Robust Topological Inference in the Presence of Outliers

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Abstract

The distance function to a compact set plays a crucial role in the paradigm of topological data analysis. In particular, the sublevel sets of the distance function are used in the computation of persistent homology—a backbone of the topological data analysis pipeline. Despite its stability to perturbations in the Hausdorff distance, persistent homology is highly sensitive to outliers. In this work, we develop a framework of statistical inference for persistent homology in the presence of outliers. Drawing inspiration from recent developments in robust statistics, we propose a median-of-means variant of the distance function $\text{MoM}_\text{Dist}$, and establish its statistical properties. In particular, we show that, even in the presence of outliers, the sublevel filtrations and weighted filtrations induced by $\text{MoM}_\text{Dist}$ are both consistent estimators of the true underlying population counterpart, and their rates of convergence in the bottleneck metric are controlled by the fraction of outliers in the data. Finally, we demonstrate the advantages of the proposed methodology through simulations and applications.

1 Introduction

Given a compact set $X \subset \mathbb{R}^d$, its persistence diagram encodes the subtle geometric and topological features which underlie $X$ as a multiscale summary, and forms the cornerstone of topological data analysis. Persistent homology serves as the backbone for computing persistence diagrams, and encodes the homological features underlying $X$ at different resolutions. The computation of persistent homology is typically achieved by constructing a filtration $V_X$, i.e., a nested sequence of topological spaces, which captures the evolution of geometric and topological features as the resolution varies. The persistent homology, which is encoded in its persistence module, $\mathbb{V}_X$, extracts the homological information from the filtration $V_X$. This is then summarized in a persistence diagram $\text{Dgm}(\mathbb{V}_X)$.

Broadly speaking, there are two different methods for obtaining filtrations. The first, and, arguably more classical method is obtained by examining the union of balls of radius $r$ centered on the points of $X$ called the $r$-offset of $X$, denoted $X(r)$, for each resolution $r > 0$. The resulting filtration $V[X] = \{X(r) : r > 0\}$, depends only on the metric properties of $X$. The second, and more general approach is based on constructing a filter function $f_X$, which reflects the topological features underlying $X$. The resulting filtration $V[f_X]$, in this case, is obtained by probing the sublevel sets $f_X^{-1}((-\infty, r])$ or the superlevel sets $f_X^{-1}([r, \infty))$ associated with $f_X$. While these two methods are vastly different, in principle, they both attempt to explore the topological features underlying $X$.

In this context, the distance function $d_X$ to the set $X$ plays a special role in topological data analysis, and satisfies the property that $V[X] = V[d_X]$. That is, the sublevel sets of the distance function encode the same topological information as the filtration from its offsets. The appeal of using the distance function in the
computation of persistence diagrams comes from the celebrated stability of persistence diagrams (Chazal et al., 2016). In a nutshell, the stability result for persistence diagrams guarantees that (i) the persistence diagrams resulting from two compact sets \( X \) and \( Y \) are close whenever the sets themselves are close in the Hausdorff distance, and, (ii) the functional persistence diagrams resulting from two filter functions \( f \) and \( g \) are close whenever \( f \) and \( g \) are close w.r.t. the \( \| \cdot \|_\infty \) metric.

In the statistical setting, one has access to \( X \) only through samples \( X_n = \{ X_1, X_2, \ldots, X_n \} \) obtained using a probability distribution \( \mathbb{P} \) which is supported on the (unknown) set \( X \). The objective, in a statistical inference framework, is to use the samples \( X_n \) to infer the true population persistence diagram \( \mathcal{D}_{\text{gm}}(\mathbb{V}[X]) \). The offset \( X_n(r) \) and filter function \( f_n \), constructed using the sample points, are themselves random quantities associated with their population counterparts \( X(r) \) and \( f_X \), respectively, and these may be used to construct a sample estimator \( \mathcal{D}_{\text{gm}}(\mathbb{V}[X_n]) \). To this end, several existing works have studied the statistical properties of \( \mathcal{D}_{\text{gm}}(\mathbb{V}[X_n]) \), e.g., constructing confidence bands and characterizing the convergence rate of \( \mathcal{D}_{\text{gm}}(\mathbb{V}[X_n]) \) to \( \mathcal{D}_{\text{gm}}(\mathbb{V}[X]) \) in the space of persistence diagrams (Fasy et al., 2014; Chazal et al., 2015a,b, 2017; Vishwanath et al., 2020).

### 1.1 Contributions

In practical settings, real-world data is likely subject to measurement errors and the presence of outliers. While some assumptions may be imposed on the noise and the outliers, in the most baneful settings, the given data may be subject to adversarial contamination. In this setting, for \( m < n/2 \), we assume that the samples \( X_n \), which we have access to, contain only \( n - m \) points obtained from the probability distribution \( \mathbb{P} \) with \( \text{supp}(\mathbb{P}) = X \), and make no further assumptions on the remaining \( m \) points. In principle, the \( m \) outliers may be carefully chosen by an adversary after examining the remaining \( n - m \) points. The overarching objective of this paper is to construct an estimator of the (unknown) population quantity \( \mathcal{D}_{\text{gm}}(\mathbb{V}[X]) \) using the corrupted sample points \( X_n \) which is, both, statistically consistent and computationally efficient.

While the stability of persistence diagrams guarantees that small perturbations in the sample points induce only small changes in the resulting persistence diagrams, even a few outliers in the samples can lead to deleterious effects. This issue is further exacerbated in the adversarial setting, where the adversary is free to place the \( m \) points where it may drastically impact the resulting topological inference.

In this paper, we introduce MoM Dist, denoted by \( d_{n,Q} \), as an outlier-robust variant of the empirical distance function which is constructed using the median-of-means principle, and we establish its theoretical properties. Notably the MoM Dist relies on a tuning parameter \( Q \) which is easy to interpret. While the persistence diagram resulting from the sublevel filtration of \( d_{n,Q} \) is a valid candidate for statistical inference, it can be expensive to compute in practice. To overcome this, we use the weighted filtrations introduced by Buchet et al. (2016) and Anai et al. (2019) to construct \( d_{n,Q} \)-weighted filtrations, \( V[X_n, d_{n,Q}] \), as computationally efficient estimators of \( \mathcal{D}_{\text{gm}}(\mathbb{V}[X]) \). Our main contributions are the following:

(I) We show that sublevel set persistence diagrams of \( d_{n,Q} \) are consistent estimators of the sublevel set persistence diagram of the true population counterpart \( d_X \) even in the presence of outliers (Theorem 3.1).

(II) We establish a stability result for the \( d_{n,Q} \)-weighted filtrations, \( V[X_n, d_{n,Q}] \), and we show that they are stable w.r.t. adversarial contamination (Theorem 3.2).

(III) Furthermore, we show that the persistence diagram \( \mathcal{D}_{\text{gm}}(\mathbb{V}[X_n, d_{n,Q}]) \) is both a computationally efficient and statistically consistent estimator of \( \mathcal{D}_{\text{gm}}(\mathbb{V}[X]) \), and we establish its convergence rate (Theorem 3.3).

(IV) Next, in a sensitivity analysis framework, we quantify the gain in robustness achieved when using the \( d_{n,Q} \)-weighted filtrations vis-à-vis its non-robust \( d_n \)-weighted counterpart (Theorem 3.4).
(V) Lastly, we propose a data-driven procedure for adaptively selecting the tuning parameter \( Q \) using Lepski’s method. For the data-driven choice \( \hat{Q} \), we show that the resulting estimator \( \mathbb{D}gm\{ \mathbb{V}[X_n, d_{n, \hat{Q}}] \} \) is statistically consistent and establish its convergence rate (Theorem 3.5).

1.2 Related Work

Several approaches have been proposed in existing literature to overcome the sensitivity of persistence diagrams to noise. The prevailing ideas in these approaches rely on constructing a filter function, \( f_P \), which reflects both the topological information and the distribution of mass underlying the support \( \text{supp}(\mathbb{P}) = X \). Replacing the population probability measure \( \mathbb{P} \) with the empirical measure \( \mathbb{P}_n \) associated with the samples \( X_n \) results in an empirical estimator \( f_{P_n} \). Some notable examples include the distance-to-measure (Chazal et al., 2011), the kernel distance (Phillips et al., 2015), and kernel density estimators (Fasy et al., 2014).

While these approaches mitigate, to some extent, the influence of noise on the resulting persistence diagrams, they are not without their drawbacks. For starters, while it may be argued that \( \mathbb{D}gm(\mathbb{V}[f_{P_n}]) \) is more resilient to noise, ultimately, this sample estimator corresponds to the population quantity \( \mathbb{D}gm(\mathbb{V}[f_P]) \), which may, nevertheless, omit some subtle geometric and topological features present in \( \mathbb{D}gm(\mathbb{V}[X]) \). Furthermore, from a statistical perspective, if \( X_n \) comprises of only \( n - m \) points from \( \mathbb{P} \) and the remaining \( m \) points constitute outliers, then the sample estimator \( \mathbb{V}[f_{P_n}] \), obtained using \( X_n \), will no longer be a valid estimator of the population quantity \( \mathbb{D}gm(\mathbb{V}[f_P]) \) which we wish to infer.

Lastly, the exact computation of these estimators can be prohibitively expensive, if not impossible in practice. For instance, the exact computation of the distance-to-measure requires computing an order-\( k \) Voronoi diagram. Moreover, in the general setting, the sublevel/superlevel filtrations arising from these approaches are computed using cubical homology, which relies on a (nuisance) grid resolution parameter. If this resolution is too coarse, then some subtle topological features are affected. On the flipside, if the resolution is too fine, then the accuracy is still impacted, as noted in Fasy et al. (2014). In the high-dimensional setting, cubical homology also falls victim to the curse of dimensionality, i.e., for a fixed grid resolution, the number of simplices in the resulting cubical complex grows exponentially with the dimension of the ambient space.

In order to overcome these computational drawbacks, Buchet et al. (2016) and Anai et al. (2019) propose weighted filtrations, \( \mathbb{V}[X_n, f_{X_n}] \), using power distances. While the weighted filtrations circumvent the need for constructing grid-based approximations, they come at the expense of exact inference, i.e., the weighted filtrations \( \mathbb{V}[X_n, f_{X_n}] \) only approximate \( \mathbb{V}[f_{X_n}] \) and do not provide valid statistical inference, even in the absence of outliers.

More recently, Vishwanath et al. (2020) propose robust persistence diagrams which are resilient to outliers using kernel density estimators (KDE), and also develop a principled framework for characterizing the sensitivity to outliers using an analogue of influence functions. Although Vishwanath et al. (2020, Theorem 1) describes the gain in robustness by considering the robust KDE \( f_{\rho, \sigma}^n \) using the persistence influence function, Vishwanath et al. (2020, Theorems 2 & 3) together establish that as \( n \to \infty \) and \( \sigma \to 0 \), the persistence diagram \( \mathbb{D}gm\left(f_{\rho, \sigma}^n\right) \) recovers the same information which underlies the sample points \( X_n \). However, if the underlying distribution is contaminated, e.g., \( \mathbb{P}^* = (1 - \pi)\mathbb{P}_{\text{signal}} + \pi\mathbb{P}_{\text{noise}} \), then the topological inference we hope to target is that of \( \mathbb{P}_{\text{signal}} \) and not that of \( \mathbb{P}^* \).

Finally, with a similar objective of mitigating the impact of noise in topological inference, recent approaches have considered multi-parameter persistent homology as a robust tool for inferring the topological features underlying \( X_n \) (Carlsson and Zomorodian, 2009). While some recent results have demonstrated some promise (e.g., Vipond et al. 2021), they are, nevertheless, computationally infeasible for most applications, in addition to being hard to interpret (Otter et al., 2017; Bjerkevik et al., 2020).
On the statistical front, robust statistics was founded on the seminal works of Tukey (1960) and Huber (1964) with the objective of developing a framework of statistical inference stable to model misspecification and the presence of extraneous errors. Over the past few decades, robust counterparts for several inference tasks have been explored in literature (Huber, 2004; Hampel et al., 2011). More recently, in the landscape of big-data and high-dimensional statistics, the field of robust statistics has witnessed a renewed interest in the statistics and computer science literature (Diakonikolas et al., 2017). In particular, the classical problem of mean and covariance estimation has been revisited in several works (Audibert and Catoni, 2011; Minsker, 2015; Devroye et al., 2016; Joly and Lugosi, 2016) with the objective of easing model assumptions to, either, the regularity of the data generating mechanism, or, the presence of outliers. See Lugosi and Mendelson (2019a) for a recent survey. A common theme underlying these works is the constant struggle to achieve a Goldilocks equilibrium: the right balance of statistical optimality, computational efficiency and robustness to model misspecification.

In this regard, the median-of-means estimator, and, more broadly, the median-of-means principle (Lecué and Lerasle, 2020), has emerged as a powerful tool for “robustifying” an existing estimator in near linear time. Although this comes slightly at the expense of statistical optimality, median-of-means estimators are, nevertheless, easier to compute than statistically optimal and robust methods such as the tournament estimators introduced by Lugosi and Mendelson (2019b). However, computing the median in high dimensions is not a well-defined task, and can be computationally burdensome. To make matters worse, robust topological summaries naïvely employing the median-of-means principle require estimating the median in infinite-dimensional space, which can be hopeless to achieve in a computationally tractable fashion. Our work overcomes this limitation by proposing a pointwise median-of-means estimator which, although computationally tractable, exhibits a concentration of measure phenomenon with respect to the true target population counterpart in the $\|\cdot\|_\infty$ metric.

**Organization.** The remainder of this paper is organized as follows. In Section 2 we present the necessary background on persistent homology and robust statistics. We first introduce the proposed methodology in Section 3.1, and then present the main results in the remainder of the section. We establish the statistical properties of the proposed estimator in Section 3.2, and we present the influence analysis in Section 3.4. Numerical results supporting the theory are provided in Section 4. The proofs of all the results are collected in Section 6.

## 2 Preliminaries

The following subsections introduce the essential ingredients used for the remaining of the paper.

**Definitions and Notations.** For two sets $A$ and $B \subseteq A$, id : $B \to A$ given by $b \mapsto b$ denotes the identity map. For $n \in \mathbb{Z}_+$, we use the notation $[n] = \{1, 2, \ldots, n\}$, and for real-valued functions $f$ and $g$ we employ the notation $f(n) \lesssim g(n)$ if $f(n) = O(g(n))$. Given a metric space $(\mathcal{M}, \rho)$ with metric $\rho : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$, the ball of radius $r$ centered at $x \in \mathcal{M}$ is denoted $B_\rho(x, r)^1$.

For a compact set $\mathcal{X} \subset \mathcal{M}$, the $r$–offset of $\mathcal{X}$ w.r.t the metric $\rho$ is given by

$$X_\rho(r) = \bigcup_{x \in \mathcal{X}} B_\rho(x, r).$$

The distance function w.r.t. the compact set $\mathcal{X}$ plays a central role in extracting the geometric and topological features underlying $\mathcal{X}$.

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1When $r < 0$ we explicitly define $B_\rho(x, r) = \emptyset$. 

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**Definition 2.1** (Distance function). For a metric space \((\mathcal{M}, \rho)\) and a compact set \(\mathcal{X} \subseteq \mathcal{M}\), the distance function to the set \(\mathcal{X}\), denoted as \(d_{\mathcal{X}}\), is given by

\[
d_{\mathcal{X}}(y) = \inf_{x \in \mathcal{X}} \rho(x, y), \quad \text{for all } y \in \mathcal{M}.
\]

For a finite collection of points \(\mathcal{X}_n\), the distance function \(d_{\mathcal{X}_n}\) is simply denoted as \(d_n\). For two compact sets \(\mathcal{X}, \mathcal{Y} \subseteq (\mathcal{M}, \rho)\) the Hausdorff distance between \(\mathcal{X}\) and \(\mathcal{Y}\) is given by

\[
H_\rho(\mathcal{X}, \mathcal{Y}) = \max \left\{ \sup_{x \in \mathcal{X}} d_\mathcal{Y}(x), \sup_{y \in \mathcal{Y}} d_\mathcal{X}(y) \right\} = \inf \{ \epsilon > 0 : \mathcal{X} \subseteq \mathcal{Y}_\rho(\epsilon), \mathcal{Y} \subseteq \mathcal{X}_\rho(\epsilon) \},
\]

and metrizes the space of all compact subsets of \((\mathcal{M}, \rho)\). Throughout the paper we assume that \((\mathcal{M}, \rho) = (\mathbb{R}^d, \|\cdot\|)\) is the usual Euclidean space with the \(\ell_2\) metric, and omit the subscript \(\rho\). However, the results here should extend to general metric spaces \((\mathcal{M}, \rho)\) with simple modifications along the lines of Chazal et al. (2015b) and Buchet et al. (2016).

We use \(\mathcal{P}(\mathcal{X})\) to denote the set of Borel probability measures defined on \(\mathbb{R}^d\) with support \(\mathcal{X} \subseteq \mathbb{R}^d\), and for \(x \in \mathbb{R}^d\), \(\delta_x\) is used to denote a Dirac measure at \(x\). A key assumption used throughout the paper is a regularity condition for the data generating mechanism. For \(a, b > 0\), the probability measure satisfies the \((a, b)\)-standard condition if

\[
\mathbb{P}\left(B(x, r)\right) > 1 \wedge ar^b \quad \text{for all } r > 0.
\]

We denote by \(\mathcal{P}(\mathcal{X}, a, b)\) the subset of \(\mathcal{P}(\mathcal{X})\) which satisfies the \((a, b)\)-standard condition in Eq. (1) for \(a, b > 0\). This regularity assumption is standard in the domain of statistical shape analysis (e.g., Cuevas and Rodríguez-Casal 2004; Chazal et al. 2015b,a, 2017). Throughout the paper, we assume that the samples \(\mathcal{X}_n\) are obtained in an adversarial contamination setting (S), as defined below.

**Sampling Setting (S).** The data comprises of \(n\) samples \(\mathcal{X}_n = \{X_1, X_2, \ldots, X_n\}\), where \(m < n/2\) samples are contaminated with unknown outliers. No distributional assumption is made on these outliers. The remaining \(n - m\) samples are observed iid from a distribution \(\mathbb{P} \in \mathcal{P}(\mathcal{X}, a, b)\), for compact \(\mathcal{X} \subseteq \mathbb{R}^d\) and \(a, b > 0\).

A glossary of notations for additional definitions and notations introduced in the subsequent sections is provided in Appendix A.

### 2.1 Background on Persistent Homology

In this section we provide the necessary background on persistent homology arising from single parameter filtrations. We refer the reader to Chazal and Michel (2017); Edelsbrunner and Harer (2010) for a detailed introduction.

Given a compact set \(\mathcal{X}\), the building block of any topological data analysis pipeline to extract meaningful information from \(\mathcal{X}\) begins with a nested sequence of filtered topological spaces called a filtration, simply denoted by \(V\). The sequence of spaces are parametrized by a resolution parameter \(t\). There are several approaches for constructing a filtration using \(\mathcal{X}\). One approach is to consider the collection of offsets built on top of \(\mathcal{X}\), i.e., \(V_t = V^t[\mathcal{X}] = \mathcal{X}(t)\). For \(s < t\), the offsets are nested \(V^s \subseteq V^t\), and \(\mathcal{V}[\mathcal{X}] = \{V^t[\mathcal{X}] : t \in \mathbb{R}\}\) is a nested sequence of topological spaces and defines the filtration built using the offsets of \(\mathcal{X}\).

The second approach to constructing a filtration is using a filter function \(f : \mathbb{R}^d \to \mathbb{R}\) which carries the topological information underlying \(\mathcal{X}\). In this scenario, one typically constructs the filtration from the sublevel...
sets associated with \( f_X \), given by \( V^t = f_X^{-1}((\infty, t]) \) for each resolution \( t \). Again, for \( s < t \), \( V^s \subset V^t \) and the sequence \( V[f_X] = \{ V^t[f_X] : t \in \mathbb{R} \} \) constitutes the sublevel filtration from \( f_X \). Mutatis mutandis a similar notion holds for the superlevel filtration.

In general, the filtration \( V[X] \) can be very different from \( V[f_X] \), although the prevailing objective is for \( V[f_X] \) to encode the same information as in \( V[X] \). In this context, the distance function \( d_X \) plays a special role owing to the fact that its sublevel filtration is the same filtration associated with the offsets, i.e., \( V[d_X] = V[X] \). This fact plays an important role in motivating the MoM Dist estimator introduced in Section 3.1, and follows by noting that for every resolution \( t > 0 \),

\[
d_X^{-1}((\infty, t]) = \left\{ x \in \mathbb{R}^d : d_X(x) \leq t \right\} = \bigcup_{x \in X} B(x, t).
\]

Let \( V = \{ V^t : t \in \mathbb{R} \} \) denote a generic filtration and let \( t^t_s : V^s \hookrightarrow V^t \) denote the inclusion map between the filtered spaces at resolutions \( s < t \). For each resolution \( t \), let \( \mathbb{V}^t = H_*(V^t; \mathbb{F}) \) be the homology\(^2\) of \( V^t \) with coefficients in a field \( \mathbb{F} \). As the resolution \( t \) varies, the evolution of topological features is captured by \( V \). Roughly speaking, new cycles (i.e., connected components, loops, holes, and higher dimensional analogues) are born, or existing cycles can merge and disappear. The collection of cycles in \( V^t \) at each resolution \( t \) is encoded as a vector space in \( \mathbb{V}^t \). The inclusion maps \( t^t_s : V^s \hookrightarrow V^t \) induce linear maps \( \phi^t_s : \mathbb{V}^s \to \mathbb{V}^t \) between the vector spaces \( \mathbb{V}^s \) and \( \mathbb{V}^t \).

