New realization of cyclotomic $q$-Schur algebras I

Kentaro Wada

Abstract. We introduce a Lie algebra $\mathfrak{g}_Q(m)$ and an associative algebra $\mathcal{U}_q, Q(m)$ associated with the Cartan data of $\mathfrak{gl}_m$ which is separated into $r$ parts with respect to $m = (m_1, \ldots, m_r)$ such that $m_1 + \cdots + m_r = m$. We show that the Lie algebra $\mathfrak{g}_Q(m)$ is a filtered deformation of the current Lie algebra of $\mathfrak{gl}_m$, and we can regard the algebra $\mathcal{U}_q, Q(m)$ as a “$q$-analogue” of $U(\mathfrak{g}_Q(m))$. Then, we realize a cyclotomic $q$-Schur algebra as a quotient algebra of $\mathcal{U}_q, Q(m)$ under a certain mild condition. We also study the representation theory for $\mathfrak{g}_Q(m)$ and $\mathcal{U}_q, Q(m)$, and we apply them to the representations of the cyclotomic $q$-Schur algebras.

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§ 0. Introduction

0.1. Let $\mathcal{H}_{n,r}$ be the Ariki-Koike algebra associated with the complex reflection group of type $G(r, 1, n)$ over a commutative ring $R$ with parameters $q, Q_0, \ldots, Q_{r-1} \in R$, where $q$ is invertible in $R$. Let $\mathcal{S}_{n,r}(m)$ be the cyclotomic $q$-Schur algebra associated with $\mathcal{H}_{n,r}$ introduced in [DJM], where $m = (m_1, \ldots, m_r)$ is an $r$-tuple of positive integers. By the result in [DJM], it is known that $\mathcal{S}_{n,r}(m)$-mod is a highest weight cover of $\mathcal{H}_{n,r}$-mod in the sense of [R] if $R$ is a field and $m$ is enough large.

In [RSVV] and [L] independently, it is proven that $\mathcal{S}_{n,r}(m)$-mod is equivalent to a certain highest weight subcategory of an affine parabolic category $\mathcal{O}$ in a dominant case of an affine general linear Lie algebra as a highest weight cover of $\mathcal{H}_{n,r}$-mod. It is also equivalent to the category $\mathcal{O}$ of rational Cherednik algebra with...
the corresponding parameters. In the argument of [RSVV], the monoidal structure on the affine parabolic category \(O\) (more precisely, the structure of \(O\) as a bimodule category over the Kazhdan-Lusztig category) has an important role.

In the case where \(r = 1\), it is known that the \(q\)-Schur algebra \(\mathcal{S}_{n,r}(m)\) is a quotient of the quantum group \(U_q(\mathfrak{gl}_m)\) associated with the general linear Lie algebra \(\mathfrak{gl}_m\), and \(\bigoplus_{n \geq 0} \mathcal{S}_{n,1}(m)\)-mod is equivalent to the category \(\mathcal{C}^\geq_{U_q(\mathfrak{gl}_m)}\) consisting of finite dimensional polynomial representations of \(U_q(\mathfrak{gl}_m)\) ([BLM], [D] and [J]). The category \(\mathcal{C}^\geq_{U_q(\mathfrak{gl}_m)}\) has a (braided) monoidal structure which comes from the structure of \(U_q(\mathfrak{gl}_m)\) as a Hopf algebra. Then the monoidal structure on \(\mathcal{C}^\geq_{U_q(\mathfrak{gl}_m)}\) is compatible with the monoidal structure on the Kazhdan-Lusztig category by [KL]. However, it is not known such structures for cyclotomic \(q\)-Schur algebras in the case where \(r > 1\) although we may expect such structures through the equivalence in [RSVV]. This is a motivation of this paper.

In [W1], we obtained a presentation of cyclotomic \(q\)-Schur algebras by generators and defining relations. The argument in [W1] are based on the existence of the upper (resp. lower) Borel subalgebra of the cyclotomic \(q\)-Schur algebra \(\mathcal{S}_{n,r}(m)\) which is introduced in [DR]. In [DR], it is proven that the upper (resp. lower) Borel subalgebra of \(\mathcal{S}_{n,r}(m)\) is isomorphic to the upper (resp. lower) Borel subalgebra of \(\mathcal{S}_{n,1}(m)\) (i.e. the case where \(r = 1\)) which is a quotient of the upper (resp. lower) Borel subalgebra of the quantum group \(U_q(\mathfrak{gl}_m)\) (\(m := \sum_{k=1}^r m_k\)) if \(m\) is enough large. The presentation of \(\mathcal{S}_{n,r}(m)\) in [W1] is applied to the representation theory of cyclotomic \(q\)-Schur algebras in [W2] and [W3]. However, this presentation is not so useful in general since, in the presentation, we need some non-commutative polynomials which are computable, but we can not describe them explicitly (see [W1, Lemma 7.2]). Hence, we hope more useful realization of cyclotomic \(q\)-Schur algebras like as the fact that the \(q\)-Schur algebra \(\mathcal{S}_{n,1}(m)\) is a quotient of the quantum group \(U_q(\mathfrak{gl}_m)\) in the case where \(r = 1\). In this paper, by extending the argument in [W1], we give a possibility of such realization of cyclotomic \(q\)-Schur algebras.

0.2. Let \(Q = (Q_1, \ldots, Q_{r-1})\) be an \(r-1\) tuple of indeterminate elements over \(\mathbb{Z}\), and \(Q(\mathbb{Q})\) be a field of rational functions with variables \(Q\). In §2, we introduce a Lie algebra \(\mathfrak{g}_Q(m)\) with parameters \(Q\) associated with the Cartan data of \(\mathfrak{gl}_m\) \((m = \sum_{k=1}^r m_k)\) which is separated into \(r\) parts with respect to \(m\) (see the paragraph 1.3). Then, in Proposition 2.13, we prove that \(\mathfrak{g}_Q(m)\) is a filtered deformation of the current Lie algebra \(\mathfrak{gl}_m[x] = Q(\mathbb{Q})[x] \otimes \mathfrak{gl}_m\) of the general linear Lie algebra \(\mathfrak{gl}_m\).

In Corollary 2.8, we see that \(\mathfrak{g}_Q(m)\) has a triangular decomposition
\[
\mathfrak{g}_Q(m) = n^- \oplus n^0 \oplus n^+.
\]

Then we can develop the weight theory to study representations of \(\mathfrak{g}_Q(m)\) in the usual manner (see §3). Let \(\mathcal{C}_Q(m)\) be the category of finite dimensional \(\mathfrak{g}_Q(m)\)-modules which have the weight space decompositions, and all eigenvalues of the action of \(n^0\) belong to \(Q(\mathbb{Q})\). Then we see that a simple \(\mathfrak{g}_Q(m)\)-module in \(\mathcal{C}_Q(m)\) is a highest weight module.

There exists a surjective homomorphism of Lie algebras \(\mathfrak{g}_Q(m) \rightarrow \mathfrak{gl}_m\) (see (2.16.1)) which can be regarded as a special case of evaluation homomorphisms (see
Remark 2.17. Let $\mathcal{C}_{\mathfrak{gl}_m}$ be the category of finite dimensional $\mathfrak{gl}_m$-modules which have the weight space decompositions. Then $\mathcal{C}_{\mathfrak{gl}_m}$ is a full subcategory of $\mathcal{C}_Q(\mathfrak{m})$ through the above surjection (see Proposition 3.7).

Let $\tilde{Q} = (Q_0, Q_1, \ldots, Q_{r-1})$ be an $r$ tuple of indeterminate elements over $\mathbb{Z}$, and $Q(\tilde{Q})$ be a field of rational functions with variables $\tilde{Q}$. Put $\mathfrak{g}_Q(\mathfrak{m}) = Q(\tilde{Q}) \otimes_Q (\mathfrak{g}_Q(\mathfrak{m}))$, and define the category $\mathcal{C}_Q(\mathfrak{m})$ in a similar way. Let $\mathcal{S}_{n,r}^1(\mathfrak{m})$ be the cyclotomic $q$-Schur algebra over $Q(\tilde{Q})$ with parameters $q = 1$ and $\tilde{Q}$. In Theorem 8.4, we prove that there exists a homomorphism of algebras

$$\Psi_1 : U(\mathfrak{g}_Q(\mathfrak{m})) \to \mathcal{S}_{n,r}^1(\mathfrak{m}),$$

where $U(\mathfrak{g}_Q(\mathfrak{m}))$ is the universal enveloping algebra of $\mathfrak{g}_Q(\mathfrak{m})$. Assume that $m_k \geq n$ for all $k = 1, 2, \ldots, r - 1$, then $\Psi_1$ is surjective. Then $\mathcal{S}_{n,r}^1(\mathfrak{m})$-mod is a full subcategory of $\mathcal{C}_Q(\mathfrak{m})$ through the surjection $\Psi_1$ (see Theorem 8.4 (ii)). We expect that the surjectivity of $\Psi_1$ also holds without the condition for $\mathfrak{m}$. (We need the condition for $\mathfrak{m}$ by a technical reason (see Remark 8.2.).)

It is known that $\mathcal{S}_{n,r}^1(\mathfrak{m})$ is semi-simple, and the set of Weyl (cell) modules $\{\Delta(\lambda) \mid \lambda \in \tilde{A}_{n,r}^+(\mathfrak{m})\}$ gives a complete set of isomorphism classes of simple $\mathcal{S}_{n,r}^1(\mathfrak{m})$-modules (see §6 and [DJM] for definitions). The characters of the Weyl modules, denoted by $\text{ch} \Delta(\lambda)$ ($\lambda \in \tilde{A}_{n,r}^+(\mathfrak{m})$), are studied in [W2]. We see that $\text{ch} \Delta(\lambda)$ ($\lambda \in \tilde{A}_{n,r}^+(\mathfrak{m})$) is a symmetric polynomial with variables $x_m$ with respect to $\mathfrak{m}$. Put $\tilde{A}_{\geq 0}(\mathfrak{m}) = \cup_{n \geq 0} \tilde{A}_{n,r}^+(\mathfrak{m})$. Then, for $\lambda, \mu \in \tilde{A}_{\geq 0}(\mathfrak{m})$, it was conjectured that

$$(0.2.1) \quad \text{ch} \Delta(\lambda) \text{ch} \Delta(\mu) = \sum_{\nu \in \tilde{A}_{\geq 0}^+(\mathfrak{m})} \text{LR}^\nu_{\lambda \mu} \text{ch} \Delta(\nu)$$

in [W2], where $\text{LR}^\nu_{\lambda \mu}$ is the product of Littlewood-Richardson coefficients with respect to $\lambda, \mu$ and $\nu$ (see §9 for details). We prove this conjecture in Proposition 9.4. We remark that the characters of Weyl modules of a cyclotomic $q$-Schur algebra do not depend on the choice of a base field and parameters.

By using the usual coproduct of the universal enveloping algebra $U(\mathfrak{g}_Q(\mathfrak{m}))$ of $\mathfrak{g}_Q(\mathfrak{m})$, we can consider the tensor product $M \otimes N$ in $U(\mathfrak{g}_Q(\mathfrak{m}))$-mod for $M, N \in U(\mathfrak{g}_Q(\mathfrak{m}))$-mod. We regard $\mathcal{S}_{n,r}^1(\mathfrak{m})$-modules $(n \geq 0)$ as a $U(\mathfrak{g}_Q(\mathfrak{m}))$-modules through the homomorphism $\Psi_1$. Take $n, n_1, n_2 \in \mathbb{Z}_{>0}$ such that $n = n_1 + n_2$. Then, in Proposition 10.1, we prove that, for $\lambda \in \tilde{A}_{n_1,r}^+(\mathfrak{m})$ and $\mu \in \tilde{A}_{n_2,r}^+(\mathfrak{m})$,

$$(0.2.2) \quad \Delta(\lambda) \otimes \Delta(\mu) = \bigoplus_{\nu \in \tilde{A}_{n_1,r}^+(\mathfrak{m})} \text{LR}^\nu_{\lambda \mu} \Delta(\nu)$$

as $U(\mathfrak{g}_Q(\mathfrak{m}))$-modules if $m_k \geq n$ for all $k = 1, 2, \ldots, r - 1$, where $\text{LR}^\nu_{\lambda \mu} \Delta(\nu)$ means the direct sum of $\text{LR}^\nu_{\lambda \mu}$ copies of $\Delta(\nu)$. In particular, we see that $\Delta(\lambda) \otimes \Delta(\nu) \in \mathcal{S}_{n,r}^1(\mathfrak{m})$-mod. The decomposition (0.2.2) gives an interpretation of the formula
in the category $\mathcal{C}_q(m)$. We expect that (0.2.2) also holds without the condition for $m$. (Note that we prove the formula (0.2.1) without the condition for $m$ in Proposition 9.4.)

0.3. Put $A = \mathbb{Z}[q, q^{-1}, Q_1, \ldots, Q_{r-1}]$, where $q, Q_1, \ldots, Q_{r-1}$ are indeterminate elements over $\mathbb{Z}$, and let $K = \mathbb{Q}(q, Q_1, \ldots, Q_{r-1})$ be the quotient field of $A$. In §4, we introduce an associative algebra $U_q,Q(m)$ with parameters $q$ and $Q$ associated with the Cartan data of $\mathfrak{gl}_m$ which is separated into $r$ parts with respect to $m$.

Let $U_{\hat{A},q,Q}(m)$ be the $A$-subalgebra of $U_q,Q(m)$ generated by defining generators of $U_q,Q(m)$ (see the paragraph 4.11). We regard $\mathbb{Q}(Q)$ as an $A$-module through the ring homomorphism $A \rightarrow \mathbb{Q}(Q)$ by sending $q$ to 1, and we consider the specialization $\mathbb{Q}(Q) \otimes_A U_{\hat{A},q,Q}(m)$ using this ring homomorphism. Then we have a surjective homomorphism of algebras

\[(0.3.1) \quad U(\mathfrak{g}_Q(m)) \rightarrow \mathbb{Q}(Q) \otimes_A U_{\hat{A},q,Q}(m)/\mathfrak{j},\]

where $\mathfrak{j}$ is a certain ideal of $\mathbb{Q}(Q) \otimes_A U_{\hat{A},q,Q}(m)$ (see (4.11.2)). We conjecture that the surjection (0.3.1) is isomorphic. Then we can regard $U_q,Q(m)$ as a “$q$-analogue” of $U(\mathfrak{g}_Q(m))$. Dividing by the ideal $\mathfrak{j}$ in (0.3.1) means that the Cartan subalgebra $U(n^0)$ of $U(\mathfrak{g}_Q(m))$ deforms to several directions in $U_q,Q(m)$ (see the paragraph 4.11 and Remark 4.12).

We see that $U_q,Q(m)$ has a triangular decomposition

\[(0.3.2) \quad U_q,Q(m) = U^- U^0 U^+\]

in a weak sense (see (4.6.1)). We conjecture that the multiplication map $U^- \otimes_K U^0 \otimes_K U^+ \rightarrow U_q,Q(m)$ gives an isomorphism as vector spaces. More precisely, we expect the existence of a PBW type basis of $U_q,Q(m)$ which is compatible with a PBW basis of $U(\mathfrak{g}_Q(m))$ through the homomorphism (0.3.1).

Anyway, thanks to the triangular decomposition (0.3.2), we can develop the weight theory to study $U_q,Q(m)$-modules in the usual manner (see §5). Let $\mathcal{C}_q,Q(m)$ be the category of finite dimensional $U_q,Q(m)$-modules which have the weight space decompositions, and all eigenvalues of the action of $U^0$ belong to $K$. Then we see that a simple $U_q,Q(m)$-module in $\mathcal{C}_q,Q(m)$ is a highest weight module.

There exists a surjective homomorphism of algebras $U_q,Q(m) \rightarrow U_q(\mathfrak{gl}_m)$ (see (4.9.1)) which can be regarded as a special case of evaluation homomorphisms (see Remark 4.10). Let $\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ be the category of finite dimensional $U_q(\mathfrak{gl}_m)$-modules which have the weight space decompositions. Then $\mathcal{C}_{U_q(\mathfrak{gl}_m)}$ is a full subcategory of $\mathcal{C}_q,Q(m)$ through the above surjection (see Proposition 5.6).

Put $\tilde{K} = K(Q_0)$ and $\tilde{A} = A[Q_0]$. We also put $U_q,Q(m) = \tilde{K} \otimes_K U_q,Q(m)$. Let $U_{\hat{A},q,Q}(m)$ be the $A$-form of $U_q,Q(m)$ taking divided powers (see the paragraph 4.13), and put $U_{\hat{A},q,Q}(m) = \tilde{A} \otimes_K U_{\hat{A},q,Q}(m)$. Let $\mathcal{J}_{n,r}(m)$ (resp. $\mathcal{J}_{n,r}^\wedge(m)$) be the cyclotomic $q$-Schur algebra over $\tilde{K}$ (resp. over $\tilde{A}$) with parameters $q$ and $\tilde{Q}$. In
Theorem 8.1, we prove that there exists a homomorphism of algebras
\[ \Psi : \mathcal{U}_q \mathcal{Q}(\mathbf{m}) \to \mathcal{S}_{r}^{\tilde{K}}(\mathbf{m}). \]

By the restriction of \( \Psi \) to \( \mathcal{U}_{\tilde{K}} \mathcal{Q}(\mathbf{m}) \), we have the homomorphism \( \Psi_{\tilde{K}} : \mathcal{U}_{\tilde{K}} \mathcal{Q}(\mathbf{m}) \to \mathcal{S}_{r}^{\tilde{K}}(\mathbf{m}) \). Then we can specialize \( \Psi_{\tilde{K}} \) to any base ring and parameters. If \( m_k \geq n \) for all \( k = 1, 2, \ldots, r - 1 \), then \( \Psi \) (resp. \( \Psi_{\tilde{K}} \)) is surjective (see also Remark 8.2 for surjectivity of \( \Psi \)). In Theorem 8.3, we prove that \( \mathcal{S}_{r}^{\tilde{K}}(\mathbf{m}) \)-mod is a full subcategory of \( \mathcal{C}_q \mathcal{Q}(\mathbf{m}) \) through the surjection \( \Psi \) if \( \mathbf{m} \) is enough large.

We conjecture that \( \mathcal{U}_q \mathcal{Q}(\mathbf{m}) \) has a structure as a Hopf algebra, and that the decomposition (0.2.2) also holds for Weyl modules of \( \mathcal{S}_{r}^{\tilde{K}}(\mathbf{m}) \) \( (n \geq 0) \) through the homomorphism \( \Psi \) and the Hopf algebra structure of \( \mathcal{U}_q \mathcal{Q}(\mathbf{m}) \). (Note that the formula (0.2.1) holds for \( \mathcal{S}_{r}^{\tilde{K}}(\mathbf{m}) \) \( (n \geq 0) \).

It is also interesting problem to obtain a monoidal structure for \( \mathcal{U}_q \mathcal{Q}(\mathbf{m}) \) (resp. \( \mathcal{U}_{\tilde{K}} \mathcal{Q}(\mathbf{m}) \) and its specialization) which should be related to the monoidal structure on the affine parabolic category \( \mathcal{O} \).

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§ 1. Notation

1.1. For a condition \( X \), put \( \delta_{(X)} = \begin{cases} 1 & \text{if } X \text{ is true,} \\ 0 & \text{if } X \text{ is false.} \end{cases} \)

We also put \( \delta_{i,j} = \delta_{(i=j)} \) for simplicity.

1.2. \( q \)-integers. Let \( \mathbb{Q}(q) \) be the field of rational functions over \( \mathbb{Q} \) with an indeterminate variable \( q \). For \( d \in \mathbb{Z} \), put \( [d] = (q^d - q^{-d})/(q - q^{-1}) \in \mathbb{Q}(q) \). For \( d \in \mathbb{Z}_{>0} \), put \( [d]! = [d][d-1] \cdots [1] \), and we put \( [0]! = 1 \). For \( d \in \mathbb{Z} \) and \( c \in \mathbb{Z}_{>0} \), put
\[ \left[ \frac{d}{c} \right] = \frac{[d][d-1] \cdots [d-c+1]}{[c][c-1] \cdots [1]} , \]
and put \( \left[ \frac{d}{0} \right] = 1 \).

It is well-known that all \( [d] \), \( [d]! \), and \( \left[ \frac{d}{c} \right] \) belong to \( \mathbb{Z}[q, q^{-1}] \). Thus we can specialize these elements to any ring \( R \) and \( q \in R \) such that \( q \) is invertible in \( R \), and we denote them by same symbols.

1.3. Cartan data. Let \( \mathbf{m} = (m_1, \ldots, m_r) \) be an \( r \)-tuple of positive integers. Put \( m = \sum_{k=1}^r m_k \). Let \( P = \bigoplus_{i=1}^m \mathbb{Z} \varepsilon_i \) be the weight lattice of \( \mathfrak{gl}_m \), and let \( P^\vee = \bigoplus_{i=1}^m \mathbb{Z} h_i \) be its dual with the natural pairing \( \langle , \rangle : P \times P^\vee \to \mathbb{Z} \) such that \( \langle \varepsilon_i, h_j \rangle = \delta_{ij} \). Put \( P_{>0} = \bigoplus_{i=1}^m \mathbb{Z}_{>0} \varepsilon_i \).

Set \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \) for \( i = 1, \ldots, m-1 \), then \( \Pi = \{\alpha_i \mid 1 \leq i \leq m-1\} \) is the set of simple roots, and \( Q = \bigoplus_{i=1}^m \mathbb{Z} \alpha_i \) is the root lattice of \( \mathfrak{gl}_m \). Put \( Q^+ = \bigoplus_{i=1}^{m-1} \mathbb{Z}_{>0} \alpha_i \).

Set \( \alpha_i^\vee = h_i - h_{i+1} \) for \( i = 1, \ldots, m-1 \), then \( \Pi^\vee = \{\alpha_i^\vee \mid 1 \leq i \leq m-1\} \) is the set of simple coroots.
We define a partial order $\geq$ on $P$, so called dominance order, by $\lambda \geq \mu$ if $\lambda - \mu \in Q^+$.

Put $\Gamma(\mathfrak{m}) = \{(i,k) | 1 \leq i \leq m_k, 1 \leq k \leq r\}$, and $\Gamma'(\mathfrak{m}) = \Gamma(\mathfrak{m}) \setminus \{(m_r,r)\}$. We identify the set $\Gamma(\mathfrak{m})$ with the set $\{1,2,\ldots,m\}$ by the bijection

$$(1.3.1) \quad \gamma : \Gamma(\mathfrak{m}) \to \{1,2,\ldots,m\} \text{ such that } (i,k) \mapsto \sum_{j=1}^{k-1} m_j + i.$$

Then, we can identify the set $\Gamma'(\mathfrak{m})$ with the set $\{1,2,\ldots,m-1\}$. Under the identification (1.3.1), for $(i,k), (j,l) \in \Gamma(\mathfrak{m})$, we define

$$(i,k) > (j,l) \text{ if } \gamma((i,k)) > \gamma((j,l)), \text{ and } (i,k) \pm (j,l) = \gamma((i,k)) \pm \gamma((j,l)).$$

We also have $(m_k+1,k) = (1,k+1)$ for $k = 1,\ldots,r-1$ (resp. $(1,1,k) = (m_{k-1},k-1)$ for $k = 2,\ldots,r$).

We may write

$$P = \bigoplus_{(i,k) \in \Gamma(\mathfrak{m})} \mathbb{Z} \varepsilon_{(i,k)}, \quad P^\vee = \bigoplus_{(i,k) \in \Gamma(\mathfrak{m})} \mathbb{Z} \omega_{(i,k)}, \quad Q = \bigoplus_{(i,k) \in \Gamma'(\mathfrak{m})} \mathbb{Z} \alpha_{(i,k)}.$$

For $(i,k) \in \Gamma'(\mathfrak{m})$, $(j,l) \in \Gamma(\mathfrak{m})$, put $a_{(i,k),(j,l)} = \langle \alpha_{(i,k)}, h_{(j,l)} \rangle$. Then, we have

$$a_{(i,k),(j,l)} = \begin{cases} 1 & \text{if } (j,l) = (i,k), \\ -1 & \text{if } (j,l) = (i+1,k), \\ 0 & \text{otherwise}. \end{cases}$$

Put

$$P^+ = \{ \lambda \in P | \langle \lambda, \alpha_{(i,k)}^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } (i,k) \in \Gamma'(\mathfrak{m}) \} \text{ and }$$

$$P^+_m = \{ \lambda \in P | \langle \lambda, \alpha_{(i,k)}^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } (i,k) \in \Gamma(\mathfrak{m}) \setminus \{(m_k,k) | 1 \leq k \leq r\} \}.$$

Then $P^+$ is the set of dominant integral weights for $\mathfrak{gl}_m$, and $P^+_m$ is the set of dominant integral weights for Levi subalgebra $\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ of $\mathfrak{gl}_m$ with respect to $\mathfrak{m} = (m_1,\ldots,m_r)$.

§ 2. Lie algebra $\mathfrak{gl}_Q(\mathfrak{m})$

In this section, we introduce a Lie algebra $\mathfrak{gl}_Q(\mathfrak{m})$ with $r-1$ parameters $\mathbf{Q} = (Q_1,\ldots,Q_{r-1})$ associated with the Cartan data in the paragraph 1.3. Then we study some basic structures of $\mathfrak{gl}_Q(\mathfrak{m})$. In particular, we prove that $\mathfrak{gl}_Q(\mathfrak{m})$ is a filtered deformation of the current Lie algebra $\mathfrak{gl}_m[\varepsilon]$ of the general linear Lie algebra $\mathfrak{gl}_m$.

2.1. Let $\mathbf{Q} = (Q_1,\ldots,Q_{r-1})$ be an $r-1$-tuple of indeterminate elements over $\mathbb{Z}$. Let $\mathbb{Z}[\mathbf{Q}] = \mathbb{Z}[Q_1,\ldots,Q_{r-1}]$ be the polynomial ring with variables $Q_1,\ldots,Q_{r-1}$, and $\mathbb{Q}(\mathbf{Q}) = \mathbb{Q}(Q_1,\ldots,Q_{r-1})$ be the quotient field of $\mathbb{Z}[\mathbf{Q}]$. 
Definition 2.2. We define the Lie algebra $\mathfrak{g} = \mathfrak{g}_Q(m)$ over $Q(Q)$ by the following generators and defining relations:

Generators: $\mathcal{X}_{(i,k),t}^\pm, \mathcal{I}_{(j,l),t}$ $((i, k) \in \Gamma'(m), (j, l) \in \Gamma(m), t \geq 0)$.

Relations:

(L1) $[\mathcal{I}_{(i,k),s}, \mathcal{I}_{(j,l),t}] = 0$,

(L2) $[\mathcal{I}_{(j,l),s}, \mathcal{X}_{(i,k),t}^\pm] = \pm a_{(j,l)_{(i,k)}} \mathcal{X}_{(i,k),s+t}^\pm$,

(L3) $[\mathcal{X}_{(i,k),t}^+, \mathcal{X}_{(j,l),s}^-] = \delta_{(i,k),(j,l)} \begin{cases} \mathcal{J}_{(i,k),s+t} & \text{if } i \neq m_k, \\ -Q_k \mathcal{J}_{(m_k,k),s+t} + \mathcal{J}_{(m_k,k),s+t+1} & \text{if } i = m_k, \end{cases}$

(L4) $[\mathcal{X}_{(i,k),t}^+, \mathcal{X}_{(j,l),s}^+] = 0$ if $(j, l) \neq (i \pm 1, k)$,

(L5) $[\mathcal{X}_{(i,k),t+1}^+, \mathcal{X}_{(i+1,k),s}^+] = [\mathcal{X}_{(i,k),t}^+, \mathcal{X}_{(i+1,k),s+1}^+]$,

$[\mathcal{X}_{(i,k),t+1}^-, \mathcal{X}_{(i+1,k),s}^-] = [\mathcal{X}_{(i,k),t}^-, \mathcal{X}_{(i+1,k),s+1}^-]$,

(L6) $[\mathcal{X}_{(i,k),s}^+, \mathcal{X}_{(i+1,k),u}^+] = [\mathcal{X}_{(i,k),s}^-, \mathcal{X}_{(i+1,k),u}^-] = 0$,

where we put $\mathcal{J}_{(i,k),t} = \mathcal{I}_{(i,k),t} - \mathcal{I}_{(i+1,k),t}$.

