Special numbers, special quaternions and special symbol elements

Diana SAVIN

Abstract. In this paper we define and we study properties of \((l, 1, p + 2q, q \cdot l)\) - numbers, \((l, 1, p + 2q, q \cdot l)\) - quaternions, \((l, 1, p + 2q, q \cdot l)\) - symbol elements. Finally, we obtain an algebraic structure with these elements.

Key Words: quaternion algebras; symbol algebras, Fibonacci numbers, Lucas numbers, Fibonacci-Lucas quaternions, Pell-Fibonacci-Lucas quaternions, \((l, 1, p + 2q, q \cdot l)\) - quaternions, \((l, 1, p + 2q, q \cdot l)\) - symbol elements.

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1. Introduction

Quaternion algebras and of symbol algebras have applications in various branches of mathematics, but also in computer science, physics, signal theory.

In this chapter we introduce special numbers, special quaternions, special symbol elements, and we present some of their properties and their applications in combinatorics, number theory and associative algebra theory.

Let \(K\) be a field with \(\text{char}(K) \neq 2\) and let \(\alpha, \beta \in K\setminus\{0\}\). We recall that the generalized quaternion algebra \(H_K(\alpha, \beta)\) is an algebra over the field \(K\) with a basis \(\{e_1, e_2, e_3, e_4\}\) (where \(e_1 = 1\) ) and the following multiplication:

| . | 1 | e_2 | e_3 | e_4 |
|---|---|---|---|---|
| 1 | 1 | e_2 | e_3 | e_4 |
| e_2 | e_2 | \alpha | e_4 | \alpha e_3 |
| e_3 | e_3 | -e_4 | \beta | -\beta e_2 |
| e_4 | e_4 | -\alpha e_2 | \beta e_2 | -\alpha \beta |

Let \(x\) be element from \(H_K(\alpha, \beta)\), \(x = x_1 \cdot 1 + x_2 e_2 + x_3 e_3 + x_4 e_4\), where \(x_i \in K\), \((\forall) i \in \{1, 2, 3, 4\}\) and let \(\overline{x}\) be the conjugate of \(x\), \(x = x_1 \cdot 1 - x_2 e_2 - x_3 e_3 - x_4 e_4\). The trace of \(x\) is \(t(x) = x + \overline{x} = 2x_1\). The norm of \(x\) is \(n(x) = x \cdot \overline{x} = x_1^2 - \alpha x_2^2 - \beta x_3^2 + \alpha \beta x_4^2\).

If \(K = \mathbb{R}\) and \(\alpha = \beta = -1\), we obtain Hamilton quaternion algebra \(\mathbb{H}_R(-1, -1)\), with the basis \(\{1, i, j, k\}\).

The generalization of a quaternion algebra is a symbol algebra.

Let \(n\) be an arbitrary positive integer, \(n \geq 3\) and let \(K\) be a field with \(\text{char}(K)\), which does not divide \(n\), containing \(\xi\), where \(\xi\) is a primitive \(n\)-th root
of unity. Let \( a, b \in K \backslash \{0\} \). The algebra \( A \) over \( K \) generated by elements \( x \) and \( y \) where
\[
x^n = a, \quad y^n = b, \quad yx = \xi xy
\]
is called a **symbol algebra** and it is denoted by \( \left( \frac{a}{b}, \kappa \right) \). Symbol algebras are also known as **power norm residue algebras**. For \( n = 2 \), we obtain the quaternion algebra over the field \( K \). Quaternion algebras and symbol algebras are associative but non-commutative algebras, of dimension \( n^2 \) over \( K \). Also, they are central simple over the field \( K \) (this means they are simple algebras and their centers are equal to \( K \)). Theoretical aspects about these algebras can be found in the books \([Pi, 82]\), \([La, 04]\), \([Mi, 71]\), \([Gi, Sz, 06]\), \([Le, 05]\), \([Als, Ba, 04]\), \([Vi, 80]\), \([Vo, 10]\), \([Ko]\). Several properties of these algebras and their applications in number theory, combinatorics, associative algebra, geometry, coding theory, mechanics can be found in the articles \([Ak, Ko, To, 14]\), \([Fla, 12]\), \([Fl, Sa, Io, 13]\), \([Fl, Sa, 14]\), \([Fl, Sa, 15]\), \([Fl, Sa, 15 (1)]\), \([Fl, Sa, 15 (2)]\), \([Fl, Sa, 17]\), \([Fl, Sa, 18]\), \([Fl, Sh, 13]\), \([Fl, Sh, 13 (1)]\), \([Ha, 12]\), \([Ho, 63]\), \([Ja, Ya, 13]\), \([Ka, Ha, 17]\), \([Li, 12]\), \([Ra, 15]\), \([Sa, Fa, Ci, 09]\), \([Sa, 14 (1)]\), \([Sa, 16 (1)]\), \([Sa, 17]\), \([Sw, 73]\), \([Tu, 13]\).

