Noncommutative Residue and Dirac operators for Manifolds with the Conformal Robertson-Walker metric

Jian Wang\textsuperscript{a,b}, Yong Wang\textsuperscript{a,*}

\textsuperscript{a}School of Mathematics and Statistics, Northeast Normal University, Changchun, 130024, P.R.China
\textsuperscript{b}Chengde Petroleum College, Chengde, 067000, P.R.China

Abstract

In this paper, we prove a Kastler-Kalau-Walze type theorem for 4-dimensional and 6-dimensional spin manifolds with boundary associated with the conformal Robertson-Walker metric. And we give two kinds of operator theoretic explanations of the gravitational action for boundary in the case of 4-dimensional manifolds with flat boundary. In particular, for 6-dimensional spin manifolds with boundary with the conformal Robertson-Walker metric, we obtain the noncommutative residue of the composition of $\pi^+ D^{-1}$ and $\pi^+ D^{-3}$ is proportional to the Einstein-Hilbert action for manifolds with boundary.

Keywords: lower-dimensional volumes; noncommutative residue; gravitational action; conformal Robertson-Walker metric.

2000 MSC: 53G20, 53A30, 46L87

1. Introduction

The noncommutative residue plays a prominent role in noncommutative geometry \textsuperscript{[1,2]}. In \textsuperscript{[2]}, Connes used the noncommutative residue to derive a conformal 4-dimensional Polyakov action analogy. Several years ago, Connes made a challenging observation that the noncommutative residue of the square of the inverse of the Dirac operator was proportional to the Einstein-Hilbert action, which was called Kastler-Kalau-Walze Theorem now. In \textsuperscript{[3]}, Kastler gave a brute-force proof of this theorem. In \textsuperscript{[4]}, Kalau and Walze proved this theorem by the normal coordinates way simultaneously. In \textsuperscript{[5]}, Ackermann gave a note on a new proof of this theorem by means of the heat kernel expansion.

On the other hand, Fedosov etc. defined a noncommutative residue on Boutet de Monvel’s algebra and proved that it was a unique continuous trace in \textsuperscript{[6]}. Wang generalized the Connes’ results to the case of manifolds with boundary in \textsuperscript{[7,8]}, and proved a Kastler-Kalau-Walze type theorem for the Dirac operator and the signature operator for lower-dimensional manifolds with boundary. In \textsuperscript{[9]}, we get the Kastler-Kalau-Walze type theorem associated to nonminimal operators by heat equation asymptotics on compact manifolds without boundary. The purpose of papers \textsuperscript{[10,11]} is to prove the Kastler-Kalau-Walze Theorem for manifolds associated with the metric near the boundary as follows: $g^M = \frac{1}{h(x)} g^{\partial M} + dx_n^2$.

In \textsuperscript{[12]}, Antoci considered the metric $g^M = e^{-2(a+1)x_n} dx_n^2 + e^{-2bx_n} g^{\partial M}$ near the boundary and studied the spectrum of the Laplace-Beltrami operator for p-forms. In \textsuperscript{[13]}, the authors dealt with a particular class of warped products, i.e. when the pseudo-metric in the base is affected by a conformal change. Motivated by Antoci and Dobarro etc., in this paper we consider the following metric near the boundary $g^M = \frac{1}{\psi(x_n)} g^{\partial M} + \psi(x_n) dx_n^2$, which we call the conformal Robertson-Walker metric. We derive the gravitational action on boundary by the noncommutative residue associated with Dirac operator for the above metric. For lower dimensional manifolds with boundary, we compute $\text{Res}[\pi^+ D^{-p_1} \circ \pi^+ D^{-p_2}]$ with the conformal

\textsuperscript{*}Corresponding author.

Email address: wangy581@nenu.edu.cn (Yong Wang)

Preprint submitted to Elsevier December 11, 2013
Robertson-Walker metric $g^M$ on $M$, and we get a generalized Kastler-Kalau-Walze theorem for lower dimensional spin manifolds. For 6-dimensional manifolds with boundary with the conformal Robertson-Walker metric, we get $\text{Wres}[\pi^+D^{-3} \circ \pi^+D^{-3}]$ is proportional to the Einstein-Hilbert action for manifolds with boundary. Which gives a operator theoretic explanations of the total gravitational action for manifolds with boundary, i.e., $I_{G_4} = \frac{-3}{8\pi^4} \text{Wres}[\pi^+D^{-1} \circ \pi^+D^{-3}]$.

This paper is organized as follows: In Section 2, we define lower dimensional volumes of spin manifolds with boundary. In Section 3, for 4-dimensional spin manifolds with boundary of the conformal Robertson-Walker metric and the associated Dirac operator $D$, we compute the lower dimensional volume $\text{Vol}^{(1,1)}_4$ and get a Kastler-Kalau-Walze type theorem in this case. In Section 4, for 6-dimensional spin manifolds with boundary of the conformal Robertson-Walker metric and the associated Dirac operator $D$ and $D^{\pm}$, we compute the lower dimensional volume $\text{Vol}^{(2,2)}_6$ and get a Kastler-Kalau-Walze type theorem in this case. In Section 5, for 6-dimensional spin manifolds with boundary of the conformal Robertson-Walker metric and the associated Dirac operator $D$ and $D^{\pm}$, we compute the lower dimensional volume $\text{Vol}^{(1,3)}_6$ for 6-dimensional spin manifolds with boundary and obtain the noncommutative residue of the composition of $\pi^+D^{-1}$ and $\pi^+D^{-3}$ is proportional to the Einstein-Hilbert action for manifolds with boundary.

2. Lower dimensional volumes of spin manifolds with boundary

In order to define lower dimensional volumes of spin manifolds with boundary, we need some basic facts and formulae about Boutet de Monvel’s calculus and the definition of the noncommutative residue for manifolds with boundary. We can find them in Section 2.3 [1] and Section 2.1 [11].

Let $M$ be a $n$-dimensional compact oriented spin manifold with boundary $\partial M$. We assume that the metric $g^M$ on $M$ has the following form near the boundary,

$$g^M = \frac{1}{\varphi(x_n)}g^{\partial M} + \psi(x_n)dx_n^2,$$

(2.1)

where $g^{\partial M}$ is the metric on $\partial M$, $\varphi(x_n), \psi(x_n) > 0$ and $\varphi(0) = \psi(0) = 1$. Let $D$ be the Dirac operator associated to $g$ on the spinors bundle $S(TM)[11]$. Let $p_1, p_2$ be nonnegative integers and $p_1 + p_2 \leq n$. From Section 2 in [11], we have

**Definition 2.1. Lower dimensional volumes of spin manifolds with boundary are defined by**

$$\text{Vol}^{(p_1,p_2)}_n M := \text{Wres}[\pi^+D^{-p_1} \circ \pi^+D^{-p_2}].$$

(2.2)

Denote by $\sigma_r(A)$ the $r$-order symbol of an operator $A$. By (2.1.4)-(2.1.8) in [11], we get

$$\text{Wres}[\pi^+D^{-p_1} \circ \pi^+D^{-p_2}] = \int_M \int_{|\xi|=1} \text{trace}_{S(TM)}[\sigma_{-n}(D^{-p_1-p_2})]\sigma(\xi)dx + \int_{\partial M} \Phi,$$

(2.3)

and

$$\Phi = \int_{|\xi'|=1} \sum_{j,k=0}^\infty \sum_{\alpha=1}^\infty \frac{(-i)^{\alpha+j+k+1}}{\alpha! (j+k+1)!} \times \text{trace}_{S(TM)}[\partial^j_{\xi_{\alpha}} \partial^{j+1}_{\xi_n} \sigma_{-p_1}(D^{-p_1})(x',0,\xi',\xi_n)]$$

$$\times \partial^k_{\xi_n} \partial^{k+1}_{\xi_n} \sigma_1(D^{-p_2})(x',0,\xi',\xi_n)]d\xi_{\alpha}\sigma(\xi')dx',$$

(2.4)

where the sum is taken over $r - k - |\alpha| + l - j - 1 = -n$, $r \geq -p_1, l \leq -p_2$.

3. A Kastler-Kalau-Walze type theorem for 4-dimensional spin manifolds with boundary of conformal warped product metric

In this section, We compute the lower dimensional volume $\text{Vol}^{(1,1)}_4$ for 4-dimensional spin manifolds with boundary of conformal warped product metric and get a Kastler-Kalau-Walze type theorem in this case.
Since $[\sigma_{-4}(D^{-2})]|_{M}$ has the same expression as $\sigma_{-4}(D^{-2})$ in the case of manifolds without boundary in \cite{4, 5, 6} and \cite{11}, we have

$$\int_M \int_{[\xi]=1} \text{tr}[\sigma_{-4}(D^{-2})]\sigma(\xi)dx = \frac{\Omega_4}{3} \int_M \text{sdvol}_M, \quad (3.1)$$

where $\Omega_n = \frac{2\pi^n}{\Gamma(n)}$. So we only need to compute $\int_{\partial M} \Phi$.

Firstly, we compute the symbol $\sigma(D^{-1})$ of $D^{-1}$. Recall the definition of the Dirac operator $D$ in \cite{11}. Let $\nabla^L$ denote the Levi-Civita connection about $g^M$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

$$\nabla^L(\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_n) = (\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_n)(\omega_{s,t}). \quad (3.2)$$

The Dirac operator is defined by

$$D = \sum_{i=1}^n c(\tilde{\epsilon}_i)\left[\tilde{\epsilon}_i - \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{\epsilon}_i)c(\tilde{\epsilon}_s)c(\tilde{\epsilon}_t)\right]. \quad (3.3)$$

where $c(\tilde{\epsilon}_i)$ denotes the Clifford action.

Then,

$$\sigma_1(D) = \sqrt{\frac{1}{4}c(\xi)}; \quad \sigma_0(D) = -\frac{1}{4} \sum_{i,s,t} \omega_{s,t}(\tilde{\epsilon}_i)c(\tilde{\epsilon}_s)c(\tilde{\epsilon}_t), \quad (3.4)$$

where $\xi = \sum_{i=1}^n \xi_i dx_i$ denotes the cotangent vector.

By Lemma 2.1 in \cite{11}, we have

\begin{equation}
\sigma_{-1}(D^{-1}) = \sqrt{\frac{1}{4}c(\xi)}; \quad \sigma_{-2}(D^{-1}) = \frac{c(\xi)\sigma_0(D)c(\xi)}{|\xi|^2} + \frac{c(\xi)}{|\xi|^4} \sum_j c(dx_j) \left(\partial_{\xi_j}[c(\xi)]|\xi|^2 - c(\xi)\partial_{\xi_j}(|\xi|^2)\right), \quad (3.5)
\end{equation}

where $\sigma_0(D) = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{\epsilon}_i)c(\tilde{\epsilon}_s)c(\tilde{\epsilon}_t)$.

