On the Probabilistic Degree of OR over the Reals*

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Abstract

We study the probabilistic degree over $\mathbb{R}$ of the OR function on $n$ variables. For $\varepsilon \in (0, 1/3)$, the $\varepsilon$-error probabilistic degree of any Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ over $\mathbb{R}$ is the smallest non-negative integer $d$ such that the following holds: there exists a distribution $\mathbf{P}$ of polynomials $P(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most $d$ such that for all $\bar{x} \in \{0, 1\}^n$, we have $\Pr_{\mathbf{P}}[P(\bar{x}) = f(\bar{x})] \geq 1 - \varepsilon$. It is known from the works of Tarui (Theoret. Comput. Sci. 1993) and Beigel, Reingold, and Spielman (Proc. 6th CCC 1991), that the $\varepsilon$-error probabilistic degree of the OR function is at most $O(\log n \cdot \log(1/\varepsilon))$. Our first observation is that this can be improved to $O\left(\log \left(\frac{n}{\varepsilon \log(1/\varepsilon)}\right)\right)$ which is better for small values of $\varepsilon$.

In all known constructions of probabilistic polynomials for the OR function (including the above improvement), the polynomials $\mathbf{P}$ in the support of the distribution $\mathbf{P}$ have the following special structure:

$$P(x_1, \ldots, x_n) = 1 - \prod_{i \in [t]} (1 - L_i(x_1, \ldots, x_n)),$$

where each $L_i(x_1, \ldots, x_n)$ is a linear form in the variables $x_1, \ldots, x_n$, i.e., the polynomial $1 - P(\bar{x})$ is a product of affine forms. We show that the $\varepsilon$-error probabilistic degree of OR when restricted to polynomials of the above form is $\Omega\left(\log \left(\frac{n}{\varepsilon \log(1/\varepsilon)}\right) / \log^2 \left(\log \left(\frac{n}{\varepsilon \log(1/\varepsilon)}\right)\right)\right)$, thus matching the above upper bound (up to poly-logarithmic factors).

1 Introduction

Low-degree polynomial approximations of Boolean functions were introduced by Razborov in his celebrated work [Raz87] on proving lower bounds for the class of Boolean functions computed by low-depth circuits. We begin by recalling this notion of approximation over $\mathbb{R}$.

Definition 1.1 (probabilistic degree). Given a Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ and $\varepsilon \in (0, 1/3)$, an $\varepsilon$-error probabilistic polynomial over $\mathbb{R}$ for $f$ is a distribution $\mathbf{P}$ of polynomials $P(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$ such that for any $\bar{x} \in \{0, 1\}^n$, we have $\Pr_{\mathbf{P}}[P(\bar{x}) \neq f(\bar{x})] \leq \varepsilon$. The $\varepsilon$-error probabilistic degree of $f$, denoted by $\text{P-deg}_\varepsilon(f)$, is the smallest non-negative integer $d$ such that the following holds: there exists an $\varepsilon$-error probabilistic polynomial $\mathbf{P}$ over $\mathbb{R}$ such that $\mathbf{P}$ is entirely supported on polynomials of degree at most $d$.

Classical results in polynomial approximation of Boolean functions [TO92, Tar93, BRS91] show that the OR function over $n$ variables, denoted by $\text{OR}_n$, has $\varepsilon$-error probabilistic degree at most $O(\log n \cdot \log(1/\varepsilon))$. This basic construction for the OR function is then recursively used to show that any function computed by an AC$^0$ circuit of size $s$ and depth $d$ has $\varepsilon$-error probabilistic degree at most $(\log s)^{O(d)} \cdot \log(1/\varepsilon)$ (see work by Harsha and Srinivasan [HS19] for recent improvements). These

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†† Similar notions over other fields are also studied. Unless otherwise specified, we will be considering probabilistic polynomials over the reals in this paper.
results can then be used to prove [Raz87, Smo87] a (slightly weaker) version of Håstad’s celebrated theorem [Hås89] that parity does not have subexponential-sized AC$^0$ circuits. These results were employed more recently by Braverman [Bra10] to prove that polylog-wise independence fools AC$^0$ functions.

Despite the fact that probabilistic polynomials for the OR function are such a basic primitive, it is surprising that we do not yet have a complete understanding of P-deg$_{\varepsilon}$($\text{OR}_n$). As mentioned above, it is known from the works of Beigel, Reingold and Spielman [BRS91] and Tarui [Tar93] that P-deg$_{\varepsilon}$($\text{OR}_n$) = $O(\log n \cdot \log(1/\varepsilon))$. The Schwartz-Zippel lemma implies that a dependence of $\Omega(\log(1/\varepsilon))$ is necessary in the above bound. However, until recently, it wasn’t clear whether any dependence on $n$ is necessary in P-deg$_{\varepsilon}$($\text{OR}_n$) over the reals$^2$. In recent papers of Meka, Nguyen and Vu [MNV16] and Harsha and Srinivasan [HS19], it was shown using anti-concentration of low-degree polynomials that the P-deg$_{1/4}$($\text{OR}_n$) = $\Omega(\sqrt{\log n})$. The main objective of this paper is to obtain a better understanding of the $\varepsilon$-error probabilistic degree of OR$_n$, P-deg$_{\varepsilon}$($\text{OR}_n$). In addition to being interesting in its own right, this question has bearing on the amount of independence needed to fool AC$^0$ circuits. Recent improvements due to Tal [Tal17] and Harsha and Srinivasan [HS19] of Braverman’s result demonstrate that $(\log s)^{2.5d+O(1)} \cdot \log(1/\varepsilon)$-wise independence fools functions computed by AC$^0$ circuits of size $s$ and depth $d$ up to error $\varepsilon$. An improvement of the upper bound on P-deg$_{\varepsilon}$($\text{OR}_n$) to $O(\log n + \log(1/\varepsilon))$ could potentially strengthen this result to $(\log s)^{d+O(1)} \cdot \log(1/\varepsilon)$, nearly matching the lower bound of $(\log s)^{d-1} \cdot \log(1/\varepsilon)$ due to Mansour [LV96].

The above discussion demonstrates that the current bounds on P-deg$_{\varepsilon}$($\text{OR}_n$) fall short of being tight in two aspects: one, the dependence on $n$ in the lower bound is $\Omega(\sqrt{\log n})$ while in the upper bound it is $O(\log n)$ and two, the joint dependence on $\varepsilon$ and $n$ in the upper bound is multiplicative, i.e., $O(\log n \cdot \log(1/\varepsilon))$ while the current lower bounds can only show an additive $\Omega(\sqrt{\log n} + \log(1/\varepsilon))$ bound.

