A NOVEL CHEBYSHEV-COLLOCATION SPECTRAL METHOD FOR SOLVING THE TRANSPORT EQUATION

ZHONGHUI LI
Business School
Shandong Normal University
Jinan, 250014, P.R. China

XIANGYONG CHEN
School of Automation and Electrical Engineering
and Key Laboratory of complex Systems and Intellignet Computing
Linyi 276005, Shandong, P.R. China
Hubei Key Laboratory of Advanced Control and Intelligent Automation of Complex Systems, and Engineering Research Center of Intelligent Geodetection Technology Ministry of Education
China University of Geosciences, Wuhan, 430074, P.R. China

JIANLONG QIU
School of Automation and Electrical Engineering
and Key Laboratory of complex Systems and Intellignet Computing
Linyi 276005, Shandong, P.R. China

TONGSHUI XIA
Business School
Shandong Normal University
Jinan, 250014, P.R. China

(Communicated by Bin Li)

Abstract. In this paper, we employ an efficient numerical method to solve transport equations with given boundary and initial conditions. By the weighted-orthogonal Chebyshev polynomials, we design the corresponding basis functions for spatial variables, which guarantee the stiff matrix is sparse, for the spectral collocation methods. Combining with direct algebraic algorithms for the sparse discretized formula, we solve the equivalent scheme to get the numerical solutions with high accuracy. This collocation methods can be used to solve other kinds of models with limited computational costs, especially for the nonlinear partial differential equations. Some numerical results are listed to illustrate the high accuracy of this numerical method.

1. Introduction. In the earlier decades, there are lots of researches focusing on the numerical solutions of transport equations. As an important applications, the transport model was firstly used to simulate an underground oil fuel leak at an old power station. Now, there are a wide range of applications of this transport
equations, such as ground flow, imaging, remote sensing, contaminant transport and seawater flow around coastal lines, target detection and nuclear reactor theory and so on. And there is a sizeable catalog of numerical methods and techniques focusing on this topic, specially how to choose or design an efficient numerical approximations. Ishimaru has investigated wave propagation in turbulence and in a random continuum where the refractive index was a random function of space and time [8]. For the one-dimensional, time-independent radiative transport equation, Kim and Ishimaru presented Chebyshev spectral approximation to treat the spatial variable, which leads to a coupled system of integral equations for the modes of the Chebyshev spectrum [9]. To get more accurate numerical solutions, Buchan et al. described a new second generation spherical wavelet method for discretized the angular dimension of the Boltzmann transport equation [2], and its approximation scheme provided a high accurate expansion of the angular domain using Chebyshev collocation polynomials mapped into a wavelet space. And Merton mainly discussed the development of two optimal discontinuous finite element Riemann methods and their application to the one-speed Boltzmann transport equation in the steady-state [14]. Meanwhile, these methods are applied to solve a series of demanding two-dimensional radiation transport problems. In recent years, a lot of attention has been devoted to the study of finite element methods to investigate various scientific models by many researchers [4, 10, 5, 6, 1] and the references therein.

Considering the exponential convergence of spectral methods [3], there are many researchers employ spectral methods to calculate the numerical solutions, such as [11, 7, 12, 16, 17] and the references therein. Wang et al. and Mao et al. studied the Second- and Sixth-order systems and modified Possion equations with some collocation methods, and the high accuracy of numerical approximations were given with details in [18, 13, 19, 21, 20]. The authors investigated an improved a-posteriori error estimate for Galerkin spectral method in one dimension in [22]. And Galerkin spectral approximations for optimal control problems with fourth-order equation were discussed in [23]. Based on the generalized Jacobi polynomials, Zhou et al. studied error estimates of spectral element methods on an interval [24]. As the applications, Niu et al. mainly gave some important results on numerical analysis of an optimal control problem governed by the stationary Navier-Stokes equations [15]. Zhou et al. researched the error estimates of spectral methods for 1-dimension singularly perturbed problem [25].

