Bayesian Lasso: Concentration and MCMC Diagnosis

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Abstract
Using posterior distribution of Bayesian LASSO we construct a semi-norm on the parameter space. We show that the partition function depends on the ratio of the $l^1$ and $l^2$ norms and present three regimes. We derive the concentration of Bayesian LASSO, and present MCMC convergence diagnosis.

Keywords: LASSO, Bayes, MCMC, log-concave, geometry, incomplete Gamma function

1. Introduction
Let $p \geq n$ be two positive integers, $y \in \mathbb{R}^n$ and $A$ be an $n \times p$ matrix with real numbers entries. Bayesian LASSO

$$c(x) = \frac{1}{Z} \exp \left( - \frac{\|Ax - y\|_2^2}{2} - \|x\|_1 \right)$$

is a typically posterior distribution used in the linear regression

$$y = Ax + w.$$

Here

$$Z = \int_{\mathbb{R}^p} \exp \left( - \frac{\|Ax - y\|_2^2}{2} - \|x\|_1 \right) dx$$

is the partition function, $\| \cdot \|_2$ and $\| \cdot \|_1$ are respectively the Euclidean and the $l_1$ norms. The vector $y \in \mathbb{R}^n$ are the observations, $x \in \mathbb{R}^p$ is the
unknown signal to recover, $\mathbf{w} \in \mathbb{R}^n$ is the standard Gaussian noise, and $\mathbf{A}$ is a known matrix which maps the signal domain $\mathbb{R}^p$ into the observation domain $\mathbb{R}^n$. If we suppose that $\mathbf{x}$ is drawn from Laplace distribution i.e. the distribution proportional to

$$\exp(-\|\mathbf{x}\|_1),$$

then the posterior of $\mathbf{x}$ known $\mathbf{y}$ is drawn from the distribution $c$. The mode

$$\arg\min \left\{ \frac{\|\mathbf{A}\mathbf{x} - \mathbf{y}\|^2}{2} + \|\mathbf{x}\|_1 : \mathbf{x} \in \mathbb{R}^p \right\}$$

of $c$ was first introduced in [18] and called LASSO. It is also called Basis Pursuit De-Noising method [4]. In our work we select the term LASSO and keep it for the rest of the article.

In general LASSO is not a singleton, i.e. the mode of the distribution $c$ is not unique. In this case LASSO is a set and we will denote by lasso any element of this set. A large number of theoretical results has been provided for LASSO. See [5], [6], [9], [12], [15] and the references herein. The most popular algorithms to find LASSO are LARS algorithm [8], ISTA and FISTA algorithms see e.g. [2] and the review article [14].

The aim of this work is to study geometry of bayesian LASSO and to derive MCMC convergence diagnosis.

2. Polar integration

Using polar coordinates $\mathbf{s} = \frac{\mathbf{x}}{\|\mathbf{x}\|} \in S, \mathbf{r} = \|\mathbf{x}\|$, the partition function (2)

$$Z_p = \int_S Z_p(\mathbf{s})d\mathbf{s},$$

where $\|\cdot\|$ denotes one of $l^2$ or $l^1$ norms in $\mathbb{R}^p$, $d\mathbf{s}$ denotes the surface measure on the unit sphere $S$ of the norm $\|\cdot\|$, and

$$Z_p(\mathbf{s}) = \int_0^{+\infty} \exp\{-f(r\mathbf{s})\}r^{p-1}dr,$$

where $f(\mathbf{x}) := \frac{\|\mathbf{A}\mathbf{x} - \mathbf{y}\|^2}{2} + \|\mathbf{x}\|_1, \mathbf{x} \in \mathbb{R}^p$.

We express the partition function (6) using the parabolic cylinder function. We also give an inequality of concentration and a geometric interpretation of the partition function $Z_p$. 

2
3. Parabolic cylinder function and partition function

We extend the function $s \in S \rightarrow Z_p(s)$

$$x \in \mathbb{R}^p \rightarrow Z_p(x) = \int_0^{+\infty} \exp\{-f(rx)\}r^{p-1}dr. \quad (7)$$

This extension is homogeneous of order $-p$.

