player who selects one of the $K$ arms at each time step $t$, and gets the reward from that arm depending on the state of that arm. The objective of the player is to maximize the undiscounted sum of the rewards for any finite horizon. However, the player does not know the set of transition probability
matrices $P$. Moreover, the player can only observe the state of the arm it chooses but not the states of the other arms. Intuitively, in order to maximize its reward, the player should explore the arms to estimate their transition probabilities and to reduce the uncertainty about the current state $x \in S$ of the system, while exploiting the information it has acquired about the system to select arms that yield high rewards. This process should be carefully balanced to yield the maximum reward for the player. In a more general sense, the player should learn to play optimally in an uncontrolled POMDP.

We denote the set of $S^k \times S^k$ stochastic matrices by $\Xi^k$. Set of sets of $K$ stochastic matrices is denoted by $\Xi = (\Xi_1, \Xi_2, \ldots, \Xi_K)$. Since $P$ is not known by the player, at time $t$, the player has an estimate of $\hat{P}$, which is given by $\hat{P}_t \in \Xi$. For two sets of transition probability matrices $P$ and $\tilde{P}$, the distance between them is defined as $||P - \tilde{P}||_1 := \sum_{k=1}^{K} ||P^k - \tilde{P}^k||_1$. At any time step, the state of the system in the next time step uncertain due to the stochastically changing states. Therefore, let $X^k_t$ be the random variable representing the state of arm $k$ at time $t$. Then, the random vector $X_t = (X^1_t, X^2_t, \ldots, X^K_t)$ represents the state of the system at time $t$.

Since the player chooses an arm in $K$ at each time step, $K$ can be seen as the action space of the player. Since the player can observe the state of the arm it selects at each time step, $Y = \cup_{k=1}^{K} S^k$ is the observation space of the player. Note that even if two states of different arms have the same reward, the player will label these states differently. Thus $S^k \cap S^l = \emptyset$ for $k \neq l$. Let $u_t \in U$ be the arm selected by the player at time $t$, where $U = K$, and $y_t \in Y$ be the state/reward observed by the player at time $t$. Then, the history at time $t$ is $z^t = (u_0, y_1, u_1, y_2, \ldots, u_{t-1}, y_t)$. Usually, for an algorithm, $u_t$ and $y_t$ depends on the history and the stochastic evolution of the arms. Therefore, we denote by $U_t$ and $Y_t$, the random variables representing the action and the observation at time $t$, respectively. We let $Q_P(y|u)$ be the sub-stochastic transition probability matrix such that $(Q_P(y|u))_{x x'} = P_P(X_{t+1} = x', Y_{t+1} = y|X_t = x, U_t = u)$. For URBP $Q_P(y|u)$ is the zero matrix for $y \notin S^u$. For $y \in S^u$ only nonzero entries of $Q_P(y|u)$ are the ones for which $x^u = y$.

Let $\alpha$ denote a learning algorithm used by the player. In order to decide which arm to select at time $t$, $\alpha$ can only use all past observations and selections by the player up to $t$. We denote by $\alpha(t)$ the arm selected by algorithm $\alpha$ at time $t$. The performance of algorithm $\alpha$ can be measured by regret which is the difference between performance of the algorithm and performance of the optimal policy with known transition probabilities up to $t$. For any algorithm $\alpha$, the regret with respect to the optimal $T$ horizon
policy is

$$R^\alpha(T) = \sup_{\gamma' \in \Gamma} \left( E_{\psi_0, \gamma'}^P \left[ \sum_{t=1}^{T} r^{\gamma'(t)}(t) \right] \right) - E_{\psi_0, \alpha}^P \left[ \sum_{t=1}^{T} r^{\alpha(t)}(t) \right],$$  \hspace{1cm} (1)

where $\Gamma$ is the set of admissible policies. In Section VI, we will propose an algorithm whose regret grows logarithmically in time. Therefore this algorithm converges to the optimal policy in terms of the average reward.

In the following sections we will frequently use results from the large deviations theory. We will relate the accuracy of the transition probability estimates of the player with the probability of deviating from the optimal action. First, we give the definition of a uniformly ergodic Markov chain.

**Definition 1:** [18] A Markov chain $X = \{X_t, t \in \mathbb{Z}_+\}$ on a measurable space $(S, B)$, with transition kernel $P(x, G)$ is uniformly ergodic if there exists constants $\rho < 1, C < \infty$ such that for all $x \in S$,

$$\|e_x P^t - \pi\| \leq C \rho^t, t \in \mathbb{Z}_+,$$  \hspace{1cm} (2)

The norm used in the above definition is the total variation norm. For finite and countable vectors this corresponds to $l_1$ norm, and the induced matrix norm corresponds to maximum absolute row sum norm. Clearly, for a finite state Markov chain, uniform ergodicity is equivalent to ergodicity. Next, we give a large deviation bound for a perturbation of a uniformly ergodic Markov chain.

**Lemma 1:** ([18] Theorem 3.1.) Let $X = \{X_t, t \in \mathbb{Z}_+\}$ be a uniformly ergodic Markov chain for which (2) holds. Let $\hat{X} = \{\hat{X}_t, t \in \mathbb{Z}_+\}$ be the perturbed chain with transition kernel $\hat{P}$. Given the two chains have the same initial distribution, let $\psi_t, \hat{\psi}_t$ be the distribution of $X, \hat{X}$ at time $t$ respectively. Then,

$$\|\psi_t - \hat{\psi}_t\| \leq C_1(P, t) \|\hat{P} - P\|,$$  \hspace{1cm} (3)

where $C_1(P, t) = \left( \hat{t} + C^\alpha \frac{\rho^t}{1-\rho^t} \right)$ and $\hat{t} = [\log_\rho C^{-1}]$.

We will frequently use the Chernoff-Hoeffding bound in our proofs, which bounds the difference between the sample mean and expected reward for distributions with bounded support.

**Lemma 2:** (Chernoff-Hoeffding Bound) Let $X_1, \ldots, X_T$ be random variables with common range $[0,1]$, such that $E[X_t|X_{t-1}, \ldots, X_1] = \mu$. Let $S_T = X_1 + \ldots + X_T$. Then for all $\epsilon \geq 0$

$$P(|S_T - T\mu| \geq \epsilon) \leq 2e^{-\frac{2\epsilon^2}{T}}.$$  

The following lemma, which will be used in the following sections, gives an upper bound on the difference between product of two equal-sized sets of numbers in the unit interval, in terms of the sum
of the absolute values of the pairwise differences between the numbers taken from different sets.

Lemma 3: for $\rho_k, \rho'_k \in [0, 1]$ we have

$$|\rho_1 \cdots \rho_K - \rho'_1 \cdots \rho'_K| \leq \sum_{k=1}^{K} |\rho_k - \rho'_k|$$

(4)

Proof: First consider $|\rho_1 \rho_2 - \rho'_1 \rho'_2|$ where $\rho_1, \rho_2, \rho'_1, \rho'_2 \in [0, 1]$. Let $\epsilon = \rho'_2 - \rho_2$. Then

$$|\rho_1 \rho_2 - \rho'_1 \rho'_2| = |\rho_1 \rho_2 - \rho'_1 (\rho_2 + \epsilon)| = |\rho_2 (\rho_1 - \rho'_1) - \rho'_1 \epsilon| \leq \rho_2 |\rho_1 - \rho'_1| + \rho'_1 |\epsilon|.$$ 

But we have

$$|\rho_1 - \rho'_1| + |\rho_2 - \rho'_2| = |\rho_1 - \rho'_1| + |\epsilon| \geq \rho_2 |\rho_1 - \rho'_1| + \rho'_1 |\epsilon|.$$ 

Thus

$$|\rho_1 \rho_2 - \rho'_1 \rho'_2| \leq |\rho_1 - \rho'_1| + |\rho_2 - \rho'_2|.$$ 

Now, we prove by induction. Clearly (4) holds for $K = 1$. Assume it holds for some $K > 1$. Then

$$|\rho_1 \cdots \rho_{K+1} - \rho'_1 \cdots \rho'_{K+1}| \leq |\rho_1 \cdots \rho_K - \rho'_1 \cdots \rho'_K| + |\rho_{K+1} - \rho'_{K+1}| \leq \sum_{k=1}^{K} |\rho_k - \rho'_k|.$$ 

IV. SOLUTIONS OF THE AVERAGE REWARD OPTIMALITY EQUATION

Assume that the transition probability matrices for the arms are known by the player. Then, URBP turns into an optimization problem (POMDP) rather than a learning problem. In its general form this problem is intractable [19], but heuristics, approximations and exact solutions under different assumptions on the arms are studied by [20], [21], [22] and many others.

One way to represent a POMDP problem is to use the belief space, i.e., the set of probability distributions over the state space. For URBP with the set of transition probability matrices $P$, the belief space is $\Psi = \{\psi : \psi^T \in \mathbb{R}^{|S|}, \psi_x \geq 0, \forall x \in S, \sum_{x \in S} \psi_x = 1\}$ which is the unit simplex in $\mathbb{R}^{|S|}$. Let $\psi_0$ denote the initial belief and $\psi_t$ denote the belief at time $t$. Then, the probability that player observes $y$ given it selects arm $u$ when the belief is $\psi$ is $V_P(\psi, y, u) := \psi Q_P(y|u) \mathbf{1}$, where $\mathbf{1}$ is the $|S|$ dimensional column vector of 1s. Given arm $u$ is chosen at belief $\psi$ and state $y$ is observed, the next belief is $T_P(\psi, y, u) = \psi Q_P(y|u)/V_P(\psi, y, u)$. Let $\Gamma$ be the set of admissible policies, i.e., any policy for which action at $t$ is a function of $\psi_0$ and $z^t$. When $P$ is known by the player, the average reward
optimality equation (AROE) is

\[ g + h(\psi) = \max_{u \in U} \{ \tilde{r}(\psi, u) + \sum_{y \in S^u} V_P(\psi, y, u) h(T_P(\psi, y, u)) \}, \tag{5} \]

where \( g \) is a constant and \( h \) is a function from \( \Psi \to \mathbb{R} \). \( \tilde{r}(\psi, u) = (\psi \cdot r(u)) = \sum_{x^u \in S^u \in S^u} x^u \phi_{u,x^u}(\psi) \) is the expected reward of action \( u \) at belief \( \psi \), \( \phi_{u,x^u}(\psi) \) is the probability that arm \( u \) is in state \( x^u \) given belief \( \psi \), \( r(u) = (r(x, u))_{x \in S} \) and \( r(x, u) = x^u \) is the reward when arm \( u \) is chosen at state \( x \).

The bound on regret, which we will derive in the following sections, will hold under the following assumption.

**Assumption 1:** \( p_{ij}^k > 0, \forall k \in K, i, j \in S^k \).

Under Assumption 1 existence of a bounded, convex continuous solution to (5) is guaranteed. Let \( E_{\psi,\gamma}[.\] denote the expectation operator when policy \( \gamma \) is used with initial state \( \psi \), and the set of transition probability matrices is \( \mathcal{P} \).

