Efficient Feature Screening for Lasso-Type Problems via Hybrid Safe-Strong Rules

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November 23, 2017

Abstract

The lasso model has been widely used for model selection in data mining, machine learning, and high-dimensional statistical analysis. However, due to the ultrahigh-dimensional, large-scale data sets collected in many real-world applications, it remains challenging to solve the lasso problems even with state-of-the-art algorithms. Feature screening is a powerful technique for addressing the Big Data challenge by discarding inactive features from the lasso optimization. In this paper, we propose a family of hybrid safe-strong rules (HSSR) which incorporate safe screening rules into the sequential strong rule (SSR) to remove unnecessary computational burden. In particular, we present two instances of HSSR, namely SSR-Dome and SSR-BEDPP, for the standard lasso problem. We further extend SSR-BEDPP to the elastic net and group lasso problems to demonstrate the generalizability of the hybrid screening idea. Extensive numerical experiments with synthetic and real data sets are conducted for both the standard lasso and the group lasso problems. Results show that our proposed hybrid rules substantially outperform existing state-of-the-art rules.

Keywords: Feature screening, Strong rules, Pathwise coordinate descent, Large-scale sparse learning
1 Introduction

The lasso model (Tibshirani, 1996) is widely used in data mining, machine learning, and high-dimensional statistics. The model is defined as the following optimization problem

$$\hat{\beta}(\lambda) = \arg\min_{\beta \in \mathbb{R}^p} \frac{1}{2n} \| y - X\beta \|^2 + \lambda \|eta\|_1,$$

where $y$ is the $n \times 1$ response vector, $X = (x_1, \ldots, x_p)$ is the $n \times p$ feature matrix, $\beta \in \mathbb{R}^p$ is the coefficient vector, and $\lambda \geq 0$ is a regularization parameter. $\| \cdot \|$ and $\| \cdot \|_1$ respectively denote the Euclidean ($\ell_2$) norm and $\ell_1$ norm.

Due to its property of automatic feature selection, the lasso model has attracted extensive studies with a wide range of successful applications to many areas, such as signal processing (Angelosante and Giannakis, 2009), gene expression data analysis (Huang and Pan, 2003), face recognition (Wright et al., 2009), text mining (Li et al., 2015) and so on. Efficiently solving the lasso model is therefore of great significance to statistical and machine learning practice.

Over the past years efficient algorithms have been developed for solving the lasso (Efron et al., 2004; Kim et al., 2007; Garrigues and Ghaoui, 2009; Boyd et al., 2011; Wu and Lange, 2008; Friedman et al., 2007; Shalev-Shwartz and Tewari, 2011). Among them the pathwise coordinate descent algorithm (Friedman et al., 2007) is simple, fast, and able to make use of the sparsity structure of the lasso and “warm start” strategy, making it very suitable and efficient to scale up to high-dimensional lasso problems (Friedman et al., 2010). With the evolving era of Big Data, however, it is increasingly common to encounter ultrahigh-dimensional, large-scale data sets. The increased number of features and observations in these data sets present added challenges to solving the lasso efficiently.

Feature screening is a powerful technique that can be used to address the Big Data challenge. Taking advantage of the sparsity assumption, feature screening aims to identify inactive features (i.e., those with zero coefficients), thereby discarding them from the lasso optimization. As a result, the dimensionality of the feature matrix – and hence the computational burden of the optimization – can be substantially reduced.

The idea of feature screening has been around for a long time, but was first stud-
ies formally by Fan and Lv (Fan and Lv, 2008), who studied the asymptotic properties of screening out features that have weak correlations with the response variable. This approach, however, does not necessarily solve the original optimization problem (1). Sequential strong rules (SSR) were proposed by Tibshirani et al. (Tibshirani, 2011) and based upon the Karush-Kuhn-Tucker (KKT) conditions for the lasso problem along with an assumption of “unit-slope” bound. These rules are simple yet very powerful in discarding a large proportion of features. However, it is possible for these rules to incorrectly screen out active features; thus, to guarantee the correctness of the solution, a post-convergence KKT checking step is required.

A separate line of research has sought to develop safe rules that are guaranteed not to discard any active features. These rules are usually based on exploiting geometric properties of the dual formulation of the lasso problem. Their main idea is to bound the dual optimal solution \( \hat{\theta}(\lambda) \) of the lasso (formally defined in Section 2.2) within a compact region \( \Theta \). Then given a feature \( x_j \), its coefficient estimate \( \hat{\beta}_j \) is guaranteed to be 0 if \( \sup_{\theta \in \Theta} |x_j^T \theta| < \lambda \).

This assertion is implied by the KKT condition: \( |x_j^T \theta(\lambda)| < \lambda \Rightarrow \beta_j = 0 \) (Boyd and Vandenberghe, 2004). The pioneering work in this direction is the SAFE rule developed by El Ghaoui et al. (Ghaoui et al., 2010). The smaller the region \( \Theta \), the more efficient the safe rule is; this has motivated other more powerful rules such as the EDPP rules (Wang et al., 2015), the Dome test (Xiang and Ramadge, 2012), and the Sphere tests (Xiang et al., 2011, 2016), which shrink \( \Theta \) according to different strategies.

In this paper, we propose combining safe and strong rules, yielding hybrid safe-strong rules (HSSR) for discarding features in lasso-type problems. The key of HSSR is to incorporate simple yet safe rules into SSR so as to remove a large amount of unnecessary post-convergence KKT checking on features that can be eliminated by safe rules. As a result, this paper will demonstrate that the total computing time for solving the lasso using these hybrid rules is substantially reduced compared to using either safe or strong rules alone. Furthermore, the idea of HSSR provides a rather general feature screening framework since (i) in principle any safe rule can be combined with SSR, resulting in a more powerful rule; and (ii) HSSR can be easily extended to other lasso-type problems, either with different loss functions or different regularization terms. In this paper we focus
on three types of lasso problems with quadratic loss, namely, the standard lasso, the group lasso, and the elastic net.

The main contributions of this research include:

1. We propose a novel feature screening framework that combines SSR with simple safe rules, resulting a family of hybrid safe-strong rules (HSSR) that are more efficient and scalable to large-scale data sets.

2. We develop two instances of HSSR, namely SSR-Dome and SSR-BEDPP, for feature screening in solving the lasso.

3. We extend SSR-BEDPP to two other lasso-type problems, the elastic-net (Zou and Hastie, 2005) and group lasso (Yuan and Lin, 2006) to demonstrate the generalizability of the hybrid screening idea.

4. We evaluate the performance of newly proposed screening rules by extensive numerical experiments on both synthetic and real data sets, and show that our rules substantially outperform state-of-the-art ones.

5. We implement all screening rules in this paper in two publicly accessible R packages. Specifically, the rules for the standard lasso and elastic net are implemented in R package biglasso\(^1\) (Zeng and Breheny, 2017), which aims to extend lasso model fitting to big data in R. The package grpreg\(^2\) (Breheny and Huang, 2015) implements screening rules for the group lasso. The underlying optimization algorithm and screening rules in the R packages are implemented in C/C++ for fast computation.

It should be noted that in this paper we assume the response vector \(y\) is centered so that the intercept term is dropped from the lasso model. We further assume the feature vectors \(\{x_j\}_{j=1}^p\) are centered and standardized to have unit variance. That is,

\[
\sum_{i=1}^{n} y_i = 0, \quad \sum_{i=1}^{n} x_{ij} = 0, \quad \frac{1}{n} \sum_{i=1}^{n} x_{ij}^2 = 1, \quad j = 1, \ldots, p. \tag{2}
\]

\(^1\)https://CRAN.R-project.org/package=biglasso
\(^2\)https://CRAN.R-project.org/package=grpreg
Standardization is a typical preprocessing step in fitting lasso models since: (1) it ensures that the penalty is applied uniformly across features with different scales of measurement; (2) it often contributes to faster convergence of the optimization algorithm; (3) as we will see in following sections, it simplifies feature screening rules and thus reduces computation complexity.

The rest of the paper is organized as follows. Section 2 reviews some existing feature screening rules related to our work. In Section 3 we propose our new hybrid screening strategy and describe two powerful rules, SSR-BEDPP and SSR-Dome, based on this strategy along with a pathwise coordinate descent algorithm to take advantage of them. In addition, this section analyzes the computational complexity of the HSSR rules and compares them to SSR and EDPP. Then in Section 4, we extend SSR-BEDPP to the elastic net and group lasso problems. Section 5 compares the performance of our rules with existing ones via extensive numerical experiments on synthetic and real data sets for both the standard lasso and the group lasso problems and conclude with some final remarks in Section 6. Proofs of theorems are given in the Appendix.

2 Existing feature screening rules

2.1 Sequential strong rules

SSR (Tibshirani et al., 2012) is a heuristic screening rule for discarding features when solving the lasso over a grid of decreasing regularization parameter values $\lambda_1 > \lambda_2 > \ldots > \lambda_K$. Specifically, after solving for $\hat{\beta}(\lambda_k)$ at $\lambda_k$, SSR discards the $j$th feature from the optimization at $\lambda_{k+1}$ if

$$\left| x_j^T r(\lambda_k) / n \right| < 2\lambda_{k+1} - \lambda_k,$$

where $r(\lambda_k) = y - X\hat{\beta}(\lambda_k)$ is the residual vector at $\lambda_k$.

