CONTINUOUS COVERS ON SYMPLECTIC MANIFOLDS

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Abstract. In this article, we first introduce the notion of a continuous cover of a manifold parametrised by any compact manifold endowed with a mass 1 volume-form. We prove that any such cover admits a partition of unity where the usual sum is replaced by integrals. We then generalize Polterovich’s notion of Poisson non-commutativity to such a context in order to get a richer definition of non-commutativity and to be in a position where one can compare various invariants of symplectic manifolds, for instance the relation between critical values of phase transitions of symplectic balls and eventual critical values of the Poisson non-commutativity. Our first main theorem claims that our generalisation of Poisson non-commutativity depends only on real one-parameter spaces since intuitively the Hilbert curve in any high dimensional parameter space fills out the entire manifold and preserves the measure. Our second main theorem states that the Poisson non-commutativity is a (not necessarily strictly) decreasing function of the size of the symplectic balls used to cover continuously any given symplectic manifold. This function has other nice properties as well that do not prevent it from undergoing singularities similar to phase transitions.

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1. Introduction

In mathematics, the notion of partition of unity is fundamental since it is the concept that distinguishes $C^\infty$ geometry from analytic geometry. In the first case, where partitions of unity apply, most objects can be decomposed in local parts, while in the second case where partitions of unity do not apply, most objects are intrinsically global and indecomposable.

It is therefore of great importance to push that notion as far as we can in order to make it more natural and applicable. Our first observation is that the right context in which one should consider partitions of unity is in the smooth category (or possibly in the measurable category if one were able to make sense of that concept for families of open sets). So here continuous covers by open subsets of a given smooth manifold $M$ will be parametrised by any smooth compact manifold endowed with a volume-form of total mass 1. We will first prove that any such continuous parametrised cover admits a continuous partition of unity made of smooth functions. This applies to all smooth manifolds.

Concentrating on an arbitrary symplectic manifold $(M,\omega)$, the covers that we will consider will be made of families of symplectically embedded balls of a given capacity $c = \pi r^2$ indexed by a measure space $(T, d\mu = dt)$. Here is our first theorem: the level of Poisson non-commutativity, as defined by Polterovich in the discrete case of partitions of unity, can be generalised to the case of our families of covers and associated partitions of unity made of smooth functions.
unity; moreover the number that we get in this general case, that depends \textit{a priori} on the parameter space, actually depends only on real one-parameter families.

Our second theorem is that, if one considers the function $f : [0, c_{\text{max}}] \rightarrow [0, \infty]$ that assigns to $c$ the Polterovich’s level of non-commutativity of the corresponding cover, as generalised by us in the continuous setting, then this function enjoys the following two properties:

1) $f$ is non-increasing, and
2) $f$ is upper semi-continuous and left-continuous.

Beside these two main results, we make precise the idea that continuous covers (and their subordinated partitions of unity) can be understood as limits of discrete covers (equipped with subordinated partitions of unity) as the number of open sets goes to infinity, and we also show that smooth covers are as good as continuous covers when it comes to covers made of symplectic balls.

We end this paper with a bold conjecture that relates phase transitions in the topology of spaces of symplectic balls with the singularities of the Poisson non-commutativity. This conjecture also happens to be in conflict with another conjecture made by Polterovich. Finally we state a final conjecture that claims that at small scales, the uncertainty is purely topological (it does not depend on the symplectic structure). Here by uncertainty, we mean the pb since after all, when the pb is applied to covers made of symplectic balls, these balls measure the level of uncertainty in a phase space and more generally in a symplectic manifold.

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\section{Continuous covers}

Throughout this article, $T$ is a compact connected smooth manifold of finite dimension endowed with a measure $\mu$ of total volume 1 coming from a volume-form $dt$. The following definition is far more restrictive than the one that we have in mind, but it will be enough for the purpose of this article.

\textit{Definition 1.} Let $M$ be a closed smooth manifold of dimension $n$ Let $U$ be a bounded open subset of Euclidean space $\mathbb{R}^n$ whose boundary is smooth and which can be extended smoothly to a collar neighbourhood of itself in such a way that the resulting open set be still embedded. A \textit{continuous cover} of $M$ (of type $(T, U)$) is a continuous map:

$$G : T \times U \rightarrow M$$

such that

1. for each $t \in T$, the map $G_t$ is a smooth embedding of $U$ to $M$ that can be extended to a smooth embedding of some (\textit{a priori} $t$-dependent) collar neighbourhood of $U$ (and therefore to the closed set $\bar{U}$), and
2. the images of $U$ as $t$ runs over the parameter space $T$, cover $M$. 
Note that, in general, the topology of $U$ could change within the $T$-family. However, to simplify the presentation, we restrict ourselves to a fixed $U$ – this is what we had in mind in the sentence preceding this definition. Note also that we could define a smooth cover in the same way by allowing $G$ to be smooth in the $T$-variable too. Since $T \times U$ is a finite dimensional smooth compact manifold, any continuous cover is $C^0$-close to a smooth one. Smooth covers of symplectic manifolds will play some role later on.

**Definition 2.** A partition of unity $F$ subordinated to a continuous cover $G$ is a continuous function

$$\tilde{F} : T \times U \to \mathbb{R}$$

such that

1. each $\tilde{F}_t : U \to \mathbb{R}$ is a smooth function with (compact) support in $U$,
2. the closure of the reunion $\bigcup_{t \in T} \text{supp}(\tilde{F}_t)$ is contained in $U$, and
3. for every $x \in M$,

$$\int_T F_t(x) dt = 1,$$

where the smooth function $F_t : M \to \mathbb{R}$ is the pushforward of $\tilde{F}_t$ to $M$ using $G_t$, extended by zero outside the image of $G_t$.

The notation $F < G$ expresses that $F$ is a partition of unity subordinated to the cover $G$.

**Remark 3.** The last definition makes sense for a measurable function $F$. However, the smoothness condition on each $F_t$ is required for the forthcoming definition of the Poisson bracket invariant. It is unclear to what extend the continuity requirement of $F$ in the variable $t$ restricts the set of partitions of unity for which the Poisson bracket invariant is well-defined; it might be superfluous. Condition (2) plays a role in the proofs of a few results of this paper by allowing us to deform $U$ a little while keeping a given $F$ fixed; we were not able to come up with arguments working without this condition. Note that we recover the usual notion of partition of unity by taking $T$ to be a finite set of points with the counting measure.

**Theorem 4.** Each continuous cover admits a partition of unity.

**Proof.** Let $G$ be a continuous cover of $M$ by open sets diffeomorphic to $U$. Cover $U$ by open balls such that their closure is always included inside $U$. Now, push forward this cover to $M$ using each $G_t$; the collection of all of these images as $t$ varies in $T$ forms an open cover of $M$ by sets diffeomorphic to the ball. Since $M$ is compact, there exists a finite subcover. Each open set in this subcover comes from some $G_t$, where $t$ is an element of a finite set $T' \subset T$.

For each $t \in T'$, consider the (finite) collection of open balls inside $U$ whose image under $G_t$ belongs to the subcover. Then consider the reunion of these collections to get another (finite) collection $C$ of open balls inside $U$. Since the closure of each ball in $C$ is contained in $U$, it is possible to contract $U$ a little bit to another open set $U'$ whose closure lies in $U$ and which also contains the closure of each ball in $C$. It follows that the restriction $G'$ of $G$ to $T \times U'$ is also a continuous cover of $M$.

