HOMOTOPY LIFTING PROPERTY IN SYMMETRIC PRODUCTS

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Abstract. In this paper we prove the homotopy lifting property for symmetric products $SP_m(X)$ and $F_m(X)$, with $X$ a Hausdorff topological space. Furthermore, we introduce a new tool, the theory of topological puzzles, to get a useful decomposition of $X^m$.

1. Introduction

Lifting a continuous function between topological spaces is a question that depends on the topological properties of the two spaces. There are known cases where the lifting can be done relatively easy. One of these situations is when the continuous function is defined over a covering space $\tilde{X}$ of a topological space $X$ (see Proposition 1.30 page 60 of [12] and Theorem 17.6 page 157 of [14]). In that case one can prove the important homotopy lifting property, but even then, lifting a continuous function between any topological space and a covering space depends strongly on the homotopy type of the spaces (see Proposition 1.33 page 61 of [12] and theorem 21.2 page 174 of [14]). Another case one has the homotopy lifting property is when working with fibrations. Fibrations in the sense of Hurewicz (see definition 1.2 page 393 of [8] and page 66 of [26]) have the homotopy lifting property by definition, beyond this, fibrations in the sense of Hurewicz are equivalent to fibrations for which one has the homotopy lifting property (see Curtis and Hurewicz’s theorem page 396 of [8]). Again, as it happens with covering maps, lifting a continuous function between any topological space and a fiber space depends strongly on the homotopy type of the spaces (see theorem 5 page 76 of [26]). In fact, fibrations can be treated in most situations as covering maps (something natural as covering spaces are fiber spaces...
with discrete fiber—see theorem 3 page 67 of [26]. Out of that cases, there are few general situations but specific ones where a kind of lifting can be done. One of this specific cases is that of the paper of Papakyriakopoulos [23]; he uses the lifting of some maps whose image lies in what he calls prismatic neighborhood and defined in the universal covering of such neighborhood; with these liftings Papakyriakopoulos is able to prove Dehn’s lemma and the sphere theorem. The lifting problem from the general point of view can be studied using obstruction theory, Postnikov towers and Moore-Postnikov towers (see pages 410 and 419 of [12]).

In this paper we want to lift continuous functions over the symmetric product of a topological space to another continuous function over the cartesian product of that space.

2. Symmetric products $SP_m(X)$ and $F_m(X)$

In this section we are going to deal with the topological spaces called symmetric products. Let $X$ be a topological space. Let $m \in \mathbb{N}$ and denote $\Sigma_m$ the set of all permutations of $m$-elements. For $x, y \in X^m$ with $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, define the relation $\varphi$ as,

$$x \varphi y \iff \exists \sigma \in \Sigma_m : (y_1, \ldots, y_m) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)}).$$

It is not difficult to see that $\varphi$ is an equivalence relation. We will denote the equivalence classes generated by $\varphi$ as $[x]_\varphi$ for every $x \in X^m$. For one hand, we are going to call the quotient space,

$$SP_m(X) = X^m/\Sigma_m = X^m/\varphi,$$

the m-symmetric product of $X$. In fact, $SP_m(X)$ is a topological space if we endowed the cartesian product $X^m$ with the product topology and the quotient space $X^m/\varphi$ with the quotient topology. Furthermore, if $X$ is a Hausdorff topological space, then so it is $SP_m(X)$.

On the other hand, we are going to call the space $F_m(X)$, of all the finite subsets of $X$, with, at least, $m$-points, the m-symmetric product of $X$ too. This notation will be clear because after the expression “m-symmetric product” we are going to write the space $SP_m(X)$ or the space $F_m(X)$ in every case. In fact, $F_m(X)$ is a topological space if we endowed it with the Vietoris finite topology (introduced by L. Vietoris in [21]), i.e., let $k \in \{1, \ldots, m\}$, $U_j$ open sets in $X$, with $j = 1, \ldots, k$, and define,

$$V(U_1, \ldots, U_k) = \{ A \in F_m(X) : A \subset \bigcup_{j=1}^k U_j \text{ with } A \cap U_j \neq \emptyset \, \forall j = 1, \ldots, k \}. $$

Then the collection of all sets $V(U_1, \ldots, U_k)$, for $j = 1, \ldots, k$, is a basis of the topology in $F_m(X)$ (see [22] 4.5 page 54), and, if $X$ is a $T_1$ topological space, that topology coincides with the relatively one of $F_m(X)$ viewed as a subspace of $2^X$, the hyperspace of all non-void close subsets of $X$ (see [3], [20], [23], [31]). Moreover, if $X$ is a $T_1$ topological
space, then so it is $F_m(X)$. In the case that $X$ is a metrizable space, so it is $F_m(X)$ with the Hausdorff distance (first introduced by Hausdorff in [11]) and, furthermore, the topology generated by the Hausdorff distance and Vietoris finite topology coincide (see [13] theorem 3.1 page 16). We explain now an important fact of the $m$-symmetric product $F_m(X)$ which is going to be very useful in the future: let $m \in \mathbb{N}$; for $x, y \in X^m$ with $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, define the relation $\phi$ as,

$$x \phi y \iff \{y_1, \ldots, y_m\} = \{x_1, \ldots, x_m\}.$$  

It is not difficult to see that $\phi$ is an equivalence relation. We will denote the equivalence classes generated by $\phi$ as $[x]_\phi$ for every $x \in X^m$. Let’s call the quotient space,

$$\tilde{F}_m(X) = X^m/\phi,$$

the **m-symmetric quotient** of $X$. By a theorem of Ganea (see [10] Hilfssatz 2) we have that,

$$F_m(X) \simeq \tilde{F}_m(X) \quad \forall m \in \mathbb{N},$$

and for all $X$ a $T_1$ topological space. It is important to realize that the binary relations $\varphi$ and $\phi$ hold the next implication, for every $m \in \mathbb{N}$ and for every $x, y \in X^m$,

$$x \varphi y \Rightarrow x \phi y.$$

In spite the fact that it is trivial to prove the last implication between the binary operations $\varphi$ and $\phi$, this remark will be crucial in our future work. Particularly, taking $p \in X^m$ we have that $[p]_\varphi \subset [p]_\phi \subset X^m$.

The symmetric products $F_m(X)$ and $SP_m(X)$ are very far to be topologically similar and it is all but trivial to see what topological properties they inherit from $X$. For one hand, the space $SP_m(X)$ was introduced by Hausdorff in [11] with the name of combinatorial product. On the other hand, the space $F_m(X)$ was introduced by K. Borsuk and S. Ulam in [4], where, above all, they research about the topological properties that these spaces inherited from $X$. We will denote the spaces $SF_m(X)$ symmetric products too. Despite the fact Hausdorff gave them the name of combinatorial products, there are other authors that call them permutation products (see [28]) and even other ones that call them symmetric products (see [11]). The complete topological classification of the symmetric products $F_m(X)$ and
$SP_m(X)$ is an open problem. This is a summary of the characterizations of $F_m(X)$ (with $I$ the close unitary interval and $S^k$ the $k$-sphere),

$F_m(I) \cong I^m$ for $m \leq 3$ by Borsuk and Ulam in [4]

$F_2(S^1) \cong \text{M"obius strip}$ by Borsuk and Ulam in [4]

$F_3(S^1) \cong S^3$ by Bott in [5]

$F_{2n-1}(S^1)$ and $F_{2n}(S^1)$ has the same homotopy type of $S^{2n-1}$ by Chinen and Koyama in [6]

$F_2(I^2) \cong I^4$ by Molski in [20]

If $X$ is a topological 2-manifold then $F_2(X)$ is a topological 4-manifold by Schori in [25].

The result obtained by Bott in [5] explains very well the difficulty to study succesfully the $m$-symmetric products $F_m(X)$ because, before Bott, Borsuk proved mistakenly in [3] that $F_3(S^1) \cong S^1 \times S^2$.

About the symmetric products $SP_m(X)$,

If $X$ is an ANR, so it is $SP_m(X)$ (see [18], [19] and [24])

$SP_m(X)$ is a topological manifold $\iff$

$$\iff \left\{ \begin{array}{l}
X \text{ is a topological manifold} \\
\text{and} \\
\dim X = 1 \text{ or } 2 \\
SP_2(X) = F_2(X) \text{ see [28]} \\
SP_m(S^2) \cong \mathbb{C}P^m \text{ see [1].}
\end{array} \right.$$  

A complete study of the characterizations of symmetric products of manifolds can be found in [28]. For a great development of algebraic topology we refer the reader to the free book of A. Hatcher [12]; to refresh the concepts related with homotopy we refer the reader to the books [1] and [30]. For an introduction to topology it is good to read the references [8], [15] and [16] and for different developments of algebraic topology see references [7], [17] and [26].

3. Exterior and interior boundaries of a set

In this section we are going to prove some results of general topology and to define two new concepts: exterior and interior boundaries of a set.

Remark 3.1. Let $X$ be a Hausdorff topological space, $U \subset X$ an open subset and $x \in U$. Then $U \setminus \{x\}$ is open. To prove that, it is enough to see that every $x' \in U \setminus \{x\}$ is an interior point, i.e., it exists $U' \subset U \setminus \{x\}$ an open neighborhood of $x'$. But this is a direct consequence of the existence of $U'_1 \subset U$ an open neighborhood of $x'$ (that exists because $U$ is an open set) and the existence of $U'_2 \subset X$ an open neighborhood of $x'$. 


that does not contain $x$ (because $X$ is Hausdorff). Taking $U' = U'_1 \cap U'_2$, we finish.

**Remark 3.2.** Let $X$ be a Hausdorff topological space, $U \subset X$ an open subset and suppose that int$U = U \cup \{x\}$ with $x \notin$ int$U$. Then $U \cup \{x\}$ is open. To prove that, let $U_x$ be an open neighborhood of $x$ contained in $U$. Like $U_x$ is open, then $U_x \subseteq$ int$U$; as int$U = U \cup \{x\}$ then $U_x \subseteq U \cup \{x\}$. Therefore $x$ is an interior point of $U \cup \{x\}$ so $U \cup \{x\}$ is an open set.

**Definition 3.3.** Let $X$ be a topological space. Take $A \subset X$. We define the **interior boundary** of $A$ as the set,
\[
\partial I A = \{ x \in \partial A : \exists \text{ an open neighborhood of } x \text{ with } \text{int}(U \setminus A) = \emptyset \}.
\]
We define the **exterior boundary** of $A$ as,
\[
\partial E A = \{ x \in \partial A : \forall \text{ an open neighborhood of } x, \text{ int}(U \setminus A) \neq \emptyset \}.
\]

**Remark 3.4.** Let $X$ be a topological space. Take $A \subset X$. By 1.3.2 theorem page 24 of [9], we have,
\[
A = \text{int}A \cup \partial I A \cup \partial E A,
\]
because $\partial A = \partial I A \cup \partial E A$ by definition 3.3.

**Remark 3.5.** Let $X$ be a topological space and $A \subset X$. Take $U \subset \text{int}A$ an open set. Then $U \cap A \neq \emptyset$.

That is easy to see. Take $x \in U$. We have two possibilities by 1.3.2 theorem (i) and (ii) page 24 of [9]: $x \in \text{int}A$ or $x \in \partial A$. If $x \in \text{int}A \subset A$ then $U \cap A \neq \emptyset$. If $x \in \partial A$, by 1.3.1 Proposition page 24 of [9] we have that $U \cap A \neq \emptyset$.

**Lemma 3.6.** Let $X$ be a topological space and $A \subset X$. Then,
\[
\partial \overline{A} = \partial E A.
\]

*Proof.* Let $x \in \partial \overline{A}$ and let $U \subset X$ an open neighborhood of $x$. Then we know that,
\[
\begin{cases}
U \cap \overline{A} \neq \emptyset \\
U \cap \overline{A} \neq \emptyset.
\end{cases}
\]
Take $x' \in U \cap \overline{A}$. Suppose that for every $U'$ an open neighborhood of $x'$, we have that $U' \notin \text{int}(U \setminus A)$. Without loss of generality, we can suppose that $U' \subset U$, so if $U' \notin \text{int}(U \setminus A)$, then $U' \cap A \neq \emptyset$. Then by the definition of accumulation point, see page 24 of [9], $x'$ is an accumulation point of $A$. Therefore $x' \in \overline{A}$ by 1.3.4 theorem page 25 of [9], and that is impossible because $x' \in \overline{A}$. So, it exists $U'$ an open neighborhood of $x'$ such that $U' \subseteq \text{int}(U \setminus A)$, so $\text{int}(U \setminus A) \neq \emptyset$ and $x \in \partial E A$. 


Take $x \in \partial \xi A$. Then, for all $U$ an open neighborhood of $x$ we have that $\text{int}(U \setminus A) \neq \emptyset$. Take $x' \in \text{int}(U \setminus A)$. Then it exists $U' \subset U \setminus A$ an open neighborhood of $x'$. If $U' \subset \partial \xi A$ we would have, 
\[ \emptyset \subset U' = \text{int} U' \subset \text{int} \partial \xi A = \emptyset, \]
and that is impossible. It is neither possible that $U' \subset \text{int} \partial \xi A$, cause, in case that $U' \cap \text{int} \partial \xi A \neq \emptyset$ we would have $U' \cap A \neq \emptyset$ by remark 3.5 but we knew that $U' \subset U \setminus A$. So we conclude that $U' \cap \text{int} \partial \xi A \neq \emptyset$. Like $x \in \partial A$ then $\emptyset \neq U \cap A \subset U \setminus A$, therefore $x \in \partial \xi A$.

