Probabilistic vs optical interpretation of quantum mechanics

Arkady L. Kholodenko

375 H.L. Hunter Laboratories, Clemson University, Clemson, SC 29634-0973, USA

Abstract

Although electrons and photons produce the same interference patterns in the two-slit experiments, the description of these patterns is markedly different. This difference was analyzed by Bohm. Later on Sanz and Miret-Artes and others were able to squeeze the differences to zero. Fortunately, they left some room for developments presented in this Letter. We noticed that in the absence of sources the electromagnetic field can be represented by the complex scalar field. It is demonstrated that the same fields are being used in the non relativistic Schrödinger equation. The connection between the electromagnetic and Schrödinger fields allows to study the topology of zero sets (Chladni patterns) of Schrödinger eigenfunctions. The existence of these patterns is contingent upon the existence of eigenvalues of multiplicity higher than one. This is permissible only in Schrödinger’s version of quantum mechanics. Presence of multiplicities is making quantum mechanical and topological entanglements equivalent.

Keywords:

Maxwell’s equations
Quantum field theory basics
Huygens’ equivalence
Chladni patterns
Quantum entanglements
Knots and links

1. Introduction

If electrons and photons\footnote{And other massive particles} produce the same interference patterns in the two-slit experiments [1-3], why then the optics formalism cannot be applied unchanged to electrons and heavier particles? What makes use of Born’s probabilistic interpretation of the wave function in quantum mechanics superior to the intensity interpretation of the Maxwellian wave function in optics? In particular, for the description of two-slit experiments in which the results are visibly coinciding for light and heavier particles? This issue was carefully investigated by...
David Bohm in his classical monograph on quantum mechanics [4], pages 97-98. Reading of these pages indicates that the differences exist just in a few places. The updated theoretical comparison was recently made by Sanz and Miret-Artes, in chapters 4 and 7 of [5]. From these chapters it follows that all objections made by Bohm in the remaining few places can be removed. Fortunately, the results of [5] along with those in references on which these results are based, and even more recent ones, can still be improved. It is the purpose of this Letter. As result, we are able to achieve the full correspondence between the description of photons and of massive particles whose Hamiltonians are manifestly time-independent. Since nowadays the sophisticated quantum mechanical experiments are typically done optically [6,7], the results of this letter may provide additional guidelines for interpretation of these optical experiments and vice versa. In the light of just mentioned correspondence, it makes sense to claim that our understanding of subtleties of quantum mechanics is contingent upon our understanding of optical formalism adopted for quantum mechanical needs. In this Letter we shall discuss several features of optical formalism which were not discussed yet in the context of quantum mechanics.

Even though the two-slits interference experiments produce the same results in both optics and quantum mechanics, it is not immediately possible to adopt word-for-word the optical formalism to quantum mechanics. This is so because of the following. Maxwell’s equations contain vector quantities, like $\mathbf{E}$ and $\mathbf{H}$. Besides, there is a polarization in optics whose analog in quantum mechanics is spin [8]. The nonrelativistic Schrödinger equation is spin-independent however. The two-slit interference fringe pattern for monochromatic light depends strongly upon the light polarization. For spinless particles in quantum mechanics used in two-slit experiment this polarization effect is absent while in optics, depending on polarization, there are four distinct cases to study. They were discovered by Fresnel and Arago at the beginning of 19th century [9,10]. For the record, these are: 1. Two rays of light polarized in the same plane. They interfere like rays of ordinary (unpolarized) light. 2. Two rays polarized at the right angles to each other. They do not interfere. 3. Two rays originally polarized at the right angles and then brought into the same plane of polarization. They do not produce interference pattern. 4 Two rays originally polarized at the right angles, if derived from the same linearly polarized wave and subsequently brought into the same plane, can interfere.

These facts complicate the comparison between the two-slit interference experiments in optics and quantum mechanics. Fortunately, there are ways out of these difficulties, e.g. those described in [5]. In this Letter we describe still another methods. This task is accomplished in several steps. The first step is discussed in Section 2. Its purpose is to introduce some known results to be used in the reminder of this Letter.

2. Derivation of the Schrödinger equation for a single photon.

We begin with writing down Maxwell’s equations without sources and currents in the vacuum. These are:
\[ \nabla \cdot E = 0, \quad \nabla \times E = \frac{1}{c} \frac{\partial H}{\partial t}, \]
\[ \nabla \cdot H = 0, \quad \nabla \times H = \frac{1}{c} \frac{\partial E}{\partial t}, \]
\tag{1}

In the above equations we keep only the speed of light \( c \) while the rest of constants we put equal to unity. They always can be restored if needed. Incidentally, the situation described by (1) is in accord with that known in standard quantum mechanics. This was emphasized by Bohm [4] and Bohm and Hiley [11]. Waves in quantum mechanics do not have sources or sinks and, accordingly, currents. In section 5 we provide arguments that the same can be achieved for the Maxwellian fields if knotted/linked field configurations are reinterpreted as charges. Next, we introduce the Riemann-Silberstein complex vector
\[ F_\lambda = \frac{1}{\sqrt{2}} (E_\lambda + i\lambda H_\lambda), \]
\tag{2}

where \( \lambda = 1(-1) \) corresponds to the positive (negative) helicity. The concept of helicity is related to the concept of polarization which for light is analogous to spin. All this is described in detail in [8]. Following Kobe [12], and Smith and Raymer [13,14], we temporarily suppress the subscript \( \lambda \) and introduce the energy density \( \varepsilon \) as
\[ F^* \cdot F = \frac{1}{2} (E^2 + H^2) \equiv \varepsilon. \]
\tag{3}

This result then allows us to introduce the photon wave function
\[ \Psi_i = \sqrt{\frac{\varepsilon}{E}} F_i, \]
\tag{4}

Here \( i=1,2,3 \) labels Euclidean coordinates while \( \varepsilon \) is defined in (6). Using (4) we require
\[ \sum_{i=1}^{3} \int d^3 x \Psi_i^* \Psi_i = 1 \]
\tag{5}

with the total energy \( \varepsilon \) defined by
\[ \int d^3 x \varepsilon = \varepsilon. \]
\tag{6}

By design, the wave function \( \Psi_i \) is satisfying the Schrödinger’s equation for the photon
\[ i \frac{\partial}{\partial t} F = c \nabla \times F, \]
\[ \text{or, } i\hbar \frac{\partial}{\partial t} F = icp \times F, \quad p = -i\hbar \nabla, \]
\tag{7a}

provided that
\[ \nabla \cdot \mathbf{F} = 0. \tag{8} \]

Equations (7) and (8) are equivalent to Maxwell’s equations (1) as required. The continuity equation for the probability now reads

\[ \frac{\partial \rho}{\partial t} + \text{div} \cdot \mathbf{j} = 0, \quad \mathbf{j} = \frac{c}{\varepsilon} \mathbf{E} \times \mathbf{H} = \frac{c}{\varepsilon} \mathbf{F}^* \times \mathbf{F}, \quad \rho = \sum_{i=1}^{3} \Psi_i^* \Psi_i. \tag{9} \]

These results are perfectly fine as far as quantization of electromagnetic field is of interest only. They are not exhibiting the universal connection with quantum mechanics of particles though. They were designed for photons only. This gives us an opportunity to describe such a connection.

