DUALITY AND CONFINEMENT IN $SO(N)$ GAUGE THEORIES\footnote{Contribution to Festschrift “Sense of Beauty in Physics”, in honor of Adriano Di Giacomo on his 70th birthday.}

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Abstract

The presence of magnetic monopole like excitations of nonabelian varieties is one of the subtlest consequences of spontaneously broken gauge symmetries. Important hints about their quantum mechanical properties, which remained long mysterious, are coming from a detailed knowledge of the dynamics of supersymmetric gauge theories which has become available recently. These developments might shed light on the problem of confinement and dynamical symmetry breaking in QCD. We discuss here some beautiful features of vortex-monopole systems, in which dual nonabelian transformations among monopoles are generated by the nonabelian vortex moduli.

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1. **Dual groups from monopole-vortex systems**

Consider a system with a hierarchical gauge symmetry breaking (with \( v_1 = |\langle \phi_1 \rangle| \gg v_2 = |\langle \phi_2 \rangle| \))

\[ G \xrightarrow{v_1 \neq 0} H \xrightarrow{v_2 \neq 0} \emptyset, \]

where \( \phi_1 \) and \( \phi_2 \) are some (elementary or composite) scalar fields in the theory.

The high-energy theory (at scales much higher than \( v_2 \)) possesses regular monopoles, associated with nontrivial elements of \( \pi_2(G/H) \): when the unbroken group \( H \) is non-abelian, the monopoles will carry some nonabelian gauge charges, generalizing non-trivially the 't Hooft-Polyakov monopole solutions \(^1\), as has been shown by Goddard-Nuyts-Olive, Bais, and others \(^2\). In fact, they are believed to form a multiplet of a group \( \tilde{H} \) which is dual \(^2\) to \( H \). Their fully quantum mechanical properties, however, have long remained mysterious.

The low-energy theory with the gauge symmetry breaking \( H \xrightarrow{v_2 \neq 0} \emptyset \) (defined at mass scales much lower than \( v_1 \)), has vortices associated with the fundamental group \( \pi_1(H) \). Existence of the vortices with exact nonabelian continuous moduli (nonabelian vortices) in models with an unbroken color-flavor type symmetry, have recently attracted considerable attention \(^4\) \(^5\) \(^6\) \(^7\).

As \( \pi_2(G) = \emptyset \) (valid for all Lie groups) no regular monopole associated with the symmetry breaking \( G \rightarrow H \) is truly stable, once the smaller vacuum expectation value (VEV) (\( v_2 \)) is taken into account (i.e., in the full theory). It means that they are confined by the vortex of the low-energy theory. On the other hand, if \( \pi_1(G) = \emptyset \) none of the vortices visible in the low-energy approximation are actually stable in the full theory, either. In fact, these vortices can be cut by the heavy monopole pair production, even though this process will be suppressed for \( v_1 \gg v_2 \). The two phenomena are, actually, the two sides of a medal.

If \( \pi_1(G) \neq \emptyset \), however, there are some vortices which are stable in the full theory. These are sourced by, so would confine, the singular Dirac monopoles once they are introduced in the theory.

One sees that these are questions closely parallel to the general idea of confinement in QCD \(^8\); these models can be regarded as concrete (dual) models in which many questions about confinement and dynamical symmetry breaking can be investigated.

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\(^1\)The dual group is defined as the group generated by the dual root vectors, \( \alpha^* = \alpha/(\alpha \cdot \alpha) \), where \( \alpha \) are the nonzero roots of \( H \).
For instance, a crucial question about confinement is whether or not it is accompanied by dynamical abelianization. Although some of the popular models for confinement in QCD do involve an abelian effective description of the system, there are general indications that in Nature and in QCD confinement is not accompanied by dynamical abelianization. The fact that in a wide class of supersymmetric theories confinement is described as a dual superconductor but of nonabelian type, makes a more careful investigation of these theories urgent. A nonabelian superconductor is characterized by vortices which carry continuous, nonabelian moduli of flux.

