LIMIT THEOREMS FOR WEIGHTED BERNOULLI RANDOM FIELDS UNDER HANNAN’S CONDITION

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Abstract. Consider a Bernoulli random field satisfying the Hannan’s condition. Recently, invariance principles for partial sums of random fields over rectangular index sets are established. In this note we complement previous results by investigating limit theorems for weighted Bernoulli random fields, including central limit theorems for partial sums over arbitrary index sets and invariance principles for Gaussian random fields. Most results improve earlier ones on Bernoulli random fields under Wu’s condition, which is stronger than Hannan’s condition.

1. Introduction

Let \( \{X_j\}_{j \in \mathbb{Z}^d} \) be a Bernoulli random field. That is, it has the form

\[
X_j = f \circ T_j(\{\epsilon_k\}_{k \in \mathbb{Z}^d})
\]

where \( f : \mathbb{R}^\mathbb{Z}^d \to \mathbb{R} \) is a measurable function, \( T_j \) is the shift operator on \( \mathbb{R}^\mathbb{Z}^d \) such that for \( w = \{w_k\}_{k \in \mathbb{Z}^d} \in \mathbb{R}^\mathbb{Z}^d \), \( [T_j(w)]_k = w_{j+k} \), and \( \{\epsilon_k\}_{k \in \mathbb{Z}^d} \) are independent and identically distributed (i.i.d.) random variables. We assume \( \mathbb{E}X_j = 0 \) and \( \mathbb{E}X_j^2 < \infty \).

We are interested in limit theorems for partial sums of weighted stationary random fields \( \{X_j\}_{j \in \mathbb{Z}^d} \), in form of

\[
S_n = \sum_{j \in \mathbb{Z}^d} b_{n,j}X_j, \quad n \in \mathbb{N},
\]

where \( \{b_{n,j}\}_{j \in \mathbb{Z}^d} \) are coefficients such that \( \sum_j b_{n,j}^2 < \infty \). We will impose further conditions on the dependence of \( \{X_j\}_{j \in \mathbb{Z}^d} \) so that \( S_n \) is well defined in the \( L^2 \) sense.

Limit theorems for stationary random fields have a long history. There is a vast literature on limit theorems for general stationary random fields, and we refer to [5, 6, 8, 9] and the references therein. On the other hand, for Bernoulli random fields, the investigation started only recently. A main motivation was to extend the well-investigated dependence conditions for stationary sequences to random fields. However, the success so far has been limited to Bernoulli random fields: Wang and Woodroofe [21] attempted to extend the Maxwell–Woodroofe condition [10, 11], but only ended up with a stronger version, El Machkouri et al. [11] extended Wu’s condition [22], and Volný and Wang [19] extended Hannan’s condition (2.1 below) [7, 10, 12].
Most of these results focused on partial sums of stationary random fields, that is \( S_n = \sum_{j \in \{1, \ldots, n\}_d} X_j \), and invariance principles for Brownian sheets have been established. Beyond this framework, limit theorems have been established for fractional Brownian sheets \[20\] and set-indexed random fields \[2, 11\]. These results can be formulated as limit theorems for weighted Bernoulli random fields as in \[12\], with Wu’s condition \[11\] on \( \{X_j\}_{j \in \mathbb{Z}^d} \) and certain assumptions on \( \{b_{n,j}\}_{n,j \in \mathbb{N}} \).

In this paper, we establish limit theorems for weighted Bernoulli random fields in form of \[12\], under Hannan’s condition. Some results here have already been established under Wu’s condition, notably \[2, 11, 20\]. It is known that Hannan’s condition is strictly weaker than Wu’s \[19\]. Therefore, our results improve the aforementioned ones.

There are two key ingredients in the proofs here. The first is a moment inequality for weighted partial sums, in form of

\[
\left\| \sum_{j \in \mathbb{Z}^d} b_{n,j}X_j \right\|_p \leq C \left( \sum_{j \in \mathbb{Z}^d} b_{n,j}^2 \right)^{1/2}
\]

for some \( p \geq 2 \). We establish such an inequality in Lemma \[2.2\] under Hannan’s condition. Such an inequality has been known under Wu’s condition \[11\] Proposition 1. The other key ingredient is the assumption of the Bernoulli random fields. Thanks to this assumption, one can construct \( m \)-dependent random fields to approximate the given ones, and the approximation error can be essentially controlled by the moment inequality above. In this way, our proof of the main result, Theorem \[2.4\], makes essential use of the two keys, and the proof is inspired by Biermé and Durieu \[2\] (see also Remark \[2.8\]). Here we present a variation of the same idea, using \( m \)-dependent approximation instead of \( m_n \)-dependent approximation.

The paper is organized as follows. The main result, Theorem \[2.4\], is established in Section \[2\]. As consequences, we present two applications. First, central limit theorems for partial sums over arbitrary index sets are investigated in Section \[3\]. Second, invariance principles in \[2, 11, 20\] are established under the (weaker) Hannan’s condition in Section \[4\].

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### 2. A CENTRAL LIMIT THEOREM

Consider i.i.d. random variables \( \{\epsilon_i\}_{i \in \mathbb{Z}^d} \) defined in a probability space \((\Omega, \mathcal{E}, \mathbb{P})\). Set \( \mathcal{F}_i = \sigma(\epsilon_j : j \in \mathbb{Z}^d, j \leq i), i \in \mathbb{Z}^d \) and \( \mathcal{F}_{iq} = \sigma(\epsilon_j : j \in \mathbb{Z}^d, j \leq i) \).
\( \mathbb{Z}^d, j_q \leq i_q \), \( q = 1, \ldots, d, i_q \in \mathbb{Z} \). As in [19], introduce the projection operator

\[
P_q = \prod_{q=1}^{d} P_{i_q}^{(q)} \quad \text{with} \quad P_{i_q}^{(q)}(\cdot) = \mathbb{E}(\cdot | \mathcal{F}_{i_q}^{(q)}) - \mathbb{E}(\cdot | \mathcal{F}_{i_q-1}^{(q)}).
\]

In this way, \( P_{i_q}^{(q)} \) and \( P_q \) are operators from \( L^2(\Omega, \mathcal{B}, \mathbb{P}) \) to \( L^2(\Omega, \mathcal{B}, \mathbb{P}) \). For more properties of these operators, see [20]. With these notations, the Han-nan’s condition states as

\[
\Delta_p(X) := \sum_{i \in \mathbb{Z}^d} \|P_0 X_i\|_p < \infty,
\]

for some \( p \geq 2 \). It is essential to assume \( \{\epsilon_j\}_{j \in \mathbb{Z}^d} \) to be i.i.d., so that the operators commute.

