Baryon Masses at Second Order in Chiral Perturbation Theory

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Abstract

We analyze the baryon mass differences up to second order in chiral perturbation theory, including the effects of decuplet intermediate states. We show that the Coleman–Glashow relation has computable corrections of order \((m_d - m_u)m_s\). These corrections are numerically small, and in agreement with the data. We also show that corrections to the \(\Sigma\) equal-spacing rule are dominated by electromagnetic contributions, and that the Gell-Mann–Okubo formula has non-analytic corrections of order \(m_s^2 \ln m_s\) which cannot be computed from known matrix elements. We also show that the baryon masses cannot be used to extract model-independent information about the current quark masses.
1. Introduction

In this paper, we analyze the octet baryon masses at second order in chiral perturbation theory. Specifically, we expand isospin-violating mass differences to $O(\varepsilon^2)$ and $O((m_d - m_u)m_s)$, and strange mass differences to $O(m_s^2)$. Our main motivation is to understand corrections to the famous $SU(3)$ relations for the baryon masses.

In chiral perturbation theory, the leading corrections to lowest-order predictions are frequently non-analytic in the quark masses. The non-analytic behavior in the chiral limit arises due to the presence of massless Nambu–Goldstone bosons in intermediate states. The non-analytic terms are therefore calculable if the necessary meson couplings are known. For example, the leading corrections to the Gell-Mann–Okubo formula are $O(m_s^{3/2})$ and $O(m_s^2 \ln m_s)$ [1]. However, we point out that the $O(m_s^2 \ln m_s)$ corrections depend on two-derivative meson–baryon couplings which are not known accurately, so only the $O(m_s^{3/2})$ corrections are calculable. On the other hand, we show that the leading corrections to the Coleman–Glashow relation are $O((m_d - m_u)m_s)$, and are calculable in terms of well-measured quantities. The predicted corrections agree with the data. We also show that the $\Sigma$ equal-spacing rule is dominated by electromagnetic corrections, not by contributions from intermediate meson states.

2. Effective Lagrangian

We will carry out our calculation in terms of an effective chiral lagrangian [2]. We will be brief in describing this lagrangian, since most of our notation is standard. The mesons are collected in the field

$$\xi(x) = e^{i\Pi(x)/f},$$

which is taken to transform under $SU(3)_L \times SU(3)_R$ as

$$\xi \mapsto L\xi U^\dagger = U\xi R^\dagger.$$

This equation implicitly defines $U$ as a function of $L$, $R$, and $\xi$. The effective lagrangian is most conveniently written in terms of

$$V_\mu \equiv \frac{i}{2} \left( \xi \partial_\mu \xi^\dagger + \xi^\dagger \partial_\mu \xi \right), \quad A_\mu \equiv \frac{i}{2} \left( \xi \partial_\mu \xi^\dagger - \xi^\dagger \partial_\mu \xi \right),$$

which transform under $SU(3)_L \times SU(3)_R$ as

$$V_\mu \mapsto UV_\mu U^\dagger + iU\partial_\mu U^\dagger, \quad A_\mu \mapsto UA_\mu U^\dagger.$$
For any field $X$ transforming like $A_\mu$, the covariant derivative
\[ \nabla_\mu X \equiv \partial_\mu X - i[V_\mu, X] \]
transforms under $SU(3)_L \times SU(3)_R$ as
\[ \nabla_\mu X \mapsto U \nabla_\mu X U^\dagger. \]

The chiral symmetry is broken explicitly by the quark masses and by electromagnetism. In the QCD lagrangian, the terms which explicitly break $SU(3)_L \times SU(3)_R$ can be written
\[ \delta \mathcal{L} = - (\bar{\psi}_L m_q \psi_R + h.c.) + e A^\mu (\bar{\psi}_L Q_L \gamma_\mu \psi_L + \bar{\psi}_R Q_R \gamma_\mu \psi_R), \]
where
\[ \psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \quad m_q = \begin{pmatrix} m_u & m_d & m_s \\ m_d & m_u & m_s \\ m_s & m_s & m_u \end{pmatrix}, \quad Q_L = Q_R = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}. \]

