A THEORY OF BASE MOTIVES

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ABSTRACT. When $A$ is a commutative local ring with residue field $k$, the derived tensor product
$$- \otimes^L_A k : D(A) \rightarrow (k - \text{Mod})$$
lifts to a functor taking values in a category of modules over the ‘Tate cohomology’ $\text{RHom}^*_A(k,k)$, which is the universal enveloping algebra of a certain Lie algebra. Under reasonable conditions this lift satisfies a spectral sequence of Adams (or Bockstein) type.

In a suitable category of ring-spectra, replacing $A \rightarrow k$ by $A(*) \rightarrow S$ or $\text{TC}(S) \rightarrow S$ yields interesting Hopf objects, with Lie algebras free after tensoring with $\mathbb{Q}$, analogous to those of motivic groups studied recently by Deligne, Connes and Marcolli, and others.

[This is a sequel to and continuation of a talk at last summer’s conference in Bonn honoring Haynes Miller [23]. I owe many mathematicians thanks for helpful conversations and encouragement, but want to single out John Rognes particularly, and thank him as well for organizing this wonderful conference.]

§1 Prologue

Historically, the first part of the stable homotopy ring to be systematically understood was the image of the $J$-homomorphism
$$J : \pi_{k-1}O = KO_k(*) \rightarrow \lim_{n \to \infty} \pi_{n+k-1}(S^n) = \pi^{S}_{k-1}(*),$$
defined on homotopy groups by the map
$$O_n \rightarrow O \rightarrow \lim_{n \to \infty} \Omega^{n-1}S^{n-1} := Q(S^0);$$
it factors through
$$KO_{4k} = \mathbb{Z} \rightarrow \zeta(1-2k) \cdot \mathbb{Z}/\mathbb{Z} \subset \pi^S_{4k-1}(*);$$
(at least, away from two).

In more geometric terms, a real vector bundle over $S^{4k}$ defines a stable cofiber sequence
$$S^{4k-1} \xrightarrow{\alpha} S^0 \xrightarrow{\text{cof}} \alpha \xrightarrow{} S^{4k} \ldots$$

Date: 20 August 2009.
This research was supported by DARPA and the NSF.
and hence an extension
\[0 \to KO(S^{4k}) \to \cdots \to KO(S^0) \to 0\]
in the group
\[\text{Ext}^{\text{Adams}}(KO(S^0), KO(S^{4k})) \cong H^1_c(\hat{\mathbb{Z}}^\times, \hat{KO}(S^{4k}))\, ,\]
where the Adams operation \(\psi^\alpha, \alpha \in \hat{\mathbb{Z}}^\times\) acts on \(\hat{KO}(S^{2k})\) by \(\psi^\alpha(b^k) = \alpha^k b^k\).
This (essentially Galois) cohomology can be evaluated, via von Staudt’s theorem, in terms of Bernoulli numbers.

In the arithmetic-geometric context, Deligne and Goncharov \([11];\) cf \([25]\) for a more homotopy-theoretic account] have constructed an abelian tensor \(\mathbb{Q}\)-linear category \(\mathcal{MTM}\) of \textbf{mixed Tate motives} over \(\mathbb{Z}\), generated by objects \(\mathbb{Q}(n)\) satisfying a (small, ie trivial when \(\ast > 1\)) Adams-style spectral sequence
\[\text{Ext}^*_{\mathcal{MTM}}(\mathbb{Q}(0), \mathbb{Q}(n)) \Rightarrow K_{2n-\ast}(\mathbb{Z}) \otimes \mathbb{Q}\]
The groups on the right have rank one in degree \(4k + 1\), with generators corresponding (via Borel regulators) to \(\zeta(1 + 2k)\).

