GLOBAL SECTIONS OF LINE BUNDLES ON A WONDERFUL COMPACTIFICATION OF THE GENERAL LINEAR GROUP

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Abstract. In a previous paper [K1] I have constructed a compactification $KGL_n$ of the general linear group $GL_n$, which in many respects is analogous to the so-called wonderful compactification of adjoint semisimple algebraic groups as studied by De Concini and Procesi. In particular there is an action of $G = GL_n \times GL_n$ on this compactification. In this paper we show how the space of global sections of an arbitrary $G$-linearized line bundle on $KGL_n$ and its orbit-closures decomposes into a direct sum of simple $G$-modules.

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1. Introduction

Let $k$ be a field of characteristic zero and let $E$ and $F$ be two $n$-dimensional vector spaces. In [K1] we have introduced a certain compactification $KGL(E, F)$ of the variety $\text{Isom}(E, F) \cong GL_n$ of linear isomorphisms from $E$ to $F$ which in many respects is analogous to De Concini and Procesi’s so-called wonderful compactification of adjoint semi-simple algebraic groups (cf. [CP]).

In particular, there is a natural action of the group $G := \text{GL}(E) \times \text{GL}(F)$ on $KGL(E, F)$ extending the one arising from right and left multiplication on $\text{Isom}(E, F)$. Furthermore $KGL(E, F)$ is smooth, the boundary, i.e. the complement of $\text{Isom}(E, F)$ in $KGL(E, F)$, is a divisor with normal crossings and the closures of the orbits of the $G$-action are precisely the nonempty intersections of the irreducible components of the boundary.

We will see in 3.7 below that the Picard group of $KGL(E, F)$ is generated by (the ideal sheaves of) the boundary components $Z_0, \ldots, Z_{n-1}$ and $Y_0, \ldots, Y_{n-1}$. Every line bundle
expressed in terms of these generators is equipped with a canonical linearization of the $G$-action and thus the space of global sections of its restriction to some orbit closure is naturally a finite dimensional $G$-module.

In this paper we show how such a space of global sections decomposes into a direct sum of simple $G$-modules. More precisely, we prove the following

**Theorem:** (Cf. Theorem 4.43 for the exact formulation). Let $L$ be a $G$-linearized line bundle of the form $L = O(\sum (m_i Z_i + l_i Y_i))$ on $KGL(E, F)$ and let $I, J \subseteq [0, n - 1]$ be subsets such that the intersection $O_{IJ} = (\cap_{i \in I} Z_i) \cap (\cap_{j \in J} Y_j)$ is nonempty. Then the decomposition of the $G$-module $H^0(O_{IJ}, i^*_L L)$ into simple submodules is given by a canonical isomorphism

$$H^0(O_{IJ}, i^*_L L) \cong \bigoplus_{(a, b) \in A_{IJ}(L)} H^0(Fl, O_{Fl}(a, b)),$$

where $A_{IJ}(L) \subset \mathbb{Z}^n \times \mathbb{Z}^n$ is a finite set defined explicitly in terms of $I$, $J$ and $L$, where $Fl$ is the product of the two complete flag manifolds associated to the vector spaces $E$ and $F$ respectively and where $O_{Fl}(a, b)$ is the product specified by $(a, b)$ of successive quotients of tautological vector bundles on $Fl$.

In [K2] we have shown the relevance of $KGL(E, F)$ for the Gieseker type degeneration of moduli stacks of vector bundles on curves: The normalization of the moduli stack of Gieseker vector bundles on an irreducible nodal curve with one singularity is isomorphic to $KGL(E, F)$, where $E$ and $F$ are certain vector bundles on the moduli stack of vector bundles on the normalization of the curve. In a forthcoming paper we will apply the results of the present paper to obtain a canonical decomposition of generalized theta functions on the moduli stack of Gieseker vector bundles (cf. [K3]).

Our proof of Theorem 4.3 is inspired by [CP] §8, where the cohomology of line bundles on complete symmetric varieties is computed. At one notable point however we have to argue differently, since to show that certain simple submodules occur in the space of global sections De Concini and Procesi make use of the fact that certain line bundles are ample (cf. [CP], Proposition 8.4), and it turns out (cf. 6.1) that the corresponding statement is false in the case of $KGL(E, F)$. Instead, we produce (in 5.3) explicit sections which generate the simple submodules in question.

After finishing this paper I have learned that A. Tchoudjem has studied the cohomology of line bundles on compactifications of arbitrary reductive groups [T]. Part of our result can probably be deduced from his, but certainly not all, since he does not deal with cohomology of the strata and does not obtain a canonical decomposition.

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2. Notation

If $p \leq q$ are two integers, we denote by $[p, q]$ the set of all integers $i$ with $p \leq i \leq q$.

3. Preliminary results

In this chapter we recall some results from [K1] which we will need in the following chapters.
Let $k$ be a field of characteristic zero. We fix two $k$-vector spaces $E$ and $F$ of rank $n$. In [K1] I have defined a compactification $KGL(E, F)$ of the scheme $\text{Isom}(E, F)$ by the following construction: Let $X^{(0)} := \mathbb{P}(\text{Hom}(E, F)^\vee \oplus k)$ and define for $i = 0, \ldots n - 1$ the closed subschemes

$$
Y_i^{(0)} := \{(f : a) \in X^{(0)} \mid \text{rk}(f) \leq i\}
$$

$$
Z_i^{(0)} := \{(f : 0) \in X^{(0)} \mid \text{rk}(f) \leq n - i\}
$$

of $X^{(0)}$. In other words, after choosing a basis for $E$ and for $F$ we can identify the scheme $X^{(0)}$ with $\text{Proj}(k[x_{00}, x_{ij}(i, j \in [1, n]))]$ and the subscheme $Y_p^{(0)}$ (the subscheme $Z_{n-p}$) belongs to the homogenous ideal generated by the $(p + 1) \times (p + 1)$-subminors of the matrix $(x_{ij})_{i,j}$ (by these minors and $x_{00}$). These subschemes satisfy inclusion relations as indicated in the following diagram:

$$
\begin{array}{cccccc}
Y_0^{(0)} & \longrightarrow & Y_1^{(0)} & \longrightarrow & \cdots & \longrightarrow Y_{n-1}^{(0)} \\
\uparrow & & & & & \uparrow \\
Z_{n-1}^{(0)} & \longrightarrow & \cdots & \longrightarrow & Z_1^{(0)} & \longrightarrow Z_0^{(0)}
\end{array}
$$

By definition, $KGL(E, F)$ is the result of successively blowing up the scheme $X^{(0)}$ as follows:

$$
X^{(0)} \longrightarrow X^{(1)} \longrightarrow X^{(2)} \longrightarrow \cdots \longrightarrow X^{(n-1)} = KGL(E, F)
$$

Here, $X^{(1)}$ is the result of blowing up $X^{(0)}$ in the (disjoint) union of the subschemes $Y_0^{(0)}$ and $Z_{n-1}^{(0)}$. Generally, in the $i$-th step we define $Y_{i-1}^{(i)}$, $Z_{n-j}^{(i)} \subset X^{(i)}$ to be the proper transforms of $Y_{j-1}^{(i-1)}$ and $Z_{n-j}^{(i-1)}$ respectively if $j \neq i$ and to be the exceptional divisors lying above $Y_{i-1}^{(i-1)}$ and $Z_{n-i}^{(i-1)}$ respectively if $j = i$. Then it turns out that the subschemes $Y_i^{(i)}$ and $Z_{n-i}^{(i)}$ are smooth and disjoint and thus the blowing up of $X^{(i)}$ in $Y_i^{(i)} \cup Z_{n-i}^{(i)}$ is a smooth projective variety $X^{(i+1)}$.

We have a natural open embedding $\text{Isom}(E, F) \subset X^{(0)}$. Since the centers of blowing up are in the complement of $\text{Isom}(E, F)$, we can regard $\text{Isom}(E, F)$ as an open subset of $KGL(E, F)$. By [K1], 4.2 the complement of $\text{Isom}(E, F)$ in $KGL(E, F)$ is a divisor with normal crossings whose irreducible components are $Z_1, \ldots, Z_{n-1}$ and $Y_1, \ldots, Y_{n-1}$, where $Z_i := Z_{i-(n-1)}^{(n-1)}$ and $Y_i := Y_{i-(n-1)}^{(n-1)}$.

