STRATIFICATIONS AND SHEAVES ON THE RAN SPACE

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Abstract. We describe poset stratifications of the product of the Ran space and the nonnegative real numbers, as a universal space for the Čech construction of simplicial complexes. This leads to a cosheaf valued in diagrams of simplicial complexes for which every restriction to \( \{ P \} \times \mathbb{R}_{\geq 0} \) recovers the persistent homology of the data set \( P \). For the stratification, we describe a partial order on isomorphism classes of abstract simplicial complexes, which allows spaces stratified by them to have entrance paths uniquely interpreted as simplicial maps.

Contents

1. Introduction 1
2. Background 3
3. Stratifications of the Ran space 5
  3.1. Natural stratifications 5
  3.2. A conical refinement 8
4. Universal spaces 9
  4.1. The homotopy category of entrance paths 9
  4.2. The Čech cosheaf 11
  4.3. The Čech sheaf 13
  4.4. Open sets and basics 15
  4.5. A worked example 16
5. Applications 17
  5.1. Persistent homology 18
  5.2. Extensions 19
  5.3. Open questions 20
References 20

1. Introduction

The Ran space \( \text{Ran}^{\leq n}(M) \) is the space of subsets \( P \) of size at most \( n \) of a Riemannian manifold \( M \). We are concerned with the product space \( \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \), where a positive real number \( r \) indicates the abstract simplicial complex we should associate to \( P \), usually via the Čech or Vietoris–Rips construction. If \( r \) or \( P \) are slightly perturbed in the appropriate topology, the associated simplicial complex
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is still the same, up to homotopy. This paper is motivated by this observation and
the goal is to make this observation precise.

The central ideas of this paper are

- posets with a minimal element,
- upsets in the poset topology, and
- posets of simplicial complexes.

We begin by defining a Čech function \( \tilde{C} \) in Definition 3.2, which associates to a
pair \( (P, r) \in \text{Ran}^{≤n}(M) \times \mathbb{R}_{≥0} \) the abstract simplicial complex given by the Čech
construction (see Remark 1.2 for more on this choice) on the points in \( P \) and the
radius \( r \). Removing the labels of vertices and defining a partial order on unlabeled
simplicial complexes, we prove

**Theorem 1 (3.6).** The unlabeled Čech map \( u\tilde{C} \) is continuous.

However, the stratification of \( \text{Ran}^{≤n}(M) \times \mathbb{R}_{≥0} \) by \( u\tilde{C} \) is not conical (see Def-
nition 2.5). Conical stratifications are nice because they tell us all neighborhoods
are stratified in the same generic way. For particular \( M \), we still have

**Theorem 2 (3.10).** If \( M \) is piecewise linear, there exists a conical semialgebraic
stratification \( c\tilde{C} \) of \( \text{Ran}^{≤n}(M) \times \mathbb{R}_{≥0} \) compatible with \( u\tilde{C} \).

Compatibility (see Definition 2.4) means every stratum of \( u\tilde{C} \) has a partition
that corresponds to strata of \( c\tilde{C} \). In Section 4.1 we interpret the strata of \( c\tilde{C} \) as equivalence classes of homotopic paths in the category of entrance paths. Mor-
pisms among classes correspond to simplicial maps, which allows us to define a
Čech functor \( \mathcal{F} \) in Section 4.2. Our main result is

**Theorem 3 (4.8, 12, 4.9, 21).** The Čech functor is a cosheaf for which

- the costalk \( \mathcal{F}(P, r) = \tilde{C}(P, r) \) recovers the Čech map,
- the restriction of \( \mathcal{F} \) to the closed subset \( \{P\} \times \mathbb{R}_{≥0} \) is also a cosheaf, and
- the homology of \( \mathcal{F}|_{\{P\} \times \mathbb{R}_{≥0}} \) is the persistent homology of the data set \( P \).

Restricting \( \mathcal{F} \) to basics (see Definition 4.13) instead of all open sets allows us
to claim in Proposition 4.14 that the cosheaf is locally constant on strata. Higher
algebraic constructions with \( \infty \)-categories allow us to define similarly a Čech sheaf
\( \mathcal{G} \) in Section 4.3 but its properties are more difficult to understand.

**Remark 1.1 (Finite manifold subsets and finite metric spaces).** Our object of
interest is a finite subset, or point cloud, or data set, of a manifold. Persistent
homology often has as input a finite metric space, and manifold subset has all
the information of a finite metric space. If the metric space can be embedded in
Euclidean space, then the approaches are the same.
Remark 1.2 (Čech and Vietoris–Rips constructions). The two most common ways to associate simplicial complexes to finite metric spaces are the Čech construction and the Vietoris–Rips construction. We take the Čech approach, because it has a shorter description and is more general. That is, a change in the input that changes the Vietoris–Rips complex must also change the Čech complex, so in the context of spaces stratified by simplicial complexes, both constructions are covered by only considering the Čech approach.

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2. Background

Let $\mathcal{SC}$ be the category of abstract simplicial complexes$^1$ and simplicial maps. We consider a simplicial complex $C$ as a pair of sets $(V(C), S(C))$, with $S(C) \subseteq P(V(C))$ closed under taking faces.

Let $sSet$ be the category of simplicial sets and $Ord$ the category of partially ordered sets and order-preserving set maps. Every simplicial complex may be viewed as a simplicial set, by first taking the set of simplices and viewing it as a poset under set inclusion, then taking the nerve. This gives a composition of functors

$$(1) \quad \mathcal{SC} \xrightarrow{\text{simp}} \text{Ord} \xrightarrow{N} sSet,$$

which we will need in Sections 4.3 and 5.2. The first step is projection to the second factor of $(V(C), S(C))$ and the second step may be thought of as barycentric subdivision of the simplex set $S(C)$, viewed as a poset by set inclusion.

Let $X,Y$ be topological spaces and $A,B$ be posets. If the partial order on the poset $A$ is not clear from context, we write $(A, \leq_A)$. Posets have the upwards-directed, or upset, or Alexandrov topology. This topology has as basis the sets $U_a := \{b \in A : a \leq_A b\}$ for all $a \in A$.

Definition 2.1. An $A$-stratification of $X$, or just stratification when $A$ is clear from context, is a continuous map $f : X \to A$. When $f$ is clear from context, we say $X$ is $A$-stratified.

For any $a \in A$, we write $A_{>a} := \{a' \in A : a < a'\}$, and $A_{a} := \{a' \in A : a \leq a'\}$, which are both posets with the induced partial order from $A$. Similarly, we write $X_a := \{x \in X : f(x) = a\}$ and call them the strata of $X$.

Definition 2.2. A sheaf $\mathcal{F}$ on an $A$-stratified space $X$ is $A$-constructible, and a cosheaf $\mathcal{F}$ is $A$-coconstructible, if $\mathcal{F}|_{X_a}$ is locally constant, for every $a \in A$.

For sets $V \subseteq X$ not necessarily open and $\mathcal{F}$ a sheaf, $\mathcal{F}|_V$ is the inverse image presheaf $V \mapsto \operatorname{colim}_{U \supseteq V} \mathcal{F}(U)$. When $\mathcal{F}$ is a cosheaf, $\mathcal{F}|_V$ is the inverse image presheaf $V \mapsto \lim_{U \supseteq V} \mathcal{F}(U)$, as in [Woo09, Appendix B].

Definition 2.3. Given an $A$-stratification $f : X \to A$ and a $B$-stratification $g : Y \to B$, a stratified map $\phi$ from $f$ to $g$ is a pair of continuous maps $\phi_{XY} : X \to Y$ and

---

$^1$All simplicial complexes will be abstract, so we will drop the adjective.
$\phi_{AB} : A \to B$ such that the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi_{XY}} & Y \\
\downarrow{f} & & \downarrow{g} \\
A & \xrightarrow{\phi_{AB}} & B
\end{array}
\]

commutes. The stratified map $\phi$ is an open embedding if both $\phi_{XY}$ and $\phi_{XY}|_{X_a}$ are open embeddings, for all $a \in A$, with an analogous description for $\phi$ a homeomorphism.

**Definition 2.4.** An $A$-stratification of $X$ is compatible with a $B$-stratification of $X$ if for every $a \in A$ and $b \in B$, either $X_a \subseteq X_b$ or $X_a \cap X_b = \emptyset$.

Stratifications define partitions of the base space, so if an $A$-stratification $f$ is compatible with a $B$-stratification $g$, there is a stratified map $\phi$ between them, with a surjective order-preserving poset map $\phi_{AB}|_f(X) : f(X) \to g(X)$.

**Definition 2.5.** Let $f : X \to A$ be an $A$-stratification of $X$. Then $X$ is conically stratified at $x \in X$ by $f$ if there exist

- a topological space $Z$,
- an $A > f(x)$-stratified topological space $Y$, and
- a stratified open embedding $Z \times C(Y) \hookrightarrow X$ whose image contains $x$.

