Signed Network Formation Games and Clustering Balance

Pedro Cisneros-Velarde Francesco Bullo

Abstract

We propose a signed network formation game, in which myopic agents dynamically and strategically change the signs of the edges in a complete network. Our agents are members of a social network who strategically reduce cognitive dissonances by changing their interpersonal appraisals. We characterize the best-response policy for this game and prove that its implementation dynamically drives the network to a sociologically meaningful sign configuration called clustering balance. In this configuration, agents in the social network form one or more clusters that have positive relationships among their members but negative relationships among members of other clusters. In the past, various researchers in the fields of psycho-sociology, political science, and physics have looked at models that explain the generation of up to two clusters. Our work contributes to these fields by proposing a simple model that generates a broader class of signed networks.

1 Introduction

1.1 Problem description

Signed graphs or networks offer a natural representation of social systems involving friendly and antagonistic relationships between their members. These relationships can be interpreted as interpersonal appraisals. In the social sciences literature, there has been several specific sign configurations that have been deemed important in a social network, including for example structural balance, clustering balance, ranked clusters of M-cliques model and transitivity model [13]. All these configurations represent a specific notion of a social balance structure. Whenever the social network is undirected, i.e., the positive or negative relationships among individuals are reciprocal, all notions of social balance reduce to structural balance or clustering balance. Consider a complete social network, i.e., one in which all individuals know each other. Historically, structural balance is the first notion that has been formulated in the seminal work by Heider [6, 7]. It characterizes the stable configurations of signs in a social network according to four rules known as “Heider’s rules” that eradicate cognitive dissonances among the appraisals of its members. As a result, in the structure of the network, one or two antagonistic clusters of individuals can appear in the social network. Each member of a cluster has positive ties with every other member of the same cluster and negative ties to individuals outside it. After Heider’s work, Davis introduced the concept of...
clustering balance in his seminal work [4], which admits any arbitrary number of clusters. Obviously, the largest number of clusters a network can have is the number of agents in the social network.

Heider’s rules provide a distinction between these two notions of social balance. Structural balance satisfies all four rules: 1) “the friend of my friend is my friend”, 2) “the enemy of my friend is my enemy”, 3) “the friend of my enemy is my enemy”, 4) “the enemy of my enemy is my enemy”. As a consequence, all triads in the network (a triad is a cycle of three nodes) are positive, i.e., the product of their edges are positive. In contrast, clustering balance satisfies all rules but the fourth one. As a consequence, the network admits positive triads and triads with three negative edges.

Since the last decade, researchers have started to incorporate dynamic models into the structural balance theory [24], aiming to explain how a network can change its sign configuration so that the network eventually satisfies structural balance. For the interested reader, we refer to the works [15, 18, 22, 12, 20] for models based on both discrete and continuous time (deterministic) dynamical systems with real-valued appraisals, to the works [1, 2, 21] on stochastic or physics related updating models, and to the works [23, 17] for models based on a game theoretical updating of the appraisals. However, despite this growing body of works, to the best of our knowledge, there has been little attention to dynamic models addressing clustering balance in the literature. In fact, only the work [23] has addressed the question of clustering balance, but, as we will see, in a different setting from ours. The work [12] has addressed the problem of dynamic models for other notions of social balance, but it does not particularly address the case of clustering balance nor has a game theoretical formulation. In this paper, we propose to fill a gap in the literature of dynamic balance theory by providing a new model of dynamic clustering balance under a game theoretical framework that has a psycho-sociological motivation in its formulation.

1.2 Statement of contribution

Our first contribution is the proposal of a novel game-theoretical model for dynamic clustering balance with sociologically meaningful interpretations and analyze the properties of its dynamics. Our model is shown to be explainable as a best-response policy to eradicate cognitive dissonances of myopic agents. Our key theoretical result is to prove finite time convergence of our model under this best-response policy to signed network structures that are related to a notion of Nash equilibrium in which clustering balance is possible to be achieved. In particular, convergence to a network satisfying clustering balance is guaranteed whenever the network has up to five agents, and, for larger networks, we present compelling numerical evidence that suggests that this also happens under generic initial conditions.

We now place our dynamic model in the context of other game theoretical signed network formation games. To the best of our knowledge, a game-theoretical interpretation to dynamic social balance was first introduced in the seminal work by van de Rijt [23]. He proposed that if a single agent in the social network can alter multiple relationships at the same time according to some strategy and randomly act “irrationally” (i.e., change her relationships against optimizing her utility function), convergence towards structural and clustering balance can be achieved. One example of proposed strategy followed by any agent \(i\) is a “copying mechanism”: \(i\) will choose another agent \(j\), and copy all the relationships \(j\) has over other agents \(k\) (e.g., if \(j\) is friend with \(k\), then \(i\) will become a friend of \(k\) too). However, \(i\) may not alter its relationship with \(j\). In contrast, our model only assumes that if \(i\) and \(j\) interact, they can only alter their mutual relationship, and that this change depends on the appraisals that any other agent \(k\) has over both of them. Thus, our work does not use a copying mechanism. The issue with having an agent continuously changing multiple relationships simultaneously (and repetitively) is that it implies a greater cognitive burden.
on the agent herself. Moreover, as pointed out by van de Rijt, multiple updating of relationships requires multiple consents from the other agents, e.g., making a positive relationship required both parties to consent to be in peace. Another difference from our model is that we assume rational agents and thus we do not introduce stochasticity as van de Rijt’s work does. As we will see later, numerical evidence suggests that our model predicts convergence to clustering balance under generic initial conditions. Therefore, adding stochasticity to avoid the convergence to networks with no clustering balance is not necessary in our model.

