Cosmological solutions of braneworlds with warped and compact dimensions

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Abstract
We study cosmological aspects of braneworld models with a warped dimension and an arbitrary number of compact dimensions. With a stabilized radion, a number of different cosmological bulk solutions are found in a general case. Both one and two brane models are considered. The Friedmann equation is calculated in each case. Particular attention is paid to six dimensional models where we find that the usual Friedmann equation can typically be recovered without fine-tuning.

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1 Introduction

The study of extra dimensions has [1], in addition to the novel features of particle phenomenology, inspired much research on the cosmological aspects of extra dimensional models. Especially the study of brane cosmology has recently been an active field [2]. Already very soon after the pioneering papers on brane world models [3], the peculiar cosmological aspect of these constructions were realized [4]. Later on, a number of papers have studied the cosmological bulk solutions of RS-type models, [5]-[7].

An important aspect of any cosmological brane world model is the question of radion stabilization, which in the RS-model corresponds to the stabilization of the distance between the two three branes. At the same time one wishes to solve the hierarchy problem introducing a fundamental, D-dimensional gravity scale $M_\star$ which is related suitably to the Planck mass $M_{Pl}$, at least at the present time. Most interesting would be if the fundamental scale $M_\star$ is near 1 TeV and hence in reach of the future colliders.

In addition to the many studies on ADD [8] and RS [9]-models, a number of papers have combined the two so that in addition to the warped dimension, a compact extra dimension is also present, see e.g. [9]-[13]. The stabilization of six dimensional models have been discussed in [14, 15]. Combining the two models can be seen as a quite natural scenario since in any case we expect to have a number compact dimensions coming from string theory. Obviously these compact dimensions do not have to be 'large' but such a possibility is worth studying. Thus of the D-1 spatial dimensions three are the usual observed dimensions, one is a Randall-Sundrum like dimension and D-5 are compact, but unconstrained dimensions. Hence the dimension of the brane is D-2. Because the Standard Model particles are free to propagate in the extra compact dimensions, their size are constrained by experiments. In particular in our models, which have the extra dimension topology of $I \times (S^1)^{(D-5)}$, where $I \subseteq \mathbb{R}$, the radii of $S^1$ factors are constrained to be smaller than about 1 TeV.$^{-1}$.

In this paper we are interested in the cosmological properties of brane world models, where the number of the extra dimensions is two or more. It is organized as follows: First we study the D-dimensional ($D \geq 5$) Einstein’s equations in order to find some cosmological bulk solutions. Writing the D-dimensional jump conditions we solve the Friedmann equation in each case. Then we concentrate on the 6-dimensional models and their properties and finally present conclusions.

2 Einstein’s equations in D-dimensions

In order to write the Einstein’s equations in D-dimension, we first define the coordinates\(^1\) as:

$$x_\mu = (t, x_1, x_2, x_3, z, \theta_1, ..., \theta_{D-5}).$$

Due to the symmetries, the metric tensor can always be chosen to have diagonal form with $b_i = b_i(t, z)$. Hence the line element can be written as

$$ds^2 = \eta(t, z)^2 dt^2 - R(t, z)^2 dx_i dx^i - a(t, z)^2 dz^2 - \sum_{i=1}^{D-5} b_i(t, z)^2 d\theta_i^2.$$  \(2\)

With this form of the metric tensor, the Einstein’s equations can be written compactly as

$$G_{AB} = -\kappa^2 T_{AB},$$  \(3\)

\(^1\)In this paper we follow a convention where capital roman letters denote the full D-dimensional space-time, small roman letters the three dimensional space and small greek letters the four dimensional space-time, e.g. $A = 0, ..., D - 1$, $a = 1, 2, 3$, $\alpha = 0, ..., 3$.\}
where \( \kappa \) is the \( D \)-dimensional gravitational constant, \( G_{AB} \) is the Einstein’s tensor in \( D \)-dimensions and \( T_{AB} \) is the \( D \)-dimensional energy-momentum tensor. It includes both the bulk and the brane energy-momentum tensors. In the bulk the energy-momentum tensor is assumed to have the form

\[
T_A^B = \text{diag}(-\Lambda_B, -\Lambda_B, -\Lambda_B, -\Lambda_B, T_4^1(t, z), ..., T_{D-1}^D(t, z)).
\]

(4)

In other words, we have a normal cosmological constant in the bulk but the cosmological constants in the compact dimensions are allowed to be functions of time and the \( z \)-coordinate. As it was already seen in [6], this is necessary in order to satisfy the continuity (and Einstein’s) equations in the bulk.