Which of these bounds is tight? A casual observer might suspect that the upper bound is, given the relatively neat expression. However, a closer look tells us that it cannot be, at least when $\varepsilon$ is quite small. For example, setting $\varepsilon = 1/2^{O(n)}$, the upper bound yields a degree of $O(n \log n)$, but it is a standard fact that any Boolean function on $n$ variables can be represented exactly (i.e., with no error) as a polynomial of degree $n$. Hence the upper bound is not tight in this regime.

Our first observation is that the upper bound of Tarui and Beigel et al. [BRS91] can indeed be slightly improved to $O\left(\log \left(\frac{n}{\log(1/\varepsilon)}\right)\right)^3$; note that this is asymptotically better than $O(\log n \cdot \log(1/\varepsilon))$ for very small $\varepsilon$. This interpolates smoothly between the construction of Tarui [Tar93] and Beigel et al. [BRS91] and the exact representation of degree $n$ mentioned above. (See Section 3 for details on this upper-bound construction.)

Given this observation, one might hope to prove a matching lower bound on the $\varepsilon$-error probabilistic degree of OR$_n$. We can indeed show such a bound (up to polylogarithmic factors) if we suitably restrict the class of polynomials being considered. While restricted, this subclass of polynomials nevertheless includes all polynomials that were used in previous upper bound constructions, including our own. Moreover, this result generalizes a result of Alon, Bar-Noy, Linial and Peleg [ABLP91], who prove such a result for a further restricted class of polynomials (mentioned at the end of this section) and for $\log(1/\varepsilon) = O(\log n)^4$. A careful reworking of their analysis shows that their lower bound extends to even smaller $\varepsilon$ to show a lower bound of $O(\log \left(\frac{n}{\log(1/\varepsilon)}\right))$ for this smaller class of polynomials.

To state our result, we first need to describe the class of polynomials for which our bounds hold. To this end, we note that all known upper-bound constructions of probabilistic polynomials for the OR function have the following structure:

$$P(x_1, \ldots, x_n) = 1 - \prod_{i \in [t]} (1 - L_i(x_1, \ldots, x_n)),$$

where each $L_i(x_1, \ldots, x_n) = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n$ is a linear form in the variables $x_1, \ldots, x_n$ (here, $a_{ij} \in \mathbb{R}$). This motivates the following definition:

$^2$For finite fields of constant size, Razborov [Raz87] showed that the $\varepsilon$-error probabilistic degree of OR$_n$ is $O(\log(1/\varepsilon))$, independent of $n$, the number of the input bits.

$^3$Here, $(\frac{N}{i})$ denotes $\sum_{0 \leq i \leq r} \binom{N}{i}$. We use the convention that $\binom{N}{i} = 0$ if $i > N$.

$^4$The result of [ABLP91] is stated in a slightly different language, but is essentially equivalent to a probabilistic degree lower bound for OR$_n$ for a suitable class of polynomials.
Theorem 1.3. Theorem 1.3 follows from definition. For \( \epsilon \in (0, 1/2) \), the \( \epsilon \)-error hyperplane covering probabilistic degree of \( f \), denoted by \( \text{hcP-deg}_\epsilon(f) \), is the smallest non-negative integer \( d \) such that the following holds: there exists an \( \epsilon \)-error probabilistic polynomial \( P \) over \( \mathbb{R} \) such that \( P \) is supported on hyperplane covering polynomials of degree at most \( d \).

We call these polynomials hyperplane covering polynomials as these polynomials have the property that the set of points in the Boolean hypercube where the polynomial evaluates to 1 (i.e., the set \( \{ \vec{x} \in \{0, 1\}^n \mid P(\vec{x}) = 1 \} \) is a union of hyperplanes not passing through the origin. We further note that all these polynomials satisfy the property that \( P(0) = 0 \). Since hyperplane covering polynomials are a subclass of probabilistic polynomials, \( \text{hcP-deg}_\epsilon(f) \geq \text{P-deg}_\epsilon(f) \). Since all our upper-bound constructions for the OR polynomials are hyperplane covering polynomials, we not only have that \( \text{P-deg}_0(\text{OR}_n) = O\left( \log\left( \frac{n}{\log(1/\epsilon)} \right) \right) \) but also that \( \text{hcP-deg}_\epsilon(\text{OR}_n) = O\left( \log\left( \frac{n}{\log(1/\epsilon)} \right) \right) \). Our main result is the following (almost) tight bound on the \( \epsilon \)-error hyperplane covering probabilistic degree of the OR function.

**Theorem 1.3 (hyperplane covering degree of \( \text{OR}_n \)).** For any any positive integer \( n \) and \( \epsilon \in (0, 1/3) \),

\[
\text{hcP-deg}_\epsilon(\text{OR}_n) = \Omega\left( \frac{\log\left( \frac{n}{\log(1/\epsilon)} \right)}{\log^2\left( \frac{n}{\log(1/\epsilon)} \right)} \right).
\]

It is open if this result can be extended to prove a tighter lower bound on the \( \epsilon \)-error probabilistic degree of \( \text{OR}_n \). The special class of hyperplane covering polynomials for which Alon, Bar-Noy, Peleg and Linial [ABLP91] proved a similar bound is the class of hyperplane covering polynomials where the linear forms are sums of variables (i.e., \( L_i(\vec{x}) = \sum_{j \in S_i} x_j \) for some \( S_i \subseteq [n] \)) ideally, one would have liked to extend their lower bound result for hyperplane covering polynomials where the linear forms are sums of variables to all polynomials. Theorem 1.3, is a step in this direction, in that, it shows that their result can be extended to a slightly larger class, the set of all hyperplane covering polynomials (modulo polylogarithmic factors). We remark that though our lower bound works for a larger class of polynomials, our proof technique is nevertheless inspired by their proof.

**Organization:** The rest of the paper is organized as follows. After some preliminaries, we prove our improved upper bound (Theorem 3.2) in Section 3 and prove the lower bound (Theorem 1.3) in Section 4.

## 2 Preliminaries

**Notation:** For a string \( x \in \{0, 1\}^n \), we denote by \( |x| \), the Hamming weight of \( x \). The \( i \)-Hamming slice will refer to the set of strings \( x \) such that \( |x| = i \). For a set \( S \), \( |S| \) denotes the cardinality of \( S \).

Recall the definition of \( \epsilon \)-probabilistic degree \( \text{P-deg}_\epsilon(f) \) from the introduction. The following propositions lists some basic properties of the probabilistic degree.