Recently such nonabelian BPS vortices, i.e., BPS vortices possessing continuous, exact zeromodes, have been explicitly constructed in the context of softly broken $\mathcal{N} = 2$ gauge theories, and in related models. The presence of an exact symmetry, typically a color-flavor diagonal symmetry $H_{C+F}$, appears to be fundamental. The cases with $G = SU(N+1)$, $H = SU(N) \times U(1)/\mathbb{Z}_N$ have been studied in some detail in the papers $[4, 5, 6, 7]$. A particularly intriguing idea $[5]$ emerging from these studies is that part of these zeromodes, due to an exact global symmetry of the system, are related to the dual nonabelian transformations among the monopoles of $\pi_2(G/H)$.

It is the main aim of this note to pursue this idea further. We shall study in particular some of the questions which arose originally with nonabelian monopoles of $SO(N)$ theories, but which are of more general nature, related to how the dual (magnetic) group transformations can be understood from the properties of joint monopole-vortex systems in the original (electric) theory.

2. Vortex moduli and monopoles in $SU(N+1) \rightarrow U(N) \rightarrow \emptyset$

Our model $[5]$ was based on softly broken $\mathcal{N} = 2$, $SU(N+1)$ theory, with symmetry breaking pattern

$$SU(N+1) \xrightarrow{\langle\phi_1\rangle \neq 0} U(N) \xrightarrow{\langle\phi_2\rangle \neq 0} \emptyset. \quad (2.1)$$

Actually, in a parallel development $[4, 5, 6, 7]$ only the system $H \xrightarrow{\tau_2 \neq 0} \emptyset$ is considered, with $H = U(N)$, and with a Fayet-Iliopoulos term in the $U(1) \subset U(N)$ part. The main interest there is the dynamics of vortex orientation modes, which turns out to be various kind of two dimensional sigma models, and their relation to the dynamics of the parent 4D theory. Monopoles discussed in these papers are abelian monopoles of certain $U(1)$ factors $\subset H$ (kinks of the sigma model).
The first breaking is due to the VEV of a scalar field in the adjoint representation, while the second, smaller VEV is the condensate of the $N_f$ squark fields, in the fundamental representation of $SU(N+1)$. The number of the flavors $N_F$ is such that the $SU(N)$ gauge interactions do not grow strong between the scales $\langle \phi_1 \rangle$ and $\langle \phi_2 \rangle$: $2N \leq N_f \leq 2N + 1$.

We shall repeat neither the construction of semiclassical monopoles of high-energy theory, nor the derivation and numerical solution of nonabelian Bogomolnyi equations for the low-energy vortices [3, 5]. We limit ourselves to a few remarks here. The crucial feature is that although the gauge symmetry is completely broken in low-energy limit by the squark VEVs, a color-flavor diagonal symmetry remains unbroken. A vortex configuration breaks this symmetry ($SU(N) \to SU(N-1) \times U(1)$), generating an exact, continuous vortex moduli (zeromodes). For the minimum vortex the moduli is $CP^{N-1}$, besides the translational modes $^4$. Note that it coincides precisely with the space of states of a quantum mechanical $N$-state system. The absence of the monopoles in the full theory is accounted for by the fact that the monopole flux can be exactly carried away by a minimum vortex (Fig. 1).

The main idea is that the dual transformations among the monopoles can be seen, in the coupled monopole-vortex system, as the nonlocal $SU(N)_{C+F}$ transformation of the combined infinitely extended monopole vortex configuration. As it turns out, in the softly broken $\mathcal{N} = 2$ model studied in Auzzi et.al. [5], the approximate semiclassical monopole configuration (nontrivial configuration of the adjoint scalar and gauge fields, with the squark fields fixed to their VEVs), transforms nontrivially only under the $SU(N)_C$. Nevertheless, the whole monopole-vortex configuration (Fig. 1) is transformed under the flavor group as well, since the low-energy vortex solution

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$^4$This result is valid for $N_f = N_c = N$. For $N_f$ larger than $N_c$, as in our case, the vortex moduli is actually larger, having the modes related to the so-called semilocal strings. We believe however that only this part of the moduli is closely connected to the dual transformations of the monopoles.
involves nontrivial squark configurations. This means that one is indeed dealing with transformations among physically distinct objects. The fact that the vortex moduli is given by $CP^{N-1}$ implies that the monopoles of minimum charge transform as a fundamental multiplet ($N$) of the dual $SU(N)$ group.

