REFUTING FEDER, KINNE AND RAFIEY

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ABSTRACT. I give an example showing that the recent claimed solution by Feder, Kinne and Rafiey to the CSP Dichotomy Conjecture is not correct.

1. The Feder-Kinne-Rafiey Algorithm

In January 2017, Feder, Kinne and Rafiey posted a manuscript [10] on the arXiv which purportedly gives a polynomial-time algorithm solving the digraph homomorphism problem $\text{Hom}(H)$, assuming only that the target digraph $H$ has a weak near unanimity (WNU) polymorphism. If correct, their algorithm would finish the proof of the CSP Dichotomy Conjecture of Feder and Vardi [6].

In my opinion, their manuscript was imprecise (or worse) at many points, making it difficult for readers to ascertain what exactly was their algorithm. The authors subsequently posted two new versions of their manuscript, the most recent (v3) on July 1, 2017. All three manuscripts attempt to articulate implementations of an algorithm-idea which roughly speaking is the following: given an instance $G \rightarrow H$ for which the target digraph $H$ has the WNU polymorphism $\phi$:

1. Do some local consistency checking. This will produce, for each vertex $x \in V(G)$, a list $L(x) \subseteq V(H)$ from which “provably impossible values for $h(x)$” (here $h$ is a hypothetical homomorphism) have been removed. (To be precise: the algorithm enforces path-consistency, a.k.a. (2,3)-consistency. The lists $L(x)$ are the cells, a.k.a. “potatoes,” of the resulting microstructure graph.)

2. Associate to each vertex $x \in V(G)$ the restriction $\phi_x := \phi|_{L(x)}$ to $L(x)$ of the WNU polymorphism.

With this preprocessing out of the way, the main part of the algorithm-idea is to iteratively “shrink” the lists $L(x)$, always maintaining the property that if solutions exist, then at least one solution will live in $\prod_x L(x)$. This will purportedly be accomplished by:

3. Allowing the maps $\phi_x$ to evolve (in some prescribed way). At all stages, each $\phi_x$ must remain a not-necessarily-idempotent WNU operation on $L(x)$, and the family $(\phi_x : x \in V(G))$ must be a “multi-sorted polymorphism” of the microstructure graph defined by $(G, H, L)$.

4. Using the evolved maps $(\phi_x : x \in V(G))$ to predict $x \in V(G)$ and $a \in L(x)$ such that $a$ can be safely deleted from the list $L(x)$.

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The novelty of the Feder-Kinne-Rafiey algorithm-idea is in steps (3) and (4). The details of step (3) are delicate (and murky). However, the authors’ criterion in step (4) is very clear: if the current list \( L(x) \) has elements \( a, b \) such that the current \( \phi_x \) satisfies \( \phi_x(b, b, \ldots, b, a) \neq a \), then \( a \) can be safely deleted from \( L(x) \).

If this algorithm-idea could be implemented, then it would reduce an instance of Hom\((\mathbb{H})\) to testing consistency of a microstructure graph with a multi-sorted WNU polymorphism \( (\phi_x \colon x \in V(G)) \) where each \( \phi_x \) satisfies \( \phi_x(b, b, \ldots, b, a) = a \) for all \( a, b \in L(x) \). This latter problem is known to CSPers as the “multi-sorted Mal’tsev case” and there are known algorithms solving it \([1, 4]\). Hence the Feder-Kinne-Rafiey algorithm-idea aims to reduce the WNU case of the general digraph homomorphism problem to the multi-sorted Mal’tsev case.

Meanwhile, Bulatov \([3]\) and Zhuk \([11]\) have independently announced solutions to the CSP Dichotomy Conjecture. Their manuscripts are much more complicated than those of Feder, Kinne and Rafiey. Thus it is of interest whether the ideas of Feder, Kinne and Rafiey are correct (or at least correctable). In this note I will show that, unfortunately, their algorithm-idea cannot be implemented, because their criterion for shrinking a list (step 4) cannot generally be done safely.