**Proposition 2.1.**

1. \( \text{P-deg}_0(\text{OR}_n) = n \).
2. If \( 0 \leq \epsilon \leq \epsilon' \leq 1/3 \), then \( \text{P-deg}_\epsilon(f) \geq \text{P-deg}_{\epsilon'}(f) \).
3. For all \( \epsilon \in [0, 1/2^n] \), \( \text{P-deg}_\epsilon(f) = \text{P-deg}_0(f) \).
4. For all constant \( k \), \( \text{P-deg}_{k\epsilon}(f) \leq k \cdot \text{P-deg}_\epsilon(f) \).
5. For all \( \epsilon \in [0, 1/2^{n-1}] \), \( \text{P-deg}_\epsilon(\text{OR}_n) = \Omega(n) \).

**Proof.** Items 1 to 4 follow from definition. For Item 5, we note that for \( \epsilon \in [0, 1/2^{n-1}] \), \( \text{P-deg}_\epsilon(\text{OR}_n) \geq \text{P-deg}_{1/2^{n-1}}(\text{OR}_n) \geq 1/2^0 \cdot \text{P-deg}_{1/2^{n-1}}(\text{OR}_n) = 1/2^0 \cdot \text{P-deg}_0(\text{OR}_n) = \Omega(n) \).

Our notion of hyperplane covering polynomials depends on the notion of a linear form.
Definition 2.2 (linear form and its support). A linear form $L(x)$ is a homogenous degree one polynomial $a_1x_1 + a_2x_2 + \cdots + a_nx_n$. Given a linear form $L(x) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ we define the support of $L$, denoted as $\text{supp}(L)$, to be the set of variables $x_i$ whose corresponding coefficient $a_i$ in $L$ is non-zero.

Recall the notion of hyperplane covering polynomials and hyperplane covering probabilistic degree $\text{hcP-deg}_\varepsilon(f)$ from the introduction. The following proposition is proved similarly to Proposition 2.1.

Proposition 2.3.  
1. $\text{hcP-deg}_0(\text{OR}_n) = n$.

2. If $0 < \varepsilon \leq \varepsilon' \leq 1/3$, then $\text{hcP-deg}_\varepsilon(\text{OR}_n) \geq \text{hcP-deg}_{\varepsilon'}(\text{OR}_n)$.

3. For all $\varepsilon \in [0, 1/2^n)$, $\text{hcP-deg}_\varepsilon(\text{OR}_n) = \text{hcP-deg}_0(\text{OR}_n)$.

4. For all constant $k$, $\text{hcP-deg}_\varepsilon(\text{OR}_n) \leq k \cdot \text{hcP-deg}_\varepsilon(\text{OR}_n)$.

5. For all $\varepsilon \in [0, 1/2^{n^{1+\varepsilon}}]$, $\text{hcP-deg}_\varepsilon(\text{OR}_n) = \Omega(n)$.

Hyperplane covering polynomials have the following closure property.

Claim 2.4. Let $t$ be a positive integer. For each $i \in [t]$, let $P_i$ be a hyperplane covering polynomial of degree $d_i$. Let $P = 1 - \prod_{i \in [t]}(1 - P_i)$. Then $P$ is a hyperplane covering polynomial of degree at most $\sum_{i \in [t]} d_i$.

Proof. For all $i \in [t]$, since $P_i$ is hyperplane covering polynomial, there exist linear forms $L_{i, 1}, \ldots, L_{i, d_i}$ such that $P_i = 1 - \prod_{j \in [d_i]}(1 - L_{i, j})$.

Therefore by Definition 1.2, $P$ is a hyperplane covering polynomial of degree at most $\sum_{i \in [t]} d_i$. \qed

The proof of our lower bound requires the following variant of the Schwartz-Zippel Lemma (due to Alon and Füredi [AF93]) and Littlewood-Offord-Erdős’ anti-concentration lemma of linear forms over the reals, which we state below.

Lemma 2.5 ([AF93, Theorem 5]). Let $P \in \mathbb{R}[x_1, \ldots, x_n]$ be a polynomial of degree at most $d$ over $\mathbb{R}$ computing a non-zero function over $\{0, 1\}^n$. Then for $x$ chosen uniformly from $\{0, 1\}^n$,

$$\Pr_{x \in \{0, 1\}^n} |P(x) \neq 0| \geq \frac{1}{2^d}.$$

Lemma 2.6 ([LO38, Erd45]). Let $L(x_1, \ldots, x_k) = \sum a_i x_i$ be a linear form which is supported on exactly $k$ variables (i.e., $a_i \neq 0, i = 1, \ldots, k$). Then, for all $a \in \mathbb{R}$ and $x$ chosen uniformly from $\{0, 1\}^n$,

$$\Pr_{x \in \{0, 1\}^n} [L_i(x) = a] \leq \frac{1}{\sqrt{k}}.$$

Our lower and upper bounds will involve expressions of the form $(\log n - \log \log(1/\varepsilon)) \cdot \log(1/\varepsilon)$. The following claim lets us rewrite this expression more compactly in terms of binomial coefficients.

Claim 2.7. For $\varepsilon \in [1/2^{n/2}, 1/2)$, we have $(\log n - \log \log(1/\varepsilon)) \cdot \log(1/\varepsilon) = \Theta \left( \log \left( \log n \right) \right)$.

Proof. Consider an integer $k \in [1, n/2]$. Then we have the following well known bounds:

$$\left( \frac{n}{k} \right)^k \leq \binom{n}{k} \leq \left( \frac{en}{k} \right)^k.$$

The claim now follows directly with $k = \lceil \log(1/\varepsilon) \rceil$, which is between 1 and $n/2$. (Notice that $\log n - \log \log(1/\varepsilon) = \Theta(\log n - k)$.) \qed
3 Upper bounds on probabilistic degree of OR

Prior to this work, the best known construction of a probabilistic polynomial of OR \( n \) in terms of degree was due to Beigel, Reingold and Spielman [BRS91] and Tarui [Tar93].

**Theorem 3.1** ([BRS91, Tar93]). For any positive integer \( n \) and \( \varepsilon \in (0, 1/3) \),

\[
P_{\text{deg}}(\text{OR}_n) \leq hP_{\text{deg}}(\text{OR}_n) = O(\log n \cdot \log 1/\varepsilon).
\]

Note that since every Boolean function can be represented exactly by a polynomial of degree \( n \), the above upper bound is meaningful only when \( \varepsilon \geq \frac{1}{2n/\log n} \). We modify the construction of Beigel, Reingold and Spielman [BRS91] and Tarui [Tar93] and give a strictly better upper bound in terms of probabilistic degree.

