3-Branes on Eguchi-Hanson $6D$ Instantons

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Abstract

We use the approach used by Eguchi-Hanson in constructing four-dimensional instanton metrics and construct a class of regular six-dimensional instantons which are nothing but $S^2 \times S^2$ resolved conifolds. We then also obtain D3-brane solutions on these EH-resolved conifolds.
1 Introduction

The AdS/CFT conjecture \cite{1, 2, 3} relates string theory on anti-de Sitter bulk spacetime to a four-dimensional conformal gauge theory on the boundary. It serves as being one of the concrete realisation of the holographic idea \cite{4}. In this picture the SU($N$) super-Yang-Mills theory lives on the boundary of the bulk $AdS_5 \times S^5$ space. Since then this gauge-gravity duality has been extensively tested and has provided a fruitful alternative in understanding of various aspects of gauge theories, both qualitatively and quantitatively. The large body of work is cited in some reviews, for example \cite{5}.

It has been rather interesting to study AdS/CFT for the class of manifolds $AdS_5 \times T^{1,1}$ which have less supersymmetry, where $T^{1,1}$ is an Einstein space. This requires constructing D3-brane solutions over $M_4 \times Y^{p,q}$ spaces where $Y^{p,q}$ is the six-dimensional Calabi-Yau cone. These cone-like geometries are singular at the tip of the cone, but deformations on these conifold geometries can be performed \cite{6} so that they become regular Calabi-Yau geometries. Several new resolved solutions have recently appeared in literature \cite{7, 8}.

In this short note we first construct a regular Calabi-Yau cone solution which we obtain by adopting standard method of constructing Eguchi-Hanson 4D instantons. We call them EH-resolved cone as they share some unique properties with Eguchi-Hanson instantons. But there are standard techniques to resolve these cones, see \cite{6, 9}. We then look for D3-brane solutions on these EH-resolved cones. The paper is organised as follows. In the section-2, we review Eguchi-Hanson 4D instanton geometry. In section-3 we construct six-dimensional regular cone-like geometry using Eguchi-Hanson ansatz. We then obtain D3-brane configurations over these spaces in the section-4. The results are summarised in the last section.

2 The 4D EH instantons

The Euclidean gravity solutions with finite action and a self-dual curvature are manifolds that are generally classified as gravitational instantons. The gravitational fields are localised in space and the metric becomes asymptotically locally flat at infinity. \cite{1} In this section we review the main aspects of the Eguchi-Hanson 4D solutions.

In the case of the Eguchi-Hanson (EH) manifold the metric ansatz is \cite{12}

$$ds^2_{EH} = \frac{dr^2}{g(r)} + \frac{r^2}{4}(d\theta^2 + \sin^2 \theta d\phi^2) + \frac{r^2}{4}g(r)(d\psi + \cos \theta d\phi)^2$$  \hspace{1cm} (1)

where $r^2 = t^2 + x^2 + y^2 + z^2$. The $t, x, y, z$ are Cartesian coordinates which define the base space which is $R^4$. There is a complete radial symmetry and one is free to pick

\footnote{It has been recently found that the information-geometry on the solution space of these instantons has a constant negative curvature \cite{13}. The results are found to be quite similar to the information-metric case of Yang-Mills instantons \cite{14}.}
the center in the base space. The above metric solves Euclidean gravity equations in flat space provided \( g(r) \) satisfies a monopole like first order equation

\[
g'(r) + \frac{4}{r}(g - 1) = 0
\]

which has an immediate solution \([12]\)

\[
g(r) = 1 - \frac{a^4}{r^4}, \quad (r \geq a)
\]

where \( a \) is an integration constant. By redefining the coordinate \( \rho^4 = r^4 - a^4 \) one can also express metric as \([12]\)

\[
ds^2_{\text{EH}} = (1 + \frac{a^4}{\rho^4})^{-\frac{1}{2}}(d\rho^2 + (\rho^2/4)(d\psi + \cos \theta d\phi)^2) + (1 + \frac{a^4}{\rho^4})\frac{3}{4}(d\theta^2 + \sin^2 \theta d\phi^2)
\]

with the coordinate ranges

\[
0 \leq \rho \leq \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi.
\]