2.3. For $\tau \in Q(Q)$, let $V_\tau = \bigoplus_{(j,l) \in \Gamma'(m)} Q(Q)v_{(j,l)}$ be the $Q(Q)$-vector space with a basis $\{v_{(j,l)} \mid (j, l) \in \Gamma(m)\}$. We can define the action of $\mathfrak{g}$ on $V_\tau$ by

$\mathcal{X}_{(i,k),t}^+ \cdot v_{(j,l)} = \begin{cases} \tau^t v_{(i,k)} & \text{if } (j, l) = (i, 1+k) \text{ and } i \neq m_k, \\ (-Q_k + \tau)\tau^t v_{(m_k,k)} & \text{if } (j, l) = (1, k+1) \text{ and } i = m_k, \\ 0 & \text{otherwise}, \end{cases}$

$\mathcal{X}_{(i,k),t}^- \cdot v_{(j,l)} = \begin{cases} \tau^t v_{(i+1,k)} & \text{if } (j, l) = (i, k), \\ 0 & \text{otherwise}, \end{cases}$

$\mathcal{I}_{(i,k),t} \cdot v_{(j,l)} = \begin{cases} \tau^t v_{(j,l)} & \text{if } (j, l) = (i, k), \\ 0 & \text{otherwise}. \end{cases}$

We can check the well-definedness of the above action by direct calculations.

2.4. For $(i, k), (j, l) \in \Gamma(m)$ and $t \geq 0$, we define the element $E_{(i,k),(j,l)}^t \in \mathfrak{g}$ by

$E_{(i,k),(j,l)}^t = \begin{cases} \mathcal{I}_{(i,k),t} & \text{if } (j, l) = (i, k), \\ [\mathcal{X}_{(i,k),0}^+, [\mathcal{X}_{(i+1,k),0}^+, \ldots, [\mathcal{X}_{(i+2,1),t}^+, \ldots, [\mathcal{X}_{(i+1,1),t}^+, \ldots, \mathcal{X}_{(j,l),t}^-]]], & \text{if } (j, l) > (i, k), \\ [\mathcal{X}_{(i-1,k),0}^-, [\mathcal{X}_{(i-2,k),0}^-, \ldots, [\mathcal{X}_{(i-2,1),t}^-, \ldots, [\mathcal{X}_{(i-1,1),t}^-, \ldots, \mathcal{X}_{(j,l),t}^-]]], & \text{if } (j, l) < (i, k), \end{cases}$

in particular, we have $E_{(i,k),(i+1,k)}^t = \mathcal{X}_{(i,k),t}^+$ and $E_{(i+1,k),(i,k)}^t = \mathcal{X}_{(i,k),t}^-$. If $(j, l) > (i, k)$, we have

$E_{(i,k),(j,l)}^t = [\mathcal{X}_{(i,k),0}^+; E_{(i+1,k),(j,l)}^t]$

$= [E_{(i,k),(j-1,1),t}; \mathcal{X}_{(j-1,1),0}^-]$. 

New realization of cyclotomic $q$-Schur algebras I
If \((j, l) < (i, k)\), we have

\[
\mathcal{E}_{(i, k), (j, l)}^t = \begin{cases} 
\mathcal{X}_{(i-1, k), 0}^t \mathcal{E}_{(i-1, k), (j, l)}^t & \text{if } (a, c) = (i-1, k), \\
\mathcal{E}_{(i, k), (j+1, l)}^t & \text{if } (a, c) = (j, l), \\
0 & \text{otherwise,}
\end{cases}
\]

\[
\mathcal{X}_{(i, k), (j, l)}^{-} = \begin{cases} 
\mathcal{E}_{(i, k), (j+1, l)}^t & \text{if } (a, c) = (i, k), \\
-\mathcal{E}_{(i, k), (j, l)}^t & \text{if } (a, c) = (j, l), \\
0 & \text{otherwise,}
\end{cases}
\]

Lemma 2.5.

(i) For \((i, k), (j, l) \in \Gamma(m)\) such that \((j, l) > (i, k)\), we have

\[
\begin{align*}
\mathcal{X}^+_{(a, c), s} \mathcal{E}^t_{(i, k), (j, l)} &= \begin{cases} 
\mathcal{E}^t_{(i-1, k), (j, l)} & \text{if } (a, c) = (i - 1, k), \\
-\mathcal{E}^t_{(i, k), (j, l)} & \text{if } (a, c) = (j, l), \\
0 & \text{otherwise,}
\end{cases} \\
\mathcal{I}^t_{(a, c), s} \mathcal{E}^t_{(i, k), (j, l)} &= \begin{cases} 
\mathcal{E}^t_{(i, k), (j+1, l)} & \text{if } (a, c) = (i, k), \\
-\mathcal{E}^t_{(i, k), (j, l)} & \text{if } (a, c) = (j, l), \\
0 & \text{otherwise,}
\end{cases}
\end{align*}
\]

where we put \(\ell = (j, l) - (i, k)\).

(ii) For \((i, k), (j, l) \in \Gamma(m)\) such that \((j, l) < (i, k)\), we have

\[
\begin{align*}
\mathcal{X}^-_{(a, c), s} \mathcal{E}^t_{(i, k), (j, l)} &= \begin{cases} 
\mathcal{E}^t_{(i+1, k), (j, l)} & \text{if } (a, c) = (i, k), \\
-\mathcal{E}^t_{(i, k), (j-1, l)} & \text{if } (a, c) = (j - 1, l), \\
0 & \text{otherwise,}
\end{cases} \\
\mathcal{I}^t_{(a, c), s} \mathcal{E}^t_{(i, k), (j, l)} &= \begin{cases} 
\mathcal{E}^t_{(i, k), (j+1, l)} & \text{if } (a, c) = (i, k), \\
-\mathcal{E}^t_{(i, k), (j, l)} & \text{if } (a, c) = (j, l), \\
0 & \text{otherwise,}
\end{cases}
\end{align*}
\]
Applying the assumption of the induction, we have

\[
\begin{align*}
\mathcal{E}_{(i-1,k),(i-1,l)}^{t+s} - \mathcal{E}_{(i,k),(i,l)}^{t+s} & \quad \text{if } \ell = 1, (a, c) = (i - 1, k) \text{ and } i - 1 \neq m_k, \\
-Q_k(\mathcal{E}_{(m,k),(m,k)}^{t+s} - \mathcal{E}_{(1,k+1),(1,k+1)}^{t+s}) + \mathcal{E}_{(m,k),(m,k)}^{t+s+1} & \quad \text{if } \ell = 1, (a, c) = (i - 1, k) \text{ and } i - 1 = m_k, \\
-Q_k\mathcal{E}_{(1,k),(j,l)}^{t+s} & \quad \text{if } \ell > 1, (a, c) = (i - 1, k) \text{ and } i - 1 \neq m_k, \\
-\mathcal{E}_{(i,k),(j+1,l)}^{t+s} & \quad \text{if } \ell > 1, (a, c) = (i, l) \text{ and } j \neq m_l, \\
Q_k\mathcal{E}_{(i,k),(1,l+1)}^{t+s} - \mathcal{E}_{(i,k),(1,l+1)}^{t+s+1} & \quad \text{if } \ell > 1, (a, c) = (j, l) \text{ and } j = m_l, \\
0 & \quad \text{otherwise},
\end{align*}
\]

where we put \( \ell = (i, k) - (j, l) \).

(iii) For \((i, k) \in \Gamma(m)\), we have

\[
\begin{align*}
& [\mathcal{Z}_{(a,c),s}, \mathcal{E}_{(i,k),(i,l)}^{t}] = 0, \\
& [\mathcal{X}_{(a,c),s}^{+}, \mathcal{E}_{(i,k),(i,l)}^{t}] = -a_{(a,c)(i,k)}\mathcal{E}_{(a,c),(a+1,c)}^{t+s}, \\
& [\mathcal{X}_{(a,c),s}^{-}, \mathcal{E}_{(i,k),(i,l)}^{t}] = a_{(a,c)(i,k)}\mathcal{E}_{(a+1,c),(a,c)}^{t+s}.
\end{align*}
\]

**Proof.** We prove (2.5.1) by the induction on \((j, l) - (i, k)\).

In the case where \((j, l) - (i, k) = 1\), it is follows from the relations (L4) and (L5).

Assume that \((j, l) - (i, k) > 1\). We have

\[
\begin{align*}
& [\mathcal{X}_{(a,c),s}^{+}, \mathcal{E}_{(i,k),(j,l)}^{t}] = [\mathcal{X}_{(a,c),s}^{+}, [\mathcal{X}_{(a,c),s}^{+}, \mathcal{E}_{(i+1,k),(j,l)}^{t}]] \\
& \quad = [\mathcal{X}_{(a,c),s}^{+}, [\mathcal{X}_{(a,c),s}^{+}, \mathcal{E}_{(i+1,k),(j,l)}^{t}]] + [[\mathcal{X}_{(a,c),s}^{+}, \mathcal{X}_{(i,k),0}^{+}], \mathcal{E}_{(i+1,k),(j,l)}^{t}].
\end{align*}
\]

Applying the assumption of the induction, we have

\[
(2.5.4) \quad [\mathcal{X}_{(a,c),s}^{+}, \mathcal{E}_{(i,k),(j,l)}^{t}] = \begin{cases} 
[\mathcal{X}_{(i,k),0}^{+}, \mathcal{E}_{(i,k),(j,l)}^{t+s}] & \text{if } (a, c) = (i, k), \\
-\mathcal{E}_{(i,k),(j+1,l)}^{t+s} & \text{if } (a, c) = (j, l), \\
[\mathcal{X}_{(i+1,k),s}^{+}, \mathcal{X}_{(i,k),0}^{+}, \mathcal{E}_{(i+1,k),(j,l)}^{t}] & \text{if } (a, c) = (i - 1, k), \\
[\mathcal{X}_{(i+1,k),s}^{+}, \mathcal{X}_{(i,k),0}^{+}, \mathcal{E}_{(i+1,k),(j,l)}^{t}] & \text{if } (a, c) = (i + 1, k), \\
0 & \text{otherwise}.
\end{cases}
\]

We also have

\[
\begin{align*}
& [\mathcal{X}_{(a,c),s}^{+}, \mathcal{E}_{(i,k),(j,l)}^{t}] = [\mathcal{X}_{(a,c),s}^{+}, [\mathcal{E}_{(i,k),(j-1,l)}^{t}, \mathcal{X}_{(j-1,l),0}^{+}]] \\
& \quad = [[\mathcal{X}_{(j-1,l),0}^{+}, \mathcal{X}_{(a,c),s}^{+}, \mathcal{E}_{(i,k),(j-1,l)}^{t}] + [\mathcal{X}_{(a,c),s}^{+}, \mathcal{E}_{(i,k),(j-1,l)}^{t}, \mathcal{X}_{(j-1,l),0}^{+}].
\end{align*}
\]
Applying the assumption of the induction, we have

\[
(2.5.5)
\]

\[
[X^+_{(a,c),s}, E_t^{(i,k),(j,l)}] = \begin{cases}
[X^+_{(j-1,l),0}, X^+_{(j,l),s}, E_t^{(i,k),(j-1,l)}] & \text{if } (a,c) = (j,l), \\
[X^+_{(j-1,l),0}, X^+_{(j-2,l),s}, E_t^{(i,k),(j-1,l)}] & \text{if } (a,c) = (j-2,l), \\
[X^+_{(i-1,k),0}, X^+_{(i,l),s}, E_t^{(i,k),(j-1,l)}] & \text{if } (a,c) = (i-1,k), \\
[X^+_{(i,k),0}, X^+_{(j-1,l),1}, E_t^{(i,k),(j-1,l)}] & \text{if } (a,c) = (j-1,l), \\
0 & \text{otherwise}.
\end{cases}
\]

By (2.5.4) and (2.5.5), we have

\[
[X^+_{(a,c),s}, E_t^{(i,k),(j,l)}] = \begin{cases}
E_t^{(i-1,k),(j,l)} & \text{if } (a,c) = (i-1,k), \\
E_t^{(i,k),(j+1,l)} & \text{if } (a,c) = (j,l), \\
X^+_{(i,k),0}, E_t^{(i,k),(i+2,k)} & \text{if } (a,c) = (i,k) = (j-2,l), \\
X^+_{(i,k),0}, X^+_{(i,l),s}, E_t^{(i+1,k),(i+3,k)} & \text{if } (a,c) = (i+1,k) = (j-2,l), \\
X^+_{(i+1,k),0}, E_t^{(i,k),(i+2,k)} & \text{if } (a,c) = (i+1,k) = (j-1,l), \\
0 & \text{otherwise}.
\end{cases}
\]

By the direct calculations using the relations (L4)-(L6), we also have

\[
[X^+_{(a,c),s}, E_t^{(i,k),(i+2,k)}] = [X^+_{(i+1,k),0}, X^+_{(i,l),s}, E_t^{(i+1,k),(i+3,k)}] = [X^+_{(i+1,k),0}, E_t^{(i,k),(i+2,k)}] = 0.
\]

Now we proved (2.5.1).

We prove (2.5.2) by the induction on \((j,l) - (i,k)\). In the case where \((j,l) - (i,k) = 1\), it is just the relation (L2). Assume that \((j,l) - (i,k) > 1\). We have

\[
[I_{(a,c),s}, E_t^{(i,k),(j,l)}] = [I_{(a,c),s}, X^+_{(i,k),0}, E_t^{(i+1,k),(j,l)}] = [X^+_{(i,k),0}, E_t^{(i+1,k),(j,l)}] = [X^+_{(i,k),0}, E_t^{(i,k),(j,l)}] + [I_{(a,c),s}, X^+_{(i,k),0}, E_t^{(i+1,k),(j,l)}].
\]

Applying the assumption of the induction, we have

\[
[I_{(a,c),s}, E_t^{(i,k),(j,l)}] = \begin{cases}
[X^+_{(i,k),0}, E_t^{(i+1,k),(j,l)}] - [X^+_{(i,k),s}, E_t^{(i+1,k),(j,l)}] & \text{if } (a,c) = (i+1,k), \\
-X^+_{(i,k),0}, E_t^{(i+1,k),(j,l)} & \text{if } (a,c) = (j,l), \\
[X^+_{(i,k),s}, E_t^{(i+1,k),(j,l)}] & \text{if } (a,c) = (i,k), \\
0 & \text{otherwise}.
\end{cases}
\]

Thus, we have (2.5.2) by applying (2.5.1).

We prove (2.5.3) by the induction on \(\ell = (j,l) - (i,k)\). In the case where \(\ell = 1,2\), we can show (2.5.3) by direct calculations. Assume that \(\ell > 2\), we have

\[
[X^+_{(a,c),s}, E_t^{(i,k),(j,l)}] = [X^+_{(a,c),s}, X^+_{(i,k),0}, E_t^{(i+1,k),(j,l)}].
\]
Thus, we have (2.5.3) by applying (2.5.1) and (2.5.2).

\[
\sum_{a,c} \text{as a} \{E_{a,c}\}
\]

Proof. where we put

\[
\text{Proposition 2.6.} \quad \{E_{(i,k),(j,l)}^t \mid (i, k), (j, l) \in \Gamma(m), t \geq 0\} \text{ gives a basis of } g = g_\mathbb{Q}(m).
\]

By Lemma 2.5, we see that \(g\) is spanned by \(\{E_{(i,k),(j,l)}^t \mid (i, k), (j, l) \in \Gamma(m), t \geq 0\}\) as a \(\mathbb{Q}(Q)\)-vector space. In fact, we see that it is a basis of \(g\) as follows.

\[
\text{Proposition 2.6.} \quad \{E_{(i,k),(j,l)}^t \mid (i, k), (j, l) \in \Gamma(m), t \geq 0\} \text{ gives a basis of } g = g_\mathbb{Q}(m).
\]

Proof. It is enough to show that \(\{E_{(i,k),(j,l)}^t \mid (i, k), (j, l) \in \Gamma(m), t \geq 0\}\) are linearly independent.

For \(\tau \in \mathbb{Q}(Q)\), let \(V_\tau = \bigoplus_{(j,l) \in \Gamma(m)} \mathbb{Q}(Q)v_{(j,l)}\) be the \(g\)-module given in 2.3. Then, we see that

\[
E_{(i,k),(j,l)}^t \cdot v_{(a,c)} = \delta_{(a,c),(j,l)} \psi_{(i,k),(j,l)}^t \tau^tv_{(i,k)},
\]

where we put

\[
\psi_{(i,k),(j,l)}^t = \left\{ \begin{array}{ll}
\prod_{p=0}^{l-k-1} (-Q_l + \tau) & \text{if } l - k > 0, \\
1 & \text{otherwise}.
\end{array} \right.
\]

Thus, if \(\sum_{(i,k),(j,l) \in \Gamma(m), t \geq 0} r_{(i,k),(j,l)}^t E_{(i,k),(j,l)}^t \in \mathbb{Q}(Q)\), we have

\[
\left( \sum_{(i,k),(j,l) \in \Gamma(m), t \geq 0} r_{(i,k),(j,l)}^t E_{(i,k),(j,l)}^t \right) \cdot v_{(a,c)} = \sum_{(i,k) \in \Gamma(m)} \psi_{(i,k),(j,l)}^t \left( \sum_{t \geq 0} r_{(i,k),(a,c)}^t \tau^t \right) v_{(i,k)} = 0.
\]
Thus, for any \((i, k), (j, l) \in \Gamma(m)\) and any \(\tau \in Q(Q)\), we have

\[
\psi_{(i, k)(j, l)} \left( \sum_{t \geq 0} r_{(i, k)(j, l)}^t \tau^t \right) = 0.
\]

This implies that \(r_{(i, k)(j, l)}^t = 0\) for any \((i, k), (j, l) \in \Gamma(m)\) and any \(t \geq 0\). \(\square\)

2.7. Let \(n^+, n^-\) and \(n^0\) be the Lie subalgebras of \(g\) generated by

\[
\{X^+_{(i, k), t} | (i, k) \in \Gamma'(m), t \geq 0\}, \{X^-_{(i, k), t} | (i, k) \in \Gamma'(m), t \geq 0\} \text{ and } \\
\{I_{(j, l), t} | (j, l) \in \Gamma(m), t \geq 0\}
\]

respectively. Then, we have the following triangular decomposition as a corollary of Proposition 2.6.

**Corollary 2.8.** We have the triangular decomposition

\[
g = n^- \oplus n^0 \oplus n^+ \quad (\text{as vector spaces}).
\]

2.9. A current Lie algebra. Let \(Q[x]\) be the polynomial ring over \(Q\), and let \(gl_m[x] = Q[x] \otimes gl_m\) be the current Lie algebra associated with the general linear Lie algebra \(gl_m\) over \(Q\). Namely, the Lie bracket on \(gl_m[x]\) is defined by

\[
[a \otimes g, b \otimes h] = ab \otimes [g, h] \quad (a, b \in Q[x], g, h \in gl_m).
\]

Let \(E_{i,j} \in gl_m\) (1 \(\leq i, j \leq m\)) be the elementary matrix having 1 at the \((i, j)\)-entry and 0 elsewhere. Put \(e_i = E_{i,i+1}\), \(f_i = E_{i+1,i}\) and \(K_j = E_{j,j}\). Then \(Q[x] \otimes gl_m\) is generated by

\[
x^t \otimes e_i, x^t \otimes f_i, x^t \otimes K_j \quad (1 \leq i \leq m - 1, 1 \leq j \leq m, t \geq 0).
\]

2.10. In the case where \(r = 1 (m = m)\), the Lie algebra \(g(m)\) over \(Q\) is generated by \(X^\pm_{i,t}\) and \(I_{j,t}\) (1 \(\leq i \leq m - 1, 1 \leq j \leq m, t \geq 0\)) with the defining relations (L1)-(L6) (for \((i, 1) \in \Gamma(m)\), we denote \((i, 1)\) by \(i\) simply). In this case, the relation (L3) is just

\[
[X^+_i, X^-_j] = \delta_{i,j}(I_{i,t} - I_{i+1,t}).
\]

Then, we have the following lemma.

**Lemma 2.11.** There exists the isomorphism of Lie algebras

\[
\Phi : g(m) \rightarrow gl_m[x] \quad (X^+_i \mapsto x^t \otimes e_i, X^-_i \mapsto x^t \otimes f_i, I_{j,t} \mapsto x^t \otimes K_j).
\]

In particular, the relations (L1)-(L6) (in the case where \(r = 1\)) give a defining relations of \(gl_m[x]\) through the isomorphism \(\Phi\).
Proof. We can show the well-definedness of the homomorphism $\Phi$ by checking the defining relations of $g(m)$ directly.

For $i, j \in \{1, \ldots, m\}$ and $t \geq 0$, we see that $\Phi(E_{i,j}^t) = x^t \otimes E_{i,j}$. Clearly, $\{x^t \otimes E_{i,j}^t \mid 1 \leq i, j \leq m, t \geq 0\}$ gives a basis of $gl_m[x]$. Thus, Proposition 2.6 implies that $\Phi$ is isomorphic.

2.12. In the case where $r \geq 2$, we can regard $g = g_Q(m)$ as a deformation of the current Lie algebra $Q(Q) \otimes_Q gl_m[x]$ as follows.

For $t \geq 0$, put

$$Y_t = \{X_{(i,k),t}^\pm, I_{(j,l),t} \mid (i,k) \in \Gamma'(m), (j,l) \in \Gamma(m)\}.$$ 

Let $g_t$ be the $Q(Q)$-subspace of $g$ spanned by

$$\{[Y_{t_1}, [Y_{t_2}, \ldots, [Y_{t_{p-1}}, Y_{t_p}] \ldots] \mid Y_{t_b} \in Y_{t_b}, \sum_{b=1}^p t_b \geq t, p \geq 1\}.$$ 

Then, we have the sequence

$$g = g_0 \supset g_1 \supset g_2 \supset \ldots.$$ 

By the defining relations (L1)-(L6), we see that

$$(2.12.1) \quad [g_s, g_t] \subset g_{s+t} \quad (s,t \geq 0).$$

For $t \geq 0$, let $\sigma_t : g_t \rightarrow g_t/g_{t+1}$ be the natural surjection. By (2.12.1), we can define the structure as a Lie algebra on $grg = \bigoplus_{t \geq 0} g_t/g_{t+1}$ by

$$[\sigma_s(g), \sigma_t(h)] = \sigma_{s+t}([g,h]) \quad (g \in g_s, h \in g_t).$$

Then we see that, $grg$ is generated by

$$\sigma_t(X_{(i,k),t}^\pm), \sigma_t(I_{(j,l),t}) \quad ((i,k) \in \Gamma'(m), (j,l) \in \Gamma(m), t \geq 0),$$

and $grg$ has a basis $\{\sigma_t(E_{(i,k),l}^t) \mid (i,k), (j,l) \in \Gamma(m), t \geq 0\}$.

Proposition 2.13. There exists the isomorphism of Lie algebras

$$\Psi : Q(Q) \otimes_Q gl_m[x] \rightarrow grg = \bigoplus_{t \geq 0} g_t/g_{t+1}$$

such that

$$x^t \otimes e_{(i,k)} \mapsto \begin{cases} 
\sigma_t(X_{(i,k),t}^+) & \text{if } i \neq m_k, \\
-Q_k^{-1} \sigma_t(X_{(m_k,k),t}^+) & \text{if } i = m_k,
\end{cases}$$

$$x^t \otimes f_{(i,k)} \mapsto \sigma_t(X_{(i,k),t}^-).$$
$x^t \otimes K_{(j,l)} \mapsto \sigma_t(I_{(j,l),t})$, where we use the identification (1.3.1) for the indices of generators of $\mathfrak{gl}_m[x]$.

**Proof.** We can show the well-definedness of the homomorphism $\Psi$ by checking the defining relations of $\mathfrak{gl}_m[x]$ directly (see Lemma 2.11). We also see that

$$
\Psi(x^t \otimes E_{(i,k),(j,l)}) = \psi_{(i,k)}(j,l) \sigma_t(E_{(i,k),(j,l)}),
$$

where we put

$$
\psi_{(i,k)}(j,l) = \begin{cases} 
\prod_{p=0}^{l-k-1} (-Q_{k+p}^{-1}) & \text{if } l - k > 0, \\
1 & \text{otherwise.}
\end{cases}
$$

Thus, we see that $\Psi$ is isomorphic. □

As a corollary of the above proposition, we have the following isomorphism between $\mathbb{Q}(Q) \otimes_{\mathbb{Q}} \mathfrak{g}(m)$ and $\mathfrak{gr}_Q \mathfrak{g}(m)$.

**Corollary 2.14.** There exists the isomorphism of Lie algebras

$$
\tilde{\Psi} : \mathbb{Q}(Q) \otimes_{\mathbb{Q}} \mathfrak{g}(m) \rightarrow \mathfrak{gr}_Q \mathfrak{g}(m) = \bigoplus_{t \geq 0} \mathfrak{g}_t / \mathfrak{g}_{t+1}
$$

such that

$$
\mathcal{X}_{(i,k),t} \mapsto \begin{cases} 
\sigma_t(X_{(i,k),t}^+) & \text{if } i \neq m_k, \\
-Q_{k}^{-1} \sigma_t(X_{(m_k,k),t}^+) & \text{if } i = m_k,
\end{cases} \quad X_{(i,k),t}^- \mapsto \sigma_t(X_{(i,k),t}^-), \quad I_{(j,l),t} \mapsto \sigma_t(I_{(j,l),t}),
$$

where we use the identification (1.3.1) for the indices of generators of $\mathfrak{g}(m)$.

**2.15.** We also have some relations between the Lie algebra $\mathfrak{g}_Q \mathfrak{g}(m)$ and the general linear Lie algebra $\mathfrak{gl}_m$ as follows. Let $\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ be a Levi subalgebra of $\mathfrak{gl}_m$ associated with $m = (m_1, \ldots, m_r)$. Then generators of $\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}$ are given by $e_{(i,k)}$, $f_{(i,k)}$ ($1 \leq i \leq m_k - 1$, $1 \leq k \leq r$) and $K_{(j,l)} ((j,l) \in \Gamma(m))$, where we use the identification (1.3.1) for indices.

**Proposition 2.16.**

(i) There exists a surjective homomorphism of Lie algebras

$$
g : \mathfrak{g}_Q \mathfrak{g}(m) \rightarrow \mathfrak{gl}_m
$$

such that

$$
g(X_{(i,k),0}^+) = \begin{cases} 
e_{(i,k)} & \text{if } i \neq m_k, \\
-Q_k \epsilon_{(m_k,k)} & \text{if } i = m_k,
\end{cases} \quad g(X_{(i,k),0}^-) = f_{(i,k)},
$$

where we use the identification (1.3.1) for the indices.
$$g(\mathcal{I}_{(j,t),0}) = K_{(j,t)}$$ and $$g(\mathcal{X}^+_{(i,k),t}) = g(\mathcal{I}_{(j,t),t}) = 0$$ for $$t \geq 1$$.

(ii) There exists an injective homomorphism of Lie algebras

$$\iota : \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r} \to \mathfrak{g}_Q(m)$$

such that $$\iota(e_{(i,k)}) = \mathcal{X}^+_{(i,k),0}$$, $$\iota(f_{(i,k)}) = \mathcal{X}^-_{(i,k),0}$$ and $$\iota(K_{(j,t)}) = \mathcal{I}_{(j,t),0}$$.

Proof. We can check the well-definedness of $$g$$ and $$\iota$$ by direct calculations. Clearly $$g$$ is surjective. Let $$\iota' : \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r} \to \mathfrak{gl}_m$$ be the natural embedding. Then, by investigating the image of generators, we see that $$\iota' = g \circ \iota$$. This implies that $$\iota$$ is injective.

Remark 2.17. The surjective homomorphism $$g$$ in (2.16.1) can be regarded as a special case of evaluation homomorphisms. However, we can not define evaluation homomorphisms for $$\mathfrak{g}_Q(m)$$ in general although we can consider $$\mathfrak{g}_Q(m)$$-modules corresponding to some evaluation modules. They will be studied in a subsequent paper.

§ 3. Representations of $$\mathfrak{g}_Q(m)$$

Thanks to the triangular decomposition in Corollary 2.8, we can develop the weight theory to study some representations of $$\mathfrak{g}_Q(m)$$ in the usual manner as follows.