**Theorem 3.2.** For any positive integer \( n \) and \( \varepsilon \in (0, 1/3) \),

\[
P_{\text{deg}}(\text{OR}_n) \leq hP_{\text{deg}}(\text{OR}_n) = O\left( \log \left( \frac{n}{\log(1/\varepsilon)} \right) \right).
\]

To begin with, we observe that the following hyperplane covering polynomial of degree \( n \) exactly computes \( \text{OR}_n \) everywhere on the Boolean hypercube:

\[
P_{\text{OR}}(x) := 1 - \prod_{i=1}^{n} \left( 1 - \frac{1}{2} \sum_{j \in [n]} x_j \right). \tag{1}
\]

For each \( i \in [n], \) the degree 1 polynomial \( \frac{1}{2} \sum_{j \in [n]} x_j \) outputs 0 on the zero input and 1 on the \( i \)-th Hamming slice. \( P_{\text{OR}} \) outputs 1 if any of these degree 1 polynomials output 1.

We now recall the construction of Beigel, Reingold and Spielman [BRS91] and Tarui [Tar93]. For each \( 0 \leq \ell \leq \log n - 1 \), they give a hyperplane covering probabilistic polynomial which outputs 0 on the zero input \( \bar{0} \) and outputs 1 with constant probability for all inputs whose Hamming weight is in the range \([2^\ell, 2^{\ell+1}]\).

**Lemma 3.3** ([BRS91, Tar93]). Let \( n \) be a positive integer. For all integers \( \ell \) such that \( 0 \leq \ell \leq \log n - 1 \) and for all \( \varepsilon \in (0, 1/3) \), there exists a distribution \( P_{\ell} \) on hyperplane covering polynomials of degree \( O(\log(1/\varepsilon)) \) such that

- \( P(\bar{0}^n) = 0 \) for all \( P \sim P_{\ell} \).
- for all inputs \( x \in \{0, 1\}^n \) whose Hamming weight is in the range \([2^\ell, 2^{\ell+1}]\),
  \[
  \Pr_{P \sim P_{\ell}} [P(x) = 1] \geq 1 - \varepsilon.
  \]

**Proof.** Fix \( 0 \leq \ell \leq \log n \) and \( \varepsilon \in (0, 1/3) \).

We begin by defining a distribution \( L_{\ell} \) of linear forms as follows: pick a random set \( S \subseteq [n] \) by picking each element of \( [n] \) independently with probability \( \frac{1}{2^\ell} \) and construct the linear polynomial

\[
L_{S}(x) := \sum_{i \in S} x_i.
\]

For a non-zero input \( x = (x_1, \ldots, x_n) \) such that the Hamming weight \( |x| \) of \( x \) is in \([2^\ell, 2^{\ell+1}]\), we have

\[
\Pr_{S} [L_{S}(x) = 1] = |x| \left( \frac{1}{2^\ell} \right) \left( 1 - \frac{1}{2^\ell} \right)^{|x|-1} \tag{where \( 0^0 = 1 \)}
\]

\[
= |x| \left( \frac{1}{2^\ell} \right) \exp(-O(1)) \quad \tag{\because (1-a)^b \geq \exp(-ab/(1-a))}
\]

\[
= \Omega(1). \tag{5}
\]

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In order to get a probabilistic polynomial $P_t$ which satisfies the requirements of Lemma 3.3, we sample $t = O(\log(1/\varepsilon))$ linear forms $L_1, \ldots, L_t$ independently from $L_t$ and construct the polynomial

$$P_{L_1, \ldots, L_t}(x) := 1 - \prod_{i \in [t]} (1 - L_i(x))$$

Note that all the polynomials in the support of $P_t$ are hyperplane covering polynomials. For any $L_1, \ldots, L_t$, degree of $P_{L_1, \ldots, L_t}$ is $t = O(\log 1/\varepsilon)$. $P_{L_1, \ldots, L_t}(0^n) = 0$ since $L_i(0^n) = 0$ for all $i \in [t]$. For any input $x$ such that $|x| \in [2^t, 2^{t+1}]$, $P_{L_1, \ldots, L_t}$ errs on $x$ only if for all $i \in [t]$, $L_i(x) \neq 1$, which happens with probability at most inverse exponential in $t$ and hence at most $\varepsilon$ (since $\Pr_{L_i \sim L_t}[L_i(x) \neq 1]$ is at most some constant less than $1$ for each $i$).

Theorem 3.1 is obtained by considering the following probabilistic polynomial $P$. For each $\ell \in \{0, \ldots, \log n - 1\}$, sample $P_\ell \sim P_\ell$ and construct

$$P := 1 - \prod_{\ell \in [\log n]} (1 - P_\ell).$$

This construction uses the probabilistic polynomial of degree $O(1/\varepsilon)$ from Lemma 3.3 for each of the $\log n$ epochs (where the $\ell$-th epoch refers to $\ell \mid x \mid \in [2^\ell, 2^{\ell+1}]$). This turns out to be wasteful for the lower epochs ($\ell \leq \log(1/\varepsilon)$). We observe that since the lower epochs have fewer slices, we can gain by using the polynomial construction from (1) instead.

Proof of Theorem 3.2. Consider the following distribution $P$ on hyperplane covering polynomials $P$: For each $\ell \in [\log(1/\varepsilon), \log n - 1]$, sample $P_\ell \sim P_\ell$ independently and construct the polynomials $P', P''$ and $P$ as follows.

$$P'(x) := 1 - \prod_{\ell = \log(1/\varepsilon)}^{\log n - 1} (1 - P_\ell(x)),$$

$$P''(x) := 1 - \prod_{i = 1}^{\log(1/\varepsilon)} \left(1 - \frac{1}{\ell} \sum_{j \in [n]} x_j\right),$$

$$P(x) := 1 - (1 - P'(x))(1 - P''(x)).$$

- By Lemma 3.3, $P_\ell$ is a hyperplane covering polynomial for each $\ell \in [\log(1/\varepsilon), \log n - 1]$. Therefore by Claim 2.4, $P'$ is hyperplane covering polynomial. Since $P''$ is also a hyperplane covering polynomial, so is $P$.
- Observe that $P''(0^n) = 0$. By Lemma 3.3, for all $\ell \in [\log(1/\varepsilon), \log n - 1]$, $P_\ell(0^n) = 0$ and hence $P''(0^n) = 0$.
- For all $i \leq \log(1/\varepsilon)$, for all inputs $x$ from the $i$-th slice, $P''(x) = 1$ and hence $P(x) = 1$. For each $\ell \in [\log(1/\varepsilon), \log n - 1]$ and each input $x$ from the $\ell$-th epoch,

$$\Pr_{P_\ell \sim P_\ell}[P_\ell(x) = 1] \geq 1 - \varepsilon.$$

Since $P_\ell(x) = 1$ implies $P(x) = 1$,

$$\Pr_{P \sim P}[P(x) = 1] \geq \Pr_{P_\ell \sim P_\ell}[P_\ell(x) = 1] \geq 1 - \varepsilon.$$

