Numerical instability of the Akhmediev breather
and a finite-gap model of it.

P. G. Grinevich \textsuperscript{1,3} and P. M. Santini \textsuperscript{2,4}

\textsuperscript{1} L.D. Landau Institute for Theoretical Physics, pr. Akademika Semenova 1a, Chernogolovka, 142432, Russia, and
Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, Russia, 119991, Moscow, GSP-1, 1 Leninskiye Gory, Main Building, and Moscow Institute of Physics and Technology, 9 Institutskiy per., Dolgoprudny, Moscow Region, 141700, Russia

\textsuperscript{2} Dipartimento di Fisica, Università di Roma ”La Sapienza”, and
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Piazz.le Aldo Moro 2, I-00185 Roma, Italy

\textsuperscript{3}e-mail: pgg@landau.ac.ru
\textsuperscript{4}e-mail: paolo.santini@roma1.infn.it

November 3, 2021

Abstract

The focusing Nonlinear Schrödinger (NLS) equation is the simplest universal model describing the modulation instability (MI) of quasi monochromatic waves in weakly nonlinear media, considered the main physical mechanism for the appearance of rogue (anomalous) waves (RWs) in Nature. In this paper we study the numerical instabilities of the Akhmediev breather, the simplest space periodic, one-mode perturbation of the unstable background, limiting our considerations to the simplest case of one unstable mode. In agreement with recent theoretical findings of the authors, in the situation in which the round-off errors are negligible with respect to the perturbations due to the discrete scheme used in the numerical experiments, the split-step Fourier method (SSFM), the numerical output is well-described by a suitable genus 2 finite-gap solution of NLS. This solution can be written in terms of different elementary functions in different time regions and, ultimately, it shows an exact recurrence of rogue waves described, at each appearance, by the Akhmediev breather. We discover a remarkable empirical formula connecting the recurrence time with the number of time steps used in the SSFM and, via our recent theoretical
findings, we establish that the SSFM opens up a vertical unstable gap whose length can be computed with high accuracy, and is proportional to the inverse of the square of the number of time steps used in the SSFM. This neat picture essentially changes when the round-off error is sufficiently large. Indeed experiments in standard double precision show serious instabilities in both the periods and phases of the recurrence. In contrast with it, as predicted by the theory, replacing the exact Akhmediev Cauchy datum by its first harmonic approximation, we only slightly modify the numerical output. Let us also remark, that the first rogue wave appearance is completely stable in all experiments and is in perfect agreement with the Akhmediev formula and with the theoretical prediction in terms of the Cauchy data.

1 Introduction

The self-focusing Nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad u = u(x, t) \in \mathbb{C}$$  \hspace{1cm} (1)

is a universal model in the description of the propagation of a quasi monochromatic wave in a weakly nonlinear medium; in particular, it is relevant in deep water [57], in nonlinear optics [47, 14, 44], in Langmuir waves in a plasma [49], and in the theory of attracting Bose-Einstein condensates [13]. It is well-known that its elementary solution

$$u_0(x, t) = e^{2it},$$  \hspace{1cm} (2)

describing Stokes waves [48] in a water wave context, a state of constant light intensity in nonlinear optics, and a state of constant boson density in a Bose-Einstein condensate, is unstable under the perturbation of waves with sufficiently large wave length [50, 10, 57, 62, 51, 45], and this modulation instability (MI) is considered as the main cause for the formation of rogue (anomalous, extreme, freak) waves (RWs) in Nature [25, 20, 12, 30, 31, 41].

The integrable nature [58] of the NLS equation allows one to construct solutions corresponding to perturbations of the background by degenerating finite-gap solutions [28, 9, 33, 34], when the spectral curve becomes rational, or, more directly, using classical Darboux [39], Dressing [59, 60] techniques. Among these basic solutions, we mention the Peregrine soliton [43], rationally localized in \(x\) and \(t\) over the background [2], the so-called Kuznetsov [35] -
Ma soliton, exponentially localized in space over the background and periodic in time; the so-called Akhmediev breather, periodic in $x$ and exponentially localized in time over the background \cite{2}. These solutions have also been generalized to the case of multi-soliton solutions, describing their nonlinear interaction, see, f.i., \cite{19,28,26,29,61}, and to the case of integrable multicomponent NLS equations \cite{8,18}.

The soliton solution over the background \cite{2} playing a basic role in this paper is the Akhmediev breather

$$A(x,t;\phi,X,T,\rho) \equiv e^{2it+i\rho \cosh[\Sigma(\phi)(t-T)+2\phi]+\sin(\phi)\cos[K(\phi)(x-X)]},$$

$$K(\phi) = 2\cos\phi, \quad \Sigma(\phi) = 2\sin(2\phi),$$

(3)

exact solution of \cite{1} for all values of the 4 real parameters $\phi, X, T, \rho$, changing the background by the multiplicative phase factor $e^{4i\phi}$:

$$A(x,t;\phi,X,T,\rho) \to e^{2it+i(\rho \pm 2\phi)}, \quad \text{as} \quad t \to \pm \infty,$$

and reaching the amplitude maximum in the point $(X,T)$, with

$$|A(X,T;\phi,X,T,\rho)| = 1 + 2\sin\phi.$$

(4)

Concerning the NLS Cauchy problems in which the initial condition consists of a perturbation of the exact background \cite{2}, if such a perturbation is localized, then slowly modulated periodic oscillations described by the elliptic solution of \cite{1} play a relevant role in the longtime regime \cite{11,12}. If the initial perturbation is $x$-periodic, numerical experiments and qualitative considerations prior to our recent works indicated that the solutions of \cite{1} exhibit instead time recurrence \cite{55,37,56,6,54,36}, as well as numerically induced chaos \cite{1,2,3}, in which solutions of Akhmediev type seem to play a relevant role \cite{15,16,17}.