For each $t$, there exists a smooth function $g_t : U' \to [0, \infty)$ which vanishes only on the boundary of $U'$ and which can be smoothly extended by zero on the complement of $U'$ in $\mathbb{R}^n$. Let

$$\tilde{g}_t : M \to [0, \infty) : x \mapsto \begin{cases} (g_t \circ G_t^{-1})(x) & \text{if } x \in G(t, U'), \\ 0 & \text{otherwise.} \end{cases}$$
This is a smooth function on $M$ which depends continuously on $t \in T$. Define

$$F_t : M \to [0, \infty) : x \mapsto \frac{\tilde{g}_t(x)}{\int_T \tilde{g}_t(x) dt}.$$ 

Going through the argument, we see that

1. each $F_t$ is well-defined, since $\int_T \tilde{g}_t(x) dt > 0$ everywhere on $M$.
2. the function $F(t, x) = F_t(x)$ is smooth in $x$ and continuous in $t$;
3. the supports of the $F_t$'s cover $M$;
4. $\int_T F_t(x) dt = 1$ for all $x \in M$.

\[ \square \]

3. The $pb$ invariant and Poisson non-commutativity

Leonid Polterovich [3, 4, 5] introduced recently the notion of the level of Poisson non-commutativity of a given classical (i.e finite) covering of a symplectic manifold. Here is the definition:

**Definition 5.** Let $(M, \omega)$ be a closed symplectic manifold and $\mathcal{U}$ a finite cover of $M$ by open subsets $U_1, \ldots, U_N$. For each partition of unity $F = (f_1, \ldots, f_N)$ subordinated to $\mathcal{U}$, take the supremum of $\parallel \sum_i a_i f_i, \sum_j b_j f_j \parallel$ when the $N$-tuples of coefficients $(a_i)$ and $(b_i)$ run through the $N$-cube $[-1, 1]^N$, where the bracket is the Poisson bracket and the norm is the $C^0$-supremum norm. Then take the infimum over all partitions of unity subordinated to $\mathcal{U}$. This is by definition the $pb$ invariant of $\mathcal{U}$. To summarize:

$$pb(\mathcal{U}) := \inf F < \mathcal{U} \sup (a_i), (b_i) \in [-1, 1]^N \parallel \left\{ \sum_i a_i f_i, \sum_j b_j f_j \right\} \parallel.$$ 

Roughly speaking, this number is a measure of the least amount of 'symplectic interaction' that sets in a cover $\mathcal{U}$ can have. It is very plausible that such a number depends on the combinatorics of the cover, but also on the symplectic properties of the (intersections of the) open sets in the cover. To illustrate this point, observe that if $\mathcal{U}$ is an open cover made of only two open sets, then $pb(\mathcal{U}) = 0$. A somewhat opposite result holds for covers constituted of displaceable open sets; let’s recall that a subset $U$ of $M$ is displaceable if there is a Hamiltonian diffeomorphism $\phi$ such that $\phi(U) \cap U = \emptyset$. The main result of Polterovich in [3, 5] is that for such a cover, the number $pb(\mathcal{U})$ (multiplied by some finite number which measures the 'symplectic size' of the sets in the cover) is bounded from below by $(2N^2)^{-1}$. In particular, $pb(\mathcal{U}) > 0$ in such a case. This result heavily relies on techniques in quantum and Floer homologies and in the theory of quasi-morphisms and quasi-states. Unfortunately, this lower bound depends on the cardinality $N$ of the open cover; as such, it does not show if one could use the $pb$-invariant in order to assign to a given symplectic manifold a (strictly positive) number that might be interpreted as its level of Poisson non-commutativity. Nevertheless, Polterovich conjectured in [4] and [5] that for covers made of displaceable open sets, there should be a strictly positive lower bound for $pb$ independent of the cardinality of the cover, an extremely hard conjecture.

One way of solving this problem might come from the extension of the $pb$-invariant from finite covers to continuous covers. Indeed, continuous covers are morally limits of finite covers as the cardinality $N$ goes to infinity, so we can expect some relation between
the minimal value of pb on such covers and the level of Poisson non-commutativity of the symplectic manifold. This extension has the advantage that one may then compare the pb invariant for continuous covers to other quantities that also depend on continuous covers, such as the critical values at which families of symplectic balls undergo a phase transition. We first need the following definition:

**Definition 6.** Let \((M,\omega)\) be a closed symplectic manifold and \(G\) a continuous cover of \(M\) of type \((T,U)\) by open subsets \(G_i(U)\). For each partition of unity \(F\) subordinated to \(G\), take the supremum of \(\left\| \left\{ \int_M a(t)F(t)dt, \int_M b(t)F(t)dt \right\} \right\|\) over all coefficients \((\text{or weights})\) \(a\) and \(b\) that are measurable functions defined on \(T\) with real values in \([-1,1]\). Then take the infimum over all partitions of unity subordinated to \(G\). This is by definition the pb *invariant* of \(G\). To summarize:

\[
\text{pb}(G) := \inf_{F < G} \sup_{\alpha, \beta : |T| \to [-1,1]} \left\| \left\{ \int_T \alpha(t)F(t)dt, \int_T \beta(t)F(t)dt \right\} \right\| .
\]

Note that we recover Polterovich’s definition by replacing \(T\) by a finite set of points. The following result shows that this pb-invariant is finite.

**Lemma 3.1.** Given a continuous cover \(G\) of type \((T,U)\), there exists a partition of unity \(F\) subordinated to \(G\) whose pb-invariant \(\text{pb} F\) is finite\(^1\).

**Proof.** Consider again the first two paragraphs in the proof of Theorem 4. We proved there that the map \(G' : T' \times U' \to M\) is a finite open cover of type \((T',U')\). Choose any partition of unity \(F'\) subordinated to this cover. Let \(U''\) be an open set such that \(U' \subset U''\) and \(\overline{U''} \subset U\). For \(t \in T'\), observe that \(t\) is an interior point of the set \(O_t := \{s \in T : G(s,U') \subset G(t,U'')\}\). Choose any smooth bump function \(\rho_t : T \to [0,\infty)\) supported in \(O_t\) which integrates to 1. Notice that the continuous function

\[
\tilde{F}_{(t)} : T \times U \to [0,\infty) : (s,u) \mapsto \rho_t(s)F'(t,G(s,u))
\]

is supported in \(O_t \times U''\) and each \(\tilde{F}_{(t)}(s,-)\) is smooth on \(U\). Set \(\tilde{F} = \sum_{t \in T'} \tilde{F}_{(t)}\). Given any point \(x \in M\) and any measurable function \(\alpha : T \to [-1,1]\), we have

\[
\int_T \alpha(s)F(s,x)ds = \sum_{t \in T'} \left( \int \alpha(s)\rho_t(s)ds \right) F'(t,x) =: \sum_{t \in T'} \alpha_tF'(t,x).
\]

The choice \(\alpha \equiv 1\) shows in particular that \(F\) is a partition of unity subordinated to \(G\). Since the pb-invariant is well-defined for the discrete partition \(F'\), it is also defined for the continuous partition \(F\).

\[\square\]

Consequently, in this paper, we can and do consider only partitions of unity \(F < G\) such that \(\text{pb} F < \infty\).

For the sake of completeness, we recall a few facts taken from Polterovich’s and Rosen’s recent book \([5]\). Further informations are available in this book and in the references therein. The setting is the following \((5,\text{chapter} \ 4)\):

- \((M^{2n},\omega)\) a compact symplectic manifold;
- \(U \subset M\) an open set;

\(^1\)The pb-invariant \(\text{pb} F\) of a partition of unity \(F\) is defined as above without the infimum over \(F < G\). That is, \(\text{pb} G = \inf_{F < G} \text{pb} F\).
functions $f, g$

**Remark 8**. Let $\phi : H(U) \rightarrow \mathbb{R}$ be a (subadditive) spectral invariant on $H(U)$ (see the definition below).

- $\phi$ is an element of $H(U)$;
- $c$ is a (subadditive) spectral invariant on $H(U)$ (see the definition below);
- $q(\phi) := c(\phi) + c(\phi^{-1})$, which is (almost) a norm on $H$;
- $w(U) := \sup_{\phi \in H(U)} q(\phi)$ the spectral width of $U$ (which may be infinite).

**Definition 7** ([5], 4.3.1). A function $c : H \rightarrow \mathbb{R}$ is called a *subadditive spectral invariant* if it satisfies the following axioms:

- **Conjugation invariance**: $c(\phi \psi) = c(\psi)$ for all $\phi, \psi \in H$;
- **Subadditivity**: $c(\phi \psi) \leq c(\phi) + c(\psi)$;
- **Stability**: 
  \[ \int_0^1 \min(f_t - g_t) dt \leq c(\phi) - c(\psi) \leq \int_0^1 \max(f_t - g_t) dt, \]
  provided $\phi, \psi \in H$ are generated by normalized Hamiltonians $f$ and $g$, respectively;
- **Spectrality**: $c(\phi) \in \text{spec}(\phi)$ for all nondegenerate elements $\phi \in H$.

**Remark 8**. The first three properties of a spectral invariant are in practice the most important ones. However, from the spectrality axiom, one can show for instance that $w(U) < \infty$ whenever $U$ is displaceable; as such, the spectrality axiom is relevant in order to tie the spectral invariant with the symplectic topology of $M$. Let’s mention that a spectral invariant exists on any closed symplectic manifold, as can be shown in the context of Hamiltonian Floer theory.

