I. INTRODUCTION

This article presents in full detail the study of the effects of a strong external magnetic field in the gap structure of a color superconductor for a system of three massless quark flavors, the main results of which were summarized in Ref. 1.

Our present knowledge of QCD at high baryonic density indicates that quark matter might be in a color superconducting phase (for reviews, see 2). This interesting possibility has originated a flurry of research. Not only with the motivation to deepen our knowledge about QCD, but mainly because of its astrophysical applications. It is very likely that quark matter occupies the inner regions of compact stars, or that even quark stars exist 3,4. On the other hand, it is well-known 5 that strong magnetic fields, as large as $B \sim 10^{12} - 10^{14}$ G, exist in the surface of neutron stars, while in magnetars they are in the range $B \sim 10^{14} - 10^{15}$ G, and perhaps as high as $10^{16}$ G 6. Moreover, the virial theorem 7 allows the field magnitude to reach values as large as $10^{18} - 10^{19}$ G. If quark stars are self-bound rather than gravitational-bound objects, the previous upper limit that has been obtained by comparing the magnetic and gravitational energies could go even higher.

In this paper we investigate how a strong magnetic field can modify color superconductivity, with the aim of further studying its possible astrophysical implications. We will start by considering three massless quark flavors. In this case, it is well established that the ground state of high-dense QCD corresponds to the CFL (color-flavor locked) phase 8. In this phase quarks form spin-zero Cooper pairs in the color-antitriplet, flavor-antitriplet representation. One can now ask whether this scenario can change when a magnetic field is switched on. Would the external field influence the pairing phenomena? As discussed in Ref. 1 and shown here in more detail, the magnetic field can drastically affect the condensation phenomenon producing a new color superconducting phase that we call magnetic color-flavor locking (MCFL) phase.

To understand how a magnetic field can affect the color superconducting pairing it is important to recall that in spin-zero color superconductivity, although the color condensate has non-zero electric charge, there is a linear combination of the photon and a gluon that remains massless 9,10. A spin-zero color superconductor may thus be penetrated by a long-range remnant “rotated” magnetic field $\vec{B}$. Although all the superconducting pairs are neutral with respect to this long-range field, a subset of them is formed by quarks of opposite rotated charges $\vec{Q}$. One would expect that this set of condensates will be strengthened by the penetrating field, since their paired quarks, having opposite $\vec{Q}$-charges and opposite spins, have parallel (rather than antiparallel) magnetic moments. As will be shown in the present paper, our intuitive expectation is confirmed by our explicit calculations. The situation here has some resemblance to the magnetic catalysis of chiral symmetry breaking 10,11, in the sense that the magnetic field strengthens the pair formation. Despite this similarity, the way the field influences the pairing mechanism in the two cases is quite different as will be shown below.

Our results signalize the color superconductor as a very peculiar kind of superconductor. Up to now this is the only physical system where magnetism and superconductivity are not at odds with each other, but on the contrary, the fermion-fermion pairing, which is the hallmark of superconductivity, is reinforced by the magnetic field.

The plan of the paper is as follows. In Sec. II we discuss the color-flavor and Dirac structure of the gap in the

PACS numbers: 12.38.Aw, 12.38.-t, 24.85.+p
presence of a constant magnetic field. The mean-field theory in the presence of the rotated magnetic field is introduced in Sec. III. Special attention is devoted to obtaining the mean-field effective action in momentum space in the presence of the external field and to the application of the Ritus transformation \[IL\] to this case. In Sec. IV we find the gap equations and solutions in the presence of the $B$ field. Starting from the general gap equations for arbitrary $B$ values, we analytically find the solutions for the limiting cases of zero and strong fields ($\tilde eB \gtrless \mu^2$). We find that at strong magnetic fields the gaps that get contributions from pairs of $\bar Q$-charged quarks are enhanced by the field. In Sec. V we sketch the derivation of the low-energy effective field theory of the Goldstone modes in the MCFL phase. The main outcomes of the paper and discussion of the results are stated in Sec. VI.

\section{The MCFL Phase}

As known, a degenerate quark system becomes unstable under any quark-quark attractive interaction at the Fermi surface. This instability leads to color-superconducting Cooper pairing. Depending on how large the density is, there are different choices for the interaction. For three different quark flavors, at densities higher than the strange quark mass, the quarks can be considered effectively massless. This system of very dense massless quarks can form a favored phase with spin-zero Cooper pairs in the color-antitriplet, flavor-antitriplet representation (the CFL phase) \[8\]. In compact stars where the macroscopic volume of quark matter must be an electrically neutral color singlet, this phase is preferred at high enough densities \[13\].

An important feature of spin-zero color superconductivity is that although the color condensate has non-zero electric charge, there is a linear combination of the photon $A_\mu$ and a gluon $G^{8}_\mu$ that remains massless \[8, 9\],

$$
\tilde A_\mu = \cos \theta A_\mu - \sin \theta G^{8}_\mu ,
$$

while the orthogonal combination \( \tilde G^{8}_\mu = \sin \theta A_\mu + \cos \theta G^{8}_\mu \) is massive. In the CFL phase the mixing angle $\theta$ is sufficiently small ($\sin \theta \sim e/g \sim 1/40$). Thus, the penetrating field in the color superconductor is mostly formed by the photon with only a small gluon admixture.

The unbroken $U(1)$ group corresponding to the long-range rotated photon (i.e. the $\bar U(1)_{\text{c.m.}}$ is generated, in flavor-color space, by $\bar Q = Q \times 1 - 1 \times Q$, where $Q$ is the electromagnetic charge operator. We use the conventions $Q = -\lambda_8/\sqrt{3}$, where $\lambda_8$ is the 8th Gell-Mann matrix. Thus our flavor-space ordering is $(s, d, u)$. In the 9-dimensional flavor-color representation that we will use in this paper (the color indexes we are using are $(1, 2, 3) = (b, g, r)$), the $\bar Q$ charges of the different quarks, in units of $\bar e = e \cos \theta$, are

\begin{equation}
\begin{pmatrix}
s_1 & s_2 & s_3 & d_1 & d_2 & d_3 & u_1 & u_2 & u_3 \\
0 & 0 & - & 0 & 0 & - & + & + & 0
\end{pmatrix}
\end{equation}

Although the interaction of an external field with dense quark matter has been investigated by several authors \[8, 13\], the effect of the penetrating $B$ field on the superconducting gap structure was not in the scope of these studies. However, as shown in our previous paper \[11\], the $B$ field can change the gap structure and lead to a new superconducting phase. To understand how this can occur notice that due to the coupling of the charged quarks with the external $B$ field, the color-flavor space is augmented by the charge operator $Q$, and consequently the order parameter of the CFL splits in new independent pieces. In this paper we show how the color superconducting pairing is modified in the presence of a magnetic field.

Let us introduce the rotated-charge projectors

$$
\Omega_0 = \text{diag}(1, 1, 0, 1, 1, 0, 0, 0, 1) ,
$$

$$
\Omega_+ = \text{diag}(0, 0, 0, 0, 0, 0, 1, 1, 0) ,
$$

$$
\Omega_- = \text{diag}(0, 0, 1, 0, 0, 0, 0, 0, 0) ,
$$

satisfying

$$
\Omega_\eta \Omega_{\eta'} = \delta_{\eta\eta'} \Omega_\eta , \quad \eta, \eta' = 0, +, - .
$$
\( \Omega_0 + \Omega_+ + \Omega_- = 1 \), \( (7) \)

In terms of \( \Omega_+ \) and \( \Omega_- \) the rotated charge operator takes the form

\[
\tilde{Q} = \sum_{\eta=0,\pm} \eta \Omega_\eta = \Omega_+ - \Omega_- .
\] (8)

Using the projectors \( \Psi \) we can express the column fermion field in the 9-dimensional color-flavor representation \((s_1, s_2, s_3, d_1, d_2, d_3, u_1, u_2, u_3)\) as the sum

\[
\psi = \psi(0) + \psi(+) + \psi(-) ,
\] (9)

where subindexes \((0)-, (+/-)\) indicate fields of zero, positive and negative rotated charge respectively defined by

\[
\psi(0) = \Omega_0 \psi , \quad \psi(+) = \Omega_+ \psi , \quad \psi(-) = \Omega_- \psi .
\] (10)

The interaction of massless quarks with the external rotated-magnetic field is described by the Lagrangian density term

\[
L_{\text{em\ quarks}}^\text{em} = \overline{\psi} (i \Pi_\mu \gamma^\mu) \psi ,
\] (11)

with

\[
\Pi_\mu = i \partial_\mu + \tilde{e} \tilde{Q} \tilde{A}_\mu .
\] (12)

Given that the quarks have different rotated charges, the Lagrangian density \( L_{\text{em\ quarks}}^\text{em} \) naturally splits in three terms

\[
L_{\text{em\ quarks}}^\text{em} = \overline{\psi}_0 (i \partial_\mu \gamma^\mu) \psi_0 + \overline{\psi}_+ (i \partial_\mu + \tilde{e} \tilde{A}_\mu) \gamma^\mu \psi_+ + \overline{\psi}_- (i \partial_\mu - \tilde{e} \tilde{A}_\mu) \gamma^\mu \psi_- .
\] (13)

There is no reason to expect that the color-flavor structure of the condensate will be the same as the CFL one. Instead, one would think that the charge-separation of the fields enforced by the external magnetic field should be somehow reflected in the color-condensate too. Below we will show that this is indeed the case.

At this point we need to define a working model where we can investigate the effects of the external magnetic field in a system of massless quarks of three different flavors. With this aim, all our calculations will be done in the context of a Nambu-Jona-Lasinio (NJL) four-fermion interaction model, abstracted from one-gluon exchange interactions in QCD \( 8 \). Although this simple model disregards the effect of the \( \tilde{B} \)-field on the gluon dynamics and assumes the same NJL couplings for the system with and without magnetic field, it keeps the main attributes of the theory, providing the correct qualitative physics.

In order to find a reasonable ansatz for the gap in the presence of a magnetic field, we will start considering the system without magnetic field and will find the most general structure allowed for the color superconducting (CS) condensate. The corresponding general structure in the presence of the magnetic field will be a particular case of the one at zero field, because the two structures have to coincide in the limit of zero field and because the field reduces the symmetry of the original theory to a subgroup of the original group. We underline that such a general structure is found using only very general symmetry arguments. The gap that minimizes the free energy is just one of many possible particular condensate patterns contained within the general structure: precisely the one that retains the highest degree of symmetry. This was the guidance principle that led to the CFL ansatz \( 8 \) in the case without magnetic field, and it will be the same principle that will ultimately take us, when a magnetic field is present, to a new ansatz: the MCFL one \( 1 \). As seen below, similarly to CFL, in the MCFL phase quarks of all three colors and all three flavors pair. The main difference is that the condensate distinguishes between gap parameters that can get contributions from rotated charged quarks and those that only get contributions from neutral quarks.

