STABILITY OF THE VOLUME PRESERVING MEAN CURVATURE FLOW IN HYPERBOLIC SPACE

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Abstract. We consider the dynamic property of the volume preserving mean curvature flow. This flow was introduced by Huisken in [Hui87] who also proved it converges to a round sphere of the same enclosed volume if the initial hypersurface is strictly convex in Euclidean space. We study the stability of this flow in hyperbolic space. In particular, we prove that if the initial hypersurface is hyperbolically mean convex and close to an umbilical sphere in the $L^2$-sense, then the flow exists for all time and converges exponentially to an umbilical sphere.

1. Introduction

1.1. Background and Main Theorem. Let $M^n$ be a smooth, embedded, closed (compact, no boundary) $n$-dimensional manifold in hyperbolic space $\mathbb{H}^{n+1}$ ($n \geq 2$), and we evolve it by the volume preserving mean curvature flow (VPMCF),

\begin{equation}
\frac{\partial F}{\partial t} = (h - H) \nu, \quad F(\cdot, 0) = F_0(\cdot),
\end{equation}

where $F_0 : M^n \to \mathbb{H}^{n+1}$ is the initial embedding, $H = H(x, t)$ is the mean curvature and $\nu = \nu(x, t)$ is the outward unit normal vector of the evolving surface $M_t = F(\cdot, t)$ at point $(x, t)$ (for simplicity, we write $(x, t) \in M_t$). The function $h$ is the average of the mean curvature on $M_t$, given by

\begin{equation}
h = h(t) = \int_{M_t} H \, d\mu = \frac{\int_{M_t} H \, d\mu}{\int_{M_t} d\mu},
\end{equation}

where $d\mu = d\mu_t$ denotes the surface area element of the evolving surface $M_t$ with respect to the induced metric $g(t)$.

In this paper we use the convention that the mean curvature is the sum of all principal curvatures. Clearly we have $H \not\equiv 0$ on $M_0$ since there is no closed minimal hypersurface in hyperbolic space. The presence of the global term $h$ in the VPMCF equation (1.1) forces the flow to behave quite differently from the usual mean curvature flow (MCF).

Hypersurfaces of constant mean curvature are critical points of the area functional under the constraint of fixed enclosed volume. These hypersurfaces are also static state for the VPMCF equation (1.1). A remarkable theorem of Huisken-Yau ([HY96]) on the existence of a foliation of spheres outside of some large compact set in asymptotic flat manifolds was achieved by studying a parameter family of VPMCFs. This flow, and the surface area preserving mean curvature flow studied in

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are special cases of so-called mixed volume preserving mean curvature flow. They are closely related to convex geometry and classical inequalities, see for instance [ES98, McC04, ACW21, WX14], also see [CRM12, EM12] for other geometric settings.

We denote $A = \{a_{ij}\}$ the second fundamental form of $M_t$ and $\hat{A} = A - \frac{H}{n} g$ its traceless part. Then we have $|\hat{A}|^2 = |A|^2 - \frac{1}{n} H^2$. This quantity measures the roundness of a (closed, immersed) hypersurface $\Sigma$ in $\mathbb{H}^{n+1}$: if $\hat{A} \equiv 0$, i.e., $\Sigma$ is umbilic at every point, then by a classical Codazzi’s theorem in differential geometry, it is a geodesic sphere, see e.g. [Spi79, Theorem 29]. We also remark that, in $\mathbb{R}^3$, De Lellis and Müller [DLM05] generalized Codazzi’s theorem by showing a version of the following quantitative rigidity

$$\inf_{\lambda \in \mathbb{R}} \|A - \lambda \text{Id}\|_{L^2(\Sigma)} \leq C \|\hat{A}\|_{L^2(\Sigma)},$$

for some universal constant $C$. Such quantitative rigidity is not available for hyperbolic space to our acknowledge.

Strict convexity (i.e., all principal curvatures are positive) plays a fundamental role in classical works of several types of MCFs, especially in Euclidean space. Huisken ([Hui84]) proved that an initial smooth closed and strictly convex hypersurface will stay convex and flow into a round point along the MCF in Euclidean space. He ([Hui87]) also showed, in the case of the VPMCF, the flow of an initial smooth closed and strictly convex hypersurface will exist for all time and flow into a round sphere in Euclidean space. The parallel result for the surface area preserving mean curvature flow is also true, showed by McCoy ([McC03]). Though natural in Euclidean geometry, this notion of convexity is not the most natural in hyperbolic space. The presence of horospheres in hyperbolic space poses strong restrictions on the geometry of hypersurfaces (via Hopf’s maximum principle): for instance any closed constant mean curvature hypersurface has mean curvature greater than $n$ in $\mathbb{H}^{n+1}$.

**Definition 1.1.** We call a hypersurface of an $(n+1)$-dimensional hyperbolic manifold (strictly) $h$-convex if every principal curvature of the hypersurface at every point is greater than 1, and call it (strictly) $h$-mean convex or hyperbolically mean convex if its mean curvature at every point is greater than $n$.

The “h-convexity” was introduced in ([CRM07]), where the authors proved that h-convexity is preserved along the VPMCF in hyperbolic space. Moreover, under the assumption of closed initial hypersurface being h-convex, they showed that the volume preserving mean curvature flow exists for all time and converges to an umbilical sphere. The “h-mean convexity”, or the notion of being hyperbolically mean convex, is much weaker than h-convexity, and it is not known to be preserved along the VPMCF. But it turns out this condition plays a very important role in proving the dynamic stability of the VPMCF.

Unlike the regular MCF, the VPMCF (1.1) has a global forcing term in the equation which greatly complicates the analysis of the flow. How the singularities of the flow may form remains elusive at the current stage of study, even in Euclidean space. Moreover in our hyperbolic space setting, the negative curvature of the ambient space presents significant challenges in analyzing the evolution equations involved in the study. As a first step to understand the long term behavior of
the flow, in this paper, we study the dynamical property of the VPMCF (1.1) in hyperbolic space in the situation that the initial hypersurface is not necessarily $h$-convex, yet close to an umbilical sphere in the $L^2$-sense. More precisely, we show the stability of the flow with initial $h$-mean convex hypersurface (namely the initial mean curvature at every point is greater than $n$) and small $L^2$-norm of the traceless part of the second fundamental form. Our main theorem is the following:

**Theorem 1.2.** Let $M^t_n \subset \mathbb{H}^{n+1}$, $n \geq 2$, be a smooth closed solution to the VPMCF (1.1) for $t \in [0, T)$ with $T \leq \infty$. Assume that $M_0$ is $h$-mean convex and

$$\max \left\{ |M_0|^2, \max_{M_0} |A|^2, \int_{M_0} |\nabla^m A|^2 \, d\mu \right\} \leq \Lambda_0^2,$$

for some $\Lambda_0 \gg 1$ and all $m \in [1, n + 3]$, where $|M_t|$ is the $n$-dimensional surface area of $M_t$ with the induced metric. Then there exists some $\epsilon_0 = \epsilon_0(n, \Lambda_0) > 0$ such that if

$$\int_{M_0} |A|^2 \, d\mu < \epsilon_0,$$

then $T = \infty$ and the flow converges exponentially to an umbilical sphere which encloses the same volume as $M_0$.

**Remark 1.3.** It is very important that $\epsilon_0$ in Theorem 1.2 does not depend on the lower bound of $H - n > 0$ on $M_0$ from $h$-mean convexity.

1.2. **Outline of the proof:** We would like to stress that there are several serious complications in order to investigate the dynamic stability for VPMCF: with a forcing term $h$, the flow is global in nature, therefore it’s difficult to localize the analysis and it is essential to keep track of $h$ along the flow; we are working in the hyperbolic space where $h$-mean convexity is likely not preserved along the flow in general and the negative curvature of hyperbolic space makes the analysis of the flow substantially more involved. To overcome these difficulties, we use iteration techniques in combination of several new tools to prove the main theorem.