Let \( V \) denote the space of bounded real-valued functions on \( \Psi \). Next, we define the undiscounted dynamic programming operator \( F : V \to V \). Let \( v \in V \), we have

\[ (Fv)(\psi) = \max_{u \in U} \left\{ \tilde{r}(\psi, u) + \sum_{y \in S^u} V_P(\psi, y, u) v(T_P(\psi, y, u)) \right\}. \tag{6} \]

**Lemma 4:** Let \( h_+ = h - \inf_{\psi \in \Psi} (h(\psi)) \), \( h_- = h - \sup_{\psi \in \Psi} (h(\psi)) \),

\[ h_{T,P}(\psi) = \sup_{\gamma \in \Gamma} \left( E_{\psi,\gamma}^{P} \left[ \sum_{t=1}^{T} r^\gamma(t) \right] \right) \]

Under Assumption 1 the following holds:

S-1 Consider a sequence of functions \( v_0, v_1, v_2, \ldots \) in \( V \) such that \( v_0 = 0 \), and \( v_l = Fv_{l-1}, l = 1, 2, \ldots \). This sequence converges uniformly to a convex continuous function \( v^* \) for which \( Fv^* = v^* + g \) where \( g \) is a finite constant. In terms of (5), this result means that there exists a finite constant \( g_P \) and a bounded convex continuous function \( h_P : \Psi \to \mathbb{R} \) which is a solution to (5).

S-2 \( h_{P^+}(\psi) \leq h_{T,P}(\psi) - T g_P \leq h_{P^+}(\psi), \forall \psi \in \Psi \).

S-3 \( h_{T,P}(\psi) = T g_P + h_P(\psi) + O(1) \) as \( T \to \infty \).

**Proof:** Sufficient conditions for the existence of a bounded convex continuous solution to the AROE is investigated in [1]. According to Theorem 4 of [1], if reachability and detectability conditions are satisfied then S-1 holds. Below, we directly prove that reachability condition in [1] is satisfied. To prove
that detectability condition is satisfied, we show another condition, i.e., subrectangular substochastic matrices, holds which implies the detectability condition.

We note that $P(X_{t+1} = x'|X_t = x) > 0$, $\forall x, x' \in S$ since by Assumption $p^k_{ij} > 0$ $\forall i, j \in S^k, \forall k \in K$.

**Condition 1:** (Reachability) There is a $\rho < 1$ and an integer $\xi$ such that

$$\sup_{\gamma \in \Gamma} \max_{0 \leq t \leq \xi} P(X_t = x | \psi_0) \geq 1 - \rho, \quad \forall \psi_0 \in \Psi.$$ 

Set $\rho = 1 - \min_{x,x'} P(X_{t+1} = x'|X_t = x)$, $\xi = 1$. Since the system is uncontrolled, state transitions are independent of the arm selected by the player. Therefore,

$$\sup_{\gamma \in \Gamma} P(X_1 = x | \psi_0) = P(X_1 = x | \psi_0) \geq \min_{x,x'} P(X_{t+1} = x'|X_t = x) = 1 - \rho.$$

**Condition 2:** (Subrectangular matrices) For any substochastic matrix $Q(y|u)$, $y \in Y, u \in U$, and for any $i, i', j, j' \in S$,

$$(Q(y|u))_{ij} > 0 \text{ and } (Q(y|u))_{i'j'} > 0 \Rightarrow (Q(y|u))_{ij'} > 0 \text{ and } (Q(y|u))_{i'j} > 0.$$ 

$Q(y|u)$ is subrectangular for $y \notin S^u$ since it is the zero matrix. For $y \in S^u$ all entries of $Q(y|u)$ is positive since $P(X_{t+1} = x'|X_t = x) > 0$, $\forall x, x' \in S$.

S-2 holds by Lemma 1 in [1], and S-3 is a consequence of S-2 and the boundedness property in S-1.

**V. COUNTABLE REPRESENTATION OF THE INFORMATION STATE**

Assume that the initial $K$ steps are such that the player select arm $k$ at the $k$th step. Then, the POMDP for the player can be written as a countable state MDP. In this way, the information state at time $t$ can be represented by $(s_t, \tau_t) = ((s^1_t, s^2_t, ..., s^K_t), (\tau^1_t, \tau^2_t, ..., \tau^K_t))$, where $s^k_t$ and $\tau^k_t$ are the last observed state from arm $k$ and time from the last observation of arm $k$ to $t$, respectively. The contribution of the initial $K$ steps to the regret is at most $Kr_{\max}$. Therefore we only analyze the time steps after this initialization, i.e., $t = 0$ when the initialization phase is completed. This way, the initial information state of the player can be written as $(s_0, \tau_0)$. Let $C$ be the set of all possible information states that the player can be in. Since the player selects a single arm at each time step, at any time $\tau^k = 1$ for the last selected arm $k$. 

The player can compute its belief state $\psi \in \Psi$ by using its transition probability estimates $\hat{P}$ together with the information state $(s_t, \tau_t)$. We let $\psi_P(s_t, \tau_t)$ be the belief that corresponds to information state $(s_t, \tau_t)$ when the set of transition probability matrices is $P$. The player knows the information state exactly, but it only has an estimate of the belief that corresponds to the information state, because it does not know the transition probabilities. The true belief computed with the knowledge of exact transition probabilities and information state at time $t$ is denoted by $\psi_t$, while the estimated belief computed with estimated transition probabilities and information state at time $t$ is denoted by $\hat{\psi}_t$.

When the belief is $\psi$ and the set of transition probability matrices is $P$, the set of optimal actions which are the maximizers of (5) is denoted by $O(\psi; P)$. Therefore, when the information state is $(s_t, \tau_t)$, and the set of transition probability matrices is $P$, the set of optimal actions is $O((s, \tau); P) := O(\psi_P((s, \tau)); P)$.

Note that even when the agent is given the optimal policy for the infinite horizon average reward problem, it may not be able to play optimally because he does not know the exact belief $\psi_t$ at time $t$. In this case, in order for the agent to play optimally, there should be an $\epsilon > 0$ such that if $||\psi_t - \hat{\psi}_t||_1 < \epsilon$, the set of actions that are optimal in $\hat{\psi}_t$ should be a subset of the set of actions that are optimal in $\psi_t$. This is indeed the case, and we prove it by exploiting the continuity of the solution to (5) under Assumption 1.

We start by defining finite partitions of the set of information states $C$.

**Definition 2:** Let $\tau^0 > 0$ be an integer. Consider a vector $i = (i^1, \ldots, i^K)$ such that either $i^k = \tau^0$ or $i^k = (s^k, \tau^k), \tau_k < \tau^0, s^k \in S^k$. Each vector will define a set in the partition of $C$ so we call $i$ a **partition vector**. Let $G_{\tau^0}$ denote the partition formed by $\tau^0$. Let $s'i^1, \ldots, i^K = \{s^k : i^k \neq \tau^0\}$ and $\tau'(i^1, \ldots, i^K) = \{\tau^k : i^k \neq \tau^0\}$. Let $M(i^1, \ldots, i^K) := \{k : i^k = \tau^0\}$ be the set of arms that are played at least $\tau^0$ time steps ago. Let $M(i^1, \ldots, i^K) := C - M(i^1, \ldots, i^K)$. Vector $(i^1, \ldots, i^K)$ forms the following set in the partition $G_{\tau^0}$.

$$G_{i^1, \ldots, i^K} = \{(s, \tau) \in C : (s, \tau)_{M(i^1, \ldots, i^K)} = s', \tau_{M(i^1, \ldots, i^K)} = \tau'), s^k \in S^k, \tau^k \geq \tau^0, \forall k \in M(i^1, \ldots, i^K)\}.$$  

Let $A(\tau^0)$ be the number of sets in partition $G_{\tau^0}$. Re-index the sets in $G_{\tau^0}$ as $G_1, G_2, \ldots, G_{A(\tau^0)}$. These sets form a partition of $C$ such that $G_l$ either consists of a single information state $(s, \tau)$ for which $\tau^k < \tau^0, \forall k \in K$, or it includes infinitely many information states of the arms in $M(i^1, \ldots, i^K)$.

Consider a set of transition probability matrices $\hat{P}$ for which Assumption 1 holds, so all arms have a stationary distribution. If we map a set $G_l$ with infinitely many information states to the belief space using $\psi_P$, for any $\delta > 0$, only a finite number of information states in $G_l$ will lie outside the radius $\delta$ ball centered at the stationary distribution of arms for which $i^k = \tau^0$. 
For a set $G_l \in \mathcal{G}_{\tau^0}$, given a set of transition probability matrices $P$, we define its center as follows. If $G_l$ only contains a single information state, then the belief corresponding to that information state is the center of $G_l$. If $G_l$ contains infinitely many information states, then the center of $G_l$ is the belief in which all arms for which $i^k = \tau^0$ are in their stationary distribution based on $P$. In both cases the belief which is the center of $G_l$ is denoted by $\psi^*(G_l; P)$. Let $O^*(G_l; P)$ be the set of optimal actions at this belief. Note that for any $\tau^0 > 0$, the number of sets with infinitely many elements is the same, and each of these sets are centered around the stationary distribution. However, as $\tau^0$ increases, the number of sets with a single information state increases. The points in belief space corresponding to these sets are shown in Figure 1.

Next, we define extensions of the sets $G_l$ on the belief space. For a set $A \in \Psi$ let $A(\epsilon)$ be the $\epsilon$ extension of that set, i.e., $A(\epsilon) = \{ \psi \in \Psi : \psi \in A \text{ or } d_1(\psi, A) < \epsilon \}$, where $d_1(\psi, A)$ is the minimum $l_1$ distance between $\psi$ and any element of $A$. The $\epsilon$-extension of $G_l \in \mathcal{G}_{\tau^0}$ corresponding to $P$ is the $\epsilon$-extension of the convex-hull of the points $\psi(s, \tau)$ such that $(s, \tau) \in G_l$. Let $J_l(\epsilon)$ denote the $\epsilon$-extension of $G_l$. Examples of $J_l(\epsilon)$ on the belief space is given in Figure V.

Let the diameter of a set $A$ be the maximum distance between any two elements of the set $A$. Another observation is that when $\tau^0$ increases, the diameter of the convex-hull of the points of the sets in $\mathcal{G}_{\tau^0}$ that contains infinitely many elements decreases. In the following lemma, we show that when $\tau^0$ is chosen large enough, there exists $\epsilon > 0$ such for all $G_l \in \mathcal{G}_{\tau^0}$, we have non-overlapping $\epsilon$-extensions in which only a subset of the actions in $O^*(G_l; P)$ is optimal.

Fig. 1. Partition of $C$ on $\Psi$ based on $P$. $G_l$ is a set with a single information state and $G_{l'}$ is a set with infinitely many information states.
Lemma 5: For any $P$, $\exists \tau^0 > 0$ and $\epsilon > 0$ such that for all $G_l \in G_{\tau^0}$, its $\epsilon$-extension $J_{l,\epsilon}$ has the following properties:

i For any $\psi \in J_{l,\epsilon}$, $O(\psi; P) \subset O^*(G_l; P)$.

ii For $l \neq l'$, $J_{l,\epsilon} \cap J_{l',\epsilon} = \emptyset$.

