Vibrational Spectrum of the B=7 Skyrme Soliton

W. K. Baskerville
Department of Mathematical Sciences, University of Durham, Science Laboratories, South Road, Durham, DH1 3LE, U.K.
(March 28, 2022)

The finite energy vibrational normal modes of the baryon number B=7 Skyrme soliton are computed. The structure of the spectrum obtained displays considerable similarity to those previously calculated for baryon numbers 2, 3 and 4. All modes expected from an approximate correspondence between skyrmions and BPS monopoles are found to be present. However, in contrast to earlier calculations, they do not all have energies below the pion mass. The remaining ‘breather-type’ modes also conform to predictions, except that one predicted multiplet is not observed.

I. INTRODUCTION

The normal mode spectra of multiskyrmions are believed to play a key role in their quantisation [1]. In the Skyrme model [2], nuclei appear as topological solitons in a classical, non-linear field theory of π-mesons. Since the model is not renormalisable, a proper treatment as a quantum field theory is very difficult. The usual approach to quantisation is based on the suggestion of Manton [3,4], that the low energy dynamics of the model is not renormalisable, a proper treatment as in a classical, non-linear field theory of π-mesons. Since the model is not renormalisable, a proper treatment as a quantum field theory is very difficult. The usual approach to quantisation is based on the suggestion of Manton [3,4], that the low energy dynamics of the model is not renormalisable, a proper treatment as in a classical, non-linear field theory of π-mesons.

W. K. Baskerville
Department of Mathematical Sciences, University of Durham, Science Laboratories, South Road, Durham, DH1 3LE, U.K.
(March 28, 2022)

The finite energy vibrational normal modes of the baryon number B=7 Skyrme soliton are computed. The structure of the spectrum obtained displays considerable similarity to those previously calculated for baryon numbers 2, 3 and 4. All modes expected from an approximate correspondence between skyrmions and BPS monopoles are found to be present. However, in contrast to earlier calculations, they do not all have energies below the pion mass. The remaining ‘breather-type’ modes also conform to predictions, except that one predicted multiplet is not observed.

I. INTRODUCTION

The normal mode spectra of multiskyrmions are believed to play a key role in their quantisation [1]. In the Skyrme model [2], nuclei appear as topological solitons in a classical, non-linear field theory of π-mesons. Since the model is not renormalisable, a proper treatment as a quantum field theory is very difficult. The usual approach to quantisation is based on the suggestion of Manton [3,4], that the low energy dynamics of the model is not renormalisable, a proper treatment as in a classical, non-linear field theory of π-mesons. Since the model is not renormalisable, a proper treatment as a quantum field theory is very difficult. The usual approach to quantisation is based on the suggestion of Manton [3,4], that the low energy dynamics of the model is not renormalisable, a proper treatment as in a classical, non-linear field theory of π-mesons. Since the model is not renormalisable, a proper treatment as a quantum field theory is very difficult. The usual approach to quantisation is based on the suggestion of Manton [3,4], that the low energy dynamics of the model is not renormalisable, a proper treatment as in a classical, non-linear field theory of π-mesons. 

W. K. Baskerville
Department of Mathematical Sciences, University of Durham, Science Laboratories, South Road, Durham, DH1 3LE, U.K.
(March 28, 2022)
dom. This is rather more than the 6B zero modes of B widely separated skyrmions, throwing open the question as to how many degrees of freedom are really relevant in the vicinity of the minimal energy solution, and therefore how many should be quantised. It is interesting to note that instanton generated approximations to Skyrme field configurations naturally have 8B − 1 degrees of freedom. Houghton recently computed the vibrational spectra of the instanton-generated 3-skyrmion, and obtained the same modes as predicted by a combination of the monopole and geometrical methods, and thus an additional triplet, with axial vector symmetry.

In this article, the vibrational normal modes of the B=7 skyrmion are computed, using the same method as in [13]. Their symmetries are then classified according to the symmetry group of the static soliton, in this case the icosahedral group, the minimal energy solution having the shape of a perfect dodecahedron [6]. The B=7 spectrum is of particular interest since a full set of predictions already exists for the normal modes. It thus provides a good test of both the monopole analogy and the geometrical ansatz for higher modes. In fact, all of the expected vibrations are observed, except for one five-fold degenerate multiplet. The B=7 vibrational spectrum may also be important for another reason. Irwin recently performed a zero mode quantisation of minimal B=7 skyrmion are computed, using the same method as in [1,9]. Their symmetries are then classified according to the SU(2) constraint, which leads to the coupling, with all its inherent numerical difficulties. However, for very small perturbations about the static solution, φ(t, x) = φst(x) + ε(t, x), the time dependent parts of K are given by terms to order at least ε 3. If ε << 1, such terms can be neglected to a good approximation, and the matrix K assigned its value at the static classical solution. The discretised equations of motion then reduce to the form

\[ \phi^a(t + 1, i, j, k) = R^a(t, i, j, k) + \lambda \phi^a(t, i, j, k) \]

where \( \lambda \) is a constant related to the Lagrange multiplier, and \( R^a \) is a function of the fields at the current and previous timesteps only. Invoking the SU(2) constraint \( \delta_{t+1}^a \phi_{t+1}^a = 1 \) yields a quadratic equation for \( \lambda \)

\[ \lambda^2 + 2\lambda R^a \phi^a + R^a R^a - 1 = 0, \]

the positive square root of which is the desired solution (the definitions of \( \lambda \) and \( R^a \) can be chosen so as to ensure this: see [4] for details). Substituting \( \lambda \) back into Eq. (3) then gives a deterministic time evolution for the Skyrme fields.

