INTEGRATION OF NONSMOOTH 2-FORMS: FROM YOUNG TO ITÔ AND STRATONOVICH

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Abstract. We show that geometric integrals of the type \( \int_{\Omega} f \, d g_1 \wedge d g_2 \) can be defined over a two-dimensional domain \( \Omega \) when the functions \( f, g_1, g_2 : \mathbb{R}^2 \to \mathbb{R} \) are just Hölder continuous with sufficiently large Hölder exponents and the boundary of \( \Omega \) has sufficiently small dimension, by summing over a refining sequence of partitions the discrete Stratonovich or Itô type terms. This leads to a two-dimensional extension of the classical Young integral that coincides with the integral introduced recently by R. Züst. We further show that the Stratonovich-type summation allows to weaken the requirements on Hölder exponents of the map \( g = (g_1, g_2) \) when \( f(x) = F(x, g(x)) \) with \( F \) sufficiently regular. The technique relies upon an extension of the sewing lemma from Rough paths theory to alternating functions of two-dimensional oriented simplices, also proven in the paper.

1. Introduction

The scope of the present paper is constructing explicitly, via the appropriate discrete approximations, the extension of the classical notion of the integral of the differential 2-form \( f \, d g_1 \wedge d g_2 \) over any sufficiently nice oriented planar domain \( \Omega \subset \mathbb{R}^2 \) (one might think for simplicity of \( \Omega \) being just an oriented polygon, or even simpler, a triangle) to the case when the maps \( f : \mathbb{R}^2 \to \mathbb{R}, \ g : (g_1, g_2) : \mathbb{R}^2 \to \mathbb{R}^2 \) are only Hölder continuous, so that one might only put the word “differential” above in quotation marks, because \( g \) might have no derivatives. If \( g \) is sufficiently smooth and \( f \) just continuous, then \( f \, d g_1 \wedge d g_2 \) can be understood in the modern differential geometry language as \( f g^*(d x_1 \wedge d x_2) \), where \( d x^i \) are coordinate 1-forms, \( i = 1, 2 \), and \( g^* \) stands for the pull-back via \( g \), or, alternatively, in a more analytic language,

\[
\int_{\Omega} f \, d g_1 \wedge d g_2 := \int_{\Omega} f(x) \det \begin{pmatrix} \partial_1 g_1^1(x) & \partial_2 g_1^1(x) \\ \partial_1 g_2^1(x) & \partial_2 g_2^1(x) \end{pmatrix} \, d x,
\]

\( \partial_i \) standing for partial derivatives in the coordinate direction \( x_i, i = 1, 2 \). The latter integral is the natural building block for integrals of classical (smooth) differential 2-forms over smooth parameterized 2-dimensional surfaces in \( \mathbb{R}^n \) via pull-back.
One comes therefore inevitably to the problem posed when trying to integrate even a very smooth differential 2-form $\omega$ in $\mathbb{R}^n$ over a parameterized Hölder surface $\varphi: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^n$, $\varphi(x) = (\varphi^i(x))_{i=1}^n$, letting formally

$$ \int_{\varphi(\Omega)} \omega := \int_{\Omega} \varphi^* \omega, $$

where $\varphi^* \omega$ stands for pull-back of $\omega$ via $\varphi$, i.e. $\varphi^* \omega := \sum_{i,j} (a_{ij} \circ \varphi) d\varphi^i \wedge d\varphi^j$ when $\omega = \sum_{i,j} a_{ij} dx^i \wedge dx^j$.

1.1. History.

1.1.1. One-dimensional integrals. The one-dimensional prototype of this problem, that is, extending the integral of a differential 1-form $uv$ over an interval $[a, b]$ of the real axis to the maps $u, v: \mathbb{R} \to \mathbb{R}$ that are only Hölder continuous, has been solved by L.C. Young [17] and independently by V. Kondurar [8]. They defined the respective integral $\int_a^b uv$ as a limit in $k$ of a converging sequence of Riemann sums of the type $\sum_{i=0}^{k-1} u(a_i)(v(a_{i+1}) - v(a_i))$ over an appropriate sequence of refining partitions of the interval $[a, b]$ by consecutive points $a_0 := a < a_1 < \ldots < a_k := b$, thus mimicking the definition of the classical Riemann integral. This provides an extension of the latter to the case $u \in C^\alpha(\mathbb{R})$, $v \in C^{\beta}(\mathbb{R})$ when $\alpha + \beta > 1$ (later several generalizations of this result for wider classes of functions were provided, see e.g. [18] as well as the recent paper [16] and references therein). It is worth remarking that the original proof of Young [17] was quite “handmade”, just by the repetitive use of Hölder inequality. Rather, nowadays it is a custom to do it in a more “automated” way by using the so-called one-dimensional sewing lemma [4, lemma 2.1], which together with the construction of this integral, now usually called Young integral, is one of the basic pillars of the modern theory of Rough paths [5, 6].

Note that in the summands $u(a_i)(v(a_{i+1}) - v(a_i))$ one could replace $u(a_i)$ by, for instance,

$$ \bar{u}_{[a,a_{i+1}]} := \frac{1}{2}(u(a_i) + u(a_{i+1})), $$

thus leading to a different notion of integral. Minding the obvious analogy with stochastic Itô (resp. Stratonovich) integration, we will further call these two constructions Itô (resp. Stratonovich) summation. The general conditions on functions $u$ and $v$ for the limits in each of these cases to exist have been studied in [11] (in the subsequent paper [14] even more general weighted averages of $u$ in place of $\bar{u}_{[a,a_{i+1}]}$ were considered). Finally, V. Matsaev and M. Solomyak constructed in [9] a similar integral substituting $\bar{u}_{[a,a_{i+1}]}$ by the integral average $\bar{f}_{[a,a_{i+1}]} u$, which extends the classical integral of a smooth differential 1-form $uv$ over an interval to the case when $v \in C^{\beta}(\mathbb{R})$ is Hölder continuous and $u$ belongs to the Besov space $B^{\alpha}_{1,1}$ with $\alpha + \beta \geq 1$. In all the mentioned cases the result is the same for $u \in C^\alpha(\mathbb{R})$, $v \in C^{\beta}(\mathbb{R})$ with $\alpha + \beta > 1$, but may be different for more general functions.

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1 A historic curiosity: the modern construction of the Young integral via sewing lemma is closer to the original one used by Kondurar in [8] although his contribution to the subject seems to be unfortunately not so well-known.
1.1.2. Multidimensional integrals. Subsequently, several ways were proposed to extend the above mentioned one-dimensional constructions to multidimensional cases, notably \([13, 3]\), which however lack the very important geometric property of the classical integral of multidimensional forms, namely, that of being alternating, i.e. changing sign with the change of domain orientation (although we also have to mention quite a different and purely geometric approach of \([7]\) allowing to treat integration of smooth differential forms over nonsmooth domains, e.g. having fractal boundary, and a quite curious recent construction of \([15]\), reducing the multidimensional integral to a one-dimensional one involving a Peano-like curve).

A different approach to the definition of a multidimensional integral of nonsmooth “differential forms” has been taken by R. Züst \([19]\). Applied to the 2D situation which is of interest in the present paper, it shows that the integral \((1.1)\) defined over smooth maps, admits the unique extension by continuity with respect to the natural topology of pointwise convergence with bounded Hölder constants to a multilinear continuous functional

\[
(f, g^1, g^2) \in C^\alpha(\mathbb{R}^2) \times C^{\beta_1}(\mathbb{R}^2) \times C^{\beta_2}(\mathbb{R}^2) \rightarrow I(f, g^1, g^2)
\]

vanishing over degenerate rectangles and triangles (namely, those having zero area) and alternating in the last two entries, if \(\alpha + \beta_1 + \beta_2 > 2\). This functional can be therefore naturally called an integral

\[
\int_{\Omega} f \, dg^1 \wedge dg^2 := I(f, g^1, g^2),
\]

and can be approximated by sums over triangles forming the sufficiently fine dyadic decomposition of \(\Omega\) of the functions of three variables (which can be better thought as functions of a triangle \((p, q, r) \in (\mathbb{R}^2)^3 \mapsto \eta_{pqr}\) defined by

\[
(1.2) \quad \eta_{pqr} := f_p \int_{[pqr]} g^1 dg^2,
\]

where \(f_p := f(p)\), the integral above being intended in the sense of Young (note that in [19] a slightly different language was used with rectangles instead of triangles; the current language is taken from [12] where a unified approach for integration of multidimensional nonsmooth “differential forms” called “rough differential forms” up to dimensions 1 and 2 was suggested). R. Züst himself has further successfully employed this integral in several remarkable geometric problems in [20].

It is easy to observe that the definition of the integral of \(f dg^1 \wedge dg^2\) through the limit of sums of terms \((1.2)\) over sequences of refining partitions, is a clear generalization of the construction of the one-dimensional Young integral described above. It is inherently based upon integration by parts, i.e. is made so that the Stokes theorem

\[
\int_{\Omega} dg^1 \wedge dg^2 = \int_{\partial \Omega} g^1 dg^2
\]

almost automatically be satisfied for appropriate \(\Omega \subset \mathbb{R}^2\) (rectangle in [19] or triangle in [12]). This is however not how one usually expects the integral to be defined: in fact, the Young integrals over the sides of the triangle \([pqr]\) in \((1.2)\) have themselves to be defined either indirectly as continuous extensions of integrals of smooth differential forms approximating the “rough differential form” \(g^1 dg^2\) or as a limit of sums of appropriate discrete approximations (on the contrary, the abstract
extension of (1.1) from spaces of smooth functions to Sobolev or Besov spaces can be done via the technique from [10, 2] dealing with weak Jacobians).

1.2. Our contribution. It seems therefore more natural to define the integral of the “rough differential forms” \( f \, dg^1 \wedge dg^2 \) by purely discrete approximations. To this aim for \( f \in C^\alpha(\mathbb{R}^2) \), \( g^i \in C^{\beta_i}(\mathbb{R}^2) \), \( i = 1, 2 \), with \( \alpha + \beta_1 + \beta_2 > 2 \), we write

\[
\text{strat}_{pq} := \frac{1}{2} \left( \frac{f_p + f_q + f_r}{3} \right) \det \begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pr} \\ \delta g^2_{pq} & \delta g^2_{pr} \end{pmatrix},
\]

\[
\text{ito}_{pq} := \frac{1}{2} f_r \det \begin{pmatrix} \delta g^1_{pq} & \delta g^1_{pr} \\ \delta g^2_{pq} & \delta g^2_{pr} \end{pmatrix}
\]

for \( [pq] \subset \mathbb{R}^2 \), where we write \( f_u \) instead of \( f(u) \) and \( \delta g^i_{uv} := g^i(v) - g^i(u), \ i = 1, 2 \). We refer to \text{strat} and \text{ito} seen as functions of three variables (better viewed as functions of a two-dimensional simplex) as Stratonovich germ and to the latter one as Itô germ because of their obvious similarity with discrete constructions of the respective integrals in stochastic calculus. The terminology of “germs”, meaning just functions of finite-dimensional simplices, is borrowed from “germs of rough differential forms” [12], which is in turn inherited from the Rough Paths theory [6].