In this paper, we investigate an Chebyshev-collocation spectral method, which is an efficient numerical method for solving the transport models. We give some numerical results to discuss the reliability and efficiency of this method to solve this kind of transport equations. We focus on this kind of transport equation.

\[
\begin{align*}
au_t - bu'' + cu &= f, & x \in (-1, 1), & t \in T > 0 \\
u \big|_{(\pm 1,t)} &= 0, \\
u(x,0) &= 0, & x \in (-1, 1)
\end{align*}
\]  

(1)

where \(a, b, c\) all are positive real value functions, the right hand side \(f\) is smooth on \((-1, 1) \times T\). Here for simplicity to analogize, we set \(a, b, c\) are positive constant and, for simplicity to state, we set them to be 1 through this paper. However it is not an essential point for numerical tests. This equation in its original form can be used to simulate the groundwater, a single solute species, and heat transport equation. The heat transport equation are usually converted by parameter transformation to simulate a conservative solute species, such as salt and other items. So how to
design some efficient and reliable numerical scheme for the numerical simulation is one of the key points within the engineering applications.

Compared with the previous work, the main contributions of this paper are summarized as follows: (1) By the weighted-orthogonal Chebyshev polynomials, we design the corresponding basis functions for spatial variables, which guarantee the stiff matrix is sparse, for the spectral collocation methods. (2) Combining with direct algebraic algorithms for the sparse discretized formula, we solve the equivalent scheme to get the numerical solutions with high accuracy. (3) Some numerical results are listed to illustrate the high accuracy of this numerical method.

The remainder of this paper is organized as follows. In Section 2, as a background, the fundamental preliminaries are given for the following parts, including the Chebyshev polynomials and the Chebyshev collocation points. The discretized formula of spectral collocation approximation is described in detail in section 3. In Section 4, some numerical examples are given to verify our conclusions. And we list the highlights in the conclusions.

2. Preliminaries and problem statement. In this section, we give some notations, which will be used in the following parts.

The Chebyshev polynomials \( T_k(x) \) are polynomials of degree \( k \) and the sequence of Chebyshev polynomials of either kind composes a polynomial sequence. We set

\[
T_k(x) = \cos(k \arccos x), \quad k = 0, 1, \ldots
\]

are the \( k \)-order Chebyshev polynomial on \((-1, 1)\).

It is easy to list the first two Chebyshev polynomials,

\[
T_0(x) = \cos(0 \arccos x) = 1,
\]

and

\[
T_1(x) = \cos(1 \arccos x) = x,
\]

One can straightforwardly determine other Chebyshev polynomials by the recurrence relation. Since the roots of this Chebyshev polynomials, which are also called Chebyshev nodes, are selected as nodes in polynomial interpolation approximations. Meanwhile, under the maximum norm, a continuous function can be approximated by this polynomial. So, in this paper we select the roots of the second kind Chebyshev polynomials as the collocation points. We choose the collocation points \( x_i \in (-1, 1) \) to formulate the numerical solution

\[
u_N = \sum_{k=0}^{N} a_k(t)T_k(x).
\]

Here, we take \( T_k(x) \) as the basis functions for spatial, and the test functions are

\[
\psi_j = \delta(x - x_j), \quad j = 1, 2, \ldots, N - 1.
\]

Usually, we select \( x_i = \cos \left( \frac{j \pi}{N} \right) \) as the collocation points.

**Remark 1:** For the collocation methods, one only needs to calculate the numerical solution \( u_N \) on the given collocation points. Hence, the test function can be proposed as the equation (4), which is valid for the solution of \( u_N \). To simplify the numerical integration, we select the collocation points as \( x_i = \cos \left( \frac{j \pi}{N} \right) \). Then we can calculate the numerical solutions on other arbitrary points with interpolations.
3. The discretized formula of spectral collocation approximation. In this section, we firstly give the discretized formula of the model with spectral collocation approximations. Secondly, we investigate the formula of the numerical solution with orthogonal Chebyshev polynomials. Note that the collocation spectral methods require the numerical solutions hold, with the integration, on the given collocation points, i.e.,

$$\int_{-1}^{1} (a(u_N)_t - bu_N'' + cu_N)\psi_j(x)dx = 0, \quad j = 1, 2, \cdots N - 1.$$  \hspace{1cm} (5)

Based on the above equality, we list the main result in following theorem.

