Mean values of divisors twisted by quadratic characters

by

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1. Introduction. As estimations for character sums have wide applications in analytic number theory, many important results have been obtained in this direction. Among the many characters studied in the literature, quadratic Dirichlet characters receive most attention due to their relations to ranks of elliptic curves, class numbers, etc. A special case of the well-known Pólya–Vinogradov inequality (see [3, Chap. 23]) asserts that for any positive $m 
eq \Box$ (where $\Box$ represents the square of a rational integer), and any $Y > 0$,

$$\sum_{n \leq Y} \left( \frac{m}{n} \right) \ll m^{1/2} \log m,$$

where $\left( \frac{m}{n} \right)$ is the Kronecker symbol.

One may regard the above Pólya–Vinogradov inequality as a first moment estimation for quadratic Dirichlet characters. In view of this, it is natural to ask about similar estimations for higher moments involving quadratic character sums. For the second moment, we note the following mean square estimate for quadratic Dirichlet characters due to M. V. Armon [1, Theorem 1]:

$$\sum_{|D| \leq X} \left| \sum_{n \leq Y} \left( \frac{D}{n} \right) \right|^2 \ll XY \log X,$$

where $D$ is the set of non-square quadratic discriminants. Weaker estimations were obtained earlier by M. Jutila [6,7] to study problems related to the mean values of class numbers of quadratic imaginary fields and the second moment of Dirichlet $L$-functions with primitive quadratic characters.

As the Pólya–Vinogradov inequality demonstrates certain square root cancellation, it is not possible to obtain an asymptotic expression for the
character sum on the left-hand side of (1.1). However, things change if we add an extra average over \( m \leq X \). In fact, it is relatively easy to obtain such an asymptotic formula when \( X, Y \) are far apart in size, as the Pólya–Vinogradov inequality (1.1) itself allows us to have a good control on the error term. On the other hand, things become much subtler when \( X \) and \( Y \) are of comparable size and it was only in 2000 that J. B. Conrey, D. W. Farmer, and K. Soundararajan [2] determined completely an asymptotic formula for all \( X, Y > 0 \) for the sum

\[
\sum_{m \leq Y} \sum_{n \leq Y} \left( \frac{m}{n} \right).
\]

The above expression motivates us to consider similar sums involving quadratic Dirichlet characters. One can certainly do so by considering mean values of any arithmetic function twisted by quadratic characters. We shall, however, consider the following more concrete sum:

\[
S(X, Y) = \sum_{m \leq X} \sum_{n \leq Y} \left( \frac{m}{n} \right) d(n).
\]

One reason that leads to our consideration of the above sum is that when we square out the sums in (1.2) and interchange the order of summations, we find that

\[
\sum_{|D| \leq X} \left| \sum_{n \leq Y} \left( \frac{D}{n} \right) \right|^2 = \sum_{n \leq Y} \sum_{|D| \leq X} \left( \frac{D}{n} \right) d(n) + \sum_{n > Y} \sum_{\substack{n = n_1 n_2 \\ n_1 \leq Y, n_2 \leq Y}} \sum_{|D| \leq X} \left( \frac{D}{n} \right).
\]

When one compares the first sum on the right-hand side above with \( S(X, Y) \), it is easy to see that they differ only by certain restrictions on the sums and this is what prompts our consideration of \( S(X, Y) \).

Another reason for considering \( S(X, Y) \) comes from the work of K. Soundararajan [9], who evaluated mollified first and second moments of quadratic Dirichlet \( L \)-functions to show that at least 87.5% of such \( L \)-functions are non-vanishing at the central point. An important part of the arguments in [9] involves evaluation of smoothed sums of the type

\[
\sum_{(d, 2) = 1} \mu^2(d) \sum_n \left( \frac{8d}{n} \right) \frac{d_j(n)}{\sqrt{n}} \omega_j \left( \frac{n}{d j^{3/2}} \right) F \left( \frac{d}{X} \right), \quad j = 1, 2.
\]

Here one may regard \( \omega_j, j = 1, 2 \), and \( F \) as Schwartz class functions and \( d_1(n) = 1, d_2(n) = d(n) \) for all \( n \). Thus, apart from the factor \( 1/\sqrt{n} \), the above sums when \( j = 1 \) can be regarded as a smoothed version of the sums considered in (1.3), except that in the inner sum above, we are essentially
summing over \(n\) going up to the size of \(d\), while in (1.3), the lengths of the summations are independent. In the same fashion, investigating \(S(X,Y)\) can be regarded as a study of the above sums for \(j = 2\) by making the lengths of the sums independent.

We are now ready to state our result.

