ASPECTS OF COULOMB GASES

DJALIL CHAFAI

Abstract. Coulomb gases are special probability distributions, related to potential theory, that appear at many places in pure and applied mathematics and physics. In these short expository notes, we focus on some models, ideas, and structures. We present briefly selected mathematical aspects, mostly related to exact solvability and first and second order global asymptotics. A particular attention is devoted to two-dimensional exactly solvable models of random matrix theory such as the Ginibre model. Thematically, these notes lie between probability theory, mathematical analysis, and statistical physics, and aim to be very accessible. They form a contribution to a volume of the Panoramas et Synthèses series around the workshop États de la recherche en mécanique statistique, organized by Société Mathématique de France, held at Institut Henri Poincaré, Paris, in the fall of 2018 (https://statmech2018.sciencesconf.org/).

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There are several introductory texts around Coulomb gases. We refer for instance to [ER05, DG09, Dei99, AGZ10, ABDF11, For10] for the relation to random matrices, to [Ser15] for the relation to analysis and Ginzburg–Landau vortices, to [Bou15, GZ19b] and references therein for a relation to geometry, to [But17] and references therein for a relation to random polynomials, to [Rou15] for a relation to Fock–Hartree quantum theory and Bose–Einstein condensates, to [Ser18b] and [Lew21] for an overview from a mathematical analysis/physics perspective, and to [LACTMS19] for a statistical physics point of view.

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Notation

The Euclidean norm of \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) is \( |x| = \sqrt{x_1^2 + \cdots + x_d^2} \). It is the modulus if \( d = 2 \) with the identification \( \mathbb{C} = \mathbb{R}^2 \). We set \( i = (0,1) \in \mathbb{C} \). The real and imaginary parts of \( z \in \mathbb{C} \) are denoted \( \Re z \) and \( \Im z \). The Lebesgue measure is denoted \( dx \). Let \( (E, \tau) \) be a topological space with Borel \( \sigma \)-field \( \mathcal{B}(E) \). We denote by \( \mathcal{C}_b(E, \mathbb{R}) \) the set of bounded continuous functions \( E \to \mathbb{R} \), and by \( \mathcal{M}_1(E) \) the set of probability measures on \( (E, \mathcal{B}(E)) \). If \( \mu_1, \mu_2, \ldots, \mu \) are in \( \mathcal{M}_1(\mathbb{R}^d) \), then \( \lim_{n \to \infty} \mu_n = \mu \) weakly, denoted

\[
\mu_n \overset{c_1}{\underset{n \to \infty}{\rightharpoonup}} \mu, \quad \text{when for all } f \in \mathcal{C}_b(E, \mathbb{R}) \text{ we have } \lim_{n \to \infty} \int f d\mu_n = \int f d\mu.
\]

This defines a sequential topology on \( \mathcal{M}_1(E) \), giving a Borel \( \sigma \)-field \( \mathcal{B}(\mathcal{M}_1(E)) \). A random probability measure on \( E \) is a random variable taking values in \( \mathcal{M}_1(E) \). By \( X \sim \mu \) we mean that the random variable \( X \) has law \( \mu \). We denote by \( \overset{d}{\Rightarrow} \) and \( \overset{d}{\rightarrow} \) the equality and the convergence in law respectively.

1. Coulomb electrostatics and equilibrium measures

The Coulomb kernel is identical to the Newton kernel. Mathematically, potential theory deals with the analysis of the Laplacian and its Green function and the behavior of harmonic functions. In some sense, it emerges naturally from the gravitational theory of Johannes Kepler and Isaac Newton, as well as from the modeling of electrostatics, namely the study of the distribution of static electric charges on conductors and their interactions. From this last point of view, it takes its historical roots in the works of Charles-Augustin de Coulomb, Joseph-Louis Lagrange, and Carl Friedrich Gauss. Some mathematical parts of potential theory were developed later on by – among others – Johann Peter Gustav Lejeune Dirichlet, Augustin de Coulomb, Joseph-Louis Lagrange, and Carl Friedrich Gauss. Some mathematical parts of potential theory were developed later on by – among others – Johann Peter Gustav Lejeune Dirichlet, Augustin de Coulomb, Joseph-Louis Lagrange, and Carl Friedrich Gauss. Some mathematical parts of potential theory were developed later on by – among others – Johann Peter Gustav Lejeune Dirichlet, Augustin de Coulomb, Joseph-Louis Lagrange, and Carl Friedrich Gauss.

\[
\text{Let } d \geq 1. \text{ The Coulomb kernel } g \text{ in } \mathbb{R}^d \text{ is given by } g(0) = +\infty, \text{ and, for all } x \in \mathbb{R}^d \setminus \{0\},
\]

\[
g(x) = \begin{cases} 
\log \frac{1}{|x|} & \text{if } d = 2, \\
\frac{1}{(d-2)|x|^{d-2}} & \text{if not.}
\end{cases}
\]

We say that \( (x, y) \mapsto G(x, y) = g(x - y) \) is the Green function of the Laplace operator \( \Delta = \partial_x^2 + \cdots + \partial_y^2 \), and \( g \) is the fundamental solution of the Poisson equation, see for instance [LL01] Theorem 6.20. Indeed, denoting \( \delta_0 \) the Dirac mass at the origin, we have, in the sense of Schwartz distributions,

\[
-\Delta g = c_d \delta_0 \quad \text{and} \quad c_d = d \omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}
\]

where \( \omega_d \) is the volume of the unit ball (its surface is \( d \omega_d \)). The case \( d = 3 \) is physical for electrostatics modeling in the ambient space. The case \( d = 2 \) also appears at many places in mathematical physics. The case \( d = 1 \) serves historically as a toy model, less singular but exactly solvable.

For simplicity, we suppose from now on that \( d \geq 2 \).
Table 1.1. The arrow of time and some of the main actors mentioned in the text or in the references. As mentioned in [Cha15], the Stigler law of eponymy states that “No scientific discovery is named after its original discoverer,” attributed by Stephen Stigler to Robert K. Merton. This is also known as the Arnold principle by some people.

Figure 1.1. Coulomb kernel in dimension 1 (solid line) 2 (dotted line) and 3 (dashed line).

For all \( \mu \in M_1(\mathbb{R}^d) \) such that \( \log(1 + |\cdot|)1_{d=2} \in L^1(\mu) \), the Coulomb energy of \( \mu \) is

\[
\mathcal{E}(\mu) = \frac{1}{2} \iint g(x - y)\,d\mu(x)d\mu(y) \in (-\infty, +\infty].
\] (1.3)

Note that if \( d = 2 \) then \( \mathcal{E}(\mu) = +\infty \) if \( \mu \) has a Dirac mass. If \( \mu \) models the distribution of unit charges (say electrons) in \( \mathbb{R}^d \) then \( \mathcal{E}(\mu) \) is the electrostatic self-interaction energy of the configuration \( \mu \).

We say that a Borel set \( B \in \mathcal{B}(\mathbb{R}^d) \) is of positive capacity when \( \text{supp}(\mu) \subset B \) and \( \mathcal{E}(\mu) < \infty \) for some \( \mu \in M_1(\mathbb{R}^d) \), and is of zero capacity when it does not carry a probability measure \( \mu \) with \( \mathcal{E}(\mu) < \infty \).

For all \( \mu \in M_1(\mathbb{R}^d) \) with \( \log(1 + |\cdot|)1_{d=2} \in L^1(\mu) \), the Coulomb potential of \( \mu \) at \( x \in \mathbb{R}^d \) is defined by

\[
U_\mu(x) = \int g(x - y)\,d\mu(y) = (g * \mu)(x).
\]
We have $U_\mu(x) \in (-\infty, +\infty]$, and $U_\mu(x) = +\infty$ if $\mu$ has a Dirac mass at point $x$. We also have
\[ \mathcal{E}(\mu) = \frac{1}{2} \int U_\mu(x) d\mu(x). \] (1.4)

Since $g \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$, the Fubini–Tonelli theorem gives $U_\mu \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$, hence $U_\mu < +\infty$ almost everywhere. Moreover $U_\mu = g * \mu$, and, in the sense of Schwartz distributions, we get from \[ \text{[Lan72]} \] that
\[ \Delta U_\mu = -c_d \mu. \] (1.5)

In particular, this gives the formula
\[ \mathcal{E}(\mu) = \frac{1}{2} \int U_\mu d\mu - \frac{1}{2c_d} \int U_\mu \Delta U_\mu dx. \] (1.6)

When $d \geq 3$, the functional $\mathcal{E}$ does not take negative values on probability measures because $g \geq 0$. However, when $d = 2$, the functional $\mathcal{E}$ may take negative values on compactly supported probability measures, due to the change of sign of $g$ when $d = 2$ inside and outside the unit ball. For instance if $\mu_r$ is the uniform law on the circle $\{x \in \mathbb{C} : |x| = r\}$ of radius $r > 0$ then, for all $x \in \mathbb{C}^2$,
\[ U_{\mu_r}(x) = -\log(r) 1_{|x| \leq r} - \log |x| 1_{|x| > r}, \quad \text{and} \quad \mathcal{E}(\mu_r) = -\frac{\log(r)}{2}, \] (1.7)
which is negative if $r > 1$. See for instance \[ \text{[ST97]} \] (0.5.5) and (1.1.6) for these computations. Similarly, if $\mu_R$ is the uniform law on the disc $\{x \in \mathbb{C} : |x| \leq R\}$ of radius $R > 0$ then we find that for all $x \in \mathbb{C}^2$,
\[ U_{\mu_R}(x) = -\frac{1}{2} \left( \frac{|x|^2}{R^2} - 1 + 2 \log R \right) 1_{|x| \leq R} - \log |x| 1_{|x| > R}, \quad \text{and} \quad \mathcal{E}(\mu_R) = \frac{1}{4} - \log(R). \] (1.8)

The functionals $U$ and $\mathcal{E}$ extend to signed measures. If $\eta = \mu - \nu$ where $\mu$ and $\nu$ are two compactly supported probability measures on $\mathbb{R}^d$, then $U_\eta$ vanishes at infinity and an integration by parts gives
\[ \mathcal{E}(\eta) = \frac{1}{2} \int U_\eta d\eta = -\frac{1}{2c_d} \int U_\eta \Delta U_\eta dx = \frac{1}{2c_d} \int |\nabla U_\eta|^2 dx, \] (1.9)
see \[ \text{[Ser15]} \]. This shows that $\mathcal{E}$ does not take negative values on signed measures with total mass zero.

The right hand side of \[ \text{[1.9]} \] is the “carré du champ” in potential theory \[ \text{[Rot76, Hir78]} \] while $-\nabla U_\mu$ is the electric field – “champ électrique” in French – generated by the configuration of charges $\mu$.

Let us introduce now $V : \mathbb{R}^d \to (-\infty, +\infty]$ such that (we say then that $V$ is an \textit{admissible potential}):
- the function $V$ is lower semi-continuous;
- the set $\{x \in \mathbb{R}^d : V(x) < +\infty\}$ has positive capacity;
- the function $V$ is not beaten by the Coulomb kernel at infinity, namely
\[ \lim_{|x| \to +\infty} (V(x) - \log |x| 1_{d=2}) > -\infty. \] (1.10)

The \textit{electrostatic energy} with \textit{external potential} $V$ is defined from $\mathcal{M}_1(\mathbb{R}^d)$ to $(-\infty, +\infty]$ by
\[ \mathcal{E}_V(\mu) = \frac{1}{2} \int \int (g(x-y) + V(x) + V(y)) \mu(dx)\mu(dy). \] (1.11)

This makes sense since the function under the double integral is bounded below on $\mathbb{R}^d \times \mathbb{R}^d$ thanks to \[ \text{[1.10]} \]. Finally, for all $\mu \in \mathcal{M}_1(\mathbb{R}^d)$, if both $\log(1 + |\cdot|) 1_{d=2}$ and $V$ are in $L^1(\mu)$, then
\[ \mathcal{E}_V(\mu) = \mathcal{E}(\mu) + \int V(x) d\mu(x). \] (1.12)

The external potential plays typically the role of a confinement.