**Corollary 3.7.** Let $X$ be a topological space and $A \subset X$. Then, 

\[ \text{int} \partial \xi A = \text{int} A \cup \partial \xi A. \]

**Proof.** We have,

\[
\begin{cases}
\partial \xi A = \text{int} \partial \xi A \cup \partial \xi A & \text{by 1.3.2 theorem (i) and (ii) page 24 of [9]} \\
\partial \xi A = \text{int} A \cup \partial \xi A \cup \partial \xi A & \text{by remark 3.4} \\
\partial \xi A = \partial \xi A & \text{by lemma 3.6}
\end{cases}
\]

Combining the three facts we finish. □

**Lemma 3.8.** Let $X$ be a topological space and $A, B \subset X$. Then, 

\[ \partial \xi (A \cap B) = \partial \xi A \cap \partial \xi B \]

\[ \partial \xi (A \cap B) = \partial \xi A \cap \partial \xi B \]

**Proof.** We are going to use the following statement, 

\[ U \setminus (A \cap B) = (U \setminus A) \cap (U \setminus B) \quad \forall U \subset X. \]

So by 1.1.6 theorem (IO3) page 15 of [9],

\[ \text{int} (U \setminus (A \cap B)) = \text{int} (U \setminus A) \cap \text{int} (U \setminus B) \quad \forall U \subset X. \]

Going to the definition of interior and exterior boundaries, definition 3.3, the proof is finished. □

**Lemma 3.9.** Let $X$ be a topological space and $A, B \subset X$. Then, 

\[ \text{int} (\overline{A \cap B}) = \text{int} \overline{A} \cap \text{int} \overline{B}. \]

**Proof.** Using the definition of the closure,

\[
\begin{cases}
A \subset \overline{A} \\
B \subset \overline{B}
\end{cases}
\]

\[ \Rightarrow A \cap B \subset \overline{A \cap B} \Rightarrow \overline{A \cap B} \subset \overline{A} \cap \overline{B}. \]

Using now 1.1.6 theorem (IO3) page 15 of [9],

\[ \text{int} \overline{A \cap B} \subset \text{int} \overline{A} \cap \text{int} \overline{B} = \text{int} \overline{A} \cap \text{int} \overline{B}. \]
By corollary 3.7 by 1.1.6 theorem (IO3) page 15 of [9] and by lemma 3.8 we have,
\[
\text{int}(A \cap B) = (\text{int}(A) \cap \text{int}(B)) \cup (\partial I(A) \cap \partial I(B))
\]
\[
\supseteq (\text{int}(A) \cup \partial I(A)) \cap (\text{int}(B) \cup \partial I(B)) = \text{int}(A) \cup \partial I(A)
\]
\[
\supseteq \text{int}(A) \cap \text{int}(B).
\]
\[
\square
\]

Remark 3.10. Let \(Y, Z\) two topological spaces and \(\gamma : Y \to Z\) a continuous map. Take \(y_0 \in Y, z_0 = \gamma(y_0)\) and \(V_0\) an open neighborhood of \(z_0\). Then,
\[
y_0 \in \text{int}\gamma^{-1}(V_0 \setminus \{z_0\}).
\]
Let's prove this statement: denote \(U_0 = \gamma^{-1}(V_0)\). Then,
\[
y_0 \in U_0 = \gamma^{-1}(V_0).
\]
So,
\[
y_0 \in \overline{U_0 \setminus \{y_0\}} = \gamma^{-1}(V_0 \setminus \{z_0\}).
\]
Suppose that \(y_0 \in \partial\gamma^{-1}(V_0 \setminus \{z_0\})\). Therefore, by lemma 3.6 we have that, \(y_0 \in \partial\gamma^{-1}(V_0 \setminus \{z_0\})\). Then,
\[
\forall U \text{ an open neighborhood of } y_0 : \text{int}(U \setminus \gamma^{-1}(V_0 \setminus \{z_0\})) \neq \emptyset.
\]
But this is clearly false just taking \(U = U_0\) so we have a contradiction supposing that \(y_0 \in \partial\gamma^{-1}(V_0 \setminus \{z_0\})\). Using 1.3.2 theorem (i) and (ii) page 24 of [9] we finish.

At this point, we are going to use the topological theory of filterbases (see chapter X of [8]) to prove that the density of a function around a point implies continuity supposing some hypotheses.

Theorem 3.11. Let \(Y, Z\) two topological spaces and \(\gamma : Y \to Z\) a map. Suppose that \(\gamma : Y \setminus \{y_0\} \to Z \setminus \{z_0\}\) is continuous and \(\gamma(y_0) = z_0\). Let the filterbase,
\[
\mathcal{B} = \{U_0 \subset Y : U_0 \text{ is an open neighborhood of } y_0\}.
\]
Then,
\[
\mathcal{B}_\gamma = \{\gamma(U_0) \subset Z : U_0 \in \mathcal{B}\}
\]
is a filterbase that accumulates at \(z_0 \in Z\) and only at \(z_0 \in Z\) if and only if it converges to \(z_0 \in Z\) and only to \(z_0 \in Z\).

Proof. \(\Rightarrow\) This is 3.2 Theorem (1) page 214 of [8].
\(\Leftarrow\) We know that \(\mathcal{B}_\gamma\) is a filterbase by chapter X section 5 first paragraph of [8]. Take some \(V_0 \subset Z\) an open neighborhood of \(z_0\) and some \(U_0 \subset Y\) an open neighborhood of \(y_0\). Define,
\[
U = U_0 \cap \gamma^{-1}(V_0 \setminus \{z_0\}).
\]
For one hand, the set $U \subset Y$ is open because of remark 3.1 and because $\gamma : Y \setminus \{y_0\} \to Z \setminus \{z_0\}$ is continuous. Let’s see that $y_0 \in \text{int}U$. Using lemma 3.9

$$\text{int}U = \text{int}U_0 \cap \text{int}\gamma^{-1}(V_0 \setminus \{z_0\})$$.

For one hand, $y_0 \in U_0 \subset \text{int}U_0$ because $U_0$ is open and by corollary 3.7 on the other hand, $y_0 \in \text{int}\gamma^{-1}(V_0 \setminus \{z_0\})$ because of remark 3.10. Therefore, remembering remark 3.2, we obtain that $U \cup \{y_0\}$ is an open neighborhood of $y_0$. Then $\gamma(U \cup \{y_0\}) \in \mathcal{B}_\gamma$. As a consequence,

$$\gamma(U) \subseteq \gamma(U_0) \cap \gamma(\gamma^{-1}(V_0 \setminus \{z_0\})) \subseteq \gamma(U_0) \cap (V_0 \setminus \{z_0\}) \subseteq V_0,$$

so

$$\gamma(U \cup \{y_0\}) \subset V_0.$$ Therefore by 2.3 definition (1) page 212 of [8], the filterbase converges to $z_0$ and only to $z_0$ (because every converging point is an accumulation point).

We need now to deepen in the behaviour of the interior and closure of a set through the inverse image of a set by a continuous function.

**Lemma 3.12.** Let $Z$ be a topological space and $B \subset Z$ a set with empty interior. Then

$$\text{int}(Z \setminus B) = Z.$$  

**Proof.** As $B$ has empty interior, by 1.3.3 Proposition page 24 and 1.3.4 Theorem (i) page 25 of [9], $Z \setminus B = Z$. Now, remembering remark 3.4 and corollary 3.7, it is enough to proof that,

$$\partial E(Z \setminus B) = \emptyset.$$  

But that is easy to see using the definition of exterior boundary and the next deduction for all $V \subset Z$ open,

$$V \setminus (Z \setminus B) = V \cap B \Rightarrow \text{int}(V \setminus (Z \setminus B)) = \text{int}(V \cap B) \subset \text{int}B = \emptyset.$$

**Corollary 3.13.** Let $Y, Z$ be topological spaces, $\gamma : Y \to Z$ a continuous function and $B \subset Z$ a set with empty interior. Then,

$$\text{int}(\gamma^{-1}(Z \setminus B)) \subset \gamma^{-1}(\text{int}(Z \setminus B)).$$  

**Proof.** This is a direct consequence of lemma 3.12.  

**Lemma 3.14.** Let $Y, Z$ be topological spaces, $\gamma : Y \to Z$ a continuous function and $B \subset Z$. Then,

$$\partial E \gamma^{-1}(B) \subset \gamma^{-1}(\partial \gamma B).$$  

**Proof.** Using 1.3.2 Theorem (i) and (ii), we have that $\overline{B} = \text{int}B \cup \partial \overline{B}$. Now by 1.4.1 Proposition (v') and (vi) of [9] we deduce that $\partial \gamma^{-1}(\overline{B}) \subset \gamma^{-1}(\partial B)$. We finish with lemma 3.6.
**Corollary 3.15.** Let $Y, Z$ be topological spaces, $\gamma : Y \to Z$ a continuous function and $B \subset Z$ a set with empty interior. Then,

$$\gamma^{-1}(Z \setminus B) = \text{int}\gamma^{-1}(Z \setminus B)$$

**Proof.** Using 1.3.2 Theorem (i) and (ii) and lemma 3.6,

$$\gamma^{-1}(Z \setminus B) = \text{int}\gamma^{-1}(Z \setminus B) \cup \partial E \gamma^{-1}(Z \setminus B).$$

But by lemma 3.14 and identity (7) we have,

$$\partial E \gamma^{-1}(Z \setminus B) \subset \gamma^{-1}(\partial E Z \setminus B) = \gamma^{-1}(\emptyset) = \emptyset.$$

□

**Corollary 3.16.** Let $Y, Z$ be topological spaces, $Y$ connected, $\gamma : Y \to Z$ a continuous function and $B \subset Z$ a set with empty interior. Then,

$$\gamma^{-1}(\text{int}(Z \setminus B)) \subset \text{int}(\gamma^{-1}(Z \setminus B)).$$

**Proof.** This is a direct consequence of corollary 3.15 and 9.1 definition page 63 of [13]. □

So, as a summary of the last part of this section we obtain the next identity,

(8) $$\gamma^{-1}(\text{int}(Z \setminus B)) = Y = \text{int}(\gamma^{-1}(Z \setminus B)),$$

with $Y, Z$ topological spaces, $Y$ connected, $\gamma : Y \to Z$ a continuous function and $B \subset Z$ a set with empty interior. As a consequence of the last identity, we have the next,

**Theorem 3.17.** Let $Y, Z$ be topological spaces, $Y$ connected, $\gamma : Y \to Z$ a continuous function and $B \subset Z$ a set with empty interior. Then,

$$\text{int}\gamma^{-1}(B) = \emptyset.$$

**Proof.** It is enough to prove that,

(9) $$\gamma^{-1}(\text{int}B) = \text{int}\gamma^{-1}(B).$$

To see that, we apply identity (8) and the next aspect,

$$\gamma^{-1}(Z \setminus B) = Y = \gamma^{-1}(Z \setminus B),$$

and the definition of the interior of a set to obtain,

$$\gamma^{-1}(\text{int}B) = \gamma^{-1}(Z \setminus \text{int}B) = Y \setminus \gamma^{-1}(Z \setminus B)$$
$$= Y \setminus \gamma^{-1}(Z \setminus B) = Y \setminus Y \setminus \gamma^{-1}(B)$$
$$= \text{int}\gamma^{-1}(B).$$

□
4. Passings-through

In this section we are going to introduce a new concept in topology, the passings-through. This tool will help us in future sections to split topological spaces in a useful way.

**Definition 4.1.** Let $X$ be a topological space. Let $X_i \subset X$, with $i = 1, \ldots, M$ and $M \in \mathbb{N}$, disjoint subspaces endowed with the relative topology, such that $X = \bigcup_{i=1}^{M} X_i$ as a set identity. Denote $\hat{X} = \bigcup_{i=1}^{M} X_i$ the topological space endowed with the disjoint union topology (see [29] or [9] or [8]). Let $Y$ be another topological space and $\gamma : Y \to \hat{X}$ a continuous map. We will say that $y \in Y$ is a **passing-through** for $\gamma$ if $\gamma(y) \in \partial X_i$ for some $i \in \{1, \ldots, M\}$ and, for every $U \subset Y$ an open neighborhood of $y$, the set $\gamma(U)$ is not contained in only one subspace $X_i$. We will denote the set of all passings-through as $P(\gamma)$.

Our aim is to show that the set $P(\gamma)$ of passings-through is nowhere dense. To prove this we need first of all some results of general topology.

**Lemma 4.2.** Let $X$ be a topological space. Let $A, B \subset X$ disjoint close subsets of $X$, both with empty interior. Then $A \cup B$ has empty interior.

**Proof.** Take $x_1 \in A \cup B$. Like $A$ and $B$ have empty interior, we have that $x_1$ is an accumulation point of, at least, two of the next three sets, $A \quad B \quad X \setminus (A \cup B)$.

As $A \cap B = \emptyset$ and $A$ and $B$ are close, then the last three sets are pairwise disjoint. Therefore, there is no $x_2 \in X$ an accumulation point for $A$ and $B$ because, as they are close sets, by 1.3.4 theorem (i) page 25 of [9], we would conclude that $x_2 \in A \cap B = \emptyset$. So $x_1$ is always an accumulation point of $X \setminus (A \cup B)$, thus, by 1.3.4 theorem (i) page 25 of [9] and, by the definition of the boundary of a set,

$$\begin{cases} x_1 \in \overline{X \setminus (A \cup B)} \\ x_1 \in A \cup B = \overline{A \cup B} = \overline{A} \cup \overline{B} \end{cases} \Rightarrow x_1 \in \overline{A \cup B} \cap \overline{X \setminus (A \cup B)} = \partial (A \cup B).$$

By 1.3.2 theorem (x) and (iii) page 24 of [9],

$$\partial A \cup \partial B = A \cup B \subseteq \partial (A \cup B) \subseteq \partial A \cup \partial B,$$

so we conclude that $A \cup B$ has empty interior. □

**Corollary 4.3.** Let $X$ be a topological space. Let $M \in \mathbb{N}$ and $\{A_j\}_{j=1}^{M}$ a collection of close subsets of $X$ pairwise disjoint, with empty interior. Then the set $\bigcup_{j=1}^{M} A_j$ has empty interior.

**Proof.** The proof is analogous to the one of lemma 42. □

**Lemma 4.4.** Let $X$ be a topological space. Let $A, B \subset X$ disjoint subsets of $X$. Then $\partial A \cup \partial B$ has empty interior.
Proof. Take the next decomposition,
\begin{equation}
\partial A \cup \partial B = (\partial A \setminus \partial B) \cup (\partial B \setminus \partial A) \cup (\partial A \cap \partial B).
\end{equation}

The sets $\partial A \setminus \partial B$ and $\partial B \setminus \partial A$ are subsets of $\partial A$ and $\partial B$ respectively, i.e., they are subsets of sets with empty interior, thus, they have empty interior and therefore they are close by 1.3.2 theorem (i) and (ii) page 24 of [9]. Like the set $\partial A \cap \partial B$ is a subset of $\partial A$ (and also of $\partial B$), then it has empty interior so it is close. Therefore, (10) is a decomposition of $\partial A \cup \partial B$ as the union of three close subsets, pairwise disjoint and with empty interior, thus, using corollary 4.3 we finish. \hfill \Box

**Corollary 4.5.** Let $X$ be a topological space. Let $M \in \mathbb{N}$ and $\{A_j\}_{j=1}^M$ a collection of subsets of $X$ pairwise disjoint. Then the set $\bigcup_{j=1}^M \partial A_j$ has empty interior.

**Proof.** The proof is analogous to the one of lemma 4.4. \hfill \Box

**Corollary 4.6.** Let $X$ be a topological space. Let $X_i \subset X$, with $i = 1, \ldots, M$ and $M \in \mathbb{N}$, disjoint subspaces endowed with the relative topology, such that $X = \bigcup_{i=1}^M X_i$ as a set identity. Denote $\hat{X} = \bigcup_{i=1}^M X_i$ the topological space endowed with the disjoint union topology (see [29] or [9] or [8]). Let $Y$ be another connected topological space and $\gamma : Y \to \hat{X}$ a continuous map. Then the set of all passings-through, $\mathcal{P}(\gamma)$, has empty interior.