**Lemma 2.1.** Let \( Y \) be a random variable measurable with respect to the \( \sigma \)-algebra \( \mathcal{F}_\infty = \sigma(\epsilon_j : j \in \mathbb{Z}^d) \), with \( \mathbb{E}Y = 0, \mathbb{E}|Y|^p < \infty \), for some \( p \geq 2 \). Then,

\[
Y = \sum_{j \in \mathbb{Z}^d} P_j Y = \lim_{m \to \infty} \sum_{j \in \{-m, \ldots, m\}^d} P_j Y \quad \text{in } L^p.
\]

**Proof.** By definition of \( P_j \),

\[
\sum_{j \in \{-m, \ldots, m\}^d} P_j Y = \mathbb{E}(Y | \mathcal{F}_m) + \sum_{\delta \in \{-1, 1\}^d \setminus \{1\}} (-1)^{|\delta|} \mathbb{E}(Y | \mathcal{F}_{m\delta})
\]

with \( |\delta| = \sum_{q=1}^{d} 1_{\{\delta_q = -1\}} \). \( 1 = (1, \ldots, 1) \in \mathbb{Z}^d \) and \( m\delta, m1 \in \mathbb{Z}^d \). By martingale convergence theorem, \( \mathbb{E}(Y | \mathcal{F}_m) \to Y \) almost surely and in \( L^p \). The other \( 2^d - 1 \) terms all converge to zero in \( L^p \). Indeed, observe that for each \( q = 1, \ldots, d \),

\[
\lim_{m \to \infty} \mathbb{E}(Y | \mathcal{F}_{-m}^{(q)}) = \mathbb{E}\left( Y \bigg\vert \bigcap_{m \in \mathbb{N}} \mathcal{F}_{-m}^{(q)} \right)
\]

almost surely and in \( L^p \), by backwards martingale convergence theorem. By Kolmogorov’s zero-one law, the limit is a constant and hence necessarily zero since \( \mathbb{E}Y = 0 \). To complete the proof, remark that \( \|\mathbb{E}(Y | \mathcal{F}_j)\|_p \leq \|\mathbb{E}(Y | \mathcal{F}_{j_q}^{(q)})\|_p \) for \( j \in \mathbb{Z}^d \).

We first give two lemmas on Bernoulli random fields under Han-nan’s condition. Throughout, infinite sums of random variables are understood as the limit in the \( L^p \) sense.

**Lemma 2.2.** Suppose \( \Delta_p(X) < \infty \) for some \( p \geq 2 \). Then, there exists a constant \( C_p \), such that for all \( \{a_i\}_{i \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d) \),

\[
(2.2) \quad \left\| \sum_{i \in \mathbb{Z}^d} a_i X_i \right\|_p \leq C_p \left( \sum_{i \in \mathbb{Z}^d} a_i^2 \right)^{1/2} \Delta_p(X).
\]
Proof. Observe that it suffices to show
\[
\left\| \sum_{i \in \Lambda} a_i X_i \right\|_p \leq C_p \left( \sum_{i \in \Lambda} a_i^2 \right)^{1/2} \Delta_p(X)
\]
for all finite \( \Lambda \subset \mathbb{Z}^d \). Then, by Lemma 2.1
\[
\sum_{i \in \Lambda} a_i X_i = \sum_{j \in \mathbb{Z}^d} \sum_{i \in \Lambda} a_i P_j X_i = \sum_{j \in \mathbb{Z}^d} P_j \left( \sum_{i \in \Lambda} a_i X_i \right).
\]
By Burkholder’s inequality and stationarity,
\[
\left\| \sum_{j \in \mathbb{Z}^d} P_j \left( \sum_{i \in \Lambda} a_i X_i \right) \right\|_p^2 \leq C_p \left\| \sum_{j \in \mathbb{Z}^d} \left( P_j \sum_{i \in \Lambda} a_i X_i \right) \right\|_p^2 \leq C_p \sum_{j \in \mathbb{Z}^d} \left( \sum_{i \in \Lambda} |a_i| \|P_0 X_{i-j}\|_p \right)^2,
\]
and the last term above is bounded by, by Cauchy–Schwarz inequality,
\[
C_p \sum_{j \in \mathbb{Z}^d} \sum_{i \in \Lambda} a_i^2 \|P_0 X_{i-j}\|_p \sum_{\ell \in \mathbb{Z}^d} \|P_0 X_{\ell-j}\|_p = C_p \Delta^2_p(X) \sum_{i \in \mathbb{Z}^d} a_i^2.
\]
Thus, we have shown (2.2). \( \Box \)

As a consequence, if \( \Delta_p(X) < \infty \), then \( S_n \) is a well-defined random variable in the \( L^p \) sense.

Lemma 2.3. Suppose \( \Delta_2(X) < \infty \). Then \( \sum_{j \in \mathbb{Z}^d} |\text{Cov}(X_0, X_j)| \leq \Delta^2_2(X) < \infty \).

Proof. Hannan’s condition enables to write \( X_i = \sum_j P_j X_i \). Since \( \{P_j\}_{j \in \mathbb{Z}^d} \) are orthogonal in the sense that \( \mathbb{E}[\langle P_j X \rangle(P_k Y)] = 0 \) for all \( j, k \in \mathbb{Z}^d, j \neq k \) and \( X, Y \in L^2(\Omega, \mathcal{F}, \mathbb{P}) \), it follows that
\[
\sum_{k \in \mathbb{Z}^d} |\mathbb{E}(X_0 X_k)| \leq \sum_{k \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d} |\mathbb{E}(P_i X_0)(P_i X_k)| \leq \sum_{k \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d} \|P_i X_0\|_2 \|P_i X_k\|_2 = \Delta^2_2(X).
\]

As a consequence, we introduce
\[
(2.3) \quad \sigma^2 := \sum_{j \in \mathbb{Z}^d} \text{Cov}(X_0, X_j)
\]
which is finite under Hannan’s condition.
To state the main result, introduce some notations. For \( \vec{b}_n = \{b_{n,j}\}_{j \in \mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d) \), set \( b_n = (\sum_{j \in \mathbb{Z}^d} b_{n,j}^2)^{1/2} \). For \( \{\vec{b}_n\}_{n \in \mathbb{N}} \subset \ell^2(\mathbb{Z}^d) \), we are interested in

\[
S_n = \sum_{j \in \mathbb{Z}^d} b_{n,j} X_j.
\]

Write \( \sigma_n^2 = \text{Var}(S_n) \). Another useful consequence of \( \Delta_2(X) < \infty \) is that

\[
\sigma_n^2 = \sum_{k \in \mathbb{Z}^d} \sum_{\ell \in \mathbb{Z}^d} b_{n,k} b_{n,\ell} \text{Cov}(X_k, X_\ell)
= \sum_{\ell \in \mathbb{Z}^d} \left( \sum_{k \in \mathbb{Z}^d} b_{n,k} b_{n,k+\ell} \right) \text{Cov}(X_0, X_\ell)
\leq b_n^2 \sum_{\ell \in \mathbb{Z}^d} |\text{Cov}(X_0, X_\ell)| \leq \Delta_2^2(X) b_n^2.
\]

Our main result is the following.

