FULL CHAINS OF TWISTS FOR ORTHOGONAL ALGEBRAS

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Abstract

We show that for some Hopf subalgebras in \( U_F(so(M)) \) nontrivially deformed by a twist \( F \) it is possible to find the nonlinear primitive copies. This enlarges the possibilities to construct chains of twists. For orthogonal algebra \( U(so(M)) \) we present a method to compose the full chains with carrier space as large as the Borel subalgebra \( B(so(M)) \). These chains can be used to construct the new deformed Yangians.

1 Introduction

Quantizations of triangular Lie bialgebras \( L \) with antisymmetric classical \( r \)-matrices \( r = -r_{21} \) are defined by a twisting element \( \mathcal{F} = \sum f(1) \otimes f(2) \in \mathcal{A} \otimes \mathcal{A} \) which satisfies the twist equations [1]:

\[
\begin{align*}
(\mathcal{F})_{12} (\Delta \otimes \text{id}) \mathcal{F} &= (\mathcal{F})_{23} (\text{id} \otimes \Delta) \mathcal{F}, \\
(\epsilon \otimes \text{id}) \mathcal{F} &= (\text{id} \otimes \epsilon) \mathcal{F} = 1.
\end{align*}
\]

(1)

Explicit form of the twisting element is quite important in applications because it provides explicit expressions for the quantum \( \mathcal{R} \)-matrix \( \mathcal{R}_F = \mathcal{F}_{21} \mathcal{F}^{-1} \) and for the twisted coproduct \( \Delta_F() = \mathcal{F} \Delta_F() \mathcal{F}^{-1} \).

The first nontrivial explicitly written twisting elements \( \mathcal{F} \) were given in the papers [2], [3], [4] and [5]. These twists can be defined on the following
carrier algebra $L$:

\[
\begin{align*}
[H, E] &= E, & [H', E] &= \gamma' E, \\
[H, A] &= \alpha A, & [H', A] &= \alpha' A, \\
[H, B] &= \beta B, & [H', B] &= \beta' B, \\
[E, A] &= [E, B] = 0, & [A, B] &= E, \\
\alpha + \beta &= 1, & \alpha' + \beta' &= \gamma'.
\end{align*}
\]

Explicit expressions for their twisting elements are

\[
\begin{align*}
\Phi_R &= e^{H \otimes H'}, \\
\Phi_J &= e^{H \otimes \sigma}, \\
\Phi_{\varepsilon J} &= \Phi_{\varepsilon} \Phi_J = e^{A \otimes B e^{-\beta \sigma}} e^{H \otimes \sigma}, \\
\sigma &= \ln (1 + E).
\end{align*}
\]

Here $r_R, r_J, r_{\varepsilon J}$ are the corresponding classical $r$-matrices.

Carrier subalgebras $L$ can be found in any simple Lie algebra $g$ of rank greater than 1.

It was demonstrated in [6] that these twists can be composed into chains. They are based on the sequences of regular injections constructed for the initial Lie algebra $g$

\[g_p \subset g_{p-1} \subset g_1 \subset g_0 = g.\]

To form the chain one must choose an initial root $\lambda_0$ in the root system $\Lambda (g)$, consider the set $\pi$ of its constituent roots

\[
\pi = \{ \lambda', \lambda'' \mid \lambda' + \lambda'' = \lambda_0; \quad \lambda' + \lambda_0, \lambda'' + \lambda_0 \notin \Lambda (g) \}
\]

and the subset $\Lambda^\perp_{\lambda_0}$ of roots orthogonal to $\lambda_0$ (the corresponding subalgebra in $g$ will be denoted by $g^\perp_{\lambda_0}$).

It was shown that for the classical Lie algebras $g$ one can always find in $g^\perp_{\lambda_0}$ a subalgebra $g_1 \subseteq g^\perp_{\lambda_0} \subset g_0 = g$ whose generators become primitive after the extended twist $\Phi_{\varepsilon J}$. Such primitivization of $g_k \subset g_{k-1}$ (called the matreshka effect [6]) provides the possibility to compose chains of extended twists of the type $\Phi_{\varepsilon J}$

\[
\begin{align*}
\mathcal{F}_{B_0 \prec \varepsilon} &= \Pi_{k=0}^{\varepsilon} \Phi_{\varepsilon_k} \Phi_{J_k}, \\
\Phi_{\varepsilon_k} \Phi_{J_k} &= \Pi_{\lambda' \in \pi_k} \exp \left\{ E_{\lambda'} \otimes E_{\lambda_k^0 - \lambda'} e^{-\frac{1}{2} \sigma_{\lambda_k^0}} \right\} \exp \left\{ H_{\lambda_k^0} \otimes \sigma_{\lambda_k^0} \right\}.
\end{align*}
\]

