Generalized Labeled Multi-Bernoulli Approximation of Multi-Object Densities

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Abstract

In multi-object tracking, statistical dependence between objects in the prediction and posterior densities is inevitable due to object interactions and/or measurement updates. Exact computation of the multi-object density is generally intractable and tractable implementations usually require statistical independence assumptions between objects. In this paper we propose a tractable multi-object density approximation that captures dependencies between targets. In particular, we derive a Generalized Labeled Multi-Bernoulli (GLMB) density that matches the cardinality distribution and the first moment of the labeled multi-object distribution of interest. In addition, based on the GLMB approximation, we propose a tractable multi-object tracking algorithm for generic measurement models. Simulation results for a multi-object Track-Before-Detect example using radar measurements in low signal-to-noise ratio (SNR) scenarios verify the applicability of the proposed approach.

Index Terms

RFS, FISST, Multi-Object Tracking, PHD.

I. INTRODUCTION

In multi-object inference the objective is the estimation of an unknown number of objects and their individual states from noisy observations. Multi-object estimation is a core problem in spatial statistics [1], [2], and multi-target tracking [3], [4], spanning a diverse range of applications. Important applications of spatial statistics include agriculture/forestry [5]–[7], epidemiology/public health [1], [2], [8], communications networks [9]–[11], while applications of multi-target tracking include radar/sonar [12]–[14], computer vision [15]–[18], autonomous vehicle [19]–[22], automotive safety [23], [24] and sensor network [25]–[28]. The multi-object probability density is important in multi-object estimation because it captures the uncertainty in the number and values of the objects as well as their interdependency. However, computing the multi-object density is usually numerically intractable and approximations are necessary.
Tractable approximations of multi-object densities usually assume statistical independence between objects even though dependence between objects in the prediction and posterior multi-object densities is unavoidable due to target interactions and measurement updates. For example, the Probability Hypothesis Density (PHD) \[29\], Cardinalized PHD (CPHD) \[30\], and multi-Bernoulli filters \[31\], are derived from multi-object densities in which objects are assumed to be independent from each other. On the other hand, multi-object estimation approaches such as Multiple Hypotheses Tracking (MHT) \[12\], \[13\] and Joint Probabilistic Data Association (JPDA) \[14\] are capable of modelling dependence between objects. However, MHT does not have the notion of multi-object density while JPDA only has the notion of multi-object density for a known number of objects. A tractable family of multi-object densities that can correctly capture dependence between objects is the recently proposed Generalized Labeled Multi-Bernoulli (GLMB) family \[32\], \[33\]. Moreover this family is conjugate with respect to the standard measurement likelihood function and closed under the Chapman-Kolmogorov equation \[32\], \[33\].

The bulk of multi-object estimation algorithms in the literature, including those discussed above, are designed for the so-called standard measurement model, where data has been preprocessed into point measurements or detections \[12\]–\[14\], \[30\], \[31\]. For a generic measurement model the GLMB density is not necessarily a conjugate prior, i.e. the multi-object posterior density is not a GLMB. This is the case in Track-Before-Detect (TBD) \[34\]–\[36\], tracking with superpositional sensors \[37\], \[38\], and visual tracking \[17\]. In general, the multi-object density is numerically intractable in applications involving non-standard measurement models. A simple strategy that drastically reduces the numerical complexity is to approximate the measurement likelihood by a separable likelihood \[39\], \[40\] for which Poisson, independently and identically distributed (iid) cluster, multi-Bernoulli and GLMB densities are conjugate priors. While this approximation has relatively good tracking performance, biased estimates arise when the separable assumption is violated.

In this paper we propose a tractable multi-object density approximation that captures the statistical dependence between objects and we apply this result develop an efficient multi-object tracking algorithm for a generic measurement model. This result is applicable to any measurement model which can be written and computed via a multi-object likelihood and hence includes the standard point detection measurement model. Our approach is based on the family of GLMB densities, which is numerically tractable and captures object dependencies \[32\]. More specifically, inspired by Mahler’s iid cluster approximation in the CPHD filter \[30\], we derive a GLMB density that matches the cardinality distribution and the first moment of the labeled multi-object density of interest. Furthermore, based on this approximation we propose a tractable multi-object tracking solution which gives formal track or label estimates for generic measurement models. As an example application, we consider a radar multi-object TBD problem with low signal to noise ratio (SNR) and closely spaced targets. Simulation results verify that the proposed approximation is able to capture target dependencies and yields effective tracking performance in challenging scenarios.

The paper is structured as follows: in Section II we recall some definitions and results for random finite sets (RFSs) and review the Bayesian approach to multi-object tracking with labeled RFS. In Section III we propose the GLMB approximation to multi-object distributions via cardinality and first moment matching. We then discuss the multi-object tracking problem for generic measurement models, review the separable likelihood approach, and detail the application of the newly proposed GLMB approximation. In Section IV we describe the
use of the GLMB approximation for the practical problem of multi-object tracking and labeling in radar TBD applications. Simulation results for challenging low SNR scenarios are collected in Section V, while conclusions and future research directions are discussed in Section VI.

II. BACKGROUND

This section briefly presents background material on multi-object filtering and labeled RFS, which form the basis for the formulation of our multi-object tracking problem.

A. Multi-object Estimation

Suppose that at time \( k \), there are \( N(k) \) object states \( x_{k,1}, \ldots, x_{k,N(k)} \), each taking values in a state space \( \mathcal{X} \). In the random finite set (RFS) framework, the multi-object state at time \( k \) is represented by the finite set \( X_k = \{x_{k,1}, \ldots, x_{k,N(k)}\} \), and the multi-object state space is the space of all finite subsets of \( \mathcal{X} \), denoted as \( \mathcal{F}(\mathcal{X}) \). An RFS is simply a random variable that takes values in the space \( \mathcal{F}(\mathcal{X}) \) that does not inherit the usual Euclidean notion of integration and density. Mahler’s Finite Set Statistics (FISST) provides powerful yet practical mathematical tools for dealing with RFSs [3], [29], [41] based on a notion of integration/density that is consistent with point process theory [42].

