SECONDARY CHARACTERISTIC CLASSES OF SURFACE BUNDLES

SØREN GALATIUS

Abstract. The Miller-Morita-Mumford classes associate to an oriented surface bundle $E \to B$ a class $\kappa_i(E) \in H^{2i}(B; \mathbb{Z})$. In this note we define for each prime $p$ and each integer $i \geq 1$ a secondary characteristic class $\lambda_i(E) \in H^{2i(p-1)-2}(B; \mathbb{Z}/p^2)$. The mod $p$ reduction $\lambda_i(E) \in H^*(B; \mathbb{F}_p)$ has zero indeterminacy and satisfies $p\lambda_i(E) = \kappa_{i(p-1)-1}(E) \in H^*(B; \mathbb{Z}/p^2)$.

1. Introduction and statement of results

Recall that any bundle $\pi : E \to B$ of oriented surfaces with finite dimensional base $B$ has an embedding $j : E \to B \times \mathbb{R}^{N+2}$ over $B$. For $N$ large, $j$ is unique up to isotopy. A choice of embedding $j$ induces a transfer map

$$B_+ \wedge S^{N+2} \xrightarrow{\pi} \text{Th}(\nu j)$$

The embedding $j : E \to B \times \mathbb{R}^{N+2}$ also induces classifying maps

$$T_{\pi}E \xrightarrow{\text{cl}(\nu j)} \text{SO}(N+2) \times_{\text{SO}(N) \times \text{SO}(2)} \mathbb{R}^2$$

$$E \xrightarrow{} \text{SO}(N+2)/\text{SO}(N) \times \text{SO}(2)$$

and

$$\nu j \xrightarrow{\text{cl}(\nu j)} \text{SO}(N+2) \times_{\text{SO}(N) \times \text{SO}(2)} \mathbb{R}^N$$

$$E \xrightarrow{} \text{SO}(N+2)/\text{SO}(N) \times \text{SO}(2)$$

For brevity, write $U = U_N = \text{SO}(N+2) \times_{\text{SO}(N) \times \text{SO}(2)} \mathbb{R}^2$ and $U_\perp = U_N^\perp = \text{SO}(N+2) \times_{\text{SO}(N) \times \text{SO}(2)} \mathbb{R}^N$. We get the composition

$$\alpha = \text{Th}(\text{cl}(\nu j)) \circ \pi_1 : B_+ \wedge S^{N+2} \to \text{Th}(U_N^\perp)$$

Recall that there is a Thom class $\lambda_{U_\perp} \in H^N(\text{Th}(U_\perp), *; \mathbb{Z})$ and that we have $H^{N*}(\text{Th}(U_\perp), *; \mathbb{Z}) = \mathbb{Z}[e(U)] \lambda_{U_\perp}$ for $* < N$. The definition of the $\kappa$-classes is

$$\kappa_i E = \alpha^*(e(U)^{i+1}.\lambda_{U_\perp}) = \pi_1^*(e(T_\pi E)^{i+1}.\lambda_{\nu j}) \in H^{2i}(B; \mathbb{Z})$$
In this paper we define secondary characteristic classes of surface bundles. The
definition involves Toda brackets. In section 2 we recall some generalities about
Toda brackets. By a surface bundle we shall mean a fibre bundle with closed
oriented smooth two-dimensional fibres.

**Lemma 1.1.** Let $p$ be a prime, and let $P^i$ denote the Steenrod power operation.
When $p = 2$, write $P^i = Sq^{2i}$ and $\beta P^i = Sq^{2i+1}$. Given a surface bundle $\pi : E \to B$,
let $\alpha : B_+ \wedge S^{N+2} \to \text{Th}(U^+_N)$ be as before and let $\lambda : \text{Th}(U^+_N) \to K(\mathbb{Z}, N)$ be
the Thom class. Then the Toda bracket

$$\{\beta P^i, \lambda, \alpha\} \subseteq H^{2i(p-1)-2+N}(B_+ \wedge S^{N+2}; \mathbb{Z}) = H^{2i(p-1)-2}(B; \mathbb{Z})$$

is defined with indeterminacy $\mathbb{Z}\kappa_{i(p-1)-1}$.