Proof: For $G_l \in G_{\tau^0}$ consider $\psi^*(G_l; P)$. For any $\psi \in \Psi$ the suboptimality gap is defined as

$$
\Delta(\psi, P) = \max_{u \in U} \left\{ \bar{r}(\psi, u) + \sum_{y \in S^u} V_P(\psi, y, u)h(T_P(\psi, y, u)) \right\} \\
- \max_{u \in U - O(\psi; P)} \left\{ \bar{r}(\psi, u) + \sum_{y \in S^u} V_P(\psi, y, u)h(T_P(\psi, y, u)) \right\}.
$$

Since $r, h, V$ and $T$ are continuous in $\psi$, we can find an $\epsilon > 0$ such that for any $\psi \in B_{2\epsilon}(\psi^*(G_l; P))$ and for all $u \in U$,

$$
\left| \bar{r}(\psi^*(G_l; P), u) + \sum_{y \in S^u} V_P(\psi^*(G_l; P), y, u)h(T_P(\psi^*(G_l; P), y, u)) \right| < \Delta(\psi^*(G_l; P), P) / 2,
$$

and $B_{2\epsilon}(\psi^*(G_l; P)) \cap B_{2\epsilon}(\psi^*(G_{l'}; P)) = \emptyset$ for $l \neq l'$. Therefore, any action $u$ which is not in $O^*(G_l; P)$ cannot be optimal for any $\psi \in B_{2\epsilon}(\psi^*(G_l; P))$. Since the diameter of the convex-hull of the sets that contains infinitely many information states decreases with $\tau^0$, there exists $\tau^0 > 0$ such that for any
$G_I \in \mathcal{G}_{\tau^0}$, the diameter of the convex-hull $J_{l,\epsilon}$ is less than $\epsilon$. Let $\tau^0$ be the smallest integer such that this holds. Then, the $\epsilon$-extension of the convex hull $J_{l,\epsilon}$ is included in the ball $B_{2\epsilon}(\psi^*(G_I; P))$ for all $G_I \in \mathcal{G}_{\tau^0}$. This concludes the proof.

Remark 1: According to Lemma 5, although we can find an $\epsilon$-extension in which a subset of $O^*(G_I; P)$ is optimal for any $\psi, \psi' \in J_{l,\epsilon}$, the set of optimal actions for $\psi$ may be different from the set of optimal actions for $\psi'$. Note that player's estimated belief $\hat{\psi}_t$ is different from the true belief $\psi_t$. If no matter how close $\hat{\psi}_t$ to $\psi_t$, the set of optimal actions for the two is different, then the player will make a suboptimal decision even if it knows the optimal policy. It appears that this is a serious problem in the design of an efficient learning algorithm. We present two different approaches. The first is an assumption on the structure of the optimal policy, and the second is a modification to the learning algorithm so that it can control the loss due to the suboptimal decisions resulting from the difference between the estimated and the true belief. This loss can be controlled in a better way if the player knows the smoothness of the $h$ function.

Assumption 2: There exists $\tau^0 \in \mathbb{N}$ such that for any $G_I \in \mathcal{G}_{\tau^0}$, there exists $\epsilon > 0$ such that the same subset of $O^*(G_I; P)$ is optimal for any $\psi \in J_{l,\epsilon} - \psi^*(G_I; P)$.

When this assumption is correct, if $\hat{\psi}_t$ and $\hat{\psi}_t$ are sufficiently close to each other, then the player will always choose an optimal arm. Assume that this assumption is false. Consider the stationary information states for which $\tau^k = \infty$ for some arm $k$. Then for any $\tau^0 > 0$, there exists a set $G_I \in \mathcal{G}_{\tau^0}$ and a sequence of information states $(s, \tau)_n, n = 1, 2, \ldots$, such that $\psi_P((s, \tau)_n)$ converges to $\psi^*(G_I; P)$ but there exists infinitely many $n$'s for which $O((s, \tau)_n; P) \neq O((s, \tau)_{n+1}; P)$.

For simplicity of analysis, we focus on the following version of Assumption 2 although our results in Section VIII will also hold when Assumption 2 is true.

Assumption 3: There exists $\tau^0 \in \mathbb{N}$ such that for any $G_I \in \mathcal{G}_{\tau^0}$, there exists $\epsilon > 0$ such that for any $\psi \in J_{l,\epsilon}$, a single action is optimal.

Remark 2: Although we don’t know a way to check if Assumption 3 holds given a set of transition probability matrices $P$, we claim that it holds for a large set of $P$s. The player’s selection does not affect state transitions of the arms, it only affects the player’s reward by changing the information state. Moreover, each arm evolves independently from each other. Assume that $P$ is arbitrarily selected from $\Xi$, and the state reward $r^k_x, x \in S^k$ is arbitrarily selected from $(0, r_{\text{max}}]$. Then at any information state $(s, \tau) \in \mathcal{C}$, the probability that the reward distribution of two arms are the same will be zero. Based on this, we claim that Assumption 3 holds with probability one, if the arm rewards and $P$ is chosen from the uniform distribution on $\Psi \times (0, r_{\text{max}}]$. In other words, the set of arm rewards and transition
probabilities for which Assumption 3 does not hold is a measure zero subset of $\Psi \times (0, r_{max}]$.

VI. AN ADAPTIVE LEARNING ALGORITHM (ALA)

Adaptive Learning Algorithm

1: Initialize: $f(t)$ given for $t \in \{1, 2, \ldots\}$, $t = 1$, $N^k = 0, N^k_{i,j} = 0, C_i^k = 0, \forall k \in K, i,j \in S^k$. Play each arm once to set the initial information state $(s, \tau)_0$. Pick $u(0)$ randomly.

2: while $t \geq 1$ do
3: \[ \bar{p}_{ij}^k = \frac{1}{|S^k|} \] (1)
4: \[ \hat{p}_{ij}^k = \frac{\bar{p}_{ij}^k}{\sum_{i \in S^k} \bar{p}_{ij}^k} \]
5: $W = \{(k, i), k \in K, i \in S^k : C_i^k < f(t)\}$. 
6: if $W \neq \emptyset$ then
7: EXPLORE
8: if $u(t - 1) \in W$ then
9: $u(t) = u(t - 1)$
10: else
11: select $u(t) \in W$ arbitrarily
12: end if
13: else
14: EXPLOIT
15: Let $\hat{\psi}_t$ be the estimate of the belief at time $t$ based on $(s_t, \tau_t)$ and $\hat{P}_t$.
16: solve $\tilde{g}_t + \hat{h}_t(\psi) = \max_{u \in U} \{\tilde{r}(\psi, u) + \sum_{y \in S_u} V(\psi, y, u)\hat{h}_t(T_{\hat{P}}(\psi, y, u))\}$, $\forall \psi \in \Psi$.
17: compute the indices of all actions at $\hat{\psi}_t$.
18: $\forall u \in U, I_t(u \hat{\psi}_t, u) = \sup_{P \in \Xi} \{\tilde{r}(\psi, u) + \sum_{y \in S_u} V(\psi, y, u)\hat{h}_t(T_P(\psi, y, u))\}$ such that
\[ \left\| \hat{P} - \bar{P} \right\|_1 \leq \sqrt{\frac{2 \log t}{N^t}}. \]
19: Let $u^*$ be the arm with the highest index (arbitrarily select one if there is more than one such arm).
20: $u(t) = u^*$.
21: end if
22: Receive reward $r_{u(t)}(t)$, i.e., state of $u(t)$ at $t$
23: Compute $(s_{t+1}, \tau_{t+1})$
24: $N^u(t) = N^u(t) + 1$
25: if $u(t - 1) = u(t)$ then
26: for $i, j \in S(u(t))$ do
27: if State $j$ is observed at $t$, state $i$ is observed at $t - 1$ then
28: $N^u_{i,j} = N^u_{i,j} + 1, C_i^u(t) = C_i^u(t) + 1$.
29: end if
30: end for
31: end if
32: $t := t + 1$
33: end while

Fig. 3. pseudocode for the Adaptive Learning Algorithm (ALA)

In this section we propose the Adaptive Learning Algorithm (ALA) given in Figure 3 as a learning
algorithm for the player. ALA consists of exploration and exploitation phases. In the exploration phase the player selects each arm for a certain time to form estimates of the transition probabilities, while in the exploitation phase the player selects an arm according to the optimal policy based on the estimated transition probabilities. At each time step, the player decides if it is an exploration phase or exploitation phase based on the accuracy of transition probability estimates. Let \( C_k^i(t) \) be the number of times a transition from state \( i \) of arm \( k \) to any state of arm \( k \) is observed by time \( t \). Let \( f(t) \) be a non-negative, increasing function which sets a condition on the accuracy of estimates. If \( C_k^i(t) < f(t) \) for some \( k \in K \), \( i \in S^k \), the player explores at time \( t \). Otherwise, the player exploits at time \( t \). In other words, the player concludes that the sample mean estimates are accurate enough to compute the optimal action correctly when \( C_k^i(t) \geq f(t) \) \( \forall k \in K, \ i \in S^k \). In an exploration step, in order to update the estimate of \( p_{ij}^k, j \in S^k \), the player does the following. It selects arm \( k \) until state \( i \) is observed, then selects arm \( k \) again to observe the next state after \( i \). It forms sample mean estimate of each \( \hat{p}_{ij}^k \), \( j \in S^k \). In order for this estimates to form a probability distribution, the player should have \( \sum_{j \in S^k} \hat{p}_{ij}^k = 1 \). Therefore, instead of the estimates \( \hat{p}_{ij}^k \), the player uses the normalized estimates \( \hat{p}_{ij}^k = \hat{p}_{ij}^k / (\sum_{l \in S^k} \hat{p}_{il}^k) \).

If ALA is in the exploitation phase at time \( t \), the player first computes \( \hat{\psi}_t \), the estimated belief at time \( t \), using the set of estimated transition probability matrices \( \hat{P}_t \). Then, it solves the average reward optimality equation using \( \hat{P}_t \), for which the solution is given by \( \hat{g}_t \) and \( \hat{h}_t \). We assume that the player can compute the solution at every time step, independent of the complexity of the problem. This solution is used to compute the indices \( I_{\psi}(\hat{\psi}_t, u) \) for each action \( u \in U \) at estimated belief \( \hat{\psi}_t \). \( I_{\psi}(\hat{\psi}_t, u) \) represents the advantage of choosing action \( u \) starting from information state \( \hat{\psi}_t \), i.e, the sum of gain and bias, inflated by the uncertainty about the transition probability estimates based on the number of times arm \( u \) is played. After computing the indices for each action, the player selects the action with the highest index. In case of a tie, one of the actions with the highest index is randomly selected. Note that it is possible to update the state transition probabilities even in the exploitation phase given that the arm selected at times \( t - 1 \) and \( t \) are the same. Thus \( C_k^i(t) \) may also increase in an exploitation phase, and the number of explorations may be smaller than the number of explorations needed in the worst case, in which the transition probability estimates are only updated at exploration steps.