\textbf{These same zeta-values} appear in differential topology \([18]\) in the classification of smooth (‘Euclidean’) cell bundles over the \(4k + 2\)-sphere. There, both even and odd zeta-values can be seen as having a common origin, summarized by a diagram (where, implicitly, \(n \to \infty\))

\[
\begin{array}{cccccc}
X
& \downarrow
\
BO_n \rightarrow B\text{Diff}(E^n) \rightarrow B\text{Diff}_c(\mathbb{R}^n)
& \downarrow
\
BQ(S^0) \rightarrow \Omega\text{Wh}(\ast).
\end{array}
\]

The space \(\text{Wh}(\ast)\) on the bottom right is Waldhausen’s smooth pseudoisotopy space, which appears in
\[K(S) = A(\ast) = S \vee \text{Wh}(\ast)\, .\]
The shift by a double suspension in the cell versus vector bundle story is explained by the factor \(B\) on the lower left, and \(\Omega\) on the lower right. The odd zeta-values appear in both geometry and topology because the natural map
\[K(\mathbb{Z}) \to K(S)\]
is a rational equivalence.
This suggests that some of the ideas of differential topology might be usefully reformulated in terms of a category of ‘motives over $S$’ analogous to the arithmetic geometers’ motives over $\mathbb{Z}$, with the algebraic $K$-spectrum of the integers replaced by Waldhausen’s $A$-theory: these zeta-values might then provide a trail of breadcrumbs leading us to some deeper insights.

In the following section we recall some machinery from homological algebra, regarding
\[
K(S) = A(*) = S \vee \text{Wh}(*) \to S
\]
and
\[
TC(S) \sim S \vee \Sigma \mathbb{C}P^\infty_1 \to S
\]
(mod completions) as analogs of local rings over $S$, with the appropriate trace maps interpreted as quotients by maximal ideals. Note that the algebraic $K$-theory spectrum of $\mathbb{Z}$ lacks such an augmentation.

Tannakian formalism identifies the category $\text{MTM}$ of mixed Tate motives as representations of a certain pro-affine $\mathbb{Q}$-groupscheme with free graded Lie algebra, conjecturally related to other areas of mathematics such as algebras of multiple zeta-values and renormalization theory [8, 10]. In the context proposed here, a similar group object
\[
\text{Spec } S \wedge_A S
\]
appears as derived automorphisms of $A$. §3 proposes to define a cycle map from arithmetic motives to their $A$-theoretic analogs, conjecturally identifying these arithmetic and geometric motivic groups.

§2 Brave new local rings

2.1 I’ll start with work on commutative local rings, eg $A \to k$ with maximal ideal $I$, with roots in the very beginnings [27] of homological algebra. Eventually $A$ will be graded, or a DGA.

The functors
\[
H_*(A,-) := \text{Tor}_*(A,k,-)
\]
and
\[
H^*(A,-) := \text{Ext}_*(A,k,-)
\]
appear in Cartan-Eilenberg; the first is covariant, and the second is contravariant, in $A$. I’ll be concerned mostly with
\[
H_*(A,k) = \text{Tor}_*(A,k,k) \quad \text{and} \quad H^*(A,k) = \text{Ext}_*(A,k,k).
\]
Under reasonable finiteness conditions, these are dual $k$-vector spaces [the associativity sseqs [7 XVI §4] degenerate]: in fact they are dual Hopf algebras, with $H^*(A,k)$ being the universal enveloping algebra of a graded Lie algebra [2].
2.1.1 **ex:** If $A = \mathbb{Z}_p \to \mathbb{F}_p$ is the residue map then

$$H_*(A, k) = E_*(Q_0)$$

is an exterior algebra on a Bockstein element of degree one. If $A = k[\epsilon]/(\epsilon^2)$ then $H_*(A, k) = k[x]$ ($|x| = 2$) is the Hopf algebra of the additive group. These are the first manifestations of Koszul duality.

2.1.2 **remarks:** For local rings this homology is closely related to Hochschild theory [7 X §2], so it may also be related to recent work [3, 16] on Hopf algebra structures on THH.

