CHARACTERIZATIONS OF ANISOTROPIC HIGH ORDER SOBOLEV SPACES

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Abstract. We establish two types of characterizations for high order anisotropic Sobolev spaces. In particular, we prove high order anisotropic versions of Bourgain-Brezis-Mironescu’s formula and Nguyen’s formula.

1. Introduction

The celebrated Bourgain-Brezis-Mironescu formula, appeared for the first time in [6, 7], and provided a new characterization for functions in the Sobolev space $W^{1,p}(\Omega)$, with $p \geq 1$ and for $\Omega \subset \mathbb{R}^N$ being a smooth bounded domain. More precisely, they proved

Theorem A. (Bourgain, Brezis and Mironescu, [6]). Let $g \in L^p(\mathbb{R}^N)$, $1 < p < \infty$. Then $g \in W^{1,p}(\mathbb{R}^N)$ iff

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(x) - g(y)|^p}{|x - y|^p} \rho(|x - y|) \, dx \, dy \leq C, \ \forall n \geq 1,$$

for some constant $C > 0$. Moreover,

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|g(x) - g(y)|^p}{|x - y|^p} \rho_n(|x - y|) \, dx \, dy = K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx.$$

Here

$$K_{N,p} = \int_{S^{N-1}} |e \cdot \sigma|^p \, d\sigma$$

for any $e \in S^{N-1}$ and $d\sigma$ is the surface measure on $S^{N-1}$. Here $(\rho_n)_{n \in \mathbb{N}}$ is a sequence of nonnegative radial mollifiers satisfying

$$\lim_{n \to \infty} \int_0^\infty \rho_n(r) \, r^{N-1} \, dr = 0, \ \forall r > 0,$$

$$\lim_{n \to \infty} \int_0^\infty \rho_n(r) \, r^{N-1} \, dr = 1.$$

Starting from the previous result and since the theory of Sobolev spaces is a fundamental tool in many branches of modern mathematics, such as harmonic analysis, complex...
analysis, differential geometry and geometric analysis, partial differential equations, etc, there has been a substantial effort to characterize Sobolev spaces in different settings and various ways depending on the situation where these spaces are used (see e.g., [1], [4], [12], [13], [21], [22], [24], [23]).

Theorem A has been extended to the high order case by Bojarski, Il'nyts'eva and Kinnunen [2] using the high order Taylor remainder and by Borghol [3] using high order differences.

We note here, as a consequence of Theorem A, that we can characterize the Sobolev space $W^{1,p}(\mathbb{R}^N)$ as follows: Let $g \in L^p(\mathbb{R}^N), \ 1 < p < \infty$. Then $g \in W^{1,p}(\mathbb{R}^N)$ iff

$$\sup_{0<\delta<1} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|g(x) - g(y)|^p}{\delta^{N+p}} \, dx \, dy < \infty. \quad (1.1)$$

Recently, Nguyen [17] (see also [18]), motivated by an estimate for the topological degree for the Ginzburg-Landau equation ([5]), established some new characterizations of the Sobolev space $W^{1,p}(\mathbb{R}^N)$ which are closely related to Theorem A. More precisely, he used the dual form of (1.1) and proved the following results:

**Theorem B. (H. M. Nguyen, [17]).** Let $1 < p < \infty$. Then the following hold:

(a) Let $g \in W^{1,p}(\mathbb{R}^N)$. Then there exists a positive constant $C_{N,p}$ depending only on $N$ and $p$ such that

$$\int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy \leq C_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx, \ \forall \delta > 0, \forall g \in W^{1,p}(\mathbb{R}^N).$$

(b) If $g \in L^p(\mathbb{R}^N)$ satisfies

$$\sup_{0<\delta<1} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy < \infty,$$

then $g \in W^{1,p}(\mathbb{R}^N)$.

(c) Moreover, for any $g \in W^{1,p}(\mathbb{R}^N)$,

$$\lim_{\delta \to 0} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta^p}{|x - y|^{N+p}} \, dx \, dy = \frac{1}{p} K_{N,p} \int_{\mathbb{R}^N} |\nabla g(x)|^p \, dx$$

The previous result has been generalized in many ways and for different spaces (see e.g. [12, 19, 20, 15]). In particular, in [20] the authors proved the following result:

**Theorem C. (H. M. Nguyen, M. Squassina [20]).** Let $1 < p < \infty$ and $K \subset \mathbb{R}^N$ be a convex, symmetric set containing the origin. Then, for every $g \in W^{1,p}_K(\mathbb{R}^N)$,

$$\lim_{\delta \to 0} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\delta^p}{\|x - y\|_K^{N+p}} \, dx \, dy = \int_{\mathbb{R}^N} \|\nabla g\|_K^p \, dx.$$
where $\| \cdot \|_K$ is the norm in $\mathbb{R}^N$ which admits as unit ball the set $K$, i.e. $\| x \|_K := \inf \{ \lambda > 0 \mid \frac{x}{\lambda} \in K \}$, $\| \cdot \|_{Z^p_K}$ is the norm associated with the $L_p$ polar body of $K$, namely

$$
\| v \|_{Z^p_K} = \left( \frac{N + p}{p} \int_K |v \cdot x|^p dx \right)^{1/p}, \quad v \in \mathbb{R}^N
$$

(1.2)

and $W^{1,p}_K(\mathbb{R}^N)$ is the associate Sobolev space.

The main purpose of this paper is to generalize Theorem A and Theorem C to high-order anisotropic Sobolev spaces. In order to describe our main results we recall the following notation ([3]): Let $f \in W^{k,p}(\Omega)$ and $\sigma = (\sigma_1, \ldots, \sigma_N) \in \mathbb{S}^{N-1}$, we denote

$$
D^k f(x) (\sigma, \ldots, \sigma) = \sum_{1 \leq i_1, \ldots, i_k \leq N} \sigma_{i_1} \ldots \sigma_{i_k} \frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}(x)
$$

and, for every $m \in \mathbb{N}$

$$
R^m f(x, y) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} f \left( \frac{m-j}{m} x + \frac{j}{m} y \right).
$$

(1.3)

**Theorem 1.1.** Let $f \in W^{m,p}(\mathbb{R}^N)$ with $m \in \mathbb{N}$ and $1 < p < \infty$. Then

$$
\lim_{\delta \to 0} \int_{|R^m f(x, y)| > \delta} \frac{\delta^p}{\| x - y \|^N_{m+\frac{p}{m}}} dx dy = \frac{N + mp}{m^{mp+1}p} \int_{\mathbb{R}^N} \int_{K} |D^m f(x)(y, \ldots, y)|^p dy dx.
$$

Notice that taking $m = 1$ in the previous theorem we get Theorem C and taking $m = 2$ and $\| \cdot \|$ the Euclidean norm we get [12, Theorem 1.1].

Our next result is the analogous of [3, Theorem 4] in our setting.

**Theorem 1.2.** Let $(\rho_\varepsilon)_\varepsilon$ be a family of functions such that $\rho_\varepsilon : [0, \infty) \to [0, \infty)$ such that

$$
\int_{0}^{\infty} \varepsilon^{N-1} \rho_\varepsilon(r) dr = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_{\delta}^{\infty} \varepsilon^{N-1} \rho_\varepsilon(r) dr = 0 \quad \forall \delta > 0
$$

(1.4)

Let $1 < p < \infty$. If $f \in W^{m,p}(\mathbb{R}^N)$ then

$$
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\| R^m f(x, y) \|^p_{m+\frac{p}{m}}}{\| x - y \|^N_K} \rho_\varepsilon(\| x - y \|_K) dx dy = \frac{N + mp}{m^{mp}} \int_{\mathbb{R}^N} \int_{K} |D^m f(x)(y, \ldots, y)|^p dy dx.
$$

(1.5)

We also prove the following results which can be considered a generalization to high-order anisotropic spaces of [12, Theorem 1.2]. More precisely, for any $f \in W^{m,p}(\mathbb{R}^N)$ and $y \in \mathbb{R}^N$ let

$$
T_y^{m-1} f(x) = \sum_{|\alpha| \leq m-1} D^\alpha f(y) \frac{(x - y)^\alpha}{\alpha!}
$$

and

$$
R_{m-1} f(x, y) = f(x) - T_y^{m-1} f(x).
$$

Then we will prove that

**Theorem 1.3.** Let $f \in W^{m,p}(\mathbb{R}^N)$, $1 < p < \infty$. Then

$$
\lim_{\delta \to 0} \int_{\mathbb{R}^N} \int_{|R_{m-1} f(x, y)| > \delta} \frac{\delta^p}{\| x - y \|^N_{m+\frac{p}{m}}} dx dy = \frac{N + mp}{(m!)^p mp} \int_{\mathbb{R}^N} \int_{K} |D^m f(x)(y, \ldots, y)|^p dy dx.
$$
Theorem 1.4. Let \( f \in W^{m,p}(\mathbb{R}^N) \), \( 1 < p < \infty \). Then
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| R_{m-1} f(x, y) \right|^p \rho_\varepsilon (\|x - y\|_K) \, dx \, dy = \frac{N + mp}{(m!)^p} \int_{\mathbb{R}^N} \int_{K} |D^m f(x, \ldots, y)|^p \, dy \, dx.
\]
Here the family \( (\rho_\varepsilon)_\varepsilon \) is as in Theorem 1.2.