3. Electromagnetic field in the absence of currents and sources as complex scalar field

Following Green and Wolf [15,16], we notice that in a region of space free of currents and charges the electromagnetic field is fully specified by the single vector potential \( \mathbf{A} \). For such a case we can write: \( \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{A} \equiv \frac{1}{c} \dot{\mathbf{A}}, \mathbf{H} = \nabla \times \mathbf{A} \). For such cases the vector potential \( \mathbf{A} \) can be rigorously derived from a single (in generally complex) scalar field potential \( V(\mathbf{x},t) \).

It can be shown that the total energy \( \mathcal{E} \), defined in (3) and (6), can be rewritten in terms of \( V(\mathbf{x},t) \) as

\[ \int d^3x \varepsilon \equiv \int d^3x \left( \frac{1}{c^2} \dot{V} V^* + \nabla V \cdot \nabla V^* \right) = \mathcal{E} \tag{10} \]

Accordingly, the Poynting flux \( \mathbf{j} \) of electromagnetic energy density is given by

\[ \mathbf{j} = -\frac{1}{2} \left( \dot{V}^* \nabla V + \dot{V} \nabla V^* \right). \tag{11} \]

Therefore, the analog of the continuity equation (9) now reads as

\[ \frac{\partial e}{\partial t} + \text{div} \cdot \mathbf{j} = 0. \tag{12} \]

It should be clear that the obtained results are equivalent to those which are presented in section 2. The advantage of having these results rewritten with help of scalar potential lies in the opportunity of bring them into correspondence with quantum mechanics. In fact, as we soon demonstrate, such scalar form of Maxwell’s equations allows us to accomplish much more. To begin, we represent \( V(\mathbf{x},t) \) in the form

\[ V(\mathbf{x},t) = \int d^3k [\alpha(\mathbf{k},t) \cos(\mathbf{k} \cdot \mathbf{x}) + \beta(\mathbf{k},t) \sin(\mathbf{k} \cdot \mathbf{x})] \equiv \int d^3k V(\mathbf{k},t), \tag{13} \]
while at the same time,

\[ A(x, t) = \int d^3k [a(k, t) \cos(k \cdot x) + b(k, t) \sin(k \cdot x)] = \int d^3k A(k, t). \] (14)

From electrodynamics it is known that the vector potential \( A \) is satisfying the vector wave equation

\[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} A - \nabla^2 A = 0. \] (15a)

Using (14) in (15a) leads to

\[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} A(k, t) + k^2 A(k, t) = 0. \] (15b)

By comparing equations (13),(14), it follows that

\[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} V(k, t) + k^2 V(k, t) = 0 \] (16a)

and, therefore,

\[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} V(x, t) - \nabla^2 V(x, t) = 0. \] (16b)

Next, suppose that the solution of (16b) can be represented in the form

\[ V(x, t) = v(x) \exp\{i\phi(x, t)\}, \quad \phi(x, t) = k\Phi(x) - \omega t. \] (17)

It should be noted that such a representation is physically motivated by its direct connection with Huygens' principle. Details are best explained in monographs by Hilbert and Courant [17] and by Luneburg [18]. Some (but not all!) of their ideas were subsequently developed by Maslov [19]. Different/alternative interpretation of (17) adopted for quantum mechanics was independently developed by Bohm in his Bohmian mechanics [11]. Using (17) in (16b) the following two equations are obtained:

\[ (\nabla \Phi)^2 - \frac{1}{k^2 v} \nabla^2 v = n^2, \] (18a)

\[ \nabla \Phi \cdot \nabla v + \frac{1}{2} (\nabla^2 \Phi) v = 0. \] (18b)

Here \( k = c/\omega, \ n^2 = 1. \) In the case of quantum mechanics, typically, \( n^2 = n^2(x) \). This fact will be discussed further below. For now, however, consider both of these equations in the limit \( k^2 \to \infty. \) Such a limit is typical for the phenomena described by methods of geometrical optics [20]. In this limit (18a) is converted into:

\[ (\nabla \Phi)^2 = n^2. \] (19a)

Physics behind equations (18a) and (18b) is predetermined by the geometric optics limit. Specifically, in this limit surfaces \( \Phi = const \) represent the wave fronts while their duals
represent light rays. These are orthogonal to the wavefronts. Following [19,20], we notice that: a) (19a) is known as the eikonal equation, b)(19b) is known as transport equation. This equation can be simplified with help of the following arguments. If $\frac{\partial}{\partial \tau}$ denotes the differentiation along a particular ray, then according to [15,19,20] we write,

$$\frac{\partial}{\partial \tau} \cdots = \nabla \Phi \cdot \nabla \cdots.$$  \hspace{1cm} (20)

Displayed identity allows us to rewrite (18b) as

$$\frac{\partial v}{\partial \tau} + \frac{1}{2} \left( v \nabla^2 \Phi \right) = 0.$$  \hspace{1cm} (19b)

Integration of the last equation is straightforward and is yielding the result:

$$v(\tau) = v(\tau_0) \exp \left[ -\frac{1}{2} \int_{\tau_0}^{\tau} d\tau' \nabla^2 \Phi \right].$$  \hspace{1cm} (19c)

Next, following [20], chr.7, we notice that the ray trajectory $x(\tau)$ can be derived from the equations of motion

$$\frac{dx}{d\tau} = \nabla \Phi.$$  \hspace{1cm} (21)

By combining equations (19b) and (21) the solution, equation (19c), can be rewritten as

$$v(x(\tau)) = v(x(0)) \exp \left[ -\frac{1}{2} \int_{\tau_0}^{\tau} d\tau' \nabla^2 \Phi(x(\tau')) \right].$$  \hspace{1cm} (22)

Notice next that $x(\tau) = x(x_0, \tau_0)$ so that $x(x_0, \tau_0) = x_0$. This observation allows us to introduce the Jacobian $J$ as follows

$$J = \det \left( \frac{\partial x^j(x_0, \tau)}{\partial x_0^i} \right).$$  \hspace{1cm} (23)

In view of (21), we can obtain as well

$$\frac{1}{J} \frac{dJ}{d\tau} = \nabla \cdot \nabla \Phi = \nabla^2 \Phi(x(\tau))$$

implying

$$J(x_0, \tau) = \exp \left[ \int_{\tau_0}^{\tau} d\tau' \nabla^2 \Phi(x(\tau')) \right] \equiv J(x(\tau)).$$  \hspace{1cm} (24)

By combining (22) and (24) we finally obtain:

$$v(x(\tau)) = \frac{v(x(\tau_0))}{\sqrt{J(x(\tau))}}.$$  \hspace{1cm} (25)
In view of this result, it is always possible to normalize \( v(x(\tau)) \), that is to require

\[
\int_{\Delta} d^3x v^2 = 1
\]  