We note that the QCD lagrangian is formally invariant under $SU(3)_L \times SU(3)_R$ provided we assign to $m_q, Q_L,$ and $Q_R$ the “spurion” transformation rules
\[ m_q \mapsto L m_q R^\dagger, \quad Q_L \mapsto L Q_L L^\dagger, \quad Q_R \mapsto R Q_R R^\dagger. \]

We therefore define the quantities
\[ m_\pm \equiv \frac{1}{2} (\xi^\dagger m_q \xi \pm \text{h.c.}), \quad Q_\pm \equiv \frac{1}{2} (\xi^\dagger Q_L \xi \pm \xi Q_R \xi^\dagger). \]

These quantities have definite parity and transform under $SU(3)_L \times SU(3)_R$ as
\[ m_\pm \mapsto U m_\pm U^\dagger, \quad Q_\pm \mapsto U Q_\pm U^\dagger. \]

The simple transformation rules of the fields defined above make it easy to write down the effective lagrangian. For example, the leading terms involving only the mesons (which respect $C, P,$ and $T$) can be written
\[ \mathcal{L}_\Pi = f^2 \text{tr}(A^\mu A_\mu) + a f^2 \Lambda_\chi \text{tr}(m_+) + \frac{e^2 f^2 \Lambda_\chi^2}{16 \pi^2} \text{tr}(Q_\pm^2). \]

The factors of $f = 93$ MeV and $\Lambda_\chi \sim 1$ GeV have been inserted to ensure that the chiral expansion is an expansion in powers of $m_q/\Lambda_\chi$ [3].
Heavy fields such as baryons can be included in the effective lagrangian as long as one only uses the lagrangian to describe processes in which the momentum transfer to the heavy particle is small compared to the scale $\Lambda_\chi$. In this case, it is convenient to write an effective lagrangian in which the baryon fields create nearly on-shell particles \[4\][5]. The basic idea is to write the baryon momentum as $p = M_B v + k$, where $M_B$ is the common octet baryon mass in the $SU(3)$ limit, and $v$ is chosen so that all of the components of the residual momentum $k$ are small compared to $\Lambda_\chi$ for the process of interest. Hence, $k/\Lambda_\chi$ acts as an expansion parameter for these processes. The effective lagrangian is then labelled by $v$, the approximately conserved baryon velocity. We will include both octet and decuplet baryon states, since the decuplet states are not much heavier than the octet states, and thus intermediate decuplet states are not suppressed \[6\]. This lagrangian is written in terms of spinor octet fields $B$ and Rarita–Schwinger decuplet fields $T^\mu$ transforming under $SU(3)_L \times SU(3)_R$ as

$$B \mapsto UBU^\dagger, \quad T^{\mu jk\ell} \mapsto U^j_m U^k_n U^\ell_p T^{mn\mu}. \quad (13)$$

The lowest-order terms in the effective lagrangian involving octet baryon fields are

$$\mathcal{L}_{B1} = \text{tr} \left( \overline{B} iv \cdot \nabla B \right) + D \text{tr} \left( \overline{B} s^\mu \{ A_\mu, B \} \right) + F \text{tr} \left( \overline{B} s^\mu [ A_\mu, B ] \right) + b_1 \text{tr} \left( \overline{B} m_+ B \right) + b_2 \text{tr} \left( \overline{B} B m_+ \right) + d_1 \frac{e^2 \Lambda_\chi}{16\pi^2} \text{tr} \left( \overline{B} Q_+^2 B \right) + d_2 \frac{e^2 \Lambda_\chi}{16\pi^2} \text{tr} \left( \overline{B} B Q_+^2 \right) + d_3 \frac{e^2 \Lambda_\chi}{16\pi^2} \text{tr} \left( Q_+ \overline{B} B Q_+ \right) + d_4 \frac{e^2 \Lambda_\chi}{16\pi^2} \text{tr} \left( \overline{B} Q_+ \right) \text{tr} \left( Q_+ B \right). \quad (14)$$

