Research Article
The Number of Blocks of a Graph with Given Minimum Degree

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A block of a graph is a nonseparable maximal subgraph of the graph. We denote by $b(G)$ the number of block of a graph $G$. We show that, for a connected graph $G$ of order $n$ with minimum degree $k \geq 1$, $b(G) \leq \lceil (2k - 3)/(k^2 - k - 1) \rceil n$. The bound is asymptotically tight. In addition, for a connected cubic graph $G$ of order $n \geq 14$, $b(G) \leq (n/2) - 2$. The bound is tight.

1. Introduction

We consider finite, undirected, simple graphs only. Let $G = (V(G), E(G))$ be a graph. The numbers of vertices and edges of $G$ are called the order and the size of $G$ and denoted by $v(G)$ and $e(G)$, respectively. A vertex $v$ is called a cut vertex if $\text{com}(G - v) > \text{com}(G)$, where $\text{com}(G)$ denotes the number of components of $G$. $c(G)$ denotes the number of cut vertices of $G$. Rao [1] proved that, for a connected graph $G$ of order $n$ and size $m$,

$$c(G) \leq \max \left\{ q : m \leq \left( \frac{n - q}{2} \right) + q \right\},$$  \hspace{1cm} (1)

characterized all extremal graphs. Rao and Rao [2] solved the corresponding problem for a strong digraph. Later, Achuthan and Rao [3] determined the maximum number of cut edges in a connected $d$-regular graph of order $p$.

Let $f(n, d) = \max \{ c(G) : G$ is a connected $k$-regular graph of order $n \}$. Rao [4] determined $f(n, d)$ for $d \leq 4$. Nirmala and Rao [5] showed that $f(n, d) = \lceil (2n - d - 5)/(d + 1) \rceil - 1$ or $\lceil (2n - d - 5)/(d + 1) \rceil - 2$ for odd $d \geq 5$ and have obtained an upper bound for $f(n, d)$ for even $d \geq 6$.

Alberten and Berman [6] proved that, for a graph $G$ of order $n$ and minimum degree $k \geq 2$,

$$c(G) \leq \frac{2k - 2}{k^2 - 2} n.$$  \hspace{1cm} (2)

This bound is asymptotically tight.

Hopkins and Staton [7] showed that every connected graph of order $n$ contains no more than $(r/(2r - 2))n$ cut vertices of degree $r$. Some related results are referred to [8, 9].

A separation of a connected graph is a decomposition of the graph into two nonempty connected subgraphs which have just one vertex in common. The common vertex is called a separating vertex of the graph. Since the graph $G$ under consideration is simple, $v \in V(G)$ is a separating vertex if and only if it is a cut vertex. A block of a graph is a nonseparable maximal subgraph of the graph. We denote by $b(G)$ the number of blocks of a graph $G$.

It is clear that any two blocks of a graph have at most one vertex in common. Recall that the block tree $B(G)$ of $G$ is the bipartite graph with bipartition $(\mathcal{B}, \mathcal{S})$, where $\mathcal{B}$ is the set of blocks of $G$ and $\mathcal{S}$, the set of separating vertices of $G$, and a block $B$, and a separating vertex $v$ is joined by an edge in $B(G)$ if and only if $B$ contains $v$. It is easy to see that if $G$ is connected, $B(G)$ is a tree. Each leaf of $B(G)$ corresponds to an end block of $G$.

Inspired from the bound for the cut vertices, in the present paper, we consider the upper bound for the number of blocks, a connected graph of order $n$ with given minimum degree. Let us begin with two easy cases when $\delta(G) = 1$ and $\delta(G) = 2$.

Proposition 1. For a connected graph $G$ of order $n \geq 2$, $b(G) \leq n - 1$, with equality if and only if $G$ is a tree.
Proof. Our proof is induction on $n$. If $n = 2$, then $G \equiv K_2$; thus, the result holds. Next, we assume that $n \geq 3$. If $G$ has no cut vertex, then $b(G) = 1 < n - 1$. Now suppose $G$ has a cut vertex. Let $B$ be an end block of $G$ and $v$ be the cut vertex, which belongs to $B$. Let $G' = G - (V(B), \{v\})$. Clearly, $G'$ is connected. By the induction hypothesis, $b(G') \leq v(G') - 1$. Since $b(G) = b(G') + 1$, $v(G') \leq v(G) - 1$, we have $b(G) \leq n - 1$, with equality only if $b(G') = v(G') - 1 = n - 2$ and $B \equiv K_2$. By the induction hypothesis, $G'$ is a tree, implying that $G$ is a tree.