We have recently started a systematic study of the Cauchy problem on the segment $[0,L]$, with periodic boundary conditions, considering, as initial condition, a generic, smooth, periodic, zero average, small perturbation of \cite{2}

$$u(x,0) = 1 + \epsilon(x),$$

$$\epsilon(x + L) = \epsilon(x), \quad ||\epsilon(x)||_{\infty} = \epsilon \ll 1, \quad \int_0^L \epsilon(x) dx = 0.$$

(6)

It is well-known that, in this Cauchy problem, the MI is due to the fact that, expanding the initial perturbation in Fourier components:

$$\epsilon(x) = \sum_{j \geq 1} \left( c_j e^{ik_j x} + c_{-j} e^{-ik_j x} \right), \quad k_j = \frac{2\pi}{L} j, \quad |c_j| = O(\epsilon),$$

(7)
and defining $N \in \mathbb{N}^+$ through the inequalities

$$\frac{L}{\pi} - 1 < N < \frac{L}{\pi}, \quad \pi < L,$$

the first $N$ modes $\pm k_j$, $1 \leq j \leq N$, are unstable, since they give rise to exponentially growing and decaying waves of amplitudes $O(\epsilon e^{\pm \sigma_j t})$, where the growing factors $\sigma_j$ are defined by

$$\sigma_j = k_j \sqrt{4 - k_j^2}, \quad 1 \leq j \leq N,$$

becoming $O(1)$ at times $T_j = O(\sigma_j^{-1} |\log \epsilon|)$, $1 \leq j \leq N$ (the first stage of MI), while the remaining modes give rise to oscillations of amplitude $O(\epsilon e^{\pm i \omega_j t})$, where $\omega_j = k_j \sqrt{k_j^2 - 4}$, $j > N$, and therefore are stable.

Using the finite gap method \[40, 27, 32\], we have established that the leading part of the evolution of a generic periodic perturbation of the constant background is described by finite-gap solutions associated with hyperelliptic genus 2 $N$ Riemann surfaces, where $N$ is the number of unstable modes. These solutions are well-approximated by an infinite time sequence of RWs described by the $N$-breather solutions of Akhmediev type, whose parameters vary at each appearance following a simple law in terms of the initial data \[22, 23, 24\]. In particular, in the simplest case of a single unstable mode $N = 1$, corresponding to the choice $\pi < L < 2\pi$, we have established that the initial condition

$$u(x, 0) = 1 + c_1 \exp(ik_1 x) + c_{-1} \exp(-ik_1 x), \quad |c_1|, |c_{-1}| = O(\epsilon), \quad k_1 = \frac{2\pi}{L}$$

(10)
evolves in the following way.

If $0 \leq t \leq O(1)$, we have the first linear stage of the MI, described by the linearized NLS around the background (2):

$$u(x, t) = e^{2it} \left\{ 1 + \frac{2}{\sigma_1} \left[ |\alpha_1| \cos \left( k_1(x - X_1^+) \right) e^{\sigma_1 t + i \phi_1} + |\beta_1| \cos \left( k_1(x - X_1^-) \right) e^{-\sigma_1 t - i \phi_1} \right] \right\} + O(\epsilon^2 |\log \epsilon|),$$

(11)

where

$$\alpha_1 = c_1 - e^{2i\phi_1} c_{-1}, \quad \beta_1 = c_{-1} - e^{-2i\phi_1} c_1,$$

$$\phi_1 = \arccos \left( \frac{k_1}{L} \right) = \arccos \left( \frac{k_1}{2} \right),$$

(12)
and $X_1^\pm$, defined as
\[ X_1^+ = \frac{\arg(\alpha_1) - \phi_1 + \pi/2}{k_1}, \quad X_1^- = -\frac{\arg(\beta_1) - \phi_1 + \pi/2}{k_1}, \tag{13} \]
are the positions of the maxima of the sinusoidal wave decomposition of the growing and decaying unstable modes. Therefore: the initial datum splits into exponentially growing and decaying waves, respectively the $\alpha$- and $\beta$-waves, each one carrying half of the information encoded in the initial datum.

If $|t - T_1(|\alpha_1|)| \leq O(1)$, where
\[ T_1(\zeta) \equiv \frac{1}{\sigma_1} \log \left( \frac{(\sigma_1)^2}{2\zeta} \right), \quad \zeta > 0, \tag{14} \]
then
\[ u(x, t) = A(x, t; \phi_1, X_1^+, T_1(|\alpha_1|), 2\phi_1) + O(\epsilon), \tag{15} \]
where $A(x, t; \phi, X, T, \rho)$ is the Akhmediev breather \[^3\]. It follows that the first RW appears in the time interval $|t - T_1(|\alpha_1|)| \leq O(1)$ and is described by the Akhmediev breather, whose parameters are expressed in terms of the initial data through elementary functions. Such a RW, appearing about the logarithmically large time $T_1(|\alpha_1|) = O(\sigma_1^{-1}|\log \epsilon|)$, is exponentially localized in an $O(1)$ time interval over the background $u_0$, changing it by the multiplicative phase factor $e^{4i\phi_1}$; in addition, the modulus of the first RW \[^{15}\] has its maximum at $t = T_1(|\alpha_1|)$, in the point $X_1^+$, and the value of this maximum has the upper bound
\[ |A(x, t; \phi_1, X_1^+, T_1(|\alpha_1|), 2\phi_1)| = 1 + 2\sin \phi_1 < 1 + \sqrt{3} \sim 2.732, \tag{16} \]
2.732 times the background amplitude, consequence of the formula $\sin \phi_1 = \sqrt{1 - (\pi/L)^2}$, $\pi < L < 2\pi$, and obtained when $L$ is close to $2\pi$. We notice that the position $x = X_1^+$ of the maximum of the RW coincides with the position of the maximum of the growing sinusoidal wave of the linearized theory; this is due to the absence of nonlinear interactions with other unstable modes. We finally remark that, in the first appearance, the RW contains informations, at the leading order, only on half of the initial wave, the half associated with the $\alpha$-wave.

It is easy to verify that the two representations \[^{11}\] and \[^{15}\] of the solution, valid respectively in the time intervals $0 \leq t \leq O(1)$ and $|t - T_1(|\alpha_1|)| \leq O(1)$, have the same behavior
\[ u(x, t) \sim e^{2it} \left( 1 + \frac{|\alpha_1|}{\sin 2\phi_1} e^{\sigma_1 t + i\phi_1} \cos[k_1(x - X_1^+)] \right) \tag{17} \]
in the intermediate region $O(1) \ll t \ll T_1(|\alpha_1|)$; therefore they match successfully.

