Stability of Poisson Equilibria and Hamiltonian Relative Equilibria by Energy Methods

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Dedicated to Klaus Kirchgässner on the occasion of his 70th birthday

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Abstract

We develop a general stability theory for equilibrium points of Poisson dynamical systems and relative equilibria of Hamiltonian systems with symmetries, including several generalisations of the Energy-Casimir and Energy-Momentum Methods. Using a topological generalisation of Lyapunov’s result that an extremal critical point of a conserved quantity is stable, we show that a Poisson equilibrium is stable if it is an isolated point in the intersection of a level set of a conserved function with a subset of the phase space that is related to the topology of the symplectic leaf space at that point. This criterion is applied to generalise the energy-momentum method to Hamiltonian systems which are invariant under non-compact symmetry groups for which the coadjoint orbit space is not Hausdorff. We also show that a $G$-stable relative equilibrium satisfies the stronger condition of being $A$-stable, where $A$ is a specific group-theoretically defined subset of $G$ which contains the momentum isotropy subgroup of the relative equilibrium. The results are illustrated by an application to the stability of a rigid body in an ideal irrotational fluid.

1. Introduction

Energy methods for determining the stability of trajectories of Hamiltonian systems are based on the general principle that an equilibrium point is Lyapunov stable if it is a strict local minimum or maximum of a conserved function, such as the Hamiltonian itself. The Energy-Casimir and Energy-Momentum Methods are extensions of this principle to, respectively, equilibrium points of Poisson systems (defined below) and relative equilibria of Hamiltonian systems with symmetry. They originate with the work of Arnold on the stability of equilibria of incompressible fluids [2]. Since then they have been used very extensively in applications to rigid bodies [19, 21, 36], elasticity theory [22, 43, 46, 47], fluids [3, 4, 9, 14, 25], and vortex structures [17, 39, 42, 49]. In this paper we first present a topological
generalisation of the energy method and then use this to obtain significant generalisations of the energy-Casimir and energy-momentum methods. We demonstrate our method by various theoretical examples and also show how it can be applied to obtain new stability results for “underwater vehicles”, modelled as rigid bodies in ideal irrotational fluids.

A Poisson manifold is a manifold $X$ with a Poisson bracket $\{ \cdot, \cdot \}$ defined on the space of smooth functions on $X$; see, e.g., [27]. A Poisson system with Hamiltonian $h : X \to \mathbb{R}$ is characterised by the fact that the time-evolution of any smooth function $f : X \to \mathbb{R}$ along trajectories of the Poisson system satisfies $\dot{f} = \{ f, h \}$. Poisson systems often arise by symmetry reduction of Hamiltonian systems with symmetry. Examples include the Euler equations of ideal fluid dynamics, for which the symmetry group is the particle-relabelling group [4], and the Kirchhoff equations for the symmetry-reduced dynamics of underwater vehicles in ideal irrotational fluids, for which the symmetry group is a Euclidean group (see, for example, [4, 18, 19] and Sections 2.1 and 2.2 of this paper).

The flow of a Poisson system on a Poisson manifold $X$ generated by a Hamiltonian $h$ preserves both $h$ and the symplectic leaves of $X$. In the case of a Poisson system which is a symmetry-reduced Hamiltonian system the invariant symplectic leaves originate from the conserved quantities associated with continuous symmetries of Hamiltonian systems by Noether’s Theorem [27]. A point $x_e$ is an equilibrium point of a Poisson system on $X$ if and only if it is a critical point of the restriction of $h$ to the leaf $L(x_e)$ through $x_e$. If $x_e$ is a local extremum of the restriction, then the standard energy method implies that $x_e$ is stable as an equilibrium point of the flow on $X$. In this case we say that $x_e$ is leafwise stable. In general $x_e$ is not a critical point of $h$ on the full space $X$. To test for stability on the whole of $X$ the energy-Casimir method supposes that there is a function $C$, the Casimir, which is constant on symplectic leaves and such that $x_e$ is a critical point of $h + C$. Stability follows if this critical point is a local extremum.

When is it possible to find a Casimir $C$ such that $x_e$ is a critical point of $h + C$? One case is when $x_e$ is a regular point of $X$, which means that locally the foliation into symplectic leaves is non-singular. Using this fact ARNOLD [2, 3] and LIBERMANN & MARLE [23] show that if $x_e$ is regular and is a local extremum of the restriction of $h$ to the leaf $L(x_e)$, then $x_e$ is stable for the full flow on $X$. Thus at regular points this test for leafwise stability is also a test for full stability. Examples show that this is not true in general, see [23, Exercise IV 15.10] and Examples 4, 5 and Section 5.5 of this paper. In such cases it is natural to ask whether there exists a space between $L(x_e)$ and $X$ such that $x_e$ is stable if it is an extremal point of the restriction of $h$ to the intermediate space. In this paper we show that there is such a space. More generally we answer a challenge posed by WEINSTEIN [51] when, referring to the interaction between Poisson structures and stability, he wrote: “As yet there is no general theory for this kind of analysis”.

Most of the results in this paper are based on a topological generalisation of the energy method (Corollary 1) which generalises a lemma of MONTALDI [32]. Corollary 1 is valid for a continuous flow on a locally compact topological space $X$ which has conserved quantities with values in another topological space. In the case of Poisson systems the conserved quantities are the Hamiltonian $h$ and the quotient
map to the space of symplectic leaves. An equilibrium $x_e$ is stable if the leafspace $L(x_e)$ is Hausdorff at $L(x_e)$ and $x_e$ is an isolated point in the fibre of the restriction of $h$ to $L(x_e)$. Thus the condition that $x_e$ be regular in the result of Arnold, Libermann and Marle can be relaxed to the leafspace being Hausdorff at $L(x_e)$. However in applications, such as Poisson systems which arise by symmetry reduction of Hamiltonian systems with non-compact, non-Abelian symmetry groups, this condition is often not satisfied. In particular it is violated in the examples from fluid dynamics we mentioned earlier. If the leafspace is not Hausdorff, then $h$ must isolate $x_e$ in a larger subset $T_2(x_e)$ which depends only on the topology of the leafspace (Theorem 1).

We recover and generalise the energy-Casimir method for Poisson equilibria: we will see that it suffices to make the assumptions of the energy-Casimir method on a subset of the Poisson manifold $X$ which contains $T_2(x_e)$. Example 7 shows that this improvement can succeed where the standard energy-Casimir method fails. Our topological results provide stability tests which are even more general.

Moreover we identify a necessary condition for the energy-Casimir method to apply, namely that the Poisson equilibrium must be tame. This condition arises from the requirement that there exists a Casimir $C$ such that the first derivative of $h + C$ at $x_e$ is zero before confinement can be established using the second derivative. If the leafspace is Hausdorff at $L(x_e)$, for example if $x_e$ is a regular point of the Poisson manifold, then $x_e$ is automatically tame. However, Poisson systems obtained by reducing Euclidean symmetry generally have equilibria which are not tame (wild). The energy-Casimir method cannot be applied to these and equilibria which are leafwise stable may be unstable in the full space, as Examples 5 and 10 below show. See also the analogous discussion of wild relative equilibria below in Section 2.2.

We now turn to the stability of relative equilibria of Hamiltonian systems with symmetry. Relative equilibria are trajectories that become equilibria after symmetry reduction. In applications they typically correspond to motions that are equilibria in appropriate rotating or translating coordinate systems. Examples that we will study in detail in this paper are relative equilibria of systems with Euclidean symmetries, and in particular rigid bodies in fluids. We consider a $G$-invariant Hamiltonian $H$ on a symplectic manifold (phase space) $P$. By Noether’s Theorem [27] the mechanical system has a conserved quantity for each continuous symmetry, and we assume that the corresponding momentum map $J : P \to g^*$ commutes with the action of the Lie group $G$ on $P$ and its coadjoint action on $g^*$, the dual of the Lie algebra $g$ of $G$. If the action of $G$ on $P$ is free and proper then the orbit space $P/G$ is a Poisson manifold and criteria for the stability of Poisson equilibria can be lifted to criteria for the $G$-stability of relative equilibria. The symplectic leaves of $P/G$ are just the Marsden-Weinstein reduced phase spaces and $J$ induces a homeomorphism between the leafspace of $P/G$ and the coadjoint orbit space $g^*/G$.

Leafwise stability of a relative equilibrium $p_e \in P$ means that it is stable to momentum-preserving perturbations of the initial condition, and is implied if the relative equilibrium is an extremal point of the reduced Hamiltonian. If the momentum $\mu_e = J(p_e)$ is regular, or more generally $g^*/G$ is Hausdorff at $G\mu_e$, then this condition also implies that $p_e$ is $G$-stable (Remark 3). When the Hessian of the reduced Hamiltonian is definite and the momentum regular, this was proved by
Libermann & Marle [23], following earlier work of Arnold [3] and Marsden and Weinstein [29]. The result for $g^* / G$ Hausdorff is due to Montaldi [32].

The energy-momentum method is a convenient lifting of these criteria to the phase space. A point $p_e$ is a relative equilibrium if and only if it is a critical point of the energy-momentum function $H_{\xi_e} = H - J_{\xi_e}$, where $\xi_e$ is the generator of the relative equilibrium, an element of the Lie algebra of $G$, and $J_{\xi_e}(p) = \langle J(p), \xi_e \rangle$. The relative equilibrium is leafwise stable if the restriction of the Hessian $d^2 H_{\xi_e}(p_e)$ to a symplectic normal space $N_1$ is definite [28, 43, 45–47]. Full stability results, i.e., stability with respect to arbitrary, non-momentum-conserving perturbations, were obtained by Patrick [37] for compact groups, and Ortega & Ratiu [34] and Lerman & Singer [20] under more general assumptions which still imply that $g^* / G$ is Hausdorff at $G\mu_e$. See also Corollary 3.

Some stability results for non-compact groups at momentum values $\mu_e$ where $g^* / G$ is not Hausdorff have been obtained by Leonard & Marsden [19] for semi-direct products of compact groups and vector spaces and applications to motions of rigid bodies in fluids. They suggested that it is necessary to test for the definiteness of $d^2 H_{\xi_e}(p_e)$ on a larger subspace of $T_{p_e} P$ than $N_1$. In this paper we sharpen and generalise their results to arbitrary groups by lifting the general stability criteria for Poisson equilibria to conditions on $H_{\xi_e}$. In particular we identify subsets containing $N_1$ on which the definiteness of $d^2 H_{\xi_e}(p_e)$ is sufficient to imply $G$-stability (Theorem 5 and Corollary 4). As in the case for Poisson equilibria, a relative equilibrium must be tame for an application of our stability tests. When a relative equilibrium is wild, stability cannot be established by energy-momentum confinement alone. However in [40] a leafwise-stable wild relative equilibrium of an idealized underwater vehicle is shown to be Lyapunov stable by KAM theory (see also Section 2.2). On the other hand Example 10 shows that leafwise-stable wild relative equilibria can be unstable via Arnold diffusion. We expect the latter behaviour to be typical near a leafwise-stable wild relative equilibrium unless the geometry of the momentum level sets permits the application of KAM stability methods.

The papers [20, 34, 37] all show, under various conditions which in particular imply that $g^* / G$ is Hausdorff at $G\mu_e$, that the definiteness of $d^2 H_{\xi_e}(p_e)$ on $N_1$ implies that $p_e$ is actually $G_{\mu_e}$-stable, i.e., trajectories that start near $p_e$ remain near $G_{\mu_e} p_e$, where $G_{\mu_e}$ is the coadjoint isotropy subgroup at $\mu_e$. However, numerical simulations of a rigid body in a fluid by Leonard & Marsden [19] suggest that this is not always true, even for a regular momentum value $\mu_e$. They prove that these relative equilibria are $\Gamma$-stable for a subgroup $\Gamma$ which lies strictly between $G_{\mu_e}$ and $G$.

In this paper we introduce the general notion of $A$-stability for any subset of $G$ and then show that any $G$-stable relative equilibrium is automatically $A$-stable where $A$ is a “cone about $G_{\mu_e}$” which can be made arbitrarily close to $G_{\mu_e}$ by restricting the perturbations from $p_e$ to be sufficiently small (Theorem 6). In many cases this result can be improved by decomposing $G_{\mu_e}$ into the product of an (essentially) compact subgroup and a non-compact submanifold and showing that the “cone” only needs to be taken about the non-compact part (Theorem 7). As a corollary we obtain a generalisation of the results of [20, 34, 37] on $G_{\mu_e}$-stability (Corollary 8). Unlike the previous results, these do not require a Hessian condition to be satisfied, only that the relative equilibrium is $G$-stable.

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Taken together, our results on the stability of Poisson equilibria and the \( G \)- and \( A \)-stability of relative equilibria provide generalisations of all previous results for finite-dimensional Poisson and Hamiltonian systems that we are aware of. We believe that our most primitive topological tests for stability (Theorems 1 and 3) are the sharpest possible general results, and that they fully explain the interaction between stability properties and Poisson structures. In this paper we restrict attention to free Hamiltonian group actions for simplicity only; the extensions to general proper actions are contained in [41].

We end this introduction with a guide to the rest of the paper. Section 2 contains an introduction to how our stability theory should be applied to relative equilibria of mechanical systems with the commonly encountered Euclidean symmetry groups \( SE(2) \) and \( SE(3) \), and compares the procedure with that for compact groups. It can be read separately from the rest of the paper. We illustrate the method by applying the theory to a rigid body submerged in an ideal irrotational fluid and conclude with a summary of what can be expected for wild relative equilibria. Section 3 contains the fundamental topological ideas that underly the rest of the paper. Topological and derivative tests for the stability of Poisson equilibria are described in Sections 4.1 and 4.2, respectively, Section 4.2 introduces the machinery of smoothings that is used to make sense of derivatives of functions on singular sets and the related notion of tame generators. It also describes the role Casimirs play in stability theory and their limitations. The topological and derivative tests for the \( G \)-stability of Hamiltonian relative equilibria are contained in Sections 5.1 and 5.3, respectively. The intermediate Section 5.2 introduces transverse Poisson structures as tools for describing local leafspace topology and uses this to discuss some special cases. In Section 5.4 we apply the stability criteria to relative equilibria of systems that are invariant under actions of the Euclidean groups \( SE(2) \) and \( SE(3) \), proving the results that we use in Section 2. In Section 5.5 we describe how our theory “explains” an example of Libermann & Marle [23] (see also [16]). Section 6 is devoted to \( A \)-stability: Section 6.1 gives the general result, Section 6.2 the improvements obtained using Lie-group decompositions and Section 6.3 the application to Euclidean groups.

Another brief introduction to the theory developed in this paper is provided by [53].

2. \( T_2 \)-energy-momentum method: A quick-start guide for typical applications

In this section we give a “quick-start” guide to how the stability theory described in the rest of this paper can be applied to mechanical systems with the Euclidean symmetry groups \( SE(2) \) and \( SE(3) \), and compare the method with the well-known energy-momentum method for systems with compact symmetry groups or, more generally, symmetry groups with \( \text{Ad} \)-invariant inner products on their Lie algebras [20, 34, 37].

We consider a mechanical system with Hamiltonian \( H \) on a symplectic manifold, the phase space \( P \), with a symmetry group \( G \) that acts freely, properly and symplectically. By Noether’s Theorem [27] the mechanical system has a conserved
quantity for each continuous symmetry, and we assume that the corresponding momentum map $J: P \rightarrow g^*$ is globally defined and $\text{Ad}^*$-equivariant. Here $\text{Ad}_g \xi = g \xi g^{-1}, g \in G, \xi \in g,$ denotes the adjoint action of $G$ on its Lie algebra $g = T_{id}G$ and $\text{Ad}^*_g$ the dual operator of $\text{Ad}_g$, i.e., $(\text{Ad}^*_g \mu)(\xi) = \mu(\text{Ad}_g \xi)$ for $\mu \in g^*$. The coadjoint action of $G$ on $g^*$ is then given by $g \mu := (\text{Ad}^*_g)^{-1} \mu$ and we make the standard assumption [20, 34, 37] that $J$ is equivariant with respect to that action of $G$ on $g^*$.

Let $X_H$ denote the Hamiltonian vector field on $P$ generated by $H$. A point $p_e$ in $P$ is a relative equilibrium with generator $\xi_e \in g$ if $X_H(p_e) = \xi_e, p_e \in T_{p_e}G_{p_e}$, i.e., the trajectory through $x_e$ is stationary in a frame moving with velocity $\xi_e \in g$. The momentum of the relative equilibrium is $\mu_e = J(p_e) \in g^*$ and we denote its isotropy with respect to the coadjoint action of $G$ on $g^*$ by $G_{\mu_e}$ and the corresponding Lie algebra by $g_{\mu_e}$.

The elements required to apply the $T_2$-energy-momentum test to $p_e$ depend crucially on the pair $(\mu_e, \xi_e)$. These elements are summarised in Table 1 for the commonly encountered cases where the symmetry group admits an Ad-invariant inner product on $g$ (e.g., if $G$ is a compact or Abelian group), or is $SE_2$ or $SE_3$.

The Table lists:

- the class of the relative equilibria, either tame or wild;
- in the cases where the relative equilibrium is tame, a subspace $T \subseteq T_{p_e}P$, called the test space, which complements the tangent space to the group orbit through $p_e$;
- a subset $A \subset G$, called the stability type, which depends on $\mu_e$ only.

The “test space” in the table is actually given as the quotient of one subspace by another subspace. By this we mean that $T$ can be taken to be any complement of the denominator in the numerator. The entries in Table 1 are proved in Section 5.4.

The energy-momentum method for leafwise stability tests definiteness of the Hessian $d^2H_{\mu_e}(p_e)$ of $H_{\mu_e} = H - J_{\mu_e}$ on the symplectic normal space $N_1 = \ker dJ(p_e)/g_{\mu_e}p_e$, i.e., on the tangent space at $p_e$ of a transverse section to the group orbit through $p_e$ inside the momentum level set $J^{-1}(\mu_e)$.