Let us recall that the gap matrix is formed by diquark condensates \( \langle q^T O q \rangle \), where \( q^T \) is the transposed field operator, and \( O = \mathcal{O}_{\text{Dirac}} \otimes \mathcal{O}_{\text{flavor}} \otimes \mathcal{O}_{\text{color}} \) is an operator acting through a direct product of the Dirac, flavor and color spaces. Fermi statistics constraints \( \mathcal{O} \) to be totally antisymmetric, i.e. \( \mathcal{O}^T = -\mathcal{O} \).
We are interested in a spin-zero condensate and will restrict our attention to the spin-zero channel that is antisymmetric in Dirac indices. Hence, the color-flavor structure of the gap should be symmetric under simultaneous exchange of both color and flavor indices. The Dirac structure is then given as $C_{\gamma_5}$, where $C = i \gamma_2 \gamma_0$ is the matrix of charge conjugation.

The one-gluon exchange interactions are attractive in the channels that are antisymmetric under the exchange of color. This fact, combined with the constraint imposed by the Fermi statistics implies that the dominant contribution to the gap comes from terms which are antisymmetric under the exchange of color indices and antisymmetric under the exchange of flavor indices. This is the color-antitriplet, flavor-antitriplet condensate. It can be shown that the antisymmetric gap can be specified in general by only three gap parameters. Color symmetric channels are repulsive, hence subdominant, but once the color-antitriplet, flavor-antitriplet condensate is generated, a color sextet, flavor sextet condensate is also induced due to mixing \cite{10}. Following a derivation similar to the one carried out in Ref. \cite{14}, one can show that the symmetric condensate can be specified in general by only six gap parameters.

Therefore, the condensate structure can be written in general as

$$\Delta = \Delta_{S_1}S_1 + \Delta_{S_2}S_2 + \Delta_{S_3}S_3 + \Delta_{S_4}S_4 + \Delta_{S_5}S_5 + \Delta_{S_6}S_6 + \Delta_{A_1}A_1 + \Delta_{A_2}A_2 + \Delta_{A_3}A_3 ,$$

where the six symmetric and three antisymmetric independent color-flavor structures are given by

$$S_1 = 2\delta^{a_1}\delta^{b_1}\delta^{i_1}\delta^{j_1}, \quad S_2 = 2\delta^{a_2}\delta^{b_2}\delta^{i_2}\delta^{j_2}, \quad S_3 = 2\delta^{a_3}\delta^{b_3}\delta^{i_3}\delta^{j_3}, \quad S_4 = (\delta^{a_1}\delta^{b_1} + \delta^{a_2}\delta^{b_1})\delta^{i_1}\delta^{j_2} + \delta^{i_2}\delta^{j_1})$$

$$S_5 = (\delta^{a_1}\delta^{b_1} + \delta^{a_2}\delta^{b_1})\delta^{i_1}\delta^{i_3}\delta^{j_3}, \quad S_6 = (\delta^{a_2}\delta^{b_2} + \delta^{a_3}\delta^{b_2})\delta^{i_2}\delta^{j_3} + \delta^{i_3}\delta^{j_2}$$

$$A_1 = \varepsilon_{a_1b_1}\varepsilon_{i_1j_1}, \quad A_2 = \varepsilon_{a_2b_2}\varepsilon_{i_2j_2}, \quad A_3 = \varepsilon_{a_3b_3}\varepsilon_{i_3j_3}$$

where $a, b$ denote color indices and $i, j$ denote flavor indices. In the case of the NJL theory, since the quark interaction is modelled by a point-like four-fermion term, the nine coefficients $\Delta$ in Eq. (14) are independent of the momentum.

As mentioned above, the energetically favored condensate should be the one retaining the highest degree of symmetry possible. In the absence of a magnetic field, that corresponds to the so-called color-flavor locked group $SU(3)_C \times SU(3)_L \times SU(3)_R \times U(1)_B \times U(1)_{e.m}$. Therefore, the original $SU(3)_C \times SU(3)_L \times SU(3)_R \times U(1)_B \times U(1)_{e.m}$ symmetry is broken by the color condensate to the diagonal subgroup $SU(3)_C \times SU(3)_L \times SU(3)_R \times U(1)_{e.m}$. This is the well-known CFL phase, which results from Eq. (14) if one demands that $\Delta_{S_1} = ... = \Delta_{S_6} = \Delta_{S}$ and $\Delta_{A_1} = ... = \Delta_{A_3} = \Delta_{A}$. The CFL structure parameter can be expressed as $\Delta_{CFL} = \Delta_{S}(U + U_0) + \Delta_{A}(U - U_0)$ with the color-flavor matrices $U_0$ and $U$ defined by

$$U_0 = \frac{1}{2} \sum_{i=1}^{6} S_i + \frac{1}{2} \sum_{i=1}^{3} A_i ,$$

$$U = \frac{1}{2} \sum_{i=1}^{6} S_i - \frac{1}{2} \sum_{i=1}^{3} A_i .$$

Let us consider now the situation with magnetic field. A background magnetic field explicitly breaks the flavor symmetry of the original theory to $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)_{e.m}$, since $u$ quarks have different electric charge than $d$ and $s$ quarks. $U(1)^{(-)}_A$ is related to the existence of an anomaly-free current given by a linear combination of the $s$, $d$, and $u$ axial currents \cite{17}. The subgroup that ensures preserving the largest degree of symmetry in the CS gap is $SU(2)_{C+L+R}$ in this case. Hence, the symmetry breaking induced by the color condensate in the presence of a magnetic field is: $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_B \times U(1)^{(-)}_A \times U(1)_{e.m} \rightarrow SU(2)_{C+L+R} \times \tilde{U}(1)_{e.m}$.

The $SU(2)_{C+L+R}$ symmetry requires the invariance of the gap under simultaneous flavor ($1 \leftrightarrow 2$) and color ($1 \leftrightarrow 2$) exchanges. This implies the following equalities among the gap parameters in Eq. (14):

$$\Delta^{S'} = \Delta_{S_1} = \Delta_{S_2} , \quad \Delta^{B} = \Delta_{S_5} = \Delta_{S_6} , \quad \Delta^{A} = \Delta_{A_1} = \Delta_{A_2} .$$
Based on the above considerations, the gap structure in the presence of a magnetic $\tilde{B}$ field takes the form

$$
\Delta = \begin{pmatrix}
2\Delta_{S'} & 0 & 0 & 0 & \Delta_A + \Delta_S & 0 & 0 & 0 & \Delta_B^A + \Delta_B^S \\
0 & 0 & 0 & \Delta_S - \Delta_A & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta_B^S - \Delta_B^A & 0 & 0 \\
\Delta_A + \Delta_S & 0 & 0 & 0 & 0 & 2\Delta_{S'} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_B^S - \Delta_B^A & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \Delta_B^S - \Delta_B^A & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \Delta_B^S - \Delta_B^A & 0 & 0 & 0 \\
\Delta_B^A + \Delta_B^S & 0 & 0 & 0 & 0 & 0 & 0 & 2\Delta_{S'} & 0 \\
\end{pmatrix}
$$

(19)

This is the so-called MCFL ansatz. In going from (11) to (19) we introduced the more convenient notation $\Delta_{S''} \equiv \Delta_{S_3}$, $\Delta_S \equiv \Delta_{S_1}$, $\Delta_A \equiv \Delta_{A_1}$. At weak magnetic field, $\tilde{B}$, the CFL and MCFL phases should be continuously connected. In the limit of zero field the gap parameters ought to satisfy $\Delta_B^A = \Delta_A$ and $\Delta_B^S = \Delta_S = \Delta_{S''} = \Delta_{S'}$.

To trace back the physical origin of the different gap parameters appearing in the MCFL structure (19) we can see that despite the overall $Q$-neutrality of all the pairs, they can be formed either by a pair of neutral or by a pair of rotated charged quarks. Any pair formed by $\bar{Q}$-charged quarks feel the background field directly through the minimal coupling of the quarks in the pair with $\tilde{B}$. The gap parameters that get contributions from pairs formed by $\bar{Q}$-charged quarks must be different from the gap parameters getting contributions only from pairs of neutral quarks. The gaps $\Delta_{A/S}$ are antisymmetric/symmetric combinations of condensates that get contributions from both charged and neutral pairs. On the other hand, the gaps $\Delta_{A/S',S''}$ are antisymmetric/symmetric combinations of condensates that only get contributions from pairs of neutral quarks. The only way the field can affect $\Delta_{A/S',S''}$ is indirectly, through the system of highly non-linear coupled gap equations.

The MCFL order parameter (19) can be rewritten in a compact way as

$$
\Delta = \Delta_{S'}(S_1 + S_2) + \Delta_{S''}S_3 + \frac{1}{2}(\Delta_A + \Delta_S)(S_4 + A_1) + \frac{1}{2}(\Delta_S - \Delta_A)(S_4 - A_1) + \\
+ \frac{1}{2}(\Delta_B^A + \Delta_B^S)(S_5 + S_6 + A_1 + A_2) + \frac{1}{2}(\Delta_B^S - \Delta_B^A)(S_5 + S_6 - A_1 - A_2).
$$

(20)

Out of the four independent, symmetric gaps, all of which are due to subleading interactions, one can single out the set of three gaps $\Delta_{S_1}, \Delta_{S_2}$ and $\Delta_S$ that get contributions only from pairs of neutral quarks. There is no reason to expect that these three symmetric gaps will differ much in magnitude. Hence, in a first approach to the problem, we will solve the gap equations assuming $\Delta_S \simeq \Delta_{S_1} \simeq \Delta_{S_2}$.

### III. EFFECTIVE ACTION IN THE PRESENCE OF A ROTATED MAGNETIC FIELD

In this Section we introduce the mean-field action in the presence of a constant rotated magnetic field in both coordinate and momentum spaces. In our derivations we will follow a parallel approach to the one developed for the case of zero field in Refs. \[12\] \[13\].

As discussed in the previous section, the rotated magnetic field naturally separates the quark fields according to their $\bar{Q}$ charge. The $B$-dependent mean-field effective action is

$$
I_B(\bar{\psi}, \psi) = \int d^4x d^4y \left\{ \frac{1}{2} \sum_{Q=0, \pm} \left[ \bar{\psi}(\bar{Q})_C(x) [G_{(0)}^{+}(\bar{Q})]^{-1}(x, y) \psi(\bar{Q})_C(y) + \bar{\psi}(\bar{Q})_C(x) [G_{(-Q)}^{-}(\bar{Q})]^{-1}(x, y) \psi(\bar{Q})_C(y) \right] \\
+ \frac{1}{2} \psi_0(x) C \Delta^{+}_0 \psi_0(y) + \bar{\psi}^{+}_0(x) C \Delta^{-}_0 \psi_0(y) + \bar{\psi}^{-}_0(x) C \Delta^{+}_0 \psi_0(y) + h.c. \right\},
$$

(21)

where $\psi_C(x) = C \bar{\psi}^{T}(x)$ is the charge-conjugate spinor of $\psi(x)$.

