Generalized Uncertainty Principle and the Conformally Coupled Scalar Field Quantum Cosmology

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We exactly solve the Wheeler-DeWitt equation for the closed homogeneous and isotropic quantum cosmology in the presence of a conformally coupled scalar field and in the context of the generalized uncertainty principle. This form of generalized uncertainty principle is motivated by the black hole physics and it predicts a minimal length uncertainty proportional to the Planck length. We construct wave packets in momentum minisuperspace which closely follow classical trajectories and strongly peak on them upon choosing appropriate initial conditions. Moreover, based on the DeWitt criterion, we obtain wave packets that exhibit singularity-free behavior.

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I. INTRODUCTION

In recent years, many investigations have been devoted to study the existence of a minimal length uncertainty in the context of the Generalized (Gravitational) Uncertainty Principle (GUP)\(^\dagger\). The notion of a fundamental limit to the resolution of structure is motivated by thought experiments as well as various proposals of quantum gravity such as string theory, loop quantum gravity, noncommutative geometry, and black hole physics. It represents an approach to effectively consider quantum gravitational aspects that cannot be treated accurately yet. This minimal length is usually considered to be proportional to the Planck length \(\ell_P \approx 10^{-35}\) m. However, it should be noted that as shown by Yoneya\(^\star\), the generalized uncertainty principle is not generally valid in string theory. In particular, the GUP introduced below is, in fact, incompatible with string theory, and there are other approaches to quantum gravity where a minimal length is not required.

Several testing schemes have been established to study the effects of the quantum gravity ranging from tabletop experiments\(^\ddagger\) to astronomical observations\(^\ddagger\)\(^\ddagger\). Some proposed trials to explain the puzzling observations of ultrahigh energy cosmic rays in the framework of quantum gravity for particle propagation\(^\ddagger\) to develop high intensity laser projects for quantum gravity phenomenology in the context of deformed special relativity\(^\ddagger\). Recently, a direct measurement method to experimentally test the existence of the fundamental minimal length scale was proposed using a quantum optical ancillary system\(^\ddagger\). This method is within reach of current technology and it is based on the detection of the possible deviations from ordinary quantum commutation relation at the Planck scale.

The interest in deformed commutation relations started from the seminal paper by Snyder in the relativistic framework\(^\ddagger\) and followed by others to investigate the effects of the minimal length on quantum mechanical and classical systems. Various problems such as harmonic oscillator with minimal uncertainty in position\(^\ddagger\)\(^\ddagger\) or in position and momentum\(^\ddagger\)\(^\ddagger\)\(^\ddagger\), Dirac oscillator\(^\ddagger\), Coulomb potential\(^\ddagger\),\(^\ddagger\), singular inverse square potential\(^\ddagger\), ultracold neutrons in gravitational field\(^\ddagger\), Lamb’s shift, Landau levels, tunneling current in scanning tunneling microscope\(^\ddagger\), cosmological problems\(^\ddagger\)\(^\ddagger\), and Casimir effect\(^\ddagger\) have been treated exactly or perturbatively in the quantum domain. On the other hand, the classical limit of the minimal length uncertainty relation, Keplerian orbits, thermostatistics, deformations of the classical systems in the phase space, and composite systems have been studied at the classical level\(^\ddagger\).

As stated above, the generalized uncertainty principle is usually interpreted as an effective description of features of a fundamental quantum theory of gravity. However, it is also possible to consider the GUP as a fundamental description of nature by itself. In this case, not only the position and momentum of a particle as variables of quantum mechanics obey such a generalized uncertainty principle, but also other variables should be quantized using a generalized quantization rule. Canonical quantum gravity as an important approach to quantize the general relativity is based on the canonical commutation relations. Similarly, in the framework of quantum field theory the canonical commutation relation between the position and momentum operators is assigned to the field and its canonical conjugated quantity. If we consider the GUP in quantum mechanics as a fundamental characterization of nature, then the quantum gravity and quantum field theory should be treated in the same way. In this direction, the quantization of gravitational fields based on the generalized uncertainty principle has been studied in many papers\(^\ddagger\)

The construction of wave packets in quantum cosmology as the solutions of the Wheeler-DeWitt (WDW)
equation and its relation with classical cosmology has attracted much attention in the literature \[24, 27\]. These wave packets are usually obtained by the superposition of the energy eigenfunctions so that they follow classical trajectories in configuration space and peak on them whenever such classical and quantum correspondence is feasible \[25, 30\]. However, since time is absent in the theory of quantum cosmology, the initial conditions can be formulated with respect to an intrinsic time parameter, which, in the case of the hyperbolic WDW equation, is taken as the scale factor for the three-geometry \[31\]. In particular, the Friedmann-Robertson-Walker (FRW) quantum cosmology in the presence of a minimally coupled scalar field is studied in Refs. \[27, 32\] and appropriate initial conditions are presented. The quantum cosmology in a \((n + 1)\)-dimensional universe with varying speed of light is investigated in Ref. \[33\]. A class of Stephani cosmological models with a spherically symmetric metric and a minimally coupled scalar field is studied in both classical and quantum domains \[34, 36\].

