Pure resolutions of unbounded complexes of modules

Abhishek Banerjee

Dept. of Mathematics, Indian Institute of Science, Bangalore-560012, India.
Email: abhishekanerjee1313@gmail.com

Abstract

Let $R$ be a commutative ring. We show that pure injective resolutions and pure projective resolutions can be constructed for unbounded complexes of $R$-modules. We use these to obtain a closed symmetric monoidal structure on the unbounded pure derived category.

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1 Introduction

Given a ring $R$, there are two well known exact structures (in the sense of Quillen [16]) on the category of $R$-modules: the usual exact structure and the pure exact structure (see [22]). The usual derived category $D(R)$, which is constructed by inverting quasi-isomorphisms in the homotopy category of $R$-modules, has been studied extensively in homological algebra. Additionally, there has been recent interest by several authors (see, for instance, [3], [5], [8], [15], [18], [22]) in studying the pure exact structure on the category of $R$-modules. This raises the question of which of the properties of the usual derived category $D(R)$ can also be extended to the “pure derived category” $D_{pur}(R)$, which is obtained by inverting pure quasi-isomorphisms in the homotopy category of $R$-modules. When $R$ is a commutative ring, the purpose of this paper is to construct pure injective and pure projective resolutions of complexes of $R$-modules. We then use these resolutions to exhibit a closed symmetric monoidal structure on the category $D_{pur}(R)$.

It is important to note that in this article, we work with unbounded complexes of $R$-modules and consider the unbounded pure derived category $D_{pur}(R)$. The question behind this paper was motivated naturally by reading the recent work of Zheng and Huang [22], where the authors study pure resolutions of bounded complexes. Further, the authors in [22] have also shown that any bounded above (resp. bounded below) complex of $R$-modules admits a pure injective (resp. pure projective) resolution. However, the general question of pure resolutions for arbitrary unbounded...
complexes is still left open in [22]. Since our methods require us to have a closed symmetric monoidal structure on the category of $R$-modules, we will limit ourselves to commutative rings. It should be mentioned here that even in the case of the classical derived category $D(R)$, the construction of projective and injective resolutions of arbitrary unbounded complexes presents some difficulties. In [19], Spaltenstein showed the existence of projective and injective resolutions for unbounded complexes, which allowed him to remove the boundedness conditions for the existence of certain derived functors of functors such as Hom and tensor product. The derived Hom and derived tensor product functors on the unbounded derived category were also constructed by Bökstedt and Neeman in [2], where the authors used the method of homotopy colimits. For more on resolutions of unbounded complexes, the reader may also see the work of Alonso Tarrío et al. [11] and Serpé [18]. Our methods in this paper are a combination of the classical method of constructing resolutions of bounded complexes (see, for instance, [12]) along with the techniques of Spaltenstein [19] for treating arbitrary unbounded complexes.

We now describe the structure of the paper in detail. In Section 2, we briefly recollect the notions of pure acyclic complexes, pure projective modules, pure injective modules, pure quasi-isomorphisms and pure resolutions that we will need in the paper. In Section 3, we show how to construct pure injective resolutions for unbounded complexes of $R$-modules. Thereafter, pure projective resolutions of unbounded complexes are constructed in Section 4. It should be noted that due to the fact that tensoring preserves cokernels but not kernels, we will need to somewhat adjust our methods in Section 4, i.e., our arguments for pure projective resolutions are not exactly the dual of our arguments for pure injective resolutions. Finally, in Section 5, we use pure projective and pure injective resolutions to obtain a “pure derived Hom functor”:

$$P\text{Hom}^\bullet_{R}(\cdot, \cdot) : D_{\text{pur}}(R)^{\text{op}} \times D_{\text{pur}}(R) \longrightarrow D_{\text{pur}}(R) \quad (1.1)$$

Then, in the case of a commutative ring $R$, [11] removes the boundedness condition for the existence of the pure derived Hom functor in [22, § 4]. We conclude by showing that we have natural isomorphisms:

$$PHom^\bullet_{R}((A^ \bullet \otimes_{R} B^ \bullet)^{\bullet}, C^ \bullet) \cong PHom^\bullet_{R}(A^ \bullet, PHom^\bullet_{R}(B^ \bullet, C^ \bullet))$$

$$Hom_{D_{\text{pur}}(R)}((A^ \bullet \otimes_{R} B^ \bullet)^{\bullet}, C^ \bullet) \cong Hom_{D_{\text{pur}}(R)}(A^ \bullet, PHom^\bullet_{R}(B^ \bullet, C^ \bullet)) \quad (1.2)$$

for any complexes $A^ \bullet$, $B^ \bullet$ and $C^ \bullet$ of $R$-modules, thus giving the unbounded pure derived category $D_{\text{pur}}(R)$ the structure of a closed symmetric monoidal category.

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2 Pure acyclic complexes

In this section, we will briefly recall some facts on pure acyclic complexes, pure projectives, pure injectives and pure resolutions that we will use in the rest of the paper. Throughout, we let $R$ be
a commutative ring with identity. We denote by $R - Mod$ the category of $R$-modules. We will denote by $C(R)$ the category of cochain complexes:

$$ (M^\bullet, d_M^\bullet) : \ldots \longrightarrow M^{i-1} \overset{d_{M}^{i-1}}{\longrightarrow} M^{i} \overset{d_{M}^{i}}{\longrightarrow} M^{i+1} \longrightarrow \ldots $$

of $R$-modules and by $K(R)$ the homotopy category of $R - Mod$. By abuse of notation, we will usually denote an object $(M^\bullet, d_M^\bullet) \in C(R)$ simply by $M^\bullet$. For any $n \in \mathbb{Z}$, we let $M[n]^\bullet$ denote the shifted cochain complex given by $M[n]^i := M^{i+n}$ with differential $d_{M[n]}^i = (-1)^n d_M^{i+n}$. The unbounded derived category of $R$-modules is denoted by $D(R)$. A cochain complex $M^\bullet$ is said to be bounded above (resp. bounded below) if $M^i$ is sufficiently small. Given complexes $(M^\bullet, d_M^\bullet)$ and $(N^\bullet, d_N^\bullet) \in C(R)$, we have an internal Hom object $(\text{Hom}_R^i(M^\bullet, N^\bullet), d^i) \in C(R)$ given by:

$$ \text{Hom}_R^i(M^\bullet, N^\bullet) := \prod_{j \in \mathbb{Z}} \text{Hom}_R(M^j, N^{i+j}) \quad \forall i \in \mathbb{Z} $$

$$ d^i(f) := d_N \circ f - (-1)^i f \circ d_M \quad \forall f \in \text{Hom}_R^i(M^\bullet, N^\bullet) $$

We now recall the following definitions.

**Definition 2.1.** (see [2])

(a) A monomorphism $f : M' \longrightarrow M$ in $R - Mod$ is said to be pure if the induced morphism $N \otimes_R f : N \otimes_R M' \longrightarrow N \otimes_R M$ is a monomorphism for each module $N \in R - Mod$.

(b) A short exact sequence:

$$ 0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 $$

in $R - Mod$ is said to be pure acyclic if the induced sequence

$$ 0 \longrightarrow N \otimes_R M' \longrightarrow N \otimes_R M \longrightarrow N \otimes_R M'' \longrightarrow 0 $$

is acyclic for each module $N \in R - Mod$. In such a situation, the morphism $M \longrightarrow M''$ is said to be a pure epimorphism.

(c) (see [2]) More generally, a complex $M^\bullet \in C(R)$ is said to be pure acyclic if the induced sequence $N \otimes_R M^\bullet$ is acyclic for each module $N \in R - Mod$. This is equivalent to the complex $\text{Hom}_R^i(F, M^\bullet)$ being acyclic for each finitely presented $R$-module $F$.

(d) An acyclic complex $(M^\bullet, d_M^\bullet) \in C(R)$ is said to be pure acyclic at some given $n \in \mathbb{Z}$ if

$$ 0 \longrightarrow \text{Im}(d_M^{n-1}) = \text{Ker}(d_M^n) \longrightarrow M^n \longrightarrow \text{Coker}(d_M^{n-1}) \longrightarrow 0 $$

is a pure short exact sequence in the sense of (b). The complex $(M^\bullet, d_M^\bullet) \in C(R)$ is pure acyclic in the sense of (c) if and only if it is pure acyclic at each $n \in \mathbb{Z}$.

Given a morphism $f^\bullet : M^\bullet \longrightarrow N^\bullet$, its mapping cone $C_f^\bullet$ is taken to be the complex $C_f^\bullet := M[1]^\bullet \oplus N^\bullet$ with differential $d_{C_f}^i$ given by

$$ d_{C_f}^i = \begin{pmatrix} d_M^{i+1} & 0 \\ -f^{i+1} & d_N^i \end{pmatrix} : M^{i+1} \oplus N^i \longrightarrow M^{i+2} \oplus N^{i+1} $$

(2.5)
We notice that the canonical projections $p^i : M^i \oplus N^{i-1} \rightarrow M^i$ determine a morphism of complexes from $C_f^*$ to $M[1]^*$.

**Definition 2.2.** *(see [22, Definition 2.7])* A morphism $f^* : M^* \rightarrow N^*$ in $C(R)$ is a pure quasi-isomorphism if its cone $C_f^*$ is pure acyclic.

Equivalently, $f^*$ is a pure quasi-isomorphism if $M' \otimes_R f^* : M' \otimes_R M^* \rightarrow M' \otimes_R N^*$ is a quasi-isomorphism for each module $M' \in R - \text{Mod}$.

**Definition 2.3.** *(see [21])* A module $P \in R - \text{Mod}$ is pure projective if the functor $\text{Hom}_R^*(P, -)$ carries pure acyclic complexes to pure acyclic complexes. Similarly, a module $I \in R - \text{Mod}$ is pure injective if the functor $\text{Hom}_R^*(I, -)$ preserves pure acyclic complexes.

The category of pure projectives in $R - \text{Mod}$ will be denoted by $\mathcal{PP}$ and the category of pure injectives in $R - \text{Mod}$ by $\mathcal{PI}$.

We mention here (see [22, Remark 2.6]) that a complex $M^* \in C(R)$ is pure acyclic if and only if $\text{Hom}_R^*(P, M^*)$ is acyclic for any pure projective $P \in \mathcal{PP}$. This is also equivalent to $\text{Hom}_R^*(M^*, I)$ being acyclic for any pure injective $I \in \mathcal{PI}$.