We will organize the iteration argument in four steps: step 1, based on the initial bounds, we derive bounds on some short time interval for several geometric quantities (Lemmas 3.1 and 3.2) such as $H$, $\nabla H$, $|A|^2$, etc. As a consequence, we show that the $h$-mean convexity is preserved on some definite time interval provided the initial hypersurface is close enough to an umbilical sphere in the $L^2$ sense; in step 2, we prove exponential decay for these quantities on the time interval obtained in previous step (Theorem 3.5), which allows us to obtain uniform bounds for these quantities on the interval; in step 3, we repeat above arguments to extend the time interval (Theorem 3.6), and finally in step 4, we prove the amount of extension for time only depends on the initial conditions. Main theorem then follows.

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2. Technical Preparations

In this section, we collect basic evolution equations for key quantities, and derive some preliminary estimates that will be used in the proof.
2.1. **Evolution equations.** Let us first fix some notations of the following geometric quantities that will be used in this study:

1. The induced metric of the evolving hypersurface $M_t$: $\{g_{ij}(t)\}$;
2. The second fundamental form of $M_t$: $A(\cdot, t) = \{a_{ij}(\cdot, t)\}$, and its square norm is given by $|A(\cdot, t)|^2 = g^{ij}g^{kl}a_{ik}a_{jl}$;
3. The mean curvature of $M_t$ with respect to the outward unit normal vector: $H(\cdot, t) = g^{ij}a_{ij}$;
4. The traceless part of the second fundamental form: $\tilde{A} = A - Hn$;
5. The area element of the evolving hypersurface $M_t$: $d\mu_t = \sqrt{\det(g_{ij})}$.

The evolution equations for these quantities are as follows:

**Lemma 2.1.** ([Hui87, HY96]) The metric of $M_t$ satisfies the evolution equation

$$\partial_t g_{ij} = 2(h - H)a_{ij}. \quad (2.1)$$

Therefore,

$$\partial_t g^{ij} = -2(h - H)a^{ij} \quad (2.2)$$

and the area element satisfies:

$$\partial_t (d\mu_t) = H(h - H)d\mu_t. \quad (2.3)$$

Moreover, the outward unit normal vector $\nu$ to $M_t$ satisfies

$$\frac{\partial \nu}{\partial t} = \nabla H, \quad (2.4)$$

where $\frac{\partial \nu}{\partial t}$ is a conventional way of writing down $\bar{\nabla} \frac{\partial}{\partial t} \nu$.

By (2.3), we have the following geometrical properties of the VPMCF:

**Corollary 2.2.** ([Hui87])

1. The $(n+1)$-dimensional volume $V_t$ of the region enclosed by $M_t$ remains unchanged along the flow, i.e.,

$$\frac{d}{dt} V_t = \int_{M_t} (h - H) \, d\mu = 0. \quad (2.5)$$

2. The $n$-dimensional surface area $|M_t|$ of $M_t$ is non-increasing along the flow, i.e.,

$$\frac{d}{dt} |M_t| = \frac{d}{dt} \int_{M_t} \, d\mu = \int_{M_t} H(h - H) \, d\mu = -\int_{M_t} (h - H)^2 \, d\mu \leq 0. \quad (2.6)$$

Following Huisken’s calculations for the MCF in general Riemannian manifolds ([Hui86]), we have the following evolution equations for key quantities in our setting. See also [HY96] for the case $n = 2$ in the setting of asymptotic flat manifolds and [CRM07] for equivalent formulas in hyperbolic space setting.

**Theorem 2.3.** We have the evolution equations for $H$ and $|A|^2$:

(i)

$$\frac{\partial}{\partial t} H = \Delta H + (H - h)(|A|^2 - n); \quad (2.7)$$
\[(2.6) \quad \frac{\partial}{\partial t}|A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|\bar{A}|^2(|A|^2 + n) - 2h \text{tr}(A^3) + 2H(h - 2H). \]

where \( \text{tr}(A^3) = g^{ij}g^{kl}g^{mn}a_{ik}a_{lm}a_{nj}. \)

We include a short proof for readers’ convenience.

**Proof.** Let \( \bar{g} = \{\bar{g}_{\alpha\beta}\} \) be the metric on \( \mathbb{H}^{n+1} \), \( \nabla \) and \( \bar{\nabla} \) be covariant derivative and Riemannian curvature tensor with respect to \( \bar{g} \). The equation (2.5) is clear since \( \text{Ric}^\nu(\nu) = -n \) in \( \mathbb{H}^{n+1} \). For (2.6), we first follow [Hui86, CRM07] to find that the second fundamental form \( \{a_{ij}\} \) of \( M_t \) satisfies the following evolution equation:

\[
\frac{\partial}{\partial t} a_{ij} = \Delta a_{ij} + (h - 2H)a_{it}a_{jt} + |\bar{A}|^2 a_{ij} + a_{ij} \bar{R}_{0l0l} - h \bar{R}_{0l0l} \\
- a_{it} \bar{R}_{ljm} - a_{lt} \bar{R}_{jim} + 2a_{im} \bar{R}_{ljm} - \bar{\nabla}_j \bar{R}_{ilt} - \bar{\nabla}_t \bar{R}_{i0l}. \tag{2.7}
\]

The last two terms which involve the covariant derivatives of the curvature tensor drop out as we are in a constant curved space. Furthermore, since \( \mathbb{H}^{n+1} \) has constant sectional curvature \(-1\), the Riemannian curvature tensor is given by:

\[
(2.8) \quad \bar{R}_{\alpha\beta\gamma\delta} = (-1) \cdot (\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\gamma\beta}).
\]

Now (2.6) follows from contraction and (2.1). \( \square \)

The covariant derivatives for \( A \) satisfy the following.

**Corollary 2.4.** We have the evolution equation for \( |\nabla^mA|^2 \) with \( m \geq 1 \):

\[
\frac{\partial}{\partial t}|\nabla^mA|^2 = \Delta |\nabla^mA|^2 - 2|\nabla^{m+1}A|^2 + \nabla^mA * \nabla^mA \\
+ \sum_{i+j+k=m} \nabla^iA * \nabla^jA * \nabla^kA + \sum_{r+s=m} \nabla^rA * \nabla^sA * \nabla^mA, \tag{2.9}
\]

where \( S * \Omega \) denotes any linear combination of tensors formed by contraction on \( S \) and \( \Omega \) by the metric \( g \). Here, in addition to constants, \( h = h(t) \) (having only time variable) may be involved in the coefficients of the contraction.

**Proof.** We have the following evolution of the second fundamental form from the proof of Theorem 2.3:

\[
\frac{\partial}{\partial t} A = \Delta A + A * A * A + A * A + * A.
\]

Meanwhile, the time derivative of the Christoffel symbols \( \Gamma^i_{jk} \) is equal to

\[
\frac{\partial}{\partial t} \Gamma^i_{jk} = \frac{1}{2} g^{il} \left\{ \nabla_j \left( \frac{\partial}{\partial t} g_{kl} \right) + \nabla_k \left( \frac{\partial}{\partial t} g_{jl} \right) - \nabla_l \left( \frac{\partial}{\partial t} g_{ij} \right) \right\} \\
= g^{il} \{ \nabla_j ((h - H)a_{kl}) + \nabla_k ((h - H)a_{lj}) - \nabla_l ((h - H)a_{jk}) \}
\]

\[
= * \nabla A + A * \nabla A, \tag{2.10}
\]

where \( *T \) denotes contraction of \( T \) by the metric \( g \). Note that we have used the evolution equation (2.1) for the metric.