**Theorem 2.4.** Let \( \{X_i\}_{i \in \mathbb{Z}^d} \) be a stationary Bernoulli random field as in (1.1) satisfying Hannan’s condition (2.1). If

\[
\lim_{n \to \infty} \sup_{j \in \mathbb{Z}^d} \frac{|b_{n,j}|}{b_n} = 0
\]

and

\[
\liminf_{n \to \infty} \frac{\sigma_n^2}{b_n^2} > 0
\]

hold, then

\[
\frac{S_n}{\sigma_n} \Rightarrow \mathcal{N}(0, 1).
\]

The condition (2.6) is subtle as it involves both the coefficients and the dependence of underlying random fields (via \( \sigma_n \)). The following corollary is more convenient, as it imposes only conditions on coefficients. However, we see later in Example 3.2 that there are examples that satisfy the conditions in Theorem 2.4 but the conclusion of Corollary 2.5 does not hold. Recall that for \( k \in \mathbb{Z}^d \), the shift operator yields \( T_k \vec{b}_n = \{b_{n,j+k}\}_{j \in \mathbb{Z}^d} \). Let \( e_1, \ldots, e_d \) be the \( d \) canonical unit vector in \( \mathbb{R}^d \).

**Corollary 2.5.** Let \( \{X_i\}_{i \in \mathbb{Z}^d} \) be a stationary Bernoulli random field as in (1.1) satisfying Hannan’s condition (2.1). Under the notations as in Theorem 2.4 if

\[
\lim_{n \to \infty} \frac{\|T_{e_q} \vec{b}_n - \vec{b}_n\|_{\ell^2}}{b_n} = 0, \text{ for all } q = 1, \ldots, d
\]

hold, then

\[
\lim_{n \to \infty} \frac{\sigma_n}{b_n} = \sigma
\]
with \( \sigma \) defined as in (2.3), and
\[
\frac{S_n}{b_n} \Rightarrow \mathcal{N}(0, \sigma^2).
\]

Proof of Corollary 2.5. We first show (2.9). Recall (2.4). Observe that
\[
-2 \sum_{k \in \mathbb{Z}^d} b_{n,k} b_{n,k+j} + j = \|T_j \tilde{b}_n - \tilde{b}_n\|_{\ell^2}^2 - \|\tilde{b}_n\|_{\ell^2}^2 - \|T_j \tilde{b}_n\|_{\ell^2}^2,
\]
and \( \|T_j \tilde{b}_n - \tilde{b}_n\|_{\ell^2} = o(b_n) \) for all fixed \( j \in \mathbb{Z}^d \), a consequence of (2.8). Therefore,
\[
\lim_{n \to \infty} \frac{1}{b_n^2} \sum_{j \in \mathbb{Z}^d} b_{n,k} b_{n,k+j} = 1 \text{ for all } j \in \mathbb{Z}^d.
\]
Thus, by the dominated convergence theorem, (2.4) and (2.10) imply (2.9).

If \( \sigma = 0 \), then \( \sigma^2_n/b_n^2 \to 0 \), and the central limit theorem is degenerate and trivially holds. If \( \sigma > 0 \), then (2.6) holds. By Cauchy–Schwarz inequality,
\[
\lim_{n \to \infty} \frac{1}{b_n^2} \sum_{j \in \mathbb{Z}^d} |b_{n,j}^2 + e_q - b_{n,j}^2| = 0, \text{ for all } q = 1, \ldots, d.
\]
It has been shown in [3, Lemma 8], using an idea from [1], that (2.11) implies (2.5). The desired result now follows from Theorem 2.4. □

Remark 2.6. Condition (2.8) was introduced in Biémé and Durieu [2, Theorem 3.1]. Condition (2.5) was also assumed there. It has been pointed out in [3, Remark 3] that (2.5) was redundant.

Proof of Theorem 2.4. We proceed an \( m \)-dependent approximation argument. For each \( m \in \mathbb{N} \), set \( S_j^{(m)} = \sigma(\epsilon_i : i \in \mathbb{Z}^d, |j - i|_\infty \leq m) \),
\[
X_j^{(m)} = \mathbb{E}(X_j \mid S_j^{(m)}), j \in \mathbb{Z}^d.
\]
In this way, \( \{X_j^{(m)}\}_{j \in \mathbb{Z}^d} \) is a \((2m + 1)\)-stationary random field. Write
\[
S_n^{(m)} = \sum_{j \in \mathbb{Z}^d} b_{n,j} X_j^{(m)} \quad \text{and} \quad \sigma_{m,n}^2 = \text{Var}(S_n^{(m)}).
\]
Observe that
\[
P_0 X_j^{(m)} = \sum_{\delta \in \{0,1\}^d} (-1)^{\delta_1 + \cdots + \delta_d} \mathbb{E} \left[ \mathbb{E}(X_j \mid S_j^{(m)}) \mid F_{j-\delta} \right]
= \sum_{\delta \in \{0,1\}^d} (-1)^{\delta_1 + \cdots + \delta_d} \mathbb{E} \left[ \mathbb{E}(X_j \mid F_{j-\delta}) \mid S_j^{(m)} \right] = \mathbb{E} \left( P_0 X_j \mid S_j^{(m)} \right),
\]
where in the second equality we used the fact that the \( \sigma \)-algebras \( S_j^{(m)} \) and \( F_\ell \) are conditionally independent and hence commuting, because they are generated by independent random variables \( \{\epsilon_j\}_{j \in \mathbb{Z}^d} \). Thus,
\[
\Delta_p(X^{(m)}) \leq \Delta_p(X),
\]
and $S^{(m)}_n$ is well defined in the $L^p$ sense if $\Delta_p(X) < \infty, p \geq 2$.

We will approximate $S_n$ by $S^{(m)}_n$. To establish a central limit theorem for $m$-dependent random variables, we will apply a result due to Heinrich [13], which requires each partial sum to be of finite number of random variables. Therefore, we introduce a finite set $V_n \subset \mathbb{Z}^d$ for each $n$ such that $|V_n| \to \infty$ and $\lim_{n \to \infty} b_n^{-2} \sum_{j \in V_n} b_n^2 b_{n,j} = 1$. Set

$$S^{(m)}_{V_n} = \sum_{j \in V_n} b_{n,j} X^{(m)}_j$$
and
$$\sigma^2_{m,V_n} = \text{Var}(S^{(m)}_{V_n}).$$

We first summarize a few estimates in the following lemma.