In the case of a symmetry group with Ad-invariant inner product, e.g., a compact or Abelian group, testing for leafwise stability is enough to establish stability with respect to arbitrary, non-momentum-conserving perturbations. Testing for leafwise stability is also sufficient if the momentum value $\mu_e$ of the relative equilibrium is regular. In these situations the leafspace is locally a $T_2$-space, i.e., it is Hausdorff at $\mu_e$ (see Corollary 3 of Section 5.3 and the paragraph thereafter). However for relative equilibria of Euclidean group actions with non-regular momenta $\mu_e$ the test spaces have to be enlarged to accommodate the “non-Hausdorff-ness” of the leafspace locally. We call the resulting test the “$T_2$-energy-momentum method”. When the leafspace is Hausdorff this reduces to the standard energy-momentum method.

In the case of a direct product of two groups (e.g., $SE_2 \times \mathbb{R}$ for underwater vehicles, see Section 2.1), the relative equilibrium is tame if both of the classes corresponding to the components of its momentum and generator under the direct
Elements for the $T_2$-energy-momentum method for commonly encountered symmetry groups. Elements of the Lie algebra of the group $SE(2) = SO(2) \rtimes \mathbb{R}^2$ are denoted $(\xi^r, \xi^a) \in \mathfrak{so}(2) \rtimes \mathbb{R}^2$, with similar notation for the dual $\mathfrak{se}(2)^*$ and for the Lie algebra of $SE(3)$ and its dual. The set $C(\mu_{ae})$ is any open cone containing $\mu_{ae}$. The set $A_{\varepsilon_0,\varepsilon_1} = \left\{(R, a) \in SE(3) : |\sin \theta_{R,\mu_e}| < \varepsilon_1 |a| + \varepsilon_0 \right\}$, where $\theta_{R,\mu_e}$ is the angle between $R\mu_e$ and $\mu_e$, and $\varepsilon_0$ and $\varepsilon_1$ are positive.

| Group | $\langle \mu_e, \xi_e \rangle$ | Class | Test Space | Stability Type |
|-------|-------------------------------|-------|------------|---------------|
| Ad-invariant metric on $g$ | any $\mu_e$, $\xi_e$ | tame | $\ker dJ(p_e)/\mathfrak{g}_{\mu_e} p_e$ | $G_{\mu_e}$ |
| $SE(2) = SO(2) \rtimes \mathbb{R}^2$ | $\mu_a^e \neq 0$ | tame | $\ker dJ(p_e)/((0) \times \mathbb{R}\mu_a^e) p_e$ | $\{1\} \times C(\mu_a^e)$ |
| | $\mu_a^e = 0, \xi_a^e = 0$ | tame | $dJ(p_e)^{-1}(\mathfrak{so}(2)^*)/\mathfrak{se}(2) p_e$ | $SE(2)$ |
| | $\mu_a^e = 0, \xi_a^e \neq 0$ | wild | Confinement fails | $SE(2)$ |
| $SE(3) = SO(3) \rtimes \mathbb{R}^3$ | $\mu_a^e \neq 0$ | tame | $\ker dJ(p_e)/((\mathbb{R}\mu_a^e) \times \mathbb{R}\mu_a^e) p_e$ | $SO(2) \times C(\mu_a^e)$ |
| | $\mu_a^e = 0, \xi_a^e = 0, \mu_e^e \neq 0$ | tame | $dJ(p_e)^{-1}(\mathfrak{so}(3)^*)/\mathfrak{se}(3) p_e$ | $A_{\varepsilon_0,\varepsilon_1}$ |
| | $\mu_a^e = 0, \xi_a^e = 0, \mu_e^e = 0$ | tame | $dJ(p_e)^{-1}(\mathfrak{so}(3)^*)/\mathfrak{se}(3) p_e$ | $SE(3)$ |
| | $\mu_a^e = 0, \xi_a^e \neq 0, \mu_e^e \neq 0$ | wild | Confinement fails | $A_{\varepsilon_0,\varepsilon_1}$ |
| | $\mu_a^e = 0, \xi_a^e \neq 0, \mu_e^e \neq 0$ | wild | Confinement fails | $SE(3)$ |

The product decomposition are tame. The test space and stability type in such a case are the products of those corresponding to the two components.

A relative equilibrium $p_e$ is said to be $A$-stable if all sufficiently small perturbations of $p_e$, including momentum-changing perturbations, initiate trajectories which remain arbitrarily close to $Ap_e$. The main practical implications of the theory developed in this paper are as follows:

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– If $p_e$ is tame, then $p_e$ is $A$-stable if the Hessian $d^2H_{\xi_e}(p_e)$ is definite when restricted to $T \subseteq T_{p_e} P$. See Corollary 3 for the $G$-stability result, Corollaries 5 and 6 for applications to Euclidean systems, Corollary 8 for $A$-stability of relative equilibria of general symmetry groups and Section 6.3 for $A$-stability in the case of Euclidean symmetry groups.

– If $p_e$ is wild, then stability cannot be established by energy-momentum confinement alone, as shown in Example 10. However, if the relative equilibrium $p_e$ can be proved to be $G$-stable by any other means (e.g., KAM methods, see Section 2.2) then $p_e$ is $A$-stable. See Section 6 and especially Theorem 6.

The next two subsections, respectively, illustrate these two aspects of the theory.

### 2.1. A rigid body in an ideal irrotational fluid

The motion of a neutrally buoyant body submerged in an inviscid irrotational fluid modelled by Kirchhoff’s equations is studied by Leonard [18] and by Leonard & Marsden [19]. The configuration manifold $SE(3)$ of the body is determined by its position $a \in \mathbb{R}^3$ and orientation $R \in SO(3)$; $(R, a) \in SE(3) = SO(3) \ltimes \mathbb{R}^3$. The Lagrangian on $TSE(3)$ is given by

$$L(R, a, \Omega, v) = \frac{1}{2} \Omega^T I \Omega + \Omega^T Dv + \frac{1}{2} v^T M v + mgl(k \cdot R k).$$

Here the tangent vectors are represented by their left translates $(\Omega, v)$ in $se(3) = so(3) \oplus \mathbb{R}^3$, the angular and linear velocities of the body in body coordinates. The operators $I$, $D$, and $M$ are constant $3 \times 3$ matrices that can be calculated from the shape and mass distribution of the body, $I$ and $M$ being symmetric. The reference configuration is such that the centre of buoyancy is at the origin and the centre of mass is at distance $l$ below the centre of buoyancy. Thus the body is bottom heavy when $l > 0$ and top heavy when $l < 0$. The Lagrangian is given by a left invariant kinetic energy on $TSE(3)$, with a nonzero potential energy function if $l \neq 0$.

For simplicity we consider an ellipsoidal body with principle axes of inertia along the axes of symmetry of the ellipsoid. In this case $I$ and $M$ are diagonal and $D = mlk\wedge$, where, for any $a \in \mathbb{R}^3$, $a\wedge$ is the antisymmetric matrix such that $a\wedge x = a \times x$. We denote the diagonal elements of $I$ by $I_1$, $I_2$ and $I_3$ and similarly denote the diagonal elements of $M$. If $l \neq 0$ the symmetries of this system are given by the left action on $SE(3)$ of the subgroup $SE(2) \times \mathbb{R}$ consisting of rotations about the vertical and translations in any direction. In the case $l = 0$, when the centres of gravity and buoyancy coincide and the potential energy vanishes, the symmetry group is the full left action of $SE(3)$ on itself. If $M_1 = M_2$ and $I_1 = I_2$ then there is a further material symmetry of the system: $SO(2) = \{ \exp(k\wedge \theta) \}$ acts as a subgroup of $SE(3)$ by inverse multiplication on the right. However we will not consider this case in this paper.

Assume that $l \neq 0$, so the symmetry group is $SE(2) \times \mathbb{R}$. We use left translation to identify the phase space $TSE(3)$ with

$$TSE(3) = SE(3) \times se(3) = SE(3) \times (\mathbb{R}^3)^2 = \{ (R, a, \Omega, v) \}.$$
The Legendre transformation, mapping $TSE(3)$ to $T^*SE(3)$ introduces the dual variables $\pi$ and $p$ to $\Omega$ and $v$ by

$$\pi = \frac{dL}{d\Omega} = I\Omega + mlk \times v, \quad p = \frac{dL}{dv} = Mv - mlk \times \Omega,$$

and the Hamiltonian

$$H = P'v + \Pi'\Omega - L = \frac{1}{2}\Omega' I\Omega + \Omega' Dv + \frac{1}{2}v' Mv - mgl(k \cdot R).$$

In these dual coordinates the momentum map $\mu = (\mu^r, \mu^a) = (\Pi_3, P) \in so(2)^* \oplus (\mathbb{R}^3)^*$ for the $SE(2) \oplus \mathbb{R}$ symmetry is

$$\Pi_3 = (R\pi + a \times Rp) \cdot k, \quad P = Rp.$$

The variable $\Gamma = R'k$ is convenient for the equations of motion, which are

$$R^{-1} \frac{dR}{dt} = \Omega, \quad R^{-1} \frac{da}{dt} = v, \quad \frac{d\pi}{dt} = \pi \times \Omega + p \times v - mg\Gamma \times k, \quad \frac{dp}{dt} = p \times \Omega, \quad \frac{d\Gamma}{dt} = \Gamma \times \Omega.$$

By substitution

$$p_e : \quad R = 1, \quad a = 0, \quad \pi = \alpha_e k, \quad p = \beta_e k, \quad \Gamma = k,$$

is a relative equilibrium with generator and momentum

$$\xi'_e = \frac{\alpha_e}{I_3}, \quad \xi''_e = \frac{\beta_e}{M_3} k, \quad \mu'_e = \alpha_e, \quad \mu''_e = \beta_e k.$$

This relative equilibrium corresponds to a falling body rotating about the axis of inertia parallel to the line joining the centres of gravity and buoyancy, which is aligned with the vertical. This case is not fully considered in [19].

The symmetry group is the direct product of $SE(2)$ and $\mathbb{R}$ and so the elements required to apply the $T_2$-energy-momentum test can be found from Table 1. Since $\mathbb{R}$ is Abelian, it trivially has an Ad-invariant inner product, all momentum-generator pairs are tame and $G_{\mu_e}$ is always the whole of $\mathbb{R}$. The component of $(\mu_e, \xi_e)$ corresponding to $SE(2)$ is tame if and only if $\alpha_e = 0$, i.e., if the rigid body only translates and does not rotate. Thus $p_e$ is tame if and only if $\alpha_e = 0$. Since energy-momentum confinement is impossible if $p_e$ is wild, we are forced to assume $\alpha_e = 0$ to proceed with the test.

Noting that the $se(2)^*$ component of the momentum is zero, we see that Table 1 gives the test space to be $T = dI(p_e)^{-1} (so(2)^* / (se(2) \times \mathbb{R})) p_e$ and the stability type to be $A = SE(2) \times \mathbb{R} = G$. Consequently the $T_2$-energy-momentum method of this paper says that if $d^2H_{\xi_e}$ is definite on any subspace $T$ such that

$$dI(p_e)^{-1} (so(2)^* / (se(2) \times \mathbb{R})) p_e = T \oplus (se(2) \times \mathbb{R}) p_e,$$

then $p_e$ is $SE(2) \times \mathbb{R}$-stability.
Since the Lagrangian $L$ is defined on $TSE(3)$, we can use the identification

$$T (TSE(3)) = \{(R, a, \Omega, v), (\delta R, \delta a, \delta \Omega, \delta v)\}$$

defined by

$$\frac{d}{dt} \bigg|_{t=0} (R \exp(t \delta R^\wedge), a + t \delta a, \Omega + t \delta \Omega, v + t \delta v).$$

We do not need the derivative of the first ($\mu_r^e = \Pi_3(p_e)$) component of the momentum map since we only have to find $dJ^{-1}(so(2))$. The derivative of the second component of the momentum map is

$$\delta P = R \delta R \times p + R \delta p$$

so that at $p_e$ the derivative is

$$\delta P = \beta_e \delta R \times k + M \delta v - mlk \times \delta \Omega.$$ 

Thus the test space is

$$T = \{(\delta R, \delta a, \delta \Omega, \delta v) : \delta v = M^{-1}(k \times (ml \delta \Omega + \beta_e \delta R)) \}/(se(2) \times \mathbb{R})p_e.$$

For $(\xi^r, \xi^a) \in se(2) \times \mathbb{R}$ the tangent vector $(\xi^r, \xi^a).p_e$ is

$$\delta R = \xi^r k, \quad \delta a = \xi^a, \quad \delta \Omega = 0, \quad \delta v = 0$$

and so the test space may be realised as

$$T = \{(\delta R, \delta a, \delta \Omega, \delta v) : \delta R \cdot k = 0, \delta a = 0, \delta v = M^{-1}(k \times (ml \delta \Omega + \beta_e \delta R)) \}.$$

Since

$$J_{\xi^r} = \frac{\beta_e}{M_3} \cdot R p = \frac{\beta_e}{M_3} \cdot R(Mv - mlk \times \Omega),$$

we must find the Hessian of the function

$$H_{\xi^r} = H - J_{\xi^r}$$

$$= \frac{1}{2} \Omega^t l \Omega + \Omega^t Dv + \frac{1}{2} v^t M v - mgl(k \cdot R k) - \frac{\beta_e}{M_3} \cdot R(Mv - mlk \times \Omega).$$

The computation requires a little care since the variable $R$ lies in the manifold $SO(3)$, but it is not hard to verify that
\[ \frac{d^2 H_\xi(p_e)}{d\delta (\delta \Omega_1, \delta \Omega_2, \delta \Omega_1, \delta \Omega_2, \delta v_1, \delta v_2)} = \delta \Omega_2 \cdot I \delta \Omega_1 + \delta v_2 \cdot M \delta v_1 + ml \delta \Omega_1 \cdot (k \times \delta v_2) + ml \delta \Omega_2 \cdot (k \times \delta v_1) \\
- mglk \cdot [\delta R_2 \times (\delta R_1 \times k)] - \frac{\beta_e^2}{M_3} k \cdot [\delta R_2 \times (\delta R_1 \times k)] \\
- \frac{\beta_e}{M_3} k \cdot [\delta v_2 \times (M \delta v_1 - mlk \times \delta \Omega_2)] \\
- \frac{\beta_e}{M_3} k \cdot [\delta R_2 \times (M \delta v_1 - mlk \times \delta \Omega_1)]. \]

(1)

and then substituting
\[ \delta v = M^{-1}(k \times (ml \delta \Omega + \beta_e \delta R)), \quad \delta R \cdot k = 0, \]

(from the definition of the test space \( T \)) and simplifying gives
\[ \frac{d^2 H_\xi(p_e)}{d\delta (\delta \Omega_1, \delta \Omega_2, \delta \Omega_1, \delta \Omega_2, \delta v_1, \delta v_2)} |_T = \delta \Omega_2 \cdot I \delta \Omega_1 - m^2 l^2 (k \times \delta \Omega_1) \cdot M^{-1} (k \times \delta \Omega_2) \\
+ \beta_e^2 M^{-1} (k \times \delta R_1) \cdot (k \times \delta R_2) \\
+ \left( mgl - \frac{\beta_e^2}{M_3} \right) (k \times \delta R_1) \cdot (k \times \delta R_2). \]

This is diagonal with entries:
\[ mlg + \frac{\beta_e^2}{M_2} - \frac{\beta_e^2}{M_3}, \quad mlg + \frac{\beta_e^2}{M_1} - \frac{\beta_e^2}{M_3}, \]
\[ I_1 M_2 - m^2 l^2 \frac{M_2}{M_1}, \quad I_2 M_3 - m^2 l^2 \frac{M_2}{M_1}, \quad I_3. \]

In the physical situation (see Leonard [18]) the last three entries are strictly positive so the test implies stability if and only if
\[ mlg > \frac{\beta_e^2}{M_2} \left( \frac{1}{M_3} - \frac{1}{M_1} \right) \quad \text{and} \quad mlg > \frac{\beta_e^2}{M_3} \left( \frac{1}{M_2} - \frac{1}{M_1} \right). \]

We note that the extra direction in the test space which is needed in addition to the kernel of the momentum mapping does not affect the results because that direction contributes only the strictly positive quantity \( I_3 \) in the Hessian. However, outside the physical regime there are parameter values for which the usual energy-momentum method (i.e., definiteness of the Hessian on ker \( dJ(p_e)/\delta t, p \)) indicates stability but energy-momentum confinement fails. Thus it is necessary to test on the larger spaces that we describe even though these do not necessarily imply stricter stability conditions.
2.2. A walk on the wild side

In the case of coincident centres the equations of motion for an underwater vehicle may be found from those for non-coincident centres by substituting $\Gamma = 0$ and $l = 0$ and removing the $\dot{\Gamma}$ equation. HOLMES et al. [15] studied the stability of relative equilibria of this system which rotate and translate with linear momentum $p \neq 0$. In this situation the momentum $\mu_e$ of the relative equilibrium is regular and the standard energy-momentum method applies. We will now consider the case of pure rotation, for which the momentum $\mu_e$ is non-regular. These rotating relative equilibria are:

$$p_e : \pi = \alpha ek, \quad p = 0.$$  

We assume $\alpha_e \neq 0$ since otherwise this is an equilibrium and a global minimum of the energy, and hence is stable. The Legendre transform dual variables are

$$\pi = I \Omega, \quad p = M v,$$

the symmetry group is $SE(3)$, and the momentum $\mu = (\mu^r, \mu^a) = (\Pi, P) \in se(3)^* = so(3)^* \oplus (\mathbb{R}^3)^*$ is

$$\Pi = P \pi + a \times Rp, \quad P = Rp.$$

The generator and momentum of $p_e$ are

$$\xi^r_e = \frac{\alpha_e}{I_3}, \quad \xi^a_e = 0, \quad \mu^r_e = \alpha_e, \quad \mu^a_e = 0.$$  

By Table 1, the relative equilibrium is wild and so energy-momentum confinement is impossible. LEONARD & MARSDEN [19] described the stability analysis of this relative equilibrium as being particularly delicate and its stability was subsequently established by PATRICK [40]. We include a short summary of this example to indicate the kind of analysis that may sometimes be possible for wild relative equilibria, and also to give an example of the “exotic” stability type $A_{\alpha_0, \epsilon_1}$.