**Claim 1.1.** There exists a finite digraph \( \mathbb{H} \) with a 3-ary WNU polymorphism \( \phi' \), and another finite digraph \( G \), such that

1. There exists a unique homomorphism \( h : G \to \mathbb{H} \).
2. If \( L(x) (x \in V(G)) \) are the lists obtained by enforcing (2, 3)-consistency for \( G \to \mathbb{H} \), then for all \( x \in V(G) \),
   a. There exist \( a, b \in L(x) \) such that \( \phi'(b, b, \ldots, b, a) \neq a \).
   b. For all \( a, b \in L(x) \), if \( \phi'(b, b, \ldots, b, a) \neq a \) then \( h(x) = a \).

Moreover, \( \mathbb{H} \) can be chosen to be core and balanced.

Note that condition (a) says that, after preprocessing the instance \( G \to \mathbb{H} \), we are not yet in the Mal’tsev case (with respect to \( \phi' \)), while condition (b) says that any attempt to apply the criterion from step (4) of the Feder-Kinne-Rafiey algorithm-idea (with \( \phi' \)) will lead to a smaller list that is disjoint from the unique solution. In particular, this implies that no matter how the details of step (3) are implemented, the algorithm-idea articulated in any of versions (v1)–(v3) of the Feder-Kinne-Rafiey manuscript will fail to give the correct answer on input \( \mathbb{H}, G \) (using this \( \phi' \)).

The proof of Claim 1.1 occupies the next section. In the final section of this note, I will give further examples that illustrate some challenges that any attempt to modify the algorithm-idea of Feder, Kinne and Rafiey must necessarily overcome.

2. Proof of Claim 1.1

2.1. High-level description for CSP experts. The construction of \( \mathbb{H}, G \) and \( \phi \) is based on a small CSP instance over the template \( A = \langle A, R \rangle \) with domain \( A = \{0, 1, 2\} \) and a single 5-ary relation

\[
R = \{(0, 0, 0, 1, 0), (0, 1, 1, 0, 0), (1, 0, 1, 0, 0), (1, 1, 0, 1, 0), (2, 2, 2, 2, 0)\}.
\]

\( A \) is core, in fact has no nontrivial endomorphisms. Note that \( R \cap \{0, 1\}^5 \) is an affine subspace of \( \mathbb{Z}_2^5 \) and so is closed under \( m(x, y, z) := x + y + z \) (mod 2) applied...
coordinately. Using this fact, it is not hard to see that the operation \( \phi : A^3 \to A \) is a polymorphism of \( \mathbb{A} \):

\[
\phi(x, y, z) = \begin{cases} 
    m(x, y, z) & \text{if } \{x, y, z\} \subseteq \{0, 1\} \\
    a & \text{if } (x, y, z) \in \{(2, a, b), (b, 2, a), (a, b, 2)\} \text{ with } a \in \{0, 1\} \\
    2 & \text{if } x = y = z = 2.
\end{cases}
\]

Clearly \( \phi(x, x, x) = x \) and \( \phi(x, y, z) = \phi(y, z, x) \) for all \( x, y, z \in A \), which means \( \phi \) is idempotent and cyclic (and hence is a WNU).

Let \( R_1 \) and \( R_2 \) be the 3-ary relations obtained by projecting \( R \) onto coordinates \((1, 2, 3)\) and \((1, 2, 4)\) respectively, and let \( \mathbb{A}' = (A, R_1, R_2) \). The CSP(\( \mathbb{A}' \)) instance \( J \) given by

\[
R_1(x_1, x_2, x_3) \ & R_1(x_1, x_5, x_6) \ & R_1(x_2, x_4, x_6) \ & R_2(x_3, x_4, x_5)
\]

is \((2,3)\)-minimal, meaning that the lists obtained after enforcing \((2,3)\)-consistency are \( L(x_i) = A \) for all \( i \). Moreover, the constant map \( \{x_1, \ldots, x_6\} \mapsto 2 \) is the unique solution to \( J \), which means that \( J \) has a solution, but has no solution satisfying \( h \) satisfying \( h(x_i) \neq 2 \). Yet for all \( a, b \in A \), \( \phi(b, b, a) \neq a \) iff \( a = 2 \) and \( b \in \{0, 1\} \).

