Employing the approximate effective action constructed from the coincidence limit of the Hadamard-Minakshisundaram-DeWitt (HaMiDeW) coefficient $a_3$, the renormalized stress-energy tensor of the quantized massive scalar field in the spacetime of a static and electrically charged dilatonic black hole is calculated. Special attention is paid to the minimally and conformally coupled fields propagating in geometries with $a = 1$, and to the power expansion of the general stress-energy tensor for small values of charge. A compact expression for the trace of the stress-energy tensor is presented.

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I. INTRODUCTION

According to our present understanding the physical content of quantum field theory formulated in a spacetime describing black hole is contained in the renormalized stress-energy tensor, $\langle T^{ab}\rangle$, evaluated in a physically motivated state [1]. And although interesting in its own, the stress-energy tensor plays a crucial role in various applications, most important of which is the problem of back reaction on the metric. Indeed, treating the stress-energy tensor as a source term of the semi-classical Einstein fields equations, one may, in principle, investigate the evolution of the system unless the quantum gravity effects become dominant. Unfortunately, this programme is hard to execute as the semi-classical field equations comprise rather complicated set of nonlinear partial differential equations, and, moreover, it requires knowledge of functional dependence of $\langle T^{ab}\rangle$ on a wide class of metrics. It is natural therefore that in order to answer – at least partially – this question, one should refer either to approximations or to numerical methods.

It seems that for the massive fields in a large mass limit, i.e., when the Compton length, $l_C$, is much smaller than the characteristic radius of curvature, $L$, (where the latter means any characteristic length scale of the spacetime), the approximation based on the asymptotic Schwinger-DeWitt expansion is of the required generality [2–4]. Since the nonlocal contribution to the effective action could be neglected it is expected that the method yields reasonable results provided the gravitational field is weak and its temporal changes remain small. Despite of the above restrictions there are still a wide class of geometries in which the approximation could be successfully applied.

For a neutral massive scalar field with an arbitrary curvature coupling satisfying

$$ (\Box - \xi R - m^2) \phi = 0, $$

where $\xi$ is the coupling constant and $m$ is the mass of the field, the approximate renormalized effective action, $W_R$, may be expanded in powers of $m^{-2}$ [5–7]. The n-th term of the expansion involves coincidence limit of the ‘HaMiDeW’ coefficient $[a_n]$ [8] constructed solely from the curvature tensor its covariant derivatives up to $2n - 2$.
order and contractions [3,9–15]. As the complexity of the ‘HaMiDeW’ coefficients rapidly grows with increasing \( n \) their practical use is limited to \( n = 3 \), perhaps \( n = 4 \). Moreover, it should be emphasised that the Schwinger-DeWitt expansion is asymptotic and adding more terms does not necessarily improve the approximation. Here we shall confine ourselves to the simplest yet calculationally involved case \( n = 3 \), in which the approximate effective action could be written as

\[
W_R = \frac{1}{32\pi^2} \int d^4x \frac{1}{m^2} [a_3].
\] (2)

Having at one’s disposal the approximation of the renormalized effective action, the stress-energy tensor could be evaluated by means of the standard formula

\[
\frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{ab}} W_R = \langle T^{ab} \rangle.
\] (3)

Since the coefficient \([a_3]\) is rather complicated so is the stress-energy tensor and the question arose of a practical applicability of the thus obtained results. Fortunately it could be used in a number of physically interesting and important cases. The method has been employed by Frolov and Zel’nikov in a series of papers [5–7] devoted to construction of \( \langle T^{ab} \rangle \) of the massive scalar, spinor and vector fields in vacuum type-D spacetimes and generalized recently to arbitrary geometries in [16,17]. General formulas describing \( \langle T^{ab} \rangle \) consist of over 100 local terms.

The effective action technique that we employ in this paper requires the metric of the spacetime to be positively defined. Hence, to obtain the physical stress-energy tensor one has to analytically continue at the final stage of calculations its Euclidean counterpart.

An alternative approach based on the WKB approximation of the solutions to the radial equation and summation of the mode functions has been developed by Anderson, Hiscock, and Samuel [18], who, among other things, succeeded in construction of the general form of the stress-energy tensor of the scalar field in a large mass limit in a static and spherically-symmetric geometry. Both approaches give, as expected, identical results and the detailed numerical analyses carried out by this authors show that for \( mM \gtrsim 2 \) (\( M \) is the black hole mass) the accuracy of the Schwinger-DeWitt approximation in the Reissner-Nordström geometry is quite good (1% or better) [19]. The Schwinger-DeWitt method has been employed in various contexts in [16–25]. The case of the massive spinor field is currently actively investigated [26].

In this article we shall study the stress-energy tensor of the quantized massive scalar field with an arbitrary curvature coupling in a background of the charged dilatonic black holes which is the solutions of the coupled system of Einstein-Maxwell-dilaton equations. A complete set of this equations may be easily derived from the action:

\[
S = \int d^4x \sqrt{-g} \left[ R - 2(\nabla \phi)^2 - e^{-2a\phi} F^2 \right],
\] (4)

where \( \phi \) is the massless dilatonic field, \( F \) is the strength of the Maxwell field \( (F_{ab} = 2\nabla_{(a}A_{b)}) \) and \( a \) is the coupling constant. For each value of the parameter \( a \) there exists a black hole solution depending on the electric charge and the mass [29,30]. The choice \( a = 1 \) corresponds to low energy limit of the string effective action, \( a = \sqrt{3} \) to the four dimensional effective model reduced from the Kaluza-Klein theory in five dimensions, and the Einstein-Maxwell system is obtained with \( a = 0 \). Here we ignore the higher curvature contribution to \( S \) [27,28], as for example the Gauss-Bonnet term.
Various properties of charged dilatonic black holes have been examined in a numerous papers. On the other hand however, quantum effects in 4D dilatonic black hole are – to the best of my knowledge – practically unexplored. This does not mean that this group of problems is uninteresting: belonging to the realm of the low-energy approximation to string theory ($a = 1$) or the Kaluza-Klein theory ($a = \sqrt{3}$), the dilatonic black holes would interact with various quantized fields. The main obstacle preventing construction of the renormalized stress-energy tensor is the computational complexity of the problem.