2.1.3 **Proposition:** The homological functor

$$M \mapsto H_*(M \otimes_{\mathbb{Z}_p} \mathbb{F}_p) := \overline{M} : D(\mathbb{Z}_p - \text{Mod}) \to \mathbb{F}_p - \text{Mod}$$

lifts to the category of $E(Q_0)$-comodules. There is a Bockstein spectral sequence

$$\text{Ext}_{E(Q_0)-\text{Comod}}^r(\overline{M}, \overline{N}) \Rightarrow \text{Hom}_{D(\mathbb{Z}_p-\text{Mod})}(M, N).$$

2.1.4 **Definition:** $G(A) := \text{Spec } H_*(A, k)$ is an affine (super) $k$-groupscheme; its grading is encoded by an action of the multiplicative group

$$G_m = \text{Spec } k[\beta^{\pm 1}],$$

and $\tilde{G}A) := G(A) \rtimes G_m$.

2.2.1 The Bockstein spectral sequence generalizes: if $M \in D(A - \text{Mod})$, let

$$\overline{M} = H_*(M \otimes^L_A k) = H_*(M \otimes_A A) \in (k - \text{Mod}),$$

where $A \to A \to k$ is a factorization of the quotient map through a cofibration and a weak equivalence (ie $A$ is a resolution of $k$, eg

$$0 \to \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \to 0.$$ 

**Proposition:** The functor $M \mapsto \overline{M}$ lifts to a homological functor

$$D(A - \text{Mod}) \to (\tilde{G}(A) - \text{reps}),$$

and there is an ‘ascent’ sseq

$$\text{Ext}_{\tilde{G}(A)-\text{reps}}^r(\overline{M}, \overline{N}) \Rightarrow \text{Hom}_{D(A - \text{Mod})}(M, N)$$

of Adams (alt: Bockstein) type . . .

The **Proof** is as in Adams’ Chicago notes [1], replacing the map $S \to MU$ with $A \to k$: thus $MU_*(X)$ becomes an $MU_*$-comodule by taking homotopy groups of the composition

$$X \wedge MU = X \wedge S \wedge MU \to X \wedge MU \wedge MU = (X \wedge MU) \wedge_{MU} (MU \wedge MU),$$

where $S$ is the suspension spectrum of $0$.
yielding
\[ MU_*(X) \to MU_*(X) \otimes_{MU_*} MU_*MU. \]
In the present context the comodule structure map comes from taking the homology of the composition
\[ M \otimes_A A = M \otimes_A A \otimes_A A \to M \otimes_A A \otimes_A A \]
\[ = (M \otimes_A A) \otimes_A (A \otimes_A A), \]
resulting in \( \overline{M} \to \overline{M} \otimes_k H_*(A,k). \) □

2.2.2 ex: The bar construction provides a cofibrant replacement for \( k \), with underlying algebra
\[ \oplus_{n \geq 0} \otimes^n I[1] \]
and a suitable differential. When \( A = k \oplus I \) is a singular \((I^2 = 0)\) extension, the differential is trivial, and \( \text{Ext}^*_A(k,k) \) is the universal enveloping algebra of the free Lie algebra on \( I^*[1] \).

2.2.3 Convergence of such generalized Adams spectral sequences is a complicated topic, related to extending the Tannakian formalism when there may be inequivalent fiber functors [22]. The stable homotopy category is very unlike that of pure motives, which is semisimple in interesting cases: instead, stable homotopy is more like the categories of \( \mathbb{F}_p \)-representations of finite \( p \)-groups, whose structure is encoded entirely through iterated extensions of trivial objects. Away from characteristic zero, it is often unrealistic to hope to recover the full structure of an abelian (or triangulated) monoidal category in terms of the automorphism group of a fiber functor; instead one usually gets at best a spectral sequence which may allow the recovery of the graded object associated to a filtration of some localization of the original category.

The generalized fiber functor defined by topological \( K \)-theory, for example, has \( \text{Gal}(\mathbb{Q}_{ab}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\times \) as (more or less) its motivic group, and the associated spectral sequence ‘sees’ only the image of the \( J \)-homomorphism; other fiber functors see different parts of (some generalization [17] of) the prime ideal spectrum of the stable homotopy category. One of the more interesting issues emerging from this picture is the relation of deformations of fiber functors (eg, taking values in categories of modules over a local ring) and their motivic groups.