Similar to the standard state space model, the multi-object system model can be specified, for each time step \( k \), via the multi-object transition density \( f_{k|k-1} \) and the multi-object likelihood function \( g_k \), using the FISST notion of integration/density. The multi-object posterior density (or simply multi-object posterior) contains all information about the multi-object states given the measurement history. The multi-object posterior recursion follows directly from the posterior recursion for vector-valued state [43], i.e. for \( k \geq 1 \)

\[
\pi_{0:k}(X_{0:k} | z_{1:k}) \propto g_k(z_k | X_k) f_{k|k-1}(X_k|X_{k-1}) \pi_{0:k-1}(X_{0:k-1} | z_{1:k-1})
\]

where \( X_{0:k} = (X_0, \ldots, X_k) \) is the multi-object state history, and \( z_{1:k} = (z_1, \ldots, z_k) \) is the measurement history with \( z_k \) denoting the measurement vector at time \( k \). Object trajectories or tracks can be accommodated in the RFS formulation by incorporating a label in the object’s state vector [3], [32], [41], [44]. The multi-object posterior then contains all information on the random finite set of tracks, given the measurement history.

In this work we are interested in the multi-object filtering density, a marginal of the multi-object posterior, which can be propagated by the multi-object Bayes filter [3], [29]

\[
\pi_{k|k-1}(X_k) = \int f_{k|k-1}(X_k|X) \pi_{k-1}(X) \delta X,
\]

\[
\pi_k(X_k) = \frac{g_k(z_k|X_k) \pi_{k|k-1}(X_k)}{\int g_k(z_k|X) \pi_{k|k-1}(X) \delta X},
\]

where \( \pi_{k|k-1} \) is the multi-object prediction density to time \( k \), and the integral is a set integral defined for any function \( f : \mathcal{F}(\mathcal{X}) \rightarrow \mathbb{R} \), and any region \( S \subset \mathcal{X} \) by

\[
\int_S f(X) \delta X = \sum_{i=0}^{\infty} \frac{1}{i!} \int_{S_i} f(\{x_1, \ldots, x_i\}) d(x_1, \ldots, x_i).
\]

Analytic solution to the multi-object Bayes filter for labeled states and track estimation from the multi-object filtering density are given in [32]. Note that a large volume of work in multi-object tracking is based on filtering, and often the term "multi-object posterior" is used in place of "multi-object filtering density". In this work we shall not distinguish between the filtering density and the posterior density.
B. Labeled RFS

To perform tracking in the RFS framework we use the labeled RFS model that incorporates a unique label in the object’s state vector to identify its trajectory [3], [4]. In this model, the single-object state space $X$ is a Cartesian product $X \times L$, where $X$ is the feature/kinematic space and $L$ is the (discrete) label space. A finite subset set $X$ of $X \times L$ has distinct labels if and only if $X$ and its labels $\{ \ell : (x, \ell) \in X \}$ have the same cardinality. An RFS on $X \times L$ with distinct labels is called a labeled RFS [3].

For the rest of the paper, we use the standard inner product notation $\langle f, g \rangle \triangleq \int f(x)g(x)dx$, and multi-object exponential notation $h^X \triangleq \prod_{x \in X} h(x)$, where $h$ is a real-valued function, with $h^\emptyset = 1$ by convention. We denote a generalization of the Kroneker delta and the inclusion function that take arbitrary arguments such as sets, vectors, etc, by

$$
\delta_Y(X) \triangleq \begin{cases} 
1, & \text{if } X = Y \\
0, & \text{otherwise}
\end{cases}
$$

$$1_Y(X) \triangleq \begin{cases} 
1, & \text{if } X \subseteq Y \\
0, & \text{otherwise}
\end{cases}
$$

We also write $1_Y(x)$ in place of $1_Y(\{x\})$ when $X = \{x\}$. Single-object states are represented by lowercase letters, e.g. $x$, $X$ while multi-object states are represented by uppercase letters, e.g. $X$, $X$, symbols for labeled states and their distributions are bolded to distinguish them from unlabeled ones, e.g. $x$, $X$, $\pi$, etc, spaces are represented by blackboard bold e.g. $\mathbb{X}$, $\mathbb{Z}$, $\mathbb{L}$, etc.

Two important statistics of an RFS relevant to this paper are the cardinality distribution $\rho(\cdot)$ and the PHD $v(\cdot)$ defined on $X$ [3]:

$$
\rho(n) = \frac{1}{n!} \int \pi(\{x_1, \ldots, x_n\})d(x_1, \ldots, x_n)
$$

$$
v(x, \ell) = \int \pi(\{(x, \ell)\} \cup X)\delta X
$$

Note the difference between the PHD in (5) and the unlabeled PHD [32] (i.e., the PHD of the unlabeled version) of the labeled RFS given by

$$
v(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \sum_{(\ell_1, \ldots, \ell_n) \in L^{n+1}} \pi(\{(x, \ell), (x_1, \ell_1), \ldots, (x_n, \ell_n)\})d(x_1, \ldots, x_n)
$$

$$
= \sum_{\ell \in L} v(x, \ell)
$$

Hence, $v(\cdot, \ell)$ can be interpreted as the PHD of the kinematic part due to label or track $\ell$. In other words, $v(\cdot, \ell)$ in (5) represents the contribution from label $\ell$ to the unlabeled PHD.

An important class of labeled RFS is the generalized labeled multi-Bernoulli (GLMB) family [32], which is the basis of an analytic solution to the Bayes multi-object filter [33]. Under the standard multi-object model, the GLMB is a conjugate prior that is also closed under the Chapman-Kolmogorov equation. If we start with a GLMB initial prior, then the multi-object prediction and posterior densities at any time are also GLMB densities.
Let $\mathcal{L} : \mathbb{X} \times \mathbb{L} \rightarrow \mathbb{L}$ be the projection $\mathcal{L}((x, \ell)) = \ell$, and $\Delta(\mathbb{X}) \triangleq \delta_{|\mathbb{X}|}(\mathcal{L}(\mathbb{X}))$ denote the distinct label indicator. A GLMB is a RFS on $\mathbb{X} \times \mathbb{L}$ distributed according to

$$\pi(\mathbb{X}) = \Delta(\mathbb{X}) \sum_{c \in \mathcal{C}} w^{(c)}(\mathcal{L}(\mathbb{X})) [p^{(c)}]^\mathbb{X}$$

(6)

where $\mathcal{C}$ is a discrete index set, $w^{(c)}(L)$ and $p^{(c)}$ satisfy:

$$\sum_{L \subseteq \mathbb{L}} \sum_{c \in \mathcal{C}} w^{(c)}(L) = 1, \quad \int p^{(c)}(x, \ell) dx = 1.$$  

(7)  

(8)

The GLMB density (6) can be interpreted as a mixture of multi-object exponentials. Each term in (6) consists of a weight $w^{(c)}(\mathcal{L}(\mathbb{X}))$ that depends only on the labels of $\mathbb{X}$, and a multi-object exponential $[p^{(c)}]^\mathbb{X}$ that depends on the entire $\mathbb{X}$.