Given a Hamiltonian function $f \in C^\infty(M)$ generating the (autonomous) Hamiltonian diffeomorphism $\phi_f = \phi_\frac{\partial f}{\partial t}$ and a spectral invariant $c$, we can define the number

$$\zeta(f) := \sigma(\phi_f) + \langle f \rangle \in \mathbb{R}$$

where $\sigma(\phi_f) := \lim_{n \to \infty} \frac{1}{n} c(\phi^n_f)$ (with $\sigma$ the homogenization of $c$) and $\langle f \rangle := V^{-1} \int_M f \omega^n$ is the mean-value of $f$, where $V = \int_M \omega^n$ is the volume of the symplectic manifold $M$.

The function $\zeta : C^\infty(M) \rightarrow \mathbb{R}$ is called the *(partial symplectic) quasi-state* associated to $c$. It has some very important properties, among which:

- **Normalization**: $\zeta(a) = a$ for any constant $a$;
- **Stability**: $\min_M (f - g) \leq \zeta(f) - \zeta(g) \leq \max_M (f - g)$;
- **Monotonicity**: If $f \geq g$ on $M$, then $\zeta(f) \geq \zeta(g)$;
- **Homogeneity**: If $s \in [0, \infty)$, then $\zeta(sf) = s\zeta(f)$;
- **Vanishing**: If the support of $f$ is displaceable, then $\zeta(f) = 0$ (this is a consequence of the spectrality axiom for $c$);
- **Quasi-subadditivity**: If $\{f, g\} = 0$, then $\zeta(f + g) \leq \zeta(f) + \zeta(g)$.

For $f, g \in C^\infty(M)$, define $S(f, g) = \min\{w(\text{supp } f), w(\text{supp } g)\} \in [0, \infty]$. It follows from Remark 8 that this number is finite whenever either $f$ or $g$ has displaceable support.

**Theorem 9** ([2], 1.4 ; [5], 4.6.1 ; the Poisson bracket inequality). For every pair of functions $f, g \in C^\infty(M)$ such that $S(f, g) < \infty$,

$$\Pi(f, g) := |\zeta(f + g) - \zeta(f) - \zeta(g)| \leq \sqrt{2S(f, g) \|\{f, g\}\|}.$$
We see that $\Pi(f,g)$ measures the default of additivity of $\zeta$. In fact, this theorem implies:

**Partial quasi-linearity**: If $S(f,g) < \infty$ and if $\{f,g\} = 0$, then

$$\zeta(f+g) = \zeta(f) + \zeta(g) \quad \text{and} \quad \zeta(sf) = s\zeta(f) \quad \forall s \in \mathbb{R}. $$

It is known that some symplectic manifolds admit a spectral invariant $c$ for which $S$ takes values in $[0, \infty)$, in which case $\zeta$ is a genuine symplectic quasi-state: it is a normalized, monotone and quasi-linear functional on the Poisson algebra $\mathcal{C}^\infty(M, \{−,−\})$.

**Theorem 10** ([3], 3.1; [5], 9.2.2). Let $(M, \omega)$ be a symplectic manifold and consider a finite cover $U = \{U_1, \ldots, U_N\}$ of $M$ by displaceable open sets. Write $w(U) := \max_i w(U_i) < \infty$.

Then

$$\text{pb}(U) w(U) \geq \frac{1}{2N^2}. $$

**Proof**: Let $F$ be a partition of unity subordinated to $U$. Set

$$G_1 = F_1, G_2 = F_1 + F_2, \ldots, G_N = F_1 + \cdots + F_N. $$

Using Theorem 1 and the vanishing property of $\zeta$, one obtains the following estimate:

$$|\zeta(G_{k+1}) - \zeta(G_k)| = |\zeta(G_k + F_{k+1}) - \zeta(G_k) - \zeta(F_{k+1})| \leq \sqrt{2 \min(w(\text{supp } G_k), w(\text{supp } F_{k+1}))} \sqrt{\{G_k, F_{k+1}\}}. $$

Using the definitions of $\text{pb}(F)$ and of $w(U)$, one gets:

$$|\zeta(G_{k+1}) - \zeta(G_k)| \leq \sqrt{2 w(U)} \sqrt{\text{pb}(F)}. $$

This inequality holds for all $k$. Using the normalization and vanishing properties of $\zeta$ and applying the triangle inequality to a telescopic sum, one gets:

$$1 = |\zeta(1) - 0| = |\zeta(G_N) - \zeta(G_1)| \leq \sum_{k=1}^{N-1} |\zeta(G_{k+1}) - \zeta(G_k)| \leq \sum_{k=1}^{N-1} \sqrt{2 w(U) \text{pb}(F)} \leq N \sqrt{2 w(U) \text{pb}(F)}. $$

Since this is true for any $F < U$, the result easily follows.

A very similar result holds in the context of continuous covers. We say that a continuous cover $G : T \times U \to (M, \omega)$ is made of displaceable sets if each set $G_t(U) = G_t(U) \subset (M, \omega)$ is displaceable. In other words, not only is each $G_t(U)$ displaceable, but so is a small neighborhood of it too.

**Theorem 11**. For any continuous cover $G$ of type $(T, U)$ made of displaceable sets, there exists a constant $c = c(G) > 0$ such that

$$\text{pb}(G) \geq c(G). $$
Proof:

The proof morally consists in a coarse-graining of the continuous cover to a finite cover. Let \( W_1, \ldots, W_N \) be any exhaustion of the compact manifold \( T \) by nested open sets with the following property: the sets \( V_1 = W_1, V_2 = W_2 - W_1, \ldots, V_N = W_N - W_{N-1} \) are such that for every \( j \) the open set \( U_j := \cup_{t \in V_j} \text{Im}(G_t) \) in \( M \) is displaceable. Assume for the moment being that such sets \( W_i \) exist. Notice that the sets \( U_j \) cover \( M \) and let \( w(G) := \sup_j w(U_j) < \infty \). Now let \( F \) be a partition of unity subordinated to \( G \) and consider the functions \( \int_{V_1} F_t dt, \ldots, \int_{V_N} F_t dt \) which form a partition of unity on \( M \) subordinated to the \( U_j \)'s. As in Theorem 2, one estimates:

\[
1 = |\zeta(1) - 0| = \left| \zeta \left( \int_{W_N} F_t dt \right) - \zeta \left( \int_{W_1} F_t dt \right) \right| \\
\leq \sum_{k=1}^{N-1} \left| \zeta \left( \int_{W_{k+1}} F_t dt \right) - \zeta \left( \int_{W_k} F_t dt \right) - \zeta \left( \int_{V_{k+1}} F_t dt \right) \right| \\
\leq \sum_{k=1}^{N-1} \sqrt{2w(G) \text{pb}(F)} \leq N \sqrt{2w(G) \text{pb}(F)}.
\]

Since this is true for all \( F < G \), and since \( 2N^2 \) depends only on \( G \) (through the choice of the \( W_j \)'s), the result follows with \( c(G) := (2N^2w(G))^{-1} \).

The sets \( W_j \)'s exist for the following reason. The closure of each \( G_t(U) \) is a compact displaceable set, so that some open neighborhood \( O_t \) of this set is displaceable. By the continuity of the cover \( G \), for any \( t \) there exists an open set \( \{t\} \in Y_t \subset T \) such that \( G(Y_t \times U) \subset O_t \). Since \( T \) is compact, only a finite number of these \( Y_t \) suffice to cover \( T \), say \( Y_1, \ldots, Y_N \). Set \( W_j = \cup_{k=1}^{j} Y_j \). Since \( V_j \subset Y_j \), the sets \( G(V_j \times U) \) are indeed displaceable. This concludes the proof.

It is natural to compare the \( \text{pb} \) invariant of different continuous covers of type \( (T, U) \), especially if there are related to each other by a continuous family of continuous covers of the same type. This might help in understanding what is the ‘optimal’ way to cover a symplectic manifold \( (M, \omega) \) by copies of a set \( U \). We are led to the following definition which lies at the heart of this article:

Definition 12. A constraint on continuous covers of \( M \) of type \( (T, U) \) is a set \( C \) of such covers; the set of all continuous covers of type \( (T, U) \) corresponds to the unconstrained case. A constrained class of continuous covers of \( M \) of type \( (T, U) \) is defined as a connected component of the given constraint. We define the \( \text{pb} \) invariant of a (constrained) class \( A \) as the infimum of \( \text{pb}(G) \) when \( G \) runs over all continuous covers in \( A \). In short:

\[
\text{pb}(A) := \inf_{G \in A} \text{pb}(G).
\]

As an instance of a constraint, we shall consider later on the one given by asking for each embedding \( G_t : (U^{2n}, \omega_0) \rightarrow (M^{2n}, \omega) \) to be symplectic. The obvious difficulty with
that $G$ such that

Given Lemma 4.1.

