ON THE ARGUMENT OF $L$-FUNCTIONS

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Abstract. For $L(\cdot, \pi)$ in a large class of $L$-functions, assuming the generalized Riemann hypothesis, we show an explicit bound for the function $S_1(t, \pi) = \frac{1}{\pi} \int_{1/2}^{\infty} \log |L(\sigma + it, \pi)| \, d\sigma$, expressed in terms of its analytic conductor. This enables us to give an alternative proof of the most recent (conditional) bound for $S(t, \pi) = \frac{1}{\pi} \arg L\left(\frac{1}{2} + it, \pi\right)$, which is the derivative of $S_1(\cdot, \pi)$ at $t$.

1. Introduction

Throughout this note, the notation $O(E)$ refers to a quantity whose absolute value is bounded by a universal constant times $E$. For any integrable function $h : \mathbb{R} \to \mathbb{C}$, its Fourier transform is defined as

$$\hat{h}(\xi) = \int_{-\infty}^{\infty} h(x) e^{-2\pi i x \xi} \, dx.$$  

We use the function $\Gamma_{\mathbb{R}}(z) = \pi^{-z/2} \Gamma\left(\frac{z}{2}\right)$, where $\Gamma$ is the meromorphic extension of $z \mapsto \int_0^{\infty} x^{z-1} e^{-x} \, dx$.

1.1. Background. Let $\zeta$ be the Riemann zeta-function and let $S(t) = \frac{1}{\pi} \arg \zeta\left(\frac{1}{2} + it\right)$, where the argument is obtained by continuous variation along the ray $\{s \in \mathbb{C} \mid \text{Re} \, s \geq \frac{1}{2} \text{ and } \text{Im} \, s = t\}$, starting from 0 at infinity. For $t \geq 1$, the number of zeros of $\zeta$ whose imaginary part is between 0 and $t$ is

$$N(t) = \frac{t}{2\pi} \log \frac{t}{2\pi} - \frac{t}{2\pi} + \frac{7}{8} + S(t) + O\left(\frac{1}{t}\right),$$

provided that $S(t)$ and $N(t)$ are defined in a consistent way when $t$ is the imaginary part of a zero of $\zeta$.

In his article [15], Littlewood considered the function

$$S_1(t) = \frac{1}{\pi} \int_{1/2}^{\infty} \log |\zeta(\sigma + it)| \, d\sigma$$

and observed that

$$\int_{t}^{u} S(v) \, dv = S_1(u) - S_1(t).$$

The proof of [15] Theorem 9] shows how it is possible to use $S_1$ to derive a bound for $S$ from a bound for $S_1$. The idea is that, since $N(t)$ is nondecreasing, $S(t)$ does not decrease faster than $-\frac{t}{2\pi} \log \frac{t}{2\pi} + \frac{t}{2\pi}$, therefore a large value of $S(t)$ would cause a large variation of $S_1$ near $t$. In the same article, he assumed the Riemann hypothesis to conclude that

$$S(t) = O\left(\frac{\log t}{\log \log t}\right).$$

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and
\[
S_1(t) = O\left(\frac{\log t}{(\log \log t)^2}\right)
\]
for \( t \geq 3 \).

The works \([9, 12, 16]\) went further by finding numerical bounds for
\[
\limsup_{t \to \infty} \left| S(t) \left(\frac{\log t}{\log \log t}\right)^{-1}\right|,
\]
while \([10, 14]\) exhibit numerical bounds for
\[
\limsup_{t \to \infty} \left| S_1(t) \left(\frac{\log t}{(\log \log t)^2}\right)^{-1}\right|.
\]
Ramachandra and Sankaranarayanan, in \([16]\), also remark that a bound of the same kind is true for Dirichlet L-functions, assuming the corresponding Riemann hypothesis. The article \([12]\) introduces the use of extremal functions of exponential type in this problem.

Currently, the best conditional bounds for \((1.2)\) and \((1.3)\) are due to Carneiro, Chandee and Milinovich. In the article \([4]\) they showed that, if the Riemann hypothesis holds,
\[
S_1(t) = \frac{1}{4\pi} \log t - \frac{1}{\pi} \sum_{\gamma} f_1(t - \gamma) + O(1),
\]
where the sum is over all \( \gamma \) such that \( \zeta\left(\frac{1}{2} + i\gamma\right) = 0 \) and
\[
f_1(x) = \frac{1}{2} \int_{1/2}^{3/2} \frac{1 + x^2}{\log \left(\sigma - \frac{1}{2}\right)^2 + x^2} \, d\sigma = 1 - x \arctan\left(\frac{1}{x}\right).
\]
Then, the tools of \([7]\) were used to find real entire minorants and majorants of exponential type for \( f_1 \), which allowed the use of the Guinand-Weil explicit formula. By this method they obtained \([4, \text{Theorem 1}]\)
\[
S_1(t) \leq \frac{\pi}{48} \frac{\log t}{(\log \log t)^2} + O\left(\frac{\log t \log \log \log t}{(\log \log t)^3}\right)
\]
and
\[
S_1(t) \geq -\frac{\pi}{24} \frac{\log t}{(\log \log t)^2} + O\left(\frac{\log t \log \log \log t}{(\log \log t)^3}\right).
\]
These inequalities bound any difference \( S_1(u) - S_1(t) = \int_u^t S(v) \, dv \). Also, \((1.3)\) may be used to compare this difference to \( S(t) \), and choosing appropriate values of \( u \) yields the inequality \([4, \text{Theorem 2}]\)
\[
|S(t)| \leq \frac{1}{4} \frac{\log t}{\log \log t} + O\left(\frac{\log t \log \log \log t}{(\log \log t)^2}\right)
\]
for sufficiently large \( t \). A shorter proof of \((1.7)\) was recently obtained in \([5, \text{Theorem 1}]\) using the classical Beurling-Selberg majorants and minorants of characteristic functions of intervals and exploiting the fact that \( \zeta \) is self-dual (i.e. \( \zeta(s) = \overline{\zeta(\overline{s})} \)).