In the following sections we will denote ALA by \( \alpha \).
VII. AN UPPER BOUND FOR REGRET

For any admissible policy $\gamma$, the regret with respect to the optimal $T$ horizon policy is given in (1), which we restate below.

$$\sup_{\gamma' \in \Gamma} \left( E_{\psi_0, \gamma}^P \left[ \sum_{t=1}^{T} r^{\gamma'}(t) \right] \right) - E_{\psi_0, \gamma}^P \left[ \sum_{t=1}^{T} r^{\gamma}(t) \right].$$

First, we derive the regret with respect to the optimal policy as a function of the number of suboptimal plays. Before proceeding, we define expressions to compactly represent the right hand side of the AROE. Let

$$L(\psi, u, h, P) = \bar{r}(\psi, u) + (V(\psi, .., u) \cdot h(T_P(\psi, .., u)))$$

$$L^*(\psi, P) = \max_{u \in U} L(\psi, u, h, P).$$

Let

$$\Delta(\psi, u; P) = L^*(\psi, P) - L(\psi, u, h, P),$$

(9)

denote the degree of suboptimality of action $u$ at information state $\psi$ when the set of transition probability matrices is $P$. From Proposition 1 of [15] we have for all $\gamma \in \Gamma$

$$R^{\gamma}_{(\psi_0; P)}(T) = \sum_{t=0}^{T-1} E_{\psi_0, \gamma}^P [\Delta(\psi_t, U_t; P)],$$

(10)

where we have used the subscript $(\psi_0; P)$ to denote the dependence of regret to the initial belief and the transition probabilities. We assume that initially all the arms are sampled once thus the initial belief is $\psi_0 = \psi_P((s_0, \tau_0))$. For the true set of transition probability matrices $P$, let $\tau^0$ and $\epsilon$ be such that Assumption 3 holds. Specifically, let $\tau^0$ be the minimum over all possible values so that Assumption 3 holds, and $\epsilon$ be the maximum over all possible values given $\tau^0$ so that Assumption 3 holds. Then the $\epsilon$-extensions of the sets $G_l \in G_{\tau^0}$ are $J_{l, \epsilon}$. Note that at any $t$ the belief $\psi_t \in J_{l, \epsilon}$ for some $l$. When $\epsilon$ is clear from the context, we simply write $J_{l, \epsilon}$ as $J_l$.

Let

$$\bar{\Delta}(J_l; u; P) = \sup_{\psi \in J_l} \Delta(\psi, u; P).$$

Note that if $U_t \in O(\psi_t; P)$ then $\Delta(\psi_t, U_t; P)=0$, else $U_t \notin O(\psi_t; P)$ then $\Delta(\psi_t, U_t; P) < \bar{\Delta}(J_l, U_t; P)$.
w.p.1. Let
\[ N_T(J_t, u) = \sum_{t=0}^{T-1} I(\psi_t \in J_t, U_t = u). \]

We have
\[
R^g_{\psi_0, P}(T) \leq \sum_{t=0}^{T-1} E^P_{\psi_0, \gamma} \left[ \sum_{l=1}^{A} \sum_{u \notin O(J_l; P)} I(\psi_t \in J_t, U_t = u) \Delta(J_t, u; P) \right] 
= \sum_{l=1}^{A} \sum_{u \notin O(J_l; P)} E^P_{\psi_0, \gamma} \left[ \sum_{t=0}^{T-1} I(\psi_t \in J_t, U_t = u) \Delta(J_t, u; P) \right] 
= \sum_{l=1}^{A} \sum_{u \notin O(J_l; P)} E^P_{\psi_0, \gamma} [N_T(J_t, u)] \Delta(J_t, u; P). \tag{11} \]

Now consider ALA, which is denoted by \( \alpha \). We will upper bound \( N_T(J_t, u) \) for suboptimal actions \( u \) for \( \alpha \) by a sum of expressions which we will upper bound individually. Let \( E_t \) be the event that ALA exploits at time \( t \) and \( F_t = \{ \| \hat{h}_t - h_P \|_\infty \leq \epsilon \} \).

\[
D_{1,1}(T, \epsilon, J_t, u) = \sum_{t=0}^{T-1} I(\hat{\psi}_t \in J_t, U_t = u, E_t, F_t, I(\hat{\psi}_t, u) \geq \mathcal{L}^*(\hat{\psi}_t, P) - 2\epsilon), \\
D_{1,2}(T, \epsilon, J_t, u) = \sum_{t=0}^{T-1} I(\hat{\psi}_t \in J_t, U_t = u, E_t, F_t, I(\hat{\psi}_t, u) < \mathcal{L}^*(\hat{\psi}_t, P) - 2\epsilon), \\
D_{1,3}(T, \epsilon) = \sum_{t=0}^{T-1} I(E_t, F^C_t), \\
D_1(T, \epsilon, J_t, u) = D_{1,1}(T, \epsilon, J_t, u) + D_{1,2}(T, \epsilon, J_t, u) + D_{1,3}(T, \epsilon), \\
D_{2,1}(T, \epsilon) = \sum_{t=0}^{T-1} I(|| \psi_t - \hat{\psi}_t ||_1 > \epsilon, E_t), \\
D_{2,2}(T, \epsilon, J_t) = \sum_{t=0}^{T-1} I(|| \psi_t - \hat{\psi}_t ||_1 \leq \epsilon, \hat{\psi}_t \notin J_t, \psi_t \in J_t, E_t), \\
D_2(T, \epsilon, J_t) = D_{2,1}(T, \epsilon) + D_{2,2}(T, \epsilon, J_t). \]

**Lemma 6:** For any \( P \) satisfying Assumption 2
\[
E^P_{\psi_0, \gamma} [N_T(J_t, u)] \leq E^P_{\psi_0, \gamma} [D_1(T, \epsilon, J_t, u)] + E^P_{\psi_0, \gamma} [D_2(T, \epsilon, J_t)] + E^P_{\psi_0, \gamma} \sum_{t=0}^{T-1} I(E^C(t)). \tag{12} \]
Proof:

\[ N_T(J_t, u) = \sum_{t=0}^{T-1} (I(\psi_t \in J_t, U_t = u, E_t) + I(\psi_t \in J_t, U_t = u, E_t^C)) \]
\[ \leq \sum_{t=0}^{T-1} I(\psi_t \in J_t, \hat{\psi}_t \in J_t, U_t = u, E_t) + \sum_{t=0}^{T-1} I(\psi_t \in J_t, \hat{\psi}_t \notin J_t, U_t = u, E_t) + \sum_{t=0}^{T-1} I(E_t^C) \]
\[ \leq D_{1,1}(T, \epsilon, J_t, u) + D_{1,2}(T, \epsilon, J_t, u) + D_{1,3}(T, \epsilon) + D_{2,1}(T, \epsilon) + D_{2,2}(T, \epsilon, J_t) \]
\[ + \sum_{t=0}^{T-1} I(E_t^C). \]

The result follows from taking the expectation of both sides.

VIII. Analysis of the Regret of ALA

In this section we show that when \( P \) is such that Assumptions 1 and 3 hold, if the player uses ALA with \( f(t) = L \log t \) with \( L \) sufficiently large, i.e., the exploration constant \( L \geq C(P) \), where \( C(P) \) is a constant that depends on \( P \), then the regret due to explorations will be logarithmic in time, while the regret due to all other terms are finite, independent of \( t \). Note that since the player does not know \( P \), it cannot know how large he should chose \( L \). For simplicity we assume that the player starts with an \( L \) that is large enough without knowing \( C(P) \). We also prove a near-logarithmic regret result, when the player sets \( f(t) = L(t) \log t \), where \( L(t) \) is a positive increasing function over time such that \( \lim_{t \to \infty} L(t) = \infty \).

Using Lemma 2, we will show that the probability that an estimated transition probability is significantly different from the true transition probability given ALA is in an exploitation phase is very small.

Lemma 7: For any \( \epsilon > 0 \), for an agent using ALA with constant \( L \geq 3/(2\epsilon^2) \),

\[ P\left( |\hat{p}_{ij}^k(t) - p_{ij}^k| > \epsilon, E_t \right) \leq \frac{2}{t^2}, \]

for all \( t > 0, i, j \in S^k, k \in K \).

Proof: See Appendix A.

Lemma 8: For any \( \epsilon > 0 \), for an agent using ALA with constant \( L \geq C_P(\epsilon) \), we have

\[ P\left( |\hat{p}_{ij,t}^k - p_{ij}^k| > \epsilon, E_t \right) \leq \frac{2}{t^2}, \]

for all \( t > 0, i, j \in S^k, k \in M \), where \( C_P(\epsilon) \) is a constant that depends on \( P \) and \( \epsilon \).
Proof: See Appendix B

A. Bounding the Expected Number of Explorations

Lemma 9:

\[
E_{\psi_0,\alpha} \left[ \sum_{t=0}^{T-1} I(E^C_t) \right] \leq \left( \sum_{k=1}^{K} |S_k^i| \right) L \log (1 + T_{\max}),
\]

where \( T_{\max} = \max_{k \in K, i, j \in S^k} E[T_{ij}^k] + 1 \), \( T_{ij}^k \) is the hitting time of state \( j \) of arm \( k \) starting from state \( i \) of arm \( k \). Since all arms are ergodic \( E[T_{ij}^k] \) is finite for all \( k \in K, i, j \in S^k \).

Proof: The number of transition probability updates that results from explorations up to time \( T - 1 \) is at most \( \sum_{k=1}^{K} \sum_{i \in S_k^i} L \log T \). The expected time spent in exploration during a single update is at most \( (1 + T_{\max}) \).

B. Bounding \( E_{\psi_0,\alpha}[D_2(T, \epsilon, J_t)] \)

Lemma 10: for \( L \geq C_P(\epsilon/(K S_{\max}^2 |S^1| \ldots |S^K| C_1(P))) \) we have

\[
E_{\psi_0,\alpha}[D_2(T, \epsilon)] \leq 2K S_{\max}^2 \beta,
\]

where \( C_1(P) = \max_{k \in K} C_1(P_k, \infty) \) and \( C_1(P_k, t) \) is given in Lemma 1.

Proof: Consider \( t > 0 \)

\[
|\hat{\psi}_t - \psi_t|_1 \leq \left( \prod_{k=1}^{K} (\hat{P}_t^k)^{\tau_k e_k^k} \right)_{x_k} - \left( \prod_{k=1}^{K} (P_k)^{\tau_k e_k^k} \right)_{x_k}
\]

\[
\leq \sum_{k=1}^{K} \left| (\hat{P}_t^k)^{\tau_k e_k^k} - (P_k)^{\tau_k e_k^k} \right|_{x_k}
\]

\[
\leq \sum_{k=1}^{K} \left\| (\hat{P}_t^k)^{\tau_k e_k^k} - (P_k)^{\tau_k e_k^k} \right\|_1
\]

\[
\leq C_1(P) \sum_{k=1}^{K} \left\| \hat{P}_t^k - P_k \right\|_1,
\]

where last inequality follows from Lemma 1.