A second relevant work is the one by Malekzadeh et al. [17], which only deals with the case of structural balance. In their game setting, the network topology is not fixed: agents update their relationships by establishing positive or negative links, as well as deleting links. In contrast to traditional network formation games, they assume that agents can unilaterally create new links (no need for bilateral consensus). In the vein of van de Rijt’s model, any selected agent needs to update her relationships with all other agents in the network simultaneously. This work introduces a utility function that each agent tries to maximize in order to reduce cognitive dissonances by enforcing the four Heider’s rules, from which our work takes inspiration with the crucial differences that we only enforce three Heider’s rules and we define it for pairs of agents. Although their model does not need a complete graph at the beginning, it is shown that after $O(n)$ steps of playing best-response dynamics (where $n$ is the number of agents in the network), the network becomes complete and satisfy structural balance. Given this precedence, we decided in our model to simply assume the network is complete and fixed since the beginning. In fact, Malekzadeh et al. show that the incorporation of both creating and deleting edges in their game made the computation of best-response policies NP-hard for any selected agent, something which is avoided in our model.

Finally, we mention the work by Hiller [8]. This work proposes a signed network formation game in the context of complete networks (agents cannot delete links), motivated by observations of how groups of people display bullying behavior and the interplay between dominance and status in conflict networks. This work does an exhaustive analysis of networks that correspond to the game’s concept of Nash equilibrium, and found that their signed structures can correspond either to networks with only positive relationships, or to a network that belongs to the notion of social balance of the transitivity model. In particular, they found that negative edges cannot be reciprocated among agents, which implies that clustering balance is not part of their analysis. They assume that agents optimize different utility functions and that any agent can alter all of its relationships. These utility functions are, as explained by Hiller, “based on agent’s incentives to bully and gang up on each other”; whereas our work is related to the eradication of cognitive dissonances. Finally, in contrast to our work and the ones mentioned above, the work by Hiller does not provide dynamics or time evolution rules that provide convergence results to a Nash equilibrium.

As mentioned before, we focus on the study of complete graphs, which has the highest density of triads. Complete graphs are important to study as a first understanding towards empirical data, besides being traditionally important as a theoretical setting for the social sciences. Triads have been playing an important role in social network analysis since many decades, and empirical evidence over datasets of users of different social media have remarked the abundance and protagonist role of triad structures in the understanding of online social networks [16, 9], even though these real-world networks are not complete. Moreover, many empirical studies have indicated the persistence and abundance of triads with all negative edges, a clear violation of the classical balance in favor of the clustering one [4, 16]. Therefore, the importance of a dynamic clustering balance model is that it allows for a theoretical explanation of such phenomena. We also remark that the study of undirected graphs, as pointed out by [17], is very important since triads with reciprocated edges with the same sign are highly abundant in the three datasets studied by [16], whereas triads with reciprocated edges with different signs have a presence of much less than 1%.
Finally, our work is also relevant for the economics and political sciences literature since signed network formation can model the generation of networks of conflict among different parties. Examples of recent studies are networks of military alliances [14], and networks of trade and international relationships among states [10].

2 Preliminary modeling

Let \( G = \{\{1, \ldots, n\}, E\} \) be an undirected complete graph composed by \( n \) nodes and an edge set \( E \) such that \( \{i, j\} \in E \) is the undirected edge between nodes \( i \) and \( j \). Then, \( G \) defines the structure or topology of a social network composed by \( n \) agents. For any \( \{i, j\} \in E \), let \( x_{ij} \in \{-1, +1\} \) denote the negative or positive interpersonal appraisal or relationship between \( i \) and \( j \). The appraisal network is defined as \( G_\chi = \{\{1, \ldots, n\}, \{x_{ij}\}_{i,j=1, i\neq j}^n\} \) with \( x_{ij} = x_{ji} \) for any \( \{i, j\} \in E \). We assume that both \( G \) and \( G_\chi \) have no self-loops. We say a triad is balanced whenever it is a positive cycle, i.e., the appraisals associated with its edges have a positive product, it is unbalanced whenever it is a negative cycle. Neutral triads are a subset of unbalanced triads such that all of its edges have associated negative appraisals. We will use the terms network and graph interchangeably.

Given a set \( A \), we denote the indicator function by \( 1_A(x) \), so that \( 1_A(x) = 1 \) if \( x \in A \), or \( 1_A(x) = 0 \) if \( x \notin A \). We omit the argument in the indicator function whenever it is clear from the context. In what follows, let \( \text{sign}(u) \in \{-1, 0, +1\} \) denote the sign of \( u \in \mathbb{R} \) (with \( \text{sign}(0) = 0 \)).

Definition 2.1 (Cognitive consonance function). The cognitive consonance function defined on the appraisal network \( G_\chi \) is given by

\[
C(G_\chi) = \sum_{\{i,j,k\} \in T} x_{ij}x_{jk}x_{ki}1\{x_{ij}=+1 \text{ OR } x_{jk}=+1 \text{ OR } x_{ki}=+1\},
\]

where \( T \) is the set of all triples of vertices that form a triad in the network, i.e., \( |T| = \binom{n}{3} \).