On the contrary, if $G$ is a tree, clearly, $b(G) = n - 1$. □

Proposition 2. For a connected graph $G$ of order $n \geq 4$ with $\delta(G) \geq 2$, $b(G) \leq n - 3$, with equality if and only if $G$ is the graph obtained from $P_{n-4}$ identifying each end with a vertex of separate $K_3$, as given in Figure 1.

Proof. If $G$ has no cut vertex, the result holds trivially. Next, we assume that $G$ has cut vertices, and thus, it has at least two end blocks and $n \geq 5$. Let $B_1, \ldots, B_q$ be all end blocks of $G$. Let $c_i$ be the cut vertex of $G_i$, which belongs to $B_i$ for each $i \in \{1, \ldots, q\}$. Clearly, $v(B_i) \geq 3$ for any $i$. If $c_i = c_j$ for any two distinct $i, j$, then $b(G) = t$ and $n = \sum_{i=1}^q (v(B_i) - 1) + 1 \geq 2t + 1$. Therefore, $b(G) \leq (n - 1)/2 \leq n - 3$.

Otherwise, $G$ has at least two cut vertices. It follows that the order $n'$ of $G' = G - \cup_{i=1}^q V(B_i), \{c_i\}$ is at least two. Hence, $n \geq n' + 2t$ and $b(G) = b(G') + t$. By Proposition 1, $b(G') \leq n' - 1$. Summing up the above, we have

$$b(G) = b(G') + t \leq n' - 1 + t \leq n - t - 1 \leq n - 3. \quad (3)$$

From the above, $b(G) = n - 3$ if and only if $t = 2$ and $G' = P_{n-4}$, $B_1 \equiv K_3 \equiv B_2$, as we promised. □

It is clear that, for a graph $G$ of order $n$, $b(G)$ decreases when $\delta(G)$ increases. For a connected graph $G$ of order $n$ and minimum degree at least $k$, we have the following result, which is asymptotically best possible.

Theorem 1. For a connected graph $G$ of order $n$ with $\delta(G) \geq k$, $b(G) < (2k - 3)/(k^2 - k - 1)n$.

We show that the bound in the above theorem is asymptotically best possible. Let $k \geq 3$. Consider a tree $T$ of order $p$ with each vertex having degree $k$ or 1. By the handshaking lemma, the number of leaves of this tree is

$$\frac{(k-2)p + 2}{k-1}. \quad (4)$$

Let $G$ be the graph obtained from identifying each leaf of $T$ with a vertex of a clique of order $k + 1$ separately. Therefore,

$$v(G) = p + \frac{(k-2)p + 2}{k-1}k,$$

$$b(G) = p - 1 + \frac{(k-2)p + 2}{k-1}. \quad (5)$$

So, we have

$$b(G) = p - 1 + \frac{(k-2)p + 2}{k-1}.$$

\[\text{Figure 1: The extremal graph attaining the upper bound in Proposition 2.}\]

However, as $p$ gets larger, $b(G)/v(G)$ gets arbitrarily close to $(2k - 3)/(k^2 - k - 1)$.

What happens for the $k$-regular graphs? The situation becomes complicated. We are just able to get an exact bound for a cubic graph $G$ of order $n$: $b(G) < n/2$ (Theorem 2), whereas by Theorem 1, we have $b(G) < (3/5)n$ for a connected graph $G$ of order $n$ with $\delta(G) \geq 3$.

Theorem 2. For a connected cubic graph $G$ of order $n$,

$$b(G) \leq \begin{cases} 1, & \text{if } n \leq 8, \\ 3, & \text{if } n = 12, \\ \frac{n}{2} - 2, & \text{otherwise}. \end{cases} \quad (7)$$

The bound is sharp.

To see the sharpness of the bound, we denote by $K^*_4$ the graph obtained from $K_4$ by replacing an edge with a path of length two, as drawn in Figure 2.

The graphs $G_n$ achieve the upper bound in Theorem 2, which are classified into three types in terms of $n \equiv 4 \pmod{6}$, $n \equiv 0 \pmod{6}$, and $n \equiv 2 \pmod{6}$, respectively.

