Improved Kernels for Tracking Path Problems

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Abstract. Given a graph $G$, terminal vertices $s$ and $t$, and an integer $k$, the Tracking Paths problem asks whether there exists at most $k$ vertices, which if marked as trackers, would ensure that the sequence of trackers encountered in each $s$-$t$ path is unique. The problem was first proposed by Banik et al. in \cite{2}, where they gave results for some restricted versions of the problem. The problem was further studied in \cite{4}, where the authors proved it to be \textit{NP}-complete and fixed-parameter tractable. They obtained a kernel (i.e. an equivalent instance) with $O(k^6)$ vertices and $O(k^7)$ edges, in polynomial time. We improve on this by giving a quadratic kernel, i.e. $O(k^2)$ vertices and edges. Further, we prove that if $G$ is a $d$-degenerate graph then there exists a linear kernel. We also show that finding a tracking set of size at most $n - k$ for a graph on $n$ vertices is hard for the parameterized complexity class $W[1]$, when parameterized by $k$.

Keywords: Graphs · Paths · Kernelization · Fixed-parameter tractable · Planar graphs · d-degenerate graphs · Tracking Paths.

1 Introduction

Graphs serve as a systematic model for modeling and analysis of many real life problems. One of the commonly studied problems in areas like networks and machine learning is tracking of moving objects. Typically a secure environment or setup that needs to be monitored, has one or more source and destination points. The requirement is usually to identify the path traced by object(s) in a network. This can be implemented by placing trackers at a subset of checkpoints.

Coordinated path tracking and framework for multi-target tracking have been discussed in \cite{11} and \cite{10}. Tracking algorithms can be used in designing debugging tools. Another useful application is the problem of leakage detection systems. In these kind of problems it would be resource efficient if a small subset of nodes/checkpoints are sufficient to trace the movements of entities in the network.

While most of the work done on these kind of problems has been heuristic-based, Banik et al. \cite{2} considered the problem of target tracking theoretically and modeled it as the following graph theoretic problem. Let $G = (V, E)$ be an undirected graph without any self loops or parallel edges and suppose $G$ has a
unique entry vertex (source) $s$ and a unique exit vertex (destination) $t$. A simple path from $s$ to $t$ is called an $s$-$t$ path. The problem is to find a set of vertices $T \subseteq V$ be a set of vertices such that for any two distinct $s$-$t$ paths, say $P_1$ and $P_2$, the sequence of vertices in $T \cap V(P_1)$ as encountered in $P_1$ is different from the sequence of vertices in $T \cap V(P_2)$ as encountered in $P_2$. Here $T$ is called a tracking set for the graph $G$, and the vertices in $T$ are called trackers. Banik et.al. [2] proved that the problem of finding a minimum-cardinality tracking set with respect to shortest $s$-$t$ paths (Tracking Shortest Paths problem) is NP-hard and APX-hard. The problem was first studied from a parameterized perspective in [4], the parameterized version of the problem being as follows.

**Tracking Paths** $(G, s, t, k)$

**Input:** An undirected graph $G = (V, E)$ with two distinguished vertices $s$ and $t$, and a non-negative integer $k$.

**Question:** Is there a tracking set $T$ of size at most $k$ for $G$?

In [4], above problem was proven to be NP-complete, and further it also shown to be fixed-parameter tractable by existence of a polynomial kernel. Specifically it was proven that an instance of Tracking Paths can be reduced to an equivalent instance of size $O(k^7)$ in polynomial time, where $k$ is the desired size of the tracking set.

**Our Results and Methods.** In this paper we give a quadratic kernel for Tracking Paths on general graphs, which is a major improvement from the $O(k^7)$ kernel given in [4]. We also give a linear kernel for Tracking Paths on $d$-degenerate graphs. Further we prove that deciding if there exists a tracking set of size at most $n - k$, where $n$ is the number of vertices in the graph, is W[1]-hard.

Given an instance $(G, s, t, k)$, we give a polynomial time algorithm that either determines that $(G, s, t, k)$ is a NO instance or produces an equivalent instance with $O(k^2)$ vertices and $O(k^2)$ edges. This polynomial time algorithm is called a kernelization algorithm and the reduced instance is called a kernel. For more details about parameterized complexity and kernelization we refer to monographs [5,7].

The kernelization algorithm works along the following lines. Let $(G, s, t, k)$ be an input instance to Tracking Paths. Two main results used to build the kernel in [4] were (i) every tracking set is also a feedback vertex set (set of vertices whose deletion removes all cycles from graph), (ii) if there exist more than $k + 4$ paths between a pair of vertices in $G$, then $G$ cannot be tracked with at most $k$ trackers. However, in this paper, though we start the algorithm with a 2-approximate solution for feedback vertex set (FVS), we use another newly introduced result. Specifically, we prove that if there exists an induced subgraph in $G$ which consists of a tree with all of its leaves adjacent to a particular vertex $v$, then the size of a minimum tracking set for $G$ is at least one less than the number of neighbors of $v$ in this tree. Then if $S$ is an FVS of size at most $2k$, we give bounds on different types of vertices in $G \setminus S$, based on how they share neighbors in $S$. Combining all these bounds we prove the existence of a quadratic kernel for general graphs.
and linear kernel for $d$-degenerate graphs. A $d$-degenerate graph is a graph in which each subgraph has a vertex whose degree is at most $d$. Observe that for such a graph $G$, the number of edges in $G'$ is $O(|V(G')|)$ for any subgraph $G'$ of $G$. Note that graphs with bounded degeneracy include planar graphs. Eppstein et al. studied Tracking Paths for planar graphs in [8], where they show that Tracking Paths remains NP-complete when the graph is planar, and give a 4-approximation algorithm for this setting.

Our linear kernel for $d$-degenerate graphs uses the fact that in a $d$-degenerate graph $G$, any subgraph $G'$ has $O(|V(G')|)$ edges. We also prove that finding a tracking set of size at most $n - k$ is $W[1]$-hard on a graph with $n$ vertices, where the parameter is $k$.

Although a tracking set is also a feedback vertex set, both are fundamentally very different. A graph may have a small FVS but the tracking set may be arbitrarily larger than the FVS. Moreover, Tracking Paths is more demanding as a problem compared to the classic covering problems studied in graph theory. While covering problems aim at hitting a particular type of structure in graphs, Tracking Paths requires distinguishing each $s$-$t$ path uniquely using a small set of vertices. In particular, even proving that Tracking Paths is NP is non-trivial. See [4] for details. A combinatorial generalization of Tracking Paths is studied in [3], where the input is a set system, and it is required to find a subset of elements from the universe that have a unique intersection with each set in the family. The problem has been shown to be a dual of the Test Cover problem. A related problem, Identifying Path Cover has been discussed in [9]. Identifying Path Cover requires finding a set of paths that cover all the vertices in a graph, and also uniquely identify each vertex by inclusion in a distinct set of paths.

Preliminaries

A kernelization algorithm is typically obtained using what are called reduction rules. These rules transform a given parameterized instance in polynomial time to an equivalent instance, and a rule is said to be safe if the resulting graph has a tracking set of size at most $k$ if and only if the original instance has one.

Throughout the paper, we assume graphs to be simple i.e. there are no self loops and multi-edges. When considering tracking set for a graph $G = (V,E)$, we assume that the given graph is an $s$-$t$ graph, i.e. the graph contains a unique source $s \in V$ and a unique destination $t \in V$ (both $s$ and $t$ are known), and we aim to find a tracking set that can distinguish between all simple paths between $s$ and $t$. Here $s$ and $t$ are also referred as the terminal vertices. In this paper, when we refer to tracking set, we mean tracking set for all $s$-$t$ paths. If $a, b \in V$, then unless otherwise stated, $\{a, b\}$ represents the set of vertices $a$ and $b$, and $(a, b)$ represents an edge between $a$ and $b$. For a vertex $v \in V$, neighborhood of $v$ is denoted by $N(v) = \{x \mid (x, v) \in E\}$. We use $\deg(v) = |N(v)|$ to denote degree of vertex $v$. For a vertex $v \in V$ and a subgraph $G'$, $N_{G'}(v) = N(v) \cap V(G')$. For a subset of vertices $V' \subseteq V$ we use $N(V')$ to denote $\bigcup_{v \in V'} N(v)$. With slight abuse of notation we use $N(G')$ to denote $N(V(G'))$.
of vertices $S \subseteq V(G)$, $G - S$ denotes the subgraph induced by the vertex set $V(G) \setminus V(S)$. If $S$ is a singleton, we may use $G - x$ to denote $G - S$, where $S = \{x\}$. $\lfloor m \rfloor$ is used to denote the set of integers $\{1, \ldots, m\}$.