After the choice of a basis for $E$ and $F$ there are canonical rational functions

$$
y_{ji}, z_{ij} \quad (1 \leq i < j \leq n)$$

$$
t_i/t_0 \quad (1 \leq i \leq n)
$$

on $KGL(E, F)$ which are related to the coordinate functions $x_{ij}/x_{00}$ on $X^{(0)}$ by the matrix equation

$$
\begin{bmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1
\end{bmatrix}
\begin{bmatrix}
x_{ij} \\
x_{00}
\end{bmatrix}
= 
\begin{bmatrix}
y_{ij} \\
\vdots
\end{bmatrix}
= 
\begin{bmatrix}
t_1/t_0 \\
\vdots \\
t_{n}/t_0
\end{bmatrix}
\begin{bmatrix}
1 \\
z_{ij} \\
\vdots
\end{bmatrix}
$$
Proposition 3.1. Fix a basis for \( E \) and for \( F \). Let \( \ell \in [0,n] \). Let \( \iota_\ell : [1,n+1] \to [0,n] \) be the bijection such that \( \iota_\ell(\ell+1) = 0 \) and such that it induces an increasing map from the complement of \( \ell+1 \) onto \([1,n]\).

There is an open subscheme \( X(\ell) \subset KGL(E,F) \) which is isomorphic to the \( n^2 \)-dimensional affine space, such that the coordinate functions on \( X(\ell) \) are the restrictions of the rational functions \( y_{ji}, z_{ij} \ (1 \leq i < j \leq n) \) and the rational functions

\[
\iota_{\ell(i+1)}/\iota_{\ell(i)} \quad (1 \leq i \leq n).
\]

Furthermore, the intersection of \( Y_i \) (of \( Z_{n-i-1} \)) with \( X(\ell) \) is empty if \( 0 \leq i \leq \ell - 1 \) (if \( \ell \leq i \leq n - 1 \)), and the equation on \( X(\ell) \) of the divisor \( Y_i \) (of the divisor \( Z_{n-i-1} \)) is \( t_{\ell(i+2)}/t_{\ell(i+1)} \) if \( \ell \leq i \leq n - 1 \) (if \( 0 \leq i \leq \ell - 1 \)).

Proof. This is a special case of \([K1], \text{Proposition 4.1.}\) \qed

In \([K1], \text{5.5}\) I have shown that \( KGL(E,F) \) can be regarded as a moduli space for certain diagrams of vector bundles. To formulate the precise statement, we have to introduce some definitions.

Let \( T \) be a \( k \)-scheme and let \( E, F \) be two locally free \( O_T \)-modules of rank \( n \). A \textit{bf-morphism} from \( E \) to \( F \) is a tuple \( \gamma = (L, \lambda, E \to F, F \to L \otimes E, r) \) where \( L \) is an invertible \( O_T \)-module, \( \lambda \) is a section of \( L \), the arrows \( E \to F \) and \( F \to L \otimes E \) are \( O_T \)-module morphisms and \( r \) is an integer between \( 0 \) and \( n \) such that locally on \( T \) there exist isomorphisms

\[
E \xrightarrow{\sim} rO_T \oplus (n-r)O_T
\]

\[
F \xrightarrow{\sim} rO_T \oplus (n-r)L
\]

with the property that via these isomorphisms the morphisms \( E \to F \) and \( F \to L \otimes E \) are expressed by the diagonal matrices

\[
\begin{bmatrix}
\mathbb{I}_r & 0 \\
0 & \lambda \mathbb{I}_{n-r}
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\lambda \mathbb{I}_r & 0 \\
0 & \mathbb{I}_{n-r}
\end{bmatrix}
\]

respectively. We will often use the following more suggestive notation for the \textit{bf-morphism} \( \gamma \):

\[
\gamma = \begin{pmatrix}
E & \circlearrowright_{(L,\lambda)} \\
\overrightarrow{E} & F
\end{pmatrix}.
\]

Let \( T, E, F \) be as above. A \textit{generalized isomorphism} \( \Phi \) from \( E \) to \( F \) is a sequence of \textit{bf-morphisms} connected as follows:

\[
E_{(M_0,\mu_0)} \xrightarrow{0} E_1 \xrightarrow{1} E_2 \cdots \xrightarrow{\iota_{n-1}} E_n \xrightarrow{\sim} F_{(L_{n-1},\lambda_{n-1})} \xrightarrow{\iota_{n-1}} \cdots \xrightarrow{1} F_{(L_1,\Lambda_1)} \xrightarrow{1} F_{(L_0,\lambda_0)} \xrightarrow{0} F
\]

which has properties for which we refer the reader to \([K1], \text{5.2.}\) since they will not be of importance here. Two generalized isomorphisms \( \Phi \) (as above) and \( \Phi' \) (with primed ingredients) from \( E \) to \( F \) are said to be \textit{equivalent}, if there exist isomorphisms \( E_i \xrightarrow{\sim} E_i', F_i \xrightarrow{\sim} F_i' \), \( M_i \xrightarrow{\sim} M_i', L_i \xrightarrow{\sim} L_i' \), such that all the obvious diagrams commute. Theorem 5.5 in \([K1]\) can now be formulated as follows:
**Theorem 3.2.** The variety $\text{KGL}(E, F)$ represents the functor which to a $k$-scheme $T$ associates the set of all equivalence classes of generalized isomorphisms from $E \otimes O_T$ to $F \otimes O_T$. In particular, there is a universal generalized isomorphism $\Phi_{\text{univ}}$:

$$E \otimes O_{(M_0, \mu_0)} \xrightarrow{\gamma} E_1 \rightarrow \cdots \rightarrow E_{n-1} \rightarrow E_n \simeq F_n \rightarrow \cdots \rightarrow F_1 \otimes O_{(L_0, \lambda_0)} \rightarrow F$$

The global sections $\mu_i$ and $\lambda_i$ of the line bundles $M_i$ and $L_i$ vanish exactly along the boundary components $Z_i$ and $Y_i$ respectively. Thus we have canonical isomorphisms $M_i = \mathcal{O}(Z_i)$ and $L_i = \mathcal{O}(Y_i)$ which identify $\mu_i$ and $\lambda_i$ with the canonical 1-sections of $\mathcal{O}(Z_i)$ and of $\mathcal{O}(Y_i)$ respectively.

In [KL], §6 I have shown that a bf-morphism $\gamma = (L, \lambda, \mathcal{E} \rightarrow \mathcal{F}, \mathcal{F} \rightarrow L \otimes \mathcal{E}, p)$ induces canonical morphisms

$$\bigwedge^r \mathcal{E} \rightarrow (L^\vee)^{\otimes \max(0, r-p)} \otimes \bigwedge^r \mathcal{F}$$

$$\bigwedge^r \mathcal{F} \rightarrow L^{\otimes \min(r, n-p)} \otimes \bigwedge^r \mathcal{E},$$

which I call the exterior powers of $\gamma$. Given a generalized isomorphism $\Phi$:

$$\xymatrix{ E \ar[r]_{\gamma} & E_1 \ar[r] & E_2 \ldots \ar[r] & E_{n-1} \ar[r] & E_n \ar[r] & F_n \ar[r] & F_{n-1} \ldots \ar[r] & F_1 \ar[r] & F}$$

over a scheme $T$ we can compose the exterior powers of the bf-morphisms occurring in it and can thus define the exterior power

$$\bigwedge^r \Phi : \bigwedge^r \mathcal{E} \rightarrow \bigotimes_{\nu=1}^r \left( \bigotimes_{i=0}^{\nu-1} L_i^\vee \otimes \bigotimes_{i=0}^{n-\nu} M_i \right) \otimes \bigwedge^r \mathcal{F}$$

of $\Phi$. If $\mathcal{E} = \mathcal{F} = n\mathcal{O}_T$ then $\bigwedge^r \mathcal{E}$ and $\bigwedge^r \mathcal{F}$ have a natural direct sum decomposition into copies of $\mathcal{O}_T$ indexed by the subsets of cardinality $r$ of $[1, n]$. Thus we have canonical inclusion and projection morphisms $\iota_A : \mathcal{O}_T \rightarrow \bigwedge^r \mathcal{E}$, $\pi_B : \bigwedge^r \mathcal{F} \rightarrow \mathcal{O}_T$ for such subsets $A, B \subseteq [1, n]$ and we can define the section det$_{A,B} \Phi := \pi_A \circ (\bigwedge^r \Phi) \circ \iota_B$ of the line bundle $\otimes_{i=1}^r (\otimes_{i=1}^{\nu-1} L_i^\vee \otimes \bigotimes_{i=0}^{n-\nu} M_i)$.

