Max-min-plus expressions for one-dimensional particle cellular automata obtained from a fundamental diagram

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Abstract. We study one-dimensional neighborhood-five conservative cellular automata (CA), referred to as particle cellular automata five (particle CA5). We show that evolution equations for particle CA5s that belong to certain types can be obtained in the form of max-min-plus expressions from a fundamental diagram. The obtained equations are transformed into other max-min-plus expressions by ultradiscrete Cole-Hopf transformation, which enable us to analyze the asymptotic behaviors of general solutions. The equations in the Lagrange representation, which describe particle motion, are also presented, which too can be obtained from a fundamental diagram. Finally, we discuss the generalization to a one-dimensional conservative neighborhood-n CA, i.e., particle CAN.

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1. Introduction

Cellular automata (CA) are dynamic systems in which space and time are discrete and physical quantities take on a finite set of discrete values. Despite their simple construction, CA exhibit complicated behavior and generate complex patterns. By virtue of these properties, CA have been used as mathematical models for complex phenomena in the fields of physics, chemistry, biology, economics, and sociology[1, 2].

Recently, one-dimensional conservative CA have attracted much attention as models for traffic flow[3]. One of the simplest models is Rule 184, which is a neighborhood-three CA (also called an elementary CA). Since Rule 184 exhibits phase transition from a free-flow state to a congestion state as real traffic flow does, it has been used as a basic model and extended to many other CA models. However, despite the intensive research on CA models for traffic flow, there are few studies dealing with all one-dimensional conservative CA in a unified way.

A powerful method for dealing with CA is the ultradiscretization method[4, 5], which connects difference equations and CA by applying the following simple formula for the transformation of variables:

$$\lim_{\varepsilon \to +0} \varepsilon \log \left( e^{A/\varepsilon} + e^{B/\varepsilon} \right) = \max(A, B).$$

Equation (1) shows that the addition of the difference equations corresponds to the max operation of the ultradiscrete equation. The multiplication and division correspond to + and −, respectively, since we have

$$\varepsilon \log \left( e^{A/\varepsilon} \times e^{B/\varepsilon} \right) = A + B, \quad \varepsilon \log \left( \frac{e^{A/\varepsilon}}{e^{B/\varepsilon}} \right) = A - B$$

Though the subtraction is not well defined for the ultradiscrete equation, we can automatically obtain the ultradiscrete equation and its solution by replacing +, × and / by max, + and −, respectively, if the subtraction is not included explicitly in the difference equations and their solutions. The algebra defined by operations of max, +, and − is called ”max-plus” algebra. Binary operations such as AND, OR, and NOT can be easily converted to max-plus operations.

In a previous study[7], we investigated one-dimensional neighborhood-four conservative CA, referred to as particle cellular automata four (particle CA4), and found that evolution equations for particle CA4-1, 4-2, and 4-3 can be obtained in the form of max-plus expressions in combination with “min,” i.e., “max-min-plus” expressions, which are the key to solving the equations and analyzing asymptotic behaviors.

In this study, we investigate particle CA5s and show that the evolution equations for particle CA5s that belong to certain types can be obtained in the form of max-min-plus expressions from fundamental diagrams. In addition, we show that the equations
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obtained are transformed into other max-min-plus expressions by ultradiscrete Cole-Hopf transformation, which enables us to analyze the asymptotic behaviors of general solutions. Furthermore, we give the Lagrange representation for the class of particle CA5s, which describes the motion of particles and can be obtained from a fundamental diagram. Finally, we discuss the generalization to the neighborhood-\( n \) CA case, that is, particle CA\( n \).

2. Particle CA

Let us consider the following equation

\[ u_j^{n+1} = f(u_{j-l}^n, u_{j-l+1}^n, \ldots, u_{j+r}^n) \quad (-\infty < j < \infty, 0 \leq n), \]

(2)

where \( n \) and \( j \) are integers representing the timestep and space site number, respectively, and \( l \) and \( r \) are positive integer constants. Assuming that \( u \) takes the value of 0 or 1, (2) is an evolution equation for a one-dimensional CA with the rule defined by \( R = l + r + 1 \) variable function \( f \). We call this CA a ”neighborhood-\( R \) CA.” Following Wolfram\[1, 2\], to each Rule \( f \), we assign a rule number \( N(f) \) such that

\[ N(f) = \sum_{(u_1, u_2, \ldots, u_R) \in \{0,1\}^R} f(u_1, u_2, \ldots, u_R) 2^{R-1} u_1 + 2^{R-2} u_2 + \ldots + 2^0 u_R. \]

In this study, we focus on the one-dimensional CAs that satisfy

\[ \sum_{j=1}^{K} u_j = \sum_{j=1}^{K} f(u_{j-l}, u_{j-l+1}, \ldots, u_{j+r}) \]

(3)

where \( K (\geq R) \) is the size of the one-dimensional lattices, and \( u_{j+K} = u_j \) holds for all \( j \). From (3), the condition

\[ \sum_{j=1}^{K} u_j^n = \sum_{j=1}^{K} u_j^{n+1}, \]

(4)

holds, which means that the sum of \( u \), that is the number of 1s at all space sites, is conserved for arbitrary \( n \). Let \( u_j^n \) denote the number of particles at the \( j \)th site and the \( n \)th timestep, and each ”1” in the solution represents a particle. Then, particles move among sites according to the evolution rule defined by (2) without creation or annihilation. We call the CA satisfying this condition (4) a ”particle CA” in this sense.

