A SCHWARZ-PICK LEMMA FOR MINIMAL MAPS

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Abstract. In this note, we prove a Schwarz-Pick type lemma for minimal maps between negatively curved Riemannian surfaces. More precisely, we prove that if $f : M \to N$ is a minimal map with bounded Jacobian between two complete negatively curved Riemann surfaces $M$ and $N$ whose sectional curvatures $\sigma_M$ and $\sigma_N$ satisfy $\inf \sigma_M \geq \sup \sigma_N$, then $f$ is area decreasing.

1. Introduction

According to the Schwarz-Pick Lemma, any non-linear holomorphic map $f : \mathbb{D} \to \mathbb{D}$ from the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ to itself must be strictly distance decreasing if we endow $\mathbb{D}$ with the Poincaré metric. Ahlfors [1] exposed in his generalization of the Schwarz-Pick Lemma the essential role played by the curvature. In the matter of fact, Ahlfors generalized this lemma to holomorphic mappings between two Riemann surfaces where the curvature of the domain manifold is bigger than the curvature of the target. Ahlfors’ result was extended by Yau [24]. Yau showed that if $M$ is a complete Kähler manifold with Ricci curvature bounded from below by a constant and $N$ is another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant, then any holomorphic mapping from $M$ into $N$ decreases distances up to a constant depending only on the curvatures of $M$ and $N$. In his proof, Yau exploited a maximum principle at infinity for bounded functions on complete non-compact Riemannian manifolds with Ricci curvature bounded from below; see for more details [23].

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In this paper, we investigate minimal maps between complete Riemann surfaces. According to the terminology introduced by Schoen [15], a smooth map $f : (M, g_M) \to (N, g_N)$ is called minimal, if its graph

$$\Gamma(f) := \{(x, f(x)) \in M \times N : x \in M\}$$

is a minimal surface of the Riemannian product $(M \times N, g_M \times g_N)$.

There are two important categories of minimal maps between Riemann surfaces. The first class contains the holomorphic and anti-holomorphic maps and the second one the minimal symplectic maps. Eells [7] proved that a holomorphic or anti-holomorphic map is automatically minimal. Notice that when both $M$ and $N$ are compact, then there are non-constant holomorphic maps between them only if the genus of $M$ is greater or equal than the genus of $N$. On the other hand, the graph of a minimal symplectic map is a minimal Lagrangian surface. Schoen [15] proved an existence and uniqueness result for minimal symplectic diffeomorphisms between hyperbolic surfaces. i.e., if $g_1$ and $g_2$ are hyperbolic metrics on a surface $M$, then there is a unique minimal map $f : (M, g_1) \to (M, g_2)$ homotopic to the identity map. If $M$ is a compact hyperbolic surface, due to results of Smoczyk [17] and Wang [22] the mean curvature flow deforms a symplectomorphism into a minimal Lagrangian map.

Let us mention here that Aiyama, Akutagawa and Wan [3] obtained a representation formula for a minimal diffeomorphism between two hyperbolic discs by means of the generalized Gauss map of a complete maximal surface in the anti-de Sitter 3-space.

In this paper, we prove a Schwarz-Pick type lemma for minimal maps between two negatively curved Riemann surfaces $(M, g_M)$ and $(N, g_N)$. Under natural assumptions, we show that a minimal map from $M$ to $N$ decreases two dimensional areas. This means that the absolute value $|J_f|$ of the Jacobian determinant $J_f := \det(df)$ of $f$, with respect to the Riemannian metrics $g_M$ and $g_N$, is less or equal than 1. Such maps are called area decreasing. In the case where the Jacobian $J_f$ is identically 1, the map $f$ is called area preserving.

