Delaunay Edge Flips in Dense Surface Triangulations *

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Abstract

Delaunay flip is an elegant, simple tool to convert a triangulation of a point set to its Delaunay triangulation. The technique has been researched extensively for full dimensional triangulations of point sets. However, an important case of triangulations which are not full dimensional is surface triangulations in three dimensions. In this paper we address the question of converting a surface triangulation to a subcomplex of the Delaunay triangulation with edge flips. We show that the surface triangulations which closely approximate a smooth surface with uniform density can be transformed to a Delaunay triangulation with a simple edge flip algorithm. The condition on uniformity becomes less stringent with increasing density of the triangulation. If the condition is dropped completely, the flip algorithm still terminates although the output surface triangulation becomes “almost Delaunay” instead of exactly Delaunay.

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1 Introduction

The importance of computing Delaunay triangulations of point sets in applications of science and engineering cannot be overemphasized. A number of different algorithms have been proposed for their computations [5, 8, 14]. Among them flip based algorithms are most popular and perhaps the most dominant approach in practice. The sheer elegance and simplicity of this approach make it attractive to implement.

Since the introduction of Delaunay flips by Lawson [17] for point sets in two dimensions, several important extensions have been made including higher dimensional point sets. Given any triangulation of the convex hull of a point set in two dimensions, it is known that Delaunay edge flips can convert the triangulation to the Delaunay triangulation. The rule for Delaunay edge flips is simple. First, check locally if the circumscribing ball of a triangle \( t \) contains a vertex of another triangle \( t' \) sharing an edge \( e \) with it. If so, replace \( e \) with the other diagonal edge contained in the union of \( t \) and \( t' \). An elegant result is that this process terminates with the output as the Delaunay triangulation [5, 8, 14]. In higher dimensions, the edge flips can be naturally extended to bi-stellar flips [14]. However, the approach extended by bi-stellar flips does not work in higher dimensions. Already in three dimensions there are examples where the flipping process can get stuck [14]. Notwithstanding this negative result, Joe [16] showed how to construct Delaunay triangulations by inserting points incrementally and applying bi-stellar flips after each point insertion. Edelsbrunner and Shah [15] extended this result to higher dimensional point sets and to weighted Delaunay triangulations. Recently, Shewchuk [19] showed that a combination of flips and some other local operation called star splay can convert an “almost Delaunay” triangulation to the Delaunay one quite efficiently.

All the aforementioned results deal with full dimensional triangulations of a point set. An important case of triangulations which are not full dimensional is surface triangulations in three dimensions. Given the increasing demand of computing surface triangulations that are sub-complexes of Delaunay triangulations [1, 7, 12], it is natural to ask if a surface triangulation can be converted to a Delaunay one by edge flips and, if so, under what conditions. Such a flip algorithm will be useful in many applications. For example, in geometric modeling, shapes are often represented with subdivision surfaces [20] or with isosurfaces [18]. These surfaces are not necessarily Delaunay. If one can convert these surfaces to a Delaunay one, a number of tools that exploit Delaunay properties can be used for further processing.

In this work we address the question of Delaunay flips in surface triangulations. Notice that our goal is to convert a surface triangulation embedded in \( \mathbb{R}^3 \) to another surface triangulation which is a sub-complex of the three dimensional Delaunay triangulation of the vertex set. This is different from the framework considered elsewhere [5, 13] where a triangulated surface endowed with a flat metric is converted into an intrinsic Delaunay triangulation comprised of simplices (not necessarily planar) embedded in the surface. In this case the embedding of the surface in \( \mathbb{R}^3 \) does not play any role whereas in our case the positions of the vertices in \( \mathbb{R}^3 \) determine the Delaunay flips.

It turns out that dense surface triangulations are amenable to a simple edge flip algorithm. A triangulation is dense if it approximates a smooth surface with sufficient resolution. We show that a dense triangulation can be flipped to a Delaunay triangulation if the density is uniform in some sense. The condition on uniformity depends on the density. The higher the density, the less stringent is the condition. The practical implication of this result is that reasonably dense triangulations can be converted to Delaunay triangulations with a simple edge flip algorithm. Such dense triangulations are numerous in practice. Subdivision and isosurface meshes are two such examples. Triangulations of moving vertices offer another such example [10, 19]. In fact,
the results in this paper have been used for a recent algorithm on maintaining deforming meshes with provable guarantees [10]. What happens if we do not have the uniformity condition? We show that the flip algorithm still terminates but the output surface may not be Delaunay. Nonetheless, this surface is “almost Delaunay” in the sense that the diametric ball of each triangle shrunk by a small amount remains empty. Bandyopadhyay and Snoeyink [4] showed the usefulness of such approximate Delaunay triangulations in molecular modeling. Because of the approximate emptiness properties of the circumscribing balls of the triangles, these approximate Delaunay triangulations may find other applications where exact Delaunay triangulations are not required.

