Low momentum nucleon-nucleon potential
and shell model effective interactions

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Abstract

A low momentum nucleon-nucleon (NN) potential $V_{\text{low}-k}$ is derived from meson exchange potentials by integrating out the model dependent high momentum modes of $V_{NN}$. The smooth and approximately unique $V_{\text{low}-k}$ is used as input for shell model calculations instead of the usual Brueckner $G$ matrix. Such an approach eliminates the nuclear mass dependence of the input interaction one finds in the $G$ matrix approach, allowing the same input interaction to be used in different nuclear regions. Shell model calculations of $^{18}$O and $^{134}$Te using the same input $V_{\text{low}-k}$ have been performed. For cut-off momentum $\Lambda$ in the vicinity of 2 $fm^{-1}$, our calculated low-lying spectra for these nuclei are in good agreement with experiments, and are weakly dependent on $\Lambda$. 

21.60.Cs; 21.30.Fe; 27.80.+j
I. INTRODUCTION

A fundamental problem in nuclear physics has been the determination of the effective nucleon-nucleon ($NN$) interaction used in the nuclear shell model, which has been successful in describing a variety of nuclear properties. There have been a number of successful approaches \[1–4\] for this determination, ranging from empirical fits of experimental data, to deriving it microscopically from the bare $NN$ potential. Despite impressive quantitative successes, the traditional microscopic approach suffers the fate of being "model dependent" owing to the fact that there is no unique $V_{NN}$ to start from. Moreover, as the Brueckner $G$ matrix has traditionally been the starting point, one obtains different input interactions for nuclei in different mass regions as a result of the Pauli blocking operator.

In this Letter, we propose a different approach to shell model effective interactions that is motivated by the recent applications of effective field theory (EFT) and the renormalization group (RG) to low energy nuclear systems \[5–8\]. Our aim is to remove some of the model dependence that arises at short distances in the various $V_{NN}$ models, and also to eliminate the mass dependence one finds in the $G$ matrix approach, thus allowing the same interaction to be used in different nuclear regions such as $^{18}$O and $^{134}$Te. A central theme of the RG-EFT approach is that physics in the infrared region is insensitive to the details of the short distance dynamics. One can therefore have infinitely many theories that differ substantially at small distances, but still give the same low energy physics if they possess the same symmetries and the "correct" long-wavelength structure \[5,8\]. The fact that the various meson models for $V_{NN}$ share the same one pion tail, but differ significantly in how they treat the shorter distance pieces illustrates this explicitly as they give the same phase shifts and deuteron binding energy. In RG language, the short distance pieces of $V_{NN}$ are like irrelevant operators since their detailed form can not be resolved from low energy data.

Motivated by these observations, we would like to derive a low-momentum NN potential $V_{low-k}$ by integrating out the high momentum components of different models of $V_{NN}$ in the sense of the RG \[5,8\], and investigate its suitability of being used directly as a model...
independent effective interaction for shell model calculations. We shall use in the present work the CD-Bonn NN potential \[V_{NN}\] for \[V_{NN}\]. In the following, we shall first describe our method for carrying out the high-momentum integration. Shell model calculations for \(^{18}\)O and \(^{134}\)Te using \(V_{low-k}\) will then be performed. Our results will be discussed, especially about their dependence on the cut-off momentum \(\Lambda\).

The first step in our method is to integrate out the model dependent high momentum components of \(V_{NN}\). In accordance with the general definition of a renormalization group transformation, the decimation must be such that low energy observables calculated in the full theory are exactly preserved by the effective theory. We turn to the model space methods of nuclear structure theory for guidance, as there has been much work in recent years discussing their similarity to the Wilson RG approach \[\text{[4,10,11]}\]. While the technical details differ, both approaches attempt to thin-out, or limit the degrees of freedom one must explicitly consider to describe the physics in some low energy regime. Once the relevant low energy modes are identified, all remaining modes or states are ”integrated” out. Their effects are then implicitly buried inside the effective interaction in a manner that leaves the low energy observables invariant. One successful model-space reduction method is the Kuo-Lee-Ratcliff (KLR) folded diagram theory \[\text{[12,13]}\]. For the nucleon-nucleon problem in vacuum, the RG approach simply means that the low momentum \(T\) matrix and the deuteron binding energy calculated from \(V_{NN}\) must be reproduced by \(V_{low-k}\), but with all loop integrals cut off at some \(\Lambda\). Therefore, we start from the half-on-shell \(T\)-matrix

\[
T(k', k, k^2) = V_{NN}(k', k) + \int_{0}^{\infty} q^2 dq V_{NN}(k', q) \frac{1}{k^2 - q^2 + i0^+} T(q, k, k^2).
\]

