0-CALABI-YAU CONFIGURATIONS AND FINITE AUSLANDER-REITEN QUIVERS OF GORENSTEIN ORDERS

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Abstract. We will revisit Wiedemann’s classification of Auslander-Reiten quivers of representation-finite Gorenstein orders in this paper. We give a simpler proof of his result [27] in which he described the Auslander-Reiten quiver of a representation-finite Gorenstein order in terms of a Dynkin diagram, a configuration and an automorphism group. A key notion in his result is configurations described in terms of Brauer relations with Straßenegenschaft [27, Satz, p. 47]. We show that configurations can be described in terms of 2-Brauer relations very briefly.

1. Introduction

Representation theory of orders is classical and has been studied by a lot of authors, see e.g. [2, 5, 6, 14, 20, 22, 24]. One of the traditional subjects is the classification of orders which are representation-finite in the sense that they have only finitely many indecomposable Cohen-Macaulay modules. The classification was given for many important classes, e.g. local orders [8, 11], tiled orders [24, 29] and Bäckström orders [18]. The important class of Gorenstein orders is an analogue of both finite dimensional self-injective algebras and commutative Gorenstein rings. The representation-finite Gorenstein orders were studied by Wiedemann [27] and Roggenkamp [21] (see also [25, 26]). Wiedemann’s work is an analogue of Riedtmann’s work on representation-finite selfinjective algebras [15, 16, 17].

One of the aims of this paper is to improve Wiedemann’s classification of Auslander-Reiten quivers of representation-finite Gorenstein orders. A key notion introduced by Wiedemann is configurations, which is similar but different from Riedtmann’s configurations [16]. He described the Auslander-Reiten quiver of a representation-finite Gorenstein order in terms of a Dynkin diagram \( \Delta \), a configuration \( C \) in \( (\mathbb{Z}\Delta)_0 \) and an automorphism group \( G \) of \( \mathbb{Z}\Delta \). In this paper, we give a simple proof of the following theorem contributed by Wiedemann [27] and Roggenkamp [21] (see Section 2.1, Definitions 2.1 and 5.3 for \( (\mathbb{Z}\Delta/G)_C \)).

\textbf{Theorem 1.1} (Theorem 5.8). Let \( R \) be a complete discrete valuation ring and \( \Lambda \) be a ring-indecomposable representation-finite Gorenstein \( R \)-order. We assume the following two conditions are satisfied:

(a) \( \text{rad } P \) is indecomposable and non-projective for any indecomposable projective \( \Lambda \)-module \( P \);
(b) \( \text{rad } P \) is not isomorphic to \( \text{rad } Q \) when the two indecomposable projective \( \Lambda \)-modules \( P, Q \) are nonisomorphic.

Then the Auslander-Reiten quiver \( \mathcal{A}(\Lambda) \) of \( \text{CM} \Lambda \) is isomorphic to \( (\mathbb{Z}\Delta/G)_C \), where \( \Delta \) is a Dynkin diagram, \( G \subset \text{Aut}(\mathbb{Z}\Delta) \) is a weakly admissible group and \( C \) is a configuration of \( \mathbb{Z}\Delta/G \).

Note that there are very few cases where the conditions (a) and (b) are not satisfied. In fact, \( \Lambda \) is a very special Bäckström-order [18] in this case [21, Remark 1]. We point out that Wiedemann’s configurations are “0-Calabi-Yau” in the sense that they are preserved by the Serre functor \( S \) in the stable category \( \text{CM} \Lambda \), while Riedtmann’s configurations are “\((-1)\)-Calabi-Yau” since they are preserved by \( S \circ [1] \). Note that a variant of Wiedemann’s and Riedtmann’s configurations appeared recently in cluster tilting theory, where \( n \)-cluster tilting objects in \( n \)-cluster categories correspond bijectively to \( n \)-Calabi-Yau configurations in the derived categories [4].

| Riedtmann’s configuration | Wiedemann’s configuration | \( n \)-cluster tilting object \( (n \geq 1) \) |
|---------------------------|---------------------------|----------------------------------|
| \((-1)\)-Calabi-Yau       | 0-Calabi-Yau               | \( n \)-Calabi-Yau               |

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For type $A$ case, Wiedemann described configurations in terms of Brauer relations with Strausseneigenschaft [27, Satz, p. 47]. In this paper, we simplify his description by introducing a much simpler notion of 2-Brauer relations and give nice correspondences with 0-Calabi-Yau configurations. Let $n \geq 1$ be an integer. We denote by $\mathbf{B}_n^2$ the set of all 2-Brauer relations of rank $n$ (see Definition 6.1) and by $\mathbf{C}(A_{n+1})$ the set of all the configurations of $\mathbb{Z}A_{n+1}$. Then our first result is the following.

**Theorem 1.2** (Theorem 6.3). We have the following one-to-one correspondence

$$\mathbf{C}(A_{n+1}) \xrightarrow{1-1} \mathbf{B}_n^2.$$  

In order to describe all the configurations for other Dynkin diagrams, we also define symmetric 2-Brauer relations of rank $2n$ and crossing 2-Brauer relations of rank $2n$, and we denote by $\mathbf{B}_n^{2,s}$ and $\mathbf{B}_n^{2,c}$ the sets of these two kinds of relations respectively (see Definitions 7.1 and 9.1). Consider the set $\mathbf{C}(B_{n+1})$ of all the configurations of $\mathbb{Z}B_{n+1}$ and the set $\mathbf{C}(C_{n+1})$ of all the configurations of $\mathbb{Z}C_{n+1}$. For $\mathbb{Z}D_{n+2}$, we consider a partition $\mathbf{C}(D_{n+2}) = \mathbf{C}^1(D_{n+2}) \sqcup \mathbf{C}^2(D_{n+2})$ (see Section 9 for details). Then we give theorems to describe all the configurations.

**Theorem 1.3** (Theorem 7.3, 8.1, 9.4). We have the following one-to-one correspondences

- $\mathbf{C}(B_{n+1}) \xrightarrow{1-1} \mathbf{B}_n^{2,s}$,
- $\mathbf{C}(C_{n+1}) \xrightarrow{1-1} \mathbf{B}_n^{2,s}$,
- $\mathbf{C}^1(D_{n+2}) \xrightarrow{1-1} \mathbf{B}_n^{2,s}$,

and a two-to-one correspondence

$$\mathbf{C}^2(D_{n+2}) \xleftarrow{2-1} \mathbf{B}_n^{2,c}.$$  

We organize this paper as follows. In Section 2, we recall some basic definitions and facts. In Section 3, we give categorical properties of the radicals of the indecomposable projective $\Lambda$-modules in the stable category $\mathbf{CM}_A$ which we call categorical configurations. In Section 4, we give a simple systematic method to calculate the length of hom-sets of the stable category $\mathbf{CM}_A$ from the Auslander-Reiten quiver $\mathbf{R}(\Lambda)$. In Section 5, we characterize the categorical configurations in the Auslander-Reiten quiver of the stable category in terms of combinatorial conditions called (C1), (C2) and we call them combinatorial configurations. In this section, we also give a simpler proof of Theorem 5.8 by using the conditions (C1), (C2) of combinatorial configurations. Later in the paper, we introduce our main results case by case: Section 6 for type $A$, Section 7 for type $B$, Section 8 for type $C$ and Section 9 for type $D$.

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## 2. Preliminaries

In this section, we recall some definitions and facts that we need later in this paper. We always denote by $f \circ g$ the composition of $X \xrightarrow{f} Y \xrightarrow{g} Z$ when they are maps or functors in this paper.

### 2.1. Valued translation quivers

Recall that we do not assume that the residue field of $R$ is algebraically closed. This is quite natural to cover important classes of orders appearing in number theory [10]. Then the Auslander-Reiten quivers of $R$-orders have the structure of valued translation quivers. First, let us recall the definition of valued translation quivers and morphisms of valued translation quivers.

A **valued translation quiver** is a quadruple $Q = (Q_0, d, \tau, T)$, where $Q_0$ is a set, $d$ and $d'$ are maps $d : Q_0 \times Q_0 \rightarrow \mathbb{N} \geq 0$ and $\tau$ is a bijection from $Q_0 \setminus Q^p$ to $Q_0 \setminus Q^q$ for subsets $Q^p$ and $Q^q$ of $Q_0$ such that for any vertices $y \in Q_0$ and $z \in Q_0 \setminus Q^p$, we have $d_{(\tau z) y} = d_{yz}$. We draw $Q$ as a quiver with valued arrows

$$x \xrightarrow{(d_{xy}, d'_{xy})} y,$$

where $x \xrightarrow{(1,1)} y$ is denoted by $x \rightarrow y$. If $Q^p = Q^q = \emptyset$, then $Q$ is called **stable**.
An automorphism \( g : Q \to Q \) of a valued translation quiver \( Q \) is a bijection \( g : Q_0 \to Q_0 \) such that \( g(Q^p) = Q^p, g(Q^i) = Q^i, g\tau = \tau g, d_{gxy} = dx_y \) and \( d'_{gxy} = d'_x \) hold for any \( x, y \in Q_0 \). We denote by \( \text{Aut}(Q) \) the automorphism group of \( Q \). A subgroup \( G \subseteq \text{Aut}(Q) \) is said to be weakly admissible [7] if for each \( g \in G \setminus \{1\} \) and \( x \in Q_0 \), we have \( x \neq gx \) and \( x^+ \cap (gx)^+ = \emptyset \), where \( x^+ \) is the set of direct successors of \( x \). Although the condition \( x \neq gx \) is not assumed in [7], it is more reasonable to assume it. Note that if \( x^+ \neq \emptyset \) and \( x^+ \cap (gx)^+ = \emptyset \), then \( x \neq gx \) holds automatically.

In such a situation, the quotient valued translation quiver \( Q/G \) is defined as follows. The set \( (Q/G)_0 \) of vertices is \( Q_0/G = \{gx \mid x \in Q_0\} \). Moreover, \( (Q/G)^p = Q^p/G, (Q/G)^i = Q^i/G \) and \( \tau, d, d' \) are defined by the following commutative diagrams:

\[
\begin{array}{ccc}
Q_0 \sim Q^p & \xrightarrow{\tau} & Q_0 \setminus Q^i \\
\downarrow & & \downarrow \\
(Q/G)_0 \sim (Q/G)^p & \xrightarrow{\tau} & (Q/G)_0 \setminus (Q/G)^i,
\end{array}
\begin{array}{ccc}
Q_0 \times Q_0 & \xrightarrow{d \ (\text{resp. } d')} & N_{\geq 0} \\
\downarrow & & \downarrow \\
(Q/G)_0 \times (Q/G)_0 & \xrightarrow{d \ (\text{resp. } d')} & N_{\geq 0},
\end{array}
\]

where the vertical maps are natural surjections.

Assume that \( \Delta \) is one of the Dynkin diagrams:

- \( A_n : \) \( \bullet \) \( \cdots \) \( \bullet \) \( \cdots \) \( \bullet \)
- \( B_n : \) \( \bullet \) \( \frac{(1,2)}{\cdots} \) \( \bullet \) \( \cdots \) \( \bullet \)
- \( C_n : \) \( \bullet \) \( \frac{(2,1)}{\cdots} \) \( \bullet \) \( \cdots \) \( \bullet \)
- \( D_n : \) \( \bullet \) \( \cdots \) \( \bullet \) \( \cdots \) \( \bullet \) \( \cdots \) \( \bullet \)
- \( E_6 : \) \( \bullet \) \( \cdots \) \( \bullet \) \( \cdots \) \( \bullet \) \( \cdots \) \( \bullet \)
- \( E_7 : \) \( \bullet \) \( \cdots \) \( \bullet \) \( \cdots \) \( \bullet \) \( \cdots \) \( \bullet \)
- \( E_8 : \) \( \bullet \) \( \cdots \) \( \bullet \) \( \cdots \) \( \bullet \) \( \cdots \) \( \bullet \)
- \( F_4 : \) \( \bullet \) \( \frac{(1,2)}{\cdots} \) \( \bullet \)
- \( G_2 : \) \( \bullet \) \( \frac{(1,3)}{\cdots} \)

Each of the above diagrams has \( n \) vertices. Let \( Q \) be a quiver with underlying Dynkin diagram \( \Delta \). The valued translation quiver \( \mathbb{Z}Q \) associated to \( Q \) is defined as follows. The vertices of \( \mathbb{Z}Q \) are indexed by \( (n, i) \) with \( n \in \mathbb{Z} \) and \( i \in Q_0 \). There are arrows

\[
(n, \alpha) : (n, i) \xrightarrow{(d_{ij}, d'_{ij})} (n, j) \quad \text{and} \quad (n, \alpha^*) : (n, j) \xrightarrow{(d'_{ij}, d_{ij})} (n + 1, i)
\]

in \( \mathbb{Z}Q \) for any arrow \( \alpha : i \xrightarrow{(d_{ij}, d'_{ij})} j \) in \( Q \), and these are all the arrows in \( \mathbb{Z}Q \). The translation \( \tau \) is defined on the whole of \( (\mathbb{Z}Q)_0 \) by \( \tau((n, i)) = (n - 1, i) \). The valued translation quiver \( \mathbb{Z}Q \) is a stable valued translation quiver. According to [9, Chapter I, 5.6], \( \mathbb{Z}Q \) only depends on the underlying Dynkin diagram \( \Delta \), not on \( Q \). Hence, we denote the corresponding valued translation quiver by \( \mathbb{Z}\Delta \).