**Proposition 3.3.**

1. Fix a basis of $E$ and $F$, let $\ell \in [0, n]$ and let $X(\ell)$ be the corresponding open subset of $\text{KGL}(E, F)$ defined in Proposition 3.1. Then $X(\ell)$ is the largest subset in the complement of the divisors $Y_0, \ldots, Y_{\ell-1}$ and $Z_0, \ldots, Z_{n-\ell-1}$ for which the sections det$_{[1,\ell],[1,\ell]} \Phi_{\text{univ}}$ are nowhere vanishing for $r = 1, \ldots, n$.

2. The morphism $\bigwedge^n \Phi_{\text{univ}}$ is nowhere vanishing.

**Proof.** The first statement follows from [KL] 7.4 and 4.3. The second statement is a consequence of loc. cit. 6.5. □

Let $T$ be a scheme, let $\mathcal{E}, \mathcal{F}$ be two vector bundles of rank $n$ on $T$, and let $\gamma$ be a bf-morphism from $\mathcal{E}$ to $\mathcal{F}$. An automorphism $g$ of $\mathcal{E}$ (an automorphism $h$ of $\mathcal{F}$) can be composed in an obvious way with $\gamma$ to give a new bf-morphism $\gamma g$ (a new bf-morphism $h \gamma$) from $\mathcal{E}$ to $\mathcal{F}$. Thus, if $\Phi$ is a generalized isomorphism from $\mathcal{E}$ to $\mathcal{F}$, we get a new generalized automorphism $h \Phi g$ from $\mathcal{E}$ to $\mathcal{F}$ by composing the two outer bf-morphisms in $\Phi$ with $h$ and $g$. 
Corollary 3.5. Let \((v_1, \ldots, v_n)\) and \((w_1, \ldots, w_n)\) be a basis for \(E\) and \(F\) respectively. Let 
\(B_1 \subset GL(E)\) and \(B_2 \subset GL(F)\) be the Borel subgroups consisting of linear automorphisms fixing the flags 
\(\{0\} \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset E\) and \(\{0\} \subset \langle w_n \rangle \subset \langle w_n, w_{n-1} \rangle \subset \cdots \subset F\) respectively. For each \(\ell \in [0, n]\) the open subset \(X(\ell)\) is invariant by the operation of 
\(B := B_1 \times B_2 \subset G\) on \(KGL(E, F)\).

Proof. Let \(R\) be a \(k\)-algebra. By 3.2 and 3.3 (1) an \(R\)-valued point of \(X(\ell)\) is given by 
generalized isomorphism \(\Phi\) from \(E \otimes R\) to \(F \otimes R\) such that the sections \(\det_{[1, r]}(1)\Phi\) \((r \in [1, n])\), \(\lambda_0, \ldots, \lambda_\ell\) and \(\mu_0, \ldots, \mu_{n-\ell}\) are nowhere vanishing. We have to show that for each 
\((g_1, g_2) \in B(R)\) the generalized isomorphism \(\Phi' = g_2 \Phi g_1^{-1}\) has again this property. Since 
the sections \(\lambda_i\) and \(\mu_i\) are the same in \(\Phi'\) and in \(\Phi\), it suffices to show that the quotients 
\(\det_{[1, r]}(1)\Phi'/\det_{[1, r]}(1)\Phi\) \((r \in [1, n])\) are in \(R^\times\). But since \(g_1 \in B_1(R)\) and \(g_2 \in B_2(R)\), 
there exist \(u_1, u_2 \in R^\times\), such that 
\[\wedge^r g_1^{-1}(v_1 \wedge \cdots \wedge v_r) = u_1 \cdot v_1 \wedge \cdots \wedge v_r,\]
\[\pi_{[1, r]}(\wedge^r g_2(w_{j_1} \wedge \cdots \wedge w_{j_r})) = \begin{cases} u_2 & \text{for } \{j_1, \ldots, j_r\} = [1, r], \\ 0 & \text{else.} \end{cases}\]
Therefore we have \(\det_{[1, r]}(1)\Phi' = u_1 u_2 \det_{[1, r]}(1)\Phi\). \(\Box\)

Definition 3.6. From now on the symbols \(L_i, M_i\) will denote the line bundles which occur 
in the universal generalized isomorphism \(\Phi_{\text{univ}}:\)

\[
\begin{array}{ccccccc}
E \otimes \mathcal{O}_{(M_0, \mu_0)} & \longrightarrow & \cdots & \longrightarrow & E_{n-1} \otimes \mathcal{O}_{(M_{n-1}, \mu_{n-1})} & \longrightarrow & F_n \otimes \mathcal{O}_{(L_{n-1}, \lambda_{n-1})} & \longrightarrow & \cdots & \longrightarrow & F_1 \otimes \mathcal{O} \\
\end{array}
\]
on \(KGL(E, F)\).

From definition 3.3 it is clear that the line bundles \(M_i\) and \(L_i\) are canonically \(G\)-linearized. Notice that also the trivial line bundles \(\det(E) \otimes_k \mathcal{O}\) and \(\det(F) \otimes_k \mathcal{O}\) carry canonical nontrivial \(G\)-linearization.

Lemma 3.7. There is a canonical isomorphism of \(G\)-linearized line bundles on \(KGL(E, F)\):

\[
(\det E)^{-1} \otimes_k \bigotimes_{i=0}^{n-1} M_i^{n-i} = (\det F)^{-1} \otimes_k \bigotimes_{i=0}^{n-1} L_i^{n-i}. \quad (*)
\]
The Picard group of the variety \(KGL(E, F)\) is generated by the isomorphism classes of the 
line bundles \(M_i\) and \(L_i\) \((i \in [0, n-1])\) and the only relations come from the isomorphism 
\((*)\).
Proof. The first statement follows from 3.3(2), since it says that the canonical morphism

$$\wedge^n \Phi_{\text{univ}} : \det(E) \otimes_k \mathcal{O} \to \bigotimes_{i=1}^n \left( \bigotimes_{j=0}^{i-1} L_j^{-1} \otimes \bigotimes_{j=0}^{n-i} M_j \right) \otimes_k \det(F)$$

is nowhere vanishing.

Recall that $\text{KGL}(E, F)$ is defined as the result of a successive blowing up $X^{(i)} \to X^{(i-1)}$ along disjoint and smooth irreducible subschemes $Y^{(i-1)}_i$ and $Z^{(i-1)}_{n-i}$ of $X^{(i-1)}$ of codimension $\geq 2$. Therefore the divisor class group of $X^{(i)}$ is the direct sum of the divisor class group of $X^{(i-1)}$ and the free abelian group generated by the two divisors $Y^{(i)}_i$ and $Z^{(i)}_{n-i}$. Now the divisor class group of $X^{(0)}$ is generated by the hyper-plane $Z^{(0)}_0$; therefore by induction it follows that the classes of $Y_0, \ldots, Y_{n-2}, Z_0, \ldots, Z_{n-1}$ freely generate the divisor class group of $\text{KGL}(E, F)$.

For each pair of subsets $I, J \subseteq [0, n-1]$ with $\min(I) + \min(J) \geq n$ we define the closed subscheme $\overline{O}_{I,J} = \overline{O}_{I,J}(E, F)$ in $\text{KGL}(E, F)$ as the intersection of the components $Z_i$ ($i \in I$) and $Y_j$ ($j \in J$). As shown in [K1] $\S$9, the subschemes $\overline{O}_{I,J}$ are precisely the closures of the orbits of $G$ acting on $\text{KGL}(E, F)$.

If $I \supseteq I'$ and $J \supseteq J'$ then we have $\overline{O}_{I,J} \subseteq \overline{O}_{I', J'}$. In particular, we have $\overline{O}_{\emptyset \emptyset} = \text{KGL}(E, F)$ and the smallest of the closed subschemes $\overline{O}_{I,J}$ are of the form

$$\overline{O}_{r,s} := \overline{O}_{[s,n-1],[r,n-1]}$$

for $r, s \in [0, n]$, $r + s = n$. (The set $[s, n-1]$ contains $r$ elements while the set $[r, n-1]$ contains $s$ elements, that’s why we write $\overline{O}_{r,s}$ instead of $\overline{O}_{s,r}$.) Let

$$i_{I,J} : \overline{O}_{I,J} \hookrightarrow \text{KGL}(E, F) \quad \text{and} \quad i_{r,s} : \overline{O}_{r,s} \twoheadrightarrow \text{KGL}(E, F)$$

denote the inclusion morphisms.