The unlabeled PHD of a generalized labeled multi-Bernoulli RFS is given by [32]

$$v(x) = \sum_{c \in \mathcal{C}} \sum_{\ell \in \mathbb{L}} p^{(c)}(x, \ell) \sum_{L \subseteq \mathbb{L}} 1_L(\ell) w^{(c)}(L)$$

(9)

The Labeled Multi-Bernoulli (LMB) family is a special case of the GLMB family with one term and a specific form for the weight $w^{(c)}(\cdot)$:

$$p^{(c)}(x, \ell) = p^{(1)}(x)$$  

$$w^{(c)}(L) = \prod_{\ell \in \mathbb{M}} (1 - r^{(\ell)}) \prod_{\ell \in \mathbb{L}} \frac{1_{M}(\ell) r^{(\ell)}}{1 - r^{(\ell)}}.$$  

where $\{(r^{(\ell)}, p^{(\ell)})\}_{\ell \in \mathbb{M}}$, $\mathbb{M} \subseteq \mathbb{L}$, is a given set of parameters with $r^{(\ell)}$ representing the existence probability of track $\ell$, and $p^{(\ell)}$ the probability density of the kinematic state of track $\ell$ given its existence [32]. Note that the index space $\mathcal{C}$ has only one element, in which case the $(c)$ superscript is not needed. The LMB family is the basis of the LMB filter, a principled and efficient approximation of the Bayes multi-object tracking filter, which is highly parallelizable and capable of tracking large number of targets [45]. The LMB filter is however a strong approximation of the Bayes multi-object tracking filter which does not capture target dependence and consequently might not be able to achieve good performance in cases of low observability and/or low signal-to-noise ratio (SNR).

### III. Multi-Object Bayesian Estimation with GLMB Priors

In this section we discuss the multi-object estimation problem with GLMB priors. Due to its conjugacy properties, we are interested in the applicability of the GLMB distribution as an approximation of general multi-object densities with statistically dependent objects. In particular, in subsection III-A we consider an approximation that leads to a GLMB distribution through a separable likelihood assumption, while in subsection III-B we describe a more principled approach to approximate a general RFS density with a GLMB distribution that matches the multi-object density of interest in both the PHD and cardinality distribution.
A. Conjugacy with respect to the Separable Likelihood

A separable multi-object likelihood of $z$ given $X$ is one of the form:

$$g(z|X) \propto \gamma_z^X$$  \hspace{1cm} (10)

If true multi-object likelihood function is separable, then the GLMB family is conjugate under the Bayes update step.

**Proposition 1.** If the multi-object prior density $\pi_+$ is a GLMB of the form (6) and the multi-object likelihood is separable of the form (10), then the multi-object posterior density $\pi$ is a GLMB of the form:

$$\pi(X|z) \propto \Delta(X) \sum_{c \in C} w_z^{(c)}(L(X)) \left[ p^{(c)}(|z) \right]^X$$  \hspace{1cm} (11)

where

$$w_z^{(c)}(L) = [\eta_z] w_z^{(c)}(L)$$  \hspace{1cm} (12)

$$p^{(c)}(x, \ell| z) = p^{(c)}(x, \ell)/\eta_z(\ell)$$  \hspace{1cm} (13)

$$\eta_z(\ell) = \left \langle p^{(c)}(\cdot, \ell)/\gamma_z(\cdot, \ell) \right \rangle$$  \hspace{1cm} (14)

**Proof:**

$$\pi(X|z) \propto g(z|X)\pi_+(X)$$

$$= \prod_{x \in X} \gamma_z(x)\pi_+(X)$$

$$= \Delta(X) \sum_{c \in C} w_z^{(c)}(L(X)) \prod_{x \in X} \gamma_z(x)p^{(c)}_+(x)$$

$$= \Delta(X) \sum_{c \in C} w_z^{(c)}(L(X)) \times$$

$$\prod_{c \in C} \left \langle p^{(c)}_+(\cdot, \ell)/\gamma_z(\cdot, \ell) \right \rangle \prod_{x \in X} \left \langle p^{(c)}_+(\cdot, \ell)/\gamma_z(\cdot, \ell) \right \rangle$$

$$= \Delta(X) \sum_{c \in C} w_z^{(c)}(L(X)) \left[ p^{(c)}(|z) \right]^X$$

In general, the true multi-object likelihood is not separable and the separable likelihood assumption constitutes a reasonable approximation if the objects do not overlap in the measurement space.

B. Approximating the Posterior

In this subsection we propose a GLMB approximation of multi-object density with statistically dependent objects. In particular we derive a GLMB density that matches the multi-object density of interest in both the PHD and cardinality distribution. The strategy is inspired by Mahler’s iid cluster approximation in the CPHD filter [30], which has proven to be very effective in practical multi-object filtering problems [4], [46], [47]. Moreover, the GLMB approximation captures object dependencies whereas the iid cluster approximation assumes that individual objects are iid from a common single-object density. While our result is used to develop
a multi-object tracking algorithm in the next section, it is not restricted to tracking and can be used in more general multi-object estimation context.

Our result follows from the observation that any labeled RFS density \( \pi \) on \( \mathcal{F}(X \times L) \) can be written as

\[
\pi(X) = w(L(X))p(X)
\]

where

\[
w(\{x_1, \ldots, x_n\}) \triangleq \int \pi(\{(x_1, \ell_1), \ldots, (x_n, \ell_n)\})d(x_1, \ldots, x_n)
\]

\[
p(\{(x_1, \ell_1), \ldots, (x_n, \ell_n)\}) \triangleq \frac{\pi(\{(x_1, \ell_1), \ldots, (x_n, \ell_n)\})}{w(\{\ell_1, \ldots, \ell_n\})}
\]

It is implicitly assumed that \( p(X) \) is defined to be zero whenever \( w(L(X)) \) is zero.

Note that since \( \pi \) is symmetric in its arguments, the integral in (16) and \( w \) is symmetric in \( \ell_1, \ldots, \ell_n \). Moreover, since \( \sum_{L \in \mathcal{F}(L)} w(L) = 1 \), it is indeed a probability distribution on \( \mathcal{F}(L) \) and can be interpreted as is the probability that the labeled RFS has label set \( \{\ell_1, \ldots, \ell_n\} \). For \( w(\{\ell_1, \ldots, \ell_n\}) > 0 \), \( p(\{(x_1, \ell_1), \ldots, (x_n, \ell_n)\}) \) is the joint probability density (on \( X^n \)) of the kinematic states \( x_1, \ldots, x_n \) given that their corresponding labels are \( \ell_1, \ldots, \ell_n \).

