CONTROLLABILITY OF THE SEMILINEAR WAVE EQUATION
GOVERNED BY A MULTIPLICATIVE CONTROL

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Dedicated to Professor Hammadi Bouslous on the occasion of his 65th birthday

Abstract. In this paper we establish several results on approximate controllability of a semilinear wave equation by making use of a single multiplicative control. These results are then applied to discuss the exact controllability properties for the one dimensional version of the system at hand. The proof relies on linear semigroup theory and the results on the additive controllability of the semilinear wave equation. The approaches are constructive and provide explicit steering controls. Moreover, in the context of undamped wave equation, the exact controllability is established for a time which is uniform for all initial states.

1. Introduction. In this paper, we study the controllability problem for a distributed parameter system governed by the following $n$-dimensional wave equation:

$$
\begin{align*}
\begin{cases}
\frac{\partial^2 w}{\partial t^2} &= \Delta w + v(x,t)w + f(t, w, w_t), &\text{in } \Omega, \ t > 0 \\
w &= 0, &\text{on } \partial\Omega, \ t > 0 \\
w(x,0) &= w_1, \ w_t(x,0) = w_2, &\text{in } \Omega
\end{cases}
\end{align*}
$$

where $\Omega$ is a bounded open set of $\mathbb{R}^n$, $n \geq 1$ with a smooth boundary $\partial\Omega$. The real valued coefficient $v(x,t)$ is the multiplicative control and $f$ is the nonlinearity. Our goal is to identify a set of states $(w(\cdot, t), w_t(\cdot, t))$ that can be achieved by system (1) at a time $T > 0$ using a suitable control $v(x, t)$. Such problems arise in various real situations (see [23] and the rich references therein). Research in the multiplicative controllability of distributed systems has been the subject of several works. The question of controllability of PDEs equations by multiplicative controls has attracted many researchers in the context of various type of equations, such as rod equation [3, 24], Beam equation [7], Schrödinger equation [6, 8, 24, 31], heat equation [11, 12, 16, 17, 18, 20, 23]. Various approaches were used to tackle the question of multiplicative controllability of hyperbolic equations like (1). The homogeneous version of (1) (i.e, $f = 0$) has been considered in [3, 9, 21, 23, 32]. The case of semilinear wave equation has been studied in [15, 22] for equilibrium-like
states of the form \((y^d, 0)\) using two controls, i.e. beside the control \(v(x, t)\), a time-dependent control has been considered in the damped part. Furthermore, research in the controllability of the semilinear wave equation by additive controls have been the subject of several works (see [27, 26, 35, 39, 40] and the references therein).

In this paper, we study the approximate and exact controllability for the system (1) by the means of a single multiplicative control, thus we will have a principal reduction in the means to control the system (1).

The paper is organized as follows: in the second section, we first consider the question of reaching approximately target states of the form \((w(0), \theta_2)\) by applying a suitable time-independent control \(v(x, t) = v_T(x)\) at a "short" time \(T\). In the second part of the same section, we define a set of target states \((\theta_1, \theta_2)\) that can be approximately achieved by using a piecewise static control in "long" time. In Section 3, we apply the result of Section 2 to define a strategy of the controller \(v(x, t)\) in order to get the exact achievement of a class of target states for both damped and undamped cases.

2. Approximate controllability.

2.1. A partial approximate controllability result. Let us consider the system (1) evolving on a time-interval \((0, T_0)\) with a nonlinear term \(f: (0, T_0) \times H^1_0(\Omega) \times L^2(\Omega) \rightarrow L^2(\Omega)\) which is globally Lipschitz. Letting \(y = (w, w_t) \in H := H^1_0(\Omega) \times L^2(\Omega)\), we obtain the following equivalent first order system:

\[
\begin{cases}
  y_t = Ay + v(t) B y + F(t, y), & t \in (0, T_0) \\
  y(0) = y_0 = (w_1, w_2)
\end{cases}
\]

where \(v(t) = v(\cdot, t) \in U := L^\infty(\Omega)\), \(B = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}\) and \(A = \begin{pmatrix} 0 & I \\ -\Delta & 0 \end{pmatrix}\) with domain \(D(A) = (H^1_0(\Omega) \cap H^2(\Omega)) \times H^1_0(\Omega)\) and where for all \(t \in (0, T_0)\) and \(y = (w_1, w_2) \in H: F(t, y) = (0, f(t, w_1, w_2))\). Here, the state space \(H\) is endowed with the following inner product: \([\langle u_1, v_1 \rangle, \langle v_1, v_2 \rangle] = \langle u_1, v_1 \rangle_{H^1_0(\Omega)} + \langle v_2, v_2 \rangle_{L^2(\Omega)}\) with corresponding norm \(\| \cdot \|\). With this Hilbert structure, the operator \(A\) generates a semigroup of isometries \(S(t)\).

For any \(\xi \in L^2(\Omega)\) we set: \(\Lambda(\xi) := \{ x \in \Omega/ \xi(x) \neq 0 \}\) and \(1_{\Lambda(\xi)}\) will denote the characteristic function of \(\Lambda(\xi)\).

Our first main result concerns the approximate controllability toward a target state \(w(T) = w_1, \partial_t w(T) = \theta_2\) within an arbitrarily small time-interval \((0, T)\), which depends on the choice of the initial state \(y_0 = (w_1, w_2)\), the target state \(y^d = (w_1, \theta_2)\) and the precision of steering. The main idea here consists on looking for a static control such that the respective solution to (2) is such that \(y(T) - y^d \rightarrow 0\), as \(T \rightarrow 0^+\). This idea was first used by Khapalov in [20] in the context of reaction-diffusion equation (see also [11, 12]).

**Theorem 2.1.** Let \((w_1, w_2) \in H\) and \(\theta_2 \in L^2(\Omega)\) and let us set \(a(x) := \frac{\theta_2 - w_2}{w_1} 1_{\Lambda(w_1)}\).

Assume that: (i) \(a \in L^\infty(\Omega)\) and (ii) for a.e. \(x \in \Omega:\ w_1(x) = 0 \Rightarrow \theta_2(x) = w_2(x)\).

Then for any \(\epsilon > 0\), there are a time \(T = T(w_1, w_2, \theta_2, \epsilon) \in (0, T_0)\) and a static control \(v(\cdot, t) = v_T(\cdot) \in W^{2, \infty}(\Omega)\) such that for the respective solution to (1), the following inequalities hold:

\[
\| w(T) - w_1 \|_{H^1_0(\Omega)} < \epsilon \text{ and } \| w_T(T) - \theta_2 \|_{L^2(\Omega)} < \epsilon.
\]
Proof. Let \( \epsilon > 0 \), and let us consider the state \( y^d = (w_1, \theta_2) \) to be achieved. For any time of steering \( 0 < T < T_0 \) we consider the control

\[
 v(x, t) = v_T(x) := \frac{a(x)}{T}, \quad t \in (0, T_0).
\]

Since \( a \in L^\infty(\Omega) \), there is a unique mild solution \( y(t) \) to (2) (see [34], p. 184), which is given by the following variation of constants formula:

\[
y(t) = S(t)y_0 + \int_0^t S(t-s)(v_T(x)By(s) + F(s, y(s)))ds, \quad \forall t \in [0, T_0].
\]

We aim to show that the control (3) guarantees the steering of system (2) to \( y^d \) at any small time \( T > 0 \), so we can assume in the sequel that \( 0 < T < T_0 := 1 \).

**Case 1.** \( a(\cdot) \in W^{2,\infty}(\Omega) \) and \( y_0 \in \mathcal{D}(A) \).

We will distinguish two subcases:

**Case 1.1.** Assume that the operator \( F \) is \( C^1 \) and globally Lipschitz from \( (0, T_0) \times H \) to \( H \).

Here, the mild solution is a classical one. In particular we have \( y(t) \in \mathcal{D}(A) = H^1_0(\Omega) \cap H^2(\Omega) \times H^1_0(\Omega), \forall t \in [0, T_0] \) (see [34], p. 187).

It comes from the assumption (i) and from (3) that: \( e^{Tv_TB} = \begin{pmatrix} I & 0 \\ a & I \end{pmatrix} \), so the assumption (ii) leads to: \( e^{Tv_TB}y_0 = y^d \).