For a path $P$, $V(P)$ denotes the vertex set of path $P$, and for a subgraph(or graph) $G'$, $V(G')$ denotes the vertex set of $G'$. For a subgraph(or graph) $G'$, $E(G')$ denotes the edge set of $G'$. Let $P_1$ be a path between vertices $a$ and $b$, and $P_2$ be a path between vertices $b$ and $c$, such that $V(P_1) \cap V(P_2) = \{b\}$. By $P_1 \cdot P_2$, we denote the path between $a$ and $c$, formed by concatenating paths $P_1$ and $P_2$ at $b$. Two paths $P_1$ and $P_2$ are said to be vertex disjoint if their vertex sets do not intersect except possibly at the end points, i.e. $V(P_1) \cap V(P_2) \subseteq \{a, b\}$, where $a$ and $b$ are the starting and end points of the paths. By distance we mean length of the shortest path, i.e. the number of edges in that path. For a graph $G = (V, E)$, an FVS is a set of vertices $S \subseteq V$ such that $G \setminus S$ is a forest.

1.1 Organization of the Paper

Section 2 analyzes graph structures that have a strict lower bound in terms of the number of trackers required in them if they appear as subgraphs in the input graph. The lemmas in this section form the basis of the reduction rules used in Sections 3 and 4 to get the respective kernels. Section 3 gives an $O(k^2)$ kernel for general graphs, where $k$ is the desired size of a tracking set. We start by finding a 2-approximate FVS $S$ for the input graph $G$. Next the vertices in $V(G) \setminus S$ are categorized on the basis of how they share neighbors in $S$ with other vertices in $V(G) \setminus S$. Section 4 uses a similar approach to derive an $O(k)$ kernel for $d$-degenerate graphs, while using the fact that a $d$-degenerate graph $G$ has $O(|V(G)|)$ edges. Section 5 discusses the $W[1]$-hardness for the problem of finding whether a graph can be tracked with $n - k$ trackers, where $n$ is the number of vertices in the graph. Finally Section 6 summarizes the results in the paper with some open problems. Section 1 explaining the terms and notations used in the paper can be found in Appendix.

2 Analyzing structures

We first give some reduction rules and basic results from 4 followed by additional ones. Then we give an important lemma based on tree like structures, which forms the base for vertex counting arguments for kernels in subsequent sections. Following reduction rules are applied in the given sequence as long as they are applicable.

Reduction Rule 1 \[4\] If there exists a vertex or an edge that does not participate in any $s$-$t$ path then delete it.

In the rest of the paper we assume that each vertex and edge participates in at least one $s$-$t$ path.

Reduction Rule 2 If degree of $s$ (or $t$) is 1 and $N(s) \neq t$ ($N(t) \neq s$), then delete $s(t)$, and label the vertex adjacent to it as $s(t)$. 
Lemma 1. Reduction Rule 2 is safe and can be implemented in polynomial time.

Proof. Consider a graph $G$ where $N(s) = \{a\}$ and $N(t) = \{a\}$. Note that since all $s$-$t$ paths pass through $a$, a minimal tracking set need not contain $a$. Also the sequence of each $s$-$t$ path starts (ends) with $s$ ($t$) followed (preceded) by $a$. Hence we can simply relabel $a$ as the source (destination) vertex $s$ ($t$), delete $s$ ($t$) from the graph $G$, and this would not affect a tracking set for the graph. Note that this reduction rule cannot be applied if $s$ and $t$ are neighbors, since we can't delete either of them from $G$. \hfill \Box

Next we give a reduction rule that bounds the number of vertices with degree two in $G$. A similar rule has been given in [4]. However, our rule is slightly tighter in the sense that we retain only one vertex of degree two as compared to two vertices being retained in the corresponding reduction rule given in [4], if multiple vertices of degree two are found connected in series (induced path).

Reduction Rule 3 If there exist $a, b, c \in V(G)$ such that $\deg(a) = \deg(b) = 2$ and $N(b) = \{a, c\}$, then delete $b$ and introduce an edge between $a$ and $c$.

Lemma 2. Reduction Rule 3 is safe and can be implemented in polynomial time.

Proof. Observe that if $(a, c) \in E(G)$, then $a$, $b$ and $c$ form a triangle such that $a$ and $b$ do not participate in any $s$-$t$ path (as they form a triangle). Such a case is not possible due to Reduction Rule 1. Next observe that the set of $s$-$t$ paths that passes through $a$ is the same as the set of $s$-$t$ paths that pass through $b$ and the set of $s$-$t$ paths that pass through $c$. For identifying one or more $s$-$t$ paths by their vertex sets, any one amongst $a$, $b$ or $c$ is sufficient as a tracker. And at most two among $a$, $b$, $c$ are sufficient to help distinguish sequence of $s$-$t$ paths that pass through these vertices. Thus a minimal tracking set would not contain all three of $a$, $b$ and $c$.

Let $G'$ be the graph obtained after applying the reduction rule. We claim that there exists a tracking set of size at most $k$ in $G'$ if and only if there exists a tracking set of size at most $k$ in $G$. First we prove the forward direction. Suppose that there exists a tracking set $T$ of size at most $k$ in $G$. If $b \notin T$, then $T$ is also a tracking set for $G'$. Let $b \in T$. If $a \notin T$, then $T \setminus \{b\} \cup \{a\}$ is a tracking set for $G'$. If $a \in T$, then it must have been the case when $a$ and $b$ both were chosen as trackers in order to distinguish sequence between some $s$-$t$ paths. In this case $T \setminus \{b\} \cup \{c\}$ is a tracking set for $G'$.

Now we prove the reverse direction. Let $T'$ be a tracking set of size at most $k$ in $G'$. We claim that $T'$ is a tracking set for $G$ as well. Suppose not. Then there exist two distinct $s$-$t$ paths, $P_1$ and $P_2$ in $G$ that contain the same sequence of trackers. If $b \notin V(P_1)$ and $b \notin V(P_2)$, then these paths exist in $G'$ as well, contradicting the assumption that $T'$ is a tracking set for $G'$. If $b \in V(P_1)$ and $b \in V(P_2)$, again it contradicts the assumption that $T'$ is a tracking set for $G'$. Now consider the case when one of these $s$-$t$ paths passes through $b$ while the other does not. Without loss of generality, let $b \in V(P_1)$ and $b \notin V(P_2)$. Since all paths that pass through $b$, also pass through $a$, there exists a path $P'_1$ in $G'$...
that is obtained by applying the reduction rule. Hence there exist paths \( P'_1 \) and \( P'_2 \) with the same sequence of trackers. This contradicts the assumption that \( T' \) is a tracking set for \( G' \). Thus the reduction rule is safe.

To apply the rule we can consider all adjacent pairs of vertices, such that both vertices in the pair are of degree 2. This takes \( O(n^2) \) time. Hence the rule is applicable in polynomial time.

\[ \square \]

**Reduction Rule 4** If \( V \setminus \{s, t\} = \emptyset \), then return a trivial YES instance.

**Lemma 3.** Reduction Rule 4 is safe and can be implemented in polynomial time.

**Proof.** Observe that if \( V \setminus \{s, t\} = \emptyset \), then \( G \) consists of only the edge \((s, t)\). Since there exists only one \( s-t \) path, no trackers are needed for distinguishing. Hence, the given instance is a YES instance. It can be seen that such a case can be identified in constant time, and thus the rule is applicable in polynomial time.

\[ \square \]

Next we recall a reduction rule from [8] that removes all those triangles from \( G \) that contain a vertex of degree two.

**Reduction Rule 5** If there exist \( a, b, c \in V(G) \) such that \( N(b) = \{a, c\} \) and \((a, c) \in E(G) \) and \( b \notin \{s, t\} \), then mark \( b \) as a tracker, delete \( b \) from \( G \) and set \( k = k - 1 \).

Next we recall a monotonicity lemma and a corollary from [4], which says that if a subgraph of \( G \) cannot be tracked with \( k \) trackers, then \( G \) cannot be tracked with \( k \) trackers either.

**Lemma 4.** [4] Let \( G = (V, E) \) be a graph and \( G' = (V', E') \) be a subgraph of \( G \) such that \( \{s, t\} \in V' \). If \( T \) is a tracking set for \( G \) and \( T' \) is a minimum tracking set for \( G' \), then \( |T'| \leq |T| \).

**Lemma 5.** [4] Any induced subgraph \( G' \) of \( G \) containing at least one edge will contain a pair of vertices \( u \) and \( v \) that satisfy following conditions: \( a \) there exists a path in \( G \) from \( s \) to \( u \), say \( P_{su} \), and another path from \( v \) to \( t \), say \( P_{vt} \), \( b \) \( V(P_{su}) \cap V(P_{vt}) = \emptyset \), \( c \) \( V(P_{su}) \cap V(G') = \{u\} \) and \( V(P_{vt}) \cap V(G') = \{v\} \).

For a subgraph \( G' \), and a pair of vertices \( u, v \in V(G') \) that satisfy the conditions of Lemma 5, we call vertex \( u \) as a local source for \( G' \) and vertex \( v \) as a local destination. Hence it follows from Lemma 5 that any subgraph containing at least one edge has a local source and a local destination. Note that a subgraph may have multiple local source-destination pairs.

Due to Lemma 4 and Lemma 5, we have the following two corollaries.

