Deformation of Curved BPS Domain Walls and Supersymmetric Flows on 2d Kähler-Ricci Soliton

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ABSTRACT

We consider some aspects of the curved BPS domain walls and their supersymmetric Lorentz invariant vacua of the four dimensional $N = 1$ supergravity coupled to a chiral multiplet. In particular, the scalar manifold can be viewed as a two dimensional Kähler-Ricci soliton generating a one-parameter family of Kähler manifolds evolved with respect to a real parameter, $\tau$. This implies that all quantities describing the walls and their vacua indeed evolve with respect to $\tau$. Then, the analysis on the eigenvalues of the first order expansion of BPS equations shows that in general the vacua related to the field theory on a curved background do not always exist. In order to verify their existence in the ultraviolet or infrared regions one has to perform the renormalization group analysis. Finally, we discuss in detail a simple model with a linear superpotential and the Kähler-Ricci soliton considered as the Rosenau solution.

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1 Introduction

Attempts to generalize the study of AdS/CFT correspondence [1] on curved spacetimes have been done, for example in the context of curved domain walls of five dimensional $N = 2$ supergravity [2, 3]. In those papers the authors have constructed curved BPS domain walls and discussed their dual description in terms of the renormalization group (RG) flow described by a beta function. By putting the supersymmetric field theory studied in [4] on a curved four dimensional AdS background they have also demonstrated in a simple model that this holographic RG flow in the field theory on the curved spacetime has indeed a description in terms of curved BPS domain walls.

So far, in the four dimensional $N = 1$ supergravity theory we only have the flat domain wall cases studied in some references, for example [5, 6, 7, 8, 9]. Therefore, the above results motivate us to apply the scenario to the case of the curved BPS domain walls of $N = 1$ supergravity in four dimensions. Our interest here is to study curved BPS domain walls together with their Lorentz invariant vacua in the context of the dynamical system and the RG flow analysis which is a generalization of the previous works [8, 9]. Particularly, we want to see how the general pattern looks like in a simple model, namely the $N = 1$ supergravity coupled to a chiral multiplet whose scalar manifold can be regarded as a solution of the Kähler-Ricci flow equation [10, 11]. This geometric soliton generates a one-parameter family of scalar manifolds, i.e. Kähler manifolds, whose the deformation parameter is $\tau \in \mathbb{R}$. In particular, this Kähler-Ricci soliton can be viewed as a volume deformation of a Kähler geometry for finite $\tau$.

Thus, defining $N = 1$ supersymmetry on the Kähler-Ricci flow means that we deform it with respect to $\tau$. As a direct implication of such treatment, all couplings such as the shifting quantities, the masses of the fields, and the scalar potential do evolve with respect to $\tau$ since those quantities depend on this geometric soliton. Such behavior is generally inherited to all solitonic solutions such as the domain walls. So, the Lorentz invariant vacua do also possess such property which can shortly be mentioned as follows.

First, near the vacua the spacetime is in general non-Einsteinian and then, becomes a space of constant curvature (which is also non-Einsteinian) related to the divergences of the RG flow. Second, the Kähler-Ricci soliton indeed affects the nature of the vacua, mapping nondegenerate vacua to other degenerate vacua and vice versa. Moreover, in a model that admits a singular geometric evolution, the vacuum structure may have a parity pair of vacua in the sense that the vacua of the index $\lambda$ turns into the other vacua of the index $2 - \lambda$ after hitting the singularity. This is an example that also occurs in general the pattern for flat domain walls [8].

Finally, in order to have a consistent picture, the eigenvalues of the first order expansion of the BPS equation have to be real. This also shows that the above vacua do not always exist in general. By performing the RG flow analysis we can further verify the existence of such vacua in the infrared or ultraviolet regions, correspond to the field theory on three dimensional curved background.

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2This Kähler-Ricci flow also appears as one loop approximation of the beta function of $N = 2$ supersymmetry in two and three dimensions, see for example [12, 13]. In this case, the parameter $\tau$ is regarded as the energy scale of the theory. However, this is not the case in higher dimension, particularly in four dimensions.

3This can be easily seen if the initial geometry is a Kähler-Einstein manifold. See Appendix C for details.
The structure of this paper is as follows. In Section 2 we review the $N = 1$ supergravity on a two dimensional Kähler-Ricci soliton and introduce some quantities which are useful for our analysis. Then, the discussion is continued in Section 3 by addressing some aspects of the curved BPS domain walls on the two dimensional Kähler-Ricci solitons. Section 4 is assigned to the discussion of the nature of the supersymmetric Lorentz invariant vacua together with their deformation on the Kähler-Ricci soliton. We put the discussion about the deformation of the supersymmetric flow on curved spacetime in Section 5. A simple example is then given in Section 6. Finally, we summarize our results in Section 7.

2 4d $N = 1$ Chiral Supergravity on 2d Kähler-Ricci Soliton

In this section we provide a review of the four dimensional $N = 1$ supergravity coupled to a chiral multiplet in which the non-linear $\sigma$-model satisfies the Kähler-Ricci flow equation defined below, which in turn implies that our $N = 1$ supergravity is defined on a one-parameter family of Kähler manifolds deformed with respect to the real parameter $\tau$ [8].

The ingredients of the $N = 1$ theory are a gravitational multiplet and a chiral multiplet. The gravitational multiplet is composed of a vierbein $e^a_\nu$ and a vector spinor $\psi_\nu$ where $a = 0, \ldots, 3$ and $\nu = 0, \ldots, 3$ are the flat and the curved indices, respectively. The member of the chiral multiplet is a complex scalar $z$ and a spin-$\frac{1}{2}$ fermion $\chi$.

We then construct a general $N = 1$ chiral supergravity Lagrangian together with its supersymmetry transformation. This construction can be found, for example, in [14]. Let us assemble the terms which are useful for our analysis. The bosonic part of the $N = 1$ supergravity Lagrangian has the form

$$\mathcal{L}^{N=1} = -\frac{M_P^2}{2} R + g_{z\bar{z}}(z, \bar{z}; \tau) \partial_\nu z \partial^\nu \bar{z} - V(z, \bar{z}; \tau),$$

(2.1)

where $M_P$ is the Planck mass and by setting $M_P \rightarrow +\infty$, the $N = 1$ global supersymmetric theory can be obtained. Next, $R$ is the Ricci scalar of the four dimensional spacetime; the pair $(z, \bar{z})$ spans a Hodge-Kähler manifold with metric $g_{z\bar{z}}(z, \bar{z}; \tau) \equiv \partial_z \partial_{\bar{z}} K(z, \bar{z}; \tau)$; and $K(z, \bar{z}; \tau)$ is a real function, called the Kähler potential. The scalar manifold satisfies the Kähler-Ricci flow equation

$$\frac{\partial g_{z\bar{z}}}{\partial \tau} = -2R g_{z\bar{z}}(\tau) = -2 \partial_z \partial_{\bar{z}} \ln g_{z\bar{z}}(\tau),$$