2 Preliminaries

2.1 Definitions and results

We need some definitions and results from \( \varepsilon \)-sampling theory [1]. Let \( \Sigma \subset \mathbb{R}^3 \) be a smooth compact surface without boundary. The medial axis is the set of centers of all maximally empty balls. The reach \( \gamma(\Sigma) \) of \( \Sigma \) is the infimum over Euclidean distances of all points in \( \Sigma \) to its medial axis. This is also the infimum of the local feature size of \( \Sigma \) as defined by Amenta and Bern [1].

A surface triangulation \( T \) is a finite simplicial 2-manifold embedded in \( \mathbb{R}^3 \). We say \( T \) is a triangulation of a surface \( \Sigma \) if vertices of \( T \) lie in \( \Sigma \) and its underlying space \( |T| \) is homeomorphic to \( \Sigma \). The triangulation \( T \) has a consistent orientation with \( \Sigma \) if the oriented normal of each triangle makes at most \( \pi/2 \) angle with the oriented normals of \( \Sigma \) at the vertices. For a triangle \( t \in T \), let \( \rho(t) \) denote the circumradius of \( t \).

2.2 Uniform dense triangulations

Definition 1 A triangulation \( T \) of a surface \( \Sigma \) is \( \varepsilon \)-dense if each triangle \( t \in T \) has \( \rho(t) \leq \varepsilon \gamma(\Sigma) \) and \( T \) has a consistent orientation with \( \Sigma \). Furthermore, for \( \delta < 1 \), if any two vertices in \( T \) has distance more than \( \delta \varepsilon \gamma(\Sigma) \), \( T \) is called \((\varepsilon, \delta)\)-dense or \( \delta \)-uniform and \( \varepsilon \)-dense.

We use notation \( B(c, r) \) to denote a ball with center \( c \) and radius \( r \). A circumscribing ball of a triangle \( t \subset \mathbb{R}^3 \) is any ball that has the vertices of \( t \) on its boundary. The diametric ball \( D_t \) is the smallest such ball; \( D_t = B(c, \rho(t)) \) where \( c \) is the circumcenter of \( t \). We say a vertex \( v \in T \) stabs a ball \( B \) if \( v \) lies inside \( B \). If \( t \) shares an edge, say \( pq \), with a triangle \( t' = pqs \), then \( s \) is a neighbor vertex of \( t \). Clearly, each triangle has three neighbor vertices.

Definition 2 A triangle \( t \in T \) is stabbed if \( D_t \) is stabbed by a vertex of \( T \). We say \( t \) is locally stabbed if the stabbing vertex is one of the three neighbor vertices of \( t \) (Figure 1).

Theorem 2.1 and Theorem 2.2 are main results for uniform dense triangulations.

Theorem 2.1 For \( \delta = 2 \sin 2\varepsilon \) and \( \varepsilon < \frac{\pi}{12} \), any \((\varepsilon, \delta)\)-dense triangulation has a stabbed triangle if and only if it has a locally stabbed triangle.

Notice that the bound on \( \delta \) is \( O(\varepsilon) \). This implies that dense triangulations require only mild restrictions on its uniformity.

We will flip edges of dense triangulations to make it Delaunay. Suppose \( pq \) is an edge in a surface triangulation \( T \). Flipping \( pq \) means replacing two triangles, say \( pqr \) and \( pqs \), incident to \( pq \) in \( T \) by the triangles \( prs \) and \( qrs \). If the new triangulation is \( T' \) we write \( T \overset{pq}{\rightarrow} T' \).
A triangulation $T$ is flipped to a triangulation $T'$ if there is a sequence of edge flips so that $T = T_0 \xrightarrow{e_0} T_1 \xrightarrow{e_1} \cdots \xrightarrow{e_{k-1}} T_k = T'$.

**Definition 3** An edge in a surface triangulation is called flippable if it is incident to a locally stabbed triangle.

One can devise an easy algorithm to convert a $(\varepsilon, \delta)$-dense triangulation to a Delaunay triangulation using Theorem 2.1. Simply flip any flippable edge existing in the triangulation. If one can prove that this flip algorithm terminates and $(\varepsilon, \delta)$-density is maintained after each flip, we will have an algorithm to flip an $(\varepsilon, \delta)$-dense triangulation to a triangulation that does not have any stabbed triangle. This means each triangle in the new triangulation has its diametric ball that does not contain any vertices of $T$. In other words, this new triangulation is Delaunay. Actually, the “Delaunayhood” is stronger since not only does each triangle have an empty circumscribing ball but the ball can be chosen to be its diametric ball. Extending the notion of Gabriel graphs of a set of points in two dimensions, we call such a triangulation Gabriel.

**Theorem 2.2** For $\delta = 2 \sin 2\varepsilon$ and $\varepsilon < \frac{\pi}{72}$, any $(\varepsilon, \delta)$-dense triangulation can be flipped to a Gabriel triangulation.

Again, notice that the condition on uniformity becomes relaxed with increasing density.

2.3 Dense triangulations

It is natural to seek similar results for dense triangulations that are not necessarily uniform. It turns out that such triangulations can be flipped to almost Delaunay triangulations but not necessarily to Delaunay triangulations. To prove this result we will need some generalizations of the concept of stabbing as well as Delaunay triangulations. We denote a ball $B = B(c, r)$ shrunk by $\alpha$ as $B_\alpha$, that is, $B_\alpha = B(c, r - \alpha)$. With this definition, $D_\alpha^t$ denotes the diametric ball of $t$ shrunk by $\alpha$.