**Theorem.** Let $f : (M, g_M) \to (N, g_N)$ be a minimal map, with bounded Jacobian determinant, between complete negatively curved Riemannian surfaces whose sectional curvatures $\sigma_M$ and $\sigma_N$ satisfy $\sigma_M \geq -\sigma \geq \sigma_N \geq -\beta$, where $\sigma$ is a positive constant. Then $f$ is area decreasing. If, additionally, there exists a point where $f$ is area preserving, then $\sigma_M = \sigma_N = -\sigma$ and the graph of $f$ is a minimal Lagrangian surface.
The author and Smoczyk proved in [13] that the mean curvature flow of maps between two compact hyperbolic Riemann surfaces preserves the graphical and the area decreasing property. Due to a result of Wan [20] minimal maps between complete Riemann surfaces satisfying the assumptions of our theorem are stable. Hence, in view of our result, the graphical mean curvature flow can be used to generate all minimal maps between compact hyperbolic Riemann surfaces.

Finally, let us mention that recently there were proved several Bernstein type results for minimal maps. In [8, 9] it is shown that a minimal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ with bounded Jacobian determinant must be affine linear. This result was recently generalized by Jost, Xin and Yang in [10]. However, such a result is not true without any assumption on the Jacobian determinant. For example, the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by
\[
f(x, y) = \frac{1}{2}(e^x - 3e^{-x})(\cos y/2, -\sin y/2)\]
is a minimal map whose Jacobian determinant takes every value in $\mathbb{R}$. Moreover, the graph of $f$ is not holomorphic with respect to any complex structure of $\mathbb{R}^4$. Furthermore, due to a result of Torralbo and Urbano [18], the graph of any minimal map $f : S^2 \to S^2$ must be holomorphic or anti-holomorphic.

2. Maximum principles at infinity

The root of the maximum principle relies on the following observation: Suppose that $u : M \to \mathbb{R}$ is a smooth function defined on a Riemannian manifold $M$ and assume that it attains at a point $x_0$ a local maximum. Then,
\[
|\nabla u|(x_0) = 0 \quad \text{and} \quad \Delta u(x_0) \leq 0.
\]
Such a point always exists in the case where the manifold is compact. However, the situation is different in the case of complete non-compact manifolds. To handle the non-compact case, Omori [11] proved the following criterion: Suppose that $M$ is a complete and non-compact Riemannian manifold, with sectional curvatures bounded from below by a constant, and $u : M \to \mathbb{R}$ is a bounded from above smooth function. Then, there exists a sequence of points $\{x_k\}_{k \in \mathbb{N}}$ such that
\[
u (x_k) \geq \sup u - 1/k, \quad |\nabla u|(x_k) \leq 1/k \quad \text{and} \quad \Delta u(x_k) \leq 1/k. \quad \text{(O-Y)}
\]
The conclusion (O-Y) is known as the Omori-Yau maximum principle. Yau [23] showed that the conclusion (O-Y) holds under the assumption Ricci curvature is bounded from below.
The result of Omori and Yau was generalized by Chen and Xin [5] to include cases where the Ricci curvature may decay at a certain rate. Later Pigola, Rigoli and Setti [12] realized that the validity of (O-Y) does not depend on curvature bounds as much as one would expect; for more details see [12, Theorem 1.9] and [2]. An important case where the Omori-Yau maximum principle can be successfully applied is for properly immersed submanifolds \( F: \Sigma \to L \) with bounded length of the mean curvature vector on a complete Riemannian manifold \( L \) with bounded sectional curvatures; see [12, Example 1.14].

3. Geometry of graphs

Let us briefly review some basic facts about the geometry of maps, following closely the presentation in [13, 14].

3.1. Notation. The graphical submanifold

\[ \Gamma(f) = \{(x, f(x)) \in M \times N : x \in M\} \]

generated by the smooth map \( f: M \to N \) between the Riemann surfaces \((M, g_M)\) and \((N, g_N)\) can be parametrized via the embedding \( F: M \to M \times N \) given by \( F = (I, f) \), where \( I: M \to M \) is the identity map. Let us denote by \( \pi_M \) and \( \pi_N \) the natural projection maps of \( M \times N \). Then, the Riemannian metric \( g_{M \times N} \) on \( M \times N \) is given by the formula

\[ g_{M \times N} = \pi_M^* g_M + \pi_N^* g_N. \]

We will denote by \( \tilde{R} \) the curvature operator of \( g_{M \times N} \). The induced Riemannian metric \( g \) on the graph \( \Gamma(f) \) of \( f \) is given by

\[ g = g_M + f^* g_N \]

and its Levi-Civita connection is denoted by \( \nabla \).