(2.2)

where $\tau$ is a real parameter related to the deformation of a Kähler surface mentioned in the previous section. The $N = 1$ scalar potential $V(z, \bar{z}; \tau)$ has the form

$$V(z, \bar{z}; \tau) = e^{K(\tau)/M_P^2} \left( g_{z\bar{z}}(\tau) \nabla_z W \nabla_{\bar{z}} \bar{W} - \frac{3}{M_P^2} W \bar{W} \right),$$

(2.3)

where $W$ is a holomorphic superpotential and $\nabla_z W \equiv (dW/dz) + (K_z(\tau)/M_P^2)W$. The Lagrangian (2.1) is invariant under the following supersymmetry transformations up to

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$^4$For an excellent review of $N = 1$ supergravity, see also for example [15, 16]

$^5$Classification of the special solutions of (2.2) using linearization method has been studied, for example, in [17].
three-fermion terms \(6\)

\[
\begin{align*}
\delta\psi_{1\nu} &= M_P \left( D_\nu \epsilon_1 + \frac{i}{2} e^{K(\tau)/2M_P^2} W^{-1} \gamma^\nu \epsilon_1 \right), \\
\delta\chi^z &= i \partial_\nu z \gamma^\nu \epsilon_1 + N^z(\tau) \epsilon_1, \\
\delta e^a_\nu &= -\frac{i}{M_P} (\bar{\psi}_1 \gamma^a \epsilon_1 + \bar{\psi}_1 \gamma^a \epsilon_1), \\
\delta z &= \bar{\chi}^z \epsilon_1,
\end{align*}
\]

where \(N^z(\tau) \equiv e^{K(\tau)/2M_P^2} g^{zz}(\tau) \bar{W}, \ g^{zz}(\tau) = (g_{zz}(\tau))^{-1}\), and the \(U(1)\) connection \(Q_\nu(\tau) \equiv -(K_z(\tau) \partial_\nu z - K_z(\tau) \partial_\nu \bar{z})\). Here, we have also defined \(\epsilon_1 \equiv \epsilon_1(x, \tau)\). The flow equation (2.2) implies that, for example, the scalar potential (2.3) and the shifting quantity \(N^z\) do evolve with respect to \(\tau\) as [8]

\[
\begin{align*}
\frac{\partial N^z(\tau)}{\partial \tau} &= 2R_z(\tau) N^z(\tau) + \frac{K_\tau(\tau)}{2M_P} N^z(\tau) + g^{zz}(\tau) \frac{K_{\bar{z}\tau}(\tau)}{M_P} e^{K(\tau)/2M_P^2} W, \\
\frac{\partial V(\tau)}{\partial \tau} &= \frac{\partial N^z(\tau)}{\partial \tau} N_z(\tau) + \frac{\partial N_z(\tau)}{\partial \tau} N^z(\tau) - \frac{3K_\tau(\tau)}{M_P} e^{K(\tau)/M_P^2} |W|^2,
\end{align*}
\]

where \(R_z \equiv g^{zz} R_{zz}\). Note that as has been studied in [18, 19] it is possible that such geometric flow (2.2) has a singular point at finite \(\tau\). For example, this situation can be directly observed in a model with Kähler-Einstein manifold as initial geometry \(^\text{7}\). In this paper we particularly consider the Rosenau solution of (2.2) which also admits such property in section 6. This special solution was firstly constructed in [20].

### 3 Curved BPS Domain Walls on 2d Kähler-Ricci Soliton

This section is assigned for the discussion of curved domain walls admitting partial Lorentz invariance. In particular, we consider curved BPS domain walls that maintain half of the supersymmetry of the parental theory. As a consequence, the background should be a three dimensional AdS spacetime.

Let us now consider the ground states which partially break the Lorentz invariance, i.e. the domain walls. The starting point is to take the ansatz metric of the four dimensional spacetime as

\[
ds^2 = a^2(u, \tau) g_{\Delta \Delta} \, dx^\Delta \, dx^\Delta - du^2,
\]

where \(\Delta, \nu = 0, 1, 2\), \(a(u, \tau)\) is the warped factor, and the parameter \(\tau\) is related to the dynamics of the Kähler metric governed by (2.2). The metric \(g_{\Delta \Delta}\) describes a three dimensional AdS spacetime. Therefore, the corresponding components of the Ricci tensor of the metric (3.1) are given by

\[
R_{\Delta \Delta} = \left[ \left( \frac{a'}{a} \right)' + 3 \left( \frac{a'}{a} \right)^2 - \frac{\Lambda_3}{a^2} \right] a^2 g_{\Delta \Delta},
\]

\[
R_{33} = -3 \left[ \left( \frac{a'}{a} \right)' + \left( \frac{a'}{a} \right)^2 \right], \tag{3.2}
\]

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\(^6\)The symbol \(D_\nu\) here is different with the one in reference [9]. \(D_\nu\) here is defined as \(D_\nu \equiv \partial_\nu - \frac{1}{4} \gamma_{ab} \omega^a_\nu\).

\(^7\)See also Appendix C.
and the Ricci scalar has the form

$$R = 6 \left[ \left( \frac{a'}{a} \right)' + 2 \left( \frac{a'}{a} \right)^2 \right] - \frac{3\Lambda_3}{a^2}, \quad (3.3)$$

where $a' \equiv \partial a/\partial u$ and $\Lambda_3$ is the negative three dimensional cosmological constant. Writing the supersymmetry transformation (2.4) and setting $\psi_{1\nu} = \chi^z = 0$ on the background (3.1), leads to

$$\frac{1}{M_P} \delta \psi_{1u} = D_u \epsilon_1 + \frac{i}{2} e^{K(\tau)/2M_P^2} W \gamma^a \epsilon_1 + \frac{i}{2M_P} Q_u(\tau) \epsilon_1, \quad (3.4)$$

$$\frac{1}{M_P} \delta \psi_{1\nu} = D_\nu \epsilon_1 + \frac{1}{2} \gamma_\nu \left( - \frac{a'}{a} \gamma_3 \epsilon_1 + i e^{K(\tau)/2M_P^2} W \epsilon_1 \right) + \frac{i}{2M_P} Q_\nu(\tau) \epsilon_1,$n

$$\delta \chi^z = i \partial_\nu \gamma^\nu \epsilon_1 + N^z(\tau) \epsilon_1. \quad (3.4)$$