In this paper, we study a closed FRW quantum cosmology with a conformally coupled scalar field and vanishing cosmological constant in the presence of a minimal length uncertainty. The basic results of the conformally coupled scalar field quantum universe have been addressed by several authors \[28, 29, 37–49\]. For instance, Page has studied the solutions of the WDW equation for the FRW universe with positive, negative and zero curvatures \[41\]. The WDW equation for the positive curvature transforms into a oscillator-ghost-oscillator differential equation which is exactly solvable. Here, both the scale factor and the scalar field obey the modified commutation relation, i.e., \(\{ Q, P \} = 1 + \beta P^2\) where \(\beta\) is the deformation parameter. We assume that the generalized quantization principle in quantum mechanics also holds for the quantities of quantum gravity and generalized representations of the operators are assigned to the observables in the canonical approach of quantum geometrodynamics. We exactly solve the GUP-corrected WDW equation in momentum space and construct wave packets with classical-quantum correspondence using appropriate choices of the initial conditions. We also obtain wave packets that avoid singularities based on the DeWitt boundary condition.

II. THE MODEL

Consider the Einstein-Hilbert action for the gravity and a conformally coupled scalar field

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{2} \nabla_{\mu} \phi \nabla^{\mu} \phi - \frac{1}{12} R \phi^2 \right],
\]

where \(\phi\) is the scalar field. Also, \(g_{\mu\nu}\), \(g\), and \(R\) denote the four-metric, its determinant, and the scalar curvature, respectively. Units are set so that \(\hbar = c = 1\). The FRW minisuperspace model with constant positive curvature and the homogeneous scalar field are given by

\[
ds^2 = -N^2(t) dt^2 + a^2(t) \left[ \frac{d\gamma^2}{1 - r^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right],
\]

\[
\dot{\phi} = \phi(t).
\]

Now, we define a new variable \(\chi = a\ell_p\phi/\sqrt{2}\) where \(\ell_p = \sqrt{8\pi G/3}\) and substitute Eq. (2) in the action (1). After integrating out the spatial degrees of freedom and discarding total time derivatives, we find the following action \[41, 47\]:

\[
S = \int dt \left[ N a - \frac{\alpha a^2}{2N} + \frac{\alpha \chi^2}{2N} - N \frac{\chi^2}{a} \right].
\]

The corresponding Hamiltonian reads

\[
H = N\mathcal{H} = N \left[ -\frac{P^2}{4a} + \frac{P^2}{4a} - a + \frac{\chi^2}{a} \right],
\]

where \(P_a = -\frac{2a^2}{N}\) and \(P_\chi = \frac{2a\chi}{N}\) are the canonical momenta conjugate to the scale factor and the scalar field, respectively.

A. The Generalized Uncertainty Principle

In ordinary quantum mechanics the position and momentum of a particle can be measured separately with arbitrary precision. One way to introduce a minimal value for the measurement of position \(Q\) is to modify the Heisenberg uncertainty relation to the so-called generalized uncertainty principle as \[6\]

\[
\Delta Q \Delta P \geq \frac{\hbar}{2} \left( 1 + \beta \left[ (\Delta P)^2 + \langle P \rangle^2 \right] \right),
\]

where \(P\) is the momentum, \(\beta = \beta_0/(M_{Pl} c)^2\), \(\beta_0\) is of the order of unity, and \(M_{Pl}\) is the Planck mass. Note that the inequality relation (6) implies an absolute minimum observable length, namely, \((\Delta Q)_{\text{min}} = \hbar / \sqrt{\beta}\). In one dimension, the above relation is given by the following deformed commutation relation:

\[
[Q, P] = i\hbar (1 + \beta P^2).
\]

One possible representation of this algebra is given by \[9\]

\[
Q \psi(p) = i\hbar (1 + \beta p^2) \partial_p \psi(p),
\]

\[
P \psi(p) = p \psi(p),
\]

where \(Q\) and \(P\) are symmetric operators on the dense domain \(S_\infty\) subject to the following scalar product:

\[
\langle \psi | \phi \rangle = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2} \psi^*(p) \phi(p).
\]
In this representation, the completeness relation and scalar product read

\[ \langle p' | p \rangle = (1 + \beta p^2) \delta(p - p'), \]
\[ 1 = \int_{-\infty}^{+\infty} \frac{dp}{1 + \beta p^2}|p\rangle \langle p|, \]

In the classical limit \( \hbar \to 0 \) the commutator in quantum mechanics is replaced by the Poisson bracket for the corresponding classical variables

\[ \frac{1}{\hbar} [X, P] \to \{X, P\}, \]

which for the deformed Heisenberg algebra \([17]\) is

\[ \{Q, P\} = 1 + \beta P^2. \]

The nonzero uncertainty in position implies that position eigenstates cannot be considered as physical states. This is due to the fact that an eigenstate of an observable necessarily has vanishing uncertainty. In fact, it is possible to construct position eigenvectors, but these states are only formal eigenvectors and they are not physical states \([19]\). Notice that this feature results from the modification of the canonical commutation relations, and the noncommutativity of space will not, in general, imply a nonzero minimal uncertainty. So, to obtain information on position, we cannot use configuration space, and we need to take into account the notion of quasi-position space.

