Hyperbolicity of orders of quaternion algebras and group rings

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Abstract

For a given division algebra of the quaternions, we construct two types of units of its \(\mathbb{Z}\)-orders: Pell units and Gauss units. Also, if \(K = \mathbb{Q}\sqrt{-d}\) for \(d\) a square free and \(R = I_K\), we classify \(R\) and \(G\) such that \(\mathcal{U}_1(RG)\) is hyperbolic. In particular, with a suitable geometric approach we prove that \(\mathcal{U}_1(RK_8)\) is hyperbolic iff \(d > 0\) and \(d \equiv 7 \pmod{8}\). In this case, the hyperbolic boundary \(\partial(\mathcal{U}_1(RG)) \cong S^2\), the two dimensional sphere.

1 Introduction

Hyperbolic groups were defined firstly by Gromov \cite{3}, from the concept of hyperbolic metric space.

Let \(G\) be a finitely generated group and \(\mathcal{G}\) its Cayley graph with the length metric, \(G\) is hyperbolic if \(\mathcal{G}\) is hyperbolic.

Gromov showed that if \(\Gamma\) is hyperbolic, then it does not contain a free abelian group of rank two, i.e., \(\mathbb{Z}^2 \not\rightarrow \Gamma\). If \(G\) is finite then \(\mathbb{Q}G\) has at most one Wedderburn component that is not a division ring and it is isomorphic to \(M_2(\mathbb{Q})\). This was first proved by Jespers in \cite{6}. Still in \cite{6}, Jespers classifies the finite groups \(G\) with non abelian free normal complement in \(\mathcal{U}_1(\mathbb{Z}G)\).

Recently, Juriaans, Passi and Prasad have classified the finite subgroups \(G\) whose group \(\mathcal{U}_1(\mathbb{Z}G)\) is hyperbolic. In the first section we extend this result, classifying the rings of algebraic integers \(R\) of a rational quadratic extensions and the finite groups \(G\) such that \(\mathcal{U}_1(RG)\) is hyperbolic.

Corrales \textit{et al}, in \cite{2}, 2004, determined generators of a subgroup of finite index of \(\mathcal{U}(\mathbb{H}(\mathbb{Z}((1+\sqrt{-7})/2)))\), whose units have norm 1.

For \(\mathbb{H}(\mathbb{Q}(\sqrt{d}))\) a division ring we construct some units of the group \(\mathcal{U}(\mathbb{H}(R))\). We obtain a Pell equation, whose solutions generate the units, which we call Pell units. Furthermore, we construct units of norm \(-1\), which gives rise to the definition of the Gauss units.
2 The rings $R$ with $\mathcal{U}_1(RG)$ hyperbolic

Throughout the text, for $d$ a square-free integer we mean that $d \in \mathcal{D} = \{d \in \mathbb{Z} \setminus \{-1, 0\} : c^2 \not| d$, for all integer $c$ which $c^2 \neq 1\}$. We let $K$ be the quadratic extension $\mathbb{Q}(\sqrt{-d})$ and $R := I_K$ be its ring of algebraic integers. The cyclic group of order $n$ is denoted by $C_n$ and the quaternion group of order 8 is denoted $K_8 := \{\pm 1, \pm i, \pm j, \pm k\}$.

If $G$ is a finite abelian group the unit group $\mathcal{U}_1(RG)$ is a hyperbolic group if, and only if, its free rank is at most 1. In [15], it is shown that it is sufficient to consider $G$ a cyclic group of order 2, 3, 4, 5, 6 or 8, and thus the free rank of $\mathcal{U}_1(RG)$ is calculated. When $G$ is one of the non-abelian groups of the Theorem 3 of [7], we show that, in case $\mathcal{U}_1(RG)$ is hyperbolic, $K$ is an imaginary quadratic extension and $G = K_8$. To prove the converse we use a geometric approach:

Definition 2.1 Let $K$ be an algebraic number field and $R$ be its ring of algebraic integers. For $a, b \in K$, we denote by $H(K) = (\frac{a, b}{K})$ the generalized quaternion algebra, i.e., $H(K)$ is the $K$-algebra

$$H(K) = K[i, j : i^2 = a, j^2 = b, -ji = ij =: k].$$

The set $\{1, i, j, k\}$ is a $K$-basis of $H(K)$. If $a, b \in R$, then

$$H(R) = R[i, j : i^2 = a, j^2 = b, -ji = ij =: k].$$

The norm of $x = x_1 + x_i + x_j + x_k k \in H(K)$ is $\eta(x) = x_1^2 - ax_i^2 - bx_j^2 + abx_k^2$.

In what follows, we consider $H(K) = K[i, j : i^2 = -1, j^2 = -1, -ji = ij =: k]$.