Proof. $\cdots$

The sought-after homotopy of covers is given by

4. Reducing the dimension of the $T$-parameter space to 1

This section is mainly devoted to the proof of Theorem 13 below which can be summarized as follows: the pb invariant of any class of $T$-families is equal to the pb invariant of an affiliated class of one-parameter families.

Before stating explicitly the theorem, let’s introduce a few preliminary notions and notations. Any constraint $C$ of type $(T,U)$ determines the subset of constrained embeddings

$$C^* := \{ G_t : U \hookrightarrow M \mid t \in T, G \in C \} \subset \text{Emb}(U,M).$$

Since $T$ is assumed to be connected, there is a well-defined map $p_T : \pi_0(C) \to \pi_0(C^*) : [G] \mapsto [G_t]$. Somewhat conversely, we have the following lemma:

**Lemma 4.1.** Given $B \in \pi_0(C^*)$, there exists a continuous cover $G$ of $M$ of type $([0,1],U)$ such that $G_s \in B$ for all $s \in [0,1]$. Moreover, any two such covers $G^0$ and $G^1$ are homotopic through such covers.

**Proof.** By definition of $C^*$, there exists a continuous cover $G'$ of $M$ of type $(T,U)$ such that $G_t \in B$ for all $t \in T$. Since $M$ is compact, there is a finite set $T' := \{ t_1, \ldots, t_N \} \subset T$ such that the $G'_t|_{T'}$ is a finite cover of $M$. Pick any continuous path $\gamma : [0,1] \to T$ such that $\gamma(k/N) = t_k$ for every $k = 1, \ldots, N$ and define $G(s) := G'((\gamma(s))$.

For the proof of the second assertion, let $G^0$ and $G^1$ be two continuous covers of $M$ of type $([0,1],U)$ such that $(G^j)_s \in B$ for any $s \in [0,1]$ and for $j = 0,1$. Since $B \subset C^*$ is connected, there is a path $\gamma : [0,1] \to B$ such that $\gamma(0) = G^0(1)$ and $\gamma(1) = G^1(1)$. Define

$$\Gamma : [0,3] \to B : s \mapsto \begin{cases} G^0(s) & \text{if } s \in [0,1], \\ \gamma(s-1) & \text{if } s \in [1,2], \\ G^1(s-2) & \text{if } s \in [2,3]. \end{cases}$$

The sought-after homotopy of covers is given by

$$G^t(s) : [0,1] \times [0,1] \to B : (s,t) \mapsto \begin{cases} \Gamma((1+4t)s) & \text{if } t \in [0,1/2], \\ \Gamma((4t-2)+(5-4t)s) & \text{if } t \in [1/2,1]. \end{cases}$$

$\square$
From this lemma, we conclude that any homotopy class \( B \in \pi_0(C^*) \) determines a unique (constrained) class of covers of type \(([0,1], U)\), namely the class of line covers by elements of \( B \). In particular, a constrained class \( A \) of covers of type \((T, U)\) determines a well-defined (constrained) class \( A^* \) of covers of type \(([0,1], U)\) such that \( p_T(A) = p_{[0,1]}(A^*) \).

In the above setting, the constraint on covers of type \(([0,1], U)\) is induced by a given constraint on covers of type \((T, U)\), so that the two types of constrained covers are on a somewhat unequal footing. However, we shall explicitly consider only constraints \( C \) which are the largest constraints such that the corresponding sets of embeddings \( C^* \) equal to sets \( C' \subset \text{Emb}(U, M) \) given beforehand, for instance the set of symplectic embeddings. In other words, such a constraint consists in all covers of type \((T, U)\) by elements of a given \( C' \) (which makes sense for any choice of \( T \)) and we shall therefore call it a prime constraint.

**Theorem 13** (Reduction of the parameter space). Let \( M \) be a symplectic manifold of dimension \( 2n \), \( U \) an open subset of \( \mathbb{R}^{2n} \) as mentioned above, and \( T \) a compact manifold of strictly positive dimension endowed with a Lebesgue measure \( \mu \) of total mass \( 1 \). Let \( A \) be a constrained class on covers of type \((T, U)\) and let \( A^* \) denotes the corresponding constrained class on covers of type \(([0,1], U)\). Then

\[
\text{pb}(A) = \text{pb}(A^*).
\]

The demonstration of this theorem will consist in proving the inequality \( \text{pb}(A) \leq \text{pb}(A^*) \) in subsection 4.2 and the inequality \( \text{pb}(A) \geq \text{pb}(A^*) \) in subsection 4.3.

### 4.2. Bipartite covers and their associated partitions of unity.

We introduce the following gadget inspired by an idea of Buhovski.

**Definition 14.** Let \( G_T \) and \( G_I \) be continuous covers of \( M \) of type \((T, U)\) and of type \( (I = [0,1], U) \) respectively. Abusing notations a little, assume that \( p_T(G_T) = p_I(G_I) \), i.e. if \( G_T \) belongs to a constrained class \( A \), then \( G_I \) belongs to the corresponding class \( A^* \). A **bipartite cover associated to** \((G_T, G_I)\), or a **bicover** in short, is a continuous cover \( \tilde{G} \) of \( M \) of type \((T, U)\) satisfying the following conditions:

1. there is a bipartition \( T = (T \setminus \text{int}\, D) \cup D \) where \( D \subset T \) is a closed embedded \( \text{dim}(T) \)-ball;
2. on \( T \setminus \text{int}\, D \), \( \tilde{G} \) is given by the pullback of \( G_T \) under a "quotient map"

\[
q : T \setminus \text{int}\, D \to (T \setminus \text{int}\, D) / \partial(T \setminus \text{int}\, D) \xrightarrow{\sim} T;
\]
3. there is an embedding \( \iota : K \hookrightarrow \text{int}\, D \) where \( K := I \times \mathbb{D}^{-1}, \) and \( \tilde{G}(\iota(u, x)) := G_I(u) \) for \((u, x) \in K\).

Given \( G_T \) and \( G_I \) as in this definition, an argument reminiscent of the proof of Lemma 4.1 shows that there exists a bicover associated to \((G_T, G_I)\) and that such a bicover belongs to the constrained class \( A \) of \( G_T \). We set \( V_T = \text{vol}(T \setminus D) \) and \( V_I = \text{vol}(\text{im}\, \iota) \).

Given a partition of unity \( F_I \) subordinated to \( G_I \), we can "extend" it to a partition of unity \( \tilde{F}_I \) subordinated to \( \tilde{G} \). More explicitly, let \( \rho : \mathbb{D}^{-1} \to [0,1] \) be a compactly supported function such that the top-form \( \rho V_I^{-1} \iota^* dt \) on \( K \) has mass \( 1 \). For \((u, x, y) \in K \times U\), we set \( \tilde{F}_I(\iota(u, x), y) = \rho(x) V_I^{-1} F_I(u, y) \) and we extend by 0 outside \( \text{im}\, \iota \).
Proposition 15 (Buhovski). Let $G_T$ and $G_I$ be continuous covers of $(M, \omega)$ of type $(T, U)$ and $(I, U)$ respectively. Given any bcover $\tilde{G}$ associated to $(G_T, G_I)$ (if it exists), we have
\[ \text{pb } G_I \geq \text{pb } \tilde{G}. \]

Proof. Let $F_I$ be a partition of unity subordinated to $G_I$. Then $\tilde{F}_I$ is a partition of unity subordinated to $\tilde{G}$. Given any $T$-weight $\alpha$, define the $I$-weight $\alpha_I$ as the fiber integral $\alpha_I(u) = \int_{[2m-1,2m[} \alpha(u,y))/2^{2n}] V^{-1} dx$. The functions $\tilde{f}(y) = \int_T \alpha(t) \tilde{F}_I(t,y)dt$ and $f(y) = \int_T \alpha_I(u) F_I(u,y)du$ on $M$ are the same. Hence $\text{pb } F_I \geq \text{pb } \tilde{F}_I$. Since this is true for any $F_I$, we get the result.

\[ \square \]

As we explained, whenever $p_T(G_T) = p_I(G_I)$, there exists a bcover $\tilde{G}_T$ associated to $(G_T, G_I)$ in the same constrained class $A$ than $G_T$. It thus follows from Proposition 15 that $\text{pb } A^* \geq \text{pb } A$.