A non-singular Maxwell field with self-dual field strength can be introduced in the Eguchi-Hanson background

\[
A_1 = \frac{a^2}{\sqrt{\rho^4 + a^4}} \sigma_z,
\]

\[
F_2 = \frac{2a^2}{\rho^4 + a^4} (e^3 \wedge e^0 + e^1 \wedge e^2).
\]

Thus for EH instanton,

\[
(F_{\mu\nu})^2 = \frac{16a^4}{(\rho^4 + a^4)^2}.
\]

Though \( a \) is an integration constant, but it is related to the presence of Maxwell field here. For the instantons located away from the origin in \( R^4 \), one can introduce 4 new parameters, \( x_0^i \), as the position variables. This is possible because the EH metric and the gauge background have complete radial symmetry, so \( \rho^2 = |x - x_0|^2 \). However, in any case the center coordinates, \( x_0^i \), of the EH metric \([4]\) and the gauge field \([7]\) must coincide in order that the solution exists. It is interesting to note that the Eq.\([7]\) is strikingly similar to the Yang-Mills instanton field strength in \([10, 11, 14]\), the only difference being the distribution of powers in the denominator. This fact did lead us to conclude that the information metric approach \([14]\) to holography can also be studied for this class of curved EH instantons also; see for more details \([13]\).

### 3 A EH-resolved 6D conifold

We construct a class of regular six-manifolds which are the Ricci-flat Kähler solutions of the Einstein equations. For this purpose, we know the existence of the conifold geometries
and corresponding $AdS/CFT$ analysis has been studied in detail by [7, 8]. The many deformed and resolved conifolds have also been worked out in [8, 9] previously. As we are interested in Eguchi-Hanson class of instantons on the conifold, so we look for a metric ansatz resembling the Eguchi-Hanson 4D metric
\[ ds^2 = \frac{dr^2}{f(r)} + \frac{r^2}{6}\left((d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2)\right) + \frac{r^2}{9} f(r)(d\psi + \sum_{i=1}^{2} \cos \theta_i d\phi_i)^2. \]
\( (8) \)

The angular coordinate ranges are taken as
\[ 0 \leq \theta_i \leq \pi, \quad 0 \leq \phi_i \leq 2\pi, \quad 0 \leq \psi \leq 2\pi. \]
\( (9) \)

One will easily note that the $r = constant$ sections of the above metric are indeed $T^{1,1}$ geometry, which is an Einstein space with positive curvature. Now all the $R_{\mu\nu} = 0$ equations are solved by the metric \( (8) \) provided $f(r)$ satisfies the following monopole type equation,
\[ f' + \frac{6}{r}(f - 1) = 0. \]
\( (10) \)

Note that this first order equation appears in various instanton solutions including the Yang-Mills and the 4D Eguchi-Hanson one. These are consequences of BPS conditions. The above differential equation has one trivial solution $f = 1$, in which case the metric becomes that of the singular Calabi-Yau cone over the base $T^{1,1} \ [6]$
\[ ds^2_{6} = dr^2 + r^2 ds^2_{T^{1,1}} \]
\( (11) \)

which is Ricci-flat and the Kähler 2-form is given by
\[ J = \frac{r^2}{6} (\sin \theta_1 d\theta_1 \wedge d\phi_1 + \sin \theta_2 d\theta_2 \wedge d\phi_2) + \frac{r}{3} (d\psi + \cos \theta_i d\phi_i) \wedge dr. \]
\( (12) \)

The nontrivial solution of \( (10) \) is
\[ f(r) = 1 - \frac{a^6}{r^6}, \quad a \leq r < \infty \]
\( (13) \)

where $a$ is an integration constant. We can take it to be the size of the instantons, or the measure of deformation. \( 2 \)

For the solution \( (13) \) the metric becomes
\[ ds^2_{6} = \frac{dr^2}{1 - \frac{a^6}{r^6}} + \frac{r^2}{6}\left((d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + (d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2)\right) + \frac{r^2}{9} (1 - \frac{a^6}{r^6})(d\psi + \cos \theta_i d\phi_i)^2. \]
\( (14) \)

which appears quite the same as Eguchi-Hanson metric \( (11) \). This EH-resolved six-manifold is topologically $R^+ \times S^2 \times S^3$. An interesting unique property of the EH-resolved metric, that is why we call it so, is that the determinant of the resolved metric remains the same