**Definition 1.2.** With notation as in Lemma 1.1 define

$$\lambda_i(E) = (-1)^i \{\beta P^i, \lambda, \alpha\} \in H^{2i(p-1)-2}(B; \mathbb{Z}) / \mathbb{Z}\kappa_{i(p-1)-1}$$

**Theorem 1.3.** The mod $p$ reduction $\lambda_i(E) \in H^*(B; \mathbb{F}_p)$ has zero indeterminacy
and satisfies

$$p\lambda_i(E) = \kappa_{i(p-1)-1} \in H^*(B; \mathbb{Z}/p^2)$$

More generally we have the following in integral cohomology

$$\kappa_{i(p-1)-1} \in p\lambda_i(E)$$

**Theorem 1.4.**

(i) If $\pi : E \to B$ and $\pi' : E' \to B$ are surface bundles, then

$$\lambda_i(E \amalg E') = \lambda_i(E) + \lambda_i(E')$$

(ii) If $\pi : E \to B$ is a surface bundles and $\pi' : E' \to B$ is obtained from $E$ by
fibrewise surgery, then

$$\lambda_i E = \lambda_i E'$$

(iii) If $\pi : E \to B$ and $\pi' : E' \to B$ are bundles of compact, non-closed surfaces
with $\partial E = S^1 \times B = \partial E'$, then

$$\lambda_i(E \cup_{S^1 \times B} E') = \lambda_i(E \cup_{S^1 \times B} (D^2 \times B)) + \lambda_i(E' \cup_{S^1 \times B} (D^2 \times B))$$

As an application of secondary classes we prove the following strengthening of
a theorem of [GMT]:

**Theorem 1.5.** Let $p$ be a prime and $s \geq 1$. Then the reduction of $\kappa_{ps(p-1)-1}$ mod
$p^2$ vanishes:

$$\kappa_{ps(p-1)-1} = 0 \in H^*(B; \mathbb{Z}/p^2)$$

Theorem 1.5 proves part of the following conjecture.

**Conjecture 1.6.** Let $s \geq 1$ and $v \geq 0$. Then

$$\kappa_{p^v s(p-1)-1} = 0 \in H^*(B; \mathbb{Z}/p^{v+1})$$
If the conjecture is true, then $\kappa_{p^s(p-1)-1}$ can be divided by $p^{v+1}$. In [GMT] we prove that this holds modulo torsion. It is also proved in [GMT] that the statement of Conjecture 1.6 is best possible in the sense that if $s \not\equiv 0 \pmod{p}$, then $\kappa_{p^s(p-1)-1} \neq 0 \in H^*(B; Z/p^{v+2})$. I hope to return to Conjecture 1.6 at a later time.

2. Secondary composition

We recall the definition of secondary compositions (Toda brackets). For further details see [Toda].

All spaces and maps are pointed. The reduced suspension $SX$ is regarded as the pushout of $X \wedge [-1, 0] \rightarrow X \wedge [0, 1]$ where $-1 \in [-1, 0]$ and $1 \in [0, 1]$ are the basepoints. Thus, two nullhomotopies $F : X \wedge [-1, 0] \rightarrow Y$ and $G : X \wedge [0, 1] \rightarrow Y$ induce a map $G - F : SX \rightarrow Y$.

For a sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

with $g \circ f \simeq 0$ and $h \circ g \simeq 0$, a choice of null-homotopies $F : g \circ f \simeq 0$ and $G : h \circ g \simeq 0$ determines a map

$$h \circ F - G \circ (f \wedge [-1, 0]) : SX \rightarrow W$$

We define the secondary composition to be the subset $\{h, g, f\} \subseteq [SX, W]$ of homotopy classes of maps obtained in this fashion, as $F, G$ ranges over all null-homotopies.

