Tangent Bundles Dynamics and Its Consequences*

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Abstract

We will consider here some dynamics of the tangent map, weaker than hyperbolicity, and we will discuss if these structures are rich enough to provide a good description of the dynamics from a topological and geometrical point of view. This results are useful in attempting to obtain global scenario in terms of generic phenomena relative both to the space of dynamics and to the space of trajectories. Moreover, we will relate these results with the study of systems that remain globally transitive under small perturbations.

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1. Introduction

A long time goal in the theory of dynamical systems is to describe the dynamics of “big sets” (generic or residual, dense, etc) in the space of all dynamical systems. It was thought in the sixties that this could be realized by the so called hyperbolic ones: systems with the assumption that the tangent bundle over the Limit set \( (L(f)) \), the accumulation points of any orbit) splits into two complementary subbundles that are uniformly forward (respectively backward) contracted by the tangent map by. The richness of this description would follow from the fact that the hyperbolic dynamic on the tangent bundle characterizes the dynamic over the manifold from a geometrical and topological point of view.

Nevertheless, uniform hyperbolicity were soon realized to be a property less universal than it was initially thought: there are open sets in the space of dynamics which are non-hyperbolic. After some initial examples of non-density of the hyperbolic systems in the universe of all systems (see \( \text{[S]} \text{[AS]} \)), two key aspects were

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focused in these examples. On one hand, open sets of non-hyperbolic diffeomorphisms which remain transitive under perturbation (existence of a dense orbit for any system). On the other, residual sets of non-hyperbolic diffeomorphisms, each one exhibiting infinitely many transitive sets. Roughly speaking, it was showed that two kind of different phenomena can appear in the complement of the hyperbolic systems: a) dynamics that robustly can be decomposed into a finite number of closed transitive sets; b) dynamics that generically exhibit infinitely many disjoint transitive sets.

The first kind of phenomena occurs in dimension higher or equal than 3 and although the examples are not hyperbolic they exhibit some kind of decomposition of the tangent bundle into invariant subbundles. The second one, was obtained by Newhouse (see [N1], [N2], [N3]), who following an early work of the non-density of hyperbolicity for $C^2$ surface maps, showed that the unfolding of a homoclinic bifurcation (non transversal intersection of stable and unstable manifolds of periodic points) leads to a very rich dynamics: residual subsets of open sets of diffeomorphisms whose elements display infinitely many sinks.

These new results naturally pushed some aspects of the theory on dynamical systems in different directions:

1. The study of the dynamical phenomena obtained from homoclinic bifurcations;
2. The characterization of universal mechanisms that could yield to robustly non-hyperbolic behavior;
3. The study and characterization of isolated transitive sets that remain transitive for all nearby system (robust transitivity);
4. The dynamical consequences that follows from some kind of the dynamics over the tangent bundle, weaker than the hyperbolic one.

As we will show, these problems are related and they indeed constitute different aspects of the same phenomena. In many cases, such relations provide a conceptual framework, as the hyperbolic theory did for the case of transverse homoclinic orbits.

In the next section, we will discuss the previous aspects for the case of surfaces maps, and in particular we will consider a dynamics in the tangent bundle weaker than hyperbolicity, called dominated splitting. In section 3 we will discuss the problems about the robust transitivity and its relation with other dynamics on the tangent bundle. Finally, in the last section, we will consider the equivalent problems for flows taking into account their intrinsic characteristic that leads to further questions and difficulties do the presence of singularities. We point out that in this survey we will focus more into topological and geometrical aspects of the dynamic rather on the ergodic ones.