Since $\Phi$ is a global form on $\partial M$, so for any fixed point $x_0 \in \partial M$, we can choose the normal coordinates $U$ of $x_0$ in $\partial M$ (not in $M$) and compute $\Phi(x_0)$ in the coordinates $U = U \times [0,1) \subset M$ and the metric $g^M = \frac{1}{\varphi(x_n)}g^{\partial M} + \psi(x_n)dx_n^2$. The dual metric of $g^M$ on $\bar{U}$ is $\varphi(x_n)g^{\partial M} + \frac{1}{\psi(x_n)}dx_n^2$. Write $g_{ij}^M = g^M(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$; $\bar{g}_{ij}^M = g^M(dx_i, dx_j)$, then

\begin{equation}
[g_{ij}^M] = \left[ \begin{array}{cc} \frac{1}{\varphi(x_n)}[\varphi g_{ij}^{\partial M}] & 0 \\ 0 & \psi(x_n) \end{array} \right]; \quad [\bar{g}_{ij}^M] = \left[ \begin{array}{cc} \varphi(x_n)[\bar{g}_{ij}^{\partial M}] & 0 \\ 0 & \frac{1}{\psi(x_n)} \end{array} \right]. \quad (3.6)
\end{equation}

and

$$\partial_{x_i}g_{ij}^M(x_0) = 0, 1 \leq i, j \leq n - 1; \quad g_{ij}^M(x_0) = \delta_{ij}. \quad (3.7)$$

Let $n = 4$ and $\{e_1, \cdots, e_{n-1}\}$ be an orthonormal frame field in $U$ about $g^{\partial M}$ which is parallel along geodesics and $e_i(x_0) = \frac{\partial}{\partial x_i}(x_0)$, then $\tilde{\epsilon}_1 = \sqrt{\varphi(x_n)}e_1, \cdots, \tilde{\epsilon}_{n-1} = \sqrt{\varphi(x_n)}e_{n-1}, \tilde{\epsilon}_n = \sqrt{\psi(x_n)}dx_n$ is the orthonormal frame field in $\bar{U}$ about $g^M$. Locally $S(TM)|_{\bar{U}} \cong \bar{U} \times \mathcal{S}_{\tilde{\epsilon}}(\frac{\partial}{\partial x_n})$. Let $\{f_1, \cdots, f_4\}$ be the orthonormal basis of $\mathcal{S}_{\tilde{\epsilon}}(\frac{\partial}{\partial x_n})$. Take a spin field $\tilde{\sigma}: \bar{U} \to \text{Spin}(M)$ such that $\pi \tilde{\sigma} = \{\tilde{\epsilon}_1, \cdots, \tilde{\epsilon}_n\}$, where $\pi: \text{Spin}(M) \to O(M)$ is a double covering, then $\{[(\sigma, f_i)], 1 \leq i \leq 4\}$ is an orthonormal frame of $S(TM)|_{\bar{U}}$. In the following, since the global form $\Phi$ is independent of the choice of the local frame, so we
can compute \( \text{tr}_{S(TM)} \) in the frame \( \{[\sigma, f_i]\}, 1 \leq i \leq 4 \). Let \( \{E_1, \ldots, E_n\} \) be the canonical basis of \( \mathbb{R}^n \) and \( c(E_i) \in \text{cl}(n) \cong \text{Hom}(\wedge^n \mathbb{C}^4, \wedge^n \mathbb{C}^4) \) be the Clifford action. By Lemma A.1 in [11], (3.10) is correct.

By the equality

\[ \frac{\partial}{\partial x_i} c(E_i) = [(\sigma, c(E_i))]; \quad c(E_i)(\sigma, f_i) = [(\sigma, c(E_i)f_i)]; \quad \frac{\partial}{\partial x_i} \left[ (\sigma, \frac{\partial}{\partial x_i}) \right], \tag{3.9} \]

then we have \( \frac{\partial}{\partial x_i} c(E_i) = 0 \) in the above frame. Therefore, we obtain

**Lemma 3.2.** For \( n \)-dimensional spin manifolds with boundary,

\[ \partial_{x_j} [(\xi^2_{gy})](x_0) = \begin{cases} 0, & \text{if } j \neq n; \\ \varphi'\left(0\right)|\xi|^2_{gy} - \psi'(0)\xi^2_n, & \text{if } j = n. \end{cases} \tag{3.10} \]

\[ \partial_{x_j} [c(\xi)](x_0) = \begin{cases} 0, & \text{if } j \neq n; \\ \partial_{x_n} [c(\xi')](x_0) + \xi_n \partial_{x_n} [c(dx_n)(x_0)], & \text{if } j = n. \end{cases} \tag{3.11} \]

where \( \xi = \xi' + \xi_n dx_n \).

**Proof.** By the equality \( \partial_{x_j} [(\xi^2_{gy})](x_0) = \partial_{x_j} (\varphi(x_0)\psi)^{1/2}_x \xi_\alpha + \psi(x_0)\xi^2_n)(x_0) \) and (3.7), then (3.9) is correct. By Lemma A.1 in [11], (3.10) is correct.

In order to compute \( \sigma_0(D)(x_0) \), we need to compute \( \omega_n,\pi(c_i)(x_0) \). By Appendix in [11], we have

**Lemma 3.3.** For \( n \)-dimensional spin manifolds with boundary. When \( i < n \) \( , \omega_n,\pi(c_i)(x_0) = \frac{1}{\sqrt{2}}\varphi'(0) \); and \( \omega_{i,n}(c_i)(x_0) = -\frac{1}{\sqrt{2}}\varphi'(0) \). In other cases, \( \omega_n,\pi(c_i)(x_0) = 0 \).

Combining (3.3) and Lemma 3.3, we obtain

**Lemma 3.4.** For \( 4 \)-dimensional spin manifolds with boundary,

\[ \sigma_0(D)(x_0) = -\frac{3}{4}\varphi'(0)c(dx_n). \tag{3.12} \]

Now we can compute \( \Phi \) (see formula (2.4) for the definition of \( \Phi \)), since the sum is taken over \( -r - l + k + j + |\alpha| = 3 \), \( r, l \leq -1 \), then we have the following five cases:

**case a)** \( I \) \( r = -1 \), \( l = -1 \), \( k = j = 0 \), \( |\alpha| = 1 \)

From (2.4) we have

\[ \text{case a)} I) = -\int_{|\xi| = 1}^{+\infty} \sum_{|\alpha| = 1} \text{trace}[(\partial_{\xi}^2 \pi^+_n \sigma_{-1}(D^{-1}) \times \partial_{\xi}^2 \partial_{\xi_n} \sigma_{-1}(D^{-1})](x_0) d\xi_n \sigma(\xi) dx'. \tag{3.13} \]

By Lemma 3.2, for \( i \neq n \), then

\[ \partial_{x_i} \sigma_{-1}(D^{-1})(x_0) = \partial_{x_i} \left( \frac{\sqrt{-1}c(\xi)}{|\xi|^2} \right)(x_0) = \sqrt{-1} \partial_{x_i} [c(\xi)](x_0) \frac{\sqrt{-1}c(\xi)}{|\xi|^2} - \frac{\sqrt{-1}c(\xi) \partial_{x_i} [c(\xi)](x_0)}{|\xi|^2} = 0. \tag{3.14} \]

Then case a) I) vanishes.

**case a)** \( II \) \( r = -1 \), \( l = -1 \), \( k = |\alpha| = 0 \), \( j = 1 \)

From (2.4) we have

\[ \text{case a)} II) = \frac{1}{2} \int_{|\xi| = 1}^{+\infty} \text{trace}[(\partial_{\xi_n} \pi^+_n \sigma_{-1}(D^{-1}) \times \partial_{\xi_n}^2 \sigma_{-1}(D^{-1})](x_0) d\xi_n \sigma(\xi) dx'. \tag{3.15} \]

4
By Lemma 3.1 and Lemma 3.2, we have

\[
\partial_{\xi_n}^2 \sigma^{-1}(D^{-1}) = \sqrt{-1} \left( -\frac{6\xi_n c(dx_n) + 2c(\xi')}{|\xi|^4} + \frac{8\xi_n^2 c(\xi)}{|\xi|^6} \right) = \frac{6i\xi_n^2 - 2i}{(1 + \xi_n^2)^3} c(\xi') + \frac{2i\xi_n - 6i\xi_n}{(1 + \xi_n^2)^3} c(dx_n),
\]

and

\[
\partial_{\xi_n} \sigma^{-1}(D^{-1})(x_0) = \frac{\sqrt{-1} \partial_{\xi_n}[c(\xi)](x_0)}{|\xi|^2} - \frac{\sqrt{-1} c(\xi) \partial_{\xi_n}(|\xi|^2)(x_0)}{|\xi|^4} = \frac{i\partial_{\xi_n}[c(\xi')](x_0)}{|\xi|^2} + \frac{i\xi_n \partial_{\xi_n}[c(dx_n)](x_0)}{|\xi|^2} - \frac{(i\varphi'(0) - i\xi_n^2 \psi'(0)) c(\xi)}{|\xi|^4}.
\]

By (2.1.1) in [11], (3.3) in [10] and the Cauchy integral formula, then

\[
\pi_+^\xi \left[ \frac{c(\xi)}{|\xi|^4} \right] (x_0)|_{\xi'|=1} = \pi_+^\xi \left[ \frac{c(\xi') + \xi_n c(dx_n)}{(1 + \xi_n^2)^2} \right] = \frac{1}{2\pi i} \lim_{u \to 0} \int_{\gamma_u} \frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i)^2(\xi - \xi_n)} d\eta_n = \left[ \frac{c(\xi') + \eta_n c(dx_n)}{(\eta_n + i)^2(\xi - \xi_n)} \right]_{|\eta_n = i} = -\frac{2 - i\xi_n}{4(\xi - i)^2} c(\xi') - \frac{i}{4(\xi - i)^2} c(dx_n).
\]

Similarly,

\[
\pi_+^\xi \left[ \frac{i\partial_{\xi_n}[c(\xi')]}{|\xi|^2} \right] (x_0)|_{\xi'|=1} = \frac{\partial_{\xi_n}[c(\xi')](x_0)}{2(\xi - i)}.
\]

\[
\pi_+^\xi \left[ \frac{i\xi_n \partial_{\xi_n}[c(dx_n)]}{|\xi|^2} \right] (x_0)|_{\xi'|=1} = \frac{i\partial_{\xi_n}[c(dx_n)](x_0)}{2(\xi - i)}.
\]

and

\[
\pi_+^\xi \left[ \frac{\xi_n c(\xi)}{|\xi|^4} \right] (x_0)|_{\xi'|=1} = \frac{-i\xi_n}{4(\xi - i)^2} c(\xi') + \frac{2\xi_n - i}{4(\xi - i)^2} c(dx_n).
\]

Combining (3.17)-(3.21), we obtain

\[
\partial_{\xi_n} \pi_+^\xi \sigma^{-1}(D^{-1})(x_0)|_{\xi'|=1} = \frac{\partial_{\xi_n}[c(\xi')](x_0)}{2(\xi - i)} + \frac{i\partial_{\xi_n}[c(dx_n)](x_0)}{2(\xi - i)} + \frac{(2i - \xi_n)\varphi'(0) + \xi_n \psi'(0)}{4(\xi - i)^2} c(\xi') + \frac{-\varphi'(0) + (1 + 2\xi_n) \psi'(0)}{4(\xi - i)^2} c(dx_n).
\]

Since \( n = 4, \text{tr}_{S(TM)}[id] = \text{dim}(\Lambda^+(2)) = 4 \). By the relation of the Clifford action and \( \text{tr}AB = \text{tr}BA \), we have the equalities:

\[
\text{tr}[c(\xi')c(dx_n)] = 0; \quad \text{tr}[c(dx_n)^2] = -4; \quad \text{tr}[c(\xi')^2](x_0)|_{\xi'|=1} = -4;
\]

\[
\text{tr}[\partial_{\xi_n}[c(\xi')^2]](x_0) = 0; \quad \text{tr}[\partial_{\xi_n}[c(\xi')c(\xi')](x_0)|_{\xi'|=1} = -2\varphi'(0);
\]

\[
\text{tr}[\partial_{\xi_n}[c(dx_n)c(\xi')]](x_0) = 0; \quad \text{tr}[\partial_{\xi_n}[c(dx_n)c(dx_n)](x_0)|_{\xi'|=1} = 2\psi'(0).
\]
From (3.16), (3.22) and (3.23), we have
\[
\text{tr}\left\{ \left[ \frac{\partial_x [c(\xi')]}{2(\xi_n - i)} + \frac{i \partial_x [c(dx_n)]} {2(\xi_n - i)} + \frac{(2i - \xi_n)\varphi'(0) + \xi_n \psi'(0)} {4(\xi_n - i)^2} c(\xi') + \frac{-\varphi'(0) + (1 + 2i\xi_n)\psi'(0)} {4(\xi_n - i)^2} c(dx_n) \right] \right\} (x_0) |_{\xi'|=1} = \frac{2(2i\varphi'(0) + \xi_n \psi'(0))}{(\xi_n - i)^2(\xi_n + i)^3}.
\] (3.24)

Substituting (3.24) into (3.15), we have
\[
\text{case a) II} = -\int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{i\varphi'(0) + \xi_n \psi'(0)} {2(\xi_n - i)^2(\xi_n + i)^3} d\xi_n \sigma(\xi') dx'.
\]
\[
= -\Omega_4 \int \frac{i\varphi'(0) + \xi_n \psi'(0)} {2(\xi_n - i)^2(\xi_n + i)^3} d\xi_n dx'.
\]
\[
= -\Omega_4 2\pi i \left[ \frac{i\varphi'(0) + \xi_n \psi'(0)} {2(\xi_n + i)^3} \right] |_{\xi_n=1} dx'.
\]
\[
= \frac{1}{8} (3\varphi'(0) + \psi'(0)) \pi \Omega_4 dx'.
\] (3.25)

where \( \Omega_4 \) is the canonical volume of \( S^4 \).

