TAUBERIAN CONDITIONS FOR $q$-CESÀRO INTEGRABILITY

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Abstract. Given a $q$-integrable function $f$ on $[0, \infty)$, we define $s(x) = \int_0^x f(t)q_t \, dt$ and $\sigma(s(x)) = \frac{1}{x} \int_0^x s(t)q_t \, dt$ for $x > 0$. It is known that if $\lim_{x \to \infty} s(x)$ exists and is equal to $A$, then $\lim_{x \to \infty} \sigma(s(x)) = A$. But the converse of this implication is not true in general. Our goal is to obtain Tauberian conditions imposed on the general control modulo of $s(x)$ under which the converse implication holds. These conditions generalize some previously obtained Tauberian conditions.

Keywords: $q$-integrable function; Tauberian conditions; $q$-derivative; $q$-integrals; quantum calculus.

1. Introduction

The first formulae of what we now call quantum calculus or $q$-calculus were introduced by Euler in the 18th century. Many notable results were obtained in the 19th century. In the early 20th century, Jackson defined the notions of $q$-derivative [9] and definite $q$-integral [10]. Also, he was the first to develop $q$-calculus in a systematic way. Following Jackson’s papers, $q$-calculus has received an increasing attention of many researchers due to its vast applications in mathematics and physics.

We will now give some concepts of the $q$-calculus necessary for the understanding of this work. We follow the terminology and notations from the book of Kac and Cheung [11]. In what follows, $q$ is a real number satisfying $0 < q < 1$.

The $q$-derivative $D_qf(x)$ of an arbitrary function $f(x)$ is defined by

$$D_qf(x) = \frac{f(x) - f(qx)}{x - qx}, \quad \text{if } x \neq 0,$$
where $D_q f(0) = f'(0)$ provided $f'(0)$ exists. If $f(x)$ is differentiable, then $D_q f(x)$ tends to $f'(x)$ as $q$ tends to 1.

Notice that the $q$-derivative satisfies the following $q$-analogue of Leibniz rule

$$D_q(f(x)g(x)) = f(qx)D_qg(x) + g(x)D_q f(x).$$

The $q$-integrals from 0 to $a$ and from 0 to $\infty$ are given by

$$\int_0^a f(x) d_qx = (1 - q)a \sum_{n=0}^{\infty} f(aq^n)q^n$$

and

$$\int_0^{\infty} f(x) d_qx = (1 - q)a \sum_{n=-\infty}^{\infty} f(q^n)q^n$$

provided the sums converge absolutely. On a general interval $[a, b]$, the $q$-integral is defined by

$$\int_a^b f(x) d_qx = \int_0^b f(x) d_qx - \int_0^a f(x) d_qx.$$

The $q$-integral and the $q$-derivative are related by the fundamental theorem of quantum calculus as follows:

If $F(x)$ is an anti $q$-derivative of $f(x)$ and $F(x)$ is continuous at $x = 0$, then

$$\int_a^b f(x) d_qx = F(b) - F(a), \quad 0 \leq a < b \leq \infty.$$  

In addition, we have

$$D_q \left( \int_0^x f(t) d_qt \right) = f(x).$$

A function $f(x)$ is said to be $q$-integrable on $\mathbb{R}_+: = [0, \infty)$ if the series $\sum_{n \in \mathbb{Z}} q^n f(q^n)$ converges absolutely. We denote the set of all functions that are $q$-integrable on $\mathbb{R}_+$ by $L_q^1(\mathbb{R}_+)$, where

$$\mathbb{R}_{q,+} = \{ q^n : n \in \mathbb{Z} \}.$$  

One may consult the recent books [2, 1] for further results and several applications of $q$-calculus.

