THE CLASSIFICATION OF ALGEBRAS OF LEVEL ONE

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Abstract. In the present paper we obtain the list of algebras, up to isomorphism, such that closure of any complex finite-dimensional algebra contains one of the algebra of the given list.

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1. Introduction

It is known that any $n$-dimensional algebra over a field $F$ may be considered as an element $\lambda$ of the affine variety $\text{Hom}(V \otimes V, V)$ via the bilinear mapping $\lambda : V \otimes V \to V$ on a vector space $V$.

Since the space $\text{Hom}(V \otimes V, V)$ form an $n^3$-dimensional affine space $B(V)$ over $F$ we shall consider the Zariski topology on this space and the linear reductive group $GL_n(F)$ acts on the space as follows:

$$(g * \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y))).$$

The orbits ($\text{Orb}(\text{-})$) under this action are the isomorphic classes of algebras. Note that algebras which satisfy identities (like, commutative, antisymmetric, nilpotency etc.) of the same dimension form also an invariant subvariety of the variety of algebras under the mentioned action.

In the study of a variety of algebras play the crucial role closures of orbit of algebras. Since closure of open set forms irreducible component of a variety, algebras whose orbits are open (so-called rigid algebras) give the description of the variety of algebras. One of the tools of finding rigid algebras are degenerations. The description of a variety of algebras by means of degenerations can be interpreted via down directed graph with the highest vertexes rigid algebras. Since any $n$-dimensional algebra degenerates to the abelian (denoted by $a_n$), any edge ends with the algebra $a_n$. For some examples of descriptions of varieties by means of degeneration graphs we refer to papers [1, 3, 4] and others.

In the paper of V.V. Gorbatsevich [2] the nearest-neighbor algebras in degeneration graph (algebras of level one) to the algebra $a_n$ are investigated. Namely, such algebras in the varieties of commutative (respectively, antisymmetric) algebras are indicated.

For the case a ground field is algebraic closed from [3] it is known that closures of orbits of algebras (denoted by $\overline{\text{Orb}(\text{-})}$) in Zariski and Euclidean topologies are coincide. That is $\lambda \in \overline{\text{Orb}(\mu)}$ can be realized by the following:

$$\exists g_t \in GL_n(\mathbb{C}(t)) \text{ such that } \lim_{t \to 0} g_t * \lambda = \mu,$$

where $\mathbb{C}(t)$ is the field of fractions of the polynomial ring $\mathbb{C}[t]$.

In this work we show that the paper [2] has some incorrectness and we describe all algebras of level one in the variety of all complex finite-dimensional algebras.

Let $\lambda$ and $\mu$ are complex algebras of the same dimension.

**Definition 1.1.** An algebra $\lambda$ is said to degenerate to algebra $\mu$, if $\text{Orb}(\mu)$ lies in Zariski closure of $\text{Orb}(\lambda)$. We denote this by $\lambda \rightarrow \mu$. 

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The degeneration $\lambda \rightarrow \mu$ is called a direct degeneration if there is no chain of non-trivial degenerations of the form: $\lambda \rightarrow \nu \rightarrow \mu$.

**Definition 1.2.** Level of an algebra $\lambda$ is the maximum length of a chain of direct degeneration. We denote the level of an algebra $\lambda$ by $\text{lev}_n(\lambda)$.

Consider the following algebras:

- $p_{\pm}^n$: $e_1 e_i = e_i, \ e_i e_1 = \pm e_i, \ i \geq 2$,
- $n_{\pm}^n$: $e_1 e_2 = e_3, \ e_2 e_1 = \pm e_3$.

**Theorem 1.3.** Let $\lambda$ be an $n$-dimensional algebra. Then

1. if the algebra $\lambda$ is skew-commutative, then $\text{lev}_n(\lambda) = 1$ if and only if it is isomorphic to $p_{-}^n$ or (with $n \geq 3$) to the algebra $n_{-}^3 \oplus a_{n-3}$. In particular, the algebra $\lambda$ is a Lie algebra.

2. if the algebra $\lambda$ is commutative, then $\text{lev}_n(\lambda) = 1$ if and only if it is isomorphic to $p_{+}^n$ or (for $n \geq 3$) to the algebra $n_{+}^3 \oplus a_{n-3}$. In particular, the algebra $\lambda$ is an Jordan algebra.

2. Main result

In this section we describe all complex finite dimensional algebras of level one.

Consider following algebras

- $\lambda_2$: $e_1 e_1 = e_2$,
- $\nu_n(\alpha)$: $e_1 e_1 = e_1, \ e_1 e_i = \alpha e_i, \ e_i e_1 = (1 - \alpha) e_i, \ 2 \leq i \leq n$.