**Theorem 1.1.** For large \(X\) and \(Y\) we have, for any \(\epsilon > 0\),

\[
S(X,Y) = \frac{XY^{1/2} \log^2 Y}{16} \eta(1) + XY^{1/2}P_1(\log Y) \\
+ X^{3/2}(\log X + 2\gamma)C_1\left(\frac{Y}{X}\right) + X^{3/2}C_2\left(\frac{Y}{X}\right) \\
+ O\left(X^{1+\epsilon}Y^{1/4+\epsilon}\left(\frac{XY^{1/2} + YX^{1/2}}{XY^{1/4}}\right)^{2/3}\right) \\
+ X^{1+\epsilon}Y^{1/2+\epsilon}\left(\frac{XY^{1/4}}{XY^{1/2} + YX^{1/2}}\right)^{1/3}\left(\frac{Y}{X}\right)^{(1+\epsilon)/2} \\
+ Y^{3/2+\epsilon}\left(\frac{XY^{1/4}}{XY^{1/2} + YX^{1/2}}\right)^{2/3} \\
+ X^{1+\epsilon}Y^{\epsilon}\left(\frac{Y}{X}\right)^2\left(\frac{XY^{1/2} + YX^{1/2}}{XY^{1/4}}\right)^2,
\]

where \(\eta(s)\) is given in (3.7) below, \(P_1(x)\) is given in (3.11) and \(C_1(\alpha), C_2(\alpha)\) are functions of \(\alpha \geq 0\) given in (3.25).

Via integration by parts one can show that for \(i = 1, 2\),

\[
C_i(\alpha) = O(\alpha^{3/2} \log \alpha), \quad \alpha \to 0.
\]

Unlike the case studied in [2], the behavior of \(C_i(\alpha)\) when \(\alpha \to \infty\) is more complicated to analyze and we shall not worry about it here.

It is easy to see that (1.4) gives a valid asymptotic formula when \(X^\epsilon \ll Y \ll X^{1-\epsilon}\) for any \(\epsilon > 0\). As the error terms given in (1.4) are somewhat involved, we apply the trivial bounds

\[
XY^{1/2} + YX^{1/2} \leq XY^{1/2}, \\
\frac{XY^{1/4}}{XY^{1/2} + YX^{1/2}} \leq Y^{-1/4}
\]

when \(X \geq Y\) to deduce from Theorem 1.1 the following more practical asymptotic formula for \(S(X,Y)\).

**Corollary 1.2.** With the notation of Theorem 1.1 for large \(X\) and \(Y\) with \(X \geq Y\), we have for any \(\epsilon > 0\),
$$S(X, Y) = \frac{XY^{1/2} \log^2 Y}{16} \eta(1) + X Y^{1/2} P_1(\log Y)$$

$$+ X^{3/2} (\log X + 2\gamma) C_1 \left( \frac{Y}{X} \right) + X^{3/2} C_2 \left( \frac{Y}{X} \right)$$

$$+ O(X^\epsilon Y^\epsilon (XY^{5/12} + Y^{5/2} X^{-1})).$$

Our strategy for proving Theorem 1.1 is similar to that in [2] proof of Theorem 1. We first replace $S(X, Y)$ by the smoothed sum

$$S(X, Y) = \sum_{m \leq X} \sum_{n \leq Y} \left( \frac{m}{n} \right) d(n) \Phi \left( \frac{n}{Y} \right) W \left( \frac{m}{X} \right).$$

Here $\Phi$ and $W$ are smooth functions supported in $(0, 1)$, satisfying $\Phi(t) = W(t) = 1$ for $t \in (1/U, 1 - 1/U)$ for a parameter $U$, and such that

$$\Phi(j)(t), W(j)(t) \ll_j U^j \quad \text{for all integers } j \geq 0.$$

We shall apply a large sieve result for quadratic Dirichlet characters to control the size of $S(X, Y) - S(X, Y)$. We then apply the Poisson summation to evaluate $S(X, Y)$. Choosing $U$ optimally leads to the assertion of Theorem 1.1.

2. Preliminaries. In this section, we gather a few auxiliary results needed in the proof of Theorem 1.1.

2.1. Gauss sums. For all odd integers $k$ and all integers $m$, we introduce the following Gauss-type sums as in [9] Sect. 2.2:

$$G_m(k) = \frac{1 - i}{2} + \left( \frac{-1}{k} \right) \frac{1 + i}{2} \sum_{a \mod k} \left( \frac{a}{k} \right) e \left( \frac{am}{k} \right),$$

where $e(x) = e^{2\pi ix}$. We quote [9] Lemma 2.3 which determines $G_m(k)$.

