Change-Point Testing and Estimation for Risk Measures in Time Series

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September 10, 2018

Abstract

We investigate methods of change-point testing and confidence interval construction for nonparametric estimators of expected shortfall and related risk measures in weakly dependent time series. A key aspect of our work is the ability to detect general multiple structural changes in the tails of time series marginal distributions. Unlike extant approaches for detecting tail structural changes using quantities such as tail index, our approach does not require parametric modeling of the tail and detects more general changes in the tail. Additionally, our methods are based on the recently introduced self-normalization technique for time series, allowing for statistical analysis without the issues of consistent standard error estimation. The theoretical foundation for our methods are functional central limit theorems, which we develop under weak assumptions. An empirical study of S&P 500 returns and US 30-Year Treasury bonds illustrates the practical use of our methods in detecting and quantifying market instability via the tails of financial time series during times of financial crisis.

Keywords: Time Series, Risk Measure, Change-Point Test, Confidence Interval, Self-Normalization

JEL classification: C14, C58, G32
1 Introduction

The quantification of risk is a central topic of study in finance. The need to guard against unforeseeable events with adverse and often catastrophic consequences has led to an extensive and burgeoning literature on risk measures. The two most popular financial risk measures, the value-at-risk (VaR) and the expected shortfall (ES), are extensively used for risk and portfolio management as well as regulation in the finance industry. Parallel to the developments in risk estimation, there has been longstanding interest in detecting changes in financial time series, especially changes in the tail structure, which is essential for effective risk and portfolio management. Indeed, empirical findings strongly suggest that financial time series frequently exhibit changes in their underlying statistical structure due to changes in economic conditions, e.g. monetary policies, or critical social events. Although there is an established literature on structural change detection for parametric time series models, including monitoring of proxies for risk such as tail index, there are no studies or tools concerning monitoring of general tail structure, or of risk measures in particular. To underscore this point, existing literature on risk measure estimation assume stationarity of time series observations over a time period of interest, with the stationarity of the risk measure being key for the estimation to make sense. However, to the best of our knowledge, no tools have been provided to verify this assumption.

We provide tools to detect general and potentially multiple changes in the tail structure of time series, and in particular, tools for monitoring for changes in risk measures such as ES and related measures such as conditional tail moments (CTM) (Methni et al., 2014) over time periods of interest. Specifically, we develop retrospective change-point tests to detect changes in ES and related risk measures. Additionally, we offer new ways of constructing confidence intervals for these risk measures. Our methods are applicable to a wide variety of time series models as they depend on functional central limit theorems for the risk measures, which we develop under weak assumptions. As will be described, our methods complement and extend the existing literature in several ways.

As mentioned previously, the literature lacks tools to monitor general tail structure or risk measures such as ES or CTM over time. This deficiency appears to be two-fold. (1) Although there are studies on VaR change-point testing, for example, Qu (2008), often
one is interested in characterizations of tail structure that are more informative than simple location measures. Indeed, the introduction of ES as an alternative risk measure to VaR was, to a great extent, driven by the need to quantify tail structure, particularly, the expected magnitude of losses conditional on losses being in the tail. Aligned with this goal, a popular measure of tail structure is the tail index, which describes tail thickness and governs distributional moments. In tail index estimation, an extreme value theory approach is typically taken with the assumption of so-called regularly-varying Pareto-type tails. However, tail index estimation is very sensitive to the choice of which fraction of sample observations is classified to be “in the tail”. Moreover, the typical regularly-varying Pareto-type tail assumption may not even be valid in some situations. And even if they are valid, the tail index is invariant to changes of the location and scale types, and thus these types of structural changes in the tail would remain undetected using tail index-based change-point tests. (2)

As mentioned before, all previous studies on risk measure estimation implicitly assume the risk measure is constant over some time period of interest—otherwise, risk measure estimation and confidence interval construction could behave erratically. For instance, if there is a sudden change in VaR (at some level) in the middle of a time series, naively estimating VaR using the entire time series could result in a wrong estimate of VaR or ES. Hence, given the importance of ES and related risk measures, a simple test for ES change over a time period of interest is a useful first step prior to follow-up statistical analysis.

To simultaneously address both deficiencies, we propose retrospective change-point testing for ES and related risk measures such as CTM. We introduce, in particular, a consistent test for a potential ES change at an unknown time based on a variant of the widely used cumulative sum (CUSUM) process. We subsequently generalize this test to the case of multiple potential ES change points, leveraging recent work by Zhang and Lavitas (2018), and unlike existing change-point testing methodologies in the literature, our test does not require the number of potential change points to be specified in advance. Our use of a risk

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1 The Hill estimator ([Hill 1975](#)) is widely used and requires the user to choose the fraction of sample observations deemed to be “in the tail” to use for estimation. However, generally, there appears to be no “best” way to select this fraction. In the specific setting of change-point testing for tail index, recommendations for this fraction in the literature range from the top 20th percentile to the top 5th percentile of observations ([Kim and Lee 2009](#), [2011](#), [Hoga 2017](#)). Such choice heavily influences the quality of tail index estimation and change detection and must be made on a case-by-case basis ([Rocco 2014](#)). It is a delicate matter as choosing too small of a fraction results in high estimation variance and choosing too large of a fraction often results in high bias due to misspecification of where the tail begins.

2 The cumulative sum (CUSUM) process was first introduced by [Page 1954](#) and is discussed in detail in [Csorgo and Horvath 1997](#).
measure such as ES is attractive in many ways. First, the fraction of observations used in ES estimation, for example, the upper 5th percentile of observations, directly has meaning, and is often comparable to the fraction of observations used in tail index estimation, as discussed previously. Second, use of ES does not require parametric-type tail assumptions, which is not the case with use of the tail index. Third, ES can detect much more general structural changes in the tail such as location or scale changes, while such changes go undetected when using tail index. Moreover, our change-point tests can be used to check in a statistically principled way whether or not ES and related risk measures are constant over a time period of interest, and provide additional validity when applying existing estimation methods for these risk measures.

Additionally, a key detail that has been largely ignored in previous studies is standard error estimation. Here, the issue is two-fold. (1) In the construction of confidence intervals for risk measures, consistent estimation of standard errors is nontrivial due to standard errors involving the entire time series correlation structure at all integer lags, and thus being infinite-dimensional in nature. A few studies in which standard error estimation has been addressed (or bypassed) are Chen and Tang (2005); Chen (2008); Wang and Zhao (2016); Xu (2016). However, confidence interval construction in these studies all require user-specified tuning parameters such as the bandwidth in periodogram kernel smoothing or window width in the blockwise bootstrap and empirical likelihood methods. Although the choice of these tuning parameters significantly influences the quality of the resulting confidence intervals, it is not always clear how best to select them. (2) Choosing the tuning parameters is not only difficult, but data-driven approaches may result in non-monotonic test power, as pointed out in numerous studies (Vogelsang, 1999; Crainiceanu and Vogelsang, 2007; Deng and Perron, 2008; Shao and Zhang, 2010).

We offer the following solutions to the above issues. (1) To address the issue of often problematic selection of tuning parameters for confidence interval construction with time series data, we make use of ratio statistics to cancel out unknown standard errors and form pivotal quantities, thereby avoiding the often difficult estimation of such nuisance parameters. We examine confidence interval construction using a technique originally referred to in the

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3For general change-point tests with time series observations, consistent estimation of standard errors is typically done using periodogram kernel smoothing, and the performance of such tests is heavily influenced by the choice of kernel bandwidth.
simulation literature as sectioning (Asmussen and Glynn, 2007), which involves splitting the data into equal-size non-overlapping sections, separately evaluating the estimator of interest using the data in each section, and relying on a normal approximation to form an asymptotically pivotal t-statistic. We also examine its generalization, referred to in the simulation literature as standardized time series (Schruben, 1983; Glynn and Iglehart, 1990) and in the time series literature as self-normalization (Lobato, 2001; Shao, 2010), which uses functionals different from the t-statistic to create asymptotically pivotal quantities.

(2) In the context of change-point testing using ES and related risk measures, to avoid potentially troublesome standard error estimation, we follow Shao and Zhang (2010) and Zhang and Lavitas (2018) and apply the method of self-normalization for change-point testing by dividing CUSUM-type processes by corresponding processes designed to cancel out the unknown standard error. The processes we divide by take into account potential change point(s) and thus avoids the problem of non-monotonic power which often plagues change-point tests that rely on consistent standard error estimation, as discussed previously.

The outline of our paper is as follows. In Section 2.2 we develop asymptotic theory for VaR and ES, specifically, functional central limit theorems, which provide the theoretical basis for the proposed confidence interval construction and change-point testing methodologies. In introducing our statistical methods, we first discuss the simpler task of confidence interval construction for risk measures in Section 3.1. Then, with several fundamental ideas in place, we take up testing for a single potential change point for ES in Section 3.2. We extend the change-point testing methodology to an unknown, possibly multiple, number of change points in Section 3.3. In Section 4 we examine the finite-sample performance of our proposed methods through simulations. We conclude with an empirical study in Section 5 of returns data for the S&P 500 exchange-traded fund (ETF) SPY and also US 30-Year Treasury bonds. Proofs of our theoretical results are delegated to the Appendix.

