On the Orders and Zeros of Solutions of a Class of Entire Function Linear Differential Equations

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Abstract. This paper is based on the theorem of entire function solutions: each nontrivial solution f of the differential equation

\[ f'' + F(z)f' + G(z)f = 0 \]

has an infinite order. The study expands its assumptions and allows some coefficients to have the equal order to prove that series of the whole transcendental solution approaches to infinity, which is to solve the order and the convergence of the zero of \( f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0 \).

1. The Basic Knowledge of Mathematics in the Paper

1.1 Entire Function

Entire function refers to the function that is holomorphic at all finite points over the whole complex plane, characterized as expanding the \((n+1)\) order derivative at the origin, which is shown as

\[ f(x) = f(x_0) + f'(x_0)(x-x_0) + f''(x_0)(x-x_0)^2/2! + f'''(x_0)(x-x_0)^3/3! + \cdots \]

with convergence in the whole plane. \( \infty \) is the only isolated singular point in entire function and when this point is moving singularity, entire function can only be a constant. The form of Lanrent expansion of the singularity \( \infty \) is the same as \( (n+1) \) order derivative at the origin.

The definition of entire function can be verified by Liouville theorem (bounded entire function must be constant), which means that if \( f(x) \) is the holomorphic function with boundedness in the whole plane, then \( f(x) \) is a constant.

Process of proof: if \( |f(x)| \leq G \) and \( x \in M \), now we make a fixed value \( a \in M \) in \( f(x) \) and make \( K \in (a,B) \), then the inequality \( |f'(a)| \leq G/B \) is obtained by Cauchy theorem. Make \( B \to \infty \), and then \( f'(a) = 0 \). Because \( a \) is any point in the whole plane, \( f'(x) = 0 \) is tenable at any point in the whole plane, which indicates that the function is entire function.

1.2 Meromorphic Function

Different from entire function, in meromorphic function, it should be holomorphic at all finite points except the pole point, and meromorphic function in open subsets of every complex plane can be shown as the comparison between two holomorphic functions (as a denominator, holomorphic function does not constantly equal to 0). The pole point is also the zero point of denominator. [1]

1.3. The theory of Nevanlinna
The theory of the value distribution can be applied to some corollaries such as differential equation in complex domain, differenced equation, complex dynamical systems, uniqueness theory of meromorphic functions and so on, where exist some definitions, including:

\[ \log^+ x = \begin{cases} \log x, x \geq 1 \\ 0, 0 \leq x < 1 \end{cases} \]

\[ m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta, \]

\[ T(r, f) = m(r, f) + N(r, f), \]

\[ m(\frac{r}{f-a}) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left( \frac{1}{g(\frac{p}{r}e^{-i\theta})} \right) d\theta, \]

\[ N(r, \frac{1}{f-a}) = \int_0^r \frac{n(t, \frac{1}{f-a}) - n(0, \frac{1}{f-a})}{t} dt + n(0, \frac{1}{f-a}) \log r \]

1.4. Borel Theorem

Borel theorem is one of the important theorems used to demonstrate the distribution of values of entire function. According to Borel theorem, let \( f \) be an entire function of finite order and its series denoted as \( p \), then \( \mathbb{C} \) can drop an exceptional value \( a \) at most.

In the equation, \( n(r, a) \) is the number of zero points of \( f(z)-a \) in \( \{z\} \text{Gr} \).

According to Borel theorem, meromorphic function can be presented as following: let \( f(z) \) be meromorphic function of finite order, and its series is denoted as \( p \). For all \( \mathbb{C} = \mathbb{C} \), the equation is tenable and it can drop 2 exceptional values \( a \) at most.

1.5. Order

The solution of order and zero in this paper is relative to the definition of entire function and the oscillatory of its solution, in which order means that the original comprehensive series constantly increases in the condition that entire function expands out of the original theorem.

