SCHRÖDINGER OPERATORS WITH PERIODIC SINGULAR POTENTIALS

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ABSTRACT. We show that formal Schrödinger operators with singular potentials from the space $W^{-1,\text{unif}}_{2,\text{loc}}(\mathbb{R})$ can be naturally defined to give selfadjoint and bounded below operators, which depend continuously in the uniform resolvent sense on the potential in the $W^{-1,\text{unif}}_{2,\text{loc}}(\mathbb{R})$-norm. In the case of periodic singular potentials we also establish pure absolute continuity and a band and gap structure of the spectrum thus generalising some classical results for singular potentials of one-dimensional quasicrystal theory.

1. Introduction

In a series of recent papers [1]–[4] Schrödinger operators have been considered with singular potentials that are distributions from the space $W^{-1,\text{loc}}_{2,\text{loc}}(\mathbb{R})$. More exactly, for a potential $q = \sigma' + \tau \in W^{-1,\text{loc}}_{2,\text{loc}}(\mathbb{R})$ with real-valued $\sigma \in L^2_{1,\text{loc}}(\mathbb{R})$ and $\tau \in L^1_{1,\text{loc}}(\mathbb{R})$ the corresponding Schrödinger operator

$$S = -\frac{d^2}{dt^2} + q$$

is defined through

$$Su = l(u) := -(u' - \sigma u)' - \sigma u' + \tau u$$

on the domain

$$\mathcal{D}(S) = \{ u \in W^1_{1,\text{loc}}(\mathbb{R}) \mid u^{[1]} := u' - \sigma u \in W^1_{1,\text{loc}}(\mathbb{R}), \ l(u) \in L^2(\mathbb{R}) \}.$$

It is easily seen that $l(u) = -u'' + qu$ in the sense of distributions, which implies, firstly, that the operator $S$ does not depend on the particular choice of $\sigma \in L^2_{1,\text{loc}}(\mathbb{R})$ and $\tau \in L^1_{1,\text{loc}}(\mathbb{R})$ in the decomposition $q = \sigma' + \tau$ and, secondly, that for regular potentials $q \in L^1_{1,\text{loc}}(\mathbb{R})$ the above definition coincides with the classical one. Moreover, the operator $S$ is shown to be selfadjoint and bounded below if $\sigma$ is compactly supported and $\tau$ is in the limit point case at $\pm \infty$ [1] or if $q \in W^{-1}_{2,\text{loc}}(\mathbb{R})$ [2].

The regularisation by quasi-derivatives procedure was first suggested in [5] for the potential $1/x$ on a finite interval (see also the books [6] and [7] for a detailed exposition of quasi-differential operators). The more general setting (1.1)–(1.3) developed in [1] allows to consider, e.g., very important cases of Coulomb $1/x$- and Dirac $\delta$-like potentials used to model short- and zero-range interactions in quantum mechanics by taking $\sigma(x) = \log |x|$ and $\sigma(x) = \chi(x)$, the Heaviside function, respectively. These two models as well as their generalizations to potentials that are singular (i.e., not locally integrable) on a discrete set were treated in many works, see, e.g., [5], [8]–[15].
and the references therein. In the present paper the potential \( q \in W_{2,\text{loc}}^{-1}(\mathbb{R}) \) is not assumed locally integrable anywhere, though the singularities cannot be too strong (say, \( \delta' \)-interactions are not allowed).

We remark that it has been realised for a long time that differential expressions of (1.1) with singular potentials do not generally determine a unique operator in \( L_2(\mathbb{R}) \). However, the operator \( S \) defined by (1.2)–(1.3) appears to be a “natural” selfadjoint operator associated with (1.1) for a potential \( q \in W_{2}^{-1}(\mathbb{R}) \) in the sense that if \( q_n \) is any sequence of regular (infinitely smooth say) potentials that converges to \( q \) in \( W_{2}^{-1}(\mathbb{R}) \), then the corresponding Schrödinger operators \( S_n \) converge to \( S \) in the uniform resolvent sense. This fact is established in [1] for regular Sturm-Liouville operators on a finite interval or Schrödinger operators with compactly supported \( \sigma \) and \( \tau \) in the limit point case at \( \pm \infty \) and in [2] for \( q \in W_{2}^{-1}(\mathbb{R}) \) or for a more general situation of polyharmonic operators in \( \mathbb{R}^n \) with potentials from an appropriate space of multipliers. See also [16] and [17] for convergence results for Schrödinger operators with potentials that are Radon measures and with \( \delta' \)-potentials, respectively, and [18] for an abstract setting of form-bounded singular perturbations.

The main aim of this note is to study Schrödinger operators with singular potentials from the space \( W_{2,\text{unif}}^{-1}(\mathbb{R}) \), and in particular with periodic singular potentials. While this problem apparently did not receive much attention in the above-cited works (potentials from the class \( W_{2,\text{unif}}^{-s}(\mathbb{R}) \), \( s > -1 \), were considered in [10]), the particular cases of periodic and quasiperiodic \( \delta \)-interactions were quite well understood within the framework of quasicrystal theory in quantum mechanics, cf. Kronig-Penney theory and its various generalizations in [9, Ch. III.2]. For instance, the Schrödinger operators with periodic \( \delta \)-interactions are shown to have purely absolutely continuous band spectrum (see also [8] and [14] for more general cases of periodic singular point-like interactions). Moreover, if \( q_n \) is a sequence of regular short-range interactions that converges to a sum \( q \) of \( \delta \)-interactions in the sense of quadratic forms (which incidentally implies \( W_{2,\text{unif}}^{-1}(\mathbb{R}) \)-convergence), then the corresponding Schrödinger operators \( S_n \) converge to \( S \) in the uniform resolvent sense. These two results are generalized in the present paper to an arbitrary potential from \( W_{2,\text{unif}}^{-1}(\mathbb{R}) \).