The plan of the paper is the following: In Section 2, we will study a helpful lemma which will be used in Section 3 to prove Theorems 1.2 and 1.1. In Section 4, we establish Theorems 1.3 and 1.4 which will give characterizations of the high-order anisotropic Sobolev spaces using the Taylor reminder.

2. A useful Lemma

Lemma 2.1. Let \( N, m \in \mathbb{N} \) and \( 1 < p < \infty \). There exists a constant \( C = C(N, m, p) > 0 \) such that for every \( 1 \leq i \leq N \) and every \( f \in L^p(\mathbb{R}^N) \) it holds
\[
\| \partial_{x_i}^{2m} f \|_{L^p(\mathbb{R}^N)} \leq C \| (\Delta)^m f \|_{L^p(\mathbb{R}^N)} \tag{2.1}
\]
and
\[
\| \partial_{x_i}^{2m+1} f \|_{L^p(\mathbb{R}^N)} \leq C \| \nabla (\Delta)^m f \|_{L^p(\mathbb{R}^N)} \tag{2.2}
\]
where \( (\Delta)^m f = (\Delta) \cdot (\Delta) \cdots (\Delta) f \).

Proof. If \( g = (\Delta)^m f \) then \( \hat{g}(\xi) = (-4\pi |\xi|^2)^m \hat{f}(\xi) \). Therefore for every \( \xi \in \mathbb{R}^N \setminus \{0\} \) we have \( \hat{f}(\xi) = \frac{\hat{g}(\xi)}{(-4\pi |\xi|^2)^m} \) and
\[
\partial_{\xi_j}^{2m} \hat{f}(\xi) = \frac{(\xi_j)^m}{(4\pi |\xi|^2)^m} \hat{g}(\xi). \tag{2.3}
\]
Since the function \( m(\xi) = \frac{(\xi_j)^m}{(4\pi |\xi|^2)^m} \) is homogeneous of order zero and smooth everywhere except at 0, it satisfies the Mikhlin multiplier theorem [11], namely there exists \( C = C(N, p) > 0 \) such that
\[
\| (m\hat{g})^\vee \|_{L^p} \leq C \|g\|_{L^p} \quad \forall g \in \mathcal{S}
\]
which together with (2.3) gives (2.1). Clearly (2.2) follows from (2.1).

Since all norms on \( \mathbb{R}^N \) are equivalent, there are \( A, B > 0 \) such that
\[
A |\cdot| \leq \| \cdot \|_K \leq B |\cdot| \tag{2.4}
\]

3. Characterizations of the higher order Sobolev spaces via m-th difference

Let \( m \geq 0 \). Set \( \Delta_h f(x) = \Delta_{\Delta_0} f(x) := f(x + h) - f(x) \), we call m-th difference the quantity \( \Delta_h^m f(x) = \Delta_h [\Delta_{\Delta_0}^{m-1} f(x)] \). By above definition, it is not difficult to show that for any positive integer \( m \), we have
\[
\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m+j} \binom{m}{j} f(x + jh)
\]
and, by [3, Lemma 8], we also have
\[
\Delta_h^m f(x) = \int_{[0,1]^m} D^m f \left( x + \sum_{j=1}^{m} t_j h \right) (h, \cdots, h) dt_1 \ldots dt_m \tag{3.1}
\]
where \([0,1]^m\) denotes the unit-cube in \(\mathbb{R}^m\). Finally, it is easy to see that for every \(x, h \in \mathbb{R}^N\)

\[ R^m f(x, x + mh) = (-1)^m \Delta^m h f(x) \]

where \(R^m f(x, y)\) is as in (1.3).

### 3.1. Nguyen’s formula

The aim of this section is to prove Theorem 1.1. We start with the following:

**Lemma 3.1.** Let \(f \in W^{m,p}(\mathbb{R}^N)\) with \(m \in \mathbb{N}\) and \(1 < p < \infty\). There holds

\[
\int \int_{\|x-y\|_{K}^{N+mp} \leq \delta} |\nabla^m f(x, x + mh)| \, dx \, dh \leq C(m, N, p) \|\nabla^m f\|_{L^p(\mathbb{R}^N)}
\]

where

\[ \nabla^m f = \begin{cases} \frac{(-\Delta)^{m/2} f}{m} & \text{if } \frac{m}{2} \text{ is even} \\ \nabla(-\Delta)^{m/2} f & \text{if } \frac{m}{2} \text{ is odd} \end{cases} \]

**Proof.** Using polar coordinates \((y = x + t\sigma, \sigma = \frac{y-x}{|y-x|} \text{ and } t = |x-y|)\) we write

\[
\int \int_{\|x-y\|_{K}^{N+mp} \leq \delta} |\nabla^m f(x, x + mh)| \, dx \, dh = \int_{S^{N-1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \int_{\|x-y\|_{K}^{N+mp} \leq \delta} |\Delta^m f(x)| \, dt \, dx \, \sigma \, d\sigma
\]

Thus, since \(A \leq \|\sigma\|_{K} \leq B\), it is enough to show that there exists a constant \(C = C(m, N, p) > 0\) such that for every \(\sigma \in \mathbb{S}^{N-1}\)

\[
\int \int_{\|x-y\|_{K}^{N+mp} \leq \delta} |\nabla^m f(x)| \, dx \, d\sigma \leq C \int_{\mathbb{R}^N} |\nabla^m f(x)|^{p} \, dx.
\]

We assume, without loss of generality, that \(\sigma = e_N = (0, \ldots, 0, 1)\). By (3.1), we have for any \(x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}\)

\[
|\Delta^m_{x_N} f(x)| \leq \int_{[0,1]^m} \int_{[0,1]^m} \frac{\partial^m_{x_N} f(x', x_N)}{m} ds_1 \ldots ds_m
\]

\[
\leq \int_{x_N}^{x_N + t} |\partial^m_{x_N} f(x', s)| ds
\]

\[
\leq \int_{x_N}^{x_N + t} \mathbb{M}_{x_N}(\partial^m_{x_N} f)(x', x_N)\]

where \(\mathbb{M}_{x_N}(f)\) denotes the maximal function of \(f\) in direction \(x_N\), namely

\[
\mathbb{M}_{x_N}(f)(x) = \mathbb{M}_{x_N}(f)(x', x_N) = \sup_{t > 0} \frac{1}{t} \int_{x_N}^{x_N + t} |f(x', s)| ds.
\]
So,
\[ \int_{\mathbb{R}^N} \int_{|\Delta^m f(x)| > \delta} \frac{\delta^p}{1 + mp} dtdx \leq \int_{\mathbb{R}^N} \int_0^\infty \frac{1}{m^p} \mathcal{M}_N(\partial^m f(x)) > \delta \frac{\delta^p}{1 + mp} dtdx \\ = \frac{1}{m^p} \int_{\mathbb{R}^N} \mathcal{M}_N(\partial^m f(x)) dx \\
\leq \frac{C}{m^p} \int_{\mathbb{R}^N} |\partial^m f(x)| dx \\
\leq \frac{C}{m^p} \|\nabla^m f\|_{L^p(\mathbb{R}^N)} \]

where in the last line we used Lemma 2.1 and in the line before we used (see [25])
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}} |\mathcal{M}_N(\partial^m f(x'))| dx'dx \leq C \int_{\mathbb{R}^N} \int_{\mathbb{R}} |\partial^m f(x', x_N)| dx'dx. \]

This gives the conclusion. \qed

We are now in position to prove Theorem 1.1.

