Geometrical formalism in gauge theories

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Abstract

We review the basic elements of the geometrical formalism for description of gauge fields and the theory of invariant connections, and their applications to the coset space dimensional reduction of Yang-Mills theories. We also discuss the problem of classification of principal fibre bundles, which is important for the quantization of gauge theories. Some results for bundles over two-dimensional spaces are presented.

Introduction

Nowadays it is a well established fact that the four known fundamental interactions of nature - the electromagnetic, weak, strong and gravitational - are gauge type interactions (see, for example, [12]). This explains the importance of gauge field theories which have been the object of intensive physical and mathematical studies in the last decades.

In the present paper we review some results from differential geometry and algebraic topology which are important for gauge theories. In particular, we discuss invariant connections in principle fibre bundles. This class of connections includes some monopole and instanton solutions and is used for description of the gauge theories dimensional base spaces. As we will show, the problem of classification important for the functional integral formulation of quantum gauge theories.

The form of the Lagrangian describing quarks and leptons is postulated to be locally gauge invariant, i.e. invariant with respect to local group transformations induced by some group $G$ called the gauge group. In models of particle physics $G$ is usually assumed to be a compact Lie group. Let us denote its Lie algebra by $\mathfrak{g}$. The adjoint actions of the group $G$ on itself and on its Lie algebra are defined in the standard way: for every element $g \in G$ we have

$$ h \rightarrow ghg^{-1} \quad \text{for any} \quad h \in G, $$

$$ X \rightarrow \text{ad}(g)X \equiv gXg^{-1} \quad \text{for any} \quad X \in \mathfrak{g}. $$

Schematically a theory of gauge interaction is constructed as follows. Let $M$ be a Riemannian manifold of dimension $D$ describing the space-time and let $\gamma$ be a metric on it. In the local coordinates $\{x^\mu\}$ it is characterized by the metric tensor $\gamma_{\mu\nu}$, where

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\( \mu = 0, 1, 2, \ldots, D - 1 \). Suppose that \( \psi(x) \) is a matter field, i.e. a field describing a lepton or a quark, and its dynamics is described by a Lagrangian \( \mathcal{L}_m(\psi, \partial_\mu \psi) \) including only the field function and its first derivatives. Depending on its physical characteristics (electric charge, leptonic charge, baryonic charge, etc.) the field transforms according to certain representation \( \rho \): for \( g \in G \)

\[
\psi(x) \rightarrow \rho(g) \psi(x).
\] (3)

The Lagrangian \( \mathcal{L}_m(\psi, \partial_\mu \psi) \) is built to be invariant under such global transformations. This invariance corresponds to the conservation of charges associated to the gauge symmetry.

However, as we already said above, the theory is postulated to be invariant under local gauge transformations, i.e. under transformations (3) with \( g \) which is a \( G \)-valued function \( g(x) \). An immediate consequence of this is that in order to realize such invariance one has to introduce a vector function \( A_\mu(x) \) with values in the Lie algebra \( G \) defined locally, i.e. for each chart of the manifold \( M \). It is called the gauge potential or gauge field. This is the field mediating the interaction of the matter particles \( \psi(x) \).

Under local gauge transformations \( A_\mu(x) \) and \( \psi(x) \) transform as follows:

\[
A_\mu(x) \rightarrow A'_\mu(x) = ad(g(x)^{-1})A_\mu(x) + g^{-1}(x)\partial_\mu g(x),
\] (4)

\[
\psi(x) \rightarrow \psi'(x) = \rho(g(x))\psi(x).
\] (5)

The Lagrangian of the theory, invariant under such transformations, is given by

\[
\mathcal{L}(\psi, \partial_\mu \psi, A) = \mathcal{L}_m(\psi, D_\mu(A)\psi) + \mathcal{L}_{YM}(A),
\] (6)

where \( \mathcal{L}_{YM} \) is the Yang-Mills Lagrangian, or pure gauge theory Lagrangian, invariant under (4), and

\[
D_\mu(A) = \partial_\mu + \rho'(A_\mu)
\] (7)

is the covariant derivative. Here \( \rho' \) is the representation of the Lie algebra \( \mathcal{G} \) corresponding to \( \rho \). It can be checked that Lagrangian (6) is invariant under simultaneous gauge transformations (4), (5). The matter fields interact through the field \( A_\mu(x) \) by exchanging quanta of this field. Thus, the form of the interaction, given by \( D_\mu(A) \), is determined by the local gauge invariance. Configurations \( (A_\mu, \psi) \) and \( (A'_\mu, \psi') \) related by (4), (5) are equivalent from the physical point of view and describe the same state of the system. In practical applications in order to fix this ambiguity one chooses a representative at each orbit of the gauge group action by imposing a gauge fixing condition. An example of such condition is the Lorentz gauge: \( \partial_\mu A^\mu(x) = 0 \), which is widely used in calculations of elementary particle processes.

In the present article we focus on the gauge part of the action \( S_{YM}[A] \) determined by the Lagrangian \( \mathcal{L}_{YM} \). It is equal to

\[
S_{YM}[A] = \int_M \sqrt{|\det \gamma_{\mu\nu}|}d^Dx \frac{1}{8e^2} \langle F_{\mu\nu}F^{\mu\nu} \rangle.
\] (8)

Here \( e \) is a gauge coupling or charge, a numerical constant characterizing the intensity of the interaction, \( \langle \cdot, \cdot \rangle \) is the Killing form in \( \mathcal{G} \) and the gauge field tensor \( F_{\mu\nu} \) is defined as

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu].
\] (9)
For example, the Maxwell electrodynamics is described by action (8) with the abelian gauge group $G = U(1)$, $A_\mu$ is the electromagnetic potential describing photons. In the case of the Weinberg-Salam-Glashow model, unifying the electromagnetic and weak interactions, the gauge group is non-abelian and is equal to $G = SU(2) \times U(1)$, certain combinations of the $A_\mu$ fields describe the $W^\pm$-bosons, $Z$-boson and photon.

The plan of the article is the following. First we discuss the geometrical formalism for the description of the gauge field. Then we introduce the notion of invariant connection and review main results on their classification. We also outline the application of these results to the dimensional reduction of multidimensional gauge theories and present a concrete example of invariant connection. Finally, we discuss briefly the classification of fibre bundles which is relevant for the quantization of gauge theories.

**Geometrical formalism for description of gauge theories**

The formalism sketched in the previous section is the one usually used in perturbative calculations of concrete physical processes and effects in the theory of particle interactions. However, some properties of the gauge theories are intimately related to the geometrical and topological structure of the space-time manifold and of the gauge group and are not accessible within the perturbation theory expansion. Instanton and monopole solutions, a complicated structure of the vacuum states, and Chern-Simons models are only few examples of this kind [8].

To study such properties of gauge theories another formalism turns out to be more adequate. We describe it in this section.

Let us introduce first the gauge 1-form $A = A_\mu dx^\mu$ for each chart of the space-time manifold $M$. Local gauge group transformations (4) are written then as

$$A \rightarrow A' = \text{ad}(g^{-1})A + g^{-1}dg,$$

where $\text{ad}$ is the adjoint action of the Lie group $G$ on its Lie algebra defined by Eq. (2). Yang-Mills action (8) can be written as

$$S_{YM}[A] = \int_M \frac{1}{4e^2} (F \wedge \ast F),$$

where the 2-form $F$ is determined by the gauge field tensor $F_{\mu\nu}$:

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu,$$

and $\ast$ denotes the Hodge star operation with respect to a given metric $\gamma$ on $M$.

