ROOT SPACE DECOMPOSITION OF $g_2$ FROM OCTONIONS

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Abstract. We describe a simple way to write down explicit derivations of octonions that form a Chevalley basis of $g_2$. This uses the description of octonions as a twisted group algebra of the finite field $F_8$. Generators of $Gal(F_8/F_2)$ act on the roots as 120-degree rotations and complex conjugation acts as negation.

1. Introduction. Let $O$ be the unique real nonassociative eight dimensional division algebra of octonions. It is well known that the Lie algebra of derivations $Der(O)$ is the compact real form of the Lie algebra of type $G_2$. Complexifying we get an identification of $Der(O) \otimes \mathbb{C}$ with the complex simple Lie algebra $g_2$.

The purpose of this short note is to make this identification transparent by writing down simple formulas for a set of derivations of $O \otimes \mathbb{C}$ that form a Chevalley basis of $g_2$ (see Theorem 10). This gives a quick construction of $g_2$ acting on $\text{Im}(O) \otimes \mathbb{C}$ because the root space decomposition is visible from the definition. The highest weight vectors of finite dimensional irreducible representations of $g_2$ can also be easily described in these terms. For an alternative construction of $g_2$ as derivations of split octonions, see [KT], pp. 104–106.

Wilson [W] gives an elementary construction of the compact real form of $g_2$ with visible $2^3 \cdot L_3(2)$ symmetry. This note started as a reworking of that paper in light of the definition of $O$ given above, namely, that $O$ can be defined as the real algebra with basis $\{e^x : x \in F_8\}$, with multiplication defined by

$$e^x e^y = (-1)^{\varphi(x,y)} e^{x+y} \quad \text{where} \quad \varphi(x, y) = \text{tr}(yx_6) \tag{1}$$

and $\text{tr} : F_8 \to F_2$ is the trace map: $x \mapsto x + x^2 + x^4$.

The definition of $O$ given above has a visible order–three symmetry $Fr$ corresponding to the Frobenius automorphism $x \mapsto x^2$ generating $Gal(F_8/F_2)$, and a visible order–seven symmetry $M$ corresponding to multiplication by a generator of $F_8^*$. Together they generate a group of order 21 that acts simply transitively on the natural basis $B = \{e^x \wedge e^y : x, y \in F_8^*, x \neq y\}$ of $\wedge^2 \text{Im}(O)$. The only element of $F_8^*$ fixed by $Fr$ is 1. Let $\{0, 1, x, y\} \subseteq F_8$ be a subset corresponding to any line of $D^2(F_2)$ containing 1. Let $B_0 \subseteq B$ be the Frobenius orbit of $e^x \wedge e^y$ and let $B_0, B_1, \ldots, B_6$ be the seven translates of $B_0$ by the cyclic group $(M)$. Then $B$ is the disjoint union of $B_0, \ldots, B_6$.

Using the well known natural surjection $D : \wedge^2 \text{Im}(O) \to \text{Der}(O)$, we get a generating set $D(B)$ of $\text{Der}(O)$. The kernel of $D$ has dimension seven with a basis $\{\sum_{b \in B_i} b : i = 0, \ldots, 6\}$. The images of $B_0, \ldots, B_6$ span seven mutually orthogonal
Cartan subalgebras transitively permuted by (M) and forming an orthogonal decomposition of Der(Ω) in the terminology of [KT]. We fix the Cartan subalgebra spanned by $D(B_0)$ because it is stable under the action of Fr. The short coroots in this Cartan are $\{ \pm D(b) : b \in B_0 \}$. At this point, it is easy to write down explicit derivations of $\mathfrak{g} \otimes \mathbb{C}$ corresponding to a Chevalley basis of $\mathfrak{g}_2$ by simultaneously diagonalizing the action of the coroots (see the discussion preceding Theorem 10).

The following symmetry considerations make our job easy. The reflections in the short roots of $\mathfrak{g}_2$ generate an $S_3$ that has index 2 in the Weyl group. In our description, the action of this $S_3$ is a-priori visible. This $S_3$ is generated by Fr acting as 120-degree rotation and the complex conjugation on $\mathfrak{g} \otimes \mathbb{C}$ acting as negation.