3. Particle CA5

In \[8\], Hattori and Takesue showed that a neighborhood-\( R \) CA with Rule \( f \) is a particle CA if and only if \( f \) satisfies

\[ f(u_1, u_2, \ldots, u_R) - u_{l+1} = q(u_1, u_2, \ldots, u_{R-1}) - q(u_2, u_3, \ldots, u_R), \]

(5)

\[ q(u_1, u_2, \ldots, u_{R-1}) = \sum_{k=1}^{l} u_k - \sum_{k=1}^{R-1} f(0, 0, \ldots, 0, u_1, u_2, \ldots, u_{R-k}), \]

(6)
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for all \((u_1, u_2, \ldots, u_R) \in \{0, 1\}^R\).

Applying Hattori and Takesue’s result to neighborhood-five CA expressed by the equation

\[ u_{j+1}^n = f(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n), \]

we obtain a system of \(2^5\) linear equations with \(2^5\) variables \(f(a, b, c, d, e)\) for all \((a, b, c, d, e) \in \{0, 1\}^5\) from (5) and (6). By finding binary solutions to the system, we obtain 428 rules for one-dimensional conservative neighborhood-five CA (particle CA5). Reflection symmetry

\[ f_1(a, b, c, d, e) = f_2(e, d, c, b, a) \]

and Boolean conjugation symmetry

\[ f_1(a, b, c, d, e) = 1 - f_2(1 - a, 1 - b, 1 - c, 1 - d, 1 - e) \]

for rules \(f_1\) and \(f_2\) reduce the 428 rules to 129 rules. Furthermore, by excluding the rules for neighborhood-three and -four CA (particle CA3 and particle CA4, respectively), we have the following 115 rules:

\[
\begin{align*}
2863377064, & \quad 2881005752, \quad 2881267852, \quad 2881398914, \quad 2944969912, \\
2945232012, & \quad 2945363074, \quad 2945428608, \quad 2947326124, \quad 2947457186, \quad 3098065832, \\
3099375756, & \quad 3099506818, \quad 3099572352, \quad 3102247912, \quad 3103295736, \quad 3103557836, \\
310368898, & \quad 3103754432, \quad 3116153216, \quad 3120335296, \quad 3132275330, \\
3136326348, & \quad 3136457410, \quad 3136522944, \quad 3137047048, \quad 3148921728, \quad 3153103808, \\
3153627912, & \quad 3163077816, \quad 3163470978, \quad 3163536512, \quad 3165434028, \quad 3165630624, \\
3167259896, & \quad 3167521996, \quad 3167653058, \quad 3167718592, \quad 3169616108, \quad 3169812704, \\
3180117376, & \quad 3182211488, \quad 3184299456, \quad 3186393568, \quad 3196108428, \quad 3198202540, \\
3200290508, & \quad 3200487104, \quad 3201011208, \quad 3202384620, \quad 3202581216, \quad 3203105320, \\
3213933712, & \quad 3214980000, \quad 3216027824, \quad 3216289924, \quad 3217067968, \quad 3217592072, \\
3218115792, & \quad 3218639896, \quad 3219162080, \quad 3219686184, \quad 3220299904, \quad 3220472004, \\
3220734008, & \quad 3220996108, \quad 3221127170, \quad 3366517672, \quad 3367827596, \quad 3370699752, \\
3372009676, & \quad 3384605056, \quad 3388787136, \quad 3400596108, \quad 3404778188, \quad 3421555648, \\
3422079752, & \quad 3431529656, \quad 3431791756, \quad 3435711736, \quad 3448569216, \quad 3450663328, \\
3452751296, & \quad 3454845408, \quad 34645060268, \quad 3482385552, \quad 3484479664, \quad 348471764, \\
348471674, & \quad 3486043912, \quad 3486567632, \quad 3487091736, \quad 3487613920, \quad 3488138024, \\
3488923844, & \quad 3489185848, \quad 3639663552, \quad 3640187656, \quad 3703627712, \quad 3704151816, \\
3705199640, & \quad 3705721824, \quad 3706245928, \quad 3706769648, \quad 3707031748, \quad 3707293752, \\
3771264248, & \quad 3822120144, \quad 3822644248, \quad 3824214256, \quad 3824738360, \quad 3888178416, \\
4040228048,
\end{align*}
\]

which is the smallest rule number among their equivalent rules. In this paper, we assign a number "\(m\)" in the range 1-115 to each rule, and call them particle CA5-\(m\).
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4. Obtaining evolution equations in the form of max-min-plus expressions from a fundamental diagram

In the field of traffic-flow theory, a fundamental diagram is a useful tool for determining the traffic state of a roadway. This diagram relates traffic flux and traffic density. Under periodic boundary conditions, the density is defined by

\[ \rho = \frac{1}{K} \sum_{j=1}^{K} u^n_j, \]

where \( K \) is the period of sites. Since the number of particles is conserved, \( \rho \) is a constant irrespective of time \( n \). The average flux is defined by

\[ \bar{q}^n = \frac{1}{K} \sum_{j=1}^{K} q(u^n_j, \ldots, u^n_{j+R-1}). \]

For large enough \( n \), the evolution often approaches a steady state where \( \bar{q}^n \) becomes a constant. In case of convergence, the constant is defined by

\[ Q = \lim_{n \to \infty} \bar{q}^n. \]

The graph of \( Q \) versus \( \rho \) is called a "fundamental diagram."