The evaporation process of the massless scalar field noninteracting with a dilaton field has been analysed in \([31,32]\) whereas the field fluctuation, $\langle \phi^2 \rangle$, of the minimally coupled massless scalar field in the vicinity of the event horizon of the dilatonic black hole has been studied by Shiraishi [33]. Specifically, it was shown that the emission rate of the Hawking radiation blows up near the extremality limit for $a > 1$. On the other hand it is finite for $a = 1$ and zero for $a < 1$. The field fluctuation diverges for $a > 0$ for the extremal configuration.

II. THE GEOMETRY

Functionally differentiating $S$ with respect to the metric tensor, dilaton field, and Maxwell field one obtains the system of equations of motion that could be solved exactly. Static and spherically-symmetric solution has been found by Gibbons and Maeda [29], and by Garfinkle, Horowitz and Strominger [30]:

$$ds^2 = A(r) \, dt^2 + \frac{dr^2}{A(r)} + B^2(r) \, d\Omega^2,$$

where

$$A(r) = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r}{r_+}\right) \frac{1-a^2}{1+a^2},$$

and

$$B^2 = r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}}.$$

The integration constants $r_+$ and $r_-$ are related to the mass and charge of the dilatonic black hole according to

$$2M = r_+ + \left(\frac{1-a^2}{1+a^2}\right) r_-$$

and

$$Q^2 = \frac{r_+ r_-}{1+a^2}.$$

The dilaton field is given by

$$e^{2a\phi} = \left(1 - \frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}},$$

whereas the electric field is simply $F = \frac{Q}{r^2} \, dt \wedge dr$. Inspection of the line element shows that the event horizon is located at $r_+$; at $r = r_-$ one has a coordinate singularity that could be ignored so long one considers region $r \geq r_+ > r_-$. The choice $a = 0$ leads to the Reissner-Nordström solution. At $|Q|/M = (1 + a^2)^{1/2}$, a case usually addressed to as an
extremal black hole, the event horizon and \( r_- \) coincides and in this limit the surface \( r = r_+ = r_- \) is zero except \( a = 0 \). Although more realistic models require massive \( \phi \) field, the dilatonic solutions (5-7) are of principal interest as they provide useful models for studies of the consequences of modifications of the geometries of the classical black holes. Finally, observe that the Kretschmann scalar \( K \) computed at the event horizon near the extremality limit behaves as

\[
K \sim (r_+ - r_-) \frac{4a^2}{r_+ r_-}.
\]

III. THE RENORMALIZED STRESS-ENERGY TENSOR

A. Approximate effective action

In the framework of the Schwinger-DeWitt approximation the first order effective action of the massive scalar field is constructed from the coincidence limit of the coefficient \( a_3 (x, x') \). Inserting \( [a_3] \) as given in Appendix into (2), integrating by parts and finally making use of the elementary properties of the Riemann tensor, after necessary simplifications one has [12–14]:

\[
W_{ren}^{(1)} = \frac{1}{192\pi^2 m^2} \int d^4x \sqrt{g} \left[ \frac{1}{2} \left( \eta^2 - \frac{\eta}{15} - \frac{1}{315} \right) R^a \Box R + \frac{1}{140} R_{pq} \Box R_{pq} - \frac{1}{135} \eta^3 R^3 + \frac{1}{30} \eta R_{pq} R_{pq} - \frac{1}{30} \eta R_{pqab} R^{pqab} \right. \\
- \frac{8}{945} R_{pq} R_{pq} + \frac{2}{315} R_{pq} R_{pq} R_{pq} + \frac{1}{1260} R_{pq} R_{pq} R_{pq} + \frac{17}{7560} R_{pq} R_{pq} R_{pq} - \frac{1}{2700} R_{pq} R_{pq} R_{pq} \\
= \frac{1}{192\pi^2 m^2} \sum_{i=1}^{10} \alpha_i W_{i}.
\]

were \( \eta = \xi - 1/6 \) and \( \alpha_i \) are equal to the numerical coefficients that stand in front of the geometrical terms in the right hand side of the equation (12).

Differentiating functionally \( W_{ren}^{(1)} \) with respect to a metric tensor one obtains rather complicated expression which schematically may be written as

\[
\langle T^{ab} \rangle = \sum_{i=1}^{10} \alpha_i \tilde{T}^{(i)ab} = \frac{1}{96\pi^2 m^2} \sqrt{g} \sum_{i=1}^{10} \alpha_i \frac{\delta W_{i}}{\delta g_{ab}} = T^{(0)ab} + \eta T^{(1)ab} + \eta^2 T^{(2)ab} + \eta^3 T^{(3)ab},
\]

where each \( \tilde{T}^{(i)ab} \) is constructed solely from the curvature tensor, its covariant derivatives and contractions. Because of the complexity of the resulting stress-energy tensor it will be not presented here and for its full form as well as the technical details the reader is referred to [16,17]. The result may be easily extended to fields of other spins as the appropriate tensors differ by numerical coefficients \( \alpha_i \) only.

The coincidence limit of \( a_4 (x, x') \) is also known: it has been calculated by Avramidi [12–14] and by Amsterdamski, Berkin and O’Connor [15]. In principle, the above procedure could be extended to include \( m^{-4} \) terms and the general structure of \( [a_4] \) indicates that the second-order stress-energy tensor divides naturally into five terms \( \sum_{i=0}^{4} \eta^i T^{(i)ab} \). Unfortunately, since the effective action constructed from \( [a_4] \) is extremely complicated, so is its functional derivative and the practical use of the thus obtained result may be a real challenge. However, \( [a_4] \) still could be employed in the analyses of the field fluctuation.

In order to simplify our discussion let us define \( q = |Q| / M, \ x_\pm = r_\pm / M \) and \( x = r / M \). The Schwinger-DeWitt technique may be used when the characteristic radius of curvature in much greater than the Compton length. Simple
considerations indicate that for \( r \gg r_+ \) it could be used for arbitrary value of \( a \). Assuming that \( \mathbb{L} \) is related to the Kretschmann scalar as

\[
R_{abcd}R^{abcd} \sim \mathbb{L}^{-4}
\]

the condition of applicability of the approximation near the event horizon could be written as

\[
\frac{2c}{M^2 x_+^3} \left( x_+ - x_- \right)^{-\frac{2a^2}{1+a^2}} \ll m^2,
\]

where \( c^2 = 2x_+^2 + \left[ x_- - (1 + a^2) x_+ \right]^2 \). It is evident that for \( a > 0 \) the Schwinger-DeWitt approximation is inapplicable for \( r_+ \) close to \( r_- \). For the extremal Reissner-Nordström black hole this condition becomes \( M^2 m^2 \gg 2\sqrt{2} \).

