Bianchi VI\(_0\)&III models: self-similar approach

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Abstract

We study several cosmological models with Bianchi VI\(_0\)&III symmetries under the self-similar approach. We find new solutions for the ‘classical’ perfect fluid model as well as for the vacuum model although they are really restrictive for the equation of state. We also study a perfect fluid model with time-varying constants, \(G\) and \(\Lambda_1\). As in other studied models we find that the behaviour of \(G\) and \(\Lambda\) are related. If \(G\) behaves as a growing time function then \(\Lambda\) is a positive decreasing time function but if \(G\) is decreasing then \(\Lambda_0\) is negative. We end by studying a massive cosmic string model, putting special emphasis in calculating the numerical values of the equations of state. We show that there is no SS solution for a string model with time-varying constants.

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1. Introduction

The current observations of the large-scale cosmic microwave background suggest to us that our physical universe is expanding isotropic and homogeneous models with a positive cosmological constant. The analysis of CMB fluctuations may confirm this picture. But other analyses reveal some inconsistency. Analysis of WMAP data sets shows us that the universe could have a preferred direction. For this reason Bianchi models are important in the study of anisotropies.

The study of SS models is quite important since a large class of orthogonal spatially homogeneous models are asymptotically self-similar at the initial singularity and are approximated by exact perfect fluid or vacuum self-similar power law models. Exact self-similar power-law models can also approximate general Bianchi models at intermediate stages of their evolution. This last point is of particular importance in relating Bianchi models to the real universe. At the same time, self-similar solutions can describe the behaviour of Bianchi models at late times, i.e. as \(t \to \infty\) (see [1]). A particular interest is in determining the exact value of the equations of state (see [2]). The geometry and physics at different points on an
The integral curve of a homothetic vector field (HVF) differ only by a change in the overall length scale and in particular any dimensionless scalar will be constant along the integral curves. In this sense, the existence of a HVF is a weaker condition than the existence of a KVF since the geometry and physics are completely unchanged along the integral curves of a Killing vector field (KVF). However, the existence of a non-trivial HVF leads to restrictions on the equations of state. We shall put special emphasis in this fact.

In modern cosmological theories, the cosmological constant remains a focal point of interest (see [3–6] for reviews of the problem). A wide range of observations now compellingly suggest that the universe possesses a non-zero cosmological constant. Some of the recent discussions on the cosmological constant ‘problem’ and on cosmology with a time-varying cosmological constant point out that in the absence of any interaction with matter or radiation, the cosmological constant remains a ‘constant’. However, in the presence of interactions with matter or radiation, a solution of Einstein equations and the assumed equation of covariant conservation of stress–energy with a time-varying $\Lambda$ can be found. This entails that energy has to be conserved by a decrease in the energy density of the vacuum component followed by a corresponding increase in the energy density of matter or radiation. Recent observations strongly favour a significant and a positive value of $\Lambda$ with magnitude $\Lambda(G\bar{h}/c^3) \approx 10^{-123}$. These observations suggest on accelerating expansion of the universe, $q < 0$ (see [7–10]).

Therefore, the paper is organized as follows. In section 2, we outline the metrics as well as the main geometrical ingredients of the models as for example all the curvature invariants. In section 3, we start by reviewing some basic notions on self-similarity (SS). We calculate the HVF for the metric and obtain the restrictions for the scale factors. Section 4 is devoted to revise the ‘classical’ solution for a perfect fluid ($\Lambda = 0$). Nevertheless, we find a new solution in each of the studied cases, i.e. Bianchi types III and VI. In section 5, we shall study the vacuum solution, as in the above section we obtain solutions for Bianchi types III and VI. Once we have finished with the classical solutions we go next to study SS solutions in different contexts. For this purpose, in section 6, we study a perfect fluid model with time-varying constants, $G$ and $\Lambda$. As we have mentioned above we shall pay special attention in calculating the numerical value of the equation of state. In section 7, we study a massive cosmic string model, while in section 8, we study its generalization i.e. by considering a model which allows time-varying constants. In the last section, we end by summarizing the main results.

2. The metrics and the geometric ingredients

Throughout the paper $M$ will denote the usual smooth (connected, Hausdorff, four-dimensional) spacetime manifold with smooth Lorentz metric $g$ of signature $(-, +, +, +)$ (see for example [11]). Thus $M$ is paracompact. A comma, semi-colon and the symbol $\mathcal{L}$ denote the usual partial, covariant and Lie derivative, respectively, the covariant derivative being with respect to the Levi-Civita connection on $M$ derived from $g$. The associated Ricci and stress–energy tensors will be denoted in component form by $R_{ij}$ ($=R^c_{jcid}$) and $T_{ij}$, respectively. We shall use a system of units where $c = 1$.

A Bianchi $\text{VI}_0$ spacetime is a spatially homogeneous spacetime which admits a group of isometries $G_3$, acting on spacelike hypersurfaces, generated by the spacelike KVs,

$$
\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = mx\partial_x - ny\partial_y + \partial_z,
$$

and therefore $C^1_1 = m, C^2_2 = -n$. In synchronous co-ordinates, the metric is

$$
\text{d}s^2 = -\text{d}t^2 + a^2(t) e^{-2mz} \text{d}x^2 + b^2(t) e^{2nz} \text{d}y^2 + c^2(t) \text{d}z^2,
$$
where the metric functions $a(t), b(t), d(t)$ are functions of the time co-ordinate only. We would like to emphasize that there are other ways to define a Bianchi VI$_0$ metric (see for example [11]), in this case, we have followed maybe the simplest one.

As it is observed, the metric (2) collapses to the following cases.

(i) If $m = -n$, then the metric collapses to Bianchi type V,

$$ds^2 = -dt^2 + a^2(t) e^{-2mc} dx^2 + b^2(t) e^{-2mc} dy^2 + d^2(t) dz^2,$$

with

$$\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = m x \partial_x + m y \partial_y + \partial_z,$$

and therefore $C_{13}^1 = m = C_{23}^2$.

(ii) If $n = 0$, then the metric collapses to Bianchi type III,

$$ds^2 = -dt^2 + a^2(t) e^{-2mc} dx^2 + b^2(t) dy^2 + d^2(t) dz^2,$$

with the following KVF (Killings vector fields):

$$\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = m x \partial_x + \partial_z,$$

and therefore $C_{13}^1 = m$.

(iii) If $m = n = 0$, then the model collapses to Bianchi I model,

$$ds^2 = -dt^2 + a^2(t) dx^2 + b^2(t) dy^2 + d^2(t) dz^2,$$

with the following KVF (Killings vector fields).

$$\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = \partial_z,$$

and therefore $C_{ij}^k = 0$.

We may define the 4-velocity as follows:

$$u^i = (1, 0, 0, 0),$$

in such a way that it is verified, $g(u^i, u^i) = -1$.

From the definition of the 4-velocity we find that

$$H = \left( \frac{a'}{a} + \frac{b'}{b} + \frac{d'}{d} \right) = \sum_i H_i,$$

$$q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1,$$

$$\sigma^2 = \frac{1}{3} \left( \sum_i H_i^2 - \sum_{i \neq j} H_i H_j \right).$$

We shall take into account the Einstein’s field equations (FE) written in the following form:

$$R_{ij} - \frac{1}{2} R g_{ij} = 8 \pi G T_{ij} - \Lambda g_{ij},$$

where $T_{ij}$ is the energy–momentum tensor.