(26)

where \( \Delta \) is the domain of integration determined by a particular problem to be solved. According to \([15]\), the energy density \( \varepsilon \) defined in (10), can be represented in view of (19a) as

\[
\varepsilon = \frac{1}{2} k^2 v^2 + \frac{k^2}{n^2} v^2 ( (\nabla \Phi)^2 + \frac{1}{k^2} (\nabla \ln v)^2 ) \rightarrow k \rightarrow \infty k^2 v^2.
\]

(27)

In the limit \( k \rightarrow \infty \) the analog of the wavefunction density \( \rho \), (9), is given now by

\[
\rho = \frac{\varepsilon}{\varepsilon} \rightarrow_{k \rightarrow \infty} \frac{v^2}{\int d^3x v^2} \rightarrow \Psi^* \Psi, \; \Psi = v(x(\tau)) \exp\{i \phi(x,t)\}.
\]

(28)

Substitution of the ansatz (17) into expression for the energy current (11) results in

\[
\frac{\mathbf{j}}{\varepsilon} = \frac{1}{2} v^2 (x) \nabla \Phi.
\]

(29)

The associated Hamilton-Jacobi (H-J) equation is given by (19a). In the limit \( k \rightarrow \infty \) its solutions are those for the Schrödinger equation. This fact is well known from the WKB theory where \( k \rightarrow \infty \) limit is the same as \( h \rightarrow 0 \) limit (the classical limit). We shall recover below Schrödinger’s equation without recourse to the \( k \rightarrow \infty \) (or \( h \rightarrow 0 \)) limit.

4. The role of electromagnetic vs vector and complex scalar fields in Schrödinger’s quantum mechanics

It is well known that the source-free Maxwell’s field equations can be recast in terms of the equations for the massless Dirac fields \([8,21]\), in terms of the massless 4-component vector fields \([22]\), and, in view of the results of this Letter, in terms of the massless complex scalar fields. There is still another option. A consistent relativistic quantum mechanics of spin 0 and 1 bosons can be developed with help of the Duffin-Kemmer equation \([22-24]\). This is Dirac-like 1st order equation in which the Dirac matrices are replaced by \( \beta \) matrices obeying commutation relation similar to those for the Dirac matrices. In dealing with the (anti) self-dual electromagnetic fields the twistorial interpretation of equations for these fields is also possible \([25]\). The comparison between Maxwell’s fields, the complex scalar, the Dirac and the vector fields is seemingly possible only for massless versions of these fields. The twistorial formalism apparently also excludes uses of the massive fields. Inclusion of masses into results

---

\( ^{2} \)With Dirac matrices being replaced by matrices for spin 1 fields
mentioned thus far does not cause additional difficulties though. All listed massive fields are reducible to the massless ones as will be explained below, in section 5.

To our knowledge, the interpretation of electromagnetic fields in terms of the complex scalar fields was not in use in physics literature thus far. The original paper by Roman [26] written in 1959 apparently was left unnoticed. This could be seen from the series of papers culminating in [23]-all using much more cumbersome Duffin-Kemmer formalism. Without repeating their content, and to avoid overlaps, we still would like to add some comments now. In standard field-theoretic notations [8,22] (making for a moment all constants equal to unity) the Lagrangian $\mathcal{L}$ for the complex massless scalar fields is given by, e.g. see [22], page 32,

$$\mathcal{L}[\varphi, \varphi^*] = \sum_n \frac{\partial \varphi}{\partial x^n} \frac{\partial \varphi^*}{\partial x^n}.$$  

(30a)

By varying the fields $\varphi$ and $\varphi^*$ in the action $\mathcal{A}$ defined by

$$\mathcal{A} = \int d^4x \mathcal{L}[\varphi, \varphi^*]$$  

(30b)

while assuming that these fields are independent and nicely decaying at infinity leads to the following equations of motion

$$\Box \varphi = 0,$$

(31a)

$$\Box \varphi^* = 0.$$  

(31b)

Here the d’Alembertian $\Box$ is defined as usual: $\Box = \frac{\partial^2}{\partial t^2} - \nabla^2$, with $\nabla^2$ being the 3-dimensional Laplacian. In his work [26] Roman was not interested in the standard field-theoretic analysis of $\mathcal{L}[\varphi, \varphi^*]$, e.g. that done in [22] on page 32. He was interested in proving that the gauge transformations of the Maxwellian fields rewritten in the formalism of complex scalar fields will keep the action $\mathcal{A}$ form-invariant (up to the total divergences vanishing at the boundary of space-time). The purpose of Roman’s work was in proving this result. With this result proven, the task of this Letter is different.

Given that the action $\mathcal{A}$ is gauge-invariant, we apply the standard field-theoretic treatment to (30) with purposes which will become obvious upon reading. From [22] we find the time component $T^{00}$ of the energy-momentum tensor (that is the energy density). It is given by

$$T^{00} = \frac{\partial \varphi}{\partial t} \frac{\partial \varphi^*}{\partial t} + \nabla \varphi \cdot \nabla \varphi^*.$$  

(32)

$T^{00}$ coincides with $\varepsilon$ defined by (10) where we temporarily put $c = 1$. Accordingly, the momentum density $T^{0i}$ given by

$$T^{0i} = -\left(\frac{\partial \varphi^*}{\partial t} \nabla_i \varphi + \frac{\partial \varphi}{\partial t} \nabla_i \varphi^*\right), i = 1, 2, 3,$$

(33)

up to a constant coincides with the flux $j$ defined in (11). Evidently, the continuity equation (12) is nothing but the law of conservation of the energy-momentum tensor [8]

$$\frac{\partial}{\partial x^\mu} T^{\mu}_\nu = 0.$$  

(34)
For the record, we use the Minkowski- space metric tensor $g_{\mu\nu}$ with signature $(+, -, -, -)$, so that $g_{\mu\nu} = g^{\mu\nu}$. For the 4-vector $a_\mu$ we have $a_\mu = g_{\mu\nu} a^\nu$, etc.

Because we are dealing with the complex scalar field, there is also a current vector $J_\mu$ responsible for carrying the charge (recall that the description of charges and currents associated with them is always associated with the existence of global gauge symmetry linked with use of complex scalar fields instead of real ones [22]). Mathematically, the conservation of current $J_\mu$ is expressible by analogy with (35) as

$$\frac{\partial}{\partial x^\mu} J_\mu = 0.$$  (35)

Explicitly, the current $J_\mu$ is given by

$$J_\mu = i (\varphi^* \frac{\partial \varphi}{\partial x^\mu} - \frac{\partial \varphi^*}{\partial x^\mu} \varphi).$$  (36)

Let $Q = \int d^3 x J_0$ be the total charge. Then (35) is the continuity equation associated with the charge conservation. It is well known [8] that $J_0$ is not always positively defined quantity. This fact is caused by the observation that at any given time $t$ both $\varphi$ and $\frac{\partial \varphi}{\partial x^0}$ may independently have arbitrary values. This fact is important because of the following. For the sake of generality and comparison, we include the mass $m^2$ term into both equations (31a,b) thus converting them into the Klein-Gordon (K-G) equations [22], page 32,

$$\left(\Box + m^2\right) \varphi = 0,$$  (37a)

$$\left(\Box + m^2\right) \varphi^* = 0.$$  (37b)

