Concatenation of Error Avoiding with Error Correcting Quantum Codes for Correlated Noise Models

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We study the performance of simple error correcting and error avoiding quantum codes together with their concatenation for correlated noise models. Specifically, we consider two error models: i) a bit-flip (phase-flip) noisy Markovian memory channel (model I); ii) a memory channel defined as a memory degree dependent linear combination of memoryless channels with Kraus decompositions expressed solely in terms of tensor products of X-Pauli (Z-Pauli) operators (model II). The performance of both the three-qubit bit flip (phase flip) and the error avoiding codes suitable for the considered error models is quantified in terms of the entanglement fidelity. We explicitly show that while none of the two codes is effective in the extreme limit when the other is, the three-qubit bit flip (phase flip) code still works for high enough correlations in the errors, whereas the error avoiding code does not work for small correlations. Finally, we consider the concatenation of such codes for both error models and show that it is particularly advantageous for model II in the regime of partial correlations.

PACS numbers: quantum error correction (03.67.Pp); decoherence (03.65. Yz).

I. INTRODUCTION

It is well known that one of the most important obstacles in quantum information processing is decoherence. It causes a quantum computer to lose its quantum properties destroying its performance advantages over a classical computer. The unavoidable interaction between the open quantum processor and its environment corrupts the information stored in the system and causes errors that may lead to wrong outputs.

There are different methods for preserving quantum coherence. One possible technique exploits the redundancy in encoding information. This scheme is known as “quantum error correcting codes” (QECCs). For a comprehensive introduction to QECCs, we refer to [1]. Within such scheme, information is encoded in linear subspaces (codes) of the total Hilbert space in such a way that errors induced by the interaction with the environment can be detected and corrected. The QECC approach may be interpreted as an active stabilization of a quantum state in which, by monitoring the system and conditionally carrying on suitable operations, one prevents the loss of information. Another possible approach is the so-called “decoherence free subspaces” (DFSs) (also known as error avoiding or noiseless codes). For a comprehensive introduction to DFSs, we refer to [2]. It turns out that for specific open quantum systems (noise models in which all qubits can be considered symmetrically coupled with the same environment), it is possible to design states that are hardly corrupted rather than states that can be easily corrected (in this sense, DFSs are complementary to QECCs). In other words, it is possible to encode information in linear subspaces such that the dynamics restricted to such subspaces is unitary. This implies that no information is lost and quantum coherence is maintained. DFS is an example of passive stabilization of quantum information.

The formal mathematical description of the qubit-environment interaction is usually given in terms of quantum channels. Quantum error correction is usually developed under the assumption of i.i.d. (identically and independently distributed) errors. These error models are characterized by memoryless communication channels Λ such that n-channel uses is given by Λ(ⁿ) = Λ⊗ⁿ. In such cases of complete independent decoherence, qubits interact with their own environments which do not interact with each other. However, in actual physical situations, qubits do interact with a common environment which unavoidably introduces correlations in the noise. For instance, there are situations where qubits in a ion trap set-up are collectively coupled to their vibrational modes [3]. In other situations, different qubits in a quantum dot design are coupled to the same lattice, thus interacting with a common thermal bath of phonons [4]. The exchange of bosons between qubits causes spatial and temporal correlations that violate the condition of error independence [5]. Memory effects introduce correlations among channel uses with the consequence that Λ(ⁿ) ≠ Λ⊗ⁿ. Recent studies try to characterize the effect of correlations on the performance of QECCs [6–10]. It appears that correlations may have negative [7] or positive [8] impact on QECCs depending on the features of the error model being considered.

Devising new good quantum codes for both independent and correlated noise models is a highly non trivial problem. However, it is usually possible to manipulate existing codes to construct new ones suitable for more general error models and with higher performances [11] [12]. Concatenation is perhaps one of the most successful quantum coding tricks employed to produce new codes from old ones. The concept of concatenated codes was first introduced in
classical error correcting schemes by Forney [13]. Roughly speaking, concatenation is a method of combining two codes (an inner and an outer code) to form a larger code. In classical error correction, Forney has carried on extensive studies of concatenated codes, how to choose the inner and outer codes, and what error probability threshold values can be achieved. The first applications of concatenated codes in quantum error correction appear in [14] [15]. In the quantum setting, concatenated codes play a key role in fault tolerant quantum computation and in constructing good degenerate quantum error correcting codes. For instance, Shor’s degenerate code can be constructed by concatenating the three-qubit bit flip and phase flip codes.

It is known that DFSs are efficient under conditions in which each qubit couples to the same environment (collective decoherence) while ordinary QECCs are designed to be efficient when each individual qubit couples to a different environment (independent decoherence) [16]. While none of the two error correction schemes is effective in the extreme limit when the other is, QECCs will still work for correlated errors [17] [18], whereas DFSs will not work in the independent error case [19]. Therefore, it would be of interest to study the performance of concatenated codes obtained by combining DFSs and QECCs for explicit noise models in the presence of partially correlated errors.

Indeed, this will be one of the main purposes in our work.

In this article, we study the performance of simple error correcting and error avoiding quantum codes together with their concatenation for correlated noise models. Specifically, we consider two error models: i) a bit-flip (phase-flip) noisy Markovian memory channel (model I); ii) a memory channel defined as a memory degree dependent linear combination of memoryless channels with Kraus decompositions expressed solely in terms of tensor products of flip noisy Markovian memory channel (model II). The performance of both the three-qubit bit flip (phase flip) and the error avoiding codes suitable for the considered error models is quantified in terms of the entanglement fidelity. We explicitly show that while none of the two codes is effective in the extreme limit when the other is, the three-qubit bit flip (phase flip) code still works for high enough correlations in the errors, whereas the error avoiding code does not work for small correlations. Finally, we consider the concatenation of such codes for both error models and show that it is particularly advantageous for model II in the regime of partial correlations.

The layout of this article is as follows. In Section II, we briefly discuss about the concatenation technique in quantum coding, the DFSs and, the entanglement fidelity as a performance measure of quantum error correction schemes. In Section III, we evaluate the performances of the three-qubit bit flip code and suitable DFSs for error models I and II. The performance of the concatenated code for both models appears in Section IV. Finally, in Section V, we present our concluding remarks.

II. ON CONCATENATION, DECOHERENCE FREE SUBSPACES AND, ENTANGLEMENT FIDELITY

In this Section, we briefly discuss about the concatenation technique in quantum coding, the DFSs and, the entanglement fidelity.

A. Quantum Concatenation Trick

For the sake of simplicity we only use two layers of concatenation and consider single qubit encoding. The generalization to arbitrary l layers of concatenation and multi-qubit encoding is relatively straightforward [20]. Assume the inner code (first layer) is a \([n_1, k_1, d_1]\) stabilizer code \(C_1\) with generators \(G_1 = \{g_1^i : i = 1, \ldots, n_1 - k_1\}\), and the outer code (second layer) is a \([n_2, 1, d_2]\) stabilizer code \(C_2\) with generators \(G_2 = \{g_2^j : j = 1, \ldots, n_j - 1\}\). The concatenated code \(C \overset{\text{def}}{=} C_1 \circ C_2\) is to map \(k_1\) qubits into \(n = n_1n_2\) qubits, with code construction parsing the \(n\) qubits into \(n_1\) blocks \(B(b) (b = 1, \ldots, n_1)\) each containing \(n_2\) qubits. In other words, given a codeword \(|c_{\text{inner}}\rangle\) for the inner code \(C_1\),

\[
|c_{\text{inner}}\rangle = \sum_{j=0}^{k_1-1} \alpha_j |\phi_j\rangle ,
\]

where \(|\phi_j\rangle\) are basis vectors for \(C_1\), the concatenated code \(C\) is constructed as follows. For any codeword \(|c_{\text{outer}}\rangle\) for the outer code \(C_2\),

\[
|c_{\text{outer}}\rangle = \sum_{i_1, \ldots, i_{n_2}} \alpha_{i_1, \ldots, i_{n_2}} |i_1\ldots i_{n_2}\rangle ,
\]
with $|i_1...i_{n_2}\rangle = |i_1\rangle \otimes ... \otimes |i_{n_2}\rangle$, replace each basis vector $|i_j\rangle$ with $i_j = 0,..., k_1 - 1$ for $j = 1,..., n_2$ by a basis vector $|\phi_{i_j}\rangle$ in $C_1$, that is

$$|e_{\text{concatenated}}\rangle \overset{\text{def}}{=} \sum_{i_1,...,i_{n_2}} \alpha_{i_1,...,i_{n_2}} |\phi_{i_1}\rangle \otimes ... \otimes |\phi_{i_{n_2}}\rangle.$$  \hspace{1cm} (3)

