Non-Minimally Coupled Einstein Gauss Bonnet Inflation Phenomenology in View of GW170817

S.D. Odintsov,1,2 V.K. Oikonomou,3,4,5 F.P. Fronimos,3
1) ICREA, Passeig Lluís Companys, 23, 08010 Barcelona, Spain
2) Institute of Space Sciences (IEEC-CSIC) C. Can Magrans s/n, 08193 Barcelona, Spain
3) Department of Physics, Aristotle University of Thessaloniki, Thessaloniki 54124, Greece
4) Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics, 634050 Tomsk, Russia (TUSUR)
5) Tomsk State Pedagogical University, 634061 Tomsk, Russia

We study the inflationary phenomenology of a non-minimally coupled Einstein Gauss-Bonnet gravity theory, in the presence of a scalar potential, under the condition that the gravitational wave speed of the primordial gravitational waves is equal to unity, that is $c_T^2 = 1$, in natural units.

The equations of motion, which are derived directly from the gravitational action, form a system of differential equations with respect to Hubble’s parameter and the inflaton field which are very complicated and cannot be solved analytically, even in the minimal coupling case. In this paper, we present a variety of different approximations which could be used, along with the constraint $c_T^2 = 1$, in order to produce an inflationary phenomenology compatible with recent observations. All the different approaches are able to lead to viable results if the model coupling functions obey simple relations, however, different approaches contain different approximations which must be obeyed during the first horizon crossing, in order for the model to be rendered correct. Models which may lead to a non-viable phenomenology are presented as well in order to understand better the inner framework of this theory. Furthermore, since the velocity of the gravitational waves is set equal to $c_T^2 = 1$, as stated by the striking event of GW170817 recently, the non-minimal coupling function, the Gauss-Bonnet scalar coupling and the scalar potential are related to each other. Here, we shall assume no particular form of the scalar potential and we choose freely the scalar functions coupled to the Ricci scalar and the Gauss-Bonnet invariant. Certain models are also studied in order to assess the phenomenological validity of the theory, but we need to note that all approximations must hold true in order for a particular model to be valid. Finally, even though each possible approach assumes different approximations, we summarize them in the last section for the sake of completeness.

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I. INTRODUCTION

The recent years have been proved to be outstanding for physicists and in particular, theoretical cosmologists. The most striking observation of the previous century that in a sense, has proved to us the incomplete perception of physicists, when it comes to understanding the secrets of the cosmos, was the realization that the Universe we live in does not only expand, but it expands with an accelerating rate [1]. As striking as it may sound, it is not in fact the only era in which the Universe may have exhibited an accelerating expansion. The inflationary era, which occurred moments after the Big Bang, also describes an accelerating expansion.

The era of inflation is very interesting and peculiar. Inflation states that the Universe experienced a drastic expansion in a tiny time interval, moments after the Big Bang occurred and before other important events such as the electroweak baryogenesis. From that moment, it is stated that the Universe throughout the years has not managed to expand as much as during the inflationary era. Furthermore, it is regarded as an essential tool which promises to shine light towards many problems which to this day remain unsolved. Inflation provides a possible explanation for the observed flatness of our Universe, the absence of magnetic monopoles which are predicted in many theories concerning high energy physics and the cosmological perturbations in matter and radiation which are currently observed. For instance the absence of magnetic monopoles can be attributed to the exponential expansion of the Universe, which led to a decrease in their average density at large scales.

Inflation however, even if it is considered a very important event in cosmology and high energy physics, it does not specify the framework of gravity that can produce such an era. In other words, it can be realized even if the theory of
gravity is not that of Einstein’s but is a different, modified theory \[2\,8\]. In common literature, there exists a plethora of models for modified gravity theories which manage to explain many observations. Such theories could also produce a plausible scenario for the aforementioned era of inflation. These possible scenarios may be endless, but in recent years, the precise observations have managed to rule out many promising models. The main observation which is the factor that decides the validity of a modified theory of gravity, and of gravity in general, is the so-called Cosmic Microwave Background (CMB). This radiation is diffused in the observable Universe and contains information which was encoded to photons during the first horizon crossing, freeze until the last scattering surface. By studying the CMB, we can discard theories which are unable to be compatible with the observations. Some of the information which can be extracted from the CMB is quantified by the spectral index of the primordial scalar perturbations and the tensor-to-scalar ratio.

There exist also many modified theories of gravity, which can manage to survive the precision tests of the CMB but still be intrinsically unrealistic. Nowadays, these theories can be tested and even discarded if deemed necessary, by studying strong sources of gravity. The era of multi-messenger astronomy is before us and provides the appropriate data to distinguish the realistic models from the unrealistic ones. Today, we are able to study cosmic events in two separate ways, firstly by witnessing the electromagnetic phenomenon, as was custom for the section of astronomy during the last century, and secondly by examining the gravitational waves emitted by strong sources of gravity. This new way of perceiving the Universe has led theorists to accept a fascinating result, that the gravitational waves, or in other words the perturbations in the metric, propagate through spacetime with their velocity being equal to that of light. For theorists who study modified theories of gravity, this is a striking result, since theoretical frameworks propose a different velocity which in fact deviates from the speed of light, must be discarded. In Ref. [9], a list of such theories is presented in detail. This result seems to be indicative of the fact that nature will always find a way to convince us whether a model for describing it is actually correct or otherwise false.

A theory belonging to the previous category is the Einstein Gauss-Bonnet gravity theory \[10\,53\] which, as a matter of fact, is the one we shall work with in this paper. In this type of theories however, the gravitational waves do not have a fixed value for their velocity and therefore can be set equal to the velocity of light, by forcing the Gauss-Bonnet coupling scalar function to obey a specific relation. This is a powerful characteristic since now these theories can survive the test of recent observations, such was the GW170817 [40]. This particular event established the term multi-messenger astronomy which we referred to previously and had a great impact on not only Cosmology, but also Nuclear Physics, as it provided also a mechanism for the creation of heavy nuclei.

In the present paper we extend the formalism of a recent previous work of ours [36], in order to study Einstein-Gauss-Bonnet gravity inflationary phenomenology, in the presence of a non-minimal coupling of the scalar field to the Ricci scalar. This is a different category of theories which contain a function of the scalar field, coupled with the Ricci scalar, which specifies the average curvature in a region. This coupled term leads to extra geometric terms in the gravitational equations of motion, or in other words it may lead to new physics, since the theory does not have an Einstein frame counterpart theory. Moreover, it can lead to simplifications or even viability of certain models which would otherwise had to be rendered as physically unrealistic. Our aim is to present several models of non-minimally coupled Einstein-Gauss-Bonnet theory, in the presence of a scalar potential too, and confront their inflationary phenomenology with the observational data.

This paper is outlined as follows: In the first section, we present the theoretical framework of the non-minimally coupled Einstein-Gauss-Bonnet theory, in the presence of a scalar potential, and we also demonstrate the constraints imposed by the condition \[c_B^2 = 1\]. It is worth mentioning that even though the GW179817 event does not refer to the inflationary era, this constraint is imposed in the present paper in order to obtain a massless primordial graviton from the perspective of particle physics. Then we introduce the slow-roll indices and assume the least possible approximations in order to make the system of the equations of motion elegant and solvable. In the following sections, we present the form of the observed quantities according to this particular framework and develop certain models for a number of possible approaches one could follow in order to solve the aforementioned system of equations of motion and thoroughly study the validity of the approximations made. Finally, in the last section we present all the possible approaches one might follow in order to solve the equations of motion and derive acceptable results. Every possible approach is accompanied by necessary approximations that must apply during the first horizon crossing, in order for the model to be called viable.

