Self-organized cooperative criticality in coupled complex systems

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Abstract – We show that the coupled complex systems can evolve into a new kind of self-organized critical state where each subsystem is not critical; however, they cooperate to be critical. This criticality is different from the classical sandpile criticality where the single system itself evolves into a critical state. We also find that the outflows can be accumulated in the coupled systems. This will lead to the emergence of spatiotemporal intermittency in the critical state.

Since it was proposed in the 1980s, the self-organized criticality (SOC) [1–3] has been one of the most popular theories in complexity science and is widely used as a way of understanding emergent complex behavior in physical [4–8], biological [9–11] and even social systems [12–15]. According to this theory, the ubiquitous power law in Nature can be interpreted as the hallmark of the critical state, a stable state that systems evolve into by themselves. SOC is usually illustrated by a simple cellular automaton, the so-called BTW sandpile model [1–3]. The dynamics of this model can be imaged as a transport phenomenon, where sand grains in a very high heap (with a local height gradient exceeding the threshold) are unstable and will tumble down the slope of this heap (against the local height gradient). We note that this dynamics is not enough to describe the natural systems where a movement along the local gradient is also possible. A well-known example is the cross effect in the transport arising in a mixture if both the concentrations and the temperature are non-uniform over the system. The cross effect can be simply illustrated in an isotropic binary mixture without viscosity when no external forces are supposed to be present and the pressure is uniform over the system [16]:

\[ J_q = -\lambda \nabla T - \alpha s_T \nabla C, \]

(1)

\[ J_d = -\rho D \nabla C - \beta s_T \nabla T. \]

(2)

the symbols \( J_q \), \( J_d \), \( T \), \( C \) and \( \rho \) are the heat flux, the diffusion flow, the temperature, the concentration of one of the components in the mixture and the density of mixture, respectively. The diffusion coefficient \( D \) and the heat conductivity \( \lambda \) are positive and are related to the normal heat conduction and diffusion, respectively. The symbol \( s_T \) is called Soret coefficient which is related to the cross effect in the transport and can be negative or positive. The coefficients \( \alpha \geq 0 \) and \( \beta \geq 0 \). From eqs. (1) and (2), one can see that the direction of heat or diffusion flow is determined by the equilibrium between the local temperature and the local concentration gradient. Two transport processes are coupled together. From the above discussions, a question arises as to what will happen if the dynamics of two SOC systems are coupled together. In this paper, we try to answer this question by using a simple model.

The model we propose here is composed of two BTW sandpiles. A BTW sandpile is represented by a two-dimensional grid with a length \( N \). To each square of the grid with coordinates \((x, y)\), we assign a number \( z(x, y) \), which represents the negative value of local height gradient at that square. The coordinates range as \( 1 \leq x \leq N \) and \( 1 \leq y \leq N \). Note that \( z \) represents in the sandpile model is not important. The more important thing is its dynamics. In real systems, \( z \) would represent any interesting dynamical variable (see examples in refs. [4–15]).

Different from the dynamics of BTW sandpile, our model couples two sandpiles in such a way that, if and only if both the negative values of local height gradients \( z_1 \) and \( z_2 \) of the two sandpiles at the same positions \((x, y)\) are greater than the threshold \( z_c \) (equaling 3 in the following simulations), the sand grains then tumble down:

\[ z_i(x, y) = z_i(x, y) - 4, \quad \text{for} \quad i = 1, 2 \]

(3)

and the tumbling down grains are transported to the nearest-neighbor grids,

\[ z_i(x \pm 1, y) = z_i(x \pm 1, y) + 1, \]

(4)

\[ z_i(x, y \pm 1) = z_i(x, y \pm 1) + 1. \]

(5)

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This model is conservative except in the boundary, for example,

\[ z_i(N,N) = z_i(N,N) - 4, \]
\[ z_i(N - 1, N) = z_i(N - 1, N) + 1, \]
\[ z_i(N, N - 1) = z_i(N, N - 1) + 1. \]

In this model, the height of one sandpile is like the temperature or concentration in the mixture. The movements of sand grains (heat/diffusion fluxes) in one sandpile are not only determined by the local height gradients of this sandpile (\( \nabla T/\nabla C \)) but also determined by the local height gradients of another sandpile (\( \nabla C/\nabla T \)). However, in the coupled sandpiles the contributions of local gradients are statistically symmetric for each sandpile while in the mixture due to different diffusion coefficients the contributions of \( \nabla T \) and \( \nabla C \) are asymmetric for different fluxes. Because of the symmetry of coupled sandpiles, we just show the statistics of one of them in the following analysis.