**Corollary 1.** [4] If a subgraph of \( G \) that contains both \( s \) and \( t \) cannot be tracked by \( k \) trackers, then \( G \) cannot be tracked by \( k \) trackers either.

**Corollary 2.** If there exists a subgraph \( G' \) of \( G \), and there exists a pair of vertices \( u, v \in V(G') \), such that \( u \) is a local source for \( G' \) and \( v \) is a local destination for \( G' \), and all paths between \( u \) and \( v \) in \( G' \) cannot be tracked by at most \( k \) trackers, then \( G \) cannot be tracked by at most \( k \) trackers.
In rest of the paper the phrase ‘subgraph cannot be tracked by \( k \) trackers’ implies that the paths between a local source and destination in a subgraph cannot be tracked with \( k \) trackers. Next corollary forms a starting point for the kernelization algorithms.

**Corollary 3.** The size of a minimum tracking set \( T \) for \( G \) is at least the size of a minimum FVS for \( G \).

For rest of the paper, we assume that the graph in context has already been preprocessed using Reduction Rules 1 to 5.

### 2.1 Vertex Disjoint Paths

Here we give a bound on the number of vertex disjoint paths that can exist between a pair of vertices in a graph \( G \), given that \( G \) can be tracked with at most \( k \) trackers. While in [4] it is proven that there can exist at most \( k + 3 \) vertex disjoint paths between a pair of vertices in \( G \), we improve the bound to \( k + 1 \). The new bound allows easy analysis and computation in future lemmas.

**Lemma 6.** If there exists two vertices \( u,v \in V \) such that there exists more than \( k + 1 \) vertex disjoint paths between \( u \) and \( v \), and the graph induced by these \( k + 1 \) paths along with \( u \) and \( v \) has \( u \) as a local source and \( v \) as a local destination, then \( G \) cannot be tracked with at most \( k \) trackers.

**Proof.** For a contradiction assume that there exist \( k + 2 \) vertex disjoint paths \( \mathcal{P} = \{P_1, \ldots, P_{k+2}\} \) between a pair of vertices \( u \) and \( v \) in \( G \), and \( G \) can be tracked with at most \( k \) trackers. Let \( G' \) be the subgraph induced by \( V(\mathcal{P}) \cup \{u,v\} \). Since \( u \) is a local source for \( G' \) and \( v \) is a local destination for \( G' \), there exists a path \( P_{su} \) that starts at \( s \) and ends at \( u \) and does not contain any vertex from \( G' - \{u,v\} \), and there exists a path \( P_{vt} \) that starts at \( v \) and ends at \( t \) and does not contain any vertex from \( G' - \{u,v\} \). See Figure 1. Consider a pair of paths \( P_i, P_j \in \mathcal{P} \). Let \( P_i \) and \( P_j \) do not contain any trackers (except for \( u \) and \( v \)). Now consider the \( s\)-\( t \) paths \( P_i' = P_{su} \cdot P_i \cdot P_{vt} \) and \( P_j' = P_{su} \cdot P_j \cdot P_{vt} \). Note that \( P_i' \) and \( P_j' \) contain the same sequence of trackers. Since these paths differ only in the vertices on paths \( P_i \) and \( P_j \), at least one vertex (except \( u \) and \( v \)) either on \( P_i \) or \( P_j \) has to be a tracker. Thus as long as there are two paths in \( \mathcal{P} \) without any trackers, there will be two \( s\)-\( t \) paths with same sequence of trackers. Hence, at least \( k + 1 \) paths in \( \mathcal{P} \) need a tracker on them. Due to Corollary 2 we know that if \( G' \) cannot be tracked with at most \( k \) trackers then \( G \) cannot be tracked with at most \( k \) trackers. This contradicts the initial assumption, and hence completes the proof. \( \square \)

**Lemma 7.** If there exists two vertices \( u,v \in V \) such that there exists more than \( k + 1 \) vertex disjoint paths between \( u \) and \( v \), then \( G \) cannot be tracked with at most \( k \) trackers.
Fig. 1. Vertex disjoint paths between a local source and destination

Fig. 2. Vertex disjoint paths between a pair of vertices

Proof. For a contradiction assume that there exist \( k + 2 \) vertex disjoint paths \( P = \{ P_1, \ldots, P_{k+2} \} \) between a pair of vertices \( u \) and \( v \) in \( G \), and \( G \) can be tracked with at most \( k \) trackers. Let \( G' \) be the subgraph induced by \( V(P) \cup \{ u, v \} \). Due to Lemma 6, there exists a local source, say \( u_1 \), and a local destination, say \( v_1 \), in \( G' \). We consider various cases possible based on position of \( u_1 \) and \( v_1 \) in \( G' \):

- When \( u = u_1 \) and \( v = v_1 \), or \( u = v_1 \) and \( v = u_1 \). Both of these cases are symmetric to each other, and have been proven in Lemma 6.

- When \( \{ u, v \} \cap \{ u_1, v_1 \} = \emptyset \). First we consider the case when \( u_1 \) and \( v_1 \) lie on different paths in \( P \). See Figure 2. We denote the path between \( u \) and \( u_1 \) (subpath of \( P_{k+2} \)) by \( \lambda_1 \), the path between \( u_1 \) and \( v \) (subpath of \( P_{k+2} \)) by \( \lambda_2 \), the path between \( u \) and \( v_1 \) (subpath of \( P_1 \)) by \( \lambda_3 \), and the path between \( v \) and \( v_1 \) (subpath of \( P_1 \)) by \( \lambda_4 \). We denote the paths in \( P \setminus \{ P_1, P_{k+2} \} \) by \( P' \). Any \( s-t \) path in \( G \) that passes through \( G' \) will be one among the following types:
  1. \( P_{su_1} \cdot \lambda_1 \cdot P_1 \cdot \lambda_4 \cdot P_{vt_1} \), where \( P_1 \in P' \)
  2. \( P_{su_1} \cdot \lambda_2 \cdot P_1 \cdot \lambda_3 \cdot P_{vt_1} \), where \( P_1 \in P' \)
  3. \( P_{su_1} \cdot \lambda_1 \cdot \lambda_3 \cdot P_{vt_1} \)
  4. \( P_{su_1} \cdot \lambda_2 \cdot \lambda_4 \cdot P_{vt_1} \)

Consider the first two cases. Let \( G'' \) be the graph induced by \( P' \). Observe that \( u \) and \( v \) are local source and destination for \( G'' \), since there exists a path \( P_{su_1} \cdot \lambda_1 \) from \( s \) to \( u \), and a path \( P_{vt_1} \cdot \lambda_4 \) from \( v \) to \( t \), and these paths intersect with \( G'' \) only at \( u \) and \( v \). Since there are \( k \) paths between \( u \) and \( v \) in \( G'' \), due to Lemma 6, these require at least \( k - 1 \) trackers in \( V(P') \setminus \{ u, v \} \).

If each of the \( k \) paths in \( P' \) has a tracker, the paths indicated in cases 3,4 have the same sequence of trackers, and this contradicts the assumption that \( G \) has a tracking set of size \( k \). Else, without loss of generality, let \( P_{k+1} \) be the path in \( P' \) that is left without a tracker.
Cases 3,4 denote two vertex disjoint paths between \(u_1\) and \(v_1\) along \(P_1\) and \(P_{k+1}\). Hence, due to Lemma 6, there must be a tracker on either \(V(\lambda_1) \cup V(\lambda_3) \setminus \{u_1, v_1\}\) or \(V(\lambda_2) \cup V(\lambda_4) \setminus \{u_1, v_1\}\). We consider following cases:

- There exists a tracker in \(V(\lambda_1) \setminus \{u_1, v_1\}\): Paths \(P_{su_1} \cdot \lambda_1 \cdot \lambda_3 \cdot P_{v_1t}\) and \(P_{su_1} \cdot \lambda_1 \cdot P_{k+1} \cdot \lambda_4 \cdot P_{v_1t}\) contain the same set of trackers.
- There exists a tracker in \(V(\lambda_2) \setminus \{u_1, v_1\}\): Paths \(P_{su_1} \cdot \lambda_2 \cdot P_{k+1} \cdot \lambda_3 \cdot P_{v_1t}\) and \(P_{su_1} \cdot \lambda_2 \cdot \lambda_4 \cdot P_{v_1t}\) contain the same set of trackers.
- There exists a tracker in \(V(\lambda_3) \setminus \{u_1, v_1\}\): Paths \(P_{su_1} \cdot \lambda_1 \cdot \lambda_3 \cdot P_{v_1t}\) and \(P_{su_1} \cdot \lambda_2 \cdot P_{k+1} \cdot \lambda_3 \cdot P_{v_1t}\) contain the same set of trackers.
- There exists a tracker in \(V(\lambda_4) \setminus \{u_1, v_1\}\): Paths \(P_{su_1} \cdot \lambda_1 \cdot P_{k+1} \cdot \lambda_4 \cdot P_{v_1t}\) and \(P_{su_1} \cdot \lambda_2 \cdot \lambda_4 \cdot P_{v_1t}\) contain the same set of trackers.

All the above cases contradict the assumption that \(G\) can be tracked with at most \(k\) trackers.

