RECONSTRUCTION FROM REPRESENTATIONS:
JACOI VOA COHOMOLOGY

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Abstract. A subalgebra of a Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ determines $\mathfrak{h}$-representation $\rho$ on $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$. In this note we discuss how to reconstruct $\mathfrak{g}$ from $(\mathfrak{h}, \mathfrak{m}, \rho)$. In other words, we find all the ingredients for building non-reductive Klein geometries. The Lie algebra cohomology plays a decisive role here.

Introduction

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a subalgebra. Let $\mathfrak{m} = \mathfrak{g}/\mathfrak{h}$ be the quotient $\mathfrak{h}$-module with representation $\rho : \mathfrak{h} \to \text{End}(\mathfrak{m})$. We address the reconstruction problem for the Lie algebra structure of $\mathfrak{g}$ from the data $(\mathfrak{h}, \mathfrak{m}, \rho)$.

In this note the spaces $(\mathfrak{h}, \mathfrak{m})$ are finite-dimensional. We do not assume the existence of an embedding $\mathfrak{m} \subset \mathfrak{g}$ as a reductive ($\mathfrak{h}$-invariant) complement to $\mathfrak{h}$, as such embeddings exist in general only if $H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}) = 0$.

We show that the Lie algebra cohomology $H^1(\mathfrak{h}, V)$ for $\mathfrak{h}$-modules $V$ plays a key role in the reconstruction. Parametrizing Lie brackets on $\mathfrak{h} \oplus \mathfrak{m}$, the Jacobi identity constrains the parameters. Our cohomological approach allows to single out linear equations, and significantly reduce the amount of quadratic constraints.

Klein geometries are homogeneous spaces $G/H$, and such have been extensively studied for reductive subgroups $H$. Our method allows to effectively handle non-reductive Klein geometries via symbolic computations.

Some other approaches to the reconstruction in the case of filtered algebras via the deformation technique can be found in [11, 4, 3]. We also mention Cartan’s procedure for construction of homogeneous models of a given geometry type [2].

1. The main result

Since $\mathfrak{h}$ is a subalgebra, the bracket $\Lambda^2 \mathfrak{h} \to \mathfrak{h} \subset \mathfrak{g}$ is the Lie algebra structure on $\mathfrak{h}$, but since $\mathfrak{m}$ is not a reductive complement the other brackets on $\mathfrak{g}$ are:

$$\mathfrak{h} \otimes \mathfrak{m} \to \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [h, u] = \varphi(h, u) + h \cdot u;$$

$$\Lambda^2 \mathfrak{m} \to \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad [u_1, u_2] = \theta_h(u_1, u_2) + \theta_m(u_1, u_2),$$

where $h \in \mathfrak{h}$, $u, u_1, u_2 \in \mathfrak{m}$, $\varphi \in \mathfrak{h}^* \otimes \mathfrak{m}^* \otimes \mathfrak{h}$, $\theta_h \in \Lambda^2 \mathfrak{h}^* \otimes \mathfrak{h}$, $\theta_m \in \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{m}$, and we denote (here and in what follows) $h \cdot u = \rho(h)u$.

The cohomology class of $\varphi$ is known from the splitting theory for modules [6]. We interpret it as the complete obstruction to the existence of a reductive complement.

Proposition. The element $\varphi$ is closed in the complex $\Lambda^* \mathfrak{h}^* \otimes \mathfrak{m}^* \otimes \mathfrak{h}$: $d\varphi = 0$, and it changes by an exact element when $\mathfrak{m} \subset \mathfrak{g}$ varies. Thus $[\varphi] \in H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h})$.

Proof. The Jacobi identity with 2 arguments from $\mathfrak{h}$ is

$$[h_1, [h_2, u]] + [h_2, [u, h_1]] + [u, [h_1, h_2]] = 0.$$
Lemma 1. For the differential \( d \) in the complex \( \Lambda^* h^* \otimes m^* \otimes h \) we have: \([\delta, d] = 0\), whence \( d\delta = 0\) and \( \delta(\varphi + d\sigma) = \delta\varphi + d\delta\sigma \).