If $Ax = 0$, then $f(rx) = \frac{\|y\|^2}{2} + r\|x\|_1$, and more if $x \neq 0$, then

$$Z_p(x) = (p-1)!\|x\|_1^{-p}\exp\left(-\frac{\|y\|^2}{2}\right).$$

If $Ax \neq 0$, then we will express $Z_p(x)$ using the parabolic cylinder function. We recall that for $b \in \mathbb{R}, z \geq 0$ the parabolic cylinder function is given by

$$D_b(z) = z^b \exp\left(-\frac{z^2}{4}\right)[1 + O(z^{-2})], \quad (8)$$

when $z \rightarrow +\infty$. We also recall the integral representation of Erdlyi [7] for the parabolic cylinder function

$$\exp\left(\frac{z^2}{4}\right)\Gamma(\nu)D_{-\nu}(z) = \int_0^{+\infty} \exp\left(-\frac{1}{2}r^2 - zr\right)r^{\nu-1}dr, \quad \nu > 0,$$

where $\Gamma(\nu) = \int_0^{+\infty} \exp(-t)t^{\nu-1}dt$ is the $\Gamma$ function.

**Proposition 3.1.** The variable

$$\omega_{\text{lasso}} := \frac{\|x\|_1 - (Ax, y)}{\|Ax\|_2} \quad (9)$$

will play an important role. It depends only on $s = \frac{x}{\|x\|_1} \in S_{p-1,1}$ and the function $s \in S_{p-1,1} \rightarrow \omega_{\text{lasso}}(s)$ is bounded below by $\lambda_{1,2} := \min\{\frac{1}{\|As\|_2} - \frac{(As, y)}{\|As\|_2} : s \in S_{p-1,1}\}$.

Now we can announce the following result.

**Proposition 3.2.** We have for $Ax \neq 0$

$$Z_p(x) = (p-1)!\exp\left(-\frac{\|y\|^2}{2}\right)\|Ax\|_2^{-p}\exp\left(\frac{\omega_{\text{lasso}}^2}{4}\right)D_{-p}(\omega_{\text{lasso}}).$$

If $Ax \rightarrow 0$, then $\omega_{\text{lasso}} \rightarrow +\infty$ and

$$Z_p(x) = (p-1)!\exp\left(-\frac{\|y\|^2}{2}\right)\|x\|_1^{-p}[1 + O(\omega_{\text{lasso}}^{-2})].$$
Corollary 3.3. If $y = 0$ then $\omega_{\text{lasso}} = \frac{1}{\|A_s\|_2}$ is bounded below by $\frac{1}{\lambda_{1,2}}$, where $\lambda_{1,2} = \max(\|A_s\|_2 : s \in S_{p-1,1})$ is the norm of the operator $A : (\mathbb{R}^p, \|\cdot\|_1) \to (\mathbb{R}^n, \|\cdot\|_2)$. The partition function

$$Z_p(s) = (p-1)!\omega_{\text{lasso}}^p \exp(\frac{\omega_{\text{lasso}}^2}{4})D_p(\omega_{\text{lasso}})$$

is $\|A_s\|_2^2$ convex and decreasing.

Proof 3.4. It suffices to remark that $Z_p(s) = \int_{0}^{+\infty} \exp\{-\frac{\|A_s\|_2^2 r^2}{2} - r\} r^{p-1} dr$.