**Definition 4** A surface triangulation $T$ is $\alpha$-Gabriel if for each triangle $t \in T$, the shrunk diametric ball $D_\alpha^t$ contains no vertex of $T$ inside.

Figure 1: (left) : $pqr$ is stabbed by $v$ and is locally stabbed by $s$; $pq$ is a flippable edge. (right) : $B_1$ and $B_2$ are $(\beta)$- and $(-\beta)$-balls of $pqr$ which is $\beta$-stabbed by $v$ and is locally $\beta$-stabbed by $s$; $pq$ is a $\beta$-flippable edge.
Let \( \mathbf{n}_t \) denote the outward normal of a triangle \( t \in T \). For a triangle \( t \in T \) and \( \beta \in \mathbb{R} \), a \( \beta \)-ball of \( t \) is a circumscribing ball of \( t \) whose center is at \( c + \beta \mathbf{n}_t \) where \( c \) is the circumcenter of \( t \). Observe that 0-ball of \( t \) is its diametric ball \( D_t \). For any \( \beta \neq 0 \), there are two balls of radius \( \sqrt{\rho(t)^2 + \beta^2} \), one is \( \beta \)-ball and another is \((-\beta)\)-ball of \( t \). See Figure 1.

**Definition 5** A triangle \( t \in T \) is \( \beta \)-stabbed if a vertex of \( T \) stabs both \( \beta \)- and \((-\beta)\)-balls of \( t \). We say \( t \) is locally \( \beta \)-stabbed if the stabbing vertex is one of the three neighbor vertices of \( t \).

Observe that if a triangle \( t \) is not \( \beta \)-stabbed, the intersection of its \( \beta \)- and \((-\beta)\)-balls cannot contain any vertex of \( T \). This intersection contains the ball \( D_t^{\alpha} \), \( \alpha = \rho(t) + \beta - \sqrt{\rho(t)^2 + \beta^2} \), which also cannot contain any vertex of \( T \). Observe that \( \alpha \leq \beta \).

**Observation 2.1** A surface triangulation \( T \) is \( \beta \)-Gabriel if it does not have any \( \beta \)-stabbed triangle.

We prove the following results.

**Theorem 2.3** For \( \varepsilon < 0.1 \), any \( \varepsilon \)-dense triangulation of a surface with reach \( \gamma \) contains a \( \beta \)-stabbed triangle only if it contains a locally \((\beta - 8\varepsilon^2\gamma)\)-stabbed triangle.

Choosing \( \beta = 8\varepsilon^2\gamma \) we conclude that there is a \( 8\varepsilon^2\gamma \)-stabbed triangle only if there is a locally stabbed triangle. Therefore, if one gets rid of all locally stabbed triangles, there cannot be any \( 8\varepsilon^2\gamma \)-stabbed triangles. In other words, the triangulation becomes \( 8\varepsilon^2\gamma \)-Gabriel by Observation 2.1.

**Theorem 2.4** Any \( \varepsilon \)-dense triangulation of a surface with reach \( \gamma \) can be flipped to a \( 8\varepsilon^2\gamma \)-Gabriel triangulation if \( \varepsilon < 0.1 \).

### 2.4 Background results

The following well known results on normal approximations will be useful in our analysis. Starting with work of Amenta and Bern [11], several versions of these results have been proved. We pick appropriate ones for our purpose. Let \( \mathbf{n}_x \) denote the outward unit normal of \( \Sigma \) at a point \( x \in \Sigma \).

**Lemma 2.1** ([9, 3]) For any two points \( x \) and \( y \) in \( \Sigma \) such that \( \|x - y\| \leq \varepsilon \gamma \) for some \( \varepsilon \leq \frac{1}{3} \), \( \angle \mathbf{n}_x, \mathbf{n}_y \leq \frac{\varepsilon}{\varepsilon \gamma} \) and \( \mathbf{n}_x, (y - x) \geq \arccos\left(\frac{1}{\sqrt{3}}\right) \).

Following lemma is an oriented version of a result in [12]. Here we use the fact that \( T \) has a consistent orientation with \( \Sigma \). Interestingly, this property does not automatically follow from triangles being small and vertex set being dense.

**Lemma 2.2** ([12]) Let \( pqr \) be a triangle in a \( \varepsilon \)-dense triangulation of a surface. Assume that \( p \) subtends a maximal angle in \( pqr \). Then, for \( \varepsilon < \frac{1}{\sqrt{2}} \), \( \angle \mathbf{n}_{pqr}, \mathbf{n}_p \leq \arcsin \varepsilon + \arcsin\left(\frac{2}{\sqrt{3}}\sin(2 \arcsin \varepsilon)\right) \).

Combining Lemmas 2.1 and 2.2 one obtains the following corollary.