The convexity of the quadratic form $\mathcal{E}_V$ is related to a Bochner positivity of the kernel $g$, see \[ \text{[Lan72, HP08, CGZ13, BHS19]} \]. Indeed for all $\lambda \in (0, 1)$ and $\mu, \nu \in \mathcal{M}_1(\mathbb{R}^d)$ with $\mathcal{E}(\mu) < +\infty$ and $\mathcal{E}(\nu) < +\infty$ and $V \in L^1(\mu) \cap L^1(\nu)$,
\[ \frac{\lambda \mathcal{E}_V(\mu) + (1-\lambda)\mathcal{E}_V(\nu) - \mathcal{E}_V(\lambda \mu + (1-\lambda)\nu)}{\lambda(1-\lambda)} = \mathcal{E}(\mu - \nu) = \frac{1}{2c_d} \int |\nabla U_{\mu-\nu}|^2 dx \geq 0. \]

We are now ready for the general concept of equilibrium measure and its properties. The following couple of theorems is a classic in potential theory. For a proof, we refer for instance to the books \[ \text{[Lan72, Hol14, ST97, Ser15]} \] and to the articles \[ \text{[BAG97, CGZ13, Ser18]} \].

\textbf{Theorem 1.1} (Equilibrium measure). The following properties hold true:
Theorem 1.2 (Properties of the equilibrium measure). The following properties hold true:

1. The equilibrium measure $\mu$ is compactly supported if
   $$ \lim_{|x| \to \infty} (V(x) - \log |x| 1_{d=2}) = +\infty; \quad (1.13) $$

2. The equilibrium measure $\mu_V$ has finite Coulomb energy $E(\mu_V) \subset \mathbb{R}$.
3. We have $\text{supp}(\mu_V) \subset \{x \in \mathbb{R}^d : V(x) \leq R\}$ for some constant $R < \infty$.
4. The following Euler–Lagrange equations hold:
   - $U_{\mu_V}(x) + V(x) \leq c_V$ for all $x \in \text{supp}(\mu_V)$,
   - $U_{\mu_V}(x) + V(x) \geq c_V$ for all $x \in \mathbb{R}^d$ except on a set of zero capacity, where $c_V$ is a quantity called the modified Robin constant defined by
     $$ c_V = E(\mu_V) - \int V \, d\mu_V. $$

In particular, for all $x \in \text{supp}(\mu_V)$ except on a set of zero capacity, we have
   $$ U_{\mu_V}(x) + V(x) = c. $$

In particular, we have the equality in the sense of distributions
   $$ \mu_V = \frac{\Delta V}{c_d}, $$
and the interior of $\text{supp}(\mu_V)$ does not intersect $\{\Delta V < 0\}$.

Remark 1.3 (Logarithmic kernels and Riesz kernels).
• The logarithmic kernel in dimension \(d\) is given by
  \[-\log |x|, \quad x \in \mathbb{R}^d, x \neq 0;\]
• The Riesz kernel \(k_s\) in \(\mathbb{R}^d\) with parameter \(s > 0\) is given by
  \[\frac{1}{s|x|^s}, \quad x \in \mathbb{R}^d, x \neq 0.\]
• The Coulomb kernel in dimension \(d \neq 2\) matches the Riesz kernel with \(s = d - 2\);
• The logarithmic kernel for all \(d \geq 1\) can be seen as the Riesz kernel with \(s = 0\). Indeed, it suffices to remove the singularity in the sense that for all \(x \in \mathbb{R}^d\) with \(x \neq 0\),
  \[\lim_{s \to 0} \frac{|x|^{-s} - 1}{s} = \partial_{s=0}|x|^{-s} = -\log |x|.
\]

In particular the Coulomb kernel in dimension \(d = 2\) is the Riesz kernel with \(s \to 0\).
• For all \(\alpha \in (0, d)\), the Riesz kernel with \(s = d - \alpha\) is the fundamental solution of the fractional Laplace operator \(\Delta_\alpha = \Delta^\frac{\alpha}{2}\), a Fourier multiplier, non-local operator if \(\alpha \neq 2\), see \([CGZ14, RS16]\).

We refer for instance to \([BHS19]\) for more analytic properties of these kernels and various applications.

2. COULOMB GASES

Let \(d \geq 2\), \(n \geq 1\), \(\beta > 0\), and let \(g\) and \(V\) as before. Suppose moreover that \(V\) is such that
\[\int_{\mathbb{R}^d} e^{-n\beta(V(x) - \log(1 + |x|)1_{x \leq 1})} \, dx < \infty.\] (2.1)
By using the fact that \(g \geq 0\) when \(d \geq 3\) and \(|x - y| \leq (1 + |x|)(1 + |y|)\) when \(d = 2\), we get then
\[Z_n = \int_{(\mathbb{R}^d)^n} e^{-\beta E_n(x_1, \ldots, x_n)} \, dx_1 \cdots dx_n < \infty\]
where
\[E_n(x_1, \ldots, x_n) = n \sum_{i=1}^n V(x_i) + \frac{1}{2} \sum_{i \neq j} g(x_i - x_j).\]
The Coulomb gas \(P_n\) is the Boltzmann–Gibbs probability measure on \((\mathbb{R}^d)^n\) given by
\[dP_n(x_1, \ldots, x_n) = \frac{e^{-\beta E_n(x_1, \ldots, x_n)}}{Z_n} \, dx_1 \cdots dx_n.\] (2.2)
It models a “gas of electrons” in \(\mathbb{R}^d\) of charge \(1/n\), at positions \(x_1, \ldots, x_n\), inverse temperature \(\beta n^2\), energy \((1/n^2)E_n(x_1, \ldots, x_n)\), subject to Coulomb pair interaction and external field of potential \(V\), namely
\[\beta E_n(x_1, \ldots, x_n) = \beta n^2 \left( \frac{1}{n} \sum_{i=1}^n V(x_i) + \frac{1}{n^2} \sum_{i < j} g(x_i - x_j) \right).\] (2.3)
Beware that we should interpret \(P_n\) as a way to model a random static configuration of charged particles. We deal here with electrostatics rather than with electrodynamics. The charged particles do not move and there is no magnetic field. We have only an electric field.

In view of Remark 1 we could also define log-gases and Riesz gases. We do not follow this idea in these notes, for simplicity and because the Coulomb case is by far the most important in applications.

2.1. One-dimensional log-gases as Coulomb gases. The formula (2.2) makes sense provided that \(Z_n > 0\). Actually the integration in (2.1) should be interpreted as with respect to the trace of the Lebesgue measure or Hausdorff measure on \(\{ V < +\infty \} \subset \mathbb{R}^d\). Similarly the integration in (2.2) should be interpreted as with respect to the trace of the Lebesgue measure or Hausdorff measure on \(\{ V < +\infty \}^n \subset (\mathbb{R}^d)^n\). This allows to incorporate in the Coulomb gas model (2.2) the one-dimensional log-gases of random matrix theory, by taking \(d = 2\) and \(V = +\infty\) on \(S^1\) where \(S\) is a one-dimensional subset of \(\mathbb{R}^2\), typically \(S = \{ x \in \mathbb{R}^2 : x_2 = 0 \}\) or \(S = \{ x \in \mathbb{R}^2 : |x| = 1 \}\). This includes all beta Hermite/Laguerre/Jacobi
ensembles, Gaussian Unitary/Orthogonal/Simplectic Ensembles, etc. For instance, the famous Gaussian Unitary Ensemble (GUE) corresponds to take $d = 2$ and

$$x \in \mathbb{R}^2 \mapsto V(x) = \begin{cases} \frac{|x|^2}{2} & \text{if } x \in S = \mathbb{R} \times \{0\} \\ +\infty & \text{if not} \end{cases}.$$  

For simplicity, we do not study further the one-dimensional log-gases, in particular the ones coming from random matrix theory. We refer to the books [Do09, Meh04, ER05, For11, AGZ10, PST11]. Actually, most of the models that we consider in the sequel are fully dimensional in the sense that $V$ is finite everywhere. The simplest models that we focus on are two-dimensional: beta-Ginibre gases.

2.2. Beta-Ginibre gas. The case $d = 2$ is known as the two-dimensional one-component plasma. We call it the beta gas for short. Its density with respect to the Lebesgue measure on $(\mathbb{R}^2)^n = \mathbb{C}^n = \mathbb{R}^{2n}$ is

$$(z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto e^{-n\beta \sum_{j=1}^n V(z_j)} \prod_{i<j} |z_i - z_j|^\beta. \tag{2.4}$$

The quadratic potential case $V = \frac{1}{2} |\cdot|^2$ is sometimes referred to as the beta-Ginibre gas. In the special case $\beta = 2$ and $V = \frac{1}{2} |\cdot|^2$, that we call the Ginibre gas, the density of $P_n$ can be written as

$$(z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto n^n \varphi_n(\sqrt{n}z_1, \ldots, \sqrt{n}z_n) \quad \text{with} \quad \varphi_n(z_1, \ldots, z_n) = e^{-\sum_{j=1}^n |z_j|^2} \prod_{i<j} |z_i - z_j|^2. \tag{2.5}$$

The beta gas (2.4) with $V = \frac{1}{2} |\cdot|^2$ and $\beta \in \{2, 4, 6, \ldots\}$ matches the squared modulus of the Laughlin wave function of the fractional quantum hall effect [Lan87]. The Ginibre gas (2.5) matches the density of the eigenvalues of Gaussian random matrices [Gin65] (see Section 7.1 for more details), the distribution of vortices in the Ginzburg–Landau modeling of superconductivity [Ser15], and rotating trapped fermions in two dimensions [LACTMS10]. The beta gas (2.4) with $\beta = 2$ such as the Ginibre gas (2.5) has a determinantal structure which provides exact solvability (see Section 7.1 for more details).

2.3. From multivariate statistics to atomic physics. Historically, Coulomb gases emerged in mathematical statistics in the years 1920/30 in the study of the spectral decomposition of empirical covariance matrices of Gaussian samples. We speak nowadays about Laguerre ensembles and Wishart random matrices. In the 1950s, Eugene P. Wigner discovered by accident this model when reading a statistics textbook, and this led him to use random matrices for the modeling of energy levels of heavy nuclei in atomic physics, see for instance [Cha14, BW11]. His work generated an enormous trend of activity in statistical physics in the 1960s, with the works of Gaudin, Mehta, Dyson, Ginibre, Marchenko, Pastur, among others. The term Coulomb gas is already in the abstract of the first seminal article of Dyson [Dys62b] and of Ginibre [Gin63]. The terms Fermi gas and one-component plasma are also used.

2.4. The Wigner jellium and electrons in metals. It turns out that Coulomb gases are related to another famous model of mathematical physics also due to Wigner. Let $S \subset \mathbb{R}^d$ be compact and let $\mu$ be a positive measure on $\mathbb{R}^d$ with $\mu(\mathbb{R}^d) = \alpha > 0$. The Coulomb gas with potential

$$V = \begin{cases} -\frac{1}{\pi} U_{\mu} & \text{on } S, \\ +\infty & \text{outside } S \end{cases}$$

is known as a Wigner jellium with background $\mu$, and is said to be charge neutral when $n = \alpha$. The background $\mu$ models a positive charge smeared out on supp($\mu$). This model, or more precisely its thermodynamic limit as $|S| \to +\infty$, was derived by Wigner in 1938 as an approximation of the Hartree–Fock quantum model in order to model electrons in metals [Wig38], see also the 1904 pre-quantum work by Thomson [Tho04] on electrons and the structure of atoms. Conversely, a Coulomb gas with smooth potential $V$ can be seen as a jellium with background $\mu$ of density $\frac{\Delta V}{\alpha}$. From this point of view, by looking at [LS], the complex Ginibre ensemble can be seen as a Jellium with full space Lebesgue background. The measure $\mu$ is positive when $V$ is sub-harmonic (meaning that $\Delta V \geq 0$). If $V$ is not sub-harmonic then $\mu$ is no-longer positive but we may interpret it as an opposite charge on the subset $\{\Delta V < 0\}$. We refer for instance to [CGZJ20a, CGZJ20b] for a bibliography and a discussion. The term jellium was apparently coined by Conyers Herring, the smeared charge being viewed as a positive "jelly", see [Hus06].
2.5. Random polynomials. The Coulomb or log gases emerging from random matrix theory describe the law of the eigenvalues of a random matrix, the roots of the characteristic polynomial. This random polynomial has random dependent coefficients. We could study the distribution of the roots of random polynomials with random independent coefficients. Actually this question emerged from various fields of research including algebraic and geometric analysis and number theory, for instance with the works of Littlewood and Offord in the 1920s, independently of the works of the statisticians on the spectral analysis of empirical covariance matrices. The simplest model that we could imagine is a random polynomial with independent and identically distributed coefficients. This model is known as Kac polynomials, and the distribution of the roots was computed in the Gaussian case by John Hammersley in 1956. There are several other natural models of random polynomials and plenty of works on such models. The gases emerging from these models are two-dimensional but differ from Coulomb gases due to the presence in several other natural models of random polynomials and plenty of works on such models. The gases distribution of the roots was computed in the Gaussian case by John Hammersley in 1956. There are independent and identically distributed coefficients. This model is known as Kac polynomials, and the probability measure

\[ \text{Random polynomials.} \]

The first proof of such a result dates back to [BAG97] and concerns one-dimensional log-gases. It is inspired by the work of Voiculescu on a Boltzmann point of view over free entropy and random matrices. Later contributions include [PH98, Har12, CGZ14]. The approach developed in [DLR20, Ber18b, Gar19] is very efficient.