For an integer $n \equiv 4 \pmod{6}$, $k = (n-4)/3$ is an even integer. Let $T_k$ be a tree in which every vertex has degree 1 or 3. It is clear that $T_k$ has exactly $(k+1)/2$ vertices of degree 1 (leaves) and $(k-1)/2$ vertices of degree 3. Let $G_n$ be a graph obtained from identifying each leaf of $T_k$ with the vertex of degree two of a separate $K^*_4$, as shown in Figure 3.

For an integer $n \equiv 0 \pmod{6}$, let $G_n$ be a cubic graph obtained from a graph $G_{n-2}$ by replacing a vertex of degree three (not belongs to any $K^*_4$) with a triangle, as shown in Figure 3.

For an integer $n \equiv 2 \pmod{6}$, let $G_n$ be a cubic graph obtained from a graph $G_{n-2}$ by inserting a $K_4-e$ into an edge of $G_{n-4}$ (not belongs to any $K^*_4$), as shown in Figure 3. It can be checked that $v(G_n) = n$ and $b(G_n) = (n/2) - 2$ for any graph $G_n$ constructed as above.

2. The Proof of Theorem 1

Suppose the result is not true and let $G$ be a counterexample of minimum order $n$, i.e., $\delta(G) \geq k$ and $b(G) \geq (2k - 3)/(k^2 - k - 1)n$, but for any connected graph $G'$ of order $n' < n$ with $\delta(G') \geq k$, $b(G') < (2k - 3)/(k^2 - k - 1)n'$. 
If $k \in \{1, 2\}$, $((2k - 3)/(k^2 - k - 1))n = n$. By Propositions 1 and 2, $b(G) \leq n - 1 < n$. Hence, $k \geq 3$. Since $n \geq k + 1$, we have $((2k - 3)/(k^2 - k - 1))n \geq 1$, and thus, $G$ has at least two blocks.

**Claim 1.** Every end block of $G$ is a clique of order $k + 1$.

**Proof** of Claim 1. If it is not, let $B$ be an end block of $G$. Let $G'$ be the graph obtained from $G$ by replacing $B$ with $B'$ of order $k + 1$. Clearly, $b(G) = b(G')$ and $\delta(G') \geq k$. By the choice of $G$,

$$b(G') < \frac{2k - 3}{k^2 - k - 1} n'. \quad (8)$$

Combining the above facts, we conclude that $b(G) < ((2k - 3)/(k^2 - k - 1))n$, contradicting the choice of $G$. $\Box$

**Claim 2.** No cut vertex of $G$ belongs to at least two end blocks of $G$.

**Proof** of Claim 2. Let $B$ and $B'$ be two end blocks of $G$ containing the same cut vertex $v$ of $G$. Let $G' = G - (V(B') \setminus \{v\})$. By Claim 1, $v(B) = v(B') = k + 1$, and thus, $v(G') = n - k$ and $\delta(G') \geq k$. By the minimality of $G$,

$$b(G') < \frac{2k - 3}{k^2 - k - 1} (n - k). \quad (9)$$

Combining (9) with the fact that $b(G) = b(G') + 1$, we have a contradiction:

$$b(G) < \frac{2k - 3}{k^2 - k - 1} (n - k) + 1 < \frac{2k - 3}{k^2 - k - 1} n, \quad (10)$$

$\Box$

**Claim 3.** Let $c$ be a cut vertex lying on an end block $B_c$. If $B \neq B_c$ is a block containing $c$, then $B \equiv K_2$.

**Proof** of Claim 3. It suffices to show that $v(B) = 2$.

First suppose that $v(B) \geq k + 1$. Let $G'$ be the graph obtained from $G$, $(V(B_c), \{c\})$ and joining $c$ to every vertex in $V(B)$, $\{c\}$. Clearly, $G'$ is a connected graph with $\delta(G') \geq k$. Moreover, by Claim 1, $v(G') = n - k$. Again, by the minimality of $G$,

$$b(G') < \frac{2k - 3}{k^2 - k - 1} (n - k). \quad (11)$$

Combining (11) with the fact that $b(G) = b(G') + 1$, we have a contradiction.

$$b(G) < \frac{2k - 3}{k^2 - k - 1} (n - k) + 1 < \frac{2k - 3}{k^2 - k - 1} n, \quad (12)$$

$\Box$

Now assume that $v(B) \in \{3, \ldots, k\}$. Let $V(B) = \{v_1, \ldots, v_r\}$, where $v_1 = c$. Since $\delta(G) \geq k$ and $r \leq k$, each $v_i$ is a cut vertex of $G$. Let $G'$ be the graph obtained from $G$ by identifying all vertices in $\{v_2, \ldots, v_r\}$. Clearly, $\delta(G') \geq k$, $n' \leq n$, and $b(G') = b(G)$. By the choice of $G$, $b(G') < ((2k - 3)/(k^2 - k - 1))n'$. Thus,
We prove that $b(G) < \frac{2k-3}{k^2-k-1}n$, contradicting the choice of $G$. This proves the claim.