The temperature of the dilatonic black hole obtained by means of standard methods is given by

\[
T_H = \frac{1}{4\pi M x_+} \left( 1 - \frac{x_-}{x_+} \right)^{\frac{1-a^2}{1+a^2}}
\]

and for given \( q \) it depends on the dilatonic coupling. Inspection of (16a) shows that

\[
T_H < (8\pi M)^{-1} \quad (a < 1),
\]

\[
T_H = (8\pi M)^{-1} \quad (a = 1),
\]

\[
T_H > (8\pi M)^{-1} \quad (a > 1).
\]

The temperature of the extremal configuration is zero for \( a < 1 \), takes the same value as for a Schwarzschild black hole for \( a = 1 \), and diverges for \( a > 0 \). Moreover, it is easily seen that the condition \( T_H \ll m \) is violated for \( a > 1 \) near the extremality limit.

**B. General case**

Solving the system (8) and (9) one easily obtains

\[
x_+ = 1 + \sqrt{1 - (1 - a^2) q^2}
\]

and

\[
x_- = \frac{1 + a^2}{1 - a^2} \left( 1 - \sqrt{1 - (1 - a^2) q^2} \right).
\]

Before proceeding further let us observe that \( R = 0 \) for \( a = 0 \), and, consequently, \( \delta W_{(1)}/\delta g_{ab} \) and \( \delta W_{(3)}/\delta g_{ab} \) is zero. The stress-energy tensor has therefore a simple form

\[
\langle T^b_{a} \rangle_{a=0} = T^{(0)b}_{a} + \eta T^{(1)b}_{a}.
\]

On the other hand, the curvature scalar vanishes at the event horizon for any \( a \) and is \( \mathcal{O} \left( q^4 \right) \) for small \( q \) elsewhere. Moreover, since \( \partial_t R \) is the only nonzero component of \( \nabla_a R \) one concludes that \( T^{(3)b}_{a}(r_+) = 0 \) and is negligible in the closest vicinity of \( r_+ \). It is because the only nonvanishing in this limit term is proportional to
\[ \nabla_a R \nabla^b R - (\nabla R)^2 \delta^b_a. \] (20)

A closer examination indicates that \( T^{(3)b}_a \) is \( O(\eta^8) \). Similarly, one expects that for small \( q \) the term \( T^{(2)b}_a \) is of order \( O(q^4) \). On the other hand, the contribution of the last two terms in the right hand side of equation (13) could be made arbitrarily large by a suitable choice of the curvature coupling. It should be noted however that such values of \( \eta \) are clearly unphysical and should be rejected.

Restricting to the exterior region and calculating components of the Riemann tensor, its contractions and covariant derivatives to the required order, after some algebra one arrives at the rather complicated result, that for obvious reasons will not be presented here. However it could be schematically written in surprisingly simple form

\[ \langle T^b_a \rangle = p \frac{1}{16 \pi^2 m^2 M^6} \sum_{ijk} d_{ijk}^b [\eta, a^2] x^i x^j x^k \] (21)

with \( 0 \leq i \leq 7, 0 \leq j \leq 3 \) and \( 0 \leq k \leq 6 \) subjected to the condition \( i + j + k = 9 \). Here

\[ p = \frac{1}{192 \pi^2 m^2 M^6}, \] (22)

and \( d_{ijk}^b \) for given \( a \) and \( \eta \) are numerical coefficients. Some extra work shows that the tensor (21) is covariantly conserved and is regular for regular geometries. Moreover, the difference \( \langle T^t_t \rangle - \langle T^r_r \rangle \) factors

\[ \langle T^t_t \rangle - \langle T^r_r \rangle = \frac{p}{(1 + a^2) x^{14}} \left[ 3 \left( \xi - \frac{1}{6} \right) \Box [a_2] - [a_3] \right], \] (23)

where the regular function \( f \sim (x_+ - x_-)^2 \) as \( x_- \to x_+ \) and consequently within the domain of applicability of the Schwinger-DeWitt approximation the stress-energy tensor is regular in a freely falling frame.

It could be demonstrated by that the trace of the stress-energy tensor of the massive scalar field has a simple form

\[ \langle T^a_a \rangle = \frac{1}{16 \pi^2 m^2} \left[ 3 \left( \xi - \frac{1}{6} \right) \Box [a_2] - [a_3] \right]. \] (24)

This equation together with

\[ \nabla_b T^b_a = 0 \] (25)

may serve as an independent check of the calculations. For conformally coupled fields the trace is proportional to the coincidence limit of \([a_3]\). We remark here that for conformally invariant massless scalar field the anomalous trace is proportional to \([a_2]\); it should be noted however, that (24) has been calculated for \( \langle T^b_a \rangle \) given by (13) whereas the trace of the conformally invariant massless fields is a general property of the regularized stress-energy tensor.

Since the practical use of the general result is severely limited, it is instructive to analyse the stress-energy tensor in some specific cases. In the latter we shall confine our analysis to \( 0 \leq a \leq \sqrt{3} \) with the special emphasis put on the case \( a = 1 \). However, before proceeding to examination of some concrete choices of \( a \) let us analyse \( \langle T^b_a \rangle \) for small \( q \).

C. Arbitrary \( a, q \ll 1 \)

Assuming \( q \ll 1 \), expanding \( \langle T^b_a \rangle \) into a power series, and finally collecting the terms with the like powers of \( q \) one has
\[ \langle T^b_a \rangle = \langle T^b_a \rangle_{a=0} + \frac{a^2}{96\pi^2m^2\times 10^6} (q^{2t(1)b} + q^{4t(2)b} + q^{6t(3)b} + \ldots), \]  

(26)

where \( \langle T^b_a \rangle_{a=0} \) is evaluated for \( a = 0 \) and coincides with the expression describing the stress-energy tensor in the geometry of the Reissner-Nordström black hole [18,16]. The explicit expressions for the coefficients \( t^{(i)b}_a \) as well as the components of \( \langle T^b_a \rangle_{a=0} \) are listed in the appendix. A closer examination shows that for \( q \lesssim 0.7 \) the expansion (26) reproduces the general result satisfactorily, and, moreover, for \( q \lesssim 1/3 \) the results weakly depend on the coupling \( a \). From (26) it is evident that for \( a = 0 \) and \( q = 0 \) the stress-energy tensor reduces to the expression derived by Frolov and Zel’nikov in the geometry of the Schwarzschild black hole [5,34].