Proof. This is because \( \delta \) is obtained from an \( h \)-homomorphism of the coefficients module. □

Given \( \theta_m \in \Lambda^2 m^* \otimes m \) satisfying \( d\delta\varphi = d\theta_m \), let us introduce the nonlinear operator

\[
Q : h^* \otimes m^* \otimes h \to h^* \otimes \Lambda^2 m^* \otimes h
\]

by \( Q\varphi = S_{\theta_2}(\varphi, \varphi) - \varphi \circ (1_h \wedge \theta_m) \), where \( S_{\theta_2} \) is the composition

\[
(h^* \otimes m^* \otimes h)^{\otimes 2} = h^* \otimes m^* \otimes h \otimes h^* \otimes m^* \otimes h \xrightarrow{\operatorname{contr}_{3,4}} h^* \otimes m^* \otimes m^* \otimes h \xrightarrow{\operatorname{alt}_{2,3}} h^* \otimes \Lambda^2 m^* \otimes h,
\]

and \( \operatorname{contr}_{3,4} \) is the contraction by the corresponding arguments. In other words,

\[
Q\varphi(h)(u_1, u_2) = \varphi(\varphi(h, u_1), u_2) - \varphi(\varphi(h, u_2), u_1) - \varphi(h, \theta_m(u_1, u_2)).
\]

Lemma 2. We have \( d(Q\varphi) = 0 \), i.e. \([Q\varphi] \in H^1(h, \Lambda^2 m^* \otimes h)\).

Proof. This follows by a straightforward computation: Let

\[
\Xi(h_1, h_2, u_1, u_2) = h_1 \cdot \varphi(h_2, u_1, u_2) - \varphi(h_2, h_1, u_1, u_2) - \varphi(h_2, u_1, h_1, u_2),
\]

\[
\Xi(h_1, h_2, u_1, u_2) = h_1 \cdot \varphi(h_2, u_1, u_2) - \varphi(h_2, h_1, u_1, u_2) - \varphi(h_2, u_1, h_1, u_2),
\]

Then using \( d\varphi = 0 \) and \( d\theta_m = \delta\varphi \) (and \( h_1 \cdot h_2 = [h_1, h_2] \)) we get

\[
d(Q\varphi)(h_1, h_2)(u_1, u_2) = \Xi(h_1, h_2, u_1, u_2) - \Xi(h_1, h_2, u_2, u_1) - \Xi(h_2, h_1, u_1, u_2) + \Xi(h_2, h_1, u_2, u_1)
\]

\[
= (h_1 \cdot \varphi)(h_2, u_1, u_2) + (h_2 \cdot \varphi)(h_1, u_1, u_2) - \varphi(h_2, [h_1, h_2], u_1, u_2) - \varphi(h_1, [h_1, h_2] \otimes u_1, u_2)
\]

\[
\Xi(h_1, h_2, u_1, u_2) = \Xi(h_1, h_2, u_1, u_2) - \Xi(h_1, h_2, u_2, u_1) - \Xi(h_2, h_1, u_1, u_2) + \Xi(h_2, h_1, u_2, u_1)
\]

\[
+ \varphi(h_1, \varphi(h_2, u_1) - \varphi(h_2, \varphi(h_1, u_2) - \varphi(h_2, [h_1, h_2], u_1, u_2) + \varphi(h_2, \varphi(h_1, u_2) - \varphi(h_2, [h_1, h_2], u_1, u_2)
\]

\[
= (h_1 \cdot \varphi)(h_2, u_1, u_2) + (h_2 \cdot \varphi)(h_1, u_1, u_2) - \varphi(h_2, [h_1, h_2], u_1, u_2) - \varphi(h_1, [h_1, h_2] \otimes u_1, u_2)
\]

\[
= \varphi(h_2, u_1) - \varphi(h_2, u_2) - \varphi(h_1, u_1) - \varphi(h_1, u_2).
\]

□
Let us define the linear operators \( q : m^* \otimes \mathfrak{h} \rightarrow h^* \otimes \Lambda^2 m^* \otimes \mathfrak{h}, \sigma \mapsto q_\sigma, \) and \( p : (\Lambda^2 m^* \otimes m)^b \rightarrow h^* \otimes \Lambda^2 m^* \otimes \mathfrak{h}, \nu \mapsto p_\nu, \) by the formulae

\[
g_\sigma(h)(u_1, u_2) = d\sigma(\varphi(h, u_1), u_2) - d\sigma(\varphi(h, u_2), u_1) + \varphi(d\sigma(h, u_1), u_2)
\]

\[
- \varphi(d\sigma(h, u_2), u_1) + d\sigma(\varphi(h, u_1), u_2) - d\sigma(\varphi(h, u_2), u_1),
\]

\[
- \varphi(h, \delta\sigma(u_1, u_2)) - d\sigma(h, \theta_m(u_1, u_2)) - d\sigma(h, \delta\sigma(u_1, u_2));
\]

\[
p_\nu(h)(u_1, u_2) = \varphi(h, \nu(u_1, u_2)).
\]