Definition 2.2 ([10]) The least natural number $s$ for which the equation

$$-1 = a_1^2 + a_2^2 + \cdots + a_s^2, a_j \in K, 1 \leq j \leq s$$

is soluble is called the stufe of $K$, say $s(K)$. When this equation admits no solution we set $s := \infty$ and $K$ is called formally real.

Rajwade, in [10], proved that if the quadratic extension $\mathbb{Q}(\sqrt{-d})$ has $s(K) = 4$ then $d \equiv 7 \pmod{8}$. Using this, in [15], we prove that the quaternion algebra $H(K)$ over $K$ is a division ring if, and only if, $d \equiv 7 \pmod{8}$ and as a corollary we obtain that if $d \not\equiv 7 \pmod{8}$ then $\mathcal{U}(RK_8)$ is not hyperbolic. Defining a proper action of the group $SL_3(H(R)) := \{x \in H : \eta(x) = 1\}$ over the three-dimensional hyperbolic space $\mathbb{H}$, and a result of Gromov about the fundamental group of a closed $n$-dimension riemannian manifold of constant negative sectional curvature, we prove that if $d \equiv 7 \pmod{8}$ then the group $\mathcal{U}(RK_8)$ is hyperbolic.
Theorem 2.3 (Theorem 1.7.5 of [15]) Let \( R \) be the integral ring of a rational quadratic extension \( K = \mathbb{Q}(\sqrt{-d}) \) and \( d \) be a square-free integer. The unit group \( \mathcal{U}(RG) \) is hyperbolic if, and only if, \( G \) is one of the groups listed below and \( R \) (or \( K \)) determined by the respective value of \( d \):

1. \( G \in \{C_2, C_3\} \) and any \( d \).
2. \( G \) is an abelian group of exponent dividing \( n \) for:
   - \( n = 2 \) and \( d > 0 \); or
   - \( n = 6 \) and \( d = 3 \); or
   - \( n = 4 \) and \( d = 1 \).
3. \( G = C_4 \) and \( d > 0 \).
4. \( G = C_8 \) and \( d = 1 \).
5. \( G = K_8 \) and \( s(K) = 4 \), that is, \( d > 0 \) and \( d \equiv 7 \pmod{8} \).

For a metric space \( X \), let the maps \( r_1, r_2 : [0, \infty[ \to X \) be proper, that is, \( r_i^{-1}(C) \) is compact for each compact \( C \subseteq X \). Two rays are equivalent if for each compact set \( C \subset X \) there exists \( N \in \mathbb{N} \), such that, \( r_i([N, \infty[), i = 1, 2 \), are in the same path connected component of \( X \setminus C \). The equivalence class of \( r \) is denoted by \( \text{end}(r) \); \( \text{End}(X) \) denotes the set of equivalence class and \( |\text{End}(X)| \) is the number of ends of \( X \). For a finitely generated group \( \Gamma \) and \( \mathcal{G} \) its Cayley graph, we define \( \text{Ends}(\Gamma) := \text{Ends}(\mathcal{G}) \) [3], [11].

Corollary 2.4 The group \( \mathcal{U}(RK_8) \) is hyperbolic if, and only if, \( d > 0 \) and \( d \equiv 7 \pmod{8} \). Furthermore, the hyperbolic boundary \( \partial(\mathcal{U}(RK_8)) \cong S^2 \), the two dimensional euclidean sphere, and \( \mathcal{U}(RK_8) \) has one end.

Observe that the previous corollary shows a class of hyperbolic groups of one end which are not virtually free.

Corollary 2.5 Let \( d \equiv 7 \pmod{8} \), if \( u_1 \cdots u_n \in \mathcal{U}(RK_8) \), then there exists \( m \in \mathbb{N} \), such that, \( \langle u_1^m, \ldots, u_n^m \rangle \) is a free group of rank less or equal to \( n \).

3 The Pell and Gauss Units

Definition 3.1 Let \( K \) be an algebraic number field and \( R \) its ring of algebraic integers. For \( a, b \in K \), we denote by \( H(K) = \left( \frac{a, b}{K} \right) \) the generalized quaternion algebra, i.e., \( H(K) \) is the \( K \)-algebra

\[
H(K) = K[i, j : i^2 = a, j^2 = b, -ji = ij =: k].
\]

The set \( \{1, i, j, k\} \) is a \( K \)-basis of \( H(K) \). If \( a, b \in R \), then

\[
H(R) = R[i, j : i^2 = a, j^2 = b, -ji = ij =: k].
\]
Proposition 3.2 Let $u = u_1 + u_2i + u_3j + u_4k \in \mathcal{U}(H(R))$ with norm $\eta(u)$. The following conditions hold:

1. $u^2 = 2u_1u - \eta(u)$
2. If $d \equiv 7 \pmod{8}$ and $\eta(u) = 1$, then $u$ is torsion if, and only if, $u_1 \in \{ -1, 0, 1 \}$. Thus, the order $o(u)$ is either $o(u) = 4, 2$ or $1$.
3. If $d \equiv 7 \pmod{8}$, and $\eta(u) = -1$ then $o(u) = \infty$.