Before we begin our study, it is worth specifying the cosmological background we shall assume. In this paper, we shall assume a flat Friedman-Robertson-Walker (FRW) metric corresponding to a line element,

\[
ds^2 = dt^2 + a^2(t) \sum_{i=1}^{3} (dx^i)^2,
\]

where \[a(t)\] denotes the scale factor.
II. ESSENTIAL FEATURES OF GW170817 NON-MINIMALLY COUPLED EINSTEIN-GAUSS-BONNET GRAVITY AND INFLATIONARY PHENOMENOLOGY

The starting point of our study is obviously the gravitational action, since all the information about the Universe at the era of inflation is contained in it. Let us assume that the action is defined as,

\[ S = \int d^4x \sqrt{-g} \left( \frac{h(\phi)R^2}{2\kappa^2} - X - V(\phi) + \mathcal{L}_c \right), \]

where \( g \) is the determinant of the metric tensor, \( \kappa = \frac{1}{M_p} \) is a constant proportional to the reduced Planck mass, \( h(\phi) \) is a dimensionless scalar function coupled to the Ricci scalar \( R \), \( X \) is the kinetic term designated as \( X = \frac{1}{2}\omega g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \), \( V \) is the scalar potential and finally, \( \mathcal{L}_c \) denotes the string corrections which are specified as \( \mathcal{L}_c = -\frac{1}{3}\xi(\phi)G \). Here, \( \xi(\phi) \) denotes the Gauss-Bonnet coupling scalar function and \( G \) signifies the Gauss-Bonnet invariant defined as \( G = R^2 - R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} \), where \( R_{\alpha\beta} \) and \( R_{\alpha\beta\gamma\delta} \) are the Ricci and Riemann tensor respectively. Since the line element corresponds to that of a flat Friedman-Robertson-Walker spacetime, then certain terms in the gravitational action are simplified. Specifically, the kinetic term is now written as \( X = -\frac{1}{2}\omega \dot{\phi}^2 \) by simply assuming that the scalar field is homogeneous and additionally, the Gauss-Bonnet scalar is written as \( G = 24H^2(\dot{H} + H^2) \). Here, the “dot” represents differentiation with respect to the cosmic time. Last but not least, we mention that even though we shall work with a canonical kinetic term, the \( \omega \) parameter will be kept undefined for the moment, instead of being replaced with \( \omega = 1 \) in order to have the phantom case \( \omega = -1 \) available too.

As mentioned before, the gravitational action contains all the information available, so the equations of motion, which are necessary in order to describe the dynamics of inflation, can be extracted from Eq. (2) by implementing the variation principle. Consequently, the equations of motion are written as

\[ \frac{3hH^2}{\kappa^2} = \frac{1}{2}\omega \dot{\phi}^2 + V - \frac{3H\dot{h}}{\kappa^2} + 12\ddot{\xi}H^3, \]

\[ -\frac{2h\dot{H}}{\kappa^2} = \omega \ddot{\phi}^2 - \frac{H\ddot{h}}{\kappa^2} - 8\dddot{\xi}H\dot{H} + \frac{\dddot{h} + h\ddot{h}}{\kappa^2} - 4H^2(\dddot{\xi} - \dddot{\xi}H), \]

\[ \dddot{\phi} + 3H\dot{\phi} + \frac{1}{\omega} \left( V' - \frac{R\dddot{h}}{2\kappa^2} + 12\xi'\dot{H}^2(\dot{H} + H^2) \right) = 0, \]

where the “prime” denotes differentiation with respect to the scalar field \( \phi \). Solving this particular system of differential equations requires finding an analytical expression for Hubble’s parameter and the scalar field \( \phi \), which should give us a complete description of the inflationary era. Unfortunately, these equations are very complicated and the system cannot be solved analytically. The solution may be extracted, only if certain approximations are made which facilitate our study and in fact make the system solvable. Before we proceed with the approximations, we shall impose a strong constraint on the velocity of the gravitational waves in order to achieve compatibility with the recent GW170817 observation.

Gravitational waves are perturbations in the metric which travel through spacetime with the speed of light [40], as it was ascertained recently. However, theories which contain string corrections lead to an expression for their cosmological tensor perturbations velocity, which in fact deviates from the speed of light. The general expression in a cosmological context is,

\[ c_T^2 = 1 - \frac{Q_f}{2Q_t}, \]

where \( Q_f = 8(\dddot{\xi} - H\dddot{\xi}) \), \( Q_t = F + \frac{Q_b}{2} \), \( Q_b = -8\dddot{\xi}H \) and \( F = \frac{\dot{h}}{h} \). If the gravitons are massless during and after the inflationary era, if we demand that \( Q_f = 0 \), meaning that \( \dddot{\xi} = H\dddot{\xi} \), we get \( c_T^2 = 1 \). As a result, Eq. (4) is greatly simplified and leads us one step closer to finding simplified solutions for the inflationary era. Apart from solving the problem with the velocity of gravitational waves, a simple differential equation is derived, which reads,

\[ H\dddot{\xi}, \]

Instead of solving this differential equation with respect to the cosmic time, like was done in Ref. [41], we can modify it properly and extract a deeper connection between the scalar field \( \phi \) and the Gauss-Bonnet coupling scalar function \( \xi \). Since \( \frac{d}{dt} = \frac{d}{d\phi} \), the differential equation can be rewritten as,

\[ \dddot{\phi} + \xi'\dddot{\phi} = H\dot{\xi}'\phi, \]
This equation can be simplified if the slow-roll approximation is considered. Let us assume that \( \ddot{\phi} \ll H\dot{\phi} \). Hence,

\[
\ddot{\phi} \simeq \frac{H\xi'}{\xi''},
\]

(9)

This is a simple correlation between the derivative of the scalar field and the first two derivatives of the Gauss-Bonnet coupling scalar function. It can be used in Eqs. (3)-(5), in order to replace the derivative of scalar field. We can further simplify the equations of motion, by using the slow-roll approximation, which is essential to inflationary phenomenology. Assuming in addition that the kinetic term is insignificant compared to the scalar potential and also Hubble’s derivative is also lesser than Hubble’s parameter squared, that is,

\[
\dot{H} \ll H^2 \quad \frac{1}{2} \omega \dot{\phi}^2 \ll V \quad \ddot{\phi} \ll H\dot{\phi},
\]

(10)

then the equations of motion can be greatly simplified as shown below,

\[
\frac{3hH^2}{\kappa^2} = V - \frac{3H^2h'}{\kappa^2} \frac{\xi'}{\xi''} - 12 \frac{\xi'^2}{\xi''} H^4,
\]

(11)

\[
- \frac{2hH}{\kappa^2} = \omega H^2 \left( \frac{\xi'}{\xi''} \right)^2 - \frac{H^2h'}{\kappa^2} \frac{\xi'}{\xi''} H^2 \dot{H} + \frac{\xi'^2}{\kappa^2} \frac{1}{\xi''} \left( \frac{\xi'}{\xi''} \right)^2,
\]

(12)

\[
V' + 3H^2 \left( \omega \frac{\xi'}{\xi''} - 2 \frac{h'}{\kappa^2} + 4\xi'H^2 \right) = 0,
\]

(13)

These are the gravitational equations of motion simplified due to the assumption of the slow-roll approximation, and due to the fact that we assumed \( c_T^2 = 1 \) for the primordial tensor perturbations. However, even simplified to this form, the system remains unsolvable. More approximations are needed in order to make the equations solvable and examine the viability of a model. The appealing characteristic of our results is that Hubble’s derivative is written proportionally to Hubble’s parameter. Hence, Eq. (12) describes as we shall see, the slow-roll index \( \epsilon_1 \), which is a powerful relation since different assumptions could lead to different approaches to the problem and therefore different results.

The dynamics of inflation can be described by studying the slow-roll indices \( \epsilon_i \), which are defined as,

\[
\epsilon_1 = \pm \frac{\dot{H}}{H^2} \quad \epsilon_2 = \frac{\ddot{\phi}}{H\dot{\phi}} \quad \epsilon_3 = \frac{\dot{F}}{2HF} \quad \epsilon_4 = \frac{\dot{E}}{2HE} \quad \epsilon_5 = \frac{\dot{F} + Q_a}{2HQ_t} \quad \epsilon_6 = \frac{\dot{Q}_t}{2HQ_t},
\]

(14)

where \( E = F \left( \omega + \frac{3(\dot{F} + Q_a)^2}{2\dot{\phi}^2 Q_t} \right) \) and \( Q_a = -4\xi'H^2 \). The sign of slow-roll index \( \epsilon_1 \) seems arbitrary but it is convenient to assume either the positive or the negative value in certain cases. For the purposes of this paper we shall take the positive value though. Furthermore, the expression of the index, as mentioned before, is depending on the approximations which will be implemented in Eq. (12) so we will refrain from writing an analytic form at this point. In the following models, we shall specify the sign of \( \epsilon_1 \) before commencing with the results. However it is worth writing the forms of the rest of the slow-roll indices,

\[
\epsilon_2 \simeq 1 + \frac{\dot{H}}{H^2} - \frac{\xi'^2}{\xi''^2} \frac{\xi'''}{\xi''},
\]

(15)

\[
\epsilon_3 \simeq \frac{1}{2} \frac{\xi'}{\xi''} \frac{h'}{h},
\]

(16)

\[
\epsilon_4 \simeq \epsilon_3 + \frac{1}{2} \frac{\xi'}{\xi''} \frac{P'}{P},
\]

(17)

\[
\epsilon_5 \simeq \frac{1}{2} \frac{\xi'}{\xi''} \left( \frac{h'}{\kappa^2} - 4\xi'H^2 \right),
\]

(18)
\[ \epsilon_6 \simeq \frac{H}{2 Q_t} \frac{\xi'}{\xi''} \left( 2(1 + \frac{\dot{H}}{H^2}) + \frac{\dot{H}^2}{H^2} \right), \]  

where we introduced \( P = \frac{E}{\dot{H}} \) for convenience. As we can see see, indices \( \epsilon_2 \) and \( \epsilon_6 \) are connected with \( \epsilon_1 \) and in addition, index \( \epsilon_4 \) with \( \epsilon_3 \). That does not mean however that indices \( \epsilon_3 \) and \( \epsilon_4 \) are equivalent.