**Corollary 2.1** Let \( q \) be any vertex in a triangle \( pqr \in T \) where \( T \) is a \( \varepsilon \)-dense triangulation of a surface. Then, for \( \varepsilon < 0.1 \), \( \angle \mathbf{n}_{pqr}, \mathbf{n}_q \leq 7\varepsilon \).

Define the dihedral angle between two adjacent triangles \( pqr \) and \( qrs \) as the angle between their oriented normals, that is, \( \angle \mathbf{n}_{pqr}, \mathbf{n}_{qrs} \). An immediate result from Corollary 2.1 is that the dihedral angle between adjacent triangles in a dense triangulation is small.

**Corollary 2.2** Let \( pqr \) and \( qrs \) be two triangles in a \( \varepsilon \)-dense triangulation of a surface. Then, for \( \varepsilon < 0.1 \), \( \angle \mathbf{n}_{pqr}, \mathbf{n}_{qrs} \leq 14\varepsilon \).
3 Flip algorithm

Our flipping algorithm is very simple. Continue flipping as long as there is a flippable edge, that is, an edge incident to a stabbed triangle.

MeshFlip(T)

1. If there is a flippable edge \( e \in T \) then flip \( e \) else output \( T \);
2. \( T := T' \) where \( T \rightarrow T' \); go to step 1.

The first issue to be settled is the termination of MeshFlip. It turns out that this simple flip algorithm terminates if \( T \) is a \( \varepsilon \)-dense triangulation of a surface for \( \varepsilon < 0.1 \).

For convenience we introduce the notion of bisectors using power distance. The power distance \( \text{pow}(B, x) \) of a point \( x \in \mathbb{R}^3 \) to a ball \( B = B(c, r) \) is \( ||c - x||^2 - r^2 \). For two balls \( B_1 \) and \( B_2 \) in \( \mathbb{R}^3 \), the bisector \( C(B_1, B_2) \) is the plane containing points with equal weighted distances to \( B_1 \) and \( B_2 \). If \( B_1 \) and \( B_2 \) intersect, the bisector \( C(B_1, B_2) \) is the plane containing the circle where the boundaries of \( B_1 \) and \( B_2 \) intersect. For two triangles \( pqr \) and \( pqs \) sharing an edge \( pq \), we write \( C_{pq} = C(D_{pq}, D_{pqs}) \). The following lemma establishes symmetry in stabbing.

**Lemma 3.1** Let \( pqr \) and \( pqs \) be two adjacent triangles where \( s \) stabs \( pqr \). If \( \angle n_{pqr}, n_{pqs} < \frac{\pi}{2} \), \( r \) stabs \( pqs \).

*Proof.* It can be shown that the bisector \( C_{pq} \) separates \( r \) and \( s \) if the planes of \( pqr \) and \( pqs \) make an angle larger than \( \frac{\pi}{2} \) or equivalently \( \angle n_{pqr}, n_{pqs} < \frac{\pi}{2} \). Let \( C_{pq}^+ \) be the half-space supported by \( C_{pq} \) and containing \( s \). Clearly, \( D_{pqs} \) lies inside \( D_{pq} \) in \( C_{pq}^+ \) as \( s \) is on the boundary of \( D_{pqs} \). On the other half-space supported by \( C_{pq} \) which does not contain \( s \), \( D_{pq} \) lies inside \( D_{pqs} \). But this half-space contains \( r \) which is on the boundary of \( D_{pqr} \). This means \( r \) is inside \( D_{pqs} \). \( \square \)

If an edge incident to a stabbed triangle is flipped in a triangulation with dihedral angles less than \( \frac{\pi}{2} \), the circumradius of each new triangle becomes smaller than the circumradius of one of the two triangles destroyed by the flip. Actually, this is the key to prove that flip sequence to get rid of all flippable edges terminate.

**Lemma 3.2** Let \( T \) be a surface triangulation with dihedral angles smaller than \( \frac{\pi}{2} \). Let \( pq \in T \) be an edge incident to a locally stabbed triangle \( pqr \) and \( pqs \) be the other triangle incident to \( pq \). We have \( \rho(qrs) \leq \max\{\rho(pqr), \rho(pqs)\} \) and \( \rho(prs) \leq \max\{\rho(pqr), \rho(pqs)\} \).

*Proof.* We prove the lemma for \( \rho(qrs) \). The case for \( \rho(prs) \) can be proved similarly. Consider the bisector \( C_{pq} \) of \( D_{pq} \) and \( D_{pqs} \), see Figure 2. Let \( C_{pq}^+ \) be the half-space supported by \( C_{pq} \) containing \( s \) and \( C_{qs}^+ \) be the half-space supported by \( C_{qs} \) containing \( p \).

By assumption the dihedral angle between \( pqr \) and \( pqs \) is at most \( \frac{\pi}{2} \). Then, Lemma 3.1 applies to claim that \( r \) stabs \( pqs \).

