A Numerical Precision Example for Teachers of Trig and Pre-Calc

Jeffrey Uhlmann
Dept. of Electrical Engineering and Computer Science
University of Missouri - Columbia

Abstract

In this paper we propose a very specific educational challenge that teachers can use to motivate ambitious and enthusiastic mathematics students who have mastered basic trigonometry and exponential functions. The objective is to lead students to a result that will hopefully surprise and entertain and – more importantly – provide a lesson about when and when not to make inferences relating to “numerical error” when using mathematical software and calculators.

1 Introduction

The availability of software tools [1, 2, 3, 4, 5, 6] and multi-function calculators has unquestionably helped students at a variety of levels to develop a stronger grasp of challenging mathematical concepts. However, these tools can also lead to an overly-casual attitude about how to interpret the effects of numerical precision. For example, students quickly learn to interpret a numerical result of 0.99999999 as being exactly 1, or 3.54e-16 as being zero. The problem, of course, is that they may become so habituated to disregarding low-order terms that they fail to recognize results that are nearly an integer – but are not.

One way to impress upon students the importance of not drawing hasty conclusions from numerical results is to develop challenge problems that defy casual expectations. This may not make sense for students who are not already confident with fundamental mathematical concepts, but it is certainly relevant for those transitioning from trigonometry to pre-calculus. In the
following two sections, we consider this issue in the context of challenging and stimulating highly motivated students who are capable of appreciating extreme examples of intuition-defying problems.

2 Fun with Trig Functions

After students have transitioned from geometry problems that are typically posed in terms of angles expressed in degrees to problems expressed in radians, they typically are introduced to a variety of trigonometric identities such as:

\[
sin^2(a) + \cos^2(a) = 1. \tag{1}
\]

Exam and homework problems can then be formulated in which students must algebraically apply the above identity to demonstrate that

\[
sin(a) \cdot \left(\frac{\cos^2(a)}{\sin(a) + \sin(a)}\right) = 1. \tag{2}
\]

After a bit more experience with various trig identities, students can be challenged to show

\[
\frac{(\sin(a) \cos(b) + \cos(a) \sin(b))}{\sin(a + b)} = 1. \tag{3}
\]

Many students find a problem of this kind very gratifying to have solved, and the result often motivates interesting questions, e.g., about the case when \(a+b=0\). Of course, not all students can be expected to enjoy and be motivated by such problems, so textbooks rarely devote much emphasis on them. This puts the onus on teachers to devise their own challenge problems to excite and challenge their top students.

One source for generating challenging problems is to introduce the general definitions of basic trig functions (and their inverses, e.g., arcsin and arccos) in terms of exponentials that allow for general arguments, e.g., which includes reals of magnitude greater than \(2\pi\). This permits formulation of a wide variety problems involving radian-argument trigonometric functions of large integers that resemble problems expressed in degrees. Although not necessarily difficult to solve, students may initially find their novelty interesting, but of course that appeal cannot last long. From a teacher’s perspective, however, the goal may simply be to maintain the interest and enthusiasm of motivated students until the entire class is ready to move on to the next topic. With that goal in mind, in the following section we propose a possible concluding challenge problem that is intended to tie the concepts of generalized trig functions and numerical precision in a surprising way.
3 A Surprising Expression

Consider a question involving the following difference expression:

\[
\arcsin(1 + \sin(11)) - \sin(11)
\]

for which students are to be asked to either determine its exact value or give an answer of “Don’t know” or “Can’t be determined precisely” or some other similar choice. Virtually all algebraic manipulations lead to ungainly expressions that provide no clear insight (though the effort is guaranteed to provide good practice dealing with exponential expressions – for better or worse!).

At some point, students can be counted on to seek a numerical result using a calculator or a more sophisticated software alternative. Some may do this at the start in hopes of avoiding the formidable amount of algebra they can foresee is inevitable, while others will resort to looking at a numerical result only after all other attacks have failed. So what will they find?

Figure 1 shows the result for Equation 4 using WolframAlpha, which is probably the most widely used online mathematical tool. As can be seen, it seems to indicate that the expression evaluates to 1. Figures 2 and 3 show results obtained from Octave/Matlab and Mathematica, respectively, that are similarly 1 to within the precision of the display. In fact, a student could be forgiven for concluding from the Mathematica result in Figure 3 that the answer given is exactly 1, i.e., not just nearly 1 up to some level of precision.

If a student were to increase the displayed precision using WolframAlpha or Mathematica, the result would be what is shown in Figure 4 with zeros out to 16 decimal places. A student might very well conclude that the string of digits beyond that long sequence of zeros is just numerical inaccuracy. This is a natural assumption to make based on practical experience with calculators and other common mathematical tools. However, students are likely to be very surprised when they discover that the number shown in Figure 4 is in fact accurate to the precision shown, i.e., that the expression of Equation 4 does not equal 1.

4 Discussion

The trigonometric difference expression given in Equation 4 is likely to be the first example a student will see in which a relatively straightforward mathematical expression produces a numerical result that is so close to 1,
Figure 1: Result of the evaluation of Equation 4 using WolframAlpha [1] with default precision (and defined in radians, not degrees). Note that the ellipsis (...) can reasonably be taken to imply that the sequence of decimal zeros continues indefinitely, or it can be taken to imply there are additional significant digits (possibly nonzero) that are not shown.

Figure 2: Result of the evaluation of Equation 4 using Octave [3] with default precision.
Figure 3: Result of the evaluation of Equation \([4]\) using Mathematica [2] with default precision. It may not be clear from the form of the result (1.) whether the answer is precisely the integer 1 or has been rounded.

Figure 4: Result of the evaluation of Equation \([4]\) using WolframAlpha with increased displayed precision. The shown digits are entirely accurate, i.e., the expression does not equal 1, but at first glance the natural inclination might be to assume that the nonzero decimal digits are simply numerical inaccuracy.
but isn’t identically 1. Thus, the example helps to reinforce two important lessons:

1. Care must be taken before assuming that near-integer results from numerical evaluations are actually integers plus some numerical error.

2. Most modern mathematical tools (e.g., WolframAlpha, Mathematica, and Maple) do not display spurious digits beyond the precision accuracy of the calculation.

At present, neither of these lessons is commonly emphasized in most mathematics curricula. Evidence of this can easily be obtained by the reader: look at Figure 4 and assess whether any level of surprise is experienced. If the lessons above are firmly grasped based on past academic experience then there should be no surprise that an expression might evaluate to a number so close to 1 without being exactly 1.

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References

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