‘Prethermalization’ in Conservative Nonsymplectic Periodically Driven Spin Systems

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We analyze a new class of time-periodic dynamics in interacting chaotic classical spin systems, whose equations of motion are conservative (phase-space volume preserving) yet possess no symplectic structure. As a result, the dynamics of the system cannot be derived from any time-dependent Hamiltonian. In the high-frequency limit, we find that the magnetization dynamics features a long-lived metastable plateau, whose duration is controlled by the fourth power of the drive frequency. However, due to the lack of an effective Hamiltonian, the system does not evolve into a strictly prethermal state. We propose a Hamiltonian extension of the system using auxiliary degrees of freedom, in which the original spins constitute an open yet nondissipative subsystem. This allows us to perturbatively derive effective equations of motion that manifestly display symplecticity breaking at leading order in the inverse frequency. We thus extend the notion of prethermal dynamics, observed in the high-frequency limit of periodically-driven systems, to a nonsymplectic setting.

Introduction—Periodically-driven closed many-body systems generically feature long-lived metastable states in the high-frequency regime, when drive frequencies are much larger than the local bandwidth [1, 2]. For sufficiently short-ranged interactions, such systems feature exponentially long-lived prethermal plateaus, where energy absorption is severely constrained and slowed down, as higher-order processes are required [3–9]. Experimentally, Floquet prethermalization has been instrumental for the realization of novel engineered properties [10–14], such as artificial gauge fields for neutral particles [15–17]; discrete time crystalline [18–22] or topologically-ordered [23–25] phases of matter without equilibrium counterparts; and as a stabilization mechanism to create long-lived coherent dynamics [26–31].

Practically, prethermalization provides a long time window to realize Trotterized dynamics on digital quantum processors [32], and establishes the parametric stability of symplectic integrators [33, 34], which conserve certain integrals of motion exactly [35]; however, energy conservation is usually lost [36].

While nonsymplectic, energy-conserving integration schemes can be implemented instead [37], the analysis of their stability is confounded by the absence of conjugate variables and Poisson brackets – conventionally required to apply the high-frequency expansion in the analysis of higher-order heating processes.

Here we pose the question of whether or not closed systems subject to a periodic, conservative (i.e., phase-space volume preserving), but nonsymplectic drive can exhibit long-lived prethermalization – or, equivalently, whether the stability of nonsymplectic integrators can be parametrically controlled. We give an affirmative answer by investigating the magnetization relaxation of a classical many-body spin chain [38–45] exposed to such a drive [Fig. 1a]. Unlike their symplectic Floquet counterparts [39, 46], nonsymplectic periodically driven equations of motion (EOM) cannot be derived from any time-dependent Hamiltonian, and, therefore, possess no well-defined quasienergy. As a consequence, they a priori

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cannot exhibit energy prethermalization and are not described by an effective Hamiltonian, even in the high-frequency regime.

Nevertheless, we show that nonsymplectic dynamics can exhibit quasi-conserved quantities which relax through a parametrically controlled long-lived prethermal plateau, the duration of which scales as the fourth power of the drive frequency. We propose a novel technique to derive an effective stroboscopic EOM in the high-frequency regime, by considering the spin chain as a subsystem of a larger Hamiltonian system, in which the nonsymplectic dynamics arises for a specific choice of initial conditions [Fig. 1b]. The leading-order inverse-frequency correction is sufficient to capture the magnetization relaxation process [Fig. 1d]. The nonsymplectic periodic drive we investigate generalizes to various classical spin models, irrespective of their dimensionality, support of interactions, and lattice geometry, and thus defines a new class of prethermal states.

Model—For the sake of concreteness, consider a bipartite chain of $L$ interacting classical spins $S_j$ governed by the time-periodic EOM

$$
\begin{align*}
\dot{S}_j &= J(S_{j-1} + S_{j+1}) \times S_j, \quad j \text{ even; for } t \in \left[0, \frac{\pi}{2}\right), \\
\dot{S}_j &= 0, \quad j \text{ odd; for } t \in \left[\frac{\pi}{2}, \tau\right), \\
\dot{S}_j &= J(S_{j-1} + S_{j+1}) \times S_j, \quad j \text{ even; for } t \in \left[\frac{\pi}{2}, \tau\right),
\end{align*}
$$

where $J$ denotes the interaction strength, and $j$ labels the lattice sites; we use periodic boundary conditions. During the first (second) half-cycle, the spins on the odd (even) sublattice are kept frozen, and produce an effective constant field in which their neighboring spins precess [Fig. 1a]; the roles of the two sublattices are then flipped, and the protocol repeats. Since the rotation axis depends on the neighboring spins – the directions of which keep changing – this protocol gives rise to chaotic nonlinear dynamics over many drive periods $\tau$; the frequency of switching is $\omega = 2\pi/\tau$.

We define the infinite-frequency limit by fixing a physical time in units of $J^{-1}$, and solving the EOM up to that time as $\tau \to 0$. In this limit, averaging over a period reduces Eq. (1) to the familiar Bloch equations, $\dot{S}_j = \{S_j, H\infty\}$, generated by the classical Heisenberg Hamiltonian,

$$
H_{\infty} = \frac{J}{2} \sum_{i=1}^{L} S_i \cdot S_{i+1},
$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket, with $\{S^a_i, S^b_j\} = \delta_{ij} \epsilon^{abc} S^c_j$. The infinite-frequency dynamics is thus symplectic, and conserves both the magnetization $M = \sum_j S_j$ and the infinite-frequency energy $H_{\infty}$. At finite frequency, the magnetization is no longer conserved, but $H_{\infty}$ remains conserved for all time.

A key feature of Eq. (1) is that, for finite drive frequencies, it cannot be derived from any Hamiltonian, time-dependent or static; this is a consequence of the lack of symplectic structure, cf. SM 1A. Nevertheless, it is curious to note that Eq. (1) still preserves the phase space volume [47], and hence the dynamics remains incompressible at all frequencies. Thus, symplecticity breaking is induced by the periodic drive; we analyze the extent to which this is controlled by the drive frequency $\omega$ below. We refer to the dynamics generated by Eq. (1) as nonsymplectic, in order to contrast with Floquet systems whose symplectic dynamics can be generated by nonlocal quasienergy operators.

The lack of symplecticity implies that the dynamics of Eq. (1) is not directly amenable to Floquet theory [but see Ref. [46]]. This raises two questions: (i) what are the similarities and differences between conservative-nonsymplectic (CNS) and symplectic periodically driven many-body systems, and (ii) how can we effectively describe their thermalizing dynamics?

Long-Lived Metastable Plateau—We address these questions by first simulating the thermalizing dynamics before presenting a theoretical description. We initialize the system in a thermal ensemble at temperature $\beta^{-1}$, magnetized along the $z$-direction; more specifically, this ensemble is thermal w.r.t. the Hamiltonian $H = 2H_{\infty} + h \sum_j S^z_j$, at $h/J = 0.7$. We then evolve each state in the ensemble up to a sufficiently long time, and measure the expectation value (ensemble-average) of the magnetization.