Clearly the center of \( D_{qrs} \) lies in the union \( C_{qr}^+ \cup C_{qs}^+ \). First, assume that \( C_{qr}^+ \) contains the center of \( D_{qrs} \). Clearly \( D_{qrs} \cap C_{qr}^+ \) is contained in \( D_{pq} \) as \( s \) is contained in \( D_{pq} \) by the assumption that \( s \) stabs \( pqr \). Therefore \( D_{qrs} \) is contained in \( D_{pq} \) in \( C_{qr}^+ \) which contains the center of \( D_{qrs} \). This implies that \( D_{qrs} \) is smaller than \( D_{pq} \) establishing the claim. If \( C_{qs}^+ \) contains the center of \( D_{qrs} \) the above argument can be repeated replacing \( D_{pq} \) with \( D_{pqs} \) and \( s \) with \( r \). \( \square \)

Since circumradii of triangles decrease by flipping flippable edges, triangles still can be oriented consistently with \( \Sigma \) and a homeomorphism using closest point map \( \Sigma \) can be established.
between $\Sigma$ and the new triangulation. In sum, the new triangulation satisfies the conditions for being $\varepsilon$-dense.

**Corollary 3.1** If $T \xrightarrow{\varepsilon} T'$ for a flippable edge $e$ and $T$ is $\varepsilon$-dense, then $T'$ is also $\varepsilon$-dense.

**Lemma 3.3** If $T$ is $\varepsilon$-dense for $\varepsilon < 0.1$, MeshFlip$(T)$ terminates.

*Proof.* Let $R_1, R_2, \ldots, R_n$ be the decreasing sequence of the radii of the diametric balls of the triangles at any instant of the flip process. First of all, an edge flip preserves the number of triangles in the triangulation. An edge flip may change the entries in this sequence of radii, but not its length. We claim that after a flip the new radii sequence $R'_1, R'_2, \ldots, R'_n$ decreases lexicographically, that is, there is a $j$ such that $R_i = R'_i$ for all $1 \leq i < j$ and $R_{j+1} > R'_{j+1}$. Let $j + 1$ be the first index where $R_{j+1} \neq R'_{j+1}$. Since each flip maintains $\varepsilon$-density (Corollary 2.2) the dihedral angles between adjacent triangles remain at most $14\varepsilon$ by Corollary 2.2. This angle is less than $\frac{\pi}{2}$ for $\varepsilon < 0.1$. One can apply Lemma 3.2 to each intermediate triangulation. By this lemma the maximum of the two radii before a flip decreases after the flip. This means the triangle corresponding to the radius $R_{j+1}$ has been flipped and its place has been taken by a triangle whose circumradius is smaller than $R_{j+1}$. So the new radii sequence is smaller lexicographically. It follows that the same triangulation cannot appear twice during the flip sequence. As there are finitely many possible triangulations with a fixed number of vertices, the flip sequence must terminate. 

## 4 Uniform dense triangulation

We prove Theorems 2.1 and 2.2 now. First, we need some technical results (Lemmas 4.2 and 4.3). We want to prove that if a vertex stabs the diametric ball of a triangle, it does not project orthogonally to a point inside that triangle. Next lemma is used to prove this fact.

**Lemma 4.1** Assume that a vertex $v$ stabs a triangle $pqr$ in a $\varepsilon$-dense triangulation of a surface where $\varepsilon < 0.1$. Let $\bar{v}$ be the point in $pqr$ closest to $v$. The angle between the segment $v\bar{v}$ and the line of $n_{pqr}$ is at least $\frac{\pi}{2} - 26\varepsilon$.

*Proof.* Let $T$ be a $\varepsilon$-dense triangulation of surface $\Sigma$ with reach $\gamma$. Since $v$ stabs $D_{pqr}$, we have $\|p - v\| \leq 2\varepsilon \gamma$ which implies that $\|v - \bar{v}\| \leq 2\varepsilon \gamma$. Walk from $v$ towards $\bar{v}$ and let $abc$ be the first triangle in $T$ that we hit. Let $y$ be the point in $abc$ that we hit. (The triangle $abc$...
could possibly be $pqr$.) We have $\|v - y\| \leq \|v - \bar{v}\| \leq 2\varepsilon \gamma$. By $\varepsilon$-density assumption, we have $\|a - y\| \leq 2\varepsilon \gamma$. It follows that $\|a - v\| \leq \|a - y\| + \|v - y\| \leq 4\varepsilon \gamma$. Then, $\angle n_v, n_a \leq 8\varepsilon$ by Lemma 2.1 and $\angle n_{abc}, n_a \leq 7\varepsilon$ by Corollary 2.1. Therefore, $\angle n_v, n_{abc} \leq 8\varepsilon + 7\varepsilon \leq 15\varepsilon$.

Let $\ell$ be an oriented line through $v$ and $\bar{v}$ such that $\ell$ enters the polyhedron bounded by $T$ at $y \in abc$ and then exits at $v$. Assume to the contrary that $\ell$ makes an angle less than $\pi - 26\varepsilon$ with $n_{pqr}$. Since $\|p - v\| \leq 2\varepsilon$, Lemma 2.1 and Corollary 2.1 imply that $\angle n_v, n_{pqr} \leq 4\varepsilon + 7\varepsilon \leq 11\varepsilon$. Thus, $\ell$ makes an angle less than $\pi - 15\varepsilon$ with $n_v$. Since $\angle n_v, n_{abc} < 15\varepsilon$, $\ell$ must make an angle less than $\pi - \frac{15\varepsilon}{2}$ with $n_{abc}$. Because $\ell$ enters at $y$ and then exits at $v$, $\angle n_v, n_{abc}$ is greater than $(\pi - \frac{15\varepsilon}{2}) - \frac{15\varepsilon}{2} = 15\varepsilon$, contradicting the previous deduction that $\angle n_v, n_{abc} < 15\varepsilon$.