We now look for solutions with a stable radion field, i.e. \( \dot{a} = 0 \). We are free to choose it to have a constant value in \( z \)-direction by redefinition of \( z \)-coordinate, thus simply \( a(t, z) = 1 \). We assume further that the compact dimensions are also stabilized, \( b_i(t, z) = b_i(z) \). Non-zero components of Einstein’s tensor are now:

\[
\begin{align*}
G_0^0 &= \sum_{i \neq j \geq 1} \frac{D-5}{i b_j} - \frac{D-5}{b_i} + \sum_{i=1}^{D-5} \frac{D-5}{b_i b_i} + 3 \sum_{i=1}^{D-5} \frac{D-5}{b_i R} + 3 \frac{R^2}{R} + 3 \frac{R''}{R} - 3 \frac{\dot{R}^2}{R^2}, \\
G_i^0 &= \sum_{i \neq j \geq 1} \frac{D-5}{i b_j} - \frac{D-5}{b_i} + \sum_{i=1}^{D-5} \frac{D-5}{b_i b_i} + \sum_{i=1}^{D-5} \frac{D-5}{b_i \eta} + \frac{2 \eta' R}{\eta R} + \frac{R''}{R} + \frac{R^2}{R} + 3 \frac{R''}{R} + 2 \frac{R''}{R} + 2 \frac{\dot{R}^2}{R^2} - \frac{\dot{R}}{R^2} (l = 1, 2, 3), \\
G_i^i &= \sum_{i \neq j \geq 1} \frac{D-5}{i b_j} - \frac{D-5}{b_i} + \sum_{i=1}^{D-5} \frac{D-5}{b_i b_i} + \sum_{i=1}^{D-5} \frac{D-5}{b_i \eta} + \frac{2 \eta' R}{\eta R} + \frac{R''}{R} + \frac{R^2}{R} + 3 \frac{R''}{R} + 3 \frac{R''}{R} + 3 \frac{R''}{R} - 3 \frac{\dot{R}^2}{R^2} - 3 \frac{\dot{R}^2}{R^2}, \\
G_n^n &= \sum_{i \neq j \neq n} \frac{D-5}{i b_j} + \sum_{i \neq j \neq n} \frac{D-5}{b_i b_i} + \frac{2 \eta' R}{\eta R} + \frac{R''}{R} + \frac{R^2}{R} + 3 \frac{R''}{R} + 3 \frac{R''}{R} + 3 \frac{R''}{R} + 3 \frac{\dot{R}^2}{R^2} - 3 \frac{\dot{R}^2}{R^2} - 3 \frac{\dot{R}^2}{R^2} (n = 5, ..., D-1), \\
G_{0i} &= -3 \frac{\eta' R}{\eta R} + 3 \frac{\dot{R}'}{R}.
\end{align*}
\]

As we see, the Einstein’s tensor does have a non-diagonal component \( G_{0i} \) whereas the energy-momentum tensor is diagonal. Thus as the first step we solve The equation \( G_{0i} = 0 \) by writing \( \eta(t, z) \) in the form

\[
\eta(t, z) = \lambda(t) \dot{R}(t, z),
\]

(6)

where \( \lambda \) is a arbitrary positive time dependent function. This is naturally the same solution that was found in the 5-dimensional case [6, 7].

Using the solution [6] for \( \eta(t, z) \), the non-zero components of \( G_A^B \) become rewritten as

\[
\begin{align*}
G_0^0 &= -3 \frac{\lambda^2 R^2}{R^2} + \sum_{i \neq j \geq 1} \frac{D-5}{i b_j} - \frac{D-5}{b_i} + \sum_{i=1}^{D-5} \frac{b'_i}{b_i} + 3 \sum_{i=1}^{D-5} \frac{b'_i R'}{b_i} + 3 \frac{R'}{R} + \frac{R''}{R} + 3 \frac{R''}{R}, \\
G_i^0 &= -1 \frac{\lambda^2 R^2}{R^2} + \sum_{i \neq j \geq 1} \frac{D-5}{i b_j} - \frac{D-5}{b_i} + \sum_{i=1}^{D-5} \frac{b'_i}{b_i} + \frac{2 R'}{R} + \frac{\dot{R}'}{R} + \frac{R''}{R} + \frac{R^2}{R} + 2 \frac{R''}{R} + 2 \frac{\dot{R}''}{R} (l = 1, 2, 3), \\
G_i^i &= -3 \frac{\lambda^2 R^2}{R^2} + \sum_{i \neq j \geq 1} \frac{D-5}{i b_j} - \frac{D-5}{b_i} + (3 \frac{R'}{R} + \frac{\dot{R}'}{R} + \frac{R''}{R}) \sum_{i=1}^{D-5} \frac{b'_i}{b_i} + 3 \frac{R^2}{R} + 3 \frac{\dot{\lambda}}{\lambda^3 R} + 3 \frac{\dot{R}'}{R R},
\end{align*}
\]