Chains of twists quantize a large variety of $r$-matrices corresponding to Frobenius subalgebras in simple Lie algebras [7].
2 Construction of a full chain of twists

The main point in the construction of a chain is the invariance of \( g_{k+1} \) with respect to \( \Phi_{E_kJ_k} \). When these subalgebras are proper the canonical chains have only a part of \( B^+ (g) \) as the twist carrier subalgebra:

\[
\cdots \subset g_{X_0}^{k+1} \subset \subset g_{X_0}^{k+1} \subset \subset g_{X_0}^{k} \subset \subset g_{X_0}^{k-1} \subset \subset \cdots \subset g_{k+1} \subset g_{k} \subset \subset g_{k-1} \subset \subset g_{k-2} \subset \subset \cdots \subset g_{0}.
\]

(4)

We would like to demonstrate that the effect of primitivization is universal and extends to the whole subalgebra \( g_{X_0}^{k} \). It was shown in [8] that the invariance of a subalgebra in \( g_{X_0}^{k} \) is only one of the forms of the primitivization. In general this is the existence (in the twisted Hopf algebra \( U_{E_kJ_k} (g_{X_0}^{k}) \)) of a primitive subspace \( V^k_{G} \) with the algebraic structure isomorphic to \( g_{X_0}^{k} \). On this subspace the subalgebra \( g_{X_0}^{k} \) is realized nonlinearly so \( V^k_{G} \) is called deformed carrier space [8].

In this context the situation with the twists for \( U (sl(N)) \) is degenerate: the subalgebra \( (sl(N))_{X_0}^{k} \) coincides with \( (sl(N))_{k+1} \), i.e. \( V^k_{G} = V_{(sl(N))_{X_0}^{k}} \).

In the case of \( U (so(M)) \) the situation is different. Let the root system \( \Lambda (so(M)) \) be

\[ \{ \pm e_i \pm e_j \mid i, j = 1, 2, \ldots M/2; i \neq j \} \]

for even \( M \) and

\[ \{ \pm e_i \pm e_j; \pm e_k \mid i, j, k = 1, 2, \ldots (M - 1)/2; i \neq j \} \]

for odd \( M \). Take \( e_1 + e_2 \) as the initial root. Here the subalgebras \( g_{X_0}^{k-1} \) and \( g_{k} \) in (4) are related as follows,

\[ g_{X_0}^{k-1} = g_{k} \oplus so^{(k)} (3) = so (M - 4k) \oplus so^{(k)} (3). \]

Consider the invariants of the vector fundamental representations of \( g_{k+1} = so(M - 4 (k + 1)) \) acting on \( g_{k} \):

\[
\begin{align*}
I_{2N+1}^{a} &= \frac{1}{2} E_a^2 + \sum_{l=3}^{N} (E_{a+l} E_{a-l}), \\
I_{2N+1}^{a \otimes b} &= E_a \otimes E_b + \sum_{l=3}^{N} (E_{a+l} \otimes E_{b-l} + E_{a-l} \otimes E_{b+l}),
\end{align*}
\]

(5)
\[I_{2N}^a = \sum_{l=3}^N (E_{a+l} E_{a-l}) ,
I_{2N}^{a \otimes b} = \sum_{l=3}^N (E_{a+l} \otimes E_{b-l} + E_{a-l} \otimes E_{b+l}) ,\]

(6)

The \(so(k)(3)\) summands are non-trivially deformed by \(\Phi_{\xi_{k-1}, J_{k-1}}\):

\[
\Delta_{\xi_{k-1}, J_{k-1}} (E_{1-2}^k) = E_{1-2}^k \otimes 1 + 1 \otimes E_{1-2}^k + (1 \otimes e^{-\frac{k}{4}k_{+1}} \pi_{M-4k}^1)
+ I_{M-4k}^1 \otimes \left(e^{-\frac{k}{4}k_{+1}} - 1\right) ,
\]

\[
\Delta_{\xi_{k-1}, J_{k-1}} (E_{2-1}^k) = E_{2-1}^k \otimes 1 + 1 \otimes E_{2-1}^k + \left(e^{\frac{k}{4}k_{+1}} - 1\right) \otimes I_{M-4k}^2 e^{-\frac{k}{4}k_{+1}}
+ (1 \otimes e^{-\frac{k}{4}k_{+1}}) I_{M-4k}^{2 \otimes 2} .
\]