1.2. L-functions. We work with a meromorphic function \( L(\cdot, \pi) \) on \( \mathbb{C} \) which meets the following requirements (for some positive integer \( m \) and some \( \theta \in [0, 1] \)). The examples include the Dirichlet L-functions \( L(\cdot, \chi) \) for primitive characters \( \chi \).

\[1\text{Our notation is motivated by L-functions arising from cuspidal automorphic representations } \pi \text{ of } GL(m) \text{ over a number field.} \]
(i) There exists a sequence \( \{\lambda_\pi(n)\}_{n \geq 1} \) of complex numbers \( \lambda_\pi(1) = 1 \) such that the series
\[
\sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}
\]
converges absolutely to \( L(s, \pi) \) on \( \{ s \in \mathbb{C} \mid \text{Re} s > 1 \} \).

(ii) For each prime number \( p \), there exist \( \alpha_{1,\pi}(p), \alpha_{2,\pi}(p), \ldots, \alpha_{m,\pi}(p) \) in \( \mathbb{C} \) such that \( |\alpha_{j,\pi}(p)| \leq p^0 \) and
\[
L(s, \pi) = \prod_p \prod_{j=1}^{m} \left( 1 - \frac{\alpha_{j,\pi}(p)}{p^s} \right)^{-1},
\]
with absolute convergence on the half plane \( \text{Re} s > 1 \).

(iii) For some positive integer \( N \) and some complex numbers \( \mu_1, \mu_2, \ldots, \mu_m \) whose real parts are greater than \(-1\) and such that \( \{\mu_1, \mu_2, \ldots, \mu_m\} = \{\overline{\mu_1}, \overline{\mu_2}, \ldots, \overline{\mu_m}\} \), the completed \( L \)-function
\[
\Lambda(s, \pi) = N^{s/2} \prod_{j=1}^{m} \Gamma_{\mathbb{R}}(s + \mu_j)L(s, \pi)
\]
is a meromorphic function of order 1 that has no poles other than 0 and 1. The points 0 and 1 are poles with the same order \( r(\pi) \in \{0, 1, \ldots, m\} \). Furthermore, the function \( \Lambda(s, \pi) = \kappa \Lambda(1 - s, \pi) \) satisfies the functional equation
\[
\Lambda(s, \pi) = \kappa \Lambda(1 - s, \pi)
\]
for some unitary complex number \( \kappa \).

Except for the assumption \( r(\pi) \leq m \), we are in the same framework as [13, Chapter 5], where many examples may be found.

### 1.3. Main results.

The theorems we prove are analogues of (1.5), (1.6) and (1.7) for \( L \)-functions. They are based on the **generalized Riemann hypothesis**, which asserts that \( \Lambda(s, \pi) \neq 0 \) if \( \text{Re} s \neq \frac{1}{2} \). The function
\[
C(t, \pi) = N \prod_{j=1}^{m} (|it + \mu_j| + 3),
\]
called the **analytic conductor** of \( L(\cdot, \pi) \), is used in their statements. Our first result is about the function
\[
S_1(t, \pi) = \frac{1}{\pi} \int_{1/2}^{\infty} \log |L(\sigma + it, \pi)| \, d\sigma.
\]

**Theorem 1.** Let \( L(\cdot, \pi) \) satisfy the generalized Riemann hypothesis. Then, for every real number \( t \),
\[
S_1(t, \pi) \leq \frac{(1 + 2\delta)^2 \pi}{48} \frac{\log C(t, \pi)}{(\log \log C(t, \pi)^{3/m})^2} + O \left( \frac{\log \log C(t, \pi) \log \log \log C(t, \pi)^{3/m}}{(\log \log C(t, \pi)^{3/m})^3} \right)
\]
and
\[
S_1(t, \pi) \geq -\frac{(1 + 2\delta)^2 \pi}{24} \frac{\log C(t, \pi)}{(\log \log C(t, \pi)^{3/m})^2} + O \left( \frac{\log C(t, \pi) \log \log \log C(t, \pi)^{3/m}}{(\log \log C(t, \pi)^{3/m})^3} \right).
\]

If \( t \) is not the imaginary part of a zero of \( L(\cdot, \pi) \) and \( t \neq 0 \), the argument function is defined by
\[
S(t, \pi) = -\frac{1}{\pi} \int_{1/2}^{\infty} \text{Im} \frac{L'}{L}(\sigma + it, \pi) \, d\sigma.
\]
Otherwise, it is
\[
S(t, \pi) = \lim_{\eta \to 0} \frac{S(t + \eta, \pi) + S(t - \eta, \pi)}{2}.
\]
We note that $S_1(t, \pi)$ is a primitive for the function $S(t, \pi)$ (details in Section 5 below). An extension of [1,7] to $L$-functions, with a good leading constant, was obtained by Carneiro, Chandee and Milinovich in [5, Theorem 5], via a direct approach using extremal majorants and minorants of exponential type for the odd function $f(x) = \arctan \left( \frac{x}{2} \right) - \frac{2x}{\pi x^2}$, available in the framework of [6]. Here we give an alternative proof of this result, deriving it from our Theorem 1.

**Theorem 2.** Let $L(\cdot, \pi)$ satisfy the generalized Riemann hypothesis. Then, for every real number $t$,

$$|S(t, \pi)| \leq \frac{1 + 2\theta}{4} \frac{\log C(t, \pi)}{\log \log C(t, \pi)^{3/m}} + O \left( \frac{\log C(t, \pi) \log \log C(t, \pi)^{3/m}}{(\log \log C(t, \pi)^{3/m})^2} \right).$$

The previous result gives information about the distribution of the zeros of $L$-functions. An example is the following corollary, related to [12, Corollary 1] and [5, Theorem 7].