For \( \sigma \in m^* \otimes \mathfrak{h} \) and \( \varphi, \theta_m \) as above define the elements \( \phi_i \in \Lambda^2 m^* \otimes \mathfrak{h}, 1 \leq i \leq 4, \) so

\[
\phi_1(u_1, u_2) = \varphi(\sigma(u_1), u_2) - \varphi(\sigma(u_2), u_1), \quad \phi_2(u_1, u_2) = [\sigma(u_1), \sigma(u_2)],
\]

\[
\phi_3(u_1, u_2) = \sigma((\sigma(u_1) \cdot u_2 - \sigma(u_2) \cdot u_1), u_1), \quad \phi_4(u_1, u_2) = \sigma(\theta_m(u_1, u_2)).
\]

**Lemma 3.** We have \( dp_\nu = 0 \) and \( q_\sigma = d(\phi_1 + \phi_2 - \phi_3 - \phi_4), \) whence \( dq_\sigma = 0. \)

**Proof.** Here the computations are a bit more involved:

\[
d\phi_1(h)(u_1, u_2) = (h \cdot \varphi)(\sigma(u_1), u_2) + \varphi(h \cdot \sigma(u_1), u_2) - \varphi(h \cdot u_1, u_2)
\]

\[
- (h \cdot \varphi)(\sigma(u_2), u_1) + \varphi(h \cdot u_2, u_1) + \varphi(\sigma(h \cdot u_2), u_1) + \varphi(\sigma(h \cdot u_1), u_2),
\]

\[
\phi_2(h)(u_1, u_2) = h \cdot (\sigma(u_1) \cdot \sigma(u_2)) - \sigma(h \cdot u_1) \cdot \sigma(u_2) + \sigma(h \cdot u_2) \cdot \sigma(u_1),
\]

\[
d\phi_3(h)(u_1, u_2) = h \cdot \sigma(u_1) \cdot u_2 - \sigma(h \cdot u_1) \cdot u_2 - \sigma(u_1) \cdot u_2 - \sigma(u_2) \cdot u_1.
\]

Therefore (we again use \( dp_\nu = 0 \) and \( db_m = d\varphi \) and also the Jacobi identity)

\[
q_\sigma(h)(u_1, u_2) = \varphi(h, u_1) \cdot \sigma(u_2) - \varphi(\sigma(h, u_1) \cdot u_2) - \varphi(h, u_2) \cdot \sigma(u_1) + \varphi(\sigma(h, u_2) \cdot u_1)
\]

\[
+ \varphi(h \cdot \sigma(u_1), u_2) - \varphi(\sigma(h \cdot u_1), u_2) - \varphi(h \cdot \sigma(u_2), u_1) - \varphi(h \cdot \sigma(h \cdot u_2), u_1) + \varphi(\sigma(h \cdot u_2), u_1) + \varphi(\sigma(h \cdot u_1), u_2),
\]

\[
+ h \cdot [\sigma(u_1) \cdot \sigma(u_2)] - \sigma(h \cdot u_1) \cdot \sigma(u_2) - \sigma(h \cdot \sigma(u_1) \cdot u_2) + \sigma(h \cdot \sigma(u_2) \cdot u_1),
\]

\[
- (h \cdot \sigma(u_2)) \cdot u_1 + \sigma(h \cdot u_1) \cdot u_2 + \sigma(h \cdot \sigma(u_1) \cdot u_2) - \sigma(h \cdot \sigma(u_2) \cdot u_1) - \sigma(h \cdot \sigma(h \cdot u_2) \cdot u_1) - \sigma(h \cdot \sigma(h \cdot u_1) \cdot u_2).
\]

The other computation is easy:

\[
d(p_\nu)(h_1, h_2)(u_1, u_2) = d(\nu(h_1, h_2)(\nu(u_1, u_2))
\]

\[
- \varphi(h_1)(h_2 \cdot \nu)(u_1, u_2) + \varphi(h_2)(h_1 \cdot \nu)(u_1, u_2) = 0.
\]

Thus \( \text{Im}(q_\sigma) \subset B^1(h, \Lambda^2 m^* \otimes \mathfrak{h}), \) \( \text{Im}(p_\nu) \subset Z^1(h, \Lambda^2 m^* \otimes \mathfrak{h}). \) Let us denote \( \Pi_\varphi = \text{Im}(p_\nu) \bmod B^1(h, \Lambda^2 m^* \otimes \mathfrak{h}) \subset H^1(h, \Lambda^2 m^* \otimes \mathfrak{h}). \)

We are now ready to state our main result.