4.3. Space-filling curves.

Definition 16. A space-filling curve of a manifold $T$ is a surjective continuous map
\[ c : [0, 1] \to T. \]

An important example of a space-filling curve is the Hilbert curve $c_H : [0, 1] \to [0, 1]^2$, which is obtained as the uniform limit of a sequence of continuous maps $c_n : [0, 1] \to [0, 1]^2$ (which we shall call "Hilbert approximants"). A special property of $c_H$ is that it sends any dyadic interval $[m/2^{2n}, (m+1)/2^{2n}]$ (which has measure $1/2^{2n}$) onto a dyadic square of measure $1/2^{2n}$ (the presence of the number 2 in the exponents is related to the fact that $[0, 1]^2$ is two-dimensional). By the density of the dyadic numbers, it follows that $c_H$ is measure-preserving, i.e. if $\mu_1$ and $\mu_2$ denote the Lebesgue measure of mass 1 on $[0, 1]$ and $[0, 1]^2$ respectively, then $\mu_1(c_H^{-1}(E)) = \mu_2(E)$ for any measurable set $E \subset [0, 1]^2$. This construction generalizes for any $d \in \mathbb{N}$ in order to construct a Hilbert space-filling curve $c_{H,d} : [0, 1] \to [0, 1]^d$ which is measure-preserving. The usual construction of the Hilbert curve is such that $c_{H,d}$ maps $\partial[0, 1]$ to $\partial([0, 1]^d)$.

Given a $d$-dimensional compact manifold $T$ endowed with a volume form $dt$ of mass 1, it is possible to pave $T$ by a finite number of closed embedded $d$-cubes $\Delta_1, \ldots, \Delta_n$ ordered in such a way that each union $\Delta_k \cup \Delta_{k+1}$ is connected. Denote by $V_k$ the volume of $\Delta_k$ with respect to $dt$. It follows from a theorem of Moser that there exists a measure-preserving diffeomorphism $\phi_k : [0, V_k^{-1/d}] \to \Delta_k$. Hence the map $c_k : [0, V_k] \to \Delta_k : t \mapsto \phi_k(V_k^{1/d} c_{H,d}(t/V_k))$ is a measure-preserving space-filling curve of $\Delta_k$. We can further choose the maps $\phi_k$ so that $c_k(V_k) = c_{k+1}(0)$, allowing us to concatenate these curves to obtain a measure-preserving space-filling curve $c : [0, 1] \to T$.

Our interest in measure-preserving space-filling curves stems from the fact that they allow a “change of integration variable”, going from $d$ integration variables to solely 1 variable. Indeed, given a measurable function $f : T \to \mathbb{R}$ and a continuous map $c : [0, 1] \to T$, the map $f \circ c : [0, 1] \to \mathbb{R}$ is also measurable. Therefore, if $f$ is integrable and if $c$ is a measure-preserving space-filling curve of $T$, we have
\[ \int_T f(t) \, dt = \int_{[0, 1]} (f \circ c)(u) \, du. \]
Let $G_T$ be a continuous cover of $M$ of type $(T, U)$ and $c : [0, 1] \to T$ be a measure-preserving space-filling curve. The map $G_I := G_T \circ (c \times id) : [0, 1] \times U \to M$ is thus a continuous cover of type $([0, 1], U)$. Given a partition of unity $F_I$ subordinated to $G_T$, the map $F_I := F_T \circ (c \times id) : [0, 1] \times U \to [0, \infty)$ is a partition of unity subordinated to $G_I$.

**Proposition 17.** Let $G_T$ be a continuous covers of $(M, \omega)$ of type $(T, U)$. Let $c : I \to T$ be a measure-preserving space-filling curve. Let $G_I = G_T \circ (c \times id)$. We have $\text{pb} G_T \geq \text{pb} G_I$.

**Proof.** Let $F_T$ be a partition of unity subordinated to $G_T$. Then, as above, we define $F_I := F_T \circ (c \times id)$, which is a partition of unity subordinated to $F_I$. Given any $I$-weight $\alpha$, we would like to pushforward it with the help of $c$ to get a $T$-weight $\alpha_T$ such that the functions $f_I(y) = \int_I \alpha(u) F_I(u, y) du$ and $f_T(y) = \int_T \alpha_T(t) F_T(t, y)dt$ on $M$ are the same. However, a space-filling curve is not injective in general, and there is no reason for the weight $\alpha$ to be constant on preimages by $c$ of singletons; some care is required in the definition of $\alpha_T$.

Consider the measure $d\mu = \alpha(u) du$. There is a well-defined notion of pushforward for measures, a notion we implicitly encountered above: the pushforward measure $c_\ast \mu$ is defined by the equation $(c_\ast \mu)(E) = \mu(c^{-1}(E))$ for any measurable set $E \subset I$. Since $c$ is measure-preserving, it is easy to see that $c_\ast \mu$ is absolutely continuous with respect to $dt$, that is, for any $E$ such that $dt(E) = 0$, we have $(c_\ast \mu)(E) = 0$. Indeed,

$$
\left| (c_\ast \mu)(E) \right| = | \mu(c^{-1}(E)) | = \left| \int_{c^{-1}(E)} \alpha(u) du \right| \leq \int_{c^{-1}(E)} | \alpha(u) | du \leq (c_\ast du)(E) = dt(E) = 0.
$$

By the Radon-Nikodym theorem, there exists a measurable function $\alpha_T$ such that $d(c_\ast \mu) = \alpha_T(t) dt$. Therefore, we have $\int_I \alpha(u)(c_\ast g)(u) du = \int_T \alpha_T(t) g(t) dt$ for any integrable function $T$. This implies that $\alpha_T$ is a $T$-weight with the properties we were seeking for.

It follows that $\text{pb} F_T \geq \text{pb} F_I$. Since this is true for all $F_T$, taking the infimum on both covers over the partitions of unity subordinated to them yields the result.

$\square$

Let $G_T$ be a cover of type $(T, U)$ in the constrained class $A$. Given any measure-preserving space-filling curve $c : I \to T$, we get a corresponding cover $G_I$ of type $(I, U)$ in the constrained class $A^\ast$. According to Proposition 17, we have $\text{pb} G_T \geq \text{pb} G_I$, whence the inequality $\text{pb} A \geq \text{pb} A^\ast$.

This concludes the proof of Theorem 13. Henceforth, except for any explicit mention to the contrary, we shall always take $T = I$.

**Remark 9.** Buhovsky (private communication) proved Theorem 13 directly in the smooth setting, where covers and partitions of unity are smooth. Proposition 15 is straightforwardly adapted to this setting, but not Proposition 17 as it uses space-filling curves which are not part of the smooth realm. Instead, Buhovsky’s argument essentially consists in using a sufficiently good embedded Hilbert approximant $c_n$ of $T$ to get a cover of type $(I, U)$ from any cover of type $(T, U)$ and to make sure that $F_I^\ast(u, y) := F_I^\ast(u, y)/ \int_I F_I^\ast(s, y) ds$ is a partition of unity, where $F_I^\ast := F_T \circ (c_n \times id) : I \times U \to [0, \infty)$. Moreover, the construction of a $T$-weight from any $I$-weight is more straightforward in
this context. This argument is well-suited for the smooth setting, but we feel that the true meaning of its success is best revealed by the use of space-filling curves.

5. Comparisons of discrete and continuous covers

Let $A$ be a constrained class of continuous covers of type $(I, U)$ and let $A' := p_I(A)$ be the associated constrained class of embeddings. Define the discrete pb-invariant associated to $A$ as (we use the notation $(1, n) = \{1, \ldots, n\}$)

$$pb_{\text{discrete}} A := \inf \{ pb G \mid G : (1, n) \times U \to M \text{ is a cover and } G_k \in A' \ \forall k \in (1, n) \}.$$  

**Theorem 18.** Given any constrained class $A$ of continuous covers of type $(I, U)$, we have $pb A \geq pb_{\text{discrete}} A$. Furthermore, if the class $A$ comes from a prime constraint $C$, then we have $pb A = pb_{\text{discrete}} A$.

