NEW OSCILLATION CLASSES AND TWO WEIGHT BUMP CONDITIONS FOR COMMUTATORS.

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ABSTRACT. In this paper we consider two weight bump conditions for higher order commutators. Given \( b \) and a Calderón-Zygmund operator \( T \), define the commutator \( T^1_b f = [T, b]f = bTf - T(bf) \), and for \( m \geq 2 \) define the iterated commutator \( T^m_b f = [b, T^{m-1}_b]f \). Traditionally, commutators are defined for functions \( b \in BMO \), but we show that if we replace \( BMO \) by an oscillation class first introduced by Pérez [31], we can give a range of sufficient conditions on a pair of weights \((u, v)\) for \( T^m_b : L^p(v) \to L^p(u) \) to be bounded. Our results generalize work of the first two authors in [10], and more recent work by Lerner, et al. [28]. We also prove necessary conditions for the iterated commutators to be bounded, generalizing results of Isralowitz, et al. [20].

1. Introduction

In this paper we will study the commutator
\[
[b, T]f(x) = b(x)Tf(x) - T(bf)(x),
\]
where \( T \) is a linear operator and \( b \) is a function. If we let \( T^1_b = [b, T] \) and for \( m \in \mathbb{N} \) define \( T^m_b = [b, T^{m-1}_b] \), then \( T^m_b \) is referred to as an iterated commutator. We will consider \( T \) to be a Calderón-Zygmund operator or CZO (see Section 2 for relevant definitions). It is well known that such operators are well behaved when \( b \in BMO \):
\[
\|b\|_{BMO} = \sup_Q \int_Q |b(x) - b_Q| \, dx < \infty
\]
where \( b_Q = \frac{1}{|Q|} \int_Q b(x) \, dx \). Coifman, Rochberg, and Weiss [6] showed that if \( T \) is a Calderón-Zygmund operator then \( [b, T] \) is bounded on \( L^p(\mathbb{R}^n) \) if \( b \in BMO \). We are interested in two weight inequalities of the form
\[
\left( \int_{\mathbb{R}^n} |T^m_b f|^p u \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^n} |f|^p v \right)^{\frac{1}{p}}
\]
where \((u, v)\) is a pair of weights.

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Two weight inequalities for CZOs are significantly more difficult than the corresponding one weight inequalities. Sawyer was the first to find a complete characterization for the Hardy-Littlewood maximal operator and operators with positive kernels in \([33, 34]\). He proved that such operators are bounded from \(L^p(v)\) to \(L^p(u)\) if and only if certain testing conditions are satisfied. Roughly speaking, the testing conditions boil down to a restricted boundedness condition on the family \(\{1_Q v^{-\frac{p'}{p}}\}_Q\) and a similar dual condition. Recently, a complete characterization was found for the Hilbert transform by Lacey, et al. \([24, 22]\). While these testing conditions are necessary and sufficient, they are difficult to verify in practice since they involve the operator itself.

We will examine conditions known as \textit{bump conditions} on the pair of weights \((u, v)\) that are sufficient and sharp. The natural two weight generalization of the one weight condition given by

\[
\sup_Q \left( \int_Q u \right)^{\frac{1}{p}} \left( \int_Q v^{-\frac{p'}{p}} \right)^{\frac{1}{p'}} < \infty,
\]

is often necessary but almost never sufficient for boundedness. However, it is possible to modify this condition to get a “universal” sufficient condition. Condition (1.1) can be written as

\[
\sup_Q \|u^{\frac{1}{p}}\|_{p,Q} \|v^{-\frac{1}{p'}}\|_{p',Q} < \infty
\]

where \(\|\cdot\|_{p,Q}\) and \(\|\cdot\|_{p',Q}\) correspond to the \(L^p\) and \(L^{p'}\) averages over \(Q\) respectively. If we enlarge or ‘bump’ these averages in the scale of Orlicz spaces, we can get two weight boundedness of many classical operators. To understand this ‘enlarging’ we need a few definitions. We say \(A(t)\) defined on \([0, \infty)\) is a Young function if it is increasing, convex, \(A(0) = 0\), and \(A(t)/t \to \infty\) as \(t \to \infty\). We will also consider \(A(t) = t\) to be a Young function by convention. Given a Young function \(A\) there exists another Young function \(\bar{A}\), called the associate function of \(A\), such that \(\bar{A}^{-1}(t)A^{-1}(t) \approx t\). If \(A\) is a Young function, then we define the Orlicz average on a cube \(Q\) of a function \(f\) by

\[
\|f\|_{A,Q} = \inf \left\{ \lambda > 0 : \int_Q A\left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\}.
\]

When \(A(t) = t^a \log(e + t)^a\) for some \(a \in \mathbb{R}\) we will write

\[
\|f\|_{A,Q} = \|f\|_{L^p(\log L)^a,Q}
\]

and when \(B(t) = \exp(t^a) - 1\) we will write

\[
\|f\|_{B,Q} = \|f\|_{\exp(L^a),Q}.
\]

We say that a Young function \(A\) belongs to \(B_p\) if the following growth condition is satisfied:

\[
\int_1^\infty \frac{A(t)}{t^p} \frac{dt}{t} < \infty.
\]
A typical $B_p$ Young function is given by $A(t) = t^{p-\delta}$ or $A(t) = t^p (\log(e+t))^{-1-\delta}$ for some $\delta > 0$.

The two weight bump theory for CZOs is fairly well developed. Early work on the bump conditions for CZOs was done by the first author and Pérez [12, 13]. They formulated the so-called two weight bump conjecture in [13], conjecturing that if $T$ is a CZO, $A$ and $B$ are Young functions with $\bar{A} \in B_p'$ and $\bar{B} \in B_p$, and the pair of weights $(u, v)$ satisfies

\begin{equation}
\sup_Q \left\| u^{\frac{1}{p}} \right\|_{A,Q} \left\| v^{-\frac{1}{p'}} \right\|_{B,Q} < \infty,
\end{equation}

then

\begin{equation}
\left( \int_{\mathbb{R}^n} |Tf|^p u \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}^n} |f|^p v \right)^{\frac{1}{p}}.
\end{equation}

This conjecture was first shown for specific operators (e.g. the Hilbert transform, Riesz transforms, etc.) or in a restricted range on $p$ [7, 9, 29]. The full conjecture was proved by Lerner in [29]. In particular, he showed that CZOs are bounded from $L_p(v)$ to $L_p(u)$ if $(u, v)$ satisfy (1.4) with $A(t) = t^p \log(e+t)^{p-1+\delta}$ and $B(t) = t^{p'} \log(e+t)^{p'-1+\delta}$ (see Section 2 for the fact that $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$). Recent work on bump conditions for CZOs involves separating the bump conditions, namely, showing that the smaller condition

\begin{equation}
\left\| u^{\frac{1}{p}} \right\|_{A,Q} \left( \int_Q v^{-\frac{1}{p'}} \right)^{\frac{1}{p'}} + \left( \int_Q u \right)^{\frac{1}{p}} \left\| v^{-\frac{1}{p'}} \right\|_{B,Q} < \infty
\end{equation}

is sufficient for the boundedness of $T$. (For an example showing this condition is weaker, see [1].) Determining the sharpest separated bump conditions for CZOs is still an active area of research (see [14, 23, 25]).

For commutators of operators the full picture is still very much incomplete. The first preliminary results for commutators were done by Cruz-Uribe and Pérez [12, 13]. The first two authors [10] showed that if $T$ is a CZO, $b \in BMO$, and

\begin{equation}
\sup_Q \left\| u^{\frac{1}{p}} \right\|_{L_p(\log L)^{2p-1+\delta},Q} \left\| v^{-\frac{1}{p'}} \right\|_{L'_{p}(\log L)^{2p'-1+\delta},Q} < \infty,
\end{equation}

then

\begin{equation}
\left( \int_{\mathbb{R}^n} |[b, T]f|^p u \right)^{\frac{1}{p}} \leq C \| b \|_{BMO} \left( \int_{\mathbb{R}^n} |f|^p v \right)^{\frac{1}{p}}.
\end{equation}

There is difference in the powers on the logarithmic factors between the bumps for CZOs versus the bumps for commutators of CZOs. That is, the powers for CZOs are $p - 1 + \delta$ and $p' - 1 + \delta$, whereas the powers for the commutators are $2p - 1 + \delta$ and $2p' - 1 + \delta$. This additional factor of 2 in the exponent of the logarithm is a consequence of the more singular nature of commutators and is sharp in the sense that one cannot take $\delta = 0$ (see [32]). In general, commutators behave worse than the corresponding CZO; for an example of this fact in the one weight setting, we refer the reader to [5].
A close examination of the proof in [10] shows that one can actually separate the bumps in a way (see Remark 2.13 and the proof Lemma 4.1 in [10]) to show that if $\tilde{A} \in B_{p'}$ and $\tilde{B} \in B_p$ and the weights satisfy
\begin{equation}
(1.6) \quad \sup_Q \| u^\frac{1}{p} \|_{L_p(\log L)^{2p-1+\delta}} \| v^{-\frac{1}{p}} \|_{B,Q} + \sup_Q \| u^\frac{1}{p} \|_{A,Q} \| v^{-\frac{1}{p}} \|_{L_p(\log L)^{2p-1+\delta}} < \infty,
\end{equation}
then inequality (1.5) holds.