In this paper we show that

(A) if \( \Omega \) is an oriented simplex (i.e. a triangle), then summing either Itô or Stratonovich germs over any sufficiently nice family of its refining triangular partitions (in particular, dyadic ones) with the appropriately chosen orientation will still lead to the same integral defined by Züst, and estimate the rate of convergence (Theorems 4.4, 5.1). The respective integral may be called both Itô and Stratonovich, and in fact generalizes the one-dimensional Young integral. It is worth emphasizing that this result might seem counterintuitive. In fact the integral should clearly vanish over degenerate triangles \( \Omega \) (i.e. those having zero area), while neither the Stratonovich nor the Itô germ possess this property (which we will further call nonatomicity), as opposed to the germ \( \eta \) defined by (1.2), nor they are in some obvious way asymptotically close to some nonatomic germ (unless of course the functions \( g_1 \) and \( g_2 \) are differentiable). It is therefore not at all clear how can one expect to be nonatomic a limit of sums of germs which are essentially not so;

(B) the integral defined in such a way can be extended to a large class of bounded open sets \( \Omega \subset \mathbb{R}^2 \) having sufficiently small box-counting dimension of the topological boundary (Theorem 6.2), and in particular can be defined in a very natural way for \( \Omega \) a simple polygon (Proposition 6.1);

(C) when \( f \) has a particular form \( f(x) = F(x, g(x)) \), then the conditions of the existence of the integral extending the classical one (for smooth forms), i.e. the requirements on Hölder exponents of \( g^i \), may be significantly relaxed at the price of requiring \( F: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) to be sufficiently regular (Theorem 7.1) by employing Stratonovich germs. This is however a very particular feature of Stratonovich but not of Itô summation as can be seen also in the one-dimensional situation (Remark 7.4). The resulting Stratonovich type integral is shown to satisfy the classical chain rule (Proposition 7.6) and may be identified with the “second order Riemann-Stieltjes” integral.
introduced in [19], the respective identification leading to a curious continuity estimate for the degree of Hölder maps (Remark 7.10). We also give an interpretation of these results in geometric terms of the existence of continuous extensions of De Rham currents associated with the graphs of smooth maps $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ to those associated with graphs of Hölder maps with sufficiently large Hölder exponents, the continuity being intended in the weak (pointwise) topology of currents (Proposition 7.7).

The key role in the proofs will be played by the observation that both the integral and the Stratonovich germ are alternating, i.e. they change sign when the triangle over which they are defined changes the orientation. In fact, our basic instrument will be the natural generalization of the two-dimensional sewing lemma and stability theorem from [12] to abstract alternating germs (Lemma A.1 and A.3 respectively).

2. Notation and preliminaries

Spaces. Let $D \subset \mathbb{R}^n$ be an open set. For an $\alpha \in (0, 1)$ we will write $C^\alpha(D)$ (abbreviated just to $C^\alpha$ when there is no possibility of confusion) for the Hölder space with exponent $\alpha$. For an $f \in C^\alpha(D)$ we denote by $|f|_\alpha$ its Hölder norm, and $\|f\|_\alpha := \|f\|_\infty + |\delta f|_\alpha$ its Hölder norm, where $\|\cdot\|_\infty$ stands for the usual supremum norm in the space of continuous function $C(D)$ (usually abbreviated to $C$). The notation $C^1(D)$ (or just $C^1$ for brevity) will stand for the usual space of continuously differentiable functions.

Simplices, chains, germs and rough differential forms. For an ordered $(k+1)$-uple of points $S = [p_0p_1 \ldots p_k] \in D^{k+1}$ we write $\text{conv } S := \text{conv } \{p_0p_1 \ldots p_k\}$ and $\text{diam } S$ for the convex envelope and the diameter of the set of points $\{p_0, \ldots, p_k\}$ respectively, and call $S$ an (oriented) $k$-simplex in $D$, if $\text{conv } S \subset D$, the set of such simplices being denoted by $\text{Simp}^k(D)$. For a $k$-simplex $S \in \text{Simp}^k(D)$ we denote by $|S|$ its $k$-dimensional volume. A (real polyhedral) $k$-chain in $D$ is an element of the real vector space $\text{Chain}^k(D)$ generated by $k$-simplices in $D$. A $k$-simplex can be identified with the “geometric” simplex $\text{conv } S$ with a chosen base point $p_0$ and the chosen orientation given by the order of the points in the list, so that $0$-simplices correspond to points, $1$-simplices to oriented segments and $2$-simplices are pointed oriented triangles.

A $k$-germ (of a $k$-differential form in $D$) is a function $\omega: \text{Simp}^k(D) \rightarrow \mathbb{R}$,

$$S = [p_0p_1 \ldots p_k] \mapsto \omega_S = \omega_{p_0p_1 \ldots p_k}.$$  

We also often write $\langle S, \omega \rangle$ instead of $\omega_S$. A $k$-cochain in $D$ is a linear functional $\omega: \text{Chain}^k(D) \rightarrow \mathbb{R}$,

$$C \mapsto \langle C, \omega \rangle.$$  

For instance, $0$-germs are just functions $p_0 \mapsto f(p_0) = f_{p_0} = \langle [p_0], f \rangle$.

The boundary $\partial S$ of an $S \in \text{Simp}^k(D)$ is the $(k-1)$-chain defined by

$$\partial[p_0p_1 \ldots p_k] := \sum_{i=0}^{k} (-1)^i [p_0 \ldots \hat{p}_i \ldots p_k],$$  

de the notation $\hat{p}_i$ standing for removal of the respective element from the list. The operator $\partial$ is naturally extended by linearity to $k$-chains. The coboundary of a
For instance, for a 0-germ $f$ one has $(\delta f)_{pq} = f_q - f_p$, and for a 1-germ $\omega$ one has $(\delta \omega)_{pqr} = \omega_{qr} - \omega_{pr} + \omega_{pq}$.

A $k$-germ $\omega$ is called

- **nonatomic**, if it vanishes on degenerate $k$-simplices $S$ (i.e. on those having zero $k$-dimensional volume $|S| = 0$). For instance, the germ $\eta$ defined by (1.2) is nonatomic, while the germs $\text{strat}$ and $\text{ito}$ defined by (1.3) are not;

- **alternating**, if

  \[ \langle [p_0 p_1 \ldots p_k], \delta \omega \rangle := (-1)^\sigma ([\sigma(p_0)\sigma(p_1)\ldots\sigma(p_k)], \omega). \]

  for every permutation of vertices $\sigma$: \{ $p_0, p_1, \ldots p_k$ $\rightarrow$ $p_0 p_1 \ldots p_k$ \}, $(-1)^\sigma$ standing for the sign of permutation (positive for even and negative for odd permutations). For instance, among the germs defined by (1.2) and (1.3), $\text{strat}$ is alternating, while $\eta$ and $\text{ito}$ are not.

Finally, a $k$-germ $\omega$ is called a **rough differential $k$-form**, if it is continuous (as a function of vertices of a simplex), and both $\omega$ and $\delta \omega$ are nonatomic. An example of a rough differential 1-form (written $g^i dg^2$ for $g^i \in C^{\beta_i}$, $i = 1, 2$, with $\beta_1 + \beta_2 > 1$) is given by the Young integral over the line segment $[pq]$, that is,

\[ \langle [pq], g^1 dg^2 \rangle := \int_{[pq]} g^1 dg^2. \]

An example of a rough differential 2-form (written $fdg^1 \wedge dg^2$ for $f \in C^\alpha$, $g^i \in C^{\beta_i}$, $i = 1, 2$, with $\alpha + \beta_1 + \beta_2 > 2$) is given by the integral defined by R. Züst in [19], namely,

\[ \langle [pqr], f dg^1 \wedge dg^2 \rangle := \int_{[pqr]} fdg^1 \wedge dg^2. \]

The **cup product** (called external product in [6]) between a $k$-germ $\omega$ and a $h$-germ $\tilde{\omega}$ is the $(k+h)$-germ $\omega \cup \tilde{\omega}$ defined by

\[ \langle [p_0 p_1 \ldots p_k p_{k+1} \ldots p_{k+h}], \omega \cup \tilde{\omega} \rangle := \langle [p_0 p_1 \ldots p_k], \omega \rangle \langle [p_k p_{k+1} \ldots p_{k+h}], \tilde{\omega} \rangle. \]

The cup product is associative but in general not commutative, and the following Leibniz rule holds [12]: for $\omega \in \text{Germ}^k(D)$, $\tilde{\omega} \in \text{Germ}^h(D)$ one has

\[ \delta(\omega \cup \tilde{\omega}) = (\delta \omega) \cup \tilde{\omega} + (-1)^k \omega \cup (\delta \tilde{\omega}). \]

3. **Estimates on germs**

We start with the following useful algebraic lemma.

**Lemma 3.1.** One has

\[
\frac{1}{2} \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{pr} \\
\delta g^2_{pq} & \delta g^2_{pr}
\end{pmatrix} = \frac{1}{2} \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^2_{pq} & \delta g^1_{pr} & \delta g^2_{pr}
\end{pmatrix} = \frac{1}{2} \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{pq} & \delta g^1_{pq} & \delta g^1_{pq}
\end{pmatrix}
\]

\[ = \mathcal{A}(\delta g^1 \cup \delta g^2)_{pqr}, \]

where $\mathcal{A}$ stands for the antisymmetrization operator

\[ \mathcal{A}(\phi \cup \psi) := \frac{1}{2} (\phi \cup \psi - \psi \cup \phi). \]
In particular,
\begin{equation}
(3.2) \quad \text{ito}_{pqr} = (f \cup A(\delta g^1 \cup \delta g^2))_{pqr}.
\end{equation}

**Proof.** It suffices to calculate
\[
\det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{pr} \\
\delta g^2_{pq} & \delta g^2_{pr}
\end{pmatrix} - \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{qr} \\
\delta g^2_{pq} & \delta g^2_{qr}
\end{pmatrix} = \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{qr} - \delta g^1_{pr} \\
\delta g^2_{pq} & \delta g^2_{qr} - \delta g^2_{pr}
\end{pmatrix} = \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{qr} \\
\delta g^2_{pq} & \delta g^2_{qr}
\end{pmatrix} = 0
\]
to show the first equality in (3.1); the second equality is quite analogous from
\[
\det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{pr} \\
\delta g^2_{pq} & \delta g^2_{pr}
\end{pmatrix} - \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{qr} \\
\delta g^2_{pq} & \delta g^2_{qr}
\end{pmatrix} = \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{qr} - \delta g^1_{pr} \\
\delta g^2_{pq} & \delta g^2_{qr} - \delta g^2_{pr}
\end{pmatrix} = \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{qr} \\
\delta g^2_{pq} & \delta g^2_{qr}
\end{pmatrix} = 0,
\]
concluding the proof. \(\square\)

Notice that \(A(\delta g^1 \cup \delta g^2) = \delta \eta\) with \(\eta = \frac{1}{2} (g^1 \delta g^2 - g^2 \delta g^1)\).

**Lemma 3.2.** One has
\begin{align}
(3.3) \quad |\text{ito}_{pqr} - \text{strat}_{pqr}| & \leq 2|\delta f|_{\alpha} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}(|pqr|)^{\alpha+\beta_1+\beta_2} \\
(3.4) \quad |\text{strat}_{pqr}| & \leq \|f\|_{\infty} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}(|pqr|)^{\beta_1+\beta_2} \\
(3.5) \quad |\text{strat}_{pqr}| & \leq 8|\delta f|_{\alpha} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}(|pqr|)^{\alpha+\beta_1+\beta_2}
\end{align}
and \(\text{strat}\) is alternating, namely,
\[
\text{strat}_{pqr} = \text{strat}_{rqp} = \text{strat}_{qpr} = -\text{strat}_{rqp} = -\text{strat}_{qpr} = -\text{strat}_{qpr}.
\]

**Remark 3.3.** Clearly, (3.4) holds even for every \(f \in M\), where \(M\) stands for the space of bounded (not necessarily measurable) functions over \(\mathbb{R}^2\) equipped with the supremum norm (still denoted by \(\|\cdot\|_{\infty}\)).