**Lemma 2.7.** With the construction described above,

$$\lim_{m \to \infty} \limsup_{n \in \mathbb{N}} \frac{\text{Var}(S_n - S^{(m)}_n)}{\sigma^2_n} = 0, \quad \lim_{m \to \infty} \limsup_{n \in \mathbb{N}} \frac{\sigma^2_{m,n} - \sigma^2_n}{\sigma^2_n} = 0,$$

and with the choice of $V_n$ described above, for every $m$ large enough,

$$\lim_{n \to \infty} \frac{\text{Var}(S^{(m)}_n - S^{(m)}_{V_n})}{\sigma^2_{m,n}} = 0, \quad \lim_{n \to \infty} \frac{\sigma^2_{m,V_n}}{\sigma^2_{m,n}} = 1.$$

**Proof of Lemma 2.7.** In the sequel, we let $C$ denote constant number independent from $n$ and $m$, but may change from line to line. We first show the first part of (2.13). Indeed, by Lemma 2.2,

$$\text{Var}(S_n - S^{(m)}_n) \leq C b_n^2 \left( \sum_{j \in \mathbb{Z}^d} \left\| P_0(X_j - X^{(m)}_j) \right\|_2^2 \right).$$

Observe that for each $j$, $\| P_0(X_j - X^{(m)}_j) \|_2 \leq \|X_j - X^{(m)}_j\|_2 \to 0$ as $m \to \infty$, and that

$$\sum_{j \in \mathbb{Z}^d} \left\| P_0(X_j - X^{(m)}_j) \right\|_2 \leq \sum_{j \in \mathbb{Z}^d} \left( \| P_0(X_j) \|_2 + \| P_0(X^{(m)}_j) \|_2 \right) \leq \Delta_2(X^{(m)}) + \Delta_2(X) \leq 2 \Delta_2(X),$$

which is finite under Hannan’s condition. By the dominated convergence theorem, $\lim_{m \to \infty} \sup_{n \in \mathbb{N}} \text{Var}(S^{(m)}_n - S_n)/b_n^2 = 0$, and the first part of (2.13) follows from the assumption (2.6). To see the second part, it suffices to observe

$$\left| \sigma^2_{m,n} - \sigma^2_n \right| \leq \text{Var}^{1/2}(S^{(m)}_n - S_n) \text{Var}^{1/2}(S^{(m)}_n + S_n).$$
We have seen that \( \sigma_n^2 \leq \Delta_2(X) \) in (2.4), and the same argument shows that
\[
\sigma_{m,n}^2 \leq \Delta_2(S_{n,m}) \sum_k |\text{Cov}(X^{(m)}_0, X^{(m)}_k)|.
\]
Moreover,
\[
\sum_{k \in \mathbb{Z}^d} |\mathbb{E}(X^{(m)}_0 X^{(m)}_k)| \leq \sum_{k \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d} |\mathbb{E}(P_i X^{(m)}_0)(P_i X^{(m)}_k)|
\]
\[
\leq \sum_{k \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d} \|P_i X^{(m)}_0\|_2 \|P_i X^{(m)}_k\|_2 = \Delta_2^2(X^{(m)}) \leq \Delta_2^2(X).
\]

Therefore, \( \text{Var}(S_{n,m} - S_n) \leq 2(\sigma_{m,n}^2 + \sigma_n^2) \leq \Delta_2^2(X) \), for all \( m, n \in \mathbb{N} \). It then follows
\[
\limsup_{n \to \infty} \frac{\sigma_{m,n}^2 - \sigma_n^2}{\sigma_{m,n}^2} \leq C \limsup_{n \to \infty} \frac{b_n \text{Var}^{1/2}(S_{n,m} - S_n)}{\sqrt{\sigma_n}}.
\]

The second part of (2.13) now follows from the first part and (2.6).

For (2.14), to show the first part, using the same argument as above it suffices to observe
\[
\frac{\text{Var}(S_{n,m}^{(m)} - S_{n,V_n}^{(m)})}{\sigma_{m,n}^2} \leq C \frac{\sum_{j \notin V_n} \sum_{i \in \mathbb{Z}^d} b_n^2 \Delta_2^2(X^{(m)})}{\sigma_{m,n}^2} \leq C \frac{\sum_{j \notin V_n} \sum_{i \in \mathbb{Z}^d} b_n^2 b_n^2 \sigma_n^2 \sigma_{m,n}^2 \sigma_n^2 \sigma_{m,n}^2 \Delta_2^2(X),}
\]
again by Lemma 2.2 and (2.12). By the second part of (2.13), for \( m \) large enough, say \( m \geq m_0 \), \( \limsup_{n \to \infty} |\sigma_{m,n}^2 - \sigma_n^2|/\sigma_n^2 \leq 1/2 \), whence
\[
\text{(2.15)} \quad \limsup_{n \to \infty} \frac{\sigma_{m,n}^2}{\sigma_{m,n}^2} \leq 2, m \geq m_0.
\]

Therefore the first part of (2.14) follows, for \( m \geq m_0 \). For the second part, observe that
\[
|\sigma_{m,n}^2 - \sigma_{m,V_n}^2| \leq \text{Var}^{1/2}(S_{n,m}^{(m)} - S_{n,V_n}^{(m)}) \text{Var}^{1/2}(S_n^{(m)} + S_{n,V_n}^{(m)}),
\]
and
\[
\sigma_{m,V_n}^2 \leq C b_n^2 \sum_{k \in V_n} |\text{Cov}(X^{(m)}_0, X^{(m)}_k)|
\]
\[
\leq C b_n^2 \sum_{k \in \mathbb{Z}^d} |\text{Cov}(X^{(m)}_0, X^{(m)}_k)| \leq C b_n^2 \Delta_2^2(X).
\]

Thus,
\[
\text{(2.16)} \quad \frac{|\sigma_{m,n}^2 - \sigma_{m,V_n}^2|}{\sigma_{m,n}^2} \leq C \left( \frac{\text{Var}(S_{n,m}^{(m)} - S_{n,V_n}^{(m)})}{\sigma_{m,n}^2} \right)^{1/2} \frac{b_n \sigma_n}{\sigma_n \sigma_{m,n}}.
\]

By (2.6), (2.15), and the first part of (2.14), for \( m \geq m_0 \) the second part of (2.14) follows.