**Proof.** By the definition of $\mathcal{P}(\gamma)$ we have that,

$$
\text{Im}_\gamma(\mathcal{P}(\gamma)) \subseteq \bigcup_{i=1}^M \partial X_i.
$$

Then the set $\text{Im}_\gamma(\mathcal{P}(\gamma))$ has empty interior because so it has the set $\bigcup_{i=1}^M \partial X_i$ by corollary 4.5. We have,

$$
\mathcal{P}(\gamma) \subseteq \gamma^{-1}(\text{Im}_\gamma(\mathcal{P}(\gamma))rack]
$$

By theorem 3.17

$$
\text{int}(\gamma^{-1}(\text{Im}_\gamma(\mathcal{P}(\gamma)))) = \gamma^{-1}(\text{int}(\text{Im}_\gamma(\mathcal{P}(\gamma)))).
$$

We conclude that $\mathcal{P}(\gamma)$ has empty interior because it is a subset of a set with empty interior. \hfill \Box

**Lemma 4.7.** Let $X$ be a topological space. Let $X_i \subset X$, with $i = 1, \ldots, M$ and $M \in \mathbb{N}$, disjoint subspaces endowed with the relative topology, such that $X = \bigcup_{i=1}^M X_i$ as a set identity. Denote $\hat{X} = \bigcup_{i=1}^M X_i$ the topological space endowed with the disjoint union topology (see [29] or [9] or [8]). Let $Y$ be another connected topological space and $\gamma : Y \to \hat{X}$ a continuous map. Then $\mathcal{P}(\gamma)$ is nowhere dense in $Y$. 


Proof. Suppose not, i.e., there is an open subset $A \subset Y$ such that $\mathcal{P}(\gamma)$ is dense in $A$. By corollary \[13\] we have that every subset of $\mathcal{P}(\gamma)$ is close. Then, as $A$ is an open set, it has to be $\mathcal{P}(\gamma) \cap A \subsetneq A$. Take $y \in A \setminus \mathcal{P}(\gamma)$. By the definition of $\mathcal{P}(\gamma)$ we have two cases,

1. $\gamma(y) \in \partial X_i$ and $\gamma(U) \subset X_i$ for some $i \in \{1, \ldots, M\}$ and for some $U \subset Y$ an open neighborhood of $y$.

2. $\gamma(y) \notin \partial X_i$ for every $i \in \{1, \ldots, M\}$.

In the first case one has $\mathcal{P}(\gamma) \cap (U \cap A) = \emptyset$, with $U \cap A \neq \emptyset$, which is a contradiction with the hypotheses that $\mathcal{P}(\gamma)$ is dense in $A$.

In the second case $\gamma(y) \in \text{int}(X_i)$ for some $i \in \{1, \ldots, M\}$. Take $V \subset X$ an open neighborhood of $\gamma(y)$ contained in $X_i$. Like $\gamma$ is continuous then $\gamma^{-1}(V)$ is an open subset of $Y$ containing $y$. But $\gamma(y') \in \text{int}(X_i)$ for every $y' \in \gamma^{-1}(V)$, then $\mathcal{P}(\gamma) \cap (\gamma^{-1}(V) \cap A) = \emptyset$, with $\gamma^{-1}(V) \cap A \neq \emptyset$, and this is contradiction with the hypotheses that $\mathcal{P}(\gamma)$ is dense in $A$. \hfill $\square$

Let’s prove in the next lemma that a continuous function has to stay in one subspace in the case that it has no passing-through.

**Lemma 4.8.** Let $X$ be a topological space. Let $X_i \subset X$, with $i = 1, \ldots, M$ and $M \in \mathbb{N}$, disjoint subspaces endowed with the relative topology, such that $X = \bigcup_{i=1}^{M} X_i$ as a set identity. Denote $\hat{X} = \bigcup_{i=1}^{M} X_i$ the topological space endowed with the disjoint union topology (see [29] or [9] or [8]). Take $Y$ a connected Hausdorff topological space and $\gamma : Y \rightarrow \hat{X}$ a continuous function. Suppose $\mathcal{P}(\gamma) = \emptyset$. Then $\gamma(y) \in X_i$ for some fixed $i \in \{1, \ldots, M\}$ and for every $y \in Y$.

**Proof.** Suppose that the image of $Y$ by $\gamma$ is, at least, in two different subspaces. Take $Y = \bigcup_{i=1}^{M} Y_i$, as a set identity, with $Y_i = \gamma^{-1}(X_i)$. As $X_i \neq \emptyset$, $i = 1, \ldots, M$, and $X_i \cap X_{i_2} = \emptyset$, for every $i_1, i_2 \in \{1, \ldots, M\}$ with $i_1 \neq i_2$, then $Y_i \neq \emptyset$ for at least two subindexes, and $Y_i \cap Y_{i_2} = \emptyset$, for every $i_1, i_2 \in \{1, \ldots, M\}$ and $i_1 \neq i_2$. We can distinguish two cases,

1. $\exists y \in Y$ such that $\gamma(y) \in \partial X_i$ for some $i \in \{1, \ldots, M\}$

2. $\forall y \in Y \gamma(y) \in \text{int}(X_i)$ for some $i \in \{1, \ldots, M\}$.

Let’s obtain a contradiction in the first case. If $\gamma(y) \in \partial X_i$ and $\gamma(U) \not\subset X_i$, for every $U$ an open neighborhood of $y$, then the image by $\gamma$ of every open neighborhood of $y$ contains points of $X_i \cap \gamma(Y)$ and points of $X_j \cap \gamma(Y)$ with $i \neq j$; in that case we would have $y \in \mathcal{P}(\gamma)$ and this is a contradiction with the hypotheses $\mathcal{P}(\gamma) = \emptyset$. Therefore, we conclude that $\gamma(y) \in \partial X_i$, $\forall y \in Y$, and it exists $U$, an open neighborhood of $y$, such that $\gamma(U) \subset X_i$; so on, $\gamma(Y)$ would not lie in two different $X_i$.  


Let’s see the second case. For every \( y \in Y \) such that \( \gamma(1) \notin \text{int}(X_i) \) take \( V_{\gamma(y)} \subset X_i \), an open neighborhood of \( \gamma(y) \) in \( X \). Define,
\[
\tilde{X}_i = \bigcup_{y \in Y_i} V_{\gamma(y)} \quad \text{for every } i \in \{1, \ldots, M\}.
\]
By definition \( \tilde{X}_i \) is open for every \( i \in \{1, \ldots, M\} \) and by hypotheses there exist, at least, two subindexes \( i \neq j \) for which \( \tilde{X}_i \neq \emptyset \) and, furthermore, \( \tilde{X}_i \cap \tilde{X}_j = \emptyset \), for every \( i, j \in \{1, \ldots, M\} \) with \( i \neq j \). All in all,
\[
\gamma(Y) = \bigcup_{i=1}^{M} \tilde{X}_i,
\]
but this is impossible because the left member is connected as \( Y \) is connected and \( \gamma \) continuous, and the right member is disconnected by construction. \( \square \)

5. Homotopy Lifting Property in \( SP_m(X) \)

To fix the notation, if \( X \) is a topological space and \( U \subset X \) is an open subset, we will denote its interior as \( \text{int}_X(U) \) or just \( \text{int}(U) \), its boundary as \( \partial_X(U) \) or just \( \partial U \), its closure as \( \text{Cl}_X(U) \) or just \( \text{Cl}(U) \) or \( \overline{U} \) and its complement as \( U^c \).

First of all, we are going to prove that the map \( \varphi \) defined in (11), is continuous, open and onto. For every \( \sigma \in \Sigma_m \) and every \( x = (x_1, \ldots, x_m) \in X^m \) we will usually denote \( x_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \).

**Lemma 5.1.** Let \( \varphi \) be the map defined by,
\[
\varphi : X^m \to SP_m(X) \quad x \mapsto \varphi(x) = [x]_{\varphi},
\]
with \([x]_{\varphi}\) defined in (4). Then \( \varphi \) is continuous, open, closed and onto.

**Proof.** Take \( \sigma \in \Sigma_m \) and define the map \( \varphi_\sigma \) as,
\[
\varphi_\sigma : X^m \to X^m \quad x = (x_1, \ldots, x_m) \mapsto \varphi_\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)}).
\]
We want to prove that \( \varphi_\sigma \) is a homeomorphism.

\( \square \) \( \varphi_\sigma \) is bijective. Suppose \( \varphi_\sigma(x) = \varphi_\sigma(x') \), so \( x_{\sigma(j)} = x'_{\sigma(j)} \) for all \( j = 1, \ldots, m \). Then \( x_j = x'_j \) for all \( j = 1, \ldots, m \), so \( x = x' \) thus \( \varphi_\sigma \) is injective. Take \( x' = (x'_1, \ldots, x'_m) \in X^m \). As \( \Sigma_m \) is a group then it exists an identity element \( \sigma_0 \) and for every \( \sigma \in \Sigma_m \) it exists \( \sigma^{-1} \in \Sigma_m \) such that \( \sigma \sigma^{-1} = \sigma_0 \). Defining \( x = (x'_{\sigma^{-1}(1)}, \ldots, x'_{\sigma^{-1}(m)}) \) we have,
\[
\varphi_\sigma(x'_{\sigma^{-1}(1)}, \ldots, x'_{\sigma^{-1}(m)}) = (x'_{\sigma_0^{-1}(1)}, \ldots, x'_{\sigma_0^{-1}(m)}) = (x'_1, \ldots, x'_m),
\]
thus \( \varphi_\sigma(x) = x' \) for \( x = (x'_{\sigma^{-1}(1)}, \ldots, x'_{\sigma^{-1}(m)}) \) and \( \varphi_\sigma \) is onto.

\( \square \) \( \varphi_\sigma \) is continuous. Take \( V \subset X^m \) an open subset. Then we can write \( V = U_1 \times \cdots \times U_m \) for some \( U_j \subset X \), \( j = 1, \ldots, m \), open subsets.
Take $V_{\sigma^{-1}} = U_{\sigma^{-1}(1)} \times \cdots \times U_{\sigma^{-1}(m)}$. $V_{\sigma^{-1}}$ is open because so they are $U_j$, $j = 1, \ldots, m$. Moreover, it is not difficult to prove that $V_{\sigma^{-1}} = \varphi_{\sigma}^{-1}(V)$, then $\varphi_{\sigma}$ is a continuous map.

$[3]$ $\varphi_{\sigma}$ is an open map. Take $V \subset X^m$ an open subset. Then we can write $V = U_1 \times \cdots \times U_m$ for some $U_j \subset X$, $j = 1, \ldots, m$, open subsets. Take $V_{\sigma} = U_{\sigma(1)} \times \cdots \times U_{\sigma(m)}$. $V_{\sigma}$ is open because so they are $U_j$, $j = 1, \ldots, m$. Moreover, it is not difficult to prove that $\varphi_{\sigma}(V) = V_{\sigma}$, then $\varphi_{\sigma}$ is an open map.

Now let’s prove the identity,

$$\varphi^{-1}(\varphi(V)) = \bigcup_{\sigma \in \Sigma_m} \varphi_{\sigma}(V),$$

for every $V \subset X^m$. As the sets of the two members of (12) are into the same topological space endowed with the same topology, we just have to prove the equality between the two sets.

$[C]$ Take $x = (x_1, \ldots, x_m) \in \varphi^{-1}(\varphi(V))$. Then it exists $\sigma \in \Sigma_m$ such that $(x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \in V$. But then $\varphi_{\sigma}(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)}) \in V$. As $\varphi_{\sigma}$ is a homeomorphism by the first part of the proof we have $x \in \varphi_{\sigma}^{-1}(V) = \varphi_{\sigma^{-1}}(V)$ so,

$$x \in \bigcup_{\sigma \in \Sigma_m} \varphi_{\sigma}(V) \Rightarrow \varphi^{-1}(\varphi(V)) \subset \bigcup_{\sigma \in \Sigma_m} \varphi_{\sigma}(V).$$

$[D]$ Take $x = (x_1, \ldots, x_m) \in \bigcup_{\sigma \in \Sigma_m} \varphi_{\sigma}(V)$. Then it exists $\sigma \in \Sigma_m$ such that $x = (x_1, \ldots, x_m) \in \varphi_{\sigma}(V)$. As $\varphi_{\sigma}$ is a homeomorphism by the first part of the proof we have $\varphi_{\sigma}^{-1}(x) = \varphi_{\sigma^{-1}}(x) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(m)}) \in V$. Like $\varphi \circ \varphi_{\sigma} = \varphi$ for every $\sigma \in \Sigma_m$, then $\varphi(x) \in \varphi(V)$. As $\varphi(V)$ is a saturated set then $x \in \varphi^{-1}(\varphi(V))$, and we conclude that,

$$\bigcup_{\sigma \in \Sigma_m} \varphi_{\sigma}(V) \subset \varphi^{-1}(\varphi(V)).$$

Thus, we have (12). As we have that $\varphi$ is continuous and onto by definition [1] putting together (12) and the fact that $\varphi_{\sigma}$ is a homeomorphism for every $\sigma \in \Sigma_m$ we conclude that $\varphi$ is also an open map.

Lemma [5.1] tells us that $\varphi$ is a homeomorphisms into every subspace of $X^m$ where it is injective and remains onto. Our work now consists on splitting $X^m$ into subspaces which we are going to use as pieces of a puzzle, so that, combined in a correct way, we can obtain a subspace of $X^m$ whom ”symmetric product” can be apply by a continuous and bijective map to $SP_m(X)$.

**Definition 5.2.** Let $X$ be a set and $m \in \mathbb{N}$. Define the subset $X_0 \subset X^m$ as,

$$X_0 = \{x = (x_1, \ldots, x_m) \in X^m : x_{j_1} \neq x_{j_2} \ \forall j_1, j_2 \in \{1, \ldots, m\}, j_1 \neq j_2\}.$$
We will denote $X_0$ as the **principal piece**. When needed we will use the notation $X_0^{(m)}$ to denote the principal piece of the cartesian product $X^m$.

**Remark 5.3.** Take $X$ a Hausdorff topological space, take $m \in \mathbb{N}$ and $X^m$ endowed with the product topology, and $X_0 \subset X^m$ endowed with the relative topology. As $X$ is Hausdorff, it is easy to prove that $X_0$ is an open subspace. Suppose now $X$ is locally path-connected. Then $X^m$ is locally path-connected and, as $X_0$ is an open subspace, it is also locally path-connected.

**Definition 5.4.** Let $X$ be a set and $k, m \in \mathbb{N}$, with $k \leq m$. Take $x = (x_1, \ldots, x_m) \in X^m$ and $j_i \in \mathbb{N}$, $j_i \geq 2$, for $i = 1, \ldots, k$ and $j_1 + \cdots + j_k \leq m$. We will say that $x$ is of **primitive type** $j_1 \ldots j_k$ if,

$$
x_j = x_1 \quad \forall j = 1, \ldots, j_1
$$

$$
x_j = x_{j_1+j_2} \quad \forall j = j_1 + 1, \ldots, j_1 + j_2
$$

$$
\vdots
$$

$$
x_j = x_{j_1+\cdots+j_k} \quad \forall j = (j_1 + \cdots + j_{k-1}) + 1, \ldots, j_1 + \cdots + j_k
$$

$$
x_1 \neq x_{j_1+j_2} \neq x_{j_1+j_2+j_3} \neq \ldots \neq x_{j_1+\cdots+j_{k-1}} \neq x_{j_1+\cdots+j_k}
$$

$$
x_r \neq x_s \quad \forall r, s \in \{j_1 + \cdots + j_k + 1, \ldots, m\} \quad \forall r \neq 1 \in \{1, \ldots, m\} \setminus \{r_1\}
$$

Now, denote $\Sigma_m$ the group of permutations of the set $\{1, \ldots, m\}$ and take $\sigma \in \Sigma_m$. We will say that $x' = (x'_1, \ldots, x'_m) \in X^m$ is of $\sigma-$**type** $j_1 \ldots j_k$ if it exists $x = (x_1, \ldots, x_m) \in X^m$ of **primitive type** $j_1 \ldots j_k$ such that $x' = (x'_1, \ldots, x'_m) = (x_{\sigma(1)}, \ldots, x_{\sigma(m)}) = x_{\sigma}$. Particularly, if we denote $e$ the identity element of $\Sigma_m$, then an element of $e-$**type** $j_1 \ldots j_k$ is in fact an element of **primitive type** $j_1 \ldots j_k$.