The vibrational spectrum can be directly computed using this algorithm. The static solution is found by numerical relaxation, using fields derived from the rational map ansatz as a starting point. A small random perturbation is then applied, after which the fields are evolved forward for a long time at constant energy. The evolving fields take the form

\[ \phi(t, x) = \phi_{st}(x) + \sum_{\text{modes}} \epsilon_n \delta_n(x) \cos(\omega_n t) + O(\epsilon^3), \]

where the functions \( \delta_n(x) \in \mathbb{R}^4 \), obeying \( \delta_n(x) \cdot \phi_{st}(x) = 0 \) are the normal modes, each excited with amplitude \( \epsilon_n \).

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \omega_0^2 \phi^1 + \lambda (\phi \cdot \phi - 1) \]

\[ + \frac{1}{4} \left( (\partial_{\mu} \phi \cdot \partial_{\nu} \phi)(\partial^{\mu} \phi \cdot \partial^{\nu} \phi) - (\partial_{\mu} \phi \cdot \partial^{\mu} \phi)(\partial_{\nu} \phi \cdot \partial^{\nu} \phi) \right) \]

where \( \lambda \) is a Lagrange multiplier field, and \( \omega_0 = \frac{2}{\sqrt{3}} \) is a dimensionless constant, the oscillation frequency of the homogeneous pion field. As in [13], the ‘standard’ value \( \omega_0 = \frac{4}{\pi} \) has been used.

The Skyrme equations of motion are derived by discretising the action \( S = \int d^4x \mathcal{L} \) on a finite lattice, then varying with respect to each field component \( \phi^a(t, i, j, k) \), at each point in the lattice. This leads in general to a coupled system of equations, due to the mixed time and space derivatives in the quartic Skyrme term. The relevant, “kinetic” part of the Lagrangian density can be written

\[ \mathcal{L}_{\text{kin}} = \frac{1}{2} \partial_{\mu} \phi^a \partial^{\mu} \phi^b \]

where \( K^{ab} = \delta^{ab}(1 + (\partial_i \phi)^2) - \partial_i \phi^a \partial_i \phi^b \) acts as a spatially dependent inertia tensor. It is the time dependence of \( K^{ab} \) which leads to the coupling, with all its inherent numerical difficulties. However, for very small perturbations about the static solution, \( \phi(t, x) = \phi_{st}(x) + \epsilon(t, x) \), the time dependent parts of \( K^{ab} \) give rise to terms of order at least \( \epsilon^3 \). If \( \epsilon \ll 1 \), such terms can be neglected to a good approximation, and the matrix \( K^{ab} \) assigned its value at the static classical solution. The discretised equations of motion then reduce to the form

\[ \phi^a(t + 1, i, j, k) = R^a(t, i, j, k) + \lambda \phi^a(t, i, j, k) \]

where \( \lambda \) is a constant related to the Lagrange multiplier, and \( R^a \) is a function of the fields at the current and previous timesteps only. Invoking the SU(2) constraint \( \delta_{t+1}^a \phi_{t+1}^a = 1 \) yields a quadratic equation for \( \lambda \)

\[ \lambda^2 + 2\lambda R^a \phi^a + R^a R^a - 1 = 0, \]

the positive square root of which is the desired solution (the definitions of \( \lambda \) and \( R^a \) can be chosen so as to ensure this: see [4] for details). Substituting \( \lambda \) back into Eq. (3) then gives a deterministic time evolution for the Skyrme fields.

The vibrational spectrum can be directly computed using this algorithm. The static solution is found by numerical relaxation, using fields derived from the rational map ansatz as a starting point. A small random perturbation is then applied, after which the fields are evolved forward for a long time at constant energy. The evolving fields take the form

\[ \phi(t, x) = \phi_{st}(x) + \sum_{\text{modes}} \epsilon_n \delta_n(x) \cos(\omega_n t) + O(\epsilon^3), \]

where the functions \( \delta_n(x) \in \mathbb{R}^4 \), obeying \( \delta_n(x) \cdot \phi_{st}(x) = 0 \) are the normal modes, each excited with amplitude \( \epsilon_n \).
The normal mode frequencies $\omega_n$ are obtained by Fourier analysis of the field $\phi(t, \mathbf{x})$, with respect to time, at any point in the box.  