Let $\text{Fl}(E)$ and $\text{Fl}(F)$ denote the full flag manifolds associated to the vector spaces $E$ and $F$ respectively and let $\text{Fl} := \text{Fl}(E) \times \text{Fl}(F)$. For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, $b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$ we define the invertible $\mathcal{O}_{\text{Fl}}$-module

$$\mathcal{O}_{\text{Fl}}(a, b) := \bigotimes_{i=1}^n (\mathcal{E}_i/\mathcal{E}_{i-1})^{\otimes a_i} \otimes \bigotimes_{i=1}^n (\mathcal{F}_i/\mathcal{F}_{i-1})^{\otimes b_i},$$

where $0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = E \otimes \mathcal{O}_{\text{Fl}}$ and $0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n = F \otimes \mathcal{O}_{\text{Fl}}$ are the two universal flags on $\text{Fl}$. The variety $\text{Fl}$ is endowed with a canonical $G$-action and the line bundles $\mathcal{O}_{\text{Fl}}(a, b)$ come with a canonical $G$-linearization.

Lemma 3.8. For each pair $r, s \in [0, n]$ with $r + s = n$ we have a canonical isomorphism $\overline{O}_{r,s} \cong \text{Fl}$, which is compatible with the $G$-action on the two varieties. Furthermore, we have a canonical isomorphism of $G$-linearized line bundles on $\overline{O}_{r,s}$:

$$i_{r,s}^*(\bigotimes_{i=0}^{n-1} (M_i^{m_i} \otimes L_i^l) \otimes (\det E)^e \otimes (\det F)^d) = \mathcal{O}_{\overline{O}_{r,s}}(a, b),$$
where \( O_{\Theta_{r,s}}(a, b) \) is the line bundle corresponding to \( O_{F_0}(a, b) \) via the isomorphism \( \overline{O}_{r,s} \rightsquigarrow F_0 \) and where \( (a, b) \in \mathbb{Z}^n \times \mathbb{Z}^n \) is defined by

\[
a_i - e = -b_{n-i+1} + d = \begin{cases} l_{n-i+1} - l_{n-i} & \text{if } i \in [1, s] \\ m_{i-1} - m_i & \text{if } i \in [s + 1, n] \end{cases}
\]

(It is understood that \( m_n = l_n = 0 \)).

**Proof.** The first part of the lemma is a special case of [K1] Theorem 9.3: Let

\[
U_i := \begin{cases} \mathcal{E}_i & \text{if } 0 \leq i \leq s \\ \mathcal{E}_{i-1} & \text{if } s + 1 \leq i \leq n + 1 \end{cases}, \quad V_i := \begin{cases} \mathcal{F}_i & \text{if } 0 \leq i \leq r \\ \mathcal{F}_{i-1} & \text{if } r + 1 \leq i \leq n + 1 \end{cases}
\]

Then in the notation of loc. cit. we have

\[
\overline{O}_{r,s} = P_0 \times_{F_0} \cdots \times_{F_0} P_r \times_{F_0} Q_r \times_{F_0} \cdots \times_{F_0} Q_1 \times_{F_0} K'
\]

where \( P_p = \overline{\text{PGL}}(V_{r-p+1}/V_{r-p}, U_{s+p+1}/U_{s+p}) \), \( Q_q = \overline{\text{PGL}}(U_{s-q+1}/U_{s-q}, V_{r+q+1}/V_{r+q}) \) and \( K' = \text{KGL}(U_{s+1}/U_s, V_{r+1}/V_r) \). Since the bundles \( V_{r-p+1}/V_{r-p}, U_{s+p+1}/U_{s+p}, U_{s-q+1}/U_{s-q}, V_{r+q+1}/V_{r+q} \) are of rank one and the bundles \( U_{s+1}/U_s, V_{r+1}/V_r \) are of rank zero, it follows that \( P_p = Q_q = K' = F_0 \) and therefore \( \overline{O}_{r,s} = F_0 \).

The second part of the lemma follows from the proof of [K1], 9.3: In the notation of that proof we have

\[
i_{r,s}^*M_i = \mathcal{M}_i = \begin{cases} O_{F_0} & \text{if } s \in [0, s - 1] \\ \mathcal{M}_0(r) & \text{if } i = s \\ \mathcal{M}_0(i-s) \otimes (\mathcal{M}_0(i-s))^\vee & \text{if } s + 1 \leq i \leq n - 1 \end{cases}
\]

where the line bundle \( \mathcal{M}_0(p) \) is ingredient of the bf-morphism

\[
\begin{pmatrix} \mathcal{E}_1(p) \\ \mathcal{M}_0(p) \end{pmatrix}_{r, s = 0} = \begin{pmatrix} \mathcal{E}_0(p) \\ \mathcal{M}_0(p) \end{pmatrix}
\]

between \( \mathcal{E}_0(p) = U_{s+p+1}/U_{s+p} \) and \( \mathcal{E}_1(0) = V_{r+p+1}/V_{r+p} \). But this means that we have a canonical isomorphism of line bundles

\[
\mathcal{M}_0(p) = (U_{s+p+1}/U_{s+p}) \otimes (V_{r+p+1}/V_{r+p})^\vee
\]

and consequently

\[
i_{r,s}^*M_i = (U_{i+2}/U_{i+1}) \otimes (U_{i+1}/U_i)^\vee \otimes (V_{n-i+1}/V_{n-i}) \otimes (V_{n-i}/V_{n-i-1})^\vee
\]

for \( i = s, \ldots, n - 1 \). Analogously we have \( i_{r,s}^*L_i = O_{F_0} \) for \( i = 0, \ldots, r - 1 \) and

\[
i_{r,s}^*L_i = (V_{i+2}/V_{i+1}) \otimes (V_{i+1}/V_i)^\vee \otimes (U_{n-i+1}/U_{n-i}) \otimes (U_{n-i}/U_{n-i-1})^\vee
\]

for \( i = r, \ldots, n - 1 \). The stated formula follows from this together with the fact that we have \( O_{F_0}(1, \ldots, 1, 0, \ldots, 0) = \det(E) \otimes O_{F_0}, \) and \( O_{F_0}(0, \ldots, 0, 1, \ldots, 1) = \det(F) \otimes O_{F_0} \). \( \square \)

**Proposition 3.9.** Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) be two elements in \( \mathbb{Z}^n \). Then

\[
H^0(F_0, O_{F_0}(a, b)) \neq 0 \text{ if and only if } a, b \text{ are increasing, i.e. if } a_1 \leq a_2 \leq \cdots \leq a_n \text{ and } b_1 \leq b_2 \leq \cdots \leq b_n.
\]

The association

\[
(a, b) \mapsto H^0(F_0, O_{F_0}(a, b))
\]
establishes a bijection between the set of all increasing \(a, b \in \mathbb{Z}^n\) and the set of simple \(G\)-modules. Furthermore, \(H^p(\mathcal{F}l, \mathcal{O}_{\mathcal{F}l}(a, b)) = 0\) for all \(p \geq 2\) and all increasing \(a, b \in \mathbb{Z}^n\).

**Proof.** This is a special case of the Borel-Bott-Weil theorem (cf. e.g. [3] II. 5.5). \(\square\)

### 4. Statement of the theorem

We keep the notations introduced in section [3]

**Definition 4.1.** Let \(L\) be a line bundle on \(\text{KGL}(E, F)\) of the form

\[
L = \bigotimes_{i=0}^{n-1} (M_i^{m_i} \otimes L_i^l) \otimes (\det E)^e \otimes (\det F)^d
\]

Let \(I, J \subseteq [0, n-1]\) and let \(i_1 := \min(I), j_1 := \min(J)\) where it is understood that \(\min(\emptyset) = n\). Assume \(i_1 + j_1 \geq n\). We denote by \(A_{I, J}(L)\) the set of all elements \((a, b) \in \mathbb{Z}^n \times \mathbb{Z}^n\), which have the following properties:

1. \(a_1 \leq a_2 \leq \cdots \leq a_n\)
2. \(\sum_{j=i+1}^{n}(a_j - e) \leq m_i\) for all \(i \in [n-j_1, n-1]\) and equality holds for \(i \in I\).
3. \(\sum_{j=1}^{n-i}(a_j - e) \geq -l_i\) for all \(i \in [n-i_1, n-1]\) and equality holds for \(i \in J\).
4. For all \(i \in [1, n]\) the equality \(a_i - e = -b_{n-i+1} + d\) holds.

For abbreviation we denote by \(A(L)\) the set \(A_{\emptyset, \emptyset}(L)\).