Further details on the construction of the stabilizer generators of $C$ can be found in \cite{20}. As a final remark, we simply point out that the above mentioned construction produces a $[n_1n_2, k_1, d]$ code with $d \geq d_1d_2$. As an illustrative example, consider a concatenated code that is based on $l$ layers of concatenation of the same code, for instance the seven-qubit \cite{7, 1, 3} CSS code. Here, an unencoded qubit is encoded into a block of seven qubits. Next each qubit is itself encoded into a block of seven qubits. Repeating this process $l$ times produces the concatenated code in which one logical qubit is recursively encoded into the state of $7^l$ physical qubits. Thus, such concatenated code is a $[7^l, 1, d]$ code with $d \geq 3^l$. Recall that a distance $d$ QECC can correct $t$ errors, where $t = \left\lfloor \frac{d-1}{2} \right\rfloor$ and $x$ is the integer part of $x$. Since $d \geq 3^l$, we see that the number of errors that a concatenated code can correct grows exponentially with the number of layers of concatenation $l$. Therefore, concatenation can be a good coding trick as long as the error model at each encoding level has the same form, that is, the same Kraus error operators with possibly different amplitude.

**B. Decoherence Free Subspaces**

Following \cite{2}, we mention few relevant properties of DFSs. Consider the dynamics of a closed system composed of a quantum system $Q$ coupled to a bath $B$. The unitary evolution of the closed system is described by the combined system-bath Hamiltonian $H_{\text{tot}}$,

$$H_{\text{tot}} = H_Q \otimes I_B + H_B \otimes I_Q + H_{\text{int}}, \quad H_{\text{int}} = \sum_{\alpha} E_{\alpha} \otimes B_{\alpha}.$$  \hspace{1cm} (4)

The operator $H_Q$ ($H_B$) is the system (bath) Hamiltonian, $I_Q$ ($I_B$) is the identity operator of the system (bath), $E_{\alpha}$ are the error generators acting solely on $Q$ while $B_{\alpha}$ act on the bath. The last term in (4) is the interaction Hamiltonian.

A subspace $\mathcal{H}_{\text{DFS}}$ of the total system Hilbert space $\mathcal{H}$ is a decoherence free subspace (unitary evolution in $\mathcal{H}_{\text{DFS}}$ for all possible bath states) if and only if:

1. $E_{\alpha} |\psi\rangle = c_\alpha |\psi\rangle$ with $c_\alpha \in \mathbb{C}$, for all states $|\psi\rangle$ spanning $\mathcal{H}_{\text{DFS}}$, and for every error operator $E_{\alpha}$ in $H_{\text{int}}$. In other words, all basis states spanning $\mathcal{H}_{\text{DFS}}$ are degenerate eigenstates of all the error generators $E_{\alpha}$;

2. $Q$ and $B$ are initially decoupled;

3. $H_Q |\psi\rangle$ has no overlap with states in the subspace orthogonal to $\mathcal{H}_{\text{DFS}}$.

To establish a direct link between QECCs and DFSs, it is more convenient to present an alternative formulation of DFSs in terms of the Kraus operator sum representation. Within such description, the evolution of the system $Q$ density matrix is written as,

$$\rho_Q (t) = Tr_B [U (\rho_Q \otimes \rho_B) U^\dagger] = \sum_k A_k \rho_Q (0) A_k^\dagger,$$  \hspace{1cm} (5)

where $U = e^{-i \frac{\pi}{\hbar} H_{\text{tot}}t}$ is the unitary evolution operator for the system-bath closed system and the Kraus operator $A_k$ (satisfying the normalization condition) are given by,

$$A_k = \sqrt{n} \langle m | U | n \rangle \text{ with } \sum_k A_k^\dagger A_k = I_Q,$$  \hspace{1cm} (6)

where $k = (n, m)$, $|m\rangle$ and $|n\rangle$ are bath states. It turns out that a $N_{\text{DFS}}$-dimensional subspace $\mathcal{H}_{\text{DFS}}$ of $\mathcal{H}$ is a DFS if and only if all Kraus operators have an identical unitary representation (in the basis where the first $N_{\text{DFS}}$ states span $\mathcal{H}_{\text{DFS}}$) upon restriction to it, up to a multiplicative constant,

$$A_k = \begin{pmatrix} g_k U_Q^{(\text{DFS})} & 0 \\ 0 & A_k \end{pmatrix},$$  \hspace{1cm} (7)
where \( \rho = \sqrt{n} \langle m | U_c | n \rangle \) and \( U_c = e^{-iH_c t} \) with \( H_c = H_B + H_{\text{int}} \). Furthermore, \( \tilde{A}_k \) is an arbitrary matrix that acts on \( \mathcal{H}_{\text{DFS}} \) (with \( \mathcal{H} = \mathcal{H}_{\text{DFS}} \oplus \mathcal{H}_{\text{DFS}}^\perp \)) and may cause decoherence there; \( U_Q^{(\text{DFS})} \) is \( U_Q \) restricted to \( \mathcal{H}_{\text{DFS}} \). Now recall that in ordinary QECCs, it is possible to correct the errors induced by a given set of Kraus operators \( \{A_k\} \) if and only if,

\[
R_r A_k = \left( \begin{array}{cc}
\lambda_{rk} I_C & 0 \\
0 & B_{rk}
\end{array} \right), \quad \forall \ r \text{ and } k,
\]

or, equivalently,

\[
A_k^\dagger A_{k'} = \left( \begin{array}{cc}
\gamma_{kk'} I_C & 0 \\
0 & \tilde{A}_k^\dagger \tilde{A}_{k'}
\end{array} \right),
\]

where \( \{R_r\} \) are the recovery operators. The first block in the RHS of (8) acts on the code space \( C \) while the matrices \( B_{rk} \) act on \( C^\perp \) where \( \mathcal{H} = C \oplus C^\perp \). From (7) and (8), it follows that DFS can be viewed as a special class of QECCs, where upon restriction to the code space \( C \), all recovery operators \( R_r \) are proportional to the inverse of the system \( \mathcal{Q} \) evolution operator,

\[
R_r \propto \left( U_Q^{(\text{DFS})} \right)^\dagger.
\]

Assuming that (10) holds, from (7) and (9) it also turns out that,

\[
A_k \propto U_{Q}^{(\text{DFS})},
\]

upon restriction to \( C \). Furthermore, from (7) and (9), it follows that \( \gamma_{kk'} = g_k^* g_{k'} \). However, while in the QECCs case \( \gamma_{kk'} \) is in general a full-rank matrix (non-degenerate code), in the DFSs case this matrix has rank 1. In conclusion, a DFS can be viewed as a special type of QECC, namely a completely degenerate quantum error correcting code where upon restriction to the code subspace all recovery operators are proportional to the inverse of the system evolution operator. As a side remark, in view of this last observation we point out that it is not unreasonable to quantify the performance of both active and passive QEC schemes by means of the same performance measure.

### C. Entanglement Fidelity

We recall the concept of entanglement fidelity as a useful performance measure of the efficiency of quantum error correcting codes. Entanglement fidelity is a quantity that keeps track of how well the state and entanglement of a subsystem of a larger system are stored, without requiring the knowledge of the complete state or dynamics of the larger system. More precisely, the entanglement fidelity is defined for a mixed state \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \) in terms of a purification \( |\psi\rangle \in \mathcal{H} \otimes \mathcal{H}_{R} \) to a reference system \( \mathcal{H}_{R} \). The purification \( |\psi\rangle \) encodes all of the information in \( \rho \). Entanglement fidelity is a measure of how well the channel \( \Lambda \) preserves the entanglement of the state \( \mathcal{H} \) with its reference system \( \mathcal{H}_{R} \). The entanglement fidelity is defined as follows \([21]\),

\[
F(\rho, \Lambda) \overset{\text{def}}{=} \langle \psi | (\Lambda \otimes I_{\mathcal{H}_{R}}) (|\psi\rangle \langle \psi|) |\psi\rangle,
\]

where \( |\psi\rangle \) is any purification of \( \rho \), \( I_{\mathcal{H}_{R}} \) is the identity map on \( \mathcal{M}(\mathcal{H}_{R}) \) and \( \Lambda \otimes I_{\mathcal{H}_{R}} \) is the evolution operator extended to the space \( \mathcal{H} \otimes \mathcal{H}_{R} \), space on which \( \rho \) has been purified. If the quantum operation \( \Lambda \) is written in terms of its Kraus operator elements \( \{A_k\} \) as, \( \Lambda (\rho) = \sum_k A_k \rho A_k^\dagger \), then it can be shown that \([22]\),

\[
F(\rho, \Lambda) = \sum_k \text{tr} (A_k \rho) \text{tr} (A_k^\dagger) = \sum_k |\text{tr} (\rho A_k)|^2.
\]