Many of the issues discussed here are consequences of works and talks with Martin Sambarino. I also want to thanks Maria J. Pacifico for her help to improve this article.
2. Surfaces maps, homoclinic tangencies and “non-critical” behaviors

After the seminal works of Newhouse, many others were developed in the direction to understand the phenomena that could appear after a bifurcation of homoclinic tangencies (tangent intersection of stable and unstable manifolds of periodic points). In fact, other fundamental dynamic prototype were found in this context, namely the so called cascade of bifurcations, the Hénon-like strange attractor ([MC], [MV]) (even infinitely many coexisting ones ([1]), and superexponential growth of periodic points ([K]). Even before these last results, Palis ([PT], [P]) conjectured that the presence of a homoclinic tangency is a very common phenomenon in the complement of the closure of the hyperbolic ones. In fact, if the conjecture is true, then homoclinic bifurcation could play a central role in the global understanding of the space of dynamics for it would imply that each of these bifurcation phenomena is dense in the complement of the closure of the hyperbolic ones. More precisely, he conjectured that Every $f \in Diff^r(M^2), r \geq 1$, can be $C^r$-approximated by a diffeomorphism exhibiting either a homoclinic tangency or by one which is hyperbolic.

The presence of homoclinic tangencies have many analogies with the presence of critical points for one-dimensional endomorphisms. Homoclinic tangencies correspond in the one dimensional setting to preperiodic critical points and it is known that its bifurcation leads to complex dynamics. On the other hand, Mañe (see [MN]) showed that for regular and generic one-dimensional endomorphisms, the absence of critical points is enough to guarantee hyperbolicity. This result raises the question about the dynamical properties of surface maps exhibiting no homoclinic tangencies. In this direction, first it is proved in that some kind of dynamic over the tangent bundle (weaker than the hyperbolic one) can be obtained in the robust lack of homoclinic tangencies. And later, it is showed that this dynamic on the tangent bundle is rich enough to describe the dynamic on the manifold. More precisely:

Theorem 1 ([PS]): Surface diffeomorphisms that can not be $C^1$-approximated by another exhibiting homoclinic tangencies, has the property that its Limit set has dominated splitting.

An $f$-invariant set $\Lambda$ has dominated splitting if the tangent bundle can be decomposed into two invariant subbundles $T_\Lambda M = E \oplus F$, such that:
\[
\|Df^n_{|E(x)}\| \cdot \|Df^{-n}_{|F(f^n(x))}\| \leq C \lambda^n, \text{ for all } x \in \Lambda, n \geq 0,
\]
with $C > 0$ and $0 < \lambda < 1$.

As, dominated splitting prevents the presence of tangencies, we could say that domination plays for surface diffeomorphisms the role that the non-critical behavior does for one dimensional endomorphisms.

To have a satisfactory description for this non-critical behavior (existence of a dominated splitting), we should describe its dynamical consequences. A natural question arises: is it possible to describe the dynamics of a system having dominated splitting?

The next result gives a positive answer (as satisfactory as in hyperbolic case) when $M$ is a compact surface. More precisely, we give a complete description of the
topological dynamics of a $C^2$ system having a dominated splitting. Actually, first, the dominated decomposition is understood under a generic assumption.

**Theorem 2 (PS1):** Let $f \in Diff^2(M^2)$ and assume that $\Lambda \subset L(f)$ is a compact invariant set exhibiting a dominated splitting such that any periodic point is a hyperbolic periodic point. Then, $\Lambda = \Lambda_1 \cup \Lambda_2$ where $\Lambda_1$ is hyperbolic and $\Lambda_2$ consists of a finite union of periodic simple closed curves $C_1, \ldots, C_n$, normally hyperbolic, and such that $f^{m_i} : C_i \to \mathbb{C}_i$ is conjugated to an irrational rotation ($m_i$ denotes the period of $C_i$).

Using this Theorem and understanding the obstruction for the hyperbolicity assuming domination, we can characterize $L(f)$ without any generic assumption.

**Theorem 3 (PS2):** Let $f \in Diff^2(M^2)$ and assume that $L(f)$ has a dominated splitting. Then $L(f)$ can be decomposed into $L(f) = J \cup \tilde{L}(f) \cup R$ such that:

1. $J$ is contained in a finite union of normally hyperbolic periodic arcs.
2. $R$ is a finite union of normally hyperbolic periodic simple closed curves supporting an irrational rotation.
3. $f/\tilde{L}(f)$ is expansive and admits a spectral decomposition (into finitely many homoclinic classes).