(Terms involving $Q_-$ do not give rise to baryon masses at tree level, and are ignored here. We have omitted terms such as $\text{tr}(m_+ \text{tr}(\overline{B} B))$ and $\text{tr}(Q_+^2 \text{tr}(\overline{B} B))$ which do not give rise to baryon mass differences. Terms containing an odd number of $Q_+$’s are forbidden by charge conjugation invariance.) The spin matrix $s^\mu$ is given by $s^\mu \equiv (\gamma^\mu - v^\mu \gamma_5)$, and the covariant derivative acts on $B$ as in eq. (5).

We now briefly consider the extraction of information about current quark masses. In chiral perturbation theory, we can only determine ratios of quark masses, since the overall scale of the quark masses can be absorbed into unknown coefficients in the chiral expansion. Furthermore, baryon mass differences are unaffected by shifting the current quark masses by a common constant. Therefore, at leading order, the baryon mass differences are sensitive only to the ratio of differences

$$r \equiv \frac{m_d - m_u}{m_s - (m_u + m_d)/2}. \quad (15)$$
However, it is easy to see that the electromagnetic terms proportional to $d_1$ and $d_2$ can exactly mimic the effect of a shift in $r$, and therefore determination of $r$ is impossible without knowing $d_1$ and $d_2$. The computation of the electromagnetic contribution to the baryon mass differences will be considered in a future publication [7].

The leading-order lagrangian involving the decuplet fields is

$$\mathcal{L}_{T_1} = -\mathcal{T}^\mu iv \cdot \nabla T_\mu + \Delta \mathcal{T}^\mu T_\mu + C(\mathcal{T}^\mu A_\mu B + \text{h.c.}) + \cdots,$$  (16)

where the $SU(3)$ indices in the last term are contracted as $\epsilon_{jmn}T^{jk\ell}A^m_k B^n_\ell$. Here, $\Delta \simeq 300$ MeV is the octet–decuplet mass difference. The omitted terms describe couplings among the decuplet fields which play no role in our analysis.

In order to work consistently to order $m_q^2$, we will need the terms in the second-order chiral lagrangian which contribute to the baryon mass differences at tree level. The terms

$$\frac{k_0}{\Lambda_x} \text{tr}(m_+) \text{tr}(\mathcal{B}iv \cdot \nabla B) + \frac{k_1}{\Lambda_x} \text{tr}(\mathcal{B}m_+ , iv \cdot \nabla B) + \frac{k_2}{\Lambda_x} \text{tr}(\mathcal{B}iv \cdot \nabla Bm_+)$$  (17)

contribute to wavefunction renormalization of the baryons. We can eliminate these terms by making a field redefinition

$$B' \equiv B + \frac{k_0}{2\Lambda_x} \text{tr}(m_+)B + \frac{k_1}{2\Lambda_x} m_+B + \frac{k_2}{2\Lambda_x} Bm_+.$$  (18)

The effective lagrangian expressed in terms of $B'$ contains no terms of the form of eq. (17). We will work with the fields $B'$ in what follows, dropping the primes for notational convenience. There are also terms

$$\frac{\sigma_1}{\Lambda_x} \text{tr}(m_+^2) \text{tr}(\mathcal{B}B) + \frac{\sigma_2}{\Lambda_x} \text{tr}(m_+) \text{tr}(m_+) \text{tr}(\mathcal{B}B),$$  (19)

which do not give rise to mass differences, and terms

$$\frac{\ell_1}{\Lambda_x} \text{tr}(m_+) \text{tr}(\mathcal{B}m_+ B) + \frac{\ell_2}{\Lambda_x} \text{tr}(m_+) \text{tr}(\mathcal{B}Bm_+),$$  (20)