Recently, Lerner, Ombrosi, and Riviera-Ríos [28] have developed the first techniques for the iterated commutators $T_b^m$. They showed that if $b \in BMO$, $\tilde{A} \in B_{p'}$ and $\tilde{B} \in B_p$, and
\begin{equation}
(1.7) \quad \sup_Q \| u^\frac{1}{p} \|_{L_p(\log L)^{m+1}p-1+\delta} \| v^{-\frac{1}{p}} \|_{B,Q} + \sup_Q \| u^\frac{1}{p} \|_{A,Q} \| v^{-\frac{1}{p}} \|_{L_p(\log L)^{m+1}p-1+\delta} < \infty,
\end{equation}
then
\begin{equation}
\left( \int_{\mathbb{R}^n} |T_b^m f|^p u \right)^{\frac{1}{p}} \leq C \| b \|_{BMO}^{m} \left( \int_{\mathbb{R}^n} |f|^p v \right)^{\frac{1}{p}}.
\end{equation}
As we were completing this project, we learned that Isralowitz, Pott and Treil [20] have independently been studying the commutators $[b,T]$ in the two-weight and matrix setting. They introduced a class of functions $b$, different from $BMO$, that interact with the weights $u$ and $v$. Namely, they are able to show that the mixed norm condition
\begin{equation}
\sup_Q \| \| \| (b(x) - b(y))u^\frac{1}{p}(x) \|_{A_Q} || v(y)^{-\frac{1}{p}} \|_{B_Q} + \sup_Q \| \| \| (b(x) - b(y))v(y)^{-\frac{1}{p}} \|_{B_Q} || u(x)^{\frac{1}{p}} \|_{A_Q} < \infty,
\end{equation}
where $\tilde{A} \in B_{p'}$ and $\tilde{B} \in B_p$, is sufficient for the boundedness $[b,T] : L^p(v) \to L^p(u)$. Such conditions are related to our condition below in Theorem 1.1; however, our results hold for general iterated commutators $T_b^m$, when $m > 1$.

We remark in passing that there are several interesting results concerning two weight results for commutators when $u,v \in A_p$ (see (1.16) below for the definition of the $A_p$ class). When the stronger side assumption that $u,v \in A_p$ is made, it is possible to provide a complete characterization of the two weight boundedness of $T_b^m$ in terms of a weighted $BMO$ space. Such results were initiated by Bloom [3] for the Hilbert transform and have since seen a resurgence. We refer the interested reader to [15, 16, 27] and the references therein.

We now turn to our results. We will consider a new bump condition on weights $u$ and $v$ that directly interact with the multiplier $b \in L^1_{loc}(\mathbb{R}^n)$. Our first main result is the following.

**Theorem 1.1.** Suppose $1 < p < \infty$, $T$ is a CZO, $m \in \mathbb{N}$ and $b \in L^1_{loc}(\mathbb{R}^n)$. If $(u,v)$ is a pair of weights and $A,B,C,D$ are Young functions with $\tilde{A},\tilde{C} \in B_{p'}$, $\tilde{B},\tilde{D} \in B_p$, and such that
\begin{equation}
(1.8) \quad K = \sup_Q \| u^\frac{1}{p} \|_{A,Q} \| (b-b_Q)^m v^{-\frac{1}{p}} \|_{B,Q} + \sup_Q \| (b-b_Q)^m u^\frac{1}{p} \|_{C,Q} \| v^{-\frac{1}{p}} \|_{D,Q} < \infty,
\end{equation}
then inequality (1.6) holds.
then
\[
\left( \int_{\mathbb{R}^n} |T_b^m f|^p u \right)^{\frac{1}{p}} \lesssim K \left( \int_{\mathbb{R}^n} |f|^p v \right)^{\frac{1}{p}}.
\]

As a consequence of this result we can prove more traditional bump conditions by assuming that the multiplier \( b \) lies in an oscillation class related to \( BMO \). Given a Young function \( \Phi \) and the associated Orlicz space \( L^\Phi \), we define the class \( \text{Osc}(\Phi) = \text{Osc}(L^\Phi) \) to be the class of functions \( b \in L^1_{\text{loc}}(\mathbb{R}^n) \) whose semi-norm
\[
\|b\|_{\text{Osc}(\Phi)} = \sup_Q \| b - b_Q \|_{\Phi,Q}
\]
is finite. The space \( BMO \) corresponds to \( \text{Osc}(\Phi) \) for \( \Phi(t) = t \); equivalently \( BMO = \text{Osc}(L^1) \). It is well-known, by the John-Nirenberg inequality, that \( BMO \) functions are exponentially integrable. Specifically, if \( b \in BMO \) then there exist constants \( c, C > 0 \) such that for any cube \( Q \),
\[
\left( \int_Q \exp \left( \frac{c|b(x) - b_Q|}{\|b\|_{BMO}} \right) \right) dx \leq C. \tag{1.9}
\]
This exponential integrability allows us to view \( BMO \) as the space of functions with exponential oscillation. By the definition of the Orlicz norm (1.2) and the John-Nirenberg inequality (1.9) we have that
\[
\|b\|_{\text{Osc}(\Phi)} \lesssim \|b\|_{BMO},
\]
where \( \Phi(t) = e^t - 1 \). Moreover, since \( \Phi(t) \) is a Young function we have \( \Phi(t) \gtrsim t \) and hence \( \text{Osc}(\Phi) \subseteq BMO \) with
\[
\|b\|_{BMO} \lesssim \|b\|_{\text{Osc}(\Phi)}.
\]
Thus, that \( BMO = \text{Osc}(\Phi) \) where \( \Phi(t) = \exp t - 1 \); for this Young function it is customary to write \( \exp L \) instead of \( \Phi^1 \), and we will write \( BMO = \text{Osc}(\exp L) \).

The class \( \text{Osc}(\Phi) \) was first introduced in [31]. The importance of condition (1.8) is that it allows us to study commutators for \( b \in \text{Osc}(\Phi) \) for a general \( \Phi \). This approach was initiated in [2]. In particular, we have the following general result.

**Theorem 1.2.** Suppose \( 1 < p < \infty \), \( T \) is a CZO, and \( m \in \mathbb{N} \). Further, suppose \( A, B, C, D, X, Y, \Phi \) are Young functions with \( \bar{A}, \bar{C} \in B_{\rho'}, \bar{B}, \bar{D} \in B_p, \) and \( X, Y \) satisfy
\[
\text{(1.10)} \quad X^{-1}(t) \lesssim \frac{B^{-1}(t)}{\Phi^{-1}(t)^m} \quad \text{and} \quad Y^{-1}(t) \lesssim \frac{C^{-1}(t)}{\Phi^{-1}(t)^m}
\]
for large \( t \). If \( b \in \text{Osc}(\Phi) \) and \( (u, v) \) is a pair of weights such that
\[
K = \sup_Q \| u^\frac{1}{p} \|_{A,Q} \| v^{-\frac{1}{p}} \|_{X,Q} + \sup_Q \| u^\frac{1}{p} \|_{Y,Q} \| v^{-\frac{1}{p}} \|_{D,Q} < \infty,
\]
then
\[
\| T_b^m f \|_{L^p(u)} \lesssim K \| b \|^m_{\text{Osc}(\Phi)} \| f \|_{L^p(v)}.
\]

When \( b \in BMO \) we may take \( \Phi(t) = \exp t - 1 \) in Theorems 1.2 and we obtain the following result from [28, Theorem 1.1].
Suppose that $1 < p < \infty$, $T$ is a CZO, $m \in \mathbb{N}$, and $A, D$ are Young functions with $\tilde{A} \in B^p_r$ and $\tilde{D} \in B^p_p$. If $b \in BMO$ and $(u, v)$ satisfy

$$K = \sup_Q \left\| u^{\frac{1}{p}} \right\| A, Q \left\| v^{\frac{1}{r'}} \right\| L^{p'}(\log L)^{(m+1)p'-1+\delta}, Q + \sup_Q \left\| u^{\frac{1}{p}} \right\| L^p(\log L)^{(m+1)p-1+\delta}, Q \left\| v^{\frac{1}{r'}} \right\| D, Q < \infty$$

for some $\delta > 0$, then

$$\| T^m_b f \|_{L^p(u)} \lesssim K\| b \|^m_{BMO} \| f \|_{L^p(v)}.$$  

However, the real power of Theorem 1.2 is that it allows us to work with a scale of subclasses of $BMO$ to obtain better bump conditions. Define $\text{Osc}(L^\infty)$ as above but using the $L^\infty$ norm, i.e., $\text{Osc}(L^\infty)$ is the space of functions such that

$$\sup_Q \| b - b_Q \|_{L^\infty(Q)} < \infty.$$  

In fact, we have that $\text{Osc}(L^\infty) = L^\infty$. Clearly, $L^\infty \subseteq \text{Osc}(L^\infty)$. Conversely, if $b \notin L^\infty$, then there exists $Q$ with $\| b \|_{\infty,Q} = \infty$. But $b_Q$ is a finite constant and so $\| b - b_Q \|_{\infty,Q} = \infty$; hence $b \notin \text{Osc}(L^\infty)$.