Recall that $[SX, W] = [X, \Omega W]$ is a group.

**Lemma 2.1.** $\{h, g, f\}$ depends only on the homotopy classes of $h$, $g$, and $f$. If $\{h, g, f\}$ is defined, then it gives a unique element in the double coset,

$$\{h, g, f\} \in h \circ [SX, Z] \setminus [SX, W]/[SY, W] \circ Sf$$

If $[SX, W]$ is abelian, then

$$\{h, g, f\} \in [SX, W]/(h \circ [SX, Z] + [SY, W] \circ Sf)$$

**Proof.** See [Toda] Lemma 1.1.

**Proposition 2.2.** For a sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \xrightarrow{k} V$$

we have

(i) $\{k, h, g\} \circ f \subseteq \{k, h, g \circ f\}$
(ii) $\{k, h \circ g, f\} \subseteq \{k, h \circ g, f\}$
(iii) $\{k \circ h, g, f\} \subseteq \{k \circ h, g, f\}$
(iv) $k \circ \{h, g, f\} \subseteq \{k \circ h, g, f\}$

**Proof.** See [Toda] Proposition 1.2.
Proposition 2.3. Let
\[ K(\mathbb{Z}, n) \xrightarrow{p} K(\mathbb{Z}, n) \xrightarrow{\rho} K(\mathbb{F}_p, n) \xrightarrow{\beta} K(\mathbb{Z}, n+1) \]
represent multiplication by \( p \), reduction mod \( p \), and the mod \( p \) Bockstein, respectively. Then
id \( \in \{ \beta, \rho, p \} \subseteq [SK(\mathbb{Z}, n), K(\mathbb{Z}, n+1)] = [K(\mathbb{Z}, n), K(\mathbb{Z}, n)] \)
\[ \square \]

Corollary 2.4. Let \( c : X \to K(\mathbb{Z}, n) \) represent a cohomology class. Let \( \rho \) and \( \beta \) be as in Proposition 2.3. Then
\[ \{ \beta, \rho, c \} = \{ 1 + p^\ast c \} \subseteq H^n(X) = [SX, K(\mathbb{Z}, n+1)] \]
where
\[ 1 + p^\ast c = \{ c' | pc' = c \} \]

Proof. Clearly the two sides have the same indeterminacy \( Zc + \beta H^{n-1}(X; \mathbb{F}_p) \), so all we need to check is that if \( pc' = c \), then \( c' \in \{ \beta, \rho, c \} \). But this follows from Proposition 2.3:
\[ \{ \beta, \rho, p \circ c' \} \supseteq \{ \beta, \rho, p \circ c \} \supseteq c' \]
\[ \square \]

3. Elementary properties of the secondary classes

Consider the oriented Grassmannian \( SO(N+2)/SO(N) \times SO(2) \). Let \( U = U_N = SO(N+2) \times SO(N) \times SO(2) \mathbb{R}^2 \) be the canonical oriented 2-dimensional vectorbundle and let \( U^\perp = U^\perp_N = SO(N+2) \times SO(N) \times SO(2) \mathbb{R}^N \) be its orthogonal complement.

Lemma 3.1 [GMT]. In \( H^*(Th(U^\perp), *, \mathbb{F}_p) \) we have that
\[ \mathcal{P}^i \lambda_{U^\perp} = (-1)^i e^{i(p-1)\lambda_{U^\perp}} \]