The space $X$ is conically stratified by $f$ if it is conically stratified at every $x \in X$ by $f$, in which case we call $f$ a conical stratification of $X$.

This is [Lur17, Definition A.5.5]. When $Y$ has an $A > f(x)$-stratification $g : Y \to A > f(x)$, its open cone $C(Y)$, understood as the quotient $Y \times [0, 1)/Y \times \{0\}$, has the $A > f(x)$-stratification $g' : C(Y) \to A > f(x)$ given by $g'(y, t \neq 0) = g(y)$ and $g'(y, 0) = f(x)$. The product $Z \times C(Y)$ is also $A > f(x)$-stratified through the projection $\pi_2$ to the cone factor. The image to have in mind is that $Z$ is an open neighborhood of $x$ in its stratum $X_{f(x)}$, and $Y$ is a collection of neighborhoods in strata directly above $X_{f(x)}$. This is extended in Section 4.4, where we describe a category of conically stratified open sets.

We will need semialgebraic geometry later, so we briefly discuss it here.

**Definition 2.6.** A set in $\mathbb{R}^N$ is semialgebraic if it can be expressed as

\[
\bigcup_{\text{finite}} \{ x \in \mathbb{R}^N : f_1(x) = 0, f_2(x) > 0, \ldots, f_m(x) > 0 \},
\]

for polynomial functions $f_1, \ldots, f_m$ on $\mathbb{R}^N$. An $A$-stratification of $X \subseteq \mathbb{R}^N$ is semialgebraic if $X_a$ is a semialgebraic set, for all $a \in A$.

It is immediate that the geometric realization of a finite abstract simplicial complex is a semialgebraic set. Conversely, every closed semialgebraic set is the homeomorphic image of the geometric realization of some abstract simplicial complex. This statement is not immediate - see [BCR98, Theorem 9.2.1] for the bounded case and [Shi97, Theorem II.4.2] for the unbounded case.

**Lemma 2.7.** Let $f$ be a semialgebraic stratification of a closed semialgebraic set $X$. Then there exists a conical semialgebraic stratification of $X$ compatible with $f$. 

Proof. Let \( f : X \to A \) be as in the statement. By [Shi97] Theorem II.4.2, there exists a simplicial complex \( K \) with homeomorphic image \( |K| \cong X \) and stratum decomposition \( f^{-1}(a) = \bigcup \sigma_{\circ}^a \), for \( \sigma^\circ \) the interior of a simplex \( \sigma \in S(K) \), for every \( a \in A \). With the partial order \( \sigma^\circ \leq \tau^\circ \) whenever \( \sigma \) is a face of \( \tau \), there is a natural stratification \( g : |K| \to \{ \sigma^\circ : \sigma \in S(K) \} \), and \( g \) is compatible with \( f \) by the mentioned result. This stratification of \( |K| \) is precisely the \( S \)-stratification of \( |K| \) given by [Lur17] Definition A.6.7, where \( S = S(K) \), which is conical by [Lur17] Proposition A.6.8. The \( S \)-stratification is semialgebraic because the interiors of simplices are semialgebraic, and finite unions of semialgebraic sets are semialgebraic. \( \square \)

3. Stratifications of the Ran space

Let \( M \) be a smooth and connected Riemannian manifold, with distance \( d_M \).

Definition 3.1. The Ran space of \( M \) is \( \text{Ran}(M) := \{ P \subseteq M : 0 < |P| < \infty \} \), with topology induced by Hausdorff distance \( d_H \) of subsets of \( M \).

For a positive integer \( n \), write \( \text{Ran}^n(M) \) and \( \text{Ran}^{\leq n}(M) \) for the subspaces of \( \text{Ran}(M) \) with elements exactly of size \( n \) and at most size \( n \), respectively.\(^2\)

Recall the Hausdorff distance between \( P, Q \in \text{Ran}(M) \) is defined as

\[
d_H(P, Q) := \max \left\{ \max_{p \in P} \min_{q \in Q} d_M(p, q), \max_{q \in Q} \min_{p \in P} d_M(p, q) \right\}.
\]

By scaling and [Lur17] Remark 5.5.1.5, the topology on \( \text{Ran}(M) \) induced by the Hausdorff metric is equivalent to another topology. From [Lur17] Remark 5.5.1.4, this is the coarsest topology that has \( \{ P \in \text{Ran}(M) : P \subseteq \bigcup U_i, P \cap U_i \neq \emptyset \ \forall \ i \} \) as open sets, for all nonempty disjoint collections of open sets \( \{ U_i \subseteq M \} \).

Definition 3.2. The \( Č \)ech map is the function \( Č : \text{Ran}^{\leq n}(M) \times \mathbb{R}_{>0} \to \text{Obj}(\mathcal{SC}) \) given by \( V(Č(P, r)) = P \) and \( P' \in S(Č(P, r)) \) whenever \( \bigcap_{p \in P'} B(p, r) \neq \emptyset \), for every \( P' \subseteq P \).

The ball \( B(p, r) \subseteq M \) is closed. We use the \( \infty \)-norm on the product space \( \text{Ran}^{\leq n}(M) \times \mathbb{R}_{>0} \), so \( d_\infty((P, r), (Q, s)) = \max\{d_H(P, Q), d(r, s)\} \).

3.1. Natural stratifications. There is a natural point-counting map \( \text{Ran}(M) \to \mathbb{Z}_{>0} \), which is a stratification by [Lur17] Remark 5.5.1.10, and is conical by [AFT17] Proposition 3.7.5.

Definition 3.3. Let \( \sim_{SC} \) be the relation on \( \text{Obj}(\mathcal{SC}) \) given by \( C \sim_{SC} C' \) whenever there is a simplicial map in \( \text{Hom}_{\mathcal{SC}}(C, C') \) that is bijective on simplices. Let \( u\mathcal{SC} := \text{Obj}(\mathcal{SC})/\sim_{SC} \) be the set of classes \( [C] \) of unlabeled simplicial complexes.

There is a natural map \( u : \text{Obj}(\mathcal{SC}) \to u\mathcal{SC} \) that removes the vertex labels of a simplicial complex. Let \( uČ := u \circ Č \) be the unlabeled Čech map.

Definition 3.4. Let \( \leq_{SC} \) be the relation on \( u\mathcal{SC} \) given by \( [C] \leq_{SC} [C'] \) whenever there is a simplicial map in \( \text{Hom}_{\mathcal{SC}}(C', C) \) that is surjective on vertices.

This relation is well-defined, irrespective of the choice of representative from \([C]\) and \([C']\), as we can compose a surjection with a bijection to still have a surjection.

\(^2\)In the former case, \( \text{Ran}^n(M) \) is also called the configuration space of \( n \) points.
Lemma 3.5. The relation \( \leq_{\text{SC}} \) defines a partial order on \( \text{uSC} \).

Proof. For reflexivity, we have a bijection \( C_1 \to C_2 \) in \( \text{SC} \), which is surjective on vertices, for any two representatives \( C_1, C_2 \) of \( |C| \). For anti-symmetry, suppose that \( |C| \leq_{\text{SC}} |C'| \) and \( |C'| \leq_{\text{SC}} |C| \). If \( |V(C')| < |V(C)| \), then we cannot have \( |C| \leq_{\text{SC}} |C'| \), and if \( |V(C)| < |V(C')| \), we cannot have \( |C'| \leq_{\text{SC}} |C| \). Hence we must have \( |V(C)| = |V(C')| \), and so any map \( C' \to C \) inducing \( |C| \leq_{\text{SC}} |C'| \) must be injective on vertices and simplices, as must any map \( C \to C' \) inducing \( |C'| \leq_{\text{SC}} |C| \). Hence we have a map \( C \to C' \) that is bijective on simplices, so \( |C| = |C'| \).

For transitivity, suppose that \( |C| \leq_{\text{SC}} |C'| \) and \( |C'| \leq_{\text{SC}} |C''| \). Then there exists a simplicial map \( C'' \to C' \) that is surjective on \( V(C') \), as well a simplicial map \( C' \to C \) that is surjective on \( V(C) \). The composition of these two simplicial maps is a simplicial map \( C'' \to C \), and as both were individually surjective on vertices, the composition must also be surjective on vertices. \( \square \)

The same arguments give that \( \leq_{\text{SC}} \) defines a preorder on \( \text{Obj}(\text{SC}) \).

We are now ready to prove the main theorem of this section.

Theorem 3.6. The unlabeled Čech map is continuous.