**D. Dilatonic black hole \( a = 1 \)**

In this subsection we shall construct and investigate the stress-energy tensor of the massive scalar field resulting from (21) for the particular combinations of couplings. Consider \( a = 1 \). Since the second factor in \( A(r) \) vanishes, we expect considerable simplifications as the event horizon is now located at \( 2M \) whereas the ‘inner’ one at \( q^2 M \). Indeed, defining \( y = r / r_+ \), equation (21) could be written in a simple form:

\[ \langle T^b_a \rangle = \frac{p}{(2y - q^2)^6} \sum_{ij} b^b_{ij} \langle \eta \rangle q^{2i} y^{-j-2} \]

(27)

with \( 0 \leq i \leq 6 \) and \( 0 \leq j \leq 7 \), where \( b^b_{ij} \) are numerical coefficients. From (23) it could be shown that for any \( \eta \) the difference \( \langle T^a_i \rangle - \langle T^r_r \rangle \) factorizes as

\[ \langle T^a_i \rangle - \langle T^r_r \rangle = \frac{1 - y}{y^5 (q^2 - 2y)^6} f(y), \]

(28)

where, for \( 0 \leq q < \sqrt{2} \) the function \( f(y) \) is regular at the event horizon. Equation (27) could be contrasted to the analogous expression evaluated in the Reissner-Nordström geometry \( (a = 0) \):

\[ \langle T^b_a \rangle = \frac{p}{y^6} \sum_{ij} e^b_{ij} \langle \eta \rangle q^{2i} y^{-j-2}, \]

(29)

where \( 0 \leq i \leq 3 \), \( 0 \leq j \leq 4 \), and \( e^b_{ij} \) are another set of numerical coefficients.

To perform quantitative analysis however, we have to refer to exact formulas. For \( \eta = 0 \) it suffices to compute only \( T_a^{(0)b} \) as the others terms do not contribute to the final result. After some algebra one has

\[ \langle T^a_i \rangle = \frac{p}{(2y - q^2)^6} \left[ \frac{313}{210 y^3} - \frac{19}{14 y^2} - q^2 \left( \frac{61}{30 y^2} - \frac{31}{70 y^3} - \frac{9}{7 y^2} \right) + q^4 \left( \frac{143}{840 y^5} + \frac{7313}{2520 y^4} - \frac{577}{210 y^3} - \frac{1}{28 y^2} \right) \right. \]
\[ \left. + q^6 \left( \frac{1381}{1120 y^6} - \frac{6607}{1680 y^5} + \frac{1813}{720 y^4} + \frac{1}{28 y^2} \right) - q^8 \left( \frac{9277}{10080 y^7} - \frac{43837}{20160 y^6} + \frac{1007}{840 y^5} + \frac{139}{10080 y^4} \right) \right] \]
\[ \left. + q^{10} \left( \frac{1817}{6720 y^8} - \frac{479}{840 y^7} + \frac{559}{1920 y^6} + \frac{7}{2880 y^5} \right) - q^{12} \left( \frac{1783}{60480 y^9} - \frac{473}{8064 y^8} + \frac{11}{384 y^7} + \frac{1}{6912 y^6} \right) \right] \]

(30)

\[ \langle T^r_r \rangle = \frac{p}{(2y - q^2)^6} \left[ \frac{1}{2 y^2} - \frac{11}{30 y^3} + q^2 \left( \frac{47}{42 y^4} - \frac{353}{210 y^3} + \frac{9}{35 y^2} \right) - q^4 \left( \frac{481}{280 y^5} - \frac{7267}{2520 y^4} + \frac{43}{45 y^3} - \frac{11}{140 y^2} \right) \right] \]
+q^6 \left( \frac{895}{672\,y^6} - \frac{11687}{5040\,y^5} + \frac{103}{112\,y^4} - \frac{11}{140\,y^3} \right) - q^8 \left( \frac{61}{112\,y^7} - \frac{2143}{2240\,y^6} + \frac{2053}{5040\,y^5} - \frac{53}{1440\,y^4} \right) \\
+q^{10} \left( \frac{2269}{20160\,y^8} - \frac{397}{2016\,y^7} + \frac{233}{2088\,y^6} - \frac{173}{20160\,y^5} \right) - q^{12} \left( \frac{139}{15120\,y^9} - \frac{91}{5760\,y^8} + \frac{1}{144\,y^7} - \frac{5}{6912\,y^6} \right) 
\tag{31}
\end{align}

\begin{align}
&\langle T^a_b \rangle = \frac{p}{(2y - q^2)^6} \left[ \frac{367}{210\,y^3} - \frac{3}{2\,y^2} - q^2 \left( \frac{367}{70\,y^4} - \frac{1129}{210\,y^3} + \frac{27}{35\,y^2} \right) + q^4 \left( \frac{65}{8\,y^5} - \frac{4379}{420\,y^4} + \frac{146}{45\,y^3} - \frac{33}{140\,y^2} \right) \\
&- q^6 \left( \frac{11099}{1680\,y^6} - \frac{11749}{1260\,y^5} + \frac{1423}{420\,y^4} - \frac{33}{140\,y^3} \right) + q^8 \left( \frac{59011}{20160\,y^7} - \frac{29209}{6720\,y^6} + \frac{1907}{1120\,y^5} - \frac{643}{5040\,y^4} \right) \\
&- q^{10} \left( \frac{13589}{20160\,y^8} - \frac{6943}{6720\,y^7} + \frac{8551}{20160\,y^6} - \frac{173}{5040\,y^5} \right) + q^{12} \left( \frac{7669}{120960\,y^9} - \frac{143}{1440\,y^8} + \frac{97}{2304\,y^7} - \frac{25}{6912\,y^6} \right) \right] 
\tag{32}
\end{align}

In fact it suffices to know only one component of the stress-energy tensor, say $\langle T^a_b \rangle$, as the remaining ones could be easily obtained solving equations (24) and (25) and putting the integration constant to zero.