Roughly speaking, the above theorem says that the dynamics of a $C^2$ diffeomorphism having a dominated splitting can be decomposed into two parts: one where the dynamic consists on periodic and almost periodic motions ($J, R$) with the diffeomorphism acting equicontinuously, and another one where the dynamics is expansive and similar to the hyperbolic case. Moreover, given a set having dominated decomposition, it is characterized its stable and unstable set, its continuation by perturbation, and their basic pieces (see [PS2]). Let us say also, that in solving the above problem, another kind of differentiable dynamical problem arose: how is affected the dynamics of a system, when its smoothness is improved?

Putting theorem 1 and 2 together, we prove the conjecture of Palis for surface diffeomorphisms in the $C^1$—topology:

**Theorem 4 (PS1):** Let $M^2$ be a two dimensional compact manifold and let $f \in Diff^1(M^2)$. Then, $f$ can be $C^1$-approximated either by a diffeomorphism exhibiting a homoclinic tangency or by an Axiom A diffeomorphism.

Similar arguments, prove that the variation of the topological entropy leads to the unfolding of homoclinic tangencies. Moreover the presence of infinitely many sinks with unbounded period also implies the unfolding of tangencies (see [PS2], [PS4]).

We want to emphasize that theorem 4 and the previous comments are strictly $C^1$ (on the other hand, theorem 2, and 3 assume that the map is $C^2$), and nothing is known in the $C^2$—topology. We would like to understand what happens in the $C^2$—topology, since many rich dynamical phenomena take place for smooth maps. About this problem we would like to make some remarks.

Recall that for smooth one-dimensional endomorphisms, the absence of critical points was enough to guarantee hyperbolicity. But for surface maps (and due to the lack of understanding) we required $C^1$-robust absence of tangencies. Many of the tools used in the $C^1$ case ($C^1$-closing Lema, perturbation of the tangent map along finite orbits) are unknown in higher topology and even in same particular
situation they are also false (see [G], [PS2]). So, taking in mind the scenario for one-dimensional dynamics, instead of ask about the $C^r$—robust absence of tangencies (for $r \geq 2$), we could try to know what is the two dimensional phenomena whose presence breaks the domination and whose absence guarantee it? In other words, what are two dimensional critical points?

To address these problems we should consider previously some other weaker questions. Observe that any dominated splitting is a continuous one. Is it true the converse, at least generically? To answer this question we would face the following: if there is a continuous invariant splitting over the Limit set for an smooth map (or even assuming an stronger hypothesis: existence of two continuous invariant foliations) can we describe the dynamic of $f$? It is clear that are dynamics exhibiting continuous splitting which are not dominated, for instance, maps exhibiting some kind of saddle connection and maps on the torus obtained as $(x, y) \rightarrow (x+\alpha, y+\beta)$. Are those dynamics the unique ones that do not exhibit domination?

On the other hand, splitting dealing with critical behaviors (tangencies or ”almost tangencies”) are well known in the measure-theoretical setting. This is the case of the non-uniform hyperbolicity (or Pesin theory), where the tangent bundle splits for points a.e. with respect to some invariant measure, and vectors are asymptotically contracted or expanded in a rate that may depend on the base point. Are the invariant measures for smooth maps on surfaces with one non zero Lyapunov exponent, non-uniformly hyperbolic? In other words, one non-zero lyapunov exponent implies that the other is also non-zero? This is not true in general, since there exist ergodic invariant measures with only one non zero Lyapunov exponent: measure supported on invariant circle normally hyperbolic; measure over a non-hyperbolic periodic point; time one map of a Cherry flows; two dimensional version of one dimension phenomena like infinitely renormalizable and absorbing cantor sets. Are those dynamics the unique counterexamples?

Now we would like to say a few words about the proof of Theorem 2. First, it is showed that the local unstable (stable) sets are one dimensional manifolds tangent to the direction $F$ ($E$ respectively). This is achieved by, given an explicit characterization of the Lyapunov stable sets under the hypothesis of domination (latter we will give a precise statement). After that, it is proved that the length of the negative iterates of the local unstable (positive for the local stable) manifolds are sumable, and using arguments of distortion hyperbolicity is concluded.