Now we proceed as in [Ham82, §13] (see also [Hui84, §7]) to obtain the equation (2.9). In particular, using (2.10), if \( S \) and \( \Omega \) are tensors satisfying the evolution
equation $\frac{\partial}{\partial t} S = \Delta S + \Omega$, then the covariant derivative $\nabla S$, which involves the Christoffel symbols, satisfies an equation of the following form:

\[(2.11) \quad \frac{\partial}{\partial t} \nabla S = \Delta (\nabla S) + S \ast \nabla A + S \ast \nabla A + A \ast \nabla A \ast \nabla S + \nabla \Omega.\]

Therefore by (2.7), we find

\[(2.12) \quad \frac{\partial}{\partial t} \nabla A = \Delta (\nabla A) + \sum_{i+j+k=1} \nabla^i A \ast \nabla^j A \ast \nabla^k A + \sum_{r+s=m} \nabla^r A \ast \nabla^s A \ast \nabla^m A.\]

Then by induction we have for $m \geq 1$,

\[(2.13) \quad \frac{\partial}{\partial t} |\nabla^m A|^2 = \Delta |\nabla^m A|^2 - 2|\nabla \nabla^m A|^2 + 2|\nabla^m A|^2 (|A|^2 + n) + 2h(|A|^2 - n \text{tr}(A^3)).\]

We also have the time derivative for the average of mean curvature $h(t)$.

**Lemma 2.5.**

\[(2.14) \quad h'(t) = \frac{\int_M (H - h)(|A|^2 - H^2 + hH) d\mu}{\int_M d\mu}.\]

**Proof.** An easy calculation using equations (2.3) and (2.5). Note that the expression does not contain terms involving $\nabla H$. □

The following inequalities for gradients are useful and we record them here.

**Lemma 2.6.** (cf. [Hui86]) The following inequalities hold:

(i) $|\nabla A|^2 \geq \frac{3}{n+2} |\nabla H|^2$;

(ii) $|\nabla \tilde{A}|^2 \geq \frac{n-1}{2n+1} |\nabla A|^2 \geq \frac{3(n-1)}{(n+2)(2n+1)} |\nabla H|^2$.

2.2. **Intuitive decay of $|\tilde{A}|^2$.** One of the key estimates for us is an exponential decay for $|\tilde{A}|^2$ on some time interval. We now give a heuristic argument to show why this is the case when $|\tilde{A}|^2$ is small and h-mean convexity is preserved.

Since $|\tilde{A}|^2 = |A|^2 - \frac{1}{n} H^2$ and $|\nabla \tilde{A}|^2 = |\nabla A|^2 - \frac{1}{n} |\nabla H|^2$, we obtain the evolution equation for $|\tilde{A}|^2$ as follows.

**Lemma 2.7.**

\[(2.15) \quad \frac{\partial}{\partial t} |\tilde{A}|^2 = \Delta |\tilde{A}|^2 - 2|\nabla \tilde{A}|^2 + 2|\tilde{A}|^2 (|A|^2 + n) + 2h \left( H|A|^2 - n \text{tr}(A^3) \right)\]

\[(2.16) \quad = \Delta |\tilde{A}|^2 - 2|\nabla \tilde{A}|^2 + 2|\tilde{A}|^2 (|A|^2 + n) - 2h \left( \text{tr}(\tilde{A}^3) + \frac{2}{n} |\tilde{A}|^2 H \right).\]
Proof. The evolution equation for $H^2$ can be easily derived from (2.5):

\[
\frac{\partial}{\partial t} H^2 = \Delta H^2 - 2|\nabla H|^2 + 2H(H-h)(|A|^2 - n). \tag{2.15}
\]

Then (2.14) follows easily from the identity (see e.g. page 335 of [Li09]):

\[
\text{tr}(A^3) - \frac{1}{n} |A|^2 H = \text{tr}(\hat{A}^3) + \frac{2}{n} |\hat{A}|^2 H.
\]

To see a heuristic argument on exponential decay of $|\hat{A}|^2$, we examine the equation (2.14) more closely, provided $|\hat{A}|^2$ is small and $|h-H|$ is also very small. Obviously, one can apply the maximum principle to (2.14) to obtain the exponential decay of $|\hat{A}|^2$, if for some small $\varepsilon > 0$ we have

\[
2|\hat{A}|^2(|A|^2 + n) - 2h\left\{\text{tr}\left(\hat{A}^3\right) + \frac{2}{n} |\hat{A}|^2 H\right\} \leq -\varepsilon|\hat{A}|^2.
\]

Since $|\text{tr}\left(\hat{A}^3\right)| \leq |\hat{A}|^3$, it suffices to show

\[
|\hat{A}|^2 + \frac{H^2}{n} + n + |h| \cdot |\hat{A}| - \frac{2hH}{n} < -\frac{\varepsilon}{2}, \tag{2.16}
\]

which can be rewritten as

\[
\frac{H^2}{n} = H + \frac{H(H-n)}{n} > |\hat{A}|^2 + n + |h| \cdot |\hat{A}| + \frac{2H(H-h)}{n} + \frac{\varepsilon}{2}.
\]

This inequality holds once we establish $H > n + \sigma$ for some $\sigma > 0$ (i.e., $h$-mean convexity) provided that $|\hat{A}|^2$ and $|h-H|$ are both sufficiently small. We will make the argument precise in §3.3.

2.3. Technical Tools. For the sake of self-containedness of the paper, we now collect tools that will be used in the proof: a version of maximum principle, Hamilton’s interpolation inequalities for tensors, a generalization of Topping’s theorem in hyperbolic space, and a $L^2$-bound for covariant derivatives of $A$ along the VPMCF. Firstly, the following version of maximum principle is useful in our iteration scheme.

**Theorem 2.8.** (Maximum Principle, see e.g. [CLN06, Lemma 2.11]) Suppose $u : M \times [0, T] \to \mathbb{R}$ satisfies

\[
\frac{\partial}{\partial t} u \leq a^{ij}(t) \nabla_i \nabla_j u + \langle B(t), \nabla u \rangle + F(u),
\]

where the coefficient matrix $(a^{ij}(t)) > 0$ for all $t \in [0, T]$, $B(t)$ is a time-dependent vector field and $F$ is a Lipschitz function. If $u \leq c$ at $t = 0$ for some $c \in \mathbb{R}$, then $u(x, t) \leq U(t)$ for all $(x, t) \in M \times \{t\}, t \in [0, T]$, where $U(t)$ is the solution to:

\[
\frac{d}{dt} U(t) = F(U) \quad \text{with} \quad U(0) = c.
\]

We also need the following Hamilton’s interpolation inequalities for tensors. These inequalities will be used inductively for us to obtain integral bounds of covariant derivatives of $\hat{A}$.
Theorem 2.9. ([Ham82]) Let $M$ be an $n$-dimensional compact Riemannian manifold and $\Omega$ be any tensor on $M$. Suppose
\[
\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \quad \text{with } r \geq 1.
\]
Then we have the estimate:
\[
\left( \int_M |\nabla^q\Omega|^{2r} \, d\mu \right)^{1/r} \leq (2r - 2 + n) \left( \int_M |\nabla^2\Omega|^{p} \, d\mu \right)^{1/p} \left( \int_M |\Omega|^q \, d\mu \right)^{1/q}.
\]

Theorem 2.10. ([Ham82]) Let $M$ and $\Omega$ be the same as in the Theorem 2.9. If $1 \leq i \leq m - 1$ and $m \geq 1$, then there exists a constant $C = C(n, m)$ independent of the metric and connection on $M$, such that:
\[
\int_M |\nabla^i\Omega|^{2m/i} \, d\mu \leq C \max_M |\Omega|^{2(m/i - 1)} \int_M |\nabla^m\Omega|^2 \, d\mu.
\]

As an application of these inequalities and Corollary 2.4, we have the following:

Lemma 2.11. For any $m \geq 0$, we have the estimate
\[
\left( \frac{d}{dt} \int_{M_t} |\nabla^m A|^2 \, d\mu \right) + 2 \int_{M_t} |\nabla^{m+1} A|^2 \, d\mu \leq C \max_{M_t} (1 + |A| + |A|^2) \int_{M_t} |\nabla^m A|^2,
\]
where $C = C(n, m, |h|)$.