A coordinate independent (and mathematically more elegant) description of gauge field is obtained within the geometrical formalism (see, for example, [10], [11], [27], [33]). The basic ingredients are the principal fibre bundle $P = P(M, G)$ with the base being the space-time manifold $M$ and the structure group being the gauge group $G$ (see, for example, [19], [21]). Let us denote the (free) action $P \rightarrow P$ of the structure group on $P$ by $\Psi_g$ ($g \in G$) and the canonical projection $P \rightarrow M$ by $\pi$. By definition, the principal fibre bundle is locally trivial, i.e. every point $x$ of $M$ has an open neighborhood
U \subset M$ such that $\pi^{-1}(U)$ is homeomorphic to $U \times G$. Let us denote the corresponding diffeomorphism $\pi^{-1}(U) \to U \times G$ by $\chi$. It is given by a formula: $\chi(p) = (\pi(p), \varphi(p)) \in U \times G$, $p \in \pi^{-1}(U) \subset P$, where $\varphi$ is a mapping of $\pi^{-1}(U)$ into $G$ satisfying $\varphi(\Psi_g p) = \varphi(p)g$ for any $p \in \pi^{-1}(U)$ and any $g \in G$. The action $\Psi_g$ of the structure group $G$ on $P$ defines an isomorphism $\sigma$ of the Lie algebra $\mathcal{G}$ of $G$ onto the Lie algebra of vertical vector fields on $P$ tangent to the fibre at each $p \in P$.

Depending on the base space and structure group the bundle $P$ admits certain connections. Gauge fields are described by the connection 1-forms $w$ of the connections in $P$. To establish a relation with the formalism in terms of gauge potentials or gauge 1-forms, described above, one chooses a local section $s$ over each open set $U$ of an open covering of the manifold $M$, $s : U \to \pi^{-1}(U) \subset P$. Choosing a family of sections for an open covering of $M$ corresponds to choosing a gauge condition. Then the $\mathcal{G}$-valued gauge 1-form $A(s)$ on $U$ is the pull-back of the connection 1-form $w$ on $P$ with respect to the section $s$: $A(s) = s^* w$. The 2-form $F$ in Eq. (10). (11) is calculated as $F(s) = s^* \Omega$, where $\Omega = Dw$ is the curvature 2-form of the connection form $w$.

It is instructive to see this relation in more detail. For this let us consider local coordinates $\{x^\mu\}$ on $U$ and the local basis of vector fields $\xi_\mu = \partial_\mu = \partial/\partial x^\mu$. Suppose now that $a$ is a fixed element of the structure group $G$ and define a section $s_a$ over $U$ as follows: for any $x \in U$, $s_a(x) = \chi^{-1}(x, a) \in \pi^{-1}(U)$. We denote by $(\xi_\mu)\chi^{-1}(x, a)$ the vector field $s_a^* \xi_\mu$ tangent to the submanifold $\chi^{-1}(U, a)$ of $P$ and by $(dx^\mu)\chi^{-1}(x, g)$ the co-vector (1-form) dual to it: $dx^\mu(\xi_\nu) = \delta_\mu^\nu$. It can be checked that the connection 1-form $w$, which corresponds to the gauge potential $A^{s_a}_\mu(x)$ on $U$, is equal to

$$w = w_0 + \sigma(A^{s_a}_\mu(x)) dx^\mu, \quad (12)$$

where $\sigma$ is the mapping from $\mathcal{G}$ onto the Lie algebra of vector fields on $P$ mentioned above. The canonical part $w_0$ of the connection form is given by the canonical $\mathcal{G}$-valued left invariant 1-form $\theta$ on group $G$ and the mapping $\varphi : \pi^{-1}(U) \to G$ which forms part of the diffeomorphism $\chi$ introduced earlier: $w_0 = \varphi^* \theta$. When the group $G$ is realized by matrices $w_0$ is often written as

$$w_0 = \varphi(p)^{-1} d \varphi(p).$$

Eq. (12) gives the explicit (local) expression of the connection 1-form in terms of the gauge potential $A^{s_a}_\mu$ in the gauge corresponding to the section $s_a$. In this way $A^{s_a}_\mu$ determines the connection in $P$. For example, the horizontal lift $\xi_\mu^s$ of the vector field $\xi_\mu$ is equal to

$$\xi_\mu^s = \xi_\mu - \sigma(A^{s_a}_\mu(x)).$$

Suppose now that $s$ is a local section in $P$ over $U$ and $g(x)$ is a $G$-valued function on $U$. Then $t(x) = \Psi_{g(x)} s(x)$ $(x \in U)$ defines another section in $P$. By straightforward calculation one can show that the gauge potentials defined by these two sections are related by gauge transformation (11) with the function $g(x)$:

$$A^t_\mu(x) = \text{ad}(g(x)^{-1})A^{s_a}_\mu(x) + g^{-1}(x) \partial_\mu g(x).$$

Thus, indeed, the choice of a local section $U \to \pi^{-1}(P)$ corresponds to the gauge fixing in the "physical" formalism described in the Introduction. Changing from one section
to another is equivalent to a gauge transformation and to changing the gauge fixing condition. Non-existence of a global cross section of a non-trivial fibre bundle $P(M, G)$ means the impossibility to introduce the gauge fixing globally. We are forced to use a local description of the gauge theory. Contrary to this, the connection 1-form $w$ exists globally. It contains all the information about the gauge field configuration in the theory. Therefore, the fibre bundle $P(M, G)$ and $w$ are the objects which define completely the gauge theory within the geometrical approach.

**Invariant connections and dimensional reduction**

The Standard Model, which describes the electromagnetic, weak and strong interactions, has been successfully tested in recent particle experiments. However, there is a number of important problems which has to be resolved and which call for consideration of more general theoretical schemes and ideas beyond the Standard Model. One of such approaches, called the Kaluza-Klein approach, is based on the hypothesis that the space-time has more than four dimensions with extra dimensions being a compact space of small enough size $R$. Then the dynamics of elementary particles is defined by a fundamental multidimensional theory, but at a larger scale the additional dimensions are not seen directly and reveal themselves in an indirect way through quantum effects. This hypothesis does not contradict the observational data if $R < 10^{-17}$ cm. By a certain procedure called dimensional reduction a multidimensional theory can be interpreted in four-dimensional terms, thus giving rise to an effective theory on the four-dimensional space-time.

Examples, interesting both from physical and mathematical points of view, are given by gauge theories on manifolds $M = M_4 \times K/H$, where $M_4$ is the macroscopic four-dimensional space-time (for example, the Minkowski space-time) and the space of extra dimensions is a compact homogeneous space realized as a coset space $K/H$, where $K$ and $H$ are compact Lie groups. Within the geometrical approach such gauge theory is described by connection forms on the principal fibre bundle $\hat{P} = P(M, G)$, where, as before, $G$ is the gauge group.

