Absolutely continuous operators on function spaces and vector measures

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Abstract Let \((\Omega, \Sigma, \mu)\) be a finite atomless measure space, and let \(E\) be an ideal of \(L^0(\mu)\) such that \(L^\infty(\mu) \subset E \subset L^1(\mu)\). We study absolutely continuous linear operators from \(E\) to a locally convex Hausdorff space \((X, \xi)\). Moreover, we examine the relationships between \(\mu\)-absolutely continuous vector measures \(m: \Sigma \to X\) and the corresponding integration operators \(T_m: L^\infty(\mu) \to X\). In particular, we characterize relatively compact sets \(\mathcal{M}\) in \(ca_\mu(\Sigma, X)\) (= the space of all \(\mu\)-absolutely continuous measures \(m: \Sigma \to X\)) for the topology \(\mathcal{T}_s\) of simple convergence in terms of the topological properties of the corresponding set \(\{T_m: m \in \mathcal{M}\}\) of absolutely continuous operators. We derive a generalized Vitali–Hahn–Saks type theorem for absolutely continuous operators \(T: L^\infty(\mu) \to X\).

Keywords Function spaces · Absolutely continuous operators · Integration operators · Countably additive vector measures · Absolutely continuous vector measures · Mackey topologies · Order-bounded topology

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1 Introduction and terminology

For terminology concerning vector lattices and function spaces we refer the reader to [1], [2], [10]. Throughout the paper we assume that \((\Omega, \Sigma, \mu)\) is a complete finite atomless measure space and \(L^0(\mu)\) denotes the corresponding space of \(\mu\)-equivalence classes of all \(\Sigma\)-measurable real valued-functions defined on \(\Omega\). Let \(E\) be an
ideal of $L^0(\mu)$ such that $L^\infty(\mu) \subset E \subset L^1(\mu)$, and let $E^\sim$ and $E_n^\sim$ stand for the order dual and order continuous dual of $E$ respectively. Then $E_n^\sim$ separates the points of $E$ and it can be identified with the Köthe dual $E'$ of $E$ through the mapping $E' \ni v \mapsto \varphi_v \in E_n^\sim$, where $\varphi_v(u) = \int_{\Omega} uv d\mu$ for all $u \in E$. It is known that the Mackey topology $\tau(E, E_n^\sim)(= \tau(E, E'))$ is a locally solid Lebesgue topology.

The so-called order-bounded topology $\tau_0$ can be defined on $E$ as the finest locally convex topology on $E$ for which every order interval in $E$ is a bounded set (see [11]). A local base $B_0$ at zero for $\tau_0$ is the class of all absolutely convex subsets of $E$ that absorb all order bounded sets in $E$. Then $\tau_0$ coincides with the Mackey topology $\tau(E, E^\sim)$. Note that if $u_n, u \in E$ and $u_n \to u$ uniformly on $\Omega$, then $u_n \to u$ for $\tau_0$.

From now on we assume that $(X, \xi)$ is a locally convex Hausdorff space (for short, l c Hs) and let $\mathcal{P}_\xi$ denote the set of all $\xi$-continuous seminorms on $X$. By $X'_\xi$ we denote the topological dual of $(X, \xi)$. We denote by $\sigma(L, K)$ and $\tau(L, K)$ the weak topology and the Mackey topology on $L$ with respect to a dual pair $(L, K)$.

Recall that a linear operator $T : E \to X$ is said to be order-bounded (resp. order-weakly compact), if for each $u \in E^+$, the set $T([-u, u])$ is $\xi$-bounded (resp. relatively $\sigma(X, X'_\xi)$-compact) in $X$ (see [6]).

**Proposition 1.1** For a linear operator $T : E \to X$ the following statements are equivalent:

(i) $T$ is order-bounded.

(ii) $T$ is $(\tau_0, \xi)$-continuous.

**Proof** (i)$\implies$(ii) Assume that $T$ is order-bounded. Let $p \in \mathcal{P}_\xi$ and $\varepsilon > 0$. We shall show that there is $V \in B_0$ such that $T(V) \subset B_p(\varepsilon)(= \{x \in X : p(x) \leq \varepsilon\})$. Indeed, let $V = T^{-1}(B_p(\varepsilon))$. Since $T(V) \subset T(T^{-1}(B_p(\varepsilon))) \subset B_p(\varepsilon)$, it is enough to show that $V$ absorbs every order interval in $E$. Given $u \in E^+$ there is $r_u > 0$ such that $T([-u, u]) \subset B_p(r_u)$. Then for $\lambda_u = \frac{\xi}{r_u}$ and for all $v \in [-u, u]$ we get $p(T(\lambda_u v)) = \lambda_u p(T(v)) \leq \varepsilon$, so $\lambda_u v \in V$. This means that $\lambda_u [-u, u] \subset V$, i.e., $V$ absorbs $[-u, u]$, as desired.