**Lemma 2.1.** If $(k_1, k_2) = 1$ then $G_m(k_1 k_2) = G_m(k_1) G_m(k_2)$. Suppose that $p^a$ is the largest power of $p$ dividing $m$ (put $a = \infty$ if $m = 0$). Then for $b \geq 1$ we have

$$G_m(p^b) = \begin{cases} 0 & \text{if } b \leq a \text{ is odd}, \\
\varphi(p^b) & \text{if } b \leq a \text{ is even}, \\
-p^a & \text{if } b = a + 1 \text{ is even}, \\
\left( \frac{m/p^a}{p} \right) p^a \sqrt{p} & \text{if } b = a + 1 \text{ is odd}, \\
0 & \text{if } b \geq a + 2. \end{cases}$$
2.2. Poisson summation. For a Schwartz function $F$, we define

$$
\tilde{F}(\xi) = \frac{1+i}{2} \hat{F}(\xi) + \frac{1-i}{2} \hat{F}(-\xi)
$$

$$
= \int_{-\infty}^{\infty} \left( \cos(2\pi \xi x) + \sin(2\pi \xi x) \right) F(x) \, dx,
$$

where $\hat{F}$ denotes the Fourier transform of $F$.

We have the following Poisson summation formula from [9, Lemma 2.6]:

**Lemma 2.2.** Let $W$ be a smooth function with compact support on the positive real numbers. For any odd integer $n$,

$$
\sum_{(d,2)=1} \left( \frac{d}{n} \right) W \left( \frac{d}{X} \right) = \frac{X}{2n} \left( \frac{2}{n} \right) \sum_{k} (-1)^k G_k(n) \tilde{W} \left( \frac{kX}{2n} \right),
$$

where $\tilde{W}$ is defined in (2.2) and $G_k(n)$ is defined in (2.1).

We shall apply the above lemma to the function $W$ defined in the Introduction. Here we recall the following estimations from [2, (4.1)]:

$$
\tilde{W}(\mu)(t) \ll_j U^{j-1} t^{-j}
$$

for all integers $\mu \geq 0$, $j \geq 1$ and all real $t > 0$.

On the other hand, we note that it follows from [2, (4.2)] that for the same $W$,

$$
\tilde{W}(t) = \frac{1 - \cos(2\pi t) + \sin(2\pi t)}{2\pi t} + O \left( \frac{1}{U} \right).
$$

2.3. A large sieve for quadratic Dirichlet characters. Another important tool needed in the proof of Theorem 1.1 is the following large sieve inequality for quadratic Dirichlet characters due to D. R. Heath-Brown [5, Theorem 1].

**Lemma 2.3.** Let $M, N$ be positive integers, and let $(a_n)_{n \in \mathbb{N}}$ be an arbitrary sequence of complex numbers. Then

$$
\sum_{m \leq M} \left| \sum_{n \leq N} a_n \left( \frac{n}{m} \right) \right|^2 \ll \varepsilon (M + N)(MN)^{\varepsilon} \sum_{n \leq N} |a_n|^2
$$

for any $\varepsilon > 0$, where the asterisks indicate that $m$ and $n$ run over positive odd square-free integers.

3. Proof of Theorem 1.1

3.1. Initial reductions. Before we evaluate $S(X,Y)$, we first want to estimate $S(X,Y) - S(X,Y)$, where $S(X,Y)$ is defined in (1.6). It is easy
to see that $S(X, Y) - S(X, Y)$ can be expressed as a linear combination of sums of the form

$$
\sum_{m \in I_1} \sum_{n \in I_2} \left( \frac{m}{n} \right) d(n) H(n, m),
$$

(3.1)

where $H(x, y) = 1$ or $\Phi(n/Y) W(m/X)$,

$I_1 \in \{[0, X/U], [X(1 - 1/U), X], [0, X]\}$,

$I_2 \in \{[0, Y/U], [Y(1 - 1/U), Y], [0, Y]\}$,

and the case $I_1 = [0, X], I_2 = [0, Y]$ is excluded.

As the arguments are similar, we only treat the sum

$$
\sum_{m \in [X(1 - 1/U), X]} \sum_{n \in [0, Y]} \left( \frac{m}{n} \right) d(n).
$$

We note that

$$
\sum_{m \in [X(1 - 1/U), X]} \sum_{n \in [0, Y]} \left( \frac{m}{n} \right) d(n) = \sum_{a^2 \leq X} \sum_{b \in [X(1 - 1/U)/a^2, X/a^2]} \sum_{e^2 \leq Y} \sum_{f \leq Y/e^2} \mu^2(b) \mu^2(f) \left( \frac{a^2 b}{e^2 f} \right) d(e^2 f)
$$

$$
= \sum_{a^2 \leq X} \sum_{e^2 \leq Y} \sum_{b \in [X(1 - 1/U)/a^2, X/a^2]} \sum_{f \leq Y/e^2} \mu^2(b) \mu^2(f) \left( \frac{b}{f} \right) d(e^2 f).
$$