Next we consider the case when both \(u_1\) and \(v_1\) lie on the same path in \(\mathcal{P}\). Without loss of generality assume that \(u_1\) and \(v_1\) both lie on \(P_1\). Here note that there exists one path between \(u_1\) and \(v_1\) that is a strict subpath of \(P_1\), and the remaining paths between \(u_1\) and \(v_1\) pass through \(\mathcal{P} \setminus P_1\), via vertices \(u, v\). Observe that \(u, v\) are a local source and destination for the subgraph \(G''\) induced by \(V(\mathcal{P} \setminus P_1)\). Since there are \(k+1\) paths between \(u, v\) in the subgraph \(G''\), due to Lemma 6 at least \(k\) trackers are required in \(V(G'')\). If there are \(k+1\) trackers in \(V(G'')\), it contradicts the assumption that \(G\) can be tracked with at most \(k\) trackers. If there are \(k\) trackers in \(V(G'')\), without loss of generality, let \(P_2\) be the path without any tracker. Now observe that there are two paths between \(u_1\) and \(v_1\), the one that does not pass through \(u, v\) (subpath of \(P_1\)) and the one that passes through \(u, v\), that do not have any trackers on them. Due to Lemma 6, at least one tracker is required on one of these paths. Hence we have a contradiction to the assumption that the graph can be tracked with at most \(k\) trackers.

- When \(u = u_1\) and \(v \neq v_1\), or, \(u \neq u_1\) and \(v = v_1\). Consider \(u = u_1\), and \(v \neq v_1\). This case can be argued similar to the second case, except that now \(\lambda_1 = u = u_1\). Similarly, if \(u \neq u_1\), and \(v = v_1\), the case is similar to the second case, except that now \(\lambda_4 = v = v_1\).

Note that the case when \(u_1 = v_1\), is not possible after application of Reduction Rule 4. Correctness of the proof follows from Corollary 1. Hence the lemma holds.

Next we give a reduction rule based on Lemma 7.

**Reduction Rule 6** Let \(G'\) be a subgraph of \(G\), consisting of a pair of vertices \(a, b\) adjacent to \(m\) vertices each of degree two. If \(a\) is a local source for \(G'\) and \(b\) is a local destination for \(G'\), then arbitrarily mark \(m - 1\) of the \(m\) vertices of degree two as trackers and delete them. If \(m > k + 1\) return a NO instance, else set \(k = k - m - 1\).

**Lemma 8.** Reduction Rule 6 is safe and can be implemented in polynomial time.
Proof. Let \( V_m \) be the set of \( m \) vertices of degree two that are adjacent to \( a \) and \( b \) and let \( V_{m-1} \) be the set of \( m - 1 \) vertices that were marked as trackers and deleted. Let \( G' \) be the newly created graph after the deletion of \( V_{m-1} \). We claim that \( G' \) has a tracking set of size \( k - m - 1 \) if and only if \( G \) has a tracking set of size \( k \). Suppose \( G' \) has a tracking set \( T' \) of size \( k - m - 1 \). If we add the vertices of \( V_{m-1} \) back to \( G' \) along with their edges, there exists \( m \) vertex disjoint paths between \( a \) and \( b \). Since \( a \) and \( b \) are the local source and destination, due to Lemma 7, at least \( m - 1 \) trackers are required on the vertices of \( V_m \). We mark all the vertices in \( V_{m-1} \) as trackers. Now all paths between \( a \) and \( b \) are tracked. Since all other paths were already being tracked by \( T' \) in \( G' \), \( T' \cup V_{m-1} \) is a tracking set of size \( k \) for \( G \).

In the other direction let \( T \) be a tracking set of size \( k \) in \( G \). Let \( x \in V_m \setminus V_{m-1} \). We claim that \( G' \) has a tracking set of size \( k - m - 1 \). Suppose not. Then there exists two \( s-t \) paths, say \( P_1 \) and \( P_2 \), in \( G' \) that have the same sequence of trackers. Observe that if both \( P_1 \) and \( P_2 \) do not intersect with edges \((a, x)\) and \((x, b)\), then \( T \) is not a tracking set of size \( k \) in \( G \). This implies that \( P_1 \) and \( P_2 \) are also two paths with same sequence of trackers in \( G \). Note that the trackers on vertices in \( V_{m-1} \) cannot be used to distinguish between \( P_1 \) and \( P_2 \), as that would leave some untracked paths between \( a \) and \( b \). Thus \( T \) is not a tracking set for \( G \), which is a contradiction. This completes the proof of safeness of Reduction Rule 6.

In order to implement Reduction Rule 6, we consider each pair of vertices \( u, v \in V(G) \), and compare all their neighbors, to check for common neighbors of degree two. This can be done in \( O(n^4) \) time. Hence the rule can be applied in polynomial time. \( \square \)

2.2 Tree-sink structure

In this section we discuss a specific graph structure, namely the tree-sink structure, and prove a lower bound for the number of trackers required if such a structure exists in an \( s-t \) graph. A tree-sink structure in a graph \( G \), is a subgraph \( G' \) such that \( V(G') = V(Tr) \cup \{x\} \), where \( Tr \) is a tree with at least two vertices, and all of its leaves are adjacent to \( x \). Here \( Tr \) is the tree while \( x \) is the sink of the tree-sink structure. Note that \( x \) may or may not be adjacent to the non-leaf vertices of \( Tr \). We prove that if the sink \( x \) is adjacent to more than \( k + 1 \) vertices in \( Tr \), then \( G \) cannot be tracked with at most \( k \) trackers.

We start with a simple case when the graph \( G \) itself is a tree-sink structure and either \( s \) or \( t \) is the sink.

Lemma 9. Let \( G \) be an \( s-t \) graph that forms a tree-sink structure with \( x \in V(G) \) as the sink and \( x \in \{s, t\} \). If \( |N(x)| = \delta \), then \( \delta - 1 \) trackers are required in \( G \), and these trackers have to be in \( V(Tr) \), where \( Tr \) is the tree induced by \( G - x \).

Proof. Without loss of generalization we assume that \( x = t \). We root \( Tr \) at the source vertex \( s \). Consider that graph \( G \) has already been preprocessed using Reduction Rules 1, 2, 3 and 4.

We prove the lemma by induction on the value of \( \delta \). Observe that due to Reduction Rule 1, 2, 3 and 4, \( \delta = 1 \) is not possible. Thus the base case for
induction is when $\delta = 2$. Note that in this case $G$ is either a triangle or a four cycle. See Figure 4.

Consider the case when $G$ is a triangle. Due to Reduction Rule 5 the vertex $v \in V(G) \setminus \{s,t\}$ is marked as a tracker and deleted. Consider the case when $G$ is a four cycle. Observe that there exist two vertices, say $u,v$, of degree two each, adjacent to $s$ and $t$. Due to Reduction Rule 6 one among $u$ and $v$ is marked as a tracker and deleted. Note that in both the cases, after application of the corresponding reduction rules, $G$ comprises only of the edge $(s,t)$. Due to Reduction Rule 4 this is a trivial YES instance. Hence, when $\delta = 2$, exactly one tracker is required in $G$. This proves that the claim holds for the base case.

Next, for induction hypothesis, we assume that the claim holds for $\delta = i$, i.e. if the sink is adjacent to $i$ vertices, then $i - 1$ trackers are required in $G$. Consider the case when $\delta = i + 1$. Note that here $\delta \geq 3$. Due to Reduction Rule 1 all leaves in $Tr$ are adjacent to $t$, $Tr$ being the tree induced by $G - t$. Consider a leaf vertex, say $v_1 \in V(Tr)$, that is at maximum distance from $s$. Since $\text{deg}(v_1) = 2$, due to Reduction Rule 3 the degree of its parent node, say $v_p$, is at least 3. Thus either $v_p$ has another child node, or $v_p$ is adjacent to $t$. We analyze both the possibilities:
– **Case I**: $v_p$ has another child node, say $v_2$. Since $v_1$ is at maximum distance possible from $s$, $v_2$ is a leaf node in $Tr$. Observe that the graph $G'$ induced by $v_1$, $v_2$, $v_p$ and $t$ has $v_p$ as a local source and $t$ as a local destination, and $\text{deg}(v_1) = \text{deg}(v_2) = 2$. Due to Reduction Rule 6, either $v_1$ or $v_2$ will be marked a tracker and deleted. This reduces the value of $\delta$ from $i + 1$ to $i$, while using one tracker.

– **Case II**: $v_p$ is adjacent to $t$. Observe that $v_1$, $v_p$ and $t$ form triangle and $\text{deg}(v_1) = 2$. Due to Reduction Rule 5, $v_1$ will be marked as a tracker and deleted. This reduces the value of $\delta$ from $i + 1$ to $i$, while using one tracker.

In both the above cases, after application of reduction rule, $\delta = i$. Due to induction hypothesis, we know that when $\delta = i$, then $i - 1$ trackers are required in $G$. Since we already used a tracker in both the above cases, the total number of trackers required when $\delta = i + 1$, is $i$. Since the sink is $t$ itself, all the trackers need to be in $V(Tr)$. This completes the proof. 