We also study the curvature behaviour of the different solutions (see for example [12–15]). The studied curvature quantities are the following ones: Ricc scalar, $I_0 = R_{ij}^i$, Krestchmann scalar, $I_1 := R_{ijkl} R^{ijkl}$, the full contraction of the Ricci tensor, $I_2 := R_{ij} R^{ij}$; the non-zero components of the Weyl tensor; the Weyl scalar, $I_3 = C^{abcd} C_{abcd} = I_1 - 2I_2 + \frac{1}{3} I_0^2$, as well as the electric scalar $I_4 = E_{ij} E^{ij}$ (see [16]) and the magnetic scalar, $I_5 = H_{ij} H^{ij}$, of the Weyl
tensor. The Weyl parameter (see Lim et al [16]) which is a dimensionless measure of the Weyl curvature tensor
\[ W^2 = \frac{W^2}{H^4} = \frac{1}{6H^4}(E_{ij}E^{ij} + H_{ij}H^{ij}) = \frac{I_3}{24H^4}. \]  
(12)

\( W \) can be regarded as describing the intrinsic anisotropy in the gravitational field. And to end, we shall calculate the gravitational entropy. From a thermodynamic point of view there is every indication that the entropy of the universe is increasing. Increasing gravitational entropy would naturally be reflected by increasing local anisotropy, and the Weyl tensor reflects this. One suggestion in this connection was Penrose’s formulation of what is called the Weyl curvature conjecture (WCC) [17]. The hypothesis is motivated by the need for a low entropy constraint on the initial state of the universe when the matter content was in thermal equilibrium. Penrose had argued that the low entropy constraint follows from the existence of the second law of thermodynamics, and that the low entropy in the gravitational field is tied to constraints on the Weyl curvature. Wainwright and Anderson [18] expressed this conjecture in terms of the ratio of the Weyl and the Ricci curvature invariants,
\[ P^2 = \frac{I_3}{I_2}. \]  
(13)

The physical content of the conjecture is that the initial state of the universe is homogeneous and isotropic. As pointed out by Rothman and Anninos [19, 20] the entities \( P^2 \) and \( I_3 \) are ‘local’ entities in contrast to what we usually think of entropy. Grøn and Hervik [13, 14] have introduced a non-local entity which shows a more promising behaviour concerning the WCC. This entity is also constructed in terms of the Weyl tensor, and it has therefore a direct connection with the Weyl curvature tensor but in a ‘non-local form’.

For SS spacetimes, Pelavas and Lake [22] have pointed out the idea that equation (13) is not an acceptable candidate for gravitational entropy along the homothetic trajectories of any self-similar spacetime. Nor indeed is any ‘dimensionless’ scalar. It is showed that the Lie derivative of any ‘dimensionless’ scalar along a homothetic vector field (HVF) is zero, and concluded that such functions are not acceptable candidates for the gravitational entropy. Nevertheless [23], since self-similar spacetimes represent asymptotic equilibrium states (since they describe the asymptotic properties of more general models), and the result \( P^2 = \text{const} \) is perhaps consistent with this interpretation since the entropy does not change in these equilibrium models, and perhaps consequently supports the idea that \( P^2 \) represents a ‘gravitational entropy’. As we shall show \( \mathcal{W}^2 \) and \( P^2 \) will be constant along homothetic trajectories, since all the dimensionless quantities remain constant along timelike homothetic trajectories.

3. The self-similar solution

In general relativity, the term self-similarity can be used in two ways. One is for the properties of spacetimes, the other is for the properties of matter fields. These are not equivalent in general. The self-similarity in general relativity was defined for the first time by Cahill and Taub [24], and Eardley [25] (see for general reviews [26–28]). Self-similarity is defined by the existence of a homothetic vector \( V \) in the spacetime, which satisfies
\[ \mathcal{L}_V g_{ij} = 2\alpha g_{ij}, \]  
(14)

where \( g_{ij} \) is the metric tensor, \( \mathcal{L}_V \) denotes Lie differentiation along \( V \) and \( \alpha \) is a constant. This is a special type of conformal Killing vectors. This self-similarity is called homothety. If \( \alpha \neq 0 \), then it can be set to be unity by a constant rescaling of \( V \). If \( \alpha = 0 \), i.e. \( \mathcal{L}_V g_{ij} = 0 \), then \( V \) is a Killing vector.
Homothety is a purely geometric property of spacetime so that the physical quantity does not necessarily exhibit self-similarity such as $\mathcal{L}_V Z = kZ$, where $k$ is a constant and $Z$ is, for example, the pressure, the energy density and so on. From equation (14) it follows that $\mathcal{L}_V R^i_{jkl} = 0$, and hence $\mathcal{L}_V R_{ij} = 0$, and $\mathcal{L}_V G_{ij} = 0$. A vector field $V$ that satisfies the above equations is called a curvature collineation, a Ricci collineation and a matter collineation, respectively. It is noted that such equations do not necessarily mean that $V$ is a homothetic vector. We consider the Einstein equations $G_{ij} = 8\pi G T_{ij}$, where $T_{ij}$ is the energy–momentum tensor. If the spacetime is homothetic, the energy–momentum tensor of the matter fields must satisfy $\mathcal{L}_V T_{ij} = 0$. For a perfect fluid case, the energy–momentum tensor takes the form $T_{ij} = (p + \rho)u_i u_j + pg_{ij}$, where $p$ and $\rho$ are the pressure and the energy density, respectively. Then, equations (14) result in

$$\mathcal{L}_V u^i = -\alpha u^i, \quad \mathcal{L}_V \rho = -2\alpha \rho, \quad \mathcal{L}_V p = -2\alpha p.$$  

(15)

As shown above, for a perfect fluid, the self-similarity of the spacetime and that of the physical quantity coincide. However, this fact does not necessarily hold for more general matter fields. Thus the self-similar variables can be determined from dimensional considerations in the case of homothety. Therefore, we can conclude homothety as the general relativistic analogue of complete similarity. From constraints (15), we can show that if we consider the barotropic equation of state, i.e., $p = f(\rho)$, then the equation of state must have the form $p = \omega \rho$, where $\omega$ is a constant. In this paper, we would like to try to show how taking into account this class of hypothesis one is able to find exact solutions to the field equations within the framework of the time varying constants.