The states that help us to recover information on positions are maximal-localization states. These states \( |\psi_{\xi}^{ML}\rangle \) as proper physical states have the properties \( \langle \psi_{\xi}^{ML} | Q | \psi_{\xi}^{ML}\rangle = \xi, \langle \Delta Q \rangle_{\psi_{\xi}^{ML}} = (\Delta Q)_{\text{min}}, \) and obey the minimal uncertainty condition \( \Delta Q \Delta P = |\langle \{Q, P\}\rangle|/2. \)

Thus, they satisfy the following equation \([19]\):

\[ (Q - \langle Q \rangle + \frac{\langle [Q, P]\rangle}{2(\Delta P)^2} (P - \langle P \rangle)) |\psi_{\xi}^{ML}\rangle = 0. \]

In momentum space, the normalized solutions read

\[ |\psi_{\xi}^{ML}(p)\rangle = \sqrt{\frac{2\sqrt{\beta}}{\pi}} (1 + \beta p^2)^{-1/2} \exp \left( -i \frac{\xi}{h\sqrt{\beta}} \tan^{-1}(\sqrt{\beta} p) \right). \]

It is now obvious that for \( \beta = 0 \) we recover the ordinary plane waves. Since maximal-localization states are normalizable, unlike the canonical case, their scalar product is a function rather than a distribution. Finally, to find quasi-position wave functions, we project an arbitrary state \(|\psi\rangle\) on the maximally localized states \( |\psi_{\xi}^{ML}\rangle\), which gives the probability amplitude for a particle being maximally localized around the position \( \xi \) with standard deviation \( (\Delta Q)_{\text{min}}. \) These projections are called quasi-position wave functions and defined by \( \psi(\xi) \equiv \langle \psi_{\xi}^{ML} | \psi \rangle. \)

Therefore, we obtain

\[ \psi(\xi) = \sqrt{\frac{2\sqrt{\beta}}{\pi}} \int_{-\infty}^{+\infty} \frac{dp}{(1 + \beta p^2)^{3/2}} \exp \left( i \frac{\xi}{h\sqrt{\beta}} \tan^{-1}(\sqrt{\beta} p) \right) \psi(p). \]

For \( \beta = 0 \), this is the well-known Fourier transformation where the ordinary position wave functions are given by \( \psi(\xi) = \langle \xi | \psi \rangle. \)

Before proceeding further, let us emphasize that it is difficult to construct a meaningful position space using GUP-corrected coordinates. Indeed, the locality problems that appear in these approaches strongly indicate that these coordinates are not physically relevant. However, they can be expressed in terms of the physical coordinates that obey the standard commutation relations, namely, \( Q = (1 + \beta P^2)q \) and \( P = p \) where \([q, p] = i\hbar. \) Here, we study the time evolution of the GUP-corrected coordinates as well as the physical coordinates in the classical domain. Also, in the quantum domain, the momentum space wave function is found in terms of the physical coordinates.

**B. Classical cosmology**

In the context of the GUP, the scale factor and the scalar field satisfy the following deformed Poisson brackets

\[ \{a, \chi\} = 0, \quad \{P_a, \chi\} = 0, \]
\[ \{a, P_a\} = 1 + \beta P_a^2, \quad \{\chi, P\chi\} = 1 + \beta P\chi^2. \]

So the equations of motion for the scale factor and the scalar field are given by

\[ \begin{align*}
\dot{a} &= \{a, H\} = -NP_a(1 + \beta P_a^2)/(2a), \\
\dot{P}_a &= \{P_a, H\} = 2N(1 + \beta P_a^2), \\
\dot{\chi} &= \{\chi, H\} = NP\chi(1 + \beta P\chi^2)/(2a), \\
\dot{P}\chi &= \{P\chi, H\} = -2N(1 + \beta P\chi^2)\chi/a. 
\end{align*} \]

Using the gauge \( N = a \), the time evolution of the scalar field reads

\[ \dot{\chi} = \{\chi, H\} = -\frac{1}{2} P\chi(1 + \beta P\chi^2), \quad \dot{P}\chi = \{P\chi, H\} = 2\chi(1 + \beta P\chi^2). \]