3.1. Let $$U(\mathfrak{g}) = U(\mathfrak{g}_Q(m))$$ be the universal enveloping algebra of the Lie algebra $$\mathfrak{g}_Q(m)$$. Then, by Corollary 2.8 together with PBW theorem, we have the triangular decomposition

$$(3.1.1)$$
$$U(\mathfrak{g}) \cong U(\mathfrak{n}^-) \otimes U(\mathfrak{n}^0) \otimes U(\mathfrak{n}^+).$$

Thanks to the triangular decomposition, we can develop the weight theory for $$U(\mathfrak{g})$$-modules as follows.

3.2. Highest weight modules. For $$\lambda \in P$$ and a multiset $$\varphi = (\varphi_{(j,l),t} | (j,l) \in \Gamma(m), t \geq 1)$$, $$\varphi_{(j,l),t} \in \mathbb{Q}(\mathbb{Q})$$, we say that a $$U(\mathfrak{g})$$-modules $$M$$ is a highest weight modules of highest weight $$(\lambda, \varphi)$$ if there exists an element $$v_0 \in M$$ satisfying the following three conditions:

(i) $$M$$ is generated by $$v_0$$ as a $$U(\mathfrak{g})$$-module,
(ii) $$\mathcal{X}^+_{(i,k),t} \cdot v = 0$$ for all $$((i,k) \in \Gamma'(m)$$ and $$t \geq 0$$,
(iii) $$\mathcal{I}_{(j,t),0} \cdot v_0 = \langle \lambda, h_{(j,l)} \rangle v_0$$ and $$\mathcal{I}_{(j,t),t} \cdot v_0 = \varphi_{(j,l),t} v_0$$ for $$(j,l) \in \Gamma(m)$$ and $$t \geq 1$$.

If an element $$v_0 \in M$$ satisfies the above conditions (ii) and (iii), we say that $$v_0$$ is a maximal vector of weight $$(\lambda, \varphi)$$. In this case, the submodule $$U(\mathfrak{g}) \cdot v_0$$ of $$M$$ is a highest weight module of highest weight $$(\lambda, \varphi)$$. If a maximal vector $$v_0 \in M$$ satisfies the above condition (i), we say that $$v_0$$ is a highest weight vector.

For a highest weight $$U(\mathfrak{g})$$-module $$M$$ of highest weight $$(\lambda, \varphi)$$ with a highest weight vector $$v_0 \in M$$, we have $$M = U(\mathfrak{n}^-) \cdot v_0$$ by the triangular decomposition
(3.1.1). Thus, the relation (L2) implies the weight space decomposition

\[
M = \bigoplus_{\mu \leq \lambda} M_{\mu} \text{ such that } \dim_{\mathbb{Q}(Q)} M_{\lambda} = 1,
\]

where \(M_{\mu} = \{v \in M \mid \mathcal{I}_{(j, l), t} \cdot v = \langle \mu, h_{(j, l)} \rangle v \text{ for } (j, l) \in \Gamma'(m)\}\).

### 3.3. Verma modules.

Let \(U(n^\geq 0)\) be the subalgebra of \(U(g)\) generated by \(U(n^0)\) and \(U(n^+)\). Then, by Proposition 2.6 together with the proof of Lemma 2.5, we see that \(U(n^+)\) (resp. \(U(n^-)\)) is isomorphic to the algebra generated by \(\{X_{(i, k), t} \mid (i, k) \in \Gamma'(m), t \geq 0\}\) with the defining relations (L4)-(L6), \(U(n^0)\) is isomorphic to the algebra generated by \(\{X_{(i, k), t} \mid (i, k) \in \Gamma'(m), (j, l) \in \Gamma(m), t \geq 0\}\) with the defining relations (L1), and that \(U(n^\geq 0)\) is isomorphic to the algebra generated by \(\{X_{(i, k), t} \mid (i, k) \in \Gamma'(m), (j, l) \in \Gamma(m), t \geq 0\}\) with the defining relations (L1)-(L6) except (L3). Then we have the surjective homomorphism of algebras

\[
U(n^\geq 0) \to U(n^0) \text{ such that } X_{(i, k), t} \mapsto 0, \mathcal{I}_{(j, l), t} \mapsto \mathcal{I}_{(j, l), t}.
\]

For \(\lambda \in P\) and a multiset \(\varphi = (\varphi_{(j, l), t})\), we define a (1-dimensional) simple \(U(n^0)\)-module \(\Theta_{(\lambda, \varphi)} = \mathbb{Q}(Q)v_0\) by

\[
\mathcal{I}_{(j, l), 0} \cdot v_0 = \langle \lambda, h_{(j, l)} \rangle v_0, \quad \mathcal{I}_{(j, l), t} \cdot v_0 = \varphi_{(j, l), t}v_0
\]

for \((j, l) \in \Gamma(m)\) and \(t \geq 1\). Then we define the Verma module \(M(\lambda, \varphi)\) as the induced module

\[
M(\lambda, \varphi) = U(g) \otimes_{U(n^\geq 0)} \Theta_{(\lambda, \varphi)},
\]

where we regard \(\Theta_{(\lambda, \varphi)}\) as a left \(U(n^\geq 0)\)-module through the surjection (3.3.1).

By definitions, the Verma module \(M(\lambda, \varphi)\) is a highest weight module of highest weight \((\lambda, \varphi)\) with a highest weight vector \(1 \otimes v_0\). Then we see that any highest weight module of highest weight \((\lambda, \varphi)\) is a quotient of \(M(\lambda, \varphi)\) by the universality of tensor products. We also see that \(M(\lambda, \varphi)\) has the unique simple top \(L(\lambda, \varphi) = M(\lambda, \varphi)/\text{rad} M(\lambda, \varphi)\) from the weight space decomposition (3.2.1).

By using the homomorphism \(\iota : U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to U(g)\) induced from (2.16.2), we have a necessary condition for \(L(\lambda, \varphi)\) to be finite dimensional as follows.

**Proposition 3.4.** For \(\lambda \in P\) and a multiset \(\varphi = (\varphi_{(j, l), t})\), if \(L(\lambda, \varphi)\) is finite dimensional, then we have \(\lambda \in P^+_m\).

**Proof.** Assume that \(L(\lambda, \varphi)\) is finite dimensional. Let \(v_0 \in L(\lambda, \varphi)\) be a highest weight vector. When we regard \(L(\lambda, \varphi)\) as a \(U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})\)-module through the injection \(\iota : U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to U(g)\), we see that \(U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})\)-submodule of \(L(\lambda, \varphi)\) generated by \(v_0\) is a (finite dimensional) highest weight \(U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})\)-module of highest weight \(\lambda\). Thus, the Lemma follows from the well-known facts for \(U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r})\)-modules. \(\square\)
3.5. Category $\mathcal{C}_Q(m)$. Let $\mathcal{C}_Q(m)$ (resp. $\mathcal{C}_Q^{>0}(m)$) be the full subcategory of $U(g)$-mod consisting of $U(g)$-modules satisfying the following conditions:

(i) If $M \in \mathcal{C}_Q(m)$ (resp. $M \in \mathcal{C}_Q^{>0}(m)$), then $M$ is finite dimensional,
(ii) If $M \in \mathcal{C}_Q(m)$ (resp. $M \in \mathcal{C}_Q^{>0}(m)$), then $M$ has the weight space decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda \quad (\text{resp. } M = \bigoplus_{\lambda \in P_{>0}} M_\lambda),$$

where $M_\lambda = \{v \in M \mid \mathcal{I}_{(j,l),0} \cdot v = \langle \lambda, h_{(j,l)} \rangle v \text{ for } (j, l) \in \Gamma(m)\}$,

(iii) If $M \in \mathcal{C}_Q(m)$ (resp. $M \in \mathcal{C}_Q^{>0}(m)$), then all eigenvalues of the action of $\mathcal{I}_{(j,l),t}$ ($(j, l) \in \Gamma(m), t \geq 0$) on $M$ belong to $\mathbb{Q}(Q)$.

By the usual argument, we have the following lemma.

**Lemma 3.6.** Any simple object in $\mathcal{C}_Q(m)$ is a highest weight module.

By using the surjection $g : U(g) \to U(\mathfrak{gl}_m)$ induced from (2.16.1), we have the following proposition.

**Proposition 3.7.** Let $\mathcal{C}_{\mathfrak{gl}_m}$ be the category of finite dimensional $U(\mathfrak{gl}_m)$-modules which have the weight space decomposition. Then, we have the followings.

(i) $\mathcal{C}_{\mathfrak{gl}_m}$ is a full subcategory of $\mathcal{C}_Q(m)$ through the surjection $g : U(g) \to U(\mathfrak{gl}_m)$.
(ii) For $\lambda \in P^+$, the simple highest weight $U(\mathfrak{gl}_m)$-module $\Delta_{\mathfrak{gl}_m}(\lambda)$ of highest weight $\lambda$ is the simple highest weight $U(g)$-module of highest weight $(\lambda, 0)$ through the surjection $g : U(g) \to U(\mathfrak{gl}_m)$, where 0 means $\varphi_{(j,l),t} = 0$ for all $(j, l) \in \Gamma(m)$ and $t \geq 1$.

§ 4. **Algebra $U_qQ(m)$**

In this section, we introduce an algebra $U_qQ(m)$ with parameters $q$ and $Q = (Q_1, \ldots, Q_{r-1})$ associated with the Cartan data in the paragraph 1.3. Then we study some basic structures of $U_qQ(m)$. In particular, we can regard $U_qQ(m)$ as a “$q$-analogue” of the universal enveloping algebra $U(\mathfrak{g}_Q(m))$ of the Lie algebra $\mathfrak{g}_Q(m)$ introduced in the section §2.

4.1. Put $A = \mathbb{Z}[Q][q, q^{-1}] = \mathbb{Z}[q, q^{-1}, Q_1, \ldots, Q_{r-1}]$, where $q, Q_1, \ldots, Q_{r-1}$ are indeterminate elements over $\mathbb{Z}$, and let $\mathbb{K} = \mathbb{Q}(q, Q_1, \ldots, Q_{r-1})$ be the quotient field of $A$.

**Definition 4.2.** We define the associative algebra $U = U_qQ(m)$ over $\mathbb{K}$ by the following generators and defining relations:

- **Generators:** $X^+_{(i,k),t}$, $\mathcal{T}^+_{(j,l),t}$, $\mathcal{K}^\pm_{(j,l)}$ ($(i, k) \in \Gamma(m), (j, l) \in \Gamma(m), t \geq 0$).

- **Relations:**
  
  \begin{align*}
  (R1) \quad K^+_{(j,l)}K^-_{(j,l)} &= K^-_{(j,l)}K^+_{(j,l)} = 1, \quad (K^\pm_{(j,l)})^2 = 1 \pm (q - q^{-1})I_{(j,l),0}, \\
  (R2) \quad [K^\pm_{(i,k)}, K^\pm_{(j,l)}] &= [K^\pm_{(i,k)}, \mathcal{T}^\sigma_{(j,l),t}] = [\mathcal{T}^\sigma_{(i,k),s}, \mathcal{T}^\sigma_{(j,l),t}] = 0 \text{ for } \sigma, \sigma' \in \{+, -\},
  \end{align*}

In the next section, we study some basic structures of $U_qQ(m)$ in §4.1.
(R3) \[ K^\pm_{(j,l)} X^\pm_{(i,k),t} = q^{\pm a_{(i,k)(j,l)}} X^\pm_{(i,k),t}, \]

(R4) \[
\begin{align*}
q^{+a_{(i,k)(j,l)}} T^\pm_{(j,l),0} X^+_{(i,k),t} &- q^{-a_{(i,k)(j,l)}} T^\pm_{(j,l),0} X^-_{(i,k),t} = a_{(i,k)(j,l)} X^\pm_{(i,k),t}, \\
q^{-a_{(i,k)(j,l)}} T^\pm_{(j,l),0} X^-_{(i,k),t} &- q^{+a_{(i,k)(j,l)}} T^\pm_{(j,l),0} X^+_{(i,k),t} = -a_{(i,k)(j,l)} X^\pm_{(i,k),t}.
\end{align*}
\]

(R5) \[
\begin{align*}
[T^\pm_{(j,l),s+1}, X^\pm_{(i,k),t}] &= q^{+a_{(i,k)(j,l)}} T^\pm_{(j,l),s} X^\pm_{(i,k),t+1} - q^{-a_{(i,k)(j,l)}} T^\pm_{(j,l),s} X^-_{(i,k),t+1} + T^\pm_{(j,l),s}, \\
[T^\pm_{(j,l),s+1}, X^-_{(i,k),t}] &= q^{-a_{(i,k)(j,l)}} T^\pm_{(j,l),s} X^-_{(i,k),t+1} - q^{+a_{(i,k)(j,l)}} T^\pm_{(j,l),s} X^+_{(i,k),t+1} + T^\pm_{(j,l),s}.
\end{align*}
\]

(R6) \[
\begin{align*}
\mathcal{J}(i,k,t, X^\pm_{(j,l),s}) &= \delta_{(i,k),(j,l)} \left\{ \begin{array}{ll}
\mathcal{K}^+_{(i,k)} \mathcal{J}(i,k,s+t) & \text{if } i \neq m_k, \\
-Q_k \mathcal{K}^+_{(m_k,k)} \mathcal{J}(m_k,k,s+t) + \mathcal{K}^+_{(m_k,k)} \mathcal{J}(m_k,k,s+t+1) & \text{if } i = m_k,
\end{array} \right.
\end{align*}
\]

(R7) \[
\begin{align*}
[X^\pm_{(i,k),t}, X^\pm_{(j,l),s}] &= 0 \quad \text{if } (j, l) \neq (i, k), (i \pm 1, k), \\
X^\pm_{(i,k),t+1} X^\pm_{(i,k),s} - q^{\pm 2} X^\pm_{(i,k),s} X^\pm_{(i,k),t+1} &= q^{\pm 2} X^\pm_{(i,k),s} X^\pm_{(i,k),t+1} - X^\pm_{(i,k),s+1} X^\pm_{(i,k),t}, \\
X^\pm_{(i,k),t+1} X^-_{(i+1,k),s} - q^{-1} X^-_{(i+1,k),s} X^\pm_{(i,k),t+1} &= X^\pm_{(i,k),t} X^\pm_{(i+1,k),s+1} - q X^\pm_{(i,k),s+1} X^\pm_{(i,k),t}, \\
X^-_{(i+1,k),s} X^-_{(i,k),t+1} - q^{-1} X^-_{(i+1,k),s+1} X^-_{(i,k),t} &= X^-_{(i+1,k),s+1} X^-_{(i,k),t} - q X^-_{(i,k),t} X^-_{(i+1,k),s+1}.
\end{align*}
\]

(R8) \[
\begin{align*}
X^\pm_{(i+1,k),u} (X^+_{(i,k),s} X^\pm_{(i,k),t} + X^+_{(i,k),t} X^\pm_{(i,k),s}) + (X^+_{(i,k),s} X^\pm_{(i,k),t} + X^+_{(i,k),t} X^\pm_{(i,k),s}) X^\pm_{(i\pm 1,k),u} &= (q + q^{-1}) (X^+_{(i,k),s} X^\pm_{(i+1,k),u} X^\pm_{(i,k),t} + X^+_{(i,k),t} X^\pm_{(i+1,k),u} X^\pm_{(i,k),s}), \\
X^-_{(i+1,k),u} (X^-_{(i,k),s} X^\pm_{(i,k),t} + X^\pm_{(i,k),t} X^-_{(i,k),s}) + (X^-_{(i,k),s} X^\pm_{(i,k),t} + X^\pm_{(i,k),t} X^-_{(i,k),s}) X^-_{(i\pm 1,k),u} &= (q + q^{-1}) (X^-_{(i,k),s} X^\pm_{(i+1,k),u} X^-_{(i,k),t} + X^\pm_{(i,k),t} X^-_{(i+1,k),u} X^-_{(i,k),s}),
\end{align*}
\]

where we put \( \mathcal{K}^+_{(i,k)} = K^+_{(i,k)} K^-_{(i+1,k)}, \) \( \mathcal{K}^-_{(i,k)} = K^-_{(i,k)} K^+_{(i+1,k)} \) and

\[
\mathcal{J}(i,k,t) = \left\{ \begin{array}{ll}
I^+_{(i,k),0} - I^-_{(i+1,k),0} + (q - q^{-1}) I^+_{(i,k),0} I^-_{(i+1,k),0} & \text{if } t = 0, \\
q^{-t} I^+_{(i,k),t} - q^{-1} I^-_{(i+1,k),t} - (q - q^{-1}) \sum_{b=1}^{t-1} q^{-t-2b} I^+_{(i,k),t-b} I^-_{(i+1,k),b} & \text{if } t > 0.
\end{array} \right.
\]

**Remark 4.3.** The relations (R4) follows from the relations (R1) and (R3) in \( \mathcal{U}_{q,Q}(m) \). Thus, we do not need the relations (R4) as a defining relations of \( \mathcal{U}_{q,Q}(m) \). However, (R4) does not follows from (R1) and (R3) in the integral forms \( \mathcal{U}^*_q, Q(m) \) and \( \mathcal{U}_{\bar{k}, q}, Q(m) \) defined below. Then, we require the relations (R4) in a defining relations of \( \mathcal{U}_{q,Q}(m) \).
4.4. By the relation (R1), for \((i, k) \in \Gamma'(m)\), we have

\[
\tilde{K}^+_{(i, k)} J_{(i, k), 0} = \frac{\tilde{K}^+_{(i, k)} - \tilde{K}^-_{(i, k)}}{q - q^{-1}}.
\]

Thus, in the case where \(s = t = 0\), we can replace the relation (R6) by

\[
[X^+_{(i, k), 0}, X^-_{(j, l), 0}] = \delta_{(i, k), (j, l)} \left\{ \begin{array}{ll}
\frac{\tilde{K}^+_{(i, k)} - \tilde{K}^-_{(i, k)}}{q - q^{-1}} & \text{if } i \neq m_k, \\
-Q_k \frac{\tilde{K}^+_k - \tilde{K}^-_k}{q - q^{-1}} + \tilde{K}^+_k J_{m_k, k, 1} & \text{if } i = m_k.
\end{array} \right.
\]

By (R8), if \(s = t\), we have

\[
\begin{align*}
X^+_{(i+1, k), u}(X^+_{(i, k), t})^2 & - (q + q^{-1})X^+_{(i, k), t}X^+_{(i, k+1, k), u}X^+_{(i, k), t} + (X^+_{(i, k), t})^2X^+_{(i+1, k), u} = 0, \\
X^-_{(i+1, k), u}(X^-_{(i, k), t})^2 & - (q + q^{-1})X^-_{(i, k), t}X^-_{(i, k+1, k), u}X^-_{(i, k), t} + (X^-_{(i, k), t})^2X^-_{(i+1, k), u} = 0.
\end{align*}
\]

By (R4) and (R5), we have

\[
[T^+_{(j, l), 1}, X^+_{(i, k), t}] = [T^-_{(j, l), 1}, X^-_{(i, k), t}] = \pm a_{(i, k), (j, l)} X^\pm_{(i, k), t+1}.
\]

By the induction on \(s\) using the relation (R6), for \(s \geq 1\), we can show that

\[
[T^\pm_{(j, l), s}, X^\pm_{(i, k), t}] = a_{(i, k), (j, l)} q^{\pm a_{(i, k), (j, l)}(s-1)} X^\pm_{(i, k), t+s} \pm a_{(i, k), (j, l)} (q - q^{-1}) \sum_{p=1}^{s-1} q^{\pm a_{(i, k), (j, l)}(p-1)} X^\pm_{(i, k), t+p} T^\pm_{(j, l), s-p},
\]

and

\[
[T^\pm_{(j, l), s}, X^-_{(i, k), t}] = -a_{(i, k), (j, l)} q^{\mp a_{(i, k), (j, l)}(s-1)} X^-_{(i, k), t+s} \mp a_{(i, k), (j, l)} (q - q^{-1}) \sum_{p=1}^{s-1} q^{\mp a_{(i, k), (j, l)}(p-1)} X^-_{(i, k), t+p} T^\pm_{(j, l), s-p}.
\]
4.5. Let \( \mathcal{U}^+ = U_q^+ \mathfrak{gl}(m) \), \( \mathcal{U}^- = U_q^- \mathfrak{gl}(m) \) and \( \mathcal{U}^0 = U_q^0 \mathfrak{gl}(m) \) be the subalgebra of \( \mathcal{U} \) generated by

\[
\{ \chi^+_{(i,k),t} \mid (i, k) \in \Gamma'(m), t \geq 0 \},
\{ \chi^-_{(i,k),t} \mid (i, k) \in \Gamma'(m), t \geq 0 \} \text{ and }
\{ T^\pm_{(j,l),t}, K^\pm_{(j,l)} \mid (j, l) \in \Gamma(m), t \geq 0 \}
\]

respectively. Then, we have the following triangular decomposition of \( \mathcal{U} \) from the relations (R1)-(R8), (4.4.5) and (4.4.6).

**Proposition 4.6.** We have

\[(4.6.1) \quad \mathcal{U} = \mathcal{U}^- \mathcal{U}^0 \mathcal{U}^+.\]

**Remark 4.7.** We conjecture that the multiplication map \( \mathcal{U}^- \otimes_{\mathbb{K}} \mathcal{U}^0 \otimes_{\mathbb{K}} \mathcal{U}^+ \to \mathcal{U} \) \( (x \otimes y \otimes z \mapsto xyz) \) gives an isomorphism as vector spaces. More precisely, we expect the existence of a PBW type basis of \( \mathcal{U} \) (cf. Proposition 2.6 and (4.11.2) with Remark 4.12).

4.8. We have some relations between the algebra \( \mathcal{U} \) and a quantum group associated with the general linear Lie algebra as follows.

Let \( U_q(\mathfrak{gl}_m) \) be the quantum group associated with the general linear Lie algebra \( \mathfrak{gl}_m \) over \( \mathbb{K} \). Namely, \( U_q(\mathfrak{gl}_m) \) is an associative algebra over \( \mathbb{K} \) generated by \( e_i, f_i \) \((1 \leq i \leq m - 1)\) and \( K_j^\pm (1 \leq j \leq m) \) with the following defining relations:

\[
\begin{align*}
(Q1) & \quad K_i^+ K_j^+ = K_j^+ K_i^+, \quad K_i^+ K_j^- = K_i^- K_j^+, \quad 1, \\
(Q2) & \quad K_j^+, e_i K_j^- = q^{a_{i,j}} e_i, \quad K_j^+, f_i K_j^- = q^{-a_{i,j}} f_i, \quad \text{where } a_{i,j} = \langle \alpha_i, h_j \rangle, \\
(Q3) & \quad e_i f_j - f_j e_i = \delta_{i,j} \frac{K_i^+ K_{i+1}^- - K_i^- K_{i+1}^+}{q - q^{-1}}, \\
(Q4) & \quad e_{i+1} e_i^2 - (q + q^{-1}) e_{i+1} e_i e_i + e_i^2 e_{i+1} = 0, \quad e_i e_j = e_j e_i (|i - j| \geq 2), \\
(Q5) & \quad f_{i+1} f_i^2 - (q + q^{-1}) f_i f_{i+1} f_i + f_i^2 f_{i+1} = 0, \quad f_i f_j = f_j f_i (|i - j| \geq 2).
\end{align*}
\]

Let \( U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \cong U_q(\mathfrak{gl}_{m_1}) \otimes \cdots \otimes U_q(\mathfrak{gl}_{m_r}) \) be the Levi subalgebra of \( U_q(\mathfrak{gl}_m) \) associated with the Levi subalgebra \( \mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r} \) of \( \mathfrak{gl}_m \). Then generators of \( U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \) are given by \( e_{(i,k)}, f_{(i,k)} \) \((1 \leq i \leq m_k - 1, 1 \leq k \leq r)\) and \( K^\pm_{(j,l)} \((j, l) \in \Gamma(m)\)) where we use the identification (1.3.1) for indices.

**Proposition 4.9.**

(i) There exits a surjective homomorphism of algebras

\[(4.9.1) \quad g : U_q \mathfrak{gl}(m) \to U_q(\mathfrak{gl}_m)\]

such that

\[
g(\mathcal{X}^{\pm}_{(i,k),0}) = \begin{cases} e_{(i,k)} & \text{if } i \neq m_k, \\ -Q_k e_{(m_k,k)} & \text{if } i = m_k, \end{cases} \quad g(\mathcal{X}^{-}_{(i,k),0}) = f_{(i,k)},
\]

\[g(\mathcal{K}^{\pm}_{(j,l)}) = \mathcal{K}^{\pm}_{(j,l)} \text{ and } g(\mathcal{X}^{\pm}_{(i,k),t}) = g(\mathcal{T}^{\pm}_{(j,l),t}) = 0 \text{ for } t \geq 1.\]

(ii) There exists an injective homomorphism of algebras

\[(4.9.2) \quad \iota : U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to U_q Q(m)\]

such that \(\iota(e_{(i,k)}) = \mathcal{X}^{+}_{(i,k),0}, \iota(f_{(i,k)}) = \mathcal{X}^{-}_{(i,k),0} \text{ and } \iota(\mathcal{K}^{\pm}_{(j,l)}) = \mathcal{K}^{\pm}_{(j,l)}.\)

**Proof.** We can check the well-definedness of \(g\) and \(\iota\) by direct calculations. Clearly \(g\) is surjective. Let \(\iota' : U_q(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \to U_q(\mathfrak{gl}_m)\) be the natural embedding. Then, by investigating the image of generators, we see that \(\iota' = g \circ \iota\). This implies that \(\iota\) is injective. \(\square\)

**Remark 4.10.** The surjective homomorphism \(g\) in (4.9.1) can be regarded as a special case of evaluation homomorphisms. However, we cannot define evaluation homomorphisms for \(U_q Q(m)\) in general although we can consider \(U_q Q(m)\)-modules corresponding to some evaluation modules. They will be studied in a subsequent paper.

**4.11.** Let \(U^*_A = U^*_{A_q,q}(m)\) be the \(A\)-subalgebra of \(U_q Q(m)\) generated by

\[\{\mathcal{X}^{\pm}_{(i,k),t}, \mathcal{T}^{\pm}_{(j,l),t}, \mathcal{K}^{\pm}_{(j,l)} \mid (i,k) \in \Gamma'(m), (j,l) \in \Gamma(m), t \geq 0\}.\]

Then, \(U^*_A\) is an associative algebra over \(A\) generated by the same generators with the defining relations (R1)-(R8). We regard \(Q(Q)\) as an \(A\)-module through the ring homomorphism \(A \to Q(Q)\) \((q \mapsto 1)\), and we consider the specialization \(Q(Q) \otimes_A U^*_A\) using this ring homomorphism. Let \(\mathfrak{J}\) be the ideal of \(Q(Q) \otimes_A U^*_A\) generated by

\[(4.11.1) \quad \{\mathcal{K}^{+}_{(j,l),t} - 1, \mathcal{T}^{+}_{(j,l),t} - \mathcal{T}^{-}_{(j,l),t} \mid (i,l) \in \Gamma(m), t \geq 0\}.\]

Let \(U(\mathfrak{g}Q(m))\) be the universal enveloping algebra of the Lie algebra \(\mathfrak{g}Q(m)\) defined in Definition 2.2. Then we can check that there exists a surjective homomorphism of algebras

\[(4.11.2) \quad U(\mathfrak{g}Q(m)) \to Q(Q) \otimes_A U^*_{A,q,q}(m)/\mathfrak{J}\]

such that \(\mathcal{X}^{\pm}_{(i,k),t} \mapsto \mathcal{X}^{\pm}_{(i,k),t}\) and \(\mathcal{T}_{(j,l),t} \mapsto \mathcal{T}^{+}_{(j,l),t} (= \mathcal{T}^{-}_{(j,l),t}).\)

**Remark 4.12.** We conjecture that the homomorphism (4.11.2) is isomorphic. Then we may regard \(U_q Q(m)\) as a \(q\)-analogue of \(U(\mathfrak{g}Q(m))\).

We also remark that we have \((\mathcal{K}^{\pm}_{(j,l)})^2 = 1\) in \(U^*_A\) by the relation (R1). On the other hand, there exists an algebra automorphism of \(U\) such that \(\mathcal{K}^{\pm}_{(j,l)} \mapsto -\mathcal{K}^{\pm}_{(j,l)}\).
and the other generators send to the same generators. Thus, the choice of signs for $K_{(j,l)}^{\pm}$ in (4.11.1) will not cause any troubles.