**Corollary 3.** Let $L(\cdot, \pi)$ satisfy the generalized Riemann hypothesis.

(i) Let $m(\gamma, \pi)$ denote the multiplicity of the zero $\frac{1}{2} + i\gamma$ of $\Lambda(\cdot, \pi)$. Then,

$$m(\gamma, \pi) \leq \frac{1 + 2\theta}{2} \frac{\log C(\gamma, \pi)}{\log \log C(\gamma, \pi)^{3/m}} + O \left( \frac{\log C(\gamma, \pi) \log \log C(\gamma, \pi)^{3/m}}{(\log \log C(\gamma, \pi)^{3/m})^2} \right).$$

(ii) Let $\frac{1}{2} + i\gamma$ and $\frac{1}{2} + i\gamma'$ be consecutive zeros of $\Lambda(\cdot, \pi)$. Then $\gamma' - \gamma$ is bounded by some universal constant and if $C(\gamma, \pi)^{3/m}$ is sufficiently large,

$$\gamma' - \gamma \leq \frac{(1 + 2\theta)\pi}{\log \log C(\gamma, \pi)^{3/m}} + O \left( \frac{\log \log \log C(\gamma, \pi)^{3/m}}{(\log \log C(\gamma, \pi)^{3/m})^2} \right).$$

We make no attempt here to estimate the universal bound for the gap between consecutive zeros of our general class of $L$-functions. Such a gap has been estimated (for a slightly restricted class of $L$-functions) in [1, Theorem 2.1]. In addition to $S(\cdot, \pi)$ and $S_1(\cdot, \pi)$, the theory of extremal functions of exponential type can be used to provide upper bounds for the modulus of an $L$-function on the critical line. This has been carried out in the work of Chandee and Soundararajan [8]:

$$\log |L \left( \frac{1}{2} + it, \pi \right)| \leq \frac{(1 + 2\theta)\log 2}{2} \frac{\log C(t, \pi)}{\log \log C(t, \pi)^{3/m}} + O \left( \frac{\log C(t, \pi) \log \log C(t, \pi)^{3/m}}{(\log \log C(t, \pi)^{3/m})^2} \right).$$

Although they considered explicitly only the case $t = 0$, their reasoning is general. Other examples of the use of bandlimited majorants to the theory of the Riemann zeta-function include [2, 3, 11].

2. Proof of Theorem 1

In this section we prove Theorem 1. We adapt the strategy of [4], where the case of the Riemann zeta-function was considered.

**Lemma 4.** Let $L(\cdot, \pi)$ satisfy the generalized Riemann hypothesis. For any real $t$,

$$S_1(t, \pi) = \frac{1}{\pi} \left( - \sum_{\gamma} F_1(t - \gamma) + \log C(t, \pi) \right) + O(m),$$

where the sum is over all values of $\gamma$ such that $L \left( \frac{1}{2} + i\gamma, \pi \right) = 0$, counted with multiplicity, and

$$F_1(x) = \frac{1}{2} \int_{1/2}^{5/2} \log \frac{4 + x^2}{\sigma - 1 + x^2} d\sigma = 2 - x \arctan \left( \frac{2}{x} \right). \quad (2.1)$$
Proof. By the product expansion of \( L(\cdot, \pi) \) and the inequality \(|\alpha_{j, \pi}(p)| \leq p\),

\[
|\log |L(s, \pi)|| \leq m \log \zeta(\Re s - 1) = O\left(\frac{m}{\log s}\right)
\]

for any \( s \) such that \( \Re s \geq \frac{5}{2} \). Because of this and of the fact that \( L(\cdot, \pi) \) is meromorphic, \( S_1(\cdot, \pi) \) is well defined and

\[
\begin{align*}
\pi S_1(t, \pi) &= \int_{1/2}^{5/2} \log |L(\sigma + it, \pi)| \, d\sigma + O(m) \\
&= \int_{1/2}^{5/2} \{ \log |L(\sigma + it, \pi)| - \log |L\left(\frac{5}{2} + it, \pi\right)| \} \, d\sigma + O(m) \\
&= \int_{1/2}^{5/2} \{ \log |\Lambda(\sigma + it, \pi)| - \log |\Lambda\left(\frac{5}{2} + it, \pi\right)| \} \, d\sigma + \int_{1/2}^{5/2} \{ \log |N^{(5/2+it)/2}| - \log |N^{(\sigma+it)/2}| \} \, d\sigma \\
&+ \sum_{j=1}^{m} \int_{1/2}^{5/2} \{ \log |\Gamma_\Re\left(\frac{5}{2} + it + \mu_j\right)| - \log |\Gamma_\Re(\sigma + it + \mu_j)| \} \, d\sigma + O(m).
\end{align*}
\]

We treat each integral separately. For the first one, we use Hadamard’s factorization formula

\[
\Lambda(s, \pi) = s^{-r(\pi)}(s - 1)^{-r(\pi)}e^{A s + B s} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},
\]

where \( A \) and \( B \) are constants and the product is over all zeros of \( \Lambda(\cdot, \pi) \). From the functional equation \( \Lambda(s, \pi) = \kappa \Lambda(1 - s, \bar{\pi}) \), one deduces that \( \Re B = -\sum_{\rho} \Re \left(\frac{1}{\pi}\right) \) (see [13, Proposition 5.7]). Hence, for \( \frac{1}{2} \leq \sigma \leq \frac{5}{2} \),

\[
\left| \frac{\Lambda(\sigma + it, \pi)}{\Lambda\left(\frac{5}{2} + it, \pi\right)} \right| = \left| \frac{\sigma + it}{\frac{5}{2} + it} \right|^{-r(\pi)} \left| \frac{\sigma - 1 + it}{\frac{5}{2} + it} \right|^{-r(\pi)} \prod_{\rho} \left| \frac{\sigma + it - \rho}{\frac{5}{2} + it - \rho} \right|,
\]

which implies, via the substitution \( \rho = \frac{1}{2} + i\gamma \), that

\[
\log \left| \frac{\Lambda(\sigma + it, \pi)}{\Lambda\left(\frac{5}{2} + it, \pi\right)} \right| = O(m + r(\pi) \log \left| \frac{\frac{5}{2} + it}{\sigma - 1 + it} \right| + \sum_{\gamma} \frac{1}{2} \log \frac{(\sigma - \frac{1}{2})^2 + (t - \gamma)^2}{4 + (t - \gamma)^2}.
\]