Proof. When $m = 0$, the inequality is obvious in light of (2.6). Now we consider $m \geq 1$. By integrating (2.9) of Corollary 2.4 and using the generalized Hölder inequality we have:
\[
\left( \frac{d}{dt} \int_{M_t} |\nabla^m A|^2 \, d\mu \right) - \int_{M_t} (h - H)H |\nabla^m A|^2 \, d\mu + 2 \int_{M_t} |\nabla^{m+1} A|^2 \, d\mu \leq C \left\{ \sum_{i + j + k = m} \left( \int_{M_t} |\nabla^i A|^{2m/i} \, d\mu \right)^{\frac{2m}{2m/i}} \left( \int_{M_t} |\nabla^j A|^{2m/j} \, d\mu \right)^{\frac{2m}{2m/j}} \left( \int_{M_t} |\nabla^k A|^{2m/k} \, d\mu \right)^{\frac{2m}{2m/k}} \right. \\
+ \left. \sum_{r + s + m = m} \left( \int_{M_t} |\nabla^r A|^{2m/r} \, d\mu \right)^{\frac{2m}{2m/r}} \left( \int_{M_t} |\nabla^s A|^{2m/s} \, d\mu \right)^{\frac{2m}{2m/s}} \left( \int_{M_t} |\nabla^m A|^2 \, d\mu \right)^{\frac{2m}{2m}} \right\} + C \int_{M_t} |\nabla^m A|^2 \, d\mu,
\]
where all the indices now take values from 1 and up and the terms in the original sums with 0 indices being absorbed by other sums and $C$’s.

Applying Theorem 2.10 for $A$, we have
\[
\left( \int_{M_t} |\nabla^q A|^{2m/q} \, d\mu \right)^{q/2m} \leq C \max_{M_t} |A|^{1-q/m} \left( \int_{M_t} |\nabla^m A|^2 \, d\mu \right)^{1/2m},
\]
where $q$ can be $i, j, k, r$ or $s$. We also notice
\[
\int_{M_t} (h - H)H |\nabla^m A|^2 \, d\mu \leq \max_{M_t} \{ |h||H| + H^2 \} \int_{M_t} |\nabla^m A|^2 \, d\mu \leq C(n, |h|) \max_{M_t} (|A|^2 + |A|) \int_{M_t} |\nabla^m A|^2 \, d\mu.
\]
Combining these inequalities, we complete the proof.
It’s known from \cite{Hui87} that $L^\infty$ bound for $|A|$ along the VPMCF in Euclidean space and the initial $L^\infty$ bounds of its covariant derivatives will give $L^\infty$ bounds for the covariant derivatives. Adapting the argument there, we have the following lemma for explicit $L^2$ bounds for our situation.

**Lemma 2.12.** Along the VPMCF, for $k \geq 0$, if

$$\max \left\{ |M_0|, \max_{M_t, t \in [0,T]} |A|^2, \max_{m \leq k} \int_{M_0} |\nabla^m A|^2 d\mu \right\} \leq \Lambda_0^2,$$

then uniformly for $t \in [0,T]$ and $m \leq k$ we have

$$\int_{M_t} |\nabla^m A|^2 d\mu \leq C(\Lambda_0, k),$$

where $C(\Lambda_0, k)$ is independent of $T$.

**Proof.** Along the VPMCF, we have $|M_t| \leq |M_0|$ by Corollary 2.2. So the conclusion is clear for $m = 0$ for any fixed $k \geq 0$. We can then prove the lemma by induction on $m$. Suppose the conclusion is true for $m \geq 0$, to see this holds for $m + 1 \leq k$, note that by Lemma 2.11, we know for $m \geq 0$,

$$\frac{d}{dt} \int_{M_t} |\nabla^m A|^2 d\mu \leq C(\Lambda_0) \int_{M_t} |\nabla^m A|^2 d\mu - 2 \int_{M_t} |\nabla^{m+1} A|^2 d\mu,$$

$$\frac{d}{dt} \int_{M_t} |\nabla^{m+1} A|^2 d\mu \leq C(\Lambda_0) \int_{M_t} |\nabla^{m+1} A|^2 d\mu.$$

Let $G(t) = C(\Lambda_0) \int_{M_t} |\nabla^m A|^2 d\mu + \int_{M_t} |\nabla^m A|^2 d\mu$. Then we have

$$G'(t) \leq C(\Lambda_0) \left( C(\Lambda_0) \int_{M_t} |\nabla^m A|^2 d\mu - \int_{M_t} |\nabla^m A|^2 d\mu \right).$$

Consider the maximum of $G(t)$ achieved at $t = \bar{t} \in [0,T]$. If $\bar{t} = 0$ then for all $t \in [0,T]$,

$$G(t) \leq G(0) \leq (C(\Lambda_0) + 1)\Lambda_0^2.$$

Otherwise by (2.17),

$$C(\Lambda_0) \int_{M_t} |\nabla^m A|^2 d\mu - \int_{M_t} |\nabla^{m+1} A|^2 d\mu \geq 0,$$

and thus we have for all $t \in [0,T]$,

$$G(t) \leq G(\bar{t}) \leq C(\Lambda_0, k).$$

Therefore by (2.18) and (2.19),

$$\int_{M_t} |\nabla^{m+1} A|^2 d\mu \leq C(\Lambda_0, k),$$

which is independent of $T$.

In the Euclidean space, Topping \cite{Top08} discovered a relation between the intrinsic diameter and the mean curvature $H$ of any closed, connected and smoothly immersed submanifold. This result has been extended to a more general Riemannian setting by Wu-Zheng \cite{WZ11}, using Hoffman-Spruck’s generalization \cite{HS74} of the Michael-Simon’s inequality \cite{MS73}. We formulate their result in our setting below.
Theorem 2.13. ([WZ11]) Let $M$ be an $n$-dimensional closed, connected manifold smoothly isometrically immersed in $\mathbb{H}^N$, where $N \geq n + 1$. There exists a constant $C = C(n)$ such that the intrinsic diameter and the mean curvature $H$ of $M$ are related by the following inequality:

$$\text{diam}(M) \leq C(n) \int_M |H|^{n-1} d\mu.$$ 

2.4. Hyperbolic mean convexity. The $h$-mean convexity is a very important geometric ingredient in our main result. Note that mean convexity and $h$-mean convexity are not known to be preserved along the VPMCF. The strict $h$-convexity is however preserved along the VPMCF in $\mathbb{H}^{n+1}$ [CRM07]. We give an alternative proof for this result by following very closely Huisken’s tensor calculations in [Hui84, Hui87] and highlighting the role of the curvature for the ambient space. Unlike the preserved mean convexity along the MCF in Euclidean space, this shows the subtlety of the $h$-mean convexity in hyperbolic space and the negative-curvature effects of the ambient space.

Proposition 2.14. ([CRM07]) Let $M^n$ be a smooth, embedded, closed hypersurface moving by the VPMCF (1.1) in a smooth, complete, hyperbolic manifold $N^{n+1}$. If the initial hypersurface $M^n$ is strictly $h$-convex, then each evolving hypersurface $M^n_t$ is also strictly $h$-convex along the flow (1.1).

Proof. Let $M_{ij} = a_{ij} - g_{ij}$. Recall the evolution equations for $a_{ij}$ and $g_{ij}$ along the mean curvature flow (1.1) as (2.7) and (2.1):

$$\frac{\partial}{\partial t} a_{ij} - \Delta a_{ij} = (h - 2H) a_{ij} + |A|^2 a_{ij} - n a_{ij} - h R_{0i0j} - a_{ij} R_{\ell m \ell m} + a_{\ell m} R_{\ell m ij},$$

where the covariant derivatives for the curvature tensor disappear since the sectional curvature is $-1$, and

$$\frac{\partial}{\partial t} g_{ij} = 2(h - H) a_{ij}.$$ 

Therefore we obtain the evolution equation for the symmetric tensor $M_{ij}$:

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + N_{ij},$$ 

where we have used $\Delta g = 0$ and

$$N_{ij} = (h - 2H) a_{ij} + |A|^2 a_{ij} - n a_{ij} - h R_{0i0j} - a_{ij} R_{\ell m \ell m} - a_{\ell m} R_{\ell m ij} + 2(H - h) a_{ij}.$$ 

Now recall from (2.8),

$$\bar{R}_{\alpha\beta\gamma\delta} = (-1) \cdot (\bar{g}_{\alpha\gamma} \bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta} \bar{g}_{\beta\gamma}).$$

Let $X$ be a null-eigenvector of $M_{ij}$ at some $(x_0, t_0)$. We arrange the coordinates such that at $(x_0, t_0)$, $X = e_1$, $g_{ij} = \delta_{ij}$ and $a_{ij} = \lambda_i \delta_{ij}$. This is justified as $\{g_{ij}\}$ is a symmetric positive-definite matrix, $\{a_{ij}\}$ is a symmetric matrix, and so they can be simultaneously diagonalized.