3. First order global asymptotics and large deviations

Let \( P_n \) be the Coulomb gas as in (2.2). If \( x_1, \ldots, x_n \) are pairwise distinct elements of \( \mathbb{R}^d \), which holds almost everywhere with respect to \( P_n \) in \( (\mathbb{R}^d)^n \), we get from (2.3) that

\[ E_n(x_1, \ldots, x_n) = n^2 \mathcal{E}_V^n(\mu_{x_1, \ldots, x_n}) \]

where

\[ \mathcal{E}_V^n(\mu) = \int V d\mu + \frac{1}{2} \int \int 1_{u \neq v} g(u - v)d\mu(u)d\mu(v) \quad \text{and} \quad \mu_{x_1, \ldots, x_n} = \frac{1}{n} \sum_{i=1}^n \delta_{x_i}. \]

The probability measure \( P_n \) is exchangeable in the sense that it is invariant by permutation of the \( n \) particles. The system is mean-field in the sense that each particle interacts with all the other particles via their empirical measure. The density of \( P_n \) at \( (x_1, \ldots, x_n) \) is a function of \( \mu_{x_1, \ldots, x_n} \) and rewrites

\[ \frac{\exp\left( -\beta n^2 \mathcal{E}_V^n(\mu_{x_1, \ldots, x_n}) \right)}{Z_n}. \]

In terms of asymptotic analysis, we expect that \( \mathcal{E}_V^n \approx \mathcal{E}_V \) as \( n \to \infty \), and the Laplace method suggests that under \( P_n \), the empirical measure \( \mu_{x_1, \ldots, x_n} \) concentrates as \( n \to \infty \) around the minimizers of \( \mathcal{E}_V \). Since there is a unique minimizer known as the equilibrium measure \( \mu_V \), we expect that the empirical measure \( \mu_{x_1, \ldots, x_n} \) under \( P_n \) converges towards \( \mu_V \) as \( n \to \infty \). More precisely, for all \( n \), let us define

\[ X_n = (X_{n,1}, \ldots, X_{n,n}) \sim P_n \quad \text{and} \quad \mu_n = \frac{1}{n} \sum_{k=1}^n \delta_{X_{n,k}}. \]

3.1. The large deviations principle. For all Borel subsets \( A \subset M_1(\mathbb{R}^d) \), \( \mathbb{P}(\mu_n \in A) = P_n(\mu_{x_1, \ldots, x_n} \in A) \). The following theorem translates mathematically the intuition above based on the Laplace principle: \( \mathbb{P}(\mu_n \in A) \approx e^{-\beta n^2 \inf_A (\mathcal{E}_V - \mathcal{E}_V(\mu_V))} \). The difficulty lies in the singularity of the Coulomb interaction.

**Theorem 3.1 (Large deviations principle).** We have

\[ \lim_{n \to \infty} \frac{\log Z_n}{\beta n^2} = -\mathcal{E}_V(\mu_V). \]

Moreover the sequence \( (\mu_n) \) satisfies to a large deviations principle of speed \( n^2 \) and good rate function \( \mathcal{E}_V - \mathcal{E}_V(\mu_V) \), in other words for all Borel subset of \( A \subset M_1(\mathbb{R}^d) \), we have

\[ \mathcal{E}_V(\mu_V) - \inf_{\text{int}(A)} \mathcal{E}_V \leq \lim_{n \to \infty} \frac{\log \mathbb{P}(\mu_n \in A)}{\beta n^2} \leq \lim_{n \to \infty} \frac{\log \mathbb{P}(\mu_n \in A)}{\beta n^2} \leq \mathcal{E}_V(\mu_V) - \inf_{\text{clo}(A)} \mathcal{E}_V \]

where \( \text{int}(A) \) and \( \text{clo}(A) \) are the interior and closure of \( A \) respectively.

**About the proof.** The first proof of such a result dates back to [BAG97] and concerns one-dimensional log-gases. It is inspired by the work of Voiculescu on a Boltzmann point of view over free entropy and random matrices. Later contributions include [PH98, Har12, CGZ14]. The approach developed in [DLR20, Ber18b, Gar19] is very efficient. □
Theorem 3.1 remains valid when \( \beta = \beta_n \) provided that
\[
\lim_{n \to \infty} n\beta_n = +\infty.
\]
This can be called the “low temperature regime”. In the “high temperature regime” \( \beta = \beta_n \) with
\[
\lim_{n \to \infty} n\beta_n = \kappa \in (0, +\infty),
\]
then Theorem 3.1 remains valid provided that we replace \( \mathcal{E}_V \) by the new functional
\[
\mathcal{E} + \frac{1}{K} \text{Entropy}(\cdot \mid \nu_{V,\kappa}) = \mathcal{E}_V + \frac{1}{K} \text{Entropy}(\cdot \mid dx) + c_{V,\kappa}
\]
where \( \nu_{V,\kappa} \) has density proportional to \( e^{-\kappa V} \), and where Entropy is the Kullback–Leibler divergence or relative entropy. Note that \(-\text{Entropy}(\cdot \mid dx)\) is by definition the Boltzmann–Shannon entropy. We should also replace \( \mu_V \) in Theorem 3.1 by the minimizer of this new functional. This is also known as the crossover regime, interpolating between \( \mu_V \) and \( \nu_{V,\kappa} \). Formally, if we turn off the interaction by taking \( g = 0 \) and if we take \( \beta_n = \kappa/n \) then \( P_n \) is the product probability measure \( \nu_{V,\kappa}^{\otimes n} \) and the large deviations principle becomes the classical Sanov theorem associated to the law of large numbers for independent random variables. The crossover regime is considered for instance in [CLMP92, BG99, Gar19, AB19, AS19].

### 3.2. First order global asymptotics

The (weak) convergence in \( \mathcal{M}_1(\mathbb{R}^d) \) is metrized by the bounded-Lipschitz distance defined by
\[
d_{BL}(\mu, \nu) = \sup \left\{ \int f(d\mu - d\nu) : \|f\|_{\infty} \leq 1, \|f\|_{\text{Lip}} \leq 1 \right\}
\]
where the supremum runs over all measurable functions \( f : \mathbb{R}^d \to \mathbb{R} \) and where
\[
\|f\|_{\infty} = \sup_x |f(x)| \quad \text{and} \quad \|f\|_{\text{Lip}} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.
\]
Now for all \( r \geq 0 \), by Theorem 3.1 with \( A = A_r = \{ \mu \in \mathcal{M}_1(\mathbb{R}^d) : d_{BL}(\mu, \mu_V) \geq r \} \), for \( n \) large enough,
\[
e^{-c_r \beta_n^2} \leq \mathbb{P}(d_{BL}(\mu_n, \mu_V) \geq r) \leq e^{-C_r \beta_n^2}, \tag{3.2}
\]
where \( c_r, C_r > 0 \) are constants depending on \( A_r \) and \( \mathcal{E}_V \) but not on \( n \). In particular, for all \( \varepsilon > 0 \),
\[
\sum_n \mathbb{P}(d_{BL}(\mu_n, \mu_V) > \varepsilon) < \infty.
\]
By the Borel–Cantelli lemma, it follows that regardless of the way we choose a common probability space to define the sequence of random vectors \( (X_n)_n \), we have, almost surely,
\[
\lim_{n \to \infty} d_{BL}(\mu_n, \mu_V) = 0. \tag{3.3}
\]
This is a sort of law of large numbers for our system of exchangeable particles. They are not independent due to the Coulomb interaction, and the information about the interaction remains in \( \mu_V \). We refer to [Ser15, Ber19a, Gar19] for the relation to the notion of “Gamma convergence”.

### 3.3. Weakly confining versus strongly confining potential

We could say that \( V \) is weakly confining when (1.10) holds, and that \( V \) is strongly confining when (1.13) holds. The integrability condition (2.1) may hold for weakly confining potentials. An example of a two dimensional Coulomb gas with a weakly confining potential is given by the Forrester–Krishnapur spherical ensemble considered in the sequel, for which the equilibrium measure is not compactly supported and is heavy-tailed.

### 3.4. Concentration of measure

The proof of Theorem 3.1 can be adapted in order to provide quantitative (meaning non-asymptotic) estimates for deviation probabilities. Namely, for all \( r \geq 0 \),
\[
\mathbb{P}(d_{BL}(\mu_n, \mu_V) \geq r) = \frac{1}{Z_n} \int_{d_{BL}(\mu_n, \mu_V) \geq r} e^{-\beta_n^2 \mathcal{E}(\mathcal{E}_V \mu_1, \ldots, \mu_n)} dx_1 \cdots dx_n.
\]
Now if we could approximate \( \mathcal{E}^\#(\mu_{x_1, \ldots, x_n}) \) with \( \mathcal{E}_V(\mu_{x_1, \ldots, x_n}) \) and use an inequality of the form
\[
d_{BL}(\mu, \mu_V) \leq c(\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)),
\]
and use a bound of the form
\[
\log Z_n \geq n^2 \mathcal{E}_V(\mu_V) + n(\beta \mathcal{E}(\mu_V) + c_V), \tag{3.4}
\]
then we would obtain a concentration of measure inequality of the form
\[ P(d(\mu_n, \mu_V) \geq r) \leq e^{-cn^r^2 + o(n^r)}. \]
for all \( n \) and all \( r \geq r_n \), for some threshold \( r_n \). Actually the quantity \( E_V(\mu_{x_1, \ldots, x_n}) \) is infinite due to the atomic nature of \( \mu_{x_1, \ldots, x_n} \) and the method requires then a regularization procedure. The details are in [CHM18]. The method, inspired by [MMS13], is related to [RS16]. See also [GZ19a, MS18, BC19a, PG20] for other variations on this topic. Moreover we could replace the bounded-Lipschitz distance by a Kantorovich–Wasserstein distance, provided a growth assumption on \( V \).

Such concentration inequalities around the equilibrium measure provide typically an upper bound on the speed of the almost sure convergence. More precisely if \( r_n \) is such that \( \sum_n e^{-cn^r^2 + o(n^r)} < \infty \), then by the Borel–Cantelli lemma, we get that almost surely, for \( n \) large enough,
\[ d_{BL}(\mu_n, \mu_V) \leq r_n. \]

On the other hand, in the case of one-dimensional log-gases with strongly convex potential \( V \) such as the Gaussian unitary ensemble, another approach is possible for concentration of measure, related to logarithmic Sobolev inequalities, see for instance [CL20] and references therein.

4. Edge behavior

We suppose in this section that \( V \) is strongly confining, in particular the equilibrium measure \( \mu_V \) is compactly supported (Theorem 1.2). The convergence (3.3) holds in a weak sense, which does not imply the convergence of the support. The most general result about the convergence of the support is probably [Ame21], and appears as a refinement of [CHM18] Theorem 1.12. When \( V \) is rotationally invariant, this provides constants \( c, r_*, p > 0 \) such that for all \( n \) and \( r \geq r_* \),
\[ P\left( \max_{1 \leq k \leq n} |X_{n,k}| \geq r \right) \leq e^{-cn^rp}. \]

The fluctuation at the edge is a difficult subject which is well understood for one-dimensional log-gases, for which it gives rise to Tracy–Widom laws. For strongly confined rotationally invariant determinantal two-dimensional Coulomb gases, it gives rise to Gumbel laws. An explicit analysis of the Ginibre Coulomb gas is presented in the sequel (Theorem 7.14), see also [CP14, JQ17, CGZJ20a, Seco20, Ame21, BGZ18, GZ18, BGZNW21, CGZJ20a] for more results in the same spirit.