(ii)$\implies$(i) Assume that $T$ is $(\tau_0, \xi)$-continuous and $p \in \mathcal{P}_\xi$. Then there is $V_p \in B_0$ such that $T(V_p) \subset B_p(1)$. Given $u \in E^+$ there exists $\lambda_u > 0$ such that $\lambda_u [-u, u] \subset V_p$. Hence $T(\lambda_u [-u, u]) \subset T(V_p) \subset B_p(1)$, so $T([-u, u]) \subset B_p(\varepsilon)$). It follows that the set $T([-u, u])$ is $\xi$-bounded in $X$. $\Box$.

Following [13] a linear operator $T : E \to X$ is said to be absolutely continuous if for each $u \in E$, $T(\llcorner_{A_n} u) \to 0$ for $\xi$ whenever $\mu(A_n) \to 0$, $(A_n) \subset \Sigma$. Absolutely continuous operators on Orlicz spaces and Frechet function spaces have been examined by Orlicz and Wnuk (see [12, 13]).

In Sect. 2 we study absolutely continuous operators $T : E \to X$. We show that a linear operator $T : E \to X$ is absolutely continuous if and only if $T$ is $(\tau(E, E_n^\sim), \xi)$-continuous. We characterize relatively compact sets in the space $L_{r, \xi}(E, X)$ of all $(\tau(E, E_n^\sim), \xi)$-continuous linear operators $T : E \to X$, provided with the topology of simple convergence. In Sect. 3 we examine the relationships between $\mu$-absolutely continuous vector measures $m : \Sigma \to X$ and the corresponding integration operators $T_m : L^\infty(\mu) \to X$. 
2 Absolutely continuous operators on function spaces

We start with the following result.

**Proposition 2.1** Assume that $T : E \to X$ is an absolutely continuous linear operator. Then $T$ is $(\tau, \xi)$-continuous.

**Proof** In view of Proposition 1.1 it is sufficient to show that $T([-u, u])$ is $\xi$-bounded in $X$ for every $u \in E^+$. For this purpose one can repeat the proof of Theorem 1 in [13].

Now we present a characterization of absolutely continuous operators on $E$. □

**Proposition 2.2** For a linear operator $T : E \to X$ the following statements are equivalent:

(i) $x' \circ T \in E_n^\sim$ for each $x' \in X'_\xi$.
(ii) $T$ is $(\sigma(E, E_n^\sim), \sigma(X, X'_\xi))$-continuous.
(iii) $T$ is $(\tau(E, E_n^\sim), \xi)$-continuous.
(iv) $T$ is smooth, i.e., $T(u_n) \to 0$ for $\xi$ whenever $u_n^{(o)} \to 0$ in $E$.
(v) $T$ is $\sigma$-smooth, i.e., $T(u_n) \to 0$ for $\xi$ whenever $u_n^{(o)} \to 0$ in $E$.
(vi) $T$ is absolutely continuous.

**Proof** (i)$\iff$(ii) See [1, Theorem 9.26].
(ii)$\implies$(iii) Assume that $T$ is $(\sigma(E, E_n^\sim), \sigma(X, X'_\xi))$-continuous. It follows that $T$ is $(\tau(E, E_n^\sim), \tau(X, X'_\xi))$-continuous (see [1, Exercise 11, p. 149]), and hence $T$ is $(\tau(E, E_n^\sim), \xi)$-continuous because $\xi \subset \tau(X, X'_\xi)$.
(iii)$\implies$(iv) Assume that $T$ is $(\tau(E, E_n^\sim), \xi)$-continuous, and let $(u_\alpha)$ be a net in $E$ such that $u_\alpha^{(o)} \to 0$ in $E$. Then $u_\alpha \to 0$ for $\tau(E, E_n^\sim)$ because $\tau(E, E_n^\sim)$ is a Lebesgue topology on $E$. Hence $T(u_\alpha) \to 0$ for $\xi$, as desired.
(iv)$\iff$(v) It is obvious.
(v)$\iff$(vi) It is enough to repeat the reasoning in the proof of Proposition 4 in [13] and use Proposition 2.1 and the fact that $u_n \to 0$ in $E$ for $\tau_0$ whenever $u_n \to 0$ uniformly on $\Omega$.
(v)$\implies$(i) It is obvious. □