We examine term by term from (2.20) to arrive at:

$$N_{11} = (h - 2H) \lambda_1^2 + (|A|^2 - n) \lambda_1 + h + 2(n - 1) \lambda_1 + 2(\lambda_1 - H) + 2(H - h) \lambda_1.$$
Meanwhile, with \( X = e_1 \) being a null-eigenvector of \( M_{ij} \), we have \( \lambda_1 = 1 \) since \( M_{11} = a_{11} - g_{11} = 0 \). Thus, we have

\[
N_{11} = |A|^2 + n - 2H \geq \frac{1}{n}H^2 - 2H + n = \frac{1}{n}(H - n)^2 \geq 0.
\]

The conclusion follows from Hamilton’s maximum principle for tensors ([Ham82]).

3. Proof of Main Theorem

We are now ready to use iteration method to prove our main theorem. It’s divided into four steps discussed in four subsections accordingly.

3.1. Step One: Short Time Bounds. We start by bounding important geometric quantities for short time, with the bounds depending on the initial conditions. This is certainly expected for a smooth flow. However, one expects such bounds to hold only for a short time, and as the flow evolves such bounds would deteriorate by extending the time interval.

The first technical lemma is as follow:

**Lemma 3.1.** Let \( M_t^n \subset H^{n+1} \), \( n \geq 2 \) be a smooth closed solution to the VPMCF (1.1) for \( t \in [0, T) \) with \( T \leq \infty \). Assume

\[
\max \left\{ |M_0|^2, \max_{M_0} |A|^2, \int_{M_0} |\nabla^m A|^2 \, d\mu \right\} \leq \Lambda_0^2
\]

for some \( \Lambda_0 \gg 1 \) and all \( m \in [1, n + 3] \), where \( |M_t| \) is the \( n \)-dimensional surface area of \( M_t \) with the induced metric. There exist constants \( \epsilon_0 = \epsilon_0(n, \Lambda_0) > 0 \) and \( t_1 = t_1(n, \Lambda_0) \in (0, 1) \) such that if

\[
\int_{M_0} |A|^2 \, d\mu \leq \epsilon < \epsilon_0,
\]

then for any \( t \in [0, t_1] \) and any \( m \in [0, n + 3] \) we have

\[
\max \left\{ \max_{M_t} |A|^2, \int_{M_t} |\nabla^m A|^2 \, d\mu \right\} \leq 2\Lambda_0^2.
\]

Moreover, there exist \( C_1 = C_1(n, \Lambda_0) \) and some universal constant \( \alpha \in (0, 1) \) such that for any \( t \in [0, t_1] \)

\[
\max_{M_t} (|\dot{A}| + |\nabla H| + |h - H|) \leq C_1 \epsilon^\alpha.
\]

**Proof.** Recall from (2.6) the evolution equation for \(|A|^2\) is given by

\[
\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2 |\nabla A|^2 + 2 |A|^2 (|A|^2 + n) - 2h \text{tr} (A^3) + 2H(h - 2H).
\]

Using the facts that \( |\text{tr} (A^3)| \leq |A|^3 \) (see Lemma 2.2 [HS99]), and \( H^2 \leq n |A|^2 \), we obtain the following inequality on \( M_t \) for all \( t \in [0, T) \):

\[
\frac{\partial}{\partial t} |A|^2 \leq \Delta |A|^2 + 2 |A|^4 + 2n |A|^2 + 2 |h|(|A|^3 + \sqrt{n}|A|).
\]

Set \( f(t) = \max_{M_t} |A|^2 \), then \( f(t) \) satisfies

\[
\frac{\partial}{\partial t} f \leq 2 f^2 + 2n f + 2 |h|(|A|^3 + \sqrt{n}|A|).\]
\[
\begin{align*}
\leq & \quad 2f^2 + 2nf + 2\sqrt{n}f^2 + 2nf \\
\leq & \quad 4nf^2 + 4nf.
\end{align*}
\] (3.6)

One solves the comparison ODE explicitly to get \( U(t) > 0 \) satisfying
\[
\log \left( 1 + \frac{1}{U(t)} \right) = \log \left( 1 + \frac{1}{U(0)} \right) - 4nt,
\]
with \( U(0) = f(0) = \max_{M_0} |A|^2 \leq \Lambda_0^2 \) by (3.1). So \( f(t) \leq U(t) \) for all \( t \in [0, T) \).

Therefore, there exists some \( t_1 = t_1(n, \Lambda_0) \in (0, 1) \) such that
\[
\max_{M_t} |A|^2 \leq 2\Lambda_0^2 \quad \text{for all} \quad t \in [0, t_1].
\] (3.7)

Moreover, by choosing \( t_1 \) sufficiently small and integrating the inequality in Lemma 2.11 over \([0, t_1]\), we have
\[
\int_{M_t} |\nabla^m A|^2 \, d\mu \leq e^{C(n, \Lambda_0)t_1} \int_{M_0} |\nabla^m A|^2 \, d\mu \leq 2\Lambda_0^2
\]
for all \( t \in [0, t_1] \) and \( m \in [1, n + 3] \). Using the Sobolev embedding on compact manifolds [Aub98], this yields
\[
|A|_{C^2(M_t)} \leq C(n, \Lambda_0) \quad \text{for all} \quad t \in [0, t_1].
\] (3.9)

In light of
\[
|h| \leq \max_{M_t} |H| \leq n \max_{M_t} |A| \leq \sqrt{2n}\Lambda_0,
\]
\[
|\mathrm{tr}(\hat{A}^3)| \leq |\hat{A}|^3 \leq \sqrt{2}\Lambda_0|\hat{A}|^2,
\]
we integrate the evolution equation (2.14) for \(|\hat{A}|^2\) over \( M_t \) for \( t \in [0, t_1] \) to get
\[
\frac{\partial}{\partial t} \int_{M_t} |\hat{A}|^2 \, d\mu \leq C(n, \Lambda_0) \int_{M_t} |\hat{A}|^2 \, d\mu,
\] (3.10)

and so using (3.2) we have
\[
\int_{M_t} |\hat{A}|^2 \, d\mu \leq e^{C(n, \Lambda_0)t} \leq C(n, \Lambda_0)e \quad \text{for all} \quad t \in [0, t_1],
\] (3.11)

where the constant \( C(n, \Lambda_0) \) can be different at places. We then apply Hamilton’s interpolation inequalities (Theorem 2.9 with \( r = 1, \ p = q = 2 \)):
\[
\int_{M_t} |\nabla^2 \hat{A}|^2 \, d\mu \leq n \left( \int_{M_t} |\hat{A}|^2 \, d\mu \right)^{\frac{1}{2}} \left( \int_{M_t} |\nabla \hat{A}|^2 \, d\mu \right)^{\frac{1}{2}} \leq C(n, \Lambda_0)e^{\frac{1}{4}},
\] (3.12)

where we use \( |\nabla \hat{A}| \leq C(n)|\nabla \hat{A}| \) and the \( L^2 \)-bound for \(|\nabla \hat{A}|\) in (3.8). In fact, applying Theorem 2.9 inductively, we have for all \( m \in [0, n + 2] \),
\[
\int_{M_t} |\nabla^m \hat{A}|^2 \, d\mu \leq C(n, \Lambda_0)e^{1/2^m} \quad \text{for all} \quad t \in [0, t_1].
\] (3.13)

