Reflected solutions of Anticipated Backward Doubly SDEs driven by Teugels Martingales.

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Abstract. We deal with reflected solutions of anticipated backward doubly stochastic differential equations (RABDSDEs) driven by Teugels martingales associated with Lévy process under a Lipschitz generator where the coefficients of these BDSDEs depend on the future and present value of the solution \((Y, Z)\). Also we study the existence of a solution for anticipated BDSDEs.

Keyword Anticipated backward doubly stochastic differential equations, random Lévy measure, comparison theorem, predictable representation, Teugels martingales, Gronwall lemma, the principle of contraction.

1 Introduction

Backward stochastic differential equations (BSDEs in short) were introduced by Bismut for the linear case [2] and by Pardoux and Peng in the general case [7]. Precisely, according to [7], given a data \((\xi, f)\) consisting of a square integrable random variable \(\xi\) and a progressively measurable process \(f\), a so-called generator, they proved the existence and uniqueness of a solution to these equations. Recently, a new type of BSDE, called anticipated BSDE (ABSDE in short), which can be regarded as a new duality type of stochastic differential delay equations, was introduced by Peng and Yang [9], see also [13, 14]. After they introduced the theory of BSDEs, Pardoux and Peng in [8] considered a new kind of BSDEs, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals. They proved existence and uniqueness of solutions for BDSDEs under Lipschitz conditions on the coefficients.

Recently, a new type of BDSDE called anticipated BDSDE (ABDSDE in short), which can be regarded as a new duality type of stochastic differential delay equations, was introduced by Xu [16], The ABDSDE is of the form

\[ Y_t = \xi + \int_t^T f(s, \Lambda_s, \Lambda_s^{\phi, \psi})ds + \int_t^T g(s, \Lambda_s, \Lambda_s^{\phi, \psi})d\overline{B}_s - \int_t^T Z_s dW_s, \quad t \in [0, T], \]

\[(Y_t, Z_t) = (\eta_t, \vartheta_t), \quad t \in [T, T + K), \]

where, \(\Lambda_s = (Y_s, Z_s)\) and \(\Lambda_s^{\phi, \psi} = (Y_s + \phi(s), Z_s + \psi(s))\), \(\phi(\cdot) : [0, T] \to \mathbb{R}_+ \cup \{0\}\) and \(\psi(\cdot) : [0, T] \to \mathbb{R}_+ \cup \{0\}\) are continuous functions and \(f\) is called a generator.

In the paper of Nualart et al. [5], a martingale representation theorem associated to Lévy processes was proved. Then it is natural to extend BDSDEs driven by Brownian motion to BDSDEs driven by a Lévy process. In the work of Ren et al. [10], the authors proved the existence and uniqueness of solutions of BDSDEs driven by Teugels martingales associated with a Lévy process, under Lipschitz conditions on the generator \(f\). These results were important from a pure mathematical point of view as well as from an application point of view in the world of finance.

The first work of Reflected BDSDEs is introduced by Bahlali et al. [1], after Y. Ren [10] introduced a special class of reflected BDSDEs (RBDSDEs, in short), which is a BDSDE but the solution is forced to stay above a lower barrier.
Motivated by the above results and by the result introduced by Xiaoming Xu [16], we establish firstly the existence and uniqueness of the solution of the Anticipated reflected BDSDE driven by Teugels Martingales (RABDSDEs, in short) in the proof we using the result of Y. Ren [11]. Let us point out that our paper extends the results of Y. Ren [11], Xiaoming Xu [16] and Gaofeng Zong [17]. The main idea of the proof is to the point fixe theorem. And we establish others results for ABDSDEs, we prove the existence and uniqueness of the solution.

The organization of the paper is as follows. In Section 2, we give some preliminaries on the martingales \( \{H^{(i)}, t \geq 0\} \) and we consider the spaces of processus also we define the Itô’s lamma. In Section 3, under certain assumptions, we obtain the existence and uniqueness solution for the associated Anticipated reflected BDSDEs (RABDSDEs) and ABDSDEs.

## 2 Preliminaries

Let \((\Omega, \mathcal{F}, P, B_t, L_t; 0 \leq t \leq T)\) be a complete Brownian-Lévy space in \(\mathbb{R} \times \mathbb{R}^*\) with Lévy mesure. For \(T > 0\), let \(\{B_t, 0 \leq t \leq T\}\) is a standard Brownian motion defined on \((\Omega, \mathcal{F}, P)\) with values in \(\mathbb{R}\) and \(\{L_t; 0 \leq t \leq T\}\) is a \(\mathbb{R}\)-valued pure jump-Lévy process of the form \(L_t = bt + L_t\) independent of \(\{B_t, 0 \leq t \leq T\}\).

Let \(\mathcal{F}^B_t := \sigma(L_s; 0 \leq s \leq t)\) and \(\mathcal{F}^P_T := \sigma(B_s - B_t; t \leq s \leq T)\), completed with \(P\)-null sets. We put, \(\mathcal{F}_t := \mathcal{F}^B_t \vee \mathcal{F}^P_T\), for each \(t \in [0, T]\), and \(\mathcal{H}_t := \mathcal{F}^B_t \vee \mathcal{F}^P_T\) for each \(t \in [0, T + K]\). It should be noted that \(\mathcal{H}_t\) is not an increasing family of sub \(\sigma\)-fields, and hence it is not a filtration.