**Theorem 1.** The Jacobi identity \( \text{Jac}(v_1, v_2, v_3) = 0 \) with 1 argument from \( \mathfrak{h} \) and the others from \( m \) gives the following constraints on the cohomology \( [\varphi] \in H^1(h, m^* \otimes \mathfrak{h}) \):

1. \( [\delta \varphi] = 0 \in H^1(h, \Lambda^2 m^* \otimes m), \) whence \( \delta \varphi = db_m; \)
2. \( [Q \varphi] \equiv 0 \in H^1(h, \Lambda^2 m^* \otimes \mathfrak{h}) \bmod \Pi_\varphi, \) so \( Q \varphi = db_\mathfrak{h} \) for some choices of \( \varphi, \theta_m. \)
Proof. Consider the Jacobi identity with 1 argument from $\mathfrak{h}$:

$$[h, [u_1, u_2]] + [u_1, [u_2, h]] + [u_2, [h, u_1]] = 0.$$ 

Taking $m$-part of this identity (this is canonical: projection along $\mathfrak{h}$), we obtain

$$\delta \varphi(h)(u_1, u_2) = (h \cdot \theta_m)(u_1, u_2) \iff \delta \varphi = d \theta_m \in B^1(\mathfrak{h}, \Lambda^2 m^* \otimes m).$$ 

This implies (1).

Two remarks are in order. First, changing $\varphi \mapsto \varphi + d \sigma$ we do not alter the property $[\delta \varphi] = 0$. Second, $\theta_m$ is determined by constraint (1) modulo $(\Lambda^2 m^* \otimes m)^h$. This changes $Q\varphi$ by an element $p_\nu \in Z^1(\mathfrak{h}, \Lambda^2 m^* \otimes m)$ due to Lemma 3.

To obtain (2) consider the $h$-part of the Jacobi identity. Note that if we change $\varphi \mapsto \varphi + d \sigma$, then $\theta_m \mapsto \theta_m + \delta \sigma$, so $Q\varphi \mapsto Q\varphi + Qd \sigma$. This leaves $Q\varphi$ closed by Lemma 2 and does not change the cohomology class by Lemma 3. However, the latter is influenced by the change of $\theta_m$ as indicated above: $[Q\varphi] \mapsto [Q\varphi] + [p_\nu]$.

The $h$-part of the Jacobi identity can now be written for some $p_\nu \in (\Lambda^2 m^* \otimes m)^h$ (the freedom in a choice of $\theta_m$)

$$Q\varphi + p_\nu = d \theta_h \iff [Q\varphi] \equiv 0 \mod \Pi_\varphi. $$

This implies (2). \qed

Now the reconstruction algorithm from the data $(\mathfrak{h}, m, \rho)$ is the following:

- Compute $H^1(\mathfrak{h}, \mathbb{V})$ for $\mathbb{V} = m^* \otimes \mathfrak{h}, \Lambda^2 m^* \otimes m, \Lambda^2 m^* \otimes \mathfrak{h}$.
- Constrain the Lie bracket parameters by (1) and (2) in the theorem.
- Constrain them further by the quadratic relations $\text{Jac}_m : \Lambda^2 m \rightarrow \mathfrak{h} \oplus m$ so:

$$\mathcal{S} [\varphi(h)(u_1, u_2), u_3] = 0,$$
$$\mathcal{S} [\varphi(h)(u_1, u_2), u_3] = 0,$$

where $\mathcal{S}$ is the cyclic summation by indices $1, 2, 3$.

2. Some specifications

In this section we discuss computation of the cohomology involved in the constraints of the theorem in several cases.

1: Reductive isotropy. Let us first note that if $\mathfrak{h}$ is a semi-simple Lie algebra then $H^1(\mathfrak{h}, \mathbb{V}) = 0$ for any $\mathfrak{h}$-module $\mathbb{V}$ (Whitehead’s lemma [12]). The same holds true if $\mathfrak{h}$ is reductive with the center acting on $\mathbb{V}$ being semi-simple and with no linear invariants. Then choosing $\varphi = 0$ the conditions of the theorem equivalently mean that

$$\theta_h : \Lambda^2 m \rightarrow \mathfrak{h}, \ \theta_m : \Lambda^2 m \rightarrow m$$

are $\mathfrak{h}$-equivariant maps. The parameters of these maps are constrained by the quadratic conditions $\text{Jac}_m = 0$. Examples of reconstruction of $\mathfrak{g}$ in the case of $\mathfrak{h} = \mathfrak{sl}(2), \mathfrak{su}(2)$ and $\dim m = 6$ can be found in [1].

2: Internal gradings. Notice that if $\mathfrak{h}$ is graded $\mathfrak{h} = \bigoplus a \mathfrak{h}_a$, with $\mathfrak{h}_0$ Abelian and $\mathbb{V}$ is a graded $\mathfrak{h}$-module $\mathbb{V} = \bigoplus \mathbb{V}_a$, then [3] the cohomology $H^*\mathbb{V}$ coincides with that of the subcomplex $(\Lambda^* \mathfrak{h}^* \otimes \mathbb{V})_0$ of total degree zero cochains, and the same is true if we use multi-grading given by $\mathfrak{h}$ (i.e. $\alpha$ takes values in a multi-dimensional lattice).