4. Geometric interpretation of the partition function

First we represent $f(rx)$ for $Ax \neq 0$ in the form

$$f(rx) = \frac{\|y\|_2^2}{2} - \frac{\omega_{\text{lasso}}^2}{2} + \frac{r\|Ax\|_2 + \omega_{\text{lasso}}^2}{2}.$$ (10)

The function $\exp\{-f(x)\}, \forall x \in \mathbb{R}^p$ is log-concav and integrable in $\mathbb{R}^p$. Observe that $Z_p^{-\frac{1}{p}}$ is a norm on the null space $\ker(A)$ of $A$. A general result [1] tells us that $x \in \mathbb{R}^p \rightarrow Z_p^{-\frac{1}{p}}(x) := \|x\|_c$ is a quasi-norm on $\mathbb{R}^p$. The unit ball of $\|\cdot\|_c$ is defined by

$$\mathcal{B}(A, y) := \{x \in \mathbb{R}^p : \|x\|_c \leq 1\}$$

$$= \{x \in \mathbb{R}^p : Z_p(x) \geq 1\}$$

$$= \{x = rs \in \mathbb{R}^p : Z_p(s) \geq r^p\}$$

$$= \{x = rs \in \mathbb{R}^p : (p-1)! \exp(-\frac{\|y\|_2^2}{2})\|A_s\|_2^{-p} \exp(\frac{\omega_{\text{lasso}}^2}{4})D_p(\omega_{\text{lasso}}) \geq r^p\}.$$  

Its contour is equal to

$$\mathcal{C}(A, y) := \{x \in \mathbb{R}^p : \|x\|_c = 1\}$$

$$= \{x \in \mathbb{R}^p : Z_p(x) = 1\}$$

$$= \{x = rs \in \mathbb{R}^p : Z_p(s) = r^p\}$$

$$= \{x = rs \in \mathbb{R}^p : (p-1)! \exp(-\frac{\|y\|_2^2}{2})\|A_s\|_2^{-p} \exp(\frac{\omega_{\text{lasso}}^2}{4})D_p(\omega_{\text{lasso}}) = r^p\}.$$  

We summarize our results in the following proposition.
Proposition 4.1. 1) For each $s \in S_{p-1,1}$, the longest segment $[0,r]s$ contained in $B(A, y)$ holds for $r = r_{\text{max}}(s)$ is solution of the equation

$$r^p = (p-1)! \exp\left(-\frac{\|y\|^2}{2}\right)\|A_s\|^{-p}_2 \exp\left(\frac{\omega_{\text{lasso}}^2}{4}\right)D_{-p}(\omega_{\text{lasso}}).$$

2) The ball

$$B(A, y) = \bigcup_{s \in S_{p-1,1}} [0, r_{\text{max}}(s)]s,$$

and its contour is equal to

$$C(A, y) = \{r_{\text{max}}(s) : s \in S_{p-1,1}\}.$$

3) The volume $B(A, y)$ is

$$\int_{S_{p-1,1}} \frac{r_{\text{max}}^p(s)}{p} ds = \frac{Z_p}{p}.$$

5. Necessary and sufficient condition to have lasso equal zero

Now we can give the necessary and sufficient condition to have $\text{lasso} = \{0\}$

Proposition 5.1. The following assertions are equivalent.

1) $0 = \text{lasso}$.

2) $\omega_{\text{lasso}}(s) \geq 0$ pour tout $s \in S_{p-1,1}$.

3) $\|A^\top y\|_{\infty} \leq 1$.

6. Concentration around the lasso

6.1. The case lasso null

The polar coordinate formula tells us that, we can draw a vector $x$ from $c(x)dx$ by drawing its angle $s$ uniformly, and then simulate its distance $r$ to the origin from

$$c(r, s)dr = \frac{1}{Z_p(s)} \exp\{-f(rs)\}r^{p-1}dr \quad (11)$$
Now let’s estimate for $r > 0$ the probability
\[
P(\|x\| > r) = \int_S \int_r^{+\infty} c(r, s) dr \frac{ds}{|S|},
\]
where $|S|$ denotes the Lebesgue measure of $S$. We introduce for each pair $a \geq 0, b \in \mathbb{R}^p$ the function
\[
g_{a,b,p}(r) := g_{a,b}(r) - (p - 1) \ln(r), \quad r > 0.
\]
(12)