In (21) symbols in parentheses indicate the correspondence to neutral (0), positively (+) or negatively (−) $\bar{Q}$-charged fermions. Propagators for fields and conjugated fields will be denoted with the customary ± supr-indexes. Explicit expressions of the inverse propagators are

$$
[G_{(0)}^{\pm}(x, y) = [i\gamma^\mu \partial_\mu \pm \mu \gamma^0] \delta^4(x - y),
$$

(22)
\[ G_{(+)0}^{\pm -1}(x,y) = [i\gamma^\mu \Pi_\mu^{(\pm)} \pm \mu \gamma^0] \delta^4(x-y), \]  
(23)

\[ G_{(-)0}^{\pm -1}(x,y) = [i\gamma^\mu \Pi_\mu^{(-)} \pm \mu \gamma^0] \delta^4(x-y), \]  
(24)

with

\[ \Pi_\mu^{(\pm)} = i \partial_\mu \pm \tilde{e} \tilde{A}_\mu, \]  
(25)

and the gap matrices are given by

\[ \Delta_0^{\pm} = [\Delta_S + \Delta_A]^{(\pm)}(U_0 - N)\Omega_0 + [\Delta_S - \Delta_A]^{(\pm)}U\Omega_0 + [\Delta_S^B + \Delta_A^B]^{(\pm)}N\Omega_0, \]  
(26)

\[ \Delta_{(\pm)}^+ = [\Delta_S^B - \Delta_A^B]^{(\pm)}U\Omega_+, \]  
(27)

\[ \Delta_{(\pm)}^- = [\Delta_S^B - \Delta_A^B]^{(\pm)}U\Omega_-, \]  
(28)

with the color-flavor matrix \( N \) defined as: \( N = \frac{1}{2}[S_S + S_6 + A_1 + A_2] \).

**A. Effective action at \( \vec{B} \neq 0 \) in momentum space**

The computation of the field-dependent quark propagators in momentum space can be managed with the use of a method, originally developed for charged fermions by Ritus [12] and later on extended to charged vector fields by Elizalde, Ferrer and Incera [19]. In Ritus’ approach the diagonalization in momentum space of the Green’s functions of charged fermions in the presence of a background magnetic field is carried out with the help of the eigenfunction matrices \( E_p(x) \). They are the wave functions of the asymptotic states of charged fermions in a uniform magnetic field and play the role in the magnetized medium of the usual plane-wave (Fourier) functions \( e^{ipx} \) at zero field. This method has also been used in the context of chiral mass generation in a magnetic field [11, 20]. Using the \( E_p(x) \) functions, we transform the propagators (23)-(24) to momentum space adequately. Below we describe the basic properties of the transformation.

The transformation functions \( E_p^{(\pm)}(x) \) for positively \((+)\), and negatively \((-)\) charged fermion fields are obtained as the solutions of the field-dependent eigenvalue equation

\[ (\Pi^{(\pm)} \cdot \gamma)E_p^{(\pm)}(x) = E_p^{(\pm)}(x)(\gamma \cdot \vec{p}^{(\pm)}), \]  
(29)

with \( \vec{p}^{(\pm)} \) given by

\[ \vec{p}^{(\pm)} = (p_0, 0, \pm \sqrt{2|e\vec{B}|}k, p_1), \]  
(30)

and

\[ E_p^{(\pm)}(x) = \sum_\sigma E_{p\sigma}^{(\pm)}(x)\Delta(\sigma), \]  
(31)

with eigenfunctions

\[ E_{p\sigma}^{(\pm)}(x) = N_{n(\pm)}e^{-i(p_0x^0 + px^x + px^3)}D_{n(\pm)}(\theta(\pm)), \]  
(32)
where \( D_{n(\pm)}(\varphi_{n(\pm)}) \) are the parabolic cylinder functions with argument \( \varphi_{n(\pm)} \) defined by

\[
\varphi_{(\pm)} = \sqrt{2|eB|}(x_1 \pm p_2/eB),
\]

and index \( n_{(\pm)} \) given by

\[
n_{(\pm)} = n_{(\pm)}(k, \sigma) = k \pm \frac{\mp B}{2|eB|} - \frac{1}{2}, \quad n_{(\pm)} = 0, 1, 2, \ldots
\]

\( k = 0, 1, 2, 3, \ldots \) is the Landau level, and \( \sigma \) is the spin projection that can take values \( \pm 1 \). The normalization constant \( N_{n_{(\pm)}} \) is

\[
N_{n_{(\pm)}} = (4\pi|eB|)^{\frac{3}{2}} / \sqrt{n_{(\pm)}!}.
\]

The spin matrices \( \Delta(\sigma) \) in (31), not to be confused with the gap coefficients, are spin projectors. They are defined as

\[
\Delta(\sigma) = \text{diag}(\delta_{\sigma 1}, \delta_{\sigma -1}, \delta_{\sigma 1}, \delta_{\sigma -1}), \quad \sigma = \pm 1,
\]

and satisfy the following relations

\[
\Delta((\pm)\dagger) = \Delta((\pm)), \quad \Delta((+) + \Delta((-)) = 1, \quad \Delta((\pm)\Delta((\pm)) = \Delta((\pm)), \quad \Delta((\pm)\Delta((\mp)) = 0,
\]

\[
\gamma'^\dagger \Delta((\pm)) = \Delta((\pm)) \gamma', \quad \gamma^- \Delta((\pm)) = \Delta((\mp)) \gamma^+.
\]

In Eq. (38), the notation \( \gamma'^\dagger = (\gamma^0, \gamma^3) \) and \( \gamma^- = (\gamma^1, \gamma^2) \) was used.

Under the \( E_p(x) \) functions, positively \((\psi_{(+)}(x))\) and negatively \((\psi_{(-)}(x))\) charged fields transform according to

\[
\psi_{(+)}(x) = \sum_{\pm} \frac{d^4p}{(2\pi)^4} E_p^{(\pm)}(x) \psi_{(\pm)}(p),
\]

\[
\overline{\psi}_{(\pm)}(x) = \sum_{\pm} \frac{d^4p}{(2\pi)^4} \overline{E_p^{(\pm)}}(x) \overline{\psi}_{(\pm)}(p),
\]

where \( E_p^{(\pm)}(x) = \gamma_0(E_p^{(\pm)}(x))^\dagger \gamma_0 \) and \( \sum_{\pm} \frac{d^4p}{(2\pi)^4} \equiv \sum_{k=0}^\infty \int \frac{dp_0 dp_1 dp_2 dp_3}{(2\pi)^4} \).

One can show that

\[
[\gamma_\mu(\Pi_{(+)}(\mu) \pm \mu_\delta_{\mu 0})] E_p^{(+)}(x) = E_p^{(+)}(x)[\gamma_\mu(\Pi_{(+)}^{(\mu)} \pm \mu_\delta_{\mu 0})],
\]

and

\[
[\gamma_\mu(\Pi_{(-)}(\mu) \pm \mu_\delta_{\mu 0})] E_p^{(-)}(x) = E_p^{(-)}(x)[\gamma_\mu(\Pi_{(-)}^{(\mu)} \pm \mu_\delta_{\mu 0})].
\]

Since the charge conjugate of a positively (negatively) charged field is a negatively (positively) charged field, the charge conjugate fields transform as

\[
\psi_{(+)}(x) = \sum_{\pm} \frac{d^4p}{(2\pi)^4} E_p^{(-)}(x) \psi_{(+)}(p),
\]

\[
\psi_{(-)}(x) = \sum_{\pm} \frac{d^4p}{(2\pi)^4} E_p^{(+)}(x) \psi_{(-)}(p).
\]
This transformation dictates what fields should form the components of positively and negatively charged Nambu-Gorkov fields.

In terms of Nambu-Gorkov fields $\Psi$, the effective action in the presence of magnetic field $\tilde{B}$ takes the form

$$I^B(\bar{\psi}, \psi) = \int d^4x \, d^4y \, \bar{\Psi}(x) S^{-1}(x, y) \Psi(y),$$

where

$$S^{-1}_{(0)}(p) = \begin{pmatrix} [G^+_{(0)0}]^{-1}(p) & \Delta^-_{(0)} \\ \Delta^+_{(0)} & [G^-_{(0)0}]^{-1}(p) \end{pmatrix},$$

$$S^{-1}_{(+)}(p) = \begin{pmatrix} [G^+_{(+0)0}]^{-1}(p) & \Delta^-_{(+)} \\ \Delta^+_{(+)} & [G^-_{(+0)0}]^{-1}(p) \end{pmatrix},$$

$$S^{-1}_{(-)}(p) = \begin{pmatrix} [G^+_{(-0)0}]^{-1}(p) & \Delta^-_{(-)} \\ \Delta^+_{(-)} & [G^-_{(-0)0}]^{-1}(p) \end{pmatrix},$$

and

$$\Delta^-_{(0)} = \gamma_0 [\Delta^+_{(0)}] \gamma_0, \quad \Delta^-_{(+)} = \gamma_0 [\Delta^+_{(+)}] \gamma_0, \quad \Delta^-_{(-)} = \gamma_0 [\Delta^+_{(-)}] \gamma_0.$$

The Nambu-Gorkov fermion fields corresponding to the different $\tilde{Q}$ charges are defined by

$$\Psi_0 = \begin{pmatrix} \psi_{(0)} \\ \psi_{(0)C} \end{pmatrix},$$

for the neutral field,

$$\Psi_+ = \begin{pmatrix} \psi_{(+)} \\ \psi_{(-)C} \end{pmatrix},$$

for the positive field, and

$$\Psi_- = \begin{pmatrix} \psi_{(-)} \\ \psi_{(+)} \end{pmatrix},$$

for the negative field.

As noticed before, the positive (negative) Nambu-Gorkov field is formed by the positive (negative) fermion field and the charge conjugate of the negative (positive) field. This means that, as it should be, the rotated charge of the up and down components of a given Nambu-Gorkov field are the same. This in turn determines the nature of the $\tilde{Q}$-charges of the fields entering in a given pair, which we know should have zero overall $\tilde{Q}$ charge.

The bare inverse propagator of the neutral field is

$$[G^\pm_{(0)0}]^{-1}(p) = \gamma_\mu (p_\mu \pm \mu \delta_\mu 0) ,$$

Here the momentum is the usual $p = (p_0, p_1, p_2, p_3)$ of the case with no background field.
For the positively and negatively charged fields the bare inverse propagators are

\[
[G_{(+)}^{±}]^{-1}(p) = \gamma_\mu (p_\mu^+ \pm i\delta_{\mu0}) ,
\]

\[
[G_{(-)}^{±}]^{-1}(p) = \gamma_\mu (p_\mu^- \pm i\delta_{\mu0}) .
\]

respectively.

**B. Nambu-Gorkov propagators in a color-flavor rotated basis**

In order to find the propagators in Nambu-Gorkov space one has to invert the matrices (46)-(48), whose color-flavor structure is rather complicated. At this point we find convenient to perform a rotation that can partially diagonalize the components of the Nambu-Gorkov propagators in color-flavor space.

Let us define a new basis for the fermion fields

\[
\psi_{ai} = \frac{1}{\sqrt{2}} \sum_{A=1}^{9} \lambda^A_i \psi^A , \quad \psi_{C,ai} = \frac{1}{\sqrt{2}} \sum_{A=1}^{9} (\lambda^A_i)^T \psi^A ,
\]

where the indices \(a\) and \(i\) refer to color and flavor respectively, and \(\lambda^A\), for \(A = 1, \ldots, 8\) are the Gell-Mann matrices, while \(\lambda^9 = \sqrt{\frac{2}{3}} \). Given that in the CFL case the Nambu-Gorkov quark propagators become color-flavor diagonal in the basis (56), we call (56) the CFL basis.