In particular, if $\mathfrak{h}$ is Abelian, and $\mathbb{V}$ completely reducible, then $H^1(\mathfrak{h}, \mathbb{V}) = \mathfrak{h}^* \otimes \mathbb{V}_0$, where $\mathbb{V}_0 = \mathbb{V}^h$ is the trivial component. Indeed, for any reductive Lie algebra $\mathfrak{h}$ with the center acting semi-simply on $\mathbb{V}$ we have $H^*(\mathfrak{h}, \mathbb{V}) = H^*(\mathfrak{h}) \otimes \mathbb{V}^h$ [4] and if $\mathfrak{h}$ is Abelian, then the differential of its cohomology complex vanishes.

3: One-dimensional space of splittings. In the case $\dim H^1(\mathfrak{h}, m^* \otimes \mathfrak{h}) = 1$ we can normalize $|\varphi| = 0 \lor 1$. In the case $|\varphi| = 0$ an $\mathfrak{h}$-invariant splitting $\mathfrak{g} = \mathfrak{h} \oplus m$ exists, and the solutions $(\theta_m, \theta_h)$ of the theorem can be found as $\mathfrak{h}$-equivariant
elements \((\Lambda^2 m^* \otimes g)^b\). If \(|c| = 1\), then the solutions space to the constraints of Theorem 1 is an affine space modelled on \((\Lambda^2 m^* \otimes g)^b\). Indeed, the system of equations \(d(\theta_m, \theta_h) = (\delta \varphi, Q\varphi)\) is linear inhomogeneous in the unknowns \((\theta_m, \theta_h)\).

With one solution (often) a-priori known it is easy to parametrize all solutions (to the constraints of Theorem 1) these parameters are yet subject to the Jacobi constraints with 3 arguments from \(m\).

4: Cartan subalgebra. Let \(\mathfrak{h}\) be a Cartan subalgebra of a semi-simple Lie algebra \(g\), and \(m = g/\mathfrak{h}\). Since \(m_0 = m^b = 0\) we have \(H^*(m^*, \otimes \mathfrak{h}) = 0\). In particular, there is a reductive complement. For the split real form \(g\), we can take the sum of root spaces \(m = \sum_{\alpha \in \mathbb{R}} \mathfrak{e}_\alpha\) as such.

The other cohomologies involved in Theorem 1 are non-trivial (this will have an implication in the next section): \(H^*(\mathfrak{h}, \Lambda^2 m^* \otimes m) = \Lambda^* \mathfrak{h}^* \otimes (\Lambda^2 m^* \otimes m)_0\) (with 0 referring to the multi-grading induced by a set of simple roots; if the real form \(g\) is not split, the complexification can be used at this step), and similarly \(H^*(\mathfrak{h}, \Lambda^2 m^* \otimes \mathfrak{h}) = \Lambda^* \mathfrak{h}^* \otimes (\Lambda^2 m^* \otimes \mathfrak{h})_0\).

Note that the submodule \((\Lambda^2 m^*)_0\) is generated by the elements \(e_\alpha \otimes \theta^\alpha\), where \(\theta^\alpha\) is the co-basis dual to the basis \(e_\alpha\) of \(m\), and similarly the submodule \((\Lambda^2 m^* \otimes m)_0\) is generated by the elements \(e_{\alpha+\beta} \otimes \theta^\alpha \wedge \theta^\beta\). Thus both submodules and hence the cohomology groups are non-trivial.

5: Nilradical of a parabolic. Let \(g = \oplus g_i\) be a semi-simple Lie algebra with the grading induced by the parabolic subalgebra \(p = \oplus_{i \geq 0} g_i\). We choose as subalgebra the nilradical of the opposite parabolic: \(\mathfrak{h} = \oplus_{i < 0} g_i\), and \(m = g/\mathfrak{h} = p\).

The cohomology \(H^*(\mathfrak{h}, \mathcal{V})\) for \(g\)-modules \(\mathcal{V}\) (restricted to \(\mathfrak{h} \subset g\)) is given (as a \(g_0\)-module) by Kostant’s version of the Bott-Borel-Weyl theorem\(^2\), however in the case \(m = p\) it is not a \(g\)-module and the computations are more involved.