Now we prove the desired central limit theorem (2.4) in three steps.
1) We first show, for $m$ large enough,

$$\lim_{n \to \infty} \frac{S_{V_n}^{(m)}}{\sigma_{m,V_n}} \Rightarrow \mathcal{N}(0,1).$$

For this purpose, we apply the central limit theorem for $m$-dependent random variables due to Heinrich [13]. We need also $\limsup_{n \to \infty} b_n^2 / \sigma_{m,V_n}^2 < \infty$, which follows from (2.6) and (2.14), for $m$ large enough. For (2.17), the required conditions in Heinrich’s theorem can be easily verified: for any $m \in \mathbb{N}$ large enough fixed,

$$\frac{1}{\sigma_{m,V_n}^2} \sum_{j \in V_n} \mathbb{E} \left( b_{n,j}^2 X_j^{(m)}^2 \right) \leq \frac{b_n^2}{\sigma_{m,V_n}^2} \operatorname{Var}(X_0^{(m)}) \leq C < \infty$$

for some constant $C$ and $n$ large enough, and for all $\varepsilon > 0$, and

$$\frac{m^{2d}}{\sigma_{m,V_n}^2} \sum_{j \in V_n} \mathbb{E} \left( b_{n,j}^2 X_j^{(m)} 1 \{|X_j^{(m)}| \geq \varepsilon m^{-2d} \} \right)$$

$$\leq \frac{m^{2d} b_n^2}{\sigma_{m,V_n}^2} \mathbb{E} \left( X_0^{(m)} 1 \{|X_0^{(m)}| \geq \varepsilon m^{-2d} / \sup_{b_{n,j}} b_{n,j} \} \right) \to 0 \text{ as } n \to \infty$$

where the last step is due to the assumption (2.5).

2) Observe that

$$\frac{S_n^{(m)}}{\sigma_{m,n}} - \frac{S_{V_n}^{(m)}}{\sigma_{m,n}} = \frac{S_n^{(m)} - S_{V_n}^{(m)}}{\sigma_{m,n}} + \frac{S_{V_n}^{(m)}}{\sigma_{m,V_n}} \frac{S_{V_n}^{(m)}}{\sigma_{m,n}}.$$

From (2.14) and (2.17), it follows that for $m$ large enough,

$$\frac{S_n^{(m)}}{\sigma_{m,n}} \Rightarrow \mathcal{N}(0,1).$$

3) At last, to show (2.7), observe that

$$\frac{S_n - S_{V_n}^{(m)}}{\sigma_{m,n}} = \frac{1}{\sigma_n} (S_n - S_{V_n}^{(m)}) + \frac{\sigma_{m,n} - \sigma_n}{\sigma_n \sigma_{m,n}} S_{V_n}^{(m)}.$$

By Lemma 2.7 it follows that

$$\lim_{m \to \infty} \limsup_{n \to \infty} \operatorname{Var} \left( \frac{S_n - S_{V_n}^{(m)}}{\sigma_{m,n}} \right) = 0.$$

Therefore, applying [4, Theorem 4.2] to (2.18) and (2.19), we have thus proved (2.7). \hfill \Box

Remark 2.8. The same $m_n$-dependent approximation as in [2, Theorem 3.1] can be applied here, once one notices that

$$\lim_{n \to \infty} \frac{\operatorname{Var}(S_n - S_{V_n}^{(m_n)})}{b_n^2} = 0$$
holds (in the same way as in the proof of the first part of (2.13)) in place of [2, Eq. (3.4)] for an appropriately chosen increasing sequence \(\{m_n\}_{n \in \mathbb{N}}\), and the the rest of the proof therein can be carried out with minor changes. In order not to introduce too much duplication, we chose to present a different proof. Our result is more general also in the sense that we consider the normalization of \(\sigma_n\) instead of \(b_n\).

3. Central limit theorems for set-indexed partial sums

In this section, we consider the case

\[ S_n \equiv S_{\Gamma_n} = \sum_{i \in \Gamma_n} X_i \]

for a sequence of subsets \(\{\Gamma_n\}_{n \in \mathbb{N}}\) of \(\mathbb{Z}^d\) with the cardinality of subsets \(|\Gamma_n| \to \infty\) as \(n \to \infty\). This corresponds to the case \(b_{n,j} = 1\{j \in \Gamma_n\}\) and \(b_n = |\Gamma_n|^{1/2}\). Then, in view of Corollary 2.5, (2.11) is automatically satisfied, and it is easy to notice that (2.11) is equivalent to

\[ \lim_{n \to \infty} \frac{|\partial \Gamma_n|}{|\Gamma_n|} = 0, \]

where \(\partial \Gamma_n = \{i \in \Gamma_n : \exists j \notin \Gamma_n, |i - j|_\infty = 1\}\) is the boundary set of \(\Gamma_n\). Indeed, if we identify \(\Gamma_n\) with an element in \(\ell^2(\mathbb{Z}^d)\) via \(b_{n,j} = 1\{j \in \Gamma_n\}\), then for each \(q = 1, \ldots, d\) we have \(\|T_{eq} \Gamma_n - \Gamma_n\|_{\ell^2} \leq 2|\partial \Gamma_n| \leq 4\sum_{m=1}^{d} \|T_{em} \Gamma_n - \Gamma_n\|_{\ell^2}\). We have thus obtained the following.

**Corollary 3.1.** For a Bernoulli random field with \(\Delta_2 < \infty\), and a sequence of subsets \(\{\Gamma_n\}_{n \in \mathbb{N}}\) of \(\mathbb{Z}^d\) satisfying \(|\Gamma_n| \to \infty\) and (3.1),

\[ \frac{S_{\Gamma_n}}{|\Gamma_n|^{1/2}} \Rightarrow \mathcal{N}(0, \sigma^2) \]

with \(\sigma^2\) given in (2.3).

In the rest of this section, we discuss what happens if we are interested in the convergence of

\[ \frac{S_n}{\sigma_n} \Rightarrow \mathcal{N}(0, 1). \]

This follows from (2.6), by Theorem 2.4 To see the role of the condition (2.6), we provide two examples. First, by Example 3.2, we show that condition (2.6) cannot be removed: otherwise (3.3) may no longer hold under Hannan’s condition. Second, by Example 3.3 we show that the assumption in Corollary 2.5 is strictly stronger than (2.6), in the sense that there are examples satisfying (2.6), but the conclusion of Corollary 2.5 does not hold. Note also that Example 3.2 also shows that when \(S_n/b_n \Rightarrow \mathcal{N}(0, \sigma^2)\) with \(\sigma^2 = 0\), one should not expect \(S_n/\sigma_n \) to converge, without further assumptions.
For the sake of simplicity, both examples are given in dimension one. Let $\{\epsilon_i\}_{i \in \mathbb{Z}}$ be the i.i.d. random variables that generate the Bernoulli random field $\Gamma_n$.

**Example 3.2.** Consider $\Gamma_n = \{0,1,\ldots,n-1\}$. We construct an example such that $S_n/\sigma_n$ converges to different limits along different subsequences.