which can be absorbed into the terms proportional to $b_1$ and $b_2$ in eq. (14) at the order to which we are working. The only nontrivial terms in the second-order effective lagrangian are

$$\mathcal{L}_{B_2} = \frac{c_1}{\Lambda_x} \text{tr}(\mathcal{B}m_+^2 B) + \frac{c_2}{\Lambda_x} \text{tr}(\mathcal{B}Bm_+^2) + \frac{c_3}{\Lambda_x} \text{tr}(m_+ \mathcal{B}m_+ B) + \frac{c_4}{\Lambda_x} \text{tr}(\mathcal{B}m_+) \text{tr}(m_+ B).$$  (21)
To count independent parameters, we must take into account the following version of the Cayley–Hamilton theorem, which holds for any $3 \times 3$ traceless matrix $X$:

$$\text{tr}(\mathcal{B}\{X^2, B\}) + \text{tr}(\mathcal{B}XBX) - \text{tr}(\mathcal{B}X)\text{tr}(XB) - \frac{1}{2}\text{tr}(X^2)\text{tr}(\mathcal{B}B) = 0.$$  (22)

For $X = Q_+$, this immediately shows that one of the electromagnetic terms in eq. (14) is redundant. One can also use this relation to eliminate one of the quark mass terms in eq. (21), since the trace part of the mass matrix can be absorbed into redefinitions of the lowest-order couplings $b_1$ and $b_2$ at the order we are working. The 7 independent baryon mass differences are therefore determined by the quark mass ratio $r$ defined in eq. (15) and 8 effective couplings: 2 $O(m_q)$ couplings, 3 independent $O(e^2)$ terms, and 3 independent $O(m_q^2)$ terms. Including the loop corrections gives rise to calculable corrections to the tree-level relations which depend on additional parameters, as discussed below. One therefore expects 1 prediction if we include only $O(m_q)$ and $O(e^2)$ terms, and no predictions if we also include the $O(m_q^2)$ terms and the loop corrections. However, we will see that the actual situation is rather different.

3. Baryon Mass Relations

If we include $O(m_q)$ and $O(e^2)$ terms, we obtain several well-known relations among the baryon masses. They are the Gell-Mann–Okubo formula [8]

$$\Delta_{\text{GMO}} \equiv \frac{3}{4} \Lambda + \frac{1}{4} \Sigma - \frac{1}{2}(N + \Xi) = O(m_{u,d}) + O(e^2),$$  (23)

and the Coleman–Glashow relation [9]

$$\Delta_{\text{CG}} \equiv \Sigma^+ - \Sigma^- + n - p + \Xi^- - \Xi^0 = 0.$$  (24)

(We use the particle names to denote the corresponding masses.) In addition, there is one combination of baryon masses which gets contributions only from the electromagnetic terms (the “$\Sigma$ equal-spacing rule” [9])

$$\Delta_{\Sigma} \equiv (\Sigma^+ - \Sigma^0) - (\Sigma^0 - \Sigma^-) = 0(e^2).$$  (25)

In the language of $SU(3)$ group theory, the Gell-Mann–Okubo relation is broken only by operators transforming under the $\Delta I = 0$ piece of a 27, the Coleman–Glashow relation by the $\Delta I = 1$ piece of a 10, and the $\Sigma$ equal-spacing rule by the $\Delta I = 2$ piece of a 27. These relations hold at this order because the large representations required to break these relations do not appear.
The experimental values are

\[ \Delta_{\text{GMO}} \simeq +6.5 \text{ MeV}, \quad \Delta_{\text{CG}} = -0.3 \pm 0.6 \text{ MeV}, \quad \Delta_{\Sigma} = 1.7 \pm 0.2 \text{ MeV}. \]  

(26)

It can be checked that the leading contributions to these quantities from higher-order terms in the effective lagrangian are

\[ \Delta_{\text{GMO}} = O(m_s^2), \quad \Delta_{\text{CG}} = O(m_u d m_s^2) + O(e^2 m_s), \quad \Delta_{\Sigma} = O(e^2) + O(m^2_{u,d}), \]  

(27)

At our present level of understanding, these corrections are not calculable and can only be estimated using power-counting arguments.