In the CFL basis the quark kinetic term (11) becomes

\[
L_{em}^{quarks} = i \bar{\psi}^A \gamma^\mu (i\partial_\mu - \tilde{Q}^{AB} \delta_\mu^C) \psi^B ,
\]

where the charge matrix is

\[
\tilde{Q}^{AB} = -\frac{1}{2} \text{Tr} (\lambda^A \lambda^B Q) .
\]

In our conventions, \(Q = -\frac{\lambda^8}{\sqrt{3}}\), and thus \(\tilde{Q}^{AB} = \frac{2i}{\sqrt{3}} f^{AB8}\) is expressed in terms of the antisymmetric structure constants of SU(3). In the basis (56) one can as well find the charge-projector operators

\[
\Omega^{AB}_0 = \text{diag}(1, 1, 1, 0, 0, 0, 0, 1, 1) ,
\]

and

\[
\Omega^{AB}_\pm = \frac{1}{2} \left( [\delta^{AB} \delta^{B4} + \delta^{AB} \delta^{B5} + \delta^{AB} \delta^{B6} + \delta^{AB} \delta^{B7}] \pm i(\delta^{AB} \delta^{B5} + \delta^{AB} \delta^{B4} - \delta^{AB} \delta^{B6} - \delta^{AB} \delta^{B7}) \right) ,
\]

which obey the conditions (6)-(8). Furthermore, one has \((\Omega_\pm) = \Omega_\mp\), so that \(\Omega_\pm \psi_C = (\psi_\mp)_C\).

From Eq. (56) one can deduce the transformation law of the gap matrix, which now reads

\[
\Delta^{AB} = \frac{1}{2} (\lambda^A)^T_{ai} \Delta^{ij} \lambda^B_{bj} .
\]

After taking into account the approximation \(\Delta_S \approx \Delta_{S'} \approx \Delta_{S''}\) in the gap matrix (19) one obtains the gap parameters in the new basis to be

\[
\Delta^{11} = \Delta^{22} = \Delta^{33} = \Delta_S - \Delta_A , \quad (62a)
\]

\[
\Delta^{44} = \Delta^{55} = \Delta^{66} = \Delta^{77} = \Delta_S - \Delta_B , \quad (62b)
\]

\[
\Delta^{88} = \frac{1}{3} (\Delta_A - 4 \Delta_A^B + 7 \Delta_S - 4 \Delta_B^S) , \quad (62c)
\]

\[
\Delta^{89} = \Delta^{98} = \frac{\sqrt{2}}{3} (\Delta_A - \Delta_A^B + \Delta_S - \Delta_B^S) , \quad (62d)
\]

\[
\Delta^{99} = \frac{2}{3} (\Delta_A + 2 \Delta_A^B + 4 \Delta_S + 2 \Delta_B^S) , \quad (62e)
\]
while all the remaining elements of $\Delta^{AB}$ are zero.

In the CFL basis the Nambu-Gorkov effective action \( I^{(\bar{\psi}, \psi)} \) can be written as

$$
I^{(\bar{\psi}, \psi)} = \int \frac{d^4 p}{(2\pi)^4} \bar{\psi}^0(p) [S^{-1}_{00}(p)]^{AB} \psi^B(p) + \sum_{\pm} \int \frac{d^4 p}{(2\pi)^4} \left[ \bar{\psi}^+(p) [S^{-1}_{(+)}(p)]^{AB} \psi^B(p) + \bar{\psi}^-(p) [S^{-1}_{(-)}(p)]^{AB} \psi^B(p) \right],
$$

(63)

As follows we find convenient to express all propagators in terms of energy projectors. For completion, in Appendix A we present the details on how they are generalized for charged particles in the presence of an external magnetic field.

For the charged fields the Nambu-Gorkov matrix reads

$$
S^{AB}_{(\pm)}(p) = \delta^{AB} \left( S_{(\pm)11}(\bar{p}^{(\pm)}) \ S_{(\pm)12}(\bar{p}^{(\pm)}) \right),
$$

(64)

where the indexes \( A, B \) take values 4, 5, 6, 7 only and the subindices 11, 12, 21, 22 specify the entry of the Nambu-Gorkov matrix. Notice that \( S^{AB}_{(\pm)}(p) \) depends on the Minkowski momenta \( \bar{p}^{(\pm)} = (p_0, \bar{p}^{(\pm)}) \) defined in Eq. \( \text{Eq. (30)} \). To avoid confusion, we will always denote with caligraphic letters the Nambu-Gorkov matrixes, and with capital letters the elements of the matrixes.

For massless quarks, the elements of the Nambu-Gorkov matrix for the positive charged quarks are

$$
S_{(+11/22)}^{(+)} = \frac{\hat{\Lambda}^+_{(+)}(\bar{p}) \gamma^0 (p_0 \mp \mu \mp |\bar{p}^{(+)}|)}{p^2_0 - (\mu - |\bar{p}^{(+)}|)^2 - (\Delta^B_A - \Delta^B_S)^2} + \frac{\hat{\Lambda}^+_{(+)}(\bar{p}) \gamma^0 (p_0 \mp \mu \mp |\bar{p}^{(+)}|)}{p^2_0 - (\mu + |\bar{p}^{(+)}|)^2 - (\Delta^B_A - \Delta^B_S)^2},
$$

(65a)

$$
S_{(+21)}^{(+)} = \gamma_5 \left\{ \frac{\hat{\Lambda}^-_{(+)}(\bar{p}) (\Delta^B_S - \Delta^B_A)}{p^2_0 - (\mu - |\bar{p}^{(+)}|)^2 - (\Delta^B_A - \Delta^B_S)^2} + \frac{\hat{\Lambda}^-_{(+)}(\bar{p}) (\Delta^B_S - \Delta^B_A)^*}{p^2_0 - (\mu + |\bar{p}^{(+)}|)^2 - (\Delta^B_A - \Delta^B_S)^2} \right\},
$$

(65b)

$$
S_{(+12)}^{(+)} = -\gamma_5 \left\{ \frac{\hat{\Lambda}^+_{(+)}(\bar{p}) (\Delta^B_S - \Delta^B_A)^*}{p^2_0 - (\mu - |\bar{p}^{(+)}|)^2 - (\Delta^B_A - \Delta^B_S)^2} + \frac{\hat{\Lambda}^-_{(+)}(\bar{p}) (\Delta^B_S - \Delta^B_A)}{p^2_0 - (\mu + |\bar{p}^{(+)}|)^2 - (\Delta^B_A - \Delta^B_S)^2} \right\},
$$

(65c)

Following the definition of the momenta \( \bar{p}^{(+)} \), we have \( |\bar{p}^{(+)}| = \sqrt{2|\tilde{e}B| k + p^2_0} \). We use the symbol \( \hat{\Lambda}^{\pm}_{(+)}(\bar{p}) \) to denote positive/negative energy projectors of positively charged quasiquarks in the presence of the external field, as defined in Appendix A.

The propagator for the negatively charged fields keeps the same structure, with the only change \( \bar{p}^{(+)} \rightarrow \bar{p}^{(-)} \). Its off-diagonal term satisfies the equation,

$$
S_{(-21)}^{(-)}(x, x) = S_{(+21)}^{(-)}(x, x),
$$

(66)

since

$$
S_{(\pm)21}(x, x) = \sum_k \int \frac{d^4 p}{(2\pi)^4} E^+_p(x) \gamma_5 \left\{ \frac{\hat{\Lambda}^-_{(\pm)}(\bar{p}) (\Delta^B_S - \Delta^B_A)}{p^2_0 - (\mu - |\bar{p}^{(\pm)}|)^2 - (\Delta^B_A - \Delta^B_S)^2} + \frac{\hat{\Lambda}^+_{(\pm)}(\bar{p}) (\Delta^B_S - \Delta^B_A)^*}{p^2_0 - (\mu + |\bar{p}^{(\pm)}|)^2 - (\Delta^B_A - \Delta^B_S)^2} \right\} E^-_p(x),
$$

(67)

so after integrating in \( p_2 \) and using the relation

$$
\int_{-\infty}^{\infty} dp_2 D_{n}(\rho_{(\pm)}) D_{n'}(\rho_{(\pm)}) = n!\sqrt{2\pi} \frac{\tilde{e}B}{\sqrt{2\tilde{e}B}} \delta_{nn'},
$$

(68)

we obtain

$$
S_{(\pm)21}(x, x) = \frac{\tilde{e}B}{2} \sum_{k=0}^{\infty} \int \frac{d p_0 d p_3}{(2\pi)^3} (\Delta^B_S - \Delta^B_A) \gamma_5 \left\{ \frac{1}{p^2_0 - (\mu - |\bar{p}|)^2 - (\Delta^B_A - \Delta^B_S)^2} + \frac{1}{p^2_0 - (\mu + |\bar{p}|)^2 - (\Delta^B_A - \Delta^B_S)^2} \right\},
$$

(69)
where \( |\vec{p}| \) stands for the common modulus of the three-momentum of positively and negatively charged quasiparticles, since \( |\vec{p}| = |\vec{p}^+| = |\vec{p}^-| \). From now on we will use the short notation \( |\vec{p}| \) for the modulus the three momentum of charged quasiparticles.

The propagator for the neutral fields in the CFL basis with indexes \( A, B = 1, 2, \ldots, 9 \) is given by

\[
S^{AB}_{(0)}(p) = \begin{pmatrix}
S^{11}_{(0)}(p) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & S^{22}_{(0)}(p) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & S^{33}_{(0)}(p) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

(70)

where each component is a \( 2 \times 2 \) Nambu-Gorkov matrix

\[
S^{AB}_{(0)}(p) = \begin{pmatrix}
S^{AB}_{(0)11}(p) & S^{AB}_{(0)12}(p) \\
S^{AB}_{(0)21}(p) & S^{AB}_{(0)22}(p)
\end{pmatrix}.
\]

(71)

One finds that \( S^{11}_{(0)} = S^{22}_{(0)} = S^{33}_{(0)} \), with Nambu-Gorkov components given by

\[
S^{11}_{(0)11/22}(p) = \frac{\Lambda^+(p)\gamma^0(p_0 + \mu \pm |\vec{p}|)}{p_0^2 - (\epsilon_p^0)^2} + \frac{\Lambda^+(p)\gamma^0(p_0 + \mu \mp |\vec{p}|)}{p_0^2 - (\epsilon_p^0)^2},
\]

(72a)

\[
S^{11}_{(0)12}(p) = \gamma_5 \left\{ \frac{\Lambda^-(p)(\Delta_S - \Delta_A)}{p_0^2 - (\epsilon_p^0)^2} + \frac{\Lambda^+(p)(\Delta_S - \Delta_A)}{p_0^2 - (\epsilon_p^0)^2} \right\},
\]

(72b)

\[
S^{11}_{(0)12}(p) = \gamma_5 \left\{ \frac{\Lambda^+(p)(\Delta_S - \Delta_A)^*}{p_0^2 - (\epsilon_p^0)^2} + \frac{-\Lambda^-(p)(\Delta_S - \Delta_A)^*}{p_0^2 - (\epsilon_p^0)^2} \right\},
\]

(72c)

with

\[
\epsilon_p^0 = \sqrt{(\mu - |\vec{p}|)^2 + (\Delta_A - \Delta_S)^2}, \quad \epsilon_p^0 = \sqrt{(\mu + |\vec{p}|)^2 + (\Delta_A - \Delta_S)^2}.
\]