In the following $a = \|A_s\|_2, \quad b = \omega_{lasso}.$

The function $r \geq 0 \rightarrow g_{a,b}(r)$ is increasing (because $b := \omega_{lasso} \geq 0$). The function $r \rightarrow g_{a,b,p}(r)$ is convex and attains its minimum at the point $r(s)$ solution de l’équation
\[
\|A_s\|_2(r\|A_s\|_2 + \omega_{lasso}) = \frac{p - 1}{r}.
\]
The positive root is given by
\[
r(s) = -\omega_{lasso} + \sqrt{\omega_{lasso}^2 + 4(p - 1)r}\frac{2\|A_s\|_2}{p}.
\]
(13)

On one hand
\[
\int_0^{+\infty} \exp\{-g_{a,b,p}(r)\} dr \geq \exp\{-g_{a,b}(r(s))\} \int_0^{r(s)} r^{p-1} dr = \exp\{-g_{a,b,p}(r(s))\} \frac{r(s)}{p}.
\]

On the other hand by using the convexity of $r \rightarrow g_{a,b}(r)$, we have for all $r > 0$,
\[
g_{a,b}(r) \geq g_{a,b}(r(s)) + \frac{(p - 1)(r - r(s))}{r(s)},
\]
because $\partial_r g_{a,b}(r(s)) = \frac{g_{a,b}(r)}{r(s)}$. We deduce for $q > 0$,
\[
\int_{q(r(s))}^{+\infty} \exp\{-g_{a,b,p}(r)\} dr \leq \exp\{-g_{a,b}(r) + p - 1\} \int_{q(r(s))}^{+\infty} \exp\{-\frac{p - 1}{r(s)}r\} r^{p-1} dr
\]
\[
\leq \exp\{-g_{a,b}(r(s)) + p - 1\} \int_{q(p-1)}^{+\infty} \exp(-r) r^{p-1} dr \frac{r(s)}{(p - 1)^p}
\]
\[
\leq \exp\{-g_{a,b}(r(s)) + p - 1\} \frac{r(s)}{(p - 1)^p} \Gamma(p, q(p - 1)),
\]
where $\Gamma(\nu, x) = \int_x^{+\infty} \exp(-t)t^{\nu-1}dt$ is the upper incomplete gamma function. Finally we get the following result.

**Proposition 6.1.** We have for all $q > 0$,

$$P(||x|| \geq qr(s)) \leq \frac{p \exp(p-1)}{(p-1)p} \Gamma(p, q(p-1)) := P(q, p). \quad (14)$$

Using the following estimate [? ]

$$x^{a-1} \exp(-x) < \Gamma(a, x) < Bx^{a-1} \exp(-x), \quad \forall a > 1, \ B > 1, \ x > \frac{B}{B-1}(a-1),$$

we get for $q > 1$,

$$\Gamma(p, q(p-1)) \leq 2q^{p-1}(p-1)^{p-1} \exp(-q(p-1)).$$

Therefore the quantity

$$P(q, p) \leq \frac{2pq^{p-1}}{(p-1)} \exp\{-(q-1)(p-1)\}.$$ 

**balance sheet**. If $x$ is drawn from the density $c$, alors $\frac{x}{r(\theta)} \in B_2(0, q)$ with a probability at least equal to $1 - P(q, p)$.

In the figure(1) we plot for $p = 2, \ n = 1, \ A = (1 \ 1)$ and $y = 0$ the density $c(r, s)dr$ for a fixed value of $s$.

We notice that the mode $c(r, s) = 0.6200$ is very close to the value of $r( s) = 0.6290 \ (13)$ for the same fixed $s$.

6.2. The general case

We take the vector $l \in \text{lasso}$. We will study the concentration of $c$ around $l$. The variable of interest is $u = x - l$. The law of $u$ has for density

$$c(u + l)du = \frac{1}{Z_p} \exp\{-f(u + l, A, y)\} du.$$ 

The change of variables formula gives for each norm $|| \cdot ||$

$$c(u + l)du = \frac{1}{Z_p} \exp\{-f(r\theta + l)\} r^{p-1} dr d\theta, \quad r > 0, \ \theta \in S.$$
By definition for any vector $x$, the convex function $r \geq 0 \rightarrow f(rs + l)$ reaches its minimum at the point $r = 0$. Therefore $r \geq 0 \rightarrow f(rs + l)$ is increasing.