We then define an effective low-momentum \(T\)-matrix by

\[
T_{low-k}(p', p, p^2) = V_{low-k}(p', p) + \int_{0}^{\Lambda} q^2 dq V_{low-k}(p', q) \frac{1}{p^2 - q^2 + i0^+} T_{low-k}(q, p, p^2),
\]

where \(\Lambda\) denotes a momentum space cut-off (such as \(\Lambda=2 fm^{-1}\)) and \((p', p) \leq \Lambda\). We require the above \(T\)-matrices satisfying the condition

\[
T(p', p, p^2) = T_{low-k}(p', p, p^2); \ (p', p) \leq \Lambda.
\]
The above equations define the effective low momentum interaction $V_{\text{low}-k}$. In the following, let us show that the above equations are satisfied by the solution

$$V_{\text{low}-k} = \hat{Q} - \hat{Q}' \int \hat{Q} + \hat{Q}' \int \hat{Q} \int \hat{Q}' \int \hat{Q} \int \hat{Q} + \ldots ,$$

which is just the KLR folded-diagram effective interaction [12,13]. A preliminary account of this result has been reported as a work in progress at a recent conference [14].

In time dependent formulation, the T-matrix of Eq.(1) can be written as $\langle k' | V \mathcal{U}(0, -\infty) | k \rangle$, $U$ being the time evolution operator. In this way we can readily perform a diagrammatic analysis of the T-matrix. A general term of it may be written as $\langle k' | (V + V \frac{1}{e(k)} V + V \frac{1}{e(k)} V \frac{1}{e(k)} V + \ldots) | k \rangle$ where $e(k) \equiv (k^2 - H_0)$, $H_0$ being the unperturbed Hamiltonian. Note that the intermediate states (represented by 1 in the numerator) cover the entire space, and $1 = P + Q$ where $P$ denotes the model space (momentum $\leq \Lambda$) and $Q$ its complement. Expanding it out in terms of $P$ and $Q$, a typical term of $T$ is of the form $V \frac{1}{e} V \frac{1}{e} V \frac{1}{e} V \frac{1}{e} V$. Let us define a $\hat{Q}$-box as $\hat{Q} = V + V \frac{1}{e} V + V \frac{1}{e} V \frac{1}{e} V + \ldots$, where all intermediate states belong to $Q$. One readily sees that the T-matrix can be regrouped as a $\hat{Q}$-box series, namely $\langle p' | T | p \rangle = \langle p' | [\hat{Q} + \hat{Q} \frac{1}{e} \hat{Q} + \hat{Q} \frac{1}{e} \hat{Q} \frac{1}{e} \hat{Q} + \ldots] | p \rangle$. Note that all the $\hat{Q}$-boxes have the same energy variable, namely $p^2$.

This regrouping is depicted in Fig. 1, where each $\hat{Q}$-box is denoted by a circle and the solid line represents the propagator $\frac{1}{e}$. The diagrams A, B and C are respectively the one- and two- and three-$\hat{Q}$-box terms of $T$, and clearly $T = A + B + C + \ldots$. Note the dashed vertical line is not a propagator; it is just a “ghost” line to indicate the external indices. We now perform a folded-diagram factorization for the T-matrix, following closely the KLR folded-diagram method [12,13]. Diagram B of Fig. 1 is factorized into the product of two parts (see B1) where the time integrations of the two parts are independent from each other, each integrating from $-\infty$ to 0. In this way we have introduced a time-incorrect contribution which must be corrected. In other words B is not equal to B1, rather it is equal to B1 plus the folded-diagram correction B2. Note that the integral sign represents a generalized folding [12,13].
Similarly we factorize the three-\(\hat{Q}\)-box term \(C\) as shown in the third line of Fig. 1. Higher-order \(\hat{Q}\)-box terms are also factorized following the same folded-diagram procedure. Let us now collecting terms in the figure in a “slanted” way. The sum of terms \(A1, B2, C3\ldots\) is just the low-momentum effective interaction of Eq.(4). (Note that the leading \(\hat{Q}\)-box of any folded term must be at least second order in \(V_{NN}\), and hence it is denoted as \(\hat{Q}'\)-box which equals to \(\hat{Q}\)-box with terms first-order in \(V_{NN}\) subtracted.) The sum \(B1, C2, D3\ldots\) is \(V_{low-k}\hat{P}\hat{Q}\). Similarly the sum \(C1+D2+E3+\cdots\) is just \(V_{low-k}\hat{P}\hat{Q}\hat{P}\hat{Q}_.\) (Note diagrams \(D1, D2, \cdots, E1, E2, \cdots\) are not shown in the figure.) Continuing this way, it is easy to see that Eqs. (1) to (3) are satisfied by the low momentum effective interaction of Eq.(4).