For fixed $a, b$, we apply the Cauchy–Schwarz inequality and Lemma 2.3 to see that

$$
\sum_{b \in [X(1 - 1/U)/a^2, X/a^2]} \sum_{f \leq Y/e^2} \mu^2(b) \mu^2(f) \left( \frac{b}{f} \right) d(e^2 f) \ll \left( \sum_{b \in [X(1 - 1/U)/a^2, X/a^2]} 1 \right)^{1/2}
$$

$$
\times \left( \sum_{b \in [X(1 - 1/U)/a^2, X/a^2]} \sum_{f \leq Y/e^2} \mu^2(b) \mu^2(f) \left( \frac{b}{f} \right) d(e^2 f) \right)^{1/2}
$$
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\[
\ll \sqrt{\frac{X}{U a^2}} \left( \left( \frac{X}{a^2} + \frac{Y}{e^2} \right) (XY)^e \sum_{f \leq Y/e^2 \atop (f,2a)=1} d^2(e^2 f) \right)^{1/2}
\]

\[
\ll (XY)^e \sqrt{\frac{X}{U a^2}} \left( \sqrt{\frac{XY}{a^2 e^2} + \frac{Y}{e^2}} \right).
\]

Applying the above estimation to the right-hand side of (3.2), we see upon summing over \(a\) and \(e\) that

\[
\sum_{m \in [X(1-1/U),X]} \sum_{n \in [0,Y]} \left( \frac{m}{n} \right) \phi \left( \frac{n}{Y} \right) \Phi \left( \frac{n}{Y} \right) \sum_k (-1)^k G_k(n) \tilde{W} \left( \frac{kX}{2n} \right)
\]

\[
\ll (XY)^e \frac{XY^{1/2} + YX^{1/2}}{\sqrt{U}}.
\]

One checks that the above estimation applies to all other types of sums given in (3.1). We then conclude that

\[
(3.3) \quad S(X,Y) - S(X,Y) \ll (XY)^e \frac{XY^{1/2} + YX^{1/2}}{\sqrt{U}}.
\]

Next, by applying Lemma 2.2 we see that

\[
S(X,Y) = X \sum_n \frac{d(n)}{2n} \left( \frac{2}{n} \right) \phi \left( \frac{n}{Y} \right) \sum_k (-1)^k G_k(n) \tilde{W} \left( \frac{kX}{2n} \right)
\]

\[
= \frac{X \tilde{W}(0)}{2} \sum_n \left( \frac{2}{n} \right) \frac{G_0(n)d(n)}{n} \phi \left( \frac{n}{Y} \right)
\]

\[
+ \frac{X}{2} \sum_{k \neq 0} (-1)^k \sum_n \left( \frac{2}{n} \right) \frac{G_k(n)d(n)}{n} \phi \left( \frac{n}{Y} \right) \tilde{W} \left( \frac{kX}{2n} \right)
\]

\[
=: M_0 + M'.
\]

### 3.2. The term \(M_0\)

We estimate \(M_0\) first. It follows straight from the definition that \(G_0(n) = \varphi(n)\) if \(n = \Box\), and \(G_0(n) = 0\) otherwise. Thus

\[
M_0 = \frac{X \tilde{W}(0)}{2} \sum_{n=\Box \atop (n,2)=1} \varphi(n)d(n) \phi \left( \frac{n}{Y} \right).
\]

By Mellin inversion, we have

\[
\phi \left( \frac{n}{Y} \right) = \frac{1}{2\pi i} \int_\gamma \left( \frac{Y}{n} \right)^s \hat{\phi}(s) \, ds,
\]

where

\[
(3.4) \quad \hat{\phi}(s) = \int_0^\infty \phi(t) t^{s-1} \, dt.
\]
Integration by parts and (1.7) show that for $\Re(s) > 0$ and for all integers $D \geq 1$,

$$\widehat{\Phi}(s) \ll (1 + |s|)^{-D}U^{-1}.$$  

Hence we deduce that

$$M_0 = \frac{X \widetilde{W}(0)}{2} \int_{(2)}^{\frac{1}{2}} Y^s \widehat{\Phi}(s) \left( \sum_{n=1}^{n=\square} \frac{\varphi(n) d(n)}{n^{1+s}} \right) ds = \frac{X \widetilde{W}(0)}{2} \int_{(2)}^{\frac{1}{2}} Y^s \widehat{\Phi}(s) \left( \sum_{n=1}^{n=\square} \frac{\varphi(n^2) d(n^2)}{n^{2+2s}} \right) ds.$$  