Around each point \( x \in \Gamma(f) \) we choose an adapted local orthonormal frame \( \{e_1, e_2; e_3, e_4\} \) such that \( \{e_1, e_2\} \) is tangent and \( \{e_3, e_4\} \) is normal to the graph. The components of the second fundamental \( A \) of the graph with respect to the adapted frame \( \{e_1, e_2; e_3, e_4\} \) are denoted as

\[ A_{ij}^\alpha := \langle A(e_i, e_j), e_\alpha \rangle. \]

Latin indices take values 1 and 2 while Greek indices take the values 3 and 4. For instance we write the mean curvature vector in the form

\[ H = H^3 e_3 + H^4 e_4. \]
From the Ricci equation we see that the curvature $\sigma_n$ of the normal bundle of $\Gamma(f)$ is given by the formula

$$\sigma_n := R^\perp_{1234} = \tilde{R}_{1234} + A^3_{11}A^4_{12} - A^3_{12}A^4_{11} + A^3_{12}A^4_{22} - A^3_{22}A^4_{12}.$$ 

The sum of the last four terms in the above formula is equal to minus the commutator $\sigma^\perp$ of the matrices $A_3 = (A^3_{ij})$ and $A_4 = (A^4_{ij})$, i.e.,

$$\sigma^\perp := \langle [A^3, A^4]e_1, e_2 \rangle = -A^3_{11}A^4_{12} + A^3_{12}A^4_{11} - A^3_{12}A^4_{22} + A^3_{22}A^4_{12}.$$ 

3.2. **Singular decomposition.** There is a natural way to diagonalize the differential $df$ of $f$. Indeed, let $\lambda^2 \leq \mu^2$ be the eigenvalues $f^*g_N$ with respect to $g_M$ at a fixed point $x \in M$. The corresponding values $0 \leq \lambda \leq \mu$ are called singular values of $f$ at $x$. Then there exists an orthonormal basis $\{\alpha_1, \alpha_2\}$ of $T_xM$, with respect to $g_M$, and $\{\beta_1, \beta_2\}$ of $T_{f(x)}N$, with respect to the metric $g_N$, such that

$$df(\alpha_1) = \lambda \beta_1 \quad \text{and} \quad df(\alpha_2) = \mu \beta_2.$$ 

Observe that the vectors

$$v_1 := \frac{\alpha_1}{\sqrt{1 + \lambda^2}} \quad \text{and} \quad v_2 := \frac{\alpha_2}{\sqrt{1 + \mu^2}}$$

are orthonormal with respect to the metric $g$ on the graph of $f$ at $x$. Hence,

$$e_1 := \frac{1}{\sqrt{1 + \lambda^2}} (\alpha_1 \oplus \lambda \beta_1) \quad \text{and} \quad e_2 := \frac{1}{\sqrt{1 + \mu^2}} (\alpha_2 \oplus \mu \beta_2)$$

form an orthonormal basis with respect to the metric $g_{M\times N}$ of the tangent space $dF(T_xM)$ of the graph $\Gamma(f)$ at $x$. Moreover,

$$e_3 := \frac{1}{\sqrt{1 + \lambda^2}} (-\lambda \alpha_1 \oplus \beta_1) \quad \text{and} \quad e_4 := \frac{1}{\sqrt{1 + \mu^2}} (-\mu \alpha_2 \oplus \beta_2)$$

form an orthonormal basis with respect to $g_{M\times N}$ of the normal space $N_xM$ of the graph $\Gamma(f)$ at the point $f(x)$.