**Corollary 2.3** Every absolutely continuous operator $T : E \to X$ is order-weakly compact.

**Proof** Note that for each $u \in E^+$, the order interval $[-u, u]$ in $E$ is relatively $\sigma(E, E_n^\sim)$-compact because $\tau(E, E_n^\sim)$ is a Lebesgue topology (see [2], Theorem 6.62). Hence by Proposition 2.2 the set $T([-u, u])$ is relatively $\sigma(X, X'_\xi)$-compact in $X$, as desired. □

Let $\mathcal{L}_{\tau, \xi}(E, X)$ stand for the space of all $(\tau(E, E_n^\sim), \xi)$-continuous linear operators from $E$ to $X$, equipped with the topology $T_\xi$ of simple convergence. Let $\mathcal{P}_\xi$ be the family of all $\xi$-continuous seminorms on $X$. Then $T_\xi$ is generated by the family $\{q_{p,u} : p \in \mathcal{P}_\xi, u \in E\}$ of seminorms, where $q_{p,u}(T) = p(T(u))$ for all $T \in \mathcal{L}_{\tau, \xi}(E, X)$.

The following result will be of importance (see [15, Theorem 2]).
Theorem 2.4 Let $\mathcal{K}$ be a $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau, \xi}$. If $C$ is a $\sigma(X^*_s, X)$-closed and $\xi$-equicontinuous subset of $X^*_s$, then $\{x' \circ T : T \in \mathcal{K}, x' \in C\}$ is a $\sigma(E_n^\sim, E)$-compact subset of $E_n^\sim$.

Now we can state a characterization of relative $\mathcal{T}_s$-compactness in $\mathcal{L}_{\tau, \xi}(E, X)$.

Theorem 2.5 Let $\mathcal{K}$ be a subset of $\mathcal{L}_{\tau, \xi}(E, X)$. Then the following statements are equivalent:

(i) $\mathcal{K}$ is relatively $\mathcal{T}_s$-compact.
(ii) $\mathcal{K}$ is $(\tau(E_n^\sim, E), \xi)$-equicontinuous and for each $u \in E$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively $\xi$-compact in $X$.

Proof (i)$\Rightarrow$(ii) Assume that $\mathcal{K}$ is relatively $\mathcal{T}_s$-compact. Let $W$ be an absolutely convex and $\xi$-closed neighbourhood of 0 for $\xi$ in $X$. Then the polar $W^0$ of $W$ (with respect to the dual pair $(E, E^*_s)$) is a $\sigma(X^*_s, X)$-closed and $\xi$-equicontinuous subset of $X^*_s$ (see [1, Theorem 9.21]). Then by Theorem 2.4 the set $H = \{x' \circ T : T \in \mathcal{K}, x' \in W^0\}$ in $E_n^\sim$ is $\sigma(E_n^\sim, E)$-compact. Hence in view of the Nakano theorem (see [2, Corollary 6.31]) the $\sigma(E_n^\sim, E)$-closed absolutely convex hull (abs conv $H^\sim$) of $H$ is $\sigma(E_n^\sim, E)$-compact in $E_n^\sim$. The the polar $V = ((\text{absconv } H)^\sim)^0$ (with respect to the dual pair $(E, E_n^\sim)$) is a $(\tau(E, E_n^\sim), E)$-neighbourhood of 0 in $E$ and $H \subseteq V^0$. Then for each $T \in \mathcal{K}$ we have that $\{x' \circ T : x' \in W^0\} \subseteq V^0$, i.e., if $x' \in W^0$, then $|x'(T(u))| \leq 1$ for all $u \in V$. This means that for each $T \in \mathcal{K}$ we have $W^0 \subseteq T(V)^0$. Hence $T(V) \subseteq T(V)^{00} \subseteq V^{00} = W$ for each $T \in \mathcal{K}$, i.e., $\mathcal{K}$ is $(\tau(E, E_n^\sim), \xi)$-equicontinuous.

Clearly, for each $u \in E$, the set $\{T(u) : T \in \mathcal{K}\}$ is relatively $\xi$-compact in $X$.

(ii)$\Rightarrow$(i) It follows from [3, Chap. 3, § 3.4, Corollary 1], [4, Chap. 3.2.2, Corollary, p. 89].