**Remark 4.2.** Notice that for \(r, s \in [0, n]\) with \(r + s = n\) the set \(A_{[s, n-1], [r, n-1]}(L)\) contains at most the single element \((a, b)\) defined in [3].

**Theorem 4.3.** Let \(L\) be a line bundle on \(\text{KGL}(E, F)\) of the form

\[
L = \bigotimes_{i=0}^{n-1} (M_i^{m_i} \otimes L_i^l) \otimes (\det E)^e \otimes (\det F)^d
\]

and let \(I, J \subseteq [0, n-1]\) be subsets with \(\min(I) + \min(J) \geq n\). Then the following holds:

1. The \(G\)-module \(H^0(\mathcal{O}_{I, J}, i_{I, J}^*L)\) comes with a canonical decomposition as follows:

\[
H^0(\mathcal{O}_{I, J}, i_{I, J}^*L) = \bigoplus_{(a, b) \in A_{I, J}(L)} H^0(\mathcal{F}l, \mathcal{O}(a, b))
\]

2. This decomposition is compatible with restriction in the sense that the following is a commutative diagram of \(G\)-modules:

\[
\begin{array}{ccc}
H^0(\text{KGL}, L) & \xrightarrow{\text{Res}} & H^0(\mathcal{O}_{I, J}, i_{I, J}^*L) \\
\bigoplus_{(a, b) \in A(L)} H^0(\mathcal{F}l, \mathcal{O}_{\mathcal{F}l}(a, b)) & \xrightarrow{\bigoplus_{(a, b) \in A(L) \cap A_{I, J}(L)}} & \bigoplus_{(a, b) \in A_{I, J}(L)} H^0(\mathcal{F}l, \mathcal{O}_{\mathcal{F}l}(a, b)) \bigoplus_{(a, b) \in A_{I, J}(L)} H^0(\mathcal{F}l, \mathcal{O}_{\mathcal{F}l}(a, b))
\end{array}
\]

where the lower arrows are the canonical projection and inclusion morphisms induced by the inclusions \(A(L) \cap A_{I, J}(L) \subseteq A(L)\) and \(A(L) \cap A_{I, J}(L) \subseteq A_{I, J}(L)\) respectively.
3. Let
\[ L' = \bigotimes_{i=0}^{n-1} (M_i^{m_i} \otimes L_i^r) \otimes f^*(\det E)^e \otimes f^*(\det F)^d, \]
where \( m_i \leq m_i \) and \( l_j' \leq l_j \) and equality holds, if \( i \in I \) and \( j \in J \) respectively. Then we have a commutative diagram of \( G \)-modules as follows:

\[
\begin{array}{ccc}
H^0(\mathcal{O}_{I,J}, i_{I,J,L}'(L')) & \otimes_{\mu^{m_i-m_i'} \otimes \lambda^i} \otimes H^0(\mathcal{O}_{I,J}, i_{I,J,L} L) \\
\bigoplus_{(a,b) \in A_{I,J}(L')} H^0(Fl, \mathcal{O}_{Fl}(a,b)) & & \bigoplus_{(a,b) \in A_{I,J}(L)} H^0(Fl, \mathcal{O}_{Fl}(a,b))
\end{array}
\]

where the upper horizontal arrow is induced by the section
\[
\left( \mu^{m_0-m_0'} \otimes \ldots \otimes \mu^{m_{n-1}-m_{n-1}'} \otimes \lambda^{l_0-l_0'} \otimes \ldots \otimes \lambda^{l_{n-1}-l_{n-1}'} \right)|_{I,J}
\]
of \( i_{I,J}(L \otimes (L')^{-1}) \) and the lower horizontal arrow is induced by the inclusion \( A_{I,J}(L') \subseteq A_{I,J}(L) \).

5. Proof of the theorem

We fix a basis \((v_1, \ldots, v_n)\) for \( E \) and \((w_1, \ldots, w_n)\) for \( F \). Let \( B_1 \subseteq \text{GL}(E) \) and \( B_2 \subseteq \text{GL}(F) \) be the Borel subgroups consisting of linear automorphisms fixing the flags

\[
\{0\} \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \ldots \subset E \quad \text{and} \quad \{0\} \subset \langle w_n \rangle \subset \langle w_n, w_{n-1} \rangle \subset \ldots \subset F
\]
respectively. Let \( U_1 \subseteq B_1 \) and \( U_2 \subseteq B_2 \) be the maximal unipotent subgroups of \( B_1 \) and \( B_2 \) respectively. Then \( B := B_1 \times B_2 \) is a Borel subgroup of \( G \) and \( U := U_1 \times U_2 \) is its maximal unipotent subgroup.

Let \( V := U \times \mathbb{A}^n \) and let \( \xi_1, \ldots, \xi_n \in H^0(V, \mathcal{O}_V) \) be the pull back of the coordinate functions on \( \mathbb{A}^n \). Let \( V^o \subset V \) be the maximal open subset where all the \( \xi_i \) are invertible. For every pair \( r, s \in [0, n] \) with \( r + s = n \) we have a morphism \( j_{r,s}^o : V^o \to \text{Isom}(E, F) \), which on \( R \)-valued points (\( R \) a \( k \)-algebra) is defined by

\[
j_{r,s}^o(x, y, z) = y \circ \zeta_r(z) \circ x
\]
for \( x \in U_1(R), \ y \in U_2(R), \ z = (z_1, \ldots, z_n) \in (R^X)^n \), where \( \zeta_r(z) : E \otimes_k R \looparrowright F \otimes_k R \) is the isomorphism defined with respect to the given basis of \( E \) and \( F \) by the diagonal matrix \( \text{diag}(\zeta_{r,1}(z), \ldots, \zeta_{r,n}(z)) \) whose entries are

\[
\zeta_{r,i}(z) = \begin{cases} 
z_i^{-1} \ldots z_r^{-1} & \text{for } i \in [1, r] \\
z_{r+1} \ldots z_i & \text{for } i \in [r+1, n]
\end{cases}
\]

It follows from 3.1 that the morphism \( j^{(r,s)} \) extends to an open immersion

\[
j^{(r,s)} : V \longrightarrow \text{KGL}(E, F)
\]
whose image is the open affine subscheme $X(r)$. We know from \[3.5\] that $X(r)$ is $B$-invariant; therefore the immersion $j^{(r,s)}$ induces a $B$-action on $V$. Explicitly, on $R$-valued points this action is given by

$$b \cdot_r (x, y, z) := (\rho x \rho^{-1} u_{1}^{-1}, u_{2} \tau y \tau^{-1}, z')$$

where $b = (u_{1} \rho, u_{2} \tau) \in B(R), u_{i} \in U_{i}(R), \rho = \text{diag}(\rho_{1}, \ldots, \rho_{n})$ and $\tau = \text{diag}(\tau_{1}, \ldots, \tau_{n})$ are $R$-valued points of the maximal torus of $B_{1}$ and $B_{2}$ respectively and $z' = (z'_{1}, \ldots, z'_{n}) \in R^{n}$ is defined by

$$z'_{i} := \begin{cases} \\
\rho_{i+1}^{-1} \rho_{i} \tau_{i+1} \tau_{i}^{-1} z_{i} & \text{if } i \in [1, r - 1] \\
\rho_{r} \tau_{r}^{-1} z_{r} & \text{if } i = r \\
\rho_{r+1}^{-1} \rho_{r+1} \tau_{r+1} z_{r+1} & \text{if } i = r + 1 \\
\rho_{i}^{-1} \rho_{i-1} \tau_{i} \tau_{i-1}^{-1} z_{i} & \text{if } i \in [r + 2, n] 
\end{cases}$$

Let $I, J$ be two subsets of $[0, n - 1]$ and assume $\min(I) + \min(J) \geq n$. By this assumption there exist $r, s \in [0, n]$ with $r + s = n$ and $I \subseteq [s, n - 1], J \subseteq [r, n - 1]$. It is clear from \[3.1\] that the closed subscheme $V_{IJ}$ of $V$ defined by the cartesian diagram

$$\xymatrix{ V_{IJ} \ar@{^{(}->}[r]^{j_{IJ}^{(r,s)}} & V \ar@{^{(}->}[r]_{j^{(r,s)}} & \text{KGL}(E, F) \ar@{^{(}->}[u]_{i_{IJ}} \ar@{^{(}->}[l]_{\overline{\mathcal{O}}_{IJ}}}$$

is cut out by the equations $\xi_{n-i} = 0$ for $i \in I$ and $\xi_{i+1} = 0$ for $i \in J$.