(73)

The remaining block matrix for the CFL coefficients \( A = 8, 9 \) in \( \text{(70)} \)

\[
\begin{pmatrix}
S^{88}_{(0)} & S^{89}_{(0)} \\
S^{98}_{(0)} & S^{99}_{(0)}
\end{pmatrix}
\]

(74)

has matrix elements which are \( 2 \times 2 \) matrices in the Nambu-Gorkov space. Given that the symmetric gap \( \Delta_S \) comes from subdominant channels, we can assume, as in the CFL case, that \( \Delta_S \ll \Delta_A \). Using this approximation the Nambu-Gorkov components of each of the matrix element of \( \text{(71)} \) can be expressed as

\[
S^{AB}_{(0)11/22} = \frac{\Lambda^+(p)\gamma^0(p_0 + \mu \pm |\vec{p}|) A_{AB}}{(p_0^2 - (\epsilon_p^0)^2)(p_0^2 - (\epsilon_p^0)^2)} + \frac{\Lambda^-(p)\gamma^0(p_0 + \mu \mp |\vec{p}|) A_{AB}}{(p_0^2 - (\epsilon_p^0)^2)(p_0^2 - (\epsilon_p^0)^2)},
\]

(75a)

\[
S^{AB}_{(0)12} = \gamma_5 \left\{ \frac{\Lambda^-(p)B^{AB}}{(p_0^2 - (\epsilon_p^0)^2)(p_0^2 - (\epsilon_p^0)^2)} + \frac{\Lambda^+(p)B^{AB}}{(p_0^2 - (\epsilon_p^0)^2)(p_0^2 - (\epsilon_p^0)^2)} \right\},
\]

(75b)

\[
S^{AB}_{(0)12} = \gamma_5 \left\{ \frac{\Lambda^+(p)(B^{AB})^*}{(p_0^2 - (\epsilon_p^0)^2)(p_0^2 - (\epsilon_p^0)^2)} + \frac{-\Lambda^-(p)(B^{AB})^*}{(p_0^2 - (\epsilon_p^0)^2)(p_0^2 - (\epsilon_p^0)^2)} \right\},
\]

(75c)

where

\[
\epsilon_p^{a/b} = \sqrt{(\mu - |\vec{p}|)^2 + (\Delta_{a/b})^2}, \quad \epsilon_p^{a/b} = \sqrt{(\mu + |\vec{p}|)^2 + (\Delta_{a/b})^2}.
\]

(76)
with
\[ \Delta^2_{3/4} = \frac{1}{2} \left( 4\Delta^2_N + \Delta^2_A \pm \Delta_A \sqrt{\Delta^2_A + 8\Delta^2_N} \right) . \] (77)
and we have defined for convenience
\[ \Delta_N = \Delta^B_A + \Delta^B_S . \] (78)

The CFL submatrices \( A \) and \( B \) read
\[ A = \left( \begin{array}{cc}
\frac{r^2}{3} (\Delta_A - \Delta_N)^2 - \frac{4}{3} (\Delta_A + 2\Delta_N)^2 & \frac{\sqrt{2}}{3} \Delta_A (\Delta_A - \Delta_N) \\
\frac{\sqrt{2}}{3} \Delta_A (\Delta_A - \Delta_N) & \frac{r^2}{3} (\Delta_A - \Delta_N)^2 - \frac{4}{3} (\Delta_A + 2\Delta_N)^2
\end{array} \right) , \] (79)
\[ B = \left( \begin{array}{cc}
\frac{r^2}{3} (\Delta_A - 4\Delta_N) + \frac{4}{3} \Delta^2_N (\Delta_A + 2\Delta_N) & \frac{\sqrt{2}}{3} (\Delta_A - \Delta_N) (r^2 - 2\Delta^2_N) \\
\frac{\sqrt{2}}{3} (\Delta_A - \Delta_N) (r^2 - 2\Delta^2_N) & 2\frac{r^2}{3} (\Delta_A + 2\Delta_N) + \frac{2\Delta^2_N}{3} (\Delta_A - 4\Delta_N)
\end{array} \right) , \] (80)
where we used the following shorthand \( r^2 \equiv p^2_0 - (|p| - \mu)^2 \). The matrices \( \tilde{A} \) and \( \tilde{B} \) are the corresponding antigap quantities. They are obtained simply by replacing \( r^2 \to \tilde{r}^2 \equiv p^2_0 - (|p| + \mu)^2 \).

IV. GAP EQUATIONS AND SOLUTIONS IN THE PRESENCE OF A ROTATED MAGNETIC FIELD

The evaluation of the different fermionic gaps can be done by writing the Schwinger-Dyson equations. In coordinate space the QCD gap equation reads
\[ \Delta^+ (x, y) = i \frac{g^2}{4} \lambda^T A \gamma^\mu S_{21} (x, y) \gamma^\nu \lambda_B D^{AB}_{\mu\nu} (x, y) , \] (81)
where, for simplicity, we have omitted explicit color and flavor indices in the gap and fermion propagator. Here \( D^{AB}_{\mu\nu} \) is the gluon propagator.

We leave for a future project the study of Eq. (81) in the high density/weak coupling limit of QCD in the presence of a magnetic field. It should be noted, though, that gluons can also feel the presence of a rotated magnetic field because some of the gluons carry \( \tilde{Q} \)-charges, and thus they minimally couple to the external \( \tilde{B} \)-field. Debye and Meissner masses, as well as Landau damping, should then be affected by \( \tilde{B} \). Hence, before studying the QCD gap equation (81) in the presence of a magnetic field, it is necessary to study the behavior of the gluon propagator for different energy scales, so as to properly assess its effect in the gap equation.

In the NJL model abstracted from one-gluon exchange that we are using in this paper the gap equation can be obtained from Eq. (81) simply by substituting the gluon propagator by
\[ D^{AB}_{\mu\nu} (x, y) = \frac{1}{\Lambda^2} g_{\mu\nu} \delta^{AB} \delta^{(4)} (x - y) . \] (82)
and using the quark propagator in the presence of the rotated magnetic field found in Section III.

The NJL model is characterized by a coupling constant \( g \) and an ultraviolet cutoff \( \Lambda \). The ultraviolet cutoff should be much larger than any of the typical energy scales of the system, such as the chemical potential \( \mu \) and the magnetic energies \( \sqrt{\epsilon B} \). In other studies of color superconductivity within the NJL model the values of \( g \) and \( \Lambda \) were chosen so to match some QCD vacuum properties, and consequently hoping to get also the correct approximated quantitative results for the gaps. We will follow the same philosophy here, however, noticing again that this completely ignores the effect of the magnetic field on the gluon dynamics.

A. Gap equations for arbitrary values of the magnetic field

Using the transformation rules given in Eq. (56) one can re-write the NJL gap equation in the CFL basis
\[ (\Delta^+)^{BC} = -i \frac{g^2}{8\Lambda^2} \sum_{B, C = 1}^9 \left\{ \text{Tr} \left( \lambda^B \lambda^C \lambda^B \gamma^\mu \right) - \frac{1}{3} \text{Tr} \left( \lambda^B \lambda^C \gamma^\mu \right) \text{Tr} \left( \lambda^B \lambda^B \right) \right\} \sigma^\mu S^{BC}_{(Q)_{121}} (x, x) \gamma^\mu . \] (83)
After computing the above $U(3)$ traces, taking into account the particular CFL-basis structure of the propagator, where $S_{11}^{ij} = S_{22}^{ij} = S_{33}^{ij}$, and keeping in mind that the relation (60) is held, we can write down more explicitly the gap equations for every element of $\Delta^{AB}$. They read

\begin{equation}
(\Delta^+)^{11} = -i \frac{g^2}{4\Lambda^2} \gamma^\mu \left\{ -\frac{5}{3} S_{11}^{(0)21}(x,x) + \frac{1}{3} \left( S_{88}^{(0)21}(x,x) + 2\sqrt{2} S_{89}^{(0)21}(x,x) + 2 S_{99}^{(0)21}(x,x) \right) \right\} \gamma^\mu , \tag{84a}
\end{equation}

\begin{equation}
(\Delta^+)^{44} = -i \frac{g^2}{4\Lambda^2} \gamma^\mu \left\{ -\frac{1}{3} \left( 2 S_{88}^{(0)21}(x,x) + \sqrt{2} S_{89}^{(0)21}(x,x) - 2 S_{99}^{(0)21}(x,x) \right) - \frac{2}{3} S_{(+)21}(x,x) \right\} \gamma^\mu , \tag{84b}
\end{equation}

\begin{equation}
(\Delta^+)^{88} = -i \frac{g^2}{4\Lambda^2} \gamma^\mu \left\{ S_{11}^{(0)21}(x,x) + \frac{1}{3} \left( S_{88}^{(0)21}(x,x) - 2\sqrt{2} S_{89}^{(0)21}(x,x) + 2 S_{99}^{(0)21}(x,x) \right) - \frac{8}{3} S_{(+)21}(x,x) \right\} \gamma^\mu , \tag{84c}
\end{equation}

\begin{equation}
(\Delta^+)^{99} = -i \frac{g^2}{4\Lambda^2} \gamma^\mu \left\{ 2 S_{11}^{(0)21}(x,x) + \frac{2}{3} S_{88}^{(0)21}(x,x) + \frac{8}{3} S_{(+)21}(x,x) \right\} \gamma^\mu , \tag{84d}
\end{equation}

\begin{equation}
(\Delta^+)^{89} = -i \frac{g^2}{12\Lambda^2} \gamma^\mu \left\{ 3 S_{11}^{(0)21}(x,x) - \left( S_{88}^{(0)21}(x,x) + \sqrt{2} S_{89}^{(0)21}(x,x) - 2 S_{99}^{(0)21}(x,x) \right) \right\} \gamma^\mu . \tag{84e}
\end{equation}

From Eqs. (82) one can immediately deduce

\begin{equation}
\Delta_A = -\frac{1}{6} \left( 5 \Delta_{11} - \Delta_{88} - \Delta_{99} \right) , \tag{85a}
\end{equation}

\begin{equation}
\Delta_S = \frac{1}{6} \left( \Delta_{11} + \Delta_{88} + \Delta_{99} \right) , \tag{85b}
\end{equation}

\begin{equation}
\Delta_N \equiv \Delta_B + \Delta_A = \frac{1}{12} \left( \Delta_{11} - 5 \Delta_{88} + 4 \Delta_{99} \right) , \tag{85c}
\end{equation}

\begin{equation}
\Delta_C \equiv \Delta_B - \Delta_A = \Delta_{44} . \tag{85d}
\end{equation}