5. Global fluctuations and Gaussian free field

Formally, from (1.9) we could write
\[ E(\mu) = \frac{1}{2} (\beta^{-1}_d \Delta U_{\mu}, U_{\mu}) + (\beta^{-1}_d \Delta V, U_{\mu}), \]
and thus
\[ \beta n^2 E(\mu) = \frac{1}{2} (\beta^{-1}_d \Delta U_{\mu}, U_{\mu}) + (\Delta V, U_{\mu}). \]
In view of the Coulomb gas formula (3.1), this suggests to interpret as \( n \to \infty \) the random function \( U_{n\mu_1, \ldots, \mu_n} \) under \( P_n \) as a Gaussian with covariance operator \( \mathcal{K} = \beta_d (\beta \Delta)^{-1} \). Actually such an object is known as a Gaussian Free Field (GFF). Next, again from (1.9), this suggests to interpret formally as \( n \to \infty \) the random measure \( n\mu_1, \ldots, \mu_n = \epsilon^{-1}_d \Delta U_{\mu_1, \ldots, \mu_n} = AU_{\mu_1, \ldots, \mu_n} \) under \( P_n \) as a Gaussian random measure with covariance operator \( A^2 \mathcal{K} = (\beta^{-1}_d \Delta)^2 (\epsilon_d (\beta \Delta)^{-1}) = (\beta_c d)^{-1} \Delta \). This argument would involve in principle a change of variable and a Jacobian, that we do not consider here. This leads naturally to conjecture that for a smooth enough test function \( f : \mathbb{R}^d \to \mathbb{R} \),
\[ n \left( \int f d\mu_n - \mathbb{E} \left( \int f d\mu_n \right) \right) \xrightarrow{n \to \infty} N\left( \int |\nabla f|^2 dx \right). \]
This can be seen as the “central limit theorem” statement associated to the “law of large numbers” statement (5.3). The limiting variance could be perturbed by edge effects depending on the relative position of the support and regularity of \( f \) and \( \mu_V \). We could have also an additional bias correction.

The Coulomb interaction together with the confinement produces a rigidity of the global configuration and reduces the variance of linear statistics. Indeed (5.4) comes with an \( n \) scaling that differs from the usual \( \sqrt{n} \) scaling for independent random variables (no interaction).
The covariance of the limiting Gaussian in (5.1) is easily guessed from the Hessian at the minimizer of the rate function in the large deviations principle of Theorem 3.1. This CLT – LDP link is well known. The GFF is an example of a log-correlated Gaussian field [DRSV17], a fashionable subject.

A statement similar to (5.1) is proved rigorously in [RV07] for the complex Ginibre ensemble by using its exact solvability (determinantal structure). See also [AHM15, AHM11]. Extensions to non-exactly solvable two-dimensional Coulomb gases are considered in [BRNY19, LS18, Ser20a, LZ20].

For one-dimensional log-gases emerging from random matrix theory, central limit theorems such as (5.1) were established using the Laplace transform and “loop equations” in [Jo98]. See also [PS11, BGG17, BLS18, HL21] and references therein for extensions and generalizations.

6. ASPECTS OF COULOMB GASES

Theorem 6.1 is taken from [Cha19a] and [CL20] (see also [CFS21]).

**Theorem 6.1** (Exact distributions for special linear statistics of general gases). Let X = (X_1, ..., X_n) be a random vector of (R^d)^n, n, d ≥ 1, with density proportional to

\[ e^{-\sum_i V(x_i)} \prod_{i<j} W(x_i - x_j) \]

where V : R^d → [0, +∞] and W : R^d → [0, +∞] are measurable.

- If V and W are homogeneous in the sense that for some a, b ≥ 0, and for all λ ≥ 0 and x ∈ R^d,

  \[ V(\lambda x) = \lambda^a V(x) \text{ and } W(\lambda x) = \lambda^b W(x), \]

  then

  \[ V(X_1) + \cdots + V(X_n) \sim \text{Gamma} \left( \frac{nd}{a} + \frac{n(n-1)b}{2a}, 1 \right). \]

- If V = γ |x|^2 for some γ > 0 then

  \[ X_1 + \cdots + X_n \sim \mathcal{N} \left( 0, \frac{n}{2\gamma} I_d \right), \]

  and moreover the orthogonal projection π on the subspace \{ (z, ..., z) : z ∈ R^d \} of (R^d)^n satisfies π(X) = X_1/n + X_2/n + ... + X_n/n (1, ..., 1), and furthermore π(X) and π^⊥(X) = X - π(X) are independent.

**Proof of Theorem 6.1.** Recall that linear change of variable is valid for integrals of measurable functions.

**First formula.** For all θ > 0, we have, with the substitution \( x_i = \left( 1 + \theta \right)^{1/a} y_i, \)

\[ \int_{(R^d)^n} e^{-\theta \sum_i V(x_i)} e^{-\sum_i V(x_i)} \prod_{i<j} W(x_i - x_j) dx = \left( \frac{1}{1 + \theta} \right)^{\frac{nd}{a} + \frac{n(n-1)b}{2a}} \int_{(R^d)^n} e^{-\sum_i V(y_i)} \prod_{i<j} W(y_i - y_j) dy. \]

We recognize the Laplace transform of Gamma \( \left( \frac{nd}{a} + \beta \frac{n(n-1)b}{2a}, 1 \right), \)

\[ \int_0^\infty e^{-\beta x} x^{\alpha-1} e^{-\lambda x} dx = \int_0^\infty x^{\alpha-1} e^{-(\lambda + \beta)x} dx = \left( \frac{\lambda}{\lambda + \beta} \right)^\alpha \Gamma(\alpha) \lambda^{-\alpha}, \]

therefore \( \sum_i V(X_i) \sim \text{Gamma} \left( \frac{nd}{a} + \frac{n(n-1)b}{2a}, 1 \right). \)

**Second formula.** For all θ ∈ R^d, we have, with the substitution \( y_i = x_i + \frac{1}{\sqrt{n}} \theta \) (a translation or shift),

\[ \int_{(R^d)^n} e^{-\theta \sum_i x_i} e^{-\sum_i V(x_i)} \prod_{i<j} W(x_i - x_j) dx = e^{\frac{\theta^2}{2}} \int_{(R^d)^n} e^{-\sum_i V(y_i)} \prod_{i<j} W(y_i - y_j) dy, \]

and we recognize the Laplace transform of the Gaussian law \( \mathcal{N} \left( 0, \frac{1}{2n} I_d \right). \)

Finally the properties related to π(X) follow from the quadratic nature of V and the shift invariance of W, and correspond to a factorization of the law of X, namely, denoting \( \pi^\perp(x) = x - \pi(x) \) and using \( |x|^2 = |\pi(x)|^2 + |\pi^\perp(x)|^2 \) (Pythagoras theorem) and \( x_1 - x_j = \pi^\perp(x_1) - \pi^\perp(x_j) \) (from the definition of π), we get

\[ e^{-\sum_i V(x_i)} \prod_{i<j} W(x_i - x_j) = e^{-\gamma |\pi(x)|^2} \times e^{-\gamma |\pi^\perp(x)|^2} \prod_{i<j} W(\pi^\perp(x_i) - \pi^\perp(x_j)). \]

This provides the independence of π(X) and π^⊥(X) as well as the fact that π(X) ~ \( \mathcal{N} \left( 0, \frac{1}{2n} I_d \right). \) □
Corollary 6.2 (Exact laws for beta-Ginibre gases). Let us consider $X_n = (X_{n,1}, \ldots, X_{n,n}) \sim P_n$ where $P_n$ is as in (2.3) with $\beta > 0$ and $V = \frac{1}{2} |\varphi|^2$. In other words, the density of $P_n$ in $\mathbb{C}^n$ is given by

$$(z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto \frac{e^{-n \varphi(|z_1|^2 + \cdots + |z_n|^2)}}{Z_n} \prod_{i<j} |z_i - z_j|^\beta.$$ 

Then

$$X_{n,1} + \cdots + X_{n,n} \sim \mathcal{N}(0, \frac{I_2}{\beta}) \quad \text{and} \quad |X_n|^2 = |X_{n,1}|^2 + \cdots + |X_{n,n}|^2 \sim \text{Gamma}\left(n + \beta \frac{n(n-1)}{4}, \beta \frac{n}{2}\right),$$

and in particular

$$\mathbb{E}(|X_{n,1}|^2 + \cdots + X_{n,n}|^2) = \frac{2}{\beta} \quad \text{and} \quad \mathbb{E}(|X_n|^2) = \mathbb{E}(|X_{n,1}|^2 + \cdots + |X_{n,n}|^2) = \frac{2}{\beta} + \frac{n-1}{2}.$$

When $\beta = 2$ we recover the Ginibre gas (2.3). Beyond this case, and up to our knowledge, it seems that there is no useful matrix model with independent entries for which the spectrum follows this beta gas.

With $\beta = \frac{1}{n}$, we get as $n \to \infty$ that the variance of the Gauss–Ginibre crossover is $2 + \frac{1}{\beta} = \frac{5}{2}$.

Proof of Corollary 6.2. It suffices to use Theorem 6.1 with $d = 2$, $V = n \varphi^2 |\varphi|^2$, $W = |\varphi|^2$, for which $\alpha = 2$ and $b = \beta$, and the scaling property $\sigma Z \sim \text{Gamma}(\alpha, \frac{\sigma^2}{\beta})$ when $Z \sim \text{Gamma}(\alpha, \lambda)$, for any $\sigma > 0$.

Note that in the determinantal case $\beta = 2$, Theorem 6.2 gives that $n|X_n|^2 \sim \text{Gamma}(1 + 2 + \cdots + n, 1)$ since it has the law of a sum of $n$ independent random variables of law Gamma(1,1), ..., Gamma(n,1). □

Remark 6.3 (Real case). For all $\beta > 0$, $n \geq 2$, let $P_n$ be the law on $\mathbb{R}^n$ with density

$$(x_1, \ldots, x_n) \in \mathbb{R}^n \mapsto \frac{e^{-n \varphi(x_1^2 + \cdots + x_n^2)}}{Z_n} \prod_{i<j} |x_i - x_j|^\beta.$$

The normalization $Z_n$ can be explicitly computed via a Mehta–Selberg integral [FW08]. It is a quadratically confined one-dimensional log-gas known as the real beta Hermite gas. The case $\beta = 2$ corresponds to GUE. If $X_n = (X_{n,1}, \ldots, X_{n,n}) \sim P_n$, then the proof of Theorem 6.2 provides

$$X_{n,1} + \cdots + X_{n,n} \sim \mathcal{N}(0, \frac{2}{\beta}) \quad \text{and} \quad X_{n,1}^2 + \cdots + X_{n,n}^2 \sim \text{Gamma}\left(n \frac{2}{\beta} + \beta \frac{n(n-1)}{4}, \beta \frac{n}{4}\right),$$

and in particular,

$$\mathbb{E}((X_{n,1} + \cdots + X_{n,n})^2) = \frac{2}{\beta} \quad \text{and} \quad \mathbb{E}(X_{n,1}^2 + \cdots + X_{n,n}^2) = \frac{2}{\beta} + n - 1.$$

Alternatively, these formulas can also be derived by using the tridiagonal random matrix model of Dumitriu and Edelman [DE02] valid for all real beta Hermite gases, see for instance [CL20].