The idea of the proof will consist on proving the following formula:

\[
y(T) - y^d = \int_0^T e^{(T-s)v_T(x)B}(Ay(s) + F(s, y(s)))ds,
\]

and showing that the term in the right-hand side of the relation (5) tends to zero as \( T \to 0^+ \).

In order for \( y(t) \) to satisfy (5), it suffices to show that \( Ay(\cdot) \in L^1(0, 1) \) (see [4]). For this end, let us apply the bounded operator \( A_\lambda = \lambda R(\lambda; A)A \) to (4), where \( R(\lambda; A) \) is the resolvent of \( A \). Thus

\[
 A_\lambda y(t) = S(t)A_\lambda y_0 + \int_0^t A_\lambda S(t-s)(v_T(x)By(s) + F(s, y(s)))ds
\]

This gives

\[
 A_\lambda y(t) = S(t)A_\lambda y_0 + \int_0^t A_\lambda S(t-s)(v_T(x)By(s))ds + \int_0^t \lambda R(\lambda; A)S'(t-s)F(s, y(s))ds
\]

where \( S'(t) \) is the derivative of \( S(t) \) with respect to \( t \). We have

\[
 \int_0^t \lambda R(\lambda; A)S'(t-s)F(s, y(s))ds = -\int_0^t \frac{d}{ds} \left( \lambda R(\lambda; A)S(t-s)F(s, y(s)) \right)ds - \int_0^t \lambda R(\lambda; A)S(t-s)A_\lambda F_s(s, y(s)) + \frac{\partial}{\partial y}(s, y(s))y_s(s)ds,
\]

where \( y_s \) refers to the derivative w.r.t "\( s \)". Thus

\[
 \int_0^t \lambda R(\lambda; A)S'(t-s)F(s, y(s))ds = \lambda R(\lambda; A)F(t, y(t)) - S(t)F(0, y_0) - \int_0^t \lambda R(\lambda; A)S(t-s)A_\lambda F_s(s, y(s)) + \frac{\partial}{\partial y}(s, y(s))y_s(s)ds.
\]
Moreover, using (4), we deduce via Gronwall’s inequality that:
\[ S_{672} \text{ MOHAMED OUZAHRA} \]
\[
\|A(\lambda t)\| \leq \|A\|_0 + \int_0^t \|A(v_T(x)B y(s))\| ds + \|F(0, y_0)\| + \|F(t, y(t))\| + \int_0^t \|\frac{\partial F}{\partial s}(s, y(s)) + \frac{\partial F}{\partial y}(s, y(s)) y_s(s)\| ds.
\]
Moreover, using (4), we deduce via Gronwall’s inequality that:
\[
\|y(t)\| \leq C\|y_0\| + C, \forall t \in [0, T] \subset [0, 1],
\]
for some positive constant \( C = C(||a||_{L^\infty(\Omega)}) > 0 \) which is independent of \( T \).

Then using the fact that \( F \) is Lipschitz we get:
\[
\|F(t, y(t))\| \leq C(1 + ||y_0||), C = C(||a||_{L^\infty(\Omega)}) > 0,
\]
and
\[
\int_0^t \|\frac{\partial F}{\partial s}(s, y(s)) + \frac{\partial F}{\partial y}(s, y(s)) y_s(s)\| ds \leq LT + L \int_0^t \|y_s(s)\| ds.
\]
Then, letting \( \lambda \to +\infty \), we deduce that:
\[
\|A(\lambda t)\| \leq \|A\|_0 + \int_0^t \|A(v_T(x)B y(s))\| ds + C(1 + ||y_0||_{D(A)}) + L \int_0^t \|y_s(s)\| ds
\]
where \( L \) is a Lipschitz constant of \( F \) and the constant \( C = C(||a||_{L^\infty(\Omega)}) > 0 \) is independent of \( T \).

In the sequel, the letter \( C \) will be used to denote a generic positive constant which is independent of \( T \).

Let us now study the terms of the right hand of inequality (9). We have \( v_T(x)B y(t) = (0, \frac{a(x, T)w(t)}{T}) \), thus since \( a \in W^{2,\infty}(\Omega) \) it comes that \( v_T(x)B y(t) \in D(A) \) for all \( t \in [0, T] \).

Moreover, we have the following second order Leibniz rule:
\[ \Delta(aw') = \Delta (aw) = 2\nabla(a) \cdot \nabla(w) + a\Delta(w), \forall w \in H^2(\Omega), \]
from which we get:
\[
\|A(v_T(x)B y(s))\| = \frac{1}{T} \||\Delta(aw(s))||_{L^2(\Omega)} \leq \frac{C}{T} \|y(s)||_{D(A)}, \forall s \in [0, T]
\]
where \( C = C(||a||_{W^{2,\infty}(\Omega)}) \) is independent of \( T \).

It follows that:
\[
\int_0^t \|A(v_T(x)B y(s))\| ds \leq \frac{C}{T} \int_0^t \|y(s)||_{D(A)} ds, \forall t \in [0, T].
\]
(10)
Since \( y(t) \) is a classical solution, we have
\[
\|y(t)\| = \|Av(t) + v_T(x)B y(t) + F(t, y(t))\| \leq \frac{C}{T} \|y(t)||_{D(A)} + T + \|F(0, 0)\|
\]
for all \( 0 < t \leq T \), where \( C = C(||a||_{L^\infty(\Omega)}) \).

Reporting (10) and (11) in (9) and taking into account (7), we deduce via Gronwall’s inequality that:
\[
\|y(t)||_{D(A)} \leq C\|y_0||_{D(A)} + C, \forall t \in [0, T],
\]
(12)
where \( C = C(||a||_{W^{2,\infty}(\Omega)}) \) is independent of \( T \). Thus \( Ay_0(\cdot) \in L^1(0, T) \), and hence the following variation of constants formula holds:
\[
y(t) = e^{\lambda v_T(x)B} y_0 + \int_0^t e^{(t-s)v_T(x)B} (Ay_0(s) + F(s, y(s))) ds, \forall t \in [0, T],
\]
(13)
from which it comes
\[ y(T) - y^d = \int_0^T e^{(T-s)v_T(x)B} (Ay(s) + F(s, y(s))) ds. \] (14)

Based on (14) and using (12) and the fact that \( F \) is Lipschitz, we deduce that:
\[ \|y(T) - y^d\| \leq C_* T (\|y_0\| + 1), \quad C_* = C_*(\|a\|_{W^{2,\infty}(\Omega)}), \] (15)

and hence \( \|y(T) - y^d\| < \epsilon \), whenever \( 0 < T < \inf(T_0 = 1, \frac{1}{C_*(1+\|y_0\|_{D(A)})}) \).

**Case 1.2.** Here, we only assume that the operator \( F \) is globally Lipschitz from \((0, T_0) \times H \rightarrow H\) (with a Lipschitz constant \( L > 0 \)), and let \( y(t) \) be the mild solution of (2) corresponding to control \( v_T(x) \) given by (3). Then we can approximate the continuous function \( t \rightarrow F(t, y(t)) \) uniformly with \( C^1 \)-functions \( (F_n) \) in \([0, T_0] = [0, 1]\). More precisely, we can consider the following Bernstein polynomial ([14], pp. 108-113):
\[ F_n(t) = \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} F(\frac{k}{n}, y(\frac{k}{n})), \quad t \in [0, 1] \] (16)

From (8) we get:
\[ \sup_{t \in [0,T]} \|F_n(t)\| \leq C(1 + \|y_0\|), \quad C = C(\|a\|_{L^\infty(\Omega)}) > 0. \]

Moreover, for all \( n \geq 1 \) we have:
\[ F'_n(t) = n \sum_{k=0}^{n-1} \binom{n-1}{k} t^k (1-t)^{n-1-k} \left( F(\frac{k+1}{n}, y(\frac{k+1}{n})) - F(\frac{k}{n}, y(\frac{k}{n})) \right), \] (16)

where \( F'_n(t) \) is the derivative of \( F_n(t) \).

Let us show that the sequence of derivative \( (F'_n) \) is uniformly bounded in \([0, T]\).

