An Analysis of Load-Balancing Algorithms on Edge-Markovian Evolving Graphs

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Abstract

Analysis of algorithms on time-varying networks (often called evolving graphs) is a modern challenge in theoretical computer science. The edge-Markovian is a relatively simple and comprehensive model of evolving graphs: every pair of vertices which is not a current edge independently becomes an edge with probability $p$ at each time-step, as well as every edge disappears with probability $q$. Clearly, the edge-Markovian graph changes its shape depending on the current shape, and the dependency refutes some useful techniques for an independent sequence of random graphs which often behaves similarly to a static random graph. It motivates this paper to develop a new technique for analysis of algorithms on edge-Markovian evolving graphs.

Specifically speaking, this paper is concerned with load-balancing, which is a popular subject in distributed computing, and we analyze the so-called random matching algorithms, which is a standard scheme for load-balancing. We prove that major random matching algorithms achieve nearly optimal load balance in $O(r \log(\Delta n))$ steps on edge-Markovian evolving graphs, where $r := \max\{p/(1-q), (1-q)/p\}$, $n$ is the number of vertices (i.e., processors) and $\Delta$ denotes the initial gap of loads unbalance. We remark that the independent sequences of random graphs correspond to $r = 1$. To avoid the difficulty of an analysis caused by a complex correlation with the history of an execution, we develop a simple proof technique based on history-independent bounds. As far as we know, this is the first theoretical analysis of load-balancing on randomly evolving graphs, not only for the edge-Markovian but also for the independent sequences of random graphs.

Keywords: load-balancing, randomized algorithms, randomly evolving graphs.

1 Introduction

In the real world, connections or relationships between individuals continue to change time by time, e.g., social relationships, peer-to-peer networks, wireless devices, etc. Such situations are naturally modeled by a graph changing its shape over time, called dynamic graph. Analysis of algorithms on dynamic graphs, including both adversarial or stochastic changes, is a modern challenge in theoretical computer science, and it is widely studied in this decade [30, 27, 33].

One of the simplest models of dynamic graphs is the dynamic Erdős-Rényi random graph: it is a time-series of Erdős-Rényi random graphs $G_0, G_1, G_2, \ldots$, where the random graphs are mutually independent. Theoretical analyses of processes related to the spreading of infection or information on the dynamic Erdős-Rényi random graph have been studied to investigate the relationship between the connectivity threshold $p$ and the speed of the spreading, for instance, SIR (susceptible-infected-removed) model [20], random walks [2], and radio broadcasting [21, 15].

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The dynamic Erdős-Rényi random graph is simple enough to mathematically analyze, while it might be a strong assumption for a model of real networks that the graph changes its shape in the next time step completely different from the current one. Clementi et al. [14] introduced the edge-Markovian evolving graph as a more general model of dynamic graphs, which includes the dynamic Erdős-Rényi random graph. Precisely, it is a random sequence of graphs $G_0, G_1, G_2, \ldots$ with the same vertex set $V$, and in an update from $G_t$ to $G_{t+1}$, each (existing) edge $e \in E(G_t)$ independently disappears with probability $q$, and each (not-existing) edge $e \in \left\{ \frac{1}{2} \right\} \setminus E(G_t)$ independently appears with probability $p$. Note that it is identical to the dynamic Erdős-Rényi random graph when $q = 1 - p$. Recently, there have been many works on the model concerning fundamental processes, e.g., flooding [14, 3, 17, 16], rumor spreading [13], and random walk [31, 12].

In this paper, we are concerned with the load-balancing problem. Suppose that each vertex $v$ initially holds $L(v) \in \mathbb{N}$ tokens. We aim to reallocate the tokens as equally as possible under the assumption that each vertex is only allowed to exchange tokens with its neighbors. The main interest of the study is the number of time steps required to reach the almost balanced configuration. There are many studies concerned with the load-balancing paradigm. The load-balancing problem naturally models the coordination of tasks in distributed processor networks and parallel machines [18]. This problem is also referred to as the distributed averaging problem, which arises in many applications, e.g., coordination of autonomous agents, estimation, and data fusion, on distributed networks such as sensors, wireless ad-hoc, and peer-to-peer networks [10]. In computational physics, load-balancing algorithms appear to simulate large and complicated correlation systems such as molecular dynamics [9] and electrostatic plasma [22].

The load-balancing problem has been well studied on static graphs. Particularly, there are many theoretical studies for a type of algorithms called random matching algorithms [25, 10, 24, 38, 11]. In a random matching algorithm, at each discrete time step, we generate a random matching $M \subseteq E$ with some property. Then, for each matching edge $\{v, u\} \in M$, we reallocate tokens on $v$ and $u$ by the random rounding: $(L(v), L(u)) \rightarrow (\lceil \frac{L(v)+L(u)}{2} \rceil, \lceil \frac{L(v)+L(u)}{2} \rceil)$ or $(\lfloor \frac{L(v)+L(u)}{2} \rfloor, \lfloor \frac{L(v)+L(u)}{2} \rfloor)$ with probability $1/2$ each. For example, the LR algorithm [25] is known as a specific algorithm to generate a random matching locally. For such algorithm, several works [24, 38, 11] studied the discrepancy, which is the maximum difference of tokens $\max_{v \in V} L(v) - \min_{v \in V} L(v)$. For example, Sauerwald and Sun [38] showed that a random matching algorithm reaches the configuration with constant discrepancy on any connected regular graph. Formally, let $\Delta$ be the initial discrepancy and $\lambda$ be the second largest eigenvalue of the diffusion matrix $P$. They showed that, for any connected regular graph, the discrepancy is at most some constant w.h.p. after $O(\frac{\log(\Delta n)}{1-\lambda})$ steps.

### 1.1 Results

We study the performance of random matching algorithms on edge-Markovian evolving graphs, although all previous works are concerned with static graphs, as far as we know. Let $\Gamma_t = (\Gamma_t(v))_{v \in V} \in \mathbb{N}^V$ denote the configuration of tokens at time $t \geq 0$ and let $\lceil \cdot \rceil$ denote the rounding operator to the nearest integer $\mathbb{N}$. Let $\Delta := \max_{v \in V} \Gamma_0(v) - \min_{v \in V} \Gamma_0(v)$ be the initial discrepancy. We study the following balancing time $T_{\text{bal}}(\Gamma_0) = T_{\text{bal}}(\Gamma_0)$ as a measure of balancing:

$$T_{\text{bal}} := \min \{ t \geq 0 : \Gamma_t(v) \in \{ \lceil \mu \rceil - 1, \lceil \mu \rceil, \lceil \mu \rceil + 1 \} \text{ for all } v \in V \}.$$  \hspace{1cm} (1)

We show the following theorem for random matching algorithms including the LR algorithm [25]. The formal condition required to random matching algorithms is in Section 2.1.

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1. The diffusion matrix $P$ is defined by $P(v, u) = 1/(2d_{\text{max}}(G))$ if $\{v, u\} \in E(G)$, $P(v, v) = 1 - d(G, v)/(2d_{\text{max}}(G))$, and $P(v, u) = 0$ otherwise.

2. $[x] := \lfloor x - 1/2 \rfloor$ for $x \in \mathbb{R}$.
Theorem 1.1. Consider a random matching algorithm on an edge-Markovian evolving graph of $\max\{p, 1-q\} = \Omega(1/n)$. For any initial configuration, $T_{bal} = O(r \log(\Delta n))$ w.h.p. where $r := \frac{\max\{p, 1-q\}}{\min\{p, 1-q\}} = \max\left\{\frac{1-q}{p}, \frac{p}{1-q}\right\}$.

Theorem 1.1 gives a simple upper bound of the balancing time for a wide range of parameters $p, q$ of edge-Markovian evolving graphs. A simple setting for $r$ to be constant is the case that both $p$ and $q$ are constants. Another condition for $r$ to be a constant is the case of $p \sim 1 - q$. This case includes dynamic Erdős-Rényi random graphs (the case of $p = 1 - q$). In this situation, even if $p = (1 - \varepsilon)/n$ for a constant $0 < \varepsilon < 1$, we can apply Theorem 1.1 and obtain $T_{bal} = O(\log(\Delta n))$. Although each $G_t$ does not contain any giant component w.h.p. in this case, it is identical with the known upper bound of the complete graph [38].

Berenbrink et al. [6] investigated the balancing time for the simple load-balancing algorithm. In this algorithm, at each time step, an edge $\{v, u\}$ is randomly picked and tokens on $v$ and $u$ are reallocated by the random rounding (see Section 5.1 for the formal definition). They showed that, on the complete graph $K_n$, $T_{bal} = O(n \log(\Delta n))$ w.h.p. for this algorithm. In this paper, we also give the following result generalizing it.

Theorem 1.2. Consider the simple load-balancing algorithm on an edge-Markovian evolving graph of $\max\{p, 1-q\} = \Omega(1/n)$. Let $r := \frac{\max\{p, 1-q\}}{\min\{p, 1-q\}}$. Then, for any initial configuration, $T_{bal} = O\left(\frac{rn \log(\Delta n)}{\min\{p, 1-q\}}\right)$ w.h.p.

Our analysis is quite simple, while analyses of load-balancing algorithms or dynamic graphs, including the edge-Markovian evolving graph, often require advanced mathematics about transition matrices [24, 38, 36, 37, 12]. Our proof technique is based on token-based analysis for static complete graphs [6]: Suppose $K = \sum_{v \in V} \Gamma_t(v)$ tokens have distinct labels and each stacked token on a vertex $v \in V$ is allocated a height in $1, 2, \ldots, \Gamma_t(v)$. In [6], the authors proposed a movement rule of tokens (called the skip mode) corresponding to an update of a configuration, where every token’s height does not increase. Furthermore, they guarantee that any token’s height sufficiently decreases w.h.p. on $K_n$. We deal with this technique more carefully to apply it to the edge-Markovian evolving graph. In particular, we take care of the imaginary tokens called inverted tokens to discuss the minimum height $\min_{v \in V} \Gamma_t(v)$ and $\max_{v \in V} \Gamma_t(v)$ together (Section 4.1). It enables us to provide a framework for analyzing the balancing time only using token-based analysis (Theorem 2.1). Our main theorem is derived from the framework and a careful estimation of conditional probabilities concerning a random matching on the edge-Markovian evolving graph (Lemma 3.1).

1.2 Related works

Several early works [18, 39, 25, 34] consider the load-balancing problem with continuous load ($L(v) \in \mathbb{R}$). In other words, each node $v \in V$ does not need any rounding but can exchange the ideal amount of load with its neighbors, e.g., $L(v)/2 \in \mathbb{R}$. In this setting, the propagation of the load on a graph is highly related to the probability distribution of Markov chains. For example, the time taken for some balancing models to reach a constant discrepancy have been shown using the second largest eigenvalue or the graph conductance [18, 39]. Compared to the continuous case, the load-balancing problem with discrete tokens ($L(v) \in \mathbb{N}$) is much harder to analyze due to the rounding errors caused in each step and each vertex. Throughout the paper, we consider the load-balancing problem with discrete tokens case.

Diffusion-based algorithms have been also well studied for the load-balancing problem [35, 23, 11, 4, 8]. Roughly speaking, in a diffusion-based algorithm on a $d$-regular graph, every vertex $v \in V$ sends $\lfloor L(v)/d \rfloor$ tokens to its all neighbors at each time step. There are many works on the discrepancy of diffusion-based algorithms. For example, Rabani et al. [35] showed that, on any $d$-regular graphs, the discrepancy of a
diffusion-based algorithm using the rounding down is at most \( O\left(\frac{d \log n}{1 - 3}\right) \) after \( O\left(\frac{\log(\Delta n)}{1 - 3}\right) \) steps. To obtain a smaller discrepancy, diffusion-based algorithms combining the rounding up and down \( [23] \), distributing tokens by the round-robin algorithm (called the rotor-router) \([1]\), and using a randomized rounding \([4]\) have been also studied. Recently, Berenbrink et al. proposed a sophisticated deterministic rounding framework and showed the \( O(d)\) discrepancy \([8]\).