2. Proof of the Theorem

According to theorem A, we assume that \( F(z) \) and \( G(z) \) which do not constantly equal to zero make up an entire function with two constants, \( \alpha > 0, \beta > 0 \), and meet the condition \( \sigma(G) < \beta \). We suppose that for any \( \varepsilon > 0 \), there exists two limited real number sets \( \{\varphi_k\} \) and \( \{\theta_k\} \), meeting the condition
\[ \phi_1 < \phi_2 < \phi_3 < \cdots < \phi_n < \phi_{n+1}, \quad \text{where} \quad \phi_{n+1} = \phi_1 + 2\pi, \quad \text{and} \quad \sum_{k=1}^{n} (\phi_{k+1} - \phi_k) < \varepsilon \]

to make
\[ |F(Z)| \geq \exp\{(1 + \sigma(\alpha)|z|^\sigma)\} \]
when value \( z \) is toward infinity within the angular domain \( \phi_k \leq \arg z \leq \theta_k \) \((k=1,2,3, \cdots, n)\). Then, the following conclusion can be drawn: each nontrivial solution \( f \) of the differential equation \( f^n + F(z)f' + G(z)f = 0 \) has infinite order.

In the past, studies about the property of solution of entire function always adopt the same assumed condition as A theorem, considering the series of an entire function equation at one coefficient is larger than other series, so the whole transcendental solution series of the obtained equation is toward infinity. In this paper, the assumed condition of A theorem is expanded, and some coefficients have the same series in the equation. In this condition, the whole transcendental solution series of the equation can be proved to be toward infinity, which means that a theorem is converted into the following theorem:

2.1. Theorem 1

Theorem 1 assumes that \( A_0, A_1, \ldots, A_{K-1} \), the entire function with finite order does not constantly equal to zero, \( K \geq 2 \). For every \( A_j \) \((j \text{ is integer}, 0 \leq j \leq k-1)\), if \( A_j \) does not constantly equal to 0, \( [2]\) then \( \lambda(A_j) < \sigma(A_j) \). Then, \( A_k \) does not equal to 0, and \( A_j \) does not equal to 0 \((i \neq j)\), \( \sigma(A_j/A_k) = \max\{\sigma(A_j), \sigma(A_k)\} \). Therefore, any transcendental solution \( f \) of the differential equation
\[ f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0 \]
meets the condition \( \sigma(f) = \infty \). Furthermore, if in the order of \( A_0, A_1, \ldots, A_{K-1} \), the first coefficient which does not constantly equal to zero is \( A_j \), then in the equation
\[ f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0 \]
the frequency of occurrence of polynomial solution is less than \( j-1 \), and other solutions are infinite series. If \( A_0 \) does not constantly equal to 0,
then any untrivial solution of the equation
\[ f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0 \]
is infinite series.

2.2. Corollary of Theorem 1

According to the relative theory of complex domain differential equation, it is inferred that any solution of
\[ f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0 \]
is entire function. From this, we can firstly assume that based on the specified condition in the theorem 1, any transcendental solution \( f \) of the equation
\[ f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0 \]
must be infinite order. But on the contrary, homogeneous equation also exists an transcendental solution \( f \) to make \( \sigma(f) = \sigma < \infty \). Here, we use lemma 1 to prove the truth of theorem 1.

Lemma 1 supposes that \( W(z) \) is the transcendental meromorphic function with finite order in the open plane. Its series is \( \rho \), and there are different pairs of integers in the finite order \( \Gamma = \{ (k_1, j_1), (k_2, j_2), \ldots, (k_m, j_m) \} \). Additionally, integer meets the condition \( k_i > j_i \geq 0 \),
i=1, 2, ⋯ , m. If there is the given constant \( \varepsilon > 0 \), then a set of measure zero \( E_i \subset [0, 2\pi) \) exists to make \( \Psi_0 \in [0, 2\pi)/E_i \), thus constant \( P_0 = P_0(\Psi_0) > 0 \) exists. If the condition \( \arg z = \Psi_0 \) is met, and all z value of \(| z | \geq P_0 \) and all pairs of integers accord with \((K, j) \in \Gamma, [3]\) then the inequation \( \left| W^{(k)}(z)/W^{(j)}(z) \right| \leq |z|^{\Gamma(kj)(\rho-1+\varepsilon)} \) exists.

