Uniqueness of Single Peak Solutions for Coupled Nonlinear Gross-Pitaevskii Equations with Potentials

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Abstract

For a couple of singularly perturbed Gross-Pitaevskii equations, we first prove that the single peak solutions, if they concentrate on the same point, are unique provided that the Taylor’s expansion of potentials around the concentration point is in the same order along all directions. Among other assumptions, our results indicate that the peak solutions obtained in [21, 31, 38] are unique. Moreover, for the radially symmetric ring-shaped potential, which attains its minimum at the spheres \( \Gamma_i := \{ x \in \mathbb{R}^N : |x| = A_i > 0 \}, i = 1, 2, \ldots, l \), and is totally degenerate in the tangential space of \( \Gamma_i \), we prove that the positive ground state is cylindrically symmetric and is unique up to rotations around the origin. As far as we know, this is the first uniqueness result for ground states under radially symmetric but non-monotonic potentials.

Keywords: Gross-Pitaevskii equations; single peak solutions; ring-shaped potentials; uniqueness.

MSC(2010): 35J47, 35J50, 46N50

1 Introduction

In this paper, we consider the following time-independent coupled nonlinear Gross-Pitaevskii equations

\[
\begin{align*}
-\varepsilon^2 \Delta u_1 + V_1(x)u_1 &= a_1 u_1^3 + \beta u_2^2 u_1 \quad \text{in } \mathbb{R}^N, \quad 1 \leq N \leq 3, \\
-\varepsilon^2 \Delta u_2 + V_2(x)u_2 &= a_2 u_2^3 + \beta u_1^2 u_2 \quad \text{in } \mathbb{R}^N.
\end{align*}
\]

System (1.1) mainly arises in the theoretical study of two-component Bose-Einstein condensation (BEC), which has been extensively studied by mathematicians and physicists, in view of the experimental realization of two component BEC for trapped alkali atomic gases [19]. Here \((u_1(x), u_2(x))\) is related to the macroscopic wave function of the ultracold atom system, \(V_i(x) \in C(\mathbb{R}^N; \mathbb{R}^+)\)(i = 1, 2,) denotes the external trapping potential, the absolute value of the parameter \(|a_i|\)(i = 1, 2,) describes the interaction strength among the atoms in each component, while \(|\beta|\) denotes the interaction strength for the atoms between different components. Meanwhile, the signs of \(a_1, a_2\) and \(\beta\) denote the interactions are attractive or repulsive, respectively.
The classification on the existence of solutions of equation (1.1) is an interesting but quite difficult topic, for which depends sensitively on the parameters \(a_1, a_2, \beta\), as well as the potentials \(V_i(x)(i = 1, 2,\ldots)\). Moreover, comparing with the case of single equation, another difficulty of studying (1.1) lies in how to distinguish the non-trivial solutions (\(u_1 \neq 0\) and \(u_2 \neq 0\)) from the semi-trivial ones (i.e., one of \(u_1\) and \(u_2\) equals to zero). As a consequence, problem (1.1) has received much attention in the last decade. In particular, after the pioneering work of Lin and Wei \[23\] which considered the existence of nontrivial solutions for the following autonomous system corresponding to (1.1) with \(V_i(x) = \mu_i > 0\) and \(\varepsilon = 1\),

\[
\begin{cases}
-\Delta u_1 + \mu_1 u_1 = a_1 u_1^3 + \beta u_2^2 u_1 & \text{in } \mathbb{R}^N, \\
-\Delta u_2 + \mu_2 u_2 = a_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbb{R}^N.
\end{cases}
\]  

For instance, Sirakov \[35\] obtained the ground states of (1.2) by searching for minimizers of the energy functional corresponding to (1.2) over the Nehari manifold. In particular, he proved that, there exist \(\beta_1 = \beta_1(a_1, a_2, \mu_1, \mu_2) \in (0, \min\{a_1, a_2\}\] and \(\beta_2 = \beta_2(a_1, a_2, \mu_1, \mu_2) \in [\max\{a_1, a_2\}, +\infty)\) (see [35, Theorem 1.2] for the explicit form of \(\beta_i\)) such that, (1.2) has a positive ground state provided that \(\beta \in (0, \beta_1) \cup (\beta_2, +\infty)\), while (1.2) has no positive solution provided that \(\beta \in [\min\{a_1, a_2\}, \max\{a_1, a_2\}\]. Moreover, for the special case of \(\mu_1 = \mu_2 = \mu > 0\), i.e., for the following system

\[
\begin{cases}
-\Delta u + \mu u = a_1 u^3 + \beta v^2 u & \text{in } \mathbb{R}^N, \\
-\Delta v + \mu v = a_2 v^3 + \beta u^2 v & \text{in } \mathbb{R}^N, \\
u, v \in H^1(\mathbb{R}^N),
\end{cases}
\]  

he showed that \(\beta_1 = \min\{a_1, a_2\}\) and \(\beta_2 = \max\{a_1, a_2\}\), and \((w_1(x), w_2(x))\) with

\[w_i(x) = \sqrt{\gamma_i \mu} w(\sqrt{\mu} x)\]

is the positive ground state of (1.3). Here \(w(|x|)\) is the unique radially symmetric positive solution of the field equation \[22\]

\[\Delta w - w + w^3 = 0, \quad w \in H^1(\mathbb{R}^N).\]  

However, as pointed out in \[35\], when \(\mu_1 \neq \mu_2\), \(\beta_1\) and \(\beta_2\) are not the optimal endpoints which determine the existence and non-existence of nonnegative nontrivial solutions of (1.2). To find out the optimal ranges for the existence of nonnegative nontrivial solutions is a quite interesting problem. Recently, the authors of \[2, 10\] make some progress on this aspect, and obtained some new intervals about \(\beta\) for the existence and nonexistence of nonnegative solutions.

Sirakov conjectured in \[35\] that, up to translations, \((w_1(x), w_2(x))\) with the form of (1.3) is the unique positive solution of (1.3) when

\[\beta \in (0, \min\{a_1, a_2\}\] or \(\max\{a_1, a_2\}, +\infty)\].

Wei and Yao \[39\] affirmed this conjecture for \(\beta \in (\max\{a_1, a_2\}, +\infty)\), or \(\beta > 0\) and small enough. Chen and Zou \[5\] proved this conjecture for the case that \(\beta > 0\) is less but close to \(\min\{a_1, a_2\}\}. Afterwards, they considered a weak version of Sirakov’s conjecture in \[4\], and proved that for all \(\beta\) satisfying (1.6), \((w_1(x), w_2(x))\) is the unique positive ground state of (1.3) up to a translation.
For the singularly perturbed problem \((1.1)\), the existing results are mainly on the existence and concentration of solutions as \(\varepsilon \to 0^+\), i.e. searching for the so called peak solutions in the natural energy space \(\mathcal{X}_\varepsilon\). Here, \(\vec{u} = (u_1, u_2) \in \mathcal{X}_\varepsilon\) if
\[
\left\{ u_i \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_i(x)|u_i(x)|^2 \, dx < \infty \right\}, \quad i = 1, 2.
\]
The norm of \(\vec{u} = (u_1, u_2) \in \mathcal{X}_\varepsilon\) is defined by
\[
\| (u_1, u_2) \|^2_{\mathcal{X}_\varepsilon} = \sum_{i=1}^2 \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u_i|^2 + V_i(x)|u_i|^2 \, dx.
\]
\(\vec{u}(x) \in \mathcal{X}_\varepsilon\) is called a solution of \((1.1)\), provided it is a critical point of the following energy functional in \(\mathcal{X}_\varepsilon\)
\[
J_\varepsilon(\vec{u}) := \frac{1}{2} \| (u_1, u_2) \|^2_{\mathcal{X}_\varepsilon} - \frac{1}{4} \int_{\mathbb{R}^N} \left( a_1 |u_1|^4 + a_2 |u_2|^4 + 2\beta |u_1|^2 |u_2|^2 \right) \, dx. \tag{1.7}
\]

Under suitable assumptions on the limit at infinity of \(V_i(x)\), Lin and Wei \cite{25} proved the existence of positive ground state for the case of \(\beta \in (-\infty, \beta_0)\), where \(\beta_0\) is a small unknown positive constant. Pomponio \cite{32} generalized the above results to the case of \(\beta > 0\) and \(\beta < 0\). In \cite{21}, Montefusco, Pellacci and Squassina considered the case of \(a_1 = a_2 = 1\) and \(\beta > 0\) under the hypothesis that
\[
0 < \min_{B(z,r)} V_i(x) < \min_{\partial B(z,r)} V_i(x), \quad \text{for some } z \in \mathbb{R}^3 \text{ and } r > 0, \quad i = 1, 2. \tag{1.8}
\]

By applying the mountain pass lemma combining with the technique of truncating the nonlinear term, they proved that equation \((1.1)\) possesses a nonnegative solution \((u_{1\varepsilon}, u_{2\varepsilon})\) and the sum \(u_{1\varepsilon} + u_{2\varepsilon}\) has a unique maximum provided \(\varepsilon > 0\) is small. Moreover, they showed that if \(\beta > 0\) is suitably small, then one of \(u_{i\varepsilon}\) vanishes as \(\varepsilon \to 0^+\). Assuming \(V_i(x)\) satisfies an abstract assumption which is a weaker version of \((1.8)\), Ikoma and Tanaka \cite{21} obtained the existence of positive solutions of \((1.1)\) if \(\beta > 0\) is suitably small. By employing the Hardy inequality in \(\mathbb{R}^3\) and some truncating technique, Chen and Zou \cite{6} extended the work of \cite{21} to the case of decaying potentials and \(\beta > 0\) is relatively large. Wang and Shi \cite{38} proposed the following universe assumption on \(V_i(x)\),
\[
0 < \inf_{|x| \to \infty} V_i(x) < \liminf_{|x| \to \infty} V_i(x), \quad \{ y \in \mathbb{R}^N : V_i(y) = \inf_{x \in \mathbb{R}^N} V_i(x), \text{for } i = 1, 2 \} \neq \emptyset, \tag{1.9}
\]
and obtained a positive ground state of \((1.1)\) for the case of \(\beta > 0\) large via the Nehari method. They also proved the existence of multiple nontrivial solutions via the Ljusternik-Schinirelmann category theory. For more results on the existence, multiplicity and concentration of solutions for \((1.1)\) in bounded domains or whole \(\mathbb{R}^N\), one can infer \cite{11, 21, 30, 33, 17, 18, 41} and the references therein.

We note that the solutions obtained by \cite{25, 31, 21, 38} are generally single peak solutions, whose precise definition can be stated as following.

**Definition 1.1.** We call \((u_{1\varepsilon}(x), u_{2\varepsilon}(x)) \in \mathcal{X}_\varepsilon\) a positive single peak solution of \((1.1)\) concentrating on some point \(x_0 \in \mathbb{R}^N\), which means that \(u_{i\varepsilon}(x) > 0 (i = 1, 2)\) and \(u_i(x) := u_{i\varepsilon}(x) + u_{2\varepsilon}(x)\) has a unique maximal point \(x_\varepsilon \to x_0 \) as \(\varepsilon \to 0^+\), while \(u_{i\varepsilon}(x) \to 0\) uniformly in \(\mathbb{R}^N \setminus B_\theta(x_0)\) for any \(\theta > 0\).
One natural question is that whether these single peak solutions are unique under certain assumptions on potentials. As far as we know, there is few result on this respect. As a consequence, in this manuscript we intend to investigate the uniqueness of peak solutions for some typical potentials provided that \( \beta > \max\{a_1, a_2\} \). For this purpose, we assume that

\[
(V) \quad V_i(x) \in C^0(\mathbb{R}^N; \mathbb{R}^+), \inf_{x \in \mathbb{R}^N} V_i(x) > 0 (i = 1, 2, \) and there exit \( \mu > 0 \) and \( z \in \mathbb{R}^3 \), such that \( V_i(z) = V_2(z) = \mu \).
\]

Our first result can be stated as follows.

**Theorem 1.1.** Let \( \beta > \max\{a_1, a_2\} \) and \( (V) \) be satisfied. Assume that there exist \( m_i, r_0 > 0 \) and \( p_i > 1 \), such that

\[
V_i(x) = \mu + m_i|x - z|^{p_i} + o(|x - z|^{p_i}) \quad \text{for all } x \in B_{r_0}(z), i = 1, 2,
\]

and there exists \( q_i > p_i - 1 \) such that

\[
\frac{\partial V_i(x + z)}{\partial x_j} = m_i p_i |x|^{p_i - 2} x_j + R_{ij}(x) \quad \text{with} \quad |R_{ij}(x)| \leq C|x|^{q_i} \quad \text{in} \quad B_{r_0}(0).
\]

Let \( \bar{u}_\varepsilon(x) = (u_{1\varepsilon}, u_{2\varepsilon}) \) and \( \bar{v}_\varepsilon(x) = (v_{1\varepsilon}, v_{2\varepsilon}) \) be two sequences of positive single peak solutions of \((1.1)\) in \( \mathcal{X}_\varepsilon \) which both concentrate on \( z \) and

\[
\lim_{\varepsilon \to 0^+} \bar{u}_\varepsilon(\varepsilon x + x_\varepsilon) = \lim_{\varepsilon \to 0^+} \bar{v}_\varepsilon(\varepsilon x + y_\varepsilon) = \bar{w}(x) \quad \text{strongly in} \quad H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N),
\]

where \( \bar{w}(x) := (w_1, w_2) \) with \( w_i(x) \) being given by \((1.4)\), and \( x_\varepsilon \) and \( y_\varepsilon \) denote the unique maximum point of \( u_{1\varepsilon} + u_{2\varepsilon} \) and \( v_{1\varepsilon} + v_{2\varepsilon} \), respectively. Then, there exists \( \varepsilon_0 > 0 \) such that \( \bar{u}_\varepsilon(x) \equiv \bar{v}_\varepsilon(x) \) provided that \( \varepsilon \in (0, \varepsilon_0) \).

The proof of Theorem 1.1 is mainly motivated by the arguments of [3, 8, 13, 14, 15] and the references therein. We also note that, as mentioned above, the existence of positive single peak solutions satisfying \((1.12)\) were derived in different articles. For example, the hypothesis \((1.10)\) indicates \((1.8)\), it then follows from [31] that there exists a positive solution concentrating on \( z \) and satisfying \((1.12)\) if \( a_1 = a_2 = 1 \) and \( \beta > 1 \). Moreover, if we assume additionally that

\[
\liminf_{|x| \to \infty} V_i(x) > \mu = \inf_{|x| \to 0^+} V_i(x) > 0, \quad i = 1, 2,
\]

then the single peak solution was also obtained by [38, Theorem 1.2], and also by [21, Theorem 1.3] provided \( \beta > 0 \) is small. Therefore, Theorem 1.1 indicates that the single peak solutions obtained in [21, 37, 38] are unique provided that \((1.10)\) and \((1.11)\) are satisfied.