2. **Definition.** Let $a, b \in \mathfrak{g}$. Write $\text{ad}_a(b) = [a, b] = ab - ba$. Define $D(a, b) : \mathfrak{g} \to \mathfrak{g}$ by

$$D(a, b) = \frac{1}{3}([\text{ad}_a, \text{ad}_b] + \text{ad}_{[a, b]}).$$

Clearly $D(a, b) = -D(b, a)$, $D(a, b)1 = 0$ and $D(1, a) = 0$. So $D$ defines a linear map from $\wedge^2 \text{Im}(\mathfrak{g})$ to $\text{End}(\mathfrak{g})$ which we also denote by $D$. So $D(a, b) = D(a \wedge b)$.

**Notation:** From here on, we shall write $\mathfrak{g} = \text{Der}(\mathfrak{g}) \otimes \mathbb{C}$. The Frobenius automorphism Fr acts on $\mathfrak{g}$, on $\mathfrak{g}$, on the roots of $\mathfrak{g}$, and so on. If $x$ is an element of any of these sets, we sometimes write $x'$ for its image under Fr. Choose $\alpha \in \mathbb{F}_8$ such that $\alpha^3 = \alpha + 1$. Write

$$e_i = e^{\alpha^i} \quad \text{and} \quad e_{ij} = D(e_i \wedge e_j).$$

Note that $\text{Fr} : e_i \mapsto e_{2i}$, that is, $e_i' = e_{2i}$, where the subscripts are read modulo 7.

3. **Lemma.** Let $x$ and $y$ be distinct elements of $\mathbb{F}_8^*$ and $z \in \mathbb{F}_8$. Then

$$D(e^x \wedge e^y)e^z = \begin{cases} 2e^y & \text{if } z = x, \\ -2e^x & \text{if } z = y, \\ 0 & \text{if } z = 0 \text{ or } z = x + y, \\ -(e^xe^y)e^z & \text{otherwise}. \end{cases}$$

**Proof.** Let $a, b \in \mathfrak{g}$. Define $R(a, b) : \mathfrak{g} \to \mathfrak{g}$ by $R(a, b) = [\text{ad}_a, \text{ad}_b] - \text{ad}_{[a, b]}$. One verifies that $R(a_1, a_2)(a_3) = -\sum_{\sigma \in S_3} \text{sign}(\sigma)[a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}]$ where $[a, b, c] = (ab)c - a(bc)$ is the associator. The properties of the associator in $\mathfrak{g}$ implies $R(a_1, a_2)(a_3) = -6[a_1, a_2, a_3]$. So

$$2D(a \wedge b) = \text{ad}_{[a, b]} + \frac{1}{2}R(a, b) = \text{ad}_{[a, b]} - 3[a, b, \cdot]. \tag{2}$$

If $z \in \mathbb{F}_2x + \mathbb{F}_2y$, then $e^z$ belongs to the associative subalgebra spanned by $e^x$ and $e^y$ and the Lemma is easily verified in this case. If $z \notin \mathbb{F}_2x + \mathbb{F}_2y$, then using equation (1) one easily verifies that $\text{ad}_{[e^x, e^y]}e^z = 2[e^x, e^y, e^z]$. The Lemma follows from this and equation (2). \qed

Let $a, b \in \mathfrak{g}$. Since the subalgebra of $\mathfrak{g}$ generated by $a$ and $b$ is associative, the maps $\text{ad}_{[a, b]}$ and $[\text{ad}_a, \text{ad}_b]$ agree on this subalgebra. Note that the restriction of $2D(a, b)$ to this subalgebra is just the inner derivation $\text{ad}_{[a, b]}$. In fact the following is well known:

4. **Lemma.** If $a, b \in \mathfrak{g}$, then $D(a \wedge b)$ is a derivation of $\mathfrak{g}$.
By linearity it suffices to show that if \( x, y \) are distinct elements of \( \mathbb{F}_8^* \), then 
\[
\delta = D(e^x \wedge e^y) \text{ is a derivation of } \mathcal{O}.
\]
Write \( L(z, u) = \delta(e^z e^u + e^z \delta(e^u) - \delta(e^z e^u)) \). It suffices to prove that \( L(z, w) = 0 \) for all \( z, w \in \mathbb{F}_8 \). Only a few cases need to be checked if one first proves the following Lemma.