2 Asymptotic Theory

2.1 Model Setup

Suppose the random variable $X$ and the stationary sequence of random variables $X_1, \ldots, X_n$ have marginal distribution function $F$. For some level $p \in (0, 1)$, we wish to estimate the
risk measures VaR and ES defined by

\[ VaR(p) = \inf \{ x \in \mathbb{R} : F(x) \geq p \} \]
\[ ES(p) = \mathbb{E} [X \mid X \geq VaR(p)] . \]

Let \( \hat{F}_n(\cdot) = n^{-1} \sum_{i=1}^{n} I(X_i \leq \cdot) \) be the sample distribution function. We consider the following “natural” sample-based nonparametric estimators.

\[
\hat{VaR}_n(p) = \inf \{ x \in \mathbb{R} : \hat{F}_n(x) \geq p \} 
\]
\[
\hat{ES}_n(p) = \frac{1}{1-p} \frac{1}{n} \sum_{i=1}^{n} X_i I X_i \geq \hat{VaR}_n(p) . \]

For \( m > l \), we will also consider the following nonparametric estimators based on samples \( X_l, \ldots, X_m \), with \( \hat{F}_{l:m}(\cdot) = (m-l+1)^{-1} \sum_{i=l}^{m} I(X_i \leq \cdot) \).

\[
\hat{VaR}_{l:m}(p) = \inf \{ x \in \mathbb{R} : \hat{F}_{l:m}(x) \geq p \} 
\]
\[
\hat{ES}_{l:m}(p) = \frac{1}{1-p} \frac{1}{m-l+1} \sum_{i=l}^{m} X_i I X_i \geq \hat{VaR}_{l:m}(p) . \]

Also, let \( F_{i}^{m} \) denote the \( \sigma \)-algebra generated by \( X_l, \ldots, X_m \), and let \( F_{i}^{\infty} \) denote the \( \sigma \)-algebra generated by \( X_l, X_{l+1}, \ldots \). The \( \alpha \)-mixing coefficient introduced by [Rosenblatt 1956] is

\[
\alpha(k) = \sup_{A \in F_{l+1}^{k}, B \in F_{j+k}^{\infty}} \left| \mathbb{P}(A) \mathbb{P}(B) - \mathbb{P}(AB) \right| ,
\]

and a sequence is said to be \( \alpha \)-mixing if \( \lim_{k \to \infty} \alpha(k) = 0 \). The dependence described by \( \alpha \)-mixing is the least restrictive as it is implied by the other types of mixing; see [Doukhan 1994] for a comprehensive discussion. In what follows, \( D[0,1] \) denotes the space of real-valued functions on \([0,1]\) that are right-continuous and have left limits, and convergence in distribution on this space is defined with respect to the Skorohod topology (Billingsley 1999). Also, for some index set \( \Delta \), \( \ell^{\infty}(\Delta) \) denotes the space of real-valued bounded functions on \( \Delta \), and convergence in distribution on this space is defined with respect to the uniform topology (Pollard 1984). We denote the integer part of a real number \( x \) by \([x]\) and the positive part by \([x]_+\). Standard Brownian motion on the real line is denoted by \( W \). In each of our theoretical results, we use one of the following two types of assumptions.
Assumption 1. There exists $a > 1$ such that the following hold.

(i) $X_1, X_2, \ldots$ is $\alpha$-mixing with $\alpha(k) = O(k^{-a})$

(ii) $X$ has positive and continuous density $f$ in a neighborhood of $VaR(p)$, and for each $k \geq 2$, $(X_1, X_k)$ has joint density in a neighborhood of $(VaR(p), VaR(p))$.

Assumption 2. There exists $\delta > 0$ such that Assumption 1 holds with $a > (2 + \delta)/\delta$ along with $E[|X|^{2+\delta}] < \infty$.

Assumption 1 condition (i) is a form of “asymptotic independence”, which ensures that the time series is not too serially dependent so that non-degenerate limit distributions are possible. A very wide range of commonly used financial time series models such as ARCH models and diffusion models satisfy this condition. Condition (ii) is a standard condition in VaR and ES estimation, and the joint density condition ensures that there are not too many ties among the time series observations. Assumption 2 indicates that the strength of the moment condition of the underlying marginal distribution and the rates of $\alpha$-mixing trade off, in that weaker $\alpha$-mixing conditions require stronger moment conditions and vice versa.

We point out, in particular, that while our results are illustrated for the most widely used risk measures, VaR and ES, they can be adapted to many other important functionals of the underlying marginal distribution. One straightforward adaptation (by assuming stronger $\alpha$-mixing and moment conditions) is to CTM: $E[X^\beta | X > VaR(p)]$ for some level $p \in (0, 1)$ and some $\beta > 0$ (Methni et al., 2014). Our results also easily extend to multivariate time series, but for simplicity of illustration, we focus on univariate time series.

2.2 Functional Central Limit Theorems

We develop functional central limit theorems for Var and ES under weak assumptions. These functional central limit theorems allow the construction of general change point statistics. Our results essentially model the “progress” of the estimation (for fixed $n$, allowing $t$ to increase from zero to one) as a random walk, which for large $n$ and after rescaling the random walk steps by $n^{-1/2}$, can be approximated by a Brownian motion.

Theorem 1. Under Assumption 2 with the modifications: $a \geq 3$ and $X$ has positive and
differentiable density at $VaR(p)$, the process

$$\{n^{1/2}t(VaR_{[nt]}(p) - VaR(p)), \ t \in [0, 1]\} \quad (5)$$

converges in distribution in $D[0, 1]$ to $\sigma_{VaR}W$, where

$$\sigma_{VaR}^2 = \frac{1}{f^2(VaR(p))} \left( \mathbb{E} [g_p^2(X_1)] + 2 \sum_{i=2}^{\infty} \mathbb{E} [g_p(X_1)g_p(X_i)] \right)$$

with $g_p(X) = \mathbb{I}(X \leq VaR(p)) - p$. Under Assumption 2, the process

$$\{n^{1/2}t(\widehat{ES}_{[nt]}(p) - ES(p)), \ t \in [0, 1]\} \quad (6)$$

converges in distribution to $\sigma_{ES}W$ in $D[0, 1]$, where

$$\sigma_{ES}^2 = \frac{1}{(1-p)^2} \left( \mathbb{E} [h_p^2(X_1)] + 2 \sum_{i=2}^{\infty} \mathbb{E} [h_p(X_1)h_p(X_i)] \right)$$

with $h_p(X) = \max(X, \text{VaR}(p)) - \mathbb{E} [\max(X, \text{VaR}(p))]$.

From this, we immediately have the following central limit theorems for VaR and ES.

**Corollary 1.** Under Assumption 1,

$$n^{1/2}(\widehat{VaR}_n(p) - \text{VaR}(p)) \overset{d}{\to} \mathcal{N}(0, \sigma_{VaR}^2),$$

where $\sigma_{VaR}^2$ is the same as in Theorem 1 for VaR. Under Assumption 2,

$$n^{1/2}(\widehat{ES}_n(p) - ES(p)) \overset{d}{\to} \mathcal{N}(0, \sigma_{ES}^2),$$

where $\sigma_{ES}^2$ is the same as in Theorem 1 for ES.

We point out that our Assumptions 1 and 2 are weaker than those of existing central limit theorems in the literature (c.f. Chen and Tang (2005); Chen (2008), who require $\alpha$-mixing coefficients to decay exponentially fast as well as additional regularity of the marginal and pairwise joint densities of the observations).

The above functional central limit theorem for VaR is a modification of a result due
to Sen (1972). The condition $a \geq 3$ for the VaR functional central limit theorem is due to our use of a so-called Bahadur representation for $\tilde{V}AR_n(p)$ by Wendler (2011). Although such a condition can likely be weakened, we do not pursue that here. Moreover, the above functional central limit theorem for ES does not require such a condition; Assumption 2 is all that is needed. In deriving the ES functional central limit theorem, we used the following Bahadur representation for $\tilde{ES}_n(p)$.

**Proposition 1.** Under Assumption 2, for any $\delta' > 0$ satisfying $-1/2 + 1/(2a) + \delta' < 0$,

$$\tilde{ES}_n(p) - \left( VAR(p) + \frac{1}{1 - p} \frac{1}{n} \sum_{i=1}^{n} [X_i - VAR(p)]_+ \right) = o_a.s. \left( n^{1-1/(2a)+\delta' \log n} \right).$$

Sun and Hong (2010) developed such a Bahadur representation in the setting of independent, identically-distributed data, but to the best of our knowledge, no such representation exists in the stationary, $\alpha$-mixing setting. Such a Bahadur representation is generally useful for developing limit theorems in many different contexts.

We also have the following extension of the functional central limit theorems for VaR and ES (Theorem 1), where we have two “time” indices instead of a single “time” index as in the standard functional central limit theorems. The standard functional central limit theorems are useful for detecting a single change point in a time series, but the following extension will allow us to detect an unknown, possibly multiple, number of change points, as we will discuss later. We point out that this result does not follow automatically from Theorem 1 and an application of the continuous mapping theorem because the estimators in Equations 3 and 4 are not additive, for instance, for $m > l$, $\tilde{ES}_{l:m}(p) \neq \tilde{ES}_{1:m}(p) - \tilde{ES}_{1:l-1}(p)$.

**Theorem 2.** Fix any $\delta > 0$ and consider the index set $\Delta = \{(s,t) \in [0,1]^2 : t - s \geq \delta\}$. Under Assumption 2, the process

$$\{n^{1/2}(t-s)(\tilde{V}AR_{[ns];[nt]}(p) - VAR(p)), (s,t) \in \Delta\}$$

converges in distribution in $\ell^\infty(\Delta)$ to $\{\sigma_{VAR}(W(t) - W(s)), (s,t) \in \Delta\}$, where $\sigma_{VAR}^2$ is the same as in Theorem 1 for VAR.

Under Assumption 2 with the modification that $(X_1, X_k)$ has a joint density for all
\( k \geq 2 \), the process

\[ \{ n^{1/2} (t - s)(\hat{ES}_{[n]}^*[n](p) - ES(p)) \}, (s, t) \in \Delta \} \tag{8} \]

converges in distribution in \( \ell^\infty(\Delta) \) to \( \{ \sigma_{\text{ES}}(W(t) - W(s)), (s, t) \in \Delta \} \), where \( \sigma_{\text{ES}}^2 \) is the same as in Theorem 1 for \( \text{ES} \).

3 Statistical Inference

3.1 Confidence Intervals

In time series analysis, confidence interval construction for an unknown quantity is often difficult, due to dependence. Indeed, from Theorem 1 and Corollary 1, we see that the standard errors appearing in the normal limiting distributions depend on the time series autocovariance at all integer lags. To construct confidence intervals using the central limit theorem in Corollary 1 directly, these standard errors must be estimated. One approach, taken in Chen and Tang (2005); Chen (2008), is to estimate using kernel smoothing the spectral density at zero frequency of the transformed time series \( g_p(X_1), g_p(X_2), \ldots \) and \( h_p(X_1), h_p(X_2), \ldots \), where \( g_p \) and \( h_p \) are from Theorem 1 and Corollary 1. Although it is known that under certain moment and correlation assumptions, spectral density estimators are consistent for stationary processes (Brockwell and Davis, 1991; Anderson, 1971), in practice it is nontrivial to obtain quality estimates due to the need to select tuning parameters for the kernel smoothing-based approach. As for other approaches, under certain conditions, resampling methods such as the moving block bootstrap (Kunsch, 1989; Liu and Singh, 1992) and the subsampling method for time series (Politis et al., 1999) bypass direct standard error estimation and yield confidence intervals that asymptotically have the correct coverage probability. However, these approaches also require user-chosen tuning parameters such as block length in the moving block bootstrap and window width in subsampling. We therefore investigate other ways of obtaining pivotal limiting distributions to construct asymptotic confidence intervals, where the quality of the obtained confidence intervals is less sensitive to the choice of tuning parameters and are thus, in some sense, more robust.