The group $K$ acts on $K/H$ naturally: for $k \in K$ and for any point $y = [k_1] \in K/H$, understood as a class of the coset space $K/H$, the transformation is given by $y \rightarrow ky = [kk_1]$. We assume that $K$ acts trivially on points of $M_4$. Therefore, the action of $K$ on $M$, which we denote by $O_k$, can be written as $O_k(x, y) = (x, ky)$ for $(x, y) \in M_4 \times K/H$. Moreover, we assume that this action can be lifted to the fibre bundle $\hat{P}$ as a left bundle automorphism\(^1\). Let us denote this action by $L_k$. It fulfills the following properties:

\[
L_{k_1k_2} = L_{k_1} \circ L_{k_2}, \quad (13)
\]

\[
L_k \circ \Psi_g = \Psi_g \circ L_k. \quad (14)
\]

It is said that $K$ is a symmetry group of the gauge theory. The transformations $L_k$

\^1 Actually, in many mathematical studies and physical applications the existence of a group bundle automorphisms $L_k$ with properties (13), (14) is taken as the initial assumption, so that the problem of the lift of $O_k$ is not addressed.
and $O_k$ are related by the following formula:

$$\pi \circ L_k = O_k \circ \pi,$$

where $\pi$ is the canonical projection in $\hat{P}$. If we denote by $o$ the origin of the coset space $K/H$, i.e. the class $o = [e] = H$ containing the group unity $e$, then $\hat{M}_4 = M_4 \times \{o\}$ is the submanifold formed by the stable points of the subgroup $H$.

A connection invariant under the action of the symmetry group $K$, or a $K$-invariant connection in the fibre bundle $\hat{P}$ is defined as a connection whose connection form $\hat{w}$ satisfies the condition

$$L_k^*\hat{w} = \hat{w}.$$ (15)

The invariant connections are interesting from the point of view of differential geometry. As we will see shortly, they are of special importance for the problem of dimensional reduction of gauge theories. For consistence one should also consider metrics $\hat{\gamma}$ on $M$ which are $K$-invariant:

$$O_k^*\hat{\gamma} = \hat{\gamma}.$$ (16)

In this case we say that the gauge theory is $K$-invariant.

Let us discuss the symmetry of gauge potentials which correspond to the invariant connections. Suppose that $U$ is an open set of a covering of the space-time manifold $M$, $s : U \to \hat{P}$ is a local section of $\hat{P}$, and $\hat{A} = s^*\hat{w}$ is the gauge 1-form on $U$. Under a transformation $O_k$ a point $\hat{x} = (x,y)$ of $M_4 \times K/H$ transforms as $\hat{x} \to O_k\hat{x}$. The invariance condition, Eq. (15), gives that

$$O_k^*\hat{A} = ad(\rho_k)^{-1}\hat{A} + (\rho_k)^{-1}d\rho_k,$$ (17)

where $\rho_k$ is a map $U \to G$ defined by the relation

$$L_k s(\hat{x}) = \Psi_{\rho_k(O_k\hat{x})} s(O_k\hat{x})$$

for any $\hat{x} = (x,y) \in U$. Gauge fields satisfying relation (17) are called symmetric gauge fields. They were first introduced and studied in papers [37], [29]. In Refs. [29] the relation between the symmetric gauge fields and the invariant connections was established and a classification of static spherically-symmetric fields in the three-dimensional space was given. From Eq. (17) it follows that for symmetric gauge fields the action of the group $K$ is equivalent to a gauge transformation. Since Yang-Mills Lagrangian (8) or (10) is gauge invariant, being calculated on symmetric gauge fields it does not depend on points of $K/H$ and, therefore, is actually a Lagrangian of an effective theory on $M_4$ only. This feature is the basis of the method of dimensional reduction of the sector of symmetric gauge fields called the coset space dimensional reduction (CSDR). In order to construct this effective theory one should express $K$-symmetric gauge fields on $M$ in terms of fields on $M_4$ and then calculate the Yang-Mills action of the initial theory in terms of these new four-dimensional fields. In this article we consider the first part of this construction. We will follow Refs. [24], [34]. In differential geometry it is usually referred to as the problem of classification of invariant connections. The invariant connections in the mathematical context were studied by Wang in Ref. [35], main results can be found in book [21]. In the context of physical
gauge theories they were extensively studied in a number of papers [13], [18] (see [24], [20] for reviews).

We consider the case when \( K \) and \( H \) are compact groups. Then the homogeneous space is reductive, i.e. the Lie algebra \( K \) of \( K \) may be decomposed into a vector space direct sum of the Lie algebra \( H \) of \( H \) and an \( ad(H) \)-invariant subspace \( M \), that is

\[
K = H \oplus M, \quad \text{(18)}
\]

\[
ad(H)M \subset M.
\]

The second property implies that \([H, M] \subset M\).

Let us denote the portion of \( \hat{P} \) over \( \tilde{M} = M_4 \times \{o\} \cong M_4 \) as \( \tilde{P} = \pi^{-1}(\tilde{M}) \subset \hat{P} \). Since transformations \( O_h, h \in H \), act trivially on \( \tilde{M} \), the transformations \( L_h \) act on \( \tilde{P} \) as vertical automorphisms. Therefore, for any \( \tilde{p} \in \tilde{P} \) there exists a map \( \tau_{\tilde{p}} : H \rightarrow G \) defined by the relation

\[
L_h \tilde{p} = \Psi_{\tau_{\tilde{p}}} \tilde{p}. \quad \text{(19)}
\]

It can be easily shown that \( \tau_{\tilde{p}} \) is a homomorphism of \( H \).

Now let us characterize a \( K \)-invariant form \( \hat{w} \) on \( \hat{P} \). We denote the inclusion map \( \tilde{P} \rightarrow \hat{P} \) by \( \iota_e \). Then \( \hat{w} = \iota_e^* \hat{w} \) is a connection 1-form on \( \tilde{P} \) invariant under the action of the group \( H \):

\[
L_h^* \hat{w} = \hat{w}. \quad \text{(20)}
\]

This form coincides with \( \hat{w} \) in points of \( \tilde{P} \) and on vector fields tangent to this subbundle. The values of \( \hat{w} \) on other vector fields are given by a \( G \)-valued 1-form on \( K \) defined by \( \alpha_{\tilde{p}} \equiv \iota_{\tilde{p}}^* \hat{w} \), where the map \( \iota_{\tilde{p}} : K \rightarrow \tilde{P} \) is induced by the \( K \)-action on \( \tilde{P} \) as follows: \( \iota_{\tilde{p}}(k) = L_k \tilde{p} \). It can be shown that since \( \alpha_{\tilde{p}} \) is \( K \)-invariant it is equal to

\[
\alpha_{\tilde{p}} = \Lambda_{\tilde{p}}(\theta), \quad \Lambda_{\tilde{p}} : K \rightarrow G,
\]

where \( \theta \) is the canonical left invariant 1-form of \( K \). The map \( \Lambda : K \rightarrow G \) is equivariant, i.e. it satisfies the property

\[
\Lambda_{\tilde{p}} \circ ad(h) = ad(\tau_{\tilde{p}}) \circ \Lambda_{\tilde{p}}. \quad \text{(21)}
\]

Since the homogeneous space \( K/H \) is reductive it is enough to specify the map \( \Lambda_{\tilde{p}} \) for each component of decomposition [13]. Its restriction to \( H \) coincides with the differential \( \tau'_{\tilde{p}} \) of the homomorphism \( \tau_{\tilde{p}} \) defined in [19]:

\[
\Lambda_{\tilde{p}}|_H = \tau'_{\tilde{p}} : H \rightarrow G,
\]

whereas its restriction to \( M \) is given by some map \( \phi_{\tilde{p}} : M \rightarrow G \) which is equivariant:

\[
\Lambda_{\tilde{p}}|_M \equiv \phi_{\tilde{p}}, \quad \phi_{\tilde{p}} \circ ad(h) = ad(\tau_{\tilde{p}}) \circ \phi_{\tilde{p}} \quad \text{(22)}
\]

(cf. [21]).