Finally, we examine the form of the \( \epsilon \)-foldings number \( N \), which is of fundamental importance in our study. By definition, the \( \epsilon \)-foldings number is written as \( N = \int_{t_i}^{t_f} H dt \) where \( t_i \) and \( t_f \) signify the initial and final moment of inflation, or to put it simply, the difference \( t_f - t_i \) signifies the duration of inflation. However, using Eq. (19), one can alter the variable and work solely with the scalar field \( \phi \). As a result, the formula for the \( \epsilon \)-foldings number is altered as shown below,

\[ N = \int_{\phi_i}^{\phi_f} \frac{\xi''}{\xi'} d\phi, \]  

This form implies that the \( \epsilon \)-foldings number is strongly dependent on the choice of the function \( \xi(\phi) \), so by choosing a simple coupling scalar function \( \xi(\phi) \), or one with appropriate characteristics, could yield in principle a simple phenomenology. In the following section we shall appropriately choose both coupling functions and study the phenomenology of the non-minimally coupled Einstein-Gauss-Bonnet model, and how viability can be achieved imposing certain necessary approximations in the equations of motion. It will also be shown that even though certain approaches seem fascinating, in the end do not result to viable models.

### III. MODELS OF NON-MINIMALLY COUPLED EINSTEIN-GAUSS-BONNET GRAVITY AND COMPATIBILITY WITH PLANCK DATA

Beginning this paper, we wrote down the gravitational action \( \mathcal{L} \), which was the starting point in deriving the equations of motion. This equation has certain unspecified functions, mainly the coupling functions \( h(\phi) \) and \( \xi(\phi) \), along with the scalar potential \( V(\phi) \). Consequently, in order to derive the expression of Hubble’s parameter, these functions must be determined. However, since the constraint in the velocity of the gravitational waves was imposed, the scalar potential depends on the other two freely chosen functions. In fact, the third equation of motion connects both coupling functions to the scalar potential. Thus, when specifying the coupling functions, the potential cannot take an arbitrarily chosen form, but must obey Eq. (13). In the following models, we shall assume that the scalar potential obeys a more simplified differential equation, which is,

\[ V' + 3H^2 \left( \frac{\omega}{\xi''} - \frac{2}{r^2} \right) = 0, \]  

This assumption is not necessary but it is convenient, since a more manageable potential may be derived, but we note that the assumption \( \xi' H^4 \ll V' \), along with the slow-roll approximations (10), must hold true in order for the model to be viable. These assumptions, in addition to the rest which shall make hereafter, will be validated if these hold true at the end of each model.

In order to ascertain the validity of a model, the results which the model produces must be confronted to the recent Planck observational data [42]. In the following models, we shall derive the values for the quantities, namely the spectral index of primordial curvature perturbations \( n_S \), the tensor-to-scalar-ratio \( r \) and finally, the tensor spectral index \( n_T \) [10]. These quantities are connected with the slow-roll indices introduced previously, as shown below,

\[ n_S = 1 + 2 \frac{\dot{H}}{H^2} - \epsilon_2 + \epsilon_3 - \epsilon_4 \]  
\[ n_T = 2 \frac{\dot{H}}{H^2} - \epsilon_6 \]  
\[ r = 16 \left| \frac{Q_a}{4HF} + \frac{\dot{H}}{H^2} - \epsilon_3 \right| \]  
\[ \frac{F_{c_A}^3}{Q_t}, \]  

where \( c_A \) is the sound wave velocity defined as,

\[ c_A^2 = 1 + \frac{Q_a(\dot{F} + Q_a)}{2\omega^2Q_t + 3(\dot{F} + Q_a)^2}, \]  

Since the sign of slow-roll index \( \epsilon_4 \) has not been specified yet, it was deemed necessary to write the ratio \( \frac{\dot{H}}{H^2} \). Moreover, spectral index \( n_T \) has not been experimentally verified to date, since no B-modes have been observed so far in the CMB. However we shall call it an observable quantity and make a prediction for its value for each model. According to the recent Planck 2018 collaboration [42], the rest observed quantities have the following values,

\[ n_S = 0.9649 \pm 0.0042 \]  
\[ r < 0.064, \]  

The sound wave velocity defined as,
\[ \frac{\dot{H}}{H^2} = 2 \left( 1 + \frac{\dot{H}}{\omega^2} \right), \]  

Finally, we examine the form of the \( \epsilon \)-foldings number \( N \), which is of fundamental importance in our study. By definition, the \( \epsilon \)-foldings number is written as \( N = \int_{t_i}^{t_f} H dt \) where \( t_i \) and \( t_f \) signify the initial and final moment of inflation, or to put it simply, the difference \( t_f - t_i \) signifies the duration of inflation. However, using Eq. (19), one can alter the variable and work solely with the scalar field \( \phi \). As a result, the formula for the \( \epsilon \)-foldings number is altered as shown below,

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This form implies that the \( \epsilon \)-foldings number is strongly dependent on the choice of the function \( \xi(\phi) \), so by choosing a simple coupling scalar function \( \xi(\phi) \), or one with appropriate characteristics, could yield in principle a simple phenomenology. In the following section we shall appropriately choose both coupling functions and study the phenomenology of the non-minimally coupled Einstein-Gauss-Bonnet model, and how viability can be achieved imposing certain necessary approximations in the equations of motion. It will also be shown that even though certain approaches seem fascinating, in the end do not result to viable models.
These values can be theoretically evaluated by inserting the wavenumber $k$ during the first horizon crossing as an input. Here, we have followed a different approach. Instead of using wavenumbers, we study the era of inflation using the inflaton field $\phi$. As a result, the previous values can be calculated by inserting the value of the scalar field at the start of inflation as an input. However, in order to evaluate the initial value of the scalar field during this era, one must find first the final value of the scalar field. This value is derived easily by equating slow-roll index $|\epsilon_1|$ with 1. This is exactly why this slow-roll index has not been properly designated so far, because different approximations yield different forms and therefore different expressions for the final value of the scalar field. In the following models, we shall firstly designate both coupling scalar functions, specify the sign of slow-roll index $|\epsilon_1|$, make certain assumptions, derive results, compare them with the observations and assess the validity of the assumptions made in each model separately. Before we proceed however, it is worth writing the analytic form of the auxiliary functions $Q_a$, $Q_b$, $Q_c$ and $Q_d$, which are,

$$Q_a \simeq -4 \frac{\xi''}{\xi'} H^3,$$

$$Q_b \simeq -8 \frac{\xi''}{\xi'} H^2,$$

$$Q_c \simeq -16 \frac{\xi''}{\xi'} \dot{H},$$

$$Q_d \simeq \frac{h}{\kappa^2} - 4 \frac{\xi''}{\xi'} H^2.$$

In the following subsections, we shall examine the phenomenology of several models by specifying the functional forms of the non-minimal coupling scalar function $h(\phi)$ and of the Gauss-Bonnet coupling function $\xi(\phi)$.

