Weak super-additivity of relative entropy and weak sub-multiplicativity of fidelity: A numerical study

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Abstract

A conjecture – weak super-additivity inequality of relative entropy – was proposed in [1]:

There exist three unitary operators $U_A \in U(H_A), U_B \in U(H_B),$ and $U_{AB} \in U(H_A \otimes H_B)$ such that

$$S(U_{AB}\rho_{AB}U_{AB}^\dagger ||\sigma_{AB}) \geq S(U_A\rho_AU_A^\dagger ||\sigma_A) + S(U_B\rho_BU_B^\dagger ||\sigma_B),$$

where the reference state $\sigma$ is required to be full-ranked. A numerical study on the conjectured inequality is conducted in this notes. The results obtained indicates that weak super-additivity inequality of relative entropy seems to hold for all qubit pairs. Similarly, the weak sub-multiplicativity of fidelity is also considered. That is, there exist three unitary operators $U_A \in U(H_A), U_B \in U(H_B),$ and $U_{AB} \in U(H_A \otimes H_B)$ such that

$$F(U_{AB}\rho_{AB}U_{AB}^\dagger,\sigma_{AB}) \leq F(U_A\rho_AU_A^\dagger,\sigma_A)F(U_B\rho_BU_B^\dagger,\sigma_B).$$

An attempt is made to give some potential applications in quantum information theory.

1 Introduction

Rau derived in [2] a monotonicity property of relative entropy and obtained the super-additivity inequality – a much stronger monotonicity – of relative entropy:

$$S(\rho_{AB}||\sigma_{AB}) \geq S(\rho_A||\sigma_A) + S(\rho_B||\sigma_B),$$

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where $\rho_{AB}$ and $\sigma_{AB}$ are bipartite states on $\mathcal{H}_A \otimes \mathcal{H}_B$. A simple counterexample [1] was provided to show that the above inequality is not correct. Moreover, it is conjectured that there exist three unitary operators $U_A \in \mathbb{U}(\mathcal{H}_A)$, $U_B \in \mathbb{U}(\mathcal{H}_B)$, and $U_{AB} \in \mathbb{U}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that

$$S(U_{AB}\rho_{AB}U_{AB}^\dagger||\sigma_{AB}) \geq S(U_A\rho_AU_A^\dagger||\sigma_A) + S(U_B\rho_BU_B^\dagger||\sigma_B),$$

where the reference state $\sigma$ is required to be full-ranked.

A numerical study on the conjectured inequality is conducted in this notes. The results obtained indicates that weak super-additivity inequality of relative entropy seems to hold for all qubit pairs. An attempt is made to give some potential applications in quantum information theory.

Before proceeding, we need to fix some notations. If the column vectors

$$p = [p_1, \ldots, p_d]^T \in \mathbb{R}^d, \quad q = [q_1, \ldots, q_d]^T \in \mathbb{R}^d$$

are two probability distributions, the **Shannon entropy** of $p$ is defined by

$$H(p) \overset{\text{def}}{=} -\sum_{i=1}^d p_i \log_2 p_i,$$

where $x \log_2 x := 0$ if $x = 0$, and the **relative entropy** of $p$ and $q$ is defined by

$$H(p||q) \overset{\text{def}}{=} \sum_{i=1}^d p_i (\log_2 p_i - \log_2 q_i).$$

Let $D(\mathcal{H}_d)$ denote the set of all the density matrices $\rho$ on a $d$-dimensional Hilbert space $\mathcal{H}$. The **von Neumann entropy** $S(\rho)$ of $\rho$ is defined by

$$S(\rho) \overset{\text{def}}{=} -\text{Tr} (\rho \log \rho).$$

In fact, this definition can be equivalently described as follows: if we denote the vector consisting of eigenvalues of $\rho$ by $\lambda(\rho) = [\lambda_1(\rho), \ldots, \lambda_d(\rho)]^T$, then we have

$$S(\rho) = H(\lambda(\rho)) = H(\lambda^\uparrow(\rho)),$$

where we write $\lambda^\uparrow(\rho)$ for a vector with components being the same as $\lambda(\rho)$ and arranged in non-increasing order, i.e.

$$\lambda^\uparrow(\rho) = [\lambda_1^\uparrow(\rho), \ldots, \lambda_d^\uparrow(\rho)]^T \quad (\lambda_1^\uparrow(\rho) \geq \cdots \geq \lambda_d^\uparrow(\rho)).$$