We have obtained that the \( K \)-invariant form \( \hat{w} \) is in one-to-one correspondence with the \( H \)-invariant connection 1-form \( \hat{w} \) on \( \hat{P} \) and the mapping \( \phi_{\tilde{p}} \). The form \( \hat{w} \)
characterizes the form $\tilde{w}$ on the space of orbits of the group $G$ in $\tilde{P}$, whereas the mapping $\phi_p$ characterizes $\tilde{w}$ along the $G$-orbit passing through $\tilde{p}$.

Since the form $\tilde{w}$ is still $H$-invariant further reduction of the fibre bundle can be carried out, namely the reduction of the structure group. This can be done in the following way. Since the structure group acts freely in each fibre, any two points $\tilde{p}, \tilde{p}'$ of the same fibre are related by a vertical transformation $\Psi_g$ for some $g \in G$. Because of this the homomorphisms at these points are conjugate to each other:

$$\tau_{\tilde{p}'}(h) = g^{-1}\tau_{\tilde{p}}(h)g$$

for all $h \in H$. Let us fix a point $\tilde{p}_0$ in $\tilde{P}$ and define

$$\tau(h) \equiv \tau_{\tilde{p}_0}(h) \text{ for all } h \in H.$$ 

Finally, we denote by $P$ all points $p$ of the fibre bundle $\tilde{P}$ for which $\tau_p = \tau$. It can be shown that $P$ is a principal fibre bundle with the base $M_4$ and the structure group $C = C_G(\tau(H))$, i.e. $P = P(M_4, C)$. Here $C_G(\tau(H))$ is the centralizer of the subgroup $\tau(H)$ in $G$,

$$C_G(\tau(H)) \equiv \{c \in C | c\tau(h)c^{-1} = \tau(h) \text{ for any } h \in H\}.$$ 

It can be shown that the restriction of the form $\tilde{w}$ to $P(M_4, C)$, $\tilde{w}|_P = w$, is a connection 1-form on $P$. It takes values in the Lie algebra $C$ of the group $C$, as it should be.

Now we summarize the main result on the classification of invariant connections. Any $K$-invariant connection form $\tilde{w}$ on $\tilde{P}(M_4 \times K/H, G)$ is in one-to-one correspondence with the pair $(w, \phi_p)$ in the reduced bundle $P(M_4, C)$, where $w$ is an arbitrary, i.e. free of additional constraints, connection 1-form in $P$ and $\phi_p$ is a linear equivariant mapping $\phi_p : M \to G$ satisfying property $[22]$. The explicit relation between $\tilde{w}$ and $(w, \phi_p)$ has been described above.

Let us see how this result is written in terms of symmetric gauge fields. For this we take an open set $U = U_{M_4} \times U_{K/H}$ of an open covering of $M$ and a local section $s : U \to \tilde{P}$ given by the formula

$$s(x, y) = L_{s_1(y)}s_2(x), \ x \in M_4, \ y \in K/H,$$

where $s_1 : U_{K/H} \to K$ and $s_2 : U_{M_4} \to P$. If $s_1$ and $s_2$ satisfy certain natural conditions, then the $K$-symmetric gauge field $\tilde{A}$, corresponding to the $K$-invariant connection form $\tilde{w}$ on $\tilde{P}$, is equal to

$$\tilde{A}(x, y) \equiv (s^*\tilde{w})_{(x,y)} = (s_2^*w)_x + \Lambda_{s_2(y)}((s_1^*\theta)_y),$$

where, as before, $\theta$ is the canonical left invariant 1-form of the group $K$. Using the properties discussed above the $K$-invariant gauge field can be written as

$$\tilde{A}(x, y) = A(x) + \tau'(\tilde{\theta}^H) + \phi_x(\tilde{\theta}_M). \quad (23)$$

Here $A = s_2^*w$ is the gauge field form on $M_4$, $\tilde{\theta} = s_1^*\theta$, $\phi_x = \phi_{s_2(x)}$, and $\tilde{\theta}^H$ and $\tilde{\theta}_M$ denote the $\mathcal{H}$- and $\mathcal{M}$-components of the form $\theta$, respectively, in accordance with
decomposition (13). Formula (23) describes a most general $K$-symmetric gauge field on $M$. This is basically Wang’s result on classification of $K$-invariant connections written in terms of forms on the base of the fibre bundle $\hat{P}(M, G)$. One can also obtain a general formula for the curvature 2-form $\hat{\Omega} = d\hat{\omega} + \frac{1}{2}[\hat{\omega}, \hat{\omega}]$ of the invariant connection or for the gauge 2-form $\hat{F}$ of the symmetric gauge field. Thus,

$$\hat{F} = s^*\hat{\Omega} = F + (D\phi)(\bar{\theta}_M) + \frac{1}{2}\left[\phi(\bar{\theta}_M), \phi(\bar{\theta}_M)\right]$$

$$- \frac{1}{2}\phi \left( [\bar{\theta}_M, \bar{\theta}_M]_M \right) - \frac{1}{2} \tau \left( [\bar{\theta}_M, \bar{\theta}_M]_H \right),$$  

(24)

where $[\cdot, \cdot]_M$ and $[\cdot, \cdot]_H$ are the $M$- and $H$-components of the product $[\cdot, \cdot]$ in the Lie algebra $K$ respectively, $F = dA + [A, A]$ is the gauge 2-form describing the tension of the gauge field $A$ on the four-dimensional space-time $M_4$, and $D\phi$ is the covariant derivative of $\phi$:

$$(D\phi)_x(\bar{\theta}_M) = d\phi_x(\bar{\theta}_M) + [A_x, \phi_x(\bar{\theta}_M)]$$

(cf. (7)).

To obtain a complete parametrization of invariant connections or symmetric gauge fields one has to resolve constraint (22) and construct the mapping $\phi_x$. This constitutes the second, algebraic part of the CSDR, or, equivalently, of the classification problem, and is not considered here. Its solution can be found in Refs. [24], [34]. Here we only mention that the main idea is to interpret constraint (22) as an intertwining condition and $\phi_x$ as the intertwining operator which intertwines equivalent representations of the group $H$ in $M$ and $G$. Thus, to construct such operator one has to decompose the linear space $M$ into the irreducible representations of $\tau(H)$ and then use Schur’s lemma to construct the most general intertwining operator.

Within the gauge theory the result on classification of invariant connections can be interpreted as follows. A Yang-Mills theory on the multidimensional space-time $M = M_4 \times K/H$ with the gauge group $G$ considered on $K$-symmetric gauge field is reduced to a theory on the four-dimensional space-time $M_4$ with the gauge group $C \subset G$ which includes a gauge field $A$ and scalar fields generated by the mapping $\phi_x$. The action of the reduced theory can be calculated from Yang-Mills action (10) explicitely using representations (23), (24). The concrete form of the reduced action, for example the scalar multiplets, the form of the potential of the scalar fields and explicit expressions of the parameters in terms of the multidimensional gauge coupling and the size of the space of extra dimensions depend on the geometry of the homogeneous space $K/H$, the homomorphism $\tau : H \rightarrow G$ and some other properties of the groups $G$ and $K$. Results of detailed studies of these features, as well as a number of interesting examples are presented in Refs. [20], [24]. In particular, it was proved that if the space of extra dimensions is a symmetric homogeneous space $K/H$, then the scalar potential of the reduced theory is always a Higgs type potential and leads to the spontaneous symmetry breaking [23], [25]. We would like to mention that the invariant connection can also be defined and constructed using the formalism of Lie derivatives with respect to the Killing vectors of the extra dimensional metric (see Ref. [20]).

