Topological and symmetry-protected non-Hermitian zero modes have attracted considerable interest in the past few years. Here, it is revealed that they can exhibit an unusual behavior when transitioning between the extended and localized regimes: When weakly coupled to a non-Hermitian reservoir, such a zero mode displays a linearly decreasing amplitude as a function of space, which is not caused by an EP of a Hamiltonian, either of the entire system or the reservoir itself. Instead, this phenomenon is due to the non-Bloch solution of a linear homogeneous recurrence relation, together with the underlying non-Hermitian particle-hole symmetry and the zeroness of its energy.

1. Introduction

Wave localization is one of the most celebrated physical phenomena of the past century.\cite{1–4} The band picture that emerged in the 1920s successfully explained localization of noninteracting or linear waves in a periodic structure,\cite{5} which revealed the distinction between normal metals and insulators. The existence of a sizable bandgap is also crucial for the robustness of topological insulators and Majorana zero modes.\cite{6–10} Take the Su-Schrieffer-Heeger (SSH) Hamiltonian\cite{11} for example, which is the simplest model that offers insight into the topological origin of edge states. As the difference between the two alternate nearest-neighbor couplings reduces, so does the bandgap between its two bands that leads to the divergence of the localization length of its zero mode(s).

A similar transition from localized states to extended states takes place in disordered systems.\cite{2,3} Again, by neglecting interaction and nonlinear effects, Philip Anderson pointed out that disorder can completely suppress diffusion, and all eigenstates of the system become exponentially localized.\cite{12} Nevill Mott also noted that in three dimensions, there exists a mobility edge that separates localized modes and extended modes in terms of their energies.\cite{13} However, unlike the straightforward transition mentioned for the SSH model above, the wavefunctions at the mobility edge display interesting features such as power law localization\cite{14} and fractal dimensionality.\cite{15}

Recently, wave localization behaviors in non-Hermitian systems\cite{16–19} also attracted considerable attention, especially due to the non-Hermitian skin effect.\cite{20–26} In the presence of a spatially uniform non-Hermitian gauge field, a significant percentage of modes in the system are localized toward one edge of the system, which holds in both 1D and higher-dimensional systems. By varying this imaginary gauge field spatially, one can also achieve localization at any target position,\cite{27} including one corner, all corners, or any interior point, as recently demonstrated using a 2D array of optical micro-ring resonators.\cite{28,29} The non-Hermitian skin effect, nevertheless, does not display a nontrivial transition to delocalization: It is exponential in nature, and it vanishes together with the imaginary gauge field, returning the system back to its underlying Hermitian properties.

Another noteworthy development in non-Hermitian systems involved topological zero modes. A zero mode, as the name suggests, is a zero-energy state that is often the result of symmetry protection. This zero energy in low-energy physics is typically set to a well-defined level, such as the Fermi level in condensed matter systems, or in the case of systems with coupled elements, the energy of a particular resonance of interest in one element. Different from their Hermitian counterparts, these non-Hermitian zero modes can live on the imaginary axis of the complex energy plane and be attributed to either non-Hermitian particle-hole (NHPH) symmetry,\cite{30–34} anti-PT symmetry,\cite{35,36} or anti-pseudo-Hermiticity,\cite{37} whether or not the bulk is periodic in construction.\cite{38} Recent experiments in photonics\cite{39–45} and other related fields demonstrated such exotic states using spatially arranged non-Hermitian elements, using for example, evanescently coupled optical waveguides\cite{46–48} and microcavities.\cite{49–51} The localization property of these non-Hermitian zero modes inside the bandgap of the hosting bulk
is well known, but related studies in a non-Hermitian reservoir have largely been lacking, especially when the reservoir does not have a localized mode itself.

In this work, we probe whether a symmetry and topologically protected zero mode exhibits exotic behaviors in a non-Hermitian reservoir. Remarkably, we observe a “linear localization” phenomenon when a zero mode transitions between localized and extended regimes in the reservoir: If weakly coupled to this reservoir, modeled by a lattice with gain and loss modulation, a zero mode can display a linearly decreasing amplitude as a function of space in the reservoir. This linear tail is clearly not the diminishing localization limit of an exponential function; a vanishing exponent in |ψ(x)|α x−β indicates an infinite localization length ξ and a uniform amplitude in space. Furthermore, a power law decay |ψ(x)|α x−β with β < 1/2 is often discarded in the study of wave localization, because such a wave function cannot be normalized. However, as we will show below, the wave function with linear localization terminates at the far end of the finite-sized non-Hermitian reservoir, and hence the lack of normalizability does not prevent it from being physical and observable. We also stress that linear localization we identify occurs away from exceptional points (EPs) of the entire lattice. It is also manifested in an eigenstate of the whole system, unlike the polynomial behavior exhibited by a Jordan chain vector in the asymptotic limit of a long time or spatial propagation.