2.3.1 Examples closer to homotopy theory appear in recent work of Dwyer, Greenlees, and Iyengar. Suppose for example that \( X \) is a connected pointed space (eg with finitely many cells in each dimension), and let \( X_+ = X \vee S^0 \) be \( X \) with a disjoint basepoint appended. Its Spanier-Whitehead dual
\[ X^D := \text{Maps}_S(\Sigma^\infty X_+, S) \]
is an \( E_\infty \) ring-spectrum, with augmentation \( X^D \to S \) given by the basepoint.
The Rothenberg-Steenrod construction \([14 \S4.22]\) then yields an equivalence
\[
\text{Hom}_{D^{-}\text{Mod}(S,S)}(\Sigma X, S) \sim \Sigma [\Omega X]
\]
of \((A_\infty, \co - E_\infty)\) Hopf algebra objects in the category of spectra.

If \(X\) is simply connected, there is a dual result with coefficients in the Eilenberg-MacLane spectrum \(k = Hk\) of a field: then the ‘double commutator’
\[
\text{Hom}_{k[\Omega X]-\text{Mod}}(k, k) \sim C^*(X, k)
\]
is homotopy equivalent to the (commutative) cochain algebra of \(X\). [This puts the homotopy groups
\[
\pi_* C^*(X, k) \cong H^{-*}(X, k)
\]
in negative dimension.] This sharpens a classical \([4]\) analogy between the homology of loopspaces and local rings.

The (generalized Koszul duality?) functor
\[
M \mapsto \text{Hom}_{X D^{-}\text{Mod}}(M, S) : (X D^{-}\text{Mod}) \to (S[\Omega X]^{-\text{Mod}})
\]
seems worth further investigation . . .

2.3.2 ex: Suspensions are formal, so if \(k = \mathbb{Q}\) and \(X = \Sigma Y\) then
\[
X D^{-} \otimes \mathbb{Q} \sim H^{-*}(\Sigma Y, \mathbb{Q})
\]
is a singular extension of \(\mathbb{Q}\), so \(\mathbb{Q}[\Omega \Sigma Y]\) is the universal enveloping algebra of the free Lie algebra on the graded dual of \(\tilde{H}^{-*-1}(Y, \mathbb{Q})[1]\).

Recent work of Baker and Richter \([5]\) identifies the Hopf algebra of non-commutative symmetric functions with the integral homology \(H_*(\Omega \Sigma CP^{\infty})\) as the universal enveloping algebra of a free graded Lie algebra. The dual Hopf algebra \(H^*(\Omega \Sigma CP^{\infty})\) is the (commutative) algebra of quasi-symmetric functions.

2.3.3 The topological cyclic homology \(\text{TC}(S; p)\) of the sphere spectrum (at \(p\)) is an \(E_\infty\) ring-spectrum, equivalent to the \(p\)-completion of \(S \vee \Sigma C P^{\infty}_{\mathbb{Z}}\) \([20]\); the subscript signifies a twisted desuspension of projective space by the Hopf line bundle.

From now on I’ll be working over the rationals, e.g. the graded algebra
\[
\text{TC}_{2n-1}(S; \mathbb{Q}_p) \cong \mathbb{Q}_p \oplus \mathbb{Q}_p(e_{2n-1}),
\]
\(n \geq 0\) (with trivial multiplication).

2.4.1 The multiplication on a ring-spectrum \(A\) defines a composition
\[
[X, A \wedge Y] \wedge [Y, A \wedge Z] \to [X, A \wedge Z]
\]
(on morphism objects in spectra) by
\[
X \to A \wedge Y \to A \wedge A \wedge Z \to A \wedge Z.
\]
The map $X \to A \wedge X$ defines a functor from the category with $S$-modules (e.g., $X, Y$) as objects, and

$$\text{Corr}_A(X, Y) := [X, A \wedge Y]$$

as morphisms, to the category of $A$-modules, because

$$\text{Corr}_A(X, Y) = [X, A \wedge Y] \to [X, [A, A \wedge Y]_A]$$

$$\cong [A \wedge X, A \wedge Y]_A.$$ Let $(A - \text{Corr})$ be the triangulated subcategory of $(A - \text{Mod})$ generated by the image of this construction. The augmentation of $A$ defines a functor from $(A - \text{Corr})$ to $S$-modules which is the identity on objects, and is given on morphisms by

$$[X, A \wedge Y] \to [X, S \wedge Y] = [X, Y].$$

I propose to inherit the composition of this functor with rationalization as an analog of the ‘fiber functor’ in §2.1:

**Corollary:** This homological functor $(A - \text{Mod}) \to (Q - \text{Mod})$ lifts to the category of $\tilde{G}(A \otimes Q)$-representations, yielding a spectral sequence

$$\text{Ext}^*_{\tilde{G}(A \otimes Q) - \text{reps}}(X, Y) \Rightarrow \text{Corr}^*_A(X, Y).$$

**Proof:** $(X \wedge A) \wedge_A S = X \ldots \Box$

2.4.2 A free Lie algebra has cohomological dimension one, so when $A$ is $\text{TC}(S; p)$ and $X$ and $Y$ are spheres, this spectral sequence degenerates to

$$\text{Ext}^1_{\tilde{G}(\text{TC} \otimes Q)}(S_{2n}^{2n}, S_0^0) \cong \text{TC}_{2n-1}(S, Q_p),$$

with left-hand side isomorphic to

$$H^1_{\text{Lie}}(\tilde{S}(\text{TC}^*[1]), S_{-2n}^{2n}) \cong \text{Hom}^0(\text{TC}^*[1], S_{-2n}^{2n}),$$

which is just the one-dimensional vector space

$$Q_p(e_{2n-1}[1]b^{-n}).$$

At a regular odd prime $p$ (cf. [13, 24]),

$$\text{Wh}(\ast)/\Sigma \text{coker } J \sim \Sigma H^\infty;$$

the cokernel of the $J$-homomorphism is a torsion space, so this yields a spectral sequence

$$\text{Ext}^*_{\tilde{G}(A \otimes Q)}(S_{-2n}^{2n}, S_0^0) \Rightarrow A_{2n-\ast}(\ast) \otimes Q$$

with $A_{4k+1}(\ast) \otimes Q \cong Q(e_{4k+1}[1]b^{-2k-1}).$
§3 \textbf{A-theoretic motives}

3.1 A retractive space \( Z \) over \( X \) is a diagram \( \xymatrix{ X \ar[r]^r & Z \ar[r]^s & X } \) which composes to the identity \( 1_X \): it’s a space over \( X \) with a cofibration section. \( Z \) is said to be finitely dominated if some finite complex is retractive over it \([15]\).

Waldhausen showed that finitely dominated retractive spaces over \( X \) form a category with weak equivalences and cofibrations, and that the \( K \)-theory spectrum \( A(X) \) of this category can be identified with \( K(S[\Omega X]) \).

More generally, \( Z \) is relatively retractive over \( X \), with respect to a map \( p : X \to Y \), if the homotopy fiber of \( p \circ r \) over any \( y \in Y \) is finitely dominated as a retractive space over the homotopy fiber of \( p \) above \( y \). The category \( \mathcal{R}(p) \) of such spaces is again closed under cofibrations and weak equivalences, with an associated \( K \)-theory spectrum \( A(X \to Y) \).

Bruce Williams \([28 \S 4]\) (using a formalism developed in algebraic geometry by Fulton and MacPherson) shows that this functor has a rich bivariant structure: compositions

\[ A(X \to Y) \wedge A(Y \to Z) \to A(X \to Z), \]

good behavior under products, &c. It behaves especially well on fibrations; in particular, the spectra

\[ \forall A(X, Y) := A(X \times Y \to X) \]

(defined by relatively retractive spaces \( Z \) over \( X \times Y \to X \)) admit good products

\[ \forall A(X, Y) \wedge \forall A(Y, Z) \to \forall A(X, Z). \]