However, $\lambda^\downarrow(\rho)$ stands for the vector with eigenvalues of $\rho$ arranged in increasing order. The **relative entropy** of two mixed states $\rho$ and $\sigma$ is defined by

$$S(\rho||\sigma) \overset{\text{def}}{=} \begin{cases} \text{Tr} (\rho (\log \rho - \log \sigma)), & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma), \\ +\infty, & \text{otherwise}. \end{cases}$$
2 Technical lemmas

The so-called quantum marginal problem, i.e. the existence of mixed states $\rho_{AB}$ two (or multi-) component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with reduced density matrices $\rho_A, \rho_B$ and given spectra $\lambda_{AB}, \lambda_A, \lambda_B$, is discussed in the literature, and a complete solution of this problem in terms of linear inequalities on the spectra is given in the following proposition.

**Proposition 2.1** (Klyachko, [3]). Assume that there is a bipartite system $AB$, described by Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. All constraints on spectra $\lambda^X (\rho_X) = \lambda^X (X = A, B, AB)$, arranged in non-increasing order, are given by the following linear inequalities:

$$\sum_{i=1}^m a_i \lambda^A_{\lambda(i)} + \sum_{j=1}^n b_j \lambda^B_{\beta(j)} \leq \sum_{k=1}^{mn} (a + b)^{\lambda AB}_{\gamma(k)},$$

where $a : a_1 \geq a_2 \geq \cdots \geq a_m, b : b_1 \geq b_2 \geq \cdots \geq b_n$ with $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j = 0$ are “test spectra”, the spectrum $(a + b)^{\lambda AB}$ consists of numbers $a_i + b_j$ arranged in non-increasing order, and $\alpha \in S_m, \beta \in S_n, \gamma \in S_{mn}$ are permutations subject to a topological condition $c^{\gamma}_{\lambda AB} (a, b) \neq 0$, where the meaning of $c^{\gamma}_{\lambda AB}$ can be found in [3].

In particular, for the simplest quantum multipartite system, i.e. two-qubit system, there is a nice solution for the quantum marginal problem:

**Proposition 2.2** (Bravyi, [4]). Mixed two-qubit state $\rho_{AB}$ with spectrum $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0$ and margins $\rho_A, \rho_B$ exists if and only if minimal eigenvalues $\lambda_A, \lambda_B$ of the margins satisfy inequalities

$$\begin{cases} \min (\lambda_A, \lambda_B) \geq \lambda_3 + \lambda_4, \\ \lambda_A + \lambda_B \geq \lambda_2 + \lambda_3 + 2\lambda_4, \\ |\lambda_A - \lambda_B| \leq \min (\lambda_1 - \lambda_3, \lambda_2 - \lambda_4). \end{cases}$$

The following result maybe gives a possibility to corrected version of superadditivity inequality. Here we give another proof in terms of matrix analysis language.

**Proposition 2.3** (Zhang, [5]). For given two quantum states $\rho, \sigma \in \mathcal{D} (\mathcal{H}_d)$, where $\sigma$ is invertible, it holds that

$$\min_{U \in U (\mathcal{H}_d)} S (U \rho U^\dagger || \sigma) = H (\lambda^{\dagger} (\rho) || \lambda^{\dagger} (\sigma)),$$

$$\max_{U \in U (\mathcal{H}_d)} S (U \rho U^\dagger || \sigma) = H (\lambda^{\dagger} (\rho) || \lambda^{\dagger} (\sigma)),$$

where $\lambda^{\dagger} (\sigma)$ stands for the eigenvalues arranged in increasing order. $U (\mathcal{H}_d)$ denotes the set of all unitary operators on $\mathcal{H}_d$. Moreover, the set $\{ S (U \rho U^\dagger || \sigma) : U \in U (\mathcal{H}_d) \}$ is identical to an interval:

$$\left[ H (\lambda^{\dagger} (\rho) || \lambda^{\dagger} (\sigma)), H (\lambda^{\dagger} (\rho) || \lambda^{\dagger} (\sigma)) \right].$$
Proof. Apparently, the unitary orbit $U_\rho$ of $\rho$ is a compact set. Moreover every differentiable curve through $\rho$ can be represented locally as $\exp(tK)\rho\exp(-tK)$ for some skew-Hermitian $K$, i.e. $K^+ = -K$. The derivative of this curve at $t = 0$ is $[K, \rho] := K\rho - \rho K [6]$.