The periodicity properties of the $\theta$-function representation of the solution \[27\] imply that, within $O(\epsilon^2 |\log \epsilon|)$ corrections, the solution of this Cauchy problem is also periodic in $t$, with period $T_p$, up to the multiplicative phase factor $\exp(2iT_p + 4i\phi_1)$ and up to the global $x$-translation of the quantity $\Delta_x$:

$$u(x, t + T_p) = e^{2iT_p + 4i\phi_1} u(x - \Delta_x, t) + O(\epsilon^2 |\log \epsilon|), \quad (18)$$

where

$$T_p = T_1(|\alpha_1|) + T_1(|\beta_1|) = \frac{2}{\sigma_1} \log \left( \frac{\sigma_1^2}{2\sqrt{|\alpha_1\beta_1|}} \right) = O(2\sigma_1^{-1} |\log \epsilon|),$$

$$\Delta_x = X_1^+ - X_1^- = \frac{\arg(\alpha_1\beta_1)}{\kappa_1}. \quad (19)$$

The time periodicity allows one to infer that the above Cauchy problem leads to an exact recurrence of RWs (of the nonlinear stages of MI), alternating with an exact recurrence of linear stages of MI \[22\]. We have, in particular, the following RW sequence. This Cauchy problem gives rise to an infinite sequence of RWs, and the $m^{th}$ RW of the sequence ($m \geq 1$) is described, in the time interval $|t - T_1(|\alpha_1|) - (m - 1)T_p| \leq O(1)$, by the analytic deterministic formula:

$$u(x, t) = A(x, t; \phi_1, x_1^{(m)}, t_1^{(m)}, \rho^{(m)}) + O(\epsilon), \quad m \geq 1, \quad (20)$$

where

$$x_1^{(m)} = X_1^+ + (m - 1)\Delta_x, \quad t_1^{(m)} = T_1(|\alpha_1|) + (m - 1)T_p, \quad \rho^{(m)} = 2\phi_1 + (m - 1)4\phi_1, \quad (21)$$

in terms of the initial data \[22\] (see Figures 1 and 2). Apart from the first RW appearance, in which the RW contains information only on half of the initial data (the one encoded in the parameter $\alpha_1$), in all the subsequent appearances the RW contains, at the leading order, informations on the full unstable part of the initial datum, encoded in both parameters $\alpha_1$ and $\beta_1$.  

6
Figure 1: The 3D plotting of $|u(x, t)|$ describing the RW sequence, obtained through the numerical integration of NLS via the Split Step Fourier Method (SSFM) [7, 52, 53]. Here $L = 6$ ($N = 1$), with $c_1 = \epsilon/2$, $c_{-1} = \epsilon(0.3 - 0.4i)/2$, $\epsilon = 10^{-4}$, and the short axis is the $x$-axis, with $x \in [-L/2, L/2]$. The numerical output is in perfect agreement with the theoretical predictions.

Figure 2: The color level plotting for the numerical experiment of Fig. 1 in which the periodicity properties of the dynamics are evident.
Two remarks are important at this point, in addition to the considerations on the instabilities we made at the beginning of this section.

a) If the initial condition (10) is replaced by a more general initial condition (6), (7) in which we excite also all the stable modes, then (11) is replaced by a formula containing also the infinitely many $O(\epsilon)$ oscillations corresponding to the stable modes. But the behavior of the solution in the overlapping region $1 \ll |t| \ll O(\sigma^{-1} |\log \epsilon|)$ is still given by (17) and matching at $O(1)$ is not affected. Therefore the sequence of RWs is still described by equations (20), (21), and the differences between the two Cauchy problems are hidden in the $O(\epsilon)$ corrections. As far as the $O(1)$ RW recurrence is concerned, only the part of the initial perturbation $\epsilon(x)$ exciting the unstable mode is relevant.

b) The above results are valid up to $O(\epsilon^2 |\log \epsilon|)$. It means that, in principle, the above RW recurrence formulae may not give a correct description for large times of $O((|\epsilon| |\log \epsilon|)^{-1})$; but since $O((|\epsilon| |\log \epsilon|)^{-1})$ is much larger than the recurrence time $O(|\log \epsilon|)$, it follows that the above formulae should give an accurate description of the RW recurrence for many consecutive appearances of the RWs.

c) Since the solution of the Cauchy problem is ultimately described by different elementary functions in different time intervals of the positive time axis, and since these different representations obviously match in their overlapping time regions, these finite gap results naturally motivate the introduction of a matched asymptotic expansions (MAE) approach, presented in the papers [23, 24], in which the initial perturbation excites all the $N \geq 1$ unstable modes “democratically”.

Since the above considerations establish the theoretical relevance of the Akhmediev breather in the description of each RW appearance in the time sequence, a natural and interesting open problem is the study of the numerical and physical instabilities of the Akhmediev breather.

In this paper we investigate experimentally the numerical instabilities, limiting our considerations to the simplest case of one unstable mode $N = 1$. In agreement with our recent theoretical findings, in the situation in which the round-off errors are negligible with respect to the perturbations due to the discrete scheme used in the numerical experiments: the Split-Step Fourier Method (SSFM) [7, 52, 53], the numerical output is well-described by a suitable genus 2 finite-gap solution of NLS. This solution can be written in terms of different elementary functions in different time regions and, ultimately,
it shows an exact recurrence of rogue waves described by the Akhmediev breather whose parameters, different at each appearance, are given in terms of the initial data via elementary functions. We discover a remarkable empirical formula connecting the recurrence time with the number of time steps used in the SSFM and, via our recent theoretical findings in [22], we establish that the SSFM opens up a vertical unstable gap whose length can be computed with high accuracy, and is proportional to the inverse of the square of the number of time steps used in the SSFM. This neat picture essentially changes when the round-off error is sufficiently large. Indeed experiments in standard double precision show serious instabilities in both the periods and phases of the recurrence. In contrast with it, as predicted by the theory, replacing the exact Akhmediev Cauchy datum by its first harmonic approximation, we only slightly modify the numerical output. Let us also remark that the first rogue wave appearance is completely stable in all experiments, in the perfect agreement with the Akhmediev formula and with the theoretical predictions [22] in terms of the Cauchy data.

2 A short summary of finite gap results

Here we summarize the classical and recent results on the NLS finite gap theory used in this paper.

The zero-curvature representation of the NLS equation [1] is given by the following pair of linear problems [58]:

\[
\begin{align*}
\vec{\Psi}_x(\lambda, x, t) &= U(\lambda, x, t)\vec{\Psi}(\lambda, x, t), \\
\vec{\Psi}_t(\lambda, x, t) &= V(\lambda, x, t)\vec{\Psi}(\lambda, x, t), \\
U &= \begin{bmatrix} -i\lambda & iu(x, t) \\ iu(x, t) & i\lambda \end{bmatrix}, \\
V(\lambda, x, t) &= \begin{bmatrix} -2i\lambda^2 + iu(x, t)u(x, t) & 2i\lambda u(x, t) - u_x(x, t) \\ 2i\lambda u(x, t) + u_x(x, t) & 2i\lambda^2 - iu(x, t)u(x, t) \end{bmatrix},
\end{align*}
\]

where

\[
\vec{\Psi}(\lambda, x, t) = \begin{bmatrix} \Psi_1(\lambda, x, t) \\ \Psi_2(\lambda, x, t) \end{bmatrix}.
\]

In the \(x\)-periodic problem with period \(L\), we have the main spectrum and the auxiliary spectrum. If \(\Psi(\lambda, x, t)\) is a fundamental matrix solution of
such that \( \Psi(\lambda, 0, 0) \) is the identity, then the monodromy matrix \( \hat{T}(\lambda) \) is defined by: \( \hat{T}(\lambda) = \Psi(\lambda, L, 0) \). The eigenvalues and eigenvectors of \( \hat{T}(\lambda) \) are defined on a two-sheeted covering of the \( \lambda \)-plane. This Riemann surface \( \Gamma \) is called the **spectral curve** and does not depend on time. The eigenvectors of \( \hat{T}(\lambda) \) are the Bloch eigenfunctions

\[
\vec{\Psi}_x(\gamma, x, t) = U(\lambda(\gamma), x, t)\vec{\Psi}(\gamma, x, t),
\]

\[
\vec{\Psi}(\gamma, x + L, t) = e^{i\lambda p(\gamma)}\vec{\Psi}(\gamma, x, t), \quad \gamma \in \Gamma,
\]

(24)

\( \lambda(\gamma) \) denote the projection of the point \( \gamma \) to the \( \lambda \)-plane.