Remark 6.4 (Langevin dynamics). The Boltzmann–Gibbs measure $P_n$ defined in (2.2) is the invariant law of the Kolmogorov diffusion process $(X_t)_{t \geq 0}$ solution of the stochastic differential equation

$$dX_t = \sqrt{\frac{2 \alpha}{\beta}} dB_t - \alpha V E_n(X_t) dt \quad (6.1)$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion on $(\mathbb{R}^d)^n$, and where $\alpha > 0$ is an arbitrary parameter which corresponds to a deterministic time change. The infinitesimal generator of the associated Markov semi-group is the second order linear differential operator without constant term

$$L = \alpha \left( \frac{1}{\beta} \Delta - V E_n \cdot \nabla \right).$$

(6.2)
that (2.4) is a gradient dynamics because the drift is the gradient of a function. From the point of view of statistical physics a stochastic differential equation such as (6.1) is also known as an overdamped Langevin dynamics, which is a degenerate version of the true (kinetic under-damped) Langevin dynamics, see for instance [CF19]. Langevin dynamics can be used for the numerical simulation of \( P_n \), see for instance [CF19, CF19, CFS21] and references therein. Dynamics such as (6.1) can be used as an interpolation device between \( X_0 \) and \( X_\infty \sim P_n \), possibly by using conservation laws related to eigenfunctions. In this spirit, and following [BCF18, CL20], we could prove Corollary (7.2) by using the fact that \( \sum_i x_i \) and \( \sum_i |x_i|^2 \) are essentially eigenfunctions of (6.2), producing Ornstein–Uhlenbeck and Cox–Ingersoll–Ross processes for which the targeted Gaussian and Gamma laws are invariant.

7. Exactly solvable two-dimensional gases from random matrix theory

The spectrum of several random matrix models are gases in dimension \( d \in \{1, 2\} \) with \( W = g \). The cases \( \beta \in \{1, 2, 4\} \) play often a special role related to algebra. We refer to [Meh04, ER05, For10] for more details on the zoology of random matrices. It is natural to ask if there exists a random matrix model with independent entries for which the spectrum is distributed according to the beta gas (2.4). Up to our knowledge, the answer is negative for (2.4) in general but positive for the Ginibre gas (2.5).

7.1. Ginibre model. A (complex) Ginibre random matrix \( M \) is an \( n \times n \) complex matrix such that

\[
\{RM_{i,j}, 3M_{i,j} : 1 \leq i, j \leq n\}
\]

(7.1)

are independent and Gaussian random variables of law \( \mathcal{N}(0, \frac{1}{n}) \). In other words the complex random variables \( \{M_{i,j} : 1 \leq i, j \leq n\} \) are independent and Gaussian of law \( \mathcal{N}(0, \frac{1}{n}) \). Note that \( E(M_{i,j}^2) = 1 \).

Let \( (\lambda_1, \ldots, \lambda_n) \) be the eigenvalues of \( M \) seen as an exchangeable random vector of \( \mathbb{C}^n \). This means that we randomize the numbering of the eigenvalues with an independent uniform random permutation of \( \{1, \ldots, n\} \). Equivalently, this corresponds to consider the random multi-set encoding the spectrum, keeping by this way the possible multiplicities but discarding the numbering of the eigenvalues.

**Theorem 7.1** (From the Ginibre random matrix to the Ginibre gas). The exchangeable random vector

\[
\left( \frac{\lambda_1}{\sqrt{n}}, \ldots, \frac{\lambda_n}{\sqrt{n}} \right)
\]

is distributed according to the Ginibre gas (2.4). In other words \( (\lambda_1, \ldots, \lambda_n) \) has density \( \varphi_n \) as in (2.4).

**Idea of the proof.** The set of \( n \times n \) complex matrices with multiple eigenvalues has zero Lebesgue measure. Since the law of \( M \) is absolutely continuous, it follows that almost surely \( M \) is diagonalizable with distinct eigenvalues. The density is proportional to \( (M^* = M^T) \) is the conjugate-transpose of \( M \)

\[
M \mapsto e^{-\sum_{i,j=1}^n |M_{i,j}|^2} = e^{-\text{Trace}(MM^*)}.
\]

In order to compute the law of the spectrum of \( M \), an idea is to use for instance the Schur unitary decomposition as a change of variable. Namely, if \( M \) is diagonalizable, then the Schur decomposition is the matrix factorization \( M = U(D + N)U^* \) where \( U \) is unitary, \( D \) is diagonal, and \( N \) is upper triangular with null diagonal (nilpotent). The matrix \( D \) carries the eigenvalues of \( M \). We have the decoupling

\[
\text{Trace}(MM^*) = \text{Trace}(DD^*) + \text{Trace}(NN^*)
\]

This allows to integrate out \((N, U)\) in the density and to get that the law of the eigenvalues of \( M \) is given by (2.4). The term \( \prod_{i<j} |x_i - x_j|^2 \) is the modulus of the determinant of the Jacobian of the change of variable. We obtain that for every symmetric bounded (or positive) measurable function \( F : \mathbb{C}^n \to \mathbb{R} \),

\[
\mathbb{E}[F(\lambda_1, \ldots, \lambda_n)] = \int_{\mathbb{C}^n} F(z_1, \ldots, z_n)\varphi_n(z_1, \ldots, z_n)dz_1 \cdots dz_n
\]

where \( dz_1 \cdots dz_n \) stands for the Lebesgue measure on \( \mathbb{C}^n = \mathbb{R}^{2n} \). The result goes back to [Gin65]. The scheme of proof that we follow here can be found in [KSI11], see also [Meh04, For10, Ch. 15].

**Remark 7.2** (Immediate properties of Ginibre random matrices).

- Since the law of \( M \) is absolutely continuous, almost surely \( MM^* \neq M^*M \) (non-normality);
- By the law of large numbers, almost surely, as \( n \to \infty \), \( \frac{M}{\sqrt{n}} \) has orthonormal rows/columns;
- The law of \( M \) is bi-unitary invariant: if \( U \) and \( V \) are unitary then \( UMV \) and \( M \) have same law;
• The Hermitian random matrices $\frac{1}{\sqrt{n}}M + \frac{1}{\sqrt{n}}M^*$ and $\frac{1}{\sqrt{n}}M - \frac{1}{\sqrt{n}}M^*$, the matrix real and imaginary parts of $M$, are independent and belong to the Gaussian Unitary Ensemble (GUE); their density is proportional to $H \mapsto e^{-\frac{1}{2}\text{Trace}(H^2)}$; Conversely, if $H_1$ and $H_2$ are independent copies of the Gaussian Unitary Ensemble then the random matrices $\frac{1}{\sqrt{n}}(H_1 + iH_2)$ and $M$ have same law.

The exact solvability of the Ginibre gas \[(2.5)\] is largely due to a determinantal structure studied below, itself related to the fact that $\beta = 2$ and $W = g$. More precisely, first of all, from (2.5) we have

$$\varphi_n(z_1, \ldots, z_n) = \prod_{k=1}^{n} \gamma(z_k) \prod_{i<j} |z_i - z_j|^2 \tag{7.2}$$

where $\gamma$ is the density of $\mathcal{N}(0, \frac{1}{2}I_2)$ given for all $z \in \mathbb{C}$ by

$$\gamma(z) = \frac{e^{-|z|^2}}{\pi}.$$

**Theorem 7.3** (Determinantal structure and marginals). For all $n \geq 1$ and $(z_1, \ldots, z_n) \in \mathbb{C}^n$,

$$\varphi_n(z_1, \ldots, z_n) = \frac{1}{n!} \det [K_n(z_i, z_j)]_{1 \leq i, j \leq n}$$

where the kernel $K_n$ is given for all $z, w \in \mathbb{C}$ by

$$K_n(z, w) = \sqrt{\gamma(z)\gamma(w)} \sum_{\ell=0}^{n-1} (z \bar{w})^\ell \ell!.$$

More generally, for all $1 \leq k \leq n$, the marginal density

$$(z_1, \ldots, z_k) \in \mathbb{C}^k \mapsto \varphi_{n,k}(z_1, \ldots, z_k) = \int_{\mathbb{C}^{n-k}} \varphi_n(z_1, \ldots, z_n) dz_{k+1} \cdots dz_n$$

satisfies, for all $(z_1, \ldots, z_k) \in \mathbb{C}^k$,

$$\varphi_{n,k}(z_1, \ldots, z_k) = \frac{(n-k)!}{n!} \det [K_n(z_i, z_j)]_{1 \leq i, j \leq k}.$$

In particular for $k = n$ we get $\varphi_{n,n} = \varphi_n$, while for $k = 1$ we get, for all $z \in \mathbb{C}$,

$$\varphi_{n,1}(z) = \frac{\gamma(z)}{n} \sum_{\ell=0}^{n-1} |z|^{2\ell} \ell!.$$

We say that the spectrum of $M$ is a Gaussian determinantal point process, see \cite[Ch. 4]{HKPV09}. The "$k$-point correlation" is $R_{n,k}(z_1, \ldots, z_k) = \frac{n!}{(n-k)!} \varphi_{n,k}(z_1, \ldots, z_k) = \det [K_n(z_i, z_j)]_{1 \leq i, j \leq k}$.

**Idea of proof.** Following for instance \cite[Sec. 5.2 and Ch. 15]{Meh04}, we get, starting with (7.2),

$$\varphi_n(z_1, \ldots, z_n) = \prod_{k=1}^{n} \gamma(z_k) \prod_{1 \leq i < j \leq n} (z_i - z_j) \prod_{1 \leq i < j \leq n} (z_i - z_j)$$

$$= \prod_{k=1}^{n} \frac{\gamma(z_k)}{n!} \det \left[ \frac{z_j^{i-1}}{\sqrt{(i-1)!}} \right]_{1 \leq i, j \leq n} \det \left[ \frac{z_j^{i-1}}{\sqrt{(i-1)!}} \right]_{1 \leq i, j \leq n}$$

$$= \frac{1}{n!} \det [K_n(z_i, z_j)]_{1 \leq i, j \leq n}.$$

On the other hand, the orthogonality of $\{ \frac{e^{\frac{1}{2}z^2}}{\sqrt{\pi}} \}$ gives the identities

$$\int_{\mathbb{C}} K_n(x, x) dx = n \quad \text{and} \quad \int_{\mathbb{C}} K_n(x, y) K_n(y, z) dy = K_n(x, z), \quad x, z \in \mathbb{C}.$$

Finally the formula for $\varphi_{n,k}$ follows by expanding the determinant in $\varphi_n$ and using these identities. \qed

**Theorem 7.4** (Mean circular Law). Let $\lambda_1, \ldots, \lambda_n$ be as in Theorem \[(2.3)\] and let us define

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k}.$$
Let \( \mu_\infty \) be the uniform distribution on the unit disc \( \{ z \in \mathbb{C} : |z| \leq 1 \} \) with density \( z \in \mathbb{C} \mapsto \frac{1_{|z|<1}}{\pi} \). Then

\[
\mathbb{E}_{\mu_n} \frac{C_n}{n} \xrightarrow{n \to \infty} \mu_\infty. 
\]

**Proof.** Let \( \varphi_{n,1} \) be as in Theorem 7.3. For all \( f \in \mathcal{C}_b(\mathbb{C}, \mathbb{R}) \), we have, using Theorem 7.1

\[
\mathbb{E} \int fd\mu_n = \frac{1}{n} \sum_{k=1}^n \int_{\mathbb{C}^n} f \left( \frac{zk}{\sqrt{n}} \right) \varphi_n(z_1, \ldots, z_n) dz_1 \cdots dz_n = n \int f(z) \varphi_{n,1}(\sqrt{n}z) dz.
\]

Thus \( \mathbb{E}_{\mu_n} \) has density \( n \varphi_{n,1}(\sqrt{n}z) \). By Theorem 7.3 and Lemma 7.5 if \( K \subset \{ z \in \mathbb{C} : |z| \neq 1 \} \) is compact,

\[
\lim_{n \to \infty} \sup_{z \in K} n \varphi_{n,1}(\sqrt{n}z) - \frac{1_{|z| \leq 1}}{\pi} = 1 \lim_{n \to \infty} \sup_{z \in K} e^{-|z|^2} e_n(n|z|^2) - 1_{|z| \leq 1} = 0.
\]

It follows then by dominated convergence that \( \mathbb{E}_{\mu_n} \xrightarrow{n \to \infty} \mu_\infty. \)

**Lemma 7.5** (Exponential series). For every \( n \geq 1 \) and \( z \in \mathbb{C} \),

\[
|e_n(nz) - e^{nz} 1_{|z| \leq 1}| \leq r_n(z)
\]

where \( e_n(z) = \sum_{k=0}^{n-1} \frac{z^k}{k!} \) is the truncated exponential series and

\[
r_n(z) = \frac{e^n}{\sqrt{2\pi n}} |z|^n \left( \frac{n+1}{n(1-|z|)} + \frac{1}{n(|z|-1)} + \frac{1}{n} \right).
\]

**Proof of Lemma 7.5** As in Mehta [Mehta, Ch. 15], for every \( n \geq 1 \), \( z \in \mathbb{C} \), if \( |z| \leq n \) then

\[
|e^z - e_n(z)| = \left| \sum_{\ell=n}^{\infty} \frac{z^\ell}{\ell!} \right| \leq \sum_{\ell=n}^{\infty} \frac{|z|^\ell}{\ell!} \leq \frac{|z|^n}{n!} \sum_{\ell=0}^{n-1} \frac{|z|^{\ell+1}}{(n-1)!} \leq \frac{|z|^n}{n!} \frac{n+1}{n+1-|z|}
\]

while if \( |z| > n \) then

\[
|e_n(z)| \leq \sum_{\ell=0}^{n-1} \frac{|z|^\ell}{\ell!} \leq \frac{|z|^{n-1}}{(n-1)!} \sum_{\ell=0}^{n-1} \frac{(n-1)^\ell}{|z|^\ell} \leq \frac{|z|^{n-1}}{(n-1)!} \frac{|z|}{|z|-n+1}.
\]

Therefore, for every \( n \geq 1 \) and \( z \in \mathbb{C} \),

\[
|e_n(nz) - e^{nz} 1_{|z| \leq 1}| \leq \frac{n^n}{n!} \left( \frac{n+1}{n+1-|nz|} 1_{|z| \leq 1} + \frac{|nz|}{|nz|-n+1} 1_{|z| > 1} \right).
\]

It remains to use the Stirling bound \( \sqrt{2\pi n} n^n \leq n! e^n \) to get the first result.