Despite its similarity with the Schwarzschild line element, the nonextremal $a = 1$ dilatonic black holes have much in common with the Reissner-Nordström solution. We shall, therefore, address the question of how the differences between the geometry of the Reissner-Nordström black hole on the one hand and the dilatonic black hole on the other are reflected in the overall behaviour of our approximate stress-energy tensors. First, from (30-32) it could be easily inferred that $\langle T^a_b \rangle$ evaluated for the extremal configuration is divergent as $y \to 1$. Indeed, for $q = \sqrt{2}$ the components of the stress-energy tensor behave as $(y - 1)^{-3}$. This is in a sharp contrast with the Reissner-Nordström case, in which the stress-energy tensor approaches

$$
\langle T^a_b \rangle = \frac{1}{2880\pi^2 m^2 M^6} \left[ \frac{16}{21} - \left( \xi - \frac{1}{6} \right) \right] \text{diag}[1, 1, -1, -1]
\tag{33}
$$

as $y \to 1$. It should be noted however, that, except $a = 0$, the region in the vicinity of the degenerate horizon of the extremal geometry is beyond the applicability of the Schwinger-DeWitt approximation. On the other hand, however, one expects that in the opposite limit, i.e. for $q \ll 1$, the appropriate components of the stress-energy tensor are almost indistinguishable.

To analyse $\langle T^a_b \rangle$ for intermediate values of $q$ let us refer to the numerical calculations. The plots of the time radial and angular components of the stress-energy tensor of the quantized massive scalar field as a function of the rescaled radial coordinate for five exemplar values of $q = 0.1, 0.2, 0.5, 0.65$ and 0.85 are displayed in figures 1–3. Inspection of the figures and comparison with the analogous results obtained for the Reissner-Nordström geometry indicates that even for the intermediate values of $q$ there are still qualitative similarities. Indeed, the time and angular components attain (positive) maximum at the event horizon, decrease with $r$ and approach (negative) minimum. The magnitude of the maximum and the modulus of the minimum increase with increasing $q$, and, consequently, so does the slope of the curves.

Before proceeding to physically interesting and important case $\eta = -1/6$, it is useful to study a role played by each $T_a^{(i)b}$ separately. First, it could be easily shown that $T_a^{(3)b}$ is negligible with respect to other terms, and, therefore,
it does not contribute to the final result for reasonable values of the curvature coupling. The run of the resulting stress-energy tensor depends on a competition between remaining components. Indeed, inspection of figure 4 in which we exhibited $T^{(1)t}_a$ as a function of the rescaled radial coordinate for four exemplar values of $q$ indicates that the term $-T^{(1)b}_a$ produces the most prominent maximum at the event horizon for $q \lesssim 0.9$ whereas for greater values of $q$ this role is played by $T^{(2)b}_a$. General features of $T^{(1)r}_a$ and $T^{(1)\theta}_a$ are essentially the same.

$\lambda \langle T^t_t \rangle$

![Graph 1](image1.png)

**FIG. 1.** This graph shows the radial dependence of the rescaled component $\lambda \langle T^t_t \rangle$, $(\lambda = 10^3 / p)$ of the stress-energy tensor of the massive conformally coupled scalar field in the geometry of the dilatonic black hole with $a = 1$. From top to bottom at the event horizon the curves are for $|Q|/M = 0.1$, $0.2$, $0.5$, $0.65$ and $0.85$. In each case $\langle T^t_t \rangle$ has its positive maximum at $r_+$ and attains negative minimum (right panel) away from the event horizon.

$\lambda \langle T^r_r \rangle$

![Graph 2](image2.png)

**FIG. 2.** This graph shows the radial dependence of the rescaled component $\lambda \langle T^r_r \rangle$, $(\lambda = 10^3 / p)$ of the stress-energy tensor of the massive conformally coupled scalar field in the geometry of the dilatonic black hole with $a = 1$. From top to bottom at the event horizon the curves are for $|Q|/M = 0.1$, $0.2$, $0.5$, $0.65$ and $0.85$. In each case $\langle T^r_r \rangle$ has its positive maximum at $r_+$ and monotonically decreases with $r$. 

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Now the run of the stress-energy tensor as a function of $q$ could be easily anticipated. The general structure remains, of course, of the form (27), but now the dominant contribution to the result is provided initially by the term $qT^{(1)b}_{a}$ and subsequently with increasing $q$ by the sum $-1/6 T^{(1)b}_{a} + 1/36 T^{(2)b}_{a}$. Moreover, since oscillatory-like behavior of $T^{(3)b}_{a}$ does not play a significant role we have also qualitative similarities with the tensor evaluated for the conformal coupling.

\[
\lambda \langle T_{\theta \theta} \rangle
\]

FIG. 3. This graph shows the radial dependence of the rescaled component $\lambda \langle T_{\theta \theta} \rangle$, ($\lambda = 10^3/p$) of the stress-energy tensor of the massive conformally coupled scalar field in the geometry of the dilatonic black hole with $a = 1$. From top to bottom at the event horizon the curves are for $|Q|/M = 0.1, 0.2, 0.5, 0.65$ and 0.85. In each case $\langle T_{i} \rangle$ has its positive maximum at + and and attains negative minimum (right panel) away from the event horizon.

Having computed $\hat{T}^{(i)b}_{a}$ and combining them with appropriate values of the coefficients $\alpha_i$ for $i = 1..10$, after simplifications and rearrangement one has

\[
\langle T^i_r \rangle = \frac{p}{(2y - q^2)^6} \left[ \frac{21}{10 y^2} - \frac{47}{30 y^4} + q^2 \left( \frac{767}{210 y^4} - \frac{1081}{210 y^3} + \frac{9}{35 y^2} \right) + q^4 \left( \frac{2951}{420 y^5} - \frac{5851}{420 y^4} + \frac{85}{12 y^3} - \frac{207}{140 y^2} \right) \right. \\
\left. + q^6 \left( \frac{20267}{3360 y^6} - \frac{1469}{70 y^4} + \frac{543}{70 y^3} - \frac{207}{140 y^2} \right) - q^8 \left( \frac{26933}{10752 y^7} - \frac{101963}{17920 y^6} + \frac{23609}{6720 y^5} - \frac{203}{320 y^4} \right) \right. \\
\left. + q^{10} \left( \frac{53287}{107520 y^8} - \frac{31429}{26880 y^6} + \frac{26559}{35840 y^5} - \frac{593}{4480 y^4} \right) - q^{12} \left( \frac{5219}{143360 y^9} - \frac{5531}{61440 y^8} + \frac{715}{12288 y^7} - \frac{127}{12288 y^6} \right) \right]
\]