**Definition 5.5.** Let $X$ be a topological space and $m, n \in \mathbb{N}$, with $n \leq m$. Take $x = (x_1, \ldots, x_m) \in X^m$ and $j_i \in \mathbb{N}$, $j_i \geq 2$, for $i = 1, \ldots, k$ and $j_1 + \cdots + j_k \leq m$. Define the next subspace,

$$(13) \quad X_{j_1 \ldots j_k} = \{x \in X^m : x \text{ is of primitive type } j_1 \ldots j_k\},$$

endowed with the relative topology. We will name $X_{j_1 \ldots j_k}$ a **primitive k-piece** of $X^m$. We will denote,

$$(14) \quad X_{j_1 \ldots j_k, \sigma} = \{x \in X^m : \text{it exists } x' \text{ of primitive type } j_1 \ldots j_k \text{ such that } x = x'_{\sigma}\},$$

a **$\sigma$ k-piece**. When needed, we will use the notation $X_{j_1 \ldots j_k}^{(m)}$ and $X_{j_1 \ldots j_k, \sigma}^{(m)}$ to denote a primitive k-piece and a $\sigma$ k-piece, respectively, of the cartesian product $X^m$. 

Remark 5.6. It is an easy exercise to prove that taking two primitive pieces \(X_{j_1 \cdots j_k}\) and \(X'_{j'_1 \cdots j'_{k'}}\), then we have,

\[
X_{j_1 \cdots j_k} \cap X'_{j'_1 \cdots j'_{k'}} \neq \emptyset \iff \begin{cases} 
k = k' 
\end{cases} \begin{cases} 
j_i = j'_i \forall i = 1, \ldots, k.
\end{cases}
\]

Definition 5.7. We define the primitive big puzzle like the subspace,

\[
BPuzzle(X^m) = \bigcup_{k=0}^{m} \bigcup_{\substack{j_i = 2 \leq m \\sum j_i = m \\sum j_i \leq j_k}} X_{j_1 \cdots j_k},
\]

endowed with the disjoint union topology (see [29] or [9] or [8]), understanding that for \(k = 0\), we are including into the union the principal piece \(X_0\). Fixing \(\sigma \in \Sigma_m\), we define the \(\sigma\)-big puzzle,

\[
BPuzzle(X^m, \sigma) = \bigcup_{k=0}^{m} \bigcup_{\substack{j_i = 2 \leq m \\sum j_i = m \\sum j_i \leq j_k}} X_{j_1 \cdots j_k, \sigma},
\]

endowed with the disjoint union topology too. Particularly we can define a new topological space \(\widehat{X}^m\),

\[
\widehat{X}^m = \bigcup_{\sigma \in \Sigma_m} BPuzzle(X^m, \sigma),
\]

endowed with the disjoint union topology.

It is easy to see that \(X^m\) and \(\widehat{X}^m\) are equal as sets but their topologies are a bit different. Let’s describe the topological relation between \(X^m\) and \(\widehat{X}^m\).

Lemma 5.8. Let \(X = (X, \mathcal{X}_X)\) be a topological space, \(A\) a set of indexes and \(X_\alpha \subset X\) pairwise disjoint subspaces endowed with the relative topology, \(\alpha \in A\). Let \(\widehat{X} = (\bigcup_{\alpha \in A} X_\alpha, \mathcal{X}_A)\) be a topological space endowed with the disjoint union topology. Then,

(a) \(\mathcal{X}_X \subset \mathcal{X}_A\).
(b) Define,

\[
\pi : \widehat{X} \rightarrow X,
\]

\[
x \mapsto \pi(x) = x.
\]

Then \(\pi\) is bijective, continuous and for every \(V \in \mathcal{X}_X\), \(\pi(V) = V \in \mathcal{X}_X\).

(c) Define,

\[
i : X \rightarrow \widehat{X},
\]

\[
x \mapsto i(x) = x.
\]

Then \(i\) is bijective, open and for every \(V \in \mathcal{X}_X\), \(i^{-1}(V) = V \in \mathcal{X}_X\).
(d) Let $Y$ be another topological space. Take $\hat{f} : Y \to \hat{X}$ a function denoting $\hat{f}(y) = x$. Then $\hat{f}$ induces a function $f : Y \to X$, with $f(y) = x$, such that:
- If $\hat{f}$ is bijective then $f$ is bijective.
- If $\hat{f}$ is continuous then $f$ is continuous.
- If $\hat{f}$ is open then $f$ is open.

(e) Every homeomorphism $\hat{f} : \hat{X} \to \hat{X}$ induces a homeomorphism $f : X \to X$ such that $\hat{f}|_X = i \circ f$.

(f) Every homeomorphism $f : X \to X$ induces a homeomorphism $\hat{f} : \hat{X} \to \hat{X}$ such that $\hat{f}|_X = i \circ f$.

Proof. To see (a) we just need the following equality for every $V \in \mathcal{T}_X$,
$$V = \bigcup_{a \in A} (V \cap X_a) \in \mathcal{T}_A.$$

Now (b) and (c) are direct consequences of (a). To prove (d) it is enough to define $f = \pi \circ \hat{f}$ and use (b). To prove (e), from $\hat{f}$ we define $f = \pi \circ \hat{f} \circ i$ and use (b) and (c). Finally, to prove (f), from $f$ we define $\hat{f} = i \circ f \circ \pi$ and use (b) and (c). \qed

Remark 5.9. Using the partition of $X^m$ in big puzzles one may take care of the difference between the product topology of $X^m$ and the disjoint union topology of $\hat{X}^m$. We will usually make the abuse of notation of denoting with the same letter the functions $\hat{f}$ and $f$ showed in lemma 5.8. This will cause no problem thanks to that lemma.

Remark 5.10. We can not prove that the continuous and bijective map $\pi$ introduce in lemma 5.8 is always a homeomorphism. However, there are cases when it is. For example, when $X$ is Hausdorff and the disjoint union is compact (see theorem 8.8 page 58 of [14]).

Notation 1. From definition 5.7 one can define two types of big puzzles: for one hand we will denote big undone puzzle, $BUPuzzle(X^m)$, to the disjoint union endowed with the disjoint union topology; on the other hand we will denote big done puzzle, $BDPuzzle(X^m)$, to the disjoint union endowed with the relative topology. We will always use the notation $BPuzzle(X^m)$ to denote the big undone puzzle.

Let’s see some relations between pieces.

Lemma 5.11. Let $X$ be a topological space and $m \in \mathbb{N}$. Then for every primitive $k$-piece $X^{(m)}_{j_1 \ldots j_k}$,
$$X^{(m)}_{j_1 \ldots j_k} \cong X^{(m-(j_1+\ldots+j_k)+k)}.$$

Proof. Define the projection $\pi_{j_1 \ldots j_k} : X^m \to X^{m-(j_1+\ldots+j_k)+k}$ for $x = (x_1, \ldots, x_m)$ like,
$$\pi_{j_1 \ldots j_k}(x) = (x_1, x_{j_1+1}, x_{j_1+j_2+1}, \ldots, x_{j_1+\ldots+j_m+1}, x_{j_1+\ldots+j_m+2}, \ldots, x_m).$$
As \( \pi_{j_1 \cdots j_k} \) is a projection then is continuous and open. Moreover, by definition 5.5 it is easy to prove that \( \pi_{j_1 \cdots j_k} |_{X_{j_1 \cdots j_k}}^{(m)} : X_{j_1 \cdots j_k}^{(m)} \rightarrow X_0^{(m-(j_1+\cdots+j_k)+k)} \) is bijective. \( \square \)

**Remark 5.12.** By definition of \( \phi \) (see [3]) one has, looking at lemma 5.11 that,
\[
\phi(x) = \phi(\pi_{j_1 \cdots j_k}(x)) \quad \forall x \in X_{j_1 \cdots j_k}^{(m)}.
\]

**Lemma 5.13.** Let \( X \) be a topological space and \( m \in \mathbb{N} \). Take two primitive pieces \( X_{j_1 \cdots j_k} \) and \( X_{j'_1 \cdots j'_{k'}} \). If \( j_1 + \cdots + j_k - k = j'_1 + \cdots + j'_{k'} - k' \) then,
\[
X_{j_1 \cdots j_k} \cong X_{j'_1 \cdots j'_{k'}}.
\]

**Proof.** It is enough to use lemma 5.11 \( \square \)

**Remark 5.14.** By remark 5.12 looking at lemma 5.13 we have that,
\[
\phi(x_1) = \phi(x_2),
\]
for every \( x_1, x_2 \in X^m \) such that \( \pi_{j_1 \cdots j_k}(x_1) = \pi_{j_1 \cdots j_k}(x_2) \) for some \( k \in \{1, \ldots, m\} \) and for some \( j_i \in \mathbb{N}, j_i \geq 2, \) for \( i = 1, \ldots, k \) and \( j_1 + \cdots + j_k \leq m \).

**Definition 5.15.** Let \( X \) be a topological space and \( m \in \mathbb{N} \). Take two \( \sigma \) \( k \)-pieces \( X_{j_1 \cdots j_k, \sigma} \) and \( X_{j'_1 \cdots j'_{k'}, \sigma} \). Define the next relation,
\[
\mathcal{R} X_{j_1 \cdots j_k, \sigma} \leftrightarrow \mathcal{R} X_{j'_1 \cdots j'_{k'}, \sigma} \Leftrightarrow j_1 + \cdots + j_k - k = j'_1 + \cdots + j'_{k'} - k'.
\]

This is an equivalent relation. Moreover, remembering homeomorphisms \( \varphi_{\sigma} \) from (11) and using lemma 5.13 we have that two \( \sigma \) \( k \)-pieces that are related by \( \mathcal{R} \) are homeomorphic.

The primitive \( k \)-piece \( X_{j_1 \cdots j_k} \) is homeomorphic to the \( \sigma \) \( k \)-piece \( X_{j_1 \cdots j_k, \sigma} \) by \( \varphi_{\sigma} \) defined in (11) so \( X_{j_1 \cdots j_k, \sigma_1} \) and \( X_{j_1 \cdots j_k, \sigma_2} \) are homeomorphic for every \( \sigma_1, \sigma_2 \in \Sigma_m \). The same deduction can be done for the \( \sigma \)-big puzzles so that we obtain that \( BPuzzle(X^m, \sigma_1) \) is homeomorphic to \( BPuzzle(X^m, \sigma_2) \) (by the homeomorphism \( \varphi_{\sigma_2 \sigma_1}^{-1} \)) for every \( \sigma_1, \sigma_2 \in \Sigma_m \). In fact, one can obtain new big puzzles just changing a \( \sigma \) \( k \)-piece \( X_{j_1 \cdots j_k, \sigma} \) by another one homeomorphic to it, and this new big puzzle would be homeomorphic to the other ones just using the universal property of the disjoint union topology (see [29] or [9] or [8]).

One has a natural action of the group \( \Sigma_{m-(j_1+\cdots+j_k)+k} \) on a primitive big \( k \)-piece \( X_{j_1 \cdots j_k} \), defined like this,
\[
(19) \quad \theta_{j_1 \cdots j_k} : X_{j_1 \cdots j_k} \ni \frac{x}{x} \mapsto \theta_{j_1 \cdots j_k}(x) = \{x\}_{j_1 \cdots j_k},
\]
defining from \( x = (x_1, \ldots, x_m) \),
\[
[x]_{j_1 \cdots j_k} = \{x' = (x'_1, \ldots, x'_m) \in X_{j_1 \cdots j_k} : \text{ for some } \sigma \in \Sigma_{m-(j_1+\cdots+j_k)+k} \}
\]
\[
(x'_1, \ldots, x'_m) = (x_1, \ldots, x_{j_1-1}, x_{j_1}(\sigma_{j_1}), x_{j_1+1}, \ldots, x_{j_1+j_2-1}, x_{\sigma(j_1+j_2)},
\]
\[
\ldots, x_{j_1+\cdots+j_k-1}, x_{\sigma(j_1+\cdots+j_k)}, x_{\sigma(j_1+\cdots+j_k+1)}, \ldots, x_{\sigma(m)})\}.
\]
defining the action of $\sigma$ over a subindex $j_1 + \cdots + j_i$ like $\sigma(j_1 + \cdots + j_i) = \sigma(i)$, for every $i = 1, \ldots, k$, and over a subindex $j_1 + \cdots + j_k + r$ like $\sigma(j_1 + \cdots + j_k + r) = \sigma(k + r)$ for every $r = 1, \ldots, m - (j_1 + \cdots + j_k)$.

Moreover, when $X$ is Hausdorff, the map $\theta$ is a covering map because it is defined as a free action of a finite group over a Hausdorff space (see theorems 17.1 and 17.2 page 154 of [14]).

**Definition 5.16.** Let $X$ be a topological space and $m \in \mathbb{N}$. From definitions 5.5 and (19) we define the topological space,

$$BPuzzle(X^m) = \bigcup_{k=0}^{m} \bigcup_{j_i = 2}^{m} \left( \frac{X_{j_1, \ldots, j_k}}{\Sigma_{m - (j_1 + \cdots + j_k) + k}} \right),$$

endowed with the disjoint union topology (see [29] or [9] or [8]), understanding that for $k = 0$, we are including into the union the quotient space $X_0/\Sigma_m$. Fixing $\sigma \in \Sigma_m$, we define,

$$BPuzzle(X^m, \sigma) = \bigcup_{k=0}^{m} \bigcup_{j_i = 2}^{m} \left( \frac{X_{j_1, \ldots, j_k, \sigma}}{\Sigma_{m - (j_1 + \cdots + j_k) + k}} \right),$$

endowed with the disjoint union topology too.

From covering maps (19) and definition 5.16 one can define the following function gluing them with the universal property of the disjoint union topology (see [29] or [9] or [8]),

$$\theta : BPuzzle(X^m) \to BPuzzle(X^m) \quad x \mapsto \theta(x) = [x]_{\sigma} = [x]_{\theta_{j_1, \ldots, j_k}}.$$

The map $\theta$ is continuous and onto. Let’s see that it is also open.

**Lemma 5.17.** Let $X, Y$ be topological spaces. Let $\{X_j\}_{j \in J}$ and $\{Y_j\}_{j \in J}$ be two families of pairwise disjoint topological subspaces of $X$ and $Y$ respectively. Let $\widehat{X} = \cup_{j \in J} X_j$ and $\widehat{Y} = \cup_{j \in J} Y_j$ be topological spaces both endowed with the disjoint union topology. Suppose we have maps $f_j : X_j \to Y_j$ for every $j \in J$. From them, define the map,

$$f : \widehat{X} \to \widehat{Y} \quad x \mapsto f(x) = f_j(x),$$

Then, $f$ is open $\iff f_j$ is open $\forall j \in J$.