For each \(t \in [0, T + K]\), we define
\[
\mathcal{G}_t := \mathcal{F}^B_t \vee \mathcal{F}^P_{T + K},
\]
the collection \(\{\mathcal{G}_t\}_{t \in [0, T + K]}\) is a filtration.

For any \(d, k \geq 1\), we consider the following spaces of processes:

- Let \(\mathcal{M}^2_{\mathcal{H}}([0, T]; \mathbb{R})\) denote the set of 1-dimensional, \(\mathcal{H}_t\)-progressively measurable stochastic processes \(\{\varphi_t; t \in [0, T]\}\) such that \(E \int_0^T |\varphi_t|^2 dt < \infty\).
- We denote by \(\mathcal{S}^2_{\mathcal{H}}([0, T]; \mathbb{R})\), the set of continuous and \(\mathcal{H}_t\)-progressively measurable stochastic processes \(\{\varphi_t; t \in [0, T]\}\) which satisfy \(E(\sup_{t \leq T} |\varphi_t|^2) < \infty\).
- \(L^2\) be the space of real valued sequences \((x_n)_{n \geq 0}\) such that \(\sum_{i=1}^{\infty} x_i^2 < \infty\), and \(||x||_p^2 = \sum_{i=1}^{\infty} x_i^2\).
- \(\mathcal{A}^2\) set of continuous, increasing, \(\mathcal{H}_t\)-measurable process \(K : [0, T] \times \Omega \to [0, +\infty(\text{with } K_0 = 0, E(K_T)^2 < \infty)\).
- \(\mathcal{M}^2_{\mathcal{H}}([0, T]; L^2)\) and \(\mathcal{S}^2_{\mathcal{H}}([0, T]; L^2)\): are the corresponding spaces of \(L^2\)-valued processes equipped with the norm \(||\varphi||_{L^2}^2 = E \int_0^T \sum_{i=1}^{\infty} |\varphi_t^{(i)}|^2 dt < \infty\).
- \(L^2(\mathcal{H}_T)\) set of \(\mathcal{H}_T\)- measurable random variables \(\xi : \Omega \to \mathbb{R}^d\) with \(E|\xi|^2 < +\infty\).
- Notice that the space \(\mathcal{B}^2_{\mathcal{H}}([0, T]; \mathbb{R}) = \mathcal{S}^2_{\mathcal{H}}([0, T]; \mathbb{R}) \times \mathcal{M}^2_{\mathcal{H}}([0, T]; L^2)\) endowd with the norm
\[
|||Y, Z|||_{\mathcal{B}^2_{\mathcal{H}}([0, T]; \mathbb{R})} = |||Y|||_{\mathcal{S}^2_{\mathcal{H}}([0, T]; \mathbb{R})} + |||Z|||_{\mathcal{M}^2_{\mathcal{H}}([0, T]; L^2)}.
\]

We denote by \((H^{(i)})_{t \geq 1}\) the Teugels martingale associated with the lévy process \(\{L_t, t \in [0, T]\}\) with is given by \(H^{(i)} = c_i, Y_t^{(i)} + c_{i-1} Y_t^{(i-1)} + \ldots + c_1 Y_t^{(1)}\), where \(Y_t^{(i)} = L_t^{(i)} - E[L_t^{(i)}] = L_t^{(i)} - tE[L_t^{(i)}]\) for all \(i \geq 1\) and \(L_t^{(i)}\) are power-jump processes. That is, \(L_t^{(1)} = L_t\) and \(L_t^{(i)} = \sum_{0 \leq s \leq t} (\Delta L_s)^i\) for \(i \geq 2\), and \([H^{(i)}, H^{(j)}], i \neq j\) and \([H^{(i)}, t], t \geq 0\) are uniformly integrable martingale with initial value 0, i.e.,
\[
(H^{(i)}, H^{(j)})_t = t\delta_{i,j},
\]
where it was shown in [6] that the coefficients $c_{i,k}$ correspond to the orthonormalization of the polynomials $1, x, x^2, \ldots$ with respect to the measure $\mu(dx) = x^3v(dx) + \sigma^2 \delta_0(dx)$, the resulting processes $H^{(i)} = \{H^{(i)}, t \geq 0\}$ are called the orthonormalized $i$th-power-jump processes.

The result depends on the following extension of the well-known Itô’s formula. Its proof follows the same way as lemma 1.3 of [8]

**Lemma 2.1.** Let $\alpha \in S^2_H([0, T]; \mathbb{R})$, $\beta$, $\gamma$ and $\sigma \in \mathcal{M}^2_H([0, T]; \mathbb{R}^2)$ such that

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \sum_{i=1}^{\infty} \int_0^t \sigma_s^{(i)} dH_s^{(i)},$$

then

$$|\alpha_t|^2 = |\alpha_0|^2 + 2 \int_0^t \alpha_s \beta_s ds + 2 \int_0^t \alpha_s \gamma_s dB_s + 2 \sum_{i=1}^{\infty} \int_0^t \alpha_s \sigma_s^{(i)} dH_s^{(i)}$$

$$- \int_0^t |\gamma_s|^2 ds + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^t \sigma_s^{(i)} \sigma_s^{(j)} d\left[H^{(i)}, H^{(j)}\right],$$

note that $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij} t$, we have

$$E |\alpha_t|^2 \leq E |\alpha_0|^2 + 2E \int_0^t \alpha_s \beta_s ds - E \int_0^t |\gamma_s|^2 ds + E \sum_{i=1}^{\infty} \int_0^t \left(\sigma_s^{(i)}\right)^2 ds.$$