4.13. The final of this section, we define the $A$-form of $U$ taking the divided powers.

For $(i,k) \in \Gamma'(m)$ and $t, d \in \mathbb{Z}_{\geq 0}$, put

$$\mathcal{X}_{(i,k),t}^{\pm(d)} = \frac{(\mathcal{X}_{(i,k),t})^d}{[d]!} \in U.$$  

For $(j,l) \in \Gamma(m)$ and $d \in \mathbb{Z}_{\geq 0}$, put

$$[K_{(j,l)}^{(d)};0] = \prod_{b=1}^d K_{(j,l)}^{+b+1} - K_{(j,l)}^{-b-1} q^b - q^{-b} \in U.$$  

Let $U_A = U_{A,q,Q}(m)$ be the $A$-subalgebra of $U$ generated by all $\mathcal{X}_{(i,k),t}^{\pm(d)}$, $I_{(j,l),t}^{\pm}$, $K_{(j,l)}^{\pm}$ and $[K_{(j,l);0}^{(d)}]$.  

§ 5. REPRESENTATIONS OF $U_{q,Q}(m)$

Thanks to the triangular decomposition (4.6.1) of $U = U_{q,Q}(m)$, we can develop the weight theory to study $U$-modules in the usual manner as follows.

5.1. Highest weight modules. For $\lambda \in P$ and a multiset $\varphi = (\varphi_{(j,l),t}^\pm | (j,l) \in \Gamma(m), t \geq 1)$ ($\varphi_{(j,l),t}^\pm \in \mathbb{K}$), we say that a $U$-module $M$ is a highest weight module of highest weight $(\lambda, \varphi)$ if there exists an element $v_0 \in M$ satisfying the following three conditions:

(i) $M$ is generated by $v_0$ as a $U$-module,
(ii) $X_{(i,k),t}^{\pm} \cdot v_0 = 0$ for all $(i,k) \in \Gamma'(m)$ and $t \geq 0$,
(iii) $K_{(j,l)}^{\pm} \cdot v_0 = q^{\lambda_h(h_{(j,l)})} v_0$ and $I_{(j,l),t}^{\pm} \cdot v_0 = \varphi_{(j,l),t}^\pm v_0$ for $(j,l) \in \Gamma(m)$ and $t \geq 1$.

If an element $v_0 \in M$ satisfies the above conditions (ii) and (iii), we say that $v_0$ is a maximal vector of weight $(\lambda, \varphi)$. In this case, the submodule $U \cdot v_0$ of $M$ is a highest weight module of highest weight $(\lambda, \varphi)$. If a maximal vector $v_0 \in M$ satisfies also the above condition (i), we say that $v_0$ is a highest weight vector.

If $v_0 \in M$ is a maximal vector of weight $(\lambda, \varphi)$, for $(j,l) \in \Gamma(m)$, we have

$$I_{(j,l),0}^{\pm} \cdot v = q^{\lambda_h(h_{(j,l)})} v, \text{ where } \lambda_{(j,l)} = \langle \lambda, h_{(j,l)} \rangle$$

by the relation (R1).

For a highest weight $U$-module $M$ of highest weight $(\lambda, \varphi)$ with a highest weight vector $v_0 \in M$, we have $M = U^- \cdot v_0$ by the triangular decomposition (4.6.1). Thus, the relation (R3) implies the weight space decomposition

$$M = \bigoplus_{\mu \in \mathcal{P}} M_{\mu} \text{ such that } \dim_{\mathbb{K}} M_{\lambda} = 1.$$
where $M_\mu = \{ v \in M \mid K_{(j,l)}^+ \cdot v = q^{(\mu,h_{(j,l)})} v \text{ for } (j,l) \in \Gamma(m) \}$.

5.2. Verma modules. Let $\tilde{U}^0$ be the associative algebra over $\mathbb{K}$ generated by $T_{(j,l),t}^\pm$ and $K_{(j,l)}^\pm$ for all $(j,l) \in \Gamma(m)$ and $t \geq 0$ with the defining relations (R1) and (R2).

We also define the associative algebra $\tilde{U}^{\geq 0}$ generated by $X_{(i,k),t}^+$, $T_{(j,l),t}^\pm$ and $K_{(j,l)}^\pm$ for all $(i,k) \in \Gamma'(m)$, $(j,l) \in \Gamma(m)$ and $t \geq 0$ with the defining relations (R1)-(R8) except (R6). Then we have the homomorphism of algebras

(5.2.1) $\tilde{U}^{\geq 0} \to U$ such that $X_{(i,k),t}^+ \mapsto X_{(i,k),t}^+$, $T_{(j,l),t}^\pm \mapsto T_{(j,l),t}^\pm$ and the surjective homomorphism of algebras

(5.2.2) $\tilde{U}^{\geq 0} \to \tilde{U}^0$ such that $X_{(i,k),t}^+ \mapsto X_{(i,k),t}^+$, $I_{(j,l),t}^\pm \mapsto I_{(j,l),t}^\pm$, $K_{(j,l)}^\pm \mapsto K_{(j,l)}^\pm$.

For $\lambda \in P$ and a multiset $\varphi = (\varphi_{(j,l),t})$, we define a (1-dimensional) simple $\tilde{U}^0$-module $\Theta(\lambda,\varphi) = \mathbb{K}_v0$ by

$$K_{(j,l)}^+ \cdot v_0 = q^{(\lambda,h_{(j,l)})}v_0, \quad I_{(j,l),t}^\pm \cdot v_0 = \varphi_{(j,l),t}^\pm v_0$$

for $(j,l) \in \Gamma(m)$ and $t \geq 1$. Then we define the Verma module $M(\lambda,\varphi)$ as the induced module

$$M(\lambda,\varphi) = U \otimes_{\tilde{U}^{\geq 0}} \Theta(\lambda,\varphi),$$

where we regard $\Theta(\lambda,\varphi)$ (resp. $U$) as a left (resp. right) $\tilde{U}^{\geq 0}$-module through the homomorphism (5.2.2) (resp. (5.2.1)).

By definitions, the Verma module $M(\lambda,\varphi)$ is a highest weight module of highest weight $(\lambda,\varphi)$ with a highest weight vector $1 \otimes v_0$. Then we see that any highest weight module of highest weight $(\lambda,\varphi)$ is a quotient of $M(\lambda,\varphi)$ by the universality of tensor products. We also see that $M(\lambda,\varphi)$ has the unique simple top $L(\lambda,\varphi) = M(\lambda,\varphi)/\mathrm{rad} M(\lambda,\varphi)$ from the weight space decomposition (5.1.1).

By using the homomorphism $\iota : U_q(g\mathfrak{gl}_{m_1} \oplus \cdots \oplus g\mathfrak{gl}_{m_r}) \to U$ in (4.9.2), we have the following necessary condition for $L(\lambda,\varphi)$ to be finite dimensional in a similar way as in the proof of Proposition 3.4.

Proposition 5.3. For $\lambda \in P$ and a multiset $\varphi = (\varphi_{(j,l),t})$, if $L(\lambda,\varphi)$ is finite dimensional, then we have $\lambda \in P_{m_1}^+$. 

5.4. Category $\mathcal{C}_{q,Q}(m)$. Let $\mathcal{C}_{q,Q}(m)$ (resp. $\mathcal{C}_{q,Q}^{\geq 0}(m)$) be the full subcategory of $\mathcal{U}$-mod consisting of $\mathcal{U}$-modules satisfying the following conditions:

(i) If $M \in \mathcal{C}_{q,Q}(m)$ (resp. $M \in \mathcal{C}_{q,Q}^{\geq 0}(m)$), then $M$ is finite dimensional,
(ii) If $M \in C_{q,Q}(m)$ (resp. $M \in C_{q,Q}^0(m)$), then $M$ has the weight space decomposition

$$M = \bigoplus_{\lambda \in P} M_\lambda$$ (resp. $M = \bigoplus_{\lambda \in P_{\geq 0}} M_\lambda$),

where $M_\lambda = \{ v \in M \mid K_{(j,l)}^+ \cdot m = q^{(\lambda,h_{(j,l)})} v \text{ for } (j, l) \in \Gamma(m) \}$.

(iii) If $M \in C_{q,Q}(m)$ (resp. $M \in C_{q,Q}^0(m)$), then all eigenvalues of the action of $T_{(j,l)}^+ ((j, l) \in \Gamma(m), t \geq 0)$ on $M$ belong to $\mathbb{K}$.

By the usual argument, we have the following lemma.

**Lemma 5.5.** Any simple object in $C_{q,Q}(m)$ is a highest weight module.

By using the surjection $g : U_q(m) \rightarrow U_q(gl_m)$ in (4.9.1), we have the following proposition.

**Proposition 5.6.** Let $C_{U_q(gl_m)}$ be the category of finite dimensional $U_q(gl_m)$-modules which have the weight space decomposition. Then we have the followings.

(i) $C_{U_q(gl_m)}$ is a full subcategory of $C_{q,Q}(m)$ through the surjection (4.9.1).

(ii) For $\lambda \in P^+$, the simple highest weight $U_q(gl_m)$-module $\Lambda U_q(gl_m)(\lambda)$ of highest weight $\lambda$ is the simple highest weight $U$-module of highest weight $(\lambda, 0)$ through the surjection (4.9.1), where $0$ means $\varphi_{(j,l),t}^\pm = 0$ for all $(j, l) \in \Gamma(m)$ and $t \geq 1$.

§ 6. **Review of cyclotomic $q$-Schur algebras**

In this section, we recall the definition and some fundamental properties of the cyclotomic $q$-Schur algebra $\mathcal{H}_{n,r}(m)$ introduced in [DJM]. See [DJM] and [M1] for details.

6.1. Let $R$ be a commutative ring, and we take parameters $q, Q_0, Q_1, \ldots, Q_{r-1} \in R$ such that $q$ is invertible in $R$. The Ariki-Koike algebra $\mathcal{H}_{n,r}$ associated with the complex reflection group $\mathcal{G}_n \simeq (\mathbb{Z}/r\mathbb{Z})^n$ is the associative algebra with 1 over $R$ generated by $T_0, T_1, \ldots, T_{n-1}$ with the following defining relations:

$$(T_0 - Q_0)(T_0 - Q_1) \ldots (T_0 - Q_{r-1}) = 0, \quad (T_i - q)(T_i + q^{-1}) = 0 \quad (1 \leq i \leq n - 1), \quad T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad (1 \leq i \leq n - 2), \quad T_i T_j = T_j T_i \quad (|i - j| \geq 2).$$

The subalgebra of $\mathcal{H}_{n,r}$ generated by $T_1, \ldots, T_{n-1}$ is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}_n$ associated with the symmetric group $\mathcal{S}_n$ of degree $n$. For $w \in \mathcal{G}_n$, we denote by $\ell(w)$ the length of $w$, and denote by $T_w$ the standard basis of $\mathcal{H}_n$ corresponding to $w$.

6.2. Put $L_1 = T_0$ and $L_i = T_{i-1} L_{i-1} T_{i-1}$ for $i = 2, \ldots, n$. These elements $L_1, \ldots, L_n$ are called Jucys-Murphy elements of $\mathcal{H}_{n,r}$ (see [M2] for properties of Jucys-Murphy elements). The following lemma is well-known, and one can easily check them from defining relations of $\mathcal{H}_{n,r}$.
**Lemma 6.3.** We have the following.

(i) $L_i$ and $L_j$ commute with each other for any $1 \leq i, j \leq n$.
(ii) $T_i$ and $L_j$ commute with each other if $j \neq i, i + 1$.
(iii) $T_i$ commutes with both $L_i, L_{i+1}$ and $L_i + L_{i+1}$ for any $1 \leq i \leq n - 1$.
(iv) $L_{i+1}L_i = (q - q^{-1})\sum_{s=0}^{t-1}L_{i+s}^L + T_iL_i^L$ for any $1 \leq i \leq n - 1$ and $t \geq 1$.
(v) $L_iT_i = -(q - q^{-1})\sum_{s=1}^{t}L_i^L + T_iL_i^L$ for any $1 \leq i \leq n - 1$ and $t \geq 1$.

6.4. Let $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}_{\geq 0}^r$ be an $r$-tuple of positive integers. Put

$$A_{n,r}(\mathbf{m}) = \left\{ \mu = (\mu^{(1)}, \mu^{(2)}, \ldots, \mu^{(r)}) \mid \mu^{(k)} = (\mu^{(k)}_1, \ldots, \mu^{(k)}_{m_k}) \in \mathbb{Z}_{\geq 0}^{m_k}, \sum_{k=1}^{r} \sum_{i=1}^{m_k} \mu^{(k)}_i = n \right\}. $$

We also put

$$A_{n,r}^+(\mathbf{m}) = \{ \mu \in A_{n,r}(\mathbf{m}) \mid \mu^{(k)}_1 \geq \mu^{(k)}_2 \geq \cdots \geq \mu^{(k)}_{m_k} \geq 0 \text{ for each } k = 1, \ldots, r \}. $$

We regard $A_{n,r}(\mathbf{m})$ as a subset of weight lattice $P = \bigoplus_{(i,k) \in \Gamma(\mathbf{m})} \mathbb{Z} \varepsilon_{(i,k)}$ by the injection $A_{n,r}(\mathbf{m}) \to P$ such that $\mu \mapsto \sum_{(i,k) \in \Gamma(\mathbf{m})} \mu^{(k)}_i \varepsilon_{(i,k)}$. Then we see that $A_{n,r}^+(\mathbf{m}) = A_{n,r}(\mathbf{m}) \cap P^+$.

For $\mu \in A_{n,r}(\mathbf{m})$, put

$$(6.4.1) \quad m_\mu = \left( \sum_{w \in \mathfrak{S}_\mu} q^{\ell(w)}T_w \right) \left( \prod_{k=1}^{r-1} \prod_{i=1}^{a_k} (L_i - Q_k) \right),$$

where $\mathfrak{S}_\mu$ is the Young subgroup of $\mathfrak{S}_n$ with respect to $\mu$, and $a_k = \sum_{j=1}^{k} |\mu^{(j)}|$. The following fact is well known:

$$(6.4.2) \quad m_\mu T_w = q^{\ell(w)}m_\mu \text{ if } w \in \mathfrak{S}_\mu.$$

The cyclotomic $q$-Schur algebra $\mathscr{S}_{n,r}(\mathbf{m})$ associated with $\mathcal{H}_{n,r}$ is defined by

$$(6.4.3) \quad \mathscr{S}_{n,r}(\mathbf{m}) = \mathrm{End}_{\mathcal{H}_{n,r}} \left( \bigoplus_{\mu \in A_{n,r}(\mathbf{m})} m_\mu \mathcal{H}_{n,r} \right).$$

For convenience in the later arguments, put $m_\mu = 0$ for $\mu \in P \setminus A_{n,r}(\mathbf{m})$.

6.5. Put $\tilde{A}_{n,r}^+(\mathbf{m}) = A_{n,r}^+(\{n, \ldots, n, m_i\})$. It is clear that $\tilde{A}_{n,r}^+(\mathbf{m}) = A_{n,r}^+(\mathbf{m})$ if $m_k \geq n$ for all $k = 1, \ldots, r - 1$. In the case where $m_k < n$ for some $k < r$, $A_{n,r}^+(\mathbf{m})$ is a proper subset of $\tilde{A}_{n,r}^+(\mathbf{m})$.

In [DJM] (see also [M1] for the case where $m_k < n$ for some $k$), it is proven that $\mathscr{S}_{n,r}(\mathbf{m})$ is a cellular algebra with respect to the poset $(\tilde{A}_{n,r}^+, \geq)$. For $\lambda \in \tilde{A}_{n,r}^+(\mathbf{m})$, let $\Delta(\lambda)$ be the Weyl (cell) module corresponding to $\lambda$ constructed in [DJM] (see also [M1] and [W3, Lemma 1.18]). By the general theory of cellular algebras
given in [GL], \( \{\Delta(\lambda) \mid \lambda \in \Lambda^+_{n,r}(m)\} \) gives a complete set of isomorphism classes of simple \( S_{n,r}(m) \)-modules if \( S_{n,r}(m) \) is semi-simple. It is also proven, in [DJM], that \( S_{n,r}(m) \) is a quasi-hereditary algebra such that \( \{\Delta(\lambda) \mid \lambda \in \Lambda^+_{n,r}(m)\} \) gives a complete set of standard modules if \( R \) is a field and \( m_k \geq n \) for all \( k = 1, \ldots, r-1 \).

From the construction of \( \Delta(\lambda) \) in [DJM], \( \Delta(\lambda) \) has a basis indexed by the set of semi-standard tableaux. Since we use them in the later argument, we recall the definition of semi-standard tableaux from [DJM].

For \( \lambda \in \Lambda^+_{n,r}(m) \), the diagram \( [\lambda] \) of \( \lambda \) is the set
\[
[\lambda] = \{(i, j, k) \in \mathbb{Z}^3 \mid 1 \leq i \leq m_k, 1 \leq j \leq \lambda_i^{(k)}, 1 \leq k \leq r\}.
\]

For \( x = (i, j, k) \in [\lambda] \), put
\[
\text{res}(x) = q^{2(j-i)}Q_{k-1}.
\]

For \( \lambda \in \Lambda^+_{n,r}(m) \) and \( \mu \in \Lambda_{n,r}(m) \), a tableau of shape \( \lambda \) with weight \( \mu \) is a map
\[
T : [\lambda] \to \{(a, c) \in \mathbb{Z} \times \mathbb{Z} \mid a \geq 1, 1 \leq c \leq r\}
\]
such that \( \mu_i^{(k)} = \sharp\{x \in [\lambda] \mid T(x) = (i, k)\} \). We define the order on \( \mathbb{Z} \times \mathbb{Z} \) by \((a, c) \geq (a', c')\) if either \( c \geq c' \), or \( c = c' \) and \( a \geq a' \). For a tableau \( T \) of shape \( \lambda \) with weight \( \mu \), we say that \( T \) is semi-standard if \( T \) satisfies the following conditions:

(i) If \( T((i, j, k)) = (a, c) \), then \( k \leq c \),
(ii) \( T((i, j, k)) \leq T((i, j+1, k)) \) if \( (i, j+1, k) \in [\lambda] \),
(iii) \( T((i, j, k)) < T((i+1, j, k)) \) if \( (i+1, j, k) \in [\lambda] \).

For \( \lambda \in \Lambda^+_{n,r}(m) \), \( \mu \in \Lambda_{n,r}(m) \), we denote by \( T_0(\lambda, \mu) \) the set of semi-standard tableaux of shape \( \lambda \) with weight \( \mu \). Then, from the cellular basis of \( S_{n,r}(m) \) in [DJM], we see that \( \Delta(\lambda) \) has the basis
\[
\{\varphi_T \mid T \in T_0(\lambda, \mu) \text{ for some } \mu \in \Lambda_{n,r}(m)\}.
\]

(See [DJM] for the definition of \( \varphi_T \).)

\section*{§ 7. Generators of cyclotomic \( q \text{-Schur algebras} \)}

In this section, we define some generators of the cyclotomic \( q \)-Schur algebra, and we obtain some relations among them which will be used to obtain the homomorphism from \( U_qQ(m) \) in the next section.

\textbf{7.1.} A partition \( \lambda \) is a non-increasing sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) of non-negative integers. For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), we denote by \( \ell(\lambda) \) the length of \( \lambda \) which is the maximal integer \( l \) such that \( \lambda_l \neq 0 \). If \( \sum_{i=1}^{\ell(\lambda)} \lambda_i = n \), we denote it by \( \lambda \vdash n \).

For a integer \( k \) and a partition \( \lambda \vdash n \) such that \( \ell(\lambda) \leq k \), put
\[
\mathcal{G}_k \cdot \lambda = \{(\mu_1, \mu_2, \ldots, \mu_k) \in \mathbb{Z}^k \geq 0 \mid \mu_i = \lambda_{\sigma(i)}, \sigma \in \mathcal{S}_k\}.
\]
7.2. For integers $t, k > 0$, we define the symmetric polynomials $\Phi^+_t(x_1, \ldots, x_k) \in R[x_1, \ldots, x_k]^S_k$ of degree $t$ with variables $x_1, \ldots, x_k$ as

$$
(7.2.1) \quad \Phi^+_t(x_1, \ldots, x_k) = \sum_{\lambda \vdash n, \ell(\lambda) \leq k} (1 - q^{t+2})^{\ell(\lambda)-1} m_{\lambda}(x_1, \ldots, x_k),
$$

where $m_{\lambda}(x_1, \ldots, x_k) = \sum_{\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in S_k^\lambda} x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}$ is the monomial symmetric polynomial associated with the partition $\lambda$. For convenience, we also define

$$
(7.2.2) \quad \Phi^+_0(x_1, \ldots, x_k) = q^{tk+1}[k].
$$

From the definition, we have

$$
(7.2.3) \quad \Phi^+_t(x_1, \ldots, x_k) = x_1 + x_2 + \cdots + x_k \quad \text{and} \quad \Phi^+_t(x_1) = x_1^t.
$$

The polynomials $\Phi^+_t(x_1, \ldots, x_k)$ satisfy the following recursive relations which will be used for calculations of some relations between generators of $\mathcal{S}_{n,r}(m)$ in later.

**Lemma 7.3.** For $t \geq 0$, we have

$$
(7.3.1) \quad \Phi^+_{t+1}(x_1, \ldots, x_k) = \sum_{s=1}^{k} \Phi^+_t(x_1, \ldots, x_s)x_{s+1} - q^{t+2} \sum_{s=1}^{k-1} \Phi^+_t(x_1, \ldots, x_s)x_{s+1}
$$

and

$$
(7.3.2) \quad \Phi^+_{t+1}(x_1, x_2, \ldots, x_k) - \Phi^+_{t+1}(x_2, \ldots, x_k)
\quad = x_1(\Phi^+_t(x_1, x_2, \ldots, x_k) - q^{t+2}\Phi^+_t(x_2, \ldots, x_k)).
$$

**Proof.** In the case where $t = 0$, we can check the statements by direct calculations. Assume that $t \geq 1$. From the definition, we have

$$
\Phi^+_t(x_1, \ldots, x_k) = \sum_{\lambda \vdash n, \ell(\lambda) \leq k} (1 - q^{t+2})^{\ell(\lambda)-1} \sum_{\mu \in S_k^\lambda} x_1^{\mu_1} x_2^{\mu_2} \cdots x_k^{\mu_k}
$$

$$
= \sum_{s=1}^{k} \sum_{\lambda \vdash n, \ell(\lambda) \leq s} (1 - q^{t+2})^{\ell(\lambda)-1} \sum_{\mu \in S_s^\lambda} x_1^{\mu_1} x_2^{\mu_2} \cdots x_s^{\mu_s}
$$

$$
= \sum_{s=1}^{k} \sum_{\lambda \vdash n, \ell(\lambda) \leq s} (1 - q^{t+2})^{\ell(\lambda)-1} \sum_{\mu \in S_s^\lambda} x_1^{\mu_1} x_2^{\mu_2} \cdots x_s^{\mu_s}.
$$
We can easily check the second equality of (7.3.1).

We prove (7.3.2) by the induction on $t$. In the case where $t = 1$, we can check (7.3.2) directly by using the relation (7.3.1) together with (7.2.3). Assume that $t > 1$. By (7.3.1), we have

$$
\Phi_{t+1}^\pm(x_1, x_2, \ldots, x_k) - \Phi_{t+1}^\pm(x_2, \ldots, x_k)
= \sum_{s=1}^{k} \Phi_{t}^\pm(x_1, \ldots, x_s)x_s - q^{2} \sum_{s=1}^{k-1} \Phi_{t}^\pm(x_1, \ldots, x_s)x_{s+1}
\quad - \sum_{s=2}^{k} \Phi_{t}^\pm(x_2, \ldots, x_s)x_s - q^{2} \sum_{s=2}^{k-1} \Phi_{t}^\pm(x_2, \ldots, x_s)x_{s+1}
= \Phi_{t}^\pm(x_1)x_1 - q^{2}\Phi_{t}^\pm(x_1)x_2 + \sum_{s=2}^{k} \left( \Phi_{t}^\pm(x_1, \ldots, x_s) - \Phi_{t}^\pm(x_2, \ldots, x_s) \right)x_s
\quad - q^{2} \sum_{s=2}^{k-1} \left( \Phi_{t}^\pm(x_1, \ldots, x_s) - \Phi_{t}^\pm(x_2, \ldots, x_s) \right)x_{s+1}.
$$

Applying the assumption of the induction, we have

$$
\Phi_{t+1}^\pm(x_1, x_2, \ldots, x_k) - \Phi_{t+1}^\pm(x_2, \ldots, x_k)
= x_1 \Phi_{t-1}^\pm(x_1)x_1 - q^{2}x_1 \Phi_{t-1}^\pm(x_1)x_2
= x_1 \Phi_{t-1}^\pm(x_1)x_1 - q^{2}x_1 \Phi_{t-1}^\pm(x_1)x_2.
$$
\[ + \sum_{s=2}^{k} x_1 \left( \Phi_{t-1}^{\pm}(x_1, x_2, \ldots, x_s) - q^{\mp 2} \Phi_{t-1}^{\pm}(x_2, \ldots, x_s) \right) x_s \]

\[ - q^{\mp 2} \sum_{s=2}^{k-1} x_1 \left( \Phi_{t-1}^{\pm}(x_1, x_2, \ldots, x_s) - q^{\mp 2} \Phi_{t-1}^{\pm}(x_2, \ldots, x_s) \right) x_{s+1} \]

\[ = x_1 \left\{ \left( \sum_{s=1}^{k} \Phi_{t-1}^{\pm}(x_1, x_2, \ldots, x_s) x_s - q^{\mp 2} \sum_{s=1}^{k-1} \Phi_{t-1}^{\pm}(x_1, x_2, \ldots, x_s) x_{s+1} \right) \right\} \]

\[ - q^{\mp 2} \left( \sum_{s=2}^{k} \Phi_{t-1}^{\pm}(x_2, \ldots, x_s) x_s - q^{\mp 2} \sum_{s=2}^{k-1} \Phi_{t-1}^{\pm}(x_2, \ldots, x_s) x_{s+1} \right) \].

Applying the relation (7.3.1), we obtain (7.3.2).

\[ \square \]

**Remark 7.4.** At first, the author defined the polynomials \( \Phi_t^{\pm}(x_1, \ldots, x_k) \) by using the relations (7.3.1) inductively. The definition of \( \Phi_t^{\pm}(x_1, \ldots, x_k) \) as in (7.2.1) was suggested by Tatsuyuki Hikita.

**7.5.** For \( \mu \in \Lambda_{n,r}(m) \) and \( (j, l) \in \Gamma(m) \), put

\[ N_{(j,l)}^{\mu} = \sum_{c=1}^{l-1} |\mu^{(c)}| + \sum_{p=1}^{j} \mu_p^{(l)}. \]

For \( (j, l) \in \Gamma(m) \) and an integer \( t \geq 0 \), we define the elements \( \mathcal{K}_{(j,l)}^{\pm}(m) \) and \( \mathcal{T}_{(j,l),t}^{\pm}(m) \) of \( \mathcal{S}_{(n,r)}(m) \) by

\[ \mathcal{K}_{(j,l)}^{\pm}(m) = \left\{ \begin{array}{ll} q^{\pm \mu_j^{(l)}} m_{\mu}, & \text{if } \mu_j^{(l)} \neq 0, \\ 0 & \text{if } \mu_j^{(l)} = 0, \end{array} \right. \]

\[ \mathcal{T}_{(j,l),t}^{\pm}(m) = \left\{ \begin{array}{ll} q^{-t+1} m_{\mu} \Phi_t^{\pm}(x_1, x_2, \ldots, x_{n+1}, \ldots, x_{n+1}, -\mu_j^{(l)} + 1) & \text{if } \mu_j^{(l)} \neq 0, \\ 0 & \text{if } \mu_j^{(l)} = 0, \end{array} \right. \]

for each \( \mu \in \Lambda_{n,r}(m) \).

It is clear that the definitions of \( \mathcal{K}_{(j,l)}^{\pm} \) are well-defined. For \( \mu \in \Lambda_{n,r}(m) \) and \( (j, l) \in \Gamma(m) \) such that \( \mu_j^{(l)} \neq 0 \), we see that \( \Phi_t^{\pm}(L_{N_{(j,l)}^{\mu}}, L_{N_{(j,l)}^{\mu}} - 1, \ldots, L_{N_{(j,l)}^{\mu}} - \mu_j^{(l)} + 1) \) commute with \( T_w \) for any \( w \in \mathcal{S}_\mu \) by Lemma 6.3 since \( \Phi_t^{\pm}(L_{N_{(j,l)}^{\mu}}, \ldots, L_{N_{(j,l)}^{\mu}} - \mu_j^{(l)} + 1) \) is a symmetric polynomials with variables \( L_{N_{(j,l)}^{\mu}}, L_{N_{(j,l)}^{\mu}} - 1, \ldots, L_{N_{(j,l)}^{\mu}} - \mu_j^{(l)} + 1 \). Thus, \( \Phi_t^{\pm}(L_{N_{(j,l)}^{\mu}}, \ldots, L_{N_{(j,l)}^{\mu}} - \mu_j^{(l)} + 1) \) commute with \( m_{\mu} \), and the definitions of \( \mathcal{T}_{(j,l),t}^{\pm} \) are well-defined.