To see the rationale of SSR, we start by noting that $\hat{\beta}(\lambda)$ satisfies the following KKT
conditions for the lasso problem (1):

\[
\begin{cases}
    x_j^T r(\lambda) / n = \lambda \text{sign}(\hat{\beta}_j), & \text{if } \hat{\beta}_j \neq 0, \\
    |x_j^T r(\lambda) / n| \leq \lambda, & \text{if } \hat{\beta}_j = 0.
\end{cases}
\]  

(4)

Let \( c_j(\lambda) = \frac{1}{n} x_j^T r(\lambda_k) \). The key idea behind SSR is to assume \( c_j(\lambda) \) is non-expansive in \( \lambda \) (or the “unit-slope” bound):

\[
|c_j(\lambda) - c_j(\lambda_\lambda)| \leq |\lambda - \lambda_\lambda|, \text{ for any } \lambda, \lambda_\lambda \in (0, \lambda_{\text{max}}].
\]  

(5)

Now, given \( \hat{\beta}(\lambda_k), \lambda_k, \lambda_{k+1} (\lambda_k \geq \lambda_{k+1}) \), if conditions (3) and (5) are satisfied, we have

\[
|c_j(\lambda_{k+1})| \leq |c_j(\lambda_{k+1}) - c_j(\lambda_k)| + |c_j(\lambda_k)| \\
< \lambda_k - \lambda_{k+1} + (2\lambda_{k+1} - \lambda_k) \\
= \lambda_{k+1},
\]

and thus \( \hat{\beta}_j(\lambda_{k+1}) = 0 \), implied by the KKT conditions (4).

SSR is simple yet able to screen out a large amount of inactive features. However, since assumption (5) may be violated, SSR requires checking KKT conditions (4) for all \( p \) coefficients after convergence has been reached at each value of \( \lambda \) to ensure that the solution is optimal. This process is time-consuming when \( p \) is large, and even more so if any violations occur, as this involves re-solving the lasso problem with the erroneously discarded features now included. Fortunately, empirical studies (Tibshirani et al., 2012; Lee and Breheny, 2015) show that the violations are quite rare.

2.2 Safe rules

As noted in the introduction, there are a number of safe rules in the literature; we focus primarily on EDPP rules, as they appear to be the most powerful safe rules developed thus far. EDPP rules are constructed by projecting the scaled response vector onto a nonempty closed and convex polytope. Here we derive simplified versions of the basic EDPP rule (BEDPP) and the sequential EDPP rule (SEDPP) under the standardization
condition (2), and refer readers to Wang et al. (2015) for the original EDPP rules and additional technical details.

The EDPP rules are based on the dual formulation of Problem (1):

\[
\hat{\theta}(\lambda) = \arg\max_{\theta \in \mathbb{R}^n} \frac{1}{2n} \|y\|^2 - \frac{n\lambda^2}{2} \|\theta - \frac{y}{n\lambda}\|^2
\]

subject to \(|x_j^T\theta| \leq 1, \quad \forall j = 1, \ldots, p,
\]

where \(\hat{\theta}(\lambda)\) is the dual optimal solution of Problem (1) under the constraints (7). The dual and primal solutions are related via:

\[
\hat{\theta}(\lambda) = \frac{y - X\hat{\beta}(\lambda)}{n\lambda}
\]

The original EDPP rules are developed by exploiting the geometric properties of the dual solutions. The simplified BEDPP and SEDPP rules are stated as the following theorems.

**Theorem 2.1** (BEDPP). For the lasso problem (1), let \(\lambda_m := \lambda_{\max} = \max_{j} |x_j^Ty/n|\) and \(x_* = \arg\max_{x_j} |x_j^Ty|\). For any \(\lambda \in (0, \lambda_m]\), under condition (2) we have \(\hat{\beta}_j(\lambda) = 0\) if

\[
|(\lambda_m + \lambda)x_j^Ty - (\lambda_m - \lambda)\text{sign}(x_j^Ty)\lambda_m x_j^T x_*| < 2n\lambda_\lambda - (\lambda_m - \lambda)\sqrt{n\|y\|^2 - n^2\lambda_m^2}.
\]

**Theorem 2.2** (SEDPP). For the lasso problem (1), let \(\lambda_m := \lambda_{\max} = \max_{j} |x_j^Ty/n|\).

Suppose we are given a sequence of \(\lambda\) values \(\lambda_m = \lambda_0 > \lambda_1 > \ldots > \lambda_K\). Then under condition (2):

1. For any \(0 < k < K\), we have \(\hat{\beta}_j(\lambda_{k+1}) = 0\) if \(\hat{\beta}(\lambda_k)\) is known and the following holds:

\[
\left|\frac{x_j^T (y - X\hat{\beta}(\lambda_k))}{\lambda_k} + \frac{c}{2} \left(x_j^Ty - \frac{ax_j^T X \hat{\beta}(\lambda_k)}{\|X \hat{\beta}(\lambda_k)\|^2}\right)\right| < n - \frac{c}{2} \sqrt{n\|y\|^2 - \frac{na^2}{\|X \hat{\beta}(\lambda_k)\|^2}}
\]

where \(c = \frac{\lambda_k - \lambda_{k+1}}{\lambda_k \lambda_{k+1}}\) and \(a = y^T X \hat{\beta}(\lambda_k)\) are two scalars.

2. For \(k = 0\), i.e., \(\lambda_k = \lambda_m\), SEDPP rule reduces to BEDPP rule. That is, we have \(\hat{\beta}_j(\lambda_{k+1}) = 0\) if rule (9) holds, in which \((\lambda_m, \lambda)\) is replaced by \((\lambda_0, \lambda_1)\).
Compared to SEDPP, the BEDPP rule is non-sequential in that screening at $\lambda_{k+1}$ via BEDPP doesn’t require the lasso solution at $\lambda_k$. As a result, BEDPP is much simpler to compute but less powerful in discarding inactive features, as shall seen in Section 3.2.

An alternative safe rule, the Dome test, is similar to BEDPP in that it is non-sequential and requires only a small computational burden; due to space constraints, we omit the details of the Dome test from this paper and refer interested readers to Xiang and Ramadge (2012) and Xiang et al. (2016). A supplementary material containing the details of the simplified Dome test can be found on the GitHub page.

### 3 Hybrid safe-strong rules

In this section, we define our newly proposed hybrid safe-strong rules (HSSR) and compare their computational complexity to the rules discussed in Section 2. In addition, we present a re-designed pathwise coordinate descent algorithm that takes advantage of these rules to increase the efficiency of solving the lasso.

#### 3.1 Definition

The motivation of HSSR is to remove a large amount of unnecessary post-convergence KKT checking, required by SSR, on features that could have been discarded by a safe screening rule. In principle, any safe rule can be combined with SSR, resulting in a family of rules which we call hybrid safe-strong rules and define as follows.

**Definition 3.1.** For solving the lasso problem (1) over a sequence of $\lambda$ values $\lambda_1 > \lambda_2 > \ldots > \lambda_K$, suppose that there exists a safe rule and that $\hat{\beta}(\lambda_k)$ is known. Let $\mathcal{S}_{k+1}$ denote the safe set, i.e., the set of features not discarded by the safe rule at $\lambda_{k+1}$. Then a corresponding hybrid safe-strong rule (HSSR) can be formulated by combining the safe rule with SSR. Specifically, HSSR discards the $j$th feature from the lasso optimization at $\lambda_{k+1}$ if

$$j \in \mathcal{S}_{k+1} \cup \{j \in \mathcal{S}_{k+1} : |x_j^T r(\lambda_k)|/n \leq 2\lambda_{k+1} - \lambda_k\},$$

3 https://github.com/YaohuiZeng/HSSR_paper_supplementary/blob/master/HSSR_supplementary_for_Dome.pdf
where \( r(\lambda_k) = y - X\hat{\beta}(\lambda_k) \).

HSSR builds upon SSR and thus enjoys all of its advantages: simple, sequential, and powerful to discard a large portion of features. As a drawback, it also requires post-convergence KKT checking. However, HSSR only needs to perform KKT checking over a subset of features since all features in the set \( S_{k+1}^c \) are discarded by the safe rule. Provided that the safe rule is simple to calculate, by which we mean that its time complexity is \( O(np) \) for obtaining the entire lasso path, HSSR should be much more efficient computationally than SSR. In addition, more powerful safe rules would result in smaller safe sets \( S_k \) and hence a larger speedup of HSSR.

In this paper, two instances of HSSR, namely SSR-BEDPP and SSR-Dome, are studied. These two rules respectively use BEDPP and the Dome test as the safe rule candidate. It’s important to mention that for a generic algorithm that solve the lasso, incorporating HSSR screening into the algorithm yields to the same global optimum. This result is stated in the following theorem.

