DISCRETE TIME QUANTUM MECHANICS

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ABSTRACT

This paper summarizes a research program that has been underway for a decade. The objective is to find a fast and accurate scheme for solving quantum problems which does not involve a Monte Carlo algorithm. We use an alternative strategy based on the method of finite elements. We are able to formulate fully consistent quantum-mechanical systems directly on a lattice in terms of operator difference equations. One advantage of this discretized formulation of quantum mechanics is that the ambiguities associated with operator ordering are eliminated. Furthermore, the scheme provides an easy way in which to obtain the energy levels of the theory numerically. A generalized version of this discretization scheme can be applied to quantum field theory problems. The difficulties normally associated with fermion doubling are eliminated. Also, one can incorporate local gauge invariance in the finite-element formulation. Results for some field theory models are summarized. In particular, we review the calculation of the anomaly in two-dimensional quantum electrodynamics (the Schwinger model). Finally, we discuss nonabelian gauge theories.

PACS numbers: 02.90.+p, 11.90.+t, 11.10.-z
I. INTRODUCTION

A classical physical system is described by a differential equation supplemented by initial conditions. The differential equation characterizes the time evolution of the system. In quantum physics one cannot specify initial conditions for an operator differential equation because specifying the values of the fundamental observables (operators) at a given time would violate the uncertainty principle. Thus, a quantum theory is described by an operator differential equation (a field equation) supplemented by an equal-time commutation relation (ETCR). The ETCR is a constraint which must hold at each time; its constancy in time expresses the unitarity (conservation of probability) of the quantum theory. The ETCR of quantum mechanics replaces the initial condition of classical mechanics.

In the study of quantum field theory there are inherent ambiguities and divergences associated with an operator-differential-equation formulation because operator-valued distributions in the continuum are so singular that products of such operators do not exist. It is well known that introducing a space-time lattice is a good way to remove these ambiguities and thus to regularize a continuum quantum field theory. The content of the theory is then contained in the continuum limit of the lattice theory.

Ordinarily, the lattice is introduced as a mathematical artifice to make sense of the functional integral representation of a quantum field-theory. In this computational scheme the lattice regularizes the functional integral as an infinite product of ordinary Riemann integrals. Then, the infinite product is approximated as a finite product of integrals which are evaluated by Monte-Carlo methods. This procedure is slow; doubling the computer time gives only minimal improvements in accuracy.

The program discussed here uses the lattice in a completely different and more fundamental way. We show how to formulate and construct a fully consistent (unitary) quantum theory on a space-time lattice. Such a theory is defined in terms of an operator difference equation (rather than a differential equation) and an ETCR that holds at equal-time lattice points. The operators in such a theory have none of the problems (infinities) that operators in the continuum have. We will see that, while there is no hope of solving operator differential equations, operator difference equations can be solved exactly. Thus, for each lattice we obtain exact closed-form solutions for the field operators rather than a slowly converging sequence of Monte-Carlo approximations.

To convert an operator differential equation to an operator difference equation we use the method of linear finite elements. This method is explained in Sec. II of this paper. There we show explicitly that for a one-particle quantum system the discrete-time operator equation obeys the constraint of unitarity. We show how to solve the operator difference equation exactly and how to use the solution to obtain accurate numerical estimates of the eigenvalues. In Sec. III we show that the method of finite elements resolves the well-known operator-ordering ambiguities one encounters in the usual Hamiltonian formulation. In Sec. IV we show that one can use higher-order finite elements to generate systems of operator difference equations. We show that the requirement of unitarity can be used to recover gaussian quadrature. In Sec. V we apply the method of finite elements to Hamiltonian systems, such as spin systems, that are associated with algebras other than the Heisenberg algebra.
The next five sections address the more difficult problem of systems having more than one degree of freedom. In Sec. VI we consider the case of two degrees of freedom and show that on a finite-element lattice the discrete quantum system is unitary. Next, we examine the question of how to apply the method of finite elements to quantum field theory problems, which are systems having an infinite number of degrees of freedom. In Sec. VII we consider the simplest case; namely, that of scalar quantum field theory. Then, in Sec. VIII we examine the Dirac equation and spinor field theories. We show that the method of finite elements resolves the fermion-doubling problem, a generic difficulty usually encountered whenever one attempts to discretize the Dirac equation. In the last two sections we use the method of finite elements to solve quantum field theories exhibiting local gauge invariance. In Sec. IX we examine quantum electrodynamics, a quantum field theory that has an abelian gauge invariance. We apply our analysis to solve the Schwinger model, massless quantum electrodynamics in two dimensions, and obtain the anomaly. In Sec. X we examine quantum field theories having local nonabelian gauge invariance.
II. QUANTUM MECHANICS WITH ONE DEGREE OF FREEDOM

A. The Method of Finite Elements

The method of finite elements\(^1\) is a technique for solving partial differential equations that is well known to applied mathematicians. The method consists of four steps. We are given a classical partial differential equation \(L\phi = 0\), which is to be solved on a region \(R\) subject to boundary conditions given on the boundary \(\partial R\). We first decompose \(R\) into a set of nonoverlapping patches, called finite elements, which completely cover \(R\). For classical (not quantum) differential equations the patches may have arbitrary sizes and shapes. On each patch we approximate the solution \(\phi\) to the partial differential equation as a polynomial. The degree of this polynomial is chosen to suit the conditions of the problem. Second, at the boundaries of contiguous patches continuity is imposed (and sometimes continuity of higher derivatives). Third, on patches that are adjacent to \(\partial R\) we impose the boundary conditions. Fourth, we impose the differential equation \(L\phi = 0\) at one point (or more than one) on each patch. Conditions two, three, and four give a system of algebraic equations satisfied by the coefficients of the polynomials. Solving this system gives a good approximation to the solution \(\phi\).

We illustrate this procedure by solving a simple classical ordinary differential equation problem:

\[
y'(x) = y(x), \quad y(0) = 1. \quad (2.1)
\]

Show that

\[
y(1) = e = 2.71828\ldots. \quad (2.2)
\]

We begin by using just one linear finite element: \(y = ax + b\), where \(0 \leq x \leq 1\). The initial condition gives one algebraic equation.

\[
y(0) = 1 \quad \Rightarrow \quad b = 1. \quad (2.3)
\]

We must impose the differential equation at one point \(x_0\) on the interval \(0 \leq x_0 \leq 1\); however, the choice of \(x_0\) remains ambiguous. Later we will see that unitarity in quantum mechanics removes this ambiguity and uniquely selects \(x_0 = 1/2\). For now we simply choose \(x_0 = 1/2\) and proceed:

\[
y'(1/2) = y(1/2) \quad \Rightarrow \quad a = \frac{a}{2} + b. \quad (2.4)
\]

Solving (2.3) and (2.4) for \(a\) and \(b\) gives \(y(x) = 2x + 1\), so that \(y(1) = 3\), which is a good result that already differs from the exact answer by only 10%.

For the case of two linear finite elements \(y_1 = at + b\), \(y_2 = ct + d\), where \(t\) is a local variable that ranges from 0 to 1/2, the initial condition gives

\[
y_1(0) = 1 \quad \Rightarrow \quad b = 1. \quad (2.5)
\]

Continuity at \(x = 1/2\) gives

\[
y_1(1/2) = y_2(0) \quad \Rightarrow \quad \frac{a}{2} + b = d \quad (2.6)
\]
and imposing the differential equation at the center of each finite element \( x = 1/4 \) and 
\( x = 3/4 \) gives
\[
\begin{align*}
y_1'(1/4) &= y_1(1/4) \quad \Rightarrow \quad a = \frac{a}{4} + b, \\
y_2'(1/4) &= y_2(1/4) \quad \Rightarrow \quad c = \frac{c}{4} + d. \\
\end{align*}
\]

(2.7)

Simultaneous solution of Eqns. (2.5–7) gives an excellent result for \( y(1) \):
\[
y(1) = y_2(1/2) = \frac{25}{9} = 2.778\ldots,
\]

(2.8)
which differs from the correct answer by 2%.

In general, for \( N \) finite elements the exact result for the approximate value of \( y(1) \) is
\[
y(1) = \left(\frac{2N + 1}{2N - 1}\right)^N,
\]

(2.9)
which for large \( N \) approximates \( e \) with a very small relative error of \( 1/(12N^2) \).

B. The Equations of Quantum Mechanics

Applying the technique of finite elements to quantum problems is much more interesting because the polynomial coefficients are operators. Consider the simple quantum-mechanical Hamiltonian
\[
H = \frac{1}{2}p^2 + V(q),
\]

(2.10)
for which Hamilton’s equations are
\[
\dot{q} = p, \quad \dot{p} = -V'(q).
\]

(2.11)

The system (2.11) constitutes a time-evolution problem for the operators \( p(t) \) and \( q(t) \). The analog of the classical initial condition is an operator constraint in the form of an ETCR
\[
[q(t), p(t)] = i.
\]

(2.12)

If (2.12) is imposed at \( t = 0 \) then, by virtue of (2.11), it holds for all \( t \).

To solve (2.11) on the interval \([0, T]\) we introduce a lattice of \( N \) linear finite elements. On each finite element \( t \) ranges from 0 to \( h \) and \( Nh = T \). Let \( q_n, (n = 0, 1, 2, \ldots, N) \) be the approximate value of \( q(nh) \). Let us examine the \( n \)th finite element, where \( p(t) \) and \( q(t) \) are approximated by the linear polynomials
\[
\begin{align*}
p(t) &= (1 - t/h)p_{n-1} + (t/h)p_n, \\
q(t) &= (1 - t/h)q_{n-1} + (t/h)q_n,
\end{align*}
\]

(2.13)
with $0 \leq t \leq h$. Substituting (2.13) into (2.11) and evaluating at the center of the finite element $t = h/2$, we obtain a pair of algebraic equations relating the operators $p_{n-1}$, $q_{n-1}$, $p_n$, and $q_n$:

\[
\frac{q_n - q_{n-1}}{h} = \frac{p_n + p_{n-1}}{2}, \tag{2.14a}
\]
\[
\frac{p_n - p_{n-1}}{h} = -V'(\frac{q_n + q_{n-1}}{2}). \tag{2.14b}
\]

The ETCR (2.12) at the lattice point $nh$ reads

\[
[q_n, p_n] = i. \tag{2.15}
\]

It is not obvious that (2.14) and (2.15) are consistent. To prove consistency we argue as follows: Equation (2.14a) implies that

\[
[q_n - q_{n-1}, p_n + p_{n-1}] = 0 \tag{2.16a}
\]

and (2.14b) implies that

\[
[q_n + q_{n-1}, p_n - p_{n-1}] = 0. \tag{2.16b}
\]

Expanding and adding together the commutators in (2.16) gives the result

\[
[q_n, p_n] - [q_{n-1}, p_{n-1}] = 0. \tag{2.17}
\]

Thus, if $[q_0, p_0] = i$ initially, then (2.15) holds for all values of $n$. The proof of the persistence of the ETCR’s holds if and only if both differential equations (2.11) are imposed on finite elements at $t = h/2$. At every other point on the finite element (2.15) ceases to be true.\(^2\) Thus, quantum-mechanical unitarity (persistence of the ETCR’s) removes a basic ambiguity that occurs in the numerical solution of classical differential equations; namely, where on the finite element to impose the differential equation.

C. Solution of the Operator Equations

We have proved the consistency of (2.14), but we must now solve these difference equations. To do so we use (2.14a) to eliminate $p_n$ from (2.14b) which now becomes

\[
\frac{4}{h^2}q_{n-1} + \frac{2}{h}p_{n-1} = V'(\frac{q_n + q_{n-1}}{2}) + \frac{4}{h^2}\left(\frac{q_n + q_{n-1}}{2}\right). \tag{2.18}
\]

If we let $x = (q_n + q_{n-1})/2$, $y = 4q_{n-1}/h^2 + 2p_{n-1}/h$, and $g(x) = V'(x) + 4x/h^2$, then (2.18) becomes

\[
y = g(x). \tag{2.19}
\]

While $x$ and $y$ are operators, (2.19) implies that they commute and thus (2.19) can be treated as if it were a c-number equation. Its exact solution is $x = g^{-1}(y)$ so

\[
q_n = -q_{n-1} + 2g^{-1}\left(\frac{2}{h}p_{n-1} + \frac{4}{h^2}q_{n-1}\right),
\]
\[ p_n = -p_{n-1} - \frac{4}{\hbar} q_{n-1} + \frac{4}{\hbar} g^{-1} \left( \frac{2}{\hbar} p_{n-1} + \frac{4}{\hbar^2} q_{n-1} \right). \]  

(2.20)

This result shows that the exact operator solution after \( N \) time steps to the lattice quantum theory in (2.14–15) is a continued (nested) function.\(^3\)

The unitarity of the lattice theory can also be demonstrated explicitly because the transfer (lattice time evolution) operator \( U \) can be expressed in closed form:

\[ q_{n+1} = U q_n U^{-1}, \quad p_{n+1} = U p_n U^{-1}, \]

where

\[ U = e^{i\hbar p_n^2/4} e^{i\hbar A(q_n)} e^{i\hbar p_n^2/4} \]  

(2.21a)

with

\[ A(x) = \frac{2}{\hbar^2} \left[ x - \frac{4}{\hbar^2} g^{-1}(x) \right] + V \left[ \frac{4}{\hbar^2} g^{-1}(x) \right]. \]  

(2.21b)

It is interesting that while the solution in (2.20) and the transfer operator \( U \) involves the function \( g^{-1} \), matrix elements of these operators only involve \( g \). For example, if we define Fock states \( |n\rangle \) at the initial time by

\[ p_0 = \frac{1}{i\gamma \sqrt{2}} (a - a^\dagger) \quad \text{and} \quad q_0 = \frac{\gamma}{\sqrt{2}} (a + a^\dagger), \]

where

\[ a|n\rangle = \sqrt{n}|n-1\rangle \quad \text{and} \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \]

then,\(^4\)

\[ \langle m|q_1|n\rangle = -\frac{\gamma}{\sqrt{2}} (\sqrt{n}\delta_{m,n-1} + \sqrt{m}\delta_{n,m-1}) \]

\[ + \frac{e^{i\theta(n-m)}}{R\sqrt{\pi} 2^{m+n} n!m!} \int_{-\infty}^{\infty} dz \ e^{-g^2(z)/(4R^2)} g'(z) H_n \left( \frac{g(z)}{2R} \right) H_m \left( \frac{g(z)}{2R} \right), \]

where \( R^2 = 4\gamma^2 h^{-4} + h^{-2}\gamma^{-2} \), \( \cos \theta = 2\gamma/(Rh^2) \), and \( H_n \) is the \( n \)th Hermite polynomial.

D. Energy Eigenvalues

It is easy to compute energy levels of quantum systems once the operator equations have been solved. There are two ways to carry out such a calculation. The quick and approximate method makes use of the one-finite-element solution to the operator difference equations. The techniques for performing this calculation are given elsewhere.\(^5\) Here are some numerical results: For the case of the harmonic oscillator, the energy gap \( \omega = E_1 - E_0 \) comes out exactly. For the anharmonic oscillator, where \( V(q) = \lambda q^4/4 \), the exact value of \( \omega \) is 1.08845... \( \lambda^{1/3} \). The one-linear-finite-element equations predict 1.14471 \( \lambda^{1/3} \) (5.2% relative error); the one-quadratic-finite-element equations (see Sec. IV) predict 1.08225 \( \lambda^{1/3} \) (-.57% relative error).

It is also possible to determine all energy differences simultaneously and to arbitrary accuracy by taking large numbers of finite elements. The procedure consists of using the
method of finite elements to calculate the matrix element $A_n = \langle 0 | q_n | 1 \rangle$ of the operator $q_n$. The sequence of numbers $A_n$ is a discrete time sequence. We can then compute the discrete Fourier transform $\tilde{A}_m$ of this sequence and search for peaks in this distribution. These peaks correspond to energy differences in the spectrum of the theory. The detailed procedure for this calculation is given elsewhere.\textsuperscript{6}

Figure 1 shows the results of a short computer calculation using 1000 finite elements for the anharmonic oscillator. The results in Fig. 1 are summarized numerically in Table 1.\textsuperscript{6}

**Figure 1.** A semilog plot of $|\tilde{A}_m|^2$ versus $m$ for the anharmonic oscillator $V(q) = 0.885q^4$. The spikes give extremely accurate approximations to energy differences $E_j - E_k$ of the exact spectrum. To read off the predicted energy differences, we note that one unit on the horizontal scale corresponds to an energy increment of $\Delta = 2\pi/[(N + 1)\hbar]$. Energy differences are measured from both the left axis and the right boundary (see Ref. 6).
Table 1. Comparison between exact eigenvalue differences for the anharmonic oscillator $V(q) = gq^4$, with $g = 0.885$ and the approximate eigenvalue differences obtained in the following manner. We find the integer values of $m$ for which $|\tilde{A}(m)|^2$ is a local maximum. The energy difference $E_j - E_k$, for some $j$ and $k$, is then predicted to be $2\pi m/[(N + 1)\hbar]$. This procedure gives relative errors of order 1–3% as the table shows. However, this procedure can be drastically refined by using by using nearby values of $|\tilde{A}(m)|^2$ to interpolate the precise noninteger value of the location of the maximum.