We say that the point $x$ is Lyapunov stable (in the future) if given $\epsilon > 0$ there exists $\delta > 0$ such that $f^n(B_\delta(x)) \subset B_\epsilon(f^n(x))$ for any positive integer $n$. Its characterization is done in any dimension whit the solely assumption that one of the subbundles is one dimensional. To avoid confusion, we call such splitting codimension one dominated splitting.

**Theorem 5:** Let $f : M \rightarrow M$ be a $C^2$-diffeomorphism of a finite dimensional compact riemannian manifold $M$ and let $\Lambda$ be a set having a codimension one dominated splitting. Then there exists a neighborhood $V$ of $\Lambda$ such that if $f^n(x) \in V$ for any positive integer $n$ and $x$ is Lyapunov stable, one of the following holds:

1. $\omega(x)$ is a periodic orbit,
2. $\omega(x)$ is a periodic curve normally attractive supporting and irrational rota-
As we said before, this theorem have important consequences related to the direction $F$. Its local invariant tangent manifold either is dynamically defined (it is a subset of the local unstable set) or there are well understood phenomena: either there are periodic curves normally contractive $\gamma$ with small length, or there are semi-attracting periodic points, or there are closed invariant curves normally hyperbolic with dynamics conjugated to an irrational rotation. With this characterization in mind, and assuming domination over the whole manifold, it is proved that $F$ is also uniquely integrable.

3. Robust transitivity

As we said in the beginning, in dimension higher or equal than 3, there exist $C^r$-open set of diffeomorphisms which are transitive and non-hyperbolic ($r \geq 1$). Observe that this phenomena take place in the $C^1$-topology, fact that it is unknown for one of the dynamics phenomena that we consider in the previous section: the residual sets of infinitely many sinks for surface maps.

The first examples of robust non-hyperbolic systems (examples of robust transitive systems which are not Anosov) were given by M. Shub (see [Sh]), who considered on the 4-torus, skew-products of an Anosov with a Derived of Anosov diffeomorphisms. Then, R. Mañé (see [M]) reduced the dimension of such examples by showing that certain Derived of Anosov diffeomorphisms on the 3-torus are robust transitive. Later, L. Díaz, (see [D1]) constructed examples obtained as a bifurcation of an heteroclinic cycle (cycle involving points of different indices). This last ideas was pushed in [BD] where it was showed a general geometric construction of robust transitive attractors.

All these systems show a partial hyperbolic splitting, which allows the tangent bundle to split into $Df$-invariant subbundles $TM = E^s \oplus E^c \oplus E^u$, where the behavior of vectors in $E^s, E^u$ under iterates of the tangent map is similar to the hyperbolic case, but vectors in $E^c$ may be neutral for the action of the tangent map. On the other hand, recently, it was proved by C. Bonatti and M. Viana that there are opens sets of transitive diffeomorphisms exhibiting a dominated splitting which do not fall into the category of partially hyperbolic ones (see [BV]).

These new situations lead to ask two natural questions: Is there a characterization of robust transitive sets that also gives dynamical information about them? Can we describe the dynamics under the assumption of either partial hyperbolicity or dominated decomposition?

The next result shows that this two questions are extremely related. In fact, some kind of dynamics on the tangent bundle is implied by the robust transitivity (see [M] for surfaces, [DPU] for three dimensional manifolds , and [BDP] for the $n$-dimensional case):

**Theorem 6:** Every robustly transitive set of a $C^1$-diffeomorphism has dominated splitting whose extremal bundles are uniformly volume contracting or expanding.

The central idea in this theorem is to show that, in the lack of domination, the
eigenspaces of a linear map (obtained by multiplying many bounded linear maps) are very unstable: by small perturbation of each of the factors, one can mix the eigenvalues in order to get a homothety, which will correspond to the creation of either a sink or a source, situation not allowed in the case of the robust transitivity. This last theorem also can be formulated in the following way:

**Theorem 7** [BDP]: There is a residual subset of $C^1$-diffeomorphisms such that for any diffeomorphism in the residual set, it is verified that for any homoclinic class of a periodic point (the closure of the intersection of the stable and unstable manifold of it) either has dominated splitting or it is contained in the closure of infinitely many sources or sinks.