The following lemma is immediate from definitions.

**Lemma 7.6.** For \( (i, k), (j, l) \in \Gamma(m) \) and \( s, t \geq 0 \), we have the following relations.
Lemma 7.7. For \((j, l) \in \Gamma(m)\), we have
\[
(K_{(j,l)}^\pm)^2 = 1 \pm (q - q^{-1})\mathcal{I}_{(j,l),0}^\pm.
\]

7.8. For \((i, k) \in \Gamma'(m)\) and an integer \(t \geq 0\), we define the element \(\tilde{K}^\pm_{(i,k)}\) and \(\mathcal{J}_{(j,l),t}\) of \(\mathcal{S}_{n,r}(m)\) by
\[
\tilde{K}^\pm_{(i,k)} = K^\pm_{(i,k)} K^\pm_{(i+1,k)}
\]
and
\[
\mathcal{J}_{(i,k),t} = \begin{cases} 
\mathcal{I}^+_{(i,k),0} - \mathcal{I}^-_{(i+1,k),0} + (q - q^{-1})\mathcal{I}^+_{(i,k),0}\mathcal{I}^-_{(i+1,k),0} & \text{if } t = 0, \\
q^{-t}\mathcal{I}^+_{(i,k),t} - q^t\mathcal{I}^-_{(i+1,k),t} - (q - q^{-1})\sum_{b=1}^{t-1} q^{-t+2b}\mathcal{I}^+_{(i,k),t-b}\mathcal{I}^-_{(i+1,k),b} & \text{if } t > 0.
\end{cases}
\]

By Lemma 7.7, we have the following corollary.

Corollary 7.9. For \((i, k) \in \Gamma'(m)\), we have
\[
\mathcal{J}_{(i,k),0} = \mathcal{I}^+_{(i,k),0} - (K^\pm_{(i,k)})^2\mathcal{I}^-_{(i+1,k),0}.
\]

7.10. For \(N \in \mathbb{Z}_{\geq 0}\) and \(\mu \in \mathbb{Z}_{>0}\), put
\[
[T; N, \mu]^+ = \begin{cases} 
1 + \sum_{h=1}^{\mu-1} q^h T_{N+1}T_{N+2} \ldots T_{N+h} & \text{if } N + \mu \leq n, \\
0 & \text{otherwise},
\end{cases}
\]
\[
[T; N, \mu]^-- = \begin{cases} 
1 + \sum_{h=1}^{\mu-1} q^h T_{N-1}T_{N-2} \ldots T_{N-h} & \text{if } n \geq N \geq \mu, \\
0 & \text{otherwise}.
\end{cases}
\]

For convenience, we put \([T; N, 0]^\pm = 0\) for any \(N \in \mathbb{Z}_{\geq 0}\).

For \(N, \mu \in \mathbb{Z}_{\geq 0}\) and \(d \in \mathbb{Z}_{>0}\), put
\[
[T; N, \mu]^+ = [T; N + (d-1), \mu - (d-1)]^+ \ldots [T; N + 1, \mu - 1]^+[T; N, \mu]^+, \\
[T; N, \mu]^-- = [T; N - (d-1), \mu - (d-1)]^- \ldots [T; N - 1, \mu - 1]^- [T; N, \mu]^--.
\]

We also put \([T; N, \mu]^+ = [T; N, \mu]^-- = 1\) for any \(N, \mu \in \mathbb{Z}_{\geq 0}\).
For $N \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, put

$$(T; N, d)^+ = \begin{cases} 1 + \sum_{h=1}^{d-1} q^h T_{N+d-h} T_{N+(h-1)} \ldots T_{N+2} T_{N+d-1} & \text{if } N + d \leq n, \\ 0 & \text{otherwise}, \end{cases}$$

$$(T; N, d)^- = \begin{cases} 1 + \sum_{h=1}^{d-1} q^h T_{N-d+h} T_{N-(h-1)} \ldots T_{N-d+2} T_{N-d+1} & \text{if } n \geq N \geq d, \\ 0 & \text{otherwise}. \end{cases}$$

We also put

$$(T; N, d)^\pm! = (T; N, d)^\pm (T; N, d-1)^\pm \ldots (T; N, 1)^\pm.$$

The following lemma follows from Lemma 6.3 immediately.

**Lemma 7.11.** For $N, \mu \in \mathbb{Z}_{\geq 0}$, we have the following.

(i) $L_i$ commute with $[T; N, \mu]^+$ unless $N + \mu \geq i \geq N + 1$.

(ii) $L_i$ commute with $[T; N, \mu]^-$ unless $N \geq i \geq N - \mu + 1$.

**Lemma 7.12.** We have the following.

(i) For $N, \mu \in \mathbb{Z}_{\geq 0}$ such that $N + \mu \leq n$ and $\mu \geq 3$, we have

$$\begin{align*}
(q^{\mu-2} T_{N+2} T_{N+3} \ldots T_{N+\mu-1})(q^{\mu-1} T_{N+1} T_{N+2} \ldots T_{N+\mu-1}) &= (q^{\mu-1} T_{N+1} T_{N+2} \ldots T_{N+\mu-1})(q^{\mu-2} T_{N+1} T_{N+2} \ldots T_{N+\mu-2}).
\end{align*}$$

(ii) For $N, \mu \in \mathbb{Z}_{\geq 0}$ such that $N \geq \mu \geq 3$, we have

$$\begin{align*}
(q^{\mu-2} T_{N-2} T_{N-3} \ldots T_{N-\mu+1})(q^{\mu-1} T_{N-1} T_{N-2} \ldots T_{N-\mu+1}) &= (q^{\mu-1} T_{N-1} T_{N-2} \ldots T_{N-\mu+1})(q^{\mu-2} T_{N-1} T_{N-2} \ldots T_{N-\mu+2}).
\end{align*}$$

(iii) For $N, \mu, c \in \mathbb{Z}_{\geq 0}$ such that $\mu \geq c \geq 1$, we have

$$\begin{align*}
[T; N+1, c]^+(q^{\mu} T_{N+1} T_{N+2} \ldots T_{N+\mu})(q^{\mu} T_{N+1} T_{N+2} \ldots T_{N+\mu}) &= (q^{\mu} T_{N+1} T_{N+2} \ldots T_{N+\mu})(q^{\mu} T_{N+1} T_{N+2} \ldots T_{N+\mu}), \\
[T; N-1, c]^-(q^{\mu} T_{N-1} T_{N-2} \ldots T_{N-\mu})(q^{\mu} T_{N-1} T_{N-2} \ldots T_{N-\mu}) &= (q^{\mu} T_{N-1} T_{N-2} \ldots T_{N-\mu})(q^{\mu} T_{N-1} T_{N-2} \ldots T_{N-\mu}).
\end{align*}$$

**Proof.** (i) and (ii) follows from the defining relations of $\mathcal{H}_{n,r}$. We can prove (iii) by the induction on $c$. \qed

**Lemma 7.13.** For $N, \mu \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{>0}$, we have

$$[T; d, N, \mu]^+ = \begin{cases} (T; N, d)^+ \left[ T; N, d-1 \right]^+ & \text{if } \mu \geq d, \\ \sum_{h=1}^{\mu-d} (q^h T_{N+d} T_{N+d+1} \ldots T_{N+d+h-1}) \left[ T; d, N, d-h \right]^+ & \text{if } \mu < d, \end{cases}$$

$$[T; d, N, \mu]^-$$. 
Applying the assumption of the induction, we have

\[ [T;N,d]^{-}\left(\frac{T;N,d-1}{d-1}\right)_{\mu-d} + \sum_{h=1}^{\mu-d} (q^{h}T_{N-d}T_{N-d-1} \cdots T_{N-d-h+1}) [T;N,d+h-1]^{-} \] if \( \mu \geq d, \)

\[ 0 \] if \( \mu < d. \)

**Proof.** In the case where \( \mu < d, \) we see that \( [T;N,d]^{\pm} = 0 \) from the definitions.

First, we prove that, if \( \mu > d, \)

\[ [T;N,d;\mu]^{+} = [T;N,d;\mu-1]^{+} + (T;N,d)^{+}(q^{\mu-d}T_{N+d}T_{N+d+1} \cdots T_{N+\mu-1}) [T;N,d-1;\mu]^{+} \]

by the induction on \( d. \) In the case where \( d = 1, \) it is clear by definitions. Assume that \( d > 1, \) then we have

\[ [T;N,d;\mu]^{+} = [T;N+(d-1),\mu-(d-1)]^{+} [T;N,d-1;\mu]^{+}. \]

Applying the assumption of the induction, we have

\[
[T;N,d;\mu]^{+} \\
= \left\{ [T;N+(d-1),\mu-d]^{+} + (q^{\mu-d}T_{N+d}T_{N+d+1} \cdots T_{N+\mu-1}) \right\} \\
\times \left\{ [T;N,d-1;\mu-1]^{+} + (T;N,d-1)^{+}(q^{\mu-d+1}T_{N+d-1}T_{N+d} \cdots T_{N+\mu-1}) [T;N,d-2;\mu-1]^{+} \right\}. \\
\]

Then, by using Lemma 7.11 and Lemma 7.12, we see that

\[
[T;N,d;\mu]^{+} \\
= [T;N+d-1,\mu-d]^{+} [T;N,d-1;\mu-1]^{+} + (q^{\mu-d}T_{N+d}T_{N+d+1} \cdots T_{N+\mu-1}) [T;N,d-1;\mu-1]^{+} \\
+ (T;N,d-1)^{+}(q^{\mu-d+1}T_{N+d-1}T_{N+d} \cdots T_{N+\mu-1}) [T;N,d-2;\mu-1]^{+} \\
+ (T;N,d-1)^{+}(q^{\mu-d+1}T_{N+d-1}T_{N+d} \cdots T_{N+\mu-1})(q^{\mu-d}T_{N+d-1}T_{N+d} \cdots T_{N+\mu-2}) \\
\times [T;N,d-2;\mu-1]^{+}. \\
\]

Note that

\[
[T;N+d-2,\mu-d]^{+} + q^{\mu-d}T_{N+d-1}T_{N+d} \cdots T_{N+\mu-2} = [T;N+d-2,\mu-d+1]^{+} \\
\]

and \([T;N+d-2,\mu-d+1]^{+} [T;N,d-1;\mu-1]^{+} = [T;N,d-1;\mu-1]^{+}, \) we have

\[
[T;N,d;\mu]^{+} \\
= [T;N,d-1;\mu-1]^{+} \\
\times \left\{ [T;N,d-1;\mu-1]^{+} + (1 + (T;N,d-1)^{+}(qT_{N+d-1}))(q^{\mu-d}T_{N+d}T_{N+d+1} \cdots T_{N+\mu-1}) [T;N,d-1;\mu-1]^{+} \right\}. \\
\]
By definition, we see that $1 + (T; N, d - 1)^+(qT_{N+d-1}) = (T; N, d)^+$. Thus, we have (7.13.1).

Next, we prove that

$$[T; N, d]^+ = (T; N, d)^+ [T; N, d-1]^+$$

by the induction on $d$. In the case where $d = 1$, it is clear from definitions. Assume that $d > 1$. Note that $[T; N, d]^+ = [T; N, d-1]^+$, by (7.13.1), we have

$$[T; N, d] = [T; N, d-1] + (T; N, d - 1)^+ (qT_{N+d-1}) [T; N, d-1]^+$$

$$= (1 + (T; N, d - 1)^+ (qT_{N+d-1}) [T; N,d-1]^+$$

$$= (T; N, d)^+ [T; N, d-1]^+.$$

Next we prove that, if $\mu \geq d$,

$$[T; N, \mu]^+ = (T; N, d)^+ \left( [T; N, d-1]^+ + \sum_{h=1}^{\mu-d} (q^h T_{N+d}T_{N+d+1} \ldots T_{N+d+h-1}) [T; N, d-1]^+ \right)$$

by the induction on $\mu - d$. In the case where $\mu = d$, it is just (7.13.2). Assume that $\mu > d$. By applying the assumption of the induction to the right-hand side of (7.13.1), we have (7.13.3).

It is similar for $[T; N, \mu]^-$. 

We have the following corollary which will be used in Theorem 8.1 to consider the divided powers in cyclotomic $q$-Schur algebras.

**Corollary 7.14.** For $N, \mu \in \mathbb{Z}_{\geq 0}$ and $d \in \mathbb{Z}_{> 0}$, there exist the elements $\mathfrak{H}^\pm (N, \mu, d) \in \mathcal{H}_{n,r}$ such that

$$[T; N, \mu] = (T; N, d)^+! \mathfrak{H}^\pm (N, \mu, d).$$

**Proof.** Note that $T_{N+d}T_{N+d+1} \ldots T_{N+d+h-1}$ (resp. $T_{N-d}T_{N-d-1} \ldots T_{N-d-h+1}$) commute with $(T; N, d - 1)^+$ (resp. $(T; N, d - 1)^-$), then we can prove the corollary by the induction on $d$ using Lemma 7.13. 

**7.15.** For $(i, k) \in I'(\mathbf{m})$, we define the elements $\mathcal{X}^+_{(i,k),0}$ and $\mathcal{X}^-_{(i,k),0}$ of $\mathcal{S}_{n,r}(\mathbf{m})$ by

$$\mathcal{X}^+_{(i,k),0}(m_\mu) = q^{-\mu_0} \alpha_{(i,k)}(m_\mu + \alpha_{(i,k)} [T; N, \mu]^{(k)}),$$

$$\mathcal{X}^-_{(i,k),0}(m_\mu) = q^{-\mu_k} \alpha_{(i,k)} h^\mu_{(i,k)} [T; N, \mu]^{(k)}.$$
for each $\mu \in \Lambda_{n,r}(m)$, where we put $\mu_{m_k+1}^{(k)} = \mu_1^{(k+1)}$ if $i = m_k$, and

$$h_{m_k - (i,k)}^\mu = \begin{cases} 1 & \text{if } i \neq m_k, \\ L_{N_{m_k-k}} - Q_k & \text{if } i = m_k. \end{cases}$$

Note that $m_{\mu \Perp \alpha_{(i,k)}} = 0$ if $\mu \Perp \alpha_{(i,k)} \notin \Lambda_{n,r}(m)$.

By [W1, Lemma 6.10], the definitions of $X_{(i,k),0}$ are well-defined. (The elements $X_{(i,k),0}$ are denoted by $\varphi_{X_{(i,k)}}$ in [W1].)

For $(i, k) \in \Gamma'(m)$ and an integer $t > 0$, we define the elements $X_{(i,k),t}$ of $\mathcal{S}_{n,r}(m)$ inductively by

\begin{equation}
X_{(i,k),t}^+ = \mathcal{I}_{(i,k),1}^+ X_{(i,k),t-1}^+ - X_{(i,k),t-1}^+ \mathcal{I}_{(i,k),1}^+; \\
X_{(i,k),t}^- = - (\mathcal{I}_{(i,k),1}^- X_{(i,k),t-1}^- - X_{(i,k),t-1}^- \mathcal{I}_{(i,k),1}^-).
\end{equation}

**Lemma 7.16.** For $(i, k) \in \Gamma'(m)$, $(j, l) \in \Gamma(m)$ and $t \geq 0$, we have

$$\mathcal{K}_{(j,l)} X_{(i,k),t}^\pm = q^{\pm \alpha_{(i,k)(j,l)}} X_{(i,k),t}^\pm.$$

**Proof.** We see the statement in the case where $t = 0$ from the definitions directly. Then we can prove the statement by the induction on $t$ using (7.15.1) together with Lemma 7.6.

We can describe the elements $X_{(i,k),t}^\pm$ of $\mathcal{S}_{n,r}(m)$ precisely as follows.

**Lemma 7.17.** For $(i, k) \in \Gamma'(m)$, $t \geq 0$ and $\mu \in \Lambda_{n,r}(m)$, we have the followings.

(i) $X_{(i,k),t}^+(m_\mu) = q^{-\mu_{i+1}^{(k)} + 1} m_\mu \alpha_{(i,k)} L_{N_{(i,k)}}^{(k)} [T; N_{(i,k)}^\mu, \mu_{i+1}^{(k)}]^+.$

(ii) $X_{(i,k),t}^-(m_\mu) = q^{-\mu_{i+1}^{(k)} + 1} m_\mu - \alpha_{(i,k)} L_{N_{(i,k)}}^{(k)} h_{m_k - (i,k)}^\mu [T; N_{(i,k)}^\mu, \mu_{i+1}^{(k)}]^-. $

**Proof.** We prove (i). We can easily show that $X_{(i,k),t}^+(m_\mu) = 0$ if $\mu_{i+1}^{(k)} = 0$ by the induction on $t$ using (7.15.1). Assume that $\mu_{i+1}^{(k)} \neq 0$. If $t = 0$, then it is just the definition of $X_{(i,k),0}^+$. We prove the equation for $t > 0$ by the induction on $t$. Note that $(\mu + \alpha_{(i,k)})^{(k)} = \mu_k^{(k)} + 1$ and $N_{(i,k)}^\mu + \alpha_{(i,k)} = N_{(i,k)}^\mu + 1$, by the assumption of the induction, we have

$$\mathcal{I}_{(i,k),1}^+ X_{(i,k),t-1}^+(m_\mu) = q^{-\mu_{i+1}^{(k)} + 1} m_\mu + \alpha_{(i,k)} L_{N_{(i,k)}}^{(k)} + L_{N_{(i,k)}}^{(k)} - \cdots - L_{N_{(i,k)}}^{(k)} / \mu_{i+1}^{(k)} + 1 \times L_{N_{(i,k)}}^{t-1} [T; N_{(i,k)}^\mu, \mu_{i+1}^{(k)}]^+.$$

On the other hand, we have

$$X_{(i,k),t-1}^+(m_\mu) = \delta_{\mu_{i+1}^{(k)} = 0} q^{-\mu_{i+1}^{(k)} + 1} m_\mu \alpha_{(i,k)} L_{N_{(i,k)}}^{t-1} [T; N_{(i,k)}^\mu, \mu_{i+1}^{(k)}]^+.$$
\[ \times (L_{N^\mu_{(i,k)}} + L_{N^\mu_{(i,k)} - 1} + \cdots + L_{N^\mu_{(i,k)} - \mu_i^{(k)} + 1}). \]

Thus, by (7.15.1) and Lemma 7.11, we have (i). (ii) is similar. \(\square\)

**Proposition 7.18.** For \((i, k), (j, l) \in \Gamma'(m)\) and \(s, t \geq 0\), we have the following relations.

(i) \([\mathcal{X}^+(i, k), t, \mathcal{X}^{(j, l)}_{(i, k), s}] = 0\) if \((j, l) \neq (i, k), (i \pm 1, k)\).

(ii) \(\mathcal{X}^+(i, k), t+1, \mathcal{X}^{(i, k), s} - q^{2} \mathcal{X}^+(i, k), s \mathcal{X}^{(i, k), t+1} = q^{2} \mathcal{X}^+(i, k), t \mathcal{X}^+(i, k), s+1 - \mathcal{X}^+(i, k), s+1 \mathcal{X}^+(i, k), t\).

(iii) \(\mathcal{X}^+(i, k), t+1, \mathcal{X}^{(i, k), s} - q^{-1} \mathcal{X}^+(i, k), s \mathcal{X}^+(i, k), t+1 = \mathcal{X}^+(i, k), t \mathcal{X}^+(i, k), s+1 - q \mathcal{X}^+(i, k), s+1 \mathcal{X}^+(i, k), t\).

Proof. (i) follows from Lemma 7.17 using Lemma 6.3.

We prove (ii). We may assume that \(t \geq s\) by multiplying \(-1\) to both sides if necessary. We prove

(7.18.1) \(\mathcal{X}^+(i, k), t+1, \mathcal{X}^{(i, k), s} - q^{2} \mathcal{X}^+(i, k), s \mathcal{X}^+(i, k), t+1 = q^{2} \mathcal{X}^+(i, k), t \mathcal{X}^+(i, k), s+1 - \mathcal{X}^+(i, k), s+1 \mathcal{X}^+(i, k), t\).

Put \(N = N^\mu_{(i, k)}\). By Lemma 7.17 together with Lemma 7.11, for \(\mu \in \Lambda_{n,r}(m)\), we have

(7.18.2) \(\mathcal{X}^+(i, k), t+1, \mathcal{X}^{(i, k), s}(m, \mu) = q^{-2}\mu_{i+1}^{(k)} + 3 m_{\mu+2\alpha_{(i, k)}} L_{N+1}^s T_{N+2}^{t+1} [T; N + 1, \mu_{i+1}^{(k)}].\)

Thus, we may assume that \(\mu_{i+1}^{(k)} \geq 2\) since \(m_{\mu+2\alpha_{(i, k)}} = 0\) if \(\mu_{i+1}^{(k)} < 2\). By the induction on \(\mu_{i+1}^{(k)}\), we can show that

(7.18.3) \(T_{N+1}[T; N + 1, \mu_{i+1}^{(k)} - 1]^{\mu_{i+1}^{(k)}} [T; N, \mu_{i+1}^{(k)}] = q[T; N + 1, \mu_{i+1}^{(k)} - 1]^{\mu_{i+1}^{(k)}} [T; N, \mu_{i+1}^{(k)}].\)

We also have, by Lemma 6.3,

(7.18.4) \(L_{N+1}^s T_{N+2}^{t+1} = (L_{N+1} L_{N+2})^s (T_{N+1} L_{N+1} T_{N+1}) L_{N+2}^{t-s} = T_{N+1} (L_{N+1} L_{N+2})^s L_{N+1} \{ L_{N+1}^{t-s-p} T_{N+1} + (q - q^{-1}) \sum_{p=1}^{t-s} L_{N+1}^{t-s-p} L_{N+2}^{p} \} = T_{N+1} L_{N+1}^{t-s} T_{N+1} + (q - q^{-1}) T_{N+1} \sum_{p=1}^{t-s} L_{N+1}^{t-s-p} L_{N+2}^{p}.\)
Thus, we have (7.18.1). Another case of (ii) is proven in a similar way. Similarly, we have
\begin{align*}
    & X_{(i,k),t+1}^+ X_{(i,k),s}^+(m_{\mu}) \\
    &= q^2 q^{-2\mu_{i+1}^{(k)}} m_{\mu+2\alpha_{(i,k)}} L_{N+1}^{l+1} L_{N+2}^s [T; N+1, \mu_{i+1}^{(k)} - 1] + [T; N, \mu_{i+1}^{(k)}] \\
    &+ q(q - q^{-1}) q^{-2\mu_{i+1}^{(k)}} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L_{N+1}^{l-p+1} L_{N+2}^{s+p} [T; N+1, \mu_{i+1}^{(k)} - 1] + [T; N, \mu_{i+1}^{(k)}] \\
    &= q^2 X_{(i,k),s}^+ X_{(i,k),t+1}^+(m_{\mu}) \\
    &+ q(q - q^{-1}) q^{-2\mu_{i+1}^{(k)}} m_{\mu+2\alpha_{(i,k)}} \sum_{p=1}^{t-s} L_{N+1}^{l-p+1} L_{N+2}^{s+p} [T; N+1, \mu_{i+1}^{(k)} - 1] + [T; N, \mu_{i+1}^{(k)}].
\end{align*}

Thus, we have (7.18.1). Another case of (ii) is proven in a similar way.

We prove (iii). Put $N = \Lambda_{(i,k)}^\mu$. In the case where $\mu_{i+1}^{(k)} = 0$, by Lemma 7.17 together with Lemma 7.11, we see that
\begin{align*}
    (X_{(i,k),t+1}^+ X_{(i+1,k),s}^+- q^{-1} X_{(i+1,k),s}^+ X_{(i,k),t+1}^+)(m_{\mu}) \\
    &= (X_{(i,k),t}^+ X_{(i+1,k),s+1}^+- q X_{(i+1,k),s+1}^+ X_{(i,k),t}^+)(m_{\mu}) \\
    &= q^{-\mu_{i+2}^{(k)}} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L_{N+1}^{s+t+1} [T; N, \mu_{i+2}^{(k)}].
\end{align*}

Assume that $\mu_{i+1}^{(k)} \neq 0$. By Lemma 7.17 together with Lemma 7.11, we have
\begin{align*}
    (X_{(i,k),t+1}^+ X_{(i+1,k),s}^+- q^{-1} X_{(i+1,k),s}^+ X_{(i,k),t+1}^+)(m_{\mu}) \\
    &= q^{-\mu_{i+1}^{(k)}-\mu_{i+2}^{(k)}} m_{\mu+\alpha_{(i,k)}+\alpha_{(i+1,k)}} L_{N+1}^{s+t+1} (q^{\mu_{i+1}^{(k)}} T_{N+1} T_{N+2} \ldots T_{N+\mu_{i+1}^{(k)}}) L_{N+\mu_{i+1}^{(k)}+1}^s \\
    &\times [T; N+\mu_{i+1}^{(k)}, \mu_{i+2}^{(k)}].
\end{align*}
By using Lemma 6.3 and (6.4.2), this equation implies Proposition 7.19. Another case of (iii) is proven in a similar way.