Many mathematicians studied the Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, generalized Fibonacci- Lucas numbers, the generalized Pell- Fibonacci- Lucas numbers, Fibonacci polynomials, Jacobsthal- Lucas polynomials, Fibonacci quaternions, the generalized Fibonacci-Lucas quaternions, the generalized Pell-Fibonacci- Lucas quaternions (see \([Ho, 63]\), \([Ca, 15]\), \([Ca, Mo, 16]\), \([Ca, 16]\), \([Fl, Sa, 15]\), \([Fl, Sa, 18]\), \([Fl, Sh, 13]\), \([Ha, 12]\), \([Sa, 14]\), \([Sw, 73]\), etc.).

In this paper we define \((l, 1, p + 2q, q \cdot l)\) – numbers, \((l, 1, p + 2q, q \cdot l)\) – quaternions, \((l, 1, p + 2q, q \cdot l)\) – symbol elements. We also study properties and applications of these elements.

This paper is organized as follow: section 2 is a preliminary section, containing theoretical notions which we will then use in our results. In section 3 we introduce two special number sequences (namely \((a_n)_{n \geq 0}\), \((b_n)_{n \geq 0}\)), we obtain some interesting properties of these sequences and we also obtain some quaternion algebras which split or some division quaternion algebras. After these, in the same section, we introduce \((l, 1, p + 2q, q \cdot l)\) – numbers, \((l, 1, p + 2q, q \cdot l)\) – quaternions, \((l, 1, p + 2q, q \cdot l)\) – symbol elements and we obtain interesting properties and applications of them.

2. Preliminaries

First of all, we recall some results about prime integers, about diophantine equations or about the Fibonacci numbers, properties which will be necessary (in the next section) for to study some quaternion algebras.
Proposition 2.1. ([Cu; 06]). Let \( m \) be a fixed positive integer. The diophantine equation \( x^2 + my^2 = z^2 \) has an infinity of solutions:

\[
x = a^2 - mb^2, \quad y = 2mab, \quad z = a^2 + mb^2, \quad a, b \in \mathbb{Z}.
\]

Theorem 2.2. ([Al, Go; 99]). Let \( n \) be a positive integer. Then, there exist integers \( x, y \) such that \( n = x^2 + y^2 \) if and only if the exponent of any prime \( p \equiv 3 \pmod{4} \) that divides \( n \) is even.

Proposition 2.3. ([Sa; 14]). For each positive integer \( n, n \equiv 7 \pmod{16} \), there exist integer numbers \( x, y \) so that, the Fibonacci number \( f_n \) can be written as \( f_n = x^2 + 9y^2 \).

Let \( K \) be a field with \( \text{char} \, (K) \neq 2 \), let \( \alpha, \beta \in K \backslash \{0\} \) and let the generalized quaternion algebra \( H_K(\alpha, \beta) \). \( H_K(\alpha, \beta) \) is a division algebra if and only if for \( x \in H_K(\alpha, \beta) \) we have \( n(x) = 0 \) only for \( x = 0 \).

We recall that \( H_K(\alpha, \beta) \) is called split by \( K \) if it is isomorphic with a matrix algebra over \( K \) (see [Pi; 82], [La;04], [Mil], [Gi, Sz; 06]). It is known the following remark about the quaternion algebras.

Remark 2.4. ([La;04], [Le; 05]). Let \( K \) be a field with \( \text{char} \, K \neq 2 \) and let \( \alpha, \beta \in K \backslash \{0\} \). Then, the quaternion algebra \( H_K(\alpha, \beta) \) is either split or a division algebra.

In the book [Gi, Sz; 06] appears the following criterion to decide if a quaternion algebra splits.

Proposition 2.5. ([Gi, Sz; 06]). Let \( K \) be a field with \( \text{char} \, K \neq 2 \) and let \( \alpha, \beta \in K \backslash \{0\} \). The quaternion algebra \( H_K(\alpha, \beta) \) splits if and only if the conic \( C(\alpha, \beta) : \alpha x^2 + \beta y^2 = z^2 \) has a rational point over \( K \) (i.e. if there are \( x_0, y_0, z_0 \in K \), not all zero such that \( \alpha x_0^2 + \beta y_0^2 = z_0^2 \)).

3. Some properties of special quaternions

Let \( l \) be a nonzero natural number. We consider the sequence \( (a_n)_{n \geq 0} \)

\[
a_n = l \cdot a_{n-1} + a_{n-2}, \; n \geq 2, \; a_0 = 0, \; a_1 = 1
\]

and let the sequence \( (b_n)_{n \geq 0} \)

\[
b_n = l \cdot b_{n-1} + b_{n-2}, \; n \geq 2, \; b_0 = 2, \; b_1 = l.
\]

Let \( \alpha = \frac{l+\sqrt{l^2+4}}{2} \) and \( \beta = \frac{l-\sqrt{l^2+4}}{2} \). It results immediately the following relations:
Binet’s formula for the sequence \((a_n)_{n \geq 0}\):
\[
a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{l^2 + 4}}, \quad (\forall) \; n \in \mathbb{N}.
\]

Binet’s formula for the sequence \((b_n)_{n \geq 0}\):
\[
b_n = \alpha^n + \beta^n, \quad (\forall) \; n \in \mathbb{N}.
\]

In the following, we show that the product of two elements belonging to the sequences \((a_n)_{n \geq 0}\), \((b_n)_{n \geq 0}\) are transformed into sums of elements belonging to the same sequences. Also, we find another properties of these sequences.