Take a longest path $P$ of $B(G)$. Let $B_1$ be an end block of $G$, which corresponds to a terminal vertex of $P = B_1c_1B_2c_2B_3c_3 \cdots$, where $c_1$ be the unique cut vertex of $G$ which belongs to $B_1$. By Claim 3, $B_2 \cong K_2$. Next, we consider three possible cases in terms of the order $v(B_3)$ of $B_3$. □

2.1. Case 1: $v(B_3) \geq k + 1$. Let $G'$ be the graph obtained from $G$, $V(B_3)$ and joining $c_2$ to each vertex of $V(B_3)$. It is clear that $\delta(G') \geq k$, and by Claim 1, $v(G') = n - k - 1$. By the minimality of $G$, $b(G') < ((2k-3)/(k^2 - k - 1))(n - k - 1)$. Since $b(G) = b(G') + 2$, we have

$$b(G) < \frac{2k-3}{k^2-k-1}(n-k-1) + 2 < \frac{2k-3}{k^2-k-1}n.$$ (14)

2.2. Case 2: $v(B_3) = 2$. By the choice of $P$, each block $B^* \neq B_3$ of $G$ containing $c_2$ is isomorphic to $K_2$. In addition, the end block containing the other end of $B^*$ is a leaf of $B(G)$. Since $\delta(c_2) \geq k$, there are $k - 1$ such end block $B_1^*, \ldots, B_{k-1}^*$, each of which are joined $c_2$, with an edge. Let $G'$ be the graph obtained from $G - \bigcup_{i=1}^{k-1} V(B_i^*)$ by identifying a vertex of a new clique of order $k + 1$ with $c_2$. It is clear that $\delta(G') \geq k$, $n' = v(G') = n - (k - 1)(k + 1) + k$, and $b(G') = b(G) - 2(k - 1) + 1$. So,

$$b(G) = b(G') + 2(k - 1) - 1 < \frac{2k-3}{k^2-k-1}n + 2(k - 1) - 1$$

$$= \frac{2k-3}{k^2-k-1}(n - (k - 1)(k + 1) + k) + 2(k - 1) - 1$$

$$= \frac{2k-3}{k^2-k-1}n$$ (15)

is a contradiction.

2.3. Case 3: $3 \leq v(B_3) \leq k$. Since $\delta(G) \geq k$, each vertex of $B_3$ is a cut vertex of $G$. We distinguish two subcases in terms of $v(B_3)$.

2.3.1. Case 3.1: $v(B_3) = k$. Since $\delta(G) \geq k$, every vertex $v \in V(B_3)$ has a neighbor not in $V(B_3)$, which belongs to distinct blocks of $G$. Let $G'$ be the graph obtained from $G$ by contracting $B_3$ to a vertex $v'$. It can be seen that $\delta(G') \geq k$, $b(G') = b(G') + 1$, and $n' = n - k + 1$. So,

$$b(G) = b(G') + 1 < \frac{2k-3}{k^2-k-1}(n - k + 1) + 1 < \frac{2k-3}{k^2-k-1}n.$$ (16)

2.3.2. Case 3.2: $3 \leq v(B_3) \leq k - 1$. Let $s = v(B_3)$ and $V(B_3) = \{u_1, \ldots, u_s\}$, where $u_1 = c_2$ and $u_s = c_3$. Since $\delta(G) \geq k$, for any $i \in \{1, \ldots, s - 1\}$, there are at least $k - s + 1$ blocks containing $u_i$, each of which is isomorphic to $K_2$, as illustrated in Figure 4.

Let $G'$ be the graph obtained from joining each component of $G - \{u_0, \ldots, u_{s-1}\}$ to $c_2$ or $c_3$ such that $d_{G'}(c_i) \geq k$ for each $j \in \{2, 3\}$. In addition, add an edge $c_2c_3$ if $c_2c_3 \notin E(G)$. Note that $G'$ is a connected graph of order $n' = n - s$ with $\delta(G') \geq k$ and $b(G') = b(G)$. By the minimality of $G$, $b(G') < (2k - 3)/(k^2 - k - 1)n'$. Therefore, $b(G) < (2k - 3)/(k^2 - k - 1)n$, contradicting the choice of $G$.