Computations involving many finite elements also give extremely accurate results in problems involving tunneling. Rather than solving the time-dependent Schrödinger equation we can obtain quantum tunneling results directly by following the time evolution of the matrix elements of operators on a discrete-time lattice.

| Energy difference | Exact | Approximate | Relative error |
|-------------------|-------|-------------|----------------|
| $E_1 - E_0$       | 1.728 | 1.674       | 3.1%           |
| $E_2 - E_1$       | 2.142 | 2.218       | -3.5%          |
| $E_3 - E_2$       | 2.537 | 2.595       | -2.3%          |
| $E_4 - E_3$       | 2.790 | 2.846       | -2.0%          |
| $E_5 - E_4$       | 3.000 | 3.097       | -3.2%          |
| $E_6 - E_5$       | 3.210 | 3.306       | -3.0%          |
| $E_3 - E_0$       | 6.407 | 6.487       | -1.2%          |
| $E_4 - E_1$       | 7.469 | 7.659       | -2.5%          |
| $E_5 - E_2$       | 8.327 | 8.454       | -1.5%          |
| $E_6 - E_3$       | 9.000 | 9.165       | -1.8%          |
| $E_5 - E_0$       | 12.20 | 12.39       | -1.6%          |
| $E_6 - E_1$       | 13.68 | 13.98       | -2.2%          |
III. GENERAL HAMILTONIAN SYSTEMS AND OPERATOR ORDERING

The Hamiltonian $H$ in (2.10) is not the most general one-particle quantum-mechanical Hamiltonian. However, the more general one-particle Hamiltonian, $H(p, q)$, suffers from an ambiguity that is not present in $H$ in (2.10); namely, the well-known problem of operator ordering. There is no such ordering problem in classical mechanics. However, quantizing a given classical-mechanical Hamiltonian $H(p, q)$ is an ambiguous procedure. The following simple example illustrates why this is so. Consider the classical Hamiltonian $H(p, q) = p^2q^2$. The corresponding quantum-mechanical Hamiltonian could consist of any of the following Hermitian operators:

$$\frac{1}{2}(p^2q^2 + q^2p^2), \quad \frac{1}{2}(pqpq + qpqp), \quad pq^2p, \quad qp^2q.$$

It could even be a linear combination of these operators. We show in this section that if we take the lattice as fundamental and use the method of finite elements, the above operator ordering ambiguity is completely eliminated.\(^8\)

With $H(p, q)$ an arbitrary function of the operators $p$ and $q$ the operator differential equations of motion in the continuum are

$$\dot{q}(t) = \frac{\partial H}{\partial p} = -i[q, H],$$

$$\dot{p}(t) = -\frac{\partial H}{\partial q} = -i[p, H]. \quad (3.1)$$

Let us address the problem of converting this system of equations in the continuum into a system of unitary operator difference equations on a time lattice. Recall that by unitary we mean that the difference equations exactly preserve the equal-time commutation relation

$$[q(t), p(t)] = i \quad (3.2)$$

at each time step. We show that if the method of finite elements is used to construct the operator difference equations then the ordering of the operators $p$ and $q$ is uniquely determined by the unitarity requirement.

The finite-element method consists of making the replacements

$$\dot{q}(t) \rightarrow (q_n - q_{n-1})/h,$$

$$\dot{p}(t) \rightarrow (p_n - p_{n-1})/h,$$

$$q(t) \rightarrow (q_n + q_{n-1})/2,$$

$$p(t) \rightarrow (p_n + p_{n-1})/2, \quad (3.3)$$

in (3.1). Thus, on the lattice, the differential equations (3.1) become

$$\dot{Q} = \frac{\partial}{\partial P}H(P, Q), \quad \dot{P} = -\frac{\partial}{\partial Q}H(P, Q), \quad (3.4)$$
where we have used the notation
\[
\dot{Q} \equiv (q_n - q_{n-1})/\hbar, \\
\dot{P} \equiv (p_n - p_{n-1})/\hbar, \\
Q \equiv (q_n + q_{n-1})/2, \\
P \equiv (p_n + p_{n-1})/2.
\] (3.5)

To establish unitarity as in Sec. II one must prove that
\[
[q_n, p_n] = [q_{n-1}, p_{n-1}], \quad (n = 0, 1, 2, \ldots).
\] (3.6)

We can establish (3.6) if we can explicitly show from the operator difference equations (3.4) that
\[
[\dot{Q}, P] + [Q, \dot{P}] = 0.
\] (3.7)

To see this we substitute the definitions in (3.5) into (3.7) and expand the commutators; this calculation directly shows that (3.7) implies (3.6). Thus, our objective is to examine the expression
\[
[\frac{\partial}{\partial P} H(P, Q), P] - [Q, \frac{\partial}{\partial Q} H(P, Q)]
\] (3.8)

and to show that it vanishes.

It is crucial to remark that in general (3.8) does not vanish. This is because the commutator
\[
\theta \equiv [Q, P]
\] (3.9)
is not a \(c\)-number even though \([q(t), p(t)]\) is; the quantity \(\theta\) is an operator because it contains the unequal-time commutators \([q_{n-1}, p_n]\) and \([q_n, p_{n-1}]\).

To investigate (3.8) we may assume that \(H\) is Hermitian and that \(H(P, Q)\) can be expanded in a series of Hermitian terms \(H_{m,n}(P, Q)\), \((m, n \geq 0)\), which consists of a sum of monomials containing \(m\) factors of \(P\) and \(n\) factors of \(Q\). We can examine each term \(H_{m,n}\) of the series independently. For example, \(H_{2,2}\) has the form
\[
H_{2,2} = aPQ^2P + bQP^2Q + c(P^2Q^2 + Q^2P^2) + d(PQPQ + QPQP),
\] (3.10)

where \(a, b, c,\) and \(d\) are real constants. To illustrate our procedure we examine \(H_{2,2}\) in detail. We compute
\[
[\frac{\partial}{\partial P} H_{2,2}, P] - [Q, \frac{\partial}{\partial Q} H_{2,2}]
\]
\[
= (2c - a - d)(\theta QP + PQ\theta) + (a - b)(Q\theta P + P\theta Q) + (b + d - 2c)(\theta PQ + QP\theta).
\] (3.11)

Thus, unitarity requires that \(2c - a - d = 0, a - b = 0,\) and \(b + d - 2c = 0\). The solution to these equations is \(a = b\) and \(d = 2c - b\) where \(b\) and \(c\) are arbitrary real constants. Thus,
it appears that there is a two-parameter family of Hamiltonians of the type $H_{2,2}$ which exhibit unitarity on the lattice. Indeed, $H_{2,2}$ can be written in the form

$$H_{2,2} = cT_{2,2} + (c - b)G_{2,2}, \quad (3.12)$$

where

$$T_{2,2}(P, Q) = PQ^2P + QP^2Q + P^2Q^2 + Q^2P^2 + PQPQ + QPQP \quad (3.13)$$

and

$$G_{2,2} = PQPQ + QPQP - P^2P - Q^2Q. \quad (3.14)$$

Thus, if we were given a continuum Hamiltonian of the general form $H_{m,n}(p, q)$ we could reorder the operators $p$ and $q$ [using the commutation relation (3.2)] to make it take the form in (3.12) before making the finite-element transcription (3.3). [Of course, this reordering of operators produces additional simpler terms of the $H_{1,1}(p, q) = a(pq + qp)$.] However, we observe that $G_{2,2}(p, q)$ is trivial; using (3.2) we see that $G_{2,2}(p, q) = -1$. The above calculation shows that the requirement of lattice unitarity forces us to preorganize the operators $p$ and $q$ in $H_{2,2}(p, q)$ in a unique way; namely, the totally symmetric sum ($T$ form) in (3.13). For example, if we are given the Hamiltonian $H_{2,2} = 5qp^2q$, this Hamiltonian must be (uniquely) reordered by using (3.2) as

$$H_{2,2}(p, q) = \frac{5}{6}T_{2,2}(p, q) + \frac{5}{2} \quad (3.15)$$

before going onto the lattice by use of (3.3).

What is remarkable is that given any Hamiltonian $H_{m,n}(p, q)$ carrying out the above procedure shows that there is always a unique form which is necessary and sufficient in order that the equal-time commutators be preserved as in (3.6). In particular, we must rewrite

$$H_{m,n}(p, q) = \alpha T_{m,n}(p, q) + H_{m-2, n-2}(p, q), \quad (3.16)$$

where $T_{m,n}$ is the totally symmetric sum ($T$ form) of all possible monomials containing $m$ factors of $p$ and $n$ factors of $q$. This process is then iterated until $H_{m,n}$ is a descending sum of totally symmetric parts:

$$H_{m,n}(p, q) = \alpha T_{m,n}(p, q) + \beta T_{m-2, n-2}(p, q) + \ldots \quad (3.17)$$

To verify this assertion we must use the fact that derivatives leave the $T$ form intact. In fact we have the identities

$$\frac{\partial}{\partial P}T_{m,n}(P, Q) = (m + n)T_{m-1,n,1}(P, Q),$$

$$\frac{\partial}{\partial Q}T_{m,n}(P, Q) = (m + n)T_{m,n-1}(P, Q). \quad (3.18)$$

In addition, we observe that commutators maintain the totally symmetric form

$$[Q, T_{m,n}(P, Q)] = T_{m-1,n,1}(P, Q, \theta),$$
\[ [T_{m,n}(P,Q), P] = T_{m,n-1,1}(P,Q,\theta), \quad (3.19) \]

where \( T_{m,n,1}(P,Q,\theta) \) is the totally symmetric sum of all monomials having \( m \) factors of \( P \), \( n \) factors of \( Q \), and one factor of \( \theta \). Using (3.18) and (3.19) it is easy to verify that the expression in (3.8) vanishes when \( H(P,Q) \) is in \( T \) form.

This ordering procedure applies to all Hamiltonians \( H(p,q) \) which are polynomials in the variables \( p \) and \( q \). However, if \( H \) is a nonpolynomial function the ordering problem is much more challenging. For example, consider a class of Hamiltonians of the form

\[ H(p,q) = H(T_{1,1}) = H(pq + qp). \quad (3.20) \]

To order the operators of this Hamiltonian we introduce a little-known set of orthonormal polynomials \( S_n(x) \) called continuous Hahn polynomials.\(^9\),\(^10\),\(^11\) These polynomials emerge from the simple observation that \( T_{n,n} \) is a polynomial function of \( T_{1,1} \); the defining equation for \( S_n(x) \) is therefore\(^12\),\(^13\)

\[ S_n(T_{1,1}) \equiv T_{n,n}/(2n-1)!!. \quad (3.21) \]

The first few polynomials \( S_n(x) \) are

\[ S_0(x) = 1, \]
\[ S_1(x) = x, \]
\[ S_2(x) = \frac{1}{2}(x^2 - 1), \]
\[ S_3(x) = \frac{1}{6}(x^3 - 5x), \]
\[ S_4(x) = \frac{1}{24}(x^4 - 14x^2 + 9), \]
\[ S_5(x) = \frac{1}{120}(x^5 - 30x^3 + 89x), \]
\[ S_6(x) = \frac{1}{720}(x^6 - 55x^4 + 439x^2 - 225). \]

These polynomials have the following properties:\(^10\)

(i) The generating function \( G(t) \) is

\[ G(t) = \frac{e^{arctant}}{(1 + t^2)^{1/2}} = \sum_{n=0}^{\infty} S_n(x)t^n. \quad (3.22) \]

(ii) The orthonormality condition is

\[ \int_{-\infty}^{\infty} dx \, w(x)S_m(x)S_n(x) = \delta_{mn}, \quad (3.23) \]
where the weight function \( w(x) \) is given by\(^{14}\)

\[
w(x) = [2 \cosh(\pi x/2)]^{-1}.
\] (3.24)

(iii) A recursion relation satisfied by \( S_n(x) \) is

\[
nS_n(x) = xS_{n-1}(x) - (n - 1)S_{n-2}(x).
\] (3.25)

The polynomials do not satisfy a second-order differential equation but they do obey the second-order functional difference eigenvalue equation

\[
(1 - ix)S_n(x + 2i) + (1 + ix)S_n(x - 2i) = (4n + 2)S_n(x).
\] (3.26)

Now we return to the problem of ordering the operator \( H \) in (3.20). Using the completeness of \( S_n(x) \) we expand \( H(x) \) as a series in \( S_n(x) \):

\[
H(x) = \sum_{n=0}^{\infty} a_n S_n(x),
\] (3.27)

where

\[
a_n = \int_{-\infty}^{\infty} dx \ w(x)H(x)S_n(x).
\] (3.28)

Thus, from (3.21) we have

\[
H(T_{1,1}) = \sum_{n=0}^{\infty} a_n T_{n,n}/(2n - 1)!!.
\] (3.29)

We have therefore represented \( H(pq + qp) \) as an infinite sum of operators in \( T \) form. In the form (3.29) \( H \) is directly amenable to lattice transcription and the resulting operator difference equations automatically preserve unitarity.

As an example, consider the Hamiltonian \( H = e^{(pq + qp)} \), where \( c \) is a constant. For the exponential function, the integral in (3.28) can be performed in closed form and the result is \( a_n = (\tan c)^n[1 + (\tan c)^2]^{1/2} \). Thus,

\[
H = [1 + (\tan c)^2]^{1/2} \sum_{n=0}^{\infty} (\tan c)^n S_n(pq + qp)
\]

\[
= [1 + (\tan c)^2]^{1/2} \sum_{n=0}^{\infty} \frac{(\tan c)^n}{(2n - 1)!!} T_{n,n}.
\] (3.30)

\( H \) is now in its unique \( T \) form and therefore the resulting Heisenberg equations can be transcribed onto the lattice.
IV. HIGHER-ORDER FINITE ELEMENTS

It is possible to generalize the linear polynomials representing $p(t)$ and $q(t)$ to polynomials in arbitrary degree $r$:

\[ p(t) = \sum_{k=0}^{r} a_k (t/h)^k, \]  
(4.1a)

\[ q(t) = \sum_{k=0}^{r} b_k (t/h)^k, \]  
(4.1b)

It is now necessary to determine $r + 1$ pairs of coefficients on each finite element interval. The procedure for doing so is evidently ambiguous; if we impose the differential equations \((2.11)\) \(d\) times on the \(n\)th interval, then it is necessary to impose \(r + 1 - d\) joining conditions (continuity, continuity of the first derivative, continuity of the second derivative, and so on) at \(t = (n - 1)h\). On the first interval, there are no joining conditions at \(t = 0\); rather we must impose \(r + 1 - d\) initial conditions, in which the values of \(q(0)\) and \(p(0)\), \(\dot{q}(0)\) and \(\dot{p}(0)\), \(\ddot{q}(0)\) and \(\ddot{p}(0)\), and so on, are specified. These values are obtained by successively differentiating the differential equations \((2.11)\). We say that as the number of joining conditions increases the approximation becomes stiffer. In one extreme, the stiffest approximation, the differential equation is imposed once on the interval, and in the other extreme, the floppy approximation, the method we will use in this paper, the differential equation is imposed \(r\) times, and we require only that the approximation be continuous.

A. Failure of the Stiff Approximation

The stiff approximation is forbidden by quantum mechanics. For a quantum-mechanical system with operators \(p\) and \(q\) the \(r\)th-degree finite-element approximation is given by equations \((4.1)\). While we could determine the coefficients \(a_k\) on the \(n\)th interval from those on the \((n - 1)\)st interval, attempting to determine the coefficients on the first interval, even in principle, leads to an inconsistency. This is because the coefficients \(a_k\) are operators.