Next we give a corollary which makes the above lemma more usable for the sake of our future arguments.

**Corollary 4.** Let $G$ be a graph and $G'$ be a subgraph of $G$ such that $G'$ induces a tree-sink structure with $v \in V(G')$ as its sink. If $|N(v) \cap V(G')| = \delta$, and $v$ is either a local source or a local destination for $G'$, then the size of a tracking set for $G$ is at least $\delta - 1$. Further these $\delta - 1$ trackers need to be in $V(G') \setminus \{v\}$.

**Proof.** Consider the subgraph $G'$. Without loss of generality, we assume that $v$ is a local destination for $G'$. Let $u \in V(G')$ be a local source corresponding to the local destination $v$. Due to Lemma 9, we have that $\delta - 1$ trackers are required in $V(G') \setminus \{v\}$ to track all paths between $u$ and $v$. From Corollary 4 if in a subgraph all paths between a local source and destination cannot be tracked with $k$ trackers then the graph cannot be tracked with $k$ trackers. Hence if $k < \delta - 1$, then $G$ cannot be tracked with at most $k$ trackers. Thus the size of a tracking set for $G$ is at least $\delta - 1$. It follows from Lemma 9 that these $\delta - 1$ trackers need to be in $V(G') \setminus \{v\}$. 

The next lemma generalizes the result in Corollary 4. We prove that regardless of where $s$ and $t$ lie in graph $G$, if $G$ forms a tree-sink structure, then the size of the tracking set for $G$ is at least the number of neighbors of the sink in the tree minus one.

**Lemma 10.** If an $s$-$t$ graph $G$ forms a tree-sink structure such that $x \in V(G)$ is the sink and $G - x$ induces a tree and $|N(x)| = \delta$, then the size of a tracking set for $G$ is at least $\delta - 1$, and at least $\delta - 2$ trackers are required in $G - x$.

**Proof.** Let $Tr$ be the tree induced by $V(G \setminus \{x\})$. The case when $x \in \{s, t\}$ has been proven in Lemma 9. Consider the case when $s, t \in V(Tr)$. We start by rooting the tree at $s$. Now create a graph $G'$ by removing the edge between $t$ and its parent vertex, say $\hat{t}$, in $Tr$. Observe that in $G'$, there exists a tree, say $Tr_1$ that can be considered rooted at $s$, consisting of all those vertices in $V(Tr)$ that
are not descendants of $t$ in $Tr$, with all its leaves adjacent to the vertex $x$. There exists another tree, say $Tr_2$, rooted at $t$, consisting of all of its descendants in $Tr$, with all of its leaves adjacent to the vertex $x$. See Figure 5. We denote the graph induced by $V(Tr_1) \cup \{x\}$ by $G_1$, and the graph induced by $V(Tr_2) \cup \{x\}$ by $G_2$. Let $\delta_1$ be the number of leaves in $Tr_1$, and $\delta_2$ be the number of leaves in $Tr_2$. Note that $\delta_1 + \delta_2 = \delta$.

![Figure 5](image_url)

Fig. 5. Removing edge $(\hat{t}, t)$ from $G$ creates two tree-sink structures in $G'$, with trees $Tr_1$ (shown with solid lines) and $Tr_2$ (shown in dashed lines) and sink $x$.

Note that $x$ is a local destination for $G_1$. Hence by Corollary 4, since $Tr_1$ has $\delta_1$ many leaves, the size of a tracking set for $G$ is at least $\delta_1 - 1$, and all these trackers must be in $V(Tr_1 - x)$.

Note that $x$ is a local source for $G_2$. Hence by Corollary 4, since $Tr_2$ has $\delta_2$ many leaves, the size of a tracking set for $G$ is at least $\delta_2 - 1$, and all these trackers must be in $V(Tr_2 - x)$.

If there exists at least $\delta_1 + \delta_2 - 1$ trackers in $G$, then the lemma holds. Else there exist $\delta_1 - 1$ trackers in $V(Tr_1 - x)$ and $\delta_2 - 1$ trackers in $V(Tr_2 - x)$. Hence, there exists exactly one path in $G_1$, say $P_1$, from $s$ to $x$ that does not contain any trackers, and exactly one path in $G_2$, say $P_2$, from $x$ to $t$ that does not contain any trackers. Consider the path $P' = \{s\} \cdot P_1 \cdot \{x\} \cdot P_2 \cdot \{t\}$. Note that if $G$ contains a total of $\delta_1 + \delta_2 - 2$ trackers, then $x$ is not a tracker and hence $P'$ does not contain any trackers. Recall the edge $e$ that was initially removed between $t$ and its parent, $\hat{t}$, in $Tr$. Consider the path in $G_1$ from $s$ to $\hat{t}$, say $P_{st}$.

We consider the following two scenarios.

- $P_{st}$ is a subpath of the path $P_1$. Consider the paths $\{s\} \cdot P_1 \cdot \{x\} \cdot P_2 \cdot \{t\}$, and $\{s\} \cdot P_{st} \cdot \{t\}$. Observe that both these paths have no trackers. Hence one more tracker is needed, either in $V(P_1)$ or $V(P_2)$ in order to distinguish them in $G$.

- $P_{st}$ is not a subpath of the path $P_1$. If $P_{st}$ does not have a tracker, both the paths $\{s\} \cdot P_1 \cdot \{x\} \cdot P_2 \cdot \{t\}$ and $\{s\} \cdot P_{st} \cdot \{t\}$ do not contain any trackers. If $P_{st}$ has a tracker, let $t_r \in V(P_{st})$ be the tracker that is closest to $\hat{t}$. Since
$\delta - 1$ is the minimum number of trackers required in $Tr_1$, there exists a path from $t_r$ to $x$ (and not passing through $s$) in $G_1$ that does not contain any trackers. Lets denote this path by $P_{t_r x}$. Let $P_{st_r}$ be the path from $s$ to $t_r$ that is a subpath of $P_{st}$. Now observe that paths $\{s\} \cdot P_{st_r} \cdot P_{t_r x} \cdot \{x\} \cdot P_2$ and $\{s\} \cdot P_{st} \cdot \{t\}$ have the same set of trackers. Hence in both the cases discussed one more tracker is required in $G$.

Thus the total number of required trackers in $G$ is at least $\delta_1 + \delta_2 - 2 + 1$, i.e. $\delta - 1$. Since the sink can be a tracker as well, a tree-sink structure requires at least $\delta - 2$ trackers in the vertex set of the tree. $\Box$

Lemma 10 along with Corollary 1 gives us the following corollary.

**Corollary 5.** In a graph $G$, if there exists a subgraph $G'$ and a vertex $v \in V(G')$, such that $G'$ forms a tree-sink structure with $v$ as a sink, and $|N(v) \cap V(G')| = \delta$ then the size of a tracking set for $G$ is at least $\delta - 1$. Further at least $\delta - 2$ trackers are required to be in $V(G') \setminus \{v\}$.

### 3 Quadratic Kernel for General Graphs

In this section we show that an instance $(G, k)$ of Tracking Paths, can be reduced to an equivalent instance $(G', k')$ such that if $(G, k)$ is a YES instance, then $|V(G')| = O(k^2)$, $|E(G')| = O(k^2)$ and $k' \leq k$. We start by applying Reduction Rules 1, 2, 3, 4 and 5. If the instance is not termed a NO instance by any of the reduction rules, we proceed with following. Recall from Corollary 3, that the size of a minimum tracking set $T$ for $G$ is at least the size of a minimum FVS for $G$. We start by finding a 2-approximate Feedback Vertex set $S$, using [1]. From Corollary 3, we have the following reduction rule.

**Reduction Rule 7.** Apply the algorithm of [1] to find a 2-approximate solution, $S$ for Feedback Vertex set. If $|S| > 2k$, then return that the given instance is a NO instance.

Observe that $F = G \setminus S$ is a forest. Now we try to bound the number of vertices and edges in $F$ for the case when all $s-t$ paths in $G$ can be tracked with at most $k$ trackers. Unless specified, we do not assume that the tree in context is a tree in $F$. When referring to a tree-sink structure, by ‘tree’ we mean the tree that forms the tree-sink structure.

We give some counting arguments to bound the vertices in $F$ and the edges incident on these vertices. We first categorize the vertices in $F$ as following:

- $V_1$: vertices that share a neighbor in $S$ with another vertex in the same tree as they belong.
- $V_2$: vertices that share a neighbor in $S$ with another vertex from some other tree in $F$.
- $V_3$: vertices that do not share any of their neighbors in $S$ with any other vertices in $F$. 
– $V_4$: vertices that do not have any neighbors in $S$.