The function

$$f(rs + l, p) := f(rs + l) - (p - 1) \ln(r), \quad r > 0,$$

is strictly convex. Its critical point $r_1(s)$ is solution of the equation

$$\partial_r f(rs + l) = \frac{p - 1}{r}.$$  

By a similar proof to that of proposition (6.1) we have the following result;

**Proposition 6.2.** If $x$ is drawn from the density $c$, and $s = \frac{x - l}{\|x - l\|}$, then for all $q > 0$,

$$P(\|x - l\| \geq q r_1(s)) \leq \frac{p \exp(p - 1)}{(p - 1)^p} \Gamma(p, q(p - 1)) := P(q, p).$$

7. **Applications**

7.1. *The contour in the case $p = 2$, $n = 1***

Let $A := (a_1, a_2)$ a matrix of order $1 \times 2$. Its null-space $\text{Ker}(A) = \{(x_1, x_2) : a_1x_1 + a_2x_2 = 0\}$. We have that $B(a_1, a_2, y)$ contains

$$\text{Ker}(A) \cap B_{2,1}.$$
This intersection is a symmetric segment noted \([ (x_1(a_1, a_2), x_2(a_1, a_2)), -(x_1(a_1, a_2), x_2(a_1, a_2)) ] \).

To determine the other points of the set \( \mathcal{B}(a_1, a_2, y) \), we will directly calculate \( Z_2(s) \). A simple calculation gives

\[
Z_2(s) = \exp\left( -\frac{y^2}{2} + \frac{\omega^2_{lasso}}{2} \right) \int_0^{+\infty} \exp\left\{-\frac{(|A_s|r + \omega_{lasso})^2}{2}\right\} r dr,
\]

and

\[
|A_s| \int_0^{+\infty} \exp\left\{-\frac{(|A_s|r + \omega_{lasso})^2}{2}\right\} r dr + \omega_{lasso} \int_0^{+\infty} \exp\left\{-\frac{(|A_s|r + \omega_{lasso})^2}{2}\right\} dr = 1.
\]

Finally we have the following proposition.

**Proposition 7.1.** 1) If \( A_s \neq 0 \), then

\[
Z_2(s) = \exp\left( -\frac{y^2}{2} + \frac{\omega^2_{lasso}}{2} \right)|A_s|^{-1}\left\{1 - \frac{\omega_{lasso}}{|A_s|} \sqrt{2\pi}(1 - F(\omega_{lasso}))\right\},
\]

where \( F \) is the distribution function of the normal law.

2) If \( A_s \neq 0 \) and \( y = 0 \), then

\[
Z_2(s) = \omega_{lasso} \exp\left( \frac{\omega^2_{lasso}}{2} \right)\left\{1 - \frac{\omega^2_{lasso}}{|A_s|} \sqrt{2\pi}(1 - F(\omega_{lasso}))\right\}.
\]

3) If \( s \in S_{1,1} \), \( A_s \neq 0 \) and \( y = 0 \), then the function \( z_2 \)

\[
z_2(b^2) = \frac{1}{b} \exp\left( \frac{1}{2b^2} \right)\left\{1 - \frac{1}{b^2} \sqrt{2\pi}(1 - F(\frac{1}{b}))\right\}
\]

defined on \((0, \lambda_{1,2}^2)\) is convex and decreasing, where \( \lambda_{1,2} = \max_{s \in S_{1,1}} |A_s| \).