Suppose that there exists a collection of mutually independent random variables $\{\zeta_n^{(k)}\}_{n,k \in \mathbb{N}}$ such that for each $k \in \mathbb{N}$, $\{\zeta_n^{(k)}\}_{n \in \mathbb{N}}$ are i.i.d., and for each $n$, $\zeta_n^{(k)}$ is $\sigma(\epsilon_n)$-measurable. We further assume that $\mathbb{E}\zeta_n^{(k)} = 0$, $\operatorname{Var}(\zeta_n^{(k)}) = 1$. A detailed construction is given at the end.

For coefficients $\{\alpha_n\}_{n \in \mathbb{N}}$ satisfying $\sum_n \alpha_n^2 < \infty$ and a sequence of increasing positive integers $\{n_k\}_{k \in \mathbb{N}}$, set

$$W_n^{(k)} = \alpha_k (\zeta_n^{(k)} - \zeta_{n-n_k}), \quad k \in \mathbb{N} \quad \text{and} \quad X_n = \sum_{k=1}^{\infty} W_n^{(k)}.$$ 

Observe that $P_0X_n = \sum_{k=1}^{\infty} P_0W_n^{(k)}$, which equals $-\alpha_k \zeta_0^{(\ell)}$ if $n = n_\ell$ for some $\ell \in \mathbb{N}$, and 0 otherwise. Thus, $\Delta_2(X) = \sum_{\ell=1}^{\infty} |\alpha_\ell| < \infty$.

Write $S_n = S_{\Gamma_n} = \sum_{i=0}^{n-1} X_i$ and $S_n(W^{(k)}) = \sum_{i=0}^{n-1} W_n^{(k)}$. So

$$S_n = \sum_{k=1}^{\infty} S_n(W^{(k)}).$$

By independence,

$$\mathbb{E}(S_{n_k}(W_\ell))^2 = \begin{cases} 2n_\ell \alpha_\ell^2 & \ell \leq k \\ 2n_k \alpha_k^2 & \ell > k \end{cases},$$

and

$$\operatorname{Var}(S_{n_k}) = \sum_{\ell=1}^{\infty} \operatorname{Var}(S_{n_k}(W_\ell)) = \sum_{\ell=1}^{k-1} 2n_\ell \alpha_\ell^2 + \sum_{\ell=k+1}^{\infty} 2n_k \alpha_k^2 + 2n_k \alpha_k^2.$$

One can choose $\alpha_k$ and $n_k$ so that

$$\text{(3.4) } \quad \operatorname{Var}(S_{n_k}) \sim \operatorname{Var}(S_{n_k}(W_k)) = 2n_k \alpha_k^2 \text{ as } k \to \infty.$$ 

For example, taking $\alpha_k = 2^{-k^2}$ and $n_k = 2^{3k^2}$ yields $\operatorname{Var}(S_{n_k}) \sim 2^{k^2+1} = o(b_{n_k})$.

Now in view of [3.4], for our purpose it suffices to choose $\zeta_k$ appropriately such that

$$Z_k := \frac{S_{n_k}(W^{(k)})}{\alpha_k \sqrt{n_k}}$$

converge to different limits along even and odd sequences.

To do so, we now give an explicit construction of $\{\zeta_n^{(k)}\}_{n,k \in \mathbb{N}}$. For the sake of simplicity, consider $(\Omega, \mathcal{B}, \mathbb{P}) = ([0,1]^\mathbb{Z}, \mathcal{B}([0,1])^\mathbb{Z}, \text{Leb}^\mathbb{Z})$, and $\epsilon_k(\omega) =$
\( \omega_k, \omega \in \Omega \). In this way, for any \( \{d_n\}_{n \in \mathbb{N}} \subset [0, 1] \), there exist a family of sets \( \{A^\pm_n\}_{n \in \mathbb{N}} \subset B([0, 1]) \) such that \( A^+_n \cap A^-_n = \emptyset \) and \( \mu(A^\pm_n) = d_n/2 \). Set

\[
\zeta^{(k)}_n(\omega) = \frac{1}{\sqrt{d_k}}(1_{A^+_k} - 1_{A^-_k})(\omega_n), n, k \in \mathbb{N}.
\]

It is clear that \( \{\zeta^{(k)}_n\}_{n,k \in \mathbb{N}} \) satisfy the conditions that we assumed at the beginning. Now set \( d_{2k} = 1, d_{2k-1} = 1/n_{2k-1} \) for \( k \in \mathbb{N} \). For \( \{A^\pm_n\}_{n \in \mathbb{N}} \) and \( \{\zeta^{(k)}_n\}_{n,k \in \mathbb{N}} \) described above, it is also clear that \( Z_{2k} \Rightarrow N(0,1) \) as \( k \to \infty \) but \( Z_{2k-1} \) converges weakly to a symmetric Poisson distribution. So \( S_n/\sigma_n \) does not converge.

**Example 3.3.** Consider \( X_i = \epsilon_i - \epsilon_{i-1} \). Observe that \( X_i \) and \( X_j \) are uncorrelated if \( |i - j| \geq 2 \). Therefore, this stochastic process satisfies \( \Delta_2(X) < \infty \). We now construct a sequence of subsets \( \{\Gamma_n\}_{n \in \mathbb{N}} \) such that \( \liminf_{n \to \infty} \sigma_n/b_n > 0 \) but \( \lim_{n \to \infty} \sigma_n/b_n \) does not exist.

We construct \( \Gamma_n \) iteratively. Set \( \Gamma_1 = \{0, 1\} \). For \( n \in \mathbb{N} \), set \( \Gamma_{n+1} = \Gamma_n \cup B_n \) with

\[
B_n = \begin{cases} 
\{a_n + 2, a_n + 3, \ldots, a_n + 2^n + 1\} & \text{even} \\
\{a_n + 2, a_n + 4, \ldots, a_n + 2 \cdot 2^n\} & \text{odd}
\end{cases}
\]

with \( a_n = \max\{j : j \in \Gamma_n\} \). By construction, \( \text{Var}(S_{\Gamma_{n+1}}) = \text{Var}(S_{\Gamma_n}) + \text{Var}(S_{B_n}) \), and \( \text{Var}(B_n) = 2\epsilon_0^2 \epsilon_0^2 \) for even, and \( 2^{n+1}\epsilon_0^2 \) for odd. At the same time, \( |\Gamma_n| = 2^n \). It is clear that the desired result follows.