$E_i$ denotes the set of edges between the set of vertices $V_i$ and $S$, where $i \in [3]$. Note that some vertices in $\mathcal{F}$ may belong to more than one of the above mentioned categories. While giving the counting arguments, we may allow this possible over counting since it does not change the asymptotic value of the bound on $V(\mathcal{F})$. Note that since each vertex in $S$ can have at most one vertex from $V_3$ adjacent to it, and $|S| \leq 2k$, it follows that $|V_3| \leq 2k$. The total number of vertices in $\mathcal{F}$ will be less than or equal to $|V_1| + |V_2| + |V_3| + |V_4|$. Now we explore each of the above categories in detail. Henceforth when we use the term neighbor(s) of a vertex for a vertex in $\mathcal{F}$, we assume that we refer to the neighbor(s) of the vertex in $S$.

### 3.1 Vertices that share a neighbor in $S$ with another vertex in the same tree

We first give a lemma that bounds the number of trees that can form a tree-sink structure with a common sink. This in turn helps us bound the number of trees in $\mathcal{F}$ whose vertices can form tree-sink structures with vertices in $S$ as sinks.

**Lemma 11.** Let there be a vertex $x$ such that $x$ is a sink for $t \geq 2$ tree-sink structures, then the numbers of trackers required is at least $t$.

![Fig. 6. Two tree-sink structures with a common sink.](image)

**Proof.** Suppose $x$ is a sink for $m$ trees. Let $G'$ be the graph induced by $x$ along with all the $m$ trees that form tree-sink structures with $x$ as the sink. Due to Lemma 5 there exists a local source and a local destination in $G'$. Note that if $x$ were either the local source or the local destination, then due to Corollary 4 each of the trees requires a tracker in their vertex set, and hence the lemma holds. Suppose not. Let $Tr_i$ denote the $i^{th}$ tree and $G_i$ denote the graph induced by the vertex set of $Tr_i$ along with the vertex $x$, for $i \in [m]$. Then due to Lemma 5 each graph $G_i$ has at least one pair of local source and destination vertices. Consider induced graphs $G_1$ and $G_2$. See Figure 6. Note that for $G_1$ there exists a path from $x$ to $t$ via $Tr_2$, that intersects with $G_1$ only at the sink $x$, thus making $x$ a local destination for $G_1$. Hence due to Corollary 6 at least one tracker is needed in $V(Tr_1)$. Next consider $G_2$. Note that there exists a path from $s$ to the
sink $x$, via $Tr_1$, that intersects at $G_2$ only at $x$, thus making $x$ a local source for $G_2$. Hence by Corollary 4 at least one tracker is needed in $V(Tr_2)$. Since these arguments can be extended for any induced graph $G_i$, it holds that at least one tracker is required in the vertex set of each of the trees. 

Next we give two lemmas to bound the vertices in $V_1$ and edges in $E_1$.

**Lemma 12.** For a vertex $f \in S$, the number of vertices in $F$ that form tree-sink structures with $f$ as a sink is at most $3k$ in a YES instance.

**Proof.** Let $f \in S$ and $V_f \subseteq V(F)$ be the set of vertices that form tree-sink structures with $f$ as a sink. Let $x$ be the number of trees that form tree-sink structures with $f$ as sink, each with $l_i$ number of vertices adjacent to $f$, where $i \in [x]$. Note that $|V_f| = \sum_{i=1}^{x} l_i$.

From Corollary 5 it is known that if a tree-sink structure is formed such that the sink is adjacent to $\delta$ vertices of the tree, then at least $\delta - 2$ trackers are required in the tree vertices. Hence, each of the trees forming tree-sink structures with $f$ as sink, require $l_i - 2$ trackers in their vertex set, $i \in [x]$. Note that a tracker in one tree of a tree-sink structure cannot act as a tracker for a tree-sink structure with a disjoint tree. Since the total budget for trackers is $k$, $\sum_{i=1}^{x} (l_i - 2) \leq k$. From Lemma 11 it follows that $f$ can be a sink for at most $k$ tree-sink structures. Thus $x \leq k$. It follows that $|V_f| = \sum_{i=1}^{x} l_i \leq 3k$. 

**Lemma 13.** The number of vertices in $F$ that share neighbors in $S$ with vertices from the same tree is at most $6k^2$ and the number of edges between these vertices and $S$ is at most $6k^2$ in a YES instance.

**Proof.** When two or more vertices from a tree in $F$ share a common neighbor, say $f \in S$, they form a tree-sink structure, the tree being the minimal connected subtree containing all neighbors of $f$ in that tree, and $f$ being the sink. Due to Lemma 12 it is known that for a vertex $f \in S$, at most $3k$ vertices from $F$ form tree-sink structures with $f$ as a sink. Since $|S| \leq 2k$, the total number of vertices in $V_1$ is at most $(2k)3k$ i.e. $6k^2$. As we considered only single edges between the sink and its neighbors in the trees of the tree-sink structures, $|E_1| \leq 6k^2$. 

**Reduction Rule 8** If the number of vertices that share neighbors in $S$ with vertices from the same tree are more than $6k^2$, then we return a NO instance.

**Lemma 14.** Reduction Rule 8 is safe and can be applied in polynomial time.

**Proof.** Safeness of the reduction rule follows from Lemma 13. To apply the rule, for each vertex $f \in S$, we consider the subgraph $G'$ induced by $G \setminus (S \setminus \{f\})$. There can be at most $n$ trees in $G' \setminus \{f\}$, and each tree can have at most $n$ vertices. For each tree we check if at least two vertices are adjacent to $f$. For tree that have at least two vertices adjacent to $f$, we count the number of such vertices. This can be done in $O(n^2)$ time. Since $|S| \leq 2k$, the total time taken will be $O(n^3)$. 

\[ \square \]
3.2 Vertices that share a neighbor with a vertex from another tree in $\mathcal{F}$

Note that these vertices may or may not share a neighbor with a vertex in the same tree. As mentioned before we allow the possible over counting of vertices of $V_1$ here as the final bound calculated is still $O(k^2)$.

Observe that if a vertex $v \in V_2$ belongs to a tree $Tr \in \mathcal{F}$ such that $|N(V(Tr)) \cap S| = 1$, then $v$ belongs to $V_1$ as well. Thus in such a case we need not count $v$ in $V_2$. Excluding such vertices, we can assume that for each vertex $v \in V_2$ it holds that $|N(V(Tr)) \cap S| \geq 2$, where $Tr \in \mathcal{F}$ is the tree to which $v$ belongs. This implies that either $v$ has at least two neighbors in $S$, or there exists a vertex in $V(Tr) - v$ that is adjacent to a vertex in $S - f$, where $f \in N(v) \cap S$ is adjacent to a vertex in another tree in $\mathcal{F}$. Since we need an upper bound on $V_2$, we assume the second case, i.e. for each vertex in $V_2$ there exists another vertex in the same tree and has a different neighbor in $S$.

Let $a, b, c \in V(Tr)$ where $Tr \in \mathcal{F}$. If $u, v \in S$ and $u, v \in (N(a) \cup N(b) \cup N(c))$ then at least two vertices among $a, b, c$ share a neighbor in $S$ (either $u$ or $v$) and thus belong to $V_1$. Hence we can assume that a pair of vertices in $S$ is adjacent to at most two vertices from $V_2$ from each tree in $\mathcal{F}$.

\textbf{Lemma 15.} The number of vertices in $\mathcal{F} \cap N(S)$ that belong to vertex disjoint paths without any trackers, between a pair of vertices in $S$ are at most $3k$ in a YES instance.

\textit{Proof.} Let $u, v \in S$ be a pair of vertices such that there exist three vertex disjoint paths (comprising of vertices from $\mathcal{F}$ between them. Let $G'$ be the subgraph induced by $u$ and $v$ along with the three vertex disjoint paths between them. If $u$ and $v$ are trackers, and are not a local source-destination pair for $G'$, then it is possible that no trackers are required on the three paths between them. See Figure 7.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{three_paths.png}
\caption{Three paths without any trackers between $u, v \in S$, when $u, v$ are trackers but do not form a local source-destination pair. Square vertices belong to $X \subseteq V_2$.}
\end{figure}

Observe that all such pairs of $u, v \in S$ need to be disjoint, else the condition that they are not local source-destination pairs for the subgraph induced by
vertex disjoint paths passing through them, will be violated. Since each such pair of \( u, v \in S \) requires two trackers, there can be at most \( \frac{k}{2} \) such pairs of vertices in \( S \). Further, each such pair, can account for 6 vertices in \( F \) that form vertex disjoint paths between the pair. Hence, the number of vertices in \( F \) that belong to vertex disjoint paths without any trackers, between a pair of vertices in \( S \) are at most \( 3k \).

\[ \square \]

Next we give a lemma to bound the vertices in \( V_2 \).

**Lemma 16.** The number of vertices in \( F \) that share a neighbor with a vertex from another tree is at most \( 20k^2 - 7k \) and the number of edges between these vertices and \( S \) is at most \( 20k^2 - 7k \) in a YES instance.

**Fig. 8.** Vertices sharing neighbors with vertices from other trees.

**Proof.** We subdivide \( V_2 \) into following two sub-categories:

- Let \( V'_2 \) be the set of vertices such that each pair of vertices from a tree share their neighbors with a pair of vertices from another tree. See Figure 8(a).