For all \( h, t \in [0, T] \) such that \( t + h \in [0, T] \), we have:
\[ y(t+h) - y(t) = \int_0^h S(t+h-s) \left\{ \frac{d}{ds} By(s) + F(s, y(s)) \right\} ds + S(t+h)y_0 - S(t)y_0 + \int_0^t S(t-s) \left\{ \frac{d}{ds} B (y(s+h) - y(s)) \right\} ds + \int_0^t S(t-s) \left\{ F(s+h, y(s+h)) - F(s, y(s)) \right\} ds, \]

from which, we derive:
\[ \|y(t+h) - y(t)\| \leq h \|Ay_0\| + h C(1 + \|y_0\|) + \frac{1}{T} \left\{ Lh + (\|a\|_{\mathcal{D}(A)} + L) \|y(s+h)-y(s)\| \right\} ds, \]

where \( C = C(\|a\|_{L^\infty(\Omega)}) \) is independent of \( T \), which by Gronwall’s inequality gives the following estimate:
\[ \|y(t+h) - y(t)\| \leq \frac{C(1 + \|y_0\|_{\mathcal{D}(A)})}{T} h, \]

where \( C = C(\|a\|_{L^\infty(\Omega)}) \) is independent of \( T \).

It follows from the expression of \( F' \) and the last inequality that:
\[ \sup_{n \geq 1} \sup_{t \in [0,T]} \|F'_n(t)\| \leq L_T := \frac{C(1 + \|y_0\|_{\mathcal{D}(A)})}{T}, \quad C = C(\|a\|_{L^\infty(\Omega)}). \]

As a consequence, \( F_n \) is \( L_T \)-Lipschitz on \([0,T]\).
In the sequel, we will apply the techniques of Case 1.1 to the following approached system:

\[
\begin{aligned}
\frac{d}{dt}y_n(t) &= Ay_n(t) + v_T(x)By_n(t) + F_n(t) \\
y_n(0) &= y_0 = y(0)
\end{aligned}
\]

Let \(y_n(t)\) denote the classical solution of the system (17), which is given by the following variation of constants formula:

\[
y_n(t) = S(t)y_0 + \int_0^t S(t-s)(v_T(x)By_n(s) + F_n(s))ds, \forall t \in [0, T_0].
\]

Based on the variation of constants formulas (4) and (18), we can show via the Gronwall’s inequality that there is \(N = N(T, \epsilon) \in \mathbb{N}\) such that:

\[
\|y_N(T) - y(T)\| < \epsilon/2.
\]

Moreover, applying the relation (6) to \(y_n(t)\) leads to:

\[
A_\lambda y_n(t) = S(t)A_\lambda y_0 + \int_0^t A_\lambda S(t-s)(v_T(x)By_n(s))ds + \int_0^t \lambda R(\lambda, A)S'(t-s)F_n(s)ds.
\]

We have

\[
\int_0^t \lambda R(\lambda, A)S'(t-s)F_n(s)ds = -\int_0^t \frac{d}{ds} (\lambda R(\lambda; A)S(t-s)F_n(s))ds + \int_0^t \lambda R(\lambda; A)S(t-s)F_n'(s)ds
\]

Then we deduce that:

\[
\|A_\lambda y_n(t)\| \leq \|Ay_0\| + \int_0^t \|A(v_T(x)By_n(s))\|ds + \|F_n(0)\| + \|F_n(t)\| + \int_0^t \|F_n'(s)\|ds.
\]

Letting \(\lambda \to +\infty\), we get

\[
\|Ay_n(t)\| \leq \|Ay_0\| + \int_0^t \|A(v_T(x)By_n(s))\|ds + C\|y_0\| + C.
\]

where \(C = C(\|a\|_{L^\infty(\Omega)})\) is a positive constant which is independent of \(T\). Then by proceeding as in the Case 1.1., we get an estimate like (15), namely:

\[
\|y_N(T) - y^d\| \leq CT(\|y_0\|_{D(A)} + 1),
\]

where \(C = C(\|a\|_{W^2,\infty(\Omega)}) > 0\) is independent of \(N\).

It follows that \(\|y_N(T) - y^d\| < \epsilon/2\), for some \(T\) small enough.

This together with (19) gives:

\[
\|y(T) - y^d\| < \epsilon.
\]

Case 2. \(a(\cdot) \in W^{2,\infty}(\Omega)\) and \(y_0 \in H\).

Let \(T > 0\), and for all \(\lambda > 0\) we set \(y_{0\lambda} := \lambda R(\lambda; A)y_0 \in D(A)\). Let \(y_\lambda\) be the mild solution of (2) corresponding to the initial state \(y_{0\lambda} = (u_{1\lambda}, u_{2\lambda})\) with the same control as in the Case 1., i.e., \(v(x, t) = v_T(x) = \frac{a}{2}\), \(0 < t < T_0\).

We have

\[
\|y(\lambda) - y^d\| \leq \|y(\lambda) - y_\lambda(T)\| + \|y_\lambda(T) - e^{A\lambda}y_{0\lambda}\| + \|e^{A\lambda}y_{0\lambda} - y^d\|.
\]

It follows from the variation of constants formula that:

\[
y_\lambda(t) - y(t) = \int_0^t S(t-s)(v_T(x)B(y_\lambda(s) - y(s)))ds + S(t)y_{0\lambda} - S(t)y_0 + \int_0^t S(t-s)(F(s, y_\lambda(s)) - F(s, y(s)))ds.
\]
Then, using the contraction property of the semigroup $S(t)$, it comes:

$$
\| y_\lambda(t) - y(t) \| \leq \| y_0 \lambda - y_0 \| + \frac{[a]_{L^\infty(\Omega)}}{T} \int_0^T \| y_\lambda(s) - y(s) \| ds + L \int_0^T \| y_\lambda(s) - y(s) \| ds, \forall t \in [0, T].
$$

Gronwall’s lemma yields

$$
\| y_\lambda(T) - y(T) \| \leq C \| y_0 \lambda - y_0 \|, \quad (C = C([a]_{L^\infty(\Omega)})).
$$

It follows from $y^d = e^{aB}y_0$ that:

$$
\| e^{aB}y_0 \lambda - y^d \| \leq \| a \|_{L^\infty(\Omega)} \| y_0 \lambda - y_0 \|.
$$

We deduce that there is a $\lambda > 0$, which is independent of $T \in (0, 1)$, such that:

$$
\| y_\lambda(T) - y(T) \| + \| e^{aB}y_0 \lambda - y^d \| < \frac{\epsilon}{2}.
$$

For such a $\lambda$, we deduce from the same arguments as in the Case 1 that there exists $0 < T < 1$ such that: $\| y_\lambda(T) - e^{aB}y_0 \| < \frac{\epsilon}{2}$. We conclude that:

$$
\| y(T) - y^d \| < \epsilon.
$$

**Case 3:** $a(\cdot) \in L^\infty(\Omega)$ and $y_0 \in H$.

Let $(a_k) \subset W^{2, \infty}(\Omega)$ be a sequence which is uniformly bounded on $\Omega$ and such that $a_k \to a$ in $L^2(\Omega)$, as $k \to +\infty$. Here, we will consider the control: $\nu_T(x) = \frac{a_k}{T}$ for a suitably selected (large enough) $k \in \mathbb{N}$, and let $y(t)$ be the corresponding solution to (2) with the initial state $y(0) = y_0 = (w_1, w_2)$.

Now, let $(w_2l) \in L^\infty(\Omega)$ be such that $w_2l \to w_2$ in $L^2(\Omega)$, as $l \to +\infty$, and let us consider the initial state $y_{0l} = (w_1, w_2l)$.