By comparing the Euler factors we find that

$$\sum_{(n,2)=1}^{\square} \frac{\varphi(n) d(n)}{n^{1+s/2}} = \sum_{(n,2)=1}^{\square} \frac{\varphi(n^2) d(n^2)}{n^{2+s}} = \zeta^3(s) \eta(s),$$

where $\eta(s) = \prod_p \eta_p(s)$ with

$$\eta_2(s) = (1 - 2^{-s})^3,$$

$$\eta_p(s) = 1 - \frac{3}{p^{1+s}} + \left( 3 - 4 \left( 1 - \frac{1}{p} \right) \right) \frac{1}{p^{2s}} - \frac{1}{p^{1+3s}} \quad \text{for } p > 2.$$  

From this we see that $\eta(s)$ is absolutely convergent in $\Re(s) > 1/2$.

We thus infer from (3.6) that

$$M_0 = \frac{X \widetilde{W}(0)}{2} \frac{1}{2\pi i} \int_{(2)}^{\frac{1}{2}} Y^s \widehat{\Phi}(s) \zeta^3(2s) \eta(2s) ds.$$  

We now shift the line of integration to $\Re(s) = 1/4 + \epsilon$. The contributions of the horizontal segments are 0 and we encounter a pole of order 3 at $s = 1/2$.

Thus we may write $M_0 = M_{0,1} + R_0$, where

$$M_{0,1} = \frac{X \widetilde{W}(0)}{2} \text{Res}_{s=1/2} Y^s \widehat{\Phi}(s) \zeta^3(2s) \eta(2s),$$

$$R_1 = \frac{X \widetilde{W}(0)}{2} \frac{1}{2\pi i} \int_{(1/4+\epsilon)}^{1/2} Y^s \widehat{\Phi}(s) \zeta^3(2s) \eta(2s) ds.$$  

To estimate $R_1$, we note that it is shown in [4] that when $0 \leq \Re(s) \leq \epsilon$,

$$L(1/2 + s, \chi_k) \ll (|k|(1 + |s|))^{3/16+\epsilon},$$

where we denote by $\chi_k$ the Kronecker symbol $(\frac{k}{\cdot})$. It follows that when $\Re(s) = 1/4 + \epsilon$,

$$\zeta(2s) \ll (1 + |s|)^{3/16+\epsilon}.$$
Using this together with (3.5) for $D = 2$, we deduce that

$$R_1 \ll XY^{1/4 + \epsilon U}.$$  

To evaluate $M_{0,1}$, we note the Laurent series expansion for $\zeta(s)$ at $s = 1$ (see [8, Corollary 1.16])

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1),$$

where $\gamma$ is Euler’s constant. We deduce from it the following Laurent series expansions at $s = 1/2$:

$$\zeta^3(2s) = \frac{1}{(2s-1)^3} + \frac{3\gamma}{(2s-1)^2} + \frac{c_1}{2s-1} + \cdots,$$

$$\eta(2s) = \eta(1) + 2\eta'(1)(s-1/2) + 2\eta''(1)(s-1/2)^2 + \cdots,$$

$$Y^s = Y^{1/2} + Y^{1/2}(\log Y)(s-1/2) + \frac{Y^{1/2}\log^2 Y}{2}(s-1/2)^2 + \cdots,$$

$$\hat{\Phi}(s) = \hat{\Phi}(1/2) + \hat{\Phi}'(1/2)(s-1/2) + \frac{\hat{\Phi}''(1/2)}{2}(s-1/2)^2 + \cdots,$$

where $c_1$ is an absolute constant. We infer from (3.4) that for some absolute constants $d_1, d_2$,

$$\hat{\Phi}(1/2) = 2 + O(1/U), \quad \hat{\Phi}'(1/2) = d_1 + O(1/U), \quad \hat{\Phi}''(1/2) = d_2 + O(1/U).$$

It follows that the residue in $M_{0,1}$ can be written as

$$\frac{Y^{1/2}\log^2 Y}{8} \eta(1) + 2Y^{1/2}P_1(\log Y) + O\left(\frac{Y^{1/2}\log^2 Y}{U}\right),$$

where $P_1$ is a polynomial of degree 1, whose coefficients involve absolute constants $c_1, d_1, d_2$. We therefore deduce from (2.4) that

$$M_{0,1} = \frac{XY^{1/2}\log^2 Y}{16} \eta(1) + XY^{1/2}P_1(\log Y) + O\left(\frac{XY^{1/2}\log^2 Y}{U}\right).$$