One of possible developments of the formalism of invariant connections would be its extension to the case of more general space-time, in particular to the case of orbifolds.
Multidimensional models on $M_4 \times S^1 / Z_2$ with a part of physical fields localized on branes situated at the fixed points of the orbifold $S^1 / Z_2$ are of much interest due to their promising physical properties. Another interesting problem to address is the calculation of characteristic classes for invariant connections.

**Example of invariant connection**

As an illustration of the general construction explained in the previous section we will consider the invariant connections in the Yang-Mills theory with the gauge group $SU(2)$ on the two-dimensional sphere $S^2$. This example is taken from Ref. [1]. The sphere is realized as a coset space $S^2 = K / H = SU(2) / U(1)$. First, let us construct the 1-forms $\theta_H$ and $\theta_M$ which appear in Eq. (23). We parametrize the sphere by angles $\vartheta$ and $\varphi$ ($0 \leq \vartheta \leq \pi$, $0 \leq \varphi < 2\pi$) and cover it with two charts $(U_1, \Phi_1)$ and $(U_2, \Phi_2)$ with the open sets $U_1$ and $U_2$ chosen as $U_1 = S^2 - \{\text{South pole}\}$ and $U_2 = S^2 - \{\text{North pole}\}$, 

\[
U_1 = \{0 \leq \vartheta < \pi, \ 0 \leq \varphi < 2\pi\}, \quad U_2 = \{0 < \vartheta \leq \pi, \ 0 \leq \varphi < 2\pi\}
\] 

and $\Phi_1, \Phi_2$ given by means of corresponding stereographic projections.

As generators of $K = SU(2)$ we take $Q_j = \sigma_j / 2i$ ($j = 1, 2, 3$), where $\sigma_j$ are the Pauli matrices. Let the subgroup $H = U(1)$ be generated by $Q_3$. Then the one-dimensional algebra $H$ is spanned by $Q_3$, and the vector space $\mathcal{M}$ in (28) is spanned by $Q_1$ and $Q_2$. Consider now the decomposition of the algebra $K$, which is the complexification of the Lie algebra of the group $K = SU(2)$. We denote by $e_\alpha$ and $e_{-\alpha}$ the root vectors and by $h_\alpha$ the corresponding Cartan element of this algebra (see, for example, [16]) and take 

\[
e_{\pm \alpha} = \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i \sigma_2), \quad h_\alpha = \sigma_3
\]

with 

\[
Ad(h_\alpha)(e_{\pm \alpha}) = [h_\alpha, e_{\pm \alpha}] = \pm 2e_{\pm \alpha},
\]

where $Ad$ denotes the adjoint action of the algebra: $Ad(X)Y \equiv [X, Y]$. Then $\mathcal{H} = Ch_\alpha$ and $\mathcal{M} = Ce_\alpha + Ce_{-\alpha}$, and the decomposition of the vector space $\mathcal{M}$ into irreducible invariant subspaces of $\mathcal{H}$ is described by the following decomposition of the representations:

\[
2 \rightarrow (2) + (-2),
\]

where in the right hand side we indicated the eigenvalues of $Ad(h_\alpha)$, and the space $\mathcal{M}$ of the reducible representation is indicated by its dimension in the left hand side.

We choose the local representatives $k^{(j)}$ ($j = 1, 2$) of points of the neighbourhood $U_j$ of the coset space $S^2 = SU(2) / U(1)$ as follows 

\[
k^{(1)}(\vartheta, \varphi) = e^{-i\varphi} e^{i\vartheta} e^{i\varphi} e^{i\vartheta}, \quad k^{(2)}(\vartheta, \varphi) = e^{i\varphi} e^{i(\vartheta - \pi)} e^{i\varphi} e^{-i\vartheta}.
\]

The functions $k^{(i)} : U_i \rightarrow SU(2)$ can also be viewed as local sections of the principal fibre bundle $K = P(K/H, H)$ over the base $K/H = S^2$ with the structure group
$H = U(1)$. By straightforward computation one obtains the forms $\bar{\theta}_H$ and $\bar{\theta}_M$:

\[
\bar{\theta}^{(i)} = (\mu^{(i)})^{-1} d\mu^{(i)} = \bar{\theta}_H^{(i)} + \bar{\theta}_M^{(i)};
\]

\[
\bar{\theta}_H^{(1)} = i\sigma_3^i (1 - \cos \vartheta) d\varphi,
\]

\[
\bar{\theta}_M^{(1)} = i\sigma_1^i (-\sin \varphi d\vartheta - \sin \vartheta \cos \varphi d\varphi) + i\sigma_2^i (\cos \varphi d\vartheta - \sin \vartheta \sin \varphi d\varphi),
\]

\[
\bar{\theta}_H^{(2)} = -i\sigma_3^i (1 + \cos \vartheta) d\varphi,
\]

\[
\bar{\theta}_M^{(2)} = i\sigma_1^i (\sin \varphi d\vartheta - \sin \vartheta \cos \varphi d\varphi) + i\sigma_2^i (\cos \varphi d\vartheta + \sin \vartheta \sin \varphi d\varphi).
\]

To construct the invariant connection on $M = S^2$ we apply general formula (23). Of course, since in the example under consideration the subspace $M_4$ is absent one should put $A = 0$.

First, let us specify the homomorphism $\tau : H = U(1) \to G = SU(2)$. Let $E_\alpha$, $E_{-\alpha}$ and $H_\alpha$ be the root vectors and the Cartan element of the algebra $G = A_1$, which appears as complexification of the Lie algebra of $G = SU(2)$. We assume that they are given by the same combinations of the Pauli matrices as the corresponding elements of complexified Lie algebra of $K = SU(2)$ described above. The group homomorphism $\tau$ is given by the expression

\[
\tau(e^{i\sigma_3^i\alpha_3}) = e^{i\kappa\sigma_3^i\alpha_3} = \cos(\kappa\alpha_3^i/2) + i\sigma_3^i \sin(\kappa\alpha_3^i/2),
\]

and it is easy to check that this definition is consistent if $\kappa$ is integer. Therefore the homomorphisms are labelled by $n \in Z$. Let us denote them by $\tau_n$. The induced algebra homomorphism is given by

\[
\tau_n(h_\alpha) = nH_\alpha. \tag{30}
\]

The three-dimensional space $\mathcal{G}$ of the adjoint representation of $A_1$ decomposes into three 1-dimensional irreducible invariant subspaces of $\tau'_n(\mathcal{H})$ and the decomposition is characterized by the following decomposition of representations:

\[
\mathbf{3} \to \mathbf{(0)} + \mathbf{(2n)} + \mathbf{(-2n)} \tag{31}
\]

(in brackets we indicate the eigenvalues of $\text{Ad}(\tau'_n(h_\alpha))$).