From a stable translation quiver and a subset of vertices, we can define a new translation quiver as follows.

**Definition 2.1.** Let \( Q \) be a stable valued translation quiver and \( C \) be a subset of \( Q_0 \). We define a valued translation quiver \( Q_C \) by adding to \( Q \) a vertex \( p_c \) and two arrows \( c \to p_c \to \tau^{-1}(c) \) for each
Let us see an example.

Example 2.2. Consider the translation quiver \( Q = \mathbb{Z} A_5 \) with the set \( C \) of all the vertices marked as follows. By adding new vertices and new arrows requested in Definition 2.1, we get the new translation quiver \( Q_C \).

\[
\begin{align*}
Q: & \quad \cdots & Q_C: & \quad \cdots \\
& \quad \cdots & & \quad \cdots \\
& \quad \cdots & & \quad \cdots
\end{align*}
\]

2.2. Cohen-Macaulay modules. In this subsection, we introduce basic facts of orders and their Cohen-Macaulay modules. Throughout this paper, we denote by \( R \) a complete discrete valuation ring, for example, \( R \) is a formal power series ring in one variable over a field. We denote the field of fractions of \( R \) by \( K \). We recall the definition of orders and Cohen-Macaulay modules first, see [5, 6, 20, 22, 24].

Definition 2.3. Let \( \Lambda \) be an \( R \)-algebra and \( M \) be a left \( \Lambda \)-module.

1. \( \Lambda \) is called an \( R \)-order if it is finitely generated and free as an \( R \)-module.
2. If \( \Lambda \) is an \( R \)-order, then a \( \Lambda \)-module \( M \) is called a Cohen-Macaulay \( \Lambda \)-module or \( \Lambda \)-lattice if it is finitely generated and free as an \( R \)-module. We denote by \( \text{CM} \Lambda \) the category of Cohen-Macaulay \( \Lambda \)-modules. It is a full subcategory of \( \text{mod} \Lambda \), which is closed under submodules and extensions.
3. We define the stable category \( \text{CM} \Lambda \) of \( \text{CM} \Lambda \) as follows.
   - The objects of \( \text{CM} \Lambda \) are the same as the objects of \( \text{CM} \Lambda \).
   - For any \( X, Y \in \text{CM} \Lambda \), the hom-set is defined by
     \[ \text{Hom}_\Lambda(X, Y) := \text{Hom}_\Lambda(X, Y)/[\Lambda](X, Y), \]
     where \([\Lambda](X, Y)\) is the set of morphisms factoring through a projective \( \Lambda \)-module.

We denote by \( \text{fl} \Lambda \) the category of finite length \( \Lambda \)-modules. Then \( (\text{fl} \Lambda, \text{CM} \Lambda) \) is a torsion pair in \( \text{mod} \Lambda \) in the sense that \( \text{Hom}_\Lambda(X, Y) = 0 \) holds for any \( X \in \text{fl} \Lambda \) and \( Y \in \text{CM} \Lambda \), and moreover, for any \( X \in \text{mod} \Lambda \), there exists an exact sequence \( 0 \to TX \to X \to FX \to 0 \) with \( TX \in \text{fl} \Lambda \) and \( FX \in \text{CM} \Lambda \). For any \( R \)-order \( \Lambda \) (e.g. \( \Lambda = R \)), we have the following exact dualities
   - \( D_1 := \text{Hom}_R(-, R) : \text{CM} \Lambda \leftrightarrow \text{CM} \Lambda^{\text{op}}, \)
   - \( D_0 := \text{Ext}^1_R(-, R) : \text{fl} \Lambda \leftrightarrow \text{fl} \Lambda^{\text{op}}. \)

We denote by \( \text{proj} \Lambda \) the category of projective \( \Lambda \)-modules. They are all projective objects in \( \text{CM} \Lambda \). Now let us define the injective objects in \( \text{CM} \Lambda \).

Definition 2.4. Let \( \Lambda \) be an \( R \)-order. We call \( I \in \text{CM} \Lambda \) an injective Cohen-Macaulay \( \Lambda \)-module if \( \text{Ext}^1_\Lambda(Y, I) = 0 \) for any \( Y \in \text{CM} \Lambda \). This is equivalent to \( I \in \text{add}(D_1(\Lambda^{\text{op}})) \). We denote by \( \text{inj} \Lambda \) the category of injective Cohen-Macaulay \( \Lambda \)-modules.

From now on, we assume that \( \Lambda \) is a ring-indecomposable \( R \)-order. We denote by \( \text{sim} \Lambda \) the set of the isomorphism classes of simple \( \Lambda \)-modules. For the category \( \mathcal{A} \), we denote the radical of \( \mathcal{A} \) by \( \text{rad} \mathcal{A}(-, -) \) and by \( \text{ind} \mathcal{A} \) the set of the isomorphism classes of indecomposable objects in \( \mathcal{A} \).

Consider the Nakayama functor
   \[ \nu := (D_1 \Lambda) \otimes_{\Lambda} - \cong D_1 \text{Hom}_\Lambda(-, \Lambda) : \text{CM} \Lambda \to \text{CM} \Lambda. \]

It induces an equivalence
   \[ \nu : \text{proj} \Lambda \sim \text{inj} \Lambda. \]
For any $X \in \text{CM} \Lambda$, let
$$\text{corad } X := D_1(\text{rad}(D_1X)) \quad \text{and} \quad \text{cotop } X := D_0(\text{top}(D_1X)).$$
We have the following proposition to give other bijections.

**Proposition 2.5.** We have one-to-one correspondences between
(a) $\text{ind} (\text{proj } \Lambda)$
(b) $\text{ind} (\text{inj } \Lambda)$
(c) $\sim \Lambda$
(a') $\text{ind} (\text{proj } \Lambda^{\text{op}})$
(b') $\text{ind} (\text{inj } \Lambda^{\text{op}})$
(c') $\sim \Lambda^{\text{op}}$.

The bijection $\text{ind} (\text{proj } \Lambda) \rightarrow \sim \Lambda$ is given by $P \mapsto \text{top } P$, the bijection $\text{ind} (\text{proj } \Lambda) \rightarrow \text{ind} (\text{inj } \Lambda)$ is given by $P \mapsto \nu P$ and the bijection $\text{ind} (\text{inj } \Lambda) \rightarrow \sim \Lambda$ is given by $I \mapsto \text{cotop } I$. Moreover, these bijections commute.

**Proof.** Since $\Lambda$ is semiperfect (i.e. every finitely generated $\Lambda$-module has a projective cover), the correspondence between (a) (resp. (a')) and (c) (resp. (c')) is classical. Since $D_1$ is the duality, (a) is in bijection with (b') (resp. (b) is in bijection with (a')).

Since $\Lambda$ is semiperfect, every finitely generated $\Lambda$-module (resp. $\Lambda^{\text{op}}$-module) has a projective cover. By using the exact duality $D_1 : \text{CM} \Lambda \rightarrow \text{CM} \Lambda^{\text{op}}$, we have the following proposition.

**Proposition 2.6.** For any $X \in \text{CM} \Lambda$, there exists a short exact sequence
$$0 \rightarrow X \rightarrow Q \rightarrow Y \rightarrow 0$$
with $Q \in \text{inj } \Lambda$ and $Y \in \text{CM} \Lambda$.

The following sequence is basic.

**Lemma 2.7.** For any $X \in \text{CM} \Lambda$, we have a short exact sequence
$$0 \rightarrow X \rightarrow \text{corad } X \rightarrow \text{cotop } X \rightarrow 0.$$

**Proof.** For $D_1X \in \text{CM} \Lambda^{\text{op}}$, we consider the exact sequence $0 \rightarrow \text{rad}(D_1X) \rightarrow D_1X \rightarrow \text{top}(D_1X) \rightarrow 0$. By applying $\text{Hom}_R(-, R)$, we have an exact sequence
$$0 \rightarrow D_1(\text{top}(D_1X)) \rightarrow D_1(X) \rightarrow D_1(\text{rad}(D_1X)) \rightarrow D_0(\text{top}(D_1X)) \rightarrow D_0(D_1X).$$
Since $D_1(D_1X) = X$, $D_1(\text{top}(D_1X)) = 0$ and $D_0(D_1X) = 0$, we have the desired sequence.

The following lemma is about the extension group from a simple module to an injective Cohen-Macaulay module over an $R$-order $\Lambda$.

**Lemma 2.8.**
1. For any $P \in \text{ind}(\text{proj } \Lambda)$ and $S \in \text{sim } \Lambda$, $\text{Hom}_\Lambda(P, S) \neq 0$ if and only if $S = \text{top } P$. In this case, $\text{Hom}_\Lambda(P, S)$ is a simple $\text{End}_\Lambda(P)$-module.
2. For any $I \in \text{ind}(\text{inj } \Lambda)$ and $S \in \text{sim } \Lambda$, $\text{Ext}_\Lambda^1(S, I) \neq 0$ if and only if $S = \text{cotop } I$. In this case, $\text{Ext}_\Lambda^1(S, I)$ is a simple $\text{End}_\Lambda(I)$-module.
3. For any $P \in \text{ind}(\text{proj } \Lambda)$ and $I \in \text{ind}(\text{inj } \Lambda)$, $\text{Ext}_\Lambda^1(top P, I) \neq 0$ if and only if $\text{cotop } I \cong \text{top } P$ if and only if $I \cong \nu P$.

**Proof.**
1. The first statement is well-known. e.g. see [13, 1.2 (3)]. We will prove that $\text{Ext}_\Lambda^1(S, I)$ is a simple $\text{End}_\Lambda(I)$-module. Apply $\text{Hom}_\Lambda(-, I)$ to $0 \rightarrow I \rightarrow \text{corad } I \rightarrow \text{cotop } I \rightarrow 0$, we have an exact sequence
$$0 = \text{Hom}_\Lambda(\text{cotop } I, I) \rightarrow \text{Hom}_\Lambda(\text{corad } I, I) \rightarrow \text{Hom}_\Lambda(I, I) \rightarrow \text{Ext}_\Lambda^1(\text{cotop } I, I) \rightarrow \text{Ext}_\Lambda^1(\text{corad } I, I) = 0,$$
where the last term is 0 since $\text{corad } I \in \text{CM} \Lambda$ and $I \in \text{inj } \Lambda$. Since $D_1I \in \text{ind}(\text{proj } \Lambda^{\text{op}})$, there is an exact sequence
$$0 \rightarrow \text{rad}(D_1I) \rightarrow D_1I \rightarrow \text{top}(D_1I) \rightarrow 0.$$
By applying $\text{Hom}_{\Lambda^\text{op}}(D_1 I, -)$, we get an exact sequence
\[
0 \to \text{Hom}_{\Lambda^\text{op}}(D_1 I, \text{rad}(D_1 I)) \to \text{Hom}_{\Lambda^\text{op}}(D_1 I, D_1 I) \to \text{Hom}_{\Lambda^\text{op}}(D_1 I, \text{top}(D_1 I)) \to 0.
\]
(2.2)
Since the left and the middle terms of (2.1) and (2.2) are isomorphic, it follows from (1) that
\[
\text{Ext}^1_\Lambda(\text{cotop} I, I) \cong \text{Hom}_{\Lambda^\text{op}}(D_1 I, \text{top}(D_1 I))
\]
is a simple $\text{End}_\Lambda(I)$-module.

(3) According to (2.8) above, the statement $\text{Ext}^1_\Lambda(\text{top} P, I) \neq 0$ is equivalent to $\text{cotop} I \cong \text{top} P$. Then by Proposition 2.5, this is equivalent to $I \cong \nu P$. □

Among orders, the class of Gorenstein orders is very important. The definition of Gorenstein orders is the following.

**Definition 2.9.** An $R$-order $\Lambda$ is Gorenstein if $\text{Hom}_R(\Lambda, R)$ is projective as a left $\Lambda$-module, or equivalently, if $\text{Hom}_R(\Lambda^\text{op}, R)$ is projective as a right $\Lambda$-module.

If $\Lambda$ is Gorenstein, then $\text{CM} \Lambda$ is a Frobenius category and therefore $\text{CM} \Lambda$ has a natural structure of a triangulated category [9]. The following lemma is about some basic property of Cohen-Macaulay modules over Gorenstein orders that we need to use later.

**Lemma 2.10.** Let $\Lambda$ be a Gorenstein $R$-order. Consider a non-split short exact sequence
\[
0 \longrightarrow X \xrightarrow{f} P \longrightarrow Y \longrightarrow 0
\]
with $P \in \text{proj} \Lambda$ and $X, Y \in \text{CM} \Lambda$. Then for any $Z \in \text{CM} \Lambda$, we have the following isomorphism
\[
\text{Hom}_\Lambda(X, Z) \cong \text{Ext}^1_\Lambda(Y, Z).
\]

**Proof.** Applying $\text{Hom}_\Lambda(-, Z)$ for $Z \in \text{CM} \Lambda$ to (2.3), we get the following long exact sequence
\[
0 \longrightarrow \text{Hom}_\Lambda(Y, Z) \longrightarrow \text{Hom}_\Lambda(P, Z) \xrightarrow{f^*} \text{Hom}_\Lambda(X, Z) \longrightarrow \text{Ext}^1_\Lambda(Y, Z) \longrightarrow 0.
\]
Since $\Lambda$ is Gorenstein, we have $\text{Ext}^1_\Lambda(Y, \Lambda) = 0$. Therefore, any morphism from $X$ to a projective $\Lambda$-module factors through $f$. In particular, we have $\text{Im} f^* \cong X(Z, Z)$. By the above exact sequence, we have $\text{Hom}_\Lambda(X, Z) \cong \text{Ext}^1_\Lambda(Y, Z)$. □

Notice that when $\Lambda$ is Gorenstein, the Nakayama functor induces an autoequivalence
\[
\nu: \text{CM} \Lambda \xrightarrow{\sim} \text{CM} \Lambda,
\]
which induces a triangle-equivalence $\nu: \text{CM} \Lambda \xrightarrow{\sim} \text{CM} \Lambda$. Also we define the syzygy functor
\[
\Omega: \text{mod} \Lambda \rightarrow \text{CM} \Lambda
\]
as the kernel of projective covers. The composition $\tau := \nu \circ \Omega: \text{CM} \Lambda \xrightarrow{\nu} \text{CM} \Lambda \xrightarrow{\Omega} \text{CM} \Lambda$ is called the Auslander-Reiten translation. If $\Lambda \otimes_R K$ is a semisimple $K$-algebra, then Auslander-Reiten theory works in $\text{CM} \Lambda$ as the following classical results show.