By contrast, the loop corrections to these relations are calculable in terms of coefficients in the effective lagrangian which may be experimentally measured. The loop corrections are generally non-analytic in \( m_q \), and when they are larger than the counterterm contributions, we can make a prediction. Loop corrections of the form of fig. 1a with a vertex coming from the \( b_1 \) and \( b_2 \) terms in eq. (14) give rise to symmetry-breaking terms that transform as an \( 8 \), and so do not affect the relations considered above. Loop corrections of the form of fig. 1b with vertices coming from the \( D \) and \( F \) terms in eq. (14) give rise to baryon mass corrections of the form

\[ \delta M_B \sim \frac{m^3_{\Pi}}{16 \pi f^2} + \frac{\Delta_B m^2_{\Pi}}{16 \pi^2 f^2} \ln \frac{m^2_{\Pi}}{\mu^2} = O(m_q^{3/2}) + O(m^2_q \ln m_q). \]  

(28)

Here \( m_{\Pi} \) is a meson mass and \( \Delta_B \) is a baryon mass difference. The coefficients \( D \) and \( F \) can be determined from semileptonic hyperon decays. The baryon masses also receive contributions from loops involving vertices from the higher-order lagrangian such as

\[ \delta \mathcal{L}_{B2} = \frac{c}{\Lambda_x} \text{tr}(\overline{B} A^\mu B A_\mu). \]  

(29)

Loop corrections of the form of fig. 1a with a vertex from terms such as these can give rise to corrections to baryon masses of the form

\[ \delta M_B \sim \frac{m^4_{\Pi}}{16 \pi^2 f^2 \Lambda_x} \ln m^2_{\Pi} = O(m^2_q \ln m_q). \]  

(30)

Since the coefficients of terms such as these are not measured, the \( O(m_q^2 \ln m_q) \) non-analytic corrections cannot be computed in general. The dependence on \( m_q \) of graphs involving decuplet intermediate states is more complicated, since these contributions also depend on the decuplet–octet mass difference \( \Delta \). In the limit \( \Delta \to 0 \), these contributions have the form of eq. (28).
We now discuss corrections to the mass relations in detail. The leading loop corrections to the Gell-Mann–Okubo relation are $O\left(\frac{m^3}{2}s\right)$ and $O(m_s^2 \ln m_s)$. These have been recently discussed in ref. [10]. One can check that there are $O(m_s^2 \ln m_s)$ corrections to $\Delta_{\text{GMO}}$ from loop diagrams involving terms with unknown coefficients such as eq. (29), so these contributions are not computable; in any case, they are not expected to be much larger than the $O(m_s^2)$ corrections from the counterterms. In ref. [10], it was found that the $O(m_s^3/2)$ corrections give $\Delta_{\text{GMO}} \approx 15$ MeV. This is in satisfactory agreement with experiment, since we expect $O(m_s^2)$ terms to give $\Delta_{\text{GMO}} \sim 10$ MeV.

The Coleman–Glashow relation has no $O(m_q^2)$ or $O(e^2)$ corrections from tree-level terms, so the leading corrections come from loop effects. Loop contributions from two-derivative terms such as the one in eq. (29) do not contribute to $\Delta_{\text{CG}}$, because these they transform as $8'$s and $27$'s. We can therefore compute the leading corrections from the couplings in the lowest-order lagrangian, eqs. (14) and (16). The contribution from octet intermediate states is

$$\Delta_{\text{CG}}^8 = \frac{(K^+)^2 - (K^0)^2}{8\pi^2 f^2} \left[ D^2(\Xi - N) + 3DF(\Lambda - \Sigma) \right] + O(m_{u,d}^2). \quad (31)$$