Proof. A basis for the upset topology on \( \text{uSC} \) consists of the sets \( U_{|C|} = \{|C'| \in \text{uSC} : |C| \leq_{\text{SC}} |C'| \} \) based at \( |C| \in \text{uSC} \), so we show the preimage of all such sets is open in \( \text{Ran}^{\leq_n}(M) \times \mathbb{R}_{>0} \). Take any \( (P, r) \in \text{Č}^{-1}(U_{|C|}) \), with \( P = \{P_1, \ldots, P_k\} \), which we will show has an open neighborhood contained in \( \text{uČ}^{-1}(U_{|C|}) \). For every \( P' \subseteq P \), let

\[
\hat{c}s(P') := \bigcap_{p \in P'} B(p, \min \{r : \bigcap_{p' \in P'} B(p', r) \neq \emptyset\}),
\]

\[
\hat{c}r(P', r) := r - d_M(P', \hat{c}s(P'))
\]

be the Čech set of \( P' \) and Čech radius of \( P' \) at \( r \), respectively. The Čech set is the smallest non-empty intersection of the closed balls of increasing radius around \( P' \) (the min in its definition exists, as the balls are closed and \( M \) is connected). The Čech radius of \( P' \) at \( r \) is given in terms of distance between sets on \( M \), which is not Hausdorff distance, but rather

\[
d_M(X, Y) = \inf_{x \in X, y \in Y} d_M(x, y).
\]

This value is positive if and only if the intersection \( \bigcap_{p \in P'} B(p, r) \) contains an open set, negative when the intersection is empty, and 0 otherwise.

Case 1: For every \( P' \subseteq P \), \( \hat{c}r(P', r) \neq 0 \). Let \( B_{\infty}^\circ((P, r), \hat{r}/4) \) be the open ball in the \( \infty \)-norm on the product \( \text{Ran} \leq_n(M) \times \mathbb{R}_{>0} \) around \( (P, r) \) of radius \( \hat{r}/4 \), where \( \hat{r} \) is the smallest of the two values

\[
r_1 := \min_{1 \leq i < j \leq k} d_M(P_i, P_j),
\]

\[
r_2 := \min_{P' \subseteq P} 2|\hat{c}r(P', r)|.
\]

Figure[] illustrates the roles of \( r_1 \) and \( r_2 \).

Let \( (Q, s) \in B_{\infty}^\circ((P, r), \hat{r}/4) \). The value \( r_1 \) guarantees that \( Q \subseteq \bigcup_{i=1}^k B^\circ(P_i, \hat{r}/4) \), and that the \( B^\circ(P_i, \hat{r}/4) \) are disjoint. Moreover, for every \( 1 \leq i \leq k \), note that \( Q \cap B^\circ(P_i, \hat{r}/4) \neq \emptyset \), as

\[
d_M(\{P_i\}, Q) = \min_{q \in Q} d_M(P_i, q) \leq d_M(P_i, Q) \leq d_\infty((P, r), (Q, s)) < \hat{r}/4.
\]

Hence the map \( \phi : Q \to P \) for which \( \phi(q) = P_i \) whenever \( q \in B^\circ(P_i, \hat{r}/4) \) is well-defined and surjective.
The value $r_2$ guarantees that the simplices in $\tilde{C}(Q, s)$ with 0-faces in $\tilde{C}(Q \cap B^\circ(P, \tilde{r}/4), s)$ correspond to simplices in $\tilde{C}(P, r)$ with $P_i$ a 0-face. Indeed, if $P' = \{P'_0, \ldots, P'_\ell\} \in S(\tilde{C}(P, r))$, there are $\prod_{i=0}^\ell |Q \cap B^\circ(P'_i, \tilde{r}/4)|$ copies of the $\ell$-simplex $P'$ in $\tilde{C}(Q, s)$. This follows by taking any $Q'_i \in Q \cap B^\circ(P'_i, \tilde{r}/4)$ for all $i = 0, \ldots, \ell$, and observing that, for $Q' = \{Q'_0, \ldots, Q'_\ell\}$,

$$\tilde{c}r(Q', s) > \tilde{c}r(P', r) - d_H(P', Q') - |r - s|$$

$$\geq r_2/2 - \tilde{r}/4 - \tilde{r}/4$$

$$\geq \tilde{r}/2 - \tilde{r}/2$$

$$= 0,$$

and so $Q' \in S(\tilde{C}(Q, s))$ is a copy of $P'$ in $\tilde{C}(Q, s)$. Similarly, suppose that $P' = \{P'_0, \ldots, P'_\ell\} \notin S(\tilde{C}(P, r))$, and take any $Q'_i \in Q \cap B^\circ(P'_i, \tilde{r}/4)$ for all $i = 0, \ldots, \ell$, with $Q' = \{Q'_0, \ldots, Q'_\ell\}$. Then

$$\tilde{c}r(Q', s) < \tilde{c}r(P', r) + d_H(P', Q') + |r - s|$$

$$\leq -r_2/2 + \tilde{r}/4 + \tilde{r}/4$$

$$\leq -\tilde{r}/2 + \tilde{r}/2$$

$$= 0,$$

and so the intersection $\bigcap_{i=0}^\ell B(Q'_i, s)$ must be empty, meaning $Q' \notin S(\tilde{C}(Q, s))$. Hence $\phi : Q \to P$ extends to a simplicial map $\tilde{C}(Q, s) \to \tilde{C}(P, r)$ that is surjective on vertices. That is, $[\tilde{C}(P, r)] \subseteq_{SC} [\tilde{C}(Q, s)]$, and so $B_{\infty}^\circ((P, r), \tilde{r}/4) \subseteq u\tilde{C}^{-1}(U_{\{C\}})$, meaning that $u\tilde{C}^{-1}(U_{\{C\}})$ is open in this case.

**Case 2:** There is some $P' \subseteq P$ with $\tilde{c}r(P', r) = 0$. Then $r_2 = 0$ from [3], so we have to make some adjustments. Consider $B_{\infty}^\circ((P, r), \tilde{r}/4)$, where $\tilde{r}$ is the smallest of the two values $r_1$ and

$$r'_2 = \min_{P' \subseteq P, \tilde{c}r(P', r) \neq 0} 2|\tilde{c}r(P', r)|.$$

As in Case 1, we claim the open neighborhood $B_{\infty}^\circ((P, r), \tilde{r}/4)$ of $(P, r)$ is contained within $(u \circ \tilde{C})^{-1}(U_{\{C\}})$. The proof of the claim proceeds as in the first case, except we may have some $\ell$-simplex in $\tilde{C}(P, r)$ that does not correspond to an $\ell$-simplex in $\tilde{C}(Q, s)$. That is, if $P' = \{P'_0, \ldots, P'_\ell\} \in S(\tilde{C}(P, r))$ and $Q'_i \in Q \cap B^\circ(P'_i, \tilde{r}/4)$ for all $i = 0, \ldots, \ell$, we may have $Q' = \{Q'_0, \ldots, Q'_\ell\} \notin S(\tilde{C}(Q, s))$, as the calculation ([7]) may have $\tilde{c}r(P', r) = 0$. However, the map $\phi$ on vertices still extends to a simplicial
map, as including faces into the larger simplex they came from is a simplicial map. The calculation \[^8\] will proceed in the same manner. Hence \(u\mathcal{C}^{-1}(U_{[C]})\) is open in this case as well. \(\square\)

It follows that \(u\mathcal{C}\) is an \(u\mathcal{SC}\)-stratification of \(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}\).

**Corollary 3.7.** Let \(\gamma : I \to \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}\) with \(u\mathcal{C}(\gamma(t))\) constant for \(t \in [0,1]\).

1. If \(u\mathcal{C}(\gamma(0)) = u\mathcal{C}(\gamma(1))\), the path \(\gamma\) induces a unique simplicial map.
2. If \(u\mathcal{C}(\gamma(0)) \neq u\mathcal{C}(\gamma(1))\), the path \(\gamma|_{[1-\varepsilon,1]}\) induces a unique simplicial map, for all \(\varepsilon \in (0,\varepsilon/4)\).

Hence \(\gamma\) induces a unique simplicial map in \(\mathcal{SC}\) from \(\mathcal{C}(\gamma(0))\) to \(\mathcal{C}(\gamma(1))\).

**Proof.** Let \(\gamma(0) = (Q,s)\) and \(\gamma(1) = (P,r)\). The set map \(\phi : Q \to P\) described in the proof of Theorem \[^3.6\] is a bijection, and \(u\mathcal{C}(\gamma(t))\) constant for all \(t \in [0,1]\) implies that every simplex \(\{Q_0,\ldots,Q_k\} \in S(\mathcal{C}(Q,s))\) corresponds to the simplex \(\{\phi(Q_0),\ldots,\phi(Q_k)\} \in S(\mathcal{C}(P,r))\), and that this correspondence is bijective. This proves the first claim.