**Proof.** The estimate (3.4) as well as the alternating property of \(\text{strat}\) is immediate from the definition of \(\text{strat}\). To show (3.3), we calculate
\[
|\text{ito}_{pqr} - \text{strat}_{pqr}| = \frac{1}{2} \left| \left( f_p + f_q + f_r \right) \frac{3}{3} - f_p \right| \det \begin{pmatrix}
\delta g^1_{pq} & \delta g^1_{pr} \\
\delta g^2_{pq} & \delta g^2_{pr}
\end{pmatrix} \\
\leq |\delta f|_{\alpha} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}(pqr)^{\alpha+\beta_1+\beta_2}
\]
as claimed. Thus, (3.5) would follow once one proves
\begin{equation}
(3.6) \quad |\delta \text{ito}_{pqr}| \leq |\delta f|_{\alpha} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}(|pqr|)^{\alpha+\beta_1+\beta_2}.
\end{equation}
To show the latter inequality, we use Lemma 3.1: namely, by (3.2) one has
\begin{equation}
(3.7) \quad \text{ito} = \frac{1}{2} \left( (f \cup \delta g^1 \cup \delta g^2) - (f \cup \delta g^2 \cup \delta g^1) \right).
\end{equation}
Therefore, using the fact that
\[
\delta(\delta g^1 \cup \delta g^2) = \delta g^1 \cup \delta(\delta g^2) - \delta(\delta g^1) \cup \delta g^2 = 0,
\]
and analogously \( \delta(\delta g^2 \cup \delta g^1) = 0 \), from (3.7) we get
\[
\delta \text{ito} = \frac{1}{2} \left( \delta(f \cup \delta g^1 \cup \delta g^2) - \delta(f \cup \delta g^2 \cup \delta g^1) \right)
\]
(3.8)
\[
= \frac{1}{2} \left( (\delta f \cup \delta g^1 \cup \delta g^2) - (\delta f \cup \delta g^2 \cup \delta g^1) \right).
\]
Since clearly,
\[
|\delta f \cup \delta g^1 \cup \delta g^2 |_{pqrs} = |\delta f |_{pq} \cdot |(\delta g^1 |_{q} \cdot |(\delta g^2 |_{rs})| \\
\leq [\delta f |_{pq}] [\delta g^1 |_{q} [\delta g^2 |_{rs}] \alpha \beta \gamma \delta \text{diam}(pqrs)^{\alpha + \beta + \gamma},
\]
and analogously
\[
|\delta f \cup \delta g^2 \cup \delta g^1 |_{pqrs} \leq [\delta f |_{pq}] [\delta g^1 |_{q} [\delta g^2 |_{rs}] \alpha \beta \gamma \delta \text{diam}(pqrs)^{\alpha + \beta + \gamma},
\]
from (3.8) we get (3.6), and therefore (3.5), hence concluding the proof.

Later in section 7 we will need also the following curious algebraic identity which is a peculiar property of only the Stratonovich germ \( \text{strat} \) and not of the Itô germ \( \text{ito} \), and could have been also used for an alternative proof of (3.5) in Lemma 3.2.

**Lemma 3.4.** One has
\[
(\delta \text{strat})_{pqrs} = \frac{1}{6} \det \begin{pmatrix}
\delta f_{pq} & \delta f_{pr} & \delta f_{ps} \\
\delta g^1_{pq} & \delta g^1_{pr} & \delta g^1_{ps} \\
\delta g^2_{pq} & \delta g^2_{pr} & \delta g^2_{ps}
\end{pmatrix}.
\]

**Proof.** By Lemma 3.1 one has
\[
6(\delta \text{strat})_{pqrs} = \delta f \cup \delta g^1 \cup \delta g^2 - \delta f \cup \delta g^2 \cup \delta g^1
\]
+ \( (\delta g^1 \cup f \cup \delta g^2 \cup \delta g^1)_{pqrs} + (\delta g^1 \cup f \cup \delta g^2 \cup \delta g^1)_{pqrs} \)
+ \( (\delta g^1 \cup \delta g^2 \cup f \cup \delta g^2 \cup \delta g^1 \cup f)_{pqrs} \).

Hence,
\[
6(\delta \text{strat})_{pqrs} = (\delta f \cup \delta g^1 \cup \delta g^2 - \delta f \cup \delta g^2 \cup \delta g^1)_{pqrs} + \\
(\delta g^1 \cup \delta f \cup \delta g^2 + \delta g^2 \cup \delta f \cup \delta g^1)_{pqrs} + \\
(\delta g^1 \cup \delta g^2 \cup \delta f - \delta g^2 \cup \delta g^1 \cup \delta f)_{pqrs}
\]
\[
= \det \begin{pmatrix}
\delta f_{pq} & \delta f_{qr} & \delta f_{rs} \\
\delta g^1_{pq} & \delta g^1_{qr} & \delta g^1_{rs} \\
\delta g^2_{pq} & \delta g^2_{qr} & \delta g^2_{rs}
\end{pmatrix} = \frac{1}{6} \det \begin{pmatrix}
\delta f_{pq} & \delta f_{pr} & \delta f_{ps} \\
\delta g^1_{pq} & \delta g^1_{pr} & \delta g^1_{ps} \\
\delta g^2_{pq} & \delta g^2_{pr} & \delta g^2_{ps}
\end{pmatrix},
\]
where the latter identity follows by adding the first column to the second one and subsequently the second column to the third one.

4. RIEMANN SUMMATION OVER DYADIC PARTITIONS

Recall [12] the dyadic decomposition of a 2-simplex \( [p_0 p_1 p_2] \in \text{Simp}^2(D) \)
\[
dya[p_0 p_1 p_2] := [q_0 q_1 q_2] + [q_1 q_2 p_2] + [q_2 p_1 q_0] + [p_0 q_2 q_1],
\]
where \( q_i := (p_i + p_\ell)/2 \) for \( \{i, j, \ell\} = \{0, 1, 2\} \). Write also cut\( [p_0 p_1] := [p_0 q] + [q p_1] \) and fill\( [p_0 p_1] := [p_0 q p_1] \), with \( q := (p_0 + p_1)/2 \) (naturally extended to chains).
For \( n \in \mathbb{N} \) define the \( n \)-th Stratonovich sum \( \text{strat}^n \), the side corrector \( S^n \) as well as the Itô sum \( \text{ito}^n \) respectively by the formulae

\[
\text{strat}^n_{pqr} := \langle \text{dya}^n[pqr], \text{strat} \rangle, \quad S^n_{pq} := \sum_{i=0}^{n-1} \langle \text{fill cut}^i[pq], \text{strat} \rangle, \quad \text{ito}^n_{pqr} := \langle \text{dya}^n[pqr], \text{ito} \rangle.
\]

**Lemma 4.1.** One has

\[
|S^n_{pq} - S^n_{pq}| \leq C \|f\|_{\infty} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}([pq])^{\beta_1 + \beta_2} 2^{n(1 - \beta_1 - \beta_2)},
\]

\[
|\langle \text{dya}^n[pqr], \text{strat} - \delta S^n \rangle - (\langle \text{dya}^{n+1}[pqr], \text{strat} \rangle - \delta S^{n+1})| \leq C[\delta f]_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}([pq])^{\alpha + \beta_1 + \beta_2} 2^{n(2 - \alpha - \beta_1 - \beta_2)}
\]

with \( C > 0 \) a universal constant. In particular, if \( \alpha + \beta_1 + \beta_2 > 2 \), then

\[
S^n_{pq} := \lim_{n \to \infty} S^n_{pq},
\]

\[
V_{pqr} := \lim_{n \to \infty} \text{strat}^n_{pqr} = \lim_{n \to \infty} (\text{strat}^n_{pqr} - \delta S^n_{pqr}) + \delta S^n_{pqr}
\]

are well defined continuous alternating germs with

\[
S^n_{pq} : C^0 \times C^{\beta_1} \times C^{\beta_2} \to \mathbb{R}, \quad V_{pqr} : C^\alpha \times C^{\beta_1} \times C^{\beta_2} \to \mathbb{R}
\]

continuous and

\[
|S^n_{pq} - S^n_{pq}| \leq \|f\|_{\infty} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}([-pq])^{\beta_1 + \beta_2} 2^{n(1 - \beta_1 - \beta_2)},
\]

\[
|\text{strat}^n_{pqr} - V_{pqr} - \delta(S^n - S)_{pqr}| \leq C[\delta f]_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}([pq])^{\alpha + \beta_1 + \beta_2} 2^{n(2 - \alpha - \beta_1 - \beta_2)},
\]

\[
|\text{strat}^n_{pqr} - V_{pqr}| \leq C\|f\|_{\infty} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}([pq])^{\beta_1 + \beta_2} 2^{n(1 - \beta_1 - \beta_2)}.
\]

**Remark 4.2.** As one easily deduces from the proof, in view of the Remark 3.3, one has, with the notation of the latter, that in fact \( S^n_{pq} \) itself may be defined over the larger space \( M \times C^{\beta_1} \times C^{\beta_2} \) and is continuous there when just \( \beta_1 + \beta_2 > 1 \).

**Proof.** We apply Lemma A.1 to our germ \( \text{strat} \) (which is continuous and alternating by construction) recalling that it satisfies both (A.1) and (A.2) with

\[
\gamma_1 := \beta_1 + \beta_2 > 1, \quad C_1 := \|f\|_{\infty} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2},
\]

\[
\gamma_2 := \alpha + \beta_1 + \beta_2 > 2, \quad C_2 := 8\|f\|_{\infty} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2}
\]

in view of Lemma 3.2. This gives (4.2) and (4.3), as well as the existence of limit germs alternating continuous \( S \) and \( V \) in (4.4) satisfying (4.5), (4.6) and (4.7). Finally, the continuity of \( S^n_{pq} \) (with fixed \([pq]\)) as a functional follows from (4.2) and implies the continuity of \( \delta S^n_{pq} : C^0 \times C^{\beta_1} \times C^{\beta_2} \to \mathbb{R} \). Continuity of

\[
V_{pqr} - \delta S^n_{pqr} := \lim_{n \to \infty} (\text{strat}^n_{pqr} - \delta S^n_{pqr}): C^\alpha \times C^{\beta_1} \times C^{\beta_2} \to \mathbb{R}
\]

follows from (4.3), hence implying the continuity of \( V \), and therefore concluding the proof.

We will need also the following Lemma already formulated in [12, example 4.7].

**Lemma 4.3.** If \( \beta_1 = \beta_2 = 1 \), then

\[
V_{pqr} = \int_{[pqr]} f \text{dya}^1 \wedge \text{dya}^2 = \int_{[pqr]} f \det(\nabla g^1, \nabla g^2).
\]
We are now at a position to prove the first principal result of this paper.

**Theorem 4.4.** If $\alpha + \beta_1 + \beta_2 > 2$, then

\[
\begin{align*}
\lim_{n \to \infty} \text{strat}^n_{pqr} &= \int_{[pqr]} f dg^1 \wedge dg^2 \\
&= \lim_{n \to \infty} \text{ito}^n_{pqr}.
\end{align*}
\]

In particular, the latter integral is

(A) nonatomic, i.e.

\[\int_{[pqr]} f dg^1 \wedge dg^2 = 0 \quad \text{when } ||[pqr]|| = 0,\]

(B) continuous and alternating in $[pqr]$, and

(C) additive, in the sense that when

\[ [pqr] = \sum_{i=1}^k \Delta_i + N + \partial R, \]

where $\Delta_i$ are oriented 2-simplices, $N$ is a polyhedral 2-chain consisting of degenerate 2-simplices (i.e. having area zero), and $R$ is a polyhedral 3-chain in $\mathbb{R}^2$, then

\[
\int_{[pqr]} f dg^1 \wedge dg^2 = \sum_{i=1}^k \int_{\Delta_i} f dg^1 \wedge dg^2.
\]

Moreover,

\[
\begin{align*}
\left| \text{strat}^n_{pqr} - \int_{[pqr]} f dg^1 \wedge dg^2 \right| &\leq C \|f\|_{\alpha} \|\delta g^1\|_{\beta_1} \|\delta g^2\|_{\beta_2} \text{diam}([pqr])^{\beta_1+\beta_2} 2^n(1-\beta_1-\beta_2), \\
(4.10) \\
\left| \text{ito}^n_{pqr} - \int_{[pqr]} f dg^1 \wedge dg^2 \right| &\leq C \|f\|_{\alpha} \|\delta g^1\|_{\beta_1} \|\delta g^2\|_{\beta_2} \text{diam}([pqr])^{\beta_1+\beta_2} 2^n(1-\beta_1-\beta_2)
\end{align*}
\]

for some $C = C(\alpha, \beta_1, \beta_2) > 0$.

**Proof.** By Lemma 4.1, the limit

\[ V_{pqr} := \lim_{n \to \infty} \text{strat}^n_{pqr} \]

exists and is a continuous multilinear functional over $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$, and

\[ V_{pqr}(f, g^1, g^2) = \int_{[pqr]} f dg^1 \wedge dg^2 := \int_{[pqr]} f \det(\nabla g^1, \nabla g^2) \, dx \]

when $f \in C^0, g^i \in C^1, i = 1, 2$. However the unique continuous extension of the latter functional defined over $C^0 \times C^1 \times C^1$ to $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$ is the Züst integral, which implies the claim (4.8), (4.10). Properties (A), (B) and (C) are now in fact the properties of the Züst integral (theorem 4.10 of [12] where they are stated by saying that the Züst germ (1.2) is *sewable*).