Therefore for all nonzero inputs $x$, $\Pr[P(x) = 1] \geq 1 - \varepsilon$.
- Since $P''$ has degree $O(1/\varepsilon)$ and by Lemma 3.3 $P_\ell$ has degree at most $\log(1/\varepsilon)$ for each $\ell \in [\log(1/\varepsilon), \log n - 1]$, $P$ has degree $(\log n - \log \log(1/\varepsilon)) \log(1/\varepsilon) = O\left(\log \left(\log(1/\varepsilon)\right)\right)$ (using Claim 2.7).

Therefore $P$ is an $\varepsilon$-error probabilistic polynomial for $\text{OR}_n$ supported of hyperplane covering polynomials of degree $O\left(\log \left(\log(1/\varepsilon)\right)\right)$. 

\[ \square \]
4 Lower bound on hyperplane covering degree of OR

We now turn to the lower bound. To prove a lower bound of $d_\varepsilon := \Omega \left( \log \left( \frac{n}{\varepsilon \log(1/\varepsilon)} \right) \right)$, by Yao’s minimax theorem (duality arguments) it suffices (and is necessary) to demonstrate a “hard” distribution $D_\varepsilon$ on $\{0, 1\}^n$ under which it is hard to approximate $OR_n$ by any hyperplane covering polynomial of degree at most $d_\varepsilon$.

Similar to previous works [MNV16, HS19], our choice of hard distribution is motivated by the polynomial constructions in the upper bound. Our hard distribution $D_\varepsilon$ is defined in terms of the following two distributions.

**Definition 4.1** ($p$-random assignment). Let $p \in [0, 1]$ and let $X = \{x_1, \ldots, x_n\}$ be a set of variables. A $p$-random assignment of $X$, denoted by $\mu_p^X$, is a random assignment $\mu: X \to \{0, 1\}$ that is chosen as follows: for each of the variables $x_i \in X$ independently set $\mu(x_i)$ to 1 with probability $p$ and 0 with probability $1 - p$.

**Definition 4.2** ($p$-random $(0, *)$-restriction). Let $p \in [0, 1]$ and let $X = \{x_1, \ldots, x_n\}$ be a set of variables. A $p$-random $(0, *)$-restriction of $X$, denoted by $\rho_p^X$, is a random restriction $\rho: X \to \{0, *\}$ chosen as follows: for each of the variables $x_i \in X$ independently set $\rho(x_i)$ to 0 with probability $(1 - p)$ and $*$ with probability $p$ (i.e., the variable is unset with probability $p$).

When the set of variables $X$ is clear from the context, we will drop the superscript $X$ from $\mu_p$ and $\rho_p$ respectively. We observe that $\mu_p^X$ can be generated by sampling a $2p$-random $(0, *)$-restriction $\rho_{2p}^X$ and setting the unset variables (i.e., $(\rho_{2p}^X)^{-1}(*)$) according to $\mu_{1/2}$. In short,

$$\mu_p^X = (\rho_{2p}^X)^{-1} \circ \rho_{2p}^X.$$

We use this observation crucially later on in the proof of the lower bound.

**Definition 4.3** (hard distribution). For $\varepsilon \in [1/2^{n/2}, 1/2)$, consider the distribution $D_\varepsilon$ on the input set $\{0, 1\}^n$ defined as follows: Let $I_\varepsilon := [1, \log n - \log \log(1/\varepsilon)] \cap \mathbb{Z}$. Pick an $\ell$ uniformly at random from $I_\varepsilon$ and output a random sample $x$ from $\mu_{1/2}^\ell$, i.e., for each $i \in [n]$, independently set $x_i \leftarrow 1$ with probability $1/2^{\varepsilon}$ and 0 otherwise.

The hard distribution $D_\varepsilon$ is a convex combination of the distributions $\mu_{1/2}^\ell$ for $\ell \in I_\varepsilon$. In other words, $D_\varepsilon := \frac{1}{|I_\varepsilon|} \sum_{\ell \in I_\varepsilon} \mu_{1/2}^\ell$. Each of the distributions $\mu_{1/2}^\ell$ roughly correspond to the epochs used in the upper-bound construction. The following claim shows that the distribution $D_\varepsilon$ puts probability at most $\varepsilon$ on the all-zeros input 0.

**Claim 4.4.** For $\varepsilon \geq 1/2^{n/2}$, we have $D_\varepsilon(0^n) \leq \varepsilon$.

**Proof.** $D_\varepsilon$ is generated by drawing an $\ell$ from $I_\varepsilon$ at random and returning a draw from $\mu_{1/2}^\ell$. Since

$$\mu_{1/2}^\ell(0^n) = \left(1 - 1/2^{\varepsilon}\right)^n \leq \varepsilon$$

for $\ell \leq \log n - \log \log(1/\varepsilon)$, $D_\varepsilon(0^n) \leq \varepsilon$.

Theorem 1.3 follows from the following “distributional” version of the theorem for $\varepsilon \in [1/2^{n/2}, 1/3]$. For smaller $\varepsilon$, Theorem 1.3 follows from Proposition 2.3:Item 5.

**Theorem 4.5.** Let $\varepsilon \in [1/2^{n/2}, 1/3]$ and $D_\varepsilon$ be the hard distribution defined in Definition 4.3 and $P = 1 - \prod_{i \in [t]} (1 - L_i)$ be a hyperplane covering polynomial of degree $t$ such that

$$\Pr_{x \sim D_\varepsilon} [P(x) \neq OR_n(x)] \leq \varepsilon$$

then, $t \geq \Omega \left( \frac{\log \left( \frac{\varepsilon \log n / \log(1/\varepsilon)}{\log(\log(1/\varepsilon))} \right) }{ \log \left( \frac{\log(1/\varepsilon)}{\log(\log(1/\varepsilon))} \right) } \right)$.