Since times of Schrödinger’s discovery of the equation bearing his name, the K-G equation was considered as relativistic analog of the Schrödinger’s equation. In analogy with nonrelativistic case by multiplying (37a) by $\varphi$ and (37b) by $\varphi^*$ and subtracting (37b) from (37a) we are repeating the same steps as for the nonrelativistic Schrödinger equation in order to obtain the continuity equation (35). In the nonrelativistic case (that is for the Schrödinger equation) this procedure yields: $j_0 = \rho = \psi^* \psi$ and $\vec{j} = \hbar \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$. The continuity equation in the nonrelativistic case is in its standard form: $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0.$

To compare this result with equations (35), (36) we must multiply (35) by $\frac{\hbar}{2m}$. Then, we obtain respectively $J_0 = \frac{\hbar}{2m}(\varphi^* \frac{\partial \varphi}{\partial x^0} - \frac{\partial \varphi^*}{\partial x^0} \varphi)$ and $J_k = \frac{\hbar i}{2m}(\varphi^* \frac{\partial \varphi}{\partial x^k} - \frac{\partial \varphi^*}{\partial x^k} \varphi) = - j_k$, $k = 1, 2, 3$.

Thus, even though (up to a sign) the currents $\vec{j}$ and $\vec{j}$ coincide, the densities $j_0$ and $J_0$ are noticeably different. This fact matters when the time-dependent problems are discussed for the K-G fields. Because of the nonpositivity of $J_0$ the full time-dependent K-G equation was discarded as the relativistic analog of the Schrödinger equation. In the time-independent case the situation is not so dramatic. Specifically, suppose that the field $\varphi$ in (37a) can be written as $\varphi(x,t) = \psi(x) \exp(-i \omega t)$, then, the K-G equation is converted into the Helmholtz equation

$$\left(\nabla^2 + m^2\right) \psi = 0.$$  (38)
Accordingly, the field $\varphi^*$ can be written now as $\varphi^*(x,t) = \psi(x) \exp(\imath \omega t)$. In view of these results, $J_0$ now acquires the form: $J_0 = \frac{\hbar \omega}{m} (\psi^* \psi)$. At the same time $J_k = 0$. This result allows us to study all kinds of stationary K-G equations [27] all making physical sense. The obtained result raises the following question. In section 3 we demonstrated that the continuity equation (12) is exactly the same as the energy-momentum conservation equation (34). At the same time, the continuity equation (9) for the photon is the same as the continuity equation (12) for the complex scalar field. This means that this equation can be used instead of the continuity equation (35) for development of the time-dependent quantum mechanical formalism for the complex scalar K-G field. It is appropriate at this point to mention that historically the massive complex scalar field was used in Yukawa theory of strong interactions where it is known as pi meson field. By extending results by Harish-Chandra [23] for this field written in the language of Duffin-Kemmer formalism already mentioned, Tokouoka [28] had studied in detail the meson-nucleon interactions. Much more recent results are discussed, for example, in [29] and references therein. Since the meson-nucleon interactions are described nowadays with help of quantum chromodynamics (QCD), the complex scalar field acts effectively as the abelianized version of the non Abelian Yang-Mills gauge field. More on this is discussed in the next section where we shall also explain how to get rid of the mass term in the K-G equation. According to [27], page 99, every spinor component of the Dirac equation with nonzero mass is satisfying the massive K-G equation. In the zero mass limit this fact creates the equivalence class between the massless K-G and Dirac equations. Apparently components of equations for higher spin particles, e.g. spin-2 gravitons, also belong to the same equivalence class [8,22,27]. In this section we still need to demonstrate how results of sections 3 and 4 are connected with the non relativistic time-independent Schrödinger equation (TISE). In the next section we shall present evidence that such an equation also belongs to the same equivalence class as the rest of basic equations for integer and half integer spins.

To connect results of sections 3 and 4 with TISE, just described treatment of the stationary K-G equation is the most helpful. After the mass term is eliminated in this equation, the K-G equation is converted into the wave equation (31). This fact allows us to use the results of Schrödinger’s second foundational paper on quantum mechanics [30], pages 13-40. In (31) we replace the speed of light $c$ by $u = c/n$, where $n = n(x,y,z)$ is the effective refractive index. Next, we use the same ansatz $\varphi(x,t) = \psi(x) \exp(\imath \omega t)$ for the wave function. Next we replace the dispersion relation $\omega = ck$ for electromagnetic waves in the vacuum by $\omega = uk$ and use the de Broglie-type relation $k = p/\hbar$ along with the fact that for the mechanical system $E = p^2/2m + V$. Thus we obtain: $\frac{p^2}{2m} = 2m(E - V)$. With the help of these results the TISE follows:

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V - E\right] \psi = 0.$$  

(39)

It follows immediately from the wave equation (31) under conditions just described. The normalized result for $J_0$ can be used immediately so that the continuity equation (35) works in this case. Nevertheless, the question remains: what to do with the continuity equation (34)? This equation is used for development of quantum mechanics of photons. Can it

---

As described below
be used in the nonrelativistic case of stationary Schrödinger equation? Equations (31)-(34) suggest that this could be possible. Nothing meaningful is obtained straightforwardly though. Indeed, by using the K-G ansatz for the wave function in (32) and (33) and restoring \( c \) in these equations yields:

\[
T^{00} = \left( \frac{\omega}{c} \right)^2 \psi^* \psi + \nabla \psi \cdot \nabla \psi^*, \tag{40a}
\]

\[
T^{0i} = 0. \tag{40b}
\]

The result obtained for \( T^{00} \) is different from that for \( J_0 \). Unlike [16], we are not going to look for arguments in favor of \( \nabla \psi \cdot \nabla \psi^* \simeq \left( \frac{\omega}{c} \right)^2 \) relation. This was done already in section 3 where results were obtained in the regime of geometrical optics. Instead, now we are going to use Schrödinger's ideas again. This time we are going to use methods developed in his 1st paper on quantum mechanics [30], pages 1-12, as well as from the already discussed 2nd paper. In particular, we again replace \( c \) by \( u \) in (40a) and then replace \( \left( \frac{\omega}{u} \right)^2 \) by \( \left( \frac{p}{\hbar} \right)^2 \). Next, by looking at (40a) we require

\[
\nabla \psi \cdot \nabla \psi^* = \left( \frac{p}{\hbar} \right)^2 \psi^* \psi
\]

and use \( \left( \frac{p}{\hbar} \right)^2 = \frac{2m(E-V)}{\hbar^2} \). These results can be rewritten in terms of the H-J equation if we temporarily use only real valued functions: \( \psi(x) = \psi^*(x) \)

\[
(\nabla \psi)^2 = \frac{2m(E-V)}{\hbar^2} \psi^2 \tag{41}
\]

Notice that this H-J equation contains explicitly Planck’s constant \( h \) while the H-J equation used in the semiclassical WKB calculations is by design \( h \)-independent. Nevertheless, (41) coincides exactly with the equation (1\textsuperscript{st}) of Schrödinger’s 1st paper on quantum mechanics, [30], pages 1-12. This difference has a profound effect on the rest of Schrödinger’s calculations. It allows him and us to restore the stationary Schrödinger equation without any approximations. For this purpose, Schrödinger introduces the functional \( J[\psi] \)

\[
J[\psi] = \frac{1}{2} \int d^3x (|\nabla \psi|^2 - \frac{2m(E-V)}{\hbar^2} \psi^2) \tag{42a}
\]

which he is minimizing under the subsidiary condition

\[
\int d^3x \psi^2 = 1. \tag{42b}
\]

Minimization produces the stationary Schrödinger equation (39) as anticipated. Thus, the continuity equations (34) and (35) both can be used in conjunction with the nonrelativistic stationary Schrödinger’s equation (39).