The claims (4.9), (4.11) follow now from (3.3).
Remark 4.5. One also has the inequality (4.6) which can be rewritten, in view of the above Theorem 4.4 as

\[ \text{strat}_{[pqr]} - \int_{[pqr]} f \, dg^1 \wedge dg^2 - \delta(S^n - S_{pqr}) \leq C\|f\|_{\alpha} |\delta g^1|_{\beta_1} |\delta g^2|_{\beta_2} \text{diam}([pqr])^{\alpha + \beta_1 + \beta_2} 2^{n(2 - \alpha - \beta_1 - \beta_2)} \]

with \( C = C(\alpha, \beta_1, \beta_2) > 0 \). Thus, in order to improve the convergence rate one should better approximate \( S^n - S \). This is the case e.g. when on the boundary of \([pqr]\) either \( f \) is null or one of the \( g^i \) is constant: in fact, in these cases \( S^n = 0 \) and hence also \( S = 0 \).

Remark 4.6. The 2-germ \( f \cup \delta g^1 \cup \delta g^2 \) in general does not provide an integral even when \( f, g^1 \) and \( g^2 \) are smooth. In fact, let \( f = 1, g^i(x_1, x_2) := x_i, i = 1, 2, \quad p = (0, 0), \quad q = (1, 0), \quad r = (0, 1) \). Then \( \langle \delta g^1[pqr], \delta g^1 \rangle \to 2||[pqr]| \) while \( \langle \delta g^1[pqr], f \cup \delta g^1 \cup \delta g^2 \rangle \to 0 \) as \( n \to \infty \), i.e. the limit is not alternating.

Remark 4.7. As mentioned in the introduction, the above theorem allows to define the integral of a differential 2-form \( \omega = f \, dg^1 \wedge dg^2 \) on \( \mathbb{R}^n \) over a parameterized Hölder surface \( \varphi : \Omega \to \mathbb{R}^n, \varphi(x) = (\varphi^i(x))_{i=1}^n \), letting

\[ \int_{[pqr]} f \, dg^1 \wedge dg^2 := \int_{[pqr]} (f \circ \varphi) d(g^1 \circ \varphi) \wedge d(g^2 \circ \varphi), \]

provided that \( f \in C^n(\mathbb{R}^n), g^i \in C_{\beta_i}(\mathbb{R}^n), i = 1, 2, \varphi \in C^\gamma(\mathbb{R}^2; \mathbb{R}^n) \) with

\[ \gamma(\alpha + \beta_1 + \beta_2) > 2. \]

Notice however that the above integral differs from the integral obtained partitioning the triangle \([\varphi(p)\varphi(q)\varphi(r)]\) with an order \( \text{diam}([pqr])^{(\beta_1 + \beta_2)} \) and not \( \text{diam}([pqr])^{(\alpha + \beta_1 + \beta_2)} \), see [12, proposition 4.29].

Corollary 4.8. If there is an \( h \in C^{\beta_3}, \beta_3 \in (0, 1] \), such that both \( g^1 \) and \( g^2 \) are \( h \)-differentiable in the sense

\[ |(\delta g^i)_{pq} - a^i_p(\delta h)_{pq}| \leq C |p - q|^{1 + \gamma_i} \]

for every \( p \in D \) as \( q \to p \), and, moreover,

\[ |(\delta g^i)_{pq} - a^i_p(\delta h)_{pq}| \leq C |p - q|^{1 + \gamma_i} \]

for some \( \gamma_i > 1 - \beta_3, i = 1, 2, \) and \( C > 0 \), then \( dg^1 \wedge dg^2 = 0 \) in the sense

\[ \int_{[pqr]} f \, dg^1 \wedge dg^2 = 0 \]

for every \( f \in C^\alpha \) with \( \alpha + \beta_1 + \beta_2 > 2 \) and every \([pqr] \subset D\).

Proof. Let

\[ \rho_{pq}^i := (\delta g^i)_{pq} - a^i_p(\delta h)_{pq}. \]

Then

\[ \frac{1}{2} \det \left( \begin{array}{cc} \delta g^1_{pq} & \delta g^1_{pq} \\ \delta g^2_{pq} & \delta g^2_{pq} \end{array} \right) = \frac{1}{2} a^1_p a^2_p \det \left( \begin{array}{cc} \delta h_{pq} & \delta h_{pq} \\ \delta h_{pq} & \delta h_{pq} \end{array} \right) + \frac{1}{2} a^1_p a^2_p \det \left( \begin{array}{cc} \rho_{pq}^1 & \delta h_{pq} \\ \rho_{pq}^1 & \delta h_{pq} \end{array} \right) \]

\[ + \frac{1}{2} a^1_p a^2_p \det \left( \begin{array}{cc} \delta h_{pq} & \rho_{pq}^1 \\ \delta h_{pq} & \rho_{pq}^1 \end{array} \right) + \frac{1}{2} \det \left( \begin{array}{cc} \rho_{pq}^1 & \rho_{pq}^1 \\ \rho_{pq}^1 & \rho_{pq}^1 \end{array} \right). \]
Letting $\gamma := \gamma_1 + \gamma_2$, from (4.13) we get
\[
\det \begin{pmatrix} \rho_{pq}^1 & \delta h_{pr} \\ \rho_{pq}^2 & \delta h_{pr} \end{pmatrix} \leq 2C[h]_{\beta_3} \text{diam}([pqr])^{1+\gamma+\beta_3},
\]
\[
\det \begin{pmatrix} \delta h_{pq} & \rho_{pr}^1 \\ \delta h_{pq} & \rho_{pr}^2 \end{pmatrix} \leq 2C[h]_{\beta_3} \text{diam}([pqr])^{1+\gamma+\beta_3},
\]
\[
\det \begin{pmatrix} \rho_{pq}^1 & \rho_{pr}^1 \\ \rho_{pq}^2 & \rho_{pr}^2 \end{pmatrix} \leq 2C \text{diam}([pqr])^{2+\gamma_1+\gamma_2},
\]
so that by (4.14) one has
\[
|\text{strat}_{pqr}| \leq 2C\|f\|_\infty (\|a^1\|_\infty \|a^2\|_\infty [h]_{\beta_3} \text{diam}([pqr])^{1+\gamma+\beta_3} + \text{diam}([pqr])^{2+\gamma_1+\gamma_2}),
\]
which concludes the proof since $1 + \gamma_1 + \beta_3 > 2$ and $2 + \gamma_1 + \gamma_2 > 2$. 

**Remark 4.9.** In particular, if $g^1$ is $g^2$-differentiable and, moreover,
\[
|\delta g^1|_{pq} - a_p (\delta g^2)_{pq} | \leq C|p - q|^{1+\gamma},
\]
for some $\gamma > 1 - \beta_2$ and $C > 0$, then $d(g^1 \wedge g^2) = 0$.

## 5. General Partitions

Theorem 4.4 shows that the integral $\int_{[pqr]} f dg^1 \wedge dg^2$ can be obtained as a limit of sums of the Stratonovich germs over dyadic partitions of the the simplex $[pqr]$. Here we show that it can be obtained by a similar summation of such germs over more general partitions.

**Theorem 5.1.** Assume that the simplex $[pqr]$ be partitioned in a finite number of disjoint simplices $\{\Delta_i\}_{i=1}^N$ not belonging to the sides of $[pqr]$ so that
\[
[pqr] - \sum_{i=1}^N \Delta_i = \partial P + \sum_{j=1}^M Q_j,
\]
where $P \in \text{Chain}^3(D)$ and each $Q_j \in \text{Simp}^2(D)$ is a degenerate simplex reduced to a line segment belonging to some side of $[pqr]$ such that two sides of each $Q_j$ are sides of some $\Delta_i$ (with opposite direction). Then
\[
\sum_{i=1}^N \langle \Delta_i, \text{strat} \rangle - \int_{[pqr]} f dg^1 \wedge dg^2 \leq C \|d\|_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \left( \sum_{i=1}^N \text{diam}(\Delta_i)^{\alpha+\beta_1+\beta_2} + \sum_{j=1}^M \text{diam}(Q_j)^{\beta_1+\beta_2} \right).
\]

**Proof.** The estimate (4.12) applied to each $\Delta_i$ with $n := 0$ gives
\[
\langle \Delta_i, \text{strat} \rangle - \int_{\Delta_i} f dg^1 \wedge dg^2 - \langle \Delta_i, \delta(S^0 - S) \rangle \leq C \|d\|_\alpha [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2} \text{diam}(\Delta_i)^{\alpha+\beta_1+\beta_2}.
\]
Summing the latter estimates over $i = 1, \ldots, N$, and recalling that
\[
\int_{[pqr]} f dg^1 \wedge dg^2 = \sum_{i=1}^N \int_{\Delta_i} f dg^1 \wedge dg^2,
\]
we get the desired result.
in view of (5.1), we get
\begin{align*}
\left| \sum_{i=1}^{N} \langle \Delta_i, \text{strat} \rangle - \int_{[pqr]} f dg^1 \wedge dg^2 - \sum_{j=1}^{M} \langle q_j, S^0 - S \rangle \right| \\
\leq C \|f\|_\alpha [\delta g^1]_{\beta_1}[\delta g^2]_{\beta_2} \sum_{i=1}^{N} \text{diam}(\Delta_i)^{\alpha + \beta_1 + \beta_2},
\end{align*}
where $q_j \in \text{Simp}^1(D)$ is the side of $Q_j$ which is not a side of any $\Delta_i$: in fact, when summing the terms
\[ \langle \Delta_i, \delta(S^0 - S) \rangle = \langle \partial \Delta_i, S^0 - S \rangle \]
over $i$, we have that every side of some simplex of the partition which is not one of $q_j$ (i.e. does not belong to a side of $[pqr]$) appears in this sum twice and in opposite directions, and hence is cancelled out from this sum. Moreover, from (4.5) applied with $q_j$ instead of $[pq]$ and $n := 0$ we get
\[ |\langle q_j, S^0 - S \rangle| \leq \|f\|_\infty [\delta g^1]_{\beta_1}[\delta g^2]_{\beta_2} \text{diam}(q_j)^{\beta_1 + \beta_2}, \]
which together with (5.3) gives (5.2) since $\text{diam} q_j = \text{diam} Q_j$. \qed

6. Integration over general domains

In section 4 we defined the integral of the “rough differential form” $f dg^1 \wedge dg^2$ over an arbitrary oriented simplex $[pqr]$ in the domain of definition of $f$ and $g$. Here we show how the latter can be naturally extended to more general domains $\Omega \subset \mathbb{R}^2$.

First, consider the case when $\Omega$ is an oriented simple (i.e. not self-intersecting) polygon with vertices $a_0, \ldots, a_k$, enumerated according to the orientation of $\Omega$ (say, counterclockwise). We will write in this case $\Omega = [a_0 \ldots a_k]$. Consider the triangulation of $\Omega$ in two-dimensional simplices $\{\Delta_i\}_{i=1}^{m}$ oriented in the same direction of $\Omega$. We set then by definition
\begin{align*}
\hat{\int}_{[a_0 \ldots a_k]} f dg^1 \wedge dg^2 := \sum_{i=1}^{m} \int_{\Delta_i} f dg^1 \wedge dg^2.
\end{align*}
The following statement is valid.