The cognitive consonance function is equal to the difference between the total number of balanced triads and unbalanced triads without considering the neutral ones in the network. Then, we can interpret this function as the total level of cognitive consonance in the appraisal network: the more balanced triads, the more cognitive consonances we expect among members of the social network.

Definition 2.2 (Clustering balance [4]). Consider a signed undirected complete graph \( G \). The following statements are equivalent:

(i) \( G_\chi \) has clustering balance;
(ii) The number of balanced triads plus the number of neutral triads is equal to \( |T| \), i.e., \( C(G_\chi) \) is equal to the number of balanced triads in \( G_\chi \);
(iii) There exists a partition of the \( n \) agents into \( k \) sets called clusters, with \( k \in \{1, \ldots, n\} \), such that all appraisals between members of the same cluster are positive and all appraisals between members of different clusters are negative. Whenever \( k \in \{1, 2\} \), the network also satisfies structural balance.

In our work, any time step \( t \) belongs to the countable infinite set \( \{0, 1, \cdots\} \).

3 Game theoretical formulation and static analysis

Definition 3.1 (Signed network formation game). We define the following signed network formation game:
At any time, a single edge from $E$ is randomly and independently picked for its updating, with all edges having a positive probability of being picked according to some fixed time-invariant distribution.

For any chosen $\{i, j\} \in E$ at any time of the game, agents $i$ and $j$ jointly

(ii) have an action space $A = \{0, 1\}$ such that, for any action $a_{ij} \in A$ performed by the agents: $a_{ij} = 1$ means flipping the sign of the appraisal $x_{ij}$ for the next time step, and $a_{ij} = 0$ means not flipping its sign for the next time step;

(iii) want to maximize the payoff or utility function

$$u_{ij} = \Delta^b_{ij} - \Delta^u_{ij} - \lambda_{ij}x_{ij},$$

where $\Delta^b_{ij}$ and $\Delta^u_{ij}$ are the number of balanced and unbalanced triads the agents $i$ and $j$ are part of, and $\lambda_{ij}$ is the number of other agents that are enemies of both $i$ and $j$, i.e., that have negative relationships with both $i$ and $j$.

In the definition of our game, it is assumed that agents are myopic because they only want to maximize their current utility. Thus, the agents ignore how their actions can affect future decisions of other agents in the network, i.e., they ignore the global effect of their actions in the future evolution of the appraisal network. We point out that the assumption of selecting a single edge of $E$ per time step is a common assumption found in the literature of dynamic network formation games (e.g., see [11]). The payoff (2) associated with each edge $\{i, j\}$ can be interpreted as a cognitive consonance between the agents $i$ and $j$ resulting from the relationships with some other agents in the network. Notice that for the computation of $u_{ij}$, both agents $i$ and $j$ effectively count the number of balanced triads they belong to, and then subtract the number of unbalanced triads that do not contain an agent who has negative relationships with both $i$ and $j$ (i.e., an agent who is a common enemy). If $u_{ij} \geq 0$, then both agents see that their current interpersonal appraisal or relationship is appropriate to relieve most of the cognitive dissonances since the number of balanced triads they are part of is greater or equal than the number of unbalanced ones. On the other hand, if $u_{ij} < 0$, then both agents have a cognitive discomfort due to the majority of unbalanced triads they are part of. For example, consider a game with only one triad formed by the agents $i, j$ and $k$ such that $x_{ij} = x_{ik} = +1$ and $x_{kj} = -1$. In the perspective of $i$, this triad violates the Heider’s rule “the enemy of my friend is my enemy”, since the enemy of $k$, which is $j$, has a positive relationship with $i$. Similarly, in the perspective of $j$, this triad violates the rule “the friend of my enemy is my enemy”. According to our game, $i$ and $j$ will switch their interpersonal appraisal to being negative and thus obey Heider’s rule and reduce cognitive dissonances (or, equivalently, increase cognitive consonances).

As a solution concept for our proposed signed network formation game, we adapt a notion of pure Nash equilibrium network based on the work [3]. A Nash equilibrium is a pure strategy profile $a^* = (a^*_{ij})$ such that the utility $u_{ij}$ under $a^*$ is greater or equal than $u_{ij}$ under strategy profile $(a_{ij}, a^*_{-ij})$ for any $a_{ij} \in A$.

Definition 3.2 (Nash equilibrium network). An appraisal network $G_X$ is a Nash equilibrium network if there exists a Nash equilibrium strategy profile $a^*$ that supports $G_X$.

Intuitively, a Nash equilibrium network is a network such that no pair of agents have an incentive to unilaterally change the sign of their interpersonal appraisals. We assume that, whenever $u_{ij} = 0$, both agents $\{i, j\}$ have no incentive to change their action (or pure strategy) pertaining edge $\{i, j\}$.

From the literature on network formation games [11], we use the term efficient network for any appraisal network such that $\sum_{\{i, j\} \in E} u_{i,j}$ is maximized.
Lemma 3.1 (Characterization of Nash equilibrium networks). The set of Nash equilibrium networks is the set of appraisal networks such that $u_{ij} \geq 0$ for any $\{i, j\} \in E$.