Since $H_{\infty}$ is conserved for each state in the ensemble, the system cannot heat w.r.t. $H_{\infty}$ – nevertheless, it can still absorb energy w.r.t. the Hamiltonian $\hat{H}$ that generated the initial ensemble; this gives rise to magnetization relaxation, and we therefore colloquially refer to the dynamics as “thermalizing”.

To quantify the rate at which this happens as a function of $\omega$, we study the magnetization relaxation for a few initial values of $\beta$. Figure 2 [inset] shows the time evolution of $\langle M_z \rangle$ for different values of the drive frequency $\omega$. Despite the lack of a symplectic structure in Eq. (1), we observe a behavior similar to symplectic Floquet drives. The system first prethermalizes to a frequency-dependent plateau above the infinite-frequency magnetization. The plateau lasts until a time $t_M$, parametrically controlled by $\omega$, when the ensemble starts approaching the $\langle M_z \rangle = 0$ state. Curiously, we observe a power-law scaling $t_M \propto \omega^\alpha$ with $\alpha = 4$, independently of $\beta$. This contrasts with both the exponential scaling expected to hold in locally-interacting Floquet systems [39], and the $\alpha = 2$ Fermi’s Golden rule regime characteristic of long-range systems [5]. We emphasize that the proofs of rigorous upper bounds on energy absorption explicitly use the Hamiltonian formalism [1, 2, 48], and hence do not directly apply to CNS systems.

To contrast the CNS drive from its symplectic counterpart, we consider the symplectic system $\dot{H}(t) = \sum_j J_j(t) S_{j+1} \cdot S_j$ with
Figure 2. Magnetization relaxation in the nonsymplectic drive, Eq. (1), occurs via a ‘prethermal’ plateau whose lifetime is controlled by the drive frequency $\omega$. We set $J=1$. The inset shows the curves at $\beta=1$ for drive periods $\tau=0.1$ (blue, rightmost) to 1.0 (cyan, leftmost), in steps of 0.1 – their intersection with the horizontal line defines $t_M$. The main plot shows that the relaxation timescale $t_M \sim \omega^4$ scales with the fourth power of the drive frequency $\omega$ (dotted lines). The applied field used to generate the initial thermal ensembles is $h_z/J=0.7$. The system size is $L=128$, with ensemble averages taken over 2000 initial states.

The Hamiltonian (3) sheds new light on our problem, as it allows us to think of the $S$-spins as an open system. Note, however, that the dynamics described by Eq. (1) is conservative since it satisfies the Poincaré recurrence theorem [49]. Recently, it was demonstrated that periodically-driven open quantum systems do not always possess a Floquet Lindbladian [50], which might be related to symplecticity breaking.

Adopting this view, we find that, for $a_j(t)=-S_j(t)$, the total energy of the system vanishes identically, $\mathcal{H}(t) \equiv 0$. Since the energy of the $S$-spin subsystem is independently conserved, it follows that the energy absorbed from the periodic drive in the total system remains trapped in the interaction term between the two systems. Moreover, the magnetization of the total system also vanishes identically, $\sum_j S_j + a_j = 0$. Thus, we can interpret the slow magnetization relaxation in the high-frequency limit as scrambling dynamics within the $M=0$ shell of the full system.

Let us emphasize that the specific choice of initial condition for the total system should not be viewed as fine-tuning. Indeed, we are here interested in describing the dynamics of the $S$-subsystem, for which the initial condition is arbitrary; the $a$-subsystem helps us to make analytic progress in a familiar and structured way. Due to the nonlinearity of the EOM generated by Eq. (3), we expect that small deviations from this initial condition lead to unstable dynamics that leaves the $a=-S$ manifold and features fundamentally different properties.

Floquet-Magnus Expansion—The Hamiltonian (3) enables the application of Floquet theory. Since prethermalization requires high frequencies, let us consider the application of inverse-frequency expansions (IFE). For the purpose of deriving an effective EOM, it suffices to focus on the stroboscopic dynamics which governs the motion of the slow degrees of freedom. Out of the different variants, we choose the Floquet-Magnus expansion since it does not require kick operators that modify the initial conditions [51].

A straightforward calculation of the Floquet Hamiltonian yields [SM 1B]

$$\mathcal{H}_F = \mathcal{H}_F^{(0)} + \mathcal{H}_F^{(1)} + \mathcal{O}(\omega^{-2}), \quad \mathcal{H}_F^{(n)} \propto \omega^{-n},$$

$$\mathcal{H}_F^{(0)} = J \sum_j S_j \cdot S_{j+1} + \frac{1}{2} a_j \cdot (S_{j-1} + S_{j+1}),$$

$$\mathcal{H}_F^{(1)} = -\frac{J^2}{8} \sum_j (-1)^j a_j \cdot \left[ (S_j + S_{j-2}) \times S_{j-1} + (S_{j+2} + S_j) \times S_{j+1} \right].$$

The zeroth-order term is the period-averaged Hamiltonian, which now includes static interactions between the original and the auxiliary spins. The first-order correction contains $a_j$-mediated nearest-neighbor interactions.
Interestingly, the first-order terms \( (\omega^1) \) is also captured at first-order in the IFE. Simulation dynamics does not capture heating to infinite temperature in the symplectic drive \([SM 2]\), where the effective dynamics and the exact dynamics. Main figure shows the power-law scaling of the relaxation time, \( t_M \sim \omega^4 \), (dotted lines) is also captured at first-order in the IFE. Simulation parameters are the same as in Fig. 2, with \( \beta = 1 \). The curves in the main figure correspond to \( \tau = 0.8 \).

between the spins \( \mathbf{S}_j \), and breaks time-reversal symmetry.

**Effective Stroboscopic Dynamics**—Equation 4 immediately yields the stroboscopic equations of motion, \( \mathbf{a}_j = \{ a_j, \mathcal{H}_F^{(0+1)} \} \) and \( \mathbf{S}_j = \{ \mathbf{S}_j, \mathcal{H}_F^{(0+1)} \} \). Again, it may be shown that \( a_j(0) = -S_j(0) \) implies \( a_j(t) = -S_j(t) \) within the effective dynamics \([SM 1 B]\). Using this to eliminate the \( a \)-spins, we obtain an effective EOM for the \( S \)-spins:

\[
\dot{\mathbf{S}}_j = \frac{J}{2} (\mathbf{S}_{j-1} + \mathbf{S}_{j+1}) \times \mathbf{S}_j - \frac{J^2 \tau}{8} (-1)^j \cdot
\]

\[
\left[ (\mathbf{S}_j + \mathbf{S}_{j-2}) \times \mathbf{S}_{j-1} + (\mathbf{S}_{j+2} + \mathbf{S}_j) \times \mathbf{S}_{j+1} \right] \times \mathbf{S}_j.
\]

As expected, the zeroth-order term corresponds to the time-averaged Heisenberg dynamics, Eq. (2). More interestingly, the first-order terms \( (\mathbf{S}_j \times \mathbf{S}_{j+1}) \times \mathbf{S}_j \) represent nonsymplectic corrections, as they cannot be derived from any \( S \)-subsystem Hamiltonian \( \mathcal{H}_{\text{eff}} \) via \( \langle \partial \mathcal{H}_{\text{eff}} / \partial \mathbf{S}_j \times \mathbf{S}_j \rangle \). Symplecticity is thus already broken at leading order in \( \omega^{-1} \). On the other hand, the \( O(\omega^{-1}) \) dynamics conserves the infinite-frequency energy \( H_{\infty} \). Since the exact dynamics also conserves \( H_{\infty} \), we conjecture that this is true at all orders in the IFE.