In the following proposition we prove that algebras $p_{n}^+ \oplus a_{n-2}$ and $n_{3}^+ \oplus a_{n-3}$ are not of level one.

**Proposition 2.1.** $p_{n}^+ \rightarrow \lambda_2 \oplus a_{n-2}$ and $n_{3}^+ \oplus a_{n-3} \rightarrow \lambda_2 \oplus a_{n-2}$.

**Proof.** The first degeneration is given by the family of transformations $g_t$:

$$g_t(e_1) = t^{-1} e_1 - \frac{t^{-2}}{2} e_2, \quad g_t(e_2) = \frac{t^{-2}}{2} e_2, \quad g_t(e_i) = t^{-2} e_i, \quad 3 \leq i \leq n.$$ 

The second one is realized by the family $f_t$:

$$f_t(e_1) = t^{-1} e_1 - t^{-2} e_3, \quad f_t(e_2) = t^{-2} e_3, \quad f_t(e_3) = \frac{t^{-2}}{2} e_2, \quad f_t(e_i) = e_i, \quad 4 \leq i \leq n.$$ 

The above Proposition shows that the second assertion of result of Theorem 1.3 is not correct.

In order to prove the main theorem we need the following interim result.

**Proposition 2.2.** Any $n$-dimensional ($n \geq 3$) non-abelian algebra degenerates to one of the following algebras

- $p_{-}^n$,
- $n_{-}^n \oplus a_{n-3}$,
- $\lambda_2 \oplus a_{n-2}$,
- $\nu_n(\alpha)$, $\alpha \in \mathbb{C}$.

**Proof.** Let $A$ be an $n$-dimensional non-abelian algebra.

Firstly we consider the case when $A$ is antisymmetric algebra. Clearly, $xx = 0$ for any $x$ of $A$. If there exist elements $x, y \in A$ such that $xy \notin <x, y>$, then we can consider a basis of $A$:

$$e_1 = x, \ e_2 = y, \ e_3 = xy, \ e_4, \ldots, \ e_n.$$
It is easy to check that algebra $A$ degenerates to the algebra $n_3^- \oplus a_{n-3}$ by the use of the family $g_t$:

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_i) = t^{-2}e_i, \quad 3 \leq i \leq n.$$  

Consider now the contrary case, i.e. $xy \not{\in} <x, y>$ for all $x, y \in L$. Then the table of multiplication of the algebra $A$ have the form

$$e_i e_j = \frac{1}{\gamma_i} e_i + \frac{\gamma_{ij}}{\gamma_i} e_j, \quad 1 \leq i, j \leq n.$$  

Since the algebra $A$ is non-abelian, without lost of generality, we can assume $\gamma_{1,2}^2 \neq 0$.

Taking the change $e_i' = \frac{1}{\gamma_i} e_i, e_2' = e_2 + \frac{\gamma_{1,2}^2}{\gamma_i} e_1$, we can suppose $e_1 e_2 = e_2$.

Consider the product

$$e_1 (e_2 + e_i) = \gamma_1 e_1 + (e_2 + e_i) + (\gamma_i - 1) e_i.$$  

Taking into account $e_1 (e_2 + e_i) \not{\in} <e_1, e_2 + e_i>$, we deduce $\gamma_i = 1, \quad 3 \leq i \leq n$. Setting $e_i' = e_i + \gamma_1 e_1, \quad 3 \leq i \leq n$, we obtain $e_1 e_i = e_i, \quad 2 \leq i \leq n$.

Putting $g_t$ as follows:

$$g_t(e_1) = e_1, \quad g_t(e_i) = t^{-1}e_i, \quad 2 \leq i \leq n,$$

we get $\lim_{t \to 0} g_t \ast A = p_n^-.$

Now we assume that the algebra $A$ is not antisymmetric. Then there exists an element $x$ of $A$ such that $xx \not{\in} 0$.

**Case 1.** Let there exists $x$ of the algebra $A$ such that $xx \not{\in} <x>$ . Then we can chose a basis $e_1 = x, \ e_2 = xx, \ldots, e_n$. The degeneration $A \to \lambda_2 \oplus a_{n-2}$ is realized by the family $g_t$:

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_i) = t^{-2}e_i, \quad 2 \leq i \leq n.$$  

**Case 2** Let $xx \in <x>$ for all $x \in A$. Then for any $x, y \in A$ we have $(x + y)(x + y) = xx + xy + yx + yy \in <x, y>$. Therefore, $xy + yx \in <x, y>$. Then we chose a basis $\{e_1 = x, e_2 = y, e_3 = e_1 e_2, \ldots, e_n\}$ of the algebra $A$. The following family

$$g_t : g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_i) = t^{-2}e_i, \quad 3 \leq i \leq n$$

derives the degeneration $A \to n_3^- \oplus a_{n-3}$.