This expression is $O(m_s(m_d - m_u))$; in particular, it is analytic in the quark masses. This arises as follows: The loop corrections to $\Delta_{\text{CG}}$ are $O(m_s(m_d - m_u) \ln m_s)$, where the logarithm involves the renormalization scale $\mu$. Because there are no $O(m_q^2)$ counterterms for $\Delta_{\text{CG}}$, changing $\mu$ changes the result by $O(m_q^3)$. We therefore chose $\mu = m_{K^0}$, which corresponds to neglecting $O(m_q^3 \ln m_s)$ contributions. With this choice, the logarithms can be expanded in meson mass differences, giving rise to an analytic result.

We now consider the contributions from decuplet intermediate states. Because the octet–decuplet mass splitting $\Delta \approx 300$ MeV is of the order of strange baryon mass differences, we will not treat $\Delta$ as a small parameter. This makes the expressions for the decuplet contribution more complicated:
\[ \Delta_{CG}^{10} = \frac{C^2}{32\pi^2 f^2} \left\{ (n - p) \left[ G_1(\Sigma^* - N, K) + 4G_1(\Delta - N, \pi) \right] \\
+ (\Xi^- - \Xi^0) \left[ G_1(\Sigma^* - \Xi, K) + 2G_1(\Omega - \Xi, K) \right. \\
+ G_1(\Xi^* - \Xi, \eta) + G_1(\Xi^* - \Xi, \pi) \left. \right] \\
+ \frac{1}{3}(\Sigma^+ - \Sigma^-) \left[ 8G_1(\Delta - \Sigma, K) + 2G_1(\Xi^* - \Sigma, K) \right. \\
+ 3G_1(\Sigma^* - \Sigma, \eta) + 2G_1(\Sigma^* - \Sigma, \pi) \left. \right] \\
+ \frac{1}{3}(\Sigma^*+ - \Sigma^*-) \left[ 2G_1(\Sigma^* - N, K) + 2G_1(\Sigma^* - \Xi, K) \right. \\
- 3G_1(\Sigma^* - \Sigma, \eta) - G_1(\Sigma^* - \Sigma, \pi) \left. \right] \\
+ \frac{20}{3}(\Xi^0 - \Xi^- - \Sigma^+- + \Sigma^-) \left[ G_1(\Delta - \Sigma, K) - G_1(\Delta - N, \pi) \right] \\
+ \frac{1}{3}(\Xi^- - \Xi^0) \left[ 2G_1(\Xi^* - \Sigma, K) - 3G_1(\Xi^* - \Xi, \eta) \right. \\
+ G_1(\Xi^* - \Xi, \pi) \left. \right] \\
+ \frac{1}{3}((K^0)^2 - (K^+)^2) \left[ 4G_2(\Delta - \Sigma, K) + G_2(\Sigma^* - N, K) \right. \\
- G_2(\Sigma^* - \Xi, K) + 2G_2(\Xi^* - \Sigma, K) \left. \right] \\
- 6G_2(\Omega - \Xi, K) \\
+ r \left[ G_3(\Sigma^* - \Sigma, \eta) - G_3(\Sigma^* - \Sigma, \pi) \right. \\
- G_3(\Xi^* - \Xi, \eta) + G_3(\Xi^* - \Xi, \pi) \left. \right] \right\} . \]