To proceed further, it is convenient to use the variable \( \Pi = \frac{1}{\sqrt{\beta}} \arctan(\sqrt{\beta} \chi), \) which results in

\[ \dot{\Pi} = -\frac{\tan(\sqrt{\beta}\Pi) \sec^2(\sqrt{\beta}\Pi)}{2\sqrt{\beta}}, \quad \dot{\chi} = 2\chi. \]

So we have

\[ \dot{\Pi} + \frac{\tan(\sqrt{\beta}\Pi) \sec^2(\sqrt{\beta}\Pi)}{\sqrt{\beta}} = 0. \]

If we set initial conditions so that \( \chi(0) = 0 \) and \( P\chi(0) = \Pi_0 \), the solutions are given by

\[ \begin{align*}
\chi(t) &= \frac{1}{2} \frac{\Pi_0 \sqrt{1 + \eta^2} \tan \left( \sqrt{1 + \eta^2} t \right)}{\sqrt{1 + (1 + \eta^2) \tan^2 \left( \sqrt{1 + \eta^2} t \right)}}, \\
P\chi(t) &= \frac{\Pi_0}{\sqrt{1 + (1 + \eta^2) \tan^2 \left( \sqrt{1 + \eta^2} t \right)}}.
\end{align*} \]
where $\eta = \sqrt{\beta} \Pi_0$. Also, if we set $P_a(0) = 0$, the Hamiltonian constraint $H = 0$ implies $\psi(0) = \Pi_0/2$, and the solutions read

$$a(t) = \frac{1}{2} \frac{\Pi_0 \sqrt{1 + \eta^2} \cot (\sqrt{1 + \eta^2} t)}{\sqrt{1 + (1 + \eta^2) \cot^2 (\sqrt{1 + \eta^2} t)}}.$$  

(25)

$$P_a(t) = \frac{\Pi_0}{\sqrt{1 + (1 + \eta^2) \cot^2 (\sqrt{1 + \eta^2} t)}}.$$  

(26)

It is straightforward to check that for $\beta \to 0$, the above solutions coincide with the harmonic oscillator solutions which represent a Lissajous ellipsis in configuration space. Note that this solution as well as the $\beta = 0$ case is singular in the classical domain; i.e., the scale factor goes to zero as $t \to \Pi/(2\sqrt{1 + \eta^2})$.

Now, let us find the time evolution of the physical coordinates. For our problem, these coordinates are given by

$$a = (1 + \beta p^2)q_a,$$

$$\chi = (1 + \beta \rho^2)q_\chi,$$

$$P_a = p,$$

$$P_\chi = \tilde{\rho},$$

where $\{q_a, p\} = \{q_\chi, \tilde{\rho}\} = 1$. Therefore, after solving $\dot{q}_a = \{q_a, \Pi\}$ and $\dot{p}_\chi = \{p_\chi, \Pi\}$ for $p_\in \{p, \tilde{\rho}\}$, we find

$$q_a(t) = \frac{\Pi_0}{4} \sqrt{1 + (1 + \eta^2)^{-1} + \tan^2 (\sqrt{1 + \eta^2} t)} \times \sin \left(2\sqrt{1 + \eta^2} t\right),$$

(28)

$$\tilde{\rho}(t) = \frac{\Pi_0}{\sqrt{1 + (1 + \eta^2) \tan^2 (\sqrt{1 + \eta^2} t)}}.$$  

(29)

and

$$q_\chi(t) = \frac{\Pi_0}{4} \sqrt{1 + (1 + \eta^2)^{-1} + \cot^2 (\sqrt{1 + \eta^2} t)} \times \sin \left(2\sqrt{1 + \eta^2} t\right),$$

(30)

$$p(t) = \frac{\Pi_0}{\sqrt{1 + (1 + \eta^2) \cot^2 (\sqrt{1 + \eta^2} t)}}.$$  

(31)

C. Quantum cosmology

Let us briefly present the minisuperspace quantization of the $\beta = 0$ universe model. Following the Dirac quantization procedure, namely, using the canonical replacement $P_a \to -i \partial \Pi_a$ and $P_\chi \to -i \partial \Pi_\chi$ and assuming a particular factor ordering, one arrives at the Wheeler-DeWitt equation for the conformally coupled scalar field model $[40, 41, 47, 48]$ ($h = 1$),

$$\mathcal{H} \Psi(\chi, a) = \left\{ P_a^2 + P_\chi^2 + 4 (a^2 - \chi^2) \right\} \Psi(\chi, a)$$

$$= \left\{ -\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial \chi^2} + 4 (a^2 - \chi^2) \right\} \Psi(\chi, a) = 0.$$  

(32)

Note that this form of the WDW equation also appears in the context of the minimally coupled scalar field $[32]$, quantum Stephani universe $[34]$, and varying speed-of-light theories $[33]$. The WDW equation (32) is separable and exactly solvable in the minisuperspace variables, and the solutions are given by

$$\Psi_n(\chi, a) = \psi_n(\chi) \psi_n(a),$$  

(33)

where $\psi_n(z) = \left(\frac{2}{\pi}\right)^{1/4} \left[ \frac{H_n(\sqrt{2}z)}{\sqrt{2^n n!}} \right] e^{-z^2}$ and $H_n(z)$ is the Hermite polynomial of degree $n$.