Further, we have that

$$L^\infty = \text{Osc}(L^\infty) \subseteq \text{Osc}(\exp L) = BMO.$$  

We will consider Orlicz spaces $L^\Phi$ such that

$$L^\infty \not\subseteq \text{Osc}(\Phi) \subset BMO.$$  

One such class of functions is $\text{Osc}(\exp L^r)$ for $r > 1$, which corresponds to $\text{Osc}(\Phi)$ where $\Phi(t) = \exp(t^r) - 1$. An alternative definition of $\text{Osc}(\exp L^r)$ is all $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ that satisfy

$$\int_Q \exp(c|b(x) - b_Q|^r) \, dx \leq C$$

for some $c, C > 0$ and all cubes $Q$. Placing $b$ in these better oscillation classes will improve the size of the bump conditions on the weights.

Suppose that $1 < p < \infty$, $T$ is a CZO, $m \in \mathbb{N}$, and $A, D$ are Young functions with $\tilde{A} \in B^p_r$ and $\tilde{D} \in B^p_p$. If $b \in \text{Osc}(\exp L^r)$ for some $\epsilon > 0$ and $(u, v)$ satisfy

$$K = \sup_Q \left\| u^{\frac{1}{p}} \right\| A, Q \left\| v^{\frac{1}{r'}} \right\| L^{p'}(\log L)^{(m+1)p'-1+\epsilon}, Q$$

$$+ \sup_Q \left\| u^{\frac{1}{p}} \right\| L^p(\log L)^{(m+1)p-1+\epsilon}, Q \left\| v^{\frac{1}{r'}} \right\| D, Q < \infty$$

for some $\delta > 0$, then

$$\| T^m_b f \|_{L^p(u)} \lesssim K\| b \|^m_{\text{Osc}(\exp L^r)} \| f \|_{L^p(v)}.$$  

We note in passing that the requirement that $r > 1$ is necessary to get something new. If $r < 1$ and we let $\Phi(t) = \exp(t^r) - 1$, then $t \lesssim \Phi(t) \lesssim e^t - 1$, and so $\text{Osc}(\Phi) = BMO$.  

For $a > 0$ the class $\text{Osc}(\exp(L^a))$ is related to

$$\sqrt[\infty]{BMO} = \{ b \in L^1_{\text{loc}}(\mathbb{R}^n) : b \geq 0 \text{ and } b^a \in BMO \}.$$ 

An example of $b \in \sqrt[\infty]{BMO}$ is given by $b(x) = |\log x|^{1/a}$. The space $\sqrt[\infty]{BMO}$ was introduced by Johnson and Neugebauer in [21], but does not seem to have been studied in depth. The following theorem is sketched in [21], but for completeness we give a full proof.

**Theorem 1.5.** Given $a > 1$, $\sqrt[\infty]{BMO} \subseteq \text{Osc}(\exp(L^a))$, and

$$\|b\|_{\text{Osc}(\exp(L^a))} \lesssim \|b^a\|^{1/a}_{BMO}.$$ 

The reverse inequality is true for smaller powers: more precisely, if $0 < a \leq 1$, then

$$b \in \text{Osc}(\exp(L^a)) \Rightarrow b \in BMO \Rightarrow \|b^a\| \in BMO.$$ 

It is not clear whether there is a converse to Theorem 1.5.

If we assume $b \in \sqrt[\infty]{BMO}$ for large $a$, for example if $b \in \bigcap_{k=1}^{\infty} \sqrt[\infty]{BMO}$, then we obtain the same sufficient bump conditions for commutators that hold for CZOs themselves.

**Corollary 1.6.** Suppose that $1 < p < \infty$, $m \geq 1$, and $A, D$ are Young functions with $\tilde{A} \in B_{p'}$ and $\tilde{D} \in B_p$. Suppose further that $(u, v)$ satisfy

$$K = \sup_Q \|u^{1/p}_Q\|_{A,Q}\|v^{-1/p}_Q\|_{L^{p/(\log L)^{p'-1+\delta},Q}}$$

$$+ \sup_Q \|u^{1/p}_Q\|_{L^{p/(\log L)^{p-1+\delta},Q}}\|v^{-1/p}_Q\|_{D,Q} < \infty$$

for some $\delta > 0$. If $b \in \sqrt[\infty]{BMO}$ for $a > \max\{p, p'\} m/\delta$, then

$$\|T^m_b f\|_{L^p(u)} \lesssim K\|b^a\|^{1/\infty}_{BMO}\|f\|_{L^p(v)}.$$ 

If we take $A(t) = t^p \log(e + t)^{p-1+\delta}$ and $D(t) = t^{p'} \log(e + t)^{p'-1+\delta}$ (so that $\tilde{A} \in B_{p'}$ and $\tilde{D} \in B_p$), then condition (1.11) becomes

$$\|u^{1/p}_Q\|_{L^{p/(\log L)^{p-1+\delta},Q}}\|v^{-1/p}_Q\|_{L^{p/(\log L)^{p'-1+\delta},Q}} < \infty,$$

which is the optimal logarithmic bump condition that is sufficient for $T$ itself to satisfy $T : L^p(v) \rightarrow L^p(u)$. This result is striking given the fact, noted above, that commutators in general are more singular and so require stronger weight conditions.

We also consider the $b$ in the oscillation class $\text{Osc}(\exp(\exp(L^r)))$ associated with the Young function $\Phi(t) = \exp(\exp(t^r)) - e$. In particular, $b \in \text{Osc}(\exp(\exp(L^r)))$ if and only if there exist constants $c, C$ such that

$$\int_Q \exp \{ \exp(c|b(x) - b_Q|^r) \} \, dx \leq C.$$ 

Since $\exp(t^r) \lesssim \exp(\exp t)$ for large $t$ we have that $\text{Osc}(\exp(\exp L)) \subseteq \bigcap_{r > 1} \text{Osc}(\exp L^r)$ and hence $T^m_b$ will satisfy two weight bounds when $b \in \text{Osc}(\exp(\exp L))$ and the pair of weights satisfies (1.11). However, we can prove a stronger condition in the scale of the so-called log-log bumps.
Corollary 1.7. Suppose that $1 < p < \infty$, $m \geq 1$, and that $A, D$ are Young functions with $\bar{A} \in B_{p'}$ and $\bar{D} \in B_p$. Suppose $b \in \text{Osc}(\exp(\exp(L^{1/\epsilon})))$ for some $\epsilon > 0$. If the pair of weights $(u, v)$ satisfy

\begin{align}
(1.12) \quad & \sup_Q \|u^{\frac{1}{p}}\|_{A,Q} \|v^{-\frac{1}{p}}\|_{L^{p'}(\log L)^{p'-1}(\log \log L)(1+\epsilon)} Q \\
& + \sup_Q \|u^{\frac{1}{p}}\|_{L^p(\log L)^{p'-1}(\log \log L)(1+\epsilon)} \|v^{-\frac{1}{p}}\|_{D,Q} < \infty,
\end{align}

then

\[ \|T^m_b f\|_{L^p(u)} \leq C \|b\|_{\text{Osc}(\exp(\exp(L^{1/\epsilon})))}^m \|f\|_{L^p(v)}. \]

We note in passing that one can continue to restrict the class of $BMO$ and get better bump conditions on the weights, passing to "log-log-log bumps", etc. We leave the details to the interested reader.