For the reader’s convenience, in what follows we give a simple proof for the existence of positive single peak solution (which is indeed ground state) in the first part of Theorem 1.2 under the hypothesis \((1.13)\). Moreover, for the so called ring-shaped potentials, i.e., \( V_i(x) = V_i(|x|) \) attains its minimum at some spheres, we derive that the ground state must concentrate on one flattest common minimal point of \( V_i(x) \) and \( V_j(x) \). In addition, we discuss the uniqueness (up to rotations) of ground state if more information of \( V_i(x) \) near its minimal points is given. Potentials of ring-shaped appear in many different areas, for instance, which are frequently used in BEC experiments, see e.g., [9, 10, 34].
In the sequel, we assume that $V_1(x)$ and $V_2(x)$ satisfy (1.13) and the common minimal points set
\[ \mathcal{Z} := \{ x \in \mathbb{R}^N : V_1(x) = V_2(x) = \mu \} \neq \emptyset \] (1.14)
consists of exactly $l(l \in \mathbb{N}^+)$ common spheres, i.e.,
\[ \mathcal{Z} = \{ x \in \mathbb{R}^N : |x| = A_j > 0, 1 \leq j \leq l; A_j \neq A_k, \forall 1 \leq j \neq k \leq l \}. \] (1.15)
Moreover, we also assume that there exist $r_0 > 0$, $b_{ij} > 0$ and $p_{ij} > 1$ such that for $i = 1, 2$ and $j = 1, 2, \ldots, l$,
\[ V_i(x) = V_i(|x|) = \mu + b_{ij} |x| - A_j^{p_{ij}} + o(|x| - A_j^{p_{ij}}) \quad \text{for any } |x| - A_j < r_0. \] (1.16)

We note that any rotation of a single peak solution for (1.1) around the origin is still a single peak solution because $V_i(x) = V_i(|x|)$ is radially symmetric. Therefore, to discuss the uniqueness, we must modulate the rotations. We note that since $\{ x : |x| = A_j > 0 \}$ is a $N-1$-dimensional manifold in $\mathbb{R}^N$, this causes that the potential is very degenerate in the tangential space of the sphere $\{ x : |x| = A_j > 0 \}$. As a consequence, one cannot derive the uniqueness via the arguments of Theorem 1.1 for the ‘step 2’ in its proof is not valid anymore. In this case to obtain the uniqueness (up to rotations) for general single peak solutions becomes very challenge. Indeed, we are even uncertain whether single peak solution is unique or not. However, for the ground state for the second best, by proceeding some refined energy estimates, we can prove that they preserve some symmetric properties, which is helpful for proving their uniqueness. For this reason, we define
\[ c_\varepsilon := \inf_{\bar{u} \in \mathcal{N}_\varepsilon} J_{\varepsilon}(\bar{u}), \] (1.17)
where the energy functional $J_{\varepsilon}(\bar{u})$ is given by (1.7), and $\mathcal{N}_\varepsilon$ denotes the Nehari manifold and is defined as
\[ \mathcal{N}_\varepsilon := \left\{ (u_1, u_2) \in \mathcal{X}_\varepsilon \setminus \{0\} : \|(u_1, u_2)||^2_{\mathcal{X}_\varepsilon} = \int_{\mathbb{R}^N} \left( a_1|u_1|^4 + a_2|u_2|^4 + 2\beta|u_1|^2|u_2|^2 \right) dx \right\}. \] (1.18)
We call $(u_1, u_2)$ a ground state of (1.1) if
\[ (u_1, u_2) \text{ is a solution of (1.1) and } J_{\varepsilon}(u_1, u_2) = c_\varepsilon. \] (1.19)
Assume that $V_i(x)$ satisfies (1.15) and (1.16). Denote $\lambda_j (1 \leq j \leq l)$ by
\[ \bar{\lambda}_j = \begin{cases} \mu^{1 - \frac{N + p_{1j}}{2}} \gamma_1 b_{1j} \int_{\mathbb{R}^N} |x_N|^{p_{1j}} w^2(x) dx, & \text{if } p_{1j} < p_{2j}, \\ \mu^{1 - \frac{N + p_{1j}}{2}} (\gamma_1 b_{1j} + \gamma_2 b_{2j}) \int_{\mathbb{R}^N} |x_N|^{p_{1j}} w^2(x) dx, & \text{if } p_{1j} = p_{2j}, \\ \mu^{1 - \frac{N + p_{2j}}{2}} \gamma_2 b_{2j} \int_{\mathbb{R}^N} |x_N|^{p_{2j}} w^2(x) dx, & \text{if } p_{1j} > p_{2j}, \end{cases} \] (1.20)
where $\gamma_i > 0$ is given by (1.4). Let $p_j := \min \{ p_{1j}, p_{2j} \}$ and set
\[ p_0 := \max_{1 \leq j \leq l} p_j, \quad \Gamma := \{ 1 \leq j \leq l : p_j = p_0 \} \quad \text{and} \quad \bar{\lambda}_0 := \min_{j \in \Gamma} \bar{\lambda}_j. \] (1.21)
Denote
\[ \mathcal{Z}_0 := \{ x \in \mathbb{R}^N : |x| = A_j \text{ with } j \in \Gamma \text{ and } \bar{\lambda}_j = \bar{\lambda}_0 \} \] (1.22)
the set of the flattest common minimum points of $V_1(x)$ and $V_2(x)$.

Under the above assumptions, we have the following refined estimates for ground states of (1.1) as $\varepsilon \to 0^+$. 5
Therefore, the arguments of [27] is not applicable for our case.

Remark 1.1. From above theorem we see that if the flattest common minimal set of 
$V_1(x)$ and $V_2(x)$ contains only one sphere, namely, $Z_0 = \{ x \in \mathbb{R}^N : |x| = A_{j_0} \}$, then the ground states of (1.11) is unique up to rotations. It also deserves to finger out that in [27], Luo, etc. investigated the existence of normalized multiple-peak solutions for the single Schrödinger equation, where the potential is also assumed to attain its minimum at a $N-1$-dimensional $C^4$ manifold. Especially, if the potential satisfies some non-degenerate assumptions near the concentration point and does not preserve any symmetry in $\mathbb{R}^N$, they verified the normalized multiple-peak solution is unique. Unfortunately, the ring-shaped potential is radially symmetric and does not satisfy the assumptions of [27]. Therefore, the arguments of [27] is not applicable for our case.
This paper is organized as follows. In Section 2, we shall first establish the local Pohozaev identity in Lemma 2.1 for nontrivial solutions of (1.1). Then, we finish the proof of Theorem 1.1 in Subsection 2.1. In Section 3, we give refined concentration of ground states of (1.1) under ring-shaped potentials and finish the proof of Theorem 1.2. In Section 4, we first prove in Lemma 4.2 that the ground state of (1.1) must be cylindrically symmetric provided the potential is ring-shaped, drive the uniqueness (up to rotations) of ground states, which completes the proof of Theorem 1.3.

2 Uniqueness of single peak solutions for polynomial potentials

In this section, we intend to prove Theorem 1.1, i.e., the uniqueness of single peak solutions of (1.1) under polynomial potentials. For this purpose, we first establish the local Pohozaev identity in the following lemma.

**Lemma 2.1.** Let \( \bar{u}_\varepsilon(x) = (u_{1\varepsilon}, u_{2\varepsilon}) \) be a nontrivial solution of (1.1), then for any \( \Omega \subset \mathbb{R}^N \), we have for \( j = 1, 2, \ldots, N \),

\[
2 \sum_{i=1}^{2} \int_{\Omega} \frac{\partial V_i(x)}{\partial x_j} u_{i\varepsilon}^2(x) dx = \sum_{i=1}^{2} \int_{\partial \Omega} \left( V_i(x) u_{i\varepsilon}^2(x) - \frac{a_i}{2} u_{i\varepsilon}^4 \right) \nu_j dS - \beta \int_{\partial \Omega} u_{1\varepsilon} u_{2\varepsilon}^2 dS \\
+ \varepsilon^2 \sum_{i=1}^{2} \int_{\partial \Omega} \left( |\nabla u_{i\varepsilon}|^2 \nu_j - 2 \frac{\partial u_{i\varepsilon}}{\partial \nu} \frac{\partial u_{i\varepsilon}}{\partial x_j} \right) dS,
\]

(2.1)

where \( \nu = (\nu_1, \nu_2, \ldots, \nu_N) \) denotes the outward unit normal of \( \partial \Omega \).

**Proof.** Multiply the first equation of (1.1) by \( \frac{\partial u_{1\varepsilon}}{\partial x_j} \) and integrate over \( \Omega \), we then have

\[
-\varepsilon^2 \int_{\Omega} \frac{\partial u_{1\varepsilon}}{\partial x_j} \Delta u_{1\varepsilon} + \int_{\Omega} V_1(x) \frac{\partial u_{1\varepsilon}}{\partial x_j} u_{1\varepsilon} = a_1 \int_{\Omega} \frac{\partial u_{1\varepsilon}}{\partial x_j} u_{1\varepsilon}^3 + \beta \int_{\Omega} \frac{\partial u_{1\varepsilon}}{\partial x_j} u_{1\varepsilon} u_{2\varepsilon} \\
= \frac{a_1}{4} \int_{\partial \Omega} u_{1\varepsilon}^4 \nu_j dS + \beta \int_{\Omega} \frac{\partial u_{1\varepsilon}}{\partial x_j} u_{1\varepsilon} u_{2\varepsilon}.
\]

(2.2)

Note that

\[
- \int_{\Omega} \frac{\partial u_{1\varepsilon}}{\partial x_j} \Delta u_{1\varepsilon} = - \int_{\partial \Omega} \frac{\partial u_{1\varepsilon}}{\partial x_j} \frac{\partial u_{1\varepsilon}}{\partial \nu} dS + \frac{1}{2} \int_{\partial \Omega} |\nabla u_{1\varepsilon}|^2 \nu_j dS,
\]

and

\[
\int_{\Omega} V_1(x) \frac{\partial u_{1\varepsilon}}{\partial x_j} u_{1\varepsilon} = \frac{1}{2} \int_{\partial \Omega} V_1(x) u_{1\varepsilon}^2 \nu_j dS - \frac{1}{2} \int_{\Omega} \frac{\partial V_1(x)}{\partial x_j} u_{1\varepsilon}^2.
\]

We then derive from (2.2) that

\[
\int_{\Omega} \frac{\partial V_1(x)}{\partial x_j} u_{1\varepsilon}^2 + \beta \int_{\Omega} \frac{\partial u_{1\varepsilon}^2}{\partial x_j} u_{2\varepsilon} = -2\varepsilon^2 \int_{\Omega} \frac{\partial u_{1\varepsilon}}{\partial x_j} \frac{\partial u_{1\varepsilon}}{\partial \nu} dS + \varepsilon^2 \int_{\partial \Omega} |\nabla u_{1\varepsilon}|^2 \nu_j dS \\
+ \int_{\partial \Omega} V_1(x) u_{1\varepsilon}^2 \nu_j dS - \frac{a_1}{2} \int_{\partial \Omega} u_{1\varepsilon}^4 \nu_j dS.
\]

(2.3)
Similarly, we derive from the second equation of (1.1) that
\[
\int_{\Omega} \frac{\partial V_2(x)}{\partial x_j} u_{2\varepsilon}^2 + \beta \int_{\Omega} \frac{\partial u_{2\varepsilon}^2}{\partial x_j} u_{2\varepsilon}^2 dx = -2\varepsilon^2 \int_{\partial \Omega} \frac{\partial u_{2\varepsilon}}{\partial x_j} \frac{\partial u_{2\varepsilon}}{\partial \nu} dS + \varepsilon^2 \int_{\partial \Omega} |\nabla u_{2\varepsilon}|^2 \nu_j dS + \int_{\Omega} V_2(x) u_{2\varepsilon}^2 \nu_j dS - \frac{a_2}{2} \int_{\partial \Omega} u_{2\varepsilon}^2 \nu_j dS.
\]  
(2.4)

Note that
\[
\int_{\Omega} \frac{\partial u_{2\varepsilon}^2}{\partial x_j} u_{1\varepsilon} dx = -\int_{\Omega} \frac{\partial u_{1\varepsilon}^2}{\partial x_j} u_{2\varepsilon} dx + \int_{\partial \Omega} u_{2\varepsilon}^2 \nu_j dS.
\]
This together with (2.3) and (2.4) gives (2.1). \(\square\)

We next derive the following analytic properties for nonnegative solutions of (1.1).

**Lemma 2.2.** Suppose that \(V_i(x) \in C^1(\mathbb{R}^N)\) satisfies (V) for \(i = 1, 2\). Let \((u_{1\varepsilon}, u_{2\varepsilon})\) be a positive solution of (1.1) concentrating on \(z\) and denote the unique maximum point of \(u_{1\varepsilon} + u_{2\varepsilon}\) by \(x_{\varepsilon}\). Then, up to subsequence,

(i) \(\bar{u}_{i\varepsilon}(x) := u_{i\varepsilon}(\varepsilon x + x_{\varepsilon}) \to w_i(x)\) uniformly in \(\mathbb{R}^N\),

where \(w_i(x)\) is given by (1.4).

(ii) There exist \(R > 0\) and \(C > 0\) independent of \(\varepsilon\) such that

\[\bar{u}_{i\varepsilon}(x) \leq Ce^{-\frac{|z|}{2\varepsilon}} \text{ if } |x| > R, \ i = 1, 2,\]

and

\[|\nabla \bar{u}_{i\varepsilon}(x)| \leq Ce^{-\frac{|z|}{2\varepsilon}} \text{ if } R < |x| < \frac{R}{\varepsilon}, \ i = 1, 2.\]

(iii) If \(V_i(x) (i = 1, 2, )\) satisfies (1.11) additionally, then

\[\lim_{\varepsilon \to 0^+} |x_{\varepsilon} - z|/\varepsilon = 0.\]

**Proof.** (2.5) and (2.6) can be proved similarly as (4.2) and (4.6) in [15], so we omit their proof here. In view of \(x_{\varepsilon} \to z\) as \(\varepsilon \to 0^+\), we deduce from (2.6) that

\[|V_i(\varepsilon x + x_{\varepsilon}) \bar{u}_{i\varepsilon}(x)| \leq Ce^{-\frac{|z|}{2\varepsilon}} \text{ for } R < |x| < \frac{R}{\varepsilon},\]

where \(C > 0\) is independent of \(\varepsilon\). Note that \(\bar{u}_{1\varepsilon}\) satisfies

\[-\Delta \bar{u}_{1\varepsilon} + V_i(\varepsilon x + x_{\varepsilon}) \bar{u}_{1\varepsilon} = a_1 \bar{u}_{1\varepsilon}^3 + \beta \bar{u}_{2\varepsilon}^2 \bar{u}_{1\varepsilon} \text{ in } \mathbb{R}^N.\]

Applying the elliptic estimates (see (3.15) in [24]) to above equation, it then follows from (2.6) and (2.9) that

\[|\nabla \bar{u}_{1\varepsilon}(x)| \leq Ce^{-\frac{|z|}{2\varepsilon}} \text{ for } R < |x| < \frac{R}{\varepsilon}.\]

The estimate for \(|\nabla \bar{u}_{2\varepsilon}|\) can be proved similarly.