In the presence of the minimal length, since there are no position eigenstates in the Heisenberg algebra representation, the Heisenberg algebra finds no Hilbert space representation in the space of position wave functions $[9]$. So, we use a proper representation of the commutation relations on wave functions in the momentum space, i.e., Eqs. (3) and (9). The GUP-corrected Wheeler-DeWitt equation in momentum space now reads

$$\left\{ -(1 + \beta p^2) \frac{\partial}{\partial p} \right\}^2 + \left\{ (1 + \beta \rho^2) \frac{\partial}{\partial \rho} \right\}^2$$

$$+ \frac{1}{4} (p^2 - \rho^2) \} \Psi(p, \rho) = 0,$$  

(34)

and a similar equation for $\psi(\rho)$ where $\epsilon$ is the separation constant. The above equation is the Schrödinger equation for a simple harmonic oscillator with minimal length uncertainty, and it can be cast into an exactly solvable differential equation $[3, 10, 20]$. Using the new variable $\rho = \frac{1}{\sqrt{\beta}} \tan^{-1} (\sqrt{\beta} p)$, which maps the domain $-\infty < p < \infty$ to $-\frac{\pi}{2\sqrt{\beta}} < \rho < \frac{\pi}{2\sqrt{\beta}}$, we find

$$\left[ \frac{\partial^2}{\partial \rho^2} - \frac{\tan^2 (\sqrt{\beta} \rho)}{4\beta} + \epsilon \right] \psi(\rho) = 0,$$  

(36)

which can be written as

$$\left[ \frac{\partial^2}{\partial \rho^2} - \frac{1}{4 \beta \epsilon^2} + \epsilon \right] \psi(\rho) = 0,$$  

(37)

where $s = \sin (\sqrt{\beta} \rho)$ and $c = \cos (\sqrt{\beta} \rho)$. Now, we take $\psi(\rho) = e^{\lambda f(s)}$, where $\lambda$ is a constant that will be determined. So, $f(s)$ satisfies

$$(1 - s^2) f'' - (2\lambda + 1) s f' + \left\{ \frac{\epsilon}{\beta} - \lambda \right\}$$

$$+ \left\{ \lambda(\lambda - 1) - \frac{1}{4\beta^2} \right\} s^2 \frac{f''}{c^2} = 0,$$  

(38)
where $-1 < s < 1$. Since the wave function should be nonsingular at $c = 0$, the coefficient of the tangent squared is required to vanish, namely, $\lambda(\lambda - 1) - \frac{1}{4\beta^2} = 0$, or equivalently,
\[
\lambda = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{\beta^2}} \right).
\]
Therefore, Eq. (38) becomes
\[
(1 - s^2) f'' - (2\lambda + 1) s f' + \left( \frac{\epsilon}{\beta} - \lambda \right) f = 0.
\]
Similarly, the nonsingularity $f(s)$ at $s = \pm 1$ and requiring a polynomial solution to Eq. (38) imply
\[
\frac{\epsilon}{\beta} - \lambda = n(n + 2\lambda),
\]
where $n$ is a non-negative integer. Thus, Eq. (40) reads
\[
(1 - s^2) f'' - (2\lambda + 1) s f' + n(n + 2\lambda) f = 0.
\]
The Gegenbauer polynomial is the solution of the above equation
\[
f(s) = C_n^\lambda(s).
\]
Also, the energy eigenvalues are given by Eqs. (41) and (39) as
\[
\epsilon_n = \left( n + \frac{1}{2} \right) \left[ \sqrt{1 + \beta^2} + \beta \right] + \beta n^2,
\]
and the normalized energy eigenfunctions read
\[
\psi_n(p) = 2^n \Gamma(\lambda) \sqrt{\frac{n!}{2\pi \Gamma(n + 2\lambda)}} c^n C_n^\lambda(s),
\]
where
\[
c = \cos \sqrt{\beta^2 p} = \frac{1}{\sqrt{1 + \beta^2}},
\]
\[
s = \sin \sqrt{\beta^2 p} = \frac{\sqrt{\beta^2 p}}{\sqrt{1 + \beta^2}}.
\]
Since $\psi(\bar{p})$ also obeys Eq. (35), the solution of the WDW equation is given by $\Psi(p, \bar{p}) = \psi_n(p)\psi_n(\bar{p})$.