**Proof of Theorem 1.1:** By changing variables (writing \( y = x + \sqrt{\delta} h\sigma, \sigma = \frac{y - x}{|y - x|} \)), we obtain
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}} \frac{\delta^p}{|x - y|^{N+mp} K} dtdy = \int_{\mathbb{R}^N} \int_{\mathbb{S}^N} \frac{1}{\|\sigma\|^{N+mp} h^{1+mp}} dhdxds \]

Following [17], we start proving that there exists \( C = C(m, N, p) > 0 \) such that for every \( \sigma \in \mathbb{S}^{N-1} \),
\[ \int_{\mathbb{R}^N} \int_0^\infty \frac{1}{h^{mp+1}} dhdx \leq C \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx \] \hspace{1cm} (3.2)

and
\[ \lim_{\delta \to 0} \int_{\mathbb{R}^N} \int_0^\infty \frac{1}{h^{mp+1}} dhdx = \frac{1}{m^p} \int_{\mathbb{R}^N} |D^m f(x, \sigma, ..., \sigma)|^p dx \] \hspace{1cm} (3.3)

Without loss of generality we assume \( \sigma = e_N = (0, 0, \ldots, 1) \) and by (3.1) we have
\[ \Delta^m_{h e_N} f(x) = h^m \int_{[0,1]^m} \partial^m_{x_N} f(x', x_N + h \sum_{j=1}^m s_j) ds \ldots ds_m \\
= h^{m-1} \int_{x_N}^{x_N+h} \partial^m_{x_N} f(x', s) ds \]
for all \((x_N, h) \in \mathbb{R} \times (0, \infty)\) and for almost everywhere \(x' \in \mathbb{R}^{N-1}\). Now given \(x' \in \mathbb{R}^{N-1}\) and \(\delta \in (0, 1)\) we define

\[
A(x', \delta) := \left\{ (x_N, h) \in \mathbb{R} \times (0, \infty) \mid \frac{\Delta^{m_{\mathbb{R}^N}} h_{x_N} f(x', x_N)}{h^m \delta} > h^m > 1 \right\},
\]

\[
A(x') := \left\{ (x_N, h) \in \mathbb{R} \times (0, \infty) \mid \left| \partial^m_{x_N} f(x', x_N) \right| h^m > m^m \right\},
\]

and

\[
B(x') := \left\{ (x_N, h) \in \mathbb{R} \times (0, \infty) \mid \left| M_N \left( \partial^m_{x_N} f \right)(x', x_N) \right| h^m > m^m \right\}.
\]

We claim that for all \((x_N, h) \in \mathbb{R} \times (0, \infty)\) and for all \(x' \in \mathbb{R}^{N-1}, \delta \in (0, 1)\)

\[
1_{A(x', \delta)}(x_N, h) \leq 1_{B(x')(x_N, h)}.
\]

Indeed, fix \((x_N, h) \in A(x', \delta)\), that is \(h > 0\) and \(\frac{\Delta^{m_{\mathbb{R}^N}} h_{x_N} f(x)}{h^m \delta} \geq h^m > 1\). Using (3.1) we get

\[
1 \leq \left| \frac{\Delta^{m_{\mathbb{R}^N}} h_{x_N} f(x)}{h^m \delta} \right| h^m \leq \left( \frac{h}{m} \right)^m \int_{[0, 1]^m} \left| \partial^m_{x_N} f(x', x_N + \frac{\sqrt{\delta}}{m} \sum_{j=1}^m t_j) \right| dt_1 ... dt_m
\]

\[
\leq \frac{h^{m-1}}{m^m} \int_{x_N}^{x_N + \frac{\sqrt{\delta}}{m} h} |\partial^m_{x_N} f(x', s)| ds
\]

\[
\leq \frac{h^m}{m^m} M_N \left( \partial^m_{x_N} f \right)(x', x_N).
\]

which implies (3.4). Moreover,

\[
\int_{\mathbb{R}^N} \int_{0}^{\infty} 1_{B(x')(x_N, h)} \frac{1}{h^{mp+1}} dh dx_N dx' = \int_{\mathbb{R}^N} \int_{0}^{\infty} \left( \frac{m^m}{h^{mp+1}} \right) \frac{1}{h^m} dh dx_N dx'
\]

\[
= \frac{1}{m^{mp+1} p} \int_{\mathbb{R}^N} |M_N \left( \partial^m_{x_N} f \right)(x)|^p dx_N dx'
\]

\[
\leq \frac{C}{m^{mp+1} p} \int_{\mathbb{R}^N} |\partial^m_{x_N} f(x)|^p dx.
\]

Putting together (3.4) and (3.5), we get

\[
\int_{\mathbb{R}^N} \int_{0}^{\infty} 1_{A(x', \delta)}(x_N, h) \frac{1}{h^{mp+1}} dh dx_N dx' \leq \frac{C}{m^{mp+1} p} \int_{\mathbb{R}^N} |\partial^m_{x_N} f(x)|^p dx,
\]

which in turn implies (3.2). To prove (3.3), we define \(F_{\delta} : \mathbb{S}^{N-1} \rightarrow \mathbb{R}\) by

\[
F_{\delta}(\sigma) := \frac{1}{\|\sigma\|_{K}^{N+mp}} \int_{\mathbb{R}^N} \int_{0}^{\infty} \frac{1}{h^{mp+1}} dh dx.
\]

We start by proving that there exists \(C = C(N, p, m) > 0\) such that for all \(\sigma \in \mathbb{S}^{N-1}\) and for all \(\delta > 0\):

\[
F_{\delta}(\sigma) \leq \frac{C}{A^{N+mp}} \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx.
\]

Since

\[
\frac{1}{\|\sigma\|_{K}^{N+mp}} \leq \frac{1}{A^{N+mp}},
\]

...
it is enough to show
\[
\int_{\mathbb{R}^N} \int_0^\infty \frac{1}{h^{mp+1}} dh dx \leq C \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx.
\]

Indeed, without loss of generality, we assume that \( \sigma = e_N = (0, ..., 0, 1) \). Hence, we need to verify that
\[
\int_{\mathbb{R}^N} \int_0^\infty \frac{1}{h^{mp+1}} dh dx \leq C \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx. \tag{3.7}
\]

By (3.5) we have
\[
\int_{\mathbb{R}^N} \int_0^\infty \frac{1}{h^{mp+1}} dh dx \leq C \int_{\mathbb{R}^N} |\partial_{x_N}^m f(x)|^p dx
\]
which is (3.6). Next we will show that
\[
F_\delta(\sigma) \to \frac{1}{m^{mp+1}p} \frac{1}{\|\sigma\|_K^{N+mp}} \int_{\mathbb{R}^N} |D^m f(x)(\sigma, ..., \sigma)|^p dx \text{ as } \delta \to 0 \text{ for every } \sigma \in \mathbb{S}^{N-1}. \tag{3.8}
\]