Proof. Let \( \mathcal{P} = \sum_i \mathcal{P}^i \). Then \( \mathcal{P}(\lambda_U) = (1 + e(U))^{p-1} \lambda_U \). Since \( \lambda_{U \oplus U^\perp} = \lambda_U \cup \lambda_{U^\perp} \) we get
\( \lambda_U \cup \lambda_{U^\perp} = \mathcal{P}(\lambda_{U \oplus U^\perp}) = \mathcal{P}(\lambda_U) \cup \mathcal{P}(\lambda_{U^\perp}) = (1 + e(U))^{p-1} \lambda_U \cup \mathcal{P}(\lambda_{U^\perp}) \)
and hence
\[ \mathcal{P}(\lambda_{U^\perp}) = (1 + e(U))^{p-1} \lambda_{U^\perp} = \left( \sum_i (-1)^i e(U)^i(p-1) \right) \lambda_{U^\perp} \]
\[ \square \]

Proof of Lemma 3.1. Clearly \( l \circ \alpha \simeq 0 \). The cohomology of the Grassmannian \( SO(N+2)/SO(N) \times SO(2) \) vanishes in odd degrees (when \( N \) is larger than the degree), so \( \beta \mathcal{P}^i \circ \lambda \simeq 0 \). Therefore \( \{ \beta \mathcal{P}^i, \lambda, \alpha \} \) is defined. It follows from Lemma 2.1 that the indeterminacy is \( \mathbb{Z}K_{i(p-1)} \).
Proof of Theorem 1.3. This follows from Proposition 2.2 and Corollary 2.4 and the diagram:

\[
\begin{array}{ccc}
B_+ \wedge S^{N+2} & \xrightarrow{\alpha} & \text{Th}(U_\infty) \\
\downarrow_{\kappa_i(p-1)-1} & & \downarrow^{e^i(p-1)\lambda} \\
K(Z, N+2i(p-1)) & \xrightarrow{p} & K(F_p, N+2i(p-1)) \\
\downarrow^\beta & & \\
K(Z, N+2i(p-1)+1)
\end{array}
\]

Indeed, Proposition 2.2 gives the inclusions

\[
\{\beta, \rho, \kappa_i(p-1)\} = \{\beta, \rho, (e^i(p-1)\lambda) \circ \alpha\} \subseteq \{\beta, \rho \circ (e^i(p-1)\lambda), \alpha\} = (-1)^i\{\beta, \rho^i\lambda, \alpha\} \supseteq (-1)^i\{\beta \rho^i, \lambda, \alpha\} = \lambda_i(E).
\]

Then Lemma 2.1 proves that the first inclusion is an equality since the two sides have the same indeterminacy \(\text{Im}(\beta) + Z\kappa_i(p-1)-1\). Therefore by Corollary 2.4

\[
\lambda_i(E) \subseteq \{\beta, \rho, \kappa_i(p-1)-1\} = \frac{1}{p}\kappa_i(p-1)-1 + Z\kappa_i(p-1)-1,
\]

and hence

\[
p\lambda_i(E) \subseteq (1 + pZ)\kappa_i(p-1)-1.
\]

Since they have the same indeterminacy, they are equal. \(\square\)

Proof of Theorem 1.4. (i) follows from the additivity of \(\alpha\), i.e. the property that \(\alpha(E \amalg E') = \alpha(E) + \alpha(E') \in [B_+ \wedge S^{N+2}, \text{Th}(U_\infty)]\). Similarly (ii) follows from the property that \(\alpha(E) = \alpha(E')\) when \(E'\) is obtained from \(E\) by fibrewise surgery. (iii) follows from (i) and (ii) since \(E \cup S^1 \times_B E'\) is obtained from \((E \cup S^1 \times_B (D^2 \times B)) \amalg (E \cup S^1 \times_B (D^2 \times B))\) by fibrewise surgery. \(\square\)

4. A VARIANT OF \(\lambda_{ps}\)

The goal of this section is to prove Theorem 1.5. The definition and properties of \(\lambda_i\) proves that \(\kappa_i(p-1)\) is divisible by \(p\). When \(i = ps\), a variant of \(\lambda_{ps}\) can be used to prove that \(\kappa_{ps}(p-1)\) is divisible by \(p^2\).