Corollary 2.6 Assume that $\mathcal{K}$ is a relatively $\mathcal{T}_s$-compact subset of $\mathcal{L}_{\tau, \xi}(E, X)$. Then $\mathcal{K}$ is uniformly $\mu$-absolutely continuous, i.e., for each $u \in E$ and $p \in \mathcal{P}_\xi$ we have

$\sup_{T \in \mathcal{K}} p(T(1_{A_n} u)) \rightarrow 0$ whenever $\mu(A_n) \rightarrow 0$, $(A_n) \subseteq \Sigma$.

Proof In view of Theorem 2.4, $\mathcal{K}$ is $(\tau(E, E_n^\sim), \xi)$-equicontinuous. Let $p \in \mathcal{P}_\xi$ and $\varepsilon > 0$ be given. Then there exists a $\tau(E, E_n^\sim)$-neighbourhood $V$ of 0 in $E$ such that for each $T \in \mathcal{K}$ we have $p(T(u)) \leq \varepsilon$ for all $u \in V$. Let $u \in E$ and $\mu(A_n) \rightarrow 0$ and let $u_n = 1_{A_n} u$ for $n \in \mathbb{N}$. Note that $u_n \rightarrow 0(\mu)$ and $|u_n(\omega)| \leq |u(\omega)|\mu$-a.e. for all $n \in \mathbb{N}$. Hence by the Riesz theorem for every subsequence $(u_{k_n})$ of $(u_n)$ there exists a subsequence $(u_{l_{k_n}})$ of $(u_{k_n})$ such that $u_{l_{k_n}}(\omega) \rightarrow 0\mu$-a.e. This means that $u_{l_{k_n}} \xrightarrow{(o)} 0$ in $E$ (see [10, Chap. 10, §1]). Hence $u_{l_{k_n}} \rightarrow 0$ for $\tau(E, E_n^\sim)$ because $\tau(E, E_n^\sim)$ is a Lebesgue topology. It follows that $u_n \rightarrow 0$ for $\tau(E, E_n^\sim)$. Then there exists $n_\varepsilon \in \mathbb{N}$ such that $u_n \in V$ for $n \geq n_\varepsilon$, and hence sup$_{T \in \mathcal{K}} p(T(1_{A_n} u)) \leq \varepsilon$ for $n \geq n_\varepsilon$. □

3 Absolutely continuous vector measures

Let $(X, \xi)$ be a quasicomplete $\mathcal{L}_H$s and $m : \Sigma \rightarrow X$ be a $\xi$-bounded vector measure (i.e., the range of $m$ is $\xi$-bounded in $X$) and $m(A) = 0$ if $\mu(A) = 0$, $A \in \Sigma$ (in symbols, $m \ll \mu$).
For \( u \in L^\infty(\mu) \) let \( \|u\|_\infty = \text{ess sup}_{\omega \in \Omega} |u(\omega)| \). Given \( u \in L^\infty(\mu) \), let \( (s_n) \) be a sequence in \( S(\mu) (= \text{the space of all } \mu\text{-simple functions on } \Omega) \) such that \( \|u - s_n\|_\infty \to 0 \) (see [10, Chap. 1, §6, Theorem 3]). Define

\[
\int u \, dm := \xi - \lim_{\Omega} \int s_n \, dm.
\]

Then the integral \( \int u \, dm \) is well defined and the corresponding integration operator \( T_m : L^\infty(\mu) \to X \) given by \( T_m(u) = \int u \, dm \) is \((\|\cdot\|_\infty, \xi)\)-continuous and linear, and for each \( x' \in X' \),

\[
x' \left( \int u \, dm \right) = \int u \, d(x' \circ m) \quad \text{for } u \in L^\infty(\mu),
\]

(see [9], [14, Lemma 6]). Conversely, let \( T : L^\infty(\mu) \to X \) be a \((\|\cdot\|_\infty, \xi)\)-continuous linear operator, and let \( m(A) = T(1_A) \) for \( A \in \Sigma \). Then \( m : \Sigma \to X \) is a \( \xi \)-bounded vector measure such that \( m \ll \mu \) (called the representing measure of \( T \)) and \( T_m(u) = T(u) \) for all \( u \in L^\infty(\mu) \).

An important example of a quasicomplete lcHs is the space \( \mathcal{L}(Y, Z) \) of all bounded linear operators between Banach spaces \( Y \) and \( Z \), provided with the strong operator topology.

Recall that a vector measure \( m : \Sigma \to X \) is said to be \( \mu \)-absolutely continuous \( m(A_n) \to 0 \) for \( \xi \) whenever \( \mu(A_n) \to 0 \), \( (A_n) \subset \Sigma \) (see [5, Definition 3, p. 11]).

Now we characterize \( \mu \)-absolutely continuous measures in terms of the properties of the corresponding integration operators.