**case a) III** \( r = -1, \ l = -1, \ j = |a| = 0, \ k = 1 \)

From (2.4) we have
\[
\text{case a) III} = -\frac{1}{2} \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi_n} \pi_{e_n}^{-1}(D^{-1}) \times \partial_{\xi_n} \partial_{x_n} \sigma^{-1}(D^{-1})] (x_0) d\xi_n \sigma(\xi') dx'.
\] (3.26)

By (2.29) in [11], we have
\[
\partial_{\xi_n} \pi_{e_n}^{-1}(D^{-1}) (x_0) |_{|\xi'|=1} = \frac{-c(\xi')}{2(\xi_n - i)^2} + \frac{-ic(dx_n)}{2(\xi_n - i)^2}.
\] (3.27)

From (3.5) we have
\[
\partial_{\xi_n} \partial_{x_n} \sigma^{-1}(D^{-1}) (x_0) |_{|\xi'|=1} = \frac{-2i\xi_n}{(1 + \xi_n^2)^2} \partial_{x_n} [c(\xi') | (x_0) + \frac{i - i\xi_n^2}{(1 + \xi_n^2)^2} \partial_{x_n} [c(dx_n)] (x_0) + \frac{(3i\xi_n - i)\varphi'(0) + (3i\xi_n^2 - i\xi_n^4)\psi'(0)} {2(1 + \xi_n^4)^3} c(dx_n)
\]
\[
+ \frac{4i\xi_n \varphi'(0) + (2i\xi_n^2 - 2i\xi_n^4)\psi'(0)} {2(1 + \xi_n^4)^3} c(\xi').
\] (3.28)

Combining (3.27) and (3.28), we obtain
\[
\text{tr}\left\{ \left[ \frac{-c(\xi') - ic(dx_n)} {2(\xi_n - i)^2} \right] \times \left[ \frac{-2i\xi_n}{(1 + \xi_n^2)^2} \partial_{x_n} [c(\xi')] | (x_0) + \frac{i - i\xi_n^2}{(1 + \xi_n^2)^2} \partial_{x_n} [c(dx_n)] (x_0) + \frac{(3i\xi_n - i)\varphi'(0) + (3i\xi_n^2 - i\xi_n^4)\psi'(0)} {2(1 + \xi_n^4)^3} c(dx_n)
\]
\[
+ \frac{4i\xi_n \varphi'(0) + (2i\xi_n^2 - 2i\xi_n^4)\psi'(0)} {2(1 + \xi_n^4)^3} c(\xi') \right] \right\} (x_0) |_{|\xi'|=1} = \frac{-2i\varphi'(0) + (\xi_n - i)\psi'(0)} {2(\xi_n - i)^2(\xi_n + i)^3}.
\] (3.29)
Substituting (3.29) into (3.26), one sees that

\[
\text{case a) III) } = \frac{-1}{2} \int_{|\xi'|=1}^{\frac{2}{2}} \int_{-\infty}^{+\infty} \frac{-2i\varphi'(0) + (\xi_n - i)\psi'(0)}{\xi_n - i}^{1} d\xi_n \sigma(\xi') dx'.
\]

\[
= \frac{-1}{2} \times 2\pi i \left[ -2i\varphi'(0) + (\xi_n - i)\psi'(0) \right] \bigg|_{\xi_n=0} dx'.
\]

\[
= \frac{1}{8} (3\varphi'(0) + \psi'(0)) \pi \Omega_3 dx'.
\]

(3.30)

\[
\text{case b) } r = -2, l = 1, k = j = |\alpha| = 0
\]

From (2.4) we have

\[
\text{case b) } = -i \int_{|\xi'|=1}^{\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^{+} \sigma_{-2} (D^{-1}) \times \partial_{\xi_n} \sigma_{-1} (D^{-1})] (x_0) d\xi_n \sigma(\xi') dx'.
\]

(3.31)

By Lemma 3.1 and Lemma 3.2, we have

\[
\partial_{\xi_n} \sigma_{-1} (D^{-1}) = \frac{-2i\xi_n}{(1 + \xi_n)^2} \varphi(\xi') + \frac{i - i\xi_n^2}{(1 + \xi_n)^2} c(dx_n),
\]

and

\[
\sigma_{-2} (D^{-1}) (x_0) = \frac{c(\xi) \sigma_0(D) (x_0) c(\xi)}{|\xi|^4} + \frac{1}{|\xi|^4} c(\xi) c(dx_n) \left[ \partial_{\xi_n} [c(\xi')] (x_0) + \xi_n \partial_{\xi_n} [c(dx_n)] (x_0) \right]
\]

\[
- \frac{1}{|\xi|^6} c(\xi) c(dx_n) c(\xi) \left[ \varphi'(0) - \xi_n^2 \psi'(0) \right].
\]

(3.33)

Then

\[
\pi_{\xi_n}^{+} \sigma_{-2} (D^{-1}) (x_0) |_{|\xi'|=1} = \pi_{\xi_n}^{+} \left[ \frac{c(\xi) \sigma_0(D) (x_0) c(\xi)}{|\xi|^4} \right.
\]

\[
+ \pi_{\xi_n}^{+} \left[ \frac{1}{|\xi|^4} c(\xi) c(dx_n) \left[ \partial_{\xi_n} [c(\xi')] (x_0) + \xi_n \partial_{\xi_n} [c(dx_n)] (x_0) \right] \right]
\]

\[
+ \pi_{\xi_n}^{+} \left[ \frac{1}{|\xi|^6} c(\xi) c(dx_n) c(\xi) \left[ \xi_n^2 \varphi'(0) - \psi'(0) \right] \right]
\]

\[
=: A + B + C
\]

(3.34)

Similarly to (3.18), by Lemma 3.1 we have

\[
A = \frac{-1}{4(\xi_n - i)^2} \left[ \frac{-3\varphi'(0)}{4} (2 + i\xi_n) c(\xi') c(dx_n) c(\xi') - \frac{3i\xi_n \varphi'(0)}{4} c(dx_n) + \frac{3i\varphi'(0)}{2} c(\xi') \right].
\]

(3.35)

And

\[
B = \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n) c(\xi') c(dx_n) \partial_{\xi_n} [c(\xi')] (x_0) + ic(\xi') c(dx_n) \partial_{\xi_n} [c(dx_n)] (x_0)
\]

\[
- i\partial_{\xi_n} [c(\xi')] (x_0) - i\xi_n \partial_{\xi_n} [c(dx_n)] (x_0),
\]

(3.36)

\[
C = \frac{\xi_n}{16(\xi_n - i)^2} \left[ -3\varphi'(0) - i\xi_n \varphi'(0) + \psi'(0) + 3i\xi_n \psi'(0) \right] c(dx_n)
\]

\[
+ \frac{1}{8(\xi_n - i)^3} \left[ -3\varphi'(0) - i\xi_n \varphi'(0) + \psi'(0) + 3i\xi_n \psi'(0) \right] c(\xi')
\]

\[
+ \frac{1}{16(\xi_n - i)^3} \left[ -8i\varphi'(0) + 9\xi_n \varphi'(0) + 3i\xi_n \varphi'(0) - 3\xi_n \psi'(0) - i\xi_n \psi'(0) \right] c(\xi') c(dx_n) c(\xi').
\]

(3.37)
By (3.32) and (3.35)-(3.37), we obtain
\[
\text{tr}[A \times \partial_{\xi_n} q_{-1}(x_0)]|_{\xi_n=1} = \frac{3i\varphi'(0)}{2(1 + \xi_n^2)},
\]
(3.38)
\[
\text{tr}[B \times \partial_{\xi_n} q_{-1}(x_0)]|_{\xi_n=1} = \frac{-2\varphi'(0) - i\xi_n\varphi'(0) + \xi_n^2 \varphi'(0)}{2(\xi_n - i)^3(\xi_n + i)^2},
\]
(3.39)
\[
\text{tr}[C \times \partial_{\xi_n} q_{-1}(x_0)]|_{\xi_n=1} = \frac{4\varphi'(0) + i\xi_n\varphi'(0) - \xi_n^2 \varphi'(0) - 3i\xi_n\varphi'(0) - \xi_n^2 \varphi'(0)}{2(\xi_n - i)^3(\xi_n + i)^2}.
\]
(3.40)

Combining (3.31), (3.38), (3.39) and (3.40), we obtain
\[
\text{case b) } = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \frac{-5i\varphi'(0) + 3\xi_n\varphi'(0) - 2\xi_n\varphi'(0)}{-2i(\xi_n - i)^3(\xi_n + i)^2} d\xi_n d\xi'
\]
\[
= -i \Omega_3 \int_{\Gamma^+} \frac{-5i\varphi'(0) + 3\xi_n\varphi'(0) - 2\xi_n\varphi'(0)}{-2i(\xi_n - i)^3(\xi_n + i)^2} d\xi_n d\xi'
\]
\[
= -i \Omega_3 \int_{\Gamma^+} \int_{-\infty}^{+\infty} \left[ -5i\varphi'(0) + 3\xi_n\varphi'(0) - 2\xi_n\varphi'(0) \right] d\xi_n d\xi'
\]
\[
= \frac{1}{8} (9\varphi'(0) - \psi'(0)) \pi \Omega_3 dx'.
\]
(3.41)

\section*{case c) $r = -1$, $l = -2$, $k = j = |\alpha| = 0$}

From (2.4) we have
\[
\text{case c) } = -i \int_{|\xi'|=1} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \times \partial_{\xi_n} \sigma_{-2}(D^{-1})](x_0) d\xi_n d\xi'(3.42)
\]

By (2.244) in [11], we have
\[
\pi_{\xi_n}^+ \sigma_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)},
\]
(3.43)