**Theorem 3.1** (Convergence). Suppose the lasso problem (1) at a given \( \lambda \) is strictly convex such that the sequence of solutions produced by an iterative algorithm \( a(\cdot) \) (such as coordinate descent) converges to the unique global optimum, \( \hat{\beta}(\lambda) \). Then that algorithm with HSSR screening converges to the same solution \( \hat{\beta}(\lambda) \).

**Proof.** Let \( X_S \) denote the submatrix of \( X \) consisting only of the features in \( S(\lambda) \). By the definition of a safe rule, the global optimum \( \hat{\beta}(\lambda) \) can be decomposed as \( \hat{\beta}(\lambda) = \left(0, \hat{\beta}_S^T(\lambda)\right)^T \), where \( \hat{\beta}_S(\lambda) \) is the solution to the following optimization problem:

\[
\hat{\beta}_S(\lambda) = \arg\min_{\beta_S \in \mathbb{R}^{\left|S(\lambda)\right|}} \frac{1}{2n} \|y - X_S\beta_S\|^2 + \lambda \|\beta_S\|_1.
\]  

(12)

Furthermore, it’s easy to verify that the algorithm \( a(\cdot) \) with SSR screening for solving (12) converges to the global optimum \( \hat{\beta}_S(\lambda) \). This is because the KKT checking procedure required by SSR guarantees the final solution satisfies the KKT optimality conditions and hence is the global optimum. Therefore, the algorithm with HSSR screening converges to \( \hat{\beta}(\lambda) \).  

\(\square\)
3.2 Performance analysis

Intuitively, the computational savings achieved by feature screening will be negated if the screening rule itself is too complicated to execute. Therefore, an efficient rule needs to balance the trade-off between its computational complexity and rejection power (i.e., how many features can be discarded). That is, an ideal screening rule should be powerful enough to discard a large portion of features and also relatively simple to compute.

To show the advantages of HSSR, we compare the aforementioned screening rules for obtaining the entire lasso path in terms of the rejection power and computational complexity of the rules themselves.

3.2.1 Screening power

Here we present a empirical comparison of different rules in terms of the power to discard features. Figure 1 depicts the results based on the GENE data (See details in Section 5.1.2). First, it’s important to note that HSSR, by construction, discards at least as many features as SSR does. Second, HSSR, SSR and SEDPP discard far more features than the non-sequential rules BEDPP and Dome. In particular, the screening power of BEDPP and Dome decreases rapidly as $\lambda$ decreases. For example, BEDPP cannot discard any features when $\lambda/\lambda_{max}$ is smaller than 0.45 in this case, whereas Dome is the least powerful and discards virtually no features when $\lambda/\lambda_{max}$ is less than 0.6.

3.2.2 Computational complexity

Table 1 presents the complexity of computing these rules for the entire path of $K$ values of $\lambda$.

For SSR (3), it’s important to observe that the quantities needed to check the KKT conditions (4), $x_j^T r(\lambda_k)$, can be re-used for executing SSR at $\lambda_{k+1}$ for that feature. Therefore, SSR requires $O(np)$ operations, as the dominant computation is calculating $X^T r(\lambda_k)$. However, since $r(\lambda_k)$ changes as a function of $\lambda_k$, the total complexity of SSR is $O(npK)$ over the entire solution path.

HSSR, on the other hand, only needs to perform KKT checking over the features not discarded by the safe screening step. Thus, $x_j^T r(\lambda_{k-1})$ must be calculated only for features
in the safe set $S_k$, yielding $O(n \sum_{k=1}^{K} |S_k|)$ operations. When the safe rule is effective (e.g. when $\lambda$ is relatively large, as shown in Figure 1), HSSR would avoid a large amount of unnecessary KKT checking and hence be much more efficient than SSR.

The complexity of SEDPP (10) is more involved. During coordinate descent, the residuals $r(\lambda_k)$ are continually updated and stored. Thus, $X\hat{\beta}(\lambda_k)$ can be obtained at a cost of $O(n)$ operations since $X\hat{\beta}(\lambda_k) = y - r(\lambda_k)$. Furthermore, only $O(n)$ calculations are needed to update $\|X\hat{\beta}(\lambda_k)\|$ and $a$, while quantities like $x_j^T y$ and $\|y\|$ can be pre-computed to avoid duplicated calculations. The more demanding parts are on the left hand side of (10), specifically, the two terms $x_j^T r(\lambda_k)$ and $x_j^T X\hat{\beta}(\lambda_k)$. Since these must be calculated for all features, this essentially involves calculating $X^T r(\lambda_k)$ and $X^T X\hat{\beta}(\lambda_k)$, both of which require $O(np)$ calculations. Thus, similar to that of SSR, the total complexity of SEDPP is $O(npK)$ for obtaining the entire solution path.

Finally, the complexity of executing BEDPP (9) over the solution path is only $O(np)$ as its dominant calculations are $X^T y$ and $X^T x_*$, which only need to be calculated once. After these initial calculations, only $O(p)$ operations are needed to compute the rule, resulting in a complexity of $O(pK)$ over the entire path. Hence the total complexity is $O(np)$ provided that $n$ is larger than $K$. The Dome test also has complexity of $O(np)$, as can be analyzed in the same fashion based on results in Xiang and Ramadge (2012).
Table 1: Complexity of computing screening rules along the entire path of $K$ values of $\lambda$. $|S_k|$ is the cardinality of safe set $S_k$ obtained by the safe rule used in HSSR.

| Rule  | Dome   | BEDPP  | SEDPP  | SSR    | HSSR    |
|-------|--------|--------|--------|--------|---------|
| Complexity | $O(np)$ | $O(np)$ | $O(npK)$ | $O(npK)$ | $O(n \sum_{k=1}^{K} |S_k|)$ |

3.2.3 Advantages of HSSR

The advantages of HSSR can be summarized as follows:

1. **Computational efficiency**: Solving the lasso with HSSR screening, as compared to other rules, involves the least computational burden. As we will see in Section 5, the result is that HSSR is the fastest of the approaches considered here.

2. **Memory efficiency**: HSSR can be much more memory-efficient. This is because both SSR and SEDPP have to fully scan the feature matrix $K$ times, while HSSR only needs to do so for the portion of the lasso path where the safe rule is not able to discard any features. This advantage of HSSR is particularly appealing in out-of-core computing, where fully scanning the feature matrix requires disk access and therefore becomes the computational bottleneck.

3. **Generalizability**: HSSR is a rather general feature screening framework, and can be easily extended to other lasso-type problems such as the elastic net and the group lasso.

3.3 Pathwise coordinate descent with HSSR

The pathwise coordinate descent (PCD) algorithm (Friedman et al., 2007) solves the lasso solution path along a grid of decreasing parameter values $\lambda_1 > \lambda_2 > \ldots > \lambda_K$. When solving for $\hat{\beta}(\lambda_k)$, PCD utilizes previous solution $\hat{\beta}(\lambda_{k-1})$ as warm starts. This “warm start” strategy makes the algorithm very efficient.

In this section, we re-design the PCD algorithm by incorporating HSSR, as described in Algorithm 1. The algorithm starts by initializing the safe sets $S$ and $S_{prev}$, which saves the safe set at previous iteration. Another set $H$, called the strong set, is also initialized to store the features in the safe set not discarded by SSR screening. The Flag variable
indicates whether the safe rule screening should be turned off or not. The rationale of this
design is to stop using the safe rule once it is no longer capable of discarding any features
(See Figure 1). It’s also important to point out that the algorithm only needs to update $z_j$
for those “newly-entered” features in the safe set (line 4) before conducting SSR screening.
This is because all $z_j$’s associated with features in $S$ must have already been computed
during post-convergence KKT checking at the previous $\lambda$ (line 15).

---

**Algorithm 1: PCD algorithm with HSSR screening**

Input : $\{x_j\}_{j=1}^p$, $y$, $\lambda_{max} = \lambda_0 > \lambda_1 > \ldots > \lambda_K$

Initialize: $S = S_{prev} = \emptyset$, $H = \emptyset$, $r = y$, $\{z_j = 0 : j = 1, 2, \ldots, p\}$, Flag = FALSE

for $k \leftarrow 1$ to $K$ do

if Flag = FALSE then

  Safe Screening: $S \leftarrow \{j : x_j \text{ survives safe screening}\}$

  Update $z_j = x_j^T r / n$ over set $\{j : j \in S \setminus S_{prev}\}$

  $S_{prev} \leftarrow S$

  if $|S| = p$ then

    Flag $\leftarrow$ TRUE

  end

end

SSR screening: $H \leftarrow \{j \in S : |z_j| \geq 2\lambda_k - \lambda_{k-1}\}$

while not converged do

  Solve (1) for $\hat{\beta}(\lambda_k)$ via coordinate descent iteration over features only in $H$
  and keep updating $r$

end

Update $z_j = x_j^T r / n$ over set $\{j : j \in S \setminus H\}$, and check KKT violations:

$V \leftarrow \{j \in S \setminus H : |z_j| \geq \lambda_k\}$

if $V \neq \emptyset$ then

  $H \leftarrow H \cup V$

  go to 11 with current solution as a warm start

end

save $\hat{\beta}_k$

end

Output : $\{\hat{\beta}\}_{k=1}^K$

---

After SSR screening, the algorithm then solves the lasso problem for $\hat{\beta}(\lambda_k)$ via coordinate
descent iterations using features only in the strong set $H$, as described by the while loop, until a predefined convergence criterion is met.