What about the converse of theorem 6? Is it true that generically a transitive system exhibiting some kind of splitting is robust transitive? On the other hand, all the examples of robust transitivity are based in either a property of the initial system or in a geometrical construction. But, does exist a necessary and sufficient condition among the partial hyperbolic system such that transitivity is equivalent to robust transitivity? Can this property be characterized in terms of the dynamic of the tangent map? This is clear for Anosov maps, where transitivity implies robust transitivity, but what about for the non-hyperbolic?. In the direction to understand this problem, in [PS5] was introduced a dynamic on the tangent bundle enough to guarantee robustness of transitivity.

**Theorem 8:** Let $f \in \text{Diff}^r(M)$ be a transitive partial hyperbolic system verifying that there is $n_0 > 0$ such that for any $x \in M$ there are $y^u(x) \in W^u(x)$ and $y^s(x) \in W^s(x)$ with:

1. $|Df^m|_{E^c(f^m(y^u(x)))} > 2$ for any $m > 0$,
2. $|Df^{-m}|_{E^s(f^{-m}(y^s(x)))} > 2$ for any $m > 0$,

then, $f$ is a non-hyperbolic robust transitive system.

Is this property generically necessary?

All these questions naturally push in the direction to understand the dynamic induced by either a partial hyperbolic system or a dominated splitting. In particular, do they exhibit (generically) spectral decomposition, as was showed for a hyperbolic system and for domination on surfaces? Observe that the non-hyperbolicity of these systems is related with the presence of points of different index. Is this (generically) a necessary condition for non-hyperbolicity? In other words, assuming that there are only points of the same index, can one conclude (generically) hyperbolicity? For the case of domination, it is possible to show that if the extremal directions are one-dimensional, then they behave topologically as a hyperbolic one (see [PS3]). Moreover, it is showed that homoclinic classes with codimension one dominated splitting and contractive bundle are generically hyperbolic. But, what happens if the external directions are one-dimensional? And what about the central directions? Of course, this question can be considered in a simpler situation: a partial hyperbolic splitting with only one dimensional central direction. Does the dynamic over the central direction characterize the kind of partial hyperbolic systems?

Many of the questions done for partial hyperbolic systems can be formulated for Iterated Function Systems. In same sense, these systems works as a model of partial hyperbolic ones. And its solution, could give an indication how to deal in
the general case.

In dimension higher than two, another kind of homoclinic bifurcation breaks the hyperbolicity: the so called heteroclinic cycles (intersection of the stable and unstable manifolds of points of different indices, see [D1] and [D2]). In particular, the unfold of these cycles imply the existence of striking dynamics being the more important, the appearance of non-hyperbolic robust transitive sets (see [KP] also for superexponential growth of periodic point associated to the unfolding of heteroclinic cycles). Moreover, any non-hyperbolic robust transitive sets exhibits generically heteroclinic cycles. In same sense, these cycles play the role for the partial hyperbolic theory as transversal intersection play for the hyperbolic theory.

A similar conjecture as the one for surfaces, was formulated by Palis in any dimension: Every \( f \in Diff^r(M), r \geq 1 \), can be \( C^r \)-approximated by a diffeomorphism exhibiting either a homoclinic tangency, a heteroclinic cycle or by one which is hyperbolic.

A similar approach as the one done in dimension 2 could be done: first, try to find the dynamic on the tangent bundle for systems \( C^1 \) – far from tangencies. About this, in [LW] it was proved a similar result as the one for surfaces: far from tangencies implies domination. Does far from heteroclinic cycles imply hyperbolicity? Does this imply that sets with periodic points of different index can not accumulate one on the other? And as we asked before: sets showing dominated decomposition exhibiting points of the same index are generically hyperbolic? These problems are also related with the problems involving tangencies and sinks: can a systems showing infinitely many sinks be approximated by another one showing tangencies? It was showed that this is true for surfaces maps (see [PS3]), and in the case of higher dimension in [PS4] is given a positive answer assuming that the sinks accumulate on a sectional dissipative homoclinic class.