According to lemma 1, if there is the given constant \( \varepsilon > 0 \), then a set of measure zero \( E_i \subset [0, 2\pi) \) exists, and the assumption \( \Psi_0 \in [0, 2\pi)/E_i \) can be put forward, thus the constant \( P_0 = P_0(\Psi_0) > 0 \) exists. For the conditions of satisfaction \( \arg z = \Psi_0 \), and \(| z | \geq P_0 \), we can substitute the given condition of theorem 1 into it and then transform it into the inequation \( \left| f^{(j)}(z)/f^{(i)}(z) \right| < |z|^{\Gamma(kj)} \), \( i=0, 1, \cdots, k-1, j=i+1, \cdots, k \).

Supposed that \( A_0, A_1, \ldots, A_{k-1} \) does not constantly equal to zero, then integer meets the condition \( 0 \leq j \leq k-1 \). If \( A_j \) does not constantly equal to 0, then \( \lambda(A_j) < \rho(A_j) \). According to Borel theorem, \( A_j(z) = B_j(z)e^{P_j(z)} \), where polynomial \( P_j(z) \) is non-constant, and \( B_j(z) \) is entire function. The inequation \( \sigma(B_j) - \lambda(A_j) < \sigma(A_j) = \deg P_j \) can be obtained to make \( \deg P_j = d_j \).

When \( i \neq j \), \( \sigma(A_j/A_i) = \deg(P_j - P_i) = \max\{d_i, d_j\} \).

It is verified that when \( i=2, 3, E_2 = \{\theta \in [0, 2\pi) : \delta(P_j, \theta) = 0, j = 0, 1, \ldots, k-1\} \cup \{\theta \in [0, 2\pi) : \delta(P_j, \theta) = 0, 0 \leq i \leq j \leq k-1\} \). From this, it can be seen that \( E_2 \) is finite order. During the collation of the equation \( A = \{d_0, d_1, \ldots, d_{k-1}\} \), when there occurs the same integers, one of them should be dropped while the other should be retained. A value of the collative integers which are different is denoted as \( A' = \{d_1, d_2, \ldots, d_m\} \), where integers are in the descending relation and \( m \) is an integer, meeting the condition \( 1 \leq m \leq k \). As for \( A_j \), gather zero measure \( H_j \subset [0, 2\pi) \) should be assumed firstly and then the second lemma should be introduced.

Lemma 2 supposes that \( P(z) \) is the non-constant polynomial with \( n \) times, and \( W(z) \), not constantly equal to 0, is a meromorphic function, whose series is less than \( n \) to make \( g = We^P \). Therefore, a set of measure zero \( H_j \subset [0, 2\pi) \) (\( j=1 \)) exists. For every \( \theta \in [0, 2\pi)/(H_1 \cup H_2) \) and the given constant \( \varepsilon : (0 < \varepsilon < 1) \), when the condition \( \tau > r_0(\theta, \varepsilon) \) exists, there are two conclusions shown as following.

1. If \( \delta(P, \theta) < 0 \), then \( \exp((1+\varepsilon)\delta(P, \theta)r^n) \leq |g(re^{\theta})| \leq \exp((1-\varepsilon)) \)

2. If \( \delta(P, \theta) > 0 \), then \( \exp((1-\varepsilon)\delta(P, \theta)r^n) \leq |g(re^{\theta})| \leq \exp((1+\varepsilon)\delta(P, \theta)r^n) \)

where \( H_2 = \{\theta : \delta(P, \theta) = 0, 0 \leq \theta \leq 2\pi\} \) is a finite order.