**Proposition 3.1** Assume that \((X, \xi)\) is a quasicomplete lcHs. Let \( m : \Sigma \to X \) be a \( \xi \)-bounded vector measure such that \( m \ll \mu \). Then the following statements are equivalent:

(i) \( x' \circ m \in \text{ca}_\mu(\Sigma) \) for each \( x' \in X'_{\xi} \).
(ii) \( x' \circ T_m \in L^\infty(\mu)_{\sim} \) for each \( x' \in X'_{\xi} \).
(iii) \( T_m \) is \((\tau(L^\infty(\mu), L^1(\mu)), \xi)\)-continuous.
(iv) \( T_m \) is \( \sigma \)-smooth.
(v) \( T_m \) is absolutely continuous.
(vi) \( m \) is \( \mu \)-absolutely continuous.

**Proof** (i)\(\implies\)(ii) Let \( x' \in X'_{\xi} \) and \( x' \circ m \in \text{ca}_\mu(\Sigma) \). Then by the Radon–Nikodym theorem there exists \( v_{x'} \in L^1(\mu) \) such that \( (x' \circ m)(A) = \int_A v_{x'} \, d\mu \) for all \( A \in \Sigma \). It follows that

\[
(x' \circ T_m)(u) = \int \Omega u \, d(x' \circ m) = \int \Omega uv_{x'} \, d\mu \quad \text{for all } u \in L^\infty(\mu),
\]

and this means that \( x' \circ T_m \in L^\infty(\mu)_{\sim} \).
(ii) $\iff$ (iii) $\iff$ (iv) $\iff$ (v) See Proposition 2.1.

(v) $\implies$ (vi) Assume that $T_m$ is absolutely continuous, and let $\mu(A_n) \to 0$, $(A_n) \subset \Sigma$. Then $m(A_n) = T_m(1_{A_n}) \to 0$ for $\xi$, as desired.

(vi) $\implies$ (i) It is obvious. □

As a consequence of Proposition 3.1 we get the following Pettis type theorem for countably additive measures (see [5, Theorem 1, p. 10]).

**Corollary 3.2** Assume that $(X, \xi)$ is a quasicomplete lcHs. Let $m : \Sigma \to X$ be a $\xi$-countably additive measure. Then the following statements are equivalent:

(i) $m \ll \mu$.

(ii) $m$ is $\mu$-absolutely continuous.

Let $ca(\Sigma, X)$ stand for the space of all $\xi$-countably additive measures $m : \Sigma \to X$. By $ca_\mu(\Sigma, X)$ we denote the subspace of $ca(\Sigma, X)$ consisting of all $m \in ca(\Sigma, X)$ that are $\mu$-absolutely continuous. Denote by $T_s$ the topology of simple convergence in $ca(\Sigma, X)$. Then $T_s$ is generated by the family $\{q_{p, A} : p \in \mathcal{P}_\xi, A \in \Sigma\}$ of seminorms, where

$$q_{p, A}(m) : = p(m(A)) \quad \text{for all} \quad m \in ca(\Sigma, X).$$

**Proposition 3.3** $ca_\mu(\Sigma, X)$ is a closed set in $(ca(\Sigma, X), T_s)$.

**Proof** Let $m \in ca(\Sigma, X)$ and $m \in cl_{T_s}(ca_\mu(\Sigma, X))$. Then there is a net $(m_\alpha)$ in $ca_\mu(\Sigma, X)$ such that $m_\alpha \to m$ for $T_s$, i.e., for each $p \in \mathcal{P}_\xi$ and $A \in \Sigma$ we have $q_{p, A}(m - m_\alpha) = p(m(A) - m_\alpha(A)) \to 0$. Assume that $\mu(A) = 0$. Then $m_\alpha(A) = 0$ for all $\alpha$, and it follows that $p(m(A)) = 0$ for each $p \in \mathcal{P}_\xi$, i.e., $m(A) = 0$. In view of Corollary 3.2 $m \in ca_\mu(\Sigma, X)$. □

Now we establish some terminology (see [14, pp. 92–93]). For $p \in \mathcal{P}_\xi$ let $X_p = (X, p)$ be the associated seminormed space. Denote by $(\tilde{X}_p, \| \cdot \|_{\tilde{p}})$ the completion of the quotient normed space $X/p^{-1}(0)$. Let $\Pi_p : X_p \to X/p^{-1}(0) \subset \tilde{X}_p$ be the canonical quotient map.