Below we first discuss briefly NHPH symmetry and the resulting non-Hermitian zero modes in our combined lattice. We then apply a first-order perturbation theory to verify the linearly localized zero mode inside the reservoir. By utilizing the zero-energy property, we further derive a linear homogeneous recurrence relation

\[ \Psi_n = 2a \Psi_{n-2} - \Psi_{n-4} \] (1)

for the zero-mode wave function inside the reservoir, where \( a \in \mathbb{R} \) is a constant independent of the lattice site index \( n \). Despite the periodic construction of the reservoir, the characteristic equation of this recurrence relation allows a non-Bloch solution when it has a single root of multiplicity 2, which we show is responsible for linear localization.

### 2. Results

#### 2.1. Model

A symmetry-protected zero mode in a Hermitian system is often found when the spectrum of the system is symmetric, satisfying \( \omega_p = -\omega_p \). A zero mode then emerges when the two mode indices \( \mu, \nu \) are identical, resulting in \( \omega_p = 0 \). Such a symmetric spectrum can be the result of chiral symmetry or particle-hole symmetry. The spectrum of a non-Hermitian system, on the other hand, is generally complex, and hence it is necessary to restore the complex conjugation in the manifestation of particle-hole symmetry, i.e., \( \omega_p = -\omega_p^* \), which are symmetric about the imaginary axis. This relation leads to the definition of a non-Hermitian zero mode, i.e., \( \text{Re}[\omega_p] = 0 \) when \( \mu = \nu \), which was shown to be a general property of many gain and loss modulated lattices (see also Note S1, Supporting Information).

The “system” we consider below is a Hermitian SSH chain [left half in Figure 1a] with real on-site potential \( \omega_0 \):

\[ H_S = \sum_n \omega_0 |n⟩⟨n| + (t_n |n⟩⟨n + 1| + h.c.) \] (2)

here the nearest-neighbor (NN) coupling \( t_n \) is given by \( t_\lambda (t_\beta) \) when \( n \) is odd (even). This system is coupled from the right via the NN coupling \( t^* \) to a non-Hermitian “reservoir” [right half in Figure 1a]:

\[ H_R = \sum_n \omega_n |n⟩⟨n| + (t|n⟩⟨n + 1| + h.c.) \] (3)

t is its NN coupling, and \( \omega_n \) is the complex on-site potential, given by \( \omega_n = \omega_0 - i \gamma (\omega_n = \omega_0 + i \gamma) \) when \( n \) is odd (even) representing loss (gain) with \( \gamma > 0 \). Note that a tight-binding model for a finite lattice does not require an explicit boundary condition; by definition, this lattice is isolated from its environment except for the assumed gain and loss (e.g., \( \gamma \) in our case). Due to the challenge of fabricating and coupling many nearly identical elements in micro- and nano-phononics, here we consider a small number of lattice sites, i.e., 9 in the system and 10 in the reservoir. Note that the entire lattice does not have parity-time symmetry, as reflected by the absence of up and down symmetry in Figure 2a. This property is independent of the number of lattice sites.

#### 2.2. Non-Hermitian Zero Modes and Linear Localization

Henceforth, we take \( \omega_0 = 0 \) as the zero energy. It is well known that topological and (chiral) symmetry-protected zero mode(s) exist in the SSH model, and in our case it is exponentially localized on the right side of the system when \( t_\lambda > t_\beta \). When coupled to the non-Hermitian reservoir, chiral symmetry of the SSH chain is broken, but non-Hermitian particle-hole symmetry still holds for the combined lattice. It pins the original Hermitian zero mode on the imaginary axis [mode 1 in Figure 2a]. As the gain and loss coefficient \( \gamma \) increases, more non-Hermitian zero modes are generated (modes 2 to 11): A pair of non-zero modes satisfying \( \omega_p = -\omega_p^* \) can coalesce at their EP on the imaginary axis, then repel each other along the imaginary axis (see Figure S1, Supporting Information). This process takes place five times in this combined lattice, giving rise to five EPs [see red squares in Figure 2a].

While all these eleven non-Hermitian zero modes are similarly localized in the system half [as indicated by the inverse participation ratio IPR = \( \langle |\Psi|^2 \rangle / \sum |\Psi|^4 \) in the range [1,1.1]; see Figure 2c], there is a clear lineage across the odd-numbered ones with an IPR ≈ 1.08. This phenomenon is a non-Hermitian analogy of avoided crossings in Hermitian systems but now on the imaginary axis [see Figure 2a], through which a non-Hermitian zero mode passes its identity to the next mode, including its Im[\( \omega_n \)] and IPR in the system.