Supersymmetry further demands that the right hand side of equations (3.4) vanish on the ground states. Then, on the three dimensional AdS spacetime there exists a Killing spinor [21]

$$D_\nu \epsilon_1 + \frac{i}{2} \ell \hat{\gamma}_\nu \epsilon_1 = 0, \quad (3.5)$$

where $\ell \equiv \sqrt{-\Lambda_3/2}$. Here, $\hat{\gamma}_\nu$ means the gamma matrices in the three dimensional AdS spacetime, and therefore $\gamma_\nu = a \hat{\gamma}_\nu$. Thus by taking $z = z(u, \tau)$ the first equation in (3.4) shows that $\epsilon_1$ depends on the spacetime coordinates, while the second equation gives a projection equation

$$\frac{a'}{a} \gamma_3 \epsilon_1 = i \left( e^{K(\tau)/2M_P^2} W(z) - \frac{\ell}{a} \right) \epsilon_1, \quad (3.6)$$

which leads to

$$\frac{a'}{a} = \pm \left| e^{K(\tau)/2M_P^2} W(z) - \frac{\ell}{a} \right|. \quad (3.7)$$

So, the warped factor $a$ is indeed $\tau$ dependent which is consistent with our ansatz (3.1). Next, since $z = z(u, \tau)$ the third equation in (3.4) becomes simply

$$z' = \mp 2 e^{i\theta(\tau)} g^{zz}(\tau) \bar{\partial}_z W(\tau), \quad \bar{z}' = \mp 2 e^{-i\theta(\tau)} g^{\bar{z}\bar{z}}(\tau) \partial_{\bar{z}} W(\tau), \quad (3.8)$$

where we have introduced the phase function $\theta(z, \bar{z}; u, \tau)$ via

$$e^{i\theta(\tau)} = \frac{1 - \ell e^{-K(\tau)/2M_P^2} (aW)^{-1}}{1 - \ell e^{-K(\tau)/2M_P^2} (aW)^{-1}}, \quad (3.9)$$

and the real function

$$W(z, \bar{z}; \tau) \equiv e^{K(\tau)/2M_P^2} |W(z)|. \quad (3.10)$$

Note that at $\theta = 0$ the flat domain wall case is regained, which corresponds to $\ell = 0$. Thus, we have the gradient flow equations (3.8), called the BPS equations in a curved

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8This warped factor $a$ is related to the c-function in the holographic correspondence [4, 7].
spacetime. Another supersymmetric flow related to the analysis is the renormalization group (RG) flow given by the beta functions

$$\beta(\tau) \equiv a \frac{\partial z}{\partial a} = -\frac{2e^{i\theta(\tau)}}{|e^{K(\tau)/2M_p^2} W(z) - \ell/a|} g^{\bar{z}\bar{z}}(\tau) \partial_{\bar{z}} W(\tau) ,$$

$$\bar{\beta}(\tau) \equiv a \frac{\partial \bar{z}}{\partial a} = -\frac{2e^{-i\theta(\tau)}}{|e^{K(\tau)/2M_p^2} W(z) - \ell/a|} g^{z\bar{z}}(\tau) \partial_z W(\tau) ,$$

(3.11)

after using (3.7) and (3.8). These functions give a description of a conformal field theory (CFT) on the three dimensional AdS spacetime. Therefore, the scalars behave as coupling constants and the warped factor $a$ can be viewed as an energy scale [4, 7, 22], and the scalar potential (2.3) can be written down as

$$V(z, \bar{z}; \tau) = 4 g^{\bar{z}\bar{z}}(\tau) \partial_{\bar{z}} W(\tau) \partial_{\bar{z}} W(\tau) - \frac{3}{M_p^2} W^2(\tau) .$$

(3.12)

To see the deformation of the domain walls clearly, we have to consider a case where, the Kähler-Ricci flow has a singularity at finite $\tau = \tau_0 < \infty$. Such a property may cause a topological change of the scalar manifold. The simplest example of this case is when the initial manifold is Kähler-Einstein manifold which has been studied in [8]. Here, our interest is only to look at another non trivial solution of (2.2) where

$$g_{\bar{z}\bar{z}}(\tau) = \begin{cases} g_1(z, \bar{z}; \tau) ; & \tau < \tau_0 , \\ 0 ; & \tau = \tau_0 , \\ -g_2(z, \bar{z}; \tau) ; & \tau > \tau_0 , \end{cases}$$

(3.13)

with $g_1(z, \bar{z}; \tau)$ and $g_2(z, \bar{z}; \tau)$ are both positive definite functions. For the case at hand the $N = 1$ theory (2.1) and the walls described by (3.8) and (3.11) diverge at $\tau = \tau_0$. This singularity disconnects two different theories, namely for $\tau < \tau_0$ we have $N = 1$ theory on positive definite metric, while for $\tau > \tau_0$ it becomes $N = 1$ theory on negative definite metric. As we will see in section 6 the Rosenau solution also has similar property and produces a singularity at $\tau_0 = 0$.

Concluding this section, we will look at the gradient flow equation (3.8). The critical points of the equation (3.8) satisfy the following conditions

$$\partial_z W = \partial_{\bar{z}} W = 0 ,$$

(3.14)

which imply that the first derivative of the scalar potential (3.12) vanishes. In other words, there is a correspondence between the critical points of $W(\tau)$ and the vacua of the $N = 1$ scalar potential $V(\tau)$ 9. Moreover, the Kähler-Ricci flow, specifically the metric (3.13), causes an evolution of these critical points (vacua) which are characterized by the beta functions (3.11) splitting into two region. If $a \to +\infty$ we have an ultraviolet (UV) region, while if $a \to 0$ we have an infrared (IR) region. Several aspects of the vacuum and the supersymmetric flow deformation will be discussed in detail in section 4 and section 5 respectively.

9The vacuum of the scalar potential (2.3) means a Lorentz invariant vacuum (ground state). In the following we will mention Lorentz invariant vacuum just as vacuum or ground state.
4 Properties of the Supersymmetric Vacua

Our attention in this section will be mainly drawn to the discussion of the supersymmetric vacua of the theory described by the scalar potential (3.12). As mentioned in the previous section these vacua are related to the critical points of the real function $W(\tau)$ and changing with respect to $\tau$. We firstly show that in general such critical points are correspond to a four dimensional non-Einsteinian spacetime. Then, a second order analysis of the vacua related to the critical points of $W(\tau)$ will be carried out. Note that the discussion here is incomplete since it does not involve a supersymmetric flow analysis which is provided in section 5. A short review about critical points of surfaces is given in Appendix B which maybe useful for the analysis.