By (3.33) we have
\[
\partial_{\xi_n} \sigma_{-2}(D^{-1})(x_0)|_{|\xi'|=1} = \frac{1}{(1 + \xi_n^2)} \left[ (3\xi_n^2 - 1)\partial_{\xi_n} [c(\xi')](x_0) + (2\xi_n^2 - 2\xi_n)\partial_{\xi_n} [c(dx_n)](x_0) \right.
\]
\[
+ (1 - 3\xi_n^2) c(\xi') c(dx_n) \partial_{\xi_n} [c(dx_n)](x_0) - 4\xi_n c(\xi') c(dx_n) \partial_{\xi_n} [c(dx_n)](x_0) \left. + \frac{1}{(1 + \xi_n^2)^2} \left[ \xi_n (9\varphi'(0) + 3\xi_n^2 \varphi'(0) + 2\psi'(0) - 4\xi_n^2 \varphi'(0)) c(\xi') c(dx_n) c(\xi') \right.\right.
\]
\[
- \frac{1}{2} \left( -7\varphi'(0) + 26\xi_n^2 \varphi'(0) + 9\xi_n^4 \varphi'(0) + 12\xi_n^2 \varphi'(0) - 12\xi_n^4 \varphi'(0) \right) c(\xi') \right.
\]
\[
- \xi_n \left( -7\varphi'(0) + 8\xi_n^2 \varphi'(0) + 3\xi_n^4 \varphi'(0) + 8\xi_n^2 \varphi'(0) - 4\xi_n^4 \varphi'(0) \right) c(dx_n) \right] \right).
\]
(3.44)

Then similarly to computations of the case b), we have
\[
\text{trace}[\pi_{\xi_n}^+ \sigma_{-1}(D^{-1}) \times \partial_{\xi_n} \sigma_{-2}(D^{-1})](x_0)|_{|\xi'|=1} = \frac{1}{i(\xi_n - i)(\xi_n + i)^2} \left[ 6i\varphi'(0) + 3\xi_n \varphi'(0) + i\psi'(0) - 2\xi_n \varphi'(0) \right]
\]
(3.45)

Combining (3.42) and (3.45), we obtain
\[
\text{case c) } = -\frac{1}{8} (9\varphi'(0) - \psi'(0)) \pi \Omega_3 dx'.
\]
(3.46)

Since $\Phi$ is the sum of the cases a), b) and c), so $\Phi = 0$. Therefore
Lemma A.2 in [11],

Let \( M \) be a 4-dimensional compact spin manifold with the boundary \( \partial M \) and the metric \( g^M \) as above and \( D \) be the Dirac operator on \( \hat{M} \), then

\[
\text{Vol}_4^{(1,1)} = \widetilde{\text{Wres}}[\pi^+D^{-1}\circ\pi^+D^{-1}] = -\frac{\Omega_4}{3} \int_M \text{sdvol}_M. \tag{3.47}
\]

Now, we recall the Einstein-Hilbert action for manifolds with boundary in [11],

\[
I_{Gr} = \frac{1}{16\pi} \int_M \text{sdvol}_M + 2 \int_{\partial M} K \text{dvol}_{\partial M} := I_{Gr,i} + I_{Gr,b}, \tag{3.48}
\]

where

\[
K = \sum_{1 \leq i,j \leq n-1} K_{i,j} \theta_{i,j}^{1/2}; \quad K_{i,j} = -\Gamma_{i,j}^{n}, \tag{3.49}
\]

and \( K_{i,j} \) is the second fundamental form, or extrinsic curvature. Take the metric in Section 2, then by Lemma A.2 in [11], \( K_{i,j}(x_0) = -\Gamma_{i,j}^{n}(x_0) = -\frac{1}{2} \phi'(0) \), when \( i = j < n \), otherwise is zero. For \( n = 4 \), we obtain

\[
K(x_0) = \sum_{i,j} K_{i,j}(x_0) \theta_{i,j}^{1/2}(x_0) = \sum_{i=1}^{3} K_{i,i}(x_0) = -\frac{3}{2} \phi'(0). \tag{3.50}
\]

So

\[
I_{Gr,b} = -3\phi'(0) \text{Vol}_M. \tag{3.51}
\]

Let \( M \) be a 4-dimensional manifold with boundary and \( P, P' \) be two pseudodifferential operators with transmission property (see [3]) on \( M \). From (4.4) in [11], we have

\[
\pi^+P \circ \pi^+P' = \pi^+(PP') + L(P, P') \tag{3.52}
\]

and \( L(P, P') \) is leftterm which represents the difference between the composition \( \pi^+P \circ \pi^+P' \) in Boutet de Monvel algebra and the composition \( PP' \) in the classical pseudodifferential operators algebra. By (2.4), we define locally

\[
\text{res}_{1,1}(P, P') := -\frac{1}{2} \int_{[\xi'] = 1} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi'_n} \pi^+_{\xi'_n} \sigma_{-1}(P) \times \partial_{\xi'_n} \sigma_{-1}(P')] d\xi_n \sigma(\xi') dx'; \tag{3.53}
\]

\[
\text{res}_{2,1}(P, P') := -i \int_{[\xi'] = 1} \int_{-\infty}^{+\infty} \text{trace}[\pi^+_{\xi'_n} \sigma_{-2}(P) \times \partial_{\xi'_n} \sigma_{-1}(P')] d\xi_n \sigma(\xi') dx'. \tag{3.54}
\]

Hence, they represent the difference between the composition \( \pi^+P \circ \pi^+P' \) in Boutet de Monvel algebra and the composition \( PP' \) in the classical pseudodifferential operators algebra partially. Then

\[
\text{case a) } \Pi = \text{res}_{1,1}(D^{-1}, D^{-1}); \quad \text{case b) } = \text{res}_{2,1}(D^{-1}, D^{-1}). \tag{3.55}
\]

Now, we assume \( \partial M \) is flat , then \{d_1, \ldots, d_n\}, \( g_{\partial M}^{\partial M} = \delta_{i,j}, \partial_x g_{\partial M} = 0. \) So \( \text{res}_{1,1}(D^{-1}, D^{-1}) \) and \( \text{res}_{2,1}(D^{-1}, D^{-1}) \) are two global forms locally defined by the aboved oriented orthonormal basis \{d_1, \ldots, d_n\}. Let \( \psi'(0) = \varphi'(0) \), from case a) \( \Pi \) and case b), then we obtain:

**Theorem 3.5.** Let \( M \) be a 4-dimensional flat compact connected foliation with the boundary \( \partial M \) and the metric \( g^M \) as above , and \( D \) be the Dirac operator on \( \hat{M} \), then

\[
\int_{\partial M} \text{res}_{1,1}(D^{-1}, D^{-1}) = \frac{\pi}{6} \Omega_3 I_{Gr,b}; \tag{3.56}
\]

\[
\int_{\partial M} \text{res}_{2,1}(D^{-1}, D^{-1}) = -\frac{\pi}{3} \Omega_3 I_{Gr,b}. \tag{3.57}
\]
Nextly, for 3-dimensional spin manifolds with boundary, we compute $\text{Vol}_3^{(1,1)}$. By Section 5 in [11], we have

$$\text{Wres}[\pi^+ D^{-1} \circ \pi^+ D^{-1}] = \int_{\partial M} \Phi.$$  

(3.58)

By (2.4), when $n = 3$, we have $r - k - |\alpha| + l - j - 1 = -3$, $r \leq -1, l \leq -1$, so we get $r = -1, l = -1, k = |\alpha| = j = 0$, then

$$\Phi = \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}_{S(TM)}[\pi_+ \sigma_1(D^{-1})(x', 0, \xi', \xi_n) \times \partial_{\xi_n} \sigma_1(D^{-1})(x', 0, \xi', \xi_n)] d\xi_n \sigma(\xi') dx'.$$  

(3.59)

By (2.2.44) in [11], we have

$$\text{Vol}_{\partial M}.$$  

Then from (3.59), (3.60) and (3.61), we have the equalities:

$$\sigma_1(D^{-1})(x_0)|_{|\xi'|=1} = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - i)}.$$  

(3.60)

From (3.5) we have

$$\partial_{\xi_n} \sigma_1(D^{-1}) = \frac{-2i \xi_n}{(1 + \xi_n^2)^2} c(\xi') + \frac{i - i \xi_n'}{(1 + \xi_n^2)^2} c(dx_n).$$  

(3.61)

Since $n = 3$, $\text{tr}(id) = \dim(S(TM)) = 2$. By the relation of the Clifford action and $\text{tr}AB = \text{tr}BA$, we have the equalities:

$$\text{tr}[c(\xi')c(dx_n)] = 0; \, \text{tr}[c(dx_n)^2] = -2; \, \text{tr}[c(\xi')^2]|_{|\xi'|=1} = -2.$$  

(3.62)

Hence from (3.59), (3.60) and (3.61), we have

$$\text{trace}[\sigma_1(D^{-1}) \times \partial_{\xi_n} \sigma_1(D^{-1})](x_0)|_{|\xi'|=1} = \frac{-1}{(\xi_n + i)^2(\xi_n - i)}.$$  

(3.63)

Then

$$\Phi = \frac{i\pi}{2} \Omega_2 \text{Vol}_{\partial M}.$$  

(3.64)

where $\text{vol}_{\partial M}$ denotes the canonical volume form of $\partial M$. Then

**Theorem 3.7.** Let $M$ be a 3-dimensional compact spin manifold with the boundary $\partial M$ and the metric $g^M$ as in Section 2 and $D$ be the Dirac operator on $M$, then

$$\text{Vol}_3^{(1,1)} = \frac{i\pi}{2} \Omega_2 \text{Vol}_{\partial M}.$$  

(3.65)

where $\text{Vol}_{\partial M}$ denotes the canonical volume of $\partial M$.

**Remark 3.8.** When $\psi(x_n) = 1$, we get Theorem 2.5 and Theorem 5.1 in [11].

4. A Kastler-Kalau-Walze type theorem for 6-dimensional spin manifolds with boundary of conformal warped product metric associated with $D^2$

In this section, We compute the lower dimensional volume $\text{Vol}_6^{(2,2)}$ for 4-dimensional spin manifolds with boundary of warped product metric $g^M = \frac{\psi(x_n)}{\psi(x_n)} g^M + \psi(x_n) dx_n^2$ and get a Kastler-Kalau-Walze type theorem in this case.

Since $[\sigma_6(D^{-4})]|_M$ has the same expression as $\sigma_6(D^{-4})$ in the case of manifolds without boundary in [12], we have

$$\int_M \int_{|\xi'|=1} \text{tr}[\sigma_6(D^{-4})] \sigma(\xi) dx = -\frac{5 \Omega_6}{3} \int_M s \text{vol}_M,$$  

(4.1)

where $\Omega_6 = \frac{2\pi^3}{\Gamma(\frac{3}{2})}$. So we only need to compute $\int_{\partial M} \Phi$. By Lemma 1 in [12], we have
Lemma 4.1.

\[
\begin{align*}
\sigma_{-2}(D^{-2}) &= |\xi|^{-2}; \quad (4.2) \\
\sigma_{-3}(D^{-2}) &= -\sqrt{-1}|\xi|^{-4}\xi_k(\Gamma^k - 2\partial^k) - \sqrt{-1}|\xi|^{-6}2\xi^j\xi_\beta\partial_jg^{\beta}. \quad (4.3)
\end{align*}
\]

Now we can compute $\Phi$ (see formula (2.4) for the definition of $\Phi$), since the sum is taken over $-r-l+k+j+|\alpha| = 5$, $r$, $l \leq -2$, then we have the following five cases:

**case a) I) $r = -2$, $l = -2$, $k = j = 0$, $|\alpha| = 1$**

From (2.4) we have

\[
\begin{align*}
\text{case a) I)} &= - \int_{|\xi'|=1}^{+\infty} \int_{|\xi|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi}^2\pi_+^{\alpha} \sigma_{-2}(D^{-2}) \times \partial_{\xi}^2\partial_{\xi_n}\sigma_{-2}(D^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \quad (4.4)
\end{align*}
\]

By Lemma 3.2, for $i < n$, then

\[
\partial_{x_i}\sigma_{-2}(D^{-2})(x_0) = \partial_{x_i}\left(\frac{1}{|\xi|^2}\right)(x_0) = - \frac{\partial_{x_i}(|\xi|^2)(x_0)}{|\xi|^4} = 0.
\]

Then case a) I) vanishes.