Let $f(U) := S(U\rho U^+ || \sigma)$ be defined over the unitary group $U(\mathcal{H}_d)$. Clearly

$$f(U) = -S(\rho) - \text{Tr} \left( U\rho U^+ \log \sigma \right).$$

Since the unitary group $U(\mathcal{H}_d)$ is a path-connected and compact space [7], it suffices to show that $f(U)$ is a continuous function.

Let $U_t = \exp(tK)$ for an arbitrary skew-Hermitian $K$. Thus

$$\frac{df(U_t)}{dt} = \text{Tr} \left( U_t\rho U_t^+[K,\log\sigma] \right), \quad (2.5)$$

implying

$$\frac{df(U_t)}{dt}|_{t=0} = \text{Tr} \left( \rho[K,\log\sigma] \right),$$

which means that $f(U)$ is continuously over $U(\mathcal{H}_d)$.

Without loss of generality, we assume that $U_0 \in U(\mathcal{H}_d)$ is the extreme point of $f$. Consider an arbitrary differentiable path $\{\exp(tK)U_0\}$ through $U_0$ in $U(\mathcal{H}_d)$ for arbitrary skew-Hermitian $K$, it follows that

$$\frac{df(\exp(tK)U_0)}{dt}|_{t=0} = \text{Tr} \left( U_0\rho U_0^+[K,\log\sigma] \right)$$

$$= \text{Tr} \left( K[\log\sigma, U_0\rho U_0^+] \right)$$

$$= 0.$$

Therefore $[\log\sigma, U_0\rho U_0^+] = 0$ follows from the arbitrariness of $K$. That is $[\sigma, U_0\rho U_0^+] = 0$. By the rearrangement inequality in mathematics, the desired conclusion is obtained.

In fact, partial results in the above proposition has already been reported in [1]. It was employed to study a modified version of super-additivity inequality of relative entropy [1].

The above theorem also gives rise to the following inequality:

$$H(\lambda^+(\rho)||\lambda^+(\sigma)) \leq S(\rho||\sigma) \leq H(\lambda^+(\rho)||\lambda^+(\sigma)). \quad (2.6)$$

If we denote $\Delta S = S(\rho_{AB}||\sigma_{AB}) - S(\rho_A||\sigma_A) - S(\rho_B||\sigma_B)$, then we have the following inequality:

$$\bar{\Delta} \leq \Delta S \leq \Delta, \quad (2.7)$$
where
\[
\Delta \overset{\text{def}}{=} H(\lambda^\dagger(\rho_{AB})||\lambda^\dagger(\sigma_{AB})) - H(\lambda^\dagger(\rho_A)||\lambda^\dagger(\sigma_A)) - H(\lambda^\dagger(\rho_B)||\lambda^\dagger(\sigma_B)).
\] (2.8)

In order to study the sign of \(\Delta S\), we propose now to study the following four differences:

\[
\begin{align*}
\Delta_{\text{min}} & \overset{\text{def}}{=} H(\lambda^\dagger(\rho_{AB})||\lambda^\dagger(\sigma_{AB})) - H(\lambda^\dagger(\rho_A)||\lambda^\dagger(\sigma_A)) - H(\lambda^\dagger(\rho_B)||\lambda^\dagger(\sigma_B)), \tag{2.9} \\
\Delta_{\text{max}} & \overset{\text{def}}{=} H(\lambda^\dagger(\rho_{AB})||\lambda^\dagger(\sigma_{AB})) - H(\lambda^\dagger(\rho_A)||\lambda^\dagger(\sigma_A)) - H(\lambda^\dagger(\rho_B)||\lambda^\dagger(\sigma_B)), \tag{2.10} \\
\Delta_{\text{mix}} & \overset{\text{def}}{=} H(\lambda^\dagger(\rho_{AB})||\lambda^\dagger(\sigma_{AB})) - H(\lambda^\dagger(\rho_A)||\lambda^\dagger(\sigma_A)) - H(\lambda^\dagger(\rho_B)||\lambda^\dagger(\sigma_B)), \tag{2.11} \\
\Delta & \overset{\text{def}}{=} H(\lambda^\dagger(\rho_{AB})||\lambda^\dagger(\sigma_{AB})) - H(\lambda^\dagger(\rho_A)||\lambda^\dagger(\sigma_A)) - H(\lambda^\dagger(\rho_B)||\lambda^\dagger(\sigma_B)). \tag{2.12}
\end{align*}
\]