**Proposition 2.11.** If $\Lambda$ is Gorenstein and $\Lambda \otimes_R K$ is a semisimple $K$-algebra, then
\begin{enumerate}
  
  (1) $\text{CM} \Lambda$ is a $R$-finite triangulated category;
  
  (2) we have a functorial isomorphism $\text{Hom}_\Lambda(X, Y) \cong \text{D}_0 \text{Ext}^1_\Lambda(Y, \nu X)$ for any $X, Y \in \text{CM} \Lambda$.
\end{enumerate}

**Proof.** (1) Let us prove the Hom-finiteness. For any $X, Y \in \text{CM} \Lambda$, since $\Lambda \otimes_R K$ is semisimple by our assumption, we have
\[
\text{Ext}^1_\Lambda(X, Y) \otimes_R K = \text{Ext}^1_\Lambda(X \otimes_R K, Y \otimes_R K) = 0.
\]
Therefore, $\text{Ext}^1_\Lambda(X, Y)$ is an $R$-module of finite length. We conclude by using Lemma 2.10.

(2) Note that we have $\tau = \text{Tr} \circ \Omega \circ D_1: \text{CM} \Lambda \rightarrow \text{CM} \Lambda$, where $\text{Tr}$ is the transpose: $\text{mod} \Lambda \rightarrow \text{mod} \Lambda^\text{op}$ [1, 3]. As $R$ has Krull dimension 1, according to [2, Chapter I, Proposition 8.7], we have a functorial isomorphism $\text{Hom}_\Lambda(X, Y) \cong \text{D}_0 \text{Ext}^1_\Lambda(Y, \tau X)$. Therefore, $\text{Hom}_\Lambda(X, Y) \cong \text{D}_0 \text{Hom}_\Lambda(Y, \nu X)$ holds for any $X, Y \in \text{CM} \Lambda$. □
Given an \( R \)-order \( \Lambda \), we can define a valued Auslander-Reiten quiver as follows.

**Definition 2.12** ([20, 3.8 Definition]). Let \( X, Y \in \text{ind}(\text{CM} \; \Lambda) \). Denote
\[
\text{Irr}(X, Y) := \frac{\text{rad}_\Lambda(X, Y)}{\text{rad}_\Lambda^2(X, Y)} \quad \text{and} \quad k_X := \frac{\text{End}_\Lambda(X)}{\text{rad}_\Lambda(X, X)}.
\]
We define
\[
d_{XY} := \dim_{k_X}(\text{Irr}(X, Y)) \quad \text{and} \quad d'_{XY} := \dim(\text{Irr}(X, Y))_{k_Y}.
\]
The **Auslander-Reiten quiver** \( \mathfrak{A}(\Lambda) \) of \( \Lambda \) is a valued translation quiver, where the translation \( \tau \) is the Auslander-Reiten translation, the vertices are the objects in \( \text{ind}(\text{CM} \; \Lambda) \) (more precisely, \( \mathfrak{A}(\Lambda)^p = \text{ind}(\text{proj} \; \Lambda) \) and \( \mathfrak{A}(\Lambda)^i = \text{ind}(\text{inj} \; \Lambda) \)) and the valued arrows are
\[
X \xrightarrow{(d_{XY}, d'_{XY})} Y,
\]
for any \( X, Y \in \text{ind}(\text{CM} \; \Lambda) \) with \( \text{Irr}(X, Y) \neq 0 \). The **stable Auslander-Reiten quiver** \( \mathfrak{A}(\Lambda) \) is the full subquiver of \( \mathfrak{A}(\Lambda) \) whose vertices are the objects in \( \text{ind}(\text{CM} \; \Lambda) \setminus \text{ind}(\text{proj} \; \Lambda) \).

**Remark 2.13** ([20, 3.8]). For any \( X \in \text{ind}(\text{CM} \; \Lambda) \), the right minimal almost split map is of the form
\[
\bigoplus_{Y \in \text{ind}(\text{CM} \; \Lambda)} Y^d_{XY} 
\rightarrow X,
\]
and the left minimal almost split map is of the form
\[
X
\rightarrow \bigoplus_{Y \in \text{ind}(\text{CM} \; \Lambda)} Y^d_{XY}.
\]

We have the following analogue of Riedtmann’s structure theorem of stable Auslander-Reiten quivers [15] for orders.

**Theorem 2.14.** Let \( R \) be a complete discrete valuation ring and \( \Lambda \) be a representation-finite Gorenstein \( R \)-order. The stable Auslander-Reiten quiver \( \mathfrak{A}(\Lambda) \) is isomorphic to a disjoint union of \( \mathbb{Z}\Delta/G \), where \( \Delta \) is a Dynkin diagram and \( G \subset \text{Aut}(\mathbb{Z}\Delta) \) is a weakly admissible group.

**Proof.** By Proposition 2.11, \( \text{CM} \; \Lambda \) is a Hom-finite triangulated category. The statement is a result by Xiao and Zhu [28]. Note that in [28, Theorem 2.3.5], triangulated categories over a field are discussed, but all arguments work in our settings. \( \square \)

### 3. Categorical configuration

**Assumption 3.1.** In this section, we assume that the following constraints are satisfied.
- \( \Lambda \) is a ring-indecomposable representation-finite Gorenstein \( R \)-order;
- \( \text{rad} \; P \) is indecomposable and non-projective for any \( P \in \text{ind}(\text{proj} \; \Lambda) \);
- \( \text{rad} \; Q \not\cong \text{rad} \; P \) for any \( Q \not\cong P \in \text{ind}(\text{proj} \; \Lambda) \).

Under Assumption 3.1, we will show in two main theorems that the set of isomorphism classes of the radicals of indecomposable projective \( \Lambda \)-modules satisfy two conditions. The first one is the following theorem.

**Theorem 3.2.** For any non-projective \( X \in \text{ind}(\text{CM} \; \Lambda) \), there exists \( P \in \text{ind}(\text{proj} \; \Lambda) \) such that
\[\text{Hom}_\Lambda(X, \text{rad} \; P) \neq 0.\]

**Proof.** For any non-projective \( X \in \text{ind}(\text{CM} \; \Lambda) \), thanks to Proposition 2.6, there exists a non-split exact sequence
\[
0 \rightarrow X \rightarrow P \rightarrow Y \rightarrow 0
\]
with \( P \in \text{inj} \; \Lambda \) and \( Y \in \text{CM} \; \Lambda \). According to Lemma 2.10, we have
\[\text{Hom}_\Lambda(X, \text{rad} \; P') \cong \text{Ext}_\Lambda^1(Y, \text{rad} \; P')\]
for any \( P' \in \text{ind}(\text{proj} \; \Lambda) \).
Assume that \( \text{Ext}^1_A(Y, \text{rad} P') = 0 \) holds for any \( P' \in \text{ind}(\text{proj} \Lambda) \). Without losing generality, suppose that \( Y \in \text{ind}(\text{CM} \Lambda) \). Consider the projective cover \( u : P \twoheadrightarrow \text{top} Y \). Since \( \pi : Y \twoheadrightarrow \text{top} Y \) is surjective, we have \( h_1 : P \rightarrow Y \) such that \( h_1 \circ \pi = u \) holds. Since \( \text{Ext}^1_A(Y, \text{rad} P) = 0 \), there exists \( h_2 : Y \rightarrow P \) with \( h_2 \circ u = \pi \). We have the following diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & \text{rad} P & \rightarrow & P & \rightarrow & \text{top} Y & \rightarrow & 0 \\
& & \downarrow{h_1} & \circ & \downarrow{h_2} & & \circ & \downarrow{\pi} & \rightarrow & \text{top} Y & \rightarrow & 0.
\end{array}
\]

Since \( u \) and \( \pi \) are right minimal and \( h_1 \circ h_2 \circ u = u \), \( h_2 \circ h_1 \circ \pi = \pi \), it follows that \( h_1 \) and \( h_2 \) are isomorphisms. Thus, \( P \cong Y \) holds, and (3.1) splits. This contradicts our assumption. \( \square \)

The second categorial condition of configurations is described in the following theorem.

**Theorem 3.3.** For any \( P, Q \in \text{ind}(\text{proj} \Lambda) \), we have

\[
\text{length}_{\text{rad}_A(\text{rad} P)} \text{Hom}_A(\text{rad} P, \text{rad} Q) = \begin{cases}
2, & \text{if } P \cong Q \cong \nu^{-1} Q; \\
1, & \text{if } P \cong Q \not\cong \nu^{-1} Q; \\
1, & \text{if } P \not\cong \nu^{-1} Q \not\cong Q; \\
0, & \text{if } P \not\cong Q, \nu^{-1} Q.
\end{cases}
\]

First, we consider diagrams of the following form with \( P, Q \in \text{ind}(\text{proj} \Lambda) \)

\[
\begin{array}{ccc}
0 & \rightarrow & \text{rad} P & \rightarrow & P & \rightarrow & \text{top} P & \rightarrow & 0 \\
& & \downarrow{f} & \circ & \downarrow{g} & \circ & \downarrow{h} & & \\
0 & \rightarrow & \text{rad} Q & \rightarrow & Q & \rightarrow & \text{top} Q & \rightarrow & 0
\end{array}
\]

(3.2)

which are discussed in the following lemma.

**Lemma 3.4.**

1. For any \( g \in \text{Hom}_A(P, Q) \), there exist \( f \in \text{Hom}_A(\text{rad} P, \text{rad} Q) \) and \( h \in \text{Hom}_A(\text{top} P, \text{top} Q) \) such that (3.2) is commutative.

2. Assume \( Q \not\cong \nu P \). For any \( f \in \text{Hom}_A(\text{rad} P, \text{rad} Q) \), there exist \( g \in \text{Hom}_A(P, Q) \) and \( h \in \text{Hom}_A(\text{top} P, \text{top} Q) \) such that (3.2) is commutative.

3. If \( h = 0 \) in (3.2), then \( f \) factors through \( \text{proj} \Lambda \).

**Proof.**

1. Since any morphism \( g : P \rightarrow Q \) induces a morphism \( f : \text{rad} P \rightarrow \text{rad} Q \), then there exists \( h \in \text{Hom}_A(\text{top} P, \text{top} Q) \) such that (3.2) is commutative.

2. It is sufficient to find \( g \in \text{Hom}_A(P, Q) \) such that \( i_P \circ g = f \circ i_Q \). Since \( Q \not\cong \nu P \), we have \( \text{Ext}^1_A(\text{top} P, Q) = 0 \) by Lemma 2.8. By applying \( \text{Hom}_A(-, Q) \) to the short exact sequence

\[
0 \rightarrow \text{rad} P \rightarrow \nu P \rightarrow \text{top} P \rightarrow 0,
\]

we get the following long exact sequence

\[
0 \rightarrow \text{Hom}_A(\text{top} P, Q) \rightarrow \text{Hom}_A(P, Q) \rightarrow \text{Hom}_A(\text{rad} P, Q) \rightarrow \text{Ext}^1_A(\text{top} P, Q) = 0.
\]

Therefore, there exists \( g \in \text{Hom}_A(P, Q) \) such that \( i_P \circ g = f \circ i_Q \).

3. Since \( h = 0 \), we have \( g \circ \pi_Q = 0 \). Thus, there exists \( t : P \rightarrow \text{rad} Q \) such that \( g = t \circ i_Q \) holds. It follows \( f \circ i_Q = i_P \circ t \circ i_Q \), therefore, \( f = i_P \circ t \in [\Lambda](\text{rad} P, \text{rad} Q) \). \( \square \)

**Lemma 3.5.** For any \( P, Q \in \text{ind}(\text{proj} \Lambda) \), we have an isomorphism:

\[
H : \text{Hom}_A(\text{rad} P, \text{rad} Q) \rightarrow \text{Hom}_A(\text{rad} P, Q).
\]
Proof. By applying $\text{Hom}_\Lambda(\text{rad } P, -)$ to the short exact sequence $0 \to \text{rad } Q \xrightarrow{i_Q} Q \xrightarrow{\pi_Q} \text{top } Q \to 0$, we get the monomorphism $H : \text{Hom}_\Lambda(\text{rad } P, \text{rad } Q) \to \text{Hom}_\Lambda(\text{rad } P, Q)$. We show that $H$ is surjective. Indeed, for any $g \in \text{Hom}_\Lambda(\text{rad } P, Q)$, since $i_Q$ is minimal right almost split and $\text{rad } P \not\cong Q$ (as $\text{rad } P$ is non-projective), there exists $f \in \text{Hom}_\Lambda(\text{rad } P, \text{rad } Q)$ satisfying $g = f \circ i_Q$. □

Now we are ready to give the proof of Theorem 3.3.