**case a) II) $r = -1$, $l = -1$, $k = |\alpha| = 0$, $j = 1$**

From (2.4) we have

\[
\begin{align*}
\text{case a) II) } &= - \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{|\xi|=1}^{+\infty} \sum_{|\alpha|=1} \text{trace}[\partial_{\xi_n\xi}^2\pi_+^{\alpha} \sigma_{-2}(D^{-2}) \times \partial_{\xi_n}\sigma_{-2}(D^{-2})](x_0) d\xi_n \sigma(\xi') dx'. \quad (4.5)
\end{align*}
\]

By Lemma 3.1 and Lemma 3.2, we have

\[
\begin{align*}
\partial_{\xi_n}\sigma_{-2}(D^{-2})(x_0) &= \partial_{\xi_n}^2\left(\frac{1}{|\xi|^2}\right)(x_0) = - \frac{2 + 6\xi_n^2}{(1 + \xi_n^2)^3}. \quad (4.6)
\end{align*}
\]

and

\[
\begin{align*}
\partial_{\xi_n\xi}\sigma_{-2}(D^{-2})(x_0) &= \frac{\xi_n^2\psi'(0) - \psi'(0)}{(1 + \xi_n^2)^3}. \quad (4.7)
\end{align*}
\]

Similarly to (3.18), we have

\[
\pi_+^{\alpha_n} \left[ \partial_{\xi_n\xi} \sigma_{-2}(D^{-2}) \right](x_0)|_{|\xi'|=1} = \frac{(i\xi_n^2 + 2)\psi'(0) - i\xi_n\psi'(0)}{4(\xi_n - i)^2}. \quad (4.8)
\]

Combining (4.6) and (4.8), we obtain

\[
\begin{align*}
\int_{-\infty}^{+\infty} \frac{(i\xi_n^2 + 2)\psi'(0) - i\xi_n\psi'(0)}{4(\xi_n - i)^2} d\xi_n &= - \frac{1}{2} \int_{|\xi|} \frac{(3\xi_n^2 - 1)(-2\phi'(0) - i\xi_n\phi'(0) + i\xi_n\psi'(0))}{(\xi_n - i)^3(\xi_n + i)^3} d\xi_n dx' \\
&= - \frac{1}{2} \int_{|\xi|} \frac{(3\xi_n^2 - 1)(-2\phi'(0) - i\xi_n\phi'(0) + i\xi_n\psi'(0))}{(\xi_n + i)^3} \bigg|_{\xi_n = i} dx' \\
&= \frac{1}{32}(5\phi'(0) + \psi'(0)), \quad (4.9)
\end{align*}
\]

Since $n = 6$, tr$_{S(TM)}[id] = \dim(\wedge^*(3)) = 8$. So by (4.5),(4.9), we get

\[
\text{case a) II) } = - \frac{1}{8}(5\phi'(0) + \psi'(0))\pi\Omega_4 dx'. \quad (4.10)
\]
where $\Omega_4$ is the canonical volume of $S^4$.

**case a) III)** $r = -2$, $l = -2$, $j = |\alpha| = 0$, $k = 1$

From (2.4) and an integration by parts, we get

\[
\text{case a) III) } = -\frac{1}{2} \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\{\partial_\xi^+ \pi^+_{\xi_n} \sigma_2(D^{-2}) \times \partial_\xi \partial_\xi_n \sigma_2(D^{-2})\}(x_0) d\xi_\sigma(\xi') dx'
\]

\[
= \frac{1}{2} \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\{\partial_\xi^2 \pi^+_{\xi_n} \sigma_2(D^{-2}) \times \partial_\xi_n \sigma_2(D^{-2})\}(x_0) d\xi_\sigma(\xi') dx'.
\]

(4.11)

By Lemma 3.1 and Lemma 3.2, we have

\[
\partial_\xi^2 \pi^+_{\xi_n} \sigma_2(D^{-2})(x_0) = \frac{-i}{(\xi_n - i)^3}.
\]

(4.12)

Substituting (4.7) and (4.12) into (4.11), one sees that

\[
\text{case a) III) } = \frac{1}{2} \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \frac{8 i \varphi'(0) - 8 i \zeta^2 \varphi'(0)}{(\xi_n - i)^3(\xi_n + i)^2} d\xi_\sigma(\xi') dx'
\]

\[
= \frac{1}{8} (5 \varphi'(0) + \psi'(0)) \pi \Omega_4 d\xi'.
\]

(4.13)

**case b)** $r = -2$, $l = -3$, $k = j = |\alpha| = 0$

From (2.4) and an integration by parts, we get

\[
\text{case b) } = -i \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\{\pi^+_{\xi_n} \sigma_2(D^{-2}) \times \partial_\xi_\sigma \sigma_3(D^{-2})\}(x_0) d\xi_\sigma(\xi') dx'
\]

\[
= i \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}\{\partial_\xi \pi^+_{\xi_n} \sigma_2(D^{-2}) \times \sigma_3(D^{-2})\}(x_0) d\xi_\sigma(\xi') dx'.
\]

(4.14)

By Lemma 3.2, we have

\[
\partial_\xi \pi^+_{\xi_n} \sigma_2(D^{-2})(x_0) = \frac{i}{2(\xi_n - i)^2}.
\]

(4.15)

In the normal coordinate, $g^{ij}(x_0) = \delta^j_1$ and $\partial_{x_j}(g^{\alpha\beta})(x_0) = 0$, if $j < n$; $\gamma^{(0)} = \varphi'(0)\delta^0_3 - \psi'(0)\delta^0_n$, if $j = n$. So by Lemma A.2 in [11], we have $\Gamma^a(x_0) = \frac{3}{2} \varphi'(0) + \frac{1}{2} \psi'(0)$ and $\Gamma^k(x_0) = 0$ for $k < n$. By the definition of $\delta^k$ and Lemma 2.3 in [11], we have $\delta^n(x_0) = 0$ and $\delta^k = \frac{1}{4} \varphi'(0)c(\xi_k)c(\xi_n)$ for $k < n$. So

\[
\sigma_3(D^{-2})(x_0)(|\xi'| = 1)
\]

\[
= \frac{-i}{(1 + \xi_n^2)^2} \left( - \frac{1}{2} \varphi'(0) \sum_{k < n} \xi_k c(\xi_k) c(\xi_n) + \xi_n \left( \frac{5}{2} \varphi'(0) + \frac{1}{2} \psi'(0) \right) \right) - \frac{2 i \zeta_n (\varphi'(0) a - \psi'(0))}{(1 + \xi_n^2)^3}.
\]

(4.16)

We note that $\int_{|\xi'| = 1} \xi_1 \cdots \xi_{2q+1}\sigma(\xi') = 0$, so the first term in (4.16) has no contribution for computing case b. Then

\[
\text{case b) } = \int_{|\xi'| = 1}^{+\infty} \int_{-\infty}^{+\infty} \frac{2 i \xi_n (9 \varphi'(0) + 5 \zeta^2 \varphi'(0) - 5 \psi'(0) + \psi'(0) \xi_n^2)}{(\xi_n - i)^3(\xi_n + i)^2} d\xi_\sigma(\xi') dx'
\]

\[
= \Omega_4 \int_{\Gamma^+} \frac{2 i \xi_n (9 \varphi'(0) + 5 \zeta^2 \varphi'(0) - 5 \psi'(0) + \psi'(0) \xi_n^2)}{(\xi_n - i)^3(\xi_n + i)^2} d\xi_\sigma dx'
\]

\[
= \frac{3}{8} (5 \varphi'(0) - \psi'(0)) \pi \Omega_4 d\xi'.
\]

(4.17)
case c) $r = -3$, $l = -2$, $k = j = |\alpha| = 0$

From (2.4) we have

\[
\text{case c)} = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi_+^{\xi_1} \sigma_{-3}(D^{-2}) \times \partial_{\xi_1} \sigma_{-2}(D^{-2})](x_0)d\xi_1 \sigma(\xi')dx'.
\]

(4.18)

By (24) in [12], we have

\[
\text{case c)} = \text{case b)} - i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_1} \sigma_{-2}(D^{-2}) \times \sigma_{-3}(D^{-2})]d\xi_1 \sigma(\xi')dx'.
\]

(4.19)

From (4.2) we have

\[
\partial_{\xi_1} \sigma_{-2}(D^{-2})(x_0) = -\frac{2\xi_1}{(1+\xi_1^2)^2}.
\]

(4.20)

Combining (4.16) and (4.20), we obtain

\[
-i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{tr}[\partial_{\xi_1} \sigma_{-2}(D^{-2}) \times \sigma_{-3}(D^{-2})]d\xi_1 \sigma(\xi')dx' = \frac{3}{4}(5\varphi'(0) - \psi'(0))\pi\Omega_4 dx'.
\]

(4.21)

From (4.19) and (4.21), we obtain

\[
\text{case c)} = \frac{3}{8}(5\varphi'(0) - \psi'(0))\pi\Omega_4 dx'.
\]

(4.22)

Since $\Phi$ is the sum of the cases a), b) and c), so $\Phi = 0$. Therefore

**Theorem 4.2.** Let $M$ be a 6-dimensional compact spin manifold with the boundary $\partial M$ and the metric $g^M$ as above and $D$ be the Dirac operator on $\tilde{M}$, then

\[
\text{Vol}_6^{(2,2)} \equiv \text{Wres}[\pi^+ D^{-2} \circ \pi^+ D^{-2}] = -\frac{5\Omega_6}{3} \int_M \text{sdvol}_M.
\]

(4.23)

Now, we recall the Einstein-Hilbert action for manifolds with boundary. Let $\psi'(0) = \varphi'(0)$, from case a) II and case b) in section 4, then we obtain:

**Theorem 4.3.** Let $M$ be a 6-dimensional compact spin manifold with the boundary $\partial M$ and the metric $g^M$ as above and $D$ be the Dirac operator on $\tilde{M}$, then

\[
\int_{\partial M} \text{res}_{2,2}(D^{-2}, D^{-2}) = \frac{3\pi}{20} \Omega_4 \text{Gr}_b;
\]

(4.24)

\[
\int_{\partial M} \text{res}_{2,3}(D^{-2}, D^{-2}) = \frac{3\pi}{10} \Omega_4 \text{Gr}_b.
\]

(4.25)

5. A Kastler-Kalau-Walze type theorem for 6-dimensional spin manifolds with boundary of conformal warped product metric associated with $D$ and $D^3$

In this section, We compute the lower dimensional volume $\text{Vol}_6^{(1,3)}$ for 6-dimensional spin manifolds with boundary and get a Kastler-Kalau-Walze type theorem in this case.