An observation is made here:
\[
\bar{\Delta} \leq \Delta_{\text{min}}, \quad \bar{\Delta} \leq \Delta_{\text{max}} \leq \Delta_{\text{mix}} \leq \Delta.
\]

We can still know that the suitably chosen qubit pair \((\rho_{AB}, \sigma_{AB})\) such that \(\Delta S\) can take arbitrary values in the interval \([\bar{\Delta}, \Delta]\), which is guaranteed by Proposition 2.3.

In fact, by Proposition 2.3 if we can show that at least one of the above-mentioned four quantities is nonnegative, then our conjectured inequality is correct.

Analytical proof concerning the above inequalities are expected. Proving these seems to be very difficult. Thus we turn to another method – a numerical study in lower dimensions.

Consider a two-qubit pair \(\rho_{AB}, \sigma_{AB}\). Let \(\lambda^\dagger(\rho_{AB}) = [\lambda_1, \lambda_2, \lambda_3, \lambda_4]\) with \(\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq 0\) and \(\sum_j \lambda_j = 1\); \(\lambda^\dagger(\sigma_{AB}) = [\mu_1, \mu_2, \mu_3, \mu_4]\) with \(\mu_1 \geq \mu_2 \geq \mu_3 \geq \mu_4 > 0\) and \(\sum_j \mu_j = 1\). Then the eigenvalue vectors corresponding their reduced density matrices, i.e. margins, are \(\lambda^\dagger(\rho_X) = [1 - \lambda_X, \lambda_X]^{\top}\) with \(\lambda_X \in [0, \frac{1}{2}]\). Similarly, \(\lambda^\dagger(\sigma_X) = [1 - \mu_X, \mu_X]^{\top}\) with \(\mu_X \in (0, \frac{1}{2}]\). Note that \(X = A, B\) in the above formulations.

In what follows, we make a numerical study of each quantity defined by Eq. (2.9)–Eq. (2.12) under the constraints (2.2) for two-qubit pair \(\rho_{AB}\) and \(\sigma_{AB}\).

### 3 Numerical study

In this section, we investigate the numerical performance of the weaker superadditivity inequality of the relative entropy to verify the correctness of our conjecture. Our tests were conducted using MATLAB R2010b, and the random data was generated by the function "rand" in MATLAB.

We test two scenarios with respect to one thousand and one million groups of random data for each quantity defined by Eq. (2.9)–Eq. (2.12). The corresponding plots are listed in Fig. 1–Fig. 4. Obviously, from Fig. 1, we can see that the difference \(\Delta_{\text{min}}\) defined by Eq. (2.9) is less
than zero in many cases. Note that in Fig. 2, there is only one negative value of $\Delta_{\text{max}}$ for the one thousand scenario, and a very small number of points are located below the X-axis for the second scenario. However, from Fig. 3 and Fig. 4, it is clear that all the differences $\Delta_{\text{mix}}$ and $\Delta$, respectively, defined by Eq. (2.11) and Eq. (2.12) are greater than zero, which supports our conjecture.

Therefore analytical proofs for the following two inequalities are expected:

\begin{align}
H(\lambda^+(\rho_{AB})||\lambda^+(\sigma_{AB})) &\geq H(\lambda^+(\rho_A)||\lambda^+(\sigma_A)) + H(\lambda^+(\rho_B)||\lambda^+(\sigma_B)), \\
H(\lambda^+(\rho_{AB})||\lambda^+(\sigma_{AB})) &\geq H(\lambda^+(\rho_A)||\lambda^+(\sigma_A)) + H(\lambda^+(\rho_B)||\lambda^+(\sigma_B)).
\end{align}

(3.1) (3.2)

In fact, if Eq. (3.1) holds, then Eq. (3.2) a fortiori holds. Based on these numerical studies, we can make a bold conjecture:

**Conjecture 3.1.**

\[
S(U_A \otimes U_B \rho_{AB} U_A^\dagger \otimes U_B^\dagger || \sigma_{AB}) \geq S(U_A \rho_A U_A^\dagger || \sigma_A) + S(U_B \rho_B U_B^\dagger || \sigma_B)
\]

for some unitaries $U_X \in U(\mathcal{H}_X)$, where $X = A, B$.