The second claim follows from Case 2 of Theorem \[^3.6\]. By reparametrization of the first claim and path concatenation, we get the stated result. \(\square\)

**Remark 3.8.** The point-counting stratification \(\text{Ran}^{\leq n}(M) \to \mathbb{Z}_{>0}\) extends to a stratification \(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \to \mathbb{Z}_{>0}\) by projection to the first factor. It is immediate that \(u\mathcal{C}\) is compatible with this stratification, as a simplicial complex \(C\) has a fixed number of points, so the elements \((P,r)\) of every stratum \((\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0})|_{[C]}\) always have \(|P|\) constant.

### 3.2. A conical refinement.

The unlabeled Čech map is not a conical stratification of \(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}\).

**Example 3.9.** It suffices to consider the case \(n = 2\). Take \((P,r) \in \text{Ran}^{\leq 2}(M) \times \mathbb{R}_{\geq 0}\) with \(r = d(P_1,P_2)/2\), and assume there is an open embedding \(\phi : Z \times C(Y) \hookrightarrow \text{Ran}^{\leq 2}(M) \times \mathbb{R}_{\geq 0}\). Using notation as in Definition \[^2.5\], the stratifications in this context are

\[
\begin{align*}
f &: \text{Ran}^{\leq 2}(M) \times \mathbb{R}_{\geq 0} \to \{\bullet, \dashrightarrow, \bullet \bullet \}, \\
g &: Y \to \{\bullet \bullet \}, \\
g' &: C(Y) \to \{\dashrightarrow, \bullet \bullet \}, \\
g' \circ \pi_2 &: Z \times C(Y) \to \{\dashrightarrow, \bullet \bullet \}.
\end{align*}
\]

For \(\dim(M) = m\), the dimensions of \(\text{Ran}^{\leq 2}(M) \times \mathbb{R}_{\geq 0}\) and the strata \(f^{-1}(\dashrightarrow)\) and \(f^{-1}(\bullet \bullet)\) are all \(2m+1\). Since \(g'\) maps the cone point \(Y \times \{0\}\) to \(\dashrightarrow\) and everything else to \(\bullet \bullet\), \(Z\) must be an open set in \(f^{-1}(\dashrightarrow)\). As \(\phi\) must be an open embedding, \(2m+1 = \dim(\text{Ran}^{\leq 2}(M) \times \mathbb{R}_{\geq 0})\)

\[
= \dim(Z \times C(Y))
= \dim(Z) + \dim(C(Y))
= 2m + 1 + \dim(C(Y)),
\]

so \(\dim(C(Y)) = 0\), meaning \(C(Y) = \ast\). Then \(g' \circ \pi_2\) has image \(\{\dashrightarrow\}\), but every open set \(U \subseteq \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}\) containing \((P,r)\) has \(\{\dashrightarrow, \bullet \bullet \}\) in its image through \(f\). Without loss of generality, \(U \subseteq \phi(Z \times C(Y))\), and since \(\phi\) is a stratified map,
Theorem 3.10. If $M$ is piecewise linear, there exists a conical semialgebraic stratification of $\text{Ran}_{\leq}^{\leq}(M) \times \mathbb{R}_{\geq 0}$ compatible with $u\mathcal{C}$.

**Proof.** Since $M$ is piecewise linear, it is semialgebraic, so [Shi97, I.2.9.1] gives that $M^n \times \mathbb{R}_{\geq 0}$ is semialgebraic. By [Bru87, Corollary 1.5], $\text{Ran}_{\leq}^{\leq}(M) \times \mathbb{R}_{\geq 0}$ is semialgebraic. The set $u\mathcal{C}-1([C]) \subseteq \mathbb{R}^n$ is semialgebraic for every $[C] \in u\mathcal{SC}$, as it is described by equalities and (strict or weak) inequalities of distances from elements of representatives $C$ to the Čech set $\check{c}s(C)$. That is, since $M$ is piecewise linear, distance on $M$ is the same as Euclidean distance, and the square of the distance function is polynomial, and we can take the square to be the defining inequality. Hence $u\mathcal{C}$ is a semialgebraic $u\mathcal{SC}$-stratification of $\text{Ran}_{\leq}^{\leq}(M) \times \mathbb{R}_{\geq 0}$. Apply Lemma 2.7 to get a conical semialgebraic stratification of $\text{Ran}_{\leq}^{\leq}(M) \times \mathbb{R}_{\geq 0}$ compatible with $u\mathcal{C}$. \hfill \Box

In particular, this applies when $M = \mathbb{R}^d$.

**Remark 3.11.** To apply [Bru87, Corollary 1.5] in Theorem 3.10, we need to find a semialgebraic set $E \subseteq M^n \times M^n$ such that $M^n/E = \text{Ran}_{\leq}^{\leq}(M)$. This is not the usual way to construct $\text{Ran}_{\leq}^{\leq}(M)$ from $M^n$, which is most often done by setting $(M^n \setminus \Delta)/S_n = \text{Ran}(M)$, where $\Delta \subseteq M^n$ contains all $n$-tuples with at least two identical entries. For example, when $M = \mathbb{R}^d$ and $n = 3$, $E$ contains the set

$$\{(x_1, \ldots, x_6) \in (\mathbb{R}^d)^3 \times (\mathbb{R}^d)^3 : x_1 - x_5 = 0, x_2 - x_4 = 0\}$$

for the symmetric group action on $(\mathbb{R}^d)^3$, identifying $(a, b, c)$ with $(b, a, c)$, and

$$\{(x_1, \ldots, x_6) \in (\mathbb{R}^d)^3 \times (\mathbb{R}^d)^3 : x_1 - x_4 = x_1 - x_5 = 0, x_2 - x_6 = x_3 - x_6 = 0\}$$

for coincidences, identifying $(a, b, b)$ with $(a, a, b)$.

4. **Universal spaces**

From now on we assume $M$ is piecewise linear. Let $c\mathcal{C} : \text{Ran}_{\leq}^{\leq}(M) \times \mathbb{R}_{\geq 0} \to \mathcal{C}$ be the conical semialgebraic stratification given by Theorem 3.10, for some appropriate poset $c\mathcal{SC}$ refining $u\mathcal{SC}$.

4.1. **The homotopy category of entrance paths.** For $X$, a topological space, recall $\text{Sing}(X)$ is the simplicial set of continuous maps $|\Delta^k| \to X$. Let $A$ be a poset and $f : X \to A$ a stratification.

**Definition 4.1.** An exit path in $X$ is a continuous map $\sigma : |\Delta^k| \to X$ for which there exists a chain $a_0 \leq \cdots \leq a_k$ in $A$ such that $f(\sigma(t_0, \ldots, t_i, 0, \ldots, 0)) = a_i$ and $t_i \neq 0$, for all $i$. An entrance path for $\sigma$ is the same, but with $f(\sigma(0, \ldots, 0, t_i, \ldots, t_k)) = a_{k-i}$ and $t_i \neq 0$, for all $i$. 

Remark 4.2. It is tempting to think \( \text{Sing}^4(X) \) complexes, not in the morphisms. The functor from \( \text{Sing}^4(X) \) to \( \text{Sing}_A(X) \) ence between exit and entrance paths is only in the indexing of the underlying \( \sigma \) precomposes every \( \rho \), so \( \text{Sing}_A(X) \) itself, so \( \text{Sing}^4(X) \) has \( \text{Sing}_A(X) \).

By \cite{Lur17} Theorem A.6.4, \( \text{Sing}_{\mathcal{SC}}(\text{Ran}^{\leq n}(M) \times R_{\geq 0}) \) is an \( \infty \)-category. We follow \cite{Lur09} Section 1.2.3 in constructing the homotopy category of an \( \infty \)-category.

Definition 4.3. Let \( \rho, \sigma \in \text{Sing}_{\mathcal{SC}}(\text{Ran}^{\leq n}(M) \times R_{\geq 0}) \) with \( \rho(0) = \sigma(0) = (P, r) \) and \( \rho(1) = \sigma(1) = (Q, s) \). Then \( \rho \) and \( \sigma \) are homotopic if there exists a 2-simplex \( \tau \in \text{Sing}_{\mathcal{SC}}(\text{Ran}^{\leq n}(M) \times R_{\geq 0}) \) for which \( d_2 \tau = \rho, d_1 \tau = \sigma, \) and \( d_0 \tau = s_0(Q, s) \).

Recall \( d_\ast \colon S_i \to S_{i-1} \) are the degeneracy maps and \( s_\ast \colon S_i \to S_{i+1} \) are the face maps in a simplicial set \( S \). By \cite{Lur09} Proposition 1.2.3.5, homotopy of 1-simplices with common endpoints is an equivalence relation, so let \([\sigma]\) denote the equivalence class of 1-simplices in \( \text{Sing}_{\mathcal{SC}}(\text{Ran}^{\leq n}(M) \times R_{\geq 0}) \) homotopic to \( \sigma \).