### A. A Model with Power-Law $h(\phi)$ and Exponential $\xi(\phi)$ Functions

This model is supposedly one of the easiest imaginable. The coupling scalar functions are defined as,

$$h(\phi) = \Lambda_1(\kappa\phi)^{n_1},$$

$$\xi(\phi) = \lambda_1 e^{\gamma_1 \kappa \phi},$$

The reasons behind choosing such functions is because the first is a very simple case of power-law, while the latter has an appealing characteristic. In the previous equations, the coupling functions appear in ratios of $\xi'/\xi''$, $h'/h$ and $h''/h$, so it is only reasonable to try and use functions which simplify greatly these ratios. This is exactly why the exponential function was chosen along with the power-law, since,

$$h' = \frac{n_1}{\phi} h \quad h'' = \frac{n_1(n_1 - 1)}{\phi^2} h \quad \xi'' = \kappa \gamma_1 \xi',$$

Now, all that remains is to specify the forms of Hubble's parameters. In this models, we shall assume that,

$$H^2 \simeq \frac{k^2 V}{3h},$$

$$\dot{H} \simeq \frac{H^2}{2} \left( \frac{h'}{h} \frac{\xi'}{\xi''} - \frac{h''}{h} \left( \frac{\xi'}{\xi''} \right)^2 \right).$$

These relations are derived from Eqs. (11) and (12) respectively. In the end, we shall examine whether the assumptions made hold true.

From the expression of the Hubble’s derivative, we see that it is appropriate to choose the positive sign for index $\epsilon_1$, that is, $\epsilon_1 = \frac{\dot{H}}{H^2}$. Thus, from Eq. (33), it can be inferred that the slow-roll index $\epsilon_1$ and subsequently the form
of the final value of the scalar field depend strongly on the ratios of the coupling functions with their derivatives respectively. Once again, the selection of a power-law and an exponential function seems appropriate since they lead to simplified form. Before we proceed with the expressions of the slow-roll indices, let us first derive the expression for the scalar potential. From Eq. (24),

\[ V(\phi) = V_1(\kappa\phi)^{2n_1} \exp \left( \alpha_1(\kappa\phi)^{1-n_1} \right), \]

where \( \alpha_1 = \frac{\omega}{\gamma_1(\kappa\phi)^{1-1}} \) and \( V_1 \) is a constant with mass dimensions \([m]^4\). As a result, the slow-roll indices can be evaluated, in certain cases having simple expressions, as shown below,

\[ \epsilon_1 \simeq \frac{n_1}{2\gamma_1\kappa\phi} \left( 1 - \frac{n_1 - 1}{\kappa\phi} \right), \]

\[ \epsilon_2 \simeq \frac{n_1}{2\gamma_1\kappa\phi} \left( 1 - \frac{n_1 - 1}{\kappa\phi} \right), \]

\[ \epsilon_3 \simeq \frac{n_1}{2\gamma_1\kappa\phi}, \]

\[ \epsilon_5 \simeq \frac{3\Lambda_1^2n_1(\kappa\phi)^{2n_1} - 4\Lambda_1^2\kappa\phi\lambda_1^2\kappa^4V(\phi)e^{\gamma_1\kappa\phi}}{6\gamma_1\kappa\Lambda_1^2(\kappa\phi)^{2n_1} - 8\gamma_1\kappa\phi\lambda_1^2\kappa^4V(\phi)e^{\gamma_1\kappa\phi}}, \]

\[ \epsilon_6 \simeq \frac{4\Lambda_1^2n_1\kappa^4V(\phi)e^{\gamma_1\kappa\phi} - 4\Lambda_1^2\kappa\phi e^{\gamma_1\kappa\phi} \kappa^3 V'(\phi) - 4\gamma_1\kappa\phi\lambda_1^2\kappa^4V(\phi)e^{\gamma_1\kappa\phi} + 3\Lambda_1^2n_1(\kappa\phi)^{2n_1}}{6\gamma_1\kappa\Lambda_1^2(\kappa\phi)^{2n_1} - 8\gamma_1\kappa\phi\lambda_1^2\kappa^4V(\phi)e^{\gamma_1\kappa\phi}}, \]

It is obvious that the first three indices have simple functional forms, while the rest are more complicated, especially the index \( \epsilon_4 \), which is omitted due to this reason. Let us now continue with the evaluation of the necessary values of the inflaton field. Firstly, as mentioned before, the final value of the scalar field can be extracted by equating index \( \epsilon_1 \) to unity. As a result, the final value of the scalar field has the following form,

\[ \phi_f = \frac{n_1\gamma_1 + \sqrt{\gamma_1^2n_1(8 - 7n_1)}}{4\gamma_1^2\kappa}. \]

Utilizing the form of the \( e \)-foldings number in Eq. (20), the initial value of the scalar field is extracted and subsequently the observed quantities. The initial value reads,

\[ \phi_i = \frac{\gamma_1(n_1 - 4N) + \sqrt{\gamma_1^2n_1(8 - 7n_1)}}{4\gamma_1^2\kappa}, \]

Specifying the free parameters of the theory could produce results compatible with the experimental values for the spectral indices and the tensor-to-scalar ratio introduced in Eq. (24). Assuming that \((\omega, V_1, \Lambda_1, \lambda_1, N, \gamma_1, n_1)=(1, 1, 100, 1, 60, -100, 0.5)\), in reduced Planck units, so for \( \kappa^2 = 1 \), the model at hand produces acceptable results, since \( n_s = 0.967045 \) and \( r = 0.000551 \) are both compatible with the latest observations\(^{42}\). Moreover, the tensor spectral index takes the value \( n_T = 0.000069 \). Finally we mention that the scalar field in equations (41) and (40) takes the values \( \phi_i = 0.6025 \) and \( \phi_f = 0.0025 \) and clearly shows that the field is decreasing as time flows. In Fig. 1, we present the contour plot of two observable quantities, namely \( n_s \) and \( r \), which is indicative of the existence of more than a single set of values for the free parameter which can lead to phenomenologically viable results. Finally, we examine each approximation which was made in order to derive the previous results. According to the previous set of parameters in reduced Planck units always, during the first horizon crossing, \( H \sim \mathcal{O}(10^{-5}) \) and \( H^2 \sim \mathcal{O}(10^{-3}) \) so the slow-roll assumption holds true. In addition, \( \frac{1}{2}\omega\dot{\phi}^2 \sim \mathcal{O}(10^{-7}) \) while \( V \sim \mathcal{O}(10^{-1}) \) and lastly, \( \dot{\phi} \sim \mathcal{O}(10^{-7}) \) and \( H\dot{\phi} \sim \mathcal{O}(10^{-5}) \). Hence, the slow-roll conditions \(^{10}\) are valid. All that remains is to ascertain the validity of the rest approximations. It turns out that \( \xi' H^2 \sim \mathcal{O}(10^{-30}) \) which is negligible compared to \( V' \sim \mathcal{O}(1) \) and therefore, the differential equation of the scalar potential is justified. Furthermore, \( h' H^2 \sim \mathcal{O}(10^{-3}) \) so in principle, it could also be omitted from the differential equation \(^{21}\). For Hubble’s parameter, \( \xi H^3 \sim \mathcal{O}(10^{-32}) \) and \( 3H\dot{h} \sim \mathcal{O}(10^{-3}) \), which are both much lower in magnitude, compared to the scalar potential. Lastly, we verify the last approximations made
for the Hubble’s rate derivatives. We note that $\dot{H}\dot{H} \sim O(10^{-34})$, $H\dot{h} \sim O(10^{-3})$ and $h''\phi^2 \sim O(10^{-5})$ hence all the approximations made are in fact valid.

Lastly we note that even if in Eq. (32) we included also the term $3H\dot{h}\kappa^2$, meaning that if we were to use the following equation

$$H^2 \simeq \frac{\kappa^2 V}{3h\left(1 + \frac{\phi}{\kappa}\right)},$$

(42)

for the Hubble rate, we would end up with the exact same values for the observed quantities. This property reassures us about the chosen approach.

This is an interesting model due to the fact that in the minimal case where $h(\phi) = 1$ [36], it can be shown that the model may lead to either eternal inflation or no inflation at all. That result implies that the non-minimal case provides a possible way of producing viable phenomenology, for a coupling function which in the minimally coupled Einstein-Gauss-Bonnet theory would lead to non-viable results.