**Proposition 2.** The GLMB that matches the cardinality distribution, and the PHD (hence the unlabeled PHD) of a labeled RFS with density \( \pi \) is given by:

\[
\hat{\pi}(X) = \sum_{L \in \mathcal{F}(L)} w^{(1)}(L(X)) \left[ p^{(1)} \right]^X
\]

where,

\[
w^{(1)}(L) = \delta_L w(I)
\]

\[
p^{(1)}(x, \ell) = 1_{\ell}(L)p_{L}(x, \ell)
\]

\[
p(\{x_1, \ldots, x_n\})(x, \ell) = \int p(\{(x, \ell), (x_1, \ell_1), \ldots, (x_n, \ell_n)\})d(x_1, \ldots, x_n)
\]

**Proof:** We first show that the cardinality distributions of \( \pi \) and \( \hat{\pi} \) are the same. Following [32] the cardinality distribution of the GLMB in (18) is given by:

\[
\hat{\rho}(n) = \sum_{L \in \mathcal{F}_n(L)} \sum_{I \in \mathcal{F}(L)} w^{(1)}(L)
\]

\[
= \sum_{L \in \mathcal{F}_n(L)} w(L)
\]

\[
= \frac{1}{n!} \sum_{(x_1, \ldots, x_n) \in X^n} w(\{x_1, \ldots, x_n\})
\]

\[
= \frac{1}{n!} \sum_{(x_1, \ldots, x_n) \in X^n} \int \pi(\{(x_1, \ell_1), \ldots, (x_n, \ell_n)\})d(x_1, \ldots, x_n)
\]

\[
= \frac{1}{n!} \int \pi(\{(x_1, \ldots, x_n)\})d(x_1, \ldots, x_n) = \rho(n)
\]

We now show that the PHD corresponding to \( \pi \) and \( \hat{\pi} \) are the same. Note from the definition of \( p^{(1)}(x, \ell) \) that

\[
p^{(1)(x_1, \ldots, x_n)}(x, \ell) = 1_{(x_1, \ldots, x_n)}(\ell)p_{(x_1, \ldots, x_n)}(x, \ell)
\]

\[
= \int p(\{x, \ell), (x_1, \ell_1), \ldots, (x_n, \ell_n)\})d(x_1, \ldots, x_n)
\]

Substituting the GLMB density (19) into eq. (5) we have
\[
\hat{v}(x, \ell) = \int \sum_{I \in \mathcal{F}(L)} w^{(I)} (\mathcal{L}((x, \ell) \cup X)) [p^{(I)}((x, \ell)) \cdot X] \delta X \\
= \sum_{I \in \mathcal{F}(L)} \int w^{(I)} (\{\ell\} \cup \mathcal{L}(X)) [p^{(I)}] \delta X p^{(I)}(x, \ell) \\
= \sum_{I \in \mathcal{F}(L)} \sum_{L \in \mathcal{F}(L)} w^{(I)} (\{\ell\} \cup L) \left[p^{(I)}(\cdot) \right] dx L p^{(I)}(x, \ell)
\]

where the last step follows from Lemma 3 in [32]. Noting that \( p^{(I)}(\cdot, \ell) \) is a probability density, and using [19] gives

\[
\hat{v}(x, \ell) = \sum_{L \in \mathcal{F}(L)} \sum_{\ell \in \mathcal{F}(L)} \delta_I (\{\ell\} \cup L) w(I) p^{(I)}(x, \ell) \\
= \sum_{L \in \mathcal{F}(L)} w (\{\ell\} \cup L) p^{(I)}(x, \ell) \\
= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\ell_1, \ldots, \ell_n) \in L^n} w (\{\ell, \ell_1, \ldots, \ell_n\}) \int p (\{x, \ell_1, \ldots, \ell_n\}) d(x_1, \ldots, x_n) \\
= \int \pi (\{(x, \ell)\} \cup X) \delta X = v(x, \ell)
\]

Remark 3. Note that in [48] Sec. V] a GLMB was proposed to approximate a particular family of labeled RFS densities that arises from multi-target filtering with merged measurements. By applying Proposition 2 it can be shown that the approximation used in [48] Sec. V] preserves the cardinality distribution and 1st moment.

Remark 4. In multi-object tracking, the matching of the PHDs \( \hat{v}(\cdot, \ell) \) and \( v(\cdot, \ell) \) in Proposition 2 is a stronger result than simply matching the unlabeled PHD alone. Notice that this property does not hold for the LMB filter, as shown in [45] (Section III), due to the imposed multi-Bernoulli structure for the cardinality distribution.

IV. APPLICATION TO MULTI-OBJECT TRACKING

In this section we propose a multi-object tracking filter for generic measurement models by applying the GLMB approximation result of Proposition 2. Specifically, we detail the prediction and update of the Bayes multi-object filter [2]-[5] for the standard multi-object dynamic model as well as a generic measurement model.

A. Prediction

Following [32], [33], to ensure distinct labels we assign each target an ordered pair of integers \( \ell = (k, i) \), where \( k \) is the time of birth and \( i \) is a unique index to distinguish targets born at the same time. The label space for targets born at time \( k+1 \) is denoted as \( \mathbb{L}_{k+1} \), and a target born at time \( k+1 \), has state \( x \in \mathcal{X} \times \mathbb{L}_{k+1} \). The label space for targets at time \( k+1 \) (including those born prior to \( k+1 \), denoted as \( \mathbb{L}_{0:k+1} \), is constructed recursively by \( \mathbb{L}_{0:k+1} = \mathbb{L}_{0:k} \cup \mathbb{L}_{k+1} \) (note that \( \mathbb{L}_{0:k} \) and \( \mathbb{L}_{k+1} \) are disjoint). A multi-object state \( X \) at time \( k+1 \), is a finite subset of \( \mathcal{X} = \mathcal{X} \times \mathbb{L}_{0:k+1} \).