Let us begin our discussion by mentioning that from (3.14) a critical point of the real function $W(\tau)$, say $p_0$, is in general $p_0 = (z_0(\tau), \bar{z}_0(\tau))$ due to the geometric flow (2.2). Such a point exists in the asymptotic regions, namely around $u \to \pm \infty$. The form of the scalar potential (3.12) at $p_0$ is

$$V(p_0; \tau) = -\frac{3}{M_P^2} W^2(p_0; \tau) \equiv -\frac{3}{M_P^2} W_0^2.$$  (4.1)

For the case of the flat domain walls discussed in [9] the equation (4.1) can be viewed as the cosmological constant of the spacetime at the vacuum. However, in general this is not the case. Therefore, we have to consider the behavior of the warped factor $a$ near the vacua, which is related to the shape of the spacetime. Around $p_0$, the solution of equation (3.7) tends to

$$a(u, \tau) = \frac{l}{W_0} \pm \left( \frac{\ell^2}{W_0^2} - \frac{l^2}{W_0^2} \right)^{1/2} \left[ A_0 e^{\pm W_0 u} - A_0^{-1} e^{\mp W_0 u} \right],$$  (4.2)

where $l \equiv \ell e^{K(p_0; \tau)/2M_P^2} \text{Re}W(z_0)$ and $A_0 \neq 0$. Since $a$ is real, then $W_0 > |l|/\ell$. Moreover, in this case we have $(a'/a)' \neq 0$ near $p_0$. So, defining

$$a'/a = \pm \left| e^{K(p_0; \tau)/2M_P^2} W(z_0) - \frac{\ell}{a} \right| \equiv \pm k,$$  (4.3)

with $k \equiv k(u, \tau) \geq 0$, the Ricci tensor (3.2) and the Ricci scalar (3.3) become

$$R_{\lambda\nu} = \left( \pm k' + 3k^2 + 2\ell^2 e^{\mp 2\int k du} \right) e^{\pm 2\int k du} g_{\lambda\nu},$$

$$R_{33} = -3 \left( \pm k' + k^2 \right),$$

$$R = \pm 6k' + 12k^2 + 6\ell^2 e^{\mp 2\int k du},$$  (4.4)

respectively, which confirms that in general the spacetime is non-Einsteinian. Let us consider some special cases as follows. For $\ell = 0$ case, we have a four dimensional AdS spacetime for $k \neq 0$ with $k' = 0$ and the cosmological constant given by (4.1), appeared in

\[\text{In the limit of flat walls we have } \ell \to 0 \text{ and } A_0 \to \pm \infty.\]

\[\text{It is important to notice that the analysis using supersymmetric flows, namely the gradient and the RG flows, shows that such a spacetime does not always correspond to a CFT in three dimensional AdS, see the discussion in section 5.}\]
the flat domain walls. Next, it is possible to have a case where $k = 0$. Here, the spacetime is four dimensional non-Einsteinian space of constant curvature, where

$$e^{K(p_0; \tau)/2M_P^2} W(z_0) = \frac{\ell}{a},$$

or in other words,

$$\text{Im}W(z_0) = 0.$$  \hspace{1cm} (4.6)

These facts tell us that the first order expansion of the beta function (3.11) would be ill defined 12. Hence, this vacuum does not correspond to the CFT on a three dimensional AdS spacetime.

Another singularity could occur at $\tau = \tau_0$ which is caused by the divergence of the geometric flow (3.13). The detail of this aspect depends on the model in which both the form of the geometric flow and the superpotential are involved. Also, in this case some quantities would diverge. We give a simple model in section 6.

The last case is a static case in which $W(z_0) \neq 0$ and the $U(1)$ connection vanishes 13. This means that $p_0$ does not depend on $\tau$, but rather is determined by the holomorphic superpotential $W(z)$. In other words, $p_0$ is a critical point of $W(z)$. However, as we will see in the following, although $p_0$ static, the second order analysis does depend on $\tau$ because the geometric flow described by the Kähler potential $K(z, \bar{z}; \tau)$ is involved in the analysis. An example of this situation is discussed in section 6.

The second part of the discussion is to study the properties of the critical points of the real function $W(\tau)$ which is $\tau$ dependent in the second order analysis. Here, the eigenvalues of its Hessian matrix are also $\tau$ dependent and have the form 14

$$\lambda^W_{1,2}(\tau) = \frac{g_{zz}(p_0; \tau)}{M_P^2} W_0 \pm 2|\partial^2_z W_0|,$$  \hspace{1cm} (4.7)

where

$$\partial^2_z W_0 \equiv \frac{e^{K(p_0; \tau)/M_P^2} W(z_0)}{2W(p_0; \tau)} \left( \frac{d^2 W}{d z^2}(z_0) + \frac{K_{zz}(p_0; \tau)}{M_P^2} W(z_0) + \frac{K_z(p_0; \tau)}{M_P^2} dW/dz(z_0) \right).$$  \hspace{1cm} (4.8)

Since the metric $g_{zz}(\tau)$ satisfies (3.13), we split the discussion into two parts. First, in the interval $\tau < \tau_0$ the metric is positive definite, namely $g_{zz}(p_0; \tau) = g_1(p_0; \tau)$, and the possible cases for $p_0$ are a local minimum if

$$|\partial^2_z W_0| < \frac{1}{2M_P^2} g_1(p_0; \tau) W_0,$$ \hspace{1cm} (4.9)

or a saddle if

$$|\partial^2_z W_0| > \frac{1}{2M_P^2} g_1(p_0; \tau) W_0.$$ \hspace{1cm} (4.10)

Furthermore, $p_0$ turns out to be degenerate when

$$|\partial^2_z W_0| = \frac{1}{2M_P^2} g_1(p_0; \tau) W_0.$$ \hspace{1cm} (4.11)

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12See equations (5.9) and (5.10) in the next section for a detail.

13Note that in the $W(z_0) = 0$ case the warped factor $a(u, \tau)$ in (4.2) becomes singular.

14Note that since we have curved BPS domain walls, this eigenvalue has a restriction coming from the eigenvalues of the first order expansion of the gradient flow (3.8). Again, see section 5 for a detail.
holds. Second, for $\tau > \tau_0$ we have a negative definite metric and $g_{zz}(p_0; \tau) = -g_2(p_0; \tau)$. In this region the conditions (4.9) and (4.10) are modified by replacing $g_1(p_0; \tau)$ with $g_2(p_0; \tau)$ which further state that $p_0$ is either a local maximum for

$$|\partial_2^2 W_0| < \frac{1}{2M_P^2} g_2(p_0; \tau) W_0,$$

or a saddle for

$$|\partial_2^2 W_0| > \frac{1}{2M_P^2} g_2(p_0; \tau) W_0.$$

For degenerate case, we use the same procedure on (4.11). Since the point $p_0$ is dynamic with respect to $\tau$, we can then summarize the above results as follows. A non degenerate critical point can be changed into a degenerate critical point and vice versa by the special geometric flow (3.13). Moreover, this flow also affects the index of the critical points, namely the critical points of index $\lambda$ turn to another critical points of index $2 - \lambda$ after passing the singularity at $\tau = \tau_0$. In other words, this is a parity transformation of the Hessian matrix of $W$ that maps a critical point in $\tau < \tau_0$ to its parity partner in $\tau > \tau_0$, such as a local minimum to a local maximum and vice versa [8].