**Proposition 3.1.** Let \((a_n)_{n \geq 0}\), \((b_n)_{n \geq 0}\) be the sequences previously defined. Then, the following equalities are true:

i) \[
b_n b_{n+m} = b_{2n+m} + (-1)^n b_m, \quad (\forall) \; n, m \in \mathbb{N};
\]

ii) \[
a_n b_{n+m} = a_{2n+m} + (-1)^{n+1} a_m, \quad (\forall) \; n, m \in \mathbb{N};
\]

iii) \[
a_{n+m} b_n = a_{2n+m} + (-1)^n a_m, \quad (\forall) \; n, m \in \mathbb{N};
\]

iv) \[
a_n a_{n+m} = \frac{1}{l^2 + 4} \left[ b_{2n+m} + (-1)^{n+1} b_m \right], \quad (\forall) \; n, m \in \mathbb{N};
\]

v) \[
b_n + b_{n+2} = (l^2 + 4) \cdot a_{n+1}, \quad (\forall) \; n \in \mathbb{N};
\]

vi) \[
a_n^2 + a_{n+1}^2 = a_{2n+1}, \quad (\forall) \; n \in \mathbb{N};
\]

vii) \[
b_n^2 + b_{n+1}^2 = (l^2 + 4) \cdot a_{2n+1}, \quad (\forall) \; n \in \mathbb{N};
\]

**Proof.** Let \(n, m\) be two positive integers. Applying Binet’s formula, we have:

i) \[
b_n b_{n+m} = (\alpha^n + \beta^n) \cdot (\alpha^{n+m} + \beta^{n+m}) = \alpha^{2n+m} + \beta^{2n+m} + \alpha^n \beta^n (\alpha^m + \beta^m) = b_{2n+m} + (-1)^n b_m.
\]

ii) \[
a_n b_{n+m} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \cdot (\alpha^{n+m} + \beta^{n+m}) = \frac{\alpha^{2n+m} - \beta^{2n+m}}{\alpha - \beta} - \alpha^n \beta^n \frac{\alpha^m - \beta^m}{\alpha - \beta} = a_{2n+m} + (-1)^{n+1} a_m.
\]
iii) \[
\alpha^{n+m} = \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} (\alpha^n + \beta^n) = \\
= a_{2n+m} + (-1)^n a_m.
\]

iv) \[
a_n a_{n+m} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \cdot \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} = \\
= \frac{1}{(\alpha - \beta)^2} \cdot [\alpha^{2n+m} + \beta^{2n+m} - \alpha^n \beta^n (\alpha^m + \beta^m)] = \\
= \frac{1}{l^2 + 4} \left[ b_{2n+m} + (-1)^{n+1} b_m \right].
\]

v) \[
b_n + b_{n+2} = \alpha^n + \beta^n + \alpha^{n+2} + \beta^{n+2} = \\
= \alpha^{n+1} \cdot \left( \alpha + \frac{1}{\alpha} \right) + \beta^{n+1} \cdot \left( \beta + \frac{1}{\beta} \right) = \\
= \alpha^{n+1} \cdot \sqrt{l^2 + 4} - \beta^{n+1} \cdot \sqrt{l^2 + 4} = (l^2 + 4) \cdot a_{n+1}.
\]

vi) Applying iv) for \( m = 0 \), we have:
\[
a_n^2 + a_{n+1}^2 = \frac{1}{l^2 + 4} \left[ b_{2n} + (-1)^{n+1} b_0 + b_{2n+2} + (-1)^{n+2} b_0 \right] = \\
= \frac{1}{l^2 + 4} \left[ b_{2n} + b_{2n+2} \right].
\]

Applying v) we obtain:
\[
a_n^2 + a_{n+1}^2 = a_{2n+1}.
\]

vii) Applying v) we have:
\[
b_n^2 + b_{n+1}^2 = (\alpha^n + \beta^n)^2 + (\alpha^{n+1} + \beta^{n+1})^2 = \\
= \alpha^{2n} + \beta^{2n} + 2 (-1)^n + \alpha^{2n+2} + \beta^{2n+2} + 2 (-1)^{n+1} = \\
= b_{2n} + b_{2n+2} = (l^2 + 4) \cdot a_{2n+1}.
\]

Let \((f_n)_{n \geq 0}\) be the Fibonacci sequence and let \((l_n)_{n \geq 0}\) be the Lucas sequence. There are well known the Cassini's identities for Fibonacci and Lucas numbers:
\[
f_{n+1}f_{n-1} - f_n^2 = (-1)^n, \quad (\forall) \quad n \in \mathbb{N}^*,
\]
and
\[
l_{n+1}l_{n-1} - l_n^2 = 5 (-1)^{n-1}, \quad (\forall) \quad n \in \mathbb{N}^*.
\]
Now, we obtain similarly results for the sequences \((a_n)_{n \geq 0}, (b_n)_{n \geq 0}\).