Let us now compare decompositions (27) and (31). For $n \neq \pm 1, 0$ there are no equivalent representations in the decomposition of $\mathcal{M}$ and $\mathcal{G}$ and the intertwining operator $\phi : \mathcal{M} \to \mathcal{G}$ is zero. It also turns out to be zero for $n = 0$. In these cases according to (23)

\[
A_n^{(1)} = \frac{i}{2} n\sigma_3 (1 - \cos \vartheta) d\varphi,
\]

\[
A_n^{(2)} = -\frac{i}{2} n\sigma_3 (1 + \cos \vartheta) d\varphi. \tag{32}
\]

If $n = 1$ or $n = -1$ the results are more interesting. Let us consider the case $n = 1$ first. Comparing (27) and (31) we see that there are pairs of representations
with the same eigenvalues and, therefore, the intertwining operator $\phi$ is non-trivial. It is determined by its action on the basis elements of $\mathcal{M}$:

$$\phi(e_\alpha) = f_1 E_\alpha, \quad \phi(-e_\alpha) = f_2 E_{-\alpha},$$

(33)

where $f_1$, $f_2$ are complex numbers. The fact that the initial groups and algebras are compact implies a reality condition [21] which tells that $f_1 = f_2^*$. Thus, the operator $\phi$ and the invariant connection are parametrized by one complex parameter $f_1$ (we will suppress its index from now on). Using Eqs. (28), (29), (30) and (33) we obtain from (23) that

$$A_1^{(1)} = i \left( \begin{array}{cc} (1 - \cos \vartheta) d\varphi & f e^{-i\varphi}(-id\vartheta - \sin \vartheta d\varphi) \\ f^* e^{i\varphi}(id\vartheta - \sin \vartheta d\varphi) & -(1 - \cos \vartheta) d\varphi \end{array} \right),$$

(34)

$$A_1^{(2)} = i \left( \begin{array}{cc} -(1 + \cos \vartheta) d\varphi & f e^{i\varphi}(-id\vartheta - \sin \vartheta d\varphi) \\ f^* e^{-i\varphi}(id\vartheta - \sin \vartheta d\varphi) & (1 + \cos \vartheta) d\varphi \end{array} \right).$$

(35)

The curvature form $F = dA + \frac{1}{2}[A, A]$ is described by the unique expression on the whole sphere and is equal to

$$F = -i\sigma_3 \left( |f|^2 - 1 \right) \sin \vartheta d\vartheta \wedge d\varphi.$$  

(36)

Action (10) evaluated for such configuration on $M = S^2$ is equal to

$$S_{YM}(f) = \frac{\pi}{2e^2 R^2} \left( |f|^2 - 1 \right)^2,$$

(37)

where $R$ is the radius of the sphere. Due to the $K$-invariance any extrema of the action found withing the subspace of invariant connections is also an extremum in the space of all connections [21]. From Eq. (37) we see that there are two types of extrema in the theory: the maximum at $f = 0$ and the minima at $f$ satisfying $|f| = 1$. Only one of them, the trivial extremum, was found in Ref. [28] as a spontaneous compactification solution in six-dimensional Kaluza-Klein theory.

A similar situation takes place for $\kappa = -1$. Again there exists a 1-parameter family of invariant connections parametrized by a complex parameter, say $h$, analogous to $f$. The action possesses two extrema: at $h = 0$ and for $|h| = 1$.

It turns out that potentials (32), (34) and (35) are related to known non-abelian monopole solutions in this theory. Namely, for $n \neq \pm 1$ and for $n = \pm 1$ with $f = 0$ these expressions coincide with the monopole solutions with the monopole number $\kappa = n$. In fact the solution with $\kappa = n > 0$ can be transformed to the solution with $\kappa = -n$ by a gauge transformation $A \rightarrow S^{-1}AS$ with the constant matrix $S = -i\sigma_1$. Eqs. (32) and Eqs. (34) and (35) with $f = 0$ describe all the monopoles in the $SU(2)$ gauge theory [15]. As it was shown in [7], all of them, except the trivial configuration with $n = 0$, are unstable. This is in accordance with the topological classification of monopoles by elements of $\pi_1(G)$ and also agrees with the fact that there is only one bundle (up to equivalence) with the base space $S^2$ and the structure group $SU(2)$. The latter will be shown in the next section.

Thus, all the monopoles are described by connections in the trivial principal fibre bundle $P(S^2, SU(2))$ and can be represented by a unique form on the whole sphere [7].
This is indeed the case. Namely there exist gauge transformations, different for $U_1$ and $U_2$ patches, so that the transformed potentials coincide. Let us demonstrate this for the case $n = 1$. In fact this property is true for the whole family of the invariant connections (32), (34). The group elements of these gauge transformations of the potentials on $U_1$ and $U_2$ are

$$V_1 = i \left( \cos \frac{\theta}{2} e^{-i\varphi \sin \frac{\theta}{2}} \right)$$

$$V_2 = i \left( e^{i\varphi \cos \frac{\theta}{2}} \sin \frac{\theta}{2} - e^{-i\varphi \cos \frac{\theta}{2}} \right)$$

By calculating

$$A^{(i)'}_1 = V_i^{-1} A^{(i)}_1 V_i - V_i^{-1} dV_i$$

for $i = 1$ and $i = 2$ one can easily check that the transformed potentials are equal to each other and are given by

$$A^{(1)'}_1 = A^{(2)'}_1 = \frac{i}{2} \left( \sigma_+ c_+ + \sigma_- c_- + \sigma_3 c_3 \right)$$

where

$$c_+ = c_\ast = e^{-i\varphi} \left\{ \left[ -\cos \vartheta + \left( f \cos^2 \frac{\theta}{2} - f^* \sin^2 \frac{\theta}{2} \right) \right] \sin \vartheta d\varphi \right. + \left. i \left( -1 + f \cos^2 \frac{\theta}{2} + f^* \sin^2 \frac{\theta}{2} \right) d\theta \right\}$$

$$c_3 = \left( 1 - \frac{f + f^*}{2} \right) \sin^2 \vartheta d\varphi - \frac{f - f^*}{2} \sin \vartheta d\varphi$$

Note that in general expressions (34), (35) and (38) the phase of the complex parameter $f$ can be rotated by residual gauge transformations which form the group $U(1)$.

For $f = 0$ this formula gives the known expression for the $\kappa = 1$ SU(2)-monopole [7]:

$$A^{(1)'}_1 = A^{(2)'}_1 = \frac{i}{2} \left( e^{i\varphi} (2i d\varphi - \sin 2\vartheta d\varphi) \right)$$

Of course, forms (38) and (40) do not have singularities on the whole sphere. For $f = f^* = 1$ forms (39) vanish. This shows that this configuration, which is also the extremum of the action, describes the trivial case of the $SU(2)$-monopole with $\kappa = 0$. Note that in the original form potentials (32) and (34) do not seem to be trivial. Of course, one can check that they are pure gauges and correspond to a flat connection, i.e. a connection with $\Omega = 0$. Vanishing of gauge field 2-form (36) for this value of $f$ confirms this.

The picture we have obtained is the following. For different homomorphisms $\tau_n : H \to G$ we have constructed different invariant connections given by Eqs. (32), (34) and (35). For $n \neq \pm 1$ or $n = \pm 1$ with $f = 0$ the connection describes the $SU(2)$-monopole solution with the monopole number $\kappa = n$. All $SU(2)$-monopoles on $S^2$ are reproduced in this way. As it was said above the solutions with numbers $\kappa$ and $(-\kappa)$ are gauge equivalent. In addition, there is a continuous 1-parameter family of invariant
connections which passes through the configurations describing the $SU(2)$-monopoles with numbers $\kappa = -1$, $\kappa = 0$ and $\kappa = 1$ in the space of all connections of the theory. Connections from this family are described by Eqs. (34), (35), (38) and (39). Not all of these connections are gauge inequivalent. Classes of gauge equivalent invariant connections are labelled by values of $|f|$. Thus, $|f| = 0$ corresponds to the class of the $\kappa = 1$ monopole. The monopole with $\kappa = -1$ can be obtained from it by the gauge transformation with the constant matrix $S = -i\sigma_1$, as it was explained above, and, hence, belongs to the same gauge class. Connections with $|f| = 1$ form the class describing the monopole with $\kappa = 0$.