\( 2 \) The third solution with plus sign $f(r) = 1 + \frac{a^6}{r^6}$ will have a naked singularity at $r = 0$. 4
as the unresolved metric. It was also the case with the 4D instantons. The apparent coordinate singularity at \( r = a \) is removable, it can be seen as follows. Near \( r = a \) we define new radial coordinate \( u^2 = r^2(1 - \frac{a^2}{r^2}) \), so the metric (14) can be written as

\[
d s_6^2 = \frac{d u^2}{(1 + \frac{2a^6}{r^6})^2} + \frac{u^2}{9}(d \psi + \cos \theta_1 d \phi_1)^2 + \frac{r^2}{6} \left( (d \theta_1^2 + \sin^2 \theta_1 d \phi_1^2) + (d \theta_2^2 + \sin^2 \theta_2 d \phi_2^2) \right). \tag{15}
\]

So near \( r = a \) it becomes

\[
d s_6^2 \approx \frac{1}{9}(d u^2 + u^2 (d \psi + \cos \theta_1 d \phi_1)^2) + \frac{a^2}{6} \left( (d \theta_1^2 + \sin^2 \theta_1 d \phi_1^2) + (d \theta_2^2 + \sin^2 \theta_2 d \phi_2^2) \right), \tag{16}
\]

and thus the metric can be made regular with the topology being \( R^2 \times S^2 \times S^2 \) near \( r = a \), if the range of \( \psi \) is restricted to \( 0 \leq \psi \leq 2\pi \). While for \( r \gg a \) the metric (14) becomes the metric over the Kähler cone (11).

In any case the Kähler 2-form for the metric (14) is

\[
J = e^5 \wedge e^0 + e^1 \wedge e^2 + e^3 \wedge e^4
\tag{17}
\]

The vielbeins are

\[
e^0 = (1 - \frac{a^6}{r^6})^{-1/2} dr, \quad e^1 = \frac{r}{\sqrt{6}} d \theta_1, \quad e^2 = \frac{r}{\sqrt{6}} \sin \theta_1 d \phi_1, \quad e^3 = \frac{r}{\sqrt{6}} d \theta_2,
\]

\[
e^4 = \frac{r}{\sqrt{6}} \sin \theta_2 d \phi_2, \quad e^5 = \frac{r}{3} (1 - \frac{a^6}{r^6})^{1/2}(d \psi + \cos \theta_1 d \phi_1 + \cos \theta_2 d \phi_2)
\tag{18}
\]

### 3.1 Adding the tensor fields

We now wish to introduce a pair of second rank tensor fields \( B_{\mu\nu} \) and \( C_{\mu\nu} \) in the 6-dimensional manifold (14). The most plausible combined action is

\[
\int \left( R \wedge 1 - \frac{1}{2!} H_{(3)} \wedge * H_{(3)} - \frac{1}{2!} F_{(3)} \wedge * F_{(3)} - \frac{1}{2!} F_{(1)} \wedge * F_{(1)} + \text{boundary terms} \right)
\tag{19}
\]

where field strengths are given by \( H_{(3)} = dB_2 \) and \( F_{(3)} = dC_2 - \chi dB_2 \) and \( F_{(1)} = d\chi \). In our Hodge-dual convention: \(*1 = e^1 \wedge \cdots \wedge e^6 \equiv \sqrt{3!} [d^6x] \). The field equations which follow from the above action are

\[
d * F_{3} = 0, \quad d(*H_{3} - \chi * F_{3}) = 0.
\tag{20}
\]

Thus a vanishing \( \chi \) solution exists provided we ensure \( F_{\mu\nu\lambda} H_{\mu\nu\lambda} = 0 \). For \( \chi = 0 \) the tensor field equations are readily solved provided we take

\[
* F_{3} = -H_{3}, \quad * H_{3} = F_{3}
\tag{21}
\]

From here, one can construct a complex harmonic \((2,1)\)-form \( G_3 = F_3 + i H_3 \) which satisfies the following complex self-duality relation

\[
* G_{3} = i G_{3}
\tag{22}
\]

and its energy-momentum tensor identically vanishes.