**Proof.** Let’s first prove $pb A \geq pb_{\text{discrete}} A$. Let $G \in A$ be a constrained continuous cover of type $(I, U)$ and let $F$ be a partition of unity subordinated to it. Recall that each $F_\ell : M \to [0, \infty)$ has compact support included in the open set $G_\ell(U')$ for some open set $U' \subset U$. Construct an open cover $(O_\ell)_{\ell \in I}$ as follows. Given $t \in I$, the set $O_t := \{ s \in I \mid G_s(U) \subset G_t(U) \}$ is open since the function $G$ is continuous. As $I$ is compact, a finite collection $(O_{t_j})_{j=1, \ldots, n}$ suffices to cover $I$. Let $\delta > 0$ be a Lebesgue number for this cover, which means that any interval in $I$ of length $l \leq \delta$ is included in one of the $O_{t_j}$’s; such a number exists since $I$ is compact. Pick $N \in \mathbb{N}$ such that $1/N < \delta$ and consider the closed sets $V_k = [(k-1)/N, k/N]$ for $k \in (1, N)$. Choose any function $r : (1, N) \to (1, n)$ such that $V_k \subset O_{r(k)}$ for all $k \in (1, N)$ and define

$$F'_\ell : G_{t_j}(U) \to [0, \infty) : x \mapsto \sum_{k \in r^{-1}(j)} \int_{V_k} F(s, x)ds.$$  

We have constructed a discrete cover $G' := G|_{1 \times \ldots \times U}$ and a partition of unity $F' = (F'_1, \ldots, F'_n)$ subordinated to it. Given any discrete weight $\alpha' : \{t_1, \ldots, t_n\} \to [-1, 1]$, we can consider the $I$-weight $\alpha(s) := \sum_{k=1}^N \alpha'(k) \chi_{V_k}(s)$ where $\chi_S$ denotes the characteristic function of a set $S$. We easily compute

$$\sum_{j=1}^n \alpha'(j) F'(j, x) = \sum_{j=1}^n \sum_{k \in r^{-1}(j)} \int_{V_k} \alpha'(j) F(s, x)ds = \int_I \alpha(s) F(s, x)ds.$$  

It readily follows that $pb F \geq pb F'$. Given $\epsilon > 0$, it is possible to choose $G$ and $F$ such that $pb F < pb A + \epsilon$, hence $pb_{\text{discrete}} A \leq pb F' < pb A + \epsilon$. As this holds for any $\epsilon > 0$, we indeed have $pb_{\text{discrete}} A \leq pb A$.

Let’s now prove $pb_{\text{discrete}} A \geq pb A$ in the case of a prime constraint. Let $G' : (1, n) \times U \to M$ be a discrete cover such that $G'_k \in A'$ for each $k \in (1, n)$. Since $A'$ is connected, there exists a path $\gamma : [0, 1] \to A'$ such that $\gamma(k/(n+1)) = G'_k$ for all $k \in (1, n)$. We can further assume that $\gamma$ is constant in the $1/3(n+1)$-neighbourhood of each $k/(n+1)$. The map $G = ev \circ (\gamma \times id) : I \times U \to M : (t, u) \mapsto \gamma(t)(u)$ is a continuous cover of type $(I, U)$. Since the constraint is prime, it follows from Lemma 4.1 that $G \in A$. Let
\( F' = (F'_1, \ldots, F'_n) \) be a partition of unity on \( M \) subordinated to \( G' \). Define
\[
F : I \times U \to [0, \infty) : (t, u) \mapsto \begin{cases} 
\frac{3(n + 1)}{2} F'_k(G(t, u)) & \text{if } t \in \left( \frac{k-1/3}{n+1}, \frac{k+1/3}{n+1} \right), \\
0 & \text{elsewhere}.
\end{cases}
\]
We easily observe that this determines a continuous partition of unity \( F \) on \( M \) subordinated to \( G \). Given an \( I \)-weight \( \alpha \), we define the discrete weight
\[
\alpha'(k) = \frac{3(n + 1)}{2} \int_{(k-1/3)/(n+1)}^{(k+1/3)/(n+1)} \alpha(s) ds.
\]
We compute for any \( x \in M \)
\[
\sum_{j=1}^{n} \alpha'(j) F'_j(x) = \sum_{j=1}^{n} \int_{(j-1/3)/(n+1)}^{(j+1/3)/(n+1)} \alpha(s) \frac{3(n + 1)}{2} F'_j(x) ds = \int_I \alpha(s) F(s, x) ds.
\]
It readily follows that \( \text{pb} F' \geq \text{pb} F \). Taking the infimum over partitions on each side, we get \( \text{pb} G' \geq \text{pb} G \). Taking infimum over constrained covers on each side, we finally obtain \( \text{pb}_{\text{discrete}} A \geq \text{pb} A \).

Despite the fact that we did not define what is a continuous cover with varying open set \( U_i \), it is not difficult to convince oneself that the last theorem would intuitively hold in many situations where one considers this broader type of continuous covers. Indeed, on the one hand, the first part of the proof would go along almost verbatim. On the other hand, the second part of the proof could be modified to interpolate between the two consecutive sets of the discrete open cover as follows: by shrinking the first of these two open sets to an open ball of diameter smaller than the Lebesgue number of the cover, then displace this ball to bring it inside this second set and to dilate the ball to equal this second set.

As a result, continuous covers truly behave as the limit of discrete covers as the cardinality tends to \( \infty \). Incidentally, Polterovich’s conjecture could be stated as saying that for continuous cover \( G \) by displaceable open sets in the prime class \( A_E \) of open sets whose displacement energy is below a finite value \( E \), there exists a constant \( C > 0 \) independent of \( E \) and \( A_E \) such that \( \text{pb}(A_E)E \geq C \).

6. FROM CONTINUOUS TO SMOOTH COVERS

The aim of this section is to consider if given any continuous cover \( G \) of type \( (I, U) \) and any partition of unity \( F \) subordinated to it, it is possible to smoothen \( G \) to a cover which still subordinates \( F \). Such a result would allow us to compute the pb-invariant of a class by only considering smooth covers.

More precisely, at least in the unconstrained case, it is not difficult to show that we can slightly perturb \( G : I \times U \to M \) to a smooth map \( G' : I \times U \to M \) which belongs to the same class \( A \) and which subordinates the partition of unity \( F \). Indeed, given any \( \delta > 0 \) and any metric \( d \) on \( M \), Whitney approximation theorem assures that there is a smooth function (which can be obtained with the aid of a “small” isotopy) \( G_{\delta} : I \times U \to M \) such that \( d(G_{\delta}, G) < \delta \). In a similar fashion to what was done in the proof of Theorem 3, the compacity of \( I \) and the assumption that the support of each function \( F_t \) is contained in the open set \( U \) allow us to pick \( \delta > 0 \) such that \( G(\{t\} \times \text{supp } \tilde{F}_t) \subset G_{\delta}(\{t\} \times U) \). This proves the claim in the absence of constraint.
The constrained case is more subtle: a direct adaptation of the preceding argument would need us to endow the constrained class $A$ with some sort of smooth structure and to make sure that a smooth approximation theorem holds true in this context. As we need to consider smooth covers of symplectic embeddings in the next section, we shall explain here only how to smoothen continuous covers inside (non-empty) classes $A$ of the prime constraint $C' = \Emb_b(U, M)$.

Consider $G$ and $F$ as given above. By hypothesis on covers and on partitions of unity, there exists a open sets $U', U''$ such that $U' \supset U'' \supset U \supset \overline{U''}$, such that the embedding $G_t : (U', \omega_0) \hookrightarrow (M, \omega)$ is symplectic for any $t \in I$ and such that $F$ is supported in $T \times U''$. The collection

\[ \{O_t := \{s \in I \mid G_s(U) \subset G_t(U'), G_s(U) \supset G_t(\overline{U''})\}\}_{t \in I} \]

is an open cover of $I$ which has some Lebesgue number $\delta_L > 0$. Given $N \in \mathbb{N}$ such that $1/N < \delta_L$, consider the partition of $I$ into intervals of length $1/N$. Let $I_k = [k/N, (k + 1)/N]$ be anyone of these intervals. It follows that there exists $t \in I$ such that for any $s \in I_k$, $G_t(\overline{U''}) \subset G_s(U) \subset G_t(U')$. In the spirit of the relative version of Whitney approximation theorem, we can find an approximating isotopy

\[ \tilde{G} : I \times I_k \times U \to G_t(U') \]