According to the main principle formulated above (despite the deformed costructure of \(V_{g_{k-1}}\)) the primitivization is realized on its isomorphic image \(V_{G}^{k+1}\) contained in \(U_{\xi_{k-1}, J_{k-1}} (g_{X_0}^{k-1})\). To find this deformed carrier subspace \(V_{G}^{k+1}\) it is sufficient to inspect the coproducts of invariants (3) and (4),

\[
\Delta_{\xi_{k}, J_{k}} (I_{M-4k}^1) = I_{M-4k}^1 \otimes e^{-\sigma_{+1}^k} + 1 \otimes I_{M-4k}^1 + I_{M-4k}^{1 \otimes 1} \left(1 \otimes e^{-\frac{k}{4}k_{+1}}\right) ,
\]

\[
\Delta_{\xi_{k}, J_{k}} (I_{M-4k}^2 e^{-\sigma_{+1}^k}) = I_{M-4k}^2 e^{-\sigma_{+1}^k} \otimes 1 + e^{\sigma_{+1}^k} \otimes I_{M-4k}^2 e^{-\sigma_{+1}^k} + I_{M-4k}^{2 \otimes 2} \left(1 \otimes e^{-\frac{k}{4}k_{+1}}\right) .
\]

Now one can construct the following nonlinear primitive generators

\[G_{1-2}^{k+1} = E_{1-2}^k - I_{M-4k}^1 , \quad \Delta_{\xi_{k}, J_{k}} (G_{1-2}^{k+1}) = G_{1-2}^{k+1} \otimes 1 + 1 \otimes G_{1-2}^{k+1} ,\]

\[G_{2-1}^{k+1} = E_{2-1}^k - I_{M-4k}^1 e^{-\sigma_{+1}^k} , \quad \Delta_{\xi_{k}, J_{k}} (G_{2-1}^{k+1}) = G_{2-1}^{k+1} \otimes 1 + 1 \otimes G_{2-1}^{k+1} ,\]

\[H_{1-2}^{k+1} , \quad \Delta_{\xi_{k}, J_{k}} (H_{1-2}^{k+1}) = H_{1-2}^{k+1} \otimes 1 + 1 \otimes H_{1-2}^{k+1} .\]

The subspace spanned by \(\{H_{1-2}^k, G_{1-2}^{k+1}, G_{2-1}^{k+1}\}\) forms the algebra \(so_{G}^{(k+1)}(3) \approx so_{G}^{(k+1)}(3)\):

\[
[H_{1-2}^k, G_{1-2}^{k+1}] = G_{1-2}^{k+1} ,
[H_{1-2}^k, G_{2-1}^{k+1}] = -G_{2-1}^{k+1} ,
[G_{1-2}^{k+1}, G_{2-1}^{k+1}] = 2H_{1-2}^k .
\]

Therefore we obtain the deformed primitive space

\[V_{G}^{k+1} (g_{X_0}^{k+1}) = V (g_{k+1}) \oplus V \left(so_{G}^{(k+1)}(3)\right) ,\]
that can be considered as a carrier for the twists (8). The next extended
Jordanian twist in the chain (that is defined on \( g_{k+1} \)) does not touch the
space \( V\left( so_{G}^{(k+1)}(3) \right) \). Consequently after all the steps of the chain we will
still have a primitive subalgebra

\[
\mathcal{D} = \sum_{k=0}^{p} \oplus so_{G}^{(k+1)}(3)
\]

defined on the sum of deformed spaces \( V\left( so_{G}^{(k+1)}(3) \right) \).

Thus in the twisted Hopf algebra \( U_{B_{0}\prec p} \left( so(M) \right) \) one can perform further
twist deformations with the carrier subalgebra in \( \mathcal{D} \). The most interesting
among them are the Jordanian twists defined by

\[
\Phi_{J_{k}} = \exp \left( H_{1-2}^{k} \otimes \sigma^{k}_{G} \right) \quad \text{with} \quad \sigma^{k}_{G} \equiv \ln \left( 1 + G^{k+1}_{1-2} \right)
\]