Proof of Theorem 3.3. (i) Assume $P \ncong \nu^{-1}Q$, or equivalently, $Q \ncong \nu P$. Fix $f \in \text{Hom}_\Lambda(\text{rad } P, \text{rad } Q)$. By Lemma 3.4, we have the commutative diagram (3.2).

(i-i) Assume $P \ncong Q$, or equivalently, $\text{top } P \ncong \text{top } Q$. Then $h = 0$ holds, and $f$ factors through $\text{proj } \Lambda$ according to Lemma 3.4(3). Thus, $\text{Hom}_\Lambda(\text{rad } P, \text{rad } Q) = 0$.

(i-ii) Assume $P \cong Q$. There is an isomorphism

$$\phi : \text{End}_\Lambda(\text{rad } P)/\text{rad}_\Lambda(\text{rad } P, \text{rad } P) \to \text{Hom}_\Lambda(\text{rad } P, \text{rad } P).$$

In fact, if $f$ is a non-isomorphism in (3.2), then $g$ is not an isomorphism. Hence, since $P$ and $Q$ are projective, $h$ is not an isomorphism, in particular $h = 0$ and $f$ factors through $\text{proj } \Lambda$ by Lemma 3.4(3). This means that $\phi$ is well-defined and surjective. If $f$ is an isomorphism in (3.2), we have $f \notin [\Lambda](\text{rad } P, \text{rad } P)$ since $\text{rad } P$ is non-projective. Thus, $\phi$ is injective. We have $\text{length}_\text{End}_\Lambda(\text{rad } P) \text{Hom}_\Lambda(\text{rad } P, \text{rad } P) = 1$.

(ii) Assume $P \cong \nu^{-1}Q$. We will prove that there is the following commutative diagram:

\[
\begin{array}{cccccc}
\text{rad}_\Lambda(P, Q) & \xrightarrow{i^*} & \text{Hom}_\Lambda(P, Q) & \xrightarrow{i^*_Q} & \text{Hom}_\Lambda(\text{rad } P, Q) & \xrightarrow{d^*} & \text{Ext}_{\Lambda}^1(\text{top } P, Q) & \to 0 \\
0 & \xrightarrow{i} & [\Lambda](\text{rad } P, \text{rad } Q) & \xrightarrow{i^*_i} & \text{Hom}_\Lambda(\text{rad } P, \text{rad } Q) & \xrightarrow{\nu^{-1}} & \text{Hom}_\Lambda(\text{rad } P, \text{rad } Q) & \to 0,
\end{array}
\]

where $H$ is an isomorphism (by Lemma 3.5) and $F$ is surjective. We get the first row by applying $\text{Hom}_\Lambda(-, Q)$ to $0 \to \text{rad } P \to \text{rad } Q \to \text{top } P \to 0$. And the second row comes from the definition of $\text{Hom}_\Lambda(-, -)$. We will prove the above statement in two steps.

First, we construct the morphism $F$. By Lemma 3.4 (1), for any $g \in \text{rad}_\Lambda(P, Q)$, we have the commutative diagram (3.2). Since $g$ is a nonisomorphism, it follows that $h = 0$. By Lemma 3.4 (3), we have $f$ factors through $\text{proj } \Lambda$. Therefore, we define the morphism

$$F : \text{rad}_\Lambda(P, Q) \to [\Lambda](\text{rad } P, \text{rad } Q) \quad \text{by} \quad g \to f.$$

Since $(i^*_i \circ i^*_Q \circ H^{-1})(g) = f = (F \circ i_*) (g)$, the first square of (3.3) is commutative.

Secondly, we show that $F$ is surjective. For any $f \in [\Lambda](\text{rad } P, \text{rad } Q)$, there exists $I \in \text{proj } \Lambda$ such that the following diagram commutes:

\[
\begin{array}{cccccc}
0 & \to & \text{rad } P & \xrightarrow{i_P} & P & \xrightarrow{\pi_P} & \text{top } P & \to 0 \\
0 & \to & \text{rad } Q & \xrightarrow{i_Q} & Q & \xrightarrow{\pi_Q} & \text{top } Q & \to 0.
\end{array}
\]

Let us prove that the morphism from $\text{Ext}_{\Lambda}^1(\text{top } P, I)$ to $\text{Ext}_{\Lambda}^1(\text{top } P, Q)$ induced by $t \circ i_Q$ is 0. Indeed, thanks to Lemma 2.8, either $\text{Ext}_{\Lambda}^1(\text{top } P, I) = 0$ holds or $I \cong Q \cong \nu P$ holds. In the first case, the conclusion is obvious. In the second case, as the morphism $t \circ i_Q : I \to Q$ is not an isomorphism, it induces a morphism $\chi$ in $\text{rad}_\Lambda(Q, Q) \subset \text{End}_\Lambda(Q)$. Moreover, by the same lemma, $\text{Ext}_{\Lambda}^1(\text{top } P, Q)$ is a simple $\text{End}_\Lambda(Q)$-module. Thus, the morphism from $\text{Ext}_{\Lambda}^1(\text{top } P, I)$ to $\text{Ext}_{\Lambda}^1(\text{top } P, Q)$ induced by $t \circ i_Q$ has its image included in $\chi \cdot \text{Ext}_{\Lambda}^1(\text{top } P, Q) = 0$. Then $d^*(s \circ t \circ i_Q) = d^*(f \circ i_Q) = 0$, thus, there exists $g : P \to Q$ such that $f \circ i_Q = i_P \circ g$. Since $\text{rad } P$ and $\text{rad } Q$ are non-projective, it follows that $f$ is not an isomorphism which means that $g \in \text{rad}_\Lambda(P, Q)$. The morphism $F$ is surjective.
(ii-i) Assume \( P \not\cong Q \). Consider the diagram (3.3) in this case. The morphism \( i'_P : \text{rad}_\Lambda(P, Q) \to \text{Hom}_\Lambda(P, Q) \) is an isomorphism. Then

\[
G : \text{Ext}^1_\Lambda(\text{top} P, Q) \to \text{Hom}_\Lambda(\text{rad} P, \text{rad} Q)
\]

is an isomorphism. According to Lemma 2.8, \( \text{Ext}^1_\Lambda(\text{top} P, Q) \) is a simple \( \text{End}_\Lambda(Q) \)-module, so of dimension 1 over \( k_Q \). Since we have \( k_Q = k_{P} \cong k_{\text{rad} P} \) (because each automorphism of \( P \) induces an automorphism of \( \text{rad} P \)), it follows that \( \text{length}_{\text{End}_\Lambda(\text{rad} P)} \text{Hom}_\Lambda(\text{rad} P, \text{rad} Q) = 1 \).

(ii-ii) Assume \( P \cong Q \). Consider the diagram (3.3). We show that \( F \) is also injective. Indeed, for any \( g \in \text{rad}_\Lambda(P, Q) \) in (3.2), if \( F(g) = f = 0 \), then \( g = 0 \) since \( h = 0 \). Hence, \( F \) is an isomorphism. Then we have

\[
\text{End}_\Lambda(\text{rad} P) = \frac{\text{End}_\Lambda(\text{rad} P)}{[\Lambda](\text{rad} P, \text{rad} P)} \cong \frac{\text{Hom}_\Lambda(\text{rad} P, Q)}{\text{Ext}^1_\Lambda(\text{rad} P, Q)}.
\]

According to Lemma 2.8, \( \text{Ext}^1_\Lambda(\text{top} P, P) \) is a \( k_P \)-vector space of dimension 1. As \( k_P \subset k_{\text{rad} P} \), then

\[
\text{length}_{\text{End}_\Lambda(\text{rad} P)} \text{Hom}_\Lambda(\text{rad} P, P) = \text{length}_{\text{End}_\Lambda(\text{rad} P)} \text{Ext}^1(\text{top} P, P) = 1.
\]

Since we have \( \text{Hom}_\Lambda(\text{rad} P, P) \supset i'_P(\text{End}_\Lambda(P)) \supset i'_P(\text{rad}_\Lambda(P, P)) \), it follows that

\[
\text{length}_{\text{End}_\Lambda(\text{rad} P)} \text{Hom}_\Lambda(\text{rad} P, P) \text{Ext}^1(\text{top} P, P) + \text{length}_{\text{End}_\Lambda(\text{rad} P)} i'_P(\text{End}_\Lambda(P)/\text{rad}_\Lambda(P, P)) = 2. \]

\[
\square
\]

4. Reading hom-sets from Auslander-Reiten quivers

In this section, we give a simple method to read the hom-set between two objects in \( \text{CMA} \) by looking the related positions of this two objects in the Auslander-Reiten quiver. First, we introduce some combinatorial notions which will be used to calculate the length of the hom-sets.

For \( x, y \in (\mathbb{Z}\Delta)_0 \), we define

\[
\delta(y, x) = \begin{cases} 1, & \text{if } y = x; \\ 0, & \text{otherwise}. \end{cases}
\]

**Definition 4.1.** Let \( Q \) be a stable valued translation quiver and \( x \) be a fixed vertex in \( Q_0 \).

1. For each \( n \in \mathbb{Z} \), we define a map

\[
h_{Q, n}(-, x) : Q_0 \to \mathbb{N}_0
\]

as follows.

- \( h_{Q, n}(-, x) := 0 \), if \( n < 0 \);
- \( h_{Q, 0}(-, x) := \delta(-, x) \);
- For \( n > 0 \), we let

\[
h'_{Q, n}(y, x) := \sum_{v \in Q_0} d_{yv} h_{Q, n-1}(v, x) - h_{Q, n-2}(\tau^{-1} y, x)
\]

and \( h_{Q, n}(-, x) := \max\{h'_{Q, n}(-, x), 0\} \).

2. By using \( h_{Q, n}(-, x) \), we get a map

\[
h_Q(-, x) := \sum_{n \geq 0} h_{Q, n}(-, x)
\]

and then a set

\[
H_Q(x) = \{ y \in Q_0 \mid h_Q(y, x) > 0 \}.
\]

If there is no danger of confusion, we denote \( h_{Q, n} \) by \( h_n \), \( h_Q \) by \( h \) and \( H_Q \) by \( H \) respectively.

Let us see an example of type \( A \).
Example 4.2. Let $x$ be a vertex in $(\mathbb{Z}\Delta)_0$ for a Dynkin quiver $\Delta$ of type $A$. The set $H(x)$ is the set consisting of all the vertices in the rectangle of Figure 1 and the values on the vertices are all the non-zero results of $h(\cdot, x)$ acting on the vertices.

![Figure 1](image)

The following theorem is about how to read hom-sets from Auslander-Reiten quiver by using the map $h$ defined in Definition 4.1.

**Theorem 4.3.** Let $\Lambda$ be a representation-finite $R$-order (not necessarily Gorenstein) and $\mathfrak{X}(\Lambda)$ the Auslander-Reiten quiver of $\text{CM}_\Lambda$. For any $X, Y \in \text{ind}(\text{CM}_\Lambda)$, we have

$$\text{length}_{\text{End}_\Lambda(Y)} \text{Hom}_\Lambda(Y, X) = h_{\mathfrak{X}(\Lambda)}(Y, X).$$

In particular, for any $X \in \text{ind}(\text{CM}_\Lambda)$, we have

$$H_{\mathfrak{X}(\Lambda)}(X) = \{ Y \in \text{ind}(\text{CM}_\Lambda) \mid \text{Hom}_\Lambda(Y, X) \neq 0 \}.$$

The remaining part of this section is devoted to prove Theorem 4.3. For any $P = \sum_{x \in Q_0} p_x x \in \mathbb{Z}Q_0$ where $\mathbb{Z}Q_0$ is the free Abelian group generated by $Q_0$, we denote $P_+ := \sum_{x \in Q_0} \max\{p_x, 0\} x$ and denote by $\text{supp} P$ the set $\{ x \in Q_0 \mid p_x \neq 0 \}$.

**Definition 4.4.** Let $Q$ be a stable valued translation quiver and $N_0Q_0$ the free Abelian monoid generated by $Q_0$.

1. We define a map $\theta : N_0Q_0 \to N_0Q_0$ by $\theta(x) := \sum_{y \rightarrow x \in Q_1} d_{yx} y$.
2. For $n \in N_0$ and $x \in Q_0$, we define $\theta_n(x) \in N_0Q_0$ by

$$\theta_n(x) := \begin{cases} x, & \text{if } n = 0; \\ \theta(x), & \text{if } n = 1; \\ (\theta(\theta_{n-1}(x)) - \tau(\theta_{n-2}(x)))_+, & \text{if } n \geq 2. \end{cases}$$

**Lemma 4.5.** Let $Q$ be a stable valued translation quiver. For any $x \in Q_0$ and $n \in N_0$, we have

$$\theta_n(x) = \sum_{y \in Q_0} h_n(y, x)y \quad \text{and} \quad H(x) = \bigcup_{i \geq 0} \text{supp} \theta_i(x).$$

**Proof.** This can be shown inductively by using the definitions. \hfill \square

Consider an $R$-order $\Lambda$. The Auslander-Reiten quiver $\mathfrak{X}(\Lambda)$ of $\text{CM}_\Lambda$ is a stable translation quiver. We consider the bijection that sends the set of isomorphism classes of objects in $\text{CM}_\Lambda$ to $N_0\mathfrak{X}(\Lambda)_0$ by

$$X \cong \bigoplus_{i=1}^l M_i \mapsto X = \sum_{i=1}^l t_i M_i,$$

where $M_i \in \mathfrak{X}(\Lambda)_0$. Hence, for any $X \in \text{CM}_\Lambda$, we have $\theta_n(X) \in \text{CM}_\Lambda$ naturally.