As before, it is enough to show that
\[
\int_{\mathbb{R}^N} \int_0^\infty G_\delta(x, h) dh dx \to \frac{1}{m^{mp+1}p} \int_{\mathbb{R}^N} |D^m f(x)(\sigma, ..., \sigma)|^p dx \text{ as } \delta \to 0,
\]
where
\[
G_\delta(x, h) := \frac{1}{h^{mp+1}} \left\{ \frac{\Delta^m_{h^\sigma} f(x)}{h^\sigma} \right\}_{h^\sigma > 1}(x, h).
\]
Without loss of generality, we suppose that \( \sigma = e_N = (0, ..., 0, 1) \). Using (3.1) it is easy to see that
\[
\lim_{\delta \to 0} 1_{A(x', \delta)}(x_N, h) = 1_{A(x')}(x_N, h) \quad \text{a.e. } (x', x_N, h) \in \mathbb{R}^{N-1} \times \mathbb{R} \times [0, \infty),
\]
thus
\[
G_\delta(x, h) \to \frac{1}{h^{mp+1}} 1_{\{\partial_{x_N}^m f(x) > m^m\}}(x, h) \quad \text{as } \delta \to 0
\]
for a.e. \((x, h) \in \mathbb{R}^N \times [0, \infty)\), and using (3.5), we have
\[
G_\delta(x, h) \leq \frac{1}{h^{mp+1}} 1_{\{h^{m\sigma} (\partial_{x_N}^m f(x)) > m^m\}}(x, h) \in L^1(\mathbb{R}^N \times [0, \infty)).
\]
Hence, by the Lebesgue dominated convergence theorem, we get (3.8). Using (3.6), (3.8) and the Lebesgue dominated convergence theorem again, we can conclude that
\[
\lim_{\delta \to 0} \int_{\mathbb{R}^N} \int_{|D^m f(x, y)| > \delta} \frac{\delta^p}{m^{mp+1}} dxdy = \int_{\mathbb{S}^{N-1}} \frac{1}{m^{mp+1}p} \frac{1}{\|\sigma\|_K^{N+mp}} \int_{\mathbb{R}^N} |D^m f(x)(\sigma, ..., \sigma)|^p dx d\sigma
\]
\[
= \frac{1}{m^{mp+1}p} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \frac{1}{\|\sigma\|_K^{N+mp}} |D^m f(x)(\sigma, ..., \sigma)|^p d\sigma dx.
\]
Now, notice that
\[
\int_{\mathbb{S}^{N-1}} \frac{1}{\| \sigma \|^N_k} |D^m f(x)(\sigma, ..., \sigma)|^p \, d\sigma
\]
\[
= (N + mp) \int_{\mathbb{S}^{N-1}} \int_0^{\frac{1}{\| \sigma \|^N_k}} |D^m f(x)(\sigma, ..., \sigma)|^p r^{N+mp-1} \, dr \, d\sigma
\]
\[
= (N + mp) \int_{\mathbb{S}^{N-1}} \int_0^{\frac{1}{\| \sigma \|^N_k}} |D^m f(x)(r\sigma, ..., r\sigma)|^p r^{N-1} \, dr \, d\sigma
\]
\[
= (N + mp) \int_K |D^m f(x)(y, ..., y)|^p \, dy.
\]
(3.9)

Hence the thesis follows.

**Remark 3.1.** We explicitly note that Lemma 1.1 generalizes some already known results: for example taking \( m = 1 \) and \( K = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \} \) we get [17, Lemma 3] and taking \( m = 2 \) and \( K = \{ x \in \mathbb{R}^n \mid |x| \leq 1 \} \) we get [13, Theorem 1.1]. Moreover, Lemma 1.1 generalizes [20, Theorem 1.1] to the case \( A = 0 \) and \( m \geq 1 \).

3.2. BBM formula. In this Section we prove Theorem 1.2.

Let \( \rho \) be a positive real function satisfying (1.4) The following result is proved in [3, Lemma 8]

**Lemma 3.2.** Let \( f \in W^{m,p}(\mathbb{R}^N) \) with \( m \geq 2 \) and \( 1 \leq p < \infty \) and let \( \rho \in L^1(\mathbb{R}) \). Then
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\Delta_h^m f(x)|^p |h|^{-mp} \rho(|h|) \, dx \, dh \leq \frac{\| \rho \|_{L^1(\mathbb{R})}}{\| \mathbb{S}^{N-1} \|} \int_{\mathbb{R}^N} \left( \int_{\mathbb{S}^{N-1}} |D^m f(x)(\sigma, ..., \sigma)|^p \, d\sigma \right) \, dx.
\]

The following result is the analogous of [3, Lemma 9] in our setting.

**Lemma 3.3.** Fix \( m \in \mathbb{N} \) and \( 1 < p < \infty \). If \( f \in C^{m+1}_{c}(\mathbb{R}^N) \) then
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R^m f(x,y)|^p}{\|x-y\|^m_K} \rho_\varepsilon(\|x-y\|_K) \, dx \, dy = \frac{(N + mp)}{m^{mp}} \int_{\mathbb{R}^N} \int_K |D^m f(x)(y, ..., y)|^p \, dy \, dx.
\]
(3.10)

**Proof.** Let \( S = \| f \|_{W^{m+1,\infty}} \). Since \( t \to |t|^p \) is uniformly continuous in \([0,(m+1)S] \) then for any \( \delta > 0 \) there exists \( C = C(\delta) > 0 \) such that
\[
\|s|^p - |t|^p \leq C|s-t| + \delta \quad \forall s, t \in [0,(m+1)S].
\]
(3.11)

Using (3.11), (2.4) and proceeding as in [3] we get
\[
\left| |\Delta_h^m f(x)|^p |h|^{-mp} - |D^m f(x)\left(\frac{h}{\|h\|_K}, ..., \frac{h}{\|h\|_K}\right)|^p \right| 
\]
\[
\leq C\|h\|_K^{m-1} \left| |\Delta_h^m f(x) - D^m f(x)(h, ..., h)| + \delta \right|
\]
\[
\leq CA^{-m-1}S \|h\|_K + \delta,
\]
for every \( x \in \mathbb{R}^N \) and for all \( h \in \mathbb{R}^N \setminus \{0\} \). In particular, when \( h = \frac{y-x}{m} \), we get
\[
|m^{mp} |R^m f(x,y)|^p \|y-x\|^{-mp} - |D^m f(x)\left(\frac{y-x}{\|y-x\|_K}, ..., \frac{y-x}{\|y-x\|_K}\right)|^p | 
\]
\[
\leq CA^{-m-1}S \|y-x\|_K + \delta.
\]
Let \( A(x, y) = \{(x, y) : \text{at least one of } \frac{m}{m_j} x + \frac{1}{m_j} y, j = 0, \ldots, m, \text{ is in } \text{supp}(f) \} \). Then we note that
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R^m f(x, y)|^p}{\|y - x\|^p_{\mathbb{K}}} \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) \, dxdy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R^m f(x, y)|^p}{\|y - x\|^p_{\mathbb{K}}} \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) \, dxdy.
\]

Using the above estimate, we get
\[
\int \int \frac{m^{mp}}{\|y - x\|^p_{\mathbb{K}}} |R^m f(x, y)|^p \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) \, dxdy \\
\leq \int \int D^m f(x) \left( \frac{y - x}{\|y - x\|_{\mathbb{K}}}, \ldots, \frac{y - x}{\|y - x\|_{\mathbb{K}}} \right) ^p \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) \, dxdy \\
+ CA^{-m-1} S \int \int \|y - x\|_{\mathbb{K}} \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) + \delta \int \int \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) \, dxdy
\]

Note that
\[
\int \int \|y - x\|_{\mathbb{K}} \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) = \int \|h\|_{\mathbb{K}} \rho_\varepsilon(\|h\|_{\mathbb{K}}) \int A(x, x+h) \, dxdh \\
\leq \int \|h\|_{\mathbb{K}} \rho_\varepsilon(\|h\|_{\mathbb{K}}) (m + 1) |\text{supp}(f)| \, dh \\
\to 0 \text{ as } \varepsilon \to 0.
\]

Similarly,
\[
\delta \int \int \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) \, dxdy \leq (m + 1) |\text{supp}(f)| \delta
\]

On the other hand,
\[
\int \int \frac{m^{mp}}{\|y - x\|^p_{\mathbb{K}}} |R^m f(x, y)|^p \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) \, dxdy \\
\lesssim \|f\|_p \int \rho_\varepsilon(\|h\|) \, dh \to 0 \text{ as } \varepsilon \to 0.
\]