Throughout this paper we assume that $f(x)$ is $q$-integrable on $\mathbb{R}_+$ and $s(x) = \int_0^x f(t) d_qt$. The symbol $s(x) = o(1)$ means that $\lim_{x \to \infty} s(x) = 0$. The $q$-Cesàro mean of $s(x)$ are defined by

$$\sigma(x) = \sigma(s(x)) = \frac{1}{x} \int_0^x s(t) d_qt.$$
Tauberian Conditions for $q$-Cesàro Integrability

The integral $\int_0^\infty f(t)\,dq\,t$ is said to be $q$-Cesàro integrable (or $(C_q, 1)$ integrable) to a finite $A$, in symbols: $s(x) \to A(C_q, 1)$, if

$$\lim_{x \to \infty} \sigma(x) = A.$$  

(1.1)

If the $q$-integral

$$\int_0^\infty f(t)\,dq\,t = A$$

(1.2)

exists, then the limit (1.1) also exists [6]. That is, $q$-Cesàro integrability method is regular. The converse is not necessarily true (see [15], Example 1). Adding some suitable condition to (1.1), which is called a Tauberian condition, may imply (1.2). Any theorem which states that the convergence of the $q$-integral follows from its $q$-Cesàro integrability and some Tauberian condition is called a Tauberian theorem.

The difference between $s(x)$ and its $q$-Cesàro mean is given by the identity [6]

$$s(x) - \sigma(x) = qv(x),$$

(1.3)

where $v(x) = \frac{1}{x} \int_0^x tf(t)\,dq\,t$. The identity (1.3) will be used in the various steps of proofs.

For each integer, $m \geq 0$, $\sigma_m(x)$ and $v_m(x)$ are defined by

$$\sigma_m(x) = \begin{cases} \frac{1}{x} \int_0^x \sigma_{m-1}(t)\,dq\,t, & m \geq 1 \\ s(x), & m = 0 \end{cases}$$

and

$$v_m(x) = \begin{cases} \frac{1}{x} \int_0^x v_{m-1}(t)\,dq\,t, & m \geq 1 \\ v(x), & m = 0 \end{cases}$$

The relationship between $\sigma_m(x)$ and $v_m(x)$ can be easily obtained by (1.3) as follows:

$$\sigma_m(x) - \sigma_{m+1}(x) = qv_m(x).$$

(1.4)

The classical control modulo of $s(x) = \int_0^x f(t)\,dq\,t$ is denoted by

$$\omega_0(x) = xD_q(s(x)) = xf(x),$$

and the general control modulo of integer order $m \geq 1$ of $s(x)$ is defined by

$$\omega_{m-1}(x) - \sigma(\omega_{m-1}(x)) = q\omega_m(x).$$

Note that the concepts of classical and general control modulo were first introduced by Çanak and Totur [3] for the integrals in standard calculus.

A function $f(x)$ is said to satisfy the property $(P)$ (see [7]), if for all $\epsilon > 0$ there exists $K > 0$ such that

$$|f(x) - f(qx)| < \epsilon$$
for all $x > K$.

Recently, Fitouhi and Brahim [7], Çanak et al. [6] and Totur et al. [15] have determined Tauberian conditions using this property. Moreover, Çanak et al. [6] showed that if $s(x)$ satisfies the property $(P)$, its $q$-Cesàro mean $\sigma(x)$ then also satisfies the property $(P)$.

Slowly oscillating real-valued functions were introduced by Schmidt [14]. A function $f(x)$ is said to be slowly oscillating, if for every $\varepsilon > 0$ there exists $K > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x > y > K$ and $x/y \to 1$. Slow oscillation condition were used in a number of Tauberian theorems for the Cesàro integrability [4, 5], logarithmic integrability [12, 16] and weighted mean integrability [13, 17] in standard calculus. Consider that, as $q$ tends to 1, the property $(P)$ corresponds to slow oscillation of a function.

The following theorems are the $q$-analogues of classical Tauberian theorems due to the Hardy [8] and Schmidt [14], respectively.