The spectrum is exactly the projection of the set \( \{ \gamma \in \Gamma, \text{Im} p(\gamma) = 0 \} \) to the \( \lambda \)-plane. The end points of the spectrum are the branch points and the double points (obtained merging pairs of branch points) of \( \Gamma \), at which \( e^{i\lambda p(\gamma)} = \pm 1 \):

\[
\vec{\Psi}(\gamma, x + L, t) = \pm \vec{\Psi}(\gamma, x, t), \quad \gamma \in \Gamma.
\]

A potential \( u(x, t) \) is called **finite-gap** if the spectral curve \( \Gamma \) is algebraic; i.e., if it can be written in the form

\[
\nu^2 = \prod_{j=1}^{2g+2}(\lambda - E_j).
\]

(25)

It means that \( \Gamma \) has only a finite number of branch points and non-removable double points. These potentials can be written in terms of Riemann theta-functions [27]. Any smooth, periodic in \( x \) solution admits an arbitrarily good finite gap approximation, for any fixed time interval.

The auxiliary spectrum can be defined as the set of zeroes of the first component of the Bloch eigenfunction: \( \Psi_1(\gamma, x, t) = 0 \), therefore it is called **divisor of zeroes**. The zeroes of \( \Psi_1(\gamma, x, t) \) depend on \( x \) and \( t \), and the \( x \) and \( t \) dynamics become linear after the Abel transform.

The spectral curve \( \Gamma_0 \) corresponding to the background (2) is rational, and a point \( \gamma \in \Gamma_0 \) is a pair of complex numbers \( \gamma = (\lambda, \mu) \) satisfying the quadratic equation \( \mu^2 = \lambda^2 + 1 \).

The Bloch eigenfunctions can be easily calculated explicitly:

\[
\psi^\pm(\gamma, x) = \left[ \frac{1}{\lambda(\gamma) \pm \mu(\gamma)} \right] e^{\pm i\mu(\gamma)x},
\]

(26)
and are periodic (antiperiodic) iff $\frac{L}{2\pi} \mu$ is an even (an odd) integer. Let us introduce the following enumeration of the periodic and antiperiodic spectral points:

$$\mu_n = \frac{\pi n}{L} = \frac{k_n}{2}, \quad \lambda_n^\pm = \pm \sqrt{\mu_n^2 - 1}, \quad n \in \mathbb{N}. \quad (27)$$

The curve $\Gamma_0$ has two branch points $E_0 = i$, $\bar{E}_0 = -i$ corresponding to $n = 0$. If $n > 0$, we have only double points. The first $N$, such that $|\mu_n| < 1$, $1 \leq n \leq N$, are unstable, where $N$ is defined by (8); correspondingly $\lambda_n^\pm$ are pure imaginary and one can introduce the following convenient parametrization

$$\mu_n = \cos \phi_n, \quad \lambda_n^\pm = \pm i \sqrt{1 - \mu_n^2} = i \sin \phi_n, \quad 1 \leq n \leq N, \quad (28)$$

implying that

$$k_n = 2 \cos \phi_n, \quad \sigma_n = 2 \sin 2\phi_n, \quad 1 \leq n \leq N. \quad (29)$$

The remaining modes are such that $|\mu_n| > 1$, $n > N$ and are stable, and the corresponding $\lambda_n^\prime$s are real. In addition, the divisor of the unperturbed problem is located at the double points.

An initial $O(\epsilon)$ perturbation of the type (6) perturbs this picture. The branch points $\lambda_0^\pm$ become $E_0 = i + O(\epsilon^2)$ and $\bar{E}_0 = -i + O(\epsilon^2)$, and all double points generically split into a pair of square root branch points, generating infinitely many gaps. If $1 \leq n \leq N$, $\lambda_n^+$ splits into the pair of branch points $(E_{2n-1}, E_{2n})$, and $\lambda_n^-$ into the pair of branch points $(\bar{E}_{2n-1}, \bar{E}_{2n})$; if $n > N$, each $\lambda_n$ splits into a pair of complex conjugate eigenvalues. In the simplest case in which $N = 1$ and one excites only the unstable mode as in (10), $E_1$, $E_2$ are the branch points obtained through the splitting of the excited mode $\lambda_1 = i \sin \phi_1$, and

$$E_1 - E_2 = \frac{\sqrt{\alpha_1 \beta_1}}{\lambda_1^+} + O(\epsilon^2), \quad (30)$$

while the infinitely many gaps associated with the stable modes are $O(\epsilon^n)$, $n > 1$; therefore they give a negligible contribution to the solution of the Cauchy problem and can be erased from the picture.

The initial position of the divisor points $\gamma_{\pm 1}$, associated with the unstable modes $\lambda_1^\pm$, are

$$\lambda(\gamma_1) = \lambda_1 + \frac{1}{4\pi} \left[ \alpha_1 + \beta_1 \right] + O(\epsilon^2), \quad \lambda(\gamma_{-1}) = -\lambda_1 + \frac{1}{4\pi} \left[ e^{2i\phi_1} \alpha_1 + e^{-2i\phi_1} \beta_1 \right] + O(\epsilon^2). \quad (31)$$
Switching on time, the divisor points $\lambda(\gamma_{\pm 1})$ start moving in time and, after the period $T_p$ defined in (19), they replace each other.