The rest of this section is devoted to proving Theorem 4.5. We begin with a proof outline in Section 4.1 followed by the proof in Section 4.2.
4.1 Proof outline

We would like to show that hyperplane covering polynomial $P$ that approximates OR$_n$ with respect to the distribution $D$, as in Theorem 4.5, must have large degree. Let $\mathcal{L}$ denote the set of linear forms that appear in $P$, i.e., $\mathcal{L} := \{ L_i \mid i \in [t] \}$.

Let us see how $P$ behaves on the distribution $\mu_{1/\varepsilon}$. or equivalently $\mu_{1/\varepsilon} \circ \rho_{1/\varepsilon-1}$. Let us see what happens to the linear forms in $\mathcal{L}$ when the restriction $\rho \sim \rho_{1/\varepsilon-1}$ is first applied. We first consider two extreme cases.

**Very few linear forms survive:** Suppose all but $\log(1/\varepsilon)$ linear forms trivialize on the restriction $\rho$ (i.e., the corresponding linear form $L_i|_{\rho}$ becomes 0). Then, $(1 - P)|_{\rho}$ is a polynomial of degree at most $\log(1/\varepsilon)$ computing a non-zero function (since $1 - P(0) = 1$). Hence, by Lemma 2.5, it is not equal to 0 with probability at least $2\varepsilon$. This implies that the polynomial $P$ errs with probability at least $\varepsilon$ on the distribution $\mu_{1/\varepsilon}$.

**All linear forms that survive have large support:** Suppose all the linear forms that survive post restriction $\rho$ have large support, say $4t^2$. Then, by the anti-concentration of linear forms over reals (Lemma 2.6), we have that each linear form is 1 with probability at most $1/\sqrt{4t^2} = 1/2$. Since there are at most $t$ linear forms, the probability that any of them is 1 is at most $t/2t = 1/2$. Thus, $P$ errs with probability $1/2$ on the distribution $\mu_{1/\varepsilon}$.

Note that the actual situation for each distribution $\mu_{1/\varepsilon}$ will most likely be a combination of the above two. We can then show that a combination of the above two arguments will still work if the surviving linear forms have the following nice structure. Let $\mathcal{L}_\rho$ be the set of surviving linear forms subsequent to the restriction $\rho$, i.e., $\mathcal{L}_\rho := \{ L_i |_{\rho} \mid i \in [t], L_i|_{\rho} \neq 0 \}$. Suppose $\mathcal{L}_\rho$ can be partitioned into 2 sets $\mathcal{L}'_{\rho} \cup \mathcal{L}''_{\rho}$ such that the number of linear forms in $\mathcal{L}'_{\rho}$ is small (less than $O(\log(1/\varepsilon))$) and each of the linear forms in $\mathcal{L}''_{\rho}$ have large support even after removing $\cup_{L \in \mathcal{L}''_{\rho}} \text{supp}(L)$ from their support. How does one then show that a constant fraction of $\rho$’s satisfy that the corresponding linear forms $\mathcal{L}_\rho$ have this nice structure? For this, we draw inspiration from the proof of Alon, Bar-Noy, Linial and Peled [ABLP91], where they prove similar bounds for hyperplane covering polynomials supported entirely on linear forms arising as sums of variables. They construct an appropriate potential function that guarantees a similar property in their lower-bound argument.

We use a slightly different potential function $\Phi_t(\mathcal{L})$, which has the following nice property. If the total number of linear forms is $t$, then $E_t[\Phi_t(\mathcal{L})] = O(t/(\log n - \log(1/\varepsilon)))$ and furthermore, whenever $\Phi_t(\mathcal{L})$ is small then the corresponding set $\mathcal{L}_t$ of surviving linear forms post restriction $\rho_{1/\varepsilon-1}$ can be partitioned as indicated above. This shows that for most $\ell$, $P$ errs on computing OR$_n$ unless $t$ is large.

4.2 Proof of Theorem 4.5

We now turn to defining the potential function $\Phi_t(\mathcal{L})$, indicated in the proof outline.

**Definition 4.6 (potential function).** The weight of a linear form $L$, denoted by $w(L)$, is defined as follows:

$$w(L) := \begin{cases} 0 & \text{if } \text{supp}(L) = \emptyset, \\ \frac{1}{\log^2(2|\text{supp}(L)|)} & \text{otherwise.} \end{cases}$$

Given a collection $\mathcal{L} = \{ L_1, \ldots, L_t \}$ of linear forms and $\ell$ a positive integer, the potential function $\Phi_t(\mathcal{L})$ is defined as follows

$$\Phi_t(\mathcal{L}) := \sum_{i=1}^{t} \rho_{\rho_{1/\varepsilon-1}} \mathbb{E} [w(L_i|_{\rho})],$$

where $\rho_{1/\varepsilon-1}$ is a $(0,*)$-restriction as defined in Definition 4.2.

The potential function $\Phi_t(\mathcal{L})$ satisfies the following two properties, given by Propositions 4.7 and 4.8

**Proposition 4.7.** There exists a universal constant $C$ such that the following holds. Let $\mathcal{L} = \{ L_1, \ldots, L_t \}$ be any collection of $t$ linear forms, then

$$\mathbb{E}_{\ell \in I_t} [\Phi_t(\mathcal{L})] \leq \frac{Ct}{|I_t|}.$$
Proposition 4.8 (partition of linear forms). Let $\mathcal{L} = \{L_1, \ldots, L_t\}$ be a collection of $t$ non-zero linear forms and $K, R$ be two positive integers such that

$$\sum_{i=1}^{t} w(L_i) < \frac{R}{\log^2(2RK)}.$$ 

Then, there exists a partition $\mathcal{L} = \mathcal{L}' \cup \mathcal{L}''$ of the set of linear forms $\mathcal{L}$ such that

- $|\mathcal{L}'| \leq R$,
- For all $L \in \mathcal{L}'$, $|\text{supp}(L) \setminus \bigcup_{L' \in \mathcal{L}'} \text{supp}(L')| \geq K$.

Before proving these two propositions, we first show how they imply Theorem 4.5.