The standard multi-object dynamic model is described as follows. Given the current multi-object state \( X' \), each state \((x', \ell') \in X' \) either continues to exist at the next time step with probability \( p_S(x', \ell') \) and evolves to a new
state \((x, \ell)\) with probability density \(f_{k+1|k}(x'|x', \ell')\delta_{x}(x')\), or dies with probability \(q_{S}(x', \ell') = 1 - p_{S}(x', \ell')\). The set of new objects born at the next time step is distributed according to
\[
f_{B}(Y) = \Delta(Y)w_{B}(L(Y))\left[p_{B}\right]^{Y}
\]
(23)
The birth density \(f_{B}\) is defined on \(X \times \mathbb{L}_{k+1}\) and \(f_{B}(Y) = 0\) if \(Y\) contains any element \(y\) with \(L(y) \notin \mathbb{L}_{k+1}\). The birth model (23) covers both labeled Poisson and labeled multi-Bernoulli. For a labeled multi-Bernoulli

Birth model:

\[
w_{B}(L) = \prod_{i \in L_{k}} \left(1 - r_{B(i)}\right) \prod_{\ell \in L_{k}} \frac{1_{\mathbb{L}}(\ell)q_{B}(\ell)}{1 - r_{B}(\ell)} ,
\]
(24)
\[
p_{B}(x, \ell) = p_{B}(\ell)(x).
\]
(25)
The multi-object state at the next time \(X\) is the superposition of surviving objects and new born objects. It was shown in [32] that the multi-object transition kernel is given by
\[
f_{k+1|k}(X|X') = f_{S}(X \cap (X \times \mathbb{L}_{0:k}))f_{B}(X - (X \times \mathbb{L}_{0:k}))
\]
For a GLMB prior density given by [5], the predicted multi-object density from eq. (2) is a GLMB given by [32]
\[
\pi_{k+1|k}(X) = \Delta(X) \sum_{f \in \mathcal{F}[L_{k+1}]} w_{k+1|k}^{(f)}(L(X)) \left[p_{k+1|k}^{(f)}\right]^{X}
\]
(26)
where
\[
w_{k+1|k}^{(f)}(L) = w_{B}(L - \mathbb{L}_{0:k})w_{S}^{(f)}(L \cap \mathbb{L}_{0:k}) ,
\]
(27)
\[
p_{k+1|k}^{(f)}(x, \ell) = 1_{\mathbb{L}_{0:k}}(\ell)p_{k}^{(f)}(x, \ell) + (1 - 1_{\mathbb{L}_{k+1}}(\ell))p_{B}(x, \ell) ,
\]
(28)
\[
p_{k}^{(f)}(x, \ell) = \frac{\langle p_{B}(\cdot, \ell)f_{k+1|k}(x|\cdot, \ell), p_{k}^{(f)}(\cdot, \ell) \rangle}{\eta_{k}^{(f)}(\ell)} ,
\]
(29)
\[
\eta_{k}^{(f)}(\ell) = \langle p_{B}(\cdot, \ell), p_{k}^{(f)}(\cdot, \ell) \rangle ,
\]
(30)
\[
w_{S}^{(f)}(L) = [\eta_{S}^{(f)}]^{L} \sum_{J \subseteq \mathbb{L}_{k}} J(L) [\eta_{S}^{(f)}]^{L - J} \alpha^{(f)}(J) ,
\]
(31)
\[
q_{S}^{(f)}(\ell) = \langle q_{S}(\cdot, \ell), p_{k}^{(f)}(\cdot, \ell) \rangle .
\]
(32)
Eqs. (26)-(32) explicitly describes how to calculate the parameters of the predicted multi-object density from the parameters of the previous multi-object density. Notice that in a practical implementation of a multi-object tracker an equivalent \(\delta\)-GLMB form is used since it facilitates implementation (for more details see [33]).

B. Update

In this section we specify the application of the proposed GLMB approximation to multi-object estimation problems using a generic measurement model given by a likelihood \(g(z_{k}|X)\) of the measurement \(z_{k}\) at time \(k\). We do not assume any particular structure for the multi-object likelihood function \(g(\cdot|\cdot)\) and hence the approach in this section is applicable to any measurement model including point detections, superpositional sensors and
matches cardinality and PHD is given by:

$$\pi_k(X|z_k) \propto g(z_k|X)p_{k|k-1}(X)$$  \hspace{1cm} (33)$$

which is a joint distribution over the multi-object state $X$ and does not retain the GLMB form in general.

If targets are well separated we can approximate the likelihood as being separable, i.e. $g(z_k|X) \approx \gamma_{z_k}X$, and obtain a GLMB posterior distribution from proposition 1, i.e.

$$\pi_k(X|z_k) \propto \Delta(X) \sum_{I \in F_{(\text{I.o.k.)}}} w_k^{(I)}(\mathcal{L}(X)) \left[ p_k^{(I)}(\cdot|z_k) \right]^X$$  \hspace{1cm} (34)$$

where

$$w_k^{(I)}(L) = [\eta_{z_k}]^L w_{k|k-1}^{(I)}(L)$$  \hspace{1cm} (35)$$

$$p_k^{(I)}(x,\ell|z_k) = p_{k|k-1}^{(I)}(x,\ell) \prod_{i=1}^m \gamma_{z_k}(x_i,\ell_i)/\eta_{z_k}(\ell)$$  \hspace{1cm} (36)$$

$$\eta_{z_k}(\ell) = \left\{ p_{k|k-1}^{(I)}(\cdot,\ell), \prod_{i=1}^m \gamma_{z_k}(\cdot,\ell) \right\}$$  \hspace{1cm} (37)$$

If instead targets are closely spaced the separable likelihood approach typically leads too a poor approximation.

Given the multi-object prediction density $\pi_{k|k-1}$ and the non-separable likelihood $g(z_k|X)$, by applying Bayes update in eq. (3) we obtain the following mixture of joint distributions $p_k^{(I)}(\cdot|z_k)$

$$\pi_k(X|z_k) = \Delta(X) \sum_{I \in F_{(\text{I.o.k.)}}} w_k^{(I)}(\mathcal{L}(X))p_k^{(I)}(X|z_k)$$  \hspace{1cm} (38)$$

where

$$p_k^{(I)}(X|z) = g(z_k|X) \left[ p_{k|k-1}^{(I)} \right]^X / \eta_{z_k}(\mathcal{L}(X))$$  \hspace{1cm} (39)$$

$$w_k^{(I)}(\mathcal{L}(X)) = \delta_I(\mathcal{L}(X))\eta_{z_k}(\mathcal{L}(X)) w_{k|k-1}^{(I)}(\mathcal{L}(X))$$  \hspace{1cm} (40)$$

$$\eta_{z_k}(\{\ell_1,\ldots,\ell_n\}) = \int g(z_k|\{x_1,\ell_1),\ldots,(x_n,\ell_n\}) \prod_{i=1}^n p_{k|k-1}^{(I)}(x_i,\ell_i)$$  \hspace{1cm} (41)$$

$$K = \int \Delta(X) \sum_{I \in F_{(\text{I.o.k.)}}} \delta_I(\mathcal{L}(X))w_{k|k-1}^{(I)} \left[ p_k^{(I)} \right]^X g(z_k|X)$$  \hspace{1cm} (42)$$