**Self-organized cooperative criticality (SOCC).** This is the most important feature we found in the coupled sandpiles. Starting from two sandpiles with the same length \( N \) (equaling 50 in the following simulations) and zero local height gradients in the whole sandpiles, the critical states are built by adding sand grains randomly to each sandpile simultaneously according to the rule \((i = 1, 2)\)

\[ z_i(x, y) = z_i(x, y) + 1, \]  

which is called the non-conservative perturbation in the BTW sandpile model [17]. After each perturbation, the two coupled sandpiles are allowed to tumble down (if needed) according to eqs. (3), (4) and (5). A local perturbation would spread to the nearest-neighbor grids. These grids would tumble down again and the perturbation is transported to the next-nearest-neighbor grids and so on until unstable grids \((i.e. z_i(x, y) > z_c)\) are not found in the sandpiles. Then other perturbations are randomly added into each sandpile to trigger a new possible avalanche.

As time elapses, the coupled sandpiles will evolve into a self-organized critical state. A signal for the critical state is the outflow which is defined as the total number of sand grains flowing out of one sandpile after each perturbation. Both for the BTW sandpile (squares in fig. 1) and for the coupled sandpiles (circles in fig. 1), the outflow will fluctuate around a constant once the critical states are built. There are other signals for the critical state, such as the power-law distributions of avalanche size (also called the total energy release) and lifetime of avalanches [1–3,17]. The avalanche size \( s \) is defined as the total number of tumbles in one sandpile induced by each perturbation and the lifetime of avalanche \( t \) is defined as the total number of simultaneous tumbles in one sandpile induced by each perturbation. In the simulation, perturbations are not simultaneously added into the two sandpiles and we will randomly choose a sandpile for each perturbation. We count the avalanche size \( s \) and the lifetime \( t \) induced by each perturbation and find that in the critical states the distribution of avalanche sizes \( D(s) \) and the distribution of lifetimes \( D(t) \) follow the same power laws both in the coupled sandpiles and the BTW sandpile (fig. 2(a) and (b)). The finite-size scalings of the distributions which have been found in the BTW sandpile are also observed in the coupled sandpiles (figs. 2(c) and (d)). For the distribution of the avalanche size, one could find that

\[ D(s) = s^{-\tau} F'(N^d/s), \]  

where \( \tau = 1.06, d = 1.92 \) and \( F' \) is a general scaling function of the single scaled variable \( N^d/s \). For the distribution of lifetime, one could find that

\[ D(t) = t^{-\alpha} F(N^\sigma/t), \]

where \( \alpha = 1.01 \) and \( \sigma = 0.75 \). Note that the values of \( d \) and \( \sigma \) are the same as those in the BTW model [2].

Although the coupled sandpiles and the BTW sandpile have the same distributions of the avalanche sizes and lifetimes, their self-organized critical states are very different qualitatively. For the BTW sandpile, all \( z(x, y) \leq z_c \) in the critical state. For the coupled sandpiles, there are many grids with \( z_1 > z_c \) or \( z_2 > z_c \), which mean that each separate sandpile in the coupled model is not critical in view of the BTW sandpile. However, they are indeed in the self-organized critical state not separately but cooperatively. Cooperative phenomena are common in the complex systems. Many species, such as bees and ants, can behave cooperatively in a self-organized way (see ref. [18] and references therein). Our model joins the two important notions of the cooperation and the SOC together in a simple manner. Thus, we call the new critical mode the self-organized cooperative criticality (SOCC).