On the other hand, (see [22, Remark 2.8]) a morphism $f^* : M^* \rightarrow N^*$ of complexes is a pure quasi-isomorphism if and only if $\text{Hom}_R^*(P, f^*) : \text{Hom}_R^*(P, M^*) \rightarrow \text{Hom}_R^*(P, N^*)$ (resp. $\text{Hom}_R^*(f^*, I) : \text{Hom}_R^*(N^*, I) \rightarrow \text{Hom}_R^*(M^*, I)$) is a quasi-isomorphism for each $P \in \mathcal{PP}$ (resp. for each $I \in \mathcal{PI}$).

**Definition 2.4.** *(see [22])* (a) Let $M^* \in C(R)$. A morphism $f^* : P^* \rightarrow M^*$ is said to be a pure projective resolution of $M^*$ if it satisfies the following conditions:

(i) $P^*$ is a complex of pure projective modules and $f^*$ is a pure quasi-isomorphism.

(ii) The functor $\text{Hom}_R^*(P^*, -)$ preserves pure acyclic complexes.

(b) Dually, a morphism $g^* : M^* \rightarrow I^*$ is said to be a pure injective resolution of $M^*$ if it satisfies:

(i) $I^*$ is a complex of pure injective modules and $g^*$ is a pure quasi-isomorphism.

(ii) The functor $\text{Hom}_R^*(I^*, -)$ preserves pure acyclic complexes.

Given an $R$-module $M$, it is known (see [21, Corollary 6]) that there exists a pure injective module $I \in \mathcal{PI}$ and a pure monomorphism $M \hookrightarrow I$. In fact, this holds more generally in any locally finitely presented additive category (see Herzog [9]). The aim of this paper is to show that any unbounded complex of $R$-modules admits a “pure projective resolution” and a “pure injective resolution”. In [22], Zheng and Huang have already shown that any bounded above (resp. bounded below) complex in $C(R)$ admits a pure injective resolution (resp. a pure projective resolution). The proof of Zheng and Huang in [22] uses the techniques of homotopy (co)limits due to Bökstedt and Neeman [2] (see also [14]). However, our proof will use a combination of the classical technique for constructing resolutions along with the methods of Spaltenstein [19] for treating arbitrary unbounded complexes.
3 Pure injective resolutions of unbounded complexes

In this section, we will need one more concept, that of a “$K$-pure injective complex”, which will be analogous to the classical notion of a $K$-injective complex (see, for instance, [19 § 1]). Although this notion already appears implicitly in Definition 2.4, we propose the following definition, which does not seem to have appeared explicitly before in the literature.

**Definition 3.1.** We will say that a cochain complex $M^\bullet \in C(R)$ is $K$-pure injective if the functor $\Hom^\bullet_R(-, M^\bullet) : C(R) \to C(R)$ takes pure acyclic complexes to pure acyclic complexes.

**Proposition 3.2.** Let $M^\bullet \in C(R)$. Then, the following statements are equivalent:

(a) $M^\bullet$ is a $K$-pure injective complex.

(b) For any pure acyclic complex $A^\bullet \in C(R)$, we have $\Hom_K(R)(A^\bullet, M^\bullet) = 0$.

**Proof.** (a) $\Rightarrow$ (b): For any two complexes $B^\bullet, C^\bullet \in C(R)$, it is well known (see, for instance, [19 §0.4]) that

$$H^k(\Hom^\bullet_R(B^\bullet, C^\bullet)) = \Hom_K(R)(B^\bullet, C[k]^\bullet) \quad \forall k \in \mathbb{Z} \quad (3.1)$$

Let $A^\bullet \in Ch(R)$ be pure acyclic. Since $M^\bullet$ is $K$-pure injective, it follows that $\Hom^\bullet_R(A^\bullet, M^\bullet)$ is pure acyclic and, in particular, acyclic. Then, $\Hom_K(R)(A^\bullet, M^\bullet) = H^0(\Hom^\bullet_R(A^\bullet, M^\bullet)) = 0$.

(b) $\Rightarrow$ (a): Shifting the indices in (3.1), we obtain:

$$H^k(\Hom^\bullet_R(B^\bullet, C^\bullet)) = \Hom_K(R)(B[-k]^\bullet, C^\bullet) \quad \forall k \in \mathbb{Z} \quad (3.2)$$

for any two complexes $B^\bullet, C^\bullet \in C(R)$. Now let $M^\bullet \in C(R)$ be such that $\Hom_K(R)(A^\bullet, M^\bullet) = 0$ for every pure acyclic $A^\bullet \in C(R)$. We have to show that $\Hom^\bullet_R(A^\bullet, M^\bullet)$ is pure acyclic, or equivalently that $\Hom^\bullet_R(F, \Hom^\bullet_R(A^\bullet, M^\bullet)) \cong \Hom^\bullet_R(F \otimes_R A^\bullet, M^\bullet)$ is acyclic for every finitely presented $R$-module $F$. If $A^\bullet$ is pure acyclic, it is immediate from Definition 2.4 that $(F \otimes_R A^\bullet)[-k]^\bullet$ is also pure acyclic for any $k \in \mathbb{Z}$. It now follows from (3.2) that:

$$H^k(\Hom^\bullet_R(F \otimes_R A^\bullet, M^\bullet)) = \Hom_K(R)((F \otimes_R A^\bullet)[-k]^\bullet, M^\bullet) = 0 \quad \forall k \in \mathbb{Z} \quad (3.3)$$

This proves the result.

Let $K_{pac}(R)$ denote the full subcategory of $K(R)$ consisting of complexes that are also pure acyclic. The cone of a morphism of pure acyclic complexes is still pure acyclic and hence $K_{pac}(R)$ is a triangulated subcategory. Then, since $K_{pac}(R)$ is closed under direct summands, it follows from Rickard’s criterion [17 Proposition 1.3] (see also [13]) that $K_{pac}(R)$ is a thick subcategory. We then consider the “pure derived category” $D_{pur}(R)$ as in [22 §3] given by the Verdier quotient:

$$D_{pur}(R) := K(R)/K_{pac}(R) \quad (3.4)$$
Consider a “right roof” in $K(R)$ given by a pair of morphisms $(f^\bullet, u^\bullet)$ of the form:

$$A^\bullet \xrightarrow{f^\bullet} C^\bullet \xleftarrow{u^\bullet} B^\bullet$$

(3.5)

where $u^\bullet$ is a pure quasi-isomorphism. Then, the morphisms in $D_{\text{pur}}(R)$ from $A^\bullet$ to $B^\bullet$ can be given by equivalence classes of right roofs as in (3.5) (see [9] for details). We will now characterize $K$-pure injective complexes in terms of the pure derived category $D_{\text{pur}}(R)$.

**Proposition 3.3.** Let $M^\bullet \in C(R)$. Then, the following statements are equivalent:

(a) $M^\bullet$ is a $K$-pure injective complex.

(b) For any complex $A^\bullet \in C(R)$, we have $\text{Hom}_{K(R)}(A^\bullet, M^\bullet) \cong \text{Hom}_{D_{\text{pur}}(R)}(A^\bullet, M^\bullet)$.

**Proof.** (b) $\Rightarrow$ (a): Let $A^\bullet \in C(R)$ be pure acyclic. Then, $A^\bullet = 0$ in the pure derived category $D_{\text{pur}}(R)$. Hence, $\text{Hom}_{K(R)}(A^\bullet, M^\bullet) \cong \text{Hom}_{D_{\text{pur}}(R)}(A^\bullet, M^\bullet) = 0$ and it follows from Proposition 3.2 that $M^\bullet$ is $K$-pure injective.

(a) $\Rightarrow$ (b): Suppose that $M^\bullet$ is a $K$-pure injective complex. We first show that if $u^\bullet : M^\bullet \rightarrow N^\bullet$ is a pure quasi-isomorphism, it must have a left inverse up to homotopy, i.e., we must have some $v^\bullet : N^\bullet \rightarrow M^\bullet$ such that $v^\bullet \circ u^\bullet \sim 1$. If $C_u^\bullet$ denotes the cone of $u^\bullet$, applying the functor $\text{Hom}_{K(R)}(-, M^\bullet)$ to the distinguished triangle $M^\bullet \xrightarrow{u^\bullet} N^\bullet \rightarrow C_u^\bullet$ gives an induced exact sequence:

$$\text{Hom}_{K(R)}(C_u^\bullet, M^\bullet) \rightarrow \text{Hom}_{K(R)}(N^\bullet, M^\bullet) \rightarrow \text{Hom}_{K(R)}(M^\bullet, M^\bullet) \rightarrow \text{Hom}_{K(R)}(C_u[-1]^\bullet, M^\bullet)$$

It follows that there exists $v^\bullet : N^\bullet \rightarrow M^\bullet$ such that $v^\bullet \circ u^\bullet \sim 1$. Now, given a morphism $(f^\bullet, u^\bullet) \in \text{Hom}_{D_{\text{pur}}(R)}(A^\bullet, M^\bullet)$ having the form of a right roof

$$A^\bullet \xrightarrow{f^\bullet} N^\bullet \xleftarrow{u^\bullet} M^\bullet$$

(3.6)

we associate $(f^\bullet, u^\bullet) \in \text{Hom}_{D_{\text{pur}}(R)}(A^\bullet, M^\bullet)$ to $v^\bullet \circ f^\bullet \in \text{Hom}_{K(R)}(A^\bullet, M^\bullet)$. Conversely, any morphism $g^\bullet \in \text{Hom}_{K(R)}(A^\bullet, M^\bullet)$ is associated to the roof $(g^\bullet, 1) \in \text{Hom}_{D_{\text{pur}}(R)}(A^\bullet, M^\bullet)$. Since the right roofs $(v^\bullet \circ f^\bullet, 1), (f^\bullet, u^\bullet)$ are equivalent in $\text{Hom}_{D_{\text{pur}}(R)}(A^\bullet, M^\bullet)$, it is clear that these two associations are inverse to each other. Hence, we have $\text{Hom}_{K(R)}(A^\bullet, M^\bullet) \cong \text{Hom}_{D_{\text{pur}}(R)}(A^\bullet, M^\bullet)$.