Therefore,

\begin{equation}
\Delta_S^+ = -i \frac{g^2}{18\Lambda^2} \gamma^\mu \left\{ S_{11}^{(0)21}(x,x) + S_{88}^{(0)21}(x,x) + S_{99}^{(0)21}(x,x) \right\} \gamma^\mu , \tag{86a}
\end{equation}

\begin{equation}
\Delta_A^+ = i \frac{g^2}{12\Lambda^2} \gamma^\mu \left\{ \frac{17}{3} S_{11}^{(0)21}(x,x) + \frac{1}{3} \left( S_{88}^{(0)21}(x,x) + 6 \sqrt{2} S_{89}^{(0)21}(x,x) + 4 S_{99}^{(0)21}(x,x) \right) \right\} \gamma^\mu , \tag{86b}
\end{equation}

\begin{equation}
(\Delta_S^B)^+ = -i \frac{g^2}{72\Lambda^2} \gamma^\mu \left\{ S_{11}^{(0)21}(x,x) - 5 S_{88}^{(0)21}(x,x) + 4 S_{99}^{(0)21}(x,x) + 12 S_{(+)21}(x,x) \right\} \gamma^\mu , \tag{86c}
\end{equation}

\begin{equation}
(\Delta_S^B)^+ = -i \frac{g^2}{72\Lambda^2} \gamma^\mu \left\{ S_{11}^{(0)21}(x,x) + 7 S_{88}^{(0)21}(x,x) + 6 \sqrt{2} S_{89}^{(0)21}(x,x) - 8 S_{99}^{(0)21}(x,x) + 24 S_{(+)21}(x,x) \right\} \gamma^\mu . \tag{86d}
\end{equation}

Using the explicit expressions of the propagators given in Sec. 111E we can write the gap equations in momentum space. In the limit $\Delta_S \ll \Delta_A$, and dropping the contribution of antiparticles, Eqs. (86a), (86b), (86c), and (86d) are respectively given by

\begin{equation}
\Delta_S = i \frac{g^2}{9\Lambda^2} \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{\Delta_A}{q_0 - (|q| - \mu)^2 - \Delta_A^2} - \frac{C_A + M}{q_0 - (|q| - \mu)^2 - \Delta_A^2} - \frac{C_B - M}{q_0^2 - (|q| - \mu)^2 - \Delta_B^2} \right\} , \tag{87a}
\end{equation}
\[
\Delta_A = i \frac{g^2}{9A^2} \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{17}{2} \frac{\Delta_A}{q_0^2 - (|q| - \mu)^2 - \Delta_A^2} + \frac{1}{2} \frac{\frac{7C_a - 2M}{q_0^2 - (|q| - \mu)^2 - \Delta_a^2} + \frac{1}{2} \frac{7C_b + 2M}{q_0^2 - (|q| - \mu)^2 - \Delta_b^2} \right\}, \tag{87b}
\]

\[
\Delta_S^B = -i \frac{g^2}{36A^2} \left\{ \int \frac{d^4q}{(2\pi)^4} \left( \frac{\Delta_A}{q_0^2 - (|q| - \mu)^2 - \Delta_A^2} + \frac{F_a + 9D_a}{q_0^2 - (|q| - \mu)^2 - \Delta_a^2} + \frac{F_b + 9D_b}{q_0^2 - (|q| - \mu)^2 - \Delta_b^2} \right) \right\} + 12\tilde{e}B \sum_{k=0}^{\infty} \int \frac{dq_0 dq_3}{(2\pi)^3} \frac{\Delta_C}{q_0^2 - (|q| - \mu)^2 - \Delta_C^2}, \tag{87c}
\]

\[
\Delta_A^B = -i \frac{g^2}{36A^2} \left\{ \int \frac{d^4q}{(2\pi)^4} \left( \frac{-\Delta_A}{q_0^2 - (|q| - \mu)^2 - \Delta_A^2} + \frac{F_a - 9D_a}{q_0^2 - (|q| - \mu)^2 - \Delta_a^2} + \frac{F_b - 9D_b}{q_0^2 - (|q| - \mu)^2 - \Delta_b^2} \right) \right\} + 24\tilde{e}B \sum_{k=0}^{\infty} \int \frac{dq_0 dq_3}{(2\pi)^3} \frac{\Delta_C}{q_0^2 - (|q| - \mu)^2 - \Delta_C^2}, \tag{87d}
\]

where \(\Delta_{a/b}^2\) was defined in Eq. (77), and

\[
C_{a/b} = \frac{\Delta_A}{2} \left( 1 \pm \frac{4\Delta_N^2 + \Delta_A^2}{\Delta_A \sqrt{\Delta_A^2 + 8\Delta_N^2}} \right), \quad M = \frac{2\Delta_N^2}{\sqrt{\Delta_A^2 + 8\Delta_N^2}}, \tag{87e}
\]

\[
F_{a/b} = \frac{\Delta_A - 6\Delta_N}{2} \pm \frac{\Delta_A^2 - 4\Delta_N^2 - 6\Delta_N \Delta_A}{2\sqrt{\Delta_A^2 + 8\Delta_N^2}}, \tag{87f}
\]

\[
D_{a/b} = \Delta_N \left( 1 \pm \frac{\Delta_A}{\sqrt{\Delta_A^2 + 8\Delta_N^2}} \right). \tag{87g}
\]

These are highly coupled non-linear equations, that can only be studied numerically. However, one can find a limiting situation where an analytical solution can be found. It corresponds to the situation of moderately high \(\tilde{B}\) fields \((\mu^2 < \tilde{e}B < \Lambda^2)\).

\section{Zero magnetic field limit}

For completion we show here that in the limit of zero magnetic field the gap equations (87a)-(87d) reduce, as it should be, to the CFL gap equations.

In the \(\tilde{B} \to 0\) limit, one should have \(\Delta^B_{A/S} \to \Delta_{A/S}\). Using

\[
\Delta^B_A = \Delta_A \gg \Delta^B_S = \Delta_S, \tag{88}
\]

in the definitions (77) and (87a)-(87g), they take a more simplified form given by

\[
\Delta_a^2 \approx 4\Delta_A^2, \quad \Delta_b^2 \approx \Delta_A^2, \quad C_a \approx \frac{4}{3}\Delta_A, \quad C_b \approx -\frac{1}{3}\Delta_A, \quad F_a \approx -4\Delta_A, \quad F_b \approx -\Delta_A, \quad D_a \approx \frac{4}{3}\Delta_A, \quad D_b \approx \frac{2}{3}\Delta_A, \quad M \approx \frac{2}{3}\Delta_A. \tag{89}
\]

Making use of (87a) and (87b) in the right hand side of Eqs. (87a)-(87d) and taking into account that when \(\tilde{B} \to 0\) the Landau level summation should be replaced by an integral following the prescription

\[
\tilde{e}B \sum_{n=0}^{\infty} f(2\tilde{e}Bn) \to \int_{-\infty}^{\infty} \frac{dq_1 dq_2}{2\pi} f(q_1^2) \tag{90}
\]
we can easily see that both Eqs. (S7a) and (S7c) reduce to the CFL gap equation for the symmetric gap

$$\Delta_S = i \frac{g^2}{9\Lambda^2} \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{2\Delta_A}{q_0^2 - (|q| - \mu)^2 - \Delta_A^2} - \frac{2\Delta_A}{q_0^2 - (|q| - \mu)^2 - 4\Delta_A^2} \right\},$$  \hfill (91a)

while both Eqs. (S7b) and (S7d) reduce to the CFL gap equation for the antisymmetric gap

$$\Delta_A = i \frac{g^2}{9\Lambda^2} \int \frac{d^4q}{(2\pi)^4} \left\{ \frac{8\Delta_A}{q_0^2 - (|q| - \mu)^2 - \Delta_A^2} + \frac{4\Delta_A}{q_0^2 - (|q| - \mu)^2 - 4\Delta_A^2} \right\}. \hfill (91b)$$

Therefore, starting from the Eqs. (S7a)-(S7d) we recover the correct CFL gap equations in the NJL theory. The quark propagators in the CFL case can be straightforwardly found, (see for example Ref. [22]), assuming that $\Delta_S \ll \Delta_A$. The solution reads ($\delta = \Lambda - \mu$)

$$\Delta_{CFL}^A \approx 2\sqrt{\delta \mu} \exp \left( - \frac{3\Lambda^2\pi^2}{2g^2\mu^2} \right),$$  \hfill (92)

and

$$\Delta_{CFL}^S \approx \Delta_{CFL}^A \frac{g^2\mu^2}{9\Lambda^2\pi^2} \ln 2.$$  \hfill (93)

C. Strong magnetic field limit

Let us now consider the MCFL gap equations for the case $\tilde{e}B \gtrsim \mu^2$. Taking into account that the main contribution to the gap comes from quark energies near the Fermi level, it follows that in the strong field region the leading contribution comes from the lowest Landau level (LLL).

Assuming $\Delta_A^B \gg \Delta_A^S, \Delta_A, \text{ and } \Delta_A \gg \Delta_S$, the definitions (77), (87e)-(87g) reduce to

$$\Delta_a^2 \approx 2(\Delta_B^A)^2, \quad \Delta_b^2 \approx 2(\Delta_B^A)^2, \quad C_{a/b} \approx \frac{\Delta_A}{2} \pm \frac{\sqrt{2}}{2} \Delta_B^A,$$

$$F_a \approx -(3 + \frac{\sqrt{2}}{2})\Delta_B^A, \quad F_b \approx -(3 - \frac{\sqrt{2}}{2})\Delta_B^A, \quad D_a \approx \Delta_B^A, \quad D_b \approx \Delta_B^A, \quad M \approx \frac{\sqrt{2}}{2} \Delta_B^A.$$  \hfill (94)

In this approximation the gap equations decouple and the equation (S7a1) for $\Delta_B^A$ becomes

$$\Delta_B^A \approx \frac{g^2}{3\Lambda^2} \int_A \frac{d^3q}{(2\pi)^3} \frac{\Delta_B^A}{\sqrt{(|q| - \mu)^2 + 2(\Delta_B^A)^2}} + \frac{g^2\tilde{e}B}{3\Lambda^2} \int_A \frac{dq_3}{(2\pi)^2} \frac{\Delta_B^A}{\sqrt{(|q_3| - \mu)^2 + (\Delta_B^A)^2}}. \hfill (95)$$

Its solution reads

$$\Delta_B^A \approx 2\sqrt{\delta \mu} \exp \left( - \frac{3\Lambda^2\pi^2}{g^2(\mu^2 + \tilde{e}B)} \right).$$  \hfill (96)

The exponent in (96) has the typical BCS form exp $[1/N\tilde{G}]$, where $\tilde{G} = g^2/3\Lambda^2$ is the characteristic effective coupling constant of the $S$ channel [23], and $N$ represents the total density of states at the Fermi surface of the four quasiquarks of a single chirality contributing to the antisymmetric gap parameter $\Delta_A^B$. If no magnetic field is present (CFL case), $N = 4N_\mu$, where $N_\mu = \mu^2/2\pi^2$ is the density of states of one quasiquark. At nonzero magnetic field the density splits in two terms $N = 2N_\mu + 2N_\tilde{B}$, because two of the four quasiquarks are neutral, thus they have density $N_\mu$, while two are charged and their density of states depends on the magnetic field. In the zero Landau level the density of states of a charged quasiquark is $N_\tilde{B} = \tilde{e}B/2\pi^2$. Notice that the gap parameter (96) increases with the magnetic field. Comparing Eqs. (92) and (96) one can see that $\Delta_B^A > \Delta_{CFL}^A$ if $\tilde{e}B > \mu^2$, as assumed in the derivation of Eq. (94). This means that the field not just changes the gap structure, but it also enhances the gaps that get contributions from pairs formed by charged quarks.
As mentioned in the Introduction, although this situation has some similarity with the magnetic catalysis of chiral symmetry breaking [10], the way the field influences the pairing mechanism in the two cases is quite different. The particles participating in the chiral condensate are near the surface of the Dirac sea. The effect of a magnetic field there is to effectively reduce the dimension of the particles at the lowest Landau level, which in turn strengthens their effective coupling, catalyzing the chiral condensate. Color superconductivity, on the other hand, involves quarks near the Fermi surface, with a pairing dynamics that is already (1 + 1)-dimensional. Therefore, the $\bar{B}$ field does not yield further dimensional reduction of the pairing dynamics near the Fermi surface and hence the lowest Landau level does not have a special significance here. Nevertheless, the field modifies the density of states of the $\bar{Q}$-charged quarks, and it is through this effect, as shown in Eq. (10), that the pairing of the charged particles is reinforced by a penetrating strong magnetic field.