**Definition 4.1.** Let \(s \geq 0\) and consider the Steenrod algebra \(\mathcal{A}_p\). When \(p = 2\) we write \(\mathcal{P}^i = \text{Sq}^{2i}\) and \(\beta \mathcal{P}^i = \text{Sq}^{2i+1}\) as before. Define \(\theta_s \in \mathcal{A}_p\) by

\[
\theta_s = \sum_{j=0}^{s} (-1)^j \binom{(p-1)(s-j)}{j} \mathcal{P}^{ps-j} \mathcal{P}^j = \mathcal{P}^{ps} + \text{terms of length 2}
\]

Define vectors \(v_s, w_s \in \mathcal{A}_p\) by

\[
v_s = (\mathcal{P}^0, \ldots, \mathcal{P}^s), \quad w_s = (\mathcal{P}^{ps}, \ldots, (-1)^j \binom{(p-1)(s-j)-1}{j} \mathcal{P}^{ps-j}, \ldots, \mathcal{P}^{(p-1)s}).
\]
Lemma 4.2. (i) In $H^*(\text{Th}(U^\perp),_*; \mathbb{F}_p)$ we have that $\theta_*\lambda_{U^\perp} = e^{ps(p-1)}\lambda_{U^\perp}$.
(ii) $v^T_s \beta w_s = \beta \theta_s$.

Proof. (i) This is similar to Lemma 3.1, using the fact that the admissible terms of length 2 act trivially on $\lambda_{U^\perp}$. Formula (ii) is the Adem relation for $P^{(p-1)s}\beta P^s$. □

Definition 4.3. Let $\alpha, \lambda, \theta$ be as above. Define the secondary characteristic class

$$\tilde{\lambda}_{ps} = (-1)^s \{\beta \theta_s, \lambda, \alpha\} \in H^{2ps(p-1) - 2}(B, \mathbb{Z})/\mathbb{Z}\kappa_{ps(p-1)-1}$$

Notice that $\tilde{\lambda}_{ps}$ satisfies the same formal properties as $\lambda_{ps}$. In particular $p\tilde{\lambda}_{ps} = (1 + p\mathbb{Z})\kappa_{ps(p-1)-1}$. In general $\tilde{\lambda}_{ps} \neq \lambda_{ps}$.

Proof of Theorem 1.5. We have

$$(-1)^s \rho \circ \{\beta \theta_s, \lambda, \alpha\} \subseteq (-1)^s \{\rho \circ \beta \theta_s, \lambda, \alpha\} = (-1)^s \{v^T_s \beta w_s, \lambda, \alpha\} \supseteq (-1)^s v^T_s \{\beta w_s, \lambda, \alpha\}$$

and it is seen that all the inclusions are equalities since the indeterminacy vanishes. Since

$$(-1)^s \{\beta w_s, \lambda, \alpha\} \in \prod_{i=0}^{s} H^{N+2i(p-1)}(B_+ \wedge S^{N+2}, \mathbb{F}_p) = \prod_{i=0}^{s} H^{2i(p-1)-2}(B; \mathbb{F}_p),$$

$v^T$ will vanish since $H^*(B; \mathbb{F}_p)$ is an unstable $A_p$-module.

Hence the mod $p$ reduction of $\tilde{\lambda}_{ps}$ vanishes, so $\kappa_{ps(p-1)-1} = p\tilde{\lambda}_{ps} = 0 \in H^*(B; \mathbb{Z}/p^2)$. □

References

[GMT] S. Galatius, I. Madsen, U. Tillmann: Divisibility of the stable Miller-Morita-Mumford classes, in preparation.

[Toda] H. Toda: Composition methods in homotopy groups of spheres, Ann. Math. Studies, No. 49. Princeton University Press, 1962.

AARHUS UNIVERSITY, AARHUS, DENMARK
E-mail address: galatius@imf.au.dk