### 4. Invariance principles for Gaussian random fields

In this section, we present two invariance principles for weighted Bernoulli random fields. Let \( \mathbb{T} \) be an index set equipped with a pseudo-metric. Consider random fields in form of

\[
S_n(t) = \sum_{j \in \mathbb{Z}^d} b_{n,j}(t)X_j, t \in \mathbb{T}.
\]

Under Hannan’s condition on \( \{X_j\}_{j \in \mathbb{Z}^d} \) and appropriate assumptions on the coefficients \( b_{n,j}(t) \), we shall establish, for an increasing sequence of positive numbers \( \{b_n\}_{n \in \mathbb{N}} \),

\[
\left\{ \frac{S_n(t)}{b_n} \right\} \Rightarrow \{G_t\}_{t \in \mathbb{T}}
\]

where \( G \) is a zero-mean Gaussian process. The space of weak convergence will be specified below. The results improve earlier ones \([2, 11, 20]\), in the sense that Wu’s condition is replaced by Hannan’s condition.

We first provide an overview on how to establish (4.2), illustrating how previous proofs can be adapted without much changes. To establish such an invariance principle, we proceed as in the standard two-step proof: we first show convergence of finite-dimensional distributions and then tightness.
To show the convergence of finite-dimensional distributions, we first remark that marginally, for 
\( b_n(t) := (\sum_{j \in \mathbb{Z}^d} b_{n,j}^{2}(t))^{1/2} \), one should expect

\[ S_n(t) \frac{b_n(t)}{b_n} \Rightarrow N(0, \sigma^2), \text{ for all } t \in \mathbb{T} \]

with \( \sigma^2 \) as in (2.3) as a consequence of Theorem 2.4. Comparing this with (4.2), it suggests that \( \lim_{n \to \infty} b_n^2(t)/b_n^2 = \text{Var}(G_t)/\sigma^2 \). Moreover, by Cramer–Wold’s device, for the weak convergence to hold, we need to show, for all \( \lambda \in \mathbb{R}^m, t \in \mathbb{T}^m, m \in \mathbb{N} \),

\[ \frac{1}{b_n} \sum_{r=1}^{m} \lambda_r S_n(t_r) \Rightarrow N(0, \Sigma_{\lambda,t}^{2}) \text{ with } \Sigma_{\lambda,t}^{2} = \text{Var} \left( \sum_{r=1}^{m} \lambda_r G_{t_r} \right). \]

The linear combinations of finite-dimensional distributions can again be represented as a linear random field via

\[ \sum_{r=1}^{m} \lambda_r S_n(t_r) = \sum_{j \in \mathbb{Z}^d} \tilde{b}_{n,j} X_j \text{ with } \tilde{b}_{n,j} = \sum_{r=1}^{m} \lambda_r b_{n,j}(t_r), \]

to which one can apply Theorem 2.4 again. This is a standard procedure to establish finite-dimensional convergence of linear random fields. In our setup we have thus proved the following as a consequence of Theorem 2.4.

Write \( \tilde{b}_n = (\sum_{j} \tilde{b}_{n,j}^2)_{1/2} \).

**Proposition 4.1.** Consider random fields in form of (4.1) with \( \{X_j\}_{j \in \mathbb{Z}^d} \) satisfying Hannan’s condition \( \Delta_2 < \infty \). Suppose there exists a sequence of real numbers \( \{b_n\}_{n \in \mathbb{N}} \) such that

(i) for all \( \lambda \in \mathbb{R}^m, t \in \mathbb{T}^m, m \in \mathbb{N} \), \( \{\tilde{b}_{n,j}\}_{j \in \mathbb{Z}^d, n \in \mathbb{N}} \) satisfy the assumptions in Theorem 2.4 and that \( \tilde{b}_n/b_n \) converges to a constant as \( n \to \infty \), and

(ii) for a zero-mean Gaussian process \( G \),

\[ \lim_{n \to \infty} \frac{1}{b_n^{2}} \mathbb{E}(S_n(t)S_n(\tau)) = \mathbb{E}(G_tG_\tau), \text{ for all } t, \tau \in \mathbb{T}. \]

Then, the convergence of finite-dimensional distributions (4.4) holds.

We highlight that to apply Proposition 4.1 the essential work consists of verifying the assumptions on \( \tilde{b}_{n,j} \), and computing the covariance (4.5). Both of these two steps are independent from the choice of dependence assumption on \( \{X_j\}_{j \in \mathbb{Z}^d} \). For invariance principles to be established below, these computations have been carried out in earlier proofs (under stronger assumptions on \( \{X_j\}_{j \in \mathbb{Z}^d} \)) and can be borrowed here without any changes.

For the tightness, the moment inequality (2.2) in Lemma 2.2 plays an important role. Similar inequality have been used to establish tightness in the aforementioned work, and the proofs can be adapted with little extra effort.
Below we present two improvements of earlier results. We only sketch the proofs in order not to introduce too much duplications.

4.1. Invariance principles for self-similar set-indexed Gaussian fields. Let \( \mu \) be a \( \sigma \)-finite measure on \( \mathbb{R}^d \). Consider

\[
S_n(A) := \sum_{j \in \mathbb{Z}^d} b_{n,j}(A) X_j \quad \text{with} \quad b_{n,j}(A) := \mu(nA \cap R_j)^{1/2}, \ A \in \mathcal{A}
\]

where \( R_j \) is the set of unit cube in \( \mathbb{R}^d \) with lower corner \( j \in \mathbb{Z}^d \), and \( \mathcal{A} \) is a class of Borel sets of \( \mathbb{R}^d \), equipped with pseudo-metric \( \rho(A, B) = \mu(A \Delta B)^{1/2} \). For \( \mu \) being the Lebesgue measure, this framework has been considered for example in [1, 9, 11]. The generalization to other measures was proposed by Bierné and Durieu [2]. In particular, they assume the measure \( \mu \) to satisfy the following.

Assumption 4.2. \( \mu \) is a \( \sigma \)-finite measure on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \), absolutely continuous with respect to the Lebesgue measure, and such that

(i) There exists \( \beta > 0 \) such that \( \mu(nA) = n^\beta \mu(A) \) for all \( n \in \mathbb{N}, A \in \mathcal{B}(\mathbb{R}^d) \).

(ii) \( \limsup_{\pi(j) \to \infty} \mu(R_j) < \infty \) and

\[
\lim_{\pi(j) \to \infty} \frac{|\mu(R_{j+\epsilon q}) - \mu(R_j)|}{\mu(R_j)} = 0, \ q = 1, \ldots, d
\]

with \( \pi(j) = \min_{q=1,\ldots,d} |j_q|, j \in \mathbb{Z}^d \).

Furthermore, they also worked with regular Borel sets \( A \), that is, for the boundary set \( \partial A \) of \( A \subset \mathbb{R}^d \), \( \text{Leb}(\partial A) = 0 \).