According to lemma 2, an entire functional expression exists an exceptional set of zero measure in the item of \( A_j \), so the measure of \( E_j = \bigcup_{j=0}^{k-1} H_j \) is zero. Now randomly take a ray
arg = Ψ₀ ∈ [0, 2π)/(E₁ ∪ E₂ ∪ E₃) , and then δ(Pᵢ, Ψ₀) ≠ 0 and there exists δ(Pᵢ, Ψ₀) ≠ δ(Pᵢ, Ψ₀) (i<j and deg(Pᵢ − Pⱼ) = 0)

As for

\[ \delta_i = \max \{ \delta(P_j, Ψ₀) : j \in \{0,1, \ldots, k-1\} \text{ and } \deg P_j = d_{ym} \}, \]

and then the only set

\[ S_i = \{0,1, \ldots, k-1\} \]

exists to make \( \delta(P_{i}, Ψ₀) = \delta_i \). In the following, some cases of \( \delta \) will be discussed.

1) If \( \delta_1, \delta_2, \delta_3, \ldots, \delta_m \) are not all less than 0, assumed \( \delta_i = \delta(P_{St}, Ψ₀) \) as the first value which is more than zero to make \( \delta = \max \{0,\delta(P_j, Ψ₀) : j \in \{0,1,2, \ldots, k-1\} \text{ and } \deg P_j = d_{ym}\}, \delta(P_i, Ψ₀) ≠ \delta_i \) , then \( 0 ≤ \delta ≤ \delta t \). In the following, according to lemma 2, we can know the equation

\[ |A_{St}(re^{iq})| ≥ \exp((1-ε_i)\delta(P_i, Ψ₀) r^{\overline{dm}}) . \]

Now, we should consider other items, \( A_q = B_q e^{iq} \) where \( q \neq St \) and \( \deg P_q = d_{ym} \). According to lemma, when \( 1 ≤ i ≤ t-1 \), \( \delta_i = \delta(P_{St}, Ψ₀) < 0 \). When there is \( r → ∞ \), there exists an inequation. But when \( t+1 ≤ i ≤ m \), and \( r → ∞ \), the inequation turns into

\[ |A_q(re^{iq})| ≤ \exp((1−ε_i)\delta(P_q, Ψ₀) r^{\overline{dm}}) ≤ \exp(0(1)r^{\overline{dm}}) . \]

However, if \( \delta_i = \delta(P_{St}, Ψ₀) > 0 \), because \( \delta_i = \delta(P_{St}, Ψ₀) > δ ≥ \delta(P_q, Ψ₀) \), we can take \( ε : 0 < ε < \frac{\delta_i - \delta}{2\delta_i} ≤ \frac{1}{2} \) from \( \delta : 0 < \delta < \frac{\delta_i - \delta}{2\delta_i} ≤ \frac{1}{2} \), and then apply the equation of lemma 2. When \( r → ∞ \), the relation of inequation

\[ |A_q(re^{iq})| ≤ \exp(1+ε') \delta(P_q, Ψ₀) r^{\overline{dm}} < \exp(1-2ε_i) \delta(P_{St}, Ψ₀) r^{\overline{dm}} \]

is obtained.

In the following, we will prove \( |f^{(S)}(z)| \) is bounded in the constructed ray \( \arg = Ψ₀ ∈ [0, 2π)/(E₁ ∪ E₂ ∪ E₃) \). Firstly, we need to suppose that the ray is boundless, and then construct \( M(r, f^{(S)}, Ψ₀) = \max \{ |f^{(S)}(z)| : 0 ≤ |z| ≤ r, \arg z = Ψ₀ \} \) to make the point range \( Z_n = r_n e^{iq}, r_n → ∞ \) exist in it, where \( n \) meets the condition \( M (r_n, f^{(S)}, Ψ₀) = |f^{(S)}(r_n e^{iq})| \). For each existing \( n \), we take its path of integration \( C : z = 0 ≤ r ≤ |Z_n| \). The function of constructed \( M \) forms the equation