Random matching algorithms are originally introduced by Ghosh and Muthukrishnan \([25]\), with a motivation of a more efficient way to send tokens than the diffusive way. They proposed an algorithm referred to as the LR algorithm, that generates random matching in a distributed way. Note that the LR algorithm uses the degree information on the adjacent vertices if the graph is irregular. Boyd et al. \([10]\) proposed an algorithm generating random matching called the distributed synchronous algorithm, which uses the maximum degree information. The discrepancy of these algorithms has been studied in \([24, 38, 11]\), e.g., Friedrich and Sauerwald \([24]\) showed that the discrepancy after \( O\left(\frac{\log(\Delta n)}{1 - 3}\right) \) steps is at most \( O\left(\sqrt{\frac{\log^3 n}{1 - 3}}\right) \) w.h.p. on any \( d\)-regular graph. As mentioned above, a constant discrepancy on any \( d\)-regular graph has been shown in \([38]\). Several works focus on deterministic (periodic) matching algorithms, called balancing circuit models \([35, 24, 38]\).

The simple load-balancing introduced in \([6]\) appears as a subroutine in the population protocol \([7, 5]\). Recently, Huang and Wang \([26]\) study the balancing time of the simple load-balancing on complete bipartite graphs.

## 2 Preliminaries

### 2.1 Edge-Markovian graph, and other terminologies about (static) graphs

An edge-Markovian graph is a sequence of (static) graphs \( G = G_0, G_1, G_2, \ldots \) where every graph \( G_t = (V, E_t) \) \( t = 0, 1, 2, \ldots \) is undirected and simple. An edge-Markovian graph \( G \) is characterized by \( G_0 = (V, E_0), p \in (0, 1) \) and \( q \in [0, 1) \). The vertex set \( V \) is invariant with respect to \( t \), where let \( n = |V| \) for convenience. The edge set \( E_t = (t = 1, 2, \ldots) \) is a random variable depending only on \( E_{t-1} \): when a distinct vertex pair \( \{u, v\} \) is NOT an edge of \( E_{t-1} \), the pair \( \{u, v\} \) becomes an edge of \( E_t \) with probability \( p \); when \( \{u, v\} \) is an edge of \( E_{t-1} \), the pair \( \{u, v\} \) withdraws from \( E_t \) with probability \( q \). In other words, let

\[
X_t(\{u, v\}) = \begin{cases} 
0 & \text{if } \{u, v\} \notin E_t \\
1 & \text{if } \{u, v\} \in E_t 
\end{cases}
\]

for any distinct pair of vertices \( \{u, v\} \in \binom{V}{2} \) and \( t = 0, 1, 2, \ldots \), and let \( P = (p_{ij}) \in \mathbb{R}^{2 \times 2} \) be a probability matrix given by

\[
p_{ij} = \Pr[X_{t+1}(\{u, v\}) = j - 1 \mid X_t(\{u, v\}) = i - 1]
\]

for \( i \in \{1, 2\} \) and \( j \in \{1, 2\} \), then

\[
P = \begin{pmatrix} 1 - p & p \\ q & 1 - q \end{pmatrix}
\]

holds (see also Fig. \([1]\)). It might be worth to mention the fact, though we do not use it in this paper, that \( \Pr[\{u, v\} \in E_t] \) approaches \( p/(p + q) \) asymptotic to \( t \). Furthermore, if \( p + q = 1 \) then \( \Pr[\{u, v\} \in E_t] = p/(p + q) \) always hold for \( t = 1, 2, \ldots \), otherwise \( \Pr[\{u, v\} \in E_t] \) is enough close to \( p/(p + q) \) in \( O(1/\log(1 - p - q)) \) time steps. See Appendix \([9]\) for more detail.

\(^3\)We emphasis that \( p + q \neq 0 \) since \( p \in (0, 1] \). The condition \( p + q \neq 0 \) excludes the case of \( G \) being static, i.e., \( G_0 = G_1 = G_2 = \cdots \) hold if and only if \( p = q = 0 \).
Figure 1: Table of conditional probabilities.

Since this paper is concerned with an arbitrary initial graph $G_0$ as an worst case analysis, we in this paper let $G(n, p, q) = G_0, G_1, G_2, \ldots$ represent an edge-Markovian graph, i.e., $G_0$ in the characterization of an edge-Markovian graph is replaced by just the number of vertices $n$.

Other terminologies about static graphs

Let $G = (V, E)$ be a (static) undirected simple graph with $n = |V|$ vertices. Let $G$ denote the entire set of graphs with $n$ vertices. Let $\delta_G(v) = |\{\{v, u\} \in E \mid u \in V\}|$ denote the degree of a vertex $v \in V$. A set of edges $M \subseteq E$ is a matching if every pair of edges never shares the end vertices, i.e., $e = \{u, v\}$ and $f = \{w, x\}$ satisfies $e \cap f = \emptyset$ for any distinct $e, f \in M$. For convenience, let $M(v)$ denote the vertex matched with a vertex $v \in V$ in a matching $M$, i.e., $u = M(v)$ if $\{v, u\} \in M$.

### 2.2 Random matching algorithm on edge-Markovian graph

Random matching is a comprehensive method for load balancing on graphs. This section describes the random matching algorithm on an edge-Markovian graph, and describes the main theorem of the paper.

#### 2.2.1 Algorithm description

Let $G(n, p, q) = G_0, G_1, G_2, \ldots$ be an edge-Markovian graph, where $G_t = (V, E_t)$ denotes the graph at time $t$. Let $\Gamma_t \in \mathbb{N}^V$ denote the configuration of tokens at time $t = 0, 1, 2, \ldots$, where $\Gamma_t(v)$ denotes the number of tokens on $v \in V$. The initial configuration $\Gamma_0$ is given arbitrarily. For convenience, let $K := \sum_{v \in V} \Gamma_0(v)$ denote the total number of tokens, which is an invariant of $t$, and let

$$\Delta := \max_{v \in V} \Gamma_0(v) - \min_{v \in V} \Gamma_0(v).$$

(3)

The random matching algorithm stochastically updates the token configuration $\Gamma_t \mapsto \Gamma_{t+1}$ as follows. At time $t$, the algorithm randomly chooses a random matching $M_t \subseteq E_t$ according to some probability distribution $D_t$ (see Section 5 for examples). For every matching edge $\{v, u\} \in M_t$, we choose either

$$(\Gamma_{t+1}(v), \Gamma_{t+1}(u)) = \begin{cases} \left( \left\lfloor \frac{\Gamma_t(v) + \Gamma_t(u)}{2} \right\rfloor, \left\lceil \frac{\Gamma_t(v) + \Gamma_t(u)}{2} \right\rceil \right) & \text{(i), or} \\ \left( \left\lfloor \frac{\Gamma_t(v) + \Gamma_t(u)}{2} \right\rfloor, \left\lfloor \frac{\Gamma_t(v) + \Gamma_t(u)}{2} \right\rfloor \right) & \text{(ii)} \end{cases}$$

(4)

with probability $1/2$\footnote{This probability $1/2$ is just for simplicity of the algorithm description (by symmetry of $v$ and $u$), and it is not essential. We can replace the probability $1/2$ with any other probability, and it is easy to apply the argument of the paper to the variant.}. For all unmatched vertices $w \in V$, i.e., $\{w, w'\} \not\in M_t$ for any $w' \in V$, set $\Gamma_{t+1}(w) = \Gamma_t(w)$.

A very simple example of the random matching algorithm\footnote{“Choose a matching randomly” is a profound problem in contrast to it looks: for instance, choose a matching (almost) uniformly at random in bipartite graph had been investigated for a long time, see e.g., [28, 29]. Of course, it is easy for some certain distributions.} is the simple load balancing, which is analyzed for static complete graphs by [6] and for complete bipartite graph by [26]: the algorithm chooses a
Algorithm 1: Random matching algorithm on $G$

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Input : $G_0 = (V, E_0), p \in (0, 1], q \in [0, 1)$ and $\Gamma_0 \in \mathbb{N}^V$
Output: $\Gamma_T$