(7.18.7)
\[
(X^+_{(i,k),t}X^+_{(i+1,k),s+1} - qX^+_{(i+1,k),s+1}X^+_{(i,k),t})(m_\mu)
= -(q - q^{-1})q^{-\mu_{i+1}^{(k)}-\mu_{i+2}^{(k)}+2}m_{\mu + \alpha_{(i+1,k)} + \alpha_{(i+1,k)}}L_{N+1}^t L_{s+1}^{s+1} + N + \mu_{i+1}^{(k)} + 1
\times [T; N, \mu_{i+1}^{(k)} + 1] + [T; N + \mu_{i+1}^{(k)} + 1] + 1 + \mu_{i+1}^{(k)} + 2
+ q^{-\mu_{i+1}^{(k)}-\mu_{i+2}^{(k)}+1}m_{\mu + \alpha_{(i+1,k)} + \alpha_{(i+1,k)}}L_{N+1}^t (q^{\mu_{i+1}^{(k)}}T_{N+1}T_{N+2} \ldots T_{N + \mu_{i+1}^{(k)}}) L_{N + \mu_{i+1}^{(k)}}^{s+1} + N + \mu_{i+1}^{(k)} + 1
\times [T; N + \mu_{i+1}^{(k)} + 1] + 1.
\]

By the induction on \(\mu_{i+1}^{(k)}\) using Lemma 6.3, we can prove that
\[
(T_{N+1}T_{N+2} \ldots T_{N + \mu_{i+1}^{(k)}}) L_{N + \mu_{i+1}^{(k)}} + 1
= L_{N+1}(T_{N+1}T_{N+2} \ldots T_{N + \mu_{i+1}^{(k)}}) + \delta_{(\mu_{i+1}^{(k)} \geq 2)}(q - q^{-1})L_{N+2}(T_{N+2}T_{N+3} \ldots T_{N + \mu_{i+1}^{(k)}})
+ (q - q^{-1}) \sum_{p=1}^{\mu_{i+1}^{(k)}-2} (T_{N+1}T_{N+2} \ldots T_{N+p})L_{N+p+2}(T_{N+p+2}T_{N+p+3} \ldots T_{N + \mu_{i+1}^{(k)}})
+ (q - q^{-1})(T_{N+1}T_{N+2} \ldots T_{N + \mu_{i+1}^{(k)} - 1})L_{N + \mu_{i+1}^{(k)} + 1}.
\]

By using Lemma 6.3 and (6.4.2), this equation implies
\[
(7.18.8)
\]
\[
m_{\mu + \alpha_{(i,k)} + \alpha_{(i+1,k)}} L_{N+1}^t (q^{\mu_{i+1}^{(k)}}T_{N+1}T_{N+2} \ldots T_{N + \mu_{i+1}^{(k)}}) L_{N + \mu_{i+1}^{(k)}}^{s+1} + N + \mu_{i+1}^{(k)} + 1
= m_{\mu + \alpha_{(i,k)} + \alpha_{(i+1,k)}} L_{N+1}^t (q^{\mu_{i+1}^{(k)}}T_{N+1}T_{N+2} \ldots T_{N + \mu_{i+1}^{(k)}}) + q(q - q^{-1})m_{\mu + \alpha_{(i,k)} + \alpha_{(i+1,k)}} L_{N+1}^t [T; N, \mu_{i+1}^{(k)} + 1] + L_{N + \mu_{i+1}^{(k)} + 1}.
\]

Thus, (7.18.7) and (7.18.8) imply
\[
(7.18.9)
\]
\[
(X^+_{(i,k),t}X^+_{(i+1,k),s+1} - qX^+_{(i+1,k),s+1}X^+_{(i,k),t})(m_\mu)
= q^{-\mu_{i+1}^{(k)}-\mu_{i+2}^{(k)}+1}m_{\mu + \alpha_{(i,k)} + \alpha_{(i+1,k)}} L_{N+1}^t (q^{\mu_{i+1}^{(k)}}T_{N+1}T_{N+2} \ldots T_{N + \mu_{i+1}^{(k)}}) L_{N + \mu_{i+1}^{(k)}}^{s+1} + N + \mu_{i+1}^{(k)} + 1
\times [T; N + \mu_{i+1}^{(k)} + 1] + 1.
\]

By (7.18.5), (7.18.6) and (7.18.9), we have
\[
X^+_{(i,k),t+1}X^+_{(i+1,k),s} - q^{-1}X^+_{(i+1,k),s}X^+_{(i,k),t+1} = X^+_{(i,k),t}X^+_{(i+1,k),s+1} - qX^+_{(i+1,k),s+1}X^+_{(i,k),t}.
\]

Another case of (iii) is proven in a similar way. □

**Proposition 7.19.** For \((i, k) \in \Gamma'(m)\) and \(s, t, u \geq 0\), we have the followings.
\( \mathcal{X}^\pm_{(i \pm 1, k), u}(X^\pm_{(i, k), s}X^\pm_{(i, k), t} + X^\pm_{(i, k), t}X^\pm_{(i, k), s}) + (X^\pm_{(i, k), s}X^\pm_{(i, k), t} + X^\pm_{(i, k), t}X^\pm_{(i, k), s}) \mathcal{X}^\pm_{(i \pm 1, k), u} \)

\( = (q + q^{-1})(X^\pm_{(i, k), s}X^\pm_{(i \pm 1, k), u}X^\pm_{(i, k), t} + X^\pm_{(i, k), t}X^\pm_{(i \pm 1, k), u}X^\pm_{(i, k), s}). \)

\( \mathcal{X}^\pm_{(i \pm 1, k), u}(X^-_{(i, k), s}X^-_{(i, k), t} + X^-_{(i, k), t}X^-_{(i, k), s}) + (X^-_{(i, k), s}X^-_{(i, k), t} + X^-_{(i, k), t}X^-_{(i, k), s}) \mathcal{X}^\pm_{(i \pm 1, k), u} \)

\( = (q + q^{-1})(X^-_{(i, k), s}X^-_{(i \pm 1, k), u}X^-_{(i, k), t} + X^-_{(i, k), t}X^-_{(i \pm 1, k), u}X^-_{(i, k), s}). \)

**Proof.** By Lemma 7.17 together with Lemma 7.11, we have

\[
(7.19.1)
\]

\[
(\mathcal{X}^\pm_{(i \pm 1, k), u}X^\pm_{(i, k), s}X^\pm_{(i, k), t} - qX^\pm_{(i, k), s}X^\pm_{(i \pm 1, k), u}X^\pm_{(i, k), t})(m_\mu)
\]

\[
= -\delta_{(\mu_{i+1}^{(k)})}q^{-\mu_{i+1}^{(k)}+2}m_{\mu + 2\alpha_{(i, k)} + \alpha_{(i+1, k)}}L_{N+1}^{T}L_{N+2}^{s}[T; N + 1, \mu_{i+1}^{(k)}] + \delta_{(\mu_{i+1}^{(k)}) \geq 2}q^{-2\mu_{i+1}^{(k)} - \mu_{i+2}^{(k)} + 4}m_{\mu + 2\alpha_{(i, k)} + \alpha_{(i+1, k)}}L_{N+1}^{t}L_{N+2}^{s}
\]

\[
\times (q^{\mu_{i+1}^{(k)}-1}T_{N+2}T_{N+3} \ldots T_{N+\mu_{i+1}^{(k)}})[T; N, \mu_{i+1}^{(k)}] + L_{N+\mu_{i+1}^{(k)}+1}^{u}[T; N + \mu_{i+1}^{(k)}, \mu_{i+1}^{(k)}].
\]

and

\[
(\mathcal{X}^\pm_{(i, k), s}X^\pm_{(i, k), t}X^\pm_{(i \pm 1, k), u} - q^{-1}X^\pm_{(i, k), s}X^\pm_{(i \pm 1, k), u}X^\pm_{(i, k), t})(m_\mu)
\]

\[
= q^{-2\mu_{i+1}^{(k)} - \mu_{i+2}^{(k)} + 2}m_{\mu + 2\alpha_{(i, k)} + \alpha_{(i+1, k)}}L_{N+1}^{t}L_{N+2}^{s}
\]

\[
\times [T; N + 1, \mu_{i+1}^{(k)}] + (q^{\mu_{i+1}^{(k)}-1}T_{N+1}T_{N+2} \ldots T_{N+\mu_{i+1}^{(k)}})[T; N + \mu_{i+1}^{(k)}, \mu_{i+1}^{(k)}].
\]

Applying Lemma 7.12 (iii), we have

\[
(7.19.2)
\]

\[
(\mathcal{X}^\pm_{(i, k), s}X^\pm_{(i, k), u}X^\pm_{(i \pm 1, k), t} - q^{-1}X^\pm_{(i, k), s}X^\pm_{(i \pm 1, k), u}X^\pm_{(i, k), t})(m_\mu)
\]

\[
= \delta_{(\mu_{i+1}^{(k)})}q^{-\mu_{i+1}^{(k)}+1}m_{\mu + 2\alpha_{(i, k)} + \alpha_{(i+1, k)}}L_{N+1}^{t}L_{N+2}^{s}L_{N+1}^{u}[T; N + \mu_{i+1}^{(k)}, \mu_{i+1}^{(k)}] + \delta_{(\mu_{i+1}^{(k)}) \geq 2}q^{-2\mu_{i+1}^{(k)} - \mu_{i+2}^{(k)} + 3}m_{\mu + 2\alpha_{(i, k)} + \alpha_{(i+1, k)}}L_{N+1}^{t}L_{N+2}^{s}L_{N+1}^{u}
\]

\[
\times (q^{\mu_{i+1}^{(k)}-1}T_{N+2}T_{N+3} \ldots T_{N+\mu_{i+1}^{(k)}})[T; N + \mu_{i+1}^{(k)}, \mu_{i+1}^{(k)}].
\]

We see that

\[
m_{\mu + 2\alpha_{(i, k)} + \alpha_{(i+1, k)}}(L_{N+1}^{t}L_{N+2}^{s}L_{N+1}^{u})T_{N+1}
\]

\[
= m_{\mu + 2\alpha_{(i, k)} + \alpha_{(i+1, k)}}T_{N+1}(L_{N+1}^{t}L_{N+2}^{s}L_{N+1}^{u})
\]

\[
= qm_{\mu + 2\alpha_{(i, k)} + \alpha_{(i+1, k)}}(L_{N+1}^{t}L_{N+2}^{s}L_{N+1}^{u})
\]
by Lemma 6.3 and (6.4.2). Then (7.19.1) and (7.19.2) imply
\[
\begin{align*}
\mathcal{X}^+_{(i+1,k),u}(\mathcal{X}^+_{(i,k),s} + \mathcal{X}^+_{(i,k),t} + \mathcal{X}^+_{(i,k),s}) + (\mathcal{X}^+_{(i,k),s} + \mathcal{X}^+_{(i,k),t}) &= (q + q^{-1})(\mathcal{X}^+_{(i,k),s} + \mathcal{X}^+_{(i+1,k),u} + \mathcal{X}^+_{(i,k),t})\mathcal{X}^+_{(i+1,k),u}.
\end{align*}
\]

The other cases of the proposition are proven in a similar way. □

By direct calculations, we have the following lemma.

Lemma 7.20. For \((i,k) \in \Gamma'(\mathbf{m}), (j,l) \in \Gamma'(\mathbf{m}), t \geq 0,\) we have the followings.

(i) \(q^{\pm a_{(i,k)(j,l)}}\mathcal{T}^\pm_{(j,l),0}(\mathcal{X}^+_{(i,k),t}) = q^{\pm a_{(i,k)(j,l)}}\mathcal{T}^\pm_{(j,l),t}\mathcal{X}^+_{(i,k),t}\).

(ii) \(q^{\pm a_{(i,k)(j,l)}}\mathcal{T}^\pm_{(j,l),0}(\mathcal{X}^-_{(i,k),t}) = q^{\pm a_{(i,k)(j,l)}}\mathcal{T}^\pm_{(j,l),t}\mathcal{X}^-_{(i,k),t}\).

We also have the following proposition.

Proposition 7.21. For \((i,k) \in \Gamma'(\mathbf{m}), (j,l) \in \Gamma'(\mathbf{m}), s, t \geq 0,\) we have the followings.

(i) \([\mathcal{T}^\sigma_{(j,l),s+t1}(\mathcal{X}^+_{(i,k),t})] = q^{\pm a_{(i,k)(j,l)}}\mathcal{T}^\pm_{(j,l),s+t1}(\mathcal{X}^+_{(i,k),t}) - q^{\pm a_{(i,k)(j,l)}}\mathcal{T}^\pm_{(j,l),t+1}\mathcal{X}^+_{(i,k),s+t1}.

(ii) \([\mathcal{T}^\sigma_{(j,l),s+t1}(\mathcal{X}^-_{(i,k),t})] = q^{\pm a_{(i,k)(j,l)}}\mathcal{T}^\pm_{(j,l),s+t1}(\mathcal{X}^-_{(i,k),t}) - q^{\pm a_{(i,k)(j,l)}}\mathcal{T}^\pm_{(j,l),t+1}\mathcal{X}^-_{(i,k),s+t1}.

Proof. By Lemma 7.17 together with Lemma 6.3, we see that
\([\mathcal{T}^\sigma_{(j,l),s+t1}(\mathcal{X}^+_{(i,k),t})] = 0\) if \((j,l) \neq (i,k), (i+1,k),\)

where \(\sigma, \sigma' \in \{+, -\}.\) Thus, it is enough to prove the cases where \((j,l) = (i,k)\) or \((j,l) = (i+1,k)\). We prove

(7.21.1) \(q^{\mathcal{T}^\pm_{(j,l),s+t1}(\mathcal{X}^+_{(i,k),t})} = q^{\mathcal{T}^\pm_{(j,l),s+t1}(\mathcal{X}^+_{(i,k),t})} - q^{-1}\mathcal{X}^+_{(i,k),t+1}\mathcal{T}^\pm_{(j,l),s+t1}.

For \(\mu \in \Lambda_{n,r}(\mathbf{m}),\) put \(N = N_{(i,k)}^\mu.\) Then, by Lemma 7.17 together with Lemma 7.11, we have

\[
(\mathcal{T}^\pm_{(i,k),s+t1}(\mathcal{X}^+_{(i,k),t}) - \mathcal{X}^+_{(i,k),t} + \mathcal{T}^\pm_{(i,k),s+t1})(m_{\mu}) = q^{s-\mu_{i+1}^{(k)}+1}m_{\mu + \alpha_{(i,k)}}
\]
\[
= \left(\Phi^+_{s+1}(L_{N+1}, L_N, L_{N-1}, \ldots, L_{N-\mu_{i+1}^{(k)}+1}) - \Phi^+_{s+1}(L_{N}, L_{N-1}, \ldots, L_{N-\mu_{i}^{(k)}+1})\right)
\]
\[
\times L_{N+1}^\mu[T; N, \mu_{i+1}^{(k)}]^+.
\]

By (7.3.2), we have

\[
(\mathcal{T}^\pm_{(i,k),s+t1}(\mathcal{X}^+_{(i,k),t}) - \mathcal{X}^+_{(i,k),t} + \mathcal{T}^\pm_{(i,k),s+t1})(m_{\mu}) = q^{s-\mu_{i+1}^{(k)}+1}m_{\mu + \alpha_{(i,k)}}
\]
\[
= L_{N+1}^\mu(\Phi^+_{s}(L_{N+1}, L_N, \ldots, L_{N-\mu_{i+1}^{(k)}+1}) - q^{-1}\Phi^+_{s}(L_{N}, L_{N-1}, \ldots, L_{N-\mu_{i}^{(k)}+1}))
\]
\[
\times L_{N+1}^\mu[T; N, \mu_{i+1}^{(k)}]^+.
\]
Proposition 7.22. For $(i, k), (j, l) \in \Gamma'(m)$ such that $(i, k) \neq (j, l)$ and $s, t \geq 0$, we have

\[ [\mathcal{X}^+_{(i,k), t}, \mathcal{X}^-_{(j,l), s}] = 0. \]

Proof. By Lemma 7.17, for $\mu \in A_{n,r}(m)$, we have

\begin{align*}
\mathcal{X}^+_{(i,k), t} \mathcal{X}^-_{(j,l), s} (m_\mu) &= q^{(s-1)\mu -(k)_{i+1} + 1} m_{\mu + \alpha_{i,k}} \\
&\times \left\{ q \Phi_s^+ (L_{N+1}, L_N, \ldots, L_{N-\mu_{i+1}^{(k)}} L_{N+1}^{(k)} [T; N, \mu_{i+1}^{(k)}] + \\
- q^{-1} L_{N+1}^{(k)} [T; N, \mu_{i+1}^{(k)}] \Phi_s^+ (L_N, L_{N-1}, \ldots, L_{N-\mu_{i+1}^{(k)}}) \right\} \\
&= (q \mathcal{I}_{(i,k), s,t+1}^+ \mathcal{X}^+_{(i,k), t+1} - q^{-1} \mathcal{X}^+_{(i,k), t+1} \mathcal{I}_{(i,k), s}) (m_\mu).
\end{align*}

Now we proved (7.21.1). Other cases are proven in a similar way. \qed

Since $(i, k) \neq (j, l)$, we have

\begin{align*}
N_{i,k}^\mu &= N_{i,k}^{\mu - \alpha_{j,l}} , \\
N_{j,l}^\mu &= N_{j,l}^{\mu + \alpha_{i,k}} , \\
(\mu - \alpha_{j,l})_{i+1} &= \begin{cases} \\
\mu_{i+1}^{(k)} & \text{if } (j, l) \neq (i+1, k), \\
\mu_{i+1}^{(k)} - 1 & \text{if } (j, l) = (i+1, k), \\
\end{cases} \\
(\mu + \alpha_{i,k})_j &= \begin{cases} \\
\mu_{j}^{(l)} & \text{if } (j, l) \neq (i+1, k), \\
\mu_{j}^{(l)} - 1 & \text{if } (j, l) = (i+1, k). \\
\end{cases} \\
h_{-j,l}^\mu &= h_{-j,l}^{\mu + \alpha_{i,k}} = \begin{cases} \\
1 & \text{if } j \neq m_j , \\
L_{N}^{\mu} - Q_t & \text{if } j = m_t. \\
\end{cases}
\end{align*}

Then, by Lemma 7.11, we have

\begin{align*}
[T; N_{i,k}^{\mu - \alpha_{j,l}}, (\mu - \alpha_{j,l})_{i+1}^{(k)} + L_{N_{j,l}^\mu} h_{-j,l}^\mu = L_{N_{j,l}^\mu} h_{-j,l}^\mu [T; N_{i,k}^{\mu - \alpha_{j,l}}, (\mu - \alpha_{j,l})_{i+1}^{(k)}] + \\
\end{align*}
and

\[ [T; N_{(i,k)}^{\mu+\alpha(i,k)}], (\mu + \alpha(i,k))_j^{(l)}] - L_{N_{(i,k)}^{\mu}}^{t} = L_{N_{(i,k)}^{\mu}}^{t+1} + [T; N_{(i,k)}^{\mu+\alpha(i,k)}], (\mu + \alpha(i,k))_j^{(l)}]. \]

Thus, in order to prove the proposition, it is enough to show that

\[ (7.22.1) \]

\[ [T; N_{(i,k)}^{\mu-\alpha(j,l)}, (\mu - \alpha(j,l))_i^{(k+1)}] + [T; N_{(i,k)}^{\mu}], (\mu)_{j}^{(l)}] - [T; N_{(i,k)}^{\mu+\alpha(j,l)}, (\mu + \alpha(j,l))_{i+1}^{(k)}]. \]

If \((j, l) \neq (i+1, k)\), we see easily that (7.22.1) holds since the product is commutative in each side. In the case where \((j, l) = (i+1, k)\), we can prove that (7.22.1) by the induction on \(\mu_{i+1}^{(k)}\). Now we proved the proposition. \(\square\)

**Remark 7.23.** There is an error in the proof of [W1, Proposition 6.11 (i)] (see the case where \((j, l) = (i+1, k)\)). The above proof also gives a fixed proof of [W1, Proposition 6.11 (i)] as a special case.

We prepare some technical lemmas.

**Lemma 7.24.** For \(\mu \in A_n, r(m)\) and \((i, k) \in \Gamma_m\), we have the followings.

1. For \(t \geq 0\) and \(1 \leq p \leq \mu_{i}^{(k)}\), we have

\[ m_\mu L_{N_{(i,k)}^{\mu}}^t [T; N_{(i,k)}^{\mu}], p^- = q^{2p-2}m_\mu \Phi_1^+(L_{N_{(i,k)}^{\mu}}, L_{N_{(i,k)}^{\mu}}^1, \ldots, L_{N_{(i,k)}^{\mu}}^p). \]

2. For \(t \geq 0\) and \(1 \leq p \leq \mu_{i+1}^{(k)}\), we have

\[ m_\mu L_{N_{(i,k)}^{\mu}}^t [T; N_{(i,k)}^{\mu}], p^+ = m_\mu \Phi_1^-(L_{N_{(i,k)}^{\mu}}^1, L_{N_{(i,k)}^{\mu}}^2, \ldots, L_{N_{(i,k)}^{\mu}}^p). \]

**Proof.** In the case where \(t = 0\), we have (i) and (ii) from (6.4.2).

We prove (i) for \(t > 0\). Put \(N = N_{(i,k)}^{\mu}\). For \(1 \leq h \leq \mu_{i}^{(k)} - 1\), by the induction on \(h\) together with Lemma 6.3 and (6.4.2), we can show that

\[ (7.24.1) \]

\[ m_\mu L_{N}^t(T_{N-1}T_{N-2} \ldots T_{N-h}) \]

\[ = m_\mu \{ (q - q^{-1})q^{h-1}L_{N}^t + \sum_{s=2}^{h} (q - q^{-1})q^{h-s}L_{N}^{t-1}(T_{N-1}T_{N-2} \ldots T_{N-s+1})L_{N-s+1} \]

\[ + L_{N}^{t-1}(T_{N-1}T_{N-2} \ldots T_{N-h})L_{N-h} \}. \]

We prove that

\[ (7.24.2) \]

\[ m_\mu L_{N}^t(T_{N-1}T_{N-2} \ldots T_{N-h}) \]

\[ = m_\mu (q^h \Phi_1^+(L_N, L_{N-1}, \ldots, L_{N-h}) - q^{h-2} \Phi_1^+(L_N, L_{N-1}, \ldots, L_{N-h+1})). \]
by the induction on \( t \). In the case where \( t = 1 \), by (7.24.1) together with (6.4.2), we have

\[
m_\mu L_N(T_{N-1}T_{N-2}\ldots T_{N-h}) = m_\mu\left\{(q - q^{-1})q^{h-1}L_N + \sum_{s=2}^{h}(q - q^{-1})q^{h-s}q^{s-1}L_{N-s+1} + q^hL_{N-h}\right \}
\]

\[
= m_\mu\left\{q^h \Phi_t^+(L_N, L_{N-1}, \ldots, L_{N-h}) - q^{-2}\Phi_{t-1}^+(L_N, L_{N-1}, \ldots, L_{N-h+1})\right\}.
\]

Assume that \( t > 1 \). Applying the assumption of the induction to (7.24.1), we have

\[
m_\mu L_N^t(T_{N-1}T_{N-2}\ldots T_{N-h}) = m_\mu\left\{(q - q^{-1})q^{h-1}L_N^t \right.
\]

\[
+ \sum_{s=2}^{h}(q - q^{-1})q^{h-s}(q^{s-1}\Phi_{t-1}^+(L_N, L_{N-1}, \ldots, L_{N-s+1})
\]

\[
- q^{s-3}\Phi_{t-1}^+(L_N, L_{N-1}, \ldots, L_{N-s+2})L_{N-s+1}
\]

\[
+ (q^h \Phi_{t-1}^+(L_N, L_{N-1}, \ldots, L_{N-h}) - q^{-2}\Phi_{t-1}^+(L_N, L_{N-1}, \ldots, L_{N-h+1})L_{N-h})\}.
\]

Put \( s' = s - 1 \), we have

\[
m_\mu L_N^t(T_{N-1}T_{N-2}\ldots T_{N-h}) = m_\mu\left\{q^h\left(L_N^t + \sum_{s=1}^{h}\Phi_{t-1}^+(L_N, L_{N-1}, \ldots, L_{N-s'})L_{N-s'}
\right.
\]

\[
- q^{-2}\Phi_{t-1}^+(L_N, L_{N-1}, \ldots, L_{N-s'+1})L_{N-s'}\}
\]

\[
- q^{h-2}\left(L_N^t + \sum_{s=1}^{h-1}\Phi_{t-1}^+(L_N, L_{N-1}, \ldots, L_{N-s'})L_{N-s'}
\right.
\]

\[
- q^{-2}\Phi_{t-1}^+(L_N, L_{N-1}, \ldots, L_{N-s'+1})L_{N-s'}\}
\]
\[ = q^{2p-2}m_\mu \Phi_t^+ (L_N, L_{N-1}, \ldots, L_{N-p}). \]

Now we obtained (i).

For \( t > 0 \) and \( 1 \leq h \leq \mu(i)_{i+1} - 1 \), by the induction on \( h \) using Lemma 6.3 and (6.4.2), we can show that

\[
m_\mu L_{N+1}^t (T_{N+1}T_{N+2} \cdots T_{N+h})
= q^{-h}m_\mu L_{N+1}^{t-1} \left\{ (1 - q^2)(1 + \sum_{s=1}^{h-1} q^s T_{N+1}T_{N+2} \cdots T_{N+s}) \right.
+ q^h T_{N+1}T_{N+2} \cdots T_{N+h} \right\} L_{N+h+1}.
\]

We prove (ii) by the induction on \( t \). We have already proved (ii) in the case where \( t = 0 \).

Assume that \( t > 0 \). By (7.24.3), we have

\[
m_\mu L_{N+1}^t [T; N, p]^+
= m_\mu L_{N+1}^{t-1} \left\{ \sum_{h=1}^{p-1} (T; N, h)^+ L_{N+h} - q^2 \sum_{h=1}^{p-1} (T; N, h)^+ L_{N+h+1} \right\}.
\]

Applying the assumption of the induction, we have

\[
m_\mu L_{N+1}^t [T; N, p]^+
= m_\mu \left\{ \sum_{h=1}^{p-1} \Phi_t^{i-1} (L_{N+1}, L_{N+2}, \ldots, L_{N+h}) L_{N+h}
- q^2 \sum_{h=1}^{p-1} \Phi_t^{i-1} (L_{N+1}, L_{N+2}, \ldots, L_{N+h}) L_{N+h+1} \right\}.
\]

Applying (7.3.1), we have

\[
m_\mu L_{N+1}^t [T; N, p]^+
= m_\mu \Phi_t^-(L_{N+1}, L_{N+2}, \ldots, L_{N+p}).
\]

**Lemma 7.25.** For \( \mu \in A_{n,r}(m) \) and \( (i, k) \in \Gamma'(m) \), put \( N = N_{(i,k)}^\mu \). Then we have the followings.

\[ \square \]
(i) If $\mu_i^{(k)} \neq 0$, we have

$$m_\mu L_N^t[T; N - 1, \mu_i^{(k)} + 1]^+ L_N[T; N, \mu_i^{(k)}]^-
= q^{2\mu_i^{(k)} - 2} m_\mu \Phi_t^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1})
+ \delta_{(\mu_i^{(k)} \neq 0)} m_\mu L_N^t([T; N + 1, \mu_i^{(k)} + 1]^- - 1)[T; N, \mu_i^{(k)}]^+]$$

(ii) If $\mu_i^{(k)} \neq 0$, we have

$$m_\mu L_N^t[T; N - 1, \mu_i^{(k)} + 1]^+ L_N[T; N, \mu_i^{(k)}]^-
= q^{2\mu_i^{(k)} - 2} m_\mu \Phi_t^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1})
- \delta_{(\mu_i^{(k)} \neq 0)} (q - q^{-1}) q^{2\mu_i^{(k)} - 1} m_\mu \Phi_t^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1})
\times \Phi^-_t(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_i^{(k)}})
+ m_\mu L_N^t L_{N+1}([T; N - 1, \mu_i^{(k)} + 1]^- - 1)[T; N, \mu_i^{(k)}]^+]$$

(iii) If $\mu_i^{(k)} \neq 0$, we have

$$m_\mu [T; N + 1, \mu_i^{(k)} + 1]^+ L_{N+1}^t[T; N, \mu_i^{(k)}]^-
= (1 + \delta_{(t \neq 0)} (q^{2\mu_i^{(k)}} - 1)) m_\mu \Phi_t^{-}(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_i^{(k)}})
+ \delta_{(\mu_i^{(k)} \neq 0)} (q - q^{-1}) \sum_{b=1}^{t-1} m_\mu q^{2\mu_i^{(k)} - 1} \Phi_{t-b}^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1})
\times \Phi_b^{-}(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_i^{(k)}})
+ m_\mu L_N^t ([T; N + 1, \mu_i^{(k)} + 1]^- - 1)[T; N, \mu_i^{(k)}]^+]$$

(iv) If $\mu_i^{(k)} \neq 0$, we have

$$m_\mu L_{N+1}^t[T; N + 1, \mu_i^{(k)} + 1]^+ L_{N+1}^t[T; N, \mu_i^{(k)}]^-
= (1 + \delta_{(t \neq 0)} (q^{2\mu_i^{(k)}} - 1)) m_\mu \Phi_{t+1}^{-}(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_i^{(k)}})
+ \delta_{(\mu_i^{(k)} \neq 0)} (q - q^{-1}) \sum_{b=1}^{t-1} m_\mu q^{2\mu_i^{(k)} - 1} \Phi_{t-b}^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)}+1})
\times \Phi_{b+1}^{-}(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_i^{(k)}})
+ m_\mu L_N^t L_{N+1}^t([T; N + 1, \mu_i^{(k)} + 1]^- - 1)[T; N, \mu_i^{(k)}]^+]$$
Proof. By the induction on $\mu_{i+1}^{(k)}$, we can prove that

$$
[T; N - 1, \mu_{i+1}^{(k)} + 1] + [T; N, \mu_{i}^{(k)}] = [T; N, \mu_{i}^{(k)}] - \delta_{(\mu_{i+1}^{(k)} \neq 0)}([T; N + 1, \mu_{i}^{(k)} + 1] - 1) [T; N, \mu_{i+1}^{(k)}] +
$$

(7.25.1)

Thus we have

$$
m_{\mu} L_{N}^{t} [T; N - 1, \mu_{i+1}^{(k)} + 1] + [T; N, \mu_{i}^{(k)}] = m_{\mu} L_{N}^{t} [T; N, \mu_{i}^{(k)}] - \delta_{(\mu_{i+1}^{(k)} \neq 0)}([T; N + 1, \mu_{i}^{(k)} + 1] - 1) [T; N, \mu_{i+1}^{(k)}] +
$$

Applying Lemma 7.24 (i), we have (i).