Now we consider the case when $xy \in <x, y>$ for all $x, y \in A$. Then for a basis $\{e_1, e_2, e_3, \ldots, e_n\}$ of $A$ we have $e_i e_i = \alpha_i e_i, \quad 1 \leq i \leq n$. Taking into account that algebra $A$ is non-antisymmetric, we can suppose $\alpha_3 \neq 0$. Without loss of generality we can assume $\alpha_3 \neq 0, 1 \leq i \leq k$ and $\alpha_3 = 0, k + 1 \leq i \leq n$. By scaling of basis elements we get $e_i e_i = e_i, \quad 1 \leq i \leq k, \quad \alpha_i = 0, k + 1 \leq i \leq n$.

The including of the following products

$$(e_1 \pm e_i)(e_1 \pm e_i) = e_1 \pm e_1 e_i \pm e_i e_1 + e_i \in <e_1 \pm e_i >,$$

imply $e_1 e_i + e_i e_1 = e_1 + e_i, \quad 1 \leq i \leq k$.

Similarly, we obtain

$$e_1 e_i + e_i e_1 = e_i, \quad k + 1 \leq i \leq n.$$  

Making the the change of basis

$$e_i' = e_i - e_1, \quad 2 \leq i \leq k \quad e_i' = e_i, \quad k + 1 \leq i \leq n,$$
we get the following products
\[ e_i e_1 = e_1, \quad e_i e_i = 0, \quad 2 \leq i \leq n, \quad e_i e_i = \alpha_i e_i + \beta_i e_1, \quad e_i e_1 = (1 - \alpha_i) e_i - \beta_i e_1, \quad 2 \leq i \leq n, \]
for some \( \alpha_i, \beta_i \in \mathbb{C} \).

The product
\[ e_1 (e_i + e_j) = (\alpha_j - \alpha_i) e_j + \alpha_i (e_i + e_j) + (\beta_j + \beta_i) e_1 \]
and \( e_i (e_i + e_j) \) imply \( \alpha_i = \alpha, \quad 2 \leq i \leq n. \)

The degeneration \( A \to \nu_n(\alpha) \) which is realized by using the family
\[ g_t : g_t(e_1) = e_1, \quad g_t(e_i) = t^{-1} e_i, \quad 2 \leq i \leq n, \]
complete the proof of proposition.

\[ \square \]

**Theorem 2.3.** Let \( A \) be an \( n \)-dimensional \((n \geq 3)\) algebra of level one, then it is isomorphic to one of the following algebras:
\[ p_n^{-}, \quad n_3^{-} \oplus a_{n-3}, \quad \lambda_2 \oplus a_{n-2}, \quad \nu_n(\alpha), \quad \alpha \in \mathbb{C}. \]

**Proof.** Due to Proposition 2.2 it is sufficient to prove that these four algebras do not degenerate to each other.

Since \( p_n^{-} \) and \( n_3^{-} \oplus a_{n-3} \) are antisymmetric algebras, but \( \lambda_2 \oplus a_{n-2} \) is commutative, we obtain
\[ \text{Orb}(p_n^{-}) \cap \text{Orb}(\lambda_2 \oplus a_{n-2}) = \{a_n\} \quad \text{and} \quad \text{Orb}(n_3^{-} \oplus a_{n-3}) \cap \text{Orb}(\lambda_2 \oplus a_{n-2}) = \{a_n\}. \]
Moreover, algebras \( n_3^{-} \oplus a_{n-3} \) and \( \lambda_2 \oplus a_{n-2} \) are nilpotent, but \( p_n^{-} \) and \( \nu(\alpha) \) are not nilpotent. Therefore, \( n_3^{-} \oplus a_{n-3} \) and \( \lambda_2 \oplus a_{n-2} \) do not degenerate to algebras \( p_n^{-} \) and \( \nu(\alpha) \).

Let us show that \( \text{Orb}(p_n^{-}) = \{p_n^{-}, a_n\} \). Consider a family of basis transformation \( g_t \) of the algebra \( p_n^{-} \). Then we have
\[
e_i e_j = \lim_{t \to 0} g_t(g_t^{-1}(e_i)g_t^{-1}(e_j)) = \lim_{t \to 0} g_t(\sum_{k=1}^{n} \beta_{i,k}(t)e_k - \sum_{k=1}^{n} \beta_{i,k}(t)e_k) = \lim_{t \to 0} g_t(\sum_{k=1}^{n} \beta_{1,k}(t)e_k - \sum_{k=1}^{n} \beta_{1,k}(t)e_k) = \beta_{1,1}(t) \sum_{k=1}^{n} \beta_{1,k}(t)e_k = \lim_{t \to 0} g_t(\beta_{1,1}(t)g_t^{-1}(e_j) - \beta_{1,1}(t)g_t^{-1}(e_j)) = \lim_{t \to 0} (\beta_{1,1}(t)e_j - \beta_{1,1}(t)e_j).
\]

If \( \lim_{t \to 0} \beta_{1,1}(t) = 0 \) for any \( i \), then we get the algebra \( a_n \).