**Proof.** The right implication is obvious. Let’s see the left one. By an easy argument (see [29] or [9] or [8]) one has that the inclusions $i_j : Y_j \to \widehat{Y}$ are open (and, in fact, close too). Take $U \subset \widehat{X}$ an open
subset. Then,
\[ f(U) = f(\bigcup_{j \in J} (U_j \cap X_j)) = \bigcup_{j \in J} f(U_j \cap X_j) = \bigcup_{j \in J} \text{i}_j(f(U_j)), \]
that is a union of open sets in \( \tilde{X} \).

**Definition 5.18.** Let \( X \) be a topological space and \( n \in \mathbb{N} \). A continuous map \( \gamma : [0,1]^n \to X \) will be denoted as an \( n \)-region.

From this point we will lead our efforts to prove that the lifting of an \( n \)-region in \( SP_m(X) \) to another one in \( X^m \) is equivalent to the lifting of an \( n \)-region in \( B\text{Puzzle}(X^m) \) to another one in \( B\text{Puzzle}(X^m) \).

**Lemma 5.19.** Let \( X \) be a Hausdorff topological space, \( m \in \mathbb{N} \). Then it exists a continuous and bijective map \( f \),

\[
\begin{align*}
B\text{Puzzle}(X^m) & \xrightarrow{\theta} \widetilde{B\text{Puzzle}}(X^m) \\
\phi & \mapsto \phi,
\end{align*}
\]

such that for every open subset \( V \subset B\text{D Puzzle}(X^m) \), \( f(\theta(\pi^{-1}(V))) \) is an open subset of \( SP_m(X) \), being \( \theta \) the map defined in (24) and \( \pi \) defined in lemma 5.8 (b).

**Proof.** Denoting \( \chi = \phi|_{BD\text{Puzzle}(X^m)} \), consider the next diagram,

\[
\begin{array}{ccc}
B\text{D Puzzle}(X^m) & \xrightarrow{\chi} & SP_m(X) \\
\pi & \downarrow \quad \downarrow \frown & \\
B\text{Puzzle}(X^m) & \xrightarrow{f} & \widetilde{B\text{Puzzle}}(X^m)
\end{array}
\]

with,

\[
\begin{align*}
f : \widetilde{B\text{Puzzle}}(X^m) & \to SP_m(X) \\
[x]_{\theta} & \mapsto f([x]_{\theta}) = [x]_{\varphi},
\end{align*}
\]

\( \pi \) the continuous and bijective map defined in lemma 5.8 (b) and \( \chi \) the restriction of \( \varphi \) (see definition 11). The map \( \theta \) is well defined because of (15) and \( f \) is well defined because \( [x]_{\theta} \subset [x]_{\varphi} \subset X^m \) for all \( x \in X^m \).

From diagram (24), lemma 5.8 (b) and definitions (23) and (24) we get \( \chi \circ \pi = f \circ \theta \). We know that \( \varphi \) is continuous, so \( \chi \) is continuous, and \( \pi \) is continuous. With the last equality, we just need to know that \( \theta \) is continuous to conclude the same for \( f \). But \( \theta \) is continuous by the universal property of the disjoint union topology (see [29] or [9] or [8]) so it is \( f \).

Let’s see now \( f \) is bijective. By decomposition (18), by lemma 5.8 and as the \( \sigma \)-big puzzles are homeomorphic among them (by the paragraph that follows definition 5.15), for every \( [x]_{\varphi} \in SP_m(X) \) it exists \( x' \in [x]_{\varphi} \cap B\text{Puzzle}(X^m) \), so \( [x']_{\varphi} = [x]_{\varphi} \) and we obtain that \( f \) is onto. Suppose now \( [x]_{\varphi} = [x']_{\varphi} \) for some \( x, x' \in B\text{Puzzle}(X^m) \); then it exists \( \sigma \in \Sigma_m \) such that \( x' = x_{\sigma} \). As \( x' \in B\text{Puzzle}(X^m) \) then it
has to be \( \sigma \in \Sigma_{m-(j_1+\ldots+j_k)+k} \) (for some \( k \in \{1, \ldots, m\} \)) and for some \( j_i \in \mathbb{N}, j_i \geq 2, \) for \( i = 1, \ldots, k \) and \( j_1 + \ldots + j_k \leq m \) and, in this case \( [x]_\theta = [x']_\theta \), concluding that \( f \) is injective.

Finally, let’s prove that for every open subset \( V \subset BDPuzzle(X^m) \), \( f(\theta(\pi^{-1}(V))) \) is an open subset of \( SP_m(X) \). But by the previous paragraphs \( \chi \circ \pi = f \circ \theta \). Thus \( f(\theta(\pi^{-1}(V))) = \chi(V) \) and \( \chi \) is open. \( \square \)

With lemma 5.19 and with the next diagram,

\[
\begin{array}{cccccc}
Y \xrightarrow{g_1} & X^m \xrightarrow{i} & BDPuzzle(X^m) \xrightarrow{\pi} & BUPuzzle(X^m) \\
g \downarrow \quad & \quad \varphi \downarrow \quad & \quad \theta \downarrow \\
SP_m(X) \xrightarrow{f} & BPUZZLE(X^m) & \\
\end{array}
\]

one can reduce the proof of the homotopy lifting property for \( SP_m(X) \) to the proof of the homotopy lifting property for \( \widetilde{BPuzzle}(X^m) \). More precisely, we just need to lift a path in \( \widetilde{BPuzzle}(X^m) \) to a path in \( BUPuzzle(X^m) \) to get the lift of a path in \( SP_m(X) \) to a path in \( X^m \). In the last diagram, \( Y \) is a topological space, \( g_1 \) is a continuous map in \( X^m \), \( g_2 \) is a continuous map in \( BUPuzzle(X^m) \), \( \tilde{g} \) is a continuous map in \( SP_m(X) \), \( \varphi \) is the map defined in \( (1) \), \( \pi \) the continuous and bijective map defined in lemma \( 5.8 \) (b), \( \theta \) is the map defined in \( (22) \), \( f \) is the continuous and bijective map defined in \( (25) \) and \( i \) is the natural inclusion of \( BDPuzzle(X^m) \) in \( X^m \).

**Lemma 5.20.** Let \( X \) and \( Y \) be Hausdorff topological spaces and \( m \in \mathbb{N} \). Let \( \tilde{g} : Y \to SP_m(X) \) a continuous map in \( SP_m(X) \). Then,

It exists a continuous map \( g_1 : Y \to X^m \) such that \( \tilde{g} = \varphi \circ g_1 \).

Proof. Remembering that the next diagram is commutative,

\[
\begin{array}{cccccc}
X^m \xrightarrow{i} & BDPuzzle(X^m) \xrightarrow{\pi} & BUPuzzle(X^m) \\
\varphi \downarrow \quad & \quad \theta \downarrow \\
SP_m(X) \xrightarrow{f} & BPUZZLE(X^m) & \\
\end{array}
\]

we obtain,

\( \tilde{g} = f \circ \theta \circ g_2 \iff \tilde{g} = \varphi \circ i \circ \pi \circ g_2 \iff \tilde{g} = \varphi \circ g_1 \).

\( \square \)

So now, our efforts will be focused to the proof of the homotopy lifting property for \( BPUZZLE(X^m) \), because the map \( f|_{BDPUZZLE(X^m)} \) is
a homeomorphism by lemma 5.19. We are going to work with the next commutative diagram, for $n \in \mathbb{N}$,

\[
\begin{array}{ccc}
[0,1]^n & \xrightarrow{\gamma} & BPuzzle(X^m) \\
\downarrow{\tilde{\gamma}} & & \downarrow{\theta} \\
BPuzzle(X^m).
\end{array}
\]

Lemma 5.20 gives us an advantage we didn’t have before: instead of working with the map $\varphi$, we are going to work with the map $\theta$. Both are continuous, open and surjective maps ($\theta$ is open by its definition and by lemma 5.17) but $\theta$ is also what we call a covering-by-parts map. This is a direct consequence of the definition (22) of theta because $\theta|_{X_{j_1 \cdots j_k}} = \theta_{j_1 \cdots j_k}$ and $\theta_{j_1 \cdots j_k}$ is a covering map as stated after its definition (19).

Our aim is to ”lift by parts” the n-region $\tilde{\gamma} : [0,1]^n \to BPuzzle(X^m)$ and then glue carefully the lifted pieces.

Remark 5.21. Let $X$ and $Y$ be topological spaces and $m \in \mathbb{N}$. Let $\tilde{\gamma} : Y \to BPuzzle(X^m)$ be a continuous function. Let $\theta$ be the map defined in (22). Take $y_0 \in Y$. Then for every $p \in \theta^{-1}(\tilde{\gamma}(y_0)) \subset BPuzzle(X^m)$ the set $\{(\theta^{-1} \circ \tilde{\gamma})(y) : y \in U_0\}$ is dense in $p$ for all $U_0 \subset Y$ an open neighborhood of $y_0$. To prove this, suppose not; then it exists $y_0$ and $U_0 \subset Y$ an open neighborhood of $y_0$ such that for some $p \in \theta^{-1}(\tilde{\gamma}(y_0)) \subset BPuzzle(X^m)$ and for some $V_0 \subset BPuzzle(X^m)$ an open neighborhood of $p$, the set $\{(\theta^{-1} \circ \tilde{\gamma})(y) : y \in U_0\}$ is not dense in $V_0$, i.e.,

\[
\{(\theta^{-1} \circ \tilde{\gamma})(y) : y \in U_0\} \cap V_0 = \{p\}.
\]

Having account that $\theta^{-1}(\tilde{\gamma}(U_0))$ is a saturated set (see 21 page 155) and from the last equality (29) we have,

\[
U_0 \cap (\tilde{\gamma}^{-1} \circ \theta)(V_0) = (\tilde{\gamma}^{-1} \circ \theta)(\{p\}),
\]

but this is impossible because the left member is an open set and the right one is a close set as: $U_0$ is open, $(\tilde{\gamma}^{-1} \circ \theta)(V_0)$ is open like $V_0$ is open, $\tilde{\gamma}$ continuous and $\theta$ open (as stated after diagram (28)), and $(\tilde{\gamma}^{-1} \circ \theta)(\{p\})$ is close as $\theta(p) = [p]_\theta$ is a point in $BPuzzle(X^m)$ and $\tilde{\gamma}$ is continuous.

Now let $Y_1 \subset Y$ be a subspace of $Y$. Suppose $\gamma : Y_1 \to BPuzzle(X^m)$ is a continuous function such that $\tilde{\gamma}|_{Y_1} = \theta \circ \gamma$. Take $y_0 \in \partial Y_1$. We want to prove that the set $\{\gamma(y) : y \in U_0 \cap Y_1\}$, with $U_0 \subset Y$ an open neighborhood of $y_0$, is dense in some $p \in \theta^{-1}(\tilde{\gamma}(y_0)) \subset BPuzzle(X^m)$. Take any $U_0 \subset Y$ an open neighborhood of $y_0$. Denote $Y_2 = Y_1 \cup \{y_0\}$ endowed with the relative topology. By the previous paragraph we have that the set

\[
\{(\theta^{-1} \circ \tilde{\gamma})|_{Y_2}(y) : y \in U_0 \cap Y_2\}
\]

dense in every $p \in \theta^{-1}(\tilde{\gamma}(y_0))$, 
so, for every $V_p \in BPuzzle(X^m)$ an open neighborhood of $p$ it exists $y_p \in Y_1$ such that $(\theta^{-1} \circ \tilde{\gamma}|_{Y_2})(y_p) \in V_p$, i.e.,
\begin{equation}
(\theta^{-1} \circ \tilde{\gamma}|_{Y_1})(y_p) \in V_p.
\end{equation}
If $\gamma(y_p) \notin V_p$ for every $p \in \theta^{-1}(\tilde{\gamma}(y_0))$, then $(\theta \circ \gamma)(y_p) \notin \theta(V_p)$, i.e., $\tilde{\gamma}|_{Y_1}(y_p) \notin \theta(V_p)$, so, $(\theta^{-1} \circ \tilde{\gamma}|_{Y_1})(y_p) \notin (\theta^{-1} \circ \theta)(V_p)$. But as $V_p \subseteq (\theta^{-1} \circ \theta)(V_p)$ then $(\theta^{-1} \circ \tilde{\gamma}|_{Y_1})(y_p) \notin V_p$ and this is a contradiction with (31). Furthermore, in the case that the set $\theta^{-1}(\tilde{\gamma}(y_0))$ is finite, then the set $\{\gamma(y) : y \in U_0 \cap Y_1\}$, with $U_0 \subseteq Y$ an open neighborhood of $y_0$, is dense in exactly one and only one $p \in \theta^{-1}(\tilde{\gamma}(y_0)) \subseteq BPuzzle(X^m)$ when $X$ is Hausdorff.

**Theorem 5.22.** Let $X$ be a Hausdorff topological space and $m,n \in \mathbb{N}$. Let $\tilde{\gamma} : [0,1]^n \to BPuzzle(X^m)$ be a continuous function. Then it exists $\gamma : [0,1]^n \to BPuzzle(X^m)$ continuous such that diagram (23) commutes.

**Proof.** First of all, we will split every piece $X_{j_1...j_k}$. Denote,
\[ \tilde{\gamma}_{j_1...j_k} = \tilde{\gamma}|_{\tilde{\gamma}^{-1}(X_{j_1...j_k}/\Sigma_{m-(j_1+...+j_k)+k})}. \]
Consider the next diagram,
\begin{equation}
\tilde{\gamma}^{-1}(X_{j_1...j_k}/\Sigma_{m-(j_1+...+j_k)+k}) \xrightarrow{\tilde{\gamma}_{j_1...j_k}} X_{j_1...j_k} \xrightarrow{\theta_{j_1...j_k}} \frac{X_{j_1...j_k}}{\Sigma_{m-(j_1+...+j_k)+k}}.
\end{equation}
Take,
\begin{equation}
\tilde{\gamma}^{-1}(X_{j_1...j_k}/\Sigma_{m-(j_1+...+j_k)+k}) = \bigcup_{\lambda_{j_1...j_k} \in \Lambda_{j_1...j_k}} \Gamma_{\lambda_{j_1...j_k}}^{j_1...j_k},
\end{equation}
the decomposition of $\tilde{\gamma}^{-1}(X_{j_1...j_k}/\Sigma_{m-(j_1+...+j_k)+k})$ in connected and locally path-connected (at the same time) components. To make simpler the notation, when possible, we will denote $\Gamma_{j_1...j_k}$ a connected and locally path-connected component. Denoting as,
\[ \tilde{\gamma}_{j_1...j_k} = \tilde{\gamma}(\Gamma_{j_1...j_k}^{j_1...j_k}), \text{ and, } \Theta_{j_1...j_k}^{j_1...j_k} = \theta^{-1}_{j_1...j_k}(\tilde{\gamma}(\Gamma_{j_1...j_k}^{j_1...j_k})), \]
we can rewrite diagram (22),
\begin{equation}
\Gamma_{j_1...j_k}^{j_1...j_k} \xrightarrow{\Theta_{j_1...j_k}^{j_1...j_k}} \frac{\Gamma_{j_1...j_k}^{j_1...j_k}}{\Sigma_{m-(j_1+...+j_k)+k}} \xrightarrow{\theta_{j_1...j_k}} \frac{\Gamma_{j_1...j_k}^{j_1...j_k}}{\Sigma_{m-(j_1+...+j_k)+k}}.
\end{equation}
As $\Gamma_{j_1...j_k}^{j_1...j_k}$ is connected and $\tilde{\gamma}_{j_1...j_k}$ is continuous therefore using 6.1.3. theorem page 352 of [9], $\Gamma_{j_1...j_k}^{\lambda_{j_1...j_k}}$ is connected. In fact, $\Gamma_{j_1...j_k}^{\lambda_{j_1...j_k}}$ is a
connected component of \(X_{j_1...j_k}/\Sigma_{m-(j_1+...+j_k)+k}\) because so it is \(\Gamma_{j_1...j_k}^{\lambda_{j_1...j_k}}\) of \(\tilde{\gamma}^{-1}(X_{j_1...j_k}/\Sigma_{m-(j_1+...+j_k)+k})\). As a consequence, every \(y \in \tilde{\Gamma}_{j_1...j_k}^{\lambda_{j_1...j_k}}\) has got a neighborhood that is disjoint with every \(\tilde{\Gamma}_{j_1...j_k}^{\lambda'_{j_1...j_k}}\) for every \(\lambda'_{j_1...j_k} \neq \lambda_{j_1...j_k} \).