We end this section remarking that, from formulas (19) and (30) discovered in [22], one can write the following relations, at leading order, between the unstable gap details and the RW recurrence period $T_p$ and $x$-shift $\Delta_x$ discussed in the previous section:

$$T_p = \frac{2}{\sigma_1} \log \left( \frac{\sigma_1^2}{2\text{Im} \lambda_1 |E_1 - E_2|} \right),$$
$$\Delta_x = \frac{\text{arg}(- (E_1 - E_2)^2)}{k_1}.$$  

(32)

3 The numerical instability of the Akhmediev breather

Due to the above instabilities, every time we have theoretical formulas describing time evolutions over the unstable background, it is important to test their stability under perturbations. Indeed: a) in any numerical experiment one uses numerical schemes approximating NLS; in addition, round off errors are not avoidable. All these facts are expected to cause the opening of basically all gaps and, due to the instability, no matter how small are the gaps associated with the unstable modes, they will cause $O(1)$ effects during the evolution (see also [1, 2, 15]). b) In physical phenomena involving weakly nonlinear quasi monochromatic waves, NLS is a first approximation of the reality, and higher order corrections have the effect of opening again all gaps, with $O(1)$ effects during the evolution caused by the unstable ones. At last, in a real experiment, a monochromatic initial perturbation like (10) should be replaced by a quasi-monochromatic approximation of it, often with random coefficients, opening again all the gaps associated with the unstable modes, with $O(1)$ effects during the evolution. The genericity of the Cauchy problems investigated in [22, 23, 24] and the associated numerical experiments seem to imply that the RW recurrences, analytically described in the previous section by a sequence of Akhmediev breathers, are expected to be stable under the above perturbations.

An interesting problem is to understand if, choosing instead at $t = 0$ the highly non generic Akhmediev breather (3) (we assume that $T > 0$ in (3)) as initial datum, numerical or physical perturbations yield $O(1)$ changes in the evolution with respect to the theoretical expectation (3). In this paper we concentrate on the instabilities generated by numerics, in the simplest case.
of a single unstable mode, postponing to a subsequent paper the study of instabilities due to the physical perturbations of the NLS equation.

The Akhmediev initial condition corresponds to a very special perturbation generating zero splitting for all the unstable and stable resonant points, and, in agreement with (32), it corresponds to \( T_p = \infty \). Indeed, the Akhmediev breather (3) describes the appearance of a RW only in the time interval \( |t - T| \leq O(1) \). In addition, for the Akhmediev solution, \( \alpha_1 = O(\epsilon) \), and \( |\beta_1| \ll \epsilon \) (but \( \beta_1 \neq 0 \)). In this non generic case, formula (30) is not valid, since we have an exact compensation between the leading term and the correction.

According to the above considerations, numerics introduces small perturbations to this non generic picture, opening small gaps for all stable and unstable modes. Again we erase from the picture the infinitely many stable gaps, and we are left with the gaps associated with the unstable mode \( \lambda_1^+ \) and with its complex conjugate. Therefore the numerical perturbation of the Akhmediev breather generates the finite gap configuration described in [22] and summarized in the previous section, and one expects that the genus 2 solution constructed there and describing the exact recurrence (20), (21) of RWs be the analytical model for the numerical instability of the Akhmediev breather. In some symmetrical cases these genus 2 solutions can be written in terms of elliptic functions [46].

Let us point out that, if we talk about numerical perturbations, we should distinguish two different sources: the difference between the continuous NLS model and the discretization used in the numerical scheme, and the round-off errors due to the finite number of digits used in the computations.

To study numerical instabilities, we used as numerical integrator the Split-Step Fourier Method (SSFM), also known as the split-step spectral method [7, 52, 53]. It uses the following algorithm. First of all, the \( x \)-boundary conditions are chosen to be periodic with period \( L \). Then one introduces a regular Cartesian lattice in the \( x, t \)-plane with the steps \( \delta x = L/N_x \) and \( \delta t \). Then, at each basic time step \( \delta t \), the NLS evolution is split into linear and non-linear parts, which are executed subsequently (each basic step is split into two steps in the asymmetric version, or three steps in the symmetric version).

In the asymmetric version, the following algorithm is used:

\[ \text{1} \] The authors are very grateful to M. Sommacal for introducing us to this method and providing us with his personalized MatLab code.
1. Non-linear step (one erases the dispersive term):

\[ u_1(x_m, t_n) = u(x_m, t_n) \exp \left( 2i|u(x_m, t_n)|^2 \delta t \right). \]

2. Linear step (one erases the nonlinear term). To do it one goes from the physical space to the Fourier space by applying the discrete Fourier transform with respect to \( x \). In our simulation we used the “Fastest Fourier Transform in the West” FFTW library [21]. In Fourier space:

\[ \hat{u}_2(p_m, t_n) = \hat{u}_1(p_m, t_n) \exp \left( -ip_m^2 \delta t \right). \]

Therefore the solution at time \( t_n + \delta t \) is calculated as the inverse discrete Fourier transform of \( \hat{u}_2(p_m, t_n) \).

Our goal was to see how well the Akhmediev solution can be reproduced in numerical experiments. We used the following settings:

1. We chose the simplest possible settings of the problem, with exactly one unstable mode \( \pi < L < 2\pi \). We fixed \( L = 6 \).
2. We used \( N_x = 512 \). We made some attempts to modify \( N_x \), and it appears that this change does not essentially affect the picture.
3. We chose the initial perturbation of order \( 10^{-4} \). The first appearance time \( T_1 \) in our experiments was \( \sim 4 - 6 \).
4. We chose the global time interval \( T_{max} = 60 \), which is one order of magnitude greater than the first appearance time.
5. We used as time step in the integration procedure \( \delta t = \frac{1}{10N} \), where \( N \) was varied from 50 to 15810.

To compare the effect of the round-off error with the effect of the numerical scheme discretization, we repeated some experiments twice, using C++ double and C++ quadruple precision. The typical round-off is \( 10^{-17} \) for double precision and \( 10^{-34} \) for quadruple precision; therefore, in the last case, it can be neglected.

Since the theory predicts that, to the leading order, the time evolution is determined by the excited unstable mode, i.e. by the first harmonics of the perturbation, we verified this prediction numerically.
4 Numerical experiments

4.1 Effect of the time step

In the first series of numerical experiments, we studied how the quality of the approximation of the Akhmediev solution depends on $\delta t$. In all these experiments $L = 6$, $T_{\text{max}} = 60$, $N_x = 512$, $\delta t = 1/(10N)$. The Cauchy datum is the Akhmediev soliton (3) at $t = 0$, with $T = 5.8738$. All experiments were proceeded with quadruple precision.