**Theorem 1.1.** ([7]) If $s(x)$ is $q$-Cesàro integrable to $A$ and
\begin{equation}
\omega_0(x) = o(1),
\end{equation}
then $\int_0^\infty f(t)d_qt = A$.

**Theorem 1.2.** ([6],[7]) If $s(x)$ is $q$-Cesàro integrable to $A$ and satisfies the property $(P)$, then $\int_0^\infty f(t)d_qt = A$.

The purpose of this study is to generalize the above theorems by imposing Tauberian conditions on the general control modulo of integer order $m \geq 1$.

## 2. Main Results

In this paper, we shall prove the following Tauberian theorems.

**Theorem 2.1.** If $s(x)$ is $q$-Cesàro integrable to $A$ and
\begin{equation}
\omega_m(x) = o(1)
\end{equation}
for some integer $m \geq 0$, then $\int_0^\infty f(t)d_qt = A$.

**Remark 2.1.** It follows from the definition of the general control modulo that condition (1.5) implies the condition (2.1).

**Theorem 2.2.** If $s(x)$ is $q$-Cesàro integrable to $A$ and $\sigma(\omega_m(x))$ satisfies the property $(P)$ for some integer $m \geq 0$, then $\int_0^\infty f(t)d_qt = A$.

**Remark 2.2.** Let the function $s(x)$ satisfy the property $(P)$, then so does the function $\sigma(\omega_m(x))$ for any non-negative integer $m$.

**Remark 2.3.** For the case $m = 0$ in Theorem 2.2, we observe that $v(x)$ satisfies the property $(P)$ which means that it is a Tauberian condition for the $q$-Cesàro integrability [6].
3. Auxiliary Results

In this section we state and prove some lemmas which are needed for the brevity of proofs of our main results.

**Lemma 3.1.** For every integer \( m \geq 1 \),
\[
(3.1) \quad xD_q(\sigma_m(x)) = v_{m-1}(x).
\]

**Proof.** Taking the \( q \)-derivative of \( \sigma_m(x) \) gives
\[
D_q(\sigma_m(x)) = \frac{1}{x} \left( \frac{1}{x} \int_0^x \sigma_{m-1}(t) dt \right)
= \frac{1}{qx} \sigma_{m-1}(x) - \frac{1}{qx^2} \int_0^x \sigma_{m-1}(t) dt
= \frac{1}{qx} (\sigma_{m-1}(x) - \sigma_m(x)).
\]

Hence, applying the identity (1.3) to \( \sigma_{m-1}(x) \), we get
\[
xD_q(\sigma_m(x)) = \frac{v_{m-1}(x)}{x},
\]
which completes the proof. \( \square \)

**Lemma 3.2.** For every integer \( m \geq 1 \),
\[
\begin{align*}
(i) \quad \ & xf(x) - v(x) = qx D_q(v(x)) \\
(ii) \quad \ & v_{m-1}(x) - v_m(x) = qx D_q(v_m(x)).
\end{align*}
\]

**Proof.** (i) Taking the \( q \)-derivative and then multiplying both sides of identity (1.3) by \( x \), we get
\[
xD_q(s(x)) - xD_q(\sigma(x)) = qx D_q(v(x)).
\]
It follows from Lemma 3.1 that
\[
xf(x) - v(x) = qx D_q(v(x)).
\]
(ii) Taking the \( q \)-derivative of both sides of (1.4), we have
\[
D_q(\sigma_m(x)) - D_q(\sigma_{m+1}(x)) = qD_q(v_m(x)).
\]
Then, multiplying (3.2) by \( x \) yields
\[
xD_q(\sigma_m(x)) - xD_q(\sigma_{m+1}(x)) = qx D_q(v_m(x)).
\]
Using Lemma 3.1, we prove that
\[
v_{m-1}(x) - v_m(x) = qx D_q(v_m(x)).
\]
\( \square \)
Lemma 3.3. For every integer \( m \geq 1 \),
\[
(3.3) \quad \sigma(xD_q(v_{m-1}(x))) = xD_q(v_m(x)).
\]