We give a little remark on the above conjecture. To prove it, we need to characterize local unitary equivalence between two bipartite states. We say that $\rho_{AB}$ is local unitary equivalent to $\rho'_{AB}$ if there are unitaries $U_X \in U(\mathcal{H}_X) (X = A, B)$ such that

$$\rho'_{AB} = (U_A \otimes U_B) \rho_{AB} (U_A \otimes U_B)^\dagger.$$ 

Along with this line, the readers, for instance, can be referred to [8].

![Figure 1](image1.png)

**Figure 1:** Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing $\Delta_{\text{min}}$. 
Figure 2: Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing $\Delta_{\text{max}}$.

Figure 3: Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing $\Delta_{\text{mix}}$. 
Figure 4: Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing $\Delta$.

A variant of strong super-additivity of the relative entropy

At the end of [9], the authors pointed out a fact given by Jenčová as follows. For convenience, we give the details of the proof.

**Proposition 4.1.** Let $\rho_{ABC}, \mu_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C)$ and $\mu_{ABC}$ be a Markov state, i.e. $I(A : C|B)_\mu = 0$. Then the conditional mutual information satisfies

$$I(A : C|B)_\rho = S(\rho_{ABC}||\mu_{ABC}) + S(\rho_B||\mu_B) - S(\rho_{AB}||\mu_{AB}) - S(\rho_{BC}||\mu_{BC}).$$

It holds that

$$S(\rho_{ABC}||\mu_{ABC}) + S(\rho_B||\mu_B) \geq S(\rho_{AB}||\mu_{AB}) + S(\rho_{BC}||\mu_{BC}).$$

**Proof.** Since $S(\rho||\sigma) = -S(\rho) - \text{Tr}(\rho \log \sigma)$, it follows that

$$\Delta(\rho_{ABC}, \sigma_{ABC}) \stackrel{\text{def}}{=} S(\rho_{ABC}||\sigma_{ABC}) + S(\rho_B||\sigma_B) - S(\rho_{AB}||\sigma_{AB}) - S(\rho_{BC}||\sigma_{BC})$$

$$= [-S(\rho_{ABC}) - \text{Tr}(\rho_{ABC} \log \sigma_{ABC}) - S(\rho_B) - \text{Tr}(\rho_B \log \sigma_B)]$$

$$- [-S(\rho_{AB}) - \text{Tr}(\rho_{AB} \log \sigma_{AB}) - S(\rho_{BC}) - \text{Tr}(\rho_{BC} \log \sigma_{BC})]$$

$$= [S(\rho_{AB}) + S(\rho_{BC}) - S(\rho_{ABC}) - S(\rho_B)] + \text{Tr}(\rho_{AB} \log \sigma_{AB})$$

$$+ \text{Tr}(\rho_{BC} \log \sigma_{BC}) - \text{Tr}(\rho_{ABC} \log \sigma_{ABC}) - \text{Tr}(\rho_B \log \sigma_B)$$

$$= I(A : C|B)_\rho + \langle \rho_{ABC}, \log \sigma_{AB} + \log \sigma_{BC} - \log \sigma_{ABC} - \log \sigma_B \rangle.$$

Moreover, $I(A : C|B)_\sigma = 0$ if and only if $\log \sigma_{AB} + \log \sigma_{BC} - \log \sigma_{ABC} - \log \sigma_B = 0$. Therefore, $\Delta(\rho_{ABC}, \mu_{ABC}) = I(A : C|B)_\rho$ if and only if $I(A : C|B)_\mu = 0$. \hfill $\Box$
A natural question arises: Is there any other candidate states for the reference states \( \mu_{ABC} \) (without Markovity) validating the strong super-additivity?