![Figure 3: Five level plots corresponding to $N = 158, 500, 1581, 5000, 15810$ respectively.](image)

In all these experiments, the numerical output shows the recurrence phenomena predicted by the finite-gap formulas, and not described by the exact Akhmediev solution. The position of the maxima at all appearances remains constant up to the grid step. According to the second formula in (32) it means that the perturbation due to the SSFM discretization opens a vertical gap. The size of the gap can be estimated using the first formula in (32):

$$|E_1 - E_2| = \frac{\sigma_1^2}{2 \text{Im} \lambda_1} e^{-\frac{\sigma_1 T p}{2}}.$$  (33)
In the numerical experiments we assume $T_p = T_2 - T_1$, where $T_1$ and $T_2$ are respectively the times of the first and second appearances. In the experiments the second and the third recurrence times do not sensibly change. The results of the experiments are presented in the following table:

| $N$  | $T_2 - T_1$ | $\exp\left(-\frac{\sigma_1(T_2 - T_1)}{2}\right)$ | $N^2 \exp\left(-\frac{\sigma_1(T_2 - T_1)}{2}\right)$ | $|E_1 - E_2|$ | $N^2|E_1 - E_2|$ |
|------|-------------|----------------------------------|----------------------------------|-------------|-----------------|
| 158  | 13.7        | $4.9 \cdot 10^{-6}$              | 0.123                            | $9.19 \cdot 10^{-6}$ | 0.229           |
| 500  | 16.3        | $4.8 \cdot 10^{-7}$              | 0.120                            | $9.03 \cdot 10^{-7}$ | 0.226           |
| 1581 | 18.9        | $4.8 \cdot 10^{-8}$              | 0.119                            | $8.88 \cdot 10^{-8}$ | 0.222           |
| 5000 | 21.4        | $5.1 \cdot 10^{-9}$              | 0.128                            | $9.54 \cdot 10^{-9}$ | 0.239           |
| 15810| 24.0        | $5.0 \cdot 10^{-10}$             | 0.126                            | $9.38 \cdot 10^{-10}$ | 0.235           |

We see that the combination $N^2 \exp\left(-\frac{\sigma_1(T_2 - T_1)}{2}\right)$ remains approximately constant, implying the following empirical law relating the recurrence time with the time step in the SSFM:

$$T_2 - T_1 \sim \frac{2}{\sigma_1} \log \left(\frac{N^2}{0.125}\right)$$  \hspace{1cm} (34)

It is an interesting problem in numerical analysis to explain this observation analytically.

We see that the recurrence in numerical solutions is well described by the finite-gap solutions. Thanks to Formula (33), one can estimate the size of the gap opened by the numerical perturbation, and the fact that such size is proportional to $1/N^2$ (see Table):

$$|E_1 - E_2| \sim \frac{0.125 \sigma_1^2}{2 \text{Im} \lambda_1} \frac{1}{N^2}.$$  \hspace{1cm} (35)

We conclude that the numerical output corresponding to the Akhmediev initial conditions is well described by the analytic formulas (20) and (21), where $T_1(|\alpha_1|) = T_1$, $\Delta_x = 0$, $T_p = T_2 - T_1$ given by (34).

For completeness, we also provide two 3D plots of the numerical solutions corresponding to the extreme values of $N$: $N = 158$ and $N = 15810.$
Figure 4: \( N = 158 \). First appearance time = 5.8734, maximum of \(|u|\) at the first appearance = 2.7039, position of the maximum = 0.527, recurrence times between consecutive appearances: 13.73, 13.49, 13.97 respectively

Figure 5: \( N = 15810 \). First appearance time = 5.8738, maximum of \(|u|\) at the first appearance = 2.7039, position of the maximaum = 0.527, recurrence times between consecutive appearances: 24.0, 23.4 respectively
4.2 Effect of round-off errors

To study the effect of the round-off errors, we repeated the above experiments with double precision.

We see that, for \( N = 500 \), the round-off errors do not change dramatically the recurrence times, but the spatial positions of the maxima have a notable change after few recurrences. In the right pictures we see that, for \( N = 5000 \), the round-off error dramatically changes the recurrence time. We conclude that, in double precision experiments with sufficiently large \( N \), the numerical perturbation due to round-off becomes more relevant than the numerical perturbation due to numerical scheme.

To illustrate it we also provide three numerical experiments made with double precision and time steps of the same order of magnitude.
Figure 7:
Level plots for $N = 5000$, $N = 7500$ and $N = 10000$ respectively.

We see that the finite-gap approximation is still relevant, but the lengths and the orientation of the gap become more or less random. In fact, increasing the number of steps, we increase the influence of the round-off error.

4.3 Effect of the Cauchy data

As it was shown in [22] theoretically, small stable harmonics of the Cauchy data do not seriously affect the leading order approximation. In this Section we verify that this situation is also relevant in numerical experiments. To do it, we check how stable is the numerical output when one replaces the exact Akhmediev initial condition by its unstable part, containing only the zero and the first harmonics:

$$u(x, 0) = 1 + c_1 e^{i k_1 x} + c_{-1} e^{-i k_1 x},$$

where the coefficients $c_1$, $c_{-1}$ are the first harmonics Fourier coefficients of the Akhmediev initial data. In our experiments

$$c_1 = 0.22341792182984515786378155403997297 \cdot 10^{-4} + 0.11311151504280589935931368075404486 \cdot 10^{-4} i,$$

$$c_{-1} = -1.760161767595421517918172784073977523 \cdot 10^{-14} + 0.2504192137797052240360210535522459765 \cdot 10^{-4} i;$$
(we slightly corrected them to have $\beta_1 = 0$ up to the round-off error).

Figure 8:
The left pair of pictures shows the $N = 500$ experiments (the left one for the Akhmediev Cauchy datum and right one for the Cauchy datum containing only the first harmonics of the Akhmediev solution). The right pair of pictures shows the $N = 5000$ experiments (the left one for the Akhmediev Cauchy datum and the right one for the Cauchy datum containing only the first harmonics of the Akhmediev solution). Both numerical experiments were made with quadruple precision.

We see that, also for high-precision calculations, the difference between these two numerical outputs is very small (much less relevant than the double precision round-off error in the previous Section).

4.4 Effect of the $x$-grid size

To estimate the effect of the spatial discretization size on the numerical simulations, we made the following experiment: we took $N = 5000$ and chose quadruple precision (from the previous experiments, it corresponds to a very high accuracy), and we repeated the above experiments with $N_x$ reduced 4 times: $N_x = 128$. At the level of the 3D output as well as at the level of the recurrence times and phase shift, the effect was negligible. We
Therefore we conclude that the size of the space discretization does not play an important role in this study. The theoretical explanation is clearly due to the fact that the Akhmediev solution is very smooth for all $t$; therefore the higher harmonics are extremely small.