From now till the end of the proof, we will work with the sets \(\Theta_{j_1...j_k}^{\lambda_{j_1...j_k}}\) and as \(\mu_{j_1...j_k}\) an index of \(\Lambda_{j_1...j_k}\), finally, we will denote \(\hat{\Theta}_{j_1...j_k}^{\mu_{j_1...j_k}}\) a path-connected component of \(\Theta_{j_1...j_k}^{\lambda_{j_1...j_k}}\) or just \(\hat{\Theta}_{j_1...j_k}\) when possible.

First of all, let’s prove a property of the sets \(\Theta_{j_1...j_k}^{\lambda_{j_1...j_k}}\).

There exists no \(x \in \Theta_{j_1...j_k}^{\lambda_{j_1...j_k}} \cup \Theta_{j_1...j_k}^{\lambda'_{j_1...j_k}}\) such that,

\[
x \in \Theta_{j_1...j_k}^{\lambda_{j_1...j_k}} \cap \Theta_{j_1...j_k}^{\lambda'_{j_1...j_k}}.
\]

Suppose not. Then \(\tilde{\gamma}^{-1}(\theta(x)) \cap (\tilde{\Gamma}_{j_1...j_k}^{\lambda_{j_1...j_k}} \cap \tilde{\Gamma}_{j_1...j_k}^{\lambda'_{j_1...j_k}}) \cap (\tilde{\Gamma}_{j_1...j_k}^{\lambda_{j_1...j_k}} \cup \tilde{\Gamma}_{j_1...j_k}^{\lambda'_{j_1...j_k}}) \neq \emptyset\). But that is impossible because \(\Gamma_{j_1...j_k}^{\lambda_{j_1...j_k}}\) and \(\Gamma_{j_1...j_k}^{\lambda'_{j_1...j_k}}\) are locally path-connected components.

Take now \(x_0 \in \Theta_{j_1...j_k}\) and \(\hat{\Theta}_{j_1...j_k}\) the path-connected component including \(x_0\). Let \(\overline{x}_0 = \theta_{j_1...j_k}(x_0) \in \tilde{\Gamma}_{j_1...j_k}\) and \(\overline{x}_1 \in \hat{\Gamma}_{j_1...j_k}\). As \(\Gamma_{j_1...j_k}\) is connected and locally path connected, then by 21.1 lemma page 175 of [12], \(\hat{\Gamma}_{j_1...j_k}\) is path connected. Therefore take a path \(\delta : [0, 1] \to \tilde{\Gamma}_{j_1...j_k}\) which holds \(\delta(0) = \overline{x}_0\) and \(\delta(1) = \overline{x}_1\); like \(\theta_{j_1...j_k}\) is a covering map as stated after its definition (19), we can apply 17.6 theorem page 157 of [13] to obtain a path \(\hat{\delta} : [0, 1] \to \Theta_{j_1...j_k}\) which holds \(\hat{\delta} = \theta_{j_1...j_k} \circ \delta\) and \(\delta(0) = x_0\); moreover, it exists \(x_1 \in \Theta_{j_1...j_k}(\overline{x}_1)\) so that \(\delta(1) = x_1\). As the last deduction can be done with every \(\overline{x}_1 \in \hat{\Gamma}_{j_1...j_k}\), we conclude that \(\hat{\Theta}_{j_1...j_k}\) holds \(\theta_{j_1...j_k}(\hat{\Theta}_{j_1...j_k}) = \tilde{\Gamma}_{j_1...j_k}\). We obtain the next diagram,

\[
\begin{array}{ccc}
\Gamma_{j_1...j_k} & \xrightarrow{\gamma_{j_1...j_k}} & \hat{\Theta}_{j_1...j_k} \\
\tilde{\gamma}_{j_1...j_k} \downarrow & & \theta_{j_1...j_k} \\
\hat{\Gamma}_{j_1...j_k} = \Theta_{j_1...j_k}/\Sigma_{m-(j_1+...+j_k)+k} & \xrightarrow{} & \Theta_{j_1...j_k}^{\lambda_{j_1...j_k}}/\Sigma_{m-(j_1+...+j_k)+k}
\end{array}
\]

Our aim with diagram (36) is to apply 21.2 theorem page 176 of [13] to obtain a lift of \(\tilde{\gamma}_{j_1...j_k}\). To do that, like \(\Gamma_{j_1...j_k}\) is connected and locally path connected by its definition, we also need the next condition,

\[
\tilde{\gamma}_{j_1...j_k}^*\Big(\pi(\Gamma_{j_1...j_k})\Big) \subseteq \theta_{j_1...j_k}^*\Big(\pi(\hat{\Theta}_{j_1...j_k})\Big),
\]

where \(\tilde{\gamma}_{j_1...j_k}^*\) and \(\theta_{j_1...j_k}^*\) are the induced maps between the fundamental groups. Let \(y_0 \in \hat{\Theta}_{j_1...j_k}/\Sigma_{m-(j_1+...+j_k)+k}\) and,

\[
\omega : \pi\Big(\hat{\Theta}_{j_1...j_k}/\Sigma_{m-(j_1+...+j_k)+k}; y_0\Big) \to \Sigma_{m-(j_1+...+j_k)+k}
\]


defined in page 165 of [14] (there is defined as \( \varphi \)). For one hand, like \( \tilde{\Theta}_{j_1 \ldots j_k} \) is path connected, we can apply 19.2 theorem page 166 of [14] to obtain that \( \theta'_{j_1 \ldots j_k} (\pi (\tilde{\Theta}_{j_1 \ldots j_k})) = \ker \omega \). On the other hand, looking at the definition of \( \gamma^*_{j_1 \ldots j_k} \), we deduce that every loop in \( \Gamma_{j_1 \ldots j_k} \) goes to a loop in \( \Gamma_{j_1 \ldots j_k} \) by the action of \( \gamma_{j_1 \ldots j_k} \), so, by definition of \( \omega \), we conclude that,

\[
\gamma^*_{j_1 \ldots j_k} (\pi (\Gamma_{j_1 \ldots j_k})) \subseteq \ker \omega,
\]

that is \( (37) \).

All in all, we use 21.2 theorem page 176 of [14] to obtain a continuous lift \( \gamma_{j_1 \ldots j_k} \) of \( \tilde{\gamma}_{j_1 \ldots j_k} \) which holds,

\[
(38) \quad \tilde{\gamma}_{j_1 \ldots j_k} = \theta_{j_1 \ldots j_k} \circ \gamma_{j_1 \ldots j_k}.
\]

From \( \gamma_{j_1 \ldots j_k} \), we want to build a continuous extension \( \tau_{j_1 \ldots j_k} : \Gamma_{j_1 \ldots j_k} \to \tilde{\Theta}_{j_1 \ldots j_k} \) in this way: denote \( \mathcal{P}_{\Gamma_{j_1 \ldots j_k}} (\tilde{\gamma}) \) the set of passings-through of \( \tilde{\gamma} \) which change from a quotient piece to another one from or towards \( \Gamma_{j_1 \ldots j_k} \) (from now til the end of the proof we will use the notation \( \mathcal{P}(\tilde{\gamma}) = \mathcal{P}_{\Gamma_{j_1 \ldots j_k}} (\tilde{\gamma}) \)). Taking \( t \in \mathcal{P}_{\Gamma_{j_1 \ldots j_k}} (\tilde{\gamma}) \) and applying remark 5.21, theorem 5.11 and 5.1 theorem page 215 of [8] we get a continuous extension of \( \gamma_{j_1 \ldots j_k} \) on \( t \). Let’s make it with every \( t \in \mathcal{P}_{\Gamma_{j_1 \ldots j_k}} (\tilde{\gamma}) \) so that we obtain a function,

\[
(39) \quad \tau_{j_1 \ldots j_k} : \Gamma_{j_1 \ldots j_k} \cup \mathcal{P}_{\Gamma_{j_1 \ldots j_k}} (\tilde{\gamma}) \to \tilde{\Theta}_{j_1 \ldots j_k},
\]

such that its restrictions to \( \Gamma_{j_1 \ldots j_k} \cup \{ t \} \) are continuous functions, for every \( t \in \mathcal{P}_{\Gamma_{j_1 \ldots j_k}} (\tilde{\gamma}) \). That function can be defined as \( \tau_{j_1 \ldots j_k} : \Gamma_{j_1 \ldots j_k} \to \tilde{\Theta}_{j_1 \ldots j_k} \) because every \( t \in \partial \Gamma_{j_1 \ldots j_k} \) on which \( \gamma_{j_1 \ldots j_k} \) is not defined, has to be a passing-through by definition 14.1 (if \( \Gamma_{j_1 \ldots j_k} \) includes no passing-through in its boundary, then by lemma 4.3 \( \gamma_{j_1 \ldots j_k} \) would be the lifting of the whole continuous function \( \tilde{\gamma} \)). Let’s prove that \( \tau_{j_1 \ldots j_k} \) is a continuous function. For one hand, \( \tau_{j_1 \ldots j_k} \) is well defined; this is true due to the next two facts: the first one is that there is no \( p \in \text{Im}(\tau_{j_1 \ldots j_k} (\mathcal{P}_{\Gamma_{j_1 \ldots j_k}} (\tilde{\gamma}))) \) such that \( p \) is in another \( \tilde{\Theta}_{j_1 \ldots j'_k} \) by definition. The second fact is that there is no \( t \in \mathcal{P}_{\Gamma_{j_1 \ldots j_k}} (\tilde{\gamma}) \) such that \( \tau_{j_1 \ldots j_k} (t) \) can be associated to two or more different points of the same equivalent class. Suppose not; then \( \tau_{j_1 \ldots j_k} (t) \) would not be well defined in the set \( \Gamma_{j_1 \ldots j_k} \cup \{ t \} \), but this is a contradiction with the previous lines. On the other hand, we want to prove that \( \tau_{j_1 \ldots j_k} \) is continuous; take the next notation,

\[
\tau_{j_1 \ldots j_k} = \tilde{\tau}_{j_1 \ldots j_k} \quad \text{and} \quad \tau_{j_1 \ldots j_k} = \theta |_{\tilde{\Theta}_{j_1 \ldots j_k}}.
\]

Let’s prove the following equality,

\[
(40) \quad \tau_{j_1 \ldots j_k}^{-1} (V) = \tilde{\tau}_{j_1 \ldots j_k}^{-1} (\tilde{\tau}_{j_1 \ldots j_k} (V)) \quad \forall V \subset \tilde{\Theta}_{j_1 \ldots j_k}.
\]
But identity (41) is true by construction of \( \pi_{j_1 \ldots j_k} \); thus, like \( \tilde{\gamma} \) is continuous and \( \theta \) is open by lemma 5.1, then for every open \( V \subset \tilde{\Theta}_{j_1 \ldots j_k} \) we have that \( \pi_{j_1 \ldots j_k}^{-1}(V) \) is open concluding that \( \pi_{j_1 \ldots j_k} \) is continuous.

Our aim now is to glue carefully the liftings obtained in the previous paragraphs. Let’s define a new concept: a shire. We will say that a set \( S = \{(\Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}}, \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}}) \in \Lambda_{j_1 \ldots j_k} \} \) is a shire if it holds \( \Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} = \theta_{j_1 \ldots j_k}(\Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}}) \) and the next two conditions,

\[ \forall (\Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}}, \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}}) \in S \text{ there exists } (\Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}}, \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}}) \in S \text{ such that } \exists \tilde{\gamma}_{i_1 \ldots i_t} \text{ which holds,} \]

\[ \Im \tilde{\gamma}_{i_1 \ldots i_t} (P(\tilde{\gamma}_{i_1 \ldots i_t})) \cap \partial \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}} \cap \partial \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}} \neq \emptyset. \]

\[ \text{[C1]} \]

The set of couples \( S \) cannot be split in a disjoint way with respect to condition \([C1]\).

We will say that \( S \) is a complete shire if,

\[ \bigcup_{\forall (\Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}}, \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}}) \in S} \Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} = [0, 1]^n. \]

In other case, we will say that \( S \) is an incomplete shire. We will say that \( S \) is an univalent shire if,

\[ \forall \Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \exists \Gamma_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}} \text{ such that } (\Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}}, \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}}) \in S. \]

We want now to prove the next result.

\[ \text{If } S = \{(\Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}}, \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}}) \} \text{ is an univalent shire } \Rightarrow \]

\[ \text{it exists a continuous function } \gamma_S : \bigcup \Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \rightarrow \bigcup \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}}. \]

First of all, take account that the family \( \{(\Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}}) \} \), of ‘first coordinates’ of the shire, is locally finite; like \( \theta \) is open and \( \tilde{\gamma} \) continuous, it is enough to prove that the family \( \{(\Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}}) \} \), of ‘second coordinates’ of the shire, is locally finite; let’s prove it: take \( x \in \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \subset \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \) and \( U_x \subset BPuzzle(X^n) \) an open neighborhood of \( x \); take another \( \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \subset \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \); it is impossible that \( x \in \partial \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \) because \( \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \) and \( \Theta_{j_1 \ldots j_k}^{\mu_{j_1 \ldots j_k}} \) are path-connected components of \( \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \).