The main results of our paper are as follows. In Section 2 we define the space \( W_{2,\text{unif}}^{-1}(\mathbb{R}) \) and show that any real-valued \( q \in W_{2,\text{unif}}^{-1}(\mathbb{R}) \) can be represented (not uniquely) in the form \( q = \sigma' + \tau \), where \( \sigma \) and \( \tau \) are real-valued functions from \( L_{2,\text{unif}}(\mathbb{R}) \) and \( L_{1,\text{unif}}(\mathbb{R}) \), respectively, i.e.,

\[
\|\sigma\|_{2,\text{unif}}^2 := \sup_{t \in \mathbb{R}} \int_t^{t+1} |\sigma(s)|^2 ds < \infty,
\]

\[
\|\tau\|_{1,\text{unif}} := \sup_{t \in \mathbb{R}} \int_t^{t+1} |\tau(s)| ds < \infty,
\]

and the derivative is understood in the sense of distributions. Then in Section 3 we prove that the operator \( S \) with \( q = \sigma' + \tau \in W_{2,\text{unif}}^{-1}(\mathbb{R}) \) as defined by (1.2)–(1.3) coincides with the form-sum operator in (1.1) and hence is selfadjoint and bounded below. In Section 4 we establish the uniform resolvent convergence result for \( W_{2,\text{unif}}^{-1}(\mathbb{R}) \)-convergence of potentials. Finally, in the last section we consider a periodic singular potential \( q \in W_{2,\text{unif}}^{-1}(\mathbb{R}) \) and prove that the corresponding operator \( S \) has an absolutely continuous spectrum.

Our results can be illustrated by the following model example.
Example 1.1. Consider the operator $-\Delta_{\alpha,\Lambda}$ of the Kronig-Penney theory with periodic lattice $\Lambda = \{na \mid n \in \mathbb{Z}\}$ [9, Ch. III.2.3]; it corresponds formally to the potential
\[ q = \sum_{n \in \mathbb{Z}} \alpha \delta(\cdot - na) \]
and is defined rigorously by acting as $-\frac{d^2}{dt^2}$ on the domain
\[ \mathcal{D}(-\Delta_{\alpha,\Lambda}) = \{ u \in W^2_2(\mathbb{R}) \cap W^2_2(\mathbb{R} \setminus \Lambda) \mid u'(na+) - u'(na-) = \alpha u(na), \ n \in \mathbb{Z} \}. \]
We represent $q$ above as $\sigma' + \tau$ with $\tau \equiv \alpha/a$ and $a$-periodic $\sigma$ equal to $\alpha/2 - \alpha t/a$ on $[0, a)$; then the corresponding operator $S$ is easily seen to be exactly $-\Delta_{\alpha,\Lambda}$. Our result implies absolute continuity of the spectrum of $-\Delta_{\alpha,\Lambda}$. Although this statement is well known in the Kronig-Penney theory, its proof within this theory heavily uses an explicit form of the resolvent of $-\Delta_{\alpha,\Lambda}$, which would not be possible for more general periodic $q \in W^{-1}_{2,\text{unif}}(\mathbb{R})$.

Throughout the paper $W^s_2(\mathbb{R})$, $s \in \mathbb{R}$, will denote the standard Sobolev space, $\| \cdot \|$ without any subscript will always stand for the $L^2(\mathbb{R})$-norm and $f^{[1]}$ for the quasi-derivative $f' - \sigma f$ of a function $f$.

2. Structure of the space $W^{-1}_{2,\text{unif}}(\mathbb{R})$

We recall that $W^{-1}_{2}(\mathbb{R})$ is the dual space to the Sobolev space $W^1_2(\mathbb{R})$, i.e., it consists of those distributions [19] that define continuous functionals on $W^1_2(\mathbb{R})$. With $\langle \cdot, \cdot \rangle$ denoting the duality, we have for $f \in W^{-1}_{2}(\mathbb{R})$
\[ \|f\|_{W_{2}^{-1}(\mathbb{R})} := \sup_{0 \neq \psi \in W^1_2(\mathbb{R})} \frac{|\langle \psi, f \rangle|}{\|\psi\|_{W^1_2(\mathbb{R})}}. \]

The local uniform analogue of this space is defined as follows. Put
\[ \phi(t) := \begin{cases} 2(t+1)^2 & \text{if } t \in [-1, -1/2), \\ 1 - 2t^2 & \text{if } t \in [-1/2, 1/2), \\ 2(t-1)^2 & \text{if } t \in [1/2, 1], \\ 0 & \text{otherwise}, \end{cases} \quad (2.1) \]
and $\phi_n(t) := \phi(t - n)$ for $n \in \mathbb{Z}$. We say that $f$ belongs to $W^{-1}_{2,\text{unif}}(\mathbb{R})$ if $f \phi_n$ is in $W^{-1}_{2}(\mathbb{R})$ for all $n \in \mathbb{Z}$ and
\[ \|f\|_{W_{2,\text{unif}}^{-1}(\mathbb{R})} := \sup_{n \in \mathbb{Z}} \|f \phi_n\|_{W_{2}^{-1}(\mathbb{R})} < \infty. \]

Our main aim of this section is to prove the following structure theorem.