From equation (14), we find the following homothetic vector field for the BVI$_0$ metric (2):

$$V = (t + t_0)\partial_t + \left(1 - (t + t_0)\frac{d^t}{a}\right) x\partial_x + \left(1 - (t + t_0)\frac{b'}{b}\right) y\partial_y + \left(1 - (t + t_0)\frac{d'}{d}\right) z\partial_z,$$  

(16)

where $t_0 \in \mathbb{R}$ is a constant, and with the following constrains for the scale factors:

$$a(t) = a_0(t + t_0)^{a_1}, \quad b(t) = b_0(t + t_0)^{b_1}, \quad d'(t) = d_0(t + t_0),$$  

(17)

where $a_1, b_1 \in \mathbb{R}$ and we set $d_0 = 1$, therefore the resulting homothetic vector field is

$$V = (t + t_0)\partial_t + (1 - a_1)x\partial_x + (1 - a_2)y\partial_y,$$  

(18)

so the metric (2) collapses to the following one:

$$ds^2 = -dt^2 + a^2(t) e^{-2mz} dx^2 + b^2(t) e^{2nz} dy^2 + (t + t_0)^2 dz^2.$$  

(19)

4. Perfect fluid solutions

The energy–momentum tensor for a perfect fluid model, $T_{ij}$, is defined as follows:

$$T_{ij} = (\rho + p)u_i u_j + pg_{ij},$$  

(20)

with the usual equation of state $p = \omega \rho$, $\omega \in \mathbb{R}$. The resulting FE for the metric (2) with a perfect fluid model yield (with $\Lambda = 0$):

$$\frac{a'}{a} + \frac{a'}{a} + \frac{b'}{b} - \frac{K}{d^2} = 8\pi G \rho,$$  

(21)

$$m \left(\frac{d'}{a} - \frac{d'}{d}\right) + n \left(\frac{d'}{d} - \frac{b'}{b}\right) = 0,$$  

(22)
the following algebraic system of equations:

\[
\begin{align*}
\frac{d^\prime}{b} + \frac{d^\prime}{d} + \frac{d^\prime}{b^2} - \frac{n^2}{d^2} &= -8\pi \text{G} \rho, \\
\frac{d^\prime}{d} + \frac{a^\prime}{a} + \frac{a^\prime}{d} - \frac{m^2}{d^2} &= -8\pi \text{G} \rho, \\
\frac{b^\prime}{a} + \frac{a^\prime}{a} + \frac{a^\prime}{b} + \frac{mn}{d^2} &= -8\pi \text{G} \rho, \\
\rho^\prime + \rho(1 + \omega) \left( \frac{a^\prime}{a} + \frac{b^\prime}{b} + \frac{d^\prime}{d} \right) &= 0, 
\end{align*}
\]

where \( K = (mn - m^2 - n^2) \).

Therefore, taking into account the restrictions for the scale factors given by equation (17), the solution for a perfect fluid model, equations (21)–(26), is the following one.

Form (26) we get

\[ \rho = \rho_0(t + t_0)^{-\left(1 + \omega)(a_1 + a_2 + 1)\right)} = \rho_0(t + t_0)^{-\gamma}, \]

with \((1 + \omega)(a_1 + a_2 + 1) = (1 + \omega)\alpha = \gamma\). Now, taking into account equation (21) we get \( a_1, a_2 + a_1 + a_2 + K = 8\pi \text{G}\rho_0 \) and \((1 + \omega)(a_1 + a_2 + 1) = 2 \), and hence (21)–(26) collapse to the following algebraic system of equations:

\[
\begin{align*}
m(a_1 - 1) + n(1 - a_2) &= 0, \\
a_2(a_2 - 1) + a_2 - n^2 &= -\Lambda \omega, \\
a_1(a_1 - 1) + a_1 - m^2 &= -\Lambda \omega, \\
a_2(a_2 - 1) + a_1(a_1 - 1) + a_1a_2 + mn &= -\Lambda \omega, \\
(1 + \omega)(a_1 + a_2 + 1) &= 2,
\end{align*}
\]

with

\[ \rho_0 = \frac{A}{8\pi \text{G}}, \quad A = a_1a_2 + a_1 + a_2 + K, \]

obtaining the following set of solutions.

4.1 Bianchi type III

Our first solution is the following one:

\[ a_1 = 1, \quad a_2 = \frac{-2\omega}{\omega + 1}, \quad m = \frac{\sqrt{-3\omega^2 + 2\omega + 1}}{\omega + 1}, \quad n = 0, \]

note that the solution only has sense if \( \omega \in (-\frac{1}{3}, 1) \), and therefore we obtain the following values: \( a_1 = 1, a_2 \in (0, 1) \), if \( \omega < 0 \), \( m \in (0, 1) \) and \((1 + \omega)(2 + a_2) = \gamma = 2 \), so \( \rho = \rho_0 t^{-2} \), and therefore the metric collapses to the following one:

\[ ds^2 = -dt^2 + (t + t_0)^2 e^{-2mz} dx^2 + (t + t_0)^2 dy^2 + (t + t_0)^2 dz^2, \]

which stands for a Bianchi III metric. Note that (33) admits the KVF given by equation (6). This solution looks really restrictive since it only has physical meaning if \( \omega < 0 \). To the best of our knowledge this solution is new.

For this solution, we have obtained the following behaviour for the main curvature quantities. We start by calculating the curvature invariants, \((l_i^2)_{i=0}^{\infty}\), they yield \( I_0 = 2 \left(a_1^2 + a_2 - m^2 + 1\right)(t + t_0)^{-2} \), \( I_1 = 4 \left(a_1^4 - 2a_1^2 + 3a_2^2 + m^4 - 2m^2 + 1\right)(t + t_0)^{-4} \) and
\[ I_2 = 2 \left( \alpha_1^4 + 2\alpha_3^2 + 2\alpha_2 + m^4 - 2m^2 + 1 - 2m^2\alpha_2 \right) (t + t_0)^{-4}. \]

As it is observed we have obtained a non-singular solution. The non-zero components of the Weyl tensor are as follows: \( C_{txx} = -(1/6)K e^{-2mz}, C_{txy} = (1/3)K(t + t_0)^{2(\alpha_1 - 1)}, C_{txz} = -(1/6)K e^{-2mz}(t + t_0)^2, C_{xzx} = -(1/3)K e^{-2mz}(t + t_0)^3 \) and \( C_{xyy} = (1/6)K(t + t_0)^{2\alpha_1} \), where \( K = (m - 1 + \alpha_2)(m + 1 - \alpha_2) \). Therefore the Weyl invariant yields \( I_3 = (4/3)K^2(t + t_0)^{-4} \). The electric invariant yields \( I_4 = (1/6)K^2(t + t_0)^{-2} \), while the magnetic invariant, \( I_5 \), vanishes. The Weyl parameter yields, \( \mathcal{W}^2 = K^2/36(2 + \alpha_2)^4 \) = const, while the gravitational entropy yields \( P^2 = \) const, as it is expected.

### 4.2. Bianchi type VI\(_0\), solution I

The ‘modified’ solution given by Collins [29] and Hsu et al (see [32]) is

\[
a_1 = a_2 = \frac{1 - \omega}{2(\omega + 1)}, \quad m = n = \frac{\sqrt{-3\omega^2 + 2\omega + 1}}{2(\omega + 1)},
\]

with \( \omega \in (-1, 1) \), and therefore the metric collapses to the following form:

\[
dx^2 = -dt^2 + (t + t_0)^{2\alpha_1} e^{-2mz} dx^2 + (t + t_0)^{2\alpha_1} e^{2mz} dy^2 + (t + t_0)^2 dz^2,
\]

finding in this way that, \( \alpha_2 \in (0, 1) \), \( m \in (0, 1/2) \) and \( (1 + \omega)(1 + 2\alpha_2) = \gamma = 2 \), so \( \rho = \rho_0(t + t_0)^{-2} \). Hence, we find that

\[
H = \frac{1 + 2a_2}{(t + t_0)}, \quad q = \frac{-2a_2}{1 + 2a_2} < 0, \quad \sigma^2 = \frac{(a_2 - 1)^2}{3(t + t_0)^2}.
\]

As it is observed we have obtained a solution lightly different with respect to the obtained one by Collins and Hsu et al, since in this case, our scale factors behaves as \( a(t) = a_0(t + t_0)^{\alpha_1}, b(t) = b_0(t + t_0)^{\alpha_2} \) and \( d(t) = d_0(t + t_0), \) i.e., \( t_0 \neq 0 \).