We first examine a technique called sectioning from the simulation literature (As-
mussen and Glynn [2007]), which can be used to construct confidence intervals for risk measures in time series. Although the method can be used to construct confidence intervals for general risk measures, in light of our results from Section 2 we apply the method to VaR and ES in particular. As a general overview of the method, let $Y_1(\cdot), Y_2(\cdot), \ldots$ be a sequence of random bounded real-valued functions on $[0, 1]$. For some user-specified integer $m \geq 2$, suppose we have the joint convergence in distribution

$$
\left( Y_n(1/m) - Y_n(0), Y_n(2/m) - Y_n(1/m), \ldots, Y_n((m-1)/m) \right) \overset{d}{\to} \frac{\sigma}{m^{1/2}} (N_1, N_2, \ldots, N_m),
$$

where $\sigma > 0$ and $N_1, \ldots, N_m$ are independent standard normal random variables. Then, with

$$
\tilde{Y}_n = m^{-1} \sum_{i=1}^{m} (Y_n(i/m) - Y_n((i-1)/m))
$$

$$
S_n = \left( (m-1)^{-1} \sum_{i=1}^{m} (Y_n(i/m) - Y_n((i-1)/m) - \bar{Y}_n)^2 \right)^{1/2},
$$

by the continuous mapping theorem, as $n \to \infty$ with $m$ fixed, $m^{1/2} \tilde{Y}_n/S_n$ converges in distribution to the Student’s $t$-distribution with $m-1$ degrees of freedom. With the limiting Student’s $t$-distribution, we may construct confidence intervals for VaR or ES by taking the random functions $Y_n(\cdot)$ to be the processes in Equations 5 or 6, respectively. The distributional convergence result in Equation 9 is easily obtained by modifying the proof of Corollary 1 or by directly applying Theorem 1.

Next, we examine a generalization of sectioning, called self-normalization, which has been studied recently in the time series literature (Lobato 2001; Shao 2010) (and earlier in the simulation literature, where it is known as standardization (Schruben 1983; Glynn and Iglehart 1990)). As with sectioning, the method applies to confidence interval construction for general risk measures, but we specialize to the case of VaR and ES. In self-normalization, the idea is to use a ratio-type statistic where the unknown standard error appears in both the numerator and the denominator and thus cancels, resulting in a pivotal limiting distribution. Once again, let $Y_1(\cdot), Y_2(\cdot), \ldots$ be a sequence of random bounded real-valued functions on $[0, 1]$. Suppose we have the distributional convergence in $D[0, 1]$ (for example, via a functional
central limit theorem): $Y_n(\cdot) \xrightarrow{d} \sigma W(\cdot)$, where $\sigma > 0$. For some positive homogeneous functional $T : D[0, 1] \mapsto \mathbb{R}$ (i.e., $T(\sigma Y) = \sigma T(Y)$ for $\sigma > 0$ and $Y \in D[0, 1]$) to which the continuous mapping theorem applies, we get

$$\frac{Y_n(1)}{T(Y_n)} \xrightarrow{d} \frac{W(1)}{T(W)},$$

which is a pivotal limiting distribution whose critical values may be computed via simulation and are tabulated in Lobato (2001). As before, we may construct confidence intervals for VaR or ES by taking the random functions $Y_n(\cdot)$ to be the processes in Equations 5 or 6 respectively. The distributional convergence result in Equation 10 follows directly from Theorem 1. As an example, considering the functional $T(Y) = \left( \int_0^1 (Y(t) - tY(1))^2 dt \right)^{1/2}$, we have the following result for ES.

$$\frac{\hat{ES}_n(p) - ES(p)}{\left( \int_0^1 t^2 \left( \hat{ES}_{[nt]}(p) - \hat{ES}_n(p) \right)^2 dt \right)^{1/2}} \xrightarrow{d} \frac{W(1)}{\left( \int_0^1 (W(t) - tW(1))^2 dt \right)^{1/2}}.$$

### 3.2 Change-Point Testing

As is the case with confidence interval construction with time series data, change-point testing in time series based on statistics constructed from functional central limit theorems and the continuous mapping theorem is often nontrivial due to the need to estimate standard errors such as $\sigma$ in Theorem 1. Motivated by the maximum likelihood method in the parametric setting, variants of the Page (1954) CUSUM statistic are commonly used for nonparametric change-point tests (Csorgo and Horvath, 1997), and generally rely on asymptotic approximations (via functional central limit theorems and the continuous mapping theorem) to supply critical values of pivotal limiting distributions under the null hypothesis of no change. As discussed in Vogelsang (1999); Shao and Zhang (2010); Zhang and Lavitas (2018), testing procedures where standard errors are estimated directly, for example, by estimating the spectral density of transformed time series via a kernel-smoothing approach, can be biased under the change-point alternative. Such bias can result in nonmonotonic power, i.e., power can decrease in some ranges as the alternative deviates from the null. To avoid this issue, Shao and Zhang (2010) and Zhang and Lavitas (2018) propose using self-normalization techniques.
to general change-point testing. We adopt this idea as discussed in Section 3.1 to our specific problem of detecting changes in tail risk measures.

As motivated in the Introduction (Section 1), it is important to perform hypothesis tests for abrupt changes of risk measures in the time series setting. We introduce the methodology for ES, but note that it can be applied to risk measures in general. For simplicity, we consider the case of at most one change point; the case of an unknown, potentially multiple, number of change points can be treated using the approach in Zhang and Lavitas (2018). For time series sample $X_1, \ldots, X_n$, let $ES_{X_i}(p)$ be the ES at level $p$ for the marginal distribution of $X_i$. We test the following null and alternative hypotheses.

$H_0 : X_1, \ldots, X_n$ is stationary, and in particular, $ES_{X_1}(p) = \cdots = ES_{X_n}(p)$

$H_1 :$ There is $t^* \in (0, 1)$ such that $ES_{X_1}(p) = \cdots = ES_{X_{nt^*}}(p) \neq ES_{X_{[nt^*]+1}}(p) = \cdots = ES_{X_n}(p)$ and $X_1, \ldots, X_{[nt^*]}$ and $X_{[nt^*]+1}, \ldots, X_n$ are separately stationary

We base our change-point test on the following variant of the CUSUM process.

$$\left\{ n^{1/2}t(1-t) \left( \hat{ES}_{1:[nt]}(p) - \hat{ES}_{[nt]+1:n}(p) \right) , t \in [0, 1] \right\}$$

Note that we split the above process for all possible break points $t \in (0, 1)$ into a difference of two ES estimators, an estimator using samples $X_1, \ldots, X_{[nt]}$ and an estimator using samples $X_{[nt]+1}, \ldots, X_n$. As the ES estimator in Equation 4 involves the VaR estimator in Equation 3 splitting the process as in Equation 12 avoids potential VaR estimation using a sample sequence containing the change point, which could have undesirable behavior. In Proposition 2 below, we self-normalize the process in Equation 4 using the approach of Shao and Zhang (2010). Note the denominator of Equation 13 below takes into account the potential change point and is split into two separate integrals involving samples $X_1, \ldots, X_{[nt]}$ and $X_{[nt]+1}, \ldots, X_n$.

**Proposition 2.** Assume Assumption 2 holds. Under the null hypothesis $H_0$

$$G_n := \sup_{t \in [0,1]} \frac{t^2(1-t)^2 \left( \hat{ES}_{1:[nt]}(p) - \hat{ES}_{[nt]+1:n}(p) \right)^2}{n^{-1} \sum_{i=1}^{[nt]} \left( \frac{1}{n} \right)^2 \left( \hat{ES}_{1:i}(p) - \hat{ES}_{1:[nt]}(p) \right)^2 + n^{-1} \sum_{i=[nt]+1}^{n} \left( \frac{n-i+1}{n} \right)^2 \left( \hat{ES}_{i:n}(p) - \hat{ES}_{[nt]+1:n}(p) \right)^2}$$

(13)
converges in distribution to

\[
G := \sup_{t \in [0, 1]} \frac{(W(t) - tW(1))^2}{\int_0^t (W(s) - \frac{2}{t} W(t))^2 \, ds + \int_t^1 \left( W(1) - W(s) - \frac{1}{1-t} (W(1) - W(t)) \right)^2 \, ds}. \tag{14}
\]

Assume the alternative hypothesis \( H_1 \) is true with the change point occurring at some fixed (but unknown) \( t^* \in (0, 1) \). For any fixed difference \( ES_{X_{(n)}}(p) = c_1 \neq c_2 = ES_{X_{(n+1)}}(p) \), we have \( G_n \xrightarrow{P} \infty \) as \( n \to \infty \). Furthermore, if the difference varies with \( n \) according to \( c_1 - c_2 = n^{-1/2+\epsilon} L \) for some \( L \neq 0 \) and \( \epsilon \in (0, 1/2) \), then \( G_n \xrightarrow{P} \infty \) as \( n \to \infty \).

The distribution of \( G \) is pivotal, and its critical values may be obtained via simulation and are tabulated in Lobato (2001). For testing \( H_0 \) versus \( H_1 \) at some level, we reject \( H_0 \) if the test statistic \( G_n \) exceeds some corresponding critical value of \( G \). In subsequent discussions concerning Equation 13, we will refer to the process appearing in the numerator as the “CUSUM process”, the process appearing in the denominator as the “self-normalizer process”, and the entire ratio process as the “self-normalized CUSUM process”.

3.3 Extension to Multiple Change Points

We extend our single change-point testing methodology to the case of multiple change points. Typically, the number of potential change points in the alternative hypothesis must be prespecified. However, we leverage the recent work of Zhang and Lavitas (2018) and introduce change-point tests of ES and related risk measures that can accommodate an unknown, possibly multiple, number of change points in the alternative hypothesis. For illustration, we introduce the method using ES for univariate time series, but the method extends easily to related risk measures and also to multivariate time series. We fix some small \( \delta > 0 \) and consider the following null and alternative hypotheses (following the notation from Section 4).