Proof. Taking Cesàro means of both sides of the identity in Lemma 3.2 (ii), we find
\[
\begin{align*}
\sigma(xD_q(v_{m-1}(x))) &= q^{-1} \sigma(v_{m-2}(x)) - \sigma(v_{m-1}(x)) \\
&= q^{-1}(v_{m-1}(x) - v_m(x)) \\
&= xD_q(v_m(x)).
\end{align*}
\]

\( \square \)

For a function \( f(x) \), we define
\[
(xD_q)_m(f(x)) = (xD_q)_{m-1}(xD_q(f(x))) = xD_q((xD_q)_{m-1}(f(x))),
\]
where \((xD_q)_0(f(x)) = f(x)\) and \((xD_q)_1(f(x)) = xD_q(f(x))\).

Lemma 3.4. For every integer \( m \geq 1 \),
\[
(3.4) \quad \omega_m(x) = (xD_q)_m(v_{m-1}(x)).
\]

Proof. We prove the assertion by using mathematical induction. From the definition of the general control modulo for \( m = 1 \) and Lemma 3.2 (i), we get
\[
\omega_1(x) = q^{-1}(\omega_0(x) - \sigma(\omega_0(x))) = q^{-1}(xf(x) - \nu(x)) = xD_q(v(x)).
\]
Assume the assertion holds for some positive integer \( m = k \). That is, assume that
\[
(3.5) \quad \omega_k(x) = (xD_q)_k(v_{k-1}(x)).
\]
We show that the assertion is true for \( m = k + 1 \). That is,
\[
\omega_{k+1}(x) = (xD_q)_{k+1}(v_k(x)).
\]
By definition of the general control modulo for \( m = k + 1 \), we have
\[
\omega_{k+1}(x) = q^{-1}(\omega_k(x) - \sigma(\omega_k(x))).
\]
Considering Lemma 3.2 (ii) and Lemma 3.3 together with (3.5), we obtain
\[
\begin{align*}
\omega_{k+1}(x) &= q^{-1}[(xD_q)_k(v_{k-1}(x)) - (xD_Q)_k(v_k(x))] \\
&= q^{-1}(xD_q)_k(v_{k-1}(x) - v_k(x)) \\
&= (xD_q)_{k+1}(v_k(x)).
\end{align*}
\]
Therefore, we conclude that Lemma 3.4 is true for each integer \( m \geq 1 \). \( \square \)

Lemma 3.5. If \( s(x) \) is \( q \)-Cesàro integrable to some finite number \( A \), then for each non-negative integer \( m \), \( \sigma(\omega_m(x)) \) is \( q \)-Cesàro integrable to \( 0 \).
Proof. If \( s(x) \to A(C_q, 1) \), then it is known that \( \sigma(x) \to A(C_q, 1) \). Thus, it follows from the identity (1.3) that \( v(x) = \sigma(\omega(x)) \to 0(C_q, 1) \). Replacing \( s(x) \) with \( v(x) \) in (1.3), we write

\[
(3.6) \quad v(x) - v_1(x) = qx D_q(v_1(x)) = q\sigma(\omega_1(x)).
\]

Then, (3.6) implies \( \sigma(\omega_1(x)) \to 0(C_q, 1) \). Now, applying (1.3) to \( xD_q(v_1(x)) \), we get

\[
(3.7) \quad xD_q(v_1(x)) - xD_q(v_2(x)) = q(xD_q)v_2(x) = q\sigma(\omega_2(x)).
\]

Hence from (3.7), \( \sigma(\omega_2(x)) \to 0(C_q, 1) \). Continuing in the same manner, we obtain \( \sigma(\omega_m(x)) \to 0(C_q, 1) \) for each non-negative integer \( m \).

\[\Box\]

**Lemma 3.6.** For every non-negative integer \( m \) and \( k \),

\[
(3.8) \quad \sigma_k(\omega_m(x)) = \omega_m(\sigma_k(x)).
\]