Theorem 2.1. For any $f \in W^{-1}_{2,\text{unif}}(\mathbb{R})$ there exist functions $\sigma \in L^1_{2,\text{unif}}(\mathbb{R})$ and $\tau \in L^1_{1,\text{unif}}(\mathbb{R})$ such that $f = \sigma' + \tau$ and
\[ C^{-1}(\|\sigma\|_{2,\text{unif}} + \|\tau\|_{1,\text{unif}}) \leq \|f\|_{W_{2,\text{unif}}^{-1}(\mathbb{R})} \leq C(\|\sigma\|_{2,\text{unif}} + \|\tau\|_{1,\text{unif}}) \]
with some constant $C$ independent of $f$. Moreover, the function $\tau$ can be chosen uniformly bounded.

We say that $f \in W_{2}^{-1}(\mathbb{R})$ vanishes on an open set $U$ if $\langle f, \psi \rangle = 0$ whenever $\psi \in W^1_2(\mathbb{R})$ has its support in $U$. The support supp$f$ of $f$ is the complement of the largest open set on which $f$ vanishes. It follows that $\langle f, \psi \rangle$ only depends on the values of $\psi$ in a neighbourhood of supp$f$, i.e., $\langle f, \psi_1 \rangle = \langle f, \psi_2 \rangle$ whenever $\psi_1 = \psi_2$ on some open set
containing $\text{supp } f$. In particular, for $f$ with compact support the number $\langle f, 1 \rangle$ can be defined as $\langle f, \psi \rangle$ for any test function $\psi$ that is identically one on a neighbourhood of $\text{supp } f$.

The crux of the proof of Theorem 2.1 is the following

**Lemma 2.2.** Suppose that $f \in W^{-1}_2(\mathbb{R})$, supp $f \subset [-1, 1]$, and that $\langle f, 1 \rangle = 0$. Then there exists a function $\sigma \in L_2(\mathbb{R})$ with supp $\sigma \subset [-1, 1]$ such that $f = \sigma'$ and $\|\sigma\| \leq C\|f\|_{W^{-1}_2(\mathbb{R})}$ for some positive constant $C$ independent of $f$.

**Proof.** Denote by $\psi_0$ any test function with supp $\psi_0 \subset (-1, 1)$ and $\langle 1, \psi_0 \rangle = 1$. We define a distribution $\sigma$ by the identity
\[
\langle \sigma, \psi \rangle = -\langle f, J\psi \rangle \tag{2.3}
\]
where $\psi$ runs over all test functions and
\[
(J\psi)(t) := \int_{-\infty}^{t} \left(\psi(s) - \langle 1, \psi \rangle \psi_0(s + 2)\right) ds.
\]
It is easily seen that $J\psi(t) = 0$ for all $|t|$ sufficiently large and therefore $J\psi$ is a test function and $\sigma$ is well defined. Observe that $J\psi' = \psi$ for any test function $\psi$, hence (2.3) yields $f = \sigma'$ by definition.

Suppose next that supp $\psi \cap [-1, 1] = \emptyset$; then $J\psi \equiv \text{const}$ on some neighbourhood of $[-1, 1]$, whence $\langle f, J\psi \rangle = 0$ by assumption and supp $\sigma \subset [-1, 1]$.

Finally we show that $\sigma \in L_2(\mathbb{R})$ and that $\|\sigma\| \leq C\|f\|_{W^{-1}_2(\mathbb{R})}$ for some constant $C$ independent of $f$. To this end it suffices to show that the operator $J$ acts boundedly from $L_2[-1, 1]$ into $W^1_2(\mathbb{R})$ as (2.3) then implies
\[
\|\sigma\| \leq \|J\|\|f\|_{W^{-1}_2(\mathbb{R})}
\]
and we can take $C = \|J\|$.

Suppose that $\psi$ is a test function with support in $[-1, 1]$. Then supp $J\psi \subset [-3, 1]$ and since $\left| \int_{-\infty}^{t} \psi(s) ds \right|^2 \leq 2\|\psi\|^2$, we have
\[
\|J\psi\|^2 \leq 4\|\psi\|^2 + 8\|\psi_0\|^2\|\psi\|^2
\]
and
\[
\|(J\psi)'\|^2 = \|\psi\|^2 + 2\|\psi_0\|^2\|\psi\|^2.
\]
This shows that the norm of $J$ as an operator from $L_2[-1, 1]$ into $W^1_2(\mathbb{R})$ does not exceed $4(1 + \|\psi_0\|)$, and the proof is complete. \hfill $\Box$

**Proof of Theorem 2.1.** For a given $f \in W^{-1}_{2,\text{unif}}(\mathbb{R})$ and $n \in \mathbb{Z}$ we put
\[
f_n := f\phi_n - a_n\chi_{[n-1/2,n+1/2]},
\]
where $a_n := \langle f\phi_n, 1 \rangle$ and $\chi_{\Delta}$ is the characteristic function of an interval $\Delta$. Then $f_n$ satisfies the assumptions of Lemma 2.2 with the interval $[-1, 1]$ replaced by $[n-1, n+1]$, and therefore for every $n \in \mathbb{Z}$ there exists a function $\sigma_n \in L_2(\mathbb{R})$ such that $f_n = \sigma'_n$ and supp $\sigma_n \subset [n-1, n+1]$. It is easily seen that with
\[
\sigma := \sum_{n \in \mathbb{Z}} \sigma_n \quad \text{and} \quad \tau := \sum_{n \in \mathbb{Z}} a_n\chi_{[n-1/2,n+1/2]}
\]
we have $f = \sigma' + \tau$, so it remains to show that $\sigma$ and $\tau$ belong to $L_{2,\text{unif}}(\mathbb{R})$ and $L_{1,\text{unif}}(\mathbb{R})$ respectively and that inequality (2.2) holds.
Denote by $\psi_n$ a $W^1_2(\mathbb{R})$-function with support in $[n-2, n+2]$ that is identically one on $[n-3/2, n+3/2]$ and linear on $[n-2, n-3/2]$ and $[n+3/2, n+2]$. Then $\|\psi_n\|_{W^1_2(\mathbb{R})} \leq 3$, whence

$$|a_n| = \langle f \phi_n, \psi_n\rangle \leq 3\|f \phi_n\|_{W^{-1}_2(\mathbb{R})} \leq 3\|f\|_{W^{-1}_2(\mathbb{R})}$$

and

$$\|f_n\|_{W^{-1}_2(\mathbb{R})} \leq \|f \phi_n\|_{W^{-1}_2(\mathbb{R})} + |a_n|\|\chi_{[n-1/2, n+1/2]}\|_{W^{-1}_2(\mathbb{R})} \leq 4\|f\|_{W^{-1}_2(\mathbb{R})}.$$ 