---

4 To theoretically evaluate the efficiency of statistical tests, an analysis based on sequences of so-called local limiting alternatives (for example, sequences \( c_1 - c_2 = O(n^{-1/2}) \) in Proposition 2 above) can be considered (see, for example, van der Vaart (1998)). However, such an analysis would be considerably involved, and we save it for future study.
$\mathcal{H}_0: X_1, \ldots, X_n$ is stationary, and in particular, $ES_{X_1}(p) = \cdots = ES_{X_n}(p)$

$\mathcal{H}_1$: There are $0 = t_0^* < t_1^* < \cdots < t_k^* < t_{k+1}^* = 1$ with $t_j^* - t_{j-1}^* > \delta$ for $j = 1, \ldots, k + 1$

such that $ES_{X_{[nt_j^*]}}(p) \neq ES_{X_{[nt_j^*]+1}}(p)$ and $X_{[nt_j^*]+1}, \ldots, X_{[nt_{j+1}^*]}$ are separately stationary

for $j = 0, \ldots, k$

Consider the index set $\Delta = \{(s, t) \in [\delta, 1 - \delta]^2 : t - s \geq \delta\}$ and the test statistic

$$H_n = \sup_{(s_1, s_2) \in \Delta} \frac{C_{n,f}(s_1, s_2)}{D_{n,f}(s_1, s_2)} + \sup_{(t_1, t_2) \in \Delta} \frac{C_{n,b}(t_1, t_2)}{D_{n,b}(t_1, t_2)},$$

where

$$C_{n,f}(s_1, s_2) = \frac{[ns_1]^2([ns_2] - [ns_1])^2}{[ns_2]^3} \left( \hat{ES}_{[ns_1] \downarrow}(p) - \hat{ES}_{[ns_1]+1:[ns_2]}(p) \right)^2$$

$$D_{n,f}(s_1, s_2) = \sum_{i=1}^{[ns_1]} i^2([ns_1] - i)^2 \left( \hat{ES}_{[ns_1] \downarrow}(p) - \hat{ES}_{[ns_1]+1:[ns_1]}(p) \right)^2$$

$$+ \sum_{i=[ns_1]+1}^{[ns_2]} \frac{(i - 1 - [ns_1])^2([ns_2] - i + 1)^2}{[ns_2]^2([ns_2] - [ns_1])^2} \left( \hat{ES}_{[ns_1]+1:[ns_2]}(p) - \hat{ES}_{[ns_2]}(p) \right)^2$$

$$C_{n,b}(t_1, t_2) = \frac{([nt_1] - [nt_2])[n - [nt_2] + 1]^2}{(n - [nt_1] + 1)^3} \left( \hat{ES}_{[nt_2]}(p) - \hat{ES}_{[nt_1]:[nt_2]-1}(p) \right)^2$$

$$D_{n,b}(t_1, t_2) = \sum_{i=[nt_1]}^{[nt_2]-1} \frac{(i - [nt_1] + 1)([nt_2] - 1 - i)^2}{(n - [nt_1] + 1)^2([nt_2] - [nt_1])^2} \left( \hat{ES}_{[nt_1]:[nt_2]}(p) - \hat{ES}_{[nt_1]:[nt_2]-1}(p) \right)^2$$

$$+ \sum_{i=[nt_2]}^{n} \frac{(i - [nt_2])^2(n - i + 1)^2}{(n - [nt_1] + 1)^2(n - [nt_2] - 1)^2} \left( \hat{ES}_{[nt_2]}(p) - \hat{ES}_{[nt_2]:[nt_2]-1}(p) \right)^2.$$

Then, under $\mathcal{H}_0$, applying Theorem 3 of Zhang and Lavitas (2018), our Theorem 2 above yields

**Corollary 2.** Assume Assumption 2 holds. Under the null hypothesis $\mathcal{H}_0$, it holds

$$H_n \overset{d}{\to} \sup_{(s_1, s_2) \in \Delta} \frac{C(0, s_1, s_2)}{D(0, s_1, s_2)} + \sup_{(t_1, t_2) \in \Delta} \frac{C(t_1, t_2, 1)}{D(t_1, t_2, 1)} = H,$$
where
\[
C(r_1, r_2, r_3) = \frac{1}{(r_3 - r_1)^2} \left( W(r_2) - W(r_1) - \frac{r_2 - r_1}{r_3 - r_1}(W(r_3) - W(r_1)) \right)^2
\]
\[
D(r_1, r_2, r_3) = \frac{1}{(r_3 - r_1)^2} \left( \int_{r_1}^{r_2} \left[ W(s) - W(r_1) - \frac{s - r_1}{r_2 - r_1}(W(r_2) - W(1)) \right]^2 ds 
+ \int_{r_2}^{r_3} \left[ W(r_3) - W(s) - \frac{r_3 - s}{r_3 - r_2}(W(r_3) - W(r_2)) \right]^2 ds \right).
\]

Under the alternative \( \mathcal{H}_1 \), \( H_n \xrightarrow{P} \infty \) as \( n \to \infty \).

We reject \( \mathcal{H}_0 \) if \( H_n \) exceeds the critical value corresponding to a desired test level of the pivotal quantity \( H \), which may be obtained via simulation. Moreover, under \( \mathcal{H}_1 \), our test is asymptotically consistent, and the analogous version of Proposition 2 holds.

To reduce the computational burden of the method, we use a grid approximation suggested by Zhang and Lavitas (2018), where in the doubly-indexed set \( \Delta \), one index is reduced to a coarser grid. Specifically, let \( G_\delta = \{(1 + k\delta)/2 : k \in \mathbb{Z}\} \cap [0, 1] \) and consider the modified statistic
\[
\tilde{H}_n = \sup_{(s_1, s_2) \in \Delta \cap ([0, 1] \times G_\delta)} \frac{C_n,f(s_1, s_2)}{D_n,f(s_1, s_2)} + \sup_{(t_1, t_2) \in \Delta \cap (G_\delta \times [0, 1])} \frac{C_n,b(t_1, t_2)}{D_n,b(t_1, t_2)}, \tag{15}
\]
As before, under \( \mathcal{H}_0 \), we have
\[
\tilde{H}_n \xrightarrow{d} \sup_{(s_1, s_2) \in \Delta \cap ([0, 1] \times G_\delta)} \frac{C(0, s_1, s_2)}{D(0, s_1, s_2)} + \sup_{(t_1, t_2) \in \Delta \cap (G_\delta \times [0, 1])} \frac{C(t_1, t_2, 1)}{D(t_1, t_2, 1)} := \tilde{H}. \tag{16}
\]
Note that simply using the original doubly-indexed set \( \Delta \), for a sample of size \( n \) and an arbitrary number of change points, we would need to search for maxima over \( O(n^2) \) points. However, using the grid approximation, we need only search for maxima over \( O(n) \) points. In contrast, if we were to use a direct extension of the single-change point detection methodology of Section 3.2 with \( m \) change points (which needs to be specified in advance), we would need to search for maxima over \( O(n^m) \) points. Hence, the methodology introduced in this section offers significant computational savings.

To obtain critical values for hypothesis testing, we simulate standard Brownian motion paths on \([0, 1]\), with each path consisting of appropriately scaled 5,000 independent stan-
standard normal random variables. Figure 1 below shows the approximate distribution based on 10,000 samples of the test statistic from Equation 16 and the estimated 0.95 quantile is 138.19.

![Graph showing the estimated density of the test statistic from Equation 16 based on 10,000 samples, with each sample utilizing 5,000 independent standard normal random variables to approximate standard Brownian motion on [0, 1]. The 0.95 quantile is 138.19 and is indicated by the vertical red dashed line.]

4 Simulations

We perform a simulation study to investigate the finite sample performance of ES confidence interval construction using the sectioning and self-normalization methods (Section 3.1) as well as upper tail change detection (Section 3.2) using ES. We consider two data generating processes, AR(1): $X_{i+1} = \phi X_i + \epsilon_i$ and ARCH(1): $X_{i+1} = \sqrt{\beta + \lambda X_i^2 \epsilon_i}$. We take the innovations $\epsilon_i$ to be i.i.d. standard normal, and we use parameters $\phi = 0.5$, $\beta = 1$ and $\lambda = 0.3$. The stationary distribution of the AR(1) process is mean-zero normal with variance $1/(1 - \phi^2)$. According to Embrechts et al. (1997), the above choice of parameters for the ARCH(1) process yields a stationary distribution $F$ with right tail $1 - F(x) \sim x^{-8.36}$ as $x \to \infty$.

4.1 Confidence Intervals

In Figure 2 below, we vary the time series sample size from 200 to 2000 and examine the widths and empirical coverage probability of the 95% confidence intervals for ES at the 0.95 level (in the upper tail past the 95th percentile) produced by the sectioning and self-
normalization methods (Section 3.1) for the AR(1) and ARCH(1) processes introduced above. Each data point is the averaged result over 10,000 replications. For each replication, we initialize the AR(1) process with its stationary distribution, and for the ARCH(1) process we use a burn-in period of 5,000 to approximately reach stationarity.

Generally, the sectioning method produces smaller confidence intervals, but yields lower empirical coverage probability compared to the self-normalization method. This is especially pronounced for small sample sizes such as 200 or 400. As sample size increases, the performance of the two methods becomes more similar, and the desired coverage probability of 0.95 is approximately reached.

![Figure 2: Left: relationship between sample size and empirical coverage probability of 95% confidence intervals for ES at the 0.95 level computed for stationary AR(1) and ARCH(1) processes. Right: relationship between time series sample size and width of 95% confidence intervals for ES at the 0.95 level computed for stationary AR(1) and ARCH(1) processes. Each plotted point is the averaged result over 10,000 replications.](image)

To further examine the finite-sample performance of the sectioning method, in Figure 3 below, we show normalized histograms (with total area one) of the pivotal t-statistics formed when using the method with $m = 10$ sections, as discussed following Equation 9. We show histograms for different time series sample sizes (200-1,200). Each histogram uses 10,000 t-statistics. We compare the histograms with the density of a t-distribution with 9 degrees of freedom, which according to our theory, is the asymptotic (in the sense of time series sequence length tending to infinity) sampling distribution of the t-statistics.
formed when using the sectioning method with 10 sections. In accordance with our theory, for both the AR(1) and ARCH(1) processes, the sampling distribution of the t-statistics are well approximated by the asymptotic sampling distribution, even at small time series sample sizes. However, there is a small, but noticeable bias at small sample sizes where the sampling distribution of the t-statistics appears to be shifted to the left of the asymptotic sampling distribution. This bias becomes less noticeable as sample size increases.