### B. A Model with Power-Law $h(\phi)$ and Error $\xi(\phi)$ Functions

Suppose now that the coupling functions are defined as,

$$h(\phi) = \Lambda_2(\kappa\phi)^{n_2},$$

(43)

$$\xi(\phi) = \frac{2\lambda_2}{\sqrt{\pi}} \int_0^{\gamma_2\kappa\phi} e^{-x^2}dx,$$

(44)

where $x$ is an auxiliary integration variable. Similar to the previous case, the coupling functions are chosen in a specific way so that they lead to simplified ratios as shown below,

$$h' = \frac{n_2}{\phi}h \quad \xi'' = -2\gamma_2^2\kappa^2\phi\xi'.$$

(45)

Let us assume that Hubble’s parameter and its derivative are approximately equal to,

$$H^2 \simeq \frac{k^2 V}{3\left(h + \frac{h'\xi'}{\xi''}\right)},$$

(46)
\[
\dot{H} \simeq \frac{H^2}{2} \frac{h' \xi'}{h \xi''},
\]

This is a different approximation in comparison to the one used in the previous subsection, in which we kept more terms in Hubble’s parameter and less in its derivative. However we shall see that for this particular model, either Eq. \(32\) or Eq. \(40\) lead to viable results. Furthermore, due to the approximations made in Hubble’s derivative, the sign of slow-roll index \(\epsilon_1\) will be positive throughout this model.

The difference in the definition of Hubble’s parameter in these two models is that it leads to a much more complicated form of the scalar potential. Assuming the previous definition, Eq. \(21\) produces the following form,

\[
V(\phi) = V_2(n_2 - 2(\gamma_2 \kappa \phi)^2)^n \exp \left( 2F_1 \left( 1, 1 - \frac{n_2}{2}, 2 - \frac{n_2}{2}; \frac{2(\gamma_2 \kappa \phi)^2}{n_2} \right) \alpha_2(\kappa \phi)^{2-n_2} \right),
\]

where \(\alpha_2 = \frac{\omega}{\Lambda_2 n_2(n_2 - 2)}\), \(V_2\) is a constant with mass dimensions \([m]^4\) and \(2F_1 \left( 1, 1 - \frac{n_2}{2}, 2 - \frac{n_2}{2}; \frac{2(\gamma_2 \kappa \phi)^2}{n_2} \right)\) is the hypergeometric function. This is a very complicated potential and is a direct result of the extra term participating in Hubble’s parameter. As in the previous model, the slow-roll indices have either extremely simple or very perplexed forms as shown below,

\[
\epsilon_1 \simeq \frac{-n_2}{2(\gamma_2 \kappa \phi)^2},
\]

\[
\epsilon_2 \simeq \frac{2 - n_2}{2(\gamma_2 \kappa \phi)^2},
\]

\[
\epsilon_3 \simeq \frac{-n_2}{2(\gamma_2 \kappa \phi)^2},
\]

\[
\epsilon_5 \simeq \frac{16(\gamma_2 \kappa \phi)^2 \lambda_2 \kappa^4 V(\phi) - 3\sqrt{\pi} \Lambda_2^2 n_2 e^{(\gamma_2 \kappa \phi)^2}(\kappa \phi)^{2n_2} (2(\gamma_2 \kappa \phi)^2 - n_2)}{(2\gamma_2 \kappa \phi)^2 (8\gamma_2 \kappa \phi \lambda_2 \kappa^4 V(\phi) + 3\sqrt{\pi} \Lambda_2^2 e^{(\gamma_2 \kappa \phi)^2}(\kappa \phi)^{2n_2} (2(\gamma_2 \kappa \phi)^2 - n_2))},
\]

In this model, indices \(\epsilon_4\) and \(\epsilon_6\) are not quoted, due to their complicated form. In contrast to the previous model, now indices \(\epsilon_1\) and \(\epsilon_3\) are equivalent and moreover, the form of the first three slow-roll indices appears to be a very simple equation, especially compared to index \(\epsilon_5\) which is also depending on the scalar potential. In addition, the form of \(\epsilon_1\) greatly constrains the values available for the exponent \(n\) of the coupling scalar function \(h(\phi)\) since it can take only negative values.

Similar to the previous model, the value of the scalar field at the end of inflation is derived from the equation \(\epsilon_1 = 1\) and therefore it reads,

\[
\phi_f = \pm \sqrt{\frac{-n_2}{2\gamma_2 \kappa}},
\]

As a result, the equation of the e-foldings number generates the formula of the value of the scalar field at the initial stage of inflation, which in turn is written as,

\[
\phi_i = \pm \sqrt{\frac{4N - n_2}{2\gamma_2 \kappa}},
\]

Assuming that in reduced Planck Units, the free parameters have the following values \((\omega, V_2, \Lambda_2, \lambda_2, N, \gamma_2, n_2) = (1, -1, -4, 1.5 \cdot 10^{31}, 60, -10, -3)\), we obtain viable results for the observational quantities, which are in good agreement with experimental evidence \([12]\), since \(n_s = 0.964397\) and \(r = 8.3836 \cdot 10^{-6}\). In addition, the tensor spectral index generates the value \(n_T = -0.779423\). However, the compatibility with the observational data for the model at hand can come for a wide range of the free parameters values, as can also be seen in Figs. \([2]\). The fascinating result about this model is that with an appropriate fine-tuning, the value of the tensor-to-scalar ratio can drop drastically. For instance, changing only the values for parameters \(\Lambda_2\) and \(\lambda_2\) to \(\Lambda_2 = -4 \cdot 10^{10}\) and \(\lambda_2 = 1.5 \cdot 10^{13}\) leads to the values \(n_s = 0.96748\) and \(r = 8.38195 \cdot 10^{-44}\). This implies that the model can survive many, if not any restriction on the tensor-to-scalar ratio generated by better experiments conducted in the future, although it is rather impossible to detect any tensor modes for such a small \(r\).
Finally, we note that all the approximations which were made for this model apply. For instance, we mention that slow-roll indices are depending solely on the ratios of the coupling functions. This is also the reason why we choose such coupling functions in the first place. On the other hand, the indices \( \epsilon_5 \) and \( \epsilon_6 \) certainly change, not just due to the difference in the Hubble rate, but also due to the scalar potential, since the latter is derived directly from the first. Instead of explaining it, it is better to show it using equations. Using equations (32) and (47), meaning that, 

\[
\dot{\phi}^2 \sim O(10^{-12}), \quad \text{in reduced Planck units, while } H\dot{\phi} \sim O(10^{-10}),
\]

hence the slow-roll approximations (10) are still valid even if the magnitudes are not separated by many orders. Concerning the approximations in the differential equation of the scalar potential and Hubble’s parameter, we mention that 12\( \xi H^2 \sim O(10^{-9}) \) and 12\( \dot{\xi}H^3 \sim O(10^{-11}) \), so compared to \( V' \sim O(10^{-6}) \) and \( V \sim O(10^{-7}) \) respectively, our approximations are once again justified. Finally, we note that \( \dot{\phi}^2 \) and \( 8\dot{\phi}H\dot{H} \) are both or order \( O(10^{-13}) \) and \( H\dot{\phi} \sim O(10^{-10}) \) and \( H\dot{H} \sim O(10^{-8}) \) hence all the approximations which were made for this model apply.

As a last comment, it is worth mentioning the selection of Hubble’s parameter in Eq. (40). Genuinely, speaking, it was chosen not because it is necessary in order to achieve viability, but in order to deviate from the approach of the previous model, although if one were to use equations (32) and (47) could also generate results compatible with the observations, with a slightly different specification of the free parameters of the theory. Many relations however shall remain the same. For instance, we mention that slow-roll indices \( \epsilon_3 \) through \( \epsilon_6 \) remain invariant since they are depending solely on the ratios of the coupling functions. This is also the reason why we choose such coupling functions in the first place. Instead of explaining it, it is better to show it using equations. Using equations (32) and (47), meaning that,

\[
H^2 \approx \frac{\kappa^2 V}{3h}, \tag{55}
\]

\[
\dot{H} = \frac{H^2 h' \xi'}{2h \xi''}, \tag{56}
\]

the scalar potential which is derived from Eq. (21) is,

\[
V(\phi) = V_2(\kappa\phi)^{2n_2} \exp \left( \frac{\omega(\kappa\phi)^{-n_2}}{2n_2\gamma_2^2\Lambda_2} \right). \tag{57}
\]