The post-convergence KKT checking takes place in line 15 after a solution is obtained:
KKT checking is applied to features that are outside of the strong set $H$ but in the safe
set $S$. If any violations are detected, the strong set is updated by adding in the features which violate the KKT conditions, and the lasso then needs to be re-solved (line 18) with the updated strong set.

4 Extensions to other lasso-type problems

4.1 SSR-BEDPP for the elastic net

The elastic net problem (Zou and Hastie, 2005) is defined as

$$
\hat{\beta}(\lambda, \alpha) = \arg\min_{\beta \in \mathbb{R}^p} \frac{1}{2n} \|y - X\beta\|^2 + \alpha \lambda \|\beta\|_1 + (1 - \alpha) \lambda \|\beta\|^2.
$$

SSR can be applied to the elastic net with minimal changes, as shown in Tibshirani et al. (2012). Specifically, let $r(\lambda_k) = y - X\hat{\beta}(\lambda_k)$, SSR discards the $j$th feature from the elastic net optimization at $\lambda_{k+1}$ if

$$
|x_j^T r(\lambda_k)/n| < \alpha(2\lambda_{k+1} - \lambda_k).
$$

Moreover, it can be shown that the KKT conditions for (14) are

$$
x_j^T r(\lambda)/n - (1 - \alpha)\lambda \beta_j = \lambda \text{sign}(\hat{\beta}_j) \quad \text{if} \quad \hat{\beta}_j \neq 0,
$$

$$
|x_j^T r(\lambda)/n - (1 - \alpha)\lambda \beta_j| \leq \lambda \quad \text{if} \quad \hat{\beta}_j = 0.
$$

The BEDPP rule in Wang et al. (2015) is not directly applicable to the elastic net problem. Here we extend BEDPP to the elastic net as the following theorem.

**Theorem 4.1** (BEDPP for elastic net). For the elastic net problem (13), let $\lambda_m := \lambda_{\text{max}} = \max_j |x_j^T y|/\alpha n$ and $x_* = \arg\max_{x_j} |x_j^T y|$. Under condition (2), for any $\lambda \in (0, \lambda_m]$ and $x_j \neq x_*$, we have $\hat{\beta}_j(\lambda) = 0$ if

$$
|\lambda_m + \lambda)x_j^T y - (\lambda_m - \lambda) \frac{\text{sign}(x_*^T y) \alpha \lambda_m}{1 + \lambda(1 - \alpha)} x_j^T x_*| < 2n\alpha \lambda \lambda_m - (\lambda_m - \lambda)\sqrt{n\|y\|^2(1 + \lambda(1 - \alpha))} - n^2 \alpha^2 \lambda_m^2
$$

$$
= \frac{1}{\sqrt{n\|y\|^2(1 + \lambda(1 - \alpha))}}
$$

$$
\frac{1}{\sqrt{n\|y\|^2(1 + \lambda(1 - \alpha))}}
$$

14
Analogous to (9), the complexity of (17) for solving the elastic net over an entire solution path is $O(np)$ since, again, $O(np)$ calculations are needed to pre-compute quantities $X^T y$, $X^T x_*$, and $\|y\|$. After that, only $O(p)$ operations are required to execute the rule. Moreover, given (14), (15), and (17), Algorithm 1 may be used for the elastic net, with appropriate modifications to the screening rules, KKT checking, and coordinate descent update.

4.2 SSR-BEDPP for the group lasso

As another example, we extend SSR-BEDPP to the group lasso problem. Suppose we have $p$ features assigned into $G$ non-overlapping groups. Let $W_g$ denote the number of features in the $g$th group. The group lasso problem (Yuan and Lin, 2006) is defined as

$$
\hat{\beta}(\lambda) = \arg\min_{\beta \in \mathbb{R}^p} \frac{1}{2n} \left\| y - \sum_{g=1}^{G} X_g \beta_g \right\|^2 + \lambda \sum_{g} \sqrt{W_g} \| \beta_g \|,
$$

where $\beta = (\beta_1^T, \ldots, \beta_G^T)^T$, $X_g$ is the $n \times W_g$ sub-matrix whose columns correspond to features in group $g$, and $\beta_g = (\beta_{g,1}, \ldots, \beta_{g,W_g})^T$ is the associated coefficient vector. Here, in addition to the standardization described in Section 1, we apply an additional level of standardization at the group level (Breheeny and Huang, 2015):

$$
\frac{1}{n} X_g^T X_g = I, \quad g = 1, \ldots, G.
$$

Given $\hat{\beta}(\lambda_k)$, it can be shown (Tibshirani et al., 2012) that SSR discards the $g$th group of coefficient vector $\hat{\beta}_g(\lambda_{k+1})$ from the group lasso optimization at $\lambda_{k+1}$ if

$$
\left\| \frac{1}{n} X_g^T r(\lambda_k) \right\| < \sqrt{W_g} (2\lambda_{k+1} - \lambda_k),
$$

where $r(\lambda_k) = y - \sum_{\ell=1}^{G} X_\ell \hat{\beta}_\ell(\lambda_k)$. Moreover, the KKT conditions for (18) are,

$$
X_g^T r(\lambda)/n = \lambda \sqrt{W_g} \theta_g, \quad g = 1, \ldots, G,
$$

where $\theta_g$ is a subgradient of $\|\hat{\beta}_g\|$.
Wang et al. (2015) also derives EDPP rules for the group lasso. With some algebra, we derive a simplified BEDPP under condition (19) for the group lasso as the following theorem.

**Theorem 4.2 (BEDPP for group lasso).** For the group lasso problem (18), let \( \lambda_m := \lambda_{\max} = \max_g \frac{\|X_g^T y\|}{n \sqrt{W_g}}, W_* = \arg \max_{W_g} \frac{\|X_g^T y\|}{n \sqrt{W_g}}, X_* \) is the data matrix of the group associated with \( W_* \), and \( \bar{v} = X_* X_*^T y \). For any \( \lambda \in (0, \lambda_m] \) and \( g = 1, 2, \ldots, G \), under condition (19) we have \( \hat{\beta}_g(\lambda) = 0 \) if

\[
\sqrt{(\lambda + \lambda_m)^2 \|X_g^T y\|^2 - \frac{2(\lambda_m^2 - \lambda^2) y^T X_g X_g^T \bar{v}}{n}} + \frac{(\lambda_m - \lambda)^2 \|X_g^T \bar{v}\|^2}{n^2} < 2n\lambda \lambda_m \sqrt{W_g} - (\lambda_m - \lambda) \sqrt{n\|y\|^2 - n^2 \lambda_m^2 W_*}
\]  

Analogous to the lasso and elastic net, the complexity of executing (22) for an entire solution path costs \( O(np) \). To see this, note that \( \bar{v} \) only needs to be calculated once and requires \( O(nW_*) \) operations. Thus, the most computationally intensive step of (22) is calculating \( X_g^T \bar{v} \) and \( y^T X_g \), each of which require \( O(nW_g) \) operations, or \( O(np) \) operations to calculate these intermediate quantities for all \( G \) groups. Once this is done, executing BEDPP rule for group lasso costs only \( O(p) \) at each \( \lambda_k \).

Given SSR and BEDPP rules for the group lasso, we can formulate the SSR-BEDPP rule and solve the group lasso based on Algorithm 1 with appropriate modifications to the screening rules and KKT checking given by (20), (22), and (21). The coordinate descent update must also be modified to a group descent update (also known as a blockwise coordinate descent update) as described in Qin et al. (2013); Breheny and Huang (2015); Meier et al. (2008).

## 5 Experiments

In this section, we conduct experiments to show that our proposed hybrid safe-strong rules significantly outperform the existing screening rules SSR and SEDPP. In addition to the two rules, we take into comparison the “Active-set Cycling” (AC) strategy (Lee et al., 2007). AC is somewhat similar to SSR in that they both begin by solving the lasso over
a subset of features and then check KKT conditions to verify the solution. AC, however, merely cycles over the nonzero coefficients. The idea of AC is simple and effective, and has been commonly applied to large-scale sparse learning problems with considerable speedup observed (Garrigues and Ghaoui, 2009; Tibshirani et al., 2012; Lee and Breheny, 2015; Meier et al., 2008).

In all numerical experiments, we focus on solving the lasso or the group lasso problems over the entire path of 100 values of \( \lambda \) which are equally spaced on the scale of \( \lambda/\lambda_{max} \) from 0.1 to 1. All experiments in this section are conducted with 20 replications, and the average computing times (in seconds) are reported. The benchmarking platform is a MacBook Pro with Intel Core i7 @ 2.3 GHz and 16 GB RAM.