We prove (ii). By Lemma 6.3, we have

$$
[T; N - 1, \mu_{i+1}^{(k)} + 1] + L_{N} = L_{N} + L_{N+1}([T; N - 1, \mu_{i+1}^{(k)} + 1] - 1) - \delta_{(\mu_{i+1}^{(k)} \neq 0)} q(q - q^{-1}) L_{N+1} [T; N, \mu_{i+1}^{(k)}].
$$

Thus, we have

$$
m_{\mu} L_{N}^{t} [T; N - 1, \mu_{i+1}^{(k)} + 1] + L_{N} [T; N, \mu_{i}^{(k)}] = m_{\mu} L_{N}^{t+1} [T; N, \mu_{i}^{(k)}] + m_{\mu} L_{N}^{t} L_{N+1}([T; N - 1, \mu_{i+1}^{(k)} + 1] - 1) [T; N, \mu_{i}^{(k)}] - \delta_{(\mu_{i+1}^{(k)} \neq 0)} q(q - q^{-1}) m_{\mu} L_{N}^{t} L_{N+1} [T; N, \mu_{i+1}^{(k)}] + [T; N, \mu_{i}^{(k)}].
$$

Applying (6.4.2), Lemma 7.11, Lemma 7.24 and (7.25.1), we have (ii).

We prove (iii). By Lemma 6.3, we have

$$
[T; N + 1, \mu_{i}^{(k)} + 1] - L_{N+1}^{t} = L_{N+1}^{t} + L_{N}^{t}([T; N + 1, \mu_{i}^{(k)} + 1] - 1) + \delta_{(\mu_{i}^{(k)} \neq 0)} q(q - q^{-1}) \sum_{b=1}^{t} L_{N}^{t-b} L_{N+1}^{b} [T; N, \mu_{i}^{(k)}].
$$

Thus, we have

$$
m_{\mu} [T; N + 1, \mu_{i}^{(k)} + 1] - L_{N+1}^{t} [T; N, \mu_{i+1}^{(k)}] = m_{\mu} L_{N+1}^{t}[T; N, \mu_{i+1}^{(k)}] + m_{\mu} L_{N}^{t}((T; N + 1, \mu_{i}^{(k)} + 1] - 1) [T; N, \mu_{i+1}^{(k)}] + \delta_{(\mu_{i}^{(k)} \neq 0)} q(q - q^{-1}) \sum_{b=1}^{t} m_{\mu} L_{N}^{t-b} L_{N+1}^{b} [T; N, \mu_{i}^{(k)}] - [T; N, \mu_{i+1}^{(k)}] +
$$

$$
m_{\mu} L_{N+1}^{t}[T; N, \mu_{i+1}^{(k)}] + \delta_{(\mu_{i}^{(k)} \neq 0)} q(q - q^{-1}) m_{\mu} L_{N+1}^{t}[T; N, \mu_{i}^{(k)}] [T; N, \mu_{i+1}^{(k)}] +
$$
Proposition 7.26. □

Applying (6.4.2), Lemma 7.11 and Lemma 7.24, we have (iv).

We prove (iv). By Lemma 6.3, we have

\[ m_\mu L_{N+1} [T; N + 1, \mu_i^{(k)} + 1]^{-1} L_{N+1}^t [T; N, \mu_i^{(k)}] + \]

\[ + m_\mu L_{N}^t ([T; N + 1, \mu_i^{(k)} + 1]^{-1} - 1) [T; N, \mu_i^{(k)}] + \]

Applying (6.4.2), Lemma 7.11 and Lemma 7.24, we have (iv).

\[ + \delta_{(\mu_i^{(k)} \neq 0)} q(q - 1)^{-1} \sum_{b=1}^{t-1} m_\mu L_{N}^t - b L_{N+1}^b [T; N, \mu_i^{(k)}] - [T; N, \mu_i^{(k)}] + \]

Proposition 7.26. For \((i, k) \in \Gamma'(\mathbf{m})\) and \(s, t \geq 0\), we have

\[ [\chi^+, (i, k), t, \chi^-_{(i, k), s}] = \begin{cases} \tilde{K}^+_{(i, k)} J_{(i, k), s} & \text{if } i \neq m_k, \\ -Q_k \tilde{K}^+_{(m_k, k)} J_{(m_k, k), s} + \tilde{K}^+_{(m_k, k)} J_{(m_k, k), s+t+1} & \text{if } i = m_k. \end{cases} \]

Proof. Assume that \(s = 0\) and \(t \geq 0\). For \(\mu \in A_{n, r}(\mathbf{m})\), put \(N = N_{(i, k)}^\mu\). By Lemma 7.17, we have

\[ (\chi^+, (i, k), t, \chi^-_{(i, k), 0})(m_\mu) \]

\[ = \delta_{(\mu_i^{(k)} \neq 0)} q^{-\mu_i^{(k)} - \mu_i^{(k)} + 1} m_\mu L_N^t [T; N - 1, \mu_i^{(k)} + 1] + h^\mu_{(i, k)} [T; N, \mu_i^{(k)}] - \]

and

\[ (\chi^-_{(i, k), 0}, \chi^+, (i, k), 0)(m_\mu) \]

\[ = \delta_{(\mu_i^{(k)} \neq 0)} q^{-\mu_i^{(k)} - \mu_i^{(k)} + 1} m_\mu h^{\mu + \alpha_{(i, k)}} [T; N + 1, \mu_i^{(k)} + 1] - L_{N+1}^t [T; N, \mu_i^{(k)}] + \]

Assume that \(i \neq m_k\). By (7.26.1) and (7.26.2) together with Lemma 7.25, we have

\[ (\chi^+, (i, k), 0 - \chi^-_{(i, k), 0}, \chi^+, (i, k), 0)(m_\mu) \]

\[ = q^{-\mu_i^{(k)} - \mu_i^{(k)} + 1} m_\mu \left\{ \delta_{(\mu_i^{(k)} \neq 0)} q^{2\mu_i^{(k)} - 2} \Phi_i^+ (L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)} + 1}) \right. \]

\[ - \left. \delta_{(\mu_i^{(k)} \neq 0)} (1 + \delta_{(\mu \neq 0)} (q^{2\mu_i^{(k)} - 1}) \Phi_i^- (L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_i^{(k)} + 1}) \right. \]

\[ - \left. \delta_{(\mu_i^{(k)} \neq 0)} \delta_{(\mu_i^{(k)} \neq 0)} (q - q^{-1}) \sum_{b=1}^{t-1} q^{2\mu_i^{(k)} - 1} \Phi_{t-b}^+ (L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)} + 1}) \quad \Box \]
Thus, we have \[ \text{Lemma 7.27.} \]
\[
\begin{align*}
\Phi_b^-(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_{i+1}^{(k)}}) &= q^{\mu_i^{(k)} - \mu_{i+1}^{(k)}} \delta_{\mu_i^{(k)} \neq 0} q^{-1} q^{-1} \Phi_t^+(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)} + 1}) \\
&- \delta_{(\mu_i^{(k)} \neq 0)} (q^{-2\mu_i^{(k)}} + q^{-1} (1 - q^{-2\mu_i^{(k)}})) q^{-1} q^{-1} \Phi_t^+(L_{N+1}, L_{N+2}, \ldots, L_{N+\mu_{i+1}^{(k)}}) \\
&- \delta_{(\mu_i^{(k)} \neq 0)} \Phi_{t-b}^-(L_N, L_{N-1}, \ldots, L_{N-\mu_i^{(k)} + 1}) \\
&= \tilde{K}^+_{(i,k)} J_{(i,k),t}(m_\mu).
\end{align*}
\]

Thus, we have \([X_{(i,k),t}^+, X_{(i,k),0}^-] = \tilde{K}^+_{(i,k)} J_{(i,k),t} \) if \(i \neq m_k\). (Note Corollary 7.9 in the case where \(t = 0\).)

In a similar way, by (7.26.1) and (7.26.2) together with Lemma 7.25, we also have \([X_{(m_k,k),t}^+, X_{(m_k,k),0}^-] = -Q_k \tilde{K}^+_{(m_k,k)} J_{(m_k,k),s+t} + \tilde{K}^+_{(m_k,k)} J_{(m_k,k),s+t+1} \) if \(i = m_k\).

Now we proved the proposition in the case where \(s = 0\) and \(t \geq 0\).

Finally, we prove the proposition by the induction on \(s\). In the case where \(s = 0\), we have already proved. Assume that \(s > 0\), by (7.15.1), we have
\[
[X_{(i,k),t}^+, X_{(i,k),s}^-] = X_{(i,k),t}^- (T_{(i,k),1} X_{(i,k),s-1}^- + X_{(i,k),s-1}^- T_{(i,k),1}) \\
- (T_{(i,k),1} X_{(i,k),s-1}^- + X_{(i,k),s-1}^- T_{(i,k),1}) X_{(i,k),t}^+.
\]

Applying Proposition 7.21 together with Lemma 7.20, we have
\[
[X_{(i,k),t}^+, X_{(i,k),s}^-] = -T_{(i,k),1} X_{(i,k),t}^+ X_{(i,k),s-1}^- + X_{(i,k),t}^+ X_{(i,k),t}^- + X_{(i,k),t}^- T_{(i,k),1} \\
+ T_{(i,k),1} X_{(i,k),s-1}^+ X_{(i,k),t}^+ - X_{(i,k),s-1}^+ X_{(i,k),t}^- T_{(i,k),1} - X_{(i,k),s-1}^+ X_{(i,k),t}^+ \\
= [X_{(i,k),t+1}^+, X_{(i,k),s-1}^-] \\
- T_{(i,k),1} [X_{(i,k),t}^+, X_{(i,k),s-1}^-] + [X_{(i,k),t}^+, X_{(i,k),s-1}^-] T_{(i,k),1}.
\]

Then, by the assumption of the induction together with Lemma 7.6, we have the proposition. \(\Box\)

**Lemma 7.27.** For \((i,k) \in \Gamma'(m)\), we have the followings.

(i) If \((q - q^{-1})\) is invertible in \(R\), we have
\[
\tilde{K}^+_{(i,k)} J_{(i,k),0}^- = \frac{\tilde{K}^+_{(i,k)} - \tilde{K}^-_{(i,k)}}{q - q^{-1}}.
\]

(ii) If \(q = 1\), we have
\[
\tilde{K}^+_{(i,k)} J_{(i,k),0}^- = J_{(i,k),0}^- - J_{(i+1,k),0}^-.
\]
Proof. For $\mu \in A_{n,r}(m)$, by the definitions together with Corollary 7.9, we have
\[
\hat{K}^+_{(i,k)}J_{(i,k),0}(m_\mu) = \hat{K}^+_{(i,k)}(I^+_{(i,k),0} - (K^-_{(i,k)})^2I^-_{(i+1,k),0})(m_\mu)
= q^{\mu_i^{(k)}-\mu_i^{(k)+1}}(q^{-\mu_i^{(k)}}[\mu_i^{(k)}] - q^{-2\mu_i^{(k)}}q^{\mu_i^{(k)+1}[\mu_i^{(k)+1}]})m_\mu
= [\mu_i^{(k)} - \mu_i^{(k)+1}]m_\mu.
\]
If $(q - q^{-1})$ is invertible in $R$, we have
\[
[\mu_i^{(k)} - \mu_i^{(k)+1}]m_\mu = \frac{q^{\mu_i^{(k)}-\mu_i^{(k)+1}} - q^{-\mu_i^{(k)}+\mu_i^{(k)+1}}}{q - q^{-1}}m_\mu = \frac{\hat{K}^+_{(i,k)} - \hat{K}^-_{(i,k)}}{q - q^{-1}}(m_\mu).
\]
Thus, we have (i).

If $q = 1$, we have
\[
[\mu_i^{(k)} - \mu_i^{(k)+1}]m_\mu &= (\mu_i^{(k)} - \mu_i^{(k)+1})m_\mu = (I^+_{(i,k),0} - I^-_{(i+1,k),0})(m_\mu).
\]
Thus, we have (ii). \qed

In the case where $q = 1$, we have the following lemma.

Lemma 7.28. Assume that $q = 1$. Then, for $(j, l) \in \Gamma(m)$ and $t \geq 0$, we have the followings.

(i) $K^\pm_{(j,l)} = 1$.

(ii) $I^+_{(j,l), t} = I^-_{(j,l), t}$.

Proof. If $q = 1$, we see that
\[
(7.28.1) \quad \Phi^+_t(x_1, \ldots, x_k) = x_1^t + x_2^t + \cdots + x_k^t,
\]
in particular we have $\Phi^+_t(x_1, \ldots, x_k) = \Phi^-_t(x_1, \ldots, x_k)$. Thus, we have the lemma from the definitions. \qed

§ 8. THE CYCLOTOMIC $q$-SCHUR ALGEBRA AS A QUOTIENT OF $\mathcal{U}_{q,Q}(m)$

Let $\tilde{Q} = (Q_0, Q_1, \ldots, Q_{r-1})$ be an $r$-tuple of indeterminate elements over $\mathbb{Z}$, and $Q(\tilde{Q}) = Q(Q_0, Q_1, \ldots, Q_{r-1})$ be the quotient field of $\mathbb{Z}[\tilde{Q}] = \mathbb{Z}[Q_0, Q_1, \ldots, Q_{r-1}]$. Put $\tilde{A} = \mathbb{Z}[q, q^{-1}, Q_0, Q_1, \ldots, Q_{r-1}]$, and let $\tilde{K} = Q(q, Q_0, Q_1, \ldots, Q_{r-1})$ be the quotient field of $\tilde{A}$, where $q$ is indeterminate over $\mathbb{Z}$. Put
\[
\mathcal{g}_Q(m) = Q(\tilde{Q}) \otimes_{Q(\tilde{Q})} \mathcal{g}_Q(m),
\]
\[
\mathcal{U}_{q,Q}(m) = \tilde{K} \otimes_{\tilde{A}} \mathcal{U}_{q,Q}(m) \text{ and } \mathcal{U}_{\tilde{A},Q}(m) = \tilde{A} \otimes_{\tilde{A}} \mathcal{U}_{\tilde{A},Q}(m).
\]
We define a full subcategory $\mathcal{C}_q^0(m)$ and $\mathcal{C}_q^{2\mathbb{Q}}(m)$ (resp. $\mathcal{C}_q,\mathcal{Q}(m)$ and $\mathcal{C}_q^{2\mathbb{Q}}(m)$) of $U(\mathfrak{g}_q(m))$-mod (resp. $U_q,\mathcal{Q}(m)$-mod) in a similar manner as $\mathcal{C}_q(m)$ and $\mathcal{C}_q^{2\mathbb{Q}}(m)$ (resp. $\mathcal{C}_q,\mathcal{Q}(m)$ and $\mathcal{C}_q^{2\mathbb{Q}}(m)$).

Let $\mathcal{H}_{n,r}$ (resp. $\mathcal{H}_{n,r}^\mathbb{K}$) be the Ariki-Koike algebra over $\mathbb{K}$ (resp. over $\mathbb{A}$) with parameters $q, Q_0, Q_1, \ldots, Q_{r-1}$, and $\mathcal{I}_{n,r}^{\mathbb{K}}(m)$ (resp. $\mathcal{I}_{n,r}^\mathbb{K}(m)$) be the cyclotomic $q$-Schur algebra associated with $\mathcal{H}_{n,r}$ (resp. $\mathcal{H}_{n,r}^\mathbb{K}$). Then, we have the following theorem.

**Theorem 8.1.** We have a homomorphism of algebras

(8.1.1) \[ \Psi : U_q,\mathcal{Q}(m) \to \mathcal{I}_{n,r}^{\mathbb{K}}(m) \]

by taking $\Psi(\mathcal{X}_{(i,j),t}^+)^t = \mathcal{X}_{(i,j),t}^+$. $\Psi(\mathcal{I}_{(i,j),t}^+)^t = \mathcal{I}_{(i,j),t}^+$ and $\Psi(\mathcal{K}_{(j,l),t}^+) = \mathcal{K}_{(j,l)}^+$.

The restriction of $\Psi$ to $U^\mathbb{K}_{k,q},\mathcal{Q}(m)$ gives a homomorphism of algebras

\[ \Psi_{\mathbb{K}} : U^\mathbb{K}_{k,q},\mathcal{Q}(m) \to \mathcal{I}_{n,r}^{\mathbb{K}}(m). \]

Moreover, if $m_k \geq n$ for all $k = 1, 2, \ldots, r-1$, the homomorphism $\Psi$ (resp. $\Psi_{\mathbb{K}}$) is surjective.

**Proof.** The well-definedness of $\Psi$ follows from Lemma 7.6, Lemma 7.7, Lemma 7.16, Proposition 7.18, Proposition 7.19, Lemma 7.20, Proposition 7.21, Proposition 7.22, and Proposition 7.26.

Note that $\mathcal{H}_{n,r}$ (resp. $\mathcal{I}_{n,r}^{\mathbb{K}}(m)$) is an $\mathcal{A}$-subalgebra of $\mathcal{H}_{n,r}^\mathbb{K}$ (resp. $\mathcal{I}_{n,r}^{\mathbb{K}}(m)$) by definitions. In particular, in order to see that $\varphi \in \mathcal{I}_{n,r}^{\mathbb{K}}(m)$ belong to $\mathcal{I}_{n,r}^{\mathbb{K}}(m)$, it is enough to show that $\varphi(m_\mu) \in \mathcal{H}_{n,r}^\mathbb{K}$ for any $\mu \in \Lambda_{n,r}(m)$.

For $\mu \in \Lambda_{n,r}(m)$ and $d \in \mathbb{Z}_{\geq 0}$, we see that,

(8.1.2) \[ \begin{bmatrix} \mathcal{K}_{(j,l)}^+ \end{bmatrix}^d (m_\mu) = \begin{cases} \begin{bmatrix} \mu_j^{(l)} \\ d \end{bmatrix} m_\mu & \text{if } d \leq \mu_j^{(l)}, \\ 0 & \text{if } d > \mu_j^{(l)} \end{cases} \] in $\mathcal{I}_{n,r}^{\mathbb{K}}(m)$. This implies that $\Psi( [\mathcal{K}_{(j,l)}^+ ]^d ) \in \mathcal{I}_{n,r}^{\mathbb{K}}(m)$.

For $(i,k) \in \Gamma'(m)$ and $t, d \in \mathbb{Z}_{\geq 0}$, we see that

\[
\begin{align*}
(\mathcal{X}_{(i,k)}^+)^d (m_\mu) & = q^{-d \mu_{(k)}^{(i)} + d(d+1)/2} m_\mu \sigma_{i,k} (L_{N_{(i,k)}^\mu} + 1) L_{N_{(i,k)}^\mu} + 2 \ldots L_{N_{(i,k)}^\mu} + d \begin{bmatrix} t; N_{(i,k)}^\mu, \mu_{i+1}^{(k)} \end{bmatrix} \\
& = q^{-d \mu_{(k)}^{(i)} + d(d+1)/2} m_\mu \sigma_{i,k} (L_{N_{(i,k)}^\mu} + 1) L_{N_{(i,k)}^\mu} + 2 \ldots L_{N_{(i,k)}^\mu} + d \begin{bmatrix} t; N_{(i,k)}^\mu, \mu_{i+1}^{(k)} \end{bmatrix} \\
& \times (T; N_{(i,k)}^\mu, d + 1) \mathcal{Y}_+ (N_{(i,k)}^\mu, \mu_{i+1}^{(k)}, d)
\end{align*}
\]
by Lemma 7.17 together with Lemma 7.11 and Corollary 7.14. We also see that $(T; N_{(i,k)}^\mu, d)^+!$ commute with $(L_{N_{(i,k)}^\mu+1}^\mu \cdot \ldots \cdot L_{N_{(i,k)}^\mu+d}^\mu)^t$ by Lemma 6.3 (iii), and see that $m_{\mu+d\alpha_{(i,k)}} (T; N_{(i,k)}^\mu, d)^+! = q^{d(d-1)/2}[d]! m_{\mu+d\alpha_{(i,k)}}$ by (6.4.2). Thus we have

$$(X_{(i,k),t}^+)^d(m_\mu) = [d]! q^{-d\alpha_{i+1} + d^2} m_{\mu+d\alpha_{(i,k)}} (L_{N_{(i,k)}^\mu+1}^\mu \cdot \ldots \cdot L_{N_{(i,k)}^\mu+d}^\mu)^t S (N_{(i,k)}^\mu, \mu_{i+1}, d)$$

in $\mathcal{F}_{n,r}^\mathbb{K} (m)$. This implies that $\Psi(X_{(i,k),t}^+(d)) \in \mathcal{F}_{n,r}^\mathbb{K} (m)$ since $S (N_{(i,k)}^\mu, \mu_{i+1}, d) \in \mathcal{H}^\mathbb{K}_{n,r}$ by the argument in the proof of Corollary 7.14. Similarly, we see that $\Psi(X_{(i,k),t}^+(d)) \in \mathcal{F}_{n,r}^\mathbb{K} (m)$. Thus, the restriction of $\Psi$ to $U_{k,q,Q} (m)$ gives a homomorphism $\Psi_k$.

The last assertion follows from [W1, Proposition 6.4].

**Remark 8.2.** In order to prove the surjectivity of $\Psi$ (resp. $\Psi_k$), we use the result of [W1, Proposition 6.4]. In fact, we considered only the case where $m_k = n$ for all $k = 1, 2, \ldots, r$ in [W1]. However, we can apply the result to the case where $m_k \geq n$ for all $k = 1, 2, \ldots, r - 1$ without any change since the surjectivity in [W1, Proposition 6.4] follows from the result in [DR]. The reason why we assume the condition $m_k \geq n$ for all $k = 1, 2, \ldots, r - 1$ to state the surjectivity of $\Psi$ is just the using results of [DR]. We expect that $\Psi$ is also surjective without this condition.

**Theorem 8.3.** Assume that $m_k \geq n$ for all $k = 1, 2, \ldots, r - 1$. Then we have the followings.

(i) $\mathcal{F}_{n,r}^\mathbb{K} (m)$-mod is a full subcategory of $\mathcal{C}_{Q_\mathbb{K}}^\mathbb{Q} (m)$ through the surjection $\Psi$ in (8.1.1).

(ii) The Weyl module $\Delta (\lambda) \in \mathcal{F}_{n,r}^\mathbb{K} (m)$-mod $(\lambda \in \Lambda_{n,r}^+ (m))$ is the simple highest weight $U_{Q_\mathbb{K}} (m)$-module of highest weight $(\lambda, \varphi)$ through the surjection $\Psi$, where the multiset $\varphi = (\varphi_{(j,l),t}^\pm) \in \mathbb{K}^\mathbb{Q} | (j, l) \in \Gamma (m), t \geq 1$ is given by

$$\varphi_{(j,l),t}^+ = Q_{t-1}^l q^{(2t-1)(\lambda_j^{(l)} - l(2j-1)[\lambda_j]^{(l)})}$$

and

$$\varphi_{(j,l),t}^- = Q_{t-1}^l q^{-l(2j-1)[\lambda_j]^{(l)}}.$$

**Proof.** For $\lambda \in \Lambda_{n,r} (m)$, let $1_\lambda$ be an element of $\mathcal{F}_{n,r}^\mathbb{K} (m)$ such that the identity on $M^\lambda$ and $1_\lambda (M^\mu) = 0$ for any $\mu \neq \lambda$. Then we have $1_\lambda 1_\mu = \delta_{\lambda\mu} 1_\lambda$ and $\sum_{\lambda \in \Lambda_{n,r} (m)} 1_\lambda = 1$.

Thus, for $M \in \mathcal{F}_{n,r}^\mathbb{K}$-mod, we have the decomposition

$$(8.3.1) \quad M = \bigoplus_{\mu \in \Lambda_{n,r} (m)} 1_\mu M. $$

Moreover, we see that

$$1_\mu M = \{ m \in M | K_{(j,l),t}^+ \cdot m = q^{\lambda_j^{(l)}} m \text{ for } (j, l) \in \Gamma (m) \}$$
from the definition of $\Psi$. Thus, any object $M$ of $\mathcal{S}_{n,r}^\circ\circ$-mod has the weight space decomposition (8.3.1) as a $U_q\tilde{Q}(m)$-module, where we remark that $\Lambda_{n,r}(m) \subset P_{\geq 0}$.

For $M \in \mathcal{S}_{n,r}^{\circ\circ\circ\circ}(m)$-mod, in order to see that all eigenvalues of the action of $\mathcal{I}_{(j,l),t}^\pm ((j,l) \in F(m), t \geq 0)$ on $M$ belong to $\tilde{K}$, it is enough to show them for $\Delta(\lambda)$ ($\lambda \in \Lambda_{n,r}(m)$) since $\mathcal{S}_{n,r}^{\circ\circ\circ\circ}(m)$ is semi-simple and $\{\Delta(\lambda) \mid \lambda \in \Lambda_{n,r}^+(m)\}$ gives a complete set of isomorphism classes of simple $\mathcal{S}_{n,r}^{\circ\circ\circ\circ}(m)$-modules. Recall that $\{\varphi_T \mid T \in T_0(\lambda, \mu)\}$ for some $\mu \in \Lambda_{n,r}(m)$} gives a basis of $\Delta(\lambda)$.

Note that $\Phi_t^\pm(L_{n,r}^{\mu_1}, L_{n,r}^{\mu_2}, \ldots, L_{n,r}^{\mu_{j-1}})$ commute with $T_w$ for any $w \in \mathfrak{S}_\mu$ by Lemma 6.3, for $T \in T_0(\lambda, \mu)$, we have

\begin{equation}
\mathcal{I}_{(j,l),t}^\pm \cdot \varphi_T = \begin{cases}
q^{t-1} \Phi_t^\pm(\res(j,l); T) \varphi_T + \sum_{S \triangleright T} r_S \varphi_S & \text{if } \mu_j^{(l)} \neq 0, \\
0 & \text{if } \mu_j^{(l)} = 0
\end{cases}
\end{equation}

in a similar argument as in the proof of [JM, Theorem 3.10], where

$$
\Phi_t^\pm(\res(j,l); T) = \Phi_t^\pm(\res(x_1), \res(x_2), \ldots, \res(x_{\mu_j^{(l)}}))
$$

with \( \{x_1, x_2, \ldots, x_{\mu_j^{(l)}}\} = \{ x \in [\lambda] \mid T(x) = (j,l) \} \), and $\triangleright$ is a partial order on $T_0(\lambda, \mu)$ defined in [JM, Definition 3.6]. This implies that all eigenvalues of the action of $\mathcal{I}_{(j,l),t}^\pm$ on $\Delta(\lambda)$ belong to $\tilde{K}$. Now we proved (i).

We prove (ii). For $\lambda \in \Lambda_{n,r}^+(m)$, let $T^\lambda$ be the unique semi-standard tableau of shape $\lambda$ with weight $\lambda$. Then, we see easily that $\varphi_T^\lambda$ is a highest weight vector of $\Delta(\lambda)$. Note that there is no tableau such that $S \triangleright T^\lambda$, then we have

\begin{equation}
\varphi_{(j,l),t}^\pm = q^{t-1} \Phi_t^\pm(Q_kq^{2(1-j)}, Q_kq^{2(2-j)}, \ldots, Q_kq^{2(\lambda_j^{(l)} - j)})
\end{equation}

by (8.3.2). Then we can prove (ii) by the induction on $t$ using (8.3.3) and (7.3.1). \qed

Let $\mathcal{S}_{1,n,r}(m)$ be the cyclotomic $q$-Schur algebra over $\mathbb{Q}(\tilde{Q})$ with parameters $q = 1$, $Q_0, Q_1, \ldots, Q_{r-1}$. Then we have the following theorem.

Theorem 8.4.

(i) We have a homomorphism of algebras

\begin{equation}
\Psi_1 : U(\mathfrak{g}_Q(m)) \to \mathcal{S}_{1,n,r}(m)
\end{equation}

by taking $\Psi_1(\chi_{(i,k),t}^\pm) = \chi_{(i,k),t}^\pm$ and $\Psi_1(\mathcal{I}_{(j,l),t}) = \mathcal{I}_{(j,l),t}^\pm (= \mathcal{I}_{(j,l),t}^-)$.

Moreover, if $m_k \geq n$ for all $k = 1, 2, \ldots, r - 1$, the homomorphism $\Psi_1$ is surjective.

(ii) Assume that $m_k \geq n$ for all $k = 1, 2, \ldots, r - 1$. Then $\mathcal{S}_{1,n,r}(m)$-mod is a full subcategory of $\mathcal{C}^{\geq 0}_{\tilde{Q}}(m)$ through the surjection $\Psi_1$. 