In the following we finally perform general analysis on the supersymmetric vacua of the scalar potential (3.12) and their relation to the critical points of $W(\tau)$. As has been mentioned in the preceding section, the critical point $p_0$ of $W(\tau)$ defines a vacuum of the theory. At $p_0$ the $\tau$ dependent eigenvalues of the Hessian matrix of the scalar potential (3.12) are given by

$$\lambda^{V_{1,2}}(\tau) = -4 \left( \frac{g_{zz}(p_0; \tau)}{M_P^2} W_0^2 - 2g_{zz}(p_0; \tau)|\partial_2^2 W_0|^2 \right) \pm \frac{2}{M_P^2} |\partial_2^2 W_0|.$$  

(4.14)

The first step is to look for $\tau < \tau_0$. Local minimum of the scalar potential (3.12) exists if

$$|\partial_2^2 W_0| > \frac{g_1(p_0; \tau)}{2M_P^2} W_0.$$  

(4.15)

On the other side, local maximum is ensured by

$$|\partial_2^2 W_0| < \frac{g_1(p_0; \tau)}{2M_P^2} W_0,$$  

(4.16)

while the inequality

$$\frac{g_1(p_0; \tau)}{2M_P^2} W_0 < |\partial_2^2 W_0| < \frac{g_1(p_0; \tau)}{M_P^2} W_0.$$  

(4.17)

shows the existence of a saddle point. The vacua become degenerate if

$$|\partial_2^2 W_0| = \frac{g_1(p_0; \tau)}{2M_P^2} W_0,$$

$$|\partial_2^2 W_0| = \frac{g_1(p_0; \tau)}{M_P^2} W_0.$$  

(4.18)

\[15\] We have similar discussion as in footnote 14.
Since the metric $g_{\bar{z}z}(p_0; \tau)$ is negative definite for $\tau > \tau_0$, one gets a local maximum by replacing $g_1(p_0; \tau)$ with $g_2(p_0; \tau)$ in (4.15). By using the same procedure to (4.16) and (4.17), conditions for a local minimum and a saddle point are obtained, respectively. Again, we have degenerate vacua if

$$|\partial^2 W_0| = \frac{g_2(p_0; \tau)}{2M_P^2} W_0,$$

$$|\partial^2 W_0| = \frac{g_2(p_0; \tau)}{M_P^2} W_0.$$  \hfill (4.19)

Here, some comments are in order. In $\tau < \tau_0$ region, the analysis of equations (4.9)-(4.11) and (4.15)-(4.18) gives the same results as in [9]. The local minimum of $W(\tau)$, given by (4.9), is mapped into the local maximum of the scalar potential (4.16). The other vacua, namely the local minimum, the saddle point, and the degenerate case, described by the second equality in (4.18), are coming from the saddle of $W(\tau)$. Lastly, there is a case where both eigenvalues of the Hessian matrix of the scalar potential and the real function $W(\tau)$ become singular. This means that this degenerate critical point of $W(\tau)$ is mapped into the degenerate vacuum and such a case is called intrinsic degenerate vacuum.

For $\tau > \tau_0$ case, since the parity transformation exists, we have the following situations. The existence of the local minimum of the scalar potential is guaranteed by the local maximum of $W(\tau)$ given by (4.12). The saddle of $W(\tau)$ are mapped into the local maximum, the saddle point, and the degenerate vacua given by the second equation in (4.19) of the scalar potential. But still, the first equation in (4.19) describes intrinsic degenerate vacua.

## 5 Supersymmetric Flows On a Curved Spacetime

This section provides the analysis of the supersymmetric flows, namely the gradient flow equations (3.8) and the RG flow described by the beta function (3.11) around the vacuum in the presence of geometric soliton (3.13). As we will see, such a soliton affects the flows by a minus sign which is, in other words, the parity map mentioned above.

First of all, we employ the dynamical system analysis on the gradient flows (3.8). A vacuum $p_0$ is an equilibrium point of (3.8) if it is also a critical point of $W(\tau)$. Around $p_0$ the first order expansion of (3.8) gives the eigenvalues

$$\Lambda_{1,2} = \pm \frac{W_0}{M_P^2} \cos \theta_0(u, \tau)$$

$$-2g_{\bar{z}z}(p_0; \tau) \left[ |\partial^2 W_0|^2 - \frac{g_2^2(p_0; \tau)}{4M_P^2} W_0^2 \sin^2 \theta_0(u, \tau) \right]^{1/2},$$  \hfill (5.1)

where $\theta_0(u, \tau) \equiv \theta(p_0; u, \tau)$ and the function $\theta(z, \bar{z}; u, \tau)$ is defined in (3.9). In general the eigenvalues (5.1) are complex because the second term in the square root could be negative. Therefore in order to have a consistent model we simply set that they must have a real value in which

$$|\partial^2 W_0| \geq \frac{|g_{\bar{z}z}(p_0; \tau)|}{2M_P^2} W_0 |\sin \theta_0(u, \tau)|,$$  \hfill (5.2)

$^{16}$If we take the limit $\ell \to 0$, then $\sin \theta(p_0; \tau) \to 0$. So, we regain the flat domain wall case with $|\partial^2 W(p_0; \tau)| \geq 0$ [9].
holds. This inequality gives a restriction of the critical points of the function $W(\tau)$ and the vacua of the theory described by (4.7) and (4.14), respectively. In other words, in order to have some vacua related to a CFT on the curved spacetime, the condition (5.2) must be fulfilled \(^{17}\).

For $\tau < \tau_0$ and $\cos \theta_0(u, \tau) \neq 0$ case we obtain that the stable nodes require

$$|\partial_z^2 W_0| > \frac{g_1(p_0; \tau)}{2M_p^2} W_0,$$

while saddles are ensured by the condition

$$\frac{g_1(p_0; \tau)}{2M_p^2} W_0 |\sin \theta_0(u, \tau)| \leq |\partial_z^2 W_0| < \frac{g_1(p_0; \tau)}{2M_p^2} W_0. \quad (5.4)$$

Looking at (5.3), (4.15), and (4.17) we find that the dynamic of the walls described by (3.8) is stable, flowing along the local minimum and the stable direction of the saddles of the scalar potential (3.12). Along local maximum the walls become unstable and the gradient flow is an unstable saddle. In other words, the dynamic of the walls is on an unstable curve flowing away from $p_0$. Moreover, in this linear analysis there is also a possibility of having $p_0$ as a bifurcation point, namely one of the eigenvalue in (5.1) vanishes. Such a case occurs if the condition (4.11) holds and it takes place on the intrinsic degenerate vacua \(^{18}\).

These conclusions are similar as in the flat domain wall case \([9]\). For $\cos \theta_0(u, \tau) = 0$, only the stable nodes survive and hence we have stable walls.

