Variational necessary conditions for optimal control problems

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Abstract

In this work, we consider infinite spatiotemporal systems. We use a more regular control function to reach the desired target prescribed on the system domain. After presenting the existence results, we characterize the solution to our problem through variations method.

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1. Introduction

Let $T > 0$ and $\Theta = \Omega \times ]0, T[ \cup \Sigma = \partial \Omega \times ]0, T[$, where $\Omega \subset \mathbb{R}^n (n = 1, 2, 3)$ is an open bounded domain with regular boundary $\partial \Omega$. The control space is $u \in L^2(0, T; H^0_\omega(\Omega))$, and our considered bi-linear equation is

\[
\begin{align*}
    y_t(x, t) &= y_{xx}(x, t) - u(x, t)y_x(x, t) \quad \Theta, \\
    y(x, 0) &= y_0(x) \quad \Omega, \\
    y &= y_x = 0 \quad \Sigma,
\end{align*}
\]

with $y_0(x) \in L^2(\Omega)$ and $W = \{ y \in L^2(0, T; H^0_\omega(\Omega))/y_t \in L^2(0, T; H^{-2}(\Omega)) \}$ represents the state space (see [5]). The system (1.1) admits a unique solution $y_u$ in $W \cap L^\infty(0, T; L^2(\Omega))$ (see [9, 10]).

The considered problem of the system (1.1) is

\[
\min_{u \in L^2(0, T; H^0_\omega(\Omega))} J_\varepsilon(u).
\]

Let $\varepsilon > 0$, the considered minimizing function $J_\varepsilon$ is

\[
J_\varepsilon(u) = \frac{1}{2}\left\| y - y_d \right\|^2_{L^2(0, T; L^2(\Omega))} + \frac{\varepsilon}{2} \int_\Theta \left[ u_t^2(x, t) + u_x^2(x, t) \right] \, dx \, dt,
\]
with the desired profile is $y^d$.

For background, quadratic costs such (1.3) used by several authors like Bradly, Lenhart in [3], Bradly et all in [4], and Lenhart in ([8]). They control the coefficient term in different types of one-dimensional bilinear equations minimizing a spatiotemporal velocity. Joshi in ([6]) studies the control of velocity in a class of bi-linear systems. Boulaaras in [2], studies a parabolic quasi-variational inequalities. Zakeri and Abchouyeh in [11], consider quadratic optimal control problem constrained by elliptic systems. Ayalaa et all in [1], gives controllability results of two dimensional bilinear systems. Kafash and Delavarkhalafi in [7] use parameterization method for optimal control problems. Zerrik and Ould Sidi in [12–14]) use quadratic control problems to steer the position of infinite-dimensional bilinear systems to the desired state on specific sub-region. Zine and Ould Sidi in [15, 16] treated quadratic control problems in the case of hyperbolic bi-linear systems.

In this work, our goal is to command the state of (1.1) to a prescribed target $y^d(x)$ by minimizing the cost (1.3), and using more regular control space $u \in L^2(0,T; H_0^1(\Omega))$. After presenting the existence results, we characterize the solution to our problem through variations method.

2. Existence results

Next theorem ensures a solution to the penalty problem (1.2).

**Theorem 2.1.** There exist a pair $(u, y)$, where $y$ is the output of the system

$$
\begin{align*}
y_t &= y_{xx} - uy_x \quad \Theta, \\
y(x,0) &= y_0(x) \quad \Omega, \\
y &= y_x = 0 \quad \Sigma,
\end{align*}
$$

and $u$ is the minimum of (1.2).

**Proof.** Let the positive nonempty set $\{J_\epsilon(u) \mid u \in L^2(0,T; H_0^1(\Omega))\} \subset \mathbb{R}$, and choosing $(u_n)_n$ a minimum which verifies

$$J^* = \lim_{n \to +\infty} J(u_n) = \inf_{u \in L^2(0,T; H_0^1(\Omega))} J_\epsilon(u).$$

It follows that, the cost $J_\epsilon(u_n)$ is bounded, consequently $\|u_n\|_{L^2(0,T; H_0^1(\Omega))} \leq C$, with $C$ is a positive constant.