5. Role of knotted/link null electromagnetic fields in the theory
of nonrelativistic Schrödinger equation

Our readers at this point might ask a question: How the obtained results are affected by currents and charges present in electromagnetic field? Recall, Bohm [4,11] was concerned exactly with this issue when he compared quantum mechanical and optical formalisms. But currents are made of moving charges! Therefore, the above question should be modified accordingly. In [31, 32] we further developed results of Ranada [33,34] who skillfully using electric-magnetic duality for source-free electromagnetic fields obtained stable solutions which we reinterpreted as Dirac monopoles, electrons and dyons (particles possessing simultaneously both the electric and magnetic charges). Topologically all these objects are represented by the interlinked (Hopf) rings made of closed electric (electron) or magnetic (Dirac monopole) lines and, for dyons, by linked and closed two electric and two magnetic lines. More complicated objects are also possible [32] but dynamically they are less stable [35]. Many of these objects belong to the so called null fields to be defined below but some are made of fields which are not null [36]. The question remains: How the masses enter into this purely topological picture? This issue was discussed in our works [31,35]. We argued that complements of knots and links make the ambient Minkowski space-time curved in the same way as caused by masses in Einsteinian gravity. Since masses should be positive, not all knots and links are allowed to exist. This restriction excludes all hyperbolic knots and links from consideration. Incidentally, mathematical methods described in [36] for knot/link generation do not involve creation of hyperbolic knots/links. The rationale for absence of hyperbolic knots and links in mathematical formalism described in [36] was not explained. The same results as in [36] were obtained in our work [32] where knots and links were generated dynamically. This allowed us in the same paper to develop mathematical and physical explanation of exclusion of hyperbolic knots and links from consideration. Introduction of masses into general relativity as well as charges into non Abelian Yang-Mills (NAYM) theory is associated with very serious technical problems. These are described in [37], page 97, and references therein. In the case on NAYM theories the problem of charges is by-passed by treating only source-free NAYM fields allowing existence of monopoles, dyons, etc. [37]. In [31] we reobtained Ranada’s results for Abelian Yang-Mills fields using standard instanton formalism. At the same time, methods for knot/link generation described in [36] do not involve uses of instantons. Although it is commonly believed that instanton methods are applicable only to the NAYM fields, many years ago Trautman obtained the Dirac monopole solution with help of the Abelian YM fields [38]. Incidentally, the same result is obtained by using the NAYM fields. The Abelian reduction method [31,37] permits to obtain results for the Dirac monopole starting with the non Abelian t’Hooft-Polyakov monopole. This is shown in detail in [39], page174. Since gravity can be rewritten in the language of NAYM fields as it was shown in 1956 by Utiyama [40], it is only natural that Robinson in 1961 was able to connect gravity with source-free Maxwell’s fields [36,41].

Now we would like to shed some light on the following issue: How presence or absence of masses is related to topics just described? Surprisingly, the link between these topics can be established with help of the Huygens’ principle. This principle can be discussed purely mechanically with methods of contact geometry and topology as it was done, for example, by
Arnol’d [42]. Reader’s friendly basics of contact geometry and topology is provided in [37].

The same principle could be discussed using theory of partial differential equations discussed, for example, by Hilbert and Courant [17], Luneburg [18], Hadamard [43] and later, by others. For the sake of space we shall not go into details pertained to Huygens’ principle in this Letter. We only mention that equations (31a,b) do obey the Huygens’ principle so that all equations which can be reduced to (31 a,b) do obey this principle. According to Hadamard, only 3 operations are allowed for conversion of a given PDE equation to the “trivial” equation(s) (31a,b). Specifically, the Huygens’ equivalence principle can be formulated as follows.

Let \( L[\phi] \) be the Huygens-equivalent operator (that is operator which is equivalent to (31a,b)) and let \( \tilde{L}[\phi] \) be another operator. Then, they are Huygens-equivalent if:

a) \( \tilde{L}[\phi] \) can be obtained from \( L[\phi] \) by the nonsingular transformations of independent variables.

b) \( \tilde{L}[\phi] = \lambda^{-1} L[\lambda \phi] \) for some positive, smooth function \( \lambda \) of independent variables.

c) \( \tilde{L}[\phi] = \rho L[\phi] \) for some positive smooth function \( \rho \) of independent variables.

With these rules in our hands, let us consider now an invertible sequence of transformations from the d’Alembert equation (31) to the massive K-G equation. Let us use the transformation \( \tilde{L}[\phi] = \lambda^{-1} L[\lambda \phi] \) with \( \lambda = e^{\alpha t} \), with \( \alpha \) some constant. Then, we obtain:

\[
e^{-\alpha t} \left( \frac{\partial^2}{\partial t^2} e^{\alpha t} \phi - [\nabla^2 e^{\alpha t} \phi] \right) = \frac{\partial^2}{\partial t^2} \phi + 2a \frac{\partial}{\partial t} \phi - \nabla^2 \phi + a^2 \phi = 0.
\]

Next, let \( \lambda_1 = e^{ibx} \) and \( \lambda_2 = e^{-icx} \). Upon substitution of these factors into previous equation, while using the rule b), we obtain after some calculation

\[
\frac{\partial^2}{\partial t^2} \phi + 2a \frac{\partial}{\partial t} \phi - \nabla^2 \phi + 2ib \frac{\partial}{\partial x} \phi - 2ic \frac{\partial}{\partial x} \phi + (a^2 - b^2 - c^2) \phi = 0.
\]

If now \( b = c \) and \( a^2 = 2b^2 \), we obtain the telegrapher equation

\[
\frac{\partial^2}{\partial t^2} \phi + 2a \frac{\partial}{\partial t} \phi - \nabla^2 \phi = 0.
\]

The K-G equation is obtained now upon substitution \( \phi = e^{mt} \psi \) into telegrapher’s equation with subsequent replacement of \( m \) by \( im \). Thus the K-G and d’Alembert equations are Huygens’-equivalent. If this is so, it would be possible to use the twistor formalism [36] for the K-G equation. The question arises: Is the stationary Schrödinger equation Huygens-equivalent to the d’Alembert equation?