4.2 Detection of Location Change in Tail

We also investigate through simulations the detection of abrupt location changes using ES at the 0.9 level (in the upper tail past the 90th percentile), as discussed in Section 3.2. We consider the same AR(1) and ARCH(1) processes introduced previously. Figure 4 below shows the approximate power of change-point tests using the self-normalized CUSUM statistic (Equations 13 and 14) at the 0.05 significance level as the magnitude of the abrupt location change varies between 0 (the null hypothesis of no change) and 3. For each data point, we perform 1,000 replications of change-point testing using times series sequences of length 400 with the potential changes occurring in the middle of the sequences. For each replication, we initialize the AR(1) process with its stationary distribution, and for the ARCH(1) process we use a burn-in period of 5,000 to approximately reach stationarity. As expected, for both processes, the power is a monotonic function of the magnitude of location change, and moreover the power curves are very similar and almost coincide. In accordance with the desired 0.05 significance level of our procedure, for both processes, the probability of false positive detection is approximately 0.05 (0.044 and 0.042 for the AR(1) and ARCH(1) processes, respectively, as indicated by the points with zero magnitude of location change in Figure 4).
Figure 3: “Goodness-of-fit” in constructing confidence intervals for ES at the 0.95 level for (1) stationary AR(1) process (top 6 histograms) and (2) approximately stationary ARCH(1) process (bottom 6 histograms) using the sectioning method with 10 sections. Each normalized histogram: 10,000 pivotal t-statistics resulting from the sectioning method with 10 sections. “Samples” refers to the sample sizes of the time series used. Red curve: density of t-distribution with 9 degrees of freedom.
Figure 4: Relationship between empirical detection probability and magnitude of location change for change-point detection with 0.05 significance level using ES at the 0.9 level. For both the AR(1) and ARCH(1) processes, the abrupt location change occurs in the middle of the time series sequence. Each plotted point is the average over 1,000 replications with time series sequences of length 400 in each replication.

In Figure 5 below, we show the sample path behavior of the self-normalized CUSUM process from Equation 13. Here, we consider a single realization of the ARCH(1) process used previously. The red lines correspond to the process sample paths under the alternative hypothesis with a unit magnitude location change in the middle of the ARCH(1) sequence of length 400. The blue dotted lines correspond to the process sample paths for the same realized ARCH(1) path, but under the null hypothesis of no location change. The horizontal black dashed line is the threshold for a 0.05 significance level, which if exceeded by the maximum of the self-normalized CUSUM process, results in rejection of the null hypothesis of no location change. As discussed in Section 3.2, the ratio form of the self-normalized CUSUM process allows unknown nuisance scale parameters to be canceled out, thereby allowing us to avoid the often problematic task of estimating standard errors in the setting of dependent data. In the case there is a location change, both the CUSUM process in the numerator and the self-normalizer process in the denominator contribute to the threshold exceedance of the self-normalized CUSUM process.
4.3 Detection of General Change in Tail

We also investigate detection of general structural changes in the upper tail of the underlying marginal distribution. Although the relationship between power and the “magnitude” of the change in the upper tail is not as simple as in the case of pure location changes, nevertheless, with Proposition 2 we will detect the change with high probability as our sample size increases. In our simulations, we study the detection of general structural changes in the tail using ES at the 0.95 level (in the upper tail past the 95th percentile), as discussed in Section 3.2. We consider variants of the AR(1) and ARCH(1) processes introduced previously.

\[
\text{AR(1) process: } X_{i+1} = \begin{cases} 
0.5X_i + t_i(16.5) & \text{for } i \leq [n/2] \\
0.5X_i + t_i(v) & \text{for } i > [n/2]
\end{cases}
\]
ARCH(1) process:  \[ X_{i+1} = \begin{cases} 
\sqrt{1 + 0.2X_i^2}\epsilon_i & \text{for } i \leq [n/2] \\
\sqrt{1 + \lambda X_i^2}\epsilon_i & \text{for } i > [n/2] 
\end{cases} \]

Here, each \( t_i(v) \) is a sample from the t-distribution with \( v \) degrees of freedom, and each \( \epsilon_i \) is a sample from the standard normal distribution. The parameter values \( v \) and \( \lambda \) after the change point in the two processes are adjusted, and we examine the effect on power. Figure 6 below shows the approximate power of change-point tests using the self-normalized CUSUM statistic (Equation 13) at the 0.05 significance level. For each data point, we perform 100 replications using times series sequences of length 1,000 and 2,000. For each replication of the AR(1) and ARCH(1) processes, we use an initial burn-in period of 5,000 to approximately reach stationarity. As expected, for both processes, the power is a monotonic function of the magnitude of change. In accordance with the desired 0.05 significance level of our procedure, for both processes, the probability of false positive detection is below 0.05 (0.03 and 0.026 for the AR(1) and ARCH(1) processes, respectively, as indicated by the leftmost point in each of the two plots of Figure 6).

Figure 6: Left: Change-point tests for AR(1) process. Relationship between empirical detection probability and degrees of freedom \( v \) of t-distributed innovations after the change point. Before the change point, \( v = 16.5 \). Right: Change-point tests for ARCH(1) process. Relationship between empirical detection probability and ARCH(1) parameter \( \lambda \) after the change point. Before the change point, \( \lambda = 0.2 \). In both cases, change-point testing is conducted with 0.05 significance level using ES at the 0.95 level. For both the AR(1) and ARCH(1) processes, the abrupt change occurs in the middle of the time series sequence. Each plotted point is an average over 100 replications, and \( n \) refers to the time series sequence length.
4.4 Detection of Multiple Changes in Tail

We additionally investigate detection of multiple structural changes in the upper tail of the underlying marginal distribution. Here, we again use ES at the 0.95 level (in the upper tail past the 95th percentile) and compare the practical performance of the single change-point methodology discussed in Section 3.2 versus the unsupervised multiple change-point methodology discussed in Section 3.3. We consider the following variant of the AR(1) process introduced previously.

\[
AR(1) \text{ process:} \quad X_{i+1} = \begin{cases} 
0.5X_i + t_i(16.5) & \text{for } i \leq \lfloor n/3 \rfloor \\
0.5X_i + t_i(v) & \text{for } \lfloor n/3 \rfloor < i \leq \lfloor 2n/3 \rfloor \\
0.5X_i + t_i(16.5) & \text{for } i > \lfloor 2n/3 \rfloor,
\end{cases}
\]  

(17)

where, as before, each \( t_i(v) \) is a sample from the t-distribution with \( v \) degrees of freedom. In this AR(1) process, the innovations initially have relatively light tails, then change to heavier tails (\( v < 16.5 \)), and finally revert back to the original lighter tails. Figure 7 below shows the approximate detection power as \( v \) is varied using the single change-point methodology (Equations 13 and 14) versus the unsupervised multiple change-point methodology (Equations 15 and 16) at the 0.05 significance level. For each data point, we perform 100 replications using times series sequences of length 1,500. For each replication, we use an initial burn-in period of 5,000 to approximately reach stationarity in the AR(1) process. We see that the single change-point testing method is unable to detect a change in the process in Equation 17, even for extremely strong deviations from the null such as the case \( v = 2.1 \). In fact, its power decays to zero as the magnitude of the change increases. On the other hand, the unsupervised multiple change-point testing method exhibits the desired performance with increasing power as the magnitude of the change increases. Hence, it is a promising candidate for detecting more complex patterns of changes in the tails of time series. In accordance with the desired 0.05 significance level of our procedure, the probability of false positive detection is below 0.05 (0.03 and 0.01 for the single change-point and unsupervised multiple change-point methodologies, respectively, as indicated by the leftmost points in Figure 7).
5 Empirical Applications

We first apply the two methods of confidence interval construction from Section 3.1 to daily log returns of the SPY ETF, which tracks the S&P 500 Index, for the years 2004-2016. In Figure 8 below, we show ES estimates of the lower 10th percentile of log returns throughout this time period along with 95% confidence bands computed using the sectioning and self-normalization methods. We use a rolling window of 100 days with 80 days of overlap between successive windows. The self-normalization method appears to be more conservative and yields a wider confidence band compared to the sectioning method, which agrees with the results presented in Figure 2. Overall, the ES estimates and confidence bands appear to capture well the increased volatility of returns during periods of financial instability such as during the 2008 Financial Crisis.
Figure 8: Top: Log returns for SPY ETF between January 7, 2004 and December 30, 2016, along with ES estimate and 95% confidence bands for lower 10th percentile. ES is computed using a rolling window of 100 days with 10 day shifts. Bottom: Cumulative log returns between January 7, 2004 and December 30, 2016.

We next apply the two methods of confidence interval construction from Section 3.1 to monthly log returns of US 30-Year Treasury bonds for the years 1942-2017. In Figure 9 below, we show ES estimates of the lower 20th percentile of log returns throughout this time period along with 95% confidence bands computed using the sectioning and self-normalization methods. We use a rolling window of 40 months with 20 months of overlap between successive windows. Again, the self-normalization method appears to be more conservative and yields a wider confidence band compared to the sectioning method, which agrees with the results presented in Figure 2. Although time series for government bond returns have a greater degree of autocorrelation compared to S&P 500 returns, our methods are still applicable.
Next, we apply our single change-point testing methodology at the 0.05 significance level to detect ES changes in the lower 5th percentile of SPY ETF log returns and also the lower 5th percentile of US 30-Year Treasury log returns. Five time series are shown in Figure 10 below, and for each, a change was detected using our method from Equations 13 and 14. First, we examine change-point tests for SPY ETF time series. Figure 10 (a) and (b) contain the 2008 Financial Crisis and 2011 August Stock Markets Fall, respectively. Figure 10 (c) also contains the 2011 August Stock Markets Fall, but with the change point located towards the end of the time series. Our methods are generally robust in situations where change points are located near the extremes of the time series. We also examine change-point tests for US 30-Year Treasury time series. Figure 10 (d) contains the 1980-1982 US recession (due in part to government restrictive monetary policy, and to a less degree the Iranian Revolution of 1979, which resulted in significant oil price increases). Figure 10 (e) contains the 2008 Financial Crisis.
Figure 10: Log returns for (a) SPY ETF between May 15, 2008 and December 17, 2008 (b) SPY ETF between May 6, 2011 and September 28, 2011 (c) SPY ETF between February 24, 2011 and August 16, 2011 (d) US 30-Year Treasury between June 29, 1973 and June 30, 1983 (e) US 30-Year Treasury between April 30, 2004 and August 31, 2012.