### 3 Main result

#### 3.1 Anticipated BDSDE with Lower barrier.

In this subsection, we consider the following 1-dimensional anticipated reflected backward doubly stochastic differential equation

$$\begin{align*}
Y_t &= \xi + \int_0^T f(s, \Lambda_s, \Lambda_s^{\phi, \psi}) ds + \int_0^T g(s, \Lambda_s, \Lambda_s^{\phi, \psi}) dB_s \\
&+ \int_0^T dK_s - \sum_{i=1}^{\infty} \int_0^T Z_s^{(i)} dH_s^{(i)}, & t \in [0, T], \\
(Y_t, Z_t) &= (\eta_t, \vartheta_t), & t \in [T, T + K],
\end{align*}$$

(3.1)

where $f$ called the generator, $\Lambda_s = (Y_{s-}, Z_s)$ and $\Lambda_s^{\phi, \psi} = (Y_{s+\phi(s)-}, Z_{s+\psi(s)})$.

Let $\phi : [0, T] \to \mathbb{R}^+_*$ and $\psi : [0, T] \to \mathbb{R}^+_*$ are continuous functions satisfying:

(A) There exists a constant $K \geq 0$ such that for all $t \in [0, T]$,

$$t + \phi(t) \leq T + K, \quad t + \psi(t) \leq T + K.$$

(B) There exists a constant $M \geq 0$ such that for each $t \in [0, T]$ and for all nonnegative integrable functions $h(\cdot)$,

$$\begin{align*}
\int_t^T h(s + \phi(s)) ds &\leq M \int_t^{T+K} h(s) ds, \\
\int_t^T h(s + \psi(s)) ds &\leq M \int_t^{T+K} h(s) ds.
\end{align*}$$

**Definition 3.1.** A solution of equation (3.1) is a triple $(Y, Z, K)$ which belongs to the space $\mathcal{B}^2_H([0, T + K], \mathbb{R}) \times \mathcal{A}^2$ and satisfies (3.1) such that:

$$\begin{align*}
S_t &\leq Y_t, \quad 0 \leq t \leq T + K, \\
\int_0^T (Y_{s-} - S_{s-}) dK_s &= 0.
\end{align*}$$
In this subsection we study the ABDSDEs with reflection under Lipschitz continuous generator. We consider the following assumptions (H1):

**H1.1** (i) There exist a constant \( c > 0 \) such that for any \((r, \tilde{r}) \in [0, T + K]^2, (t, \omega, y, z, \pi, \zeta) \),

\[
(t, \omega, y, z, \tilde{\pi}, \tilde{\zeta}) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{B}^2_{\mathbb{F}} ([0, T + K], \mathbb{R}),
\]

\[
\left| f(t, \omega, y, z, \pi (r), \zeta (\tilde{r})) - f(t, \omega, \tilde{y}, \tilde{z}, \tilde{\pi} (r), \tilde{\zeta} (\tilde{r})) \right|^2 \\
\leq c \left( |y - \tilde{y}|^2 + ||z - \tilde{z}||^2_{L^2} + \mathbb{E}^F _t \left[ |\pi (r) - \hat{\pi} (r)|^2 + ||\zeta (\tilde{r}) - \tilde{\zeta} (\tilde{r})||^2_{L^2} \right] \right).
\]

(ii) There exists a constant \( c > 0, 0 < \alpha_1 < \frac{1}{2} \) and \( 0 < \alpha_2 < \frac{1}{M} \) satisfying \( 0 < \alpha_1 + \alpha_2 M < \frac{1}{2} \), such that

\[
\left| g(t, \omega, y, z, \pi (r), \zeta (\tilde{r})) - g(t, \omega, \tilde{y}, \tilde{z}, \tilde{\pi} (r), \tilde{\zeta} (\tilde{r})) \right|^2 \\
\leq c \left( |y - \tilde{y}|^2 + \mathbb{E}^F _t |\pi (r) - \hat{\pi} (r)|^2 \right) + \alpha_1 ||z - \tilde{z}||^2_{L^2} + \alpha_2 \mathbb{E}^F _t \left| \zeta (\tilde{r}) - \tilde{\zeta} (\tilde{r}) \right|^2_{L^2}.
\]

**H1.2** For any \((t, \omega, y, z, \pi, \zeta)\),

\[
\mathbb{E} \int_0^T |f(s, \omega, 0, 0, 0, 0)| ds < \infty,
\]

\[
\mathbb{E} \int_0^T |g(s, \omega, y, z, \pi, \zeta)| ds < \infty.
\]

**H1.3** The terminal value \( \xi \) be a given random variable in \( L^2 \).

Also we consider the following assumptions (H2):

**H2.1** \((S_t)_{t \geq 0}\) is a continuous progressively measurable real valued process satisfying

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T+K} (S_t^+)^2 \right) < +\infty, \quad \text{where } S^+_t := \max (S_t, 0).
\]

**H2.2** For any \( t \in [T, T + K] \), \( S_t \leq \eta_t, \mathbb{P}\)-almost surely.

**H2.3** \((\eta_t, \phi_t) \in \mathcal{B}^2_{\mathbb{F}} ([T, T + K], \mathbb{R})\).

**H2.4** \((K_t)_{t \in [0, T]}\) is a continuous, increasing process with \( K_0 = 0 \) and \( \mathbb{E} (K_T)^2 < +\infty \).