As such, the collection \( V \) can be described more succinctly as the category \( V = \{ V^t, t^t_s : s \leq t \} \) with the inclusion maps \( t^t_s \) representing the morphisms for \( s \leq t \). The image of \( V \) under the homology functor \( \text{Hom}_* : V \to \mathbb{V} \), gives us the persistence module

\[
\mathbb{V} \doteq \left\{ \mathbb{V}^t, \phi^t_s : s \leq t \right\},
\]

where the induced maps \( \phi^t_s : \mathbb{V}^s \to \mathbb{V}^t \) are homomorphisms between two vector spaces. For \( r < s < t \), the persistence module can equivalently be represented as

\[
\ldots \to \mathbb{V}^r \xrightarrow{\phi^t_r} \mathbb{V}^s \xrightarrow{\phi^t_s} \mathbb{V}^t \to \ldots
\]

Informally, a new topological feature is born at resolution \( b \in \mathbb{R} \) if the cycle associated with that feature is not present in \( \mathbb{V}^{b-\epsilon} \) for all \( \epsilon > 0 \). The same feature is said to die at resolution \( d > b \) if the cycle associated with this feature disappears from \( \mathbb{V}^{d+\epsilon} \) for all \( \epsilon > 0 \), resulting in the (ordered) persistence pair \((b, d)\). By collecting all the persistence pairs, the persistence module \( \mathbb{V} \) may be succinctly represented by a persistence diagram,

\[
\mathbb{Dgm}(\mathbb{V}) \doteq \left\{ (b, d) \in \mathbb{R}^2 : b \leq d \leq \infty \right\}.
\]

### 2.2 Interleaving of Persistence Modules

Given two persistence modules \( \mathbb{V} = \{ \mathbb{V}^t, \phi^t_s \}_{s \leq t} \) and \( \mathbb{W} = \{ \mathbb{W}^t, \psi^t_s \}_{s \leq t} \), they are said to be equivalent (or isomorphic) if there exists a family of isomorphisms \( \{ \xi_t \}_{t \in \mathbb{R}} \) such that each \( \xi^t : \mathbb{V}^t \to \mathbb{W}^t \) is an isomorphism. This notion can be extended to define two collection of maps \( \{ \alpha_t : t \in \mathbb{R} \} \) and \( \{ \beta_t : t \in \mathbb{R} \} \) which weave the two persistence modules together.

\(^2\)Where, as per convention, the order of homology, denoted by \( * \), is an arbitrary non-negative integer.
**Definition 2.2 (Interleaving of persistence modules).** Given two persistence modules $\mathbb{V}$ and $\mathbb{W}$, and two monotone increasing maps $\alpha, \beta : \mathbb{R} \to \mathbb{R}$, $\mathbb{V}$ and $\mathbb{W}$ are said to be $(\alpha, \beta)$– interleaved if the following diagrams commute for all $s \leq t$

$$
\begin{align*}
\mathbb{V}^s & \xrightarrow{\phi_s^\alpha} \mathbb{V}^t \\
\mathbb{W}^s & \xrightarrow{\psi_s^\alpha} \mathbb{W}^\alpha(t) \\
\mathbb{V}^\alpha(s) & \xrightarrow{\alpha_s} \mathbb{V}^\alpha(t) \\
\mathbb{W}^\alpha(s) & \xrightarrow{\psi_s^{\alpha(t)}} \mathbb{W}^\alpha(t) \\
\mathbb{V}^\beta(t) & \xrightarrow{\beta_t} \mathbb{V}^\beta(s) \\
\mathbb{W}^\beta(t) & \xrightarrow{\phi_s^{\beta(t)}} \mathbb{W}^\beta(s) \\
\mathbb{W}^\beta(s) & \xrightarrow{\psi_s^{\beta\circ\alpha(t)}} \mathbb{W}^\beta(t) \\
\end{align*}
$$

**Remark 2.1.** The persistence modules $\mathbb{V}$ and $\mathbb{W}$ are purely algebraic objects, and their underlying filtrations $V$ and $W$ are not necessarily compatible. However, when the filtrations $V$ and $W$ arise as filtered subsets of the same underlying space (e.g., $\mathbb{R}^d$), we can similarly define an $(\alpha, \beta)$– interleaving between the filtrations $V$ and $W$ by replacing all linear maps in Definition 2.2 by inclusion maps.

The resulting persistence diagrams $\text{Dgm}(V)$ and $\text{Dgm}(W)$ are elements of the space of persistence diagrams $\Omega = \{(x, y) : x \leq y\}$, which is endowed with the family of $q$–Wasserstein metrics $W_q(\cdot, \cdot)$ for $1 \leq q \leq \infty$. We refer the reader to Edelsbrunner and Harer (2010); Mileyko et al. (2011) for more details. In special case of $q = \infty$, the resulting metric $W_\infty$ is commonly referred to as the bottleneck distance, and is given as follows.

**Definition 2.3 (Bottleneck distance).** Given two persistence diagrams $D_1, D_2 \in \Omega$, the bottleneck distance is given by

$$W_\infty(D_1, D_2) = \inf_{\gamma \in \Gamma} \sup_{p \in D_1 \cup \Delta} \|p - \gamma(p)\|_\infty,$$

where $\Gamma = \{\gamma : D_1 \cup \Delta \to D_2 \cup \Delta\}$ is the set of all multi-bijections from $D_1$ to $D_2$ including the diagonal $\Delta = \{(x, y) : x = y\}$ with infinite multiplicity.

Although the space of persistence diagrams $(\Omega, W_q)$, together with the $q$–Wasserstein distance, presents a challenging mathematical structure for refined statistical analyses (Mileyko et al., 2011; Turner et al., 2014), the stability of persistence diagrams provides a handle on this space by allowing us to directly work on the space generating the filtrations.

**Lemma 2.1 (Stability of persistence diagrams; Cohen-Steiner et al. 2007; Chazal et al. 2016).** Given two compact sets $X, Y \subset \mathbb{R}^d$,

$$W_\infty(\text{Dgm}(\mathbb{V}[X]), \text{Dgm}(\mathbb{V}[Y])) \leq H(X, Y).$$

Alternatively, for two filter functions $f, g : \mathbb{R}^d \to \mathbb{R}$,

$$W_\infty(\text{Dgm}(\mathbb{V}[f]), \text{Dgm}(\mathbb{V}[g])) \leq \|f - g\|_\infty.$$
Remark 2.2.

(i) When the interleaving maps \((\alpha, \beta)\) are additive, i.e., of the form \(\alpha : t \mapsto t + \epsilon\) and \(\beta : t \mapsto t + \delta\), then persistence diagrams \(Dgm(V)\) and \(Dgm(W)\) obtained from the persistence modules satisfy the following relationships:

\[
Dgm(V) \in Dgm(W) + [-\delta, \epsilon]^2 \quad \text{and} \quad Dgm(W) \in Dgm(V) + [-\epsilon, \delta]^2,
\]

where \(\oplus\) denotes the Minkowski sum in \(\mathbb{R}^2\). A coarser bound is obtained from the stability theorem (Cohen-Steiner et al., 2007) which guarantees that

\[
W_\infty(Dgm(V), Dgm(W)) \leq \max \{\epsilon, \delta\}.
\]

(ii) Furthermore, when the interleaving maps are identical, i.e., \(\alpha \equiv \beta : t \mapsto t + \epsilon\), this notion can be extended to define an interleaving pseudo-distance between persistence modules,

\[
d_I(V, W) \doteq \inf \{\epsilon > 0 : V and W are (\alpha, \alpha) - interleaved for \alpha : t \mapsto t + \epsilon\}.
\]

From the isometry theorem (Chazal et al., 2016) the interleaving distance is identical to the bottleneck distance, i.e., \(W_\infty(Dgm(V), Dgm(W)) = d_I(V, W)\). In such cases, it is equivalent to say that \(V\) and \(W\) are \((\alpha, \alpha)\)-interleaved or \(d_I(V, W) \leq \epsilon\). Similarly, for filtrations \(V\) and \(W\) comprising of subsets of \(\mathbb{R}^d\),

\[
d_I(V, W) \doteq \inf \{\epsilon > 0 : V_t \subseteq W_{t+\epsilon} \quad \text{and} \quad W_t \subseteq V_{t+\epsilon}\}.
\]

By functoriality, \(d_I(V, W) \leq \epsilon \implies d_I(V, W) \leq \epsilon \implies W_\infty(Dgm(V), Dgm(W)) \leq \epsilon\).

2.3 Weighted Rips Filtrations

In practice, given a compact set \(X \subseteq \mathbb{R}^d\) or a filter function \(f\), the persistence modules \(V[X]\) and \(V[f]\) are computed using simplicial complexes. In particular:

(i) For each \(t \in \mathbb{R}\), one may use the Čech or Alpha complex to compute the nerve of the cover, \(\text{nerve}\{B(x, t) : x \in X\}\). Since the Nerve lemma (Edelsbrunner and Harer, 2010) guarantees that \(V[X] \cong \text{nerve}\{B(x, t) : x \in X\}\), the resulting persistence module \(V[X]\) may be computed exactly using simplicial homology.

(ii) In the case of \(V[f]\), this is typically achieved by choosing a grid resolution parameter \(\epsilon\), and constructing a cubical complex \(\mathcal{K}_\epsilon\) on the underlying space. The function \(f : \mathbb{R}^d \to \mathbb{R}\) may be extended to define \(f : \mathcal{K}_\epsilon \to \mathbb{R}\), and at each resolution \(t \in \mathbb{R}\), the sublevel sets \(V[f,x]\) can be approximated using the lower-star filtration \(\mathcal{K}_t\) = \(\{\sigma \in \mathcal{K}_\epsilon : \max_{x \in \sigma} f(x) \leq t\}\). Therefore, the filtration \(V[f]\) can be approximated by the filtration \(\{\mathcal{K}_t : t \in \mathbb{R}\}\), and the resulting persistence module is computed using cubical homology.

Note that (i) is able to compute the exact persistence module in practice, but is unable to weight points according to \(f\). On the other hand, (ii) is only an approximate computation and depends on the nuisance parameter \(\epsilon\). Furthermore, the size of the cubical complex is \(|\mathcal{K}_\epsilon| = O(\epsilon^{-d})\), making it scale poorly in high dimensions. To overcome this limitation, Buchet et al. (2016) proposed the \(f\)-weighted filtrations, which was subsequently generalized by Anai et al. (2019).
Given a non-negative weight function $f : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ and power $1 \leq p \leq \infty$, the weighted radius function of resolution $t > 0$ at $x$ is given by

$$r_{f,x}(t) = \begin{cases} 
(t^p - f(x)^p)^{1/p} & \text{if } t \geq f(x) \\
-\infty & \text{if } t < f(x).
\end{cases}$$

Consequently, $B_{f,\rho}(x,t)$ is the weighted ball of resolution $t$ at $x$ w.r.t. the metric $\rho$, which is illustrated in Figure 1, and is given by

$$B_{f,\rho}(x,t) \doteq B_{\rho}(x, r_{f,x}(t)) = \left\{ y \in \mathbb{R}^d : \rho(x,y) \leq r_{f,x}(t) \right\}.$$  

Given $X \subseteq \mathbb{R}^d$, the collection of weighted balls $V^t[X, f] = \{ B_f(x, t) : x \in X \}$, is called the weighted cover of $X$. The $f$-weighted offset at resolution $t$ is given by the union of balls in $V^t[X, f]$,

$$V^t[X, f] \doteq \bigcup_{x \in X} B_f(x,t).$$

Together with the inclusion maps $\iota_s : V^s[X, f] \hookrightarrow V^t[X, f]$, the $f$-weighted filtration is given by

$$V[X, f] \doteq \{ V^t[X, f], \iota_s : s \leq t \}.$$  

The image of $V[X, f]$ under the homology functor $\mathbf{Hom}_* : V[X, f] \to \mathcal{V}[X, f]$, results in the weighted persistence module $\mathcal{V}[X, f] \doteq \{ \mathcal{V}^t[X, f], \phi^t_s : s \leq t \}$, where the induced maps $\phi^t_s : \mathcal{V}^s[X, f] \to \mathcal{V}^t[X, f]$ are linear maps between vector spaces. The weighted-simplicial complexes

$$\mathcal{C}^t[X, f] = \text{nerve}\{ \mathcal{V}^t[X, f] \} \quad \text{and} \quad \mathcal{R}^t[X, f] = \text{Rips}\{ \mathcal{V}^t[X, f] \}$$

denote the weighted-Čech complex and weighted-Rips complex associated with the weighted cover $V^t[X, f]$ respectively. Without loss of generality $V^t[X, f] = H_*\{ V^t[X, f] \}$ is the homology of the offset $V^t[X, f]$, which, by the nerve lemma, is the same as the homology of the weighted-Čech complex. Furthermore, if $f(x) \equiv 0$ for all $x \in \mathbb{R}^d$ then the resulting filtrations are the usual unweighted filtrations. In particular, $V[X_n] \cong \mathcal{C}[X_n, f]$ and $\mathcal{R}[X_n, f]$ correspond to Čech and Rips filtrations, respectively. The following structural results appear in Anai et al. (2019), and serve as analogues of the stability result for $f$-weighted filtrations.
Lemma 2.2 (Anai et al., 2019, Propositions 3.2 & 3.3). Given $\mathbb{X} \subset \mathbb{R}^d$ and $f, g : \mathbb{X} \rightarrow \mathbb{R}_+$

(i) $\mathbb{V}[\mathbb{X}, f]$ and $\mathbb{V}[\mathbb{X}, g]$ are $(\alpha, \alpha)$–interleaved for $\alpha : t \mapsto t + \|f - g\|_\infty$.

Additionally, given $\mathbb{Y} \subset \mathbb{R}^d$ and $h : \mathbb{X} \cup \mathbb{Y} \rightarrow \mathbb{R}_+$, if $h$ is $L$–Lipschitz and $H(\mathbb{X}, \mathbb{Y}) \leq \epsilon$, then

(ii) $\mathbb{V}[\mathbb{X}, h]$ and $\mathbb{V}[\mathbb{Y}, h]$ are $(\beta, \beta)$–interleaved for $\beta : t \mapsto t + \epsilon(1 + L^p)^{1/p}$.

2.4 Median-of-means Estimators

Median-of-means (MoM) estimators have gained popularity in the robust machine learning owing to recent success, both theoretically and experimentally. See, for example, Devroye et al. (2016); Lugosi and Mendelson (2019a); Lécué and Lerasle (2020). The background for MoM estimators in the context of mean estimation is as follows: samples $\mathbb{X}_n = \{X_1, X_2, \ldots, X_n\}$ are observed and we wish to construct an estimator for the population mean $\theta$. The sample mean $\bar{\theta} = \mathbb{X}_n$ is known to achieve sub-Gaussian estimation error only when the samples $\mathbb{X}_n$ themselves are observed from a sub-Gaussian distribution.

Robust statistics deals with two important relaxations to this model: (i) the samples $\mathbb{X}_n$ are observed iid from $\mathbb{P}$, but $\mathbb{P}$ is no longer sub-Gaussian and is assumed to have heavy tails; and (ii) a fraction $\pi < \frac{1}{2}$ of the samples are assumed to be contaminated with outliers, and the remaining $(1 - \pi)n$ samples are observed from a well-behaved distribution $\mathbb{P}$.

The median-of-means estimator $\hat{\theta}_{\text{MOM}}$, originally introduced by Nemirovskij and Yudin (1983), addresses these relaxations by constructing a robust estimator of location as follows: For $1 \leq Q \leq n$, the sample $\mathbb{X}_n$ is partitioned into subsets $\{S_1, S_2, \ldots, S_Q\}$ such that each subset $S_q \subset \{1, 2, \ldots, n\}$ with $|S_q| = \lfloor n/Q \rfloor$. The MoM estimator $\hat{\theta}_{\text{MOM}}$ is, then, defined as

$$\hat{\theta}_{\text{MOM}} = \text{median}\{\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_Q\},$$

where $\{\hat{\theta}_q : q \in [Q]\}$ are the sample means computed for each subset $\{S_q : q \in [Q]\}$. Audibert and Catoni (2011) showed that, in the univariate setting, $\hat{\theta}_{\text{MOM}}$ achieves sub-Gaussian rates of convergence for heavy tailed data. Minsker (2015) and Devroye et al. (2016) extend these results to the multivariate setting by considering the geometric median. The MoM idea has subsequently been extended in several other directions, e.g., U-statistics (Joly and Lugosi, 2016), kernel mean embeddings (Lerasle et al., 2019) and general M-estimators (Lécué and Lerasle, 2020) among others. Most importantly, these extensions move away from the heavy-tailed framework and provide significant insights on how $\hat{\theta}_{\text{MOM}}$ can overcome the second relaxation, i.e., estimation in the presence of outliers contamination. While the MoM estimators are not unique in their ability to recover the signal under heavy tailed noise, or in the presence of contamination, they are very simple to construct in most cases, and provide a clear characterization of the effect of noise on the estimation error.

3 Main Results

In the following, we present a MoM estimator to obtain outlier robust persistence diagrams in Section 3.1, and its statistical properties along with the influence analysis are presented in Sections 3.2—3.4. In Section 3.5 we present a method for adaptively calibrating the MoM tuning parameter using a data-driven procedure. The proofs for all results are deferred to Section 6.

3.1 Empirical distance function using the Median-of-Means principle

Let $\mathbb{X}_n = \{X_1, X_2, \ldots, X_n\} \subset \mathbb{R}^d$ be a sample of $n$ observations. We assume that the samples are obtained under sampling setting $(\mathcal{S})$. We emphasize that this setting encompasses the following scenarios:
(a) The samples $X_n$ are obtained i.i.d. from $P \in \mathcal{P}(X, a, b)$ for compact $X \subset \mathbb{R}^d$.

(b) The samples are obtained from a distribution $P = (1 - \pi)P_{signal} + \pi P_{noise}$, where $\pi \in (0, 1/2)$ and $P_{signal} \in \mathcal{P}(X, a, b)$.

(c) $\{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n\}$ is first sampled i.i.d. from $P \in \mathcal{P}(X, a, b)$, and then handed over to an adversary. The adversary is then free to examine the $n$ points, and replace any $m < n/2$ of them with some points of their choice. The modified dataset, $\tilde{X}_n$, is then shuffled and handed to the topologist for inference, who has no prior knowledge of the original $\{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_n\}$.

The central objective is to derive a statistically consistent and computationally efficient estimator of $\mathcal{D}_{gm}(V[X])$ which is robust to the misspecification scenarios detailed above, using the samples $\tilde{X}_n$. To this end, the MoM Distance (MoM Dist) function $d_{n,Q}$ is defined as follows.

**Definition 3.1 (MoM Dist).** Given a collection of points $X_n \subset \mathbb{R}^d$ and $1 \leq Q \leq n$, let $\{S_1, S_2, \ldots, S_Q\}$ be a partition of $X_n$ into $Q$ disjoint blocks, such that each subset $S_q \subset X_n$ comprises of $|S_q| = \lfloor n/Q \rfloor$ samples. The MoM distance function $d_{n,Q} : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ is defined to be

$$d_{n,Q}(y) = \text{median} \left\{ d_{n,S_q}(y) : q \in [Q] \right\} = \text{median} \left\{ \inf_{x \in S_q} \| x - y \| : q \in [Q] \right\}. \quad (3)$$

The proposed outlier robust persistence diagram $\mathcal{D}_{gm}(V[X_n, d_{n,Q}])$ is then obtained using $d_{n,Q}$-weighted filtration $V[X_n, d_{n,Q}]$.

Note that we recover the usual empirical distance function, i.e., $d_{n,1} \equiv d_n$ when $Q = 1$.