Lie-Poisson reduction of the phase space $T^*SE(3)$ yields the Poisson reduced space as $T^*SE(3)/SE(3) = se(3)^*$ and the symplectic reduced spaces as the coadjoint orbits of this. The generic coadjoint orbits are all diffeomorphic to $TS^2$ and so all have dimension 4. The relative equilibrium $p_e$ descends to an equilibrium $x_e \in se(3)^*$ which is in a nongeneric orbit of dimension 2 diffeomorphic to $S^2$. The problem is to determine the Lyapunov stability of the Poisson equilibrium $x_e$ on the Poisson space $se(3)^*$.

The analysis of [40] proceeds directly on the Poisson reduced space, where the first step is to find a “blow-up” space consisting of the product of $TS^2$ and a parameter space, and a “blow-down” map from this into $se(3)^*$. The blow-up space has a “generic sector” and a “nongeneric sector”: the blow-down map maps the generic sector diffeomorphically into the union of the generic leaves of $se(3)$ in such a way that each copy of $TS^2$ in the generic sector is mapped diffeomorphically to a unique coadjoint orbit of $se(3)^*$. However on the nongeneric sector the blow-down map is many-to-one into the nongeneric leaves of $se(3)^*$. This many-to-one feature can
be related to an extra $SO(2)$ symmetry in the blow-up space: the Hamiltonian on the blow-up space is symmetric with respect to this symmetry on the nongeneric sector but not on the generic sector. The $SE(3)$-stability of the relative equilibrium follows from the stability of a set of relative equilibria of the new $SO(2)$ symmetry in the blow-up space which are found by pulling back the equilibrium $x_e$ via the blow-down map. However it is stability under perturbations to the nongeneric sector that is required. Since those are symmetry breaking, the analysis reduces to the stability of a periodic orbit under symmetry-breaking perturbations. The relevant Poincaré maps can be estimated using an extension of a normal form from [38] and stability follows from the Moser Twist Theorem, under the condition that $I_3$ is not between $I_1$ and $I_2$.

Thus the stability problem for the pure-spin, coincident-centre relative equilibrium has been shown to be essentially a symmetry-breaking question and the confinement mechanism is far more delicate than the Lyapunov-function energy-momentum mechanism.

From Table 1 the stability type is $A_{\varepsilon_0,\varepsilon_1}$, which implies that the orientation of the rotation axis can drift arbitrarily far from its original direction as the translational drift increases. This loss of orientational stability through large excursions has been confirmed by numerical simulations presented in [40]. For an ordinary rigid body (i.e., one not immersed in a fluid) orientational stability of rotation about a long or short axis does, of course, follow from energy-momentum confinement since there the symmetry group $SO(3)$ is compact.

Wild relative equilibria also occur in planar point vortex systems. A system of $N$ point vortices in the plane is a Hamiltonian system on $\mathbb{C}^N = \{z_n\}$ where $z_n$ denotes the location of the $n$-th vortex, which has a strength denoted by $\Gamma_n$. These systems have $SE(2)$ as symmetry groups and momentum mappings which are $Ad^\ast$-equivariant when the total vorticity $\sum \Gamma_n$ is zero. As shown in [39], three vortices of strength $-\Gamma/3$ symmetrically placed around a single vortex of strength $\Gamma$ is a relative equilibrium of the 4-vortex system which is purely rotational (i.e., $\xi^a = \mu^a = 0$ and $\xi^r \neq 0$) and hence wild. These are interesting relative equilibria since they have nothing stronger than $SE(2)$ stability, and under perturbation they move about the plane as coherent “particles” with a specific, calculable mass. Countless simulations suggest that these relative equilibria are $SE(2)$ stable. Since 4-vortex systems have generic symplectic reduced spaces of dimension 4, while the relative equilibria occur on symplectic reduced spaces of dimension 2, the dimensions are favourable for a proof by KAM confinement, after a blow-up. However this is a project for future work.

3. Topology and stability

In this section we give a stability criterion for equilibria of continuous flows on a topological space $X$ with conserved quantity $f: X \to Y$, a continuous map from $X$ into another topological space $Y$ (Corollary 1). This result follows from a general topological stability lemma (Lemma 1) and is the foundation for the stability theory developed in subsequent sections.
We begin with a measure of the extent to which a space is not Hausdorff at a point.

**Definition 1.** Let $Y$ be a topological space and $y \in Y$. Define

$$T_2(y) \equiv \{ y' \in Y : U \cap U' \neq \emptyset \text{ for all neighbourhoods } y \in U \subseteq Y \text{ and } y' \in U' \subseteq Y \}.$$  

We say that a topological space $X$ is Hausdorff at $x \in X$ if $T_2(x) = \{x\}$.

The following is frequently useful and easily proved.

**Proposition 1.**

(i) Let $X$ and $Y$ be topological spaces and $(x, y) \in X \times Y$. Then $T_2(x, y) = T_2(x) \times T_2(y)$.

(ii) If $X$, $Y_1$ and $Y_2$ are topological spaces and $f_i : X \to Y_i$, $i = 1, 2$, are continuous, then $(f_1 \times f_2)^{-1}(T_2(y_1, y_2)) = f_1^{-1}(T_2(y_1)) \cap f_2^{-1}(T_2(y_2))$. In particular, if $Y_2$ is Hausdorff, then $(f_1 \times f_2)^{-1}(T_2(y_1, y_2))$ is the $y_2$ level set of $f_1 f_2^{-1}(T_2(y_1))$.

The next result gives a sufficient condition for continuous curves in $X$ preserving $f$ and starting near a point $x$ to be confined to small neighbourhoods of $x$. It is inspired by Lemma 1.4 of Montaldi [32], in which $Y$ is Hausdorff, and will be applied below to trajectories of vector fields.

**Lemma 1** (Topological Stability Lemma). Let $X$ and $Y$ be topological spaces, $f : X \to Y$ a continuous map, $x \in X$ and $y = f(x)$. Assume that:

(i) $X$ is locally compact at $x$;

(ii) There exists a neighbourhood $U$ of $x$ in $X$ such that $f^{-1}(T_2(y)) \cap U = \{x\}$.

Then for every neighbourhood $U$ of $x$ there is a neighbourhood $V$ of $x$ such that if $c : [0, 1] \to X$ is a continuous curve for which $f \circ c$ is constant and $c(0) \in V$, then $c(t) \in U$ for all $t \in [0, 1]$.

If the second condition holds we say that $x$ is an isolated point of $f^{-1}(T_2(y))$.

**Proof.** We will prove the lemma by showing that if $f$-constant curves are not confined to small neighbourhoods of $x$, then $x$ is not an isolated point in $f^{-1}(T_2(y))$.

Let $B$ be a neighbourhood base at $x$ consisting of compact sets. For any $U \in B$ and any $V \in B$ such that $V \subseteq U$, let $c_{U, V} : [0, 1] \to X$ be a continuous curve such that $f \circ c$ is constant, $c_{U, V}(0) \in V$, and $c_{U, V}(1) \not\in U$. By connectedness of $[0, 1]$ there exists $t_{U, V} \in (0, 1]$ such that $x_{U, V} = c_{U, V}(t_{U, V}) \in \partial U$.

For fixed $U \in B$, the set $\{x_{U, V}\}_{V \in B}$ is a net in $\partial U$ by reverse inclusion of the $V$’s.

Since $\partial U$ is compact there is a subnet, $\{x_{U_i, V_i}\}$, which converges to a point $z_U \in \partial U$. The continuity of $f$ implies that $\{f(x_{U_i, V_i})\}$ converges to $f(z_U)$. Since the curves $c_{U_i, V_i}$ preserve $f$ we also have:

$$f(x_{U_i, V_i}) = f(c_{U_i, V_i}(t_{U_i, V_i})) = f(c_{U_i, V_i}(0)).$$
The net \( c_{U/V\lambda}(0) \) converges to \( x \) and so \( f(x_{U/V\lambda}) \) also converges to \( y = f(x) \).

Thus, every neighbourhood of \( f(z_U) \) meets every neighbourhood of \( y \), and so \( f(z_U) \in T_2(y) \). This is the required contradiction, since \( z_U \in f^{-1}(T_2(y)) \) and \( \{z_U\}_{U \in B} \) is a net converging to \( x \). □

Now consider a continuous flow \( \phi: X \times \mathbb{R} \rightarrow X \) on \( X \) which preserves \( f \),

\[
\phi(t)(x) = f(x),
\]

and which has an equilibrium point at \( x \in X \). The equilibrium point is stable if, for every open neighbourhood \( U \) of \( x \) in \( X \), there exists a neighbourhood \( V \) of \( x \) such that if \( x \in V \), then \( \phi(t)(x) \in U \) for all \( t \). Lemma 1 then implies the following stability criterion for flows with a conserved quantity \( f \).

**Corollary 1.** Let \( X \) and \( Y \) be topological spaces, \( X \) locally compact, and \( f: X \rightarrow Y \) a continuous map. Let \( x \) be an equilibrium point of a continuous flow on \( X \) which preserves \( f \) and \( y = f(x) \). Then \( x \) is stable if it is an isolated point in \( f^{-1}(T_2(y)) \).

The examples in Section 4.3 show that Corollary 1 is false in general if \( x \) is only isolated in \( f^{-1}(y) \). The following example shows that it is necessary to assume that \( X \) is locally compact in the results above.

**Example 1.** For \( x = (q_n, p_n)_{n \geq 0} \in X = \ell^2(\mathbb{R}) \times \ell^2(\mathbb{R}) \) let

\[
h(x) = \frac{1}{2} \sum_{n=0}^{\infty} p_n^2 + \frac{1}{2} \sum_{n=0}^{\infty} 4^{-n} q_n^2.
\]

Then \( h \) is differentiable on \( X \) and \( h(0) = 0 \). Clearly \( T_2(0) = \{0\} \) and 0 is isolated in \( h^{-1}(0) \subset X \). The standard linear symplectic structure \( J((q_n, p_n)) = (p_n, -q_n) \) gives the Hamiltonian system \( \dot{x} = J dh(x) \) and defines a flow \( \phi_t \) on \( X \). For \( m \geq 1 \), the solution of this Hamiltonian system with initial value such that \( p_m = 2^{-m} \), \( p_n = 0 \) for \( n \neq m \) and \( q_n = 0 \) for all \( n \), has \( q_m = -1 \) at time \( t = 2^{m-1} \pi \), as is easily verified. Hence 0 is an unstable equilibrium. Note that all eigenvalues of \( d^2 h(0) \) are positive, but that 0 is an accumulation point of the spectrum of \( d^2 h(0) \) and is therefore in its continuous spectrum.

In applications, the flow \( \phi_t \) on \( X \) typically comes from a differential equation whose phase space \( X \) is a manifold. Then \( X \) is locally compact if and only if it is finite dimensional. Thus the requirement that \( X \) is locally compact implies that Corollary 1 cannot be applied directly to deduce the nonlinear stability of equilibria of partial differential equations. Indeed, it is well known that for partial differential equations positivity of the second variation need not imply stability of an equilibrium, a phenomenon which is related to the non-equivalence of norms on Banach spaces [5].

**4. Stability of Poisson equilibria**

In this section we apply the topological stability result of Section 3 to equilibria of Poisson systems to obtain generalisations of the energy-Casimir method.
4.1. Topological tests

A finite-dimensional Poisson manifold $X$ is partitioned into immersed submanifolds by its symplectic leaves, see e.g. [50], which without loss of generality we will always assume to be connected. Define two points in $X$ to be equivalent if they belong to the same symplectic leaf, let $Z$ be the quotient of $X$ by this equivalence relation and $L : X \rightarrow Z$ the quotient map. We will regard the symplectic leaf $L(x)$ through $x$ as both a subset of $X$ and as a point in $Z$.

A function $h : X \rightarrow \mathbb{R}$ generates a vector field on $X$ which is uniquely defined by the requirement that

$$\dot{f} = \{h, f\}$$

for all differentiable functions $f$ on $X$. The flow $\phi_t$ of $X$ preserves the fibres of both $h$ and $L$, and hence those of the product map $f = L \times h : X \rightarrow Y = Z \times \mathbb{R}$. We can therefore apply the topological stability result Corollary 1.

First note that $x_e \in X$ is an equilibrium point of $\phi_t$ if and only if it is an equilibrium point of the restriction of the flow to the invariant submanifold $L(x_e)$ or, equivalently, if and only if the restriction of $h$ to $L(x_e)$ has a critical point at $x_e$. An equilibrium $x_e$ is said to be leafwise stable if it is stable for the restricted flow on $L(x_e)$. Clearly stability in the full space $X$ implies leafwise stability.

For $x \in X$, let $T_2(x) = L^{-1}(T_2(L(x)))$. Note that $L(x) \subseteq T_2(x)$. More generally, for any open neighbourhood $U$ of $x$ in $X$, let $L^U(x)$ denote the symplectic leaf of $U$ through $x$. This is the connected component of $L(x) \cap U$ which contains $x$. Denote the space of these symplectic leaves by $Z^U$ and the corresponding quotient map by $L^U : U \rightarrow Z^U$. Define

$$T^U_2(x) = (L^U)^{-1} \left( T_2(L^U(x)) \right),$$

where $T_2(L^U(x))$ is taken in $Z^U$. The set $T^U_2(x)$ is contained in $T_2(x) \cap U$, but may be strictly smaller, for example if $L(x)$ accumulates on itself at $x$.

Recall that a point $x_e$ in a Poisson manifold $X$ is said to be regular (or minimal) if there exists an open neighbourhood $U$ of $x_e$ in $X$ such that $\dim L(x) = \dim L(x_e)$ for all $x$ in $U$. The set of regular points is open and dense in $X$ since in local coordinates it corresponds to the set where the matrix of the Poisson tensor has locally constant rank.

The following result provides topological conditions for leafwise stability and stability and states that for regular equilibria these conditions coincide.

**Theorem 1.** Let $x_e$ be an equilibrium point of the flow generated by a Hamiltonian $h$ on a Poisson manifold $X$, and $U$ be an open neighbourhood of $x_e$. Then $x_e$ is

(i) leafwise stable if there is an open neighbourhood $U$ of $x_e$ in $X$ such that

$$h^{-1}(h(x_e)) \cap L^U(x_e) = \{x_e\};$$

(ii) stable if there is an open neighbourhood $U$ of $x_e$ in $X$ such that

$$h^{-1}(h(x_e)) \cap T^U_2(x_e) = \{x_e\}.$$
Moreover, if \( x_e \) is a regular point of \( X \), then there is an open neighbourhood \( U \) of \( x_e \) in \( X \) such that \( T_2^U(x_e) = L^U(x_e) \) and part (i) implies stability.

These statements remain true if \( h \) is replaced by any conserved quantity with values in a Hausdorff space.

Proof.

(i) This follows from Corollary 1 with \( X \) replaced by \( L^U(x_e) \) and \( f \) by \( h \). Clearly, \( T^2(h(x_e)) = \{ h(x_e) \} \) and the hypothesis says that \( x_e \) is an isolated point in the \( h(x_e) \) level set of the restriction of \( h \) to \( L^U(x_e) \).

(ii) Apply Corollary 1 to the neighbourhood \( U \) and map \( f = L^U \times h \), noting that by Proposition 1 we have \( T_2(y) = T_2(L^U(x_e), h(x_e)) = T_2(L^U(x_e)) \times \{ h(x_e) \} \) and so \( f^{-1}(T_2(y)) = h^{-1}(h(x_e)) \cap T_2^U(x_e) \).

If \( x_e \) is regular, then there exists an open neighbourhood \( U \) of \( x_e \) in \( X \) for which the symplectic leaves provide a regular foliation [50, Corollary 2.3]. The quotient of \( U \) by this foliation is Hausdorff and so \( T_2^U(x_e) \) is equal to \( L^U(x_e) \).

In Example 4 we show that there is an open subset of Hamiltonians on \( \mathfrak{sl}(2)^* \cong \mathbb{R}^3 \) for which Theorem 1 implies that the origin is a stable equilibrium, and another open set for which the origin is leafwise stable but not stable.

The next result is a simple corollary of Theorem 1 for a very special case.

Corollary 2. Let \( x_e \) be an equilibrium point of a Hamiltonian \( h \) on the Poisson manifold \( X \). Suppose that \( T_2^U(x_e) \) is a one-dimensional submanifold of \( X \). Then \( x_e \) is stable if \( dh(x_e) \) is nonzero on \( T_2(x_e) \).

Proof. If \( dh(x_e) \) is nonzero on \( T_2(x_e) \) then the level set of \( h \) containing \( x_e \) intersects \( T_2(x_e) \) transversely and hence in an isolated point.

We now discuss conditions on the derivative and Hessian of \( h \) at the Poisson equilibrium \( x_e \) for stability to hold. Since \( x_e \) is an equilibrium, the restriction of \( h \) to \( L(x_e) \) has a critical point at \( x_e \), and \( x_e \) is leafwise stable if the second derivative of the restriction is definite. The following result, a special case of Theorem 1, states that for generic points in \( X \) this condition also implies that \( x_e \) is stable.

4.2. \( T_2 \)-energy-Casimir method

We now discuss conditions on the derivative and Hessian of \( h \) at the Poisson equilibrium \( x_e \) for stability to hold. Since \( x_e \) is an equilibrium, the restriction of \( h \) to \( L(x_e) \) has a critical point at \( x_e \), and \( x_e \) is leafwise stable if the second derivative of the restriction is definite. The following result, a special case of Theorem 1, states that for generic points in \( X \) this condition also implies that \( x_e \) is stable.