**Theorem 3.1.** For the numerical solution $u_N(x, t)$ of (1), it is readily to get that

$$\frac{\partial u_N}{\partial t}|_{x_j} = \sum_{k=0}^{N} (\hat{a}_k(t) + a_k(t)) \cos\left(\frac{j k \pi}{N}\right).$$  \hspace{1cm} (6)

**Proof.** In fact, we only need to investigate the first part within (6). By (5), one directly knows that

$$T'_k(x) = \frac{k \sin(k \arccos x)}{\sin(\arccos x)}.$$

And following the recursive relation between the trigonometrical polynomials

$$\cos(k \arccos x) = \frac{1}{2} \left( \sin((k + 1) \arccos x) - \sin((k - 1) \arccos x) \right),$$

we have the relationship between $T_k(x)$ and $T'_k(x)$, which reads

$$T_k(x) = \frac{1}{2} \left( \frac{1}{k + 1} T'_{k+1}(x) - \frac{1}{k - 1} T'_{k-1}(x) \right), \quad k \geq 2$$

Similarly, we employ $T_k(x)$ to expand $\frac{\partial u_N(x, t)}{\partial x}$, there holds

$$\frac{\partial u_N(x, t)}{\partial x} = \sum_{k=0}^{N-1} \tilde{a}_k T_k(x)$$

$$= \sum_{k=1}^{N-1} \tilde{a}_k - \tilde{a}_{k+1} T'_k(x) - \frac{\tilde{a}_2}{2} T'_1(x)$$

$$+ \frac{\tilde{a}_{N-2}}{2(N - 2)} T'_{N-1}(x) + \frac{\tilde{a}_{N-1}}{2(N - 2)} T'_N(x) + \tilde{a}_0 T'_1(x)$$

$$= \sum_{k=1}^{N} a_k T'_k(x),$$

which means for $k = 0, 1, \cdots, N - 1$, there hold

$$\tilde{a}_{N+1} = 0, \quad \tilde{a}_N = 0,$$

and

$$\bar{c}_k \tilde{a}_k = \tilde{a}_{k+2} + 2(k + 1) a_{k+1},$$

where

$$\bar{c}_0 = 2, \quad \bar{c}_k = 1, \quad (1 \leq k \leq N - 1),$$
By the same techniques, we expand \( \frac{\partial^2 u_N(x,t)}{\partial x^2} \) with the second kind of Chebyshev polynomials, we have for \( k = 0, 1, \ldots, N-1 \), there hold
\[
\hat{a}_{N+1} = 0, \quad \hat{a}_N = 0,
\]
and
\[
c_k a_k = \hat{a}_{k+2} + 2(k+1) \hat{a}_{k+1},
\]
which states the relationship between \( \hat{a}_k \) and \( a_k \). Combining the expansion of in (3) and the above results, we declare that the equation (6) holds. Therefore the theorem is proved. \( \square \)

**Remark 2:** For the numerical solution \( u_N(x,t) \), we give the expression of \( \frac{\partial u_N}{\partial t} \big|_{x_j} \), and we also ensure \( \frac{\partial^2 u_N(x,t)}{\partial x^2} \) with the second kind of Chebyshev polynomials, then the relationship between \( \hat{a}_k \) and \( a_k \) is given, then Theorem 3.1 has shown the core result on the solution of the Transport Equation.

4. **Numerical examples.** In this section, we list two numerical examples to verify the high accurate numerical solutions of this transport equations. We choose the finite difference methods (FDM) and our Chebyshev collocation spectral methods (CCSM) to depict the high accuracy of the CCSM.