Hence, by sending \( \varepsilon \to 0 \) and then \( \delta \to 0 \), we can now conclude that
\[
\lim_{\varepsilon \to 0} \int \int \frac{|R^m f(x, y)|^p}{\|y - x\|^p_{\mathbb{K}}} \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) \, dxdy \\
\leq \frac{1}{m^{mp}} \lim_{\varepsilon \to 0} \int \int \frac{|D^m f(x) \left( \frac{y - x}{\|y - x\|_{\mathbb{K}}}, \ldots, \frac{y - x}{\|y - x\|_{\mathbb{K}}} \right)|^p}{\|y - x\|^p_{\mathbb{K}}} \rho_\varepsilon(\|y - x\|_{\mathbb{K}}) \, dxdy.
\]
We next compute the limit of the quantity on the right-hand side. We have

\[
\int_{\mathbb{R}^N} \left| D^m f(x) \left( \frac{x - y}{\|x - y\|_K}, \ldots, \frac{x - y}{\|x - y\|_K} \right) \right|^p \rho_\varepsilon(\|x - y\|_K) \, dy
\]

\[
= \int_{S^{N-1}} \int_0^\infty \left| D^m f(x)(\sigma, \ldots, \sigma) \right|^p \frac{1}{\|\sigma\|_K^{N-mp}} \rho_\varepsilon(r\|\sigma\|_K) r^{N-1} \, dr \, d\sigma
\]

\[
= \int_{S^{N-1}} \|\sigma\|_K^{N+mp} \left| D^m f(x)(\sigma, \ldots, \sigma) \right|^p \, d\sigma
\]

\[
= (N + mp) \int_K \left| D^m f(x)(y, \ldots, y) \right|^p \, dy,
\]

where the last equality follows from (3.9). Hence, we have that,

\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R^m f(x, y)|^p}{\|y - x\|_K^{mp}} \rho_\varepsilon(\|y - x\|_K) \, dxdy \leq \frac{N + mp}{m^{mp}} \int_{\mathbb{R}^N} \int_K \left| D^m f(x)(y, \ldots, y) \right|^p \, dydx.
\]

Now assume that \(\text{supp}(f) \subset B_R\), then

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \frac{D^m f(x)(\sigma, \ldots, \sigma)}{\|\sigma\|_K^{N-mp}} \right|^p \rho_\varepsilon(\|y - x\|_K) \, dxdy
\]

\[
= \int_{B_{R} \mathbb{R}^N} \int_{\{|y-x| \leq 1\}} \left| D^m f(x) \left( \frac{y - x}{\|y - x\|_K}, \ldots, \frac{y - x}{\|y - x\|_K} \right) \right|^p \rho_\varepsilon(\|y - x\|_K) \, dxdy
\]

\[
\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{m^{mp} \left| R^m f(x, y) \right|^p}{\|y - x\|_K^{mp}} \rho_\varepsilon(\|y - x\|_K) \, dxdy
\]

\[
+ \int_{B_{R} \mathbb{R}^N} \int_{\{|y-x| \leq 1\}} C_\delta S \|y - x\|_K \rho_\varepsilon(\|y - x\|_K) \, dydx + \int_{B_{R} \mathbb{R}^N} \int_{\{|y-x| \leq 1\}} \rho_\varepsilon(\|y - x\|_K) \, dxdy
\]

As above

\[
\int_{B_{R} \mathbb{R}^N} \int_{\{|y-x| \leq 1\}} \|y - x\|_K \rho_\varepsilon(\|y - x\|_K) \, dydx \to 0 \text{ as } \varepsilon \to 0,
\]

\[
\int_{B_{R} \mathbb{R}^N} \int_{\{|y-x| \leq 1\}} \rho_\varepsilon(\|y - x\|_K) \, dxdy \lesssim \delta.
\]
We also note that

\[
\int \int_{B_{R|\mathbb{R}^N}} \left| D^m f(x) \left( \frac{y - x}{\|y - x\|_K}, \ldots, \frac{y - x}{\|y - x\|_K} \right) \right|^p \rho_\varepsilon (\|y - x\|_K) \, dy \, dx
\]

\[
= \int \int_{B_{R|\mathbb{S}^{N-1}}} \int_0^{\sigma} \left| D^m f(x) (\sigma, \ldots, \sigma) \right|^p \rho_\varepsilon (s) \left( \frac{s}{\|\sigma\|_K} \right)^{N-1} \frac{1}{\|\sigma\|_K} \, ds \, d\sigma
\]

\[
\geq \int \int_{B_{R|\mathbb{S}^{N-1}}} \int_0^A \left| D^m f(x) (\sigma, \ldots, \sigma) \right|^p \frac{1}{\|\sigma\|_K^{N+mp}} \rho_\varepsilon (s) s^{N-1} \, ds \, d\sigma
\]

\[
\to \int \int_{B_{R|\mathbb{S}^{N-1}}} \left| D^m f(x) (\sigma, \ldots, \sigma) \right|^p \frac{1}{\|\sigma\|_K^{N+mp}} \, d\sigma
\]

\[
= (N + mp) \int \int_{\mathbb{R}^N} \left| D^m f(x) (y, \ldots, y) \right|^p \, dy \, dx \text{ as } \varepsilon \to 0.
\]

Thus,

\[
\frac{N + mp}{m^{mp}} \int \int_{\mathbb{R}^N} \left| D^m f(x)(y, \ldots, y) \right|^p \, dy \, dx \leq \lim_{\varepsilon \to 0} \frac{1}{R} \int \int_{\mathbb{R}^N} \left| R^m f(x, y) \right|^p \rho_\varepsilon (\|y - x\|_K) \, dx \, dy.
\]

Hence, we now can conclude that

\[
\lim_{\varepsilon \to 0} \frac{1}{R} \int \int_{\mathbb{R}^N} \left| R^m f(x, y) \right|^p \rho_\varepsilon (\|y - x\|_K) \, dx \, dy = \frac{N + mp}{m^{mp}} \int \int_{\mathbb{R}^N} \left| D^m f(x)(y, \ldots, y) \right|^p \, dy \, dx.
\]

\[\square\]

**Proof of Theorem 1.2.** So now we consider \( f \in W^{m,p} (\mathbb{R}^N) \) and let \( f_n \in C^\infty_c (\mathbb{R}^N) \) such that \( f_n \to f \) in the \( W^{m,p} (\mathbb{R}^N) \) norm. Then one has

\[
\left( \int \int_{\mathbb{R}^N} \int \int_{\mathbb{R}^N} \left| R^m f(x, y) \right|^p \|x - y\|_K^{-mp} \rho_\varepsilon (\|x - y\|_K) \, dx \, dy \right)^{\frac{1}{p}}
\]

\[
- \left( \int \int_{\mathbb{R}^N} \int \int_{\mathbb{R}^N} \left| R^m f_n(x, y) \right|^p \|x - y\|_K^{-mp} \rho_\varepsilon (\|x - y\|_K) \, dx \, dy \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int \int_{\mathbb{R}^N} \int \int_{\mathbb{R}^N} \left| R^m (f - f_n)(x, y) \right|^p \|x - y\|_K^{-mp} \rho_\varepsilon (\|x - y\|_K) \, dx \, dy \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int \int_{\mathbb{R}^N} \left( \int \int_{\mathbb{S}^{N-1}} \left| D^m (f_n - f)(x)(\sigma, \ldots, \sigma) \right|^p \, d\sigma \right) \, dx \right)^{\frac{1}{p}}.
\]

Where we used Lemma 3.2 in the last inequality. Thus

\[
\left( \int \int_{\mathbb{R}^N} \int \int_{\mathbb{R}^N} \left| R^m f(x, y) \right|^p \|x - y\|_K^{-mp} \rho_\varepsilon (\|x - y\|_K) \, dx \, dy \right)^{\frac{1}{p}} =
\]

\[
\left( \int \int_{\mathbb{R}^N} \int \int_{\mathbb{R}^N} \left| R^m f_n(x, y) \right|^p \|x - y\|_K^{-mp} \rho_\varepsilon (\|x - y\|_K) \, dx \, dy \right)^{\frac{1}{p}} + o(1)
\]
here \( o(1) \to 0 \) as \( n \to \infty \) uniformly on \( \varepsilon \). So fix \( \varepsilon > 0 \), then there exists \( n_0 \) big enough so that for \( n \geq n_0 \), we have \( \| f - f_n \|_{W^{m,p}} < \varepsilon \) and

\[
\left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |R^m_f(x, y)|^p \|x - y\|^{-mp} \rho \varepsilon (\|x - y\|) \, dy \, dx \right)^{\frac{1}{p}}
\]

\[
- \left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |R^m f_n(x, y)|^p \|x - y\|^{-mp} \rho \varepsilon (\|x - y\|) \, dy \, dx \right)^{\frac{1}{p}} < \varepsilon.
\]