However, the \( \omega^{-1} \)-correction breaks magnetization conservation. Figure 3 shows a comparison between the exact and effective magnetization dynamics up to and including \( O(\omega^{-2}) \). Note that magnetization relaxation, including the prethermal plateau, is already captured by the \( O(\omega^{-1}) \) terms. This contrasts with energy relaxation in the symplectic drive \([SM 2]\), where the effective dynamics does not capture heating to infinite temperature at any order \([52]\), and the Floquet-Magnus expansion diverges \([2]\). Curiously, we find that the \( O(\omega^{-1}) \) dynamics relaxes magnetization faster than the exact dynamics. This is peculiar, since taking into account all higher-order corrections (i.e., considering the exact dynamics) adds more long-range and multi-body terms to the effective EOM, which one would expect to lead to faster relaxation. At the same time, our analysis reveals that the scaling of the relaxation time is \( t_M \sim \omega^4 \) in both the first- and second-order effective EOM \([Fig. 3 inset]\); the differences in relaxation times are, therefore, caused by a nonuniversal truncation-order-dependent prefactor \( c^{(n)} \): \( \Gamma_M^{(n)} = c^{(n)} \omega^4 \). We observe that the second-order dynamics leads to magnetization relaxation that is slower than the exact dynamics, hinting at the possibility of an oscillatory convergence to the exact magnetization curve as we include higher-order terms of the IFE \([53]\).

To see how the \( \omega^4 \)-scaling might arise, consider instead of Floquet-Magnus — the van Vleck IFE \([11]\) for the Floquet Hamiltonian \( \mathcal{H}_{\text{eff}} \), which is manifestly independent of the initial time \([SM 1 B 2]\). For the linearly polarized, real-valued Hamiltonian \( \mathcal{H}(t) \), a straightforward calculation shows that the \( O(\omega^{-1}) \) terms vanish, and we have \( \mathcal{H}_{\text{eff}} \sim \mathcal{H}_1^{(0)} + \mathcal{H}_1^{(2)} \). In the high-frequency regime, \( \mathcal{H}_1^{(2)} \in O(\omega^{-2}) \) plays the role of a weak \( S \)-magnetization breaking perturbation. A Fermi’s golden rule-type argument then implies that the magnetization relaxation rate cannot be faster than \( \Gamma_M \sim \omega^{-4} \), the inverse of which defines the ‘prethermal’ lifetime scaling as seen in Fig. 2.

Finally, we mention that a Hamiltonian description is not required to derive the effective EOM \([SM 1 C]\). We have independently derived Eq. (5) by using two-times perturbation theory and the phase-space density approach. We believe that an alternative Lagrangian description exists as well \([54]\).

**Discussion & Outlook**—In summary, we have identified a novel example of prethermalizing dynamics in classical bipartite periodically-driven spin systems, characterized by conservative nonsymplectic chaotic dynamics: the system we studied thus differs from conventional periodically-driven dissipative systems. The long-time behavior of the magnetization dynamics features a prethermal plateau, whose lifetime scales as the fourth power of the drive frequency. By considering the spins to be part of a larger system, we have derived an approximate description for the effective dynamics using the IFE, which captures the magnetization relaxation, and provides an argument for the observed \( \omega^4 \)-scaling law of the magnetization relaxation time.

Our \( \langle \mathbf{S}, \mathbf{a} \rangle \) model can be viewed as an interesting example of nonergodic scarred dynamics \([55–58]\) in classical many-body systems: the attainable phase space of the total \( \langle \mathbf{S}, \mathbf{a} \rangle \)-system is constrained via the initial condition for all time \([Fig. 1c]\). It will be intriguing to explore the information spreading in such constrained systems \([59–63]\). Analyzing the semiclassical and fully quantum limits of the dynamics are topics of current research. Other in-
teresting directions that go beyond the strict periodicity of the drive include generalizations to random multipolar driving and quasiperiodic extensions [7, 64–66].

Furthermore, our analysis is in practice directly relevant to classical ODE solvers designed to conserve integrals of motion exactly. In particular, our results indicate that symplectic integrators can afford a larger time step compared to nonsymplectic schemes that appear to be only power-law stable. The exact conservation of $H_\infty$, however, suggests that the nonsymplectic scheme may be a better choice for initial states in thermal equilibrium. Similar to recent work on Floquet Trotterization in quantum systems [32], the analysis of prethermal plateaus can improve the techniques for simulating equations of motion, allowing us to probe the hydrodynamic regimes of these systems with simulations of larger systems and longer times.

Conservative-nonsymplectic periodic drives, such as the one we analyzed in detail in this work, have straightforward generalizations to other spin models (e.g., XY, XXZ, etc.), including long-range interactions, higher dimensions, and various lattice types, and therefore define a new class of thermalizing systems. Extensions to rotor models or discrete periodic maps can likely be constructed as well. Our work thus opens up a new direction in the study of nonequilibrium systems without a Hamiltonian description.

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Supplemental Material
‘Prethermalization’ in Conservative Nonsymplectic Periodically Driven Spin Systems

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SM 1. ANALYTICAL PROPERTIES OF THE NONSYMPLECTIC DRIVE

A. Proof of the lack of symplectic structure in the dynamics of Eq. (1)

Here we demonstrate explicitly that the EOM in Eq. (1) are not symplectic [33, 34]. To this end, we first introduce
the relevant notions from differential geometry.

The phase space for a system of $L$ classical spins is defined as
\[ \mathcal{P} = \{(S_1, \ldots, S_N) : |S_j|^2 = 1, \ \forall j \in \{1, \ldots, L\}\}, \] (S.1)
with $S_j$ a unit vector in three-dimensional space. To incorporate the norm constraint, we can parameterize each spin
using its azimuthal angle $\varphi_j$ and its projection on the $z$-axis, $z_j$:
\[ S_j = \left( \sqrt{1 - z_j^2} \cos \varphi_j, \sqrt{1 - z_j^2} \sin \varphi_j, z_j \right)^t. \] (S.2)

Using these coordinates, the symplectic form $\omega$ w.r.t. the conjugate variables $(\varphi_j, z_j)$, can be locally defined as
\[ \omega = \sum_{j=1}^L d\varphi_j \wedge dz_j. \] (S.3)

A smooth function $f : \mathcal{P} \to \mathcal{P}$ is called symplectic, if and only if it preserves the symplectic form, i.e., $f^* \omega = \omega$, where the asterisk denotes the pullback. Given an energy function $H$, Hamilton’s equations of motion read as
\[ \dot{\varphi}_j = \partial_{z_j} H, \quad \dot{z}_j = -\partial_{\varphi_j} H. \] (S.4)

A vector field $X$ generates a flow on phase space defined as the solution to:
\[ \frac{d}{dt} S_j(t) = X(S_j(t)). \] (S.5)
Hamilton’s equation are associated with the Hamiltonian vector field $X_H$, defined implicitly by $\iota_{X_H} \omega = dH$, where $\iota$ is the exterior derivative. The flow generated by $X_H$ is called the Hamiltonian flow.