(7)
\[ G^n_n = -\frac{3}{\lambda^2 R^2} + \sum_{i \neq j \neq n}^{D-5} \frac{b'_i b'_j}{b_i b_j} + \sum_{n \neq i}^{D-5} \frac{b''_i}{b_i} + (3 \frac{R'}{R} + \frac{\dot{R'}}{R}) \sum_{n \neq i}^{D-5} \frac{b'_i}{b_i} + 3 \frac{R'^2}{R^2} + 3 \frac{R''}{R} + \frac{3}{\lambda^3 R} \frac{\dot{\lambda}}{R} + 3 \frac{R'}{R} + \frac{\dot{R'}}{R} (n = 5, ..., D - 1). \]

The Einstein’s equations which are directly related to the 4-dimensional space-time properties via couplings to 4-dimensional matter are the equations for \( G^0_0 \) and \( G^i_i \). In contrast the other components of the Einstein’s tensor are related to the unknown components \( T^4_4, ..., T^{D-1}_{D-1} \) of the energy-momentum tensor and remain, at this stage, less constrained.

Besides the Einstein’s equations, the equations to be satisfied include also the continuity equations in D-dimensional space-time, \( T^{AB} ;_B = 0 \). However, in the bulk only one of these continuity equations is non-trivial giving

\[ T^{4B} ;_B = T^4_4 + (T^4_4 - \Lambda_B)(3 \frac{R'}{R} + \frac{\dot{R}'}{R}) + \sum_{i=4}^{D-1} (T^4_i - T^i_i) \frac{b'_i}{b_i} = 0. \]  

We remind the reader, that the continuity equations are automatically satisfied due to Bianchi identities whenever the Einstein’s equations have been solved.

3 Cosmological bulk solutions

To reach our goal to study the cosmological properties of the brane models we have to solve first the Einstein equations in the bulk. The whole set of equations nor even the first four are hardly solvable in general. However, our special choices allows us to relate these to each other.

Thus in order to solve the Einstein equations we simplify the set of equations further. We first note that since \( \eta(t, z) \) is given by \( \eta = \frac{1}{\lambda^2 R^2} \), we can write \( G^0_0 \) in terms of \( G^i_i \):

\[ G^0_0 = G^0_0 + \frac{1}{3} \frac{R}{R} \frac{\dot{\phi}}{R} G^0_0 \]  

hence in the bulk, as long as \( \Lambda_B \) is constant, we only need to look for solutions of the first Einstein’s equation \( G^0_0 = \kappa^2 \Lambda_B \) in order to solve the first four Einstein’s equations.

Even now, the general solution of the equation \( G^0_0 = \kappa^2 \Lambda_B \) is not known. We can, however, try to solve it in two simple special cases: We can use either exponential Ansätze where it is assumed that \( b_i(z) = \exp(k_i z) \) or a \( \cosh \) -Ansatz, which we call the bowl-model, where \( b_i(z) = b_i(0) \cosh \frac{dz}{k_i} \) and \( d = D - 5 \) is the number of compact extra dimensions. These two choices carry some essential and interesting properties as will be seen. We first study the exponential model and then the bowl-model.

3.1 Exponential models

We first assume the exponential Ansätze for the scale factors of the compact dimensions, \( b_i(z) = b_i(0) \exp(k_i z) \). The \((00)\)-component of the Einstein’s equations in the bulk is then

\[ -\frac{3}{\lambda^2 R^2} + \sum_{i \neq j = 1}^{D-5} k_i k_j + 3 \frac{R'}{R} \sum_{i=1}^{D-5} k_i + 3\left(\frac{R'}{R}\right)^2 + 3 \frac{R''}{R} = \kappa^2 \Lambda_B, \]  

which is easily solvable. The nature of the solution depends, however, on the relations between the parameters of the equation. Indeed, there are four distinct classes of solutions, in addition to the trivial case \( k_i = 0 \ \forall i \), to equation Eq. (10), depending on the values of the constants \( k_i \).
By defining the combinations of the constants \( k_i \) as \( A = \sum_{i,j=1}^{D-5} k_i k_j \) and \( B = \sum_{i}^{D-5} k_i \) the solutions to Eq. (10) are given by

\[
\text{A) } B^2 \neq \frac{8}{3}(A - \kappa^2 \Lambda_B) \neq 0: \quad R^2(t, z) = \frac{3}{(\lambda^2(t)(A - \kappa^2 \Lambda_B)) + d_1(t) \exp \left( \frac{z}{6}(-3B + \sqrt{9B^2 - 24(A - \kappa^2 \Lambda_B)}) \right) + d_2(t) \exp \left( \frac{z}{6}(-3B - \sqrt{9B^2 - 24(A - \kappa^2 \Lambda_B)}) \right)
\]