The left plot of fig. 3 shows a snapshot of grids in the minimally stable states. We define the minimally stable state by the least stable state where one more adding grain will make tumbles happen. Thus, the grids with “\( z_1 = z_c \) and \( z_2 > z_c \)” in the sandpile labeled by “1” or with “\( z_1 > z_c \) and \( z_2 = z_c \)” in the sandpile labeled by “2” are in the minimally stable state. The geometric dimension of
Fig. 2: Distribution of (a) avalanche sizes $D(s)$ and (b) lifetimes $D(t)$ in critical states. The slopes of lines in (a) and (b) are about $-1.06$ and $-1.11$, respectively. Panels (c) and (d) are the test of finite-size scaling of $D(s)$ and $D(t)$, respectively. In the above figures, blank points show the results of SOCC and black dots show the results of BTW sandpile with $N = 50$.

Fig. 3: (Colour on-line) Left: the snapshot of the SOCC state in one sandpile. The minimally stable states with "$z_1 > z_c$ and $z_2 = z_c$" or "$z_2 > z_c$ and $z_1 = z_c$" are denoted by grey grids. Right: the relation between the number of minimally stable states $n$ in the chosen rectangle (as the red rectangle in the left plot) and the length of this rectangle $L$, averaged over 5000 samples. The slope of the dashed line is 2.

clusters in the minimally stable state can be measured as follows [19]. We set a square in the center of the whole sandpile and count the number of minimally stable states $n$ in this square. Then, a power-law relation between the length of the square $L$ and $n$ can be set up. That is the length-number relation: $n \sim L^D$, where $D$ is called the cluster fractal dimension. Measured result shows that $D = \log n / \log L \approx 2$ (the right plot of fig. 3), which means that the structure of the minimally stable state fills densely the space. More interesting, the dimension of the structure of the minimally stable state (with $z = z_c$) in the BTW sandpile is also approximate to 2. Thus, the geometric structures of the minimally stable state are statistically similar both in the SOC and SOCC sandpiles. That is possibly why we obtain similar distributions of avalanche size and lifetimes and their finite-size scalings in the two models.

Accumulative effect and spatiotemporal intermittency. – We note that the outflow in the SOC state fluctuates around a value of 1 (squares in fig. 1). This means that the inflow and the outflow in the SOC state are in statistical equilibrium and the average of $z(x, y)$ will also fluctuate around a constant (squares in fig. 4(a)). However, in the SOCC state the outflow fluctuates around a constant smaller than 1 (squares in fig. 1). This means that the inflow is greater than the outflow and the adding grains will be accumulated in the sandpile at a constant rate (squares in fig. 4(a)).
The increase or non-stationarity of $⟨z⟩$ in the SOCC state would not be a transient phenomenon. We also study the smaller sandpile ($N = 10$) which is expected to reach the stationarity more quickly. It is found that a non-stationarity of $⟨z⟩$ can also last up to a time magnitude of $10^6$ (fig. 4(b)). In the following discussions, we will see that the non-stationarity is mainly caused by the accumulations in the minimally stable state. In the minimally stable state, even if local height gradients in one sandpile are greater than $z_c$, the tumbling will not happen until the gradients of the corresponding sites in another sandpile are also greater than $z_c$. Compared with the BTW model, the accumulative effect will greatly reduce the number of avalanches, including the great ones which could cross the sandpile and transport sand grains out of the sandpile.

In fig. 5(a), we analyze the averages of $z$ with $z > z_c$ and $z \leq z_c$, respectively, and find that the adding grains are mainly accumulated to the grids with $z > z_c$. To see the accumulative process intuitively, we observe the evolution of the distribution of $z$ in the diagonal direction in one sandpile (fig. 5(b)). One can see that as time elapses the grids with $z > z_c$ accumulate more and more adding grains. At the same time, a more and more intermittent distribution of $z$ appears in space (fig. 5(b)). The intermittency will also appear in the time series of $z$ (fig. 5(c)). From these figures, one can see that the fluctuations of $z_i$ between any two grids or time periods would be up to about 40 while in SOC fluctuations would not exceed $z_c (= 3)$. Qualitatively, this phenomenon about large fluctuations in time and space is called the spatiotemporal intermittency. Many complex systems show spatiotemporal intermittency, such as the fluid turbulence [20–22], the earthquakes [3,23] and the traffic jam [14]. The accumulative effect of the SOCC sandpiles provides another possible mechanism for the emergency of spatiotemporal intermittency.