Proof. Consider an appraisal network such that $u_{ij} \geq 0$ for any $\{i, j\} \in E$. Then, its pure strategy profile is $a_{ij}^* = 0$ for any $\{i, j\} \in E$. Now, let some $\{m, n\} \in E$ choose any action $a_{mn} \in A$ and define the new pure strategy $a = (a_{mn}, a_{-mn}^*)$. Then, $u_{mn}$ under $a^*$, which is non-negative, is greater or equal than $u_{mn}$ under $a$, which is non-positive. Then, by Definition 3.2 this appraisal network is a Nash equilibrium network.

To prove the converse, consider a Nash equilibrium network. Since no pair of agents want to deviate their pure strategy, it follows that they have the strategy profile $a_{ij}^* = 0$ for any $\{i, j\} \in E$; and therefore, $u_{ij} \geq 0$ for any $\{i, j\} \in E$. \(
\)

Lemma 3.2 (Nash equilibrium networks and clustering balance). All appraisal networks that satisfy clustering balance are Nash equilibrium networks. Moreover, for $n \leq 5$, the set of Nash equilibrium networks is the set of networks that has clustering balance; and, for $n > 5$, there exist Nash equilibrium networks that do not have clustering balance.

Proof. We first prove the first statement of the lemma. From Definition 2.2, an appraisal network $G_X$ satisfying clustering balance is such that, for any $\{i, j\} \in E$, $\Delta_{ij}^b = 0$. Now, if $x_{ij} = -1$ and $\{i, j\}$ belongs to a triad with all negative edges, then $\Delta_{ij}^b = 0$; otherwise, $\Delta_{ij}^b > 0$. From this, it follows that $u_{ij} = \Delta_{ij}^b \geq 0$ for any $\{i, j\} \in E$; and by Lemma 3.1, that $G_X$ is a Nash equilibrium network. This finishes the proof for the first statement of the lemma.

We proceed to prove that, for $n \leq 5$, the set of Nash equilibrium networks is the set of networks that has clustering balance. The case $n = 3$ is immediate. Consider $n = 4$. Let us arbitrarily label the vertices by elements of the set $\{1, 2, 3, 4\}$. By contradiction, assume there exists at least one unbalanced triad. We will try to construct a Nash equilibrium network containing at least one unbalanced triad. Without loss of generality, let $\{1, 2, 3\}$ be an unbalanced triad and let $x_{12} = -1$, so that $x_{23} = x_{31} = +1$. We have two cases to analyze: the case in which the network only has unbalanced triads, and the one in which it has at least one balanced triad. For the first case, assume all triads are unbalanced, i.e., all triads are negative. Then, it follows that

\[
\begin{align*}
(-1)x_{24}x_{14} &= -1, \\
(+1)x_{14}x_{34} &= -1, \\
(+1)x_{24}x_{34} &= -1. 
\end{align*}
\]

So that, letting $x_{24} = s$ for any $s \in \{-1, +1\}$, we get $x_{14} = s$, $x_{34} = -s$. If $s = +1$, then the appraisal network does not have triads composed by three negative edges and it follows that $u_{ij} < 0$ for any $\{i, j\} \in E$, from which we conclude that the constructed network cannot be a Nash equilibrium network. Now, if $s = -1$, then it turns out that there is only one triad with all negative edges (the one composed by agents 1, 2 and 4), and we immediately see that $u_{12} < 0$ so that also this network cannot be a Nash equilibrium network. We conclude from this analysis that the network we attempt to construct must have (at least) one balanced triad that shares one edge with the unbalanced triad composed by agents 1, 2 and 3. Then, it is only sufficient to consider two cases in our analysis: whether the shared edge is positive or negative. Since we have a complete graph, these two cases translate into just analyzing whether the shared edge is $\{1, 2\}$ or $\{2, 3\}$. If the shared edge is $\{1, 2\}$, then there is only one edge that is missing its associated appraisal value: $\{3, 4\}$. However, for any value $x_{34} \in \{-1, +1\}$, it is easy to check that either $u_{23} < 0$ or $u_{13} < 0$. Then, the appraisal network we attempted to construct cannot be a Nash equilibrium network. Now, if the shared edge is $\{2, 3\}$, then the edge that is missing its associated appraisal value is
Figure 1: The left image shows all seven possible configurations for the (complete) appraisal network with five agents that has clustering balance. The right image shows an example of a Nash equilibrium network for six agents that does not have clustering balance. Whenever an edge is present, it corresponds to a positive appraisal; otherwise, we assume there is a negative appraisal.

\{1, 4\}. However, for any value \(x_{14} \in \{-1, +1\}\), it is easy to check that either \(u_{12} < 0\) or \(u_{13} < 0\). Then, this cannot be a Nash equilibrium network. Therefore, we conclude that it is not possible to construct a Nash equilibrium network that contains at least one unbalanced triad. We conclude by contradiction, that all Nash equilibrium networks have clustering balance.

From the previous analysis we make the following observation: if a complete graph of four vertices contains an unbalanced and a balanced triad, then it must contain one more additional balanced triad. With this observation, we can also prove the nonexistence of Nash equilibrium networks that do not satisfy clustering balance for the case \(n = 5\) after some careful algebraic and combinatorial analysis. See Figure 1 for all the possible balanced cases for the appraisal network.