**Remark 7.6** (Probabilistic view). There is a probabilistic interpretation of Lemma 7.5. For all \( z \in \mathbb{C} \),

\[
\lim_{n \to \infty} n \varphi_{n,1}(\sqrt{n}z) = \frac{1}{\pi} \left( 1_{|z|<1} + \frac{1}{2} 1_{|z|=1} \right).
\]

Namely, by rotational invariance, it suffices to consider the case \( z = r > 0 \). Next, if \( Y_1, \ldots, Y_n \) are independent and identically distributed random variables following the Poisson law of mean \( r^2 \), then

\[
e^{-nr^2} e_n(nr^2) = \mathbb{P}(Y_1 + \cdots + Y_n < n) = \mathbb{P} \left( \frac{Y_1 + \cdots + Y_n}{n} < 1 \right).
\]

Now \( \lim_{n \to \infty} \frac{Y_1 + \cdots + Y_n}{n} = r^2 \) almost surely by the strong law of large numbers, and thus the probability in the right-hand side above tends as \( n \to \infty \) to 0 if \( r > 1 \) and to 1 if \( r < 1 \). In other words, for all \( r \neq 1 \),

\[
\lim_{n \to \infty} e^{-nr^2} e_n(nr^2) = 1_{r<1}.
\]

It remains to note that for \( r = 1 \) by the central limit theorem we get

\[
\mathbb{P} \left( \frac{Y_1 + \cdots + Y_n}{n} < 1 \right) = \mathbb{P} \left( \frac{Y_1 + \cdots + Y_n - n}{\sqrt{n}} < 0 \right) \xrightarrow{n \to \infty} \frac{1}{2}.
\]
Remark 7.7 (Incomplete gamma function). It is well known that the Gamma and the Poisson laws are connected. Namely, if \(X \sim \text{Gamma}(n, \lambda)\) with \(n \geq 1\) and \(\lambda > 0\) and \(Y \sim \text{Poisson}(r)\) with \(r > 0\) then

\[
\mathbb{P}(X \geq \lambda r) = \frac{1}{(n-1)!} \int_0^\infty x^{n-1} e^{-x} dx = e^{-r} \sum_{\ell=0}^{n-1} \frac{r^\ell}{\ell!} = \mathbb{P}(Y \geq n).
\]

Also we could use Gamma random variables instead of Poisson random variables in Remark 7.6. Note also that the integral in the middle of the formula above is the incomplete Gamma function \(\Gamma(n, r)\). This allows to benefit from the asymptotic analysis of this special function, see [KS11] and references therein.

Theorem 7.8 (Strong circular law). With the notations of Theorem 7.4, almost surely,

\[
\mu_n \xrightarrow{c_s} \mu_\infty.
\]

Note that this convergence holds regardless of the way we define the random matrices on the same probability space when \(n\) varies. This is an instance of the concept of complete convergence, see [Yuk98].

Idea of the proof. The argument, due to Jack Silverstein, is in [Hwa86]. It is similar to the quick proof of the strong law of large numbers for independent random variables with bounded fourth moment. It suffices to establish the result for an arbitrary compactly supported \(f \in C_b(\mathbb{C}, \mathbb{R})\). Let us define

\[
S_n = \int_{\mathbb{C}} f d\mu_n \quad \text{and} \quad S_\infty = \frac{1}{\pi} \int_{|z| \leq 1} f(z) dz.
\]

Suppose for now that we have

\[
\mathbb{E}[(S_n - ES_n)^4] = \mathcal{O}\left(\frac{1}{n}\right), \tag{7.3}
\]

By monotone convergence or by the Fubini–Tonelli theorem,

\[
\mathbb{E} \sum_{n=1}^\infty (S_n - ES_n)^4 = \sum_{n=1}^\infty \mathbb{E}[(S_n - ES_n)^4] < \infty
\]

and thus \(\sum_{n=1}^\infty (S_n - ES_n)^4 < \infty\) almost surely, which implies \(\lim_{n \to \infty} S_n - ES_n = 0\) almost surely. Since \(\lim_{n \to \infty} ES_n = S_\infty\) by Theorem 7.4, we get that almost surely

\[
\lim_{n \to \infty} S_n = S_\infty.
\]

Finally, one can swap the universal quantifiers on \(\omega\) and \(f\) thanks to the separability of the set of compactly supported continuous bounded functions \(\mathbb{C} \to \mathbb{R}\) equipped with the supremum norm. To establish the fourth moment bound (7.3), we set

\[
S_n - ES_n = \frac{1}{n} \sum_{k=1}^n Z_k \quad \text{with} \quad Z_k = f\left(\frac{\lambda_k}{\sqrt{n}}\right) - \mathbb{E}f\left(\frac{\lambda_k}{\sqrt{n}}\right).
\]

Next, we obtain, with \(\sum_{k_1, \ldots, k_n}\) running over distinct indices in \(1, \ldots, n\),

\[
\mathbb{E}\left[(S_n - ES_n)^4\right] = \frac{1}{n^4} \sum_{k_1} \mathbb{E}[Z_{k_1}^4] + \frac{4}{n^4} \sum_{k_1, k_2} \mathbb{E}[Z_{k_1} Z_{k_2} Z_{k_2}^3] + \frac{3}{n^4} \sum_{k_1, k_2} \mathbb{E}[Z_{k_1}^2 Z_{k_2}^2] + \frac{6}{n^4} \sum_{k_1, k_2, k_3} \mathbb{E}[Z_{k_1} Z_{k_2} Z_{k_3} Z_{k_3}^2] + \frac{1}{n^4} \sum_{k_1, k_2, k_3, k_4} \mathbb{E}[Z_{k_1} Z_{k_2} Z_{k_3} Z_{k_4}].
\]

The first three terms of the right are \(\mathcal{O}(n^{-2})\) since \(\max_{1 \leq k \leq n} |Z_k| \leq \|f\|_\infty\). The expressions of \(\varphi_{n,3}\) and \(\varphi_{n,4}\) from Theorem 7.3 allow to show that the remaining two terms are also \(\mathcal{O}(n^{-2})\), see [Hwa86].

The following theorem includes Theorem 7.4, which corresponds to the case \(k = 1\).
Theorem 7.9 (Chaoticity). Let $\mu_\infty$ be the uniform distribution on the unit disc $\{ z \in \mathbb{C} : |z| \leq 1 \}$. For all $1 \leq k \leq n$, denoting by $P_{n,k}$ the $k$-dimensional marginal distribution of the Ginibre gas (2.1), we have

$$P_{n,k} \xrightarrow{n \to \infty} \mu_\infty^k.$$ 

Idea of the proof. The measures $P_{n,k}$ and $\mu_\infty$ have densities $\varphi_{n,k}$ and $\varphi_\infty(z) = \pi^{-1} 1_{|z| \leq 1}$.

The case $k = 1$ is nothing else but Theorem 7.3, namely $P_{n,1} \xrightarrow{n \to \infty} \mu_\infty$. This comes via dominated convergence from the fact that $\lim_{n \to \infty} \varphi_{n,1} = \varphi_\infty$ uniformly on compact subsets of $\{ z \in \mathbb{C} : |z| \neq 1 \}$.

Let us consider now the case $k = 2$. Here again, by dominated convergence, it suffices to show that

$$\lim_{n \to \infty} \varphi_{n,2}^2 = \varphi_\infty^2$$

uniformly on compact subsets of $\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| \neq 1, |z_2| \neq 1, z_1 \neq z_2 \}$.

By Theorem 7.8, for all $z_1, z_2 \in \mathbb{C}$,

$$\varphi_{n,2}(z_1, z_2) = \frac{n}{n - 1} e^{-n(|z_1|^2 + |z_2|^2)} (e_n(n|z_1|^2)e_n(n|z_2|^2) - |e_n(nz_1\overline{z_2})|^2)$$

$$= \frac{n}{n - 1} \varphi_{n,1}(z_1)\varphi_{n,1}(z_2) - \frac{n}{n - 1} \frac{e^{-n(|z_1|^2 + |z_2|^2)}}{\pi^2} |e_n(nz_1\overline{z_2})|^2$$

(7.4)

where $e_n$ is as in Lemma 7.5. It follows that for any $n \geq 2$ and $z_1, z_2 \in \mathbb{C}$,

$$\Delta_n(z_1, z_2) = \varphi_{n,2}(z_1, z_2) - \varphi_{n,1}(z_1)\varphi_{n,1}(z_2)$$

$$= \frac{1}{n - 1} \varphi_{n,1}(z_1)\varphi_{n,1}(z_2) - \frac{n}{n - 1} \frac{e^{-n(|z_1|^2 + |z_2|^2)}}{\pi^2} |e_n(nz_1\overline{z_2})|^2.$$ (7.5)

In particular, using $\varphi_{n,2} \geq 0$ for the lower bound,

$$-\varphi_{n,1}(z_1)\varphi_{n,1}(z_2) \leq \Delta_n(z_1, z_2) \leq \frac{1}{n - 1} \varphi_{n,1}(z_1)\varphi_{n,1}(z_2).$$

From this and Lemma 7.5, we first deduce that for any compact subset $K$ of $\{ z \in \mathbb{C} : |z| > 1 \}$

$$\lim_{n \to \infty} \sup_{z_1 \in K} |\Delta_n(z_1, z_2)| = \lim_{n \to \infty} \sup_{z_1 \in K} |\Delta_n(z_1, z_2)| = 0.$$

It remains to show that $\Delta_n(z_1, z_2) \to 0$ as $n \to \infty$ when $z_1$ and $z_2$ are in compact subsets of $\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1 \}$. In this case $|z_1z_2| \leq 1$, and Lemma 7.5 gives

$$|e_n(nz_1\overline{z_2})|^2 \leq 2e^{2nR(z_1\overline{z_2})} + 2r_n^2(z_1\overline{z_2}).$$

Next, using the elementary identity $2R(z_1\overline{z_2}) = |z_1|^2 + |z_2|^2 - |z_1 - z_2|^2$, we get

$$e^{-n(|z_1|^2 + |z_2|^2)}|e_n(nz_1\overline{z_2})|^2 \leq 2e^{-n|z_1 - z_2|^2} + 2e^{-n|z_1|^2 - |z_2|^2} r_n^2(z_1\overline{z_2}).$$ (7.6)

Since $|z_1z_2| \leq 1$, the formula for $r_n$ in Lemma 7.5 gives

$$e^{-n(|z_1|^2 + |z_2|^2)} r_n^2(z_1\overline{z_2}) \leq e^{-n(|z_1|^2 + |z_2|^2 - 2\log|z_1|^2 - \log|z_2|^2)} \frac{(n + 1)^2}{2\pi n}.$$ (7.5)

Using (7.5), (7.6) and the bounds $\varphi_{n,1} \leq \pi^{-1}$ and $u - 1 - \log u > 0$ for $0 < u < 1$, it follows that $\Delta_n(z_1, z_2)$ tends to 0 as $n \to \infty$ uniformly in $z_1, z_2$ on compact subsets of

$$\{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1, z_1 \neq z_2 \}.$$ (7.7)

This finishes the proof of the case $k = 2$. The case $k \geq 3$ follows from the case $k = 2$ by Lemma 7.11. □

Remark 7.10 (Impossibility of global uniform convergence of densities). The convergence of $\varphi_{n,1}$ cannot hold uniformly on arbitrary compact sets of $\mathbb{C}$ since the point-wise limit is not continuous on the unit circle. Similarly, the convergence of $\varphi_{n,2}$ cannot hold on $\{ (z, z) : z \in \mathbb{C}, |z| < 1 \}$ since $\varphi_{n,2}(z, z) = 0$ for any $n \geq 2$ and $z \in \mathbb{C}$ while $\varphi_\infty(z)\varphi_\infty(z) = \pi^{-2} \neq 0$ when $|z| < 1$. 