From eq. (39), we see that each component $\left[ p_{k|k-1}^{(I)} \right]^X$ from the prior GLMB is transformed into a joint density $p^{(I)}(X|z_k)$ and not into a multi-object exponential. From Proposition 2 the approximate GLMB $\hat{\pi}_k(X|z_k)$ that matches cardinality and PHD is given by:

$$\hat{\pi}_k(X|z_k) = \Delta(X) \sum_{I \in F_{(\text{I.o.k.)}}} \hat{w}_k^{(I)}(\mathcal{L}(X)) \left[ \hat{p}_k^{(I)}(\cdot|z_k) \right]^X$$  \hspace{1cm} (43)$$

$$\hat{p}_k^{(I)}(\{\ell_1,\ldots,\ell_n\})(\{\ell_1,\ldots,\ell_n\}|z_k) = \int p_k^{(I)}(\{\ell_1,\ldots,\ell_n\})(\{x_1,\ell_1),\ldots,(x_n,\ell_n\})|z_k)$$  \hspace{1cm} (44)$$

$$\hat{w}_k^{(I)}(\mathcal{L}(X)) = w_k^{(I)}(\mathcal{L}(X))$$  \hspace{1cm} (45)$$

Notice that we only approximate the spatial distribution in each component of the GLMB and retain the weighting terms $w_k^{(I)}(\mathcal{L}(X))$ from the joint update. This allows us to use the GLMB recursion for the approximate posterior while capturing the targets correlation effects due to the joint update. Eqs. (43)-(45) are
obtained applying the results in Proposition 2 to the joint posterior GLMB in eq. (38). Detailed derivations are reported in the Appendix.

V. NUMERICAL RESULTS

In this section we verify the proposed GLMB approximation technique via an application to recursive multi-object tracking with radar power measurements. Target tracking is usually performed on data that have been preprocessed into point measurements or detections. The bulk of multi-object tracking algorithms in the literature are designed for this type of data [3], [12], [49], [50]. Compressing information from the raw measurement into a finite set of points is very effective for a wide range of applications. However, for applications with low SNR, this approach may not be adequate as the information loss incurred in the compression becomes significant. Consequently, it becomes necessary to make use of all information contained in the pre-detection measurements, which in turn requires more advanced sensor models and algorithms.

We first describe the dynamic model and multi-object measurement equation used to simulate the radar power measurements. We then report numerical results for the separable likelihood approximation and GLMB posterior approximation. Throughout this section our recursive multi-object tracker is implemented with a particle filter approximation [43], [51] of the GLMB density [33].

A. Dynamic Model

The kinematic part of the single-target labeled state vector \( x_k = (x_k, \ell_k) \) at time \( k \) is described by
\[
x_k = \begin{bmatrix} \tilde{x}_k^T, \zeta_k \end{bmatrix}^T,
\]
which comprises the planar position and velocity vectors \( \tilde{x}_k = [p_x,k, \dot{p}_x,k, p_y,k, \dot{p}_y,k]^T \) in 2D Cartesian coordinates, respectively, and the unknown modulo of the target complex amplitude \( \zeta_k \in \mathbb{R} \). A Nearly Constant Velocity (NCV) model is used to describe the target dynamics, while a zero-mean Gaussian random walk is used to model the fluctuations in time of the target complex amplitude, i.e.,
\[
x_{k+1} = F x_k + v_k, \quad v_k \sim \mathcal{N}(0; Q)
\]
where \( v_k \) is a zero-mean Gaussian noise, \( T_s \) the radar sampling time, and the matrices are defined as:
\[
F = \text{diag}(F_1, F_1, 1), \quad F_1 = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix},
\]
\[
Q = \text{diag}(a_{\text{max}}Q_1, a_{\text{max}}Q_1, a_{\zeta}T_s), \quad Q_1 = \begin{bmatrix} T_s^2 & T_s^2 \\ T_s^2 & T_s \end{bmatrix}
\]
where \( a_{\text{max}} \) determines the process noise intensity for the motion, and \( a_{\zeta} \) the amplitude fluctuation in linear domain.

B. TBD Measurement Equation

A target \( x \in X \) illuminates a set of cells \( C(x) \), where \( C(x) \) is usually referred to as the target template. A radar positioned at the Cartesian origin collects a vector measurement \( z = [z^{(1)} \ldots z^{(m)}] \) consisting of the power signal returns, i.e.
\[
z^{(i)} = |z_A^{(i)}|^2
\]
where \( z_A^{(i)} \) is the complex signal in cell \((i)\), i.e.

\[
z_A^{(i)} = \sum_{x \in X : i \in C(x)} A(x) h_A^{(i)}(x) + w^{(i)}
\]  

(47)

Here,

- \( w^{(i)} \) is a zero-mean white complex Gaussian noise with variance \( 2\sigma_w^2 \)
- \( h_A^{(i)}(x) \) is the point spread function in cell \((i)\) from a target with state \( x \)

\[
h_A^{(i)}(x) = \exp\left(-\frac{(r_i - r(x))^2}{2R} - \frac{(d_i - d(x))^2}{2D} - \frac{(b_i - b(x))^2}{2B}\right)
\]  

(48)

where \( R, D \) and \( B \) are constants related to the radar cell resolution; \( r(x), d(x), \) and \( b(x) \) are the target coordinates in the measurement space; and \( r_i, d_i, b_i \) are the cell centroids.
- \( A(x) \) is the complex echo of target \( x \), which for Swerling 0 model is given by

\[
A(x) = \bar{A}_x e^{j\theta}, \quad \theta \sim U[0, 2\pi)
\]  

(49)

Let \( \hat{z}^{(i)} \) denote the deterministic part of the signal in cell \((i)\):

\[
\hat{z}^{(i)} = \left| z_A^{(i)} \right|^2 = \sum_{x \in X : i \in C(x)} \bar{A}_x h_A^{(i)}(x)
\]

and \( SNR \) be the signal-to-noise ratio defined by

\[
SNR = 10 \log\left(\frac{\bar{A}_x^2}{2\sigma_w^2}\right)
\]  

(50)

where we can choose \( \sigma_w^2 = 1 \) so that \( \bar{A}_x = \sqrt{2 \cdot 10^{SNR/10}} \). The measurement in each cell \( z^{(i)} \) is described by a non-central chi-squared distribution with 2 degrees of freedom and non-centrality parameter \( \hat{z}^{(i)} \), and simplifies to a central chi-squared distribution with 2 degrees of freedom in the case \( \hat{z}^{(i)} = 0 \). Then, the likelihood ratio for cell \((i)\) is given by:

\[
\ell(z^{(i)} | X) = \exp\left(-0.5\hat{z}^{(i)}\right) I_0\left(\sqrt{z^{(i)}\hat{z}^{(i)}}\right)
\]  