Since \( 1 \leq \left| \frac{\frac{5}{2} + it}{\sigma - 1 + it} \right| \leq \frac{5}{\sigma - 1} \), integrating we get

\[
\int_{1/2}^{5/2} \{ \log |\Lambda(\sigma + it, \pi)| - \log |\Lambda\left(\frac{5}{2} + it, \pi\right)| \} \, d\sigma = - \sum_{\gamma} F_1(t - \gamma) + O(m).
\]

Our considerations on the last \( m \) integrals use Stirling’s formula

\[
\frac{\Gamma'(z)}{\Gamma(z)} = \log(1 + z) - \frac{1}{z} + O(1)
\]

in the form

\[
\frac{\Gamma'_R(z)}{\Gamma_R(z)} = \frac{1}{2} \log(2 + z) - \frac{1}{z} + O(1),
\]

valid for \( \Re z > -\frac{1}{2} \). For any \( \mu \) such that \( \Re \mu > 0 \), integration by parts yields

\[
\begin{align*}
\int_{1/2}^{5/2} \{ \log |\Gamma_\Re\left(\frac{5}{2} + \mu + it\right)| - \log |\Gamma_\Re(\sigma + \mu + it)| \} \, d\sigma \\
&= \int_{1/2}^{5/2} (\sigma - \frac{1}{2}) \Re \frac{\Gamma'_R}{\Gamma_R}(\sigma + \mu + it) \, d\sigma
\end{align*}
\]
\[
\frac{1}{2} \int_{1/2}^{5/2} (\sigma - \frac{1}{2}) \left( \frac{1}{2} \log |2 + \sigma + \mu + 3it| - \Re \frac{1}{\sigma + \mu + 3it} \right) d\sigma + O(1)
\]
\[
= \int_{1/2}^{5/2} (\sigma - \frac{1}{2}) \log |2 + \sigma + \mu + 3it| d\sigma + O(1)
\]
\[
= \int_{1/2}^{5/2} (\sigma - \frac{1}{2}) \log(|\mu + 3it| + 3) d\sigma + O(1)
\]
\[
= \log(|\mu + 3it| + 3) + O(1).
\]

If \(-1 < \Re \mu \leq 0\), the relation \(\Gamma_R(z + 2) = \frac{2}{\pi} \Gamma_R(z)\) brings us back to the previous case. Indeed,
\[
\int_{1/2}^{5/2} \log \left\{ \Gamma_R \left( \frac{5}{2} + \mu + 3it \right) \right\} - \log \left\{ \Gamma_R \left( \sigma + \mu + 3it \right) \right\} d\sigma
\]
\[
= \int_{1/2}^{5/2} \left\{ \log \left[ \Gamma_R \left( \frac{5}{2} + \mu + 3it \right) \right] - \log \left[ \Gamma_R \left( 2 + \sigma + \mu + 3it \right) \right] - \log \left[ \frac{5}{2} + \mu + 3it \right] \right\} d\sigma
\]
\[
= \log(|\mu + 3it| + 3) + O(1),
\]
as before.

Finally, \(\int_{1/2}^{5/2} \log |N^{(5/2+3it)/2}| - \log |N^{(\sigma+3it)/2}| d\sigma = \log N\). Combining our computations we get
\[
\pi S_1(t, \pi) = - \sum_{\gamma} F_1(t - \gamma) + \log N + \sum_{j=1}^{m} \log(|\mu_j + 3it| + 3) + O(m)
\]
\[
= \sum_{\gamma} F_1(t - \gamma) + \log C(t, \pi) + O(m).
\]