The following subsection shows that the evolution equations for particle CA5s that belong to certain types can be obtained in the form of max-min-plus expressions from a fundamental diagram.

4.1. Type-A

Let us take particle CA5-34 (Rule 3163536512) as an example. The evolution equation for particle CA5-34 is given by

\[ u^{n+1}_j = f(u^n_{j-2}, u^n_{j-1}, u^n_j, u^n_{j+1}, u^n_{j+2}) \] (7)

together with the following rule table of \( f \).

| abcd | f(a, b, c, d, e) |
|------|----------------|
| 1111 | 1 0 1 1 1 1 0 0 1 0 0 0 1 1 1 1 |
| 1110 | 0 1 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 1101 | 0 0 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 1100 | 0 0 0 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 1011 | 0 0 0 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 1010 | 0 0 0 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 1001 | 0 0 0 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 1000 | 0 0 0 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 0111 | 0 1 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 0110 | 0 1 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 0101 | 0 1 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 0100 | 0 1 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 0011 | 0 1 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 0010 | 0 1 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 0001 | 0 1 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |
| 0000 | 0 1 1 1 1 0 0 1 0 1 0 0 1 1 1 1 |

The upper row of the table shows all possible combinations of binary arguments, and the lower row gives the value of \( f \).

From (5) and (6), the evolution equation for particle CA5-34 can be rewritten as

\[ u^{n+1}_j = u^n_j + q(u^n_{j-2}, u^n_{j-1}, u^n_j, u^n_{j+1}, u^n_{j+2}) - q(u^n_{j-1}, u^n_j, u^n_{j+1}, u^n_{j+2}) \] (8)

together with the following rule table of \( q \).

| abcd | q(a, b, c, d) |
|------|--------------|
| 1111 | 1 1 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 1110 | 0 1 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 1101 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 1100 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 1011 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 1010 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 1001 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 1000 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 0111 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 0110 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 0101 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 0100 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 0011 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 0010 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 0001 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
| 0000 | 0 0 1 1 1 0 1 1 0 1 0 1 0 0 0 0 |
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The upper row of the table shows all possible combinations of binary arguments, and the lower row gives the value of $q$.

Employing numerical simulation for (8), we obtain the fundamental diagram and examples of evolutions for particle CA5-34 shown in Figure 1.

Since the fundamental diagram in Figure 1 is a piecewise linear curve, the relationship between $Q$ and $\rho$ can be expressed in terms of the max-min-plus expression,

$$Q(\rho) = \max(\min(2\rho, 1-\rho), \min(\rho, 2-2\rho)),$$

which is a piecewise linear function composed of four linear functions $2\rho$, $1-\rho$, $\rho$, and $2-2\rho$. (See Figure 2)

Let us consider replacing variables $m\rho$ in (9) by

$$m\rho \rightarrow \begin{cases} 
\sum_{k=1}^{m} u_{j-k}^n & (m > 0) \\
-\sum_{k=1}^{-m} u_{j+k-1}^n & (m < 0)
\end{cases}.$$

Then, we obtain

$$Q(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n) = \max(\min(u_{j-2}^n + u_{j-1}^n, 1 - u_j^n), \min(u_{j-1}^n, 2 - u_j^n - u_{j+1}^n))$$

from (9). (See Figure 3)
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\[ Q(\rho) = \begin{cases} 1 - \rho & \text{if } 0 < \rho < 1 \ominus (2 - 2\rho) \\ 2\rho & \text{if } \rho = 0 \oplus 2 - 2\rho \\ 1 & \text{if } \rho = 1 \ominus (2 - 2\rho) \end{cases} \]

Figure 2. Four linear functions of \( \rho \) and the composed piecewise linear function

\[ u_{n+1}^j = u_n^j + \max(\min(u_{n-2}^j + u_{n-1}^j, 1 - u_n^j), \min(u_{n-1}^j, 2 - u_n^j - u_{n+1}^j)) \]
\[ - \max(\min(u_{n-1}^j + u_n^j, 1 - u_{n+1}^j), \min(u_n^j, 2 - u_{n+1}^j - u_{n+2}^j)) \]

(12)

This is the max-min-plus expression for particle CA5-34. Introducing the ultradiscrete Cole-Hopf transformation from \( u \) to \( F \),

\[ u_n^j = F_n^j - F_{n-1}^j, \]