\[
\text{B) } A = \kappa^2 \Lambda_B, \ B \neq 0: \quad R^2(t, z) = 2z/(\lambda^2(t)B) + d_1(t) \exp(-Bz) + d_2(t)
\]

\[
\text{C) } A = \kappa^2 \Lambda_B, \ B = 0: \quad R^2(t, z) = d_1(t) + d_2(t)z + z^2/\lambda^2(t)
\]

\[
\text{D) } B^2 = \frac{8}{3}(A - \kappa^2 \Lambda_B) \neq 0: \quad R^2(t, z) = \frac{3}{(\lambda^2(t)(A - \kappa^2 \Lambda_B)) + \exp(-\frac{B}{2}z)(d_1(t) + d_2(t)z)}
\]

Any of these solutions will satisfy the first four Einstein’s equations. In order that the rest of the Einstein’s equations are also valid in the bulk, we must generally have non-trivial components of the energy momentum tensor \( \mathbb{T} \). In six dimensions \( i.e. \ T^i_i = T^i_i(t, z), \ i = 4, ..., D - 1 \).

3.2 Bowl-models

Of particular interest are models with only a single brane with matter, hence one does not have to be concerned with the properties and implications of matter on a hidden brane. This means that either we have no second brane at all or it only has tension and no matter. Clearly, if we wish to have only a single brane, we must have more than one warped extra dimension so that we can wrap the brane along a compact extra dimension. The coordinate point that we have to be concerned with in such a construction is the origin, \( z = 0 \). Having a vanishing stress tensor at \( z = 0 \) means that the jumps of \( R \) and \( b_i \) must vanish at the origin \( i.e. \) the first derivative of the functions must be continuous. Clearly, the exponential Ansätze for the \( b_i \)s do not satisfy these conditions and in general it is not easy to see whether solutions where \( R'(t, 0) = b'_i(t, 0) = 0 \), exist or not.

As a special case we can construct a solution by choosing an essentially common Ansatz for the compact extra dimensions:

\[
b_i(z) = b_i(0) \cosh^{2/\alpha}(kz).
\]

With this Ansatz, we can solve for \( R(z, t) \):

\[
R^2(t, z) = \frac{2}{\beta \lambda^2} + d_1(t)e^{(z/2(\alpha - \sqrt{\alpha^2 - 4\beta}))}2F_1\left(\frac{\alpha - \sqrt{\alpha^2 - 4\beta}}{2k}, \frac{\alpha}{2k}, 1 - \sqrt{\alpha^2 - 4\beta}, e^{2kz}\right) + d_2(t)e^{(z/2(\alpha + \sqrt{\alpha^2 - 4\beta}))}2F_1\left(\frac{\alpha + \sqrt{\alpha^2 - 4\beta}}{2k}, \frac{\alpha}{2k}, 1 + \sqrt{\alpha^2 - 4\beta}, e^{2kz}\right)
\]

where \( \alpha = \frac{2\kappa}{d+1}, \beta = \frac{1}{2}(\alpha k - \kappa^2 \Lambda_B) \) and \( 2F_1(a, b; c; z) \) is the hypergeometric function. One can now choose one of the \( d_i(t) \)’s in such a way that \( R'(0) \) vanishes.

Unfortunately, this solution is not very intuitive, and so, in order to study the properties of the cosmological bowl solutions, we restrict ourselves to the case \( k^2 = \frac{2(d+1)}{d+3}\kappa^2 \Lambda_B \). In six dimensions \( d = 1 \) this implies

\[
R^2(t, z) = \frac{2}{\lambda^2(t)\kappa^2 \Lambda_B} \ln(\cosh(kz)) + d(t),
\]

whereas for \( d > 1 \) we write

\[
R^2(t, z) = d_1(t) \cosh^\nu(kz) - \frac{2}{\lambda^2(t)k^2 \nu^2}.
\]
where the exponent is given by $\nu = 1 - \frac{2d}{d+1}$. Note, that the condition $R'(0) = 0$ has already been exploited so that there is only one brane in the scenario which, technically, removes one of arbitrary functions $d_1, d_2$ of the Eq. (13). Note also, that these solutions correspond necessarily to a positive bulk cosmological constant. This model will be studied in more detail in Section 4.

### 3.3 Jump equations

Next we move forward to study the properties of the models on the brane(s). We relate the energy momentum tensor of the D-1 -dimensional brane to the solutions of D-dimensional Einstein equations. We idealize the brane to be a zero width object in z-direction regardless how the brane has been formed. Thus, any brane locates at an incontinuity of the z-derivative of the metric tensor, in particular at the edge of the space.