Firstly, we compute the symbol $\sigma(D^{-3})$ of $D^{-3}$. Recall the definition of the Dirac operator $D$ [1][16]. Let $\nabla^L$ denote the Levi-Civita connection about $g^M$. In the local coordinates $\{x_i; 1 \leq i \leq n\}$ and the fixed orthonormal frame $\{\tilde{e}_1, \cdots, \tilde{e}_n\}$, the connection matrix $(\omega_{s,t})$ is defined by

\[
\nabla^L(\tilde{e}_1, \cdots, \tilde{e}_n) = (\bar{e}_1, \cdots, \bar{e}_n)(\omega_{s,t}).
\]

(5.1)
The Dirac operator is defined by

\[ D = \sum_{i=1}^{n} c(\tilde{e}_i) \left[ \tilde{c}_i - \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i) c(\tilde{e}_s)c(\tilde{e}_t) \right]. \] (5.2)

where \( c(\tilde{e}_i) \) denotes the Clifford action.

Recall the definition of the Dirac operator \( D^2 \) in \([4], [5] \) and \([12]\), we have

\[ D^2 = - \sum_{i,j} g^{i,j} \left[ \partial_i \partial_j + 2\sigma_i \partial_j + (\partial_i \sigma_j) + \sigma_i \sigma_j - \Gamma_{i,j}^k \partial_k - \Gamma_{i,j}^k \sigma_k \right] + \frac{1}{4} s. \] (5.3)

where \( \sigma_i = -\frac{1}{4} \sum_{s,t} \omega_{s,t}(\partial_i) e_s e_t. \)

Combining (5.2) and (5.3), we have

\[ D^3 = \sum_{i=1}^{n} c(\tilde{e}_i) (e_i, dx_i) \left\{ - \sum_{i,j} g^{i,j} \partial_i \partial_j \partial_k - \sum_{i,j} g^{i,j} (4\sigma_i \partial_j - 2\Gamma_{i,j}^k \partial_k) \partial_i + \frac{1}{4} s \partial_i \right\} - \sum_{i,j} g^{i,j} \left( \partial_i \sigma_j + \sigma_i \partial_j - \Gamma_{i,j}^k \partial_k + \frac{1}{4} s \partial_i \right) \]

\[ - \sum_{i,j} g^{i,j} (2\partial_i \partial_j + 2\partial_i \sigma_j + 3\sigma_i \partial_j + 3\Gamma_{i,j}^k \partial_k + \frac{1}{4} s \partial_i) \]

\[ - \sum_{i,j} \left[ \omega_{s,t}(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t) - \sum_{s,t} g^{i,j} (2\sigma_i \partial_j - \Gamma_{i,j}^k \partial_k) \right] \]

By Section in \([14]\), we obtain

\[ \sigma_3(D^3) = \sqrt{-1} c(\xi)|\xi|^2, \] (5.5)

\[ \sigma_2(D^3) = c(\xi)(4\sigma^k - 2\Gamma^k) \xi^k - \frac{1}{4} \xi^2 \sum_{s,t} \omega_{s,t}(\tilde{e}_s)c(\tilde{e}_s)c(\tilde{e}_t), \] (5.6)

\[ \sigma_1(D^3) = \sum_{i=1}^{n} c(\tilde{e}_i) (e_i, dx_i) \left\{ - \sum_{i,j} g^{i,j} \left( 2\partial_i \sigma_j + 2\partial_i \partial_j + 3\sigma_i \partial_j + 3\Gamma_{i,j}^k \partial_k \right) + \frac{1}{4} s \partial_i \right\} \]

\[ - \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t) - \sum_{i,j} g^{i,j} (2\sigma_i \partial_j - \Gamma_{i,j}^k \partial_k) \right\], \] (5.7)

\[ \sigma_0(D^3) = \sum_{i=1}^{n} c(\tilde{e}_i) (e_i, dx_i) \left\{ - \sum_{i,j} g^{i,j} \left( \partial_i \partial_j + \partial_i \sigma_j + \sigma_i \partial_j - \Gamma_{i,j}^k \partial_k \right) + \frac{1}{4} s \partial_i \right\} \]

\[ - \frac{1}{4} \sum_{s,t} \omega_{s,t}(\tilde{e}_i)c(\tilde{e}_s)c(\tilde{e}_t) - \sum_{i,j} g^{i,j} \left( \partial_i \sigma_j + \sigma_i \sigma_j - \Gamma_{i,j}^k \partial_k \right) + \frac{1}{4} s \partial_i \right\}. \] (5.8)

Write

\[ D'' = (-\sqrt{-1})^{i,j} \partial_{x_i}^{a_j}; \quad \sigma(D^3) = p_3 + p_2 + p_1 + p_0; \quad \sigma(D^{-3}) = \sum_{j=3}^{\infty} q_{-j}. \] (5.9)
By the composition formula of pseudodifferential operators, we have

\[
1 = \sigma(D^3 \circ D^{-3}) = \sum_{\alpha} \frac{1}{\alpha!} \partial^{\alpha}_x [\sigma(D)] D^{\alpha}_x [\sigma(D^{-1})]
\]
\[
= (p_3 + p_2 + p_1 + p_0)(q_{-3} + q_{-4} + q_{-5} + \cdots)
+ \sum_j (\partial_{c_j} p_3 + \partial_{c_j} p_2 + \partial_{c_j} p_1 + \partial_{c_j} p_0)(D_{x_j} q_{-3} + D_{x_j} q_{-4} + D_{x_j} q_{-5} + \cdots)
= p_3 q_{-3} + (p_3 q_{-4} + p_2 q_{-3} + \sum_j \partial_{c_j} p_3 D_{x_j} q_{-3}) + \cdots, \quad (5.10)
\]

Then we obtain

\[
q_{-3} = p_3^{-1}; \quad q_{-4} = -p_3^{-1} [p_3 q_{-3} + \sum_j \partial_{c_j} p_3 D_{x_j} (p_3^{-1})]. \quad (5.11)
\]

By Lemma 2.1 in [11] and (5.4)-(5.11), we obtain

**Lemma 5.1.**

\[
\sigma_{-3}(D^{-3}) = \frac{\sqrt{-1} \psi(\tilde{\omega})}{|\xi|^4}, \quad (5.12)
\]

\[
\sigma_{-4}(D^{-3}) = \frac{c(\xi) p_2(D^2 c(\xi))}{|\xi|^8} + \frac{c(\xi)}{\xi^{10}} \sum_j c(d x_j)|\xi|^2 + 2 \xi c(\xi) \left[ \partial_{x_j}[c(\xi)]|\xi|^2 - 2 c(\xi) \partial_{x_j}(|\xi|^2) \right], \quad (5.13)
\]

where \(\sigma_0(D) = -\frac{1}{4} \sum_{x, s} \psi_{s, t}(\tilde{e}_i) c(\tilde{e}_i) c(\tilde{e}_i) c(\tilde{e}_i)\).

Since \(\Phi\) is a global form on \(\partial M\), so for any fixed point \(x_0 \in \partial M\), we can choose the normal coordinates \(U\) of \((x_0)\) in the coordinates \(\tilde{U} = U \times [0, 1) \subset M\) and the metric \(g^M = \frac{1}{\psi(x_0)} g^{\partial M} + (\psi(x_0)) dx_n^2\). The dual metric of \(g^M\) on \(\tilde{U}\) is \(\varphi(x_0) g^{\partial M} + \frac{1}{\psi(x_0)} dx_n^2\). Write \(g_0^{\partial M} = g^M(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})\); \(g_{ij}^{\partial M} = g^M(dx_i, dx_j)\), then

\[
[g_{i,j}^{\partial M}] = \left[\begin{array}{cc} 0 & 1 \\
\psi(x_0) & 0 \end{array}\right]; \quad [g_{i,j}^{\partial M}] = \left[\begin{array}{cc} \varphi(x_0) & 0 \\
0 & \frac{1}{\psi(x_0)} \end{array}\right], \quad (5.14)
\]

and

\[
\partial_{x_i} g_{ij}^{\partial M}(x_0) = 0, 1 \leq i, j \leq n - 1; \quad g_{ij}^{\partial M}(x_0) = \delta_{ij}. \quad (5.15)
\]

Let \(n = 6\) and \(\{e_1, \ldots, e_{n-1}\}\) be an orthonormal frame field in \(U\) about \(g^{\partial M}\) which is parallel along geodesics and \(e_i(x_0) = \frac{\partial}{\partial x_i}(x_0)\), then \(\{\tilde{e}_i = \sqrt{\varphi(x_0)} e_1, \ldots, \tilde{e}_{n-1} = \sqrt{\varphi(x_0)} e_{n-1}, \tilde{e}_n = \frac{1}{\psi(x_0)} \psi(x_0) \partial_x\}\) is the orthonormal frame field in \(\tilde{U}\) about \(g^M\). Locally \(S(TM)|_{\tilde{U}} \cong \tilde{U} \times \bigwedge^2 C^\infty(\frac{\tilde{U}}{U}\)). Let \(\{f_1, \ldots, f_8\}\) be the orthonormal basis of \(\bigwedge^2 C^\infty(\frac{\tilde{U}}{U}\)). Take a spin frame field \(\sigma : \tilde{U} \rightarrow Spin(M)\) such that \(\pi \sigma = \{\tilde{e}_i, \cdots, \tilde{e}_n\}\), where \(\pi : Spin(M) \rightarrow O(M)\) is a double covering, then \(\{[(\sigma, f_i)], 1 \leq i \leq 8\}\) is an orthonormal frame of \(S(TM)|_{\tilde{U}}\). In the following, since the global form \(\Phi\) is independent of the choice of the local frame, so we can compute \(\Phi|_{S(TM)}\) in the frame \(\{[(\sigma, f_i)], 1 \leq i \leq 8\}\) is an orthonormal frame of \(S(TM)|_{\tilde{U}}\). Let \(\{E_1, \ldots, E_8\}\) be the canonical basis of \(R^n\) and \(c(E_i) \in Cl_n(\psi(x_0) \partial_x) \cong \text{Hom}(\wedge^2 C^\infty(\frac{\tilde{U}}{U}), \bigwedge^2 C^\infty(\frac{\tilde{U}}{U}))\) be the Clifford action. By [11], [12], and [16], we have

\[
c(\tilde{e}_i) = [(\sigma, c(E_i))]; \quad c(\tilde{e}_i)[(\sigma, f_i)] = [(\sigma, c(E_i) f_i)]; \quad \frac{\partial}{\partial x_i} = [(\sigma, \frac{\partial}{\partial x_i})], \quad (5.16)
\]

then we have \(\frac{\partial}{\partial x_i} c(\tilde{e}_i) = 0\) in the above frame.

Combining (3.3) and Lemma 3.3, we obtain

**Lemma 5.2.** For 4-dimensional spin manifolds with boundary,

\[
\sigma_0(D)(x_0) = -\frac{5}{4} \varphi'(0) c(dx_n). \quad (5.17)
\]
From (3.22), (5.21) and (5.22), we have

\[
\Delta = \sum_{|\alpha|=1} \text{trace} \left[ \partial_x^2 \pi^+_{\xi_n} \sigma_{-3}(D^{-3}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
\] (5.18)

By Lemma 3.2, for \( i < n \), then

\[
\partial_x \sigma_{-1}(D^{-1})(x_0) = \partial_x \left( \frac{\sqrt{-1} c(\xi)}{\sqrt{\xi'}} \right)(x_0) = \frac{\sqrt{-1} \partial_x c(\xi)(x_0)}{\sqrt{\xi'}} - \frac{\sqrt{-1} c(\xi) \partial_x (\sqrt{\xi'})}{\sqrt{\xi'}}(x_0) = 0.
\] (5.19)

Then case a) I) vanishes.