By following this lineage, we find that the tails of the odd-numbered zero modes inside the non-Hermitian reservoir experience a transition from delocalized [Figure 1b] to localized [Figure 1d] when \( \gamma \) increases, also verified by the IPR calculated in the reservoir [Figure 2d]. During this transition, we observe a linear tail in mode 11 at \( \gamma \approx 2t \) [Figure 1c], where not just \( \text{Re}[\omega_p] \)
= 0 (which defines a non-Hermitian zero mode) but also \( \text{Im}[\omega]\) is approximately zero \( (\omega/\tau = 0.0356i) \). Its tail in the reservoir has an \( R^2 \) value of 0.9997, close to being perfectly linear. This qualitative change of the wave function is confirmed using a first-order perturbation theory (see solid lines in Figure 1), where we treat the weak coupling \( t' \) as the small perturbation parameter (see Methods Section). In addition, we also find that the correction to the zero-mode energy is zero, which is qualitatively correct and shines the light on our later analysis using the recurrence relation (1).

#### 2.3. Origin of Linear Localization

To probe whether this linear tail is related to an EP of a Hamiltonian, we first note that the critical value \( \gamma = 2t \) does not produce an EP of the combined lattice; the latter, as Figure 2a shows, all take place in \( \gamma < 2t \), and hence their corresponding wave functions are delocalized in the reservoir (see Figure S1, Supporting Information). Furthermore, if we treat the reservoir as periodic, its Bloch Hamiltonian

\[
H_k = \begin{pmatrix}
  i\gamma & t(1 + e^{i\gamma}) \\
  t(1 + e^{-i\gamma}) & -i\gamma
\end{pmatrix}
\]

has two bands \( \omega_k(k) = \pm \sqrt{2t^2(1 + \cos k) - \gamma^2} \), where \( k \in [-\pi, \pi] \) is the momentum. These two bands exhibit an EP ring in the \( k-\gamma \) parameter space (see Figure S2, Supporting Information), at \( \omega = 0 \) and given by \( k = \pm \cos^{-1}[\gamma^2/(2t^2) - 1] \). At \( \gamma = 2t \), this EP ring gives an EP at \( k = 0 \), where \( H_k \) becomes the Hamiltonian of the classical parity-time symmetric dimer \(^{17}\); its coalesced eigenstate \( \psi = [i, 1]^T \) with \( \omega_k(0) = 0 \) or the corresponding Jordan vector \( J = [0, i]^T \) (defined by \( H_k J = \psi \)) does not imply a linear tail, either. Finally, the eigenstates of the finite-sized reservoir do not display a linear tail at this critical value of \( \gamma \). The corresponding eigenvalues are not EPS either, with the ones closest to 0 at \( \omega/\tau = \pm 0.563 \); their wave functions are best approximated by sine functions of a half period (see Figure S3, Supporting Information).

Having excluded the cause of linear localization from an EP of a Hamiltonian, either of the combined lattice or the reservoir itself, we focus on the recurrence relation (1). To derive this relation, we first rewrite Equation (3) in the non-Hermitian reservoir as

\[
\Psi_n = i \frac{\kappa_{A,B}}{t} \psi_{n-1} - \psi_{n-2}
\]

for an eigenstate, where \( \kappa_{A,B} \equiv -i(\omega - \omega_{A,B}) \). For a non-Hermitian zero mode we have \( \text{Re}[\omega] = 0 \), and hence \( \kappa_{A,B} = \text{Im}[\omega] \pm \gamma \) are real and they represent the effective loss and gain coefficients on the \( A \) and \( B \) sublattices in the reservoir. Next, by applying Equation (5) to five consecutive lattice sites and eliminating \( \Psi_{n-1}, \Psi_{n-2} \), we arrive at Equation (1) with

\[
\alpha = -\left(1 + \frac{\kappa_A \kappa_B}{2t^2}\right) = -\left(1 + \frac{\text{Im}[\omega]^{-2}}{2t^2}\right) \in \mathbb{R}
\]
Besides the trivial solution \( \Psi_n = 0 \) on all lattice sites, the recurrence relation (1) also permits the Bloch solution

\[
\Psi_n = \beta_1 b_1^n + \beta_2 b_2^n
\]  

(7)

where \( \beta_1, \beta_2 \) are two constants and the integer \( m \) is the index on a sublattice, i.e., \( m = (n+1)/2 \) on the \( A \) sublattice (where \( n \) is odd) and \( m = n/2 \) on the \( B \) sublattice (where \( n \) is even). \( b_\pm \) are the two roots of the characteristic equation

\[
b^2 - 2ab + 1 = 0
\]

(8)
of Equation (1), or more explicitly, \( b_\pm = \alpha \pm i\sqrt{1 - \alpha^2} \equiv e^{\pm i\theta} \) where \( \theta = \cos^{-1} \alpha \).