Now again by the Sobolev embedding [Aub98], we have:
\[
|\hat{A}|_{C^2(M_t)} \leq C(n, \Lambda_0)e^\alpha,
\] (3.14)

for all \( t \in [0, t_1] \) and some universal constant \( \alpha \in (0, 1) \). Now by (ii) of Lemma 2.6, for all \( t \in [0, t_1] \) we have
\[
\max_{M_t} |\nabla H| \leq C(n) \max_{M_t} |\nabla \hat{A}| \leq C(n, \Lambda_0)e^\alpha.
\] (3.15)
Furthermore, by Corollary 2.2, the surface area $|M_t|$ is non-increasing along the flow, i.e.

$$|M_t| \leq |M_0| \leq \Lambda_0^2. \tag{3.16}$$

Using Theorem 2.13, (3.7), (3.15) and (3.16), we arrive at

$$|h(t) - H(x,t)| = \left( \int_{M_t} d\mu \right)^{-1} \left| \int_{M_t} H(y,t) - H(x,t) d\mu(y) \right| 
\leq \text{diam}(M_t) \max_{M_t} |\nabla H| 
\leq C(n, \Lambda_0) \epsilon^\alpha \tag{3.17}$$

for all $(x,t) \in M_t$ and $t \in [0,t_1]$. This together with (3.14) and (3.15) give (3.4), and we conclude the proof.

With the above control of geometric quantities, we next show that the h-mean convexity is preserved for short time if the initial hypersurface is close to an umbilical sphere in the $L^2$-sense.

**Lemma 3.2.** Let $M^n_t \subset \mathbb{H}^{n+1}$ for $n \geq 2$ be a smooth closed solution to the VPMCF (1.1) as in Lemma 3.1 with the initial condition (3.1). Suppose

$$\min_{M_0}(H - n) \geq c_0 > 0. \tag{3.18}$$

Then there exist $\epsilon_1 = \epsilon_1(n, \Lambda_0) \in (0, \epsilon_0)$ and $T_1 = T_1(n, \Lambda_0) \in (0, t_1]$, where $\epsilon_0$ and $t_1$ are as in Lemma 3.1, such that if

$$\int_{M_0} |A|^2 d\mu \leq \epsilon < \epsilon_1,$$

then for $t \in [0,T_1]$ we have

$$\min_{M_t}(H - n) \geq \frac{c_0}{2} > 0. \tag{3.19}$$

**Proof.** We start with the evolution equation for $H$ (2.5):

$$H_t = \Delta H + (H - h)(|A|^2 - n).$$

By (3.7) and (3.9), for any $(x,t) \in M_t$, $t \in [0,t_1]$, we have:

$$\left| \frac{\partial}{\partial t} H \right| (x,t) \leq C(n, \Lambda_0), \tag{3.20}$$

where we have also used $|\nabla^2 H| \leq C(n)|\nabla^2 A|$. Using (3.15) and (3.20) and choosing $T_1 = T_1(n, \Lambda_0) \in (0, t_1]$ and $\epsilon_1 = \epsilon_1(n, \Lambda_0) \in (0, \epsilon_0)$ sufficiently small, we have

$$\min_{M_t}(H - n) \geq \frac{1}{2} \min_{M_0}(H - n) \geq \frac{c_0}{2} > 0.$$
3.2. Step Two: Reduction. In the previous subsection we have obtained estimates \((3.3)\) and \((3.4)\) on some time interval \([0, t_1]\), provided that the initial hypersurface is close to a umbilical sphere in the \(L^2\)-sense (see \((1.4)\)). In this step, we make a key reduction. Namely, we show it suffices to prove the main theorem when the mean curvature \(H\) of the evolving hypersurface is close to \(n\). In particular, we have the following.

**Proposition 3.3.** Let \(M^n_t \subset \mathbb{H}^{n+1}\) for \(n \geq 2\) be a smooth closed solution to the VPMCF \((1.1)\) on \(t \in [0, t_1]\) with \(t_1 = t_1(n, \Lambda_0) \in (0, 1)\), where \(t_1\) and \(\Lambda_0\) are as in Lemma 3.1. If \((3.1)\) and \((3.2)\) hold, then

1. either the evolving hypersurface \(M_t\) becomes strictly \(h\)-convex, and the flow \((1.1)\) exists for all time and converges exponentially to an umbilical sphere,
2. or there is a constant \(C_2 = C_2(n, \Lambda_0) > 0\) such that for all \((x, t) \in M_t\), \(t \in [0, t_1]\), we have

\[
|H(x, t) - n| \leq C_2 \epsilon^\frac{2}{n}
\]

where \(\epsilon\) is from \((3.2)\) and \(\alpha \in (0, 1)\) is from \((3.4)\).

**Proof.** On the time interval \([0, t_1]\), we recall the estimate \((3.4)\) from Lemma 3.1:

\[
\max_{M_t} (|A| + |\nabla H| + |h - H|) \leq C_1 \epsilon^\alpha
\]

for some \(C_1 = C_1(n, \Lambda_0) > 0\). Let \(\lambda_i\) be the principal curvatures of \(M_t\) at \((x, t) \in M_t\). Direct algebra gives

\[
|A|^2 = \frac{1}{n} \sum_{i<j} (\lambda_i - \lambda_j)^2,
\]

so there exists \(C_3 = C_3(n, \Lambda_0) > 0\) such that for all \((x, t) \in M_t, t \in [0, t_1]\),

\[
|\lambda_i(x, t) - \lambda_j(x, t)| \leq C_3 \epsilon^\alpha.
\]

Therefore for all \((x, t) \in M_t, t \in [0, t_1]\) and any fixed \(i \in \{1, 2, \ldots, n\}\), we have

\[
|H(x, t) - n\lambda_i(x)| \leq C_4 \epsilon^\alpha,
\]

for some \(C_4 = C_4(n, \Lambda_0) > 0\).

For some \(C_5 = C_5(n, \Lambda_0) > 0\) which will be fixed shortly, suppose there is \(\eta_0 = C_5 \epsilon^\frac{2}{n} > 0\) where \(\epsilon \in (0, \epsilon_0)\) and some \((x_0, t_0) \in M_{t_0}\) where \(t_0 \in [0, t_1]\) such that \(H(x_0, t_0) < n - \eta_0\). Then from \((3.24)\) we have:

\[
n\lambda_i(x_0, t_0) - C_4 \epsilon^\alpha \leq H(x_0, t_0) < n - \eta_0 = n - C_5 \epsilon^\frac{2}{n}.
\]

Since \(\epsilon \in (0, \epsilon_0)\) is small, for properly chosen \(C_5\) and \(C_6 = C_6(n, \Lambda_0) > 0\), we have \(\lambda_i(x_0, t_0) < 1 - C_6 \epsilon^\frac{2}{n}\) for all \(i \in \{1, 2, \ldots, n\}\). In light of \(\max_{M_t} |\nabla H| \leq C_1 \epsilon^\alpha\), the smallness of \(\epsilon\) and the diameter bound from Theorem 2.13, we have \(H < n\) at every point of \(M_{t_0}\). However, this contradicts the fact that any smooth closed hypersurface has at least one point whose mean curvature is greater than \(n\) in \(\mathbb{H}^{n+1}\) by comparing with horospheres.