(51)

and the likelihood function for the vector measurement \( z \) takes the form

\[
g(z | X) \propto \prod_{i \in C(X)} \ell(z^{(i)} | X)
\]  

(52)

In practice, the modified Bessel function \( I_0(\cdot) \) is evaluated using the approximation given in [52]. Notice that eqs. [51]-[52] capture the superpositional nature of the power returns for each measurement bin due to the possibility of closely spaced targets target, i.e. overlapping target templates. The separable likelihood assumption is obtained from eqs. [51]-[52] by assuming that at most one target contributes to the power return from each cell \((i)\),

\[
\hat{z}^{(i)} = \left| \hat{z}^{(i)} \right|^2 = \begin{cases} \left| \bar{A}_x h_A^{(i)}(x) \right|^2, & \exists x \in X : i \in C(x) \\ 0, & \text{otherwise} \end{cases}
\]
Table I
PARAMETERS USED IN SIMULATIONS

| Parameter                              | Symbol | Value          |
|----------------------------------------|--------|----------------|
| Range Resolution                       | $R$    | {5, 10} m     |
| Azimuth Resolution                     | $B$    | $1^\circ$      |
| Doppler Resolution                     | $D$    | 1 m/sec        |
| Signal-to-Noise Ratio                  | $SNR$  | 7 dB           |
| Target Acceleration                    | $a_{max}$ | 3 m/s         |
| Amplitude Fluctuation                  | $a_{p}$ | 1              |
| Sampling Time                          | $T_s$  | {2, 1} sec     |
| 1st Birth Point Coordinates            | $x_{1B}$ | [1250, -10, 1000, -10] |
| 2nd Birth Point Coordinates            | $x_{2B}$ | [1000, -10, 1250, -10] |
| 3rd Birth Point Coordinates            | $x_{3B}$ | [1250, -10, 1250, -10] |
| Birth Probability                      | $P_B$  | 0.01           |
| Survival Probability                   | $P_S$  | 0.99           |
| $n^\circ$ of particles per target      | $N_p$  | 3000           |

C. Separable Likelihood Results

In this section we report simulation results for a radar TBD scenario under the separable likelihood assumption. In practice the assumption is valid when targets do not overlap at any time or only for few scans. Also a birth density which is relatively peaky compared to the targets kinematics is required to verify the assumption. This apparently obvious requirement is very important otherwise the algorithm will propose new target hypotheses that always violate the separable likelihood assumption yielding to a bias in the estimated number of targets.

The considered scenario is depicted in fig. 1: we have a time varying number of targets due to various births and deaths with a maximum of 11 targets present mid scenario. Fig. 2 shows a snapshot of the received power returns for a Doppler bin containing 4 targets. The parameters are reported in Table I: the range resolution is $R = 5$ m, the sampling time is $T_s = 2$ sec, and the Gaussian birth covariance is $Q_B = \text{diag} \left( (5 \text{ m})^2, (2 \text{ m/sec})^2, (5 \text{ m})^2, (2 \text{ m/sec})^2 \right)$. Results are collected in figs. 3-4 where we report the estimation results for a single trial along the $x$ and $y$ coordinates in fig 3 as well as the Monte Carlo results for the estimated number of targets and position OSPA distance in fig. 4 respectively. Notice that the average estimated number of targets slightly differs from the true number due to closely spaced targets (as verified by fig. 1), but the overall performance is satisfactory given the low SNR of 7 dB.

D. Non-Separable Likelihood Results

In this section we consider a more difficult radar TBD scenario where the separable likelihood assumption would lead to a bias on the estimated number of targets. Specifically, compared to the previous scenario we consider a larger uncertainty for the birth density $Q_B = \text{diag} \left( (15 \text{ m})^2, (5 \text{ m/sec})^2, (15 \text{ m})^2, (5 \text{ m/sec})^2 \right)$, a worse range resolution $R = 10$ m, and a smaller sampling time $T_s = 1$ sec. The results show that a GLMB filter using the joint likelihood update followed by the proposed GLMB approximation guarantees satisfactory tracking performance. The estimation results for a single trial along the $x$ and $y$ coordinates are given in fig 6.
Figure 1. Separable likelihood simulations scenario. Targets appear from the top right corner and move closer to the radar positioned at the Cartesian origin. $SNR = 7dB$, $T_s = 2$ sec and range resolution $R = 5m$.

Figure 2. Snapshot of the received power returns for a Doppler bin containing 4 targets and true targets amplitude and position in the Range-Azimuth plane.

and the Monte Carlo results for the estimated number of targets and position OSPA distance are given in fig. 7.

VI. CONCLUSIONS

This paper has proposed a tractable class of GLMB approximations for labeled RFS densities. Unlike traditional PHD/CPHD and cardinality balanced or labeled Multi-Bernoulli approximations which force target independence assumptions, our proposed approximations allows for object dependence in addition to matching the cardinality distribution and the first moment or PHD. The result has particular significance in multi-target tracking since it leads to tractable recursive filter implementations with formal track estimates for a wide range of measurement models. A radar based TBD example with low SNR and a time varying number of closely spaced targets was presented to verify the theoretical result. The key result presented in Section III is not only important to recursive multi-object filtering but is also generally applicable for statistical estimation problems.
Figure 3. Separable likelihood simulations. Trajectories along the $x$ and $y$ coordinates. $SNR = 7dB$, $T_s = 2$ sec and range resolution $R = 5m$.

Figure 4. Separable likelihood simulations. Monte Carlo results for estimated number of targets (top) and the OSPA distance (bottom) taken on the positional components with parameters $c = 50m$ and $p = 2$. $SNR = 7dB$, $T_s = 2$ sec and range resolution $R = 5m$.

involving point processes or random finite sets.

APPENDIX

In this section we give detailed derivations of eqs. (44)-(45) that define the posterior approximate GLMB in eq. (43), and are obtained applying the results in Proposition 2 to the joint posterior GLMB in eq. (38).