After crossing the singularity at $\tau = \tau_0$, i.e. the $\tau > \tau_0$ case, we obtain the following inequality

$$|\partial_z^2 W_0| > \frac{g_2(p_0; \tau)}{2M_p^2} W_0,$$

describing unstable nodes, whereas saddles need

$$\frac{g_2(p_0; \tau)}{2M_p^2} W_0 |\sin \theta_0(u, \tau)| \leq |\partial_z^2 W_0| < \frac{g_2(p_0; \tau)}{2M_p^2} W_0,$$

assuming $\cos \theta_0(u, \tau) \neq 0$. We have unstable walls flowing along the local maximum, and the unstable direction of the saddles of the scalar potential (3.12). On the other hand, along local minima the walls become stable approaching $p_0$ on the stable curve of the saddle flow. Again, a similar situation is obtained for a bifurcation point which requires

$$|\partial_z^2 W_0| = \frac{g_2(p_0; \tau)}{2M_p^2} W_0. \quad (5.7)$$

If $\cos \theta_0(u, \tau) = 0$, then we have only unstable nodes which means that the model admits only unstable walls.

Now let us perform an analysis on the RG flows described by the beta function (3.11) for finding out the nature of the vacuum $p_0$ in the UV and IR regions. Our starting point is to expand the beta function (3.11) around $p_0$. We obtain the matrix

$$U \equiv - \begin{pmatrix} \partial \beta / \partial z(p_0) & \partial \beta / \partial z(p_0) \\ \partial \bar{\beta} / \partial z(p_0) & \partial \bar{\beta} / \partial z(p_0) \end{pmatrix}, \quad (5.8)$$

\(^{17}\)See also the discussion on the RG flow below.

\(^{18}\)To see the type of this fold bifurcation one has to check the higher order terms. At least one of these terms is non vanishing \([23]\).
whose eigenvalues are
\[ \lambda_1^U = k^{-1} \left( \frac{\mathcal{W}_0}{M_P^2} \cos \theta_0(u, \tau) \right. \]
\[ + 2g_{z\bar{z}}(p_0; \tau) \left[ \frac{1}{2} \left( g_{z\bar{z}}(p_0; \tau) - \frac{g_{z\bar{z}}(p_0; \tau)}{4M_P^4} \mathcal{W}_0^2 \sin^2 \theta_0(u, \tau) \right) \right]^{1/2} \), (5.9) \]
\[ \lambda_2^U = k^{-1} \left( \frac{\mathcal{W}_0}{M_P^2} \cos \theta_0(u, \tau) \right. \]
\[ - 2g_{z\bar{z}}(p_0; \tau) \left[ \frac{1}{2} \left( g_{z\bar{z}}(p_0; \tau) - \frac{g_{z\bar{z}}(p_0; \tau)}{4M_P^4} \mathcal{W}_0^2 \sin^2 \theta_0(u, \tau) \right) \right]^{1/2} \). (5.10) \]

Similar as in the gradient flow, since the condition (5.2) is fulfilled, then both eigenvalues (5.9) and (5.10) are real. In the UV region where \( a \to +\infty \), it demands that at least one of the above eigenvalues have to be positive along which the RG flow can depart from the vacuum. Let us first discuss the \( \tau < \tau_0 \) region. In \( \cos \theta_0(u, \tau) > 0 \) case, all possibilities of the vacua are allowed. But, for \( \cos \theta_0(u, \tau) \leq 0 \) only stable nodes are permitted and therefore saddle, local minimum, and non-intrinsic stable degenerate vacua could exist. In other words, in this situation the gradient flow is only flowing along the stable direction of such vacua. In \( \tau > \tau_0 \) region we have the same result for \( \cos \theta_0(u, \tau) > 0 \) case. However, for \( \cos \theta_0(u, \tau) \leq 0 \) case only unstable nodes are allowed which tell us the existence of unstable vacua such as saddle, local maximum, and non-intrinsic unstable degenerate vacua. In addition, the gradient flow is taking unstable direction of the vacua and hence we have here unstable situation.

On the other side in the IR region, where \( a \to 0 \), requires at least a negative eigenvalue of (5.8) which is the direction of the RG flow approaching the vacuum. In \( \tau < \tau_0 \) and \( \cos \theta_0(u, \tau) \geq 0 \) we find that only stable nodes survive and again, we have only stable situation in which local maximum and intrinsic degenerate vacua are forbidden here. Conversely, we have all possible vacua for \( \cos \theta_0(u, \tau) < 0 \). Second, in \( \tau > \tau_0 \) and \( \cos \theta_0(u, \tau) \geq 0 \) it turns out that only unstable nodes exist, therefore the theory admits only the unstable vacua mentioned above. Lastly, everything is allowed for \( \cos \theta_0(u, \tau) < 0 \).

### 6 A Model with the Rosenau Soliton

In this section we give an example in which the geometric flow is called the Rosenau soliton satisfying (3.13). This soliton was firstly studied in the context of fluid dynamics [20] and has been considered also by geometrician as a toy model in two dimensional complex surfaces. For a review see for example [24].

The Rosenau soliton has the form
\[ g_{z\bar{z}}(\tau) = -\frac{4c^2}{b} \frac{\sinh(2b\tau)}{\cosh[2c(z + \bar{z})] + \cosh(2b\tau)} , \] (6.1)
whose corresponding \( U(1) \) connection is given by
\[ Q(\tau) = -\frac{ic}{b} \ln \left( \frac{\cosh[\nu + 2b\tau]}{\cosh \nu} \right) (dz - d\bar{z}) , \] (6.2)
where \( c \in \mathbb{R}, b > 0, \) and \( v \equiv c(z + \bar{z}) - b\tau \) which diverges at \( \tau = 0 \) or \( c = 0 \). It is easy to see that the metric (6.1) is invariant under parity transformation, \( c \leftrightarrow -c \). We can further get its Kähler potential

\[
K(\tau) = -\frac{2}{b} \int \ln \left( \frac{\cosh[v + 2b\tau]}{\cosh v} \right) dv = 2b\tau v + 2 \Re \left[ \text{dilog} (1 + i e^{v+2b\tau}) - \text{dilog} (1 + i e^v) \right] ,
\]

(6.3)

where dilogarithm function have been introduced [25]

\[
\text{dilog}(x) \equiv \int_1^x \frac{\ln t}{1 - t} dt = \sum_{k=1}^{+\infty} \frac{x^k}{k^2} .
\]

(6.4)

Looking at (6.1), we find that

\[
g_1(\tau) = g_2(\tau) = \frac{4c^2}{b} \frac{|\sinh(2b\tau)|}{\cosh[2c(z + \bar{z})] + \cosh(2b\tau)} ,
\]

(6.5)

for all \( \tau \) but not at singularity \( \tau = 0 \) and \( c \neq 0 \).