We illustrate this problem by a simple example for which \(r = 2\). On the first interval we represent

\[ p(x) = p_0 + a_1 x + a_2 x^2 \]  
(4.2a)

and

\[ q(x) = q_0 + b_1 x + b_2 x^2. \]  
(4.2b)

For the sake of complete generality we impose the differential equations \((2.11)\) at \(\alpha h\) and \(\beta h\), respectively, where \(0 \leq \alpha \leq \beta \leq 1\) are as yet undetermined:

\[ \frac{b_1}{h} + \frac{2b_2}{h} \alpha = p_0 + a_1 \alpha + a_2 \alpha^2, \]  
(4.3a)

\[ \frac{a_1}{h} + \frac{2a_2}{h} \beta = -V'(q_0 + b_1 \beta + b_2 \beta^2). \]  
(4.3b)
Next, we impose the initial conditions. The condition (2.12) reads

$$[q_0, p_0] = i,$$  \hspace{1cm} (4.4)

We also impose two more commutator conditions which we obtain from the continuum equations at $t = 0$.

$$[q(0), \dot{p}(0)] = [q_0, a_1] = 0,$$  \hspace{1cm} (4.5)

$$[\dot{q}(0), p(0)] = [b_1, p_0] = 0.$$  \hspace{1cm} (4.6)

There are two more commutator conditions that follow from the equations of motion (4.3):

$$[b_1 + 2\alpha b_2, p_0 + \alpha a_1 + \alpha^2 a_2] = 0,$$  \hspace{1cm} (4.7)

$$[a_1 + 2\beta a_2, q_0 + \beta b_1 + \beta^2 b_2] = 0.$$  \hspace{1cm} (4.8)

The five commutators (4.4–8) are kinematical in nature; they make no reference to the dynamical content of the theory, which is embodied in the function $V$.

For this quantum system to be internally consistent, (4.4–6), the analogs of the three equal-time commutators, must hold again at $t = h$; that is

$$[q_0 + b_1 + b_2, p_0 + a_1 + a_2] = i,$$  \hspace{1cm} (4.9)

$$[q_0 + b_1 + b_2, a_1 + 2a_2] = 0,$$  \hspace{1cm} (4.10a)

$$[p_0 + a_1 + a_2, b_1 + 2b_2] = 0.$$  \hspace{1cm} (4.10b)

We can show that if (4.10) is assumed to hold, then (4.9) holds so long as $\alpha + \beta = 1$. However, (4.10) does not hold in general unless $\alpha = 1$ and $\beta = 1$, which implies the failure of (4.9).

This kind of demonstration can be given for any stiff approximation to a quantum system. Thus, on the basis of quantum-mechanical consistency we reject any kind of stiff finite-element scheme in which more than a single initial commutator is imposed.

Also, even if a successful stiff scheme could be found, we would prefer not to use it because stiff schemes are not as accurate as floppy approximations. A maximally stiff approximation yields a relative error of $N^{-r}$ between the exact solution and the finite-element approximation to the exact solution. On the other hand, for a floppy approximation, the relative error between these two quantities is $N^{-2r}$.

B. Consistency of the Floppy Approximation

The failure of the stiff approximation discussed the previous subsection is not very surprising and it is all the more remarkable that the floppy approximation is successful. We begin by examining the case $r = 1$, using the notation of the preceding subsection.

1. Case $r = 1$

Imposing equations (2.11) at $t = \alpha h$ and $t = \beta h$, respectively, yields [here $q_1 = q(1h)$ and $p_1 = p(1h)$]

$$\frac{q_1 - q_0}{h} = (1 - \alpha)p_0 + \alpha p_1,$$  \hspace{1cm} (4.11)
\[
\frac{p_1 - p_0}{h} = -V'(1 - \beta)q_0 + \beta q_1, \quad (4.12)
\]
in the first finite element. Since \( p(0) = p_0 \) and \( q(0) = q_0 \) we have
\[
a_0 = p_0, \quad b_0 = q_0
\]
and the continuity conditions at \( t = h \) are
\[
a_1 = p_1 - p_0, \quad b_1 = q_1 - q_0.
\]
Is there a choice for \( \alpha \) and \( \beta \) such that (4.11) and (4.12) together with (2.11), the equal-time commutator at \( t = 0 \), imply that \([q_1,p_1] = i\)? Equations (4.11) and (4.12) yield the following commutators:
\[
[q_1 - q_0, (1 - \alpha)p_0 + \alpha p_1] = 0, \quad (4.13)
\]
\[
[(1 - \beta)q_0 + \beta q_1, p_1 - p_0] = 0. \quad (4.14)
\]
Combining the three commutation relations (2.11), (4.13) and (4.14) does indeed yield \([q_1,p_1] = i\) provided that \( \alpha \) and \( \beta \) satisfy the constraint
\[
\alpha + \beta = 1. \quad (4.15)
\]
Having shown consistency with quantum mechanics on the first finite element it follows on all finite elements by virtue of the continuity condition on \( p(t) \) and \( q(t) \) at the boundaries of adjacent finite elements, \( t = nh \).

In this section we are primarily interested in the symmetric choice \( \alpha = \beta = 1/2 \), where the equations of motion are imposed at the midpoints of the finite elements. Any other choice for \( \alpha \) and \( \beta \) breaks time-reversal symmetry and leads to numerical approximations which are not as accurate as in the symmetric case.

2. Case \( r = 2 \)

Here we impose the equations of motion (2.10) twice on each finite element. In view of our above remarks with regard to symmetry and numerical accuracy, we will restrict our attention to the symmetric case where both of the equations of motion are imposed at the same points \( x = \alpha_1 h \) and \( x = \alpha_2 h \).

On the \( n = 1 \) finite element we have
\[
p(x) = p_0 + a_1 \frac{x}{h} + a_2 \frac{x^2}{h^2},
\]
\[
q(x) = q_0 + b_1 \frac{x}{h} + b_2 \frac{x^2}{h^2}.
\]
Imposing (2.11) at \( x = \alpha_1 h \) and \( x = \alpha_2 h \) gives
\[
\frac{b_1}{h} + \frac{2b_2}{h} - \alpha_1 = p_0 + a_1 \alpha_1 + a_2 \alpha_1^2,
\]
\[
\frac{a_1}{h} + \frac{2a_2}{h} \alpha_1 = -V'(q_0 + b_1 \alpha_1 + b_2 \alpha_1^2), \\
\frac{b_1}{h} + \frac{2b_2}{h} \alpha_2 = p_0 + a_1 \alpha_2 + a_2 \alpha_2^2, \\
\frac{a_1}{h} + \frac{2a_2}{h} \alpha_2 = -V'(q_0 + b_1 \alpha_2 + b_2 \alpha_2^2). 
\]  \quad (4.16)

From these equations we obtain the kinematical commutators:

\[
[p_0 + a_1 \alpha_1 + a_2 \alpha_1^2, b_1 + 2b_2 \alpha_1] = 0,
\]
\[
[a_1 + 2a_2 \alpha_1, q_0 + b_1 \alpha_1 + b_2 \alpha_1^2] = 0,
\]
\[
[p_0 + a_1 \alpha_2 + a_2 \alpha_2^2, b_1 + 2b_2 \alpha_2] = 0,
\]
\[
[a_1 + 2a_2 \alpha_2, q_0 + b_1 \alpha_2 + b_2 \alpha_2^2] = 0. 
\]  \quad (4.17)

By adding these commutators we can prove that

\[
[q_1, p_1] = [q_0, p_0] = i
\]

if and only if \( \alpha_1 \) and \( \alpha_2 \) are given by

\[
\alpha_1 = \frac{1}{2} - \frac{1}{\sqrt{12}}, \quad \alpha_2 = \frac{1}{2} + \frac{1}{\sqrt{12}}. 
\]  \quad (4.18)

This is the condition for quantum consistency.

3. Case \( r = 3 \)

Now we impose the equations of motion three times on each finite element at \( x = \alpha_1 h, \)
\( x = \alpha_2 h, \) and \( x = \alpha_3 h. \) Taking

\[
p(x) = p_0 + a_1 \frac{x}{h} + a_2 \frac{x^2}{h^2} + a_3 \frac{x^3}{h^3}
\]

and

\[
q(x) = q_0 + b_1 \frac{x}{h} + b_2 \frac{x^2}{h^2} + b_3 \frac{x^3}{h^3}
\]

we obtain six equations analogous to (4.16) from which we derive six commutator conditions analogous to (4.17). Once again we add the six commutator conditions together. However, now the two commutators at \( x = \alpha_2 h \) are weighted by the factor \( 8/5. \) The condition for quantum consistency is now

\[
\alpha_1 = \frac{1}{2} - \sqrt{3/20},
\]
\[
\alpha_2 = \frac{1}{2}.
\]

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\[ \alpha_3 = \frac{1}{2} + \sqrt{3/20}. \]  
(4.19)

4. General Case

The sequence of points \( \alpha \) at which the operator equations of motion must be imposed, \( 1/2 \) for \( r = 1 \), \( 1/2 \pm 1/\sqrt{12} \) for \( r = 2 \), \( 1/2 \) and \( 1/2 \pm \sqrt{3/20} \) for \( r = 3 \), fits a well-known pattern. These numbers are the zeros of the \( r \)th Legendre polynomial \( P_r(2\alpha - 1) \). The first three such polynomials are

\[
\begin{align*}
P_1(2\alpha - 1) &= 2\alpha - 1, \\
P_2(2\alpha - 1) &= 6\alpha^2 - 6\alpha + 1, \\
P_3(2\alpha - 1) &= (2\alpha - 1)(10\alpha^2 - 10\alpha + 1).
\end{align*}
\]

These zeros are the so-called Gaussian knots or nodes which are used to perform quadrature integration. The weighting of the commutators necessary to derive the consistency condition (the factor of \( 8/5 \) mentioned above) is exactly the weighting used in Gaussian quadrature.\textsuperscript{15}

We conclude this section by reemphasizing that the only way to preserve the equal-time commutation relations is to impose the operator equations of motion at the Gaussian nodes. If the commutator relations are preserved at successive intervals of time, then the theory is unitary; that is, there exists a transfer operator [like that in (2.21) for linear finite elements] that is unitary and therefore probability is conserved. This same point has been observed in a totally different context by Durand,\textsuperscript{16} who showed that a lattice discretization of the Schrödinger equation preserves orthonormality of the wave functions only if the lattice points lie at the Gaussian knots.
V. OTHER ALGEBRAS

The successful discretization of operator equations arising from Hamiltonian systems associated with the Heisenberg algebra (2.12) raises an obvious question; namely, is it possible to discretize the operator equations associated with other algebras? For example, consider the algebra associated with the rotation group $SO(3)$:

$$[X, Y] = iZ, \quad [Y, Z] = iX, \quad [Z, X] = iY. \quad (5.1)$$

If we construct a Hamiltonian, $H(X, Y, Z)$, associated with this algebra, then the continuum equations of motion for the operator variables $X(t)$, $Y(t)$, and $Z(t)$ are

$$\dot{X}(t) = -i[X, H],$$
$$\dot{Y}(t) = -i[Y, H],$$
$$\dot{Z}(t) = -i[Z, H]. \quad (5.2)$$

The exact solution to the operator equations (5.2) satisfies the ETCR in (5.1).

It is not easy to find a way to approximate such a system by a set of operator difference equations that preserve the commutators in (5.1) at each time step. For example, consider the Hamiltonian spin system

$$H(X, Y, Z) = XY + YX. \quad (5.3)$$

For this system, the continuum equations of motion are

$$\dot{X}(t) = XZ + ZX,$$
$$\dot{Y}(t) = -YZ - ZY,$$
$$\dot{Z}(t) = 2Y^2 - 2X^2. \quad (5.4)$$

The linear finite-element prescription for discretizing continuum equations of motion replaces undifferentiated variables by averages and first derivatives by forward differences. Here we consider two adjacent lattice sites which we label 1 and 2. In terms of the discrete $X$, $Y$, and $Z$ variables the equations of motion (5.4) become

$$\frac{X_2 - X_1}{h} = \left( \frac{X_2 + X_1}{2} \right) \left( \frac{Z_2 + Z_1}{2} \right) + \left( \frac{Z_2 + Z_1}{2} \right) \left( \frac{X_2 + X_1}{2} \right),$$
$$\frac{Y_2 - Y_1}{h} = -\left( \frac{Y_2 + Y_1}{2} \right) \left( \frac{Z_2 + Z_1}{2} \right) - \left( \frac{Z_2 + Z_1}{2} \right) \left( \frac{Y_2 + Y_1}{2} \right),$$
$$\frac{Z_2 - Z_1}{h} = 2 \left[ \left( \frac{Y_2 + Y_1}{2} \right)^2 - \left( \frac{X_2 + X_1}{2} \right)^2 \right]. \quad (5.5)$$
where $h$ is the lattice spacing. The statement of unitarity is that the SO(3) commutation relations hold at each lattice site. That is, if $X_1$, $Y_1$, and $Z_1$ satisfy

$$[X_1, Y_1] = iZ_1, \quad [Y_1, Z_1] = iX_1, \quad [Z_1, Z_1] = iY_1,$$

then as a consequence of (5.5), $X_2$, $Y_2$, and $Z_2$ satisfy the same equations.

It is easy to see that Eqs. (5.5) violate unitarity. We merely solve (5.5) perturbatively in powers of the lattice spacing $h$. We seek solutions for $X_2$, $Y_2$, and $Z_2$ in the form

$$X_2 = X_1 + Ah + Bh^2 + Ch^3 + \ldots,$$
$$Y_2 = Y_1 + Dh + Eh^2 + Fh^3 + \ldots,$$
$$Z_2 = Z_1 + Gh + Hh^2 + Ih^3 + \ldots,$$

(5.7)

We insert (5.7) into (5.5) and compare powers of $h$ to obtain explicit solutions for the operator coefficients $A$, $B$, $C$, ... . The results are

$$A = X_1Z_1 + Z_1X_1,$$
$$B = 2Y_1X_1Y_1 + 2Z_1X_1Z_1 - 2X_1^3 + \frac{3}{2}X_1,$$
$$C = Z_1^2X_1Z_1 + Z_1X_1Z_1^2 - 4(X_1^2Z_1X_1 + X_1Z_1X_1^2) - \frac{3}{4}(X_1Z_1 + Z_1X_1),$$
$$D = -Y_1Z_1 - Z_1Y_1,$$
$$E = 2Z_1Y_1Z_1 + 2X_1Y_1X_1 - 2Y_1^3 + \frac{3}{2}Y_1,$$
$$F = -Z_1^2Y_1Z_1 + Z_1Y_1Z_1^2 + 4(Y_1^2Z_1Y_1 + Y_1Z_1Y_1^2) + \frac{3}{4}(Y_1Z_1 + Z_1Y_1),$$
$$G = 2(Y_1^2 - X_1^2),$$
$$H = -4Y_1Z_1Y_1 - 4X_1Z_1X_1 - 2Z_1,$$
$$I = 4(X_1^4 - Y_1^4) + 6Y_1Z_1^2Y_1 - 6X_1Z_1^2X_1 + \frac{5}{2}(Y_1^2 - X_1^2).$$

(5.8)

We can now compute the commutator of $X_2$ with $Y_2$ to order $h^3$:

$$[X_2, Y_2] = iZ_2 + 4i(Y_1Z_1^2Y_1 - X_1Z_1^2X_1)h^3 + i(X_1^2 - Y_1^2)h^3 + O(h^4).$$

(5.9)

Observe that the difference between $[X_2, Y_2]$ and $iZ_2$ is not zero but rather of order $h^3$, showing that the lattice equations (5.5) violate unitarity. It is not surprising that the violation first occurs in order $h^3$ because we already know that for any differential equation the expansion (5.7) agrees with the continuum result through order $h^2$, and that the continuum equations, of course, do not violate unitarity. We do not know of any simple way to avoid this violation of unitarity.
Nevertheless, there is an indirect technique for obtaining a unitary set of lattice equations of motion. We merely convert the spin system to an equivalent Heisenberg system having two degrees of freedom by means of the Schwinger transformation:

\[
X = \frac{1}{4} \left[ (q_1 - i p_1)(q_2 + i p_2) + (q_2 - i p_2)(q_1 + i p_1) \right],
\]

\[
Y = \frac{i}{4} \left[ -(q_1 - i p_1)(q_2 + i p_2) + (q_2 - i p_2)(q_1 + i p_1) \right],
\]

\[
X = \frac{1}{4} \left[ (q_1 - i p_1)(q_1 + i p_1) - (q_2 - i p_2)(q_2 + i p_2) \right].
\] (5.10)

The sorts of equations that result from this transcription will be discussed in the next section.
VI. SYSTEMS WITH MANY DEGREES OF FREEDOM

For systems having more than one degree of freedom it is not as easy to show that the method of finite elements is consistent with unitarity. We illustrate this situation with a system having two degrees of freedom \((p, q)\) and \((\pi, \phi)\). For the Hamiltonian

\[
H = \frac{p^2}{2} + \frac{\pi^2}{2} + V(q, \phi),
\]

(6.1)

Hamilton’s equations in the continuum read

\[
\dot{q} = p, \quad (6.2a)
\]
\[
\dot{\phi} = \pi, \quad (6.2b)
\]
\[
\dot{p} = -\frac{\partial}{\partial q} V(q, \phi), \quad (6.2c)
\]
\[
\dot{\pi} = -\frac{\partial}{\partial \phi} V(q, \phi). \quad (6.2d)
\]

On the lattice, the linear finite-element transcription of these equations of motion is

\[
\frac{q_1 - q_0}{h} = \frac{p_1 + p_0}{2}, \quad (6.3a)
\]
\[
\frac{\phi_1 - \phi_0}{h} = \frac{\pi_1 + \pi_0}{2}, \quad (6.3b)
\]
\[
\frac{p_1 - p_0}{h} = -\frac{\partial}{\partial q} V\left(\frac{q_1 + q_0}{2}, \frac{\phi_1 + \phi_0}{2}\right), \quad (6.3c)
\]
\[
\frac{\pi_1 - \pi_0}{h} = -\frac{\partial}{\partial \phi} V\left(\frac{q_1 + q_0}{2}, \frac{\phi_1 + \phi_0}{2}\right). \quad (6.3d)
\]

Note that while there is no operator-ordering problem in (6.2c) and (6.2d) because \([q(t), \phi(t)] = 0\), there appears to be a serious ordering problem in (6.3c) and (6.3d) because it is not clear that \((q_1 + q_0)/2\) and \((\phi_1 + \phi_0)/2\) commute. To resolve this problem we define

\[
\alpha = \frac{2p_0}{h} + \frac{4q_0}{h^2}, \quad \beta = \frac{2\pi_0}{h} + \frac{4\phi_0}{h^2}, \quad (6.4)
\]

and

\[
\sigma = \frac{q_0 + q_1}{2}, \quad \tau = \frac{\phi_0 + \phi_1}{2}. \quad (6.5)
\]

Now, (6.3c) and (6.3d) become

\[
\alpha = \frac{\partial}{\partial \sigma} V(\sigma, \tau) + \frac{4}{h} \sigma, \quad (6.6a)
\]
\[
\beta = \frac{\partial}{\partial \tau} V(\sigma, \tau) + \frac{4}{h} \tau. \quad (6.6b)
\]
The simultaneous solution of (6.6) has the form
\[ \sigma = \sigma(\alpha, \beta), \quad \tau = \tau(\alpha, \beta), \quad (6.7) \]

But, \( \alpha \) and \( \beta \) involve only operators at the initial time, so \([\alpha, \beta] = 0\). Thus, \( \sigma \) and \( \tau \) also commute, and there is, in fact, no ordering problem in (6.3).