Given a vector measure $m : \Sigma \to X$ with $m \ll \mu$, let $m_p : \Sigma \to \tilde{X}_p$ be given by

$$m_p(A) : = (\Pi_p \circ m)(A) \quad \text{for} \quad A \in \Sigma.$$ 

Then $m_p$ is a Banach space-valued measure on $\Sigma$. We define the $p$-variation $\|m\|_p$ of $m$ by

$$\|m\|_p(A) : = \|m_p\|(A) \quad \text{for} \quad A \in \Sigma,$$

where $\|m_p\|$ denotes the semivariation of $m_p : \Sigma \to \tilde{X}_p$. Note that $m$ is $\xi$-bounded if and only if $\|m\|_p(\Omega) < \infty$ for each $\xi$-continuous seminorm $p$ on $X$. Moreover, we have (see [14, Lemma 7]):

$$\|m\|_p(\Omega) = \|T_m\|_p := \sup \left\{ p \left( \int_\Omega u \, dm \right) : u \in L^\infty(\mu), \|u\|_\infty \leq 1 \right\}. \quad (3.1)$$
For a subset $\mathcal{M}$ of $ca_{\mu}(\Sigma, X)$ let

$$
\mathcal{K}_\mathcal{M} = \{ T_m \in \mathcal{L}_{r, \xi}(L^\infty(\mu), X) : m \in \mathcal{M} \}.
$$

Now we are ready to state a characterization of relative compactness in the space $(ca_{\mu}(\Sigma, X), T_s)$ in terms of the topological properties of the set $\mathcal{K}_\mathcal{M}$ (see [8, Theorem 7], [15, Theorem 8], [16, Theorem 2.1]).

**Theorem 3.4** Let $(X, \xi)$ be a quasicomplete lcHs. Then for a set $\mathcal{M}$ in $ca_{\mu}(\Sigma, X)$ the following statements are equivalent:

(i) $\mathcal{K}_\mathcal{M}$ is a relatively compact set in $(\mathcal{L}_{r, \xi}(L^\infty(\mu), X), T_s)$.

(ii) $\mathcal{K}_\mathcal{M}$ is $(\tau(L^\infty(\mu), L^1(\mu)), \xi)$-equicontinuous and for each $u \in L^\infty(\mu)$, the set $\{T_m(u) : m \in \mathcal{M}\}$ is relatively $\xi$-compact in $X$.

(iii) $\mathcal{M}$ is uniformly $\mu$-absolutely continuous and for each $A \in \Sigma$, the set $\{m(A) : m \in \mathcal{M}\}$ is relatively $\xi$-compact in $X$.

(iv) $\mathcal{M}$ is a relatively compact set in $(ca_{\mu}(\Sigma, X), T_s)$.

**Proof** (i)$\iff$(ii) See Theorem 2.5.

(ii)$\implies$(iii) Assume that (ii) holds and let $\mu(A_n) \to 0$, $(A_n) \subset \Sigma$. Then using Proposition 3.1 and Corollary 2.6 for each $p \in \mathcal{P}_\xi$ we have

$$
\sup_{m \in \mathcal{M}} p(m(A_n)) = \sup_{m \in \mathcal{M}} p(T_m(\mathbb{1}_{A_n})) \to 0.
$$

This means that the family $\mathcal{M}$ is uniformly $\mu$-absolutely continuous.

(iii)$\implies$(iv) Assume that (iii) holds. Then $\mathcal{M} \subset ca_{\mu}(\Sigma, X) \subset ca(\Sigma, X)$ and $\mathcal{M}$ is a uniformly $\xi$-countably additive set in $ca(\Sigma, X)$. Hence by [8, Theorem 7] $\mathcal{M}$ is a relatively compact set in $(ca(\Sigma, X), T_s)$. Since $ca_{\mu}(\Sigma, X)$ is closed in $(ca(\Sigma, X), T_s)$, we obtain that $\mathcal{M}$ is a relatively compact set in $(ca_{\mu}(\Sigma, X), T_s)$.