In 1935 Fock initiated study of this problem (having different goals in mind though). While studying the (accidental) degeneracy of the stationary Schrödinger equation for the hydrogen atom he converted this equation into the integral equation looking very similar to the Poisson integral in the theory of functions of one complex variable. Recall that harmonic functions (that is functions obeying the Laplace equation) inside the circle (and, thus, of any domain which can be conformally mapped into circle) are represented via Poisson integral. Fock initiated study of what is known now as dynamical symmetry groups. These groups are allowing to solve quantum mechanical problems group-theoretically thus by-passing uses
of the Schrödinger equation. This direction of research has grown into a large field [44] nowadays. Fock’s work is presented in Vol.1, pages 400-410 of [44]. Group-theoretic analysis had lead Fock to conclusions that the bound states of hydrogen atom should be studied in 4-dimensional Euclidean space while scattering states should be studied in 3+1 dimensional hyperbolic (Lobachevski) space. The Euclidean 4-dimensional version of the Schrödinger equation (for the bound states) was reduced by Fock to study of solutions of the 4-dimensional Laplacian. The description of the scattering states was only outlined by Fock. He suggested that this case would require study of solutions of the d’Alembertian. This suggestion was waiting for its solution for 31 years. In 1966 it was finally solved by Itzykson and Bander [45]. At the level of classical mechanics all finite dimensional exactly integrable systems are equivalent to each other since their trajectories in phase space are Liouville tori [42]. Thus, Fock results for the hydrogen atom are canonically extendable to all two–body exactly solvable nonrelativistic mechanical problems. Although Itzykson and Bander [45] obtained the d’Alembertian for scattering states thus making the stationary Schrödinger equation Huygens-equivalent to the d’Alembertian equation, the results of [45] happened to be very cumbersome. The transparency of results for both bound and scattering states of hydrogen atom was achieved only in 2008 in the paper by Frenkel and Libine [46]. Using quaternions-analogs of complex numbers in 4 dimensions these authors extended the Poisson formula for harmonic functions from 2 to 4 dimensions. This allowed them to extend effortlessly the obtained results from Euclidean 4 dimensional to Minkowski 3+1 spaces.

With these results in our hands now we are in a position to explain quantum mechanical significance of the null fields. These fields were known since 1787, when Chladni, a German physicist, studied nodal lines of a vibrating metal plate by stroking this plate covered by sand with a violin bow [47]. After studying a variety of nodal patterns systematically, he wrote a book in 1802 where all these patterns were systematized. Remarkably, the 1802 edition of Chladni book was translated into English and published in 2015 [48]. Nowadays, Chladni patterns can be seen on YouTube [49].

With help of results of previous sections now we are in the position to conclude this Letter with explanation of the connection between Chladni patterns, null fields, knots/links and quantum mechanics. To set up the notations, we begin with the description of two dimensional Chaladni problems following [50]. It is convenient to think about a given Riemannian surface $\Sigma$ with metric $g_\Sigma$ as a vibrating membrane with $u(\mathbf{x}, t)$, ($\mathbf{x} \in \Sigma$) being a displacement at time $t$ of the membrane from its original position. The function $u$ is a solution to the wave equation

$$\frac{\partial^2}{\partial t^2} u = \nabla^2_{\Sigma} u.$$  \hspace{1cm} (43)

By representing solution in the form $u(\mathbf{x}, t) = v(t) w(\mathbf{x})$, the above equation splits into two equations, e.g.

$$\frac{\partial^2}{\partial t^2} v = \lambda v$$  \hspace{1cm} (44a)

and

$$\nabla^2_{\Sigma} w = \lambda w.$$  \hspace{1cm} (44b)
The (null) zero set, \( \Xi(w) := \{ x \in \Sigma : w(x) = 0 \} \), is called nodal (Chladni) set. The definitions just described need to be supplemented with the boundary (e.g., Dirichlet or Neumann) conditions so that the above nodal problem is known as fixed membrane problem. In the case when the membrane is a closed surface the above problem is known as free membrane problem. The first detailed calculation of Chladni patterns was done by Poisson in 1829. The chronology of subsequent developments along with detailed numerous examples can be found in encyclopedic book [51] by Lord Rayleigh in ch.-rs 9 and 10. The same results were recently reproduced in the book [52]. At much more advanced level Chladni patterns were studied by Cheng [53] who proved that for an arbitrary smooth Riemannian surface \((\Sigma, g_\Sigma)\) the nodal set is a collection of immersed closed curves. These curves are easily seen in [49] at sufficiently high frequencies, provided the periodic boundary conditions on the plate are imposed. Cheng’s result is remarkable since later on analogous results were obtained in 3 dimensions in [54]. This time, since the nodal curves are closed, they can generate knots and links, of any complexity. The results of [54] can be reobtained with help of results obtained in this Letter superimposed with [31,32,37]. Following [55] we write

\[
\mathbf{F}(r, t) = \mathbf{F}_+ + \mathbf{F}_-. \quad \mathbf{F}_\pm(r, t) = \int_0^\infty d\omega e^{\mp i\omega t} \mathbf{F}_\pm(\omega, r).
\]

In such notations (7),(8) can be rewritten as

\[
\nabla \cdot \mathbf{F}_\pm = 0, \quad (46a)
\]
\[
\nabla \times \mathbf{F}_\omega = k\mathbf{F}_\omega \quad (46b)
\]

with \( k = \omega/c \). In plasma physics (46b) is known as ”force-free equation” while in hydrodynamics it is known as ”Beltrami” equation. It was discussed in detail in our book [37] in the context of methods of contact geometry/topology while in [31] and [32] these methods were used for generation of all kinds of nonhyperbolic knots and links, including those of Ranada-type, that is Hopf-like links. Applying operator \( \nabla \times \) to (46b) while taking into account (46a) results in

\[
\nabla^2\mathbf{F}_\omega + k^2\mathbf{F}_\omega = 0 \quad (47)
\]

to be compared with (44). This (vector) version of the Helmholtz equation is known as Chandrasekhar-Kendall (CK) equation [37]. These authors noticed that every solution of (46b) is solution of (47) but the converse is not true. This happens to be of fundamental importance for our tasks. In [37], page 30, we stated that solution of (46b) is a composition of fields of 3 types: a) solenoidal (46a), b) lamellar \( \mathbf{F}_\omega \cdot (\nabla \times \mathbf{F}_{-\omega}) = 0 \), c) Beltrami \( \mathbf{F}_\omega \times (\nabla \times \mathbf{F}_{-\omega}) = 0 \). The null fields used in creation of ”linked and closed beams of light” [36] are lamellar. In terms of notations set up in [55] they are defined as

\[
\mathbf{F}_\omega \cdot \mathbf{F}_{-\omega} = 0. \quad (48a)
\]
As explained in [37], the same classification of fields exist in physics of liquid crystals. In [37], pages 32-34, it was demonstrated that the Faddeev-Skyrme (F-S) knot/link generating model is of the same liquid crystalline origin. It also can be looked upon as originating from the Abelian reduction of NAYM fields. Therefore, all results of [36] as well as of [31,32] are compatible with those originating from the F-S model [56]. Now we can study lamellar (null) solutions of the C-K equation. Being logically guided by [54,55] our results differ in details from results of these works. This permits us to inject new physics absent in these references. We begin with (48a). It can be rewritten as

\[ |\mathbf{E}_\omega|^2 - |\mathbf{H}_\omega|^2 + 2(\mathbf{E}_\omega \cdot \mathbf{H}_\omega + \bar{\mathbf{E}}_\omega \cdot \mathbf{H}_\omega) = 0. \]  