**Proposition 3.4.** Let $(I^\bullet, d_I^\bullet) \in C(R)$ be a bounded below complex of pure injective modules. Then, $I^\bullet$ is $K$-pure injective.
Proof. For the sake of definiteness, we suppose that \( I^j = 0 \) for every \( j < 0 \) and \( I^0 \neq 0 \). Using Proposition 3.2, it suffices to show that \( \text{Hom}_{C(R)}(A^\bullet, I^\bullet) = 0 \) for any pure acyclic \((A^\bullet, d_{A}^\bullet) \in C(R)\). We will show that any morphism \( f : (A^\bullet, d_{A}^\bullet) \rightarrow (I^\bullet, d_{I}^\bullet) \) in \( C(R) \) is null-homotopic. For some given integer \( K \), we suppose that we have constructed maps \( t^k : A^k \rightarrow I^k \) for all integers \( k \leq K \) such that \( t^k \circ d_{A}^k + d_{I}^{k-2} \circ t^{k-1} = f^k - 1 \). This is already true for \( K = 0 \). We now note that:

\[
(f^k - d_{I}^{k-1} \circ t^k) \circ d_{A}^k = f^k \circ d_{A}^k - d_{I}^{k-1} \circ (t^k \circ d_{A}^k) = (f^k \circ d_{A}^k - d_{I}^{k-1} \circ f^{k-1}) + d_{I}^{k-1} \circ d_{I}^{k-2} \circ t^{k-1} = 0
\]

Hence, the morphism \((f^k - d_{I}^{k-1} \circ t^k) : A^k \rightarrow I^k\) factors through \( A^k/\text{Im}(d_{A}^{k-1}) = A^k/\text{Ker}(d_{A}^k) \cong \text{Im}(d_{A}^{k}) \). From Definition 2.1, it follows that \( \text{Im}(d_{A}^{k-1}) \hookrightarrow A^{k+1} \) is a pure monomorphism. Since \( I^k \) is pure injective, the morphism \( \text{Im}(d_{A}^{k}) \hookrightarrow I^k \) extends to a morphism \( t^{k+1} : A^{k+1} \rightarrow I^k \) that satisfies \((f^k - d_{I}^{k-1} \circ t^k) = t^{k+1} \circ d_{A}^k\). This proves the result.

\[
\square
\]

As mentioned in Section 2, it is well known (see [21, Corollary 6]) that any \( R \)-module \( M \) admits a pure monomorphism \( M \hookrightarrow I \) into a pure injective module \( I \). We will now show that any bounded below complex of \( R \)-modules admits a pure injective resolution.

**Proposition 3.5.** Let \( M^\bullet \in C(R) \) be a cochain complex of \( R \)-modules that is bounded below. Then, there exists a pure quasi-isomorphism \( u^\bullet : M^\bullet \rightarrow I^\bullet \) such that \( I^\bullet \) is a bounded below complex of pure injective modules.

**Proof.** For the sake of definiteness, we suppose that the complex \((M^\bullet, d^\bullet)\) satisfies \( M^j = 0 \) for each \( j < 0 \) but \( M^0 \neq 0 \). We put \( I^j = 0 \) for each \( j < 0 \). We choose a pure monomorphism \( u^0 : M^0 \hookrightarrow I^0 \) with \( I^0 \) pure injective. Then, for every \( i < 1 \), we have already constructed pure injective modules \( I^i \), morphisms \( u^i : M^i \rightarrow I^i \) and differentials \( e^{i-1} : I^{i-1} \rightarrow I^i \) such that we have induced isomorphisms:

\[
H^{i-1}(M^\bullet \otimes_R N) \cong K(e^{i-1} \otimes_R N)/\text{Im}(e^{i-2} \otimes_R N) \quad \forall \ N \in R-\text{Mod}
\]  

(3.7)

and monomorphisms:

\[
\text{Coker}(d^{i-1} \otimes_R N) \hookrightarrow \text{Coker}(e^{i-1} \otimes_R N) \quad \forall \ N \in R-\text{Mod}
\]

(3.8)

We suppose that we have already done this for all \( i \in \mathbb{Z} \) less than some given integer \( k \geq 1 \). We will now show that we can choose a pure injective \( I^k \), a morphism \( u^k : M^k \rightarrow I^k \) and a differential \( e^{k-1} : I^{k-1} \rightarrow I^k \) such that:

\[
H^{k-1}(M^\bullet \otimes_R N) \cong K(e^{k-1} \otimes_R N)/\text{Im}(e^{k-2} \otimes_R N) \quad \forall \ N \in R-\text{Mod}
\]

(3.9)

\[
\text{Coker}(d^{k-1} \otimes_R N) \hookrightarrow \text{Coker}(e^{k-1} \otimes_R N) \quad \forall \ N \in R-\text{Mod}
\]

We consider the colimit \( C^k \) defined by the following pushout square:

\[
\begin{array}{ccc}
M^{k-1} & \xrightarrow{d^{k-1}} & M^k \\
\downarrow & & \downarrow \\
\text{Coker}(e^{k-2}) & \longrightarrow & C^k
\end{array}
\]

(3.10)
and choose a pure monomorphism $C^k \hookrightarrow I^k$ with $I^k$ pure injective. Then, for any $N \in R - Mod$, $C^k \otimes_R N \rightarrow I^k \otimes_R N$ is a monomorphism and the following square is still a pushout:

\[
\begin{array}{ccc}
M^{k-1} \otimes_R N & \xrightarrow{d^{k-1} \otimes_R N} & M^k \otimes_R N \\
\downarrow & & \downarrow \\
Coker(e^{k-2}) \otimes_R N = Coker(e^{k-2} \otimes_R N) & \longrightarrow & C^k \otimes_R N
\end{array}
\]  

(3.11)

We now define $e^{k-1} : I^{k-1} \rightarrow I^k$ to be the composition $I^{k-1} \rightarrow Coker(e^{k-2}) \rightarrow C^k \hookrightarrow I^k$ and $u^k : M^k \rightarrow I^k$ to be the composition $M^k \rightarrow C^k \rightarrow I^k$. Applying the dual of [12, Lemma 68] to the pushout square (3.11) along with the monomorphism $C^k \otimes_R N \hookrightarrow I^k \otimes_R N$ gives us a monomorphism $Coker(d^{k-1} \otimes_R N) \hookrightarrow Coker(e^{k-1} \otimes_R N)$.

We now put $d^i_N := d^i \otimes_R N$ and $e^j_N := e^j \otimes_R N$ for any integer $j$ and any $R$-module $N$. Since the morphisms $M^{k-1} \otimes_R N \rightarrow M^k \otimes_R N$ and $M^{k-1} \otimes_R N \rightarrow Coker(e^{k-2} \otimes_R N)$ both factor through the epimorphism $M^{k-1} \otimes_R N \twoheadrightarrow (M^{k-1} \otimes_R N)/Im(d^{k-2}_N)$, we can simply replace $M^{k-1} \otimes_R N$ by $(M^{k-1} \otimes_R N)/Im(d^{k-2}_N)$ in (3.11) and still obtain a pushout square. Now if we let $C_N$ be the colimit of the system $Coker(e^{k-2}_N) \leftarrow (M^{k-1} \otimes_R N)/Im(d^{k-2}_N) \twoheadrightarrow Im(d^{k-1}_N)$, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
(M^{k-1} \otimes_R N)/Im(d^{k-2}_N) = Coker(d^{k-2}) & \longrightarrow & Im(d^{k-1}) \longrightarrow M^k \otimes_R N \\
\downarrow & & \downarrow \\
Coker(e^{k-2}_N) & \longrightarrow & C_N \longrightarrow C^k \otimes_R N
\end{array}
\]  

(3.12)

where all the squares are pushouts. The pushout of the epimorphism $(M^{k-1} \otimes_R N)/Im(d^{k-2}_N) \twoheadrightarrow Im(d^{k-1}_N)$ gives an epimorphism $Coker(e^{k-2}_N) \twoheadrightarrow C_N$. On the other hand, since $R - Mod$ is an abelian category, it follows that the pushout of the monomorphism $Im(d^{k-1}_N) \hookrightarrow M^k \otimes_R N$ gives a monomorphism $C_N \hookrightarrow C^k \otimes_R N$. Accordingly, the morphism $e^{k-1}_N : I^{k-1} \otimes_R N \rightarrow I^k \otimes_R N$ can be factored as the epimorphism:

\[
I^{k-1} \otimes_R N \rightarrow (I^{k-1} \otimes_R N)/Im(e^{k-2}_N) = Coker(e^{k-2}_N) \twoheadrightarrow C_N
\]  

(3.13)

followed by the monomorphism:

\[
C_N \hookrightarrow C^k \otimes_R N \hookrightarrow I^k \otimes_R N
\]  

(3.14)

From (3.13) and (3.14) it follows that $C_N \cong (I^{k-1} \otimes_R N)/Ker(e^{k-1}_N) = Im(e^{k-1}_N)$. By assumption, we have a monomorphism $Coker(d^{k-2}_N) \hookrightarrow Coker(e^{k-2}_N)$. This gives us the following pushout square:

\[
\begin{array}{ccc}
(M^{k-1} \otimes_R N)/Im(d^{k-2}_N) = Coker(d^{k-2}) & \xrightarrow{epic} & Im(d^{k-1}) = (M^{k-1} \otimes_R N)/Ker(d^{k-1}_N) \\
\downarrow \text{monic} & & \downarrow \\
(I^{k-1} \otimes_R N)/Im(e^{k-2}_N) = Coker(e^{k-2}) & \xrightarrow{epic} & Im(e^{k-1}) = (I^{k-1} \otimes_R N)/Ker(e^{k-1}_N)
\end{array}
\]  

(3.15)

8
The fact that (3.15) is a pushout square and that $\text{Coker}(d_{k-2}^N) \rightarrow \text{Coker}(e_{k-2}^N)$ is a monomorphism shows that we have an isomorphism of the kernels of the two horizontal morphisms. It is also clear that the kernels of these horizontal morphisms are $H^{k-1}(M^* \otimes R N)$ and $\text{Ker}(e_{k-1} \otimes_R N)/\text{Im}(e_{k-2} \otimes_R N)$ respectively. Thus, we can construct inductively a pure quasi-isomorphism $M^* \rightarrow I^*$ where $I^*$ is a bounded below complex of pure injectives.