3.1.1 Existence and uniqueness of solutions.

**Theorem 3.1.** Let \( f, g \) satisfies the hypothesis (H1), (H2) and (A), (B) are hold. Then the Anticipated RBDSDEs (3.1) has a unique solution \((Y_t, Z_t, K_t)_{t \in [0, T+K]}\).

3.1.2 Proof of the existence and uniqueness result.

**Proof**: Uniqueness. Let \((Y^j, Z^j, K^j) \in \mathcal{B}^2_{\mathbb{F}} ([0, T + K], \mathbb{R}) \times \mathcal{A}^2 \) for \( j = 1, 2 \) be any two solutions.

Define \( \Delta Y_t = Y_t^1 - Y_t^2, \Delta Z_t^{(i)} = Z_t^{(i)} - Z_t^{(i)} \) and \( \Delta K_t = K_t^1 - K_t^2 \) and for a function:

\[
\begin{align*}
\Delta f (s) &= f(s, \Lambda_s^1, \Lambda_s^{1, \phi}, \psi) - f(s, \Lambda_s^2, \Lambda_s^{2, \phi}, \psi), \\
\Delta g (s) &= g(s, \Lambda_s^1, \Lambda_s^{1, \phi}, \psi) - g(s, \Lambda_s^2, \Lambda_s^{2, \phi}, \psi).
\end{align*}
\]

We consider the following equation

\[
\begin{align*}
&\Delta Y_t = \xi_T + \int_t^T \Delta f (s) ds + \int_t^T \Delta g (s) d\tilde{B}_s + \int_t^T d (\Delta K_s) - \sum_{i=1}^{\infty} \int_t^T \Delta Z_s^{(i)} \, dH_s^{(i)}, \quad t \in [0, T], \\
&\left(\Delta Y_t, \Delta Z_t\right) = (0, 0), \quad t \in [T, T + K],
\end{align*}
\]
where $\Lambda^j_i = \left(Y^j_{s_i}, Z^j_{s_i}\right)$ and $\Lambda^j,\psi = \left(Y^j_{s+\psi(s)}, Z^j_{s+\psi(s)}\right)$ for $j = 1, 2$. It follows from Itô’s formula that

$$
\mathbb{E}\left(e^{\beta t}|\Delta Y_t|^2\right) + \beta \mathbb{E}\int_t^T e^{\beta s}|\Delta Y_s|^2 ds + \sum_{i=0}^{\infty} \mathbb{E}\int_t^T e^{\beta s}|\Delta Z_s^{(i)}|^2 ds = 2\mathbb{E}\int_t^T e^{\beta s}\Delta Y_s \Delta f(s) ds + \mathbb{E}\int_t^T e^{\beta s}|\Delta g(s)|^2 ds + 2\mathbb{E}\int_t^T e^{\beta s}\Delta Y_{s-}d(\Delta K_s).
$$

Since $\int_t^T e^{\beta s}\Delta Y_{s-}d(\Delta K_s) \leq 0$, we have

$$
\mathbb{E}\left(e^{\beta t}|\Delta Y_t|^2\right) + \beta \mathbb{E}\int_t^T e^{\beta s}|\Delta Y_s|^2 ds + \sum_{i=0}^{\infty} \mathbb{E}\int_t^T e^{\beta s}|\Delta Z_s^{(i)}|^2 ds \leq 2\mathbb{E}\int_t^T e^{\beta s}\Delta Y_s \Delta f(s) ds + \mathbb{E}\int_t^T e^{\beta s}|\Delta g(s)|^2 ds.
$$

Using Young’s inequality $2ab \leq \epsilon_1a^2 + \frac{\lambda_1^2}{\epsilon_1}$ and hypothesis (H.1), we have

$$
2\mathbb{E}\int_t^T e^{\beta s}\Delta Y_s \Delta f(s) ds \leq \epsilon_1 \mathbb{E}\int_t^T e^{\beta s}|\Delta Y_s|^2 ds + \left(\frac{c}{\epsilon_1} + cM\right) \mathbb{E}\int_t^T e^{\beta s}\left(|\Delta Y_s|^2 + ||\Delta Z_s||^2\right) ds,
$$

and also

$$
\mathbb{E}\int_t^T e^{\beta s}|\Delta g(s)|^2 ds \leq (c + cM) \mathbb{E}\int_t^T e^{\beta s}|\Delta Y_{s-}|^2 ds + (\alpha_1 + \alpha_2M) \mathbb{E}\int_t^T e^{\beta s}|\Delta Z_s|^2 ds.
$$