Using a construction of Bulín, Delić, Jackson and Niven [5] (improving Feder and Vardi [6]), one can translate \( \mathbb{A}, J \) and \( \phi \) to a digraph \( \mathbb{H} \), an instance \( \mathbb{G} \) of Hom(\( \mathbb{H} \)), and a polymorphism \( \phi' \) of \( \mathbb{H} \) with essentially the same properties. I will give a detailed description of the resulting digraphs \( \mathbb{H}, \mathbb{G} \) in sections 2.3 and 2.4.

2.2. Additional remarks. \( \mathbb{A} \) and \( \mathbb{A}' \) are examples of a kind of template that Bulatov [2] calls *semilattice block Mal’tsev*. This refers to the fact that the domain \( \{0, 1, 2\} \) has a partition, namely \( 01|2 \), which is invariant under \( \phi \) and is such that \( \phi \) is a Mal’tsev operation on each block; and \( \mathbb{A} \) (or \( \mathbb{A}' \)) additionally has a 2-ary polymorphism \( f \) (namely, \( f(x, y) := \phi(x, x, y) \)) which respects this partition and has the property that, modulo the partition, \( f \) is a semilattice operation. Semilattice block Mal’tsev templates have been a sticking point for researchers attempting to solve the CSP Dichotomy Conjecture. In particular, Maróti has circulated but never published some important partial results [8], and Marković and McKenzie [7] and Payne [9] have some further unpublished results. It was only earlier this year that Bulatov [2] finally solved the Dichotomy Conjecture for semilattice block Mal’tsev templates, two months before he announced his solution to the full Dichotomy Conjecture. So it is perhaps not a surprise that semilattice block Mal’tsev templates should be the source of counter-examples to the algorithm-idea of Feder, Kinne and Rafiey.

2.3. The digraph \( \mathbb{H} \). The following construction based on \( \mathbb{A} \) is due to Bulín, Delić, Jackson and Niven [5]. Define a balanced digraph \( \mathbb{H} \) as follows. The vertex set \( V(\mathbb{H}) \) consists of the union of \( \{0, 1, 2\} \), \( \{\alpha, \beta, \gamma, \delta, \tau\} \), and a set of 190 auxiliary vertices lying on 15 pairwise disjoint oriented paths of net length 7 connecting each pair in \( \{0, 1, 2\} \times \{\alpha, \beta, \gamma, \delta, \tau\} \). These paths are:
For $a \in \{0, 1, 2\}$ and $\lambda \in \{\alpha, \beta, \gamma, \delta, \tau\}$, let $P_{a, \lambda}$ denote the oriented path from $a$ to $\lambda$ (pictured above and on the previous page). Thus $H$ can be roughly pictured as
$H$ is core, because the template $A$ on which it is based is core [5, Corollary 4.2], and has a 3-ary idempotent cyclic polymorphism $\phi'$ extending $\phi$, because both $A$ and the digraph $\bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet$ have one [5, Theorem 5.1]. I will partially describe $\phi'$ in section 2.5.

2.4. The instance $G$. Let $Q_1, \ldots, Q_4$ denote the following oriented paths:

(There is also a $Q_5$ completing the pattern, but we will not need it.) If we define the bijection $\lambda \mapsto \overline{\lambda}$ from $\{\alpha, \beta, \gamma, \delta, \tau\}$ to $R$ via

$\overline{\alpha} = (0, 0, 0, 1, 0)$
$\overline{\beta} = (0, 1, 1, 0, 0)$
$\overline{\gamma} = (1, 0, 1, 0, 0)$
$\overline{\delta} = (1, 1, 0, 1, 0)$
$\overline{\tau} = (2, 2, 2, 2, 0)$

then the following is true (see [5, proof of Lemma 3.6]).

Claim 2.1. for all $a \in \{0, 1, 2\}$, $i \in \{1, 2, 3, 4\}$ and $\lambda \in \{\alpha, \beta, \gamma, \delta, \tau\}$,

$Q_i \rightarrow \mathbb{P}_{a, \lambda}$ iff $\overline{\lambda}[i] = a$.