**Remark 3.1.** For each block $S_q$, distance function $d_{n,q} \in L_\infty(\mathbb{R}^d)$ can be viewed as the Kuratowski embedding of $S_q$. The most natural generalization of the multivariate median-of-means estimators proposed by Minsker (2015) and Lerasle et al. (2019) would suggest the following estimator as the natural candidate for MoM Dist:

$$\tilde{d}_{n,Q} = \arg \inf_{f \in L_\infty(\mathbb{R}^d)} \sum_{q=1}^{Q} \| f - d_{n,S_q} \|_\infty,$$

where the median under consideration corresponds to the geometric median in $L_\infty(\mathbb{R}^d)$. Although $\tilde{d}_{n,Q}$ has its appeal from a theoretical perspective, the computation of $\tilde{d}_{n,Q}$ involves an infinite-dimensional optimization problem, making it infeasible in practice. In contrast, the proposed estimator in Definition 3.1, is a pointwise median-of-means estimator with a tractable computational cost. This has the promise of being highly modular, and widely applicable in many practical settings. The technical difficulty arises in showing that the pointwise estimator $d_{n,Q}$ achieves an exponential concentration bound around $d_X$ in the $L_\infty(\mathbb{R}^d)$ metric.

Similar to the proposed methodology in Definition 3.1, the procedure of partitioning the data $X_n$ into smaller subsets, and then aggregating them as an estimator of persistent homology has been shown to satisfy several favorable properties by Solomon et al. (2021) and Gómez and Mémoli (2021), albeit in a different context. We argue that a similar principle, in our setting, also leads to provably robust estimators.

**Computational considerations.** Given a weighting function $f$, the first step in constructing the $f$-weighted filtration begins with estimating the weights associated with the sample points, i.e., $w_i = f(\tilde{X}_i)$ for all $i \in [n]$. After this step, the computational complexity of constructing the $f$-weighted filtration $V[X_n, f]$ is independent

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3 Without loss of generality, we may assume that $n$ is divisible by $Q$, so that $n/Q \in \mathbb{Z}_+$. 
of the choice of the weighting function \( f \). Table 1 compares the computational complexity for three robust filtrations: (i) the MoM Dist \( d_{n,Q} \), (ii) the distance-to-measure \( \delta_{n,k} \) (DTM, Anai et al., 2019), and (iii) the robust kernel density estimator \( f^R_{n,\sigma} \) (RKDE, Vishwanath et al., 2020). Given a test point \( x \in \mathbb{R}^d \), the distance from \( x \) to each block \( S_q \) is optimally computed using a \( k-d \) tree. The pre-processing step, which involves the construction of the \( k-d \) tree (Wald and Havran, 2006), typically has time complexity \( O(|S_q| \log |S_q|) \) for each block \( q \in [Q] \) with \( |S_q| = n/Q \). Thereafter \( O(\log |S_q|) \) time is needed for a single query (Cormen et al., 2009, Chapter 10). The results for each block \( q \in [Q] \) are then aggregated to compute the median, which takes an additional \( O(Q) \) time per query. This results in a total evaluation time of \( O(n \cdot (Q + \log n/Q)) \) for \( n \) samples.

The distance-to-measure with parameter \( m \) requires the evaluation of the distance to the \( k \)-th nearest neighbor for \( k = \lfloor m n \rfloor \). This is, again, optimally computed using a \( k-d \) tree; however, unlike \( d_{n,Q} \), the \( k-d \) tree needs to be constructed for all \( n \) samples, resulting in a time complexity of \( O(n \log n) \) for pre-processing. Thereafter, the evaluation time takes \( O(k \log n) \) for each query point, resulting in \( O(n \cdot k \log n) \) for evaluation over \( n \) samples. The robust KDE \( f^R_{n,\sigma} \), on the other hand, requires \( O(n^2) \) time to compute the Gram-matrix in each iteration of the KIRWLS algorithm, and takes \( O(n^2) \) for \( \ell \) outer loops. After this pre-processing step, the coefficients of \( f^R_{n,\sigma} \) may be used to evaluate each query in \( O(n) \) time. The three weighted filtrations \( V[X_n, d_{n,Q}] \), \( V[X_n, \delta_{n,k}] \) and \( V[X_n, f^R_{n,\sigma}] \) are illustrated in Figure 2.

![Comparison of computational complexity for robust weighted filtrations.](image)

We conclude this section with the following result, which establishes that MoM Dist is 1–Lipschitz.

**Lemma 3.1.** Given samples \( \mathcal{X}_n = \{X_1, X_2, \ldots, X_n\} \) and \( Q < n \),

\[
|d_{n,Q}(x) - d_{n,Q}(y)| \leq \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^d.
\]
3.2 Statistical properties of $V[\mathcal{d}_{n,Q}]$

We begin our analysis by characterizing the persistence diagrams obtained using the sublevel filtration of $\mathcal{d}_{n,Q}$. The following result (proved in Section 6.3), establishes that $D_{gm}(V[\mathcal{d}_{n,Q}])$ is a statistically consistent estimator of target population quantity $D_{gm}(V[X])$ under sampling setting $(\mathcal{S})$, and establishes its rate of convergence in the $W_\infty$ metric.

**Theorem 3.1 (Sublevel filtration).** Suppose $\mathbb{P} \in \mathcal{P}(\mathbb{X}, a, b)$ is a probability distribution with support $\mathbb{X}$ satisfying the $(a,b)$–standard condition, and $\mathbb{X}_n$ is obtained under sampling condition $(\mathcal{S})$. For $2m < Q < n$ and for all $\delta < e^{-(1+b)Q}$,

$$\Pr \left\{ W_\infty \left( D_{gm}(V[\mathcal{d}_{n,Q}]), D_{gm}(V[X]) \right) \leq g(n, Q, a, b) \right\} \geq 1 - \delta, \quad (4)$$

where

$$g(n, Q, a, b) = \left( \frac{Q \log(n/Q)}{a n} + 4Q \log(1/\delta) \right)^{1/b} \frac{a(Q-2m) n}{a(Q-2m) n}.$$  

Furthermore, if the number of outliers grows with $n$ as $m = cn^{\epsilon}$ for $c > 0$ and $\epsilon \in [0, 1)$ then

$$\mathbb{E} \left[ W_\infty \left( D_{gm}(V[\mathcal{d}_{n,Q}]), D_{gm}(V[X]) \right) \right] \lesssim \left( \frac{\log n}{n^{1-\epsilon}} \right)^{1/b}. \quad (5)$$

**Remark 3.2.** The following salient observations can be made from Proposition 3.1.

(i) In addition to characterizing the uniform rate of convergence of $\mathcal{d}_{n,Q}$, Eq. (4) also provides a uniform confidence band for $D_{gm}(V[X])$ in the presence of outliers. The two terms appearing in $g(n, Q, a, b)$ may be interpreted as follows: The first term is similar to the term appearing in Chazal et al. (2015b, Theorem 2) with an effective sample size of $n/Q$ instead of $n$, which is a consequence of the Median-of-Means procedure. The second term incorporates the desired confidence level $\delta$ adaptive to the volume dimension $b > 0$, with an effective sample size of $n/Q$. Notably, as the number of outliers $m$ increases, the number of blocks $Q$ must also increase; thereby widening the resulting confidence band.

(ii) The complex inter-dependence of the parameters $m, Q$ and $\delta$ in Eq. (4) is simplified in Eq. (5). In the absence of outliers, i.e., when $m = 0$ and $Q = 1$, we recover the same convergence rate as in Chazal et al. (2015b, Theorem 4),

$$\mathbb{E} \left[ W_\infty \left( D_{gm}(V[\mathcal{d}_{n,Q}]), D_{gm}(V[X]) \right) \right] \lesssim \left( \frac{\log n}{n} \right)^{1/b}. \quad (6)$$

Specifically, it becomes apparent that accommodating for more adverse noise conditions comes at the price of an attenuated rate of convergence.

(iii) The admissible confidence level $\delta$ for constructing the confidence band is implicitly dependent on the parameter $Q$. This phenomenon is unavoidable with estimators based on the median-of-means principle. We refer the reader to Lugosi and Mendelson (2019a, Section 2.4) for a comprehensive discussion on how robustness must come at the price of the confidence level $\delta$ being restricted.
The proof of Proposition 3.1 relies on the following lemma, which allows us to control the deviation of a pointwise median-of-means estimator from its uncontaminated population counterpart in terms of a Binomial tail probability.

**Lemma 3.2.** Suppose \( \mathbb{P} \in \mathcal{P}(\mathbb{X}) \) for \( \mathbb{X} \subset \mathbb{R}^d \) and \( \mathbb{X}_n = \mathbb{X}_{n,m}^* \cup \mathbb{Y}_m \) is obtained under sampling condition (8) with \( \mathbb{X}_{n,m}^* \) observed i.i.d. from \( \mathbb{P} \). Let \( \mathbb{P}_n \) denote the empirical measure associated with \( \mathbb{X}_n \) and for \( 2m < Q < n \), let \( \mathbb{P}_q \) be the empirical measure associated with the block \( S_q \) for all \( q \in [Q] \). Given a statistical functional \( T : \mathcal{P}(\mathbb{R}^d) \to L_\infty(\mathbb{R}^d) \), let \( T_Q(\mathbb{P}_n) \in L_\infty(\mathbb{R}^d) \) be the pointwise MoM estimator given by

\[
T_Q(\mathbb{P}_n)(x) = \text{median}\{ T(\mathbb{P}_q)(x) : q \in [Q]\}, \quad \text{for all } x \in \mathbb{R}^d.
\]

Then, for \( t > 0 \)

\[
\mathbb{P}\left( \|T_Q(\mathbb{P}_n) - T(\mathbb{P})\|_\infty > t \right) \leq \mathbb{P}\left( \sum_{q \in A} \xi_q(t; n, Q) > \frac{Q - 2m}{2} \right),
\]

where \( A = \{ q \in [Q] : S_q \cap \mathbb{Y}_m = \emptyset \} \) are the indices for the blocks containing no outliers, and

\[
\xi_q(t; n, Q) = 1\left( \|T(\mathbb{P}_q) - T(\mathbb{P})\|_\infty > t \right) \quad \text{for all } q \in A.
\]

The statement of Lemma 3.2 holds for empirical processes arising from general classes of pointwise median-of-means estimators. In particular, by taking \( T(\mathbb{P}_q) = d_{n,q} \) to be the distance function w.r.t. block \( S_q \), the estimator \( d_{n,Q} \) satisfies the conditions of Lemma 3.2. We also point out that the exponential concentration bound in Proposition 3.1 is strictly better than similar bounds appearing in other pointwise MoM estimators, e.g., Humbert et al. (2020, Theorem 2). This is owing to the Chernoff bound (instead of a Hoeffding bound) used for bounding the Binomial tail probability appearing in Lemma 3.2. This provides a significant gain for Binomial random variables with shrinking probability (Hagerup and Rüb, 1990).

### 3.3 Statistical properties of \( V[\mathbb{X}_n, d_{n,Q}] \)

In practice, the sublevel filtration \( V[d_{n,Q}] \) cannot be computed exactly, and one must rely on approximations using cubical homology. To this end, we now turn our attention to \( d_{n,Q} \)-weighted filtrations computed on the sample points directly. Before we study the statistical properties of the \( d_{n,Q} \)-weighted filtration, we provide a useful characterization of the persistence diagram obtained using the sublevel sets of \( d_{n,Q} \).

**Lemma 3.3.** Given samples \( \mathbb{X}_n \) and \( Q < n \), \( V[d_{n,Q}] \) and \( V[\mathbb{R}^d, d_{n,Q}] \) are (id, \( \alpha \))-interleaved for \( \alpha : t \mapsto 2^{\frac{p-1}{p}} t \) for all \( p \geq 1 \). In particular, \( V[d_{n,Q}] = V[\mathbb{R}^d, d_{n,Q}] \) when \( p = 1 \).

We now turn our attention to the \( d_{n,Q} \)-weighted filtration \( V[\mathbb{X}_n, d_{n,Q}] \). The following result establishes that the persistence module \( \Pi[\mathbb{X}_n, d_{n,Q}] \) is sufficiently regular.

**Lemma 3.4 (Regularity).** For \( \mathbb{X}_n \) obtained under sampling setting (8) and \( d_{n,Q} \) defined in Eq. (3), the persistence module \( \Pi[\mathbb{X}_n, d_{n,Q}] \) is \( q \)-tame and pointwise finite-dimensional.

The proof of Lemma 3.4 is a direct consequence of Anai et al. (2019, Proposition 3.1), and ensures that the persistence diagram \( \mathcal{D}\Pi[\mathbb{X}_n, d_{n,Q}] \) is well-defined.

Next, in order to establish that \( \mathcal{D}\Pi[\mathbb{X}_n, d_{n,Q}] \) is a consistent estimator of \( \mathcal{D}\Pi[\mathbb{X}] \) and to construct uniform confidence bands in the space of persistence diagrams \( (\Omega, \mathcal{W}_\infty) \), we need a tighter control for how
the two persistence modules are interleaved. To this end, Lemmas 3.5 and 3.6 will be of assistance, and serve as generalizations of Anai et al. (2019, Lemma 4.8 & Proposition 4.9). The following result, which holds for a general metric space \((M, \rho)\) and an arbitrary weight function \(f\), provides a handle for the interleavings between \(f\)-weighted filtrations computed on two nested sets using the same function \(f\).

**Lemma 3.5.** Given a metric space \((M, \rho)\), two compact subsets \(X, Y\) of \(M\) such that \(X \subseteq Y\), and a weight function \(f : M \to \mathbb{R}_{\geq 0}\), let \(V^f_\rho[X, f]\) and \(V^f_\rho[Y, f]\) be their respective \(f\)-weighted filtrations. If \(f\) satisfies the property that

\[
\inf_{x \in X} \rho(x, y) \leq f(y) + a,
\]

for \(a > 0\) and for all \(y \in Y\), then the filtrations are \((id, \alpha)\)-interleaved, i.e.,

\[
V^t[X, f] \subseteq V^t[Y, f] \subseteq V^\alpha(t)[X, f],
\]

for \(\alpha : t \mapsto 2^{-1/\beta} t + a + \sup_{x \in X} f(x)\).

Since map \(\alpha\) appearing in Lemma 3.5 is not purely a translation map, it does not lead to a bound in the interleaving metric as per Eq. (2), and, therefore, a bound in the \(W_\infty\) metric cannot be characterized using Lemma 3.5 alone. The next result, which is stated only for the Euclidean space \((\mathbb{R}^d, ||\cdot||)\), establishes that for sufficiently large values of \(t\), the map \(\alpha\) may be replaced by a translation map.

**Lemma 3.6.** Let \((M, \rho) = (\mathbb{R}^d, ||\cdot||)\). Suppose \(X, Y \subseteq \mathbb{R}^d\) are compact sets such that \(X \subseteq Y\), and \(f\) satisfies the same conditions as in Lemma 3.5 for \(a > 0\). Let \(t(X)\) be the filtration value for the simplex corresponding to \(X\) in nerve\(\{V^f_\rho[X, f]\}\), i.e.,

\[
t(X) = \inf \left\{ t > 0 : \bigcap_{x \in X} B_{f, \rho}(x, t) \neq \emptyset \right\},
\]

and \(\beta : t \mapsto t + c(X)\) be a non-decreasing map with

\[
c(X) = a + \sup_{x \in X} f(x) + \left(1 - \frac{1}{p}\right) t(X).
\]

Then for all \(t \geq t(X)\), the homomorphisms \(\phi^\beta(t) : V^f_\rho[X, f] \to V^\beta(t)[X, f]\) are trivial, i.e.,

\[
\text{Im}(\phi^\beta(t)) \cong \begin{cases} F & \text{if } V^f_\rho[X, f] = H_0(V^f_\rho[X, f]) \\ \{0\} & \text{if } V^f_\rho[X, f] = H_k(V^f_\rho[X, f]), \ k > 0 \end{cases}.
\]

Furthermore, the bottleneck distance between the resulting \(f\)-weighted persistence diagrams is bounded above as

\[
W_\infty(\text{Dgm}(V^f_\rho[X, f]), \text{Dgm}(V^f_\rho[Y, f])) \leq c(X).
\]

**Remark 3.3.** Unlike Lemma 3.5, which is stated for general metric spaces, restricting ourselves to the Euclidean space \((\mathbb{R}^d, ||\cdot||)\) in Lemma 3.6 is sufficient for the objective of this work. However, as outlined in the proof, the only issue arises when Anai et al. (2019, Lemma B.1) is invoked. While Anai et al. (2019, Lemma B.1) (which holds for affine spaces satisfying the parallelogram identity) extends naturally to Banach spaces, the extension to general metric spaces will require some care on a case-by-case basis.
In essence, the preceding two results enable us to control the filtrations in two separate stages, and, then, “stitch” the results together. See Figure 3 for an illustration. This forms the crux of the next result, which establishes an analogue of the stability result for $d_{n,Q}$-weighted filtrations, but unlike the stability for the usual distance function $d_n$, it is also robust to outliers.

**Theorem 3.2.** (Stability & robustness of $d_{n,Q}$-weighted filtrations) Let $X_n = X_{n,m} \cup Y_m$ be a collection of points obtained under sampling condition (S). For $Q > 2m$ let $d_{n,Q}$ be the MoM Dist function computed on the contaminated points $X_n$ and let $d_{n-m}$ be the distance function w.r.t. the inliers $X_{n,m}^*$. Then

$$W_\infty \left( \bigvee_{X_n, d_{n,Q}} \bigvee_{X_{n,m}^*, d_{n-m}} \right) \leq \sup_{x \in X_{n,m}} d_{n,Q}(x) + \|d_{n,Q} - d_{n-m}\|_\infty + \left( 1 - \frac{1}{p} \right) t(X_{n,m}^*),$$

where $t(X_{n,m}^*)$ is the filtration value of the simplex associated with the inliers $X_{n,m}^*$ in the filtration $V[X_{n,m}^*, d_{n,Q}]$. In particular, when $p = 1$ we have

$$W_\infty \left( \bigvee_{X_n, d_{n,Q}} \bigvee_{X_{n,m}^*, d_{n-m}} \right) \leq \sup_{x \in X_{n,m}} d_{n,Q}(x) + \|d_{n,Q} - d_{n-m}\|_\infty. \quad (7)$$

**Remark 3.4.** The following observations follow from Theorem 3.2.

(i) In contrast to what would follow from Lemma 2.2 (ii) for the standard unweighted filtration, the term appearing in the r.h.s. of Eq. (7) completely eliminates the dependence on the Hausdorff distance between $X_n$ and $X_{n,m}^*$ in the $d_{n,Q}$-filtration. More generally, the same bound in Proposition 3.2 holds even when $V[X_n, d_{n,Q}]$ is replaced by $V[M, d_{n,Q}]$ for any set $M \supseteq X_{n,m}^*$.

(ii) Notably, $V[X_n, d_{n,Q}]$ remains resilient to outliers. To see this, observe that the first term appearing in the r.h.s. of Eq. (7) may be bounded as

$$\sup_{x \in X_{n,m}} d_{n,Q}(x) = \sup_{x \in X_{n,m}^*} |d_{n,Q}(x) - d_X(x)| \leq \|d_{n,Q} - d_X\|_\infty,$$

where the first equality follows from the fact that $d_X(x) = 0$ for all $x \in X_{n,m}^*$. Therefore, from the proof of Theorem 3.1, the r.h.s. of Eq. (7) vanishes with high probability for sufficiently large sample sizes.
(iii) For \( p = 1 \), a similar analysis for the DTM-filtrations appears in Anai et al. (2019, Theorem 4.5) and the bottleneck distance is bounded above as

\[
W_{\infty}\left(\mathcal{D}gm(\mathbb{V}[\mathbb{X}_n, \delta_{n,k}]), \mathcal{D}gm(\mathbb{V}[\mathbb{X}_{n-m}^*, \delta_{n-m,k}])\right) \leq \sqrt{\frac{n}{k}} W_2(\mathbb{X}_{n-m}^*, \mathbb{X}_n) + \sup_{x \in \mathbb{X}_{n-m}^*} \delta_{n-m,k}.
\]

While the last term on the r.h.s. converges to the uncontaminated population analogue with high probability, the first term involving the Wasserstein distance \( W_2(\mathbb{X}_{n-m}^*, \mathbb{X}_n) \) can be large even for a few extreme outliers. In contrast, the r.h.s. of Eq. (7) converges to zero with high probability with no assumptions on the outliers \( \mathbb{Y}_m \).

With this background we are now in a position to state our main result, which characterizes the rate of convergence for the \( d_{n,Q} \)-weighted filtration on the contaminated sample points, \( V[\mathbb{X}_n, d_{n,Q}] \), to the counterfactual population analogue \( V[\mathbb{X}] \) in the \( W_{\infty} \) metric.