Finally, we examine some necessary conditions for the iterated commutators to be bounded. In Theorem 1.1 the unbumped condition for a Calderón-Zygmund operator is obtained by taking $A(t) = C(t) = t^p$ and $B(t) = D(t) = t^{p'}$. With these Young functions, condition (1.8) becomes

\begin{align}
(1.13) \quad & \sup_Q \left( \int_Q |b - b_Q|^{mp} u \right)^{\frac{1}{p}} \left( \int_Q v^{-\frac{1}{p'}} \right)^{\frac{1}{p'}} + \sup_Q \left( \int_Q u \right)^{\frac{1}{p}} \left( \int_Q |b - b_Q|^{mp'} v^{-\frac{1}{p'}} \right)^{\frac{1}{p'}} < \infty.
\end{align}

This condition turns out to be necessary for the sparse operators that dominate the commutators (see Theorem 2.3 below). However, for the iterated commutators themselves we can only prove that a weaker condition is necessary. Recall that the Hilbert transform is given by

\[ Hf(x) = \text{p.v.} \int_R \frac{f(y)}{x - y} dy. \]

For first order commutators of the Hilbert transform, or higher order commutators of even orders, we obtain the following necessary conditions.

Theorem 1.8. Suppose $1 < p < \infty$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, $(u, v)$ is a pair of weights, and $H$ is the Hilbert transform on $\mathbb{R}$. If

\[ \left( \int_R |[b, H]f|^p u \right)^{\frac{1}{p}} \leq C \left( \int_R |f|^p v \right)^{\frac{1}{p}}, \]

then

\begin{align}
(1.14) \quad & \sup_I \left( \frac{1}{v(I)} \int_I |b - b_I|^p u \right)^{\frac{1}{p}}, \sup_I \left( \frac{1}{u^{-\frac{1}{p'}}(I)} \int_I |b - b_I|^p' v^{-\frac{1}{p'}} \right)^{\frac{1}{p'}} \leq C.
\end{align}

Our techniques extend to even order iterated commutators when $b$ is real.
Theorem 1.9. Suppose $1 < p < \infty$, $m \in 2\mathbb{N}$, $b$ is a real valued locally integrable function, $(u, v)$ is a pair of weights, and $H$ is Hilbert transform on $\mathbb{R}$. If

$$
\left( \int_{\mathbb{R}} |H_b^m f|^p u \right)^{\frac{1}{p}} \leq C \left( \int_{\mathbb{R}} |f|^p v \right)^{\frac{1}{p}}
$$

then

$$
\sup_I \left( \frac{1}{v(I)} \int_I |b - b_I|^{mp} u \right)^{\frac{1}{p}}, \sup_I \left( \frac{1}{u^{-\frac{1}{p'}}(I)} \int_I |b - b_I|^{mp'} v^{-\frac{1}{p'}} \right)^{\frac{1}{p'}} \leq C.
$$

The necessity of condition (1.14) was discovered independently by Isralowitz, Pott and Treil [20] in higher dimensions. We provide a different proof for the Hilbert transform, which has the advantage that it yields (1.15) when $m > 1$ is even. On the other hand, while our approach works for the iterated commutators of even orders, it does not seem to work for odd orders.

It was also observed in [20] that condition (1.15) for $m = 1$ is equivalent to the weighted $BMO$ spaces considered in [3, 15, 16, 27] when $u, v \in A_p$. Thus Theorem 1.8 provides a new proof of the necessity condition in Bloom’s original result.

We now consider the one weight case $u = v$. We say a weight $w \in A_p$ if

$$
[w]_{A_p} = \sup_I \left( \int_I w \right) \left( \int_I w^{-\frac{1}{p'}} \right)^{\frac{1}{p'}} < \infty.
$$

If we combine Theorems 1.1 and 1.8 in the one weight case and use some well-known properties of $A_p$ weights, then we obtain the following characterization for the Hilbert transform. When assuming $w \in A_p$ it is well known that $H_b^m$ is bounded on $L^p(w)$ when $b \in BMO$. However, the necessity of $BMO$ seems to be new for higher order commutators (see [4] for the necessity when $m = 1$).

Corollary 1.10. Suppose $m \in \{1\} \cup 2\mathbb{N}$, $1 < p < \infty$, $b$ is a real valued function in $L^1_{\text{loc}}(\mathbb{R})$, $w \in A_p$, and $H$ is the Hilbert transform on $\mathbb{R}$. Then $H_b^m$ is bounded on $L^p(w)$ if and only if $b \in BMO$.

The remainder of the paper is organized as follows. In Section 2 we give the necessary definitions and machinery about approximation of commutators by so-called sparse operators. We also state two results, Theorem 2.3 and Corollary 2.4, for the sparse operators that together implies Theorem 1.1. We prove all our main results in Section 3. In Section 4 we prove Theorem 1.5 for the class $\sqrt{BMO}$. Finally, in Section 5 we prove the necessary conditions for the Hilbert transform, Theorems 1.8 and 1.9, and Corollary 1.10.

2. Preliminaries

2.1. Calderón-Zygmund operators. We say that $K(x, y)$ defined on $(x, y) \in \mathbb{R}^{2n}$ with $x \neq y$ is a standard kernel if

$$
|K(x, y)| \lesssim \frac{1}{|x - y|^n}, \quad x \neq y,
$$
and there exists some $\delta > 0$ such that

$$|K(x + h, y) - K(x, y)| + |K(x, y + y) - K(x, y)| \lesssim \frac{|h|^\delta}{|x - y|^{n+\delta}}$$

whenever $|x - y| > 2|h|$. An operator $T$ is said to be a Calderón-Zygmund operator if $T$ is bounded on $L^2(\mathbb{R}^n)$ and has the integral representation

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y) \, dy$$

for all $f \in L^2(\mathbb{R}^n)$ and $x \notin \text{supp } f$.

2.2. Orlicz spaces. We will need some basic facts and notation for Young functions and Orlicz spaces. Here we follow the treatment given in [8, Chapter 5]. For functions $A$ and $B$ we will use the notation $A(t) \lesssim B(t)$ to mean that there exist constants $c, t_0 > 0$ such that $A(t) \leq cB(t)$ for $t \geq t_0$ and $A(t) \approx B(t)$ to mean that $A(t) \lesssim B(t)$ and $B(t) \lesssim A(t)$. As mentioned in Section 1, every Young function has an associate Young function. Given $\Phi$ the associate Young function is defined by

$$\Phi(t) = \sup_{s > 0} (st - \Phi(s))$$

and satisfies

$$s \leq \Phi^{-1}(s)\Phi^{-1}(s) \leq 2s.$$

Recall that the Orlicz average is given by

$$\|f\|_{\Phi,Q} = \inf \left\{ \lambda > 0 : \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) \, dx \leq 1 \right\}.$$

We will need the general version of Hölder’s inequality for Orlicz spaces (see [8, 26] for details).

**Lemma 2.1.** Let $A, B$ be continuous and strictly increasing functions on $[0, \infty)$ and $C$ be Young function that satisfies $A^{-1}(t)B^{-1}(t) \lesssim C^{-1}(t)$ for $t$ large. Then

$$\|fg\|_{C;Q} \lesssim \|f\|_A\|g\|_B.$$

Moreover, we define the Orlicz maximal function

$$M_\Phi f(x) = \sup_{Q \ni x} \|f\|_{\Phi,Q}.$$

Pérez [30] showed that $M_\Phi$ is bounded on $L^p$ if and only if $\Phi \in B_p$:

$$\int_1^\infty \frac{\Phi(t) \, dt}{t^{p-1}} < \infty.$$

As mentioned in the introduction, $\Phi(t) = t^{p-\epsilon}$ and $\Phi(t) = t^{p/(\log(e + t))^{-1-\delta}}$ are typical $B_p$ Young functions. The Young function $\Psi(t) = t^p \log(e + t)^q$ for $p > 1$ and $q \in \mathbb{R}$ is called a log-bump and $\Theta(t) = t^p \log(e + t)^q \log[\log(e^e + t)]^r$ for $p > 1$ and
$q, r \in \mathbb{R}$ is known as a log-log bump. Will use the following calculations several times in what follows:

\[(2.1) \quad \Psi^{-1}(t) \approx \frac{t^\frac{1}{p}}{\log(e + t)^\frac{1}{p}}.\]

\[(2.2) \quad \bar{\Psi}(t) \approx \frac{t^{\frac{1}{p'}}}{\log(e + t)^\frac{1}{p'q}}.\]

\[(2.3) \quad \Theta^{-1}(t) \approx \frac{t^\frac{1}{p}}{\log(e + t)^\frac{1}{p}\log\log(e^{e^e} + t)^\frac{1}{p'}}.\]

\[(2.4) \quad \bar{\Theta}(t) \approx \frac{t^{\frac{1}{p'}}}{\log(e + t)^\frac{1}{p'q}\log\log(e^{e^e} + t)^\frac{1}{p'}}.\]

These estimates are well-known in the literature on bump conditions (see for example [11, p. 424]) and we refer the reader to the book [8, Chapter 5] for more details.