This is obviously a much simpler form compared to Eq. (48) which can be attributed to the simpler expression of the Hubble rate. Consequently,

\[
\epsilon_5 \approx \frac{8\gamma_2k_2\kappa^2\lambda_2\kappa^4V(\phi) - 3\sqrt{\pi}\Lambda_2^{n_2}\gamma_2e^{(\gamma_2\kappa^2\phi)^2(\kappa\phi)^{2n_2}}}{16\gamma_2\kappa\phi\lambda_2\kappa^4V(\phi) + 12\sqrt{\pi}(\Lambda_2\gamma_2\kappa\phi)^2e^{(\gamma_2\kappa\phi)^2(\kappa\phi)^{2n_2}}}, \tag{58}
\]
\[ \epsilon_6 \simeq \frac{4\lambda_2 k^4 V(\phi) \left(2(\gamma_2 \kappa \phi)^2 + n_2 + 1\right) - \kappa \phi \left(4\lambda_2 k^3 V'(\phi) + 3\sqrt{\pi} \gamma_2 \Lambda_2^2 h_2 e^{(\gamma_2 \kappa \phi)^2} (\kappa \phi)^{2n_2}\right)}{\left(2\gamma_2 \kappa \phi\right)^2 \left(4\lambda_2 k^4 V(\phi) + 3\sqrt{\pi} \Lambda_2^2 \gamma_2 \kappa \phi e^{(\gamma_2 \kappa \phi)^2} (\kappa \phi)^{2n_2}\right)} \], \quad (59)

The change of the Hubble rate, instead of altering the potential and the last two slow-roll indices, will lead to a different set of free parameters which lead to compatibility. For instance, assuming that the only change is \( \lambda_2 = 1.5 \cdot 10^{21} \), the spectral indices and the tensor-to-scalar ratio \( r \) take the values \( n_s = 0.967698, \ n_T = 3.72809 \cdot 10^{-8} \) and \( r = 7.39423 \cdot 10^{-9} \) which are accepted values \([42]\) as well. Lastly, the approximations made still hold true but instead of maintaining the same order of magnitude, the keep their relative order. For instance, \( X/V \sim \mathcal{O}(10^{-6}) \) and \( \dot{H}/H^2 \sim \mathcal{O}(10^{-2}) \), exactly as in the previous approach.

Hence, both approaches may lead to viable results. Neglecting the second term in the denominator of Eq. \([60]\) leads only to a change of a single parameter and therefore, the numerical value of quantities such as the Hubble parameter itself. However a single redefinition of a parameter is capable of restoring the viability.

C. A Model with Trigonometric \( h(\phi) \) and Power-law \( \xi(\phi) \) Functions

Let us now present an apparently elegant model, however non-viable since the slow-roll approximation breaks down. In this particular model, we shall assume basic functions as coupling functions, which at first sight might seem odd. Let,

\[ h(\phi) = \Lambda_3 \sin(\gamma_3 \kappa \phi + \theta), \quad (60) \]

\[ \xi(\phi) = \lambda_3 (\kappa \phi)^{n_3}. \quad (61) \]

This choice benefits us due to the connection between the derivatives of such functions as shown below,

\[ h'' = - (\gamma_3 \kappa)^2 h \quad \xi'' = \frac{n_3 - 1}{\phi} \xi', \quad (62) \]

In this model, we shall assume that Hubble’s parameter and its derivative are given by the following expressions,

\[ H^2 \simeq \frac{k^2 V}{3 h \xi}, \quad (63) \]

\[ \dot{H} \simeq - \frac{H^2 h''}{2 h} \left(\frac{\xi'}{\xi''}\right)^2. \quad (64) \]

Despite the odd choice for the coupling functions, it is obvious that it facilitates our study since the ratios are simplified and as a result, slow-roll index \( \epsilon_1 \) shall be simplified as well. Furthermore, we shall consider this particular index has a positive value, i.e \( \epsilon_1 = \frac{\dot{H}}{H^2} \). Before continuing to the evaluation of the slow-roll indices, let us first derive the formula for the scalar potential from Eq. \([21]\). According to the previous designations, the scalar potential must have the following form,

\[ V(\phi) = V_3 (\kappa \phi)^{2(n_3 - 1)} \left[ \cos\left(\frac{\gamma_3 \kappa \phi + \theta}{2}\right) - \sin\left(\frac{\gamma_3 \kappa \phi + \theta}{2}\right) \right]^{\alpha_3} \left[ \cos\left(\frac{\gamma_3 \kappa \phi + \theta}{2}\right) + \sin\left(\frac{\gamma_3 \kappa \phi + \theta}{2}\right) \right]^{-\alpha_3}, \quad (65) \]

where \( \alpha_3 = \frac{\omega}{\Lambda_3 \gamma_3} \). As shown, the scalar potential has again a very complicated form and cannot be used easily. Despite the form, the scalar potential participates only in slow-roll indices \( \epsilon_4 \) through \( \epsilon_6 \) and only the first three indices should concern us. More specific, the slow-roll indices can be written as,

\[ \epsilon_1 \simeq \frac{1}{2} \left(\frac{\gamma_3 \kappa \phi}{n_3 - 1}\right)^2, \quad (66) \]

\[ \epsilon_2 \simeq \frac{1}{n_3 - 1} + \frac{1}{2} \left(\frac{\gamma_3 \kappa \phi}{n_3 - 1}\right)^2, \quad (67) \]
expression of slow-roll index $\epsilon$ derivatives are approximated as follows, are shown below respectively, frequency of the oscillations is obviously depending on parameter $\gamma$. Apparently, in this approach the slow-roll indices $\epsilon$ are positive and dependent on parameter $\gamma$.

As it was demonstrated in a previous subsection, this choice leads to simplified ratios of the derivatives of the coupling functions. This time however, instead of a power-law model, we chose a linear model simply because the second derivative of the coupling function $h(\phi)$ is set equal to zero. Let us assume that the Hubble rate and its derivatives are approximated as follows,

$$H^2 \simeq \frac{k^2 V}{3h \left(1 + \frac{h}{k} \frac{\xi'}{\xi''}\right)},$$

$$\dot{H} \simeq \frac{H^2 \xi'}{2 \xi''} \left(\frac{h'}{h} - \frac{\omega k^2 \xi'}{h \xi''}\right).$$

Similarly, index $\epsilon_4$ was omitted due to its complicated form. In this model, we see that index $\epsilon_3$ has a $\phi$-dependence. Apparently, in this approach the slow-roll indices $\epsilon_3$ through $\epsilon_6$ keep oscillating during the era of inflation. The frequency of the oscillations is obviously depending on parameter $\gamma$.

According to our previous statements, the final value of the scalar field has a particularly simple form due to simple expression of slow-roll index $\epsilon_1$, and similarly the initial value presumably is described in simple terms. Both values are shown below respectively.

$$\phi_f = \pm \frac{\sqrt{2}}{\kappa} \left| \frac{n_3 - 1}{\gamma_3} \right|,$$

$$\phi_i = \pm \frac{\sqrt{2}}{\kappa} \left| \frac{n_3 - 1}{\gamma_3} \right| e^{-\frac{\kappa}{\gamma_3}}.$$

In the following, we shall choose positive signs for both values.

Assuming that in reduced Planck Units, ($\omega, V_3, A_3, \lambda_3, N, \theta, \gamma_3, n_3$)=(1, 10$^{-6}$, 10$^3$, 10$^{13}$, 60, $\frac{4}{3}$, 1, 14.5) the values for the observed quantities derived from Eq. (22) are in agreement with observations [12], as $n_S = 0.96405$ and $r = 0.0388044$ are both compatible values. Moreover, the tensor spectral index takes the value $n_T = -0.00485$ and the scalar field takes the values $\phi_i = 0.224208$ and $\phi_f = 19.0919$. In this case, the field increases with time.

Despite the elegance or the accuracy of this model, it is in fact a non-viable model due to the approximations imposed. For simplicity, we shall not mention the order of magnitude of each object, but we shall state that even though the slow-roll approximations (10) do apply, the term $H h$ is greater than $h'' \phi^2$ and therefore, our approach with Eq. (54) is rendered invalid. Numerically speaking, $H h \sim O(10^{-25})$ while $h'' \phi^2 \sim O(10^{-26})$. Perhaps a different set of parameters or a complete different model could under the same assumptions yield a viable description for the inflationary era. Such possibility was not further studied.