In Figure 11 below, we show the plots associated with change-point testing for the time series in Figure 10 (d). Similar to our findings for Figure 5, we see that the CUSUM process in the numerator and the self-normalizer process in the denominator (recall Equation 13) contribute to the threshold exceedance of the self-normalized CUSUM process.
Finally, we compare the effectiveness of the unsupervised multiple change-point testing and single change-point testing methodologies for detecting ES changes in the lower 5th percentile of SPY ETF log returns for longer time horizons. In agreement with our findings from the simulation study in Section 4.4, at the 0.05 significance level, we confirm that the unsupervised multiple change-point test is able to detect the presence of one or more change points in the SPY ETF time series between the start of 2007 and the end of 2010 (thus including the 2008 Financial Crisis) as shown in Figure 12, while the single change-point test is unable to detect any change.\(^5\) In particular, the value of the unsupervised multiple change-point test statistic in Equation (15) is 299.4, which well exceeds the test’s rejection threshold of 138.2 (corresponding to the 0.05 significance level) and gives a strong signal for rejection of the null hypothesis that there is no change point. However, the single change-point test performs poorly on this longer time series, and the value of the test statistic in Equation (13) is 1.915, which is well below the test’s rejection threshold of 40.1 (correspond-\(^5\)This could be due to the effect of multiple change points canceling out over the this longer time period.
ing to the 0.05 significance level). Thus, the single change-point test is reliable on shorter time horizons where only one change point occurs, but can fail on longer time horizons with multiple change points. Our unsupervised multiple change-point test provides a solution to this problem.

Figure 12: Log returns for SPY ETF between January 3, 2007 and December 20, 2010. A change in the tail distribution of this time series is clearly present between the end of 2008 through early 2009 (during the 2008 Financial Crisis). At the 0.05 significance level, the single change-point test is unable to detect a change in ES in the lower 5th percentile of the log returns, while the unsupervised multiple change-point test finds strong evidence of one or more changes.

6 Conclusion

We propose methodology to perform confidence interval construction and change-point testing for fundamental nonparametric estimators of risk such as ES. This allows for evaluation of the homogeneity of ES and related measures such as the conditional tail moments, and in particular allows the investigator to detect general tail structural changes in time series observations. While current approaches to tail structural change testing typically involve quantities such as the tail index and thus require parametric modeling of the tail, our approach does not require such assumptions. Moreover, we are able to detect more general structural changes in the tail using ES, for example, location and scale changes, which are
undetectable using tail index. Hence, we advocate the use of ES for general purpose monitoring for tail structural change. We note that our proposed sectioning and self-normalization methods for confidence interval construction and change-point testing still require some user choice, for example, the number of sections to use in sectioning or which particular functional to use in the self-normalization. Simulations suggest that our method is robust to these user choices. Therefore, we view our method as more robust compared to extant approaches which involve consistent estimation of standard errors or blockwise versions of the bootstrap or empirical likelihood, which can be more sensitive to tuning parameters. Our simulations illustrate the promising finite-sample performance of our procedures. Furthermore, we are able to construct pointwise ES confidence bands for SPY ETF returns in the period 2004-2016 and for US 30-Year Treasury returns in the period 1942-2017 that well capture periods of market distress such as the 2008 Financial Crisis. In similar spirit, our change-point tests are able to detect tail structural changes through the ES for both the SPY ETF returns and the US 30-Year Treasury returns during key times of financial instability in the recent past.

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**Appendix**

In what follows, let $p \in (0, 1)$ be a fixed probability level of interest for VaR, ES and their estimators. To keep notation simple in our proofs, we will use the shorthand $\hat{Q}_n(\cdot) := \hat{VaR}_n(\cdot)$ and $q(\cdot) := VaR(\cdot)$. When we are working at the probability level $p$ of interest, we
will omit the argument of $\hat{Q}_n(\cdot)$ and $q(\cdot)$ and simply write $\hat{Q}_n$ and $q$. We let $\hat{Q}_{tm}$ denote the VaR estimator and $\hat{F}_{tm}$ the empirical distribution function based on samples $X_1, \ldots, X_m$.

For a sample $X_1, X_2, \ldots, X_n$, denote the order statistics by $X_{n,1} \leq X_{n,2} \leq \cdots \leq X_{n,n}$.

**Proof of Theorem 1**

Denote $Y_n(t) = n^{-1/2}[nt](\hat{Q}_{nt} - q)$ and $Y_n'(t) = n^{-1/2} \sum_{j=1}^{[nt]}(1(X_j \leq q) - p)$ for $t \in [0, 1]$. Under Assumption 1, Theorem 0 of Herrndorf (1985) yields the convergence $Y_n \xrightarrow{d} \sigma' W$ for some $\sigma' \geq 0$. We will simply show that $\sup_{t \in [0,1]} |Y_n(t)f(q) - Y_n'(t)| \xrightarrow{d} 0$, from which it follows that $Y_n \xrightarrow{d} \sigma W$, where $\sigma = \sigma' / f(q)$. As in the proof of Theorem 6.2 of Sen (1972), for every positive integer $k$, define

$$k^* = |q(k^{-1/\delta})| + |q(1 - k^{-1/\delta})| + 1$$

for some $\delta > 0$. Consider a sequence $k_n$ of positive integers such that $k_n \to \infty$, but $n^{-1/2}k_n k_n^* \to 0$. We then have

$$\sup_{t \in [0,1]} |Y_n(t) - Y_n'(t)| \leq \sup_{t \in [0,n^{-1}k_n]} |Y_n(t)| + \sup_{t \in [0,n^{-1}k_n]} |Y_n'(t)| + \sup_{t \in [n^{-1}k_n, 1]} |Y_n(t) - Y_n'(t)|. $$

For the first term on the right hand side, notice that

$$\sup_{t \in [0,n^{-1}k_n]} |Y_n(t)| = \max_{1 \leq k \leq k_n} \frac{k}{n^{1/2} \sigma} |\hat{Q}_k - q| \leq \frac{k_n}{n^{1/2} \sigma} (|X_{k_n,k_n} - q| + |X_{k_n,1} - q|).$$

Then, with $n(1 - F(X_{n,n}))$ converging in distribution to a standard exponential, it holds

$$\mathbb{P} \left( q \leq X_{k_n,k_n} \leq q(1 - k_n^{-1/\delta}) \right) = \mathbb{P} \left( k_n(1 - p) \geq k_n(1 - F(X_{k_n,k_n})) \geq k_n^{-\delta} \right) \to 1.$$ 

Similarly, with $nF(X_{n,1})$ converging in distribution to a standard exponential,

$$\mathbb{P} \left( q(k_n^{-1/\delta}) \leq X_{k_n,1} \leq q \right) = \mathbb{P} \left( k_n^{-\delta} \leq k_n F(X_{k_n,1}) \leq k_n p \right) \to 1.$$ 

So asymptotically with probability one,

$$|q(1 - k_n^{-1/\delta}) - q| \geq |X_{k_n,k_n} - q|$$

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\[ |q(k_n^{-1-\delta}) - q| \geq |X_{k_n,1} - q| . \]

Thus, asymptotically with probability one,

\[
\sup_{t \in [0,n^{-1}k_n]} |Y_n(t)| \leq \frac{k_n}{n^{1/2}\sigma}(k_n^* + 2|q|) \rightarrow 0 .
\]

The functional convergence \( Y_n' \overset{d}{\rightarrow} \sigma'W \) for some \( \sigma' \geq 0 \), due to Herrndorf (1985) gives us

\[
\sup_{t \in [0,n^{-1}k_n]} |Y_n'(t)| \overset{d}{\rightarrow} 0 .
\]

Lastly, by the Bahadur representation for \( \hat{Q}_n \) in Theorem 1 of Wendler (2011), which holds under Assumption 1,

\[
\sup_{t \in [n^{-1}k_n,1]} |Y_n(t) - Y_n'(t)| = \max_{k_n \leq k \leq n} kn^{-1/2} \left| \hat{Q}_k - q - \frac{1}{f(q)}k^{-1} \sum_{j=1}^{k} (\mathbb{I}(X_j \leq q) - p) \right|
\]

\[
= \max_{k_n \leq k \leq n} kn^{-1/2} o_{a.s.}(k^{-5/8}(\log k)^{3/4}(\log \log k)^{1/2})
\]

\[
= o_{a.s.}(1) .
\]

This concludes the proof of the VaR functional central limit theorem.