### 2.3. Sparse Families.

Over the past decade sparse families have played a central role in the study of Calderón-Zygmund operators. For our purpose we will follow the approach from [19, 26]. A cube is a subset of $\mathbb{R}^n$ of the form $Q = a + [0, h)^n$ where $a \in \mathbb{R}^n$ and $h > 0$. We call $h$ the side length of $Q$ and write $\ell(Q) = h$. A collection of cubes $D$ is said to be a dyadic grid if for each $Q \in D$, $\ell(Q) = 2^k$ for some $k \in \mathbb{Z}$, for each $k \in \mathbb{Z}$ the set $\{Q \in D : \ell(Q) = 2^k\}$ forms a partition of $\mathbb{R}^n$, and given $Q, P \in D$ one has $Q \cap P \in \{\emptyset, P, Q\}$.

Given a dyadic grid $\mathcal{D}$, a family of cubes $S \subset \mathcal{D}$ is a sparse family if there exists $0 < \delta < 1$, such that for each $Q \in S$ there exists a measurable subset $E_Q \subset Q$ with $|E_Q| \geq \delta|Q|$ and the family $\{E_Q : Q \in S\}$ is disjoint. Recently, Lerner, Ombrosi, and Rivera-Ríos [26] proved a sparse domination formula for commutators of Calderón-Zygmund operators. This was later extended to the iterated commutators by Ibañez-Firnkorn and Rivera-Ríos [19]. In particular they showed that given a CZO $T$, $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, and $f \in L^\infty(\mathbb{R}^n)$, there exist $3^n$ sparse families $\mathcal{S}_j \subseteq \mathcal{D}_j$, $j = 1, \ldots, 3^n$, such that

\[(2.5) \quad |T_b^m f(x)| \lesssim \sum_{j=1}^{3^n} \sum_{Q \in \mathcal{S}_j} \sum_{k=0}^m |b(x) - b_Q|^{m-k} \left(\int_Q |b - b_Q|^k f\right) 1_Q(x).\]

Our first lemma shows that we can simplify inequality (2.5) by only having to work with the endpoints of the sum corresponding to $k = 0$ and $k = m$. More precisely, given a sparse family $S$ and $b \in L^1_{\text{loc}}(\mathbb{R}^n)$, let

\[(2.6) \quad T^m_{S,b} f(x) = \sum_{Q \in S} \left(\int_Q |b - b_Q|^m f\right) 1_Q(x).\]
The adjoint operator \((T_{S,b}^m)^*\), defined by \(\int (T_{S,b}^m)^* g = \int f (T_{S,b}^m)^* g\), is given by

\[
(T_{S,b}^m)^* f(x) = \sum_{Q \in S} |b(x) - b_Q|^m \left( \int_Q f \right) 1_Q(x).
\]

**Lemma 2.2.** Suppose \(T\) is a CZO, \(b \in L_{\text{loc}}^1(\mathbb{R}^n)\), and \(f \in L_c^\infty(\mathbb{R}^n)\). Then there exist \(3^n\) sparse families \(S_j \subseteq \mathcal{D}_j, j = 1, \ldots, 3^n\), such that

\[
|T_b^m f(x)| \lesssim \sum_{j=1}^{3^n} T_{S_j,b}^m f(x) + (T_{S_j,b}^m)^* f(x).
\]

**Proof.** Fix a sparse family of cubes \(S\) and let \(x \in \mathbb{R}^n\) and \(Q \in S\) be such that \(x \in Q\). Then, for each \(k, 1 < k < m\),

\[
\sum_{k=0}^{m} |b(x) - b_Q|^m \left( \int_Q |b(y) - b_Q|^k f(y) dy \right) \leq m \int_Q \max\{|b(x) - b_Q|^m, |b(y) - b_Q|^m\} f(y) dy \approx |b(x) - b_Q|^m \int_Q f(y) dy + \int_Q |b(y) - b_Q|^m f(y) dy.
\]

Inequality (2.8) now follows from (2.5). \(\square\)

We now state our main result for a general sparse operator \(T_{S,b}^m\).

**Theorem 2.3.** Suppose \(m \in \mathbb{N}\), \(b \in L_{\text{loc}}^1(\mathbb{R}^n)\), \(S\) is a sparse family, and \(A, B\) are Young functions with \(\hat{A} \in B_{p'}\) and \(B \in B_p\). If \((u, v)\) are a pair of weights that satisfy

\[
\sup_{Q \in S} \|u^{\frac{1}{p}} 1_{A,Q}\|(b - b_Q)^m v^{-\frac{1}{p}} \|_{B,Q} < \infty,
\]

then the sparse operator \(T_{S,b}^m\) (2.6) satisfies

\[
\|T_{S,b}^m f\|_{L^p(u)} \leq C \|f\|_{L^p(v)}
\]

for all \(f \in L^p(v)\).

Conversely, if \(T_{S,b}^m\) satisfies (2.10), then the pair of weights \((u, v)\) satisfies (2.9) with \(A(t) = t^p\) and \(B(t) = t^{p'}\): that is,

\[
\sup_{Q \in S} \left( \int_Q u \right)^{\frac{1}{p}} \left( \int_Q |b - b_Q|^m v^{-\frac{1}{p}} \right)^{\frac{1}{p'}} < \infty.
\]
As a corollary we obtain the corresponding result for the adjoint operator \((T_{S,b}^m)^*\), since any linear operator \(S\) satisfies \(S : L^p(v) \to L^p(u)\) if and only if \(S^* : L^{p'}(u^{-\frac{1}{p'}}) \to L^{p'}(v^{-\frac{1}{p'}})\) by duality.

**Corollary 2.4.** Suppose \(m \in \mathbb{N}, b \in L^1_{\text{loc}}(\mathbb{R}^n)\), \(S\) is a sparse family, and \(C, D\) are Young functions with \(\bar{C} \in B_{p'}\) and \(\bar{D} \in B_p\). If \((u, v)\) are a pair of weights that satisfy

\[
\sup_{Q \in \mathcal{S}} \|(b - b_Q)^m u^\frac{1}{p}\|_{C,Q} \|v^{-\frac{1}{p'}}\|_{D,Q} < \infty,
\]

then the sparse operator \((T_{S,b}^m)^*\) (2.7) satisfies

\[
\|(T_{S,b}^m)^* f\|_{L^p(u)} \leq C\|f\|_{L^p(v)}
\]

for all \(f \in L^p(v)\).

Conversely, if \((T_{S,b}^m)^*\) satisfies (2.11), then the pair of weights \((u, v)\) satisfies

\[
\sup_{Q \in \mathcal{S}} \left( \int_Q |b - b_Q|^{mp} u^{\frac{1}{p}} \left( \int_Q v^{\frac{p'}{p'}} \right)^{\frac{1}{p'}} \right) \leq C.
\]

3. Proofs

In this section we prove our main results. Theorem 1.1 follows immediately from Lemma 2.2, Theorem 2.3, and Corollary 2.4. We now prove Theorem 2.3.

**Proof of Theorem 2.3.** Let \(S\) be a sparse family and let \(T_{S,b}^m\) be the associated operator (2.6). Furthermore, let

\[
K = \sup_{Q \in \mathcal{S}} \|u^\frac{1}{p}\|_{A,Q} \|(b - b_Q)^m v^{-\frac{1}{p'}}\|_{B,Q}.
\]