This expression for the entanglement fidelity is very useful for explicit calculations. Finally, assuming that

\[
\Lambda : \mathcal{M}(\mathcal{H}) \ni \rho \mapsto \Lambda (\rho) = \sum_k A_k \rho A_k^\dagger \in \mathcal{M}(\mathcal{H}), \ dim_C \mathcal{H} = N
\]

and choosing a purification described by a maximally entangled unit vector \( |\psi\rangle \in \mathcal{H} \otimes \mathcal{H} \) for the mixed state \( \rho = \frac{1}{\dim_C \mathcal{H}} I_{\mathcal{H}} \), we obtain

\[
F \left( \frac{1}{N} I_{\mathcal{H}}, \Lambda \right) = \frac{1}{N^2} \sum_k |\text{tr} A_k|^2.
\]

The expression in (15) represents the entanglement fidelity when no error correction is performed on the noisy channel \( \Lambda \) in (14).
III. THREE-QUBIT BIT FLIP CODE AND DFS

In this Section, we consider two correlated error models. Error correction is performed by means of the three-qubit bit flip code and a suitable decoherence free subspace. The code performance measure used is the entanglement fidelity.

**Model I.** The first model is a bit flip noisy quantum Markovian memory channel \( \Lambda^{(n)}(\rho) \) (model I). In explicit terms, we consider \( n \) qubits and Markovian correlated errors in a bit flip quantum channel,

\[
\Lambda^{(n)}(\rho) \equiv \sum_{i_1, \ldots, i_n=0}^{1} p_{i_n|i_{n-1}} p_{i_{n-1}|i_{n-2}} \ldots p_{i_2|i_1} p_{i_1} \left( A_{i_n} \otimes \ldots \otimes A_{i_1} \right) \rho \left( A_{i_n} \otimes \ldots \otimes A_{i_1} \right)^{\dagger},
\]

where \( A_0 \equiv I, A_1 \equiv X \) are Pauli operators. Furthermore the conditional probabilities \( p_{i_k|i_j} \) are given by,

\[
p_{i_k|i_j} = (1 - \mu)p + \mu \delta_{i_k, i_j}, \quad p_{i_k=0} = 1 - p, \quad p_{i_k=1} = p,
\]

with,

\[
\sum_{i_1, \ldots, i_n=0}^{1} p_{i_n|i_{n-1}} p_{i_{n-1}|i_{n-2}} \ldots p_{i_2|i_1} p_{i_1} = 1. \tag{18}
\]

To simplify our notation, we may choose to omit the symbol of tensor product "\( \otimes \)" in the future, \( A_{i_n} \otimes \ldots \otimes A_{i_1} \equiv A_{i_n} \ldots A_{i_1} \). Furthermore, we may choose to omit the bar "\( | \)" in \( p_{i_k|i_j} \) and simply write the conditional probabilities as \( p_{i_k,i_j} \).

**Model II.** The second quantum communication channel \( \Lambda_{\mu}^{(n)}(\rho) \) (model II) that we consider is a memory quantum channel defined in terms of a linear combination of simple memoryless channels with Kraus decompositions expressed in terms of bit-flip error operators. The coefficients of such combination are dependent on the memory parameter \( \mu \).

In explicit terms, we consider the following channel \( \Lambda_{\mu}^{(n)}(\rho) \),

\[
\Lambda_{\mu}^{(n)}(\rho) \equiv a_0(\mu) \Lambda_0^{(n)}(\rho) + a_1(\mu) \Lambda_1^{(n)}(\rho), \tag{19}
\]

where \( a_0(\mu) \equiv 1 - \mu, a_1(\mu) \equiv \mu, \Lambda_0^{(n)}(\rho) \equiv \Lambda^{(n)}(\rho) \) in (16) in the limiting case of \( \mu = 0 \) (\( n \)-uses of a memoryless bit flip channel). Finally, \( \Lambda_1^{(n)}(\rho) \) describes \( n \)-uses of a memoryless channel whose Kraus decomposition is characterized only by weight-0 and weight-\( n \) error operators with amplitudes \( (1 - p)^{\frac{n}{2}} \) and \( \left[ 1 - (1 - p)^n \right]^{\frac{n}{2}} \), respectively,

\[
\Lambda_1^{(n)}(\rho) \equiv (1 - p)^n \rho + \left[ 1 - (1 - p)^n \right] X^1 X^2 \ldots X^{n-1} X^n \rho X^1 X^2 \ldots X^{n-1} X^n. \tag{20}
\]

In this Section, QEC is performed via the three-qubit bit flip code and a suitable decoherence free subspace. Although the error models considered are not truly quantum in nature, from this preliminary work we hope to gain useful insights for extending error correction techniques to quantum error models in the presence of partial correlations. The performance of quantum error correcting codes is quantified by means of the entanglement fidelity as function of the error probability \( p \) and degree of memory \( \mu \).

**A. The three-qubit bit flip code**

1. **Model I.**

**Error Operators.** In the simplest example, we consider the limiting case of (16) with \( n = 3 \),

\[
\Lambda^{(3)}(\rho) \equiv \sum_{i_1, i_2, i_3=0}^{1} p_{i_3|i_2} p_{i_2|i_1} p_{i_1} \left[ A_{i_3} A_{i_2} A_{i_1} \rho A_{i_1}^\dagger A_{i_2}^\dagger A_{i_3}^\dagger \right], \quad \text{with} \quad \sum_{i_1, i_2, i_3=0}^{1} p_{i_3|i_2} p_{i_2|i_1} p_{i_1} = 1. \tag{21}
\]

Substituting (17) in (21), it follows that the error superoperator \( \mathcal{A} \) associated to channel (21) is defined in terms of the following error operators,

\[
\mathcal{A} \leftrightarrow \{ A_0^\prime, A_1^\prime \} \quad \text{with} \quad \Lambda^{(3)}(\rho) \equiv \sum_{k=0}^{7} A_k^\prime \rho A_k^\dagger \quad \text{and} \quad \sum_{k=0}^{7} A_k^\dagger A_k^\prime = I_{8 \times 8}. \tag{22}
\]
In an explicit way, the error operators \( \{ A'_0, \ldots, A'_7 \} \) are given by,

\[
A'_0 = \sqrt{p_0^{(3)}} I^1 \otimes I^2 \otimes I^3, \quad A'_1 = \sqrt{p_1^{(3)}} X^1 \otimes I^2 \otimes I^3, \quad A'_2 = \sqrt{p_2^{(3)}} I^1 \otimes X^2 \otimes I^3, \\
A'_3 = \sqrt{p_3^{(3)}} I^1 \otimes I^2 \otimes X^3, \quad A'_4 = \sqrt{p_4^{(3)}} X^1 \otimes X^2 \otimes I^3, \quad A'_5 = \sqrt{p_5^{(3)}} X^1 \otimes I^2 \otimes X^3, \\
A'_6 = \sqrt{p_6^{(3)}} I^1 \otimes X^2 \otimes X^3, \quad A'_7 = \sqrt{p_7^{(3)}} X^1 \otimes X^2 \otimes X^3,
\]

where the coefficients \( p_k^{(3)} \) for \( k = 1, \ldots, 7 \) are given by,

\[
\begin{align*}
p_0^{(3)} &= p_0^2, \quad p_1^{(3)} = p_{00} p_{10} p_0, \quad p_2^{(3)} = p_{01} p_{10} p_0, \quad p_3^{(3)} = p_{00} p_{01} p_1, \\
p_4^{(3)} &= p_{10} p_{11} p_0, \quad p_5^{(3)} = p_{01} p_{10} p_1, \quad p_6^{(3)} = p_{01} p_{11} p_1, \quad p_7^{(3)} = p_{11}^2 p_1,
\end{align*}
\]

with,

\[
\begin{align*}
p_0 &= (1 - p), \quad p_1 = p, \quad p_{00} = ((1 - \mu)(1 - p) + \mu), \\
p_{01} &= (1 - \mu)(1 - p), \quad p_{10} = (1 - \mu)p, \quad p_{11} = ((1 - \mu)p + \mu).
\end{align*}
\]