If there exist \( i_0 \) \((1 \leq i_0 \leq n)\) such that \( \lim_{t \to 0} \beta_{i_0,1}(t) = \beta_{i_0} \neq 0 \), then without loss of generality we can suppose \( i_0 = 1 \).

Taking the change of basis \( e'_1 = \frac{1}{\beta_{1,1}} e_1, \quad e'_i = e_i - \frac{\beta_{i,1}}{\beta_{1,1}} e_1 \) in the algebra \( \lim_{t \to 0} g_t * p_n^{-} \) we have
\[ e'_i e'_i = -e'_i e'_1 = e'_i. \]

Thus, we obtain \( \lim_{t \to 0} g_t * p_n^{-} = p_n^{-} \).

In a similar way we show that \( \text{Orb}(\nu(\alpha)) = \{\nu(\alpha), a_n\} \).

Consider
\[
e_i e_i = \lim_{t \to 0} g_t(g_t^{-1}(e_i)g_t^{-1}(e_i)) = \lim_{t \to 0} g_t(\sum_{k=1}^{n} \beta_{i,k}(t)e_k - \sum_{k=1}^{n} \beta_{i,k}(t)e_k) = \lim_{t \to 0} g_t(\beta_{i,1}(t)^2 e_1 + \beta_{1,1}(t) \sum_{k=2}^{n} \beta_{i,k}(t)e_k) = \lim_{t \to 0} g_t(\beta_{i,1}(t) \sum_{k=1}^{n} \beta_{i,k}(t)e_k) = \lim_{t \to 0} \beta_{i,1}(t)e_i.
\]
\[ e_i e_j = \lim_{t \to 0} g_t^{-1}(e_i, g_t^{-1}(e_j)) = \lim_{t \to 0} g_t(\sum_{k=1}^{n} \beta_{i,k}(t)e_k \sum_{k=1}^{n} \beta_{j,k}(t)e_k) = \]

\[ \lim_{t \to 0} g_t(\beta_{i,1}(t)\beta_{j,1}(t)e_1 + \alpha \beta_{i,1}(t) \sum_{k=2}^{n} \beta_{j,k}(t)e_k + (1 - \alpha) \beta_{j,1}(t) \sum_{k=2}^{n} \beta_{i,k}(t)e_k) = \]

\[ \lim_{t \to 0} g_t(\alpha \beta_{i,1}(t) \sum_{k=1}^{n} \beta_{j,k}(t)e_k + (1 - \alpha) \beta_{j,1}(t) \sum_{k=1}^{n} \beta_{i,k}(t)e_k) = \lim_{t \to 0}(\alpha \beta_{i,1}(t)e_j + (1 - \alpha) \beta_{j,1}(t)e_i). \]

If \( \lim_{t \to 0} \beta_{i,1}(t) = 0 \) for all \( i \), \( 1 \leq i \leq n \) then we have the algebra \( a_n \).

If there exist \( i_0 \) \( (1 \leq i_0 \leq n) \) such that \( \lim_{t \to 0} \beta_{i_0,1}(t) = \beta_{i_0} \neq 0 \), then, without lost of generality, we can assume that \( \beta_i \neq 0 \) for \( 1 \leq i \leq k \) and \( \beta_i = 0 \) for \( k + 1 \leq i \leq n \).

Taking the change

\[ e'_1 = \frac{1}{\beta_1} e_1, \quad e'_i = \frac{1}{\beta_i} e_i - \frac{1}{\beta_1} e_1, \quad 1 \leq i \leq k, \quad e'_i = e_i, \quad k + 1 \leq i \leq n \]

in the algebra \( \lim_{t \to 0} g_t * \nu_n(\alpha) \), we derive the table of multiplication:

\[ e'_1 e'_1 = e'_1, \quad e'_1 e'_i = \alpha e'_i, \quad 2 \leq i \leq n, \quad e'_i e'_1 = (1 - \alpha) e_i, \quad 2 \leq i \leq n. \]

Remark that two-dimensional algebras of level one are the following

\[ p_2, \quad \lambda_2, \quad \nu_2(\alpha). \]

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