Finally, assume \(n = 6\). We now construct a Nash equilibrium network that does not have clustering balance. Assume the appraisal network has all its initial appraisals positive. Then, set the following: \(x_{12} = x_{23} = x_{35} = x_{51} = -1\). It is easy to check that this network is a Nash equilibrium network that does not have clustering balance (e.g., observe that \(u_{12} = 0\) with the triad formed by nodes \(\{1, 2, 4\}\) being unbalanced with two positive appraisals). See Figure 1 for an illustration of this appraisal network. Now, for any \(n > 6\), set again all appraisals in the appraisal network to be negative and then choose six of its nodes. For these nodes, construct a subgraph exactly as the example we just did for \(n = 6\), and we immediately see that this appraisal network is a Nash equilibrium network.

**Lemma 3.3 (Efficient networks).** All efficient networks are Nash equilibrium networks and they satisfy structural balance.

**Proof.** For any \(\{i, j\} \in E\), we observe that this edge belongs to \(n - 2\) triads in the network, and so \(u_{ij} \leq n - 2\). Then, \(u_{ij} = n - 2\) if and only if \(\Delta^b_{ij} = n - 2\) and \(\Delta^u_{ij} = 0\). Since this must hold for all edges in the appraisal network in order to maximize \(\sum_{\{i,j\}\in E} u_{ij}\) (which reaches the value \(\binom{n}{2}(n - 2)\)), it follows that such network is a Nash equilibrium network. From Definition 2.2, it immediately follows that this network has structural balance.

**4 Dynamic analysis**

**Definition 4.1 (Influence dynamics).** Given an initial appraisal network \(x_{ij}(0) \in \{-1, +1\}\), at each time \(t\), for any selected \(\{i, j\} \in E\), the influence dynamics is defined by

\[
x_{ij}(t + 1) = \begin{cases} 
\text{sign}(f_1(\{i, j\}, G_X(t))), & \text{if } f_1(\{i, j\}, G_X(t)) \neq 0, \\
x_{ij}(t), & \text{otherwise}, 
\end{cases}
\]  

(3)
where \( f_t \{i,j\}, G_X(t) = \text{sign} \left( \sum_{k \neq i,j}^n x_{ik}(t)x_{kj}(t)1\{x_{ik}(t) = +1 \text{ or } x_{jk}(t) = +1\} \right) \). The appraisals associated to the rest of the unselected edges are left unchanged.

Notice that the dynamics (3) is well defined because \( x_{ij}(t) = \pm 1 \) for any \( \{i,j\} \in E \) and any \( t > 0 \) with probability one.

The term “influence dynamics” comes from a sociological interpretation of the fact that the updating of \( x_{ij} \) depends on \( x_{ik}x_{kj} \) with \( k \neq i,j \). For example, in the perspective of agent \( i \), the updating of the interpersonal appraisal or relationship she has with \( j \) (by the term \( x_{ij} \)) considers the influence that \( k \) has over \( i \) subject to what \( k \) thinks of \( j \) (by the product term \( x_{ik}x_{kj} \)). Notice that whenever both \( i \) and \( j \) are enemies of \( k \), the influences that \( k \) has over \( i \) and \( j \) are not considered (i.e., \( k \) cannot be trusted, since she is a common enemy). Then, the fourth Heider’s rule “the enemy of my enemy is my enemy” is not enforced by the agents, i.e., this rule does not elicit any cognitive dissonance.

Now, since agents are rational and myopic, they should play the game according to some policy such that they maximize the payoffs associated to their interpersonal appraisals in the present time. Every pair of agents want to optimally change their current relationship in order to reduce cognitive dissonance. Recall that whenever \( u_{ij} = 0 \), \( i \) and \( j \) have no incentive to flip the sign of their appraisal since this will not change their utilities. Then, the following theorem provides a characterization of a best-response policy taken by the agents.

**Theorem 4.1** (Best-response policy). For any \( \{i,j\} \in E \) chosen at any time step \( t > 0 \), let \( a_{ij}(t) = \text{sign}(|x_{ij}(t) - x_{ij}(t-1)|) \), where the appraisal \( x_{ij}(t) \) is updated according to the influence dynamics (3). Then, the strategy \( a(t) \) is a best-response policy for the proposed signed network formation game.

**Proof.** Let us first note that, from the rationality of the agents and the definition of the game and our assumptions, for any chosen \( \{i,j\} \in E \), the following is the best-response policy by agents \( i \) and \( j \) in order to maximize the utility (2): if \( u_{ij} \geq 0 \) then \( a_{ij} = 0 \); and if \( u_{ij} < 0 \), then \( a_{ij} = 1 \).

Now, we observe that (3) can be expressed as:

\[
x_{ij}(t+1) = \begin{cases} 
\arg \max_{x_{ij}(t) \in \{+1,-1\}} u_{ij}(t), & \text{if } u_{ij}(t) \neq 0, \\
 x_{ij}(t), & \text{otherwise},
\end{cases}
\]

and from the proposed expression for \( a_{ij}(t) \) in the theorem statement, we have just constructed a best-response policy. \( \square \)

Theorem 4.1 states that, if agents update their appraisals according to the influence dynamics, they are indeed playing a best-response policy for the game. Then, it only remains to verify that the trajectories of the dynamics (3) converge to a Nash equilibrium network.