To estimate the infinite sum that appears in the preceding lemma, we employ the Guinand-Weil explicit formula. Its statement depends on noting that, by the product expansion of \(L(\cdot, \pi)\),
\[
\frac{L'(s, \pi)}{L(s, \pi)} = - \sum_{p} \sum_{j=1}^{m} \alpha_{j, \pi}(p) \frac{1}{p^s} \left( 1 - \frac{\alpha_{j, \pi}(p)}{p^s} \right)^{-1} \log p,
\]
where the right-hand side converges absolutely if \(\Re s > 1\). This shows that the logarithmic derivative of \(L(\cdot, \pi)\) has a Dirichlet series:
\[
\frac{L'(s, \pi)}{L(s, \pi)} = - \sum_{n=2}^{\infty} \frac{\Lambda(\pi)(n)}{n^s},
\]
where \(\Lambda(\pi)(n) = 0\) if \(n\) is not a power of prime and \(\Lambda(\pi)(p^k) = \sum_{j=1}^{m} \alpha_{j, \pi}(p^k) \log p\) if \(p\) is prime and \(k\) is a positive integer.
Lemma 5. Let \( h \) be an analytic function defined on a strip \( \{ z \in \mathbb{C} \mid \frac{1}{2} - \varepsilon < \text{Im} \, z < \frac{1}{2} + \varepsilon \} \) such that 
\[ h(z)(1 + |z|)^{1+\delta} \text{ is bounded for some positive } \delta. \] Then
\[
\sum_{\rho} h\left( \frac{\rho - \frac{1}{i}}{i} \right) = r(\pi) \left\{ h\left( \frac{1}{2i} \right) + h\left( -\frac{1}{2i} \right) \right\} + \frac{\log N}{2\pi} \int_{-\infty}^{\infty} h(u) \, du
\]
\[
+ \frac{1}{\pi} \sum_{j=1}^{m} \int_{-\infty}^{\infty} h(u) \Re \left( \frac{\Gamma_{\mathbb{R}}}{\Gamma} \left( \frac{1}{2} + \mu_j + iu \right) \right) \, du
\]
\[
- \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \left\{ \Lambda_{\pi}(n) \hat{h}\left( \frac{\log n}{2\pi} \right) + \overline{\Lambda_{\pi}(n)} \hat{h}\left( -\frac{\log n}{2\pi} \right) \right\}
\]
\[
- \sum_{-1 < \Re \mu_j < \frac{1}{2}} \left\{ \hat{h}\left( -\frac{\mu_j - \frac{1}{2}}{i} \right) + \hat{h}\left( \frac{\mu_j + \frac{1}{2}}{i} \right) \right\} - \frac{1}{2} \sum_{\Re \mu_j = \frac{1}{2}} \left\{ \hat{h}\left( -\frac{\mu_j - \frac{1}{2}}{i} \right) + \hat{h}\left( \frac{\mu_j + \frac{1}{2}}{i} \right) \right\},
\]
\[(2.4)\]

where the sum runs over all zeros \( \rho \) of \( \Lambda(\cdot, \pi) \) and the coefficients \( \Lambda_{\pi}(n) \) are defined by \eqref{2.3}.

Proof. This is a modification of the proof of \cite[Theorem 5.12]{13}. The idea is to consider the integral
\[
\frac{1}{2\pi i} \int h\left( \frac{s - \frac{1}{i}}{i} \right) \frac{\Lambda'(s, \pi)}{\Lambda(s, \pi)} \, ds
\]
over the rectangular contour connecting the points \( 1 + \eta + iT_1, -\eta + iT_1, -\eta - iT_2, 1 + \eta - iT_2 \), say with \( \eta = \varepsilon/2 \). Then one sends \( T_1, T_2 \to \infty \) over an appropriate sequence of heights that keep the zeros as far as possible (recall that at height \( T \), we have \( O(\log C(t, \pi)) \) zeros, see \cite[Proposition 5.7]{13}). One then uses the functional equation \eqref{2.3} to replace the integral over the line \( \Re s = -\eta \) by an integral over the line \( \Re s = 1 + \eta \), and finally one moves the remaining integrals to the line \( \Re s = \frac{1}{2} \), picking up possibly some additional poles at the \( \mu_j \)'s.

The function \( F_1 \) defined in \eqref{2.1} is not analytic. However, if we take \( h \) equal to an analytic minorant or majorant of \( F_1 \), we obtain lower and upper bounds for \( \sum \gamma F_1(t - \gamma) \). To estimate the right-hand side of \eqref{2.4}, it is convenient to choose \( \hat{h} \) compactly supported and to minimize the \( L^1 \)-norm of \( h - F_1 \). Carneiro, Littmann and Vaaler studied this minimization problem in a more abstract setting (see \cite{2}) and it was shown in \cite{2} that their result could be applied to the function \( f_1 \) defined by \eqref{2.1}. The following lemma is a rescaling of the obtained conclusion.

Lemma 6. For every \( \Delta \geq 1 \), there is a unique pair of real entire functions \( G^-_\Delta : \mathbb{C} \to \mathbb{C} \) and \( G^+_\Delta : \mathbb{C} \to \mathbb{C} \) satisfying the following properties:

(i) For real \( x \) we have
\[
\frac{-c}{1 + x^2} \leq G^-_\Delta(x) \leq F_1(x) \leq G^+_\Delta(x) \leq \frac{c}{1 + x^2},
\]
for some positive constant \( c \). Moreover, for any complex number \( z \) we have
\[
|G^+_\Delta(z)| = O\left( \frac{\Delta^2}{1 + \Delta |z|} e^{2\pi \Delta |\text{Im} \, z|} \right).
\]

(ii) The Fourier transforms of \( G^\pm_\Delta \) are continuous functions supported on the interval \( [-\Delta, \Delta] \) and satisfy
\[
|\hat{G}_\Delta^\pm(\xi)| = O(1)
\]
for all \( \xi \in [-\Delta, \Delta] \).
(iii) The $L^1$-distances of $G^+_{\Delta}$ to $F_1$ are given by
\[
\int_{-\infty}^{\infty} \{ F_1(x) - G^+_{\Delta}(x) \} \, dx = \frac{2}{\Delta} \int_{1/2}^{3/2} \left\{ \log \left( 1 + e^{-4\pi\Delta (\sigma - 1/2)} \right) - \log \left( 1 + e^{-4\pi\Delta} \right) \right\} \, d\sigma
\]
and
\[
\int_{-\infty}^{\infty} \{ G^+_{\Delta}(x) - F_1(x) \} \, dx = -\frac{2}{\Delta} \int_{1/2}^{3/2} \left\{ \log \left( 1 - e^{-4\pi\Delta (\sigma - 1/2)} \right) - \log \left( 1 - e^{-4\pi\Delta} \right) \right\} \, d\sigma.
\]