5.1 Results for the lasso

In this section, we compare SSR-BEDPP and SSR-Dome with existing methods AC, SSR, and SEDPP in solving the standard lasso problem. Basic pathwise coordinate descent (“Basic PCD”) with no screening or active cycling is used as baseline for the comparison. Our R package biglasso (Version 1.3-2) implements all these methods and is used for all the numerical studies.

5.1.1 Synthetic data

We first demonstrate with synthetic data that SSR-BEDPP is more scalable in both \( n \) and \( p \) (i.e., number of observations and features). We adopt the same model in Wang et al. (2015) to simulate data: \( y = X\beta + 0.1\epsilon \), where \( X \) and \( \epsilon \) are i.i.d. sampled from \( N(0,1) \). Here we consider two different cases:

(a) **Case 1: varying** \( p \). We set \( n = 1,000 \) and vary \( p \) from 1,000 to 10,000. We randomly select 20 true features, and sample their coefficients from Unif[-1, 1]. After simulating \( X \) and \( \beta \), we then generate \( y \) according to the true model;

(b) **Case 2: varying** \( n \). We set \( p = 10,000 \) and vary \( n \) from 200 to 10,000. \( \beta \) and \( y \) are generated in the same way as in Case 1.
Figure 2: Average computing time as a function of $p$ (left) and $n$ (right) for solving the lasso along a sequence of 100 values of $\lambda$. Note that the lines for SSR and SEDPP overlap and cannot be distinguished.

Figure 2 compares the average computing time of solving the lasso over a sequence of 100 values of $\lambda$ by the different methods. In all settings, our rule SSR-BEDPP is uniformly 5x faster than Basic PCD. More importantly, SSR-BEDPP is around 2x faster than state-of-the-art rules SSR and SEDPP. Note that the computing times of SSR and SEDPP are almost the same so the lines of these two cannot be distinguished in the plots. Note that SSR and SEDPP provide only a small advantage over AC, while SSR-BEDPP achieves more than a 2x additional speedup compared to AC.

It’s worth mentioning that the new rule SSR-Dome can also provide substantial speedup - 1.6x faster than AC and 1.4x faster than SSR or SEDPP, proving the effectiveness of hybrid screening. Since the Dome test itself is less powerful than the BEDPP rule (Wang et al., 2015) and takes equally long to compute, it is not surprising that SSR-BEDPP is the faster of the two approaches.

5.1.2 Real data

Real-world data sets often have complicated signals and correlation structures which affect the performance of the screening rules. In this section, we compare the aforementioned
methods using diverse real data sets:

(a) **Breast cancer gene expression data**\(^4\) (GENE): this data set contains gene expression measurements of 17,322 genes of 536 breast cancer patients from The Cancer Genome Atlas project. The goal is to identify genes with expression levels related to that of a tumor suppressor gene, namely BRCA1.

(b) **MNIST handwritten image data**\(^5\) (MNIST): this data set contains gray images of handwritten digits whereas 60,000 images for training and 10,000 for testing. Each image is of \(28 \times 28\) dimension. Following Wang et al. (2015), we first use the training set to construct a feature matrix \(X \in \mathbb{R}^{784 \times 60000}\). We then randomly choose an image in the test set as the response vector \(y \in \mathbb{R}^{784}\) for each of the 20 replications.

(c) **Cardiac fibrosis genome-wide association data**\(^6\) (GWAS): this data set contains the single nucleotide polymorphisms (SNPs) data collected on 313 human hearts. The goal of the study is to discover SNPs that are associated with increased fibrosis. The response vector \(y \in \mathbb{R}^{313}\) is the log of the cardiomyocyte:fibroblast ratio, and the feature matrix \(X \in \mathbb{R}^{313 \times 660,496}\) records the data for the 660,496 SNPs.

(d) **Subset of New York Times bag-of-words data**\(^7\) (NYT): this data set is from the UCI Machine Learning Repository Lichman (2013). The raw data matrix contains 300,000 documents represented as rows of 102,660 words, where the cell \((i, j)\) records the number of occurrences of word \(j\) in article \(i\). Following Xiang et al. (2016), we preprocess the raw data by first removing documents with low word counts and then randomly selecting a subset of 5,000 documents and 55,000 words to form the feature matrix \(X \in \mathbb{R}^{5000 \times 55000}\). At each replication, a word column is randomly chosen from the rest set to be the response \(y \in \mathbb{R}^{5000}\).

Table 2 summarizes the dimensions of the real data sets and the average computing times. The speedup of different methods relative to Basic PCD is depicted in Figure 3.

\(^4\)http://myweb.uiowa.edu/pbreheny/data/bcTCGA.html
\(^5\)http://yann.lecun.com/exdb/mnist/
\(^6\)https://arxiv.org/abs/1607.05636
\(^7\)https://archive.ics.uci.edu/ml/datasets/Bag+of+Words
Table 2: Average computing time (standard error) for solving the lasso along a sequence of 100 values of $\lambda$ on real data sets.

| Method       | GENE ($n = 536$, $p = 17,322$) | MNIST ($n = 784$, $p = 60,000$) | GWAS ($n = 313$, $p = 660,495$) | NYT ($n = 5,000$, $p = 55,000$) |
|--------------|--------------------------------|---------------------------------|---------------------------------|---------------------------------|
| Basic PCD    | 12.84 (0.06)                   | 91.73 (6.32)                    | 266.22 (1.14)                   | 246.87 (24.12)                  |
| AC           | 1.54 (0.01)                    | 6.48 (0.11)                     | 43.59 (0.19)                    | 44.57 (1.96)                    |
| SSR          | 1.13 (0.01)                    | 5.58 (0.04)                     | 21.89 (0.10)                    | 33.64 (0.64)                    |
| SEDPP        | 1.26 (0.02)                    | 5.57 (0.04)                     | 21.47 (0.07)                    | 35.26 (1.21)                    |
| SSR-Dome     | 0.86 (0.01)                    | 2.92 (0.07)                     | 18.87 (0.10)                    | 23.01 (1.59)                    |
| SSR-BEDPP    | **0.69 (0.01)**                | **1.74 (0.09)**                 | **16.27 (0.08)**                | **17.88 (1.75)**                |

Again, SSR-BEDPP outperforms all other methods with the most speedup on all data sets, ranging from 13.8x (NYT) to 52.7x (MNIST) faster than Basic PCD.

In comparison to AC, SSR-BEDPP results in additional speedup ranging from 2.2x on GENE data to 3.7x on MNIST data. SSR and SEDPP, however, provide a meaningful improvement over AC only for the GWAS data; for the other three data sets, the speedup is quite small. Overall, SSR-BEDPP is 1.3x to 3.2x faster than SSR and SEDPP based on the four real data sets.

### 5.2 Results for the group lasso

In this section, we conduct experiments via our R package grpreg\(^8\) (Version 3.1-1) to compare SSR-BEDPP with existing methods AC, SSR, and SEDPP in solving the group lasso problem. Note again that basic group descent algorithm (“Basic GD”) with no screening or active cycling is used as baseline.

#### 5.2.1 Synthetic data

To generate the synthetic data, we again use the model: $y = X\beta + 0.1\varepsilon$, where $X$ and $\varepsilon$ are i.i.d. sampled from $N(0, 1)$. Here we fix the number of observations $n$ to be 1,000, and the number of features in all groups to be 10. We vary the number of total groups from 100 to 10,000. In all settings, we randomly select 10 nonzero groups (i.e., groups of features that

---

\(^8\)See Version 3.1-1 at: https://github.com/YaohuiZeng/grpreg
Figure 3: The speedup relative to Basic PCD for solving the lasso along a sequence of 100 values of $\lambda$ on real data sets.

having nonzero coefficients), and sample the 100 coefficients in these groups from Unif[-1, 1]. After simulating $X$ and $\beta$, we then obtain $y$ according to the true model.

Figure 4 depicts the average computing time of solving the group lasso over a sequence of 100 $\lambda$ values. Again, the computing times by SSR and SEDPP are so close that the corresponding two lines cannot be distinguished. Similar conclusions as for lasso case can be drawn here: (1) our new rule SSR-BEDPP provides remarkable reduction of computing time uniformly across all settings by more than 7x speedup compared to Basic GD, and by around 2x speedup compared to SSR and SEDPP; (2) SSR and SEDPP performs almost identically, and offer only a small advantage over AC.

5.2.2 Real data

We evaluate the performance of different rules using the following real data sets.

(a) Genetic rare variant study data (GRVS): this data set contains real exon sequencing data from the 1000 Genomes Project Consortium et al. (2010) on 697 subjects and 24,487 genetic variants. The genetic variants are grouped into 3205 genes (i.e., $n = 697$, $p = 24,487$, and $G = 3,205$). 20 different response vectors containing the quantitative phenotypes are simulated according to a plausible genetic model of variant-disease
Figure 4: Average computing time as a function of the number of groups for solving the group lasso along a sequence of 100 values of $\lambda$. Note that the lines for SSR and SEDPP overlap and cannot be distinguished.

association described in Almasy et al. (2011).