We have the following triangular inequality:

$$
\| y(T) - e^{aB}y_0 \| \leq \| y(T) - e^{a_kB}y_0 \| + \| e^{a_kB}y_0 - e^{a_lB}y_{0l} \| + \| e^{a_lB}y_{0l} - e^{aB}y_{0l} \| + \| e^{aB}y_{0l} - e^{aB}y_0 \|.
$$

From the relation $e^{a_kB} = \begin{pmatrix} I & 0 \\ a_k & I \end{pmatrix}$, we deduce that:

$$
\| e^{a_kB}y_0 - e^{a_kB}y_{0l} \| + \| e^{aB}y_{0l} - e^{aB}y_0 \| \leq \sup_{k \in \mathbb{N}} \left( 1, \| a_k \|_{L^\infty(\Omega)}, \| a \|_{L^\infty(\Omega)} \right) \| y_{0l} - y_0 \|
$$

and $e^{a_kB}y_{0l} - e^{aB}y_{0l} = (0, (a_k - a)w_{2l})$. Let $l \in \mathbb{N}$ be such that

$$
\sup_{k \in \mathbb{N}} \left( 1, \| a_k \|_{L^\infty(\Omega)}, \| a \|_{L^\infty(\Omega)} \right) \| w_{2l} - w_2 \| < \frac{\epsilon}{3},
$$

and for such value of $l$, we consider a $k$ such that

$$
\| a_k - a \|_{L^2(\Omega)} \| w_{2l} \|_{L^\infty(\Omega)} < \frac{\epsilon}{3}.
$$

Then, for this value of $k$, it comes from the Case 2 that there exists $T > 0$ such that:

$$
\| y(T) - e^{a_kB}y_0 \| < \frac{\epsilon}{3}.
$$

We conclude that

$$
\| y(T) - e^{aB}y_0 \| < \epsilon.
$$

Finally, since $e^{aB}y_0 = y_d$, it comes

$$
\| w(T) - w_1 \|_{H^1_0(\Omega)} < \epsilon \quad \text{and} \quad \| w_1(T) - \theta_2 \|_{L^2(\Omega)} < \epsilon.
$$

\[\square\]
2.2. Global approximate controllability. In this subsection, we will consider the following equation:

\[
\begin{cases}
    w_{tt} = \Delta w + v(x,t)w - h(x)w_t + f(w), & \text{in } \Omega \times (0,T) \\
    w = 0, & \text{on } \partial\Omega \times (0,T) \\
    w(x,0) = w_1, w_t(x,0) = w_2, & \text{in } \Omega
\end{cases}
\]  

(23)

where \( T > 0, \ h \in L^\infty(\Omega) \) and the nonlinear term \( f : L^2(\Omega) \to L^2(\Omega) \) is a globally Lipschitz function. Here, we will study the approximate controllability problem for \( T > 0 \) using robustness results on the observability property (see [32]), we can see the estimate (24) holds for \( \zeta \) large enough provided there is a subset \( O \) of the support of \( h \) satisfying the following so-called geometrical control condition (GCC): "there exists \( x_0 \in \mathbb{R}^n \) such that \( O \) is a neighborhood of the closure of the set \( \Gamma(x_0) := \{ x \in \partial\Omega/ (x - x_0) \nu(x) > 0 \} \), where \( \nu(x) \) denotes the unit outward normal at \( x \in \partial\Omega \) (see [5]). In particular, for \( n = 1 \) and \( g = 1_\omega \) the estimate (24) holds for \( \omega = (a,b) \subset \Omega = (0,1) \) and \( T > 2 \inf(a,1-b) \) (see [40]).

We have the following remarks regarding the estimate (24).

Remark 1. 1. For \( f = b_\zeta = 0 \), the inequality (24) was established for \( T \) large enough provided there is a subset \( O \) of the support of \( h \) satisfying the following so-called geometrical control condition (GCC): "there exists \( x_0 \in \mathbb{R}^n \) such that \( O \) is a neighborhood of the closure of the set \( \Gamma(x_0) := \{ x \in \partial\Omega/ (x - x_0) \nu(x) > 0 \} \), where \( \nu(x) \) denotes the unit outward normal at \( x \in \partial\Omega \) (see [5]). In particular, for \( n = 1 \) and \( g = 1_\omega \) the estimate (24) holds for \( \omega = (a,b) \subset \Omega = (0,1) \) and \( T > 2 \inf(a,1-b) \) (see [40]).

2. Using robustness results on the observability property (see [32]), we can see that (24) also holds under the geometrical control condition for small Lipschitz constant \( L_\zeta \) of the operator: \( y \in L^2(\Omega) \to f(y) + b_\zeta(x)y \). Indeed, let the above (GCC) hold, so that:

\[
\int_0^T \int_\Omega h(x)|\phi_1(x)|^2dxdt \geq \delta \| \phi_1, \phi_2 \|_H^2, \ \forall (\phi_1, \phi_2) \in H,
\]  

(25)

(26)

(some \( T, \delta > 0 \), where \( \phi \) is the solution of \( \phi_{tt} = \Delta \phi, \phi(0) = \phi_1 \in H^1_0(\Omega), \phi_2 \in L^2(\Omega) \). We can easily show that the solution \( T(t)y_0 = (\varphi(t), \varphi_1(t), y_0 = (\varphi_1, \varphi_2) \) of (25) verifies \( \| T(t)y_0 \| \leq e^{L_\zeta t}\| y_0 \|, \ \forall t \geq 0 \). Then using the variation of constants formula, we get:

\[
|\langle CS(t)y_0, S(t)y_0 \rangle| \leq \| h \|_{L^\infty(\Omega)} T L_\zeta (1 + e^{L_\zeta T}) e^{L_\zeta T} \| y_0 \|^2 + |\langle CT(t)y_0, T(t)y_0 \rangle| \leq C |
\]

\[
where C = \left( \begin{array}{cc} 0 & 0 \\ 0 & h(x)I \end{array} \right). \text{ From this and (26) it comes:}
\]

\[
\int_0^T |\langle CT(t)y_0, T(t)y_0 \rangle|d\tau \geq (\delta - \beta)\| y_0 \|^2 \text{ with } \beta = \| h \|_{L^\infty} T e^{L_\zeta T} L_\zeta (1 + e^{L_\zeta T}).
\]
Hence (24) holds whenever \( L_ζ < γ^{-1}(δ) \), where \( γ^{-1} \) is the inverse function of \( γ : s → ||h||_∞ T^2 se^{−T}(1 + e^{−T}) \).

3. An other situation in which (24) holds is the case of functions: \( f(y)(x) = k(y(x)) \), where \( k : R → R \) is such that \( k(0) = 0 \) and \( sk(s) ≤ −s^3 ||b_ζ||_L^∞(Ω) \), \( ∀s ∈ R \) (see [36]).

The following result concerns the approximate controllability toward target states of the form \((θ_1, 0)\).

**Theorem 2.2.** Let \( θ_1 ∈ H^1_0(Ω) \cap H^2(Ω) \) and let assumptions \((P_1) − (P_4)\) hold for \( ζ = θ_1 \). Then for every initial state \((w_0, w_1) ∈ H \) and for every \( ϵ > 0 \), there is a time \( T = T(w_0, w_2, θ_1, ϵ) > 0 \) and a static control \( v(⋅, t) = v_T(⋅) ∈ L^∞(Ω) \) such that the respective solution to (23) satisfies:

\[
||w(T) − θ_1||_{H^1_0(Ω)} + ||w_t(T)||_{L^2(Ω)} < ϵ. \tag{27}
\]

**Proof.** Let \( ϵ > 0 \) be fixed. Let us consider the control:

\[
v(x, t) = v_T(x) = −\frac{Δθ_1 + f(θ_1)}{θ_1} 1_{Λ(θ_1)}, \tag{28}
\]

and let us set \( z = w − θ_1 \). Thus the system (23) takes the form:

\[
\begin{align*}
z_{tt} & = Δz + v(x, t)(z + θ_1) + Δθ_1 − h(x)z_t + f(z + θ_1), & \text{in } Ω × (0, T) \\
z & = 0, & \text{on } ∂Ω × (0, T) \\
z(0) & = z_1, z_t(0) = z_2, & \text{in } Ω
\end{align*} \tag{29}
\]

where \( z_1 = w_1 − θ_1 \) and \( z_2 = w_2 \).

We have: \( Δθ_1 − (Δθ_1/θ_1) 1_{Λ(θ_1)} θ_1 = 0 \) (see [2, 32]). Then, using this and the fact that for almost every \( x \) in \( Ω; : θ_1(x) = 0 ⇒ f(θ_1)(x) = 0 \), the system (29) (controlled with (28)) becomes:

\[
\begin{align*}
z_{tt} & = Δz + b_{θ_1}(x)z − h(x)z_t + fθ_1(z), & \text{in } Ω × (0, T) \\
z & = 0, & \text{on } ∂Ω × (0, T) \\
z(0) & = z_1, z_t(0) = z_2, & \text{in } Ω
\end{align*} \tag{30}
\]

where \( f_{θ_1}(z) = f(z + θ_1) − f(θ_1), z ∈ L^2(Ω) \).