Let \( A - \text{Corr} \) be the triangulated envelope \([6]\) of the symmetric monoidal additive category with finite CW complexes \( X, Y \) as objects, and \( \forall A_0(X, Y) = \pi_0 \forall A(X, Y) \) as morphisms. Composition

\[ A(X \times Y \to Y) \to [X, A(Y)] \to [X, A \wedge Y] \]

of the standard assembly map with a slightly less familiar relative co-assembly map \([12 \S 5]\) defines a monoidal stabilization functor

\[ (A - \text{Corr}) \to (A - \text{Corr}) \]

analogous to inverting the Tate motive, or to the introduction of desuspension in classical homotopy theory. However, \( A \)-theory of spaces is a highly nonlinear functor, and might possess other interesting stabilizations.

3.2 The motivic constructions of Suslin and Voevodsky \([27]\) begin with a category whose objects are schemes of finite type over some nice base, and whose morphism groups \( \text{SmCorr}(V, W) \) of (roughly) sums of irreducible subvarieties \( Z \) of \( V \times W \) which are finite with respect to the projection \( V \times W \to V \), and surjective on components of \( V \).
When $V$ and $W$ are defined over a number field (e.g., $\mathbb{Q}$), classical arguments [cf. eg [21]] show that

$$Z(\mathbb{C}) \cup V(\mathbb{C}) \times W(\mathbb{C}) \to V(\mathbb{C}) \times W(\mathbb{C})$$

is finitely dominated relatively retractive with respect to $V(\mathbb{C}) \times W(\mathbb{C}) \to V(\mathbb{C})$, defining a cycle class homomorphism

$$\text{SmCorr}(V, W) \to \forall A_0(V(\mathbb{C}), W(\mathbb{C})),$$

and hence a functor

$$V \mapsto V(\mathbb{C}) : (\text{SmCorr}) \to (A - \text{Corr}).$$

My hope is that this will lead to an identification of the motivic group for the category of mixed Tate motives with $\hat{G}(A \otimes \mathbb{Q})$. It seems at least possible that $\hat{G}(TC \otimes \mathbb{Q})$ is the larger motivic group seen in physics [8, 9 §3.1] by Connes and Marcolli.

3.3 I don’t want to end this sketch without mentioning one last possibility. Dundas and Østvær have proposed a bivariant $K$-theory based on categories $\mathcal{E}(E, F)$ of suitably exact functors between the categories of (cell) modules over (associative) ring-spectra $E$ and $F$.

These module categories are to be understood as categories with weak equivalences and cofibrations; the exact functors are to preserve these structures, and be additive in a certain sense. $\mathcal{E}(E, F)$ is again a Waldhausen category, which suggests that the category $(\text{Alg}_A)$ with associative ring-spectra $E, F$ as its objects, and

$$\text{Alg}_A(E, F) := K(\mathcal{E}(E, F))$$

as morphisms, is an interesting analog of categories of noncommutative correspondences proposed by various research groups [9 §6, 19 §4]. It seems reasonable to expect that this category will naturally be enriched over $A$.