Once the frequencies have been identified, the time evolution process is repeated. Maps of the normal modes $\delta_n(x)$ can be constructed by performing discrete Fourier sums on each component of the evolving field. The space of perturbations has a very useful inner product

$$\langle \delta_1 | \delta_2 \rangle = \int_{\text{box}} \delta_1^a(x) K^{ab}(x) \delta_2^b(x) \, d^3x, \quad (6)$$

the matrix $K^{ab}(x)$ being included in this definition to ensure orthogonality between modes of different frequency: $\langle \delta_1 | \delta_2 \rangle = 0$ if $\omega_1 \neq \omega_2$. In general, the normal modes fall into degenerate multiplets at discrete frequencies. Each such multiplet forms a matrix representation of the symmetry group of the static soliton, via the inner product $\langle \delta | S | \delta \rangle$. The symmetry operations $S$ in general consist of a physical operation, for example a rotation or reflection, which leaves the energy and baryon densities of the static soliton invariant, plus an isospin transformation of the pion fields ($\phi^2$, $\phi^3$, $\phi^4$), which “undoes” the effect of the physical operation on the static fields. Modes of the same frequency always form an irreducible representation (irrep) of the symmetry group, which places strong constraints on the allowed degeneracies. In the case of the $B=7$ multiskyrmion, the symmetry group of the static soliton is the icosahedral group, which has irreducible representations of degeneracy one, three, four and five only.

Determining the degeneracy and symmetry of each multiplet is in principle straightforward. For degenerate modes, each random initial perturbation will produce a different linear combination $\sum_\delta \epsilon_\delta \delta_i(x)$, when one projects out the relevant frequency. Applying the symmetries of the static soliton will yield yet other, different combinations. The degeneracy of a given frequency can be found by computing the rank of the matrix of inner products between different linear combinations of the degenerate modes. It is also possible to project any given linear combination out of all the others. In this way, a library of orthonormal modes can be built up. Once a full orthonormal basis has been obtained for a given frequency, it is easy to calculate its symmetry. Irreducible representations are uniquely labelled by the characters $\sum_\delta \langle \delta | S | \delta \rangle$ of certain symmetry operations $S$.

Unfortunately, several difficulties arise in this procedure for the particular case of $B=7$. The main problem is that the simulations are carried out on a finite, cubic lattice, which breaks the dodecahedral symmetry of the static solution. The most immediate consequence of this is that some multiplets may be artificially split, for example a five-fold degenerate multiplet into a doublet and triplet of close, but not identical, frequency. The breaking may also cause certain modes to be generated more easily than others, if they fit better onto the finite grid. Thus different initial conditions do not really produce randomly different linear combinations of degenerate modes. This can effectively ‘hide’ some of the degeneracy, since the method outlined above depends on finding enough independent linear combinations. A closely related technical difficulty is that not all of the symmetry operations of the icosahedral group can be exactly realised on a cubic lattice. If the cubic lattice does not map onto itself under some physical operation, then interpolation must be used to find the new field values. This can be done accurately enough to compute the character of such a symmetry, given an orthonormal basis; but is not suitable for separating out such a basis in the first place. Since only 24 (out of 120) physical symmetries can be performed exactly on the cubic lattice, this is a severe limitation. The last (but certainly not least) problem is contamination between modes of different frequency. Forty-three vibrational modes are predicted for $B=7$, lying in at least eleven different multiplets. These are all expected to occur within a fairly small range of frequencies. The finite time scale of the simulation leads inevitably to finite-width peaks in the power spectrum. Considerable overlap between different multiplets is therefore possible, and does in fact occur, as can immediately be seen from Figure ().

The resolution of these difficulties requires a closer examination of the icosahedral symmetry group $I_h$, as well as the particular alignment of the dodecahedral static solution on the cubic lattice. The character table for $I_h$ is given in Table III. The physical symmetries can be divided into ten ‘conjugacy classes’ of like operations. The identity $\mathds{1}$ and inversion $i$ both form classes on their own. There are six five-fold rotational axes in a dodecahedron, running through the centres of opposite faces. Rotations by $\frac{2\pi}{5}$ about these, in either direction, form one conjugacy class (with 12 elements), labelled $C_5$, while rotations by $\frac{4\pi}{5}$ form another, $C_5^2$, also with 12 elements. There are ten three-fold axes, through pairs of opposite corners of the dodecahedron. The 20 $\frac{2\pi}{3}$ rotations associated with these form another conjugacy class, $C_3$. The final rotational symmetry is two-fold, about axes which bisect pairs of opposite edges of the dodecahedron. This gives 15 rotations by $\pi$ (there are 30 edges, since each edge is shared by two of the twelve pentagonal faces), forming the conjugacy class $C_2$. The remaining symmetry operations are all combinations of inversion with one of the rotations already mentioned, and are denoted $iC_n$, for rotation $C_n$. (Note that $iC_2$ gives pure reflections, in planes which contain two opposite edges, and bisect four pentagonal faces.) The characters for these classes are exactly the same as for the equivalent rotations without inversion, except that negative parity representations pick up a minus sign. This is because the full icosahedral group $I_h$ is just the group of rotations of an icosahedron (dodecahedron) $I$, extended by inversion: $I_h = I \times Z_2$. The irreducible representations are labelled by their degeneracies and parities, where this is sufficient to distinguish them (eg. $5^-$ is a five-fold degenerate multiplet with negative parity). The only exceptions are the four three-dimensional irreps, which are labelled $F_1$ and $F_2$,}
FIG. 1. Schematic diagram of one cube within a dodecahedron. The edges of the dodecahedron are drawn as dotted lines on the faces of the cube. In reality, they would project slightly, and the edges of the cube would lie slightly behind the pentagonal faces.

again with superscripts denoting parity. With this notation, a vector has symmetry $F_1^+$, while an axial vector transforms as $F_2^+$. 