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3 Note that it can be written as \( J \equiv dA = d(-\frac{a^2}{3}(d \psi + \cos \theta_1 d \phi_1)) \). Also \( J \wedge J \equiv \text{Vol}[M_6] \).
3.2 The singular $B$-field solution

We now construct a solution of the coupled field equations of Einstein and 2-rank tensor fields. It is found that a solution exists where the metric is taken to be the Eq. (14) and the $B$ field as

$$B_2 = \frac{m}{2} \log \frac{r^6 - a^6}{r_0^6} \, \omega_2,$$

(23)

where $\omega_2$ is a closed 2-form over $S^2$'s

$$\omega_2 = \frac{1}{2} (\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2).$$

Then the corresponding 3-form field strength becomes

$$H_{(3)} = \frac{3mr^5}{(r^6 - a^6)} dr \wedge \omega_2$$

(24)

where $m$ is a constant parameter. The field strength $F_{(3)}$ is determined by the Hodge-dual relations (21) and it is

$$F_{(3)} = m \, \omega_3$$

(25)

where $\omega_3 = (d\psi + \cos \theta_i d\phi_i) \wedge \omega_2$. There is a constant flux through $S^3$

$$\int_{S^3} F_3 = m$$

(26)

which will be quantized. Due to the self-dual property the field strength $G_3$ is harmonic and the corresponding Euclidean energy-momentum tensor identically vanishes. This has been explicitly checked by us. This is the reason that the Einstein equations retain their empty-space form $R_{\mu\nu} = 0$. Also we have

$$F_{\mu\nu\lambda} H^{\mu\nu\lambda} = 0$$

for this background which is required for axion to be vanishing.

However, the field expression

$$\frac{1}{2.3!} (H_{\mu\nu\lambda})^2 = \frac{(9m)^2}{(r^6 - a^6)}$$

(27)

is singular at $r = a$. Since $(F)^2 = (H)^2 \sim \frac{1}{r^6}$ for large $r$, the asymptotic behavior of these $B$-instantons is different from that of the regular 4D instantons discussed in the last section. Note that, the tensor-fields remain divergent at $r = a$ even though the metric (14) is regular there. The action for the tensor fields diverges when integrated over the range $a \leq r \leq \infty$ because of the singularity of the tensor fields at $r = a$. So this cannot represent a regular instanton background unlike Eguchi-Hanson background. This singularity at $r = a$ is similar to the Klebanov-Strassler conifold singularity where the fields are singular at the tip of the cone $r = 0$. Whole of this structure reduces to the singular conifold case when $a = 0$. One had to deform the conifold in order to avoid the singularity [7].
4 Embedding in ten dimensions

The six-dimensional conifolds can also be embedded in ten-dimensional IIB string theory quite elegantly [7]. It is of main interest to insert D3-branes on the above resolved conifold (14). The full ten-dimensional ansatze are then

\[
\begin{align*}
 ds^2 &= h(r)^{-\frac{1}{2}}(-dt^2 + dx \cdot dx) + h(r)^{\frac{1}{2}} \left( \frac{dr^2}{f(r)} + \frac{r^2}{9}f(r)(\tilde{\psi})^2 + \frac{r^2}{6}\sum_{i=1}^{2}(d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) \right) \\
 F_5 &= dh^{-1} \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 + Q(r)\omega_2 \wedge \omega_3 \\
 \Phi &= 0, \quad \chi = 0 \\
 F_3 &= m\omega_3, \quad B_2 = \frac{m}{2}\log \frac{r^6}{r_0^6} - \frac{a}{2} \omega_2
\end{align*}
\]

where \( f(r) = 1 - \frac{a}{r} \) and we defined \( \tilde{\psi} \equiv d\psi + \sum_i \cos \theta_i d\phi_i \). We have to ensure that field strength \( F_5 \equiv dC_4 + B_2 \wedge F_3 \) satisfies its equations of motion

\[
dF_5 = d^* F_5 = H_3 \wedge F_3
\]

and is also self-dual. Thus all field equations are solved provided

\[
 M(r) = \frac{m^2}{2}\log \frac{r^6}{r_0^6} - \frac{a}{2}
\]

\[
 fr^5 \partial_r h = -Q(r) = -M(r) - c_0
\]

where \( c_0, r_0 \) are constants. An exact solution exists when \( a = 0 \) (unresolved case)

\[
h(r) = b_0 + \frac{1}{r^4} \left( \frac{c_0}{4} + \frac{3m^2}{4}\log \left( \frac{r}{r_0} \right) + \frac{3m^2}{16} \right)
\]

which is the solution of [7] provided we set \( c_0 = 16\pi g_s N \). In the absence of 3-form flux, \( m = 0 \), the above solution represents \( N \) D3-branes on a singular conifold.