(34)

\[
\langle T^{(2)}_r \rangle = \frac{p}{(2y - q^2)^6} \left[ \frac{21}{10 y^2} - \frac{47}{30 y^4} + q^2 \left( \frac{767}{210 y^4} - \frac{1081}{210 y^3} + \frac{9}{35 y^2} \right) + q^4 \left( \frac{2951}{420 y^5} - \frac{5851}{420 y^4} + \frac{85}{12 y^3} - \frac{207}{140 y^2} \right) \right. \\
\left. + q^6 \left( \frac{20267}{3360 y^6} - \frac{1469}{70 y^4} + \frac{543}{70 y^3} - \frac{207}{140 y^2} \right) - q^8 \left( \frac{26933}{10752 y^7} - \frac{101963}{17920 y^6} + \frac{23609}{6720 y^5} - \frac{203}{320 y^4} \right) \right. \\
\left. + q^{10} \left( \frac{53287}{107520 y^8} - \frac{31429}{26880 y^6} + \frac{26559}{35840 y^5} - \frac{593}{4480 y^4} \right) - q^{12} \left( \frac{5219}{143360 y^9} - \frac{5531}{61440 y^8} + \frac{715}{12288 y^7} - \frac{127}{12288 y^6} \right) \right]
\]

(35)

and
\[ \langle T^\mu_\nu \rangle = \frac{p}{(2y - q^2)^6} \left[ \frac{1543}{210 y^3} - \frac{63}{10 y^2} - q^2 \left( \frac{649}{35 y^4} - \frac{3509}{210 y^3} + \frac{27}{35 y^2} \right) + q^4 \left( \frac{40}{y^5} - \frac{2049}{35 y^4} + \frac{311}{12 y^3} - \frac{621}{140 y^2} \right) 
- q^6 \left( \frac{262747}{6720 y^6} - \frac{210673}{3360 y^5} + \frac{1049}{35 y^4} - \frac{621}{140 y^3} \right) + q^8 \left( \frac{342549}{17920 y^7} - \frac{1695223}{53760 y^6} + \frac{68787}{4480 y^5} - \frac{607}{280 y^4} \right) 
- q^{10} \left( \frac{33375}{7168 y^8} - \frac{69651}{8960 y^7} + \frac{410303}{107520 y^6} - \frac{593}{1120 y^5} \right) \right] \]

The qualitative behaviour of the stress-energy tensor of the minimally coupled scalar field is similar to the conformally coupled case, and, once again, for the intermediate values of \( q \) one has quantitative similarities with the Reissner-Nordström case. Moreover, from figure 4 one can easily deduce the general behaviour of the stress-energy tensor for arbitrary coupling for \( q < 0.9 \).

![Graphs showing radial dependence of the rescaled \( T^{(0)}_{tt} \) (panel A), \( T^{(1)}_{tt} \) (panel B), \( T^{(2)}_{tt} \) (panel C) and \( T^{(3)}_{tt} \) (panel D) for \( q = 0.1, 0.5, 0.7 \) and 0.9. The scaling factor is \( 960 \pi^2 m^2 M^6 \). The magnitude of \( T^{(i)}_{tt} \) grows with increasing \( q \) for \( i = 0, 2, \) and 3.](image_url)

Finally we remark, that the dilatonic black holes with \( a = 1 \) or \( a = 0 \) do not exhaust physically important solutions. For example for \( a = \sqrt{3} \) one has a four dimensional effective model reduced from the Kaluza-Klein theory in five dimensions. By (21) and the approximate stress-energy tensor expressed in term of \( x, x_+ \) and \( x_- \) could be schematically written as
\[ \langle T^b_a \rangle = p \frac{1}{[x \cdot (x - x^-)]^{15/2}} \sum_{ijk} d_{ijk}^b \eta \ x^i x^j x^k \] (37)

where \( 0 \leq i \leq 7, 0 \leq j \leq 3 \) and \( 0 \leq k \leq 6 \) subjected to the condition \( i + j + k = 9 \). The qualitative behaviour of the stress-energy tensor for both \( \eta = 0 \) and \( \eta = -1/6 \) is similar to \( \langle T^b_a \rangle \) constructed in the geometry of a dilatonic black hole with \( a = 1 \) and its run for small \( q \) could be easily inferred from (26).

**IV. CONCLUDING REMARKS**

In this paper we have constructed and examined the approximate renormalized stress-energy tensor of the massive scalar field in the spacetime of the static electrically charged dilatonic black hole with the special emphasis put on the string inspired case \( a = 1 \). The method employed here is based on the observation that the lowest order of the expansion of the effective action in \( m^{-2} \) could be expressed in terms of the integrated coincidence limit of coefficient \( a_3 (x, x') \). Although the line element of the dilatonic black hole has a simple form, the analytical formulas describing the stress-energy tensor for a general \( a \) constructed within the Schwinger-DeWitt framework are extremely complicated and hence hard to utilize. Fortunately, for a concrete choice of \( a \) there are considerable simplification.

Expanding for \( q \ll 1 \) the stress-energy tensor into a power series it is possible to analyse the influence of \( a \) on \( \langle T^b_a \rangle \). For \( q = 0 \) it reduces to the result derived by Frolov and Zel'nikov whereas for small values of \( \tilde{q} \) the stress-energy tensor resembles that evaluated in the Reissner-Nordström geometry. The discrepancies between the tensors grow with \( \tilde{q} \). It should be stressed however that in the opposite limit the Schwinger-DeWitt technique is inapplicable.

The problem of the massless fields certainly deserves separate treatment, this however would require extensive numerical calculations as even for simplest case of the Schwarzschild geometry existing analytical approximations give, at best, only qualitative agreement with the exact results. At the moment we only know that the horizon value of the field fluctuation [33]

\[ \langle \phi^2 \rangle = \frac{1}{48 \pi^2 M^2 x_+^2} \left[ 1 - \frac{x_-}{(1 + a^2) x_+} \right] \left( 1 - \frac{x_-}{x_+} \right)^{-2 a^2} \] (38)

which is divergent in the extremality limit for \( a > 0 \). This suggests that the stress-energy tensor is also divergent at \( r_+ \) of the extremal case. On the other hand, a first nonvanishing term of the approximation to the field fluctuation for a massive field is simply

\[ \langle \phi^2 \rangle = \frac{1}{16 \pi^2 m^2} \left[ a_2 \right] + \mathcal{O} \left( m^{-4} \right), \] (39)

and it could be easily shown that

\[ \langle \phi^2 \rangle = \frac{f (a, r_+, r_-)}{720 \pi^2 m^2 M^4 x_+^6} (1 + a^2)^{-2} \left( 1 - \frac{x_-}{x_+} \right)^{-2 a^2} + \mathcal{O} \left( m^{-4} \right), \] (40)

where

\[ f (a, r_+, r_-) = \left[ 4 + 3 a^2 (1 - 5 \xi) \right] x_-^2 - 6 (1 + a^2) x_+ x_- + 3 (1 + a^2)^2 x_+^3. \] (41)