Let \( ϑ \) be a solution of the system:

\[
ϑ_{tt} = Δϑ + b_{θ_1}(x)ϑ + f_{θ_1}(ϑ), \quad ϑ(0) = ϑ_1 ∈ H^1_0(Ω), \quad ϑ_1(0) = ϑ_2 ∈ L^2(Ω). \tag{31}
\]

Then, remarking that \( ϑ := ϑ + θ_1 \) satisfies the equation: \( ϑ_{tt} = Δϑ + b_{θ_1}(x)ϑ + f(ϑ) \), we deduce by taking \( ϑ(0) = ϑ_1 \) and \( ϑ_t(0) = ϑ_2 \) in (25) that:

\[
\int_0^T \int_Ω h(x)|ϑ_t(x)|^2 dx dt ≥ δ||ϑ_0, ϑ_1||_{H^1}^2, \quad ∀(ϑ_1, ϑ_2) ∈ H. \tag{32}
\]

Moreover, since \( f_{θ_1} \) is Lipschitz and satisfies \( f_{θ_1}(0) = 0 \) and \( ⟨f_{θ_1}(z) + b_{θ_1}(x)z, z⟩ ≤ 0 \), for all \( z ∈ L^2(Ω) \), we deduce that the solution of (30) can be defined for all \( t ≥ 0 \) and satisfies the following exponential decay (see [36]):

\[
||z(t), z_t(t)|| ≤ Me^{−σ}||z(0), z_t(0)||, \quad ∀t ≥ 0,
\]

for some constants \( M, σ > 0 \) which are independent of \( T \).

We deduce that for \( T > \frac{1}{δ} \ln\left(\frac{M||w_0 − θ_1, w_2||}{ε}\right) \), the solution of (23) satisfies the following estimate:

\[
||w(T) − θ_1||_{H^1_0(Ω)} + ||w_t(T)||_{L^2(Ω)} < ϵ. \tag{33}
\]

\( □ \)
Our main result in this section concerns the case of a full target state \( \theta = (\theta_1, \theta_2) \) and is stated as follows:

**Theorem 2.3.** Let \( (\theta_1, \theta_2) \in H^1_0(\Omega) \cap H^2(\Omega) \times L^2(\Omega) \) be such that: \( d(x) := \frac{\theta_1}{\theta_1^2 1_{A(\theta_1)}} \in L^\infty(\Omega) \) and that for a.e. \( x \in \Omega \), we have \( \theta_1(x) = 0 \Rightarrow \theta_2(x) = 0 \). We further assume that assumptions \((P_1)-(P_4)\) hold for \( \zeta = \theta_1 \). Then for every initial state \((w_1, w_2) \in H\) and for every \( \epsilon > 0 \), there is a time \( T = T(w_1, w_2, \theta_1, \theta_2, \epsilon) > 0 \) and a piecewise static control \( v(\cdot, t) = v_T(\cdot) \in L^\infty(\Omega) \) such that the respective solution to (23) satisfies:

\[
\|w(T) - \theta_1\|_{H^1(\Omega)} + \|w_T(T) - \theta_2\|_{L^2(\Omega)} < \epsilon. \tag{34}
\]

**Proof.** From Theorem 2.2, we deduce that for any \( \epsilon > 0 \) there is a time \( T_1 = T_1(w_1, w_2, \theta_1) \) such that the control: \( v(x, t) = v_1(x) = -\frac{\Delta \theta_1 + f(\theta_1)}{\theta_1^2} 1_{A(\theta_1)}, \ t \in (0, T_1) \) guarantees the following estimate for the corresponding solution of (23):

\[
\|w(T_1) - \theta_1\|_{H^1(\Omega)} + \|w_T(T_1)\|_{L^2(\Omega)} < \epsilon. \tag{35}
\]

We will continue to control our initial system (23) on \((T_1, T)\) until the achievement of the full target state \( \theta = (\theta_1, \theta_2) \), where \( T > T_1 \) is to be determined.

Consider the following system:

\[
\begin{align*}
\begin{cases}
    w_T &= \Delta w + v(x, t)w - h(x)w_T + f(w), & \text{in } \Omega \times (T_1, T) \\
    w &= 0, & \text{on } \partial \Omega \times (T_1, T) \\
    w(x, T_1) &= w(T_1^-), \quad w_t(x, T_1) = w_t(T_1^-), & \text{in } \Omega
\end{cases}
\]

(36)

We will use Theorem 2.1 to reach \((w(T_1^-), \theta_2)\) at a time \( T > T_1 \) which is close to \( T_1 \). For this end, let us observe that by virtue of (35) the system (36) can be approximated by the following one:

\[
\begin{align*}
\begin{cases}
    \tilde{w}_T &= \Delta \tilde{w} + v(x, t)\tilde{w} - h(x)\tilde{w}_T + f(\tilde{w}), & \text{in } \Omega \times (T_1, T) \\
    \tilde{w} &= 0, & \text{on } \partial \Omega \times (T_1, T) \\
    \tilde{w}(x, T_1) &= \theta_1, \quad \tilde{w}_t(x, T_1) = 0, & \text{in } \Omega
\end{cases}
\]

(37)

Applying Theorem 2.1 to system (37), we deduce the existence of a static control \( v(x, t) = v_2(x) \in L^\infty(\Omega) \) such that the corresponding state \((\tilde{w}(T), \tilde{w}_t(T))\) is close to \((\theta_1, \theta_2)\) at some \( T = T(\theta_1, T_1) \) which is sufficiently close to \( T_1^+ \).

Using the same control \( v(x, t) = v_2(x) \) for (36), we can see by Gronwall’s inequality and the variation of constants formula that:

\[
\|w(t) - \tilde{w}(t)\| \leq \|w(T_1^-) - \theta_1\| e^{C(t-T_1)}, \quad T_1 \leq t \leq T,
\]

where \( C = L + \|v_2\|_{L^\infty(\Omega)} \) and \( L \) is a Lipschitz constant of the function \( F : H \to H; \ y = (w_1, w_2) \mapsto (0, f(w_1) - hw_2) \). Thus

\[
\|w(t) - \tilde{w}(t)\| < e^C \epsilon, \quad \forall t \in (T_1, T), \tag{38}
\]

whenever \( 0 < T - T_1 < 1 \).

We deduce that (34) holds. Then, we conclude that the initial system (23) can be approximately steered to \((\theta_1, \theta_2)\) at \( T \) by using the control:

\[
v(x, t) = \begin{cases}
    v_1(x), & t \in (0, T_1) \\
    v_2(x), & t \in (T_1, T)
\end{cases}
\]

This completes the proof. \( \square \)
Remark 2. According to the proof of Theorem 2.1, we can see that in the case: \(\psi_1, \psi_2 \in W^{2,\infty}(\Omega)\), one can take the control \(v_2(x) = \frac{\partial}{\partial t_1} \psi_1, \psi_2 \) in the time-interval \((T_1, T)\).

3. Exact controllability. In this section, we study the set of target states that can be exactly achieved at a finite time by the system (23) for \(n = 1\). The idea in this part consists first, thanks to the continuity of the Sobolev embedding \(H^1(\Omega) \hookrightarrow C^0(\Omega)\) for \(n = 1\) and \(\Omega = (0, l), l > 0\), in applying the results of Section 2 in order to make the state closer to the desired one at a time \(T_1\) with respect to \(L^\infty\)-norm. Then one can exploit the results of the exact additive controllability of semilinear wave equation to construct a time \(T\) and a control \(v(x, t)\) on \((T_1, T)\) that guarantee the exact steering of the target state at \(T\).

3.1. Damped case. In this part, we will study the exact controllability of the one-dimensional version of the equation (23) evolving in a time-interval \((0, T)\).

For any \(\xi \in H^2(\Omega)\) and \(0 < t_0 < T\), we consider the following system:

\[
\begin{cases}
\psi_{tt} &= \Delta \psi + b \psi - h(x) \psi_t + f(\psi) + 1_{\Omega} u(x, t), \quad \text{in } \Omega \times (t_0, T) \\
\psi(0, t) &= \psi(l, t) = 0, \quad \text{in } (t_0, T) \\
\psi(\cdot, t_0) &= \psi_1, \quad \text{in } \Omega
\end{cases}
\]

(39)

where \(\Omega\) is a sub-domain of \(\Omega\), and let us consider the following property:

\((P_5)\): For every \(t_0 > 0\), the system (39) is exactly null controllable at some time \(T > t_0\) with a control \(u(x, t)\) satisfying

\[
\left( \int_{t_0}^{T} \|u(\cdot, t)\|^2_{L^2(\Omega)} dt \right)^{\frac{1}{2}} \leq c_{T-t_0} \|\psi_1\|_{H^1_0(\Omega)},
\]

(40)

where \(c_{T-t_0} > 0\) is a constant depending on \(T - t_0\).