**Proposition 3.6.** Let $f^* : M_2^* \rightarrow M_1^*$ be a morphism of bounded below complexes in $C(R)$. Let $u_1^* : M_1^* \rightarrow I_1^*$ be a pure quasi-isomorphism from $M_1^*$ to a bounded below complex $I_1^*$ of pure injectives. Then, there exists a bounded below complex $I_2^*$ of pure injectives, a pure quasi-isomorphism $u_2^* : M_2^* \rightarrow I_2^*$ and a morphism $g^* : I_2^* \rightarrow I_1^*$ fitting into a commutative diagram:

$$
\begin{array}{ccc}
M_2^* & \xrightarrow{u_2^*} & I_2^* \\
\downarrow f^* & & \downarrow g^* \\
M_1^* & \xrightarrow{u_1^*} & I_1^* \\
\end{array}
\quad (3.16)
$$

**Proof.** It is clear that the mapping cone $C_{u_1f}^*$ of the composition $u_1^* \circ f^* : M_2^* \rightarrow I_1^*$ is a bounded below complex in $C(R)$. Applying Proposition 3.5, we choose a pure quasi-isomorphism $v^* : C_{u_1f}^* \rightarrow I^*$ to a bounded below complex of pure injectives. Thereafter, we consider the composition $h^* : I_1^* \rightarrow C_{u_1f}^* \xrightarrow{v^*} I^*$ and its mapping cone $C_h^*$. We now have the following commutative diagram in $K(R)$:

$$
\begin{array}{ccc}
M_2^* & \xrightarrow{u_2^*} & I_1^* \\
\downarrow u_2^* & & \downarrow g^* \\
C_h[-1]^* & \xrightarrow{h^*} & I_1^* \\
\downarrow 1 & & \downarrow 1 \\
M_1^* & \xrightarrow{u_1^*} & I_1^* \\
\end{array}
\quad (3.17)
$$

Since the horizontal rows in (3.17) are distinguished triangles, the triangulated structure on $K(R)$ implies that we have a morphism $u_2^* : M_2^* \rightarrow C_h[-1]^*$ making the diagram commute. It is clear that $C_h[-1]^*$ is a bounded below complex of pure injectives and we set $I_2^* := C_h[-1]^*$. Now, for any module $N \in R - \text{Mod}$, we have an induced commutative diagram:

$$
\begin{array}{ccc}
N \otimes_R M_2^* & \xrightarrow{N \otimes_R (u_2^* \circ f^*)} & N \otimes_R I_1^* \\
\downarrow N \otimes_R u_2^* & & \downarrow N \otimes_R v^* \\
N \otimes_R I_2^* = N \otimes_R C_h[-1]^* & \xrightarrow{N \otimes_R g^*} & N \otimes_R I_1^* \\
\downarrow N \otimes_R v^* & & \downarrow N \otimes_R v^* \\
N \otimes_R I_2^* = N \otimes_R C_h[-1]^* & \xrightarrow{N \otimes_R v^*} & N \otimes_R I_1^* \\
\end{array}
\quad (3.18)
$$

Since $v^*$ is a pure quasi-isomorphism, $N \otimes_R v^*$ is a quasi-isomorphism. Since the mapping cone commutes with the functor $N \otimes_R$, the horizontal rows in (3.18) are still distinguished triangles. Now, since $1 : N \otimes_R I_1^* \rightarrow N \otimes_R I_1^*$ and $N \otimes_R v^* : N \otimes_R C_{u_1f}^* \rightarrow N \otimes_R I^*$ are quasi-isomorphisms
(and hence isomorphisms in the derived category \(D(R)\)), the third morphism \(N \otimes_R u_2^\bullet : N \otimes_R M_2^\bullet \to N \otimes_R I_2^\bullet\) is also a quasi-isomorphism (see [20, Tag 014A]). Hence, \(u_2^\bullet : M_2^\bullet \to I_2^\bullet\) is a pure quasi-isomorphism that fits into the commutative square (3.16).

We will now proceed to construct pure injective resolutions for arbitrary, unbounded complexes of \(R\)-modules. We will first need the notion of a “special inverse system”.

**Definition 3.7.** (see [19, Definition 2.1]) Let \(T\) be a class of complexes in \(C(R)\) that is closed under isomorphisms.

(a) A \(T\)-special inverse system of complexes is an inverse system \(\{I_n^\bullet\}_{n \in \mathbb{Z}}\) of complexes in \(C(R)\) satisfying the following conditions for each \(n \in \mathbb{Z}\):

1. The cochain map \(I_n^\bullet \to I_{n-1}^\bullet\) is surjective.
2. The kernel \(K_n^\bullet := \ker(I_n^\bullet \to I_{n-1}^\bullet)\) lies in the class \(T\).
3. The short exact sequence of complexes:

\[
0 \to K_n^\bullet \xrightarrow{i_n^\bullet} I_n^\bullet \xrightarrow{p_n^\bullet} I_{n-1}^\bullet \to 0 \tag{3.19}
\]

is “semi-split”, i.e., it is split in each degree.

(b) The class \(T \subseteq C(R)\) is said to be closed under special inverse limits if the inverse limit of every \(T\)-special inverse system in \(C(R)\) is contained in \(T\).

By slight abuse of notation, we will refer to \(\{I_n^\bullet\}_{n \geq 0}\) as a \(T\)-special inverse system if setting \(I_n^\bullet = 0\) for all \(n < 0\) makes \(\{I_n^\bullet\}_{n \in \mathbb{Z}}\) into a \(T\)-special inverse system in the sense of Definition 3.7 above.

**Proposition 3.8.** (a) Let \(C \subseteq C(R)\) be a class of complexes. Let \(T(C)\) denote the class of complexes \(M^\bullet \in C(R)\) such that \(\text{Hom}_R^\bullet(A^\bullet, M^\bullet)\) is pure acyclic for each \(A^\bullet \in C\). Then, \(T(C)\) is closed under special inverse limits.

(b) The class of all \(K\)-pure injective complexes is closed under special inverse limits.

Proof. (a) We begin by setting:

\[
\mathcal{T}' := \{B^\bullet \in C(R) \mid B^\bullet = F \otimes_R A^\bullet \text{ for some finitely presented } R\text{-module } F \text{ and some } A^\bullet \in C\}
\]

Now since \(\text{Hom}_R^\bullet(F \otimes_R A^\bullet, M^\bullet) \cong \text{Hom}_R^\bullet(F, \text{Hom}_R^\bullet(A^\bullet, M^\bullet))\), a complex \(M^\bullet \in T(C)\) if and only if \(\text{Hom}_R^\bullet(B^\bullet, M^\bullet)\) is acyclic for each \(B^\bullet \in \mathcal{T}'\). It now follows from [19, Corollary 2.5] that \(T(C)\) is closed under special inverse limits. The result of (b) follows directly from (a) by taking \(C\) to be the class of all pure acyclic complexes in \(C(R)\).
Given a complex \((M^\bullet, d^\bullet) \in C(R)\), we recall that for any \(n \in \mathbb{Z}\), its truncation \(\tau^{\geq n}M^\bullet\) is given by setting
\[
(\tau^{\geq n}M)^i = \begin{cases} 
M^i & \text{if } i > n \\
\text{Coker}(d^{n-1}) & \text{if } i = n \\
0 & \text{if } i < n
\end{cases}
\] (3.20)

Then, it is clear that \(H^i(\tau^{\geq n}M^\bullet) = H^i(M^\bullet)\) for all \(i \geq n\) and \(H^i(\tau^{\geq n}M^\bullet) = 0\) otherwise. Further, the canonical morphisms \(\tau^{\geq n-1}M^\bullet \rightarrow \tau^{\geq n}M^\bullet\) can be used to express \(M^\bullet\) as an inverse limit
\[M^\bullet = \varprojlim_{n \geq 0} \tau^{\geq n}M^\bullet.\]

**Proposition 3.9.** For any complex \(M^\bullet \in C(R)\) there exists a special inverse system \(\{I^\bullet_n\}_{n \geq 0}\) of \(K\)-pure injective complexes and a morphism \(\{f_n : \tau^{\geq n}M^\bullet \rightarrow I^\bullet_n\}_{n \geq 0}\) of inverse systems satisfying the following conditions:

(a) Each \(I^\bullet_n\) is a bounded below complex of pure injectives.

(b) Each \(f_n\) is a pure quasi-isomorphism.

**Proof.** Using Proposition 3.5, we choose a pure quasi-isomorphism \(f_0 : \tau^{\geq 0}M^\bullet \rightarrow I^\bullet_0\) with \(I^\bullet_0\) a bounded below complex of pure injectives. For some \(n \geq 1\), we assume that we have already chosen pure quasi-isomorphisms \(f_j : \tau^{\geq -j}M^\bullet \rightarrow I^\bullet_j\) for all \(0 \leq j \leq n - 1\) satisfying the required conditions. We now set:
\[I^\bullet := I^\bullet_{n-1}, \quad N^\bullet := \tau^{\geq -n}M^\bullet, \quad f : N^\bullet \rightarrow \tau^{\geq -n+1}M^\bullet, \quad f_{n-1} : I^\bullet_{n-1} \rightarrow I^\bullet_n = I^\bullet\] (3.21)

We let \(C^\bullet_j\) denote the cone of \(f\). Again using Proposition 3.5, we choose a pure quasi-isomorphism \(g : C^\bullet_j \rightarrow J^\bullet\) with \(J^\bullet\) a bounded below complex of pure injective modules. Since \(C^\bullet_j = N[1]^\bullet \oplus I^\bullet\) as a \(\mathbb{Z}\)-graded module, \(g\) induces morphisms \(g' : N[1]^\bullet \rightarrow C^\bullet_j \xrightarrow{g} J^\bullet\) and \(g'' : I^\bullet \rightarrow C^\bullet_j \xrightarrow{g} J^\bullet\) of graded modules and \(g''\) is actually a morphism of complexes. We rewrite \(g' : N[1]^\bullet \rightarrow J^\bullet\) as a morphism \(g' : N^\bullet \rightarrow J[\mathbb{Z}]\) and consider the following morphism for each \(i \in \mathbb{Z}\):
\[h^i : N^i \rightarrow C_{-g''[-1]^i} = I^i \oplus J[\mathbb{Z}]^i, \quad n \mapsto (f(n), g'(n)) = (f(n), g(n, 0))\] (3.22)

where \(C_{-g''[-1]^i}\) is the cone of \(-g''\). We claim that \(h^\bullet = \{h^i\}_{i \in \mathbb{Z}}\) is a morphism of complexes. For this, we note that for some \(i \in \mathbb{Z}\) and for any \(n \in N^i\) we have:
\[
h \circ d_N(n) = (f \circ d_N(n), g(d_N(n), 0))
\]
\[
d_{C_{-g''[-1]} \circ h(n)} = (d_I \circ f(n), g'' \circ f(n) - d_J \circ g(n, 0))
\]
\[
= (d_I \circ f(n), g'' \circ f(n) - g \circ d_C(n, 0))
\]
\[
= (d_I \circ f(n), g'' \circ f(n) - g(-d_N(n), f(n)))
\]
\[
= (f \circ d_N(n), g(d_N(n), 0))
\]

where \(d_N\) is the differential on \(N^\bullet\). As a graded module, it is immediate that the cone \(C^\bullet_h\) of \(h^\bullet\) satisfies:
\[C^\bullet_h = N[1]^\bullet \oplus C_{-g''[-1]^i} = N[1]^\bullet \oplus I^\bullet \oplus J[\mathbb{Z}]^i = (N[2]^\bullet \oplus I[1]^\bullet \oplus J^\bullet)[-1] = C_{-g[-1]^i}\] (3.24)