Then, we have the following inequality

$$
\mathbb{E}\left(e^{\beta t}|\Delta Y_t|^2\right) + \beta \mathbb{E}\int_t^T e^{\beta s}|\Delta Y_s|^2 ds + \mathbb{E}\int_t^T e^{\beta s}||\Delta Z_s||^2 ds \leq \epsilon_1 \mathbb{E}\int_t^T e^{\beta s}|\Delta Y_s|^2 ds + \left(\frac{c}{\epsilon_1} + 2cM + c\right) \mathbb{E}\int_t^T e^{\beta s}|\Delta Y_{s-}|^2 ds
$$

$$
+ \left(\alpha_1 + \alpha_2M + \frac{c}{\epsilon_1} + cM\right) \mathbb{E}\int_t^T e^{\beta s}||\Delta Z_s||^2 ds,
$$

choosing $\epsilon_1 > 0$ such that, $\left(\alpha_1 + \alpha_2M + \frac{c}{\epsilon_1} + cM\right) < 1$ and $\beta - \epsilon_1 > 0$, we get

$$
\mathbb{E}\left(e^{\beta t}|\Delta Y_t|^2\right) \leq CE\int_t^T e^{\beta s}|\Delta Y_{s-}|^2 ds,
$$

where $C = \frac{c}{\epsilon_1} + 2cM + c$. The uniqueness of solution follows from Gronwall’s lemma.

**Existence.** Before we start proving equation (3.1) has a unique solution with $f$, $g$ independent on the value and the future value of $(Y, Z)$, i.e., P-a.s., $f(t, \omega, y, z, \pi, \zeta) = f(t, \omega)$ and $g(t, \omega, y, z, \pi, \zeta) = g(t, \omega)$, for any $(t, y, z, \pi, \zeta)$. Then by Y. Ren [11] and the previous proof, we deduce that the equation (3.1) where $f$ and $g$ independent on the value and the future value of $(Y, Z)$ has a unique solution.

Now, we shall prove the existence in the general case. For all $(r, \tilde{r}) \in [t, T + K]^2$, $\forall t \in [0, T + K]$

$$
f(t, \omega, y, z, \pi, \zeta) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^2 \to \mathcal{S}^2([0, T + K], \mathbb{R}) \to \mathcal{M}^2_{\mathbb{R}}([0, T + K], \mathbb{R}).
$$

Let $\mathcal{S}^2_{\mathbb{R}}([0, T + K]; \mathbb{R}^d) \times \mathcal{M}^2_{\mathbb{R}}([0, T + K]; \mathbb{R}^2)$ endowed with the norm

$$
|||Y, Z|||_\beta = \left(\mathbb{E}\left[\int_0^{T+K} e^{\beta s}\left(|Y_{s-}|^2 + \sum_{i=1}^{\infty} |Z_s^{(i)}|^2\right) ds\right]\right)^{\frac{1}{2}}.
$$
Therefore, for given $U \in S^1_{\mathcal{A}}([0, T + K], \mathbb{R})$, $V \in \mathcal{M}^2_{\mathcal{B}}([0, T + K]; \mathbb{R})$, there exists a unique solution $(Y, Z, H)$ for the following ABDSDEs with reflection

\[
\begin{cases}
Y_t = \xi_T + \int_t^T f(s, \theta_s, \phi_{s}^0)ds + \int_t^T g(s, \theta_s, \phi_{s}^0)\,d\tilde{B}_s + \int_t^T dK_s - \sum_{i=1}^{\infty} \int_t^T Z^{(i)}_s \,dH^{(i)}_s, \quad t \in [0, T], \\
(Y_t, Z_t) = (\eta_t, \vartheta_t), \quad t \in [T, T + K],
\end{cases}
\]  

(3.2)

We can construct the mapping $\Phi$ is well defined, let $(Y_t, Z_t)$ and $(\tilde{Y}_t, \tilde{Z}_t)$ be two solution of system (3.2) such that $(Y_t, Z_t) = \Phi(U_{t-}, V_t)$ and $(\tilde{Y}_t, \tilde{Z}_t) = \Phi(\tilde{U}_{t-}, \tilde{V}_t)$.

For $\beta \in \mathbb{R}$. The couple $(\Delta Y_t, \Delta Z_t)$ solve the ABDSDEs with reflection

\[
\begin{cases}
\Delta Y_t = \int_t^T \Delta f(s)\,ds + \int_t^T \Delta g(s)\,d\tilde{B}_s + \int_t^T \Delta dK_s - \sum_{i=1}^{\infty} \int_t^T \Delta Z^{(i)}_s \,dH^{(i)}_s, \quad t \in [0, T], \\
(\Delta Y_t, \Delta Z_t) = (0, 0), \quad t \in [T, T + K],
\end{cases}
\]

where for a function $h \in \{f, g\}$, $\Delta h(s) = h(s, \theta_s, \phi_{s}^0) - h(s, \tilde{\theta}_s, \tilde{\phi}_{s}^0)$, $\theta_s = (U_{s-}, V_s)$, $\phi_{s}^0 = (U_{s+\phi(s)-}, V_{s+\phi(s)+})$, $\tilde{\theta}_s = (\tilde{U}_{s-}, \tilde{V}_s)$, $\tilde{\phi}_{s}^0 = (\tilde{U}_{s+\phi(s)-}, \tilde{V}_{s+\phi(s)+})$ and $\Delta \Psi_s = \Psi_s - \tilde{\Psi}_s$.