The above inequalities yield the estimates

$$\|\tau\|_{1,\text{unif}} \leq \sup |\tau| \leq 3\|f\|_{W^{-1}_2(\mathbb{R})}$$

and

$$\|\sigma\|_{2,\text{unif}} \leq 8\|J\|\|f\|_{W^{-1}_2(\mathbb{R})},$$

which establishes one part of the inequality required. For the second one we observe that for $\sigma \in L_{2,\text{unif}}(\mathbb{R})$ and $\tau \in L_{1,\text{unif}}(\mathbb{R})$ it holds

$$\langle \sigma' \phi_n, \psi \rangle = \langle \sigma, (\phi_n \psi)' \rangle \leq 2\|\sigma\|_{2,\text{unif}}\|\phi_n \psi\| \leq 6\|\sigma\|_{2,\text{unif}}\|\psi\|_{W^1_2(\mathbb{R})}$$

and

$$\langle \tau \phi_n, \psi \rangle \leq 2\|\tau\|_{1,\text{unif}} \sup |\psi| \leq 2\|\tau\|_{1,\text{unif}}\|\psi\|_{W^1_2(\mathbb{R})}.$$ 

Here we have used the inequality $\sup |\psi| \leq \|\psi\|_{W^1_2(\mathbb{R})}$, which follows from the relations

$$|\psi(t)|^2 = 2 \int_{-\infty}^{t} \text{Re} \psi^* \overline{\psi} \leq \int_{-\infty}^{t} (|\psi'|^2 + |\psi|^2) \leq \|\psi\|^2_{W^1_2(\mathbb{R})}.$$ 

Therefore $\sigma'$ and $\tau$ also belong to the space $W^{1-1}_{2,\text{unif}}(\mathbb{R})$ and, moreover,

$$\|\sigma'\|_{W^{1-1}_{2,\text{unif}}(\mathbb{R})} \leq 6\|\sigma\|_{2,\text{unif}},$$

$$\|\tau\|_{W^{1-1}_{2,\text{unif}}(\mathbb{R})} \leq 2\|\tau\|_{1,\text{unif}},$$

and the theorem is proved. \(\square\)

**Remark 2.3.** We say that a distribution $f$ is $T$-periodic if $\langle f, \psi(t) \rangle = \langle f, \psi(t+T) \rangle$ for any test function $\psi$. It is easily seen that for a 1-periodic potential $f \in W^{1-1}_{2,\text{unif}}(\mathbb{R})$ the above construction gives a 1-periodic function $\sigma$ and a constant function $\tau \equiv \langle \hat{f} \phi_0, 1 \rangle$. If $f$ is $T$-periodic, we first apply the construction to the 1-periodic potential $\hat{f}(t) := f(Tt)$ to write $\hat{\sigma} = \hat{\sigma}' + \hat{\tau}$ with 1-periodic $\hat{\sigma}$ and $\hat{\tau} \equiv \langle \hat{f} \phi_0, 1 \rangle$, and then after rescaling we get $f = \sigma' + \tau$ with $T$-periodic $\sigma(t) := T\hat{\sigma}(t/T)$ and $\tau := \hat{\tau}$.

### 3. Selfadjointness of the Operator $S$

In this section, we shall prove that the operator $S$ as given by (1.2) and (1.3) is selfadjoint and bounded below. In fact, we shall show that the quadratic form of the operator $S$ coincides with

$$t(u) := (u', u') - (\sigma u', u) - (\sigma u, u') + (\tau u, u)$$

and the latter is a relatively bounded perturbation of the form $t_0(u) := (u', u') + (u, u)$ with relative bound zero. Therefore $t$ is closed and bounded below on the domain $W^1_2(\mathbb{R})$, whence $S$ is a selfadjoint bounded below operator and $\mathcal{D}(S) \subset \mathcal{D}(t) = W^1_2(\mathbb{R})$. 

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Lemma 3.1. For any $f \in W^1_2(0,1]$ and any $\varepsilon \in (0,1]$ the following inequalities hold:

$$\max_{t \in [0,1]} |f(t)|^2 \leq \varepsilon \int_0^1 |f'|^2 dt + 8\varepsilon^{-1} \int_0^1 |f|^2 dt, \quad (3.1)$$

$$\left( \int_0^1 |f|^2 dt \right)^{1/2} \leq \varepsilon \int_0^1 |f'|^2 dt + 4\varepsilon^{-3} \int_0^1 |f|^2 dt. \quad (3.2)$$

Proof. For an arbitrary function $\phi \in W^1_2(\mathbb{R})$ and any $\eta > 0$ we find that

$$|\phi(t)|^2 = \int_{-\infty}^t \frac{d}{ds} |\phi(s)|^2 ds = 2 \int_{-\infty}^t \text{Re} \phi \overline{\phi} ds \leq \eta \|\phi\|^2 + \eta^{-1} \|\phi\|^2.$$