Combining the above estimate with (3.9), we conclude that

$$M_0 = \frac{XY^{1/2}\log^2 Y}{16} \eta(1) + XY^{1/2}P_1(\log Y)$$

$$+ O\left(\frac{XY^{1/2}\log^2 Y}{U} + XY^{1/4 + \epsilon U}\right).$$

3.3. The term $M'$. Now suppose $k \neq 0$. By Mellin inversion, we have

$$\Phi\left(\frac{n}{Y}\right)\tilde{W}\left(\frac{kX}{2n}\right) = \frac{1}{2\pi i} \int \left(\frac{Y}{n}\right)^s \tilde{f}(s, k) ds,$$
where

\[ \tilde{f}(s, k) = \int_0^\infty \Phi(t) \tilde{W} \left( \frac{kX}{2Y} \right) t^{s-1} \, dt. \]

Integration by parts and (2.3) show that for \( \Re(s) > 0 \) and for all integers \( D, E > 0 \),

\[ \tilde{f}(s, k) \ll (1 + |s|)^{-D} \left( 1 + \frac{|k|X}{Y} \right)^{-E+D} U^{E-1}. \]

We apply (3.13) to recast \( M' \) as

\[ M' = \frac{X}{2} \sum_{k \neq 0} (-1)^k \frac{1}{2\pi i} \int_{(2)} \tilde{f}(s, k) Y^s G(1 + s, k) \, ds, \]

where

\[ G(1 + s, k) = \sum_n \left( \frac{2}{n} \right) \frac{G_k(n) d(n)}{n^{1+s}} = \sum_{(n, 2) = 1} \frac{G_{2k}(n) d(n)}{n^{1+s}}, \]

where the last equality above follows from the observation that, by changing variables in (2.1), we have \( \left( \frac{2}{n} \right) G_k(n) = G_{2k}(n) \) since \( n \) is odd.

Write \( 2k = k_1 k_2^2 \) where \( k_1 \) is square-free and \( k_2 \) is positive. In the region \( \Re(s) > 1 \), we recast \( G(1 + s, k) \) as

\[ G(1 + s, k) = L(1/2 + s, \chi_{k_1})^2 \prod_p G_p(s, k) =: L(1/2 + s, \chi_{k_1})^2 G(s, k), \]

where \( G_p(s, k) \) is defined as follows:

\[ G_p(s, k) = \begin{cases} \left( 1 - \frac{1}{p^{1/2+s}} \left( \frac{k_1}{p} \right) \right)^2 & \text{if } p = 2, \\ \left( 1 - \frac{1}{p^{1/2+s}} \left( \frac{k_1}{p} \right) \right)^2 \sum_{r=0}^\infty \frac{d(p^r)}{p^{r(1/2+s)}} \frac{G_{2k}(p^r)}{p^{r/2}} & \text{if } p \neq 2. \end{cases} \]

By Lemma 2.1 we see that for a generic \( p \nmid 2k \),

\[ G_p(s, k) = 1 - \frac{3}{p^{1+2s}} + \frac{2}{p^{3/2+3s}} \left( \frac{k_1}{p} \right). \]

This shows that \( G(s, k) \) is holomorphic in \( \Re(s) > 0 \). From our evaluation of \( G_p(s, k) \) for \( p \nmid 2k \) we see that for \( \Re s \geq \epsilon \),

\[ G(s, k) \ll |k|^{\epsilon} \prod_{p\mid 2k} |G_p(s, k)|. \]

To derive a bound for \( G_p(s, k) \) when \( p \mid 2k \), we suppose that \( p^a \mid 2k \). By the trivial bound \( |G_k(p^r)| \leq p^r \), it follows that \( |G_p(s, k)| \leq (a + 1)^2 \) for those \( p \). We then conclude that for \( \Re s \geq \epsilon \),

\[ G(s, k) \ll |k|^{\epsilon}. \]
Using this, we see that

\[
M' = \frac{X}{2} \sum_{k \neq 0} (-1)^k \frac{1}{2\pi i} \left( \int_{(2)} \tilde{f}(s, k) Y^s L(1/2 + s, \chi_{k_1})^2 G(s, k) \, ds \right)
\]