The joint, monopoles-vortex configuration (Fig. 1) is neither topologically stable nor BPS saturated. The monopoles and vortices are only approximately so, under the condition, $v_1 \gg v_2$. The existence of certain parameters upon which the theory depends in a holomorphic way, which allows us to study the properties of the monopoles and vortices separately in appropriate effective theories, are one of the useful features of supersymmetric systems.

In the simplest case with the symmetry breaking,

$$SU(3) \xrightarrow{\langle \phi_1 \rangle \neq 0} U(2) \xrightarrow{\langle \phi_2 \rangle \neq 0} \emptyset,$$

the moduli space of the minimum vortices is $SU(2)/U(1) \sim S^2 = CP^1$, and is parametrized by two Euler angles. In other words the energy of the whole configuration (Fig. 1) is invariant under color-flavor diagonal transformations of the form, $U_{C+F} = e^{ir_2 \beta} e^{ir_3 \alpha}$. The degenerate $(1,0)$ and $(0,1)$ monopoles are transformed to each other by these transformations, so do the $(1,0)$ and $(0,1)$ vortices.

An apparent problem arises when one considers the monopoles and vortices of higher winding numbers. It is not difficult to see that the vortices $(2,0)$ and $(0,2)$ are indeed connected by some $U_{C+F}$ transformations but that no $U_{C+F}$ transformations exist which connect, for instance, $(2,0)$ to $(1,1)$ vortex $^5$. What is going wrong? Is the $SU_{C+F}(2)$ group unrelated to the dual $SU(2)$, after all? What this shows actually is the fact that the action of the dual $SU(2)$ group is not so simply related to the electric $SU(2)$ group. The vortices of winding number two would confine the monopoles of charge two, either in a triplet or a singlet of (dual) $SU(2)$. The singlet does not transform. As for the triplet monopoles, the space of the quantum states of a three-state system is $CP^2$. It is interesting that the moduli of the co-axis vortex of winding number two in an $SU(2)$ theory, has recently been studied $^9$: it is given by $CP^2$. This appears to provide for a further consistency of our picture.

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$^5$We thank W. Vinci for this observation $^7,^9$. 


3. Monopole-vortex system in $SO(N)$ theories

3.1. Monopoles

For concreteness, we consider here the case of $SO(N)$ theory with odd $N$. Let us consider first the breaking pattern,

$$
SO(2N + 3) \xrightarrow{\langle \phi_1 \rangle \neq 0} SO(2N + 1) \times U(1) \xrightarrow{\langle \phi_2 \rangle \neq 0} \emptyset.
$$

which can be realized, for instance by an adjoint scalar field acquiring a vacuum expectation value (VEV) of the form,

$$
\langle \phi_1 \rangle = v_1 \begin{pmatrix}
0 & i & 0 \\
-i & 0 & \\
0 & \cdots & 0
\end{pmatrix},
$$

and by the (much smaller) squark VEVs discussed below. To find the monopoles, choose $N \ SO(4) \sim SU(2) \times SU(2)$ subgroups, living in $(1, 2, 2i + 1, 2i + 2)$ spaces, $i = 1, 2, \ldots, N$. They are all broken to $U(1) \times U(1)$ by the VEV above. The $2N$ nonabelian monopoles can be constructed by embedding the ’t Hooft-Polyakov monopoles in these $2N$ $SU(2)$ subgroups broken to $U(1)$’s. One finds thus a natural candidate for the set of monopoles, to be transformed as the fundamental multiplet of a dual $USp(2N)$ group. In order to construct the transformations among them, we make full use of the unbroken $SO(2N + 1)$ group. The idea is to make a map between the $SO(2N + 1)$ generators (antisymmetric matrices) and the $USp(2N)$ generators which have the form,