Moreover, if $\overline{\lambda}[i] = a$ then the homomorphism $Q_i \rightarrow \mathbb{P}_{a, \lambda}$ is unique and surjective.
Next I define two gadgets: $S(x,y,z)$ and $S'(x,y,z)$. These are the digraphs with three distinguished vertices $x, y, z$ given in the next figure:

![Diagram of gadgets S(x,y,z) and S'(x,y,z)](image)

The 3-ary relations on $V(H)$ pp-defined by $S(x,y,z)$ and $S'(x,y,z)$,

$S := \{ h(x), h(y), h(z) : S(x,y,z) \rightarrow H \}$

$S' := \{ h(x), h(y), h(z) : S'(x,y,z) \rightarrow H \}$,

turn out to be

$S = \text{pr}_{1,2,3}(R) = \{(u,v,w) \in (Z_2)^3 : u + v + w = 0 \} \cup \{(2,2,2)\}$

$S' = \text{pr}_{1,2,4}(R) = \{(u,v,w) \in (Z_2)^3 : u + v + w = 1 \} \cup \{(2,2,2)\}$.

Moreover, for each $(a,b,c) \in S$ there exists a unique homomorphism $S(x,y,z) \rightarrow H$ satisfying $(h(x), h(y), h(z)) = (a,b,c)$. A similar remark holds for $S'$ and $S'(x,y,z)$.

Now define $G$ to be the digraph obtained by starting from vertices $\{x_1, x_2, \ldots, x_6\}$ and connecting them with the following four gadgets:

$S(x_1, x_2, x_3), \ S(x_1, x_5, x_6), \ S(x_4, x_2, x_6), \ S'(x_4, x_5, x_3)$.

(Of course one takes pairwise disjoint isomorphic copies of the gadgets.) Thus $G$ is the digraph depicted schematically in the next figure:

![Diagram of digraph G](image)

Using Claim 2.1 and the definition of $G$, one can show that there is exactly one homomorphism $h : G \rightarrow H$ and it satisfies $h(x_i) = 2$ for all $i = 1, 2, \ldots, 6$. 
2.5. **The unary lists and** $\phi'$. In this section I describe the unary lists $L(x)$ which are produced by enforcing $(2,3)$-consistency on $G \rightarrow \mathbb{H}$. I also describe the restrictions $\phi^3_{\,L(x)}$ to these lists of any of the idempotent cyclic polymorphisms $\phi'$ of $\mathbb{H}$ produced by the recipe in [5].

For each $a \in \{0,1,2\}$, $i \in \{1,2,3,4\}$, and $\lambda \in \{\alpha, \beta, \gamma, \delta, \tau\}$ with $\overline{a}[i] = a$, let $h_{i}^{a,\lambda}$ denote the unique homomorphism $Q_i \rightarrow \mathbb{P}_{a,\lambda}$. (See Claim 2.1.)

Let $B = \{\alpha, \beta, \gamma, \delta, \tau\}$. For each $i = 1, \ldots, 4$ and $v \in V(Q_i)$, define $\sigma_{i,v} : B \rightarrow V(\mathbb{H})$ by

$$\sigma_{i,v}(\lambda) := h_{i}^{a,\lambda}(v) \text{ where } a = \overline{a}[i],$$

and note that $\sigma_{i,v}(\lambda)$ is a vertex of the path $\mathbb{P}_{a,\lambda}$ from $a$ to $\lambda$, and is at the same height as $v$ is in $Q_i$. In particular, $\sigma_{i,t}$ is the identity map $B \rightarrow B$ for all $i$.

Every vertex $\overline{v}$ of $G$ with $\overline{v} \not\in \{x_1, \ldots, x_6\}$ is the image of some $v \in V(Q_i)$ in the copy of $Q_i$ ending at $t_j$ for some unique $v$ and $j$. If in addition $\overline{v} \not\in \{t_1, t_2, t_3, t_4\}$ (i.e., $v \neq t$), then $i$ is also uniquely determined. In all cases I'll let $\sigma_{\overline{v}} := \sigma_{i,v}$.