For this solution we have obtained the following behaviour for the main curvature quantities. We start by calculating the curvature invariants, \( (I_i^j)_{i=0} \), they yield \( I_0 = 2 \left( 3\alpha_1^4 - m^2 \right) (t + t_0)^{-2}, I_1 = 4 \left( 3\alpha_1^4 - 4\alpha_3^2 + 4\alpha_2^2 - 2\alpha_1^2m^2 + 3m^4 + 4m^2\alpha_1 - 4m^2 \right) (t + t_0)^{-4} \) and \( I_2 = 2 \left( 3\alpha_1^4 - 2\alpha_3^2 + 2\alpha_2^2 + m^4 - 2m^2\alpha_2 \right) (t + t_0)^{-4} \). As it is observed we have obtained a non-singular solution. The non-zero components of the Weyl tensor are \( C_{txx} = (1/3)m^2(t + t_0)^{2(\alpha_1 - 1)} e^{-2mz}, C_{txy} = m(1 - \alpha_1)(t + t_0)^{3\alpha_1 - 1} e^{-2mz}, C_{txz} = (1/3)m^2(t + t_0)^{3\alpha_1 - 1} e^{-2mz}, C_{xyy} = (2/3)m^2(t + t_0)^{3\alpha_1 - 1}, C_{xzx} = -(1/3)(t + t_0)^{3\alpha_1 - 1} e^{-2mz} \) and \( C_{xzy} = (1/3)(t + t_0)^{3\alpha_1 - 1} e^{2mz} \). Therefore the Weyl invariant yields \( I_3 = (1/3)(-3\alpha_1^4 + 6\alpha_3 - 3 + m^2) (t + t_0)^{-4} \). The electric invariant yields \( I_4 = (2/3)m^2(t + t_0)^{-4} \), while the magnetic invariant, \( I_5 \), behaves as \( I_5 = 2m^2(\alpha_1 - 1)^2(t + t_0)^{-4} \). The Weyl parameter yields \( \mathcal{W}^2 = (m^2 \left( 3\alpha_1^4 - 6\alpha_3 + 3 + m^2 \right) )/(9(1 + 2\alpha_1)^4 < \) const, while the gravitational entropy yields \( P^2 = < \) const, as it is expected.

### 4.3. Bianchi type VI\(_0\), solution II

We find a new solution (to the best of our knowledge this solution is new), where

\[
a_1 = \frac{1 - a_2(\omega + 1) - \omega}{\omega + 1}, \quad a_2 = a_2, \quad n = \frac{b(\alpha_2(\omega + 1) + 2\omega)}{(\omega + 1)}, \quad m = b(1 - a_2),
\]

with \( b = \sqrt{\frac{1 - \omega}{2(\omega + 1)}} \), and at a first look we may say that \( \omega \in (-1, 1) \), and therefore \( b \in [0, \infty) \).

The metric collapses to the following form:

\[
dx^2 = -dt^2 + (t + t_0)^{2\alpha_1} e^{-2mz} dx^2 + (t + t_0)^{2\alpha_1} e^{2mz} dy^2 + (t + t_0)^2 dz^2,
\]

\[
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finding in this way that, $a_1 = a_1(a_2, \omega)$, so depending on the different values of $\omega$ we get

| $\omega$ | $a_1$ | $a_2$ | $n$ | $m$ |
|----------|-------|-------|-----|-----|
| 1        | $-a_2$ | $a_2$ | 0   | 0   |
| $1/3$    | $\frac{1}{3} - a_2$ | $(0, 1/2)$ | $\frac{\sqrt{3}}{3}(2a_2 + 1)$ | $\frac{\sqrt{3}}{3}(1 - a_2)$ |
| 0        | $1 - a_2$ | $(0, 1)$ | $a_2$ | $1 - a_2$ |
| $(-1/3)^+$ | $2 - a_2$ | $(0, 2)$ | $n$  | $m$  |

Note that the case $\omega = 1$ looks nonphysical since the scale factor $a_1 < 0$, for this reason we shall not take into account this particular case. Therefore, the solution is only valid if $\omega \in (-\frac{1}{3}, 1)$ and hence we get $(1 + \omega)(1 + a_1 + a_2) = \gamma = 2$, so $\rho = \rho_0(t + t_0)^{-2}$. Hence, we find that

$$H = \frac{1 + a_1 + a_2}{t}, \quad q = -\frac{(a_1 + a_2)}{1 + a_1 + a_2} < 0$$

and

$$\sigma^2 = \frac{2}{3t^2} \left(a_1^2 + a_2^2 + 1 - a_1a_2 - a_1 - a_2\right).$$

For simplicity we calculate the curvature behaviour for the case $\omega = 0$. We start by calculating the curvature invariants. $(I_i)_{i=0}^2$, they yield $I_0 = 4a_2(a_2 - 1)(t + t_0)^{-2}$, $I_1 = 16a_2^2(a_2 - 1)^2(t + t_0)^{-4} = I_2$. As it is observed we have obtained a non-singular solution. The non-zero components of the Weyl tensor are $C_{111} \sim (t + t_0)^{-2a_2}$, $C_{122} \sim (t + t_0)^{-1 - 2a_2}$, $C_{133} \sim (t + t_0)^{2(a_2 - 1)}$, $C_{222} \sim (t + t_0)^{2a_2 - 3}$, $C_{122} \sim (t + t_0)^{2(a_2 - 1)}$ and $C_{333} \sim (t + t_0)^{2a_2 - 1}$. Therefore the Weyl invariant yields $I_3 = 32a_2^2(a_2 - 1)^2/(3(t + t_0)^3)$. The electric invariant yields $I_4 = 2a_2^2(a_2 - 1)^2/3(t + t_0)^3$, while the magnetic invariant, $I_5$, behaves as $I_5 = 2a_2^2(a_2 - 1)^2(t + t_0)^{-3}$. The Weyl parameter yields $V^2 = a_2^2(a_2 - 1)^2/36 = const$, while the gravitational entropy yields $P^2 = 2/3$, as it is expected.

5. SS vacuum solution

In this case, the algebraic system to solve is the following one:

$$a_1a_2 + a_1 + a_2 + (mn - m^2 - n^2) = 0, \quad (38)$$
$$m(a_1 - 1) + n(1 - a_2) = 0, \quad (39)$$
$$a_2(a_2 - 1) + a_2 - n^2 = 0, \quad (40)$$
$$a_1(a_1 - 1) + a_1 - m^2 = 0, \quad (41)$$
$$a_2(a_2 - 1) + a_1(a_1 - 1) + a_1a_2 + mn = 0, \quad (42)$$

obtaining the following solutions (as well as the trivial one).