5. Lemma. (a) Suppose \( u + x + y \) and \( v \) are distinct elements of \( \mathbb{F}_8^* \). Then \( \delta(e^v) \) and \( e^v \) anticommute.

(b) Suppose \( u, v \) and \( x + y \) are three distinct elements of \( \mathbb{F}_8^* \). If \( L(u, v) = 0 \), then \( L(u + v, u) = 0 \).

One can directly prove Lemmas 4 and 5 using Lemma 3. Since Lemma 4 is well-known (see [S]), we shall omit the details of the proof and move on to describe the kernel of \( D : \wedge^2 \text{Im}(\mathcal{O}) \to \text{Der}(\mathcal{O}) \). Let \( M : \mathbb{F}_8 \to \mathbb{F}_8 \) be the automorphism \( M(x) = ax \). Let \( \tau = M \) or \( \tau = \text{Fr} \). Recall the multiplication rule of \( \mathcal{O} \) from equation (1). Note that \( \varphi(\tau x, \tau y) = \varphi(x, y) \). It follows that \( (ab)^x = a^x b^x \) for \( a, b \in \mathcal{O} \) where \( x \) acts on \( \mathcal{O} \) by \( e^x \mapsto (e^x)^x = e^{3x} \). Since the derivations \( D(a \wedge b) \) are defined in terms of multiplication in \( \mathcal{O} \), it follows that \( D(a \wedge b) c = D(a^x b^x) c^x \) for all \( a, b, c \in \mathcal{O} \) and thus, by linearity,
\[
(D(w)c^x = D(w^x)c^x \text{ for all } w \in \wedge^2 \text{Im}(\mathcal{O}), \ c \in \mathcal{O}.
\]

Let \( \{0, 1, x, y\} \subseteq \mathbb{F}_8 \) be the subset corresponding to any line of \( \mathbb{F}_2^2(\mathbb{F}_2) \) containing 1. Define
\[
\Delta = e^x \wedge e^y + (e^x \wedge e^y) + (e^x \wedge e^y)^y \in \wedge^2 \text{Im}(\mathcal{O}).
\]

Note that the element \( \pm \Delta \) is independent of choice of the line and choice of the ordered pair \((x, y)\), since the Frobenius action permutes the three lines containing 1, and interchanging \((x, y)\) changes \( \Delta \) by a sign. To be specific, we choose \((x, y) = (\alpha, \alpha^3)\). Then
\[
\Delta = e_1 \wedge e_3 + e_2 \wedge e_6 + e_4 \wedge e_5.
\]

6. Lemma. (a) \( \ker(D) \) has a basis given by \( \Delta, \Delta^M, \cdots, \Delta^M^6 \).

(b) One has \([e_{13}, e_{26}] = 0\).

Proof. (a) Let \( w \in \wedge^2 \text{Im}(\mathcal{O}) \) and \( c \in \mathcal{O} \). Since \( D(w)c = 0 \) implies \( D(w^M)c^M = 0 \), it suffices to show that \( D(\Delta) = 0 \). Lemma 3 implies that if \( \{0, 1, x, y\} \subseteq \mathbb{F}_8 \) is a subset corresponding to a line in \( \mathbb{F}_2^2(\mathbb{F}_2) \), then \( D(e^x \wedge e^y) c^1 = 0 \), since \( x + y = 1 \). So \( D(\Delta)c^1 = 0 \). Since \( \Delta' = \Delta \), the equation \( D(\Delta)e^x = 0 \) implies \( 0 = D(\Delta)(e^x)^y = D(\Delta)e^{x^2} \). So it suffices to show that \( D(\Delta) \) kills \( e^x \) and \( e^{x^2} \). This is an easy calculation using Lemma 3. This proves that \( \Delta, \Delta^M, \cdots, \Delta^M^6 \in \ker(D) \). One verifies that these seven elements are linearly independent.

(b) Write \( X = [e_{13}, e_{26}] \). From part (a), we know that \( e_{13} + e_{26} + e_{45} = 0 \). It follows that \([e_{13}, e_{26}] = [e_{26}, e_{45}] \), that is, \( X \) is Frobenius invariant. So it suffices to show that \( X \) kills \( e_0, e_1, e_3 \). The equation \( Xe_0 = 0 \) is immediate. Verifying \( Xe_1 = 0 \) is an easy calculation using Lemma 3. The calculation for \( e_3 \) is identical to the calculation for \( e_1 \) since \( e_{13} = -e_{31} \) and \( e_{26} = -e_{62} \).