Next, we need to examine the alignment of the dodecahedral soliton on the cubic grid. A dodecahedron contains within itself five ‘cubes’; that is, there are five sets of eight vertices, each of which form the corners of a cube. Each of the 20 vertices of the dodecahedron is a member of two such cubes. In the B=7 static soliton used in simulations, one of these cubes is aligned with the axes of the cubic lattice. See Figure (1) for a schematic diagram. It should be obvious, looking at this Figure, which symmetry operations map this cube to itself (and hence preserve the grid structure): the identity, inversion, three $C_2$ rotations (about the Cartesian axes), three reflections, eight $C_3$ rotations (about the body diagonals of the cube), and the same eight rotations combined with inversion. The vital question is how to use these symmetries to produce orthonormal bases for all possible multiplets. It is very useful, at this stage, to keep in mind some sort of physical model for representations of various degeneracies. For example, the Cartesian axes will obviously transform as a triplet (in fact $F_1^-$). Given a general mixture of all three (a vector in a random direction), the symmetry operations $(\mathbb{I} + C_x)$, $(\mathbb{I} + C_y)$ and $(\mathbb{I} + C_z)$, or equivalently $(C_y + C_z)$, $(C_x + C_z)$ and $(C_x + C_y)$, will project out the $x$, $y$ and $z$ components respectively, where $C_x$ is a $180^\circ$ rotation about the $x$-axis, etc. Degeneracies four and five are a little trickier. The five cubes mentioned above form a representation, since all symmetries of the dodecahedron will rotate them among themselves. This representation is not irreducible, since the sum or symmetric combination of all five transforms trivially (as $1^+$), i.e. this combination is invariant under all physical symmetries of the dodecahedron. However, the four remaining antisymmetric configurations, orthogonal to the symmetric combination and to each other, do give an irreducible $4^+$ representation. Labelling the five cubes $\{1\} \cdots \{5\}$, a suitable orthonormal basis for the degeneracy 4 irrep can be written

$$|A\rangle = \frac{1}{2} (|1\rangle + |2\rangle - |3\rangle - |4\rangle)$$

$$|B\rangle = \frac{1}{2} (|1\rangle + |3\rangle - |2\rangle - |4\rangle)$$

$$|C\rangle = \frac{1}{2} (|1\rangle + |4\rangle - |2\rangle - |3\rangle)$$

$$|D\rangle = \frac{1}{2\sqrt{5}} (4|5\rangle - |1\rangle - |2\rangle - |3\rangle - |4\rangle).$$

If the cube which is aligned with the lattice is taken to be $|5\rangle$, then the $C_2$ rotations about the Cartesian axes act as follows:

$C_x : (A, B, C, D) \rightarrow (A, -B, -C, D)$

$C_y : (A, B, C, D) \rightarrow (-A, B, -C, D)$

$C_z : (A, B, C, D) \rightarrow (-A, -B, C, D).$

Using these results, it is quite simple to construct projection operators for the orthonormal basis

$$P_A = \mathbb{1} + C_x - C_y - C_z$$

$$P_B = \mathbb{1} - C_x + C_y - C_z$$

$$P_C = \mathbb{1} - C_x - C_y + C_z$$

$$P_D = \mathbb{1} + C_x + C_y + C_z.$$
It should be noted at this stage that $P_A$, $P_B$ and $P_C$ also act as projectors for the triplet formed by the Cartesian axes. What now for a five-fold basis? There are six $C_5$ rotational axes, giving a 6-dimensional representation which decomposes in a similar way to the 5 cubes: again the symmetric combination transforms trivially as $1^+$, while five orthogonal antisymmetric combinations form $5^-$. Labelling the six $C_5$ axes \{1\}$\cdots$6\}, a possible basis for the latter is

$$\begin{align}
|a\rangle &= \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle), \\
|b\rangle &= \frac{1}{\sqrt{2}} (|3\rangle - |4\rangle), \\
|c\rangle &= \frac{1}{\sqrt{2}} (|5\rangle - |6\rangle), \\
|d\rangle &= \frac{1}{2} (|1\rangle + |2\rangle - |3\rangle - |4\rangle), \\
|e\rangle &= \frac{1}{2\sqrt{3}} (2|5\rangle + 2|6\rangle - |1\rangle - |2\rangle - |3\rangle - |4\rangle).
\end{align}$$