We compute the cohomology \(H^1(\mathfrak{h}, m^* \otimes \mathfrak{h})\) in the case of \([1]\)-grading\(^2\): \(g = h \oplus g_0 \oplus h^*\) (where \(g_1\) is identified with \(h^* = g_{-1}\) via the Killing form, and similarly \(g_0 = g_0\)). We have: \(m = g_0 \oplus h^*, m^* = g_0 \oplus h\) (this decomposition of modules is not \(\mathfrak{h}\)-invariant; to get an invariant computation one should use the spectral sequences based on the \(h\)-invariant filtration, but we skip doing so).

The cochain complex for the cohomology is:

\[0 \to (g_0 \oplus h) \otimes \mathfrak{h} \xrightarrow{d_0} h^* \otimes (g_0 \oplus h) \otimes \mathfrak{h} \xrightarrow{d_1} \Lambda^2 h^* \otimes (g_0 \oplus h) \otimes \mathfrak{h} \to \ldots\]

where \(d_0\) projects the first term to \(g_0 \otimes \mathfrak{h} \subset h^* \otimes \mathfrak{h} \otimes \mathfrak{h}\) and \(d_1\) projects the second term to \(h^* \otimes g_0 \otimes h^* \otimes \mathfrak{h}\), with \(\delta : h^* \otimes g_0 \subset h^* \otimes h^* \otimes h \to \Lambda^2 h^* \otimes \mathfrak{h}\) being the Spencer operator (skew-symmetrization). Thus \(B^1 = g_0 \otimes \mathfrak{h} \subset h^* \otimes h \otimes \mathfrak{h}\), \(Z^1 = Z^1/B^1 = (g_0 \otimes h \cap S^2 h^* \otimes \mathfrak{h})\) is the Sternberg-Spencer prolongation of \(g_0\), whence \(H^1 = Z^1/B^1 = [g_0^{(1)} + h^* \otimes h] / g_0 \otimes h\).

By Yamaguchi’s prolongation theorem\(^3\) we have \(g^{(1)} = g_1 = h^*\) in all \([1]\)-graded cases except \(A_t/P_t\) (and dually \(A_t/P_t\)). In the latter case \(g_0 = gl(h)\), so \(g^{(1)} = S^2 h^* \otimes h\). Thus we conclude

\[H^1(\mathfrak{h}, m^* \otimes \mathfrak{h}) = \begin{cases} S^2 h^* \otimes h \otimes h & \text{when the grading is } A_t/P_t, \\ h^* + \frac{h^* \otimes h}{g_0} \otimes h & \text{otherwise.} \end{cases}\]

\(^2\)The \([1]\)-gradings of simple complex Lie algebras are: \(A_t/P_t, B_t/P_t, C_t/P_t, D_t/P_t, D_t/P_t, E_6/P_t, E_7/P_t\) in Bourbaki’s ordering of the nodes on the Dynkin diagram. Which of those extend to the real versions is decided by the Satake diagram.
3. Applications

In this section we consider some simple applications, illustrating the developed technique. More substantial outcomes can be extracted from [9, 10] (the first reference is in retrospective, while the second essentially uses the results of this paper). An application to reconstruction from another technique can be found in [5].

We start with a toy example: one-dimensional subalgebra \( \mathfrak{h} \subset \mathfrak{sl}_2 \). Let \( \mathfrak{m} = \mathfrak{sl}_2 / \mathfrak{h} \) and \( \rho \) be the isotropy representation. If \( \mathfrak{h} \) is a (non-compact) Cartan subalgebra, then the triple \((\mathfrak{h}, \mathfrak{m}, \rho)\) recovers either the original Lie algebra \( \mathfrak{sl}_2 \) or the algebra \( \mathbb{R} \times \mathbb{R}^2 \), where the action of \( \mathfrak{h} = \mathbb{R} \) on the Abelian piece \( \mathbb{R}^2 \) is by the matrix \( \text{diag}(-1,1) \). If, on the other hand, \( \mathfrak{h} \) is nilpotent, then the triple \((\mathfrak{h}, \mathfrak{m}, \rho)\) recovers either the original semi-simple Lie algebra or a solvable algebra \( \mathfrak{g} \) with \([\mathfrak{g}, \mathfrak{g}] \supset \mathfrak{h} \).

Now if \( \mathfrak{h} \) is two-dimensional (Borel), then \((\mathfrak{h}, \mathfrak{m}, \rho)\) recovers either the original \( \mathfrak{g} = \mathfrak{sl}_2 \) or the above \( \mathbb{R} \times \mathbb{R}^2 = \langle e_1, e_2, e_3 : [e_1, e_2] = e_2, [e_1, e_3] = -e_3 \rangle \), \( \mathfrak{h} = \langle e_1, e_2 \rangle \). Note that for the corresponding homogeneous space \( G/H \) the isotropy has a kernel: in the Lie subalgebra \( \mathfrak{h} \) the element \( e_2 \) generates an ideal and thus acts non-effectively. As a result we obtain an effective homogeneous representation \( G/H = G'/H' \) with Lie algebras corresponding to the groups: \( \mathfrak{g}' = \langle e_1, e_3 \rangle \), \( \mathfrak{h} = \langle e_1 \rangle \).