We finally prove (2.8). We only give the proof for the case of \(p_1 < p_2\), while the case of \(p_1 \geq p_2\) can be obtained in a similar way. Taking \(\Omega = B_{r_0}(x_{\varepsilon})\), it then follows from Lemma 2.1 that, for any \(j = 1, 2, \ldots, N\),
\[
\sum_{i=1}^{2} \int_{B_{\rho_i}(x)} \frac{\partial V_i(x)}{\partial x_j} u_{ie}^2(x) \, dx = \sum_{i=1}^{2} \int_{\partial B_{\rho_i}(x)} \left( V_i(x) u_{ie}^2(x) - \frac{a_i}{2} u_{ie}^4 \right) \nu_j \, dS
\]

\[
- \beta \int_{\partial B_{\rho_i}(x)} u_{ie}^4 u_{je}^2 \, dS + \varepsilon^2 \sum_{i=1}^{2} \int_{\partial B_{\rho_i}(x)} \left( |\nabla u_{ie}|^2 \nu_j - 2 \frac{\partial u_{ie}}{\partial \nu} \frac{\partial u_{ie}}{\partial x_j} \right) \, dS.
\]

(2.10)

Note that \( x_\varepsilon \xrightarrow{\varepsilon \to 0^+} z \), we then derive from (2.6) and (2.7) that

\[
e^{-N} \cdot \text{RHS of (2.10)}
\]

\[
= \sum_{i=1}^{2} \int_{B_{\rho_i}(0)} \left( V_i(\varepsilon x + x_\varepsilon) u_{ie}^2 - \frac{a_i}{2} u_{ie}^4 \right) \nu_j \, dS - \beta \int_{\partial B_{\rho_i}(0)} \bar{u}_{ie}^2 \bar{u}_{je}^2 \, dS
\]

\[
+ \sum_{i=1}^{2} \int_{\partial B_{\rho_i}(0)} \left( |\nabla \bar{u}_{ie}|^2 \nu_j - 2 \frac{\partial \bar{u}_{ie}}{\partial \nu} \frac{\partial \bar{u}_{ie}}{\partial x_j} \right) \, dS
\]

\[
= O(e^{-N}).
\]

On the other hand, it follows from (1.11) that

\[
e^{-N} \cdot \text{LHS of (2.10)} = \sum_{i=1}^{2} \int_{B_{\rho_i}(0)} \frac{\partial V_i(\varepsilon x + x_\varepsilon)}{\partial x_j} \bar{u}_{ie}^2(x) \, dx
\]

\[
= \sum_{i=1}^{2} \int_{B_{\rho_i}(0)} \left[ m_i p_i \varepsilon x + x_\varepsilon - z \right]^{p_i-2} \varepsilon x + x_\varepsilon - z \right] j + R_{ij}(\varepsilon x + x_\varepsilon - z) \right] \bar{u}_{ie}^2(x) \, dx
\]

\[
= \sum_{i=1}^{2} \int_{B_{\rho_i}(0)} m_i p_i \varepsilon x + x_\varepsilon - z \right]^{p_i-2} \varepsilon x + x_\varepsilon - z \right] j \bar{u}_{ie}^2(x) \, dx + O((|x_\varepsilon - z| + \varepsilon)^{\min\{q_1,q_2\}}).
\]

(2.12)

We claim that

\[
|x_\varepsilon - z|/\varepsilon \text{ is uniformly bounded as } \varepsilon \to 0^+.
\]

(2.13)

For otherwise, if (2.13) is false, we can extract a subsequence such that

\[
\lim_{\varepsilon \to 0^+} \frac{|x_\varepsilon - z|}{\varepsilon} = +\infty, \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \frac{x_\varepsilon - z}{|x_\varepsilon - z|} = \xi \text{ for some } \xi \in S^{N-1}.
\]

(2.14)

Since \( p_1 < p_2 \), it then follows from (2.10),-- (2.12) and (2.14), that

\[
\left| \int_{B_{\rho_i}(0)} \frac{\varepsilon x + x_\varepsilon - z}{|x_\varepsilon - z|}^{p_1-2} \left( \frac{\varepsilon x + x_\varepsilon - z}{|x_\varepsilon - z|} \right) \bar{u}_{ie}^2(x) \, dx \right|
\]

\[
= \varepsilon^{-1} |x_\varepsilon - z|^{1-p_1} O(e^{-N}) + O(|x_\varepsilon - z|^{\min\{q_1,q_2\}-p_1+1}) + O(|x_\varepsilon - z|^{p_2-p_1}).
\]

(2.15)

Letting \( \varepsilon \to 0^+ \), it then follows from (2.14) and (2.15) that

\[
\xi_j \int_{\mathbb{R}^N} u_j^2(x) \, dx = 0, \ j = 1, 2, \ldots, N.
\]
This indicates that ξ = 0, which however contradicts (2.14). (2.13) is thus proved.

From (2.13), we see that, up to a subsequence, \( \lim_{\varepsilon \to 0^+} \frac{x - z}{\varepsilon} = y \in \mathbb{R}^N \). Proceeding the similar argument as that of (2.15), we deduce that

\[
\left| \int_{B_{\varepsilon x}(0)} |x + \frac{x - z}{\varepsilon}| p - 2 (x + \frac{x - z}{\varepsilon}) \bar{u}_i(x) dx \right| = \varepsilon^{p - 1 - p_1} O(\varepsilon^{-\frac{N}{2}}) + O(\varepsilon^{\min(p_1, p_2) - p_1} + O(\varepsilon^{p_2 - p_1})).
\]

Letting \( \varepsilon \to 0^+ \), it follows that

\[
\int_{\mathbb{R}^N} \left| x + \frac{x - y}{\varepsilon} |p - 2| (x + \frac{x - y}{\varepsilon}) w_i(x) dx = 0, \ j = 1, 2, \ldots, N.
\]

Noting that \( w_1(x) = w_1(|x|) \) is strictly decreasing in \( |x| \), one can easily deduce from above that \( y = 0 \), and therefore (2.15) holds.

### 2.1 Proof of Theorem 1.1

This subsection is devoted to the proof of Theorem 1.1 on the uniqueness of positive solutions for the case of \( \beta > \max\{a_1, a_2\} \). We first note from [7, Lemma 2.2 and Theorem 3.1] that, the positive solution \((w_1, w_2)\) of (1.3) is non-degenerate in the sense that, the solution space of the linearized system for (1.3) about \((w_1, w_2)\) satisfying

\[
\begin{align*}
L_1(\phi_1, \phi_2) := & \Delta \phi_1 - \mu \phi_1 + 3 a_1 w_1^2 \phi_1 + \beta w_2^2 \phi_1 + 2 \beta w_1 w_2 \phi_2 = 0 \text{ in } \mathbb{R}^N, \\
L_2(\phi_2, \phi_1) := & \Delta \phi_2 - \mu \phi_2 + 3 a_2 w_2^2 \phi_2 + \beta w_1^2 \phi_2 + 2 \beta w_1 w_2 \phi_1 = 0 \text{ in } \mathbb{R}^N,
\end{align*}
\]

is exactly \( N \)-dimensional, namely,

\[
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix} = \sum_{j=1}^{N} b_j \begin{pmatrix} \frac{\partial w_i}{\partial x_j} \\ x_j \end{pmatrix}
\]  

for some constants \( b_j \). Assume that \((u_{1\varepsilon}, u_{2\varepsilon})\) and \((v_{1\varepsilon}, v_{2\varepsilon})\) are two different positive solutions of (1.1) concentrating on \( z \). Let \( x_\varepsilon \) and \( y_\varepsilon \) be the unique maximum point of \( u_{1\varepsilon} + u_{2\varepsilon} \) and \( v_{1\varepsilon} + v_{2\varepsilon} \), respectively. Define

\[
\bar{u}_{i\varepsilon}(x) := u_{i\varepsilon}(\varepsilon x + x_\varepsilon) \text{ and } \bar{v}_{i\varepsilon}(x) := v_{i\varepsilon}(\varepsilon x + x_\varepsilon), \ i = 1, 2.
\]

It then follows from Lemma 2.2 that

\[
\bar{u}_{i\varepsilon}(x), \bar{v}_{i\varepsilon}(x) \xrightarrow{\varepsilon \to 0^+} w_i(x) := \sqrt{\gamma_i} u_i(\sqrt{\varepsilon}x) \text{ uniformly in } \mathbb{R}^N, \ i = 1, 2.
\]

From (1.1) and (2.14), we see that \((\bar{u}_{1\varepsilon}, \bar{u}_{2\varepsilon})\) and \((\bar{v}_{1\varepsilon}, \bar{v}_{2\varepsilon})\) both satisfy the system

\[
\begin{align*}
- \Delta u_1 + V_1(\varepsilon x + x_\varepsilon) u_1 &= a_1 u_1^3 + \beta u_2^2 u_1 \text{ in } \mathbb{R}^N, \\
- \Delta u_2 + V_2(\varepsilon x + x_\varepsilon) u_2 &= a_2 u_2^3 + \beta u_1^2 u_2 \text{ in } \mathbb{R}^N.
\end{align*}
\]

**Lemma 2.3.** There exist \( C_1 > 0 \) and \( C_2 > 0 \) independent of \( \varepsilon \), such that

\[
C_1 \| \bar{v}_{2\varepsilon} - \bar{u}_{2\varepsilon} \|_{L^\infty(\mathbb{R}^N)} \leq \| \bar{v}_{1\varepsilon} - \bar{u}_{1\varepsilon} \|_{L^\infty(\mathbb{R}^N)} \leq C_2 \| \bar{v}_{2\varepsilon} - \bar{u}_{2\varepsilon} \|_{L^\infty(\mathbb{R}^N)} \text{ as } \varepsilon \to 0^+.
\]
From the first equation of (2.19), we have

\[
\frac{\partial v}{\partial t} - \Delta v + V_1(\varepsilon x + x_\varepsilon)(v_\varepsilon - \bar{u}_\varepsilon) = a_1(v_\varepsilon^3 - \bar{u}_\varepsilon^3) + \beta \left[ \frac{\partial^2 v}{\partial x_\varepsilon^2}(v_\varepsilon - \bar{u}_\varepsilon) + \frac{v_\varepsilon^2 - \bar{u}_\varepsilon^2}{|v_\varepsilon - \bar{u}_\varepsilon|} \right] \quad \text{in } \mathbb{R}^N.
\]

Set

\[
\zeta_\varepsilon(x) = \frac{v_\varepsilon(x) - \bar{u}_\varepsilon(x)}{\|v_\varepsilon - \bar{u}_\varepsilon\|_{L^\infty(\mathbb{R}^N)}},
\]

It then yields from (2.22) that

\[
- \Delta \zeta_\varepsilon + V_1(\varepsilon x + x_\varepsilon)\zeta_\varepsilon = a_1\left(\frac{v_\varepsilon^3}{v_\varepsilon^2} + v_\varepsilon \bar{u}_\varepsilon + \bar{u}_\varepsilon^2\right)\zeta_\varepsilon \\
+ \beta\left[ \frac{\partial^2 \zeta_\varepsilon}{\partial x_\varepsilon^2} + \frac{v_\varepsilon^2 - \bar{u}_\varepsilon^2}{\|v_\varepsilon - \bar{u}_\varepsilon\|_{L^\infty(\mathbb{R}^2)}} \right] \quad \text{in } \mathbb{R}^N.
\]

Note that \(\|\zeta_\varepsilon\|_{L^\infty(\mathbb{R}^N)} \leq 1\), one can apply the standard elliptic regularity theory to derive from (2.24) and (2.21) that, there exists \(C > 0\) independent of \(\varepsilon\), such that \(\|\zeta_\varepsilon\|_{C^{1,\alpha}(\mathbb{R}^N)} \leq C\) for some \(\alpha \in (0, 1)\). This indicates that, up to subsequence, \(\zeta_\varepsilon \xrightarrow{\varepsilon \to 0^+} \tilde{\zeta}(x)\) for some \(\tilde{\zeta}(x)\). From (2.24) we see that \(\tilde{\zeta}\) satisfies

\[
\left\{ - \Delta + \mu - (3a_1 w_1^2 + \beta w_2^2) \right\} \tilde{\zeta}(x) = 0 \quad \text{in } \mathbb{R}^N.
\]

It then follows from (1.1) that

\[
\left\{ - \Delta + 1 - (3a_1 \gamma_1 + \beta \gamma_2)w^2(x) \right\} \tilde{\zeta}\left(\frac{x}{\sqrt{R}}\right) = 0 \quad \text{in } \mathbb{R}^N.
\]

Noting that \((3a_1 \gamma_1 + \beta \gamma_2) \in (1, 3)\), we thus conclude from (2.25) and [7, Lemma 2.2] that \(\tilde{\zeta} \equiv 0\) in \(\mathbb{R}^N\).

On the other hand, let \(z_\varepsilon\) be a point satisfying \(|\zeta_\varepsilon(z_\varepsilon)| = |\zeta_\varepsilon|_{L^\infty(\mathbb{R}^N)} = 1\). Since both \(\bar{u}_\varepsilon\) and \(v_\varepsilon\) decay exponentially as \(|x| \to \infty\), it then follows from (2.24) and standard elliptic regularity theory that \(|z_\varepsilon| \leq C\) uniformly in \(\varepsilon\). Consequently, we deduce that \(\zeta_\varepsilon \to \tilde{\zeta} \neq 0\) uniformly on \(\mathbb{R}^N\), which leads to a contradiction. Hence (2.21) is false and the inequality on the right hand side of (2.20) holds true.

Applying the same argument as above, one can deduce from the second equation of (2.19) that the inequality on the left hand side of (2.20) also holds. The proof of Lemma 2.3 is completed.