**Proof.** Using Lemma 3.4 and Lemma 3.3 respectively, it follows

\[
(3.9) \quad \sigma_k(\omega_m(x)) = \sigma_k((xD_q)_m v_{m-1}(x)) = (xD_q)_{m+1} \sigma_{m+k}(x).
\]

On the other hand, taking Lemma 3.4 and Lemma 3.1 into account we find

\[
(3.10) \quad \omega_m(\sigma_k(x)) = (xD_q)_m v_{m-1}(\sigma_k(x)) = (xD_q)_{m+1} (\sigma_{m+k}(x)).
\]

Therefore, the proof is completed from the equality of (3.9) and (3.10). \[\Box\]

The following lemma shows a different representation of the difference \( s(x) - \sigma(x) \).

**Lemma 3.7.** For any function \( s(x) \) defined on \( (0, \infty) \), we have the identity

\[
(3.11) \quad s(x) - \sigma(x) = \frac{q}{1 - q} (\sigma(x) - \sigma(qx)),
\]

where \( \sigma(qx) = \frac{1}{qx} \int_0^{qx} s(t) dq t \).

**Proof.** By the definition of the \( q \)-integral, we may write

\[
\int_0^{qx} s(t) dq t = (1 - q) qx \sum_{n=0}^{\infty} s(xq^{n+1}) q^{n} = (1 - q) x \sum_{n=1}^{\infty} s(xq^n) q^n
\]

\[
= (1 - q) x \left( \sum_{n=0}^{\infty} s(xq^n) q^n - s(x) \right) = \int_0^{x} s(t) dq t - (1 - q) xs(x).
\]
Dividing the both sides of the last equality by $q x$, we get
\[
\frac{q}{1 - q}(\sigma(x) - \sigma(qx)) = s(x) - \sigma(x).
\]
\]

It is clear from Lemma 3.7 that, even if $\sigma(x)$ is convergent, $\sigma(x)$ and $\sigma(qx)$ do not tend to same value when $s(x)$ is not convergent.

4. Proofs

In this section, we give proofs of our main results.

4.1. Proof of Theorem 2.1

From the hypothesis we have
\[
\omega_m(x) = x D_q \sigma(\omega_{m-1}(x)) = o(1),
\]
for some integer $m \geq 1$. On the other hand, from Lemma 3.5, $\sigma(\omega_{m-1}(x)) \rightarrow 0(C_q, 1)$. Hence, applying Theorem 1.1 to $\sigma(\omega_{m-1}(x))$ we obtain
\[
\sigma(\omega_{m-1}(x)) = o(1).
\]
Considering (4.1) and (4.2) together with the identity
\[
\omega_{m-1}(x) - \sigma(\omega_{m-1}(x)) = q \omega_m(x),
\]
we get
\[
\omega_{m-1}(x) = x D_q \sigma(\omega_{m-2}(x)) = o(1).
\]
By Lemma 3.5, we also have $\sigma(\omega_{m-2}(x)) \rightarrow 0(C_q, 1)$. Now, applying Theorem 1.1 to $\sigma(\omega_{m-2}(x))$ we obtain
\[
\sigma(\omega_{m-2}(x)) = o(1).
\]
From (4.3), (4.4) and the identity
\[
\omega_{m-2}(x) - \sigma(\omega_{m-2}(x)) = q \omega_{m-1}(x),
\]
we find
\[
\omega_{m-2}(x) = x D_q \sigma(\omega_{m-3}(x)) = o(1).
\]
Taking (4.1), (4.3) and (4.5) into account and proceeding likewise, we observe that $\omega_0(x) = o(1)$. Therefore, the proof follows from Theorem 1.1. \qed
4.2. Proof of Theorem 2.2