Given a function $f \in W^1_2[0,1]$, we extend it to $\phi \in W^1_2(\mathbb{R})$ through

$$\phi(t) := \begin{cases} f(t) & \text{if } t \in [0,1], \\ f(2-t)(2-t) & \text{if } t \in (1,2), \\ f(-t)(1+t) & \text{if } t \in [-1,0), \\ 0 & \text{otherwise}; \end{cases}$$

then

$$\|\phi\|^2 \leq 3\|f\|_{L^2_2(0,1)}^2 + 4\|f\|_{L^2_2(0,1)}^2 \quad \text{and} \quad \|\phi\|^2 \leq 2\|f\|_{L^2_2(0,1)}^2.$$

Therefore

$$\max_{t \in [0,1]} |f(t)|^2 \leq \eta \|\phi\|^2 + \eta^{-1} \|\phi\|^2 \leq 3\eta \|f\|^2_{L^2_2(0,1)} + (4\eta + 2\eta^{-1}) \|f\|_{L^2_2(0,1)}^2,$$

which implies (3.1) upon setting $\varepsilon = 3\eta \leq 1$. Using (3.1) with $\varepsilon^2$ instead of $\varepsilon$, we derive the inequality

$$\|f\|_{L^2_2(0,1)}^2 \leq \max_{t \in [0,1]} |f(t)|^2 \|f\|^2_{L^2_2(0,1)} \leq \varepsilon^2 \|f\|^4_{L^2_2(0,1)} + 8\varepsilon^{-2} \|f\|^2_{L^2_2(0,1)} \|f\|_{L^2_2(0,1)}^2$$

$$\leq \left( \varepsilon \|f\|^2_{L^2_2(0,1)} + 4\varepsilon^{-3} \|f\|_{L^2_2(0,1)}^2 \right)^2,$$

and the proof is complete. \hfill  \Box

Lemma 3.2. The quadratic form $t$ is closed and bounded below on $W^1_2(\mathbb{R})$.

Proof. Suppose that $\sigma \in L_{2,\text{unif}}(\mathbb{R})$, $\tau \in L_{1,\text{unif}}(\mathbb{R})$, and $u \in W^1_2(\mathbb{R})$. Using the relations (3.1) and (3.2), we find that, with an arbitrary $\varepsilon \in (0,1]$ and $\eta \in (0,1]$,

$$|(\sigma u', u)| \leq \sum_{n \in \mathbb{Z}} \int_n^{n+1} |\sigma u'| dt \leq \sum_{n \in \mathbb{Z}} \left( \int_n^{n+1} |\sigma|^2 dt \right)^{1/2} \left( \int_n^{n+1} |u' u|^2 dt \right)^{1/2}$$

$$\leq \|\sigma\|_{L^2_{2,\text{unif}}} (\varepsilon \|u'\|^2 + 4\varepsilon^{-3} \|u\|^2)$$

and

$$|(\tau u, u)| \leq \sum_{n \in \mathbb{Z}} \int_n^{n+1} |\tau u| dt \leq \sum_{n \in \mathbb{Z}} \int_n^{n+1} |\tau| dt \max_{t \in [n,n+1]} |u(t)|^2$$

$$\leq \|\tau\|_{L^1_{1,\text{unif}}} (\eta \|u'\|^2 + 8\eta^{-1} \|u\|^2).$$

This shows that the quadratic form $(\sigma u', u) + (\sigma u, u') + (\tau u, u)$ is bounded with respect to the form $t_0$ with relative bound zero. Therefore by the KLMN theorem (see [20, Theorem X.17]) the quadratic form $t$ is closed and bounded below on the domain $D(t) = D(t_0) = W^1_2(\mathbb{R})$. The lemma is proved. \hfill  \Box
Remark 3.3. Using the above inequalities with $\varepsilon = \min \{1, (4\|\sigma\|_{2, \text{unif}})^{-1}\}$ and $\eta = \min \{1, (2\|\tau\|_{1, \text{unif}})^{-1}\}$, we find that the quadratic form $t$ is bounded below by
\[
\gamma(t) := -(2(4\|\sigma\|_{2, \text{unif}})^4 + 16\|\tau\|_{1, \text{unif}}^2 + 6).
\]
Recalling Theorem 2.1 we can recast this as
\[
\gamma(t) \geq -(a\|q\|_{W_{2, \text{unif}}^{-1}(\mathbb{R})} + b)^4
\]
with some $a, b > 0$ independent of $q$.

Denote by $T$ a selfadjoint operator that is associated with the form $t$ according to the second representation theorem (see, e. g., [21, Theorem VI.2.23]). Recall that $T$ is the selfadjoint operator for which $D(T) \subset D(t)$ and the equality $(Tu, v) = t(u, v)$ holds for all $u \in D(T)$ and all $v \in D(t)$.

Theorem 3.4. The operator $S$ coincides with $T$. In particular, the operator $S$ is selfadjoint, bounded below, and $D(S) \subset W_{2}^{1}(\mathbb{R})$.