Suppose now that \( x \in \partial \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \) with \( \lambda_{j_1 \ldots j_k} \neq \lambda_{j_1 \ldots j_k} \); in that case \( \exists t \in \partial \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \cap \partial \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \); with \( t \in \Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \), being \( \Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \) associated to a path-connected component of \( \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \); and \( \Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \) associated to a path-connected component of \( \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \); but that is impossible because \( \Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \) and \( \Gamma_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \) are connected and locally-path connected components. All in all, every \( \Theta_{j_1 \ldots j_k}^{\lambda_{j_1 \ldots j_k}} \) can share its boundary with just one
of every piece \( X_{j_1\ldots j_k'} \) different from \( X_{j_1\ldots j_k} \). As the number of pieces of the puzzle \( BPuzzle(X^m) \) is finite, we conclude that the family \( \{ \Theta_{j_1\ldots j_k}^{\mu_{j_1\ldots j_k}} \}_{j_1\ldots j_k} \) is locally finite, thus, so is the family \( \{ \Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}} \}_{j_1\ldots j_k} \). Therefore, the family \( \{ \Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}} \}_{j_1\ldots j_k} \) is locally finite. Applying now exercise 9(c) page 127 of [21], that is a generalization of Pasting lemma (theorem 18.3 page 123 of [21]), and using continuous functions \( \gamma_{j_1\ldots j_k} \) defined in (39), we obtain the continuous function \( \gamma_S \) predicted in (43), that is a function because the shire is univalent, and continuous by construction.

Finally, we need to prove that a maximal univalent shire associated to an \( x \in BPuzzle(X^m) \) is, in fact, a complete shire. Let’s begin to prove the next statement,

Take \( \mathcal{S} \) an incomplete shire. Then it exists a shire \( \mathcal{S}' \) such that \( \mathcal{S} \subset \mathcal{S}' \). Furthermore, if \( \mathcal{S} \) is univalent, \( \mathcal{S}' \) can be built as univalent. As \( \mathcal{S} \) is an incomplete shire, then,

\[
[0, 1]^n \setminus \bigcup_{(\Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}}, \Theta_{j_1\ldots j_k}^{\mu_{j_1\ldots j_k}}) \in \mathcal{S}} \Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}} \neq \emptyset.
\]

Like,

\[
[0, 1]^n = \widetilde{\gamma}^{-1}(BPuzzle(X^m)) = \bigcup_{k=0}^{m} \bigcup_{\begin{array}{c} j_1+\cdots+j_k \leq m \\ j_1 \leq \cdots \leq j_k \end{array}} \bigcup_{i=1}^{m} \Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}}(X_{j_1\ldots j_k}/\Sigma_{m-(j_1+\cdots+j_k)+k})
\]

remembering (33) we can take a connected and locally path-connected component,

\[
\Gamma' \subset [0, 1]^n \setminus \bigcup_{(\Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}}, \Theta_{j_1\ldots j_k}^{\mu_{j_1\ldots j_k}}) \in \mathcal{S}} \Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}}
\]

that holds \( \partial \Gamma' \cap \partial (\bigcup_{(\Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}}, \Theta_{j_1\ldots j_k}^{\mu_{j_1\ldots j_k}}) \in \mathcal{S}} \Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}}) \neq \emptyset \). Take now \( \hat{\Theta}' \) a path-connected component of \( BPuzzle(X^m) \) associated to \( \Gamma' \) (in the sense of diagram (33)), that holds, \( \partial \hat{\Theta}' \cap \partial (\bigcup_{(\Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}}, \Theta_{j_1\ldots j_k}^{\mu_{j_1\ldots j_k}}) \in \mathcal{S}} \Gamma_{j_1\ldots j_k}^{\lambda_{j_1\ldots j_k}}) \neq \emptyset \). That path-connected component exists by construction. Define now the new shire like this,

\[
\mathcal{S}' = \mathcal{S} \cup \{ (\Gamma', \hat{\Theta}') \}.
\]

Therefore \( \mathcal{S}' \) is a (univalent if so it is \( \mathcal{S} \)) shire by construction and because so it is \( \mathcal{S} \). At this point, we have proved statement (11).

To finish the proof of the theorem, take \( x \in BPuzzle(X^m) \) and \( \mathcal{S} \) a maximal univalent shire containing \( x \) (in its second coordinates). Using
statement (14) we conclude that $S$ is a complete univalent shire and applying statement (13) we finish the proof.

**Theorem 5.23.** Let $X$ be a Hausdorff topological space and $m, n \in \mathbb{N}$. Let $\tilde{\gamma}$ be an $n$-region over $SP_m(X)$. Then it exists $\gamma$ an $n$-region over $X^m$ such that $\tilde{\gamma} = \varphi \circ \gamma$.

**Proof.** This theorem is a direct consequence of lemma 5.20 and theorem 5.22.

**Remark 5.24.** Looking back to the theory developed in this section 5, one realizes that the lifting of a continuous map $\tilde{\gamma} : Y \to SP_m(X)$ can be done analogously just asking $Y$ to be connected and locally path connected (these conditions are needed to assure the existence of some complete shire).

**Remark 5.25.** One can also prove the homotopy lifting property for the space,

$$SP_G(X) = X^m/G,$$

the symmetric product generated by the action of a subgroup $G$ of $\Sigma_m$. It is just necessary to develop all the theory showed in this section 5 substituting $\Sigma_m$ by $G$ and every subgroup $S$ of $\Sigma_m$ by $G \cap S$.

### 6. Homotopy lifting property in $F_m(X)$

As in the previous section, first of all, we want to prove that the map defined in (3) is continuous, open and onto. To do this, we are going to deal with the theory of puzzles introduced in section 5. Moreover, we will need some notation related to the theory of partitions of an entire number.

**Notation 2.** In definition 5.5 we introduced the notion of $\sigma$ k-piece with the notation $X_{j_1...j_k,\sigma}$. In this section we will turn that notation on,

$$X_{j_0j_1...j_k,\sigma}.$$

Looking carefully definition 5.5 it is obvious that the coordinates that fullfill the positions from $j_1 + \cdots + j_k + 1$ to $m$ of a point in $X_{j_1...j_k,\sigma}$ are not repeated, so the positive entire number $j_0$ represents the amount of coordinates that are not repeated (in that positions) in every point of $X_{j_1...j_k,\sigma}$ (i.e., $j_0 = m - (j_1 + \cdots + j_k)$).

Further than the last notation, we need to introduce some notions of the theory of partitions of an entire number $m$. We will follow the development of this theory from [2].

**Definition 6.1.** Let $m, \alpha_i \in \mathbb{N}$, with $i = 1, \ldots, m$. We will use the notation $\tau_{\alpha} = [1^{\alpha_1} \cdots m^{\alpha_m}]$ to denote a partition of $m$ that has $\alpha_i$ parts of size $i$, $1 \leq i \leq m$. We will denote the set of all partitions of $m$ as $\mathbb{P}(m)$.
Remark 6.2. From definition 6.1 one can see in an easy way the relation between partitions and the pieces of our puzzle: let $X_{j_0j_1\ldots j_k\sigma}$ be a $\sigma$-k-piece. By definition 5.1 one has that $j_0 + \cdots + j_k = m$, so $j_0j_1\ldots j_k$ represents a partition of $m$. Thus, one can relate the set $\mathcal{J} = \{(j_0, \ldots, j_k) \in \mathbb{N}^k : j_0 + \cdots + j_k = m, \quad 2 \leq j_1 \leq \cdots \leq j_k \leq m\}$ with the set of partitions assigning every $\overline{j} = (j_0, \ldots, j_k)$ to its associated partition $\tau_{\overline{j}} = [1^{a_1} \ldots m^{a_m}]$. This is a bijective relation that will be denoted as $\tau_{\overline{j}} = \tau_{\alpha}$. Particularly one has that $j_0 = \alpha_1$, so the equivalent relation defined in 5.13 can be translated to the notation of partitions to,

$$
\begin{align*}
\left\{ \begin{array}{l}
    j_1 + \cdots + j_k = 2\alpha_2 + \cdots + m\alpha_m \\
    k = \alpha_2 + \cdots + \alpha_m
\end{array} \right. \\
\Leftrightarrow (j_1 + \cdots + j_k) - k = \sum_{i=2}^{m} (i-1)\alpha_i = m - \sum_{i=1}^{m} \alpha_i,
\end{align*}
$$

then,

$$
X_{j_1\ldots j_k} \sim X_{j'_1\ldots j'_k} \Leftrightarrow (j_1 + \cdots + j_k) - k = (j'_1 + \cdots + j'_k) - k' \Leftrightarrow \sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \alpha'_i.
$$

Definition 6.3. Let $\tau_{\alpha}, \tau_{\alpha'} \in \mathbb{P}(m)$. We define the next relation,

$$
\tau_{\alpha} \sim \tau_{\alpha'} \Leftrightarrow \sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \alpha'_i.
$$

This is an equivalent relation and we will denote the quotient set as $\mathbb{P}(m)/\sim$.

Lemma 6.4. Let $X$ be a topological space. Let $\mathcal{R}$ be the equivalent relation between $\sigma$-k-pieces defined in 5.13. Let $\sim$ be the equivalent relation between partitions defined in 6.2. Then,

$$
X_{j_1\ldots j_k} \sim X_{j'_1\ldots j'_k} \Leftrightarrow \tau_{\alpha} \sim \tau_{\alpha'},
$$

with $\tau_{\alpha}$ and $\tau_{\alpha'}$ the partitions associated to $j_1 \ldots j_k$ and $j'_1\ldots j'_k$ respectively.

Proof. This is a direct consequence of remark 6.2. \qed

Notation 3. We will denote $M = |\mathbb{P}(m)/\sim|$ and $m_{\alpha} = |[\tau_{\alpha}]_{\sim}|$. By lemmas 5.13 and 6.3 one has $m_{\alpha}$ homeomorphisms between two pieces related by $\mathcal{R}$. So we will denote the set of that homeomorphisms as $\{\psi_{\tau_{\alpha}}\}_{i=1}^{m_{\alpha}}$. Therefore, for every $\mathcal{T} = (\tau_{\alpha_1}, \ldots, \tau_{\alpha_M})$, with $[\tau_{\alpha_{i_1}}]_{\sim} \neq [\tau_{\alpha_{i_2}}]_{\sim}$ for all $i_1, i_2 \in \{1, \ldots, M\}$ and $i_1 \neq i_2$, one has a homeomorphism,

$$
\psi_{\mathcal{T}} : \bigcup_{\sigma \in \Sigma_m} BPuzzle(X^m, \sigma) \to \bigcup_{\sigma \in \Sigma_m} BPuzzle(X^m, \sigma),
$$

(45)
built composing the homeomorphisms $\varphi_\sigma$ (defined in (11)) with $\psi_\varphi$, with $i \in \{1, \ldots, m_\alpha\}$ and $i' = 1, \ldots, M$, and gluing that compositions with the universal property of the disjoint union topology, see [29] or [11] or [8]. Finally, we will denote $\mathbb{P}^M(m)$ to the set $\{(\tau_{\alpha_1}, \ldots, \tau_{\alpha_M}) : [\tau_{\alpha_1}]_\sim \neq [\tau_{\alpha_2}]_\sim \text{ for all } i_1, i_2 \in \{1, \ldots, M\} \text{ and } i_1 \neq i_2\}$. That implies that we have $|\mathbb{P}^M(m)|$ homeomorphisms $\psi_\varphi$. By remark 5.9 we will denote the homeomorphism induced by $\psi_\varphi$ from $X^m$ to $X^m$ with the same notation.

With notations 2 and 3 we can prove a result, analogous to lemma 5.1 for the function $\phi$. From this point we are going to work with the topological space introduced in (4). We have still remarked that this space is homeomorphic to $F_m(X)$, for every $X$ a $T_1$ topological space, by a theorem of Ganea (see [10] Hilfssatz 2) so every topological result for $\tilde{F}_m(X)$ can be immediately translated to $F_m(X)$.

**Lemma 6.5.** Let $\phi$ be the map defined by,

$$\phi : X^m \to \tilde{F}_m(X)$$

$$x \mapsto \phi(x) = [x]_\phi,$$

with $[x]_\phi$ defined in (3). Then $\phi$ is continuous, open, closed and onto.

**Proof.** Let’s prove first the identity,

(46) $$\phi^{-1}(\phi(V)) = \bigcup_{\sigma \in \Sigma_m, \varphi \in \mathbb{P}^M(m)} \psi_\varphi(\varphi_\sigma(V)),$$

for every $V \subset X^m$. As the sets of the two members of (46) are into the same topological space endowed with the same topology, we just have to prove the equality between the two sets.

Let $x = (x_1, \ldots, x_m) \in \phi^{-1}(\phi(V))$. Suppose $x = (x_1, \ldots, x_m)$ has exactly $k$ different coordinates (for some $k \in \{1, \ldots, m\}$). Take those $k$ coordinates and build all the points with the first $k$-th coordinates equals to those ones and fulfilled the rest of coordinates with one of those of the set of $k$ ones. Afterwards, take all the points generated by $\varphi_\sigma$, for every $\sigma \in \Sigma_m$, over all the last set. Denote the final set as $\{x_j\}_{j \in J}$. By construction we have that $\phi^{-1}(\phi(\{x_j\}_{j \in J})) \subset \phi^{-1}(\phi(V))$ and,

$$x \in \phi^{-1}(\phi(\{x_j\}_{j \in J})) \subset \bigcup_{\sigma \in \Sigma_m, \varphi \in \mathbb{P}^M(m)} \psi_\varphi(\varphi_\sigma(V)),$$

then,

$$\phi^{-1}(\phi(V)) \subset \bigcup_{\sigma \in \Sigma_m, \varphi \in \mathbb{P}^M(m)} \psi_\varphi(\varphi_\sigma(V)).$$

Take $x = (x_1, \ldots, x_m) \in \bigcup_{\sigma \in \Sigma_m, \varphi \in \mathbb{P}^M(m)} \psi_\varphi(\varphi_\sigma(V))$. Therefore $x = (x_1, \ldots, x_m) \in \psi_\varphi(\varphi_\sigma(V))$ for some $\sigma \in \Sigma_m$ and some $\varphi \in \mathbb{P}^M(m)$. 


As $\psi_\tau$ and $\varphi_\sigma$ are homeomorphisms by notation $3$ and the first part of the proof of lemma $5.1$, respectively, we have that $\varphi_\sigma^{-1}(\psi_\tau^{-1}(x)) \in V$. So, by $6$ and by remark $5.14$ we have that $\phi(\varphi_\sigma^{-1}(\psi_\tau^{-1}(x))) = \phi(x)$. Therefore $x \in \phi^{-1}(\phi(\varphi_\sigma^{-1}(\psi_\tau^{-1}(x))))$ then $x \in \phi^{-1}(\phi(V))$. Thus,

$$\bigcup_{\sigma \in \Sigma_m, \tau \in \mathbb{P}^M(m)} \psi_\tau(\varphi_\sigma(V)) \subset \phi^{-1}(\phi(V)).$$

All in all, we have (46). Like we have that $\phi$ is continuous and onto by definition $3$, putting together (46) and the facts that $\psi_\tau$ are homeomorphisms by remark $3$ and $\varphi_\sigma$ is a homeomorphism for every $\sigma \in \Sigma_m$ by the proof of lemma $5.1$, we conclude that $\phi$ is also an open map. □

**Corollary 6.6.** Let $X$ be a topological space and $m \in \mathbb{N}$. Then it exists a continuous, open and onto map from $SP_m(X)$ to $F_m(X)$.