When \( |\alpha| < 1 \), \( \theta \) is real and we have \( \Psi_n = (\beta_1 + \beta_2)\cos m\theta + i (\beta_1 - \beta_2)\sin m\theta \), leading to a \( |\Psi_n| \) oscillating between the upper bound \( |\beta_1| + |\beta_2| \) and the lower bound \( ||\beta_1| - |\beta_2|| \) as \( n \) (and \( m \)) varies. In other words, the non-Hermitian zero modes are delocalized in the reservoir, as we have seen in Figure 1b where \( \alpha = -0.875 \). When \( |\alpha| > 1 \), \( \theta \) is imaginary and \( \Psi_n \) is the linear superposition of two exponential functions, with one decaying toward the left and the other toward the right. The latter then leads to the localization in the reservoir, as we have shown in Figure 1d where \( \alpha = 3.5 \).

The critical behavior takes place at \( \alpha = \pm 1 \), where the characteristic equation (8) has a single root \( b = \alpha \) of multiplicity 2. Besides the (now) single solution given by equation (7), i.e., \( \Psi_n \propto a^n \), there also exists a non-Bloch solution proportional to \( m\alpha^n \) with which we write the general solution as

\[
\Psi_n = a^n (\rho_1 + \rho_2 m) \quad \rho_{1,2} = \text{const}
\]

(9)
The term linear in \( m \) can then potentially lead to linear localization if \( \rho_2 \neq 0 \). We note though \( \alpha = -1 \) does not cause linear localization, because it requires one or both of \( \kappa_{A, B} \) to be zero, and Equation (5) tells us immediately that \( |\Psi_n| \) is a constant on at least one sublattice and hence delocalized. Therefore, the linear localization shown in Figure 1c is the result of \( \alpha = 1 \), which is achieved in mode 11 at \( \gamma \approx 2t \) with \( \text{Im}[\omega_n] \approx 0 \) [Figure 2b].

To determine \( \rho_{1,2} \) in Equation (9), i.e., the offset and slope of the linear tail, we study the recurrence relation at both ends of the reservoir for \( \gamma = 2t \) and \( \text{Im}[\omega_n] = 0 \). For convenience, we relabel the rightmost site as 1 and the leftmost site in the reservoir as \( N_R \) (10 in our case). By revisiting Equation (5) for the last three sites on the right (i.e., \( \Psi_j = -2i \Psi_j - \Psi_{j+1} \)) and that for just the last two (i.e., \( \Psi_2 = 2i \Psi_1 \)), we find \( \rho_1 = 0 \) (i.e., no offset of the linear tail) and

\[
\Psi_n = \begin{cases} 
\rho_2 n/2, & (n \text{ odd}) \\
\rho_2 n/2, & (n \text{ even}) 
\end{cases}
\]

(10)
inside the reservoir. Similarly, by studying the equation for \( \Psi_{N_{R}} = i \rho_{2} N_{R}/2 \) that couples to the rightmost site in the system with a normalized wave function \( \Psi_{0} = e^{i\phi} (\phi \in \mathbb{R}) \), i.e.,

\[
i \partial_{t} \Psi_{N_{R}} = 0 = i\gamma(\rho_{2} N_{R}/2) + t' e^{i\phi} + t \rho_{2}(N_{R} - 1)/2
\]

we find \( \phi = 0 \) or \( \pi \) and

\[
\rho_{2} = \pm \frac{2}{1 + N_{R}} t'
\]  

(11)

In other words, the slope of the linear tail is proportional to the relative strength of the system-reservoir coupling \( t' \) over that in the reservoir (t). It is also inversely proportional to the size of the reservoir in the large \( N_{R} \) limit, where we find that \( |\Psi_{N_{R}}| = |\rho_{2} N_{R}/2 \) is fixed at \( t'/t \), and the linear tail simply stretches as the reservoir becomes longer until the effect of the EP between modes 10 and 11 in Figure 2 becomes significant (see Figure S4, Supporting Information and the discussion at the end of the next subsection).

### 2.4. Strong Coupling Regime

The results above for \( \sigma = 1 \), again, are only valid if we also have \( \text{Im}[\omega_{0}] = 0 \) at \( \gamma = 2t \) for the non-Hermitian zero mode. As we have mentioned, the prediction \( \text{Re}[\omega_{0}] = \text{Im}[\omega_{0}] = 0 \) from the perturbation theory holds only when the system-reservoir coupling \( t' \) is weak. When \( t' \) is strong and comparable to \( t \), then \( \text{Im}[\omega_{0}] \approx 0 \) breaks down (and even \( \text{Re}[\omega_{0}] = 0 \) breaks down), and linear localization disappears.