III. WAVE PACKETS IN MOMENTUM SPACE

The general solution of the WDW equation (34) is
\[
\Psi(p, \bar{p}) = \sum_{n=\text{even}} A_n \psi_n(p)\psi_n(\bar{p}) + i \sum_{n=\text{odd}} B_n \psi_n(p)\psi_n(\bar{p}),
\]
where the separability of the eigenfunctions into even and odd categories is due to the parity symmetry of the WDW equation. Also, the initial wave function and the initial derivative of the wave function are given by
\[
\Psi(0, \bar{p}) = \sum_{n=\text{even}} A_n \psi_n(0)\psi_n(\bar{p}),
\]
\[
\left. \frac{\partial \Psi(p, \bar{p})}{\partial p} \right|_{p=0} = i \sum_{n=\text{odd}} B_n \psi_n(\bar{p})\psi_n'(0).
\]
So, $A_n$ determines the initial wave function and $B_n$ determines its initial derivative. Notice that we are free to choose arbitrary expansion coefficients. However, since we are interested in constructing wave packets with classical and quantum correspondence, these coefficients will not be independent anymore. To address the issue of the initial conditions, consider the WDW equation (34) near $p = 0$, namely,
\[
\left\{ \left( 1 + \beta^2 p^2 \right) \frac{\partial}{\partial p} \right\}^2 \psi(p, \bar{p}) = 0.
\]
The solution to this equation can be written as $\psi(p, \bar{p}) = \xi(p)\psi(\bar{p})$, which results in
\[
\frac{d^2 \xi(p)}{dp^2} + \epsilon \xi(p) = 0,
\]
\[
- \left( 1 + \beta^2 p^2 \right) \frac{d^2}{dp^2} \psi(\bar{p}) + \frac{1}{4} \beta^2 \psi(\bar{p}) = \epsilon \psi(\bar{p}),
\]
where $\epsilon$ is the separation constant with discrete values. Thus, the general solution to equation (51) is
\[
\psi(p, \bar{p}) = \sum_{n=\text{even}} A_n \cos(\sqrt{\epsilon_n p})\psi_n(\bar{p}) + i \sum_{n=\text{odd}} B_n \sin(\sqrt{\epsilon_n p})\psi_n(\bar{p}),
\]
which is valid for small values of $p$. Now, the initial wave function and its initial slope read
\[
\psi(\bar{p}, 0) = \sum_{n=\text{even}} A_n \psi_n(\bar{p}),
\]
\[
\psi'(\bar{p}, 0) = i \sum_{n=\text{odd}} B_n \psi_n(\bar{p}),
\]
where the prime denotes the derivative with respect to $p$. The prescription for choosing the coefficients is that they have the same functional form, namely
\[
\left\{ \begin{array}{ll}
A_n = f(n), & \text{for } n \text{ even}, \\
B_n = f(n), & \text{for } n \text{ odd},
\end{array} \right.
\]
where $f(n)$ is chosen so that the initial wave function has a proper classical description. In terms of $A_n$ and $B_n$, we obtain
\[
\left\{ \begin{array}{ll}
A_n &= \frac{1}{\psi_n'(0)} f(n), & \text{for } n \text{ even}, \\
B_n &= \frac{1}{\psi_n'(0)} f(n), & \text{for } n \text{ odd}.
\end{array} \right.
\]
Now, using Eqs. (48) and (58), the wave packet takes the following form in momentum space.
\[
\Psi(p, \beta) = \frac{2^{\lambda/4} \Gamma(\lambda)}{\sqrt{2\pi}} \frac{(1 + \beta p^2)^{-\lambda/2}}{(1 + \beta^2)^{-\lambda/2}} \left\{ \sum_{n=\text{even}} \frac{1}{C_n^\lambda(0)} \sqrt{n!} (n + \lambda) \left( \frac{\sqrt{\beta p}}{\sqrt{1 + \beta^2}} \right)^{\lambda} C_n^\lambda \left( \frac{\sqrt{\beta p}}{\sqrt{1 + \beta^2}} \right) \right\} + i \sum_{n=\text{odd}} \frac{\sqrt{c_n}}{2 \lambda \sqrt{2} \beta C_{n-1}^\lambda} \sqrt{n! (n + \lambda)} \left( \frac{\sqrt{\beta p}}{\sqrt{1 + \beta^2}} \right)^{\lambda} C_n^\lambda \left( \frac{\sqrt{\beta p}}{\sqrt{1 + \beta^2}} \right). \tag{59}
\]