such that $\tilde{G}(0, -,-) = G$, such that $\tilde{G}(1, -,-)$ is smooth, such that $\tilde{G}(-, k/N, -) = G_{k/N}$ and $\tilde{G}(-, (k+1)/N, -) = G_{(k+1)/N}$ and which satisfies $\tilde{G}(-, s, \overline{U''}) \subset G_s(U)$. This is most easily seen by considering the composition $G_{s}^{-1} \circ \tilde{G}$ which takes values in $U' \subset \mathbb{R}^{2n}$, as the ambient linear structure allows for a regularization by convoluting $G_s$ with a $(r,s)$-dependent $\delta$-approximating function. Incidentally, but very importantly, we can make sure that each $\tilde{G}(r, s, -)$ is a (nonsymplectic in general) embedding of $U$ which is arbitrary $C^{\infty}$-close to $G_s$. For each $(r, s) \in I \times I_k$, consider the 2-form $\omega_{r,s} := \tilde{G}_{r,s}^* \omega$ on $U$; it is without lost of generality sufficiently $C^0$-close to the form $\omega_0$ for the forms $\omega_{v,r,s} = (1-v)\omega_{r,s} + v\omega_0$ with $v \in I$ to all be symplectic. The Moser argument then defines a smooth isotopy $\phi : I \times I \times I_k \times U \to G_t(U') : (v, r, s, u) \mapsto \phi(v, r, s, u)$ such that $\phi(0, 0, r, s) = \tilde{G}$, such that (without lost of generality) each $\phi_{v,r,s}$ is an embedding which are furthermore symplectic when $v = 1$. It can be chosen such that it acts trivially on the values $(r, s)$ for which $\tilde{G}(r, s, -)$ is already symplectic. Set $G'_v = \phi(v, 1, r, s)$. We can further assume without lost of generality that $\text{supp}(F_s) \subset G'_v(\overline{U''})$, which proves the claim.

Although we will not have any specific need for it, we can ask whether given $\epsilon > 0$, a continuous cover $G$ and a partition of unity $F$ subordinated to it (with $\text{pb} F < \infty$) we can smoothen $F$ (which is a priori only continuous in the $t$-variable) to a smooth partition of unity $F'$ subordinated to $G$ in such a way that $|\text{pb} F - \text{pb} F'| < \epsilon$. The difficulty here is that the absolute Poisson bracket $|{-,-}|$ is not continuous under $C^0$-perturbations of its arguments (even though it certainly is under $C^1$-perturbations), so that it is unclear if any regularization of $F$ would suffice to prove such an assertion. This difficulty nevertheless suggests that continuous partitions should have generically a higher pb-invariant than smooth ones. It therefore seems plausible that the pb-invariants computed with the aid of continuous or smooth data coincide.
7. The behaviour of \( \text{pb} \) on symplectic balls

For the rest of this article, we only consider \( U = B^{2n}(r) \), that is the standard symplectic ball of radius \( r \) (or of capacity \( c = \pi r^2 \)). We also only consider (smooth) symplectic covers, that is covers \( G \) of type \((T, U)\) satisfying the symplectic prime constraint \( C \) given as follows: \( G \in C \) if \( G_t \in \text{Emb}_\omega(U, M) \) for every \( t \in T \). We shall write \( U(c), C(c) \) and \( C'(c) \) when we want to stress the dependence on \( c \).

Of special interest are the cases when \( T = S^n \) for some \( n \geq 1 \). A constrained class \( A \) of \( C \) determines a connected component \( A' \subset C' \) and thus corresponds to an element of the \( n \)-th homotopy group \( \pi_n(A') \). The \( \text{pb} \)-invariants of such constrained classes hence allow for the study of the homotopic properties of \( C'(c) \), properties which might change with \( c \). Consequently, it appears important to better understand how the \( \text{pb} \)-invariants depend on the capacity \( c \). This behavior of \( \text{pb} \) on \( c \) is the main question raised in this paper.

Theorem \[\text{18}\] proves that \( \text{pb}(A) \) is the same as \( \text{pb}(A') \) where \( A' \) is the symplectic constrained class of (smooth) covers of type \((I, U)\) determined by \( p_I(A') = A' \). We therefore lose no generality in assuming that \( T = I \).

We denote by \( c_{\max} \) the largest capacity of a symplectic ball embedded in \( M \) (that can be much small than the one implied by the volume constraint \( \text{Vol}(U(c)) \leq \text{Vol}(M, \omega) \), according to the Non-Squeezing Theorem).

Definition 19. We define the Poisson bracket function \( \text{pb} : (0, c_{\max}) \to [0, \infty) \) to be
\[
\text{pb}(c) := \inf \{ \text{pb}(A) \mid A \in \pi_0(C(c)) \} = \inf \{ \text{pb}(G) \mid G : I \times U(c) \to M \text{ is symplectic} \}.
\]

Theorem 20. The Poisson bracket function \( \text{pb} \) is a non-increasing, upper semi-continuous and left-continuous function.

Proof. In order to avoid any confusion between the many functionals \( \text{pb} \) defined, we shall here denote \( \text{pb}(c) \) as \( f(c) \).

(a) Let us first show that the function is non-increasing.

Consider a smooth symplectic cover \( G : I \times U(c) \to M \). Given \( c' \leq c \), the standard inclusion \( U(c') \subset U(c) \) induces a smooth map \( G' : I \times U(c') \to M \) by restriction of \( G \). In general, \( G' \) is not a cover of \( M \), but it can be extended to a smooth symplectic cover \( G'' : I \times U(c') \to M \) in the following way. Let \( p \in \text{Im} G'_1 \). By the transitivity of the action of Hamiltonian symplectomorphims, there exists for any \( q \in M \) a Hamiltonian flow \( \phi^t_q : M \to M \) such that \( \phi^1_q(p) = q \). The composition \( G'_q := \phi^1_q \circ G'_1 : U(c') \to M \) is a smooth symplectic embedding. The collection \( \{ G'_q(U(c')) \}_{q \in M} \) is an open cover of \( M \); by compactness, there exists a finite open subcover \( G'_{q_1}, \ldots, G'_{q_N} \). We can define a smooth Hamiltonian isotopy \( \Phi : I \times M \to M \) by smoothly reparametrising the concatenation of the flows \( \phi^t_{q_1}, \phi^t_{q_2}, \phi^t_{q_3}, \phi^t_{q_4}, \ldots, \phi^t_{q_N} \). The map \( \tilde{G} : I \times U(c) \to M \) defined by smoothly concatenating \( G \) and the smooth symplectic cover \( \Phi \circ (\text{id} \times G_1) : I \times U(c) \to M \) extends \( G \) and its restriction to \( U(c') \) defines the aforementioned symplectic cover \( G'' := (\tilde{G})' \).

\[\text{It is still a conjecture, that we shall dub the symplectic camel conjecture, whether } C'(c) \text{ is connected for any compact symplectic manifold } (M, \omega) \text{ and any } c.\]
Against our previous notation, let $A'$ denote the constrained class of the cover $G'$. Given $\epsilon > 0$, consider a smooth cover $G'' \in A'$ such that $\text{pb} G'' < \text{pb} A' + \epsilon$. Let $G'_\epsilon$ be a (smooth) bipartite cover associated to $(G'', G'_\epsilon)$: Proposition 15 implies that $\text{pb} G'_\epsilon < \text{pb} A' + \epsilon$. As $G'_\epsilon$ can be understood as a smooth isotopy of symplectic embeddings of $U(c')$ into $M$, it has the symplectic extension property: there exists a Hamiltonian isotopy $\Psi : I \times M \to M$ such that $G'_\epsilon = \Psi \circ (\text{id} \times (G'_\epsilon)_0)$.

Set $G'_\epsilon := \Psi \circ (\text{id} \times G_0)$ and observe that its restriction to $I \times U(c')$ is $G'_\epsilon$. Put differently, $G'_\epsilon$ is a refinement of $G_\epsilon$. As such, any partition of unity subordinated to the former is also subordinated to the latter, and it follows readily that $\text{pb} G'_\epsilon \leq \text{pb} G'_\epsilon$. Consequently, $f(c) < f(c') + \epsilon$ for all $\epsilon > 0$, hence the claim: $f(c') \geq f(c)$ whenever $c' \leq c$.

(b) Now let us show that for every $c \in [0, c_{\text{max}}]$, the function $f$ is upper semi-continuous at $c$, i.e. $\limsup_{c' \to c} f(c') \leq f(c)$.

On the one hand, it follows from part (a) that $f(c)$ is greater or equal to all limits of $f$ from the right.