**Proposition 3.2.** Let \((a_n)_{n \geq 0}, (b_n)_{n \geq 0}\) be the sequences previously defined. Then, the following identities are true:

i) \(a_n a_{n-1} - a_n^2 = (-1)^n, \quad (\forall) \ n \in \mathbb{N}^*;\)

ii) \(b_n b_{n-1} - b_n^2 = (-1)^n \cdot (l^2 + 4), \quad (\forall) \ n \in \mathbb{N}^*.

**Proof.** i)

\[
a_{n+1}a_{n-1} - a_n^2 = \frac{\alpha^{n+1} - \beta^{n+1}}{\sqrt{l^2 + 4}} \cdot \frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{l^2 + 4}} - \frac{(\alpha^n - \beta^n)^2}{l^2 + 4} = \frac{(-1)^{n-1} \cdot (\alpha^2 + \beta^2) + 2 (-1)^n}{l^2 + 4} = \frac{(-1)^n \cdot (b_2 + 2)}{l^2 + 4} = (-1)^n.
\]

ii)

\[
b_{n+1}b_{n-1} - b_n^2 = (\alpha^{n+1} + \beta^{n+1}) \cdot (\alpha^{n-1} + \beta^{n-1}) - (\alpha^n + \beta^n)^2 = (-1)^{n-1} \cdot (\alpha^2 + \beta^2) - 2 (-1)^n = (-1)^{n-1} \cdot (b_2 + 2) = (-1)^{n-1} \cdot (l^2 + 4).
\]

**Proposition 3.3.** Let \((b_n)_{n \geq 0}\) be the sequence previously defined. Then, the followings are true:

i) if \(l\) is even, then \(b_n\) is even \((\forall) \ n \in \mathbb{N};\)

ii) if \(l\) is odd, then \(b_n\) is even if and only if \(n \equiv 0 \pmod{3};\)

iii) if \(n \equiv 0 \pmod{6}\), then \(b_{n-1} \cdot b_{n+1} \equiv 3 \pmod{4};\)

iv) if \(n \equiv 3 \pmod{6}\), then \(b_{n-1} \cdot b_{n+1} \equiv 1 \pmod{4}.

**Proof.** For i), ii), iii) and iv) the proof is immediate, using the principle of mathematics induction (after \(n \in \mathbb{N}\). □

**Proposition 3.4.** Let \((a_n)_{n \geq 0}, (b_n)_{n \geq 0}\) be the sequences previously defined. Then, the followings are true:

i) The quaternion algebra \(\mathbb{H}_Q\) \((-1, f_{2n+1})\) splits, \((\forall) \ n \in \mathbb{N}^*;\)

ii) The quaternion algebra \(\mathbb{H}_Q\) \((-1, 5f_{2n+1})\) splits, \((\forall) \ n \in \mathbb{N}^*;\)

iii) The quaternion algebra \(\mathbb{H}_Q\) \((-1, a_{2n+1})\) splits, \((\forall) \ n \in \mathbb{N}^*;\)

iv) The quaternion algebra \(\mathbb{H}_Q\) \((-1, (l^2 + 4) \cdot a_{2n+1})\) splits, \((\forall) \ n \in \mathbb{N}^*;\)

v) The quaternion algebra \(\mathbb{H}_Q\) \((-1, f_{2n+1}f_{2n-1})\) splits, \((\forall) \ n \in \mathbb{N}^*;\)

vi) The quaternion algebra \(\mathbb{H}_Q\) \((-1, a_{2n+1}a_{2n-1})\) splits, \((\forall) \ n \in \mathbb{N}^*;\)

vii) The quaternion algebra \(\mathbb{H}_Q\) \((-1, -b_{n+1}b_{n-1})\) is a division algebra, \((\forall) \ n \in \mathbb{N}^*;\)

viii) The quaternion algebra \(\mathbb{H}_Q\) \((1, b_{n+1}b_{n-1})\) splits, \((\forall) \ n \in \mathbb{N}^*;\)

ix) If \(l\) is odd and \(n \equiv 0 \pmod{6}\), then the quaternion algebra \(\mathbb{H}_Q\) \((-1, b_{n+1}b_{n-1})\) is a division algebra, \((\forall) \ n \in \mathbb{N}^*;\)

x) If \(6 \nmid n\) and the exponent of any prime \(p \equiv 3 \pmod{4}\) that divides \(b_{n+1}b_{n-1}\) is even, then the quaternion algebra \(\mathbb{H}_Q\) \((-1, b_{n+1}b_{n-1})\) splits, \((\forall) \ n \in \mathbb{N}^*;\)
The quaternion algebra \( H \) splits, \( \forall \) \( n \in \mathbb{N}^* \), \( n \equiv 7 \) (mod 16).