Proof. Fix $u \in D(T)$ and take an arbitrary $v \in D(t)$. Then $u \in W_{2}^{1}(\mathbb{R})$ and by (3.1) with $\varepsilon = \sqrt{8}$
\[
\int_{\mathbb{R}} |\sigma u|^2 \leq \sum_{n \in \mathbb{Z}} \int_{n}^{n+1} |\sigma u|^2 \leq \|\sigma\|_{2, \text{unif}}^2 \sum_{n \in \mathbb{Z}} \max_{t \in [n, n+1]} |u(t)|^2 \leq \sqrt{8}\|\sigma\|_{2, \text{unif}}^2 (\|u'||^2 + \|u||^2) < \infty.
\]
Therefore $\sigma u \in L_{2}(\mathbb{R})$ and $u^{[1]} \in L_{2}(\mathbb{R})$; now
\[
(Tu, v) = t(u, v) = (u', v) - (\sigma u, v') - (\sigma u', v) + (\tau u, v)
\]
and hence $Tu = -(u^{[1]}' - \sigma u' + \tau u$ in the sense of distributions. Observe that $Tu \in L_{2}(\mathbb{R}) \subset L_{1, \text{loc}}(\mathbb{R})$ and $\sigma u', \tau u \in L_{1, \text{loc}}(\mathbb{R})$; it follows that $(u^{[1]})' \in L_{1, \text{loc}}(\mathbb{R})$ and so $u^{[1]} \in W_{1, \text{loc}}^{1}(\mathbb{R})$. Therefore $u \in D(S)$ and $T \subset S$; since $S$ is evidently a symmetric operator, we conclude that $T = S$. The proof is complete. \hfill $\Box$

4. Continuous dependence on the potential

In this section, we shall prove that the Schrödinger operator $S$ defined by (1.2)–(1.3) depends continuously (in the sense of the uniform resolvent convergence) on the potential $q$ in the $W_{2, \text{unif}}^{-1}(\mathbb{R})$-norm. We remark that this generalizes the convergence results with respect to the topology of $W_{2}^{-1}(\mathbb{R})$ of [2] and the $\ast$-weak topology of Radon measures of [16].

Theorem 4.1. Suppose that $q_{n} \in W_{2, \text{unif}}^{-1}(\mathbb{R})$, $n \in \mathbb{N}$, is a sequence of potentials that converges to $q$ in $W_{2, \text{unif}}^{-1}(\mathbb{R})$-norm and that $S_{n}$ and $S$ are the corresponding Schrödinger operators. Then $S_{n}$ converge to $S$ as $n \to \infty$ in the uniform resolvent sense, i. e.,
\[
\| (S_{n} - \lambda)^{-1} - (S - \lambda)^{-1} \| \to 0 \quad \text{as} \quad n \to \infty
\]
for any $\lambda$ in the resolvent set of $S$ and $S_{n}$, $n \in \mathbb{N}$. 


Proof. Observe first that the corresponding quadratic forms \( t_n \) and \( t \) have the same domain \( W_2^1(\mathbb{R}) \) and are uniformly bounded below (see (3.3)). We choose sequences \( \sigma_n \in L_{2,\text{unif}}(\mathbb{R}) \) and \( \tau_n \in L_{1,\text{unif}}(\mathbb{R}) \) so that \( q_n - q = \sigma_n' + \tau_n \) and

\[
\|\sigma_n\|_{2,\text{unif}} + \|\tau_n\|_{1,\text{unif}} \leq C\|q_n - q\|_{W_2^{-1}\text{unif}}(\mathbb{R})
\]

with the constant \( C \) of Theorem 2.1. Repeating the arguments of the proof of Lemma 3.2 we find that

\[
|t_n(u) - t(u)| \leq |(\sigma_n u', u)| + |(\sigma_n u, u')| + |(\tau_n u, u)|
\]

\[
\leq C_1 (\|\sigma_n\|_{2,\text{unif}} + \|\tau\|_{1,\text{unif}}) \|u\|^2_{W_2^1(\mathbb{R})} \leq C_1 C \|q_n - q\|_{W_2^{-1}\text{unif}}(\mathbb{R}) \|u\|^2_{W_2^1(\mathbb{R})}
\]

for all \( u \in W_2^1(\mathbb{R}) \) and some constant \( C_1 > 0 \). The claim follows now from [22, Theorem VIII.25c], and the proof is complete. \( \square \)

Remark 4.2. An alternative way to prove Theorem 4.1 is to use the results of Section 3 to show that the space \( W_2^{-1}\text{unif}(\mathbb{R}) \) is embedded into the space of multipliers \( M_0[1] \); then the claim of Theorem 4.1 follows from Theorem 9 of [2].

5. Periodic potentials

Suppose now that the potential \( q \in W_2^{-1}\text{unif}(\mathbb{R}) \) is 1-periodic; then (recall Remark 2.3) \( q = \sigma' + \tau \) with 1-periodic \( \sigma \in L_{2,\text{unif}}(\mathbb{R}) \) and \( \tau \equiv \langle q\phi_0, 1 \rangle \). The purpose of this section is to show that in this case the spectrum of \( S \) is absolutely continuous and has a band and gap structure.

Proof of absolute continuity of \( S \) follows the standard spectral analysis of periodic Schrödinger operators (cf. [23, Ch. XIII.16]). We decompose the space \( L_2(\mathbb{R}) \) into a direct integral

\[
\int_{[0,2\pi]} \mathcal{H}' \frac{d\theta}{2\pi} =: \mathcal{H}
\]

with identical fibres \( \mathcal{H}' := L_2[0,1] \); then the operator \( U : L_2(\mathbb{R}) \rightarrow \mathcal{H} \) defined by

\[
(Uf)(x, \theta) = \sum_{n=-\infty}^{\infty} e^{-in\theta} f(x + n)
\]

is unitary. Now the operator \( \tilde{S} := USU^{-1} \) is unitarily equivalent to \( S \) and can be decomposed into the direct integral

\[
\tilde{S} = \int_{[0,2\pi]} S_\theta \frac{d\theta}{2\pi},
\]

where \( S_\theta \) is the operator in \( \mathcal{H}' \) defined by

\[
S_\theta f = l(f)
\]

on the domain

\[
\mathcal{D}(S_\theta) = \{ f \in W_1^{1,\text{loc}}[0,1] \mid f^{[1]} \in W_1^{1,\text{loc}}[0,1], \ l(f) \in \mathcal{H}', \ f^{[1]}(1) = e^{i\theta} f^{[1]}(0), f(1) = e^{i\theta} f(0) \}.
\]