Then we have

\[
\lim_{\varepsilon \to 0} \left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |R^m f(x, y)|^p \|x - y\|^{-mp} \rho \varepsilon (\|x - y\|) \, dy \, dx \right)^{\frac{1}{p}}
\]

\[
- \left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |D^m f(x) (y, \cdots, y)|^p \, dy \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \lim_{\varepsilon \to 0} \left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |R^m f(x, y)|^p \|x - y\|^{-mp} \rho \varepsilon (\|x - y\|) \, dy \, dx \right)^{\frac{1}{p}}
\]

\[
- \left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |R^m f_n(x, y)|^p \|x - y\|^{-mp} \rho \varepsilon (\|x - y\|) \, dy \, dx \right)^{\frac{1}{p}}
\]

\[
+ \left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |D^m f_n(x) (y, \cdots, y)|^p \, dy \, dx \right)^{\frac{1}{p}}.
\]

\[
\leq \lim_{\varepsilon \to 0} \left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |R^m f_n(x, y)|^p \|x - y\|^{-mp} \rho \varepsilon (\|x - y\|) \, dy \, dx \right)^{\frac{1}{p}}
\]

\[
- \left( \frac{1}{2} \right) \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |D^m f_n(x) (y, \cdots, y)|^p \, dy \, dx \right)^{\frac{1}{p}} + 2\varepsilon.
\]

So the conclusion follows from Theorem 3.3.

\[
\square
\]

4. CHARACTERIZATIONS OF THE HIGHER ORDER SOBOLEV SPACES VIA THE TAYLOR REMAINDER

We recall that

\[
T^{m-1}_y f(x) = \sum_{|\alpha| \leq m-1} D^\alpha f(y) \frac{(x - y)^\alpha}{\alpha!}
\]

and

\[
R_{m-1} f (x, y) = f (x) - T^{m-1}_y f (x).
\]

Proceeding as in [13] and by an easy induction we get

\[
R_{m-1} f (x, x + he_N) = h^m \int_{[0,1]^m} \partial^m_{x_N} f (x', x_N + \sum_{i=1}^{m} t_i h) \prod_{i=1}^{m} t_i^{m-i} \, dt_1 \cdots dt_m
\]

(4.1)
4.1. Nguyen’s formula.

**Lemma 4.1.** Let \( f \in W^{m,p}(\mathbb{R}^N), 1 < p < \infty \). Then there exists a constant \( C = C(m, N, p) > 0 \) such that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+mp}_K} dxdy \leq C \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx, \forall \delta > 0.
\]

**Proof.** By (2.4), we have

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+mp}_K} dxdy \leq \frac{1}{A N + mp} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+mp}_K} dxdy.
\]

Hence we now will show that there exists \( C = C(m, N, p) > 0 \) such that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+mp}_K} dxdy \leq C \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx.
\]

We note that

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{|x-y|^{N+mp}_K} dxdy = \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^N} \int_{0}^{\infty} \frac{\delta^p}{h^{mp+1}} dh dx d\sigma.
\]

Hence it is enough to prove that for every \( \sigma \in \mathbb{S}^{N-1} \):

\[
\int_{\mathbb{R}^N} \int_{0}^{\infty} \frac{\delta^p}{h^{mp+1}} dh dx \leq C \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx.
\]

Moreover, we can assume without loss of generality that \( \sigma = e_N \). In this case, by direct calculation, we have

\[
|R_{m-1} f(x, x + he_N)| = \left| h^m \int_0^1 \cdots \int_0^1 \partial_{x_N}^m f(x', x_N + s_m s_{m-1} \cdots s_1 h) s_1^{m-1} s_2^{m-2} \cdots s_{m-2} s_{m-1} ds_m \cdots ds_1 \right| \\
\leq h^m \int_0^1 \cdots \int_0^1 M_N \left( \partial_{x_N}^m f \right)(x) s_1^{m-1} s_2^{m-2} \cdots s_{m-2} s_{m-1} ds_m \cdots ds_1 \\
\leq \frac{1}{m!} h^m M_N \left( \partial_{x_N}^m f \right)(x).
\]

Hence

\[
\int_{\mathbb{R}^N} \int_0^{\infty} \frac{\delta^p}{h^{mp+1}} dh dx \leq \frac{1}{m!} h^m M_N \left( \partial_{x_N}^m f \right)(x) \\
= \frac{1}{(m!)^p m} \int_{\mathbb{R}^N} \left| M_N \left( \partial_{x_N}^m f \right)(x) \right|^p dx \\
\leq C \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx.
\]
We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3:** Using a change of variables (writing \( y = x + \sqrt{\delta} h \sigma, \sigma = \frac{y - x}{||y - x||} \)), we obtain

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\delta^p}{\|x - y\|^{|N+mp|}_K} \, dx \, dy = \int_{S^{N-1}} \int_{\mathbb{R}^N} \int_{0}^{\infty} \frac{1}{\|\sigma\|^{|N+mp|}_K} \, h^{mp+1} \, dh \, dx \, ds.
\]

Again we define the auxiliary function \( F_\delta: S^{N-1} \rightarrow \mathbb{R} \) by

\[
F_\delta(\sigma) := \frac{1}{\|\sigma\|^{|N+mp|}_K} \int_{\mathbb{R}^N} \int_{0}^{\infty} \frac{1}{h^{mp+1}} \, dh \, dx.
\]

We first prove that for all \( \sigma \in S^{N-1}, \forall \delta > 0 \)

\[
F_\delta(\sigma) \leq \frac{1}{A^{N+mp}} C(m, N, p) \int_{\mathbb{R}^N} |\nabla^m f(x)|^p \, dx.
\]  

(4.2)

Since

\[
\frac{1}{\|\sigma\|^{|N+mp|}_K} \leq \frac{1}{A^{N+mp}},
\]

it is enough to show that

\[
\int_{\mathbb{R}^N} \int_{0}^{\infty} \frac{1}{h^{mp+1}} \, dh \, dx \leq C(m, N, p) \int_{\mathbb{R}^N} |\nabla^m f(x)|^p \, dx.
\]

Indeed, without loss of generality, we assume that \( \sigma = e_N = (0, ..., 0, 1) \). Hence, we need to verify that

\[
\int_{\mathbb{R}^N} \int_{0}^{\infty} \frac{1}{h^{mp+1}} \, dh \, dx \leq C(m, N, p) \int_{\mathbb{R}^N} |\nabla^m f(x)|^p \, dx.
\]  

(4.3)

But one has

\[
|\frac{R_{m-1} f(x, x + \sqrt{\delta} h e_N)}{h^{m} \delta}| \leq \frac{1}{m!} M_N \left( \partial_{xN}^m f \right)(x).
\]

By (4.1) we get

\[
\int_{\mathbb{R}^N} \int_{0}^{\infty} \frac{1}{h^{mp+1}} \, dh \, dx \leq \int_{\mathbb{R}^N} \int_{0}^{\infty} \frac{1}{h^{mp+1}} \, dh \, dx \leq \frac{(m!)^p}{mp} \int_{\mathbb{R}^N} \left| M_N \left( \partial_{xN}^m f \right)(x) \right|^p \, dx
\]

\[
\leq C(m, N, p) \int_{\mathbb{R}^N} |\nabla^m f(x)|^p \, dx.
\]
Leading to

\[ F_\delta(\sigma) \leq \frac{1}{A^{N+mp}C(m, N, p)} \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx. \]

Next we will show that

\[ F_\delta(\sigma) \rightarrow \frac{1}{(m!)^p m p \|\sigma\|^{N+mp}K} \int_{\mathbb{R}^N} |D^m f(x)(\sigma, \ldots, \sigma)|^p dx \quad \text{as} \quad \delta \rightarrow 0 \quad \text{for every} \quad \sigma \in \mathbb{S}^{N-1}. \quad (4.4) \]

Again, it is enough to show

\[ \int_{\mathbb{R}^N} \int_0^\infty G_\delta(x, h) dh dx \rightarrow \frac{1}{(m!)^p m p} \int_{\mathbb{R}^N} |D^m f(x)(\sigma, \ldots, \sigma)|^p dx \quad \text{as} \quad \delta \rightarrow 0, \]

where

\[ G_\delta(x, h) := \frac{1}{h^{mp+1}} \{ \int_{|D^m f(x)(\sigma, \ldots, \sigma)| h^m > 1} \} (x, h). \]

With loss of generality, we suppose that \( \sigma = e_N = (0, \ldots, 0, 1) \). Noting that for all \( \sigma \in \mathbb{S}^{N-1} \): \( G_\delta(x, h) \rightarrow \frac{1}{h^{mp+1}} \chi_{\{h^m M_N(\partial^m_N f) > m!\}} (x, h) \in L^1(\mathbb{R}^N \times [0, \infty)) \).