As an example, we show in Figures 3a,b the evolution of two modes near \( \omega = 0 \) as a function of \( t' \) with a fixed \( \gamma = 2t \). When \( t' = 0 \), i.e., the system and the reservoir are decoupled, mode I is the original Hermitian zero mode of the SSH chain localized at site 9, and mode II is the non-Hermitian zero mode of the reservoir mentioned before in Figure S3 (Supporting Information), with its wave function given approximately by sine functions of a half period. They move along the imaginary axis and coalesce at an EP when \( t'/t \approx 0.547 \), after which they move off the imaginary axis and are no longer non-Hermitian zero modes. Figure 3c shows the identical spatial profile of modes I and II at \( t'/t = 0.6 \), and clearly they do not display a linear tail in the reservoir.

We note that unlike the case of varying \( \gamma \) shown in Figure 2a, there is no succession of EPs on the imaginary axis as \( t' \) increases. In fact, the EP mentioned above is the only one that exists in \( t' > 0 \) (the situation in \( t' < 0 \) is the same as the sign of \( t' \) does not change the eigenvalues of the modes). This observation can be attributed to the fact that at \( t' = 0 \), all modes of the system are on the real axis while those of the reservoir are on the imaginary axis. They have minute couplings except between modes I and II discussed above, and this pair are simply given by \( \omega_{0} = \pm t' + iy/2 \) in the large \( t' \) limit.

Nevertheless, we can still find a proper \( \gamma \) that realizes \( \sigma = 1 \) but away from \( \gamma = 2t \) with a finite \( \text{Im}[\omega_{0}] \). In this case, Equation (9)
only indicates that the wave function of the non-Hermitian zero mode is linear on each of the two sublattices (given by even and odd numbered sites in the reservoir), because the recurrence condition at the right boundary that leads to Equation (10) is now different. An example is given in Figure 3d, where \( \alpha = 1 \) is achieved at \( t' = 0.6t \) with \( \gamma = 2.036t \). Such zigzag linear localization can also be found when the reservoir itself is a non-Hermitian SSH chain (see Figure S5, Supporting Information).

We also note that although \( t' = 0.2t \) is weak compared to other relevant energy scales with the reservoir shown in Figure 1, it becomes strong effectively near an EP. This is similar to what happens near an avoided crossing in quantum mechanics, where the wave functions of two coupled modes become highly hybridized. The same situation near an EP can also destroy linear localization. For example, as the reservoir becomes much longer, the EP between modes 10 and 11 shown in Figure 2 approaches \( \gamma = 2t \), and the system becomes highly sensitive to the value of \( \gamma \). At exactly \( \gamma = 2t \), the spatial profiles of modes 10 and 11 are both hybridized of the original edge state in the system and the sine-like mode in the reservoir (see Figure S4, Supporting Information). The approximation \( \text{Im}[\omega_n] \approx 0 \), while still true (e.g., \( \text{Im}[\omega_n] = 0.0415 \) with \( N = 134 \)), no longer justifies Equation (10) due to the enhanced sensitivity near the EP. However, if we slightly change \( \gamma/t \) to \( 3.78 \times 10^{-4} + 2 \), we find \( \alpha = 1 \) and achieve zigzag linear localization (see Figure S4, Supporting Information). In this case, the two linear tails on the two sublattices are well aligned with each other, giving rise to an \( R^2 \) value of 0.9996.

2.5. Photonic Simulations

The tight-binding model, similar to those given by Equations (2) and (3) [see also Methods Section], has been employed successfully to capture various phenomena in coupled waveguide systems. Some references include, for example, the experimental demonstration of parity-time symmetric photonics,[47] the observation of supersymmetric mode converters,[48] and the creation of a non-Hermitian zero-mode.[43]

To showcase the validity of the tight-binding model in our study, i.e., linear localization can be demonstrated on a realistic photonic platform, we consider in Figure 4 a evanescently coupled InP waveguides.[57,58] The electric field propagating...
4. Methods Section