To show this, we denote by \( \tilde{S} \) the operator given by the right hand side of (5.2). For any compactly supported \( f \in \mathcal{D}(S) \) the sum in (5.1) is finite and hence the relations \( Uf \in \mathcal{D}(\tilde{S}) \) and \( \tilde{SU}f = USf \) are straightforward. Next we observe that the functions \( f \in \mathcal{D}(S) \) with compact support constitute a core of \( S \). Therefore for any \( f \in \mathcal{D}(S) \)
there exist compactly supported $f_n \in \mathcal{D}(S)$ such that $f_n \to f$ and $S f_n \to S f$ in $L_2(\mathbb{R})$ as $n \to \infty$; then $U f_n \to U f$ and $S U f_n = U S f_n \to U S f$ in $\mathcal{H}$ as $n \to \infty$. As $\tilde{S}$ is closed we get $U f \in \mathcal{D}(\tilde{S})$ and $S U f = U S f$; therefore $\tilde{S} = US^{-1} \subset \tilde{S}$ and $\tilde{S} = \tilde{S}$ since both these operators are selfadjoint.

It follows from decomposition (5.2) that a number $\lambda$ belongs to the spectrum $\sigma(\tilde{S})$ of the operator $\tilde{S}$ if and only if for any $\epsilon > 0$

$$d\mu\{\theta \in [0, \pi] \mid \sigma(S_\theta) \cap (\lambda - \epsilon, \lambda + \epsilon) \neq \emptyset\} > 0,$$

(5.3)

where $d\mu$ denotes the Lebesgue measure on $\mathbb{R}$ ([23, Theorem XIII.85d]).

Observe [1] that all operators $S_\theta$ have discrete spectra. We shall prove the following result.

**Lemma 5.1.** For every fixed nonreal $\lambda$ the resolvent $(S_\theta - \lambda)^{-1}$ is an analytic operator function of $\theta$ in a neighbourhood of $(0, 2\pi)$.

**Proof.** Denote by $u_1 = u_1(t, \lambda)$ and $u_2 = u_2(t, \lambda)$ solutions of equation $l(u) = \lambda u$ satisfying the boundary conditions $u_1(0) = 0$ and $u_2(1) = 0$. We recall that $u$ being a solution of $l(u) = \lambda u$ means that

$$\frac{d}{dt} \begin{pmatrix} u^{[1]} \\ u \end{pmatrix} = \begin{pmatrix} -\sigma & -\sigma^2 + \tau - \lambda \\ 1 & \sigma \end{pmatrix} \begin{pmatrix} u^{[1]} \\ u \end{pmatrix},$$

(5.4)

and hence $u$ enjoys standard uniqueness properties of solutions to second order differential equations with regular (i.e. locally integrable) coefficients.

Observe that $u_1(1) \neq 0$ and $u_2(0) \neq 0$ as otherwise $\lambda$ would be an eigenvalue of the operator $S_D$ determined in $\mathcal{H}'$ by the differential expression $l$ and the Dirichlet boundary conditions [1]; since $S_D$ is a selfadjoint operator, this is impossible. Moreover, it follows from the Cayley-Hamilton theorem that the Wronskian

$$W(t) := u_1(t) u_2^{[1]}(t) - u_1^{[1]}(t) u_2(t)$$

does not depend on $t \in [0, 1]$ and hence can be normalized to be 1. It is easily seen that the resolvent $(S_D - \lambda)^{-1}$ is given then by an integral operator $(K_\lambda f)(t) = \int_0^1 K(t, s) f(s) \, ds$ with the kernel

$$K(t, s) = \begin{cases} u_1(t) \overline{u}_2(s) & \text{if } t \leq s, \\ \overline{u}_1(s) u_2(t) & \text{if } t \geq s. \end{cases}$$

Consider the difference $v := (S_\theta - \lambda)^{-1} f - (S_D - \lambda)^{-1} f$; this function solves the equation $l(v) = \lambda v$ and hence equals $\alpha_1(\theta, f) u_1 + \alpha_2(\theta, f) u_2$ for some coefficients $\alpha_1$ and $\alpha_2$ dependent on $\theta$ and $f$. The function

$$w := (S_\theta - \lambda)^{-1} f = (S_D - \lambda)^{-1} f + \alpha_1(\theta, f) u_1 + \alpha_2(\theta, f) u_2$$

satisfies the quasiperiodic boundary conditions $w(1) = e^{i\theta} w(0)$ and $w^{[1]}(1) = e^{i\theta} w^{[1]}(0)$, which means that $\alpha_1$ and $\alpha_2$ must solve the system

$$\alpha_1 u_1(1) - e^{i\theta} \alpha_2 u_2(0) = 0;$$

$$\alpha_1 \left\{ u_1^{[1]}(1) - e^{i\theta} u_1^{[1]}(0) \right\} + \alpha_2 \left\{ u_2^{[1]}(1) - e^{i\theta} u_2^{[1]}(0) \right\} = \beta(f, \theta),$$

where $\beta(f, \theta) := e^{i\theta} u_1^{[1]}(0) \int_0^1 f \overline{u}_2 - u_1^{[1]}(1) \int_0^1 f \overline{u}_1$. We observe that for $f = 0$ and any $\theta \in [0, 2\pi)$ the homogeneous system above has only the trivial solution $\alpha_1 = \alpha_2 = 0$ as otherwise the function $w = \alpha_1 u_1 + \alpha_2 u_2$ would satisfy the quasiperiodic boundary conditions and hence the nonreal number $\lambda$ would be an eigenvalue of the selfadjoint
operator $S_\theta$, which is impossible. Therefore the discriminant $d(\theta)$ of the above system is nonzero and its solution equals
\[
\alpha_1(f, \theta) = \frac{e^{i\theta}u_2(0)}{d(\theta)}\beta(f, \theta), \quad \alpha_2(f, \theta) = \frac{u_1(1)}{d(\theta)}\beta(f, \theta).
\]
We see that $\alpha_1$ and $\alpha_2$ are continuous linear functionals of $f$ depending analytically on $\theta \in (0, 2\pi)$. This completes the proof of analyticity of $(S_\theta - \lambda)^{-1}$.

