Stochastic Successive Convex Approximation for Non-Convex Constrained Stochastic Optimization

An Liu, Senior Member, IEEE, Vincent Lau, Fellow IEEE and Borna Kananian, Student Member, IEEE
Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology

Abstract—This paper proposes a constrained stochastic successive convex approximation (CSSCA) algorithm to find a stationary point for a general non-convex stochastic optimization problem, whose objective and constraint functions are non-convex and involve expectations over random states. The existing methods for non-convex stochastic optimization, such as the stochastic (average) gradient and stochastic majorization-minimization, only consider minimizing a stochastic non-convex objective over a deterministic convex set. To the best of our knowledge, this paper is the first attempt to handle stochastic non-convex constraints in optimization problems, and it opens the way to solving more challenging optimization problems that occur in many applications. The algorithm is based on solving a sequence of convex objective/feasibility optimization problems obtained by replacing the objective/constraint functions in the original problems with some convex surrogate functions. The CSSCA algorithm allows a wide class of surrogate functions and thus provides many freedoms to design good surrogate functions for specific applications. Moreover, it also facilitates parallel implementation for solving large scale stochastic optimization problems, which arise naturally in today’s signal processing such as machine learning and big data analysis. We establish the almost sure convergence of the CSSCA algorithm and customize the algorithmic framework to solve several important application problems. Simulations show that the CSSCA algorithm can achieve superior performance over existing solutions.

Index Terms—Non-convex stochastic optimization, Successive convex approximation, Parallel optimization

I. INTRODUCTION

A. Background

Deterministic convex optimization theory is very powerful and allows low complexity solutions for large scale problems. However, stochastic processes and effects appear naturally in the real physical world and in many cases, their effects cannot be neglected. For example, in wireless communications, we have random channel noise, random channel fading as well as random interference, and we would be able to have an infinite data rate without errors if the effect of channel noise were to be ignored. In signal processing applications, such as radar detection or signal recovery, we need to extract useful signals and data from those that are contaminated in noisy observations. In all these examples, the physical system is not deterministic and it is naturally important to take into account the underlying random process in modeling optimization problems. This motivates the study of stochastic optimization. In fact, stochastic optimizations play a critical role in various key application areas such as wireless resource optimizations, compressive sensing and (sparse) signal recovery, machine learning, etc.

Despite the important role of stochastic optimization in many applications, it is still far from mature compared to its deterministic counterpart. For example, we still lack an efficient algorithm to solve non-convex stochastic optimization problems that occur in many applications, especially when the constraint is also non-convex and involves expectations over random states. Moreover, many applications dealing with large systems require solving large scale (non-convex) stochastic optimization problems. In this case, it is desirable to design parallel algorithms that can distribute the computational load across a number of computation nodes. In this paper, we propose a constrained stochastic successive convex approximation (CSSCA) method for general non-convex stochastic optimization problems whose objective and constraints contain expectations of non-convex functions. The CSSCA method is also suitable for parallel implementation.

B. Related works

There are three major existing methods on non-convex stochastic optimization.

Stochastic Gradient-based methods: Stochastic gradient/subgradient [1] is a common method to solve unconstrained stochastic optimization problems. In each iteration, an unbiased estimation of the gradient of the objective function is obtained and a gradient-like update is performed. Under some technical conditions, almost sure convergence to stationary points can be established [2]. Various variations of the stochastic gradient method have been proposed [3]–[6]. For convex stochastic optimization problems with a simple convex feasible set, the stochastic gradient projection method has been proposed and been shown to converge to the optimal solution almost surely [7], [8]. However, the convergence of the stochastic gradient projection is no longer guaranteed when it is applied to non-convex stochastic optimization problems. To handle the non-convexity, a gradient averaging method [9], [10] is proposed where the gradient projection update at each iteration is based on the average of the current and past gradient samples. Intuitively, the average sample gradient tends to converge to the true gradient of the objective function and thus the convergence follows a similar analysis to that of the gradient projection method for deterministic non-convex problems. Under some technical conditions, one can indeed prove the convergence of the gradient averaging method to a stationary point [11]. There are also algorithms with averaging in both gradients and iterates (optimization variables), for example those in [12], [13].

This work was supported by the National Science Foundation of China, project no. 61571383.
Stochastic Majorization-Minimization: Majorization-minimization (MM) [14] is a powerful optimization principle that includes many well-known optimization methods as special cases, such as proximal gradient method [15], expectation-maximization (EM) algorithm [16], cyclic minimization [17], variational Bayes techniques [18], and DC programming [19] (where “DC” stands for difference of convex functions). The basic idea of MM is to iteratively minimize a surrogate function that upper-bounds the objective (but matches the value of the objective function and its derivative at the current iterate). MM monotonically decreases the objective value until convergence to a stationary point. Stochastic MM [20], [21] is an extension of MM to solve stochastic non-convex optimization problems. Specifically, at each iteration, a sample surrogate function is first obtained as an upper bound of the sample objective function. Then the updated optimization variable is obtained by minimizing the average surrogate function (the average of the current and past sample surrogate functions). Intuitively, the average surrogate function tends to converge to a deterministic upper bound of the objective function that matches the value of the objective function and its derivative at a limiting point, from which it can be shown that any limiting point of the algorithm is a stationary point. Please refer to [20], [21] for the formal convergence proof of the stochastic MM.

Stochastic Successive Convex Approximation (SCA): SCA [22] is similar to MM in the sense that it also iteratively minimizes a sequence of surrogate functions. However, the conditions on the surrogate functions are different. SCA requires the surrogate function to be convex but not necessarily an upper bound of the objective function. On the other hand, MM requires the surrogate function to be an upper bound of the objective function but not necessarily convex. Since there is no upper bound constraint, we have more freedom to choose a surrogate function at each iteration that can better approximate the objective function. As a result, SCA may yield a faster convergence speed with properly chosen surrogate functions. In [23], a stochastic parallel SCA method is proposed for non-convex stochastic sum-utility optimization problems in multi-agent networks. In this method, all agents update their optimization variables in parallel by solving a sequence of convex subproblems. Almost sure convergence to stationary points is also proved.

C. Contributions

All of the above existing works on non-convex stochastic optimization have assumed simple constraints where the feasible set of the problem can be represented by a deterministic convex set. However, in many applications, such as those considered in Section II, the constraints may involve expectations of non-convex functions. Moreover, there are few works on parallel algorithms that are suitable for large scale non-convex stochastic optimization, and the existing parallel algorithms often impose some restrictions on the objective/constraints. For example, the parallel SCA method in [23] assumes that the constraint can be represented by a Cartesian product of deterministic convex sets. These restrictions significantly limit the application of existing methods. In this paper, we propose a more general non-convex stochastic optimization method to avoid many of the above restrictions on the objective/constraints. The main contributions are summarized below.

- **A general stochastic SCA method and its convergence proof:** We propose a CSSCA method which can be applied to more general non-convex stochastic optimization problems whose objective and constraint contain expectations of non-convex functions. Moreover, we establish the convergence of the CSSCA method to stationary points. This opens the door to solving more difficult stochastic optimization problems that occur in many new applications.

- **Parallel CSSCA:** We propose a parallel CSSCA algorithm where the minimization of the surrogate function is decomposed into independent subproblems and each subproblem is solved by a user (computation node) in a parallel way. Such a parallel CSSCA algorithm is suitable for solving large-scale (non-convex) stochastic optimization problems arising in machine learning and signal processing.

- **Specific CSSCA algorithm design for some important applications:** We apply the CSSCA to solve several important application problems in wireless communications. We show that it is crucial to choose application specific surrogate functions for different applications. We believe that the proposed CSSCA-based solutions for these application problems alone are of great interest to the community.

The rest of the paper is organized as follows. The problem formulation is given in Section II together with some application examples. The CSSCA algorithm and the convergence analysis are presented in Section III and IV, respectively. The parallel CSSCA algorithm is proposed in Section V. Section VI applies the CSSCA method to solve several important application problems. Finally, the conclusion is given in Section VII

II. PROBLEM FORMULATIONS

Consider the following non-convex constrained stochastic optimization problem:

\[
\min_{x} f_{0}(x) \triangleq \mathbb{E} [g_{0}(x, \xi)]
\]

\[
s.t. f_{i}(x) \triangleq \mathbb{E} [g_{i}(x, \xi)] \leq 0, i = 1, ..., m,
\]

where \(x \in \mathcal{X}\) is the optimization variable with \(\mathcal{X}\) being the domain of the problem; and \(\xi\) is a random state defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), with \(\Omega\) being the sample space, \(\mathcal{F}\) being the \(\sigma\)-algebra generated by subsets of \(\Omega\), and \(\mathbb{P}\) being a probability measure defined on \(\mathcal{F}\). We assume that the functions \(g_{i} : \mathcal{X} \times \Omega \rightarrow \mathbb{R}, i = 0, ..., m\) are continuously differentiable (and possibly non-convex) functions in \(x\). Moreover, \(\mathbb{E} [g_{i}(x, \xi)]\), \(i = 0, ..., m\) are Lipschitz continuous and \(\mathcal{X}\) is compact. As a result, problem (1) embraces a lot of
important applications including chance constraint problems [24]. In the following, we give some important application examples of the problem formulation in (1).