We say that an initial condition at \( t = 0 \) is generic for the appraisal matrix \( G_X(0) \) if every initial appraisal is independently sampled with probability \( 0 < p < 1 \) of taking the value \( +1 \) and probability \( 1-p \) of taking the value \( -1 \).

**Theorem 4.2** (Convergence of the influence dynamics). Consider the initial appraisals \( x_{ij}(0) \in \{-1,+1\} \) for any \( \{i,j\} \in E \) that define the appraisal network \( G_X(0) \). Then, under the influence dynamics (3), \( G_X(t) \) converges with probability one to a Nash equilibrium network in finite time. Moreover, \( G_X(t) \) converges to clustering balance with probability one for \( n \leq 5 \), and, under generic initial conditions, with positive probability for \( n > 5 \).
Proof. We first claim that the influence dynamics is equivalent to a time-homogeneous finite-state Markov chain $\mathcal{M}$ with state space $\mathcal{B}$ of all appraisal networks $G_X$, with the state $G_X(t) \in \mathcal{B}$ being the current appraisal network at time $t$ and defined by the set $\{x_{ij}(t)\}_{\{i,j\} \in E}$. To see this, we first notice that for any time step $t > 0$ and any appraisal network $G_X$ described by the influence dynamics, it follows from (i) of Definition 3.1 that it is completely possible to determine all the possible outcomes for the appraisal network in the next time-step. Then, the Markov property is satisfied since the (distribution of) possible outcomes for the appraisal matrix at $t + 1$ depends only on time index $t$. Moreover, the appraisal network $G_X(t)$ can have up to $|E|$ possible different outcomes. Then, the Markov chain $\mathcal{M}$ is well-posed.

From the definition of the Markov chain $\mathcal{M}$, it immediately follows that the set of Nash equilibrium networks is equal to the set of absorbing states for $\mathcal{M}$. Let $\mathcal{W}_a$ be the set of absorbing states. Then, to prove convergence to the absorbing states, we need to show that $\mathcal{B} \setminus \mathcal{W}_a$ is composed of transient states [5]. Since $\mathcal{M}$ has finite states, the convergence will be achieved in finite time with probability one.

Assume $G_X(t) \in \mathcal{B} \setminus \mathcal{W}_a$ and some selected edge $\{i, j\}$. If $u_{ij}(t) \geq 0$, then $G_X(t + 1) = G_X(t)$, and so $\mathcal{C}(G_X(t + 1)) = \mathcal{C}(G_X(t))$. Now, assume $u_{ij}(t) < 0$. Let $\Delta_{ij}^n$ be the number of neutral triads in which agents $i$ and $j$ are part of. Then, since the change of $x_{ij}$ can only affect the nature (i.e., being balanced or unbalanced) of the $n - 2$ triads that $i$ and $j$ are part of, it follows that:

(i) If $x_{ij}(t) = +1$, then it is possible that $k \geq 0$ new neutral triads might appear in $G_X(t + 1)$, and so

$$\mathcal{C}(G_X(t + 1)) - \mathcal{C}(G_X(t)) = (\Delta_{ij}^u(t + 1) - (\Delta_{ij}^u(t + 1) - \Delta_{ij}^n(t + 1)) - (\Delta_{ij}^b(t) - (\Delta_{ij}^u(t) - \Delta_{ij}^n(t))))$$

$$= (\Delta_{ij}^u(t) - (\Delta_{ij}^u(t) - \Delta_{ij}^n(t + 1)) - (\Delta_{ij}^b(t) - (\Delta_{ij}^u(t) - \Delta_{ij}^n(t))))$$

$$= 2(\Delta_{ij}^u(t) - \Delta_{ij}^b(t)) + (\Delta_{ij}^u(t + 1) - \Delta_{ij}^n(t))$$

$$= -2u_{ij}(t) + k$$

$$> 0.$$

(ii) If $x_{ij}(t) = -1$, then it is possible that $k \geq 0$ new neutral triads might disappear in $G_X(t + 1)$, and so

$$\mathcal{C}(G_X(t + 1)) - \mathcal{C}(G_X(t)) = (\Delta_{ij}^b(t + 1) - (\Delta_{ij}^b(t) - \Delta_{ij}^n(t)) - (\Delta_{ij}^b(t) - (\Delta_{ij}^u(t) - \Delta_{ij}^n(t))))$$

$$=((\Delta_{ij}^u(t) + \Delta_{ij}^b(t)) - (\Delta_{ij}^u(t) - \Delta_{ij}^n(t + 1)) - (\Delta_{ij}^b(t) - (\Delta_{ij}^u(t) - \Delta_{ij}^n(t))))$$

$$= (\Delta_{ij}^u(t) + k - (\Delta_{ij}^b(t) - \Delta_{ij}^n(t + 1)) - (\Delta_{ij}^b(t) - (\Delta_{ij}^u(t) - k))$$

$$= 2(\Delta_{ij}^u(t) - \Delta_{ij}^b(t)) + \Delta_{ij}^n(t + 1)$$

$$= -2u_{ij}(t) + \Delta_{ij}^u(t + 1)$$

$$> 0.$$

From this analysis, we conclude that $\mathcal{C}(G_X(t + 1)) > \mathcal{C}(G_X(t))$.