**Proof.** Since ii) is a generalization of i), we are proving directly iii).

iii) If we consider the equation \(-x^2 + a_{2n+1} \cdot y^2 = z^2\), we apply Proposition 3.1 (vi) and we obtain that it has the following solution in \( \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} : \)

\[
(x_0, y_0, z_0) = (a_n, 1, a_{n+1}).
\]

According to Proposition 2.5, it results that the quaternion algebra \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, a_{2n+1}) \) splits, \( \forall \) \( n \in \mathbb{N}^* \).

iv) Using Proposition 3.1 (vii), it results that the equation \(-x^2 + (l^2 + 4) \cdot a_{2n+1} \cdot y^2 = z^2\) has a solution in \( \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \), namely \((x_0, y_0, z_0) = (b_n, 1, b_{n+1})\). Applying Proposition 2.5, it results that the quaternion algebra \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, (l^2 + 4) \cdot a_{2n+1}) \) splits, \( \forall \) \( n \in \mathbb{N}^* \).

ii) This is a particular case of ii) (for \( l = 1 \)).

vi) This is a generalization of v), so we are proving only vi).

Using Proposition 3.2 i), we find the following solution in \( \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \) for the equation \(-x^2 + a_{2n+1} \cdot a_{2n-1} \cdot y^2 = z^2\) : \((x_0, y_0, z_0) = (((-1)^n \cdot 1, a_{2n})\). Applying Proposition 2.5, we obtain that the quaternion algebra \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, a_{2n+1}a_{2n-1}) \) splits, \( \forall \) \( n \in \mathbb{N}^* \).

vii) Let the quaternion algebra \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, -b_{n+1}b_{n-1}) \) and let \( \{1, e_2, e_3, e_4\} \) a basis in this algebra. Let \( x = x_1 \cdot 1 + x_2 \cdot e_2 + x_3 \cdot e_3 + x_4 \cdot e_4 \in \mathbb{H}_{\mathbb{Q}} (-1, -b_{2n+1}b_{2n-1}) \). The norm of \( x \) is \( (x) = x_1^2 + x_2^2 + b_{n-1}b_{n+1}x_3^2 + b_{n-1}b_{n+1}x_4^2 \). Since \( b_n \in \mathbb{N}^* \), for \( \forall \) \( n \in \mathbb{N}^* \), it results that \((x) = 0\) if and only if \( x = 0 \). So, \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, -b_{n+1}b_{n-1}) \) is a division algebra for \( \forall \) \( n \in \mathbb{N}^* \).

Similarly, it results immediately that \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, -f_{n+1}f_{n-1}) \), \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, -a_{n+1}a_{n-1}) \), \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, -l_{n+1}l_{n-1}) \) are division algebras for \( \forall \) \( n \in \mathbb{N}^* \).

viii) We study if the equation \( x^2 + b_{n+1}b_{n-1} \cdot y^2 = z^2 \) has rational solutions. Applying Proposition 2.1, it results that the equation \( x^2 + b_{n+1}b_{n-1} \cdot y^2 = z^2 \) has solutions in integer numbers, so it has solutions in the set of rational numbers. Using Proposition 2.5, we obtain that the quaternion algebra \( H \equiv \mathbb{H}_{\mathbb{Q}} (1, b_{n+1}b_{n-1}) \) splits.

ix) If \( n \equiv 0 \) (mod 6), according to Proposition 3.3, \( b_{n-1} \cdot b_{n+1} \equiv 3 \) (mod 4). We study if the equation \(-x^2 + b_{n+1}b_{n-1} \cdot y^2 = z^2 \) has integer solutions. We suppose that this equation has a solution \((x_0, y_0, z_0) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \setminus \{0, 0, 0\}\), g.c.d \((x_0, y_0) = g.c.d \((y_0, z_0) = g.c.d \((y_0, z_0) = 1\). We have: \( b_{n+1}b_{n-1} \cdot y_0^2 \equiv 0 \) or 3 (mod 4), but \( x_0^2 + z_0^2 \equiv 1 \) or 2 (mod 4), so we cannot have \( b_{n+1}b_{n-1} \cdot y_0^2 = x_0^2 + z_0^2 \). It results that the equation \(-x^2 + b_{n+1}b_{n-1} \cdot y^2 = z^2 \) does not have integer solutions. We obtain immediately that the equation \(-x^2 + b_{n+1}b_{n-1} \cdot y^2 = z^2 \) does not have solutions in the set of rational numbers, so the quaternion algebra \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, b_{n+1}b_{n-1}) \) does not split.

Applying Remark 2.4, we obtain that the quaternion algebra \( H \equiv \mathbb{H}_{\mathbb{Q}} (-1, b_{n+1}b_{n-1}) \) is a division algebra, \( \forall \) \( n \in \mathbb{N}^* \).

x) Case 1 : \( l \) is odd.

If \( n \equiv 3 \) (mod 6), according to Proposition 3.3 (iv) \( b_{n-1} \cdot b_{n+1} \equiv 1 \) (mod 4). If \( n \equiv 1 \) or 2 or 4 or 5 (mod 6), according to Proposition 3.3 (ii) \( b_{n-1} \cdot b_{n+1} \) is even.