3.3. **Jacobians of the projection maps.** Let $\omega_M$ denote the Kähler form of the Riemann surface $(M, g_M)$ and $\omega_N$ the Kähler form of $(N, g_N)$. Let us define the parallel forms

$$\omega_1 := \pi_M^* \omega_M \quad \text{and} \quad \omega_2 := \pi_N^* \omega_N.$$ 

Consider now two smooth functions $u_1$ and $u_2$ given by

$$u_1 := *(F^*\omega_1) = *\{ (\pi_M \circ F)^* \omega_M \} = *(f^* \omega_M)$$

and

$$u_2 := *(F^*\omega_2) = *\{ (\pi_N \circ F)^* \omega_N \} = *(f^* \omega_N)$$


where here $*$ stands for the Hodge star operator with respect to the metric $g$. Note that $u_1$ is the Jacobian of the projection map from $\Gamma(f)$ to the first factor of $M \times N$ and $u_2$ is the Jacobian of the projection map of $\Gamma(f)$ to the second factor of $M \times N$. With respect to the basis $\{e_1, e_2, e_3, e_4\}$ of the singular decomposition, we have

$$u_1 = \frac{1}{\sqrt{(1 + \lambda^2)(1 + \mu^2)}} \quad \text{and} \quad |u_2| = \frac{\lambda \mu}{\sqrt{(1 + \lambda^2)(1 + \mu^2)}}.$$

The Jacobian determinant $J_f$ of $f$, with respect to the metrics $g_M$ and $g_N$, is the function defined by the formula

$$f^* \omega_N = J_f \omega_M.$$

Therefore,

$$J_f = \frac{u_2}{u_1}.$$

The difference $u_1 - |u_2|$ measures how far the map $f$ is from being area preserving. In particular, we say that $f$ is area decreasing if $u_1 - |u_2| \geq 0$ and strictly area decreasing if $u_1 - |u_2| > 0$. If $u_1 - |u_2| \equiv 0$, then $f$ is called area preserving. Clearly, in the latter situation $f$ is symplectic.

3.4. The Kähler angles. There are two natural complex structures associated to the product space $(M \times N, g_{M \times N})$, i.e.,

$$J_1 := \pi^*_M J_M - \pi^*_N J_N \quad \text{and} \quad J_2 := \pi^*_M J_M + \pi^*_N J_N,$$

where $J_M$ and $J_N$ are the complex structures on $M$ and $N$ defined by

$$\omega_M(\cdot, \cdot) = g_M(J_M \cdot, \cdot) \quad \text{and} \quad \omega_N(\cdot, \cdot) = g_N(J_N \cdot, \cdot).$$

Chern and Wolfson in [6] introduced a function which measures the deviation of $dF(T_xM)$ from a complex line of the space $T_{F(x)}(M \times N)$. More precisely, if we consider $(M \times N, g_{M \times N})$ as a complex manifold with respect to $J_1$ then its corresponding Kähler angle $a_1$ is given by

$$\cos a_1 = \varphi := g_{M \times N}(J_1 dF(v_1), dF(v_2)) = u_1 - u_2.$$

For our convenience we may require that $a_1 \in [0, \pi]$. Observe that, although $\varphi$ is smooth, in general $a_1$ is not smooth at points where $\varphi = \pm 1$. If there exists a point $x \in M$ where $a_1(x) = 0$ then $dF(T_xM)$ is a complex line of $T_{F(x)}(M \times N)$ and $x$ is called a complex point of $F$. If $a_1(x) = \pi$ then $dF(T_xM)$ is an anti-complex line of $T_{F(x)}(M \times N)$ and $x$ is said anti-complex point of $F$. In the case where $a_1(x) = \pi/2$, the point $x$ is called Lagrangian point of the map $F$. In this case $u_1 = u_2$. 
Similarly, if we regard \((M \times N, g_{M \times N})\) as a Kähler manifold with respect to the complex structure \(J_2\), then its corresponding Kähler angle \(a_2\) is defined by the formula
\[
\cos a_2 = \vartheta := g_{M \times N}(J_2 \, dF(v_1), dF(v_2)) = u_1 + u_2.
\]
Notice that \(f\) is area decreasing if and only if both functions \(\varphi\) and \(\vartheta\) are non-negative. Moreover, observe that there are no points on \(M\) where \(\varphi = -1\) or \(\vartheta = -1\). If \(M\) is complete and non-compact, then \(\inf \varphi = -1\) or \(\inf \vartheta = -1\) if and only if both singular values \(\lambda\) and \(\mu\) of \(f\) tends to infinity.