**Proposition 6.1.** Under conditions of Theorem 4.4 for every $b \in \mathbb{R}^2$ one has
\begin{align*}
\int_{[a_0 \ldots a_k]} f dg^1 \wedge dg^2 = \sum_{j=0}^{k} \int_{[a_0 \ldots a_{j+1}b]} f dg^1 \wedge dg^2,
\end{align*}
where $k+1 := 0$. In particular, the definition (6.1) is correct (i.e. independent on the particular triangulation $\{\Delta_i\}$), the above integral is alternating (i.e. preserves/resp. changes sign with odd/resp. even permutation of the vertices), nonatomic (i.e. zero on polygons of zero area), and the map
\begin{align*}
(f, g^1, g^2) \mapsto \int_{[a_0 \ldots a_k]} f dg^1 \wedge dg^2
\end{align*}
is a continuous multilinear functional over $C^0 \times C^{\beta_1} \times C^{\beta_2}$ continuous also in the vertices $a_0, \ldots, a_k$ (i.e. continuous with respect to the simultaneous convergence of both functions involved and of the vertices).
Proof. Writing $\Delta_i := [a_i^1 a_i^2 a_i^3]$, one has
\[
\sum_{i=1}^{m} \partial [b \alpha_i^1 \alpha_i^2 \alpha_i^3] = \sum_{i=1}^{m} [\alpha_i^1 \alpha_i^2 \alpha_i^3] - \sum_{i=1}^{m} [b \alpha_i^1 \alpha_i^3] + \sum_{i=1}^{m} [b \alpha_i^2 \alpha_i^3] - \sum_{i=1}^{m} [b \alpha_i^1 \alpha_i^2],
\]
so that taking into account (6.1), and recalling that
\[
\langle \partial | p q r s \rangle, f d g^1 \wedge d g^2 \rangle = 0,
\]
we get
\[
\int_{[a_0 \ldots a_k]} f d g^1 \wedge d g^2 = \sum_{i=1}^{m} \left( \int_{[b \alpha_i^1 \alpha_i^2]} f d g^1 \wedge d g^2 - \int_{[b \alpha_i^1 \alpha_i^2]} f d g^1 \wedge d g^2 + \int_{[b \alpha_i^1 \alpha_i^3]} f d g^1 \wedge d g^2 \right) = \sum_{i=1}^{m} \left( \int_{[\alpha_i^1 \alpha_i^2 \beta]} f d g^1 \wedge d g^2 + \int_{[\beta \alpha_i^2 \alpha_i^3]} f d g^1 \wedge d g^2 + \int_{[\alpha_i^2 \alpha_i^3 \beta]} f d g^1 \wedge d g^2 \right),
\]
the latter equality being due to the alternating property of the integral. Every one-dimensional edge $[p q]$ of the triangulation not belonging to the boundary of $\Omega$ belongs to exactly two simplices of the triangulation leading to two terms in the right-hand side of the latter equality, $\int_{[p q b]} f d g^1 \wedge d g^2$ and $\int_{[p q b]} f d g^1 \wedge d g^2$ which cancel out due to the alternating property of the integral. Therefore, the right-hand side of the latter equality contains only terms of the type $\int_{[p q b]} f d g^1 \wedge d g^2$ with $[p q]$ belonging to the boundary of $\Omega$; due to the additivity property of the integral they all sum up to the right-hand side of (6.2). The rest of the statement follows now immediately from (6.2) together with the respective properties of the integral over simplices.

If $\Omega$ is a finite union of disjoint simple oriented polygons $\Omega_1, \ldots, \Omega_l$ then it is natural to set
\[
\int_{\Omega} f d g^1 \wedge d g^2 := \sum_{i=1}^{l} \int_{\Omega_i} f d g^1 \wedge d g^2,
\]
so that the above integral clearly exists under the conditions of Theorem 4.4.

Finally, we able to define naturally the $\int_{\Omega} f d g^1 \wedge d g^2$ for quite general bounded open sets $\Omega \subset \mathbb{R}^2$ with a chosen orientation. To this aim for every $k \in \mathbb{N}$ let $P_k$ be the union of open squares with vertices in $2^{-k} \mathbb{Z}^2$ contained in $\Omega$. Clearly this is a bounded open set which is a finite union of simple polygons. We assume all $P_k$ to be oriented in the same way as $\Omega$. The following result holds true.

**Theorem 6.2.** Under conditions of Theorem 4.4, if additionally $\Omega \subset \mathbb{R}^2$ is a bounded open set satisfying
\[
\overline{\dim}_{\text{box}} \partial \Omega < \beta_1 + \beta_2,
\]
where $\overline{\dim}_{\text{box}}$ stands for the upper box-counting dimension, there is the limit
\[
\int_{\Omega} f d g^1 \wedge d g^2 := \lim_{k} \int_{P_k} f d g^1 \wedge d g^2.
\]
In this case the map
\[
(f, g^1, g^2) \mapsto \int_{\Omega} f d g^1 \wedge d g^2
\]
is a continuous multilinear functional over $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$.

Proof. Take a $d \in (\dim_{\text{box}} \partial \Omega, \beta_1 + \beta_2)$. The set $P_{k+m} \setminus P_k$ can be covered by triangles by dividing along the diagonal each of the squares of side length $2^{-(k+m)}$ with disjoint interiors composing it. The total number of such squares is estimated from above by the number of squares with vertices in $2^{-k} \mathbb{Z}^2$ touching $\partial \Omega$, hence by $C(2^k)^d$ where $C > 0$ depends only on $\partial \Omega$. Hence the number of triangles in the chosen cover of $P_{k+m} \setminus P_k$ is estimated by $2C(2^k)^d(2^m)^2$. Each triangle $\Delta$ in this cover has diameter $D := 2^{-(k+m)}$, and therefore by (4.10) together with (3.4) one has

$$\left| \int_{\Delta} f dg^1(x) \wedge dg^2(x) \right| \leq C'D^{\beta_1+\beta_2},$$

where $C' > 0$ depends only on $\|f\|_{\alpha}, |g^1|_{\beta_1}, |g^2|_{\beta_2}$. Thus

$$\left| \int_{P_{k+m}} f dg^1(x) \wedge dg^2(x) - \int_{P_k} f dg^1(x) \wedge dg^2(x) \right| \leq 2C(2^k)^d(2^m)^2C'^22^{-(k+m)(\beta_1+\beta_2)} \to 0 \quad \text{as } k \to +\infty$$

(even uniformly over bounded sets of $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$) because of the assumption $\beta_1 + \beta_2 > d$. This shows that the sequence of integrals $\{\int_{P_k} f dg^1 \wedge dg^2\}_k$ is Cauchy, and hence the existence of the limit as claimed. This limit is clearly multilinear on $(f, g^1, g^2)$ since so is the integral over simple polygons, and its continuity over $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$ follows from that of the integral over polygons and of the fact that the above convergence is uniform over bounded sets of $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$. \qed

Remark 6.3. Clearly under the condition (6.4) the integral $\int f dg^1 \wedge dg^2$ coincides with the classical one if $f$, $g^1$ and $g^2$ are smooth.

Remark 6.4. Combining Theorem 5.1 and Proposition 6.1, we have that the integral $\int f dg^1 \wedge dg^2$ in Theorem 6.2 may be also approximated directly by sums of either Stratonovich or Itô germs over sufficiently fine triangulations of $P_k$ (for sufficiently large $k$).

Remark 6.5. If in the construction used in Theorem 6.2 one substitutes the dyadic grids $2^{-k}\mathbb{Z}^2$ with some other ones (e.g. rotated and/or with side length of the cubes converging to zero with different speed), one would obtain under conditions of Theorem 6.2 in exactly the same way the existence of the limit in (6.5) (but now with different meaning of $P_k$), and its continuity and multilinearity over $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$. Since this limit for smooth $f$, $g^1$ and $g^2$ still coincides with the classical integral, we get therefore that it also coincides with $\int f dg^1 \wedge dg^2$ over the whole $C^\alpha \times C^{\beta_1} \times C^{\beta_2}$, and hence the role of the particular sequence of grids in the definition (6.5) is not essential.

7. STRATONOVICH TYPE INTEGRALS OF MORE IRREGULAR FORMS

We consider in this section the integrals of the type

$$\int f(x, g(x)) dg^1(x) \wedge dg^2(x)$$

defined for Hölder functions $g := (g^1, g^2) : \mathbb{R}^2 \to \mathbb{R}^2$ when $F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$. In fact, it happens that if one uses a Stratonovich-type construction, i.e. employs alternating germs $\text{strat}_{gr}$ defined for $f(x) := F(x, g(x))$, then the above integral may be defined
under much less restrictive requirements than those given by Theorem 4.4. In particular, we are able to trade regularity of \( g \) for the higher regularity of \( F \). Here we only limit ourselves to the case when the domain of integration \( \Omega \subset \mathbb{R}^2 \) is an oriented simplex (i.e. triangle \([pqr]\)), since the case of more general domains can be easily treated as in section 6.

**Theorem 7.1.** Let \( F: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R} \) such that

(i) \( u \mapsto F(u, \cdot) \in C(\mathbb{R}^2; C^{1,\gamma}(\mathbb{R}^2)), \gamma \in (0,1], \)

(ii) \( u \mapsto F(\cdot, u) \in C(\mathbb{R}^2; C^{\alpha}), \)

and let \( f(x) := F(x, g(x)) \), where \( g(x) := (g^1(x), g^2(x)) \). If \( \beta_1 + \beta_2 > 1 \) and

\[
\alpha + \beta_1 + \beta_2 > 2, \quad (1 + \gamma)\beta_i + \beta_1 + \beta_2 > 2, \quad i = 1, 2,
\]

then, with the notation of (4.1) the limit

\[
V_{pqr}(g) := \lim_{n \to \infty} \text{strat}^n_{pqr}
\]

exists. Moreover, it is continuous and alternating as a function of \([pqr]\) fixed \( g^1 \) and \( g^2 \), nonatomic in the sense that

\[
V_{pqr}(g) = 0 \quad \text{when} \quad ||pqr|| = 0,
\]

and continuous as the functional of \( g \), so that it is reasonable to denote

\[
\int_{[pqr]} F(x, g(x)) \, dg^1(x) \wedge dg^2(x) := V_{pqr}(g).
\]

**Remark 7.2.** It is worth observing that (7.1) implies \( \beta_i > 1/3, \quad i = 1, 2 \). In fact, assuming without loss of generality \( \beta_1 < \beta_2 \), we get from (7.1) \( (2 + \gamma)\beta_1 + \beta_2 > 2 \), and hence

\[
\beta_1 > \frac{2 - \beta_2}{2 + \gamma} \geq \frac{1}{3}.
\]

On the other hand, \( \beta_i > 1/2, \quad i = 1, 2 \), is clearly sufficient for the second inequality in (7.1) to hold. Note also that if \( \beta_1 = \beta_2 = \beta \), and \( F(x, y) := F(y) \) for every \((x, y) \in \mathbb{R}^2 \times \mathbb{R}^2\), then the first inequality of (7.1) is automatically satisfied since we may take \( \alpha \) to be arbitrarily close to 1, and therefore (7.1) is equivalent to \( \beta > 2/(3 + \gamma) \) (e.g. \( \beta > 1/2 \) when \( F \in C^{1,1} \)), which is far less restrictive than what is asserted in Theorem 4.4 (the latter requires in this case \( \beta > 2/3 \), since \( f \in C^{3} \)).

**Remark 7.3.** It follows from the proof that the limit germ

\[
V_{pqr} := \int_{[pqr]} F(x, g(x)) \, dg^1(x) \wedge dg^2(x)
\]

is continuous also with respect to \( F \) (with respect to a topology compatible with (i) and (ii)).