In Fig. 3 we have depicted the resulting wave packet for \( \beta = 0.1 \) and \( f(n) = \frac{\kappa^n e^{-|\kappa|^2/4}}{\sqrt{2^n n!}} \), where \( \kappa = |\kappa|e^{-\theta} \), \( \theta = 0 \), and \( |\kappa| = 4 \). This choice of the coefficients is due to the fact that \( f(n) \) results in the initial wave function with two well-separated peaks. One peak corresponds to the initial values of classical momenta \((p_n, \chi_n)\), and the other corresponds to their final values. As the figure shows, the wave packet is smooth, and the crest of the wave packet closely follows the corresponding classical trajectory. Figure 2 shows the wave packet corresponding to a nonsymmetric classical trajectory, i.e., \( t \to t + \Delta \) in Eq. (24),

\[
P_\chi(t) = \frac{\Pi_0}{\sqrt{1 + (1 + \eta^2) \tan^2 \left( \sqrt{1 + \eta^2} (t + \Delta) \right)}} \tag{60}
\]

where we set \( \Delta = -1 \). For this case, we have \( \beta = 0.1 \), \( f(n) = \frac{\kappa^n e^{-|\kappa|^2/4}}{\sqrt{2^n n!}} \), \( \theta = \pi/8 \), and \( |\kappa| = 4 \). As the figures show, for both cases the wave packets closely follow the classical trajectories and peak on them. Note that this behavior is due to the proper adjustment of the expansion coefficients in Eq. (59). Also, the initial conditions \( P_\chi(0) = \Pi_0 \) and \( P_\chi(0) < \Pi_0 \) correspond to \( \theta_0 = 0 \) and \( \theta_0 \neq 0 \), respectively.

IV. THE SINGULARITY PROBLEM

Cosmological singularities are one of the puzzling phenomena in modern physics. In fact, because of the extreme conditions at this stage of evolution of the Universe, the laws of physics must break down at the singularities. In this case, the predictive power of the theory is lost, and much efforts has been devoted to find physical mechanisms that eliminate the offending singularities. In the semiclasical domain, some phenomena, such as particle production, negative vacuum stresses, and the presence of massive scalar fields, are proposed to escape from the classical collapse predicament. However, all these proposals violate various positive-energy conditions of the singularity theorems. On the other hand, it is conjectured that quantum effects can resolve this fundamental dilemma. Now, we present three proposals for quantum singularity avoidance and discuss the singular nature of the obtained wave packets. We indicate that the singularity problem can be resolved following the DeWitt criterion.

A. The DeWitt boundary condition

DeWitt suggested the following boundary condition\[31,\]

\[
\left. \Psi \left[ (3) G \right] \right|_{\beta=\tilde{\beta}_0} = 0, \tag{61}
\]

for all three-geometries \((3) G\) related with singular three-geometries. So, the criterion for the quantum universe to be singularity free is that the wave function vanishes at the classical singularity. For our case, at fixed \( \tilde{p} \), the momentum space wave function transforms to the quasi-position wave function as follows\[17,\]

\[
\psi(\xi) \bigg|_{\beta=\tilde{\beta}_0} = \frac{2\sqrt{\beta}}{\pi} \int_{-\infty}^{+\infty} dp \frac{dp}{(1 + \beta p^2)^{3/2}} \times e^{\frac{2\sqrt{\beta}}{\pi} \tan^{-1}(\sqrt{\beta p})} \Psi(\tilde{p}_0, p), \tag{62}
\]

where \( \xi = \langle a \rangle \). By taking \( A_n = 0 \), \( \Psi(p, \beta) \) is an odd function of \( p \) and we have \( \psi(0) = 0 \). Consequently, the wave function which satisfies the DeWitt boundary condition reads

\[
\Psi(p, \beta) = \frac{2^{\lambda-1} \Gamma(\lambda)}{\lambda \sqrt{2\pi} \sqrt{\beta}} \frac{(1 + \beta p^2)^{-\lambda/2}}{(1 + \beta^2)^{-\lambda/2}} \sum_{n=\text{odd}} \frac{i \sqrt{c_n}}{C_n^{\lambda+1}(0)} \sqrt{n!} (n + \lambda) f(n) C_n^{\lambda} \left( \frac{\sqrt{\beta p}}{\sqrt{1 + \beta^2}} \right)^{\lambda} C_n^{\lambda} \left( \frac{\sqrt{\beta p}}{\sqrt{1 + \beta^2}} \right). \tag{63}
\]

This result shows that the wave packet (59) violates the DeWitt boundary condition. In Fig. 3 we have plotted the contour plot of the wave packet (63), which, unlike
Eq. (59), it does not show the classical behavior; i.e., it is oscillatory and vanishes at several points along a classical trajectory (Fig. 4). Note that the significance of this boundary condition is still controversial and it has been argued that the DeWitt boundary condition has little to do with the quantum singularity avoidance [52–54].