In writing this result, we have used SU(3) relations valid to \( O((m_d - m_u)m_s) \) for the decuplet masses to eliminate the dependence on the poorly-measured \( \Delta \) isospin splittings [11]. The terms proportional to \( r \) arise from \( \pi^0 - \eta \) mixing, which is proportional to \( r \). Here

\[ G_1(M, m) \equiv 2(M^2 - m^2)F(m/M) + (2M^2 - m^2)\ln \frac{m^2}{\mu^2} \]  \hspace{1cm} (33)

\[ G_2(M, m) \equiv -\frac{1}{M}(M^2 - m^2)F(m/M) - M\ln \frac{m^2}{\mu^2} \]  \hspace{1cm} (34)

\[ G_3(M, m) \equiv \frac{2}{3M}(M^2 - m^2)^2F(m/M) + M\left( \frac{2}{3}M^2 - m^2 \right)\ln \frac{m^2}{\mu^2} \]  \hspace{1cm} (35)

where
\[ F(x) \equiv \begin{cases} 
\frac{1}{\sqrt{1-x^2}} \ln \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}} & \text{for } x < 1, \\
\frac{2}{\sqrt{x^2-1}} \tan^{-1} \sqrt{x^2-1} & \text{for } x \geq 1.
\end{cases} \quad (36)\]

In these expressions, \( \mu \) is the renormalization scale. In the limit \( M \gg m \), we have

\[ G_1(M, m) \rightarrow (2M^2 - m^2) \ln \frac{4M^2}{\mu^2} - \frac{1}{2} m^2, \quad (37) \]
\[ G_2(M, m) \rightarrow -M \ln \frac{4M^2}{\mu^2}, \quad (38) \]
\[ G_3(M, m) \rightarrow M \left( \frac{2}{3} M^2 - m^2 \right) \ln \frac{4M^2}{\mu^2} - \frac{1}{6} M m^2, \quad (39) \]

up to terms that vanish as \( M \to \infty \). This shows that the decuplet contributions decouple in the limit where the octet–decuplet splitting \( \Delta \) gets large, since the only terms which do not vanish as \( \Delta \to \infty \) are analytic in the quark masses. In this limit, the decuplet contributions can be absorbed into counterterms in an effective lagrangian which does not contain decuplet fields, and the \( \mu \) dependence in eqs. (37)–(39) simply renormalizes the couplings in this effective lagrangian. In the opposite limit \( M \ll m \), we have

\[ G_1(M, m) \rightarrow -m^2 \ln \frac{m^2}{\mu^2}, \quad (40) \]
\[ G_2(M, m) \rightarrow \frac{\pi m}{2}, \quad (41) \]
\[ G_3(M, m) \rightarrow \frac{2\pi m^3}{3}, \quad (42) \]

up to terms that vanish as \( M \to 0 \). In this limit, \( \Delta_{10}^{CG} \) has the same non-analytic dependence on the quark masses as the contributions from octet intermediate states. Changing the renormalization scale \( \mu \) changes \( \Delta_{10}^{CG} \) by \( O(m^3_q) \), so we again take \( \mu = m_{K^0} \) for purposes of numerical evaluation.

Numerically, we find

\[ \Delta_{\text{theory}}^{CG} = -2.2D^2 + 1.3DF + C^2(0.5 \pm 0.5 + 8.1r) \text{ MeV}. \quad (43) \]

The numerical uncertainty in the coefficient of \( C^2 \) is due to the uncertainty in the decuplet isospin splittings. We use the lowest-order fit values for \( D \) and \( F \), since for these values the non-analytic corrections appear to be under control [12]: \( D = 0.85 \pm 0.06, F = 0.52 \pm 0.04 \). The errors are based on estimates of higher-order corrections, and may be larger. The coupling \( C \) can be determined from decuplet strong decays to be \( |C| = 1.2 \pm 0.1 \) [13]. The
value for \( r \) is currently rather controversial, since it is closely related to the problem of whether \( m_u = 0 \) \cite{14}. We will use the range \( 0.025 \leq r \leq 0.043 \) which allows \( m_u = 0 \) (upper value) as well as the value from lowest-order chiral perturbation theory (lower value). We then find

\[ \Delta_{\text{CG}}^\text{theory} = 0.2 \pm 0.7 \text{ MeV} \]  \hspace{1cm} (44)

where the quoted error is dominated by the uncertainty on the decuplet isospin splittings. This prediction is in agreement with the experimental result \( \Delta_{\text{CG}} = -0.3 \pm 0.6 \) MeV. We note that there is substantial cancellation between the octet and decuplet contributions: the octet contribution alone would give \( \Delta_{\text{CG}}^\text{theory} = -1.0 \pm 0.2 \) MeV.