Now let us first choose for simplicity the holomorphic superpotential

\[W(z) = a_0 + a_1 z ,\]

(6.6)

with \( a_0, a_1 \in \mathbb{R} \). So to find a supersymmetric vacuum one has to solve the condition (3.14), which in this model becomes

\[a_1 - \frac{2c}{b M_P^2} \ln \left( \frac{\cosh[c(z_0 + \bar{z}_0) + b\tau]}{\cosh[c(z_0 + \bar{z}_0) - b\tau]} \right) (a_0 + a_1 z_0) = 0 .\]

(6.7)

Nondegeneracy requires \( W(z_0) \neq 0 \). We then obtain the solution of (6.7) as

\[
tanh(2cx_0) \coth \left( \frac{ba_1 M_P^2}{4c(a_0 + a_1 x_0)} \right) = \coth(b\tau) ,
\]

(6.8)

which follow that the imaginary part of \( W(z_0) \) vanishes. Thus, from (4.6) we find that this model admits only singular vacua which do not related to the CFT in three dimensions. In addition, it is easy to see that the origin belongs to this class of vacuum for \( a_1 = 0 \) and \( a_0 \neq 0 \) at which the \( U(1) \) connection disappears.

The next case is to replace \( a_1 \) by \( ia_1' \) with \( a_1' \in \mathbb{R} \) in the superpotential (6.6). Then, we obtain

\[
y_0 = \frac{a_0}{a_1'} ,
\]

\[
tanh(2cx_0) \coth \left( \frac{bM_P^2}{4cx_0} \right) = \coth(b\tau) ,
\]

(6.9)

in which the superpotential evaluated at \( p_0 \) has the form

\[W(z_0) = ia_1' x_0 .\]

(6.10)
Hence, the warped factor (4.2) simplifies to
\[ a(u, \tau) = \pm \frac{\ell}{W_0} \left[ A_0 e^{+W_0u} - A_0^{-1} e^{-W_0u} \right], \]
(6.11)
where
\[ W_0 = e^{K(x_0; \tau)/2M_P^2} |a'_1x_0|. \]
(6.12)
We close this section by providing the rest quantities related to the analysis of the eigenvalues, namely \( \lambda^W_{1,2}, \lambda^V_{1,2}, \lambda^U_{1,2} \) and \( \lambda^U_{1,2} \). These are
\[
|\partial_z^2 W_0| = \frac{W_0}{2M_P^2} \left| g_{zz}(p_0; \tau) - \frac{1}{M_P^2} |K_z|^2(p_0; \tau) \right|, \\
\cos \theta_0(u, \tau) = \left(1 + \ell^2(aW_0)^2\right)^{-1/2}, \\
\sin \theta_0(u, \tau) = \pm \ell(aW_0)^{-1} \left(1 + \ell^2(aW_0)^2\right)^{-1/2}.
\]
(6.13)
Therefore, we have a model where there may be a possibility of having vacua which correspond to the CFT. Moreover, in this model the singularities are at \( \tau = 0 \) and at \( a'_1 = 0 \).

**7 Conclusions**

We have studied the nature of the \( N = 1 \) supergravity curved BPS domain walls and their vacuum structure. In particular, we have considered a four dimensional \( N = 1 \) supergravity coupled to a chiral multiplet whose the scalar manifold can be viewed as the Kähler-Ricci soliton satisfying the geometric flow equation (2.2). Some consequences are emerged as follows. First of all, all couplings such as the scalar potential and the shifting quantity are evolving with respect to \( \tau \) which are given in (2.5). Next, the warped factor, the BPS equations, and the beta function describing the domain walls and the three dimensional CFT, respectively, also depend on \( \tau \).

The analysis on the vacua of the theory shows that in general the spacetime is non-Einsteinian, deformed with respect to \( \tau \) whose Ricci tensor and Ricci scalar have the form provided in (4.4). This corresponds to the CFT on the three dimensional AdS spacetime which is ensured by the beta function (3.11). In this paper we just mentioned three special cases. First, for \( \ell = 0 \) and \( k \neq 0 \) with \( k' = 0 \) we regain four dimensional AdS spacetim es which appear in the flat domain walls. Second, there is a singularity at \( k = 0 \) and \( \ell \neq 0 \) related to the RG flow analysis in which the spacetime has a constant curvature but non-Einsteinian. Note that the singularity caused by the geometric flow is trivial since some quantities in this case would become singular. Third, if \( W(z_0) \neq 0 \) and the \( U(1) \) connection vanishes, then we have a static case. For this case, the vacua are defined by the critical points of the superpotential \( W(z) \).

Since our ground states generally depend on \( \tau \) and additionally the theory does admit the existence of singular point at \( \tau = \tau_0 \) described in (3.13), then we have to split the region in order to analyze the function \( W(\tau) \) and the scalar potential \( V(\tau) \). In \( \tau < \tau_0 \) region we reproduce the previous results in [9] as follows. Our analysis confirms that the deformation of the critical points of \( W(\tau) \) can only be in the following types, namely local minima, saddles, and degenerate critical points. The local minima verify the existence
of local maximum vacua, whereas the saddles are mapped into local minimum, saddle, or non-intrinsic degenerate vacua. There is possibly a special situation where we have intrinsic degenerate vacua coming from the degenerate critical points of $W(\tau)$. Hence, these also prove that the vacua certainly deform with respect to $\tau$.

For $\tau > \tau_0$ case, similar as mentioned above, $W(\tau)$ also admits the evolution of its critical points which are local maxima, saddles, and degenerate critical points. In this case, however, the local maxima are mapped into local minimum vacua, while the saddles imply the existence of local maximum, saddle, or non-intrinsic degenerate vacua. In addition, intrinsic degenerate vacua could possibly exist. So, these results show that the geometric flow (3.13) indeed changes the Hessian matrix of the real function $W(\tau)$ by a minus sign. Or in other words, the vacua of the index $\lambda$ change to the other vacua of the index $2 - \lambda$ caused by Kähler-Ricci flow. The ground states of index $2 - \lambda$ in $\tau > \tau_0$ region are called the parity pair of those with the index $\lambda$ in $\tau < \tau_0$ region.

Furthermore, the analysis using the gradient and the RG flows show that the above vacua do not always exist. First, the first order expansion of the BPS equations yields the condition (5.2) that ensures the existence of the vacua related to the three dimensional CFT. Then, in the interval $\tau < \tau_0$ we have stable nodes flowing along the local minimum, and the stable direction of the saddle vacua, whereas unstable saddles flow along the local maximum vacua. On the other side, for $\tau > \tau_0$ case, the gradient flow turns into the unstable nodes flowing along the local maximum and the unstable direction of the saddle vacua, while along the local minimum the gradient flow is flowing on the stable curve of the saddle flow which is called the stable saddles. In both region there is a possibility of having a bifurcation point occurring near intrinsic degenerate vacua.

In order to check the existence of a ground state one has to carry out the analysis on the beta function. Particularly in the UV region and $\cos \theta_0(u, \tau) \leq 0$, for $\tau < \tau_0$ we have only stable nodes which further imply that the theory admits only stable vacua, whereas for $\tau > \tau_0$ the gradient flow turns into unstable nodes. On the other hand, in the IR region and $\cos \theta_0(u, \tau) \geq 0$, for $\tau < \tau_0$ again only stable situation is allowed, while for $\tau > \tau_0$ everything becomes unstable.