A major result in symplectic geometry is that any Hamiltonian flow is symplectic; conversely, if the flow of a complete vector field $X$ is symplectic, then $\iota_X \omega = dK$ is a closed form ($ddK = 0$), and hence the flow is locally generated by some Hamiltonian $K$ [33].

1. Nonsymplecticity of the exact EOM in Eq. (1)

Having introduced these definitions, we can now demonstrate that the flow generated by the EOM (1) is not symplectic. To do this, without loss of generality we set $J = 1$, and consider the vector field $X$ that generates the first half-cycle motion:

\[
\begin{cases}
\dot{S}_j = (S_{j-1} + S_{j+1}) \times S_j, & j \text{ even} \\
\dot{S}_j = 0, & j \text{ odd}.
\end{cases}
\]  

(S.6)

We will demonstrate that the form $\iota_X \omega$ is not closed, i.e., $d\iota_X \omega \neq 0$. Thus, the flow is not locally generated by a Hamiltonian, and hence it is not symplectic.

To see this, we use the definition in Eq. (S.3) to calculate

\[
\iota_X \omega = \sum_{j=1}^L \iota_X (d\varphi_j \wedge dz_j) = \sum_{j=1}^L \varphi_j dz_j - \dot{z}_j d\varphi_j = \sum_{j \text{ even}} \varphi_j dz_j - \dot{z}_j d\varphi_j \\
= \sum_{j \text{ even}} \left( \frac{z_{j+1} + z_{j-1} - \frac{z_j}{\sqrt{1 - z_j^2}} \left( \sqrt{1 - z_{j-1}^2} \cos(\varphi_j - \varphi_{j-1}) + \sqrt{1 - z_{j+1}^2} \cos(\varphi_j - \varphi_{j+1}) \right) }{\sqrt{1 - z_j^2}} \right) dz_j \\
- \sqrt{1 - z_j^2} \left( \sqrt{1 - z_{j-1}^2} \sin(\varphi_j - \varphi_{j-1}) + \sqrt{1 - z_{j+1}^2} \sin(\varphi_j - \varphi_{j+1}) \right) d\varphi_j,
\]  

(S.7)

where in the second line we used Eq. (S.6) written in the coordinate representation from Eq. (S.2). A straightforward calculation now gives $d\iota_X \omega \neq 0$, and hence the flow of $X$ is not symplectic. The same argument applies to the second half-cycle. Thus, we conclude that the EOM in Eq. (1) cannot be generated by a Hamiltonian function.

2. Conservation of phase-space volume

Before, we conclude the discussion, let us also prove that the phase space volume remains conserved under Eq. (1). Intuitively, for a fixed half-cycle, each spin is subject to a rotation about a fixed axis, which conserves the phase-space volume; since this is true for both half-cycles, it follows that the entire dynamics preserves the phase space volume, and hence the dynamics are conservative. In the following, we prove this mathematically.

Formally, the phase space volume is defined by the volume element

\[
dV = \bigwedge_{j=1}^L \omega_j = d\varphi_1 \wedge dz_1 \wedge \cdots \wedge d\varphi_L \wedge dz_L.
\]  

(S.8)

Liouville’s theorem reads as

\[
0 = \mathcal{L}_X dV = \iota_X ddV + ddX dV = ddX dV,
\]  

(S.9)

where we used that the volume form is closed, $ddV = 0$, since the symplectic form itself is closed. Here $\mathcal{L}_X$ denotes the Lie derivative along the flow of $X$. 

To demonstrate the validity of Liouville’s theorem for Eq. (S.6), it suffices to focus on the second spin $S_2$:

$$\mathrm{d} x \mathrm{d} V = \mathrm{d} x \left( \mathrm{d} \varphi_1 \wedge \mathrm{d} z_1 \wedge \cdots \wedge \mathrm{d} \varphi_L \wedge \mathrm{d} z_L \right)
$$

$$\begin{aligned}
&= \mathrm{d} \left( z_3 + z_1 - \frac{z_2}{\sqrt{1 - z_2^2}} \left[ 1 - z_2^2 \cos(\varphi_2 - \varphi_1) + \sqrt{1 - z_2^4 \cos(\varphi_2 - \varphi_3)} \right] \right) \mathrm{d} \varphi_1 \wedge \mathrm{d} z_1 \wedge \mathrm{d} \varphi_2 \wedge \mathrm{d} z_2 \wedge \mathrm{d} \varphi_3 \wedge \mathrm{d} z_3 \wedge \cdots \wedge \mathrm{d} \varphi_L \wedge \mathrm{d} z_L \\
&\quad + \mathrm{d} \left( -\frac{z_2}{\sqrt{1 - z_2^2}} \left[ 1 - z_2^2 \sin(\varphi_2 - \varphi_1) + \sqrt{1 - z_2^4 \sin(\varphi_2 - \varphi_3)} \right] \right) \mathrm{d} \varphi_1 \wedge \mathrm{d} z_1 \wedge \mathrm{d} \varphi_2 \wedge \mathrm{d} z_2 \wedge \mathrm{d} \varphi_3 \wedge \mathrm{d} z_3 \wedge \cdots \wedge \mathrm{d} \varphi_L \wedge \mathrm{d} z_L \\
&\quad + \text{all other even spins} \\
&= 0,
\end{aligned}$$

(S.10)

where in the last equality we used the antisymmetric property of the wedge product.

Every symplectic map preserves the phase space volume, but as we have seen, the converse is not true.

**B. Details of the auxiliary spin Hamiltonian description**

1. **Derivation of the EOM for the coupled $S$-$a$ system**

To hold just the odd sites fixed, we require something which can cancel the effective field of the even sites – but without simply measuring the even spins and then applying a site-dependent field. We achieve this by coupling the physical spins $S$ to a set of auxiliary spins $a$. The full system is illustrated in Fig. 1b, with the time evolution generated by the Hamiltonian:

$$\mathcal{H} = J \sum_j S_j \cdot S_{j+1} + J \sum_j \left( \frac{1}{2} + g(t)(-1)^j \right) a_j \cdot (S_{j-1} + S_{j+1}).$$

(S.11)

where

$$g(t) = \frac{1}{2} \text{sgn}(\sin \omega t).$$

(S.12)

It should be noted that we do not fix the dynamics of the $a$ spins – they evolve as unit-length spins under the above Hamiltonian dynamics with their own equations of motion, $\dot{a} = (\partial \mathcal{H}/\partial a) \times a$. We will, however, impose a particular set of initial conditions.