(iv)$\implies$(i) Assume that $\mathcal{M}$ is a relatively compact set in $(ca_{\mu}(\Sigma, X), T_s)$, and let $(T_m)_{\alpha}$ be a net in $\mathcal{K}_\mathcal{M}$. Without loss of generality, we can assume that $m_{\alpha} \to m$ for $T_s$, where $m \in ca_{\mu}(\Sigma, X)$. We shall show that $T_{m_{\alpha}} \to T_m$ in $(\mathcal{L}_{r, \xi}(L^\infty(\mu), X), T_s)$. Indeed, let $p \in \mathcal{P}_\xi$ and fix $\varepsilon > 0$. Since $\mathcal{M}$ is a $T_s$-bounded subset of $ca_{\mu}(\Sigma, X)$, for each $A \in \Sigma$ we have

$$
sup_{\alpha} p(m_{\alpha}(A)) = sup_{\alpha} q_{p, \xi}(m_{\alpha}) < \infty.
$$

Hence, since the mapping $\Pi_p : X \to X^*_p$ is $(p, \| \cdot \|_p)$-continuous, we obtain that

$$
sup_{\alpha} \| (m_{\alpha}(A)) \|_p = sup_{\alpha} \| (\Pi_p \circ m_{\alpha})(A) \|_p < \infty.
$$

In view of the Nikodym boundedness theorem (see [5, Theorem 1, p. 14]) and 3.1 we get

$$
c = sup_{\alpha} \| T_{m_{\alpha}} \|_p = sup_{\alpha} \| m_{\alpha} \|_p (\Omega) < \infty.
$$

Let $u \in L^\infty(\mu)$ be given and choose $s_0 \in S(\mu)$ such that $\|u - s_0\|_\infty \leq \frac{\varepsilon}{3a}$, where

$$
a = max(c, \| T_m \|_p).$$

Then there exists $a_0$ such that $p(T_{m_{\alpha}}(s_0) - T_m(s_0)) \leq \frac{\varepsilon}{3}$ for $\alpha \geq a_0$. Hence for $\alpha \geq a_0$ we get

$$
p(T_{m_{\alpha}}(u) - T_m(u)) \leq p(T_m(u - s_0)) + p(T_m(s_0) - T_{m_{\alpha}}(s_0)) + p(T_{m_{\alpha}}(s_0) - T_{m_{\alpha}}(u))
$$
Theorem 3.5 Assume that

\[ T_m = \frac{a \cdot \varepsilon}{3a} + \frac{a \cdot \varepsilon}{3a} = \varepsilon. \]

This means that \( T_m \xrightarrow{\alpha} T_m \) in \((L_{\tau, \xi}(L^\infty(\mu), X), T)\), as desired.

Recall that the general Vitali–Hahn–Saks theorem (see [7, Theorem 2.14’]) says that if \((m_k)\) is a sequence of \(\mu\)-absolutely continuous measures on a \(\sigma\)-algebra \(\Sigma\) taking values in a lHs \((X, \xi)\), and \(m(A) := \xi - \lim m_k(A)\) for each \(A \in \Sigma\), then \(m : \Sigma \to X\) is a \(\mu\)-absolutely continuous measure and the family \(\{m_k : k \in \mathbb{N}\}\) is uniformly \(\mu\)-absolutely continuous.

Now we shall state a generalized Vitali–Hahn–Saks theorem for operators from \(L^\infty(\mu)\) to a quasicomplete lHs \((X, \xi)\).

**Theorem 3.5** Assume that \((X, \xi)\) is a quasicomplete lHs. Let \(m_k : \Sigma \to X\) be \(\mu\)-absolutely continuous measures for \(k \in \mathbb{N}\) and assume that \(m(A) := \xi - \lim m_k(A)\) exists for each \(A \in \Sigma\). Then the following statements hold:

1. \(m : \Sigma \to X\) is a \(\mu\)-absolutely continuous measure, and the integration operator \(T_m : L^\infty(\mu) \to X\) is absolutely continuous.
2. \(T_m(u) = \xi - \lim_k T_{m_k}(u)\) for all \(u \in L^\infty(\mu)\).
3. The family \(\{T_{m_k} : k \in \mathbb{N}\}\) is \((\tau(L^\infty(\mu), L^1(\mu)), \xi)\)-equicontinuous.
4. The family \(\{T_{m_k} : k \in \mathbb{N}\}\) is uniformly absolutely continuous.

**Proof** In view of the general Vitali–Hahn–Saks theorem (see [7, Theorem 2.14’]) \(m : \Sigma \to X\) is \(\mu\)-absolutely continuous, and by Proposition 3.1 \(T_m : L^\infty \to X\) is absolutely continuous.