Finally, we make some comments regarding applications and generalizations of the results presented in this paper. The question of the massless field has been addressed above. A careful analysis carried out for \( a = 0 \) in ref. [36]...
shows that at least up to $O(m^{-4})$ the adapted method approximates well the field fluctuation of the massive field in the thermal state of temperature $T_H$. It would be interesting to extend this analysis for any value of $a$. We also remark that the derived stress-energy tensors may be employed as a source term of the semiclassical Einstein field equations. Indeed, preliminary calculations indicate that it is possible to construct the solution to the linearized semiclassical Einstein-Maxwell-dilaton equations. We hope that because of their simplicity presented results will be of use in subsequent applications. We intend to return to this group of problems elsewhere.

**APPENDIX**

**A. Coincidence limits of the coefficients $a_2(x, x')$ and $a_3(x, x')$**

In this appendix we list coincidence limits of the coefficients $a_2(x, x')$ and $a_3(x, x')$ for the scalar field equation (1). With the normalization employed in this paper the coefficient $[a_2]$ reads

$$[a_2] = \frac{1}{6} \left( \xi - \frac{1}{5} \right) \Box R + \frac{1}{2} \left( \xi - \frac{1}{6} \right)^2 R^2 + \frac{1}{180} \left( R_{abcd} R^{abcd} - R_{ab} R^{ab} \right), \tag{42}$$

whereas $[a_3]$ could be written as

$$[a_3] = \frac{b_3}{7!} + \frac{c_3}{360}, \tag{43}$$

where

$$b_3 = \frac{35}{9} R^3 + 17 R_p R^p - R_{pq} R^{pq} - 4 R_{pq} R^{pq},$$

$$+ 9 R_{abc:p} R^{abc:p} + 2 R \Box R + 18 \Box^2 R - 8 R_{pq} \Box R^q - \frac{14}{3} R_{pq} R^{pq},$$

$$+ 24 R_{pq:a} R^{pq} - \frac{208}{9} R_p R_{pq} R^p + 12 R_{pq} R_{pq} + \frac{64}{3} R_{pq} R_{ab} R^{pq} + \frac{16}{3} R_{pq} R_{abc} R^{abc} + \frac{80}{9} R_{pq} R_{ab} R_{cd} R^{abcd} \tag{44}$$

and

$$c_3 = -(5 \xi - 30 \xi^2 + 60 \xi^3) R^3 - \left(12 \xi - 30 \xi^2 \right) R_p R^p - \left(22 \xi - 60 \xi^2 \right) R \Box R$$

$$- 6 \xi \Box^2 R - 4 \xi R_{pq} R^{pq} + 2 \xi R R_{pq} R^{pq} - 2 \xi R R_{pq} R_{pqab}. \tag{45}$$

**B. $\langle T^a_b \rangle$ of the massive scalar fields in the spacetime of the Reissner-Nordström black hole**

Inserting curvature tensor and its covariant derivatives into the general formulas obtained from functional differentiation of the effective action (12) with respect to the metric tensor one obtains the approximate stress-energy tensor of massive fields. Since the curvature scalar of the Reissner-Nordström geometry is zero, one expects considerable simplifications. Indeed, it could be easily shown that the tensors $\tilde{T}_a^{(1)b}$ and $\tilde{T}_a^{(3)b}$ do not contribute to the final result. The stress-energy tensor of the massive scalar field with arbitrary coupling with curvature in the Reissner-Nordström has the form (19), where
and necessary simplifications (26), where the coefficients of the dilatonic black hole. Assuming \( q \ll 1 \) and expanding the result into a power series, after the necessary simplifications (26), where the coefficients are given by

\[
T_{(0)t} = \frac{313}{7} x^3 + \frac{285}{14} x^4 + q^2 \left( -\frac{769}{14} x^2 - \frac{192}{7} x^3 + \frac{135}{7} x^4 \right) + q^4 \left( \frac{514}{7} x - \frac{101}{21} x^2 \right) - \frac{208 q^6}{7},
\]

\[
T_{(0)r} = -11 x^3 + \frac{15}{2} x^4 + q^2 \left( \frac{709}{14} x^2 + \frac{248}{7} x^3 + \frac{27}{7} x^4 \right) + q^4 \left( -\frac{46}{7} x + \frac{421}{21} x^2 \right) + \frac{74 q^6}{7},
\]

\[
T_{(0)\theta} = \frac{367}{7} x^3 - \frac{45}{2} x^4 + q^2 \left( -\frac{3303}{14} x^2 + \frac{814}{7} x^3 - \frac{81}{7} x^4 \right) + q^4 \left( \frac{1726}{7} x - \frac{1522}{21} x^2 \right) - \frac{73 q^6}{7},
\]

\[
T_{(1)t} = -792 x^3 + 360 x^4 + q^2 \left( 2604 x^2 - 1008 x^3 \right) + q^4 \left( -2712 x + 728 x^2 \right) + 819 q^6,
\]

\[
T_{(1)r} = 216 x^3 - 144 x^4 + q^2 \left( -588 x^2 + 336 x^3 \right) + q^4 \left( 504 x - 208 x^2 \right) - 117 q^6
\]

and

\[
T_{(1)\theta} = -1008 x^3 + 432 x^4 + q^2 \left( 3276 x^2 - 1176 x^3 \right) + q^4 \left( -3408 x + 832 x^2 \right) + 1053 q^6.
\]