It follows from Lemma 5.1 that the eigenvalues $\lambda_k(\theta)$ and the eigenvectors $v_k(\theta)$, $k \in \mathbb{N}$, can be labelled to be analytic in $\theta$. By (5.3) the spectrum of $S$ is now just the union of ranges of the functions $\lambda_k(\theta)$, $k \in \mathbb{N}$, when $\theta$ varies over $[0, 2\pi)$; this establishes the so-called band and gap structure of $\sigma(S)$. Absolute continuity of $S$ will follow from [23, Theorem XIII.86] as soon as we show that all $\lambda_k(\theta)$ are nonconstant. To this end we shall give an alternative description of the spectra of $S_\theta$.

Denote by $v_1(t, \lambda)$ and $v_2(t, \lambda)$ solutions of the system (5.4) satisfying the following initial conditions:
\[
v_1^{[1]}(0, \lambda) = 1, \quad v_1(0, \lambda) = 0, \quad v_2^{[1]}(0, \lambda) = 0, \quad v_2(0, \lambda) = 1.
\]
Then the fundamental matrix $M(t, \lambda)$ given by
\[
M(t, \lambda) := \begin{pmatrix} v_1^{[1]}(t, \lambda) & v_2^{[1]}(t, \lambda) \\ v_1(t, \lambda) & v_2(t, \lambda) \end{pmatrix}
\]
depends continuously on $t \in [0, 1]$ and analytically on $\lambda \in \mathbb{R}$, $\det M(t, \lambda) \equiv 1$ by the Cayley-Hamilton theorem, and for any solution $X(t)$ of equation (5.4) the following equality holds
\[
X(t) = M(t, \lambda)X(0).
\]
If $X(t) = (x^{[1]}(t), x(t))^T$ and $x(t)$ is an eigenfunction of $S_\theta$, then the vector $X$ satisfies the boundary condition $X(1) = e^{i\theta}X(0)$, which implies that $M(1, \lambda)X(0) = e^{i\theta}X(0)$. Therefore $\lambda \in \mathbb{R}$ is an eigenvalue of the operator $S_\theta$ if and only if $e^{i\theta}$ is an eigenvalue of the matrix $M(1, \lambda)$. Since $\det M(1, \lambda) \equiv 1$, the latter condition is equivalent to the equality
\[
\text{tr} M(1, \lambda) = 2 \cos \theta. \tag{5.5}
\]

**Lemma 5.2.** The function $\text{tr} M(1, \lambda)$ is strictly monotone at a point $\lambda_0$ whenever $|\text{tr} M(1, \lambda_0)| < 2$.

**Proof.** The statement of the lemma will follow from the results of [24] as soon as we show that the matrix $M(1, \lambda)$ is positively rotating. We recall that this requires the function $\arg \left( M(1, \lambda)X \right)$ to be strictly increasing in $\lambda \in \mathbb{R}$ for any nonzero vector $X \in \mathbb{C}^2$; here for a vector $X = (x_1, x_2)$ we put $\arg X := \arg(x_1 + ix_2)$ measured continuously in $X$.

Observe that if $X(t, \lambda) = (x_1(t, \lambda), x_2(t, \lambda)) := M(t, \lambda)X$, then by definition $X(t, \lambda)$ solves the system (5.4) and hence $x_2$ satisfies the equation $\frac{d}{dt}x_2(t, \lambda) = \lambda x_2(t, \lambda)$. Put $\theta(t, \lambda) := \arg X(t, \lambda)$; then cot $\theta(t, \lambda) \equiv x_2^{[1]}(t, \lambda) / x_2(t, \lambda)$. After differentiating both sides in $t$ we get
\[
- \frac{\theta'}{\sin^2 \theta} = -\sigma x_2' x_2 - x_2^{[1]} x_2' + \tau - \lambda = -(\cot \theta + \sigma)^2 + \tau - \lambda,
\]
or
\[ \theta' = \lambda \sin^2 \theta - \tau \sin^2 \theta + (\cos \theta + \sigma \sin \theta)^2. \]

It follows from [25, proof of Theorem XI.3.1] that the function \( \theta(1, \lambda) \) is strictly increasing in \( \lambda \), and therefore \( M(1, \lambda) \) is positively rotating and the claim of the lemma follows.

Recalling now relation (5.5), we derive the following

**Corollary 5.3.** Every eigenvalue \( \lambda_k(\theta), k \in \mathbb{N}, \) when chosen continuous in \( \theta \), is analytic and strictly monotone in \( \theta \) on the intervals \((0, \pi)\) and \((\pi, 2\pi)\).

Now we combine the results obtained and apply [23, Theorem XIII.86] to arrive at the following conclusion.

**Theorem 5.4.** Suppose that the potential \( q \in W_{2,\text{unif}}^{-1}(\mathbb{R}) \) is periodic and let \( S \) denote the corresponding Schrödinger operator of (1.1) constructed by (1.2)–(1.3). Then the spectrum of \( S \) is purely absolutely continuous and has a band and gap structure.

**Acknowledgements.** The authors thank Prof. V. A. Mikhailets for enlightening discussions and many useful remarks and comments.

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