It is now necessary to verify that the ETCR’s are preserved in time. From the solutions
\[
\begin{align*}
q_1 &= -q_0 + 2\sigma(\alpha, \beta), \\
p_1 &= -p_0 - \frac{4}{\hbar} q_1 + \frac{4}{\hbar} \sigma(\alpha, \beta), \\
\phi_1 &= -\phi_0 + 2\tau(\alpha, \beta), \\
\pi_1 &= -\pi_0 - \frac{4}{\hbar} \phi_1 + \frac{4}{\hbar} \tau(\alpha, \beta),
\end{align*}
\]

it is necessary to verify that
\[ [q_1, p_1] = [\phi_1, \pi_1] = i \quad (6.9a) \]

and
\[ [q_1, \phi_1] = [q_1, \pi_1] = [\phi_1, p_1] = [p_1, \pi_1] = 0. \quad (6.9b) \]

It is easy to verify (6.9a):
\[
[q_1, p_1] = [q_0, p_0] + \frac{4}{\hbar} \{q_0, \sigma(\alpha, \beta)\} + 2\{p_0, \sigma(\alpha, \beta)\} \\
= i + \frac{4}{\hbar} \frac{\partial \sigma}{\partial \alpha} \frac{2i}{\hbar} + 2\frac{\partial \sigma}{\partial \alpha} \left( -\frac{4i}{\hbar^2} \right) \\
= i.
\]

However, it is harder to verify (6.9b):
\[
[q_1, \phi_1] = -2[\phi_0, \tau(\alpha, \beta)] - 2[\sigma(\alpha, \beta), \phi_0] \\
= -\frac{4i}{\hbar} \frac{\partial \tau}{\partial \alpha} + \frac{4i}{\hbar} \frac{\partial \sigma}{\partial \beta}.
\]

We can show that \([q_1, \phi_1] = 0\) by verifying that \(\partial \tau / \partial \alpha = \partial \sigma / \partial \beta\) because the system defined by (6.3) is Hamiltonian. Explicitly, we have
\[ \frac{\partial \tau}{\partial \alpha} = \frac{\partial \sigma}{\partial \beta} = \frac{-\frac{\partial^2 V}{\partial \sigma \partial \tau}}{\left( \frac{\partial^2 V}{\partial \sigma^2} + \frac{4}{\hbar^2} \right) \left( \frac{\partial^2 V}{\partial \sigma \partial \tau} + \frac{4}{\hbar^2} \right) - \frac{\partial^2 V}{\partial \sigma \partial \tau}}. \quad (6.12) \]

The other commutators are evaluated similarly. For details and discussion of the time evolution operator see Ref. 18.

We conclude this section with a discussion of a coupled fermion-boson system. Consider the system governed by the continuum Hamiltonian
\[
H = \frac{p^2}{2} + S(x) + \bar{\psi}\psi W(x). \quad (6.13)
\]
The continuum Heisenberg equations of motion for this system are

\[
\begin{align*}
\dot{x} &= p, \\
\dot{p} &= -S'(x) - \bar{\psi}W'(x)\psi, \\
i\dot{\psi} &= W(x)\psi, \\
-i\dot{\bar{\psi}} &= \bar{\psi}W(x).
\end{align*}
\tag{6.14a-6.14d}
\]

To derive these equations we make use of the canonical equal-time commutation and anticommutation relations for the dynamical variables

\[
\begin{align*}
[x, p] &= i, \\
[\psi, \bar{\psi}]_+ &= 1, \\
[x, \psi] &= [x, \bar{\psi}] = [p, \psi] = [p, \bar{\psi}] = 0, \\
\psi^2 &= \bar{\psi}^2 = 0.
\end{align*}
\tag{6.15a-6.15d}
\]

In (6.14) we have ordered the operators in order to make the equations of motion manifestly Hermitian, which ordering is irrelevant in the continuum, but not on the lattice.

If we were now to put the system (6.14) on the lattice using our finite-element prescription, we would find that the resulting lattice theory is not unitary. In anticipation of this difficulty we will replace the function \(W'\) in (6.14b) by a function \(Z\), which will be determined by the requirement that the system be unitary. We will see that \(Z\) satisfies an interesting nonlinear equation. We will, of course, find that as the lattice spacing \(h\) tends to zero, \(Z\) approaches \(W'\).

We take, then, for our lattice difference equation

\[
\begin{align*}
\frac{x_1 - x_0}{h} &= \frac{p_1 + p_0}{2}, \\
\frac{p_1 - p_0}{h} &= -S'(\sigma) - \bar{\sigma}Z(\sigma)\phi, \\
i\frac{\psi_1 - \psi_0}{h} &= W(\sigma)\phi, \\
-i\frac{\bar{\psi}_1 - \bar{\psi}_0}{h} &= \bar{\phi}W(\sigma),
\end{align*}
\tag{6.16a-6.16d}
\]

where

\[
\begin{align*}
\sigma &= \frac{x_1 + x_0}{2}, \\
\phi &= \frac{\psi_1 + \psi_0}{2}.
\end{align*}
\tag{6.17a-6.17b}
\]

We will now simply quote the results found in Ref. 18. There is no problem with unitarity in the pure fermion sector of the theory, because it is easy to show that

\[
\psi_1 = \frac{1 - ihW(\sigma)/2}{1 + ihW(\sigma)/2}\psi_0.
\tag{6.18}
\]
Similarly, there is no problem in the pure boson sector. The constraint comes when we examine the mixed commutator \([x_1, \psi_1]\). Requiring this to vanish leads to the following equation for \(Z\):

\[
Z(x) = \frac{W'(x)}{1 + \hbar^2 S''(x)/4} + [1 + \hbar^2 W^2(x)/4] \left( S'(x) - S' \left( x - \frac{h^2 W'(x)/4}{(1 + h^2 W^2(x)/4)(1 + h^2 S''(x)/4)} \right) \right). \tag{6.19}
\]

For small lattice spacing this approaches

\[
Z(x) = W'(x) - \frac{S'''(x)(W'(x))^2 h^4}{32} + O(h^6). \tag{6.20}
\]

Notice that if \(S\) is a polynomial of degree two, \(Z\) is exactly \(W'\). This suggests that electrodynamics will not present subtleties when analyzed using finite elements, while supersymmetry, for example, may require more care.
VII. SCALAR QUANTUM FIELD THEORY

It is straightforward to generalize the above discussion to quantum field theory. In this section we will apply the method of finite elements to self-interacting, two-dimensional scalar field theory, and in particular, calculate the mass renormalization for the $(\phi^{2N})_2$ and the sine-Gordon field theories.

Consider the Hamiltonian
\[ H = \frac{1}{2} \phi^2_t + \frac{1}{2} \phi^2_x + V(\phi). \] (7.1)

This gives rise to the operator Klein-Gordon equation
\[ \phi_{tt} - \phi_{xx} = f(\phi), \] (7.2a)

where
\[ f(\phi) = -V'(\phi). \] (7.2b)

Because we will be using linear finite elements, we rewrite (7.2a) as a system of first-order equations:
\[ \phi_t = \pi, \] (7.3a)
\[ \phi_x = \Gamma, \] (7.3b)
\[ \pi_t - \Gamma_x = f(\phi). \] (7.3c)

We introduce a rectangular finite-element lattice with the time-lattice spacing being $h$ and the space-lattice spacing being $\Delta$. The spatial extent of the lattice is $L$. If we approximate the field in the finite element by a polynomial linear in $x$ and $t$,
\[ \phi(x, t) = \left(1 - \frac{t}{h}\right) \left(1 - \frac{x}{\Delta}\right) \phi_{m-1,n-1} + \left(1 - \frac{t}{h}\right) \frac{x}{\Delta} \phi_{m,n-1} + \frac{t}{h} \left(1 - \frac{x}{\Delta}\right) \phi_{m-1,n} + \frac{t}{h} \frac{x}{\Delta} \phi_{m,n}, \] (7.4)

and impose the equations of motion (7.3) at the center of the finite element, we obtain the system of difference equations
\[ \frac{1}{2h}(\phi_{m+1,n+1} + \phi_{m+1,n+1} - \phi_{m,n} - \phi_{m+1,n}) = \frac{1}{4}(\pi_{m+1,n+1} + \pi_{m,n+1} + \pi_{m+1,n} + \pi_{m,n}), \] (7.5a)
\[ \frac{1}{2\Delta}(\phi_{m+1,n+1} + \phi_{m+1,n} - \phi_{m,n+1} - \phi_{m,n}) = \frac{1}{4}(\Gamma_{m+1,n+1} + \Gamma_{m,n+1} + \Gamma_{m+1,n} + \Gamma_{m,n}), \] (7.5b)
\[ \frac{1}{2h}(\pi_{m,n+1} + \pi_{m+1,n+1} - \pi_{m,n} - \pi_{m+1,n}) - \frac{1}{2\Delta}(\Gamma_{m+1,n+1} + \Gamma_{m,n} + \Gamma_{m,n+1} - \Gamma_{m,n}) = f\left(\frac{\phi_{m+1,n+1} + \phi_{m,n+1} + \phi_{m+1,n} + \phi_{m,n}}{4}\right). \] (7.5c)
The first index, \( m \), on the fields represents the spatial lattice site and the second index, \( n \), represents the temporal lattice site. **Note the significant mnemonic for linear finite elements:** corresponding to directions in which derivatives are taken, a forward difference is taken, while in other directions, a forward average is taken. The index \( m \) ranges from 0 to \( M = L/\Delta \); all fields are taken to be periodic in the spatial lattice:

\[
\phi_{0,n} = \phi_{M,n}. \tag{7.6}
\]

For technical reasons, which will be discussed below, we will take \( M \) to be odd. As shown in Ref. 19, it is the space-averaged operators,

\[
\Phi_{m,n} \equiv \frac{1}{2}(\phi_{m,n} + \phi_{m-1,n}), \tag{7.7a}
\]

\[
\Pi_{m,n} \equiv \frac{1}{2}(\pi_{m,n} + \pi_{m-1,n}), \tag{7.7b}
\]

which obey the canonical equal-time commutation relations

\[
[\Phi_{m,n}, \Phi_{m',n}] = [\Pi_{m,n}, \Pi_{m',n}] = 0,
\]

\[
[\Phi_{m,n}, \Pi_{m',n}] = \frac{i}{\Delta} \delta_{m,m'}.	ag{7.8}
\]

These commutation relations are the discrete analogs of the continuum equal-time commutation relations.

In principle, we can solve the system of difference equations (7.5); for the purposes here, however, it is sufficient to expand in powers of the temporal lattice spacing \( h \). We do not expand in \( \Delta \). The expansions for \( \phi_{m,n+1}, \pi_{m,n+1}, \) and \( \Gamma_{m,n+1} \) are

\[
\phi_{m,n+1} = \phi_{m,n} + hA_m + h^2 B_m + O(h^3),
\]

\[
\pi_{m,n+1} = \pi_{m,n} + hC_m + h^2 D_m + O(h^3), \tag{7.9}
\]

\[
\Gamma_{m,n+1} = \Gamma_{m,n} + hE_m + h^2 F_m + O(h^3).
\]

Inserting (7.9) into (7.5) leads to a set of difference equations for the operators \( A_m, B_m, C_m, \ldots \). These equations all have the same generic form:

\[
x_m + x_{m+1} = R_m, \tag{7.10}
\]

which, for periodic boundary conditions and \( M \) odd, has the general solution

\[
x_m = \frac{1}{2} \left[ \sum_{k = m}^{M-1} (-1)^{k+m} R_k - \sum_{k = 0}^{m-1} (-1)^{k+m} R_k \right] = \frac{1}{2} \sum_{k = m}^{M+m-1} (-1)^{k+m} R_k. \tag{7.11}
\]

Next, following the quantum-mechanical discussion in Sec. II, we introduce a Fock-space representation for the canonical operators \( \Phi_{m,n} \) and \( \Pi_{m,n} \) defined in (7.7):

\[
\Phi_{m,n} = \sum_{k=1}^{M} \gamma_k (a_k e^{ikm2\pi/m} + a_k^\dagger e^{-ikm2\pi/M}), \tag{7.12a}
\]

\[
\Pi_{m,n} = \sum_{k=1}^{M} \frac{i}{2\gamma_k L} (-a_k e^{ikm2\pi/m} + a_k^\dagger e^{-ikm2\pi/M}), \tag{7.12b}
\]
where
\[ [a_k, a_l] = \delta_{k,l}, \quad (7.13a) \]
\[ [a_k, a_l] = [a_k^*, a_l^*] = 0. \quad (7.13b) \]

In (7.12) $\gamma_k$, $k = 1, 2, \ldots, M$, are arbitrary parameters which later will be fixed by a variational argument, similar to that given in Sec. II. Recall that the spatial size of the lattice is $L = M\Delta$.

As a consequence of (7.13), the equal-time commutation relations (7.8) are satisfied at the initial time $n$ provided that
\[ \gamma_k^2 = \gamma_{M-k}^2. \quad (7.14) \]

It is crucial that as a consequence of the operator difference equations (7.5) the equal time commutation relations (7.8) are preserved at all subsequent lattice sites, in particular those at $n + 1$.

Using (7.12) we find easily
\[ \frac{1}{2}(\Gamma_{m-1,n} + \Gamma_{m,n}) = \frac{2i}{\Delta} \sum_{k=1}^{M} \gamma_k \tan \frac{k\pi}{M} (a_k e^{ikm2\pi/m} - a_k^* e^{-ikm2\pi/M}), \quad (7.15) \]

It is easy to verify the expected equal-time commutation relation between $\frac{1}{2}(\Gamma_{m-1,n} + \Gamma_{m,n})$ and $\Phi_{m,n}$:
\[ [\Phi_{m,n}, \frac{1}{2}(\Gamma_{m-1,n} + \Gamma_{m,n})] = 0. \quad (7.16) \]

We will merely display the order-$h$ coefficients:
\[ A_m = \Pi_m, \quad (7.17a) \]
\[ \frac{1}{2}(E_{m-1} + E_m) = \frac{1}{\Delta L} \sum_{k=1}^{M} \frac{1}{\gamma_k} \tan \frac{k\pi}{M} (a_k e^{ikm2\pi/m} + a_k^* e^{-ikm2\pi/M}), \quad (7.17b) \]
\[ \frac{1}{2}(C_{m-1} + C_m) = -\frac{4}{\Delta^2} \sum_{k=1}^{M} \gamma_k \tan^2 \frac{k\pi}{M} (a_k e^{ikm2\pi/m} + a_k^* e^{-ikm2\pi/M}) \]
\[ + f(\Phi_{m,n}). \quad (7.17c) \]

We do not bother to display the solutions for $B_m$, $D_m$, and $F_m$; however, they can be calculated similarly.