**Example 1** (MIMO Transmit Signal Design with Imperfect CSI [25]). Consider a downlink system that consists of a multiple-antenna base station (BS) and $K$ single-antenna users. The BS is equipped with $n$ antennas, and it simultaneously transmits $K$ data streams to the $K$ users using MIMO signaling based on the estimated channel state information (CSI) $\hat{h}_k, k=1, ..., K$. The true channel vectors $h_k$’s can be modeled as $h_k = \hat{h}_k + e_k$, where $e_k$ represents the channel estimation error. With channel estimation error, the BS can no longer guarantee the desired rate for each user. In this case, the BS may improve the average MIMO transmission performance under the channel estimation error by ensuring that the expected rate of each user must exceed a target value. Specifically, the MIMO transmit signal design problem with imperfect CSI can be formulated as the following power minimization problem subject to the expected rate requirement:

$$\begin{align*}
\min_{\{Q_k\geq 0\}} \sum_{k=1}^{K} Tr(Q_k) \\
n.s.t. \quad \mathbb{E}\left[\log\left(1 + \frac{h_k^H Q_k h_k}{\sum_{j \neq k} h^H_k Q_j h_j + \sigma^2_k}\right)\right] \geq r_k, \forall k,
\end{align*}$$

where $Q_k$ is the covariance matrix of the transmit signal for user $k$, $\sigma^2_k$ is the variance of the thermal noise at user $k$, and $r_k$ is the expected rate requirement for user $k$. The expectation is taken w.r.t. the channel estimation error $e_k$ conditioned on $\hat{h}_k$. In Problem (2), the random state is $\xi = [e_1, ..., e_K]^T$. The sample objective function $g_0(x, \xi)$ is convex, and the sample constraint functions $g_i(x, \xi), i = 1, ..., K$ are non-convex.

**Example 2** (Robust Beamforming Design [26]). Consider the same MIMO downlink system with channel estimation error as in Example 1. However, unlike Example 1 where the expected rate of each user is guaranteed under the channel estimation error, we consider a stronger quality of service requirement where the rate of each user must exceed a target value with high probability. To be more specific, we consider the following robust beamforming design formulation:

$$\begin{align*}
\min_{\{w_k\}} \sum_{k=1}^{K} \|w_k\| \\
n.s.t. \quad \mathbb{E}\left[\eta_k \left(\sum_{i \neq k} |h^H_i w_i|^2 + \sigma^2_k\right) - |h^H_k w_k|^2\right] \leq \epsilon,
\end{align*}$$

where $w_k \in \mathbb{C}^n$ is the beamforming vector for user $k$, $\sum_{k=1}^{K} \|w_k\|$ is the total transmit power at the BS, and the constraint ensures that the SINR of user $k$ exceeds a target value $\eta_k$ with probability no less than $1 - \epsilon$. Note that the BS only knows $\hat{h}_k$. Therefore, (3) is a chance constraint with the random state given by the channel estimation error vector $\xi = [e_1, ..., e_K]^T$.

Problem (3) is a chance constrained problem [24] and is not exactly an instance of Problem (1). However, we can transform Problem (3) into an approximate formulation which is an instance of Problem (1) as follows. First, note that $\Pr\{SINR_k \leq \eta_k\} = \mathbb{E}[u(\eta_k - SINR_k)]$, where $u(x)$ is the step function. There are many smooth approximations of the step function. Let $\tilde{u}_\theta(x)$ denote a smooth approximation of the step function $u(x)$ with smooth parameter $\theta$, e.g., one possible form of a smooth approximate function is

$$\tilde{u}_\theta(x) = \frac{1}{1 + e^{-\theta x}},$$

where the smooth parameter $\theta$ can be used to control the approximation error. By replacing the step $u(x)$ with its smooth approximation $\tilde{u}_\theta(x)$, we can obtain an approximation of Problem (3):

$$\begin{align*}
\min_{\{w_k\}} \sum_{k=1}^{K} \|w_k\| \\
n.s.t. \quad \tilde{u}_\theta\left(\eta_k \left(\sum_{i \neq k} |h^H_i w_i|^2 + \sigma^2_k\right) - |h^H_k w_k|^2\right) \leq \epsilon,
\end{align*}$$

which is an instance of Problem (1). Using the above approximation, a general chance constrained problem can also be transformed into Problem (1).

**Example 3** (Massive MIMO Hybrid Beamforming Design [27]). Consider a multi-user massive MIMO downlink system where a BS serves $K$ single-antenna users. The BS is equipped with $M \gg 1$ antennas and $S$ transmit RF chains, where $K \leq S < M$. Hybrid beamforming [27], [28] is employed at the BS to support simultaneous transmissions to the $K$ users. Specifically, the precoder is split into a baseband precoder and an RF precoder as $FG$, where $G = [g_1, ..., g_K] \in \mathbb{C}^{S \times K}$ is the baseband precoder using the $S$ RF chains, and $F \in \mathbb{C}^{M \times S}$ is the RF precoder using, for example, the RF phase shifting network [29]. Hence, all elements of $F$ have equal magnitude, i.e., $F_m = e^{j\theta_m}$, where $\theta_m$ is the phase of the $(m, s)$-th element $F_m$ of $F$. For given RF precoder $F$, a zero-forcing baseband precoder is used to mitigate the multi-user interference, i.e.,

$$G = FF^H \left(HFF^H H^H\right)^{-1} P^{1/2},$$

where $H = [h_k]_{k=1}^{K} \in \mathbb{C}^{K \times M}$ is the composite channel matrix, $h_k \in \mathbb{C}^M$ is the channel vector of user $k$, and $P = \text{Diag}(p_1, ..., p_K)$ with $p_k$ representing a parameter to control the tradeoff between the transmit power allocated to user $k$ and the data rate of user $k$. Consider the maximization of the ergodic sum rate in the above massive MIMO system with hybrid beamforming:

$$\max_{\Theta, p} \sum_{k=1}^{K} \log_2 \left(1 + p_k\right)$$

$$\text{s.t.} \quad f(\Theta, p) := \mathbb{E}\left[Tr(FGG^H F^H)\right] - P \leq 0,$$

where $\Theta \in \mathbb{C}^{M \times S}$ and the $(m, s)$-th element of $\Theta$ is $\theta_{m,s}$, $p = [p_1, ..., p_K]^T$, and $P$ is the average transmit power.
III. CONSTRAINED STOCHASTIC SUCCESSIVE CONVEX APPROXIMATION

A. Challenges of Solving Problem (1)

Since Problem (1) is, in general, non-convex, we focus on designing an efficient algorithm to find a stationary point of Problem (1). There are two major challenges in solving Problem (1): 1) the non-convexity of the constraint functions; and 2) the stochastic nature of the constraint functions (i.e., it is difficult to accurately calculate the expectations in the constraint functions).

For the special case when $\xi$ is a deterministic vector, (1) reduces to a deterministic optimization problem with non-convex constraint. In this case, an MM algorithm has been proposed in [30] to find a stationary point. The MM algorithm in [30] starts from a feasible point. Due to the property of MM, it can be shown that all the subsequent iterates generated by the MM algorithm are still feasible, and the algorithm will eventually converge to a stationary point. However, in the stochastic case, it is not easy to find a feasible point since the constraint functions are non-convex and may not have closed-form expressions. Moreover, even if we can find a feasible initial point, the stochastic MM algorithm can no longer ensure that all the subsequent iterates are still feasible due to the randomness caused by $\xi$. As a result, it is much more challenging to design an algorithm for Problem (1) which involves stochastic non-convex constraints. Indeed, to the best of our knowledge, there lacks an efficient algorithm in the literature to handle stochastic non-convex constraints. Most existing algorithms for non-convex stochastic optimization only consider deterministic and convex constraints.

Challenge 1 (Challenges of Algorithm Design). Design an efficient algorithm to find a stationary point of Problem (1) with stochastic non-convex objective and constraint functions. Due to noisy estimate of the constraints, the sequence of iterates generated by the algorithm is not always feasible. How to ensure the limiting point of the algorithm is feasible almost surely? Moreover, both the constraint and objective functions contain expectation and are not necessarily convex; how to ensure a limiting point of the algorithm is a stationary point almost surely?

B. Summary of Algorithm

We propose a constrained stochastic successive convex approximation (CSSCA) algorithm to solve Problem (1), where at each iteration, $\bar{x}$ is updated by solving a convex optimization problem obtained by replacing the objective and constraint functions $f_i(x), i = 0, \ldots, m$ with their convex surrogate functions $\bar{f}_i(x), i = 0, \ldots, m$.