Lemma 4.2 Assume that a vertex $v$ stabs $D_{pqr}$ of a triangle $pqr$ in an $\varepsilon$-dense triangulation where $\varepsilon < 0.1$. There exists an edge, say $pq$, such that $r$ and $v$ are separated by the plane $H_{pq}$ that contains $pq$ and is perpendicular to $pqr$.

Proof. By Lemma 4.1 $\bar{v}v$ makes a positive angle with the line of $n_{pqr}$. It follows that $v$ does not project orthogonally onto a point inside $pqr$. Hence, there exists an edge $pq$ such that $H_{pq}$ separates $r$ and $v$.

Next lemma leads to Theorem 2.1. This is where we require bounded aspect ratios of triangles which ultimately lead to the uniformity condition. The aspect ratio of a triangle $t$ is the ratio of $p(t)$ to its smallest edge length.

Lemma 4.3 Assume that a vertex $v$ stabs a triangle $pqr$ in an $\varepsilon$-dense triangulation $T$ where each triangle has aspect ratio $\alpha < \frac{1}{2\sin 2\varepsilon}$. If $\varepsilon < \frac{\pi}{72}$, either $pqr$ is locally stabbed or $v$ stabs a triangle $t$ such that $\text{pow}(v, D_t) < \text{pow}(v, D_{pqr})$.

Proof. By Lemma 4.2 there is a plane $H_{pq}$ through the edge $pq$ and perpendicular to $pqr$ such that $H_{pq}$ separates $r$ and $v$. Let $pq$ be the other triangle incident to $pq$. If $s$ lies inside $D_{pqr}$, $pq$ is locally stabbed and we are done. So assume that $s$ does not lie inside $D_{pqr}$. By Corollary 2.2 $\angle n_{pqr}, n_{pq} \leq 14\varepsilon$, which is less than $\frac{\pi}{2}$ for $\varepsilon < \frac{\pi}{72}$. Therefore, $H_{pq}$ separates $r$ and $s$ too. It means that $v$ and $s$ lie on the same side of $H_{pq}$; see Figure 3.

Let $C_{pq}$ denote the bisector of $D_{pqr}$ and $D_{pq}$. Suppose that $C_{pq}$ contains $v$ and $s$ on the same side. It follows that $D_{pq}$ contains $D_{pqr}$ inside on this side as $s$ lies outside $D_{pqr}$. Also $v$ lies inside $D_{pq}$ since $v$ lies inside $D_{pqr}$. It immediately implies that $v$ stabs $pq$ and $\text{pow}(v, D_{pq}) < \text{pow}(v, D_{pqr})$. Therefore, we can establish the lemma if we prove that $C_{pq}$ contains $v$ and $s$ on the same side. This is exactly where we need bounded aspect ratios for triangles.

Let $\bar{s}$ and $\bar{v}$ be the orthogonal projections of $s$ and $v$ respectively onto the line of $pq$. Consider the following facts.

(i) The acute angle between $s\bar{s}$ and $n_{pqr}$ is equal to $\frac{\pi}{2} - \angle n_{pqr}, n_{pq}$, which is at least $\frac{\pi}{2} - 14\varepsilon$ by Corollary 2.2.

(ii) The angle between $H_{pq}$ and $C_{pq}$ cannot be larger than $\angle n_{pqr}, n_{pq}$, which is at most $14\varepsilon$.

(iii) We prove that $\angle n_{pqr}, v\bar{v} > \angle n_{pqr}, n_{pq} = \angle H_{pq}, C_{pq}$.

The above three facts together imply that $C_{pq}$ contains $v$ and $s$ on the same side as $H_{pq}$. Therefore, only thing remains to prove is fact (iii).
Figure 3: (left) : triangle pqr is stabbed by v. Both v and s lie on the same side of Hpq and Cpq. The case of v being in the thin wedge between Hpq and Cpq is eliminated if pqr has bounded aspect ratio. (middle) : the worst case for angle ∠vpq. (right): the planes of Hpq and vpq make large angle ensuring v and s are on the same side of Cpq.