Let $L := \mathbb{Q}(\sqrt{d})$ and $\xi \neq \psi \in \{ 1, i, j, k \}$. For $\epsilon = x + y\sqrt{d} \in \mathcal{U}(I_L)$, we denote $u_{(\epsilon)} := x\sqrt{-d}\xi + y\psi \in H(K)$.

Proposition 3.3 Let $d \equiv i \pmod{4}, i \in \{ 2, 3 \}$ and $\xi \neq \psi \in \{ 1, i, j, k \}$. The following conditions hold:

1. $u_{(\epsilon)} \in \mathcal{U}(H(R))$ if, and only if, $\epsilon = p + m\sqrt{d} \in \mathcal{U}(I_L)$.
2. If $1 \notin \text{supp}(u)$ then $u_{(\epsilon)}$ is torsion.
3. If $\mu, \nu \in \mathcal{U}(I_L)$ and $1 \in \text{supp}(u_{(\mu)}) \cap \text{supp}(u_{(\nu)})$, then $u_{(\mu)}u_{(\nu)} = u_{(\mu\nu)}$.
4. If $1 \in \text{supp}(u_{(\epsilon)})$, then $\langle u_{(\epsilon)} \rangle = \{ u_{(\epsilon^n)}, n \in \mathbb{Z} \}$.
5. For $d \equiv 3 \pmod{4}$ and $F := \mathbb{Q}(\sqrt{2d})$.

$$u = m\sqrt{-d}\xi + p\psi + (1 - p)\phi \in \mathcal{U}(H(R)) \iff \epsilon = (2p - 1) + m\sqrt{2d} \in \mathcal{U}(I_F)$$

Theorem 3.4 Let $H(K)$ be a division ring. If $x + y\sqrt{d} \in \mathcal{U}(I_L)$, then

$$u = \begin{cases} \frac{x}{2}\sqrt{-d} + \frac{y}{2}(x + y)i + \frac{x}{2}(x - y)k & \text{if } y \equiv 0 \pmod{2} \\ x\sqrt{-d} + (xy\sqrt{d})i + (\frac{x^2 - y^2}{2}d)j + (\frac{1 + (x^2 - y^2)}{2}d)k & \text{if } y \equiv 1 \pmod{2} \end{cases}$$

are units in $H(R)$.

Definition 3.5 The given units above are called Pell Units. For $l \in \{ 2, 3 \}$, a Pell $l$-unit is a unit whose support has cardinality $l$, and the unique non integer coefficient is of the form $m\sqrt{-d}$.
Theorem 3.6 Let $H(K)$ be a division ring. If $m \equiv 2 \pmod{4}$, then there exist integers $p, q, r$, such that, $u = m\sqrt{-d} + pi + qj + rk \in U(H(R))$.

Definition 3.7 A unit $u$ of $H(R)$ whose support has cardinality $l := |\text{supp}(u)| > 1$, the unique non integer coefficient of $u$ is of the form $m\sqrt{-d}$ and $m^2 \pm 1$ is a sum of three square integers is called a Gauss unit, or a Gauss $l$-unit.

Proposition 3.8 Let $u$ be a unit of norm $\eta(u) = 1$, $l \in \{2, 3\}$, and $H(K)$ a division ring.

$u$ is a Pell $l$-unit if, and only if, $u$ is a Gauss $l$-unit

Theorem 3.9 Let $d \equiv 7 \pmod{8}$. If $u, v \in U(H(R))$ are Gauss 2-units, and $\text{supp}(u) \cap \text{supp}(v) = \{1\}$, then there exists $m \in \mathbb{N}$, such that, $\langle u^m, v^m \rangle$ is a free group of rank two.

In [2], the authors exhibit a set of generators $S$ of $SL_1(H(\mathbb{Z}(\frac{1+i\sqrt{-7}}{2})))$. The gauss unit $v = 6\sqrt{-7} + 15i + 5j + k$ has norm $\eta(v) = -1$, therefore $U(H((\frac{1+i\sqrt{-7}}{2}))) = \langle S, v \rangle$. The elements of the set $S \setminus \{i, j\}$ are units of the form $m\sqrt{-7} + (m - \sqrt{-7}^2)i + pj$, possibly with a permutation of the coefficients. If the condition $d \equiv 7 \pmod{8}$ is assumed. Then the solutions of the equation $m^2 + 2p^2 = 2 + d$, give rise to these units.

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