To estimate \(\|T_{S,b}^m f\|_{L^p(u)}\) we bound the bilinear form \(\int (T_{S,b}^m f)g u\) for \(g \in L^{p'}(u)\). We have

\[
\int_{\mathbb{R}^n} (T_{S,b}^m f)g u \, dx = \sum_{Q \in \mathcal{S}} \left( \int_Q |b - b_Q|^m f \right) \left( \int_Q g u \right) |Q|
\]

\[
\leq \sum_{Q \in \mathcal{S}} \|u^\frac{1}{p}\|_{A,Q} \|(b - b_Q)^m v^{-\frac{1}{p'}}\|_{B,Q} \|f v^{\frac{p'}{p}}\|_{B,Q} \|u^\frac{1}{p}\|_{A,Q} \|g u^{\frac{1}{p'}}\|_{A,Q} |Q|
\]

\[
\leq K \sum_{Q \in \mathcal{S}} \|f v^{\frac{1}{p'}}\|_{B,Q} \|g u^{\frac{1}{p'}}\|_{A,Q} |Q|
\]

\[
\leq K \int_{\mathbb{R}^n} M_B(f v^{\frac{1}{p'}}) M_A(g u^{\frac{1}{p'}}) \, dx
\]

\[
\leq K \|M_B(f v^{\frac{1}{p'}})\|_{L^{p'}} \|M_A(g u^{\frac{1}{p'}})\|_{L^{p'}}
\]

\[
\leq K \|M_B\|_{B(L^{p'})} \|M_A\|_{B(L^{p'})} \|f\|_{L^{p'}(v)} \|u\|_{L^{p'}(u)}.
\]
To prove necessity, let \( \sigma = v^{-\frac{1}{p'}} \), fix \( Q \in \mathcal{S} \) such that \( \int_Q |b - b_Q|^{|m|} \sigma \, dx > 0 \) (since any other cube will not contribute to the supremum), and define \( f \) by
\[
f = |b - b_Q|^{|m|}\sigma 1_Q.
\]
Then \( f \geq 0 \) and for \( x \in Q \),
\[
T_{S,b}^m f(x) \geq \int_Q |b - b_Q|^m f \, dx = \int_Q |b - b_Q|^{|mp'|} \sigma \, dx.
\]
If we plug this into the norm inequality, we have
\[
\left( \int_Q |b - b_Q|^{|mp'|} \sigma \, dx \right)^{\frac{1}{p'}} \leq C \left( \int_{\mathbb{R}^n} |f|^p v \, dx \right)^{\frac{1}{p}} \leq C \left( \int_Q |b - b_Q|^{|mp'|} \sigma \, dx \right)^{\frac{1}{p'}}.
\]
If we rearrange terms, we get
\[
\left( \int_Q u \, dx \right)^{\frac{1}{p'}} \left( \int_Q |b - b_Q|^{|mp'|} \sigma \, dx \right)^{\frac{1}{p'}} \leq C.
\]

**Proof of Theorem 1.2.** This result is a corollary of Theorem 1.1 and the Orlicz Hölder inequality, Lemma 2.1. Indeed, since \( B, X, \Phi \) satisfy
\[
\Phi^{-1}(t)^m X^{-1}(t) \lesssim B^{-1}(t)
\]
for \( t \) large, if we let \( \Phi_m(t) = \Phi(t^\frac{1}{m}) \), we have that
\[
\| (b - b_Q)^m v^{-\frac{1}{p'}} \|_{B, Q} \lesssim \| (b - b_Q)^m \|_{\Phi_m, Q} \| v^{-\frac{1}{p'}} \|_{X, Q} \approx \| (b - b_Q)^m \|_{\Phi, Q} \| v^{-\frac{1}{p'}} \|_{X, Q}.
\]
Here we used that
\[
\| f^m \|_{\Phi_m, Q} = \| f \|_{\Phi, Q}^m,
\]
which holds for any Young function when \( \Phi_m(t) = \Phi(t^\frac{1}{m}) \). Hence,
\[
\sup_Q \| u^\frac{1}{p} \|_{A, Q} \| (b - b_Q)^m v^{-\frac{1}{p'}} \|_{B, Q} \lesssim \| b \|_{\text{osc}(\Phi)} \sup_Q \| u^\frac{1}{p} \|_{A, Q} \| v^{-\frac{1}{p'}} \|_{X, Q} < \infty.
\]
A similar argument shows that
\[
\sup_Q \| (b - b_Q)^m u^\frac{1}{p} \|_{C, Q} \| v^{-\frac{1}{p'}} \|_{D, Q} \lesssim \| b \|_{\text{osc}(\Phi)} \sup_Q \| u^\frac{1}{p} \|_{Y, Q} \| v^{-\frac{1}{p'}} \|_{D, Q} < \infty,
\]
and thus the hypotheses of Theorem 1.1 are satisfied. \( \square \)

**Proof of Corollary 1.4.** Define
\[
B(t) = t^{p'} \log(e + t)^{p'-1+\delta},
\]
\[
C(t) = t^p \log(e + t)^{p-1+\delta},
\]
for some $\delta > 0$. Then by (2.2) we have that

$$\bar{B}(t) \simeq \frac{t^p}{\log(e + t)^{1 + \frac{1}{p'}}} \quad \text{and} \quad \bar{C}(t) \simeq \frac{t^{p'}}{\log(e + t)^{1 + \frac{1}{p'} \delta}},$$

so that $\bar{B} \in B_p$ and $\bar{C} \in B_{p'}$. Now define $X, Y,$ and $\Phi$ by

$$X(t) = t^{p'} \log(e + t)^{(1 + m)e^{p' - 1 + \delta}},$$

$$Y(t) = t^p \log(e + t)^{(1 + m)e^{p - 1 + \delta}},$$

$$\Phi(t) = \exp(t^{\frac{1}{p'}}).$$

Then $\text{Osc}(\Phi) = \text{Osc}(\exp L^{\frac{1}{2}})$. We will show that the two conditions in (1.10) in Theorem 1.2 hold: that is, that

$$\Phi - 1(t)^m X - 1(t) \lesssim B^{-1}(t) \quad \text{and} \quad \Phi - 1(t)^m Y - 1(t) \lesssim C^{-1}(t)$$

hold for large $t$. We will prove the required inequality for the triple $B, X,$ and $\Phi$; the proof of the other inequality for $C, Y,$ and $\Phi$ is the same. By (2.1) we have that

$$B^{-1}(t) \simeq \frac{t^{\frac{1}{p'}}}{\log(e + t)^{\frac{1}{p'} + \frac{1}{p'}}},$$

$$\Phi^{-1}(t) \simeq \log(e + t)^{\epsilon},$$

$$X^{-1}(t) \simeq \frac{t^{\frac{1}{p'}}}{\log(e + t)^{me^{p' - 1 + \delta} + \frac{1}{p'} + \frac{1}{p'}}},$$

and hence,

$$\Phi^{-1}(t)^m X^{-1}(t) \simeq \log(e + t)^{me} \frac{t^{\frac{1}{p'}}}{\log(e + t)^{me^{p' - 1 + \delta} + \frac{1}{p'} + \frac{1}{p'}}} = \frac{t^{\frac{1}{p'}}}{\log(e + t)^{\frac{1}{p'} + \frac{1}{p'}}} \simeq B^{-1}(t).$$

We now prove Corollary 1.6. This result will follow from the fact that $\sqrt{BMO} \subseteq \text{Osc}(\exp L^\alpha)$ (Theorem 1.5) and the following result, which roughly says that we may take $\epsilon = 0$ in Corollary 1.4.

**Theorem 3.1.** Suppose that $1 < p < \infty$, $m \geq 1$, and $A, D$ are Young functions with $\bar{A} \in B_{p'}$ and $D \in B_p$. Suppose further that the pair $(u, v)$ satisfies

$$K = \sup_Q \|u^{\frac{1}{p'}}\|_{A,Q} \|v^{-\frac{1}{p'}}\|_{L^{p'}(\log L)^{p' - 1 + \delta}, Q} + \sup_Q \|u^{\frac{1}{p'}}\|_{L^p(\log L)^{p - 1 + \delta}, Q} \|v^{-\frac{1}{p'}}\|_{D,Q} < \infty$$

for some $\delta > 0$. If $b \in \text{Osc}(\exp L^\alpha)$ for $0 < \epsilon < \frac{\delta}{m \max\{p, p'\}}$, then

$$\|T_b^m f\|_{L^p(u)} \lesssim K \|b\|_{\text{Osc}(\exp L^\alpha)}^m \|f\|_{L^p(v)}.$$

**Proof of Theorem 3.1.** Again we will use Theorem 1.2. Let $\delta > 0$ be as in the statement of theorem. Define

$$\alpha = \delta - \epsilon m' \quad \text{and} \quad \beta = \delta - \epsilon m,$$
so that $\alpha, \beta > 0$. Now define

\[ B(t) = t^{p'} \log(e + t)^{p'-1+\alpha} \]
\[ C(t) = t^{p} \log(e + t)^{p-1+\beta} \]

so that

\[ B(t) \approx \frac{t^p}{\log(e + t)^{1+\frac{1}{p}\alpha}} \quad \text{and} \quad C(t) \approx \frac{t^p}{\log(e + t)^{1+\frac{1}{p'}}\beta}, \]

which satisfy $\bar{B} \in B_p$ and $\bar{C} \in B_{p'}$. Further, define $X$, $Y$, and $\Phi$ by

\[ X(t) = t^{p'} \log(e + t)^{p'-1+\delta}, \]
\[ Y(t) = t^{p} \log(e + t)^{p-1+\delta}, \]
\[ \Phi(t) = \exp(t^{\frac{1}{p}}) - 1. \]