Specifically, at iteration $t$, a new realization of the random vector $\xi^t$ is obtained and the surrogate functions $\bar{f}_i(x), i = 0, \ldots, m$ are updated based on $\xi^t, \bar{x}^t$. The surrogate function $\bar{f}_i(x)$ can be viewed as a convex approximation of $f_i(x)$. Note that in order to allow maximum freedom for surrogate function design in different applications, we do not specify the exact form of the surrogate functions $\bar{f}_i(x), \forall i$ in this framework algorithm. In Section III.C, we will give conditions for the surrogate functions $\bar{f}_i(x), \forall i$ under which the convergence of the algorithm is guaranteed, and a few common methods to construct the surrogate functions that satisfy the convergence conditions.

Then the optimal solution $\bar{x}^t$ of the following problem is solved:

$$\bar{x}^t = \arg\min_{\bar{x}} \bar{f}_0(\bar{x})$$

$$s.t. \bar{f}_i(\bar{x}) \leq 0, i = 1, \ldots, m, \tag{7}$$

which is a convex approximation of (1). Note that Problem (2) is not necessarily feasible. If Problem (2) turns out to be infeasible, the optimal solution $\bar{x}^t$ of the following convex problem is solved:

$$\bar{x}^t = \arg\min_{x, \alpha} \alpha$$

$$s.t. \bar{f}_i(x) \leq \alpha, i = 1, \ldots, m, \tag{8}$$

which minimizes the constraint functions. Given $\bar{x}^t$ in one of the above two cases, $\bar{x}$ is updated according to

$$\bar{x}^{t+1} = (1 - \gamma^t) \bar{x}^t + \gamma^t \bar{x}^t. \tag{9}$$

where $\gamma^t \in (0, 1]$ is a sequence to be properly chosen. The overall algorithm is summarized in Algorithm I and the block diagram of the algorithm is given in Fig. 1.

C. Smooth Surrogate Function Construction

To guarantee the convergence of Algorithm 1, we need to make the following assumptions on the surrogate functions.

Assumption 1 (Assumptions on properties of surrogate functions). For all $i \in \{0, \ldots, m\}$ and $t = 0, 1, \ldots$, we have

1) $\bar{f}_i(x)$ is uniformly strongly convex in $x$. 
Algorithm 1. Constrained stochastic successive convex approximation

Input: \{\gamma_i\}.
Initialize: \(x^0 \in X\); \(t = 0\).

Step 1:
The random vector \(\xi^t\) is realized.
Update the surrogate functions \(f_i^t(x)\), \(\forall i\) using \(\xi^t\), \(x^t\).

Step 2:
//Objective update
If Problem (17) is feasible
Solve (17) to obtain \(\bar{x}^t\).

//Feasible update
Else
Solve (8) to obtain \(\bar{x}^t\).
End if

Step 3:
Update \(x^{t+1}\) according to (9).

Step 4:
Let \(t = t + 1\) and return to Step 1.

2) \(f_i^t(x)\) is a Lipschitz continuous function w.r.t. \(x\). Moreover, \(f_i^t(x) - f_i^2(x) \leq B \|x_i - x_i^2\| + e(t_1, t_2), \forall x \in X\) for some constant \(B > 0\), where \(\lim_{t_1, t_2 \rightarrow \infty} e(t_1, t_2) = 0\).

3) For any \(x \in X\), the function \(f_i^t(x)\), its derivative, and its second order derivative are uniformly bounded.

Assumption 2 (Asymptotic consistency of surrogate functions). For all \(i \in \{0, ..., m\}\), we have
\[
\lim_{t \rightarrow \infty} \|f_i^t(x^t) - f_i(x^t)\| = 0,
\lim_{t \rightarrow \infty} |f_i^t(x^t) - f_i(x^t)| = 0.
\]

These assumptions are quite standard and are satisfied for a large class of surrogate functions. In the following, we give some common examples of surrogate functions \(f_i^t(x)\) that satisfy the above assumptions.

1) Recursive Surrogate Function: In this case, the surrogate function \(f_i^t(x)\) can be expressed using a recursive formula as
\[
f_i^t(x) = (1 - \rho_i) f_i^{t-1}(x) + \rho_i \hat{g}_i(x, x^t, \xi^t),
\]
where \(\rho_i \in (0, 1]\) is a sequence to be properly chosen, \(\hat{g}_i(x, x^t, \xi^t)\) is a convex approximation of the function \(g_i(x, \xi)\) around the point \(x^t\) and it is called the \textit{sample surrogate function} at the \(t\)-th iteration. The initial value \(f_i^{t-1}(x) = 0\).

Assumption 3 (Assumptions on \(\hat{g}_i(x, x^t, \xi^t)\)). For all \(i \in \{0, ..., m\}\), we have
1) \(\hat{g}_i(x, x, \xi) = g_i(x, \xi)\) and \(\nabla \hat{g}_i(x, x, \xi) = \nabla g_i(x, \xi), \forall x \in X, \forall \xi \in \Omega\).
2) \(\hat{g}_i(x, y, \xi)\) is uniformly strongly convex in \(x\).
3) For any \(\xi \in \Omega\) and \(y \in X\), the function \(\hat{g}_i(x, y, \xi)\) is Lipschitz continuous in both \(x\) and \(y\).\)
4) The function \(\hat{g}_i(x, y, \xi)\), its derivative, and its second order derivative w.r.t. \(x\) are uniformly bounded.

An example of first order sample surrogate function \(\hat{g}_i(x, y, \xi)\) that satisfies Assumption 3 is
\[
\hat{g}_i(x, y, \xi) = g_i(y, \xi) + \nabla y g_i(y, \xi) (x - y) + \tau_i \|x - y\|^2, \tag{11}
\]
where \(\tau_i > 0\) can be any constant. The surrogate function in (11) includes the Lipschitz gradient surrogate function in [20] for stochastic MM as a special case. In the Lipschitz gradient surrogate function, \(\tau_i\) must be sufficiently large to ensure that \(\hat{g}_i(x, y, \xi) \geq g_i(x, \xi), \forall x \in X\). However, (11) does not have such a restriction and thus provides more freedom to design better surrogate functions.

2) Structured Surrogate Function in [23]: Suppose \(g_i(x, \xi)\) can be divided into two components as
\[
g_i(x, \xi) = g_i^c(x, \xi) + g_i^g(x, \xi),
\]
where \(g_i^c(x, \xi)\) is convex and \(g_i^g(x, \xi)\) can be either convex or non-convex. Then the structured surrogate function \(f_i^t(x)\) is given by [23]
\[
f_i^t(x) = (1 - \rho_i) f_i^{t-1} + \rho_i g_i^c(x, \xi^t) + \rho_i^g \nabla g_i^g(x, \xi^t) (x - x^t) + \tau_i \|x - x^t\|^2, \tag{12}
\]
where \(\tau_i > 0\) can be any constant, \(f_i^t\) is an approximation for \(\nabla \mathbb{E}[g_i(x^t, \xi)]\) and it is updated recursively according to
\[
f_i^t = (1 - \rho_i) f_i^{t-1} + \rho_i g_i^c(x^t, \xi^t),
\]
with \(f_i^{-1} = 0\), and \(f_i^t\) is an approximation for the gradient \(\nabla \mathbb{E}[g_i(x^t, \xi)]\), which is updated recursively according to
\[
f_i^t = (1 - \rho_i) f_i^{t-1} + \rho_i \nabla g_i(x^t, \xi^t),
\]
with \(f_i^{-1} = 0\). The structured surrogate function in [12] contains the convex component \(g_i^c(x, \xi)\) of the original sample objective function \(g_i(x, \xi)\), which helps to reduce the approximation error and potentially achieve a faster initial convergence speed [23].

3) Validity of the above Surrogate Functions: We formally prove that the above two surrogate functions satisfy the conditions in Assumption 1 and 2 under the following conditions on the step sizes.

Assumption 4 (Assumptions on step sizes).
1) \(\rho_i \rightarrow 0, \sum \rho_i = \infty, \sum (\rho_i)^2 < \infty\),
2) \(\gamma_i \rightarrow 0, \sum \gamma_i = \infty, \sum (\gamma_i)^2 < \infty\),
3) \(\lim_{t \rightarrow \infty} \gamma_i / \rho_i = 0\).

Proposition 1 (Validity of the recursive surrogate). Under Assumption 3 and 2 if we choose the surrogate functions \(f_i^t(x), \forall i\) as in (10), then Assumption 1 and 2 are satisfied. Please refer to Appendix A for the proof.

Proposition 2 (Validity of the structured surrogate). Under Assumption 2 if we choose the surrogate functions \(f_i^t(x), \forall i\) as in (12), then Assumption 1 and 2 are satisfied. The proof is similar to that of Proposition 1 and is omitted for conciseness.
D. Key Differences from the Conventional Stochastic SCA

The conventional stochastic SCA algorithms in [20], [23] only consider deterministic and convex constraints. There are two key differences between the conventional stochastic SCA and the proposed CSSCA due to the consideration of stochastic non-convex constraints.

First, in the conventional stochastic SCA, the constraints are deterministic and convex. As a result, there is no need to construct and update the surrogate functions for constraints. In CSSCA, however, we need to construct and update the surrogate functions for constraints.

