q-Calculus Framework For Entropy In Multifractal Distributions

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Abstract

The connection between Tsallis entropy for a multifractal distribution and Jackson’s q-derivative is established. Based on this derivation and definition of a homogeneous function, a q-analogue of Shannon’s entropy is discussed. q-additivity of this entropy is shown. We also define q-analogue of Kullback relative entropy. The implications of lattice structure beneath q-calculus are highlighted in the context of q-entropy.

Non-extensive Tsallis Thermostatics (NTT) generalizes the Boltzmann-Gibbs (BG) statistics, to treat the non-extensivity of physical systems. It has been applied with success to many different situations (for complete reviews, see ref. [3]). One non-extensive quantity which is playing a useful role is Tsallis entropy. Given a probability distribution \( \{p_i\}_{i=1,...,W} \) where \( i \) is the index for system configuration, Tsallis entropy is given by

\[
S_q^T = \frac{1 - \sum_{i=1}^{W} (p_i)^q}{q - 1}.
\]  

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is a real parameter, assumed to be positive. \( W \) is the number of accessible configurations. Boltzmann’s constant \( k_B \) has been set equal to unity. As \( q \to 1 \), \( S_q^T \to -\sum p_i \ln p_i \), which is Shannon entropy. Thus parameter \( q \) describes deviations of Tsallis entropy from Shannon entropy.

On a different side, quantum algebras and in general, \( q \)-deformed physical theories \( ^3 \) have been subject of great attention in the last decade. An important feature of these theories is the presence of one (or more) deformation parameter \( q \), which describes deviation from standard Lie symmetries. Usually, for \( q \to 1 \), the formalism reverts to the standard one.

Recently, Tsallis noted \(^4 \) a similarity between \( q \)-numbers used in \( q \)-deformed theories and the entropy of eq. (1). Notably the pseudo-additive property of both quantities is alike. Abe \(^6 \) provided a more analytic foundation to this connection, based on \( q \)-calculus. Also in \(^7 \), \( q \)-deformed Landau diamagnetism was studied to strengthen the relation between quantum groups and NTT.

Originally, Tsallis’ proposal was aimed to accommodate scale invariance in a system with multifractal properties to the thermodynamic formalism. In this communication, we clarify the relation between \( q \)-calculus and multifractal properties of a probability distribution. We also propose a more general definition of \( q \)-entropy based on \( q \)-calculus. It is shown that pseudo-additivity of this entropy follows from \( q \)-additivity of \( q \)-numbers. Moreover, Tsallis entropy can also be accommodated in this definition. The lattice structure
behind the $q$-calculus framework is also highlighted.

For a multifractal distribution, we assume a local scaling for the probabilities,

$$p_i = R^{\alpha_i},$$

where $\{\alpha_i\}_{i=1,\ldots,W}$ is the set of scaling indices. $R$ is the size of each box used to cover the phase space. Understandably, $p_i$ can be interpreted as the probability of visiting the box labelled with $i$ and having a scaling index $\alpha_i$. Thus, in a manner similar to suggested in [6], we can write Shannon’s entropy as

$$-\sum_i \alpha_i \frac{d}{d\alpha_i} p_i = -\sum_i p_i \ln p_i.$$  \hspace{1cm} (3)

Note however, that the variable $\alpha_i$ here can be given a suitable interpretation which was a dummy variable in [6].

If we replace the ordinary derivative in (3) with Jackson’s $q$-derivative [8], we get

$$-\sum_i \alpha_i D^q_{\alpha_i} p_i = \frac{1 - \sum_{i=1}^{W} (p_i)^q}{q - 1},$$ \hspace{1cm} (4)

which is Tsallis entropy. Instead of Jackson’s derivative, if we use the symmetric $q$-derivative which has $q \leftrightarrow q^{-1}$ invariance, we obtain the alternate entropy suggested in [9]. In the following, we concentrate on the entropy based on Jackson’s derivative.

We argue that although use of ordinary derivative w.r.t. $\alpha_i$ in (3) is mathematically correct, it is not proper in an operational sense. Note that even though the size of the boxes is taken to be small, both the size and
their number is finite. So it looks more reasonable to use $q$-derivative which involves dilatation of the argument $\alpha_i$ than the ordinary derivative which takes into account infinitesimal changes of argument.