In the strong field approximation the remaining gap equations [56], [101], [103] are respectively given by

$$
\Delta_S \approx \frac{g^2}{18\Lambda^2} \int_A \left( \frac{\Delta_A}{\sqrt{(|q| - \mu)^2}} - \frac{\Delta_A}{\sqrt{(|q| - \mu)^2 + 2(\Delta_A^B)^2}} \right),
$$

$$
\Delta_A \approx \frac{g^2}{4\Lambda^2} \int_A \left( \frac{\Delta_A}{9\sqrt{(|q| - \mu)^2 + \Delta_A^2}} + \frac{7}{9} \frac{\Delta_A}{\sqrt{(|q| - \mu)^2 + 2(\Delta_A^B)^2}} \right),
$$

and

$$
\Delta_A^B \approx -\frac{g^2}{6\Lambda^2} \int_A \frac{\Delta_A^B}{\sqrt{(|q| - \mu)^2 + 2(\Delta_A^B)^2}} + \frac{g^2\bar{e}B}{6\Lambda^2} \int_{-\Lambda}^{\Lambda} \frac{dq_3}{(2\pi)^3} \frac{\Delta_A^B}{\sqrt{(|q_3| - \mu)^2 + (\Delta_A^B)^2}},
$$

The corresponding solutions are

$$
\Delta_S \approx \frac{2}{17}(1 - \frac{2}{1 + y})\Delta_A,
$$

$$
\Delta_A \approx \sqrt{4\mu\delta} \exp\left[-\frac{36}{17x} + \frac{21}{17x(1 + y)}\right],
$$

$$
\Delta_A^B \approx \frac{y}{1 + y} - \frac{1}{2}\Delta_A^B,
$$

where $x \equiv g^2\mu^2/\Lambda^2\pi^2$, and $y \equiv \bar{e}B/\mu^2$. As expected, the solutions [106], [109], [110] are consistent with the initial assumptions $\Delta_A^B \gg \Delta_A$, and $\Delta_A \gg \Delta_S$ if the field satisfies $\bar{e}B > \mu^2$. When the field increases within this strong field region $\Delta_A^B$ grows and $\Delta_A$ decreases, and they clearly split. Even for fields just slightly larger than $\mu^2$ the results are reliable. For example, for $y = 3/2$ and $x \sim 0.3$ [24], one finds $\Delta_A \sim 0.2\Delta_A^B$ for $y = 3/2$, while for $x \sim 1$ then $\Delta_A \sim 0.5\Delta_A^B$. For $\mu \sim 350 - 400$ MeV, one can estimate the field strength for $y = 3/2$ to be $\bar{e}B \sim (1.2 - 1.6) \times 10^9$ G.

V. EFFECTIVE FIELD THEORY FOR THE GOLDSTONE BOSONS OF THE MCFL PHASE

In the absence of a magnetic field, three-flavor massless quark matter at high baryonic density is in the CFL phase on which diquark condensates lock the $SU(3)$ color and $SU(3)$ flavor transformations thereby breaking both symmetries according to the following symmetry breaking pattern

$$
SU(3)_C \times SU(3)_L \times SU(3)_R \times U(1)_B \rightarrow SU(3)_{A+L+R}.
$$

Nine Goldstone bosons appear as the result of the Anderson-Higgs mechanism. One is a singlet, scalar mode, associated to the breaking of the baryonic symmetry, and the remaining octet is associated to the axial $SU(3)_A$ group, just like the octet of mesons in vacuum.

Strictly speaking, once electromagnetic effects are taken into account, the flavor symmetries $SU(3)_L \times SU(3)_R$ of the original three-flavor massless quark matter are reduced to $SU(2)_L \times SU(2)_R \times U(1)^{11}_A$ with $U(1)^{11}_A \subset SU(3)_A$,
since the electromagnetic charge of $d$ and $s$ quarks is different from that of the $u$ quark. However, because the electromagnetic structure constant $\alpha_{\text{e.m.}}$ is so small, this effect is really tiny, a small perturbation, and one can still consider the original flavor as an approximated symmetry of the theory.

On the other hand, we know that in the CFL phase a so-called “rotated” electromagnetism remains, allowing the penetration of the color superconductor by a conventional magnetic field in the form of a “rotated” magnetic field. If the penetrating magnetic field is strong enough, the effect of electromagnetism cannot be treated anymore as a small perturbation since, as shown in previous sections, it affects the pairing phenomena and leads to the development of the MCFL phase, whose symmetry breaking pattern is

$$ SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)^{(1)} \times U(1)_B \times U(1)_{\text{e.m.}} \rightarrow SU(2)_{C+L+R} \times \tilde{U}(1)_{\text{e.m.}}. $$

(104)

As mentioned in Section II, the locked $SU(2)$ group corresponds to the maximal unbroken symmetry, such that it maximizes the condensation energy. The counting of broken generators here, after taking into account the Anderson-Higgs mechanism, tells us that there are only five Goldstone bosons, all of which are neutral with respect to the rotated charge. As in the CFL case, one is associated to the breaking of $SU(2)_A$, and another one is associated to the breaking of $U(1)^{(1)}_A$. Therefore, apart from modifying the structure and magnitude of the gap, the applied strong magnetic field does also affect the low energy properties of the color superconductor, since it reduces the number of Goldstone bosons from nine to five, none of which is charged.

Following an approach similar to the one discussed in [21] and [25], we can derive the low-energy effective theory for the Goldstone bosons in the MCFL phase. Here we will just sketch the main steps leaving detailed calculations for a future work. The main idea is to introduce left and right coset fields

$$ X^{ia} \sim \epsilon^{ijk} \epsilon^{abc} \langle \psi_L^b \psi_L^c \rangle^*, \quad Y^{ia} \sim \epsilon^{ijk} \epsilon^{abc} \langle \psi_R^b \psi_R^c \rangle^* $$

(105)

where $a, b, c$ denote flavor indices, $i, j, k$ denote color indices, and $L/R$ denote left/right chirality, respectively; which transform under the global transformations of the original group as

$$ X \rightarrow U_L X U_C^\dagger, \quad Y \rightarrow U_R Y U_C^\dagger. $$

(106)

where $U_C$ is a color transformation, while $U_L \in SU(2)_L \times U(1)_L^{(1)}$ and $U_R \in SU(2)_R \times U(1)_R^{(1)}$ are left and right flavor transformations respectively. We are ignoring in our discussion the baryon symmetry, but its Goldstone mode can be incorporated in a similar way.

Then, keeping in mind that the Goldstone modes are related to motions along the degenerate minima of the effective potential, one can obtain the effective action for the Goldstone modes just by factoring out the modulus of the condensate and allowing the phases of $X$ and $Y$ to depend on time and space.

At this point it is convenient to introduce the flavor singlet

$$ \Sigma = X Y^\dagger = \exp \left( i \Phi \frac{f_{\pi,B}}{f_{\pi,B}} + i \phi_0 \right), \quad \Phi = \phi_A \sigma^A, \quad A = 1, 2, 3. $$

(107)

where $\sigma_A$ are the $SU(2)$ generators and the fields $\phi_A$ are associated to the breaking of $SU(2)_A$, while $\phi_0$ is associated to the breaking of $U(1)^{(1)}_A$.

Taking into account that the effective low energy Lagrangian of the Goldstone bosons in the MCFL should be symmetrical under the rotations of the original group $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)^{(1)}_A$, we can write it down up to leading order in the derivatives as

$$ \mathcal{L} = \frac{f_{\pi,B}^2}{4} \left( \text{Tr} \left( \partial_0 \Sigma \partial_0 \Sigma^\dagger \right) + v_\perp^2 g_{ij}^\perp + v_\parallel^2 g_{ij}^\parallel \right) \text{Tr} \left( \partial_i \Sigma \partial_j \Sigma^\dagger \right). $$

(108)

Notice that we introduced different longitudinal and transverse velocities. This should be done because besides the usual Lorentz symmetry breaking proper of the theory at finite density, the strong magnetic field induces extra symmetry reduction, since only the spacial $SO(2)$ rotations in the plane perpendicular to the magnetic field are allowed. The constants $f_{\pi,B}$, $v_\perp$ and $v_\parallel$ should be computed from the microscopic theory. We leave such a computation for a future project.

It is remarkable that although the external magnetic field does not explicitly enter in the low-energy Lagrangian [108], since it does not couple minimally to the MCFL Goldstone bosons because these are all neutral with respect to the rotated charge, its presence is manifested anyway through the anisotropy of the velocities.
The effective field theory for these Goldstone fields is totally analogous to that found for QCD in a strong magnetic field in Ref. \cite{17}. Were it not for the presence of the extra Goldstone boson associated to the baryon symmetry breaking in the MCFL phase, the two low-energy theories, at low and high densities, in an external magnetic field would be completely equivalent, and one would expect them to be connected by a crossover. We might think that because the existence of the extra Goldstone mode, this is not possible. However, the situation is similar to the case without magnetic field. There also the high and low density effective theories of the Goldstone modes are identical with the exception of the extra Goldstone mode in the color superconductor due to baryon symmetry breaking, and yet it has been argued \cite{24} that the extra mode in the CFL phase may have a counterpart in the chiral phase, the dibaryon state $H$ found in Ref. \cite{27}. Therefore, the question about the possible equivalence between high and low density low energy theories in a magnetic field still remains open.

VI. CONCLUDING REMARKS

In this paper we investigated the effect of a magnetic field on the color superconducting gap of a dense, three massless quark flavor system. Our main result is that the magnetic field leads to the formation of a new color-flavor locking phase, that we have called the MCFL phase. We found that the magnetic field affects essentially the gap by modifying the density of states of the charged quarks on the Fermi surface.

In the analytic calculations performed in this paper we assumed a strong field approximation ($\epsilon B \gtrsim \mu^2$), which corresponds to fields $\sim 10^{18}$G. The only reason we considered such strong fields was to simplify our task so we could avoid to sum in Landau levels, and hence could perform analytical calculations. However, we underline that there is nothing special about the LLL here, because the main effect of the field is to modify the density of states and that is not restricted only to the LLL. On the other hand, there are indications, based on the study of the low energy degrees of freedom of the CFL effective field theory in the presence of an external magnetic field \cite{28}, that for $\epsilon B^{ext} \sim 5 \cdot 10^{16}$ G the symmetry pattern of the theory qualitatively separates from that of the CFL phase and becomes characteristic of the new MCFL phase. These estimates suggest that a distinguishable separation between the gaps $\Delta_A$ and $\Delta_B$ may take place already at those magnetic field orders. Of course, a definitive determination of the field strength required to separate the two phases can only be obtained through a numerical calculation on which the effect of all the higher Landau levels in the gap equations are included.