Next, we have considered a simple model in which the superpotential is linear and the Kähler-Ricci soliton has a special form called the Rosenau solution. In the model where $a_0, a_1 \in \mathbb{R}$ we find that the model has only singular ground states unrelated to the three dimensional CFT. However, if we set at least $a_1 \in \mathbb{C}$, we may obtain the vacua related to the CFT.

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A Convention and Notation

The purpose of this appendix is to assemble our conventions in this paper. The spacetime metric is taken to have the signature \((+,-,-,-)\) while the Christoffel symbol is given by
\[
\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho})
\]
where \(g_{\mu\nu}\) is the spacetime metric. The Riemann curvature has the form
\[
R^\mu_{\nu\rho\lambda} = \partial_\rho \Gamma^\mu_{\nu\lambda} - \partial_\lambda \Gamma^\mu_{\nu\rho} + \Gamma^\sigma_{\nu\lambda} \Gamma^\mu_{\rho\sigma} - \Gamma^\sigma_{\nu\rho} \Gamma^\mu_{\sigma\lambda}
\]
and the Ricci tensor is defined to be \(R_{\nu\lambda} = R^\mu_{\nu\mu\lambda}\).

The following indices are given:
\[
\nu, \lambda = 0, 1, 2, \quad \text{label three dimensional curved spacetime indices}
\]
\[
a, b = 0, 1, 2, \quad \text{label three dimensional flat spacetime indices}
\]
\[
\mu, \nu = 0, \ldots, 3, \quad \text{label four dimensional curved spacetime indices}
\]
\[
a, b = 0, \ldots, 3, \quad \text{label four dimensional flat spacetime indices}
\]

B Critical Point of A Function

The structure and the logic of this section are similar to those in [9] which are useful for our analysis in the paper. Firstly, we consider any arbitrary (real) \(C^\infty\)-function \(f(z, \bar{z})\) of \(f(z, \bar{z})\) satisfies
\[
\frac{\partial f}{\partial z}(p_0) = 0, \quad \frac{\partial f}{\partial \bar{z}}(p_0) = 0. \quad (B.1)
\]

The point \(p_0\) is said to be a non-degenerate critical point if the Hessian matrix of \(f(z, \bar{z})\)
\[
H_f \equiv 2 \begin{pmatrix} \frac{\partial^2 f}{\partial z \partial \bar{z}}(p_0) & \frac{\partial^2 f}{\partial z^2}(p_0) \\ \frac{\partial^2 f}{\partial \bar{z}^2}(p_0) & \frac{\partial^2 f}{\partial \bar{z} \partial z}(p_0) \end{pmatrix} \quad (B.2)
\]
is non-singular, \textit{i.e.},
\[
\det H_f = 4 \left[ \left( \frac{\partial^2 f}{\partial z \partial \bar{z}}(p_0) \right)^2 - \frac{\partial^2 f}{\partial \bar{z}^2}(p_0) \frac{\partial^2 f}{\partial \bar{z} \partial z}(p_0) \right] \neq 0. \quad (B.3)
\]
The eigenvalues of the Hessian matrix \(H_f\) are given by
\[
\lambda_1^f = \frac{1}{2} \left( \text{tr} H_f + \sqrt{(\text{tr} H_f)^2 - 4 \det H_f} \right),
\]
\[
\lambda_2^f = \frac{1}{2} \left( \text{tr} H_f - \sqrt{(\text{tr} H_f)^2 - 4 \det H_f} \right). \quad (B.4)
\]
The eigenvalues defined in (B.4) can be used to classify the critical point \(p_0\) of the function \(f\) as follows:
1. If $\lambda_1 > 0$ and $\lambda_2 > 0$, then $p_0$ is a local minimum describing a stable situation.
2. If $\lambda_1 < 0$ and $\lambda_2 < 0$, then $p_0$ is a local maximum describing an unstable situation.
3. If $\lambda_1 > 0$ and $\lambda_2 < 0$ or vice versa, then $p_0$ is a saddle point.
4. If at least one of its eigenvalue vanishes, then $p_0$ is said to be degenerate.

C A Quick Review of the 2d Kähler-Ricci Flow

This section is devoted to give a short review of the two dimensional Kähler-Ricci flow equation

$$\frac{\partial g_{z \bar{z}}}{\partial \tau} = -2R_{z \bar{z}}(\tau) = -2 \partial_z \partial_{\bar{z}} \ln g_{z \bar{z}}(\tau), \quad \text{(C.1)}$$

discussed in Section 2 where $\tau \in \mathbb{R}$. Here, we particularly consider a simple case where the initial geometry at $\tau = 0$ is a Kähler-Einstein surface satisfying

$$R_{z \bar{z}}(x, 0) = \Lambda_2 g_{z \bar{z}}(x; 0), \quad \text{(C.2)}$$

where $g_{z \bar{z}}(x; 0)$ is an Einstein metric and $\Lambda_2 \in \mathbb{R}$. Then, we show that a Kähler-Ricci soliton can be viewed as an area deformation of a Kähler geometry for finite $\tau$.

Let us simply choose that the constant $\Lambda_2 > 0$ and the initial metric $g_{z \bar{z}}(x; 0)$ is definite positive. Taking the metric ansatz

$$g_{z \bar{z}}(x; \tau) = \rho(\tau) g_{z \bar{z}}(x; 0), \quad \text{(C.3)}$$

and then the definition of Ricci tensor, we have

$$R_{z \bar{z}}(x; \tau) = R_{z \bar{z}}(x; 0) = \Lambda_2 g_{z \bar{z}}(x; 0). \quad \text{(C.4)}$$

Inserting (C.3) into (C.1), we get

$$g_{z \bar{z}}(x; \tau) = (1 - 2\Lambda_2 \tau) g_{z \bar{z}}(x; 0). \quad \text{(C.5)}$$

This geometric soliton has a singularity at $\tau = 1/2\Lambda_2$. After hitting the singularity, namely for $\tau > 1/2\Lambda_2$, the geometry changes such that the metric is negative definite with a negative "cosmological" constant. This shows that the Kähler-Ricci flow interpolates two different $N = 1$ theories disconnected by the singularity at $\tau = 1/2\Lambda_2$.

Finally, we show that the Kähler-Ricci soliton can be viewed as a volume deformation of a Kähler manifold for finite $\tau$. To be precise, the area of the soliton (C.5) has the form

$$\sqrt{\det g(x; \tau)} d^2 x = |1 - 2\Lambda_2 \tau| \sqrt{\det g(x; 0)} d^2 x. \quad \text{(C.6)}$$

For $0 \leq \tau < 1/2\Lambda_2$, the geometry is diffeomorphic to the initial geometry endowed with a positive definite metric with $\Lambda_2 > 0$, while for $\tau > 1/2\Lambda_2$ one has a geometry admitting a negative definite metric with negative "cosmological" constant.
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