D. A Model with Linear $h(\phi)$ and Exponential $\xi(\phi)$ Functions

As a final model, we shall make the almost the same choices for the functions $h(\phi)$ and $\xi(\phi)$ as in the first model we presented, but in this case, we shall assume a linear form for $h(\phi)$. The reason is that this specific linear form of $h(\phi)$ has a direct effect on the tensor-to-scalar ratio as we show shortly. The coupling functions shall take the form,

$$h(\phi) = A_4 \kappa \phi,$$

$$\xi(\phi) = \lambda_4 e^{\gamma_4 \kappa \phi}.$$
Due to the form of $\dot{H}$, it is easier to assume the positive value of index $\epsilon_1$, meaning that $\epsilon_1 = \frac{\dot{H}}{H}$. Furthermore, this approach in fact could be categorized as a special case of Eq. (12), where we neglect only the term $8\xi H \dot{H}$ as the term proportional to $h''$ disappears due to the linear choice for the coupling function. Let us proceed with the evaluation of the scalar potential. According to equation (21), the potential must be equal to,

$$V(\phi) = V_4 \Lambda_4 (1 + \gamma_4 \kappa \phi)^{2-\alpha_4},$$  

(77)

where $V_4$ is the integration constant with mass dimensions $[m]^4$ and $\alpha_4 = \frac{\omega}{\gamma_4 \Lambda_4}$. This is by far the simplest scalar potential that was derived in this paper, due to the choice of the coupling scalar functions. In the following, we present the slow-roll indices which are expected to have very simplified forms, compared to the previous models, and indeed these are,

$$\epsilon_1 \approx \frac{1}{\gamma_4} \frac{1 - \omega}{\gamma_4 \Lambda_4 \kappa \phi},$$  

(78)

$$\epsilon_2 \approx \frac{\gamma_4 \Lambda_4 - \omega}{2 \gamma_4^2 \kappa \Lambda_4 \phi},$$  

(79)

$$\epsilon_3 \approx \frac{1}{2 \gamma_4 \kappa \phi},$$  

(80)

$$\epsilon_5 \approx \frac{4 \gamma_4^2 \Lambda_4^4 \kappa^4 V(\phi) e^{\gamma_4 \kappa \phi} - 3 \Lambda_4^4 (\gamma_4 \kappa \phi + 1)}{2 \gamma_4 (4 \gamma_4 \Lambda_4^4 \kappa^4 V(\phi) e^{\gamma_4 \kappa \phi} - 3 \Lambda_4^4 \kappa \phi (\gamma_4 \kappa \phi + 1))},$$  

(81)

$$\epsilon_6 \approx \frac{(\gamma_4 \kappa \phi + 1) (4 \gamma_4 \Lambda_4 \kappa^3 V'(\phi) - 3 \Lambda_4^2 (\gamma_4 \kappa \phi + 1)) + 4 \gamma_4^2 \kappa \phi \Lambda_4^4 \kappa^4 V(\phi) e^{\gamma_4 \kappa \phi}}{2 \gamma_4 (4 \gamma_4 \Lambda_4^4 \kappa^4 V(\phi) e^{\gamma_4 \kappa \phi} - 3 \Lambda_4^4 \kappa \phi (\gamma_4 \kappa \phi + 1))},$$  

(82)

Even in the simple linear form of the scalar coupling function $h(\phi)$, the index $\epsilon_4$ has a quite lengthy form so we did not quote it here. However, the first three indices are simple as expected and thus the values of the scalar field can be easily derived. Thus, similar to previous models,

$$\phi_f = \frac{\gamma_4 \Lambda_4 - \omega}{2 \gamma_4^2 \kappa \Lambda_4},$$  

(83)

$$\phi_i = \frac{\gamma_4^2 \Lambda_4 - 2 \gamma_4 \Lambda_4 N - \omega}{2 \gamma_4^2 \kappa \Lambda_4},$$  

(84)

Assuming that the free parameters of the model obtain the values ($\omega, V_4, \Lambda_4, \lambda_4, N, \gamma_4$) = (1, 1.4, -40, 1, 60, 10) in reduced Planck Units, meaning $\kappa = 1$, then it turns out that Eq. (22) produces compatible results since $n_S = 0.964796$ and $r = 0.000363645$. Furthermore the tensor spectral index is equal to $n_T = -0.00004587$. In Fig. 3 we present the parametric plot of the $n_S$ and $r$. This case is quite different from the first model since here there exists only one one-on-one correlation between $n_S$ and $r$.

Finally, we examine the validity of our approximations. For the slow-roll approximations (10), we note that $\dot{H} \sim \mathcal{O}(10)$ and $H^2 \sim \mathcal{O}(10^6)$, the kinetic term is of order $\frac{1}{4} \omega \dot{\phi}^2 \sim \mathcal{O}(10)$ while the scalar potential is of order $V \sim \mathcal{O}(10^6)$ and finally $\dot{\phi} \sim \mathcal{O}(1)$ while $H \dot{\phi} \sim \mathcal{O}(10^5)$ so clearly the slow-roll conditions do apply, even marginally. The rest approximations concern this specific approach and in fact do apply since for the differential equation of the scalar potential, $V' \sim \mathcal{O}(10^6)$ while $12 \dot{H}^2 \sim \mathcal{O}(10^{15})$, for Hubble’s parameter, we note that $V \sim \mathcal{O}(10^6)$ while $12 \dot{H}^2 \sim \mathcal{O}(10^{15})$ and finally for Hubble’s derivative, we mention that $3 \dot{H} \dot{H} \sim \mathcal{O}(10^{17})$ while $\dot{\phi}^2 \sim \mathcal{O}(10^2)$. The orders of magnitude indicate that each approximation made in this approach is in fact valid. In addition, this model contained the least amount of extra approximations necessary to make the system of equations of motion solvable due to the linear choice of the function coupled to the Ricci scalar, as only a single extra approximation, apart from the slow-roll approximations, was made in each equation of motion.
FIG. 3: Parametric plot of spectral index of scalar perturbations $n_s$ (x axis) and tensor-to-scalar ratio $r$ (y axis) depending on parameters $\Lambda_4$ and $\gamma_4$ ranging from [-10, -1] and [1, 20] respectively. It shows a clearly a one-on-one connection between the observed quantities which is attributed to the model of the coupling functions $h(\phi)$ and $\xi(\phi)$ and not the approach on Hubble’s parameters.

IV. OVERVIEW OF GENERALIZED SLOW-ROLL CONDITIONS FOR GW170817-COMPATIBLE NON-MINIMALLY COUPLED EINSTEIN-GAUSS-BONNET GRAVITY

In the final section of this paper, we shall present all the possible approximations that can be made for an GW170817-compatible non-minimally coupled Einstein-Gauss-Bonnet gravity, in order to obtain a viable inflationary era. Apart from the ones used in the models presented in the previous section, there exist also other possible arrangements of approximations which we shall present without supporting them with a model but state the necessary conditions under which one could work. Firstly, the third equation of motion, Eq. (13) in this framework is more or less acting like an auxiliary equation which generates the scalar potential once the coupling functions and the form of Hubble’s parameter have been specified. Therefore, one may wish to keep that equation as it is, without altering it. Obviously this is acceptable, but we made the approximation $12\xi' H^4 \ll V'$ in order to simplify the expression. In consequence, there exist two possibilities. One can either work with the full expression,

$$V' + 3H^2 \left( \omega \frac{\xi'}{\xi''} - 2\frac{h'}{\kappa^2} + 4\xi' H^2 \right) = 0,$$

or the simplified expressions,

$$V' + 3H^2 \left( \omega \frac{\xi'}{\xi''} - 2\frac{h'}{\kappa^2} \right) = 0,$$

$$V' - 6H^2 \frac{h'}{\kappa^2} = 0,$$

Only the first form was used in this letter. For the sake of consistency, we mention that the second model is in fact viable in both cases. This approximation simplifies the differential equation greatly and the scalar potential can be extracted easier. Furthermore, as it was demonstrated in the previous models, it is a decent assumption since compared to the rest terms, $12\xi' H^4$ is in fact negligible.

Concerning Hubble’s Parameter in Eq. (11), the relation can be simplified in three ways,

$$H^2 \simeq \frac{\kappa^2 V}{3h},$$

$$H^2 \simeq \frac{\kappa^2 V \xi''}{3h \xi' \xi''},$$

$$H^2 \simeq \frac{\kappa^2 V}{3h \left( 1 + \frac{\omega \xi'}{\kappa \xi''} \right)}.$$
The first approach is similar to the second one but is also simplified and therefore leads to a better solution for the scalar potential. There is no point in adding the kinetic term since it is many orders smaller, than the potential in the slow-roll approximation. Furthermore, the term $\xi'\dot{\phi}H^4$ is also negligible compared to the other terms. The dominant contribution comes of course from the scalar potential and the squared Hubble rate. For the sake of consistency however, we mention that since the slow-roll conditions are assumed to hold true, then $\epsilon_1 \ll 1$ and as a result Eq. (89) is invalid in this regime. Indeed, in the trigonometric choice it was shown that even though compatible with the observations results where produced, the model was intrinsically unrealistic. This is an expected feature under the slow-roll assumptions.