(b) **B-spline regression on GENE data (GENE-SPLINE):** here we revisit the GENE data in Section 5.1.2 and fit a B-spline regression model using the group lasso. Specifically, 5-term basis expansions are first applied to each of the 17,322 features in GENE data, resulting in 86,610 new features in total. The 5 basis expansions for the same raw feature are treated as in the same group. Therefore, $n = 536$, $p = 86,610$ and $G = 17,322$.

Table 3 presents the average computing time and the speedup relative to Basic GD for solving the group lasso along a sequence of 100 values of $\lambda$ on the above two real data sets. SSR-BEDPP again outperforms other methods on the two real data sets with 6.3x and 33.4x speedup compared to Basic GD. In addition, it’s around 1.4x faster than SSR and SEDPP, which again show similar performance. Finally, SSR-BEDPP is over 1.5x faster than AC for GRVS data, and nearly 2x faster for GENE-SPLINE data.
Table 3: Average computing time (standard error) and the speedup relative to Basic GD for solving group lasso along a sequence of 100 values of $\lambda$ on real data sets.

| Method      | Time     | Speedup | Time     | Speedup |
|-------------|----------|---------|----------|---------|
| Basic GD    | 15.84 (0.41) | 1.0     | 147.78 (1.21) | 1.0 |
| AC          | 3.84 (0.08)  | 4.1     | 8.19 (0.08)   | 18.0  |
| SSR         | 3.30 (0.11)  | 4.8     | 6.34 (0.05)   | 23.3  |
| SEDPP       | 3.51 (0.10)  | 4.5     | 6.89 (0.05)   | 21.4  |
| SSR-BEDPP   | 2.51 (0.10)  | 6.3     | 4.42 (0.04)   | 33.4  |

6 Conclusion

In this paper, we propose novel, efficient hybrid safe-strong rules (HSSR) for lasso-type models. The key of HSSR is to incorporate a simple, safe rule into SSR to alleviate a large amount of unnecessary post-convergence KKT checking required by SSR. We demonstrate that this hybrid of two very different types of rules is substantially more efficient than either type alone. This innovative idea is motivated by the insights from careful complexity analysis. The idea is simple yet remarkably powerful in reducing the computing time of solving for the entire solution path in lasso-type problems. As a result, the newly proposed rules are much more scalable and suitable to large-scale sparse learning problems.

For the standard lasso problem, we develop two instances of HSSR: SSR-Dome and SSR-BEDPP. Moreover, we extend SSR-BEDPP to the elastic net and the group lasso to illustrate the generalizability of the HSSR framework. Extensive studies on synthetic and real data sets demonstrate that the newly proposed rules substantially outperform the existing state-of-the-art screening rules SSR and SEDPP.

The basic idea proposed in this manuscript can be further generalized in several ways. We are currently working on extending the hybrid screening idea to other lasso-type problems such as sparse logistic regression and sparse support vector machines. Another variation on the idea proposed here would be to “re-hybridize” SSR with another safe rule once BEDPP is no longer effective. For example, as illustrated in Figure 1, for this data set BEDPP becomes useless at $\lambda_{60}$. At that point, we could apply the EDPP rule (10) to obtain a new safe rule, effective for $\lambda_{61}, \lambda_{62}, \ldots$ by only varying $\lambda_{k+1}$. This rule would require $O(np)$ calculations at $\lambda_{61}$, but only $O(p)$ calculations at future as the computationally
expensive terms only need to be computed once and saved, as in the proposed algorithm. This approach may offer additional computational savings beyond SSR-BEDPP, especially in the latter part of the solution path.

All screening rules presented in this manuscript are implemented in two publicly accessible R packages, \texttt{biglasso} (for the standard lasso and elastic net) and \texttt{grpreg} (for the group lasso). Benchmarking experiments (Zeng and Breheny, 2017) show that \texttt{biglasso} is considerably faster than existing popular packages of its kind, including the popular R package \texttt{glmnet}, as a result of the hybrid screening rules proposed here.

\section*{Appendix}

\appendix

\section{Proof of Theorem 2.1}

\begin{proof}
Since at $\lambda_m$ the dual optimal solution is known: $\theta(\lambda_m) = \frac{y}{n\lambda_m}$, Theorem 19 in in Wang et al. (2015) is applicable. Let $v_1(\lambda_m) = \text{sign}(x^T y)x$, $v_2(\lambda, \lambda_m) = \frac{y}{n\lambda} - \frac{y}{n\lambda_m}$, $v_2^\bot(\lambda, \lambda_m) = v_2(\lambda, \lambda_m) - \frac{(v_1(\lambda_m), v_2(\lambda, \lambda_m))}{\|v_1(\lambda_m)\|^2}v_1(\lambda_m)$, then the BEDPP rule for the lasso (1) (note our lasso formulation has a factor $1/n$) rejects the $j$th feature if

$$|x_j^T \left( \frac{y}{n\lambda_m} + \frac{1}{2}v_2^\bot(\lambda, \lambda_m) \right)| < 1 - \frac{1}{2}\|v_2^\bot(\lambda, \lambda_m)\|\|x_j\|.$$ 

Note that: (a) under conditions (2) $\|x_j\| = \sqrt{n}$, $\forall j$; (b) $x_j^T y = \text{sign}(x^T y)n\lambda_m$. With some algebra, it’s easy to show that $v_2^\bot(\lambda, \lambda_m) = \left( \frac{1}{n\lambda} - \frac{1}{n\lambda_m} \right)(y - \text{sign}(x^T y)\lambda_m x)$, and hence $\|v_2^\bot(\lambda, \lambda_m)\|$ can be simplified as $\left( \frac{1}{n\lambda} - \frac{1}{n\lambda_m} \right)\sqrt{y^Ty - n\lambda_m^2}$. Substituting these two pieces into the above inequality with some rearrangement yields to the simplified BEDPP.
\end{proof}

\section{Proof of Theorem 2.2}

\begin{proof}
In view of Corollary 20 in Wang et al. (2015), for $k = 0$ case, $v_1(\lambda_k)$, $v_2(\lambda_{k+1}, \lambda_k)$ and $v_2^\bot(\lambda_{k+1}, \lambda_k)$ reduce to those in Appendix A. So the SEDPP rule becomes the BEDPP rule.

For $0 < k < K$, let $v_1(\lambda_k) = \frac{x^T \hat{\theta}(\lambda_k)}{n\lambda_k}$, $v_2(\lambda_{k+1}, \lambda_k) = \frac{y - x^T \hat{\theta}(\lambda_k)}{n\lambda_k}$, $v_2^\bot(\lambda_{k+1}, \lambda_k) = v_2(\lambda_{k+1}, \lambda_k) - \frac{(v_1(\lambda_k), v_2(\lambda_{k+1}, \lambda_k))}{\|v_1(\lambda_k)\|^2}v_1(\lambda_k)$. According to Corollary 20 in Wang et al. (2015), the
(sequential) EDPP rule for the lasso (1) rejects the $j$th feature if
\[
\left| x_j^T \left( \frac{y - \hat{x} \beta(\lambda_k)}{n \lambda_m} + \frac{1}{2} \tilde{v}_2 (\lambda_{k+1}, \lambda_k) \right) \right| < 1 - \frac{1}{2} ||v_2^\perp (\lambda_{k+1}, \lambda_k)|| \|x_j\|.
\]
Denote $c = \frac{\lambda_k - \lambda_{k+1}}{\lambda_k \lambda_{k+1}}$, $a = y^T \hat{X} \hat{\beta}(\lambda_k)$. We first note that $v_2^\perp (\lambda_{k+1}, \lambda_k)$ can be simplified as:
\[
v_2^\perp (\lambda_{k+1}, \lambda_k) = v_2 (\lambda_{k+1}, \lambda_k) - \frac{\langle v_1 (\lambda_k), v_2 (\lambda_{k+1}, \lambda_k) \rangle}{\| v_1 (\lambda_k) \|^2} v_1 (\lambda_k)
= \frac{y}{n \lambda_{k+1}} - \frac{y - \hat{x} \beta(\lambda_k)}{n \lambda_k} - \frac{\hat{\beta}(\lambda_k)^T X^T (\lambda_k - \lambda_{k+1}) y + \lambda_{k+1} \hat{x} \beta(\lambda_k)}{n \lambda_k \lambda_{k+1} \| X \hat{\beta}(\lambda_k) \|^2}
= \frac{y}{n \lambda_{k+1}} - \frac{y - \hat{x} \beta(\lambda_k)}{n \lambda_k} - \frac{ac X \hat{\beta}(\lambda_k)}{n \| X \hat{\beta}(\lambda_k) \|^2} - \frac{X \hat{\beta}(\lambda_k)}{n \lambda_k}
= \frac{c}{n} \left( y - \frac{a X \hat{\beta}(\lambda_k)}{\| X \hat{\beta}(\lambda_k) \|^2} \right).
\]
Then with some algebra, $||v_2^\perp (\lambda_{k+1}, \lambda_k)||$ can be simplified to be $\frac{c}{n} \sqrt{||y||^2 - a^2/\| X \hat{\beta} \|^2}$.
Plugging the two terms back into the inequality of the SEDPP rule and with some rearrangement gives the simplified SEDPP rule and completes the proof.