Case 2 : \( l \) is even, according to Proposition 3.3 (i) \( b_{n-1} \cdot b_{n+1} \) is even.
In all these cases, it is possible to exist a prime \( p \equiv 3(\text{mod } 4) \) that divides \( b_{n+1}b_{n-1} \). If the exponent of any prime \( p \equiv 3(\text{mod } 4) \) that divides \( b_{n+1}b_{n-1} \) is even, according to Theorem 2.2 there exist integers \( x_0; z_0 \) such that \( b_{n+1}b_{n-1} = x_0^2 + z_0^2 \). This implies that \((x_0, 1, z_0)\) is a solution to integer numbers for the equation \( -x^2 + b_{n+1}b_{n-1} \cdot y^2 = z^2 \), so, according Proposition 2.5, the quaternion algebra \( \mathbb{H}_\mathbb{Q}(-1, b_{n+1}b_{n-1}) \) splits, \((\forall)\ n \in \mathbb{N}^*\).

Let \( n \) be a positive integer number, \( n \equiv 7 \pmod{16} \). Using Proposition 2.3 we obtain that there are \( x_0, z_0 \in \mathbb{Z} \) such that \((x_0, 1, z_0)\) is a solution of the equation \( -9x^2 + f_n \cdot y^2 = z^2 \). Applying Proposition 2.5, we obtain that the quaternion algebra \( \mathbb{H}_\mathbb{Q}(-9, f_n) \) splits, \((\forall)\ n \in \mathbb{N}^*\), \( n \equiv 7 \pmod{16} \).

Let \( p, q \) be two arbitrary integers and \( (a_n)_{n \geq 0}, (b_n)_{n \geq 0} \) are the sequences previously defined. If \( n \in \mathbb{N}^* \), \( a_n = (-1)^{n+1} \cdot a_n \).

Let the sequence \( (u_n)_{n \geq 0} \),

\[
u_{n+1} = pa_n + qb_{n+1}, \ n \geq 0.\]

To avoid confusion, we will use the notation \( u^{p,q}_n \) for \( u_n \).

We remark that \( u_n = lu_{n-1} + u_{n-2}, \ (\forall)\ n \in \mathbb{N}, n \geq 2, \)

We calculate \( u_0 = pa_0 + qb_0 = p + 2q, u_1 = pa_0 + qb_1 = q \cdot l \). We call the elements of the sequence \( (u_n)_{n \geq 0} \) the \( (l, 1, p + 2q, q \cdot l) \)-numbers.

**Remark 3.5.** Let \( p, q \) be two arbitrary integers, and let \( (u^{p,q}_n)_{n \geq 1} \) the sequence previously defined. Then, we have:

\[
pu_{n+1} + qbu_n = u^{p,q}_n + u^{pl,o}_n, \ \forall \ n \in \mathbb{N} - \{0\}.\]

**Proof.** We compute

\[
pu_{n+1} + qbu_n = pla_n + pa_{n-1} + qb_n = u^{p,q}_n + u^{pl,o}_n.\]

Let \( \alpha, \beta \in \mathbb{Q}^* \). We consider the generalized quaternion algebra \( \mathbb{H}_\mathbb{Q}(\alpha, \beta) \) with basis \( \{1, e_1, e_2, e_3\} \). We define the \( n \)-th \( (l, 1, p + 2q, q \cdot l) \)-quaternion to be the element of the form

\[
U^{p,q}_n = u^{p,q}_n \cdot 1 + u^{p,q}_{n+1} \cdot e_1 + u^{p,q}_{n+2} \cdot e_2 + u^{p,q}_{n+3} \cdot e_3.\]

**Remark 3.6.** Let \( U^{p,q}_n \) be the \( n \)-th \( (l, 1, p + 2q, q \cdot l) \)-quaternion. Then, we have:

\[
U^{p,q}_0 = 0 \text{ if and only if } p = q = 0.\]

**Proof.** “\( \Rightarrow \)” It is trivial.

“\( \Rightarrow \)” If \( U^{p,q}_0 = 0 \), using the fact that \( \{1, e_1, e_2, e_3\} \) is a basis in quaternion algebra \( \mathbb{H}_\mathbb{Q}(\alpha, \beta) \), we obtain that \( u^{p,q}_n = 0, u^{p,q}_{n+1} = 0, u^{p,q}_{n+2} = 0, u^{p,q}_{n+3} = 0. \)

From the recurrence relation of the sequence \( (u^{p,q}_n)_{n \geq 1} \), it results that \( u^{p,q}_{n-1} = 0, \)
\(u_n^{p,q} = 0, \ldots, s_1^{p,q} = 0, u_0^{p,q} = 0.\) So, \(q = 0\) and \(p = 0.\)

About the generalized Fibonacci-Lucas quaternions \((G_n^{p,q})_{n \geq 0}\), in the paper [Fl, Sa; 15] (Theorem 3.5), we proved that:

i) The set

\[
M = \left\{ \sum_{i=1}^{n} 5G_{n_i}^{p_i,q_i} | n \in \mathbb{N}^+, p_i, q_i \in \mathbb{Z}, \forall i = 1, n \right\} \cup \{1\}
\]

has a ring structure with quaternion addition and multiplication.

ii) The set \(M\) is an order of the quaternion algebra \(\mathbb{H}_Q(\alpha, \beta)\).