**Lemma 2.4.** Assume that \(a_1, a_2 \in (0, a^*)\), and let

\[
\xi_\varepsilon(x) = \frac{v_\varepsilon(x) - u_\varepsilon(x)}{\|v_\varepsilon - u_\varepsilon\|_{L^\infty(\mathbb{R}^2)}^2}, \quad i = 1, 2.
\]

Then for any fixed \(x_0 \in \mathbb{R}^N\) and \(\delta > 0\), there exits \(\delta(\varepsilon) \in (\delta, 2\delta)\) such that

\[
\int_{\partial B_\delta(x_0)} \left( \varepsilon^2 |\nabla \xi_\varepsilon|^2 + V_1(x)|\xi_\varepsilon|^2 \right) dS = O(\varepsilon^N) \quad \text{as } \varepsilon \to 0^+, \quad i = 1, 2.
\]

\[\Box\]
Proof. We first note from Lemma 2.3 that there exists $C > 0$ independent of $\varepsilon$ such that

$$0 \leq |\xi_{1\varepsilon}(x)|, |\xi_{2\varepsilon}(x)| \leq C < \infty \text{ and } |\xi_{1\varepsilon}(x)\xi_{2\varepsilon}(x)| \leq 1 \text{ in } \mathbb{R}^N. \quad (2.28)$$

Recalling from (1.1), we see that $(\xi_{1\varepsilon}, \xi_{2\varepsilon})$ satisfies

$$
\begin{align*}
&-\varepsilon^2 \Delta \xi_{1\varepsilon} + V_1(x)\xi_{1\varepsilon} = a_1 \left( v_{1\varepsilon}^2 + v_1 u_{1\varepsilon} + u_{1\varepsilon}^2 \right) \xi_{1\varepsilon} + \beta \left[ u_{2\varepsilon}^2 \xi_{1\varepsilon} + v_1 (v_{2\varepsilon} + u_{2\varepsilon}) \xi_{2\varepsilon} \right] \\
&-\varepsilon^2 \Delta \xi_{2\varepsilon} + V_2(x)\xi_{2\varepsilon} = a_2 \left( v_{2\varepsilon}^2 + v_2 u_{2\varepsilon} + u_{2\varepsilon}^2 \right) \xi_{2\varepsilon} + \beta \left[ u_{1\varepsilon}^2 \xi_{2\varepsilon} + v_2 (v_{1\varepsilon} + u_{1\varepsilon}) \xi_{1\varepsilon} \right]
\end{align*}
$$

Multiplying the first equation of above by $\xi_{1\varepsilon}$ and integrating over $\mathbb{R}^N$, we obtain that

$$
\varepsilon^2 \int_{\mathbb{R}^N} |\nabla \xi_{1\varepsilon}|^2 + \int_{\mathbb{R}^N} V_1(x)|\xi_{1\varepsilon}|^2 \\
= a_1 \int_{\mathbb{R}^N} \left( v_{1\varepsilon}^2 + v_1 u_{1\varepsilon} + u_{1\varepsilon}^2 \right) |\xi_{1\varepsilon}|^2 + \beta \int_{\mathbb{R}^N} \left[ u_{2\varepsilon}^2 |\xi_{1\varepsilon}|^2 + v_1 (v_{2\varepsilon} + u_{2\varepsilon}) |\xi_{2\varepsilon}| \right] \\
\leq C \int_{\mathbb{R}^N} \left( v_{1\varepsilon}^2 + v_1 u_{1\varepsilon} + u_{1\varepsilon}^2 \right) + C \int_{\mathbb{R}^N} \left[ u_{2\varepsilon}^2 + v_1 (v_{2\varepsilon} + u_{2\varepsilon}) \right] \\
\leq C \varepsilon^N \int_{\mathbb{R}^N} \left( \bar{v}_{1\varepsilon}^2 + \bar{v}_{1\varepsilon} \bar{u}_{1\varepsilon} + \bar{u}_{1\varepsilon}^2 \right) + C \varepsilon^N \int_{\mathbb{R}^N} \left[ \bar{u}_{2\varepsilon}^2 + \bar{v}_{1\varepsilon} (\bar{v}_{2\varepsilon} + \bar{u}_{2\varepsilon}) \right] \\
\leq C \varepsilon^N \text{ as } \varepsilon \to 0^+.
$$

In view of (2.28) and the exponentially decay estimate of $\bar{u}_{1\varepsilon}$ and $\bar{u}_{2\varepsilon}$ in (2.6), similar to Lemma A.4 of [26], we can apply the mean value theorem of integrals to derive that, for any $x_0 \in \mathbb{R}^N$ and $\delta > 0$, there exits $\delta = \delta(\varepsilon) \in (\delta, 2\delta)$ such that

$$
\int_{\partial B_\delta(x_0)} \left( \varepsilon^2 |\nabla \xi_{1\varepsilon}|^2 + V_1(x)|\xi_{1\varepsilon}|^2 \right) dS \leq C \delta^{-1} \varepsilon^N \text{ as } \varepsilon \to 0^+.
$$

This means that (2.27) holds for $i = 1$. One can prove (2.27) for $i = 2$ in a similar way.

Based on above lemmas, we are ready to proving Theorem 1.1. We only give the detailed proof for the case of $N = 3$, since the cases of $N = 1, 2$, can be derived similarly.

Proof of Theorem 1.1. Step 1. Let

$$\tilde{\xi}_{ie}(x) = \xi_{ie}(\varepsilon x + x_0), \quad i = 1, 2, \quad (2.29)$$

where $\xi_{ie}$ is defined by (2.26). We claim that, up to a subsequence,

$$(\tilde{\xi}_{1\varepsilon}, \tilde{\xi}_{2\varepsilon}) \to (\tilde{\xi}_{10}, \tilde{\xi}_{20}) \text{ in } C_{loc}^1(\mathbb{R}^3) \text{ as } \varepsilon \to 0^+,$$

where $(\tilde{\xi}_{10}, \tilde{\xi}_{20})$ satisfies

$$
\begin{pmatrix}
\tilde{\xi}_{10} \\
\tilde{\xi}_{20}
\end{pmatrix} = \sum_{j=1}^{3} d_j \begin{pmatrix}
\frac{\partial \omega_1}{\partial x_j} \\
\frac{\partial \omega_2}{\partial x_j}
\end{pmatrix} \quad (2.30)
$$

for some constants $d_j$ with $j = 1, 2, 3$.

Following (2.19), one can actually check that $(\tilde{\xi}_{1\varepsilon}, \tilde{\xi}_{2\varepsilon})$ satisfies

$$
\begin{align*}
&-\Delta \tilde{\xi}_{1\varepsilon} + V_1(\varepsilon x + x_0)\tilde{\xi}_{1\varepsilon} = a_1 \left( \bar{v}_{1\varepsilon}^2 + \bar{v}_{1\varepsilon} \bar{u}_{1\varepsilon} + \bar{u}_{1\varepsilon}^2 \right) \tilde{\xi}_{1\varepsilon} \\
&\quad + \beta \left[ u_{2\varepsilon}^2 \tilde{\xi}_{1\varepsilon} + \bar{v}_{1\varepsilon} (v_{2\varepsilon} + \bar{u}_{2\varepsilon}) \tilde{\xi}_{2\varepsilon} \right] \text{ in } \mathbb{R}^3,
\end{align*}
$$

$$
\begin{align*}
&-\Delta \tilde{\xi}_{2\varepsilon} + V_2(\varepsilon x + x_0)\tilde{\xi}_{2\varepsilon} = a_2 \left( \bar{v}_{2\varepsilon}^2 + \bar{v}_{2\varepsilon} \bar{u}_{2\varepsilon} + \bar{u}_{2\varepsilon}^2 \right) \tilde{\xi}_{2\varepsilon} \\
&\quad + \beta \left[ u_{1\varepsilon}^2 \tilde{\xi}_{2\varepsilon} + \bar{v}_{2\varepsilon} (v_{1\varepsilon} + \bar{u}_{1\varepsilon}) \tilde{\xi}_{1\varepsilon} \right] \text{ in } \mathbb{R}^3.
\end{align*}
$$

(2.31)
From (2.28) and (2.29) we see that 
\[ \xi_{1\varepsilon}(x) \text{ and } \xi_{2\varepsilon}(x) \text{ are both bounded uniformly in } \mathbb{R}^3. \] (2.32)

The standard elliptic regularity theory then implies that, there exists \( C > 0 \) independent of \( \varepsilon \) such that \( \| \xi_{1\varepsilon} \|_{C_{\text{loc}}^{1,\alpha}(\mathbb{R}^3)} \leq C \) for some \( \alpha \in (0, 1) \). Therefore, up to a subsequence, it yields from (2.31) that \((\xi_{1\varepsilon}, \xi_{2\varepsilon}) ) \xrightarrow{\varepsilon \to 0^+} (\xi_{10}, \xi_{20}) \) in \( C_{\text{loc}}^{1,\alpha}(\mathbb{R}^3) \), where \((\xi_{10}, \xi_{20}) \) satisfies

\[
\begin{align*}
-\Delta \xi_{10} + \mu \xi_{10} - 3a_1 w_2^2 \xi_{10} - \beta w_1 w_2 \xi_{10} &= 0 \quad \text{in } \mathbb{R}^3, \\
-\Delta \xi_{20} + \mu \xi_{20} - 3a_2 w_2^2 \xi_{20} - \beta w_1 w_2 \xi_{10} &= 0 \quad \text{in } \mathbb{R}^3.
\end{align*}
\]

This together with (2.16) indicates that \((\xi_{10}, \xi_{20}) \) satisfies (2.30) for some constants \( d_j \) with \( j = 1, 2, 3 \).

**Step 2.** The constants \( d_1 = d_2 = d_3 = 0 \) in (2.30), i.e., \( \xi_{10} = \xi_{20} = 0 \).

Let \( \Omega = B_\delta(x_\varepsilon) \) with \( \delta = \delta(\varepsilon) > 0 \) being given by Lemma 2.4. Applying Lemma 2.1 to \((u_{1\varepsilon}, u_{2\varepsilon}) \) and \((v_{1\varepsilon}, v_{2\varepsilon}) \) on \( \Omega = B_\delta(x_\varepsilon) \), respectively, one can easily derive that

\[
\int_{B_\delta(x_\varepsilon)} \left[ \frac{\partial V_1(x)}{\partial x_j}(v_{1\varepsilon} + u_{1\varepsilon})\xi_{1\varepsilon} + \frac{\partial V_2(x)}{\partial x_j}(v_{2\varepsilon} + u_{2\varepsilon})\xi_{2\varepsilon} \right] dx := \sum_{i=1}^{2} I^i_{\varepsilon} + J_{\varepsilon},
\] (2.33)

where we denote

\[
I^i_{\varepsilon} = -2\varepsilon^2 \int_{\partial B_\delta(x_\varepsilon)} \left[ \frac{\partial v_{ie}}{\partial x_j} \frac{\partial \xi_{ie}}{\partial \nu} + \frac{\partial \xi_{ie}}{\partial x_j} \frac{\partial u_{ie}}{\partial \nu} \right] dS + \varepsilon^2 \int_{\partial B_\delta(x_\varepsilon)} \nabla \xi_{ie} \cdot \nabla (v_{ie} + u_{ie}) \nu_j dS + \varepsilon^2 \int_{\partial B_\delta(x_\varepsilon)} V_i(x) (v_{ie} + u_{ie}) \xi_{ie} \nu_j dS
\]

and

\[
J_{\varepsilon} = -\beta \int_{\partial B_\delta(x_\varepsilon)} [v_{1\varepsilon}^2 (v_{2\varepsilon} + u_{2\varepsilon}) \xi_{2\varepsilon} + u_{2\varepsilon}^2 (v_{1\varepsilon} + u_{1\varepsilon}) \xi_{1\varepsilon}] dS.
\]

In view of the fact that \( \nabla \xi_{ie} \) satisfies the exponential decay (2.4), we deduce from Lemma 2.4 that

\[
\varepsilon^2 \int_{\partial B_\delta(x_\varepsilon)} \left[ \frac{\partial v_{ie}}{\partial x_j} \frac{\partial \xi_{ie}}{\partial \nu} \right] dS \leq \varepsilon \left( \int_{\partial B_\delta(x_\varepsilon)} \left| \frac{\partial v_{ie}}{\partial x_j} \right|^2 dS \right)^{1/2} \left( \int_{\partial B_\delta(x_\varepsilon)} \left| \frac{\partial \xi_{ie}}{\partial \nu} \right|^2 dS \right)^{1/2} \leq C \varepsilon^{1 + 4\alpha} + \epsilon \frac{C}{\varepsilon} \text{ as } \varepsilon \to 0^+.
\]

Similarly, employing (2.15), (2.7) and Lemma 2.4 again, we can prove that other terms of \( I^i_{\varepsilon} \) and \( J_{\varepsilon} \) can also be controlled by the order \( o(e^{-\frac{C\alpha}{\varepsilon}}) \). Therefore, we conclude from above that

\[
\sum_{i=1}^{2} I^i_{\varepsilon} + J_{\varepsilon} = o(e^{-\frac{C\alpha}{\varepsilon}}) \text{ as } \varepsilon \to 0^+.
\] (2.34)
It then follows from (1.11), (2.33) and (2.34) that,

\[
o(\varepsilon^{-\frac{C\delta}{\varepsilon}}) = \sum_{i=1}^{2} \int_{B_{\frac{\varepsilon}{2}}(0)} \frac{\partial V_i(x)}{\partial x_j}(\bar{v}_{ie} + \bar{u}_{ie}) \xi_{ie} \, dx
\]

\[
= \varepsilon^3 \sum_{i=1}^{2} \int_{B_{\frac{\varepsilon}{2}}(0)} \frac{\partial V_i(\varepsilon x + (x_{\varepsilon} - z)/\varepsilon + z)}{\partial x_j}(\bar{v}_{ie} + \bar{u}_{ie}) \xi_{ie} \, dx
\]

\[
= \varepsilon^3 \sum_{i=1}^{2} \int_{B_{\frac{\varepsilon}{2}}(0)} \left\{ m_i p_i \varepsilon^{p_i-1} |x + \frac{x_{\varepsilon} - z}{\varepsilon}|^{p_i-2} (x + \frac{x_{\varepsilon} - z}{\varepsilon})_i (\bar{v}_{ie} + \bar{u}_{ie}) \xi_{ie}
\right.

\[+ R_{ij} (\varepsilon x + (x_{\varepsilon} - z)) (\bar{v}_{ie} + \bar{u}_{ie}) \xi_{ie}\}

\]

(2.35) \]

Moreover, we deduce from (1.11), (2.6), (2.8) and (2.32) that

\[
\left| \int_{B_{\frac{\varepsilon}{2}}(0)} R_{ij} (\varepsilon x + (x_{\varepsilon} - z)) (\bar{v}_{ie} + \bar{u}_{ie}) \xi_{ie} \, dx \right|
\]

\[
\leq C \varepsilon^h \int_{B_{\frac{\varepsilon}{2}}(0)} |x + \frac{x_{\varepsilon} - z}{\varepsilon}| q_i (\bar{v}_{ie} + \bar{u}_{ie}) \xi_{ie} \, dx \leq C \varepsilon^h,
\]

where \( q_i > p_i - 1 \) for \( i = 1, 2 \).