Proof. This is a slightly different version of [4, Lemma 4]. The definitions (1.4) and (2.1) imply that $G(\varepsilon) > \log \left( \frac{1}{1 + \varepsilon} \right)$ because
\[
\log(1 + \varepsilon) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \varepsilon^n - 1 \quad \text{for any } \varepsilon \in (0, 1) \text{ and the convergence is uniform.}
\]
Similarly,
\[
\int_{-\infty}^{\infty} \{ G^+_{\Delta}(x) - F_1(x) \} \, dx = \frac{1}{2\pi\Delta^2} \int_0^{4\pi\Delta} \left\{ \log \left( 1 - e^{-\varepsilon} \right) - \log \left( 1 - e^{-4\pi\Delta} \right) \right\} \, dx
\]
\[
\leq \frac{1}{2\pi\Delta^2} \int_0^{4\pi\Delta} \log \left( 1 - e^{-\varepsilon} \right) \, dx
\]
\[
= \frac{1}{2\pi\Delta^2} \int_0^{1} \log \left( 1 - \frac{y}{y} \right) \, dy
\]
\[
= \frac{\pi}{12\Delta^2},
\]
because $\frac{\log(1+y)}{y} - \sum_{n=1}^{\infty} \frac{1}{n} y^{n-1}$ for any $y \in (0, 1)$ and the convergence is monotone.

Observe that the $L^1$-distances given in Lemma 8 (iii) are of magnitude $1/\Delta^2$. Indeed,
\[
\int_{-\infty}^{\infty} \{ F_1(x) - G^+_{\Delta}(x) \} \, dx = \frac{1}{2\pi\Delta^2} \int_0^{4\pi\Delta} \left\{ \log \left( 1 + e^{-z} \right) - \log \left( 1 + e^{-4\pi\Delta} \right) \right\} \, dx
\]
\[
\leq \frac{1}{2\pi\Delta^2} \int_0^{4\pi\Delta} \log \left( 1 + e^{-z} \right) \, dx
\]
\[
= \frac{1}{2\pi\Delta^2} \int_0^{1} \log \left( 1 + \frac{y}{y} \right) \, dy
\]
\[
= \frac{\pi}{12\Delta^2},
\]
because $\frac{\log(1+y)}{y} - \sum_{n=1}^{\infty} \frac{1}{n} y^{n-1}$ for any $y \in (0, 1)$ and the convergence is monotone.

We are now ready to prove Theorem 1. The strategy is to apply Lemma 5 to the functions $G^+_{\Delta}(t - \cdot)$ and $G^-_{\Delta}(t - \cdot)$, to find bounds for $S_1(t, \pi)$ that depend on $\Delta$ and to optimize the choice of $\Delta$.

Proof of Theorem 1. We first prove the upper bound. For each $\Delta \geq 1$, take $G^-_{\Delta}$ as in Lemma 8 and let $h(z) = G^-_{\Delta}(t - z)$. By Lemma 8,
\[
S_1(t, \pi) \leq \frac{1}{\pi} \left( -\sum_{\gamma} h(\gamma) + \log C(t, \pi) \right) + O(m).
\]
By Lemma 8 (i), the function $G^-_{\Delta}(z)(1 + z^2)$ is bounded on the real line and $G^-_{\Delta}(z) = O(\Delta^2 e^{2\pi\Delta |\text{Im}z|})$. An application of the Phragmén-Lindelöf principle for the function $G^-_{\Delta}(z)(1 + z^2)e^{2\pi\Delta |z|}$ tells us that this function is bounded on the upper half plane. Hence $z \mapsto G^-_{\Delta}(z)(1 + z^2)$ is bounded on the strip $0 \leq \text{Im} z \leq \frac{1}{2} + \varepsilon$ (for any $\varepsilon > 0$), and since it is real entire, it is bounded on the strip $-\frac{1}{2} - \varepsilon \leq \text{Im} z \leq \frac{1}{2} + \varepsilon$. Therefore, $h$
Inserting this in (2.7) and using (2.5) together with the fact that

\[
\sum_{\gamma} h(\gamma) = \frac{\log N}{2\pi} \int_{-\infty}^{\infty} h(u) \, du + \frac{1}{\pi} \sum_{j=1}^{m} \int_{-\infty}^{\infty} h(u) \Re \left( \frac{\Gamma'(u)}{\Gamma(u)} \left( \frac{1}{2} + \mu_j + iu \right) \right) \, du
\]

\[
- \frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \left\{ \Lambda_\pi(n) \hat{h} \left( \frac{\log n}{2\pi} \right) + \Lambda_{\pi^{-1}}(n) \hat{h} \left( -\frac{\log n}{2\pi} \right) \right\}
\]

\[+ O(m\Delta^2 e^{\pi\Delta}).\]

For each index \( j = 1, 2, \ldots, m \), Stirling’s formula \([2.2]\) yields

\[
\int_{-\infty}^{\infty} h(u) \Re \left( \frac{\Gamma'(u)}{\Gamma(u)} \left( \frac{1}{2} + \mu_j + iu \right) \right) \, du = \frac{1}{2} \int_{-\infty}^{\infty} G_\Delta(t-u) \log \left| \frac{1}{2} + \mu_j + iu \right| \, du
\]

\[
- \frac{1}{2} \int_{-\infty}^{\infty} G_\Delta(t-u) \Re \left( \frac{1}{\mu_j + \frac{1}{2} + iu} \right) \, du + O(1).
\]

Combining this with the inequality

\[
\left| \int_{-\infty}^{\infty} G_\Delta(t-u) \Re \left( \frac{1}{\mu_j + \frac{1}{2} + iu} \right) \, du \right| \leq c \int_{-\infty}^{\infty} \left| \Re \left( \frac{1}{\mu_j + \frac{1}{2} + iu} \right) \right| \, du
\]

\[
= c \int_{-\infty}^{\infty} \left| \Re \left( \frac{1}{\mu_j + \frac{1}{2}} \right) \right| \, du
\]

\[
\leq \pi c
\]

we find that

\[
\int_{-\infty}^{\infty} h(u) \Re \left( \frac{\Gamma'(u)}{\Gamma(u)} \left( \frac{1}{2} + \mu_j + iu \right) \right) \, du = \frac{1}{2} \int_{-\infty}^{\infty} G_\Delta(t-u) \log \left| \frac{1}{2} + \mu_j + iu \right| \, du + O(1)
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} G_\Delta(u) \log \left| \frac{1}{2} + \mu_j + it - iu \right| \, du + O(1)
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} G_\Delta(u) \left\{ \log(|\mu_j + it| + 3) + O(\log(|u| + 2)) \right\} \, du + O(1)
\]

\[= \frac{1}{2} \log(|\mu_j + it| + 3) \int_{-\infty}^{\infty} G_\Delta(u) \, du + O(1).\]