We refer the reader to [19, 27, 35, 37, 38, 39, 40] for some results on the exact controllability problem for equations like (39).

We are ready to state our first main result of this section.

Theorem 3.1. Let \(n = 1, (w_1, w_2) \in H^1_0(\Omega) \times L^2(\Omega) - \{(0, 0)\}\) and let \(\theta_1 \in H^1_0(\Omega) \cap H^2(\Omega)\) be such that: \(\theta_1 \neq 0\), a.e. in \(\overline{\Omega}\) for some open subset \(O\) of \(\Omega\). Assume that assumptions \((P_1) - (P_5)\) hold for \(\xi = \theta_1\).

Then there exist \(T = T(w_1, w_2, \theta_1) > 0\) and a control \(v(\cdot, \cdot) \in L^2(0, T; L^2(\Omega))\) such that the respective solution to the system (23) satisfies \(w(T) = \theta_1\) and \(w_1(T) = 0\).

Proof. Let us set \(z = w - \theta_1\) in the system (23). We have:

\[
z_t = \Delta z + v(x, t)(z + \theta_1) - h(x)z_t + f(z + \theta_1) + \Delta \theta_1, \quad \forall t \in (0, T), \text{ a.e. } x \in \Omega.
\]

(41)

Then for any fixed \(0 < \epsilon < 1\), it comes from Theorem 2.2 that there is a time \(T_1 > 0\) (large enough) such that the control defined by \(v(x, t) = b_{\theta_1} = -\frac{\Delta \theta_1 + f(\theta_1)}{\theta_1} 1_{\Lambda(\theta_1)}\) guarantees the following estimate:

\[
\| (z(T_1), z_t(T_1)) \|_{H} < \epsilon.
\]

(42)

Let \(T > T_1\), and let us consider the following system:

\[
\begin{cases}
\psi_{tt} &= \Delta \psi + b \psi - h(x) \psi_t + f(\psi) + 1_{\Omega} u(x, t), \quad \text{in } \Omega \times (T_1, T) \\
\psi(0, t) &= \psi(l, t) = 0, \quad \text{in } (T_1, T) \\
\psi(\cdot, T_1) &= z(T_1^-), \quad \text{in } \Omega
\end{cases}
\]

(43)
where \( u(x, t) \) is an additive control.

By assumption, there exists \( T > T_1 \) and \( u(\cdot) \in L^2(T_1, T; L^2(\Omega)) \) satisfying (40) and such that the respective solution to system (43) satisfies: \((\psi(T), \psi(\tau)) = (0, 0)\). Then, in order to construct a control that steers (23) to \((\theta_1, 0)\), it suffices to look for a control \( v(x, t) = v_1(x, t) + b_\theta_1(x) \) on \((T_1, T)\) such that:

\[
v_1(x, t)(\psi(x, t) + \theta_1(x)) = 1_O u(x, t).
\]

For this purpose, we will show that \( \psi(x, t) + \theta_1(x) \neq 0 \), a.e. \( x \in O \times (T_1, T) \), and then take for \( t \in (T_1, T) \):

\[
v_1(x, t) = \begin{cases} \frac{u(x, t)}{\psi(x, t) + \theta_1(x)}, & \text{a.e.} \ x \in O \\ 0, & \text{a.e.} \ x \in \Omega \setminus O \end{cases}
\]

The solution of (43) satisfies the following integral formula:

\[
(\psi, \psi_t)(t) = S(t - T_1)(z(T_1), z_t(T_1)) + \int_{T_1}^{t} S(t - \tau)(0, -h(x)\psi(x, \tau) + b_\theta_1(\tau) + f_\theta_1(\psi(\tau)) + 1_O u(x, \tau))d\tau, \forall t \in [T_1, T].
\]

(44)

Since \( f_\theta_1 \) is Lipschitz and vanishes at 0, we deduce from the formula (44) and by using (40) and (42) that:

\[
\|(\psi, \psi_t)(t)\|_H \leq C \epsilon + C \int_{T_1}^{t} \|(\psi, \psi_t)(\tau)\|_H d\tau, \forall t \in [T_1, T] \text{ (for some } C > 0 \text{),}
\]

which by using the Gronwall’s inequality gives:

\[
\|(\psi, \psi_t)(t)\|_H \leq C \epsilon, \forall t \in [T_1, T], \ (C > 0)
\]

and so

\[
\|(\psi(t))\|_{H_0^1(\Omega)} \leq C \epsilon, \forall t \in [T_1, T].
\]

This together with the continuity of the embedding for \( n = 1 \ H_0^1(\Omega) \to L^\infty(\Omega) \) (see e.g. [1]) gives:

\[
\|(\psi(t))\|_{L^\infty(\Omega)} \leq C \epsilon, \ (C > 0).
\]

(45)

Moreover, since \( \theta_1 \neq 0 \), a.e. in \( \Omega \), we deduce from the fact that the embedding \( H^1(\Omega) \to C^0(\overline{\Omega}) \) is continuous (recall that \( n = 1 \)) that \( |\theta_1| \geq \mu > 0 \), a.e. in \( O \).

Then taking \( 0 < \epsilon < \frac{\mu}{2C} \) in (45), we deduce that for all \( t \in (T_1, T) \) we have

\[
|\psi(t) + \theta_1| \geq |\theta_1| - |\psi(t)| \geq \frac{\mu}{2}, \text{ a.e. in } O.
\]

(46)

Then, one can choose the control \( v_1 \) as follows:

\[
v_1(x, t) = \frac{u(x, t)}{\psi(x, t) + \theta_1(x)} 1_{O \times (T_1, T)}.
\]

(47)

From (46) and the fact that \( u \in L^2(T_1, T; L^2(\Omega)) \), it comes that \( v_1 \in L^2(T_1, T; L^2(\Omega)) \).

With this control, the system (41) becomes:

\[
\begin{cases}
\varepsilon t = Az + b_\theta_1, z - h(x)z_t + f_\theta_1(z) + (z + \theta_1) \frac{u(\cdot, t)}{u(\cdot, t) + \theta_1} 1_O, & \text{in } \Omega \times (T_1, T) \\
z(0, t) = z(t, t) = 0, & \text{in } (T_1, T) \\
z(\cdot, T) = z(T_1^-), z(\cdot, T_1) = z_t(T_1^-), & \text{in } \Omega
\end{cases}
\]

(48)

It is obvious that \( \psi \) is a solution of (48). Let us show that this is the unique one.
Let $z \in H^1_0(\Omega)$ be a solution of (48). The Hölder’s inequality leads to:

$$
\int_{T_1}^T \|u(s)(z(s) - \psi(s))\|_{L^2(\Omega)} ds \leq \|u(\cdot, t)\|_{L^2(T_1, T; L^2(\Omega))} \|z(s) - \psi(s)\|_{L^2(T_1, T; H^1_0(\Omega))} ds
$$

for some constant $C > 0$. This together with (46) and the variation of constants formula enables us to establish the following inequality:

$$
\|z(t) - \psi(t)\| \leq (C_1 + C_2\|u(\cdot, t)\|_{L^2(T_1, T; L^2(\Omega))}) \left( \int_{T_1}^T \|z(s) - \psi(s)\|^2 ds \right)^{\frac{1}{2}},
$$

for some constants $C_1, C_2 > 0$. As a consequence we have $z(t) - \psi(t) = 0$ for all $t \in [T_1, T]$. Then solution of the system (41) is such that $z(T) = 0$ and $z_t(T) = 0$ and hence $w(T) = \theta_1$ and $w_t(T) = 0$.

We conclude that the control defined by:

$$
v(\cdot, t) = \begin{cases} 
-\frac{\Delta \theta_1 + f(\theta_1)}{u(\cdot, t)}, \theta_1 + 1_{O} - \frac{\Delta \theta_1 + f(\theta_1)}{\theta_1} 1_{\Lambda(\theta_1)}, & t \in (0, T_1) \\
\frac{u(\cdot, t)}{\theta_0}, & t \in (T_1, T)
\end{cases}
$$

steers the system (23) from $(w_1, w_2)$ to $(\theta_1, 0)$ at $T$.