1 for $t = 0$ to $T - 1$
   2 generate a random matching $M_t \subseteq E_t$;  // see Algorithms 2, 3 and 4
   3 for $\{u, v\} \in M_t$ do                   
   4      | Eq. (4);                                
   5      end                                    
   6 // we obtain $\Gamma_{t+1}$                 
   7 $(G_{t+1}$ is generated from $G_t$);          // see Section 2.1
8 end                                          
9 return $\Gamma_T$;
```

vertex $v \in V$ uniformly at random, then chooses a single edge incident to $v$ as far as exist, i.e., the random matching consists of just a single edge as $M_t = \{\{u, v\}\}$, or empty. An edge $\{u, v\} \in E_t$ is chosen with probability $\frac{1}{n} \left( \frac{1}{d_G(v)} + \frac{1}{d_G(u)} \right)$. See Section 5 for other examples.

An execution of a random matching on an edge-Markovian graph is represented by a sequence of a triplet $(\Gamma_t, G_t, M_t)$. For convenience, let $\mathcal{E}_t = (\Gamma_0, G_0, M_0), (\Gamma_1, G_1, M_1), \ldots, (\Gamma_{t-1}, G_{t-1}, M_{t-1}), \Gamma_t$ denote the history of an execution until time $t$. In the execution, we remark, $G_t$ depends only on $G_{t-1}$, and $M_t$ is chosen according to $\mathcal{D}_t$ depending on $G_t$ (and possibly depends on $\mathcal{E}_t$, too). Depending on $\Gamma_t$ and $M_t$, the configuration $\Gamma_{t+1}$ is probabilistically determined (recall Eq. (4)).

2.2.2 $F$-fair condition and main theorem

We say the distribution $\mathcal{D}_t$, which a random matching $M_t \subseteq E_t$ follows, satisfies the $F$-fair condition (for $\mathcal{E}$) if there exists $F > 0$ such that

$$
\Pr \left[ \{u, v\} \in M_t \mid G_t = G, \mathcal{E} \right] \geq \frac{F}{\max\{d_G(v), d_G(u)\}}
$$

holds for any $\{u, v\} \in E_t$, for any graph $G \in \mathcal{G}$ and for any other events $\mathcal{E}$, where we assume the execution $\mathcal{E}_t$ as $\mathcal{E}$ but not limited to. We also say a random matching algorithm satisfies the $F$-fair condition if $M_t$ satisfies the condition Eq. (5) at any time $t$ in the algorithm. We remark that $F$ can be a function of $n$ such as $1/n$: we will show two examples of specific random matching algorithms in Section 5 where the algorithms respectively satisfy $F = 1/8$ and $F = 1/n$ conditions. Notice that $F$ cannot be more than 1 as far as $M_t$ is a matching.

Now, we are ready to describe our main theorem.

**Theorem 2.1.** Suppose a pair of $p \in (0, 1]$ and $q \in [0, 1)$ satisfy $\max\{p, 1 - q\} \geq \theta / n$ for a constant $\theta > 0$. Let $G(n, p, q) = G_0, G_1, G_2, \ldots$ be an edge Markovian graph, and let $\Gamma_0 \in \mathbb{N}^V$ be an initial configuration of tokens. If the random matching algorithm satisfies the $F$-fair condition ($0 < F \leq 1$) given

Proof: Suppose $v \in V$ satisfies $d_G(v) = \max_{u \in V} d_G(u)$. Under [5], the expected number of partners of $v$ in a random matching $M_t$ satisfies $E[|\{u, v\} \in E \mid M_t(v) = u|] = \sum_{u, v} \Pr [\{u, v\} \in M_t \mid G_t = G, \mathcal{E}] \geq \sum_{u, v} \frac{F}{d_G(v)} = F$. Since any matching $M_t$ must satisfy $|\{u, v\} \in E \mid M_t(v) = u| \leq 1$, we get $F \leq 1$.

The assumption of a constant $\theta$ is just for simplicity of the arguments. We can establish a similar theorem for $\theta = o(n)$ if we allow $c$ to be a function of $p$ and $q$. See the supplementary argument of the theorem just below, and the definition of $c_*$.
in (5) then its balancing time \( T_{bal} \) satisfies

\[
\Pr \left[ T_{bal} \leq \frac{cr \log \left( \frac{\Delta u}{\epsilon} \right)}{F} \right] \geq 1 - \epsilon
\]

for any \( \epsilon (0 < \epsilon < 1/4) \), where \( c \) is an appropriate constant, \( r := \max \left\{ \frac{p}{1-q}, \frac{1-q}{p} \right\} \geq 1 \) and \( \Delta := \max_{v \in V} \Gamma_0(v) - \min_{v \in V} \Gamma_0(v) \) (cf., (3)).

In fact, we give a constant \( c = 91/c_* \) for Theorem 2.1 in our proof, where

\[
c_* := \left(1 - \exp \left( -\frac{\theta}{3} \right) \right)^2
\]

This constant \( c \) is not optimized at all due to the simplification of the arguments. We remark that \( c_* = c_*(\theta) \) is monotone increasing with respect to \( \theta (\theta > 0) \), such that \( c_*(+0) = 0, c_*(1) \approx 0.02678 \) and \( c_*(\infty) = 0.5 \).

3 A Lemma for Theorem 2.1

As a preliminary step of the proof of Theorem 2.1, this section establishes a key lemma about a probability of a random matching in an edge-Markovian graph. A random matching \( M_t \subseteq E_t \) clearly depends on \( G_t \), and the graph \( G_t \) depends on \( G_{t-1} \). It makes the analysis of \( \Gamma_t \) complex compared with some simpler models of dynamic graphs, so-called independent random graph model, where all \( G_t \) are mutually independent. To avoid the difficulty caused by the history-dependence of the edge-Markovian model, we give a useful lower bound of the probability that a specific vertex \( v \) is matched with a desired vertex of \( U_t \subseteq V \), where \( U_t \) is given randomly depending on the history of an execution \( \mathcal{E}_t \). The lower bound plays a key role in the proof of Theorem 2.1 in Section 4.

Lemma 3.1. Suppose a pair of \( p \in (0, 1] \) and \( q \in [0, 1) \) satisfy \( \max\{p, 1-q\} \geq \theta/n \) for a constant \( \theta > 0 \). Let \( G(n, p, q) = G_0, G_1, G_2, \ldots \) be an edge Markovian graph, and let \( M_t \subseteq E_t \) be a random matching of \( G_t = (V, E_t) \) according to a distribution \( \mathcal{D}_t \). Let \( U_t \subseteq V \) be any random variable. Note that \( M_t \) and \( U_t \) may depend on each other as well as any other random variables; for convenience let \( \mathcal{E} \) denotes any possible event, e.g., an execution \( (\Gamma_0, G_0, M_0), (\Gamma_1, G_1, M_1), (\Gamma_2, G_2, M_2), \ldots \) in Section 2.2. If \( \mathcal{D}_t \) satisfies the F-fair condition (5) for \( \mathcal{E} \) then

\[
\Pr [M_t(v) \in U_t | \mathcal{E}] \geq \frac{c_* F}{r} E \left[ \frac{|U_t|}{n} | \mathcal{E} \right]
\]

holds, where \( c_* \) is a constant given by Eq. (7), and \( r := \max \left\{ \frac{p}{1-q}, \frac{1-q}{p} \right\} \).

Proof. For convenience, let \( D := 2n \max\{p, 1-q\} \), and let

\[
\mathcal{G}_{v,u} = \left\{ G = (V, E) \in \mathcal{G} \mid \{v,u\} \in E, d_G(v) \leq D + 1, d_G(u) \leq D + 1 \right\}
\]

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\footnote{The assumption of a constant \( \theta \) is just for \( c_* \) to be a constant to \( p \) and \( q \). We can establish a similar lemma for \( \theta = o(n) \) if we allow \( c_* \) to be a function of \( p \) and \( q \). See the definition of \( c_* \) in Eq. (7).}
for \( v, u \in V \). Considering the marginal probabilities, we see that

\[
\Pr [M_t(v) \in U_t | \mathcal{E}] = \sum_{t \in 2^V} \Pr [M_t(v) \in U | U_t = U, \mathcal{E}] \cdot \Pr [U_t = U | \mathcal{E}]
\]

\[
= \sum_{t \in 2^V} \sum_{u \in U} \Pr [M_t(v) = u | U_t = U, \mathcal{E}] \cdot \Pr [U_t = U | \mathcal{E}]
\]

\[
= \sum_{t \in 2^V} \Pr \{\{v, u\} \in M_t | U_t = U, \mathcal{E}\} \cdot \Pr [U_t = U | \mathcal{E}]
\]

\[
= \sum_{t \in 2^V} \sum_{u \in U} \Pr \{\{v, u\} \in M_t | U_t = U, G_t = G, \mathcal{E}\} \cdot \Pr [U_t = U, G_t = G | \mathcal{E}]
\]

\[
\geq \frac{F}{D+1} \sum_{t \in 2^V} \sum_{u \in U} \Pr [U_t = U, G_t = G | \mathcal{E}] \quad \text{(by Eq. (5) and } \max\{d_G(u), d_G(v)\} \leq D\}
\]

\[
= \frac{F}{D+1} \sum_{t \in 2^V} \sum_{u \in U} \Pr [U_t = U, G_t \in \mathcal{G}, u | \mathcal{E}]
\]

\[
= \frac{F}{D+1} \sum_{t \in 2^V} \sum_{u \in U} \Pr [G_t \in \mathcal{G}, u | U_t = U, \mathcal{E}] \Pr [U_t = U | \mathcal{E}]
\]

(9)

hold.

Concerning the term \( \Pr [G_t \in \mathcal{G}, u | U_t = U, \mathcal{E}] \) in Eq. (9), we will claim

\[
\Pr [G_t \in \mathcal{G}, u | U_t = U, \mathcal{E}] \geq \min\{p, 1-q\} \left(1 - \exp \left(-\frac{\theta}{3}\right)\right)^2
\]

(10)

holds for any \( U \in 2^V \) and \( u \in U \). In fact,

\[
\Pr [G_t \in \mathcal{G}, u | U_t = U, \mathcal{E}]
\]

\[
= \Pr \left[ \{v, u\} \in E_t, \left. \begin{array}{l}
d_{G_t}(v) \leq D+1, \\
d_{G_t}(u) \leq D+1 \end{array} \right| U_t = U, \mathcal{E} \right]
\]

\[
= \Pr \left[ \{v, u\} \in E_t | U_t = U, \mathcal{E} \right] \cdot \Pr \left[ \begin{array}{l}
d_{G_t}(v) \leq D+1, \\
d_{G_t}(u) \leq D+1 \end{array} \right| \{v, u\} \in E_t, U_t = U, \mathcal{E} \right]
\]

\[
\geq \min\{p, 1-q\} \Pr \left[ \begin{array}{l}
d_{G_t}(v) \leq D+1, \\
d_{G_t}(u) \leq D+1 \end{array} \right| \{v, u\} \in E_t, U_t = U, \mathcal{E} \right]
\]

(11)

holds, where the last inequality follows \( \Pr \left[ \{v, u\} \in E_t | U_t = U, \mathcal{E} \right] \geq \min\{p, 1-q\} \) since \( \{v, u\} \in E_t \) depends only on \( E_{t-1} \) in the edge-Markovian model (cf., Section 2.1) and the probability is \( 1-q \) (if \( \{v, u\} \in E_{t-1} \) or \( \{v, u\} \not\in E_{t-1} \)).

To evaluate the second term of Eq. (11), let \( X_t(\{w, w'\}) \) for \( \{w, w'\} \in \binom{V}{2} \) be independent binary random variables given by \( X_t(\{w, w'\}) = 1 \) if \( \{w, w'\} \in E_t \), otherwise \( X_t(\{w, w'\}) = 0 \). Then, \( d_G(w) = \)

\footnote{We in this paper assume in a marginal probability \( \sum_{x \in \Omega} \Pr[A | X = x] \Pr[X = x] \) that \( \Pr[A | X = x] \Pr[X = x] = 0 \) if \( \Pr[X = x] = 0 \). In other words, "\( x \in \Omega \)" in the subscription of \( \sum \) is an abbreviation of "\( x \in \{x' \in \Omega | \Pr[X = x'] \neq 0\} \)."}
\[ \sum_{w' \in V \setminus \{w\}} X_t(\{w, w'\}) \] holds. Then,

\[
\Pr \left[ \begin{array}{l} d_{G_t}(v) \leq D + 1, \\