(13)

we obtain the evolution equation in the form of max-min-plus expressions for \( F \),

\[ F_{n+1}^j = \min(\max(F_{n-2}^j, F_{n-1}^j - 1), \max(F_{n-1}^j, F_{n+2}^j - 2)) \]

(14)

which is composed of linear functions of \( F_{n+k}^j \) \((k \in \mathbb{Z})\). Being able to express evolution equations for \( F \) as a combination of linear functions of \( F \) is critical for analyzing the asymptotic behavior of general solutions of particle CA. We used this fact for the analysis of the asymptotic behavior of particle CA4 in a previous study[7].
Note here that (14) can be directly obtained from (9) by applying the following replacements. (See Figure 4)

\[ m \rho + a \rightarrow F^n_{j-m} - a \]
\[ \text{max} \rightarrow \text{min} \]
\[ \text{min} \rightarrow \text{max} \]

**Figure 4.** Obtaining equations for \( F \) from the fundamental diagram

It is known that particle CAs allow two different representations: Euler representation and Lagrange representation. In the Euler representation, particles are observed at a certain fixed point in space as dependent(field) variables, while in the Lagrange representation, we trace each particle and follow its trajectory. Thus, a dependent variable represents each particle’s position in the Lagrange representation.

In previous studies [9, 10], we proposed an Euler-Lagrange transformation for particle CAs by developing the following transformation formulas for the variable of Euler representation \( u^n_j \), which denotes the number of particles at the \( j \)th site and \( n \)th timestep, and the variable of Lagrange representation \( x^n_i \), which denotes the position of the \( i \)th particle at the \( n \)th timestep,

\[ u^n_j = F^n_j - F^n_{j-1}, \quad (15) \]
\[ F^n_j = \sum_{i=1}^{N} H(j-x^n_i), \quad (16) \]

where \( H(x) \) is the step function defined by \( H(x) = 1 \) if \( x \geq 0 \), and \( H(x) = 0 \) otherwise. Applying the transformation (16) to (14) and using the formulas,

\[ \sum_{i=1}^{N} H(j - \min(a_i, b_i)) = \max\left(\sum_{k=1}^{N} H(j - a_i), \sum_{k=1}^{N} H(j - b_i)\right), \quad (17) \]
\[ \sum_{i=1}^{N} H(j - \max(a_i, b_i)) = \min\left(\sum_{k=1}^{N} H(j - a_i), \sum_{k=1}^{N} H(j - b_i)\right) \quad (18) \]
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\[
\max\left(\sum_i H(j - a_i) - m, 0\right) = \sum_i H(j - a_{i+m}),
\]

(19)

where we assume that \(a_1 < a_2 < \cdots < a_N\) and \(b_1 < b_2 < \cdots < b_N\), we obtain

\[
x_i^{n+1} = \max(\min(x_i^n + 2, x_{i+1}^n - 1), \min(x_i^n + 1, x_{i+2}^n - 2)),
\]

(20)

which is the Lagrange representation for particle CA5-34 in the form of a max-min-plus expression. Note here that (20) can be directly obtained from (9) by applying the following replacements. (See Figure 5.)

\[
m\rho + a \rightarrow x_{i+a} + m
\]

(21)

Figure 5. Obtaining equations for \(x\) from the fundamental diagram

To summarize, the procedure we have performed above is as follows.

(i) Employ numerical simulation for particle CA and obtain the fundamental diagram.

(ii) If the fundamental diagram is a piecewise linear curve, construct \(Q(\rho)\) from the fundamental diagram.

(iii) Construct flux \(q\) from \(Q(\rho)\) and obtain the evolution equation (Euler representation),

\[
u_j^{n+1} = u_j^n + q(u_{j-2}^n, u_{j-1}^n, u_j^n, u_{j+1}^n) - q(u_{j-1}^n, u_j^n, u_{j+1}^n, u_{j+2}^n).
\]

(iv) Obtain the evolution equation for \(F\),

\[
F_j^{n+1} = \phi(F_{j-2}^n, F_{j-1}^n, F_j^n, F_{j+1}^n, F_{j+2}^n).
\]

(v) Obtain the evolution equation for \(x\) (Lagrange representation),

\[
x_i^{n+1} = p(x_{i-2}^n, x_{i-1}^n, x_i^n, x_{i+1}^n, x_{i+2}^n).
\]

Among the 115 rules for particle CA5, there are 17 for which the evolution equations for \(u_j^n\), \(F_j^n\), and \(x_i^n\) can be obtained in the form of max-min-plus expressions from fundamental diagrams by using the procedure described above. These equations are given in Appendix A. In the tables of the Appendix, \(m\) denotes the number of particle CA5-\(m\) and \(N(f)\) denotes the rule number defined by Wolfram. Hereafter, we call the 17 rules type-A.
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\[
\text{Fundamental diagram} \quad \downarrow \quad \text{piecewise linear function } Q(\rho) \quad \downarrow \quad \text{Replacing variables from } \rho \text{ to } u \\
\quad \text{max-min-plus expression for flux} \quad q(u_{j-2}, u_{j-1}, u_j, u_{j+1}) \\
\quad \text{Ultradiscrete} \\
\quad \text{Cole-Hopf} \\
\quad \text{Transformation} \quad u^n_j = F^n_j - F^n_{j-1} \\
\quad \text{max-min-plus expression for } F \\
\quad \phi(F_{j-2}, F_{j-1}, F_j, F_{j+1}, F_{j+2}) \\
\quad \downarrow \quad \text{Euler-Lagrange transformation} \\
\quad \text{max-min-plus expression for } x \\
\quad p(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})
\]