We also considered corrections to the \( \Sigma \) equal-spacing rule. We find that all loop contributions to \( \Delta_{\Sigma} \) are at most \( O(m_{u,d}^2) \), and are numerically negligible compared to the experimental value. This can be understood from the fact that the only \( \Delta I = 2 \) operators formed from the quark mass matrix have coefficient \( (m_d - m_u)^2 \). We therefore conclude that \( \Delta_{\Sigma} \) is dominated by the electromagnetic contribution, which is expected to be of order

\[ \delta M_B^{\text{EM}} \sim \frac{e^2 \Lambda \chi}{16\pi^2} \sim 0.5 \text{ MeV}. \]  \hspace{1cm} (45)

This is clearly the right order-of-magnitude to explain the experimental value.

4. Conclusions

We have considered the chiral expansion of the baryon mass differences to \( O(m_q^2) \) and \( O(e^2) \) in the chiral expansion. We have found that we cannot extract model-independent information about the current quark masses, and that the \( O(m_s^2 \ln m_s) \) corrections to the Gell-Mann–Okubo relation are not calculable. On the other hand, we showed that there are calculable corrections to the Coleman–Glashow relation which are in agreement with the data, and that the corrections to the \( \Sigma \) equal-spacing rule are dominated by electromagnetic contributions.

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6. References

[1] L.-F. Li and H. Pagels, *Phys. Rev. Lett.* **26**, 1204 (1971); P. Langacker and H. Pagels, *Phys. Rev.* **D10**, 2904 (1974).

[2] J. Schwinger, *Phys. Lett.* **24B**, 473 (1967); S. Coleman, J. Wess, and B. Zumino, *Phys. Rev.* **177**, 2239 (1969); C. G. Callan, S. Coleman, J. Wess, and B. Zumino, *Phys. Rev.* **177**, 2247 (1969); S. Weinberg, *Physica* **96A**, 327 (1979).

[3] H. Georgi and A. V. Manohar, *Nucl. Phys.* **B234**, 189 (1984)

[4] H. Georgi, *Phys. Lett.* **240B**, 447 (1990); T. Mannel, W. Roberts, and Z. Ryzak, *Nucl. Phys.* **B368**, 204 (1992).

[5] E. Jenkins and A. V. Manohar, *Phys. Lett.* **255B**, 558 (1991).

[6] E. Jenkins and A. V. Manohar, *Phys. Lett.* **259B**, 353 (1991). For a review, see E. Jenkins and A. V. Manohar, *Proceedings of the Workshop on Effective Field Theories of the Standard Model*, edited by U.-G. Meißner (World Scientific, 1992).

[7] M. A. Luty and R. Sundrum, in preparation.

[8] M. Gell-Mann, *Phys. Rev.* **125**, 1067 (1962); S. Okubo, *Prog. Theor. Phys.* **27**, 949 (1962).

[9] S. Coleman and S. L. Glashow, *Phys. Rev. Lett.* **6**, 423 (1961).

[10] E. Jenkins, *Nucl. Phys.* **B368**, 190 (1992).

[11] R. F. Lebed, LBL-34704/UCB-PTH-93/27.

[12] M. A. Luty and M. White, LBL-34040/CfPA-TH-93-10, to be published in *Phys. Lett.* **B**.

[13] M. N. Butler, M. J. Savage, and R. P. Springer, *Nucl. Phys.* **B399**, 69 (1993).

[14] See e.g. D. B. Kaplan and A. V. Manohar, *Phys. Rev. Lett.* **56**, 2004 (1986); H. Leutwyler, *Nucl. Phys.* **B337**, 108 (1990).
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