This motivates the following

**Definition.** We call a pair \((\mathfrak{h}, \mathfrak{g})\) reconstruction rigid, if \( \mathfrak{g} \) is the unique non-flat algebra with the isotropy data \((\mathfrak{h}, \mathfrak{m}, \rho)\) and no nontrivial \( \mathfrak{g} \)-ideals supported in \( \mathfrak{h} \).

Recall that the flat algebra is \( \mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m} \), where \( \mathfrak{m} \) is Abelian and the bracket \( \mathfrak{h} \wedge \mathfrak{m} \rightarrow \mathfrak{m} = \rho \). Now we consider three examples.

1: **Borel subalgebra in \( \mathfrak{sl}_3 \).** Let \( \mathfrak{h} = \mathfrak{b} \) be a minimal parabolic (maximal solvable) subalgebra in \( \mathfrak{sl}_3 \) (realized as the set of upper-triangular matrices) and \( \mathfrak{m} = \mathfrak{sl}_3 / \mathfrak{h} \). The cohomology group \( H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}) = \mathbb{R} \), so there exists a unique (one-dimensional) obstruction to finding a reductive complement to \( \mathfrak{h} \). If the obstruction vanishes, then the Lie algebra structure on \( \mathfrak{h} \oplus \mathfrak{m} \) is encoded by \( (\Lambda^2 \mathfrak{m}^\ast \otimes (h \oplus m))^h = 0 \). So this case is flat: the only possible Lie algebra structure is \( \mathfrak{g} = \mathfrak{h} \ltimes \mathfrak{m}^3 \).

If the obstruction \( [\mathfrak{z}] \subset \mathfrak{h} \) is non-zero, it can be normalized to 1. In this case we compute the other cohomologies \( H^1(\mathfrak{h}, \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{m}) = \mathbb{R}^3 \), \( H^1(\mathfrak{h}, \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{h}) = 0 \). The first constraint of our theorem, \( \delta \varphi = d\varphi_m \), is non-trivial and locks all the structure constants to be equal to that of \( \mathfrak{sl}_3 \). Thus \((\mathfrak{h}, \mathfrak{sl}_3)\) is reconstruction rigid. This shows the uniqueness of (effective) homogeneous representation with 5D isotropy of the flag variety \( F(1,2,3) = \text{SL}_3(\mathbb{R}) / \mathbb{O}(3)/\mathbb{Z}_2^3 \) (this Klein geometry is non-reductive unless we reduce the symmetry group: \( F(1,2,3) = \text{O}(3)/\mathbb{Z}_2^3 \)).

Note that the isotropy data of the homogeneous space \( M^3 = G/H \) include the invariant contact 2-distribution (it is integrable in the flat case) that is split as a sum of two line fields \( \Pi^2 = L^1 \oplus L_3^1 \subset TM \). This geometric structure corresponds to a second order ODE with respect to point transformations.

To verify this, we note that there exists a unique ODE with symmetry of the maximal dimension 8 (this is indeed the ODE \( y''(x) = 0 \) viewed as a pair of line fields – vertical and the total derivative – on \( J^1(\mathbb{R}, \mathbb{R}) = M^3(x, y, y') \)). In fact, it is known that the other homogeneous geometries encoding second order ODEs can exist on 3-dimensional Lie groups only (trivial isotropy). Our approach allows us to independently verify this.

2: **Cartan subalgebra in \( \mathfrak{sl}_3 \).** Let \( \mathfrak{h} = \mathfrak{c} \) be a Cartan subalgebra in \( \mathfrak{sl}_3 \) (realized as the set of diagonal matrices) and \( \mathfrak{m} = \mathfrak{sl}_3 / \mathfrak{h} \). Since we know from the previous section that \( H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}) = 0 \) and \( \mathfrak{m} \subset \mathfrak{g} \) can be \( \mathfrak{h} \)-invariant, the reconstruction is reduced to classifying \( \mathfrak{h} \)-equivariant maps \( \Lambda^2 \mathfrak{m} \rightarrow \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \). A simple computation shows that the results of reconstruction is as follows:

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\(^3\)In this and subsequent computations the DifferentialGeometry package of Maple was used.
This uniquely normalizes the representation in some complex basis of $h$

In cases (3)-(7) the action of $\mathfrak{gl}_2$ is not reconstruction rigid, and the real version $\text{SL}_3(\mathbb{R})/((\mathbb{R}^\times)^2 \oplus \mathbb{R})$ of the complex flag variety $F^C(1,2,3) = \text{SL}_3(\mathbb{C})/B^C = U(3)/T^3 = SU(3)/T^2$ is not (uniquely) recoverable from its isotropy data.