**Figure 6.** Procedure for obtaining evolution equations of type-A particle CA5 in max-min-plus expression

### 4.2. Type-B

Other than type-A, there exist particle CA5 rules for which the fundamental diagram is a piecewise linear curve, and the evolution equations for \(u^n_j, F^n_j\), and \(x^n_i\) can be obtained.

Let us consider particle CA5-15 (rule 3099572352). Employing numerical simulation, we obtain the fundamental diagram for particle CA5-15, as shown in figure 7. The piecewise linear curve can be composed from two linear functions \(2\rho\) and \(2 - 2\rho\) as follows.

\[Q(\rho) = \min(2\rho, 2 - 2\rho) \quad (22)\]

Following the same procedure as in the case of type-A, we obtain a flux as follows:

\[q(u_{j-2}, u_{j-1}, u_j, u_{j+1}) = \min(u_{j-2} + u_{j-1}, 2 - u_j - u_{j+1}) \quad (23)\]

Evaluating the values of (23) for all possible combinations of binary values, we obtain the following table.

| abcd | 111 | 1110 | 1101 | 1100 | 1011 | 1010 | 1001 | 1000 | 0111 | 0110 | 0101 | 0100 | 0011 | 0010 | 0001 | 0000 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| q(a, b, c, d) | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |

However, the flux of particle CA5-15 is given by

| abcd | 111 | 1110 | 1101 | 1100 | 1011 | 1010 | 1001 | 1000 | 0111 | 0110 | 0101 | 0100 | 0011 | 0010 | 0001 | 0000 |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| q(a, b, c, d) | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
Max-min-plus expressions for 1D particle CA obtained from a fundamental diagram and the value of $q(1, 1, 0, 0)$ is found to be different. Thus, $(23)$ is not the flux of particle CA5-15.

To solve this problem, let us suppose that the piecewise linear curve in Figure 7 is not composed of two but more than two linear functions. Let us consider

$$Q(\rho) = \min(1, 2\rho, 2 - 2\rho)$$

(24)

instead of (22). Note here that (22) and (24) are identical functions for real values of $\rho$. (See Figure 8.)

![Figure 7. Fundamental diagram of particle CA5-15](image)

If we apply variable replacement from $\rho$ to $u$ to the above expression, we obtain

$$q(u_{j-2}, u_{j-1}, u_j, u_{j+1}) = \min(1, u_{j-2} + u_{j-1}, 2 - u_j - u_{j+1})$$

(25)

which gives the following table:

| $abcd$ | 1111 | 1110 | 1101 | 1100 | 1011 | 1010 | 1000 | 0111 | 0110 | 0101 | 0100 | 0011 | 0010 | 0001 | 0000 |
|--------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| $q(a, b, c, d)$ | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
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which is identical to the flux of particle CA5-15. Thus, the evolution equation in the
form of a max-min-plus expression for particle CA5-15 is

\[
q_{ij}^{n+1} = q_{ij}^{n} + \min(1, u_{j-2}^{n} + u_{j-1}^{n}, 2 - u_{j}^{n} - u_{j+1}^{n}) - \min(1, u_{j-1}^{n} + u_{j}^{n}, 2 - u_{j+1}^{n} - u_{j+2}^{n}).
\]

(26)

We can also obtain the evolution equations for \( F \) and \( x \) as follows.

\[
F_{j}^{n+1} = \max(F_{j}^{n} - 1, F_{j-2}^{n}, F_{j+2}^{n} - 2)
\]

(27)

\[
x_{i}^{n+1} = \min(x_{i+1}^{n}, x_{i}^{n} + 2, x_{i+2}^{n} - 2) = x_{i}^{n} + \min(2, x_{i+1}^{n} - x_{i}^{n}, x_{i+2}^{n} - x_{i}^{n} - 2)
\]

(28)

There are nine rules for particle CA5, of which the evolution equations for \( u \), \( F \), and \( x \) are obtained using the procedure described above. These equations are given in Appendix B. We call the nine rules type-B.

4.3. Particle CA5 rules other than type-A and type-B

We have employed numerical simulations for all 115 particle CA5 rules to obtain fundamental diagrams, from which we have obtained evolution equations in the form of max-min-plus expressions for 17 rules of type-A and nine rules of type-B. There are 89 rules for particle CA5, for whom evolution equations have not yet been obtained. From the results of numerical simulations, the 89 rules are classified into the following two cases.

- Although the obtained fundamental diagram is a piecewise linear curve, replacing variables from \( \rho \) to \( u \) does not give the correct flux expression for \( q \). In addition, the type-B procedure does not seem to work well.
- As the obtained fundamental diagram is not a piecewise linear curve, we cannot use the procedure of replacing variables from \( \rho \) to \( u \).