Second, the sequence of iterates generated by the conventional stochastic SCA is always feasible. In contrast, the sequence of iterates generated by the CSSCA may not be feasible, and thus it is necessary to perform the feasible update by solving (8) to ensure that the algorithm converges to a feasible point. Specifically, in Step 2 of CSSCA, when Problem (7) is feasible, we do an objective update to reduce the objective function by solving a convex approximation of (1) in Problem (7). Otherwise, we do a feasible update to reduce the constraint functions by solving an approximate feasibility problem in (8).

In summary, due to the stochastic non-convex constraints, the sequence of iterates generated by the CSSCA may not be feasible and we have to do a feasible update as well. As a result, the convergence analysis of the CSSCA is also more challenging than that of the conventional stochastic SCA. We shall provide the convergence proof in the next section.

IV. CONVERGENCE ANALYSIS

There are several challenges in the convergence proof for Algorithm 1, as explained below.

**Challenge 2** (Challenges of Convergence Proof). We need to show that at every limiting point, all constraints are satisfied, which is non-trivial since Algorithm 1 may oscillate between the feasible update and objective update. Moreover, the limiting point is obtained by averaging over all the previous outputs from either feasible updates or objective updates, which makes it difficult to show that the limiting point is a stationary point of the original problem (1).

To prove the convergence of Algorithm 1, we further make the following assumptions on the problem structure.

**Assumption 5** (Assumptions on Problem (1)).
1) For any \( i \in \{0, \ldots, m\} \) and \( \xi \in \Omega \), the function \( g_i(x, \xi) \), its derivative, and its second order derivative are uniformly bounded.
2) Let \( x^*_p \) be any stationary point of the following feasibility problem:

\[
\min_{\alpha} \alpha \\
\text{s.t. } f_i(x) \leq \alpha, \forall i = 1, \ldots, m.
\]

We assume that \( f_i(x^*_p) \leq 0, i = 1, \ldots, m. \)

The first assumption is quite standard and is satisfied for a large class of problems. The second assumption ensures that Problem (1) is feasible. If there is a stationary point \( x^*_p \), which is not feasible, then Algorithm 1 may get stuck at this stationary point \( x^*_p \). Therefore, the second assumption is necessary for the algorithm to converge to a feasible point of the problem.

To state the convergence result, we need to prove the convergence of surrogate functions, and introduce the concept of Slater condition for the converged surrogate functions.

**Lemma 1** (Convergence of the surrogate functions). Suppose Assumptions 1, 2, 4 and 5 are satisfied. Consider a subsequence \( \{x^i\}_{j=1}^\infty \) converging to a limit point \( x^* \). There exists uniformly continuous functions \( \hat{f}_i(x) \) such that

\[
\lim_{j \to \infty} f^i_j(x) = \hat{f}_i(x), \forall x \in \mathcal{X},
\]

almost surely. Moreover, we have

\[
\|\nabla \hat{f}_i(x^*) - \nabla \hat{f}_i(x^*)\| = 0.
\]

Please refer to Appendix [E] for the proof.

**Slater condition for the converged surrogate functions**: Given a subsequence \( \{x^i\}_{j=1}^\infty \) converging to a limit point \( x^* \) and let \( \hat{f}_i(x) \), \( \forall i \) be the converged surrogate functions as defined in Lemma 1. We say that the Slater condition is satisfied at \( x^* \) if there exists \( x \in \text{int} \mathcal{X} \) such that

\[
\hat{f}_i(x) < 0, \forall i = 1, \ldots, m.
\]

A similar Slater condition is also assumed in [30] to prove the convergence of a deterministic MM algorithm with non-convex constraints.

Before the introduction of the main convergence theorem, we give a lemma that is crucial for the convergence proof.

**Lemma 2**. Let \( \{x^i\}_{i=1}^\infty \) denote the sequence of iterates generated by Algorithm 1. We have

\[
\lim_{t \to \infty} \max_{i \in \{1, \ldots, m\}} f_i(x^t) = 0, \text{ w.p.1.}
\]

The lemma states that the algorithm will converge to the feasible region, and the gap between \( \bar{x}^t \) and \( x^t \) converges to zero, almost surely. Please refer to Appendix [E] for the proof.

**Theorem 1** (Convergence of Algorithm 1). Suppose Assumptions 1, 2, 4 and 5 are satisfied. For any subsequence \( \{x^i\}_{j=1}^\infty \) converging to a limit point \( x^* \), if the Slater condition is satisfied at \( x^* \), then \( x^* \) is a stationary point of Problem (1) almost surely.

Please refer to Appendix [E] for the proof.

V. PARALLEL IMPLEMENTATION FOR DECOUPLED CONSTRAINTS

In this section, we consider a parallel implementation of Algorithm 1 over a distributed system for stochastic optimization problems with decoupled constraints. There are \( K \) nodes in the system. The optimization variables are partitioned into...
K blocks $\bar{x} = (x_k)_{k=1}^K$ and node $k$ needs to optimize the $k$-th block $x_k$. Specifically, the stochastic optimization problem with decoupled constraints is formulated as

$$\min_{x \in \mathbb{R}^n} f_0(x) = \mathbb{E}[g_0(x, \xi)]$$

$$s.t. f_{i,k}(x_k) = \mathbb{E}[g_{i,k}(x_k, \xi)] \leq 0, \quad i = 1, \ldots, m_k, k = 1, \ldots, K.$$  

In Problem (16), there are $K$ groups of constraints, where the $k$-th constraint group contains $m_k$ constraints with the constraint functions $f_{i,k}(x_k)$, $i = 1, \ldots, m_k$ only depending on the $k$-th block $x_k$. Problem (16) includes many distributed optimization problems, such as the multi-agent optimization problems considered in [23], as special cases.

We use the recursive surrogate function in (10) as an example to illustrate the parallel implementation of Algorithm 1. The parallel implementation for the structured surrogate function is similar. In this case, the sample surrogate function for each function $g_{i,k}(x_k, \xi)$ in the constraint in (16) is denoted by $\hat{g}_{i,k}(x_k, x^t_k, \xi^t)$, which is naturally decoupled over the $K$ blocks $(x_k)_{k=1}^K$. To facilitate parallel implementation of Algorithm 1, we consider the decoupled sample surrogate function for the function $g_0(x, \xi)$ in the objective, which has the following form:

$$\hat{g}_0(x, x^t, \xi^t) = \sum_{k=1}^K \hat{g}_{0,k}(x_k, x^t_k, \xi^t_k).$$

One example of the decoupled sample surrogate function is

$$\hat{g}_{0,k}(x_k, x^t_k, \xi^t_k) = \frac{1}{K} g_0(x_k, x^t_k, \xi^t_k) + \nabla x_k g_0(x_k, x^t_k, \xi^t_k)(x_k - x^t_k)$$

$$+ \tau_k \|x_k - x^t_k\|^2, \forall k$$

where $\tau_k > 0$ is some constant.

By choosing a decoupled sample surrogate function for $g_0(x, \xi)$, the surrogate function $\hat{f}_0(x)$ for the objective $f_0(x)$ is given by

$$\hat{f}_0(x) = \sum_{k=1}^K \hat{f}_{0,k}(x_k),$$

where

$$\hat{f}_{0,k}(x_k) = (1 - \rho^t) \hat{f}_{0,k}^{t-1}(x_k) + \rho^t \hat{g}_{0,k}(x_k, x^t_k, \xi^t_k),$$

with $\hat{f}_{0,k}^{t-1}(x_k) = 0$. The surrogate function $\hat{f}_{i,k}(x_k)$ for the $i$-th constraint in the $k$-th constraint group is given by

$$\hat{f}_{i,k}(x_k) = (1 - \rho^t) \hat{f}_{i,k}^{t-1}(x_k) + \rho^t \hat{g}_{i,k}(x_k, x^t_k, \xi^t_k),$$

with $\hat{f}_{i,k}^{t-1}(x_k) = 0$. Note that in the surrogate update step (Step 1 of Algorithm 1), the surrogate functions $\hat{f}_{i,k}(x_k)$, $i = 0, 1, \ldots, m_k$ corresponding to the $k$-th block $x_k$ can be performed distributedly at node $k$.

In the objective update in Step 2, the optimization problem in (7) can be decoupled into $K$ independent subproblems as

$$\tilde{x}^t_k = \arg\min_{x_k} \hat{f}_{0,k}(x_k)$$

$$s.t. \hat{f}_{i,k}(x_k) \leq 0, i = 1, \ldots, m_k,$$

for $k = 1, \ldots, K$, which can be solved by the $K$ nodes in a distributed and parallel way. Similarly, in the constraint update in Step 2, the optimization problem in (8) can be decoupled into $K$ independent subproblems as

$$\tilde{x}^t_k = \arg\min_{x_k} \hat{f}_{i,k}(x_k)$$

$$s.t. \hat{f}_{i,k}(x_k) \leq 0, i = 1, \ldots, m_k,$$

for $k = 1, \ldots, K$, which can be solved by the $K$ nodes in a distributed and parallel way. The optimal solution of (8) is given by $x^t = (\tilde{x}^t_k)_{k=1}^K$ and the optimal value of (8) is given by $\alpha = \min_{k=1}^K \alpha_k$. The update of $x$ in Step 3 is also decoupled as

$$x^{t+1}_k = (1 - \gamma^t) x^t_k + \gamma^t \tilde{x}^t_k.$$  

VI. APPLICATIONS

In this section, we shall apply the proposed CSSCA to solve the three application problems described in Section III. As discussed in the introduction, there are only a few algorithms that can handle the non-convex stochastic constraints. Among them, sample average approximation (SAA) is a common method to solve a general stochastic optimization problem with non-convex stochastic constraints. The online primal-dual algorithm in [31] may also be used to solve a non-convex stochastic optimization problem, although the convergence is not guaranteed. On the other hand, the Bernstein approximation and its variations [26] are the state-of-the-art algorithms to handle the chance constraint in Example 2. Therefore, we compare the performance of the CSSCA with the SAA and online primal-dual (for Example 1 and 3), as well as the Bernstein approximation (for Example 2). The step sizes/parameters in all algorithms are tuned such that they can achieve their best empirical convergence speed. The simulation results clearly show the advantage of the proposed CSSCA over these baseline algorithms.