\sum_{w \in V \setminus \{u\}} X_t(\{v, w\}) \leq D + 1 \end{array} \right] \\
= \Pr \left[ \begin{array}{l} \sum_{w \in V \setminus \{v\}} X_t(\{v, u\}) \leq D + 1, \\
\sum_{w \in V \setminus \{v, u\}} X_t(\{v, w\}) \leq D + 1 \end{array} \right] \\
= \Pr \left[ \begin{array}{l} \sum_{w \in V \setminus \{v\}} X_t(\{v, u\}) \leq D, \\
\sum_{w \in V \setminus \{u\}} X_t(\{v, w\}) \leq D \end{array} \right] \\
\cdot \Pr \left[ \begin{array}{l} \sum_{w \in V \setminus \{u\}} X_t(\{v, w\}) \leq D \end{array} \right] \\
= \Pr \left[ \sum_{w \in V \setminus \{v, u\}} X_t(\{v, w\}) \leq D \right] \tag{12}
\]
holds where both of the last two equalities follow the fact that \(X_t(\{w, w'\})\) for \(\{w, w'\} \in E\) are independent.

Concerning Eq. (12), we remark that its expectation satisfies

\[
E \left[ \sum_{w \in V \setminus \{v, u\}} X_t(\{v, w\}) \right] \quad \left| \begin{array}{l} U_t = U, E \end{array} \right] = \sum_{w \in V \setminus \{v, u\}} E \left[ X_t(\{v, w\}) \right] \left| \begin{array}{l} U_t = U, E \end{array} \right] \\
= \sum_{w \in V \setminus \{v, u\}} \Pr \left[ X_t(\{v, w\}) = 1 \left| \begin{array}{l} U_t = U, E \end{array} \right] \right] \\
\leq n \max\{p, 1 - q\} \tag{13}
\]
where the last inequality follows the edge-Markovian model Eq. (2). Thus, we have

\[
\Pr \left[ \sum_{w \in V \setminus \{v, u\}} X_t(\{v, w\}) \leq D \left| \begin{array}{l} U_t = U, E \end{array} \right] \right] \\
= 1 - \Pr \left[ \sum_{w \in V \setminus \{v, u\}} X_t(\{v, w\}) > D \left| \begin{array}{l} U_t = U, E \end{array} \right] \right] \\
\geq 1 - \Pr \left[ \sum_{w \in V \setminus \{v, u\}} X_t(\{v, w\}) \geq 2n \max\{p, 1 - q\} \left| \begin{array}{l} U_t = U, E \end{array} \right] \right] \quad \text{(since } D = 2n \max\{p, 1 - q\} \text{)} \\
\geq 1 - \exp \left( -\frac{n \max\{p, 1 - q\}}{3} \right) \quad \text{(by Lemma A.2(i) using Eq. (13))} \\
\geq 1 - \exp \left( -\frac{\theta}{3} \right) \quad \text{(since } n \max\{p, 1 - q\} \geq \theta \text{ by assumption)} \tag{14}
\]
hold. By Eqs. Eqs. (11), (12) and (14), we obtain the desired claim Eq. (10).
Now, combining Eqs. (9) and (10), we obtain
\[
\Pr \left[ M_t(v) \in U_t \mid \mathcal{E} \right] \\
\geq \frac{F}{D + 1} \sum_{U \in 2^V} \sum_{u \in U} \Pr \left[ G_t \in \mathcal{G}_{v,u} \mid U_t = U, \mathcal{E} \right] \Pr \left[ U_t = U \mid \mathcal{E} \right] 
\text{(by Eq. (9))}
\geq \frac{F \min \{ p, 1 - q \} \left( 1 - \exp \left( -\frac{\theta}{3} \right) \right)^2}{D + 1} \sum_{U \in 2^V} \sum_{u \in S} \Pr \left[ U_t = U \mid \mathcal{E} \right] 
\text{(by Eq. (10))}
= \frac{F \min \{ p, 1 - q \} \left( 1 - \exp \left( -\frac{\theta}{3} \right) \right)^2}{D + 1} \sum_{U \in 2^V} \left( |U| \cdot \Pr \left[ U_t = U \mid \mathcal{E} \right] \right) 
= \frac{F \min \{ p, 1 - q \} \left( 1 - \exp \left( -\frac{\theta}{3} \right) \right)^2}{D + 1} \mathbb{E} \left[ |U_t| \mid \mathcal{E} \right] 
\text{(15)}
\]
hold. Finally, we remark that the coefficient of Eq. (15) satisfies
\[
\frac{F \min \{ p, 1 - q \} \left( 1 - \exp \left( -\frac{\theta}{3} \right) \right)^2}{D + 1} \geq \frac{F \min \{ p, 1 - q \} \left( 1 - \exp \left( -\frac{\theta}{3} \right) \right)^2}{2n \max \{ p, 1 - q \} + 1} 
\text{(since } D = 2n \max \{ p, 1 - q \} \text{ by definition)}
\geq \frac{F \min \{ p, 1 - q \} \left( 1 - \exp \left( -\frac{\theta}{3} \right) \right)^2}{2n \max \{ p, 1 - q \} \left( 1 + \frac{1}{2\theta} \right)} 
\text{(since } 2n \max \{ p, 1 - q \} \geq 2\theta \text{ by assumption)}
= \frac{\left( 1 - \exp \left( -\frac{\theta}{3} \right) \right)^2}{2 + \frac{1}{\theta}} \frac{F}{\max \{ p, 1 - q \} / \min \{ p, 1 - q \}} \frac{1}{n} 
= \frac{c \cdot \frac{F}{r} \frac{1}{n}}{2 + \frac{1}{\theta}} \left( \text{where } \frac{\max \{ p, 1 - q \}}{\min \{ p, 1 - q \}} = \max \left\{ \frac{1 - q}{p}, \frac{p}{1 - q} \right\} = r \right) 
\text{(16)}
\]
where \( c = \frac{\left( 1 - \exp \left( -\frac{\theta}{3} \right) \right)^2}{2 + \frac{1}{\theta}} \) is given by Eq. (7). Eq. (8) is clear from Eqs. (15) and (16). \( \square \)

4 Proof of Theorem 2.1

We prove Theorem 2.1 by a version of the token-based analysis developed by [6]. As a preliminary step, Section 4.1 introduces definitions for our token-based analysis. Section 4.2 briefly explains our proof strategy based on the token-based analysis, and the detail of the proof follows.

4.1 Preliminary for a token-based analysis

The idea of the token-based analysis is to track the place and height for each token in an execution of a load-balancing algorithm, where we assume that every token has a unique ID, tokens on a vertex are stacked in a pile, and tokens are orderly reallocated in each time step by the load-balancing algorithm under a refined description of the procedures.

Suppose we have an initial token configuration \( \Gamma_0 \in \mathbb{N}^V \) of \( K = \sum_{v \in V} \Gamma_0(v) \) tokens. The \( K \) tokens have distinct labels (i.e., unique IDs) \( a_1, a_2, \ldots, a_K \). For convenience, let \( A \) denote the entire set of tokens, i.e., \( A = \{ a_1, \ldots, a_K \} \). Every token \( a \in A \) is allocated \( (P_t(a), H_0(a)) \in V \times \mathbb{N}_{>0} \), where \( P_t(a) \) denotes
Figure 2: An example of reallocation of tokens and complementary tokens from $\Gamma_t$ to $\Gamma_{t+1}$. Here, $\Gamma_t(v) = 6$ and $\Gamma_t(u) = 1$ (left figure), and assume that the total number of tokens $K = \sum_{v \in V} \Gamma_t(v') = 8$, meaning that one more token places another vertex $w$ behind the figure. Suppose $\{u, v\} \in M_t$. Then, either $\Gamma_{t+1}(v) = 4$ and $\Gamma_{t+1}(u) = 3$ (middle figure), or $\Gamma_{t+1}(v) = 3$ and $\Gamma_{t+1}(u) = 4$ (right figure) are obtained with equally probability $1/2$. For an example of the individual token’s move, token $a_5$ which places $(P_t(a_5), H_t(a_5)) = (v, 5)$ at time $t$ (left fig.) moves to $(P_{t+1}(a_5), H_{t+1}(a_5)) = (u, 3)$ (middle fig.) or $(v, 3)$ (right fig.) at time $t + 1$. For an example of the complementary token’s move, token $b_5$ places $(P_t(b_5), H_t(b_5)) = (u, 3)$ at time $t$, where notice that $H_t(b_5) = K + 1 - H(b_5)$ holds. Then, $b_5$ moves to $(P_{t+1}(a_5), H_{t+1}(a_5)) = (u, 3)$ (middle fig.) or $(v, 3)$ (right fig.) at time $t + 1$.

the vertex on which the token $a$ places at time $t (t = 0, 1, 2, \ldots)$, and $H_t(a) (1 \leq H_t(a) \leq \Gamma_t(v))$ represents the “height” of the token $a$ in the “pile” of $\Gamma_t(v)$ tokens at vertex $v = P_t(a)$; thus $\{H_t(a') | P_t(a') = v (a' \in A)\} = \{1, 2, \ldots, \Gamma_t(v)\}$ must hold for any $v \in V$ (see Fig.2 for example).

Then, we define the procedure to update $(P_t(a), H_t(a))$, meaning that it is a refinement of the random-matching algorithm (on an edge-Markovian graph) given in Section 2.2. Suppose that $\Gamma_t$ is the token configuration at time $t$, and that $M_t$ is the random matching given by the random matching algorithm. Let $\{u, v\}$ be an edge of $M_t$. Without loss of generality, we may assume that $\Gamma_t(v) \geq \Gamma_t(u)$. If the token $a$ satisfies $H_t(a) \leq \Gamma_t(u)$, then the token $a$ stays as it is, i.e., $(P_{t+1}(a), H_{t+1}(a)) = (P_t(a), H_t(a))$ (see e.g., token $a_1$ in Fig.2). Suppose $H_t(a) > \Gamma_t(u)$ (see e.g., token $a_5$ in Fig.2), which implies $P_t(a) = v$ since $\Gamma_t(v) \geq \Gamma_t(u)$. Then, the token $a$ moves to

$$\begin{align*}
(P_{t+1}(a), H_{t+1}(a)) & = \begin{cases} 
(v, \Gamma_t(u) + k) & \text{(if } H_t(a) - \Gamma_t(u) = 2k - 1 \text{ ) } \left( k \in \{1, 2, \ldots\} \right) \\
(u, \Gamma_t(u) + k) & \text{(if } H_t(a) - \Gamma_t(u) = 2k \text{ ) } \left( k \in \{1, 2, \ldots\} \right)
\end{cases}
\end{align*}$$

in case (i) of Eq. (4) (middle in Fig.2), while

$$\begin{align*}
(P_{t+1}(a), H_{t+1}(a)) & = \begin{cases} 
(v, \Gamma_t(u) + k) & \text{(if } H_t(a) - \Gamma_t(u) = 2k \text{ ) } \left( k \in \{1, 2, \ldots\} \right) \\
(u, \Gamma_t(u) + k) & \text{(if } H_t(a) - \Gamma_t(u) = 2k - 1 \text{ ) } \left( k \in \{1, 2, \ldots\} \right)
\end{cases}
\end{align*}$$

in case (ii) (right in Fig.2).\(^{10}\) It is easy to see that this procedure provides the configuration $\Gamma_{t+1}$ defined by the random matching algorithm. It is also not difficult to see the following facts.

**Observation 4.1.** The function $H_t (t = 0, 1, 2, \ldots)$, which is sequentially provided by the above procedure, has the following two properties.

\(^{10}\)When $\Gamma_t(v) - \Gamma_t(u)$ is even, cases (i) and (ii) of Eq. (4) provides the same configuration $\Gamma_{t+1}(v)$ and $\Gamma_{t+1}(v)$ for $v$ and $u$. Concerning token reallocation, either way of (i) and (ii) is fine: Each provides the following property (A1) and (A2), and it does not cause any trouble in the following our analysis.
(A1) \( H_t \) is monotone nonincreasing with respect to \( t \), i.e., \( H_{t+1}(a) \leq H_t(a) \) for any token \( a \in A \) and any time \( t = 0, 1, 2, \ldots \).

(A2) Suppose \( \{v, u\} \in M_t \) and \( \Gamma_t(v) \geq \Gamma_t(u) \). If a token \( a \in A \) satisfies both \( P_t(a) = v \) and \( H_t(a) \geq \Gamma_t(u) \), then \( H_{t+1}(a) - \Gamma_t(u) = \left\lceil \frac{H_t(a) - \Gamma_t(u)}{2} \right\rceil \) holds.

Next, we introduce a gadget of complementary tokens, the concept of which is similar to the one used in \[6\], in order to use a symmetric argument to simplify our token-based analysis. Let \( \overline{\Gamma}_t \in \mathbb{N}^V \) for \( t = 0, 1, 2, \ldots \) be defined by \( \overline{\Gamma}_t(v) := K - \Gamma_t(v) \) for all \( v \in V \), where we call \( \Gamma_t \) the configuration of complementary tokens at time \( t \). For convenience, let \( K := \sum_{v \in V} \overline{\Gamma}_0(v) = K(n - 1) \) denote the total number of complementary tokens, and let \( \overline{\mu} := \frac{1}{n} \sum_{v \in V} \overline{\Gamma}_0(v) = K(n - 1)/n \) denote its average.