Considering Lemma 3.7 we may construct the identity

$$\sigma(\omega_m(x)) - \sigma_2(\omega_m(x)) = \frac{q}{1-q} [\sigma_2(\omega_m(x)) - \sigma_2(\omega_m(qx))].$$

Since $\sigma(\omega_m(x))$ satisfies the property (P), its $q$–Cesàro mean $\sigma_2(\omega_m(x))$ also satisfies the property (P). Let $\epsilon > 0$ be given. Then, there exists $K > 0$ such that

$$-\epsilon < \sigma_2(\omega_m(x)) - \sigma(\omega_m(x)) < \epsilon$$

for every $x > K$. By (4.6), we write

$$\sigma(\omega_m(x)) - \epsilon < \sigma_2(\omega_m(x)) < \sigma(\omega_m(x)) + \epsilon.$$ 

Since $s(x) \rightarrow A(C_q,1)$, we have by using Lemma 3.5 that $\lim_{x \rightarrow \infty} \sigma_2(\omega_m(x)) = 0$. Thus, it follows from (4.7)

$$-\epsilon < \liminf_{x \rightarrow \infty} \sigma(\omega_m(x)) < \limsup_{x \rightarrow \infty} \sigma(\omega_m(x)) < \epsilon,$$

which is equivalent to

$$\lim_{x \rightarrow \infty} \sigma(\omega_m(x)) = 0.$$ 

It yields from the equality

$$\sigma(\omega_m(x)) = \sigma((xD_q)m_{m-1}(x)) = xD_q(xD_q)m_{m-2}(x) = xD_q\sigma_2(\omega_{m-1}(x)),$$

that $xD_q\sigma_2(\omega_{m-1}(x)) = o(1)$. Also, by Lemma 3.5, $\sigma(\omega_{m-1}(x)) \rightarrow 0(C_q,1)$. Further, regularity of $q$–Cesàro integrability implies $\sigma_2(\omega_{m-1}(x)) \rightarrow 0(C_q,1)$. Then, if we apply Theorem 1.1 to $\sigma_2(\omega_{m-1}(x))$ we obtain

$$\lim_{x \rightarrow \infty} \sigma_2(\omega_{m-1}(x)) = 0.$$ 

From the definition of the general control modulo, it is easy to see

$$\sigma(\omega_{m-1}(x)) - \sigma_2(\omega_{m-1}(x)) = q\sigma(\omega_m(x)).$$

Combining (4.8), (4.9) and (4.10), we reach

$$\lim_{x \rightarrow \infty} \sigma(\omega_{m-1}(x)) = 0.$$ 

Now, since

$$\sigma(\omega_{m-1}(x)) = \sigma((xD_q)m_{m-1}v_m(x)) = xD_q(xD_q)m_{m-2}v_{m-1}(x) = xD_q\sigma_2(\omega_{m-2}(x)), $$

we find \( xD_q \sigma_2(\omega_{m-2}(x)) = o(1) \). Besides, we have \( \sigma_2(\omega_{m-2}(x)) \to 0(C_q, 1) \) from Lemma 3.5 and the regularity of \( q \)-Cesàro integrability. Now, applying Theorem 1.1 to \( \sigma_2(\omega_{m-2}(x)) \) we get

\[
\lim_{x \to \infty} \sigma_2(\omega_{m-2}(x)) = 0. \tag{4.12}
\]

Considering (4.11), (4.12) and the identity

\[
\sigma(\omega_{m-2}(x)) - \sigma_2(\omega_{m-2}(x)) = q\sigma(\omega_{m-1}(x)),
\]

we have

\[
\lim_{x \to \infty} \sigma(\omega_{m-2}(x)) = 0. \tag{4.14}
\]

In the light of (4.8), (4.11) and (4.14), continuing in the same fashion we conclude

\[
\lim_{x \to \infty} \sigma(\omega_0(x)) = \lim_{x \to \infty} v(x) = 0.
\]

Therefore, since \( s(x) \to A(C_q, 1) \), we obtain via (1.3) that \( \lim_{x \to \infty} s(x) = A. \)

5. Extensions

In this section, we will present the \( q \)-Hölder or \((H_q, k)\) integrability method which is an obvious generalization of the \( q \)-Cesàro integrability. Later, we extend our main results to this method.