Similarly for some \(C'_5 = C'_5(n, \Lambda_0) > 0\) which will be fixed shortly, suppose there is some \(\eta'_0 = C'_5 \epsilon^\frac{2}{n} > 0\) where \(\epsilon \in (0, \epsilon_0)\) and some \((x'_0, t'_0) \in M_{t_0}\) such that \(H(x'_0, t'_0) > n + \eta'_0\). We have

\[
n\lambda_i(x'_0, t'_0) + C_4 \epsilon^\alpha \geq H(x'_0, t'_0) > n + \eta'_0 = n + C'_5 \epsilon^\frac{2}{n}.
\]
Using again the smallness of \( \epsilon \), for properly chosen \( C_5^i \) and \( C_6^i \) we have \( \lambda_i(x_0^i, t_0^i) > 1 + C_6^i \epsilon^2 \) for any \( i \in \{1, 2, \cdots, n\} \). Using again the fact that \( \max_{M_t} |\nabla H| \leq C_1 \epsilon^\alpha \), smallness of \( \epsilon \) and the diameter bound from Theorem 2.13, we find \( \lambda_i(x, t_0^i) > 1 \) for all \( i \in \{1, 2, \cdots, n\} \) and all \( (x, t_0^i) \in M_0^i \). Namely, \( M_0^i \) is strictly \( h \)-convex. By the main theorem of [CRM07], the VPMCF then exists for all time after \( t = t_0^i \), stays strictly \( h \)-convex and converges exponentially to an umbilical sphere in \( \mathbb{H}^{n+1} \).

Finally, we are left with (3.21), which completes the proof. 

\[ \Box \]

Remark 3.4. By Proposition 3.3, we can now assume \( H \) of \( M_t \) is very close to \( n \) on time interval \([0, t_1]\), namely the inequality (3.21), for the remaining proof for Theorem 1.2, and therefore we now have \( H > 0 \) (hence \( h > 0 \)).

3.3. Step Three: Precise Decay. In the previous subsection we have obtained estimates (3.3), (3.4) and (3.19) on some short time interval \([0, T_1]\), provided that the initial hypersurface is close to an umbilical sphere in the \( L^2 \) sense (see (1.4)) and \( h \)-mean convex (see (3.18)). These bounds will likely deteriorate along the flow if we iterate for later time intervals. For an iteration argument to work, we need to establish time-independent bound on these quantities for this short time interval.

In this subsection, we show that, if estimates similar to (3.3), (3.4) and (3.19) hold on some time interval \([0, T_1]\), then we can choose sufficiently small \( \epsilon \) in the initial \( L^2 \)-bound (1.4) on \( \hat{A} \), such that \( |\hat{A}|, |\nabla H| \) and \( |h - H| \) exponentially decay on this time interval \([0, T_1]\). More precisely, we establish the following theorem.

**Theorem 3.5.** Let \( M_t^n \subset \mathbb{H}^{n+1} \) for \( n \geq 2 \) be a smooth closed solution to the VPMCF (1.1) with the initial condition

\[ \int_{M_0^n} |\hat{A}|^2 \, d\mu \leq \epsilon. \]

Suppose for any \( t \in [0, T_1] \) with \( T_1 \leq \infty \) and all \( m \in [1, n + 3] \) we have

\[ \max \left\{ |M_0^n|, \max_{M_t^n} |\hat{A}|^2, \int_{M_0^n} |\nabla^m \hat{A}|^2 \, d\mu \right\} \leq \Lambda_1^2, \quad \min_{M_t^n} (H - n) \geq \sigma, \]

\[ \max_{M_t^n} (|\hat{A}| + |\nabla H| + |h - H|) \leq C_1 e^{\beta}, \]

for constants \( \Lambda_1 > 0, \sigma > 0, \beta \in (0, 1) \) and \( C_1 > 0 \). Then there exists some \( \epsilon_2 = \epsilon_2(n, \Lambda_0, \beta, C_1) > 0 \) such that if \( \epsilon < \epsilon_2 \), then for all \( t \in [0, T_1] \) we have

\[ \max_{M_t^n} |\hat{A}| \leq \max_{M_0^n} |\hat{A}|, \]

\[ \max_{M_t^n} (|\hat{A}| + |\nabla H| + |h - H|) \leq C_2(n, \Lambda_1, C_1) \left( \max_{M_0^n} |\hat{A}| \right)^\alpha e^{-\alpha \sigma t}, \]

where \( \alpha \in (0, 1) \) is the universal constant from Lemma 3.1.

**Proof.** To start with, by Lemma 2.12 and (3.25), for \( m \in [1, n + 3] \) and \( t \in [0, T_1] \) we have

\[ \int_{M_t^n} |\nabla^m \hat{A}|^2 \, d\mu \leq C(n, \Lambda_1), \]
which works as the replacement of (3.3) as in the proof of Lemma 3.1. Now using (3.25) we compute
\[
\frac{1}{n} H^2 - \frac{hH}{n} = n - \frac{H \int_{M_t} d\mu}{n \int_{M_t} d\mu} \leq n - \frac{(n + \sigma)^2}{n} < -2\sigma,
\]
(3.29)
and
\[
\left| \frac{1}{n} H^2 - \frac{hH}{n} \right| (x, t) = \left| H(x, t) \cdot \frac{\int_{M_t} [H(x, t) - H(y, t)] d\mu(y)}{n \int_{M_t} d\mu} \right| \\
\leq \frac{1}{n} \max_{M_t} H \cdot \text{diam}(M_t) \cdot \max_{M_t} |\nabla H| \\
\leq C(n, \Lambda_1, C_1)e^\beta,
\]
where we have used \(|H| \leq \sqrt{n}|A| \leq \sqrt{n}\Lambda_1\) and Theorem 2.13.

Now by (2.14), (3.29) and (3.30), we have
\[
\frac{\partial}{\partial t} |\hat{A}|^2 = \Delta |\hat{A}|^2 - 2|\nabla \hat{A}|^2 + 2|\hat{A}|^2(|\hat{A}|^2 + n) - 2h \left\{ \text{tr} (\hat{A}^3) + \frac{2}{n} |\hat{A}|^2 e^H \right\} \\
\leq \Delta |\hat{A}|^2 + 2|\hat{A}|^2(|\hat{A}|^2 + \frac{1}{n} H^2 + n) + 2h |\hat{A}|^3 - \frac{4hH}{n} |\hat{A}|^2 \\
= \Delta |\hat{A}|^2 + 2 \left( |\hat{A}|^2 + h|\hat{A}| + \frac{1}{n} H^2 + n - \frac{2hH}{n} \right) |\hat{A}|^2 \\
\leq \Delta |\hat{A}|^2 - (4\sigma - \hat{C}e^\beta)|\hat{A}|^2 \\
\leq \Delta |\hat{A}|^2 - \sigma|\hat{A}|^2.
\]
where \(\hat{C} = \hat{C}(n, \Lambda_1, C_1) > 0\) and for the the last step we choose \(\epsilon\) to be sufficiently small. Therefore, we conclude the exponential decay of \(|\hat{A}|\) from the maximum principle, i.e. Theorem 2.8,
\[
\max_{M_t} |\hat{A}|^2 \leq e^{-\sigma t} \max_{M_0} |\hat{A}|^2,
\]
and the estimate (3.27) also follows. This is where the \(h\)-mean convexity is essentially involved in our arguments, see (3.29). Afterwards, we can prove (3.28) by the exact arguments in the proof of Lemma 3.1, namely (3.12)–(3.17).

3.4. Step Four: Time Extension. In this step, we use the exponential decay of \(|\hat{A}|, |\nabla H|\) and \(|h - H|\) on some short time interval obtained in previous step to extend the time interval of interest.

**Theorem 3.6.** Let \(M^n_t \subset \mathbb{H}^{n+1}\) for \(n \geq 2\) be a smooth closed solution to the VPMCF (1.1) with the initial hypersurface satisfying
\[
|M_0| \leq \Lambda_0, \quad \max_{M_0} |H| \leq \Lambda_0, \quad \int_{M_0} |\nabla m A|^2 d\mu \leq \Lambda_0^2, \quad \min_{M_0} (H - n) \geq \frac{1}{\Lambda_0^2} > 0
\]
for all \(m \in [1, n + 3]\). Suppose for any \(t \in [0, T]\) with \(T < \infty\) we have
\[
\max_{M_t} |A|^2 \leq \Lambda_0^2, \quad \min_{M_t} (H - n) \geq \frac{1}{2\Lambda_0} > 0
\]
(3.31)
and
\[ \max_{M_t} (|A| + |\nabla H| + |h - H|) \leq C_* e^{\frac{3}{2} \alpha^2} e^{-\alpha \sigma t} \leq C_* e^{\frac{3}{2}}, \]
where \( \alpha \in (0, 1) \) is the universal constant from Lemma 3.1 and \( \sigma = \frac{1}{2\Lambda_0} \) is as in Theorem 3.5. Then there exist \( \epsilon_3 = \epsilon_3(n, \Lambda_0, \alpha, C_*) > 0 \) and \( T_2 = T_2(n, \Lambda_0) > 0 \) such that if
\[ \int_{M_0} |A|^2 d\mu \leq \epsilon < \epsilon_3, \]
then (3.31) and (3.32) hold for \( t \in [0, T + T_2] \).