Now applying Itô’s formula for $e^{\beta t} |\Delta Y_t|^2$, we get

\[
e^{\beta t} |\Delta Y_t|^2 + \beta \int_t^T e^{\beta s} |\Delta Y_{s-}|^2 \,ds = 2 \int_t^T e^{\beta s} \Delta Y_{s-} \Delta f(s)\,ds + 2 \int_t^T e^{\beta s} \Delta Y_{s-} \Delta g(s)\,d\tilde{B}_s + 2 \int_t^T e^{\beta s} \Delta Y_{s-} \,d\Delta K_s - 2 \int_t^T \sum_{i=1}^{\infty} e^{\beta s} \Delta Y_{s-} \Delta Z^{(i)}_s \,dH^{(i)}_s + \int_t^T e^{\beta s} |\Delta g(s)|^2 \,ds - \int_t^T e^{\beta s} \Delta Y_{s-} \sum_{i,j=1}^{\infty} \Delta Z^{(i)}_s \Delta Z^{(j)}_s \,d[H_s^{(i)}, H_s^{(j)}].
\]

Note that $\int_t^T e^{\beta s} \Delta Y_{s-} \Delta g(s)\,d\tilde{B}_s$, $\int_t^T \sum_{i=1}^{\infty} e^{\beta s} \Delta Y_{s-} \Delta Z^{(i)}_s \,dH^{(i)}_s \not\in (\mathcal{F}_t)$ and hypothesis (H.1), we have

\[
\begin{align*}
2E \int_t^T e^{\beta s} |\Delta Y_{s-} \Delta f(s)| \,ds &\leq \epsilon_1 E \int_t^T e^{\beta s} |\Delta Y_{s-}|^2 \,ds + \frac{c}{\epsilon_1} E \int_t^T e^{\beta s} \left(||\Delta U_{s-}||^2 + ||\Delta V_s||^2_s\right) \,ds, \\
E \int_t^T e^{\beta s} |\Delta g(s)|^2 \,ds &\leq (c + cm) E \int_t^T e^{\beta s} |\Delta U_{s-}|^2 \,ds + (\alpha_1 + \alpha_2 m) E \int_t^T e^{\beta s} ||\Delta V_s||^2_s \,ds.
\end{align*}
\]

Then, we have

\[
\begin{align*}
Ee^{\beta t} |\Delta Y_t|^2 &\leq \left(\frac{c}{\epsilon_1} + c + 2cm\right) E \int_t^T e^{\beta s} |\Delta Y_{s-}|^2 \,ds + \left(\alpha_1 + \alpha_2 m\right) E \int_t^T e^{\beta s} ||\Delta V_s||^2_s \,ds, \\
&\leq \left(\frac{c}{\epsilon_1} + c + 2cm\right) E \int_t^T e^{\beta s} |\Delta U_{s-}|^2 \,ds + \left(\alpha_1 + \alpha_2 m\right) E \int_t^T e^{\beta s} ||\Delta V_s||^2_s \,ds.
\end{align*}
\]
we noting that $\Delta Y_t = \Delta Z_t = 0$, for all $t \in [T, T + K]$

$$(\beta - \epsilon_1) \mathbb{E} \int_t^{T+K} e^{\beta s} |\Delta Y_s| \Delta Y_s - \mathbb{E} \int_t^{T+K} e^{\beta s} |\Delta Z_s| \Delta Z_s \leq \left( \frac{c}{\epsilon_1} + c + 2cM \right) \mathbb{E} \int_t^{T+K} e^{\beta s} |\Delta U_{s-}| \Delta Y_s - \mathbb{E} \int_t^{T+K} e^{\beta s} |\Delta V_s| \Delta Z_s \leq \left( \frac{c}{\epsilon_1} + c + 2cM \right) \mathbb{E} \int_t^{T+K} e^{\beta s} |\Delta U_{s-}| \Delta Y_s - \mathbb{E} \int_t^{T+K} e^{\beta s} |\Delta V_s| \Delta Z_s$$

where $\epsilon_2 = \frac{2cM}{(\alpha_1 + \alpha_2M)}$. Hence, if we choose $\epsilon_0 = \epsilon_1$ satisfying $\hat{e} = \alpha_1 + \alpha_2M + \left( \frac{\hat{\epsilon}}{\epsilon_0} + cM \right) < 1$, choose $\beta = \epsilon_0 + \epsilon_2$. Then, we deduce

$$\mathbb{E} \int_t^{T+K} \mathbb{E} e^{\beta s} |\Delta Y_{s-}|^2 \Delta Y_s + \mathbb{E} \int_t^{T+K} e^{\beta s} |\Delta Z_s| \Delta Z_s \leq \hat{e} \mathbb{E} \int_t^{T+K} e^{\beta s} (\epsilon_2 |\Delta U_{s-}|^2 + |\Delta V_s|^2) ds.$$ 

Thus, the mapping $\Phi$ is a strict contraction on $S_{\mathbb{H}}^2 ([0, T + K]; \mathbb{R}^d) \times \mathcal{M}_{\mathbb{H}}^2 ([0, T + K]; \mathbb{R}^d)$ and it has a unique fixed point $(Y_t, Z_t) \in S_{\mathbb{H}}^2 ([0, T + K]; \mathbb{R}^d) \times \mathcal{M}_{\mathbb{H}}^2 ([0, T + K]; \mathbb{R}^d)$. 