C Proof of Theorem 4.1

Proof. Denote $\bar{X} = \left( \begin{array}{c} X \\ \sqrt{n(1 - \alpha)} \lambda \cdot I \end{array} \right)$, $\bar{y} = \left( \begin{array}{c} y \\ 0 \end{array} \right)$. The the elastic net problem can then be rewritten as,
\[
\hat{\beta}(\lambda, \alpha) = \arg\min_{\beta \in \mathbb{R}^p} \frac{1}{2n} (\bar{y} - \bar{X} \beta)' (\bar{y} - \bar{X} \beta) + \alpha \lambda \| \beta \|_1,
\]
which is in the form of the standard lasso with original $\lambda$ reparameterized with $\alpha \lambda$. Hence Theorem 19 in Wang et al. (2015) is applicable, provided $(X, y, \lambda_m, \lambda)$ is replaced by $(\bar{X}, \bar{y}, \alpha \lambda_m, \alpha \lambda)$. That is, the BEDPP rule rejects the $j$th feature if
\[
\left| \bar{x}_j^T \left( \frac{\bar{y}}{n \alpha \lambda_m} + \frac{1}{2} \tilde{v}_2^\perp (\lambda, \lambda_m) \right) \right| < 1 - \frac{1}{2} ||\tilde{v}_2^\perp (\lambda, \lambda_m)|| \|\bar{x}_j\|.
\]
Here $\lambda_m$ is reparameterized as $\lambda_m = \max_j |\mathbf{x}_j^T \mathbf{y}|/(n\alpha)$. \( \bar{v}_2^+(\lambda, \lambda_m) = \bar{v}_2(\lambda, \lambda_m) - \frac{(\mathbf{x}_1^T \mathbf{y}, \mathbf{v}_2^+(\lambda, \lambda_m))}{\|\bar{v}_1(\lambda_m)\|^2} \bar{v}_1(\lambda_m) \), where $\bar{v}_1(\lambda_m) = \text{sign}(\mathbf{x}_1^T \mathbf{y}) \mathbf{x}_*$, $\bar{v}_2(\lambda, \lambda_m) = \left( \frac{\bar{y}}{n\alpha} - \frac{\bar{y}}{n\alpha\lambda_m} \right)$.

On the other hand, it’s easy to verify that $\bar{x}_j^T \bar{y} = x_j^T y$, $\forall j$; $\|\bar{y}\| = \|y\|$; $\|\bar{x}_j\|^2 = \|x_j\|^2 + n\lambda(1 - \alpha)$; $\bar{x}_j^T \bar{x}_k = x_j^T x_k$, $\forall j \neq k$. With some algebra, $\bar{v}_2^+(\lambda, \lambda_m)$ can be simplified as follows,

\[
\bar{v}_2^+(\lambda, \lambda_m) = \frac{\bar{y}}{n\alpha\lambda} - \frac{\bar{y}}{n\alpha\lambda_m} - \frac{\text{sign}(x_j^T y)x_j^T y}{\|x_*\|^2} \left( \frac{1}{n\alpha\lambda} - \frac{1}{n\alpha\lambda_m} \right) \text{sign}(x_j^T y)x_*
\]

\[
= \bar{y} \left( \frac{1}{n\alpha\lambda} - \frac{1}{n\alpha\lambda_m} \right) - \frac{\text{sign}(x_j^T y)x_j^T y}{n(1 + \lambda(1 - \alpha))} x_*
\]

\[
= \left( \frac{1}{n\alpha\lambda} - \frac{1}{n\alpha\lambda_m} \right) \left( \bar{y} - \frac{\text{sign}(x_j^T y)\alpha\lambda_m}{1 + \lambda(1 - \alpha)} x_* \right),
\]

followed by which yields to that $\|\bar{v}_2^+(\lambda, \lambda_m)\| = \left( \frac{1}{n\alpha\lambda} - \frac{1}{n\alpha\lambda_m} \right) \sqrt{\|y\|^2 - \frac{n\alpha^2 \lambda_m^2}{1 + \lambda(1 - \alpha)}}$.

- If $\bar{x}_j = \bar{x}_*$, $x_j^T \bar{x}_* n(1 + \lambda(1 - \alpha))$, it can be shown that

\[
\left|\bar{x}_j^T \left( \frac{\bar{y}}{n\alpha\lambda_m} + \frac{1}{2} \bar{v}_2^+(\lambda, \lambda_m) \right) \right| = \frac{1}{2n\alpha\lambda\lambda_m} |(\lambda + \lambda_m) x_j^T y - (\lambda_m - \lambda) \text{sign}(x_j^T y) n\alpha\lambda_m| = 1,
\]

which is always larger than the RHS of (23). In other words, $x_*$ won’t be rejected.

- If $\bar{x}_j \neq \bar{x}_*$, $x_j^T \bar{x}_* = x_j^T x_*$. We have

\[
\left|\bar{x}_j^T \left( \frac{\bar{y}}{n\alpha\lambda_m} + \frac{1}{2} \bar{v}_2^+(\lambda, \lambda_m) \right) \right| = \frac{1}{2n\alpha\lambda\lambda_m} |(\lambda + \lambda_m) x_j^T y - (\lambda_m - \lambda) \frac{\text{sign}(x_j^T y)\alpha\lambda_m}{1 + \lambda(1 - \alpha)} x_j^T x_*|.
\]

Plugging this piece and the simplified $\|\bar{v}_2^+(\lambda, \lambda_m)\|$ into (23) with some additional algebra yields to the BEDPP rule for the elastic net (17).

\[\Box\]

**D Proof of Theorem 4.2**

*Proof.* We first note that it can be easily shown the dual optimal solution to the group lasso problem (18) at $\lambda_m$ is $\theta_{\lambda_m}^* = \frac{\mathbf{v}}{n\lambda_m}$. Denote $\mathbf{v} = X_* \mathbf{x}_*$, $\bar{v}_2(\lambda, \lambda_m) = \frac{\mathbf{y}}{n\lambda} - \theta_{\lambda_m}^*$, $\bar{v}_2^+(\lambda, \lambda_m) = \left( \frac{\mathbf{y}}{n\alpha\lambda} - \frac{\mathbf{y}}{n\alpha\lambda_m} \right) - \frac{\text{sign}(x_j^T y)x_j^T y}{n(1 + \lambda(1 - \alpha))} x_*$. We have

\[
\bar{v}_2(\lambda, \lambda_m) = \frac{\mathbf{y}}{n\alpha\lambda} - \frac{\mathbf{y}}{n\alpha\lambda_m} - \frac{\text{sign}(x_j^T y)x_j^T y}{n(1 + \lambda(1 - \alpha))} x_*.
\]

Plugging this piece and the simplified $\|\bar{v}_2^+(\lambda, \lambda_m)\|$ into (23) with some additional algebra yields to the BEDPP rule for the elastic net (17).
\( \mathbf{v}_2(\lambda, \lambda_m) - \frac{\langle \mathbf{v}_2(\lambda, \lambda_m), \mathbf{v} \rangle}{||\mathbf{v}||^2} \mathbf{v} \). According to Theorem 20 in Wang et al. (2015), for any \( \lambda \in (0, \lambda_m] \), we have the BEDPP rule that rejects the \( g \)th group of features (i.e., \( \hat{\beta}_g(\lambda) = 0 \)) if,

\[
\left\| \mathbf{X}_g^T \left( \mathbf{\theta}^*_{\lambda_m} + \frac{1}{2} \mathbf{v}_2^\top(\lambda, \lambda_m) \right) \right\| < \sqrt{W_g} - \frac{1}{2} ||\mathbf{v}_2^\top(\lambda, \lambda_m)|| ||\mathbf{X}_g||. \tag{24}
\]

Note \( \mathbf{v}_2^\top(\lambda, \lambda_m) \) can be simplified as follows.