Hence, by the Lebesgue dominated convergence theorem, we get (4.4). Once again, by the Lebesgue dominated convergence theorem, we can conclude that

\[
\lim_{\delta \to 0} \int_{\mathbb{R}^N} \int_{|D^m f(x,y)| > \delta} \frac{\delta^p}{\|y\|^N_{N+mp}} dxdy = \int_{\mathbb{S}^{N-1}} \frac{1}{(m!)^p m p \|\sigma\|^{N+mp}K} \int_{\mathbb{R}^N} |D^m f(x)(\sigma, \ldots, \sigma)|^p dx d\sigma = \frac{1}{(m!)^p m p} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} \frac{1}{\|\sigma\|^{N+mp}K} |D^m f(x)(\sigma, \ldots, \sigma)|^p d\sigma dx = \frac{N + mp}{(m!)^p m p} \int_{\mathbb{R}^N} \int_{\mathbb{S}^{N-1}} |D^m f(x)(y, \ldots, y)|^p dy dx
\]

\[ \square \]

4.2. BBM formula. Now we will focus on the proof of Theorem 1.4. We note that the mollifiers \( \rho \in L^1_{\text{loc}}(0, \infty) \) are a family of functions such that \( \rho \geq 0 \),

\[ \int_0^\infty \rho_\varepsilon(r) r^{-N+\varepsilon} dr = 1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_0^\infty \rho_\varepsilon(r) r^{-N+\varepsilon} dr = 0 \quad \text{for all} \quad \delta > 0. \]

Lemma 4.2. Let \( f \in W^{m,p}(\mathbb{R}^N) \), \( 1 < p < \infty \). Then there exists a constant \( C = C(m, N, p) > 0 \) such that

\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f(x, y)|^p}{\|x - y\|^N_{N+mp}K} \rho_\varepsilon(\|x - y\|_K) dx dy \leq C \int_{\mathbb{R}^N} |\nabla^m f(x)|^p dx, \forall \varepsilon > 0. \]
Proof. By density we can assume that $f \in C^\infty_c(\mathbb{R}^N)$. We have

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x, y)|^p}{\|x - y\|_K^{mp}} \rho_\varepsilon (\|x - y\|_K) \, dx \, dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x + h, x)|^p}{\|h\|_K^{mp}} \rho_\varepsilon (\|h\|_K) \, dx \, dh
$$

$$
= \int_{\mathbb{R}^N} \rho_\varepsilon (\|h\|_K) \int_{\mathbb{R}^N} |R_{m-1} f (x + h, x)|^p \, dx \, dh.
$$

Since by [2, (2.17)] there exists $C = C(m, N, p) > 0$ such that

$$
\int_{\mathbb{R}^N} |R_{m-1} f (x + h, x)|^p \, dx \leq C |h|^{mp} \int_{\mathbb{R}^N} |x|^{mp} \, dx,
$$

we have

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x, y)|^p}{\|x - y\|_K^{mp}} \rho_\varepsilon (\|x - y\|_K) \, dx \, dy \leq \frac{1}{A^{mp}} C \int_{\mathbb{R}^N} \rho_\varepsilon (\|h\|_K) \, dh \int_{\mathbb{R}^N} |\nabla^m f (x)|^p \, dx.
$$

But

$$
\int_{\mathbb{R}^N} \rho_\varepsilon (\|h\|_K) \, dh = \int_{S^{N-1}} \int_0^\infty \rho_\varepsilon (r \|\sigma\|_K) r^{N-1} \, dr \, d\sigma
$$

$$
= \int_{S^{N-1}} \int_0^\infty \rho_\varepsilon (s) \left( \frac{s}{\|\sigma\|_K} \right)^{N-1} \frac{1}{\|\sigma\|_K} \, ds \, d\sigma
$$

$$
\leq \frac{1}{A^{mp}}.
$$

□

Proof of Theorem 1.4: First, we assume that $f \in C^\infty_c(\mathbb{R}^N)$. We have

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x, y)|^p}{\|x - y\|_K^{mp}} \rho_\varepsilon (\|x - y\|_K) \, dx \, dy = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x + h, x)|^p}{\|h\|_K^{mp}} \rho_\varepsilon (\|h\|_K) \, dx \, dh
$$

By Taylor’s formula, we have that for every $\delta > 0$, there exists $C_\delta > 0$ such that

$$
|R_{m-1} f (x + h, x)|^p \leq (1 + \delta) \frac{1}{(m!)^p} |D^m f (x)(h, ..., h)|^p + C_\delta |h|^{(m+1)p}.
$$
Hence
\[ \int_{|h| \leq 1} |h|^{(m+1)p} \frac{\rho_\varepsilon (\|h\|_K)}{\|h\|_{mp}^p} \, dh \]
\[ \leq \frac{1}{(m!)^p} (1 + \delta) \int_{|h| \leq 1} |D^m f(x)(h, \ldots, h)|^p \frac{\rho_\varepsilon (\|h\|_K)}{\|h\|_{mp}^p} \, dh \]
\[ + C_\delta \int_{|h| \leq 1} |h|^{(m+1)p} \frac{\rho_\varepsilon (\|h\|_K)}{\|h\|_{mp}^p} \, dh \]
\[ = \frac{1}{(m!)^p} (1 + \delta) \int_{|h| \leq 1} \bigg| D^m f(x) \bigg|^{(m+1)p} \frac{\rho_\varepsilon (\|h\|_K)}{\|h\|_{mp}^p} \, dh \]
\[ + 2C_\delta |\text{supp } (f)| \int_{|h| \leq 1} |h|^{(m+1)p} \frac{\rho_\varepsilon (\|h\|_K)}{\|h\|_{mp}^p} \, dh. \]

We first note that
\[ \int_{|h| \leq 1} |h|^{(m+1)p} \frac{\rho_\varepsilon (\|h\|_K)}{\|h\|_{mp}^p} \, dh \lesssim \int_{|h| \leq 1} \|h\|_K^p \, dh \]
\[ = \int_{S^{N-1}} \int_0^1 r^p \|\sigma\|_K^p \rho_\varepsilon (r \|\sigma\|_K) \, r^{N-1} \, dr \, d\sigma \]
\[ = \int_{S^{N-1}} \int_0^1 s^p \rho_\varepsilon (s) \left( \frac{s}{\|\sigma\|_K} \right)^{N-1} \, ds \, d\sigma \]
\[ \sim \int_0^1 s^p \rho_\varepsilon (s) \, s^{N-1} \, ds \to 0 \text{ as } \varepsilon \to 0. \]