**Remark 7.4.** We notice that an analogous result is easy to obtain in the one-dimensional case. Namely, roughly speaking, if \( g \in C^3(\mathbb{R}) \) is Hölder continuous and \( F: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is \( C^{\alpha}(\mathbb{R}) \) in the first variable and \( C^{1,\gamma}(\mathbb{R}) \) in the second one, then the Stratonovich-type sums

\[
\sum_i \frac{1}{2} (F(x_i, g(x_i)) + F(x_{i+1}, g(x_{i+1}))) (\delta g)_{x_i, x_{i+1}}
\]
over a sequence of partitions \((x_i)_i\) of \([a, b]\) converge as \(\sup_i |x_{i+1} - x_i| \to 0\) when

\[
\alpha + \beta > 1 \text{ and } \beta(2 + \gamma) > 1.
\]

This can be deduced at once starting from the calculation

\[
\delta\theta_{pq} = \frac{1}{2} \det \begin{pmatrix} \delta f_{pq} & \delta f_{pr} \\ \delta g_{pq} & \delta g_{pr} \end{pmatrix}
\]

with \(f_p := F(p, g_p)\) and \(\theta_{pq} := \frac{1}{2} (f_p + f_q) \delta g_{pq}\). The assumptions on \(f\) give the Taylor expansion

\[
\delta f_{pq} = a_p \delta g_{pq} + O(|q - p|^{\alpha} + |q - p|^{\beta(1+\gamma)})
\]

so that a cancelation occurs in the determinant providing \(|\delta\theta_{pq}| = O(|q - p|^{\alpha + \beta} + |q - p|^{\beta(2 + \gamma)})\), which gives the possibility to apply the one-dimensional sewing lemma [4, lemma 2.1] if (7.2) holds. In particular, we notice that if \(\alpha = \gamma = 1\), then \(\beta > 1/3\) is allowed, which is well below the threshold of H"older exponents for the existence the Young integral (defined for \(\beta > 1/2\)). It is worth emphasizing that this is the peculiar feature of the Stratonovich integral, not of the Itˆ o one. In fact, if \(f\) with \(\delta f \) not of Itˆ o germs. This is the case for instance when \(g\) with 

\[
\text{Proof.}\] Let \(f_u(t) := F(u, x + t(y - x))\) for \(\{u, x, y\} \in \mathbb{R}^2\). Writing

\[
F(u, y) = f_u(1) = f_u(0) + \int_0^1 (f_u)'(s) \, ds = f_u(0) + (f_u)'(0) + \int_0^1 ((f_u)'(s) - (f_u)'(0)) \, ds
\]

\[
= F(u, x) + \nabla F(u, \cdot)(x) \cdot (y - x)
\]

\[
+ \int_0^1 (\nabla F(u, \cdot)(x + s(y - x)) - \nabla F(u, \cdot)(x)) \cdot (y - x) \, ds,
\]

we get with \(x := g_u, y := g_v\) the relationship

\[
(\delta F)_{uv} = (\delta F(\cdot, g_v))_{uv} + (\delta F(u, \cdot))_{g_u, g_v}
\]

\[
= (\delta F(\cdot, g_v)_{uv} + \delta g^2_{uv} \partial_1 F(u, \cdot)(g^1_u, g^2_u) + \delta g^2_{uv} \partial_2 F(u, \cdot)(g^1_u, g^2_u)) + R_{uv},
\]

where

\[
R_{uv} := \delta g^1_{uv} \int_0^1 \left( \partial_1 F(u, \cdot)(g^1_u + s\delta g^1_{uv}, g^2_u + s\delta g^2_{uv}) - \partial_1 F(u, \cdot)(g^1_u, g^2_u) \right) \, ds
\]

\[
+ \delta g^2_{uv} \int_0^1 \left( \partial_2 F(u, \cdot)(g^1_u + s\delta g^1_{uv}, g^2_u + s\delta g^2_{uv}) - \partial_2 F(u, \cdot)(g^1_u, g^2_u) \right) \, ds,
\]

so that

\[
|\delta F(\cdot, g_v)|_{uv} \leq C|v - u|^\alpha,
\]

\[
|R_{uv}| \leq C \left( |\delta g^1_{uv}| + |\delta g^2_{uv}| \right) \left( (\delta g^1_{uv})^2 + (\delta g^2_{uv})^2 \right)^{\gamma/2}
\]
for \((u,v)\) in a bounded set (the constant \(C > 0\) depending on this set). From Lemma 3.4 one gets therefore
\[
(\delta \text{strat})_{pqrs} = \frac{1}{6} \det \begin{pmatrix}
(\delta F(\cdot, g_q))_{pq} & (\delta F(\cdot, g_r))_{pr} & (\delta F(\cdot, g_s))_{ps} \\
\delta g^1_{pq} & \delta g^1_{pr} & \delta g^1_{ps} \\
\delta g^2_{pq} & \delta g^2_{pr} & \delta g^2_{ps}
\end{pmatrix} +
\]
(7.4)

and hence
\[
|\delta \text{strat}_{pqrs}| \leq C \left(\text{diam}(pqrs)^{\alpha + \beta_1 + \beta_2} + \text{diam}(pqrs)^{(1 + \gamma)(\beta_1 + \beta_2) + \beta_1 + \beta_2}\right)
\]
(7.5)

\[
\leq C \text{diam}(pqrs)^d
\]

with \(d := (\alpha \wedge (1 + \gamma)(\beta_1 + \beta_2)) + \beta_1 + \beta_2\) and \(C > 0\) depending continuously on \(F\) (with respect to the topology compatible with (i) and (ii)) and on \([\delta g^i]_{\beta_i}, i = 1, 2\).

Recalling (3.4) from Lemma 3.2, and that \text{strat} is alternating by the same Lemma, while \(d > 2\) because of (7.1), we have that Lemma A.1 applies with
\[
\gamma_1 := \beta_1 + \beta_2 > 1, \quad C_1 := \|f\|_{\infty} [\delta g^1]_{\beta_1} [\delta g^2]_{\beta_2},
\]
\[
\gamma_2 := d > 2, \quad C_2 := C,
\]
yielding the existence of continuous alternating germs
\[
S_{pq} := \lim_{n \to \infty} S_{pq}^n,
\]
\[
V_{pqr} := \lim_{n \to \infty} \text{strat}^n_{pqrs} = \lim_{n \to \infty} (\text{strat}^n_{pqrs} - \delta S^n_{pqrs}) + \delta S^n_{pqrs}.
\]

It remains now to prove that fixed \([pq]\), the map
\[
g \in C^{\beta_1} \times C^{\beta_2} \mapsto V_{pq}(g)
\]
is continuous. To this aim let \(\{g_k\} \subset C^{\beta_1} \times C^{\beta_2}\), converging to \(g\) pointwise as \(k \to \infty\), and \([\delta g^i_k]_{\beta_i} + [\delta g^i_k]_{\beta_2} < C < +\infty\) for all \(k \in \mathbb{N}\). Let \(f_k, R_k, S_k, \text{strat}^k, \text{strat}_k, V_k\) be the same as \(f, R, S^n, S, \text{strat}, V\) respectively but with \(g_k, g_k^2\) instead of \(g^1, g^2\). Clearly, as in (7.5) we have
\[
|\delta \text{strat}_{pqrs}| \leq C \text{diam}(pqrs)^d.
\]

The claim follows now by Lemma A.3 with \(\gamma_2 := d, \gamma_1 = \beta_1 + \beta_2\) (in fact, (A.2) is given by (7.6), and (A.1) is just (3.4) from Lemma 3.2).

Remark 7.5. One could strengthen the above Theorem 7.1 by proving the existence and continuity with respect to the data of a more general Stratonovich type integral
\[
\int_{[pq]} F(x, h(x)) dg^1(x) \wedge dg^2(x),
\]
where \(F\) is as in Theorem 7.1, \(\psi \in C^{2+\gamma}(\mathbb{R}^2; \mathbb{R}^2), \gamma \in (0, 1), h^i \in C^{\beta_i}(\mathbb{R}^2), g^i := \psi^i \circ h, i = 1, 2\) with \(h := (h^1, h^2)\), \(\psi := (\psi^1, \psi^2)\), \(\beta_i > 1/2, i = 1, 2\) and satisfy the first inequality of (7.1). In fact, letting \(f(x) := F(x, h(x))\), and using the notation of (4.1) we would have the existence of the limit
\[
\lim_{n \to \infty} \text{strat}^n_{pqrs} = \int_{[pq]} F(x, h(x)) dg^1(x) \wedge dg^2(x).
\]
To show this, we adapt the arguments of the proof of the above Theorem 7.1, changing (7.4) with (7.7)

\[
\delta_{\text{strat}}_{pqrs} = \frac{1}{6} \det \begin{pmatrix}
\frac{\delta F(\cdot, g_q)}{\delta q} & \frac{\delta F(\cdot, g_r)}{\delta r} & \frac{\delta F(\cdot, g_s)}{\delta s} \\
\frac{\delta g_q}{\delta g_q} & R_{pq} & R_{pr} \\
\frac{\delta g_r}{\delta g_r} & R_{pr} & R_{ps}
\end{pmatrix} + \frac{1}{6} \det \begin{pmatrix}
\frac{\nabla_h F(p, h_p) \cdot \delta h_p}{\delta h_p} & \frac{\nabla_h F(p, h_p) \cdot \delta h_p}{\delta h_p} & \frac{\nabla_h F(p, h_p) \cdot \delta h_p}{\delta h_p} \\
\frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p} \\
\frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p}
\end{pmatrix} + \frac{1}{6} \det \begin{pmatrix}
\frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p} \\
\frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p} \\
\frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p} & \frac{\nabla h_p \cdot \delta h_p}{\delta h_p}
\end{pmatrix},
\]

where

\[r^i_{uv} := \delta g^i_{uv} - (\nabla \psi)_{uv} \cdot \delta h_{uv}, \quad i = 1, 2.\]

Then the first two terms in (7.7) are estimated by \(C \text{diam}([pqrs])^{d_1}\) with \(d_1 > 2\) as in (7.5) because of (7.1) (the second inequality of which is automatically satisfied in view of Remark 7.2 due to the requirement \(\beta_i > 1/2, i = 1, 2\)), while the other two are estimated by \(C \text{diam}([pqrs])^{d_2}\) with \(d_2 := 4(\beta_1 \wedge \beta_2) > 2\), because

\[|r^i_{uv}| \leq C |u - v|^{2(\beta_i \wedge \beta_j)},\]

and thus \(|\delta_{\text{strat}}_{pqrs}| \leq C \text{diam}([pqrs])^d\), the constants in all the above estimates depending continuously with respect to the data. This allows to proceed as in the proof of Theorem 7.1 showing the existence and continuity with respect to the data of the above integral.

**Proposition 7.6** (chain rule). Let \(F\) be as in Theorem 7.1, \(\psi \in C^2(\gamma(\mathbb{R}^2; \mathbb{R}^2), \gamma \in (0, 1], h^i \in C^\beta(\mathbb{R}^2), \) and \(g^i := \psi^i \circ h, i = 1, 2, \) where \(h := (h^1, h^2), \psi := (\psi^1, \psi^2).\) If \(\beta_i > 1/2, i = 1, 2\) and the first inequality of (7.1) holds, then

\[
\int_{[pqr]} F(x, h(x)) \, dg^1(x) \wedge dg^2(x)
\]

(7.8)  
\[
= \int_{[pqr]} F(x, h(x)) \det D\psi(h^1(x), h^2(x)) \, dh^1(x) \wedge dh^2(x)
\]

Note that the integral on the right-hand side of (7.8) exists, is continuous and alternating as a function of \([pqr]\) fixed \(h^1\) and \(h^2\), and continuous as the functional of \(h^1, h^2\) by Theorem 7.1.

**Proof.** The equality (7.8) is true when \(g^i\) are smooth. The general case follows from continuity of the integrals on the left and right-hand sides of (7.8) with respect to the pointwise convergence of \(g^i, i = 1, 2\) with uniformly bounded Hölder constants. \(\square\)

We may give an interpretation of the above results in the spirit of theorem 3.2 from [1]. Namely, a smooth (say, \(C^1\)) function \(g = (g_1, g_2): [pqr] \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2\) can be naturally identified with the smooth surface representing its graph, and
therefore, with the De Rham 2-current $T_g$ over $[pqr] \times \mathbb{R}^2$ (endowed with orthogonal coordinates $(x, y) := (x^1, x^2, y^1, y^2)$) defined by

(7.9) \[ T_g(Fdx^1 \wedge dx^2) := \int_{[pqr]} F(x, g(x)) \, dx^1 \wedge dx^2, \]

(7.10) \[ T_g(Fdx^i \wedge dy^j) := \int_{[pqr]} F(x, g(x)) \, dx^i \wedge dy^j(x), \]

(7.11) \[ T_g(Fdy^1 \wedge dy^2) := \int_{[pqr]} F(x, g(x)) \, dy^1(x) \wedge dy^2(x), \]

for every $f \in C^2([pqr] \times \mathbb{R}^2)$.

**Proposition 7.7.** If $g^i \in C^{\beta_i}$, $i = 1, 2$, with

(7.12) \[ 3\beta_1 + \beta_2 > 2, \quad 3\beta_2 + \beta_1 > 2, \]

then the map $g \mapsto T_g$ between $C^1([pqr]; \mathbb{R}^2)$ and the space $D_2([pqr] \times \mathbb{R}^2)$ of 2-currents in $[pqr] \times \mathbb{R}^2$ endowed with its weak (pointwise) topology admits the unique continuous extension to the space $C^{\beta_1} \times C^{\beta_2}$ (the continuity being intended, as usual, with respect to pointwise convergence with uniformly bounded Hölder constants).