It is not difficult to see that if the time series \( \Gamma_0, \Gamma_1, \Gamma_2, \ldots \) follows the random matching algorithm then the time series \( \overline{\Gamma}_0, \overline{\Gamma}_1, \overline{\Gamma}_2, \ldots \) itself also follows the random matching algorithm with exactly the same matchings \( M_t \ (t = 0, 1, 2, \ldots) \). Then, we will define the procedure for complementary tokens, which is essentially the same as the procedure for \( A \). We assume that complementary tokens also have distinct labels \( b_1, b_2, \ldots, b_K \), and let \( B \) denote the entire set of complementary tokens. Every complementary token \( b \in B \) is allocated \( \overline{P}_0(b), \overline{H}_0(b) \in V \times \mathbb{N}_{\geq 0} \) where \( \overline{P}_t(b) \) denotes the vertex on which the token \( b \) places at time \( t \) (\( t = 0, 1, 2, \ldots, \)), and \( \overline{H}_t(b) \) \( (1 \leq \overline{H}_t(b) \leq \overline{\Gamma}_t(v)) \) represents the “height” of the token \( b \) at \( v = \overline{P}_t(b) \). For convenience, we define \( \overline{H}_t(b) := K + 1 - \overline{H}_t(b) \) for \( b \in B \). Then, \( \overline{\Gamma}(v) + 1 \leq \overline{H}_t(b) \leq K \) holds. It looks that complementary tokens are stacked on tokens \( \Gamma(v) \), in the inverse order of \( \overline{H}_t(b) \). Thus, we call \( \overline{H}_t \) inverted height.

Then, we define the procedure of the random matching algorithm for complementary tokens. Let \( \overline{\Gamma}_t \) and \( \overline{P}_t \) be respectively the configurations of tokens and complementary tokens at time \( t \). Suppose that \( M_t \) is a random matching, and \( \{u, v\} \in M_t \). Without loss of generality we may assume \( \overline{\Gamma}_t(v) \geq \overline{\Gamma}_t(u) \). Then \( \overline{\Gamma}_t(v) \leq \overline{\Gamma}_t(u) \). If the token \( b \) satisfies \( \overline{H}_t(a) \leq \overline{\Gamma}_t(u) \), then the token \( b \) stays as it is i.e., \( (\overline{P}_{t+1}(b), \overline{H}_{t+1}(b)) = (\overline{P}_t(a), \overline{H}_t(b)) \). Suppose \( \overline{H}_t(b) > \overline{\Gamma}_t(u) \). Notice that \( \overline{P}(b) = u \). Then, the token \( b \) moves to
\[
(\overline{P}_{t+1}(b), \overline{H}_{t+1}(b)) = \begin{cases} 
(v, \overline{\Gamma}_t(v) + k) & \text{if } \overline{H}_t(b) - \overline{\Gamma}_t(v) = 2k - 1 \ (k \in \{1, 2, \ldots\}) \\
(u, \overline{\Gamma}_t(v) + k) & \text{if } \overline{H}_t(b) - \overline{\Gamma}_t(v) = 2k \ (k \in \{1, 2, \ldots\})
\end{cases}
\]
in case (i) of Eq. \([4]\) (see also middle in Fig. \([2]\), while
\[
(\overline{P}_{t+1}(b), \overline{H}_{t+1}(b)) = \begin{cases} 
(v, \overline{\Gamma}_t(v) + k) & \text{if } \overline{H}_t(b) - \overline{\Gamma}_t(v) = 2k - 1 \ (k \in \{1, 2, \ldots\}) \\
(u, \overline{\Gamma}_t(v) + k) & \text{if } \overline{H}_t(b) - \overline{\Gamma}_t(v) = 2k \ (k \in \{1, 2, \ldots\})
\end{cases}
\]
in case (ii) of Eq. \([4]\) (see also right in Fig. \([2]\)). We can see that this procedure provides the configuration \( \overline{\Gamma}_{t+1} \) defined by the random matching algorithm. The following observation is essentially the same as Observation \([4,1]\).

Observation 4.2. The function \( \overline{H}_t \ (t = 0, 1, 2, \ldots) \), which is sequentially provided by the above procedure, has the following two properties.

(B1) \( \overline{H}_t \) is monotone nonincreasing with respect to \( t \), i.e., \( \overline{H}_{t+1}(b) \leq \overline{H}_t(b) \) for any token \( b \in B \) and any time \( t = 0, 1, 2, \ldots \).

(B2) Suppose \( \{v, u\} \in M_t \) and \( \overline{\Gamma}_t(v) \leq \overline{\Gamma}_t(u) \). If a token \( b \in B \) satisfies both \( \overline{P}_t(b) = v \) and \( \overline{H}_t(b) \geq \overline{\Gamma}_t(u) \), then \( \overline{H}_{t+1}(b) - \overline{\Gamma}_t(u) = \left\lceil \frac{\overline{H}_t(b) - \overline{\Gamma}_t(u)}{2} \right\rceil \) holds.

From Observations \([4,1]\) and \([4,2]\), we can show the following lemma, which forms the basis of our analysis in Section \([4] \). Lemma \([4,3]\) will be used as \( x = \mu, \pi, \lceil \mu \rceil \) and \( \lceil \pi \rceil \) in the following sections.
Lemma 4.3. Let \( x \in \mathbb{R} \) be an arbitrary real. The following holds for any \( t \in \mathbb{N} \).

(i) Let \( a \in A \) be a token satisfying \( H_t(a) \geq x \). Suppose that the vertex \( P_t(a) \) is matched with \( v \in V \) in \( M_t \), i.e., \( v = M_t(P_t(a)) \). If \( \Gamma_t(v) \leq x \) then \( H_{t+1}(a) - x \leq \left\lfloor \frac{H_t(a) - x}{2} \right\rfloor \).

(ii) Let \( b \in B \) be a complementary token satisfying \( \overline{H}_t(b) \geq x \). Suppose that the vertex \( \overline{P}_t(b) \) is matched with \( u \in V \) in \( M_t \), i.e., \( u = M_t(\overline{P}_t(b)) \). If \( \overline{\Gamma}_t(u) \leq x \) then \( \overline{H}_{t+1}(b) - x \leq \left\lfloor \frac{\overline{H}_t(b) - x}{2} \right\rfloor \).

Proof. We prove (i). To begin with, we see that

\[
H_{t+1}(a) - x = H_{t+1}(a) - \Gamma_t(v) + \Gamma_t(v) - x
= \left\lfloor \frac{H_t(a) - \Gamma_t(v)}{2} \right\rfloor + \Gamma_t(v) - x
\]

holds by Observations 4.1. If \( H_t(a) - \Gamma_t(v) \) is even then

\[
\left(17\right) = \frac{H_t(a) - \Gamma_t(v)}{2} + \Gamma_t(v) - x
= \frac{H_t(a) + \Gamma_t(v) - 2x}{2}
\]

holds, and we obtain the claim in this case.

Suppose \( H_t(a) - \Gamma_t(v) \) is odd. Then, we have

\[
\left(17\right) = \frac{H_t(a) - \Gamma_t(v)}{2} + \frac{1}{2} + \Gamma_t(v) - x
= \frac{H_t(a) + \Gamma_t(v) - 2x + 1}{2}
\]

holds. If \( \Gamma_t(v) - x \leq -1 \) then

\[
\left(18\right) \leq \frac{H_t(a) - x}{2} \leq \left\lfloor \frac{H_t(a) - x}{2} \right\rfloor
\]

holds, and we obtain the claim in this case.

The remaining case, that is \( H_t(a) - \Gamma_t(v) \) is odd and \( -1 < \Gamma_t(v) - x \leq 0 \). Since \(-1 < \Gamma_t(v) - x \leq 0\),

\[
\left\lfloor \frac{H_t(a) - \Gamma_t(v) - 1}{2} \right\rfloor < \frac{H_t(a) - x}{2} \leq \left\lfloor \frac{H_t(a) - \Gamma_t(v)}{2} \right\rfloor
\]

hold. Since \( H_t(a) - \Gamma_t(v) \) is odd, the strict inequality and the integrality of \(\left(19\right)\) imply

\[
\left\lfloor \frac{H_t(a) - \Gamma_t(v)}{2} \right\rfloor = \left\lfloor \frac{H_t(a) - x}{2} \right\rfloor
\]

(20) holds. Then,

\[
\left(17\right) = \left\lfloor \frac{H_t(a) - x}{2} \right\rfloor + \Gamma_t(v) - x \quad \text{(by (20))}
\]

holds. We obtain (i). The proof of (ii) is similar.
4.2 Proof strategy

Roughly speaking, Theorem 2.1 claims $\mu - 2 < \Gamma_{T_{\text{bal}}}(v) < \mu + 2$ for any $v \in V$ w.h.p. (see Lemma 4.8 in Section 4.3 for precise). We prove the claim in two phases: In Phase I (see Section 4.3), we prove at least one of $\mu - 1 \leq \min_{v \in V} \Gamma_{T_1}(v)$ or $\max_{v \in V} \Gamma_{T_1}(v) \leq \mu + 1$ holds w.h.p. for a sufficiently large $T_1$ (see Lemma 4.4 for more detail). In Phase II (see Section 4.4), we prove that if $\mu - 1 \leq \min_{v \in V} \Gamma_{T_1}(v)$ then $\max_{v \in V} \Gamma_{T_1+T_2}(v) < \mu + 2$ also holds w.h.p., as well as if $\max_{v \in V} \Gamma_{T_1}(v) \leq \mu + 1$ then $\mu - 2 < \min_{v \in V} \Gamma_{T_1+T_2}(v)$ also holds w.h.p. for a sufficiently large $T_2$ (see Lemma 4.6 for more detail).

We remark that $\min_{v \in V} \Gamma_{t}(v)$ and $\max_{v \in V} \Gamma_{t}(v)$ are respectively monotone non-decreasing/non-increasing with respect to $t$, which imply $\mu - 2 < \Gamma_{T_1+T_2}(v) < \mu + 2$ holds for any $v \in V$.

We prove the claims by the token-based analysis introduced in Section 4.1, using Lemmas 3.1 and 4.3. In the following arguments, we assume that an edge-Markovian evolving graph $G(n,p,q)$ satisfies $\max \{p, 1-q\} \geq \theta /n$ for a constant $\theta > 0$, and that a random matching algorithm satisfies the $F$ condition Eq. (5), according to the hypothesis of Theorem 2.1. For a technical reason, we also assume $\Delta \geq 2$ (recall Eq. (3)); otherwise it is already balanced, i.e., $T_{\text{bal}} = 0$, and the theorem is trivial.

4.3 Phase I

Let us begin with Phase I analysis. This section establishes the following lemma.

Lemma 4.4 (Phase I). Let $T_1 \geq \frac{360}{\varepsilon^2} \log(\frac{\Delta n}{\varepsilon})$ for $\varepsilon (0 < \varepsilon < 1/4)$. Then,

$$\Pr \left[ \max_{v \in V} \Gamma_{T_1}(v) \leq \mu + 1 \right] \geq 1 - \varepsilon^2$$

holds.$^{11}$

Proof. We prove the claim by the token-based analysis given in Section 4.1. For convenience, let $A^+ = \{a \in A \mid H_0(a) > \mu + 1\}$ and $B^+ = \{b \in B \mid H_0(b) > \mu + 1\}$. Firstly, we remark that $\max_{a \in A^+} H_t(a) \leq \mu + 1$ implies (if and only if, in fact) $\max_{v \in V} \Gamma_t(v) \leq \mu + 1$ since $H_t(a)$ is monotone non-increasing with respect to $t$, by Observation 4.1 (A1). Remark $\max_{b \in B^+} H_t(b) \leq \mu + 1$ implies $\max_{v \in V} \Gamma_t(v) \leq \mu + 1$ as well. Thus, our desired event is rephrased by

$$\left[ \max_{v \in V} \Gamma_{T_1}(v) \leq \mu + 1 \right] \quad \iff \quad \left[ \max_{a \in A^+} H_{T_1}(a) \leq \mu + 1 \right] \quad \iff \quad \left[ \bigwedge_{a \in A^+} \left[ H_{T_1}(a) - \mu \leq 1 \right] \right] \quad \iff \quad \left[ \bigwedge_{(a,b) \in A^+ \times B^+} \left[ \left[ H_{T_1}(a) - \mu \leq 1 \right] \lor \left[ H_{T_1}(b) - \mu \leq 1 \right] \right] \right]$$