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To show that (3.24) is an isomorphism of complexes, for some given \( i \in \mathbb{Z} \), we choose any \( x \in N^{i+1}, y \in I^i \) and \( z \in J^{i-1} \). Then, we see that:

\[
\begin{align*}
  d_{C_h}(x, y, z) &= (-d_N(x), h(x) + d_{C_{g'}}[-1](y, z)) \\
  &= (-d_N(x), f(x) + d_I(y), g(x, 0) + g''(y) - d_J(z)) \\
  &= (-d_N(x), f(x) + d_I(y), g(x, y) - d_J(z))
\end{align*}
\]

Accordingly, we have an isomorphism \( C_h^* \cong C_{g'}[-1]^* \) of complexes. Since \( g \) is a pure quasi-isomorphism, it now follows that \( C_h^* \) is pure acyclic. In other words, \( h^* : N^* = \tau^{\geq -n}M^* \rightarrow C_{g'}[-1]^* \) is a pure quasi-isomorphism. From the definitions, it is clear that \( I_n^* := C_{g'}[-1]^* \) is a bounded below complex of pure injectives. Finally, the definition in (3.22) makes it clear that \( \{I_n^*\}_{n \geq 0} \) is a special inverse system of \( K \)-pure injective complexes. This proves the result.

\[
(3.25)
\]

**Lemma 3.10.** Let \( \{g_n^* : X_n^* \rightarrow Y_n^*\}_{n \geq 0} \) be a morphism of inverse systems \( \{X_n^*\}_{n \geq 0} \) and \( \{Y_n^*\}_{n \geq 0} \) of complexes of \( R \)-modules such that each \( g_n^* \) is a quasi-isomorphism. Suppose that for each \( i \geq 0 \), we can choose a positive integer \( N(i) \) such that the morphisms

\[
\tau^{\geq -i}X_{n+1}^* \rightarrow \tau^{\geq -i}X_n^* \quad \tau^{\geq -i}Y_{n+1}^* \rightarrow \tau^{\geq -i}Y_n^*
\]

are epimorphisms in each degree for each \( n \geq N(i) \). Then, the induced morphism on the inverse limits \( g^* : X^* := \lim_{n \geq 0} X_n^* \rightarrow Y^* := \lim_{n \geq 0} Y_n^* \) is a quasi-isomorphism.

**Proof.** First we fix some \( i \geq 0 \). It is clear from the definition of the truncations in (3.20) that the quasi-isomorphisms \( g_n^* : X_n^* \rightarrow Y_n^* \) induce quasi-isomorphisms \( \tau^{\geq -i}g_n^* : \tau^{\geq -i}X_n^* \rightarrow \tau^{\geq -i}Y_n^* \) of complexes. Further, the transition maps of the inverse systems \( \{\tau^{\geq -i}X_n^*\}_{n \geq N(i)} \) and \( \{\tau^{\geq -i}Y_n^*\}_{n \geq N(i)} \) are all surjections. It now follows from [11, Corollary 3.11] that the induced morphism

\[
\lim_{n \geq 0} \tau^{\geq -i}X_n^* = \lim_{n \geq N(i)} \tau^{\geq -i}X_n^* \rightarrow \lim_{n \geq N(i)} \tau^{\geq -i}Y_n^* = \lim_{n \geq 0} \tau^{\geq -i}Y_n^*
\]

is a quasi-isomorphism. From the definition of truncations in (3.20) it is also clear that:

\[
H^j(\lim_{n \geq 0} \tau^{\geq -i}X_n^*) = H^j(\lim_{n \geq 0} X_n^*) \quad H^j(\lim_{n \geq 0} \tau^{\geq -i}Y_n^*) = H^j(\lim_{n \geq 0} Y_n^*)
\]

for all integers \( j \geq -i + 2 \). Choosing \( i \) to be arbitrarily large, we now see that \( H^j(X^*) \xrightarrow{\sim} H^j(Y^*) \) for all \( j \in \mathbb{Z} \), i.e., \( g^* : X^* \rightarrow Y^* \) is a quasi-isomorphism.
We now make one more observation. Let \( \{ I^{(j)} \}_{j \geq 0} \) be a family of pure injective modules and consider the product \( I := \prod_{j \geq 0} I^{(j)} \). If \( F \) is a finitely presented \( R \)-module and \( A^\bullet \in C(R) \) is pure acyclic, we observe that \( \text{Hom}_R^\bullet(F, \text{Hom}_R^\bullet(A^\bullet, I)) = \text{Hom}_R^\bullet(F \otimes_R A^\bullet, I) = \prod_{j \geq 0} \text{Hom}_R^\bullet(F \otimes_R A^\bullet, I^{(j)}) \) is acyclic. It follows that the product \( I \) is also pure injective. We now prove the main result of this section.

**Proposition 3.11.** Let \( R \) be a commutative ring and let \( M^\bullet \in C(R) \) be a cochain complex of \( R \)-modules. Then, \( M^\bullet \) has a pure injective resolution.

**Proof.** We consider the special inverse system \( \{ I^*_n \}_{n \geq 0} \) of \( K \)-pure injective complexes and the pure quasi-isomorphisms \( \{ f_n : \tau^{\geq -n} M^\bullet \rightarrow I^*_n \}_{n \geq 0} \) given by Proposition 3.9. We choose any pure projective \( R \)-module \( P \). Then, we have quasi-isomorphisms:

\[
\text{Hom}_R^\bullet(P, f_n) : \text{Hom}_R^\bullet(P, \tau^{\geq -n} M^\bullet) \rightarrow \text{Hom}_R^\bullet(P, I^*_n)
\]  

(3.29)

We first consider the inverse system \( \{ \text{Hom}_R^\bullet(P, \tau^{\geq -n} M^\bullet) \}_{n \geq 0} \). Choose any \( i \geq 0 \). Then, it is clear that for \( n \geq i + 2 \), the canonical morphisms

\[
\tau^{\geq -i} \text{Hom}_R^\bullet(P, \tau^{\geq -n} M^\bullet) \rightarrow \tau^{\geq -i} \text{Hom}_R^\bullet(P, \tau^{\geq -n} M^\bullet)
\]

are all identity. Further, from the proof of Proposition 3.9 we know that each \( I^*_{n+1} \) can be expressed as \( I^*_{n+1} = C_{x_n}[-1]^\bullet \), where \( C_{x_n}^\bullet \) is the mapping cone of a morphism \( x_n^\bullet : I^*_n \rightarrow f_n^\bullet \). Since the functor \( \text{Hom}_R^\bullet(P, -) \) commutes with mapping cones (see, for instance, [1, (A.2.1.2)]), we get

\[
\text{Hom}_R^\bullet(P, I^*_{n+1}) = \text{Cone}(\text{Hom}_R^\bullet(P, x_n^\bullet))[-1]^\bullet
\]

(3.31)

Accordingly, the morphisms \( \text{Hom}_R^\bullet(P, I^*_{n+1}) = \text{Cone}(\text{Hom}_R^\bullet(P, x_n^\bullet))[-1]^\bullet \rightarrow \text{Hom}_R^\bullet(P, I^*_n) \) are all surjective for \( n \geq 0 \). It follows easily that for any \( i \geq 0 \), the induced morphism on the truncations:

\[
\tau^{\geq -i} \text{Hom}_R^\bullet(P, I^*_{n+1}) \rightarrow \tau^{\geq -i} \text{Hom}_R^\bullet(P, I^*_n)
\]

(3.32)

is surjective for each \( n \geq 0 \). We now set \( I^\bullet := \varprojlim I^*_n \) and consider the induced morphism \( f^\bullet : M^\bullet = \varprojlim \tau^{\geq -n} M^\bullet \rightarrow I^\bullet \). From (3.30) and (3.32) and applying Lemma 3.10 it follows that we have a quasi-isomorphism on inverse limits:

\[
\text{Hom}_R^\bullet(P, M^\bullet) = \varprojlim_{n \geq 0} \text{Hom}_R^\bullet(P, \tau^{\geq -n} M^\bullet) \\
\text{Hom}_R^\bullet(P, f^\bullet) \\
\varprojlim_{n \geq 0} \text{Hom}_R^\bullet(P, I^*_n) = \text{Hom}_R^\bullet(P, I^\bullet)
\]

(3.33)

for any pure projective \( R \)-module \( P \). From (3.33), we conclude that \( f^\bullet : M^\bullet \rightarrow I^\bullet \) is a pure quasi-isomorphism. Further, we know from Proposition 3.9 that \( \{ I^*_n \}_{n \geq 0} \) is a special inverse system of
injectives. For this, we notice that for any $j \in \mathbb{Z}$, $I^j$ is the limit of the following inverse system:

$$... \rightarrow I^j_{n+1} = I^j_n \oplus J^{j-1}_n \xrightarrow{p_{n+1}} I^j_n = I^j_{n-1} \oplus J^{j-1}_{n-1} \xrightarrow{p_n} ... \rightarrow I^j_0$$

where each $p_n$ is the canonical projection onto the direct summand. Then, $I^j$ can be expressed as the direct product $I^j = I^j_0 \oplus \prod_{n \geq 1} J^{j-1}_{n-1}$ of pure injectives. Hence, $I^j$ is pure injective.