Definition 4.4. The homotopy category of \( \text{Sing}_{\mathcal{SC}}(\text{Ran}^{\leq n}(M) \times R_{\geq 0}) \) has

- pairs \((P, r) \in \text{Ran}^{\leq n}(M) \times R_{\geq 0}\) as objects, and
- homotopy classes \([\sigma]\) as morphisms from \(\sigma(0)\) to \(\sigma(1)\).

By \cite{Lur09} Proposition 1.2.3.8, this description defines a category. We denote this category by \( \text{Ho}(\text{Sing}_{\mathcal{SC}}(\text{Ran}^{\leq n}(M) \times R_{\geq 0})) \).

Lemma 4.5. Every morphism in \( \text{Ho}(\text{Sing}_{\mathcal{SC}}(\text{Ran}^{\leq n}(M) \times R_{\geq 0})) \) induces a unique simplicial map in \( \mathcal{SC} \).

Proof. Take \([\sigma] \in \text{Hom}_{\text{Ho}(\text{Sing}_{\mathcal{SC}}(\text{Ran}^{\leq n}(M) \times R_{\geq 0}))}((P, r), (Q, s))\) and choose a representative \( \sigma \in [\sigma] \). By Corollary 3.7, we have a unique simplicial map, call it \( \hat{\sigma} \in \text{Hom}_{\mathcal{SC}}(\hat{C}(P, r), \hat{C}(Q, s)) \).

For uniqueness, take some other \( \rho \in [\sigma] \), so there exists \( \tau \in \text{Sing}_{\mathcal{SC}}(\text{Ran}^{\leq n}(M) \times R_{\geq 0}) \) with \( d_2 \tau = \rho, d_1 \tau = \sigma, \) and \( d_0 \tau = s_0(Q, s) \). Write \( P = \{P_1, \ldots, P_k\} \).

\footnote{The choice of “entrance” instead of “entry” comes from interpreting “exit” as a noun rather than a verb.}
the endpoints of $\sigma$ and $\rho$ are both fixed, the homotopy between the two extends to $k$ path homotopies from $\sigma_i: I \to M$ to $\rho_i: I \to M$ on $M$, with $\sigma_i(0) = \rho_i(0) = P_i$ and $\sigma_i(1) = \rho_i(1)$. Hence the set maps $P \to Q$ induced by both $\sigma$ and $\rho$ are the same, and as simplicial maps are determined by where vertices are sent, $\tilde{\sigma} = \tilde{\rho}$. □

**Definition 4.6.** Let $F: \text{Ho} (\text{Sing}_{\mathcal{C}} (\text{Ran}^{\le n}(M) \times \mathbb{R}_{\ge 0})) \to \mathcal{C}$ be the functor given by $F(P, r) = \tilde{C}(P, r)$ and $F([\sigma]) = \tilde{\sigma}$.

This assignment is well-defined by Lemma 4.5 and functorial by [Lur09, Proposition 1.2.3.7]. Restriction induces functors $F_U: \text{Ho} (\text{Sing}_{\mathcal{C}} (U)) \to \mathcal{C}$ for every open or semialgebraic subset $U \subseteq \text{Ran}^{\le n}(M) \times \mathbb{R}_{\ge 0}$.

**4.2. The Čech cosheaf.** In this section we describe a cosheaf built from $F$ whose costalks recover the Čech map $\tilde{C}$. Let $\text{Cat}$ be the category of small categories and $\text{Cat}_{/\mathcal{C}}$ the overcategory of functors into $\mathcal{C}$.

**Definition 4.7.** Let $F: \text{Op} (\text{Ran}^{\le n}(M) \times \mathbb{R}_{\ge 0}) \to \text{Cat}_{/\mathcal{C}}$ be the functor given by $F(U) = F_U$ and $F(V \subseteq U)$ the inclusion.

Since $\text{Ho} (\text{Sing}_{\mathcal{C}} (V))$ is a (not necessarily full) subcategory of $\text{Ho} (\text{Sing}_{\mathcal{C}} (U))$ whenever $V \subseteq U$, this definition makes sense. For every $U$, the image of $F(U)$ is a diagram of simplicial complexes and simplicial maps in $\mathcal{C}$, whose unlabeled representatives are described in Figure 3. Note that some simplicial maps may be the identity, and there are not always unique simplicial maps between strata.

**Figure 3.** A visual description of the functor $F$.

Algebraically, the functor $F$ may be thought of as the composition

$$
\begin{align*}
U &\longrightarrow \text{Sing}_{\mathcal{C}} (U) \longrightarrow \text{Ho} (\text{Sing}_{\mathcal{C}} (U)) \longrightarrow \left( \text{Ho} (\text{Sing}_{\mathcal{C}} (U)) \right) \\
\text{Top} &\longrightarrow \text{Cat}_{\infty} \longrightarrow \text{Cat} \longrightarrow \text{Cat}_{/\mathcal{C}}
\end{align*}
$$

of functors. The homotopy category functor $\text{Ho} (\cdot)$ preserves colimits, as it is a left adjoint. Also $\text{Cat} \to \text{Cat}_{/\mathcal{C}}$ preserves colimits, as in an overcategory colimits are computed in the underlying category. Since we have no such results for the functor $\text{Sing}_{\mathcal{C}} (\cdot)$, we resort to a direct construction to prove

**Theorem 4.8.** The functor $F$ is a cosheaf.
Proof. Let $U \in \text{Op}(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0})$ with a cover $\{U_i\}$ of open sets. We will construct an inverse to the natural map $\alpha : \text{colim}_i \mathcal{F}(U_i) \to \mathcal{F}(U)$. Here we write $\text{colim}_i \mathcal{F}(U_i)$ for the colimit of the functor $N(\{U_i\}) \to \text{Cat}_{/SC}$ from the nerve of the open cover.

For an arbitrary $U_i$ in the cover $\{U_i\}$, consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{F}(U_i) & \xleftarrow{\text{inclusion}} & \text{Ho}(\text{Sing}_{\text{cSC}}(U_i)) \\
 & & \text{Ho}(\text{Sing}_{\text{cSC}}(U)) = \mathcal{F}(U) \\
\text{colim}_i \mathcal{F}(U_i) & \xrightarrow{\text{c}} & \mathcal{F}(U) \\
\end{array}
$$

The functor $c$ is the canonical map that comes with the colimit. The functor $\alpha$ is a bijection on objects, as $\{U_i\}$ covers $U$ and the maps into $\text{Ho}(\text{Sing}_{\text{cSC}}(U))$ from the diagram $\{\mathcal{F}(U_i)\}$ are inclusions. On morphisms, $\alpha$ takes $\phi$ to the homotopy class $[\tilde{\phi}]$, whenever a morphism $\tilde{\phi}$ in $\text{Ho}(\text{Sing}_{\text{cSC}}(U_i))$ maps to $\phi$ in $\text{colim}_i \mathcal{F}(U_i)$.

Let $\beta : \mathcal{F}(U) \to \text{colim}_i \mathcal{F}(U_i)$ on objects be the reverse bijection of $\alpha$. To make the diagram

$$
\begin{array}{ccc}
\text{Ho}(\text{Sing}_{\text{cSC}}(U_i)) & \xleftarrow{\text{inclusion}} & \text{Ho}(\text{Sing}_{\text{cSC}}(U)) \\
 & & \text{colim}_i \mathcal{F}(U_i) \\
\end{array}
$$

commute, we note that a homotopy class $[\sigma]$ in $\text{Ho}(\text{Sing}_{\text{cSC}}(U))$ has representatives contained in arbitrarily small neighborhoods of its starting point, as simplices are contractible and $\text{cSC}$ is conical. Hence every morphism in $\text{Ho}(\text{Sing}_{\text{cSC}}(U))$ is the image of some arrow from the diagram $\{\mathcal{F}(U_i)\}$, so we let $\beta$ match this image with the image through $c$. By Lemma 4.5 every representative of $[\sigma]$ induces the same simplicial map in $\text{SC}$, so commutativity extends when diagram (11) is extended to the overcategory $\text{Cat}_{/\text{SC}}$. Functoriality of $\beta$ follows from the functoriality of $\alpha$, surjectivity of the diagram $\{\mathcal{F}(U_i)\}$ into $\mathcal{F}(U)$, and the commutativity of (10).