We now move the line of integration to \( \Re(s) = \epsilon \) to see that we encounter poles only when \( k_1 = 1 \) (so that \( L(s, \chi_{k_1}) = \zeta(s) \)), in which case there is a pole of order 2 at \( s = 1/2 \). Thus we may write \( M' = M_{1,2} + R_2 \), where (by an obvious change of notation, writing \( 2k^2 \) in place of the corresponding \( k \) and observing that \( k_1^2 = k_2^2 \) if and only if \( k_1 = \pm k_2 \))

\[
M_{1,2} = X \text{Res}_{s=1/2} \sum_{k=1}^{\infty} Y^s \tilde{f}(1/2, 2k^2) G(1/2, 2k^2) \zeta(1/2 + s)^2,
\]

and

\[
R_2 = \frac{X}{2} \sum_{k \neq 0} (-1)^k \frac{1}{2\pi i} \left( \int_{(\epsilon)} \tilde{f}(s, k) Y^s L(1/2 + s, \chi_{k_1})^2 G(s, k) \, ds \right).
\]

To evaluate \( M_{1,2} \), we note that it follows from (3.10) that near \( s = 1/2 \),

\[
\zeta(1/2 + s)^2 = \frac{1}{(s - 1/2)^2} + \frac{2\gamma}{s - 1/2} + O(s - 1/2).
\]

It follows that

\[
M_{1,2} = XY^{1/2} \log Y \sum_{k=1}^{\infty} \tilde{f}(1/2, 2k^2) G(1/2, 2k^2)
\]

\[
+ XY^{1/2} \sum_{k=1}^{\infty} \tilde{f}(1/2, 2k^2) G(1/2, 2k^2) \left( \frac{2\gamma}{\tilde{f}(1/2, 2k^2)} + \frac{G'(1/2, 2k^2)}{G(1/2, 2k^2)} \right).
\]

To proceed further, we need to first estimate \( G'(1/2, 2k^2) \) in terms of \( k \). To do so, we note that it follows from (3.18) that it suffices to bound \( \frac{G'(1/2, 2k^2)}{G(1/2, 2k^2)} \).

We recast \( G(1/2, 2k^2) \) as

\[
G(1/2, 2k^2) = \mathcal{H}(1/2, 2k^2) \prod_{p \mid k} G_p(1/2, 2k^2) \left( 1 - \frac{3}{p^2} + \frac{2}{p^3} \right)^{-1},
\]

where

\[
\mathcal{H}(s, 2k^2) = G_2(s, 2k^2) \prod_{p \mid k} \left( 1 - \frac{3}{p^{1+2s}} + \frac{2}{p^{3+3s}} \right) \cdot \prod_{p \mid k} G_p(s, 2k^2).
\]

Observe that \( \frac{G'(1/2, 2k^2)}{G(1/2, 2k^2)} \) equals the logarithmic derivative of \( G(s, 2k^2) \) at \( s = 1/2 \). Combining (3.16) and (3.17), we see that \( \mathcal{H}(s, 2k^2) \) is independent
of $k$. We further define, for all primes $p$,

$$S_p(s, 2k^2) = \sum_{r=0}^{\infty} \frac{d(p^r)}{p^{r(1/2+s)}} \frac{G_{4k^2}(p^r)}{p^{r/2}}.$$ 

We deduce from Lemma 2.1 that $S_p(s, 2k^2) > 1$. It follows from the above discussion together with the trivial bound $|G_k(p^r)| \leq p^r$ that for any $\epsilon > 0$,

$$\frac{G'(1/2, 2k^2)}{G(1/2, 2k^2)} \ll \sum_{p|k} S'_p(1/2, 2k^2) \pm |k| \epsilon \ll \sum_{p|k} S'_p(1/2, 2k^2) + |k| \epsilon \ll |k| \epsilon.$$ 

Combining this with (3.18), we see that for any $\epsilon > 0$,

(3.21) \hspace{1cm} G'(1/2, 2k^2) \ll |k| \epsilon.

Now, we apply the definition of $\tilde{f}(s, k)$ in (3.14) together with the bounds (3.18) and (3.21) to see that

(3.22)

$$\sum_{k=1}^{\infty} \tilde{f}(1/2, 2k^2)G(1/2, 2k^2) = \int_0^\infty \frac{\Phi(t)}{\sqrt{t}} \sum_{k=1}^{\infty} \tilde{W}\left(\frac{k^2X}{Yt}\right)G(1/2, 2k^2) \, dt,$$

$$\sum_{k=1}^{\infty} \tilde{f}'(1/2, 2k^2)G(1/2, 2k^2) = \int_0^\infty \frac{\Phi(t) \log t}{\sqrt{t}} \sum_{k=1}^{\infty} \tilde{W}\left(\frac{k^2X}{Yt}\right)G(1/2, 2k^2) \, dt,$$

$$\sum_{k=1}^{\infty} \tilde{f}(1/2, 2k^2)G'(1/2, 2k^2) = \int_0^\infty \frac{\Phi(t)}{\sqrt{t}} \sum_{k=1}^{\infty} \tilde{W}\left(\frac{k^2X}{Yt}\right)G'(1/2, 2k^2) \, dt.$$