3.2. Example. Here, we will present an illustrating example for Theorems 2.3 & 3.1. Let us consider the following semilinear and linear systems respectively with additive control:

$$
\begin{cases}
\psi_{tt} = \Delta \psi + \mu(x)\psi - h(x)\psi_t + k(\psi) + 1_O u(x, t), & \text{in } Q_T := \Omega \times (0, T) \\
\psi(0, t) = \psi(t, t) = 0, & \text{in } (0, T) \\
\psi(0, 0) = \psi_1, \psi_t(0, 0) = \psi_2,
\end{cases}
$$

and

$$
\begin{cases}
\varphi_{tt} = \Delta \varphi + \mu(x)\varphi + 1_O u_0(x, t), & \text{in } Q_T \\
\varphi(0, t) = \varphi(t, t) = 0, & \text{in } (0, T) \\
\varphi(0, 0) = \varphi_1, \varphi_t(0, 0) = \varphi_2,
\end{cases}
$$

where $O$ is an open subset of $\Omega$, the functions $\mu$ and $h$ are such that $\mu, h \in L^\infty(\Omega)$, the nonlinear term $k : R \to R$ is Lipschitz, $u_0(x, t)$ and $u(x, t)$ are the additive controls and belong to $L^2(O \times (0, T))$.

Let us first examine the exact controllability of (49). For this end, we start with proving that under the assumption of exact controllability of the linear part (50), the semilinear system (49) is exactly controllable over the same time interval as the linear version (50).

The following elementary controllability result for the system (49) is sufficient for our purpose.

**Lemma 3.2.** Assume that:

(i) $\text{supp}(h) \subset O$, and

(ii) for all $y \in L^2(\Omega)$, we have $\text{supp}(k \circ y) \subset O$.

If the linear system (50) is null exactly controllable with a control satisfying (40), then so is the semilinear system (49).
Proof. Let us consider the system (49) and the following one:
\begin{align}
\begin{cases}
\varphi_{tt} & = \Delta \varphi + \mu(x) \varphi + 1_\Omega u_0(x,t), \quad \text{in } Q_T \\
\varphi(0,t) & = \varphi(t,0) = 0, \quad \text{in } (0,T) \\
\varphi(.,0) & = \psi_1, \varphi(.,0) = \psi_2, \quad \text{in } \Omega
\end{cases}
\end{align}
(51)

For any couple of control $u_0(x,t)$ and corresponding solution $\varphi$ of (51), we consider the control: $u(x,t) = u_0(x,t) + h(x) \varphi_t(x,t) - k(\varphi(x,t))$.

From the assumptions on $u_0(x,t)$, $h$ and $k$ we can see that $u(x,t)$ satisfies (40) and $1_\Omega u(x,t) = 1_\Omega u_0(x,t) + h(x) \varphi_t(x,t) - k(\varphi(x,t))$.

Then with the control $u(x,t)$, the system (49) takes the form:
\begin{align}
\begin{cases}
\psi_{tt} & = \Delta \psi + \mu(x)(\psi - \varphi_t) + (k(\psi) - k(\varphi)) + 1_\Omega u_0(x,t), \quad \text{in } Q_T \\
\psi(0,t) & = \psi(t,0) = 0, \quad \text{in } (0,T) \\
\psi(.,0) & = \psi_1, \psi(.,0) = \psi_2, \quad \text{in } \Omega
\end{cases}
\end{align}
(52)

which admits $\varphi$ as a particular solution, and by uniqueness it comes $\psi = \varphi$. Hence the null exact controllability of the semilinear system (49) follows from the one of its linear part (51).

Let us now describe our illustrative example. Consider the system (23) with $h = 1_\Omega$ for some proper open subset $O = (a,b)$ of $\Omega = (0,3)$ such that $[1,2] \subset O$ and let $f$ be such that $f(y)(x) = k(y(x))$, where $k(s) = -\alpha(s)$ and $\alpha(s) = -c(s-a)^2(s-b)^21_\Omega$, $c > 0$. Here, the function $k : \mathbb{R} \rightarrow \mathbb{R}$ is $C^1$ and $\text{supp}(k) \subset O$, so $k$ is Lipschitz.

Now in order to define our target state, consider the function defined by:
\[\xi_1(x) = \begin{cases}
x, & x \in [0,1] \\
(1 - (x-1)(x-2))e^{(x-1)^2(x-2)^2}, & x \in [1,2] \\
(-x+3), & x \in [2,3]
\end{cases}\]
and for each $\eta > 0$ we set $\xi_\eta = \eta \xi_1$, and consider the target state $(\theta_1, \theta_2) = (\xi_\eta, -\Delta \xi_\eta)$. We have $(\theta_1, \theta_2) \in \left(H^1_0(0,3) \cap H^2(0,3)\right) \times L^2(0,3)$ and $\frac{\Delta \theta_1}{\theta_1} \in L^\infty(0,3)$.

Moreover for $\eta > 0$ small enough i.e. $\eta < \frac{a}{\|\xi_1\|_{L^\infty(\Omega)}}$, we have $k(\xi_\eta(x)) = 0$, $\forall x \in \Omega$ so that $b_{\theta_1}(x) = -\frac{\Delta \theta_1}{\theta_1} = \frac{b_\theta}{\theta_1}$.

Let us show that (24) holds. For this end, let us write: $b_{\theta_1}(x)y(x) + f(y)(x) = p(x)y(x) + q(y(x))$ with $p(x) = b_{\theta_1}(x) - \|b_{\theta_1}\|_{L^\infty(\Omega)}1_{(1,2)} \leq 0$ and $q(s) = k(s) + s\|b_{\theta_1}\|_{L^\infty(\Omega)}1_{(1,2)}$.

Observing that $sq(s) = sk(s) \leq 0$ for all $s \in O \setminus (1,2)$, we can see that $sq(s) \leq 0$, $\forall s \in \mathbb{R}$, whenever $c > \|b_{\theta_1}\|_{L^\infty(\Omega)} \max_{s \in (1,2)} \frac{a}{(s-a)^2(s-b)^2}$.

From [19, 36], we deduce that the estimate (24) holds. Moreover, we have $\langle b_{\theta_1}(x)y + f(y), y \rangle \leq 0$, $y \in L^2(\Omega)$. Then according to Theorem 2.3, one can approximately achieve $(\theta_1, \theta_2)$ (for $\eta < \frac{a}{\|\xi_1\|_{L^\infty(\Omega)}}$) using the control:
\[v(x,t) = \begin{cases}
v_1(x) = -\frac{\Delta \xi_1}{\xi_1}1_\Omega, & t \in (0, T_1) \\
v_2(x), & t \in (T_1, T_2)
\end{cases}\]
for $T_1$ large enough ($T_1 \rightarrow +\infty$), $T_2$ sufficiently close to $T_1$ and for $a,b,c$ satisfying the above mentioned conditions.

Here, the control $v_2 \in W^{2,\infty}(\Omega)$ is an approximation of $-\frac{\Delta \xi_1}{(T_1-T_1)}1_\Omega$ in $L^2(\Omega)$.

Let us establish the null exact controllability of the additive-control system (43). From the definition of $\theta_1$, $h$ and $f$, it is clear that the assumptions of Lemma 3.2
are satisfied for $\mu = b_0$. Moreover, we know that (see [30, 39, 40]) there exist a $T > T'_1$ and a control $u_0 \in L^2(T'_1; T; L^2(O))$ satisfying (40) for $t_0 = T'_1$ that steers the linear system:

\[
\begin{align*}
\varphi_H &= \Delta \varphi + b_0 \varphi + 1_O u_0(x, t), & \text{in } \Omega \times (T'_1, T) \\
\varphi(0, t) &= \varphi(3, t) = 0, & \text{in } (T'_1, T) \\
\varphi(\cdot, T'_1) &= w(T'_1 - \theta_1, \varphi(\cdot, T'_1) = w_1(T'_1 - \theta_1), & \text{in } \Omega}
\end{align*}
\]

(52)

to $(0, 0)$ at $T$. Then it follows from Lemma 3.2 that the control $u(x, t) = u_0(x, t) + h(x) \varphi_t - f_{\theta_1}(\varphi)$ guarantees the null exact steering of system (43) to $(0, 0)$ and satisfies (40). Then applying Theorem 3.1, we deduce that the control:

\[
v(x, t) = \begin{cases} \\
\frac{-\Delta \xi_1}{\xi_1} 1_O, & t \in (0, T_1) \\
v_2(x), & t \in (T_1, T'_1) \\
\frac{\Delta \xi_1}{\varphi(x, t) + \Delta \xi_1} 1_O, & t \in (T'_1, T)
\end{cases}
\]

guarantees the exact steering of system (23) to $(\theta_1, 0)$, where $\psi$ is the solution of system (43) corresponding to the control $u(x, t)$.