Proof of Theorem 4.5. Let

$$t = \frac{\log(1/\varepsilon) \cdot (\log n - \log(1/\varepsilon))}{2C \log^2 \left( \frac{1}{\varepsilon^2} \cdot \frac{\log^4 (1/\varepsilon)}{(\log n - \log(1/\varepsilon))^3} \right)},$$

where $C$ is the universal constant in Proposition 4.7. Note that $t = \Omega \left( \log \left( \frac{n}{\log(1/\varepsilon)} \right) / \log^2 \left( \log \left( \frac{n}{\log(1/\varepsilon)} \right) \right) \right)$ (via Claim 2.7). Let $P = 1 - \prod_{i=0}^{t} (1 - L_i)$ be any hyperplane covering polynomial of degree $t$. Recall that $\varepsilon \geq \sqrt{2^{n/2}}$. Let $\mathcal{L} = \{L_1, \ldots, L_t\}$. To prove Theorem 4.5 it suffices to show the following

$$\Pr_{x \in D_\varepsilon} [P(x) \neq \text{OR}_n(x)] > \varepsilon. \quad (2)$$

We have from Claim 4.4 that $D_\varepsilon(0^n) = \Pr_{x \sim D_\varepsilon} [x = 0] < \varepsilon$. Since $\Pr [P(x) \neq 1] \leq \Pr [P(x) \neq \text{OR}_n(x)] + \Pr [x \neq 0^n]$, in order to show inequality (2), it suffices to prove

$$\Pr_{x \sim D_\varepsilon} [P(x) \neq 1] \geq 2\varepsilon. \quad (3)$$

Since $D_\varepsilon = \frac{1}{|\mathcal{L}|} \sum_{i \in [t]} \mu_{i/2}^{[n]}$ and $\mu_{p}^{[n]} = \mu_{i/2} \circ \rho_{2p}^{[n]}$, this is equivalent to showing

$$\mathbb{E}_{i \in [t]} \mathbb{E}_{\rho \sim \mu_{i/2}^{[n]}} \left[ \Pr_{x \sim \mu_{i/2}} [P(x) \neq 1] \right] \geq 2\varepsilon. \quad (4)$$

To this end, we first apply Proposition 4.7 to the set $\mathcal{L}$ of $t$ linear forms in the polynomial $P$ to obtain that

$$\mathbb{E}_{i \in [t]} \mathbb{E}_{\rho \sim \mu_{i/2}^{[n]}} \left[ \sum_{j \in [t]} w(L_{i|j}\rho) \right] = \mathbb{E}_{i \in [t]} [\Phi_{\varepsilon}(\mathcal{L})] \leq \frac{Ct}{|\mathcal{L}|}.$$ 

Applying Markov’s inequality to the above, we have

$$\Pr_{i, \rho} \left[ \sum_{j \in [t]} w(L_{i|j}\rho) \leq \frac{2Ct}{|\mathcal{L}|} \right] \geq \frac{1}{2}.$$ 

We call an $(\ell, \rho)$ pair good if the above event holds, i.e., $\sum_{i=1}^{t} w(L_{i|j}\rho) \leq 2Ct/|\mathcal{L}|$. Thus,

$$\Pr_{i, \rho} [(\ell, \rho) \text{ is good}] \geq \frac{1}{2}. \quad (5)$$

Now given a good $(\ell, \rho)$-pair, let $\mathcal{L}_\rho$ be the set of surviving linear forms subsequent to the restriction $\rho$, i.e., $\mathcal{L}_\rho := \{L_{i|\rho} | i \in [t], L_{i|\rho} \neq 0\}$. We thus have $\sum_{L \in \mathcal{L}_\rho} w(L) \leq 2Ct/|\mathcal{L}|$. Let $K := 4t^2$ and $R := \log(1/\varepsilon)$. It can be checked that for this choice of parameters we have $2Ct/|\mathcal{L}| < R/\log^2(2RK)$. We can now apply Proposition 4.8 to obtain a partition $\mathcal{L}_\rho = \mathcal{L}_\rho' \cup \mathcal{L}_\rho''$ such that

- $|\mathcal{L}_\rho'| \leq R = \log(1/\varepsilon)$,
• for all \( L \in \mathcal{L}' \), we have \(|\text{supp}(L) \setminus \bigcup_{L' \in \mathcal{L}'} \text{supp}(L')| \geq K = 4t^2\).

Consider the polynomial \( P|_{\rho} = 1 - \prod_{x \in \{0\}} (1 - L_i|_{\rho}) = 1 - \prod_{L \in \mathcal{L}_\rho} (1 - L) \) subsequent to the restriction \( \rho \). We will rewrite this polynomial as \( P|_{\rho} = 1 - Q|_{\rho} \cdot Q''\) where the polynomials \( Q|_{\rho} \) and \( Q''\) are defined as follows (using the sets \( \mathcal{L}_\rho \) and \( \mathcal{L}''_\rho \) respectively).

\[
Q|_{\rho}(x) := \prod_{L \in \mathcal{L}_\rho} (1 - L(x)), \\
Q''(x) := \prod_{L \in \mathcal{L}''_\rho} (1 - L(x)).
\]

Note that \( P|_{\rho} = 1 - Q|_{\rho} \cdot Q''\).

Since \( |\mathcal{L}_\rho| \leq \log(1/\varepsilon)\), we have that the degree of \( Q|_{\rho} \) is at most \( \log(1/\varepsilon) \). Furthermore \( Q|_{\rho}(x) \neq 0 \) (since \( Q|_{\rho}(0) = 1 \)). Thus applying Lemma 2.5, we have

\[
\Pr_{x \sim \mu_{1/2}} [Q|_{\rho}(x) \neq 0] \geq 8\varepsilon.
\]

Consider any setting of variables in \( \bigcup_{L \in \mathcal{L}_\rho} \text{supp}(L) \) such that \( Q|_{\rho}(x) \neq 0 \). Even conditioned on setting all these variables, we know that each \( L \in \mathcal{L}_\rho \) still has surviving support of size at least \( 4t^2 \). Thus, by Lemma 2.6, we have for each \( L \in \mathcal{L}_\rho ''\),

\[
\Pr_{x \sim \mu_{1/2}} [L(x) = 1 \mid Q|_{\rho}(x) \neq 0] \leq \frac{1}{\sqrt{4t^2}} = \frac{1}{2t}.
\]

By a union bound, we have

\[
\Pr_{x \sim \mu_{1/2}} [Q|_{\rho}(x) = 0 \mid Q|_{\rho}(x) \neq 0] = \Pr_{x \sim \mu_{1/2}} [\exists L \in \mathcal{L}_\rho, L(x) = 1 \mid Q|_{\rho}(x) \neq 0] \leq \frac{t}{2t} = \frac{1}{2}.
\]

Hence,

\[
\Pr_{x \sim \mu_{1/2}} [P|_{\rho}(x) \neq 1] = \Pr_{x \sim \mu_{1/2}} [Q|_{\rho}(x) \neq 0] \cdot \Pr_{x \sim \mu_{1/2}} [Q'|_{\rho}(x) = 0 \mid Q|_{\rho}(x) \neq 0] \geq 8\varepsilon \cdot \frac{1}{2} = 4\varepsilon.
\]

Finally averaging over all \( (\ell, \rho) \) we have from above and (5)

\[
\Pr_{x \sim \mathcal{D}_\nu} [P(x) \neq 1] \geq \Pr_{\ell, \rho} [(\ell, \rho) \text{ is good }] \cdot \Pr_{x \sim \mathcal{D}_\nu} [P|_{\rho}(x) \neq 1 \mid (\ell, \rho) \text{ is good }] \geq \frac{1}{2} \cdot 4\varepsilon = 2\varepsilon.
\]

This proves (3) and thus completes the proof of Theorem 4.5. \(\square\)

We are now left with the proofs of Propositions 4.7 and 4.8. We begin with the proof of Proposition 4.8.