$$
\begin{pmatrix}
B & A \\
A^\dagger & -B^\dagger
\end{pmatrix},
$$

where $A, B$ are $N \times N$ matrices with the constraints, $A^\dagger = A$, $B^\dagger = B$, in an appropriate basis of monopoles. The $i$th $SO(4) \sim SU(2) \times SU(2)$ subgroup is generated by (with a simplified notation $(1, 2, 3, 4) \equiv (1, 2, 2i + 1, 2i + 2)$)

$$
T_1^{\pm} = -\frac{i}{2} (\Sigma_{23} \pm \Sigma_{41}), \quad T_2^{\pm} = -\frac{i}{2} (\Sigma_{31} \pm \Sigma_{42}), \quad T_3^{\pm} = -\frac{i}{2} (\Sigma_{12} \pm \Sigma_{43}).
$$

The two monopoles from this $SO(4)$ group are taken to be $i$-th and $N + i$-th components of the fundamental representation of $USp(2N)$ group. The pair can be transformed to each other by rotations in the $(2i + 2, 2N + 3)$ plane ($\subset SO(2N + 1)$), thus

$$
\Sigma_{2i+2,2N+3} \longrightarrow A_{i,i}.
$$
On the other hand, the two monopoles associated with subgroups $T^\pm$ living in the $(1, 2, 2i + 1, 2i + 2)$ subspace and those living in the $(1, 2, 2j + 1, 2j + 2)$ subspace, $j \neq i$, are transformed into each other by rotations in the $(2i+1, 2i+2, 2j+1, 2j+2)$ space: they transform in $SO(2N)$ (in the subspace $i = 3, 4, \ldots, 2N+2$). In order to see that they actually transform as a pair of $U(N)$ representations, let us go to the bases where a $SO(2N)$ vector naturally breaks to $N + \bar{N}$ under $U(N)$. This can be done by going from the original $SO(2N)$ basis,

$$\begin{pmatrix} E & F \\ -F^t & D \end{pmatrix},$$

where $D, E, F$ are all purely imaginary $N \times N$ matrices with the constraints $E^t = -E, D^t = -D$, to a new basis by

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -i/\sqrt{2} \\ -i/\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} E & F \\ -F^t & D \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & i/\sqrt{2} \\ i/\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (E + D) + i(F + F^t) & i(E - D) + (F - F^t) \\ -i(E - D) + (F - F^t) & (E + D) - i(F + F^t) \end{pmatrix},$$

that is,

$$\psi_a^i = \frac{1}{\sqrt{2}} (\hat{\psi}_a^{2i-1} + \hat{\psi}_a^{2i}), \quad \psi_a^{N+i} = \frac{1}{i\sqrt{2}} (\hat{\psi}_a^{2i-1} - \hat{\psi}_a^{2i}). \quad (i = 1, 2, \ldots, N). \quad (3.8)$$

It can then be seen that the $U(N)$ elements (acting on $\psi_a^i, \psi_a^{N+i}$) are generated by the set of $SO(2N)$ infinitesimal transformations with $E = D, F = F^t$, so

$$(\Sigma_{2i,2i} + \Sigma_{2i-1,2j-1}), \quad i (\Sigma_{2i,2j-1} - \Sigma_{2i-1,2j}) \rightarrow B_{i,j}. \quad (3.9)$$

Nondiagonal elements of $A_{ij}$ can be generated by combining the actions of (3.5) and (3.9).

### 3.2. Vortices

Let us now consider the low-energy theory, with the symmetry breaking

$$SO(2N + 1) \times U(1) \xrightarrow{\langle \phi_2 \rangle \neq 0} \emptyset. \quad (3.10)$$

As $\pi_1(SO(2N + 1) \times U(1)) = \mathbb{Z}_2 \times \mathbb{Z}$, there are two types of vortices in this theory: a vortex carrying a $\mathbb{Z}_2$ flux and the $U(1)$ vortices with a $\mathbb{Z}$ flux. A vortex carrying
the minimum flux of one type only, corresponds to $\pi_1(2N + 3) = \mathbb{Z}_2$; it confines the singular, Dirac monopole of the fundamental theory, if it is introduced in the theory.