4. Bochner formulas for the Jacobians

We will derive here the derivative and the Laplacian of a parallel 2-form on the product manifold \(M \times N\). The proofs are straightforward, make use of the Gauss-Codazzi equations and can be found in [22] (see also [13]). For this reason we omit them. From now we will always assume that \(f\) is a minimal map.

**Lemma 4.1.** Let \(\omega\) be a parallel 2-form on the product manifold \(M \times N\). Then the covariant derivative of the form \(F^* \omega\) is given by
\[
(\nabla_{e_i} F^* \omega)_{ij} = \sum_{\alpha} \left( A_{ki}^\alpha \omega_{\alpha j} + A_{kj}^\alpha \omega_{\alpha i} \right),
\]
for any adapted orthonormal frame field \(\{e_1, e_2; e_3, e_4\}\).

Again by a direct computation we can show the following formula on the Laplacian of a parallel 2-form on the product manifold \(M \times N\).

**Lemma 4.2.** Let \(\Omega\) be a parallel 2-form on the product manifold \(M \times N\). The Laplacian of the form \(F^* \omega\) is given by the following formula
\[
-(\Delta F^* \omega)_{ij} = \sum_{\alpha, k, l} \left( A_{ki}^\alpha A_{li}^\alpha \omega_{kj} + A_{kj}^\alpha A_{li}^\alpha \omega_{ki} \right) - 2 \sum_{\alpha, \beta, k} A_{ki}^\alpha A_{kj}^\beta \omega_{\alpha \beta}
+ \sum_{\alpha, k} \left( \tilde{R}_{k \alpha \omega a} + \tilde{R}_{k j a} \omega_{\alpha \beta} \right)
\]
where \(\{e_1, e_2; e_3, e_4\}\) is an arbitrary adapted local orthonormal frame.

From Lemma 4.2 we can compute the Laplacian of the Jacobians \(u_1\) and \(u_2\).

**Lemma 4.3.** The Jacobian functions \(u_1\) and \(u_2\) satisfy the following coupled system of partial differential equations
\[
-\Delta u_1 = ||A||^2 u_1 + 2 \sigma_1 u_2 + \sigma_M (1 - u_1^2 - u_2^2) u_1 - 2 \sigma_N u_1 u_2^2,
-\Delta u_2 = ||A||^2 u_2 + 2 \sigma_1 u_1 + \sigma_N (1 - u_1^2 - u_2^2) u_2 - 2 \sigma_M u_1^2 u_2.
\]
Using the special frames introduced in subsection \textsection{3.2} from Lemma \textsection{4.1} and Lemma \textsection{4.3}, by a direct computation we deduce the following:

\textbf{Lemma 4.4.} \textit{The gradients of the functions }$\varphi$ \textit{and }$\vartheta$ \textit{are given by the equations}

\begin{align*}
2\|\nabla \varphi\|^2 &= (|A|^2 - 2\sigma^\perp)(1 - \varphi^2) & \& 2\|\nabla \vartheta\|^2 &= (|A|^2 + 2\sigma^\perp)(1 - \vartheta^2).
\end{align*}

Moreover, the functions $\varphi$ and $\vartheta$ satisfy the following coupled system of partial differential equations

\begin{align*}
-\Delta \varphi &= (|A|^2 - 2\sigma^\perp)\varphi + \frac{1}{2} (\sigma_M(\varphi + \vartheta) + \sigma_N(\varphi - \vartheta))(1 - \varphi^2), \\
-\Delta \vartheta &= (|A|^2 + 2\sigma^\perp)\vartheta + \frac{1}{2} (\sigma_M(\varphi + \vartheta) - \sigma_N(\varphi - \vartheta))(1 - \vartheta^2).
\end{align*}

Observe that away from complex or anti-complex points the second fundamental form quantities $|A|^2 + 2\sigma^\perp$ and $|A|^2 - 2\sigma^\perp$ are expressed in terms of the cosines of the Kähler angles of the graph and of their gradients.