Proposition 2 ([23, Theorem III.12.4]). If an equilibrium \( x_e \) is regular and the second derivative of the restriction of the Hamiltonian \( h \) to \( L(x_e) \) is positive or negative definite, then \( x_e \) is both leafwise stable and stable.
We will notationally suppress reference to $U$. We call Poisson equilibria for which the first derivative of $T_U(x_e)$ contains a smooth curve and so $h$ does not isolate $x_e$ in $T^U_U(x_e)$. In such a case stability cannot be concluded from the mechanism of confinement by energy level sets. However, if $T^U_U(x_e)$ is a manifold, if the first derivative of $h$ vanishes on $T^U_U(x_e)$ and if the second derivative of the restriction of $h$ to $T^U_U(x_e)$ is definite, then $h$ again isolates $x_e$ and $x_e$ is stable. We call Poisson equilibria for which the first derivative of $h$ satisfies this condition, and which are therefore amenable to Hessian-type stability tests, tame. To extend this to cases for which $T^U_U(x_e)$ is not a manifold (see Examples 4, 6, 7 below) we first define the notions of the tangent space and a smoothing of a singular set.

**Definition 2.** Let $M$ be a manifold, $S \subseteq M$ and $m \in S$.

1. The tangent space $T_mS$ of $S$ at $m$ is the subset of $T_mM$ consisting of the derivatives $c'(0)$ of all $C^1$ curves $c(t)$ in $M$ with $c(0) = m$ and $c(t) \in S$ for $t \geq 0$.
2. A smoothing of $S$ at $m$ is a finite number of submanifolds $B_i \subseteq M$ such that $S \subseteq \cup_{i=1}^n B_i$ and $T_mB_i \subseteq \text{span}(T_mS)$ for $1 \leq i \leq n$.

If $T^U_U(x_e)$ is a manifold, then clearly it is its own smoothing. Examples 4 and 6 feature some smoothings of singular $T^U_U(x_e)$ sets. Weak smoothings which satisfy the first condition of part 2, but not the second, always exist since it is possible to take $n = 1$ and $B_1 = M$. In Example 9 the $T^U_2$-set does not have a smoothing.

We now define tame equilibria at points with arbitrary $T^U_2$-sets.

**Definition 3.** Let $X$ be a Poisson manifold.

1. A generator at $x$ is an element $\xi \in T^*_xX$ which annihilates $T_xL(x)$.
2. A generator $\xi$ at $x$ is tame if it annihilates $T_x\{T^U_U(x)\}$. Generators that are not tame are said to be wild.
3. The generator of an equilibrium $x_e$ of a Poisson system with Hamiltonian $h$ is $dh(x_e)$.
4. An equilibrium is tame if its generator is tame, and wild otherwise.

We will notationally suppress reference to $U$ even though the property of being tame is $U$ dependent. The set of tame generators at $x$ is a vector subspace of $T^*_xX$. If $\xi \in T^*_xX$ is tame, then it annihilates every smoothing $\{B_i\}_{i=1}^n$ of $T^U_2(x)$, i.e., $\xi$ annihilates $T_xB_i$ for all $i$.

If $x$ is regular, then $T^U_2(x) = L^U(x)$ for some neighbourhood $U$ and every generator is tame. In Corollary 2 the hypothesis that $dh(x_e)$ is nonzero on $T_{x_e}(T^U_2(x_e))$ implies that $dh(x_e)$ is wild. In Example 4 below, for which $T^U_2(0)$ is a cone, $T_0\{T^U_2(0)\}$ spans the whole of $X = \mathbb{R}^3$ and so every nonzero derivative at 0 is wild.

A Casimir on $U$ is a continuous function $C : U \to \mathbb{R}$ which is constant on the symplectic leaves of $U$. This condition implies that $C$ is also constant on every set $T^U_2(x)$ for every $x \in U$. Casimirs are conserved quantities along integral curves contained in $U$, since any Poisson flow preserves the symplectic leaves. Since Casimirs are constant on $T^U_2(x)$, the derivatives of smooth Casimirs at $x$ are tame. It follows that if $x_e$ is an equilibrium and $C$ is a smooth Casimir at $x_e$, then $dh(x_e)$ is tame if and only if $d(h + C)(x_e)$ is tame.
Theorem 2 (Poisson $T_2$-Energy-Casimir Method). Let $x_e$ be a tame equilibrium point of the Hamiltonian $h$. Let $\{B_i\}_{i=1}^n$ be a smoothing of $T_2^U(x_e)$ at $x_e$. Then $x_e$ is stable if for each $i$ there is a smooth Casimir $C_i$ such that the Hessian $d^2((h + C_i)|_{B_i})(x_e)$ is positive or negative definite on $T_{x_e}B_i$.

Proof. Set $\hat{h} = (h + C_1, \ldots, h + C_n)$. Since $T_2^U(x_e) \subseteq \bigcup_{i=1}^n B_i$, by Theorem 1 it suffices to show that $\hat{h}$ isolates $x_e$ on $\bigcup_{i=1}^n B_i$. As there are only finitely many $B_i$, this follows if $h + C_i$ isolates $x_e$ on each $B_i$. This in turn is implied by the definiteness of the Hessians and the Morse lemma. $\square$

The standard energy-Casimir method, see e.g. [27], states that if $x_e$ is an equilibrium point of the flow generated by a Hamiltonian $h$ and there exists a smooth Casimir $C$ such that $x_e$ is a critical point of $h + C$ on the whole of $X$ and $d^2(h + C)(x_e)$ is definite, then $x_e$ is stable. Theorem 2 is a strict generalisation of this result because it only requires $x_e$ to be a definite critical point for a function on a subset of $X$. Examples 3 and 7 describe Poisson systems for which the theorem can be used to prove stability, though the standard energy-Casimir method fails.

The standard energy-Casimir method and its generalisation, Theorem 2, can only be applied to tame equilibria, but the topological stability result Theorem 1 can sometimes be applied to wild equilibria. Examples include the cases covered by Corollary 2 and Examples 3 and 4.

Smoothings and Casimirs are both implements designed to handle singularities of $T_2^U(x_e)$ for the purpose of constructing Hessian tests for stability. Fine smoothings can eliminate the necessity of including Casimirs when calculating the Hessians, while coarse smoothings may require their inclusion. Example 6 illustrates the play available in choosing smoothings versus Casimirs.

If there exists a smoothing such that $T_2^U(x_e) = \bigcup_{i=1}^n B_i$ (for example, if $T_2^U(x_e)$ is a submanifold) then every Casimir is constant on each of the $B_i$. In this case the second derivatives of the restrictions of the Casimirs to the $B_i$ vanish, and the inclusion of the Casimirs in the Hessians in Theorem 2 is unnecessary. However, Example 4 shows that inclusion of Casimirs can be necessary in cases when $T_2^U(x_e)$ is singular.

More generally, the inclusion of the Casimir $C_i$ is unnecessary when all Casimirs restricted to $B_i$ have vanishing second derivative at $x_e$. This follows if $B_i \subseteq T_2^U(x_e)$ but can also be implied by an infinitesimal relation between the smoothing and the set $T_2^U(x_e)$, as follows. For any Casimir $C$ we have $d^2C(x_e)(v, v) = 0$ for all $v \in T_{x_e}(T_2^U(x_e))$, since Casimirs are constant on curves in $T_2^U(x_e)$. Regarding the symmetric bilinear form $d^2C(x_e)$ as a linear map on the tensor product $T_{x_e}B_i \otimes T_{x_e}B_i$, it follows that $d^2C(x_e)$ vanishes on the whole of $T_{x_e}B_i$ if

$$\text{span} \left\{ v \otimes v : v \in T_{x_e}(T_2^U(x_e)) \cap T_{x_e}B_i \right\} = T_{x_e}B_i \otimes T_{x_e}B_i. \quad (2)$$

Consequently, in Theorem 2, the inclusion of $C_i$ is unnecessary for any $i$ such that this spanning condition holds. An application of this is given in Example 7.

Remark 1. If $\{B_i\}_{i=1}^n$ is only a weak smoothing of $T_2(x_e)$, but there exist Casimirs $C_i$ such that for $i = 1, \ldots, n$ we have $d(h + C_i)(x_e)|_{B_i} = 0$ and the Hessians...
If \( d^2(h + C_i)(x_e) \) are definite, then \( x_e \) is again stable. This can be deduced as in the proof of Theorem 2. The standard energy-Casimir method is recovered by taking \( n = 1 \) and \( B_1 = M \).

**Remark 2.** Ortega & Ratiu have obtained a version of the standard energy-Casimir method [35, Corollary 4.11] which states that if \( C = c_1 + \cdots + c_m \), where all the \( c_j \)'s are Casimirs, then definiteness of \( d^2 (h + C)(x_e) \) is only required on the intersection \( W = \ker dc_1(x_e) \cap \cdots \cap \ker dc_m(x_e) \). This result can be obtained from Theorem 2 and Remark 1 by applying the implicit-function theorem to the equation \( c(x) = c(x_e) \), where the map \( c : X \to \mathbb{R}^m \) is defined by \( c(x) = (c_1(x), \ldots , c_m(x)) \), to produce a weak smoothing \( B \) with \( T_{x_e}^2(x_e) \subset c^{-1}(c(x_e)) \subset B \) and \( T_{x_e} \theta B = W \). It is an improvement on the standard energy-Casimir method in that it is only necessary to test for definiteness on the subspace \( W \) of \( T_{x_e} X \). However, this subspace may be strictly larger than those provided by Theorem 2, as is shown in Example 3. Moreover the set of Poisson equilibria for which the result of Ortega and Ratiu can be used to prove stability is the same as that for the standard energy-Casimir method.

### 4.3. Examples

Here we collect together a number of examples to illustrate the stability theory described above. Many of the examples are of equilibria on Poisson spaces which are duals of Lie algebras, for which the symplectic leaves are coadjoint orbits. Other examples are Poisson structures on \( X = \mathbb{R}^3 \) with Poisson brackets of the form

\[
\{ h, f \} = \nabla A \cdot (\nabla h \times \nabla f),
\]

where \( A = A(x,y,z) \) is a smooth function. The vector field generated by a Hamiltonian \( h \) is

\[ \dot{x} = \nabla A \times \nabla h. \]

We may assume that \( A(0) = 0 \). For these structures, \( A \) is a Casimir since \( \{ A, f \} = 0 \) for all functions \( h \), and the two-dimensional symplectic leaves of \( X \) are the connected components of the level sets of regular values of \( A \). Each critical point \( x \) of \( A \) is also a symplectic leaf since there \( \{ h, f \}(x) = 0 \) for all smooth functions \( h \) and \( f \). For any open neighbourhood \( U \) of \( 0 \) in \( X \), the set \( T_{x_e}^2(x) \) is contained in the connected component of \( A^{-1}(0) \cap U \) which contains \( 0 \). If \( A = \frac{1}{2}(x^2 + y^2 + \varepsilon z^2) \), then \( X \) is isomorphic to the dual of the Lie algebra, \( g^* \), of \( SO(3) \) (\( \varepsilon = 1 \)), \( SE(2) \) (\( \varepsilon = 0 \)) or \( SL(2, \mathbb{R}) \) (\( \varepsilon = -1 \)) with their standard Poisson structures. These three Poisson structures are described in [27] and are also used in [51] to illustrate the interdependence of stability and Poisson structure.

**Example 2.** The configuration space of a rigid body is \( SO(3) \), and the corresponding phase space \( T^*SO(3) \). After symmetry reduction we obtain a Poisson system on \( X \cong so(3)^* \), and so \( A = \frac{1}{2}(x^2 + y^2 + z^2) \). For any neighbourhood \( U \) of \( 0 \) we have \( L(0) = T_{x_e}^2(0) = \{ 0 \} \) and so \( 0 \) is an isolated point of \( h^{-1}(h(0)) \cap T_{x_e}^2(0) \), and
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hence a stable equilibrium, for any Hamiltonian $h$. Every generator at 0 is tame. Every point $x_e \neq 0$ is regular, since the symplectic leaf through each nearby point (a sphere) is two-dimensional. Thus all generators at these points are also tame. In fact every generator is tame at every point in $g^*$ for any compact group, since in this case the symplectic leaves are the coadjoint orbits of $G$ on $g^*$ and the quotient space $g^*/G$ is always Hausdorff.

The Hamiltonian for the standard rigid body is $h = x^2/2I_1 + y^2/2I_2 + z^2/2I_3$, where the $I_j$ are the principal moments of inertia of the body. The restriction of $h$ to each regular symplectic leaf has critical points where the sphere intersects the coordinate axes, corresponding to rotations about the principal axes. These critical points are definite for the rotations about the principal axes with largest and smallest moments of inertia.

**Example 3.** If $A = \frac{1}{2}(x^2 + y^2)$, then $X \cong se(2)^*$. The symplectic leaves are points on the $z$-axis and cylinders of nonzero radius about the $z$-axis.

If $x_e$ does not lie on the $z$-axis, then it is regular and every generator is tame. If $x_e$ is a point on the $z$-axis, then $T_2^U(x_e)$ is the intersection of the $z$-axis with $U$, a one-dimensional manifold. The set of tame generators at $x_e$ is equal to the subspace $\mathbb{R}^2 \subseteq se(2)$ consisting of infinitesimal translations. If $dh(x_e)$ is tame, the point $x_e$ is stable if the restriction of $h$ to the $z$-axis is positive or negative definite. Generators containing nonzero rotational components are wild. However, in this case this implies that $dh(x_e)$ is nonzero on $T_{x_e}(T_2^U(x_e)) = \{x = y = 0\}$ and so 0 is again stable by Corollary 2.

The Casimirs of this Poisson space are all functions of $A = \frac{1}{2}(x^2 + y^2)$ and so have critical points on the $z$-axis. It follows that both the standard energy-Casimir method and its generalisation due to Ortega and Ratiu (see Remark 2) can only be applied to Hamiltonians for which $dh(x_e) = 0$. Note also that the kernel of the derivative of every Casimir is the whole space, so the result of Ortega and Ratiu does not provide an improvement to the standard energy-Casimir method in this case.

This example lies at the heart of applications to Hamiltonian systems with Euclidean symmetries. Phase-space reduction typically reduces such systems to Poisson systems on spaces which contain $se(2)^*$ as a component. See also Example 5.

**Example 4.** Next consider $A = \frac{1}{2}(x^2 + y^2 - z^2)$, giving $X \cong sl(2, \mathbb{R})^*$. The symplectic leaves are the connected components of the hyperboloids $A = a$ for $a \neq 0$, the connected components on the complement of 0 in the cone $A = 0$, and the origin itself. Every nonzero point is regular and all generators at these points are tame.

At the origin, $T_2^U(0)$ is the intersection of the cone $A = 0$ with $U$. Since $T_0(T_2^U(0))$ spans the whole of $\mathbb{R}^3$ the only tame generator is $\xi = 0$. Nevertheless Hamiltonians with wild generators $dh(0) \neq 0$ can again have stable equilibria at 0. Theorem 1 implies that 0 is stable if $dh(0)$ “points into the cone $A = 0$”, so that ann($dh(0)$) intersects the cone only at the origin. If $dh(0)$ points out of the cone, then the intersection is (infinitesimally) a pair of lines and the equilibrium is
unstable. The instability follows from the fact that, if \( dh(0) = (\xi_1, \xi_2, \xi_3) \), the linearised equations of motion at the origin have eigenvalues 0 and \( \pm \sqrt{\xi_1^2 + \xi_2^2 - \xi_3^2} \). Since 0 is trivially leafwise stable for any Hamiltonian, this is an example of an equilibrium that is leafwise stable but not stable.

Now consider the tame case \( dh(0) = 0 \). For a smoothing we have to take \( B = U \), a full neighbourhood of the origin. Without including Casimirs we can only use Theorem 2 to conclude stability if \( d^2 h(0) \) is definite. However by using the Casimir \( A \) it can be seen that, for example, the Hamiltonian \( ax^2 + by^2 + cz^2 \) has a stable equilibrium whenever \( c > -a \) and \( c > -b \). Thus in general Casimirs cannot be dispensed with completely.

**Example 5.** Consider an 8-dimensional Hamiltonian system with \( SE(2) \), the Euclidean symmetries of the plane, as symmetry group. Suppose that after reduction by \( SE(2) \) we obtain a Poisson system on the Poisson space \( X = se(2)^* \times \mathbb{R}^2 \), with coordinates \( x, y, z \) on \( se(2)^* \), modelling the momentum of the rigid motion in body coordinates, and with coordinates \( p, q \) on \( \mathbb{R}^2 \) satisfying \( \{p, q\} = 0 \), modelling the “shape dynamics”. Then the symplectic leaf through 0 is the two-dimensional manifold \( L(0) = \{x = y = z = 0\} \) and \( T^U_0(0) \) is the intersection of \( U \) with the three-dimensional manifold \( \{x = y = 0\} \). The set of generators can be identified with \( T^U_0(se(2)^*) \cong se(2) \) and the tame generators are the pure translations, as in Example 3.

Assume 0 is a wild equilibrium, i.e., \( d_s h(0) \neq 0 \). In contrast to Example 3, Corollary 2 does not apply because \( T^U_0(0) \) is three-dimensional, and 0 can be unstable. To see this consider the Hamiltonian \( h = az - qy + \frac{1}{2}(q^2 + p^2) \). This has a wild generator at 0. Its restriction to the symplectic leaf has a strict local minimum at 0, so 0 is leafwise stable. The equations of motion are:

\[
\dot{x} = ay, \quad \dot{y} = -ax, \quad \dot{z} = -qx, \quad \dot{q} = p, \quad \dot{p} = -q + y.
\]

When \( a = 1 \), for arbitrarily small \( \delta \) the curve

\[
x = \delta \sin t, \quad y = \delta \cos t, \quad z = \frac{\delta^2}{8}(t \sin 2t - \sin^2 t - t^2),
\]

\[
q = \frac{\delta}{2} t \sin t, \quad p = \frac{\delta}{2} (\sin t + t \cos t)
\]

is a solution which leaves any neighbourhood of 0 for \( t \) sufficiently large. This instability is already present in the linearised equations at 0 and is caused by the \( 1:1 \) resonance between the \( x, y \) and \( q, p \) frequencies.