4) We have for \( s \in S_{1,1} \)

\[
Z_2(s) = z_2\left( \frac{1}{\omega^2_{lasso}} \right), \quad \forall \omega \in S_{1,1}.
\]

the ball

\[
\mathcal{B}(A, 0) = \{ rs : \quad Z_2(s) \geq r^2 \}.
\]

is contained in the unit disk \( \|x\|_1 \leq 1 \) for the norm \( l^1 \). The contour is defined by the equation

\[
Z_2(s) = r^2.
\]
The norm of the linear operator $A : (\mathbb{R}^2, \| \cdot \|_1) \to (\mathbb{R}, \| \cdot \|_2)$ is defined by

$$\lambda_{1,2} = \sup_{s : \| s \|_1 = 1} \| As \|_2.$$ 

the function $s \to Z_2(s) = z_2(\lambda^2_{1,2})$ is constant on est constante sur

$$\Omega_{1,2} = \{ s : \| s \|_1 = 1, \| As \| = \lambda_{1,2} \}.$$ 

If $A = (1, 1)$ then

$$\Omega_{1,2} = \{ s : \| s \|_1 = 1, \| As \| = \lambda_{1,2} \} = [(1, 0), (0, 1)] \cup [(-1, 0), (0, -1)].$$

If $A = (a_1, a_2)$ with $|a_1| < |a_2|$, then

$$\Omega_{1,2} = \{ s : \| s \|_1 = 1, \| As \|_2 = \lambda_{1,2} \} = \{(0, \text{sgn}(a_2)), (0, -\text{sgn}(a_2))\}.$$ 

In both case

$$\{ z_2(\lambda^2_{1,2}) \}^{\frac{1}{2}} \Omega_{1,2}$$

is part of the contour. The other points of the contour are deduced from the equation

$$z_2(b^2) = a^2, \quad b \in (0, \lambda_{1,2}).$$

Each pair $(a, b)$ generate four points of $B((a_1, a_2), 0)$ of the form $as$ where

$$|s_1| + |s_2| = 1, \quad |a_1s_1 + a_2s_2| = b.$$ 

We plot in the figure 2 the contour of $B(a_1, a_2, 0)$ for different choices of the matrix $(a_1, a_2)$. We notice that the surface of $B((a_1, a_2), 0)$ is decreasing function relatively the norm $\lambda_{1,2}$ of the matrix $A$.

**Remark 7.2.** The numerics show that $Z(\omega_{\text{lasso}})$ explodes for the large values of $\omega_{\text{lasso}}$, it means that $\omega$ is closes to the null-space of $A$. to eleminate that explosion we need to estimate the tail of the gaussian density. Using the Gordon estimation [10]

$$\frac{\exp(-\frac{x^2}{2})}{x + \frac{1}{x}} \leq \sqrt{2\pi(1 - F(x))} \leq \frac{\exp(-\frac{x^2}{2})}{x}, \quad x > 0,$$

we have the following approximation

$$\frac{1}{b^2} - \frac{1}{b} \leq z_2(b^2) \leq \frac{1}{b^2} - \frac{1}{1 + b^2}, \quad \text{near to } 0.$$ (17)
8. MCMC diagnosis

Here we take $p = 7$, $n = 4$, $A \sim \mathcal{B}(\pm \frac{1}{\sqrt{n}})$ and for simplicity we consider $y = 0$. We sample from the distribution $c$ using Hastings-Metropolis algorithm ($x^{(t)}$) and propose the test $\|x^{(t)}\|_2 \leq qr(\theta^{(t)})$ as a criterion for the convergence. Here $\theta^{(t)} := \frac{x^{(t)}}{\|x^{(t)}\|_2}$. We recall that if $x$ is drawn from the target distribution $c$, then $\|x\|_2 \leq qr(\theta)$ with the probability at least equal to $P(q, p)$. Table 2 gives the values of the probability $P(q, p)$. Note that for $q \geq 2.5$ the criterion $\|x^{(t)}\|_2 \leq qr(\theta^{(t)})$ is satisfied with a large probability.

| $q$  | 2    | 2.5  | 3    | 3.5  | 4    | 4.5  | 5    |
|------|------|------|------|------|------|------|------|
| $P(q, p)$ | 0.6672 | 0.9446 | 0.9924 | 0.9991 | 0.9999 | 1.0000 | 1.0000 |

Table 1: Values of the probability $P(q, p)$ for $p = 7$.