**Claim 2.2.** Let $L(v)$ be the unary lists produced by enforcing $(2,3)$-consistency on $G \rightarrow \mathbb{H}$.

1. $L(x_i) = \{0,1,2\}$ for all $i = 1, \ldots, 6$.
2. For every $\overline{v} \in V(G) \setminus \{x_1, \ldots, x_6\}$, $L(\overline{v}) = \text{ran}(\sigma_{\overline{v}})$. Moreover, $\sigma_{\overline{v}} : B \rightarrow L(\overline{v})$ is a bijection.

In particular, $L(t_j) = B$ for each $j = 1, \ldots, 4$. As another example, if $v = v_1^{2R} \in V(Q_1)$ and $\overline{v}$ is its image in the copy of $Q_1$ from $x_4$ to $t_3$ in $G$, then

$$L(\overline{v}) = \text{ran}(\sigma_{1,v}) = \{ \sigma_{1,v}(\alpha), \sigma_{1,v}(\beta), \sigma_{1,v}(\gamma), \sigma_{1,v}(\delta), \sigma_{1,v}(\tau) \}$$

$$= \{ h_{1}^{0,\alpha}(v), h_{1}^{0,\beta}(v), h_{1}^{1,\gamma}(v), h_{1}^{1,\delta}(v), h_{1}^{2,\tau}(v) \}$$

$$= \{ u_{0a}, u_{0b}^{2R}, u_{1\gamma}, u_{1\delta}^{2R}, u_{2\tau}^{2} \}.$$

The recipe from [5] for producing a 3-ary idempotent cyclic polymorphism $\phi'$ of $\mathbb{H}$ extending $\phi$ is not canonical; it depends on a choice of such a polymorphism of the digraph $\bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet$. Luckily, the restrictions of $\phi'$ to the lists $L(v)$ are independent of this choice.

**Claim 2.3.** Let $\phi'$ be one of the 3-ary idempotent cyclic polymorphisms of $\mathbb{H}$ extending $\phi$ given by the construction in [5].

1. $\phi'|_{\{0,1,2\}} = \phi$.
2. The bijection $\lambda \mapsto \overline{\lambda}$ from $B$ to $R \subseteq A^5$ is an isomorphism from the algebra $\langle B, \phi'|_{B} \rangle$ to the subalgebra of $\langle A, \phi \rangle^5$ with domain $R$.
3. Suppose $\overline{v} \in V(G) \setminus \{x_1, \ldots, x_6\}$. Then $\sigma_{\overline{v}}$ is an isomorphism from $\langle B, \phi'_B \rangle$ to $\langle L(\overline{v}), \phi'_{L(\overline{v})} \rangle$.

At this point the reader has enough information to verify that $\mathbb{H}$, $G$, and $\phi'$ satisfy Claim 1.1.

2.6. **The binary lists.** For completeness, I describe the binary lists that arise from enforcing $(2,3)$-consistency on $G \rightarrow \mathbb{H}$. I first define some binary relations on $A (= \{0,1,2\})$ and $B$. 

**Definition 2.4.** Here are some binary relations on $B$:

$$
E_1 = \{(\alpha, \beta)^2 \cup \{\gamma, \delta\}^2 \cup \{(\tau, \tau)\} \}
$$

$$
E_2 = \{(\alpha, \gamma)^2 \cup \{\beta, \delta\}^2 \cup \{(\tau, \tau)\} \}
$$

$$
E_3 = \{(\alpha, \delta)^2 \cup \{\beta, \gamma\}^2 \cup \{(\tau, \tau)\} \}
$$

$$
E_{34} = \{(\alpha, \delta) \times \{\beta, \gamma\} \} \cup \{(\beta, \gamma) \times \{\alpha, \delta\} \} \cup \{(\tau, \tau)\}.
$$

Here are some relations between $B$ and $A$ (subsets of $B \times A$):

$$
P_1 = \{(\alpha, 0), (\beta, 0), (\gamma, 1), (\delta, 1), (\tau, 2)\}
$$

$$
P_2 = \{(\alpha, 0), (\beta, 1), (\gamma, 0), (\delta, 1), (\tau, 2)\}
$$

$$
P_3 = \{(\alpha, 0), (\beta, 1), (\gamma, 1), (\delta, 0), (\tau, 2)\}
$$

$$
P_4 = \{(\alpha, 1), (\beta, 0), (\gamma, 0), (\delta, 1), (\tau, 2)\}
$$

$$
\Delta_{BA} = \{(\alpha, \beta, \gamma, \delta) \times \{0, 1\}\} \cup \{(\tau, 2)\}.
$$

Finally, I need the relation $\Delta = \{0, 1\}^2 \cup \{(2, 2)\}$ on $A$.