$\square$

4 Pure projective resolutions of unbounded complexes

In this section, we will construct pure projective resolutions for arbitrary complexes of $R$-modules. As in the previous section, our methods are an adaptation of the classical method for constructing projective resolutions of bounded above complexes (see, for example, [12]) along with the techniques of Spaltenstein [19] for treating unbounded complexes. Unfortunately, the proofs in this section are not always the dual of the arguments in Section 3. However, we will try to be as concise as possible by pointing out all those arguments that are dual to the case of pure injective resolutions.

**Definition 4.1.** We will say that a cochain complex $M^\bullet \in C(R)$ is $K$-pure projective if the functor $\text{Hom}_R^\bullet(M^\bullet, \_): C(R) \rightarrow C(R)$ carries pure acyclic complexes to pure acyclic complexes.

We make the following observation: given a pure acyclic complex $A^\bullet \in C(R)$ and any finitely presented $R$-module $F$, we consider the complex $\text{Hom}_R^\bullet(F, A^\bullet)$. Now if $F'$ is any other finitely presented $R$-module, the tensor product $F' \otimes_R F$ is still finitely presented. Hence, $\text{Hom}_R^\bullet(F', \text{Hom}_R^\bullet(F, A^\bullet)) \cong \text{Hom}_R^\bullet(F' \otimes_R F, A^\bullet)$ is acyclic for any finitely presented $R$-module $F'$ and we conclude that $\text{Hom}_R^\bullet(F, A^\bullet)$ is actually pure acyclic.

**Proposition 4.2.** For a complex $M^\bullet \in C(R)$, the following statements are equivalent:

(a) $M^\bullet$ is $K$-pure projective.

(b) For any pure acyclic complex $A^\bullet \in C(R)$, we have $\text{Hom}_{K(R)}(M^\bullet, A^\bullet) = 0$.

(c) For any complex $A^\bullet \in C(R)$, we have $\text{Hom}_{K(R)}(M^\bullet, A^\bullet) \cong \text{Hom}_{D_{\text{pur}}(R)}(M^\bullet, A^\bullet)$.

**Proof.** (a) $\Rightarrow$ (b): This is dual to the corresponding argument in the proof of Proposition 3.2.

(b) $\Rightarrow$ (a): Let $A^\bullet \in C(R)$ be pure acyclic. We have to show that $\text{Hom}_R^\bullet(M^\bullet, A^\bullet)$ is pure acyclic. From the observation above, we know that for any finitely presented $R$-module $F$, $\text{Hom}_R^\bullet(F, A^\bullet)$ (and hence $\text{Hom}_R^\bullet(F, A^\bullet)[k]$ for any $k \in \mathbb{Z}$) is pure acyclic. Then, we get:

$$H^k(\text{Hom}_R^\bullet(F, \text{Hom}_R^\bullet(M^\bullet, A^\bullet))) = H^k(\text{Hom}_R^\bullet(M^\bullet, \text{Hom}_R^\bullet(F, A^\bullet))) = \text{Hom}_{K(R)}(M^\bullet, \text{Hom}_R^\bullet(F, A^\bullet)[k]) = 0$$

(4.1)
for any \( k \in \mathbb{Z} \). Hence, \( \text{Hom}_R^\bullet(M^\bullet, A^\bullet) \) is pure acyclic.

(c) \( \Rightarrow \) (b): This is clear because any pure acyclic \( A^\bullet \) is 0 in \( D_{pur}(R) \).

(a) \( \Rightarrow \) (c): The proof is dual to the corresponding argument in the proof of Proposition 3.3 if we consider the morphisms in the derived category \( D_{pur}(R) \) as “left roofs” in place of the “right roofs” appearing in (3.6).

\[
\]

Proposition 4.3. Let \( P^\bullet \in C(R) \) be a bounded above complex of pure projective modules. Then, \( P^\bullet \) is \( K \)-pure projective.

Proof. Following Proposition 4.2, it suffices to show that \( \text{Hom}_{K(R)}(P^\bullet, A^\bullet) = 0 \) for any pure acyclic \( A^\bullet \in C(R) \). The rest of the argument is now dual to that in the proof of Proposition 3.4.

It is known (see [21, Proposition 1]) that for any \( R \)-module \( M \), there exists a pure epimorphism \( P \twoheadrightarrow M \) with \( P \) a pure projective module. We are now ready to show that any bounded above complex of \( R \)-modules admits a pure projective resolution. Now, a key step in the proof of Proposition 3.5, which gives the corresponding result for pure injective resolutions, is the fact that the functor \( \_ \otimes_R N \) preserves cokernels for any \( N \in R-\text{Mod} \). Since this no longer holds for kernels, we must modify our approach somewhat to obtain pure projective resolutions, i.e., we cannot simply dualize the proof of Proposition 3.5 here.

Proposition 4.4. (a) Let \( M^\bullet \in C(R) \) be a cochain complex that is bounded above. Then, there exists a pure quasi-isomorphism \( v^\bullet : P^\bullet \rightarrow M^\bullet \) such that \( P^\bullet \) is a bounded above complex of pure projective modules.

(b) Let \( f^\bullet : M^\bullet_2 \rightarrow M^\bullet_1 \) be a morphism of bounded above complexes in \( C(R) \). Let \( v^\bullet_2 : P^\bullet_2 \rightarrow M^\bullet_2 \) be a pure quasi-isomorphism to \( M^\bullet_2 \) from a bounded above complex \( P^\bullet_2 \) of pure projectives. Then, there exists a bounded above complex \( P^\bullet_1 \) of pure projectives, a pure quasi-isomorphism \( v^\bullet_1 : P^\bullet_1 \rightarrow M^\bullet_1 \) and a morphism \( g : P^\bullet_2 \rightarrow P^\bullet_1 \) fitting into a commutative diagram:

\[
\begin{array}{ccc}
P^\bullet_2 & \xrightarrow{v^\bullet_2} & M^\bullet_2 \\
\downarrow{g^\bullet} & & \downarrow{f^\bullet} \\
P^\bullet_1 & \xrightarrow{v^\bullet_1} & M^\bullet_1 
\end{array}
\] (4.2)

Proof. (a) For the sake of definiteness, we suppose that the complex \( (M^\bullet, d^\bullet) \) satisfies \( M^0 \neq 0 \) and \( M^i = 0 \) for each \( i > 0 \). We set \( P^i = 0 \) for each \( i > 0 \) and choose a pure epimorphism \( v^0 : P^0 \rightarrow M^0 \) with \( P^0 \) pure projective. Then, for every \( i \geq 0 \), we have already obtained pure projectives \( P^i \), morphisms \( v^i : P^i \rightarrow M^i \) along with differentials \( e^i : P^i \rightarrow P^{i+1} \) such that we have induced isomorphisms:

\[
\text{Ker}(\text{Hom}_R(Q, e^{i+1})) / \text{Im}(\text{Hom}_R(Q, e^i)) \xrightarrow{\sim} H^{i+1}(\text{Hom}_R^\bullet(Q, M^\bullet)) \quad \forall Q \in \mathcal{P}
\] (4.3)
and epimorphisms:

\[ \text{Ker}(\text{Hom}_R(Q, e^i)) \to \text{Ker}(\text{Hom}_R(Q, d^i)) \quad \forall Q \in \mathcal{PP} \quad (4.4) \]

We suppose that we have already done this for all integers \( i \) greater than some given integer \( k \geq -1 \). We will show that there exists a pure projective \( P^k \), a morphism \( v^k : P^k \to M^k \) and a differential \( e^k : P^k \to P^{k+1} \) such that:

\[ \text{Ker}(\text{Hom}_R(Q, e^{k+1})) / \text{Im}(\text{Hom}_R(Q, e^k)) \sim \text{H}^{k+1}(\text{Hom}_R(Q, M^*)) \quad \forall Q \in \mathcal{PP} \quad (4.5) \]

We now consider the object \( L^k \) defined by the following pullback square:

\[
\begin{array}{ccc}
L^k & \longrightarrow & \text{Ker}(e^{k+1}) \\
\downarrow & & \downarrow \\
M^k & \underset{d^k}{\longrightarrow} & M^{k+1}
\end{array}
\quad (4.6)
\]

and choose a pure epimorphism \( P^k \to L^k \) with \( P^k \) pure projective. Then, for any \( Q \in \mathcal{PP} \), the induced morphism \( \text{Hom}_R(Q, P^k) \to \text{Hom}_R(Q, L^k) \) is still an epimorphism and we obtain a pullback square

\[
\begin{array}{ccc}
\text{Hom}_R(Q, L^k) & \longrightarrow & \text{Hom}_R(Q, \text{Ker}(e^{k+1})) = \text{Ker}(\text{Hom}_R(Q, e^{k+1})) \\
\downarrow & & \downarrow \\
\text{Hom}_R(Q, M^k) & \underset{\text{Hom}_R(Q, d^k)}{\longrightarrow} & \text{Hom}_R(Q, M^{k+1})
\end{array}
\quad (4.7)
\]

We set \( e^k : P^k \to P^{k+1} \) to be the composition \( P^k \to L^k \to \text{Ker}(e^{k+1}) \to P^{k+1} \) and \( v^k : P^k \to M^k \) to be the composition \( P^k \to L^k \to M^k \). Now applying to the pullback square (4.7) arguments that are dual to those applied to the pushout square (4.11) in the proof of Proposition 3.5, we obtain the required results in (4.5). The induced morphisms \( \text{Hom}_R(Q, v^*) : \text{Hom}_R(Q, P^*) \to \text{Hom}_R(Q, M^*) \) being quasi-isomorphisms for each pure projective \( Q \in \mathcal{PP} \), it follows that \( v^* : P^* \to M^* \) is a pure quasi-isomorphism.

(b) As in part (a), we use the functors \( \text{Hom}_R(Q, -) \) with \( Q \in \mathcal{PP} \) in place of the functors \( - \otimes_R N \) with \( N \in R - \text{Mod} \) appearing in the proof of Proposition 3.6. Since \( \text{Hom}_R(Q, -) \) also preserves mapping cones, we can now apply arguments dual to those in the proof of Proposition 3.6 to prove this result.

In order to proceed to pure projective resolutions of unbounded complexes, we will need to consider “special direct systems”.