5 Conclusions

In this paper we have studied the numerical instabilities of the Akhmediev exact solution of the self-focusing Nonlinear Schrödinger equation, describing the simplest one-mode, $x$-periodic perturbation of the unstable constant background solution, limiting our considerations to the simplest case of one unstable mode. In agreement with the theoretical predictions associated with the theory developed in [22], in the situation in which the round-off errors are negligible with respect to the perturbations due to the discrete numerical scheme, the numerical output shows that the Akhmediev breather is unstable, and that this instability is well-described by genus 2 finite-gap solutions. These solutions are well-approximated by different elementary functions in different time regions, describing a time sequence of Akhmediev one-breathers.

We discover the remarkable formulas (34), (35) connecting the recurrence time and the gap opening to the number of time steps of the SSFM used as the numerical scheme. In particular, the length of the two gaps opened by the SSFM is proportional to the inverse of the square of the number of time steps. Since the RW sequence generated by the numerical scheme has no phase shifts, it follows that these gaps are open vertically $\Re E_1 = \Re E_2 = 0$.

This clean picture essentially changes when the round-off error is sufficiently large. Indeed, the standard double precision experiments show serious instabilities in both periods and phases of the recurrence. In particular, increasing the number of time steps, we increase the instability. In contrast with it, replacing the exact Akhmediev Cauchy datum by the first harmonic approximation, we only slightly modify the numerical output, as predicted by the theory.

Let us also remark that the first appearance time, the position of the maximum and the value of it are completely stable in all experiments, and in perfect agreement with the Akhmediev formula, as well as with the theoretical predictions coming from [22] in terms of the Cauchy data.
6 Acknowledgments

Two visits of P. G. Grinevich to Roma were supported by the University of Roma “La Sapienza”, and by the INFN, Sezione di Roma. P. G. Grinevich and P. M. Santini acknowledge the warm hospitality and the local support of CIC, Cuernavaca, Mexico, in December 2016. P.G. Grinevich was also partially supported by RFBR grant 17-51-150001.

We acknowledge useful discussions with F. Briscese, F. Calogero, C. Conti, E. DelRe, A. Degasperis, A. Gelash, I. Krichever, A. Its, S. Lombardo, A. Mikhailov, D. Pierangeli, M. Sommacal and V. Zakharov.

References

[1] M. Ablowitz, B. Herbst, On homoclinic structure and numerically induced chaos for the nonlinear Schrodinger equation, SIAM Journal on Applied Mathematics (1990) 339351.

[2] M. J. Ablowitz, C. M. Schober, and B. M. Herbst, Numerical Chaos, Roundoff Errors and Homoclinic Manifolds, Phys. Rev. Lett. 71, 2683 (1993).

[3] M. J. Ablowitz, J. Hammack, D. Henderson, C. M. Schober, Long-time dynamics of the modulational instability of deep water waves, Physica D 152153 (2001) 416433.

[4] N. N. Akhmediev and V.I. Korneev, Theor. Math. Phys. 69,1089 (1986).

[5] N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, Exact first order solutions of the Nonlinear Schödinger equation, Theor. Math. Phys, 72, 809 (1987)

[6] N. N. Akhmediev, Nonlinear physics: Déjà vu in optics, Nature (London) 413, 267 (2001).

[7] G. P. Agrawal, Nonlinear Fiber Optics (3rd ed.). San Diego, CA, USA: Academic Press. ISBN 0-12-045143-3, 2001.

[8] F. Baronio, A. Degasperis, M. Conforti, S. Wabnitz, Solutions of the vector nonlinear Schrödinger equations: evidence for deterministic rogue waves, Physical Review Letters 109 (4) (2012) 44102.
[9] E.D. Belokolos, A.I. Bobenko, V.Z. Enolski, A.R. Its, V.B. Matveev, Algebro-geometric Approach in the Theory of Integrable Equations, Springer Series in Nonlinear Dynamics, Springer, Berlin, 1994.

[10] T. B. Benjamin, J. E. Feir, The disintegration of wave trains on deep water. Part I. Theory, Journal of Fluid Mechanics 27 (1967) 417430.

[11] G. Biondini and G. Kovacic, Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions, J. Math. Phys. 55, (2014), 031506.

[12] G. Biondini, S. Li, and D. Mantzavinos, Oscillation structure of localized perturbations in modulationally unstable media, Phys. Rev. E 94, 060201(R) (2016).

[13] Yu. V. Bludov, V. V. Konotop, N. Akhmediev, Physical Review A 80, 033610 (2009)

[14] U. Bortolozzo, A. Montina, F.T. Arecchi, J.P. Huignard, S. Residori, Spatiotemporal pulses in a liquid crystal optical oscillator, Physical Review Letters 99 (2) (2007) 36.

[15] A. Calini, N. M. Ercolani, D. W. McLaughlin, C. M. Schober, Mel’nikov analysis of numerically induced chaos in the nonlinear Schrödinger equation, Physica D 89 (1996) 227-260.

[16] A. Calini, C. Schober, Homoclinic chaos increases the likelihood of rogue wave formation, Physics Letters A 298 (56) (2002) 335349.

[17] A. Calini and C. M. Schober, Dynamical criteria for rogue waves in nonlinear Schrödinger models, Nonlinearity 25 (2012) R99R116.

[18] A. Degasperis and S. Lombardo, Integrability in action: solitons, instability and rogue waves, in Rogue and Shock Waves in non linear Dispersive Waves”, Lecture Notes in Physics, M. Onorato, S. Resitori, F. Baronio (Eds.), 2016. [http://www.springer.com/us/book/9783319392127].

[19] P. Dubard, P. Gaillard, C. Klein, V. B. Matveev, Eur. Phys. J. Special Topics 185, 247 (2010).

[20] K. B. Dysthe and K. Trulsen, Note on Breather Type Solutions of the NLS as Models for Freak-Waves, Physica Scripta. T82, (1999) 48-52.
[21] http://www.fftw.org/

[22] P.G. Grinevich and P.M. Santini, The finite gap method and the analytic description of the exact rogue wave recurrence in the periodic NLS Cauchy problem. 1. arXiv:1707.05659.

[23] P.G. Grinevich and P.M. Santini, The analytic description of the exact rogue wave recurrence in the NLS periodic Cauchy problem via matched asymptotic expansion 1. The case of one and two unstable modes; preprint 2017.

[24] P.G. Grinevich and P.M. Santini, The analytic description of the exact rogue wave recurrence in the NLS periodic Cauchy problem via matched asymptotic expansion 1. The case of one and two unstable modes; preprint 2017.

[25] K. L. Henderson and D. H. Peregrine and J. W. Dold, Unsteady water wave modulations: fully nonlinear solutions and comparison with the nonlinear Schrödinger equation, Wave Motion 29, 341 (1999).

[26] R. Hirota, Direct Methods for Finding Exact Solutions of Nonlinear Evolution Equations, Lecture Notes in Mathematics, Vol. 515, Springer, New York, 1976.