**Encoding Operator.** Consider a three-qubit bit flip code that encodes 1 logical qubit into 3-physical qubits. The codewords are given by,

\[
\begin{align*}
|000\rangle \overset{\text{tensoring}}{\longrightarrow} |000\rangle \otimes |000\rangle &\equiv |0_L\rangle, \quad |100\rangle \overset{\text{tensoring}}{\longrightarrow} |100\rangle \otimes |000\rangle &\equiv |100\rangle, \\
U_{\text{CNOT}}^{ij} &\overset{\text{U}}{\longrightarrow} |110\rangle, \quad U_{\text{CNOT}}^{ij} &\overset{\text{U}}{\longrightarrow} |111\rangle &\equiv |1_L\rangle.
\end{align*}
\]

The operator \( U_{\text{CNOT}}^{ij} \) is the CNOT gate from qubit \( i \) to \( j \) defined as,

\[
U_{\text{CNOT}}^{ij} = \frac{1}{2} [(I^i + Z^i) \otimes I^j + (I^i - Z^i) \otimes X^j].
\]

Finally, the encoding operator \( U_{\text{enc}} \) such that \( U_{\text{enc}} |000\rangle = |000\rangle \) and \( U_{\text{enc}} |100\rangle = |111\rangle \) is defined as,

\[
U_{\text{enc}} \overset{\text{def}}{=} (U_{\text{CNOT}}^{13} \otimes I^3) \circ (U_{\text{CNOT}}^{12} \otimes I^3).
\]

**Correctable Errors and Recovery Operators.** The set of error operators satisfying the detectability condition \( \mathbb{P}_C \mathbb{A}_C \mathbb{P}_C = \lambda \mathbb{A}_C \mathbb{P}_C \), where \( \mathbb{P}_C = |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L| \) is the projector operator on the code subspace \( \mathcal{C} = \text{Span}\{ |0_L\rangle, |1_L\rangle \} \) is given by,

\[
\mathbb{A}_{\text{detectable}} = \{ A'_0, A'_1, A'_2, A'_3, A'_4, A'_5, A'_6, A'_7 \} \subseteq \mathbb{A}.
\]

The only non-detectable error is \( A'_7 \). Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that \( \mathbb{A}^t_{\text{correctable}} \mathbb{A}_{\text{correctable}} \) is detectable. It follows then that,

\[
\mathbb{A}_{\text{correctable}} = \{ A'_0, A'_1, A'_2, A'_3 \} \subseteq \mathbb{A}_{\text{detectable}} \subseteq \mathbb{A}.
\]

The action of the correctable error operators \( \mathbb{A}_{\text{correctable}} \) on the codewords \( |0_L\rangle \) and \( |1_L\rangle \) is given by,

\[
|0_L\rangle \rightarrow A'_0 |0_L\rangle = \sqrt{p_0^{(3)}} |000\rangle, \quad A'_1 |0_L\rangle = \sqrt{p_1^{(3)}} |100\rangle, \quad A'_2 |0_L\rangle = \sqrt{p_2^{(3)}} |010\rangle, \quad A'_3 |0_L\rangle = \sqrt{p_3^{(3)}} |001\rangle, \\
|1_L\rangle \rightarrow A'_0 |1_L\rangle = \sqrt{p_0^{(3)}} |111\rangle, \quad A'_1 |1_L\rangle = \sqrt{p_1^{(3)}} |011\rangle, \quad A'_2 |1_L\rangle = \sqrt{p_2^{(3)}} |101\rangle, \quad A'_3 |1_L\rangle = \sqrt{p_3^{(3)}} |110\rangle.
\]

The two four-dimensional orthogonal subspaces \( \mathcal{V}^{0_L} \) and \( \mathcal{V}^{1_L} \) of \( \mathcal{H}_2^3 \) generated by the action of \( \mathbb{A}_{\text{correctable}} \) on \( |0_L\rangle \) and \( |1_L\rangle \) are given by,

\[
\mathcal{V}^{0_L} = \text{Span}\{ |e_1^{0_L}\rangle = |000\rangle, |e_2^{0_L}\rangle = |100\rangle, |e_3^{0_L}\rangle = |010\rangle, |e_4^{0_L}\rangle = |001\rangle \},
\]

\[
\mathcal{V}^{1_L} = \text{Span}\{ |e_1^{1_L}\rangle = |000\rangle, |e_2^{1_L}\rangle = |100\rangle, |e_3^{1_L}\rangle = |010\rangle, |e_4^{1_L}\rangle = |001\rangle \}.
\]
Therefore, the entanglement fidelity $F$ results, and, correction scheme provided by the three-qubit bit flip code here considered. The quantum operation

$$
R_l \overset{\text{def}}{=} V_l \sum_{i=0}^{1} \left| e_l^{i} \right\rangle \left\langle e_l^{i} \right|,
$$

where the unitary operator $V_l$ is such that $V_l \left| e_l^{i} \right\rangle = \left| i_L \right\rangle$ for $i \in \{0, 1\}$. Substituting (32) and (33) into (34), it follows that the four recovery operators $\{R_1, R_2, R_3, R_4\}$ are given by,

$$
R_1 = |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L|, \quad R_2 = |0_L\rangle \langle 001| + |1_L\rangle \langle 011|,
$$

$$
R_3 = |0_L\rangle \langle 010| + |1_L\rangle \langle 101|, \quad R_4 = |0_L\rangle \langle 000| + |1_L\rangle \langle 110|.
$$

Using simple algebra, it turns out that the $8 \times 8$ matrix representation $[R_l]$ with $l = 1,..,4$ of the recovery operators is given by,

$$
[R_1] = E_{11} + E_{88}, \quad [R_2] = E_{12} + E_{87}, \quad [R_3] = E_{13} + E_{86}, \quad [R_4] = E_{14} + E_{85},
$$

where $E_{ij}$ is the $8 \times 8$ matrix where the only non-vanishing element is the one located in the $ij$-position and it equals 1. It follows that $\mathcal{R} \leftrightarrow \{R_l\}$ is indeed a trace preserving quantum operation since,

$$
\sum_{l=1}^{4} R_l^\dagger R_l = I_{8\times8}.
$$

The action of this recovery operation $\mathcal{R}$ on the map $\Lambda^{(3)}(\rho)$ in (22) leads to,

$$
\Lambda^{(3)}_{\text{recover}}(\rho) \equiv \left(\mathcal{R} \circ \Lambda^{(3)}\right)(\rho) \overset{\text{def}}{=} \sum_{k=0}^{7} \sum_{l=1}^{4} (R_l A_k) \rho (R_l A_k)^\dagger.
$$

**Entanglement Fidelity.** We want to describe the action of $\mathcal{R} \circ \Lambda^{(3)}$ restricted to the code subspace $\mathcal{C}$. Therefore, we compute the $2 \times 2$ matrix representation $[R_l A_k]_{\mathcal{C}}$ of each $R_l A_k$ with $l = 1,..,4$ and $k = 0,..,7$ where,

$$
[R_l A_k]_{\mathcal{C}} \overset{\text{def}}{=} \begin{pmatrix} \langle 0_L| R_l A_k |0_L\rangle & \langle 0_L| R_l A_k |1_L\rangle \\ \langle 1_L| R_l A_k |0_L\rangle & \langle 1_L| R_l A_k |1_L\rangle \end{pmatrix}.
$$

Substituting (31) and (32) into (39), it turns out that the only matrices $[R_l A_k]_{\mathcal{C}}$ with non-vanishing trace are given by,

$$
[R_1 A_0]_{\mathcal{C}} = \sqrt{p_0^{(3)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [R_2 A_1]_{\mathcal{C}} = \sqrt{p_1^{(3)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
$$

$$
[R_3 A_2']_{\mathcal{C}} = \sqrt{p_2^{(3)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad [R_4 A_3']_{\mathcal{C}} = \sqrt{p_3^{(3)}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Therefore, the entanglement fidelity $F_{\text{bit}}^{(3)}(\mu, p)$ defined as,

$$
F_{\text{bit}}^{(3)}(\mu, p) \overset{\text{def}}{=} \mathcal{F}^{(3)} \left(\frac{1}{2} I_{2 \times 2}, \mathcal{R} \circ \Lambda^{(3)}\right) = \frac{1}{(2)^d} \sum_{k=0}^{7} \sum_{l=1}^{4} \left| \text{tr} \left([R_l A_k]_{\mathcal{C}}\right)\right|^2,
$$

results,

$$
F_{\text{bit}}^{(3)}(\mu, p) = \tilde{p}_0^{(3)} + \tilde{p}_1^{(3)} + \tilde{p}_2^{(3)} + \tilde{p}_3^{(3)}.
$$

The expression for $F_{\text{bit}}^{(3)}(\mu, p)$ in (41) represents the entanglement fidelity quantifying the performance of the error correction scheme provided by the three-qubit bit flip code here considered. The quantum operation $\mathcal{R} \circ \Lambda^{(3)}$ appearing
in (41) is defined in equation (38) and the recovery operators \( R_l \) are explicitly given in (35). The action of \( R_l A'_k \) in (41) is restricted to the code space \( C \) defined in (26).