**Theorem 3.3 (\( d_{n,Q} \)-weighted filtration).** Let \( p = 1 \). Suppose \( \mathbb{P} \in \mathcal{P}(\mathbb{X}, a, b) \) is a probability distribution with support \( \mathbb{X} \) satisfying the \((a, b)\)-standard condition, and \( \mathbb{X}_n = \mathbb{X}_{n-m}^* \cup \mathbb{Y}_m \) is obtained under sampling condition (S). Then, for \( 2m < Q < n \) and for all \( \delta \in (0, 1) \),

\[
\mathbb{P}\left\{ W_{\infty}\left(\mathbb{V}[\mathbb{X}_{n-m}^* \cup \mathbb{Y}_m, d_{n,Q}], \mathbb{V}[\mathbb{X}]\right) \leq \mathfrak{f}(n, m, Q, \delta_1, \delta_2) \right\} \geq 1 - \delta,
\]

where

\[
\mathfrak{f}(n, m, Q, a, b) = \left( \frac{Q \log(n/Q)}{a(n/Q)} + \frac{4Q \log(1/\delta_1)}{a(Q - 2m)n} \right)^{1/b} + \left( \frac{\log(n - m)}{a(n - m)} + \frac{4 \log(1/\delta_2)}{a(n - m)} \right)^{1/b},
\]

for \( \delta_1, \delta_2 \in (0, 1) \) such that \( \delta_1 \leq e^{-(1+b)Q} \) and \( \delta_1 + \delta_2 = \delta \). In particular, if \( m_n = cn^\epsilon \) for \( 0 \leq \epsilon < 1 \), then

\[
\mathbb{E}\left[ W_{\infty}\left(\mathbb{V}[\mathbb{X}_{n-m}^* \cup \mathbb{Y}_m, d_{n,Q}], \mathbb{V}[\mathbb{X}]\right) \right] \lesssim \left( \frac{\log n}{n^{1-\epsilon}} \right)^{1/b}.
\]

**Remark 3.5.** We make the following observations from Theorem 3.3.

(i) The term appearing in the r.h.s. of Eq. (8) is identical to the term appearing in the r.h.s. of Eq. (6) in Theorem 3.1. Therefore, the \( d_{n,Q} \)-weighted filtration and the \( d_{n,Q} \) sublevel filtration converge to the same population limit with identical convergence rates. They both differ from the minimax rate without outliers (Chazal et al., 2015b, Theorem 4) by a factor of \( n^{-\epsilon/b} \).

(ii) The uniform confidence band we obtain from Theorem 3.3 can, in principle, be computed for any confidence level \( \delta \in (0, 1) \). However, the restriction on \( \delta_1 \) makes the confidence band obtained using \( V[\mathbb{X}_n, d_{n,Q}] \) wider than that obtained using Proposition 3.1. This is, ultimately, the price we have to pay for choosing the computationally tractable \( d_{n,Q} \)-weighted filtration as the estimator as opposed to the \( d_{n,Q} \) sublevel filtration.

We conclude this section with the following result, which relates the sublevel filtration \( V[d_{n,Q}] \) to \( V[\mathbb{X}_n, d_{n,Q}] \).

**Proposition 3.1.** Given samples \( \mathbb{X}_n = \{X_1, X_2, \ldots, X_n\} \) and \( Q < n \), the filtrations \( V[d_{n,Q}] \) and \( V[\mathbb{X}_n, d_{n,Q}] \) are \((\eta, \xi)\)-interleaved, where

\[
\eta: t \mapsto 2^{p-1} \mathfrak{g}_n(t) + \sup_{x \in \mathbb{X}_{n-m}^*} d_{n,Q}(x), \quad \xi: t \mapsto 2^{p-1} \eta(t),
\]
and $p \geq 1$. Specifically, when $p = 1$,

$$W_\infty\left(\operatorname{Dgm}(\mathcal{V}[d_n,Q]), \operatorname{Dgm}(\mathcal{V}[\mathcal{X}_n, d_n,Q])\right) \leq \sup_{x \in \mathcal{X}_{n-m}} d_{n,Q}(x).$$

The above result characterizes the error incurred when using $\mathcal{V}[\mathcal{X}_n, d_n,Q]$ to approximate the sublevel filtration $\mathcal{V}[d_n,Q]$. In light of Remark 3.4 (ii), this error vanishes with increasing sample size. In contrast, the approximation error for the DTM-filtration is non-vanishing (Anai et al., 2019, Proposition 4.6).

### 3.4 Influence analysis

The statistical analysis in the previous sections establishes that, even in the presence of outliers, as the number of samples increases we can eventually mitigate the effect of the outliers. In this section, we provide a more precise characterization for the influence the outliers have on the resulting $d_{n,Q}$-weighted filtrations, in contrast to the non-robust counterpart—the $d_{n}$-weighted filtrations.

Given a probability measure $\mathbb{P} \in \mathcal{P}(\mathbb{X}, a, b)$, Vishwanath et al. (2020, Definition 4.1) characterized the influence an outlier at $x \in \mathbb{R}^d$ has on a persistence diagram $\operatorname{Dgm}(\mathcal{V}[f_{\mathbb{P}}])$—obtained using the sublevel sets of $f_{\mathbb{P}}$—using the persistence influence function

$$\Psi(f_{\mathbb{P}}; x) \doteq \lim_{\epsilon \to 0} W_\infty\left(\operatorname{Dgm}(\mathcal{V}[f_{\mathbb{P}}]), \operatorname{Dgm}(\mathcal{V}[f_{\mathbb{P}}_{\epsilon}])\right),$$

where $f_{\mathbb{P}}' = (1 - \epsilon)\mathbb{P} + \delta_x$ is the perturbation curve w.r.t. $x$ in the space of probability measures. The persistence influence is a generalization of the influence function in robust statistics (Hampel et al., 2011) to general metric spaces. The analysis in this section is similar in spirit to the analysis based on the persistence influence, but differs in two important aspects. First, the $d_{n,Q}$-weighted filtration is computed purely on the sample points—by partitioning the samples into $Q$ disjoint blocks—and, therefore, the notion of persistence influence is adapted to the samples, in contrast to Eq. (9), which is based on the data-generating distribution $\mathbb{P}$. Additionally, unlike the case of the persistence influence function—where the influence of outliers in the resulting persistence diagram is quantified in terms of the bottleneck distance—here we directly examine the influence the outlying point has on the resulting persistence diagram itself. This provides a more tractable interpretation for how outliers impact the resulting topological inference.

The discussion in the previous section focused on the weighted filtrations, which can be approximated using the weighted-Čech complex. Here, we will explicitly restrict ourselves to the case of the weighted Rips filtrations. Firstly, a majority of the computational applications of persistent homology are performed using the Rips complex, with several optimized implementations widely available, e.g. Ripser, Gudhi, GiottoTDA. Furthermore, since the Rips complex $\mathcal{R}^{t}[\mathbb{X}, f]$ is defined to be the flag complex associated with the 1–skeleton of $\mathcal{C}^t[\mathbb{X}, f]$, the weighted Rips persistence diagram is entirely characterized by its $0$– and $1$–simplices.

With this background, we now introduce the empirical persistence influence framework. Suppose we are given a collection of observations $\mathbb{X}_n$, which is sampled i.i.d. from a probability distribution $\mathbb{P}$ of interest. Let $\operatorname{Dgm}(\mathcal{V}[\mathbb{X}_n, f_n])$ be its weighted–Rips persistence diagram, where the weight function $f_n$ is constructed using the samples $\mathbb{X}_n$. Suppose $\mathbb{X}_n$ is contaminated with $m < \frac{n}{2}$ outliers to obtain the contaminated dataset $\mathbb{X}_{n+m}$. In particular, we may assume that the $m$–points are placed at an outlying location $x_0$, i.e.,

$$\mathbb{X}_{n+m} = \mathbb{X}_n \bigcup \{x_0\}.$$

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such that the factor $m$ and the location $x_0$ together control the relative influence the outliers have. This is similar to the role played by the factor $\epsilon$ in the perturbation curve associated with the persistence influence. Note that when $m = 0$, the influence of the outliers is non-existent in the dataset.

Let $Dgm(V[X, f_{n+m}])$ be the weighted–Rips persistence diagram constructed on $X_{n+m}$ using the weight function $f_{n+m}$. This gives rise to a collection of spurious topological features in the resulting persistence diagram. If $b_n(\{x_0\})$ is the birth time associated with a hypothetical topological feature with mass 0 at $x_0$ (i.e. $0\delta_{x_0}$) in $Dgm(V[X_n, f_n])$, and $b_{n+m}(\{x_0\})$ is the birth time associated with the observed topological feature associated with the $m$–points at $x_0$ (i.e. $m\delta_{x_0}$), then the empirical persistence influence of $x_0$ can be characterized by

$$\text{influence}(b; X_n, f_n, m, x_0) = \Delta b_{n,m}(\{x_0\}) = b_n(\{x_0\}) - b_{n+m}(\{x_0\}).$$

Indeed, when $b_n(\{x_0\}) - b_{n+m}(\{x_0\})$ is small, the resulting weighted-Rips persistence diagram $Dgm(V[X_{n+m}, f_{n+m}])$ is more robust, and vice versa.

In a similar vein as Vishwanath et al. (2020, Definition 4.1) we may also characterize the influence the outliers have on the persistence diagrams resulting from the sublevel filtrations as

$$\text{influence}(W_\infty; X_n, f_n, m, x_0) = W_\infty\left(Dgm(V[f_{m+n}]), Dgm(V[f_n])\right) \leq \|f_{n+m} - f_n\|_\infty. \quad (11)$$

The following result establishes that, under some mild conditions and with high probability, the $d_{n,Q}$–weighted Rips persistence diagrams are more robust than their non-robust counterpart.

**Theorem 3.4** (Influence analysis of $d_{n,Q}$–weighted filtrations). For $X_n$ observed i.i.d. from $P \in \mathcal{P}(X, a, b)$ and $x_0 \in \mathbb{R}^d$, let $X_{n+m}$ be given by

$$X_{n+m} = X_n \cup \left\{\bigcup_{j=1}^m \{x_0\}\right\}.$$ 

For $2m < Q < n + m$, let $d_{n+m}$ and $d_{n+m,Q}$ denote the distance and MoM distance function w.r.t. $X_{n+m}$, and let $\Delta b_{n,m}(\{x_0\})$ and $\Delta b_{n,m,Q}(\{x_0\})$ be as defined in Eq. (10) for $d_{n+m}$ and $d_{n+m,Q}$ respectively. Then

$$\Delta b_{n,m,Q}(\{x_0\}) \leq \Delta b_{n,m}(\{x_0\}) \quad \text{a.s.}$$

Furthermore, for $n_Q = (n + m)/Q$ and $c = \min\{a2^{-(1+b)}, a2^{-2b}\}$, if

$$\omega(x_0) = cd_x(x_0)^b > \log \frac{n_Q}{n_Q} + \frac{4(1 + b)^2 Q^2}{n_Q^2}, \quad (I)$$

then, for all $\delta \in (0, 1)$ satisfying

$$(1 + b)^2 Q^2 \leq \log(2/\delta) \leq \frac{n_Q\omega(x_0) - \log n_Q}{4Q}, \quad (II)$$

with probability greater than $1 - \delta$,

$$\|d_{n+m} - d_n\|_\infty - \|d_{n+m,Q} - d_n\|_\infty \geq \left(\frac{2 \log n_Q}{an_Q} + \frac{8 \log(2/\delta)}{an_Q}\right)^{1/b}.$$
Remark 3.6. The result from Theorem 3.4 may be interpreted as follows.

(i) The first part guarantees that the $d_{n,Q}$-weighted persistence diagram always has a smaller influence on the birth time in comparison to the non-robust counterpart. Since the Rips persistence diagram is entirely determined by the filtration values associated with the 0− and 1− simplices, this provides a partial picture for the influence $x_0$ has on the resulting persistence diagrams. Characterizing the influence on the 1−simplices is far more challenging owing to the combinatorial complexity in characterizing their lifetimes.

(ii) The second part compares the upper bounds on the empirical persistence influence from Eq. (11). When conditions (I) and (II) hold, then with high probability, persistence diagrams obtained using $d_{n,Q}$ are closer to the truth than those obtained using $d_n$. Therefore, the interplay between $n, m$ and $x_0$ is better understood by characterizing when conditions (I) and (II) hold.

(iii) For fixed $n$ observe that (I) is satisfied whenever $d_{X}(x_0)$ is sufficiently large, i.e., $x_0$ is sufficiently far away from the support. On the other hand, if $x_0$ is fixed, then (I) is satisfied when $\log n_Q/n_Q$ is sufficiently small, i.e., $n$ is sufficiently large. Together, this implies that for condition (I) to be satisfied, either (a) we need the outliers to be sufficiently well-separated from the support $X$ such that we are able to distinguish outliers $x_0$ from the inliers $X_n$, or (b) for outliers placed very close to the support $X$ we need sufficiently many inliers $n$ for us to be able to distinguish them from the outliers. On the other hand, note that if $n$ and $m$ are fixed, then the r.h.s. of (I) is directly proportional to $Q$. Although $Q$ can take any values between $2m < Q < (n + m)$, choosing a value of $Q$ much larger than $2m + 1$ will likely breach condition (I) for a fixed $x_0$. Equivalently, for a suboptimal choice of $Q$, we need the outliers to be sufficiently far away from the inliers in order to be able to distinguish them.

(iv) The l.h.s. of (II) is equivalent to the constraint that $\delta \leq e^{-(1+b)Q}$, which appears in Theorems 3.1 and 3.3. The r.h.s. of (II) specifies a lower-bound on the confidence level $\delta$. Condition (I) guarantees that the admissible values of $\delta \in (0, 1)$ satisfying (II) is nonempty. For fixed $m, Q$ and $x_0$, the r.h.s. of (II) is directly proportional to $n$, i.e., the lower bound vanishes as $n \to \infty$.

(v) When conditions (I) and (II) are satisfied, we have the following lower bound from the l.h.s. of (II):

$$\|d_{n+m} - d_n\|_\infty - \|d_{n+m,Q} - d_n\|_\infty \gtrsim \left( \frac{\log(n + m)/Q}{a(n + m)/Q} + \frac{Q^2}{(n + m)/Q} \right)^{1/b}. \tag{12}$$

In the regime when $n, m \to \infty$, and for the optimal choice of $Q$, i.e., $Q = km$ for $k > 2$, the r.h.s. of Eq. (12) is non-trivial when $m = \Omega(n^{1/3})$. Therefore, under conditions (I) and (II), when there are sufficiently many outliers, there is greater evidence to support the robustness of $d_{n,Q}$.

3.5 Auto-tuning the parameter $Q$

The result in Theorem 3.3 relies on the crucial assumption that the number of outliers $m^*$ is known a priori. While this assumption may hold in certain adversarial settings, in general, this information may be unavailable. In order to make Theorem 3.3 more useful in practical settings, we discuss two solutions for calibrating the parameter $Q$. The first procedure is based on Lepski’s method (Lepski, 1991), which is a powerful data-driven method for adaptive parameter selection. In this case, we also provide theoretical guarantees for the adaptively tuned estimator. The second procedure—which is based on some heuristic observations regarding the sample estimator $\mathbb{V}[X_n, d_{n,Q}]$—works well in practice, and may be used as a precursor to Lepski’s method.

When the number of outliers $m^*$ is known, choosing $Q^* = 2m^* + 1$ results in the rate of convergence in Theorem 3.3. However, without access to $m^*$, Lepski’s method provides a systematic procedure for selecting
a parameter $\hat{\theta}$ which provides the same error guarantees as $Q^*$ (Birgé, 2001). The procedure is as follows. Let $m_{\min}$ and $m_{\max}$ be two coarse bounds on (unknown) $m^*$ such that $m_{\min} \leq m^* \leq m_{\max}$. For a choice of $\theta > 1$, let $m(j) = \theta^j m_{\min}$ and define

$$J = \{ j \geq 1 : m_{\min} \leq m(j) < \theta m_{\max} \}.$$  

For $\mathbb{P} \in \mathcal{P}(X, a, b)$ and $X_n$ obtained under sampling condition (8), let $V_n(j) = \mathbb{V}[X_n, d_n, Q(j)]$ be the persistence module obtained using the MoM Dist-weighted filtration with $Q(j) = 2(m(j) + 1)$. For $\delta \in (0, 1)$ and $\delta_{\max} = \delta - e^{-(1+b)(2m_{\max}+1)}$, let $h(n, m, \delta)$ be defined as follows:

$$h(n, m, \delta) = 2 \left( \frac{2m + 1}{an} \right)^{1/b} \left( \frac{ne^{(1+b)(2m_{\max}+1)}}{m+1} \right)^{1/b} + \left( \frac{1}{an-m} \right)^{1/b} e^{4 \log(1/\delta_{\max})},$$

where for $z > 0$, $W_0(z)$ is the Lambert $W_0$ function given by the identity $W_0(z) e^{W_0(z)} = z$. With this background, let $j$ be the output of the following procedure:

$$j = \min \left\{ j \in J : W_\infty(V_n(j), V_n(j')) \leq 2h(n, m(j'), \delta) \quad \text{for all} \quad j' \in J, j' > j \right\}, \tag{13}$$

the resulting weighted persistence module $\hat{V}_n = V_n(j) = \mathbb{V}[X_n, d_n, Q(j)]$ is the Lepski estimator for $\mathbb{V}[X]$. The following result establishes that the adaptive selection of $Q$ results in an estimator with the same convergence guarantees as in Theorem 3.3.

**Theorem 3.5** (Adaptive $d_n, Q$-weighted filtration). Suppose $X_n$ is obtained under sampling condition (8) for $\mathbb{P} \in \mathcal{P}(X, a, b)$, and suppose $m_{\min}$ and $m_{\max}$ are known such that unknown number of outliers $m^* \in [m_{\min}, m_{\max}]$, and $m^* < n/2$. For a chosen $\theta > 1$ let $\hat{j}$ be the output of data-driven procedure in Eq. (13) and let $V_n = V_n(j)$. Then, for all $\delta \in (0, 1)$,

$$\mathbb{P} \left( W_\infty \left( \mathcal{D}g_\mathcal{M}(\hat{V}_n), \mathcal{D}g_\mathcal{M}(\mathbb{V}[X]) \right) \leq 3h(n, \theta m^*, \delta) \right) \geq 1 - \delta \log_\theta \left( \frac{\theta m_{\min}}{m_{\max}} \right).$$

**Remark 3.7.** We make the following useful observations from Theorem 3.5.

(i) We make the distinction that the output $\hat{V}_n$ of Lepski’s method does not necessarily correspond to the optimal choice $V^*_n$ if $m^*$ were known. Instead, Theorem 3.5 guarantees that error associated with $\hat{V}_n$ is of the same order (up to constants) as that of $V^*_n$.

(ii) While Lepski’s method guarantees optimal errors for the adaptive estimator without any knowledge of the true $m^*$; in practice, however, the empirical performance depends on several factors. Since the procedure in Theorem 3.5 is designed to match the guarantee of Theorem 3.3, the success of the procedure crucially depends on the tightness of the bound $f(n, m, Q, \delta_1, \delta_2)$ in Theorem 3.3. Furthermore, the implementation described in Eq. (13) requires knowledge of the parameters $a, b > 0$ arising from the $(a, b)$—standard condition. While the calibration of $a$ and $b$ in practice is more of an art and beyond the scope of the paper, we emphasize here that it is possible to construct a statistically consistent estimator of the true population quantity $\mathbb{V}[X]$ in a purely data-adaptive fashion, even in the presence of adversarial contamination.

(iii) Unlike a standard grid search, Lepski’s method adapts to the true noise level $m^*$ in an efficient manner. Given a reasonable estimate for $m_{\min}$ and $m_{\max}$, Lepski’s method has a computational cost of $O\left(\log^2 \left(\frac{m_{\max}}{m_{\min}}\right)\right)$. However, the choice of $\theta > 1$ must also be made judiciously, e.g., replacing $\theta$ with $\sqrt{\theta}$ for the procedure in Eq. (13) will require $\sim 4$ times more computational time.
(iv) In the worst case, when there are no reasonable estimates for $m_{\min}$ and $m_{\max}$, choosing $m_{\min} = 1$ and $m_{\max} = n/2$ requires $O(\log n)$ computational time. Notably, more than just the additional computational price, a suboptimal choice of $m_{\min}$ and $m_{\max}$ leads to poor performance. To see this, note that the term $h(n, m, \delta)$ is a lower bound for the term $\ell(n, m, Q, \delta_1, \delta_2)$ in Theorem 3.3 when $Q = 2m + 1$ and $\delta_1 = e^{-((1+\beta)/2m_{\max}+1)} \leq e^{-(1+\beta)/Q}$. Therefore, when the number of outliers grows with $n$ as $m^* = cn^q$ for $c > 0$ and $\epsilon \in [0, 1)$, a similar analysis to that in Theorem 3.1 and Theorem 3.3 yields that

$$\mathbb{E}\left[ W_\infty \left( \mathcal{D}_{gm}(\bar{V}_n), \mathcal{D}_{gm}(\mathbb{V}[\mathbb{X}]) \right) \right] \lesssim \left( \frac{\log n}{n/m_{\max}} \right)^{1/b}.$$ 

Therefore, if the bound $m_{\max}$ is not tight, i.e., $m_{\max} = Cn^\beta$ for $\epsilon < \beta$, then asymptotically, the output of Lepski’s method is not adaptive to the true noise $m^*$, and, instead, reflects the suboptimal choice of $m_{\max}$.