Hamiltonian of the Entire Lattice. The Hamiltonian $H_\mathrm{s}$ of the Hermitian “system” and the effective Hamiltonian $H_\mathrm{e}$ of the non-Hermitian “reservoir” are specified using Equations (2) and (3) in their second quantization forms. The corresponding matrix forms are given by

$$H_\mathrm{s} = \begin{pmatrix} \begin{pmatrix} a_0 & t_B \\ t_B & a_0 \end{pmatrix} & \begin{pmatrix} t_B & a_0 \\ a_0 & t_B \end{pmatrix} & \cdots \\ \begin{pmatrix} a_0 & t_B \\ t_B & a_0 \end{pmatrix} & \begin{pmatrix} t_B & a_0 \\ a_0 & t_B \end{pmatrix} & \cdots \\ \cdots & \cdots & \cdots \end{pmatrix}$$

and the effective Hamiltonian of the entire lattice can then be expressed as:

$$H = H_\mathrm{s} + t' \lambda (\gamma H_\mathrm{e} + H_\mathrm{e})$$

Perturbation theory. In Figure 1, the results of the tight-binding model are verified using a first-order perturbation theory. Here, we first outline this approach. We treat the weak coupling $t'$ between the system and the reservoir as the small perturbation parameter, and we denote $H_\mathrm{p} = H_\mathrm{s} + H_\mathrm{e}$ ($\gamma$) as the unperturbed (and uncoupled) Hamiltonian of the system ($H_\mathrm{s}$) and the reservoir ($H_\mathrm{e}$). In the coupled system, the Hamiltonian becomes

$$H = H_\mathrm{s} + t' H'$$

where $H'$ contains only two nonzero elements in its matrix form, which are 1’s coupling the rightmost lattice site in the system and the leftmost site in the reservoir [see Equation (14)]. The right and left eigenstates of $H_\mathrm{p}$ are denoted by $|\psi_\mu^{(0)}\rangle$ and $\langle \psi_\nu^{(0)}|$ and they satisfy the biorthogonal relation

$$\langle \psi_\nu^{(0)}|\psi_\mu^{(0)}\rangle = \delta_{\mu\nu}$$

away from an EP of $H_\mathrm{e}$. Because $H_\mathrm{e}$ is symmetric, i.e., $H_\mathrm{e} = H_\mathrm{e}^\dagger$, we find $\psi_\mu^{(0)} = \psi_\mu^{(0)}$ for all modes.

Note that because there is no coupling between the system and the reservoir in $H_\mathrm{p}$, $|\psi_\mu^{(0)}\rangle$, $\langle \psi_\nu^{(0)}|$ are both zero either in the system or the reservoir. There are then two consequences. First, the biorthogonal relation above is nontrivial only when both modes $\mu$ and $v$ belong to the system or the reservoir simultaneously. Second, an EP of $H_\mathrm{e}$ must be an EP of either $H_\mathrm{s}$ or $H_\mathrm{e}$. Clearly, there is no EP in the (Hermitian) SSH part of the Hamiltonian, and we show in Figure S3 (Supporting Information) that there is no EP in a finite-sized reservoir either when linear localization takes place (i.e., at $\gamma \approx 2t$). Therefore, we can apply Equation (16) in the perturbation theory to study linear localization.
Similar to the perturbation theory in (Hermitian) quantum mechanics, the first-order correction to the energy is given by

$$\omega^{(1)}_{\mu} = t' \mathcal{S} \langle \phi^{(0)}_{\mu} | H' | \phi^{(0)}_{\mu} \rangle \equiv t' H'_{\mu\mu}$$  \hspace{1cm} (17)

Due to the spatial properties of $| \psi^{(0)}_{\mu} \rangle$, $| \phi^{(0)}_{\mu} \rangle$ mentioned above and the structure of $H'$, it is straightforward to show that $H'_{\mu\mu}$ is zero if both $| \psi^{(0)}_{\mu} \rangle$, $| \phi^{(0)}_{\mu} \rangle$ are in the system or reservoir. As a result, $H'_{\mu\mu}$ (and hence $\omega^{(1)}_{\mu}$) are zero for all eigenstates of $H$, i.e., the first-order correction to the energy of the original SSH zero mode is zero, which we have used in the derivation of Equation (10) in the main text.

In the meanwhile, the first-order correction to the wave function is given by

$$| \psi^{(1)}_{\mu} \rangle = t' \sum_{\nu, \nu' \neq \mu} \frac{H'_{\mu\nu}}{\omega^{(0)}_{\mu} - \omega^{(0)}_{\nu}} | \psi^{(0)}_{\nu} \rangle$$  \hspace{1cm} (18)

which is plotted in Figure 1 of the main text for different values of $\gamma$. We note that this correction does not change the zero-mode wave function in the system; it only changes the latter in the reservoir, again due to the property of $H'_{\mu\mu}$ mentioned above.

The result for the second-order correction to the energy is shown in Figure 3b. It displays the characteristic quadratic dependence on the perturbation parameter (i.e., $t'$), which is given by

$$\omega^{(2)}_{\mu} = t'^2 \sum_{\nu, \nu' \neq \mu} \frac{H'_{\mu\nu} H'_{\nu\mu}}{\omega^{(0)}_{\mu} - \omega^{(0)}_{\nu}}$$  \hspace{1cm} (19)

Note that it does not take the absolute value of the numerator, which differs from the (Hermitian) perturbation theory in quantum mechanics. This feature is required to perturb the energy to become complex, even with real-valued $\omega^{(0)}_{\mu\nu}$ as we have here.