(i) We find our first solution as

$$a_2 = a_2, \quad n = -a_2, \quad a_1 = \frac{1}{2}(1 + \sqrt{-2a_2^2 + 4a_2 + 1}) = m, \quad (43)$$

and where the solution only has math sense if

$$a_2 \in (1 - \frac{1}{2}\sqrt{6}, 1 + \frac{1}{2}\sqrt{6})$$

the negative solution may have sense in the study of singularities. We only consider the positive range i.e. $a_2 \in (0, 2.2247)$, and therefore we get $a_1 \in (0, 1, 35)$. 


Equivalently we find

$$a_2 = a_2, \quad n = a_2, \quad a_1 = \frac{1}{2} \left(1 + \sqrt{-2a_1^2 + 4a_2 + 1}\right) = -m,$$

so the metric collapses to the following form:

$$ds^2 = -dt^2 + (t + t_0)^{2n} e^{2a_2 z} \ dx^2 + (t + t_0)^{2a_2} e^{2a_2 z} \ dy^2 + (t + t_0)^2 \ dz^2,$$

it describes a metric belonging to Bianchi type VIo, i.e. it admits the KVF given by

$$\xi_1 = \partial_x, \quad \xi_2 = \partial_y, \quad \xi_3 = -a_1 x \partial_x - a_2 y \partial_y + \partial_z,$$

and hence $C_{13} = -a_1, C_{23} = -a_2$, and to the best of our knowledge is new.

For this solution we have found a really curious curvature behaviour. The curvature invariants $(I_i)_{i=0}^2$ vanish. The non-zero components of the Weyl tensor are $C_{111} = -(1/2)K e^{2a_1 z}(t + t_0)^{2a_1 - 1}, C_{222} = (1/2)K e^{2a_1 z}(t + t_0)^{2a_1 - 1}, C_{1122} = (1/2)K e^{2a_1 z}(t + t_0)^{2a_1 - 1}$, and $C_{3333} = (1/2) e^{2a_1 z}(t + t_0)^{2a_1}$, where $K = (a_1 - 1 + a_2)(a_1 - a_2)$. Therefore the Weyl invariant yields $I_3 = 0$. The electric invariant yields $I_4 = (1/2)K^2(t + t_0)^{-4}$, while the magnetic invariant, $I_5$, behaves as $I_5 = (1/2)K^2(t + t_0)^{-4}$. The Weyl parameter yields $W^2 = K^2/(6(a_1 + a_2 + 1)^3) = \text{const}$, while the gravitational entropy yields $P^2 = \infty$, as it is expected, since $I_2 = 0$.

(ii) And the set of solutions

$$a_2 = n = 0, \quad a_1 = m = 1,$n

$$a_2 = n = 1, \quad a_1 = m = 0, \quad a_2 = n = 0, \quad a_1 = 1, \quad m = -1,$$

so the metric collapses to the following form (see Hsu et al [32], section 2.6.3 with $t_0 = 0$)

$$ds^2 = -dt^2 + (t + t_0)^2 e^{2z} \ dx^2 + dy^2 + (t + t_0)^2 \ dz^2.$$

This solution describes a Bianchi III metric, admitting three KVF given by equation (6) with $m = -1$. For this solution we have found a really pathological behaviour. All the curvature quantities vanish. So this solution lacks of any physical interest.

### 6. Perfect fluid model with variable constants

As we have pointed out in section 1, models with a dynamic cosmological term $\Lambda(t)$ are becoming popular as they solve the cosmological constant problem in a natural way. There are significant observational evidence for the detection of Einstein’s cosmological constant, $\Lambda$ or a component of material content of the universe that varies slowly with time and space to act like $\Lambda$. Recent cosmological observations by high-Z Supernova Team and Supernova Cosmological Project suggest the existence of a positive cosmological constant. These observations on magnitude and redshift of type Ia supernova suggest that our universe may be an accelerating function of the cosmological density in the form of the cosmological $\Lambda$-term. This motivates us to study the cosmological models in which $\Lambda$ varies with time.

The resulting FE for the metric (2) with a perfect fluid matter model (20) and with $G$ and $\Lambda$ time varying are

$$\frac{d' b'}{a} + \frac{d' d'}{a d} + \frac{d' b'}{d b} + \frac{K}{d^2} = 8\pi G \rho + \Lambda,$$

$$m \left( \frac{d'}{a} - \frac{d'}{d} \right) + n \left( \frac{d'}{d} - \frac{b'}{b} \right) = 0,$$
where

\[ \rho = \rho_0 t^{-\gamma}, \]

From equation (51) we get

\[ \gamma = \omega + 1 \alpha \text{ and } \alpha = (a_1 + a_2 + 1). \]

From equation (46) we obtain

\[ \Lambda(t) = \left[ A t^{-2} - 8 \pi G \rho_0 t^{-(\omega + 1) \alpha} \right], \]

where \( A = a_1 a_2 + a_1 + a_2 + K \) and \( K = (m n - m^2 - n^2) \).

Now, taking into account equations (52) and (54), algebra brings us to obtain

\[ G(t) = G_0 t^{\omega - 2}, \quad G_0 = \frac{A}{4 \pi \rho_0 (\omega + 1) \alpha}. \]

While the cosmological 'constant' behaves as

\[ \Lambda(t) = \frac{A}{c^2} \left( 1 - \frac{2}{\gamma} \right) t^{-2} = \Lambda_0 t^{-2}. \]

With all these results we find that the system to solve is the following one:

\[ m(a_1 - 1) + n(1 - a_2) = 0, \quad (57) \]

\[ a_2(a_2 - 1) + a_2 - n^2 = A, \quad (58) \]

\[ a_1(a_1 - 1) + a_1 - m^2 = A, \quad (59) \]

\[ a_2(a_2 - 1) + a_1 a_2 + a_1(a_1 - 1) + mn = A, \quad (60) \]

where \( A = A \left( \frac{n^2}{\alpha} \right) \) whose solutions are the following ones.

### 6.1. Bianchi type VI_0

As it is observed we obtain a solution that is valid for all equation of state \( \omega \), with

\[ a_2 = a_2, \quad a_1 = \frac{a_2^2 + 1}{a_2 + 1}, \quad m = \frac{a_2^2 + 1 - 2 a_2^3}{2 a_2 + 1}, \quad n = \frac{m a_2}{a_2 + 1}. \]

So this solution only has physical meaning if \( a_2 \in (0, 1) \), and therefore we have \( a_1 \in (0.82, 1), m \in (0, 1), n \in (0, 1) \), so the metric takes the following form:

\[ ds^2 = -dt^2 + (t + t_0)^2(dr^2 - c^{-2} dr^2) + (t + t_0)^2b^2 d\chi^2 + (t + t_0)^2b^2 d\gamma^2 + (t + t_0)^2 d\varphi^2. \quad (61) \]

Therefore, we have the following behaviour for the main quantities: \( \gamma = (\omega + 1) \alpha \in (0.6, 6) \), \( \alpha = (a_1 + a_2 + 1) \in (1.82, 3) \) and \( A = a_1 a_2 + a_1 + a_2 + K > 0 \), therefore, \( \rho = \rho_0 (t + t_0)^{- \gamma} \), \( \gamma \in (0.6, 6) \), is a positive decreasing time function.
G behaves as \( G(t) = G_0(t + t_0)^{\gamma-2} \), so it will be a growing function if \( \gamma > 2 \), constant if \( \gamma = 2 \) and a decreasing time function if \( \gamma < 2 \).