**Claim 2.5.** Let $L(v, v')$ be the binary lists produced by enforcing $(2, 3)$-consistency on $G \rightarrow H$.

1. $L(x_i, x_j) = \Delta$ for all $i \neq j$.
2. If $x_k$ and $t_j$ are the endpoints of a copy of $Q_i$, then $L(t_j, x_k) = P_i$.
3. If $x_k$ and $t_j$ are not the endpoints of a copy of any $Q_i$, then $L(t_j, x_k) = \Delta_{BA}$.
4. If $t_j$ and $t_k$ ($j \neq k$) are connected by two copies of $Q_i$ extending from some $x_t$, then $L(t_j, t_k) = E_i$. In the remaining case, $L(t_1, t_4) = E_{34}$.
5. Suppose $\sigma \in V(G) \setminus \{x_1, \ldots, x_6\}$ such that $\sigma$ is in a copy of $Q_i$ ending at $t_j$. Then $L(t_j, \sigma) = \text{graph}(\sigma_{|_\sigma})$.

All other values of $L(v, v')$ can be deduced from this claim.

### 3. Discussion

Using Claims 2.2 and 2.5, one can see that each unary list $L(v)$ produced by the example in Claim 1.1 can be partitioned into two sublists $L_{01}(v)$ and $L_2(v)$, so that for all $v, v' \in V(G)$, $L(v, v')$ is contained in $(L_{01}(v) \times L_{01}(v')) \cup (L_2(v) \times L_2(v'))$. In other words, the microstructure graph of $(G, H, L)$ disconnects into two components.

The components can be easily found, and the question $G \rightarrow H$ can thus be reduced to searching in the lists $L_{01}(v)$ and (separately) in the lists $L_2(v)$. $\phi'$ restricted to either family of sublists is already in the Mal'tsev case, so one can invoke the Bulatov-Dalmau algorithm on each sublist to determine that $G \rightarrow H$. (An analogous fact is true of the original CSP$(\mathbb{A})$ instance defined in Section 2.1.)

Any reasonable algorithm can be assumed to first look for this kind of decomposition. In particular, Feder, Kinne and Rafiey could insert a test for this kind of decomposition in their algorithm-idea; the instance constructed in Claim 1.1 would then not refute their (modified) algorithm.

In the next section I will sketch the construction of some more complicated examples which do not decompose in this trivial way.
4. A more complicated example

This second counter-example is a variation of the first, so I won’t describe it quite as thoroughly. Here is the idea on which it is based. Take the template \(A = \langle A; E, R_0 \rangle\) with domain \(A = \{0, 1, 2\}\) and two relations \(E\) (2-ary) and \(R_0\) (4-ary) given by
\[
E = \{0, 1, 2\}^2 \setminus \{(2, 2)\}
\]
\[
R_0 = \{(0, 0, 0, 1), (0, 1, 1, 0), (1, 0, 1, 0), (1, 0, 1, 1), (1, 0, 1, 2), (2, 2, 2, 2)\}.
\]
\(A\) is core, and the operation \(\phi : A^3 \to A\) defined below is a WNU polymorphism:
\[
\phi(x, y, z) = \begin{cases} m(x, y, z) & \text{if } \{x, y, z\} \subseteq \{0, 1\} \\ a & \text{if } (x, y, z) \in \{(2, a, b), (b, 2, a), (a, b, 2)\} \text{ with } a \in \{0, 1\} \\ 2 & \text{if } x = y = z = 2. \end{cases}
\]