**Classification of fibre bundles**

So far we have been discussing the classical aspects of gauge theories. One of the ways to quantize them consists in considering a functional integral of the type

$$Z = \mathcal{N} \int_{\mathcal{A}} \mathcal{D}A e^{-S_{YM}[A]} T(A), \quad (41)$$

where $S_{YM}[A]$ is the classical action given by (8) or (10), $\mathcal{N}$ is a normalization factor and $T(A)$ is some function or functional of the gauge field $A$. For example, $T(A)$ can be a product of $A$-fields at different points or the traced holonomy for a closed path in $M$. In the first case integral (41) defines a Green function, whereas in the second case it gives the vacuum expectation value of the Wilson loop variable [8]. The measure $\mathcal{D}A$ is understood in a heuristic sense adopted in quantum field theory. The integral is taken over the space $\mathcal{A}$ of connections. In general $\mathcal{A}$ may consist of a number of components (or sectors) $\mathcal{A}_\alpha$ labelled by elements $\alpha$ of an index set $\mathcal{B}$, $\alpha \in \mathcal{B}$. Then functional integral (41) is given by a sum over the elements of $\mathcal{B}$, each term of the sum being the functional integral over the subspace of connections $\mathcal{A}_\alpha$.

The set of connected components of $\mathcal{A}$ is in 1-1 correspondence with the set of non-equivalent (i.e. which cannot be mapped one into another by a bundle isomorphism) principal $G$-bundles $P(M,G)$ over manifold $M$. Let us denote this set as $\mathcal{B}_G(M)$, i.e. $\mathcal{B} \cong \mathcal{B}_G(M)$. The problem of characterization of this set and classification of such bundles is considered in a number of books and articles. A method, which in many cases gives a solution to this problem and which we closely follow here, is discussed in lectures [2] and in Refs. [1], [20]. Relevant results from algebraic topology can be found in [4, 30, 32]. Here we only outline the construction. We would like to note that in fact the considerations and the results in the rest of this section are valid for a more general case, namely when $M$ is a path-connected CW-complex and not just a smooth manifold, and $G$ is a topological group (see definitions, for example, in Ref. [32])\(^2\).

It turns out that for any topological group $G$ there exists a space $BG$, called the classifying space, and a principal $G$-bundle $EG = P(BG,G)$, called the universal $G$-bundle, such that every principal $G$-bundle $P(M,G)$ is induced from $EG$, i.e. $P = f^*(EG)$ for some map $f : M \to BG$ [32]. Two homotopic maps $M \to BG$ induce

\(^2\)The CW-complexes and the homotopy equivalences considered here are actually pointed CW-complexes and homotopy equivalences between pointed spaces. For the sake of simplicity of notations we do not indicate the base points explicitly. The reader interested in complete formulas is referred to Ref. [20].
equivalent bundles. The total space $EG$ of the universal bundle is $\infty$-universal, i.e. $\pi_q(EG) = 0$ for all $q \geq 1$. The universal bundle is unique up to homotopy equivalence. Brown’s theorem [32] implies that for any $G$ there exists a CW-complex $BG$ and a principal $G$-bundle $EG = P(BG, G)$ such that for any CW-complex $M$ there is an equivalence

$$B_G(M) \cong [M ; BG].$$

By examining a certain exact homotopy sequence it can be shown that the base $BG$ of the universal bundle satisfies the property [2], [32]

$$\pi_q(BG) = \pi_{q-1}(G), \quad q \geq 1.$$  \hfill (42)

As an immediate application of this relation we obtain the classification of principal $G$-bundles over the sphere $S^n$ in terms of the $(n-1)$th homotopy group of $G$:

$$B_G(S^n) \cong [S^n ; BG] \cong \pi_n(BG) \cong \pi_{n-1}(G).$$  \hfill (43)

In particular, $B_{SU(2)}(S^2) \cong \pi_1(SU(2)) = 0$, i.e. there is only one $SU(2)$-bundle with the base $S^2$, namely the trivial one $P = S^2 \times SU(2)$. This result has already been mentioned in the previous section.

The question is how one can construct such universal bundles and characterize $[M; BG]$ in terms of objects which can be calculated in a relatively easy way. It turns out that the Eilenberg-MacLane spaces play important role for this problem because of their special homotopic properties. Such spaces are often denoted as $K(\pi, n)$, where $\pi$ is a group and $n$ is a positive integer, and are defined as follows:

i) $K(\pi, n)$ is path connected;

ii) $\pi_q(K(\pi, n)) = \begin{cases} \pi, & \text{if } q = n, \\ 0, & \text{if } q \neq n. \end{cases}$

If $n \geq 1$ and $\pi$ is Abelian, the space $K(\pi, n)$ exists as a CW-complex and can be constructed uniquely up to a homotopy equivalence [30]. The property, which is crucially important for us, is the following:

$$[M; K(\pi, n)] \cong H^n(M, \pi),$$  \hfill (44)

where $H^n(M, \pi)$ is the $n$th singular cohomology group with coefficients in $\pi$ [6], [30], [32]. A simple example is $K(Z, 1) = S^1$.

Often the Eilenberg-Maclane spaces are infinite dimensional. The example important for us is the space $CP^\infty$ which is a CW-complex. It is understood as the union (direct limit) of the complex projective spaces $CP^n$ of the sequence $CP^1 \subset CP^2 \subset \ldots$ [30]. Then $\pi_q(CP^\infty) = \lim_{j \to \infty} \pi_q(CP^j)$ and

$$\pi_2(CP^\infty) = \mathbb{Z}, \quad \pi_q(CP^\infty) = 0 \text{ for } q \neq 2.$$ 

Thus, $CP^\infty = K(Z, 2)$.

This space serves for the classification of principal fiber bundles with $G = U(1)$ [2], [19]. Indeed, consider the Hopf bundle $S^{2n-1} = P(CP^n, U(1))$, where the sphere is realized as

$$S^{2n-1} = \left\{ (z_1, \ldots, z_n) \in C^n \mid \sum_{i=1}^{n} |z_i|^2 = 1 \right\}.$$
and the bundle projection \( p : S^{2n-1} \rightarrow CP^n \) is given by

\[
p(z_1, \ldots, z_n) = \left( \frac{z_2}{z_1}, \frac{z_3}{z_1}, \ldots, \frac{z_n}{z_1} \right)
\]

for a neighbourhood with \( z_1 \neq 0 \), etc. Since \( \pi_q(S^{2n-1}) = 0 \) for \( q < 2n - 1 \), the bundle is \((2n-2)\)-universal. Then one takes the direct limits \( S^\infty = \lim_{\rightarrow} S^n, CP^\infty = \lim_{\leftarrow} CP^n \). The bundle \( S^\infty = P(CP^n, U(1)) \) is \( \infty \)-universal, and, therefore, \( EU(1) = S^\infty \) and \( BU(1) = CP^\infty = K(Z, 2) \). Thus, for a CW-complex of any dimension

\[
\mathcal{B}_{U(1)}(M) \cong [M; BU(1)] \cong [M; CP^\infty] = [M; K(Z, 2)] \cong H^2(M; Z).
\]

This result gives the classification of principal \( U(1) \)-bundles \( P(M, U(1)) \). For the case when \( M \) is a smooth manifold it was discussed in \([22] \).