**Theorem 4.6** ([12, Theorems 4.1, 7.1]). Let $\Lambda$ be an $R$-order and $X \in \text{CM}_\Lambda$. We have the following isomorphism of functors

$$\text{rad}_{\text{CM}_\Lambda}^n(-, X)/\text{rad}_{\text{CM}_\Lambda}^{n+1}(-, X) \cong \text{Hom}_\Lambda(-, \theta_n(X))/\text{rad}_{\text{CM}_\Lambda}(-, \theta_n(X)).$$
Proposition 5.1. Let \( \Delta \) be a Dynkin diagram and \( x \in (\mathbb{Z}\Delta)_0 \). There exist a positive integer \( m \) and a vertex \( y \in (\mathbb{Z}\Delta)_0 \) such that

\[
h_{m-1}(-, x) = \delta(-, y) \quad \text{and} \quad h_n(-, x) = \begin{cases} h'_n(-, x), & \text{if } n < m; \\ 0, & \text{if } n \geq m. \end{cases}
\]

Proof. For Dynkin type \( A_n, B_n, C_n, D_n, E_6, E_7 \) and \( E_8 \), it is easy to check by calculation. Computations of Sections 6, 7, 8 and 9 will make it clear. See Figure 2 for type \( A \) in Section 6, Figure 3 for type \( B, C \) in Sections 7 and 8, Case 1 and Case 2 in Section 9 for type \( D \). \( \square \)

Definition 5.2. Let \( \Delta \) be a Dynkin diagram and \( G \subset \text{Aut}(\mathbb{Z}\Delta) \) a weakly admissible group.

(1) For each \( x \in (\mathbb{Z}\Delta)_0 \), we define \( \omega(x) := y \), where \( y \) is the vertex satisfying \( h_{m-1}(-, x) = \delta(-, y) \) in Proposition 5.1. Then \( \omega \) gives an automorphism of \( \mathbb{Z}\Delta \).

(2) Since \( \omega(gx) = g(\omega(x)) \) holds for each \( g \in G \) and \( x \in (\mathbb{Z}\Delta)_0 \), then \( \omega \) induces an automorphism of the quotient quiver \( \mathbb{Z}\Delta/G \) defined in Subsection 2.1 given by \( \omega(Gx) = G(\omega(x)) \) for any \( x \in (\mathbb{Z}\Delta)_0 \).

For any \( x \in (\mathbb{Z}\Delta)_0 \), we have

\[
h_{\mathbb{Z}\Delta}(\omega(x), x) = 1.
\]

(5.1)

By using \( \omega \) and the map \( h \), we define combinatorial configurations as follows.

Definition 5.3. Let \( Q \) be a stable valued translation quiver of the form \( \mathbb{Z}\Delta/G \), where \( \Delta \) is a Dynkin diagram and \( G \subset \text{Aut}(\mathbb{Z}\Delta) \) is a weakly admissible group. A configuration \( C \) of \( Q \) is a set of vertices satisfying the following two conditions:

(C1) For any vertex \( x \in Q_0 \), there exists a vertex \( c \in C \) such that \( h(x, c) > 0 \) which equivalently means

\[
Q_0 = \bigcup_{c \in C} H(c).
\]

(C2) \( \omega(C) = C \) holds. Moreover, for any vertices \( c, d \in C \), we have

\[
h(d, c) = \begin{cases} 2, & \text{if } d = c = \omega(c); \\ 1, & \text{if } d = c \neq \omega(c); \\ 1, & \text{if } d = \omega(c) \neq c; \\ 0, & \text{otherwise}. \end{cases}
\]
Remark 5.4. When \( Q = \mathbb{Z}\Delta \), then (C2) is equivalent to \( C \cap H(c) = \{ c, \omega(c) \} \) by (5.1).

Now we compare the maps \( h \) for \( \mathbb{Z}\Delta \) and \( \mathbb{Z}\Delta/G \) which we used to define configurations in the following lemma.

Lemma 5.5. Let \( \Delta \) be a Dynkin diagram and \( G \subset \text{Aut}(\mathbb{Z}\Delta) \) be a weakly admissible group. Consider the canonical morphism \( \pi : \mathbb{Z}\Delta \to \mathbb{Z}\Delta/G \) associated to the translation quiver \( \mathbb{Z}\Delta \) and \( G \). For any \( x, y \in (\mathbb{Z}\Delta)_0 \), we have

\[
 h_{\mathbb{Z}\Delta/G}(\pi y, \pi x) = \sum_{y' \in G y} h_{\mathbb{Z}\Delta}(y', x).
\]

Proof. According to Definition 4.1, it is sufficient to prove that

\[
 h_{\mathbb{Z}\Delta/G,n}(\pi y, \pi x) = \sum_{y' \in G y} h_{\mathbb{Z}\Delta,n}(y', x)
\]

holds for any \( x, y \in (\mathbb{Z}\Delta)_0 \) and \( n \geq 0 \). We use induction on \( n \).

When \( n = 0 \), it is easy to check.

By the definition of quotients of valued translation quivers and weakly admissible automorphism groups in Subsection 2.1, we have

\[
 d_{\pi y\pi v} = \sum_{y' \in G y} d_{y'\pi v}.
\]

Assume that \( n \geq 1 \). Suppose that \( h_{\mathbb{Z}\Delta/G,i}(\pi y, \pi x) = \sum_{y' \in G y} h_{\mathbb{Z}\Delta,i}(y', x) \) holds for any \( i \leq n - 1 \). We have

\[
 \sum_{\pi v \in (\mathbb{Z}\Delta/G)_0} d_{\pi y\pi v} h_{\mathbb{Z}\Delta/G,n-1}(\pi v, \pi x) = \sum_{\pi v \in (\mathbb{Z}\Delta/G)_0} d_{\pi y\pi v} \sum_{v' \in G v} h_{\mathbb{Z}\Delta,n-1}(v', x)
\]

\[
 = \sum_{v \in (\mathbb{Z}\Delta)_0} d_{\pi y\pi v} h_{\mathbb{Z}\Delta,n-1}(v, x)
\]

\[
 = \sum_{v \in (\mathbb{Z}\Delta)_0} \left( \sum_{y' \in G y} d_{y'} \right) h_{\mathbb{Z}\Delta,n-1}(v, x)
\]

\[
 = \sum_{y' \in G y} \sum_{\pi v \in (\mathbb{Z}\Delta)_0} d_{y'\pi v} h_{\mathbb{Z}\Delta,n-1}(\pi v, \pi x),
\]

and

\[
 h_{\mathbb{Z}\Delta/G,n-2}(\pi^{-1} y, \pi x) = \sum_{y' \in G(\pi^{-1} y)} h_{\mathbb{Z}\Delta,n-1}(y', x)
\]

\[
 = \sum_{y' \in G y} h_{\mathbb{Z}\Delta,n-2}(\pi^{-1} y, x).
\]

By definition, we get

\[
 h'_{\mathbb{Z}\Delta/G,n}(\pi y, \pi x) = \sum_{\pi v \in (\mathbb{Z}\Delta/G)_0} d_{\pi y\pi v} h_{\mathbb{Z}\Delta/G,n-1}(\pi v, \pi x) - h_{\mathbb{Z}\Delta/G,n-2}(\pi^{-1} y, \pi x)
\]

\[
 = \sum_{y' \in G y} \left( \sum_{v \in (\mathbb{Z}\Delta)_0} d_{y'\pi v} h_{\mathbb{Z}\Delta,n-1}(\pi v, \pi x) - h_{\mathbb{Z}\Delta,n-2}(\pi^{-1} y, x) \right)
\]

\[
 = \sum_{y' \in G y} h'_{\mathbb{Z}\Delta,n}(y', x).
\]

By Proposition 5.1, there exists an integer \( m \) such that

\[
 \begin{cases} 
 h'_{\mathbb{Z}\Delta,n}(-, x) \geq 0, & \text{if } n < m; \\
 h'_{\mathbb{Z}\Delta,n}(-, x) \leq 0, & \text{if } n \geq m.
\end{cases}
\]
Therefore, we have
\[
h_{Z\Delta/G,n}(\pi y, \pi x) = \max\{h'_{Z\Delta/G,n}(\pi y, \pi x), 0]\}
\[
= \max\left\{\sum_{y' \in G} h'_{Z\Delta,n}(y', x), 0\right\}
\[
= \sum_{y' \in G} \max\{h'_{Z\Delta,n}(y', x), 0\}
\[
= \sum_{y' \in G} h_{Z\Delta,n}(y', x).
\]

By Lemmas 4.5, 5.5 and Proposition 5.1, we get the following lemma by direct calculation.

**Lemma 5.6.** Let $Q$ be a stable translation quiver of the form $\mathbb{Z}\Delta/G$, where $\Delta$ is a Dynkin diagram and $G \subset \text{Aut} (\mathbb{Z}\Delta)$ is a weakly admissible group. For any $X \in Q_0$, we have
\[
h_{Z\Delta/G,m-1}(-,X) = \delta(-,\omega X) \quad \text{and} \quad \omega X = \theta_{m-1}(X),
\]
where $m$ is the same integer as in Proposition 5.1 for $Z\Delta$ and $x \in \pi^{-1}(X)$. Moreover, we get
\[
h_{Z\Delta/G,k}(-,X) = 0 \quad \text{and} \quad \theta_k(X) = 0
\]
for any $k \geq m$.

The definitions about configurations and $\omega$ are completely combinatorial, we show in the following proposition that the automorphism $\omega$ also has some categorical meaning. Recall that in Theorem 2.14, for any representation-finite Gorenstein $R$-order $\Lambda$, the stable Auslander-Reiten quiver $\mathfrak{A}(\Lambda)$ is isomorphic to $\mathbb{Z}\Delta/G$ for some Dynkin diagram $\Delta$ and some weakly admissible group $G \subset \text{Aut}(\mathbb{Z}\Delta)$.

**Proposition 5.7.** Let $\Lambda$ be a representation-finite Gorenstein $R$-order with $\mathfrak{A}(\Lambda) \cong \mathbb{Z}\Delta/G$. For any $X \in \text{ind}(\text{CM}_\Lambda)$, we have $\omega X = \nu^{-1}X$.

**Proof.** Fix any $X \in \text{ind}(\text{CM}_\Lambda)$. We denote by $M$ the direct sum $\bigoplus_{Y \in \text{ind}(\text{CM}_\Lambda)} Y$, by $E_Y : M \to M$ the idempotent at $Y$ for any $Y \in \text{ind}(\text{CM}_\Lambda)$ and by $\iota : X \to M$ the canonical inclusion. Since all the morphisms from $X$ to $M$ factor through $\iota$, it follows that $\iota$ generates $\text{Hom}_\Lambda(X,M)$ as an $\text{End}_\Lambda(M)^{\text{op}}$-module. Moreover, $\iota E_X = \iota$ holds. Therefore, we have
\[
\text{top}_{\text{End}_\Lambda(M)^{\text{op}}} \text{Hom}_\Lambda(X,M) = \left(\text{top}_{\text{End}_\Lambda(M)^{\text{op}}} \text{Hom}_\Lambda(X,M)\right) E_X,
\]
where $\text{top}_{\text{End}_\Lambda(M)^{\text{op}}} \text{Hom}_\Lambda(X,M)$ is the top of $\text{Hom}_\Lambda(X,M)$ as an $\text{End}_\Lambda(M)^{\text{op}}$-module.