By (15)

\[
\left\| \hat{\psi}_t - \psi_t \right\|_1 \leq |S^1| \ldots |S^K| C_1(P) \sum_{k=1}^{K} \left\| \hat{P}_t^k - P_k \right\|_1.
\]
Thus we have

\[
P \left( \| \hat{\psi}_t - \psi_t \|_1 > \epsilon, E_t \right) \leq P \left( \sum_{k=1}^{K} \| \hat{P}^k_t - P^k \|_1 > \epsilon / (|S^1| \ldots |S^K| C_1(P)), E_t \right) \leq K \sum_{k=1}^{K} \mathbb{P} \left( \| \hat{P}^k_t - P^k \|_1 > \epsilon / (K |S^1| \ldots |S^K| C_1(P)), E_t \right) \leq 2 K S_{\max}^2 \frac{1}{t^2},
\]

where last inequality follows from Lemma 8. Then,

\[
E^{P_{\psi,0,\alpha}}_{\psi_t}[D_{2,1}(T, \epsilon)] = \sum_{t=0}^{T-1} P_{\psi_t,0}(\| \psi_t - \hat{\psi}_t \|_1 > \epsilon, E_t) \leq 2 K S_{\max}^2 \beta.
\]

Next we will bound \( E^{P_{\psi,0,\alpha}}_{\psi_t}[D_{2,2}(T, \epsilon, J_l)] \).

**Lemma 11:** Let \( \tau^0 \) be such that Assumption 2 holds. Then for \( \epsilon < \xi/2 \), \( E^{P_{\psi,0,\alpha}}_{\psi_t}[D_{2,2}(T, \epsilon, J_l)] = 0, l = 1, \ldots, A. \)

**Proof:** By Assumption 3 any \( \psi_t \in J_l \) is at least \( \xi \) away from the boundary of \( J_l \). Thus given \( \hat{\psi}_t \) is at most \( \epsilon \) away from \( \psi_t \), it is at least \( \xi/2 \) away from the boundary of \( J_l \).

C. **Bounding** \( E^{P_{\psi,0,\alpha}}_{\psi_t}[D_1(T, \epsilon, J_l, u)] \)

Define the following function:

\[
\text{MakeOpt}(\psi, u; P, \epsilon) := \left\{ \tilde{P} = (\tilde{P}^1, \ldots, \tilde{P}^K) : \tilde{P}^k \in \Xi^k, \mathcal{L}(\psi, u, h_P, \tilde{P}) \geq \mathcal{L}^*(\psi, P) - \epsilon \right\},
\]

For any \( u \) and \( \psi \), if \( \text{MakeOpt}(\psi, u; P, \epsilon) \neq \emptyset \) for all \( \epsilon > 0 \), then the pair \((\psi, u)\) is called a critical information state-action pair. Let

\[
\text{Crit}(P) := \{ (\psi, u) : u \text{ is suboptimal at } \psi, \text{ MakeOpt}(\psi, u; P, \epsilon) \neq \emptyset, \forall \epsilon > 0 \}.
\]

Let

\[
J_{\psi,0}(\tilde{P}; P, \epsilon) := \inf \{ \| \tilde{P} - \tilde{P} \|_1^2 : \tilde{P} \in \text{MakeOpt} (\psi, u; P, \epsilon) \}.
\]
Lemma 12: Let $\tau^0, \delta > 0$ be such that and $\delta < J_{\psi,u}(P; P, 3\epsilon)/2$, for any $\psi \in J_l, l = 1, \ldots, A(\tau^0), (\psi, u) \in \text{Crit}(P)$. Then for $L \geq C_P(f_P, \beta, \delta) / (KS_{\text{max}}^2)$ we have

$$E^P_{\psi_0, \alpha}[D_{1,1}(T, \epsilon, J_l, u)] \leq (2KS_{\text{max}}^2 + 4\delta)\beta,$$

(16)

where $f_P, \epsilon$ is a function such that $f_P, \epsilon(\delta) > 0$ for $\delta > 0$ and $\lim_{\delta \to 0} f_P, \epsilon(\delta) = 0$.

Proof: See Appendix C.

Lemma 13: For $L$ large enough, $E^P_{\psi_0, \alpha}[D_{1,2}(T, \epsilon, J_l, u)] \leq 2KS_{\text{max}}^2\beta$.

Proof: If suboptimal action $u$ is chosen at information state $\hat{\psi}_t$ this means that for the optimal action $u^* \in O(\hat{\psi}_t; P)$

$$I_t(\hat{\psi}_t, u^*) \leq I_t(\hat{\psi}_t, u) < L^*(\hat{\psi}_t; P) - 2\epsilon.$$  

This implies

$$\forall P \in \Xi, \|\hat{P} - \tilde{P}_t\|_1 \leq \sqrt{\frac{2\log t}{N_t(u)}},$$

$$\Rightarrow (V(\hat{\psi}_t, u^*) \cdot \hat{h}_t(\hat{T}_P(\hat{\psi}_t, u))) < (V(\hat{\psi}_t, u^*) \cdot h_P(\hat{T}_P(\hat{\psi}_t, u))) - 2\epsilon.$$  

Since on $F_t (33)$ holds,

$$\{I_t(\hat{\psi}_t, u^*) \leq L^*(\hat{\psi}_t; P) - 2\epsilon\} \subset \left\{ \forall P \in \Xi, \|\hat{P} - \tilde{P}_t\|_1 \leq \sqrt{\frac{2\log t}{N_t(u)}}, (V(\hat{\psi}_t, u^*) \cdot h_P(\hat{T}_P(\hat{\psi}_t, u))) < (V(\hat{\psi}_t, u^*) \cdot h_P(\hat{T}_P(\hat{\psi}_t, u))) - \epsilon \right\}. \tag{17}$$

But since $h_P$ is continuous there exists $\delta_1 > 0$ such that the event in (17) implies

$$\|T_P(\hat{\psi}_t, y, u^*) - T_P(\hat{\psi}_t, y, u^*)\|_1 > \delta_1, \forall y \in S^{u^*}.$$  

Again since $T_P(\psi, y, u)$ is continuous in $P$, there exists $\delta_2 > 0$ such that above equation implies

$$\|\tilde{P} - P\|_1 > \delta_2.$$
Thus
\[
\{I_t(\hat{\psi}_t, u^*) \leq L^*(\hat{\psi}_t; P) - 2\epsilon\} \subset \left\{ \forall \tilde{P} \in \Xi, \left\| \tilde{P} - \hat{P}_t \right\|_1 \leq \sqrt{\frac{2 \log t}{N_t(u)}} \Rightarrow \left\| \tilde{P} - P \right\|_1 > \delta_2 \right\}
\]
\[
\subset \left\{ \left\| \tilde{P}_t - P \right\|_1 > \delta_2 + \sqrt{\frac{2 \log t}{N_t(u)}} \right\}
\]
\[
\subset \left\{ \left\| \tilde{P}' - P \right\|_1 > \delta_2 \right\}.
\]

Therefore
\[
E^P_{\psi_0, \alpha} [D_{1,2}(T, \epsilon, J_t, u)] \leq \sum_{t=0}^{T-1} P \left( \left\| \tilde{P}_t - P \right\|_1 > \delta_2, E_t \right)
\]
\[
\leq \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{(i,j) \in S^k \times S^k} P \left( |\tilde{p}_{i,j,t}^k - p_{i,j}^k| \geq \frac{\delta_2}{KS_{\max}^2}, E_t \right)
\]
\[
\leq 2KS_{\max}^2 \beta,
\]
for \( L \geq CP(\delta_2/(KS_{\max}^2)) \).

Next, we consider bound \( E^P_{\psi_0, \alpha} D_{1,3}(T, \epsilon) \).

**Lemma 14:** For any \( \epsilon > 0 \), there exists \( \varsigma > 0 \) depending on \( \epsilon \) such that if \( \left\| P^k - \hat{P}^k \right\|_1 < \varsigma, \forall k \in K \) then \( \left\| h_P - h_{\hat{P}} \right\|_\infty < \epsilon \).

**Proof:**

Since \( h_{\hat{P}} \) is continuous in \( \psi \) by lemma 4 for any \( \hat{P} \) such that Assumption 1 holds, and \( \tilde{r}(\psi), V_P, T_{\hat{P}} \) are continuous in \( \hat{P} \), for any \( \psi \in \Psi \).

\[
g_P + h_P(\psi) = \arg max_{u \in U} \left\{ \tilde{r}(\psi, u) + \sum_{y \in S^*} V_P(\psi, y, u)h_P(T_P(\psi, y, u)) \right\}
\]
\[
= \arg max_{u \in U} \left\{ \tilde{r}(\psi, u) + \sum_{y \in S^*} V_P(\psi, y, u)h_{\hat{P}}(T_P(\psi, y, u)) + q(P, \hat{P}, \psi, u) \right\},
\]
for some function \( q \) such that \( \lim_{P \to P} q(P, \hat{P}, \psi, u) = 0, \forall \psi \in \Psi, u \in U \).

Let \( \tilde{r}(P, \hat{P}, \psi, u) = \tilde{r}(\psi, u) + q(P, \hat{P}, \psi, u) \). We can write (18) as

\[
g_P + h_P(\psi) = \arg max_{u \in U} \left\{ \tilde{r}(P, \hat{P}, \psi, u) + \sum_{y \in S^*} V_P(\psi, y, u)h_{\hat{P}}(T_P(\psi, y, u)) \right\}
\]
(19)
Note that (19) is the average reward optimality equation for a system with set of transition probability matrices $P$, and perturbed rewards $\bar{r}(P, \hat{P}, \psi, u)$. Since $\lim_{\hat{P} \to P} r(P, \hat{P}, \psi, u) = \bar{r}(\psi, u), \forall \psi \in \Psi, u \in U$, we expect $h_{\hat{P}}$ to converge to $h_P$. Next, we prove that this is true. Let $F_{\hat{P}}$ denote the dynamic programming operator defined in (6), with transition probabilities $P$ and rewards $r(P, \hat{P}, \psi, u)$. Then, by S-1 of Lemma (4), there exists a sequence of functions $v_{0,\hat{P}}, v_{1,\hat{P}}, v_{2,\hat{P}}, \ldots$ such that $v_{0,\hat{P}} = 0, v_{l,\hat{P}} = F_{\hat{P}} v_{l-1,\hat{P}}$ and another sequence of functions $v_{0,P}, v_{1,P}, v_{2,P}, \ldots$ such that $v_{0,P} = 0, v_{l,P} = F_{P} v_{l-1,P}$, for which

$$\lim_{l \to \infty} v_{l,\hat{P}} = h_{\hat{P}}, \quad (20)$$

$$\lim_{l \to \infty} v_{l,P} = h_P, \quad (21)$$

uniformly in $\psi$.