In a similar vein, Lepski’s method may be used to adaptively select the parameter $Q$ to obtain a statistically consistent sublevel set persistence module. The following result outlines a data-driven procedure to obtain $\bar{j} \in \mathcal{J}$ such that the resulting sublevel persistence module $\bar{V}_n = V_n(\bar{j}) = V[d_n, Q(\bar{j})]$ has the same convergence guarantee as Theorem 3.1.

Theorem 3.6 (Adaptive sublevel filtration). For $\mathbb{P} \in \mathcal{P}(\mathbb{X}, a, b)$, suppose $\mathbb{X}_n$ is obtained under sampling condition (S), and suppose $m_{\min}$ and $m_{\max}$ are known such that unknown number of outliers $m^* \in [m_{\min}, m_{\max}]$ and $m^* < n/2$. Let $\bar{V}_n(\bar{j}) = V[d_n, Q(\bar{j})]$ be the sublevel persistence module obtained using $d_n, Q(\bar{j})$ with $Q(\bar{j}) = 2m(\bar{j}) + 1$ for all $\bar{j} \in \mathcal{J}$. For a chosen $\theta > 1$, let $\bar{j}$ be the output of data-driven procedure,

$$\bar{j} = \min \left\{ j \in \mathcal{J} : W_\infty (V_n(j), V_n(j')) \leq 2p(n, m(j'), \delta) \quad \text{for all} \quad j' \in \mathcal{J}, j' > j \right\},$$

where

$$p(n, m, \delta) = \left( \frac{2m + 1}{an} \right) W_0 \left( \frac{n e^{(1+\delta)/2m + 1}}{2m + 1} \right)^{1/b}.$$

Then, for all $\delta \leq e^{-(1+\beta)/2m_{\max}+1}$ and $\bar{V}_n = V_n(\bar{j})$,

$$\mathbb{P} \left( W_\infty \left( \mathcal{D}_{gm}(\bar{V}_n), \mathcal{D}_{gm}(\mathbb{V}[\mathbb{X}]) \right) \leq 3h(n, \theta m^*, \delta) \right) \geq 1 - \delta \log_0 \left( \frac{\theta m_{\min}}{m_{\max}} \right).$$

The proof is identical to that of Theorem 3.5, and is, therefore, omitted. The success of Lepski’s method depends on the tightness of the probabilistic bounds, knowledge of the (nuisance) parameters (i.e. $a, b$) appearing in these bounds, and a prudent choice for $m_{\min}$ and $m_{\max}$. While the calibration of $a$ is beyond the scope of this paper, in $\mathbb{R}^d$ a conservative choice for $b$ would be the dimension $d$ of the ambient space. We refer the reader to (Chazal et al., 2015b, Section 4) for further details.

To address the last bottleneck in Lepski’s method, we describe a heuristic method to select the parameter $Q$, which may be used to obtain reasonable choices for $m_{\min}$ and $m_{\max}$. The method is based on the observation that the blocks $\{S_q : q \in [Q]\}$ may be resampled by shuffling the sample points $\mathbb{X}_n$ prior to partitioning it. The resulting estimator $\mathbb{V}[\mathbb{X}_n, d_n, Q]$ is an unbiased estimator of the same population quantity when $2m < Q < n$. Therefore, we may choose the smallest value of $Q$ for which the pairwise bottleneck distance over permutations
of the data is minimized. Specifically, suppose \( X_n^\sigma = \{ X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(n)} \} \) is a permutation of \( X_n \), then

\[
\hat{Q}_R = \arg\min_{Q \geq 1} \sum_{1 \leq i < j \leq N} W_\infty \left[ \bigvee_{i=1}^N |X_n^{\sigma_i}, d_{n,Q}|, \bigvee_{j=1}^N |X_n^{\sigma_j}, d_{n,Q}| \right],
\]

where, for a chosen number of replicates \( N, \sigma_i, \sigma_j \) are permutations of \([n]\) for each \( i, j \in [N] \). Furthermore, for \( \hat{m}_R = \lceil \hat{Q}_R/2 \rceil \) and for a constant \( C > 1 \), the bounds \( m_{\min} \) and \( m_{\max} \) may be taken to be \( C^{-1} \hat{m}_R \) and \( C \hat{m}_R \), respectively.

4 Experiments

In the following section, we supplement the theory through illustration of the performance of the robust filtrations \( \bigvee[d_{n,Q}] \) and \( \bigvee[X_n, d_{n,Q}] \) in synthetic experiments. The tools for data-adaptive construction of \( d_{n,Q} \)-weighted filtrations, in addition to the code for all experiments, are made publicly available in the \texttt{RobustTDA.jl} Julia package\(^4\). In all experiments, the persistence diagrams are computed using the \texttt{Ripserer.jl} backend (Čufar, 2020), and we set the parameter \( p = 1 \) for the weighted-filtrations.

4.1 Adaptive calibration of \( Q \)

For \( n = 500, K = 30 \) replicates and for each \( i \in [K] \), point clouds \( X_n^{(i)} \) are generated on a circle, and \( m^{(i)} \sim \text{Unif}([50, 150]) \) outliers added from a Matérn cluster process. This is illustrated in Figure 4 (a). Taking \( m_{\min} = 20, m_{\max} = 200 \) and \( \theta = 1.07 \), the adaptive estimate \( \hat{m}^{(i)} \) is computed using Lepski’s method, and \( \hat{m}_R^{(i)} \) is computed using the heuristic method described in Section 3.5 with \( N = 50 \). For a single replicate \( i \in [K] \), Figure 4 (b) plots \( \sum_{1 \leq i < j \leq N} W_\infty \left[ \bigvee_{i=1}^N |X_n^{\sigma_i}, d_{n,Q}|, \bigvee_{j=1}^N |X_n^{\sigma_j}, d_{n,Q}| \right] \) vs. \( Q \). In most cases, we have observed that the resampled bottleneck distance criterion stabilizes shortly before the optimal value of \( m \). Figure 4 (c) shows a boxplot for the relative errors \( \{ \hat{m}^{(i)} - m^{(i)} / m^{(i)} : i \in [K] \} \) and \( \{ \hat{m}_R^{(i)} - m^{(i)} / m^{(i)} : i \in [K] \} \) for Lepski’s method and the heuristic procedure, respectively. Lepski’s method is fairly robust to the choice of the hyperparameters, and, consistently selects \( \hat{m}^{(i)} \geq m^{(i)} \). In contrast, since the resampled bottleneck distance from the heuristic procedure often stabilizes before \( m^{(i)} \), we observe that \( \hat{m}_R^{(i)} < m^{(i)} \).

4.2 Comparison of \( \bigvee[d_{n,Q}] \) and \( \bigvee[X_n, d_{n,Q}] \)

The objective of this experiment is to illustrate that the \( d_{n,Q} \)-weighted filtration \( \bigvee[X_n, d_{n,Q}] \) reasonably approximates the sublevel filtration \( \bigvee[d_{n,Q}] \). For the same setup as 4.1, \( X_n \) comprises of \( n = 550 \) points obtained by sampling 500 points on a circle with additive Gaussian noise (\( \sigma = 0.01 \)) and \( m = 50 \) outliers added from a Matérn cluster process. For \( Q = \hat{Q} \) selected using Lepski’s method, Figure 5 (a) depicts the MoM Dist function \( d_{n,Q} \). Figure 5 (b) illustrates the scatter plot for \( X_n \) with the points colored by the weights \( d_{n,Q}(x_i) \) for each \( x_i \in X_n \). The shaded regions show the \( d_{n,Q} \)-weighted offsets \( V^t[X_n, d_{n,Q}] \) for \( t \in \{1.5, 1.75, 2, 2.25\} \) colored from white to blue. Figure 5 (c) depicts the sublevel persistence diagram \( \Delta gm(\bigvee[d_{n,Q}]) \) computed using cubical homology on a grid of resolution 0.5. As expected by the result of Proposition 3.1, the \( d_{n,Q} \)-weighted persistence diagram \( \Delta gm(\bigvee[X_n, d_{n,Q}]) \) in Figure 5 (d) captures the essential topological information in \( \Delta gm(\bigvee[d_{n,Q}]) \).

\(^4\)https://www.github.com/sidv23/RobustTDA.jl/
Figure 4: Comparison of Lepski’s method and the heuristic procedure for selecting the parameter $Q$.

Figure 5: Comparison of sublevel filtrations with the $d_{n,Q}$-weighted filtration.
4.3 High dimensional topological inference

In this experiment, we illustrate the advantage of using $d_{n,Q}$-weighted filtrations for high dimensional topological inference. Points are uniformly sampled in $\mathbb{R}^3$ from two interlocked circles. Using a random rotation matrix $Q \in SO(100)$, the points are transformed to an arbitrary configuration in $\mathbb{R}^{100}$. The samples $X_n \subset \mathbb{R}^{100}$ are obtained by replacing 12.5% of the points in $\mathbb{R}^{100}$ with outliers sampled from $\text{Uniform}([-0.2, 0.2])$. A scatterplot for $X_n$ projected to 3 arbitrary coordinates is shown in Figure 6 (a). Since the point cloud is embedded in $\mathbb{R}^{100}$, computing sublevel filtrations using cubical homology with the same resolution as earlier requires $(10/0.5)^{100} \approx 10^{131}$ simplices to be stored in memory. In contrast, computing the $d_{n,Q}$-weighted filtrations requires is less intensive. Figure 6 (b) shows the persistence diagram $\mathcal{D}gm(\hat{V}_n)$ obtained using $d_{n,Q}$-weighted filtrations, where the parameter $Q$ is adaptively selected using Lepski’s method. The two 1st order homological features underlying the interlocked circles are recovered. Figure 6 (c) illustrates the persistence diagram $\mathcal{D}gm(\mathcal{V}[X_n, \delta_{n,k}])$ obtained using DTM-weighted filtrations. Since the DTM parameter $k \in [1, n]$ results in a smoothing similar to the parameter $Q \in [1, n]$ for the MoM Dist, the parameter $k$ is set to the value of $Q$ obtained using Lepski’s method.

![Figure 6: Robust persistence diagrams for interlocked circles in $\mathbb{R}^{100}$ using $d_{n,Q}$ and $\delta_{n,k}$ weighted filtrations.](image)

4.4 Recovering the true signal under adversarial contamination

In this experiment, we illustrate how $\mathcal{V}[X_n, d_{n,Q}]$ can be used to recover the true topological features in the presence of adversarial contamination. In Figure 7 (a), we consider a $28 \times 28$ image for the digit “6” from the MNIST database (Deng, 2012). We consider the setting in which an adversary is allowed to manipulate 10% of the image by modifying the pixel intensities. Figure 7 (b) depicts the adversarially contaminated version of the image by transforming the “6” to an “8”.

For each pixel $p$ with pixel intensity $\iota(p)$, we convert the image to a point cloud $X_n \subset \mathbb{R}^2$ by sampling $10 \times \iota(p)$ points uniformly from the region enclosed by the pixel. Figures 7(d, e) illustrate the point clouds obtained from the true and contaminated images with $n - m \approx 1100$ and $n \approx 1300$, respectively. The persistence diagrams constructed using the distance function $d_n$ for the two point clouds are reported in Figures 7(g, h). The persistence diagram in Figure 7 (h) indicates the presence of the additional loop introduced by the adversary. To account for the adversarial contamination, we compute the MoM Dist function $d_{n,Q}$ with the parameter $Q$ selected using the contamination budget, i.e., $Q = 1 + 2(1100 \times 10\%) = 221$. Figure 7 (f) shows the adversarially contaminated point cloud with each point $x_i \in X_n$ colored by the value of $d_{n,Q}(x_i)$. The resulting $d_{n,Q}$-weighted persistence diagram $\mathcal{D}gm(\mathcal{V}[X_n, d_{n,Q}])$ is reported in Figure 7 (f). We note that
recover the prominent features of Figure 7 (g) up to a rescaling. Additionally, for each pixel \( p \) we compute a rescaled version of \( d_{n,Q} \), given by

\[
f_{n,Q}(p) = \frac{\max_x d_{n,Q}(x) - d_{n,Q}(p)}{\max_x d_{n,Q}(x)},
\]

as a proxy for the pixel intensity obtained using \( d_{n,Q} \). In Figure 7 (c), we plot the level sets \( \{p : f_{n,Q} = t\} \) on the original image for \( t \geq 0.8 \).

4.5 Empirical influence analysis

In this experiment, we examine the influence of outliers on \( d_{n,Q} \)-weighted filtrations. For \( n = 500 \), points \( X_n \) are sampled uniformly from a circle. We compute the unweighted persistence diagram \( D_n = \mathcal{D}gm(\mathbb{V}[X_n]) \).

In a small neighborhood around the center of the circle, outliers \( Y_m \) are sampled uniformly from \([-0.1, 0.1]^2\). For the composite sample \( X_n \cup Y_m \) and a fixed value of \( Q = 100 \) & \( k = 50 \), we compute the MoM Dist weighted persistence diagram \( D_{n+m,k}^{\text{MoM}} = \mathcal{D}gm(\mathbb{V}[X_n \cup Y_m, d_{n+m,k}]) \), the DTM weighted persistence diagram \( D_{n+m,k}^{\text{DTM}} = \mathcal{D}gm(\mathbb{V}[X_n \cup Y_m, \delta_{n+m,k}]) \), and the RKDE weighted persistence diagram \( D_{n+m,p,\sigma}^{\text{RKDE}} \) from the RKDE \( f_{p,\sigma}^{n+m} \) using the Hampel loss function \( \rho \) and a Gaussian kernel \( K_\sigma \). Since the RKDE \( f_{p,\sigma}^{n+m} \) does not behave like a distance function, we convert \( f_{p,\sigma}^{n+m} \) to a distance-like function \( d_{n+m,p,\sigma} \) using a similar approach as Phillips et al. (2015) to obtain

\[
d_{n+m,p,\sigma}(x) = \| K_\sigma (\cdot, x) - f_{p,\sigma}^{n+m} \|_{L_1} = \sqrt{\sum_{1 \leq i,j \leq n+m} w_i w_j K_\sigma (X_i, X_j) + K_\sigma (x, x) - 2 f_{p,\sigma}^{n+m}(x)}.
\]

The RKDE-weighted persistence diagram \( D_{n+m,p,\sigma}^{\text{RKDE}} = \mathcal{D}gm(\mathbb{V}[X_n \cup Y_m, d_{n+m,p,\sigma}]) \) is then computed using the \( d_{n+m,p,\sigma} \)-weighted filtration on the composite sample. The bandwidth of the kernel and the parameters for the Hampel loss function are selected using the same approach as in Vishwanath et al. (2020). For each diagram, we compute the birth time \( b(\{x_0\}) \) for the first outlier \( x_0 \in Y_m \), and the bottleneck influence \( \mathcal{W}_\infty (D_{n+m}, D_n) \), as described in Section 3.4. We generate 10 such samples for each value of \( m \), and report the average in Figure 8.

From Figure 8 (a), we note that \( D_{n+m,Q}^{\text{MoM}} \) and \( D_{n+m,k}^{\text{DTM}} \) show similar behavior, although the outliers consistently appear earlier in the DTM persistence diagram \( D_{n+m,k}^{\text{DTM}} \). Since the birth time \( b(\{x_0\}) \) alone does not fully characterize the impact an outlier has on inferring the topological feature underlying the circle, we also compute the maximum persistence for the first order persistence diagram in Figure 8 (b). We point out that the behavior of \( b(\{x_0\}) \) with respect to \( m \) largely reflects the influence an outlier has on the relevant topological signal. Furthermore, for \( D_{n+m,Q}^{\text{MoM}} \), we observe the sharp transition which occurs when \( m = 50 \) and \( m = 80 \), which is due to the fact that the theoretical guarantees for \( d_{n,Q} \) from Theorem 3.4 are valid only when \( 2m < Q = 100 \). Similarly, from Theorem 3.3, the outliers are guaranteed to have little influence on \( D_{n+m,Q}^{\text{MoM}} \) whenever \( m \leq 50 \), as seen in Figure 8 (c).

On the other hand, while the RKDE remains resilient to uniform outliers, we note that \( D_{n+m,p,\sigma}^{\text{RKDE}} \) is significantly impacted by the outliers placed at a single point in center of the circle. This is evidenced by the sharp transitions for \( D_{n+m,p,\sigma}^{\text{RKDE}} \) in Figures 8 (b, c). However, unlike \( d_{n+m,Q} \) and \( \delta_{n+m,k} \), by construction \( \| d_{n+m,p,\sigma} \|_\infty \leq \sup_{x} \sqrt{2} K_\sigma (x, x) < \infty \). Therefore, the impact the outliers have on \( D_{n+m,p,\sigma}^{\text{RKDE}} \) are bounded; and despite being more sensitive to the outliers, the resulting influence the outliers have on \( D_{n+m,p,\sigma}^{\text{RKDE}} \) in Figures 8 (a, b, c) is bounded.
Figure 7: Recovering the topological information underlying the signal in the presence of adversarial contamination.
In this paper, we introduce a methodology for constructing filtrations which are computationally efficient, provably robust, and statistically consistent even in the presence of outliers. To our knowledge, our results are the first of this type.

To elaborate, we introduced MoM Dist, $d_{n,Q}$, as a computationally efficient and outlier-robust variant of the distance function based on the median-of-means principle, and established some of its theoretical properties. In particular, when the samples contain outliers in the adversarial contamination setting, we (i) showed that the $d_{n,Q}$-weighted filtrations are statistically consistent estimators of the true (uncontaminated) population counterpart, (ii) characterized its convergence rate in the bottleneck metric, and (iii) provided uniform confidence bands in the space of persistence diagrams. Furthermore, we used an empirical influence analysis framework to quantify the robustness of the $d_{n,Q}$-filtrations, and provide a framework for selecting the parameter $Q$.

Topological inference in the presence of outliers is a topic which has received considerable attention in recent years, and with good reason. We would like to highlight that the objective in this paper has been to develop a framework of topological inference in which the population target is the persistence diagram $\mathcal{Dgm}(\mathcal{V}[X])$. 

5 Conclusion & Discussion

Figure 8: Influence analysis for $d_{n,Q}$-weighted filtrations vis-à-vis DTM-based filtrations and unweighted filtrations.

(a) influence($b; \mathcal{X}_n, f_n, m, x_0$) 
(b) max Persistence for the first order diagram 
(c) influence($W_\infty; \mathcal{X}_n, f_n, m, x_0$)
Therefore, the proposed methodology disregards, to a large extent, the distribution of mass on the support. As a future direction, we would like to explore a framework of inference which incorporates information from, both, the geometry of the underlying space and the structure of the probability measure generating the data. As noted in Anai et al. (2019, Section 5), their results follow only from a few simple properties of the distance-to-measure. We build off their foundation to provide some useful generalizations which we hope will be useful in the analysis of other estimators using this framework.

6 Proofs

In this section, we present the proofs for the results in Section 3.

6.1 Proof for Lemma 3.1

We begin by noting that for each \( q \in [Q] \), the distance function \( d_{n,q} \) associated with the block \( S_q \) is 1–Lipschitz (Boissonnat et al., 2018, Chapter 9.1). Thus, for each \( q \in [Q] \) and for all \( x, y \in \mathbb{R}^d \) we have that

\[
0 \leq d_{n,q}(x) \leq d_{n,q}(y) + \|x - y\|,
\]

and, therefore, it follows that

\[
\text{median}\{d_{n,q}(x) : q \in [Q]\} \leq \text{median}\{d_{n,q}(y) : q \in [Q]\} + \|x - y\|.
\]

As a result, we obtain that \( d_{n,Q}(x) \leq d_{n,Q}(y) + \|x - y\| \). Exchanging \( x \) and \( y \) in the steps above yields the desired result.