where the last converse implication is for some technical reason of the arguments below.$^{12}$

Roughly speaking, Lemma 4.3 claim that $H_{t+1}(a) - \mu$ is reduced to almost a half of $H_t(a) - \mu$ when the vertex $P_{t}(a)$ is matched with a vertex $v$ satisfying $\Gamma_{t}(v) \leq \mu$ in $M_t$, in case that $H_t(a) - \mu > 0$. For

$^{11}$We here just remark that $\max_{v \in V} \Gamma_t(v) \leq \mu + 1$ is equivalent to $\min_{v \in V} L(v) \geq \mu - 1$. See Observation 4.5 for detail.

$^{12}$We are going to use a dichotomy in Eq. (26), below.
convenience, let $S(\Gamma_t) := \{ v \in V \mid \Gamma_t(v) \leq \mu \}$ and $\overline{S}(\Gamma_t) := \{ v \in V \mid \Gamma_t(v) \geq \mu \} = \{ v \in V \mid \overline{\Gamma}_t(v) \leq \overline{\mu} \}$ for each time $t = 0, 1, 2, \ldots$. Then, let

$$Y_t(a, b) = \begin{cases} 1 & \text{if } [M_t(P_t(a)) \in S(\Gamma_t)] \vee [M_t(P_t(b)) \in \overline{S}(\Gamma_t)] \\ 0 & \text{(otherwise)} \end{cases}$$  \hspace{1cm} (23)

for $(a, b) \in A^+ \times B^+$, where we remark that $[M_t(P_t(a)) \in S(\Gamma_t)]$ means “$\exists u \in S(\Gamma_t)$ such that $\{u, v\} \in M_t$ where $v = P_t(a)$.” In fewer words, $Y_t(a, b)$ denotes the indicator random variable of a desired matching at time $t$. To circumvent the effect of ceiling function in Lemma 4.3, we remark a fact that $13$ holds for $x \geq 2$, which implies $H(a) - \mu$ is reduced to half or less if the event $[M_t(P_t(a)) \in S(\Gamma_t)]$ occurs TWICE. Clearly, $\lceil \frac{x}{2} \rceil \leq 1$ holds for $1 \leq x \leq 2$, thus we get $H(a) - \mu \leq 1$ if we got the event $[M_t(P_t(a)) \in S(\Gamma_t)]$ at most $2 \log_2 \Delta + 1 \leq 3 \log_2 \Delta$ times$^{14,15}$, where we remark $H_0(a) - \mu \leq \max_{v \in V} \Gamma_0(v) - \min_{v \in V} \Gamma_0(v) = \Delta$. $[H_T_1(b) - \overline{\mu} \leq 1] \leq 1$ is as well. These imply that it is sufficient to get $[H_T_1(a) - \mu \leq 1] \vee [H_T_1(b) - \overline{\mu} \leq 1]$, that is, $\sum_{t=0}^{T_1-1} Y_t(a, b) \geq 6 \log_2 \Delta$ holds. In summary, we obtain

$$\Pr \left[ \max_{v \in V} \Gamma_T(v) \leq \mu + 1 \right] \vee \max_{v \in V} \overline{\Gamma}_T(v) \leq \overline{\mu} + 1 \right] \right]
\geq \Pr \left[ \sum_{t=0}^{T_1-1} Y_t(a, b) \geq 6 \log_2 \Delta \right]
\geq \sum_{(a, b) \in A^+ \times B^+} \Pr \left[ \sum_{t=0}^{T_1-1} Y_t(a, b) < 6 \log_2 \Delta \right] \right]$$

and the remaining task is to prove (24) is enough large.

To evaluate $\Pr[\sum_{t=0}^{T_1-1} Y_t(a, b) < 6 \log_2 \Delta]$, we estimate the probability of $Y_t(a, b) = 1$ for $t = 0, 1, \ldots, T_1 - 1$. Notice that the event $[Y_t(a, b) = 1]$ definitely depends on the history of the execution

$^{13}$Proof: To begin with, $\lceil \frac{x+1}{2} \rceil < \lceil \frac{x+2}{2} \rceil = \lceil \frac{x+2}{2} \rceil \rceil \leq \lceil \frac{x+1}{2} \rceil - 1 + 1 = \frac{x+6}{2} = \frac{x}{2} + \frac{x+6}{2} \leq \frac{x}{2} + 1$ where the last inequality follows the assumption $x \geq 2$. Now it is not difficult to see that $\lceil \frac{x+1}{2} \rceil \leq \frac{x}{2}$.

$^{14}$Let $y_i = \frac{x}{2}$ for $i = 1, 2, \ldots, n$ then $y_i \log_2 x = \frac{2x}{2^i \log_2 x} \leq 2$. Note $2(\log_2 x) - 1 + 1 = 2(\log_2 x) - 1 \leq 2 \log_2 x + 1$.

$^{15}$Here we use the technical assumption that $\Delta \geq 2$ for the inequality $2 \log_2 \Delta + 1 \leq 3 \log_2 \Delta$. 

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We give a lower bound of the probability of $Y_t(a, b) = 1$ independent of $\mathcal{E}_t$. To be precise,

$$\mathbb{P}[Y_t(a, b) = 1 \mid \mathcal{E}_t] = \mathbb{P}[M_t(P_t(a)) \in S(\Gamma_t) \vee M_t(P_t(b)) \in \overline{S}(\Gamma_t) \mid \mathcal{E}_t] \geq \max \{\mathbb{P}[M_t(P_t(a)) \in S(\Gamma_t) \mid \mathcal{E}_t], \mathbb{P}[M_t(P_t(b)) \in \overline{S}(\Gamma_t) \mid \mathcal{E}_t]\} \geq \frac{c_4 F}{r} \max \left\{ \mathbb{E}\left[\frac{|S(\Gamma_t)|}{n} \mid \mathcal{E}_t\right], \mathbb{E}\left[\frac{|\overline{S}(\Gamma_t)|}{n} \mid \mathcal{E}_t\right]\right\} \quad \text{(by Lemma 3.1)} \quad (25)$$

$$\geq \frac{c_4 F}{2r} \quad \text{(because $|S(\Gamma_t)| + |\overline{S}(\Gamma_t)| \geq n$, by the definitions)} \quad (26)$$

hold.\(^{16}\)

Then, we evaluate $\mathbb{P}[\sum_{t=0}^{T_1-1} Y_t(a, b) < 6 \log_2 \Delta]$. For convenience, let $Z_t$ ($t = 0, 1, \ldots, T_1 - 1$) be independent binary random variables such that $\mathbb{P}[Z_t = 1] = \frac{c_4 F}{2r}$, and let $Z = \sum_{t=0}^{T_1-1} Z_t$. Then, it is not difficult to see that

$$\mathbb{P} \left[ \sum_{t=0}^{T_1-1} Y_t(a, b) < 6 \log_2 \Delta \right] \leq \mathbb{P} \left[ \sum_{t=0}^{T_1-1} Y_t(a, b) \leq 6 \log_2 \Delta \right] \leq \mathbb{P} \left[ Z \leq 6 \log_2 \Delta \right] \quad (27)$$

holds. Note that

$$\mathbb{E}[Z] = T_1 \mathbb{P}[Z_t = 1] \geq \frac{36r}{c_4 F} \log(\Delta n/\varepsilon) \cdot \frac{c_4 F}{2r} \quad \text{(since $T_1 \geq \frac{36r}{c_4 F} \log(\Delta n/\varepsilon)$)}$$

$$\geq 18 \log(\Delta n/\varepsilon)$$

$$\geq 12 \log_2 \Delta \quad \text{(since $n \geq 1, \varepsilon < 1$ and $\log_2 e \leq 1.5$)} \quad (28)$$

holds. Then, we have

$$\mathbb{P} \left[ \sum_{t=0}^{T_1-1} Y_t(a, b) \leq 6 \log_2 \Delta \right] \leq \mathbb{P} \left[ Z \leq \mathbb{E}[Z] \right] \quad (by \ (27))$$

$$\leq \mathbb{P} \left[ Z \leq \frac{\mathbb{E}[Z]}{2} \right] \quad (by \ (29))$$

$$\leq \exp \left( -\frac{(1/2)^2 \mathbb{E}[Z]}{2} \right) \quad \text{(by Lemma A.2 (ii))}$$

$$\leq \exp \left( -\frac{\mathbb{E}[Z]}{8} \right)$$

$$\leq \exp \left( -2 \log(\Delta n/\varepsilon) \right) \quad \text{(by (28))}$$

$$= \left( \frac{\varepsilon}{\Delta n} \right)^2. \quad (30)$$

\(^{16}\)We emphasize that the inequality Eq. (25) (and similarly Eq. (39), appearing later) is the heart of the analysis of the paper: Due to the edge-Markovian model, $M_t(P_t(a))$ and $S(\Gamma_t)$ (or $M_t(P_t(b))$ and $\overline{S}(\Gamma_t)$, as well) may have a correlation through the history $\mathcal{E}_t$, while Lemma 3.1 proves a history-independent lower-bound.
Finally,

\[ (24) = 1 - \sum_{(a,b) \in A^+ \times B^+} \Pr \left[ \sum_{t=0}^{T_1-1} Y_t(a,b) < 6 \log_2 \Delta \right] \]

\[ \geq 1 - \sum_{(a,b) \in A \times B} \left( \frac{\varepsilon}{\Delta n} \right)^2 \]  

(by (30))

\[ \geq 1 - (\Delta n)^2 \left( \frac{\varepsilon}{\Delta n} \right)^2 \]

\[ = 1 - \varepsilon^2 \]

and we obtain the claim. \qed

Before going to Phase II, we give the following remark concerning Lemma 4.4.

**Observation 4.5.** About the left-hand-side of (21) in Lemma 4.4

\[
\Pr \left[ \max_{v \in V} \Gamma_{T_1}(v) \leq \mu + 1 \right] \lor \left[ \max_{v \in V} \Gamma_{T_1}(v) \leq \bar{\mu} + 1 \right] = \Pr \left[ \min_{v \in V} \Gamma_{T_1}(v) \geq \mu - 1 \right] \lor \left[ \min_{v \in V} \Gamma_{T_1}(v) \geq \mu - 1 \right]
\]

(31)

(32)

hold.

**Proof.** By definition of \( \overline{\Gamma}_t(v) = K - \Gamma_t(v) \) and a fact \( \bar{\mu} = K - \mu \), we remark

\[
[\max_{v \in V} \Gamma_{T_1}(v) \leq \mu + 1] \iff [\min_{v \in V} \overline{\Gamma}_{T_1}(v) \geq \bar{\mu} - 1], \quad \text{as well as} \quad (33)
\]

\[
[\max_{v \in V} \overline{\Gamma}_{T_1}(v) \leq \bar{\mu} + 1] \iff [\max_{v \in V} \Gamma_{T_1}(v) \geq \mu - 1] \quad (34)
\]

hold. In fact, concerning Eq. (34),

\[
\max_{v \in V} \overline{\Gamma}_{T_1}(v) = \max_{v \in V} (K - \Gamma_{T_1}(v)) = K + \max_{v \in V} (-\Gamma_{T_1}(v)) = K - \min_{v \in V} \Gamma_{T_1}(v)
\]

\[
\bar{\mu} + 1 = K - \mu + 1
\]

hold, which implies

\[
[\max_{v \in V} \Gamma_{T_1}(v) \leq \bar{\mu} + 1] \iff [K - \min_{v \in V} \Gamma_{T_1}(v) \leq K - \mu + 1]
\]

\[
\iff [\min_{v \in V} \Gamma_{T_1}(v) \geq \mu - 1]
\]

and we obtain Eq. (34) which implies Eq. (31). Eq. (34) is similar, and hence Eq. (32) holds, too. \qed

As we stated in the proof strategy in Section 4.2, we got Eq. (31) is enough large by Lemma 4.4. In the following sections, we will use Lemma 4.4 in the form of Eq. (32).
4.4 Phase II

By Lemma 4.4 and Observation 4.5, we got a situation that \( \min_{v \in V} \Gamma(v) \leq \mu + 1 \) \( \lor \) \( \min_{v \in V} \bar{\Gamma}(v) \leq \bar{\mu} + 1 \) w.h.p. in Phase I. We prove the following lemma as Phase II analysis, in mind the Markov property of the execution of the random matching algorithm on an edge-Markovian graph.

**Lemma 4.6 (Phase II).** Let \( T_2 \geq \frac{54}{\varepsilon^2} \log \left( \frac{\Delta n}{\varepsilon} \right) \) for \( \varepsilon (0 < \varepsilon < 1/4) \).

(i) If \( \min_{v \in V} \Gamma_0(v) \geq \mu - 1 \) then \( \Pr \left[ \max_{v \in V} \Gamma_{T_2}(v) \leq \lceil \mu \rceil + 1 \right] \geq 1 - \varepsilon^2 \).

(ii) If \( \min_{v \in V} \Gamma_0(v) \geq \bar{\mu} - 1 \) then \( \Pr \left[ \max_{v \in V} \bar{\Gamma}_{T_2}(v) \leq \lfloor \bar{\mu} \rfloor + 1 \right] \geq 1 - \varepsilon^2 \).

Before the proof, we remark that Lemma 4.6 implies that \( \Pr \left[ \Gamma_{T_2}(v) \leq \lceil \mu \rceil + 1 \right] \geq 1 - \varepsilon^2 \) holds for any \( v \in V \), where we remark \( \min_{u \in V} \Gamma_0(u) \geq \mu - 1 \) implies \( \min_{u \in V} \Gamma_t(u) \geq \mu - 1 \) for \( t \geq 0 \) by the monotone non-decreasing property of \( \Gamma_t(v) \) with respect to \( t \). The Eq. (35) is the goal of Lemma 4.6 as we stated in the proof strategy in Section 4.2. In Section 4.5, we will finalize the proof of Theorem 2.1 based on Lemmas 4.4 and 4.6.

**Proof of Lemma 4.6 (Phase II).** We prove (35) by the token-based analysis given in Section 4.1. For convenience, let \( A' = \{ a \in A \mid H_0(a) > \lfloor \mu \rfloor \} \). Then, it is easy to see that

\[