Setting $f_j(t) = \frac{1}{2} + g(t)(-1)^j$, the Hamiltonian equations of motion are:

$$\begin{aligned}
\dot{S}_j &= J (S_{j-1} + S_{j+1}) a_{j-1} + f_{j-1}(t) a_{j+1} + f_{j+1}(t) a_j \times S_j \\
\dot{a}_j &= J f_j(t) (S_{j-1} + S_{j+1}) \times a_j.
\end{aligned}$$

(S.13)

We fix the initial conditions as $a_j(0) = -S_j(0)$. Now, over the first half-period, $0 < t < \tau/2$, we have $g(t) = +1/2$, which implies

$$\dot{a}_j = \begin{cases} J(S_{j-1} + S_{j+1}) \times a_j, & j \text{ even} \\ \text{0}, & j \text{ odd} \end{cases}$$

(S.14)

and

$$\dot{S}_j = \begin{cases} J(S_{j-1} + S_{j+1}) \times S_j, & j \text{ even} \\ J(S_{j-1} + S_{j+1}) + a_{j-1} + a_{j+1} \times S_j, & j \text{ odd} \end{cases}$$

(S.15)

We will now show that, over this half-period, $\dot{S}_j = 0$ for odd $j$ (and so for even $j$, $S_j$ evolves in a constant effective field), and, moreover, that the initial condition is preserved, i.e., $\forall j, \forall t : S_j(t) = -a_j(t)$. First, observe that for
even \( j \), \( \mathbf{S}_j \) and \(-\mathbf{a}_j\) have the same equation of motion with the same initial condition. Thus \( \mathbf{S}_j(t) = -\mathbf{a}_j(t) \) for even \( j \). This then implies that, for odd \( j \), the effective field seen by \( \mathbf{S}_j \) vanishes at all times in the half-period, and thus \( \dot{\mathbf{S}}_j = 0 \) for all odd \( j \). The equations of motion directly establish \( \dot{\mathbf{a}}_j = 0 \) for odd \( j \), and so the initial conditions are preserved.

This argument is clearly symmetric with respect to the parity of \( j \), and so the opposite situation holds over the next half-period. Since the initial condition is preserved throughout, the Hamiltonian (S.11) exactly reproduces the nonsymplectic drive.

2. Derivation of the Floquet-Magnus Hamiltonian

Our goal here is to find the time-independent Floquet Hamiltonian \( \mathcal{H}_F[t_0] \) which governs the stroboscopic dynamics of the combined system consisting of the original spins \( \mathbf{S} \) and the auxiliary spins \( \mathbf{a} \). Note that \( \mathcal{H}_F[t_0] \) depends explicitly on the initial choice for the phase of the drive or, equivalently, on the initial time \( t_0 \), and we keep track of this dependence below.

Then, using the Floquet-Magnus expansion [11], we can construct the Floquet Hamiltonian as a series in the drive-period \( \tau \):

\[
\mathcal{H}_F[t_0] = \sum_{n=0}^{\infty} \mathcal{H}_F^{(n)}[t_0],
\]

where the superscript \((n)\) means \( \mathcal{H}_F^{(n)} \propto \mathcal{O}(\tau^n) = \mathcal{O}(\omega^{-n}) \). The lowest-order contributions to the Floquet Hamiltonian are, explicitly,

\[
\mathcal{H}_F^{(0)}[t_0] = \mathcal{H}_0
\]

\[
\mathcal{H}_F^{(1)}[t_0] = \frac{1}{2\tau} \int_{t_0}^{t_0+\tau} dt_1 \int_{t_0}^{t_1} dt_2 \{H(t_1), H(t_2)\}
\]

\[
\mathcal{H}_F^{(2)}[t_0] = \frac{1}{3!\tau} \int_{t_0}^{t_0+\tau} dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 \{\{H(t_1), H(t_2), H(t_3)\} + \{H(t_3), H(t_2), H(t_1)\}\}.
\]

In the subsequent discussion, it will be convenient to separate out the time-averaged term \( \mathcal{H}_0 \) and the time-dependent term \( \mathcal{V} \),

\[
\mathcal{H}(t) = J \sum_j \left[ \mathbf{S}_j \cdot \mathbf{S}_{j+1} + \frac{1}{2} \mathbf{a}_j \cdot (\mathbf{S}_{j-1} + \mathbf{S}_{j+1}) \right] + g(t)J \sum_j (-1)^j \mathbf{a}_j \cdot (\mathbf{S}_{j-1} + \mathbf{S}_{j+1}) = \mathcal{H}_0 + g(t)\mathcal{V}.
\]

The explicit time-dependence is now only carried by the function \( g(t) \), and the first-order term becomes

\[
\mathcal{H}_F^{(1)}[t_0] = \frac{1}{2\tau} \int_{t_0}^{t_0+\tau} dt_1 \int_{t_0}^{t_1} dt_2 (g(t_1) - g(t_2)) \{\mathcal{V}, \mathcal{H}_0\} = \left(\frac{t_0}{2} - \frac{\tau}{8}\right)\{\mathcal{V}, \mathcal{H}_0\}.
\]

A straightforward calculation yields the required Poisson bracket,

\[
\{\mathcal{V}, \mathcal{H}_0\} = J^2 \sum_j (-1)^j \varepsilon^{j\mu\nu\lambda} (\mathbf{a}_j^\mu + \mathbf{a}_{j-2}^\mu) (\mathbf{S}_j^\nu + \mathbf{S}_{j-2}^\nu) \mathbf{S}_{j+1}^\lambda.
\]

Rewriting this in terms of dot and cross products, and shifting some site labels, we arrive at the first-order term,

\[
\mathcal{H}_F^{(1)}[t_0] = \left(\frac{t_0}{2} - \frac{\tau}{8}\right) J^2 \sum_j (-1)^j \mathbf{a}_j \cdot [(\mathbf{S}_j + \mathbf{S}_{j-2}) \times \mathbf{S}_{j-1} + (\mathbf{S}_{j+2} + \mathbf{S}_j) \times \mathbf{S}_{j+1}].
\]

3. Effective EOM

The equation of motion to order \( \omega^{-1} \) can be derived from the effective Hamiltonian \( \mathcal{H}_F^{(0)} + \mathcal{H}_F^{(1)} \). We fix the Floquet gauge by setting \( t_0 = 0 \), and thus obtain

\[
\dot{\mathbf{a}}_j = \frac{\partial \mathcal{H}_F^{(0)}}{\partial \mathbf{a}_j} \times \mathbf{a}_j + \frac{\partial \mathcal{H}_F^{(1)}}{\partial \mathbf{a}_j} \times \mathbf{a}_j
\]

\[
= \left(\frac{J}{2} (\mathbf{S}_{j-1} + \mathbf{S}_{j+1}) - \frac{\tau J^2}{8} (-1)^j [(\mathbf{S}_j + \mathbf{S}_{j-2}) \times \mathbf{S}_{j-1} + (\mathbf{S}_{j+2} + \mathbf{S}_j) \times \mathbf{S}_{j+1}]\right) \times \mathbf{a}_j.
\]
Using the conditions $a_j(t) = -S_j(t)$, we find
\[ \dot{S}_j = \frac{J}{2} (S_{j-1} + S_{j+1}) \times S_j + \frac{\tau J^2}{8} (-1)^j \left[ (S_j + S_{j-2}) \times S_{j-1} + (S_{j+2} + S_j) \times S_{j+1} \right] \times S_j. \] (S.25)