We generalized these results for \((1, a, p + 2q, q)\)quaternions \((S_n^{p,q})_{n \geq 0}\), in the paper [Fl, Sa; 17] (Proposition 5.4), namely:

Let \(a\) be a nonzero natural number and let \(O\) be the set

\[
O = \left\{ \sum_{i=1}^{n} (1 + 4a) S_{n_i}^{p_i,q_i} | n \in \mathbb{N}^+, p_i, q_i \in \mathbb{Z}, \forall i = 1, n \right\} \cup \{1\}.
\]

Then \(O\) is an order of the quaternion algebra \(\mathbb{H}_Q(\alpha, \beta)\).

Similarly, in the paper [Fl, Sa; 18] we introduced the generalized Pell- Fibonacci-Lucas numbers \((r_n^{p,q})_{n \geq 0}\), the generalized Pell- Fibonacci-Lucas quaternions \((R_n^{p,q})_{n \geq 0}\), and we proved that (Proposition 3.7. from the paper [Fl, Sa; 18]) the set

\[
O = \left\{ \sum_{i=1}^{n} 8R_{n_i}^{p_i,q_i} | n \in \mathbb{N}^+, p_i, q_i \in \mathbb{Z}, \forall i = 1, n \right\} \cup \{1\}
\]

is an order of the quaternion algebra \(\mathbb{H}_Q(\alpha, \beta)\).

Here, we generalized these numbers and these quaternions: the sequence \((a_n)_{n \geq 0}\) is the generalization for the Pell sequence \((P_n)_{n \geq 0}\), and the sequence \((b_n)_{n \geq 0}\) is the generalization for the Pell-Lucas sequence \((Q_n)_{n \geq 0}\). Also, the sequence \((u_n^{p,q})_{n \geq 0}\) is the generalization for the sequence \((r_n^{p,q})_{n \geq 0}\) and the sequence of the 
\((1, 1, p + 2q, q - 1)\) quaternions \((U_n^{p,q})_{n \geq 0}\) is the generalization for the sequence of the generalized Pell- Fibonacci-Lucas quaternions \((R_n^{p,q})_{n \geq 0}\).

Let \(\epsilon\) be a primitive root of the unity of order 3 and let \(K\) be a field with the property \(\epsilon \in K\). Let \(\alpha_1, \alpha_2 \in K^*\) and let \(A = \left(\frac{\alpha_1, \alpha_2}{K^*}\right)\) be the symbol algebra of degree 3. \(A\) has a \(K^*\) basis \(\\{x^{j_1}y^{j_2} | 0 \leq j_1, j_2 < 3\}\), with \(x^3 = \alpha_1, y^3 = \alpha_2, xy = \epsilon xy\).

In the paper [Fl, Sa; 14], we defined the \(n\)-th Fibonacci symbol element

\[
F_n = f_n \cdot 1 + f_{n+1} \cdot x + f_{n+2} \cdot x^2 + f_{n+3} \cdot y + f_{n+4} \cdot xy + f_{n+5} \cdot x^2 y + f_{n+6} \cdot y^2 + f_{n+7} \cdot xy^2 + f_{n+8} \cdot x^2 y^2.
\]
In the paper [Fl, Sa, Io; 13] we defined the $n$-th Lucas symbol element

$$L_n = l_n \cdot 1 + l_{n+1} \cdot x + l_{n+2} \cdot x^2 + l_{n+3} \cdot y + l_{n+4} \cdot xy + l_{n+5} \cdot x^2y + l_{n+6} \cdot y^2 + l_{n+7} \cdot xy^2 + l_{n+8} \cdot x^2y^2.$$ 

Now, we define the $n$-th $(l, 1, p + 2q, q \cdot l)$ – symbol element to be the element of the form

$$U_{n}^{p,q} = u_{n}^{p,q} \cdot 1 + u_{n+1}^{p,q} \cdot x + u_{n+2}^{p,q} \cdot x^2 + u_{n+3}^{p,q} \cdot y +
+ u_{n+4}^{p,q} \cdot xy + u_{n+5}^{p,q} \cdot x^2y + u_{n+6}^{p,q} \cdot y^2 + u_{n+7}^{p,q} \cdot xy^2 + u_{n+8}^{p,q} \cdot x^2y^2.$$ 

**Remark 3.7.** Let $U_{n}^{p,q}$ be the $n$-th $(l, 1, p + 2q, q \cdot l)$ – symbol element. Then, we have:

$$U_{n}^{p,q} = 0 \text{ if and only if } p = q = 0.$$ 

The proof of this remark is similar to the proof of Remark 3.6.

With proof ideas similar to those in the Theorem 3.5 from the paper [Fl, Sa; 15], Proposition 5.4 from the paper [Fl, Sa; 17], Proposition 3.7. from the paper [Fl, Sa; 18], we obtain the following results:

**Proposition 3.8.** Let $l$ be a nonzero natural number and let $M_1$ be the set

$$M_1 = \left\{ \sum_{i=1}^{n} \left( l^2 + 4 \right) U_{n_i}^{p_i,q_i} | n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, (\forall) i = 1, n \right\} \cup \{1\}.$$ 

Then $M_1$ is an order of the quaternion algebra $\mathbb{H}_Q(\alpha, \beta)$.