Next, we prove that for any $l \in \{1, 2, \ldots\}$, $\lim_{\hat{P} \to P} v_{l,\hat{P}} = v_{l,P}$ uniformly in $\psi$. Let $q_{\text{max}}(P, \hat{P}) = \sup_{u \in U, \psi \in \Psi} |q(P, \hat{P}, \psi, u)|$. By Equation 2.27 of [1], we have

$$\sup_{\psi \in \Psi} \left\{|v_{l,\hat{P}}(\psi) - v_{l,P}(\psi)|\right\} = \sup_{\psi \in \Psi} \left\{|F^{l-1} v_{l,\hat{P}}(\psi) - F^{l-1} v_{l,P}(\psi)|\right\} \leq \sup_{\psi \in \Psi} \left\{|v_{l,\hat{P}}(\psi) - v_{l,P}(\psi)|\right\} \leq 2q_{\text{max}}(P, \hat{P}), \quad (22)$$

where the last inequality follows form $v_{0,P} = 0, v_{0,\hat{P}} = 0$, and

$$v_{l,P}(\psi) = \max_{u \in U} \{\bar{r}(\psi, u)\}$$

$$v_{l,\hat{P}}(\psi) = \max_{u \in U} \left\{\bar{r}(\psi, u) + q(P, \hat{P}, \psi, u)\right\}.$$

Consider a sequence $\{\hat{P}_n\}_{n=1}^\infty$ which converges to $P$. Since $\lim_{n \to \infty} q_{\text{max}}(P, \hat{P}_n) = 0$, for any $\epsilon > 0$, there exists $N_0$ such that for all $n > N_0$ we have $q_{\text{max}}(P, \hat{P}_n) < \epsilon/2$, which implies by (22) that

$$\sup_{\psi \in \Psi} \left\{|v_{l,\hat{P}_n}(\psi) - v_{l,P}(\psi)|\right\} < \epsilon,$$

for all $\psi \in \Psi$. Therefore, for any $l \in \{1, 2, \ldots\}$, we have

$$\lim_{\hat{P} \to P} v_{l,\hat{P}} = v_{l,P}, \quad (23)$$
uniformly in $\psi$. Using (20) and (21), for any $\epsilon > 0$ and any $n \in \{1, 2, \ldots\}$, there exists $N_1(n)$ such that for any $l > N_1(n)$ and $\psi \in \Psi$, we have

$$|v_{l,P_n}(\psi) - h_{\hat{P}_n}(\psi)| < \epsilon/3,$$

$$|v_{l,P}(\psi) - h_P(\psi)| < \epsilon/3.$$

Similarly using (23), for any $\epsilon > 0$, there exists $N_0$ such that for all $n > N_0$ and $\psi \in \Psi$, we have

$$|v_{l,P_n}(\psi) - v_{l,P}(\psi)| \leq \epsilon/3.$$  \hspace{1cm} (24)

These imply that for any $\epsilon > 0$, there exists $N_2 \geq N_0$ and such that for all $n > N_2$, such that for all $\psi \in \Psi$, we have

$$|h_P(\psi) - h_{\hat{P}_n}(\psi)| \leq |h_P(\psi) - v_{l,P}(\psi)| + |v_{l,P_n}(\psi) - v_{l,P}(\psi)| + |v_{l,P_n}(\psi) - h_{\hat{P}_n}(\psi)| < \epsilon,$$

since there exists some $l > N_1(n)$ such that (24) holds. Therefore, for any $\epsilon > 0$ there exists some $\eta > 0$ such that $|P - \hat{P}| < \eta$ implies $|h_P(\psi) - h_{\hat{P}(\psi)}|_\infty \leq \epsilon$.

\hspace{1cm} ■

**Lemma 15:** For any $\epsilon > 0$, let $\varsigma > 0$ be such that Lemma 14 holds. Then for an agent using ALA with $L \geq C_P(\varsigma/S_{\max}^2)$, we have

$$E_{\psi_0,\alpha}^P[D_{1,3}(T, \epsilon)] \leq 2K S_{\max}^2 \beta .$$  \hspace{1cm} (25)

**Proof:** We have by Lemma 14

$$\left\{ \|P^k - \hat{P}_t^k\|_1 < \varsigma, \forall k \in K \right\} \subset \left\{ \|h_P - h_t\|_\infty < \epsilon \right\},$$

which implies

$$\left\{ \|P^k - \hat{P}_t^k\|_1 \geq \varsigma, \text{ for some } k \in K \right\} \supset \left\{ \|h_P - h_t\|_\infty \geq \epsilon \right\}.$$
Then

\[
E_{\psi_0,\alpha}^P D_{1,\delta}(T, \epsilon) = E_{\psi_0,\alpha}^P \left[ \sum_{t=0}^{T-1} I(E_t, F_t^C) \right]
\]

\[
\leq \sum_{t=0}^{T-1} P\left( \left| P^k - \hat{P}_t^k \right|_1 \geq \varsigma, \text{ for some } k \in K, E_t \right)
\]

\[
\leq \sum_{k=1}^{K} \sum_{(i,j) \in S^k \times S^k} \sum_{t=0}^{T-1} P\left( \left| p_{ij}^k - \hat{p}_{ij,t}^k \right| > \frac{\varsigma}{S_{\max}^2}, E_t \right)
\]

\[
\leq 2K S_{\max}^2 \beta .
\]

\[\blacksquare\]

D. Logarithmic regret upper bound

**Theorem 1:** Let \( \tau^0 \) be a mixing time such that Assumptions [1] [2] and ?? are true. Under these assumptions, for an agent using ALA with \( L \) sufficiently large, for any action \( u \in U \) which is suboptimal for the belief vectors in \( J_l \)

\[
E_{\psi_0,\alpha}^P \left[ N_T(J_l, u) \right] \leq \left( \sum_{k=1}^{K} |S^k| \right) L \log T(1 + T_{\max}) + (8K S_{\max}^2 + 4/\delta) \beta ,
\]

for some \( \delta > 0 \) depending on \( L \). Therefore

\[
R_{\psi_0;P}(T) \leq (L \log T(1 + T_{\max}) + (8K S_{\max}^2 + 4/\delta) \beta ) \times \sum_{l=1}^{A(\tau^0)} \sum_{u \in \Omega(J_l; P)} \Delta(J_l, u; P).
\]

**Proof:** The result follows from Lemmas [9] [10] [11] [12] [13] [15] and (11).

\[\blacksquare\]

IX. EXTENSIONS TO THE ADAPTIVE LEARNING ALGORITHM (ALA)

In this section we propose several extensions to ALA and relaxation of certain assumptions. Firstly, we consider an adaptive exploration function for ALA, by which the player can achieve near-logarithmic regret without knowing a sufficient bound on the distance between the true and estimated transition probabilities (such that the exploration constant \( L \) can be chosen large enough). Secondly, we present a modified algorithm for which Assumption 2 can be relaxed. We prove that the modified algorithm can achieve logarithmic regret without Assumption 2.

A. An adaptive exploration function

We note that the analysis in Section VIII holds when ALA is run with a sufficiently large exploration constant \( L \) such that at each exploitation step the estimated transition probabilities \( \hat{P} \) is close enough
to \( P \) to guarantee that all regret terms in (12) is finite except the regret due to explorations which is logarithmic in time. In other words, there is an \( L_0(P) > 0 \) such that when ALA is run with \( L \geq L_0(P) \), at the end of each exploration step we have \( ||\hat{P} - P||_1 \leq \delta(P) \), where \( \delta(P) > 0 \) is a constant for which Theorem 1 holds.

However, in our learning model we assumed that the player does not know the transition probabilities initially, therefore it is impossible for the player to check if \( L \geq L_0(P) \). Let \( \Xi \subset \Xi \) be the set of transition probability matrices where \( P \) lies in. If the player knows \( \Xi \), then it can compute \( \tilde{L}_0 = \sup_{\hat{P} \in \Xi} L_0(\hat{P}) \), and choose \( L > \tilde{L}_0 \).

In this section, we present another exploration function for ALA such that the player can achieve near-logarithmic regret even without knowing \( L_0(P) \) or \( \tilde{L}_0 \). Let \( f(t) = L(t) \log t \) where \( L(t) \) is an increasing function such that \( L(1) = 1 \) and \( \lim_{t \to \infty} L(t) = \infty \). The intuition behind this exploration function is that after some time \( T_0 \), \( L(t) \) will be large enough so that the estimated transition probabilities are sufficiently accurate, and the regret due to incorrect calculations is a constant independent of time.

**Theorem 2:** When \( P \) is such that Assumptions 1 and 3 hold. If the player uses ALA with \( f(t) = L(t) \log t \), for some \( L(t) \) such that \( L(1) = 1 \) and \( \lim_{t \to \infty} L(t) = \infty \), then there exists constant \( \delta(L) > 0 \), \( \tau^0(P) > 0 \), \( T_0(L, P) > 0 \) such that the regret is upper bounded by

\[
R_{\psi, P}(T) \leq r_{\max} \left( T_0(L, P) + \left( \sum_{k=1}^{K} |S^k| \right) L(T) \log (1 + T_{\max}) \right.

\[
+ \left( 8KS_{\max}^2 + 4/\delta(L) \right) \beta \left( \sum_{l=1}^{A(\tau^0)} \sum_{u \in O(J_l; P)} \tilde{\Delta}(J_l, u; P) \right)
\]

\[
\leq r_{\max} \left( T_0(L, P) + \left( \sum_{k=1}^{K} |S^k| \right) L(T) \log (1 + T_{\max}) \right.

\[
+ \left( 8KS_{\max}^2 + 4/\delta(L) \right) \beta(\tau^0)^M \left( \sum_{k=1}^{K} |S^k| \right) \max_{l \in \{1, \ldots, A(\tau^0)\}} \tilde{\Delta}(J_l, u; P) \right)
\]

**Proof:** The regret up to \( T_0(L, P) \) can be at most \( r_{\max} T_0(L, P) \). After \( T_0(L, P) \), since \( L(t) \geq L_0(P) \), transition probabilities at exploitation steps sufficiently accurate so that all regret terms in (12) except the regret due to explorations is finite. Since time \( t \) is an exploration step whenever \( C_t^k(t) < L(t) \log t \), the regret due to explorations is

\[
r_{\max} \left( \sum_{k=1}^{K} |S^k| \right) L(T) \log (1 + T_{\max})
\]
Remark 3: There is a tradeoff between choosing a rapidly increasing $L$ or a slowly increasing $L$. The regret of ALA up to time $T_0(L, P)$ is linear. Since $T_0(L, P)$ is decreasing in function $L$, a rapidly increasing $L$ will have better performance when the considered time horizon is small. However, in terms of asymptotic performance, i.e., as $T \to \infty$, $L$ should be a slowly diverging sequence. For example if $L = \log(\log t)$, then the asymptotic regret will be $O(\log(\log t) \log t)$.