The cosmological constant behaves as \( \Lambda(t) = \Lambda(1 - \frac{2}{3}) (t + t_0)^{-2} \), hence its sign will depend on the value of \( \gamma \). If \( \gamma < 2 \) then we get a negative \( \Lambda \), vanishes if \( \gamma = 2 \), and we find that it is a positive time decreasing function if \( \gamma > 2 \). Note that if \( \gamma > 2 \), then \( G \) is growing and \( \Lambda_0 > 0 \).

The rest of the quantities behaves as
\[
H = \frac{\alpha}{(t + t_0)}, \quad q = \frac{1 - \alpha}{\alpha} < 0, \quad \forall \alpha, \quad \sigma^2 = \frac{(a_1^2 + a_2^2 + 1 - a_1 a_2 - a_1 - a_2)}{3(t + t_0)^2}.
\]
For example, we may see in the following table the behaviour of the main quantities for a particular value of \( a_2 \):

| \( a_2 \) | \( a_1 \) | \( \omega \) | \( \gamma \) | \( G \) | \( \Lambda \) |
|----------|----------|----------|----------|----------|----------|
| 1/2      | 0.833    | 1        | > 2      | \( \nearrow \) | > 0      |
| 1/2      | 0.833    | 1/3      | > 2      | \( \nearrow \) | > 0      |
| 1/2      | 0.833    | 0        | > 2      | \( \nearrow \) | > 0      |
| 1/2      | 0.833    | -1/3     | < 2      | \( \nearrow \) | < 0      |

Choosing other parameters we find another behaviour.

For this solution we have obtained the following behaviour for the main curvature quantities. We start by calculating the curvature invariants, \((t_0)^3\omega_a\), they yield \( I_0 = 2K_0(t + t_0)^{-2} \), \( I_1 = 4K_1(t + t_0)^{-4} \) and \( I_2 = 2K_2(t + t_0)^{-4} \), where \( K_i = K(a_1, a_2, m, n) \) are numerical constants. As it is observed we have obtained a non-singular solution. The non-zero components of the Weyl tensor are \( C_{111x} = K_1(t + t_0)^{2(a_1-1)}e^{-2mc}, C_{122x} = K_2(t + t_0)^{2(a_2-1)}e^{-2mc} \), \( C_{133x} = K_3(t + t_0)^{2(a_1+a_2+1)}e^{-2mc} \), \( C_{144x} = K_4(t + t_0)^{2(a_1+a_2+1)}e^{-2mc} \), \( C_{222x} = K_5, C_{233x} = K_6 \), \( C_{333x} = K_7(t + t_0)^{2(a_1+a_2+1)}e^{-2mc} \), \( C_{344x} = K_8(t + t_0)^{2(a_1+a_2+1)}e^{-2mc} \), \( C_{444x} = K_9(t + t_0)^{2(a_1+a_2+1)}e^{-2mc} \). Therefore the Weyl invariant yields \( I_1 = 2t + t_0)^{-4} \). The electric invariant yields \( I_4 = (1/6)K_4(t + t_0)^{-4} \), while the magnetic invariant, \( I_5 \), behaves as \( I_5 = (1/6)K_5(t + t_0)^{-4} \). The Weyl parameter yields, \( W^2 = \text{const} \) (note that \( W^2 \rightarrow 0 \) is really small, so the model isotropize) while the gravitational entropy yields \( P^2 = \text{const} \), as it is expected.

6.2. Bianchi III solution

This solution is a Bianchi III metric with
\[
a_1 = a_1, \quad a_2 = 1, \quad m = 0, \quad n = \sqrt{1 - a_1^2},
\]
so \( a_1 \in (0, 1) \). In this way, the metric collapses to the following form:
\[
dx^2 = -dr^2 + (t + t_0)^{2m} dx^2 + (t + t_0)^2(e^{2mc} dy^2 + dz^2), \tag{62}
\]
note that if \( a_1 \rightarrow 1 \), then we get
\[
dx^2 = -dr^2 + (t + t_0)^2(dx^2 + dy^2 + dz^2),
\]
this is fact the self-similar solution obtained from a Bianchi V metric, which is only valid for \( \omega = -1/3 \), (in fact, the SS Bianchi V is the FRW solution with \( k = -1 \) and \( \omega = -1/3 \).)

With regard to the main quantities we find that
\[
H = \frac{(a_1 + 2)}{(t + t_0)} = \frac{\alpha}{(t + t_0)}, \quad q = \frac{1 - \alpha}{\alpha}, \quad \sigma^2 = \frac{1}{3} \frac{(a_1 - 1)^2}{(t + t_0)^2}.
\]
Hence we have the following behaviour for the main quantities: \( \gamma = (\omega + 1)\alpha \in (1, 6) \), \( \alpha = (a_1 + 2) \in (2, 3) \) and \( A = a_1a_2 + a_1 + a_2 + K > 0 \), therefore, \( \rho = \rho_0(t + t_0)^{-\gamma} \). As \( \gamma \in (1, 6) \), is a positive decreasing time function, \( \gamma < 2 \iff \omega < 0 \).

\( G \) behaves as \( G(t) = G_0(t + t_0)^{\gamma-2} \), so it will be a growing function if \( \gamma > 2 \), constant if \( \gamma = 2 \) and a decreasing time function if \( \gamma < 2 \).

The cosmological constant behaves as \( \Lambda(t) = \Lambda(1 - \frac{2}{\gamma})^{-2} \), hence its sign will depend on the value of \( \gamma \). If \( \gamma < 2 \) then we get a negative \( \Lambda \), vanishes if \( \gamma = 2 \) and we find a positive time decreasing function if \( \gamma > 2 \).

For this solution we have obtained the following behaviour for the main curvature quantities. We start by calculating the curvature invariants, \( (I_i^j)^{a,b} \), they yield \( I_0 = 2 (a_1^2 + a_1 + 1 - n^2) (t + t_0)^{-2} \), \( I_1 = 4 (3a_1^4 - 2a_1^3 + 2a_1^2 + n^4 - 2n^2 + 1) (t + t_0)^{-3} \) and \( I_2 = 2 (a_1^2 + 2a_1 + 2a_1 + n^4 - 2n^2a_1 - 2n^2 + 1) (t + t_0)^{-4} \). As it is observed we have obtained a non-singular solution. The non-zero components of the Weyl tensor are the following ones: \( C_{ixix} = (1/3)K(t + t_0)^{2(\alpha-1)} \), \( C_{ixiy} = -(1/6)Ke^{2n}, C_{i222} = -(1/6)Ke^{2n}, C_{xyxy} = (1/6)Ke^{2n}(t + t_0)^{2n} \), \( C_{2xy} = (1/6)Ke^{2n}(t + t_0)^{2n} \) and \( C_{2y22} = -(1/3)Ke^{2n}(t + t_0)^2e^{2n} \), where \( K = (a_1 - 1 + a_1)(a_1 - 1 - a_1) \). Therefore the Weyl invariant yields \( I_3 = (4/3)K^2(t + t_0)^{-4} \). The electric invariant yields \( I_4 = (1/6)K^2(t + t_0)^{-4} \), while the magnetic invariant, \( I_5 \), vanishes. The Weyl parameter yields \( W^2 = K^2/36(2 + a_1)^2 \) = const (note that \( W^2 \to 0 \) is really small, so the model isotropize) while the gravitational entropy yields, \( P^2 = \text{const} \), as it is expected.