When \( p_1 = p_2 \), it then follows from (2.36) that

\[
o(1) = p_1 \int_{B_{\frac{\varepsilon}{2}}(0)} \left| x + \frac{x_{\varepsilon} - z}{\varepsilon} \right|^{p_1-2} (x + \frac{x_{\varepsilon} - z}{\varepsilon})_j \sum_{i=1}^{2} \left[ m_i (\bar{v}_{ie} + \bar{u}_{ie}) \xi_{ie} \right] \, dx.
\]

On the other hand, we deduce from (2.6) and (2.32) that

\[
o(e^{-\varepsilon\xi}) = \int_{B_{\frac{\varepsilon}{2}}(0)} \left| x + \frac{x_{\varepsilon} - z}{\varepsilon} \right|^{p_1-2} (x + \frac{x_{\varepsilon} - z}{\varepsilon})_j \sum_{i=1}^{2} \left[ m_i (\bar{v}_{ie} + \bar{u}_{ie}) \xi_{ie} \right] \, dx.
\]

Therefore,

\[
\int_{\mathbb{R}^{3}} |x + \frac{x_{\varepsilon} - z}{\varepsilon}|^{p_1-2} (x + \frac{x_{\varepsilon} - z}{\varepsilon})_j \sum_{i=1}^{2} \left[ m_i (\bar{v}_{ie} + \bar{u}_{ie}) \xi_{ie} \right] \, dx.
\]

(2.37)

Noting that \( w_i(x) = w_i(|x|), \frac{\partial w_i}{\partial x_i} = w'(|x|) \frac{x_i}{|x|} \) and \( w'(|x|) < 0 \). Setting \( \varepsilon \to 0^+ \), we then derive from (2.18), (2.8) and (2.37) that

\[
0 = \int_{\mathbb{R}^{3}} |x|^{p_1-2} x_j \left[ m_1 w_1 \tilde{\xi}_{10} + m_2 w_2 \tilde{\xi}_{20} \right] \, dx
\]

\[
= \int_{\mathbb{R}^{3}} |x|^{p_1-2} x_j \sum_{i=1}^{3} d_i (m_1 w_1 \frac{\partial w_1}{\partial x_i} + m_2 w_2 \frac{\partial w_2}{\partial x_i}) \, dx
\]

\[
= \frac{d_1}{3} \int_{\mathbb{R}^{3}} |x|^{p_1-1} (m_1 w_1 w'_1(|x|) + m_2 w_2 w'_2(|x|)) \, dx, \quad j = 1, 2, 3.
\]

This indicates that \( d_1 = d_2 = d_3 = 0 \). Similarly, if \( p_1 \neq p_2 \), we can apply the same arguments as above to derive that \( d_1 = d_2 = d_3 = 0 \). Consequently, we conclude that \( \tilde{\xi}_{10} = \tilde{\xi}_{20} = 0 \).
Step 3. $\xi_{10} = \tilde{\xi}_{20} = 0$ cannot occur. Let $(\bar{x}_\varepsilon, \bar{y}_\varepsilon)$ satisfy $|\tilde{\xi}_{1\varepsilon}(\bar{x}_\varepsilon)\tilde{\xi}_{2\varepsilon}(\bar{y}_\varepsilon)| = \|\tilde{\xi}_{1\varepsilon}\tilde{\xi}_{2\varepsilon}\|_{L^\infty(\mathbb{R}^3)} = 1$. By the exponential decay (2.6), one can easily deduce from (2.31) that $|\bar{x}_\varepsilon| \leq C$ and $|\bar{y}_\varepsilon| \leq C$ uniformly in $\varepsilon$. We thus conclude that $\xi_{1\varepsilon} \to \bar{\xi}_1 \neq 0$ uniformly on $\mathbb{R}^3$ as $\varepsilon \to 0^+$. This however contradicts to the fact that $\xi_{10} = \xi_{20} = 0$ on $\mathbb{R}^3$. We thus complete the proof of Theorem 1.1.

3 Refined Concentration of Ground States for Ring-shaped Potentials

In this section we intend to study the existence and limit behavior of ground states of (1.1) under ring-shaped potentials. For simplicity of notations, we denote

$$B(u_1, u_2) := \int_{\mathbb{R}^N} (a_1 |u_1|^4 + a_2 |u_2|^4 + 2\beta |u_1|^2 |u_2|^2) \, dx.$$  (3.1)

The Nehari manifold corresponds to equation (1.3) is defined as

$$\mathcal{N}^\mu := \{ (u_1, u_2) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \setminus (0,0) : \sum_{i=1}^2 \int_{\mathbb{R}^N} |\nabla u_i|^2 + \mu |u_i|^2 \, dx = B(u_1, u_2) \}.$$  (3.2)

Set

$$c^\mu := \frac{1}{4} \inf_{(u_1, u_2) \in \mathcal{N}^\mu} B(u_1, u_2) = \frac{1}{4} \inf_{(u_1, u_2) \in \mathcal{N}^\mu} \sum_{i=1}^2 \int_{\mathbb{R}^N} |\nabla u_i|^2 + \mu |u_i|^2 \, dx,$$  (3.3)

and

$$\bar{c}_1 = \inf_{(u_1, 0) \in \mathcal{N}^\mu} \frac{1}{4} B(u_1, 0), \quad \bar{c}_2 = \inf_{(0, u_2) \in \mathcal{N}^\mu} \frac{1}{4} B(0, u_2).$$  (3.4)

Then we have the following lemma, which addresses the relationship between $c^\mu$, $\bar{c}_1$ and $\bar{c}_2$.

**Lemma 3.1.** If $\beta > \max\{a_1, a_2\}$, then $c^\mu$ can be attained by $(w_1, w_2)$, where $(w_1, w_2)$ is the unique positive solution of (1.3) and is given by (1.4). Moreover, there holds that

$$c^\mu < \min\{\bar{c}_1, \bar{c}_2\},$$  (3.5)

and

$$c^\mu = \frac{1}{4} B(w_1, w_2) = \mu^{\frac{2-N}{2}} \left(a_1 \gamma_1^2 + a_2 \gamma_2^2 + 2\beta \gamma_1 \gamma_2\right) \int_{\mathbb{R}^N} |w|^4 \, dx.$$  (3.6)

**Proof.** One can easily prove that each $\bar{c}_i$ can be attained by $\sqrt{\frac{\mu^{\frac{2-N}{2}}}{a_i}} w(\sqrt{\mu x})$ $(i = 1, 2)$, with $w(x)$ being the unique positive solution of (1.5) and

$$\bar{c}_i = \frac{\mu^{\frac{2-N}{2}}}{4a_i} \int_{\mathbb{R}^N} |w|^4 \, dx, \quad i = 1, 2.$$  (3.7)

Using $\beta > \max\{a_1, a_2\}$ and noting that $(w_1, w_2) \in \mathcal{N}^\mu$, one can check that

$$c^\mu \leq \frac{1}{4} B(w_1, w_2) = \mu^{\frac{2-N}{2}} \left(a_1 \gamma_1^2 + a_2 \gamma_2^2 + 2\beta \gamma_1 \gamma_2\right) \int_{\mathbb{R}^N} |w|^4 \, dx < \min\{\bar{c}_1, \bar{c}_2\}.$$
This implies (3.5). As a consequence, there holds that
\[ c^u := \frac{1}{4} \inf_{(u_1, u_2) \in N^u, u_1, u_2 \neq 0} B(u_1, u_2). \]

It then follows from [35, Theorem 1] that (3.6) holds.

**Lemma 3.2.** Let \( c_\varepsilon \) be defined by (1.17). Then, as \( \varepsilon \to 0^+ \), there holds that
\[ 0 < c^u \leq c_\varepsilon / \varepsilon^N \leq c^u + o(1). \]

Moreover, if \( V_i(x) \) (\( i = 1, 2 \)) satisfies (1.15) and (1.16), we further have
\[ c_\varepsilon / \varepsilon^N \leq c^u + \frac{\bar{\lambda}_0}{2} \varepsilon^p_0 + o(\varepsilon^p_0), \]
where \( \bar{\lambda}_0 \) is given by (1.21).

**Proof.** Choose a cutoff function \( 0 \leq \varphi \in C^\infty_0(\mathbb{R}^2) \) such that \( \varphi(x) = 1 \) for \( |x| \leq 1 \), and \( \varphi(x) = 0 \) for \( |x| \geq 2 \). Set
\[ \tilde{w}_{i\varepsilon}(x) = w_i\left(\frac{|x - x_0|}{\varepsilon}\right)\varphi\left(\frac{|x - x_0|}{R}\right), \quad i = 1, 2, \quad (3.10) \]
where \( x_0 \in \mathcal{Z} \) and \( R > 0 \) will be determined later. Recalling that
\[ \|w\|_2^2 = \|\nabla w\|_2^2 = \frac{1}{2} \|w\|_4^4, \quad (3.11) \]
and it follows from [11, Proposition 4.1] that \( w(x) \) decays exponentially
\[ w(x), \ |\nabla w(x)| = O(|x|^{-\frac{1}{4}} e^{-|x|}) \] as \( |x| \to \infty. \quad (3.12) \]

Using (3.11) and (3.12), direct calculations show that
\[ \int_{\mathbb{R}^N} |\nabla \tilde{w}_{i\varepsilon}|^2 \, dx = \varepsilon^{N-2} \left[ \int_{\mathbb{R}^N} |\nabla w_i(x)|^2 \, dx + O(e^{-\sqrt{\lambda_R} \varepsilon}) \right], \]
\[ \int_{\mathbb{R}^N} |\tilde{w}_{i\varepsilon}|^4 \, dx = \varepsilon^N \left[ \int_{\mathbb{R}^N} |w_i(x)|^4 \, dx + O(e^{-\sqrt{\lambda_R} \varepsilon}) \right], \quad (3.13) \]
and
\[ \int_{\mathbb{R}^N} V_i(x)|\tilde{w}_{i\varepsilon}|^2 \, dx = \varepsilon^N \int_{\mathbb{R}^N} V_i(\varepsilon x + x_0)|w_i(x)|^2 \varphi^2(\varepsilon x / R) \, dx \]
\[ = \varepsilon^N \left( \mu \int_{\mathbb{R}^N} |w_i|^2 \, dx + o(1) \right) \quad (3.14) \]

Taking \( t_\varepsilon > 0 \) such that \( t_\varepsilon(\tilde{w}_{1\varepsilon}, \tilde{w}_{2\varepsilon}) \in N_\varepsilon \), where \( N_\varepsilon \) is given by (1.18). It then follows from (3.13) and (3.14) that \( \lim_{\varepsilon \to 0^+} t_\varepsilon = 1 \). Therefore, we recall from (1.4) and (3.6) that
\[ c_\varepsilon \leq J_\varepsilon(t_\varepsilon(\tilde{w}_{1\varepsilon}, \tilde{w}_{2\varepsilon})) = \frac{t_\varepsilon^4}{4} B(\tilde{w}_{1\varepsilon}, \tilde{w}_{2\varepsilon}) \]
\[ = \frac{\varepsilon^N}{4} \left( 1 + o(1) \right) \left[ B(w_1, w_2) + O(e^{-\sqrt{\lambda_R} \varepsilon}) \right] \]
\[ = (c^u + o(1)) \varepsilon^N. \quad (3.15) \]
On the other hand, for any $\delta > 0$, taking $(u_{1\varepsilon}, u_{2\varepsilon}) \in \mathcal{N}_\varepsilon$ such that

$$
\frac{1}{4} \sum_{i=1}^{2} \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u_{i\varepsilon}|^2 + V_i(x)|u_{i\varepsilon}|^2 \, dx \leq c_\varepsilon + \delta \varepsilon^N.
$$

From (1.13), we deduce that there exists $\bar{t}_\varepsilon \in (0, 1]$ such that $\bar{t}_\varepsilon(u_{1\varepsilon}, u_{2\varepsilon}) := \bar{t}_\varepsilon(u_{1\varepsilon}(\varepsilon x), u_{2\varepsilon}(\varepsilon x)) \in \mathcal{N}^\mu$. Thus,

$$
\varepsilon^\mu \leq \frac{\bar{t}_\varepsilon^2}{4} \sum_{i=1}^{2} \int_{\mathbb{R}^N} |\nabla u_{i\varepsilon}|^2 + \mu |u_{i\varepsilon}|^2 \, dx \leq \frac{\bar{t}_\varepsilon^2}{4} \sum_{i=1}^{2} \int_{\mathbb{R}^N} |\nabla u_{i\varepsilon}|^2 + V_i(\varepsilon x)|u_{i\varepsilon}|^2 \, dx
$$

$$
= \varepsilon^{-\mu} \sum_{i=1}^{2} \int_{\mathbb{R}^N} \varepsilon^2 |\nabla u_{i\varepsilon}|^2 + V_i(x)|u_{i\varepsilon}|^2 \, dx = \varepsilon^{-\mu} c_\varepsilon + \delta.
$$

Letting $\delta \to 0^+$, this together with (3.15) indicates (3.8).

If $V_i(x)$ satisfies (1.15) and (1.16), we choose some $x_0 \in Z_0$, where the set $Z_0$ is given by (1.22). Thus, $|x_0| = A\beta_0$ for some $1 \leq j_0 \leq \beta$. Taking $R = \frac{A}{2}$ with $r_0 > 0$ being given by (1.16), then,

$$
\int_{\mathbb{R}^N} V_i(x)|\bar{w}_{i\varepsilon}|^2 \, dx = \varepsilon^N \int_{\mathbb{R}^N} V_i(\varepsilon x + x_0)|w_i(x)|^2 \varphi^2(\varepsilon x/R) \, dx
$$

$$
= \varepsilon^N \bigg[ \mu \int_{\mathbb{R}^N} |w_i|^2 \, dx + b_{j_0}\varepsilon^{p_{j_0}} \int_{\mathbb{R}^N} |x_N|^{p_{j_0}} |w_i|^2 \, dx + o(\varepsilon^{p_{j_0}}) \bigg].
$$

Furthermore,

$$
t_{\varepsilon}^2 = \frac{\| (\bar{w}_{1\varepsilon}, \bar{w}_{2\varepsilon}) \|^2_{L^2}}{B(\bar{w}_{1\varepsilon}, \bar{w}_{2\varepsilon})} = \frac{\sum_{i=1}^{2} \int_{\mathbb{R}^N} (|\nabla w_i|^2 + \mu w_i^2) \, dx + \tilde{\lambda}_0 \varepsilon^{p_0} + o(\varepsilon^{p_0})}{B(w_1, w_2) + O(\varepsilon^{-\frac{\sqrt{\lambda}_0}{\varepsilon}})}
$$

$$
= 1 + \frac{\tilde{\lambda}_0 \varepsilon^{p_0}}{B(w_1, w_2)} + o(\varepsilon^{p_0}).
$$

Therefore, we have

$$
c_\varepsilon \leq J_\varepsilon(\bar{t}_\varepsilon(\bar{w}_{1\varepsilon}, \bar{w}_{2\varepsilon})) = \frac{t_{\varepsilon}^4}{4} B(\bar{w}_{1\varepsilon}, \bar{w}_{2\varepsilon})
$$

$$
= \varepsilon^N \left[ 1 + \frac{2\tilde{\lambda}_0 \varepsilon^{p_0}}{B(w_1, w_2)} + o(\varepsilon^{p_0}) \right] \left[ B(w_1, w_2) + O(\varepsilon^{-\frac{\sqrt{\lambda}_0}{\varepsilon}}) \right]
$$

$$
= \left[ e^\mu + \frac{\tilde{\lambda}_0 \varepsilon^{p_0}}{2} + o(\varepsilon^{p_0}) \right] \varepsilon^N.
$$

This finishes the proof of (3.9). \qed

**Lemma 3.3.** Let $V_i(x)$ and $V_2(x)$ satisfy (1.13) and (1.14), and $\beta > \max\{a_1, a_2\}$. Assume that (1.17) has a nonnegative ground state $(u_{1\varepsilon}, u_{2\varepsilon})$, and $u_{i\varepsilon}(x) \neq 0$ for both $i = 1, 2$. Then,

(i) $u_{1\varepsilon}(x) + u_{2\varepsilon}(x)$ admits at least one global maximum point, denoted it by $x_\varepsilon$. Furthermore, up to a subsequence, there holds that

$$
x_\varepsilon \to \bar{x}_0 \text{ for some } \bar{x}_0 \in Z \text{ as } \varepsilon \to 0^+.
$$
(ii) Define
\[ w_{i\varepsilon}(x) := u_{i\varepsilon}(\varepsilon x + x_\varepsilon) \geq 0, \]  
(3.20)
then
\[ \lim_{\varepsilon \to 0^+} w_{i\varepsilon}(x) = w_i(x) \text{ strongly in } H^1(\mathbb{R}^N), \quad i = 1, 2. \]  
(3.21)