By Proposition 2.11 (2), we have $\text{Hom}_\Lambda(X,M) \cong D_0 \text{Hom}_\Lambda(\nu^{-1}M,X)$. So we have
\[
\text{soc}_{\text{End}_\Lambda(M)} \text{Hom}_\Lambda(\nu^{-1}M,X) \cong \text{soc}_{\text{End}_\Lambda(M)} D_0 \text{Hom}_\Lambda(\nu^{-1}M,X) \cong D_0 \text{top}_{\text{End}_\Lambda(M)^{\text{op}}} \text{Hom}_\Lambda(X,M),
\]
where $\text{soc}_{\text{End}_\Lambda(M)} \text{Hom}_\Lambda(\nu^{-1}M,X)$ is the socle of $\text{Hom}_\Lambda(\nu^{-1}M,X)$ as an $\text{End}_\Lambda(M)$-module. Thus, by (5.2), the following equation holds:
\[
E_X \left(\text{soc}_{\text{End}_\Lambda(M)} \text{Hom}_\Lambda(\nu^{-1}M,X)\right) = \text{soc}_{\text{End}_\Lambda(M)} \text{Hom}_\Lambda(\nu^{-1}M,X).
\]
As $\nu^{-1}M$ can be identified to $M$ and $\nu^{-1}E_X = E_{\nu^{-1}X}$ under this identification, we get
\[
E_{\nu^{-1}X} \left(\text{soc}_{\text{End}_\Lambda(M)} \text{Hom}_\Lambda(M,X)\right) = \text{soc}_{\text{End}_\Lambda(M)} \text{Hom}_\Lambda(M,X).
\]
Thanks to Theorem 4.6, we have the following isomorphism of functors
\[
F_n(-) := \text{rad}_{\text{CM}_\Lambda}^n(-,X)/\text{rad}_{\text{CM}_\Lambda}^{n+1}(\nu^{-1}M,X) \cong \text{Hom}_\Lambda(-,\theta_n(X))/\text{rad}_{\text{CM}_\Lambda}(\nu^{-1}M,X).
\]
By Lemma 5.6 and (5.3), we know $F_{m-1}(-,X) \neq 0$ and $F_k(M) = 0$ for any $k \geq m$ since $\theta_k(X) = 0$. As $\Lambda$ is a representation-finite $R$-order, there exists an integer $l$ such that $\text{rad}_{\text{CM}_\Lambda}(M,-,X) = 0$. Hence, for any $k \geq m$, we have $\text{rad}_{\text{CM}_\Lambda}^k(M,X) = 0$. 
Let $f$ be a nonzero morphism in $\text{rad}_{\text{CM}}(\theta_{m-1}(X), X) \subset \text{rad}_{\text{CM}}(M, X)$. For any $g \in \text{rad}_{\text{CM}}(M, M)$, $g \circ f = 0$ holds since $g \circ f \in \text{rad}_{\text{CM}}(M, X) = 0$. Hence, we get

$$f \in \text{soc}_{\text{End}_{A}(M)}\text{Hom}_{A}(M, X) = E_{\nu^{-1}X}(\text{soc}_{\text{End}_{A}(M)}\text{Hom}_{A}(M, X)).$$

Since $f = E_{\nu^{-1}X} \circ f = E_{\nu^{-1}X} \circ E_{\theta_{m-1}(X)} \circ f$, it follows that $E_{\nu^{-1}X} \circ E_{\theta_{m-1}(X)} \neq 0$. Therefore, we get $\nu^{-1}X = \theta_{m-1}(X) = \omega X$. \hfill $\Box$

By using the above proposition and the categorial properties of the radicals we showed in Section 3, we can prove the following main theorem now.

**Theorem 5.8.** Let $R$ be a complete discrete valuation ring and $\Lambda$ be a ring-indecomposable representation-finite Gorenstein $R$-order. Assume the following conditions satisfied:

- $\text{rad } P$ is indecomposable and non-projective for any $P \in \text{ind}(\text{proj } \Lambda)$;
- $\text{rad } P \neq \text{rad } Q$ when $P \neq Q \in \text{ind}(\text{proj } \Lambda)$.

Then the Auslander-Reiten quiver $\mathfrak{A}(\Lambda)$ of CM $\Lambda$ is isomorphic to $(\mathbb{Z}\Delta/G)_C$, where $\Delta$ is a Dynkin diagram, $G \subset \text{Aut}(\mathbb{Z}\Delta)$ is a weakly admissible group and $C$ is a configuration of $\mathbb{Z}\Delta/G$.

**Proof.** By Theorem 2.14, the Auslander-Reiten quiver $\mathfrak{A}(\Lambda)$ of CM $\Lambda$ is isomorphic to the stable translation quiver $\mathbb{Z}\Delta/G$ for some Dynkin diagram $\Delta$ and some weakly admissible group $G \subset \text{Aut}(\mathbb{Z}\Delta)$.

Consider the set

$$C_\Lambda := \{\text{rad } P \mid P \text{ is indecomposable projective}\}.$$ 

According to Theorem 3.2, for any $X \in \mathfrak{A}(\Lambda)$, there exists an indecomposable projective $\Lambda$-module $P$ such that $\text{Hom}_{\Lambda}(X, \text{rad } P) \neq 0$. It follows Theorem 4.3 that $X \in \text{H}_{\mathfrak{A}(\Lambda)}(\text{rad } P)$. Hence, $C_\Lambda$ satisfies the condition (C1) in Definition 5.3. By Theorem 3.3, we know that

$$\text{length}_{\text{End}_{A}(\text{rad } P)}\text{Hom}_{A}(\text{rad } P, \text{rad } Q) = \begin{cases} 2, & \text{if } P \cong Q \cong \nu^{-1}Q; \\ 1, & \text{if } P \cong Q \not\cong \nu^{-1}Q; \\ 1, & \text{if } P \not\cong \nu^{-1}Q \not\cong Q; \\ 0, & \text{otherwise}. \end{cases}$$

Since $\text{rad } P$ and $\text{rad } Q$ are non-projective and indecomposable, according to Theorem 4.3, it means that

$$h_{\mathfrak{A}(\Lambda)}(\text{rad } P, \text{rad } Q) = \begin{cases} 2, & \text{if } \text{rad } P \cong \text{rad } Q \cong \omega(\text{rad } Q); \\ 1, & \text{if } \text{rad } P \cong \text{rad } Q \not\cong \omega(\text{rad } Q); \\ 1, & \text{if } \text{rad } P \not\cong \omega(\text{rad } Q) \not\cong \text{rad } Q; \\ 0, & \text{otherwise}. \end{cases}$$

Since $\omega(C_\Lambda) = \nu^{-1}(C_\Lambda) = C_\Lambda$, it follows that (C2) holds. Therefore, $C_\Lambda$ is a configuration of $\mathbb{Z}\Delta/G$ by Definiton 5.3.

By Definition 2.1, we get a new translation quiver $(\mathbb{Z}\Delta/G)_{C_\Lambda}$ by adding to $\mathbb{Z}\Delta/G$ a vertex $P$ and arrows

$$\text{rad } P \to P \to \tau^{-1}(\text{rad } P)$$

for each $\text{rad } P \in C_\Lambda$. By definition of $\text{rad } P$ and the fact that any epimorphism to $P$ splits,

$$\text{rad } P \to P$$

represents all the irreducible morphisms ending at $P$. Then by Auslander-Reiten theory, since $\text{rad } P$ is non-projective and indecomposable for each indecomposable projective $\Lambda$-module $P$,

$$P \to \tau^{-1}(\text{rad } P)$$

represents all the irreducible morphisms starting from $P$. And because $\text{rad } P \not\cong \text{rad } Q$ when $P \not\cong Q \in \text{ind}(\text{proj } \Lambda)$, the Auslander-Reiten quiver $\mathfrak{A}(\Lambda)$ is isomorphic to $(\mathbb{Z}\Delta/G)_{C_\Lambda}$. \hfill $\Box$

The remaining part of the section is devoted to study connections between combinatorial configurations of $\mathbb{Z}\Delta$ and combinatorial configurations of $\mathbb{Z}\Delta/G$. By using Lemma 5.5, we show in the following proposition that configurations of $\mathbb{Z}\Delta/G$ are essentially the same as configurations of $\mathbb{Z}\Delta$. 
Proposition 5.9. Let $\Delta$ be a Dynkin diagram and $G$ an weakly admissible automorphism group of $\mathbb{Z}\Delta$. Consider the canonical morphism $\pi : \mathbb{Z}\Delta \to \mathbb{Z}\Delta/G$. We have the following one-to-one correspondence

$$\{ \text{configurations in } \mathbb{Z}\Delta/G \} \leftrightarrow \{ \text{configurations in } \mathbb{Z}\Delta \text{ preserved by } G \}.$$

Proof. Let show that a set $C$ of vertices of $\mathbb{Z}\Delta/G$ is a configuration of $\mathbb{Z}\Delta/G$ if and only if $\pi^{-1}(C)$ is a configuration of $\mathbb{Z}\Delta$. According to Lemma 5.5, for any $x, y \in (\mathbb{Z}\Delta)_0$, we have

$$h_{\mathbb{Z}\Delta/G}(\pi x, \pi y) = \sum_{x' \in Gx} h_{\mathbb{Z}\Delta}(x', y).$$

Let $x \in (\mathbb{Z}\Delta)_0$. If there exists a vertex $c \in \pi^{-1}(C)$ such that $h_{\mathbb{Z}\Delta}(x, c) > 0$, then $h_{\mathbb{Z}\Delta/G}(\pi x, \pi c) > 0$. If there exists $\pi c \in C$ such that $h_{\mathbb{Z}\Delta/G}(\pi x, \pi c) > 0$, then there exists $x' \in Gx$ such that $h_{\mathbb{Z}\Delta}(x', c) > 0$. Since $h_{\mathbb{Z}\Delta}(gx, gc) = h_{\mathbb{Z}\Delta}(x, c)$ holds for any $g \in G$, it follows that $h_{\mathbb{Z}\Delta}(x, c') > 0$ holds for some $c' \in Gc$. Hence, $C$ satisfies (C1) if and only $\pi^{-1}(C)$ satisfies (C1).

Let $c, d \in \pi^{-1}(C)$. If

$$h_{\mathbb{Z}\Delta/G}(\pi d, \pi c) = \begin{cases} 2, & \text{if } \pi d = \pi c = \omega(\pi c); \\ 1, & \text{if } \pi d = \pi c \neq \omega(\pi c); \\ 1, & \text{if } \pi d = \omega(\pi c) \neq \pi c; \\ 0, & \text{otherwise}, \end{cases}$$

(5.4)

holds, then since $h_{\mathbb{Z}\Delta}(\omega c, c) \geq h_{\mathbb{Z}\Delta,m-1}(\omega c, c) = 1$ by Proposition 5.1, $h_{\mathbb{Z}\Delta}(c, c) = h_{\mathbb{Z}\Delta,0}(c, c) = 1$ and $\nu(c) \neq c$ hold for any $c \in (\mathbb{Z}\Delta)_0$, it follows that

$$h_{\mathbb{Z}\Delta}(d, c) = \begin{cases} 1, & \text{if } d = c \neq \omega(c); \\ 1, & \text{if } d = \omega(c) \neq c; \\ 0, & \text{otherwise}, \end{cases}$$

(5.5)

holds. On the other hand, if (5.5) holds, then (5.4) holds by Lemma 5.5 and the definition of weakly admissible automorphism groups in Subsection 2.1. We also have that $\omega(C) = C$ if and only $\omega(\pi^{-1}(C)) = \pi^{-1}(\omega(C)) = \pi^{-1}(C)$. Hence, $C$ satisfies (C2) if and only $\pi^{-1}(C)$ satisfies (C2). Therefore, $C$ is a configuration of $\mathbb{Z}\Delta/G$ if and only if $\pi^{-1}(C)$ is a configuration of $\mathbb{Z}\Delta$.

6. Type A

In this section, we will introduce a simple method to describe all the configurations of Dynkin type A. The idea comes from works by Riedtmann [16] and Wiedemann [27]. We start by introducing the definition of 2-Brauer relations.

Definition 6.1. First, we recall the notion of Brauer relations and then give the definition of 2-Brauer relations.

(1) A Brauer relation $B$ of rank $n$ is an equivalence relation on the set $\{1, 2, \ldots, n\}$ such that the convex hulls (when we identify the set $\{1, 2, \ldots, n\}$ to the $n$-th roots of the unity) of distinct equivalence classes are disjoint.

(2) A 2-Brauer relation $B$ of rank $n$ is a Brauer relation of rank $n$ which only allows at most two elements in any equivalence class. We define a permutation $\sigma_B$ as follows

$$\sigma_B(i) := \begin{cases} i, & \text{if } i \text{ itself is an equivalence class}; \\ j, & \text{if } \{i, j\} \text{ is an equivalence class}. \end{cases}$$

We denote by $B^2_n$ the set of 2-Brauer relations of rank $n$ and by $i \sim j$ if $i$ is equivalent to $j$.

Here are some examples of 2-Brauer relations.

Example 6.2. All the 2-Brauer relations of rank 4 are shown as follows.
In this example, we have $\sigma_{B_6}(1) = 3$, $\sigma_{B_6}(2) = 2$, $\sigma_{B_6}(3) = 1$ and $\sigma_{B_6}(4) = 4$.

In the rest, we consider the following coordinates of the vertices of $\mathbb{Z}A_{n+1}$ in $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$.

Notice that some different vertices have same coordinates, in fact, we have $[i \ j] = \omega([j \ i])$ as shown in Figure 2. By Definition 5.3, a vertex $c$ is in a configuration if and only if $\omega(c)$ is in. Therefore, the configuration contains all the vertices with the same coordinates as $c$. As a consequence, we can describe a configuration by these coordinates. By calculation, the set $H([j \ i])$ defined in 4.1 consists of all the vertices covered by the rectangle with the four vertices $[j \ i]$, $[j \ j]$, $[i \ j]$ and $[i \ i]$ in Figure 2. By using this,
Theorem 6.3. We denote by $\mathbf{C}(A_{n+1})$ the set of configurations of $\mathbb{Z}A_{n+1}$. Then there is a one-to-one correspondence

$$\mathbf{C}(A_{n+1}) \leftrightarrow \mathbf{B}_n^2, \quad C \mapsto B_C$$

where $B_C$ is the equivalence relation generated by $i \sim j$ for each $[i, j] \in C$.

The following is an example.