**Proposition 5.1.** Let $L$ be a $G$-linearized line bundle on $\text{KGL}(E, F)$ and let $I, J \subseteq [0, n - 1]$ with $\min(I) + \min(J) \geq n$. Then for each simple $G$-module $W$ the $G$-module $H^{0}(\overline{\mathcal{O}}_{IJ}, i_{IJ}^{*}L)$ contains $W$ at most with multiplicity one as a submodule.

**Proof.** (Analogous to the proof of Lemma 8.2 in \[CP\].) Let $s_{1}$ and $s_{2}$ be two global sections of $i_{IJ}^{*}L$ which generate $B$-invariant lines in $H^{0}(\overline{\mathcal{O}}_{IJ}, i_{IJ}^{*}L)$ on which $B$ operates by the same character. I claim that $\overline{\mathcal{O}}_{IJ}$ contains a dense open $B$-orbit $\Omega$. Indeed, let $z = (z_{1}, \ldots, z_{n}) \in \mathbb{A}^{n}$, where $z_{i} = 0$ if $n - i \notin I$ or $i - 1 \notin J$, and $z_{i} = 1$ else. Then by the preceding formulae it follows easily that if we choose $r, s \in [0, n]$ such that $r + s = n, I \subseteq [s, n - 1], J \subseteq [r, n - 1]$, then the image of the point $(1, z) \in V_{IJ} \subseteq U \times \mathbb{A}^{n}$ by the morphism $j_{IJ}^{(r,s)}$ is contained in a dense open $B$-orbit in $\overline{\mathcal{O}}_{IJ}$. Therefore $s_{1}/s_{2}$ is a rational function on $\overline{\mathcal{O}}_{IJ}$, which is necessarily constant, since its restriction to $\Omega$ is constant. \square

**Proposition 5.2.** Let $L = \bigotimes_{i=0}^{n-1}(M_{i}^{\mu_{i}} \otimes L_{i}^{\lambda})$ and let $I, J \subseteq [0, n - 1]$ with $\min(I) + \min(J) \geq n$. If the $G$-module $H^{0}(\overline{\mathcal{O}}_{IJ}, i_{IJ}^{*}L)$ contains an irreducible $G$-module $W$ as a submodule, then there exists an element $(a, b) \in A_{IJ}(L)$ such that $W$ is isomorphic to the $G$-module $H^{0}(\text{Fl}, \mathcal{O}_{\text{Fl}}(a, b))$.

**Proof.** (Analogous to the proof of Proposition 8.2 in \[CP\].) Let $r, s \in [0, n]$ with $r + s = n$ and $I \subseteq [s, n - 1], J \subseteq [r, n - 1]$. Since $V_{IJ}$ is isomorphic to an affine space $\mathbb{A}^{N}$ for some $N$, there exists a nowhere vanishing section $s_{0}$ of the line bundle $L_{IJ} := (j_{IJ}^{(r,s)})^{*}i_{IJ}^{*}L$ on $V_{IJ}$. The group $B$ acts on $H^{0}(V_{IJ}, L_{IJ})$ and for any $b \in B$ the section $b \cdot s_{0}$ is again nowhere vanishing and thus a scalar multiple of $s_{0}$, since invertible functions on $V_{IJ}$ are constant. Therefore $B$ acts by a character on the line generated by $s_{0}$. 

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\[3.5\] This reference is likely to the Proposition 3.5 in a related work. 

\[CP\] This reference is likely to a joint work by the authors, referring to a specific Proposition 8.2.
Now let $s \in W$ be a highest weight vector with respect to $B$. Thus $s$ is a global section of $i^*_{IJ}L$ which generates a $B$-invariant line inside the space $H^0(\overline{O}_{IJ}, i^*_{IJ}L)$. Let $s_1$ be its pull back by $j_{IJ}(s) : V_{IJ} \to \overline{O}_{IJ}$. Then we have $s_1 = f s_0$ for some regular function $f$ on $V_{IJ}$. Clearly $f$ generates a $B$-invariant line in $H^0(V_{IJ}, \mathcal{O})$, therefore $f$ is left unchanged by the action of the maximal unipotent subgroup $U$ and it follows that $f$ must be a polynomial in the $\xi_i$, where $i \in [1, n]$, $n - i \notin I$, $i - 1 \notin J$. In fact $f$ must be a monomial in these $\xi_i$, since otherwise the $B$-translates of $f$ would generate a subspace of dimension $\geq 2$ of $H^0(V_{IJ}, \mathcal{O})$.

It follows from the above that there is a divisor $D = \sum_{i \in [0, n-1]} \beta_i Z_i + \sum_{i \in [0, n-1]} \alpha_i Y_i$ on $KGL(E, F)$, where $\beta_i \geq 0$, $\alpha_i \geq 0$ for all $i$ and $\beta_i = 0$ if $i \in I$, $\alpha_i = 0$ if $i \in J$, such that the pull back of $D$ to $\overline{O}_{IJ} \cap X(r)$ coincides with the restriction of the vanishing divisor of $s$ to this open subscheme of $\overline{O}_{IJ}$. Therefore there is a global section $s'$ of $i^*_{IJ}L(-D)$ whose image under the canonical map $i^*_{IJ}L(-D) \to i^*_{IJ}L$ is $s$ and whose restriction to $\overline{O}_{IJ} \cap X(r)$ is nowhere vanishing. Since the intersection $\overline{O}_{r,s} \cap X(r)$ is nonempty, it follows that the restriction of $s'$ to the closed subscheme $\overline{O}_{r,s} \subseteq \overline{O}_{IJ}$ is a nonzero section of $i^*_{r,s}L(-D)$.

Let $m'_i := m_i - \beta_i$ and $l'_i := l_i - \alpha_i$ and let $(a, b) \in \mathbb{Z}^n \times \mathbb{Z}^n$ be defined by
\[
a_i = -b_{n-i+1} = \begin{cases} \frac{l'_n}{l_i} - l_{n-i} & \text{if } i \in [1, s] \\ m'_{i-1} - m'_i & \text{if } i \in [s+1, n]\end{cases}
\]
with the convention that $m'_n = l'_n = 0$. By Lemma 3.8 we have $i^*_{r,s}L(-D) = \mathcal{O}_{\overline{O}_{r,s}}(a, b)$.

Consider the following diagram of $G$-modules:
\[
H^0(\overline{O}_{IJ}, i^*_{IJ}L) \leftarrow H^0(\overline{O}_{IJ}, i^*_{IJ}L(-D)) \to H^0(\overline{O}_{r,s}, \mathcal{O}_{\overline{O}_{r,s}}(a, b))
\]

The left arrow is injective and maps $s'$ to $s$. The right arrow maps $s'$ to a non-zero element and by Lemma 3.8 and 3.9 the object on the right is a simple $G$-module. Therefore $H^0(\overline{O}_{r,s}, \mathcal{O}_{\overline{O}_{r,s}}(a, b))$ is isomorphic to $W$ as $G$-module. Let us gather what we know about $(a, b)$:

1. Since, as we have seen above, the line bundle $\mathcal{O}_{\overline{O}_{r,s}}(a, b)$ has a non-vanishing global section, it follows from Lemma 3.9 that $a_1 \leq \cdots \leq a_n$.
2. We have $\sum_{j=i+1}^{n} a_j = m'_i \leq m_i$ for $i \in [s, n-1]$ and equality holds if $i \in I$.
3. We have $\sum_{j=1}^{n-i} a_j = -l'_i \geq -l_i$ for $i \in [r, n-1]$ and equality holds if $i \in J$.
4. By definition, $a_i = -b_{n-i+1}$.

Let $i_1 := \min(I)$ and $j_1 := \min(J)$. In the above argument we can choose any $r, s$ with $r \in [n - i_1, j_1]$ and a priori $(a, b)$ depends on $r, s$ but by Lemma 3.9 the fact that $H^0(\overline{O}_{r,s}, \mathcal{O}_{\overline{O}_{r,s}}(a, b))$ and $W$ are isomorphic as $G$-modules determines $(a, b)$ which is therefore independent of $r, s$. It follows that the inequality in 2. holds for all $i \in [n - i_1, n - 1]$ and the inequality in 3. holds for all $i \in [n - i_1, n - 1]$, i.e. we have $(a, b) \in A_{IJ}(L)$. 