By Lemma \([3]\) (ii), the Fourier transform \( \hat{h}(\xi) = e^{-2\pi i \xi} \tilde{G}_{\Delta}(-\xi) \) is supported on \([-\Delta, \Delta]\) and is uniformly bounded. Also, \( |\Lambda_\pi(n)| \leq m\Delta(n)n^{\theta} \), and therefore

\[
\frac{1}{2\pi} \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}} \left\{ \Lambda_\pi(n) \hat{h} \left( \frac{\log n}{2\pi} \right) + \Lambda_{\pi^{-1}}(n) \hat{h} \left( -\frac{\log n}{2\pi} \right) \right\} = O \left( \sum_{n \leq e^{2\pi\Delta}} \Lambda(n)n^{\theta - \frac{1}{4}} \right) = O \left( me^{(1+2\theta)\pi\Delta} \right),
\]

where the last equality follows by the Prime Number Theorem and summation by parts.

In view of \([2.9]\) and \([2.10]\), equation \([2.8]\) becomes

\[
\sum_{\gamma} h(\gamma) = \frac{\log C(t, \pi)}{2\pi} \int_{-\infty}^{\infty} G_\Delta(u) \, du + O \left( me^{(1+2\theta)\pi\Delta} \right) + O \left( m\Delta^2 e^{\pi\Delta} \right).
\]

Inserting this in \([2.7]\) and using \([2.5]\) together with the fact that \( \int_{-\infty}^{\infty} F_1(x) \, dx = 2\pi \), we obtain

\[
S_1(t, \pi) \leq \frac{\log C(t, \pi)}{48\pi\Delta^2} + O(m\Delta^2 e^{(1+2\theta)\pi\Delta})
\]
for any \( t \) and any \( \Delta \geq 1 \). Choosing

\[
\Delta = \max \left\{ \frac{\log \log C(t, \pi)^{3/m} - 5 \log \log C(t, \pi)^{3/m}}{(1 + 2\gamma)\pi}, 1 \right\},
\]

we arrive at the desired conclusion.

The proof of the lower bound follows the same lines, using \( h(z) = G_{\lambda}^{-1}(t - z) \) and inequality (2.6). \( \square \)

3. Theorem 1 implies Theorem 2

For real numbers \( t < u \) let us denote by \( N(t, u, \pi) \) the number of nontrivial zeros of \( L(\cdot, \pi) \) with ordinates \( \gamma \) such that \( t \leq \gamma \leq u \), counted with multiplicity (zeros with ordinates equal to the endpoints \( t \) or \( u \) are counted with half of their multiplicities). The following fact connects the variation of \( S(\cdot, \pi) \) to the nontrivial zeros of \( L(\cdot, \pi) \), like equation (1.1) in the case of \( \zeta \).

**Lemma 7.** Let \( L(\cdot, \pi) \) satisfy the generalized Riemann hypothesis. Let \( t \) and \( u \) be real numbers such that \( t < u \leq t + 5 \). Then

\[
N(t, u, \pi) = S(u, \pi) - S(t, \pi) + \frac{u - t}{2\pi} \log C(t, \pi) + O(m).
\]

**Proof.** If \( v \neq 0 \) and \( v \) is not the imaginary part of a zero of \( L(\cdot, \pi) \), then \( S'(v, \pi) = \frac{1}{2i} \text{Re} \frac{L'}{L}(\frac{1}{2} + iv, \pi) \). At each zero (trivial or non-trivial) \( \rho = \frac{1}{2} + i\gamma \) of \( L(\cdot, \pi) \) the function \( S(\cdot, \pi) \) jumps by the multiplicity of this zero; at 0 it jumps by \( -2r(\pi); \) and for each \( j \) such that \( -1 < \text{Re} \mu_j < -\frac{1}{2} \), it jumps by 2 at \( -\text{Im} \mu_j \). Therefore

\[
N(t, u, \pi) = S(u, \pi) - S(t, \pi) - \frac{1}{\pi} \int_t^u \text{Re} \frac{L'}{L} \left( \frac{1}{2} + iv, \pi \right) dv + O(m).
\]

By the definition of \( \Lambda(\cdot, \pi) \),

\[
\frac{\Lambda'}{\Lambda}(s, \pi) = \frac{\log N}{2} + \sum_{j=1}^{m} \frac{\Gamma'_{\text{Re}}(s + \mu_j)}{\Gamma_{\text{Re}}(s + \mu_j)} + \frac{L'}{L}(s, \pi).
\]

By the functional equation (1.8), the real part of \( \frac{\Lambda'}{\Lambda}(\cdot, \pi) \) vanishes on the line \( \frac{1}{2} + iv \). Therefore

\[
- \int_t^u \text{Re} \frac{L'}{L} \left( \frac{1}{2} + iv, \pi \right) dv = \int_t^u \left\{ \frac{\log N}{2} + \sum_{j=1}^{m} \text{Re} \frac{\Gamma'_{\text{Re}}(\mu_j + \frac{1}{2} + iv)}{\Gamma_{\text{Re}}(\mu_j + \frac{1}{2} + iv)} \right\} dv
\]

\[= \left( \frac{1}{2} \right) \int_t^u \left\{ \log N + \sum_{j=1}^{m} \log |\frac{5}{2} + \mu_j + iv| \right\} dv + O(m)
\]

\[= \frac{u - t}{2} \log C(t, \pi) + O(m).
\]