So, the energy content of the brane is related to the global solutions through the jump equations. The jump equations in the D-dimensional case can be read from the Einstein’s equations as usual. The brane contribution to the energy-momentum tensor is supposed to be confined to some $z = z'$, i.e. it is assumed to have the form

$$(T^B_A)_{Br} = \delta(z - z') \hat{T}^B_A = \delta(z - z') \text{diag}(\Lambda + \rho, \Lambda - p, \Lambda - p, \Lambda - p, 0, A_1 - P_1, ..., A_{D-5} - P_{D-5}). \quad (16)$$

Thus there is a uniform matter distribution also in the directions of the compact dimensions. Solving for the different components we get (for general $b_i = b_i(t, z)$):

$$\left[\frac{R'}{R}\right]_{z'} = \frac{\kappa^2}{D-2} \left( (D-5)(\rho + p) - (D-6)(\Lambda + \rho) + \sum_{i=1}^{D-5} (A_i - P_i) \right)$$

$$\left[\frac{\eta'}{\eta}\right]_{z'} = \frac{\kappa^2}{D-2} (\rho + p)$$

$$\left[\frac{b'_j}{b_j}\right]_{z'} = \frac{\kappa^2}{2(D-2)} \left( 4\Lambda + \rho - 3p + \sum_{i=1, i \neq j}^{D-5} (A_i - P_i) + (3 - D)(A_j - P_j) \right) \quad (17)$$

From these, we see that there are two special cases, $D = 5$ where there is no pressure ($p$) (nor compact dimensions) dependence on the jump of $R$, and $D = 6$, where the brane tension $\Lambda$ does not contribute either to the jump of $R$ or $\eta$.

One of the continuity equations $(T^{0,A}_{;A} = 0)$, takes on the brane the usual form

$$\dot{\rho} + 3H(\rho + p) = 0, \quad (18)$$

where $H = \dot{R}/R|_{z'}$. Because we have the freedom to choose $\lambda$, we may use the standard time at a brane by taking $\lambda(t) = 1/\hat{R}(t, z')$. Note, however, that if there is more than one brane, standard time can be chosen only at one of these. The other constraints can be obtained by noting that from the solution of $\eta$, it follows that $\eta'/\eta = R'/R + R/R\partial_t(R'/R)$. On the brane, this relates the jumps by

$$\left[\frac{\eta'}{\eta}\right]_{z_0} = \left[\frac{R'}{R}\right]_{z_0} + \frac{1}{H} \partial_t \left( \left[\frac{R'}{R}\right]_{z_0} \right). \quad (19)$$

Using the jump conditions and the continuity equation, we get a relation that binds the brane energy density, $\rho$, and brane pressures, $p, P_i$ by

$$(D - 5)(\dot{\rho} - \frac{1}{3}\dot{\rho}) - \sum_{i=1}^{D-5} \dot{P}_i = 0. \quad (20)$$
Hence, if the jump condition for \( R' \) is satisfied and Eq. (20) holds, the jump condition for \( \eta' \) is automatically satisfied. Note that in cosmological 5D-models \[5, 6\], relation (19) reduces to the usual continuity equation and hence a solution which satisfies the jump condition for \( R' \) will also satisfy the condition for \( \eta' \), which was noted in \[7\]. It is clear that in higher dimensional models, some energy can be flowing in/out of the compact dimensions when the universe is not radiation dominated (\( p \neq \rho/3 \)).

By assuming that the compact dimensions are static, \( \dot{b}_j = 0 \), we get by differentiating the jump equation for \( b_j' / b_j \) with respect to time, an equation for each \( j = 1, \ldots, d \) that relates the energy and pressures in different dimensions on the brane,

\[
\dot{\rho} - 3\dot{\rho} - \sum_{i=1, \ i \neq j}^{D-5} \dot{P}_i - (3 - D)\dot{P}_j = 0.
\] (21)

Note that these equations and Eq. (20) are not linearly independent, so that the system is not overconstrained. By summing the equations for each \( j \), we get again Eq. (20) and hence we only have \( d \) equations for \( d \) unknowns.

### 3.4 Friedmann equations

With the global bulk solutions and the jump conditions, we can now calculate the Friedmann equation on the brane in each case. At first we shall be concerned with models with two branes separated by distance \( L \), which may be negative. The branes are located at \( z = z_0 \) and \( z = z_1 = z_0 + L \). We normalize the scale factor \( R(t, z) \) so that \( R(t, z_0) = a_0(t) \) i.e. we live on the brane at \( z_0 \). The Friedmann equation in each case can be calculated by first calculating the jump of the first derivative at the two branes, which allows us to express \( \lambda(t) \) in terms of \( \sigma_i \equiv -[R'/R]_{z_i} \). Since the Hubble constant on our brane is given by \( H^2 = (\dot{a}_0/a_0)^2 = 1/(a_0^2 \lambda^2) \), we can then express \( H \) as a function of \( \sigma_i \).