Let $x_{ij}/x_{00}$ $(i, j \in [1, n])$ denote the coordinate functions on $GL_n = \text{Isom}(E, F)$ interpreted as rational functions on $KGL(E, F)$. For each integer $p \in [1, n]$ we define the rational function $d^p$ on $KGL_n$ as the determinant of the $p \times p$ sub-matrix of $(x_{ij}/x_{00})_{i,j \in [1, n]}$ with indices in $[1, p] \times [1, p]$, i.e. we set $d^p := \det_{[1,p][1,p]} (x_{ij}/x_{00})$. For convenience we define $d^0 := 1$. 

Lemma 5.3. Let $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$. Then we have the following equality of divisors on $KGL(E, F)$:

$$\text{div} \left( \prod_{i=1}^{n} \left( \frac{d_i}{d_{i-1}} \right)^{a_{n-i+1}} \right) = \sum_{i=0}^{n-1} \left( - \sum_{j=i+1}^{n} a_j \right) Z_i + \sum_{i=0}^{n-1} \left( \sum_{j=i+1}^{n} a_j \right) Y_i + \sum_{i=1}^{n-1} (a_{n-i+1} - a_{n-i}) \Delta_i,$$

where $\Delta_i$ denotes the closure in $KGL(E, F)$ of the subscheme $\{d_i = 0\} \subset GL_n$. Furthermore, for every $I, J \subseteq [0, n-1]$ with $\min(I) + \min(J) \geq n$ the closed subscheme $\mathcal{O}_{IJ}$ is not contained in any of the $\Delta_i$.

Proof. The subvariety $\Delta_i$ is the locus of vanishing of the global section $\det_{[1,i][1,i]} \Phi_{\text{univ}}$ of the line bundle $\bigotimes_{i=1}^{j} \left( \bigotimes_{j=0}^{n} M_j \otimes \bigotimes_{j=0}^{n-1} L_{-1} \right)$ and by Lemma 5.3 the complement of the union of all $\Delta_i$ is precisely the union of the open sets $X(\ell)$ where $\ell$ runs through $[0, n]$. In the notation introduced before 3.1 we have $d_i/d_{i-1} = t_i/t_0$. Now using 3.1 a simple calculation shows that for each $\ell$ the divisor of $\prod_{i=1}^{n} (t_i/t_0)^{a_{n-i+1}}$ on $X(\ell)$ is a linear combination of the restrictions of the $Z_i$ and $Y_i$ with coefficients as given in the formula.

For the second part of the lemma we choose $\ell$ such that $I \subseteq [n-\ell, n-1]$ and $J \subseteq [\ell, n-1]$ (which is always possible). Then the intersection of $\mathcal{O}_{IJ}$ with $X(\ell)$ is clearly nonempty; therefore $\mathcal{O}_{IJ}$ is not contained in the complement of $X(\ell)$. In particular it is not contained in any of the $\Delta_i$.

We now come to the proof of Theorem 4.3

Proof of the first statement: Let $L$ and $I, J$ be as in the theorem and assume first that $e = d = 0$. If $A_{IJ}(L)$ is empty, then $H^0(\mathcal{O}_{IJ}, i^*_{IJ}L) = (0)$ by 5.2 and therefore the statement 1 of the theorem trivially holds in this case.

Assume $A_{IJ}(L)$ is non-empty, let $(a, b) \in A_{IJ}(L)$ and let $r, s \in [0, n]$ with $r + s = n$ and $I \subseteq [s, n-1]$, $J \subseteq [r, n-1]$. Let $L' := \bigotimes_{i=1}^{n} (M_i^{m_i} \otimes L_i^{l_i})$, where $m_i := \sum_{j=i+1}^{n} a_j$ and $l_i := -\sum_{j=i+1}^{n} a_j$. From Lemma 5.3 it follows that there exists a global section of $L'$ whose restriction to $\mathcal{O}_{r,s}$ is nonzero. By 5.8 we have $i_{r,s}^* L' = \mathcal{O}_{\mathcal{O}_{r,s}^\vee}(a, b)$. Therefore we have a non-vanishing restriction morphism

$$H^0(\mathcal{O}_{IJ}, i_{IJ}^* L') \to H^0(\mathcal{O}_{r,s}, \mathcal{O}_{\mathcal{O}_{r,s}^\vee}(a, b)).$$

Together with 5.9 and 5.11 it follows that the $G$-module $H^0(\mathcal{O}_{IJ}, i_{IJ}^* L')$ contains an irreducible submodule $W \cong H^0(\mathcal{O}_{r,s}, \mathcal{O}_{\mathcal{O}_{r,s}^\vee}(a, b))$ exactly with multiplicity one. In particular, by the above restriction morphism the submodule $W \subseteq H^0(\mathcal{O}_{IJ}, i_{IJ}^* L')$ is canonically identified with $H^0(\mathcal{O}_{r,s}, \mathcal{O}_{\mathcal{O}_{r,s}^\vee}(a, b))$ which in turn is canonically isomorphic to $H^0(\mathcal{O}_{r,s}, \mathcal{O}_{\mathcal{O}_{r,s}^\vee}(a, b)).$

I claim that the identification of $W$ with $H^0(\mathcal{O}_{r,s}, \mathcal{O}_{\mathcal{O}_{r,s}^\vee}(a, b))$ is independent of the choice of the numbers $r, s$. For this it is clearly sufficient to show that the composite morphism

$$H^0(KGL, L') \to H^0(\mathcal{O}_{r,s}, \mathcal{O}_{\mathcal{O}_{r,s}^\vee}(a, b)) \to H^0(\mathcal{O}_{r,s}, \mathcal{O}_{\mathcal{O}_{r,s}^\vee}(a, b))$$

does not depend on $r, s$. This will be shown below. For the moment we assume this fact.

It is clear from the definition of $A_{IJ}(L)$ that

$$\left( \mu_0^{m_0 - m_0} \otimes \ldots \otimes \mu_{n-1}^{m_{n-1} - m_{n-1}} \otimes \lambda_0^{l_0 - l_0} \otimes \ldots \otimes \lambda_{n-1}^{l_{n-1} - l_{n-1}} \right) \bigg|_{\mathcal{O}_{IJ}}$$
is a non-vanishing global section of $i^*_i(L \otimes (L')^{-1})$, and therefore defines a canonical injective morphism

$$H^0(O_{ij}, i^*_i L') \to H^0(O_{ij}, i^*_i L).$$

It follows that $H^0(O_{ij}, i^*_i L)$ contains an irreducible $G$-submodule (the image of $W$), which is canonically isomorphic to $H^0(Fl, O_{Fl}(a, b))$. This together with 5.1 and 5.2 clearly implies statement 1 of the theorem in the case $e = d = 0$.

For arbitrary $e, d$ the result is easily deduced from that special case, the key observation being that we have a canonical isomorphism

$$O_{Fl}(a, b) \otimes (\det E)^e \otimes (\det F)^d = O_{Fl}(a + (e, \ldots, e), b + (d, \ldots, d))$$

of $G$-linearized line bundles on $Fl$.

**Independence of $(r, s)$:** It remains to be shown that the morphism $(*)$ does not depend on $r, s$. In fact, since $H^0(Fl, O_{Fl}(a, b))$ is a simple $G$-module, it suffices to produce a point $z$ in $Fl$ and a global section $s$ of $L'$, whose image in $H^0(Fl, O_{Fl}(a, b))$ evaluates to a nonzero element in in the fiber $O_{Fl}(a, b)[z]$ of $O_{Fl}(a, b)$ at $z$, which is independent of $r, s$.

By 5.3 the rational function $\prod_{i=1}^n(d_i/d_{i-1})^{a_{n-i+1}} = \prod_{i=1}^n(t_i/t_0)^{a_{n-i+1}}$ gives rise to a global section of $O_{KGL}(D)$, where

$$D := \sum_{i=0}^{n-1} (m'_i Z_i + l'_i Y_i)$$

Let $s \in H^0(KGL, L')$ be the element, which corresponds to this section via the canonical isomorphism $O_{KGL}(D) \sim L'$ and let $z \in Fl$ be the point given by the pair of flags

$$\{0\} \subset \langle v_n, \ldots, v_{n-1} \rangle \subset \cdots \subset E \quad \text{and} \quad \{0\} \subset \langle w_1, \ldots, w_2 \rangle \subset \cdots \subset F.$$

Then the image of $s$ by the morphism

$$H^0(KGL, L') \to H^0(O_{r,s}, i^*_r s L') \sim H^0(Fl, O_{Fl}(a, b)) \to O_{Fl}(a, b)[z] =$$

$$= \bigotimes_{i=1}^n \left( \frac{(v_n, \ldots, v_{n-i+1})}{(v_n, \ldots, v_{n+i-2})} \right) \otimes \bigotimes_{i=1}^n \left( \frac{(w_1, \ldots, w_i)}{(w_1, \ldots, w_{i-1})} \right) = \left( \bigotimes_{i=1}^n (v_i \otimes w_i^{-1})^{a_{n-i+1}} \right)$$

is precisely the generator $\otimes_{i=1}^n (v_i \otimes w_i^{-1})^{a_{n-i+1}}$ of the fiber of $O_{Fl}(a, b)$ at $z$.