Consider first a free theory, for which $f = -\mu^2 \phi$. We extract spectral information by taking matrix elements of (7.9) between the Fock vacuum and a one-particle state with lattice momentum $l$:
\[ \langle 1, l | = \langle 0 | a_l. \quad (7.18) \]

We compare these matrix elements with an assumed approximate exponential time dependence with a single frequency:
\[ \langle 1, l | \Phi_{m,n+1} | 0 \rangle \approx e^{i\omega_l h} \langle 1, l | \Phi_{m,n} | 0 \rangle, \]
\[ \langle 1, l | \Pi_{m,n+1} | 0 \rangle \approx e^{i\omega_l h} \langle 1, l | \Pi_{m,n} | 0 \rangle. \quad (7.19) \]
The $O(h^0)$ and $O(h^1)$ equations give a relation between the frequency $\omega_l$ and the variational parameter $\gamma_l$,

$$\omega_l = \frac{1}{2L \gamma_l}, \quad (7.20)$$

and the dispersion relation

$$\omega_l^2 = \mu^2 + \frac{4}{\Delta^2} \tan^2 \frac{\pi l}{M}. \quad (7.21)$$

In the continuum limit, $\Delta \to 0$, $M \to \infty$, and $L = M \Delta \to \infty$, the lattice equivalent of the continuum momentum $p$ is

$$p = \frac{2\pi l}{M \Delta}. \quad (7.22)$$

Thus, we recover the continuum dispersion relation

$$\omega^2 = \mu^2 + p^2. \quad (7.23)$$

It is remarkable that if we include the two additional equations coming from the $O(h^2)$ terms in $\Phi_{m,n+1}$ and $\Pi_{m,n+1}$ as well as the three further equations coming from $\Gamma_{m,n+1}$, only redundant information is supplied.

Consider now an interacting theory for which

$$V = \mu^2 \phi^2 + \frac{g}{2N} \phi^{2N}. \quad (7.24)$$

Following the procedure described above, we find that (7.20) still holds, but the dispersion relation (7.21) is replaced by

$$\omega_l^2 = m_{\text{ren}}^2 + \frac{4}{\Delta^2} \tan^2 \frac{\pi l}{M}, \quad (7.25)$$

where

$$m_{\text{ren}}^2 = \mu^2 + (2N - 1)!gX^{N-1}, \quad X = \sum_{k=1}^{M} \gamma_k^2, \quad (7.26)$$

In the continuum limit, an asymptotic analysis\textsuperscript{20} serves to evaluate $X$:

$$X = \frac{1}{2\pi} \ln \left( \frac{4}{m_{\text{ren}} \Delta} \right). \quad (7.27)$$

An equation for the renormalized mass is thus obtained when we substitute (7.27) into (7.26):

$$m_{\text{ren}}^2 = \mu^2 + \frac{(2N - 1)!!}{(2\pi)^{N-1}} g \left[ \ln \left( \frac{4}{m_{\text{ren}} \Delta} \right) \right]^{N-1}. \quad (7.28)$$

This nonperturbative result closely resembles the formula in continuum perturbation theory:

$$m_{\text{ren}}^2 = \mu^2 + \frac{(2N - 1)!!}{(2\pi)^{N-1}} g \left[ \ln \left( \frac{\Lambda}{\mu} \right) \right]^{N-1}. \quad (7.29)$$
The correspondence is provided by the identification of the momentum cutoff $\Lambda$ with $\pi/\Delta$.

We close this section by extending this calculation to the sine-Gordon model, for which

$$V = \frac{\mu^2}{g^2} \cos g\phi.$$  \hfill (7.30)

The corresponding force is

$$f = -\frac{\mu^2}{g} \sin g\phi = -\mu^2 \sum_{N=0}^{\infty} \frac{(-1)^N g^{2N} \phi^{2N+1}}{(2N + 1)!}. \hfill (7.31)$$

Each term in the sum in (7.31) gives a contribution to the renormalized mass of the form given in (7.26), so the formula for the renormalized mass is therefore

$$m_{\text{ren}}^2 = \mu^2 \sum_{N=0}^{\infty} \frac{(-1)^N g^{2N} (2N + 1)!! X^N}{(2N + 1)!} = \mu^2 e^{-g^2 X/2}, \hfill (7.32)$$

where $X$ is given in (7.26). The dispersion relation is

$$\omega_l^2 = \frac{4}{\Delta^2} \tan^2 \frac{\pi l}{M} + \mu^2 e^{-g^2 X/2}. \hfill (7.33)$$

An asymptotic analysis in the continuum limit leads once again to (7.27), so a simple calculation yields the following relation between the renormalized and the unrenormalized masses:

$$m_{\text{ren}} = \mu \left( \frac{\mu \Delta}{4} \right)^{g^2/(8\pi-g^2)}. \hfill (7.34)$$

This is the characteristic power-law renormalization found in the conventional treatments of the sine-Gordon model, and reduces to the perturbative result of Coleman when $g^2/8\pi$ is small.
VIII. THE DIRAC EQUATION AND FERMION DOUBLING

The finite-element lattice Dirac equation in 3 + 1 dimensions is
\[ i\gamma^0 \frac{\hbar}{\Delta} (\psi_{\vec{m},n+1} - \psi_{\vec{m},n}) + \frac{i\gamma^j}{\Delta} (\psi_{m+1,\vec{m}_\perp,\vec{n}} - \psi_{m,\vec{m}_\perp,\vec{n}}) + \mu \psi_{\vec{m},\vec{n}} = 0 \]  
(8.1)

Here the overbar represents a forward average over that coordinate:
\[ x_{\vec{n}} = \frac{1}{2}(x_{m+1} + x_m), \]
and the notation \( \vec{m}_\perp \) means that all spatial coordinates but \( m_j \) are averaged. Let us begin by finding the momentum-space spinors, the eigenvectors of the transfer matrix. That is, write for a plane wave at time \( n \)
\[ \psi_{\vec{m},n} = u_n e^{-i\mathbf{p} \cdot \mathbf{m} 2\pi/M}, \]  
(8.2a)
and at time \( n + 1 \)
\[ \psi_{\vec{m},n+1} = u_n e^{-i\mathbf{p} \cdot \mathbf{m} 2\pi/M}. \]  
(8.2b)

The transfer matrix \( T \) is defined by
\[ u_{n+1} = Tu_n. \]  
(8.3)

By substituting (8.2a) and (8.2b) into the Dirac equation (8.1) we easily find that
\[ T = \left( \frac{i\gamma^0}{\hbar} + \frac{\vec{\gamma} \cdot \mathbf{t}}{\Delta} + \frac{\mu}{2} \right)^{-1} \left( \frac{i\gamma^0}{\hbar} - \frac{\vec{\gamma} \cdot \mathbf{t}}{\Delta} - \frac{\mu}{2} \right) \]
\[ = \left( 1 + \frac{\mu^2 h^2}{4} + \frac{h^2}{\Delta^2 t^2} \right)^{-1} \left( 1 - \frac{\mu^2 h^2}{4} - \frac{h^2}{\Delta^2 t^2} + \frac{2h}{\Delta} \gamma^0 \vec{\gamma} \cdot \mathbf{t} + \mu hi\gamma^0 \right), \]  
(8.4)

where
\[ \mathbf{t} = \mathbf{t}_p, \quad (\mathbf{t}_p)_i = \tan p_i \pi/M. \]  
(8.5)

Let us adopt a representation of the Dirac matrices in which
\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad i\gamma^0 \gamma^j = \sigma^{0j} = i \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix}. \]  
(8.6)

Then the eigenvalues of \( T \) are easily found:
\[ \lambda = \frac{1 \pm i\hbar \tilde{\omega}/2}{1 \mp i\hbar \tilde{\omega}/2} \]  
(8.7)

Here \( \tilde{\omega} \) is an abbreviation for
\[ \tilde{\omega} = \sqrt{\frac{4t^2}{\Delta^2} + \mu^2}, \]  
(8.8)
which is exactly the same as the dispersion relation (7.21). It is obvious that $\lambda$ has modulus unity, so it can be written in the form $\lambda = \exp(\pm i\omega h)$, where $\omega$ is, of course, a function of $h$. (The relation between $\omega$ and $\tilde{\omega}$ is $\tilde{\omega} = \frac{2}{\pi} \tan \frac{h\omega}{2}$.) The corresponding eigenvectors may also be found straightforwardly. They are to be normalized according to

$$u_\pm^\dagger \gamma^0 u_\pm = \pm 1.$$  \hspace{1cm} (8.9)

They are

$$u_\pm = \left( \frac{\pm \left(\tilde{\omega} \pm \mu\right)/2\mu}{\left[(\tilde{\omega} \mp \mu)/2\mu\right]^{1/2}} \right) \chi,$$  \hspace{1cm} (8.10)

where $\chi$ is a two-component, rest-frame spinor, normalized by $\chi^\dagger \chi = 1$. Thus, with

$$i\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$  \hspace{1cm} (8.11)

we have

$$u_\pm = \left[ \left( \frac{\tilde{\omega} + \mu}{2\mu} \right)^{1/2} \pm \gamma_5 \frac{\tilde{\sigma} \cdot \mathbf{t}}{t} \left( \frac{\tilde{\omega} - \mu}{2\mu} \right)^{1/2} \right] u_\pm^{(0)},$$  \hspace{1cm} (8.12)

where $u_\pm^{(0)}$ is a four-component rest-frame spinor with $\gamma^0$ eigenvalue of $\pm 1$. Therefore, in terms of the spinors $\tilde{u}_+(\mathbf{p}) = u_+ (\mathbf{p})$, $\tilde{u}_- (\mathbf{p}) = u_- (-\mathbf{p})$, we have the completeness relations

$$\sum_{\text{spins}} \tilde{u}_\pm u_\pm^{\dagger} \gamma^0 = \pm \frac{1}{2\mu} \left( \mu \pm \gamma^0 \tilde{\omega} \mp \frac{2\gamma \cdot \mathbf{t}}{\Delta} \right),$$  \hspace{1cm} (8.13)

which in the continuum limit reduces to $\pm (\mu \mp \gamma \cdot \mathbf{p})/2\mu$. We have the same result on the lattice, provided we define

$$\tilde{p}^0 = \tilde{\omega}, \quad \tilde{\mathbf{p}} = \frac{2\mathbf{t}}{\Delta}. $$  \hspace{1cm} (8.14)

All of this tells us that the momentum expansion of the Dirac field has the form

$$\psi_{\mathbf{m},n} = \sum_{s,\mathbf{p}} \sqrt{\frac{\mu}{\omega}} \left( b_{p,s} u_{p,s} e^{i\mathbf{p} \cdot \mathbf{m} 2\pi/M} + d_{p,s}^\dagger v_{p,s} e^{-i\mathbf{p} \cdot \mathbf{m} 2\pi/M} \right),$$  \hspace{1cm} (8.15)

where we now use the standard notation $u = i\gamma_5 \tilde{u}_-$, $v = i\gamma_5 \tilde{u}_+$, with the usual interpretation that $d^\dagger$ creates a positive energy positron, while $b$ annihilates a positive energy electron. The canonical lattice anticommutation relations

$$\{\psi_{\mathbf{m},n}, \psi_{\mathbf{m}',n}^\dagger\} = \frac{1}{L^3} \delta_{\mathbf{m},\mathbf{m}'} \delta_{n,n'},$$  \hspace{1cm} (8.16)

will now be satisfied if

$$\{b_{p,s}, b_{p',s'}^\dagger\} = \frac{1}{L^3} \delta_{\mathbf{p},\mathbf{p}'} \delta_{s,s'}, \quad \{d_{p,s}, d_{p',s'}^\dagger\} = \frac{1}{L^3} \delta_{\mathbf{p},\mathbf{p}'} \delta_{s,s'},$$  \hspace{1cm} (8.17)
and all other anticommutators of these operators vanish.

The unitarity of $T$ is sufficient to establish the unitarity of the fermion sector in the noninteracting theory. For further details see Refs. 23 and 24.

It is apparent that the dispersion relation (8.8) solves the fermion doubling problem; that is, $\omega^2$ assumes the value $\mu^2$ at $p = 0 \pmod{M}$ and nowhere else. In this discussion we have assumed, as in Sec. VII, that $M$ is odd so that the Dirac field is periodic; however, the same conclusion would hold if antiperiodic boundary conditions were used. In the dispersion relation one would simply replace $p$ by $p + \frac{1}{2}$.

Let us conclude this section by summarizing the properties of the linear finite-element Dirac equation (8.1):

1. It is unitary in that the equal-time anticommutation relations are exactly preserved in time.
2. It may be derived from an Hermitian action.
3. There is no fermion doubling.
4. The difference equation is local, in that only nearest-neighbor terms appear.
5. It is chirally symmetric in the massless limit.

For a complete discussion of these points see Ref. 23.

The no-go theorems of Karsten and Smit, Nielsen and Ninomiya, and Rabin are avoided because the time development operator is nonlocal, which arises because undifferentiated fields appear as averages.
IX. DIRAC EQUATION WITH INTERACTIONS AND THE SCHWINGER MODEL

In the last two sections of this review, we will discuss interactions of fermions with gauge fields. In this section we will concentrate on the Dirac equation, interacting with either Abelian or nonabelian gauge fields, and make application to electrodynamics, in particular to the Schwinger model. In the next section the nonabelian interactions of the gauge fields among themselves will be derived. (The original treatment of finite-element electrodynamics was given in Ref. 27.) For simplicity, initially our discussion will be restricted to (1+1) dimensions.

A. Equations of Motion in the Continuum

We begin by recalling the form of the continuum field equations of a nonabelian gauge field \( A_\mu \) coupled to a fermion field \( \psi \). Let us start with the free Dirac equation

\[
(i\partial + \mu)\psi = 0.
\]

Equation (9.1) is invariant under an infinitesimal gauge transformation

\[
\psi \rightarrow \psi + \delta \psi, \quad \delta \psi = ig\delta \omega \psi, \quad \delta \omega = \delta \omega_a T_a,
\]

provided \( \delta \omega \) is constant. Here \( T \) is the generator of the gauge group. If \( \delta \omega \) is not constant, we can restore the invariance by adding an interaction term to the Dirac equation,

\[
(i\partial + gA + \mu)\psi = 0, \quad A^\mu = A^\mu_a T_a,
\]

provided \( A \) transforms according to

\[
A_\mu \rightarrow A_\mu + \delta A_\mu, \quad \delta A_\mu = \partial_\mu \delta \omega + ig[\delta \omega, A_\mu].
\]

Under (9.4) the curl of \( A \) is not covariant, so we must add a suitable interaction term to construct the field strength:

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu].
\]

The field strength transforms covariantly:

\[
F_{\mu\nu} \rightarrow F_{\mu\nu} + \delta F_{\mu\nu}, \quad \delta F_{\mu\nu} = ig[\delta \omega, F_{\mu\nu}].
\]

Finally, because the current

\[
j^\mu = g\bar{\psi}T\gamma^\mu \psi
\]

transforms covariantly,

\[
j^\mu \rightarrow j^\mu + \delta j^\mu, \quad \delta j^\mu = ig[\delta \omega, j^\mu],
\]
we must include the interaction term in the Yang-Mills equation,

\[ D_\mu F^{\mu\nu} = j^\nu, \quad \text{where} \quad D_\mu = \partial_\mu - ig[A_\mu, \cdot]. \quad (9.9) \]

**B. Free Lattice Equations of Motion**

In 1 + 1 dimensions, the free Dirac equation (8.1) reads

\[ \frac{i\gamma^0}{\hbar}(\phi_{m,n+1} - \phi_{m,n}) + \frac{i\gamma^1}{\Delta}(\theta_{m+1,n} - \theta_{m,n}) + \frac{\mu}{2}(\theta_{m+1,n} + \theta_{m,n}) = 0. \quad (9.10) \]

Here we have used the abbreviations

\[ \phi_{m,n} = \frac{1}{2}(\psi_{m+1,n} + \psi_{m,n}), \quad (9.11a) \]

\[ \theta_{m,n} = \frac{1}{2}(\psi_{m,n+1} + \psi_{m,n}). \quad (9.11b) \]

Similarly, the field strength \( E = F_{01} \) is constructed as

\[ \tilde{E}^{(0)}_{m,n} = \frac{1}{\hbar}(C_{m+1,n+1} - C_{m+1,n}) - \frac{1}{\Delta}(B_{m+1,n+1} - B_{m,n+1}), \quad (9.12) \]

where

\[ B_{m,n} = \frac{1}{2}[(A_0)_{m,n} + (A_0)_{m,n-1}], \quad (9.13a) \]

\[ C_{m,n} = \frac{1}{2}[(A_1)_{m,n} + (A_1)_{m-1,n}]. \quad (9.13b) \]

In (9.12) the tilde signifies the average over the four adjacent lattice sites:

\[ \tilde{E}^{(0)}_{m,n} = \frac{1}{4}(E^{(0)}_{m+1,n+1} + E^{(0)}_{m+1,n} + E^{(0)}_{m,n+1} + E^{(0)}_{m,n}) \equiv E^{(0)}_{m,n}, \quad (9.14) \]

and the \((0)\) superscript is a reminder that this is the free field strength.