A. MIMO Transmit Signal Design with Imperfect CSI

Consider the MIMO transmit signal design problem with imperfect CSI as in (2). The objective function is a linear convex function, and the constraints can be rewritten as $\mathbb{E}[g_k(Q, H)] \leq 0, \forall k$ with

$$g_k(Q, H) = g_k^0(Q, H) + g_k^\xi(Q, H),$$

$$g_k^0(Q, H) = r_k - \log \left( \sum_{j=1}^K h_{kj} Q h_j + \sigma_k^2 \right),$$

$$g_k^\xi(Q, H) = \log \left( \sum_{j \neq k} h_{kj}^* Q h_j + \sigma_k^2 \right),$$

where $Q = \{Q_i\}_{i=1}^K$ is the set of all covariance matrices, and $H = [h_{kj}]_{k=1}^K \in \mathbb{C}^{K \times n}$ is the composite channel matrix. Note that $g_k^0(Q, H)$ and $g_k^\xi(Q, H)$ are the convex and non-convex components, respectively, of $g_k(Q, H)$. This motivates us to choose a structured surrogate function. Specifically,
we first calculate the gradient of the non-convex component with respect to $Q_i$ as

$$\nabla Q_i g_k (Q_i, H) = -\frac{h_k h_k^H}{\sum_{j \neq k} h_j^H Q_j h_j + \sigma_k^2}, \forall i \neq k,$$

and \( \nabla Q_i g_k (Q_i, H) = 0 \), and the gradient of the convex component with respect to $Q_i$ as

$$\nabla Q_i g_k (Q_i, H^t) = -\frac{h_k h_k^H}{\sum_{j=1}^K h_j^H Q_j h_j + \sigma_k^2}, \forall i.$$

Then the surrogate function is given by

$$f^t_k (Q) = (1 - \rho^t) f^{t-1}_k + \rho^t g_k (Q, H^t) + \rho^t \bar{g}_k (Q^t, H^t)$$

$$+ \rho^t \sum_{i \neq k} \Re \left[ T r \left( \nabla Q_i g_k (Q^t, H^t) (Q_i - Q^t) \right) \right]$$

$$+ (1 - \rho^t) \sum_{i=1}^K \Re \left[ T r \left( (\mathbf{F}^{t-1}_i)^H (Q_i - Q^{t_i}) \right) \right]$$

$$+ \tau_k \sum_{i=1}^K T r \left( (Q_i - Q^t) (Q_i - Q^{t_i})^H \right), \quad (20)$$

where $\Re [\cdot]$ is the real operator, $T r (\cdot)$ is the trace operator, $H^t = [h^t_k]_{k=1,...,K} \in \mathbb{C}^{K \times n}$ with $h^t_k = h_k + e^t_k$, and $e^t_k, k = 1, ..., K$ denotes the channel estimation error observed (generated) at iteration $t$. The matrices $F^{t-1}_i$ can be calculated recursively as

$$F^t_i = (1 - \rho^t) F^{t-1}_i + \rho^t \nabla Q_i g_k (Q^t, H^t),$$

where $\nabla Q_i g_k (Q^t, H^t) = \nabla Q_i g_k (Q^t, H^t) + \nabla Q_i \bar{g}_k (Q^t, H^t)$, and the constant $f^t_k$ can be calculated recursively as

$$f^t_k = (1 - \rho^t) f^{t-1}_k + \rho^t g_k (Q^t, H^t).$$

With the surrogate functions in (20), we can implement the proposed CSSCA for Problem 2.

We compare the proposed CSSCA with the SAA and online primal-dual algorithms. After applying the SAA on the constraint functions using $N = 200$ realizations of channel estimation errors, the problem becomes a deterministic optimization problem with non-convex constraints. We apply the deterministic MM method in (20) to solve the resulting non-convex problem. Similarly, the SAA of the constraint function also consists of a convex component plus a concave component, and in the deterministic MM, only approximation for the concave component is required. Specifically, we use linear approximation (i.e., first order Taylor expansion) as the (upper bound) surrogate function for the concave component in the deterministic MM method.

**Numerical Results:** In the simulations, there are $n = 8$ antennas and $K = 4$ users. The estimated channel coefficients $h_k$ are generated according to i.i.d. complex Gaussian distributions with zero mean and unit variance. The channel estimation error $e_k$ also has i.i.d. complex Gaussian entries with zero mean and variance 0.002. The target average rate for all users is set to be the same as $r_k = 1$. Finally, the noise variance for all users is set to be 0.1.

In Fig. 2 and 3 we plot the objective function (average transmit power) and maximum constraint function (target average rate minus achieved average rate) versus the iteration number respectively. The CSSCA and SAA converge to the same average transmit power with all target average rates satisfied with high accuracy. However, the online primal-dual algorithm cannot converge properly and has much higher average transmit power. The number of iterations required to achieve a good convergence accuracy in the proposed CSSCA is only about 3 times larger than that in the SAA. However, the per iteration complexity of the proposed CSSCA is much smaller than the SAA (CPU time: 0.47 s versus 4.95 s). Therefore, the proposed CSSCA is more efficient than the SAA.

**B. Robust Beamforming Design**

The original robust beamforming design problem in (3) is a chance constraint problem. To apply the proposed CSSCA, we first approximate the step function using the smooth
function in \(4\), where a parameter \(\theta\) is used to control the approximation error, and then obtain a smooth approximation of \(g\) in \(5\). Problem \(5\) is an instance of \(1\) and the constraint can be written as \(\mathbb{E}[g_k(w, H)] \leq 0, \forall k\) with \(g_k(w, H) = \tilde{u}_\theta(s_k(w, H))\), where

\[
s_k(w, H) = \sum_{i \neq k} |h_k^i w_i|^2 + \sigma_k^2 - |h_k^i w_k|^2,
\]

and \(w = \{w_i\}_{i=1}^K\) is the set of all beamforming vectors.

We choose to use the recursive surrogate function in \(10\). Specifically, we first calculate the gradient of \(g_k(w, H)\) with respect to \(w_i\) as

\[
\nabla_{w_i} g_k(w, H) = \begin{cases} 2\tilde{u}_\theta'(s_k(w, H)) h_k^i w_i h_k & i \neq k \\ -2\tilde{u}_\theta'(s_k(w, H)) h_k^i w_i h_k & i = k \end{cases}.
\]

Then we can obtain the expression of the recursive surrogate function using \(10\), \(11\) and \(21\), and implement the proposed CSSCA for Problem \(5\).

As for the baseline algorithms, we use the Bernstein method proposed in \(26\). The Bernstein method usually achieves an SINR outage probability that is less than the target and thus is conservative. In the simulations, we also consider another baseline which combines the Bernstein method with a bisection search to further improve the performance. The details of this combined method can be found in \(26\).

**Numerical Results:** We use a similar simulation configuration as that in \(26\). There are \(n = 3\) antennas and \(K = 3\) users. The SINR targets for all users are the same: \(\eta_k = 5\) dB, \(\forall k\). We set the value of the smooth parameter \(\theta = 200\). The channel estimates \(\{\hat{h}_k\}\) and channel estimation error \(\{e_k\}\) have the same distributions as that in Example 2. The noise variances for all users are set to be 0.01.

In Table \(1\) we examine the feasibility rates and the average transmit power of the three algorithms. To this end, 500 sets of channel estimates \(\{\hat{h}_k\}\) were generated. It can be seen that all algorithms exhibit a similar feasibility rate (a solution found by an algorithm is feasible if it satisfies the SINR outage probability constraint in \(3\) with finite transmit power). The combined method achieves the best performance and the proposed CSSCA achieves a better performance than the Bernstein method. Note that we have used a fixed \(\theta = 200\) in the simulations. It is possible to design a better algorithm based on the CSSCA by adjusting the parameter \(\theta\) dynamically over the iterations, which is left for future work. The proposed CSSCA works for any channel estimation error distributions, while the Bernstein methods only work for Gaussian error distributions.