When \( a \neq 0 \), an exact solution is a bit cumbersome. We can however make following suitable radial coordinate choice,

\[
 \rho^6 = r^6 - a^6, \quad \frac{a}{r^6} = \left( \frac{\rho}{r} \right)^6, \quad \frac{a^6}{r^6} = \left( \frac{\rho}{r} \right)^6
\]

so that the EH-resolved metric (14) in the new coordinates becomes

\[
 ds^2 = \frac{dr^2 + \frac{\rho^2}{9}(\tilde{\psi})^2}{q(\rho)^{2/3}} + \frac{\rho^2}{6}q(\rho)^{1/3}\sum_{i=1}^{2}(d\theta_i^2 + \sin^2 \theta_i d\phi_i^2)
\]

with the function \( q(\rho) = 1 + a^6/\rho^6 \) and the new coordinate range \( 0 \leq \rho \leq \infty \). The singular conifold is obtained whenever we set \( a = 0 \). In these coordinates 10-dimensional metric (28) will look like

\[
 ds^2 = h(\rho)^{-\frac{1}{2}}(dx_{\mu}^2) + h(\rho)^{\frac{1}{2}} \left( \frac{dr^2 + \frac{\rho^2}{9}(\tilde{\psi})^2}{q(\rho)^{\frac{1}{3}}} + \frac{\rho^2}{6}q(\rho)^{\frac{1}{3}}\sum_{i=1}^{2}(d\theta_i^2 + \sin^2 \theta_i d\phi_i^2) \right)
\]
We find that the solution of the equation (30) in the region $\rho \gg a$ can be expressed as

$$h(\rho) = b_0 + \frac{4\pi g_s N}{\rho^4} \left(1 + \frac{4}{15} \frac{a^6}{\rho^6} + \cdots\right) + \frac{3m^2}{\rho^4} \left(\frac{1}{16} \log\left(\frac{10\rho^4}{\rho^0_0}\right) + \frac{1}{150} \frac{a^6}{\rho^6} \log\left(\frac{10\rho^10}{\rho^0_0}\right) + \cdots\right) \tag{34}$$

which gives exactly the metric in [7] once $a = 0$. But the conifold is the resolved conifold now. Even in the absence of H-flux ($m = 0$) the brane geometry does not exactly look like $AdS_5 \times T^{1,1}$ if $b_0$ is dropped, that is when $4\pi g_s N \gg \rho$. The $AdS$ space will be obtained only when we take $4\pi g_s N \gg \rho \gg a$, that is the branes must be located far away from the origin of the conifold. Thus if large number of coincident D3-branes are placed in the region $\rho \gg a$, the near horizon geometry would become more or less $AdS_5 \times T^{1,1}$. That could be seen by dropping the constant $b_0$ in (34) and setting $m = 0$

$$h(\rho) \sim \frac{4\pi g_s N}{\rho^4} \left(1 + O\left(\frac{a^6}{\rho^6}\right) + \cdots\right) \tag{35}$$

When $m \neq 0$ the same will apply but instead there will be flux dependent deformations, which are logarithmic, of the $AdS_5$ geometry with interesting consequences in CFT. The D3-Branes on resolved conifolds have several interesting results in the dual CFT [7].

## 5 Summary

In this short note we constructed regular six-dimensional conifolds simply starting from Eguchi-Hanson like ansatz and solving the resultant first order monopole like equations. We then switched on $F_3$ flux, the field strengths are found to be singular even for the resolved case. We then construct solutions in ten-dimensions and obtain D3-brane solutions which spread over EH-resolved conifold. The solutions are singular with the presence of 3-form flux background. Switching off the flux will make the solutions regular. However taking near horizon limit does not immediately give us $AdS_5 \times T^{1,1}$ space. The $AdS$ geometry exists only for the branes located away from the deformed region of the cone.

While we were finishing this work two simultaneous works have appeared [16, 17] which also discuss $AdS/CFT$ on the resolved $S^2$ and $S^2 \times S^2$ -conifolds. To add, the generalised Eguchi-Hanson solitons in odd dimensions have been constructed in [18].

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