\( \square \)

To derive Theorem 2 from Theorem 1 we recall the fact that \( S_1(\cdot, \pi) \) is a primitive for \( S(\cdot, \pi) \). Indeed, for almost every real \( v \),

\[
S(v, \pi) = -\frac{1}{\pi} \int_{1/2}^{\infty} \text{Im} \frac{L'}{L} (\sigma + iv, \pi) \, d\sigma.
\]

The function \( \frac{L'}{L}(\sigma + iv, \pi) \) is absolutely integrable in the region \( \{ s \in \mathbb{C} \mid \text{Re} \, s \geq \frac{1}{2} \text{ and } t \leq \text{Im} \, s \leq u \} \) since it has only simple poles and decays exponentially as \( \sigma \to \infty \). So we can apply Fubini’s theorem to get

\[
\int_t^u S(v, \pi) dv = -\frac{1}{\pi} \int_t^u \int_{1/2}^{\infty} \text{Im} \frac{L'}{L} (\sigma + iv, \pi) \, d\sigma \, dv
\]
Proof of Corollary 3. (i) If \( t < u \leq t + 5 \) and thus \( S(t, \pi) = S_1(u, \pi) - S_1(t, \pi), \)

as claimed.

Proof of Theorem 2. Let \( \nu \) be a real number such that \( 0 < \nu \leq 5 \) to be chosen later. The inequality

\[
| \log C(t + \nu, \pi)^{3/m} - \log C(t, \pi)^{3/m} | \leq 3 \log 3
\]

implies that

\[
\frac{\log C(t + \nu, \pi)^{3/m}}{(\log \log C(t + \nu, \pi)^{3/m})^2} = \frac{\log C(t, \pi)^{3/m}}{(\log \log C(t, \pi)^{3/m})^2} + O(1).
\]

Therefore, by Theorem 3 at heights \( t \) and \( t + \nu, \)

\[
| S_1(t + \nu, \pi) - S_1(t, \pi) | \leq \frac{(1 + 2\theta)^2 \pi}{16} \frac{\log C(t, \pi)}{(\log \log C(t, \pi)^{3/m})^2} + \frac{\nu^2}{4\pi} \log C(t, \pi) + O(m)
\]

Applying Lemma 7 we see that

\[
S_1(t + \nu, \pi) - S_1(t, \pi) = \int_{t}^{t + \nu} (u, \pi) du \\
\geq \int_{t}^{t + \nu} \left( \log C(t, \pi) + O(m) \right) du \\
= \nu S(t, \pi) - \frac{\nu^2}{4\pi} \log C(t, \pi) + O(m)
\]

and thus

\[
S(t, \pi) \leq \frac{(1 + 2\theta)^2 \pi}{16 \nu} \frac{\log C(t, \pi)}{(\log \log C(t, \pi)^{3/m})^2} + \frac{\nu^2}{4\pi} \log C(t, \pi) + O\left( \frac{\log C(t, \pi) \log C(t, \pi)^{3/m}}{\nu \log \log C(t, \pi)^{3/m}} \right)
\]

The upper bound for \( S(t, \pi) \) is obtained with the choice

\[
\nu = \frac{(1 + 2\theta)\pi}{2 \log \log C(t, \pi)^{3/m}}
\]

(note that \( 0 < \nu \leq 5 \)) and the lower bound can be established by the same method, considering \( S_1(t, \pi) - S_1(t - \nu, \pi) \).

Proof of Corollary 3. (i) If \( \rho = \frac{1}{2} + i\gamma \) is a zero of \( \Lambda(\cdot, \pi) \), part (i) follows directly from Theorem 2 and identity (3.1) with \( t = \gamma^- \) and \( u = \gamma^+ \).

(ii) By Lemma 7 if \( t \) and \( u \) are real numbers such that \( t < u \leq t + 5 \) and \( \Lambda(\cdot, \pi) \) has no zeros between \( \frac{1}{2} + it \) and \( \frac{1}{2} + iu, \)

\[
\frac{u - t}{2\pi} \log C(t, \pi) = -S(u, \pi) + S(t, \pi) + O(m)
\]

By Theorem 2

\[
u - t \leq \frac{(1 + 2\theta)\pi}{\log \log C(t, \pi)^{3/m}} + a \frac{\log \log C(t, \pi)^{3/m}}{(\log \log C(t, \pi)^{3/m})^2}
\]

for some universal constant \( a \). If \( \gamma' - \gamma \leq 5 \), it is enough to let \( t \to \gamma \) and \( u \to \gamma' \). Otherwise, we let \( u = t + 5 \) and \( t \to \gamma \). The obtained inequality is possible only if \( \log C(\gamma, \pi)^{3/m} \leq e^{\max\{a, 3\}} \). Taking \( \Lambda(\cdot, \tilde{\pi}) \) in place of
\( \Lambda(\cdot, \pi), \) we get \( \log C(\gamma', \pi) \leq \varepsilon_{\text{max} (a, \delta)} \). Then
\[
\frac{1}{m} (\log C(\gamma, \pi) + \log C(\gamma', \pi)) \leq \frac{2}{3} \varepsilon_{\text{max} (a, \delta)},
\]
and for some index \( j \) we must have
\[
\log(|i\gamma + \mu_j| + 3) + \log(|i\gamma' + \mu_j| + 3) \leq \frac{2}{3} \varepsilon_{\text{max} (a, \delta)}.
\]
This implies that \( i\gamma + \mu_j \) and \( i\gamma' + \mu_j \) are bounded by some universal constant. \( \square \)

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