**Example 4.1** Select the transport equation (1) with the exact solution
\[
u(x,t) = (1-x^2)(\sin t), \quad x \in (-1,1), \quad t \in T = (0,5)
\]
and
\[
f = (1-x^2)(\sin t + \cos t) + 2(\sin t).
\]
Select \( t = 0.5 \), the maximum absolute errors are given in the following table.

| N  | CCSM          | FDM          |
|----|---------------|--------------|
| 8  | 2.58952e-4    | 7.92233e-1  |
| 10 | 3.51652e-6    | 5.35228e-1  |
| 12 | 2.93379e-7    | 3.71949e-2  |
| 14 | 4.67534e-9    | 2.68015e-2  |
| 16 | 2.5433e-2     | 9.58506e-2  |

In this Table 1, we find that, with the high regularity of the exact solution, the CCSM arrives the machine precision quickly, but not the FDM. Meanwhile, the convergence is profiled in the following Figure 1.

**Example 4.2** We choose the exponential function to state the spatial part and cosine-function for the time, i.e.,
\[
u(x,t) = (\exp x)(\cos t), \quad x \in (-1,1), \quad t \in T = (0,10),
\]
and
\[
f = -(\exp x)(\sin t).
\]
We fix \( t = 1 \), the maximum absolute errors are given in the following Table 2 and Figure 2.

In the Table 2, for the exponential function can be accurately approximated by the polynomials, we know the numerical errors of CCSM rapidly decay into machine precision. However, the FDM just stays at a lower-order accuracy.
Figure 1. The maximum errors of $u - u_N$ with log10 at $t = 0.5$

Table 2. The $L^\infty$-error of numerical solutions at $t = 1$

| N  | CCSM         | PDM          |
|----|--------------|--------------|
| 8  | 2.99237e-4  | 8.00453e-1  |
| 10 | 7.33715e-7  | 5.56804e-1  |
| 12 | 1.66371e-9  | 4.01949e-2  |
| 14 | 9.97109e-12 | 3.08050e-2  |
| 16 | 5.74238e-14 | 1.00513e-2  |

5. Conclusions. This paper has studied the Chebyshev collocation spectral method for the transport equations. We have investigated the discretized formula of the numerical solution with the collocation approximations, which was our one important highlight. And some numerical results, compared with the results of finite difference methods, have been given to clarify the efficiency of our Chebyshev collocation spectral methods. For future work, how to give a more effective algorithm and apply it to optimal control or dynamic game of complex system is one of the key points.

Acknowledgments. The authors would like to thank the anonymous referees for their valuable comments and suggestions that improved the paper. This work was supported by the 111 project under Grant B17040, and in part supported by the Project of Shandong Province Higher Educational Science and Technology Program (Nos. J12LJ67 and J18KA354), and by the Project of National Natural Science Foundation of China under (Grant Nos. 11571157, 61877033 and 11805091), and the National Natural Science Foundation of Shandong Province under Grant Nos. ZR2019YQ28, ZR2019BF045, ZR2019QF004 and ZR2019YQ05. This work was also
Figure 2. The maximum errors of $u - u_N$ with log10 at $t = 1$

supported in part by the Development Plan of Youth Innovation Team of University in Shandong Province under Grants No. 2019KJN007.

REFERENCES

[1] B. Bialecki, Sinc-collection methods for two-point boundary value problems, *IMA Journal of Numerical Analysis*, 11 (1991), 357–375.

[2] A. G. Buchan, C. C. Pain, M. D. Eaton, R. P. Smedley-Stevenson and A. J. H. Goddard, Chebyshev spectral hexahedral wavelets on the sphere for angular discretisations of the boltzmann transport equation, *Annals of Nuclear Energy*, 35 (2008), 1098–1108.

[3] C. Canuto, M. Y. Hussaini, A. Quarteroni and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York, 1988.

[4] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Classics in Applied Mathematics, 40. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.

[5] J. D. Dockery, Numerical solution of travelling waves for reaction-diffusion equations via the sinc-galerkin method, In Bowers K., Lund J. (eds) *Computation and Control II. Progress in Systems and Control Theory*, 11 (1991), 95–113.

[6] M. El-Gamel, A comparison between the Sinc-Galerkin and the modified decomposition methods for solving two-point boundary-value problems, *Journal of Computational Physics*, 223 (2007), 369–383.

[7] P. Heidelberger and P. D. Welch, A spectral method for confidence interval generation and run length control in simulations, *Communications of the ACM*, 24 (1981), 233–245.