**Definition 4.5.** (see [17, Definition 2.6]) Let \( \mathcal{T} \) be a class of complexes in \( C(R) \) that is closed under isomorphisms.
(a) A $\mathcal{T}$-special direct system of complexes is a direct system $\{P^\bullet_n\}_{n \in \mathbb{Z}}$ of complexes in $C(R)$ satisfying the following conditions for each $n \in \mathbb{Z}$:

1. The cochain map $P^\bullet_{n-1} \to P^\bullet_n$ is injective.
2. The cokernel $C^\bullet_n := \text{Coker}(P^\bullet_{n-1} \to P^\bullet_n)$ lies in the class $\mathcal{T}$.
3. The short exact sequence of complexes:

$$0 \to P^\bullet_{n-1} \xrightarrow{\iota_n} P^\bullet_n \xrightarrow{p^\bullet_n} C^\bullet_n \to 0$$

is “semi-split”, i.e., it is split in each degree.

(b) The class $\mathcal{T} \subseteq C(R)$ is said to be closed under special direct limits if the direct limit of every $\mathcal{T}$-special direct system in $C(R)$ is contained in $\mathcal{T}$.

By slight abuse of notation, we will refer to $\{P^\bullet_n\}_{n \geq 0}$ as a special direct system if setting $P^\bullet_n = 0$ for all $n < 0$ makes $\{P^\bullet_n\}_{n \in \mathbb{Z}}$ into a special direct system in the sense of Definition 4.5 above.

Proposition 4.6. (a) Let $\mathcal{C} \subseteq C(R)$ be a class of complexes. Let $\mathcal{T}(\mathcal{C})$ denote the class of complexes $M^\bullet \in C(R)$ such that $\text{Hom}^\bullet_R(M^\bullet, A^\bullet)$ is pure acyclic for each $A^\bullet \in \mathcal{C}$. Then, $\mathcal{T}(\mathcal{C})$ is closed under special direct limits.

(b) The class of all $K$-pure projective complexes is closed under special direct limits.

Proof. (a) We set:

$$\mathcal{T}' := \{B^\bullet \in C(R) \mid B^\bullet = \text{Hom}^\bullet_R(F, A^\bullet) \text{ for some finitely presented } R\text{-module } F \& \text{some } A^\bullet \in \mathcal{C}\}$$

Now since $\text{Hom}^\bullet_R(F, \text{Hom}^\bullet_R(M^\bullet, A^\bullet)) \cong \text{Hom}^\bullet_R(M^\bullet, \text{Hom}^\bullet_R(F, A^\bullet))$, it follows that a complex $M^\bullet \in \mathcal{T}(\mathcal{C})$ if and only if $\text{Hom}^\bullet_R(M^\bullet, B^\bullet)$ is acyclic for each $B^\bullet \in \mathcal{T}'$. From [19 Corollary 2.8], we now see that $\mathcal{T}(\mathcal{C})$ is closed under special direct limits. Finally, the result of (b) follows from (a) by letting $\mathcal{C}$ be the class of all pure acyclic complexes in $C(R)$.

For a complex $(M^\bullet, d^\bullet) \in C(R)$ and any given integer $n$, we now recall that its truncation $\tau_{\leq n} M^\bullet$ is given by:

$$\tau_{\leq n} M^\bullet = \begin{cases} 0 & \text{if } i > n \\ \text{Ker}(d^n) & \text{if } i = n \\ M^i & \text{if } i < n \end{cases}$$

It is clear that the complex $M^\bullet$ may be expressed as the direct limit $M^\bullet = \lim_{\longrightarrow} \tau_{\leq n} M^\bullet$. 

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Proposition 4.7. For any complex $M^\bullet \in C(R)$ there exists a special direct system $\{P^\bullet_n\}_{n \geq 0}$ of $K$-pure projective complexes and a morphism $\{f_n : P^\bullet_n \to \tau \leq n M^\bullet\}_{n \geq 0}$ of direct systems satisfying the following conditions:

(a) Each $P^\bullet_n$ is a bounded above complex of pure projectives.
(b) Each $f_n$ is a pure quasi-isomorphism.

Proof. The proof of this is dual to that of Proposition 3.9.

Lemma 4.8. Let $\{P^{(j)}\}_{j \geq 0}$ be a family of pure projective modules. Then, the direct sum $\bigoplus_{j \geq 0} P^{(j)}$ is pure projective.

Proof. We set $P := \bigoplus_{j \geq 0} P^{(j)}$ and consider a pure acyclic complex $A^\bullet$. We have to check that $\text{Hom}_R^\bullet(P, A^\bullet)$ is pure acyclic. For this, we choose a finitely presented $R$-module $F$ and see that:

$$\text{Hom}_R^\bullet(F, \text{Hom}_R^\bullet(P, A^\bullet)) = \text{Hom}_R^\bullet(F, \prod_{j \geq 0} \text{Hom}_R^\bullet(P^{(j)}, A^\bullet)) = \prod_{j \geq 0} \text{Hom}_R^\bullet(F, \text{Hom}_R^\bullet(P^{(j)}, A^\bullet))$$

Since each $P^{(j)}$ is pure projective, each of the complexes $\text{Hom}_R^\bullet(P^{(j)}, A^\bullet)$ is pure acyclic and hence each $\text{Hom}_R^\bullet(F, \text{Hom}_R^\bullet(P^{(j)}, A^\bullet))$ is acyclic. Since the product of acyclic complexes in $R-\text{Mod}$ must be acyclic, we conclude that $\text{Hom}_R^\bullet(F, \text{Hom}_R^\bullet(P, A^\bullet))$ is acyclic for any finitely presented $R$-module $F$. This proves the result.

Proposition 4.9. Let $R$ be a commutative ring and let $M^\bullet \in C(R)$ be a cochain complex of $R$-modules. Then, $M^\bullet$ has a pure projective resolution.

Proof. We consider the special direct system $\{P^\bullet_n\}_{n \geq 0}$ of $K$-pure projective complexes along with the pure quasi-isomorphisms $\{f_n : P^\bullet_n \to \tau \leq n M^\bullet\}_{n \geq 0}$ from the proof of Proposition 4.7. We set $P^\bullet := \lim_{n \geq 0} P^\bullet_n$. It is clear that pure quasi-isomorphisms commute with direct limits. This gives us an induced pure quasi-isomorphism:

$$f^\bullet : P^\bullet = \lim_{n \geq 0} P^\bullet_n \to \lim_{n \geq 0} \tau \leq n M^\bullet = M^\bullet \quad (4.10)$$

Using Proposition 4.6(b), we see that the direct limit $P^\bullet$ is $K$-pure projective. It remains to show that $P^\bullet$ is a complex of pure projective modules. From the construction of each $P^\bullet_n$ in Proposition 4.7 which is dual to the construction in Proposition 3.9 it follows that each term in the direct limit $P^\bullet$ is actually a direct sum of a family of pure projective modules. It now follows from Lemma 4.8 that each term in $P^\bullet$ is pure projective.
5 Closed monoidal structure on the pure derived category

In this section, we will use the pure projective and pure injective resolutions developed so far to give a closed monoidal structure on the pure derived category $D_{pur}(R)$. Now, if $u^\bullet : M^\bullet \rightarrow N^\bullet$ is a pure quasi-isomorphism and $A$ is any $R$-module, it is immediate from the definitions that the induced morphism $A \otimes_R u^\bullet : A \otimes_R M^\bullet \rightarrow A \otimes_R N^\bullet$ is a pure quasi-isomorphism. In order to produce a tensor structure on $D_{pur}(R)$, we will need to extend this fact to tensor products of (possibly unbounded) complexes of $R$-modules.

Given cochain complexes $(M^\bullet, d^\bullet_M)$, $(A^\bullet, d^\bullet_A) \in C(R)$, we recall that their tensor product $(M^\bullet \otimes_R A^\bullet)^\bullet$ is the total complex associated to the double complex $(M^\bullet \otimes_R A^\bullet)^{ij} := M^i \otimes_R A^j$ with differentials $d^{ij} := d^i_M \otimes_R d^j_A + (M^\bullet \otimes_R A^\bullet)^{ij} \rightarrow (M^\bullet \otimes_R A^\bullet)^{i+1,j}$ and $d^{ij} := M^i \otimes_R d^j_A : (M^\bullet \otimes_R A^\bullet)^{ij} \rightarrow (M^\bullet \otimes_R A^\bullet)^{ij+1}$. 

**Lemma 5.1.** Let $u^\bullet : M^\bullet \rightarrow N^\bullet$ be a pure quasi-isomorphism. Then, for any given integer $n \in \mathbb{Z}$, the induced morphism on the truncations $\tau_{\leq n}u^\bullet : \tau_{\leq n}M^\bullet \rightarrow \tau_{\leq n}N^\bullet$ is a pure quasi-isomorphism.

**Proof.** We will show that $\text{Hom}^\bullet_R(P, \tau_{\leq n}u^\bullet) : \text{Hom}^\bullet_R(P, \tau_{\leq n}M^\bullet) \rightarrow \text{Hom}^\bullet_R(P, \tau_{\leq n}N^\bullet)$ is a quasi-isomorphism for any pure projective $P \in \mathcal{P}$. Since $u^\bullet$ is a pure quasi-isomorphism, we already know that $\text{Hom}^\bullet_R(P, u^\bullet) : \text{Hom}^\bullet_R(P, M^\bullet) \rightarrow \text{Hom}^\bullet_R(P, N^\bullet)$ is a quasi-isomorphism. This induces a quasi-isomorphism on the truncations:

$$\tau_{\leq n}\text{Hom}^\bullet_R(P, u^\bullet) : \tau_{\leq n}\text{Hom}^\bullet_R(P, M^\bullet) \rightarrow \tau_{\leq n}\text{Hom}^\bullet_R(P, N^\bullet) \quad \text{(5.1)}$$

Further, since the functor $\text{Hom}^\bullet_R(P, -)$ preserves kernels, it is clear from the definition in (4.9) that the truncations satisfy:

$$\text{Hom}^\bullet_R(P, \tau_{\leq n}M^\bullet) = \tau_{\leq n}\text{Hom}^\bullet_R(P, M^\bullet) \quad \text{Hom}^\bullet_R(P, \tau_{\leq n}N^\bullet) = \tau_{\leq n}\text{Hom}^\bullet_R(P, N^\bullet) \quad \text{(5.2)}$$

for any integer $n \in \mathbb{Z}$. Combining (5.1) and (5.2), the result follows. $\square$

**Proposition 5.2.** Let $u^\bullet : M^\bullet \rightarrow N^\bullet$ be a pure quasi-isomorphism of complexes. Then, for any cochain complex $A^\bullet \in C(R)$, the induced morphism $A^\bullet \otimes_R u^\bullet : (A^\bullet \otimes_R M^\bullet)^\bullet \rightarrow (A^\bullet \otimes_R N^\bullet)^\bullet$ is a pure quasi-isomorphism.