Let \(R_1\) and \(R_2\) be the 3-ary relations obtained by projecting \(R_0\) onto coordinates \((1, 2, 3)\) and \((1, 2, 4)\) respectively. Let \(A' = \langle A, E, R_1, R_2 \rangle\). The CSP\((A')\) instance \(I\) given by
\[
E(x_0, x_1) \& R_1(x_1, x_2, x_3) \& R_1(x_1, x_5, x_6) \& R_1(x_2, x_4, x_6) \& R_2(x_3, x_4, x_5)
\]
is \((2, 3)\)-minimal, the microstructure graph of its lists is connected, and it has exactly two solutions: the maps which send \(\{x_1, \ldots, x_6\} \mapsto 2\) and \(x_0 \mapsto 0\) or \(1\). Thus one cannot delete 2 from the list for any of \(x_1, \ldots, x_6\) without losing all solutions, yet (from the perspective of \(\phi\)) the appearance of 2 in any of the lists is a violation Mal’tsev property. When this template and instance are translated to balanced digraphs via Bulín, Dolić, Jackson and Niven [5], the result is an instance of a digraph homomorphism problem whose lists do not decompose in the trivial way, and which has a solution but step (4) of the algorithm-idea of Feder, Kinne and Rafiey will fail with high probability (assuming that the first list to be shrunk at step (4) is chosen randomly).

4.1. The translation. Define a balanced digraph \(H\) as follows. Let \(R\) be the following 5-ary relation on \(\{0, 1, 2\}\):
\[
R = \{00010, 00011, 00012, 01100, 01101, 01102, 10100, 10101, 10102, 11010, 11011, 11012, 22220, 22221\}.
\]
(Note that \(pr_{1,2,3,4}(R) = R_0\) and \(pr_{1,5}(R) = E\), and that \(\phi\) is a polymorphism of \(R\).)

The vertex set \(V(H)\) consists of the union of \(A = \{0, 1, 2\}\), \(R\), and 672 auxiliary vertices lying on 42 pairwise disjoint oriented paths of net length 7 connecting each pair in \(A \times R\). Given \(a \in A\) and \(\lambda \in R\), the oriented path \(P_{a,\lambda}\) connecting \(a\) to \(\lambda\) is defined as follows:
- \(P_{a,\lambda}\) starts with \(a \rightarrow u_{a\lambda}^0\) and ends with \(u_{a\lambda}^5 \rightarrow \lambda\).
- Intermediate vertices of \(P_{a,\lambda}\) are \(u_{a\lambda}^i\) for \(i = 1, \ldots, 4\).
- For each \(i = 1, \ldots, 5\), the oriented path from \(u_{a\lambda}^{i-1}\) to \(u_{a\lambda}^i\) is \(u_{a\lambda}^{i-1} \rightarrow u_{a\lambda}^i\) if \(\lambda[i] = a\) and is \(u_{a\lambda}^{i-1} \rightarrow u_{a\lambda}^i \leftarrow u_{a\lambda}^{i-1} \rightarrow u_{a\lambda}^i\) otherwise.

See Figure 1 for a picture of \(P_{a,\lambda}\) when \(a = 1\) and \(\lambda = 01102\), and also a schematic diagram of \(H\). \(H\) has a 3-ary WNU polymorphism \(\phi'\) extending \(\phi\), as shown in [5].

For each \(j = 1, \ldots, 5\), let \(Q_j\) be the oriented path of net length 7 defined as follows:
\[ \lambda = 01102 \]

\[
\begin{array}{c}
\text{Figure 1.} \\
\end{array}
\]

\[
\begin{array}{c}
\text{Figure 2.} \\
\end{array}
\]

- \( Q_i \) starts as \( b \to v_i^0 \) and ends as \( v_i^5 \to t \).
- Intermediate vertices of \( Q_i \) are \( v_i^j \) for \( j = 1, \ldots, 4 \).
- For each \( j = 1, \ldots, 5 \), the oriented path from \( v_i^{j-1} \) to \( v_i^j \) is \( v_i^{j-1} \to v_i^j \) if \( i = j \) and is \( v_i^{j-1} \to v_i^{jL} \leftarrow v_i^{j-1R} \to v_i^j \) otherwise.