### C. Massive MIMO Hybrid Beamforming Design

In the massive MIMO hybrid beamforming design problem in \(6\), the constraint can be written as \(\mathbb{E}[g(\Theta, p, H)] \leq 0\) with \(g(\Theta, p, H) = T_r \left( FGG^H F^H \right) - P\), which is a linear function of \(p\) but a complicated function of \(\Theta\). In the proposed CSSCA, we consider the following surrogate function for the constraint function:

\[
\hat{f}^t(\Theta, p) = f^t + T_r \left( (\mathbf{F}_\Theta^t)^T (\Theta - \Theta^t) \right) + (\mathbf{F}_p^t)^T (p - p^t),
\]

\[
+ \tau T_r \left( (\Theta - \Theta^t) (\Theta - \Theta^t)^T \right) + \tau \|p - p^t\|^2,
\]

where \(\mathbf{F}_\Theta^t\) and \(\mathbf{F}_p^t\) can be calculated recursively as

\[
\mathbf{F}_\Theta^t = (1 - \rho^t) \mathbf{F}_\Theta^{t-1} + \rho^t \nabla_{\Theta} g(\Theta^t, p^t, H^t),
\]

\[
\mathbf{F}_p^t = (1 - \rho^t) \mathbf{F}_p^{t-1} + \rho^t \nabla_p g(\Theta^t, p^t, H^t),
\]

\(H^t\) is the channel sample obtained at the \(t\)-th iteration, and the constant \(f^t\) can be calculated recursively as

\[
f^t = (1 - \rho^t) f^{t-1} + \rho^t g(\Theta^t, p^t, H^t).
\]

\(22\) is a special case of the structured surrogate function in \(12\) with zero convex component \(g_k^0(\Theta, p, H) = 0\). The gradients of \(g(\Theta, p, H)\) w.r.t. \(\Theta\) and \(p\) in \(23\) are given by

\[
\nabla_{\Theta} g(\Theta, p, H) = \text{Re} \left[ F^* \circ \left( 2H^t_F AH_F F - 2BF \right) \right],
\]

\[
\nabla_p g(\Theta, p, H) = \text{Diag} \left[ H_F FF^H F F^H H_F \right],
\]

where \(\circ\) denotes the Hadamard product, \(\text{Diag}(M)\) denotes a vector consisting of the diagonal elements of the matrix \(M\), and

\[
H_F = \left( H_F FF^H H_F \right)^{-1},
\]

\[
A = H \left( FF^H \right)^2 H_F^2 P + \left( H \left( FF^H \right)^2 H_F^2 P \right)^H,
\]

\[
B = FF^H H_F^2 PH_F + \left( FF^H H_F^2 PH_F \right)^H.
\]

With the surrogate function in \(22\), the feasible update in \(8\) is a quadratic programming with a closed-form solution. On the other hand, the objective update in \(7\) is a simple optimization problem with a logarithm objective function and a quadratic constraint, which can be easily solved by the Lagrange dual method. Specifically, for given Lagrange multiplier, the optimal primal variable that maximizes the Lagrange function has a closed-form solution. Then we can use a bisection method to find the optimal Lagrange multiplier. The details are omitted for conciseness.

We consider the SAA with \(N = 200\) channel samples as the baseline algorithm and the resulting deterministic optimization problem has a non-convex constraint, which is again solved using the deterministic MM method in \(30\). The MM method uses a surrogate function which has similar form as that in \(22\) but with \(\tau\) chosen to be sufficiently large to make the surrogate function an upper bound of the constraint function. The online primal-dual algorithm is also included as a baseline.

**Numerical Results:** In the simulations, the massive MIMO BS is equipped with \(M = 64\) antennas and \(S = 8\) transmit
The CSSCA and SAA converge to the same sum rate with per iteration complexity of the proposed CSSCA is much more efficient than the SAA. Moreover, since the SAA is an offline method, it requires a channel sample collection phase to obtain a sufficiently large number of channel samples before calculating the optimized RF precoder. As a result, the performance will be bad at the channel sample collection phase, which may last for a few hundreds channel coherence intervals. On the other hand, the proposed CSSCA is an online method which can update the RF precoder whenever a new channel sample is obtained. As a result, it can achieve a better overall performance compared to the SAA.

VII. Conclusions

We consider a general stochastic optimization problem where both objective and constraint functions are non-convex and involve expectations over random states. We propose a CSSCA algorithm to find a stationary point of the problem. At each iteration, the algorithm first updates the convex surrogate functions for the objective and constraints based on the observed random state and current iterate. If the convex approximation problem constructed from the surrogate functions is feasible, the algorithm performs an objective update by solving the convex approximation problem. Otherwise, it performs a feasibility update by minimizing the maximum of the surrogate functions for constraints. We show that under some technical conditions, the algorithm converges to a stationary point of the original problem almost surely. We also give a parallel implementation for the algorithm when the constraint function is decoupled. The parallel version of the CSSCA is desirable for solving large-scale stochastic optimization problems such as those that rise in machine learning and big data. Finally, we use several important application examples to illustrate the effectiveness of the proposed algorithm.

Appendix

A. Proof of Proposition 1

Assumption 1-1 and Assumption 1-3 follow immediately from Assumption 3. The rest of the proof relies on the following lemma.

Lemma 3. Under Assumption 3 we have

\[
\lim_{t \to \infty} |f_i^t(x^t) - f_i(x^t)| = 0,
\]

\[
\lim_{t \to \infty} \|\nabla f_i^t(x^t) - \nabla f_i(x^t)\| = 0,
\]

\[
\lim_{t \to \infty} |\bar{f}_i^t(x) - \bar{g}_i(x, x^t)| = 0, \forall x \in \mathcal{X},
\]

for \(i = 0, \ldots, m\) w.p.1., where \(\bar{g}_i(x, x^t) \equiv \mathbb{E}[g_i(x, x^t, \xi)]\).

Proof: Lemma 3 is a consequence of (12), Lemma 1. We only need to verify that all the technical conditions therein are satisfied by the problem in Lemma 3 and the proof is similar to that of (23), Lemma 1). The details are omitted for conciseness.

Assumption 2 follows immediately from Lemma 3. To prove Assumption 1-2, it follows from Lemma 3 that

\[
f_i^t(x) = \bar{g}_i(x, x^t) + e_i(t),
\]

(24)
where \( \lim_{t \to \infty} e_t(t) \to 0 \). From Assumption 5 \( \tilde{g}_i(x, x^t) \) is Lipschitz continuous in \( x^t \) and thus
\[
| \tilde{g}_i(x, x^{t_1}) - \tilde{g}_i(x, x^{t_2}) | \leq B \| x^{t_1} - x^{t_2} \|,
\]
for some constant \( B > 0 \). Combining (24) and (25), we have
\[
\tilde{f}_i^{(t)}(x) - \tilde{f}_i^{(t)}(x) \leq B \| x^{t_1} - x^{t_2} \| + e_g(t_1, t_2),
\]
where \( \lim_{t_1, t_2 \to \infty} e_g(t_1, t_2) = 0 \), from which Assumption 12 follows.

B. Proof of Lemma 7

Due to Assumption 1, the families of functions \( \{ \tilde{f}_i^{(t)}(x) \} \) are equicontinuous. Moreover, they are bounded and defined over a compact set \( \mathcal{X} \). Hence the Arzela–Ascoli theorem implies that, by restricting to a subsequence, there exists uniformly continuous functions \( f_i(x) \) such that (14) is satisfied. Finally, (15) follows immediately from (14) and Lemma 5.

C. Proof of Lemma 2

1. We first prove \( \limsup_{t \to \infty} f(x^t) \leq 0 \) w.p.1., where \( f(x) = \max_{i \in \{1, \ldots, m\}} f_i(x) \).

Let \( T_{\epsilon} = \{ t : f(x^t) \geq \epsilon \} \) for any \( \epsilon > 0 \). We show that \( T_{\epsilon} \) is a finite set by contradiction.

Suppose \( T_{\epsilon} \) is infinite. We first show that
\[
\liminf_{t \to \infty} \| x^t - x^\gamma \| > 0
\]
by contradiction. Suppose \( \liminf_{t \to \infty} \| x^t - x^\gamma \| = 0 \). Then there exists a subsequence \( t^j \in T_{\epsilon} \) such that \( \| x^{t^j} - x^\gamma \| = 0 \). Let \( x^\circ \) denote a limiting point of the subsequence \( \{ x^{t^j} \} \), and let \( f_i(x^\circ), \forall i \) be the converged surrogate functions as defined in Lemma 1. According to the update rule of Algorithm 1, there are two cases.

Case 1: \( x^\circ \) is the optimal solution of the following convex optimization problem:

\[
\begin{align*}
\min_{x^\circ} & \quad \hat{f}_0(x) \\
\text{s.t.} & \quad \hat{f}_i(x) \leq 0, \; i = 1, \ldots, m.
\end{align*}
\]

In this case, we have \( f(x^{\circ}) = \max_{i \in \{1, \ldots, m\}} f_i(x^{\circ}) \leq 0 \), which contradicts the definition of \( T_{\epsilon} \).