Now we continue with the proof of the ES functional central limit theorem. First, under Assumption 2 by Theorem 1.7 of Ibragimov (1962) and Theorem 0 of Herrndorf (1985), we have the convergence in distribution of the process

\[
\left\{ n^{-1/2}\left([nt]q + \frac{1}{1-p} \sum_{k=1}^{[nt]} [X_k - q]_+ - [nt]\mathbb{E}[X \mid X \geq q] \right) : t \in [0,1] \right\}
\]

to \( \sigma W \) in \( D[0,1] \), where \( \sigma \geq 0 \). Now we show that

\[
\sup_{t \in [0,1]} n^{-1/2} \left| \frac{1}{1-p} \sum_{k=1}^{[nt]} X_k \mathbb{I}(\hat{Q}_{[nt]} \leq X_k) - \left([nt]q + \frac{1}{1-p} \sum_{k=1}^{[nt]} [X_k - q]_+ \right) \right| = o_p(1) ,
\]
from which the convergence in distribution of the process
\[
\left\{ n^{-1/2} \frac{1}{1-p} \sum_{k=1}^{[nt]} X_k \mathbb{I}\left( \hat{Q}_{[nt]} \leq X_k \right) - \mathbb{E}[X | X \geq q] : t \in [0,1] \right\}
\]
to \( \sigma W \) in \( D[0,1] \) follows immediately. Using the inequalities at the very end of the proof of Proposition 1 below, we have
\[
\sup_{t \in [0,1]} n^{-1/2} \left| \frac{1}{1-p} \sum_{k=1}^{[nt]} X_k \mathbb{I}\left( \hat{Q}_{[nt]} \leq X_k \right) - \left( [nt]q + \frac{1}{1-p} \sum_{k=1}^{[nt]} [X_k - q]_+ \right) \right|
\leq \sup_{t \in [0,1]} \frac{1}{1-p} \sum_{k=1}^{[nt]} \left| [q] \left| \hat{F}_k[\hat{Q}_{[nt]}] - p \right| + \hat{Q}_{[nt]} - q \left( 3 \left| \hat{F}_k(\hat{Q}_{[nt]}) - p \right| + \hat{F}_k(q) - p \right) \right|
\leq \sup_{1 \leq k \leq n} \frac{1}{1-p} \frac{k}{n^{1/2}} \left( \left| [q] \hat{F}_k(\hat{Q}_k) - p \right| + \hat{Q}_k - q \left( 3 \left| \hat{F}_k(\hat{Q}_k) - p \right| + \hat{F}_k(q) - p \right) \right).
\]

Observe that for some \( C > 0 \), we have
\[
\sup_{1 \leq k \leq n} \frac{1}{1-p} \frac{k}{n^{1/2}} \left| [q] \hat{F}_k(\hat{Q}_k) - p \right| \overset{a.s.}{\leq} C \frac{1}{1-p} \frac{1}{n^{1/2}} |q| = O(n^{-1/2}).
\]

Let \( k_n \rightarrow \infty \) be the sequence from the proof of the VaR functional central limit theorem. Then observe that
\[
\sup_{1 \leq k \leq n} \frac{1}{1-p} \frac{k}{n^{1/2}} \left| \hat{Q}_k - q \right| \left( 3 \left| \hat{F}_k(\hat{Q}_k) - p \right| + \hat{F}_k(q) - p \right) \leq \sup_{1 \leq k \leq k_n} \frac{4}{1-p} \frac{k}{n^{1/2}} \left| \hat{Q}_k - q \right| + \sup_{k_n \leq k \leq n} \frac{1}{1-p} \frac{k}{n^{1/2}} \left| \hat{Q}_k - q \right| \left( 3 \left| \hat{F}_k(\hat{Q}_k) - p \right| + \hat{F}_k(q) - p \right).
\]

As was shown in the proof of the VaR functional central limit theorem,
\[
\sup_{1 \leq k \leq k_n} \frac{4}{1-p} \frac{k}{n^{1/2}} \left| \hat{Q}_k - q \right| = o_{a.s.}(1).
\]
Finally, using the Bahadur representation from Proposition [I], we have

\[
\sup_{k_n \leq k \leq n} \frac{1}{1 - p n^{1/2}} \left| \hat{Q}_k - q \right| \left( 3 \left| \hat{F}_k(\hat{Q}_k) - p \right| + \left| \hat{F}_k(q) - p \right| \right) \\
\leq \sup_{k_n \leq k \leq n} \frac{1}{1 - p n^{1/2}} k \left( k^{-1/(2a)} + \delta' \log k \right) \\
\leq \sup_{k_n \leq k \leq n} \frac{1}{1 - p} o_{a.s.} \left( k^{-1/(2a)} + \delta' \log k \right) \\
= o_{a.s.}(1).
\]

Note from the proof of Proposition 1 below that \(-1/2 + 1/(2a) + \delta' < 0\), and so the last equality holds. We have thus shown the stronger version

\[
\sup_{t \in [0,1]} n^{-1/2} \left| \frac{1}{1 - p} \sum_{k=1}^{\lceil nt \rceil} X_k I(\hat{Q}_{\lceil nt \rceil} \leq X_k) - \left( \lceil nt \rceil q + \frac{1}{1 - p} \sum_{k=1}^{\lceil nt \rceil} [X_k - q]_+ \right) \right| = o_{a.s.}(1),
\]

and the proof is complete.

**Proof of Proposition 1**

First, note the following.

\[
\left| \frac{1}{1 - p} \sum_{k=1}^{n} X_k I(\hat{Q}_{n} \leq X_k) - \left( \hat{Q}_n + \frac{1}{1 - p} \sum_{k=1}^{n} [X_k - \hat{Q}_n]_+ \right) \right| \\
\leq \frac{1}{1 - p} \left| \hat{Q}_n \right| \left| \hat{F}_n(\hat{Q}_n) - p \right|.
\]
Next, observe the following.

\[
\left| \left( \hat{Q}_n + \frac{1}{1-p} \frac{1}{n} \sum_{k=1}^{n} [X_k - \hat{Q}_n]_+ \right) - \left( q + \frac{1}{1-p} \frac{1}{n} \sum_{k=1}^{n} [X_k - q]_+ \right) \right|
\]

\[
= \left| (\hat{Q}_n - q) + \frac{1}{1-p} \frac{1}{n} \sum_{k=1}^{n} (q - \hat{Q}_n) \mathbb{I} (\hat{Q}_n \leq X_k) + \frac{1}{1-p} \frac{1}{n} \sum_{k=1}^{n} (X_k - q) \left( \mathbb{I} (\hat{Q}_n \leq X_k) - \mathbb{I} (q \leq X_k) \right) \right|
\]

\[
\overset{a.s.}{=} \frac{1}{1-p} (\hat{Q}_n - q) (\hat{F}_n(\hat{Q}_n) - p) + \frac{1}{1-p} \frac{1}{n} \sum_{k=1}^{n} (X_k - q) \left( \mathbb{I} (\hat{Q}_n \leq X_k) - \mathbb{I} (q \leq X_k) \right)
\]

\[
\leq \frac{1}{1-p} |\hat{Q}_n - q| \left| \hat{F}_n(\hat{Q}_n) - p \right| + \frac{1}{1-p} |\hat{Q}_n - q| \left| \frac{1}{n} \sum_{k=1}^{n} \mathbb{I} (\hat{Q}_n \leq X_k) - \frac{1}{n} \sum_{k=1}^{n} \mathbb{I} (q \leq X_k) \right|
\]

\[
\overset{a.s.}{=} \frac{1}{1-p} |\hat{Q}_n - q| \left| \hat{F}_n(\hat{Q}_n) - p \right| + \frac{1}{1-p} |\hat{Q}_n - q| \left| \hat{F}_n(\hat{Q}_n) - \hat{F}_n(q) \right|
\]

\[
\leq \frac{1}{1-p} |\hat{Q}_n - q| \left( 2 \left| \hat{F}_n(\hat{Q}_n) - p \right| + \left| \hat{F}_n(q) - p \right| \right)
\]

We claim that under Assumption 1, \( \left| \hat{F}_n(\hat{Q}_n) - p \right| = O_{a.s.}(n^{-1}) \). Simply notice the following.

\[
\left| \hat{F}_n(\hat{Q}_n) - p \right| \leq n^{-1} \sum_{k=1}^{n} \mathbb{I} \left( \hat{Q}_n = X_k \right)
\]

\[
\leq n^{-1} + \mathbb{I} \left( \hat{Q}_n = X_i = X_j, i \neq j \right)
\]

\[
= n^{-1} + \mathbb{I} \left( \hat{Q}_n = X_i = X_j, i \neq j, \hat{Q}_n \in \mathcal{O} \right) + \mathbb{I} \left( \hat{Q}_n = X_i = X_j, i \neq j, \hat{Q}_n \notin \mathcal{O} \right)
\]

\[
\leq n^{-1} + \mathbb{I} \left( X_i = X_j, i \neq j, X_i \in \mathcal{O} \right) + \mathbb{I} \left( \hat{Q}_n \notin \mathcal{O} \right)
\]

\[
\overset{a.s.}{=} n^{-1} + \mathbb{I} \left( \hat{Q}_n \notin \mathcal{O} \right) \quad \text{by Assumption 1 (ii)}
\]

\[
\overset{a.s.}{=} n^{-1} \quad \text{for } n \text{ sufficiently large.}
\]

Here, \( \mathcal{O} \) is the neighborhood of \( q \) in Assumption 1 (ii), and the last line of the above is due to the strong consistency of \( \hat{Q}_n \), which holds by Theorem 2.1 of Xing et al. (2012) under Assumption 1. Also by Theorem 2.1 of Xing et al. (2012), \( |\hat{Q}_n - q| = o_{a.s.}(n^{-1/2} \log n) \).

Now we examine the term \( \left| \hat{F}_n(q) - p \right| \). Let \( \delta' > 0 \) such that \(-1/2 + 1/(2a) + \delta' < 0\).
Note that

\[ P \left( \left| \hat{F}_n(q) - p \right| > n^{-1/2+1/(2a)+\delta'} \right) = P \left( \left| \sum_{k=1}^{n} (\mathbb{I}(X_k \leq q) - p) \right| > n^{1/2+1/(2a)+\delta'} \right) \]

\[ \leq \frac{E \left[ \left| \sum_{k=1}^{n} (\mathbb{I}(X_k \leq q) - p) \right|^{2a} \right]}{n^{a+1+2a\delta'}} \]

\[ \leq \frac{Kn^a}{n^{a+1+2a\delta'}} = Kn^{1-2a\delta'} \]

Given Assumption (i), the second inequality above is due to Theorem 4.1 of Shao and Yu (1996), where \( K \) is a global constant depending only on \( C > 0 \) and \( a > 1 \) such that \( \alpha(n) \leq Cn^{-a} \) for all \( n \). Since \( \sum_{n=1}^{\infty} Kn^{1-2a\delta'} < \infty \), the First Borel-Cantelli Lemma yields

\[ \left| \hat{F}_n(q) - p \right| = O_{a.s.}(n^{-1/2+1/(2a)+\delta'}). \]