Thus the image of $s$ in $O_{Fl}(a, b)[z]$ does not depend on $r, s$ as was to be shown.

**Proof of the second statement:** Let $(a, b) \in A(L)$ and let $(a', b') \in A_{ij}(L)$. Consider the composite morphism

$$H^0(Fl, O_{Fl}(a, b)) \to H^0(KGL, L) \to H^0(O_{ij}, i^*_i L) \to H^0(Fl, O_{Fl}(a', b'))$$

(†)

If $(a, b) \neq (a', b')$, then this morphism is clearly 0, since then the domain and the target are non-isomorphic simple $G$-modules. If $(a, b) = (a', b')$, let $L' := \bigotimes_{i=0}^{n-1} (M'_i \otimes L'_i) \otimes f^*(\det E)^e \otimes f^*(\det F)^d$, where $m'_i := \sum_{j=i+1}^n a_j$ and $l'_i := -\sum_{j=1}^{n-i} a_j$, let $r, s \in [0, n]$ with
Proposition 8.4 in [CP] holds. We will see below that the answer is negative.

In [CP], it is natural to ask whether also the analogue of the ampleness result stated in

\[ \text{composite morphism} \]

defines an injective homomorphism

in [CP].

where the horizontal arrows are the restriction morphisms and the vertical arrows are defined as in the proof of the first statement of the theorem. Since all \( \text{G-modules} \) in this diagram contain the simple submodule \( H^0(\text{Fl}, \mathcal{O}_{\text{Fl}}(a, b)) \) with multiplicity one, it follows that the morphism \((\dagger)\) has to be the identity in this case.

Proof of the third statement: Let \((a', b') \in A_{l, j}(L')\) and let \((a, b) \in A_{l, j}(L)\). Consider the composite morphism

\[
H^0(\text{Fl}, \mathcal{O}_{\text{Fl}}(a', b')) \rightarrow H^0(\mathcal{O}_{l, j}, i^*_{l, j} L') \rightarrow H^0(\mathcal{O}_{l, j}, i^*_{l, j} L) \rightarrow H^0(\text{Fl}, \mathcal{O}_{\text{Fl}}(a, b))
\]

If \((a', b') \neq (a, b)\), this morphism vanishes by the same argument as above. If \((a', b') = (a, b)\), then the assertion that \((\dagger\dagger)\) is the identity morphism follows similarly as above from the commutative diagram

\[
\begin{array}{c}
\text{commutative diagram} \\
H^0(\mathcal{O}_{l, j}, i^*_{l, j} L') \rightarrow H^0(\mathcal{O}_{l, j}, i^*_{l, j} L) \\
\downarrow \quad \downarrow \\
H^0(\mathcal{O}_{l, j}, i^*_{l, j} L'') \rightarrow H^0(\mathcal{O}_{l, j}, i^*_{l, j} L'') \\
\end{array}
\]

where \(r, s \in [0, n]\) with \(I \subseteq [s, n - 1]\) and \(J \subseteq [r, n - 1]\), and \(m''_i := \sum_{j=1}^n a_j, \quad l''_i := -\sum_{j=1}^{n-i} a_j\) and where \(L'' := \bigotimes_{i=0}^{n-1} (M''_i \otimes L''_i) \otimes f^*(\det E) \otimes f^*(\det F)^d\).

6. Non-ampleness

Since large parts of our proof of Theorem 4.3 in [CP], it is natural to ask whether also the analogue of the ampleness result stated in Proposition 8.4 in [CP] holds. We will see below that the answer is negative.

Adopting the notation of [CP] let \(\bar{X}\) be the complete symmetric variety associated to the data \((G, \sigma)\), where \(G\) is a semi-simple simply connected algebraic group over the complex numbers and \(\sigma\) is a nontrivial involution on \(G\). Let \(S_1, \ldots, S_{\ell}\) be the closures of the 1-codimensional orbits of the natural action of \(G\) on \(\bar{X}\). By [CP], 8.1 the unique closed orbit \(Y = \cap_{i=1}^{\ell} S_i\) is isomorphic to \(G/P\) for some parabolic \(P \subset G\) and the restriction of line bundles defines an injective homomorphism \(i^* : \text{Pic}(\bar{X}) \rightarrow \text{Pic}(Y)\). Proposition 8.4 in [CP] can be formulated as follows:
Let $L$ be a line bundle on $\text{Pic}(\overline{X})$ such that $H^0(Y, i^*L) \neq 0$ and let $\omega_{\overline{X}}$ denote the dualizing line bundle on $\overline{X}$. Then the line bundle $\omega_{\overline{X}}^{-1}(-S_1 - \cdots - S_t) \otimes L$ is ample.

The following result shows that the analogue of this Proposition in our context is false.

**Proposition 6.1.** Let $\omega_{KGL} := \text{det} \Omega^1_{KGL(E, F)}$ denote the dualising line bundle on $KGL(E, F)$. Let $a_1 \leq \cdots \leq a_n$, $a_i \in \mathbb{Z}$ and let $L = \bigotimes_{i=0}^{n-1} (M_i^{m_i} \otimes L_i^l)$, where $m_i := \sum_{j=i+1}^{n} a_j$ and $l_i := -\sum_{j=1}^{n-i} a_j$. Then $H^0(\mathcal{O}_{r,s}, i_{r,s}^* L)$ is nonzero for every $r, s \in [0, n]$ with $r + s = n$, but neither the line bundle $\omega_{KGL}^{-1}(-\sum_{i=0}^{n-1}(Z_i + Y_i)) \otimes L$ nor the line bundle $L$ itself is ample.

**Proof.** Using the inductive blowing up procedure which defines $KGL(E, F)$ it is easy to see that $\omega_{KGL} \cong \bigotimes_{i=1}^{n-1} (M_i \otimes L_i)^{(i-n)-1}$.

Let $e_1, \ldots, e_n$ be the canonical basis of $N := \mathbb{Z}^n$. Let $\Delta$ be the smooth complete fan in $N \mathbb{Q} = \mathbb{Q}^n$ whose one-dimensional cones are generated by the vectors $\pm \sum_{i \in I} e_i$, where $I$ runs through the nonempty subsets of $[1, n]$ and whose $n$-dimensional cones are the sets $\sigma(\alpha, \ell) := \{ x \in \mathbb{Q}^n \mid x_{\alpha(1)} \leq \cdots \leq x_{\alpha(\ell)} \leq 0 \leq x_{\alpha(\ell+1)} \leq \cdots \leq x_{\alpha(n)} \}$ where $\alpha$ runs through the set of permutations of $[1, n]$ and $\ell$ runs through the set $[0, n]$. Let $KT$ be the smooth complete torus embedding associated to $\Delta$.

Let $\tilde{T}$ be the torus embedding defined in [K1] p 563, and let $Z_{i, \tilde{T}}, Y_{i, \tilde{T}}$ be the divisors on $\tilde{T}$ defined there. The variety $\tilde{T}$ can be identified with an open subscheme of $KT$ and $KT$ can be identified with the closure in $KGL(E, F)$ of a maximal torus in $\text{GL}_n \cong \text{Isom}(E, F)$ such that the restriction of the line bundles $M_i$ and $L_i$ to $\tilde{T}$ are $\mathcal{O}_{\tilde{T}}(Z_{i, \tilde{T}})$ and $\mathcal{O}_{\tilde{T}}(Y_{i, \tilde{T}})$ respectively.

From [3.8] and [3.9] it is immediate that for any $r, s \in [0, n]$ with $r + s = n$ the restriction of $L$ to $\mathcal{O}_{r, s}$ has non-vanishing global sections. On the other hand, with the help of criterion [C], 3.1 it is easy to see that neither the restriction of $L$ nor that of $\omega_{KGL}^{-1}(-\sum_{i=0}^{n-1}(Z_i + Y_i)) \otimes L$ to the closed subscheme $KT$ are ample. Therefore the bundles $L$ and $\omega_{KGL}^{-1}(-\sum_{i=0}^{n-1}(Z_i + Y_i)) \otimes L$ cannot be ample. □

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