Case 2: \( x^\circ \) is the optimal solution of the following convex optimization problem:

\[
\begin{align*}
\min_{x, \alpha} & \quad \alpha \\
\text{s.t.} & \quad \hat{f}_i(x) \leq \alpha, \; i = 1, \ldots, m.
\end{align*}
\]

Since the Slater condition is satisfied (by choosing a sufficiently large \( \alpha \), we can always find a point \( x \in \mathcal{X} \) such that \( f_i(x) < \alpha, \; i = 1, \ldots, m \)), the KKT condition of the problem (27) implies that there exist \( \lambda_1, \ldots, \lambda_m \) such that
\[
\begin{align*}
\sum_i \lambda_i \nabla \hat{f}_i(x^{\circ}) &= 0 \\
1 - \sum_i \lambda_i &= 0 \\
f_i(x^{\circ}) &\leq \alpha, \; \forall i = 1, \ldots, m \\
\lambda_i (f_i(x^{\circ}) - \alpha) &= 0, \; \forall i = 1, \ldots, m.
\end{align*}
\]

It follows from Lemma 1 and (28) that \( x^{\circ} \) also satisfies the KKT condition of Problem (13). By Assumption 5 we have \( f_i(x^{\circ}) \leq 0, \; i = 1, \ldots, m \), which again contradicts the definition of \( T_{\epsilon} \).

Therefore, \( \liminf_{t \to \infty} \| x^t - x^\gamma \| > 0 \), i.e., there exists a sufficiently large \( t_c \) such that
\[
\| x^t - x^\gamma \| \geq \epsilon', \forall t \in T_{\epsilon}'
\]
where \( \epsilon' > 0 \) is some constant and \( T_{\epsilon}' = T_{\epsilon} \cap \{ t \geq t_c \} \).

Define function \( f_i(x) = \max_{i \in \{1, \ldots, m\}} f_i(x) \). From Assumption 1, \( f_i^{(t)}(x^t) \) is strongly convex, and thus
\[

(27)
\]
\[
\begin{align*}
f_i(x^{t+1}) &\leq f_i(x^t) + \gamma' \nabla f_i(x^t) d_i + L_f(\gamma')^2 \| d_i \|^2 \\
&= f(x^t) + L_f(\gamma')^2 \| d_i \|^2 + f_i(x^t) - f(x^t) \\
&+ \gamma' (\nabla f_i(x^t) - \nabla f_i(x^t)) d_i \\
&\leq f(x^t) + f_i(x^t) - f(x^t) - \gamma' \| d_i \|^2 \\
&+ \gamma' (\tilde{f}_i^{(t)}(x^t) - \tilde{f}_i^{(t)}(x^t)) + o(\gamma') \\
&\leq f(x^t) - \gamma' \| d_i \|^2 + o(\gamma'), \forall i = 1, \ldots, m
\end{align*}
\]

where \( o(\gamma') \) means that \( \lim_{t \to \infty} o(\gamma')/\gamma' = 0 \). In (31)-a, we used (30) and \( \lim_{t \to \infty} \| \tilde{f}_i^{(t)}(x^t) - \nabla f_i(x^t) \| = 0 \), and the last inequality follows from \( f_i(x^t) \leq f(x^t), \liminf_{t \to \infty} f_i(x^t) = f_i(x^t) \geq 0, \) and \( \lim_{t \to \infty} \| f_i(x^t) - \tilde{f}_i^{(t)}(x^t) \| = 0 \). Since (31) holds for all \( i = 1, \ldots, m \), by choosing a sufficiently large \( t_c \), we have
\[

f(x^{t+1}) - f(x^t) \leq -\gamma' \| d_i \|^2 \\
\leq -\gamma' \| d_i \|^2, \forall t \in T_{\epsilon}'.
\]

for some \( \gamma' > 0 \). Moreover, from Assumption 5 the directional derivative of \( f(x) \) is uniformly bounded, and thus there exists a constant \( B \) such that
\[
| f(x^{t+1}) - f(x^t) | \leq B \| x^{t+1} - x^t \| \leq B' \gamma',
\]

for some \( B' > 0 \). Finally, it follows from (32) and (33) that
\[
f(x^t) \leq 2\epsilon, \forall t \geq t_c
\]

Since (34) is true for any \( \epsilon > 0 \), it follows that \( \limsup_{t \to \infty} f(x^t) \leq 0 \).

2. Then we prove that \( \liminf_{t \to \infty} \| x^t - x^\gamma \| = 0 \), w.p.1.

2.1: We first prove that \( \liminf_{t \to \infty} \| x^t - x^\gamma \| = 0 \) w.p.1.

Note that the feasible problem in (33) is strictly convex and thus the solution is uniquely given by \( \bar{x}^t \). Therefore, when a feasible update is performed at iteration \( t \), we have \( \tilde{f}_i^{(t)}(x^t) \geq 0 \) and
\[
\bar{x}^t = \arg\min_{x^t} \hat{f}_0(x) \\
\text{s.t.} \quad \tilde{f}_i^{(t)}(x) \leq \bar{x}^t(\bar{x}^t) \quad i = 1, \ldots, m.
\]
As a result, $\tilde{x}^t$ can be expressed in a unified way as

$$\tilde{x}^t = \arg \min_{\bar{x}} f_0^t(\bar{x})$$

subject to $f_i^t(\bar{x}) \leq \alpha_i^t, i = 1, ..., m.$

where $\alpha_i^t = 0$ when an objective update is performed and

$$\alpha_i^t = f_i^t(\tilde{x}^t)$$

when a feasible update is performed. Since

$$\lim_{t \to \infty} [f_i^t(\tilde{x}^t) - f(x^t)] = 0,$$

we have proved that $\limsup_{t \to \infty} f(\bar{x}^t) \leq 0$, and it follows that

$$\lim_{t \to \infty} \alpha_i^t = 0.$$  

where $\tilde{x}^t$ denote the projection of $\bar{x}^t$ on to the feasible set of Problem (35). Then it follows from $\lim_{t \to \infty} \alpha_i^t = 0$, $\limsup_{t \to \infty} f_i^t(\bar{x}^t) = \limsup_{t \to \infty} f(\bar{x}^t) \leq 0$, and the strong convexity of $f_i^t(\bar{x}^t)$ that

$$\lim_{t \to \infty} \| \bar{x}^t - \tilde{x}^t \| = 0.$$  

(36)

From Assumption 1, $f_0^t(\bar{x})$ is uniformly strongly convex, and thus

$$\nabla^T f_0^t(\bar{x}) d^t \leq -\eta \| d^t \|^2 + \tilde{f}_0^t(\bar{x}) - f_0^t(\bar{x})$$

$$= -\eta \| d^t \|^2 + \tilde{f}_0^t(\bar{x}) - f_0^t(\bar{x})$$

$$\bar{x}^t = \tilde{x}^t - \alpha_i^t \xi_j(\bar{x}^t)$$

$$\leq -\eta \| d^t \|^2 + e(t),$$  

(37)

for some $\eta > 0$, where $d^t = \bar{x}^t - \tilde{x}^t$, $\lim_{t \to \infty} e(t) = 0$, and the last equality follows from (36). From Assumption 1, the gradient of $f_0(x)$ is Lipschitz continuous, and thus there exists $L_0 > 0$ such that

$$f_0(\bar{x}^{t+1}) \leq f_0(\bar{x}^t) + \gamma \| \nabla^T f_0(\bar{x}) \| d^t + L_0 \gamma^t \| d^t \|^2$$

$$= f_0(\bar{x}^t) + L_0 \gamma^t \| d^t \|^2$$

$$+ \gamma \left( \| \nabla^T f_0(\bar{x}) - \nabla^T f_0(\bar{x}) \| + \nabla^T f_0(\bar{x}) \right) d^t$$

$$\leq f_0(\bar{x}^t) - \gamma \eta \| d^t \|^2 + o(\gamma^t)$$

where in the last inequality, we used (37) and $\lim_{t \to \infty} \| \nabla^T f_0(\bar{x}) - \nabla^T f_0(\bar{x}) \| \leq 0$ with a positive probability. Then we can find a realization such that $\| d^t \| \geq \gamma$ at the same time for all $t$. We focus next on such a realization. By choosing a sufficiently large $\gamma_0$, there exists $\bar{\gamma} > 0$ such that

$$f_0(\bar{x}^{t+1}) - f_0(\bar{x}^t) \leq -\gamma \| d^t \|^2, \forall t \geq \gamma_0.$$  

(38)

It follows from (38) that

$$f_0(\bar{x}^t) - f_0(\bar{x}^{t+1}) \leq -\gamma \sum_{j=t_0}^t \gamma_j^t \| d^t \|^2,$$

which, in view of $\sum_{j=t_0}^t \gamma_j^t = \infty$, contradicts the boundedness of $\{ \bar{x}_j^t \}$. Therefore it must be

$$\lim sup_{t \to \infty} \| \bar{x}^t - x^t \| = 0 \text{ w.p.1.}$$

2.2: Then we prove that $\lim sup_{t \to \infty} \| \bar{x}^t - x^t \| = 0 \text{ w.p.1.}$

We first prove a useful lemma.