B. Adaptive Learning Algorithm with Finite Partitions

In this section, we consider a variant of ALA which is called adaptive learning algorithm with finite partitions (ALA-FP). We show that ALA-FP achieves logarithmic regret even when Assumption \[2\] does not hold. Basically, ALA-FP takes as input the mixing time $\tau^0$, then forms $\mathcal{G}_{\tau^0}$ partition of the set of information states $\mathcal{C}$. At each exploitation step ALA-FP solves the estimated AROE based on the transition probability estimated $\hat{P}$, and if the belief $(s, \tau)$ is in $G_l$, the player arbitrarily picks an arm in $O^*(G_l; \hat{P})$, instead of picking an arm in $O(\psi_P((s, \tau)); \hat{P})$. When time $t$ is an exploitation step, the player plays optimally by choosing an arm in $O(\psi_P((s, \tau)); P)$, or near-optimally by choosing an arm in $O^*(G_l; P)$ different from $O(\psi_P((s, \tau)); P)$ such that $(s, \tau) \in G_l$, or suboptimally by choosing an arm that is neither in $O(\psi_P((s, \tau)); P)$ nor in $O^*(G_l; P)$. By Lemma \[5\] we know that when $\tau^0$ is chosen large enough, for any $G_l \in \mathcal{G}_{\tau^0},$ and $(s, \tau)$, $O(\psi_P((s, \tau)); P)$ is a subset of $O^*(G_l; P)$. Since the solution to the AROE is a continuous function, by choosing a large enough $\tau^0$, we can control the regret due to near-optimal actions. The regret due to suboptimal actions can be bounded the same way as in Theorem \[1\]. The following theorem gives a logarithmic upper bound on the regret of ALA-FP.

**Theorem 3:** When the true set of transition probabilities $P$ is such that Assumption \[7\] is true, for a player using ALA-FP with exploration constant $L$, and mixing time $\tau^0$ sufficiently large such that for any $(s, \tau) \in G_l$, $G_l \in \mathcal{G}_{\tau^0}$, we have $|h_P(\psi_P((s, \tau))) - h_P(\psi^*(G_l; P))| < C/2T$, where $C$ is a constant and $T$ is the time horizon, the regret of ALA-FP is upper bounded by

$$R^\alpha_{\psi_P; P}(T) \leq C + (L \log T(1 + T_{\text{max}}) + (8KS^2_{\text{max}} + 4/\delta)\beta) \times \sum_{l=1}^{A(\tau^0)} \sum_{u \not\in O(J_l; P)} \Delta(J_l, u; P),$$

for some $\delta > 0$ which depends on $L$ and $\tau^0$.

**Proof:** The regret at time $T$ is upper bounded by \[10\]. Consider any $t$ which is an exploitation step. Let $l$ be such that $(s_l, \tau_l) \in G_l$. If the selected arm $\alpha(t) \in O(\psi_P((s, \tau)); P)$, then an optimal decision is made at $t$, so the contribution to regret in time step $t$ is zero. Next, we consider the case when $\alpha(t) \not\in O(\psi_P((s, \tau)); P)$. In this case there are two possibilities. Either $\alpha(t) \in O^*(G_l; P)$ or not. We know that when $O^*(G_l; \hat{P}_l) \subset O^*(G_l; P)$ we have $\alpha(t) \in O^*(G_l; P)$. Since $|h_P(\psi_P((s, \tau))) -$
\[ h_P(\psi^*(G_l; P)) | < C/2T \text{ for all } (s, \tau) \in G_l, \text{ we have by } (\theta). \]

\[ \Delta(\psi_t, \alpha(t); P) = \mathcal{L}^*(\psi_t, P) - \mathcal{L}(\psi_t, \alpha(t), h_P, P) \leq C/T. \tag{26} \]

Therefore, contribution of a near-optimal action to regret is at most \( C/T \).

Finally, consider the case when \( \alpha(t) \notin O^*(G_l; P) \). This implies that either the estimated belief \( \hat{\psi}_t \) is not close enough to \( \psi_t \) or the estimated solution to the AROE \( \hat{h}_t \) is not close enough to \( h_P \). Due to the non-vanishing suboptimality gap at any belief vector \( \psi^*(G_l; P) \), and since decisions of ALA-FP is only based on belief vectors corresponding to \( (s, \tau) \in C \), the regret due to suboptimal actions can be bounded by Theorem [1]. We get the regret bound by combining all these results.

Note that the regret bound in Theorem [3] depends on \( \tau^0 \) which depends on \( T \) since \( \tau^0 \) should be chosen large enough so that for every \( G_l \) in the partition created by \( \tau^0 \), function \( h_P \) should vary by at most \( C/2T \). Clearly since \( h_P \) is a continuous function, the variation of \( h_P \) on \( G_l \) decreases with the diameter of \( G_l \) on the belief space. Note that there is a term in regret that depends linearly on the number of sets \( A(\tau^0) \) in the partition generated by \( \tau^0 \), and \( A(\tau^0) \) increases proportional to \( (\tau^0)^M \).

This tradeoff is not taken into account in Theorem [3]. For example, if \( (\tau^0)^M \geq T \) then the regret bound in Theorem [3] is useless. Another approach is to jointly optimize the regret due to suboptimal and near-optimal actions by balancing the number of sets \( A(\tau^0) \) and the variation of \( h_P \) on sets in partition \( \mathcal{G}_{\tau^0} \). For example given \( 0 < \theta \leq 1 \), we can find a \( \tau^0(\theta) \) such that for any \( (s, \tau) \in G_l \), \( G_l \in \mathcal{G}_{\tau^0(\theta)} \) we have \( |h_P(\psi_P((s, \tau))) - h_P(\psi^*(G_l; P))| < C/(2T^\theta) \). Then, the regret due to near-optimal decisions will be proportional to \( CT^{1-\theta} \), and the regret due to suboptimal decision will be proportional to \( (\tau^0(\theta))^M \). Let \( C = \sup_{\psi \in \Psi} h_P(\psi) - \inf_{\psi \in \Psi} h_P(\psi) \). Since \( T^{1-\theta} \) is decreasing in \( \theta \) and \( \tau^0(\theta) \) is increasing in \( \theta \), there exists \( \theta \in [0, 1] \), such that \( \theta = \arg\min_{\theta \in [0, 1]} |T^{1-\theta} - (\tau^0(\theta))^M| \) and \( |T^{1-\theta} - (\tau^0(\theta))^M| \leq (\tau^0(\theta) + 1)^M - (\tau^0(\theta))^M \). If the optimal value of \( \theta \) is in \((0, 1)\), then given \( \theta \), the player can balance the tradeoff, and achieve sublinear regret proportional to \( T^{1-\theta} \). However, since the player does not know \( P \) initially, it may not know the optimal value of \( \theta \). Online learning algorithms for the player to estimate the optimal value of \( \theta \) is a future research direction.

X. Conclusion

In this paper we proved that given the transition probabilities of the arms are positive for any state, there exists index policies which gives logarithmic regret with respect to the optimal finite horizon policy uniformly in time. Our future research includes finding approximately optimal policies with polynomial computation time, and study the short term performance of the proposed algorithms.
APPENDIX A

PROOF OF LEMMA 7

Let \( t(l) \) be the time \( C_t^{k(l)}(i) = l \). We have,

\[
\tilde{p}_{ij}^k(t) = \frac{N_t^k(i,j)}{C_t^k(i)} = \sum_{l=1}^{C_t^k(i)} \frac{I(X_{t(l)-1}^k = i, X_{t(l)}^k = j)}{C_t^k(i)}.
\]

Note that \( I(X_{t(l)-1}^k = i, X_{t(l)}^k = j), l = 1, 2, \ldots, C_t^k(i) \) are IID random variables with mean \( \tilde{p}_{ij}^k \). Then

\[
P\left( |\bar{p}_{ij}^k(t) - \tilde{p}_{ij}^k| > \epsilon, E_t \right) = P \left( \left| \sum_{l=1}^{C_t^k(i)} \frac{I(X_{t(l)-1}^k = i, X_{t(l)}^k = j)}{C_t^k(i)} - \tilde{p}_{ij}^k \right| \geq \epsilon, E_t \right)
\]

\[
= \sum_{b=1}^{t} P \left( \sum_{l=1}^{C_t^k(i)} I(X_{t(l)-1}^k = i, X_{t(l)}^k = j) - C_t^k(i)p_{ij}^k \geq C_t^k(i)\epsilon, C_t^k(i) = b, E_t \right)
\]

\[
\leq \sum_{b=1}^{t} 2e^{-2(a\log t)\epsilon^2} = \sum_{b=1}^{t} e^{-2a\log t\epsilon^2} = 2t - 2a\log t\epsilon^2 \leq \frac{2}{t^2}.
\]

where we used Lemma 2 and the fact that \( C_t^k(i) \geq L \log t \) w.p.1. in the event \( E_t \).

APPENDIX B

PROOF OF LEMMA 8

From symmetry we have

\[
P\left( |\bar{p}_{ij,t}^k - \tilde{p}_{ij}^k| > \epsilon, E_t \right) = P\left( \bar{p}_{ij,t}^k - \tilde{p}_{ij}^k > \epsilon, E_t \right) + P\left( \bar{p}_{ij,t}^k - \tilde{p}_{ij}^k < -\epsilon, E_t \right)
\]

\[
= 2P\left( \bar{p}_{ij,t}^k - \tilde{p}_{ij}^k > \epsilon, E_t \right).
\]

Then

\[
P\left( \tilde{p}_{ij}^k > \epsilon, E_t \right) = P\left( \frac{\tilde{p}_{ij}^k}{\sum_{l \in S^k} \tilde{p}_{il}^k} - \tilde{p}_{ij}^k > \epsilon, E_t \right)
\]

\[
= P\left( \frac{\tilde{p}_{ij}^k}{\sum_{l \in S^k} \tilde{p}_{il}^k} - \tilde{p}_{ij}^k > \epsilon, \sum_{l \in S^k} \tilde{p}_{il}^k - 1 < \delta, E_t \right)
\]

\[
+ P\left( \frac{\tilde{p}_{ij}^k}{\sum_{l \in S^k} \tilde{p}_{il}^k} - \tilde{p}_{ij}^k > \epsilon, \sum_{l \in S^k} \tilde{p}_{il}^k - 1 \geq \delta, E_t \right)
\]

\[
\leq P\left( \frac{\tilde{p}_{ij}^k}{1 - \delta} - \tilde{p}_{ij}^k > \epsilon, E_t \right) + P\left( \sum_{l \in S^k} \tilde{p}_{il}^k - 1 \geq \delta, E_t \right).
\]
We have
\[ P \left( \frac{\bar{p}_{ij}^k}{1 - \delta} - p_{ij}^k > \epsilon, E_t \right) = P \left( \bar{p}_{ij}^k - p_{ij}^k > \epsilon (1 - \delta) - \delta, E_t \right). \tag{30} \]

Note that \( \epsilon (1 - \delta) - \delta \) is decreasing in \( \delta \). We can choose a \( \delta \) small enough such that \( \epsilon (1 - \delta) - \delta > \epsilon/2 \).