Proof of Proposition 4.8. Consider the following algorithm to obtain the partition \( \mathcal{L} = \mathcal{L}' \cup \mathcal{L}'' \).

1. Initialize \( \mathcal{L}' \leftarrow \emptyset \) and \( \mathcal{L}'' \leftarrow \mathcal{L} \).

2. While there exists an \( L \in \mathcal{L}'' \) such that \(|\text{supp}(L) \setminus \bigcup_{L' \in \mathcal{L}'} \text{supp}(L')| \leq K\),

   • Move such an \( L \) from \( \mathcal{L}'' \) to \( \mathcal{L}' \) (i.e., \( \mathcal{L}' \leftarrow \mathcal{L}' \cup \{L\} \) and \( \mathcal{L}'' \leftarrow \mathcal{L}'' \setminus \{L\} \)).

Let \( \text{supp}(\mathcal{L}') \) be the union of supports of all linear forms in \( \mathcal{L}' \) (i.e., \( \text{supp}(\mathcal{L}') = \bigcup_{L \in \mathcal{L}'} \text{supp}(L) \)). When the algorithm terminates, we have \(|\text{supp}(\mathcal{L}') \setminus \text{supp}(\mathcal{L})| \geq K \) for all \( L \in \mathcal{L}'' \).

We now argue that \( |\mathcal{L}'| \leq R \). Each iteration of the while loop adds a linear form \( L \) to \( \mathcal{L}' \) with at most \( K \) new variables. If the while loop is performed for \( T \) iterations, then the support of each \( L \) added to \( \mathcal{L}' \) is at most \( TK \). We now argue that \( T < R \). If not, then after exactly \( R \) iterations of the while loop, we have that

\[
\sum_{L \in \mathcal{L}} \omega(L) \geq \sum_{L \in \mathcal{L}'} \omega(L) \geq \frac{R}{\log^2(2RK)},
\]

contradicting the hypothesis of the proposition. Hence \( T < R \). The size of \( \mathcal{L}' \) is the number of iterations of the while loop and is thus bounded above by \( R \). This completes the proof of the proposition. \(\square\)
Proof of Proposition 4.7.

\[ E_{\ell \in I_s} [\Phi_\ell(L)] = E_{\ell \in I_s} \left[ \frac{E_{\rho \sim \rho_{j,\zeta^{-1}} \in \ell} \left( \sum_{i \in [\ell]} w(L_i|\rho) \right)}{\sum_{i \in [\ell]} E[w(L_i|\rho)]} \right] \]

\[ = \frac{1}{|I_s|} \sum_{\ell \in [t]} \sum_{\rho \in T_0} E[w(L_i|\rho)] \]

\[ = \frac{1}{|I_s|} \sum_{\ell \in [t]} \left( \sum_{\ell > \log |\text{supp}(L_i)|} E[w(L_i|\rho)] + \sum_{\ell \leq \log |\text{supp}(L_i)|} E[w(L_i|\rho)] \right). \]

\( T_1 \) and \( T_2 \) are bounded using Claim 4.9 and Claim 4.10 respectively. Hence,

\[ E_{\ell \in I_s} [\Phi_\ell(L)] \leq \frac{1}{|I_s|} \sum_{i=1}^{t} \left( 2 + \frac{\pi^2}{6} + \frac{e}{e-1} \right) \leq \frac{t}{|I_s|} \left( 4 + \frac{\pi^2}{6} \right). \]

Claim 4.9. Let \( L \) be a linear form such that \( |\text{supp}(L)| = k \). Then

\[ \sum_{\ell \text{ log } k} E_{\rho \sim \rho_{j,\zeta^{-1}}} [w(L|\rho)] \leq 2. \]

Proof. \( \sum_{\ell \text{ log } k} E_{\rho \sim \rho_{j,\zeta^{-1}}} [w(L|\rho)] \leq \sum_{\ell \text{ log } k} \left( \Pr_{\rho} [\text{supp}(L|\rho) = 0] \cdot 0 + \Pr_{\rho} [\text{supp}(L|\rho) \geq 1] \cdot 1 \right) \]

\[ \leq \sum_{\ell \text{ log } k} \left( 1 - \left( 1 - \frac{1}{2^{\ell-1}} \right)^k \right) \]

\[ \leq \sum_{i \geq 0} \frac{k}{2^i} = 2. \]

Claim 4.10. Let \( L \) be a linear form such that \( |\text{supp}(L)| = k \). Then

\[ \sum_{\ell=0}^{\log k} E_{\rho \sim \rho_{j,\zeta^{-1}}} [w(L|\rho)] \leq \frac{\pi^2}{6} + \frac{e}{e-1}. \]

Proof. \( \sum_{\rho} E_{\rho} [w(L|\rho)] \leq \Pr_{\rho} \left[ |\text{supp}(L|\rho)| \geq \frac{k}{2^T} \right] \frac{1}{\log^2 (2k/2^T)} + \Pr_{\rho} \left[ |\text{supp}(L|\rho)| \leq \frac{k}{2^T} \right] \]

\[ \leq \frac{1}{(\log k - \ell + 1)^2} + \exp \left( -\frac{k}{4 \cdot 2^{\ell-1}} \right) \]

[By Chernoff bound]

\[ \sum_{\ell=0}^{\log k} E_{\rho \sim \rho_{j,\zeta^{-1}}} [w(L|\rho)] \leq \sum_{\ell=0}^{\log k} \frac{1}{(\log k - \ell + 1)^2} + \sum_{\ell=0}^{\log k} \exp \left( -\frac{k}{2^{\ell+1}} \right) \]

\[ = \sum_{i=1}^{\log k} \frac{1}{i^2} + \sum_{i=1}^{\log k} \exp \left( -2^{i-1} \right) \]

\[ \leq \frac{\pi^2}{6} + \frac{e}{e-1}. \]

\[ \square \]
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