Example 6.4. The correspondence $\mathbf{B}_n^2 \leftrightarrow \mathbf{C}(A_5)$ is the following, where the 2-Brauer relations $B_i$ for $1 \leq i \leq 9$ are listed in Example 6.2.
Proof of Theorem 6.3. First, we show that the map $\Phi : C(A_{n+1}) \to B^2_n$ is well-defined, i.e. $B_C \in B^2_n$ holds for any $C \in C(A_{n+1})$. According to Definition 5.3 (C1), for each $k \in \{1, 2, \ldots, n\}$, there exists a vertex $[j \ i] \in C$ such that $[k \ k] \in H([j \ i])$. 
This means \( i = k \) or \( j = k \) by Figure 2. Then each \( k \in \{1, 2, \ldots, n\} \) is in an equivalence class of \( B_C \). By Definition 5.3 (C2), for any \([i \ j] \in C\), we have

\[
H([i \ j]) \cap C = \{(i \ j), [j \ i]\} \quad \text{and} \quad H([j \ i]) \cap C = \{(i \ j), [j \ i]\}. \tag{6.1}
\]

Assume that there is an equivalence class of \( B_C \) contains at least three elements. Then there exist \([i \ j] \in C\) and \([i \ k] \in C\). Since \([i \ k] \in H([i \ j]) \cup H([j \ i])\), this contradicts with (6.1). Hence, each equivalence class of \( B_C \) only contains at most two elements. Now, we start to prove that the convex hulls of the equivalence classes of \( B_C \) are disjoint. Without losing generality, assume that the convex hull of \( i \sim j \) (\(1 \leq i \leq j \leq n\)) is joint with the convex hull of \( r \sim s \), then

\[
r \in \{j, \ldots, n, 1, \ldots, i\}, s \in \{i, \ldots, j\} \quad \text{or} \quad r \in \{i, \ldots, j\}, s \in \{j, \ldots, n, 1, \ldots, i\} \tag{6.2}
\]

holds. This means \([r \ s] \in H([i \ j]) \cup H([j \ i])\) (see Figure 2), which contradicts (6.1). Hence, the convex hull of the equivalence classes of \( B_C \) are disjoint. Therefore, the equivalence relation \( B_C \) is a 2-Brauer relation of rank \( n \).

We construct a map \( \Psi : B^2_n \to C(A_{n+1}) \) by \( B \mapsto C_B := \{[1 \ \sigma_B(1)], [2 \ \sigma_B(2)], \ldots, [n \ \sigma_B(n)]\} \). Now, we prove that the map is well-defined, i.e. the set \( C_B \) is a configuration of \( ZA_{n+1} \) for each \( B \in B^2_n \). For any \([i \ j] \in C_B\), we have \([j \ i] \in C_B\) since \( \sigma^2_j = \text{Id} \). Assume that \([r \ s] \in C_B \) belongs to \( H([i \ j]) \cup H([j \ i])\). Then (6.2) holds. This means the convex hull of \( i \sim j \) is joint with the convex hull of \( r \sim s \), which contradicts with \( B \in B^2_n \). Hence, we have

\[
H([i \ j]) \cap C_B = \{(i \ j), [j \ i]\} \quad \text{and} \quad H([j \ i]) \cap C_B = \{(i \ j), [j \ i]\}.
\]

(C2) holds. For any \([r \ s] \in (ZA_{n+1})_0\), there exists \( t \in \{1, 2, \ldots, n\} \) such that \( \sigma_B(r) = t \) and \( \sigma_B(t) = r \). Then \([r \ s] \in H([r \ t]) \cup H([t \ r]), \) (C1) holds. We have \( C_B \in C(A_{n+1}) \).

Since each equivalence class of 2-Brauer relation has at most two elements, it follows that \( \Phi \circ \Psi = \text{Id} \) and \( \Psi \circ \Phi = \text{Id} \) by our construction. \( \square \)

7. Type B

In this section, we will describe all the configurations of type \( B \). In order to realize the description easily, we need to introduce symmetric 2-Brauer relations which are similar but different from 2-Brauer relations given in Definition 6.1 for type \( A \).

**Definition 7.1.** A symmetric 2-Brauer relation \( B^s \) of rank \( 2n \) is a 2-Brauer relation of rank \( 2n \) which is symmetric with respect to rotation by \( \pi \). We denote by \( B^2_n^s \) the set of all symmetric 2-Brauer relations of rank \( 2n \).

Let us see the example of symmetric 2-Brauer relations of rank 4 as follows.

**Example 7.2.** All the symmetric 2-Brauer relations of rank 4 are shown as follows, where

\[
B^2_n^s = \{B^s_1, B^s_2, B^s_3, B^s_4, B^s_5\}.
\]

\[
\begin{align*}
&\begin{array}{ccc}
3 & 4 & \cdots \\
2 & 1 & \cdots \\
\end{array} & \begin{array}{ccc}
3 & 4 & \cdots \\
2 & 1 & \cdots \\
\end{array} & \begin{array}{ccc}
3 & 4 & \cdots \\
2 & 1 & \cdots \\
\end{array} & \begin{array}{ccc}
3 & 4 & \cdots \\
2 & 1 & \cdots \\
\end{array} & \begin{array}{ccc}
3 & 4 & \cdots \\
2 & 1 & \cdots \\
\end{array} \\
& B^s_1 & B^s_2 & B^s_3 & B^s_4 & B^s_5
\end{align*}
\]

In the rest, we consider the following coordinates of the vertices of \( ZB_{n+1} \) in \( \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z} \).
Remark that we use only coordinates \([r, s]\) such that \(0 \leq s - r (\mod 2n) \leq n\). Notice that some different vertices have the same coordinates. Let \([j + n, i + n] \in (\mathbb{Z}B_{n+1})_0\). By calculation, we have

\[
\omega([j + n, i + n]) = [j, i].
\]  

(7.1)

\[
H([j + n, i + n]) = \{ [s, t] \mid \forall s \in \{j, j + 1, \ldots, i\}, \forall t \in \{i, i + 1, \ldots, i + n\}, t - s (\mod 2n) \leq n \}\cup
\]

\[
\{ [s, t] \mid \forall s \in \{i + 1, i + 2, \ldots, j + n\}, \forall t \in \{j + n, j + n + 1, \ldots, i + n\}\}
\]  

(7.2)

is the set consisting of all the vertices in Figure 3.

By using the above analysis, we can get the following theorem. Note as before that two vertices having the same coordinates are in the same configurations.

**Theorem 7.3.** We denote by \(C(B_{n+1})\) the set of all the configurations of \(\mathbb{Z}B_{n+1}\), then there is a one-to-one correspondence

\[
C(B_{n+1}) \xrightarrow{1-1} B_{2n}^{2s}, \quad C \leftrightarrow B_C,
\]

where \(B_C\) is the equivalence relation generated by \(i \sim j\) for each \([j, i] \in C\).

First, let us see the example of configurations of \(\mathbb{Z}B_3\) and symmetric 2-Brauer relations of rank 4.

**Example 7.4.** The symmetric 2-Brauer relations of rank 4 are listed in Example 7.2. Then the one-to-one correspondence \(C(B_3) \leftrightarrow B_{4}^{2s}\) is given by:

- \(B_{1}^{s} \rightarrow 42 \quad (1, 2) \quad 13 \quad (1, 2) \quad 24 \quad (1, 2) \quad 31 \quad (1, 2) \quad 42 \quad (1, 2) \quad 13 \quad (1, 2) \quad 24 \)
- \(11 \rightarrow 12 \rightarrow 22 \rightarrow 33 \rightarrow 44 \rightarrow 11 \rightarrow 12 \rightarrow 22 \rightarrow 33 \)
Proof of Theorem 7.3. First of all, we will prove that the map $\Phi : C(B_{n+1}) \to B_{2n}$ is well-defined, i.e. $B_C$ is a symmetric 2-Brauer relation for each $C \in C(B_{n+1})$. For each $[j i] \in C$, according to (7.1), we know that

$$\omega([j i]) = [j - n \ i - n] = [j + n \ i + n] \in C.$$  

So the equivalence relation $B_C$ is symmetric with respect to rotation by $\pi$. According to Definition 5.3 (C1), for each $t \in \{1,2,\ldots,2n\}$, there exists a vertex $[j + n \ i + n] \in C$ such that

$$[t t] \in H([j + n \ i + n]).$$

This means that $i = t$ or $j + n = t$ by Figure 3. Then by definition and the symmetry of $B_C$, each $[t i] \in [C]$ is in an equivalence class of $B_C$. Assume that there is an equivalence class of $B_C$ contains at least three elements. Then there exist $[j i] \in C$ and $[t i] \in C$ (or $[j s] \in C$). Since

$$[t i], [j s] \in H([j i]) \cup H([j + n \ i + n]),$$

this contradicts (C2). Hence, each equivalence class in $B_C$ contains at most two elements. Assume that that the convex hull of $j \sim i$ is joint with the convex hull of $r \sim s$ (we know $[r s], [j i] \in C$), then by (7.2), we have $[r s] \in H([j i]) \cup H([j + n \ i + n])$, which contradicts (C2). Hence, the convex hulls of the corresponding equivalence classes are disjoint. Therefore, $B_C \in B_{2n}^2$.

We construct a map

$$\Psi : B_{2n}^2 \to C(B_{n+1}) \quad \text{given by} \quad B \mapsto C_B := \{[1 \sigma_B(1)], \ldots, [2n \sigma_B(2n)]\} \cap (ZB_{n+1})_0.$$  

In fact, for any $s, t \in \{1,2,\ldots,2n\}$ with $s \neq t \pm n$, there is only one of $[s t], [t s]$ representing a vertex in $(ZB_{n+1})_0$ for our coordinates. And when $s = t \pm n$, both $[s t]$ and $[t s]$ are in $(ZB_{n+1})_0$. Since $B$ is symmetric with respect to rotation by $\pi$, for each $[j i] \in C$, it follows that

$$\omega([j i]) = [j + n \ i + n] \in C_B.$$  

Assume that $[r s] \in C_B$ belongs to $H([j i]) \cup H([j + n \ i + n])$. Then by (7.2), it means that the convex hull of $r \sim s$ is joint with the convex hull of $i \sim j$ or the convex hull of $i + n \sim j + n$. A contradiction. Hence, by (7.2) and Figure 3, we have

$$H([j i]) \cap C_B = \{[j i], [j + n \ i + n]\} \quad \text{and} \quad H([j + n \ i + n]) \cap C_B = \{[j i], [j + n \ i + n]\}.$$  

Thus, (C2) holds. For any $[r s] \in (ZB_{n+1})_0$, there exists $t$ such that $\sigma_B(t) = r$ in $B$. Hence, $[r s] \in H([r t]) \cup H([r + n \ t + n])$, and (C1) holds. Therefore, $C_B \in C(B_{n+1})$. 

Thus, (C2) holds. For any $[r s] \in (ZB_{n+1})_0$, there exists $t$ such that $\sigma_B(t) = r$ in $B$. Hence, $[r s] \in H([r t]) \cup H([r + n \ t + n])$, and (C1) holds. Therefore, $C_B \in C(B_{n+1})$. 

"Thus, (C2) holds. For any $[r s] \in (ZB_{n+1})_0$, there exists $t$ such that $\sigma_B(t) = r$ in $B$. Hence, $[r s] \in H([r t]) \cup H([r + n \ t + n])$, and (C1) holds. Therefore, $C_B \in C(B_{n+1})$. 

Thus, (C2) holds. For any $[r s] \in (ZB_{n+1})_0$, there exists $t$ such that $\sigma_B(t) = r$ in $B$. Hence, $[r s] \in H([r t]) \cup H([r + n \ t + n])$, and (C1) holds. Therefore, $C_B \in C(B_{n+1})$."
Since each equivalence class of symmetric 2-Brauer relations contains at most two elements and there is at least one of \([s \mid t], [t \mid s]\) in \((ZB_{n+1})_0\) for any \(s, t \in \{1, 2, \ldots, 2n\}\), it follows that \(\Phi \circ \Psi = \text{Id}\) and \(\Psi \circ \Phi = \text{Id}\) by our construction. \(\square\)

8. Type C

In this section, we will describe all the configurations of type \(C\). We consider the following coordinates of the vertices of \(ZC_{n+1}\) in \(\mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}\).

By similar analysis with type \(B\) in Section 7, we have the following theorem.

**Theorem 8.1.** We denote by \(C(C_{n+1})\) the set of all configurations of \(ZC_{n+1}\), then there is a one-to-one correspondence

\[\begin{align*}
C(C_{n+1}) & \leftrightarrow B_{2n}^s, \\
1 \leftrightarrow B_C
\end{align*}\]

where \(B_C\) is the equivalence relation generated by \(i \sim j\) for each \([j \mid i]\) \(\in C\), \(B_{2n}^s\) and \(\sigma\) are defined in Sections 6 and 7.

We omit the proof because it is identical with the one for Type \(B\).

9. Type D

In this section, we describe configurations of type \(D\) by using crossing 2-Brauer relations. First, let us introduce the definition of crossing 2-Brauer relations.

**Definition 9.1.** A crossing 2-Brauer relation \(B_c^e\) of rank \(2n\) is an equivalence relation which only allows at most two elements in any equivalence class on the set \(\{1, 2, \ldots, 2n\}\) satisfying the following two conditions:

1. exactly two of the convex hulls of distinct equivalence classes are joint;
2. the equivalence relation is symmetric with respect to rotation by \(\pi\).

We define a permutation \(\sigma_{B_c^e}\) as follows

\[\sigma_{B_c^e}(i) := \begin{cases} i, & \text{if } \{i\} \text{ is an equivalence class;} \\
j, & \text{if } \{i, j\} \text{ is an equivalence class.}\end{cases}\]

Denote by \(B_{2n}^{2c}\) the set of crossing 2-Brauer relations of rank \(2n\) and \(i \sim j\) if \(i\) is equivalent with \(j\).

Let us see an example of crossing 2-Brauer relations.