The Friedmann equations for the four types of the exponential models \[10\] are:

- **A)**
  \[
  H^2 = \frac{A + \kappa^2 \Lambda_B}{3} \frac{B^2 + 4B(\sigma_0 + \sigma_1) - 4(\gamma^2 + 2(e^{2\gamma L} - 1)\gamma(\sigma_0 - \sigma_1) - 4\sigma_0 \sigma_1)}{B^2 + 4B\sigma_1 - 4\gamma\left(e^{2\gamma L} + 1) - 2(e^{(B+2\gamma)L}/2)\sigma_1(e^{2\gamma L} - 1)^{-1}\right)}
  \]

- **B)**
  \[
  H^2 = \frac{B}{2(e^{BL} - 1)} (\sigma_0 \sigma_1 - B(\sigma_0 - \sigma_1 e^{BL}))
  \]

- **C)**
  \[
  H^2 = \frac{2\sigma_0 \sigma_1 L - \sigma_0 + \sigma_1}{L(1 - \sigma_1 L)}
  \]

- **D)**
  \[
  H^2 = \frac{B^2 B^2 L + 4\sigma_1 (2 + BL) + 4\sigma_0 (BL - 2 + 4\sigma_1 L)}{8(B^2 L + 4\sigma_1 (2 + BL - 2 e^{BL/2}))},
  \]

where \( \gamma = \sqrt{9B^2 - 24(A - \kappa^2 \Lambda_B)} \). By using the jump equations, one can write these in terms of the quantities defined on the branes.

For the bowl-model the Friedmann equation has a remarkably simple form. We obtain

\[
H^2 = \frac{k}{2} [2\sigma_0 \text{coth}(kz_0) - k\nu],
\] (23)

which is valid for all \( d \geq 1 \).

We study these equations in more detail in the next Section, where the six-dimensional case is considered. Generalization to higher dimensions can then be done straightforwardly.
4 Six-dimensional case

As we have seen from the general expressions, in six dimensions there is no contribution to the jumps of \( R \) or \( \eta \) from the brane tension. Also, with only one compact dimension, the extra dimension can be identified with \( \mathbb{R}^2 \) and we can write the cosmological solution for a model with only a single brane wrapped around the origin with no additional matter or tension carrying brane. In this Section we study these properties in more detail.

The \((00)\)-component of the Einstein’s equations in six dimension, after substituting \( \eta(t, z) = \lambda(t) R(t, z) \), is

\[
- \frac{3}{\lambda^2 R^2} + \frac{b''}{b} + 3 \frac{b'}{b} \frac{R'}{R} + 3 \left( \frac{R''}{R} \right)^2 + 3 \frac{R''}{R} = \kappa^2 \Lambda_B. \tag{24}
\]

Let us first look for solutions with the exponential Ansatz, \( b(t, z) = b_0 \exp(kz) \). Again, depending on the value of the parameters we have different types of non-trivial solutions, which can be read out from Eqs. (11). However, the case \( C \) is possible only if \( k = \Lambda_B = 0 \), corresponding constant metric component \( b \) and the solution given in Eq. (11). Thus, because \( A = B^2 = k^2 \), the cases that remain are:

A) \( k^2 \neq \kappa^2 \Lambda_B, k^2 \neq 8\kappa^2 \Lambda_B/5 \):

\[
R^2(t, z) = d_1(t) \exp\left( \frac{z}{6} (-3k + \sqrt{24\kappa^2 \Lambda_B - 15k^2}) \right) + d_2(t) \exp\left( \frac{z}{6} (-3k - \sqrt{24\kappa^2 \Lambda_B - 15k^2}) \right) + \frac{3}{(k^2 - \kappa^2 \Lambda_B) \lambda^2}
\]

B) \( k^2 = \kappa^2 \Lambda_B \neq 0 \):

\[
R^2(t, z) = d_1(t) e^{-kz} + d_2(t) + \frac{2z}{k \lambda^2}
\]

D) \( k^2 = -8\kappa^2 \Lambda_B/5 \neq 0 \):

\[
R^2(t, z) = -\frac{5}{\lambda^2(t) \kappa^2 \Lambda_B} + \exp\left(-\frac{k}{2z}\right) \left( d_1(t) + d_2(t) z \right)
\]