If

\[
\lim_{x \to \infty} \sigma_k(x) = A,
\]

then \( \int_0^\infty f(t)d_q t \) is said to be integrable by the \( q \)-Hölder method of order \( k \in \mathbb{N}_0 \) (shortly, \((H_q, k)\) integrable) to \( A \), and this fact is denoted by \( s(x) \to A(H_q, k) \). In particular, the method \((H_q, 0)\) indicates the convergence in the ordinary sense and the method \((H_q, 1)\) is equivalent to \((C_q, 1)\). The \((H_q, k)\) methods are regular for any \( k \) and are compatible for all \( k \). The power of the method increases with increasing \( k \): The \((H_q, k)\) integrability implies \((H_q, k')\) integrability for any \( k' > k \).

**Theorem 5.1.** Let \( s(x) \to A(H_q, k + 1) \). If

\[
\omega_m(x) = o(1) \tag{5.1}
\]

for some integer \( m \geq 0 \), then \( \int_0^\infty f(t)d_q t = A \).

**Proof.** By (5.1) and the regularity of the \((C_q, 1)\) method, we obtain \( \sigma_k(\omega_m(x)) = o(1) \) for each integer \( k \geq 0 \). Then, from Lemma 3.6 it is clear that

\[
\omega_m(\sigma_k(x)) = o(1) \quad \text{for each} \quad k \in \mathbb{N}_0. \tag{5.2}
\]
Besides, from the assumption since \( \sigma_k(x) \to A(C_q, 1) \), Theorem 2.1 implies
\[
\lim_{x \to \infty} \sigma_k(x) = A
\]
which is also equivalent to \( \sigma_{k-1}(x) \to A(C_q, 1) \). From (5.2), we know that \( \omega_m(\sigma_{k-1}(x)) = o(1) \). Now, applying Theorem 2.1 to \( \sigma_{k-1}(x) \) yields
\[
\lim_{x \to \infty} \sigma_{k-1}(x) = A
\]
which is also equivalent to \( \sigma_{k-2}(x) \to A(C_q, 1) \). Repeating the same steps \( k \)-times we conclude
\[
\lim_{x \to \infty} \sigma_0(x) = \int_0^\infty f(t)d_qt = A.
\]
\( \square \)

**Theorem 5.2.** Let \( s(x) \to A(H_q, k + 1) \). If \( \sigma(\omega_m(x)) \) satisfies the property \((P)\) for some integer \( m \geq 0 \), then \( \int_0^\infty f(t)d_qt = A \).

**Proof.** If \( \sigma(\omega_m(x)) \) satisfies the property \((P)\), then so does \( \sigma_k(\omega_m(x)) \) for every non-negative integer \( k \). From Lemma 3.6, since
\[
\sigma_k(\omega_m(x)) = \omega_m(\sigma_k(x))
\]
we find that \( \sigma(\omega_m(\sigma_k(x))) \) also satisfies \((P)\) for all \( k \in \mathbb{N}_0 \). Considering the hypothesis \( \sigma_k(x) \to A(C_q, 1) \) and Theorem 2.2 we obtain
\[
s(x) \to A(H_q, k)
\]
which requires \( \sigma_{k-1}(x) \to A(C_q, 1) \). Moreover, since \( \sigma(\omega_m(\sigma_{k-1}(x))) \) satisfies \((P)\), we get
\[
s(x) \to A(H_q, k - 1)
\]
which requires \( \sigma_{k-2}(x) \to A(C_q, 1) \). Applying the same reasoning \( k \)-times we reach that
\[
s(x) \to A(H_q, 0)
\]
which means \( \lim_{x \to \infty} s(x) = A \). \( \square \)

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