Four of these paths are pictured in Figure 2. As in the first counter-example, we have \( Q_i \to P_{a, \lambda} \) iff \( \lambda[i] = a \), and when both conditions hold, the homomorphism is unique and surjective.
The gadgets \( S(x, y, z) \) and \( S'(x, y, z) \) are defined exactly as before. A new gadget \( E(x, y) \) is defined by the scheme \( x \xrightarrow{Q_5} t \xleftarrow{Q_1} y \).

Now define \( G \) to be the digraph obtained by starting from vertices \( \{x_0, x_1, \ldots, x_6\} \) and connecting them with the following five gadgets:

\[
E(x_0, x_1), \quad S(x_1, x_2, x_3), \quad S(x_1, x_5, x_6), \quad S(x_4, x_2, x_6), \quad S'(x_4, x_5, x_3).
\]

(Of course one takes pairwise disjoint isomorphic copies of the gadgets.) Thus \( G \) is the digraph depicted schematically in Figure 3.

By construction, there exist exactly two homomorphisms \( G \rightarrow H \), which are uniquely determined by their values on \( x_0, \ldots, x_6 \). These homomorphisms correspond to the two solutions to the CSP(\( A' \)) instance \( J \) defined at the beginning of this section. Furthermore, when the instance \( G \rightarrow H \) is preprocessed, the microstructure graph of the resulting unary and binary lists is highly connected.

This example is problematic for the Feder-Kinne-Rafiey algorithm-idea for the following reason. The two solutions agree on the vertices \( x_1, \ldots, x_6 \) as well as on the paths connecting \( x_1, \ldots, x_6 \) to \( t_1, \ldots, t_4 \). The common value of the two solutions at each of these vertices is also the unique element of the corresponding list which fails the Mal'tsev property. Thus if the Feder-Kinne-Rafiey algorithm-idea is executed on this instance, and the first list chosen by step (4) to be shrunk is not on the path from \( x_1 \) to \( x_0 \), then the new lists will be disjoint from both solutions and thus the algorithm will fail. Apparently, if the algorithm is not to fail at the very beginning of step (4), it must do some processing to determine which list, of the several satisfying the criterion of step (4), is the “correct” list to shrink.

This example can be beefed up a bit, by extending \( G \) “on the left” with another “pyramid” of gadgets, but this time so that the linear constraints on \( \{0, 1\} \) are consistent (see Figure 4). This new system has 8 solutions, all of which equal 2 on \( x_1, \ldots, x_6 \) and are in \( \{0, 1\} \) on \( x'_1, \ldots, x'_6 \). At the start of step (4), the Feder-Kinne-Rafiey algorithm-idea can safely remove the unique violation of the Mal’tsev property from the list of any of the variables from the left-hand pyramid, but cannot remove the unique violation of the Mal’tsev property from the list of any variable from

**Figure 3.** The instance digraph \( G \)

The gadgets \( S(x, y, z) \) and \( S'(x, y, z) \) are defined exactly as before. A new gadget \( E(x, y) \) is defined by the scheme \( x \xrightarrow{Q_5} t \xleftarrow{Q_1} y \).

Now define \( G \) to be the digraph obtained by starting from vertices \( \{x_1, x_1, \ldots, x_6\} \) and connecting them with the following five gadgets:

\[
E(x_0, x_1), \quad S(x_1, x_2, x_3), \quad S(x_1, x_5, x_6), \quad S(x_4, x_2, x_6), \quad S'(x_4, x_5, x_3).
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By construction, there exist exactly two homomorphisms \( G \rightarrow H \), which are uniquely determined by their values on \( x_0, \ldots, x_6 \). These homomorphisms correspond to the two solutions to the CSP(\( A' \)) instance \( J \) defined at the beginning of this section. Furthermore, when the instance \( G \rightarrow H \) is preprocessed, the microstructure graph of the resulting unary and binary lists is highly connected.

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the right-hand pyramid. How is the algorithm to decide which list to reduce? Such are the challenges facing any attempt to save the Feder–Kinne–Rafiey algorithm–idea.

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