Substituting (24) and (25) into (42), we finally obtain

\[
F_{\text{bit}}^{(3)}(\mu, p) = \mu^2 (2p^3 - 3p^2 + p) + \mu (-4p^3 + 6p^2 - 2p) + (2p^3 - 3p^2 + 1) \text{ (model I).} \tag{43}
\]

Notice that for a vanishing degree of memory \( \mu \), the entanglement fidelity becomes,

\[
F_{\text{bit}}^{(3)}(0, p) = 2p^3 - 3p^2 + 1. \tag{44}
\]

Remarks on the coding for phase flip memory channels. The code for the phase flip channel has the same characteristics as the code for the bit flip channel. These two channels are unitarily equivalent since there is a unitary operator, the Hadamard gate \( H \), such that the action of one channel is the same as the other, provided the first channel is preceded by \( H \) and followed by \( H^\dagger \).

\[
\Lambda_{\text{phase}}(\rho) \overset{\text{def}}{=} (H \circ \Lambda_{\text{bit}} \circ H^\dagger)(\rho) = (1 - p) \rho + pZ\rho Z, \tag{45}
\]

where,

\[
\Lambda_{\text{bit}}(\rho) \overset{\text{def}}{=} (1 - p) \rho + pX\rho X \text{ and, } H \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{46}
\]

These operations may be trivially incorporated into the encoding and error-correction operations. The encoding for the phase flip channel is performed in two steps: i) first, we encode in three qubits exactly as for the bit flip channel; ii) second, we apply a Hadamard gate to each qubit,

\[
|0\rangle \overset{\text{tensoring}}{\longrightarrow} |000\rangle \overset{U_{\text{enc}}^{\text{bit}}}{\longrightarrow} |000\rangle \overset{R_l^{\otimes 3}}{\longrightarrow} |0_L\rangle \overset{\text{def}}{=} |++\rangle, \quad |1\rangle \overset{\text{tensoring}}{\longrightarrow} |100\rangle \overset{U_{\text{enc}}^{\text{bit}}}{\longrightarrow} |111\rangle \overset{H^{\otimes 3}}{\longrightarrow} |1_L\rangle \overset{\text{def}}{=} |--\rangle, \tag{47}
\]

where,

\[
\begin{pmatrix} |+\rangle \\ |--\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}. \tag{48}
\]

The unitary encoding operator \( U_{\text{enc}}^{\text{phase}} \) is given by \( U_{\text{enc}}^{\text{phase}} \overset{\text{def}}{=} H^{\otimes 3} \circ U_{\text{enc}}^{\text{bit}} \) with \( U_{\text{enc}}^{\text{bit}} \) defined in (28). Furthermore, in the phase flip code, the recovery operation is the Hadamard conjugated recovery operation from the bit flip code, \( R_k^{\text{phase}} \overset{\text{def}}{=} H^{\otimes 3} R_k^{\text{bit}} H^{\otimes 3} \).

2. Model II

We consider \( \Lambda_{\mu}^{(n)}(\rho) \) in (19) with \( n = 3 \). Technical details will be omitted. They can be obtained by following the line of reasoning presented for the three-qubit bit flip error correction scheme applied to the model I. It turns out that,

\[
F_{\text{bit}}^{(3)}(\mu, p) = \mu (-3p^3 + 6p^2 - 3p) + (2p^3 - 3p^2 + 1) \text{ (model II).} \tag{49}
\]

B. DFS for bit flip noise

1. Model I

Error Operators. Consider the limiting case of (16) with \( n = 2 \) qubits and correlated errors in a bit flip quantum channel,

\[
\Lambda^{(2)}(\rho) \overset{\text{def}}{=} \frac{1}{\sum_{i_1, i_2=0}^1} p_{i_2|i_1} p_{i_1} (A_{i_2} \otimes A_{i_1}) \rho (A_{i_2} \otimes A_{i_1})^\dagger, \tag{50}
\]
The error superoperator $A$ associated to channel (50) is defined in terms of the following error operators,

$$A \leftrightarrow \{A'_0, \ldots, A'_3\} \text{ with } \Lambda^{(2)}(\rho) \equiv \sum_{k=0}^{3} A'_k \rho A'_k^\dagger \text{ and, } \sum_{k=0}^{3} A'_k A'_k^\dagger = I_{4 \times 4}. \quad (51)$$

In an explicit way, the error operators $\{A'_0, \ldots, A'_3\}$ are given by,

$$A'_0 = \sqrt{p_0^{(2)}} I_1 \otimes I_2, \quad A'_1 = \sqrt{p_1^{(2)}} X_1 \otimes I_2, \quad A'_2 = \sqrt{p_2^{(2)}} I_1 \otimes X_2, \quad A'_3 = \sqrt{p_3^{(2)}} X_1 \otimes X_2, \quad (52)$$

where the coefficients $p_k^{(2)}$ for $k = 0, \ldots, 3$ are given by,

$$p_0^{(2)} = p_{00}p_0, \quad p_1^{(2)} = p_{01}p_1, \quad p_2^{(2)} = p_{01}p_1, \quad p_3^{(2)} = p_{11}p_1. \quad (53)$$

**Encoding Operator.** We encode our logical qubit with a simple decoherence free subspace of two qubits given by (2),

$$|0\rangle \rightarrow |0_L\rangle = |+-\rangle \text{ and, } |1\rangle \rightarrow |1_L\rangle = |---\rangle, \quad (54)$$

with $|\pm\rangle$ defined in (45). As a side remark, we point out that a suitable DFS encoding for a phase flip Markovian correlated noise model is given by $|0\rangle \rightarrow |0_L\rangle = |01\rangle$ and, $|1\rangle \rightarrow |1_L\rangle = |10\rangle$.

**Correctable Errors and Recovery Operators.** The set of error operators satisfying the detectability condition (23), $P_C A_k P_C = \lambda A_k P_C$, where $P_C = |0\rangle \langle 0| \oplus |1\rangle \langle 1|$ is the projector operator on the code subspace $C = \text{Span } \{|0_L\rangle, |1_L\rangle\}$ is given by,

$$A_{\text{detectable}} = \{A'_0, A'_3\} \subseteq A. \quad (55)$$

Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that $A_{\text{correctable}} A_{\text{correctable}}$ is detectable. It follows then that,

$$A_{\text{correctable}} = A_{\text{detectable}} \subseteq A. \quad (56)$$

The action of the correctable error operators $A_{\text{correctable}}$ on the codewords $|0_L\rangle$ and $|1_L\rangle$ is given by,

$$|0_L\rangle \rightarrow A'_0 |0_L\rangle = \sqrt{p_0^{(2)}} |+-\rangle, \quad A'_3 |0_L\rangle = -\sqrt{p_3^{(2)}} |---\rangle, \quad |1_L\rangle \rightarrow A'_0 |1_L\rangle = \sqrt{p_0^{(2)}} |---\rangle, \quad A'_3 |1_L\rangle = -\sqrt{p_3^{(2)}} |---\rangle. \quad (57)$$

The two one-dimensional orthogonal subspaces $\mathcal{V}^{0_L}$ and $\mathcal{V}^{1_L}$ of $\mathcal{H}_2$ generated by the action of $A_{\text{correctable}}$ on $|0_L\rangle$ and $|1_L\rangle$ are given by,

$$\mathcal{V}^{0_L} = \text{Span } \{|v^{0_L}_+\rangle = |+-\rangle\} \text{ and, } \mathcal{V}^{1_L} = \text{Span } \{|v^{1_L}_-\rangle = |---\rangle\}. \quad (58)$$