Acknowledgements

The authors thank Vadim Oganessian, Sarang Gopalakrishnan, Bo Zhen, Steven Johnson, and Aaron Welters for helpful discussions. This project was supported by the NSF under Grant No. PHY-1847240. A brief summary of an early version was included in the submission of an unpublished poster.\(^{62}\)

Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Keywords

linear localization, Non-Hermitian photonics, optical micro-cavities, optical waveguides, topological edge states

[1] M. Imada, A. Fujimori, Y. Tokura, Rev. Mod. Phys. 1998, 70, 1039.
[2] P. A. Lee, T. V. Ramakrishnan, Rev. Mod. Phys. 1985, 57, 287.
[3] A. Lagendijk, B. van Tiggelen, D. S. Wiersma, Phys. Today 2009, 62, 24.
[4] B. Kramer, A. Mackinnon, Rep. Prog. Phys. 1993, 56, 1469.
[5] F. Bloch, Z. Phys. 1929, 57, 545.
[6] C. Nayak, S. H. Simon, A. Stern, M. Freedman, S. Das Sarma, Rev. Mod. Phys. 2008, 80, 1083.
[7] M. Z. Hasan, C. L. Kane, Rev. Mod. Phys. 2010, 82, 3045.
[8] X.-L. Qi, S.-C. Zhang, Rev. Mod. Phys. 2011, 83, 1057.
[9] J. Alicea, Rep. Prog. Phys. 2012, 75, 076501.
[10] C. W. J. Beenakker, Rev. Mod. Phys. 2015, 87, 1037.
[11] W. P. Su, J. R. Schrieffer, A. J. Heeger, Phys. Rev. Lett. 1979, 42, 1698.
[12] P. W. Anderson, Phys. Rev. 1958, 109, 1492.
[13] N. F. Mott, Adv. Phys. 1967, 16, 49.
[14] I. Varga, J. Pipek, B. Vásári, Phys. Rev. B 1992, 46, 4978.
[15] H. Aoki, Phys. Rev. B 1986, 33, 7370.
[16] Y. V. Konotop, J. Yang, D. A. Zysyulin, Rev. Mod. Phys. 2016, 88, 035002.
[17] L. Feng, R. El-Ganainy, L. Ge, Nat. Photonics 2017, 11, 752.
[18] R. El-Ganainy, K. G. Makris, M. Khajavikhan, Z. H. Musslimani, S. Rotter, D. N. Christodoulides, Nat. Phys. 2018, 14, 11.
[19] Ş. K. Özdemir, S. Rotter, F. Nori, L. Yang, Nat. Mater. 2019, 18, 783.
[20] N. Hatio, D. R. Nelson, Phys. Rev. Lett. 1996, 77, 570.
[21] S. Longhi, D. Gatti, G. D. Valle, Sci. Reports 2015, 5, 13376.
[22] F. Song, S. Yao, Z. Wang, Phys. Rev. Lett. 2019, 123, 170401.
[23] S. Weidemann, M. Kremer, T. Helbig, T. Hofmann, A. Stegmaier, M. Greiter, R. Thomale, A. Szameit, Science 2020, 368, 311.
[24] L. Li, C. H. Lee, J. Gong, Phys. Rev. Lett. 2020, 124, 250402.
[25] L. Zhang, Y. Yang, Y. Ge, Y.-J. Guan, Q. Chen, Q. Fan, F. Chen, R. Xi, Y.-D. Jia, S.-Q. Yuan, H.-X. Sun, H. Chen, B. Zhang, Nat. Commun. 2021, 12, 6297.
[26] W. Wang, X. Wang, G. Ma, Nature 2022, 608, 7921.
[27] J. D. H. Rivero, L. Ge, Phys. Rev. B 2021, 103, 041111.
[28] Z. Zhang, X. Qiao, B. Midya, K. Liu, J. Sun, T. Wu, W. Liu, R. Agarwal, J. Miquel Jornet, S. Longhi, N. M. Litchinitser, L. Feng, Science 2020, 368, 760.
[29] Z. Gao, X. Qiao, M. Pan, S. Wu, J. Yim, K. Chen, B. Midya, L. Ge, L. Feng, Phys. Rev. Lett. 2023, 130, 261301.
[30] B. Qi, L. Zhang, L. Ge, Phys. Rev. Lett. 2018, 120, 093901.
[31] L. Ge, Photon. Res. 2018, 6, A10.
[32] K. Kawabata, S. Higashikawa, Z. Gong, Y. Ashida, M. Ueda, Nat. Commun. 2019, 10, 1.