The following result generalizes [2, Theorem 4.5 (i)], by replacing Wu’s condition by Hannan’s condition. For concrete examples on self-similar set-indexed random fields as applications, see [2].

Theorem 4.3. Let \( \mu \) be a measure on \( \mathbb{R}^d \) satisfying Assumption 4.2, and let \( \mathcal{A} \) be a class of regular Borel sets of \( \mathbb{R}^d \) such that \( \mu(A) < \infty \) for all \( A \in \mathcal{A} \). Suppose there exists \( p \geq 2 \) such that \( \Delta_p < \infty \) and

\[
\int_{0}^{1} N(A, \rho, \epsilon)^{1/p} d\epsilon < \infty,
\]

where \( N(A, \rho, \epsilon) \) denotes the covering number of \( A \). Then,

\[
\left\{ \frac{S_n(A)}{n^{\beta/2}} \right\}_{A \in \mathcal{A}} \Rightarrow \sigma \{ \mathbb{G}(A) \}_{A \in \mathcal{A}}
\]

in the space of continuous functions on \( \mathcal{A} \) equipped with supremum norm, where \( \sigma \) is as in [2,3] and \( \mathbb{G} \) is a zero-mean Gaussian process with covariance \( \text{Cov}(\mathbb{G}(A), \mathbb{G}(B)) = \mu(A \cap B) \).
Proof. First, by [2, Proposition 4.2], for each \(A \in \mathcal{A}\), \(\{b_{n,j}(A)\}_{n,j}\) satisfy the assumptions of Theorem 2.4, and \(b_n^2(A) = n^\beta \mu(A)\). This tells the order of normalization should be \(n^{\beta/2}\).

To show the convergence of finite-dimensional distributions, we apply Proposition 4.1. The verifications of conditions and the computations of covariance, all based on definitions of \(b_{n,j}(A)\) and properties of \(\mu\) only, have been carried out in the proof of [2, Theorem 4.3].

To show the tightness, as in [2], we apply [15, Theorem 11.6], which states, if for some constant \(C > 0, p \geq 2\),

\[
\frac{1}{n^{\beta/2}} \|S_n(A) - S_n(B)\|_p \leq C \rho(A, B) \text{ for all } n \in \mathbb{N}, A, B \in \mathcal{A},
\]

and (4.6) holds, then

\[
\lim_{\eta \downarrow 0} \mathbb{E} \left( \sup_{A, B \in \mathcal{A} \atop \rho(A, B) < \eta} \frac{|S_n(A) - S_n(B)|}{n^{\beta/2}} \right) = 0,
\]

which yields the tightness. Now (4.7) follows from (2.2), and the proof is thus completed. \(\square\)

Remark 4.4. Other invariance principles in [2, 11], with different criteria on tightness, can be established in similar ways. All these proofs were based on a moment inequality similar to (2.2) under Wu’s condition [11, Proposition 1]. The adaptation would consist of replacing it by (2.2), without further changes. We omit these results.

4.2. An invariance principle for fractional Brownian sheet. Consider a linear random field \(\{Y_j\}_{j \in \mathbb{Z}^d}\) in form of

\[
Y_j = \sum_{k \in \mathbb{Z}^d} a_k X_{j-k}, j \in \mathbb{Z}^d,
\]

with \(\sum_k a_k^2 < \infty\). Invariance principles for

\[
S_n(t) = \sum_{1 \leq j \leq nt} Y_j, t \in [0, 1]^d
\]

with \(nt = (nt_1, \ldots, nt_d)\) have been studied in the literature. Observe that

\[
S_n(t) = \sum_{j \in \mathbb{Z}^d} b_{n,j}(t) X_j \quad \text{with} \quad b_{n,j}(t) = \sum_{1 \leq i \leq nt} a_{i-j}.
\]

The following theorem generalizes [20, Theorem 3]. In particular, [20] considered the case that \(\{a_j\}_{j \in \mathbb{Z}^d}\) is of the product form: there exist real numbers \(\{a_{j_q}^{(q)}\}_{j_q \in \mathbb{Z}}, q = 1, \ldots, d\) such that

\[
a_j = \prod_{q=1}^d a_{j_q}^{(q)}.
\]
Introduce also $b_{n,j}^{(q)} = \sum_{i=1}^{n} a_{i-j}^{(q)}$ and $b_n(q) = (\sum_{j \in \mathbb{Z}} b_{n,j}^{(q)})^{1/2}$. Examples on coefficients satisfying the assumption below can be found in [20, Example 2].

**Theorem 4.5.** Suppose there exists $H \in (0, 1)^d$ such that

$$
\lim_{n \to \infty} \frac{b_{n,s}^{(q)}}{b_n^{(q)}} = s^{2H_q}, \quad \text{for all } s \in [0, 1], q = 1, \ldots, d,
$$

and there exists $p$ such that

$$
p \geq 2, p > \max_{q=1,\ldots,d} \frac{1}{H_q} \text{ and } \Delta_p(X) < \infty.
$$

Then, $\{S_n(t)/b_n\}_{t \in [0,1]^d}$ converges weakly in $D([0,1]^d)$ to a fractional Brownian sheet $\mathbb{G}^H$ with Hurst index $H$, a zero-mean Gaussian process with covariance

$$
\text{Cov}(\mathbb{G}^H_s, \mathbb{G}^H_t) = \frac{1}{2d} \prod_{q=1}^{d} \left( s_{q}^{2H_q} + t_{q}^{2H_q} - |t_{q} - s_{q}|^{2H_q} \right), s, t \in [0, 1]^d.
$$

**Proof.** To show the convergence of finite-dimensional distributions, the conditions in Proposition 4.1 have been verified as in [20, proof of Proposition 1]; actually, there a different set of conditions in [20, Definition 1] on $b_{n,j}$ were verified. The equivalence between conditions there and ours were pointed out by Biermé and Durieu [2, Remark after Theorem 3.1] (see also Remark 2.6). We also point out that the conditions in [20] were actually redundant: in [20, Definition 1], Eq.(8) implies Eq.(9) by Cauchy–Schwarz inequality.

To show the tightness, by [14, Corollary 3], it suffices to show, for some $\beta > 1, p > 0$,

$$
\|S_n(t)\|_p \leq C b_n \prod_{q=1}^{d} t_{q}^{\beta/p}, t \in [0, 1]^d.
$$

For this purpose, by (2.2),

$$
\|S_n(t)\|_p \leq C \left( \sum_{j \in \mathbb{Z}^d} b_{n,j}^{2}(t) \right)^{1/2} \Delta_p(X),
$$

which could lead to the desired condition (4.8). This plan can be carried out as in [20, Proposition 2], with Eq. (23) therein replaced by (4.9) above and no other changes. We omit the details.

\[\square\]

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