8.1. Independent sampler (IS)

The proposal distribution

$$Q(x_2, x_1) = p(x_2) = \frac{1}{2^p} \exp(-\|x_2\|_1), \quad \forall x_1, x_2.$$
The ratio
\[ \frac{c(x)}{p(x)} \leq \frac{2p}{Z}, \quad \forall x. \]

It's known that MCMC \((x^{(t)})\) with the target distribution \(c\) and the proposal distribution \(p\) is uniformly ergodic \([13]\):

\[
\sup_{A \subset B(R^p)} \|P(x^{(t)} \in A | x^{(0)}) - \int_A c(x)dx\| \leq (1 - \frac{Z}{2^p})^t.
\]

Here \(Z \approx 2.2142\) and then \((1 - \frac{Z}{2^p}) = 0.9827\). Figure 4(a) shows respectively the plot of \(t \rightarrow 5r(\theta^{(t)})\) and \(t \rightarrow \|x^{(t)}\|_2\).

8.2. Random-walk (RW) Metropolis algorithm

We do not know if the target distribution \(c\) satisfies the curvature condition in \([17]\) Section 6. Here we propose to analyse the convergence of the Random walk Metropolis algorithm \((x^{(t)})\) using the criterion \(\|x^{(t)}\|_2 \leq qr(\theta^{(t)})\). Figure 4(b) shows respectively the plot of \(t \rightarrow 5r(\theta^{(t)})\) and \(t \rightarrow \|x^{(t)}\|_2\).

Figures 4 show that contrary to independent sampler algorithm, the random walk (RW) algorithm satisfies early the criterion \(\|x^{(t)}\|_2 \leq 5r(\theta)\). More precisely

1) the independent sampler (IS) algorithm begins to satisfy the criterion \(\|x^{(t)}\|_2 \leq 5r(\theta^{(t)})\) at \(t = 8 \times 10^5\) iteration.

2) The RW algorithm begins to satisfy the criterion \(\|x^{(t)}\|_2 \leq 3.5r(\theta^{(t)})\) at \(t = 939065\) iteration, but the IS algorithm never satisfies the criterion \(\|x^{(t)}\|_2 \leq 3.5r(\theta^{(t)})\).

We finally compare IS and RW algorithms using the fact that \(\int_{R^p} xc(x)dx = 0\). The best algorithm will furnish the best approximation of the integral \(\int_{R^p} xc(x)dx\). Table 3 gives the estimators \(\frac{1}{N} \sum_{t=1}^N x^{(t)}_{IS} \approx \int_{R^p} xc(x)dx\) and \(\frac{1}{N} \sum_{t=1}^N x^{(t)}_{RW} \approx \int_{R^p} xc(x)dx\). It follows that \(\|\frac{1}{N} \sum_{t=1}^N x^{(t)}_{IS}\|_2 = 0.0187\) and \(\|\frac{1}{N} \sum_{t=1}^N x^{(t)}_{RW}\|_2 = 0.0041\). We conclude that the random walk algorithm wins for both criteria against independent sampler algorithm.
Table 2: \( IS \) and \( RW \) estimators using \( N = 10^6 \) iterations.

| \( x_{IS} \) | \( x_1 \) | \( x_2 \) | \( x_3 \) | \( x_4 \) | \( x_5 \) | \( x_6 \) | \( x_7 \) |
|---|---|---|---|---|---|---|---|
| \( x_{RW} \) | 0.0005 | -0.0019 | -0.0002 | 0.0012 | -0.0005 | 0.0031 | -0.0011 |

Figure 3: (a): Test of convergence of MCMC algorithm with proposal distribution \( p(x_2) \).
(b): Test of convergence of MCMC algorithm with \( N(0, 0.5I_p) \) proposal distribution. \( N = 10^6 \) iterations.

9. Conclusion

We studied the geometry of bayesian LASSO using polar coordinates and calculated the partition function. We obtained a concentration inequality and derived MCMC convergence diagnosis for the convergence of Hasting Metropolis algorithm. We showed that the random walk MCMC with the variance 0.5 wins again the independent sampler with Laplace proposal distribution.
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