Let \(p \in \mathcal{P}_\xi\) and fix \(\varepsilon > 0\). We show that \(p(T_{m_k}(u) - T_m(u)) \to 0\) for each \(u \in L^\infty(\mu)\). Indeed, since \(p(m_k(A) - m(A)) \to 0\) for all \(A \in \Sigma\), we have

\[ \| \Pi_p (m_k(A) - m(A)) \|_p ^\sim \to 0, \text{ i.e., } \| (m_k)_p(A) - m_p(A) \|_p ^\sim \to 0 \text{ for all } A \in \Sigma. \]

It follows that \(\sup_k \| (m_k)_p(A) \|_p ^\sim < \infty\) for all \(A \in \Sigma\), and in view of the Nikodym boundedness theorem (see [5, Theorem 1, p. 14]) and 3.1 we get

\[ a = \sup_k \| T_{m_k} \|_p = \sup_k \| m_k \|_p (\Omega) < \infty. \]

Let \(u \in L^\infty(\mu)\) be given and choose \(s_0 \in S(\mu)\) such that \(\| u - s_0 \|_{\infty} \leq \frac{\varepsilon}{3a}\), where \(a = \max(c, \| T_m \|_p)\). Then there is \(k_0 \in \mathbb{N}\) such that \(p(T_{m_k}(s_0) - T_m(s_0)) \leq \frac{\varepsilon}{3} \) for \(k \geq k_0\). Hence for \(k \geq k_0\) we have

\[ p(T_{m_k}(u) - T_m(u - s_0)) \leq p(T_m(u - s_0)) + p(T_m(s_0) - T_{m_k}(s_0)) + p(T_{m_k}(s_0) - T_m(u)) \]
\[ \leq \| T_m \|_p \cdot \| u - s_0 \|_{\infty} + p(T_m(s_0) - T_{m_k}(s_0)) + \| T_{m_k} \|_p \cdot \| s_0 - u \|_{\infty} \]
\[ \leq a \cdot \frac{\varepsilon}{3a} + \frac{\varepsilon}{3} + a \cdot \frac{\varepsilon}{3a} = \varepsilon. \]
It follows that $T_{m_k} \to T$ for $T_s$ in $L_{\tau,\xi}(L^\infty(\mu), X)$. Since $\{T_{m_k} : k \in \mathbb{N}\} \cup \{T\}$ is a $T_s$-compact subset of $L_{\tau,\xi}(L^\infty(\mu), X)$, by Theorem 2.5 the set $\{T_{m_k} : k \in \mathbb{N}\}$ is $(\tau(L^\infty(\mu), L^1(\mu)), \xi)$-equicontinuous, and by Corollary 2.6 it is uniformly absolutely continuous.

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References

1. Aliprantis, C.D., Burkinshaw, O.: Positive Operators. Academic Press, New York (1985)
2. Aliprantis, C.D., Burkinshaw, O.: Locally Solid Riesz Spaces with Applications to Economics, 2nd edn. In: Math. Surveys and Monographs, no. 105 (2003)
3. Bourbaki, N.: Elements of Mathematics, Topological Vector Spaces, chaps. 1–5, Springer Verlag, Berlin (1987)
4. Cristescu, R.: Topological Vector Spaces. Editura Academiei, Romania (1977)
5. Diestel, J., Uhl, J.J.: Vector Measures, Amer. Math. Soc., Math. Surveys 15, Providence, RI (1977)
6. Dodds, P.G.: $\sigma$-weakly compact mappings of Riesz spaces. Trans. Amer. Math. Soc. 214, 389–402 (1975)
7. Drewnowski, L.: Decompositions of set functions. Studia Math. 48, 23–48 (1973)
8. Graves, W.H., Ruess, W.: Compactness in spaces of vector-valued measures and a natural Mackey topology for spaces of bounded measurable functions. Contemp. Math. 2, 189–203 (1980)
9. Hoffman-Jorgensen, J.: Vector measures. Math. Scand. 28, 5–32 (1971)
10. Kantorovich, L.V., Akilov, A.V.: Functional Analysis. Pergamon Press, Oxford-Elmsford, New York (1982)
11. Namioka, I.: Partially ordered linear topological spaces. Mem. Amer. Math. Soc. 24 (1957)
12. Orlicz, W.: Operations and linear functionals in spaces of $\varphi$-integrable functions, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8, 563–565 (1960)
13. Orlicz, W., Wnuk, W.: Absolutely continuous and modularly continuous operators defined on spaces of measurable functions. Ricerche di Matematica 15(2), 243–258 (1991)
14. Panchapagesan, T.V.: Applications of a theorem of Grothendieck to vector measures. J. Math. Anal. Appl. 214, 89–101 (1997)
15. Schaefer, H., Zhang, X-D.: On the Vitali–Hahn–Saks theorem, Operator Theory: Advances and Applications, vol. 75, pp. 289–297. Birkhäuser, Basel (1995)
16. Zhang, X-D.: On weak compactness in spaces of measures. J. Funct. Anal. 143, 1–9 (1997)