Then we have that

\[ \Phi^{-1}(t)^m X^{-1}(t) \approx \log(e + t)^m e \frac{t^{\frac{1}{p'}}}{\log(e + t)^{\frac{1}{p}+\frac{1}{p'}}} = \frac{t^{\frac{1}{p'}}}{\log(e + t)^{\frac{1}{p}+\frac{1}{p'}}} \approx B^{-1}(t); \]

similarly,

\[ \Phi^{-1}(t)^m Y^{-1}(t) \approx C^{-1}(t), \]

and so the hypotheses of Theorem 1.2 are satisfied. \qed

The proof of Corollary 1.7 follows a similar argument to that of Corollary 1.4 and we will only sketch it here. Define the Young functions

\[ B(t) = t^{p'} \log(e + t)^{p'-1+\alpha}, \]
\[ C(t) = t^{p} \log(e + t)^{p-1+\beta}, \]
\[ X(t) = t^{p'} \log(e + t)^{p'-1+\delta}(1+me)^{p'-1+\delta}, \]
\[ Y(t) = t^{p} \log(e + t)^{p-1+\delta}(1+me)^{p-1+\delta}, \]
\[ \Phi(t) = \exp[\exp(t^{\frac{1}{p}})] - e. \]

Then (2.3) and (2.4) imply

\[ B^{-1}(t) \approx \frac{t^{\frac{1}{p'}}}{\log(e + t)^{\frac{1}{p}+\frac{1}{p'}}}, \]
\[ X^{-1}(t) \approx \frac{t^{\frac{1}{p'}}}{\log(e + t)^{\frac{1}{p}+\frac{1}{p'}}}(1+me)^{\frac{1}{p}+\frac{1}{p'}}, \]
\[ \Phi^{-1}(t) \approx \log[\log(e + t)]^e. \]

Hence,

\[ \Phi^{-1}(t)^m X^{-1}(t) \approx \log[\log(e + t)]^{-me} \frac{t^{\frac{1}{p'}}}{\log(e + t)^{\frac{1}{p}+\frac{1}{p'}}}(1+me)^{\frac{1}{p}+\frac{1}{p'}} \approx B^{-1}(t). \]
4. Roots of BMO functions

In this section we prove Theorem 1.5. Recall that for $a > 1$ we define the space

$$\sqrt[2]{BMO} = \{b \in L^1_{\text{loc}}(\mathbb{R}^n) : b \geq 0 \text{ and } b^a \in BMO\}.$$

Some of what we do is implicit in [21], however, here we give a more complete and systematic treatment. We will make extensive use of the function $F(x) = x^{1/a}$, defined on $x \geq 0$, which satisfies $F(x^a) = x$. Since $a > 1$, $F$ is Hölder continuous of order $1/a$, that is,

(4.1) $$|F(x) - F(y)| \leq |x - y|^{1/a}.$$

We first observe that $\sqrt[2]{BMO} \subseteq BMO$, when $a > 1$. Indeed, given a cube $Q$ and $b \in \sqrt[2]{BMO}$ let $c_Q = F((b^a)_Q)$. Then

$$\int_Q |b(x) - c_Q| \, dx = \int_Q |F(b(x)^a) - F((b^a)_Q)| \, dx$$

$$\leq \int_Q |b(x)^a - (b^a)_Q|^{1/a} \, dx \leq \|b^a\|_{BMO}^{1/a}.$$

Proof of Theorem 1.5. Let $b \in \sqrt[2]{BMO}$; without loss of generality we may assume that $\|b^a\|_{BMO} = 1$. The general case when $\|b^a\|_{BMO} \neq 0$ now follows by homogeneity if we replace $b$ by $b/\|b^a\|_{BMO}^{1/a}$. Let $u = b^a$ so that $u \in BMO$. By (1.9) there exist constants $c, C > 0$ such that

$$\int_Q \exp(c|u - u_Q|) \leq C$$

for all cubes $Q$. Let $F(x) = x^{1/a}$, $x \geq 0$; then $F(u) = b$. Now fix a cube $Q$ and let $A = \frac{c}{2}$. Then

$$\int_Q \exp(A|b - b_Q|^a) = \int_Q \exp(A|b - F(u_Q) + F(u_Q) - b_Q|^a)$$

$$\leq \int_Q \exp(c|b - F(u_Q)|^a + c|F(u_Q) - b_Q|^a)$$

$$= \exp(c|F(u_Q) - b_Q|^a) \int_Q \exp(c|u - F(u_Q)|^a)$$

$$\leq \exp(c|F(u_Q) - b_Q|^a) \int_Q \exp(c|u - u_Q|)$$

$$\leq C \exp(c|F(u_Q) - b_Q|^a)|Q|,$$

where we used inequality (4.1) in the second to last inequality. Then, since $a > 1$,

$$|F(u_Q) - b_Q|^a = \left|\int_Q (F(u_Q) - b(x)) \, dx\right|^a \leq \int_Q |b(x) - F(u_Q)|^a \, dx.$$
\[ = \int_Q |F(u) - F(u_Q)|^a \leq \int_Q |u - u_Q|, \]

where we again used inequality (4.1). Hence, by Jensen’s inequality,

\[
\exp(c|F(u_Q) - b_Q|) \leq \exp \left( c \int_Q |u - u_Q| \right) \leq \int_Q \exp(c|u - u_Q|) \leq C.
\]

Therefore,

\[
\int_Q \exp(A|b - b_Q|^a) \leq C^2|Q|,
\]

which in turn implies that

\[
\|b - b_Q\|_{\exp(L^a),Q} \leq K,
\]

where \( K \) depends on \( A \) and \( C \). Thus \( b \in \text{Osc}(\exp L^a) \).

\[ \square \]

5. NECESSARY CONDITIONS FOR THE HILBERT TRANSFORM

In this section we prove the necessity conditions for the commutators of the Hilbert transform. We will follow the approach of Hytönen [17] for the Beurling transform (see also [18]), although we note that these proofs are for unweighted estimates.

Proof of Theorems 1.8 and 1.9. We first prove the case when \( m = 1 \). Suppose \((u, v)\) are weights and \( b \in L^1_{\text{loc}}(\mathbb{R})\) such that \([b, H] : L^p(v) \to L^p(u)\) with

\[
\| [b, H] f \|_{L^p(u)} \leq C \| f \|_{L^p(v)}.
\]

Let \( I = [a, b] \) be a fixed interval and \( c = (a + b)/2 \) be its center. Define

\[
f(x) = \text{sgn}(b(x) - b_I)|b(x) - b_I|^{p-1} 1_I(x);\]

then

\[
\int_I |b(x) - b_I|^p u(x) \, dx
\]

\[
= \int_I (b(x) - b_I) f(x) u(x) \, dx
\]

\[
= \int_I \int_I (b(x) - b(y)) f(x) u(x) \, dy \, dx
\]

\[
= \int_I \int_I \frac{(b(x) - b(y))}{x - y} (x - y) f(x) u(x) \, dy \, dx
\]

\[
= \int_I \int_I \frac{(b(x) - b(y))}{x - y} (x - c + c - y) f(x) u(x) \, dy \, dx
\]

\[
= \int_\mathbb{R} \left( \int_\mathbb{R} \frac{(b(x) - b(y))}{x - y} 1_I(y) \, dy \right) \frac{x - c}{|I|} 1_I(x) f(x) u(x) \, dx
\]

\[
+ \int_I \left( \int_\mathbb{R} \frac{(b(x) - b(y)) c - y}{x - y} 1_I(y) \, dy \right) 1_I(x) f(x) u(x) \, dy \, dx
\]

\[
= \int_\mathbb{R} [b, H](1_I(x) g_c(x) f(x)) u(x) \, dx - \int_\mathbb{R} [b, H](g_c(x) f(x)) u(x) \, dx,
\]

where \( g_c(x) = \frac{x - c}{|I|} 1_I(x) \).
where
\[ g_c(u) = \frac{u - c}{|I|} \mathbf{1}_I(u). \]

Note that \( \text{supp}(g_c) \subseteq I \) and \( g \in L^\infty \) with \( \|g_c\|_\infty \leq \frac{1}{2} \). Hence,
\[
\left| \int_\mathbb{R} [b, H](\mathbf{1}_I)(x)g_c(x)f(x)u(x) \, dx \right| \leq \int_\mathbb{R} \| [b, H](\mathbf{1}_I)(x)g_c(x) \| f(x)u(x) \, dx \\
\leq \| [b, H](\mathbf{1}_I) \|_{L^p(u)} \| g_c f \|_{L^p'(u)} \\
\leq \frac{C}{2} \| \mathbf{1}_I \|_{L^p(v)} \| f \|_{L^p'(u)} \\
\leq \frac{C}{2} v(I)^{1 \over p'} \left( \int_I |b - b_I|^p u \right)^{1 \over p'}.
\]