The compositions $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity on their respective categories by construction, so, in particular, $\alpha$ is an isomorphism. □

Note that the openness of $U$ did not play a role in the proof. Hence dualizing the colimit argument to a limit argument, we get that for any semialgebraic $^4 V \subseteq \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}$,

$$
\lim_{U \supseteq V} \mathcal{F}(U) = F_V.
$$

In particular, this implies the costalk $\mathcal{F}(\mathcal{P}, r)$ is the Čech complex $\check{\mathcal{C}}(\mathcal{P}, r)$. Slightly more generally, this implies that

**Proposition 4.9.** The inverse image precosheaf $\mathcal{F}|_{\mathcal{P} \times \mathbb{R}_{\geq 0}}$ is a cosheaf.

$^4$We must have $V$ semialgebraic because the existence of a conical stratification is only guaranteed for semialgebraic sets.
Proof. Since open sets in $\mathbb{R}_{\geq 0}$ are semialgebraic, \textcolor{red}{[12]} gives that $\lim_{U \supseteq V} F(V) = F_V$. Repeat Theorem \textcolor{red}{4.8} with the stratified space $\{P\} \times \mathbb{R}_{\geq 0}$ instead of $\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}$ to show that the assignment $V \mapsto F_V$ defines a cosheaf. The naturally induced $\infty$-category structure on $\text{Sing}_{\text{cSC}}(\{P\} \times \mathbb{R}_{\geq 0})$ is retained, as the spaces are semialgebraic. □

Given any two different contractible neighborhoods $U, U'$ of an element $(P, r) \in \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}$, the entrance path categories $\text{Sing}_{\text{cSC}}(U)$ and $\text{Sing}_{\text{cSC}}(U')$ are different on the nose. Passing to homotopy categories provides some stability.

**Proposition 4.10.** Any two small enough contractible open neighborhoods $U, U'$ of every $(P, r) \in \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}$ have isomorphic values $F(U) \cong F(U')$.

**Proof.** This follows from the assumption that $\tilde{C}$ is a conical $\text{cSC}$-stratification By Definition \textcolor{red}{2.5} every small enough neighborhood of $(P, r)$ is a neighborhood of the cone point of the stratified space $Z \times C(Y) \to A^q$ (for an appropriate space $Z$ and an appropriate stratified space $Y \to A$), using the notation of [Lur17, Definition A.5.3]. Let $U, U' \to A^q \subseteq \text{cSC}$ be two such stratified contractible neighborhoods. Being contractible neighborhoods of the cone point, there is a stratified open embedding between them, with a homeomorphism between the spaces and the identity maps between the posets, making the diagram

\begin{equation}
\begin{array}{ccc}
U & \cong & U' \\
\downarrow & & \downarrow \\
A^q & \rightarrow & A^q \\
\end{array}
\end{equation}

commute. Hence the categories $\text{Ho}(\text{Sing}_{\text{cSC}}(U))$ and $\text{Ho}(\text{Sing}_{\text{cSC}}(U'))$ are equivalent, via the stratified map and the order embedding $A^q \to \text{cSC}$. Since the functors $F_U$ and $F_{U'}$ from Definition \textcolor{red}{4.6} are invariant under such stratified maps, the same map induces a natural isomorphism between them, making the diagram

\begin{equation}
\begin{array}{ccc}
\text{Ho}(\text{Sing}_{\text{cSC}}(U)) & \cong & \text{Ho}(\text{Sing}_{\text{cSC}}(U')) \\
F_U & & F_{U'} \\
\text{SC} & \rightarrow & \text{SC} \\
\downarrow & & \downarrow \\
\end{array}
\end{equation}

commute. □

The proof of Theorem \textcolor{red}{3.6} gives the precise size of “small enough” for a pair $(P, r)$ as the ball of radius $\bar{r}/4$. This Proposition becomes a locally constant statement when viewed in a more restricted context (see Remark \textcolor{red}{4.14}).

### 4.3. The Čech sheaf

In this section we describe a sheaf built from $F$. Let $\text{Kan} \subseteq \text{sSet}$ be the $\infty$-category of Kan complexes and $\mathcal{S}$ the $\infty$-category of spaces, constructed as the simplicial nerve (also called the homotopy coherent nerve) $N'(\text{Kan})$ of the simplicial category of Kan complexes, as in [Lur09, Definition 1.2.16.1].

**Theorem 4.11.** The functor $F$ induces an $\mathcal{S}$-valued $\text{cSC}$-constructible sheaf on $\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}$.
**Proof.** Begin with the functor

\[ F : \text{Ho}(\text{Sing}_{SC}(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0})) \to \text{SC} \]

from Definition 4.6. Drop the vertex set and only take simplices to get a functor

\[ \text{Ho}(\text{Sing}_{SC}(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0})) \to \text{simp}(\text{SC}) \subseteq \text{Ord} \]

into posets, as in (1). Apply the homotopy category-nerve adjunction \( \text{Ho} : \text{sSet} \cong \text{Cat : N} \) from [Lur09, Proposition 1.2.3.1] to get a functor

\[ \text{Sing}_{SC}(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}) \to N(\text{Ord}) \subseteq \text{sSet} \]

into simplicial sets. Invert weak equivalences of the Quillen model structure (morphisms inducing weak homotopy equivalences through geometric realization) on \( \text{sSet} \) to get a functor

\[ \text{Sing}_{SC}(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}) \to \text{sSet}[W^{-1}] \cong \text{Kan}. \]

Finally, apply the simplicial nerve to \( \text{Kan} \) to get a functor

\[ (15) \quad \tilde{F} : \text{Sing}_{SC}(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}) \to N'(\text{Kan}) \cong S. \]

We finish by checking the conditions of [Lur17, Theorem A.9.3].

Since \( M \) is paracompact, \( \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \) is paracompact. For the singular shape condition, note that every open \( U \subseteq \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \) may be described as a union \( U = \bigcup_{i} U_{i} \), where \( U_{i} \) sits in the homeomorphic image of Euclidean space. This follows from [Shi97, Theorem II.4.2] and by using the open star cover on the underlying simplicial complex describing \( \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \). As Euclidean space and its submanifolds are locally of singular shape (by [Lur17, Lemma A.4.14] and the existence of good open covers), [Lur17, Remark A.4.16] gives that \( \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \) is locally of singular shape. By assumption, \( cC \) is a conical \( cSC \)-stratification. By construction, the image of \( \tilde{C} \) in \( \text{SC} \) is finite, so no infinite ascending chain exists in \( \text{SC} \). As [Shi97, Theorem II.4.2] gives a locally finite simplicial complex describing \( \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \), \( cSC \) also satisfies the ascending chain condition. \( \square \)

Write \( \mathcal{X} \) for \( \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \) and let \( \text{Shv}^{cSC}(\mathcal{X}) \) be the category of \( S \)-valued \( cSC \)-constructible sheaves on \( \mathcal{X} \), as in Definition 2.2. We now describe the promised sheaf, which we call \( \mathcal{G} \). The \( \infty \)-category functor

\[ \Psi_{\mathcal{X}} : N(\text{sSet}_{/\text{Sing}_{SC}(\mathcal{X})}) \to \text{Shv}^{cSC}(\mathcal{X}) \]

from [Lur17, Section A.9] takes \( \tilde{F} \) and assigns to an open set \( U \subseteq \mathcal{X} \) the category

\[ (16) \quad \mathcal{G}(U) := \Psi_{\mathcal{X}}(\tilde{F})(U) = \text{Fun}_{\text{Sing}_{SC}(\mathcal{X})}(\text{Sing}_{SC}(U), \text{Un}_{\op}(\tilde{F})). \]

The simplicial set \( \text{Un}_{\op}(\tilde{F}) \) is the Grothendieck construction, or unstraightening, of \( \tilde{F} \) via the \( \op \)-equivalence of \( \mathcal{C}(\text{Sing}_{SC}(\mathcal{X})) \). The simplicial category \( \mathcal{C}(\text{Sing}_{SC}(\mathcal{X})) \) is defined through the adjunction \( \mathcal{C} : \text{sSet} \cong \text{Cat : N} \) from [Lur09, Definition 1.1.5.5], analogous to the Ho \( \cong N \) adjunction used in the proof of Theorem 4.11. More precisely, the 0-simplices of \( \text{Un}_{\op}(\tilde{F}) \) are pairs \( (\sigma, s) \), where \( \sigma \) is an object of \( \text{Sing}_{SC}(\mathcal{X}) \) and \( s \) is a simplex of \( \tilde{F}((\sigma)) \). The 1-simplices \( (\sigma, s) \rightarrow (\tau, t) \) are defined as pairs of morphisms \( (\sigma \xrightarrow{f} \tau, \tilde{F}(f)(s) \rightarrow t) \).
Hence $G(U)$ is the $\infty$-category of functors $\varphi$ that make the diagram

$$\begin{align*}
\text{Sing}_{c\text{SC}}(U) & \xrightarrow{\varphi} \text{Un}_{\text{op}}(\tilde{F}) \\
\text{Sing}_{c\text{SC}}(X) \quad \text{inclusion} \quad \text{forgetful} \quad \text{forgetful} \\
\text{Sing}_{c\text{SC}}(X) & \xrightarrow{\varphi} \text{Un}_{\text{op}}(\tilde{F})
\end{align*}$$

(17)

commute. The forgetful functor to $\text{Sing}_{c\text{SC}}(A')$ drops the second components.