(10a)  

(10b)  

(10c)

The action of $C_x$, $C_y$, and $C_z$ on these states is as follows

$$
\begin{align}
C_x : (a, b, c, d, e) &\rightarrow (a, -b, -c, d, e) \\
C_y : (a, b, c, d, e) &\rightarrow (-a, b, -c, d, e) \\
C_z : (a, b, c, d, e) &\rightarrow (a, -b, c, d, e). 
\end{align}
$$

(11a)  

(11b)  

(11c)

It is then apparent that the projection operators $P_A$, $P_B$ and $P_C$ defined in Eq. (4) also act as projectors for states $|a\rangle$, $|b\rangle$ and $|c\rangle$ respectively, while $P_D$ will project out some combination of states $|d\rangle$ and $|e\rangle$. It may seem an incredible coincidence that the same symmetries project out the basis states for multiplets of degeneracy three, four and five, but in fact this is deeply related to the way in which the cubic lattice breaks the icosahedral symmetry. Furthermore, these same projectors work for all degeneracy three, four and five multiplets, not just $F_1^-$, $4^+$ and $5^+$, from which they were derived. This last is not obvious, in fact is perhaps counter-intuitive, but nevertheless has been found to work in practice.

The projectors defined in Eq. (4) are the main tool required to separate out the various multiplets, when contamination occurs between them. States of different parity can easily be separated, using the projectors

$$
P_+ = 1 + i, \quad P_- = 1 - i.
$$

(12)

However, contamination between degenerate multiplets of the same parity is more awkward. The coincidence between the projection operators means that mode $|A\rangle$ of a $4^+$ cannot be distinguished from mode $|a\rangle$ of a $5^+$ for example, or from the “$x$” component of $F_1^-$ or $F_2^-$. However, it is possible to tell whether the three modes projected out by $P_A$, $P_B$ and $P_C$ are derived purely from a single multiplet, or represent some mixture, by considering their behaviour under $C_5$ rotations. A $120^\circ$ degree rotation in physical space causes a cyclic permutation of the three projections. This is also a $120^\circ$ degree rotation in the ‘phase space’ of the three modes, and since the modes may be aligned or anti-aligned with the Cartesian axes of this space, two of the modes may pick up minus signs. This occurs randomly, and the chance of it happening the same way for two different multiplets which are mixed together is only $1/3$. Hence if the overlap $\langle \delta|P_A|\delta\rangle$ is exactly equal to $\pm \langle \delta|C_5P_B|\delta\rangle$, we are almost certainly dealing with a pure multiplet. If, furthermore, $\langle \delta|P_D|\delta\rangle = 0$, then $|\delta\rangle$ represents a pure triplet. Otherwise, the mode must have degeneracy four or five. The latter possibilities can be distinguished by the behaviour of $P_D|\delta\rangle$ under $C_5$ rotations. If the multiplet has degeneracy four, then $\langle \delta|P_D^3C_5P_D|\delta\rangle = 1$, while for a degeneracy 5 multiplet $\langle \delta|P_D^4C_5P_D|\delta\rangle = -\frac{1}{2}$. For degeneracy 5, the operator $(C_3 - C_5^{-1})P_D$ (where $C_3$ and $C_5^{-1}$ are $120^\circ$ rotations in opposite directions about the same axis) projects out a linear combination orthogonal to that projected out by $P_D$, thus completing the orthonormal basis.

The overall procedure for interpretation of the spectrum is then as follows. First of all, any contamination by zero modes (spin or linear momentum) is projected out. A careful watch is kept during the simulation to make sure that “drift” (ie. excitation of zero modes) is kept within strict limits, to ensure the validity of the basic assumption underlying the computer algorithm. Additionally, severe damping is applied at the edges of the box for the first 1000 timesteps of the simulation, before any data collection begins. This helps to decrease the proportion of radiation relative to real vibrational modes, and also helps reduce the subsequent drift. Still slight contamination from zero modes occurs, which must be removed from the perturbations $\delta^\alpha(x)$. Next, the perturbations are separated into positive and negative parity components. The projectors $P_A$ to $P_D$ are applied to the resulting states. The overlaps $\langle \delta|P_i|\delta\rangle$ and $\langle \delta|C_5P_i|\delta\rangle$ are calculated, as is the matrix element $\langle \delta|P_D^iC_5P_D|\delta\rangle$. If a pure multiplet is found, an orthonormal basis for it is constructed. This can then be projected out of all other frequencies. Hopefully this will remove enough contamination that at least one other mode (of the same parity) will now be pure. The procedure can then be repeated, until there are no pure modes left. As the library of basis states for pure multiplets is built up, a check is kept on how much of each frequency is accounted for. If all modes are suitably normalised, then

$$
\sum_{i,n} \langle \omega_n^{(i)}|\omega_n^{(i)}\rangle = 1
$$

(13)