**Example 6.** Let \( X = \mathbb{R}^3 \) with the Poisson structure given by (3) with \( A = \frac{1}{2}(x^2 - y^2) \). The set \( T^U_2(0) \) is the intersection of \( U \) with the union of the two planes \( x = \pm y \). The only tame generator is \( \xi = 0 \). One smoothing of \( T^U_2(0) \) is provided by the two planes themselves: \( B_1 = \{x = y\} \), \( B_2 = \{x = -y\} \). Another is obtained by taking a whole open neighbourhood \( U \) of 0.
Consider the Hamiltonian $h = ax^2 - by^2 + z^2$. The restriction of $h$ to each $B_i$ is $(a - b)x^2 + z^2$ and so $0$ is stable by Theorem 2 if $a > b$, without using any Casimirs. Alternatively the Casimir $A$ can be used on the whole of $U$:

$$h - 2\lambda A = (a - \lambda)x^2 - (b - \lambda)y^2 + z^2$$

and so, if $a > b$, then taking $a > \lambda > b$ gives stability by the energy-Casimir method.

**Example 7.** This is another example of an equilibrium which is stable by Theorem 2, but for which the standard energy-Casimir method and the generalisation of Ortega and Ratiu (Remark 2) both fail. Again let $X = \mathbb{R}^3$ with the Poisson structure given by (3), but with $A = (a^2x^2 - y^2)y$ where $a \neq 0$. The set $T_2^U(0)$ is the intersection of $U$ with the union of the three planes $y = 0$ and $y = \pm ax$. The only tame generator is $\xi = 0$. The restriction of the Hamiltonian $h = x^2 - y^2 + z^2$ to $y = 0$ isolates $0$ in $y = 0$, while its restrictions to $y = \pm ax$ isolate $0$ if $|a| \leq 1$. Thus $0$ is stable by Theorem 2 if $|a| \leq 1$. However, any Casimir on $X$ must satisfy $d^2C(0) = 0$ since, as $v$ varies over the three planes forming the set $T_2(0)$, the vectors $v \otimes v$ span the symmetric part of $\mathbb{R}^3 \otimes \mathbb{R}^3$. Thus the standard and Ortega-Ratiu energy-Casimir methods cannot be used to deduce stability.

**Example 8.** This example shows that the use of $T_2^U(x_e)$ instead of the larger set $T_2(x_e)$ is necessary when the symplectic leaves accumulate upon themselves. Let $\hat{X} = \mathbb{R}^3$ with the Poisson structure (3) with $A = x - ay$. Let

$$\hat{h} = \frac{1}{2}(z^2 + (\sin y - a\sin x)^2 - (\sin x - a\sin y)^2),$$

and let $\hat{x}_e = 0$. The action of $\mathbb{Z} \times \mathbb{Z}$ on $\hat{X}$ by $(b_1, b_2) \cdot (x, y, z) = (x + 2\pi b_1, y + 2\pi b_2, z)$ is Poisson and $\hat{h}$ is invariant. Let $X = \hat{X}/\mathbb{Z} \times \mathbb{Z}$ be the Poisson quotient, and $\hat{h}$ and $x_e$ be the projections of $\hat{h}$ and $\hat{x}_e$ to the quotient. Let $a$ be irrational. Then the symplectic leaf through $x_e$ is the projection of the plane $x = ay$ in $\hat{X}$. This is dense in $X$ since it is the product of a densely winding line on a 2-torus and $\mathbb{R}$. So $T_2(x_e) = X$ and Theorem 2 fails to show that $x_e$ is stable, since the Hessian of $h$ is indefinite on $X$. However, by taking $U$ to be the projection of $(-r, r) \times (-ar, ar) \times \mathbb{R}$ for sufficiently small $r$, the set $T_2^U(x_e)$ becomes the product of the projection of the line segment $y = ax$, $x \in (-r, r)$, and $\mathbb{R}$, which is not dense. The Hessian of $h$ restricted to $T_2^U(x_e)$ is definite since $\hat{h} = \frac{1}{2}(z^2 + (1 - a^2)y^2) + \text{h.o.t.}$, on the plane $x = ay$ in $\hat{X}$.

**Example 9.** Let $X = sl(3; \mathbb{R})^*$. Using the Killing form we can identify $X$ with $sl(3; \mathbb{R})$, and hence with the space of traceless $3 \times 3$ real matrices. Let $x_e$ denote the subregular nilpotent matrix with $(x_e)_{12} = 1$ and all other entries equal to zero. We claim that $T_2^U(x_e)$ does not have a smoothing for any neighbourhood $U$. The coadjoint orbit $L(x_e)$ has codimension four. Let $\Sigma$ be a four-dimensional section through $x_e$ transverse to $L(x_e)$. This has an induced Poisson structure that is described in Section 5.2 below and $T_2^U(x_e)$ is isomorphic to the product of a neighbourhood of $x_e$ in $L(x_e)$ and $T_2^{\Sigma U}(x_e)$ where $U_\Sigma$ is a neighbourhood of $x_e$ in $\Sigma$. We will show that $T_2^{\Sigma U}(x_e)$ does not have a smoothing for any neighbourhood $U_\Sigma$. 

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Let $\pi : X \to \mathbb{R}^2$ denote the mapping defined by generators of the ring of invariants of the coadjoint action on $X$, i.e., the symmetric polynomials of the eigenvalues of the matrices. Let $\pi_X$ be the restriction of $\pi$ to $\Sigma$. Since the generators are Casimirs the set $T_x \Sigma (x_e)$ is contained in the two-dimensional fibre of $\pi_X$ through $x_e$. Brieskorn [8] has shown that this fibre has a simple singularity of type $A_2$ (see also [48]), which means that it is diffeomorphic to the variety defined by $x^2 + y^2 + z^3 = 0$. The tangent space to this variety at 0 (in the sense of Definition 2) is just the non-negative $z$-axis, ie $T_x \Sigma (x_e) = \{ (0, 0, z) : z \geq 0 \}$. However the fibre $\pi^{-1} (\Sigma (x_e))$ contains the intersection of $\Sigma$ with the coadjoint orbit through the regular nilpotent matrix defined by $x_{12} = x_{23} = 1$ and all other entries equal to zero. This intersection is a symplectic leaf of $\Sigma$ of dimension two which, by Jordan normal form theory, contains $x_e$ in its closure. It must therefore be contained in $T_x \Sigma (x_e)$. Thus a smoothing $\{ B_i \}_{i=1}^n$ must have one $B_i$ at least of dimension 2 and it is not possible that $T_x B_i \subset \text{span} T_x \Sigma (x_e)$ for each $i$.

Example 10. This example shows that the assumption in Theorem 2 that the equilibrium is tame is essential, even for Poisson systems that are reductions of Hamiltonian systems of physically recognisable forms. Take $P = T^* (SE(2)^n)$ and $G = SE(2)^n$ acting by the cotangent lift of its left action on itself. Then $X = P/G$ is the Lie algebra dual $(se(2)^n)^* \cong (\mathbb{R}^3)^n$, and the generic coadjoint orbits are products of cylinders. In cylindrical coordinates $(r_i, \theta_i, z_i)$ the Poisson bracket is

$$\{ f, g \} = \sum_{i=1}^n \left( \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial \theta^i} - \frac{\partial g}{\partial z_i} \frac{\partial f}{\partial \theta^i} \right).$$

Consider Hamiltonians $h : P/G \to \mathbb{R}$ of the form

$$h = \sum_{i=1}^n \frac{1}{2} z_i^2 + r_i^2 F_i (\theta, z) \quad (4)$$

for smooth functions $F_i$ on the product of $n$ cylinders, $z = (z_1, \ldots, z_n)$. Then the corresponding Poisson system is given by

$$\dot{r}_i = 0, \quad \dot{z}_i = -\partial_{\theta^i} h, \quad \dot{\theta}^i = \partial_{z_i} h, \quad i = 1, \ldots, n.$$

For $i = 1, \ldots, n$ the coordinate functions $r_i$ are Casimirs that parametrise the Hamiltonian (4). Any point with all $r_i = 0$ is a symplectic leaf, and hence a (leafwise stable) equilibrium. The full stability of these equilibria corresponds to stability under small perturbation to nonzero $r_i$. These are perturbations of a completely integrable system, and so when $n \geq 3$ ARNOLD drift in the variables $z_i$ is expected [1]. For $n = 3$ Bessi [7] has constructed functions $F_i$ for which instability occurs near $z_1 = z_3 = 0$ and $z_2 = 2$. The $T_2$-set corresponding to any equilibrium with all $r_i = 0$ is the set $\{ (r_1, \theta^i, z_i) : r_1 = \cdots = r_n = 0 \}$ and $dh$ restricted to the $T_2$-set is not zero unless all $z_i$ are zero. Thus the equilibrium is wild for $z \neq 0$.

We note that for $n = 1$ stability follows from Corollary 2, and that for $n = 2$ we essentially have a family of two-degree-of-freedom systems (the phase space is the product of two cylinders of widths $r_1$ and $r_2$) so that generically stability follows from KAM theory.
5. Stability of Hamiltonian relative equilibria

In this section we apply the results of Section 4 on the stability of Poisson equilibria to relative equilibria of Hamiltonian systems which are invariant under free group actions. To do this we use the fact that relative equilibria are equilibria of the symmetry-reduced dynamics on the Poisson orbit manifold.

Let \( P \) be a finite-dimensional symplectic manifold with a free symplectic action of a Lie group \( G \). Assume that the action has an \( \text{Ad}_G^* \)-equivariant momentum map \( J : P \to g^* \) with respect to the coadjoint action of \( G \) on \( g^* \). Let \( H \) be a \( G \)-invariant function on \( P \), \( X_H \) the corresponding equivariant Hamiltonian vector field and \( \Phi_t \) its equivariant flow. The flow preserves the level sets of both \( H \) and \( J \). Note that a fibre \( J^{-1}(\mu) \) is only \( G_{\mu} \)-invariant and the restriction of the flow to the fibre is \( G_{\mu} \)-equivariant.

By definition, a point \( p_e \in P \) is a relative equilibrium if there exists a generator \( \xi_e \in g \) such that \( X_H (p_e) = \xi_e p_e \). This is equivalent to \( p_e \) being a critical point of \( H_{\xi_e} = H - J_{\xi_e} \) where \( J_{\xi_e} (p) = J(p)(\xi_e) \). Note that the trajectory of \( X_H \) through \( p_e \) is \( \exp(t \xi_e) p_e \). Equivariance and conservation of \( J \) implies that \( \xi_e \in g_{\mu_e} \).

**Definition 4.** A relative equilibrium \( p_e \) is:

- \( G \)-stable if for every \( G \)-invariant neighbourhood \( U \) of \( p_e \) there is a neighbourhood \( V \) such that \( \Phi_t(p) \in U \) for all \( p \in V \) and all \( t \);
- leafwise stable if it is \( G_{\mu_e} \)-stable for the restriction of \( \Phi_t \) to the momentum level set \( J^{-1}(\mu_e) \).

Thus a relative equilibrium is leafwise stable if it is stable (mod \( G_{\mu_e} \)) to momentum-preserving perturbations and \( G \)-stable if it is stable (mod \( G \)) to all perturbations. If a relative equilibrium is \( G \)-stable, then it is also leafwise stable. An example showing that the converse is not true in general was given by Libermann & Marle [23] (see also [16] and Section 5.5).

For simplicity we restrict the discussion in this paper to free, proper group actions. In [41] we will take advantage of the generality of the topological-stability lemma to extend the stability theory to actions with nontrivial isotropy subgroups. For a free, proper action, the orbit space \( P/G \) is a smooth manifold which inherits a Poisson structure from the symplectic structure on \( P \). The Hamiltonian \( H \) descends to a function \( h \) on \( P/G \) for which the corresponding Poisson flow is the flow \( \phi_t \) on \( P/G \) induced by \( \Phi_t \). The orbit \( x_e = Gp_e \in P/G \) is an equilibrium point of \( \phi_t \). Moreover the relative equilibrium \( p_e \) is \( G \)-stable if and only if \( x_e \) is Lyapunov stable in the usual sense, and is leafwise stable in the sense of Definition 4 if and only if \( x_e \) is leafwise stable in the sense of Section 4. We can therefore expect to lift the theory developed in Section 4 to obtain both topological and derivative tests for the \( G \)-stability of relative equilibria.

The topological and derivative tests for the \( G \)-stability of Hamiltonian relative equilibria are contained in Sections 5.1 and 5.3, respectively. Section 5.2 introduces transverse Poisson structures as tools for describing local leafspace topology and uses this to discuss some special cases. In Section 5.4 we discuss applications to rel-
ative equilibria of systems that are invariant under actions of the Euclidean groups SE(2) and SE(3), proving the results that we presented in Section 2. Finally, in Section 5.5 we describe how an example of Libermann & Marle [23] fits into our theory.

5.1. G-stability by topological methods

In this section we obtain topological stability criteria for relative equilibria of G-invariant Hamiltonians on P by applying the results of Section 4.1 for Poisson equilibria to the symmetry-reduced flow on the Poisson orbit manifold P/G.

The main result of the section is the following theorem. It is a generalisation to non-compact free group actions of Montaldi’s stability theorem [32] for compact groups. An extension to general proper actions is given in [41].

**Theorem 3.** Let $H$ be a G-invariant Hamiltonian on P with a relative equilibrium at $p_e$. Let S be a slice transverse to the orbit $Gp_e$ at $p_e$. Then $p_e$ is G-stable if there exists an open neighbourhood $U_S$ of $p_e$ in S and an open neighbourhood $U_{\mu_e}$ of $\mu_e$ in $g^*$ such that

$$H^{-1}(H(p_e)) \cap J^{-1}(T^2 U_{\mu_e} (\mu_e)) \cap U_S = \{ p_e \}.$$  \hspace{1cm} (5)

This remains true if $H$ is replaced by any conserved quantity with values in any Hausdorff space.

The proof of Theorem 3 uses the slice S as a local model for the orbit space P/G. The projection from S to P/G induces a Poisson structure on S that is isomorphic to that on the corresponding open neighbourhood of $Gp_e$ in P/G. The following result shows that the $T_2$-set of the Poisson structure on S at $p_e$ is just the pullback of the $T_2$-set of the Poisson structure on $g^*$ at $\mu_e$.

**Lemma 2.** Let S denote a slice to $Gp_e$ at $p_e$.

(i) The symplectic leaves of the induced Poisson structure on S are the connected components of the intersections $J^{-1}(O) \cap S$, where O is a coadjoint orbit in $g^*$.

(ii) There exist arbitrarily small neighbourhoods $U_S$ of $p_e$ in S and $U_{\mu_e}$ of $\mu_e$ in $g^*$ such that

$$T^2 U_S (p_e) = J^{-1}(T^2 U_{\mu_e} (\mu_e)) \cap U_S.$$  \hspace{1cm} (6)

**Proof.** The first part follows from the fact that the symplectic leaves of P/G are the connected components of $J^{-1}(O)/G$.

For the second part we note that since J is a submersion there exist arbitrarily small neighbourhoods $U_{p_e}$ of $p_e$ in P and $U_{\mu_e}$ of $\mu_e$ in $g^*$, such that if M is a connected set in $U_{\mu_e}$, then $J^{-1}(M) \cap U_{p_e}$ is also connected. If $U_S = U_{p_e} \cap S$ then this also implies that $J^{-1}(M) \cap U_S$ is connected, since this last intersection is homeomorphic to the projection of $J^{-1}(M) \cap U_{p_e}$ to P/G. It follows that the momentum map induces a homeomorphism between the leaf space of $U_S$ and that of $U_{\mu_e}$, which in turn implies (6). □
Proof of Theorem 3. We may assume that the neighbourhoods $U_S$ and $U_{\mu_e}$ in (5) also satisfy the conclusions of Lemma 2. The condition (5) therefore implies that the restriction $H|_{S}$ isolates $p_e$ in $T_{2}^{U_{S}}(p_e)$ and so Theorem 1 implies that $p_e$ is Lyapunov stable for the flow on $S$ generated by $H|_{S}$ and the induced Poisson structure on $S$. Identifying $S$ with a neighbourhood of $Gp_e$ in $P/G$ and the flow on $S$ with the quotient of the flow on $P$ generated by $H$ gives the result.

The statement that $H$ can be replaced by any conserved quantity with values in any Hausdorff space follows immediately from the analogous statement in Theorem 1. □

Remark 3. It follows from Lemma 2 that the leaf space of $P/G$ is Hausdorff at $Gp_e$ if and only if $g^{\ast}/G$ is Hausdorff at $G_{\mu_e}$. If this holds, then the condition (5) reduces to

$$H^{-1}(H(p_e)) \cap J^{-1}(\mu_e) \cap U_S = \{p_e\},$$

which means that $H$ isolates $G_{\mu_e}p_e$ in its momentum level set. Sufficient conditions on $\mu_{e}$ for $g^{\ast}/G$ to be Hausdorff at $G_{\mu_e}$ are given in Proposition 5. In general condition (7) always implies that $p_e$ is leafwise stable.

5.2. Transverse Poisson structures and tame generators

Weinstein’s local splitting theorem for a Poisson manifold $X$ [50, Theorem 2.1] states that any point $x \in X$ has a neighbourhood $U$ that is isomorphic as a Poisson manifold to the product of a neighbourhood of $x$ in the leaf $L(x)$ and a transverse Poisson space $\Sigma$. The isomorphism class of this transverse Poisson structure does not depend on either the point $x$ in $L(x)$ or the choice of transverse section $\Sigma$. In Section 5.2.1 we give a detailed description of the transverse Poisson space at a point $\mu \in g^{\ast}$ and use this to give sufficient conditions for $g^{\ast}/G$ to be locally Hausdorff at $G_{\mu}$ and for a generator $\xi \in g_{\mu}$ to be tame. In Section 5.2.2 we show that the transverse Poisson structure at $Gp \in P/G$ is isomorphic to that at $\mu = J(p)$ and use this to describe the tame generators on $P/G$ at $Gp$.