  Consider a pair of vertices \( u, v \in S \). Observe that if pairs of vertices from different trees are incident to \( u \) and \( v \), they form vertex disjoint paths between \( u \) and \( v \) passing through the trees to which they belong. Let \( X \) be the set of vertices that form vertex disjoint paths without any trackers, between pairs of vertices of \( S \). From Lemma 15, \( |X| \leq 3k \).

Next we consider those pairs of vertices in \( S \) that are adjacent to at least four pairs of vertices of \( V'_2 \) leading to formation of at least four vertex disjoint paths. Observe that for such a pair of vertices in \( S \), at least one tracker is needed on the paths passing through vertices of \( V'_2 \) and the pair. Let there be \( m \) such pairs of vertices in \( S \), and \( Y \subseteq V'_2 \) be the set of vertices considered in this case. Let \( P_i \) denote the pairs of vertices from \( Y \) that are adjacent to the \( i \)th pair of vertices in \( S \).

From Lemma 4 it is known that if there are more than \( k+1 \) number of vertex disjoint paths between a pair of vertices, then \( G \) cannot be tracked with at most \( k \) trackers. Then the total number of trackers required is \( \sum_{i=1}^{m} (|P_i| - 3) \).
Since the total budget for the number of trackers is at most $k$, it follows that $\sum_{i=1}^{m}|P_i| - 3 \leq k$. Since each pair of vertices from $S$ considered here, requires at least one tracker on a vertex in $F$ (as there are at least 4 vertex disjoint paths between each pair), $m \leq k$. Hence $\sum_{i=1}^{m}|P_i| \leq 4k$, and hence the count of vertices forming the paths between the $m$ pairs, is less than or equal to $8k$. So far we assumed the vertex disjoint paths formed by vertices of $Y$ to be disjoint since we have counted trackers for each pair separately (and added them). However note that, trackers may be shared among different paths formed by vertices of $Y$. There might be two (or more) pairs of vertices in $S$ that share a neighbor $y \in Y$ (while forming vertex disjoint paths through other vertices in $F$) and $y$ may be a tracker. Since $|S| \leq 2k$, at most $2k$ vertices from $S$ may be adjacent to $2k$ vertices of $Y$ from a single tree, with only one of these vertices being a tracker. Thus for each of the $8k$ vertices counted so far in $Y$, we may have at most $(2k - 1)$ vertices of $Y$ that do not have a tracker. Hence $|Y| \leq 8k(2k - 1)$.

Let $V''_2$ be the set of vertices that share a neighbor with a vertex of another tree, but there does not exist a pair of vertices in any tree that shares their neighbors with a pair of vertices from another tree, i.e. $V''_2 = V_2 \setminus V'_2$. See Figure 8(b). Observe that in this case, a pair of vertices, say $u,v \in S$, can be adjacent to a pair of vertices of $V'_2$ from at most one tree. Since there are at most $\binom{2k}{2}$ pairs in $S$, and each pair is adjacent to two vertices from $V''_2$, it holds that $V''_2 \leq 2\binom{2k}{2}$. Hence $|V''_2| \leq 2k(2k - 1)$.

Hence $|V_2| \leq 20k^2 - 7k$. Since we have considered only single adjacencies between the vertices of $V'_2$ and $S$, it holds that $|E_2| \leq 20k^2 - 7k$.  

**Reduction Rule 9** If the number of vertices that share a neighbor in $S$ with vertices from same tree is more than $20k^2 - 7k$, then we return a NO instance.

**Lemma 17.** Reduction Rule 9 is safe and can be applied in polynomial time.

**Proof.** Safeness of the reduction rule follows from Lemma 16. In order to implement the rule, first we go through all vertices in $F$ and create a data structure that maintains which vertex belongs to which tree. This operation can be done in $O(n^2)$ time. Next for each vertex $v \in F$, if $Tr \in F$ is the tree to which $v$ belongs, we check if a vertex in $N(v) \cap S$ has a neighbor in $V(F) \setminus V(Tr)$. Since the number of trees in $F$ is $O(n)$, we can do the check for all vertices of $F$ in $O(n^2)$ time. \hfill $\square$

### 3.3 Vertices that do not have any neighbors in $S$

Note that all leaves in $F$ necessarily have a neighbor in $S$ due to Reduction Rules 1 and 2. Hence the vertices in $V_4$ are only the internal vertices of the trees in $F$.

**Lemma 18.** The number of vertices in $F$ that do not have neighbors in $S$ is at most $78k^2 - 15k$ in a YES instance.
the same tree as they belong to is $O(S)$ vertices and vertices in another category. We denote the set of vertices in this category as $V'$. We revisit the vertex categories listed in Section 3, and discuss the bound for the total number of edges in $V'$, i.e., $|E(V')| = O(k^2)$ in general graphs.

Proof. The total number of vertices in $F$ is less than or equal to $|V_1| + |V_2| + |V_3| + |V_4|$. Due to Reduction Rules [8, 9] and Lemma [18], $|F| \leq 6k^2 + 20k^2 - 7k + 2k + 78k^2 - 15k$. Hence $F \leq 104k^2 - 20k$. Thus $|V(G)| \leq 104k^2 - 18k$ after including the vertices from $S$. From Lemmas [13] and [16] it is known that the total number of edges between $F$ and $S$ is at most $26k^2 - 5k$. Since $F$ is a forest, the maximum number of edges whose both end points lie in $S$ is $\binom{2k}{2}$. Hence the total number of edges in $G$ is at most $26k^2 - 5k + 104k^2 - 20k + 3k - 2k$, i.e., $132k^2 - 27k$. Since both the number of vertices and edges in $G$ is upper bounded by $O(k^2)$, it holds that TRACKING PATHS admits a kernel of size $O(k^2)$.

4 Linear Kernel for $d$-degenerate Graphs

In this section we show that an instance $(G, k)$ of TRACKING PATHS, where $G$ is a $d$-degenerate graph, can be reduced to an equivalent instance $(G', k')$ such that if $(G, k)$ is a YES instance then $|V(G')| = O(k)$, $|E(G')| = O(k)$ and $k' \leq k$. We revisit the vertex categories listed in Section 3 and discuss the bound for each category of vertices.

4.1 Vertices that share at least one neighbor in $S$ with at least one vertex in the same tree as they belong

We denote the set of vertices in this category as $V_1^p$. As explained in Subsection 3.1, vertices from a single tree sharing a neighbor in $S$ form a tree-sink structure.

Lemma 19. The number of vertices that share a neighbor in $S$ with a vertex in the same tree as they belong to is $O(k)$ and there are $O(k)$ edges between these vertices and vertices in $S$ in a YES instance for $d$-degenerate graphs.

Proof. Observe that when a vertex in a tree in $F$ shares a neighbor in $S$ with a vertex from the same tree, a tree sink structure is formed. Consider the graph $G'$ induced by the tree-structures formed by vertices in $V_1^p$ along with vertices in $S$. Let us assume that all $k$ trackers are used in $G'$ to distinguish between the paths formed in these tree-sink structures in $G'$. Since $G'$ is a subgraph of $G$ which is a
$d$-degenerate graph, the number of edges (paths) between the $k$ trackers and $2k$ vertices in $S$ is $O(k)$. Consider an edge (path) between a tracker $x \in V^p_1$ and a vertex $y \in S$. Edge $(x, y)$ (path) can be part of only a single tree-sink structure, and each tree-sink structure can have at most two paths left without a tracker. Since in each tree-sink structure, the number of vertices adjacent to the sink has only a constant difference with the number of trackers required, the number of vertices from $V^p_1$ in these tree-sink structure is $O(k)$. Further only constant many trees can be left without a tracker, in the cases when it is a tree with only two leaves, and the vertex in $S$ serves as a tracker. Hence $|V^p_1| = O(k)$ and $|E^p_1| = O(k)$. \[\square\]

**Reduction Rule 10** If the number of vertices that share neighbors in $S$ with vertices from same tree is more than $O(k)$, then we return a NO instance for $d$-degenerate graphs.

Safety of above rule holds from Lemma 19 and from Lemma 14 it follows that the rule can be applied in polynomial time.

### 4.2 Vertices that share a neighbor with at least one vertex from some other tree in $F$

We use $V^p_2$ to denote the set of vertices that share a neighbor in $S$ with a vertex from another tree, and $E^p_2$ to denote the edges between $V^p_2$ and $S$. If a pair of vertices, say $u, v \in S$, is adjacent to more than two vertices of $V^p_1$ category from a single tree, then some of these vertices share the vertex $u$ or $v$ as a neighbor and hence form a tree-sink structure. Such vertices will be counted in $V^p_1$ as well. Hence, we restrict ourselves to the assumption that each pair of vertices in $S$ is adjacent to at most one pair of vertices from $V^p_2$ in a single tree. We assume that for each vertex of $V^p_2$ in a tree, there exists one more vertex of in the same tree that is adjacent to a different neighbor in $S$.

**Lemma 20.** The number of vertices in $F$ that share a neighbor with a vertex from another tree is $O(k)$ in a YES instance for $d$-degenerate graphs.