A space $W$ over $X \times Y$ defines an $X^D, Y^D$ bimodule $W^D$, and

$$W \mapsto \text{Hom}_{X^D-\text{Mod}}(W^D, -)$$

is a natural candidate for an exact functor, and hence a map

$$\mathcal{R}(X \times Y \to X) \to \mathcal{E}(X^D, Y^D).$$

If so, this might define another interesting stabilization of $A - \text{Corr}$, related more closely to the Waldhausen $K$-theory of Spanier-Whitehead duals than to spherical group rings.
1. J.F. Adams, Stable homotopy and generalised homology, Chicago Lectures in Mathematics. University of Chicago Press (1974)
2. M. André, M. Hopf algebras with divided powers, J. Algebra 18 (1971) 19–50
3. V. Angeltveit, J. Rognes, Hopf algebra structure on topological Hochschild homology. Algebr. Geom. Topol. 5 (2005), 1223–1290, available at [arXiv:math/0502195](http://arxiv.org/abs/math/0502195)
4. L. Avramov, S. Halperin, Through the looking glass: a dictionary between rational homotopy theory and local algebra, in Algebra, algebraic topology and their interactions (Stockholm, 1983), 1–27, Lecture Notes in Math., 1183, Springer, Berlin, 1986
5. A. Baker, B. Richter, Quasisymmetric functions from a topological point of view. Math. Scand. 103 (2008), 208–242, available at [arXiv:math/0605743](http://arxiv.org/abs/math/0605743)
6. A. Bondal, M. Larsen, V. Lunts, Grothendieck ring of pretriangulated categories. Int. Math. Res. Not. (2004) 1461–1495
7. H. Cartan, S. Eilenberg, Homological algebra, Princeton (1956)
8. P. Cartier, Cartier, Pierre A mad day's work: from Grothendieck to Connes and Kontsevich. The evolution of concepts of space and symmetry, BAMS 38 (2001) 389–408
9. A. Connes, C. Consani, M. Marcolli, Noncommutative geometry and motives: the thermodynamics of endomotives, available at [arXiv:math/0512138](http://arxiv.org/abs/math/0512138)
10. ——, M. Marcolli, Quantum fields and motives. J. Geom. Phys. 56 (2006) 55–85, available at [arXiv:hep-th/0504085](http://arxiv.org/abs/hep-th/0504085)
11. P. Deligne, A.B. Goncharov, Groupes fondamentaux motiviques de Tate mixte, Ann. Sci. École Norm. Sup. 38 (2005) 1–56, available at [arXiv:math/0302267](http://arxiv.org/abs/math/0302267)
12. W. Dorabiala, M. Johnson, The product theorem for parametrized homotopy Reide-meister torsion, J. Pure Appl. Algebra 196 (2005) 53–90
13. B. Dundas, Relative K-theory and topological cyclic homology. Acta Math. 179 (1997) 223–242
14. W. Dwyer, J. Greenlees, S. Iyengar, Duality in algebra and topology. Adv. Math. 200 (2006) 357–402, available at [arXiv:math/0510247](http://arxiv.org/abs/math/0510247)
15. ——, M. Weiss, B. Williams. A parametrized index theorem for the algebraic K-theory Euler class, Acta Math. 190 (2003) 1–104.
16. K. Hess, Homotopic Hopf-Galois extensions: foundations and examples, available at [arXiv:0902.3393](http://arxiv.org/abs/0902.3393)
17. M. Hovey, J. Palmieri, The structure of the Bousfield lattice, in Homotopy-invariant algebraic structures . . . , available at [arXiv:math/9801103](http://arxiv.org/abs/math/9801103)
18. K. Igusa, Higher Franz-Reidemeister torsion, AMS/IP Studies in Advanced Mathematics 31, AMS (2002)
19. M. Kontsevich, Notes on motives in finite characteristic, available at [arXiv:math/0702206](http://arxiv.org/abs/math/0702206)
20. I. Madsen, C. Schlichtkrull, The circle transfer and K-theory, in Geometry and topology: Aarhus (1998) 307–328, Contemp. Math., 258 AMS (2009)
21. J. Milnor, Morse Theory, Princeton (1963)
22. J. Morava, Toward a fundamental groupoid for the stable homotopy category Geometry & Topology Monographs, available at [arXiv:math/0509001](http://arxiv.org/abs/math/0509001)
23. ——, To the left of the sphere spectrum, available at [www.ruhr – uni – bochum.de/topologie/conf08/](http://www.ruhr-university-bochum.de/topologie/conf08/)
24. J. Rognes, The smooth Whitehead spectrum of a point at odd regular primes. Geom. Topol. 7 (2003) 155–184, available at [arXiv:math/0304384](http://arxiv.org/abs/math/0304384)
25. O. Röndigs, PA Østvær, Modules over motivic cohomology. Adv. Math. 219 (2008) 689–727
26. J. Tate, Homology of Noetherian rings and local rings. Illinois J. Math. 1 (1957), 14–27
27. V. Voevodsky, Cancellation theorem, available at arXiv:math/0202012
28. B. Williams, Bivariant Riemann-Roch theorems, in Geometry and topology 377–393, Contemp. Math. 258, AMS (2000)

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