\[ f^{(S-1)}(z_n) = f^{(S-1)}(0) + \int_{0}^{z_n} f^{(S)}(u)du \]

after integral and there exists \( |f^{(S)}(z_n)| ≤ |f^{(S)}(z)| \) on the \( C \) path, so the inequation
\[ |f^{(S)}(z_n)| \leq |f^{(S)}(0)| + |z_n| \cdot |f^{(S)}(z_n)| \] can be obtained. Then, we put the condition of C path into the inequation, transforming it as \[ |f^{(S)}(z_n)| / |f^{(S)}(z_n)| \leq (1 + o(1)) + |z_n| \] 

By summarizing the above content, it can be seen that when \( z_n \to \infty \), \[ |f^{(S-j)}(z_n)/f^{(S)}(z_n)| \leq (1 + o(1)) + |z_n| \] }, j=1, 2, \ldots, St. Because the function which is originally constructed does not constantly equal to zero, when \( z_n \to \infty \), we can collate the following inequation combining with the above content. 

\[ |A_0(z_n)| \cdot |f(z_n)| - f^{(S)}(z_n)| \leq \exp\{(1 - 2\varepsilon_1)\delta(P_{S_1}, \Psi_0)|z_n| d_{n,m}| \cdot |z_n|^M_1 < \exp\{(1 - \varepsilon_1)\delta(P_{S_1}, \Psi_0)|z_n| d_{n,m}| \] , where M1 is a positive constant to make the definition of the inequation and \( |A_0(re^{i\phi_0})| \geq \exp((1 - \varepsilon_1)\delta(P_{S_1}, \Psi_0)r^{d_{m}}) \) contradictory, so it can prove that the original constructor is bounded in \( \arg = \Psi_0 \in [0, 2\pi)/(E_1 \cup E_2 \cup E_3) \) .

Now we make \( |f^{(S)}(re^{i\phi_0})| \leq M_2 \), where M2 and M1 are positive constants. Then take path of integration of upper bound \( C' = \{ z : \arg z = \Psi_0, 0 \leq |z| \leq r \} \), and obtain the equation \( f^{(S-1)}(z) = f^{(S-1)}(0) + \int_0^z f^{(S)}(u)du \) after integral. When \( z = re^{i\phi_0} \) is large enough, \[ |f^{(S-1)}(z)| \leq M_3|z| \] is obtained, and M3 is a constant more than zero. From this, we can infer \( |f(z)| \leq M'|z|^k \), \( M' > 0 \).

2) We supposed that \( \delta_1, \delta_2, \delta_3, \ldots, \delta_m \) are all less than zero, and there exists integer J: \( 0 \leq j \leq k - 1 \), \( \delta_0(P_j, \Psi_0) \leq 0 \), to make \( \delta' = \max_{0 \leq j \leq k - 1} \{ \delta(P_j, \Psi_0) \}, d = \min \{ d_j \} \). Then \( \delta' < 0 \), \( d > 0 \). After that, we use lemma 2 to set \( \varepsilon_2(0 < \varepsilon_2 < \frac{1}{2}) \) in order to make \( |A_j(re^{i\phi_0})| \leq \exp((1 - \varepsilon_2)\delta(P_j, \Psi_0)r^{d_j}) \leq \exp((1 - \varepsilon_2)\delta r^{d_j}) \). If \( |f^{(S)}(z)| \) is boundless in the above constructed ray \( \arg = \Psi_0 \in [0, 2\pi)/(E_1 \cup E_2 \cup E_3) \), we construct \( M(r, f^{(k)}, \Psi_0) = \max\{ f^{(k)}(z) : 0 \leq |z| \leq r, \arg z = \Psi_0 \} \) like (1), then there exists \( z'' = r'e^{i\phi_0} \), \( r' \to \infty \), meeting the condition \( M(r', f^{(k)}, \Psi_0) = |f^{(k)}(r'e^{i\phi_0})| \).