Showing that the initial condition $a_j(0) = -S_j(0)$ is conserved by the dynamics is a simple matter of deriving the general equation of motion for the $S$-spins directly, and checking that we obtain the same effective equation of motion (S.25). We have, to first order,
\[ \dot{S}_j = \frac{\partial H_F^{(0)}}{\partial S_j} \times S_j + \frac{\partial H_F^{(1)}}{\partial S_j} \times S_j \]
\[ = \left( J (S_{j-1} + S_{j+1}) + \frac{J}{2} (a_{j-1} + a_{j+1}) + \frac{\tau J^2}{8} (-1)^j \left[ (a_j + a_{j-2}) \times S_{j-1} \right. \right. \]
\[ + (a_{j+2} + a_j) \times S_{j+1} + (a_{j+1} + a_{j-1}) \times (S_{j+1} + S_{j-1}) \left. \right] \right) \times S_j, \] (S.26)
which, upon inserting the condition $a_j = -S_j$, may be readily seen to reduce to the same effective equation of motion Eq. (S.25).

In a similar manner, we derive the second-order contribution to the Floquet-Hamiltonian, $H_F^{(2)}$. Setting $t_0 = 0$, the effective equations of motion are, to second-order,
\[ \dot{S}_j = \frac{J}{2} (S_{j-1} + S_{j+1}) \times S_j \]
\[ - \frac{J^2 \tau}{8} (-1)^j \left[ (S_j + S_{j-2}) \times S_{j-1} + (S_j + S_{j+2}) \times S_{j+1} \right] \times S_j \]
\[ + \frac{J^3 \tau^2}{96} \left[ 2 (S_{j-1} \cdot S_{j-2}) S_{j-3} \right. \]
\[ + [S_{j-1} \cdot (S_{j+1} + S_j + S_{j-2} - 2S_{j-3}) - 1] S_{j-1} \]
\[ + [S_j \cdot (S_{j+1} + S_{j-1} - S_{j-2}) + S_{j-2} \cdot (S_{j-1} - S_{j+1}) - 2] S_{j-1} \]
\[ + [S_{j+2} \cdot (S_{j-1} + S_{j+1} - S_{j+2}) + S_{j-1} \cdot (S_{j+1} - S_{j-1} - 2] S_{j+1} \]
\[ + [S_{j+1} \cdot (S_{j-1} + S_j + S_{j+2} - 2S_{j+3}) - 1] S_{j+2} \]
\[ + [2 S_{j+1} \cdot S_{j+2}] S_{j+3} \left. \right] \times S_j \]
\[ + O(\tau^3). \] (S.27)

4. Proof that the effective EOM are nonsymplectic

The effective equations of motion for the $S$-subsystem break symplecticity at first order. To see this, rather than considering the flow of vector fields, let us note that we have terms of the form
\[ \dot{S} = (S \times h) \times S + ... \] (S.28)
in the first-order EOM. For such a term to arise from Hamilton’s equations, $\dot{S} = \partial H/\partial S \times S$, we would require a Hamiltonian $H$ containing a term $K$ such that
\[ \frac{\partial K}{\partial S^\mu} = \epsilon^{\mu\nu\lambda} S^\nu h^\lambda. \] (S.29)
Up to an irrelevant constant, such a $K$ must be quadratic in $S$ and linear in $h$, for which the most generic possibility is
\[ K = A^{\mu\nu\lambda} S^\mu S^\nu h^\lambda, \] (S.30)
for some arbitrary rank-3 tensor $A$. But now we obtain
\[ \frac{\partial K}{\partial S^\mu} = (A^{\mu\nu\lambda} + A^{\nu\mu\lambda}) S^\nu h^\lambda, \] (S.31)
and since $A^{\mu\nu\lambda} + A^{\nu\mu\lambda} = \epsilon^{\mu\nu\lambda}$ is a contradiction [the left-hand-side is symmetric w.r.t. $\mu \leftrightarrow \nu$, while $\epsilon^{\mu\nu\lambda}$ on the right-hand side is anti-symmetric by definition], we conclude that such terms as (S.28) cannot be obtained from a Hamiltonian, and, thus, are nonsymplectic.
C. Alternative derivations of the effective EOM

In this appendix, we provide two alternative ways to derive the effective equations of motion using (i) two-times perturbation theory, and (ii) the equation of motion for the phase space density (Liouville’s equation). Both methods can be generalized to higher orders in a straightforward way.

1. Two-times perturbation theory

Here we illustrate an alternative method via two-times perturbation theory [67] to derive the effective EOM for the nonsymplectic drive.

We start by considering the exact EOM, given by:

\[ \dot{S}_j = -J \left[ \frac{1}{2} + (-1)^j g(t) \right] S_j \times (S_{j-1} + S_{j+1}), \]  

(S.32)

where \( g(t) = \frac{1}{2} \text{sgn}(\sin \omega t) \). When the driving period is sufficiently small, one can decompose the time evolution for spin \( S_j \) into a fast motion variable (denoted by \( \eta_j \)) and slow motion (denoted by \( \sigma_j \)) as

\[ S_j = \sqrt{1 - \eta_j^2} \sigma_j + \eta_j, \]  

(S.33)

where the square root is required to keep the norm of the spin vector \( S_j \) constant at unity at all times.

To derive an ansatz for the structure of the fast motion variable, we first take the exact EOM, and explicitly separate the time-average from the rest:

\[ \dot{S}_j = -\frac{J}{2} S_j \times (S_{j-1} + S_{j+1}) + J(-1)^j g(t) S_j \times (S_{j-1} + S_{j+1}). \]  

(S.34)

The time-average term does not contribute to the fast motion \( \eta_j \), by definition. Thus, integrating the right-hand side over the fast time, the leading order contribution to the fast motion is given by

\[ \eta_j(t) = -J(-1)^j \left( \int_0^t dt' g(t') \right) \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) := -J(-1)^j G(t) \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}). \]  

(S.35)

Note that \( G(t) \), shown in Fig. S1, has the maximum value \( \tau/4 \), and is thus \( O(\omega^{-1}) \). The fast motion \( \eta_j \) is therefore also \( O(\omega^{-1}) \) and, as expected, vanishes as \( \omega \to \infty \).