Notice that $\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L} \neq \mathcal{H}_2$. This means that the trace preserving recovery superoperator $R$ is defined in terms of one standard recovery operator $R_1$ and by the projector $R_\perp$ onto the orthogonal complement of $\bigoplus_{i=0}^{1} \mathcal{V}^{i_L}$, i.e. the part of the Hilbert space $\mathcal{H}_2$ which is not reached by acting on the code $C$ with the correctable error operators. In the case under consideration,

$$R_1 \equiv |+-\rangle \langle ++| + |--\rangle \langle --|, \quad R_\perp = \sum_{s=1}^{2} |r_s\rangle \langle r_s|, \quad (59)$$

where $\{|r_s\rangle\}$ is an orthonormal basis for $(\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L})^\perp$. A suitable basis $B_{(\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L})^\perp}$ is given by,

$$B_{(\mathcal{V}^{0_L} \oplus \mathcal{V}^{1_L})^\perp} = \{r_1 = |++\rangle, r_2 = |--\rangle\}. \quad (60)$$

Therefore, $R \leftrightarrow \{R_1, R_\perp\}$ is indeed a trace preserving quantum operation,

$$R_1^\dagger R_1 + R_\perp^\dagger R_\perp = I_{4 \times 4}. \quad (61)$$
The action of this recovery operation \( R \) with \( R_2 \equiv R_\perp \) on the map \( \Lambda^{(2)} (\rho) \) in (51) yields,

\[
\Lambda^{(2)}_{\text{recover}} (\rho) \equiv \left( R \circ \Lambda^{(2)} \right) (\rho) \overset{\text{def}}{=} \sum_{k=0}^{3} \sum_{l=1}^{2} (R_l A'_k) \rho (R_l A'_k)^\dagger,
\]

(62)

**Entanglement Fidelity.** We want to describe the action of \( R \circ \Lambda^{(2)} \) restricted to the code subspace \( C \). Therefore, we compute the \( 2 \times 2 \) matrix representation \( [R_l A'_k]_C \) of each \( R_l A'_k \) with \( l = 1, 2 \) and \( k = 0, \ldots, 3 \) where,

\[
[R_l A'_k]_C \overset{\text{def}}{=} \left( \begin{array}{cc}
0_L & 0_L
\end{array} \right) \begin{array}{cc}
\langle 0_L | & \langle 1_L |
\end{array} \begin{array}{cc}
R_l A'_k & R_l A'_k
\end{array} \begin{array}{cc}
0_L & 1_L
\end{array} \right).
\]

(63)

FIG. 1: Threshold curves for model I: concatenated code (dashed line), DFS (thin solid line).

Substituting (52) and (59) into (63), it turns out that the only matrices \( [R_l A'_k]_C \) with non-vanishing trace are given by,

\[
[R_1 A'_0]_C = \sqrt{\tilde{p}_0^{(2)}} \left( \begin{array}{cc}
1 & 0
0 & 1
\end{array} \right), \quad [R_1 A'_3]_C = -\sqrt{\tilde{p}_3^{(2)}} \left( \begin{array}{cc}
1 & 0
0 & 1
\end{array} \right)
\]

(64)

Therefore, the entanglement fidelity \( \mathcal{F}^{(2)}_{\text{DFS}} (\mu, p) \) defined as,

\[
\mathcal{F}^{(2)}_{\text{DFS}} (\mu, p) \overset{\text{def}}{=} \mathcal{F}^{(2)} \left( \frac{1}{2} I_{2 \times 2}, R \circ \Lambda^{(2)} \right) = \frac{1}{(2)^2} \sum_{k=0}^{3} \sum_{l=1}^{2} \left| \text{tr} \left( [R_l A'_k]_C \right) \right|^2,
\]

(65)

results,

\[
\mathcal{F}^{(2)}_{\text{DFS}} (\mu, p) = \tilde{p}_0^{(2)} + \tilde{p}_3^{(2)}.
\]

(66)

The expression for \( \mathcal{F}^{(2)}_{\text{DFS}} (\mu, p) \) in (65) represents the entanglement fidelity quantifying the performance of the error correction scheme provided by the error avoiding code here considered. The quantum operation \( R \circ \Lambda^{(2)} \) appearing in (65) is defined in equation (62) and the recovery operators \( R_l \) are explicitly given in (59). The action of \( R_l A'_k \) in (65) is restricted to the code space \( C \) defined in (54).

Substituting (25) and (53) into (66), we finally obtain

\[
\mathcal{F}^{(2)}_{\text{DFS}} (\mu, p) = \mu (-2p^2 + 2p) + (2p^2 - 2p + 1) \quad \text{(model I)}.
\]

(67)

We point out that error correction schemes improve the transmission accuracy only if the failure probability \( \mathcal{P} (\mu, p) \) is strictly less than the error probability \( p \) [20].
\[ P(\mu, p) \overset{\text{def}}{=} 1 - F(\mu, p) < p. \] (68)

In view of (68), we can determine threshold curves \( \bar{\mu}(p) \) that allow to select the two-dimensional parametric region where error correction schemes are useful. For instance, considering the model I in absence of correlations, it follows that the three-qubit code is effective only if \( p < 0.5 \). However for such values of the error probability, the DFS considered does not work for \( \mu \) approaching zero. The threshold curve for the DFS for the model I appear in Figure 1. The DFS works only in the parametric region above the thin solid line in Figure 1, while the three-qubit bit-flip code works for all values of the memory degree \( \mu \) when the error probability is less than 0.5.

2. Model II

We consider \( \Lambda_n^{(n)}(\rho) \) in (19) with \( n = 2 \). Technical details will be omitted. They can be obtained by following the line of reasoning presented when studying the error avoiding code applied to the model I. It turns out that,

\[ F^{(2)}_{\text{DFS}}(\mu, p) = \mu \left( -2p^2 + 2p \right) + \left( 2p^2 - 2p + 1 \right) \quad \text{(model II)} \] (69)

Notice that the entanglement fidelities in (67) and (69) are equal. Finally, the threshold curves for the three-qubit bit flip and DFS for the model II appear in Figure 2. The DFS works only in the parametric region above the dashed line while the three-qubit code works only below the thin solid line.

![Figure 2: Threshold curves for model II: DFS (dashed line), three-qubit bit flip code (thin solid line), concatenated code (thick solid line).](image)

IV. CONCATENATED CODES

A. Model I

**Error Operators.** Consider the limiting case of (16) with \( n = 6 \) qubits and correlated errors in a bit flip quantum channel,

\[ \Lambda^{(6)}(\rho) \overset{\text{def}}{=} \sum_{i_1, \ldots, i_6 = 0}^1 p_{i_6|i_5|i_4|i_3|i_2|i_1} (A_{i_6}A_{i_5}A_{i_4}A_{i_3}A_{i_2}A_{i_1}) \rho (A_{i_6}A_{i_5}A_{i_4}A_{i_3}A_{i_2}A_{i_1})^\dagger. \] (70)

The error superoperator \( \mathcal{A} \) associated to channel (70) is defined in terms of the following error operators,

\[ \mathcal{A} \leftrightarrow \{ A_{0}^\dagger, \ldots, A_{63}^\dagger \} \quad \text{with} \quad \Lambda^{(6)}(\rho) \overset{\text{def}}{=} \sum_{k=0}^{2^6-1} A_k^\dagger \rho A_k^\dagger \quad \text{and} \quad \sum_{k=0}^{2^6-1} A_k^\dagger A_k^\dagger = I_{64 \times 64}. \] (71)
The error operators in the Kraus decomposition (71) are $2^6 = 64$, 
\[
\sum_{k=0}^{6} \binom{6}{k} = 2^6,
\] (72)
where $\binom{6}{k}$ is the cardinality of weight-$k$ error operators.

**Encoding Operator.** We encode our logical qubit with a concatenated subspace obtained by combining the decoherence free subspace in (74) (inner code, $C_{DFS} = C_{inner}$) with the three-qubit bit flip code in (26) (outer code, $C_{bit} = C_{outer}$). We obtain that the codewords of the concatenated code $C = C_{DFS} \circ C_{bit}$ are given by,
\[
|0_L\rangle = \frac{1}{2}(|000000\rangle - |000111\rangle + |111000\rangle - |111111\rangle),
\]
\[
|1_L\rangle = \frac{1}{2}(|000000\rangle + |000111\rangle - |111000\rangle - |111111\rangle).
\] (73)
As a side remark, we point out that a suitable concatenated code for the case of a phase flip Markovian correlated noise model is given by $|0\rangle \rightarrow |0_L\rangle = |+++- - - -\rangle$ and, $|1\rangle \rightarrow |1_L\rangle = |--- + + +\rangle$.