\max_{v \in V} \Gamma_{T_2}(v) \leq \lceil \mu \rceil + 1 \iff \max_{a \in A'} H_{T_2}(a) \leq \lfloor \mu \rfloor + 1
\]

\[
\iff \bigwedge_{a \in A'} \left[ H_{T_2}(a) - \lfloor \mu \rfloor \leq 1 \right]
\]

(36)

holds.

In a similar way as Lemma 4.4, let \( S'(\Gamma_t) := \{ v \in V \mid \Gamma_t(v) \leq \lfloor \mu \rfloor \} \), and let

\[
Y_t'(a) = \begin{cases} 
1 & \text{(if } M_t(P_t(a)) \in S'(\Gamma_t)\text{)} \\
0 & \text{(otherwise)}
\end{cases}
\]

(37)

for \( a \in A' \), meaning that \( Y_t'(a) \) is the indicator random variable of a desired matching at time \( t \). Then, we see that it is sufficient for \( H_{T_2}(a) - \lfloor \mu \rfloor \leq 1 \) that \( \sum_{t=0}^{T_2-1} Y_t'(a) \geq 3 \log_2 \Delta \) holds, by Lemma 4.3 in a similar way as Lemma 4.4. Precisely,

\[
\Pr \left[ \max_{v \in V} \Gamma_{T_2}(v) \leq \lceil \mu \rceil + 1 \right] = \Pr \left[ \bigwedge_{a \in A'} \left[ H_{T_2}(a) - \lfloor \mu \rfloor \leq 1 \right] \right] \quad \text{(by (36))}
\]

\[
= \Pr \left[ \bigwedge_{a \in A'} \sum_{t=0}^{T_2-1} Y_t'(a) \geq 3 \log_2 \Delta \right] \quad \text{(by Lemma 4.3 with the above argument)}
\]

\[
= 1 - \Pr \left[ \bigvee_{a \in A'} \sum_{t=0}^{T_2-1} Y_t'(a) < 3 \log_2 \Delta \right]
\]

(38)

\[
= 1 - \sum_{a \in A'} \Pr \left[ \sum_{t=0}^{T_2-1} Y_t'(a) < 3 \log_2 \Delta \right] \quad \text{(union bound)}
\]
holds.

To evaluate $\Pr[\sum_{t=0}^{T_2-1} Y_t'(a) < 3 \log_2 \Delta]$, we estimate the probability of $Y_t'(a) = 1$ by giving a lower bound independent of the history of execution $E_t$. Then,

$$\Pr[Y_t'(a_t) = 1 \mid E_t] = \Pr[M_t(P_t(a)) \in S'(\Gamma_t) \mid E_t] \quad \text{(by Eq. (37))}$$

$$\geq \frac{c_4 F}{r} \mathbb{E}\left[\left|S'(\Gamma_t)\right| \mid E_t\right] \quad \text{(by Lemma 3.1)}$$

$$\geq \frac{c_4 F}{3r} \quad \text{(since $|S'(\Gamma_t)| \geq n/3$ by Lemma 4.7)}$$

hold, where the last inequality follows Lemma 4.7, which we will prove just below this proof, with the fact that $\min_{v \in V} \Gamma_t(v) \geq \min_{v \in V} \Gamma_0(v) \geq \mu - 1$ since $\min_{v \in V} \Gamma_t(v)$ is monotone non-decreasing with respect to $t$.

Then, we evaluate $\Pr[\sum_{t=0}^{T_2-1} Y_t'(a) < 3 \log_2 \Delta]$. For convenience, let $Z_t' (t = 0, 1, \ldots, T_2 - 1)$ be independent binary random variables such that $\Pr[Z_t' = 1] = \frac{c_4 F}{3r}$, and let $Z' = \sum_{t=0}^{T_2-1} Z_t'$. It is not difficult to see that $\Pr[\sum_{t=0}^{T_2-1} Y_t'(a) < 3 \log_2 \Delta] \leq \Pr[\sum_{t=0}^{T_2-1} Z_t'(a) < 3 \log_2 \Delta]$ holds. Note that

$$\mathbb{E}[Z'] = T_2 \Pr[Z_t' = 1] \geq \frac{54r}{c_4 F} \log(\Delta n/\varepsilon) \quad \text{(since $T_2 \geq \frac{54r}{c_4 F} \log(\Delta n/\varepsilon)$)}$$

$$\geq 18 \log(\Delta n/\varepsilon) \quad \text{(since $n \geq 1, \varepsilon < 1$ and $\log_2 e \leq 1.5$)} \quad \text{(41)}$$

holds. Then, we have

$$\Pr\left[\sum_{t=0}^{T_2-1} Z_t'(a) < 3 \log_2 \Delta\right] \leq \Pr[Z' \leq \frac{\mathbb{E}[Z']}{4}] \quad \text{(by (42))}$$

$$\leq \exp\left(-\frac{(3/4)^2 \mathbb{E}[Z']}{2}\right) \quad \text{(by Lemma A.2 (ii))}$$

$$\leq \exp\left(-\frac{\mathbb{E}[Z']}{3}\right) \quad \text{(by (41))}$$

$$\leq \exp\left(-6 \log(\Delta n/\varepsilon)\right) \quad \text{(by (41))}$$

Finally,

$$\frac{38}{} \geq 1 - \sum_{a \in A'} \left(\frac{\varepsilon}{4 \Delta n}\right)^6$$

$$\geq 1 - \Delta n \left(\frac{\varepsilon}{\Delta n}\right)^6$$

$$= 1 - \frac{\varepsilon^6}{(\Delta n)^5}$$

$$\geq 1 - \varepsilon^2$$

and we obtain (i). The proof of (ii) is similar.
Lemma 4.7. If $\Gamma \in \mathbb{N}^V$ satisfies $\min_{v \in V} \Gamma(v) \geq \mu - 1$ then $|S'(\Gamma)| \geq n/3$ where $S'(\Gamma) := \{ v \in V \mid \Gamma(v) \leq \lceil \mu \rceil \}$ and $\mu = \sum_{v \in V} \Gamma(v)/n$.

Proof. Notice that
\[
n\mu = \sum_{v \in V} \Gamma(v) = \sum_{v \in S'(\Gamma)} \Gamma(v) + \sum_{v \notin S'(\Gamma)} \Gamma(v) \geq |S'(\Gamma)|(\mu - 1) + (n - |S'(\Gamma)|)(\lceil \mu \rceil + 1) \quad \text{(by the definition of $S'(\Gamma)$)}
\]
\[
= \frac{3}{2}|S'(\Gamma)| + n\mu + \frac{n}{2}
\]
holds. Now, the claim is clear. \qed

4.5 Final step of the proof

By Lemmas 4.4 and 4.6, we got a situation $[\mu - 1 \leq \Gamma_{T_1 + T_2}(v) \leq \lceil \mu \rceil + 1] \vee [\overline{\mu} - 1 \leq \Gamma_{T_1 + T_2}(v) \leq \lceil \overline{\mu} \rceil + 1]$ holds for any $v \in V$ w.h.p. The next lemma remarks that this is the situation which we want.

Lemma 4.8. Let $\Gamma \in \mathbb{N}^V$ where $\mu = \sum_{v \in V} \Gamma(v)$. For convenience, let $\phi_1 := [\mu - 1 \leq \Gamma(v) \leq \lceil \mu \rceil + 1]$ and let $\phi_2 := [\overline{\mu} - 1 \leq \Gamma(v) \leq \lceil \overline{\mu} \rceil + 1]$. Then, $\phi_1 \vee \phi_2$ implies $\Gamma(v) \in \{ \lceil \mu \rceil - 1, \lceil \mu \rceil, \lceil \mu \rceil + 1 \}$.

Proof. Since $\overline{\Gamma}(v) = K - \Gamma(v)$ and $\overline{\mu} = K - \mu$ by their definitions,
\[
\phi_2 \Leftrightarrow K - \mu - 1 \leq K - \Gamma(v) \leq \lceil K - \mu \rceil + 1
\]
\[
\Leftrightarrow \mu + 1 \geq \Gamma(v) \geq -\lceil -\mu \rceil - 1 \quad \text{(43)}
\]
holds. Thus, we see
\[
\phi_1 \vee \phi_2 \Leftrightarrow \min\{\mu, -\lceil -\mu \rceil\} - 1 \leq \Gamma(v) \leq \max\{\mu, \lceil \mu \rceil\} + 1 \quad \text{(44)}
\]
holds. Concerning the most right-hand-side of Eq. (44),
\[
\max\{\mu, \lceil \mu \rceil\} + 1 < \max\{\lceil \mu \rceil + 1, \lceil \mu \rceil\} + 1 \leq \lceil \mu \rceil + 2
\]
holds. Similarly, concerning the most left-hand-side of Eq. (44),
\[
\min\{\mu, -\lceil -\mu \rceil\} - 1 > \min\{\lceil \mu \rceil - 1, \lceil \mu \rceil - 1\} - 1 \geq \lceil \mu \rceil - 2
\]
holds. Consequently, we see that
\[
\text{(44)} \quad \Rightarrow \lceil \mu \rceil - 2 < \Gamma(v) < \lceil \mu \rceil + 2. \quad \text{(45)}
\]
Since $\Gamma(v)$ is an integer, $\lceil \mu \rceil$ as well, Eq. (45) implies $\Gamma(v) \in \{ \lceil \mu \rceil - 1, \lceil \mu \rceil, \lceil \mu \rceil + 1 \}$. We obtain the claim. \qed
Now, we finalize the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Let $T_{bal} = T_1 + T_2$ where $T_1$ and $T_2$ respectively satisfy the conditions in Lemmas 4.4 and 4.6. Clearly, $T_{bal} = \frac{990}{c_2 r} \log(\frac{\Delta n}{\epsilon}) + 2$ is sufficient. Then,

\[
\Pr[\Gamma_{T_{bal}}(v) \in \{[\mu] - 1, [\mu], [\mu] + 1\}] \\
\geq \Pr \left[ \frac{\mu - 1 \leq \Gamma_{T_1} + T_2(v) \leq [\mu] + 1}{\forall [\mu] - 1 \leq \Gamma_{T_1} + T_2(v) \leq [\mu] + 1} \right] (by \ Lemma 4.8) \\
= \Pr \left[ \frac{\mu - 1 \leq \Gamma_{T_1}(v) \leq [\mu] + 1}{\forall [\mu] - 1 \leq \Gamma_{T_1}(v) \leq [\mu] + 1} \right] \cdot \Pr \left[ \frac{\mu - 1 \leq \Gamma_{T_2}(v) \leq [\mu] + 1}{\forall [\mu] - 1 \leq \Gamma_{T_2}(v) \leq [\mu] + 1} \right] \\
\geq (1 - \epsilon^2)^2 \text{ (by Eq. (35), and Lemma 4.4 with (32), respectively)} \\
\geq 1 - \epsilon
\]

holds for any $v \in V$, where the last inequality follows the assumption $0 < \epsilon < 1/4$. We obtain the claim. \hfill \Box

## 5 Implications of Theorem 2.1

The random matching (Algorithm 1) is a comprehensive method for load balancing on networks, seemingly quite natural and simple, while there is variety how to draw a random matching, which is a profound issue in fact. This section introduces three major varieties, namely simple load balancing (cf., [6, 26]), local random matching (LR) algorithm (cf., [25]), and a simple variant of the distributed synchronous algorithm (cf., [10, 24, 38, 11]). The main focus of this section is to check their $F$-fair conditions (5), and we show the upper bounds of their balancing times implied by Theorem 2.1.

**Supplemental terminologies about static graphs.** To describe the algorithms, we introduce supplemental terminologies about static graphs. Suppose $G = (V, E)$ is an undirected simple graph. Let $N_E(v)$ (or simply $N(v)$) denote the set of vertices adjacent to $v \in V$, i.e., $N(v) := \{u \in V \mid \{u, v\} \in E\}$. We remark $v \notin N(v)$ since $G$ is simple. We also remark $d(v) = |N(v)|$ clearly. Specially, we remark that we use $N_E'(v)$ and $d_E'(v)$ for an edge subset $E' \subseteq E$, i.e., $N_E'(v) := \{u \in V \mid \{u, v\} \in E'\}$ and $d_E'(v) = |N_E'(v)|$. They are also abbreviated as $N(v)$ and $d(v)$ on $E'$, without confusion.

In this section, we also deal with directed edges. Let $\vec{G} = (V, \vec{E})$ denote a directed graph where $\vec{E} \subseteq V^2$ denotes the set of directed edges. Both $(u, v)$ and $(v, u)$ can simultaneously exist in $\vec{E}$, but duplication of a direct edge is not allowed, i.e., $(u, v)$ is at most one in $\vec{E}$. A self-loop is NOT allowed here, i.e., $(v, v)$ cannot exist in $\vec{E}$. For a set of directed edges $\vec{E}$, let $N_{\vec{E}}^-(v)$ and $N_{\vec{E}}^+(v)$ (or simply $N^-(v)$ and $N^+(v)$, resp.) respectively denote the sets of in/out neighboring vertices of $v \in V$, i.e., $N_{\vec{E}}^-(v) := \{u \in V \mid (u, v) \in \vec{E}\}$ and $N_{\vec{E}}^+(v) := \{u \in V \mid (v, u) \in \vec{E}\}$. Let $d^-(v)$ and $d^+(v)$ respectively denote in-degree and out-degree on $\vec{E}$, i.e., $d^-(v) := |N^-{\vec{E}}(v)|$ and $d^+(v) := |N^+{\vec{E}}(v)|$.

### 5.1 Simple load-balancing

The simplest verity could be “$M_1$ consists of a single edge.” Berenbrink et al. [6] spotlighted this folklore technology, and they gave a simple analysis. The algorithm is formally described in Algorithm 2. It is not
Algorithm 2: Random matching in simple load-balancing

**Input**: $G = (V, E)$

**Output**: $M \subseteq E$ such that $M$ consists of a single edge, or is empty.

1. set $M := \emptyset$
2. choose $v \in V$ u.a.r.;