□
Lemma 7.11 (Chaoticity). Let $E$ be a Polish space. For all $n \geq 1$, let $P_n \in \mathcal{M}_1(E^n)$ be exchangeable, and for all $1 \leq k \leq n$, let $P_{n,k} \in \mathcal{M}_1(E^k)$ be its $k$-dimensional marginal distribution. For all $n \geq 1$, let us pick a random vector $X_n = (X_{n,1}, \ldots, X_{n,n}) \sim P_n$ and let us define the random empirical measure
\[
\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_{n,i}}.
\]
For all $\mu \in \mathcal{M}_1(E)$, the following properties are equivalent:

1. $\mu_n \xrightarrow{d} \mu$ in $\mathcal{M}_1(M_1(E))$ (here we see $\mu_n$ a random variable taking values in $\mathcal{M}_1(E)$);
2. $P_{n,k} \xrightarrow{d} \mu^{\otimes k}$ for any fixed $k \geq 1$ (note that $P_{n,k}$ has a meaning as soon as $n \geq k$);
3. $P_{n,2} \xrightarrow{d} \mu^{\otimes 2}$;
where these weak convergences are with respect to continuous and bounded test functions.

Proof. Folkloric in the domain of mean field particle systems. We refer to [BCF18] and references therein. □

Theorem 7.12 (Central limit phenomenon). Let $\mu_n$ be as in Theorem 7.11. Then, for all measurable $f: \mathbb{C} \to \mathbb{R}$ which are $C^1$ in a neighborhood of the unit disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$ of the complex plane,
\[
n \left[ \int f d\mu_n - \mathbb{E} \int f d\mu_n \right] = \sum_{k=1}^{n} \left[ f\left( \frac{\lambda_k}{\sqrt{n}} \right) - \mathbb{E} f\left( \frac{\lambda_k}{\sqrt{n}} \right) \right] \xrightarrow{n \to \infty} \mathbb{N} \left( \frac{1}{4\pi} \|f\|^2_{H^1(D)} + \frac{1}{2\pi} \|f\|^2_{H^{1/2}(\partial D)} \right)
\]
where
\[
\|f\|^2_{H^1(D)} = \int_{D} |\nabla f|^2 dz \quad \text{and} \quad \|f\|^2_{H^{1/2}(\partial D)} = \sum_{k \in \mathbb{Z}} |k||\hat{f}(k)||^2
\]
where $\hat{f}(k)$ is the $k$-th Fourier coefficient of $f$ on $\partial D = \{z \in \mathbb{C} : |z| = 1\}$, namely
\[
\hat{f}(k) = \frac{1}{2\pi} \int_{0}^{2\pi} f(e^{i\theta}) e^{-ik\theta} d\theta.
\]
Note that $\|f\|^2_{H^{1/2}(\partial D)} = 0$ if $f$ is analytic on a neighborhood of $\partial D$.

About the proof. It is known [CL05] that the cumulants of linear statistics of determinantal processes have a nice form that can be used to prove a central limit theorem. This idea is followed in [R V07] in order to produce the result, via combinatorial identities, and via reduction to polynomial test functions. The method is used for more general two-dimensional determinantal gases in [AHM11, AHM15]. □

Theorem 7.13 (Distribution of the moduli in the Gindibr model). Let $(\lambda_1, \ldots, \lambda_n)$ be the exchangeable random vector considered in Theorem 7.11. Then the following equality in distribution holds
\[
(|\lambda_1|, \ldots, |\lambda_n|) \xrightarrow{d} (Z_{\sigma(1)}, \ldots, Z_{\sigma(n)})
\]
where $Z_1, \ldots, Z_n$ are independent non-negative random variables with
\[
Z_k^2 \sim \Gamma(k, 1), \quad 1 \leq k \leq n,
\]
and where $\sigma$ is a uniform random permutation of $\{1, \ldots, n\}$ independent of $Z_1, \ldots, Z_n$. Equivalently, for all symmetric bounded measurable $F: \mathbb{R}^n \to \mathbb{R}$, we have $\mathbb{E}(F(|\lambda_1|, \ldots, |\lambda_n|)) = \mathbb{E}(F(Z_1, \ldots, Z_n))$.

This is an equality between two exchangeable laws on $\mathbb{R}^n$, in other words an equality in law between two configurations of unlabeled random points in $\mathbb{R}$ (multi-sets). Note in particular that for all $1 \leq k \leq n$, taking $F(x_1, \ldots, x_n) = \sum_{i_1, \ldots, i_k} f(x_{i_1}) \cdots f(x_{i_k})$ gives equality of $k$-point correlation functions.

The law $\Gamma(a, \lambda)$ has density $x \in \mathbb{R} \mapsto e^{-\lambda x} x^{a-1} \frac{1}{\Gamma(a)}$ for $x > 0$, and $\Gamma(a, \alpha) \ast \Gamma(b, \lambda) = \Gamma(a + b, \lambda)$.

Note that $(\sqrt{2}Z_k)^2 \sim \Gamma(k, \frac{1}{2}) \sim \Gamma(2k, \frac{1}{2}) \sim \Gamma(2k, 1) = \Gamma(2k, \frac{1}{2})$ since $\chi^2(n) = \Gamma(\frac{n}{2}, \frac{1}{2})$ for all $n \geq 1$.

For $n = 1$ we recover the Box–Muller formula $|X|^2 \sim \chi^2(2) = \Gamma(1, \frac{1}{2}) = \Gamma(1, \frac{1}{2})$ with $X \sim \mathcal{N}(0, I_2)$. 

---

Footnotes:
1. The law $\Gamma(a, \lambda)$ has density $x \in \mathbb{R} \mapsto e^{-\lambda x} x^{a-1} \frac{1}{\Gamma(a)}$ for $x > 0$, and $\Gamma(a, \alpha) \ast \Gamma(b, \lambda) = \Gamma(a + b, \lambda)$.
2. Note that $(\sqrt{2}Z_k)^2 \sim \Gamma(k, \frac{1}{2}) \sim \Gamma(2k, \frac{1}{2}) \sim \Gamma(2k, 1) = \Gamma(2k, \frac{1}{2})$ since $\chi^2(n) = \Gamma(\frac{n}{2}, \frac{1}{2})$ for all $n \geq 1$.
3. For $n = 1$ we recover the Box–Muller formula $|X|^2 \sim \chi^2(2) = \Gamma(1, \frac{1}{2}) = \Gamma(1, \frac{1}{2})$ with $X \sim \mathcal{N}(0, I_2)$. 

---
Proof. From Theorem 7.14, the exchangeable random vector \((\lambda_1, \ldots, \lambda_n)\) has density \(\varphi_n\). It follows that the density of the exchangeable random vector \(\{(|\lambda_1|, \ldots, |\lambda_n|)\}\) is obtained from \(\varphi_n\) by integrating the phases in polar coordinates. In polar coordinates \(x_k = r_k e^{i\theta_k}\), the density \(\varphi_n\) writes

\[
(r_1, \ldots, r_n, \theta_1, \ldots, \theta_n) \mapsto \frac{e^{-\sum_{j=1}^n r_j^2}}{\pi^n} \prod_{j<k} |r_j e^{i\theta_j} - r_k e^{i\theta_k}|^2.
\]

Now we have, denoting \(\Sigma_n\) the symmetric group of permutations of \(\{1, \ldots, n\}\),

\[
\prod_{j<k} |r_j e^{i\theta_j} - r_k e^{i\theta_k}|^2 = \prod_{j<k} (r_j e^{i\theta_j} - r_k e^{i\theta_k}) \prod_{j<k} (r_j e^{i\theta_j} - r_k e^{i\theta_k})
\]

\[
= \det \begin{bmatrix}
  r_j^{k-1} e^{i(k-1)\theta_j} & 1 \leq j, k \leq n \\
  \vdots & \vdots \\
  r_j e^{i\theta_j} & 1 \leq j, k \leq n
\end{bmatrix} \prod_{j<k} |r_j e^{i\theta_j} - r_k e^{i\theta_k}|
\]

\[
= \left( \sum_{\sigma \in \Sigma_n} (-1)^{\text{sign}(\sigma)} \prod_{j=1}^n r_j^{\sigma(j)-1} e^{i(\sigma(j)-1)\theta_j} \right) \left( \sum_{\sigma' \in \Sigma_n} (-1)^{\text{sign}(\sigma')} \prod_{j=1}^n r_j^{\sigma'(j)-1} e^{i(\sigma'(j)-1)\theta_j} \right)
\]

\[
= \sum_{\sigma, \sigma' \in \Sigma_n} (-1)^{\text{sign}(\sigma) + \text{sign}(\sigma')} \prod_{j=1}^n r_j^{\sigma(j) + \sigma'(j) - 2} e^{i(\sigma(j)-\sigma'(j))\theta_j}.
\]

If we integrate the phases, we note that only the terms with \(\sigma = \sigma'\) contribute to the result, namely

\[
\int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{j<k} |r_j e^{i\theta_j} - r_k e^{i\theta_k}|^2 d\theta_1 \cdots d\theta_n = (2\pi)^n \sum_{\sigma \in \Sigma_n} \prod_{j=1}^n r_j^{2(\sigma(j)-1)} = (2\pi)^n \text{per} \left[ \begin{bmatrix} r_j^{2k} \end{bmatrix} \right] \text{per} \left[ \begin{bmatrix} r_j^{2k} \end{bmatrix} \right]_{1 \leq j, k \leq n}
\]

where “per” stands for “permanent.” Therefore, the (exchangeable) density of the moduli is given by

\[
\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{e^{-\sum_{j=1}^n r_j^2}}{\pi^n} \prod_{j<k} |r_j e^{i\theta_j} - r_k e^{i\theta_k}|^2 d\theta_1 \cdots d\theta_n = \text{perm} \left[ \begin{bmatrix} r_j e^{i\theta_j} \end{bmatrix} \right]_{1 \leq j, k \leq n}
\]

But if \(f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}\) are probability density functions then \((x_1, \ldots, x_n) \mapsto \text{perm}[f_j(x_k)]_{1 \leq j, k \leq n}\) is the density of the random vector \((X_{\sigma(1)}, \ldots, X_{\sigma(n)})\) where \(X_1, \ldots, X_n\) are independent real random variables with densities \(f_1, \ldots, f_n\) and where \(\sigma\) is a uniform random permutation of \(\{1, \ldots, n\}\) independent of \(X_1, \ldots, X_n\). Also the desired result follows from the formula above and the fact that for all \(1 \leq k \leq n\), a non-negative random variable \(Z_k\) has density \(r \mapsto \frac{2}{\pi} e^{r^2} \cdot r^2 \cdot 1_{r \geq 0}\) if and only if \(Z_k \sim \Gamma(\lambda, k)\).

This proof, essentially due to Kostlan [Kos92], see also [Rid03], relies on the determinantal nature of \(\varphi_n\) in (2.3), and remains usable for general determinantal processes, see for instance [HKPV09].

Theorem 7.14 (Spectral radius). With the notation of Theorem 7.14 almost surely

\[
\rho_n = \max_{1 \leq k \leq n} |\lambda_k| \sqrt{n} \to 1.
\]

Moreover, denoting \(\kappa_n = \log \frac{\lambda_n}{2\pi} - 2 \log(\log(n))\),

\[
\sqrt{4n\kappa_n} \left( \rho_n - 1 - \sqrt{\frac{\kappa_n}{4n}} \right) \to \text{Gumbel}^4
\]

Note the the second statement (Gumbel fluctuation) implies that \(\rho_n \to 1\) in probability.