This equation is surely satisfied if \( |\mathbf{E}_\omega|^2 = |\mathbf{H}_\omega|^2 \) and \( \mathbf{E}_\omega \cdot \mathbf{H}_\omega + \bar{\mathbf{E}}_\omega \cdot \mathbf{H}_\omega = 0 \). Next, without loss of generality and following [55] it is permissible to assume that \( \mathbf{E}_\omega = \bar{\mathbf{E}}_\omega \) and \( \mathbf{H}_\omega = \bar{\mathbf{H}}_\omega \). This then leads to \( \mathbf{E}_\omega \cdot \mathbf{H}_\omega = 0 \). Should we avoid use of the time Fourier transforms given in (45), just presented two (null) equations (that is \( |\mathbf{E}_\omega|^2 = |\mathbf{H}_\omega|^2 \) and \( \mathbf{E}_\omega \cdot \mathbf{H}_\omega = 0 \)) are sufficient for generation of all kinds of torus knots and Hopf links, including those discovered by Ranada, evolving in time [36, 57,58]. These results do not let us to make a connection with Chladni patterns though and with the physics associated with these patterns. To correct this, using [59] we consider the following remarkable identity

\[ (\nabla^2 + k^2)(\mathbf{r} \cdot \mathbf{v}) = 2\nabla \mathbf{v} + \mathbf{r} \cdot (\nabla^2 + k^2)\mathbf{v}. \]  

Here \( \mathbf{v} \) is either \( \mathbf{E}_\omega \) or \( \mathbf{H}_\omega \). The scalar \( \mathbf{r} \cdot \mathbf{F}_\omega \) is convenient to rewrite in the notations of [54], i.e. \( \mathbf{r} \cdot \mathbf{F}_\omega = u = u_1 + iu_2 \). Evidently, in view of (16b) scalar \( u \) can be identified with \( V(\mathbf{x},t) \) defined in (13) so that \( \mathbf{E}_\omega \) and \( \mathbf{H}_\omega \) can be recovered if needed. Fortunately, this will not be necessary in the rest of this Letter. By combining (47) and (49) we obtain Chladni-type equations

\[ (\nabla^2 + k^2)u_{k1} = 0 \quad \text{and} \quad (\nabla^2 + k^2)u_{k2} = 0. \]  

These equations were introduced and discussed in [54] using purely mathematical arguments. Since here the same equations were reobtained with help of physical arguments, this allows us to extend results of [54]. First, we notice that both equations have the same eigenvalue \( \lambda_1 = \lambda_2 = k^2 \). Having the same eigenvalue (of multiplicity 2) the wavefunctions \( u_{k1} \) and \( u_{k2} \) are not the same though as demonstrated in [54] and below. Although the presence of i-factor is essential, the difference goes beyond this fact. This difference should not be confused with the degeneracy concept in quantum mechanics. Presence of eigenvalues having multiplicities is responsible for effects of entanglements. This is explained in detail our book [37], pages 386-395. Now we are in the position to demonstrate that: a) quantum mechanical entanglement is equivalent with the entanglement in the topological knot-theoretic sense; b) presence of eigenvalues with multiplicity makes Schrödinger’s and Heisenberg’s interpretation of quantum mechanics not equivalent. The last statement follows immediately from the detailed Heisenberg-style calculations presented in [60]. Therefore, we are only left with explanation of a).

Following [31] consider force-free equation (46b), where temporarily we replace \( \mathbf{F}_\omega \) by \( \mathbf{v} \) as in (49). Then by applying to both sides of (46b) the operator \( \nabla \cdot \) and by assuming
that $k = \text{const} = \kappa(x,y,z)$ we obtain $\nabla \cdot \kappa \mathbf{v} = \mathbf{v} \cdot \nabla \kappa = 0$. Let $\mathbf{r}(t) = \{x(t), y(t), z(t)\}$ be some trajectory on the surface $\text{const} = \kappa(x,y,z)$. In such a case $\frac{d}{dt} \kappa \{x(t), y(t), z(t)\} = v_x \kappa_x + v_y \kappa_y + v_z \kappa_z = \mathbf{v} \cdot \nabla \kappa = 0$. This means that the "velocity" $\mathbf{v}$ is always tangential to the surface $\text{const} = \kappa(x,y,z)$. Since the vector field $\mathbf{v}$ is being assumed nowhere vanishing, the surface $\text{const} = \kappa(x,y,z)$ can only be as torus $T^2$. The field lines of $\mathbf{v}$ on $T^2$ should be closed if $\text{const}$ is rational number. Thus we just demonstrated that the force-free equation (46b) supplies us with the condition of existence of all possible torus knots for rational $\kappa$'s.

Starting with work by Ranada such torus knots were explicitly designed both in [31, 32] and [36]. Now it remains to demonstrate that Chladni-type equations (50) lead to 3-dimensional Chladni patterns associated with these equations. Since this was done already in [54], our task is only to supply some physics to the results of [54]. Since the surface $\text{const} = \kappa(x,y,z)$ is $T^2$, while the solid torus is defined by $T^2 = D^2 \times S^1$, with $D^2$ being a disc, it is convenient to introduce a cylindrical system of coordinates and to consider the Neumann-type problem for the Helmholtz equations (50) written in cylindrical coordinates. This task is facilitated by the accumulated knowledge about circular waveguides in electrodynamics.

The solutions $u_{k_1} = J_1(\rho \sqrt{\lambda}) \cos \varphi$ and $u_{k_2} = J_1(\rho \sqrt{\lambda}) \sin \varphi$ discussed in [54][ here are just TE and TM-type solutions known for circular waveguides [61]. For small $\rho$'s they are represented by $u_{k_1} \simeq \frac{\sqrt{\lambda}}{2} \rho \cos \varphi$ and $u_{k_2} \simeq \frac{\sqrt{\lambda}}{2} \rho \sin \varphi$. Both functions become zero for $\rho = 0$. Thus, the individual nodal Chladni sets are respectively given by $u_{k_1}^{-1}(0)$ and $u_{k_2}^{-1}(0)$, the Chladni centerline $S^1$ for the solid torus is determined by the transversality condition: $u(0) = u_{k_1}^{-1}(0) \cap u_{k_2}^{-1}(0)$. It is indeed the transversality condition since we can plot both $\rho \cos \varphi$ and $\rho \sin \varphi$ on the complex plane $C$ so that they are transversal to each other [62]. The transversality condition needed for the validity of Thom's isotopy theorem [54, 62] (assuring that the obtained results are stable with respect to possible perturbations). These perturbations will occur because by definition every knot is an embedding of $S^1$ into $S^3 = \mathbb{R}^3 \cup \{\infty\}$. The validity of Thom’s theorem is required to assure that the embedding of solid tori $T^2$ into $S^3$ could be done in such a way that it shall produce knots of any complexity as long as they are non wild. Although [54] claims that all knots could be obtainable this way, physically, this is unrealistic since complex knots are dynamically generated from the simpler ones [35] and such process precludes the formation of hyperbolic-type knots [32].