At this point, it is not clear if there may be other procedures for obtaining max-min-plus expressions for them.

5. Particle CA\( n \)

We can extend our analysis in the previous section to neighborhood-\( n \) CA, i.e., the particle CA\( n \) case.

Let us consider the case of \( n = 6 \). One example of particle CA6 is Rule 13755053124876288240. From numerical simulation, we obtain the fundamental diagram for the rule shown in Figure [9] Following the procedure of type-A of particle CA5, we obtain

\[
q(u_{j-2}, u_{j-1}, u_{j}, u_{j+1}, u_{j+2}) = \max(-u_{j}, \min(u_{j-2} + u_{j-1} - 1, 1 - u_{j} - u_{j+1} - u_{j+2})),
\]

\[
\min(u_{j-1} - 1, 1 - u_{j} - u_{j+1}, \min(u_{j-2} + u_{j-1} - 2, 2 - u_{j} - u_{j+1} - u_{j+2})),
\]

(29)

\[
F_{j}^{n+1} = \min(F_{j+1}^{n}, \max(F_{j-2}^{n} + 1, F_{j+3}^{n} - 1), \max(F_{j-1}^{n} + 1, F_{j+2}^{n} - 1), \max(F_{j-2}^{n} + 2, F_{j+3}^{n} - 2)),
\]

(30)
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\[ x_{i+1} = \max(x_i - 1, \min(x_{i-1} + 2, x_{i+1} - 3), \min(x_{i-1} + 1, x_{i+1} - 2), \min(x_{i-2} + 2, x_{i+2} - 3)). \]

(31)

Although we have not investigated all particle CA6 rules, the analysis of particle CA5 in the previous section and the example of particle CA6 above assure us that our procedure can be applied to particle CA6 rules.

6. Summary

We have studied particle CA5 and shown that the evolution equations for type-A and type-B can be obtained in the form of max-min-plus expressions from a fundamental diagram. The obtained equations have been transformed into max-min-plus expressions that are composed of linear functions of \( F \) by ultradiscrete Cole-Hopf transformation. Furthermore, we have obtained the Lagrange representation of the evolution equations.

Although we have not obtained evolution equations for all 115 particle CA5 rules, it is important for us to have been able to introduce a unified approach using max-plus algebra, together with the fundamental diagram, for examining particle CAs.

In a previous study[7], we analyzed asymptotic behaviors of solutions and derived functions \( Q(\rho) \) for particle CA4 mathematically. We have not done this for particle CA5, but will be able to do so starting from the equations for \( F \) obtained in this study. This problem will be addressed in a future study.

Finally, investigating the generalization to the neighborhood-\( n \) case is important for the future, and we will report on this in a forthcoming paper.
### Max-min-plus expressions for 1D particle CA obtained from a fundamental diagram

#### Appendix A. Type-A

| $m$ | $N(f)$ | $q(u_{j-2}, u_{j-1}, u_j, u_{j+1})$ |
|-----|--------|----------------------------------|
| 1   | 2863377064 | $\min (u_{j-2} + u_{j-1} - 1 - u_j - u_{j+1})$ |
| 3   | 2881267852 | $\max (-u_j, \min (u_{j-2} + u_{j-1} - 1, 1 - u_j - u_{j+1}))$ |
| 4   | 2881398914 | $\max (-u_j - u_{j+1}, \min (u_{j-2} + u_{j-1} - 1, 1 - u_j - u_{j+1}))$ |
| 5   | 2881464448 | $\min (u_{j-2} + u_{j-1}, 2 - u_j - u_{j+1})$ |
| 33  | 3163470978 | $\max (-u_j - u_{j+1}, \min (u_{j-2} + u_{j-1} - 1, -u_j), \min (u_{j-1} - 1, 1 - u_j - u_{j+1}))$ |
| 34  | 3163536512 | $\max (\min (u_{j-2} + u_{j-1}, 1 - u_j), \min (u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 38  | 3167521996 | $\max (-u_j, \min (u_{j-2} + u_{j-1}, 2 - u_j - u_{j+1}))$ |
| 39  | 3167530558 | $\max (-u_j - u_{j+1}, \min (u_{j-2} + u_{j-1} - 1, -u_j), \min (u_{j-2} + u_{j-1} - 2, 1 - u_j - u_{j+1}))$ |
| 40  | 3167518592 | $\max (\min (u_{j-2} + u_{j-1}, 1 - u_j), \min (u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 50  | 3200487104 | $\max (\min (u_{j-2} + u_{j-1}, 1 - u_j - u_{j+1}), \min (u_{j-1} - 1, -u_j), \min (u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 54  | 3203105320 | $\max (\min (u_{j-2} + u_{j-1}, 1 - u_j - u_{j+1}), \min (u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 59  | 3217067968 | $\max (\min (u_{j-1}, 1 - u_j), \min (u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 64  | 3219686184 | $\max (\min (u_{j-1}, 1 - u_j - u_{j+1}), \min (u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 65  | 3220990904 | $\max (0, \min (u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 68  | 3220996108 | $\max (-u_j, \min (u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 69  | 3221127170 | $\max (-u_j - u_{j+1}, \min (u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 96  | 3488138024 | $\max (\min (u_{j-1}, 1 - u_j - u_{j+1}), \min (u_{j-2} + u_{j-1} - 1, 1 - u_j))$ |