The effective interaction of Eq.(4) can be calculated using iteration methods. A number of such iteration methods have been developed; the Krenciglowa-Kuo [15] and the Lee-Suzuki iteration methods [16] are two examples. These methods were formulated primarily for the case of degenerate \(PH_0P, H_0\) being the unperturbed Hamiltonian. For our present two-nucleon problem, \(PH_0P\) is obviously non-degenerate. Non-degenerate iteration methods [17] are more complicated. However, a recent iteration method developed by Andreozzi [18] is particularly efficient for the non-degenerate case. This method shall be referred to as the Andreozzi-Lee-Suzuki (ALS) iteration method, and has been employed in the present work.

We have carried out numerical checks to ensure that certain low-energy physics of \(V_{NN}\) are indeed preserved by \(V_{low-k}\). We first check the deuteron binding energy \(BE_d\) given by \(V_{low-k}\). For a range of \(\Lambda\), such as \(0.5fm^{-1} \leq \Lambda \leq 3fm^{-1}\), \(BE_d\) given by \(V_{low-k}\) agrees very accurately (to 4 places after the decimal) with that given by \(V_{NN}\). In Fig. 2, we present some \(^1S_0\) phase shifts calculated from the CD-Bonn \(V_{NN}\) (dotted line) and the \(V_{low-k}\) (circles) derived from it, using a momentum cut-off \(\Lambda = 2.0fm^{-1}\). As seen, the phase shifts from the former are well reproduced by the latter. We have also checked the half-on-shell T-matrix given by \(V_{NN}\) and by \(V_{low-k}\), and found very good agreement between them [14]. In short, our numerical checks have reaffirmed that the deuteron binding energy, low energy phase shifts and low momentum half-on-shell T-matrix of \(V_{NN}\) are all preserved by \(V_{low-k}\). As far
as those physical quantities are concerned, $V_{\text{low}-k}$ and $V_{NN}$ are equivalent.

Having proven the "physical equivalence" of $V_{\text{low}-k}$ and $V_{NN}$ in the sense of the RG, we turn now to microscopic shell model calculations in which we use $V_{\text{low}-k}$ as the input interaction. We have performed shell-model calculations for $^{18}$O and $^{134}$Te following the same procedure as outlined in Refs. [2,3], except that the $G$-matrix vertices used there are replaced by our present $V_{\text{low}-k}$. A model space with two valence nucleons in the $(0d_{5/2}, 0d_{3/2}, 1s_{1/2})$ shell is used for $^{18}$O, and a $(0g_{7/2}, 1d_{5/2}, 1d_{3/2}, 2s_{1/2}, 0h_{11/2})$ one for $^{134}$Te. In a concurrent paper [11] we have found that $V_{\text{low}-k}$ is almost independent of the underlying $V_{NN}$ for the values of $\Lambda$ considered here. Therefore, although the CD-Bonn potential [9] is used in our calculations, we stress that very similar results will be obtained if we calculate $V_{\text{low}-k}$ from other models such as the Paris or Argonne V-18 potentials.