for all original perturbations $|\delta\rangle$, where $\{|\omega_n^{(i)}\rangle\}$ is the full set of orthonormal basis states. It may happen that some multiplets are inextricably mixed, so that it is not possible to extract a pure set of basis states for them. This will apply mainly to the “triplet” states, as modes $|D\rangle$ of $4^\pm$ and $\{|d\rangle, |e\rangle\}$ of $5^\pm$ can always be separated at
least from each other. It is still possible to tell how many "triplets" there are. Applying $P_A$, $P_B$ and $P_C$ to any one of the remaining frequencies will give basis vectors in the combined space of the remaining multiplets. This can be projected out of the other frequencies, and the process repeated, as for the pure modes, until all modes have been accounted for. In practice, this means until all the sums in Eq. (13) add to at least 0.995. Thus it is always possible to fully classify the symmetries of all vibrational modes in the spectrum, even if they cannot all be projected out cleanly. The main difference is that frequencies can be confidently assigned to the pure multiplets, whereas mixtures will in general merely be confined within a certain range of frequencies. Sometimes a tentative assignment can be made if a pure doublet appears strongly at a given frequency.

One last technical detail should be mentioned. As was explained above, the symmetry operations that will be applied to the perturbations $\{\delta\}$, are combinations of physical symmetries and isospin transformations. The alignment of the static solution on the cubic grid is as shown in Figure 1. It is also necessary to determine the exact symmetry of the pion fields in the static solution, or at least, how they transform under the specific symmetries needed to identify the perturbations. From the foregoing discussion, it should be clear that the symmetry operations needed to project out orthonormal bases for all modes are inversion, the three $\pi$-rotations $C_x$, $C_y$ and $C_z$, plus one $\frac{2\pi}{3}$-rotation (any of the eight which can be realised on the cubic lattice will do) and its inverse. Additionally, we will need to calculate the character of one $\frac{2\pi}{3}$ rotation, to distinguish between the triplet representations $F_1$ and $F_2$. The following equations give the full symmetry operations used, both the physical operations on the Cartesian axes $(x, y, z)$, and the isospin transformations which restore the original fields of the static solution.

$$i : (x, y, z) \rightarrow (x, -y, -z)$$
$$\pi(x, \pi_2, \pi_3) \rightarrow (-\pi_1, -\pi_2, -\pi_3)$$

$$C_x : (x, y, z) \rightarrow (x, -y, -z)$$
$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & -1 & 0 \\ -\sin \theta & 0 & -\cos \theta \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$$

$$C_y : (x, y, z) \rightarrow (-x, y, -z)$$
$$\pi(x, \pi_2, \pi_3) \rightarrow (-\pi_1, \pi_2, -\pi_3)$$

$$C_z : (x, y, z) \rightarrow (-x, y, z)$$
$$\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix} \rightarrow \begin{pmatrix} -\cos \theta & 0 & \sin \theta \\ 0 & -1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$$

$$C_3 : (x, y, z) \rightarrow (y, z, x)$$

where $\alpha = 1.0206', \theta = 1.1410'$. Note also that $(\pi_1, \pi_2, \pi_3)$ are the fields $(\phi^2, \phi^3, \phi^4)$, in the notation used earlier in this Section.

**III. RESULTS.**

Simulations of the vibrations of the B=7 multiskyrmion were carried out on a finite lattice of $65^3$ points, with side length $L = 8$ in Skyrme units. The resulting power spectrum is displayed in Figure 2.

The lowest peak in the spectrum, at frequency 0.019, was found to have total overlap $\sum \langle \delta_0 | J_i | \delta_0 \rangle = 0.9911$ with angular momentum zero modes. It is thus identified as comprising the rotational modes of the static soliton, which are broken from zero to finite energy by the discrete lattice. The same thing happened in the B=2, 3 and 4 vibrational spectra. The next three peaks contain a pure $5^+$ multiplet (mainly in the middle of the three peaks) and a pure $5^-$ multiplet, which is split between the first and third peaks, with the modes $\{a|b|c\}$ mainly in one peak, and $\{d|e\}$ mainly in the other. Pictures of all the modes may be found in Figure 3.

---

**FIG. 2. B=7 Power Spectrum.**
degeneracy 5 multiplets represent scattering type modes. In the positive parity $5^+$ at frequency 0.302, two opposite pentagonal faces of the dodecahedron split away. Presumably these will become B=2 tori, which, if the mode were amplified, would move away in opposite directions, leaving a $B=3$ tetrahedron at the origin. The $5^-$ modes, split between frequencies 0.289 and 0.320, are more difficult to decipher. It appears that one edge is pulled away, suggesting that the dodecahedron may be splitting into a single skyrmion and a $B=6$ soliton. Both these scattering processes have a “line” symmetry, which may explain their degeneracy. There are six natural lines in the dodecahedron, the $C_5$ axes. Exciting modes in all six directions simultaneously however would be equivalent to the trivial breather, so that only five directions are really independent, hence the degeneracy. This argument is certainly valid for the positive parity modes. It is not as obvious for the negative parity vibrations, but the similarity between the symmetries is certainly suggestive.