Regarding the convergence, see [BCGZ20] for a random analytic function point of view, related to the central limit theorem. Regarding the fluctuation, see [Joh07] Ben10 for an interpolation with the Tracy–Widom fluctuation at the edge of GUE.

Idea of proof. By Theorem 7.13 Since \(Z_k^2 \overset{d}{=} E_1 + \cdots + E_k\) where \(E_1, \ldots, E_k\) are independent and identically distributed exponential random variables of unit mean, we get, for every \(r > 0\),

\[
\mathbb{P}(\rho_n \leq \frac{1}{\sqrt{n}r}) = \prod_{1 \leq k \leq n} \mathbb{P} \left( \frac{E_1 + \cdots + E_k}{n} \leq r^2 \right).
\]

By the law of large numbers, this tends as \(n \to \infty\) to 0 or 1 depending on the position of \(r\) with respect to 1. Moreover the central limit theorem suggests that \(\rho_n\) behaves as \(n \to \infty\) as the maximum of

\[^4\text{If } X \sim \text{Exp}(1) \text{ then } -\log(X) \text{ has Gumbel law and cumulative distribution function } \mathbb{P}(\log(X) \leq x) = e^{-e^{-x}}.\]
independent and identically distributed Gaussian random variables, a situation for which it is known that the fluctuation follows the Gumbel law. The full proof is in [Rid03] and involves crucially a quantitative central limit theorem and the Borel–Cantelli lemma. The approach is robust and remains valid beyond the Ginibre gas, for determinantal gases, see for instance [CP14, JQ17, GZ18] and references therein. □

**Remark 7.15** (Real or quaternionic Ginibre model). How about an analogue of Theorem 7.1 when the entries of $M$ are real Gaussian or real quaternionic Gaussian instead of complex Gaussian? Some answers are already in [Gin65]. In these cases, the density of the eigenvalues can be computed but it is not the beta gas (2.4) with $\beta \in \{1, 4\}$. This is in contrast with the $G(O(U(S))E$ triplet of the Hermitian random matrix Dysonian universe [Dys02]. See for instance [Edc97, Dub18b] and references therein.

**Remark 7.16** (Large deviations). The large deviations principle for the beta Ginibre gas (2.5) was established in [HP00, PH08], using a method inspired from [BAG08], itself inspired from [Voi93, Voi94]. It does not rely on the determinantal structure, and allows to extend Theorem 7.8 to all $\beta > 0$.

### 7.2. More determinantal models.

It is well known that the ratio of two independent real standard Gaussian random variables follows a Cauchy distribution. The following theorem can be seen as a matrix version of this phenomenon.

**Theorem 7.17** (Forrester–Krishnapur spherical ensemble). Let $M_1$ and $M_2$ be independent copies of the Ginibre random matrix defined in (7.1). Then as an exchangeable random vector of $\mathbb{C}^n$, the eigenvalues of $M_1M_2^{-1}$ have density

$$ (z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto \frac{1}{Z_n} \prod_{j<k} |z_j - z_k|^2 Z_n \prod_{j=1}^n (1 + |z_j|^2)^{n+1}. $$

This corresponds to the beta gas (2.4) with $V = \frac{1}{2} \log(1 + |z|^2)$ and $\beta = 2$. Moreover its push-forward on the Riemann sphere using inverse stereographic projection is the uniform law on the sphere.

See [Kri09] and [HKPV09, PK09, For10] for a proof. The set of singular $n \times n$ complex matrices is a hyper-surface of zero Lebesgue measure in $\mathbb{C}^{n^2}$ and therefore, almost surely, the Ginibre random matrix $M$ in (7.1) is invertible (its law is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}^{n^2}$).

From Theorem 7.14, the equilibrium measure of the gas is heavy tailed with density

$$ z \in \mathbb{C} \mapsto \frac{1}{\pi(1 + |z|^2)^2}. $$

A large deviations principle for the empirical measure associated to the Coulomb gas is proved in [Har12] in relation with the sphere. The convergence of the empirical measure is also considered in [Ber11].

**Theorem 7.18** (Zyczkowski–Sommers ensemble). Let $U = (U_{j,k})_{1 \leq j,k \leq m}$ be a random $m \times m$ unitary matrix following the (Haar) uniform law on this compact group of matrices. Then, for all $1 \leq n < m$, as an exchangeable random vector of $\mathbb{C}^n$, the eigenvalues of the truncation $(U_{j,k})_{1 \leq j,k \leq n}$ have density

$$ (z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto \frac{1}{Z_n} \prod_{j=1}^n (1 - |z_j|^2)^{m-n-1} \prod_{1 \leq j<k \leq n} |z_j - z_k|^2. $$

This corresponds to the beta gas (2.4) with $V = V_{n,m} = \frac{m-n-1}{n} \log \frac{1}{1 - |z|^2}$ and $\beta = 2$.

See [ZS00] for a proof, and [PK09] for the special case $m \geq 2n$ and a link with the pseudo-sphere and Schur transformation. Following [PR05] and references therein, if $\lim_{m,n \to \infty} \frac{n}{m} = \alpha \in (0, 1)$ then the empirical measure converges towards the heavy tailed probability measure with density

$$ z \in \mathbb{C} \mapsto \frac{(1 - \alpha)}{\pi \alpha(1 - |z|^2)^2} 1_{|z| \leq \sqrt{\alpha}}. $$

In a sense this law interpolates between the uniform law on the unit disc ($\alpha \to 0$ after scaling by $\sqrt{\alpha}$) and the uniform law on the unit circle ($\alpha \to 1$). A large deviations principle is obtained in [PR05]. Concentration inequalities are derived in [MS19], while the fluctuation at the edge is studied in [JQ17].
Theorem 7.19 (Product of Ginibre random matrices). Let $m \geq 1$ and let $M_1, \ldots, M_m$ be independent and identically distributed copies of the $n \times n$ Ginibre matrix $V$. Then, as an exchangeable random vector of $\mathbb{C}^n$, the eigenvalues of the scaled product $n^{-1/2}M_1 \cdots M_m$ have density
\[
\prod_{j=1}^n w_m(\sqrt{n}|z_j|) \prod_{j<k} |z_j - z_k|^2
\]
where $w_k$ is the Meijer $G$-function given by the recursive formula
\[
w_1(z) = e^{-|z|^2} \text{ and } w_k(z) = 2\pi \int_0^\infty w_{k-1}(\frac{z}{r}) e^{-r^2} dr.
\]
This corresponds to the beta gas (2.3) with $V = V_{n,m} = -\frac{1}{n} \log w_m(\sqrt{n}z)$ and $\beta = 2$.

See [AB12] for a proof. Following [GT11, Bar11], its converges to the equilibrium measure with density
\[
z \in \mathbb{C} \mapsto \frac{|z|^m - 2}{m\pi} 1_{|z| \leq 1}.
\]
We recover the uniform law on the unit disc when $m = 1$. The edge fluctuation is considered in [JQ17].

Remark 7.20 (Determinantal gases and random normal matrices). An $n \times n$ complex matrix is normal when $MM^* = M^*M$. The random matrices in theorems 7.9, 14, 15, 17, 19 are not normal. Let us comment now on models of normal random matrices. Let $N_n$ be the hyper-surface of $\mathbb{C}^{n^2}$ of all $n \times n$ normal matrices. Let $V : \mathbb{C} \to \mathbb{R}$ be $C^2$ and such that $V(z) \geq c \log(1 + |z|^2)$ for some constant $c > 0$. Following [CZ98, EP05], let us consider the probability measure on $N_n$ with density proportional to $e^{-nV(M)}$ with respect to the Hausdorff measure on $N_n$. This produces random normal matrices, and their eigenvalues, seen as an exchangeable random vector, have density given by the gas (2.3) with $\beta = 2$. This random (normal) matrix model is referred to as the random normal matrix model. The fluctuation of the empirical measure is studied in [AHM15, AHM11], while the fluctuation at the edge is studied in [CP14, JQ17, CZ18].

The power of a Ginibre matrix has also a nice determinantal structure, see [Dub18a].

8. Comments and open problems

We have skipped several important old and new results on Coulomb gases. The main themes are local versus global, first versus second order, microscopics versus macroscopics, non-universal versus universal.

Universality. The first order global convergence $\lim_{n \to \infty} \mu_n = \mu_V$, that we call macroscopics, is not universal in the sense that the limit $\mu_V$ still depends on $V$. The second order convergence provided by the central limit theorem (5.3) is universal in the sense that the limit should not depend on $V$. Similarly, for a two-dimensional Coulomb gas with radial confining potential $V$, the limit of the edge depends on $V$ but its fluctuation does not and is universal. Universality emerges often in asymptotic analysis, as for classical limit theorems of probability theory.

Microscopics. A second order analysis corresponds to the asymptotic analysis of $n(\mu_n - \nu_V)$ as $n \to \infty$. This can be seen as a microscopic analysis while the convergence $\mu_n \to \mu_V$ is a macroscopic analysis. This corresponds to a second order Taylor formula for the quadratic form $\xi_V$, in other words in a special factorization, leading to a new object called the renormalized energy. This was the subject of an article of works by Étienne Sandier and Sylvia Serfaty, and by Sylvia Serfaty and other co-authors. See for instance [Ser18a, Ser18b, LS17, Ser15] and references therein. An outcome of this refined analysis is a second order asymptotics for the free energy. More precisely, recall that the Boltzmann–Shannon entropy of the Boltzmann–Gibbs measure $P_n$ in (2.2) is defined by
\[
S(P_n) = -\int_{[Rd]^n} f_n(x_1, \ldots, x_n) \log f_n(x_1, \ldots, x_n) dx_1 \cdots dx_n
\]
where $f_n$ is the density of $P_n$. Its Helmholtz free energy is given by
\[
\int E_n dP_n \frac{S(P_n)}{\beta} = -\frac{\log Z_n}{\beta},
\]
see [Cha15]. Now following [LS17], if $\mu_V$ has density $f_V$ with a finite Boltzmann–Shannon entropy
\[
S(\mu_V) = -\int_{R^d} f_V \log f_V dx,
\]
then we have an asymptotic expansion of the free energy as $n \to \infty$ as

$$\log Z_n \beta = \begin{cases} \frac{n^2}{2} \mathcal{E}_V(\mu V) - \frac{n \log n}{4} + n(c_\beta + c'_\beta S(\mu V)) + n o_n(1) & \text{if } d = 2, \\ \frac{n^2}{2} \mathcal{E}_V(\mu V) + n(c_\beta, d, V + c'_\beta S(\mu V)) + n o_n(1) & \text{if } d \neq 2, \end{cases}$$

where $c_\beta, c'_\beta, c_\beta, d, V$ are constants which can be made explicit.

**Edge.** The most elementary open question related to Coulomb gases is perhaps the law of fluctuation at the edge, even in the case of rotationally invariant confining potential for arbitrary values of $d$ and $\beta$. The Gumbel fluctuation is known for instance to be universal for a class of two dimensional $(d = 2)$ determinant (planar) Coulomb gases with radial confining potential, see [CP14]. The same question for arbitrary $\beta$ is open, and the same question for arbitrary dimension $d \geq 3$ and $\beta > 0$ is also open.

**Crystallization.** A conjecture related to Coulomb gases is the emergence of rigid structures at low temperatures. This is known as crystallization and was proved in special cases, for instance for one-dimensional Coulomb gases. See for instance [Ser15] [BL15] [Ser18b] [Ser18a] [PS20] and references therein.

**More.** Among all the important results on Coulomb gases that we have not yet mentioned, we may cite the approximate transport maps for universality considered in [FG16] [BF15], the rigidity analysis for hierarchical Coulomb gases considered in [Cha19b], the local density for two-dimensional Coulomb gases considered in [BBNY17], the Dobrushin–Lanford–Ruelle type equations considered in [DHLM21], the Coulomb gas properties on the sphere considered in [BH19], the local laws and rigidity considered in [AS21], the quasi-Monte-Carlo method on the sphere considered in [Ber19], and the Berezinskii–Kosterlitz–Thouless transition [KP17] [GS20].

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