Robustness. – To test the robustness of the criticality in the coupled model, we randomly remove the connections between nearest-neighbor grids in each sandpile. When the connection between nearest-neighbor grids, say $z_1(x,y)$ and $z_1(x+1,y)$, is removed, the rules of eqs. (3), (4) and (5) are modified by

$$z_1(x,y) = z_1(x,y) - 3,$$
$$z_1(x+1,y) = z_1(x+1,y),$$
$$z_1(x-1,y) = z_1(x-1,y) + 1,$$
$$z_1(x,y \pm 1) = z_1(x,y \pm 1) + 1.$$

In the simulation, the numbers of removed connections for the two sandpiles are the same. Figures 6(a) and (b) show the distributions of $D(s)$ and $D(t)$ with different numbers of removed connections. We find that these distributions also have the power laws and no changes have been found in the exponents. However, the statistics on the very large avalanches deviate from the power laws. With the increase of removal, the deviation becomes large. The connectivity of the sandpile is decreased by the removal. Thus, the large avalanches occur with a lower frequency than in the unspoiled sandpile.

We also remove the correlations between the two sandpiles at the same coordinations. At these grids without correlations, if $z_1 > z_c$ or $z_2 > z_c$ the sand grains will tumble down. Simulations show that almost nothing is changed in the distributions of $D(s)$ and $D(t)$ even the proportion of the removed correlations is increased to 20% (figs. 6(c) and (d)).

**Non-conservative coupled sandpiles.** – The average of $z$ in the conservative coupled sandpiles will increase infinitely if we continually add sand grains into the sandpiles. It seems unphysical. In real systems, the

![Fig. 4: (Colour on-line) Time evolution of the $⟨z⟩$ averaged over the whole sandpile (circles for SOCC and rectangles for SOC).](image1)

![Fig. 5: (Colour on-line) Accumulative effect and spatiotemporal intermittency. (a) Time evolution of the $⟨z⟩$ averaged over the grids with $z > z_c$ (red line), $z \leq z_c$ (green line) and all $z$ (blue line). (b) The values of $z$ along the diagonal direction of one sandpile at different times. (c) Time evolution of $z$ at four randomly chosen grids with coordinates (5,5) (blue line), (5,10) (green line) and (10,10) (black line).](image2)
dissipation always exists. In ref. [24], authors proposed a non-conservative BTW sandpile to simulate the effect of dissipation. In this model, when tumbles happen at \((x, y)\), no matter how large \(z(x, y)\) is, it will become zero. Following this sprit, we modify the rules of (3) by \(z_i(x, y) = 0\). Other rules like eqs. (4) and (5) are not changed.

The non-conservative coupled sandpiles will also evolve into the SOCC states. Different from the conservative sandpiles, the inflow and outflow in this SOCC state is in statistical equilibrium (see fig. 7). It means that the accumulation effect is statistically offset by the dissipation. We stress “statistically” here because the local grids with \(z > z_c\) will also accumulate sand grains if tumbles have not happened in these grids. For the non-conservative BTW model, there is no accumulation because the largest value of \(z\) will not exceed \(z_c\) in this model. We measure the distributions of \(D(s)\) and \(D(t)\) in the critical states and find that they also have the power laws with the same exponents as in the conservative BTW sandpile (not shown).

In conclusion, we propose a model by coupling two BTW sandpiles in a simple way. This model can conjoin many notions popular in the complexity science, including the SOC, the cooperation, and the spatiotemporal

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**Fig. 6:** Distributions of the avalanche sizes and the lifetimes in the SOCC state of the spoiled couple sandpiles. In (a) and (b), the model is spoiled by randomly removing the connections between nearest-neighbor grids in each sandpile. The lines with different symbols correspond to different numbers of removed connections in each sandpile (circles for 250, squares for 500 and triangles for 1000). In (c) and (d), the model is spoiled by randomly removing correlations between the two coupled sandpiles. The meaning of symbols is the same as in (a) and (b). The slopes of dashed lines in plots of \(D(s)\) and \(D(t)\) are the same as the corresponding lines in fig. 2.

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**Fig. 7:** (Colour on-line) Time evolution of (a) the \(\langle z \rangle\) averaged over the whole sandpile and (b) the coarse-grained outflow in the SOCC state (circles) and the SOC state (rectangles), respectively.
intermittency. We believe that our model could be further extended to understand the real systems which might be composed of many coupled SOC systems.

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