Moreover, the Weyl module \( \Delta(\lambda) \in \mathcal{S}^1_{n,r}(\mathfrak{m}) \mod (\lambda \in \Lambda^+_{n,r}) \) is the simple highest weight \( U(\mathfrak{g}_Q(\mathfrak{m})) \)-module of highest weight \((\lambda, \varphi)\) through the surjection \( \Psi_1 \), where the multiset \( \varphi = (\varphi(j,t), t \in Q(\tilde{Q}) | (j,l) \in \Gamma(\mathfrak{m}), t \geq 1) \) is given by

\[
\varphi(j,t) = Q^l_{t-1} \lambda^{(l)}_j.
\]

Proof. Note Lemma 7.27 and Lemma 7.28, then we can prove the theorem in a similar way as in the proof of Theorem 8.1 and Theorem 8.3. \( \square \)

§ 9. Characters of Weyl modules of cyclotomic \( q \)-Schur algebras

In this section, we study the characters of Weyl modules of cyclotomic \( q \)-Schur algebras as symmetric polynomials. In particular, we prove the conjecture given in [W2] (the formula (9.2.1) below) which will be understood as the decomposition of the product of Weyl modules in the case where \( q = 1 \).

9.1. Characters. For \( k = 1, \ldots, r \), let \( x^{(k)}_m = (x_{(1,k)}, x_{(2,k)}, \ldots, x_{(m_k,k)}) \) be the set of \( m_k \) independent variables, and put \( x_m = \bigcup_{k=1}^r x^{(k)}_m \). Let \( \mathbb{Z}[x^\pm_m] \) (resp. \( \mathbb{Z}[x_m] \)) be the ring of Laurent polynomials (resp. the ring of polynomials) with variables \( x_m \).

For \( \lambda \in P \), we define the monomial \( x^\lambda \in \mathbb{Z}[x^\pm_m] \) by \( x^\lambda = \prod_{k=1}^r \prod_{i=1}^{m_k} x^{(\lambda, h(i,k))} \).

For \( M \in C_Q(\mathfrak{m}) \) (resp. \( M \in C_{Q,q}(\mathfrak{m}) \)), we define the character of \( M \) by

\[
(9.1.1) \quad \text{ch} M = \sum_{\lambda \in P} \dim M_{\lambda} x^\lambda \in \mathbb{Z}[x^\pm_m].
\]

It is clear that \( \text{ch} M \in \mathbb{Z}[x_m] \) if \( M \in C^\geq 0_Q(\mathfrak{m}) \) (resp. \( M \in C^\geq 0_{Q,q}(\mathfrak{m}) \)).

When we regard \( M \in C_Q(\mathfrak{m}) \) as a \( U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \)-module through the injection (2.16.2), \( \text{ch} M \) defined by (9.1.1) coincides with the character of \( M \) as a \( U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \)-module since \( M_{\lambda} \) is also the weight space of weight \( \lambda \) as a \( U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \)-module. Thus, by the known results for \( U(\mathfrak{gl}_{m_1} \oplus \cdots \oplus \mathfrak{gl}_{m_r}) \)-modules, we see that

\[
\text{ch} M \in \bigotimes_{k=1}^r \mathbb{Z}[x^{(k)}_m]^\otimes_{m_k} \text{ if } M \in C^\geq 0_Q(\mathfrak{m}),
\]

where \( \mathbb{Z}[x^{(k)}_m]^\otimes_{m_k} \) is the ring of symmetric polynomials with variables \( x^{(k)}_m \), and we regard \( \bigotimes_{k=1}^r \mathbb{Z}[x^{(k)}_m]^\otimes_{m_k} \) as a subring of \( \mathbb{Z}[x_m] \) through the multiplication map

\[
\bigotimes_{k=1}^r \mathbb{Z}[x^{(k)}_m]^\otimes_{m_k} \to \mathbb{Z}[x_m], \quad (\bigotimes_{k=1}^r f(x^{(k)}_m) \mapsto \prod_{k=1}^r f(x^{(k)}_m)).
\]

It is similar for \( M \in C_{Q,q}(\mathfrak{m}) \) through the injection (4.9.2).

9.2. The character of the Weyl module \( \Delta(\lambda) \in \mathcal{S}^1_{n,r}(\mathfrak{m}) \) \( (\lambda \in \widetilde{\Lambda}^+_{n,r}(\mathfrak{m})) \) is studied in [W2]. Note that \( \Delta(\lambda) \) \( (\lambda \in \widetilde{\Lambda}^+_{n,r}(\mathfrak{m})) \) does not depend on the choice of the base field and parameters. Put \( \widetilde{\Lambda}^+_{\geq 0,r}(\mathfrak{m}) = \cup_{n \geq 0} \widetilde{\Lambda}^+_{n,r}(\mathfrak{m}) \). For \( \lambda, \mu \in \widetilde{\Lambda}^+_{\geq 0,r}(\mathfrak{m}) \), the
following formula was conjectured in [W2, Conjecture 2]:

\[(9.2.1) \quad \text{ch} \Delta(\lambda) \text{ch} \Delta(\mu) = \sum_{\nu \in \tilde{A}^+_0(r)(m)} \text{LR}_\lambda^\mu \text{ch} \Delta(\nu) \text{ for } \lambda, \mu \in \tilde{A}^+_0(r)(m),\]

where \(\text{LR}_\lambda^\mu = \prod_{k=1}^r \text{LR}_{(\lambda(k), \mu(k))}^{(k)}\), and \(\text{LR}_{(\lambda(k), \mu(k))}^{(k)}\) is the Littlewood-Richardson coefficient for the partitions \(\lambda^{(k)}\), \(\mu^{(k)}\) and \(\nu^{(k)}\). We prove this conjecture as follows.

**Proposition 9.3.** For \(\lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \in \tilde{A}^+_n(r)(m)\), we denote

\[ (0, \ldots, 0, \lambda^{(k)}, 0, \ldots, 0) \in \tilde{A}^+_n(r)(m) \]

by \((0, \ldots, \lambda^{(k)}, \ldots, 0)\) simply, where \(n_k = \sum_{i=1}^{m_k} \lambda_i^{(k)}\) (i.e. \(\lambda^{(k)}\) appears in the \(k\)-th component in \((0, \ldots, \lambda^{(k)}, \ldots, 0)\)). Let

\[ S_{\lambda^{(k)}}(x_1 \cup \cdots \cup x_{r}) \in \mathbb{Z}[x_1 \cup \cdots \cup x_{r}]_{\mathbb{Z}[x_1] \cup \cdots \cup x_{r}] \]

be the Schur polynomial for the partition \(\lambda^{(k)}\) with variables \(x_1^{(k)} \cup \cdots \cup x_{r}^{(k)}\), where we regard \(\mathbb{Z}[x_1^{(k)} \cup \cdots \cup x_{r}^{(k)}]_{\mathbb{Z}[x_1] \cup \cdots \cup x_{r}]\) as a subring of \(\bigotimes_{k=1}^r \mathbb{Z}[x_k^{(k)}]_{\mathbb{Z}[x_k]}\) in the natural way. Put \(\tilde{S}_\lambda(x_m) = \text{ch} \Delta(\lambda) \quad (\lambda \in \tilde{A}^+_n(r)(m))\). Then we have the following proposition.

**Proposition 9.4.** For \(\lambda, \mu \in \tilde{A}^+_n(r)(m)\), we have the following formulas.

(i) \(\tilde{S}_{(0, \ldots, \lambda^{(k)}, \ldots, 0)}(x_m) = S_{\lambda^{(k)}(x_1 \cup \cdots \cup x_{r})}\).

(ii) \(\tilde{S}_\lambda(x_m) = \prod_{k=1}^r \tilde{S}_{(0, \ldots, \lambda^{(k)}, \ldots, 0)}(x_m)\).

(iii) \(\tilde{S}_\lambda(x_m)\tilde{S}_\mu(x_m) = \sum_{\nu \in \tilde{A}^+_n(r)(m)} \text{LR}_\lambda^\mu \tilde{S}_\nu(x_m)\).

**Proof.** (i). By the definition of the cellular basis of \(\mathcal{A}_{n,r}(m)\) in [DJM], for \(\lambda \in \tilde{A}^+_n(r)(m)\), we have

\[(9.4.1) \quad \tilde{S}_\lambda(x_m) = \text{ch} \Delta(\lambda) = \sum_{\mu \in A_{n,r}(m)} \sharp T_0(\lambda, \mu) x^\mu.\]

Thus, we have

\[(9.4.2) \quad \tilde{S}_{(0, \ldots, \lambda^{(k)}, \ldots, 0)}(x_m) = \sum_{\mu \in A_{n_k,r}(m)} \sharp T_0((0, \ldots, \lambda^{(k)}, \ldots, 0), \mu) x^\mu,\]

where \(n_k = \sum_{i=1}^{m_k} \lambda_i^{(k)}\). We see that

\[\mu^{(1)} = \cdots = \mu^{(k-1)} = 0 \quad \text{if} \quad T_0((0, \ldots, \lambda^{(k)}, \ldots, 0), \mu) \neq \emptyset\]
by the definition of semi-standard tableaux. Thus, we have $\tilde{S}_{(\lambda^{(1)}, \ldots, \lambda^{(r)})}(x_m) \in \otimes_{l=k}^r \mathbb{Z}[x_m] \Sigma_{m_k}$. Put

$$A_{n_k,r}^k(m) = \{\mu = (\mu^{(l)})_{l=1}^r \in A_{n_k,r}(m) | \mu^{(l)} = 0 \text{ for } l = 1, \ldots, k - 1\}.$$ 

Put $m' = m_k + \cdots + m_r$. We identify the set $A_{n_k,1}(m')$ with $A_{n_k, r}^k(m)$ by the bijection $\theta^k : A_{n_k, 1}(m') \to A_{n_k, r}^k(m)$ such that

$$(\theta^k(\mu))^{(k+l)}_i = \begin{cases} \mu_i & \text{if } l = 0, \\ \mu_{m_k + m_k + 1 + \cdots + m_k + l - 1 + i} & \text{if } 1 \leq l \leq r - k \end{cases}$$

for $\mu = (\mu_1, \mu_2, \ldots, \mu_{m'}) \in A_{n_k,1}(m')$. By the well-known fact, we can describe the Schur polynomial $S_{\lambda^{(k)}}(x_m^{(k)} \cup \cdots \cup x_m^{(r)})$ as

$$(9.4.3) \quad S_{\lambda^{(k)}}(x_m^{(k)} \cup \cdots \cup x_m^{(r)}) = \sum_{\mu \in A_{n_k,1}(m')} \nonumber \# T_0(\lambda^{(k)}, \mu) x^\mu,$$

where we put $x^\mu = \prod_{i=1}^{m_k} x_{i(i,k)}^{m_i} \prod_{l=1}^{r-k} \prod_{i=1}^{m_l} x_{i(i,k+l)}^{m_i \mu_{m_k + m_k + 1 + \cdots + m_k + l - 1 + i}}$. From the definition of semi-standard tableaux, we see that

$$\nonumber \# T_0(\lambda^{(k)}, \mu) = \# T_0((0, \ldots, \lambda^{(k)}, \ldots, 0), \theta^k(\mu))$$

for $\mu \in A_{n_k,1}(m')$. Thus, by comparing the right hand sides of (9.4.2) and of (9.4.3), we obtain (i).

(ii). First we prove that

$$(9.4.4) \quad \tilde{S}_{(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})}(x_m) = \tilde{S}_{(\lambda^{(1)}, 0, \ldots, 0)}(x_m) \tilde{S}_{(0, \lambda^{(2)}, \ldots, \lambda^{(r)})}(x_m).$$

By (9.4.1), we have

$$(9.4.5) \quad \tilde{S}_{(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})}(x_m) = \sum_{\mu \in A_{n_k,r}(m)} \# T_0(\lambda, \mu) x^\mu.$$

On the other hand, we have

$$(9.4.6) \quad \tilde{S}_{(\lambda^{(1)}, 0, \ldots, 0)}(x_m) \tilde{S}_{(0, \lambda^{(2)}, \ldots, \lambda^{(r)})}(x_m)$$

$$= \left( \sum_{\nu \in A_{n_1,r}(m)} \# T_0((\lambda^{(1)}, 0, \ldots, 0), \nu) x^\nu \right) \left( \sum_{\tau \in A_{n'_{r-1},r}(m)} \# T_0((0, \lambda^{(2)}, \ldots, \lambda^{(r)}), \tau) x^{\tau} \right)$$

$$= \sum_{\mu \in A_{n_k,r}(m)} \left( \sum_{\nu \in A_{n_1,r}(m), \tau \in A_{n'_{r-1},r}(m)} \# T_0((\lambda^{(1)}, 0, \ldots, 0), \nu) \# T_0((0, \lambda^{(2)}, \ldots, \lambda^{(r)}), \tau) \right) x^\mu.$$
Thus, (9.4.5), (9.4.6) and (9.4.7) imply (9.4.4). By applying a similar argument to

\[ \mathcal{T}_0(\lambda, \mu) = \sum_{\nu \in A_{n',r}(m), \tau \in A_{n',r}^+(m')} \mathcal{T}_0((\lambda^{(1)}, 0, \ldots, 0), \nu) \mathcal{T}_0((0, \lambda^{(2)}, \ldots, \lambda^{(r)}), \tau). \]

Thus, (9.4.5), (9.4.6) and (9.4.7) imply (9.4.4). By applying a similar argument to

\[ S_{(0,\lambda^{(2)},\ldots,\lambda^{(r)})}(x_m) \] inductively, we obtain (ii).

By (i) and (ii), we have

\[
\begin{align*}
\tilde{S}_\lambda(x_m) \tilde{S}_\mu(x_m) &= \left( \prod_{k=1}^r \tilde{S}_{(0,\ldots,\lambda^{(k)},\ldots,0)}(x_m) \right) \left( \prod_{k=1}^r \tilde{S}_{(0,\ldots,\mu^{(k)},\ldots,0)}(x_m) \right) \\
&= \left( \prod_{k=1}^r S_{\lambda^{(k)}}(x_m^{(k)} \cup \cdots \cup x_m^{(r)}) \right) \left( \prod_{k=1}^r S_{\mu^{(k)}}(x_m^{(k)} \cup \cdots \cup x_m^{(r)}) \right) \\
&= \prod_{k=1}^r \left( \sum_{\nu(k) \in A_{\geq 0,1}^+(m_k+\cdots+m_r)} LR_{\lambda^{(k)},\mu^{(k)}} S_{\nu^{(k)}}(x_m^{(k)} \cup \cdots \cup x_m^{(r)}) \right) \\
&= \sum_{\nu \in A_{\geq 0,r}(m)} \left( \prod_{k=1}^r LR_{\lambda^{(k)},\mu^{(k)}} \right) \prod_{k=1}^r S_{(0,\ldots,\nu^{(k)},\ldots,0)}(x_m) \\
&= \sum_{\nu \in A_{\geq 0,r}^+(m)} LR_{\nu} \tilde{S}_{\nu}(x_m),
\end{align*}
\]

where we note that, if \( \ell(\lambda^{(k)}) > m_k + \cdots + m_r \) for some \( k \), we have \( S_{\lambda^{(k)}}(x_m^{(k)} \cup \cdots \cup x_m^{(r)}) = 0 \) and \( \mathcal{T}_0(\lambda, \mu) = 0 \) for any \( \mu \in A_{n',r}(m) \). Now we obtained (iii). \( \Box \)

§ 10. Tensor products for Weyl modules of cyclotomic \( q \)-Schur algebras at \( q = 1 \)

By using the comultiplication \( \Delta : U(\mathfrak{g}/\mathfrak{gl}(m)) \to U(\mathfrak{gl}(m)) \otimes U(\mathfrak{gl}(m)) \) (\( \Delta(x) = x \otimes 1 + 1 \otimes x \)), we define the \( U(\mathfrak{gl}(m)) \)-module \( M \otimes N \) for \( U(\mathfrak{gl}(m)) \)-module \( M \) and \( N \). We regard \( S_{n,r}^+(m) \)-modules \( (n \geq 0) \) as a \( U(\mathfrak{gl}(m)) \)-modules through the homomorphism \( \Psi_4 \) in (8.4.1). Note that \( S_{n,r}^+(m) \) is semi-simple, and \( \{ \Delta(\lambda) \mid \lambda \in A_{n,r}^+(m) \} \) gives a complete set of isomorphism classes of simple \( S_{n,r}^+(m) \)-modules if \( m_k \geq n \) for all \( k = 1, 2, \ldots, r-1 \). Then, we have the following proposition.

**Proposition 10.1.** Assume that \( m_k \geq n \) for all \( k = 1, 2, \ldots, r-1 \). Take \( n_1, n_2 \in \mathbb{Z}_{>0} \) such that \( n = n_1 + n_2 \). For \( \lambda \in A_{n_1,r}^+(m) \) (resp \( \mu \in A_{n_2,r}^+(m) \)), let \( \Delta(\lambda) \) (resp \( \Delta(\mu) \)) be the Weyl module of \( S_{n_1,r}^+(m) \) (resp \( S_{n_2,r}^+(m) \)) corresponding \( \lambda \) (resp \( \mu \)).
Then we have

\begin{equation}
\Delta(\lambda) \otimes \Delta(\mu) \cong \bigoplus_{\nu \in \mathcal{A}_{\mu, r}(m)} LR_{\lambda \mu}^\nu \Delta(\nu) \text{ as } U(\mathfrak{g}_Q(m))-\text{modules},
\end{equation}

where $\Delta(\nu)$ is the Weyl module of $\mathcal{S}_{n, r}^1(m)$ corresponding $\nu$, and $LR_{\lambda \mu}^\nu \Delta(\nu)$ means the direct sum of $LR_{\lambda \mu}^\nu$ copies of $\Delta(\nu)$. In particular, $\Delta(\lambda) \otimes \Delta(\mu) \in \mathcal{S}_{n, r}^1(m)-\text{mod}$.

\textbf{Proof.} For $\tau \in P_{\geq 0}$, put

$$\pi_m(\tau) = (|\tau^{(1)}|, |\tau^{(2)}|, \ldots, |\tau^{(r)}|) \in \mathbb{Z}_{\geq 0}^r,$$

where $|\tau^{(l)}| = \sum_{j=1}^{m_l} (\tau, h(j, l))$ for $l = 1, \ldots, r$. We denote by $\geq$ the lexicographic order on $\mathbb{Z}_{\geq 0}^r$. Then we have the weight space decomposition

\begin{equation}
\Delta(\lambda) \otimes \Delta(\mu) = \bigoplus_{\tau \in \mathcal{A}_{\mu, r}(m) \atop \pi_m(\tau) \leq \pi_m(\lambda + \mu)} (\Delta(\lambda) \otimes \Delta(\mu))_\tau.
\end{equation}

On the other hand, it is clear that $\Delta(\lambda) \otimes \Delta(\mu) \in C_{\tilde{Q}}^{\geq 0}(m)$. Thus, we have

\begin{equation}
[\Delta(\lambda) \otimes \Delta(\mu)] = \sum_{\nu \in \mathcal{A}_{\mu, r}(m) \atop \pi_m(\nu) \leq \pi_m(\lambda + \mu)} \sum_{\varphi} d_{\nu, \varphi}[L(\nu, \varphi)] \text{ in } K_0(C_{\tilde{Q}}^{\geq 0}(m)),
\end{equation}

where $d_{\nu, \varphi}$ is the composition multiplicity of the simple highest weight $U(\mathfrak{g}_Q(m))$-module $L(\nu, \varphi)$ of highest weight $(\nu, \varphi)$ in $\Delta(\lambda) \otimes \Delta(\mu)$.

Note that $L_{i+1}T_i = T_i L_i$ and $L_{i}T_i = T_i L_{i+1}$ since $q = 1$. Then, for $(j, l) \in \Gamma(m)$ and $t \geq 1$, we see that

\begin{equation}
\mathcal{I}_{(j, l), t} \cdot v = Q_{l-1}^j v^{(l)} \text{ for any } v \in (\Delta(\lambda) \otimes \Delta(\mu))_\nu
\end{equation}

if $\pi_m(\nu) = \pi_m(\lambda + \mu)$ by the argument in the proof of [JM, Proposition 3.7 and Theorem 3.10]. This implies that

\begin{equation}
L(\nu, \varphi) \cong \Delta(\nu) \text{ if } d_{\nu, \varphi} \neq 0 \text{ and } \pi_m(\nu) = \pi_m(\lambda + \mu)
\end{equation}

by Theorem 8.4 (ii). By Proposition 9.4 (iii) together with (10.1.3) and (10.1.5), we have

\begin{equation}
\text{ch}(\Delta(\lambda) \otimes \Delta(\mu)) = \tilde{S}_\lambda(x_m) \tilde{S}_\mu(x_m)
= \sum_{\nu \in \mathcal{A}_{\mu, r}(m)} LR_{\lambda \mu}^\nu \tilde{S}_\nu(x_m)
\end{equation}
\[
= \sum_{\substack{\nu \in A^+_n, \chi(m) \in A^+_u, (m) \\ \pi(m) \nu = \pi(m, \lambda + \mu)}} d_{\nu} \widetilde{S}_\nu(x_m) + \sum_{\nu \in A^+_n, \chi(m) \in A^+_u, (m) \in A^+_u, (m)} \sum_{\varphi} d_{\nu, \varphi} \text{ch} L(\nu, \varphi),
\]

where \(d_{\nu}\) is the composition multiplicity of \(\Delta(\nu)\) in \(\Delta(\lambda) \otimes \Delta(\mu)\). Note that \(LR_{\lambda \mu}^\nu = 0\) unless \(\pi(m) \nu = \pi(m, \lambda + \mu)\), the equations (10.1.6) imply \(d_{\nu} = LR_{\lambda \mu}^\nu\) if \(\pi(m) \nu = \pi(m, \lambda + \mu)\) and \(d_{\nu, \varphi} = 0\) if \(\pi(m) \nu < \pi(m, \lambda + \mu)\). Thus, we have

\[
(10.1.7) \quad [\Delta(\lambda) \otimes \Delta(\mu)] = \sum_{\nu \in A^+_n, \chi(m) \in A^+_u, \chi(m, \lambda + \mu)} LR_{\lambda \mu}^\nu [\Delta(\nu)].
\]

By (10.1.2), for any \(k = 1, 2, \ldots, r - 1\) and any \(t \geq 0\), we have

\[
(10.1.8) \quad \mathcal{X}^+_{(m, k), t} \cdot \left( \bigoplus_{\nu \in A^+_n, \chi(m) \in A^+_u, \chi(m, \lambda + \mu)} (\Delta(\lambda) \otimes \Delta(\mu))_{\nu} \right) = 0
\]

since \(\pi(m) (\nu + \alpha(m_k, k)) > \pi(m) \nu\). Then, by (10.1.4) and (10.1.8) together with the relation (L2), we see that

\[
(10.1.9) \quad \{ v \in (\Delta(\lambda) \otimes \Delta(\mu))_{\nu} \mid X^+_{(i, k), t} \cdot v \text{ for all } (i, k) \in \Gamma'(m) \text{ and } t \geq 0 \} = \{ v \in (\Delta(\lambda) \otimes \Delta(\mu))_{\nu} \mid e_{(i, k)} \cdot v \text{ for all } (i, k) \in \Gamma(m) \setminus \{(m_k, k) \mid 1 \leq k \leq r\} \}
\]

for \(\nu \in A^+_n, \chi(m) \in A^+_u, \chi(m, \lambda + \mu)\) such that \(\pi(m) \nu = \pi(m, \lambda + \mu)\), where \(e_{(i, k)} \in U(gl_{m_1} \oplus \cdots \oplus gl_{m_r})\) acts on \(\Delta(\lambda) \otimes \Delta(\mu)\) through the injection (2.16.2). On the other hand, \(\bigoplus_{\nu \in A^+_n, \chi(m) \in A^+_u, \chi(m, \lambda + \mu)} (\Delta(\lambda) \otimes \Delta(\mu))_{\nu}\) is a \(U(gl_{m_1} \oplus \cdots \oplus gl_{m_r})\)-submodule of \(\Delta(\lambda) \otimes \Delta(\mu)\) and we have

\[
(10.1.10) \quad \bigoplus_{\nu \in A^+_n, \chi(m) \in A^+_u, \chi(m, \lambda + \mu)} (\Delta(\lambda) \otimes \Delta(\mu))_{\nu} \cong \bigoplus_{\nu \in A^+_n, \chi(m) \in A^+_u, \chi(m, \lambda + \mu)} LR_{\lambda \mu}^\nu \Delta_{gl_{m_1}}(\nu^{(1)}) \otimes \cdots \otimes \Delta_{gl_{m_r}}(\nu^{(r)})
\]

as \(U(gl_{m_1} \oplus \cdots \oplus gl_{m_r})\)-modules by comparing the character (note [W2, Lemma 2.6]). By (10.1.7), (10.1.9) and (10.1.10), we see that

\[
\Delta(\lambda) \otimes \Delta(\mu) \cong \bigoplus_{\nu \in A^+_n, \chi(m) \in A^+_u, \chi(m, \lambda + \mu)} LR_{\lambda \mu}^\nu \Delta(\nu)
\]

as \(U(g_{\mathbb{Q}}(m))\)-modules. \(\square\)

**Remarks 10.2.**

(i) For \(M, N \in C_{\mathbb{Q}}(m)\), we see that \(\text{ch}(M \otimes N) = \text{ch}(M) \text{ch}(N)\) by definition of characters. Then the decomposition (10.1.1) gives an interpretation of the formula (9.2.1) (Proposition 9.4 (iii)) in the category \(C_{\mathbb{Q}}(m)\).
(ii) We conjecture that the algebra $U_{q,\tilde{Q}}(m)$ has a structure as a Hopf algebra. Then we also conjecture the similar decomposition for the tensor product of Weyl modules of $\mathcal{J}_{n,r}^\mathbb{K}(m)$ ($n \geq 0$) as in (10.1.1).

References

[BLM] A. Beilinson, G. Lusztig and R. MacPherson, A geometric setting for the quantum deformation of $GL_n$, Duke. Math. J. 61 (1990), 655-677.

[DJM] R. Dipper, G. James and A. Mathas, Cyclotomic $q$-Schur algebras, Math. Z. 229 (1998), 385-416.

[D] J. Du, A note on quantized Weyl reciprocity at root of unity, Algebra Colloq. 2 (1995), 363–372.

[DR] J. Du and H. Rui, Borel type subalgebras of the $q$-Schur$m$ algebra, J. Algebra 213 (1999), 567-595.

[GL] J.J. Graham and G.I. Lehrer, Cellular algebras, Invent. Math. 123 (1996), 1-34.

[JM] G. James and A. Mathas, The Jantzen sum formula for cyclotomic $q$-Schur algebras, Trans Amer. Math. Soc. 352 (2000), 5381-5404.

[J] M. Jimbo, A $q$-analogue of $U(gl(N + 1))$, Hecke algebra and the Yang-Baxter equation, Lett. Math. Phys. 11 (1986), 247–252.

[KL] D. Kazhdan and G. Lusztig, Tensor structures arising from affine Lie algebras I-IV, J. Amer. Math. Soc. 6-7 (1993-1994) 905-947, 949-1011, 335-381, 383-453.

[L] I. Losev, Proof of Varagnolo-Vasserot conjecture on cyclotomic categories $\mathcal{O}$, arXiv:1305.4894.

[M1] A. Mathas, The representation theory of the Ariki-Koike and cyclotomic $q$-Schur algebras; in “Representation theory of algebraic groups and quantum groups”, Adv. Stud. Pure Math. Vol. 40, Math. Soc. Japan, Tokyo 2004, pp. 261-320.

[M2] A. Mathas, Seminormal forms and Gram determinants for cellular algebras, J. Reine Angew. Math. 619 (2008), 141–173.

[R] R. Rouquier, $q$-Schur algebras and complex reflection groups, Moscow Math. J. 8, 119-158.

[RSV] R. Rouquier, P. Shan, M. Varagnolo and E. Vasserot, Categorifications and cyclotomic rational double affine Hecke algebras, arXiv:1305.4456.

[W1] K. Wada, Presenting cyclotomic $q$-Schur algebras, Nagoya Math. J. 201 (2011), 45-116.

[W2] K. Wada, On Weyl modules of cyclotomic $q$-Schur algebras, Contemp. Math. 565 (2012), 261-286.

[W3] K. Wada, Induction and restriction functors for cyclotomic $q$-Schur algebras, Osaka J. Math. 51 (2014), 785-822.

Department of Mathematics, Faculty of Science, Shinshu University, Asahi 3-1-1, Matsumoto 390-8621, Japan
E-mail address: wada@math.shinshu-u.ac.jp