Our calculated low-lying $J^\pi$ states of $^{18}$O and $^{134}$Te are presented in Fig.3 and Fig.4. In the same figures, results of the corresponding $G$-matrix calculations are also shown; the $V_{\text{low}-k}$ results are just as good or slightly better. It may be mentioned that the $G$-matrix is energy dependent and Pauli blocking dependent, while $V_{\text{low}-k}$ is not which is a desirable feature. An important issue is what value one should use for $\Lambda$. Guided by general EFT arguments, the minimum value for $\Lambda$ must be large enough so that $V_{\text{low}-k}$ explicitly contains the necessary degrees of freedom for the physical system. Such a value for $\Lambda_{\text{min}}$ is signalled when the calculated spectra first become insensitive to $\Lambda$ [11]. Conversely, we want $\Lambda$ to be smaller than the short distance scale $\Lambda_{\text{max}}$ at which the model dependence of the different $V_{NN}$ starts to creep [11]. Systems in which these two constraints are consistent with each other (i.e., $\Lambda_{\text{min}} < \Lambda_{\text{max}}$) are amenable to EFT-RG inspired effective theories, as they possess a clear separation of scales between the relevant long wavelength modes and the model dependent short distance structure. We have found [11] that $\Lambda_{\text{max}}$ should not be much greater than 2.0-2.5 fm$^{-1}$ as this is the scale at which $V_{\text{low}-k}$ first becomes dependent on the particular $V_{NN}$ used. There is another consideration: Most NN potentials are constructed to fit empirical phase shifts up to $E_{\text{lab}} \approx 350$ MeV [11]. Since $E_{\text{lab}} \leq 2\hbar^2\Lambda^2/M$, $M$ being the nucleon mass, and one should require $V_{\text{low}-k}$ to reproduce the same empirical phase shifts,
a choice of Λ in the vicinity of 2 \text{fm}^{-1} would seem to be appropriate.

Guided by the above considerations, we have used in our calculation two values for the momentum cut-off, namely Λ = 2.0 and 2.2 \text{fm}^{-1}. It is satisfying to see that the results in Fig. 3 and Fig. 4 are rather insensitive to the choice of Λ; this suggests that Λ_{min} is around 2\text{fm}^{-1} in harmony with the EFT philosophy mentioned earlier. Perhaps more importantly, both are in satisfactory agreement with experiments. Note that all energies are relative to the ground state energy of the respective core nucleus.

We emphasize that we have used the same $V_{\text{low-k}}$ interaction in both $^{18}\text{O}$ and $^{134}\text{Te}$ calculations, and it appears to work equally well for both nuclei. This is in marked contrast to the traditional approach in which one uses $G$-matrices, as the Pauli blocking operator is very different for the two nuclear mass regions. This is an exciting result, as it suggests the possibility for a common shell-model interaction that is nearly model independent and suitable for a wide range of nuclei.

In summary, we have investigated a RG-EFT inspired approach to shell model calculations that is a ”first step” towards a model independent calculation that uses one common interaction over a wide range of nuclei. Using the KLR folded diagram approach in conjunction with the ALS iteration method, we have performed a RG decimation where the model dependent pieces of $V_{NN}$ models are integrated out to obtain a nearly unique low momentum potential $V_{\text{low-k}}$. This $V_{\text{low-k}}$ preserves the deuteron pole as well as the low energy phase shifts and half-on-shell $T$ matrix. We have used $V_{\text{low-k}}$, which is a smooth potential, directly in shell model calculations of $^{18}\text{O}$ and $^{134}\text{Te}$ without first calculating the $G$ matrix. The results are in satisfactory agreement with experiment for both nuclei, and they are insensitive to Λ in the neighborhood of Λ ≈ 2 \text{fm}^{-1}. For completeness, we add that satisfactory results have also been obtained using the same $V_{\text{low-k}}$ in shell model calculations in the $^{208}\text{Pb}$ and $^{132}\text{Sn}$ regions. Calculations of spin orbit splittings using the same $V_{\text{low-k}}$ have also led to satisfactory results. These results are a work in progress and will be reported in a subsequent paper. We do feel that $V_{\text{low-k}}$ may become a promising and reliable effective interaction for shell model calculations of few valence nucleons, over a wide range of nuclear regions.
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FIG. 1. Folded-diagram factorization of the half-on-shell T-matrix.
FIG. 2. Comparison of phase shifts given by $V_{\text{low}-k}$ and $V_{NN}$. 
FIG. 3. Low lying states of $^{18}\text{O}$.
FIG. 4. Low lying states of $^{134}$Te.