Finally, the free Yang-Mills equations driven by a current are

\[ \frac{1}{\hbar}(F_{m+1,n} - F_{m,n}) = -\tilde{j}^0_{m,n}, \quad (9.15a) \]

\[ \frac{1}{\hbar}(G_{m,n+1} - G_{m,n}) = \tilde{j}^1_{m,n}, \quad (9.15b) \]

where

\[ F_{m,n} = \frac{1}{2}(E_{m,n+1} + E_{m,n}), \quad (9.16a) \]

\[ G_{m,n} = \frac{1}{2}(E_{m+1,n} + E_{m,n}), \quad (9.16b) \]
and
\[ j^\mu_{m,n} = g\bar{\Psi}_{m,n} T\gamma^\mu \Psi_{m,n}. \] (9.17)

Here we have used \( \Psi = \tilde{\psi} \). (We have, of course, anticipated the interaction with the fermion by including \( j^\mu \).)

C. Constructing the Interaction Term Involving \( A^1 \)

We now proceed to construct the interaction terms for the Dirac equation (9.10). We begin by recognizing that the gauge transformation of the fermion on the lattice is
\[ \delta \Psi_{m,n} = ig\delta \omega_{m,n} \Psi_{m,n}, \] (9.18)
which is the appropriate lattice version of (9.2) because (9.18) guarantees that the mass term in (9.10) transforms covariantly, that is, by the same rule. (We also note that then the current in (9.17) transforms covariantly.) We can regard (9.18) as a difference equation for \( \delta \psi_{m,n} \); in particular, we find
\[ \delta \theta_{m,n} = ig\delta \omega_{m,n} \theta_{m,n} + i\frac{g}{2} \sum_{m'=m+1}^{M+m} (-1)^{m+m'} (\delta \omega_{m',n} - \delta \omega_{m'-1,n}) \theta_{m',n}. \] (9.19)

Here, we have used either periodic or antiperiodic boundary conditions depending on the size of the lattice:
\[ \psi_{m+M,n} = (-1)^{M+1} \psi_{m,n}, \quad \delta \psi_{m+M,n} = (-1)^{M+1} \delta \psi_{m,n}, \] (9.20)
where \( M \) is the number of spatial lattice sites. The boson variables such as \( \delta \omega_{m,n} \) will be assumed to be periodic.

It is evident that the second term in (9.10) does not transform covariantly under (9.19); as in the continuum, an interaction term must be added. Now the free field strength in (9.12) is invariant under the transformations
\[ (\delta A_0)_{m,n} = \frac{1}{2h} (\delta \Lambda_{m+1,n+1} + \delta \Lambda_{m,n+1} + \delta \Lambda_{m+1,n} + \delta \Lambda_{m,n}), \] (9.21a)
\[ (\delta A_1)_{m,n} = \frac{1}{2\Delta} (\delta \Lambda_{m+1,n+1} + \delta \Lambda_{m+1,n} + \delta \Lambda_{m,n+1} + \delta \Lambda_{m,n}). \] (9.21b)

The finite-element connection between \( \delta \Lambda_{m,n} \) and \( \delta \omega_{m,n} \) is
\[ \delta \omega_{m,n} = \frac{1}{4} (\delta \Lambda_{m+1,n+1} + \delta \Lambda_{m+1,n} + \delta \Lambda_{m,n+1} + \delta \Lambda_{m+1,n}). \] (9.22)

Then the scalar and vector potentials, (9.13a) and (9.13b), transform by
\[ \delta^{(0)} B_{m,n} = \frac{1}{h} (\delta \omega_{m,n} - \delta \omega_{m,n-1}), \] (9.23a)
\[ \delta^{(0)} C_{m,n} = \frac{1}{\Delta} (\delta \omega_{m,n} - \delta \omega_{m-1,n}). \] (9.23b)
Here the superscript \( (0) \) reminds us that further transformations of \( B \) and \( C \) will have to be deduced.

In the rest of this subsection we will examine that portion of the Dirac equation proportional to \( \gamma^1 \). A short calculation reveals that under (9.19)

\[
\delta \left[ \frac{i\gamma^1}{\Delta} (\theta_{m+1,n} - \theta_{m,n}) \right] = \frac{i\gamma^1}{\Delta} ig \left[ \delta \omega_{m,n} (\theta_{m+1,n} - \theta_{m,n}) \right. \\
\left. - \sum_{m' = m+1}^{m+M} (-1)^{m+m'} (\delta \omega_{m',n} - \delta \omega_{m'-1,n}) \theta_{m',n} \right].
\]

(9.24)

The first term here expresses the desired covariance of this term in the Dirac equation. The second term will be cancelled if we introduce an interaction term

\[
I_{m,n}^{(1)} = \frac{i\gamma^1}{\Delta} ig \delta \omega_{m,n} I_{m,n}^{(1)} - \delta I_{m,n}^{(1)} - \delta I_{m,n}^{(2)},
\]

(9.27)

and vary it with respect to \( C \) according to (9.23b). Indeed, in the continuum limit, this term reduces to the corresponding interaction term in (9.3) because

\[
\sum_{m' = m+1}^{m+M} (-1)^{m+m'} g_{m'} \to -g(m\Delta) \text{ as } \Delta \to 0, \ m \to \infty, \ M \to \infty,
\]

(2.26)

if \( g_{M+m} = (-1)^{M+1} g_m \). But, on the lattice, we are not finished, for we have neither yet varied (9.25) with respect to \( \theta \) according to (9.19), nor achieved the covariance of the interaction term.

A straightforward calculation now reveals that

\[
\delta \theta I_{m,n}^{(1)} = ig \delta \omega_{m,n} I_{m,n}^{(1)} - \delta I_{m,n}^{(1)} - \delta I_{m,n}^{(2)}.
\]

(9.27)

Here the first term is the required covariance term, the second term involves a new variation of \( C \),

\[
\delta^{(1)} C_{m,n} = i g \frac{1}{2} [\delta \omega_{m,n} + \delta \omega_{m-1,n}, C_{m,n}],
\]

(9.28)

and the third term is the \( \delta^{(0)} \) variation of a new interaction term

\[
I_{m,n}^{(2)} = -\frac{i\gamma^1}{2\Delta} (ig\Delta)^2 \sum_{m' = m+1}^{m+M} \sum_{m'' = m'+1}^{m'+M-1} (-1)^{m+m''} C_{m',n} C_{m'',n} \theta_{m'',n}.
\]

(9.29)

It is easy to see that (9.29) vanishes in the continuum limit, while the variation (9.28) reduces to the second term in (9.4) in that same limit. Once again, we are not finished: we still need to vary \( I^{(2)} \) with respect to \( \delta \theta \) and \( \delta^{(1)} C \), and produce the required covariance of \( I^{(2)} \).
Clearly, this process of adding successive interaction terms and evermore $C$ variations never terminates. But it is easy to discern the general pattern. The easiest way to express $I^{(N)}$, where $N$ is the order in $g\Delta C$, is by the following inductive formula: (here and in the rest of this section we delete the time index $n$, since all variables are evaluated at that time)

$$N \geq 1 : \quad I_m^{(N)} = \frac{1}{2} \sum_{m' = m+1}^{m+M} (-1)^{m+m'} \sum_{k=1}^{N} \frac{1}{k!} (-ig\Delta C_{m'})^k I_{m'}^{(N-k)}, \quad (9.30)$$

where we define $I_m^{(0)} = -2i\gamma^1 \theta_m / \Delta$. The gauge transformations are given by (9.19) and

$$k \neq 1 : \quad \delta^{(k)} C_m = \frac{(ig\Delta)^k B_k}{\Delta} \left[ \ldots [\delta \omega_m - \delta \omega_{m-1}, C_m], \ldots, C_m \right] \quad (9.31a)$$

$$\delta^{(1)} C_m = \frac{ig}{2} [\delta \omega_m + \delta \omega_{m-1}, C_m], \quad (9.31b)$$

where $B_k$ is the $k$th Bernoulli number. The required covariance statement

$$\delta_\theta I_m^{(N)} + \sum_{k=0}^{N} \delta^{(k)} I_m^{(N-k+1)} = ig\delta \omega_m I_m^{(N)} \quad (9.32)$$

is proved in Ref. 28.

From (9.30) we can derive an “integral equation” satisfied by the full interaction term for the vector potential, $I = \sum_{N=1}^{\infty} I^{(N)}$:

$$I_m = \frac{1}{2} \sum_{m' = m+1}^{m+M} (-1)^{m+m'} \left( e^{-ig\Delta C_{m'}} - 1 \right) \left( I_{m'}^{(0)} + I_{m'} \right). \quad (9.33)$$

From this, a difference equation can be immediately derived:

$$I_m + e^{ig\Delta C_m} I_{m-1} = \frac{2i\gamma^1}{\Delta} \left( 1 - e^{ig\Delta C_m} \right) \theta_m. \quad (9.34)$$

### D. Scalar Potential Part of Dirac Equation

The procedure is now clear. We start from the part of the free Dirac equation (9.10) proportional to $\gamma^0$,

$$\frac{i\gamma^0}{\hbar} (\phi_{n+1} - \phi_n). \quad (9.35)$$

Here we have dropped the spatial index $m$ because throughout this section all variables will be evaluated at the same spatial coordinate. Now, we need to solve the fermion transformation equation (9.18) for $\delta \phi$. We immediately find

$$\delta \phi_n = ig \delta \omega_n \phi_n + (-1)^n \left( \delta \phi_0 - ig \delta \omega_0 \phi_0 \right)$$

$$- ig \sum_{n'=1}^{n} (-1)^{n+n'} (\delta \omega_{n'} - \delta \omega_{n'-1}) \phi_{n'}. \quad (9.36)$$

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At this point we make a slight variation on the procedure of Ref. 27. We will choose as an initial condition on the \( \phi \) variation

\[
\delta \phi_0 = \frac{i}{2} g (\delta \omega_0 + \delta \omega_{-1}) \phi_0, \tag{9.37}
\]

which will simplify the form of subsequent formulas. The reason for the choice (9.37) (or the slightly different choice made in Ref. 27) is to ensure that the lattice variation of the time difference of \( \phi \) and hence the Dirac equation have the correct continuum limit. The latter is obtained from

\[
2 \sum_{n=1}^{n} (-1)^{n+n'} f_{n'} + (-1)^{n} f_0 \rightarrow f(nh), \quad \text{as} \quad h \to 0, \quad n \to \infty. \tag{9.38}
\]

Using (9.37) we write (9.36) as

\[
\delta \phi_n = ig \delta \omega_n \phi_n - ig \sum_{n=0}^{n} (-1)^{n+n'} (\delta \omega_{n'} - \delta \omega_{n'-1}) \phi_{n'}, \tag{9.39}
\]

where the prime on the sum signifies that the 0th term is counted with half weight:

\[
\sum_{n'=0}^{n} f_{n'} = \sum_{n=1}^{n} f_{n'} + \frac{1}{2} f_0. \tag{9.40}
\]

The variation of (9.35) is immediate:

\[
\delta \left[ \frac{i \gamma^0}{h} (\phi_{n+1} - \phi_n) \right] = \frac{i \gamma^0}{h} ig \delta \omega_n (\phi_{n+1} - \phi_n) \\
+ 2ig \frac{\gamma^0}{h} \sum_{n'=0}^{n} (-1)^{n+n'} (\delta \omega_{n'} - \delta \omega_{n'-1}) \phi_{n'}. \tag{9.41}
\]

The first term here is the required covariance of (9.35), while the second term on the right-hand side of (9.41) is cancelled by the variation of the following interaction term,

\[
K_n^{(1)} = -2(igh) \frac{i \gamma^0}{h} \sum_{n'=0}^{n} (-1)^{n+n'} B_{n'} \phi_{n'}. \tag{9.42}
\]

under (9.23a),

\[
\delta^{(0)} B_n = \frac{1}{h} (\delta \omega_n - \delta \omega_{n-1}). \tag{9.43}
\]

Of course, using (9.38), we find that \( K^{(1)} \) reduces to the appropriate interaction term in the continuum Dirac equation.
However, here again we are not finished. We must vary $K^{(1)}$ with respect to $\delta \theta$. Doing so necessitates the introduction of a new field variation,

\[ n \geq 1 : \quad \delta^{(1)}B_n = \frac{ig}{2} [\delta \omega_n + \delta \omega_{n-1}, B_n], \quad (9.44a) \]

\[ n = 0 : \quad \delta^{(1)}B_0 = \frac{ig}{2} [\delta \omega_0 + \delta \omega_{-1}, B_0] + \frac{ig}{4} [\delta \omega_0 - \delta \omega_{-1}, B_0], \quad (9.44b) \]

which reduces to the appropriate portion of the continuum variation (9.4) in the continuum limit, and a new interaction term,

\[ K^{(2)}_n = -\frac{i\gamma^0}{h} 2(igh)^2 \sum_{n' = 0}^{n} \sum_{n'' = 0}^{n'} (-1)^{n+n''} B_{n'} B_{n''} \phi_{n''}, \quad (9.45) \]

which vanishes in the continuum limit. Here

\[ n' \geq 1 : \quad \sum_{n'' = 0}^{n'} \sum_{n'' = 0}^{n'} (-1)^{n+n''} \frac{1}{2} f_0 + \sum_{n'' = 1}^{n-1} f_{n''} + \frac{1}{2} f_{n'}, \quad (9.46) \]

\[ \sum_{n'' = 0}^{0} f_{n''} = 0. \]

Once again, this iterative procedure continues indefinitely. We may again write the general result in terms of an inductive formula. The order $N$ interaction term is

\[ K^{(N)}_n = -\sum_{n' = 1}^{n} (-1)^{n+n'} \sum_{k=1}^{N} \frac{(igh)^k}{k!} B_{n'} K^{(N-k)}_{n'} + \frac{1}{h} (igh)^N \frac{1}{2N!} (-1)^n B_0^N K_0^{(0)}, \quad (9.47) \]

where $K_0^{(0)} = -2i\gamma^0 \phi_n/h$. In Ref. 28 we show that these transform according to the required law,

\[ \delta \phi K^{(N)}_n + \sum_{k=0}^{N} \delta^{(k)}K^{(N-k+1)}_n = i g \delta \omega_n K^{(N)}_n, \quad (9.48) \]

where $\delta \phi_n$ is given by (9.39) and

\[ k \neq 1 : \quad \delta^{(k)}B_n = \frac{(igh)^k}{h} \frac{B_k}{k!} \left[ \ldots [\delta \omega_n - \delta \omega_{n-1}, B_n], \ldots, B_n \right], \quad n \neq 0, \quad (9.49a) \]

\[ \delta^{(1)}B_n = \frac{ig}{2} [\delta \omega_n + \delta \omega_{n-1}, B_n], \quad n \neq 0, \quad (9.49b) \]

\[ k \neq 1 : \quad \delta^{(k)}B_0 = \frac{(igh)^k}{h} \frac{B_k}{k!} 2^k \left[ \ldots [\delta \omega_0 - \delta \omega_{-1}, B_0], \ldots, B_0 \right], \quad (9.49c) \]

\[ \delta^{(1)}B_0 = \frac{ig}{2} [\delta \omega_0 + \delta \omega_{-1}, B_0] - i g \frac{B_1}{2} [\delta \omega_0 - \delta \omega_{-1}, B_0], \quad (9.49d) \]
where, again, $B_k$ is the $k$th Bernoulli number.

As before, from (9.47) an “integral equation” can be immediately obtained for the full interaction term with the scalar potential, $K_n = \sum_{N=1}^{\infty} K_n^{(N)}$:

$$K_n = -\sum_{n'=1}^{n} (-1)^{n+n'} \left( e^{-i\hbar B_{n'}} - 1 \right) \left( K_{n'} + K_{n'}^{(0)} \right) + (-1)^n \left( e^{i\hbar B_0/2} - 1 \right) K_{0}^{(0)} ,$$

(9.50)

which is equivalent to the difference equation

$$K_n + e^{i\hbar B_n} K_{n-1} = \frac{2i\gamma^0}{\hbar} \left( 1 - e^{i\hbar B_n} \right) \phi_n .$$

(9.51)

The full lattice Dirac equation is given by (9.10), (9.30), and (9.47):

$$\frac{i\gamma^0}{\hbar} \left( \psi_{m,n+1} - \psi_{m,n} \right) + \frac{i\gamma^1}{\Delta} (\theta_{m+1,n} - \theta_{m,n}) + \frac{\mu}{2} (\phi_{m,n+1} + \phi_{m,n}) + (I_{m,n} + K_{m,n}) = 0 .$$

(9.52)

Note that (9.52) gives $\psi_{m,n+1}$ in terms of fields at time $n$ and earlier, so that this difference equation may be solved by time stepping through the lattice.