We start by showing that the mixture of joint distributions in eqs. (38)-(42) naturally arises when using a joint likelihood $g(z_k|X)$. Assume the multi-object prediction density $\pi_{k|k-1}$ is a $\delta$-GLMB of the form

$$
\pi_{k|k-1}(X) = \Delta(X) \sum_{I \in \mathcal{F}(L_{o,k})} \delta_I(L(X)) w_{k|k-1}^{(I)} \left[ p_{k|k-1}^{(I)} \right]^X
$$
Figure 5. Non-Separable likelihood simulations scenario. Targets appear from the top right corner and move closer to the radar positioned at the Cartesian origin. $SNR = 7dB$, $T_s = 1$ sec and range resolution $R = 10m$. Note the larger uncertainty $Q_B$ relative to the scenario for separable likelihood.

Figure 6. Non-Separable likelihood simulations. Trajectories along the $x$ and $y$ coordinates. $SNR = 7dB$, $T_s = 1$ sec and range resolution $R = 10m$.

then applying Bayes theorem with the joint likelihood $g(z_k|X)$ leads to:

$$
\pi_k(X|z_k) = \frac{\Delta(X) \sum_{I \in F(I_{0,k})} \delta_f(L(X)) w_k(I_{k-1}) \left[ p_k(I_{k-1}) \right] K^{(I)}}{\int \Delta(X) \sum_{I' \in F(I_{0,k})} \delta_f(L(X)) w_k(I_{k-1}) \left[ p_k(I_{k-1}) \right] K^{(I)} \ g(z_k|X) d(x_1, \ldots, x_n)}
$$
In order to use the results in Proposition 2 we need to rewrite the posterior $\pi_k(X|z_k)$ in the form

$$\pi_k(X|z_k) = w_k(\mathcal{L}(X)|z_k)p_k(X|z_k)$$

where $w_k(\mathcal{L}(X)|z_k)$ is given by

$$w_k(\{\ell_1, \ldots, \ell_n\} | z_k) = \int \pi_k(\{x_1, \ell_1\}, \ldots, \{x_n, \ell_n\})d(x_1, \ldots, x_n) = \int \Delta(X) \sum_{I \in \mathcal{F}(I_{0,k})} w_k^{(I)}(\{\ell_1, \ldots, \ell_n\})p_k^{(I)}(X|z_k)d(x_1, \ldots, x_n) = \sum_{I \in \mathcal{F}(I_{0,k})} \int \Delta(X)w_k^{(I)}(\{\ell_1, \ldots, \ell_n\})p_k^{(I)}(X|z_k)d(x_1, \ldots, x_n) = \sum_{I \in \mathcal{F}(I_{0,k})} w_k^{(I)}(\{\ell_1, \ldots, \ell_n\}) \int p_k^{(I)}(X|z_k)d(x_1, \ldots, x_n) = \sum_{I \in \mathcal{F}(I_{0,k})} w_k^{(I)}(\{\ell_1, \ldots, \ell_n\}) \sum_{I \in \mathcal{F}(I_{0,k})} \delta_I(\{\ell_1, \ldots, \ell_n\}) \frac{w_k^{(I)}(\{\ell_1, \ldots, \ell_n\})K^{(I)}}{K} = w_k^{(I)}(\{\ell_1, \ldots, \ell_n\})K^{(I)}K^{-1}$$

while for $p_k(X|z_k)$ we have

$$p_k(X|z_k) = \pi_k(X|z_k)/w_k(\mathcal{L}(X)|z_k) = \frac{\Delta(X) \sum_{I \in \mathcal{F}(I_{0,k})} w_k^{(I)}(\mathcal{L}(X)|z_k)p_k^{(I)}(X|z_k)}{\sum_{I \in \mathcal{F}(I_{0,k})} \Delta(X)w_k^{(I)}(\mathcal{L}(X)|z_k)}$$

We can now apply the results in Proposition 2 and construct a GLMB approximation using eqs. (19)-(21).
new weight $\hat{w}_k^{(I)}(L(X))$ are obtained as follows

$$\hat{w}_k^{(I)}(L(X)) = \delta_I(L(X)) w_k(L(X))$$

$$= \delta_I(L(X)) \frac{w_{k|k-1}^{(I)}}{K}$$

$$= w_k^{(I)}(L(X))$$

which proves eq. (45). For the densities $p_k^{(I)}(\cdot|z_k)$ we have

$$p_k^{(I)}((\zeta, \ell)|z_k) = 1_\ell(\ell)p_{I-\ell}(\zeta, \ell)$$

$$= 1_\ell(\ell) \int p((x, \ell), (x_1, \ell_1), \ldots, (x_n, \ell_n))|z) \cdot dx_{1, \ldots, n}$$

$$= \frac{1_\ell(\ell) \Delta((\ell, \ell_1), \ldots, \ell_n)}{w_k((\ell, \ell_1), \ldots, \ell_n)} \sum_{J \in \mathcal{F}(\ell, \ell_1, \ldots, \ell_n)} \delta_J((\ell, \ell_1), \ldots, \ell_n) w_{k|k-1}^{(J)} \frac{K^{(I)}}{K}$$

$$\times \int p_k^{(J)}((x, \ell), (x_1, \ell_1), \ldots, (x_n, \ell_n))|z_k) \cdot dx_{1, \ldots, n}$$

$$= \frac{1_\ell(\ell) \Delta((\ell, \ell_1), \ldots, \ell_n)}{K^{(\ell, \ell_1, \ldots, \ell_n)}} \frac{w_{k|k-1}^{(\ell, \ell_1, \ldots, \ell_n)} K^{(\ell, \ell_1, \ldots, \ell_n)}}{K}$$

$$\times \int p_k^{(\ell, \ell_1, \ldots, \ell_n)}((x, \ell), (x_1, \ell_1), \ldots, (x_n, \ell_n))|z_k) \cdot dx_{1, \ldots, n}$$

Hence, for a specific label set $I = \{i_1, \ldots, i_m\}$ and a specific track $i_j \in I$ we have the following

$$p_k^{(\{i_1, \ldots, i_m\})}((\zeta, i_j)|z_k) = 1_{\{i_1, \ldots, i_m\}}(i_j) \times \int p_k^{(\{i_1, i_j, \ldots, i_m\})}((\zeta, i_j), (x_1, \ell_1), \ldots, (x_n, \ell_n))|z_k) \cdot dx_{1, \ldots, n}$$

$$= \int p_k^{(\{i_1, i_j, \ldots, i_m\})}((\zeta, i_j), (x_1, \ell_1), \ldots, (x_n, \ell_n))|z_k) \cdot dx_{1, \ldots, n}$$

$$= \int p_k^{(\{i_1, i_j, \ldots, i_m\})}((x_1, i_j), \ldots, (x_m, i_m)|z_k) \cdot dx_{1, \ldots, j-1, j+1, \ldots, x_m}$$

which is exactly the expression given in eq. (44).

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