**Proof.** By lemmas $5.1$ and $6.5$ the map $\phi \circ \varphi^{-1}$ (well defined by $6$), between $SP_m(X)$ and $\tilde{F}_m(X)$, is continuous and open. It is easy to see that is also onto. □

Lemma $6.5$ tells us that $\phi$ is a homeomorphism into every subspace of $X^m$ where it is injective and remains onto. Our work now consists on splitting $X^m$ into subspaces which we are going to use as pieces of a puzzle, so that, combined in a correct way, we can obtain a subspace of $X^m$ whom "symmetric product" can be apply by a continuous and bijective map to $\tilde{F}_m(X)$.

**Definition 6.7.** Fix $\tau = (\tau_{\alpha_1}, \ldots, \tau_{\alpha_M}) \in \mathbb{P}^M(m)$ and remember the notation introduced in remark $6.2$. We define the **primitive small puzzle associated to** $\tau$ like the subspace,

$$SPuzzle(X^m, \tau) = \bigcup_{i=0}^{M} \bigcup_{j=\tau_{\alpha_i}} X_{j_1 \ldots j_k},$$

endowed with the disjoint union topology (see $29$ or $9$ or $8$), understanding that for $i = 0$, we are including into the union the principal piece $X_0$. Fixing $\sigma \in \Sigma_m$, we define the $\sigma-$ **small puzzle associated to** $\tau$,

$$SPuzzle(X^m, \tau, \sigma) = \bigcup_{i=0}^{M} \bigcup_{j=\tau_{\alpha_i}} X_{j_1 \ldots j_k, \sigma},$$
endowed with the disjoint union topology too. Particularly we can define a new topological space \( \hat{X}^m \),

\[
\hat{X}^m = \bigcup_{\sigma \in \Sigma_m} \bigcup_{\tau \in \mathcal{P}(m)} \text{SUPuzzle}(X^m, \tau, \sigma),
\]

endowed with the disjoint union topology.

**Notation 4.** From definition 6.7 one can define two types of small puzzles: for one hand we will denote **small undone puzzle**, \( \text{SUPuzzle}(X^m, \tau, \sigma) \), to the disjoint union endowed with the disjoint union topology; on the other hand we will denote **small done puzzle**, \( \text{SDPuzzle}(X^m, \tau, \sigma) \), to the disjoint union endowed with the relative topology. We will always use the notation \( \text{SPuzzle}(X^m) \) to denote the small undone puzzle.

It is obvious that the \( \text{SDPuzzle}(X^m, \tau, \sigma) \) is a topological subspace of the \( \text{BDPuzzle}(X^m, \sigma) \). That fact is not so obvious for the undone puzzles and this is what is proved in the next proposition.

**Proposition 6.8.** Let \( X \) be a topological space and \( m \in \mathbb{N} \). Take \( \tau \in \mathcal{P}(m) \). Then,

\[
\text{SUPuzzle}(X^m, \tau, \sigma) \subset \text{BDPuzzle}(X^m, \sigma) \quad \forall \sigma \in \Sigma_m,
\]

e.g., the topology of the \( \sigma \)-small puzzle as a subspace of the \( \sigma \)-big puzzle coincides with the disjoint union topology.

**Proof.** Take an open set \( U \subset \text{BDPuzzle}(X^m, \sigma) \). Then,

\[
U \cap \text{SUPuzzle}(X^m, \tau, \sigma) = \bigcup_{i=0}^{M} \bigcup_{j=\tau_{i}} \left( U \cap X_{j_1\cdots j_k, \sigma} \right),
\]

and the right member is open in the disjoint union topology.

Take now \( U \subset \text{SUPuzzle}(X^m, \tau, \sigma) \) an open subset in the disjoint union topology. Then there exist \( U_j \subset X_{j_1\cdots j_k, \sigma} \) open subsets, with \( \tau_j \in \{ \tau_i \}_{i=0}^{M} \), such that,

\[
U = \bigcup_{i=0}^{M} \bigcup_{j=\tau_{i}} \left( U_j \cap X_{j_1\cdots j_k, \sigma} \right).
\]

Choose now the sets,

\[
V_j = \begin{cases} 
U_j & \text{if } \tau_j \in \{ \tau_i \}_{i=0}^{M} \\
\emptyset & \text{if } \tau_j \notin \{ \tau_i \}_{i=0}^{M}.
\end{cases}
\]

Then we have,

\[
U = \bigcup_{i=0}^{M} \bigcup_{j=\tau_{i}} \left( U_j \cap X_{j_1\cdots j_k, \sigma} \right) = \bigcup_{k=0}^{m} \bigcup_{j_{1}=2, \ j_{1}\leq \cdots \leq j_{k}}^{m} \left( V_j \cap X_{j_1\cdots j_k, \sigma} \right) = V,
\]
and \( V \) is an open subset of the \( \text{BUPuzzle}(X^m, \sigma) \).

### Corollary 6.9

Let \( X \) be a topological space, \( m \in \mathbb{N} \) and \( \sigma \in \Sigma_m \). Then,

\[
\text{BUPuzzle}(X^m, \sigma) = \bigcup_{\varphi \in \mathbb{P}(m)} \text{SUPuzzle}(X^m, \varphi, \sigma),
\]

with the set of the right member endowed with the disjoint union topology.

**Proof.** The proof is analogue to that of proposition 6.8.

### Corollary 6.10

Let \( X \) be a topological space and \( m \in \mathbb{N} \). Then,

\[
\widetilde{X}^m = \hat{X}^m,
\]

where the left member is defined in (13) and the right one is defined in (12).

**Proof.** This is a direct consequence of corollary 6.9.

The definition of new small puzzles and of the function \( \theta \) for those ones can be stated analogue to the paragraphs around (before and after) definitions (19) and (22). In fact by proposition 6.8” our new \( \theta \) can be defined as \( \xi = \theta|_{\text{SUPuzzle}(X^m, \varphi)} \).

### Definition 6.11

Let \( X \) be a topological space and \( m \in \mathbb{N} \). From definitions 5.5 and (19) we define the topological space,

\[
\tilde{\text{SPuzzle}}(X^m, \varphi) = \bigcup_{i=0}^{M} \bigcup_{\vec{\tau} = \tau_{\alpha_i}} \left( X_{j_1:j_k}/\Sigma_{m-(j_1+\cdots+j_k)+k} \right),
\]

declared with the disjoint union topology (see [29] or [9] or [8]), understanding that for \( i = 0 \), we are including into the union the principal piece \( X_0 \). Fixing \( \sigma \in \Sigma_m \), we also define,

\[
\tilde{\text{SPuzzle}}(X^m, \varphi, \sigma) = \bigcup_{i=0}^{M} \bigcup_{\vec{\tau} = \tau_{\alpha_i}} \left( X_{j_1:j_k, \sigma}/\Sigma_{m-(j_1+\cdots+j_k)+k} \right),
\]

declared with the disjoint union topology too.

From now till the end of the section we can proof all the results of section 5 from definition 5.16 till theorem 5.23 making the next changes,

\[
\begin{align*}
\varphi & \to \phi \\
\text{BDPuzzle}(X^m, \sigma) & \to \text{SDPuzzle}(X^m, \varphi, \sigma) \\
\text{BUPuzzle}(X^m, \sigma) & \to \text{SUPuzzle}(X^m, \varphi, \sigma) \\
\theta & \to \xi = \theta|_{\text{SUPuzzle}(X^m, \varphi)}
\end{align*}
\]
Lemma 6.12. Let $X$ be a Hausdorff topological space, $m \in \mathbb{N}$. Then it exists a continuous and bijective map $f$,

$$f : SPuzzle(X^m) \rightarrow \tilde{F}_m(X),$$

such that for every open subset $V \subset SDPuzzle(X^m)$, $f(\xi(\pi^{-1}(V)))$ is an open subset of $\tilde{F}_m(X)$, being $\xi$ the map defined in (52) and $\pi$ defined in lemma 5.8 (b).

With lemma 6.12 and with the next diagram,

$$f : SPuzzle(X^m) \rightarrow \tilde{F}_m(X),$$

one can reduce the proof of the homotopy lifting property for $\tilde{F}_m(X)$ to the proof of the homotopy lifting property for $SPuzzle(X^m)$. More precisely, we just need to lift a path in $SPuzzle(X^m)$ to a path in $\tilde{F}_m(X)$ to a path in $X^m$. In the last diagram, $Y$ is a topological space, $g_1$ is a continuous map in $X^m$, $g_2$ is a continuous map in $SPuzzle(X^m)$, $\tilde{g}$ is a continuous map in $\tilde{F}_m(X)$, $\phi$ is the map defined in (3), $\xi$ is the map defined in (52), $f$ is the continuous and bijective map obtained in lemma 6.12 and defined by,

$$f : SPuzzle(X^m) \rightarrow \tilde{F}_m(X) \quad [x]_{\xi} \mapsto f([x]_{\xi}) = [x]_{\phi},$$

and $i$ is the natural inclusion of $SDPuzzle(X^m)$ in $X^m$.

Lemma 6.13. Let $X$ and $Y$ be a Hausdorff topological spaces and $m \in \mathbb{N}$. Let $\tilde{g} : Y \rightarrow \tilde{F}_m(X)$ a continuous map in $\tilde{F}_m(X)$. Then,

It exists a continuous map $g_1 : Y \rightarrow X^m$ such that $\tilde{g} = \phi \circ g_1$.

So now, our efforts will be dedicated to the proof of the homotopy lifting property for $SPuzzle(X^m)$. We are going to work with the next commutative diagram, for $n \in \mathbb{N}$,

$$[0,1]^n \xymatrix{ \gamma \ar[r] \ar[d] & SPuzzle(X^m) \ar[d]^{\xi} }$$

$$SPuzzle(X^m).$$
Remark 6.14. Let $X$ and $Y$ be topological spaces and $m \in \mathbb{N}$. Let $\tilde{\gamma} : Y \to \text{SPuzzle}(X^m)$ be a continuous function. Let $\xi$ be the map defined in (52). Take $y_0 \in Y$. Then for every $p \in \tilde{\gamma}(y_0) \subset \text{SPuzzle}(X^m)$ the set $\{(\xi^{-1} \circ \tilde{\gamma})(y) : y \in U_0\}$ is dense in $p$ for all $U_0 \subset Y$ an open neighborhood of $y_0$. To prove this, suppose not; then it exists $y_0$ and $U_0 \subset Y$ an open neighborhood of $y_0$ such that for some $p \in \tilde{\gamma}(y_0) \subset \text{SPuzzle}(X^m)$ and for some $V_0 \subset \text{SPuzzle}(X^m)$ an open neighborhood of $p$, the set $\{(\xi^{-1} \circ \tilde{\gamma})(y) : y \in U_0\}$ is not dense in $V_0$, i.e.,

$$
(57) \quad \{(\xi^{-1} \circ \tilde{\gamma})(y) : y \in U_0\} \cap V_0 = \{p\}.
$$

Having account that $\xi^{-1}(\tilde{\gamma}(U_0))$ is a saturated set (see [21] page 155) and from the last equality (57) we have,

$$
U_0 \cap (\tilde{\gamma}^{-1} \circ \xi)(V_0) = (\tilde{\gamma}^{-1} \circ \xi)(\{p\}),
$$

but this is impossible because the left member is an open set and the right one is a close set as: $U_0$ is open, $(\tilde{\gamma}^{-1} \circ \xi)(V_0)$ is open like $V_0$ is open, $\tilde{\gamma}$ continuous and $\xi$ open (as stated after diagram (56)), and $(\tilde{\gamma}^{-1} \circ \xi)(\{p\})$ is close as $\xi(p) = [p]_\xi$ is a point in $\text{SPuzzle}(X^m)$ and $\tilde{\gamma}$ is continuous.

Now let $Y_1 \subset Y$ a subspace of $Y$. Suppose $\gamma : Y_1 \to \text{SPuzzle}(X^m)$ is a continuous function such that $\tilde{\gamma}|_{Y_1} = \xi \circ \gamma$. Take $y_0 \in \partial Y_1$. Then the set $\{\gamma(y) : y \in U_0 \cap Y_1\}$, with $U_0 \subset Y$ an open neighborhood of $y_0$, is dense in some $p \in \tilde{\gamma}(y_0) \subset \text{SPuzzle}(X^m)$. Take any $U_0 \subset Y$ an open neighborhood of $y_0$. Denote $Y_2 = Y_1 \cup \{y_0\}$ endowed with the relative topology. By the previous paragraph we have that the set

$$
\{(\xi^{-1} \circ \tilde{\gamma}|_{Y_2})(y) : y \in U_0 \cap Y_2\}
$$

is dense in every $p \in \tilde{\gamma}(y_0)$,

so, for every $V_p \in \text{SPuzzle}(X^m)$ an open neighborhood of $p$ it exists $y_p \in Y_1$ such that $(\xi^{-1} \circ \tilde{\gamma}|_{Y_1})(y_p) \in V_p$, i.e.,

$$
(58) \quad (\xi^{-1} \circ \tilde{\gamma}|_{Y_1})(y_p) \in V_p.
$$

If $\gamma(y_p) \notin V_p$ for every $p \in \tilde{\gamma}(y_0)$, then $(\xi \circ \gamma)(y_p) \notin \xi(V_p)$, i.e., $\tilde{\gamma}|_{Y_1}(y_p) \notin \xi(V_p)$, so, $(\xi^{-1} \circ \tilde{\gamma}|_{Y_1})(y_p) \notin (\xi^{-1} \circ \xi)(V_p)$. But as $V_p \subset (\xi^{-1} \circ \xi)(V_p)$ then $(\xi^{-1} \circ \tilde{\gamma}|_{Y_1})(y_p) \notin V_p$ and this is a contradiction with (58).

Theorem 6.15. Let $X$ be a Hausdorff topological space and $m, n \in \mathbb{N}$. Let $\tilde{\gamma}$ be an $n$-region over $\text{SPuzzle}(X^m)$. Then it exists $\gamma$ an $n$-region over $\text{SPuzzle}(X^m)$ such that $\tilde{\gamma} = \xi \circ \gamma$.

Theorem 6.16. Let $X$ be a Hausdorff topological space and $m, n \in \mathbb{N}$. Let $\tilde{\gamma}$ be an $n$-region over $\text{SPuzzle}(X^m)$. Then it exists $\gamma$ an $n$-region over $X^m$ such that $\tilde{\gamma} = \phi \circ \gamma$.

Corolary 6.17. Let $X$ be a Hausdorff topological space and $m, n \in \mathbb{N}$. Let $\tilde{\gamma}$ be an $n$-region over $\text{SPuzzle}(X^m)$. Then it exists $\gamma$ an $n$-region over $X^m$ such that $\tilde{\gamma} = \phi \circ \gamma$. 
Proof. It is just necessary to use theorem 6.16 and a result of Ganea (see [10] Hilfssatz 2). □

Remark 6.18. Looking back to the theory develop in this section 6, one realizes that the lifting of a continuous map $\tilde{\gamma} : Y \to F_m(X)$ can be done analogously just asking $Y$ to be connected and locally path connected (these conditions are needed to assure the existence of some complete shire).

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