The next peak, at frequency 0.358, is a pure triplet, with symmetry $F_2^-$. In this scattering, two pentagons from nearby, but not adjacent, faces of the dodecahedron are pulled away. From their relative alignments, it seems likely that these two tori will later coalesce into a single $B=4$ cube. This mode probably therefore represents the breaking of the dodecahedron into two clusters, one with $B=3$ and the other $B=4$. A pure $4^+$ multiplet is split between the next two peaks, at frequencies 0.446 and 0.459. This mode has a tetrahedral symmetry, with four corners of the dodecahedron, arranged in a tetrahedron, being pulled away. Presumably these will become four single skyrmions, leaving a $B=3$ tetrahedron in the centre. The opposite extreme of the motion goes through the dual tetrahedron, hence the negative parity.

The next peak, at frequency 0.522, is a trivial breather ($1^+$), a simple size fluctuation. After that, at 0.553, is a triplet of dipole breathing modes, with symmetry $F_1^-$. The remaining peaks all contain a mixture of several modes. There is a pure $4^-$ multiplet, spread between frequencies 0.741 to 0.842, which is a breathing mode with tetrahedral symmetry. The last negative parity mode is a pure, radiative $F_2^-$ triplet. This is the expected $k = 0$ radiation, which because of the damping at the beginning of the run does not appear as a separate peak, but merely causes slight contamination (about 20%) in the peak at frequency 0.609.

One more reasonably pure multiplet was identified, a $4^+$ spread across the frequency range 0.609 to 0.716. This is the last of the scattering modes predicted by the monopole analogy. It breaks the pattern previously observed in the $B=2$, 3 and 4 vibrational spectra, in that it has a higher frequency than two ‘breathing’ type modes. This may be explained by the fact that it represents a maximal breaking up of the $B=7$ soliton, splitting it into seven single skyrmions. This mode clearly corresponds to the scattering recently proposed by Singer and Sutcliffe [14], using the instanton approximation, as a comparison to the pictures in their paper will demonstrate.

| Frequency | Degeneracy | Symmetry | Description |
|-----------|------------|----------|-------------|
| 0.019     | 3          | $F_0^+$  | Broken spin zero modes: $J^+$ |
| 0.289 - 0.320 | 5          | $5^-$    | Scattering: $1 + 6$. One edge splits away as a single skyrmion. |
| 0.302     | 5          | $5^+$    | Scattering: $2 + 3 + 2$. Two opposite faces of the dodecahedron split off as $B=2$ tori. |
| 0.358     | 3          | $F_2^-$  | Scattering: $3 + 4$. Two nearby tori split away, then recombine as a $B=4$ cube. |
| 0.446 - 0.459 | 4          | $4^+$    | Scattering: $3 + 1 + 1 + 1 + 1$. Tetrahedral symmetry. |
| 0.522     | 1          | $1^+$    | Trivial breather. |
| 0.553     | 3          | $F_1^+$  | Dipole breather. |
| 0.609     | 3          | $F_2^-$  | $k = 0$ radiation. |
| 0.609 - 0.716 | 4          | $4^+$    | Scattering: $1 + 1 + 1 + 1 + 1 + 1 + 1$. Cubic symmetry. |
| 0.697 - 0.716 | 5          | $5^+$    | Quadrupole breather. |
| 0.741 - 0.842 | 4          | $4^+$    | Breather. Tetrahedral symmetry. |
| 0.826 - 0.842 | 4          | $4^+$    | Breather. Cubic symmetry. |

TABLE II. Summary of $B=7$ vibrational modes.

The remaining positive parity modes cannot be cleanly separated, but are a mixture of one $4^+$ and one $5^+$ multiplet. In fact the mixing is not too severe (this can be seen when computing the characters), and the $5^+$ modes can be identified as quadrupole breathing motions, while the $4^+$ are breathers with a line symmetry similar to that of the $4^+$ scattering mode described above.

This completes the list of all modes found below frequency 0.9. A summary is given in Table II. Definite frequencies are given where they can be confidently assigned; where a mode is split into two or more peaks, or appears as part of an inseparable mixture, a range of frequencies is given. Modes are characterised as either scattering or breathing type, and an attempt is made to classify the former in terms of the clusters (of different baryon number) into which the dodecahedron breaks up. Some of these identifications are fairly certain, while others are more tentative. A full dynamical simulation of these attractive channels will be necessary to confirm the guesses made here.
IV. CONCLUSIONS.