**case a) II)** \( r = -1, \ l = -3, \ k = 0 \), \( j = 1 \)

From (2.4) we have

\[
\text{case a) II) } = \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace} \left[ \partial_x^2 \pi^+_{\xi_n} \sigma_{-3}(D^{-3}) \right] (x_0) d\xi_n \sigma(\xi') dx'.
\] (5.20)

By Lemma 3.1 and Lemma 3.2, we have

\[
\partial_x^2 \sigma_{-3}(D^{-3}) = \frac{20i\xi_n^2 - 4i}{(1 + \xi_n^2)^2} c(\xi') + \frac{12i\xi_n^3 - 12i\xi_n}{(1 + \xi_n^2)^2} c(d\xi_n).
\] (5.21)

Since \( n = 6 \), \( \text{tr}_{S(TM)[id]} = \text{dim}(\wedge^n(3)) = 8 \). By the relation of the Clifford action and \( \text{tr}AB = \text{tr}BA \), we have the equalities:

\[
\begin{align*}
\text{tr}[c(\xi')c(d\xi_n)] &= 0; \quad \text{tr}[c(\xi')c(d\xi_n)^2] = -8; \quad \text{tr}[c(\xi')c(\xi_0)]|_{\xi'|=1} = -8; \\
\text{tr}[\partial_x [c(\xi')c(d\xi_n)] = 0; \quad \text{tr}[\partial_x [c(\xi')c(\xi_0)]|_{\xi'|=1} = -4\varphi'(0); \\
\text{tr}[\partial_x [c(\xi_0)c(\xi')] = 0; \quad \text{tr}[\partial_x [c(\xi_0)c(\xi_0)]|_{\xi'|=1} = 4\varphi'(0).
\end{align*}
\] (5.22)

From (3.22), (5.21) and (5.22), we have

\[
\begin{align*}
\text{tr} & \left\{ \frac{i\xi_n [c(\xi')]c(\xi_0)}{2(\xi_n - i)} \int_{-\infty}^{+\infty} \left[ \frac{20i\xi_n^2 - 4i}{(1 + \xi_n^2)^2} c(\xi') + \frac{12i\xi_n^3 - 12i\xi_n}{(1 + \xi_n^2)^2} c(d\xi_n) \right] (x_0) |_{\xi'|=1} \\
& \times \left( \frac{20i\xi_n^2 - 4i}{(1 + \xi_n^2)^2} c(\xi') + \frac{12i\xi_n^3 - 12i\xi_n}{(1 + \xi_n^2)^2} c(d\xi_n) \right) (x_0) \right\} \\
& \int_{-\infty}^{+\infty} \frac{8(i\varphi'(0) + \xi_n \psi'(0))(1 - 2i\xi_n + 3i\xi_n^2)}{(\xi_n - i)^2(\xi_n + i)^4} d\xi_n.
\end{align*}
\] (5.23)

Substituting (5.23) into (5.20), we have

**case a) II)**

\[
\begin{align*}
&\quad \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{8(i\varphi'(0) + \xi_n \psi'(0))(1 - 2i\xi_n + 3i\xi_n^2)}{(\xi_n - i)^2(\xi_n + i)^4} d\xi_n \sigma(\xi') dx' \\
&= -\Omega_4 \int_1^{+\infty} \frac{4i\varphi'(0) + \xi_n \psi'(0))(1 - 2i\xi_n + 3i\xi_n^2)}{(\xi_n + i)^4} d\xi_n dx' \\
&= -\Omega_4 \frac{2\pi i}{4!} \left[ \frac{4i\varphi'(0) + \xi_n \psi'(0))(1 - 2i\xi_n + 3i\xi_n^2)}{(\xi_n + i)^4} \right] d\xi_n dx' \\
&= -\frac{1}{16} \left( 15\varphi'(0) + 7\psi'(0) \right) \pi \Omega_4 dx',
\end{align*}
\] (5.24)
where $\Omega_4$ is the canonical volume of $S^4$.

**case a) III** $r = -3$, $l = -1$, $j = |\alpha| = 0$, $k = 1$

From (2.4) we have

$$
\text{case a) III) } = -\frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\partial_{\xi^n} \pi_{-1}(D^{-1}) \times \partial_{\xi^n} \sigma_{-3}(D^{-3})](x_0) d\xi_n \sigma(\xi') dx'.
$$

By (2.2.29) in [11], we have

$$
\partial_{\xi^n} \pi_{-1}(D^{-1})(x_0)|_{|\xi'|=1} = \frac{-c(\xi')}{2(\xi_n - i)^2} + \frac{-ic(dx_n)}{2(\xi_n - i)^2}.
$$

From (5.12) we have

$$
\partial_{\xi^n} \partial_{x^n} \sigma_{-3}(D^{-3})(x_0)|_{|\xi'|=1} = -\frac{4i\xi_n}{(1 + \xi_n^2)^3} \partial_{x^n} [c(\xi')](x_0) + \frac{i - 3i\xi_n^2}{(1 + \xi_n^2)^3} \partial_{x^n} [c(dx_n)](x_0)
$$

$$
+ \frac{(10i\xi_n^2 - 2i)\psi'(0) + (6i\xi_n^2 - 6i\xi_n^4)\psi''(0)}{(1 + \xi_n^2)^4} c(dx_n)
$$

$$
+ \frac{12i\xi_n\psi''(0) + (4i\xi_n - 8i\xi_n^3)\psi''(0)}{(1 + \xi_n^2)^4} c(\xi').
$$

(5.27)

Combining (5.26) and (5.27), we obtain

$$
\text{tr} \left\{ \left[ -\frac{c(\xi')}{2(\xi_n - i)^2} \right] \times \left[ \frac{-4i\xi_n}{(1 + \xi_n^2)^3} \partial_{x^n} [c(\xi')](x_0) + \frac{i - 3i\xi_n^2}{(1 + \xi_n^2)^3} \partial_{x^n} [c(dx_n)](x_0)
$$

$$
+ \frac{(10i\xi_n^2 - 2i)\psi'(0) + (6i\xi_n^2 - 6i\xi_n^4)\psi''(0)}{(1 + \xi_n^2)^4} c(dx_n)
$$

$$
+ \frac{12i\xi_n\psi''(0) + (4i\xi_n - 8i\xi_n^3)\psi''(0)}{(1 + \xi_n^2)^4} c(\xi') \right\} (x_0)|_{|\xi'|=1}
$$

$$
= \frac{(8i - 32\xi_n - 8i\xi_n^2)\psi''(0) + (2i - 14\xi_n - 14i\xi_n^2 + 18\xi_n^3)\psi''(0)}{(\xi_n - i)^2(\xi_n + i)^4}.
$$

(5.28)

Substituting (5.28) into (5.25), one sees that

$$
\text{case a) III) } = \frac{1}{2} \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \frac{(8i - 32\xi_n - 8i\xi_n^2)\psi''(0) + (2i - 14\xi_n - 14i\xi_n^2 + 18\xi_n^3)\psi''(0)}{(\xi_n - i)^2(\xi_n + i)^4} d\xi_n \sigma(\xi') dx'.
$$

$$
= \frac{1}{2} \times \frac{2\pi i}{4!} \left[ \frac{(8i - 32\xi_n - 8i\xi_n^2)\psi''(0) + (2i - 14\xi_n - 14i\xi_n^2 + 18\xi_n^3)\psi''(0)}{(\xi_n - i)^4} \right]|_{\xi_n=\Omega_4} dx'.
$$

(5.29)

**case b) $r = -2$, $l = -3$, $k = j = |\alpha| = 0$**

From (2.4) we have

$$
\text{case b) } = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \text{trace}[\pi_{-2}(D^{-1}) \times \partial_{\xi^n} \sigma_{-3}(D^{-3})](x_0) d\xi_n \sigma(\xi') dx'.
$$

(5.30)

By Lemma 3.1, Lemma 3.2 and Lemma 5.1, we have

$$
\partial_{\xi^n} \sigma_{-3}(D^{-3}) = -\frac{4i\xi_n c(\xi') + (i - 3i\xi_n^2) c(dx_n)}{(1 + \xi_n^2)^3}.
$$

(5.31)
and
\[
\sigma_{-2}(D^{-1})(x_0) = \frac{c(\xi)\sigma_0(D)(x_0)c(\xi)}{|\xi|^4} + \frac{1}{|\xi|^4} c(\xi)c(dx_n) \left[ \partial_{x_n}[c(\xi')(x_0) + \xi_n \partial_{x_n}[c(dx_n)]](x_0) \right] - \frac{1}{|\xi|^6} c(\xi)c(dx_n)c(\xi) \left[ \varphi'(0) - \xi_n^2 \psi'(0) \right].
\] (5.32)

Then
\[
\pi_{\xi_n}^+ \sigma_{-2}(D^{-1})(x_0)|_{\xi' = 1} = \pi_{\xi_n}^+ \left[ \frac{c(\xi)\sigma_0(D)(x_0)c(\xi)}{|\xi|^4} \right. \\
+ \pi_{\xi_n}^+ \frac{1}{|\xi|^4} c(\xi)c(dx_n) \left[ \partial_{x_n}[c(\xi')(x_0) + \xi_n \partial_{x_n}[c(dx_n)]](x_0) \right] \\
\left. + \pi_{\xi_n}^+ \frac{1}{|\xi|^6} c(\xi)c(dx_n)c(\xi) \left[ \xi_n^2 \psi'(0) - \varphi'(0) \right] \right] = A + B + C.
\] (5.33)

Similarly to (3.18), by Lemma 3.4 we have
\[
A = \frac{-1}{4(\xi_n - i)^2} \left[ \frac{-3i\varphi'(0)}{4} (2 + i\xi_n)c(\xi)c(dx_n)c(\xi') - \frac{-3i\xi_n \varphi'(0)}{4} c(dx_n) + \frac{3i\varphi'(0)}{2} c(\xi') \right].
\] (5.34)

And
\[
B = \frac{-1}{4(\xi_n - i)^2} \left[ (2 + i\xi_n)c(\xi)c(dx_n)\partial_{x_n}[c(\xi')](x_0) + i\xi_n c(dx_n)\partial_{x_n}[c(dx_n)](x_0) \right] \\
- i\xi_n \partial_{x_n}[c(\xi')](x_0) - i\xi_n \partial_{x_n}[c(dx_n)](x_0).
\] (5.35)

\[
C = \frac{\xi_n}{16(\xi_n - i)^4} \left[ \frac{-3\varphi'(0) - i\xi_n \varphi'(0) + \psi'(0) + 3i\xi_n \psi'(0)}{4} c(dx_n) \\
+ \frac{1}{8(\xi_n - i)^4} \left[ -3\varphi'(0) - i\xi_n \varphi'(0) + \psi'(0) + 3i\xi_n \psi'(0) \right] c(\xi') \\
+ \frac{1}{16(\xi_n - i)^4} \left[ -8i\varphi'(0) + 9\xi_n \varphi'(0) + 3i\xi_n^2 \varphi'(0) - 3\xi_n^2 \psi'(0) - i\xi_n^2 \psi'(0) \right] c(\xi')c(dx_n)c(\xi').
\] (5.36)

By (5.31) and (5.34)-(5.36), we obtain
\[
\text{tr}[A \times \partial_{\xi_n} \sigma_{3}(D^{-3})(x_0)]|_{\xi' = 1} = \frac{(5 + 15i\xi_n) \varphi'(0)}{(\xi_n - i)^4 (\xi_n + i)^3},
\] (5.37)

\[
\text{tr}[B \times \partial_{\xi_n} \sigma_{3}(D^{-3})(x_0)]|_{\xi' = 1} = \frac{(2 - 3i\xi_n + 3\xi_n^2) \varphi'(0) + (3i\xi_n + 3\xi_n^2) \psi'(0)}{(\xi_n - i)^4 (\xi_n + i)^3},
\] (5.38)

\[
\text{tr}[C \times \partial_{\xi_n} \sigma_{3}(D^{-3})(x_0)]|_{\xi' = 1} = \frac{(-4i + 11\xi_n + 6i\xi_n^2 - 3\xi_n^3) \varphi'(0) - (5\xi_n + 6i\xi_n^2 + 3\xi_n^3) \psi'(0)}{(\xi_n - i)^3 (\xi_n + i)^3}.
\] (5.39)