**E. Unitarity of the Dirac Equation in the Temporal Gauge**

For the case of Abelian electrodynamics, the Dirac equation may be written explicitly. (For a complete discussion, see Ref. 27.) The generalization to four dimensions is immediate, and the result may be expressed in terms of the spatially averaged electron field as follows:

$$\frac{i\gamma^0}{\hbar} \left( \psi_{m,n+1} - \psi_{m,n} \right) + \frac{i\gamma^1}{\Delta} \left[ \sum_{m'j}^{M} (-1)^{m_j+m'_j} \psi_{m_j,m}_{\perp,m} - \sum_{m'j}^{m-1} (-1)^{m_j+m'_j} \psi_{m_j,m}_{\perp,m} \right] + \frac{\mu\gamma^0}{2} (\psi_{m,n+1} + \psi_{m,n}) + 2\frac{\gamma^0\gamma^j}{\Delta} \sum_{m'_j}^{M} \alpha_{m_{\perp};m_j,m'_j}^{(j)} \psi_{m_j,m'} = 0 ,$$

(9.53)

where a sum over the repeated index $j$ is understood. Here, we have adopted a temporal gauge, $A^0 = 0$, and expressed the interaction in terms of (only the $j$th index is explicit and the spatial coordinates refer to the $j$th direction)

$$\alpha_{m,m'}^{(j)} = \frac{i}{2} (-1)^{m+m'} \sec \zeta \sum_{m''}^{M} \sgn(m''-m) \sgn(m''-m') \left( e^{2i\zeta_{m''}} - 1 \right)$$

$$\times \exp \left[ i \sum_{m''}^{M} \sgn(m''-m) \sgn(m''-m') \sgn(m''-m) \zeta_{m''} \right] .$$

(9.54)
We have used the abbreviations

\[ \zeta_m = \frac{e\Delta}{2m_{j-1}} \quad \zeta = \sum_{m_j=1}^{M} \zeta_m, \quad (9.55) \]

and

\[ \text{sgn}(x) = \begin{cases} +1, & x > 0, \\ -1, & x \leq 0. \end{cases} \quad (9.56) \]

We can now carry out the sum over \( m'' \) in (9.54):

\[ \alpha^{(j)}_{m,m'} = i\epsilon_{m',m}(-1)^{m+m'} \left[ -1 + \cos \left( \sum_{m''=1}^{M} \text{sgn}(m'' - m)\text{sgn}(m'' - m')\zeta_{m''} \right) \sec \zeta \right] 
- (-1)^{m+m'} \sin \left( \sum_{m''=1}^{M} \text{sgn}(m'' - m)\text{sgn}(m'' - m')\zeta_{m''} \right) \sec \zeta, \quad (9.57) \]

where

\[ \epsilon_{m',m} = \begin{cases} 1, & m' > m, \\ 0, & m' = m, \\ -1, & m' < m. \end{cases} \quad (9.58) \]

It is obvious that \( \alpha^{(j)} \) is Hermitian.

Let us write the Dirac equation (9.53) in the form

\[ U\psi_{n+1} = V\psi_n. \quad (9.59) \]

It is apparent that \( V = 2 - U \), so the transfer matrix is

\[ T = 2U^{-1} - 1. \quad (9.60) \]

The condition that \( T \) is unitary translates into the following condition on \( U \):

\[ U + U^\dagger = 2. \quad (9.61) \]

From (9.53) the matrix \( U \) is explicitly

\[ U_{m,m'} = \delta_{m,m'} + \frac{\hbar\gamma^0}{\Delta} \delta_{m,m'} \left( -1 \right)^{m_j+m_j'} \epsilon_{m_j,m_j'} \delta_{m_\perp,m_\perp}' 
- \frac{i\hbar\mu\gamma^0}{2} \delta_{m,m'} - \frac{i\hbar\gamma^0\gamma^j}{\Delta} \alpha^{(j)}_{m_\perp,m_j,m_j'} \delta_{m_\perp,m_\perp}'. \quad (9.62) \]

Therefore, the unitarity condition (9.61) is equivalent to the condition that \( \alpha^{(j)} \) be Hermitian:

\[ \alpha^{(j)}\dagger = \alpha^{(j)}, \quad (9.63) \]

which is satisfied as noted above.
F. Solution of the Abelian Equations—The Schwinger Model

Here we wish to illustrate how the equations of motion are solved, and spectral information extracted, for the special case of the Schwinger model, electrodynamics in two space-time dimensions with zero bare fermion mass. We can work in the gauge $A_0 = 0$, and choose a square lattice, $\Delta = h$, in which case the difference equation (9.34) can be solved by

$$\phi_{m,n+1}^{(+)} = e^{iehC_{m,n}}\phi_{m-1,n}^{(+)};$$
$$\phi_{m-1,n+1}^{(-)} = e^{-iehC_{m,n}}\phi_{m,n}^{(-)},$$

(9.64a)

where the superscripts denote the chirality, the eigenvalues of $i\gamma_5 = \gamma^0\gamma^1$. We take $\phi_{m,n}^{(\pm)}$ to be the canonical fermion variables, which for free fields have the Fock–space expansion

$$\phi_{m,n} = \sum_{k=1}^{M} e^{ip_kmh}(e^{-ip_nh}\psi^{(+)}_k + e^{ip_nh}\psi^{(-)}_k),$$

(9.65)

where for $M$ even, $p_k = (2k+1)\pi/(Mh)$, $\psi^{(\pm)}$ are the eigenvectors of $\gamma_5$, and

$$[a_k^{(\pm)}, a_k^{(\mp)}]_+ = \frac{1}{Mh}\delta_{kk'},$$

(9.66)

the other anticommutators being zero. The physical interpretation of $a_k^{(\pm)}$, $a_k^{(\pm)\dagger}$ as creation and annihilation operators is as follows:

for $0 \leq k \leq \frac{M}{2} - 1$, $a_k^{(\pm)}|0\rangle = a_k^{(\pm)\dagger}|0\rangle = 0$;

for $-\frac{M}{2} \leq k < 0$, $a_k^{(\pm)\dagger}|0\rangle = a_k^{(\pm)}|0\rangle = 0$;

(9.67)

this construction then implies the correct lattice fermion Green’s function.

The only physical particle in the Schwinger model is a boson, which we denote by $B$, of mass $\mu$. We can obtain the dispersion relation for this particle in a manner analogous to that employed in Sec. VII by taking matrix elements of the equations of motion between the vacuum and a $B$ state of momentum $q_l = 2\pi l/Mh$. Using (9.15) we find

$$\langle B, l | J_{m,n}^{(+)} | 0 \rangle = \frac{1}{2} \langle B, l | J_{m,n}^0 + J_{m,n}^1 | 0 \rangle$$
$$= \frac{1}{4h} \left\{ -(e^{i\omega_l h} + 1)(e^{-iq_l h} - 1) + (e^{i\omega_l h} - 1)(e^{-iq_l h} + 1) \right\} \langle B, l | E_{m,n} | 0 \rangle$$
$$\approx \frac{i}{2} (q_l + \omega_l) \langle B, l | E_{m,n} | 0 \rangle,$$

(9.68)

as $h \to 0$. On the other hand, we evaluate the current matrix element using the solution of the Dirac equation (9.64) and the Fock–space expansion at the initial time $n = 0$. That is, since

$$\phi_{m,n+1}^{(+)} = e^{iehC_{m,n}}\phi_{m-n,0}^{(+)};$$

(9.69)

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and

$$\varphi^{(+)}_{m,0} = \sum_{k=1}^{M} e^{ip_k m h} a_k^{(+)}$$ \quad (9.70)

we have

$$J^{(+)}_{m,n} = \frac{e}{4} \sum_{k,k'=1}^{M} a_k^{(+)} \dagger e^{i(p_{k'} - p_k)(m-n)h} \left[ 1 + e^{i(p_k - p_{k'})h} \right]$$

$$+ e^{ip_k h} e^{-i e h} \sum_{r=0}^{n} (C_{m-n+r,r}-C_{m-n+1+r,r}) e^{-i e h C_{m+1,n}}$$

$$+ e^{-i p_{k'} h} e^{i e h} \sum_{r=0}^{n} (C_{m-n+r,r}-C_{m-n+1+r,r}) e^{i e h C_{m+1,n}} a_{k'}^{(+)}. \quad (9.71)$$

We now assume that the $B$ states are not created by fermion operators, that is, for all $k > 0$,

$$\langle B,l | a_k^{(+)} \dagger \rangle = 0,$$ \quad (9.72)

and that the commutator of $a_k^{(+)}$ and $C_{m,n}$ is negligible as $h \to 0$. Then, using the canonical relation (9.66) together with the vacuum definition (9.67), we find that the matrix element of (9.71) is

$$\langle B,l | J^{(+)}_{m,n} | 0 \rangle = \frac{e}{4} \frac{1}{Mh} i e h \sum_{k=-M/2}^{-1} \left( e^{-i p_k h} - e^{i p_k h} \right)$$

$$\times \langle B,l | C_{m+1,n} + \sum_{r=0}^{n} (C_{m-n+r,r} - C_{m-n+1+r,r}) | 0 \rangle,$$ \quad (9.73)

where an expansion in $h$ has been carried out. The sum on $k$ in (9.73) is immediately evaluated as $2i/\sin(\pi/M)$, while the remaining matrix element is

$$\langle B,l | C_{m,n} | 0 \rangle \left[ e^{-i q_l h + i q_l h \sum_{r=0}^{n} e^{i h (q_l - \omega_l) r}} \right] \approx \frac{\omega_l}{\omega_l - q_l} \langle B,l | C_{m,n} | 0 \rangle,$$ \quad (9.74)

where in the last summation on $r$ we have deleted a rapidly oscillating term $\sim e^{-i(\omega_l - q_l) nh} \to 0 \ (nh \to \infty)$. Finally, from (9.12) we learn, as $h \to 0$, that

$$i \omega_l \langle B,l | C_{m,n} | 0 \rangle = \langle B,l | E_{m,n} | 0 \rangle.$$ \quad (9.75)

When we put (9.68), (9.73), (9.74), and (9.75) together, we obtain the desired dispersion relation,

$$\langle B,l | E_{m,n} | 0 \rangle = \frac{2}{i} \frac{1}{\omega_l + q_l} \langle B,l | J^{(+)}_{m,n} | 0 \rangle$$

$$= \frac{2}{i} \frac{1}{\omega_l + q_l} \frac{2i}{4M} \sin \pi/M \frac{\omega_l}{\omega_l - q_l} \langle B,l | C_{m,n} | 0 \rangle$$

$$= \frac{1}{\omega_l^2 - q_l^2} \frac{2i}{M \sin \pi/M} \langle B,l | E_{m,n} | 0 \rangle.$$ \quad (9.76)
or
\[ \omega_l^2 = q_l^2 + \mu^2, \tag{9.77} \]
where the mass \( \mu \) is
\[ \mu^2 = \frac{e^2}{M \sin \pi/M}. \tag{9.78} \]

Using the solution (9.64) we may also calculate the divergences of the vector and axial currents. It is easily seen from (9.64) that the lattice divergence of the vector current
\[ J_{\mu m,n} = e\Psi_{m,n}^\dagger \gamma^0 \gamma^\mu \Psi_{m,n} \tag{9.79} \]
is
\[ (\nabla_\mu J_{\mu})_{m,n} = \frac{ie}{2h} \sin \left( \frac{eh^2}{2} \tilde{E}_{m,n} \right) e^{i\epsilon h (C_{m+1,n+1} + C_{m,n+1})/2} \phi_{m+1,n+1}^\dagger \phi_{m,n+1} + \text{h.c.}, \tag{9.80} \]
which has vanishing vacuum expectation value because
\[ \langle \phi_{m,n} \phi_{m+1,n}^\dagger \rangle = -\frac{i\gamma_5}{Mh \sin(\pi/M)}. \tag{9.81} \]

On the other hand, this same result shows that the axial-vector current
\[ J_{\mu m,n} = e\Psi_{m,n}^\dagger \gamma^0 \gamma^\mu \gamma_5 \Psi_{m,n} \tag{9.82} \]
yields the axial anomaly
\[ (\nabla_\mu J_{\mu})_{m,n} = -e^2 \left( M \sin \frac{\pi}{M} \right)^{-1} \tilde{E}_{m,n}. \tag{9.83} \]
The same anomaly emerges as in (9.78). The details of this latter calculation can be found in Ref. 27. Note that the lattice anomaly \( e^2/(M \sin(\pi/M)) \) differs from the continuum value \( e^2/\pi \) by a term of order \( 1/M^2 \), an error typical of the linear finite-element approximation.
X. LATTICE YANG-MILLS EQUATIONS

We now continue the discussion where we left off in Sec. IX.C. For simplicity, we continue to work in $1 + 1$ dimensions, and we will here assume a square lattice, $\hbar = \Delta$.

A. Construction of the Field Strength

Now that the gauge transformations for $C$ and $B$ are completely determined, it should be straightforward to work out the interaction terms in the field strength, $E = F_{01}$. We begin by recalling that the zeroth order construction (9.12)

$$\tilde{E}^{(0)}_{m,n} = \frac{1}{\hbar}(C_{m+1,n+1} - C_{m+1,n}) - \frac{1}{\hbar}(B_{m+1,n+1} - B_{m,n+1}),$$  \hspace{1cm} (10.1)

is invariant under (9.23),

$$\delta^{(0)} C_{m,n} = \frac{1}{\hbar}((\delta \omega_{m,n} - \delta \omega_{m-1,n}),$$

$$\delta^{(0)} B_{m,n} = \frac{1}{\hbar}((\delta \omega_{m,n} - \delta \omega_{m,n-1}).$$   \hspace{1cm} (10.2a)

However, under the $\delta^{(1)}$ transformations (9.28) and (9.44) the variation of $\tilde{E}^{(0)}$ is

$$\delta^{(1)} \tilde{E}^{(0)}_{m,n} = ig[\delta \omega_{m,n}, \tilde{E}^{(0)}_{m,n}]$$

$$+ \frac{ig}{2}\delta^{(0)}[[B_{m+1,n+1}, C_{m+1,n+1}] + [B_{m,n+1}, C_{m+1,n+1}] + [B_{m+1,n+1}, C_{m+1,n}]$$

$$- [B_{m,n+1}, C_{m+1,n}] + [C_{m+1,n}, C_{m+1,n+1}] - [B_{m,n+1}, B_{m+1,n+1}]].$$  \hspace{1cm} (10.3)

The total variation in (10.3) allows us to identify the first-order interaction term. Because it is already clear that the interaction terms in the field strength will be local, involving fields at the four corners of the finite element, let us simplify the following by introducing the abbreviations

$$B_1 \equiv B_{m+1,n+1}, \quad \text{and} \quad B_0 \equiv B_{m,n+1},$$  \hspace{1cm} (10.4a)

$$C_1 \equiv C_{m+1,n+1}, \quad \text{and} \quad C_0 \equiv C_{m+1,n}. $$  \hspace{1cm} (10.4b)

Then, the free term (10.1) is

$$\tilde{E}^{(0)}_{m,n} = \frac{1}{\hbar}(C_1 - C_0 - B_1 + B_0),$$  \hspace{1cm} (10.5)

and the interaction deduced from (10.3) is

$$\tilde{E}^{(1)}_{m,n} = -\frac{ig}{2}[[B_1, C_1] + [B_0, C_1] + [B_1, C_0] - [B_0, C_0] + [C_0, C_1] - [B_0, B_1]].$$  \hspace{1cm} (10.6)

Note that $E$ is antisymmetric under the interchange $B \leftrightarrow C$, an antisymmetry that will be maintained in each order. We could add here an arbitrary multiple of $(\tilde{E}^{(0)}_{m,n})^2$ because
this is invariant under $\delta^{(0)}$ variations. We choose not to do so, and thereby keep a minimal form in terms of nested commutators. This form, of course, guarantees hermiticity.