6.2 Proof for Lemma 3.2

For \( t > 0 \), define two events

\[
E_1 = \{\|T_Q(P_n) - T(P)\|_\infty \leq t\}, \quad \text{and} \quad E_2 = \left\{\#\{q \in [Q] : \|T(P_q) - T(P)\|_\infty > t\} \leq \frac{Q}{2}\right\}.
\]

First, we show that \( E_2 \subseteq E_1 \). To this end for any \( \omega \in E_2 \), we have

\[
\omega \in E_2 \implies \omega \in \left\{\#\{q \in [Q] : \|T(P_q) - T(P)\|_\infty > t\} \leq \frac{Q}{2}\right\}
\]

\[
\implies \omega \in \left\{\#\{q \in [Q] : \|T(P_q) - T(P)\|_\infty \leq t\} > Q - \frac{Q}{2}\right\}
\]

\[
\implies \omega \in \left\{\#\{q \in [Q] : \forall x \in \mathbb{R}^d, T(P)(x) - t \leq T(P_q)(x) \leq T(P)(x) + t\} > \frac{Q}{2}\right\}
\]

\[
\implies \omega \in \left\{\forall x \in \mathbb{R}^d, T(P)(x) - t \leq \text{median}\{T(P_q)(x) : q \in [Q]\} \leq T(P)(x) + t\right\}
\]

\[
\implies \omega \in \left\{\|T_Q(P_n) - T(P)\|_\infty \leq t\right\}
\]

\[
\implies \omega \in E_1.
\]

Therefore, we have \( E_2 \subseteq E_1 \). Next, note that \( E_2 \) can be written as

\[
E_2 = \left\{\sum_{q=1}^{Q} \xi_q(t; n, Q) \leq \frac{Q}{2}\right\},
\]
where, for each $q \in [Q]$,

$$\xi_q(t; n, Q) \doteq 1\left(\|T(P_q) - T(P)\|_{\infty} > t\right).$$

Since $0 \leq \xi_q(t; n, Q) \leq 1$ a.s., we have that

$$\sum_{q=1}^Q \xi_q(t; n, Q) = \sum_{q \in A} \xi_q(t; n, Q) + \sum_{q \in A^c} \xi_q(t; n, Q) \leq \sum_{q \in A} \xi_q(t; n, Q) + |A^c| \leq \sum_{q \in A} \xi_q(t; n, Q) + m.$$

As a result, we can further bound the probability of $E_2$ from below as

$$\mathbb{P}(E_2) \geq \mathbb{P}\left(\sum_{q \in A} \xi_q(t; n, Q) \leq \frac{Q}{2} - m\right) \tag{14}$$

Combining Eq. (14) with the fact that $E_2 \subseteq E_1$, we obtain

$$\mathbb{P}\left(\|T(Q(P_n) - T(P))\|_{\infty} > t\right) = \mathbb{P}(E_1^c) \leq \mathbb{P}(E_2^c) \leq \mathbb{P}\left(\sum_{q \in A} \xi_q(t; n, Q) > \frac{Q}{2} - m\right),$$

which gives us the desired result.

### 6.3 Proof of Theorem 3.1

First, we note from the stability of persistence diagrams that,

$$\mathbb{P}\left\{ \mathbf{w}_{\infty}\left(\mathcal{D}gm(d_{n,Q}), \mathcal{D}gm(d_{X})\right) > t \right\} \leq \mathbb{P}\{\|d_{n,Q} - d_{X}\|_{\infty} > t\}. \tag{15}$$

Therefore, it suffices to control the probability of the event $\{\|d_{n,Q} - d_{X}\|_{\infty} > t\}$. To this end, let $A = \{q \in [Q] : S_q \cap \mathbb{Y}_m = \emptyset\}$ be the blocks which contain no outliers. From the assumption on $Q$, i.e., $2m < Q < n$, it follows that, and $|A| > Q/2$. For $q \in [Q]$, let $\xi_q(t; n, Q)$ be given by

$$\xi_q(t; n, Q) = 1\left(\|d_{n,q} - d_{X}\|_{\infty} > t\right).$$

On application of Lemma 3.2 to the estimator $d_{n,Q}$, it follows that

$$\mathbb{P}\left\{\|d_{n,Q} - d_{X}\|_{\infty} > t\right\} \leq \mathbb{P}\left(\sum_{q \in A} \xi_q(t; n, Q) > \frac{Q}{2} - m\right). \tag{16}$$

Since $S_q \subseteq \mathbb{X}_{n,m}^*$ for all $q \in A$, it follows that $\{\xi_q(t; n, Q) : q \in A\}$ are i.i.d. Bernoulli($p(t; n, Q)$) random variables, where

$$p(t; n, Q) = \mathbb{E}(\xi_q(t; n, Q)) = \mathbb{P}\left(\|d_{n,q} - d_{X}\|_{\infty} > t\right).$$

For the remainder of the proof we need two key ingredients: (i) we need an upper bound for $\mathbb{E}(\xi_q(t; n, Q))$, and (ii) we need a tight bound for the binomial tail probability in Eq. (16).
**Bound for** \( p(t; n, Q) \). From Chazal et al. (2015b, Theorem 2), under the \((a, b)\)–standard condition it follows that

\[
p(t; n, Q) \leq \frac{2^b}{at^b} \exp \left( -\frac{n}{Q} at^b \right) = \exp \left( -\frac{n}{Q} at^b - \log (at^b) + b \log 2 \right).
\]  

(17)

**Binomial tail probability bound.** For \( 0 < \epsilon < 1 \), using the Chernoff-Hoeffding bound from Lemma B.2 yields,

\[
P \left( \frac{1}{|A|} \sum_{q \in A} \xi_q(t; n, Q) > \epsilon \right) \leq \exp \left( |A| \left( \frac{2}{e} + \epsilon \log p(t; n, Q) \right) \right).
\]

Using the bound for \( p(t; n, Q) \) from Eq. (17), we obtain

\[
P \left( \frac{1}{|A|} \sum_{q \in A} \xi_q(t; n, Q) > \epsilon \right) \leq \exp \left( |A| \left( \frac{2}{e} + \epsilon \log \left( 2 + \epsilon n \log 2 \right) \right) \right)
\]

\[
\leq \exp \left( |A| \left( 1 + \epsilon \log 2 + \epsilon n \log (at^b) \right) \right)
\]

\[
\leq \exp \left( |A| \left( 1 + \epsilon \Omega(t, n/Q) \right) \right),
\]

where, in the last line we use \( \Omega(t, n/Q) = (n/Q)at^b + \log (at^b) \) for brevity. When \( t \) satisfies the condition that

\[
\Omega(t, n/Q) \geq 2(1 + be) \frac{\epsilon}{e},
\]

then it implies that

\[
1 + be - \epsilon \Omega(t, n/Q) \leq -\frac{\epsilon}{2} \Omega(t, n/Q),
\]

and we get

\[
P \left( \frac{1}{|A|} \sum_{q \in A} \xi_q(t; n, Q) > \epsilon \right) \leq \exp \left( -\frac{|A|\epsilon}{2} \Omega(t, n/Q) \right).
\]

By setting \( \delta \) equal to the r.h.s. of the inequality above, we obtain

\[
\Omega(t, n/Q) = \frac{2 \log(1/\delta)}{|A|\epsilon}.
\]

(19)

When \( \delta \leq e^{-(1+b)Q} \), using the fact that \( Q > |A| \) and \( 0 < \epsilon < 1 \), it follows that

\[
\Omega(t, n/Q) = \frac{2 \log(1/\delta)}{|A|\epsilon} \geq \frac{2(1 + b)Q}{|A|\epsilon} \geq \frac{2(1 + be)}{\epsilon},
\]
and, therefore, the condition in Eq. (18) is satisfied. Consequently, for \( \delta \leq e^{-(1+b)Q} \), on rearranging the terms in Eq. (19) we obtain

\[
P \left( \sum_{q \in A} \xi_q(t; n, Q) > \frac{2 \log(1/\delta)}{\Omega(t, n/Q)} \right) \leq \delta. \tag{20}
\]

Comparing Eq. (16) with Eq. (20) we conclude that

\[
P \left( \sum_{q \in A} \xi_q(t; n, Q) > \frac{Q - 2m}{2} \right) = P \left( \sum_{q \in A} \xi_q(t; n, Q) > \frac{2 \log(1/\delta)}{\Omega(t, n/Q)} \right) \leq \delta,
\]

by setting

\[
\frac{2 \log(1/\delta)}{\Omega(t, n/Q)} = \frac{Q - 2m}{2} \iff \Omega(t, n/Q) = \frac{4 \log(1/\delta)}{Q - 2m}.
\]

Since \( \Omega(t, n/Q) = \frac{n}{Q} at^b + \log(at^b) \), this is equivalent to

\[
\exp \left( \frac{n}{Q} at^b \right) n \frac{at^b}{Q} = n Q \exp \left( \frac{4 \log(1/\delta)}{Q - 2m} \right).
\]

Moreover, using the fact that the Lambert \( W_0 \) function is given by the identity \( W_0(x)e^{W_0(x)} = x \) (Hoorfar and Hassani, 2008), we obtain that

\[
t = \left( \frac{Q}{an} W_0 \left( \frac{n}{Q} \exp \left\{ \frac{4 \log(1/\delta)}{Q - 2m} \right\} \right) \right)^{1/b}. \tag{21}
\]

Therefore, from Eq. (15) and (16), for \( t \) satisfying Eq. (21) and for all \( \tau \geq t \) we have that

\[
P \left\{ \omega_\infty \left( \mathcal{D} g_{\mu}(d_{n,Q}), \mathcal{D} g_{\mu}(\bar{d}_X) \right) > \tau \right\} \leq P \left\{ \omega_\infty \left( \mathcal{D} g_{\mu}(d_{n,Q}), \mathcal{D} g_{\mu}(\bar{d}_X) \right) > t \right\} \leq \delta. \tag{22}
\]

Since \( \delta \leq e^{-(1+b)Q} \), observe that

\[
\frac{4 \log(1/\delta)}{Q - 2m} \geq \frac{4(1+b)Q}{Q - 2m} \geq 4(1+b) > 1.
\]

Furthermore, using the fact that \( W_0(z) \leq \log(z) \) for \( z > e \) (Hoorfar and Hassani, 2008, Eq. 1.1), we may take \( \tau \) to be

\[
t = \left( \frac{Q}{an} W_0 \left( \frac{n}{Q} \exp \left\{ \frac{4 \log(1/\delta)}{Q - 2m} \right\} \right) \right)^{1/b} \leq \left( \frac{Q}{an} \log \left( \frac{n}{Q} \exp \left\{ \frac{4 \log(1/\delta)}{Q - 2m} \right\} \right) \right)^{1/b}
\]

\[
= \left( \frac{Q \log(n/Q)}{an} + \frac{4Q \log(1/\delta)}{a(Q - 2m)n} \right)^{1/b} \tau.
\]

Plugging this into Eq. (22), we obtain the desired result.
For the second claim in the theorem, by inverting the relationship between \( t \) and \( \delta \) in Eq. (21) and using the fact that \( W_0(z) \) is an increasing function for \( z > 0 \), observe that the constraint on \( \delta \) equivalently specifies a constraint on \( t \), i.e.,

\[
\delta \leq e^{-(1+b)Q} \iff t \geq \left( \frac{Q}{\alpha n} W_0 \left( \frac{n}{Q} \exp \left\{ \frac{4(1+b)Q}{Q - 2m} \right\} \right) \right)^{1/b}.
\]

A sufficient condition for this to hold is that

\[
t \geq t(n, Q) = \left( \frac{Q \log(n/Q)}{\alpha n} + \frac{4(1+b)Q^2}{a(Q - 2m)n} \right)^{1/b}.
\]

Therefore, from Eq. (22) we have that for all \( t \geq t(n, Q) \)

\[
\mathbb{P}\left\{ W_\infty(\mathcal{D}_m(d_{n,Q}), \mathcal{D}_m(d_X)) > t \right\} \leq \exp \left( -\left( \frac{Q - 2m}{4} \right) \Omega(t, n/Q) \right).
\]

By taking \( \mathbb{P}\left\{ W_\infty(\mathcal{D}_m(d_{n,Q}), \mathcal{D}_m(d_X)) > t \right\} \) to be its maximum value of 1 in the interval \([0, t(n, Q)]\) we have

\[
\mathbb{E}\left[ W_\infty(\mathcal{D}_m(d_{n,Q}), \mathcal{D}_m(d_X)) \right] = \int_0^\infty \mathbb{P}\left\{ W_\infty(\mathcal{D}_m(d_{n,Q}), \mathcal{D}_m(d_X)) > t \right\} dt
\]

\[
\leq t(n, Q) + \int_{t(n, Q)}^\infty \exp \left( -\left( \frac{Q - 2m}{4} \right) \Omega(t, n/Q) \right) dt.
\]

By taking \( w = \Omega(t, n/Q) \) and setting \( r_n = 4(1+b)Q/(Q - 2m) \), we further obtain

\[
\mathbb{E}\left[ W_\infty(\mathcal{D}_m(d_{n,Q}), \mathcal{D}_m(d_X)) \right]
\]

\[
\lesssim t(n, Q) + \left( \frac{Q}{n} \right)^{1/b} \int_{r_n}^{\infty} \frac{e^{-w/4} W_0 \left( \frac{n}{Q} e^w \right)^{1/b}}{w + 1} dw
\]

\[
\lesssim (ii) t(n, Q) + \left( \frac{\log(n/Q)}{n/Q} \right)^{1/b} \int_{r_n}^{\infty} \frac{e^{-w/4}}{w + 1} dw + \left( \frac{Q}{n} \right)^{1/b} \int_{r_n}^{\infty} \frac{e^{-w/4} w^{1/b}}{w + 1} dw,
\]

where (ii) follows from the fact that \( W_0(z) \leq \log(z) \) for \( z > e \) together with, either, an application of Lemma B.1 (iii) when \( b \geq 1 \), or Lemma B.1 (i) with the additional factor \( 2^{1/b-1} \) being absorbed in the symbol \( \lesssim \) when \( b < 1 \). The term \( a \) can be bounded above using the incomplete \( \Gamma \) function as,

\[ a = \int_{r_n}^{\infty} \frac{e^{-w/4}}{w + 1} dw = e^{1/4} \int_{(r_n - 1)/4}^{\infty} v^{-1} e^{-v} dv = e^{1/4} \Gamma \left( 0, (r_n - 1)/4 \right) < \infty. \]
Similarly, using the fact that \(w + 1 > 1\), the term (6) may be bounded above as,
\[
\mathbb{B} = \int_{r_n}^{\infty} e^{-w/4w_1/b} dw \leq \int_{r_n}^{\infty} e^{-w/4w_1/b} dw \leq \frac{\Gamma(1 + b^{-1})}{4^{1+1/b}} \int_{r_n}^{\infty} \pi(v)dv \leq \frac{\Gamma(1 + b^{-1})}{4^{1+1/b}} < \infty,
\]
where \(\pi\) is the probability density function of the distribution \(\Gamma(1 + b^{-1}, 1/4)\). Therefore, the inequality in Eq. (23) becomes
\[
\mathbb{E}[W_{\infty}(\mathcal{D}_{\text{gm}}(d_{n,Q}), \mathcal{D}_{\text{gm}}(d_{x}))] \leq \frac{\log(n/Q)}{n/Q} + \frac{Q}{n} + \left(\frac{Q}{n}\right)^{1/b}.
\]
When the number of outliers grows with \(n\) as \(m_n = cn^\epsilon\) where \(0 \leq \epsilon < 1\), let the number of blocks be \(Q_n = 3cn^{\beta}\), where \(\epsilon \leq \beta < 1\). Therefore,
\[
\mathbb{E}[W_{\infty}(\mathcal{D}_{\text{gm}}(d_{n,Q}), \mathcal{D}_{\text{gm}}(d_{x}))] \leq \inf_{\epsilon \leq \beta < 1} \left(\frac{\log(n/Q)}{n/Q} + \frac{n^{2\beta}}{3n^{\beta} - 2n^{\epsilon}}\right)^{1/b} + \left(\frac{n^{\beta}}{n}\right)^{1/b} \leq \left(\frac{\log n}{n^{1-\epsilon}}\right)^{1/b},
\]
which gives us the desired result.

### 6.4 Proof of Lemma 3.3

For simplicity, let \(f = d_{n,Q}\) denote the MoM Dist function. By definition, \(V[f]\) and \(V[\mathbb{R}^d, f]\) are \((\text{id}, \alpha)-\)interleaved if the following relationship holds
\[
V^t[f] \subseteq V^t[\mathbb{R}^d, f] \subseteq V^{\alpha(t)}[f].
\]
The first inclusion is straightforward since
\[
V^t[f] \subseteq \bigcup_{x \in V^t[f]} B_f(x, t) = \bigcup_{x \in \mathbb{R}^d} B_f(x, t) = V^t[\mathbb{R}^d, f].
\]
For the second inclusion, suppose \(x \in V^t[\mathbb{R}^d, f]\), i.e., there exists \(y \in \mathbb{R}^d\) such that \(\|x - y\| \leq r_{f,y}(t)\). It suffices to show that \(x \in V^{\alpha(t)}[f]\). To this end, note that since \(d_{n,Q}\) is \(1\)–Lipschitz by Lemma 3.1 it follows that
\[
f(x) \leq f(y) + \|x - y\| \\
\leq f(y) + r_{f,y}(t) \\
= f(y) + (t^p - f(y)^p)^{\frac{1}{p}} \\
\overset{(i)}{=} \leq 2^{\frac{p-1}{p}} (f(y)^p + (t^p - f(y)^p))^{\frac{1}{p}} = 2^{\frac{p-1}{p}} t,
\]
where (i) follows from an application of Lemma B.1 (iii). Since \(f(x) \leq 2^{\frac{p-1}{p}} t = \alpha(t)\), it implies that \(x \in V^{\alpha(t)}\) and the result follows. When \(p = 1\), note that \(\alpha(t) = t\), and therefore \(V[f] = V[\mathbb{R}^d, f]\).
6.5 Proof of Lemma 3.5

Since $X \subseteq \mathbb{X}$, the inclusion $V^t_\rho[X, f] \subseteq V^t_\rho[Y, f]$ holds trivially. For the next part, let $V^t = V^t_\rho[X, f]$ and $U^t = V^t_\rho[Y, f]$ denote the respective $f$–weighted filtrations, so as to avoid the notational overload. In order to show the second inclusion, i.e., $U^t \subseteq V^\alpha(t)$, consider $z \in U^t$. Then, there exists $y \in \mathbb{Y}$ such that $z \in B_{f, \rho}(y, t)$. If $y \in X \subset Y$, then it immediately follows that $z \in V^t \subseteq V^\alpha(t)$. In what remains, for $y \in \mathbb{Y} \setminus X$, it is sufficient to show that there exists $x \in X$ such that $z \in B_{f, \rho}(x, \alpha(t))$.

To this end, let $x^*_y = \arg\inf_{x \in X} \rho(x, y)$ be the projection of $y$ onto $X$ via $\rho$. Then two following cases arise: (I) $\rho(x^*_y, z) \leq \rho(x^*_y, y)$, and (II) $\rho(x^*_y, z) \geq \rho(x^*_y, y)$ (see Figure 9).

**Case I.** The distance between $x^*_y$ and $z$ will satisfy

$$\rho(x^*_y, z) \leq \rho(x^*_y, y) \leq f(y) + a \leq (t^p - \rho(y, z)^p)^{\frac{1}{p}} + a \leq t + a$$

where (i) follows from the assumption on $f$, and (ii) follows from the fact that if $z \in B_{f, \rho}(y, t)$, then $\rho(y, z) \leq r_{f, \rho}(t) = (t^p - f(y)^p)^{1/p}$. Furthermore, from Lemma B.1 (vi) we obtain

$$\rho(x^*_y, z) \leq (t + a + \sup_{x \in X} f(x))^p - f(x^*_y)^p)^{\frac{1}{p}} \leq (t + \sup_{x \in X} f(x))^p - f(x^*_y)^p)^{\frac{1}{p}} \leq (2^{1-\frac{1}{p}} t + a + \sup_{x \in X} f(x))^p - f(x^*_y)^p)^{\frac{1}{p}} = (\alpha(t)^p - f(x^*_y)^p)^{\frac{1}{p}} = r_{f, x^*_y}(\alpha(t)),$$

where the last inequality holds because $2^{1-\frac{1}{p}} \geq 1$. The last line implies that $z \in B_{f, \rho}(x^*_y, \alpha(t)) \subseteq V^\alpha(t)$.

**Case II.** For $r = \rho(x^*_y, y)$ let $y'$ be the projection of $z$ onto $\partial B(x^*_y, r)$, i.e.,

$$y' = \arg\inf_{x' \in \partial B(x^*_y, r)} \rho(x', z).$$

Figure 9: Illustration of Case I (Left) and Case II (Right).
The point \( y' \) satisfies the following three properties: (PI) \( \rho(x^*_y, y') = \rho(x^*_y, y) \), since \( y' \in \partial B_{x^*_y}(x^*_y, r); \) (PII) \( \rho(x^*_y, y') \leq \rho(z, y) \) by definition of \( y' \); and (PIII) \( \rho(x^*_y, y') + \rho(y', z) \geq \rho(x^*_y, z) \) from the triangle inequality.