**Proof.** We choose any $R$-module $B$ and set $C^\bullet := B \otimes_R A^\bullet$. Further, for any integer $m \in \mathbb{Z}$, we set $C^\bullet_m := \tau_{\leq m}C^\bullet$. Now for any $m, n \in \mathbb{Z}$, we claim that the induced morphism:

$$(C^\bullet_m \otimes_R \tau_{\leq n}M^\bullet)^\bullet \rightarrow (C^\bullet_m \otimes_R \tau_{\leq n}N^\bullet)^\bullet \quad \text{(5.3)}$$

is a quasi-isomorphism. From Lemma 5.1 we know that $\tau_{\leq n}u^\bullet : \tau_{\leq n}M^\bullet \rightarrow \tau_{\leq n}N^\bullet$ is a pure quasi-isomorphism. It follows that for any fixed $i \in \mathbb{Z}$, the morphism

$$C^i_m \otimes_R \tau_{\leq n}u^\bullet : C^i_m \otimes_R (\tau_{\leq n}M^\bullet) \rightarrow C^i_m \otimes_R (\tau_{\leq n}N^\bullet) \quad \text{(5.4)}$$

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is a quasi-isomorphism. Since \( C^\bullet_{m}, \tau_{\leq n}M^\bullet \) and \( \tau_{\leq n}N^\bullet \) are all bounded above complexes, it now follows from a standard spectral sequence argument (see, for instance, [20, Tag 0132]) that the quasi-isomorphisms in (5.4) induce a quasi-isomorphism of the total complexes in (5.3). Taking direct limits of quasi-isomorphisms over all \( m, n \in \mathbb{Z} \), we obtain a quasi-isomorphism:

\[
B \otimes_R (A^\bullet \otimes_R M^\bullet)^* = (C^\bullet \otimes_R M^\bullet)^* \rightarrow (C^\bullet \otimes_R N^\bullet)^* = B \otimes_R (A^\bullet \otimes_R N^\bullet)^* \tag{5.5}
\]

for any \( R \)-module \( B \). This proves the result.

Proof. (a) Let \( P^\bullet \) be a \( K \)-pure projective complex. Then, \( \text{Hom}_R^*(P^\bullet, u^\bullet) : \text{Hom}_R^*(P^\bullet, M^\bullet) \rightarrow \text{Hom}_R^*(P^\bullet, N^\bullet) \) is a pure quasi-isomorphism.

(b) Let \( I^\bullet \) be a \( K \)-pure injective complex. Then, \( \text{Hom}_R^*(u^\bullet, I^\bullet) : \text{Hom}_R^*(N^\bullet, I^\bullet) \rightarrow \text{Hom}_R^*(M^\bullet, I^\bullet) \) is a pure quasi-isomorphism.

Proof. (a) Let \( C^\bullet_u \) denote the pure acyclic complex that is the mapping cone of the pure quasi-isomorphism \( u^\bullet : M^\bullet \rightarrow N^\bullet \). For any complex \( B^\bullet \in C(R) \), we know that the functor \( \text{Hom}_R^*(B^\bullet, \_\_) \) on \( C(R) \) commutes with mapping cones (see, for instance, [4, (A.2.1.2)]). In particular, we see that:

\[
\text{Cone}(\text{Hom}_R^*(P^\bullet, u^\bullet)) = \text{Hom}_R^*(P^\bullet, C^\bullet_u) \tag{5.7}
\]

Since \( P^\bullet \) is a \( K \)-pure projective complex, it follows from the definition that \( \text{Hom}_R^*(P^\bullet, C^\bullet_u) \) is pure acyclic. Combining this with (5.7), we conclude that \( \text{Cone}(\text{Hom}_R^*(P^\bullet, u^\bullet)) \) is pure acyclic and hence \( \text{Hom}_R^*(P^\bullet, u^\bullet) \) is a pure quasi-isomorphism.

(b) For any complex \( B^\bullet \in C(R) \), the contravariant functor \( \text{Hom}_R^*(\_, B^\bullet) \) on \( C(R) \) preserves mapping cones up to a shift (see, for instance, [10 (1.5.3)]). Since the shift of a pure acyclic complex is still pure acyclic, the result of part (b) now follows by applying an argument dual to that in part (a).

Let \( M^\bullet, N^\bullet \in C(R) \) be two arbitrary complexes of \( R \)-modules. From the results of Section 4, we may choose a pure quasi-isomorphism \( P^\bullet_M : M^\bullet \rightarrow M^\bullet \) giving a pure projective resolution of \( M^\bullet \). Similarly, using the results of Section 3, we may choose a pure quasi-isomorphism \( N^\bullet \rightarrow I^\bullet_N \) giving a pure injective resolution of \( N^\bullet \). We can now define a “pure derived Hom functor”:

\[
P\text{Hom}_R^*(\_, \_\_ : D_{par}(R)^{\text{op}} \times D_{par}(R) \rightarrow D_{par}(R) \quad P\text{Hom}_R^*(M^\bullet, N^\bullet) := \text{Hom}_R^*(P^\bullet_M, I^\bullet_N) \tag{5.8}
\]
The functor in (5.8) is well defined due to the pure quasi-isomorphisms

\[
\text{Hom}_R^\bullet(P_M, N^\bullet) \longrightarrow \text{Hom}_R^\bullet(P_M, I_N^\bullet) \leftarrow \text{Hom}_R^\bullet(M^\bullet, I_N^\bullet)
\] (5.9)

that both follow from Proposition 5.3.

We conclude by showing that we have obtained a closed symmetric monoidal structure on the pure derived category \(D_{\text{pur}}(R)\).

**Proposition 5.4.** For any \(A^\bullet, B^\bullet, C^\bullet \in C(R)\), we have natural isomorphisms:

\[
\text{PHom}_R^\bullet((A^\bullet \otimes_R B^\bullet)^\bullet, C^\bullet) \cong \text{PHom}_R^\bullet(A^\bullet, \text{PHom}_R^\bullet(B^\bullet, C^\bullet))
\]

\[
\text{Hom}_{D_{\text{pur}}(R)}((A^\bullet \otimes_R B^\bullet)^\bullet, C^\bullet) \cong \text{Hom}_{D_{\text{pur}}(R)}(A^\bullet, \text{PHom}_R^\bullet(B^\bullet, C^\bullet))
\] (5.10)

**Proof.** We choose pure projective resolutions \(P_A^\bullet\) and \(P_B^\bullet\) of \(A^\bullet\) and \(B^\bullet\) respectively. Let \(I_C^\bullet\) be a pure injective resolution of \(C^\bullet\). From Proposition 5.2, it follows that \((P_A^\bullet \otimes_R P_B^\bullet)^\bullet\) is pure quasi-isomorphic to \((A^\bullet \otimes_R B^\bullet)^\bullet\). From the definitions, it is also clear that \((P_A^\bullet \otimes_R P_B^\bullet)^\bullet\) is a \(K\)-pure projective complex each of the terms of which is pure projective. Hence, \((P_A^\bullet \otimes_R P_B^\bullet)^\bullet\) is a pure projective resolution of \((A^\bullet \otimes_R B^\bullet)^\bullet\). It now follows that:

\[
\text{PHom}_R^\bullet((A^\bullet \otimes_R B^\bullet)^\bullet, C^\bullet) = \text{Hom}_R^\bullet((P_A^\bullet \otimes_R P_B^\bullet)^\bullet, I_C^\bullet)
\]

\[
\cong \text{Hom}_R^\bullet(P_A^\bullet, \text{PHom}_R^\bullet(P_B^\bullet, I_C^\bullet))
\]

\[
= \text{Hom}_R^\bullet(P_A^\bullet, \text{PHom}_R^\bullet(B^\bullet, C^\bullet))
\]

\[
= \text{PHom}_R^\bullet(A^\bullet, \text{PHom}_R^\bullet(B^\bullet, C^\bullet))
\] (5.11)

To prove the second isomorphism in (5.10), we proceed as follows: it is already clear that:

\[
\text{Hom}_{D_{\text{pur}}(R)}((A^\bullet \otimes_R B^\bullet)^\bullet, C^\bullet) \cong \text{Hom}_{D_{\text{pur}}(R)}((P_A^\bullet \otimes_R P_B^\bullet)^\bullet, I_C^\bullet)
\] (5.12)

Since \(I_C^\bullet\) is \(K\)-pure injective, it follows from Proposition 3.3 that:

\[
\text{Hom}_{D_{\text{pur}}(R)}((P_A^\bullet \otimes_R P_B^\bullet)^\bullet, I_C^\bullet) \cong \text{Hom}_{K(R)}((P_A^\bullet \otimes_R P_B^\bullet)^\bullet, I_C^\bullet)
\] (5.13)

The closed monoidal structure on \(K(R)\) now gives:

\[
\text{Hom}_{K(R)}((P_A^\bullet \otimes_R P_B^\bullet)^\bullet, I_C^\bullet) \cong \text{Hom}_{K(R)}(P_A^\bullet, \text{Hom}_R^\bullet(P_B^\bullet, I_C^\bullet))
\] (5.14)

We can put \(\text{PHom}_R^\bullet(B^\bullet, C^\bullet) = \text{Hom}_R^\bullet(P_B^\bullet, I_C^\bullet)\). Further, since \(P_A^\bullet\) is \(K\)-pure projective, it follows from Proposition 4.2 that:

\[
\text{Hom}_{K(R)}(P_A^\bullet, \text{Hom}_R^\bullet(P_B^\bullet, I_C^\bullet)) \cong \text{Hom}_{D_{\text{pur}}(R)}(P_A^\bullet, \text{PHom}_R^\bullet(B^\bullet, C^\bullet))
\] (5.15)

Since \(P_A^\bullet \cong A^\bullet\) in \(D_{\text{pur}}(R)\), the sequence of isomorphisms 5.12)-(5.15) proves the second isomorphism in (5.10).
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