**Proof.** We begin by applying Lemma 3.1 and Lemma 3.2 to obtain \( \epsilon_4 = \epsilon_0(n, \Lambda_0^2) \) and \( T_2 = T_1(n, \Lambda_0^2) \) such that if
\[ \int_{M_0} |A|^2 d\mu \leq \epsilon < \epsilon_4, \]
then for all \( t \in [T, T + T_2] \) we have
\[ \max_{M_t} (|A|^2, \int_{M_t} |\nabla m A|^2 d\mu) \leq 2\Lambda_0^2 \quad \text{and} \quad \min_{M_t} (H - m) \geq \frac{1}{4\Lambda_0^2}, \]
\[ \max_{M_t} (|A| + |\nabla H| + |h - H|) \leq C_1(n, \Lambda_0) \epsilon^\alpha, \]
where \( C_1 \) and \( \alpha \) are from Lemma 3.1. Then choose \( \epsilon_5 = \epsilon_5(n, \Lambda_0, \alpha, C_*) > 0 \) sufficiently small so that for any \( \epsilon < \epsilon_5 \), we have
\[ C_1(n, \Lambda_0) \epsilon^{\alpha - \frac{3}{2}} \leq C_. \]
Therefore for all \( t \in [0, T + T_2] \) we have (3.34) and also
\[ \max_{M_t} (|A| + |\nabla H| + |h - H|) \leq C_* e^{\frac{3}{2}}. \]

By Corollary 2.2 the surface area \( |M_t| \) is non-increasing along the flow, therefore \( |M_t| \leq \Lambda_0 < \Lambda_0^2 \) by the initial condition (1.3) as long as the flow exists, in particular, on \( [0, T + T_2] \). Now we apply the Theorem 3.5 on \( [0, T + T_2] \) with \( \Lambda_0^2 = 2\Lambda_0^2, C_1 = C_*, \)
\( \beta = \frac{\alpha^2}{2} \) and \( \sigma = \frac{1}{4\Lambda_0^2} \) to conclude that for some \( \epsilon_6 := \epsilon_2(n, \Lambda_0, \alpha, C_*) > 0 \) sufficiently small, if \( \epsilon < \epsilon_6 \), then for all \( t \in [0, T + T_2] \), we have
\[ \max_{M_t} (|A| + |\nabla H| + |h - H|) \leq C_2(n, \Lambda_0, C_*) \left( \max_{M_0} |A| \right)^{\alpha} e^{-\alpha \sigma t} \]
\[ \leq C_2(n, \Lambda_0, C_*) [C_1(n, \Lambda_0)]^{\alpha} \epsilon^{\alpha - \frac{3}{2}}, \]
where we’ve used (3.4) at \( t = 0 \). Now choose \( \epsilon_7 = \epsilon_7(n, \Lambda_0, \alpha, C_*) > 0 \) small enough so that
\[ C_2(n, 2\Lambda_0^2, C_*) [C_1(n, \Lambda_0)]^{\alpha} \epsilon^{\alpha - \frac{3}{2}} \leq C_*, \]
thus (3.32) holds for all \( t \in [0, T + T_2] \).

We are left to show (3.31) for \( t \in [0, T + T_2] \). Let’s examine each term in (3.31). Consider \( \max_{M_t} |A| \). Recall the time derivative formula for \( h(t) \) (2.13) is given by
\[ h'(t) = \frac{\int_{M_t} (H - h)(|A|^2 - H^2 + hH) d\mu}{\int_{M_t} d\mu}. \]
Then using (3.34) and (3.35), we have
\[
|h'(t)| \leq C_3(n, C_*, \Lambda_0)\epsilon^{\frac{2}{3}} e^{-\alpha \sigma t}
\]
for all \( t \in [0, T + T_2] \). Note that, from the initial condition (1.3), we also have
\[
h(0) = \int_{M_0} H \, d\mu = 2 \left( M_0, H \right) \leq \max_{M_0} |H| \leq \Lambda_0.
\]
By choosing \( \epsilon < \epsilon_8 = \epsilon_8(n, \Lambda_0, \alpha, C_*) \) sufficiently small, we then have for any \( t \in [0, T + T_2] \):
\[
|h(t)| \leq \frac{6}{5} \lambda_0.
\]
Then by (3.35) and \( n \geq 2 \), for sufficiently large \( \Lambda_0 \) we have
\[
\max_{M_t} |A| = \max_{M_t} \sqrt{\left| A \right|^2 + \frac{1}{n} H^2} \leq \max_{M_t} \left( \left| A \right| + \frac{1}{\sqrt{n}} |H - h| \right) + \frac{1}{\sqrt{n}} |h(t)| \leq \Lambda_0.
\]
Finally, we consider the term \( \min_{M_t} (H - n) \). Using the evolution equations for \( H \) (see (2.5)) and \( d\mu \) (see (2.3)), we have
\[
\int_{M_t} H \, d\mu - \int_{M_0} H \, d\mu = \int_0^t \int_{M_s} H^2(h - H) + (H - h)(|A|^2 - n) \, d\mu \, ds
\]
\[
\geq -C(n, \Lambda_0, C_*) \epsilon^{\frac{1}{2}} \int_0^t e^{-\alpha \sigma s} \, ds \geq -C_4(n, \Lambda_0, \alpha, C_*) \epsilon^{\frac{1}{2}},
\]
where we’ve used again the bound on \( |H - h| \) in (3.32) for \( t \in [0, T + T_2] \). Therefore,
\[
\int_{M_t} H \, d\mu \geq \left( n + \frac{1}{\Lambda_0^2} \right) |M_0| - C_4(n, \Lambda_0, \alpha, C_*) \epsilon^{\frac{1}{2}} \geq \left( n + \frac{2}{3\Lambda_0^2} \right) |M_0|,
\]
where we’ve chosen \( \epsilon < \epsilon_{10} = \epsilon_{10}(n, \Lambda_0, \alpha, C_*) \) sufficiently small and used the initial condition \( \min_{M_0} (H - n) \geq \frac{1}{\Lambda_0^2} \).

Now applying the bound on \( |\nabla H| \) in (3.32) which holds for all \( t \in [0, T + T_2] \), we conclude from (3.41) and \( |M_t| \leq |M_0| \) that if \( \epsilon < \epsilon_{11} = \epsilon_{11}(n, \Lambda_0, \alpha, C_*) \) is chosen sufficiently small, then for all \( t \in [0, T + T_2] \), we have
\[
\min_{M_t} (H - n) \geq \frac{1}{2\Lambda_0^2}.
\]
Choosing \( \epsilon_3 = \min\{\epsilon_4, ..., \epsilon_{11}\} > 0 \), we conclude the proof of the theorem.

Now we conclude the proof of our main theorem.

**Proof.** (of Theorem 1.2) In light of Lemma 3.1, Lemma 3.2 and Theorem 3.5, by choosing \( \Lambda_0 \) sufficiently large, we are in position to apply Theorem 3.6. Thus we can keep extending the VPMCF and estimates (3.31) and (3.32) for a fixed amount of time depending only on the initial condition. Hence the flow (1.1) exists for all time and converges exponentially to a closed umbilic hypersurface in \( \mathbb{R}^{n+1} \) by (3.32), i.e. an umbilical sphere ([Spi79]).
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