We then apply (2.3) with $j = 1$ for $|k| \geq \sqrt{UY/X}$ and (2.4) for smaller $k$ to see that

$$\sum_{k=1}^{\infty} \tilde{W}\left(\frac{k^2X}{Yt}\right)G(1/2, 2k^2)$$

$$= \frac{Yt}{2\pi X} \sum_{k=1}^{\infty} \frac{G(1/2, 2k^2)}{k^2} \left(1 - \cos\left(\frac{2\pi k^2X}{Yt}\right) + \sin\left(\frac{2\pi k^2X}{Yt}\right)\right) + O(U^{-(1-\epsilon)/2}(Y/X)^{(1+\epsilon)/2}(1 + |t|)),$$

(3.23)

$$\sum_{k=1}^{\infty} \tilde{W}\left(\frac{k^2X}{Yt}\right)G'(1/2, 2k^2)$$

$$= \frac{Yt}{2\pi X} \sum_{k=1}^{\infty} \frac{G'(1/2, 2k^2)}{k^2} \left(1 - \cos\left(\frac{2\pi k^2X}{Yt}\right) + \sin\left(\frac{2\pi k^2X}{Yt}\right)\right) + O(U^{-(1-\epsilon)/2}(Y/X)^{(1+\epsilon)/2}(1 + |t|)).$$
Applying (3.22) and (3.23) to (3.20), we deduce that

\[
M_{1,2} = X^{3/2} (\log X + 2\gamma) C_1(Y/X) + X^{3/2} C_2(Y/X)
+ O \left( X Y^{1/2} \log Y U^{-(1-\epsilon)/2} (Y/X)^{(1+\epsilon)/2} + \frac{Y^{3/2} \log Y}{U} \right),
\]

where we define, for \( \alpha \geq 0 \),

\[
C_1(\alpha) = \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{G(1/2, 2k^2)}{k^2} \int_0^\infty \sqrt{y} \left( 1 - \cos \left( \frac{2\pi k^2}{y} \right) + \sin \left( \frac{2\pi k^2}{y} \right) \right) dy,
\]

\[
C_2(\alpha) = \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{G(1/2, 2k^2)}{k^2} \int_0^\infty \sqrt{y} \left( 1 - \cos \left( \frac{2\pi k^2}{y} \right) + \sin \left( \frac{2\pi k^2}{y} \right) \right) dy
+ \frac{1}{2\pi} \sum_{k=1}^{\infty} \frac{G'(1/2, 2k^2)}{k^2} \int_0^\infty \sqrt{y} \left( 1 - \cos \left( \frac{2\pi k^2}{y} \right) + \sin \left( \frac{2\pi k^2}{y} \right) \right) dy.
\]

Now, using (3.8) together with (3.15) for \( D = 2 \), \( E = 4 \) for all \( k \) and (3.18), we deduce from (3.19) that

\[
R_2 \ll X Y^{\epsilon} \left( \frac{Y}{X} \right)^2 U^3 \sum_{|k| \geq 1} \frac{|k|^{3/16+\epsilon}}{|k|^2} \ll X Y^{\epsilon} \left( \frac{Y}{X} \right)^2 U^3.
\]

We then conclude from (3.24) and (3.26) that

\[
M' = X^{3/2} (\log X + 2\gamma) C_1(Y/X) + X^{3/2} C_2(Y/X)
+ O \left( X Y^{1/2} \log Y U^{-(1-\epsilon)/2} (Y/X)^{(1+\epsilon)/2} + \frac{Y^{3/2} \log Y}{U} + X Y^{\epsilon} (Y/X)^2 U^3 \right).
\]

3.4. Conclusion. Combining (3.3), (3.12) and (3.27), we see that

\[
S(X,Y) = X Y^{1/2} \log^2 Y \eta(1) + X Y^{1/2} P_1(\log Y)
+ X^{3/2} (\log X + 2\gamma) C_1(Y/X) + X^{3/2} C_2(Y/X)
+ O \left( X Y^{1/2} \log^2 Y \frac{U}{U} + X Y^{1/4+\epsilon} U + (XY)^\epsilon \frac{XY^{1/2} + XY^{1/2}}{\sqrt{U}}
+ XY^{1/2} \log Y U^{-(1-\epsilon)/2} (Y/X)^{(1+\epsilon)/2} + \frac{Y^{3/2} \log Y}{U} + XY^{\epsilon} (Y/X)^2 U^3 \right).
\]

By taking

\[
U = \left( \frac{XY^{1/2} + XY^{1/2}}{XY^{1/4}} \right)^{2/3},
\]

we derive the expression (1.4) in Theorem 1.1 and this completes the proof.


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