7. String cosmological model

The relativistic treatment of strings was initiated by Letelier (see [33, 34]) and Stachel (see [35]). For early references see [36–39]. Here we have considered gravitational effects, arise from strings by coupling of stress energy of strings to the gravitational field. Letelier (see [34]) defined the massive strings as the geometric strings (massless) with particles attached along its expansions. Recently, some authors have been working in this area in different contexts (see for example [40–42]).

The energy–momentum tensor, \( T_{ij} \), for a cloud of massive strings is given by

\[
T_{ij} = (\rho + p)u^iu_j + pg_{ij} - \lambda x^i x_j,
\]

where \( \rho(t) \) is the rest energy density of the cloud of strings with particles attached to them (p-strings). \( \lambda(t) \) is the string tension density, which may be positive or negative, \( u^i \) is the 4-velocity for the cloud particles. \( x^i \) is the 4-vector which represents the strings direction which is the direction of anisotropy and \( \rho = \rho_p + \lambda \), where \( \rho_p \) denotes the particle energy density (is the cloud rest energy density), i.e. the string tension density is connected to the rest energy \( \rho \) for a cloud of strings (p-strings) with particle attached to them by this relation (see [43, 44]).

The direction of strings satisfies the standard relations: \( u^i u_i = -x^i x_i = -1, u^i x_i = 0, x^i = (0, 0, 0, d^{-1}(t)) \). In fact, the choice of \( x^i = (0, 0, 0, d^{-1}(t)) \) is the only one possible since if we choose other direction then from the FE we get \( \lambda = 0 \). It is customary to assume a relation between \( \rho \) and \( \lambda \) in accordance with the state equation for strings. The simplest one is a proportionality relation \( \rho = \alpha \lambda \), where the most usual choices of the constant \( \alpha \) are the following ones (see [44]): geometric strings, Nambu, \( \alpha = 1 \), \( \rho = \lambda \), \( \rho_p = 0 \), p-strings or Takabayasi strings

\[
\alpha = 1 + W,
\]

with \( W \geq 0 \), iff \( \rho_p = W \lambda \). Reddy strings \( \alpha = -1 \). A more general ‘barotropic’ equation of state is \( \rho = \rho(\lambda), \rho_p = \rho - \lambda \). In the case of Takabayasi strings if \( W \) is very small, then only
geometric strings appear. On the other hand, if $W$ is infinitely large, then particles dominate over strings.

We shall consider the Takabayasi’s equation of state, i.e. $\rho = \alpha \lambda$, with $\alpha = 1 + W$, so the resulting FE are

\[
\frac{a'}{a} \frac{b'}{b} + \frac{a'}{a} \frac{d'}{d} + \frac{d'}{d} \frac{b'}{b} + \frac{K}{d^2} = 8\pi G \rho, \tag{65}
\]

\[
m \left( \frac{a'}{a} - \frac{d'}{d} \right) + n \left( \frac{d'}{d} - \frac{b'}{b} \right) = 0, \tag{66}
\]

\[
\frac{b''}{b} + \frac{d''}{d} + \frac{d'}{d} \frac{b'}{b} - \frac{n^2}{d^2} = -8\pi G \omega \rho, \tag{67}
\]

\[
\frac{d''}{d} + \frac{a''}{a} + \frac{a'}{a} \frac{d'}{d} - \frac{m^2}{d^2} = -8\pi G \omega \rho, \tag{68}
\]

\[
\frac{b''}{b} + \frac{a''}{a} + \frac{a'}{a} \frac{b'}{b} + \frac{mn}{d^2} = -8\pi G (\omega \rho - \lambda), \tag{69}
\]

\[
\rho' + \rho (1 + \omega) H = \lambda \frac{d'}{d}, \tag{70}
\]

\[
\lambda (m - n) = 0. \tag{71}
\]

We have obtained equations (70) and (71) from the condition $\text{div } T = 0$. As it is observed equation (71) brings us to simplify the FE, since $m = n$, and then from equation (66) we get a new constrain, $a = b$, so the FE collapse to the following ones:

\[
\left( \frac{a'}{a} \right)^2 + \frac{a' d'}{a d} - \frac{m^2}{d^2} = 8\pi G \rho, \tag{72}
\]

\[
\frac{d''}{d} + \frac{a''}{a} + \frac{a' d'}{a d} - \frac{m^2}{d^2} = -8\pi G \omega \rho, \tag{73}
\]

\[
2 \frac{d'}{a} + \left( \frac{a'}{a} \right)^2 + \frac{m^2}{d^2} = -8\pi G (\omega \rho - \lambda), \tag{74}
\]

\[
\rho' + \rho (1 + \omega) \left( \frac{a'}{a} + \frac{d'}{d} \right) = \lambda \frac{d'}{d}. \tag{75}
\]

So, following the above procedure, and taking into account the SS solution, given by equation (17), we may find the following solution. From (75) we get

\[
\rho = \rho_0 (t + t_0)^{-\beta}, \quad \beta = (1 + \omega)(a_1 + a_2 + 1) - \frac{1}{1 + W} = 2,
\]

and the algebraic system to solve is the following one:

\[
a_1 (a_1 - 1) + a_1 - m^2 = -A \omega, \tag{76}
\]

\[
2a_1 (a_1 - 1) + a_1^2 + m^2 = -A \left( \omega - \frac{1}{1 + W} \right), \tag{77}
\]

\[
(1 + \omega)(2a_1 + 1) - \frac{1}{1 + W} = 2, \tag{78}
\]
with
\[ \rho_0 = \frac{A}{8\pi G}, \quad A = a_1^2 + 2a_1 - m^2. \]

Therefore, we may find the following solution:
\[ a_1 = a_2, \quad m = \sqrt{a_1^2(\omega + 1) + 2a_1\omega} = n, \]  
\[ W = -\frac{\omega(2a_1 + 1) + 2(a_1 - 1)}{\omega(2a_1 + 1) + 2a_1 - 1}. \]  
\[ (79) \]
\[ (80) \]

**Discussion.** Since \( W \geq 0 \), and \( a_1 > 0, (a_1^2(\omega + 1) + 2a_1\omega \geq 0) \) we get

| \( \omega \) | \( m(\omega) \) | \( W(\omega) \) |
|---|---|---|
| 1 | \( \sqrt{a_1^2 + a_1} \geq 0, \forall a_1 \geq 0 \) | \( -\frac{2(a_1 + 1)(a_1 - 1)}{2(a_1 + 1)(a_1 - 1)} \geq 0, \forall a_1 \in (0, 1/4) \implies W \in (0, \infty) \) |
| 1/3 | \( \sqrt{a_1^2 + \frac{1}{3}a_1} \geq 0, \forall a_1 \geq 0 \) | \( -\frac{2(a_1 - 1)}{2(a_1 - 1)} \geq 0, \forall a_1 \in (0, 1/4) \implies W \in (0, \infty) \) |
| 0 | \( \sqrt{a_1^2} \geq 0, \forall a_1 \geq 0 \) | \( -\frac{2(a_1^2 - 1)}{2(a_1^2 - 1)} \geq 0, \forall a_1 \in (0, 1/4) \implies W \in (0, \infty) \) |
| -1/3 | \( \sqrt{a_1^2 - 1} \geq 0, \forall a_1 \geq 1 \) | \( -\frac{2(a_1^2 - 1)}{2(a_1^2 - 1)} \geq 0, \forall a_1 \in (2, 7/4) \implies W \in (0, \infty) \) |
| -2/3 | \( \sqrt{a_1^2 - 4} \geq 0, \forall a_1 \geq 4 \) | \( -\frac{2(a_1^2 - 4)}{2(a_1^2 - 4)} \geq 0, \forall a_1 \in (2, 7/4) \implies W \in (0, \infty) \) |