First, observe that if \( \overline{v} \) is the closest point of v in pq, we have by Lemma 4.1

\[
\angle n_{pqr}, v\overline{v} \geq \frac{\pi}{2} - 26\varepsilon \geq 14\varepsilon \geq \angle n_{pqr}, n_{pqs}.
\]

So, assume the contrary. In that case, the closest point of v in pq is one of p or q. Assume it to be p. Since \( \overline{v} \) lies outside pq, the angle \( \angle vpq \) is obtuse. We claim that this angle cannot be arbitrarily close to \( \pi \). In fact, this angle cannot be more than the maximum obtuse angle \( pq \) makes with the tangent plane of \( D_{pqr} \) at \( p \). Simple calculation (Figure 3(middle)) shows that this angle is \( \frac{\pi}{2} + \arccos \frac{\|p - q\|}{2\rho(pqr)} \) giving

\[
\angle vpq \leq \frac{\pi}{2} + \arccos \frac{1}{2a}
\]

where \( a \) is the aspect ratio of pqr. Since \( D_{pqr} \) contains v inside, \( \|v - p\| \leq 2\varepsilon \gamma \). By Lemma 2.1 \( \angle n_p, vp \geq \arccos \varepsilon \). Applying Corollary 2.1 we get

\[
\angle n_{pqr}, vp \geq \angle n_p, vp - \angle n_{pqr}, n_p \geq \arccos \varepsilon - 7\varepsilon.
\]

Let \( zp||v\overline{v} \) (Figure 3(right)). Then, \( \angle v\overline{v}, vp = \angle vpz = \angle vpq - \frac{\pi}{2} \leq \arccos \frac{1}{2a} \). One has

\[
\angle n_{pqr}, v\overline{v} \geq \angle n_{pqr}, vp - \angle v\overline{v}, vp = \angle n_{pqr}, vp - \angle vpz \geq \arccos \varepsilon - 7\varepsilon - \arccos \frac{1}{2a} \geq \frac{\pi}{2} - 10\varepsilon - \arccos \frac{1}{2a} \text{ for } \varepsilon < \frac{\pi}{12}.
\]

We are now left to show that \( \frac{\pi}{2} - 10\varepsilon - \arccos \frac{1}{2a} > \angle n_{pqr}, n_{pqs} \), requiring, \( \frac{\pi}{2} - 24\varepsilon > \arccos \frac{1}{2a} \) or, \( a < \frac{1}{2\sin 24\varepsilon} \).

This is precisely the condition required by the lemma which can be achieved for \( \varepsilon < \frac{\pi}{72} \).
Proof. [Proof of Theorems 2.4 and 2.2] The ‘if’ part of Theorem 2.1 is obvious. To prove the ‘only if’ part, let \( pqr \) be stabbed by \( v \). With \( \delta = 2\sin 24\varepsilon \) aspect ratios are at most \( 1/(2\sin 24\varepsilon) \). So, by Lemma 4.3 either \( pqr \) is locally stabbed or \( v \) stabs a triangle \( t \) where \( \text{pow}(v, D_t) < \text{pow}(v, D_{pqr}) \). In the latter case repeat the argument with \( t \). We must reach a locally stabbed triangle since the power distance of \( v \) from the diametric balls cannot decrease indefinitely. For Theorem 2.2 observe that maximum circumradius decreases after each flip and nearest neighbor distance cannot be decreased by flips. So, MeshFlip maintains \((\varepsilon, \delta)\)-density after each flip which is the only thing remained to be proved.

5 Dense triangulations

We establish Theorems 2.3 and 2.4 in this section. We drop the uniformity condition, i.e., we assume \( T \) is only \( \varepsilon \)-dense for some \( \varepsilon > 0 \). We will use the notation \( \pi_i^\beta \) to denote the plane parallel to \( t \) and passing through the point \( c + \beta \mathbf{n}_i \) where \( c \) is the circumcenter of \( t \). In other words, \( \pi_i^\beta \) is the diametric plane parallel to \( t \) in the \( \beta \)-ball of \( t \).

Lemma 5.1 For \( \varepsilon < 0.1 \), let \( T \) be a \( \varepsilon \)-dense triangulation of a surface with reach \( \gamma \). A vertex stabs a \( \beta \)-ball of a triangle \( t \in T \) only if there is a triangle \( t' \in T \) with \( v \) as a neighbor vertex and \( v \) stabs the circumscribing ball of \( t' \) that has center in the plane of \( \pi_i^\beta \).

Proof. Let \( B \) be a \( \beta \)-ball of \( t \) stabbed by \( w \). Consider the edges of \( T \) lying in \( B \) and planes passing through these edges which are orthogonal to \( \pi_i^\beta \). Let \( P_e \) denote such a plane passing through the edge \( e \). Let \( t = pqr \) and \( pq \) be the edge so that \( H_{pq} \) separates \( r \) and \( w \) according to Lemma 4.2 The line segment \( rw \) must cross \( P_{pq} = H_{pq} \) and possibly others. Let \( pq = e_1, e_2, \ldots, e_k \) be the sequence of edges so that \( rw \) crosses \( P_{e_1}, P_{e_2}, \ldots, P_{e_k} \) in this order. Consider two triangles \( t_i \) and \( t_i+1 \) incident to any edge \( e_i \) in the sequence \( e_1, e_2, \ldots, e_k \). Let \( B_i \) and \( B_{i+1} \) be the two balls circumscribing \( t_i \) and \( t_{i+1} \) respectively and having centers on the plane \( \pi_i^\beta \). Observe that the bisector of \( B_i \) and \( B_{i+1} \) is \( P_{e_i} \). If a vertex of \( t_{i+1} \) lies inside \( B_i \), we have \( t' = t_i \) satisfying the lemma. Otherwise, \( B_i \) is contained in \( B_{i+1} \) on the side of \( P_{e_i} \) which contains \( w \). So, \( \text{pow}(B_i, w) > \text{pow}(B_{i+1}, w) \). Since this relation holds for any \( i \in [1, k] \), we have either found the triangle \( t' \) satisfying the lemma or \( 0 > \text{pow}(B, w) = \text{pow}(B_1, w) \geq \text{pow}(B_k, w) \). In the latter case \( \text{pow}(B_k, w) \) is negative and hence \( B_k \) contains \( w \) inside. The ball \( B_k \) circumscribes \( t_k \) and has center in \( \pi_i^\beta \). It is stabbed by \( w \) where \( w \) is a neighbor vertex of \( t_k \) satisfying properties of \( t' \) required by lemma.