Consider now a generalized probability distribution $\{p_i\}$ where $p_i(\alpha_i)$ is homogeneous function of degree $a_i$ and $\alpha_i$ is not necessarily a scaling index. Then by definition

$$\alpha_i D_{\alpha_i}^q p_i(\alpha_i) = [a_i]p_i(\alpha_i),$$

where $[a_i] = \frac{q^{a_i} - 1}{q - 1}$ is the Jackson $q$-number. Then we define the $q$-entropy as

$$-\sum_i \alpha_i D_{\alpha_i}^q p_i(\alpha_i) = -\sum_i [a_i]p_i(\alpha_i).$$

As $q \to 1$, we get

$$-\sum_i \alpha_i \frac{d}{d\alpha_i} p_i = -\sum_i a_i p_i.$$  

If we identify, $a_i = \ln p_i$ in (7), we get Shannon entropy on the r.h.s. Alternatively, if we set $\alpha_i$ as the local scaling index, we again obtain Shannon entropy as defined in (3). Similarly, if we take $\alpha_i$ in (6) as scaling index, then equality of (4) and (6) gives

$$[a_i] = \frac{q^{a_i} - 1}{q - 1} = \frac{(p_i)^{q-1} - 1}{q - 1}$$

which gives

$$a_i = \frac{q - 1}{\ln q} \ln p_i$$

Thus we can alternatively define Tsallis entropy as the negative of mean of $[a_i]$’s over the probability distribution, where $a_i$ is given by (3).
Tsallis entropy can be looked upon as a $q$-deformed Shannon entropy \cite{6}. Thus by setting $a_i = \ln p_i$ in the definition of $q$-entropy (in order to obtain Shannon entropy in the limit $q \to 1$, we get another $q$-deformed analogue of Shannon entropy

$$S_q' = -\sum_i [\ln p_i] p_i. \quad (10)$$

Note that $S_q' > S^T_q$ for $0 < q < 1$. In fig. 1 we compare the $i$-th term of Shannon, Tsallis and $q$-entropy by taking $p_i = 0.2$. Only entropy values for $q < 1$ appear to be physically meaningful, as discussed below. Thus $S_q'$ is a concave function due to a similar property of Tsallis entropy.

To see the additive property of $S_q'$, consider two independent subsystems $I$ and $II$, described by normalized probability distributions $\{p_i\}$ and $\{p_j\}$ respectively. Then the $q$-entropy of the combined system may be written as

$$S_q'(I + II) = -\sum_{i,j} [\ln p_{ij}] p_{ij},$$

$$= -\sum_{i,j} [\ln p_i + \ln p_j] p_i p_j$$

$$= S_q'(I) + S_q'(II) + (1 - q)S_q'(I)S_q'(II). \quad (11)$$

We have made use of the pseudo-additive property of the $q$-numbers and the normalization property of the probability distributions. The similar property of Tsallis entropy can be seen to emerge because of the relation (9).

We may also define the $q$-analogue of Kullback relative entropy \cite{10}, in going from a probability distribution $p^0$ to another one $p$. Consider the difference $[a_i] - [a_i^0]$, where $a_i = \ln p_i$ and $a_i^0 = \ln p_i^0$. The average weighted
against the new probability distribution, gives the $q$-analogue of the Kullback relative entropy,

$$K_q(p, p^0) = \sum_i p_i ([a_i] - [a^0_i]).$$  \hfill (12)

In the limit of $q \to 1$, we get the standard Kullback relative entropy. Using

$$K_q(p^0, p) = \sum_i p^0_i ([a^0_i] - [a_i]),$$

we obtain the $q$-analogue of the symmetric sum,

$$D_q(p, p^0) = K_q(p, p^0) + K_q(p^0, p) = \sum_i ([a_i] - [a^0_i])(p_i - p^0_i)$$ \hfill (13)

Each term in the sum is positive and is zero iff $p_i = p^0_i$. Thus this function appears suitable for a metric in the functional space of probability distributions.

Finally, we remark on the lattice structure which underlies the $q$-calculus framework of entropy. A natural lattice already exists, because we partition the phase space into boxes of equal size $R$. The lattice constant $R$ can be identified with $|q - 1|$. Thus $q \to 1$ limit also implies $R \to 0$. The finite size of the boxes causes coarse graining of the phase space, as a result the information we would have about the structure of the distribution is also coarse grained. Thus the value of generalized $q$-entropies should be greater than the Shannon entropy, which is shown here as the limit of $q \to 1$ case. We note that both Tsallis and $q$-entropy satisfy this condition for $q < 1$ (Fig. 1). The divergence of $q$-entropy can also be explained because as $q \to 0$, the size of the boxes increases, which causes more loss of information and thus increase in entropy.
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Figure 1: Plots of $i$-th term of entropies for $p_i = 0.2$ against the deformation parameter $q$.

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