From results in [9], we have

$$
\begin{align*}
u_n &\rightharpoonup u \quad L^2(0,T; H_0^1(\Omega)), \\
y^n &\to y \quad W, \\
y^n_{xx} &\to \chi \quad W, \\
y^n_x &\to \Lambda \quad W, \\
y^n &\to \Psi \quad W.
\end{align*}
$$

We Consider the equation $y^n_t(x, t) = y^n_{xx} - u_n y^n_x$, and passing to the limit we can deduce that $y_t(x, t) = \Psi$, $y \to y_{xx}$, $y_{xx} = \chi$ and $uy_x = \Lambda$. Hence we obtain

$$y_t = y_{xx} - u(x, t)y_x.$$

The lower semi continuity of $J_\epsilon(u)$, allows us to deduce that

$$J_\epsilon(u) = \inf_{u \in L^2(0,T; L^2(\Omega))} \frac{1}{n} \left[ \|y^n - y^d\|_{L^2(0,T; L^2(\Omega))}^2 + \epsilon \int_\Theta \left[ u_t^2 + u_x^2 \right] \right]_{n} \to \|u\|_{L^2(0,T; L^2(\Omega))},$$

Consequently $u$ is the solution of (1.2).
3. Characterization of solution

We want to suggest a characterization for solution of the control problem (1.2). First we study the differential of \( J_\varepsilon(u) \).

**Lemma 3.1.** A differential of

\[
L^2(0, T; H^1_0(\Omega)) \quad \text{to} \quad W,
\]

\[ u \quad \text{to} \quad y(u), \]

is

\[
y(u + \varepsilon l) - y(l) \quad \text{to} \quad \psi,
\]

where \( \psi \) verifies the system

\[
\begin{aligned}
\psi_t &= \psi_{xx} - u \psi_x - l \psi_x & \Theta, \\
\psi(x, 0) &= 0 & \Omega, \\
\psi(x) &= 0 & \Sigma.
\end{aligned}
\] (3.1)

With \( y = y(u) \), \( u \in L^2(0, T; H^1_0(\Omega)) \), and \( d(y(u))l \) is the differential of \( u \to y(u) \) respecting \( u \).

**Proof.** The solution of equation (3.1), verifies

\[
\| \psi \|_W \leq k_1 \| y \|_{L^\infty(0, T; H^1_0(\Omega))} \| l \|_{L^2(0, T; H^1_0(\Omega))}.
\]

Also

\[
\| \psi' \|_W \leq k_2 \| y \|_{L^\infty(0, T; H^1_0(\Omega))} \| l \|_{L^2(0, T; H^1_0(\Omega))}.
\]

Thus,

\[
\| \psi \|_{C([0, T]; H^1_0(\Omega))} \leq k_3 \| l \|_{L^2(0, T; H^1_0(\Omega))}.
\]

Then, we obtain \( h \in L^2(0, T; L^2(\Omega)) \to \psi \in C([0, T]; H^1_0(\Omega)) \) is bounded, (see ([5])). Put \( y_1 = y(u + l) \) and \( \varphi = y_1 - y \), then \( \varphi \) is the state of

\[
\begin{aligned}
\varphi_t(x, t) &= \varphi_{xx} - u(x, t) \varphi_x(x, t) - l(x, t)(y_1)_x & \Theta, \\
\varphi(x, 0) &= 0 & \Omega, \\
\varphi(x) &= 0 & \Sigma.
\end{aligned}
\]

Thus

\[
\| \varphi \|_{L^\infty([0, T]; H^1_0(\Omega))} \leq k_4 \| l \|_{L^2([0, T]; H^1_0(\Omega))}.
\]

Let \( \phi = \varphi - \psi \) which verify the system

\[
\begin{aligned}
\phi_t &= \phi_{xx} + u(x, t) \phi_x(x, t) + l(x, t) \varphi_x & \Theta, \\
\phi(x, 0) &= 0 & \Omega, \\
\phi(x) &= 0 & \Sigma,
\end{aligned}
\]

where \( \phi \in C(0, T; H^1_0(\Omega)) \), consequently

\[
\| \phi \|_{C([0, T]; H^1_0(\Omega))} \leq k \| l \|_{L^2([0, T]; H^1_0(\Omega))}^2
\]

and we have

\[
\| y(u + l) - y(u) - d(y(u))l \|_{C([0, T]; H^1_0(\Omega))} = \| \phi \|_{C([0, T]; H^1_0(\Omega))} \leq k \| l \|_{L^2([0, T]; H^1_0(\Omega))}^2.
\]

\[ \square \]
The $k_\ell, \{i = 1, 2, 3, 4\}$, and $k$ are positive constants. Next, we consider the adjoint systems

$$
\begin{cases}
-p_t = p_{xx} + (up)_x + (y - y^d) & \Theta, \\
u_x(x, 0) = u_x(x, T) = 0 & \Omega, \\
p(x, T) = 0 & \Omega, \\
p = p_x = 0 & \Sigma.
\end{cases}
$$

The following lemma gives an explicit formula of the differential of $J_\varepsilon(u)$.