We have shown that the MCFL phase has a smaller vector symmetry than the CFL phase and consequently the number of Goldstone bosons reduces from 9 to 5. Our zero-temperature results imply that a propagating rotated photon with energy less than the lightest charged quark mode cannot scatter, since all the $\bar{Q}$-charged quarks acquire a gap and all the Nambu-Goldstone bosons are neutral. The anisotropy present in the background of an external magnetic field and the existence of charged Goldstone bosons in CFL but not in MCFL indicates a rather different low energy physics in these two phases and it should be reflected in the transport properties. In particular, the MCFL superconductor is transparent and behaves at $T = 0$ as an anisotropic dielectric, as opposed to the isotropic dielectric behavior of the CFL phase \cite{22, 24}. However, similar to the CFL, the medium will become optically opaque as soon as leptons are thermally excited \cite{30}.

Although the $\bar{Q}$ neutrality of the MCFL is guaranteed without having to introduce any electron density due to the absence of gapless modes and the $\bar{Q}$-neutrality of the diquark condensates, imposing the 3- and 8- color neutralities may require the introduction of small ”chemical potentials” $\mu_3$ and $\mu_8$ due to the difference between the main gaps $\Delta_A$ and $\Delta_B$.

If we introduce quark masses the scenario will be much more complicated though. First, we would have to revise the neutrality conditions and see how they are affected. In addition, one would expect that the effect of the magnetic field increasing the gaps formed by $\bar{Q}$-charged quarks might have some impact in counteracting the stress produced by the mass of the strange quark on the corresponding Fermi sphere and perhaps also in the onset of gapless modes, along with the problem of the instability of the gluon modes due to imaginary Meissner masses. Considering the consequences of incorporating quark masses, together with a careful study of the effects of the magnetic field in the low energy physics, in transport properties or in neutrino dynamics will be the subject of future investigations.

Even if only at a speculative level, it is worth to mention some possible astrophysical consequences of the main results of this paper. The equation of state of the color superconductor will be affected by the magnetic field, as different quark gaps will be affected in a different way by the field. We do not expect this to be a pronounced effect, though, but it might be interesting to see whether it affects the mass-to-radius ratio of the star. Because the low energy physics of the CFL and MCFL are so different, depending on the strength of the star’s magnetic field there may be different signatures in the cooling process of the star. Transport properties, such as viscosities or thermal conductivities, will be also affected by the presence of the magnetic field. Finally, the dynamics of the magnetic field itself might be very peculiar in quark matter, differing from that in a neutron star, which is commonly believed to be
an electromagnetic superconductor.

Acknowledgments
This work was supported in part by NSF grant PHY-0070986, by MEC under grants FPA2004-00996 and AYA 2005-08013-C03-02, and by GVA under grant GV05/164.

APPENDIX A: PROJECTORS IN THE PRESENCE OF A BACKGROUND MAGNETIC FIELD

In the absence of a background magnetic field the condensate is usually given in terms of different projectors \cite{16}. The chiral and helicity projection operators are given respectively by

$$ P_{R,L} = \frac{1 \pm \gamma_5}{2} , $$

(A1)

and

$$ H_{\pm}(p) = \frac{1}{2} [1 \pm \gamma_5 \gamma_0 \vec{\gamma} \cdot \vec{p}] . $$

(A2)

The massless quark free-energy projector for positive and negative energies is given by

$$ \Lambda_{\pm}(p) = \frac{1 \pm \gamma_0 \vec{\gamma} \cdot \vec{p}}{2} . $$

(A3)

For massless quarks, the chiral and energy projectors are enough to specify the quark propagators. Furthermore, in a NJL theory, where there are only contact interactions, the gap is a momentum independent constant, and the Dirac structure of the condensate is particularly simple in the spin zero case, where it simply reduces to $C \gamma_5$, where $C = i \gamma_2 \gamma_0$ is the matrix of charge conjugation.

In the presence of an external $\vec{B}$ field the helicity-projection operators (A2) and the free-energy projectors (A3) are not conserved quantities, since they do not commute with the field-dependent Hamiltonian. But their generalization in terms of the field dependent momentum operator (12) are. In covariant form, these conserved operators can be expressed respectively by

$$ \tilde{H}_{\pm}(\Pi) = \frac{1 \pm \frac{i}{2} \gamma_5 [u_\mu \vec{\Pi}_\nu - u_\nu \vec{\Pi}_\mu] \sigma^{\mu\nu}}{2} , $$

(A4)

$$ \tilde{\Lambda}_{\pm}(\Pi) = \frac{1 \pm \frac{i}{2} [u_\mu \vec{\Pi}_\nu - u_\nu \vec{\Pi}_\mu] \sigma^{\mu\nu}}{2} , $$

(A5)

where $u^\mu$ is the four-velocity of the center of mass of the many particle system. In the rest frame, the helicity and energy projectors can be expressed in momentum space respectively as

$$ \tilde{H}^{(\pm)}(\vec{p}) = \frac{1 \pm \gamma_5 \gamma_0 \vec{\gamma} \cdot \vec{p}}{2} , $$

(A6)

$$ \tilde{\Lambda}^{(\pm)}(\vec{p}) = \frac{1 \pm \gamma_0 \vec{\gamma} \cdot \vec{p}}{2} . $$

(A7)

Similarly to the free case, in the presence of a “rotated” magnetic field the chiral (A1), helicity (A2) and energy (A3) projectors commute.

[1] E. J. Ferrer, V. de la Incera and C. Manuel, Phys. Rev. Lett. 95, 152002 (2005).
[2] K. Rajagopal and F. Wilczek, hep-ph/0011333; M. Alford, Ann. Rev. Nucl. Part. Sci. 51, 131 (2001); G. Nardulli, Riv. Nuovo Cim. 25N3, 1 (2002); T. Schäfer, "Quark Matter," hep-ph/0304281; D. H. Rischke, Prog. Part. Nucl. Phys. 52, 197 (2004); H.-C. Ren, hep-ph/0404074; I. A. Shovkovy, Found. Phys. 35 (2005) 1309-1358.

[3] N. Itoh, Prog. Theor. Phys. 44, 291 (1970); A. R. Bodmer, Phys. Rev. D 4, 1601 (1971); E. Witten, Phys. Rev. D 30, 272 (1984).

[4] M. Dey, I. Bombaci, J. Dey, S. Ray and C. B. Samanta, Phys. Lett. B 438, 123 (1998); X. D. Li, I. Bombaci, M. Dey, J. Dey, and E. P. J. van den Heuvel, Phys. Rev. Lett. B 83, 3776 (1999); X. D. Li, S. Ray, J. Dey, M. Dey, and I. Bombaci, Ap. J. 527, L51 (1999); F. Weber, Prog. Part. Nucl. Phys. 54, 193 (2005); Sinha M., M. Dey, S. Ray, and J. Dey; "Strange Stars and Superbursts at near Eddington Mass Accretion Rates," astro-ph/0504292; D. Page, and A. Cumming, "Superbursts from strange stars," astro-ph/0508444.

[5] I. Fushiki, E. H. Gudmundsson, and C.J. Pethick, Astrophys. J. 342, 958 (1989); T.A. Mihara, et. al., Nature (London) 346, 250 (1990); G. Chanmugam, Ann. Rev. Astron. Astrophys. 30, 143 (1992); P. P. Kronberg, Rep. Prog. Phys. 57, 325 (1994); D. Lai, Rev. Mod. Phys. 73, 629 (2001); D. Grasso and H.R. Rubinstein, Phys. Rep. 348, 163 (2001).

[6] C. Thompson and R. C. Duncan, Astrophys. J. 473, 322 (1996).

[7] L. Dong and S.L. Shapiro, ApJ. 383, 745 (1991).

[8] M. Alford, K. Rajagopal and F. Wilczek, Nucl. Phys. B 537, 443 (1999).

[9] M. Alford, J. Berges, and K. Rajagopal, Nucl. Phys. B 571, 269 (2000).

[10] V. P. Gusynin, V. A. Miransky, and I. A. Shovkovy, Phys. Rev. Lett. 73, 3499 (1994); Phys. Lett. B 349, 477 (1995); Phys. Rev. D 52, 4747 (1995); Nucl. Phys. B 462, 249 (1996); K. G. Klimenko, Z. Phys. C 54, 323 (1992); Teor. Mat. Fiz. 90, 3 (1992).

[11] D.-S Lee, C. N. Leung and Y. J. Ng, Phys. Rev. D 55, 6504 (1997); E. J. Ferrer, and V. de la Incera, Phys. Rev. D 62, 054008 (1998); Phys. Lett. B 481, 287 (2000); E. Elizalde, E. J. Ferrer, and V. de la Incera, Phys. Rev. D 68, 096004 (2003).

[12] V.I. Ritus, Ann.Phys. 69, 555 (1972); Sov. Phys. JETP 48, 788 (1978) [Zh. Eksp. Teor. Fiz. 75, 1560 (1978)].

[13] M. Alford, and K. Rajagopal, JHEP 0206, 031 (2002).

[14] K. Rajagopal and A. Schmitt, Phys. Rev. D 73 (2006) 045003.

[15] D. Pisarski, and D. H. Rischke, Phys. Rev. Lett. 83, 37 (1999); Phys. Rev. D 60, 094013 (1999); Phys. Rev. D 61, 074017 (2000).

[16] V. A. Miransky, and I. A. Shovkovy, Phys. Rev. D 66, 045006 (2002).

[17] D. Bailin, and A. Love, Phys. Rep. 107, 325 (1984).

[18] E. Elizalde, E. J. Ferrer, and V. de la Incera, Ann. of Phys., 295, 33 (2002); Phys. Rev. D 70, 043012 (2004).

[19] A. Ayala, A. Bashir, A. Raya, and E. Rojas, "Dynamical mass generation in strong coupling QED with weak magnetic fields," hep-ph/0602209.

[20] D.T. Son and M.A. Stephanov,Phys. Rev. D 61, 074012 (2000).

[21] D. F. Litim and C. Manuel, Phys. Rev. D 64, 094013 (2001).

[22] D. T. Son, Phys. Rev. D 59, 094019 (1999).

[23] R. Casalbuoni, R. Gatto, G. Nardulli and M. Ruggieri, Phys. Rev. D 68, 034024 (2003).

[24] R. Casalbuoni and R. Gatto, Phys. Lett. B 464, 111 (1999).

[25] T. Schäfer and F. Wilczek, Phys. Rev. Lett. 82, 3956 (1999).

[26] R. Jaffe, Phys. Rev. Lett. 38, 195 (1977), 617(E) (1977).

[27] C. Manuel, “Low energy properties of color-flavor locked superconductors,” hep-ph/0512054; C. Manuel and M. H. Tytgat, Phys. Lett. B 501, 200 (2001).

[28] C. Manuel and K. Rajagopal, Phys. Rev. Lett. 88, 042003 (2002).

[29] I. A. Shovkovy and P.J. Ellis, Phys. Rev. C 67, 048801 (2003).