Therefore, by depending on the equation of state for the perfect fluid we have a very different behaviour. The only nonphysical solution is \( \omega = -2/3 \). The solution leads us to get the following metric:
\[ ds^2 = -dt^2 + (t + t_0)^2n e^{-2mz} dx^2 + (t + t_0)^2n e^{2mz} dy^2 + (t + t_0)^2 dz^2, \]  
\[ (81) \]
with
\[ (1 + \omega)(1 + 2a_1) - \frac{1}{1 + W} = \beta = 2, \quad \rho = \rho_0(t + t_0)^{-2}. \]  
\[ (82) \]

In this case, we have not obtained the corresponding Bianchi III solution since we get only a solution for \( n = m \neq 0 \). As it is observed the metric (81) has the same structure as the metric (35), so it will have the same curvature behaviour.

Note that equation (71) is quite strong since it says us that if we are working with a Bianchi III metric, where \( n = 0 \), then the only one (the unique) possibility is \( m = 0 \), in such a way that the metric collapses to the Bianchi type I. This means that there is no massive string Bianchi III solution.

**8. String cosmological model with time varying ‘constants’**

Following the above model, we shall try to generalize it by considering the possible time variation of the ‘constants’ \( G \) and \( \Lambda \). In this case, the FE are (taking into account the above restrictions)
\[ \left( \frac{a'}{a} \right)^2 + \left( \frac{a''}{a} \right)^2 - \frac{m^2}{a^2} = 8\pi G \rho + \Lambda, \]  
\[ (83) \]
\[ \frac{a''}{a} + \frac{d^2}{da^2} + \left( \frac{a'}{a} \right)^2 - \frac{m^2}{a^2} = -8\pi G \omega \rho + \Lambda, \]  
\[ (84) \]
\[ 2\left( \frac{a''}{a} \right)^2 + \left( \frac{a'}{a} \right)^2 - \frac{m^2}{a^2} = -8\pi G (\omega \rho - \lambda) + \Lambda. \]  
\[ (85) \]
\[ \rho' + \rho(1 + \omega) \left( \frac{2a' + d'}{a + d} \right) = \frac{\lambda d'}{d}, \quad (86) \]

\[ \Lambda' = -8\pi G' \rho, \quad (87) \]

i.e. \( m = n \), and \( a = b \), we have also taken into account the condition \( \text{div} \, T = 0 \). By considering the SS solution given by equation (17) we find from equations (86) and (64)

\[ \rho = \rho_0(t + t_0)^{-\gamma}, \quad \gamma = (2a_1 + 1)(\omega + 1) - \frac{1}{1 + W}, \quad \text{where we shall assume that } \rho_0 > 0. \]

Following the same procedure as in the above sections we find that \( G(t) = G_0(t + t_0)^{-1} \), with \( G_0 = \frac{A}{4\pi\rho_0} \), and \( A = a_1^2 + 2a_1 - m^2 \). The cosmological ‘constant’ behaves as

\[ \Lambda(t) = A(t + t_0)^{-2} = \Lambda_0(t + t_0)^{-2}. \]

With all these results, we find that the algebraic system to solve is the following one:

\[ a_1^2 - m^2 = A \left( 1 - \frac{2(\omega + 1)\gamma}{\gamma} \right), \quad (88) \]

\[ 3a_1^2 - 2a_1 + m^2 = A \left( 1 - \frac{2(\omega + 1)}{\gamma} - \frac{2}{(1 + W)\gamma} \right), \quad (89) \]

and where its solution is \( a_1 = m = 0 \), so we arrive at the conclusion that there is no SS solution for this model.

9. Conclusions

In this paper, we have studied some Bianchi types VI_0&III models under the self-similarity hypothesis. We have started by reviewing the ‘classical’ perfect fluid solution already studied by Collins, Wainwright and Hsu. Furthermore, we have been able to improve their solution since we have found a non-singular solution for the scale factors i.e. they behave as \( a(t) \sim (t + t_0)^{a_1} \). Nevertheless, we have obtained two very restrictive solutions, and to the best of our knowledge are new. The first of them corresponds to Bianchi type III and it is only valid for \( \omega \in (-1/3, 0) \) in order to make \( a_2 > 0 \). The second of them corresponds to Bianchi type VI, and it is only valid for an equation of state \( \omega \in (-1/3, 1) \). Of course we have also obtained the ‘classical’ solution, already studied by Collins and several authors. For all these solutions we have studied their curvature behaviour, i.e. we have studied their curvature invariants together with the Weyl tensor, the Weyl parameter, \( \mathcal{W} \) (showing that the models isotropize since \( \mathcal{W} \to 0 \)) and we have also calculated the gravitational entropy, \( P^2 \). In all the studied cases, \( P^2 = \text{const} \), since we are working with a SS solution.

In the second of our studied models we have found two solutions for the vacuum model. The first of the obtained solutions corresponds to Bianchi type VI, and to the best of our knowledge we believe that it is new. By calculating its curvature invariants, \( (I_{ij})_{(i = 0)} \), we have showed that all of them vanish, nevertheless the Weyl tensor does not. The Weyl parameter is really small and constant and the gravitational entropy, \( P^2 = \infty \), since \( I_2 = 0 \). This fact tells us that the definition of the gravitational entropy does not work for vacuum models. The second of the obtained solutions corresponds to Bianchi type III and it has been already obtained by Hsu and Wainwright. It is a flat solution, for this reason all its curvature invariants vanish.

With regard to a perfect fluid model with time-varying constants we have obtained two solutions. Both solutions are valid for equation of state \( \omega \in (-1, 1] \). In this case, we have been able to enlarge the range of validity for the equation of state. The first of the studied solution is of Bianchi type VI, while the second one belongs to Bianchi type III. In both
solutions we have shown that if \( G \) behaves as a growing time function then \( \Lambda \) is a 'positive' decreasing time function. In the same way, if \( G \) is decreasing then \( \Lambda \) behaves as a 'negative' decreasing time function.

For the massive string cosmological model we have been able to obtain a solution for Bianchi type VI, where the equations of state are \( \omega \in (-2/3, 1] \) and \( W \in (0, \infty) \). As we have seen it is impossible to obtain a Bianchi type III solution. Such restriction is obtained from the conservation equation, \( \text{div} \ T = 0 \). In the last of our studied models, which is a massive string cosmological model with time-varying constants, we have arrived at the conclusion that there is no SS solution for this model, which is quite surprising.

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