From the numerical study of weak super-additivity of relative entropy, we can come up with another conjecture:

**Conjecture 4.2.** For arbitrary states \( \rho_{ABC}, \sigma_{ABC} \in D(\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C) \), where \( \sigma_{ABC} \) is full-ranked, it holds that

\[
H(\lambda^\downarrow(\rho_{ABC}) || \lambda^\uparrow(\sigma_{ABC})) + H(\lambda^\downarrow(\rho_B) || \lambda^\uparrow(\sigma_B)) \geq H(\lambda^\downarrow(\rho_{AB}) || \lambda^\downarrow(\sigma_{AB})) + H(\lambda^\downarrow(\rho_{BC}) || \lambda^\downarrow(\sigma_{BC})).
\]  

(4.3)

## 5 Extended study: weak sub-multiplicativity of fidelity

The fidelity between two quantum states, represented by density operators \( \rho \) and \( \sigma \), is defined as

\[
F(\rho, \sigma) \eqdef \text{Tr}\left(\sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}\right).
\]

This is an extremely fundamental and useful quantity in quantum information theory. In [5], we obtained the following result.

**Proposition 5.1.** It holds that

\[
\max_{U \in U(\mathcal{H}_d)} F(\rho, U\sigma U^\dagger) = F(\lambda^\downarrow(\rho), \lambda^\downarrow(\sigma)), \quad (5.1)
\]

\[
\min_{U \in U(\mathcal{H}_d)} F(\rho, U\sigma U^\dagger) = F(\lambda^\downarrow(\rho), \lambda^\uparrow(\sigma)). \quad (5.2)
\]

Moreover the set \( \{ F(\rho, U\sigma U^\dagger) : U \in U(\mathcal{H}_d) \} \) is an interval:

\[
\left\{ F(\rho, U\sigma U^\dagger) : U \in U(\mathcal{H}_d) \right\} = \left[ F(\lambda^\downarrow(\rho), \lambda^\downarrow(\sigma)), F(\lambda^\downarrow(\rho), \lambda^\uparrow(\sigma)) \right]. \quad (5.3)
\]

Note that \( F(p, q) \eqdef \sum_j \sqrt{p_j q_j} \) is the classical fidelity between two probability distributions \( p = \{p_j\} \) and \( q = \{q_j\} \).

In this proposition, we obtained an inequality:

\[
F(\lambda^\downarrow(\rho), \lambda^\downarrow(\sigma)) \leq F(\rho, \sigma) \leq F(\lambda^\downarrow(\rho), \lambda^\uparrow(\sigma)).
\]

We remark here that this inequality for fidelity was already obtained in [10].

In what follows, we study the difference:

\[
F(\rho_{AB}, \sigma_{AB}) - F(\rho_A, \sigma_A)F(\rho_B, \sigma_B). \quad (5.4)
\]
To that, we study the following quantities under the constraints in Proposition 2.2:

\[
\begin{align*}
\Delta F_1 &= F(\lambda^+(\rho_{AB}), \lambda^+(\sigma_{AB})) - F(\lambda^+(\rho_A), \lambda^+(\sigma_A))F(\lambda^+(\rho_B), \lambda^+(\sigma_B)) \quad (5.5) \\
\Delta F_2 &= F(\lambda^+(\rho_{AB}), \lambda^+(\sigma_{AB})) - F(\lambda^+(\rho_A), \lambda^+(\sigma_A))F(\lambda^+(\rho_B), \lambda^+(\sigma_B)) \quad (5.6) \\
\Delta F_3 &= F(\lambda^+(\rho_{AB}), \lambda^+(\sigma_{AB})) - F(\lambda^+(\rho_A), \lambda^+(\sigma_A))F(\lambda^+(\rho_B), \lambda^+(\sigma_B)) \quad (5.7) \\
\Delta F_4 &= F(\lambda^+(\rho_{AB}), \lambda^+(\sigma_{AB})) - F(\lambda^+(\rho_A), \lambda^+(\sigma_A))F(\lambda^+(\rho_B), \lambda^+(\sigma_B)) \quad (5.8) \\
\Delta F_5 &= F(\lambda^+(\rho_{AB}), \lambda^+(\sigma_{AB})) - F(\lambda^+(\rho_A), \lambda^+(\sigma_A))F(\lambda^+(\rho_B), \lambda^+(\sigma_B)) \quad (5.9) \\
\Delta F_6 &= F(\lambda^+(\rho_{AB}), \lambda^+(\sigma_{AB})) - F(\lambda^+(\rho_A), \lambda^+(\sigma_A))F(\lambda^+(\rho_B), \lambda^+(\sigma_B)) \quad (5.10)
\end{align*}
\]

Figure 5: Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing \(\Delta F_1\).