5.2.1. Coadjoint actions. In this section we will define an explicit choice of a transverse Poisson space to the coadjoint orbit $G_{\mu}$ at $\mu \in g^{\ast}$ and describe some of its properties.

Let $G_{\mu}$ denote the isotropy subgroup of $\mu \in g^{\ast}$ and $g_{\mu}$ the isotropy subalgebra. Let $n_{\mu}$ denote a complement to $g_{\mu}$ in $g$, so that $g = g_{\mu} \oplus n_{\mu}$ and $g^{\ast} = g_{\mu}^{\circ} \oplus n_{\mu}^{\circ}$, where the superscript $^{\circ}$ denotes an annihilator. The decomposition of $g$ induces an isomorphism between $n_{\mu}^{\circ}$ and $g_{\mu}^{\ast}$. An easy calculation shows that $g_{\mu} = g_{\mu}^{\circ}$, so the affine subspace $\mu + n_{\mu}^{\circ}$ is transverse to the coadjoint orbit $G_{\mu}$ at $\mu$. It then follows from [50, Proposition 4.1] that there is an induced Poisson structure on a neighbourhood $\Sigma$ of $\mu$ in $\mu + n_{\mu}^{\circ}$. This transverse Poisson structure can be described explicitly as follows [10, 44]. Let $\pi_{g_{\mu}^{\circ}}$ denote the projection from $g^{\ast}$ to $g_{\mu}^{\circ}$ with kernel $n_{\mu}^{\circ}$. Since $\Sigma$ is affine, its tangent spaces are all equal to $n_{\mu}^{\circ}$, which is identified with $g_{\mu}^{\ast}$. Thus the derivatives of functions on $\Sigma$ can be regarded as taking values in $g_{\mu}$. 

Proposition 3 ([10, 44]). There exists a neighbourhood $V \subseteq n_\mu^0$ of 0 such that:

(i) For every $v \in V$ and every $\xi \in g_\mu$ the equation

$$\pi_{g_\mu} \left( \text{ad}^*_{\xi+v}(\mu+v) \right) = 0$$

has a unique solution $\eta = \eta_{\mu+v}(\xi) \in n_\mu$ and the map $j_\mu(v) : g_\mu \to g$ defined by $j_\mu(v)\xi = \xi + \eta_{\mu+v}(\xi)$ is linear, depends smoothly on $v$ and satisfies $j_\mu(0)\xi = \xi$.

(ii) A Poisson structure is defined on $\Sigma = \mu + V$ by the bracket

$$\{ f, g \}(\mu + v) = -[\mu + v, [j_\mu(v)d_v f(\mu + v), j_\mu(v)d_v g(\mu + v)]$$

where $f, g$ are smooth functions on $\Sigma$ and $[,]$ is the Lie bracket on $g$.

(iii) The Poisson structure on a neighbourhood of $\mu$ in $g^*$ is isomorphic to the product of this Poisson structure on $\Sigma$ and the Kostant-Kirillov-Souriau symplectic structure on a neighbourhood of $\mu$ in $G_\mu$.

The symplectic leaves of the Poisson structure of $\Sigma$ are the intersections of $\Sigma$ with the coadjoint orbits on $g^*$. Next we characterise these using some notation and results from [52].

Definition 5. For each $v$ in a neighbourhood $V \subseteq n_\mu^0$ of 0 define $Z_{\mu,v}$ to be the connected component containing the identity of $\tilde{Z}_{\mu,v} \equiv \left\{ g \in G : \text{Ad}^*_g(\mu+v) \in \mu + V \right\}$, and define $Z_{\mu,V} = \bigcup_{v \in V} Z_{\mu,v}$. Clearly $Z_{\mu,0} = G_\mu^0$ and $Z_{\mu,V} \supseteq G_{\mu+v}^0$.

Proposition 4.

(i) For each $v$ sufficiently close to 0, there exists a neighbourhood $W$ of the identity in $G$ such that $Z_{\mu,v} \cap W$ is a manifold of dimension $\dim G_\mu$. Its tangent space at the identity is $T_0 Z_{\mu,v} = j_\mu(0) g_\mu$.

(ii) For $V$ sufficiently small, the symplectic leaf through $\mu + v$ of the transverse Poisson structure on $\Sigma = \mu + V$ is equal to $Z_{\mu,v}(\mu + v) \cap \Sigma$.

The second statement says essentially that the symplectic leaves of $\Sigma$ are the “orbits” of $Z_{\mu,v}$. However in general $Z_{\mu,v}$ and $Z_{\mu,V}$ are not groups.

Proof. The first statement was proved in [52] and follows from an application of the implicit-function theorem. The second statement is an immediate consequence of Definition 5 and the fact that the symplectic leaves of $\Sigma$ are its intersections with the coadjoint orbits of $G$. □
in the leaf $G\mu$. Thus, a generator $\xi \in g_\mu$ is tame as a generator on $g^*$ if and only if it is tame as a generator on $\Sigma$, i.e., if and only if the equivalent conditions $T_\mu(T_2^*\mu) \subseteq \text{ann} g_\mu^*$ and $T_\mu(T_2^U\mu) \subseteq \text{ann} g^*$ hold. The annihilator of $\xi$ in $g^*_\mu$ is denoted by $\text{ann} g^*_\mu$, and other annihilators in a similar way. Note that the property of being tame does not depend on the choice of the transverse section $\Sigma$.

In the next section we will need a stronger form of tameness.

**Definition 6.** A generator $\xi \in T^*\Sigma \cong g_\mu$ is very tame if $T_2^*\mu \subseteq \mu + \text{ann} \xi$.

Clearly very tame generators are always tame, but in general the converse does not hold, see Remark 4. However, as we will see in Section 5.4, for the Euclidean groups $SE(2)$ and $SE(3)$ the notions of tame and very tame generators are equivalent.

There are two important special cases of transverse Poisson structures.

**Definition 7.** We say that $\mu \in g^*$ is:

- regular if $\dim g_\mu = \dim g_\nu$ for every $\nu$ in a neighbourhood of $\mu$;
- split if there exists a $G^*_\mu$-invariant complement $n_\mu$ to $g_\mu$ in $g$.

Here $G^*$ denotes the identity component of $G$. Note that $\mu$ is regular if and only if it is a regular point for the Lie-Poisson structure on $g^*$ in the sense of Section 4.2, since the symplectic leaves are the coadjoint orbits and the definition above implies that these have constant dimension near regular points.

It is clear that if there exists a $G$-invariant inner product on $g$, as in the case of compact or Abelian groups, then every $\mu$ is split, the $T_2$-sets are trivial and every generator is very tame. Some other results are given in the following proposition.

**Proposition 5.**

(i) If $\mu$ is regular, then $g_\mu$ is Abelian, $Z_{\mu,v} = G^*_{\mu+v}$ for $v$ small, the transverse Poisson structure on $\Sigma$ is trivial, $g^*/G$ is Hausdorff at $G\mu$ and every generator is very tame. The set of regular points in $g^*$ is open and dense.

(ii) Let $\mu$ be split. Then $Z_{\mu,v} = G^*_\mu$ for every $v$, the transverse Poisson structure on $\Sigma$ is isomorphic to the standard Lie-Poisson structure on $g^*_\mu$ and its symplectic leaves are the coadjoint orbits of $G^*_\mu$ on $g^*_\mu$. Moreover a generator $\xi \in g_\mu$ is tame if and only if it is very tame, and a sufficient condition for $\xi \in g_\mu$ to be tame is that its adjoint orbit $G^*_\mu \xi$ is bounded. Finally $g^*/G$ is Hausdorff at $G\mu$ if there is a $G^*_\mu$-invariant inner product on $g_\mu$ and in particular if $G\mu$ is compact.

**Proof.** The results for regular $\mu$ are due essentially to Lie [24] and Duflo & Vergne [11]. For split $\mu$ the results in the first sentence of part (ii) follow from Proposition 3. See also [50, 10] and references therein. For the remaining statements of part (ii) note that since $\mu$ is split we can assume that $\mu = 0$, $G_\mu = G$ and $\Sigma$ is an open neighbourhood of $0$ in $g^*$. The linearity of the action of $G$ on $g^*$ implies that if $v \in T_2^*\mu$, then $\lambda v \in T_2^*\mu$ for all $\lambda \in \mathbb{R}$. It follows that $v \in T_0(T_2^*\mu)$ and so $T_2^*\mu \subseteq T_0(T_2^*\mu)$. Hence, if $\xi$ annihilates $T_0(T_2^*\mu)$, then it annihilates $T_2^*\mu$, and so the sets of tame and very tame generators coincide.
If \( \nu \in T^\Sigma_2 (0) \), then there are sequences \( \nu_n \in \Sigma \) and \( g_n \in G^0_\mu \), such that \( \nu_n \rightarrow \nu \) and \( g_n \nu_n \rightarrow 0 \) as \( n \rightarrow \infty \). We need to show that \( \langle \nu, \xi \rangle = 0 \) if \( G^0_\xi \) is bounded. But \( \langle \nu, \xi \rangle = \lim_{n \rightarrow \infty} \langle \nu_n, \xi \rangle = \lim_{n \rightarrow \infty} \langle g_n \nu_n, g_n \xi \rangle = 0 \) since \( g_n \nu_n \rightarrow 0 \), \( g_n \xi \in G^0_\xi \) and \( G^0_\xi \) is bounded. \( \square \)

**Remark 4.** In general, regular momenta need not be split and their \( Z_{\mu, \nu} \) need not be equal to \( G^0_\nu \). Examples include nonzero points on the nilpotent cone of \( sl(2, \mathbb{R})^* \) (see Example 4 and Example 4.4 of [44]). If \( \mu \) is non-split and non-regular, then there may be tame generators \( \xi \in g_\mu \) which are not very tame. In Example 9, where \( \mu \) is a subregular nilpotent element of \( sl(3, \mathbb{R})^* \), the fact that \( T_\mu (T^U_2 (\mu)) \) is one-dimensional while \( T^U_2 (\mu) \) is two-dimensional implies that there exist generators \( \xi \) which annihilate \( \xi \) and \( T_\mu (T^U_2 (\mu)) \) but not \( T^U_2 (\mu) \) itself.

**5.2.2. The orbit space \( P/G \).** We now give a description of how the transverse Poisson structure at \( \mu \in g^* \) is related to the local Poisson structure of \( P/G \) near \( Gp \).

As in Section 5.1 we identify a neighbourhood of \( Gp \) in \( P/G \) with a slice \( S \) transverse to the orbit \( Gp \) in \( P \). By Lemma 2 the symplectic leaf \( L^S (p) \) of the Poisson structure on \( S \) is the connected component of \( J^{-1}(G\mu) \cap S \) containing \( p \). Since \( J \) is a submersion, the image \( \Sigma = J(S) \) is a submanifold of \( g^* \) through \( \mu \) that is transverse to \( G\mu \) and so has an induced transverse Poisson structure. The fact that \( J \) is a Poisson map [27] implies that that the transverse Poisson space at \( p \in S \) is isomorphic to that at \( \mu \in g^* \). It follows from Weinstein’s local splitting theorem that a sufficiently small neighbourhood of \( p \in S \) is isomorphic as a Poisson manifold to a neighbourhood of \( (p, \mu) \) in the product \( L^S (p) \times \Sigma \).

Recall that \( p_\mu \) is a relative equilibrium with generator \( \hat{\xi}_\mu \) if and only if it is a critical point of \( H_{\xi_\mu} \). The derivative tests which we give in Section 5.3 state that if the generator is tame (or very tame) and the Hessian \( d^2 H_{\xi_\mu} (p_\mu) \) is definite when restricted to certain subspaces of the normal space \( N \) to \( Gp_\mu \) at \( p_\mu \), then \( p_\mu \) is \( G \)-stable. The Witt decomposition [6] of the normal space to any group orbit \( Gp \) splits it into two components \( N = N_0 \oplus N_1 \), where \( N_1 \) is the *symplectic normal space*, i.e., a maximal subspace of \( N \) for which the restriction of the symplectic form on \( T_p P \) is nondegenerate. More explicitly \( N_1 \) is a complement to \( T_p (G\mu) \) in \( \ker dJ(p) \). For free group actions it is isomorphic to the tangent space at \( Gp \) to the leaf \( L(Gp) \) through \( Gp \) in \( P/G \), i.e. the Marsden-Weinstein reduced phase space [29].

The derivative \( dJ(p_\mu) \) maps a complement \( N_0 \) to \( N_1 \) in \( N \) isomorphically to \( T_\mu \Sigma \), the normal space to \( G\mu \) at \( \mu \) and hence identifies it with \( n_\mu \equiv g_\mu^* \). Using this identification it can be endowed with the transverse Poisson structure at \( \mu \) described in Section 5.2.1. The normal space \( N \) can then be given the Poisson structure obtained by taking the product of this structure on \( N_0 \) with the symplectic structure on \( N_1 \).

The following result is a version of Weinstein’s local splitting of \( P/G \), or equivalently a slice \( S \), due to Guillemin, Sternberg and Marle.

**Theorem 4** ([12, 26]). Let \( S \) be a slice in \( P \) transverse to \( Gp \) at \( p \) such that \( J(S) \subset \Sigma = \mu + n_\mu \). Then there exists a Poisson isomorphism from \( S \) to an open neighbourhood of the origin in \( N = N_0 \oplus N_1 \) with its product Poisson structure such
that the restriction of the momentum map $J$ to $S$ is given by $J_S(v, w) = \mu + v$, where $v \in N_0 \cong n^a_\mu$ and $w \in N_1$.

It follows from the local decomposition of $P/G$ given in Theorem 4 that the set of generators at $Gp_e$ can also be identified with $T^{*}_\mu \Sigma$, and hence with $g^*_\mu$, and that a generator is (very) tame for the Poisson structure on $P/G$ if and only if under this identification it is (very) tame for the transverse Poisson structure on $\Sigma \cong g^*_\mu$.

### 5.3. $T_2$-energy-momentum-Casimir method

We now turn to the task of lifting the derivative tests of Section 4.2 from $P/G$ to the symplectic phase space $P$. Our main results are Theorem 5 and Corollary 4 which, when combined with the results of Section 6, generalise all previous published results for free group actions. Their extensions to general proper actions will be given in [41].

Before giving our main result we present a simple corollary of Theorem 3. This will be generalised by Corollary 4.

**Corollary 3.** Let $p_e$ be a relative equilibrium of $H$ with generator $\xi_e$. Suppose that $g^*/G$ is Hausdorff at $\mu_e = J(p_e)$ and that the Hessian $d^2 H_{\xi_e}(p_e)$ is (positive or negative) definite when restricted to any symplectic normal space at $p_e$. Then $p_e$ is $G$-stable.

If $g^*/G$ is not Hausdorff at $\mu_e$ then the definiteness of $d^2 H_{\xi_e}(p_e)$ on a symplectic normal space only implies that $p_e$ is leafwise stable.

By Proposition 5 the space $g^*/G$ is Hausdorff at $\mu_e$ if either $\mu_e$ is regular or $\mu_e$ is split and there exists a $G_{\mu_e}$-invariant inner product on $g^*_{\mu_e}$. The $G$-stability result for $\mu_e$ regular is Theorem 8.17 of Chapter IV of [23]. The condition that $\mu_e$ is split and there exists a $G_{\mu_e}$-invariant inner product on $g^*_{\mu_e}$ is implied by the existence of a $G_{\mu_e}$-invariant inner product on the whole of $g^*$, which is always true for compact groups. Under this stronger hypothesis [20, 34] show that definiteness of $d^2 H_{\xi_e}(p_e)$ on a symplectic normal space implies that $p_e$ is actually $G_{\mu_e}$-stable, generalising a result of [37] for compact groups $G$. We recover this result by combining Corollary 3 with Corollary 8 in Section 6.

**Proof.** If $S$ is a slice to $Gp_e$ at $p_e$ the symplectic normal space can be identified with $T_{p_e}(J^{-1}(\mu_e) \cap S)$ and the definite Hessian, together with the fact that $p_e$ is a critical point of $H_{\xi_e}$, implies that $p_e$ is isolated in $H_{\xi_e}^{-1}(H_{\xi_e}(p_e)) \cap J^{-1}(\mu_e) \cap S$. Since $J_{\xi_e}$ is constant on $J^{-1}(\mu_e)$ this implies that $p_e$ is also isolated in $H^{-1}(H(p_e)) \cap J^{-1}(\mu_e) \cap S$. If $g^*/G$ is Hausdorff at $\mu_e$ then this is equivalent to condition (5) of Theorem 3 and so $p_e$ is also $G$-stable. □

In Theorem 5 below we generalise the “energy-momentum method” of Corollary 3 to obtain criteria for the $G$-stability of Hamiltonian relative equilibria for general non-compact symmetry groups and non-regular momentum values. Before formulating the theorem we make some preliminary remarks on smoothings and Casimirs. A local Casimir is a continuous function $C : U_{\mu_e} \rightarrow \mathbb{R}$ on a neighbourhood $U_{\mu_e}$ of $\mu_e$ which is constant on the connected components of the coadjoint orbits intersected with $U_{\mu_e}$.
Theorem 5

1. It follows from Lemma 2 that for a sufficiently small slice \( S \) to \( G \) at \( p_e \) we have \( T_\Sigma^2(p_e) = J^{-1}(T_\Sigma^2(\mu_e)) \cap S \) where \( \Sigma = J(S) \). Since \( J \) is a submersion at \( p_e \), this means that \( \{B_i\}_{i=1}^n \) is a smoothing of \( T_\Sigma^2(p_e) \) if and only if \( \{J(B_i)\}_{i=1}^n \) is a smoothing of \( T_\Sigma^2(\mu_e) \).