In addition to these, we also have the bowl-solutions for which \( b'(0) \) vanishes. In six dimensions the Ansatz for \( b \) is \( b(z) = \cosh(kz) \) and generally we write

\[
R^2(t, z) = d_1(t) \exp\left( \frac{kz}{2} (1 - \alpha) \right) _2F_1\left( \frac{1}{2}, \frac{1}{2}; 1, 1 - \frac{1}{2} \alpha, e^{-2kz} \right) + d_2(t) \exp\left( \frac{kz}{2} (1 + \alpha) \right) _2F_1\left( \frac{1}{2}, \frac{1}{2}; 1 + \alpha, 1 + \frac{1}{2} \alpha, e^{-2kz} \right) + \frac{3}{\lambda^2(k^2 - \kappa^2 \Lambda_B)}, \tag{26}
\]

where \( \alpha \equiv \sqrt{1 - \frac{8}{15}(1 - \kappa^2 \Lambda_B)} \) and \( _2F_1(a, b; c; z) \) is the hypergeometric function. In particular, if \( k^2 = \kappa^2 \Lambda_B \) we obtain the special solution given by Eq. (11).

4.1 Jumps

The cosmological solutions obtained by placing branes at \( z = z_0 \) and \( z = z_1 \) can now be studied as previously. From the jump equations (17) we get:

\[
\left. \frac{[b']}{b} \right|_{z_i} = \frac{\kappa^2}{4} (4\Lambda_i + \rho_i - 3p_i - 3\Lambda_i^\theta + 3P_i)
\]

\[
\left. \frac{[R']}{R} \right|_{z_i} = \frac{\kappa^2}{4} (\rho_i + p_i + \Lambda_i^\theta - P_i)
\]

\[
\left. \frac{[\eta']}{\eta} \right|_{z_i} = \frac{\kappa^2}{4} (-3\rho_i - 3p_i + \Lambda_i^\theta - P_i), \tag{27}
\]

7
where \( \Lambda_i \) and \( \Lambda_i^\theta \) denote the tensions of usual four-dimensional Minkowski manifold and \( \theta \) direction, correspondingly. Note how the brane tension, \( \Lambda_i \), does not contribute to the jump of \( R' \). The exponential Ansatz obviously relates the energy density on the two branes since \(|b'|/b|_{z_0} = \pm k = -|b'|/b|_{z_1} \), where the sign in front of \( k \) is same as the sign of the difference \( z_1 - z_0 \).

An interesting case may be found if the other brane is assumed to be empty of matter, \( i.e. \rho_1 = p_1 = P_1 = 0 \) and even \( \Lambda_i^\theta = 0 \). In this case there is only a four-dimensional cosmological constant at \( z_1 \) and \( \sigma_1 = 0 \). Also we find that \( k = \kappa^2 \Lambda_{z=0} \) and, in \( \text{D}- \) case, the negative cosmological constant of the bulk reads \( \Lambda_B = -\frac{5}{8} \kappa^2 \Lambda^2 \).

### 4.2 Friedmann equations

The Friedmann equations for the cases \( \text{A} \) and \( \text{B} \) and \( \text{D} \) can be read from the previously obtained solutions \((22)\), again with \( A = k^2 \), \( B = k \) so, that \( \gamma = \sqrt{24 \kappa^2 \Lambda_B - 15 k^2} \). These special cases of the equations are rather lengthy and not very intuitive and hence they are not explicitly presented here.

The Friedmann equations on our brane at \( z = z_0 \) have interesting limits when the two branes are well separated from each other. In case \( \text{B} \) we see that

\[
\begin{align*}
Lk \gg 1, \quad H^2 & \approx \frac{\kappa^2}{4L}(\rho_0 + p_0 + \Lambda_0^\theta - P_0) - \frac{k}{4L} \\
-Lk \gg 1, \quad H^2 & \approx |k| \frac{\kappa^2}{4}(\rho_0 + p_0 + \Lambda_0^\theta - P_0)
\end{align*}
\]

(28)

where we have assumed that \(|k| \sim |\sigma_i|\). Hence, in these limit cases the Friedmann equation is linear in \( \rho_0 \) and is not coupled to the matter on the other brane. Note also that Eqs. \((22)\) are linear in \( \sigma_0 \) when \( \sigma_1 = 0 \) recreating the conventional evolution of the scale factor. In the bowl-model, we again follow the same procedure as before and find the Friedmann equation on the brane (note, that now we only have a single brane):

\[
H^2 = \frac{\kappa^3}{2} \sqrt{\Lambda_B} \coth(kz_0)(\rho_0 + p_0 + \Lambda_0^\theta - P_0).
\]

(29)