**Example 9.2.** The set \(B_4^{2c}\) consists of only one crossing 2-Brauer relation \(B_4^c\) of rank 4 as follows.
In this case, we have $\sigma_{B_c^1}(1) = 3$, $\sigma_{B_c^1}(2) = 4$, $\sigma_{B_c^1}(3) = 1$ and $\sigma_{B_c^1}(4) = 2$.

We consider the coordinates of the vertices of $\mathbb{Z}D_{n+2}$ in Figure 4 in $\mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2n\mathbb{Z}$. For each $[r s]$, $0 \leq s - r \text{mod} 2n \leq n$ holds. If $s - r = n \text{mod} 2n$ holds, we add a sign $+\text{ or } -$.

We denote $[i i + n]_{\pm} := [i i + n]_{+} + [i i + n]_{-} \in \mathbb{N}_0(\mathbb{Z}D_{n+2})_0$ for any $i$. We give a description of the map $\omega$ and the set $H$ for the vertices in $\mathbb{Z}D_{n+2}$. We need to divide the description into two cases.

Case 1: Let $[j + n i + n] \in (\mathbb{Z}D_{n+2})_0$ with $i \neq j \pm n$. By calculation, we have

$$\omega([j + n i + n]) = [j i].$$

(9.1)

$$H([j + n i + n]) = \{ [s s + n]_{\pm} \mid \forall s \in \{j, j + 1, \ldots, i\}\} \cup$$

$$\{ [s t] \mid \forall s \in \{j, j + 1, \ldots, i\}, \forall t \in \{i, i + 1, \ldots, i + n - 1\}, t \neq s + n\} \cup$$

$$\{ [s t] \mid \forall s \in \{i + 1, i + 2, \ldots, j + n\}, \forall t \in \{j + n, j + n - 1, \ldots, i + n\}\}$$

(9.2)

is the set consisting of all the vertices in Figure 5.
Case 2: Let \([i + n,\epsilon]_\epsilon \in (Z D_{n+2})_0\) for \(\epsilon \in \{-, +\}\). By Definition 4.4 and direct calculation, we have
\[
\omega([i + n,\epsilon]_\epsilon) = [i - n,\epsilon]_\epsilon = [i + n,\epsilon]_\epsilon
\]
and
\[
H([i + n,\epsilon]_\epsilon) = \{[i - k, i + n - k,\epsilon]_\epsilon | \forall k \in \{0, 1, \ldots, n\}\} \cup \\
\{[i - n + s, i + \epsilon]_\epsilon | \forall s \in \{1, \ldots, n\}, \forall t \in \{0, 1, \ldots, n - 1\}\}.
\]

We divide the configurations into two disjoint sets \(C^1(D_{n+2})\) and \(C^2(D_{n+2})\). We denote by \(C^1(D_{n+2})\) the set of all the configurations of \(Z D_{n+2}\) which either contain one subset of vertices of the form \([i + n,\epsilon]_\epsilon\) or contain no vertices of the form \([i + n,\epsilon]_\epsilon\). And we denote the complement of \(C^1(D_{n+2})\) by \(C^2(D_{n+2})\).

We know the structure of \(C^1(D_{n+2})\). Now we study \(C^2(D_{n+2})\). It is obvious by \((C2)\) and \((9.4)\) that for any configuration \(C\) of \(Z D_{n+2}\), there exist at most two numbers \(i, j \in \{1, 2, \ldots, 2n\}\) such that
\[
[i + n,\epsilon]_\epsilon, [i + n,\epsilon]_\epsilon, [j + n,\epsilon]_\epsilon, [j + n,\epsilon]_\epsilon \in C.
\]
By using this fact, we give a description of \(C^2(D_{n+2})\) in the following lemma.

**Lemma 9.3.** \(C^2(D_{n+2}) = \{C | C is a configuration of Z D_{n+2} satisfying (9.5) with i \neq j\}\).

**Proof.** Let \(C\) be a configuration of \(Z D_{n+2}\). If \(C\) does not contain any vertex of the form \([i + n,\epsilon]_\epsilon\) with \(\epsilon \in \{-, +\}\), then \(C \in C^1(D_{n+2})\).

Since \(\omega([i + n,\epsilon]_\epsilon) = [i + n,\epsilon]_\epsilon\), it follows that \([i + n,\epsilon]_\epsilon \in C\) if and only if \([i + n,\epsilon]_\epsilon \in C\). Without losing generality, we assume that \([i + n,\epsilon]_\epsilon \in C\). By \((C1)\), there exists \(C\) such that \([i + n,\epsilon]_\epsilon \in H(c)\). If there is no other vertex of the form \([n + t,\epsilon]_\epsilon\) in \(C\), then \(c\) could only be a vertex in Case 1. However, in Case 1, \([i + n,\epsilon]_\epsilon \in H(c)\) if and only if \([i + n,\epsilon]_\epsilon \in H(c)\). This contradicts the facts that \([i + n,\epsilon]_\epsilon \in C\) and \(\omega(c) \neq [i + n,\epsilon]_\epsilon\). Hence, there exists \([j + n,\epsilon]_\epsilon \in C\) such that \([i + n,\epsilon]_\epsilon \in H([j + n,\epsilon]_\epsilon)\). Moreover, \(\omega([j + n,\epsilon]_\epsilon) = [j + n,\epsilon]_\epsilon \in C\).

When \(i = j\), we have \(C \in C^1(D_{n+2})\). Since \(C\) cannot contain more than 4 vertices of the form \([n + t,\epsilon]_\epsilon\), it follows that \(C^2(D_{n+2})\) consists of all the configurations containing \([i + n,\epsilon]_\epsilon, [i + n,\epsilon]_\epsilon, [j + n,\epsilon]_\epsilon, [j + n,\epsilon]_\epsilon\) with \(i \neq j\).

Now we define an involution \((-)^*\) on the set \(C^2(D_{n+2})\) as follows. For each \(C \in C^2(D_{n+2})\), we define
\[
C^* := (C \setminus \{[i + n,\epsilon]_\epsilon, [i + n,\epsilon]_\epsilon, [j + n,\epsilon]_\epsilon, [j + n,\epsilon]_\epsilon\}) \cup \\
\{[i + n,\epsilon]_\epsilon, [i + n,\epsilon]_\epsilon, [j + n,\epsilon]_\epsilon, [j + n,\epsilon]_\epsilon\}.
\]
It is clear that \((C^*)^* = C\) and \(C^*\) is a different configuration of \(Z D_{n+2}\) than \(C\).

There are two different kinds of configurations of type \(D\) and there are symmetric 2-Brauer relations and crossing 2-Brauer relations. We have the following theorem of correspondence between the configurations of \(Z D_{n+2}\) and those two types of Brauer relations we defined.

**Theorem 9.4.** There are a one-to-one correspondence \(C^1(D_{n+2}) \leftrightarrow B^2_{2n}^s\) and a two-to-one correspondence \(C^2(D_{n+2}) \leftrightarrow B^2_{2n}^s\), \(C, C^* \rightarrow B_C = B_{C^*}\), where \(B_C\) is an equivalence relation generated by \(i \sim j\) for each \([j i]_\epsilon \in C\) with \(\epsilon \in \{\emptyset, +, -\}\) and \(B^2_{2n}^s\) is defined in Section 7.

We use \(Z D_4\) as an example.

**Example 9.5.** The symmetric 2-Brauer relations of rank 4 are listed in Example 7.2 and the crossing 2-Brauer relation of rank 4 is listed in Example 9.2. Then the one-to-one correspondence \(C^1(D_4) \leftrightarrow B^2_{4}^s\) is the following.
The two-to-one correspondence between $C^2(D_4)$ and $B_4^2$ is given as follows.
Proof of Theorem 9.4. First, we show that the two maps

\[ \Phi_1 : C^1(D_{n+2}) \to B_{2n}^{2s} \quad \text{and} \quad \Phi_2 : C^2(D_{n+2}) \to B_{2n}^{2c} \]

are well-defined. Let \( C \) be a configuration of \( \mathbb{Z}D_{n+2} \). By a similar proof of Theorem 7.3, we know that \( B_C \) is an equivalence relation on the set \( \{1, 2, \ldots, 2n\} \), symmetric with respect to rotation \( \pi \) and each equivalence class of \( B_C \) contains at most two elements.

Assume that \( C \in C^1(D_{n+2}) \). Consider \( \{[i \ i + n]_+, [i \ i + n]_-\} \) as \( [i \ i + n]_{\pm} \). In fact, \( C \) becomes a configuration of type \( B \) with \( [i \ i + n]_{\pm} \). Thus, by Theorem 7.3, we have \( B_C \in B_{2n}^{2s} \).

Assume that \( C \in C^2(D_{n+2}) \) and \( \{[i \ i + n]_+, [i + n \ i]_+, [j \ j + n]_-, [j + n \ j]_-\} \subset C \) with \( i \neq j \). Since \( i \neq j \), we know that the convex hull of \( i \sim i + n \) and the convex hull of \( j \sim j + n \) are joint. By (9.3), (9.4) and (C2), no convex hulls of the equivalence classes of vertices in Case 1 would cross the convex hulls of \( i \sim i + n \) and \( j \sim j + n \). According to (9.6), the configuration \( C^* \) has the same equivalence relation. Therefore, \( B_C = B_{C^*} \in B_{2n}^{2c} \).

Notice that for any \( s, t \in \{1, 2, \ldots, 2n\} \) with \( s \neq t \), there is only one of \( \{[s \ t], [t \ s]\} \) representing a vertex in \( \mathbb{Z}D_{n+2} \) for our coordinates. And when \( s = t + n = t - n \), both \([t + n \ t]_\pm \) and \([t \ t + n]_\pm \) are in \( \mathbb{Z}D_{n+2} \). We construct a map

\[ \Psi_1 : B_{2n}^{2s} \to C^1(D_{n+2}) \quad \text{by} \quad B \mapsto C_B := \{[1 \ \sigma_B(1)]_\epsilon, \ldots, [2n \ \sigma_B(2n)]_\epsilon\} \cap (\mathbb{Z}D_{n+2})_0, \]

where \( \epsilon = \emptyset \) when \( \sigma_B(i) \neq i \pm n \) and \( \epsilon = \pm \) when \( \sigma_B(i) = i \pm n \) for \( 1 \leq i \leq 2n \). Similarly as we did in the proof of Theorem 7.3, we have \( C_B \in C^1(D_{n+2}) \). Hence, the map \( \Psi_1 \) is well-defined.

We construct a map

\[ \Psi_2 : B_{2n}^{2c} \to \{C, C^*\} \quad \text{by} \quad B \mapsto \{C_B, C^*_B\}, \]

where \( C_B, C^*_B \) are defined as follows. For each equivalence class \( i \sim j \) of \( B \) with \( j \neq i \), we put the vertex \( [j \ i] \) in \( C_B \). Since there exist \( s, t \) such that the equivalence classes \( s \sim s + n \) and \( t \sim t + n \) are in \( B \), we put

\[ [s \ s + n]_+ , [s + n \ s]_+ , [t \ t + n]_-, [t + n \ t]_- \in C_B. \]

By exchanging \( + \) and \( - \), we get \( C^*_B \). Now, we prove that \( \Psi_2 \) is well-defined. For any vertex \( [j \ i] \in C_B \), we have \([j + n \ i + n] \in C_B \) since \( B \) is symmetric. As the convex hull of each equivalence class \( i \sim j \) with \( j \neq i \pm n \) is disjoint with any other convex hulls, by (9.2), we have \( H(\{j + n \ i + n\}) \cap C_B = \{[j \ i], [j + n \ i + n]\} \).

Similarly, by (9.4), we have

\[ H([s \ s + n]_+ \cap C_B = \{[s \ s + n]_+, [s + n \ s]_+\} \quad \text{and} \quad H([t \ t + n]_- \cap C_B = \{[t \ t + n]_-, [t + n \ t]_-\}. \]

Therefore, (C2) holds. For the vertices of the forms \([r \ r + n]_+ \) and \([r \ r + n]_- \), by (9.4), we have

\[ [r \ r + n]_+ \in H([s \ s + n]_+) \cup H([s + n \ s]_+) \quad \text{and} \quad [r \ r + n]_- \in H([t \ t + n]_-) \cup H([t + n \ t]_-). \]

For other vertices of the form \([s \ r] \in (\mathbb{Z}D_{n+2})_0 \) with \( s \neq r \pm n \), there exists an equivalence class \( r \sim t \) in \( B \) such that \([r \ s] \in H([r \ t]) \cup H([r + n \ t + n]) \). So (C1) holds. Thus, the set \( C_B \) and similarly \( C^*_B \) are configurations of \( \mathbb{Z}D_{n+2} \). The map \( \Psi_2 \) is well-defined.

Since each equivalence class here allows at most two elements and there is at least one of \( \{[s \ t], [t \ s]\} \) in \( \mathbb{Z}D_{n+2} \) for any \( s, t \in \{1, 2, \ldots, 2n\} \), it follows that \( \Phi_1 \circ \Psi_1 = \text{Id} \) and \( \Psi_1 \circ \Phi_1 = \text{Id} \) by our construction. And since \( B_{C^*} = B_C \) for any \( C \in C^2(D_{n+2}) \), it follows that \( \Phi_2 \circ \Psi_2 = \text{Id} \) and \( \Psi_2 \circ \Phi_2(C) = \Psi_2 \circ \Phi_2(C^*) = \{C, C^*\} \).

\[ \square \]

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