During the process of the inference applied in 1), the contradiction occurs, thus \( |f^{(S)}(z)| \) is bounded in the above ray \( \arg = \Psi_0 \in [0, 2\pi)/(E_1 \cup E_2 \cup E_3) \). When \( z = re^{i\phi_0} \) is large enough, \( |f(z)| \leq M'|z|^k \) \( M' > 0 \) is a constant.
By combining the conclusions of 1) and 2), we can obtain any ray. When there is \( |z| = r \geq r_0 \) \( (\Psi_0 > 0) \) in the \( \arg z = \Psi_0 \in [0,2\pi) / (E_1 \cup E_2 \cup E_3) \), there is the inequation \( |f(z)| \leq M(\Psi_0)|z|^k \) which exists \( r_1(\varepsilon) > 0 \) for any small value \( \varepsilon > 0 \).

When it meets the condition \( r > r_1 \), \( |f(z)| \leq \exp(\varepsilon r^{\sigma + 1}) \). We take a point in the path \( \theta_j \in [0,2\pi) \setminus (E_1 \cup E_2 \cup E_3) \ (j = 1,2, \ldots, n) \), to make it meets the condition \( \theta_1 < \theta_2 < \ldots \ldots < \theta_n < \theta_{n+1} = \theta_1 + 2\pi \), \( \max \{ \theta_{j+1} - \theta_j : 1 \leq j \leq n \} < \frac{\pi}{\sigma + 1} \).

We suppose \( P = \max \{ r_0(\theta_j), r_j \} + 1 \), \( M = \max \{ M(\theta_j) \} \), so within the range of \( \theta_j \leq \arg z \leq \theta_{j+1} \ (j = 1,2, \ldots, n) \), when \( r \geq R \), there exists the inequation \( |f(z)/z^k| < |f(z)| \leq \exp(\varepsilon r^{\sigma + 1}) \).

However, in the boundary of the angular domain which is constructed by ray, when there exists \( |z| \geq R \), the angular domain \( \{ z : \theta \leq \arg z \leq \theta_j, |z| \geq R \} \) exists \( |f(z)| \leq M(\Psi_0)|z|^k \) (treat \( \Psi_0 \) as 1 within \( \{ z : |z| \geq R \} \)), and then we can know that \( |f(z)| \leq M|z|^k + M_0 \) (\( M_0 \) is a positive constant) by combining with the analyticity of \( f(z) \), thus it can be inferred that if the number of solutions of \( f(z) \)’s polynomial is over \( k \), then it contradicts \( f \) transcendental function. Therefore, within the limitation of the condition of lemma 1, any transcendental solution existing in the equation \( f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0 \) must have infinite series.

3. Conclusion

Based on the order of \( A_0, A_1, \ldots, A_{k-1} \), there exists items more than zero and less than zero. We suppose that the coefficient of one of them not constantly equal to zero is \( A_j \), the equation

\[
f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0
\]

is obtained. If there is the solution of the polynomial \( f(z) = a_pz^p + \cdots + a_0 \), then \( f(z) = 0 \) is one of the solutions of

\[
f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0.
\]

If \( f(z) \) does not constantly equal to zero, then \( a_p \neq 0 \). When \( p \geq k \), \( f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0 \). The order of left is \( \max \{ d_j \} > 0 \). When \( j \leq p < k \), \( f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0f = 0 \). The order of left is \( \max \{ d_j \} > 0 \), which contradicts 0 value of the equation. Therefore, it is proved that in the equation, the number of solutions is less than or
equal to j-1 times at most. Based on proof by contradiction, we can know that if an equation exists an untrivial solution, then its series is infinite. In consequence, if we make the series of polynomial solution finite, then there only exists a zero solution.

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