7. Definition (The roots and coroots). Lemma 6 implies that \((C e_{13} + C e_{26}) \) is an abelian subalgebra of \( \mathfrak{g} \). We fix this Cartan and call it \( H \). Fix a pair of coroots
\[
H_{\pm \beta} = \pm H_\beta = \mp i e_{13}
\]
in \( H \). Using Frobenius action, we obtain the six coroots \( \pm \{H_\beta, H'_\beta, H''_\beta\} \) corresponding to the short roots. The six coroots corresponding to the long roots are
Figure 1. The roots on the left and the coroots on the right.

\[ \pm\{H_\gamma, H'_\gamma, H''_\gamma\} \]

where

\[ H_{\pm\gamma} = \pm H_\gamma = \pm \frac{1}{3}(e_{13} - e_{26}). \]

We shall define a basis of \( g \) containing \( H_\beta \) and \( H_\gamma \). The scaling factors like \( \frac{1}{3} \) are chosen to make sure that the structure constants of \( g \) with respect to this basis are integers and are smallest possible. Define roots \( \beta, \gamma \) such that

\[
\begin{pmatrix}
\beta(H_\beta) \\
\gamma(H_\beta)
\end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}
\]

is the Cartan matrix of \( g_2 \). So \( \{\beta, \gamma\} \) is a pair of simple roots with \( \beta \) being the short root; see Figure 1. Let \( \Phi_{\text{short}} \) be the set of six short roots and let \( \Phi \) be the set of twelve roots of \( g \). Note that the Frobenius acts on \( H \) by anti-clockwise rotation of 120-degrees and complex conjugation acts by negation.

Once the roots and coroots have been fixed, the weight space decompositions of the two smallest irreducible representations of \( g \) can be found by simultaneously diagonalizing the actions of \( H_\beta \) and \( H_\gamma \). These weight spaces are described below.

8. **The standard representation:** Write \( V = \text{Im}(\mathbb{Q}) \otimes \mathbb{C} \). This is the standard representation of \( g \). Define the vectors \( v_0, v_{\pm\beta}, v'_{\pm\beta}, v''_{\pm\beta} \) in \( V \) by choosing

\[ v_0 = e_0 \quad \text{and} \quad v_{\pm\beta} = (\pm ie_1 + e_3). \]

One easily verifies that \( v_0 \) spans the weight space \( V_0 \) for each short root \( \psi \). See Figure 2. One has the weight space decomposition:

\[ V = \mathbb{C}v_0 \oplus \left( \bigoplus_{\psi \in \Phi_{\text{short}}} \mathbb{C}v_\psi \right). \]

9. **The adjoint representation:** If \( \psi \) is a short root of \( g \), define

\[ E_\psi = \frac{1}{6}D(v_0 \wedge v_\psi). \]

If \( \nu \) is a long root of \( g \), then there exists a unique short root \( \psi \) such that \( \nu = \psi - \psi' \). Define

\[ E_\nu = \frac{1}{6}D(v_\psi \wedge v'_{-\psi}). \]

One easily verifies that \( E_\rho \) spans the root space \( g_\rho \) for each root \( \rho \in \Phi \). One has the root space decomposition:

\[ g = H \oplus \left( \bigoplus_{\rho \in \Phi} \mathbb{C}E_\rho \right). \]

Note that \( E_\beta = \frac{1}{6}D(v_0 \wedge v_\beta) = \frac{1}{6}D(e_0 \wedge (ie_1 + e_3)) = \frac{1}{6}(-ie_{10} + e_{03}), \)

and

\[ E_\gamma = \frac{1}{6}D(v_{-\beta} \wedge v''_\beta) = \frac{1}{6}D((-ie_1 + e_3) \wedge (ie_2 + e_6)) = \frac{1}{6}(e_{12} + e_{36} - i(e_{23} + e_{16})). \]
To write down the other $E_\rho$’s, apply the $S_3$ symmetry generated by complex conjugation and Frobenius.