3.3. Undamped case. In this subsection, we will establish an exact controllability result for an uniform time $T$ when dealing with undamped equation. We consider the following one dimensional undamped equation:

\[
\begin{align*}
w_{tt} &= \Delta w + f(w) + v(x, t)w, & \text{in } \Omega \times (0, T) \\
w(0, t) &= w(l, t) = 0, & \text{in } (0, T) \\
w(x, 0) &= w_1, w_t(x, 0) = w_2, & \text{in } \Omega.
\end{align*}
\]

(53)

where $f$ is globally Lipschitz. In the context of additive controls, Zuazua [40] has considered the exact internal controllability of the one dimensional version of following semilinear system:

\[
y_{tt} = \Delta y + f(y) + 1_O u(x, t),
\]

(54)

for some open subset $O$ of $\Omega$. The multidimensional case has been treated in [39].

For what follows, we need some notations: For any fixed $x_0 \in \mathbb{R}^n$, we set $\Gamma_0 = \{x \in \partial \Omega / (x - x_0) \cdot \nu(x) > 0\}$, where $\nu(x)$ denotes the unit outward normal vector of $\Omega$ at $x \in \partial \Omega$. Let the zone of action for (54) be such that: $O = \Omega \cap O_e(\Gamma_0)$ for some $\epsilon > 0$, where $O_e(\Gamma_0) := \{x \in \mathbb{R}^n / |x - z| < \epsilon \text{ for some } z \in \Gamma_0\}$.

In the sequel, we set $\alpha(x_0) := 2 \max_{x \in \Omega} |x - x_0|$.

The next theorem states our second main result of this section.

**Theorem 3.3.** Let $n = 1$ and let $T > 2\alpha(x_0)$ for some fixed $x_0 \in \mathbb{R} \setminus \Omega$. Let $(w_1, w_2) \in H^1_0(\Omega) \times L^2(\Omega) - \{(0, 0)\}$, and let $\theta_1 \in H^1_0(\Omega) \cap H^2(\Omega)$ be such that $\theta_1(x) \neq 0$, a.e. $x \in \overline{O}$ and $b_0 \in L^\infty(\Omega)$.

If the target state $\theta := (\theta_1, 0)$ is approximately reachable at a time $T_1 \in (0, \alpha(x_0))$ with a control $v_{T_1} \in L^2(0, T_1; L^2(\Omega))$, then there exists a control $v \in L^2(0, T; L^2(\Omega))$ such that the corresponding solution of (53) satisfies: $w(T) = \theta_1$ and $w_1(T) = 0$.

**Proof.** Let $T > 2\alpha(x_0)$ be fixed, and let us set $z = w - \theta_1$ in the system (53). Then $z$ satisfies the following system:

\[
\begin{align*}
z_{tt} &= \Delta z + v(x, t)(z + \theta_1) + f(z + \theta_1) + \Delta \theta_1, & \text{in } \Omega \times (0, T) \\
z(0, t) &= z(l, t) = 0, & \text{in } (0, T) \\
z(x, 0) &= w_1 - \theta_1, z_t(x, 0) = w_2, & \text{in } \Omega
\end{align*}
\]

(55)
For any fixed $0 < \epsilon < 1$, there are $T_1 \in (0, \alpha(x_0))$ small enough and a control $v_{T_1}$ that provide the following estimate:

$$\|z(T_1), z_t(T_1)\|_H < \epsilon.$$  \hspace{1cm} (56)

Letting $v(x, t) = v_1(x, t) + b_{\theta_1}$ in (55), we obtain the following homogeneous system:

$$\begin{cases}
  z_{tt} & = \Delta z + b_{\theta_1} z + v_1(x, t)(z + \theta_1) + f_{\theta_1}(z), & \text{in } \Omega \times (0, T) \\
  z(0, t) & = z(l, t) = 0, & \text{in } (0, T) \\
  z(x, 0) & = w_1 - \theta_1, z_t(x, 0) = w_2, & \text{in } \Omega
\end{cases}$$  \hspace{1cm} (57)

Let us now consider the following system:

$$\begin{cases}
  \psi_{tt} & = \Delta \psi + b_{\theta_1} \psi + f_{\theta_1}(\psi) + 1_{O}u(x, t), & \text{in } \Omega \times (T_1, T) \\
  \psi(0, t) & = \psi(l, t) = 0, & \text{on } \partial \Omega \times (T_1, T) \\
  \psi(\cdot, T_1) & = z(T_1^-), \psi_t(\cdot, T_1) = z_t(T_1^-), & \text{in } \Omega
\end{cases}$$  \hspace{1cm} (58)

where $u(x, t)$ is an additive control.

From [39], the system:

$$\begin{cases}
  \tilde{\psi}_{tt} = \Delta \tilde{\psi} + b_{\theta_1} \tilde{\psi} + f_{\theta_1}(\tilde{\psi}) + 1_{O}u(x, t + T_1), & \text{in } \Omega \times (0, T - T_1) \\
  \tilde{\psi}(\cdot, t) = \psi(\cdot, t) = 0, & \text{on } \partial \Omega \times (0, T - T_1) \\
  \tilde{\psi}(\cdot, 0) = z(T_1^-), \tilde{\psi}_t(\cdot, 0) = z_t(T_1^-), & \text{in } \Omega
\end{cases}$$  \hspace{1cm} (59)

is exactly null controllable at $T - T_1 > \alpha(x_0)$ with a steering control $u(\cdot, \cdot + T_1)$ that evolves in $L^2(0, T - T_1; L^2(\Omega))$ and satisfies the estimate:

$$\int_{0}^{T - T_1} \|u(\cdot, t + T_1)\|_{L^2(\Omega)}^2 \, dt \leq C\|(z(T_1), z_t(T_1))\|_H^2,$$  \hspace{1cm} (60)

for some constant $C = C(T)$. In other words, we have $u(\cdot, \cdot) \in L^2(T_1, T; L^2(\Omega))$ and

$$\int_{T_1}^{T} \|u(\cdot, t)\|_{L^2(\Omega)}^2 \, dt \leq C\|(z(T_1), z_t(T_1))\|_H^2,$$  \hspace{1cm} (61)

where the positive constant $C$ can be chosen independent of $T_1$. Furthermore, the respective solution of the system (58) satisfies:

$$\tilde{\psi}(T - T_1), \tilde{\psi}_t(T - T_1)) = (0, 0).$$

Then, in order to construct a control that steers (53) to $(\theta_1, 0)$ at $T$, it suffices to look for a control $v(x, t)$ on $(T_1, T)$ such that:

$$v(x, t)(\psi(x, t) + \theta_1(x)) = 1_{O}u(x, t), \text{ a.e. } x \in \Omega.$$  \hspace{1cm} (53)

For the remainder part, it suffices to reproduce the corresponding part in the proof of Theorem 3.1 to deduce that the state $(\theta_1, 0)$ can be exactly achieved using the following control:

$$v(\cdot, t) = \begin{cases}
  v_{T_1}(\cdot, t), & t \in (0, T_1) \\
  b_{\theta_1} + \frac{u(\cdot, t)}{\psi(\cdot, t) + \theta_1}1_{O}, & t \in (T_1, T)
\end{cases}$$

\[\square\]

Remark 3.  
1. The same result of the above theorem remains true if $T \geq k\alpha(x_0)$ and $T_1 \in (0, (k - 1)\alpha(x_0))$, $k \geq 2$.
2. The results of Theorems 3.1 & 3.3 can be extended to several dimension in high energy spaces (see [32] for the bilinear case).
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