On the other hand, for any $\epsilon > 0$, there is a pair $(G, F)$ defined on $I \times U(c)$ that achieves the infimum up to $\epsilon$: $\text{pb} F < f(c) + \epsilon$. By our definition of a partition of unity, there is a strictly smaller capacity $c' < c$ such that the support of $F$ is compact inside the open ball $U(c') \subset U(c)$. Transporting the data to the restriction of the pair $(G, F)$ to $I \times U(c'')$ for any $c'' \in [c', c]$, one gets

$$f(c'') \leq f(c) + \epsilon.$$ 

Since the choice of $c'$ obviously depends on $\epsilon > 0$, and might get as close to $c$ when $\epsilon$ approaches to zero, we do not get $f(c'') \leq f(c)$ but only that $f$ is upper semi-continuous from the left.

(c) We wish to prove that $f$ is in fact left-continuous, that is to say that $f(c)$ is equal to the limit of $f(c')$ as $c'$ tends to $c$ from the left. Consider a sequence of capacities $c_i < c$ converging to $c$ with highest value $\lim f(c_i)$ (the value $\infty$ is not excluded). This limit cannot be smaller than $f(c)$ because otherwise it would contradict the non-increasing property. However, by upper semi-continuity, it cannot be greater than $f(c)$. Therefore, it has to be equal to $f(c)$.

\hfill \Box

8. Phase transitions and the pb function

The first phase transition discovered in Symplectic Topology, that physicists could not have discovered since it appears only starting at the third homotopy level for ruled symplectic 4-manifolds, is the following one:

**Theorem 21.** (Anjos-Lalonde-Pinsonnault) In any ruled symplectic 4-manifold $(M, \omega)$, there is a unique value $c_{\text{crit}}$ such that the infinite dimensional space $\text{Emb}(c, \omega)$ of all symplectic embeddings of the standard closed ball of capacity $c$ in $M$ undergoes the following striking property: below $c_{\text{crit}}$, the space $\text{Emb}(c, \omega)$ is homotopy equivalent to a finite dimensional manifold, while above that value, $\text{Emb}(c, \omega)$ does not retract onto any finite dimensional manifold (or CW-complex) since it possesses non-trivial homology groups in dimension as high as one wishes. Below and above that critical value, the homotopy type stays the same.
Definition 22. Given a closed symplectic manifold $(M, \omega)$, let us call an uncertainty phase transition any critical value $c$ at which the space of symplectic embeddings of balls of capacity $c$ into $(M, \omega)$ undergoes a change of its homotopy type.

The reason for the term phase transition is still debatable, but there are several physical reasons to adopt that terminology.

The main driving conjecture of this paper is the following:

Conjecture 23. (The Poisson–uncertainty conjecture) The function $pb(c)$ corresponding to a compact symplectic manifold $(M, \omega)$ is a locally constant, upper semi-continuous and left-continuous function of $c$, with jumps downwards at the uncertainty phase transitions and only at these values.

This conjecture is admittedly a very bold one, as it claims a highly specific and nontrivial behaviour for the function $pb(c)$. Our main argument in its favour rests on our belief that the constancy/transition phenomenon exhibited by Anjos-Lalonde-Pinsonnault should be true for any symplectic manifold, and from the fact discussed earlier that symplectic covers of type $(S^n, U(c))$ probe the homotopy type of $\text{Emb}(c, (M, \omega))$. We therefore expect the Poisson bracket function to be a witness of these homotopy groups, and what is more a very reliable one.

A consequence of this conjecture is that, assuming that there are only finitely many uncertainty phase transitions, the limit of $pb(c)$ as $c$ goes to zero is a finite number. This last assumption is motivated by the impression that small displaceable balls should not see the symplectic form as well as by the fact that the space of (unparametrised) symplectic balls below the critical values retracts to the topology of the manifold itself for ruled symplectic 4-manifolds. This assumption refines the symplectic camel conjecture for small capacities and it leads us to state the following conjecture, which could be understood as a sort of universal property of the Poisson bracket function:

Conjecture 24. (The Topology conjecture). The limit of the function $pb(c)$, as $c$ tends to zero, depends only on the topology of the symplectic manifold.

There are nevertheless some reasons to think that the Poisson-uncertainty conjecture is false as stated.

One of them is Theorem 13 which states that the $pb$-invariant of a cover is in fact independent of the parametrising space $T$. This implies for instance that the Poisson bracket function $pb(c)$ does not distinguish the homotopy groups of $\text{Emb}(c, (M, \omega))$. In fact, it seems to depend only on the 0-th homotopy set. However, the work of Anjos-Lalonde-Pinsonnault showed that for ruled symplectic 4-manifolds, not every homotopy group of $\text{Emb}(c, \omega)$ undergoes a transition; in particular, the symplectic camel conjecture holds for these manifolds i.e. $\text{Emb}(c, \omega)$ is connected whenever it is non-empty. It is therefore possible that $pb(c)$ does not have any feature which betrays the presence of transitions in the homotopy type of the space of balls.

Another reason is that the Poisson-uncertainty conjecture and the Topology conjecture are both incompatible in some situations with Polterovich’s conjecture we mentioned in section 3. To see this, let us consider the simple situation of $(M, \omega)$ being $S^2$ with its standard symplectic form, say of area $A$. As $M$ is a surface, it satisfies the symplectic camel conjecture, which is to say that the space $\text{Emb}(c, \omega)$ is connected. The Poisson
bracket function is then defined for any $c \in (0, A)$ and should be a constant according to
the Poisson-uncertainty conjecture, or at the very least should not have any discontinuity.

This behaviour is contrary to what Polterovich’s conjecture implies. Indeed, there exists
on any closed symplectic manifold a spectral invariant $c$ such that $c(\text{Id}) = 0$, see Theorem
4.7.1 in [5]. It follows from that and the other properties of $c$ that the spectral width $w(U)$
of any subset $U \subset M$ satisfies $w(U) \leq 4e_H(U)$ where $e_H(U)$ is the Hofer displacement
energy of $U$. For open sets in $S^2$, $e_H(U) = \text{Area}(U)$ if this area is smaller than $A/2$ and
$e_H(U) = \infty$ otherwise. In this context, Polterovich’s conjecture states that there is a
constant $C > 0$ such that for any, continuous or discrete, cover $G$ of $S^2$ by displaceable
open sets, the inequality

$$\text{pb}(G)w(G) \geq C$$

This conjecture implies that $\text{pb}(c)e_H(U(c)) \geq C$. Thus when $c < A/2$, we have $\text{pb}(c) \geq
2C/c$, which contradicts the claim we made about the constancy of $\text{pb}(c)$. Furthermore, in
this regime, Polterovich’s conjecture implies $\text{pb}(c) \geq 2C/A > 0$. However, we observe that
$\text{pb}(c) = 0$ whenever $c > A/2$; two symplectic balls of capacity $c > A/2$ suffice to cover $S^2$
and the $\text{pb}$-invariant of such a cover vanishes. Polterovich’s conjecture hence goes against
the claim that the Poisson bracket function $\text{pb}(c)$ only has discontinuities when $\text{Emb}(c, \omega)$
undergoes a transition in its homotopy type.

Even though Polterovich’s conjecture and ours are incompatible as stated, let us note
that they are not exhaustive. For instance, as far as we can tell today, $\text{pb}(c)$ might vanish
for every $c$ and every closed symplectic manifold. Indeed, despite some evidences pointed
out by Polterovich in [4] to back his conjecture, its veracity is at the moment far from clear.
This state of affairs is due to the huge difficulty that there is in studying and computing
the Poisson bracket invariants, and these competing conjectures should be taken as proofs
of the complex and subtle way in which these invariants are tied up with the symplectic
topology of manifolds.

As a concluding remark, we point out that our borrowings in the thermodynamical
and statistical mechanical terminology are explained by our insight that tools from these
subjects might play a role in the understanding of the symplectic problems we considered in
this paper. The space of symplectically embedded balls can be understood as an infinite
dimensional (pre)symplectic manifolds which is some sort of limit of finite dimensional
ones. In this paper, continuous covers have also been understood as limits of discreetes
ones. It is a recurrent theme in statistical mechanics that systems with a very large
number of degrees of freedom tend to behave in a similar and somewhat simple way.
While researches in the field of symplectic topology have mainly focused on the properties
of low-dimensional manifolds, the very existence of statistical mechanics suggests to us that
very interesting mathematics should come out from the study of very high-dimensional
symplectic manifolds and that this study could be indirectly helpful to the study of lower-
dimensional ones too, for instance through the study of embedded symplectic balls.

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