**Proof.** If $g^i \in C^{\beta_i}$, $i = 1, 2$, then the formulæ (7.9), (7.10) and (7.11) still make sense for an $F \in C^2([pqr] \times \mathbb{R}^2)$ if one interprets the integrals involved in the sense of Stratonovich. Namely, one defines the integral

(A) in (7.9), say, in the usual Riemann (or Lebesgue) sense (which in this case is equivalent to the Stratonovich integral),

(B) in (7.11) in the sense of Theorem 7.1 (with $\alpha := 1$, $\gamma := 1$), and

(C) in (7.10) again in the sense of Theorem 7.1 but with $x^i$ in place of $g^i$, $g^j$ in place of $g^2$, and $F$ in place of $F$, where $F$ is defined by

\[ F(x^1, x^2, y^1, y^2) := \begin{cases} F(x^1, x^2, g^1(x), y^2), & i = 1, j = 2, \\ F(x^1, x^2, y^1, g^2(x)), & i = 2, j = 1, \end{cases} \]

and with $\gamma := 1$, $\alpha := \beta_1$ and 1 in place of $\beta_1$ for the case $i = 1, j = 2$ or $\alpha := \beta_2$ and 1 in place of $\beta_2$ for the case $i = 2, j = 1$.

Note that (7.12) makes Theorem 7.1 to be applicable with such data.

Continuity of the map $g \mapsto T_g$ between $C^{\beta_1} \times C^{\beta_2}$ and the space of currents endowed with its weak (pointwise) topology is given by Theorem 7.1. The fact that it is the unique continuous extension of its restriction to $C^1 \times C^1$ follows from the density of $C^1$ in any Hölder space (with respect to the uniform convergence with bounded Hölder constants).

**Remark 7.8.** The proof of Proposition 7.7 shows that the formulæ (7.9), (7.10) and (7.11) still make sense for the current $T_g$ with $g \in C^{\beta_1} \times C^{\beta_2}$ when $F \in C^2([pqr] \times \mathbb{R}^2)$ (in fact, even for $F \in C^{1,1}$), if one interprets the integrals appearing there in the sense of Stratonovich, i.e. as in Theorem 7.1 (in particular, in (7.9) it may be interpreted as the usual Riemann or Lebesgue integral).

**Remark 7.9.** Theorem 3.2 from [1] says that the map $g \mapsto T_g$ defined by the formulæ (7.9), (7.10) and (7.11) between $C^1([pqr]; \mathbb{R}^2)$ and the space of currents endowed with its weak topology admits a unique continuous extension to the Sobolev space $W^{1,1}_{loc}([pqr]; \mathbb{R}^2)$ (even sequentially weakly continuous one). It is worth noting that the extended current may be then defined for continuous differential forms (i.e. with
Lemma A.1. sewing lemma.

and additive (although in fact this could be proven), as it is usually done in the properties. Note that we do not prove here that the respective germs are nonatomic necessarily the one defined by (A.3) with (A.5) (A.7)

follows by continuity and approximation.

We also notice that continuity of the right hand side in (7.13) gives the following quantitative continuity of degree of Hölder maps:

\[
\int_{\mathbb{R}^2} f(x) \left( \deg \left( (h^1, h^2), [pq], x \right) - \deg \left( (k^1, k^2), [pq], x \right) \right) \, dx \leq \|f\|_{1,\gamma} \|h - k\|_\beta
\]

APPENDIX A. EXISTENCE, UNIQUENESS AND STABILITY OF INTEGRALS

In this section we assume that \( \omega \) be an abstract 2-germ in \( D \subset \mathbb{R}^2 \) (i.e. not necessarily the one defined by (1.3) satisfying

(A.1) \(|\omega_{pq}\| \leq C_1 \text{diam}([pq])^{\gamma_1}, \)

(A.2) \(|(\delta \omega)_{pqrs}| \leq C_2 \text{diam}([pqrs])^{\gamma_2}, \)

with positive constants \( \gamma_1, \gamma_2, C_1, C_2 \) independent on \([pq]\) and \([pqrs]\). We define then \( \omega^n \) and \( S^n \) by

(A.3) \( \omega^n_{pq} := \langle \text{dya}^n[pq], \omega \rangle, \quad S^n_{pq} := \sum_{i=0}^{n-1} \langle \text{fill cut}'[pq], \omega \rangle. \)

We prove here the existence of limits \( \lim_n \omega^n \) and \( \lim_n S^n \) and their basic stability properties. Note that we do not prove here that the respective germs are nonatomic and additive (although in fact this could be proven), as it is usually done in the sewing lemma.

Lemma A.1. Under the conditions (A.1) and (A.2) if \( \omega \) is alternating, then

(A.4) \(|S^n_{pq} - S^n_{pq}| \leq C \text{diam}([pq])^{\gamma_1} 2^{n(1-\gamma_1)}, \)

(A.5) \(|([pq], (\omega^n - \delta S^n) - (\omega^{n+1} - \delta S^{n+1}))| \leq C \text{diam}([pq])^{\gamma_2} 2^{n(2-\gamma_2)} \)

with \( C > 0 \). In particular, if \( \gamma_1 > 1 \) and \( \gamma_2 > 2 \), then the germs

\( S^n_{pq} := \lim_{n \to \infty} S^n_{pq}, \)

\( V_{pq} := \lim_{n \to \infty} \omega^n_{pq} = \lim_{n \to \infty} (\omega^n_{pq} - \delta S^n_{pq}) + \delta S^n_{pq} \)

are well defined, continuous (if so is \( \omega \), alternating and

(A.6) \(|S^n_{pq} - S_{pq}| \leq C \text{diam}([pq])^{\gamma_1} 2^{n(1-\gamma_1)}, \)

(A.7) \(|\omega^n_{pq} - V_{pq} - \delta(S^n - S)_{pq}| \leq C \text{diam}([pq])^{\gamma_2} 2^{n(2-\gamma_2)}, \)

(A.8) \(|\omega^n_{pq} - V_{pq}| \leq C \text{diam}([pq])^{\gamma_1 + \gamma_2} 2^{n(1-\gamma_1 \wedge \gamma_2)} \).
Proof. For the readers’ convenience we organize the proof in several steps.

**Step 1.** To prove (A.5), observe that for some geometric map $\rho$: $\text{Simp}^2(D) \rightarrow \text{Chain}^3(D)$ one has

$$\omega^1_{p_0p_1p_2} - \omega^0_{p_0p_1p_2} = \langle \text{dyad}[p_0p_1p_2], \omega \rangle - \langle [p_0p_1p_2], \omega \rangle$$

(A.9)

$$= \langle \partial \rho([p_0p_1p_2]), \omega \rangle - \langle \text{fill}[p_0p_1], \omega \rangle + \langle \text{fill}[p_1p_2], \omega \rangle + \langle \text{fill}[p_2p_0], \omega \rangle$$

Moreover,

$$\rho([p_0p_1p_2]) = \sum_{i=1}^{4} Q_i, \quad Q_i \in \text{Simp}^3(D), \text{diam} Q_i \leq \text{diam}([p_0p_1p_2]), i = 0, \ldots, 2,$$

and therefore by (A.2) we have

$$|\langle \rho([p_0p_1p_2]), \delta \omega \rangle| \leq C \text{diam}([p_0p_1p_2])^{\gamma_2},$$

with $C := 4C_1$. Writing then $\text{dyad}[pqr] = \sum_{i=1}^{2^n} \Delta_i$ with $\Delta_i \in \text{Simp}^2(D)$ being dyadic simplices equal up to translations to to $2^{-n} [pqr]$, we get from (A.9)

$$\langle \Delta_i, \omega^1 \rangle - \langle \Delta_i, \omega^0 \rangle = \langle \rho(\Delta_i), \delta \omega \rangle + \langle \text{fill} \partial \Delta_i, \omega \rangle,$$

and summing the latter expressions over $i = 1, \ldots, 2^n$, we arrive at

$$\omega^1_{pqr} - \omega^0_{pqr} = \sum_{i=1}^{2^n} \langle \Delta_i, \omega^1 - \omega^0 \rangle$$

(A.11)

$$= \sum_{i=1}^{2^n} \langle \rho(\Delta_i), \delta \omega \rangle + \langle \text{cut}^n \partial [pqr], \omega \rangle,$$

since if $\Delta_i$ and $\Delta_j$ have a common couple of vertices, say, $p_0$ and $p_1$, then by alternating property of $\omega$ one has

$$\langle \text{fill} [p_0p_1], \omega \rangle = - \langle \text{fill} [p_1p_0], \omega \rangle,$$

i.e. the respective terms cancel out from the above sum, while the terms coming from the sides of dyadic simplices belonging to the boundary of $[pqr]$ remain, their sum giving rise to $\langle \text{cut}^n \partial [pqr], \omega \rangle$. Observing that

$$\langle \text{cut}^n \partial [pqr], \omega \rangle = \langle [pqr], \delta \text{S}^{n+1} - \delta \text{S}^n \rangle$$

and rewriting (A.11) with this help, we arrive at

$$\omega^1_{pqr} - \delta \text{S}^{n+1}_{pqr} - \omega^0_{pqr} - \delta \text{S}^n_{pqr}) = \sum_{i=1}^{2^n} \langle \rho(\Delta_i), \delta \omega \rangle.$$

Therefore,

$$|\langle \omega^1_{pqr} - \delta \text{S}^{n+1}_{pqr} \rangle - \langle \omega^0_{pqr} - \delta \text{S}^n_{pqr} \rangle | \leq \sum_{i=1}^{2^n} | \langle \rho(\Delta_i), \delta \omega \rangle |$$

$$\leq C \sum_{i=1}^{2^n} \text{diam}(\Delta_i)^{\gamma_2} \text{ by (A.10)}$$

$$\leq C 2^{2n} \left( \frac{\text{diam}([pqr])}{2^n} \right)^{\gamma_2}$$
as claimed.

Step 2. The estimate (A.4) follows with \( C := C_1 \) just observing that
\[
S_{pq}^{n+1} - S_{pq}^n = (\text{fill cut}^n[pq], \omega),
\]
while in view of (A.1) and of the definition of fill cut\(^n\) one has
\[
| (\text{fill cut}^n[pq], \omega) | \leq C_1 2^n \left( \frac{\text{diam}(pq)}{2^n} \right)^{\gamma_1}.
\]

Step 3. Existence of \( S \) and \( V \) follow now from (A.4) and (A.5) respectively. Since \( \omega \) is alternating, then so are \( \omega^n \) and \( S^n \), and therefore also \( V \) and \( S \). Now, the continuity of \( \omega \) implies that of \( S^n \) and \( \omega^n \) for each fixed \( n \in \mathbb{N} \), and hence the continuity of \( S \) and \( V \) follow from (A.6) and (A.8) respectively once they are proven. E.g. to prove continuity of \( S \), for \([pq] \subset D \) and \([rs] \subset D \) with \( D \) bounded, given an \( \varepsilon > 0 \), we choose an \( n \in \mathbb{N} \) such that \( C \text{ diam } D 2^n(1-\gamma_1) < \varepsilon / 3 \), so that
\[
|S_{pq} - S_{rs}| \leq |S_{pq} - S_p^q| + |S_{pq}^n - S_{rs}^n| + |S_{rs}^n - S_{rs}|
\leq 2\varepsilon / 3 + |S_{pq} - S_{rs}^n| \quad \text{by (A.6) and the choice of } \varepsilon,
\]
so that it is enough to find a \( \delta = \delta(n, \varepsilon) > 0 \) such that \( |S_{pq}^n - S_{rs}^n| < \varepsilon / 3 \) once \(|p-q| + |r-s| < \delta \) to get \( |S_{pq} - S_{rs}| < \varepsilon \). The proof of continuity of \( V \) is completely analogous (with the use of (A.8) instead of (A.6)).