The triangle inequality then yields

\[ \left| \frac{1}{1-p} \sum_{k=1}^{n} X_k \mathbb{I}(\hat{Q}_n \leq X_k) - \left( q + \frac{1}{1-p} \sum_{k=1}^{n} [X_k - q]_+ \right) \right| \]

\[ \leq \frac{1}{1-p} \left( \left| \hat{Q}_n \right| \left| \hat{F}_n(\hat{Q}_n) - p \right| + \left| \hat{Q}_n - q \right| (2 \left| \hat{F}_n(\hat{Q}_n) - p \right| + \left| \hat{F}_n(q) - p \right|) \right) \]

\[ \leq \frac{1}{1-p} \left( \left| q \right| \left| \hat{F}_n(\hat{Q}_n) - p \right| + \left| \hat{Q}_n - q \right| \left| \hat{F}_n(\hat{Q}_n) - p \right| + \left| \hat{Q}_n - q \right| (2 \left| \hat{F}_n(\hat{Q}_n) - p \right| + \left| \hat{F}_n(q) - p \right|) \right) \]

\[ = \frac{1}{1-p} \left( \left| q \right| \left| \hat{F}_n(\hat{Q}_n) - p \right| + \left| \hat{Q}_n - q \right| (3 \left| \hat{F}_n(\hat{Q}_n) - p \right| + \left| \hat{F}_n(q) - p \right|) \right) \]

\[ = o_{a.s.}(n^{-1+1/(2a)+\delta'} \log n). \]

**Proof of Theorem 2**

We first show the result for VaR. We apply Theorem 2.5 of Volgushev and Shao (2014). To check that the required conditions of their theorem hold, we note the following. By van der Vaart and Wellner (1996) Example 3.9.21, for fixed \( p \in (0,1) \), the mapping from a distribution function to its VaR at level \( p \): \( F \mapsto \text{VaR}(p) \) is Hadamard-differentiable tangentially to the set of functions continuous at \( \text{VaR}(p) \). Also, by Theorem 1 of Bucher
the sequential empirical process
\[
\left\{ \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} (\mathbb{I}(X_i \leq x) - F(x)), \ (t, x) \in [0, 1] \times \mathbb{R} \right\}
\]
converges in distribution in \(\ell^\infty([0, 1] \times \mathbb{R})\), with respect to the uniform topology, to a tight, centered Gaussian process. By Addendum 1.5.8 of van der Vaart and Wellner (1996), almost surely, the limit process has uniformly continuous paths, with respect to the Euclidean metric on \([0, 1] \times \mathbb{R}\). Thus, Theorem 2.5 of Volgushev and Shao (2014) yields the desired result for VaR.

We continue with the proof of the result for ES. Recall, under Assumption 2, by Theorem 1.7 of Ibragimov (1962) and Theorem 0 of Herrndorf (1985), we have the convergence in distribution of the process
\[
\left\{ n^{-1/2} \left( [nt]q + \frac{1}{1-p} \sum_{k=1}^{[nt]} [X_k - q]_+ - [nt] \mathbb{E}[X \mid X \geq q] \right) : t \in [0, 1] \right\}
\]
to \(\sigma W\) in \(D[0, 1]\), where \(\sigma \geq 0\). Since the limit process has continuous paths, almost surely, it is well known that the distributional convergence can be taken with respect to the uniform topology (see Pollard (1984)). Also, recall the definition of the index set \(\Delta = \{ s, t \in [0, 1], t - s \geq \delta \}\), for arbitrarily small \(\delta > 0\). Next, considering the continuous mapping from \(C([0, 1]) \rightarrow C(\Delta)\) (which are the spaces of continuous functions on \([0, 1]\) and \(\Delta\), equipped with their respective uniform metrics) given by \(\{ Y(t), t \in [0, 1] \} \mapsto \{ Y(t) - Y(s), (s, t) \in \Delta \}\), the continuous mapping theorem yields convergence in distribution of the process
\[
\left\{ n^{-1/2} \left( ([nt] - [ns])q + \frac{1}{1-p} \sum_{k=[ns]+1}^{[nt]} [X_k - q]_+ - ([nt] - [ns]) \mathbb{E}[X \mid X \geq q] \right) : (s, t) \in \Delta \right\}
\]
to \(\{ \sigma(W(t) - W(s)), (s, t) \in \Delta \}\) in \(\ell^\infty(\Delta)\), where \(\sigma \geq 0\). Hence, it suffices to show
\[
\sup_{(s, t) \in \Delta} \frac{1}{\sqrt{n}} \left| \frac{1}{1-p} \sum_{k=[ns]+1}^{[nt]} X_k \mathbb{I}(X_k \geq \hat{Q}_{[ns]+1:[nt]}) - \left( ([nt] - [ns])q + \frac{1}{1-p} \sum_{k=[ns]+1}^{[nt]} [X_k - q]_+ \right) \right| = o_P(1).
\]
Using the inequalities at the very end of the proof of Proposition 1 above, we have

\[
\sup_{(s,t) \in \Delta} \frac{1}{\sqrt{n}} \left| \frac{1}{1 - p} \sum_{k = [ns] + 1}^{[nt]} X_k \mathbb{I}(X_k \geq \hat{Q}_{[ns] + 1; [nt]}) - \left( ([nt] - [ns])q + \frac{1}{1 - p} \sum_{k = [ns] + 1}^{[nt]} [X_k - q]_+ \right) \right|
\leq \sup_{(s,t) \in \Delta} \frac{1}{1 - p} \frac{[nt] - [ns]}{\sqrt{n}} \left( |q| \left| \hat{F}_{[ns] + 1; [nt]}(\hat{Q}_{[ns] + 1; [nt]}) - p \right| + \hat{Q}_{[ns] + 1; [nt]} - q \right) \left( 3 \left| \hat{F}_{[ns] + 1; [nt]}(\hat{Q}_{[ns] + 1; [nt]}) - p \right| + \left| \hat{F}_{[ns] + 1; [nt]}(q) - p \right| \right).
\]

By the assumption that \((X_1, X_k)\) has joint density for all \(k \geq 2\), there can be no ties among \(X_1, X_2, \ldots\). So with \(\delta > 0\) from the definition of the index set \(\Delta\),

\[
\sup_{(s,t) \in \Delta} \left| \hat{F}_{[ns] + 1; [nt]}(\hat{Q}_{[ns] + 1; [nt]}) - p \right| \leq \frac{1}{[nt] - [ns]}.
\]

Hence, it suffices to show, as \(n \to \infty\),

\[
\sup_{(s,t) \in \Delta} \frac{[nt] - [ns]}{\sqrt{n}} \left( \hat{Q}_{[ns] + 1; [nt]} - q \right) \left| \hat{F}_{[ns] + 1; [nt]}(q) - p \right| = o_P(1).
\]

In light of our above arguments and the result for VaR established above, we have, as \(n \to \infty\),

\[
\sup_{(s,t) \in \Delta} \sqrt{n}(t - s) \left| \hat{F}_{[ns] + 1; [nt]}(q) - p \right| = O_P(1)
\]

\[
\sup_{(s,t) \in \Delta} \sqrt{n}(t - s) \left( \hat{Q}_{[ns] + 1; [nt]} - q \right) = O_P(1),
\]

from which the desired result follows immediately.

**Proof of Proposition 2**

We are concerned with the following variant of the CUSUM process:

\[
\left\{ n^{-1/2} \left( \frac{n - [nt]}{n} \sum_{k=1}^{[nt]} X_k \mathbb{I}(\hat{Q}_{1; [nt]} \leq X_k) \right) \right\}^{[nt]} \sum_{k=1}^{[nt]} \frac{X_k \mathbb{I}(\hat{Q}_{[nt] + 1; n} \leq X_k)}{1 - p} - \frac{n}{[nt] + 1} \sum_{k=1}^{[nt] + n} \frac{X_k \mathbb{I}(\hat{Q}_{[nt] + 1; n} \leq X_k)}{1 - p} : t \in [0, 1] \right\}.
\]

Under Assumption 2, using Theorem 1 it is straightforward to obtain convergence in distribution in \(D[0, 1]\) of the above process to \(\sigma B\) where \(B(t) := W(t) - tW(1)\) (for \(t \in [0, 1]\)) is
standard Brownian bridge on \([0, 1]\). Notice that under Assumption 1 by Theorem 1

\[
\left\{ n^{-1/2} \left( \frac{n - [nt]}{n} \sum_{k=1}^{[nt]} \frac{\max(X_k, q)}{1 - p} - \frac{n}{n} \sum_{k=[nt]+1}^{n} \frac{\max(X_k, q)}{1 - p} \right) : t \in [0, 1] \right\}
\]

converges in distribution in \(D[0, 1]\) to \(\sigma B\). Then we get the desired conclusion by observing that

\[
\sup_{t \in [0, 1]} n^{-1/2} \left| \frac{n - [nt]}{n} \sum_{k=1}^{[nt]} \frac{X_k \mathbb{1}[\hat{Q}_{[nt]} \leq X_k]}{1 - p} - \frac{n}{n} \sum_{k=[nt]+1}^{n} \frac{X_k \mathbb{1}[\hat{Q}_{[nt]+1:n} \leq X_k]}{1 - p} \right|
\]

\[
\leq \sup_{t \in [0, 1]} n^{-1/2} \left| \sum_{k=1}^{[nt]} \frac{X_k \mathbb{1}[\hat{Q}_{[nt]} \leq X_k]}{1 - p} - \sum_{k=1}^{[nt]} \frac{\max(X_k, q) - qp}{1 - p} \right|
\]

\[
+ \sup_{t \in [0, 1]} n^{-1/2} \left| \sum_{k=[nt]+1}^{n} \frac{X_k \mathbb{1}[\hat{Q}_{[nt]+1:n} \leq X_k]}{1 - p} - \sum_{k=[nt]+1}^{n} \frac{\max(X_k, q) - qp}{1 - p} \right|
\]

\[
\leq \sup_{t \in [0, 1]} n^{-1/2} \left| \sum_{k=1}^{[nt]} \frac{X_k \mathbb{1}[\hat{Q}_{[nt]} \leq X_k]}{1 - p} - \left( \frac{nt}{p} + \sum_{k=1}^{[nt]} \frac{[X_k - q]^+}{1 - p} \right) \right|
\]

\[
+ \sup_{t \in [0, 1]} n^{-1/2} \left| \sum_{k=[nt]+1}^{n} \frac{X_k \mathbb{1}[\hat{Q}_{[nt]+1:n} \leq X_k]}{1 - p} - \left( (n - [nt])q + \sum_{k=[nt]+1}^{n} \frac{[X_k - q]^+}{1 - p} \right) \right|
\]

\[= o_p(1).\]

The convergence in probability to zero follows by stationarity of the sequence \(X_1, X_2, \ldots\) and the last line of the proof of Theorem 1.