Our goal is to derive an EOM to order \( \omega^{-1} \) for the slow motion \( \sigma_j \). Taking the full time derivative of Eq. (S.33) (w.r.t. the time \( t \) which contains both the slow and the fast time variable), one obtains

\[ \dot{\sigma}_j = \frac{\dot{S}_j - \eta_j}{\sqrt{1 - \eta_j^2}} + \frac{\eta_j \cdot \dot{\sigma}_j}{1 - \eta_j^2}. \]  

(S.36)

We now want to eliminate the \( \eta_j \) and \( S_j \) dependence from the right-hand side. We first use the fact that \( \eta_j \) is \( O(\omega^{-1}) \) to simplify the denominators by discarding any terms of higher-order than \( \omega^{-1} \) in inverse-frequency, which yields

\[ \sqrt{1 - \eta_j^2} = 1 + O(\omega^{-2}), \quad 1 - \eta_j^2 = 1 + O(\omega^{-2}). \]  

(S.37)
Then we can find an expression for \( \dot{S}_j \) by inserting the ansatz from Eq. (S.33) into Eq. (S.32), which leads to
\[
\dot{S}_j = -J \left[ \frac{1}{2} + (-1)^j g(t) \right] (\sigma_j + \eta_j) \times \left[ (\sigma_{j-1} + \eta_{j-1}) + (\sigma_{j+1} + \eta_{j+1}) \right] + O(\omega^{-2}),
\]
where we also made use of Eq. (S.37). At the same time, taking the derivative of Eq. (S.35) w.r.t. the full time variable (fast and slow), we arrive at
\[
\eta_j = -J(-1)^j g(t) \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) - J(-1)^j G(t) \frac{d}{dt} \left[ \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) \right].
\]
The two equations above for the derivatives \( \eta_j \) and \( \dot{S}_j \) can now be inserted in Eq. (S.36):
\[
\dot{\sigma}_j = \dot{S}_j - \eta_j + \eta_j \cdot \eta_j \sigma_j + O(\omega^{-2})
\]
\[
= -J \left[ \frac{1}{2} + (-1)^j g(t) \right] \left[ \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) + \eta_j \times (\sigma_{j-1} + \sigma_{j+1}) + \sigma_j \times (\eta_{j-1} + \eta_{j+1}) \right]
+ J(-1)^j g(t) \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) + J(-1)^j G(t) \frac{d}{dt} \left[ \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) \right]
+ J^2 G(t) g(t) \sigma_j [\sigma_j \times (\sigma_{j-1} + \sigma_{j+1})]^2
+ O(\omega^{-2}),
\]
where, again, Eq. (S.37) has been used. There are two remaining sources of fast motion on the right-hand side of this equation. We eliminate the first, the variable \( \eta_j \), using Eq. (S.35). The second source is the time-dependence of the functions \( g(t) \) and \( G(t) \), which oscillate rapidly in the high-frequency regime \( \omega \gg J \) - we thus also need to average over the fast timescale.

To do this, for any function \( h(t) \), we define \( \overline{h} = \tau^{-1} \int_0^\tau h(t) dt \) as the time averaged value over one period \( \tau \), and obtain [cf. Fig. S1]
\[
\overline{g(t)} = 0, \quad \overline{G(t)} = \frac{\tau}{8}, \quad \overline{g(t)G(t)} = 0,
\]
see Fig. S1. Next, we insert Eq. (S.35) into Eq. (S.40) and perform the time average of the resulting equation over a single period \( \tau \). In the limit \( \omega \gg J \), one may assume that the slow variable \( \sigma_j \) does not change over the time \( \tau \). This yields
\[
\dot{\sigma}_j = -\frac{J}{2} \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) + \frac{(-1)^j J^2 G}{2} \left[ \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) \right] \times (\sigma_{j-1} + \sigma_{j+1})
- \frac{J^2 (-1)^j G}{2} \sigma_j \times (\sigma_{j-1} - (\sigma_{j-2} + \sigma_j) + \sigma_{j+1} \times (\sigma_j + \sigma_{j+2})]
+ J(-1)^j G \left[ \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) + \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) \right]
+ O(\omega^{-2}).
\]
Finally, in order to derive a self-consistent EOM for \( \sigma_j \), we need to eliminate \( \dot{\sigma}_j \) from the terms on right-hand side. This can be done by noting that all such terms come with an \( O(\omega^{-1}) \) prefactor, since \( G \) is \( O(\omega^{-1}) \). Hence, the derivatives \( \dot{\sigma}_j \) on the right-hand-side can be eliminated by using the EOM, Eq. (S.42) itself, but in this instance retaining only terms of \( O(1) \),
\[
\dot{\sigma}_j = -\frac{J}{2} \sigma_j \times (\sigma_{j-1} + \sigma_{j+1}) + O(\omega^{-1}).
\]
Inserting this back in Eq. (S.42), we arrive at the effective EOM for the slow variable \( \sigma_j \),
\[
\dot{\sigma}_j = \frac{J}{2} (\sigma_{j-1} + \sigma_{j+1}) \times \sigma_j - \frac{(-1)^j J^2 T}{8} \left[ (\sigma_{j-2} + \sigma_j) \times \sigma_{j-1} + (\sigma_j + \sigma_{j+2}) \times \sigma_{j+1} \right] \times \sigma_j + O(\omega^{-2}).
\]
2. Effective Liouville equation approach

The modern theoretical analysis of prethermalization is based on Floquet’s theorem, which requires the linearity of the equations of motion (EOM) [68]. Since thermalization can microscopically be traced back to chaotic trajectories,
and chaos in classical systems can only occur in nonlinear EOM \[69\], this might at first appear paradoxical. A similar “problem” with Floquet’s theorem occurs for quantum dynamics in the Heisenberg picture, where the EOM are also nonlinear. The application of Floquet’s theorem in these cases is justified by the linearity of the alternative Schrödinger picture based on the Liouville-von Neumann equation for the density operator (phase-space density) \[39\], which is defined by means of a commutator (Poisson bracket) structure in Hamiltonian mechanics \[39\]. For classical systems, it is equivalent to the existence of conjugate variables, and induces a symplectic structure on phase space.

Yet a third way to obtain the effective EOM in a systematic expansion controlled by \(\omega^{-1}\) is as follows: note that the Liouville equation, Eq. (S.9), is a linear, time-periodic ODE for the phase space density. Although, for our system, it cannot be written in its familiar form using the Poisson bracket due to the lack of a symplectic structure, the latter is not required to apply Floquet’s theorem.

Indeed, by exploiting this fact, it was shown in Ref. \[46\] that the Floquet-Magnus expansion can be applied to differential equations of the form

\[
\dot{\vec{S}} = \vec{X}(\vec{S}, t), \quad \vec{X}(\vec{S}, t + \tau) = \vec{X}(\vec{S}, t),
\]

where \(\vec{S}\) denotes all spin variables of the system, and \(\vec{X}\) is a (possibly nonlinear) but time-periodic vector field flow.