Since \( z \in B_{x^*_y}(y, t) \), when \( \rho(x^*_y, y) \leq a \) we may use the triangle inequality to obtain
\[
\rho(x^*_y, z) \leq \rho(x^*_y, y) + \rho(z, y) \leq a + (t^p - f(y)^p)^\frac{1}{p} \leq a + t \leq a + 2^{1 - \frac{1}{p}} t. \tag{24}
\]

Alternatively, when \( \rho(x^*_y, y) > a \) we obtain the following inequality,
\[
t^p \geq \rho(y, z)^p + f(y)^p
\]
\[(iii) \geq \rho(z, y)^p + (\rho(x^*_y, y) - a)^p
\]
\[(iv) = \rho(z, y)^p + (\rho(x^*_y, y') - a)^p
\]
\[(v) \geq \rho(z, y')^p + (\rho(x^*_y, y') - a)^p
\]
\[(vi) \geq \left( \rho(x^*_y, z) - \rho(x^*_y, y') \right)^p + \left( \rho(x^*_y, y') - a \right)^p
\]
\[(vii) \geq 2^{1 - p} \left( \rho(x^*_y, z) - a \right)^p, \tag{25}
\]
where (iii) holds from the assumption on \( f \), (iv–vi) follow from (PI–PIII) respectively, and (vii) uses Lemma B.1 (i). Rearranging the terms of Eq. (25) we get \( \rho(x^*_y, z) \leq a + 2^{1 - \frac{1}{p}} t \). Therefore, from Eq. (24) and Eq. (25), in case (II) we have that
\[
\rho(x^*_y, z) \leq 2^{1 - \frac{1}{p}} t + a
\]
\[(viii) \leq \left( 2^{1 - \frac{1}{p}} t + a + \sup_{x \in \mathbb{X}} f(x) \right)^p - f(x^*_y)^p \right)^\frac{1}{p} \]
\[= \tau_{f,x^*_y}(\alpha(t)),
\]
where (viii) uses Lemma B.1 (vi). Similar to case (I), we obtain \( z \in B_{x^*_y}(x, \alpha(t)) \subseteq V_{\alpha(t)} \).

### 6.6 Proof of Lemma 3.6

Let \( t(\mathbb{X}) = \inf \{ t > 0 : \bigcap_{x \in \mathbb{X}} B_{x^*_y}(x, t) \neq \emptyset \} \), and let \( x_0 \in \bigcap_{x \in \mathbb{X}} B_{x^*_y}(x, t) \). To ease the notation, let \( U^t = V^t_{\rho}[Y, f] \) denote the usual \( f \)-weighted filtration, and let \( W^t \) be defined as
\[
W^t = \left\{ \bigcup_{x \in \mathbb{X}} B_{x^*_y}(x, \beta(t)) \right\} \cup \left\{ \bigcup_{y \in \mathbb{Y} \setminus \mathbb{X}} B_{x^*_y}(y, t) \right\},
\]
such that \( U^t \subseteq W^t \subseteq U^{\beta(t)} \). With this background, the proof closely follows that of Anai et al. (2019, Proposition 4.8). Specifically, the proof is based on the following outline:

1. We first establish that for any \( y \in \mathbb{Y} \setminus \mathbb{X} \), there exists \( x = x^*_y \in \mathbb{X} \) such that for all \( t \geq t(\mathbb{X}) \), \( B_{x^*_y}(y, t) \cap B_{x^*_y}(x, \beta(t)) \) is star-shaped around \( x_0 \). Since this holds for all \( y \in \mathbb{Y} \setminus \mathbb{X} \), it also holds for \( \bigcup_{y \in \mathbb{Y} \setminus \mathbb{X}} B_{x^*_y}(y, t) \), and, therefore, \( W^t \) is star-shaped and contractible to \( x_0 \).

2. The inclusion map \( t_t : U^t \hookrightarrow U^{\beta(t)} \) can be decomposed as \( t_t = j_t \circ \kappa_t \) where \( j_t : U^t \hookrightarrow W^t \) and \( \kappa_t : W^t \hookrightarrow U^{\beta(t)} \). Since \( W^t \) is star-shaped and contractible, i.e., \( W^t \sim \{ x_0 \} \), the linear map between the homology groups induced by \( \kappa_t \), i.e., \( v_t : \mathbb{H}^t \rightarrow \mathbb{H}^{\beta(t)} \) will be trivial.
The interleavings $\alpha(t)$ (Lemma 3.5) and $\beta(t)$ are combined to provide the bound in $W_\infty$.

**Claim 1.** Let $y \in Y \setminus X$. We need to show that there exists $x \in X$ such that $B_{f,\rho}(x, \beta(t)) \cup B_{f,\rho}(y, t)$ is star-shaped around $x_0$, i.e., for any $z \in B_{f,\rho}(y, t)$ the curve $[x_0, z]$ is contained inside the set $B_{f,\rho}(x, \beta(t)) \cup B_{f,\rho}(y, t)$. See Figure 10.

To this end, let $x = \arg\inf_{z \in X} \rho(z, y)$ be the projection of $y$ onto $X$. Note that, from the definition of $x_0$, $x_0 \in B_{f,\rho}(x, t)$ for all $t \geq t(X)$. For simplicity, let $S^t = B_{f,\rho}(x, \beta(t)) \cup B_{f,\rho}(y, t)$. Additionally, let $\pi(x)$ and $\pi(y)$ be the projection of $x$ and $y$ onto $[x_0, z]$, respectively, i.e.,

$$\pi(x) = \arg\inf_{x' \in [x_0, z]} \rho(x', x),$$

mutatis mutandis, the same for $\pi(y)$. By definition, $\rho(x, \pi(x)) \leq \rho(x, x_0)$ and $\rho(y, \pi(y)) \leq \rho(y, z)$, and consequently, $\pi(y) \in B_{f,\rho}(y, t)$. This implies that $[\pi(y), z] \subseteq S^t$. What remains to be established is that $[x_0, \pi(y)] \subseteq S^t$. In order to show this, note that it is sufficient to show that $\pi(y) \in B_{f,\rho}(x, \beta(t))$. Indeed, if this holds, then $[x_0, \pi(y)] \subseteq B_{f,\rho}(x, \beta(t)) \subseteq S^t$, and it will follow that $[x_0, \pi(y)] \cup [\pi(y), z] = [x_0, z] \subseteq S^t$.

Let $\tau = \rho(y, \pi(y))$. Since $\pi(y) \in B_{f,\rho}(y, t)$, when $\rho(x, y) > a$ it follows that

$$\tau \leq r_{f,\rho}(t)$$

$$\leq \left( t^p - f(y)^p \right)^{\frac{1}{p}}$$

$$\leq \left( t^p - (\rho(x, y) - a)^p \right)^{\frac{1}{p}},$$

where the last inequality follows from the assumption on $f$. Thus, we have

$$\rho(x, y) \leq \left( t^p - \tau^p \right)^{\frac{1}{p}} + a. \quad (26)$$

Alternatively, when $\rho(x, y) \leq a$, Eq. (26) holds trivially. Since $\pi(x) \in B_{f,\rho}(x, t)$ and $\rho(x, \pi(x)) \leq \rho(x, x_0)$, it follows that

$$\rho(x, \pi(x)) \leq t(X). \quad (27)$$
Since $\rho = \|\cdot\|$, Anai et al. (2019, Lemma B.2) holds, which, combined with Eqs. (26) and (27) yields

$$\rho(x, \pi(y))^2 \leq \left( \left( t^p - \tau^p \right)^{\frac{1}{p}} + a \right)^2 + \tau(2\tau(X) - \tau)$$

$$\leq \left( t^p - \tau^p \right)^{\frac{2}{p}} + \tau(2\tau(X) - \tau) + a^2 + 2a\left( t^p - \tau^p \right)^{\frac{1}{p}}$$

where (i) is a consequence of Anai et al. (2019, Lemma B.2), (ii) follows from Anai et al. (2019, Lemma B.3) and noting that $t^p - \tau^p \leq t^p$ since $\tau \leq t$, and $\kappa = (1 - \frac{1}{p})$. Additionally, from Lemma B.1 (vi) we obtain

$$\rho(x, \pi(y)) \leq t + a + \kappa t(X) \leq \left( t + a + \kappa t(X) + \sup_{x \in X} f(x) \right)^p - f(x)^p = r_{f, \pi}(\beta(t)).$$

This implies that $\pi(y) \in B_{f, \rho}(x, \beta(t))$, and establishes claim $\mathbb{1}$.

For claim $\mathbb{2}$, note that since $W^t \sim \{x_0\}$, for the $k$th homology group $\mathbb{W}^t$, we have that $\mathbb{W}^t \simeq \mathbb{F}$ for $k = 0$, and $\mathbb{W}^t \simeq \{0\}$ for $k > 0$. Therefore, the map $w_t : \mathbb{W}^t \rightarrow \mathbb{U}^{\beta(t)}$ is trivial, and consequently, so is the linear map $\mathbb{U}^t \rightarrow \mathbb{U}^{\beta(t)}$.

In order to show claim $\mathbb{3}$, observe that the persistence modules $\mathbb{U}$ and $\mathbb{V}$ are

$$\begin{cases} (id, \alpha)-interleaved & \text{for all } t \text{ and for } \alpha : t \mapsto 2^{1 - \frac{1}{p}} t + a + \sup_{x \in X} f(x) \\
(id, \beta)-interleaved & \text{for } t \geq t(X) \text{ and for } \beta : t \mapsto t + a + \kappa t(X) + \sup_{x \in X} f(x). \end{cases}$$

When $t \leq t(X)$, from Anai et al. (2019, Lemma B.1),

$$\alpha(t) = t + \left( 2^{1 - \frac{1}{p}} - 1 \right) t + a + \sup_{x \in X} f(x) \leq t + \kappa t(X) + a + \sup_{x \in X} f(x) = \beta(t).$$

Thus, $\alpha(t) \leq \beta(t)$ for $t \leq t(X)$. Since $\beta : t \mapsto t + c(X)$ is an additive interleaving for $c(X) = \kappa t(X) + a + \sup_{x \in X} f(x)$, this implies that

$$W_\infty(\mathbb{D}\text{gm}(\mathbb{U}), \mathbb{D}\text{gm}(\mathbb{V})) \leq c(X),$$

which establishes claim $\mathbb{3}$.

6.7 Proof of Theorem 3.2

We begin by establishing the following result:

$$W_\infty \left( \mathbb{V}[X_n, d_{n,Q}], \mathbb{V}[X_n^*, d_{n,Q}] \right) \leq \sup_{x \in X_n^*} d_{n,Q}(x) + \left( 1 - \frac{1}{p} \right) t(X_n^{*}).$$

Observe that from Lemma 3.5 and Lemma 3.6, it suffices to show that for every $y \in \mathbb{Y}_m$, the MoM-Dist function $d_{n,Q}$ satisfies the property that

$$\inf_{x \in X_n^{*}} \|x - y\| \leq d_{n,Q}(y).$$
To this end, let $A = \{q \in [Q] : S_q \cap \mathbb{Y}_m = \emptyset \}$ be the blocks containing no outliers. For $y \in \mathbb{Y}_m$ and every $q \in A$, we have that $S_q \subseteq \mathbb{X}^*_m$ and therefore
\[
\inf_{x \in \mathbb{X}^*_m} \|x - y\| \leq \inf_{x \in S_q} \|x - y\| = d_{n,q}(y).
\]
Since this holds for every $q \in A$, taking the infimum on the right hand side over $Q$ yields
\[
\inf_{x \in \mathbb{X}^*_m} \|x - y\| \leq \inf_{q \in [Q]} d_{n,q}(y).
\]
Since $2m < Q$ by assumption, using the pigeonhole principle we further have that
\[
\inf_{q \in [Q]} d_{n,q}(y) \leq \text{median}\{d_{n,q}(y) : q \in [Q]\},
\]
which implies that $\inf_{x \in \mathbb{X}^*_m} \|x - y\| \leq d_{n,Q}(y)$ for every $y \in \mathbb{Y}_m$. Therefore, taking $a = 0$ in Lemma 3.5 and Lemma 3.6 we obtain
\[
W_\infty \left( \mathbb{V}[\mathbb{X}_n, d_{n,Q}], \mathbb{V}[\mathbb{X}^*_m, d_{n,Q}] \right) \leq \sup_{x \in \mathbb{X}^*_m} d_{n,Q}(x) + \left(1 - \frac{1}{p}\right) t(\mathbb{X}^*_m).
\]
Turning our attention to the quantity appearing in the statement of the theorem, note that an application of the triangle inequality yields
\[
W_\infty \left( \mathbb{V}[\mathbb{X}_n, d_{n,Q}], \mathbb{V}[\mathbb{X}^*_m, d_{n,Q}] \right) \leq W_\infty \left( \mathbb{V}[\mathbb{X}_n, d_{n,Q}], \mathbb{V}[\mathbb{X}^*_m, d_{n,Q}] \right) + W_\infty \left( \mathbb{V}[\mathbb{X}^*_m, d_{n,Q}], \mathbb{V}[\mathbb{X}^*_m, d_{n-m}] \right) \leq \sup_{x \in \mathbb{X}^*_m} d_{n,Q}(x) + \left(1 - \frac{1}{p}\right) t(\mathbb{X}^*_m) + \|d_{n,Q} - d_{n-m}\|_\infty,
\]
where the first term in $(\ast)$ follows from Eq. (28) and the last term follows from Lemma 2.2 (i). This gives us the desired result. Furthermore, when $p = 1$ note that $1 - 1/p = 0$, giving us the tighter bound in this case. ■

6.8 Proof of Theorem 3.3

We begin by noting that $\mathbb{V}[\mathbb{X}] = \mathbb{V}[\mathbb{X}, d_{\mathbb{X}}]$. Indeed, the distance function $d_{\mathbb{X}}(x) = 0$ for all $x \in \mathbb{X}$. We may further conclude that
\[
\sup_{x \in \mathbb{X}} d_{\mathbb{X}}(x) = 0.
\]
The bottleneck distance between $\mathbb{V}[\mathbb{X}^*_m \cup \mathbb{Y}_m, d_{n,Q}]$ and $\mathbb{V}[\mathbb{X}]$ may be bounded above as
\[
W_\infty \left( \mathbb{V}[\mathbb{X}^*_m \cup \mathbb{Y}_m, d_{n,Q}], \mathbb{V}[\mathbb{X}] \right) \leq W_\infty \left( \mathbb{V}[\mathbb{X}^*_m \cup \mathbb{Y}_m, d_{n,Q}], \mathbb{V}[\mathbb{X}^*_m, d_{n,Q}] \right) =: (a)
\]
\[+ W_\infty \left( \mathbb{V}[\mathbb{X}^*_m, d_{n,Q}], \mathbb{V}[\mathbb{X}^*_m, d_{\mathbb{X}}] \right) =: (b)
\]
\[+ W_\infty \left( \mathbb{V}[\mathbb{X}^*_m, d_{\mathbb{X}}], \mathbb{V}[\mathbb{X}, d_{\mathbb{X}}] \right).
\]
When \( p = 1 \), the term \((b)\) is bounded above by taking \( p = 1 \) in Eq. (28) (from the proof of Theorem 3.2) to give

\[
\begin{align*}
\text{(a)} & = \sup_{\begin{subarray}{l}
x \in \mathbb{H}^*_n \end{subarray}} d_{n,Q}(x) \leq \sup_{\begin{subarray}{l}
x \in \mathbb{H}^*_n \end{subarray}} d_{n,Q}(x) \\
& \leq \sup_{\begin{subarray}{l}
x \in \mathbb{H} \end{subarray}} d_{n,Q}(x) \\
& \leq \|d_{n,Q} - d_{\mathcal{H}}\|_\infty + \sup_{\begin{subarray}{l}
x \in \mathbb{H} \end{subarray}} d_{\mathcal{H}}(x) \\
& \leq \|d_{n,Q} - d_{\mathcal{H}}\|_\infty,
\end{align*}
\]

where \((*)\) follows from the fact \( \mathbb{H}^*_n \subset \mathbb{H} \), \((\dagger)\) uses the identity \( f(x) \leq \|f - g\|_\infty + g(x) \) for all \( x \in \mathbb{H} \), and \((\ddagger)\) follows from Eq. (29). Plugging in the bounds for the bottleneck distance we obtain

\[
W_\infty \left( \bigvee [\mathbb{H}^*_n \cup \mathbb{H}_m, d_{n,Q}], \bigvee [\mathbb{H}] \right) \leq 2\|d_{n,Q} - d_{\mathcal{H}}\|_\infty + H(\mathbb{H}^*_n, \mathbb{H}).
\]

By noting that the Hausdorff distance \( H(\mathbb{H}^*_n, \mathbb{H}) = \|d_{n-m} - d_{\mathcal{H}}\|_\infty \), for \( t_1, t_2 \) such that \( t_1 + t_2 = t \) we may bound the tail probability for the bottleneck distance as follows.

\[
\begin{align*}
\mathbb{P} \left\{ W_\infty \left( \bigvee [\mathbb{H}^*_n \cup \mathbb{H}_m, d_{n,Q}], \bigvee [\mathbb{H}] \right) > t \right\} & \leq \mathbb{P} \left( 2\|d_{n,Q} - d_{\mathcal{H}}\|_\infty > t_1 \right) + \mathbb{P} \left( \|d_{n-m} - d_{\mathcal{H}}\|_\infty > t_2 \right) \\
& \leq \delta_1 + \delta_2 = \delta,
\end{align*}
\]

where the relationship between \( \delta_1, \delta_2 \) and \( t_1, t_2 \) is given by Eq. (21), i.e., \( \delta_1 \leq e^{-(1+b)Q} \) from the condition in Theorem 3.1, \( \delta_2 = \delta - \delta_1 \).

\[
t_1 = 2 \left( \frac{Q}{a n} W_0 \left( \frac{n}{Q} \exp \left( \frac{4\log(1/\delta_1)}{Q - 2 - 2m} \right) \right) \right)^{1/b}, \quad \text{and} \quad t_2 = \left( \frac{1}{a n} W_0 \left( n e^{4\log(1/\delta_2)} \right) \right)^{1/b}.
\]

Furthermore, using the bound for the Lambert \( W_0 \) function \( W_0(z) \leq \log z \) for \( z > e \), we have

\[
t_1 \leq 2 \left( \frac{Q \log(n/Q)}{a n} + \frac{4Q \log(1/\delta_1)}{a(Q - 2 - 2m)n} \right)^{1/b}, \quad t_2 \leq \left( \frac{\log(n-m)}{a(n-m)} + \frac{4\log(1/\delta_2)}{a(n-m)} \right)^{1/b},
\]

and \( t = t_1 + t_2 \leq f(n, m, Q, \delta_1, \delta_2) \). Therefore, the bound in Eq. (30) yields

\[
\begin{align*}
\mathbb{P} \left\{ W_\infty \left( \bigvee [\mathbb{H}^*_n \cup \mathbb{H}_m, d_{n,Q}], \bigvee [\mathbb{H}] \right) \leq f(n, m, Q, a, b) \right\} \\
\geq \mathbb{P} \left\{ W_\infty \left( \bigvee [\mathbb{H}^*_n \cup \mathbb{H}_m, d_{n,Q}], \bigvee [\mathbb{H}] \right) > t_1 + t_2 \right\} \geq 1 - \delta,
\end{align*}
\]

which gives the desired result. The second part of the theorem follows directly using the identical procedure as that used in the proof of Theorem 3.1 in Section 6.3.
6.9 Proof of Proposition 3.1

We begin by noting from Lemma 3.3 that $V[d_{n,Q}]$ and $V[\mathbb{R}^d, d_{n,Q}]$ are $(\text{id}, \alpha)$—interleaved for $\alpha : t \mapsto 2^{v-1/p}t$. Furthermore, consider the intermediate filtrations $V[X_{n,m}^*, d_{n,Q}]$. From Proposition 3.2 and Lemma 3.5 we have that $V[\mathbb{R}, d_{n,Q}]$ and $V[X_{n,m}, d_{n,Q}]$ are $(\eta, \text{id})$—interleaved for

$$
\eta : t \mapsto 2^{v-1/p}t + \sup_{x \in X_{n,m}} d_{n,Q}(x).
$$

Using an identical argument, but reversing the order, we have that $V[X_{n,m}^*, d_{n,Q}]$ and $V[X_n, d_{n,Q}]$ are $(\text{id}, \eta)$—interleaved. We can now apply the “triangle inequality” for generalized interleavings (Bubenik et al., 2015, Proposition 3.11) to obtain that $V[d_{n,Q}]$ and $V[X_n, d_{n,Q}]$ are $(\text{id} \circ \eta \circ \text{id}, \alpha \circ \text{id} \circ \eta)$—interleaved. On simplifying the interleaving maps, we obtain that the two filtrations are $(\eta, \xi)$—interleaved for $\xi(t) = \alpha \circ \eta(t) = 2^{v-1/p} \eta(t)$.

6.10 Proof of Theorem 3.4

The birth time of a connected component at $x_0$ in $V[X, f]$ is given by $b_f(\{x_0\}) = f(x_0)$. Therefore, $\Delta b_{n,m}(\{x_0\})$ is given by

$$
\Delta b_{n,m}(\{x_0\}) = b_n(\{x_0\}) - b_{n+m}(\{x_0\}) = d_n(x_0) - d_{n+m}(x_0) = d_n(x_0),
$$

where the last equality follows from the fact that $d_{n+m}(x_0) = 0$, since $x_0 \in X_{n+m}$. On the other hand, from the proof of Proposition 3.2,

$$
d_n(x_0) = \inf_{x \in X_n} \|x - x_0\| \leq d_{n+m,Q}(x_0).
$$

Therefore, we have that $\Delta b_{n,m,Q}(\{x_0\}) = d_n(x_0) - d_{n+m,Q}(x_0) \leq \Delta b_{n,m}(\{x_0\})$, and the result follows.