| $m$ | $N(f)$ | $\phi(F_{j-2}, F_{j-1}, F_j, F_{j+1}, F_{j+2})$ |
|-----|--------|---------------------------------|
| 1   | 2863377064 | $\max (F_{j-2}, F_{j-2} + 1)$ |
| 3   | 2881267852 | $\min (F_{j+1}, \max (F_{j-2} + 1, F_{j+2} - 1))$ |
| 4   | 2881398914 | $\min (F_{j+2}, \max (F_{j-2} + 1, F_{j+2} - 1))$ |
| 5   | 2881464448 | $\max (F_{j-2}, F_{j+2} - 2)$ |
| 33  | 3163470978 | $\min (F_{j+2}, \max (F_{j-2} + 1, F_{j+1}), \max (F_{j-1} + 1, F_{j+2} - 1))$ |
| 34  | 3163536512 | $\max (\min (F_{j-2}, F_{j+1} - 1), \max (F_{j-1}, F_{j+2} - 2))$ |
| 38  | 3167521996 | $\min (F_{j+1}, \max (F_{j-2} + 2, F_{j+2} - 1))$ |
| 39  | 3167530558 | $\min (F_{j+2}, \max (F_{j-2} + 1, F_{j+1}), \max (F_{j-2} + 2, F_{j+2} - 1))$ |
| 40  | 3167518592 | $\max (\min (F_{j-2}, F_{j+1} - 1), \max (F_{j-2} + 1, F_{j+2} - 2))$ |
| 50  | 3200487104 | $\min (F_{j-2}, F_{j+2} - 1), \max (F_{j-1}, F_{j+1} - 1), \max (F_{j-2} + 1, F_{j+2} - 2))$ |
| 54  | 3203105320 | $\min (F_{j-2}, F_{j+2} - 1), \max (F_{j-1} + 1, F_{j+2} - 2))$ |
| 59  | 3217067968 | $\min (F_{j-1}, F_{j+1} - 1), \max (F_{j-2} + 1, F_{j+2} - 2))$ |
| 64  | 3219686184 | $\max (F_{j-1}, F_{j+2} - 1), \max (F_{j-2} + 1, F_{j+2} - 2))$ |
| 65  | 3220990904 | $\min (F_{j}, \max (F_{j-2} + 1, F_{j+2} - 2))$ |
| 68  | 3220996108 | $\min (F_{j+1}, \max (F_{j-2} + 1, F_{j+2} - 2))$ |
| 69  | 3221127170 | $\min (F_{j+2}, \max (F_{j-2} + 1, F_{j+2} - 2))$ |
| 96  | 3488138024 | $\max (\min (F_{j-1}, F_{j+2} - 1), \max (F_{j-2} + 1, F_{j+1} - 1))$ |
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| m  | $N(f)$                                                                 | $p(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$ |
|----|------------------------------------------------------------------------|---------------------------------------------|
| 1  | 2881377064                                                             | min$(x_i + 2, x_{i+1} - 2)$                 |
| 3  | 2881267852                                                             | max$(x_i - 1, min(x_{i-1} + 2, x_{i+1} - 2))$ |
| 4  | 2881398914                                                             | max$(x_i - 2, min(x_{i-1} + 2, x_{i+1} - 2))$ |
| 5  | 2881464448                                                             | min$(x_i + 2, x_{i+2} - 2)$                 |
| 33 | 3163420978                                                             | max$(x_i - 2, min(x_{i-1} + 2, x_i - 1), min(x_{i-1} + 1, x_{i+1} - 2))$ |
| 34 | 3163526512                                                             | min$(x_i + 2, x_{i+1} - 1), min(x_i + 1, x_{i+2} - 2))$ |
| 38 | 3165721996                                                             | max$(x_i - 1, min(x_{i-2} + 2, x_{i+1} - 2))$ |
| 39 | 3166753058                                                             | max$(x_i - 2, min(x_{i-1} + 2, x_i - 1), min(x_{i-2} + 2, x_{i+1} - 2))$ |
| 40 | 3167718592                                                             | min$(x_i + 2, x_{i+1} - 1), min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 50 | 3200487104                                                             | max$(x_i + 2, x_{i+1} - 2), min(x_i + 1, x_{i+1} - 1), min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 54 | 3203105320                                                             | min$(x_i + 2, x_{i+1} - 2), min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 59 | 3217072896                                                             | min$(x_i + 1, x_{i+1} - 1), min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 64 | 3219686184                                                             | max$(x_i + 1, x_{i+1} - 2), min(x_i + 1, x_{i+2} - 2))$ |
| 65 | 3220099004                                                             | max$(0, min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 68 | 3220961208                                                             | max$(x_i - 1, min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 69 | 3221127170                                                             | max$(x_i - 2, min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 96 | 3488138024                                                             | max$(min(x_i + 1, x_{i+1} - 2), min(x_{i-1} + 2, x_{i+1} - 1))$ |