[8] A. Ishimaru, Wave propagation and scattering in random media and rough surfaces, *Proceedings of the IEEE*, 79 (1991), 1359–1366.

[9] A. D. Kim and A. Ishimaru, A chebyshev spectral method for radiative transfer equations applied to electromagnetic wave propagation and scattering in a discrete random medium, *J. Comput. Phys.*, 152 (1999), 264–280.

[10] V. B. Kisselev, L. Roberti and G. Perona, An application of the finite element method to the solution of the radiative transfer equation, *Journal of Quantitative Spectroscopy and Radiative Transfer*, 51 (1994), 603–614.

[11] A. Lundbladh, D. S. Henningson and A. V. Johansson, An Efficient Spectral Integration Method for the Solution of the Navier-Stokes Equations, Aeronautical Research Institute of Sweden Bromma, 1992.
[12] X. J. Li and C. J. Xu, A space-time spectral method for the time fractional diffusion equation, *SIAM Journal on Numerical Analysis*, 47 (2009), 2108–2131.

[13] A. M. Mao, L. J. Yang, A. X. Qian and S. X. Luan, Existence and concentration of solutions of schrödinger-poisson system, *Applied Mathematics Letters*, 68 (2017), 8–12.

[14] S. R. Merton, C. C. Pain, R. P. Smedley-Stevenson, A. G. Buchan and M. D. Eaton, Optimal discontinuous finite element methods for the boltzmann transport equation with arbitrary discretisation in angle, *Annals of Nuclear Energy*, 35 (2008), 1741–1759.

[15] H. F. Niu, D. P. Yang and J. W. Zhou, Numerical analysis of an optimal control problem governed by the stationary navier-stokes equations with global velocity-constrained, *Communications in Computational Physics*, 24 (2018), 1477–1502.

[16] B. Wang, A. Iserles and X. Y. Wu, Arbitrary-order trigonometric fourier collocation methods for multi-frequency oscillatory systems, *Foundations of Computational Mathematics*, 16 (2016), 151–181.

[17] B. Wang, F. W. Meng and Y. L. Fang, Efficient implementation of rkn-type fourier collocation methods for second-order differential equations, *Applied Numerical Mathematics*, 119 (2017), 164–178.

[18] B. Wang, X. Y. Wu and F. W. Meng, Trigonometric collocation methods based on lagrange basis polynomials for multi-frequency oscillatory second-order differential equations, *Journal of Computational and Applied Mathematics*, 313 (2017), 185–201.

[19] B. Wang, H. L. Yang and F. W. Meng, Sixth-order symplectic and symmetric explicit erkn schemes for solving multi-frequency oscillatory nonlinear hamiltonian equations, *Calcolo*, 54 (2017), 117–140.

[20] B. Wang, Triangular splitting implementation of rkn-type fourier collocation methods for second-order differential equations, *Mathematical Methods in the Applied Sciences*, 41 (2018), 1998–2011.

[21] X. Y. Wu and B. Wang, Exponential fourier collocation methods for solving first-order differential equations, *In Recent Developments in Structure-Preserving Algorithms for Oscillatory Differential Equations*, Springer, Singapore, (2018), 55–84.

[22] J. W. Zhou and D. P. Yang, An improved a posteriori error estimate for the galerkin spectral method in one dimension, *Computers & Mathematics with Applications*, 61 (2011), 334–340.

[23] J. W. Zhou, J. Zhang and X. Q. Xing, Galerkin spectral approximations for optimal control problems governed by the fourth order equation with an integral constraint on state, *Computers & Mathematics with Applications*, 72 (2016), 2549–2561.

[24] J. W. Zhou, J. Zhang, H. T. Xie and Y. Yang, Error estimates of spectral element methods with generalized jacobi polynomials on an interval, *Applied Mathematics Letters*, 74 (2017), 199–206.

[25] J. W. Zhou, Z. W. Jiang, H. T. Xie and H. F. Niu, The error estimates of spectral methods for 1-dimension singularly perturbed problem, *Applied Mathematics Letters*, 100 (2020), 106001, 8 pp.

Received July 2019; revised February 2020.