The proof of Theorem 1 is completed.

3. Proof of Theorem 2

Suppose the result is not true and let $G$ be a counterexample of minimum order $n$. The following fact is clear:

1. $G$ must contain cut vertex.

Since no cubic graph of order $\leq 8$ has a cut vertex, $n \geq 10$.

2. Moreover, $n \geq 14$. If $10 \leq n \leq 12$, it is not hard to check that $b(G) \leq 3$.

Claim 4. Every end block of $G$ is a $K^*_4$.
Proof of Claim 4. If it is not, let $B$ be an end block of $G$. Let $G'$ be the graph obtained from $G$ by replacing $B$ with $K_3^1$. Clearly, $G'$ is a connected cubic graph of order $n' < n$ and $b(G') = b(G)$. By the minimality of $G$,  
\begin{equation}
\frac{n'}{2} - 2.
\end{equation}

Combining the above facts, we conclude that $b(G) \leq \frac{n}{2} - 2$, contradicting the choice of $G$.

Take a longest path $P$ of $B(G)$. Let $B_1$ be an end block of $G$, which corresponds to a terminal vertex of $P = B_1c_1B_2c_2B_3c_3 \cdots$, where $c_1$ be the unique cut vertex of $G$ which belongs to $B_1$. Since $G$ is a cubic graph, $B_2 \equiv K_2$. Next we consider three possible cases.

3.1. Case 1: $v(B_2) = k \geq 4$. If $v \in V(B_2)$ is a cut vertex of $G$, then $v$ belongs to another block which is isomorphic to $K_2$. In addition, the end block containing the other end of the $K_2$ is a leaf of $B(G)$. We may assume $B_2$ has $r$ cut vertices except $c_1$, which belong to $B_1, B_2', \ldots, B_r'$, respectively, where $B'_1 = B_1$.

Let $G'$ be the graph obtained from $G - \cup_{i=1}^{r} V(B'_i) - (V(B'_1), [c_1])$ by identifying $c_1$ with the vertex of degree two of a new $K_2^1$. It is clear that $G'$ is a cubic graph of order $n = n' - 5r + k - 5$ and $b(G') = b(G) - 2r$. By the induction hypothesis, $b(G') \leq \frac{1}{2}(n' - 5r + k - 5 - 2)$. Thus,  
\begin{align*}
 b(G) &= b(G') + 2r \\
 &\leq \frac{1}{2}(n' - 5r + k - 5 - 2) + 2r \\
 &= \frac{n}{2} - \frac{5r + k - 5}{2} \\
 &\leq \frac{n}{2} - 2.
\end{align*}

3.2. Case 2: $v(B_2) = 3$. It follows that $B_1 \equiv K_3$. Every vertex of $B_2$ is a cut vertex. Let $G'$ be the graph obtained by the same operation as in the proof of Case 1. We have $n' = n - 8$ and $b(G') = b(G) - 4$. Therefore,  
\begin{align*}
 b(G) &= b(G') + 4 \leq \frac{1}{2}(n' - 8) - 2 + 4 = \frac{n}{2} - 2.
\end{align*}

3.3. Case 3: $v(B_2) = 2$. By the choice of $P$ and $v(B_2) = 2$, one can find another longest path $P' = B'_1c'_1B'_2c'_2B'_3c'_3 \cdots$ of $B(G)$. Let $G'$ be the graph obtained from identifying $c_1$ of $G - V(B_1) - V(B'_1)$ with the vertex of degree two of a new $K_3^1$. Note that $b(G') = b(G) - 3$ and $n' = n - 6$. By the induction hypothesis, $b(G') \leq \frac{n' - 6 - 2}{2} = \frac{1}{2}n - 2$. Therefore,  
\begin{align*}
 b(G) &= b(G') + 3 \leq \frac{1}{2}(n - 6 - 2) + 3 = \frac{1}{2}n - 2.
\end{align*}

The proof is completed.

4. Conclusions and Future Work

By arguing the properties of a minimum counterexample to the assertion of the main theorems and by using several kinds of graph transformation, we arrive at a contradiction, and thereby, we show our results. However, the upper bound for $b(G)$ remains open if $G$ is a $k$-regular graph with $k \geq 4$. One of the referees pointed out the possibility of the obtained results to some real-life applications and other fields (see [10–12] for instance).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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