Combining (5.30), (5.37), (5.38) and (5.39), we obtain
\[
\text{case b) } = -i \int_{|\xi'|=1}^{+\infty} \int_{-\infty}^{+\infty} \left( 3\xi_n - i \right) \left( -7i\varphi'(0) + 5\xi_n \varphi'(0) - 2\xi_n \psi'(0) \right) d\xi_n \sigma(\xi') dx' \\
= -i\Omega_4 \int_{+\infty}^{+\infty} \left( 3\xi_n - i \right) \left( -7i\varphi'(0) + 5\xi_n \varphi'(0) - 2\xi_n \psi'(0) \right) d\xi_n dx' \\
= -i\Omega_4 \frac{2\pi i}{4!} \left[ \frac{(3\xi_n - i) - 7i\varphi'(0) + 5\xi_n \varphi'(0) - 2\xi_n \psi'(0)}{-i(\xi_n + i)^3} \right]_{\xi_n = 1} dx' \\
= \frac{1}{16} (55 \varphi'(0) - \psi'(0)) \pi \Omega_4 dx'.
\] (5.40)
case c) \( r = -1, \ l = -4, \ k = j = |\alpha| = 0 \)

From (2.4) we have

\[
\text{case c)} = -i \int_{[\xi'] = 1}^{\xi} \int_{-\infty}^{+\infty} \text{trace}[\pi_{\xi}^* \sigma_{-1}(D^{-1}) \times \partial_{\xi_n} \sigma_{-4}(D^{-3})](x_0) d\xi_n \sigma(\xi') dx'.
\]

(5.41)

By (2.2.44) in [11], we have

\[
\pi_{\xi}^* \sigma_{-1}(D^{-1})(x_0) = \frac{c(\xi') + ic(dx_n)}{2(\xi_n - t)}. \tag{5.42}
\]

From (5.13) we have

\[
\partial_{\xi_n} \sigma_{-4}(D^{-3})(x_0) = \frac{(59\xi_n + 27\xi_3)^2 \phi'(0) + (8\xi_n - 24\xi_3) \psi'(0)}{2(1 + \xi_3)^5} c(\xi') c(dx_n) c(\xi')
\]

\[
+ \frac{(33 - 180\xi_3^2 - 85\xi_3^4) \phi'(0) + (-48\xi_3^2 + 80\xi_3^4) \psi'(0)}{2(1 + \xi_3)^5} c(\xi')
\]

\[
+ \frac{(49\xi_n - 97\xi_3^3 - 50\xi_3^5) \phi'(0) + (-48\xi_3^3 + 48\xi_3^5) \psi'(0)}{2(1 + \xi_3)^5} c(dx_n)
\]

\[
+ \frac{-6\xi_n}{1 + \xi_3^2} c(\xi') c(dx_n) \partial_{\xi_n} [c(\xi')] (0) + \frac{-3 + 15\xi_3^2}{(1 + \xi_3^2)^5} \partial_{\xi_n} [c(\xi')] (0)
\]

\[
+ \frac{1 - 5\xi_3^2}{1 + \xi_3^2} c(dx_n) \partial_{\xi_n} [c(dx_n)] (0) + \frac{-6\xi_n + 12\xi_3^3}{(1 + \xi_3^2)^4} \partial_{\xi_n} [c(dx_n)] (0). \tag{5.43}
\]

Then similarly to computations of the case b), we have

\[
\text{trace}[\pi_{\xi}^* \sigma_{-1}(D^{-1}) \times \partial_{\xi_n} \sigma_{-4}(D^{-3})](x_0) |_{[\xi'] = 1}
\]

\[
= \frac{2i}{(\xi_n - i)^5 (\xi_n + i)^5} \left[ (-30 - 72i\xi_n + 96\xi_n^2 - 20i\xi_n^4 + 50\xi_n^3) \phi'(0) + (1 - 15i\xi_n + 29\xi_n^2 + 49i\xi_n^3 - 36\xi_n^4) \psi'(0) \right]. \tag{5.44}
\]

Combining (5.41) and (5.44), we obtain

\[
\text{case c)} = -\frac{3}{16} (35 \phi'(0) - 6 \psi'(0)) \pi \Omega_4 dx'. \tag{5.45}
\]

Since \( \Phi \) is the sum of the cases a), b) and c), so

\[
\Phi = -\frac{1}{16} (40 \phi'(0) - 11 \psi'(0)) \pi \Omega_4 dx'. \tag{5.46}
\]

Now recall the Einstein-Hilbert action for manifolds with boundary [11][12][16],

\[
I_{Gr} = \frac{1}{16\pi} \int_M s d\text{vol}_M + 2 \int_{\partial M} K d\text{vol}_{\partial M} := I_{Gr, i} + I_{Gr, b}, \tag{5.47}
\]

where

\[
K = \sum_{1 \leq i,j \leq n-1} K_{i,j} \gamma_{ij M}^3; \quad K_{i,j} = -\Gamma_{i,j}^n, \tag{5.48}
\]

and \( K_{i,j} \) is the second fundamental form, or extrinsic curvature. Taking the metric in Section 2, then by Lemma A.2 [11], for \( n = 6 \), then

\[
K(x_0) = -\frac{5}{2} \phi'(0); \quad I_{Gr, b} = -5 \phi'(0) \text{Vol}_{\partial M}. \tag{5.49}
\]

Let \( \psi'(0) = c \phi'(0) \), then we obtain
Remark 5.4. In [11] [12], Wang computed
\[ \text{Vol}^{(1,3)}_0 = \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}] = -\frac{5\Omega_5}{3} \int_M \sigma \text{dvol}_M + (1 - \frac{11c}{40}) \pi \Omega_4 \int_{\partial M} K d\text{Vol}_M. \]  

(5.50)

Remark 5.4. In [11] [12], Wang computed \( \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-1}] \) and \( \tilde{\text{Wres}}[\pi^+ D^{-2} \circ \pi^+ D^{-2}] \). In that cases, the boundary terms vanished, where the two operators are symmetric. Theorem 5.3 states the boundary terms is non-zero when we compute \( \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}] \). The reason is that \( D^{-1} \) and \( D^{-3} \) are not symmetric.

Let
\[ \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}] = \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}] + \tilde{\text{Wres}}_b[\pi^+ D^{-1} \circ \pi^+ D^{-3}], \]  

(5.51)

where
\[ \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}] = \int_M \int |\xi| = 1 \text{ trace}_{S(TM)} [\sigma_0 D^{-1}] \sigma(\xi) dx \]  

(5.52)

and
\[ \tilde{\text{Wres}}_b[\pi^+ D^{-1} \circ \pi^+ D^{-3}] = \int_{\partial M} \int |\xi'| = 1 \sum_{j=k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{\alpha + j + k + 1} \frac{\alpha! (j + k + 1)!}{(j + k + 1)!} \times \text{trace}_{S(TM)} (\partial^{\alpha k}\xi \sigma_0 \sigma(\xi)) \xi d\xi' \]  

(5.53)

denote the interior term and boundary term of \( \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}] \).

Combining (5.47), (5.50) and (5.51), we obtain

Corollary 5.5. Let \( M \) be a 6-dimensional compact spin manifold with the boundary \( \partial M \) and the metric \( g^M \) as above and \( D \) be the Dirac operator on \( M \), then
\[ I_{Gr,i} = -\frac{3}{80\pi \Omega_5} \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}]; \]
\[ I_{Gr,b} = \frac{80}{(40 - 11c) \pi \Omega_4} \tilde{\text{Wres}}_b[\pi^+ D^{-1} \circ \pi^+ D^{-3}]. \]  

(5.54)

In particular, when \( e = \frac{40}{11} + \frac{6400\Omega_5}{3\pi \Omega_4} \), then we obtain

Theorem 5.6. Let \( M \) be a 6-dimensional compact spin manifold with the boundary \( \partial M \) and the metric \( g^M \) as above, \( D \) be the Dirac operator on \( M \) and \( \psi(0) = c \psi'(0) \), then
\[ \text{Vol}^{(1,3)}_0 = \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}] = -\frac{80\pi \Omega_5}{3} \left[ \frac{1}{16\Omega_5} \int_M \sigma \text{dvol}_M + 2 \int_{\partial M} K d\text{Vol}_M \right]; \]
\[ I_{Gr} = -\frac{3}{80\pi \Omega_5} \tilde{\text{Wres}}[\pi^+ D^{-1} \circ \pi^+ D^{-3}]. \]  

(5.55)

Acknowledgements

This work was supported by Fok Ying Tong Education Foundation under Grant No. 121003 and NSFC. 11271062. And Jian Wang’s Email address: wangj068@gmail.com. The author also thank the referee for his (or her) careful reading and helpful comments.
References

[1] V. W. Guillemin.: A new proof of Weyl’s formula on the asymptotic distribution of eigenvalues. Adv. Math. 55, no. 2, 131-160, (1985).
[2] M. Wodzicki.: local invariants of spectral asymmetry. Invent. Math. 75(1), 143-178, (1995).
[3] A. Connes.: Quantized calculus and applications. XIIth International Congress of Mathematical Physics(Paris,1994), Internat Press, Cambridge, MA, 15-36, (1995).
[4] D. Kastler.: The Dirac Operator and Gravitation. Comm. Math. Phys. 166, 633-643, (1995).
[5] W. Kalau and M. Walze.: Gravity, Noncommutative geometry and the Wodzicki residue. J. Geom. Physics. 16, 327-344,(1995).
[6] T. Ackermann.: A note on the Wodzicki residue. J. Geom.Phys. 20, 404-406, (1996).
[7] B. V. Fedosov, F. Golse, E. Leichtnam, E. Schrohe.: The noncommutative residue for manifolds with boundary. J. Funct. Anal. 142, 1-31, (1996).
[8] Y. Wang.: Difffential forms and the Wodzicki residue for Manifolds with Boundary. J. Geom. Physics. 56, 731-753, (2006).
[9] Y. Wang.: Differential forms the Noncommutative Residue for Manifolds with Boundary in the non-product Case, Letters in Mathematical Physics. 77, 41-51, (2006).
[10] Wang, J., Wang, Y.: Nonminimal operators and non-commutative residue, J. Math. Phys. 53, 072503 (2012).
[11] Y. Wang.: Gravity and the Noncommutative Residue for Manifolds with Boundary. Letters in Mathematical Physics. 80, 37-56, (2007).
[12] Y. Wang.: Lower-Dimensional Volumes and Kastler-kalau-Walze Type Theorem for Manifolds with Boundary . Commun. Theor. Phys. Vol 54, 38-42, (2010).
[13] Antoci. F.: On the spectrum of the Laplace-Beltrami operator for p-forms for a class of warped product metrics. Adv.Math. 188. 247-93, (2004).
[14] Wang, J., Wang, Y.: The Kastler-Kalau-Walze type theorem for 6-dimensional manifolds with boundary , arXiv: math.DG/1211.6223.
[15] F. Dobarro, B. Únal.: About curvature, conformal metrics and warped products, J. Phys. A: Math. Theor. 13907-13930, 40 (2007).
[16] Y. Yu.: The Index Theorem and The Heat Equation Method, Nankai Tracts in Mathematics-Vol.2, World Scientific Publishing, (2001).