Step 3. Finally, we prove (A.6), (A.7) and (A.8). The inequality (A.5) is proven by the chain of estimates
\[
|S_{pq}^n - S_{pq}| = \left| \sum_{k=n+1}^{\infty} (S_{pq}^k - S_{pq}^{k-1}) \right| \leq C \text{ diam}(pq)^{\gamma_1} \sum_{k=n+1}^{\infty} 2^{k(1-\gamma_1)} \quad \text{by (A.4)}
\leq C \frac{2^n(1-\gamma_1)}{1 - 2^{1-\gamma_1}} \text{ diam}(pq)^{\gamma_1}.
\]

Analogously, (A.7) follows from
\[
|\omega_{pq}^n - V_{pq} - \delta(S^n - S)_{pqr}| = |(\omega_{pq}^n - \delta S^n_{pq}) - (V_{pq} - \delta S_{pq})|
\leq \left| \sum_{k=n+1}^{\infty} \left( (\omega_{pq}^k - \delta S_{pq}^k) - (\omega_{pq}^{k-1} - \delta S_{pq}^{k-1}) \right) \right|
\leq C \text{ diam}(pq)^{\gamma_2} \sum_{k=n+1}^{\infty} 2^{k(2-\gamma_2)} \quad \text{by (A.5)}
\leq C \frac{2^n(2-\gamma_2)}{1 - 2^{2-\gamma_2}} \text{ diam}(pq)^{\gamma_2}.
\]

Finally, (A.6) gives
\[
|\delta(S^n - S)_{pq}| \leq C \text{ diam}(pq)^{\gamma_1} 2^n(1-\gamma_1),
\]
which together with (A.7) implies (A.8) for \( \text{diam}(pq) \) < 1 (which is enough since \( D \) is assumed bounded), thus concluding the proof. \( \square \)

As a result of Lemma A.1 we have that \( V \) and \( S \) satisfy
\[
|S_{pq}| \leq C \text{ diam}(pq)^{\gamma_1},
|\omega_{pq} - (V - \delta S)_{pq}| \leq C \text{ diam}(pq)^{\gamma_2}.
\]
In particular, if $\gamma_1 > 1$ and $\gamma_2 > 2$ this implies
\begin{align}
|S_{pq}| &\leq o(\text{diam}([pq])) \quad \text{as diam}([pq]) \to 0, \quad (A.13) \\
|\omega_{pqr} - (V - \delta S)_{pqr}| &\leq o(\text{diam}([pqr])^2) \quad \text{as diam}([pqr]) \to 0. \quad (A.14)
\end{align}
Moreover, since
\begin{align}
S_{pq} &= \sum_{i=0}^{\infty} \langle \text{fill cut}^i[pq], \omega \rangle, \\
\text{then one has}
(\delta S)_{prq} &= \omega_{prq} \quad \text{when } r = \frac{p+q}{2}. \quad (A.15)
\end{align}
Finally,
\begin{align}
\langle \text{dyad}[pqr], V \rangle &= \langle [pqr], V \rangle. \quad (A.16)
\end{align}

The following curious result, though not used elsewhere in this paper, gives the uniqueness of such a couple $(S, V)$ for a given $\omega$.

**Lemma A.2.** Given an $\omega \in \text{Germ}^2(D)$, the couple of germs $(S, V) \in \text{Germ}^1(D) \times \text{Germ}^2(D)$ satisfying (A.13), (A.14), (A.15) and (A.16) is unique.

**Proof.** Suppose that there are two couples $(S_i, V_i) \in \text{Germ}^1(D) \times \text{Germ}^2(D), i = 1, 2$ satisfying (A.13), (A.14), (A.15) and (A.16). Then for $S := S_1 - S_2$ and $V := V_1 - V_2$ we get
\begin{align}
|S_{pq}| &\leq o(\text{diam}([pq])) \quad \text{as diam}([pq]) \to 0, \quad (A.17) \\
|(V - \delta S)_{pqr}| &\leq o(\text{diam}([pqr])^2) \quad \text{as diam}([pqr]) \to 0, \quad (A.18) \\
(\delta S)_{prq} &= 0 \quad \text{when } r = \frac{p+q}{2}. \quad (A.19)
\end{align}
For each $n \in \mathbb{N}$ dividing dyadically the line segment $[pq]$ by consecutive points
\begin{align}
r_j := \left(1 - \frac{j}{2^n}\right)p + \frac{j}{2^n}q, \quad j = 0, \ldots, 2^n,
\end{align}
we get
\begin{align}
S_{pq} &= \sum_{j=0}^{2^n} S_{r_j, r_{j+1}} \quad (A.19)
\end{align}
by (A.19), and hence,
\begin{align}
|S_{pq}| &\leq \sum_{j=0}^{2^n} |S_{r_j, r_{j+1}}| \leq 2^n o\left(\frac{|pq|}{2^n}\right) = |pq| o(1) \quad (1)
\end{align}
as $n \to 0$, by (A.17), and taking the limit in the above inequality as $n \to \infty$, we get $S_{pq} = 0$. Then (A.18) is reduced to
\begin{align}
|V_{pqr}| &= o(\text{diam}([pqr])^2) \quad \text{as diam}([pqr]) \to 0. \quad (A.20)
\end{align}
Recalling that $\langle (\text{dyad})^n[pqr], V \rangle = \langle [pqr], V \rangle$ for every $n \in \mathbb{N}$ (because both $V_1$ and $V_2$ are assumed to satisfy (A.16)), we get using (A.20) the estimate
\begin{align}
|V_{pqr}| &= |\langle (\text{dyad})^n[pqr], V \rangle| = 2^{2n} o\left(\frac{\text{diam}([pqr])^2}{2^{2n}}\right) \\
&= \text{diam}([pqr])^2 o(1) \to 0
\end{align}
as $n \to \infty$. This implies $V = 0$ concluding the proof. \qed
Consider now a sequence of continuous alternating germs \( \{ \omega_k \} \subset \text{Germ}^2(D) \) satisfying
\[
|\omega_k|_{pq} \leq C_1 \text{diam}([pq])^\gamma_1,
\]
\[
|\delta \omega_k|_{pqrs} \leq C_2 \text{diam}([pqrs])^\gamma_2,
\]
with positive constants \( \gamma_1 > 1, \gamma_2 > 2, C_1, C_2 \) independent on \([pq], [pqrs]\) and \(k\).
\[
(\omega^n_k)_{pq} := \langle \text{dya}^n[pq], \omega_k \rangle, \quad (S^n_k)_{pq} := \sum_{i=0}^{n-1} \langle \text{fill cut}^i[pq], \omega_k \rangle.
\]

Lemma A.1 guarantees the existence for each \( k \in \mathbb{N} \) of continuous alternating germs
\[
(S^n_k)_{pq} := \lim_{n \to \infty} (S^n_k)_{pq},
\]
\[
(V^n_k)_{pq} := \lim_{n \to \infty} (\omega^n_k)_{pq} = \lim_{n \to \infty} ((\omega^n_k)_{pq} - \delta(S^n_k)_{pq}) + \delta(S^n_k)_{pq}.
\]

Suppose further that \( \omega_k \to \omega \) pointwise. Then clearly the latter satisfy (A.1) and thus Lemma A.1 provides the existence of continuous alternating germs
\[
S_{pq} := \lim_{n \to \infty} S^n_{pq},
\]
\[
V_{pq} := \lim_{n \to \infty} \omega^n_{pq} = \lim_{n \to \infty} (\omega^n_{pq} - \delta(S^n_{pq}) + \delta(S^n_{pq}),
\]
where \( \omega^n \) and \( S^n \) are defined by (A.3). The following stability statement is valid.

**Lemma A.3.** Under the above conditions one has \( S = \lim_k S_k \) and \( V = \lim_k V_k \) pointwise.

**Proof.** We note first that
\[
|S^n_k - S^n_{pq}| = |\langle \text{fill cut}^n[pq], \omega - \omega_k \rangle| \leq C_12^n \left( \frac{\text{diam}([pq])}{2^n} \right)^\gamma_1 \to 0
\]
as \( n \to \infty \) uniformly in \( k \), which implies \( S = \lim_k S_k \) pointwise via the standard estimate
\[
|S_k - S_{pq}| \leq |S_k - (S^n_k)_{pq}| + |(S^n_k)_{pq} - S^n_{pq}| + |S_{pq} - S^n_{pq}|.
\]

Writing
\[
(V_k - \delta S_k) - (V - \delta S) = -(\omega^n_k - V_k - \delta(S^n_k - S_k)) + (\omega^n - V - \delta(S^n - S)) - (\omega^n - \omega^n_k - \delta(S^n - S^n_k)),
\]
and evaluating the latter relationship at \([pq]\), using
\[
|\omega^n_k_{pq} - (V_k)_{pq} - \delta(S^n_k - S_k)_{pq}| \leq C2^{n(2-\gamma_2)},
\]
\[
|\omega^n_{pq} - V_{pq} - \delta(S^n - S)_{pq}| \leq C2^{n(2-\gamma_2)}
\]
with \( C > 0 \) independent of \( n \) and \( k \), we arrive at the estimate
\[
|V_k - \delta S_k|_{pq} - (V - \delta S)_{pq}| \leq 2C2^{n(2-\gamma_2)} + |\omega^n_{pq} - (\omega^n_k)_{pq} - \delta(S^n - S^n_k)_{pq}|
\]

Given an \( \varepsilon > 0 \) we fix an \( n = n(\varepsilon) \in \mathbb{N} \) such that the first term on the right-hand side of (A.24) does not exceed \( \varepsilon/2 \), and since \( \lim_k S^n_k = S^n \) and \( \lim_k \omega^n_k = \omega^n \) pointwise, we get that also the second term on the does not exceed \( \varepsilon/2 \) for all sufficiently large \( k \). This means
\[
V - \delta S = \lim_k (V_k - \delta S_k)
\]
pointwise and therefore $V = \lim_k V_k$ pointwise since $\lim_k S_k = S$, concluding the proof. \hfill \Box

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References

[1] G. Alberti and P. Majer. Gap phenomenon for some autonomous functionals. *J. Convex Anal.*, 1(1):31–45, 1994.
[2] H. Brezis and H.-M. Nguyen. The Jacobian determinant revisited. *Invent. Math.*, 185(1):17–54, 2011.
[3] K. Chouk and M. Gubinelli. Rough sheets. *ArXiv e-prints*, June 2014.
[4] D. Feyel and A. de La Pradelle. Curvilinear integrals along enriched paths. *Electron. J. Probab.*, 11:no. 34, 860–892, 2006.
[5] P. K. Friz and M. Hairer. *A course on rough paths*. Universitext. Springer, Cham, 2014.
[6] M. Gubinelli. Controlling rough paths. *J. Funct. Anal.*, 216(1):86–140, 2004.
[7] J. Harrison. Differential complexes and exterior calculus. *ArXiv e-prints*, January 2006.
[8] V. Kondurar. Sur l’intégrale de Stieltjes. *Rec. Math. [Mat. Sbornik] N.S.*, 2 (44):361–366, 1937.
[9] V. I. Macaev and M. Z. Solomjak. Existence conditions for the Stieltjes integral. *Mat. Sb. (N.S.)*, 88(130):522–535, 1972.
[10] W. Sickel and A. Youssfi. The characterisation of the regularity of the Jacobian determinant in the framework of potential spaces. *J. London Math. Soc. (2)*, 59(1):287–310, 1999.
[11] H. L. Smith. On the existence of the Stieltjes integral. *Trans. Amer. Math. Soc.*, 27(4):491–515, 1925.
[12] E. Stepanov and D. Trevisan. Towards geometric integration of rough differential forms. 2017. http://cvgmt.sns.it/paper/3671/.
[13] N. Towghi. Multidimensional extension of L. C. Young’s inequality. *JIPAM. J. Inequal. Pure Appl. Math.*, 3(2):Article 22, 13, 2002.
[14] F. M. Wright and J. D. Baker. On integration-by-parts for weighted integrals. *Proc. Amer. Math. Soc.*, 22:42–52, 1969.
[15] S. C. P. Yam. Analytical and topological aspects of signatures, 2008. Ph.D. Thesis, University of Oxford.
[16] P. Yaskov. On pathwise Riemann-Stieltjes integrals. *Statist. Probab. Lett.*, 150:101–107, 2019.
[17] L. C. Young. An inequality of the Hölder type, connected with Stieltjes integration. *Acta Math.*, 67(1):251–282, 1936.
[18] L. C. Young. General inequalities for Stieltjes integrals and the convergence of Fourier series. *Math. Ann.*, 115(1):581–612, 1938.
[19] R. Züst. Integration of Hölder forms and currents in snowflake spaces. *Calc. Var. Partial Differential Equations*, 40(1-2):99–124, 2011.
[20] R. Züst. Some results on maps that factor through a tree. *Anal. Geom. Metr. Spaces*, 3:73–92, 2015.
INTEGRATION OF NONSMOOTH 2-FORMS

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