\section{5. Proof of the theorem}

From our assumptions, the Omori-Yau maximum principle is valid in our setting. It suffices now to prove that both $\inf \varphi$ and $\inf \vartheta$ are non-negative numbers. Suppose to the contrary that $\inf \varphi < 0$. Note that since by assumption $J_f$ is bounded, it follows that $\inf \varphi > -1$. Hence from the Omori-Yau maximum principle we have that there exists a sequence $\{x_k\}_{k \in \mathbb{N}}$, such that

\begin{align*}
\lim \varphi(x_k) = \inf \varphi, \quad \lim |\nabla \varphi|(x_k) = 0 \quad \text{and} \quad \lim \Delta \varphi(x_k) \geq 0.
\end{align*}

From Lemma \textsection{4.4} we have that

\begin{align*}
-\Delta \varphi(x_k) &= \frac{2\varphi(x_k)}{1 - \varphi^2(x_k)} |\nabla \varphi|^2(x_k) + \sigma_N(x_k) \varphi(x_k)(1 - \varphi^2(x_k)) \\
&\quad + \frac{1}{2} (\sigma_M(x_k) - \sigma_N(x_k)) (\varphi(x_k) + \vartheta(x_k))(1 - \varphi^2(x_k)).
\end{align*}

Note that the functions $1 - \varphi^2$ and $\varphi + \vartheta$ are positive. Hence, because of our curvature assumptions the last line of the above equality is non-negative. Passing to the limit we deduce that

\begin{align*}
0 &\geq -\sigma \inf \varphi (1 - (\inf \varphi)^2) > 0,
\end{align*}

which leads to a contradiction. Consequently, $\inf \varphi \geq 0$. Similarly, we prove that $\inf \vartheta \geq 0$. Hence, the map $f$ must be area decreasing. This completes the first part of the proof.
Let us suppose now that $f$ is an area decreasing map. Then both $\varphi$ and $\vartheta$ are non-negative functions. Assume that there is a point $x_0 \in M$ where $f$ is area preserving. Without loss of generality, let assume that $f$ is orientation preserving at $x_0$. Consequently,

$$\varphi(x_0) = 0 = \min \varphi.$$  

From Lemma 4.4 we deduce that

$$-\Delta \varphi = \{|A|^2 - 2\sigma^\perp + \sigma_N(1 - \varphi^2)\} \varphi + \frac{1}{2}(\sigma_M - \sigma_N)(\varphi + \vartheta)(1 - \varphi^2) \geq 2\{|A|^2 - 2\sigma^\perp + \sigma_N(1 - \varphi^2)\} \varphi.$$  

Then from Hopf’s strong minimum principle we deduce that $\varphi$ must vanish identically. Going back to the above identity we obtain that $\sigma_M = -\sigma = -\sigma_N$, everywhere. This completes the proof of the theorem.

**Remark 5.1.** Let us conclude now our paper with some final comments and remarks.

(a) It was very crucial in our proof that the second fundamental terms $|A|^2 \pm \sigma^\perp$, were expressed as gradient terms of the cosines $\varphi$ and $\vartheta$ of the Kähler angles of the graph. However, such a good structure is not available in higher dimensions and codimensions.

(b) There are various Schwarz-Pick type results for harmonic maps in the literature; see for instance [4, 16, 19]. On the other hand, a minimal map $f$ between two Riemannian manifolds $(M, g_M)$ and $(N, g_N)$ becomes harmonic if we equip $M$ with the graphical metric $g = g_M + f^*g_N$.

As one can see from the singular value decomposition, the map $f : (M, g) \to (N, g_N)$ is already length decreasing, since its singular values are $\frac{\lambda}{\sqrt{1 + \lambda^2}}$ and $\frac{\mu}{\sqrt{1 + \mu^2}}$.

Hence, one cannot deduce a Schwarz-Pick type result for minimal maps by applying directly the already known results for harmonic maps.

(c) If the map $f$ is holomorphic or anti-holomorphic then, according to the result of Yau [24], it is length decreasing without imposing apriori anything on the size of the differential of $f$. 
References

[1] L.V. Ahlfors, *An extension of Schwarz’s lemma*, Trans. Amer. Math. Soc. **43** (1938), 359–364.