On the other hand, since Hubble’s derivative is written proportionally to Hubble’s parameter squared, the slow-roll index $\epsilon_1$ can be extracted directly from the form of Eq. (12), so the inflationary phenomenology strongly depends on the approximations made in this particular equation. The possible approximations are shown below,

$$\dot{H} \sim \frac{H^2}{2} \frac{h'}{h} \frac{\xi'}{\xi''}, \quad (91)$$

$$\dot{H} \sim -\frac{H^2}{2} \frac{h''}{h} \left(\frac{\xi'}{\xi''}\right)^2, \quad (92)$$

$$\dot{H} \sim -H^2 \frac{\kappa^2 \omega}{2h} \left(\frac{\xi'}{\xi''}\right)^2, \quad (93)$$

$$\dot{H} \sim \frac{H^2}{2} \frac{\xi'}{\xi''} \left(\frac{h'}{h} - \frac{h''}{h} \frac{\xi'}{\xi''}\right), \quad (94)$$

$$\dot{H} \sim \frac{H^2}{2} \frac{\xi'}{\xi''} \left(\frac{h'}{h} - \frac{\kappa^2 \omega}{h} \frac{\xi'}{\xi''}\right), \quad (95)$$

$$\dot{H} \sim -\frac{H^2}{2} \left(\frac{\kappa^2 \omega}{h} + \frac{h''}{h}\right) \left(\frac{\xi'}{\xi''}\right)^2. \quad (96)$$

Adding more terms than these in the equations above, renders the system extremely difficult to solve, if not unsolvable. The term $8\xi H \dot{H}$ was neglected in most approximations, along with the term $\dot{\phi}h'$. The latter is easily justified from the slow-roll approximation, since $\dot{\phi} \ll H \phi$. Consequently, the term $\dot{\phi}h'$ may be neglected in every approximation, whether the term $Hh'\dot{\phi}$ participates in the equation of motion or is also neglected. However they could be used together, for instance if the constant-roll condition is used. This case however would also change the form of $\dot{\phi}$ derived from the constraint of the velocity of gravitational waves, so this is a topic of another study. The first is neglected since it does not contain the coupling function $b(\phi)$, so it will lead to difficulties. The same argument may be used for the kinetic term which participates in the last two cases. However, these cases could lead to a simple expression, perhaps a polynomial, as was the case with the first model of the previous section, so it was worth mentioning them as well. In each case, the expression of Hubble’s derivative is greatly simplified if the coupling functions are chosen in an elegant way so that the ratios of their derivatives are functionally simple. That was the reason behind choosing simple functions such as power-law and exponential coupling functions. Furthermore, it can easily be inferred that equations (92) and (96) are incompatible with the slow-roll conditions therefore they cannot be implemented in the present framework. Therefore the overall approximated forms of $\dot{H}$ are 4. We shall return to this statement in the following.

The choice of Hubble’s parameter does not alter the first three slow-roll indices, but only affects the scalar potential derived from Eq. (21) and the value of the free parameters of the theory which lead to phenomenologically viable results, as it was demonstrated in the second model of the previous section. Hence, it would be legitimate to work with a more inclusive equation, such as Eq. (91), even if the second term of the denominator is negligible, so long it leads to a manageable scalar potential and not to physical inconsistencies. For instance we mention that in the third model presented in the previous section, the choice of a more inclusive Hubble parameter led to the appearance of complex number, both in the observed quantities but also in other quantities, such as the scalar potential and consequently the Hubble rate itself. Moreover, these complex numbers could not disappear with the choice of a better selection of parameters or a different fine tuning. However, when switched to Eq. (88), the complex numbers disappeared.
Despite not being a viable model at all, the third model of the previous section is indicative of how intricate the system of the gravitational equations of motion really is.

Let us continue with our study and refer to the conditions under which the model is rendered viable in terms of the approach which was chosen from the previous possible cases. Firstly, each equation in this paper was derived by assuming the slow-roll conditions hold true. This in turn implies that no matter the choice of coupling functions or the extra approximations on the equations of motion, one must ascertain whether the following conditions are true during the first horizon crossing, or in other words the initial moment of inflation.

\[
\dot{H} \ll H^2 \quad \frac{1}{2} \omega \dot{\phi}^2 \ll V \quad \ddot{\phi} \ll H \dot{\phi}.
\]

These approximations are essential and must hold true, otherwise the whole approach would be rendered invalid. In addition to these approximations, one must check whether the approximation made in order to derive the differential equation (21) and (88) apply as well. These approximations are,

\[
12\zeta' H^4 \ll V' \quad 12\zeta H^3 \ll V \quad 3H \dot{h} \ll \kappa^2 V \quad h \ll h' \quad \frac{\xi'}{\xi''} \ll \frac{h'}{\kappa^2}.
\]

Note that the third relation may be violated only if Eq. (80) is used whereas the fourth refers only to the approach of Eq. (89) which as mentioned previously is at variance with the slow-roll conditions. Also, the last refers only to Eq. (93). Finally, we mention the approximations which must be valid in the equation of Hubble’s derivative. For equation (91),

\[
\kappa^2 \omega \dot{\phi}^2 \ll H \dot{h} \quad h'' \dot{\phi}^2 \ll H \dot{h} \quad 8\kappa^2 \xi H \dot{h} \ll H \dot{h}.
\]

Concerning Eq. (92),

\[
\kappa^2 \ll h'' \quad H \dot{h} \ll h'' \dot{\phi}^2 \quad 8\kappa^2 \xi H \dot{h} \ll h'' \dot{\phi}^2.
\]

Similarly, using Eq. (93) leads to the following approximations,

\[
h'' \ll \kappa^2 \quad H \dot{h} \ll \kappa^2 \omega \dot{\phi}^2 \quad 8\xi H \dot{h} \ll \omega \dot{\phi}^2.
\]

The rest approaches contain more terms and therefore lesser approximations must be implemented. For instance, in order to use Eq. (94), the following approximations must be valid,

\[
8\kappa^2 \xi H \dot{h} + \kappa^2 \omega \dot{\phi}^2 \ll H \dot{h} + h'' \dot{\phi}^2,
\]

and similarly, for Eq. (95),

\[
8\kappa^2 \xi H \dot{h} + h'' \dot{\phi}^2 \ll H \dot{h} + \kappa^2 \omega \dot{\phi}^2.
\]

Finally, the necessary extra condition under which Eq. (96) would be valid is,

\[
H \dot{h} + 8\kappa^2 \xi H \dot{h} \ll (\kappa^2 \omega + h'')\dot{\phi}^2.
\]

The last three approximations were written in this form for convenience, but we note that a single expression on the right hand side must be compared to each term of the left hand side separately. Furthermore, each approximation made for either model refers to the absolute value of each term. As mentioned before, the approximated forms of equations (72) and (79) cannot be used under the slow-roll assumption since they violate the expression \(\dot{h} \ll H \dot{h}\).

No matter the choice of equations, if a single approximation is invalid then the whole model will be wrong, even if the results happen to be compatible with the latest observations, as was the case with the third model the previous section.

V. CONCLUSIONS

In this paper, we presented a new approach on non-minimally coupled theories of gravity which contain string corrections, by imposing the condition that the gravitational wave speed is equal to unity. We demonstrated that when constraints on the velocity of the gravitational waves are imposed, quantities with different origins in the action, become interconnected. Specifically, the scalar potential is not freely chosen but is derived from a differential
The choice of approximations may vary as there exist a lot possible configurations of Hubble’s parameter $H^2$ and its derivatives $\dot{H}$, so this approach is extremely model dependent. As a result, the equations which are derived are fully solvable in an analytic way, and in certain cases they are also elegant. Moreover, many models which could seem to be able to manifest compatible results with the recent observations may be not valid models in fact, due to the violation of even a single approximation, so one must be very careful when working with any model and following either approach. Finally, we presented the different possible approaches and considered the corresponding approximations which must apply in order for the model to be rendered viable and easy to solve analytically. We should note that our formalism can easily be applied for the cases that the scalar potential is absent, so this would result to a constraint between the Einstein-Gauss-Bonnet scalar coupling function $\xi(\phi)$ and the function $h(\phi)$. We aim to address this interesting subclass of theories in a future work.

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