Similar to the Section 3, we test two scenarios with respect to one thousand and one million groups of random data for each quantity defined by Eq. (5.5)–Eq. (5.10). The corresponding plots are listed in Fig. 5–Fig. 10.

From Fig. 5 and 6, it is clearly that both of the differences \(\Delta F_1 \) and \(\Delta F_2\) defined by Eq. (5.5) and Eq. (5.6) are always less than zero in our numerical test. However, Fig. 7–Fig. 10 imply that the value of \(\Delta F_3 - \Delta F_6\) are distributed in \((-1, 1)\) and not always negative. All of the numerical simulation motivates us to conjecture the following inequalities:

\[
\begin{align*}
F(\lambda^+(\rho_{AB}), \lambda^+(\sigma_{AB})) &\leq F(\lambda^+(\rho_A), \lambda^+(\sigma_A))F(\lambda^+(\rho_B), \lambda^+(\sigma_B)), \\
F(\lambda^+(\rho_{AB}), \lambda^+(\sigma_{AB})) &\leq F(\lambda^+(\rho_A), \lambda^+(\sigma_A))F(\lambda^+(\rho_B), \lambda^+(\sigma_B)).
\end{align*}
\]
Figure 6: Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing $\Delta F_2$.

Figure 7: Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing $\Delta F_3$. 
Figure 8: Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing $\Delta F_4$.

Figure 9: Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing $\Delta F_5$.  

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Figure 10: Two scenarios with respect to one thousand (the left one) and one million (the right one) groups of random data for testing $\Delta F_6$.

6 Conclusion

In this context, we conducted numerical studies on weak super-additivity of relative entropy and weak sub-multiplicativity of fidelity, respectively. These data supports the following two inequalities: for qubit pair $(\rho_{AB}, \sigma_{AB})$,

$$H(\lambda^\downarrow(\rho_{AB}) || \lambda^\downarrow(\sigma_{AB})) \geq H(\lambda^\downarrow(\rho_A) || \lambda^\downarrow(\sigma_A)) + H(\lambda^\downarrow(\rho_B) || \lambda^\downarrow(\sigma_B)).$$

(6.1)

and

$$F(\lambda^\downarrow(\rho_{AB}), \lambda^\downarrow(\sigma_{AB})) \leq F(\lambda^\downarrow(\rho_A), \lambda^\downarrow(\sigma_A))F(\lambda^\downarrow(\rho_B), \lambda^\downarrow(\sigma_B))$$

(6.2)

We guess the conjectured inequalities hold for a general qudit pair $(\rho_{AB}, \sigma_{AB})$.

Our numerical studies show that the super-additivity inequality of relative entropy is indeed not valid globally even for full-ranked states:

$$S(\rho_{AB} || \sigma_{AB}) \neq S(\rho_A || \sigma_A) + S(\rho_B || \sigma_B).$$

Thus the sub-multiplicativity inequality of fidelity is not valid globally:

$$F(\rho_{AB}, \sigma_{AB}) \neq F(\rho_A, \sigma_A)F(\rho_B, \sigma_B).$$

In the future research, we will consider the following constrained optimization problems
under local unitary transformations:

\[
\begin{align*}
\text{max} & \quad \max_{U_A \in U(H_A), U_B \in U(H_B)} S(U_A \otimes U_B \rho_{AB} U^\dagger_A \otimes U^\dagger_B || \sigma_{AB}), \quad (6.3) \\
\text{min} & \quad \min_{U_A \in U(H_A), U_B \in U(H_B)} S(U_A \otimes U_B \rho_{AB} U^\dagger_A \otimes U^\dagger_B || \sigma_{AB}), \quad (6.4) \\
\text{max} & \quad \max_{U_A \in U(H_A), U_B \in U(H_B)} F(U_A \otimes U_B \rho_{AB} U^\dagger_A \otimes U^\dagger_B, \sigma_{AB}), \quad (6.5) \\
\text{min} & \quad \min_{U_A \in U(H_A), U_B \in U(H_B)} F(U_A \otimes U_B \rho_{AB} U^\dagger_A \otimes U^\dagger_B, \sigma_{AB}). \quad (6.6)
\end{align*}
\]

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