On the other hand,
\[ \frac{1}{(m!)^p} \int_{|h| \leq 1} \bigg| D^m f(x) \bigg|^{1+1p} \frac{\rho_\varepsilon (\|h\|_K)}{\|h\|_{mp}^p} \, dh \]
\[ = \frac{1}{(m!)^p} \int_{|h| \leq 1} \bigg| D^m f(x)(\sigma, \ldots, \sigma) \bigg|^p \frac{\rho_\varepsilon (r \|\sigma\|_K)}{\|\sigma\|_{mp}^p} \, r^{N-1} \, dr \, d\sigma \]
\[ = \frac{1}{(m!)^p} \int_{S^{N-1}} \int_0^\infty \bigg| D^m f(x)(\sigma, \ldots, \sigma) \bigg|^p \frac{\rho_\varepsilon (s)}{\|\sigma\|_{mp}^p} \, \left( \frac{s}{\|\sigma\|_K} \right)^{N-1} \frac{1}{\|\sigma\|_K} \, ds \, d\sigma \]
\[ = \frac{1}{(m!)^p} \int_{S^{N-1}} \int_0^\infty \bigg| D^m f(x)(\sigma, \ldots, \sigma) \bigg|^p \frac{1}{\|\sigma\|_{K}^{N+mp}} \, d\sigma \, dx \]
\[ = \frac{N + mp}{(m!)^p} \int_K \bigg| D^m f(x)(y, \ldots, y) \bigg|^p \, dy \, dx. \]
Also, since \( f \in C^\infty_c (\mathbb{R}^N) \), we have
\[
\int_{\mathbb{R}^N} \int_{|h| \geq 1} \frac{|R_{m-1} f(x + h, x)|^p}{\|h\|_{K}^{mp}} \rho_\varepsilon(\|h\|_K) \, dx \, dh \\
\lesssim \int_{\mathbb{R}^N} \rho_\varepsilon(\|h\|_K) \, dh \\
\lesssim \int_{1}^{\infty} \rho_\varepsilon(s) s^{N-1} ds \to 0 \text{ as } \varepsilon \to 0.
\]
Hence, letting \( \varepsilon \to 0 \) and \( \delta \to 0 \), we get
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f(x, y)|^p}{\|x - y\|_{K}^{mp}} \rho_\varepsilon(\|x - y\|_K) \, dxdy \leq \frac{N + mp}{(m!)^p} \int_{\mathbb{R}^N} \int_{K} |D^m f(x, y, \ldots, y)|^p \, dydx.
\]
On the other hand, on any compact set \( B \) and \( |h| \leq 1 \) we have
\[
\frac{1}{(m!)^p} |D^m f(x)(h, \ldots, h)|^p \leq (1 + \delta) |R_{m-1} f(x + h, x)|^p + C_{\delta, B} |h|^{(m+1)p}.
\]
Hence,
\[
\frac{1}{(m!)^p} \int_{B} \int_{|h| \leq 1} \frac{|D^m f(x)(h, \ldots, h)|^p}{\|h\|_{K}^{mp}} \rho_\varepsilon(\|h\|_K) \, dx \, dh \\
\leq (1 + \delta) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f(x + h, x)|^p}{\|h\|_{K}^{mp}} \rho_\varepsilon(\|h\|_K) \, dx \, dh + C_{\delta, B} \int_{B} \int_{|h| \leq 1} |h|^{(m+1)p} \rho_\varepsilon(\|h\|_K) \frac{1}{\|h\|_{K}^{mp}} \, dx \, dh.
\]
But one has
\[
\frac{1}{(m!)^p} \int_{B} \int_{|h| \leq 1} \frac{|D^m f(x)(h, \ldots, h)|^p}{\|h\|_{K}^{mp}} \rho_\varepsilon(\|h\|_K) \, dx \, dh \\
= \frac{1}{(m!)^p} \int_{B} \int_{S^{N-1}} \int_{0}^{1} |D^m f(x)(\sigma, \ldots, \sigma)|^p \rho_\varepsilon(r \|\sigma\|_K) \frac{1}{\|\sigma\|_{K}^{mp}} \, d\sigma \, dr \, dxd \\
= \frac{1}{(m!)^p} \int_{B} \int_{S^{N-1}} \int_{0}^{1} |D^m f(x)(\sigma, \ldots, \sigma)|^p \rho_\varepsilon(s \|\sigma\|_K) \frac{1}{\|\sigma\|_{K}^{mp}} \left( \frac{s}{\|\sigma\|_K} \right)^{N-1} \frac{1}{\|\sigma\|_K} \, ds \, d\sigma \, dxd \\
\geq \frac{1}{(m!)^p} \int_{B} \int_{S^{N-1}} |D^m f(x)(\sigma, \ldots, \sigma)|^p \frac{1}{\|\sigma\|_{K}^{N+mp}} \int_{0}^{A} \rho_\varepsilon(s) s^{N-1} ds \, d\sigma \, dxd \\
\to \frac{1}{(m!)^p} \int_{B} \int_{S^{N-1}} |D^m f(x)(\sigma, \ldots, \sigma)|^p \frac{1}{\|\sigma\|_{K}^{N+mp}} \, d\sigma \, dxd \text{ as } \varepsilon \to 0.
\]
By letting \( \varepsilon \to 0 \), and then \( \delta \to 0 \), we get
\[
\frac{1}{(m!)^p} \int_{B} \int_{S^{N-1}} |D^m f(x)(\sigma, \ldots, \sigma)|^p \frac{1}{\|\sigma\|_{K}^{N+mp}} \, d\sigma \\
\leq \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f(x + h, x)|^p}{\|h\|_{K}^{mp}} \rho_\varepsilon(\|h\|_K) \, dx \, dh.
\]
Hence,
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f(x + h, x)|^p}{\|h\|_{K}^{mp}} \rho_\varepsilon(\|h\|_K) \, dx \, dh = \frac{N + mp}{(m!)^p} \int_{\mathbb{R}^N} \int_{K} |D^m f(x)(y, \ldots, y)|^p \, dydx.
\]
In the general case \( f \in W^{m,p}(\mathbb{R}^N) \), we fix \( \tau > 0 \) then there exists \( C(\tau) > 1 \) such that for all \( a, b \in \mathbb{R} \):
\[
|a|^p \leq (1 + \tau)|b|^p + C(\tau)|a - b|^p.
\]
Now, by density, we can choose \( g \in C^\infty_c(\mathbb{R}^N) \) such that
\[
\int_{\mathbb{R}^N} |\nabla^m (f - g)(x)|^p dx \leq \frac{\tau}{C(\tau)}
\]
and
\[
\left| \int_{\mathbb{R}^N} \int_K |D^m g(x)(y, \ldots, y)|^p dy dx - \int_{\mathbb{R}^N} \int_K |D^m f(x)(y, \ldots, y)|^p dy dx \right| \leq \tau.
\]
Then
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x + h, x)|^p}{\|h\|^m_K} \rho_\varepsilon (\|h\|_K) dxdh
\]
\[
\leq (1 + \tau) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} g (x + h, x)|^p}{\|h\|^m_K} \rho_\varepsilon (\|h\|_K) dxdh
\]
\[
+ C(\tau) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} (f - g) (x + h, x)|^p}{\|h\|^m_K} \rho_\varepsilon (\|h\|_K) dxdh
\]
\[
\leq (1 + \tau) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} g (x + h, x)|^p}{\|h\|^m_K} \rho_\varepsilon (\|h\|_K) dxdh
\]
\[
+ C(\tau) C(m, N, p) \int_{\mathbb{R}^N} |\nabla^m (f - g)(x)|^p dx.
\]
Letting \( \varepsilon \to 0 \), we obtain
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x + h, x)|^p}{\|h\|^m_K} \rho_\varepsilon (\|h\|_K) dxdh
\]
\[
\leq (1 + \tau) \frac{N + mp}{(m!)^p} \int_{\mathbb{R}^N} \int_K |D^m g(x)(y, \ldots, y)|^p dy dx + C(m, N, p) \tau
\]
\[
\leq (1 + \tau) \frac{N + mp}{(m!)^p} \left[ \int_{\mathbb{R}^N} \int_K |D^m f(x)(y, \ldots, y)|^p dy dx + \tau \right] + C(m, N, p) \tau.
\]
Since \( \tau \) can be chosen arbitrarily, we deduce that
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x + h, x)|^p}{\|h\|^m_K} \rho_\varepsilon (\|h\|_K) dxdh
\]
\[
\leq \frac{N + mp}{(m!)^p} \int_{\mathbb{R}^N} \int_K |D^m f(x)(y, \ldots, y)|^p dy dx.
\]
Also, if we switch the role of \( f \) and \( g \) in the above argument, then we get
\[
\frac{N + mp}{(m!)^p} \int_{\mathbb{R}^N} \int_K |D^m f(x)(y, \ldots, y)|^p dy dx
\]
\[
\leq \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x + h, x)|^p}{\|h\|^m_K} \rho_\varepsilon (\|h\|_K) dxdh.
\]
Hence,
\[
\lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|R_{m-1} f (x + h, x)|^p}{\|h\|^m_K} \rho_\varepsilon (\|h\|_K) dxdh
\]
\[
= \frac{N + mp}{(m!)^p} \int_{\mathbb{R}^N} \int_K |D^m f(x)(y, \ldots, y)|^p dy dx.
\]
\( \square \)
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