Lemma 4. There exists a constant $\tilde{L} > 0$ such that

$$\| \bar{x}^t - \tilde{x}^t \| \leq \tilde{L} \| \bar{x}^t - \tilde{x}^t \| + e(t_1, t_2),$$

where $\lim_{t_1, t_2 \to \infty} e(t_1, t_2) = 0$.

Proof: From Assumption 1 and 5, we have

$$\| f_i^t(\bar{x}) - f_i^t(\bar{x}) \| \leq B \| \bar{x}^t - \tilde{x}^t \| + e(t_1, t_2),$$

(39)

for all $\bar{x} \in \mathcal{X}$ and $i = 0, 1, ..., m$, where $\lim_{t_1, t_2 \to \infty} e(t_1, t_2) = 0$. Then it follows from (39) and (55), and the Lipschitz continuity and strong convexity of $f_i^t(\bar{x})$ for all constant $B_1, B_2 > 0$. This is because for the strictly convex problem in (55) with Lipschitz continuous and strongly convex objective/constraint functions, when the objective and constraint functions in (55) are changed by some amount $e_i(\bar{x})$, $i = 0, 1, ..., m$, the optimal solution $\bar{x}^t$ will be changed by the same order, i.e., the change is within the range $\pm \mathcal{O}(\max_{i \in \mathcal{X}} |e_i(\bar{x})|)$. Finally, Lemma 4 follows from (40) immediately.

Using Lemma 4 and following the same analysis as that in (23), Proof of Theorem 1, it can be shown that $\lim sup_{t \to \infty} \| \bar{x}^t - x^t \| = 0 \text{ w.p.1.}$

This completes the proof.

D. Proof of Theorem 7

According to Lemma 1 Lemma 2 and (55), $x^*$ must be the optimal solution of the following convex optimization problem almost surely:

$$\min_{x} f_0(x)$$

subject to $\tilde{f}_i(x) \leq 0, i = 1, ..., m.$

Since the Slater condition is satisfied, the KKT condition of the problem (41) implies that there exist $\lambda_1, ..., \lambda_m$ such that

$$\nabla f_0(x) + \sum_{i=1}^m \lambda_i \nabla f_i(x^*) = 0$$

$$\tilde{f}_i(x^*) \leq 0, \forall i = 1, ..., m$$

$$\lambda_i \tilde{f}_i(x^*) = 0, \forall i = 1, ..., m.$$  

(42)

It follows from Lemma 1 and (42) that $x^*$ also satisfies the KKT condition of Problem (1). This completes the proof.

REFERENCES

[1] J. C. Spall, Introduction to Stochastic Search and Optimization: Estimation, Simulation and Control. Hoboken, NJ: Wiley, 2003.

[2] D. P. Bertsekas and J. N. Tsitsiklis, “Gradient convergence in gradient methods with errors,” SIAM J. Optim., vol. 10, no. 3, pp. 627–642, 2000.

[3] B. T. Polyak and A. B. Juditsky, “Acceleration of stochastic approximation by averaging,” SIAM J. Control Optim., vol. 30, no. 4, pp. 838–855, 1992.

[4] S. S. Ram, A. Nedic, and V. V. Veeravalli, “Stochastic incremental gradient descent for estimation in sensor networks,” in 2007 Conference Record of the Forty-First Asilomar Conference on Signals, Systems and Computers, Nov 2007, pp. 582–586.

[5] R. Johnson and T. Zhang, “Accelerating stochastic gradient descent using predictive variance reduction,” in Advances in Neural Information Processing Systems 26, C. J. C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Q. Weinberger, Eds., 2013, pp. 315–323.
[6] A. Defazio, F. Bach, and S. Lacoste-Julien, “SAGA: A fast incremental gradient method with support for non-strongly convex composite objectives,” in *Advances in Neural Information Processing Systems 27*, 2014, pp. 1646–1654.

[7] Y. Ermoliev, “On the method of generalized stochastic gradients and quasi-fejer sequences,” *Cybern.*, vol. 5, no. 2, pp. 208–220, 1972.

[8] E. Yousefi, A. Nedic, and U. V. Shanbhag, “On stochastic gradient and subgradient methods with adaptive steplength sequences,” *Automatica*, vol. 48, no. 1, pp. 56–67, 2012.

[9] A. Ruszczyński, “Feasible direction methods for stochastic programming problems,” *Math. Program.*, vol. 19, no. 1, pp. 220–229, Dec. 1980.

[10] F. Bach, “Adaptivity of averaged stochastic gradient descent to local strong convexity for logistic regression,” *J. Mach. Learn. Res.*, vol. 15, no. 1, pp. 595–627, Jan 2014.

[11] A. M. Gupal and L. G. Bazhenov, “Stochastic analog of the conjugant gradient method,” *Cybernetics*, vol. 8, no. 1, pp. 138–140, 1972.

[12] G. Yin and K. Yin, “Asymptotically optimal rate of convergence of smoothed stochastic recursive algorithms,” *Stochastics and Stochastic Reports*, vol. 47, no. 1-2, pp. 21–46, 1994.

[13] G. Yin, *Adaptive Filtering with Averaging*. New York, NY: Springer New York, 1995, pp. 375–396.

[14] Y. Sun, P. Babu, and D. P. Palomar, “Majorization-minimization algorithms in signal processing, communications, and machine learning,” *IEEE Transactions on Signal Processing*, vol. 65, no. 3, pp. 794–816, Feb 2017.

[15] D. P. Bertsekas, “Incremental gradient, subgradient, and proximal methods for convex optimization: A survey,” *MIT, Cambridge, MA, LIDS Tech. Rep.*, 2010.

[16] O. Cappe and E. Moulines, “On-line expectation-maximization algorithm for latent data models,” *Journal of the Royal Statistical Society. Series B (Statistical Methodology)*, vol. 71, no. 3, pp. 593–613, 2009.

[17] P. Stoica and Y. Selen, “Cyclic minimizers, majorization techniques, and the expectation-maximization algorithm: a refresher,” *IEEE Signal Processing Magazine*, vol. 21, no. 1, pp. 112–114, Jan 2004.

[18] M. J. Wainwright and M. I. Jordan, “Graphical models, exponential families, and variational inference,” *Found. Trends Mach. Learn.*, vol. 1, no. 1-2, pp. 1–305, Jan 2008.

[19] R. Horst and N. V. Thoai, “De programming: overview,” *Journal of Optimization Theory and Applications*, vol. 103, no. 1, pp. 1–43, 1999.

[20] J. Mairal, “Stochastic majorization-minimization algorithms for large-scale optimization,” in *Advances in Neural Information Processing Systems 26*, C. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K. Weinberger, Eds., 2013, pp. 2283–2291.

[21] E. Chouzenoux and J. C. Pesquet, “A stochastic majorize-minimize subspace algorithm for online penalized least squares estimation,” *IEEE Trans. Signal Processing*, vol. PP, no. 99, pp. 1–1, 2017.

[22] G. Scutari, F. Facchinei, P. Song, D. P. Palomar, and J. S. Pang, “Decomposition by partial linearization: Parallel optimization of multi-agent systems,” *IEEE Trans. Signal Processing*, vol. 62, no. 3, pp. 641–656, Feb 2014.

[23] Y. Yang, G. Scutari, D. P. Palomar, and M. Pesavento, “A parallel decomposition method for nonconvex stochastic multi-agent optimization problems,” *IEEE Trans. Signal Processing*, vol. 64, no. 11, pp. 2949–2964, June 2016.

[24] A. Nemirovski and A. Shapiro, “Convex approximations of chance constrained programs,” *SIAM J. Optim.*, vol. 17, no. 4, pp. 969–996, 2006.

[25] M. Ding and S. D. Blostein, “MIMO minimum total MSE transceiver design with imperfect CSI at both ends,” *IEEE Trans. Signal Processing*, vol. 57, no. 3, pp. 1141–1150, March 2009.

[26] K.-Y. Wang, T.-H. Chang, W.-K. Ma, A.-C. So, and C.-Y. Chi, “Probabilistic SINR constrained robust transmit beamforming: A Bernstein-type inequality based conservative approach,” in *Proc. IEEE ICASSP 2011*, May 2011, pp. 3080–3083.

[27] A. Liu and V. K. N. Lau, “Phase-only RF precoding for massive MIMO systems with limited RF chains,” *IEEE Trans. Signal Processing*, vol. 62, no. 17, pp. 4505–4515, Sept. 2014.

[28] ——, “Impact of CSI knowledge on the codebook-based hybrid beamforming in massive MIMO,” *IEEE Transactions on Signal Processing*, vol. 64, no. 24, pp. 6545–6556, Dec 2016.

[29] X. Zhang, A. Molisch, and S.-Y. Kung, “Variable-phase-shift-based RF-baseband codesign for MIMO antenna selection,” *IEEE Trans. Signal Processing*, vol. 53, no. 11, pp. 4091–4103, Nov. 2005.

[30] M. Razaviyayn, “Successive convex approximation: Analysis and applications,” Ph.D. dissertation, University of Minnesota, 2014.