**3.2 Anticipated BDSDE.**

In this subsection we consider the anticipated BDSDE as follows

$$\left\{ \begin{array}{l}
Y_t = \xi_T + \int_t^T f(s, \Lambda_s, \Lambda_s^{\psi}) ds + \int_t^T g(s, \Lambda_s, \Lambda_s^{\psi}) d\hat{B}_s - \sum_{i=1}^\infty \int_t^T \int_0^{\gamma_i} Z_s^{i} dH_s^{i}, t \in [0, T], \\
(Y_t, Z_t) = (\eta_t, \vartheta_t), \quad t \in [T, T + K],
\end{array} \right. \quad (3.3)$$

where $f$ the generator, $\Lambda_s = (Y_{s-}, Z_s)$ and $\Lambda_s^{\psi} = (Y_{s+\psi(s)-}, Z_{s+\psi(s)})$.

**Definition 3.2.** A solution of equation (3.3) is a couple $(Y, Z)$ which belongs to the space $S_{\mathbb{H}}^2 ([0, T + K]; \mathbb{R}) \times \mathcal{M}_{\mathbb{H}}^2 ([0, T + K]; \mathbb{R}^d)$ and satisfies (3.3).

**3.2.1 Existence and uniqueness of solutions.**

**Theorem 3.2.** Assume that (A), (B) and (H1) are satisfied. Then for given $(\eta_t, \vartheta_t) \in B_{\mathbb{H}}^2 ([T, T + K], \mathbb{R})$ l’equation (3.3) has a unique solution $(Y_t, Z_t) \in B_{\mathbb{H}}^2 ([0, T + K], \mathbb{R})$.

**3.2.2 Proof of the existence and uniqueness result.**

Before we start proving equation (3.3) has a unique solution with $f, g$ independent on the value and the futur value of $(Y, Z)$, i.e., $P$-a.s., $f(t, \omega, y, z, \pi, \zeta) = f(t, \omega)$ and $g(t, \omega, y, z, \pi, \zeta) = g(t, \omega)$, for any $(t, y, z, \pi, \zeta)$. More precisely, given $f, g$ such that

$$E \left( \int_0^T |f(t)|^2 \, dt \right) < \infty, \quad E \left( \int_0^T |g(t)|^2 \, dt \right) < \infty.$$ 

Under the above assumption on $f, g, \xi$. 

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Proposition 3.1. Given \( \xi \in L^2(\mathcal{H}_T) \), there exists a unique couple of processes \( (Y_t, Z_t) \in B^4_H([0, T + K], \mathbb{R}) \), to solve the following BDSDEs,
\[
\begin{cases}
Y_t = \xi_T + \int_t^T f(s) ds + \int_t^T g(s) d\overline{B}_s - \sum_{i=1}^{\infty} \int_t^T Z^i_s dH^i_s, & t \in [0, T], \\
(Y_t, Z_t) = (\eta_t, \vartheta_t), & t \in [T, T + K].
\end{cases}
\]

Proof: We consider the following filtration
\[
\mathcal{G}_t := \mathcal{F}^L_t \lor \mathcal{F}^B_{T+K},
\]
and the \( \mathcal{G}_t \) square integrable martingale
\[
M_t = E^{\mathcal{G}_t}\left( \xi_T + \int_t^T f(s) ds + \int_t^T g(s) d\overline{B}_s \right), \quad t \in [0, T].
\]

Thank’s to the predictable representation property in Nualart et al. [5] yields that there exists \( Z \in \mathcal{M}^2_H([0, T]; l^2) \) such that
\[
M_t = M_0 + \sum_{i=1}^{\infty} \int_0^t Z^i_s dH^i_s,
\]
hence
\[
M_T = M_t + \sum_{i=1}^{\infty} \int_t^T Z^i_s dH^i_s.
\]

Let
\[
Y_t = M_t - \int_t^T f(s) ds - \int_t^T g(s) d\overline{B}_s,
\]
\[
= E^{\mathcal{G}_t}\left( \xi_T + \int_t^T f(s) ds + \int_t^T g(s) d\overline{B}_s \right),
\]
\[
= M_T - \sum_{i=1}^{\infty} \int_t^T Z^i_s dH^i_s - \int_0^t f(s) ds - \int_0^t g(s) d\overline{B}_s,
\]
from which, we deduce that
\[
Y_t = \xi + \int_t^T f(s) ds + \int_t^T g(s) d\overline{B}_s - \sum_{i=1}^{\infty} \int_0^T Z^i_s dH^i_s.
\]

Now by the same procedure of Xu [16], we can obtain the uniqueness and \( \mathcal{H}_t \)- measurable of \( Y_t \) and \( Z_t \).

We are now in a position to give the proof of Theorem 3.2.