Appendix B. Type-B

| m  | $N(f)$                                                                 | $q(u_{j-2}, u_{j-1}, u_j, u_{j+1})$ |
|----|------------------------------------------------------------------------|---------------------------------------------|
| 2  | 2881005752                                                             | min$(max(0, u_{j-2} + u_{j-1} - 1), 1 - u_j - u_{j+1})$ |
| 13 | 3099575756                                                             | max$(-u_j, min(0, u_{j-2} + u_{j-1} - 1, 1 - u_j - u_{j+1}))$ |
| 14 | 3099506818                                                             | max$(-u_j - u_{j+1}, min(0, u_{j-2} + u_{j-1} - 1, 1 - u_j - u_{j+1}))$ |
| 15 | 3099572352                                                             | min$(1, u_{j-2} + u_{j-1}, 2 - u_j - u_{j+1})$ |
| 53 | 3202581216                                                             | max$((u_{j-2} + u_{j-1}, max(0, 1 - u_j - u_{j+1})), min(u_{j-2} + u_{j-1}, 1, 2 - u_j - u_{j+1}))$ |
| 63 | 3219162080                                                             | max$((u_{j-1}, max(0, 1 - u_j - u_{j+1})), min(u_{j-2} + u_{j-1}, 1, 2 - u_j - u_{j+1}))$ |
| 67 | 3220734008                                                             | max$((0, 1 - u_j - u_{j+1}), min(u_{j-2} + u_{j-1} - 1, 2 - u_j - u_{j+1}))$ |
| 95 | 3487613920                                                             | max$((u_{j-1}, max(0, 1 - u_j - u_{j+1})), min(u_{j-2} + u_{j-1} - 1, 1 - u_j))$ |
| 98 | 3489185848                                                             | max$(0, 1 - u_j - u_{j+1}), min(u_{j-2} + u_{j-1} - 1, 1 - u_j))$ |

| m  | $N(f)$                                                                 | $\phi(F_{j-2}, F_{j-1}, F_j, F_{j+1}, F_{j+2})$ |
|----|------------------------------------------------------------------------|--------------------------------------------------|
| 2  | 2881005752                                                             | max$(min(F_j, F_{j-2} + 1), F_{j-2} - 1))$ |
| 13 | 3099575756                                                             | min$(min(F_j+1, max(F_j, F_{j-2} + 1, F_{j+2} - 1)))$ |
| 14 | 3099506818                                                             | min$(max(F_{j+2}, max(F_j, F_{j-2} + 1, F_{j-2} - 1)))$ |
| 15 | 3099572352                                                             | min$(F_{j-1}, F_{j-2}, F_{j+2} - 2))$ |
| 53 | 3202581216                                                             | min$(max(F_{j-2}, min(F_j, F_{j-2} - 1)), max(F_{j-2} + 1, F_{j+2} - 2))$ |
| 63 | 3219162080                                                             | min$(max(F_{j-1}, min(F_j, F_{j+2} - 1)), max(F_{j-2} + 1, F_{j+2} - 2))$ |
| 67 | 3220734008                                                             | min$(max(F_{j}, F_{j+2} - 1), max(F_{j-2} + 1, F_{j+2} - 2))$ |
| 95 | 3487613920                                                             | min$(max(F_{j-1}, min(F_j, F_{j+2} - 1)), max(F_{j-2} + 1, F_{j+1} - 1))$ |
| 98 | 3489185848                                                             | min$(max(F_j, F_{j+2} - 1), max(F_{j-2} + 1, F_{j+1} - 1))$ |
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| $m$ | $N(f)$ | $p(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2})$ |
|-----|--------|---------------------------------------------|
| 2   | 2881005752 | $\min(\max(0, x_{i-1} + 2), x_{i+1} - 2)$ |
| 13  | 3099375756 | $\max(x_{i-1}, \min(0, x_{i-1} + 2), x_{i+1} - 2)$ |
| 14  | 3099506818 | $\max(x_{i-2}, \min(0, x_{i-1} + 2), x_{i+1} - 2)$ |
| 15  | 3099572352 | $\min(x_{i+1}, x_i + 2, x_{i+2} - 2)$ |
| 53  | 3202581216 | $\max(\min(x_i + 2, \max(0, x_{i+1} - 2)), \min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 63  | 3219162080 | $\max(\min(x_i + 1, \max(0, x_{i+1} - 2)), \min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 67  | 322074008  | $\max(\min(x_i, x_{i+1} - 2), \min(x_{i-1} + 2, x_{i+2} - 2))$ |
| 95  | 3487613920 | $\max(\min(x_i + 1, \max(x_i, x_{i+1} - 2)), \min(x_{i-1} + 2, x_{i+1} - 1))$ |
| 98  | 3489185848 | $\max(\min(x_i, x_{i+1} - 2), \min(x_{i-1} + 2, x_{i+1} - 1))$ |

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