**Lemma 3.2.** Consider $u \in L^2(0, T; H^1_0(\Omega))$ be the solution of (1.2), we obtain

$$
\lim_{\beta \to 0} \frac{J_\varepsilon(u + \beta l) - J_\varepsilon(u)}{\beta} = \int_\Omega \int_0^T \psi(y - y^d) dt dx + \varepsilon \int_\Omega \int_0^T \left[(u_t l_t) + (u_x l_x)\right] dt dx.
$$

Proof. The cost $J_\varepsilon(u)$ (1.3), can be expressed by

$$
J_\varepsilon(u) = \frac{1}{2} \int_\Omega \int_0^T (y - y^d)^2 dt dx + \frac{\varepsilon}{2} \int_\Omega \int_0^T \left[u_t^2 + u_x^2\right] dt dx.
$$

Put $y_\beta = y(u_\varepsilon + \beta l)$ and $y = y(u_\varepsilon)$, using (3.3) we have

$$
\lim_{\beta \to 0} \frac{J_\varepsilon(u + \beta l) - J_\varepsilon(u)}{\beta} = \lim_{\beta \to 0} \frac{1}{2} \int_\Omega \int_0^T \left(\frac{y_\beta - y}{\beta}\right)^2 dt dx
$$

$$
\quad + \lim_{\beta \to 0} \frac{\varepsilon}{2\beta} \int_\Omega \int_0^T \left[(u_t l_t) + (u_x l_x)\right] dt dx,
$$

then

$$
\lim_{\beta \to 0} \frac{J_\varepsilon(u + \beta l) - J_\varepsilon(u)}{\beta}
$$

$$
= \lim_{\beta \to 0} \frac{1}{2} \int_\Omega \int_0^T \left(\frac{y_\beta - y}{\beta}\right)^2 dt dx
$$

$$
+ \lim_{\beta \to 0} \frac{\varepsilon}{2\beta} \int_\Omega \int_0^T \left[(u_t l_t) + (u_x l_x)\right] dt dx.
$$

$$
= \int_\Omega \int_0^T \psi(x, t)(y - y^d) dt dx + \int_\Omega \int_0^T \varepsilon \left[(u_t l_t) + (u_x l_x)\right] dt dx.
$$

The next theorem characterizes the solution of (1.2).

**Theorem 3.3.** The optimal control $u \in L^2(0, T; H^1_0(\Omega))$ of (1.2) verifies the following variational formula

$$
u_{ttt} + u_{xx} + \frac{1}{\varepsilon} py_x = 0,
$$

where $y = y(u)$ is the output of (1.1) and $p$ is the one of (3.2).

Proof. Let $l \in L^2(0, T; H^1_0(\Omega))$ and $u + \beta l \in L^2(0, T; H^1_0(\Omega))$ for $\beta > 0$. The extremal of $J_\varepsilon$ is realized at $u$, then

$$
0 \leq \lim_{\beta \to 0} \frac{J_\varepsilon(u + \beta l) - J_\varepsilon(u)}{\beta}.
$$

Lemma (3.2) gives

$$
0 \leq \int_\Omega \int_0^T \psi(x, t)(y - y^d) dt dx + \int_\Omega \int_0^T \varepsilon \left[(u_t l_t) + (u_x l_x)\right] dt dx,
$$

and

$$
0 \leq \int_\Omega \int_0^T \psi(x, t)(y - y^d) dt dx + \int_\Omega \int_0^T \varepsilon \left[(u_t l_t) + (u_x l_x)\right] dt dx.
$$
then substituting the weak form of (3.2), we have
\[
0 \leq \int_0^T \int_\Omega \psi (-p_t - p_{xx} - (up)_x) \, dt \, dx + \int_0^T \int_\Omega \varepsilon \left[ (u_t l_t) + (u_x l_x) \right] \, dt \, dx.
\]

Using a simple integral by parts, we obtain
\[
0 \leq \int_0^T \int_\Omega p(\psi_t - \psi_{xx} + u\psi_x) \, dt \, dx + \int_0^T \int_\Omega \varepsilon \left[ (u_t l_t) + (u_x l_x) \right] \, dt \, dx.
\]

The system (3.1), we obtain
\[
0 \leq \int_0^T \int_\Omega -l(x, t)y_x p(x, t) \, dt \, dx + \int_0^T \int_\Omega \varepsilon \left[ (u_t l_t) + (u_x l_x) \right] \, dt \, dx = \int_0^T \int_\Omega \left[ -l(x, t)y_x p(x, t) + \varepsilon (u_t l_t) + \varepsilon (u_x l_x) \right] \, dt \, dx.
\]

Consequently, for an arbitrary control \( l = l(t) \in L^2(0, T; H^1_0(\Omega)) \) we conclude the equation
\[
-ly_x p - \varepsilon u_{tt} l - \varepsilon u_{xx} l = 0,
\]
which allow us to drive the following variational formula
\[
u_{tt} + u_{xx} + \frac{1}{\varepsilon} p y_x = 0,
\]
that the solution \( u \) of (1.2) must satisfy.

\[ \square \]

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