**Proof.** (i). For any fixed \( \varepsilon > 0 \), we deduce from (1.11) that \( u_{i\varepsilon} \) satisfies
\[ -\varepsilon^2 \Delta u_{i\varepsilon} \leq c_{i\varepsilon}(x)u_{i\varepsilon} \text{ in } \mathbb{R}^N, \quad i = 1, 2, \]
where
\[ c_{i\varepsilon}(x) = a_1 u_{1\varepsilon}^2 + \beta u_{2\varepsilon}^2 \text{ and } c_{2\varepsilon}(x) = a_2 u_{2\varepsilon}^2 + \beta u_{1\varepsilon}^2. \]
Applying the De Giorgi–Nash–Moser theory (cf. [20, Theorem 4.1] or [12, Theorem 8.15]), we then see that
\[ u_{i\varepsilon}(x) \to 0 \text{ as } |x| \to \infty, \quad i = 1, 2. \]
This implies that \( u_{1\varepsilon}(x) + u_{2\varepsilon}(x) \) admits at least one global maximum point, denoted it by \( x_\varepsilon \). Let \( (w_{1\varepsilon}, w_{2\varepsilon}) \) be defined by (3.20), then it satisfies
\[
\begin{aligned}
&-\Delta w_{1\varepsilon} + V_1(\varepsilon x + x_\varepsilon)w_{1\varepsilon} = a_1 u_{1\varepsilon}^3 + \beta u_{2\varepsilon}^2 w_{1\varepsilon} \text{ in } \mathbb{R}^N, \\
&-\Delta w_{2\varepsilon} + V_2(\varepsilon x + x_\varepsilon)w_{2\varepsilon} = a_2 u_{2\varepsilon}^3 + \beta u_{1\varepsilon}^2 w_{2\varepsilon} \text{ in } \mathbb{R}^N.
\end{aligned}
\]
(3.22)
Since \( w_{1\varepsilon}(x) + w_{2\varepsilon}(x) \) attains its maximum at \( x = 0 \), we thus derive from above that
\[ \mu(w_{1\varepsilon}(0) + w_{2\varepsilon}(0)) \leq a_1 u_{1\varepsilon}(0)^3 + a_2 u_{2\varepsilon}(0)^3 + \beta w_{1\varepsilon}(0)w_{2\varepsilon}(0)(w_{1\varepsilon}(0) + w_{2\varepsilon}(0)). \]
This indicates that there exists \( C > 0 \) independent of \( \varepsilon \), such that
\[ w_{i\varepsilon}(0) \geq C > 0 \text{ holds for } i = 1 \text{ or } 2. \]
Without loss of generality, we assume that the above estimate holds as least for \( i = 1 \). As a consequence, there exists \( 0 \neq \tilde{w}_1(x) \geq 0 \) and \( \tilde{w}_2(x) \geq 0 \), such that, up to a subsequence,
\[ (w_{1\varepsilon}(x), w_{2\varepsilon}(x)) \rightharpoonup (\tilde{w}_1(x), \tilde{w}_2(x)) \text{ weakly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N). \]
(3.23)
We claim that \( \tilde{w}_2(x) \neq 0 \). For otherwise if \( \tilde{w}_2(x) \equiv 0 \), it then follows from (1.13) that
\[ -\Delta \tilde{w}_1 + \mu \tilde{w}_1 \leq a_1 \tilde{w}_1^3 \text{ in } \mathbb{R}^N. \]
(3.24)
This indicates that there exists \( t_0 \in (0, 1) \) such that \( t_0(\tilde{w}_1, 0) \in \mathcal{N}^\mu \). Therefore, from (3.4) and (3.5) we have
\[
\begin{aligned}
\lim_{\varepsilon \to 0^+} c_\varepsilon / \varepsilon^N &= \frac{1}{4} \lim_{\varepsilon \to 0^+} \sum_{i=1}^2 \int_{\mathbb{R}^N} |\nabla w_{i\varepsilon}|^2 + V_i(\varepsilon x + x_\varepsilon)|w_{i\varepsilon}|^2 \, dx \\
&\geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla \tilde{w}_1|^2 + \mu |\tilde{w}_1|^2 \, dx \geq \tilde{c}_1 > c^\\n\end{aligned}
\]
(3.25)
This leads to a contradiction in view of (3.3). Thus \( \tilde{w}_2(x) \neq 0 \), and
\[ 0 \leq \tilde{w}_i(x) \neq 0 \quad \text{for both } i = 1, 2. \]
(3.26)
Assume that (3.19) is incorrect, then for any sequence \( \{ \varepsilon_k \} \), up to a subsequence, there always holds that
\[
\lim_{k \to \infty} |x_{\varepsilon_k}| = \infty \quad \text{or} \quad \lim_{k \to \infty} x_{\varepsilon_k} = z_0 \notin \mathbb{Z}.
\] (3.27)
But in any case, one can deduce from (1.13) that
\[
\lim_{\varepsilon \to 0^+} \frac{1}{4} \sum_{i=1}^{2} \int_{\mathbb{R}^N} |\nabla w_{ie}|^2 + V_i(\varepsilon x + x) |w_{ie}|^2 \, dx > \int_{\mathbb{R}^N} |\nabla \bar{w}_1|^2 + \mu |\bar{w}_1|^2 \, dx.
\] (3.28)
Moreover, from (3.22) and (3.27), we derive that there exists \( t_1 \in (0, 1) \) such that \( t_1(\bar{w}_1, \bar{w}_2) \in \mathcal{N}^\mu \). It then yields from (3.3) and (3.28) that
\[
\lim_{\varepsilon \to 0^+} \frac{c_e}{\varepsilon^N} = \frac{1}{4} \lim_{\varepsilon \to 0^+} \sum_{i=1}^{2} \int_{\mathbb{R}^N} |\nabla w_{ie}|^2 + V_i(\varepsilon x + x) |w_{ie}|^2 \, dx
\]
\[
> \frac{1}{4} \sum_{i=1}^{2} \int_{\mathbb{R}^N} |\bar{w}_1|^2 + \mu |\bar{w}_1|^2 \, dx \geq c^\mu.
\] (3.29)
This contradicts (3.8). Therefore (i) is proved.

(ii). From (3.19) and (3.23), we see that \( (\bar{w}_1(x), \bar{w}_2(x)) \) solves (1.3). Moreover, it follows from (3.28) and the strong maximum principle that both \( \bar{w}_i(x) > 0 \) in \( \mathbb{R}^N \). Therefore, by the uniqueness of positive solutions for (1.3), we see that \( \bar{w}_i(x) \equiv w_i(x) \) \( (i = 1, 2) \), where \( w_i(x) \) is given by (1.4). Hence, it follows from (3.8) that
\[
c^\mu = \lim_{\varepsilon \to 0} \frac{c_e}{\varepsilon^N} = \frac{1}{4} \lim_{\varepsilon \to 0^+} \sum_{i=1}^{2} \int_{\mathbb{R}^N} |\nabla w_{ie}|^2 + V_i(\varepsilon x + \varepsilon y) |w_{ie}|^2 \, dx
\]
\[
> \frac{1}{4} \sum_{i=1}^{2} \int_{\mathbb{R}^N} |\bar{w}_1|^2 + \mu |\bar{w}_1|^2 \, dx \geq c^\mu.
\] (3.30)
This yields (3.21), and the proof of the lemma is complete. □

**Proof of Theorem 1.2.** *Step 1: Existence of positive ground states.* For any \( \varepsilon > 0 \) fixed, it yields from Theorem 3.1 in the appendix that (1.1) has at least one nonnegative ground state \((u_{1\varepsilon}, u_{2\varepsilon}) \in \mathcal{X}_\varepsilon \). Set \( v_{1\varepsilon}(x) := u_{1\varepsilon}(\varepsilon x) \) \( (i = 1, 2) \), then \((v_{1\varepsilon}(x), v_{2\varepsilon}(x))\) satisfies
\[
\begin{align*}
-\Delta v_{1\varepsilon} + V_1(\varepsilon x) v_{1\varepsilon} &= a_1 v_{1\varepsilon}^3 + \beta v_{2\varepsilon}^2 v_{1\varepsilon} \quad \text{in} \ \mathbb{R}^N, \\
-\Delta v_{2\varepsilon} + V_2(\varepsilon x) v_{2\varepsilon} &= a_2 v_{2\varepsilon}^3 + \beta v_{1\varepsilon}^2 v_{2\varepsilon} \quad \text{in} \ \mathbb{R}^N.
\end{align*}
\] (3.31)
We next prove that \( u_{1\varepsilon}(x) \neq 0 \) and \( u_{2\varepsilon}(x) \neq 0 \) provided that \( \varepsilon > 0 \) is small enough. Indeed, if \( u_{2\varepsilon}(x) \equiv 0 \), then \((v_{1\varepsilon}, 0)\) satisfies (3.31). From (1.13), we see that there exists \( t_\varepsilon \in (0, 1] \) such that \( t_\varepsilon(v_{1\varepsilon}, 0) \in \mathcal{N}^\mu \). As a consequence of Lemma 3.1 we further have
\[
\varepsilon^{-N} c_e = \frac{\varepsilon^{-N}}{4} \int_{\mathbb{R}^N} |\nabla u_{1\varepsilon}|^2 + V(x) |u_{1\varepsilon}|^2 \, dx = \frac{1}{4} \int_{\mathbb{R}^N} |\nabla v_{1\varepsilon}|^2 + V(\varepsilon x) |v_{1\varepsilon}|^2 \, dx
\]
\[
\geq \frac{1}{4} \int_{\mathbb{R}^N} |\nabla v_{1\varepsilon}|^2 + \mu |v_{1\varepsilon}|^2 \, dx \geq c_1 > c^\mu.
\] (3.32)
This however contradicts (3.8). Thus, \( u_{2\varepsilon}(x) \neq 0 \). Similarly, we also deduce that \( u_{1\varepsilon}(x) \neq 0 \). Furthermore, by applying the strong maximum principle, we see that \( u_{1\varepsilon}(x) > 0 \) and \( u_{2\varepsilon}(x) > 0 \) in whole \( \mathbb{R}^N \).
Applying (3.22) again, we have
\[ w_{1\varepsilon}(x) \xrightarrow{\varepsilon \to 0^+} w_i(x) \] for \( i = 1, 2 \). (3.33)

Since the origin \( x = 0 \) is the unique maximum point of \( w_1 + w_2 \), proceeding similar arguments of [15, Theorem 1.3], one can prove that \( x = 0 \) is the unique maximum point of \( w_{1\varepsilon} + w_{2\varepsilon} \), which indicates that \( x_\varepsilon \) is the unique maximum point of \( u_{1\varepsilon}(x) + u_{2\varepsilon}(x) \).

**Step 3: Refined energy estimate for ring-shaped potentials.** Let \( (u_{1\varepsilon}, u_{2\varepsilon}) \) be the convergent subsequence, and \( (w_{1\varepsilon}, w_{2\varepsilon}) \) be given by (3.20). Choose \( t_\varepsilon > 0 \) such that \( t_\varepsilon(w_{1\varepsilon}, w_{2\varepsilon}) \in B^{\mu} \), i.e.,
\[
t_\varepsilon^2 = \frac{\sum_{i=1}^2 \int_{\mathbb{R}^N} |\nabla w_{1\varepsilon}|^2 + \mu |w_{1\varepsilon}|^2 \, dx}{B(w_{1\varepsilon}, w_{2\varepsilon})} \in (0, 1).
\] (3.34)

It then follows from (1.13) and (3.22) that
\[
t_\varepsilon^2 = \frac{\sum_{i=1}^2 \int_{\mathbb{R}^N} |\nabla w_{1\varepsilon}|^2 + \mu |w_{1\varepsilon}|^2 \, dx}{B(w_{1\varepsilon}, w_{2\varepsilon})} \in (0, 1).
\]

Applying (3.22) again, we have
\[
c_\varepsilon = \frac{1}{4} \|(u_{1\varepsilon}, u_{2\varepsilon})\|_{\dot{X}_0}^2 = \frac{\varepsilon^N}{4} \sum_{i=1}^2 \int_{\mathbb{R}^N} |\nabla w_{1\varepsilon}|^2 + V_i(\varepsilon x + x_\varepsilon) |w_{1\varepsilon}(x)|^2 \, dx
\[
= \frac{\varepsilon^N}{4} \left[ \sum_{i=1}^2 \int_{\mathbb{R}^N} \left( |\nabla w_{1\varepsilon}|^2 + \mu |w_{1\varepsilon}|^2 \right) + (V_i(\varepsilon x + x_\varepsilon) - \mu) |w_{1\varepsilon}(x)|^2 \, dx \right]
\[
\geq \frac{\varepsilon^N}{4} \left[ \sum_{i=1}^2 \int_{\mathbb{R}^N} t_\varepsilon^2 (|\nabla w_{1\varepsilon}|^2 + \mu |w_{1\varepsilon}|^2) + (V_i(\varepsilon x + x_\varepsilon) - \mu) |w_{1\varepsilon}(x)|^2 \, dx \right]
\[
\geq c_\mu \varepsilon^N + \frac{\varepsilon^N}{4} \sum_{i=1}^2 \int_{\mathbb{R}^N} (V_i(\varepsilon x + x_\varepsilon) - \mu) |w_{1\varepsilon}(x)|^2 \, dx, \tag{3.36}
\]

where the fact \( t_\varepsilon(w_{1\varepsilon}, w_{2\varepsilon}) \in B^{\mu} \) is used in the last inequality.

Since \( \lim_{\varepsilon \to 0^+} x_\varepsilon = \bar{x}_0 \in Z \), where \( Z \) is defined in (1.14), there exists \( 1 \leq j_0 \leq l \) such that \( |\bar{x}_0| = A_{j_0} \). Without loss of generality, suppose that \( p_{j_0} = p_{1j_0} \leq p_{2j_0} \). We now claim that
\[
\frac{|x_{\varepsilon} - A_{j_0}|}{\varepsilon} \quad \text{is uniformly bounded as} \quad \varepsilon \to 0^+, \tag{3.37}
\]

and
\[
p_{j_0} = p_0 \quad \text{where} \quad p_0 \quad \text{is defined by (1.21)}. \tag{3.38}
\]

Indeed, since \( |\varepsilon x + x_\varepsilon| - A_{j_0} \leq r_0 \) for any \( x \in B_{r_0/2\varepsilon}^\mu(0) \) provided \( \varepsilon > 0 \) is small enough,
we deduce from (1.16) that

$$\int_{\mathbb{R}^N} (V_1(\varepsilon x + x) - \mu) |w_{1\varepsilon}(x)|^2 \, dx \geq \frac{b_{1j_0}}{2} \int_{B_{2\varepsilon}(0)} \left| \frac{\varepsilon x^2 + 2x \cdot x + (x_1^2 - A_{j_0}^2)}{\varepsilon x + x + A_{j_0}} \right| p_{j_0} |w_{1\varepsilon}(x)|^2 \, dx. $$

(3.39)

On the contrary, suppose that either $p_{j_0} < p_0$ or (3.37) does not hold. Then, for any given $M > 0$, we obtain from (3.39) that

$$\liminf_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} (V_1(\varepsilon x + x) - \mu) |w_{1\varepsilon}(x)|^2 \, dx \geq \frac{b_{1j_0}}{2} \int_{B_{2\varepsilon}(0)} \frac{\varepsilon x^2 + 2x \cdot x + (x_1^2 - A_{j_0}^2)}{\varepsilon x + x + A_{j_0}} |p_{j_0} |w_{1\varepsilon}(x)|^2 \, dx \geq M. $$

This together with (3.36) gives that

$$c_\varepsilon \geq \varepsilon^{-N}(\varepsilon^0 + M \varepsilon^{p_0}),$$

which however contradicts (3.9). Thus, the claims (3.37) and (3.38) are proved.