Note that for every open $U \subseteq X$, Theorem 4.11 proceeds in the same way if we begin instead with the functor $F_U: \text{Ho}(\text{Sing}_{c\text{SC}}(U)) \to \text{SC}$. Hence we denote by $\tilde{F}_U = \tilde{F}(U)$ the associated functor constructed at (15).

**Remark 4.12.** The relation between $F$ and $G$ comes from both being defined in terms of the functor $F$. This relation is strengthened by observing that every $\varphi \in G(U)$ factors naturally through the unstraightening of $\tilde{F}_U$. This follows by the inclusion natural transformation among diagrams of the type (17), making the diagram

$$\begin{align*}
\text{Sing}_{c\text{SC}}(U) & \xrightarrow{\varphi} \text{Un}_{\text{op}}(\tilde{F}_X) \quad \text{inclusion} \\
\text{Sing}_{c\text{SC}}(X) \quad \text{inclusion} \quad \text{forgetful} \quad \text{forgetful} \\
\text{Sing}_{c\text{SC}}(U) & \xrightarrow{\varphi} \text{Un}_{\text{op}}(\tilde{F}_U)
\end{align*}$$

commute. It is unclear if the connection between $F$ and $G$ extends to the level of (co)stalks.

### 4.4. Open sets and basics

Presheaves and precosheaves on a topological space $X$ are given by functors from $\text{Op}(X)$. When $X$ is stratified, it is common (for example [AFT17, CP16]) to define presheaves and precosheaves as functors from a different category.

**Definition 4.13.** A basic is a conically stratified space with a stratified homeomorphism to the conically stratified space $R^i \times C(Y)$, for some non-negative integer $i$ and some stratified space $Y$. A basic of $X$ is an element $U \in \text{Op}(X)$ with a stratified homeomorphism to a basic.

That is, a basic of $X$ requires the existence of a stratified homeomorphism from $g: R^i \times C(Y) \to A' \cup \{\ast\}$, induced by $g': Y \to A' \subseteq A$, to $f_{|U}: U \to A$. This setup follows Definitions 2.3 and 2.5. The category of all basics is denoted $\text{Bsc}$, and the category of all basics of $X$ is denoted $\text{Bsc}(X)$. The morphisms in these categories are stratified embeddings. There is a natural inclusion functor $\text{Bsc}(X) \to \text{Op}(X)$, which forgets the stratification of basics.
For any open set $U \in \mathsf{Op}(X)$ with a cover $\{U_i\}$ of opens, the sheaf condition for a precosheaf $\mathcal{F}$ on $X$ and a presheaf $\mathcal{G}$ on $X$ are the conditions that
\[
\operatorname{colim}_{N(\{U_i\})} \mathcal{F}(U_i) \xrightarrow{\cong} \mathcal{F}(U) \quad \text{and} \quad \mathcal{G}(U) \xrightarrow{\cong} \lim_{N(\{U_i\})} \mathcal{G}(U_i),
\]
respectively, where the (co)limit is taken over the diagram of the nerve of the covering, with inclusion maps as arrows. Hence for any $U \in \mathsf{Bsc}(X)$ we require the same condition for pre(co)sheaves from $\mathsf{Bsc}(X)$, noting that the inclusion morphisms are not the only morphisms in the category $\mathsf{Bsc}(X)$, but are the only ones we consider in the diagram $N(\{U_i\})$. This gives Propositions 4.9 and 4.10 more meaning, allowing us to restate them as

**Proposition 4.14.** Considering $\mathcal{F}$ as a functor from the category $\mathsf{Bsc}(X)$,

- $\mathcal{F}$ is locally constant, and
- $\mathcal{F}|_{\{P\} \times \mathbb{R}_{\geq 0}}$ is $\mathsf{cSC}$-coconstructible and locally constant.

Being locally constant is more restrictive than being coconstructible (from Definition 2.2), as coconstructibility demands being locally constant only on some subsets.

### 4.5. A worked example.

Let $M = S^1$ with the induced distance when embedded as the unit circle in $\mathbb{R}^2$. Let $\mathcal{C} : \text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0} \to \mathsf{cSC}$ be a conical refinement of the Čech map, with image $A = \{\cdot, \bigcirc, \cdots, \cdot, \cdot\} \subseteq \mathsf{cSC}$, and strata

\[
\begin{align*}
&c\mathcal{C}^{-1}(\cdot) = \{(P, r) \in \text{Ran}^1(S^1) \times \mathbb{R}_{\geq 0} : r = 0\}, \\
&c\mathcal{C}^{-1}(\bigcirc) = \{(P, r) \in \text{Ran}^1(S^1) \times \mathbb{R}_{\geq 0} : r > 0\}, \\
&c\mathcal{C}^{-1}(\cdots) = \{(P, r) \in \text{Ran}^2(S^1) \times \mathbb{R}_{\geq 0} : d(P_1, P_2) = 2r\}, \\
&c\mathcal{C}^{-1}(\cdot \cdot) = \{(P, r) \in \text{Ran}^2(S^1) \times \mathbb{R}_{\geq 0} : d(P_1, P_2) > 2r\}, \\
&c\mathcal{C}^{-1}(\bigcirc \bigcirc) = \{(P, r) \in \text{Ran}^2(S^1) \times \mathbb{R}_{\geq 0} : d(P_1, P_2) < 2r\}.
\end{align*}
\]

Distance $d$ is the same as Euclidean distance. Figure 5 gives a visual representation of $\text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0}$ as a subset of $S^1 \times S^1 \times \mathbb{R}_{\geq 0}$, with $S^1$ is described as $[0, 2\pi]/0 \sim 2\pi$. The red curve is $\sin(\theta/2)$, and is part of the 1-dimensional stratum $c\mathcal{C}^{-1}(\cdots)$. The right and front sides of the prism are identified.
Figure 5. The stratified space $\text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0}$ and its strata via $c\hat{C}$.

The cosheaf $\mathcal{F}$ on the whole space is a functor $\text{Ho}(\text{Sing}_{\leq 2}(\text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0})) \to \mathcal{SC}$ given by

$$\mathcal{F}(P, r) = \begin{cases} \cdot \cdot \cdot & P \in \text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0}, d(P_1, P_2) > 2r, \\ \cdot \cdot \cdot & P \in \text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0}, d(P_1, P_2) \leq 2r, \\ \cdot \cdot \cdot & P \in \text{Ran}^1(S^1) \times \mathbb{R}_{\geq 0}. \end{cases}$$

on objects and

$$\mathcal{F}(\gamma) = \begin{cases} \cdot \cdot \cdot \xrightarrow{i} & \gamma(0) \in c\hat{C}^{-1}(\cdot \cdot \cdot), \gamma(1) \in c\hat{C}^{-1}(\cdot \cdot \cdot), \\ \cdot \cdot \cdot \xrightarrow{c} & \gamma(0) \in c\hat{C}^{-1}(\cdot \cdot \cdot), \gamma(1) \in c\hat{C}^{-1}(\cdot \cdot \cdot), \\ \cdot \cdot \cdot \xrightarrow{c} & \gamma(0) \in c\hat{C}^{-1}(\cdot \cdot \cdot), \gamma(1) \in c\hat{C}^{-1}(\cdot \cdot \cdot) \cup c\hat{C}^{-1}(\cdot \cdot \cdot), \end{cases}$$

on entrance paths, where $i$ is the inclusion simplicial map and $c$ is the constant simplicial map. The sheaf $\mathcal{G}$ on the whole space is a category of functors

$$\varphi: \text{Sing}_{\leq 2}(\text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0}) \to \text{Un}_{\text{op}}(\tilde{\mathcal{F}})$$

over $\text{Sing}_{\leq 2}(\text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0})$, where $F = \mathcal{F}(\text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0})$. Every such functor takes the image $(P, r)$ of a 0-simplex to a pair $\{\hat{C}(P, r), s\}$, where $s$ is a simplex of the associated simplicial set $N(\text{simp}(\hat{C}(P, r)))$, here one of

$$N(\text{simp}(\cdot \cdot \cdot)) = \{S_0 = \{\cdot, \cdot\}, S_n\geq 1 = \emptyset\},$$

$$N(\text{simp}(\cdot \cdot \cdot)) = \{S_0 = \{\cdot, \cdot, \cdot\}, S_1 = \{\cdot \rightarrow \cdot, \cdot \rightarrow \cdot\}, S_n\geq 2 = \emptyset\},$$

$$N(\text{simp}(\cdot \cdot \cdot)) = \{S_0 = \{\cdot\}, S_n\geq 1 = \emptyset\}.$$

On open subsets $U \subseteq \text{Ran}^{\leq 2}(S^1) \times \mathbb{R}_{\geq 0})$, the values $\mathcal{F}(U)$ and $\mathcal{G}(U)$ of the cosheaf and sheaf, respectively, are induced by the restriction of the respective values on the whole space.