Proof. [Proof of Theorem 2.3] Let \( t \in T \) be \( \beta \)-stabbed. By definition, the \( \beta \)-ball and \( (-\beta) \)-ball of \( t \) are stabbed by a vertex \( w \). Apply Lemma 5.1 to both of these balls. Observe that the planes \( (P_e) \) that we construct in the lemma remain same for both of these balls. It means that the segment \( rw \) in the proof crosses same set of planes. In other words, the triangle \( t' \) guaranteed by Lemma 5.1 remains same. Let \( B \) and \( B' \) be the two circumscribing balls of \( t' \) which have their centers in \( \pi_i^\beta \) and \( \pi_i'^{-\beta} \) respectively. If we prove that \( B \) and \( B' \) are larger than \( (\beta - 88\varepsilon^2\gamma) \)-ball of \( t' \), we will be done since then \( t' \) will be locally \((\beta - 88\varepsilon^2\gamma)\)-stabbed.

Let \( c \) and \( c' \) be the circumcenters of \( t \) and \( t' \) respectively. Since \( t' \) has an edge in the diametric ball \( D_t \) and all triangles have circumradius less than \( \varepsilon \gamma \), the distance \( \|c - c'\| \) is at most \( 2\varepsilon \gamma \). We have \( \angle \mathbf{n}_t, \mathbf{n}_p \leq 7\varepsilon \) and \( \angle \mathbf{n}_t, \mathbf{n}_s \leq 7\varepsilon \) where \( p \) and \( s \) are vertices of \( t \) and \( t' \) respectively. Also the distance between \( p \) and \( s \) cannot be more than \( 4\varepsilon \gamma \) which gives \( \angle \mathbf{n}_p, \mathbf{n}_s \leq 8\varepsilon \). In all, \( \angle \mathbf{n}_t, \mathbf{n}_t' \leq 22\varepsilon \) when \( \varepsilon < 0.1 \). We want to estimate the distance of \( c' \) from the plane of \( t \). In the
worst case this distance is
\[ \|c - c'\| \sin \angle n_t, n_{t'} \leq 4\varepsilon\gamma \sin 22\varepsilon \leq 88\varepsilon^2\gamma. \]

It means if we choose \( \beta > 88\varepsilon^2\gamma \), the center \( c' \) lies inside the slab made by offsetting \( \pi_t \) by \( \beta \) on both sides. The distance of \( c' \) from these planes is at least \((\beta - 88\varepsilon^2\gamma)\). Therefore, with \( \beta > 88\varepsilon^2\gamma \) we have \( B \) and \( B' \) larger than \((\beta - 88\varepsilon^2\gamma)-ball \) of \( t' \) proving the claim.

**Proof.** [Proof of Theorem 2.4] We apply algorithm MeshFlip on the \( \varepsilon \)-dense triangulation \( T \) of a surface whose reach is \( \gamma \). According to Theorem 2.3 output triangulation cannot have any \( 88\varepsilon^2\gamma \)-stabbed triangle. By Observation 2.1 the output is \( 88\varepsilon^2\gamma \)-Gabriel.

Instead of flipping all locally stabbed triangles, one may flip more conservatively. If we go on flipping edges that are incident to \( \beta \)-stabbed triangles, we get a triangulation which is \((\beta + 88\varepsilon^2)\)-Gabriel. We flip less edges than MeshFlip does and hence obtain a worse triangulation in terms of approximation to Gabriel triangulation.

### 6 Conclusions

In this work we showed that a uniform dense surface triangulation can be flipped to a Delaunay one using simple Delaunay-like flips. If uniformity condition is dropped, we get almost Delaunay surface triangulation.

This research ensues some open questions. Can the dense triangulations be flipped to exact Delaunay triangulation? It is unlikely that such triangulations can be flipped to exact Gabriel triangulation. It might very well be that they cannot be flipped to exact Delaunay triangulations. Our flip algorithm converts dense triangulations to almost Gabriel triangulations. Is it true that such triangulations are actually a weighted Delaunay triangulation of its vertex set weighted appropriately? Or, is it possible to assign weights to the vertices and carry out edge flips to convert a dense surface triangulation to a weighted Delaunay one? We plan to address these questions in future work.

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