2. Every local Casimir \( C : U_{\mu_e} \to \mathbb{R} \) with \( \Sigma \subset U_{\mu_e} \) is constant on \( T_\Sigma^2(\mu_e) \). By Lemma 2 the function \( C \circ J \) is a Casimir on every slice \( S \) satisfying \( J(S) \subseteq U_{\mu_e} \). Consequently, as in Section 4, if \( \{B_i\}_{i=1}^n \) is a smoothing of \( T_\Sigma^2(p_e) \), the function \( C \circ J |_{B_i} \) has a critical point at \( p_e \) for each \( i \).

3. If \( \{B_i\}_{i=1}^n \) is a smoothing of \( T_\Sigma^2(p_e) \), a relative equilibrium \( p_e \) with generator \( \xi_e \in \mathfrak{g}_{\mu_e} \) is a critical point of \( H|_{B_i} \) for each \( i \) if and only if \( \xi_e \) is tame. To see this, note that \( dH(p_e) = \langle dJ(p_e), \xi_e \rangle \) annihilates \( T_{p_e}T_\Sigma^2(p_e) = dJ(p_e)^{-1}(T_{\mu_e}T_\Sigma^2(\mu_e)) \) if and only if \( \xi_e \) annihilates \( T_{p_e}T_\Sigma^2(\mu_e) \).

The next theorem provides two Hessian tests for the \( G \)-stability of relative equilibria with tame generators. The first uses the second derivatives of the restrictions of \( H \) to the manifolds \( B_i \) in a smoothing of \( T_\Sigma^2(p_e) = J^{-1}(T_\Sigma^2(\mu_e)) \) \( \cap S \). In general it is not possible to calculate the second derivative of \( H|_{B_i} \) and then restrict it to the tangent spaces \( T_{p_e}B_i \) because \( H \) need not have a critical point at \( p_e \). If \( J^{-1}(T_\Sigma^2(\mu_e)) = J^{-1}(\mu_e) \), then this can be circumvented by replacing the second derivative of \( H \) by that of \( H_{\xi_e} = H - J_{\xi_e} \), which does have a critical point at \( p_e \). This is valid because \( J_{\xi_e} \) is constant on this level set of \( J \). We used this idea in the proof of Corollary 3.

If the set \( J^{-1}(T_\Sigma^2(\mu_e)) \) is larger than the level set of \( J \), then it may not be possible to replace \( H \) by \( H_{\xi_e} \). However, if \( \xi_e \) is very tame, as defined in Definition 6, then \( J_{\xi_e} \) is constant on the whole of \( J^{-1}(T_\Sigma^2(\mu_e)) \) and it is possible to proceed as in the Hausdorff case. This gives the second test. Both tests can be strengthened by the addition of Casimirs, as stated in the theorem.

Theorem 5 (\( T_2 \)-Energy-Momentum-Casimir Method). Let \( p_e \) be a relative equilibrium of \( H \) with generator \( \xi_e \). Let \( S \) be a slice to \( G \) at \( p_e \) and \( \{B_i\}_{i=1}^n \) a smoothing of \( T_\Sigma^2(p_e) \).

(i) If \( \xi_e \) is a tame generator, then \( p_e \) is G-stable if for each \( 1 \leq i \leq n \) there is a smooth local Casimir \( C_i \) such that the Hessian \( d^2((H + C_i \circ J)|_{B_i})(p_e) \) is definite on \( T_{p_e}B_i \).

(ii) If \( \xi_e \) is a very tame generator for the transverse Poisson structure on \( \Sigma = J(S) \), then \( p_e \) is G-stable if for each \( 1 \leq i \leq n \) there is a smooth local Casimir \( C_i \) with a critical point at \( \mu_e \) and a Hessian \( d^2((H_{\xi_e} + C_i \circ J)(p_e) \) that is definite when restricted to \( T_{p_e}B_i \).

Proof.

(i) By Remark 5.2 the functions \( C_i \circ J \) are Casimirs on \( S \) and so \( I. \) follows from Theorem 2.

(ii) If \( \xi_e \) is very tame, then for all \( p \in T_\Sigma^2(p_e) = J^{-1}(T_\Sigma^2(\mu_e)) \) we have
\[ J_{\xi_e}(p) = \langle J(p), \xi_e \rangle = \langle J(p) - \mu_e, \xi_e \rangle + \langle \mu_e, \xi_e \rangle = \langle \mu_e, \xi_e \rangle \]

since \( J(p) \in T^2_\Sigma(\mu_e) \subseteq \mu_e + \text{ann} \xi_e \). Let \( f_i = H|_{S^\Sigma} + C_i \circ J|_S \), \( \tilde{f}_i = H|_{S^\Sigma} + C_i \circ J|_S \), \( f = (f_1, \ldots, f_n) \) and \( \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_n) \).

Then on \( J^{-1}(T^2_\Sigma(\mu_e)) \) the restrictions of \( f \) and \( \tilde{f} \) differ by a constant vector.

If \( C_i \) has a critical point at \( \mu_e \) then \( \tilde{f}_i \) has a critical point at \( p_e \) and

\[ d^2 \tilde{f}_i(p_e)|_{T_{p_e}B_i} = d^2(\tilde{f}_i|_{B_i})(p_e). \]

It follows that if the restriction of the Hessian of \( \tilde{f}_i \) to \( T_{p_e}B_i \) is definite, then \( p_e \) is an isolated point in \( (\tilde{f}_i|_{B_i})^{-1}(\tilde{f}_i(p_e)) \) for each \( i \) and so an isolated point in \( \tilde{f}^{-1}(\tilde{f}(p_e)) \cap J^{-1}(T^2_\Sigma(\mu_e)) \). Since on \( J^{-1}(T^2_\Sigma(\mu_e)) \) the functions \( f \) and \( \tilde{f} \) differ only by a constant, this implies that \( p_e \) is an isolated point of \( f^{-1}(f(p_e)) \cap J^{-1}(T^2_\Sigma(\mu_e)) \). The proof now follows from Theorem 3. \( \square \)

We will extend the \( T^2 \)-energy-momentum-Casimir method to general proper actions in [41].

**Remark 6.**

1. By Remark 5.1, inclusion of Casimirs in the theorem is unnecessary if the smoothing \( \{ J(B_i) \}_{i=1}^n \) of \( T^2_\Sigma(\mu_e) \) satisfies conditions similar to those stated at the end of Section 4.2. In particular, Casimirs are unnecessary if \( T^2_\Sigma(\mu_e) \) is a manifold, and the inclusion of \( C_i \) is unnecessary for any \( i \) such that (2) holds for the Poisson space \( \Sigma \).
2. If \( T^2_\Sigma(p_e) \) is a one-dimensional submanifold of \( S \), then it is not difficult to formulate a derivative test for the \( G \)-stability of \( p_e \) analogous to that of Corollary 2 for Poisson equilibria.
3. Theorem 5 can be extended to weak smoothings, as in Remark 1.

We announced the following corollary of Theorem 5 in [53, Theorem 2]. Although the result is not always optimal, it gives a generalisation of the energy-momentum method of [20, 37, 34] to non-compact symmetry groups that, as will be seen in Corollaries 5 and 6, is optimal for the Euclidean symmetry groups that are most likely to arise in applications. Recall from Section 5.2.2 that the normal space \( N \) to \( Gp_e \) at \( p_e \) decomposes as \( N_0 \oplus N_1 \), where \( N_1 \) is a symplectic normal space and \( N_0 \cong g^*_{\mu_e} \).

**Corollary 4 (Simple Energy-Momentum Method).** Let \( t_{\mu_e} \subseteq g^*_{\mu_e} \) be the space of very tame generators at \( \mu_e \) and \( w^*_{\mu_e} = \text{ann} g^*_{\mu_e} t_{\mu_e} \). Then the relative equilibrium \( p_e \) with momentum \( \mu_e \) is \( G \)-stable if its generator \( \xi_e \) is very tame and \( d^2 H|_{\xi_e}(p_e)|_{w^*_{\mu_e} \oplus N_1} \) is definite.

We call the subspace \( w^*_{\mu_e} = \text{ann} g^*_{\mu_e} t_{\mu_e} \) of \( g^*_{\mu_e} \) the space of wild momenta.

**Proof.** Choose the slice \( S \) at \( p_e \) as in Theorem 4 and identify it with the normal space \( N = N_0 \oplus N_1 \). By definition, very tame generators annihilate the set \( T^2_\Sigma(\mu_e) \) and so the space \( B = w^*_{\mu_e} \oplus N_1 \subseteq N \) is a weak smoothing of \( T^2_\Sigma(p_e) \). The corollary therefore follows from Remark 6.3. \( \square \)
For the case of semidirect products of compact groups and vector spaces Leonard & Marsden [19] also identified an intermediate set between \( N_1 \) and \( T_p P \) on which definiteness of \( d^2 H_p(p_e) \) implies stability. Corollary 4 generalises their results.

In [53] we restricted our attention to the case where \( \mu_e \) is split and defined generators \( \xi \in g_{\mu_e} \) to be tame if \( G_{\mu_e}^0 \xi \) is bounded. By Proposition 5(ii) this definition of tame generators is stronger than the notion of very tame generators used in this paper.

5.4. Applications to Euclidean equivariant systems

We now show how the energy-momentum method of this section applies to relative equilibria of Euclidean invariant Hamiltonian systems. We treat the cases where the symmetry group is the special Euclidean group \( G = SE(2) = SO(2) \times \mathbb{R}^2 \) of rotations and translations of the plane and the special Euclidean group \( G = SE(3) = SO(3) \times \mathbb{R}^3 \) of three-space, proving most of Table 1 from Section 2; the result on \( A \)-stability in the last column of Table 1 will be addressed in Section 6.3. Relative equilibria of Euclidean invariant Hamiltonian systems have been studied for several systems, including the dynamics of underwater vehicles [18, 19] and systems of point vortices [33], as described in detail in Section 2, and the statics of elastic rods [31].

In the case of \( SE(2) \) the coadjoint action is

\[
(R, a) \cdot (\mu^r, \mu^a) = (\mu^r + R\mu^a \cdot a, R\mu^a)
\]

where we have represented elements of \( SO(2) \) as \( 2 \times 2 \) matrices and

\[
\mathbb{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

From this it can be seen that \((\mu^r, \mu^a) \mapsto |\mu^a| \) is a Casimir, and that the coadjoint orbits are the cylinders about the \( \mu^r \) axis together with the points on the \( \mu^r \) axis. Consequently \( \mu \in se(2)^* \) is regular if \( \mu^a \neq 0 \), in which case \( G_\mu = \{ (1, t\mu^a) : t \in \mathbb{R} \} \); and \( \mu \in se(2)^* \) is non-regular if \( \mu^a = 0 \), in which case \( G_\mu = SE(2) \); see also Example 3. Any regular momentum \((\mu^r, \mu^a)\) is split since the transverse section \( \{ (\mu^r, t\mu^a) : t \in \mathbb{R} \} \) is \( G_\mu \) invariant, while the non-regular momenta are trivially split since they are at zero-dimensional coadjoint orbits. Any two neighbourhoods of two non-regular momenta both meet sufficiently narrow cylinders about the \( \mu^r \)-axis, so if \( \mu \) is non-regular, then \( T_2(\mu) \) is the \( \mu^r \)-axis. Given a non-regular momentum \((\mu^r, \mu^a)\), a generator \((\xi^r, \xi^a)\) is tame if it annihilates the tangent space to this \( T_2 \)-set or equivalently if and only if \( \xi^r = 0 \), and every tame generator is very tame. Casimirs are not required for an application of Theorem 5 in this case since the \( T_2 \)-sets are manifolds. The \( T_2 \)-set at the non-regular momenta is equal to the annihilator of the (very) tame generators and so Theorem 5 is equivalent to Corollary 4 here. Summarising all this gives the following corollary.

**Corollary 5.** Assume \( G = SE(2) \). Let \( p_e \) be a relative equilibrium of \( H \) with generator \( \xi_e = (\xi^r, \xi^a) \) and momentum \( \mu_e = (\mu^r, \mu^a) \).
(i) If $\mu^a_e \neq 0$, then $\mu_e$ is regular, $G_{\mu_e} = \{(1, t\mu^a_e) : t \in \mathbb{R}\}$, and $p_e$ is $G$-stable if $d^2H_{\xi_e}(p_e)$ is definite on any complement to $g_{\mu_e}p_e$ in $\ker dJ(p_e)$ (i.e., definite on any symplectic normal space).

(ii) If $\mu^a_e = 0$, then $\mu_e$ is non-regular, $G_{\mu_e} = SE(2)$, $\xi_e$ is tame if and only if $\xi^r_e = 0$, and $p_e$ is $G$-stable if $\xi_e$ is tame and $d^2H_{\xi_e}(p_e)$ is definite on any complement to $gp_e$ in $dJ^{-1}(p_e)(so(2)^*)$.

The coadjoint action of $SE(3)$ is explicitly computed in [27] and the analysis proceeds in a similar way to the $SE(2)$ case. Again the $T_2$-sets are subspaces of $se(3)^*$, so that every tame generator is also very tame, even though the momenta which are nonzero and non-regular are not split. The result is the following corollary.

**Corollary 6.** Assume $G = SE(3)$. Let $p_e$ be a relative equilibrium of $H$ with generator $\xi_e = (\xi^r_e, \xi^a_e)$ and momentum $\mu_e = (\mu^r_e, \mu^a_e)$.

(i) If $\mu^a_e \neq 0$, then $\mu_e$ is regular, $G_{\mu_e} = \{(R, t\mu^a_e) : R\mu^a_e = \mu^a_e, t \in \mathbb{R}\} \cong SO(2) \times \mathbb{R}$, and $p_e$ is $G$-stable if $d^2H_{\xi_e}(p_e)$ is definite on any complement to $g_{\mu_e}p_e$ in $\ker dJ(p_e)$ (i.e., definite on any symplectic normal space).

(ii) If $\mu^a_e = 0$, then $\mu_e$ is non-regular,

$$G_{\mu_e} = \left\{ (R, a) : R\mu^r_e = \mu^r_e, a \in \mathbb{R}^3 \right\} \cong SO(2) \times \mathbb{R}^3$$

$\xi_e$ is tame if and only if $\xi^r_e = 0$, and $p_e$ is $G$-stable if $\xi_e$ is tame and $d^2H_{\xi_e}(p_e)$ is definite on any complement to $gp_e$ in $dJ^{-1}(p_e)(so(3)^*)$.

These two corollaries prove all the entries of Table 1 except for the last column. For both the $SE(2)$ and $SE(3)$ symmetry groups our theory shows that, in the case of zero translational momentum, only for purely translating relative equilibria of $SE(2)$ invariant systems is $G$-stability accessible by energy-momentum methods, since only then is the generator tame. We presented a detailed application to relative equilibria of the Kirchhoff model for underwater vehicles in Section 2.1. For spinning relative equilibria which have zero translational momentum, $G$-stability is accessible in low-dimensional phase spaces by a blow-up argument coupled with KAM theory, as we explained in Section 2.2; see also [40].

### 5.5. An example of Libermann and Marle

This example is Exercise (15.10) on page 274 of [23] and was worked out in detail by KRISHNAPRASAD in the appendix of [16]. It was presented as an example of a relative equilibrium which is leafwise stable, but not $G$-stable. Here we show how it relates to the theory of this paper.

The group $G$ is a semidirect product $G = \mathbb{R} \rtimes \mathbb{R}$ with multiplication

$$(a, b)(a', b') = (a + a', b + e^ab').$$
The phase space $P$ is $T^* G = \mathbb{R}^4 = \{(q^1, q^2, p_1, p_2)\}$ with its standard symplectic form $dq^1 \wedge dp_1 + dq^2 \wedge dp_2$. The action of $G$ on $P$ is the cotangent lift of the left translation action of $G$ on itself:

$$(a, b)(q^1, q^2, p_1, p_2) = (a + q^1, b + e^a q^2, p_1, e^{-a} p_2).$$

The Lie algebra dual is $\mathfrak{g}^* = \mathbb{R}^2 = \{(\nu_1, \nu_2)\}$, the coadjoint action is

$$\text{Ad}^a_{(a, b)^{-1}} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & e^{-a} b \\ 0 & e^{-a} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

and the $\text{Ad}^*$-equivariant momentum mapping is

$$J(q^1, q^2, p_1, p_2) = \begin{pmatrix} p_1 + p_2 q^2 \\ p_2 \end{pmatrix}.$$
6. Beyond $G$-stability

In this section we show that $G$-stable relative equilibria typically satisfy a stronger stability property. To motivate this, suppose $p(t)$ is an integral curve of $X_H$ starting at $p(0) \approx p_e$. If $p_e$ is $G$-stable, then there is a curve $g(t)$ in $G$ such that $g(t)^{-1}p(t) \approx p_e$. By continuity and by equivariance and conservation of the momentum $J$, we have

$$\mu_e = J(p_e) \approx J(g(t)^{-1}p(t)) = Ad_{g(t)}^* J(p(t)) = Ad_{g(t)}^* J(p(0)) \approx Ad_{g(t)}^* \mu_e.$$ 

In other words, conservation of momentum ought to imply that trajectories which start close to a $G$-stable relative equilibrium $p_e$ should remain close to the $G\mu_e$-orbit through $p_e$, and not just the $G$-orbit. Results of this type on $G\mu_e$-stability, all of which assume Hessian conditions and compactness-related invariant inner products or norms on $g$, first appeared in [37], with extensions to non-free actions in [20, 30, 34]. That $G$-stable relative equilibria are not generally $G\mu_e$-stable in the non-compact case was noticed in [37] and then by LEONARD & MARSDEN [19] in the context of the stability of underwater vehicles. In this section, assuming only the $G$-stability of a relative equilibrium, we obtain results on stability which lie between $G\mu_e$-stability and $G$-stability. The results are proved for free actions, but extend to general proper actions [41].