Therefore, we conclude that the sequence $(\mathcal{C}(G_X(t)))_t$ is non-decreasing, and that, for any $G_X(t) \in \mathcal{B} \setminus \mathcal{W}_a$, there is always a positive probability of selecting an edge such that $\mathcal{C}(G_X(t + 1)) > \mathcal{C}(G_X(t))$. This let us conclude that $\mathcal{B} \setminus \mathcal{W}_a$ is composed of transient states.

Finally, the second statement of the theorem follows directly from Lemma 3.2. □
5 Additional discussion and results

5.1 Connection to optimization and energy functions

We observe from our convergence theorem that the influence dynamics attempts to solve the following combinatorial optimization problem

\[
\min_{\{x_{ij}\} \in \{-1, +1\}^{|E|}} -\mathcal{C}(G_X).
\]

Clearly, a global maximum for this function corresponds to an efficient network. A similar discrete optimization formulation has been previously proposed in the physics community for the case of dynamic structural balance models [19] with the interpretation of the minimization of a potential energy function associated to generalized Ising models and complete social networks.

5.2 Numerical evidence

In this section we show some numerical results about the convergence of the influence dynamics to clustering balance. We analyze the evolution of appraisal networks for different network sizes \(n \in \{3, \ldots, 25\}\). For each fixed \(n\), we generate 10000 generic initial conditions in which each entry of the initial appraisal network \(G_X(0)\) takes the value +1 or −1 with equal probability 0.5. The results are shown in Figure 2. As expected, for \(n \leq 5\), all appraisal networks converged to a balanced network. Remarkably, we find that the success rate was no less than 99.98% for \(n \in \{15, \ldots, 25\}\), and no less than 99.72% for \(n \in \{6, \ldots, 15\}\). In Figure 3 we show the empirical frequency of the final number of clusters for those networks that successfully converged to clustering balance for \(n \in \{11, 20\}\).

Observing these and other numerical simulations for other generic initial conditions and greater values of \(n\) (not shown here), we propose the following informal conjecture for \(n > 5\): under generic initial conditions, there is a very high probability of convergence to clustering balance and this probability goes to 1 as \(n\) increases. In other words, the basin of attraction of appraisal networks that have clustering balance is much larger than the ones for other Nash equilibrium networks.

5.3 Connection to structural balance theory

We know that structural balance enforces the fourth Heider’s rule, so we can modify our model in this paper so that this rule is enforced. This would mean changing our signed network formation game to consider as unbalanced triads all negative triads, so that we do not need to define the concept of neutral triads. Finally, we need to modify both our influence dynamics and cognitive consonance function by replacing any indicator function by a constant number 1. Under this new setting, \(\mathcal{C}\) is the sum of all triads with positive and negative cycles in the network, and the (modified) influence dynamics will seek to maximize this quantity, aiming to attain the maximum value \(\mathcal{C}(G_X) = \binom{n}{3}\) for some appraisal network \(G_X\).

We analyze the evolution of appraisal networks in the same settings as in the previous subsection. The results are shown in Figure 2. We also find that for \(n \leq 5\), all appraisal networks converged to a network that has structural balance. For \(n > 5\), we found that the success rate was no less than 93.47%. It seems that the success rate stabilize and oscillate around a value with the success rate for even number of agents slightly better than for an odd number of them. We cannot conclude if this observed behavior will be the case for much larger values of \(n\), but we expect to.

As mentioned in the introduction of this paper, empirical evidence has suggested the strong presence of triads with all negative edges in online social networks, which favors the presence
Figure 2: Success rate of convergence to (a) clustering balance (note range 99% – 100%) or (b) structural balance (note range 99% – 100%) for appraisal networks of different sizes. For each fixed size \( n \), 10000 simulations were performed under generic initial conditions. For clustering balance, the appraisal networks evolve according to the influence dynamics (Definition 3), and for structural balance, they evolve according to a modification on these dynamics as stated in subsection 5.3.

of the notion of clustering balance over structural balance. The contrast between the obtained numerical convergence results for clustering balance compared to ones for structural balance may be a theoretical predictor of the empirical observations.

6 Conclusion

We propose, to the best of our knowledge, the first model of a signed network formation game for the notion of clustering balance, whereby agents can update only one interpersonal appraisal at a time. We have formally shown how, in our proposed game, finding a Nash equilibrium can provide a model for dynamic clustering balance. Moreover, our model has a psycho-sociological interpretation in which the best-response policy results in the eradication of cognitive dissonances among individuals in a social network. However, broader interpretations can be given using the same underlying model; for example, the interpretation of how countries or communities change their positive or negative diplomatic relationships in order to avoid or create conflict respectively. This makes our work relevant to the fields of economics of conflict and political science. Finally, our model’s relationship to potential energy functions and combinatorial optimization may make this work relevant to the physics and mathematical communities. As future work, we plan to study extensions of the proposed game such as unilateral updating of appraisals. We believe such unilateral updating may allow for the generation of dynamic models for other notions of social balance.

References

[1] T. Antal, P. L. Krapivsky, and S. Redner. Dynamics of social balance on networks. Physical Review E, 72(3):036121, 2005. doi:10.1103/PhysRevE.72.036121.
Figure 3: Empirical frequency of the final number of clusters for (complete) appraisal networks of sizes 11 and 20. For each plot, 10000 simulations were performed with a randomly generated initial appraisal network.