\[
\mathbf{v}_2^\top(\lambda, \lambda_m) = \frac{\mathbf{v}}{\|\mathbf{v}\|^2} - \frac{\langle \mathbf{v}, \mathbf{v}_2(\lambda, \lambda_m) \rangle}{\|\mathbf{v}\|^2} \mathbf{v} = \frac{\mathbf{y}}{n} \left( \lambda - \frac{1}{\lambda} \right) - \frac{\mathbf{y}^T \mathbf{X}_* \mathbf{X}^T \mathbf{y} \mathbf{X}_* \mathbf{X}^T \mathbf{y}}{n} \left( \lambda - \frac{1}{\lambda} \right)
\]

\[
= \frac{\mathbf{y}}{n} \left( \lambda - \frac{1}{\lambda} \right) - \frac{\mathbf{X}_* \mathbf{X}^T \mathbf{y} \mathbf{X}_* \mathbf{X}^T \mathbf{y}}{n} \left( \lambda - \frac{1}{\lambda} \right)
\]

\[
= \frac{\mathbf{y}}{n} \left( \lambda - \frac{1}{\lambda} \right) \left( \mathbf{I} - \frac{\mathbf{X}_* \mathbf{X}^T \mathbf{y}}{n} \right),
\]

where the second equality is because \( \mathbf{X}^T \mathbf{X}_* = n\mathbf{I} \) under the condition (19). The left hand side of the rule then becomes,

\[
\left\| \mathbf{X}_g^T \left( \mathbf{\theta}^*_{\lambda_m} + \frac{1}{2} \mathbf{v}_2^\top(\lambda, \lambda_m) \right) \right\|
\]

\[
= \left\| \mathbf{X}_g^T \left( \frac{\mathbf{y}}{n \lambda_m} + \frac{1}{2n} \left( \lambda - \frac{1}{\lambda} \right) \left( \mathbf{I} - \frac{\mathbf{X}_* \mathbf{X}^T}{n} \right) \mathbf{y} \right) \right\|
\]

\[
= \left\| \frac{\mathbf{X}_g^T \mathbf{y}}{2n} \left( \lambda + \frac{1}{\lambda_m} \right) - \frac{1}{2n} \left( \lambda - \frac{1}{\lambda} \right) \frac{\mathbf{X}_g^T \mathbf{v}}{n} \right\|
\]

\[
= \frac{1}{2n \lambda \lambda_m} \left\| (\lambda + \lambda_m) \mathbf{X}_g^T \mathbf{y} - (\lambda_m - \lambda) \frac{\mathbf{X}_g^T \mathbf{v}}{n} \right\|
\]

\[
= \frac{1}{2n \lambda \lambda_m} \sqrt{(\lambda + \lambda_m)^2 \|\mathbf{X}^T_g \mathbf{y}\|^2 - \frac{2(\lambda^2 - \lambda_m^2)}{n} \mathbf{y}^T \mathbf{X}_g^T \mathbf{y} \mathbf{X}_g^T \mathbf{v} + \frac{(\lambda_m - \lambda)^2 \|\mathbf{X}^T_g \mathbf{v}\|^2}{n^2}}.
\]
The right hand side of the rule is,

\[ \sqrt{W_g} - \frac{1}{2} \| \bar{v}_2^+(\lambda, \lambda_m) \| \| X_g \| \]

\[ = \sqrt{W_g} - \frac{1}{2n} \left( \frac{1}{\lambda} - \frac{1}{\lambda_m} \right) \| X_g \| \sqrt{y^T \left( \frac{I - X_* X_*^T}{n} \right)^T \left( \frac{I - X_* X_*^T}{n} \right) y} \]

\[ = \sqrt{W_g} - \frac{1}{2n} \left( \frac{1}{\lambda} - \frac{1}{\lambda_m} \right) \| X_g \| \sqrt{\| y \|^2 - n \lambda^2 W_*} \]

Note that the second equality is due to that \( I - X_* X_*^T / n \) is idempotent; the last equality is because: (i) \( \| X_*^T y \| = n \sqrt{W_* \lambda_m} \), implied by the definitions of \( \lambda_m \) and \( X_* \); (ii) \( \| X_g \| = n \), again implied by the standardization condition (19). Here \( \| X_g \| \) is the matrix \( L_2 \) norm, which is equal to the largest singular value of \( X_g \).

Substituting the simplified results into (24) with some rearrangement yields the BEDPP rule for the group lasso.

\[ \square \]

References

Almasy, L., T. D. Dyer, J. M. Peralta, J. W. Kent, J. C. Charlesworth, J. E. Curran, and J. Blangero (2011). Genetic analysis workshop 17 mini-exome simulation. In BMC proceedings, Volume 5, pp. S2. BioMed Central.

Angelosante, D. and G. B. Giannakis (2009). Rls-weighted lasso for adaptive estimation of sparse signals. In IEEE International Conference on Acoustics, Speech and Signal Processing, pp. 3245–3248.

Boyd, S., N. Parikh, E. Chu, B. Peleato, and J. Eckstein (2011). Distributed optimization and statistical learning via the alternating direction method of multipliers. Foundations and Trends® in Machine Learning 3(1), 1–122.

Boyd, S. and L. Vandenberghe (2004). Convex optimization. Cambridge university press.
Breheny, P. and J. Huang (2015). Group descent algorithms for nonconvex penalized linear and logistic regression models with grouped predictors. *Statistics and Computing* 25(2), 173–187.

Consortium, G. P. et al. (2010). A map of human genome variation from population-scale sequencing. *Nature* 467(7319), 1061–1073.

Efron, B., T. Hastie, I. Johnstone, R. Tibshirani, et al. (2004). Least angle regression. *The Annals of statistics* 32(2), 407–499.

Fan, J. and J. Lv (2008). Sure independence screening for ultrahigh dimensional feature space. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 70(5), 849–911.

Friedman, J., T. Hastie, H. Höfling, R. Tibshirani, et al. (2007). Pathwise coordinate optimization. *The Annals of Applied Statistics* 1(2), 302–332.

Friedman, J., T. Hastie, and R. Tibshirani (2010). Regularization paths for generalized linear models via coordinate descent. *Journal of statistical software* 33(1), 1.

Garrigues, P. and L. E. Ghaoui (2009). An homotopy algorithm for the lasso with online observations. In *Advances in neural information processing systems*, pp. 489–496.

Ghaoui, L. E., V. Viallon, and T. Rabbani (2010). Safe feature elimination for the lasso and sparse supervised learning problems. *arXiv preprint arXiv:1009.4219*.

Huang, X. and W. Pan (2003). Linear regression and two-class classification with gene expression data. *Bioinformatics* 19(16), 2072–2078.

Kim, S.-J., K. Koh, M. Lustig, S. Boyd, and D. Gorinevsky (2007). An interior-point method for large-scale-regularized least squares. *IEEE journal of selected topics in signal processing* 1(4), 606–617.

Lee, H., A. Battle, R. Raina, and A. Y. Ng (2007). Efficient sparse coding algorithms. *Advances in neural information processing systems* 19, 801.
Lee, S. and P. Breheny (2015). Strong rules for nonconvex penalties and their implications for efficient algorithms in high-dimensional regression. *Journal of Computational and Graphical Statistics* 24(4), 1074–1091.

Li, Y., A. Algarni, M. Albathan, Y. Shen, and M. A. Bijaksana (2015). Relevance feature discovery for text mining. *IEEE T. Knowl. Data En.* 27(6), 1656–1669.

Lichman, M. (2013). UCI machine learning repository.

Meier, L., S. Van De Geer, and P. Bühlmann (2008). The group lasso for logistic regression. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 70(1), 53–71.

Qin, Z., K. Scheinberg, and D. Goldfarb (2013). Efficient block-coordinate descent algorithms for the group lasso. *Mathematical Programming Computation* 5(2), 143–169.

Shalev-Shwartz, S. and A. Tewari (2011). Stochastic methods for 11-regularized loss minimization. *Journal of Machine Learning Research* 12(Jun), 1865–1892.

Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society. Series B (Methodological)* 58(1), 267–288.

Tibshirani, R. (2011). Regression shrinkage and selection via the lasso: a retrospective. *Journal of the Royal Statistical Society. Series B (Statistical Methodology)* 73(3), 273–282.

Tibshirani, R., J. Bien, J. Friedman, T. Hastie, N. Simon, J. Taylor, and R. J. Tibshirani (2012). Strong rules for discarding predictors in lasso-type problems. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 74(2), 245–266.

Wang, J., P. Wonka, and J. Ye (2015). Lasso screening rules via dual polytope projection. *Journal of Machine Learning Research* 16, 1063–1101.

Wright, J., A. Y. Yang, A. Ganesh, S. S. Sastry, and Y. Ma (2009). Robust face recognition via sparse representation. *IEEE T. Pattern Anal.* 31(2), 210–227.

Wu, T. T. and K. Lange (2008). Coordinate descent algorithms for lasso penalized regression. *The Annals of Applied Statistics*, 224–244.
Xiang, Z. J. and P. J. Ramadge (2012). Fast lasso screening tests based on correlations. In *2012 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 2137–2140. IEEE.

Xiang, Z. J., Y. Wang, and P. J. Ramadge (2016). Screening tests for lasso problems. *IEEE Transactions on Pattern Analysis and Machine Intelligence* PP(99), 1–1.

Xiang, Z. J., H. Xu, and P. J. Ramadge (2011). Learning sparse representations of high dimensional data on large scale dictionaries. In *Advances in Neural Information Processing Systems*, pp. 900–908.

Yuan, M. and Y. Lin (2006). Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 68(1), 49–67.

Zeng, Y. and P. Breheny (2017). The biglasso package: A memory- and computation-efficient solver for lasso model fitting with big data in r. *ArXiv e-prints*.

Zou, H. and T. Hastie (2005). Regularization and variable selection via the elastic net. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 67(2), 301–320.