It is straightforward to find the higher-order interaction terms. We find $\tilde{E}^{(2)}$ from the equation

$$\delta^{(2)} \tilde{E}^{(0)}_{m,n} + \delta^{(1)} \tilde{E}^{(1)}_{m,n} + \delta^{(0)} \tilde{E}^{(2)}_{m,n} = ig [\delta \omega_{m,n}, \tilde{E}^{(1)}_{m,n}], \quad (10.7)$$

Up to the above-mentioned ambiguity it is

$$\tilde{E}^{(2)}_{m,n} = -\frac{g^2 \hbar}{12} \left\{ [B_1, [B_0, C_1]] - [C_1, [B_1, C_0]] - [B_1, [B_1, C_0]] + [C_1, [C_1, B_0]] ight.$$  
$$+ 2[B_1, [B_0, C_1]] - 2[C_1, [C_0, B_1]] + 2[B_0, [B_1, C_1]] - 2[C_0, [C_1, B_1]]$$  
$$- 2[B_1, [B_0, C_0]] + 2[C_1, [C_0, B_0]] + 4[B_0, [B_1, C_0]] - 4[C_0, [C_1, B_0]]$$ 
$$+ [B_0, [B_0, C_1]] - [C_0, [C_0, B_1]] - [B_0, [B_0, C_0]] + [C_0, [C_0, B_0]]$$  
$$- [B_0, [B_0, B_1]] + [C_0, [C_0, C_1]] - [B_1, [B_0, B_1]] + [C_1, [C_0, C_1]] \right\}. \quad (10.8)$$

Our final explicit example is $\tilde{E}^{(3)}$, obtained from

$$\delta^{(3)} \tilde{E}^{(3)}_{m,n} + \delta^{(2)} \tilde{E}^{(2)}_{m,n} + \delta^{(1)} \tilde{E}^{(1)}_{m,n} = ig [\delta \omega_{m,n}, \tilde{E}^{(2)}_{m,n}], \quad (10.9)$$

(Recall that $\delta^{(3)} = 0$.) It has fewer terms than might have been anticipated:

$$\tilde{E}^{(3)}_{m,n} = \frac{ig^3 \hbar^2}{24} \left\{ [B_1, [B_0, [B_0, B_1]]] - [C_1, [C_0, [C_0, C_1]]] + 2[B_0, [C_0, [B_0, B_1]]] ight.$$  
$$- 2[C_0, [B_0, [C_0, C_1]]] - 2[B_1, [C_0, [B_0, B_1]]] + 2[C_0, [B_1, [B_0, B_1]]]$$ 
$$+ 2[C_1, [B_0, [C_0, C_1]]] - 2[B_0, [C_1, [C_0, C_1]]] + 2[B_1, [B_0, [C_1, B_1]]]$$ 
$$- 2[C_1, [C_0, [B_1, C_1]]] - 2[B_0, [B_1, [B_0, C_1]]] + 2[C_0, [C_1, [C_0, B_1]]]$$ 
$$- [B_1, [C_1, [B_1, C_1]]] - [B_0, [C_0, [B_0, C_1]]] - 2[B_0, [C_1, [B_1, C_1]]]$$ 
$$+ 2[B_1, [B_0, [B_1, C_1]]] + 2[C_0, [B_1, [C_0, B_1]]] - 2[B_1, [C_0, [B_1, B_1]]]$$  
$$- 2[B_0, [C_0, [B_0, B_1]]] + 2[C_0, [B_0, [C_0, B_1]]] + [B_1, [C_0, [B_1, B_1]]]$$ 
$$- [C_1, [B_0, [C_1, B_0]]] - 2[B_0, [C_1, [B_1, C_0]]] + 2[C_1, [B_0, [B_1, C_0]]]$$ 
$$- 2[C_1, [C_0, [B_1, B_0]]] + 2[C_0, [C_1, [B_0, B_1]]] - 2[B_0, [C_0, [B_1, C_1]]]$$  
$$+ 2[C_0, [B_0, [C_1, B_1]]]] \right\}. \quad (10.10)$$

Although we can compute $\tilde{E}^{(N)}$ to any required order $N$, we have not yet discovered a general iterative formula for $\tilde{E}^{(N)}$. A notable feature is the appearance in $\tilde{E}^{(N)}$ of the term

$$(igh)^N \frac{B_N}{2} \sum_{i=1}^{4} [A_i, [A_i, \ldots, [A_i, \tilde{E}^{(0)}] \ldots]], \quad (10.11)$$

where $\{A_i\} = \{B_0, B_1, C_0, C_1\}$, which, for example, results in a significant simplification of (10.8), and explains the absence of many terms in (10.10), because $B_3 = 0$. The calculational methods so far developed for extracting information about matrix elements
require the knowledge of only the first few orders in any case, so the first few \( E^{(N)} \) should be sufficient for at least the initial stages of the finite-element solution.

The generalization to four dimensions is nontrivial but straightforward. Of course there are more Yang-Mills field components, and new structures emerge. The lowest-order interaction terms are given in Ref. 30.

**B. Construction of Yang-Mills Equations**

We finally must construct the lattice analogue of the continuum Yang-Mills equations, (9.9). Upon a moment’s reflection, however, it is clear that the structure of the interactions in the Dirac equation carries over to the Yang-Mills equations, with multiplication by powers of the potentials replaced by nested commutators. The reason for this is as follows.

We must solve the covariance equation

\[
\delta \tilde{E}_{m,n} = ig[\delta \omega_{m,n}, \tilde{E}_{m,n}] \tag{10.12}
\]

for \( \delta F \) and \( \delta E \), where \( F \) and \( G \) are given by (9.16). Because necessarily the boson fields must be periodic, to do this we require that the number of spatial lattice sites, \( M \), be odd. Then we find

\[
\delta F_{m,n} = ig[\delta \omega_{m,n}, F_{m,n}] + \frac{ig}{2} \sum_{m'=m+1}^{m+M} (-1)^{m+m'} [\delta \omega_{m',n} - \delta \omega_{m'+1,n}, F_{m',n}], \tag{10.13}
\]

and

\[
\delta G_{m,n} = ig[\delta \omega_{m,n}, G_{m,n}] - ig \sum_{n'=0}^{n'} (-1)^{n+n'} [\delta \omega_{m,n'} - \delta \omega_{m,n'-1}, G_{m,n'}]. \tag{10.14}
\]

These have just the form as the transformation of \( \theta \) and \( \phi \), (9.19) and (9.39), respectively. Here, as in (9.37), we have adopted the initial conditions, at fixed spatial coordinate \( m \),

\[
\delta G_0 = \frac{ig}{2} [\delta \omega_0 + \delta \omega_{-1}, G_0]. \tag{10.15}
\]

Without more ado, we transcribe the form of the interaction terms. For \( F \) we suppress the local variable \( n \) and write, for the term to be added to the left-hand side of (9.15a),

\[
\mathcal{I}_m^{(N)} = \frac{1}{2} \sum_{m'=m+1}^{m+M} (-1)^{m+m'} \sum_{k=1}^{N} \frac{(igh)^k}{k!} \left[ C_{m', [C_{m', [\ldots, [C_{m', \mathcal{I}_{m'}^{(N-k)}]_{\text{k nested commutators}]_{\text{k nested commutators}]_{\text{k nested commutators}]_{\text{k nested commutators}]]_{\text{k nested commutators}]_{\text{k nested commutators}]]}_{\text{k nested commutators}]]\cdots]]} \right], \tag{10.16}
\]

and for \( G \) we suppress the local variable \( m \) and write, for the term to be added to the left-hand side of (9.15b),

\[
\mathcal{K}_n^{(N)} = -\sum_{n'=1}^{n} (-1)^{n+n'} \sum_{k=1}^{N} \frac{(igh)^k}{k!} \left[ B_{n', [B_{n', [\ldots, [B_{n', \mathcal{K}_{n'}^{(N-k)}]_{\text{k nested commutators}]_{\text{k nested commutators}]_{\text{k nested commutators}]_{\text{k nested commutators}]]}_{\text{k nested commutators}]]_{\text{k nested commutators}]]_{\text{k nested commutators}]} \right] + (igh)^N \frac{1}{2^N N!} (-1)^n \left[ B_0, [B_0, [\ldots, [B_0, \mathcal{K}_0^{(0)}]_{\text{N nested commutators}]_{\text{N nested commutators}]_{\text{N nested commutators}]_{\text{N nested commutators}]}_\cdots]] \right]. \tag{10.17}
\]
Here $\mathcal{I}_m^{(0)} = -2F_m/h$ and $\mathcal{K}_n^{(0)} = -2G_n/h$. In Ref. 29 we prove the required covariance statements:

$$\delta_F \mathcal{I}_m^{(N)} + \sum_{k=0}^{N} \delta^{(k)} \mathcal{I}_m^{(N-k+1)} = ig[\delta \omega_m, \mathcal{I}_m^{(N)}], \quad (10.18)$$

and

$$\delta_G \mathcal{K}_n^{(N)} + \sum_{k=0}^{N} \delta^{(k)} \mathcal{K}_n^{(N-k+1)} = ig[\delta \omega_n, \mathcal{K}_n^{(N)}]. \quad (10.19)$$

The Yang-Mills equations are given by (9.15), (10.16), and (10.17):

$$\frac{1}{\hbar}(F_{m+1,n} - F_{m,n}) + \mathcal{I}_{m,n} = -\mathcal{j}_{m,n}^0, \quad (10.20a)$$

$$\frac{1}{\hbar}(G_{m,n+1} - G_{m,n}) + \mathcal{K}_{m,n} = \mathcal{j}_{m,n}^1, \quad (10.20b)$$

where

$$\mathcal{I}_{m,n} = \sum_{N=1}^{\infty} \mathcal{I}_{m,n}^{(N)}, \quad (10.21a)$$

$$\mathcal{K}_{m,n} = \sum_{N=1}^{\infty} \mathcal{K}_{m,n}^{(N)}. \quad (10.21b)$$

Here, we easily derive from (10.16) and (10.17) first

$$\mathcal{I}_{m} = \frac{1}{2} \sum_{m'=m+1}^{m+M} (-1)^{m+m'} \left[ e^{-i\hbar C_{m'}} \left( \mathcal{I}_{m'}^{(0)} + \mathcal{I}_{m'} \right) e^{i\hbar C_{m'}} - \mathcal{I}_{m'}^{(0)} - \mathcal{I}_{m'} \right], \quad (10.22a)$$

and

$$\mathcal{K}_{n} = -\sum_{n'=1}^{n} (-1)^{n+n'} \left[ e^{-i\hbar B_{n'}} \left( \mathcal{K}_{n'}^{(0)} + \mathcal{K}_{n'} \right) e^{i\hbar B_{n'}} - \mathcal{K}_{n'}^{(0)} - \mathcal{K}_{n'} \right]$$

$$+ (-1)^{n} \left[ e^{i\hbar B_0/2} \mathcal{K}_0^{(0)} e^{-i\hbar B_0/2} - \mathcal{K}_0^{(0)} \right], \quad (10.22b)$$

and then the difference equations

$$\mathcal{I}_m e^{i\hbar C_m} + e^{i\hbar C_m} \mathcal{I}_{m-1} = -\frac{2}{\hbar} \left[ e^{i\hbar C_m}, F_m \right], \quad (10.23a)$$

and

$$\mathcal{K}_n e^{i\hbar B_n} + e^{i\hbar B_n} \mathcal{K}_{n-1} = -\frac{2}{\hbar} \left[ e^{i\hbar B_n}, G_n \right]. \quad (10.23b)$$

We are currently studying the solutions of these equations in two and four space-time dimensions.

We thank G. Dunne for many extended and useful conversations. LRM thanks the Physics Department at Washington University for its hospitality. We thank the U. S. Department of Energy for funding this research.
REFERENCES

1 Strictly speaking, we are using the collocation method, a technique closely resembling the method of finite elements.

2 If one imposes (14a) at one point on the finite element, say \( t = ah \), and (14b) at another point on the finite element, say \( t = bh \), then the unitarity condition (15) is satisfied for all \( n \) if \( a + b = 1 \). However, the resulting difference scheme is only correct to order \( 1/N \) and not to order \( 1/N^2 \).

3 The inverse function \( g^{-1} \) is unique if \( V''(x) \) is positive (that is, if \( V \) is a single potential well). The case of double wells for which \( g^{-1} \) is not unique is an extremely interesting one deserving further investigation.

4 C. M. Bender, L. M. Simmons, Jr., and R. Stong, Phys. Rev. D 33, 2362 (1986).

5 C. M. Bender, K. A. Milton, D. H. Sharp, L. M. Simmons, Jr., and R. Stong, Phys. Rev. D 32, 1476 (1985). An alternative approach to quantum mechanics in which one attempts to solve the operator equations of motion in the continuum is considered in C. M. Bender and G. V. Dunne, Phys. Lett. B 200, 520 (1988); Phys. Rev. D 40, 2739 (1989); Phys. Rev. D 40, 3504 (1989).

6 C. M. Bender and M. L. Green, Phys. Rev. D 34, 3255 (1986).

7 C. M. Bender, F. Cooper, J. E. O'Dell, and L. M. Simmons, Jr., Phys. Rev. Lett. 55, 901 (1985); C. M. Bender, F. Cooper, V. P. Gutschick, and M. M. Nieto, Phys. Rev. D 32, 1486 (1985).

8 C. M. Bender, L. R. Mead, and S. S. Pinsky, Phys. Rev. Lett. 56, 2445 (1986).

9 The polynomials described here are special cases of continuous Hahn polynomials of imaginary argument. These polynomials were recently discussed by N. M. Atakishiyev and S. K. Suslov, J. Phys. A 18, 1583 (1985), and R. Askey, J. Phys. A 18, L1017 (1985).

10 C. M. Bender, L. R. Mead, and S. S. Pinsky, J. Math. Phys. 28, 509 (1987).

11 There is an exact one-to-one correspondence between all possible sets of polynomials (both orthogonal and nonorthogonal) and rules for operator orderings. See C. M. Bender and G. V. Dunne, J. Math. Phys. 29, 1727 (1988).

12 This definition of \( S_n(x) \) in (3.21) very closely resembles that of the Chebyshev polynomials \( T_n(x) \). Using the fact that \( \cos(n\theta) \) is a polynomial in \( \cos(\theta) \) one defines \( T_n(\cos \theta) \equiv \cos(n\theta) \).

13 Equation (3.21) can be generalized to include off-diagonal \( T \) forms:

\[
T_{m,m+k} = \frac{(2m+k)!}{(m+k)!2^{m+1}} \left\{ q^k, S_m(T_{1,1}) \right\}_+. 
\]

14 There is an interesting connection between these polynomials and the Euler numbers \( E_n : \int_{-\infty}^{\infty} dx \ w(x) x^{2n} = |E_{2n}| \).
15 P. M. Prenter, *Splines and Variational Methods* (Wiley, New York, 1975).

16 L. Durand, Proc. Int. Symp. on Orthogonal Polynomials and Their Applications, Bar-le-Duc, France, 1984.

17 C. M. Bender, F. Cooper, K. A. Milton, S. S. Pinsky, L. M. Simmons, Jr., Phys. Rev. D 35, 3081 (1987).

18 C. M. Bender, K. A. Milton, S. S. Pinsky, and L. M. Simmons, Jr., Phys. Rev. D 33, 1692 (1986).

19 C. M. Bender and D. H. Sharp, Phys. Rev. Lett. 50, 1535 (1983). For an alternative application of finite elements to the evaluation of functional integrals in quantum field theory see C. M. Bender, G. S. Guralnik, and D. H. Sharp, Nucl. Phys. B207, 54 (1982).

20 C. M. Bender and K. A. Milton, Phys. Rev. D 34, 3149 (1986).

21 R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 11, 3424 (1975).

22 S. Coleman, Phys. Rev. D 11, 2088 (1975).

23 C. M. Bender, K. A. Milton, and D. H. Sharp, Phys. Rev. Lett. 51, 1815 (1983).

24 D. Miller, K. A. Milton, and S. Siegemund-Broka, Phys. Rev. D 46, 806 (1993).

25 Equation (8.8) was also discovered by R. Stacey; see Phys. Lett. 129B, 239 (1983) and Phys. Rev. D 26, 468 (1982). The finite-element result was generalized to arbitrary dimension by T. Matsuyama (1984 preprint).

26 L. H. Karsten and J. Smit, Nucl. Phys. B183, 103 (1981); H. B. Nielsen and M. Ninomiya, Nucl. Phys. B185, 20 (1981); J. M. Rabin, Phys. Rev. D 24, 3218 (1981).

27 C. M. Bender, K. A. Milton, and D. H. Sharp, Phys. Rev. D 31, 383 (1985).

28 K. A. Milton and T. Grose, Phys. Rev. D 41, 1261 (1990).

29 T. Grose and K. A. Milton, Phys. Rev. D 37, 1603 (1988).

30 K. A. Milton, in *Proceeding of the XXVth International Conference on High-Energy Physics*, Singapore, 1990, edited by K. K. Phua and Y. Yamaguchi (World Scientific, Singapore, 1991), p. 432.