**Correctable Errors and Recovery Operators.** Recall that the detectability condition is given by $P_C A_k^\dagger P_C = \lambda A_k P_C$ where the projector operator on the code space $C$ is $P_C = |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L|$. Observe that,
\[
P_C A_k^\dagger P_C = \langle 0_L | A_k^\dagger | 0_L \rangle |0_L\rangle\langle 0_L| + \langle 0_L | A_k^\dagger | 1_L \rangle |1_L\rangle\langle 0_L| + \langle 1_L | A_k^\dagger | 0_L \rangle |0_L\rangle\langle 1_L| + \langle 1_L | A_k^\dagger | 1_L \rangle |1_L\rangle\langle 1_L|. \] (74)
Therefore, it turns out that for detectable error operators we must have,
\[
\langle 0_L | A_k^\dagger | 0_L \rangle = \langle 1_L | A_k^\dagger | 1_L \rangle \quad \text{and}, \quad \langle 0_L | A_k^\dagger | 1_L \rangle = \langle 1_L | A_k^\dagger | 0_L \rangle = 0. \] (75)
In the case under consideration, it follows that the only error operators (omitting for the sake of simplicity the proper error amplitudes) not fulfilling the above conditions are proportional to,
\[
X^1 X^2 X^3 \quad \text{and} \quad X^4 X^5 X^6.
\] (76)
For such operators, we get
\[
\langle 0_L | X^1 X^2 X^3 | 0_L \rangle = 1, \quad \langle 1_L | X^1 X^2 X^3 | 1_L \rangle = -1, \quad \text{and}, \quad \langle 0_L | X^4 X^5 X^6 | 0_L \rangle = -1, \quad \langle 1_L | X^4 X^5 X^6 | 1_L \rangle = 1.
\] (77)
Therefore $X^1 X^2 X^3$ and $X^4 X^5 X^6$ are not detectable. Thus, the cardinality of the set of detectable errors $A_{detectable}$ is 62. Furthermore, recall that the set of correctable errors $A_{correctable}$ is such that $A_{correctable} \subseteq A_{detectable}$ (in the hypothesis of invertible error operators). Therefore, after some reasoning, we conclude that the set of correctable errors is composed by 32 error operators. The correctable weight-0, 1 and 2 correctable error operators are (omitting the proper error amplitudes),
\[
\{I\}_{\text{weight-0}}, \quad \{X^1, X^2, X^3, X^4, X^5, X^6\}_{\text{weight-1}},
\] (78)
and,
\[
\{X^1 X^4, X^1 X^5, X^1 X^6, X^2 X^4, X^2 X^5, X^2 X^6, X^3 X^4, X^3 X^5, X^3 X^6\}_{\text{weight-2}},
\] (79)
respectively. The correctable weight-4 errors are,
\[
\left\{X^1 X^2 X^4 X^5, X^1 X^2 X^4 X^6, X^1 X^2 X^5 X^6, X^1 X^3 X^4 X^5, X^1 X^3 X^4 X^6, X^1 X^3 X^5 X^6, X^2 X^3 X^4 X^5, X^2 X^3 X^4 X^6, X^2 X^3 X^5 X^6\right\}_{\text{weight-4}}.
\] (80)
Finally, weight-5 and weight-6 error operators are given by,
\[
\{X^1 X^2 X^3 X^4 X^5, X^1 X^2 X^3 X^4 X^6, X^1 X^2 X^4 X^5 X^6, X^1 X^3 X^4 X^5 X^6, X^1 X^2 X^3 X^5 X^6, X^2 X^3 X^4 X^5 X^6\}_{\text{weight-5}},
\] (81)
and,
\[
\{X^3 X^2 X^3 X^4 X^5 X^6\}_{\text{weight-6}}.
\] (82)
where, the Hilbert space $\mathcal{H}_2^6$ can be decomposed in two 32-dimensional orthogonal subspaces $\mathcal{V}^{0L}$ and $\mathcal{V}^{1L}$. In other words, $\mathcal{H}_2^6 = \mathcal{V}^{0L} \oplus \mathcal{V}^{1L}$ where

$$\mathcal{V}^{0L} = \text{Span} \left\{ |v_{0L}^k \rangle = \frac{1}{\sqrt{p_k^{(6)}}} A_k^0 |0_L \rangle \right\} \quad \text{and} \quad \mathcal{V}^{1L} = \text{Span} \left\{ |v_{1L}^k \rangle = \frac{1}{\sqrt{p_k^{(6)}}} A_k^1 |1_L \rangle \right\},$$

(83)

with $A_k^j \in \mathcal{A}_{\text{correctable}}$, $\forall k = 0, ..., 31$ (numbering the correctable error operators from 0 to 31). Notice that $\langle v_{k}^{i_L} | v_{k'}^{j_L} \rangle = \delta_{kk'} \delta_{ij}$, with $k, k' \in \{0, ..., 31\}$ and $i, j \in \{0, 1\}$ since,

$$\langle v_{k}^{i_L} | v_{k'}^{j_L} \rangle = \left( i_L | A_{k-1}^{j}\frac{A_{k-1}^{iL}}{\sqrt{p_k^{(6)}}} \sqrt{p_k^{(6)}} | j_L \right) = \frac{1}{\sqrt{p_k^{(6)}}} \langle i_L | A_k^{iL} A_{k'}^{jL} | j_L \rangle = \frac{1}{\sqrt{p_k^{(6)}}} \alpha_{kk'}^{iL} \delta_{ij},$$

(84)

where we have used the fact that the square (Hermitian) matrix $\alpha_{kk'}^{iL}$ equals $\sqrt{p_k^{(6)}} \sqrt{p_k^{(6)}} \delta_{kk'}$. The recovery superoperator $\mathcal{R} \leftrightarrow \{ R_l \}$ with $l = 1, ..., 32$ is defined as [13],

$$R_l \overset{\text{def}}{=} V_l \sum_{i=0}^{1} | v_{iL}^{l} \rangle \langle v_{iL}^{l} |,$$

(85)

where the unitary operator $V_l$ is such that $V_l | v_{iL}^{l} \rangle = | i_L \rangle$ for $i \in \{0, 1\}$. Notice that,

$$R_l \overset{\text{def}}{=} V_l \sum_{i=0}^{1} | v_{iL}^{l} \rangle \langle v_{iL}^{l} | = |0_L\rangle \langle 0_L | + |1_L\rangle \langle 1_L |.$$

(86)

If turns out that the 32 recovery operators are given by,

$$R_{l+1} = R_l \frac{A_l^T}{\sqrt{p_l}} = \left( |0_L\rangle \langle 0_L | + |1_L\rangle \langle 1_L | \right) \frac{A_l^T}{\sqrt{p_l}},$$

(87)

with $l \in \{0, ..., 31\}$. Notice that $\mathcal{R} \leftrightarrow \{ R_l \}$ is a trace preserving quantum operation since,

$$\sum_{l=1}^{32} R_l \sum_{l=1}^{32} \left( \sum_{i_L} | i_L \rangle \langle v_{iL}^{l} | \right) \left( \sum_{j_L} | j_L \rangle \langle v_{jL}^{l} | \right) = \sum_{l=1}^{32} \sum_{i_L} \sum_{j_L} | v_{iL}^{l} \rangle \langle i_L | j_L \rangle \langle v_{jL}^{l} | = \sum_{l=1}^{32} \sum_{i_L} \langle v_{iL}^{l} | = I_{64 \times 64},$$

(88)

since $\{| v_{iL}^{l} \rangle \}$ with $l = 1, ..., 32$ and $i_L \in \{0, 1\}$ is an orthonormal basis for $\mathcal{H}_2^6$. Finally, the action of this recovery operation $\mathcal{R}$ on the map $\Lambda^{(6)}(\rho)$ in [71] leads to,

$$A_{\text{recover}}^{(6)}(\rho) \overset{\text{def}}{=} \left( \mathcal{R} \circ \Lambda^{(6)} \right)(\rho) = \sum_{k=0}^{2^6-1} \sum_{l=1}^{32} (R_l A_k^l) \rho (R_l A_k^l)^\dagger.$$

(89)

**Entanglement Fidelity.** We want to describe the action of $\mathcal{R} \circ \Lambda^{(6)}$ restricted to the code subspace $\mathcal{C}$. Recalling that $A_l^k = A_l^{kL}$, it turns out that,

$$\langle i_L | R_{l+1} A_k^l | j_L \rangle = \frac{1}{\sqrt{p_l}} \langle i_L | 0_L \rangle \langle 0_L | A_l^{iL} A_k^j | j_L \rangle + \frac{1}{\sqrt{p_l}} \langle i_L | 1_L \rangle \langle 1_L | A_l^{iL} A_k^j | j_L \rangle.$$