This means that in the general expression for the twisting element \( \mathcal{F}_{B_{0}\prec p} \) one
can insert in the appropriate \( k \geq 0 \) places the Jordanian twisting factors
defined on the deformed carrier spaces, i.e. to perform a substitution

\[
\Phi_{\varepsilon_{k}} \Phi_{J_{k}} \Rightarrow \Phi_{G_{k}}^{j} \Phi_{\varepsilon_{k}} \Phi_{J_{k}} = \Phi_{g_{k}}
\]

\[
\text{exp} \left\{ I_{M-4k}^{\otimes 2} \left( 1 \otimes e^{-\frac{1}{2} \sigma^{k}_{1+2}} \right) \right\} \cdot \text{exp} \left\{ H_{1+2}^{k} \otimes \sigma^{k}_{1+2} \right\} \Rightarrow
\]

\[
\text{exp} \left( H_{1-2}^{k} \otimes \sigma^{k}_{G} \right) \cdot \text{exp} \left\{ I_{M-4k}^{\otimes 2} \left( 1 \otimes e^{-\frac{1}{2} \sigma^{k}_{1+2}} \right) \right\} \cdot \text{exp} \left( H_{1+2}^{k} \otimes \sigma^{k}_{1+2} \right)
\]

This gives the full chain in the following form

\[
\mathcal{F}_{\tilde{g}_{0}\prec p} = \prod_{k=p}^{0} \Phi_{g_{k}} = \prod_{k=p}^{0} \left( \exp \left( H_{1-2}^{k} \otimes \sigma^{k}_{G} \right) \cdot \exp \left\{ I_{M-4k}^{\otimes 2} \left( 1 \otimes e^{-\frac{1}{2} \sigma^{k}_{1+2}} \right) \right\} \cdot \exp \left( H_{1+2}^{k} \otimes \sigma^{k}_{1+2} \right) \right).
\]

(7)

Obviously the additional twistings by \( \Phi_{G_{k}}^{j} \) cannot be performed before the
deformation of the corresponding spaces \( V_{G}^{k+1} \) by the extended Jordanian
twists \( \Phi_{\varepsilon_{k}} \Phi_{J_{k}} \).

3 Applications

The previous result means that we have constructed explicit quantizations

\[
\mathcal{R}_{g_{0}\prec p} = \left( \mathcal{F}_{g_{0}\prec p} \right)_{21} \left( \mathcal{F}_{g_{0}\prec p} \right)^{-1}
\]
of the following set of classical $r$-matrices:

$$r_{\mathcal{G}_0 \prec p} = \sum_{k=0}^{p} \eta_k \left( H_{1+2}^k \wedge E_{1+2}^k + \xi_k H_{1-2}^k \wedge E_{1-2}^k + I_{M-4k}^{1\wedge 2} \right)$$

Here all the parameters are independent.

The dimensions of the nilpotent subalgebras $N^+ (so (M))$ in the sequence $g_{X_0}^+ \subset g_{X_0}^{1} \subset \ldots \subset g_{X_0}^{1} \subset g$ are subject to the simple relation:

$$\dim \left( N^+ (so (M)) \right) - \dim \left( N^+ (so (M-4)) \right) = 2 \left( \dim d^v_{so(M-4)} + 1 \right).$$

Taking this into account we see that the chains (7) are full in the sense that for $p = p^{\text{max}} = [M/4] + [(M + 1)/4]$ their carrier spaces contain all the generators of $N^+ (so (M))$. When $M$ is even-even or odd the total number of Jordanian twists in a maximal full chain $\mathcal{F}_{\mathcal{G}_0 \prec p}^{\text{max}}$ is equal to the rank of $so (M)$. Thus in the latter case the carrier subalgebra is equal to $B^+ (so (M)).$

It was demonstrated in [9] how to construct new Yangians using the explicit form of the twisting element. These new Yangians are defined by the corresponding rational solution of the matrix quantum Yang-Baxter equation (YBE). In particular, for the orthogonal classical Lie algebras $so (M)$ one needs the twisting element $F$ in the defining (vector) representation $d^v$ and the auxiliary operators: the flip $P : v \otimes w \rightarrow w \otimes v$ ($P \in \text{Mat} (M) \otimes \text{Mat} (M)$) and the operator $K$, which is obtained from $P$ by transposing its first tensor factor. The following expression gives the corresponding deformed rational solution of the YBE:

$$ud^v \left( F_{21} F^{-1} \right) + P - \frac{u}{u - 1 + M/2} d^v (F_{21}) K d^v \left( F^{-1} \right)$$

Here $u$ is a spectral parameter. In [10] such deformed solutions were obtained in the explicit form for the canonical chains $\mathcal{F} = \mathcal{F}_{\mathcal{B}_0 \prec p}$.

All the calculations can be reproduced for the twisting elements $\mathcal{F} = \mathcal{F}_{\mathcal{G}_0 \prec p}$ of the full chains. This will lead to a new set of so called deformed Yangians [11].

This work was partially supported by the Russian Foundation for Basic Research under the grant 00-01-00500 (VDL) and 98-01-00310 (PPK).
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