[2] L.J. Alías, P. Mastrolia, and M. Rigoli, *Maximum principles and geometric applications*, Springer Monographs in Mathematics, Springer, Cham, 2016.

[3] R. Aiyama, K. Akutagawa, and T.Y. Wan, *Minimal maps between the hyperbolic discs and generalized Gauss maps of maximal surfaces in the anti-de Sitter 3-space*, Tohoku Math. J. (2) **52** (2000), 415–429.

[4] Q. Chen and G. Zhao, *A Schwarz lemma for $V$-harmonic maps and their applications*, Bull. Aust. Math. Soc. **96** (2017), 504–512.

[5] Q. Chen and Y.-L. Xin, *A generalized maximum principle and its applications in geometry*, Amer. J. Math. **114** (1992), 355–366.

[6] S.-S. Chern and J.G. Wolfson, *Minimal surfaces by moving frames*, Amer. J. Math. **105** (1983), no. 1, 59–83.

[7] J. Eells, *Minimal graphs*, Manuscripta Math. **28** (1979), 101–108.

[8] Th. Hasanis, A. Savas-Halilaj, and Th. Vlachos, *On the Jacobian of minimal graphs in $\mathbb{R}^4$*, Bull. Lond. Math. Soc. **43** (2011), 321–327.

[9] Th. Hasanis, A. Savas-Halilaj, and Th. Vlachos, *Minimal graphs in $\mathbb{R}^4$ with bounded Jacobians*, Proc. Amer. Math. Soc. **137** (2009), 3463–3471.

[10] J. Jost, Y.-L. Xin, and L. Yang, *Curvature estimates for minimal submanifolds of higher codimension and small $G$-rank*, Trans. Amer. Math. Soc. **367** (2015), 8301–8323.

[11] H. Omori, *Isometric immersions of Riemannian manifolds*, J. Math. Soc. Japan **19** (1967), 205–214.

[12] S. Pigola, M. Rigoli, and A. Setti, *Maximum principles on Riemannian manifolds and applications*, Mem. Amer. Math. Soc. **174** (2005), no. 822, x+99.

[13] A. Savas-Halilaj and K. Smoczyk, *Mean curvature flow of area decreasing maps between Riemann surfaces*, Ann. Global Anal. Geom. **53** (2018), 11–37.

[14] A. Savas-Halilaj and K. Smoczyk, *Bernstein theorems for length and area decreasing minimal maps*, Calc. Var. Partial Differential Equations **50** (2014), 549–577.

[15] R.M. Schoen, *The role of harmonic mappings in rigidity and deformation problems*, Complex geometry (Osaka, 1990), Lecture Notes in Pure and Appl. Math., vol. 143, Dekker, New York, 1993, pp. 179–200.

[16] C.-L. Shen, *A generalization of the Schwarz-Ahlfors lemma to the theory of harmonic maps*, J. Reine Angew. Math. **348** (1984), 23–33.

[17] K. Smoczyk, *Angle theorems for the Lagrangian mean curvature flow*, Math. Z. **240** (2002), 849–883.

[18] F. Torralbo and F. Urbano, *Minimal surfaces in $S^2 \times S^2$*, J. Geom. Anal. **25** (2015), 1132–1156.

[19] V. Tosatti, *A general Schwarz lemma for almost-Hermitian manifolds*, Comm. Anal. Geom. **15** (2007), 1063–1086.

[20] T.Y. Wan, *Stability of minimal graphs in products of surfaces*, Geometry from the Pacific Rim (Singapore, 1994), de Gruyter, Berlin, 1997, pp. 395–401.

[21] M.-T. Wang, *Deforming area preserving diffeomorphism of surfaces by mean curvature flow*, Math. Res. Lett. **8** (2001), 651–661.

[22] M.-T. Wang, *Mean curvature flow of surfaces in Einstein four-manifolds*, J. Differential Geom. **57** (2001), 301–338.
[23] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201-228.

[24] S.-T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. **100** (1978), 197–203.

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