5. Applications
5.1. **Persistent homology.** Persistent homology takes in a finite subset $P$ of a manifold often of $\mathbb{R}^d$, and produces a collection of intervals of $\mathbb{R}$ paired with a homology dimension $d$. Alternatively, it is a collection of functors $PH_{P,d}: (\mathbb{R}, \leq) \to \text{Vect}$, for Vect the category of finite-dimensional vector spaces over a field $k$.

**Proposition 5.1.** For every $P \in \text{Ran}^{\leq n}(M)$, the image of $F(P) \times \mathbb{R}_{\geq 0}$ is isomorphic to the diagram $D_1 \to \cdots \to D_k$ in $\text{SC}$, such that

- $D_i \leq \text{SC} D_{i+1}$ and $D_i \neq D_{i+1}$ for all $i = 1, \ldots, k - 1$,
- $D_1$ is $|P|$ disconnected 0-simplices, and
- $D_k$ is $\Delta^{|P|}$.

**Proof.** Since $\check{c}C$ is a conical $c\text{SC}$-stratification of $\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}$, every semialgebraic subset of $\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}$ inherits a conical stratification by restriction. Since $\{P\} \times \mathbb{R}_{\geq 0}$ is 1-dimensional, the only possible conical stratification is

$$ (18) \quad \cdots \quad \bullet \quad \cdots $$

which means the diagram $F(P) \times \mathbb{R}_{\geq 0}$ looks like

$$ (19) \quad \bullet \quad \cdots \quad \bullet \quad \bullet \quad \cdots \quad \bullet \quad \cdots $$

where every node represents the collection of objects of $\text{Ho}(\text{Sing}_{\leq n}(\{P\} \times \mathbb{R}_{\geq 0}))$ connected by a degenerate exit path, that is, one completely within a single stratum. Recall this stratification is compatible with the restriction of the $\text{SC}$-stratification $\check{C}$ to $\{P\} \times \mathbb{R}_{\geq 0}$, which is

$$ (20) \quad \cdots \quad \bullet \quad \cdots \quad \bullet \quad \cdots \quad \bullet \quad \cdots \quad \bullet \quad \cdots $$

where every half-open interval is where the Čech map $\check{C}$ is constant. As the closed endpoint is at the lower end of every half-open interval, every backwards arrow in (19) is the identity simplicial map on the appropriate simplicial complex. By compatibility, every forward arrow in (19) is either the identity simplicial map (if the right endpoint is in the same interval of (20)) or the inclusion map (if the right endpoint is in a different interval of (20)), induced by the identity map on vertices. Since $|P| \leq n$ and by collapsing all the identity simplicial maps, we get the stated properties. \hfill $\square$

Hence $H_d(F(P) \times \mathbb{R}_{\geq 0}; k) \cong PH_{P,d}$. More precisely, and by Proposition 4.9

$$ (21) \quad PH_{P,d}(t) = H_d \left( F(P,t); k \right), $$

$$ PH_{P,d}(t \leq s) = H_d \left( \lim \left( F(P \times [t,s]); k \right) \to \text{colim} \left( F(P \times [t,s]); k \right). $$

In other words, the persistent homology functor of any finite point sample $P$ is completely described by the cosheaf $F$.

**Remark 5.2.** Homology preserves colimits of filtered diagrams, so

$$ H_d \circ F: \text{Bsc}(\text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0}) \to \text{Cat}/\text{Vect} $$

is a cosheaf valued in functors (that is, diagrams) of homology groups. By Proposition 4.9 the same holds for the restriction cosheaves $F| \{P\} \times \mathbb{R}_{\geq 0}$.

---

5See Remark 1.1 for a discussion on the type of input.
5.2. **Extensions.** In this section we relate the presented ideas to other areas of research which suggest reasonable extensions.

**Remark 5.3 (Simplicial sets and chain complexes).** Alternatively to the association (1) of simplicial sets to simplicial complexes, we can send a \( k \)-simplex to the \( k \)-simplices of all its possible orderings. We retain functoriality and the lack of ordering, but the geometric realization is not homotopy equivalent to the original simplex. Let \( E \) be this functor. For \( \text{Free} : \text{sSet} \to \text{Ab} \) the free simplicial abelian group functor and \( \text{DK} : \text{Ab} \to \text{Ch} \) the Dold–Kan correspondence, we have a colimit-preserving composition

\[
\begin{align*}
\text{SC} & \xrightarrow{E} \text{sSet} \xrightarrow{\text{Free}} \text{Ab} \xrightarrow{\text{DK}} \text{Ch}.
\end{align*}
\]

The functor \( E \) preserves colimits by construction, \( \text{Free} \) as it is a left adjoint, and \( \text{DK} \) because it is an equivalence. Following the same equivalence from the overcategory \( \text{Cat}_{/\text{SC}} \), we get a cosheaf over \( \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \) valued in \( \text{Cat}_{/\text{Ch}} \), the over category of chain complexes.

**Remark 5.4 (Generalized persistence).** Bubenik, de Silva, and Scott in [BdSS15] extend the framework of persistent homology to generalized persistence modules, or functors from a poset into another category. These are filtered diagrams in our setting, as we often have posets with a minimal element (the single 0-simplex).

In the proof of Proposition 5.1 every path in \( \text{Ran}^{\leq n}(M) \times \mathbb{R}_{\geq 0} \) was a zigzag diagram \([19]\), and so two such diagrams form parts of a poset diagram. That is, take a path \( \gamma : I \to \text{Ran}^{\leq n}(M) \) and consider the “ribbon” \( \gamma(I) \times \mathbb{R}_{\geq 0} \) to compare the persistent homologies of the data sets \( \gamma(0) \) and \( \gamma(1) \), which form the “edges” of the ribbon. For example, we can compare

![Diagram](example.png)

with no trivial simplicial maps, to see that the persistence modules of \( \gamma(0) \) and \( \gamma(1) \) will be related by maps in not necessarily one direction. Now the persistent homology of \( \gamma(0) \) and \( \gamma(1) \) is contained within a generalized persistence module.

**Remark 5.5 (Mapper).** Singh, Memoli, and Carlsson in [SMC07] introduce a method that associates an abstract simplicial complex of at most \( k \) dimensions to a finite point cloud \( P \) and a map \( P \to \mathbb{R}^k \). Since all maps on discrete spaces are smooth, this method may be viewed as a function

\[
\hat{\mathcal{C}}_k : \text{Ran}^{\leq n}(M) \times C^\infty(*^n, \mathbb{R}^k) \to \text{SC}.
\]

Our setting had \( k = 1 \) and smooth maps that send every point to the same value. With this description we would need to describe the stratification of \( C^\infty(*^n, \mathbb{R}^k) \),
which would yield a better understanding of the clustering methods proposed by Mapper.

5.3. Open questions. We conclude by posing some natural questions that have been left unanswered.

(1) Is the Hausdorff distance on \(\text{Ran}^\leq n(M)\) an upper bound for the interleaving distance \(d_I\) of persistence modules?

See, for example, [BL17] for more on interleaving distance. A positive answer would say that for every \(P, Q \in \text{Ran}^\leq n(M)\), we have \(d_I(PH_{P,d}, PH_{Q,d}) \leq d_H(P, Q)\). The opposite direction fails spectacularly: given a point sample \(P \subseteq M\), the point samples \(P \cup \{\epsilon\}, P \cup \{-\epsilon\} \subseteq M \times \mathbb{R}\) have interleaving distance 0, but Hausdorff distance (in \(M \times \mathbb{R}\)) at least \(\epsilon\), for every \(\epsilon > 0\).

(2) Which results can be extended to the infinite-dimensional \(\text{Ran}(M) \times \mathbb{R}_{\geq 0}\)?

Theorem 4.11 depends on the non-existence of infinite ascending chains, so the same argument cannot be used everywhere. The stratification by and continuity of the Čech map extend naturally, but semialgebraic sets require finitely many equations and inequalities defining them.

(3) What is the stalk of the sheaf \(\mathcal{G}\)?

(4) Does \(\mathcal{G}\) become simpler when restricted to basics?

The sheaf is hard to get a grasp on because of the high-powered machinery used to describe it. On a small enough basic \(U\), the category \(\mathcal{G}(U)\) does not seem to contain more information than the image of \(\mathcal{F}(U)\).

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