**Proposition 3.9.** Let $l$ be a nonzero natural number and let $M_2$ be the set

$$M_2 = \left\{ \sum_{i=1}^{n} \left( l^2 + 4 \right) U_{n_i}^{p_i,q_i} | n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, (\forall) i = 1, n \right\} \cup \{1\}.$$ 

Then $M_2$ is an order of the symbol algebra $A = \left( \frac{\alpha_1, \alpha_2}{K, \varepsilon} \right)$.

Since the proofs of Proposition 3.8 and Proposition 3.9 are similar, we only prove one of them.

**Proof of Proposition 3.9.**

We prove that $M_2$ is a free $\mathbb{Z}$– submodule of rank 9 of the symbol algebra $A = \left( \frac{\alpha_1, \alpha_2}{K, \varepsilon} \right)$.

According to Remark 3.7, $U_{n}^{0,0} = 0 \in O$.

Let $n, m \in \mathbb{N}^*, p, q, p', q', c, d \in \mathbb{Z}$. We have:

$$cu_{n}^{p,q} + du_{m}^{p',q'} = u_{n}^{c,cq} + u_{m}^{d,p',dq'}.$$
This implies that 
\[
\mathcal{U}_p,q + d\mathcal{U}_m, q' = \mathcal{U}_p, cq + \mathcal{U}_m, dq' 
\]
(5.1.)

So, $M_2$ is a free $\mathbb{Z}$-submodule of rank 9 of the symbol algebra $A$.

We consider the set $M_3 = \left\{ \sum_{i=1}^{\infty} l^{2+4} u_{n, q_i} | n \in \mathbb{N}^*, p_i, q_i \in \mathbb{Z}, (\forall)i = 1, n \right\}$.

We are proving that $M_2$ is a subring of the symbol algebra $A$. Taking into account the relation (5.1.), it is enough to prove that $(l^{2+4} \mathcal{U}_p, q) (l^{2+4} \mathcal{U}_m, q') \in M_2$.

For this, it is enough to prove that $(l^{2+4} u_{n, q}) (l^{2+4} u_{m, q'}) \in M_3$.

Let $m, n$ be two integers, $n < m$. We calculate:

\[
(l^{2+4} u_{n, q}) (l^{2+4} u_{m, q'}) = 
\]
\[
= (l^{2+4} (p a_{n-1} + q b_n) (l^{2+4} (p' a_{m-1} + q' b_m)) = 
\]
\[
= (l^{2+4} p p' a_{n-1} a_{m-1} + (l^{2+4} p q' a_{n-1} b_m + 
+ (l^{2+4} p' q a_{m-1} b_n + (l^{2+4} q q' b_n b_m).
\]

Using Proposition 3.1, we have:

\[
(l^{2+4} u_{n, q}) (l^{2+4} u_{m, q'}) = 
\]
\[
= (l^{2+4})^2 p p' \frac{1}{l^{2+4}} [b_{n+m-2} + (-1)^n b_{m-n}] + (l^{2+4})^2 p q' [a_{m+n-1} + (-1)^n a_{m-n+1}] + 
+ (l^{2+4})^2 p' q [a_{n+m-1} + (-1)^n a_{m-n-1}] + (l^{2+4})^2 q q' [b_{n+m} + (-1)^n b_{m-n}] = 
\]
\[
= (l^{2+4})^2 [p q' a_{n+m-1} + q q' b_{m+n}] + (l^{2+4})^2 [(-1)^n p' q a_{m-n-1} + (-1)^n q q' b_{m-n}] + 
+ (l^{2+4}) [(l^{2+4} p' q a_{n+m-1} + (-1)^n p p' b_{n+m-2}]
\]

Applying the definition of the sequence $(u_n)_{n \geq 0}$ and Remark 3.5, we obtain:

\[
(l^{2+4} u_{n, q}) (l^{2+4} u_{m, q'}) = 
\]
\[
= (l^{2+4} u_{n+m}^{(2+4)p q' .(2+4)q q'} + (l^{2+4}) (l^{2+4} u_{m-n}^{(1)n(2+4)p q' .(-1)n(2+4)q q'} + 
+ (l^{2+4} u_{m-n}^{(1)n-1(2+4)p q' .(-1)n-1(2+4)q q'} + (l^{2+4} u_{m-n}^{(1)n-1(2+4)p q' .0} + 
+ (l^{2+4} u_{m+n-2}^{(1)(2+4)p q' .0}) + (l^{2+4} u_{m+n-1}^{(1)(2+4)p q' .0} .
\]

So, $(l^{2+4} u_{n, q}) (l^{2+4} u_{m, q'}) \in M_3$.

It results that $M_2$ is an order of the symbol algebra $A = \left( \frac{21+pq}{K} \right)$.
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Diana SAVIN
Faculty of Mathematics and Computer Science,
Ovidius University,
Bd. Mamaia 124, 900527, CONSTANTANTA, ROMANIA
[http://www.univ-ovidius.ro/math/](http://www.univ-ovidius.ro/math/)
e-mail: savin.diana@univ-ovidius.ro, dianet72@yahoo.com