Following (3.37), up to a subsequence of $\{\varepsilon\}$, there exists $\kappa \in \mathbb{R}$ such that

$$\lim_{\varepsilon \to 0^+} \frac{|x_1| - |A_{j_0}|}{\varepsilon} = \kappa. $$

(3.40)

Without loss of generality, we assume that $x_0 = (0, 0, \cdots, A_{j_0})$. Applying the Fatou’s Lemma, it then follows from (1.16), (1.20), (3.38) and (3.40) that

$$\liminf_{\varepsilon \to 0^+} \sum_{i=1}^{2} \int_{\mathbb{R}^N} (V_i(\varepsilon x + x) - \mu) |w_{i\varepsilon}(x)|^2 \, dx \geq \liminf_{\varepsilon \to 0^+} \sum_{i=1}^{2} \varepsilon^{p_{j_0}} b_{ij_0} \int_{\mathbb{R}^N} \frac{(V_i(\varepsilon x + x) - \mu) |x + x_{j_0}| + (x_1^2 - A_{j_0}^2)}{|x + x_{j_0} + A_{j_0}|} |p_{j_0} |w_{i\varepsilon}(x)|^2 \, dx \geq \liminf_{\varepsilon \to 0^+} \sum_{i=1}^{2} \varepsilon^{p_{j_0}} b_{ij_0} \int_{\mathbb{R}^N} |x_1| + \kappa |p_{j_0} |w_{i}(x)|^2 \, dx \geq \liminf_{\varepsilon \to 0^+} \sum_{i=1}^{2} \varepsilon^{p_{j_0}} b_{ij_0} \int_{\mathbb{R}^N} |x_1| p_{j_0} |w_{i}(x)|^2 \, dx = \bar{\lambda}_{j_0} \geq \bar{\lambda}_0,$$

(3.41)

where $\bar{\lambda}_{j_0}$ and $\bar{\lambda}_0$ are defined in (1.20) and (1.21), respectively. Here it needs to note that “=” hold in last two inequalities of (3.41) if and only if $\kappa = 0$ and $x_0 \in Z_0$, accordingly. Moreover, from (1.23), (3.34) and (3.31), we have

$$\ell_\varepsilon^2 \geq \frac{\sum_{i=1}^{2} \int_{\mathbb{R}^N} \left| \nabla w_{i\varepsilon} \right|^2 + \mu |w_{i\varepsilon}|^2 \, dx}{B(w_{1\varepsilon}, w_{2\varepsilon})} \leq 1 - \frac{\sum_{i=1}^{2} \int_{\mathbb{R}^N} (V_i(\varepsilon x + x) - \mu) |w_{i\varepsilon}|^2 \, dx}{B(w_1, w_2) + o(1)} \leq 1 - \frac{\bar{\lambda}_0 \varepsilon^{p_0} + o(\varepsilon^{p_0})}{B(w_1, w_2) + o(1)}.$$
This indicates that
\[
\frac{1}{t_\varepsilon^2} \geq 1 + \frac{\lambda_0 \varepsilon + o(\varepsilon)}{B(w_1, w_2) + o(1)} (1 + o(1)).
\]

Noting that \(t_\varepsilon(w_{1\varepsilon}, w_{2\varepsilon}) \in N^\mu\), it then follows from (3.33) that
\[
\sum_{i=1}^{2} \int_{\mathbb{R}^N} (|\nabla w_{1\varepsilon}|^2 + \mu |w_{1\varepsilon}|^2) \, dx \geq \frac{4e^\mu}{t_\varepsilon^2} = 4c^\mu + \lambda_0 \varepsilon + o(\varepsilon).
\]
Together with (3.35) and (3.41), this yields that
\[
c_\varepsilon \geq \varepsilon^N \left[ c^\mu + \frac{\lambda_0}{2} \varepsilon + o(\varepsilon) \right].
\]

Combining with (3.34), we deduce that (3.42) is indeed an identity. Hence, all equalities in (3.41) hold, which implies that \(\kappa = 0\) and \(x_0 \in Z_0\). (1.24) then follows from (3.40). 

\[\square\]

4 Uniqueness of ground state for ring-shaped potentials

In this section, we investigate the uniqueness of ground state for ring-shaped potentials. We first introduce the following well known Borsuk-Ulam theorem (see, for example, [36, Theorem 9]).

**Lemma 4.1** (Borsuk-Ulam theorem). For any given continuous function \(f : S^{n_1} \rightarrow \mathbb{R}^{n_2}\) with \(n_1 \geq n_2 \geq 1\), there exists \(x \in S^{n_1}\) such that \(f(x) = f(-x)\).

Motivated by the argument of Theorem 1 in [29], we have the following cylindrical symmetry result for ground states, which is the core for the proof of Theorem 1.3.

**Lemma 4.2.** Suppose that \(N = 2\) or \(3\), and \(V_i(x) = V_i(|x|)\) is radially symmetric about the origin. Let \(\bar{u}_\varepsilon(x) = (u_{1\varepsilon}(x), u_{2\varepsilon}(x))\) be one ground state of (1.1). Assume that \(u_{1\varepsilon}(x) + u_{2\varepsilon}(x)\) has a unique nonzero maximal point, denoted it by \(x_\varepsilon \neq 0\). Then, \(u_{1\varepsilon}(x)\) and \(u_{2\varepsilon}(x)\) are cylindrically symmetric with the line \(Ox\).

**Proof.** We only give the proof when \(N = 3\), for the case of \(N = 2\) can be derived similarly. In view of \(V_i(x) = V_i(|x|)\), one see that any rotation of \(\bar{u}_\varepsilon(x)\) is still a ground state of (1.1). Therefore, without loss of generality, we assume that the point \(x_\varepsilon\) lies on the \(X_3\)-axis.

Let \(OX_1X_2\) be the vector space spanned by \(X_1\) and \(X_2\) axes. For any \(e \in S^2 \cap OX_1X_2\), taking \(v \in e^\perp \cap S^2\), and setting
\[
\Pi_v = \{ x \in \mathbb{R}^3 : x \cdot v = 0 \}, \quad \Pi_v^+ := \{ x \in \mathbb{R}^3 : x \cdot v > 0 \} \quad \text{and} \quad \Pi_v^- := \{ x \in \mathbb{R}^3 : x \cdot v < 0 \}.
\]
Let
\[
G(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) := \sum_{i=1}^{2} (\varepsilon^2 |\nabla u_{i\varepsilon}| + V_i(x) u_{i\varepsilon}^2) + B(\bar{u}_\varepsilon),
\]
and
\[
\varphi(v) := \int_{\Pi_v^+} G(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon) - \int_{\Pi_v^-} G(x, \bar{u}_\varepsilon, \nabla \bar{u}_\varepsilon).
\]
One can easily see that
\[ \varphi(v) = -\varphi(-v) \quad \forall v \in e^\perp \cap S^2(\cong S^1). \]

It then follows from Lemma 4.1 that, there exists \( v \in e^\perp \cap S^2 \) such that \( \varphi(v) = 0 \). Thus,
\[ \int_{\Pi_v^+} G(x, \tilde{u}_e, \nabla \tilde{u}_e) = \int_{\Pi_v^-} G(x, \tilde{u}_e, \nabla \tilde{u}_e) = 0. \tag{4.1} \]

Let
\[ \tilde{u}_e^+ = \begin{cases} \tilde{u}_e(x) & x \in \Pi_v^+ \cup \Pi_v, \\ \tilde{u}(P) & x \in \Pi_v^- \cup \Pi_v \end{cases}, \quad \tilde{u}_e^- = \begin{cases} \tilde{u}_e(x) & x \in \Pi_v^+ \cup \Pi_v, \\ \tilde{u}(P) & x \in \Pi_v^- \cup \Pi_v \end{cases}, \]
where \( P \) denotes the orthogonal projection with respect to the hyperplane \( \Pi_v \). It then follows from Lemma 4.1 that, there exists \( \epsilon \in 0 \cap S^2 \) such that \( \varphi(\epsilon) = 0 \) and
\[ \int_{\Pi_v} G(x, \tilde{u}_e, \nabla \tilde{u}_e) = 0, \]
and
\[ \int_{\Pi_v} G(x, \tilde{u}_e^- \nabla \tilde{u}_e) = \int_{\Pi_v} G(x, \tilde{u}_e^- \nabla \tilde{u}_e) = 0. \]

This indicates that \( \tilde{u}_e^+ \in \mathcal{N}_e \) and thus
\[ c_e = J_e(\tilde{u}_e^-) = \frac{1}{2} J_e(\tilde{u}_e^+) + J_e(\tilde{u}_e^-) \geq c_e. \]

Therefore, we deduce that \( \tilde{u}_e^+ \) and \( \tilde{u}_e^- \) are both ground states of (4.1). Set
\[ \tilde{w}_e(x) := \tilde{u}_e - \tilde{u}_e(x) \quad \text{and} \quad F(\tilde{w}) = \frac{a_1 u_1^4 + a_2 u_2^4}{4} + \frac{\beta u_1^2 u_2^2}{2}. \]

We then derive from above that
\[ -\Delta \tilde{w}_e(x) = \mathcal{V}(x)\tilde{w}_e(x) + \mathcal{A}(x)\tilde{w}_e(x), \]
where the matrices \( \mathcal{V}(x) = diag\{V_1(x), V_2(x)\} \) and \( \mathcal{A}(x) \) is given by
\[ \mathcal{A}(x) = \begin{pmatrix} \int_0^1 \frac{\partial^2 F}{\partial u_1^2}(t\tilde{u}_e + (1-t)\tilde{u}_e^+)dt & \int_0^1 \frac{\partial^2 F}{\partial u_1\partial u_2}(t\tilde{u}_e + (1-t)\tilde{u}_e^+)dt \\ \int_0^1 \frac{\partial^2 F}{\partial u_2^2}(t\tilde{u}_e + (1-t)\tilde{u}_e^+)dt & \int_0^1 \frac{\partial^2 F}{\partial u_2^2}(t\tilde{u}_e + (1-t)\tilde{u}_e^+)dt \end{pmatrix}. \]

Since \( \mathcal{V}(x) \in L^\infty_{loc}(\mathbb{R}^3) \) and \( \mathcal{A}(x) \in L^\infty(\mathbb{R}^3) \), and note that \( \tilde{u}_e(x) \equiv \tilde{u}_e^+(x) \quad \forall x \in \Pi_v^+ \), we thus can derive from the unique continuation principle (see for instance, the appendix in [28]) that
\[ \tilde{u}_e(x) = \tilde{u}_e^+(x) \quad \forall x \in \mathbb{R}^3. \]

This indicates that \( \tilde{u}_e(x) \) is symmetric with respect to the hyperplane \( \Pi_v \). Moreover, recalling from Theorem 4.2 that \( u_{1e}(x) + u_{2e}(x) \) has a unique maximal point \( x_e \neq 0 \), we then see that \( x_e \in \Pi_v \) and thus \( \Pi_v = \overline{Oe x_e} \), where \( \mathcal{O}e x_e \) denotes the hyperplane consists of the line \( \overline{Oe} \) and \( \overline{Oe x_e} \). Therefore, we have obtained that
\[ \tilde{u}_e(x) \] is symmetric with respect to \( \overline{Oe x_e} \) for all \( e \in S^2 \cap OX_1 X_2 \).
As a consequence, we know that \( \bar{u}_\varepsilon(x) \) is cylindrically symmetric with respect to the line \( O\varepsilon \).

Based on Lemma 4.2 as well as some techniques carry out in the proof of Theorem 1.1 we finally finish the proof of Theorem 1.3

**Proof of Theorem 1.3.** We only give the proof for the case of \( N = 3 \). Similar to the proof of Theorem 1.1 we assume that \( \bar{u}(x) = (u_{1\varepsilon}, u_{2\varepsilon}) \) and \( \bar{v}(x) = (v_{1\varepsilon}, v_{2\varepsilon}) \) are two different nonnegative ground states of (1.1), and let \( x_\varepsilon \) and \( y_\varepsilon \) be the unique maximum point of \( u_{1\varepsilon} + u_{2\varepsilon} \) and \( v_{1\varepsilon} + v_{2\varepsilon} \), respectively. We first assume that \( x_\varepsilon \) and \( y_\varepsilon \) are both lie on the \( X_3 \) axis, namely,

\[
 x_\varepsilon = (0, 0, x_{3\varepsilon}) \quad \text{and} \quad y_\varepsilon = (0, 0, y_{3\varepsilon}).
\]

From Theorem 1.2, we have

\[
 \lim_{\varepsilon \to 0^+} \frac{|x_\varepsilon| - A_{j_0}}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{|y_\varepsilon| - A_{j_0}}{\varepsilon} = 0. \tag{4.3}
\]

Let \( \bar{u}_{i\varepsilon}(x) \) and \( \bar{v}_{i\varepsilon}(x) \) be given by (2.17). \( \xi_{i\varepsilon}(x) \) and \( \bar{\xi}_{i\varepsilon}(x) \) are still defined by (2.26) and (2.29), respectively. Similar to the the Step 1 in the proof of Theorem 1.1 we know that (2.30) still holds. We next prove that \( d_3 = 0 \). Repeating the arguments of (2.33) to (2.35), we see that

\[
 \left( \sum_{i=1}^{2} \int_{B_3(x_{3\varepsilon})} \frac{\partial V_i(x)}{\partial x_j} (v_{i\varepsilon} + u_{i\varepsilon}) \xi_{i\varepsilon} \, dx \right) = o(e^{-\frac{C\varepsilon}{x}}). \tag{4.4}
\]