In Section 6.1 we give the main general theorem. This is specialised to split momenta in Section 6.2, and then applied to Euclidean invariant Hamiltonian systems in Section 6.3, improving results of LEONARD & MARSDEN [19] and verifying the entries of the last column of Table 1.

6.1. $A$-stability

We first define a very general stability property for equivariant flows.

**Definition 8.** Let $P$ be a topological space with an action of a group $G$ and a $G$-equivariant flow $\Phi_t : P \to P$. For any subset $A \subseteq G$, a point $p_e \in P$ is said to be $A$-stable with respect to the flow $\Phi_t$ if, for any open neighbourhood $U \subseteq P$ of $p_e$, there exists an open neighbourhood $U' \subseteq P$ of $p_e$, such that if $p \in U'$ then $\phi_t(p) \in AU$ for all $t \in \mathbb{R}$.

If $A \subseteq B \subseteq G$, then if $p_e$ is $A$-stable it is also $B$-stable. It is easy to show that a point $p_e$ is $A$-stable if it is $AW$-stable for any neighbourhood $W$ of $1 \in G$.

Let $\Sigma \subseteq \mu_e + n_{\mu_e}^0$ be a transverse section through $G\mu_e$, at $\mu_e = J(p_e)$. For any small neighbourhood $V$ of $0$ in $n_{\mu_e}^0$ and any point $v \in V$, let $Z_{\mu_e,v}$ and $Z_{\mu_e,V}$ be the subsets of $G$ as in Definition 5. For any neighbourhood $W$ of $1$ in $G$, define

$$AV,W(\mu_e) = \bigcup_{g \in W} gZ_{\mu_e,v}v^{-1}.$$ 

Since $Z_{\mu_e,0} = G_{\mu_e}$, we have $G_{\mu_e} \subseteq AV,W(\mu_e) \subseteq G$ for any pair of neighbourhoods $V$ and $W$. 
As in Section 5.1, we identify $S$ if

We may choose $U$ coordinates the momentum map is given by $J(g, ν, w)

\{\text{open neighbourhoods of } S\}

Proof. Let $S$ be a slice at $p_e$ which is mapped by $J$ to the transverse section $\Sigma \subseteq \mu_e + n_{\mu_e}^0$ at $\mu_e$. By Theorem 4 the slice $S$ can be identified with the product of open neighbourhoods of $[0]$ in $N_0$ and $N_1$, and in these coordinates the restriction of $J$ to $S$ is given by $J_3(v, w) = \mu_e + v$, where $v \in N_0 = n_{\mu_e}^0$. By equivariance, a $G$-invariant neighbourhood of $Gp_e$ can be parametrised by $G \times S$ and in these coordinates the momentum map is given by $J(g, v, w) = \text{Ad}^*_{g^{-1}}(\mu_e + v)$.

Let $Φ_t$ denote the $G$-equivariant flow on $P$ and $φ_t$ the induced flow on $P/G$. As in Section 5.1, we identify $S$ with an open neighbourhood of $Gp_e$ in $P/G$ and denote the flow on $S$ by $φ_t$. Let $U$ be a sufficiently small neighbourhood of $p_e$. We then have $W = (\mu_e + U_0) \times U_1$ where $W$ is a neighbourhood of $1$ in $G$, and $U_0$ and $U_1$ are neighbourhoods of $[0]$ in $N_0$ and $N_1$, respectively.

Since $Gp_e$ is Lyapunov stable for the flow $φ_t$ on $P/G$, there exists a neighbourhood $U'_0$ of $p_e$ in $U_S = (\mu_e + U_0) \times U_1$ such that $φ_t(U'_0) \subseteq U_5$ for all $t$. We may choose $U'_1$ to be a product $(\mu_e + U_0^0) \times U'_1$ where $U_0^0$ and $U'_1$ are again neighbourhoods of $[0]$ in $N_0$ and $N_1$, respectively. Since $S$ is a slice, for $g_0 \in G$, $v_0 \in U'_0$ and $w_0 \in U'_1$ we have

$$Φ_t(g_0, v_0, w_0) = (g(t), v(t), w(t)) = g(t)(1, v(t), w(t)) = g(t)φ_t(v_0, w_0).$$

The momentum map $J$ is preserved by the flow $Φ_t$ and so $\text{Ad}^*_{g(t)^{-1}}(\mu_e + v(t)) = \text{Ad}^*_{g(t)^{-1}}(\mu_e + v_0) = (\mu_e + v(t)) \in \mu_e + U_0$. If the neighbourhood $U_0 \subseteq V$ is chosen small enough, this implies that $g(t) \in g_0Z_{\mu_e, v_0}$ for all $t$ (see Definition 5). It follows that

$$Φ_t(g_0, v_0, w_0) = \gamma(t)(g_0, v(t), w(t)) = \gamma(t)(g_0, φ_t(v_0, w_0))$$

where $γ(t) = g(t)g_0^{-1} \in g_0Z_{\mu_e, v_0}^{-1}$. Hence $Φ_t(W \times U_5) \subseteq A_{W, V}(\mu_e)U$, as required. □

6.2. Split momenta

Theorem 6 can be improved when $μ_e$ is split. In this case Proposition 5 says $Z_{\mu_e, V} = G_{\mu_e}^0$, and so $A_{V, W}(μ_e) = G_{\mu_e}^{0, W} = \bigcup_{g \in W} gG_{\mu_e}^{0, g}g^{-1}$. Let $G_{\mu_e}^0 = L_{\mu_e}K_{\mu_e}$, where $L_{\mu_e}$ is a submanifold of $G_{\mu_e}^0$ and $K_{\mu_e}$ a subgroup of $G_{\mu_e}^0$ for which there exists a $K_{\mu_e}$-invariant inner product on $g^*$. Such a splitting exists for any connected Lie group with $K_{\mu_e}$ a maximal compact subgroup of $G_{\mu_e}^0$ [13, Theorem 3.1]. However, in some cases it may be possible to take $K_{\mu_e}$ larger than this. For any neighbourhood $W$ of $1$ in $G$ define $L_{\mu_e}^W = \bigcup_{g \in W} gL_{\mu_e}g^{-1}$.

Corollary 7. If $p_e$ is a $G$-stable relative equilibrium and $μ_e = J(p_e)$ is split, then $p_e$ is a $L_{\mu_e}^WK_{\mu_e}$-stable equilibrium for every neighbourhood $W$ of $1$ in $G$. 
The following corollary generalises the results of [37, 20, 34] by requiring only that $\bar{G}$ acts on $g^*$, and hence on $g$. It follows that $T$ must lie in the centre of $G^o$.

For any neighbourhood $W$ of $1$ in $G$, define $K_{\mu_e}^W = \bigcup_{p_e \in W} gK_{\mu_e}g^{-1}$. For any neighbourhood $U$ of $p_e$ in $P$, we claim that there exist neighbourhoods $W$ of $1$ in $G$ and $U'$ of $p_e$ in $P$ such that $K_{\mu_e}^W U' \subseteq K_{\mu_e} U$. Indeed, consider the continuous map $\tau : G^o \times \bar{G} \times P \rightarrow P$ defined by $\tau(g, kT, p) = k^{-1} g k g^{-1} p$. Note that this is well defined because $T$ lies in the centre of $G^o$. So given $U$ and $kT$ there exist open neighbourhoods $W_k$ of $1$ in $G^o$, $V_k$ of $kT$ in $\bar{G}$, and $U_k$ of $p$ in $P$ such that $\tau(W_k, V_k, U_k) \subseteq U$. Since $\bar{G}$ is compact, there exists a finite subcover $\{V_k\}_{i=1}^n$ of $\bar{G}$. Put $W = \bigcap_{i=1}^n W_k$ and $U' = \bigcap_{i=1}^n U_k$ to prove the claim.

For such a choice of $U'$ and $W$ we have

$$G^o_{\mu_e} W \subseteq L^W_{\mu_e} K_{\mu_e}^W U' \subseteq L^W_{\mu_e} K_{\mu_e} U.$$ 

It follows that if $p_e$ is $G^o_{\mu_e}$-stable, then it is also $L^W_{\mu_e} K_{\mu_e}$-stable. □

The following corollary generalises the results of [37, 20, 34] by requiring only that $p_e$ is $G$-stable and not that the restriction of $d^2 H_{\mu_e}(p_e)$ to the symplectic normal space $N_1$ is definite.

**Corollary 8.** If $p_e$ is a $G$-stable relative equilibrium and there exists a $G^o_{\mu_e}$-invariant inner product on $g^*$, then $p_e$ is $G^o_{\mu_e}$-stable.

We end this section by applying Corollary 7 to the case of a semidirect product $G = K \ltimes \mathbb{V}$ with $K$ a compact Lie group acting linearly on a finite-dimensional real vector space $\mathbb{V}$. Assume that $\mu_e$ is split and $G_{\mu_e}$ has the form $G_{\mu_e} = K_{\mu_e} \ltimes \mathbb{V}_{\mu_e}$, where $K_{\mu_e}$ is a closed subgroup of $K$ and $\mathbb{V}_{\mu_e}$ is a $K_{\mu_e}$-invariant subspace of $\mathbb{V}$. Then $G_{\mu_e} = \mathbb{V}_{\mu_e} K_{\mu_e}$. Let $W$ denote an open neighbourhood of $(1, 0)$ in $K \ltimes \mathbb{V}$ of the form $W_K \times W_\mathbb{V}$, where $W_K$ is a neighbourhood of $1$ in $K$ and $W_\mathbb{V}$ is a neighbourhood of $0$ in $\mathbb{V}$. Then a calculation shows $\mathbb{V}_{\mu_e} = W_K^{-1} \mathbb{V}_{\mu_e} \subseteq \mathbb{V}$, a generalised “cone” in $\mathbb{V}$. Set theoretically, $A = W_K^{-1} \mathbb{V}_{\mu_e} K_{\mu_e}$ is equal to $K_{\mu_e} \times W_K^{-1} \mathbb{V}_{\mu_e} \subseteq K \times \mathbb{V}$, so we obtain the following result.

**Corollary 9.** Let $p_e$ be a $K \ltimes \mathbb{V}$-stable relative equilibrium for which $\mu_e$ is split and $G_{\mu_e} = K_{\mu_e} \ltimes \mathbb{V}_{\mu_e}$. Then $p_e$ is a $K_{\mu_e} \times W_K^{-1} \mathbb{V}_{\mu_e}$-stable relative equilibrium for any open neighbourhood $W_K$ of $1$ in $K$.

### 6.3. Euclidean invariant Hamiltonian systems

Finally, we apply the $A$-stability results of this section to relative equilibria of Euclidean invariant Hamiltonian systems and briefly discuss their implications for the stability of rigid bodies in fluids, see Section 2 and [18, 19]. As in Corollaries 5 and 6 we treat the cases where the symmetry group is the special Euclidean group $G = SE(2)$ of the plane or the special Euclidean group $G = SE(3)$ of three-space, thereby proving the last column of Table 1. We assume that the relative equilibria are $G$-stable and compute subsets $A \subseteq G$ for which the relative equilibria are $A$-stable.
First consider the symmetry group \( G = SE(2) \). For a regular momentum value \( \mu_e \) (case 1 of Corollary 5) let \( C \) be any open cone containing \( \mu_e^r \) and \( A = \{1\} \times C \). Then, for some open neighbourhood \( W_{SO(2)} \) of \( 1 \in SO(2) \), \( C \subseteq \bigcup_{R \in W_{SO(2)}} R R \mu_e^a \) and so \( p_e \) is \( A \)-stable by Corollary 9. The cone can be made arbitrarily “thin” but smaller cones will require initial conditions closer to \( p_e \). For a non-regular momentum value (case (ii) of Corollary 5) we have \( G_{\mu_e} = G \) so, trivially, \( A = G \) since \( A \) must contain \( G_{\mu_e} \).

Now let \( G = SE(3) \). As in the case of \( SE(2) \) symmetry, for a regular momentum value \( \mu_e \) (case (i) of Corollary 6) we can apply Corollary 9 and conclude that \( p_e \) is \( A \)-stable for \( A = SO(2) \times C \) where \( C \) is any open cone containing \( \mu_e^r \) and \( SO(2) \) consists of rotations about \( \mu_e^a \). For zero momentum \( A = SE(3) \) trivially. Case (ii) of Corollary 6 with \( \mu_e^r \neq 0 \) is more interesting because \( \mu_e \) is not split and so Corollary 9 does not apply and a direct analysis of the sets \( A_V, W(\mu_e) = \bigcup_{\varphi \in W} g Z_{\mu_e, \varphi} \circ \overline{g}^{-1} \) used in Theorem 6 is required. The coadjoint orbit through \((\mu_e', 0)\) is the two-sphere \( \{ (\mu', 0) : |\mu'| = |\mu_e'| \} \) and so a transverse section to the orbit can be taken as

\[
\Sigma = \left\{ \mu_e + (v', \nu^a) : v' = t \mu_e, t \in \mathbb{R}, \nu^a \in \mathbb{R}^3 \right\} \cong \mathbb{R} \times \mathbb{R}^3.
\]

Let \( \theta_{R, \mu_e'} \) denote the angle between \( R \mu_e^r \) and \( \mu_e' \). We claim that, if \( \varepsilon_0 > 0 \) and \( \varepsilon_1 > 0 \), and

\[
A_{\varepsilon_0, \varepsilon_1} \equiv \left\{ (R, a) \in SE(3) : |\sin \theta_{R, \mu_e'}| < \varepsilon_1 |a| + \varepsilon_0 \right\},
\]

then there are neighbourhoods \( V \) of \( \mu_e \) and \( W \) of \( 1 \in SE(3) \) such that \( A_V, W \subseteq A_{\varepsilon_0, \varepsilon_1} \). Therefore by Theorem 6 an \( SE(3) \) stable relative equilibrium with a momentum \( A \) as in case (ii) of Corollary 6 with \( \mu_e' \neq 0 \) is \( A_{\varepsilon_0, \varepsilon_1} \)-stable for any \( \varepsilon_0 > 0 \) and \( \varepsilon_1 > 0 \). To prove the claim, using the \( SE(3) \) coadjoint action [27] it can be shown that \( Z_{\mu_e, \nu} \) is contained in the set of \( (R, a) \in SE(3) \) satisfying

\[
\mu_e' \times (R(\mu_e^r + v') + a \times R \nu^a) = 0.
\]

(9)

In this equation, \( v' = t \mu_e^r \) where \( t \) and \( \nu^a \) are small. If \( a \) is small, then \( \mu_e' \times R \mu_e^r \approx 0 \), so \( R \) is nearly a rotation about \( \mu_e' \), but the equation admits solutions for \( R \in SO(3) \) arbitrary as long as \( a \) is large enough. We rewrite (9) as

\[
(1 + t) \mu_e' \times R \mu_e^r = -\mu_e' \times (a \times R \nu^a),
\]

whereupon

\[
|\sin \theta_{R, \mu_e'}| = \frac{|\mu_e' \times R \mu_e^r|}{|\mu_e'|^2} \leq \frac{1}{(1 - |t|)} |\nu^a| |a|.
\]

(10)

The action of conjugation is

\[
(R', a') = (\tilde{R}, \tilde{a})(R, a)(\tilde{R}, \tilde{a})^{-1} = (\tilde{R} \tilde{R}^{-1}, \tilde{R} a - \tilde{R} \tilde{R}^{-1} \tilde{a} + \tilde{a}).
\]

By continuity, for arbitrary \( \varepsilon_0 \) there is a neighbourhood \( W \) of \( 1 \in SE(3) \) such that, if \( (\tilde{R}, \tilde{a}) \in W \), then

\[
|\sin \theta_{R', \mu_e'}| = |\sin \theta_{\tilde{R} \tilde{R}^{-1}, \mu_e'}| < |\sin \theta_{R, \mu_e'}| + \frac{\varepsilon_0}{2}.
\]

(11)
The required $R$ uniformity of this estimate follows from compactness of $SO(3)$. Combining (10) and (11), and choosing $V$ so that $|v| < (1 - |t|)|\mu_r|\varepsilon_1$ gives

$$|\sin \theta_{\mu_r'}| < \frac{|v|}{|\mu_r'|}|a| + \frac{\varepsilon_0}{2} < \varepsilon_1|a| + \frac{\varepsilon_0}{2}.$$  

Also, by shrinking $W$, there is the uniform (in $a$ and $R$) estimate

$$|a'| = |\tilde{R}a - \tilde{R}R\tilde{R}^{-1}\tilde{a} + \tilde{a}| \geq |a| - |\tilde{R}R\tilde{R}^{-1}\tilde{a} - \tilde{a}| \geq |a| - \frac{\varepsilon_0}{2\varepsilon_1},$$  

from which follows

$$|\sin \theta_{\mu_r'}| < \varepsilon_1\left(|a'| + \frac{\varepsilon_0}{2\varepsilon_1}\right) + \frac{\varepsilon_0}{2} = \varepsilon_1|a'| + \varepsilon_0,$$

as required.

This $A_{\varepsilon_0,\varepsilon_1}$-stability implies orientation stability only when translation is a priori confined. Notice that for the set $A$ given by Corollary 7 for split $\mu_e$ the projection of $A$ to the compact part of $G$ is the same as that of $G_{\mu_e}$. Equation (9) shows that in the non-split case these projections can be very different.

The results described in this section were inspired by the stability analysis of Léonard & Marsden [19] of the Kirchhoff model for the dynamics of a rigid body in a fluid. For $G$ a semidirect product of a compact group and a vector space, they prove $\Gamma$-stability results for certain groups $\Gamma$ between $G_{\mu_e}$ and $G$. For regular momentum values of $SE(2)$-invariant systems their group $\Gamma$ is $\mathbb{R}^2$, while for regular momentum values of $SE(3)$-invariant systems it is $SO(2) \times \mathbb{R}^3$. However, numerical integration of an example with $SE(3)$ symmetry suggests that the drift in the translational direction only occurs within a cone, in accord with the results of this section.

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