[33] R. Okugawa, T. Yokoyama, Phys. Rev. B 2019, 99, 041202.
[34] Y.-J. Wu, J. Hou, Phys. Rev. A 2019, 99, 062107.
[35] L. Ge, H. E. Türeci, Phys. Rev. A 2013, 88, 053810.
[36] F. Zhang, Y. Feng, X. Chen, L. Ge, W. Wan, Phys. Rev. Lett. 2020, 124, 053901.
[37] G. Scolarici, J. Phys. A: Math. Gen. 2002, 35, 7493.
[38] L. Ge, Phys. Rev. A 2017, 95, 023812.
[39] F. Baboux, L. Ge, T. Jacqumin, M. Biondi, E. Galopin, A. Lemaitre, L. Le Gratiet, I. Sagnes, S. Schmidt, H. E. Türeci, A. Amo, J. Bloch, Phys. Rev. Lett. 2016, 116, 066402.
[40] C. Poli, M. Bellec, U. Kuhl, F. Mortessagne, H. Schomerus, Nat. Commun. 2015, 6, 7710.
[41] P. St-Jean, V. Goblot, E. Galopin, A. Lemaitre, T. Ozawa, L. Le Gratiet, I. Sagnes, J. Bloch, A. Amo, Nat. Photonics 2017, 11, 651.
[42] M. A. Bandres, S. Wittek, G. Harari, M. Parto, J. Ren, M. Segev, D. N. Christodoulides, M. Khajavikhan, Science 2018, 359, eaar4005.
[43] M. Pan, H. Zhao, P. Miao, S. Longhi, L. Feng, Nat. Commun. 2018, 9, 1308.
[44] H. Zhao, P. Miao, M. H. Teimourpour, S. Malzard, R. El-Ganainy, H. Schomerus, L. Feng, Nat. Commun. 2018, 9, 981.
[45] M. Parto, S. Wittek, H. Hodaei, G. Harari, M. A. Bandres, J. Ren, M. C. Rechtsman, M. Segev, D. N. Christodoulides, M. Khajavikhan, Phys. Rev. Lett. 2018, 120, 113901.
[46] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, D. Kip, Nat. Phys. 2010, 6, 192.
[47] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, D. N. Christodoulides, Phys. Rev. Lett. 2009, 103, 039302.
[48] M. Heinrich, M.-A. Miri, S. Stützer, R. El-Ganainy, S. Nolte, A. Szameit, D. N. Christodoulides, Nat. Commun. 2014, 5, 3698.
[49] B. Peng, Ş. K. Özdemir, F. Lei, F. Monifi, M. Gianfreda, G. L. Long, S. Fan, F. Nori, C. M. Bender, L. Yang, Nat. Phys. 2014, 10, 394.
[50] L. Chang, X. Jiang, S. Hua, C. Yang, J. Wen, L. Jiang, G. Li, G. Wang, M. Xiao, Nat. Photon. 2014, 8, 524.
[51] H. Hodaei, A. U. Hassan, S. Wittek, H. Garcia-Gracia, R. El-Ganainy, D. N. Christodoulides, M. Khajavikhan, Nature 2017, 558, 187.
[52] G. Corrielli, G. D. Valle, A. Crespi, R. Osellame, S. Longhi, Phys. Rev. Lett. 2013, 111, 220403.
[53] M.-A. Miri, A. Alú, Science 2019, 363, 42.
[54] S. Longhi, G. Della Valle, Phys. Rev. A 2014, 89, 052132.
[55] M. C. Zheng, D. N. Christodoulides, R. Fleischmann, T. Kottos, Phys. Rev. A 2010, 82, 010103.
[56] K. H. Rosen, Discrete Mathematics and its Applications, 7th ed. McGraw-Hill, New York 2012.
[57] P. Miao, Z. Zhang, J. Sun, W. Walasik, S. Longhi, N. M. Litchinitser, L. Feng, Science 2016, 353, 464.
[58] Z. J. Wong, Y.-L. Xu, J. Kim, K. O’Brien, Y. Wang, L. Feng, X. Zhang, Nat. Photon. 2016, 10, 796.
[59] S. Klaiman, U. Günther, N. Moiseyev, Phys. Rev. Lett. 2008, 101, 080402.
[60] http://ab-initio.mit.edu/wiki/index.php/Main_Page (accessed: March 2018).
[61] L. Feng, Z. J. Wong, R.-M. Ma, Y. Wang, X. Zhang, Science 2014, 346, 972.
[62] B. Qi, L. Ge, in Frontiers in Optics/Laser Science 2018, JW4A.41.