Proof: Let \( S^4_H([0, T + K]; \mathbb{R}^d) \times \mathcal{M}^2_H([0, T + K]; l^2) \) endowed with the norm
\[
\| (Y, Z) \|_\beta = \left( E \left[ \int_0^{T+K} e^{\beta s} \left( |Y_s|^2 + \sum_{i=1}^{\infty} |Z^i_s|^2 \right) ds \right] \right)^{\frac{1}{2}}.
\]

Let we consider the following mapping:
\[
\Phi : S^4_H([0, T + K]; \mathbb{R}) \times \mathcal{M}^2_H([0, T + K]; l^2) \to S^4_H([0, T + K]; \mathbb{R}) \times \mathcal{M}^2_H([0, T + K]; l^2),
\]
where the couple \( (Y_t, Z_t)_{T \leq t \leq T + K} \in S^4_H([0, T + K]; \mathbb{R}) \times \mathcal{M}^2_H([0, T + K]; l^2) \) such that
\[
(Y_t, Z_t)_{T \leq t \leq T + K} = (\eta_t, \vartheta_t)
\]
and satisfies the equation (3.3). Thanks to Proposition (3.1), the mapping \( \Phi \) is well defined. Let \( (Y_t, Z_t) \) and \( (\tilde{Y}_t, \tilde{Z}_t) \) be two solution of (3.3) such that \( (Y_t, Z_t) = \Phi(y_{t-}, z_t) \) and \( (\tilde{Y}_t, \tilde{Z}_t) = \Phi(\tilde{y}_{t-}, \tilde{z}_t) \).
For $\beta \in \mathbb{R}$. The couple $(\Delta Y_t, \Delta Z_t)$ solve the ABDSDEs with tangles martingale
\[
\begin{aligned}
\Delta Y_t &= \int_t^T \Delta f(s) \, ds + \int_t^T \Delta g(s) \, dB_s - \sum_{i=1}^{t+i} \int_t^T \Delta Z_s^{(i)} \, dH_s^{(i)}, \quad t \in [0, T), \\
(\Delta Y_t, \Delta Z_t) &= (0, 0), \quad t \in [T, T+K].
\end{aligned}
\]

where for a function $h \in \{ f, g \}$, $\Delta h(s) = h(s, \theta_s, \theta_s^{\phi, \psi}) - h(s, \bar{\theta}_s, \bar{\theta}_s^{\phi, \psi})$, $\theta_s = (y_{s-}, z_s)$, $\theta_s^{\phi, \psi} = (y_{s+}(s-), z_{s+}(s-))$, $\bar{\theta}_s = (\bar{y}_{s-}, \bar{z}_s)$, $\bar{\theta}_s^{\phi, \psi} = (\bar{y}_{s+}(s-), \bar{z}_{s+}(s-))$ and $\Delta \Psi_s = \Psi_s - \bar{\Psi}_s$. Applying Itô’s formula to $e^{\beta t} |\Delta Y_t|^2$, we obtain
\[
e^{\beta t} |\Delta Y_t|^2 + \beta \int_t^T e^{\beta s} |\Delta Y_{s-}|^2 \, ds = 2 \int_t^T e^{\beta s} \Delta Y_s \Delta f(s) \, ds + 2 \int_t^T e^{\beta s} \Delta Y_s \Delta g(s) \, dB_s - 2 \int_t^T \sum_{i=1}^{t+i} e^{\beta s} \Delta Y_s \Delta Z_s^{(i)} \, dH_s^{(i)} + \int_t^T e^{\beta s} |\Delta g(s)|^2 \, ds - \int_t^T \sum_{i,j=1}^{t+i,j} e^{\beta s} \Delta Z_s^{(i)} \Delta Z_s^{(j)} \, d \langle H_s^{(i)}, H_s^{(j)} \rangle,
\]

note that $\int_0^t e^{\beta s} \Delta Y_s \Delta g(s) \, dB_s$, $\int_0^t \sum_{i=1}^{t+i} e^{\beta s} \Delta Y_s \Delta Z_s^{(i)} \, dH_s^{(i)} \forall i \geq 1$ and $\int_0^t \sum_{i,j=1}^{t+i,j} e^{\beta s} \Delta Z_s^{(i)} \Delta Z_s^{(j)} \, d \langle H_s^{(i)}, H_s^{(j)} \rangle$ for $i \neq j$ are uniformly integrable martingales.

Now taking the mathematical expectation on both sides, we obtain
\[
\mathbb{E}e^{\beta t} |\Delta Y_t|^2 + \beta \mathbb{E} \int_t^T e^{\beta s} |\Delta Y_{s-}|^2 \, ds + \mathbb{E} \int_t^T \sum_{i=1}^{t+i} e^{\beta s} |\Delta Z_s^{(i)}|^2 \, ds
\]
\[
= 2 \mathbb{E} \int_t^T e^{\beta s} \Delta Y_s \Delta f(s) \, ds + \mathbb{E} \int_t^T e^{\beta s} |\Delta g(s)|^2 \, ds.
\]

Now by the same computation of Lipschitz coefficient for Anticipated reflected BDSDEs in general case, we deduce that
\[
\mathbb{E} \int_t^T e^{\beta s} \left( \epsilon_2 |\Delta Y_{s-}|^2 + ||\Delta Z_s||^2 \right) \, ds \leq \epsilon \mathbb{E} \int_t^{T+K} e^{\beta s} \left( \epsilon_2 |\Delta Y_{s-}|^2 + ||\Delta Z_s||^2 \right) \, ds,
\]

where $0 < \epsilon < 1$. Thus, the mapping $\Phi$ is a strict contraction on $S^2_{\mathbb{R}}([0, T+K]; \mathbb{R}) \times M^2_{\mathbb{R}}([0, T+K]; \mathbb{R})$ and it has a unique fixed point $(Y_T, Z_T) \in S^2_{\mathbb{R}}([0, T+K]; \mathbb{R}) \times M^2_{\mathbb{R}}([0, T+K]; \mathbb{R})$.

Finally, we complete the proof of theorem 3.2. 

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