(90)

We now need to compute the $2 \times 2$ matrix representation $[R_l A_k^l]_{\mathcal{C}}$ of each $R_l A_k^l$ with $l = 0, ..., 31$ and $k = 0, ..., 2^6 - 1$ where,

$$[R_{l+1} A_k^l]_{\mathcal{C}} \overset{\text{def}}{=} \left( \begin{array}{cc} \langle 0_L | R_{l+1} A_k^l | 0_L \rangle & \langle 0_L | R_{l+1} A_k^l | 1_L \rangle \\ \langle 1_L | R_{l+1} A_k^l | 0_L \rangle & \langle 1_L | R_{l+1} A_k^l | 1_L \rangle \end{array} \right).$$

(91)

For $l, k = 0, ..., 31$, we note that $[R_{l+1} A_k^l]_{\mathcal{C}}$ becomes,

$$[R_{l+1} A_k^l]_{\mathcal{C}} = \begin{pmatrix} \langle 0_L | A_l^{iL} A_k^j | 0_L \rangle & 0 \\ 0 & \langle 1_L | A_l^{iL} A_k^j | 1_L \rangle \end{pmatrix} = \sqrt{p_l} \delta_{lk} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(92)
while for any pair \((l, k)\) with \(l = 0, \ldots, 31\) and \(k > 31\), it follows that,
\[
\langle 0_L | R_{l+1} A'_k | 0_L \rangle + \langle 1_L | R_{l+1} A'_k | 1_L \rangle = 0.
\] (93)
We conclude that the only matrices \([R_l A'_k]_C\) with non-vanishing trace are given by \([R_{l+1} A'_k]_C\) with \(l = 0, \ldots, 31\) where,
\[
[R_{l+1} A'_k]_C = \sqrt{\tilde{p}'_l} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (94)
Therefore, the entanglement fidelity \(\mathcal{F}_{\text{conc}}^{(6)} (\mu, p)\) defined as,
\[
\mathcal{F}_{\text{conc}}^{(6)} (\mu, p) \overset{\text{def}}{=} \mathcal{F}_{\text{conc}}^{(6)} \left( \frac{1}{2} I_{2 \times 2}, R \circ \Lambda^{(6)} \right) = \frac{1}{2} \sum_{k=0}^{2^6-1} \sum_{l=1}^{32} | \text{tr} \left( [R_l A'_k]_C \right) |^2,
\] (95)
becomes,
\[
\begin{align*}
\mathcal{F}_{\text{conc}}^{(6)} (\mu, p) &= p_{00}^5 p_0 + 2 p_{00}^4 p_{10} p_0 + p_{00}^3 p_{01} p_{10} (4 p_0 + p_1) + 4 p_{00}^2 p_{01} p_{10} p_0 + p_{00}^2 p_{01} p_{10} p_{11} p_0 + 3 p_{00} p_{01}^2 p_{10} p_0 + \\
&\quad + 4 p_{01} p_{10}^2 p_{11} p_0 + 3 p_{01}^2 p_{10}^2 p_{11} p_1 + p_{00} p_{01} p_{10} p_{11}^2 p_1 + p_{01} p_{10}^3 p_{11} (p_0 + 4 p_1) + 2 p_{10}^4 p_{11} p_0 + p_{11}^5 p_1.
\end{align*}
\] (96)
Substituting (25) into (96), we finally get
\[
\begin{align*}
\mathcal{F}_{\text{conc}}^{(6)} (\mu, p) &= \mu^5 \left( -8 p^6 + 24 p^5 - 24 p^4 + 8 p^3 \right) + \mu^4 \left( 40 p^6 - 120 p^5 + 130 p^4 - 60 p^3 + 10 p^2 \right) + \\
&\quad + \mu^3 \left( -80 p^6 + 240 p^5 - 264 p^4 + 128 p^3 - 26 p^2 + 2 p \right) + \\
&\quad + \mu^2 \left( 80 p^6 - 240 p^5 + 252 p^4 - 104 p^3 + 10 p^2 + 2 p \right) + \\
&\quad + \mu \left( -40 p^6 + 120 p^5 - 112 p^4 + 24 p^3 + 12 p^2 - 4 p \right) + \left( 8 p^6 - 24 p^5 + 18 p^4 + 4 p^3 - 6 p^2 + 1 \right).
\end{align*}
\] (97)
The threshold curves for the concatenated code defined in (73) for the model I appear in Figure 1. It turns out that the concatenated code does not work in the region delimited by the two dashed lines. We emphasize that in view of equations (43), (67) and (97), it turns out that the concatenated code outperforms the bit-flip code in regions with very high memory parameter values and outperforms the DFS in regions with low memory parameter values. In
particular, the relevance of the concatenation trick shines where the weaknesses of the inner and outer codes (yet not
concatenated) are more pronounced, that is in regions with both high error probability and low memory parameter
values. It is within this area that we can identify the region where the concatenated code outperforms both the inner
and the outer codes. However, there is a big parametric region characterized by intermediate values of the degree of
memory (see Figure 1) where the concatenation trick does not work well for model I. On the contrary, we will show
that the quantum coding trick is particularly useful for the model II in the presence of partial correlations.

B. Model II

We consider $\Lambda_m^{(n)}(\rho)$ in (19) with $n = 6$. Technical details will be omitted. They can be obtained by following the
line of reasoning presented when studying the concatenated code applied to the model I. It turns out that,

$$F_{\text{conc}}(\mu, p) = \mu \left(-8p^6 + 24p^5 - 18p^4 + 6p^3 \right) + \left(8p^6 - 24p^5 + 18p^4 + 4p^3 - 6p^2 + 1\right).$$  \hspace{1cm} (98)

The threshold curve for the concatenated code defined in (73) for the model II appears in Figure 2. It follows that
while none of the two codes is effective in the extreme limit when the other is, the three-qubit bit flip (phase flip) code
still works for correlated errors, whereas the error avoiding code does not work in the absence of correlations. The
concatenated code works everywhere except below the thick solid curve in Figure 2. Here there is a parametric region
characterized by partial correlations and delimited by the dashed and thin solid lines where only the concatenated code
is effective. Furthermore, from (49), (69) and (98), it follows that the concatenated code is especially advantageous
for the model II for partially correlated error operators. For the sake of clarity, in Figure 3 we plot the entanglement
fidelities (69) (thin solid line), (98) (dashed line) and (49) (tick solid line) for $p = 10^{-1}$. For such value of the error
probability, the error avoiding code only works for $\mu \gtrsim 0.30$ (threshold value obtained from Figure 2), the three-qubit
bit flip code only works for $\mu \lesssim 0.44$ (threshold value obtained from Figure 2) while the concatenated code with
entanglement fidelity given in (98) is is efficient for any value of $\mu \in [0, 1]$.

V. FINAL REMARKS

In this article, we studied the performance of simple error correcting and error avoiding quantum codes together with
their concatenation for correlated noise models. For model I, a bit-flip (phase-flip) noisy Markovian memory channel,
we have applied both the three-qubit bit flip (phase flip) and a suitable error avoiding code. The performance of the
codes was quantified in terms of the entanglement fidelities in (43) and (67). The performance of the concatenated
code applied to model I appears in (97). In Figure 1, we have plotted the parametric regions where error correction
is effective. We have presented a similar analysis for the model II, a memory channel defined as a memory degree
dependent linear combination of memoryless channels with Kraus decompositions expressed solely in terms of tensor
products of X-Pauli (Z-Pauli) operators (model II). The performance of the codes was quantified in terms of the
entanglement fidelities in (49) and (69). The performance of the concatenated code applied to model II appears in
(98). In Figure 2, we have plotted the parametric regions where error correction schemes are effective.

Our analysis explicitly shows that while none of the two codes is effective in the extreme limit when the other is,
the three-qubit bit flip (phase flip) code still works for correlated errors, whereas the error avoiding code does not
work in the absence of correlations. Finally, our final finding leads to conclude that the concatenated code in (73) is
particularly advantageous for model II in the regime of partial correlations (see Figure 3).

Acknowledgments

C. C. thanks S. L’Innocente and C. Lupo for useful discussions. This work was supported by the European
Community’s Seventh Framework Program under grant agreement 213681 (CORNER Project; FP7/2007-2013).

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