Finally we wish to give the relations using the observed 4-dimensional quantities. By using Eq. \((20)\) we can express the pressure \( P_0 \) in terms of the 5-dimensional energy density: By integration we obtain \( \rho_i = 3(p_i - P_i) + C \), where \( C \) is a constant. Moreover, in order to recover the usual Friedmann equation, we need to express the energy density in terms of the four-dimensional quantity. Because the components of the 5-dimensional energy-momentum tensor are related to the 4-dimensional counterpart by \((T^B_A)_{4D} = \int dz d\theta b(z) \delta(z-z_0) T^B_A\). We obtain for the exponential models \((T^B_A)_{4D} = 2\pi b(0)e^{kz_0} T^B_A\) and for the bowl model \((T^B_A)_{4D} = 2\pi b(0) \coth(kz_0) T^B_A\).

Thus the Friedman equations \((28)\) and \((29)\) may be written in terms of 4-dimensional quantities. We find that they are of the form

\[
H^2 = \frac{8\pi}{3M_{eff}^2}(\rho_0 + \Lambda_{eff}),
\]

(30)

where the effective mass scale \( M_{eff} \) and effective cosmological constant \( \Lambda_{eff} \) depend on the model. In particular, for the bowl model

\[
\frac{8\pi}{3M_{eff}^2} = \frac{\kappa^3}{2} \frac{\sqrt{\Lambda_B}}{2\pi b(0) \sinh(kz_0)}
\]

(31)

and \( \Lambda_{eff} = \Lambda^\theta - \frac{1}{8} C \). It is, however, clear, that the effective scale \( M_{eff} \) is a subject of constraints appearing from standard cosmology, nucleosynthesis, recombination \textit{etc.} because Hubble rate may change. Therefore \( M_{eff} \) has to be close enough to \( M_{Pl} \).
As the eq. (30) gives a relation between the parameter of the model and observed quantities, so does the equation, which determines the strength of gravitational interaction. Namely, the 4-dimensional gravitational constant $G_4 = M_{Pl}^{-2}$ is now given by

$$\frac{1}{G_4}R(z_0, t)^3 = \frac{2\pi}{G_6} \int dz \frac{R(z, t)^3 b(z)}{\eta(z, t)},$$

where $G_6 = \kappa^2 / 8\pi = M_s^{-4}$. The requirement is thus that the Planck mass $M_{Pl}$ equals now the usual value observed in the gravitational experiments.

## 5 Conclusions

In this paper we have studied the cosmological evolution of brane world models with a warped and a number of compact dimensions. By studying how the bulk evolves with time we find that the induced evolution on the brane(s) is quite naturally linearly proportional to the energy density on the brane. Moreover, in a number cases, the usual Friedmann equation is reproduced without any dependence on the energy density on the so-called hidden brane.

A particularly nice scenario is the bowl model where only one brane is present and hence there is no hidden brane contribution. The standard Friedmann equation is recreated with an effective cosmological constant on the brane. Interesting models can also be realized by assuming that the other brane carries only brane tension along the non-compact dimensions since then the Friedmann equation again has the usual form. Furthermore, standard cosmological evolution can be reached in some two-brane models by letting the two branes be well separated from each other.

An obvious omission in the analysis is the question of radion stabilization. Here we have to, in addition to the stabilization of the distance between the brane, be concerned with stabilization of the extra compact dimensions. Such considerations are clearly needed when one wishes to fit the models considered here into a larger theoretical framework. Here our main concern has been to consider these models from a purely phenomenological point of view and see how cosmological evolution is affected. In this paper we have also omitted the analysis of the graviton spectrum in each case which would be needed in order to study the correction to the Newtonian potential. In the six-dimensional case this has been done in [11, 13].

With both warped and, possibly large, compact extra dimensions, one also needs to keep a number of experimental constraints in mind. The appearance of the KK-tower of standard model particles sets an upper limit for the size of the extra compact dimensions on the brane, $b_i(0)^{-1} > 1$ TeV. The corrections to the Newtonian potential must also be small enough. In order to alleviate the hierarchy problem, the size of the extra dimensional volume must also be large enough, so that the effective gravitational constant on our brane is small as given by Eq. (32). In six dimensions these experimental constraints were analyzed in detail in [13] and these results are probably also essentially valid for more than six dimensions. Anyway, we have found that, at this level, adding compact dimensions surely increases the number of viable models, despite the new experimental constraints set by the KK-particle spectrum.

Brane world models with warped and compact dimensions are an interesting possibility whose implications for e.g. particle phenomenology are not yet fully known. Cosmologically they seem appealing since the standard Friedmann evolution with linear dependence on the energy density and no appearance of hidden brane quantities, is reached in many cases naturally without any fine tuning.

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