Now let us consider the generalization of this construction for the case of other gauge groups. The idea is that for classification of bundles with the base \( M \) with \( \dim M \leq n \) only homotopy groups of low dimensions are important. Then instead of \( BG \) one can use some other space \( BG_n \) which is related to it and which may be easier to construct and to study. To define \( BG_n \) let us introduce the notion of \( p \)-equivalence.

Consider two path connected spaces \( X \) and \( Y \) and a continuous map \( f : X \rightarrow Y \) such that for all \( x \in X \) the induced map \( f_* : \pi_q(X) \rightarrow \pi_q(Y) \) is an isomorphism for \( 0 < q < p \) and an epimorphism for \( q = p \). Such map \( f \) is called \( p \)-equivalence. Then, if there exists some space \( BG_n \) and a map \( f : BG \rightarrow BG_n \) which is \((n+1)\)-equivalence, it can be proved that

\[
[M; BG] \cong [M; BG_n]
\]

for any CW-complex \( M \) with \( \dim M \leq n \) \([4], [32] \).

Another important element which is used for the classification of principal fibre bundles is the Postnikov decomposition (called also the Postnikov factorization). It allows to construct certain exact sequences which give the required \((n+1)\)-equivalences.

From now on we restrict the discussion to the case of two-dimensional spaces. In addition we assume certain properties of \( G \), they are specified below. We would like to mention that these properties are not too restrictive and are usually fulfilled in physical applications. Results for \( \mathcal{B}_G(M) \) with \( M \) of higher dimensions can be found in \([2] \).

Since \( \dim M = 2 \) for the classification of principal \( G \)-bundles it suffices to find a 3-equivalence \( BG \rightarrow BG_2 \). Its existence is guaranteed by the Postnikov factorization theorem \([3], [30] \). Then \( \mathcal{B}_G(M) \cong [M; BG] \cong [M; BG_2] \). The term \([M; BG_2]\) appears in an extended Puppe sequence in the corresponding Postnikov diagram (see details in \([26] \)). Finally, we arrive at the following result on the classification of principal \( G \)-bundles over two-dimensional spaces.

**Theorem 1.** \([26] \) Let \( M \) be a path-connected pointed CW-complex of \( \dim M = 2 \). Let \( G \) be a group such that \( \pi_0(G) \) is abelian and discrete and acts trivially on the higher homotopy groups \( \pi_n(G) \) for \( n \geq 1 \). Then for the set \( \mathcal{B}_G(M) \) of equivalence classes of principal \( G \)-bundles over \( M \) there exists the following short exact sequence of pointed sets:

\[
0 \rightarrow H^2(M; \pi_1(G)) \rightarrow \mathcal{B}_G(M) \rightarrow H^1(M; \pi_0(G)) \rightarrow 0.
\]

(46)
Further specification of $\mathcal{B}_G(M)$ requires the knowledge of additional information about $M$ or $G$. Let us consider two particular cases.

1. $\pi_1(G) = 0$. It follows from (46) that

$$\mathcal{B}_G(M) \cong H^1(M; \pi_0(G)).$$

This case includes the class of discrete groups. If $G$ is discrete, $\pi_0(G) \cong G$ and the formula above simplifies further:

$$\mathcal{B}_G(M) \cong H^1(M; G).$$

2. $G$ is path-connected. Then $\pi_0(G) = 0$ and

$$\mathcal{B}_G(M) \cong H^2(M; \pi_1(G)).$$

Let us make a few comments. Eq. (48), giving the classification of principal $G$-bundles with a discrete structure group, is in fact a particular case of a more general result valid for CW-complexes $M$ of any dimension, see Ref. [26] for further details. Relation (49) for $G = U(1)$ is, of course, in agreement with result (45) for the base space of any dimension.

The results in cases 1 and 2 above can be obtained, in fact, from the following general theorem.

**Theorem 2.** Let $X$ be a pointed CW-complex, $Y$ is an $(n - 1)$-connected pointed space, and $H^{q+1}(X; \pi_q(Y)) = 0 = H^q(X; \pi_q(Y))$ for all $q \geq n$. Then there exists a one-to-one correspondence

$$[X; Y] \cong H^n(X; \pi_n(Y)).$$

Let us apply this theorem to $X = M$ with $	ext{dim} M = 2$. In the case when $Y = BG$ is 0-connected with $\pi_2(BG) = \pi_1(G) = 0$, we obtain result (47). If $Y = BG$ is 1-connected, i.e. $\pi_1(BG) = \pi_0(G) = 0$, then (50) gives (49). The latter case is also a result of the Hopf-Whitney classification theorem [36].

Relations (47) and (49) become more concrete if further properties of the group $G$ are known. For completeness of the discussion, we present a list of the first homotopy groups $\pi_1(G)$ for some connected Lie groups:

1. Simply connected, $G = SU(n)$, $Sp(n)$: $\pi_1(G) = 0$.

2. $G = SO(n)$, $n = 3$ and $n \geq 5$: $\pi_1(G) = Z_2$.

3. $G = U(n)$: $\pi_1(G) = Z$.

In particular, using Eq. (49) one can obtain that $\mathcal{B}_{SU(2)}(S^2) \cong H^2(S^2; \pi_1(SU(2))) = 0$, the result already mentioned above.
If the space $M$ is of certain type, sequence (46) and relations (47), (49) become more concrete. Thus, if $M$ is a two-dimensional differentiable manifold one can use known formulas for cohomology groups. For example, if $\pi$ is abelian then (see [6])

1) for $M$ compact and orientable

$$H^2(M; \pi) \cong \pi,$$  

(51)

2) for $M$ compact and not orientable

$$H^2(M; \pi) \cong \pi/2\pi$$

(52)

More expressions for cohomology groups of various two-dimensional surfaces can be found in Ref. [17]. In particular, for $M = S^2$ it is known that its first cohomology group $H^1(S^2, \pi) = 0$. Then from Theorem 1, Eq. (44) and the definition of the Eilenberg-MacLane spaces it follows that

$$B_G(S^2) \cong H^2(S^2; \pi_1(G)) \cong [S^2; K(\pi_1(G), 2)] \cong \pi_2(K(\pi_1(G), 2)) = \pi_1(G).$$

This is in accordance with Eq. (48).

To finish our discussion of the result let us mention a classification of principal fibre bundles over two-dimensional compact orientable manifolds obtained by Witten in Ref. [38]. It is known that any connected Lie group $G$ can be obtained as a quotient group $G = \tilde{G}/\Gamma$ (see, for example, [3], [14]). Here $\tilde{G}$ is the unique (up to isomorphism) connected and simply connected Lie group, called a universal covering group of $G$. $\Gamma$ is a discrete subgroup of the center $Z(\tilde{G})$ of $\tilde{G}$. Witten showed that principal $G$-bundles over $M$ are classified by elements of $\Gamma$, i.e. $B_G(M) \cong \Gamma$. This agrees with Eq. (49). Indeed, taking into account that $\pi_1(G) \cong \pi_0(\Gamma) \cong \Gamma$ and using Eq. (51), from (49) we get

$$B_G(M) \cong H^2(M; \pi_1(G)) \cong H^2(M; \Gamma) \cong \Gamma.$$

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