[27] A. R. Its, and V. P. Kotljarov, Explicit formulas for solutions of a nonlinear Schrödinger equation, Dokl. Akad. Nauk Ukrain. SSR Ser. A 1051, 965-968 (1976).

[28] A. R. Its, A. V. Rybin and M. A. Sall, Exact integration of nonlinear Schrödinger equation, Theor. Math. Phys. 74 20-32 (1988).

[29] D. J. Kedziora, A. Ankiewicz, and N. Akhmediev, Second-order nonlinear Schrödinger equation breather solutions in the degenerate and rogue wave limits, Phys. Rev. E 85, 066601 (2012).

[30] C. Kharif and E. Pelinovsky, Physical mechanisms of the rogue wave phenomenon, Eur. J. Mech. B/ Fluids J. Mech. 22, 603-634 (2004).

[31] C. Kharif and E. Pelinovsky, Focusing of nonlinear wave groups in deep water, JETP Lett. 73, 170 - 175 (2001).
[32] I. M. Krichever, Methods of algebraic Geometry in the theory on non-linear equations, Russian Math. Surv. 32, 185-213 (1977).

[33] I. M. Krichever, Spectral theory of two-dimensional periodic operators and its applications, Russian Math. Surveys, 44:2,(1989), 145–225.

[34] I. M. Krichever, Perturbation Theory in Periodic Problems for Two-Dimensional Integrable Systems. Sov. Sci. Rev., Sect. C, Math. Phys. Rev. 9:2 (1992), 1–103.

[35] E. A. Kuznetsov, Solitons in a parametrically unstable plasma, Sov. Phys. Dokl. 22, 507-508 (1977).

[36] E. A. Kuznetsov, Fermi-Pasta-Ulam recurrence and modulation instability, JETP Letters, 105 2 (2017), 125-129.

[37] B. M. Lake, H. C. Yuen, H. Rungaldier, and W. E. Ferguson, Nonlinear deep-water waves: Theory and experiment. Part 2. Evolution of a continuous wave train J. Fluid Mech. 83, 49 (1977).

[38] Y.-C. Ma, The perturbed plane wave solutions of the cubic Schrödinger equation, Stud. Appl. Math. 60, 43 - 58 (1979).

[39] V. B. Matveev and M. A. Salle, Darboux transformations and solitons, (Berlin, Heidelberg: Springer Series in Nonlinear Dynamics, Springer-Verlag), 1991.

[40] S. P. Novikov, The periodic problem for the Korteweg-de Vries equation, Funct. Anal. Appl., 8:3 (1974), 236–246.

[41] M. Onorato, S. Residori, U. Bortolozzo, A. Montina, F.T. Arecchi, Rogue waves and their generating mechanisms in different physical contexts, Physics Reports 528 (2013) 4789.

[42] A. Osborne, M. Onorato and M. Serio, The nonlinear dynamics of rogue waves and holes in deep-water gravity wave trains, Phys. Lett. A 275, 386, (2000).

[43] D. H. Peregrine, Water waves, nonlinear Schrödinger equations and their solutions, J. Austral. Math. Soc. Ser. B 25 (1983), 16-43.
[44] D. Pierangeli, F. Di Mei, C. Conti, A. J. Agranat and E. DelRe, Spatial Rogue Waves in Photorefractive Ferroelectrics, PRL 115, 093901 (2015).

[45] L. Salasnich, A. Parola, and L. Reatto, Modulational Instability and Complex Dynamics of Confined Matter-Wave Solitons, Phys. Rev. Lett. 91, 080405 (2003).

[46] A. O. Smirnov, Periodic Two-Phase Rogue Waves, Mathematical Notes, 2013, Vol. 94, No. 6, pp. 897907.

[47] D. R. Solli, C. Ropers, P. Koonath and B. Jalali, Nature 450, 1054 (2007).

[48] G. Stokes, On the Theory of Oscillatory Waves, Transactions of the Cambridge Philosophical Society VIII (1847) 197229, and Supplement 314326.

[49] C. Sulem and P-L. Sulem, The nonlinear Schrödinger equation (Self focusing and wave collapse) (Springer-Verlag, New York, Berlin, Heidelberg, 1999).

[50] V. I. Vespalov and V. I. Talanov, Filamentary structure of light beams in nonlinear liquids, JETP Letters. 3 (12), 307, 1966.

[51] T. Taniuti and H. Washimi, Self-Trapping and Instability of Hydromagnetic Waves along the Magnetic Field in a Cold Plasma, Phys. Rev. Lett. 21, 209 (1968).

[52] J. A. C. Weideman and B. M. Herbst, Split-step methods for the solution of the nonlinear Schrödinger equation. SIAM Journal on Numerical Analysis, 23, 485507, 1986.

[53] T R Taha and X Xu. Parallel Split-Step Fourier Methods for the coupled nonlinear Schrödinger type equations. The Journal of Supercomputing, 5, 523, 2005.

[54] G. Van Simaeys, P. Emplit, and M. Haelterman, Experimental Demonstration of the Fermi-Pasta-Ulam Recurrence in a Modulationally Unstable Optical Wave, Phys. Rev. Lett. 87, 033902 (2001).
[55] H. C. Yuen and W. E. Ferguson, Relationship between Benjamin-Feir
instability and recurrence in the nonlinear Schrödinger equation, Phys.
Fluids 21, 1275 (1978).

[56] H. Yuen, B. Lake, Nonlinear dynamics of deep-water gravity waves,
Advances in Applied Mechanics 22 (67) (1982) 229.

[57] V. E. Zakharov, Stability of period waves of finite amplitude on surface
of a deep fluid, Journal of Applied Mechanics and Technical Physics,
9(2) (1968) 190-194.

[58] V. E. Zakharov and A. B. Shabat, Exact theory of two-dimensional
self-focusing and one-dimensional self-modulation of waves in nonlinear
media, Sov. Phys. JETP 34, 1, 62-69, 1972.

[59] V. E. Zakharov and A. B. Shabat, A scheme for integrating the non-
linear equations of mathematical physics by the method of the inverse
scattering transform I, Funct. Anal. Appl. 8, 226-235 (1974).

[60] V. E. Zakharov and A. V. Mikhailov, Relativistically invariant two-
dimensional models of field theory which are integrable by means of
the inverse scattering problem method, Sov. Phys. - JETP 47, 1017-27,
1978.

[61] V. E. Zakharov and A. A. Gelash, On the nonlinear stage of Modulation
Instability, PRL 111, 054101 (2013).

[62] V. Zakharov, L. Ostrovsky, Modulation instability: the beginning, Phys-
ica D: Nonlinear Phenomena 238 (5) (2009) 540548.