Similarly, the second term satisfies
\[
\left| \int_\mathbb{R} [b, H](g_c)(x)f(x)u(x) \, dx \right| \leq \| [b, H](g_c) \|_{L^p(u)} \| f \|_{L^p'(u)} \\
\leq C \| g_c \|_{L^p(v)} \| f \|_{L^p'(u)} = \frac{C}{2} v(I)^{1 \over p'} \left( \int_I |b - b_I|^p u \right)^{1 \over p'}.
\]

Thus, we have
\[
\int_I |b - b_I|^p u \leq \left| \int_\mathbb{R} [b, H](\mathbf{1}_I)(x)g_c(x)f(x)u(x) \, dx \right| + \left| \int_\mathbb{R} [b, H](g_c)(x)f(x)u(x) \, dx \right| \\
\leq C v(I)^{1 \over p'} \left( \int_I |b - b_I|^p u \right)^{1 \over p'}.
\]

It now follows that
\[
\left( \frac{1}{v(I)} \int_I |b - b_I|^p u \right)^{1 \over p'} \leq C.
\]

The other estimate,
\[
\left( \frac{1}{u^{-\mu'}(I)} \int_I |b - b_I|^{p' \over \mu'} v^{-\mu'} \right)^{1 \over p'} \leq C,
\]
follows from duality by interchanging the roles of the weights \((u, v)\) with \((v^{-\mu'}, u^{-\mu'})\).

For the case \( m = 2k \) we will assume that \( b \) is real valued. The even iterated commutators,
\[ H_b^{2k} f(x) = p.v. \int_\mathbb{R} \frac{(b(x) - b(y))^{2k}}{x - y} f(y) \, dy, \]
have a positivity that we will exploit. Let
\[ f(x) = |b(x) - b_I|^{2k(p - 1)} \mathbf{1}_I(x), \]
and notice that \( f \geq 0 \). Then
By Hölder’s inequality and the boundedness of $H^2_0$

\[ \int_I |b(x) - b_I|^{2kp} u(x) \, dx = \int_I |b(x) - b_I|^{2k} f(x) u(x) \, dx \]

\[ = \int_I \left( \int_R \frac{(b(x) - b(y))^{2k}}{x-y} \, dy \right) f(x) u(x) \, dx \leq \int_I \left( \int_R \frac{(b(x) - b(y))^{2k}}{x-y} \, dy \right) f(x) u(x) \, dy \, dx. \]

It is exactly at this point that we have used that $m = 2k$ is even and $b$ is real, since

\[ |b(x) - b(y)|^{2k} = (b(x) - b(y))^{2k}. \]

Then

\[ \int_I |b(x) - b_I|^{2kp} u(x) \, dx \]

\[ \leq \int_I \left( \int_R \frac{(b(x) - b(y))^{2k}}{x-y} \, dy \right) f(x) u(x) \, dx \]

\[ = \int_I \left( \int_R \frac{(b(x) - b(y))^{2k}}{x-y} \, dy \right) f(x) u(x) \, dx \]

\[ + \int_I \left( \int_R \frac{(b(x) - b(y))^{2k}}{x-y} \, dy \right) \chi_I(x) f(x) u(x) \, dx \]

\[ = \int_R H^2_b(1_I)(x) g_c(x) f(x) u(x) \, dx - \int_R H^2_b(1_I)(x) g_c(x) f(x) u(x) \, dx. \]

By Hölder’s inequality and the boundedness of $H^2_b : L^p(v) \to L^p(u)$, we get that

\[ \int_I |b - b_I|^{2kp} u \leq \|H^2_b(1_I)\|_{L^p(u)} \|g_c\|_{L^p(u)} + \|H^2_b(g_c)\|_{L^p(u)} \|f\|_{L^p(u)} \]

\[ \leq C\|1_I\|_{L^p(v)} \|g_c\|_\infty \|f\|_{L^p(u)} + C\|g_c\|_{L^p(v)} \|f\|_{L^p(u)} \leq C\nu(I)\frac{1}{\nu} \left( \int_I |b - b_I|^{2kp} u \right)^{\frac{1}{\nu}}. \]

The rest of the proof follows as in the case $m = 1$. \qed

Finally, we prove Corollary 1.10. First note that if $w \in A_p$, then given any cube $Q$

\[ \left( \frac{|Q|}{w(Q)} \right)^{\frac{1}{p'}} \leq \left( \int_Q w^{-\frac{p'}{p}} \right)^{\frac{1}{p'}} \leq \left[ w \right]_{A_p} \left( \frac{|Q|}{w(Q)} \right)^{\frac{1}{p'}}. \]

Therefore, we have

\[ \left( \frac{1}{w(Q)} \int_I |b - b_Q|^{mp} w \right)^{\frac{1}{p'}} \leq \left( \int_Q |b - b_Q|^{mp} w \right)^{\frac{1}{p'}} \left( \int_Q w^{-\frac{p'}{p}} \right)^{\frac{1}{p'}} \leq \left[ w \right]_{A_p} \left( \frac{1}{w(Q)} \int_Q |b - b_Q|^{mp} w \right)^{\frac{1}{p'}}; \]

similarly,

\[ \left( \frac{1}{w^{-\frac{p'}{p}}(Q)} \int_Q |b - b_Q|^{mp} w^{-\frac{p'}{p}} \right)^{\frac{1}{p'}}. \]
Proof of Corollary 1.10. First suppose that $H_b^m$ is bounded on $L^p(w)$ for $m \in \{1\} \cup 2\mathbb{N}$, with operator norm $\|H_b^m f\|_{L^p(w)} \leq C\|f\|_{L^p(w)}$. If $I$ is a fixed interval, then by Hölder’s inequality and (1.14) we have

$$\int_I |b - b_I|^m \leq \left( \int_I |b - b_I|^{mp} w \right)^{\frac{1}{p}} \left( \int_I w^{-\frac{p'}{p}} \right)^{\frac{1}{p'}} \leq [w]_{A_p}^{\frac{1}{p}} \left( \frac{1}{w(I)} \int_I |b - b_I|^{mp} w \right)^{\frac{1}{p}} \leq C[w]_{A_p}^{\frac{1}{p}}.$$  

Since $I$ is arbitrary, we get that $b \in BMO$ with

$$\|b\|_{BMO}^m \leq C[w]_{A_p}^{\frac{1}{p}}.$$ 

On the other hand, suppose $b \in BMO$. We will find Young functions $A, B, C, D$ such that the weight $w$ and $b$ satisfy the hypothesis of Theorem 1.1, thus showing that $H_b^m$ is bounded on $L^p(w)$. Since $w \in A_p$, there exists $r > 1$, a reverse Hölder exponent, such that there exists a constant $C$ with

$$\left( \int_I w^r \right)^{\frac{1}{r}} \leq C \int_I w, \quad \text{and} \quad \left( \int_I w^{-r'} \right)^{\frac{1}{r'}} \leq C \int_I w^{-\frac{r'}{r}}$$

for any interval $I$. If $1 < s < r$, then

$$\left( \int_I |b - b_I|^m w^s \right)^{\frac{1}{mp}} \left( \int_I w^{-s'p'} \right)^{\frac{1}{mp}} \leq \left( \int_I |b - b_I|^{msr} w^{s'p'} \right)^{\frac{1}{msrp}} \left( \int_I w^r \right)^{\frac{1}{mp}} \left( \int_I w^{-s'p'} \right)^{\frac{1}{mp'}} \lesssim \|b\|_{BMO}^m [w]_{A_p}^{\frac{1}{p}},$$

similarly,

$$\left( \int_I w^s \right)^{\frac{1}{sp}} \left( \int_I |b - b_I|^{msr'} w^{-s'p'} \right)^{\frac{1}{msrp'}} \lesssim \|b\|_{BMO}^m [w]_{A_p}^{\frac{1}{p}}.$$ 

In particular, condition (1.8),

$$\sup_Q \| (b - b_Q)^m w^\frac{1}{p} \|_{A,Q} w^{-\frac{1}{p}} \|_{B,Q} + \sup_Q \| w^\frac{1}{p} \|_{C,Q} \| (b - b_Q)^m w^{-\frac{1}{p}} \|_{D,Q} < \infty,$$

is satisfied with $A(t) = C(t) = t^{sp}$ and $B(t) = D(t) = t^{sp'}$. Moreover, we have $A, C \in B_p$ and $B, D \in B_p$ since $s > 1$. Theorem 1.1 now implies that if $b \in BMO$, then $H_b^m : L^p(w) \to L^p(w)$.  \qed
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