References

[1] R.Feynman and R.Leighton, The Feynman Lectures on Physics, Vol.3,Basic Books, New York, 2011.

[2] M.Arndt, O.Nairz, J.Vos-Andreae, C.Keller, G.Zoun and A.Zellinger, Nature, 401 (1999) 680.

[3] S. Eibenberger, S. Gerlich, M. Arndt, M. Mayor and J. Tüxen, Phys. Chem. Chem. Phys. 15 (2013) 14696.

[4] D. Bohm, Quantum Theory, Dover Publications Inc., New York, 1989.

---

4Here $J_1(x)$ is the standard Bessel function and $\lambda$ adjusted (with account of cylindrical symmetry) eigenvalue.
[5] A. Sanz, and S. Miret-Artés, A Trajectory Description of Quantum Processes. I. Fundamentals, Springer-Verlag, Berlin, 2012.
[6] U. Leonhardt, Measuring the Quantum State of Light, Cambridge U. Press, Cambridge, UK, 1997.
[7] M. Fox, Quantum Optics: An Introduction, Oxford U. Press, Oxford, UK, 2006.
[8] V. Berestetskii, E. Lifshitz and L. Pitaevskii, Relativistic Quantum Theory, Pergamon Press, Oxford, UK, 1971.
[9] E. Collett, Am. J. Phys. 39, (1971) 1483.
[10] B. Kanseri, N. Bisht and H. Kandpal, Am. J. Phys. 76 (2008) 39.
[11] D. Bohm and B. Hiley, Undivided Universe, Rutledge Publ. Co., London, 1993.
[12] D. Kobe, Found. Phys. 29 (1999) 1203.
[13] M. Raymer and B. Smith, SPIE Conference on Optics and Photonics, Conference number 5866, (2005).
[14] M. Raymer and B. Smith, New J. Phys. 9 (2007) 414.
[15] H. Green and E. Wolf, Proc. Phys. Soc. A 66 (1953) 1129.
[16] E. Wolf, Proc. Phys. Soc. A 74 (1959) 269.
[17] D. Hilbert and R. Courant, Methods of Mathematical Physics, Vol. 2, Interscience Publishers, New York, 1962.
[18] R. Luneburg, Mathematical Theory of Optics, U. of California Press, Berkeley, CA, 1966.
[19] V. Maslov and M. Fedoriuk, Semiclassical Approximation in Quantum Mechanics, Reidel Publ. Co., Boston, MA, 1981.
[20] H. Römer, Theoretical Optics, Wiley-VCH, Hoboken, NJ, 2005.
[21] A. Akhiezer and V. Berestetsky, Quantum Electrodynamics, Interscience Publishers, New York, 1965.
[22] N. Bogoliubov and D. Shirkov, Introduction to the Theory of Quantized Fields, John Wiley & Sons, New York, 1976.
[23] Harish-Chandra, Proc. Roy. Soc. London, A186 (1946) 502.
[24] P. Ghose, A. Majumdar, S. Guha and J. Sau, Phys. Lett. A 290 (2001) 205.
[25] B. Shabat, Introduction to Complex Analysis. Part II, American Mathematical Society, Providence, RI, 1992.
[26] P. Roman, Proc. Phys. Soc. A 74 (1959) 281.
[27] W. Greiner, Relativistic Quantum Mechanics. Wave Equations, Springer-Verlag, Berlin, 2000.
[28] Z. Tokuoka, Prog. Theor. Phys. 10 (1953) 137.
[29] E. Rojas, B. El-Bennich, J. de Melo and M. Paracha, Few-Body Syst. 56 (2015) 639.
[30] E. Schrödinger, Collected Papers on Wave Mechanics, Chelsea Publ. Co., New York, 1978.
[31] A. Kholodenko, Analysis & Math. Phys. 6 (2016) 163.
[32] A. Kholodenko, Ann. Phys. 371 (2016) 77.
[33] A. Ranada, Lett. Math. Phys. 18 (1989) 97.
[34] A. Ranada, J. Phys. A 25 (1992) 1621.
[35] A. Kholodenko, Int. J. Mod. Phys. A 30 (2015) 1550189.
[36] A. Arrayas, D. Bouwmeester and J. Trueba, Phys. Rep. 667 (2017) 1.
[37] A. Kholodenko, Applications of Contact Geometry and Topology in Physics, World Scientific, Singapore, 2013.
[38] A. Trautman, Int. J. Theor. Phys. 16 (1977) 561.
[39] M. Göckeler and T. Shuker, Differential Geometry, Gauge Theories and Gravity, Cambr. U. Press, Cambridge, UK, 1987.
[40] R. Utiyama, Phys. Rev. 101 (1956) 1597.
[41] I. Robinson, J. Math. Phys. 2 (1961) 290.
[42] V. Arnol’d, Mathematical Methods of Classical Mechanics, Springer-Verlag, Berlin, 1989.
[43] P. Günter, Huygens Principle and Hyperbolic Equations, Academic Press Inc., Boston, MA, 1988.
[44] A. Bohm, Y. Ne’eman and A. Barut, Dynamical Groups and Spectrum Generating Algebras, Vol. 1 & 2, World Scientific, Singapore, 1988.
[45] C. Itzykson and M. Bander, Group theory of the hydrogen atom, I & II, Rev. Mod. Phys. 38 (1966), 330 and 346.
[46] I. Frenkel and M. Libine, Adv. Math. 218 (2008) 1806.
[47] D. Jacobson, N. Nadirashvili and J. Toth, Russian Math. Surveys 56 (2001) 1085.
[48] E. Chaladni, Tretease on Acoustic, Springer-Verlag, Berlin, 2015.
[49] https://www.youtube.com/watch?v=3idd3GEmOwK
[50] R. Komendarczyk, AMS Transactions, 358 (2005) 2399.
[51] J. Rayleigh, Theory of Sound, Vol. 1, Macmilland and Co. Ltd, London, 1896.
[52] T. Rossing and N. Fletcher, Principles of Vibrations and Sound, Springer-Verlag, Berlin, 2004.
[53] S-Y. Cheng, Comm. Math. Helvetici 51 (1976) 43.
[54] A. Enciso, D. Hartley and D. Peralta-Salas, J. Funct. Analysis 271 (2016) 182.
[55] G. Kaiser, J. Opt. A 6 (2004) S243.
[56] http://hopfion.com/faddee.html
[57] I. Besieris and A. Shaarawi, Optics Lett. 34 (2009) 3887.
[58] J. Waite, The Hopf Fibration and Encoding Torus Knots into Light Fields, Ms of Sci. thesis, U. of Nevada, 2016.
[59] C. Bouwkamp and H. Casimir, Physica 20 (1954) 539.
[60] A. Kholodenko, Int. Math. Forum 4 (2009) 509.
[61] http://nptel.ac.in/courses/112105165/lec14.pdf
[62] A. Majthay, Foundations of Catastrophe Theory, Pitman Publishers, Boston, 1985.