The slow dynamics of the system is then captured by the effective EOM

\[
\dot{\vec{S}} = \vec{X}_{\text{eff}}(\vec{S});
\]

To leading order in the inverse-frequency, the Floquet-Magnus expansion, we have

\[
\vec{X}_{\text{eff}} = \sum_{n=0}^{\infty} \vec{X}_{\text{eff}}^{(n)}, \quad \vec{X}_{\text{eff}}^{(n)} \propto \omega^{-n},
\]

\[
\vec{X}_{\text{eff}}^{(0)} = \frac{1}{\beta} \int_0^t \mathrm{d}t \vec{X}(\vec{S}, t),
\]

\[
\vec{X}_{\text{eff}}^{(1)} = \frac{1}{4\pi\omega} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \left[ \vec{X}(\vec{S}, t_1), \vec{X}(\vec{S}, t_2) \right],
\]

where the Lie bracket \([\cdot, \cdot]_L\) of two vector fields \(\vec{X}(\vec{S})\) and \(\vec{Y}(\vec{S})\) is defined as

\[
L_\vec{X}\vec{Y} = \left[ \vec{X}(\vec{S}), \vec{Y}(\vec{S}) \right]_L = \vec{X}(\vec{S}) \cdot \nabla_\vec{S} \vec{Y}(\vec{S}) - \vec{Y}(\vec{S}) \cdot \nabla_\vec{S} \vec{X}(\vec{S}).
\]

Performing the derivatives and calculating the time-ordered integral, we arrive at the effective EOM in Eq. (5).

### SM 2. COMPARISON BETWEEN SYMPLECTIC AND NONSYMPLECTIC DRIVES

In the main text we have analysed a nonsymplectic drive for the classical Heisenberg chain, given by the successive evolution of even and odd-numbered sites. A natural complement of this is to consider a drive given by the successive evolution of even and odd-numbered bonds.

In contrast to the nonsymplectic site-based drive considered in the main text, the bond-based drive is symplectic – and we therefore refer to it as the symplectic drive. It is generated by the Hamiltonian

\[
H_{\text{sympl}}(t) = J \sum_j \left( \frac{1}{2} + (-1)^j g(t) \right) S_j \cdot S_{j+1},
\]

which has the same infinite-frequency limit \(H_\infty\) as the nonsymplectic drive. However, the two drives differ in the way they approach the infinite-frequency limit.

Similarly to the nonsymplectic drive, each step of the symplectic drive is exactly solvable. In contrast to the nonsymplectic drive, the symplectic drive exactly conserves the magnetization; it does not, however, conserve the energy \(H_\infty\) (at finite frequency). This is the canonical situation to which rigorous estimates for the rate of energy absorption \[39\] apply, and we numerically verify these predictions.

We take the same initial ensemble at \(\beta = 1\) as we used for the nonsymplectic evolution in the main text. In Fig. S2, we find that the system heats up to the maximum entropy state consistent with the conserved magnetization (i.e., \(\langle E \rangle \to J \langle M \rangle^2\)), and that, as expected from Ref. \[39\], the heating time \(t_E\) is exponentially suppressed with the driving frequency, \(t_E \sim \exp(-c\omega)\), for some model-dependent constant \(c\). This stands in stark contrast to the algebraic suppression observed for the nonsymplectic drive in the main text.
Figure S2. Comparison between the magnetization loss in the nonsymplectic drive (left) and Floquet heating in the symplectic drive (right), for the same initial ensemble (2000 states, $\beta = 1$, $L = 128$). (Upper panels) Both drives exhibit a prethermal plateau (see insets), but heating in the symplectic drive is suppressed exponentially, compared to the power-law suppression in the nonsymplectic drive. Inset for the nonsymplectic drive shows the curves from $\tau = 0.1$ (rightmost) to $\tau = 1.0$ (leftmost); inset for the symplectic drive shows the curves from $\tau = 0.8$ (rightmost) to $\tau = 1.8$ (leftmost). The functional form of the suppression is independent of the threshold value (controlled by $\kappa$) used to define $t_M$ and $t_E$ – the dotted lines in the insets correspond to $\kappa = 0.5$ (top) to $\kappa = 0.9$ (bottom). (Lower panels) show convincingly that the nonsymplectic drive is not described by an exponential suppression, and, conversely, that the symplectic drive is not described by a power-law suppression.

SM 3. DETAILS OF THE NUMERICAL SIMULATIONS

In this appendix we provide further details of our numerical procedures. We first discuss the construction of the initial states in the thermal ensembles, and then give an overview of the integration of the equations of motion.

A. Initial ensemble

The initial states are constructed using heatbath Monte Carlo (MC) simulations [70], which uses the fact that the thermal distribution of a single spin $S_i$ in its local field $S_{i-1} + S_{i+1} + h\hat{z}$ is exactly invertible (see also supplementary material of Ref. [71]). We use ensembles of 2000 states, and each state begins as a completely independent random configuration. We then perform $L \times 10^5$ heatbath updates (randomly selecting the spin to be redrawn from its local thermal distribution) to cool the state to the desired temperature. We use the same set of 2000 initial states for all dynamical evolution protocols (for a fixed $L$ and $\beta$), to ensure a fair comparison between the symplectic and nonsymplectic drives, and between the exact nonsymplectic dynamics and the dynamics given by the effective Floquet-Magnus Hamiltonian; we have checked that using different initial ensembles does not change the reported results.
B. Dynamical evolution

We now turn to the details of the dynamical evolution. The nonsymplectic drive, Eq. (1), and the symplectic drive in Eq. (S.49), can be integrated to machine precision, since the exact solution can be written in closed form for each step of the drive: a single spin in a constant magnetic field evolves as

\[ \dot{S} = M \times S \Rightarrow S(t) = \exp(M \cdot R t)S(0), \]  

where \( R \) denotes the (vector of) the generators of rotations. For the nonsymplectic drive, half of the spins \( S_i \) evolve in the constant (over the half-period) field \( S_{i-1} + S_{i+1} \); for the symplectic drive, the spins on a bond \( \{i, i+1\} \) evolve in the constant field \( S_i + S_{i+1} \).

The fact that each step of the drive is exactly solvable means that the numerical evolution is very efficient (since the timestep is \( \tau/2 \)) and accumulates only machine precision errors, allowing us to evolve to long times \( t_f = 10^8 \), or even \( t_f = 10^9 \). The values of the energy \( H_\infty \) and the magnetization \( M_z \) are stored at \( 10^4 \) stroboscopic times on a log-spaced (to the nearest stroboscopic time \( t \in \tau \mathbb{Z} \)) grid.

In contrast, the dynamics generated by the effective Hamiltonians \( H_{1F}^{(1)} \) and \( H_{1F}^{(2)} \) are not exactly integrable, and we use the standard fourth-order Runge-Kutta (RK4) method with a timestep of \( \delta t = 0.001 \) (in units of \( |J|=1 \)). We store the values of the observables at the stroboscopic times on the log-spaced grid used for the exact dynamics, up to the final time of the RK4 simulations, \( t_f = 10^6 \). With these values of \( \delta t \) and \( t_f \), the typical error in the energy density over the simulations (which should be conserved by the nonsymplectic drive and its effective Hamiltonians) is \( \sim 10^{-12} \).