From (1.26), we see that

\[
 \varepsilon^{-3} \cdot L.H.S.\, of \, (4.3) \tag{4.5}
\]

\[
 = \sum_{i=1}^{2} \int_{B_3(0)} \frac{\partial V_i(\varepsilon x + x_\varepsilon)}{\partial x_3} (\bar{v}_{i\varepsilon} + \bar{u}_{i\varepsilon}) \bar{\xi}_{i\varepsilon} \, dx
\]

\[
 = \sum_{i=1}^{2} \int_{B_3(0)} \left\{ p_{j_0} \varepsilon^{p_{j_0} - 1} \frac{\varepsilon x_3^2 + 2 x_3 \varepsilon x_3 (x_3^2 - A_{j_0}^2)}{\varepsilon} \frac{\varepsilon x_3^2 + 2 x_3 \varepsilon x_3 (x_3^2 - A_{j_0}^2)}{\varepsilon} \right\} \left( \bar{v}_{i\varepsilon} + \bar{u}_{i\varepsilon} \right) \bar{\xi}_{i\varepsilon} \, dx. \tag{4.6}
\]

Noting that

\[
 \lim_{\varepsilon \to 0^+} \frac{\varepsilon x_3^2 + 2 x_3 \varepsilon x_3 (x_3^2 - A_{j_0}^2)}{\varepsilon} = x_3 \text{ for a.e. } x \in \mathbb{R}^3, \tag{4.7}
\]

it then follows from (1.26) that

\[
 \left| \int_{B_3(0)} R_i (\varepsilon x + x_\varepsilon - A_{j_0}) (\bar{v}_{i\varepsilon} + \bar{u}_{i\varepsilon}) \bar{\xi}_{i\varepsilon} \, dx \right| \cdot e^{-r_i}
\]

\[
 \leq C \int_{B_3(0)} \frac{\varepsilon x_3^2 + 2 x_3 \varepsilon x_3 (x_3^2 - A_{j_0}^2)}{\varepsilon} \left( \bar{v}_{i\varepsilon} + \bar{u}_{i\varepsilon} \right) \bar{\xi}_{i\varepsilon} \, dx \leq C < \infty. \tag{4.8}
\]

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Assume that \( p_{1j_0} = p_{2j_0} \), similar to the proof of (2.37), we further deduce from (4.8) that
\[
o(1) = \int_{\mathbb{R}^3} \frac{\varepsilon x^2 + 2x_3 x + (x_3^2 - A_{j_0}^2) / \varepsilon}{|x + x_3| + A_{j_0}} |p_{j_0} - 2(\frac{\varepsilon x^2 + 2x_3 x + (x_3^2 - A_{j_0}^2) / \varepsilon}{|x + x_3| + A_{j_0}})} dx.
\]
(4.9)

Letting \( \varepsilon \to 0^+ \), we then deduce from (4.7) and (4.9) that
\[
0 = \int_{\mathbb{R}^3} |x_3|^{p_{j_0} - 2} x_3 \left[ w_1 \tilde{\xi}_{10} + w_2 \tilde{\xi}_{20} \right] dx.
\]
and
\[
d_3 \int_{\mathbb{R}^3} x_3 |x|^{p_{j_0} - 2} x_3 \left( w_1 \frac{\partial u_1}{\partial x_3} + m_2 w_2 \frac{\partial u_2}{\partial x_3} \right) dx
\]
\[
d_3 \int_{\mathbb{R}^3} |x_3|^{p_{j_0} + 1} \left( w_1 u_1(|x|) + w_2 u_2(|x|) \right) dx.
\]
This indicates that \( d_3 = 0 \). For the case of \( p_{1j_0} \neq p_{2j_0} \), proceeding similar arguments as above, one also have \( d_3 = 0 \). Moreover, from (1.2) and Lemma 4.2 we see that \((\tilde{\xi}_{1e}, \tilde{\xi}_{2e})\), and thus \((\tilde{\xi}_{10}, \tilde{\xi}_{20})\) is cylindrically symmetric with the \( X_3 \) axis. This implies that \( d_1 = d_2 = 0 \). Therefore, we have \( \tilde{\xi}_{10} = \tilde{\xi}_{20} = 0 \), which however cannot occur by applying the arguments of step 3 for the proof of Theorem 1.1.

From above, we see that the ground state which is cylindrically symmetric with the \( X_3 \) axis is unique. We denote it by \( \tilde{u}_e(x) \). Then, for any nonnegative ground state \( \tilde{\omega}_e(x) \), there exists \( T_\omega \in O(3) \) such that \( \tilde{\omega}(T_\omega x) \) is symmetric with the \( X_3 \) axis. Note that \( \tilde{\omega}(T_\omega x) \) is still a ground state of (1.1), we thus deduce that \( \tilde{u}_e(x) = \tilde{\omega}(T_\omega x) \) in \( \mathbb{R}^3 \). The proof of Theorem 1.3 is completed.

A Appendix

Motivated by the arguments of [58, 51], we intend to prove the existence of non-negative ground states for (1.1) under general potentials in appendix. The following is our main result.

**Theorem A.1.** Suppose that \( V_1(x) \) and \( V_2(x) \) satisfy (1.13) and (1.14), and \( \beta > \max\{a_1, a_2\} \). Then, for \( \varepsilon > 0 \) is small enough, equation (1.1) has at least one nonnegative ground state.

Denote \( V_i^\infty := \liminf_{|x| \to \infty} V_i(x) \) \((i = 1, 2)\), and consider the following problem
\[
\begin{aligned}
-\varepsilon^2 \Delta u_1 + V_1^\infty u_1 = a_1 u_1^3 + \beta u_1^2 u_2 & \quad \text{in } \mathbb{R}^N, \ N \leq 3, \\
-\varepsilon^2 \Delta u_2 + V_2^\infty u_2 = a_2 u_2^3 + \beta u_1^2 u_2 & \quad \text{in } \mathbb{R}^N.
\end{aligned}
\]
(A.1)

The corresponding energy functional is defined by
\[
J_\varepsilon^\infty(\tilde{u}) := \frac{1}{2} \sum_{i=1}^2 \int_{\mathbb{R}^N} \left( \varepsilon^2 |\nabla u_i|^2 + V_i^\infty |u_i|^2 \right) dx - \frac{1}{4} B(u_1, u_2) dx.
\]
(A.2)

Set
\[
c_\varepsilon^\infty := \inf_{\tilde{u} \in N_\varepsilon^\infty} J_\varepsilon^\infty(\tilde{u}), \ \text{where } N_\varepsilon^\infty := \{ \tilde{u} \neq 0 : \langle J_\varepsilon^\infty(\tilde{u}), \tilde{u} \rangle = 0 \}.
\]
(A.3)

Then, we have the following lemma.
Lemma A.2. Let (1.13) be satisfied, we have
\[ c^\infty_\varepsilon / \varepsilon^N > c^\mu \text{ for any } \varepsilon > 0. \]  
(A.4)

Proof. Consider
\[ \begin{cases} 
-\Delta u_1 + V_1^\infty u_1 = a_1 u_1^3 + \beta u_1^2 u_1 \text{ in } \mathbb{R}^N, \quad N \leq 3, \\
-\Delta u_2 + V_2^\infty u_2 = a_2 u_2^3 + \beta u_2^2 u_2 \text{ in } \mathbb{R}^N,
\end{cases} \]  
(A.5)

and define
\[ c^\infty_\varepsilon := \inf_{\vec{u} \in \mathcal{N}^\infty_\varepsilon} \mathcal{J}^\infty_\varepsilon (\vec{u}), \]  
where $\mathcal{N}^\infty_\varepsilon := \{ \vec{u} \neq 0 : \langle \mathcal{J}^\infty_\varepsilon(\vec{u}), \vec{u} \rangle = 0 \}$.
(A.6)

Therefore, it suffices to prove that
\[ c^\infty_\varepsilon > c^\mu. \]  
(A.8)

Let \( \{ \vec{u}_n \} \subset \mathcal{N}^\infty_\varepsilon \) be a minimizing sequence of \( c^\infty_\varepsilon \), we then have
\[ \sum_{i=1}^{2} \int_{\mathbb{R}^N} (|\nabla u_{in}|^2 + V_i^\infty |u_{in}|^2) dx = B(u_{1n}, u_{2n}) \xrightarrow{n \to \infty} 4c^\infty > 0. \]  
(A.9)

From (A.9) one can deduce that there exists \( \delta > 0 \) such that
\[ \int_{\mathbb{R}^N} |u_{1n}|^2 + |u_{2n}|^2 dx \geq \delta > 0. \]

Choosing \( t_n > 0 \) such that \( t_n \vec{u}_n \in \mathcal{N}^\mu \), it then follows from (A.9) that
\[ t_n^2 = \sum_{i=1}^{2} \int_{\mathbb{R}^N} (|\nabla u_{in}|^2 + V_i^\infty |u_{in}|^2) dx / B(\vec{u}_n) 
= 1 - \sum_{i=1}^{2} (V_i^\infty - \mu) \int_{\mathbb{R}^N} |u_{in}|^2 dx/(4c^\infty + o(1)) 
\leq 1 - \delta_1 \text{ for some } \delta_1 > 0. \]

As a consequence, we have
\[ c^\mu \leq J_\mu(t_n \vec{u}_n) = \frac{t_n^4}{4} B(\vec{u}_n) \leq (1 - \delta_1)^2 (c^\infty + o(1)) < c^\infty. \]

This implies (A.8) and finishes the proof of this lemma.

Lemma A.3. Assume that there exists
\[ \{ \vec{u}_n \} \subset \mathcal{N}_\varepsilon, \text{ such that } J_\varepsilon(\vec{u}_n) \xrightarrow{n} c_\varepsilon, \text{ and } J_\varepsilon'(\vec{u}_n) \xrightarrow{n} 0. \]  
(A.10)

Then, if \( \varepsilon > 0 \) is small enough,
\[ \vec{u}_n \xrightarrow{n} \vec{u}_0 \text{ strongly in } \mathcal{X}_\varepsilon, \text{ where } \vec{u}_0 \text{ is a ground state of (1.1)}. \]  
(A.11)
Proof. From (A.10) we see that \( \{ \vec{u}_n \} \) is bounded in \( X_\varepsilon \). Thus, there exists \( \vec{u}_0 \in X_\varepsilon \) such that \( \vec{u}_n \xrightarrow{n} \vec{u}_0 \) weakly in \( X_\varepsilon \). This implies (A.11). Therefore, to finish the proof of this lemma it remains to prove that \( \vec{u}_0 \neq 0 \). On the contrary, if \( \vec{u}_0 = 0 \), then

\[
\vec{u}_n \xrightarrow{n} 0 \text{ in } L^p_{loc}(\mathbb{R}^N) \times L^p_{loc}(\mathbb{R}^N), \text{ for all } 2 \leq p < 2^*. \tag{A.12}
\]

Claim:

\[
\liminf_{n \to \infty} \sum_{i=1}^{2} \int_{\|x\| \geq R(\delta)} (V_i(x) - V_i^\infty) |u_{in}|^2 dx \geq 0. \tag{A.13}
\]

Actually, for any \( \delta > 0 \), there exists \( R(\delta) > 0 \) such that \( V_i(x) - V_i^\infty > -\delta \) if \( |x| \geq R(\delta) \). This indicates that, there exists \( C > 0 \) such that

\[
\int_{|x| \geq R(\delta)} (V_i(x) - V_i^\infty) |u_{in}|^2 dx \geq -C\delta.
\]

From (A.12) we also have

\[
\liminf_{n \to \infty} \sum_{i=1}^{2} \int_{|x| < R(\delta)} (V_i(x) - V_i^\infty) |u_{in}|^2 dx = 0.
\]

The above two estimates yield the claim (A.13) holds.

Noting that \( \vec{u}_n \in N_\varepsilon \), we thus deduce from (A.13) that \( \limsup_{n \to \infty} J_\varepsilon(t_n \vec{u}_n) \leq 0 \). Hence, there exists \( t_n > 0 \) satisfying \( \limsup_{n \to \infty} t_n \leq 1 \), such that \( t_n \vec{u}_n \in N_\varepsilon^\infty \). This implies

\[
c_{\varepsilon}^\infty \leq \lim_{n \to \infty} J_\varepsilon(t_n \vec{u}_n) = \frac{1}{4} \lim_{n \to \infty} t_n^4 B(\vec{u}_n) = \frac{1}{4} \lim_{n \to \infty} t_n^4 J_\varepsilon(t_n \vec{u}_n) \leq c_{\varepsilon}.
\]

Together with (3.8), (A.4) and (A.7), we derive the following contradiction,

\[
c_{\mu} < c_{\varepsilon}^\infty = \lim_{\varepsilon \to 0^+} c_{\varepsilon}^\infty / \varepsilon^N \leq \lim_{\varepsilon \to 0^+} c_{\varepsilon} / \varepsilon^N = c_{\mu}.
\]

Therefore, there holds that \( \vec{u}_0 \neq 0 \) and the proof is finished. \( \square \)

Lemma A.4. There exists nonnegative \( \{ \vec{u}_n \} \subset N_\varepsilon \) satisfying (A.11).

Proof. Let

\[
\eta_\varepsilon := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_\varepsilon(\gamma(t))
\]

be the mountain pass level of (1.1), where

\[
\Gamma := \{ \gamma(t) \in C(H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)) : \gamma(0) = 0, J_\varepsilon(\gamma(1)) < 0 \}.
\]

We claim

\[
\eta_\varepsilon = c_{\varepsilon}. \tag{A.14}
\]
On the one hand, for any $\vec{u} \in \mathcal{N}_\varepsilon$, there exists $t_0 > 1$ such that $J_\varepsilon(t_0 \vec{u}) < 0$. Let $\gamma(t) = t(t_0 \vec{u})$, then $\gamma(t) \in \Gamma$ and $J_\varepsilon(\gamma(t))$ attains its maximum at the unique point $t = 1/t_0$, i.e., $\max_{t \in [0,1]} J_\varepsilon(\gamma(t)) = J_\varepsilon(\vec{u})$. This indicates that

$$c_\varepsilon = \inf_{\vec{u} \in \mathcal{N}_\varepsilon} J_\varepsilon(\vec{u}) \geq \eta_\varepsilon. \quad (A.15)$$

On the other hand, it follows from [37, Theorem B.1] that there exists a nonnegative $(PS)_{\eta_\varepsilon}$ sequence $\{\vec{u}_n \geq 0\} \subset \mathcal{X}_\varepsilon$ such that

$$J_\varepsilon'\left(\vec{u}_n\right) \to 0 \quad \text{and} \quad J_\varepsilon(\vec{u}_n) \to \eta_\varepsilon > 0.$$

This indicates that $\{\vec{u}_n\}$ is uniformly bounded from below and above in $\mathcal{X}_\varepsilon$, and there exits $t_n \to 1$ such that $\{t_n \vec{u}_n\} \subset \mathcal{N}_\varepsilon$. Therefore, we have

$$J_\varepsilon'(t_n \vec{u}_n) = t_n J_\varepsilon'(\vec{u}_n) + (t_n - t_n^3)B'(\vec{u}_n) \overset{n \to \infty}{\to} 0, \quad (A.16)$$

and

$$\eta_\varepsilon = \lim_{n \to \infty} J_\varepsilon(\vec{u}_n) = \lim_{n \to \infty} J_\varepsilon(t_n \vec{u}_n) \geq c_\varepsilon. \quad (A.17)$$

This together with (A.15) indicates (A.14). Moreover, (A.14), (A.16) and (A.17) complete the proof of this lemma.

Proof of Theorem A.1: Theorem A.1 follows directly from Lemmas A.3 and A.4.

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