3. if $N(v) \neq \emptyset$ then choose one $u \in N(v)$ u.a.r., and put $\{u, v\} \in M$;
4. return $M$;

Difficult to see that

\[
\Pr[M = \{u, v\}] := \frac{1}{n} \frac{1}{d(u)} + \frac{1}{n} \frac{1}{d(v)} = \frac{1}{n} \left( \frac{1}{d(u)} + \frac{1}{d(v)} \right) \geq \frac{1}{n} \left( \frac{1}{\max\{d(u), d(v)\}} \right) \tag{46}
\]

holds for any $\{u, v\} \in E$, meaning that Algorithm 2 satisfies $F = 1/n$-fair condition (5). Thus, Theorem 2.1 and Eq. (46) imply that the algorithm achieves $T_{bal} = O(rn \log(n\Delta))$ with high probability on edge-Markovian graphs.

**A variant: Draw from $E$ u.a.r.** To avoid the situation that Algorithm 2 output $M_t = \emptyset$, a reader may think why not choose an edge of $E$ uniformly at random. In this variant,

\[
\Pr[M = \{u, v\}] := \frac{1}{|E|} \geq \frac{1}{n^2} \left( \frac{1}{\max\{d(u), d(v)\}} \right) \tag{47}
\]

holds, where the last inequality is tight in an order of magnitude, in a cerebrated lollipop graph for an instance, or in the graph $K_{n-2} + K_2$ for another instance. Theorem 2.1 and Eq. (47) imply that the algorithm achieves $T_{bal} = O(rn^2 \log(n\Delta))$ with high probability on edge-Markovian graphs. Of course, this upper bound might not be tight.

### 5.2 LR algorithm

To draw a random matching, it could be a natural idea to choose a random subset of edges and then to edit it to a matching. Ghosh and Muthukrishnan in [25] originally proposed the random matching scheme for load balancing (cf., Algorithm 1), referred to as the local randomized (LR) algorithm, where they gave such an algorithm to generate a random matching as well. The algorithm is summarized in Algorithm 3, and it could be regarded in line with the above natural idea. For Algorithm 3, Ghosh and Muthukrishnan [25] proved the following lemma.

**Lemma 5.1** (25). $\Pr[\{u, v\} \in M] \geq \frac{1}{8 \max\{d(v), d(u)\}}$ holds for any $\{u, v\} \in E$.

Theorem 2.1 and Lemma 5.1 imply that the algorithm achieves $T_{bal} = O(r \log(n\Delta))$ with high probability on edge-Markovian graphs.
that for each in the original algorithm \[10\]. The lines 4 and 5 of Algorithm 4 realize the probability based on the Metropolis-Hastings technique (cf., \[32\]) instead of using the globally maximum degree and analyzed it.

Some works \[24, 38, 11\] about the random matching algorithm for load balancing employ this algorithm algorithm pretty well on bipartite graphs, while it needs attention for non-bipartite graphs. The

Another natural idea for a random matching may be to let vertices choose random partners. This idea works

5.3 (A localized version of) distributed synchronous algorithm

Another natural idea for a random matching may be to let vertices choose random partners. This idea works pretty well on bipartite graphs, while it needs attention for non-bipartite graphs. The distributed synchronous algorithm, given by Boyd et al. \[10\] in a bit different context, could be regarded in line with the above idea. Some works \[24, 38, 11\] about the random matching algorithm for load balancing employ this algorithm and analyzed it.

Here, we describe a variant of the algorithm in Algorithm \[4\] where we localize the choice of a partner (II. 4–5) based on the Metropolis-Hastings technique (cf., \[32\]) instead of using the globally maximum degree in the original algorithm \[10\]. The lines 4 and 5 of Algorithm \[4\] realize the probability

\[
\Pr[(u, v) \in \vec{M} | v \in I] = \begin{cases} \frac{1}{d(v)} & \text{if } d(u) \leq d(v) \\ \frac{1}{d(v)} \frac{d(u)}{d(v)} & \text{if } d(u) > d(v) \end{cases} = \frac{1}{\max\{d(u), d(v)\}}
\]

for each \(u \in N(v)\), which provides the following lemma.

\[\text{Formally, this could be } [v \in I] \land [(v, u) \in \vec{M}]. \text{ However, we remark } [(v, u) \in M'] \text{ implies } [v \in I] \text{ by the algorithm, meaning that } [v \in I] \text{ is redundant.}\]
Lemma 5.2. Algorithm 4 satisfies $F = 1/4$-fair condition, i.e.,

$$\Pr\{\{u, v\} \in M\} \geq \frac{1}{4\max\{d(u), d(v)\}}$$

holds for any $\{u, v\} \in E$.

Proof. By the line 7 of Algorithm 4,

$$\Pr\{\{v, u\} \in M\} = \Pr[v \in I] \cdot \Pr[(v, u) \in \tilde{M} | v \in I] \cdot \prod_{v' \in N(u) \setminus \{v\}} \Pr[(v', u) \notin M]$$

$$+ \Pr[u \in I] \cdot \Pr[(u, v) \in \tilde{M} | u \in I] \cdot \prod_{u' \in N(v) \setminus \{u\}} \Pr[(u', v) \notin M]$$

$$= \frac{1}{2} \cdot \frac{1}{\max\{d(u), d(v)\}} \cdot \frac{1}{2} \cdot \prod_{v' \in N(u) \setminus \{v\}} \Pr[(v', u) \notin M]$$

$$+ \frac{1}{2} \cdot \frac{1}{\max\{d(u), d(v)\}} \cdot \frac{1}{2} \cdot \prod_{u' \in N(v) \setminus \{u\}} \Pr[(u', v) \notin M]$$

(49)

holds. Here,

$$\prod_{w \in N(v) \setminus \{v\}} \Pr[(w, v) \notin M] = \prod_{w \in N(v) \setminus \{v\}} (1 - \Pr[w \in I] \Pr[(w, v) \in M | w \in I])$$

$$= \prod_{w \in N(v) \setminus \{v\}} \left(1 - \frac{1}{2\max\{d(w), d(v)\}}\right)$$

$$\geq 1 - \sum_{w \in N(v) \setminus \{v\}} \frac{1}{2\max\{d(w), d(v)\}}$$

$$\geq 1 - \sum_{w \in N(v) \setminus \{v\}} \frac{1}{2d(v)}$$

$$\geq 1 - \frac{d(v)}{2d(v)}$$

$$= \frac{1}{2}$$

(50)

holds. By Eqs. (49) and (50), we obtain the claim.

Theorem 2.1 and Lemma 5.2 imply that the algorithm achieves $T_{\text{bal}} = O(r \log(n\Delta))$ with high probability on edge-Markovian graphs.

6 Concluding Remark

Motivated by a technique for an analysis of algorithms on dynamic graphs, this paper gave an upper bound of the balancing time of random matching algorithms for load balancing on edge-Markovian graphs, which is a major topic in the context of distributed computing. To avoid the difficulty caused by the complicated correlation in the history of executions, we have developed a technique of history-independent bound Lemma 3.1 and Eqs. (25) and (39), focusing on the $F$-fair factor which existing algorithms have. Concerning our bound for the load-balancing algorithms, the $r = \max\left\{\frac{1-q}{p}, \frac{p}{1-q}\right\}$ factor in Theorem 2.1 could be improved by more careful arguments. Concerning the random matching algorithm, an
extension of the analysis technique to random edge-subset algorithm is an interesting future work. Concerning the edge-Markovian graph, an extension to more general model, particularly vertex increasing model, is an important future work.

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A Tools

**Lemma A.1** (Lemma 1.8.7 in [19]). Let $X_1, \ldots, X_n$ be arbitrary binary random variables and $X_1^*, \ldots, X_n^*$ be independent binary random variables. Let $X = \sum_{i=1}^n X_i$ and $X^* = \sum_{i=1}^n X_i^*$. Suppose that $\Pr[X_i = 1 | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \geq \Pr[X_i^* = 1]$ for all $1 \leq i \leq n$ and all $x_1, \ldots, x_{i-1} \in \{0, 1\}$ with $\Pr[X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] > 0$. Then, $\Pr[X \leq \lambda] \leq \Pr[X^* \leq \lambda]$ for all $\lambda \in \mathbb{R}$.

**Lemma A.2** (The Chernoff inequality (see, e.g., Theorem 1.10.21 in [19])). Let $X_1, \ldots, X_n$ be $n$ independent random variables taking values in $[0, 1]$. Let $X = \sum_{i=1}^n X_i$. Let $\mu^- \leq \mathbb{E}[X] \leq \mu^+$. Then, we have the following:

(i) $\Pr[X \geq (1 + \epsilon)\mu^+] \leq \exp\left(-\frac{\min(\epsilon^2, \epsilon)\mu^+}{3}\right)$ for $\epsilon \geq 0$.

(ii) $\Pr[X \leq (1 - \epsilon)\mu^-] \leq \exp\left(-\frac{\epsilon^2\mu^-}{2}\right)$ for $0 \leq \epsilon \leq 1$.

B Convergence behavior of edge-Markovian graphs

For a technical reason, we assume $p \in (0, 1)$ and $q \in (0, 1)$ here.

**Proposition B.1.** $\Pr[\{u, v\} \in E_t]$ approaches $p/(p + q)$ asymptotic to $t$.

$^{18}$The case of $p \in \{0, 1\}$ or $q \in \{0, 1\}$ is similar, but it needs some treatment in individual cases.
Proof. Clearly $P$ is ergodic, meaning that the unique limit distribution is the stationary distribution. Thus, just check $(\frac{q}{p+q}, \frac{p}{p+q})P = (\frac{q}{p+q}, \frac{p}{p+q})$ holds.

Another proof. Note that

$$P - (1 - p - q)I = \begin{pmatrix} 1 - p - (1 - p - q) & p \\ q & 1 - q - (1 - p - q) \end{pmatrix}$$

holds, where $I$ is the identity matrix. In other words,

$$P = (1 - p - q)I + (p + q) \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{pmatrix}$$

holds. Thus,

$$(\frac{q}{p+q}, \frac{p}{p+q})P = (\frac{q}{p+q}, \frac{p}{p+q}) \begin{pmatrix} 1 - p - (1 - p - q) & p \\ q & 1 - q - (1 - p - q) \end{pmatrix}$$

holds, where $I$ is the identity matrix. In other words,

$$P = (1 - p - q)I + (p + q) \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{pmatrix}$$

holds. Thus,

$$P = (1 - p - q)I + (p + q) \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{q}{p+q} & \frac{p}{p+q} \end{pmatrix}$$

holds for any distribution $x$. 

Proposition B.2. Let $\pi = \left(\frac{q}{p+q}, \frac{p}{p+q}\right)$, for convenience. If $p + q = 1$ then $\frac{1}{2} \|xP^t - \pi\|_1 = 0$ for any $t = 0, 1, 2, \ldots$ and for any probability distribution $x = (x_1, x_2)$, i.e., $x_1 \geq 0, x_2 \geq 0$ and $x_1 + x_2 = 1$. Otherwise, for any $\epsilon (0 < \epsilon < 1)$, $\frac{1}{2} \|xP^t - \pi\|_1 \leq \epsilon$ holds for $t \geq \log \epsilon / \log |1 - p - q|$ and for any probability distribution $x$.

Proof. When $p + q = 1$, the claim is clear by Eq. (51). Suppose $p + q \neq 1$. Let $t \geq \log \epsilon / \log |1 - p - q|$, then Eq. (51) implies

$$\frac{1}{2} \|xP^t - \pi\|_1 = \|1 - p - q\|^t \left|\frac{x_1 - \frac{q}{p+q}}{2} + \frac{x_2 - \frac{p}{p+q}}{2}\right|$$

$$\leq |1 - p - q|^t$$

$$\leq |1 - p - q|^{\log \epsilon / \log |1 - p - q|}$$

$$= \epsilon$$

holds for any distribution $x$. 

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