[2] T. Antal, P.L. Krapivsky, and S. Redner. Social balance on networks: The dynamics of friendship and enmity. *Physica D: Nonlinear Phenomena*, 224(1):130–136, 2006. doi:10.1016/j.physd.2006.09.028.

[3] A. Calvó-Armengol and R. İlıkılıç. Pairwise-stability and Nash equilibria in network formation. *International Journal of Game Theory*, 38(1):51–79, 2009. doi:10.1007/s00182-008-0140-7.

[4] J. A. Davis. Clustering and structural balance in graphs. *Human Relations*, 20(2):181–187, 1967. doi:10.1177/001872676702000206.

[5] G. Grimmett and D. Stirzaker. *Probability and Random Processes*. Oxford University Press, 2001.

[6] F. Heider. Social perception and phenomenal causality. *Psychological Review*, 51(6):358–374, 1944. doi:10.1037/h0055425.

[7] F. Heider. Attitudes and cognitive organization. *The Journal of Psychology*, 21(1):107–112, 1946. doi:10.1080/00223980.1946.9917275.

[8] T. Hiller. Friends and enemies: A model of signed network formation. *Theoretical Economics*, 12(3):1057–1087, 2017. doi:10.3982/TE1937.

[9] H. Huang, J. Tang, L. Liu, J. Luo, and X. Fu. Triadic closure pattern analysis and prediction in social networks. *IEEE Transactions on Knowledge and Data Engineering*, 27(12):3374–3389, 2015. doi:10.1109/TKDE.2015.2453956.

[10] M. O. Jackson and S. Nei. Networks of military alliances, wars, and international trade. *Proceedings of the National Academy of Sciences*, 112(50):15277–15284, 2015. doi:10.1073/pnas.1520970112.

[11] M. O. Jackson and A. Watts. The evolution of social and economic networks. *Journal of Economic Theory*, 106(2):265 – 295, 2002. doi:10.1006/jeth.2001.2903.
P. Jia, N. E. Friedkin, and F. Bullo. The coevolution of appraisal and influence networks leads to structural balance. *IEEE Transactions on Network Science and Engineering*, 3(4):286–298, 2016. doi:10.1109/TNSE.2016.2600058.

E. C. Johnsen. The micro-macro connection: Exact structure and process. In F. Roberts, editor, *Applications of Combinatorics and Graph Theory to the Biological and Social Sciences*, pages 169–201. Springer, 1989. doi:10.1007/978-1-4684-6381-1_7.

M. D. Konig, D. Rohner, M. Thoenig, and F. Zilibotti. Networks in conflict: Theory and evidence from the great war of Africa. *Econometrica*, 85(4):1093–1132, 2017. doi:10.3982/ECTA13117.

K. Kulakowski, P. Gawroński, and P. Gronek. The Heider balance: A continuous approach. *International Journal of Modern Physics C*, 16(05):707–716, 2005. doi:10.1142/S012918310500742X.

J. Leskovec, D. Huttenlocher, and J. Kleinberg. Signed networks in social media. In 28th *International Conference on Human Factors in Computing Systems*, pages 1361–1370, Atlanta, USA, 2010. doi:10.1145/1753326.1753532.

M. Malekzadeh, M. Fazli, P. Jalaly Khalidabadi, H. R. Rabiee, and M. A. Safari. Social balance and signed network formation games. In *Proceedings of 5th KDD Workshop on Social Network Analysis (SNA-KDD)*, San Diego, USA, August 2011.

S. A. Marvel, J. Kleinberg, R. D. Kleinberg, and S. H. Strogatz. Continuous-time model of structural balance. *Proceedings of the National Academy of Sciences*, 108(5):1771–1776, 2011. doi:10.1073/pnas.1013213108.

S. A. Marvel, S. H. Strogatz, and J. M. Kleinberg. Energy landscape of social balance. *Physical Review Letters*, 103:198701, 2009. doi:10.1103/PhysRevLett.103.198701.

W. Mei, P. Cisneros-Velarde, G. Chen, N. E. Friedkin, and F. Bullo. Dynamic social balance and convergent appraisals via homophily and influence mechanisms. *Automatica*, October 2017. Submitted. URL: https://arxiv.org/pdf/1710.09498.pdf.

F. Radicchi, D. Vilone, S. Yoon, and H. Meyer-Ortmanns. Social balance as a satisfiability problem of computer science. *Physical Review E*, 75:026106, 2007. doi:10.1103/PhysRevE.75.026106.

V. A. Traag, P. Van Dooren, and P. De Leenheer. Dynamical models explaining social balance and evolution of cooperation. *PLOS ONE*, 8(4):e60063, 2013. doi:10.1371/journal.pone.0060063.

A. van de Rijt. The micro-macro link for the theory of structural balance. *Journal of Mathematical Sociology*, 35(1-3):94–113, 2011. doi:10.1080/0022250X.2010.532262.

X. Zheng, D. Zeng, and F.-Y. Wang. Social balance in signed networks. *Information Systems Frontiers*, 17(5):1077–1095, 2015. doi:10.1007/s10796-014-9483-8.