On the other hand, there is a vast works about conservative partial hyperbolic systems describing, in same particular cases, their ergodic properties. The description of the dynamics strongly use the invariance of the volume measure, information that it is not available in the general case that we would like describe. For references about it see the complete review on this subject done by Burns, Pugh, Shub and Wilkinson ([BPSW]). Moreover it is showed in [Bo] and [BoV] some kind of dichotomy (as the one done in theorem 8) for conservative maps in terms of Lyapunov exponents and domination. Also I would like to mention a recent and remarkable work of F. Rodriguez Hertz ([R]) where is proved that many Linear automorphisms on \( T^4 \) are stable ergodic, using different kinds of techniques that even if only work in the conservative case, they could be useful to understand the general case.

4. Flows

For flows, a striking example is the Lorenz attractor [Lo], given by the solutions of the polynomial vector field in \( R^3 \):

\[
X(x, y, z) = \begin{cases} 
\dot{x} = -\alpha x + \alpha y \\
\dot{y} = \beta x - y - xz \\
\dot{z} = -\gamma z + xy
\end{cases}
\]

(1)
where $\alpha, \beta, \gamma$ are real parameters. Numerical experiments performed by Lorenz (for $\alpha = 10, \beta = 28$ and $\gamma = 8/3$) suggested the existence, in a robust way, of a strange attractor toward which tends a full neighborhood of positive trajectories of the above system. That is, the strange attractor could not be destroyed by any perturbation of the parameters. Most important, the attractor contains an equilibrium point $(0,0,0)$, and periodic points accumulating on it, and hence can not be hyperbolic. Notably, only now, three and a half decades after this remarkable work, was it proved [Tu] that the solutions of (1) satisfy such a property for values $\alpha, \beta, \gamma$ near the ones considered by Lorenz.

However, already in the mid-seventies, the existence of robust non-hyperbolic attractors was proved for flows introduced in [ABS] and [Gu], which we now call geometric models for Lorenz attractors. In particular, they exhibit, in a robust way, an attracting transitive set with an equilibrium (singularity). Moreover, the properties of this geometrical models, allow one to extract very complete dynamical information. A natural question raises, is such features present for any robust transitive set?

In [MPP] a positive answer for this question is given:

**Theorem 9:** $C^1$ robust transitive sets with singularities on closed 3-manifolds verifies:

1. there are either proper attractors or proper repellers;
2. the eigenvalues at the singularities satisfy the same inequalities as the corresponding ones at the singularity in a Lorenz geometrical model;
3. there are partially hyperbolic with a volume expanding central direction.

The presence of a singularity prevents these attractors from being hyperbolic. But they exhibit a weaker form of hyperbolicity singular hyperbolic splitting. This class of vector fields contains the Axiom A systems, the geometric Lorenz attractors and the singular horseshoes in (LP), among other systems. Currently, there is a rather satisfactory and complete description of singular hyperbolic vector fields defined on 3-dimensional manifolds (but the panorama in higher dimensions remains open). More precisely, it is proved in a sequel of works that a singular hyperbolic set for flow is $K^\ast$-expansive, the periodic orbits are dense in its limit set, and it has a spectral decomposition (see [PP], [K]).

On the other hand, for the case of flows, appears a new kind of bifurcation that leads to a new dynamics distinct from the ones for diffeomorphism: the so called singular cycles (cycles involving singularities and periodic orbits, see [BLMP], [Mc], [MP] and [MPPT] for examples of dynamics in the sequel of the unfolding of it). Systems exhibiting this cycles are dense among open set of systems exhibiting a singular hyperbolic splitting. Moreover, recently A. Arroyo and F. Rodriguez Hertz (see [AR]) studying the dynamical consequences of the dominated splitting for the Linear Poincare flow, proved that any three dimensional flow can be $C^1$-approximated either by a flow exhibiting tangency or singular cycle, or by a hyperbolic one.
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