Then
\[ P \left( \bar{p}_{ij}^k - p_{ij}^k > \epsilon (1 - \delta) - \delta, E_t \right) \leq \frac{2}{t^2}, \tag{31} \]

for \( L \geq 6/(\epsilon^2) \). We also have
\[ P \left( | \sum_{l \in S^k} \bar{p}_{il}^k - 1 | \geq \delta, E_t \right) \leq P \left( \sum_{l \in S^k} | \bar{p}_{il}^k - p_{il}^k | \geq \delta, E_t \right). \]

Consider the events
\[ A = \{ | \bar{p}_{il}^k - p_{il}^k | < \delta/|S^k|, \forall k \in K \} \]
\[ B = \{ \sum_{l \in S^k} | \bar{p}_{il}^k - p_{il}^k | < \delta \} \]

If \( \omega \in A \) then \( \omega \in B \). Thus \( A \subset B \), \( B^C \subset A^C \). Then
\[ P \left( \sum_{l \in S^k} | \bar{p}_{il}^k - p_{il}^k | \geq \delta, E_t \right) = P(B^C, E_t) \leq P(A^C, E_t) \]
\[ = P \left( \bigcup_{k=1}^K | \bar{p}_{il}^k - p_{il}^k | \geq \delta/|S^k|, E_t \right) \]
\[ \leq \sum_{k=1}^K P \left( | \bar{p}_{il}^k - p_{il}^k | \geq \delta/S_{\max} \right) \]
\[ \leq \frac{2K}{t^2}, \tag{32} \]

for \( L \geq S_{\max}^2/(2\delta^2) \). Combining (28), (31) and (32) we get
\[ P \left( | \bar{p}_{ij,t}^k - p_{ij,t}^k | > \epsilon, E_t \right) \leq \frac{2K + 2}{t^2}, \]

for \( L \geq \max \{6/(\epsilon^2), S_{\max}^2/(2\delta^2) \} = C_P(\epsilon). \)
APPENDIX C

PROOF OF LEMMA [12]

First consider the case $(\hat{\psi}_t, u) \notin \text{Crit}(P)$. Then $\exists \epsilon_0 > 0$ such that $\text{MakeOpt}(\psi, u; P, \epsilon_0) = \emptyset$. On the event $F_t$ we have

$$\left| \sum_{y \in S^u} V(\hat{\psi}_t, y, u)(\hat{h}_t(T_P(\hat{\psi}_t, y, u)) - h_P(T_{\hat{P}}(\hat{\psi}_t, y, u))) \right| \leq \epsilon, \quad \forall u \in U, \hat{P} \in \Xi. \quad (33)$$

Thus for $\epsilon < \epsilon_0/3$,

$$I_t(\hat{\psi}_t, u) \leq \sup_{\hat{P} \in \Xi} \{ \bar{r}(\hat{\psi}_t, u) + \sum_{y \in S^u} V(\hat{\psi}_t, y, u)\hat{h}_t(T_{\hat{P}}(\hat{\psi}_t, y, u)) \}$$

$$\leq \sup_{\hat{P} \in \Xi} \{ \bar{r}(\hat{\psi}_t, u) + \sum_{y \in S^u} V(\hat{\psi}_t, y, u)h_P(T_P(\hat{\psi}_t, y, u)) \}$$

$$< \mathcal{L}^* (\hat{\psi}_t, P) - \epsilon_0 + \epsilon$$

$$< \mathcal{L}^* (\hat{\psi}_t, P) - 2\epsilon,$$

which implies that

$$I(\hat{\psi}_t \in J_t, U_t = u, E_t, F_t, I(\hat{\psi}_t, u) \geq \mathcal{L}^* (\hat{\psi}_t, P) - 2\epsilon) = 0.$$

Next, we consider the case when $(\hat{\psi}_t, u) \in \text{Crit}(P)$. Note that by Lemma [??] $\tau^0$ can be selected such that the suboptimality gap for any belief in $J_l$, $l \in \{1, \ldots, A(\tau^0)\}$ is at least $\delta'' > 0$. Consider $(\psi, u) \in \text{Crit}(P), \psi \in J_l$ for some $l \in \{1, \ldots, A(\tau^0)\}$. Let $3\epsilon = C\delta''$ where $0 < C < 1$ is some constant. We have

$$\mathcal{L}(\psi, u, h_P, P) \leq \mathcal{L}^* (\psi, P) - \delta''.$$

Since MakeOpt$(\psi, u; P, C\delta'') \neq \emptyset$, for any $\hat{P}$ such that

$$\mathcal{L}(\psi, u, h_P, \hat{P}) \geq \mathcal{L}^* (\psi, P) - C\delta''.$$
we have
\[ L(\psi, u, h_\mathcal{P}, \tilde{\mathcal{P}}) - L(\psi, u, h_\mathcal{P}, \mathcal{P}) \geq (1 - C)\delta'' \]
\[ \Rightarrow \sum_{y \in S^u} V(\psi, y, u)(h_\mathcal{P}(T_\mathcal{P}(\psi, y, u)) - h_\mathcal{P}(T_\mathcal{P}(\psi, y, u))) \geq (1 - C)\delta'' \]
\[ \Rightarrow \max_{y \in S^u}|(h_\mathcal{P}(T_\mathcal{P}(\psi, y, u)) - h_\mathcal{P}(T_\mathcal{P}(\psi, y, u))| \geq (1 - C)\delta'' \]
\[ \Rightarrow \|\mathcal{P} - \tilde{\mathcal{P}}\|_1 \geq 2\delta, \]
for some \( \delta > 0 \) by continuity of \( h_\mathcal{P} \) and \( T_\mathcal{P} \). Therefore \( J_{\psi, u}(\mathcal{P}; \mathcal{P}, 3\epsilon) > 2\delta \) for any \((\psi, u) \in \text{Crit}(\mathcal{P}), \psi \in J_l \) for some \( l \in \{1, \ldots, A(l_0)\} \). Before proceeding, we also note that \( J_{\psi, u}(\mathcal{P}; \mathcal{P}, \epsilon) \) is continuous in its first argument. Therefore there exists a function \( f_{\mathcal{P}, \epsilon} \) such that \( f_{\mathcal{P}, \epsilon}(\delta) > 0 \) for \( \delta > 0 \) and \( \lim_{\delta \to 0} f_{\mathcal{P}, \epsilon}(\delta) = 0 \), for which \( |J_{\psi, u}(\mathcal{P}; \mathcal{P}, \epsilon) - J_{\psi, u}(\mathcal{P}; \mathcal{P}, \epsilon)| > \delta \) implies \( \|\mathcal{P} - \tilde{\mathcal{P}}\|_1 > f_{\mathcal{P}, \epsilon}(\delta) \).

The event \( \{\mathcal{I}_t(\hat{\psi}_t, u) \geq \mathcal{L}^*(\hat{\psi}_t, \mathcal{P}) - 2\epsilon\} \) is equivalent to
\[ \exists \hat{\mathcal{P}} \in \Xi : \left( \|\hat{\mathcal{P}} - \mathcal{P}\|^2 \leq \frac{2\log t}{N_t(u)} \right), \]
\[ (\bar{r}(\hat{\psi}_t, u) + (V(\hat{\psi}_t, , u) \cdot \hat{h}_t(T_\mathcal{P}(\hat{\psi}_t, , u))) \geq \mathcal{L}^*(\hat{\psi}_t, \mathcal{P}) - 2\epsilon) . \tag{34} \]

On the event \( F_t \) we have
\[ \sum_{y \in S^u} V(\hat{\psi}_t, y, u)(\hat{h}_t(T_\mathcal{P}(\hat{\psi}_t, y, u)) - h_\mathcal{P}(T_\mathcal{P}(\hat{\psi}_t, y, u))) \leq \epsilon, \forall u \in U, \hat{\mathcal{P}} \in \Xi. \]

Thus (34) implies
\[ \exists \hat{\mathcal{P}} \in \Xi : \left( \|\hat{\mathcal{P}} - \mathcal{P}\|^2 \leq \frac{2\log t}{N_t(u)} \right), \]
\[ (\bar{r}(\hat{\psi}_t, u) + (V(\hat{\psi}_t, , u) \cdot h_\mathcal{P}(T_\mathcal{P}(\hat{\psi}_t, , u))) \geq \mathcal{L}^*(\hat{\psi}_t, \mathcal{P}) - 3\epsilon) . \tag{35} \]

From the definition of \( J_{\psi, u}(\hat{\mathcal{P}}; \mathcal{P}, \epsilon) \) (35) implies
\[ J_{\psi, u}(\hat{\mathcal{P}}; \mathcal{P}, 3\epsilon) \leq \frac{2\log t}{N_t(u)}. \]
Thus we have

\[
D_{1,1}(T, \epsilon, J_l, u) \leq \sum_{t=0}^{T-1} I \left( \hat{\psi}_t \in J_l, U_t = u, J_{\hat{\psi}_t,u}(\hat{P}_t; P, 3\epsilon) \leq 2 \frac{\log t}{N_l(u)}, (\hat{\psi}_t, u) \in \text{Crit}(P), E_t \right)
\]

\[
\leq \sum_{t=0}^{T-1} I \left( \hat{\psi}_t \in J_l, U_t = u, E_t, (\hat{\psi}_t, u) \in \text{Crit}(P) J_{\hat{\psi}_t,u}(P; P, 3\epsilon) \leq 2 \frac{\log t}{N_l(u)} + \delta \right)
\]

\[
+ \sum_{t=0}^{T-1} I \left( \hat{\psi}_t \in J_l, U_t = u, E_t, (\hat{\psi}_t, u) \in \text{Crit}(P) J_{\hat{\psi}_t,u}(P; P, 3\epsilon) > J_{\hat{\psi}_t,u}(\hat{P}_t; P, 3\epsilon) + \delta \right).
\]

(36)

Note that (36) is less than or equal to

\[
\frac{4 \log T}{\delta}.
\]

(38)

By the continuity argument we mentioned above \( J_{\hat{\psi}_t,u}(P; P, 3\epsilon) > J_{\hat{\psi}_t,u}(\hat{P}_t; P, 3\epsilon) + \delta \) implies \( \| \hat{P}_t - P \|_1 > f_{P,3\epsilon}(\delta) \). Thus (37) is upper bounded by

\[
\sum_{t=0}^{T-1} I \left( \| \hat{P}_t - P \|_1 > f_{P,3\epsilon}(\delta), E_t \right).
\]

Taking expectation we have

\[
\sum_{t=0}^{T-1} P \left( \| \hat{P}_t - P \|_1 > f_{P,3\epsilon}(\delta), E_t \right) \leq \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{(i,j) \in S^k \times S^k} P \left( |\hat{p}_{ij,t}^k - p_{ij}^k| \geq f_{P,3\epsilon}(\delta) \right)
\]

\[
\leq KS_{\max}^2 \beta.
\]

(39)

Combining (38) and (39) we have

\[
E_{\psi_0,\alpha}^P[D_{1,1}(T, \epsilon, J_l, u)] \leq (KS_{\max}^2 + 4/\delta)\beta.
\]

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