For the second part, we begin by observing that $\|d_{n+m} - d_n\|_{\infty}$ can be bounded from below as follows:

$$
\|d_{n+m} - d_n\|_{\infty} \geq d_n(x_0) - d_{n+m}(x_0)
= d_n(x_0) - 0
= \inf_{x \in X_n} \|x - x_0\|
\geq \inf_{x \in X} \|x - x_0\| = d_X(x_0).
$$

Furthermore, for $\delta \leq e^{-(1+b)}Q$ and $k = \max \{1, 2^{1/b}\}$, with probability greater than $1 - \delta$,

$$
\|d_{n+m,Q} - d_n\|_{\infty}
\leq (i) \|d_{n+m,Q} - d_X\|_{\infty} + \|d_n - d_X\|_{\infty}
\leq (i) \left[ \frac{1}{anQ} W_0 \left( nQ \exp \left( \frac{4Q \log(2/\delta)}{Q - 2m} \right) \right) \right]^{1/b} + \left[ \frac{1}{an} W_0 (n \exp \{4 \log(2/\delta)\}) \right]^{1/b}
\leq (ii) \left[ \frac{k}{a^{1/b}} \left[ \frac{1}{nQ} W_0 \left( nQ \exp \left( \frac{4Q \log(2/\delta)}{Q - 2m} \right) \right) + \frac{1}{n} W_0 (n \exp \{4 \log(2/\delta)\}) \right] \right]^{1/b}
\leq (iii) \left[ \frac{k}{a^{1/b}} \left[ \frac{\log nQ}{nQ} + \frac{\log n}{n} + \frac{4 \log(2/\delta)}{nQ(Q - 2m) + 1/n} \right] \right]^{1/b}
\leq (iv) \left[ \frac{k}{a^{1/b}} \left[ \frac{\log nQ}{nQ} + \frac{\log n}{n} + 4 \log(2/\delta) \left( \frac{Q}{nQ(Q - 2m)} + \frac{1}{n} \right) \right] \right]^{1/b}
$$

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where, for \( n_Q = (n + m)/Q \), (i) is a consequence of the triangle inequality and (ii) follows from the proofs of Theorem 3.1 and Theorem 3.3, (iii) uses Lemma B.1, (iv) follows from the fact that \( W_0(z) < \log(z) \) for \( z > e \), and (v) uses the fact that \( n_Q < n \) and \( (Q - 2m)^{-1} \leq 1 \) for \( 2m > Q \).

Observe that if \( 2\eta(n, m, Q, \delta) \leq d_{\mathcal{X}}(x_0) \), then with probability greater than \( 1 - \delta \),

\[
\|d_{n+m} - d_n\|_{\infty} - \|d_{n+m,Q} - d_n\|_{\infty} \geq d_{\mathcal{X}}(x_0) - \eta(n, m, Q, \delta) \geq \eta(n, m, Q, \delta),
\]

and the result follows. Therefore, in order to establish the claim for the second part it suffices to check that

\[
2\eta(n, m, Q, \delta) \leq d_{\mathcal{X}}(x_0)
\]

under conditions (I) and (II). To this end, note that

\[
d_{\mathcal{X}}(x_0) \geq 2\eta(n, m, Q, \delta) \iff \varpi(x_0) \geq \frac{\log n_Q + 4Q \log(2/\delta)}{n_Q},
\]

which is satisfied whenever \( \delta \) satisfies the r.h.s. of condition (II), i.e.,

\[
\log(2/\delta) \leq \frac{n_Q \varpi(x_0) - \log n_Q}{4Q}.
\]

Furthermore, the l.h.s. of condition (II), i.e., \( \delta \leq e^{-(1+b)Q} \), is satisfied only when

\[
(1 + b)^2Q^2 \leq \frac{n_Q \varpi(x_0) - \log n_Q}{4Q},
\]

or, equivalently, when condition (I) is satisfied:

\[
\varpi(x_0) \geq \frac{\log n_Q}{n_Q} + \frac{4(1 + b)^2Q^3}{n_Q}.
\]

The result now follows from Eq. (31).

6.11 Proof of Theorem 3.5

Let \( j^* = \min \{ j \in J : m(j) > m^* \} \). By definition of \( J \) we have that \( |J| \leq 1 + \log_\theta(m_{\max}/m_{\min}) \) and \( m(j^*) < \theta m^* \) for \( \theta > 1 \). The outline of the proof is as follows. First, we show that \( h(n, m, \delta) \) is non-decreasing in \( m \), from which it follows that \( h(n, m(j), \delta) \leq h(n, m(j + 1), \delta) \). Next, we show that the event \( \{ j \leq j^* \} \) contains the event \( \mathcal{E} \) given by

\[
\mathcal{E} = \bigcap_{j \in J} \left\{ W_\infty(\mathbb{V}_n(j), \mathbb{V}[X]) \leq h(n, m(j), \delta) \right\}.
\]

Then, using a standard procedure for obtaining the Lepski bound (e.g., Theorem 5.1 of Minsker 2018 and Theorem 3.1 of Chen and Zhou 2020), we show that the event \( \mathcal{E} \), and, therefore the event \( \{ j \leq j^* \} \), holds with probability at least \( 1 - \delta \log_\theta(m_{\max}/m_{\min}) \). Lastly, we use the bound on the event \( \{ j \leq j^* \} \) to obtain the desired result.

1. Monotonicity of \( h(n, m, \delta) \) in \( m \). Consider the function \( f(z; \alpha, \beta) = \alpha W_0(\beta z)/z \) for fixed constants \( \alpha, \beta > 0 \). The derivative of \( f \) is given by

\[
f'(z; \alpha, \beta) = \frac{d}{dz} \left( \frac{\alpha}{z} W_0(\beta z) \right) = \alpha \left( \frac{\beta}{z} W'_0(\beta z) - \frac{1}{z^2} W_0(\beta z) \right).
\]
\[
\alpha \left( \frac{\beta \left( \frac{W_0(\beta z)}{\beta z(1 + W_0(\beta z))} \right)}{z^2} - \frac{1}{z^2} W_0(\beta z) \right)
\]

\[
= -\frac{\alpha W_0(\beta z)}{z^2(1 + W_0(\beta z))} < 0 \quad \text{for all} \quad z > 0.
\]

Note that in (i) we have used the fact that the derivative of the Lambert \(W_0\) function is given by \(W'_0(z) = \frac{W_0(z)}{z(1 + W_0(z))}\). Therefore, it follows that \(f\) is non-increasing in \(z\). The claim follows by noting that the function \(h\) is given by

\[
h(n, m, \delta) = f(n/(2m + 1); \alpha_1, \beta_1)^{1/b} + f(n - m; \alpha_2, \beta_2)^{1/b},
\]

for constants \(\alpha_1, \beta_1, \alpha_2, \beta_2 > 0\) not depending on \(n\) or \(m\).

2. \(\mathcal{E}\) is a subset of \(\{j \leq j^*\}\). We begin by noting that since \(m^* \leq m(j^*)\), it follows that \(2m^* < Q(j) = 2m(j) + 1\) for all \(j \geq j^*\) and satisfies the first condition for Theorem 3.3. By taking

\[
\delta_1 = e^{-2(1+b)(2m_{\max}+1)} \leq e^{-2(1+b)Q(j)},
\]

and \(\delta_2 = \delta - \delta_1\), note that \(h(n, m(j), \delta) \leq f(n, m(j), Q(j), \delta_1, \delta_2)\). Therefore, we may use Theorem 3.3 to obtain

\[
\mathbb{P}\left( W_\infty(\mathbb{V}(n, j), \mathbb{V}[X]) > h(n, m(j), \delta) \right) < \delta \quad \text{for all} \quad j \geq j^*.
\]

Furthermore, by definition of \(j\), it follows that for all \(j < j\), there exists at least one \(i > j\) such that \(W_\infty(\mathbb{V}(i), \mathbb{V}(j)) > 2h(n, m(i), \delta)\). Therefore,

\[
\{j > j^*\} \subseteq \bigcup_{\{j \in \mathcal{J}; j > j^*\}} \left\{ W_\infty(\mathbb{V}(n, j), \mathbb{V}(j^*)) > 2h(n, m(j), \delta) \right\}
\]

\[
\subseteq \bigcup_{\{j \in \mathcal{J}; j > j^*\}} \left\{ W_\infty(\mathbb{V}(n, j), \mathbb{V}[X]) > h(n, m(j), \delta) \right\} \cup \left\{ W_\infty(\mathbb{V}(n, j^*), \mathbb{V}[X]) > h(n, m(j^*), \delta) \right\}
\]

\[
= \bigcup_{\{j \in \mathcal{J}; j \geq j^*\}} \left\{ W_\infty(\mathbb{V}(n, j), \mathbb{V}[X]) > h(n, m(j), \delta) \right\} = \mathcal{E}^c,
\]

where, in (ii) we have used the fact that \(h(n, m(j^*), \delta) \leq h(n, m(j), \delta)\) for all \(j > j^*\), and, therefore

\[
\left\{ W_\infty(\mathbb{V}(n, j), \mathbb{V}[X]) \leq h(n, m(j), \delta) \right\} \cap \left\{ W_\infty(\mathbb{V}(n, j^*), \mathbb{V}[X]) \leq h(n, m(j^*), \delta) \right\}
\]

\[
\subseteq \left\{ W_\infty(\mathbb{V}(n, j), \mathbb{V}(n(j^*))) \leq 2h(n, m(j), \delta) \right\}.
\]

By inverting the above inclusion we get the inclusion in (ii). Therefore, we obtain that \(\mathcal{E} \subseteq \{j \leq j^*\}\).

3. Tail bound for the event \(\mathcal{E}\). Applying a union bound to Eq. (33), we obtain

\[
\mathbb{P}(\mathcal{E}^c) = \mathbb{P}\left( \bigcup_{\{j \in \mathcal{J}; j \geq j^*\}} \left\{ W_\infty(\mathbb{V}(n, j), \mathbb{V}[X]) > h(n, m(j), \delta) \right\} \right)
\]

\[
\leq \sum_{\{j \in \mathcal{J}; j \geq j^*\}} \mathbb{P}\left( W_\infty(\mathbb{V}(n, j), \mathbb{V}[X]) > h(n, m(j), \delta) \right)
\]

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$\delta \log_{\theta} \left( \frac{\theta m_{\text{max}}}{m_{\text{min}}} \right)$.

where (iv) follows from Eq. (32) and (v) uses the fact that $|\mathcal{J}| \leq 1 + \log_{\theta}(m_{\text{max}}/m_{\text{min}})$.

4. Bound for $W_{\infty}(\mathbb{V}_n(j), \mathbb{V}[X])$. We begin by noting that when the event $\mathcal{E}$ holds, we have that

$$W_{\infty}(\mathbb{V}_n(j), \mathbb{V}[X]) \leq W_{\infty}(\mathbb{V}_n(j), \mathbb{V}(j^*)) + W_{\infty}(\mathbb{V}_n(j^*), \mathbb{V}[X])$$

where the first term in (vi) follows from the definition of $j$, which is guaranteed to hold because $\mathcal{E} \subseteq \{ j \leq j^* \}$, and the second term in (vi) follows from the definition of $\mathcal{E}$. The inequality in (vii) uses the fact that $m(j^*) < \theta m^*$ and the fact that $h(n, m, \delta)$ is non-decreasing in $m$. Therefore, we have the inclusion

$$\mathcal{E} \subseteq \left\{ W_{\infty}(\mathbb{V}_n(j), \mathbb{V}[X]) \leq 3h(n, \theta m^*, \delta) \right\}.$$

Using the tail bound on $\mathcal{E}$ we obtain

$$P\left( W_{\infty}(\mathbb{V}_n(j), \mathbb{V}[X]) \leq 3h(n, \theta m^*, \delta) \right) \geq P(\mathcal{E}) \geq 1 - \delta \log_{\theta} \left( \frac{\theta m_{\text{max}}}{m_{\text{min}}} \right),$$

which is the desired result. ■

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A Glossary of Notations

- $H_\rho(X, \mathcal{Y})$ : Hausdorff distance between $X \subseteq \mathcal{M}$ and $\mathcal{Y} \subseteq \mathcal{M}$ measured w.r.t. metric $\rho$.
- $V^t[f]$ : Sublevel set of $f$ at level $t$ given by $\{ x \in \mathbb{R}^d : f(x) \leq t \}$
- $V[f]$ and $\mathcal{V}[f]$ : Sublevel filtration $\{V^t[f] : t \in \mathbb{R} \}$ and its persistence module
- $r_f(x,t)$ : The $f$-weighted radius function of resolution $t$ at $x$. $r_f(x,t) = (tp - f(x))^\frac{1}{p}$
- $B_f(x,t)$ : $f$-weighted ball at $x$ with radius $r_f(x,t)$ w.r.t the metric $\rho$.
- $V^t[X, f]$ : The $f$-weighted offset of $X$ at resolution $t$ given by $V^t[X, f] = \bigcup_{x \in X} B_f(x,t)$
- $V[X, f]$ : $f$-weighted filtration, i.e., $\cdots \rightarrow V^t[X, f] \rightarrow V^t_{2}[X, f] \rightarrow \cdots \rightarrow V^t_{n}[X, f] \rightarrow \cdots$
- $\mathcal{V}[X, f]$ : $f$-weighted persistence module, i.e., $\mathcal{V}[X, f] = \text{Hom}(V[X, f])$
- $\mathcal{D}_{gm}(\mathcal{V})$ : Persistence diagram associated with the persistence module $\mathcal{V}$
- $\hat{\theta}_{n,Q}$ : MoM-estimator, median of $\hat{\theta}_1, \ldots, \hat{\theta}_Q$, where $\hat{\theta}_q$ is the estimator from block $S_q$. 
- $d_{n,Q}$ : MoM Dist function given by $d_{n,Q}(x) = \text{median}\{\inf_{y \in S_q} \|x - y\| : q \in [Q]\}$
- $d_X$ : Distance function to a compact set $X$ given by $d_X(y) = \inf_{x \in X} \|x - y\|$

B Supplementary Results

The following lemma is a collection of well-known inequalities (and their slight variants). We state them here for reference, as they are used frequently in the proofs.

**Lemma B.1.** For $0 < y \leq x$ and $p \geq 1$, the following inequalities hold:

1. $x^p + y^p \leq (x + y)^p \leq 2^{p-1}(x^p + y^p)$;
2. $2^{1-p}x^p - y^p \leq (x - y)^p \leq x^p - y^p$;
3. $(x + y)^\frac{1}{p} \leq x^\frac{1}{p} + y^\frac{1}{p} \leq 2^{\frac{p-1}{p}}(x + y)^\frac{1}{p}$;
4. $x^\frac{1}{p} - y^\frac{1}{p} \leq (x - y)^\frac{1}{p} \leq 2^{\frac{p-1}{p}} x^\frac{1}{p} - y^\frac{1}{p}$;
5. $y^{1-p}x^\frac{1}{p} \leq x \leq y^{1-p}x^p$;
6. $x \leq \left((x + y)^p - y^p\right)^\frac{1}{p}$.

**Proof.** Part (i). Let $f(y) = (x + y)^p - x^p - y^p$ on the interval $0 < y \leq x$. The derivative,

$$f'(y) = p(x + y)^{p-1} - py^{p-1} \geq 0$$

for all $0 < y \leq x$ and $p \geq 1$. Therefore $f(y)$ is strictly non-decreasing, and $f(y) \geq f(0) = 0$. This gives us the first inequality. For the second inequality, note that $g(z) = z^p$ is convex for $z \geq 0$. This follows from the
fact that \( g''(z) = p(p - 1)z^{p-2} \geq 0 \) for all \( z \geq 0 \) and \( p \geq 1 \). By convexity, we obtain

\[
2^{-p}(x + y)^p = \left( \frac{1}{2}x + \frac{1}{2}y \right)^p \leq \frac{x^p + y^p}{2},
\]

which leads to the second inequality.

**Part (ii).** Let \( z = (x - y) \). Applying the first inequality from the preceding part to \( z \) and \( y \) we get \( z^p \leq (y + z)^p - y^p \), i.e., \((x - y)^p \leq x^p - y^p\). Similarly, from the second inequality, \((z + y)^p \leq 2^{p-1}(z^p + y^p)\), which is the same as \( 2^{1-p}x^p - y^p \leq (x - y)^p \).

**Part (iii).** Note that \( f(z) = z^{\frac{1}{p}} \) is concave for all \( z \geq 0 \) and \( p \geq 1 \), since

\[
f''(z) = -\left( \frac{p - 1}{p^2} \right) z^{\frac{1-2p}{p}} \leq 0,
\]

for all \( z \geq 0 \), \( p \geq 1 \). Therefore, by concavity,

\[
2^{-\frac{1}{p}}(x + y)^{\frac{1}{p}} \geq \frac{x^p + y^p}{2},
\]

which leads to the right hand side inequality, i.e., \( x^{\frac{1}{p}} + y^{\frac{1}{p}} \leq 2^{1-\frac{1}{p}}(x + y)^{\frac{1}{p}} \). For the left hand side inequality, let \( f(y) = x^{\frac{1}{p}} + y^{\frac{1}{p}} - (x + y)^{\frac{1}{p}} \) on the interval \( 0 < y \leq x \). The derivative is given by

\[
f'(y) = \frac{1}{p} \left( y^{\frac{1}{p}-1} - (x + y)^{\frac{1}{p}-1} \right) \geq 0,
\]

since \( 0 < 1/p \leq 1 \) and \( 0 < y \leq x \). Thus, \( f(y) \) is increasing on the interval \([0, x]\), and, therefore, \( f(y) \geq f(0) = 0 \). This leads to the desired result.

**Part (iv).** The proof is identical to the proof in Part (ii). The inequalities are obtained by taking \( z = (x - y) \), and applying the results of Part (iii).

**Part (v).** Since \( y \leq x \), it follows that \( 1 \leq (x/y)^{\frac{1}{p}} \leq x/y \leq (x/y)^p \) for \( p \geq 1 \). By rearranging the terms, we get \( x \leq y^{1-p}x^p \) and \( x \geq y^{1-\frac{1}{p}}x^{\frac{1}{p}} \).

**Part (vi).** We have \( x = (x + y - y) = \left( (x + y - y)^p \right)^{\frac{1}{p}} \). From Part (ii) we have

\[
(x + y - y)^p \leq (x + y)^p - y^p,
\]

which, on rearrangement, yields \( x \leq \left( (x + y)^p - y^p \right)^{\frac{1}{p}} \).

**Lemma B.2** (Chernoff-Hoeffding bound simplified). Suppose \( Z_1, Z_2, \ldots, Z_N \) are i.i.d. Bernoulli(\( p \)) random variables. Then, for \( 0 < \epsilon < 1 \),

\[
P \left( \frac{1}{N} \sum_{i=1}^{N} Z_i > \epsilon \right) \leq \exp \left( N \left( \frac{2}{\epsilon} + \epsilon \log(p) \right) \right).
\]
Proof. For $0 < \epsilon < 1$, using the Chernoff-Hoeffding bound for binomial random variables (Hoeffding, 1963, Theorem 1) we have

$$
\mathbb{P}\left(\frac{1}{N} \sum_{1 \leq i \leq N} Z_i > \epsilon\right) \leq \exp\left(-N \cdot \text{KL}(\text{Ber}(\epsilon) \| \text{Ber}(p))\right),
$$

(34)

where $\text{Ber}(\epsilon)$ and $\text{Ber}(p)$ are Bernoulli distributions with parameters $\epsilon$ and $p$ respectively, and $\text{KL}(\mathbb{P} \| \mathbb{Q})$ is the Kullback-Leibler divergence of $\mathbb{Q}$ w.r.t $\mathbb{P}$. Simplifying the quantity in the exponent, we get

$$
\text{KL}(\text{Ber}(\epsilon) \| \text{Ber}(p)) = \epsilon \log\left(\frac{\epsilon}{p}\right) + (1 - \epsilon) \log\left(\frac{1 - \epsilon}{1 - p}\right)
= \epsilon \log(\epsilon) + (1 - \epsilon) \log(1 - \epsilon) - \epsilon \log(p) - (1 - \epsilon) \log(1 - p)
\geq -\frac{2}{e} - \epsilon \log(p),
$$

where the last inequality uses the fact that $x \log(x) \geq -1/e$ for all $0 \leq x \leq 1$, and $-(1 - \epsilon) \log(1 - p) \geq 0$ for all $0 \leq \epsilon, p \leq 1$. Substituting this in Eq. (34) yields the result. •