The regular monopoles of minimum charge considered in the preceding subsection are confined by a double-vortex, with the minimum flux with respect to both factors in $\mathbb{Z}_2 \times \mathbb{Z}$. In analogy with the case of $U(N)$ models discussed before, we wish to construct a model in which an unbroken global $SO(2N + 1)_{C+F}$ group remains, broken only by individual vortex configurations, so that continuous zeromodes (moduli) develop.

It turns out that it is not as straightforward as in the $SU(N)$ case to construct such a model, with a hierarchical gauge symmetry breaking (3.1), in the context of softly broken $\mathcal{N} = 2$ supersymmetric gauge theories, with the superpotential,

$$W = \{\sqrt{2} \tilde{Q}_i \Phi Q^i + m_i \tilde{Q}_i Q^i\} + \mu \text{Tr} \Phi^2, \quad m_i \to m. \quad (3.11)$$

It is however quite easy to add an $\mathcal{N} = 1$ superpotential

$$\Delta W = M (Q_i Q_i + \tilde{Q}_i \tilde{Q}_i), \quad M = m \quad (3.12)$$

so that the vacua with desired properties do appear. For instance, in the case of the $SO(5)$ theory, the adjoint VEVs can be taken as

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & i m_1 \\ -i m_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \phi_1 = \sqrt{2} m_1, \quad (3.13)$$

which breaks the gauge symmetry as $SO(5) \to SO(3) \times U(1)$, at a high mass scale, $v_1 = m_1$, while the squark VEVs take the form,

$$Q_1 = \begin{pmatrix} d_1 \\ -i d_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{Q}_1 = \begin{pmatrix} \tilde{d}_1 \\ i \tilde{d}_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |\tilde{d}_1|^2 = d_1^2 \ll m_1^2, \quad \text{Re} \, d_1 \tilde{d}_1 = -\frac{\mu m_1}{2}, \quad (3.14)$$

and

$$Q_2 = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}, \quad Q_4 = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}, \quad (3.15)$$
The squark VEVs with $w, \sqrt{m_1 \mu} \sim v_2 \ll v_1$ break the gauge symmetry completely at low energies, leaving an exact $SO(3)_{C+F}$ symmetry. \footnote{Another possibility for constructing a model of this sort, with nonabelian vortex moduli, is to introduce more than one adjoint scalar fields, $\Phi_i$ (L. Ferretti, unpublished).}

In conclusion, the idea of nonabelian confinement, with dual group transformations among the monopoles generated by vortex zeromodes (moduli), can be naturally generalized to this system, although in this case the regular monopoles transform in the dual, $USp(2N)$ group and the confining string consists of two types of vortices. (Fig. 2)

\section{SO(2N+1) → U(N) → ∅}

The magnetic monopoles in these cases, according to the Goddard-Nuyts-Olive construction, belong to the second rank (symmetric) tensor representation of the dual $SU(N)$ group, and should transform as such. Again, by constructing the base of degenerate $\frac{N(N+1)}{2}$ monopoles following E. Weinberg \cite{2}, it is a simple matter to check
that not all of these monopoles are connected by the unbroken \(SU(N)_C\) transformations. This is another example of the nontrivial relation between the actions of the electric group \(H\) or \(H_{C+F}\) under which the monopole-vortex complex transforms, and the dual group \(\tilde{H}\) under which the nonabelian monopoles are believed to transform. In the simplest case \(N=2\), again the result of Hashimoto et. al. and Auzzi et. al. [9] that the vortex moduli of the winding number two be \(CP^2\), shows the consistency of our picture, the monopoles being in a triplet in this case.

5. Conclusion

Although some very detailed quantum properties of nonabelian monopoles are known now (for instance, [10]), the precise relation between the dual transformation of the monopoles and the electric, unbroken symmetries, has yet to be fully elucidated. We hope that studies along the line of this note will help in clarifying these issues, and eventually in understanding the confinement itself.

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