**B. The criterion of the expectation value of observables**

Based on the proposal by Lund [53] and Gotay and Isenberg [56], a quantum state $\psi$ is singular if and only if $\langle \psi | Q f | \psi \rangle = 0$ for any quantum observable $Q f$ where its classical counterpart $f$ vanishes at the singularity. One of the advantages of this criterion is that it is straightforward to check. For our case, if we write the wave function as a sum of even and odd parts, namely $\Psi = \Psi_e + i \Psi_o$, the expectation value of the scale factor observable reads

$$
\langle a \rangle = \langle \Psi | i \left( 1 + \beta p^2 \right) \frac{\partial}{\partial p} | \Psi \rangle_{| p = \tilde{p}_0 }
$$

$$
= i \int_{-\infty}^{+\infty} dp \Psi^* (p, \tilde{p}_0) \frac{\partial}{\partial p} \Psi (p, \tilde{p}_0)
$$

$$
= \int_{-\infty}^{+\infty} dp \left\{ \Psi_e^* (p, \tilde{p}_0) \Psi_e (p, \tilde{p}_0) - \Psi_o^* (p, \tilde{p}_0) \Psi_o (p, \tilde{p}_0) \right\} (64)
$$

which is identically zero for the solution (63) and vanishes at $\tilde{p}_0 = 0$ for the solution (59). Therefore, this test for quantum collapse shows that both solutions (59) and (63) cannot escape the quantum mechanical singularity. In the next subsection, we show that this result may be due to the “choice of time” on the classical level.
FIG. 3: The contour plot of the wave packet in momentum space: Eq. (63) (left) and Eq. (59) (right). We set $\beta = 0.1$, $f(n) = \frac{n^n}{\sqrt{2n^n}} e^{-|\kappa|^2/4}$, $\theta = 0$, and $|\kappa| = 4$.

FIG. 4: The square of the wave packet $|\Psi(p, \tilde{p})|^2$ along classical trajectory for $\beta = 0.2$, $f(n) = \frac{n^n}{\sqrt{2n^n}} e^{-|\kappa|^2/4}$, $\theta = 0$, and $|\kappa| = 4$.

C. Fast- and slow-time gauges

It is shown that the quantum collapse is predetermined by the choice of time on the classical domain. In this regard, Gotay and Demaret conjectured that slow-time quantum dynamics is always nonsingular, while fast-time quantum dynamics is inevitably singular, i.e., leads to the collapse. A time variable $t$ is called fast time if the singularities occur at either $t = -\infty$ or $t = \infty$. Otherwise, $t$ is called slow time. Indeed, fast-time gauge dynamics is complete and can be considered as a regularization of a slow-time gauge dynamics which is incomplete. This distinction is particularly useful in the quantum mechanical domain.

For our case, classical solutions show that the scale factor runs from $a = 0$ at $t = -\Pi/(2\sqrt{1 + \eta^2})$ and then collapses to $a = 0$ at $t = \Pi/(2\sqrt{1 + \eta^2})$. So, the time $t$ here is slow time. Since the above conjecture implies that the slow-time quantum dynamics is nonsingular and as we showed before $\langle \psi | Q f | \psi \rangle = 0$ at the classical singularity, we conclude that the quantum dynamics (34) is not quantized in the slow-time gauge. Note that for $\beta = 0$ and at the classical level, the scale factor $a$ expands monotonically from $a = 0$ at $\phi = -\infty$ to its maximum value and then collapses monotonically back to $a = 0$ at $\phi = +\infty$. Thus, if we take $t = \phi$ as a time choice, it is a fast-time gauge which runs from $-\infty$ to $+\infty$, and its corresponding quantum dynamics based on the effective Hamiltonian $H = -P_\phi$ will be singular as well. For our case, although this conjecture does not determine the time gauge of the quantum dynamics, it only indicates that the model is quantized in a fast-time gauge.

V. CONCLUSIONS

We have studied a closed Friedmann-Robertson-Walker quantum cosmology model in the presence of a conformally coupled scalar field and in the context of the generalized uncertainty principle. In this framework, both the scale factor and the scalar field satisfy the modified commutation relation $[Q, P] = i(1 + \beta P^2)$ where $\beta$ is the GUP parameter. We exactly solved the Wheeler-DeWitt equation in momentum space and obtained the solutions in terms of the Gegenbauer polynomials. In
principle, since the WDW equation is a second-order differential equation, the initial wave function and its initial derivative, i.e., the expansion coefficients, can be chosen freely. However, the classical and quantum correspondence imposes a particular relation between the expansion coefficients. Here, we proposed a particular relation between the even and odd expansion coefficients that determine the initial wave function and its initial derivative, respectively. The resulting wave packets closely followed their corresponding classical trajectories and peaked on them in the momentum space. For $f(n) = \kappa^n e^{-|\kappa|^2/4} \sqrt{2^n n!}$ where $\kappa = |\kappa| e^{-i\theta}$, we showed that the symmetric and nonsymmetric classical solutions correspond to $\theta = 0$ and $\theta \neq 0$, respectively. These wave packets are also singular in the quantum domain based on the DeWitt boundary condition. This problem can be avoided (even in the absence of the GUP) by taking $A_\mu = 0$, namely, vanishing the even expansion coefficients. However, the criterion of the expectation value of observables and the conjecture by Gotay and Demaret showed that the singularity problem still exists due to the fast-time gauge of the quantum dynamics.

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