3: Homogeneous 4D almost complex spaces with $\text{Sol}_2$ isotropy. Let us investigate 4-dimensional almost complex homogeneous spaces with 2-dimensional solvable (non-abelian) isotropy, i.e. $\mathfrak{h}$ preserves a complex structure on $\mathfrak{m}$. We assume the isotropy representation $\rho$ to be effective. If the almost complex structure $J$ is non-integrable, then the isotropy representation $\rho: \mathfrak{so}_2 \rightarrow \mathfrak{gl}_2(\mathbb{C})$ also leaves invariant the Nijenhuis tensor $0 \neq N_J \in \Lambda^2 \mathbb{C}^{2*} \otimes \mathbb{C}^2$: $\rho(v) \cdot N_J = 0 \forall v \in \mathfrak{so}_2$. This uniquely normalizes the representation in some complex basis of $\mathbb{C}^2$ so:

$$e_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad [e_1, e_2] = e_2.$$  

Now we compute the cohomology:

$$H^1(\mathfrak{h}, \mathfrak{m}^* \otimes \mathfrak{h}) = \mathbb{R}^2, \quad H^1(\mathfrak{h}, \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{m}) = \mathbb{R}^5, \quad H^1(\mathfrak{h}, \Lambda^2 \mathfrak{m}^*_\mathfrak{h} \otimes \mathfrak{h}) = \mathbb{R}^4.$$

Thus a-priori we get a non-reductive decomposition and the bracket $\mathfrak{h} \wedge \mathfrak{m} \rightarrow \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ takes values in both summands, being parametrized by 2 real numbers. These two parameters are however forced to vanish by the further homological constraints of Theorem [H] (to be precise, by $[d\varphi] = 0 \in H^1(\mathfrak{h}, \Lambda^2 \mathfrak{m}^* \otimes \mathfrak{m})$ forming 8 scalar linear equations). Thus a-posteriori the $\mathfrak{h}$-module $\mathfrak{g}$ splits, and the remaining brackets can be chosen as elements of the module $(\Lambda^2 \mathfrak{m}^* \otimes \mathfrak{g})^\mathfrak{h}$ satisfying the Jacobi identity.

These are computable, however another look at the Jacobi identity shows that the complement $\mathfrak{m} \subset \mathfrak{g}$ can be chosen in such a way that both $\mathfrak{h}$ and $\mathfrak{m}$ are subalgebras, though the bracket $[\mathfrak{h}, \mathfrak{m}]$ takes values in the whole $\mathfrak{g}$. This still allows to conclude that the almost complex structure $J$ on the homogeneous space $M = G/H$ coincides with the left-invariant structure on some Lie group $M^I$. Such groups are classified, but we do not include this rather long list.

Finally notice that by [S] a symmetry of a (non-integrable) almost complex structure $J$ on a 4-manifold $M$ is at most 4D (in this case $J$ is the left-invariant structure on a Lie group) unless the 2-distribution $\text{Im}(N_J) \subset TM$ is integrable and $J, N_J$ are projectible along its leaves. We may also observe this from the obtained structure equations of $\mathfrak{h} \oplus \mathfrak{m}$ in our case.

Appendix A. Lie algebra cohomology

For completeness recall the formula for the Lie algebra differential in the complex $\Lambda^* \mathfrak{h}^* \otimes \mathfrak{V}$ for $H^* \mathfrak{(h, V)}$. If $\varphi \in \Lambda^k \mathfrak{h}^* \otimes \mathfrak{V}$ and $h_i \in \mathfrak{h}$, then

$$d\varphi(h_0, \ldots, h_k) = \sum (-1)^i h_i \cdot \varphi(h_0, \ldots, \hat{h}_i, \ldots, h_k)$$

$$+ (-1)^{i+j} \varphi([h_i, h_j], h_0, \ldots, \hat{h}_i, \ldots, \hat{h}_j, \ldots, h_k).$$
Note that if $d\varphi = 0$, then $h \cdot \varphi = d_i h \varphi$, where $i_h \varphi = \varphi(h, \cdot) \in \Lambda^{k-1} \mathfrak{h}^* \otimes \mathfrak{V}$ is the result of substitution of $h$ as the first argument. Thus $H^*(\mathfrak{h}, \mathfrak{V}) = H^*(\mathfrak{h}, \mathfrak{V})^h$ and the equivariancy (of e.g. $[\varphi]$) is not a constraint on the cohomology classes.

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