**Proof.** We subdivide $V^p_2$ into following two sub-categories:

- Let $V^p_2'$ be the set of vertices such that each pair of vertices from a tree share their neighbors with a pair of vertices from another tree. See Figure 8(a). Consider a pair of vertices $u, v \in S$. Observe that if pairs of vertices from different trees are incident to $u$ and $v$, they form vertex disjoint paths between $u$ and $v$ passing through the trees to which they belong. Since the location of trackers depends on the position of local source and local destination, it is possible that both $u$ and $v$ are themselves trackers. Let $X^p$ be the set of vertices that form vertex disjoint paths without any trackers, between pairs of vertices of $S$. From Lemma 15, $|X^p| \leq 3k$.

Next we consider those pairs of vertices in $S$ that are adjacent to at least four pairs of vertices of $V^p_2$ leading to formation of at least four vertex disjoint...
paths. Let \( Y^p \subseteq V_2' \) denote the set of these vertices. Observe that for such a pair of vertices in \( S \), at least one vertex among the set of vertices of \( Y^p \) (or a vertex from the path joining a pair of vertices in \( Y^p \) inside a tree) that it is adjacent to, has to be a tracker.

Now consider the graph \( G' \) induced by the paths created by the vertices in \( Y^p \) and their neighbors in \( S \). Let us assume that \( k \) trackers are required to distinguish between the paths formed by vertices in \( Y^p \) along with their neighbors in \( S \). Since it is a \( d \)-degenerate graph, the number of edges (or paths) between \( k \) trackers in \( Y^p \) and the \( 2k \) vertices is bounded by \( O(k) \).

Next if we consider that each of these trackers is used to track more than one paths between pairs of vertices from \( S \). Let \( w \in Y^p \) be a vertex that lies on one of these paths, say \( P \). If more paths passing through the vertices of \( Y^p \) share the vertex \( w \) as a tracker, with no other vertex on them as a tracker, they either share an edge (subpath) with path \( P \) or not. If these paths share an edge (subpath), then this leads to formation of multiple vertex disjoint paths between endpoints of that edge (subpath). If these paths just share the vertex \( w \) as a tracker, it leads to multiple tree-sink structures being formed with \( w \) as a tracker. Hence such paths, not having any other tracker than \( w \), can only be constant in number. Thus the total count of such vertices in \( V_2'' = O(k) \). Observe that \( |E_2''| \leq O(k) \) as well since we have considered only single adjacencies between the vertices of \( V_2'' \) and \( S \).

- Let \( V_2''' \) be the set of vertices that share a neighbor with a vertex of another tree, but there does not exist a pair of vertices in any tree that shares their neighbors with a pair of vertices from another tree, i.e. \( V_2''' = V_2'' \setminus V_2' \).

See Figure 8(b). Let \( E_2''' \) be the set of edges between \( V_2''' \) and \( S \). Observe that, a pair of vertices, say \( u, v \in S \), can be adjacent to a unique pair of vertices of \( V_2''' \). If \( u \) and \( v \) are adjacent to two pairs of vertices of \( V_2''' \) from different trees, these vertices would have already been counted for in \( V_2' \).

Note that if for each pair of vertices in \( S \), the unique path between them that passes through a pair of vertices from \( V_2''' \) is shrunk to a single edge, then all these unique paths would reduce to edges between the vertices of \( S \). Since it is a \( d \)-degenerate graph, there can be at most \( O(k) \) such edges. Since each edge might have originally contained two vertices of \( V_2''' \), \( |V_2'''| = O(k) \).

Since single adjacencies have been assumed between vertices of \( V_2''' \) and \( S \), \( |E_2'''| = O(k) \) as well.

Hence \( |V_2'''| = O(k) \) and \( |E_2'''| = O(k) \).

\[ \square \]

**Reduction Rule 11** If the number of vertices that share a neighbor in \( S \) with vertices from same tree is more than \( O(k) \), then we return a NO instance for \( d \)-degenerate graphs.

Safeness of above rule holds from Lemma 20 and from Lemma 17 it follows that the rule can be applied in polynomial time.
4.3 Vertices that do not have any neighbors in $S$

We denote the set of vertices of this category as $V^p$. Here we consider the vertices in $F$ that do not have any neighbors in $S$. Note that all leaves in $F$ necessarily have a neighbor in $S$ due to Reduction Rules 1 and 2. Hence the vertices in $V^p$ are only the internal vertices of the trees in $F$.

Lemma 21. The number of internal (non-leaf) vertices in $F$ is $O(k)$ in a YES instance for $d$-degenerate graphs.

Proof. Since each vertex in $S$ can be adjacent to a single vertex of $V^p$, $|V^p| \leq 2k$. Due to structural properties of trees and Reduction Rule 3 and Lemmas 19 and 20, $|V^p| \leq 3(|V^p| + |V^p| + |V^p|)$. Thus $|V^p| = O(k)$. $\square$

4.4 Final Kernel

Theorem 2. Tracking Paths admits a kernel of size $O(k)$ in $d$-degenerate graphs.

Proof. The total number of vertices in $F$ is equal to $|V^p \cup V^p \cup V^p \cup V^p|$. It is known that $|V^p| \leq 2k$. From Lemmas 19, 20 and 21, it follows that $|F| = O(k)$. Hence $|V(G)| = O(k)$, after including the $2k$ vertices from $S$ as well. From Lemmas 19 and 20, it is known that the total number of edges between $F$ and $S$ is at most $O(k)$. The maximum number of edges whose both end points lie in $S$ is $O(k)$ since it is a $d$-degenerate graph. Hence $|E(G)| = O(k)$. This proves that Tracking Paths admits a kernel of size $O(k)$ in $d$-degenerate graphs. $\square$

Independently a linear kernel for planar graphs can be derived from the 4 approximation algorithm given in [8] as follows. Let the $G(V,E)$ be a planar graph with $F$ as the set of faces/regions in $G$. It is shown in [8] that the number of faces $|F| \leq 2OPT + 1$, where $OPT$ is the number of trackers in an optimum tracking set. If $k = OPT$, $|F| \leq 2k + 1$. Let $V_{\geq 3}$ be the set of vertices with degree greater than or equal to 3 and $V_2$ be the set of vertices with degree equal to 2. It is shown in [8] that $|V_{\geq 3}| \leq 2(|F| - 2)$. Thus, $|V_{\geq 3}| \leq 2((2k + 1) - 2) = 4k - 2$. After application of the reduction rules suggested by the authors there will be no vertices of degree one left in the graph, and there can exist at most one vertex of degree of degree two between a pair of vertices with degree higher than two. In order to bound the vertices in $V_2$, assume that all degree two vertices are short-circuited (the vertex is deleted and an edge is introduced between its neighbors). Observe that the number of edges in this new graph $G(V', E')$ will be an upper bound for $|V_2|$. Since short-circuited does not affect planarity of graph, due to Euler’s rule, $|E'| = |V'| + |F'| - 2$. Note that $|V'| = |V_{\geq 3}|$ and $|F'| = |F|$. Using the bounds for $F$ and $V_{\geq 3}$, we have $|E'| \leq 4k - 2 + 2k + 1$. Hence $|V_2| \leq 6k - 1$. Combining the bounds for $V_2$ and $V_{\geq 3}$, it follows that $|V| \leq 6k - 1 + 4k - 2$. Thus if a planar graph has a tracking set of size at most $k$, the vertex set is bounded by $10k - 3$. It follows from Euler’s rule that the number of edges will also be linear in $k$.
5 Hardness Result

Here we show that finding a tracking set of size at most \(n - k\) for a graph \(G\) with \(n\) vertices is \(W[1]\)-hard.

**Theorem 3.** For general graphs the problem of finding a tracking set of size at most \(n - k\) in a graph of \(n\) vertices, is \(W[1]\)-hard.

**Proof.** Tracking Paths has been shown NP-hard by a reduction from Vertex Cover in [4]. Specifically it has been shown that given a graph \(G(V,E)\) on \(n\) vertices one can construct in polynomial time a graph \(G'(V',E')\), where \(|V'| = n' = |V| + |E| + 5\), such that \(G\) has a vertex cover of size \(k\) if and only if \(G'\) has a tracking set of size \(k + |E| + 2\). It follows that \(G\) has an independent set of size \(k\) if and only if \(G'\) has a tracking set of size \(n - k + |E| + 2\), i.e. \(n' - k - 3\). Hence \(G\) has an independent set of size \(k + 3\) if and only if \(G'\) has a tracking set of size \(n' - k\). Since Independent Set is known to be \(W[1]\)-hard [7], it follows that the problem of finding a tracking set of size at most \(n - k\) is \(W[1]\)-hard as well. \(\sqcup \sqcap\)

6 Conclusions

In this paper we give improved kernels for the Tracking Paths problems. This is achieved via exploiting the connection between a feedback vertex set and tracking set, structural properties of trees, and by some counting arguments. An open problem is to find a \(O(c^k)\) FPT algorithm for the problem. Other directions to explore are approximation algorithms and studying the problem for other graph classes like directed graphs and weighted graphs.

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