The vibrational spectrum of the B=7 dodecahedral Skyrme soliton has been calculated, and a full identification of the symmetries for all modes lying below frequency 0.9 has been made. The sums defined in Eq. (13) were computed for all peaks, and all exceed 0.995, so it seems certain that nothing has been missed at these frequencies. All attractive channel scattering modes predicted by the monopole analogy [1] were observed numerically. Four of the five predicted complex breathing motions [11] were also observed. The ‘absentee’ is a predicted 5^* breathing mode. It is possible that this multiplet may be lurking at a frequency higher than those considered here. If so, it could be rather difficult to find it amongst the radiation which becomes more predominant as the frequency increases. While this mode is missing, the conjecture of Baskerville and Michaels [11], that there should be a total of 8B-4 modes for a multiskyrmion of baryon number B, cannot be said to have been conclusively proved. However, thirty eight vibrational modes have been identified, giving a total of forty seven modes on inclusion of the nine zero modes. This is definitely more than 6B or 6B+1 (forty two or forty three), and while it is possible that more vibrational modes may be discovered in the future, it is difficult to see how one could get rid of some already found. The fact that the symmetries of the breathing modes observed were correctly predicted by [1] is also persuasive. Note that the instanton inspired counting suggested by Houghton [12], giving 8B-1 modes, is not ruled out either. This ansatz gives the same vibrational modes as the geometrical method of [1], plus an extra triplet. If one multiplet may have been missed due to its high frequency, perhaps a second could be hiding up there too. The evidence would seem to favour the mode counting scaling roughly like 8B, at any rate; whether precisely as 8B-4 or 8B-1 remains to be decided. However, on the whole, the predictions made for the vibrational spectrum of the B=7 Skyrme soliton have proved to be accurate.

As to the problem of the wrong spin obtained by zero mode quantisation of the B=7 skyrmion [13], there certainly exist vibrational modes of low enough frequency to permit hope of its resolution. In general, the more symmetry that is exhibited by a particular solution, the greater are the constraints imposed upon its possible spin states. Therefore, to find states with lower spin than the \( \frac{7}{2} \) permitted for the static solution, the vibrations must break its dodecahedral symmetry. The two lowest frequency modes found both have rather low symmetry, which is promising. Indeed, one can only hope that these vibrations still possess enough symmetry to forbid spin \( \frac{7}{2} \) states. This calculation is beyond the scope of this paper, but the results presented here certainly point to it as a promising direction for future research.

ACKNOWLEDGMENTS.

I would like to thank N. Manton, C. Houghton, R. Battye, P. Sutcliffe and W.J. Zakrzewski for useful discussions of this work. In particular, P. Sutcliffe provided the computer code which generated the approximate static solution, and N. Manton suggested some of the scattering interpretations offered above. All simulations were performed on COSMOS, the Origin 2000 supercomputer owned by the UK-CCC and supported by HEFCE and PPARC. Many thanks to S. Rankin for his help with a variety of computer-related problems. This work was supported financially by PPARC.

[1] C. Barnes, W.K. Baskerville and N. Turok, Phys. Rev. Lett. 79 (1997) 367-370; Phys. Lett. B411 (1997) 180-186.
[2] T. H. R. Skyrme, Proc. Roy. Soc. B 260 (1961) 127.
[3] G.S. Adkins, C.R. Nappi and E. Witten, Nucl. Phys. B228 (1983) 552.
[4] G. Adkins, C.R. Nappi, Nuc. Phys. B233 (1984) 109.
[5] E. Braaten, S. Townsend and L. Carson, Phys. Lett. B235 (1990) 147.
[6] R.A. Battye and P.M. Sutcliffe, Phys. Rev. Lett. 79 (1997) 363-366.
[7] R.A. Leese, N.S. Manton and B.J. Schroers, Nucl. Phys. B442 (1995) 228.
[8] N. Walet, Nucl. Phys. A606 (1996) 429.
[9] W.K. Baskerville, C. Barnes and N. Turok, talk given at “Solitons” conference: Kingston, Ontario, July 1997; submitted to conference proceedings.
[10] C.J. Houghton, N.S. Manton and P.M. Sutcliffe, Nucl. Phys. B510 (1998) 507.
[11] W.K. Baskerville and R. Michaels, Phys. Lett. B448 (1999) 275-280.
[12] C.J. Houghton, ‘Instanton vibrations of the 3-Skyrmion’, DAMTP-1999-52, hep-th/9905009.
[13] P. Irwin, ‘Zero mode quantisation of multi-Skyrmions’, DAMTP-1998-26, hep-th/9804142.
[14] U. Pen, D. Spergel and N. Turok, Phys. Rev. D49 (1994) 692.
[15] M. Hamermesh, Group Theory and its Application to Physical Problems, Dover, 1962.
[16] M.A. Singer and P.M. Sutcliffe, ‘Symmetric Instantons and Skyrme Fields’, hep-th/9901075.
[17] T.H.R. Skyrme, Nucl. Phys. 31 (1962) 556.
FIG. 3. Contours of constant baryon density for the B=7 soliton, combined with its normal modes, as indexed by their frequencies in Table II. Where a frequency range is given in this Table, modes are labelled by the lower limit of that range. Note that the radiation at frequency 0.609 is not plotted, so that the mode labelled $\delta_{609}$ is the $4^+$ Singer-Sutcliffe scattering mode. All other modes are uniquely determined by their frequency or lower frequency limit, and appear in the same order as in Table II. Two different basis modes are shown for frequency 0.302, as the break-up into separate clusters shows up better in one, and the overall symmetry in the other.