**C. Power expansion of the stress-energy tensor for** \( q \ll 1 \)

Repeating the calculations for the line element (5-6) one obtains components of the stress-energy tensor in the geometry of a general dilatonic black hole. Assuming \( q \ll 1 \) and expanding the result into a power series, after the necessary simplifications (26), where the coefficients \( t_{(i)b} \) are given by

\[
t_{(1)t} = \frac{939}{35} \times 7 - x^2 + \eta \left( 192 x - \frac{2376}{5} x \right),
\]

\[
t_{(1)r} = 4 x - \frac{33}{5} + \eta \left( \frac{648}{5} - \frac{384 x}{5} \right),
\]

\[
t_{(1)\theta} = \frac{1101}{35} - 12 x - \eta \left( \frac{3024}{5} - \frac{1152 x}{5} \right),
\]

\[
t_{(2)t} = \frac{1207}{140} - \frac{4359 a^2}{140} - \frac{9773}{210 x} + \frac{939 a^2}{14 x} + \frac{181 x}{105} + \frac{19 a^2 x}{7} - \frac{x^2}{14}
\]

\[
+ \eta \left( \frac{2754 a^2}{5} - \frac{1594}{5} + \frac{888}{x} - \frac{1188 a^2}{x} - \frac{436 x}{5} - \frac{48 a^2 x + 12 x^2}{5} \right)
\]

\[
+ \eta^2 \left( -6076 + \frac{6000}{x} + 1908 x - 180 x^2 \right),
\]

\[
i_{(2)r} = -\frac{12091}{1260} + \frac{213 a^2}{20} + \frac{3793}{210 x} - \frac{33 a^2}{2 x} + \frac{83 x}{315} - a^2 x + \frac{11 x^2}{70}
\]

\[
+ \eta \left( 58 - \frac{1026 a^2}{5} - \frac{1032}{5 x} + \frac{324 a^2}{x} + \frac{388 x}{15} + \frac{96 a^2 x - 24 x^2}{5} \right)
\]

\[
+ \eta^2 \left( 1556 - \frac{1200}{x} - 612 x + 72 x^2 \right)
\]
\[ t^{(2)\theta}_\theta = \frac{6841}{252} - \frac{4881 a^2}{140} - \frac{18139}{210 x} + \frac{1101 a^2}{14 x} + \frac{227 x}{315} - \frac{3 a^2 x}{70} + \frac{3348 a^2}{5} + \frac{5572}{5 x} + \frac{1512 a^2}{x} - \frac{1628 x}{15} - \frac{288 a^2 x}{5} + \frac{72 x^2}{5} - \frac{296}{3} \]
\[ + \eta^2 \left( -7232 + \frac{7200 x}{x} + 2268 x - 216 x^2 \right), \]

(57)

\[ t^{(3)t}_t = \frac{4847}{840} - \frac{4138 a^2}{315} + \frac{4359 a^4}{280} + \frac{20149}{420 x^2} - \frac{24971 a^2}{210 x^2} + \frac{3443 a^4}{28 x^2} - \frac{9059}{420 x} + \frac{6026 a^2}{105 x} - \frac{297 a^4}{4 x} - \frac{79 x}{210} + \frac{19 a^4}{14} - \frac{x^2}{28} + \frac{2 a^2}{14} \]
\[ + \frac{152 a^2 x}{x^2} + \frac{a^2 x^2}{28} + \eta \left( \frac{5642 a^2}{5 x^2} + \frac{1294}{15} - \frac{662 a^2}{5} \right) - \frac{1377 a^4}{5 x^2} - \frac{4346}{x^2} + \frac{2178 a^4}{x^2} + \frac{1936}{5 x} - \frac{278 a^2}{15 x} + \frac{1314 a^2}{x} - \frac{78 x}{5} + \frac{198 a^2}{5} + \frac{24 a^4 x}{x} + \frac{6 x^2 - 6 a^2 x^2}{x} \]
\[ + \frac{7634 a^2}{x^2} - \frac{658}{5 x} - \frac{2232}{x} + \frac{21984 a^2}{x^2} + \frac{86}{5} + \frac{198 a^2}{x} + \frac{6 x^2 - 6 a^2 x^2}{5 x} + \frac{21596 a^2}{x^2} + \frac{534 x - 1254 a^2 x - 90 x^2 + 90 a^2 x^2}{x} \]

(58)

\[ t^{(3)r}_r = -\frac{1783}{840} - \frac{187 a^2}{70} - \frac{213 a^4}{40} + \frac{5009}{420 x^2} + \frac{5111 a^2}{210 x^2} - \frac{121 a^4}{x^2} + \frac{401}{60 x} - \frac{4352 a^2}{315 x} + \frac{93 a^4}{4 x} + \frac{109 x}{2 a^2 x} - \frac{2 a^2}{45} + \frac{401}{140} - \frac{11 x^2}{14} - \frac{11 a^2 x^2}{140} - \frac{662 a^2}{x^2} - \frac{18 x}{5} + \frac{450 a^4}{x} + \frac{12 a^2 x^2}{5} + \frac{12 a^2 x^2}{5} \]
\[ + \eta^2 \left( \frac{130 - 2442 a^2}{x^2} + \frac{288}{60 x} + \frac{4656 a^2}{x^2} - \frac{60 x}{x^2} + \frac{5764 a^2}{x} - \frac{166 x + 454 a^2 x + 36 x^2}{x} - \frac{36 a^2 x^2}{x} \right). \]

(59)

and

\[ t^{(3)\phi}_\phi = \frac{2411}{360} - \frac{7267 a^2}{1260} + \frac{4881 a^4}{280} + \frac{29963}{420 x^2} - \frac{8857 a^2}{70 x^2} + \frac{4037 a^4}{28 x^2} - \frac{11617}{420 x} - \frac{2792 a^2}{63 x} - \frac{2361 a^4}{28 x} - \frac{13 x}{31 a^2 x} - \frac{3 a^4 x}{3} - \frac{33 x^2}{33 a^2 x^2} + \frac{33 a^2 x^2}{28 x^2} - \frac{11178}{420 x} \]
\[ + \frac{2052}{x^2} - \frac{976 a^2}{5 x} + \frac{160 a^4}{10 x} - \frac{252 a^2 x}{5} + \frac{144 a^4}{x} + \frac{36 x^2}{5} - \frac{36 a^2 x^2}{x} + \frac{2772 a^4}{x^2} + \frac{644 x}{x} - \frac{1508 a^2 x - 108 x^2 + 108 a^2 x^2}{5} \]
\[ + \eta^2 \left( 9182 a^2 / x^2 - 870 - \frac{2970}{x^2} + \frac{26640 a^2}{x^2} + \frac{1098}{x} - \frac{2608 a^2}{x} + 644 x - 1508 a^2 x - 108 x^2 + 108 a^2 x^2 \right) \]

(60)

The terms containing \( \eta^3 \) appear starting from \( q^8 \)

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