10. **Theorem.** The set $\{H_\beta, H_\gamma\} \cup \{E_\rho: \rho \in \Phi\}$ is a Chevalley basis of $\mathfrak{g}$.

**Remark on proof.** Checking that these generators of $\mathfrak{g}$ obey the commutation rules dictated by the root space decomposition is a routine verification using their action on the standard representation $V$ as described in remark 11. Because of the visible $S_3$ symmetry of our construction, only few cases need to be checked. □

**Warning:** Identify $(\wedge^2 \text{Im}(\mathfrak{O}) \otimes \mathbb{C})$ with $\mathfrak{so}_7(\mathbb{C})$ in the standard manner (see [FH], page 303) so that $e_i \wedge e_j$ gets identified with the skew symmetric matrix $2(E_{ij} - E_{ji})$ where $E_{ij}$ is the matrix with rows and columns indexed by $\mathbb{Z}/7\mathbb{Z}$ whose only nonzero entry is 1 in the $(i,j)$-th slot. Let $\nu$ be a long root. Write $\nu = \psi - \psi'$ for a short root $\psi$. It is curious to note that

$$[D(v_0 \wedge v_\psi), D(v_0 \wedge v_\psi')]_{\mathfrak{so}_7(\mathbb{C})} = 4[E_\psi, E_{-\psi'}]_{\mathfrak{so}_7(\mathbb{C})} = 12E_\nu = -D[v_0 \wedge v_\psi, v_0 \wedge v_\psi']_{\mathfrak{so}_7(\mathbb{C})},$$

even though $-D$ is not a Lie algebra homomorphism.

11. **Remark (Action of the Chevalley basis on the standard representation).** The action of the vectors $\{E_\rho: \rho \in \Phi\}$ on the weight vectors $\{v_0\} \cup \{v_\psi: \psi \in \Phi_{\text{short}}\}$ is determined up to scalars by weight consideration since $[\mathfrak{g}_\rho, V_\rho'] \subseteq V_{\rho+\rho'}$ and each weight space $V_\rho$ is at most one dimensional. The non-trivial scalars are determined by the following rules: Let $\psi$ be a short root and let $\rho$ be a root such that $\psi + \rho$ is also a short root. Then

$$E_\psi v_0 = v_\psi, \quad E_\psi v_{-\psi} = -2v_0, \quad \text{and} \quad E_{\rho+\psi} = \pm v_\rho$$

where the plus sign holds if and only if $v_{\rho+\psi}$ is equal to $v_\psi'$ or $-v_\psi''$. In other words, the plus sign holds if and only if the movement from $\psi$ to $(\rho + \psi)$ in the direction of $\rho$ defines an anti-clockwise rotation of angle less than $\pi$ around the origin. The relations in equation (3) are easily verified using Lemma 3. Only a few relations need to be checked, because of the $S_3$ symmetry. The nontrivial scalars involved in this action are indicated in Figure 2 next to the dashed arrows. For example, the $-2$ next to the horizontal arrow means that $E_{\beta_1}v_{-\beta} = -2v_0$. 

![Figure 2](image-url)
12. The irreducible representations of $g_2$: We finish by describing the finite dimensional irreducible representations of $g$ in terms of the standard representation $V$. This was worked out in [HZ]. The description given below follows quickly from the results of [HZ].

Fix the simple roots $\{ \beta, \gamma \}$ as in Figure 1. Then the fundamental weights of $g$ are $\mu_1 = -\beta''$ and $\mu_2 = -\gamma'$. For each non-negative integer $a, b$, let $\Gamma_{a,b}$ denote the finite dimensional irreducible representation of $g$ with highest weight $(a\mu_1 + b\mu_2)$. The two smallest ones are the standard representation $V = \Gamma_{1,0}$ and the adjoint representation $g = \Gamma_{0,1}$.

Let $\lambda$ be the Young tableau having two rows, corresponding to the partition $(a + b, b)$. Then $\Gamma_{a,b}$ can be realized as a subspace of the Weyl module $S_\lambda(V)$; see [HZ], Theorem 5.5. From [F], chapter 8, recall that the vectors in $S_\lambda(V)$ can be represented in the form

$$w = w_{1,1}w_{1,2} \cdots w_{1,a}w_{2,1}w_{2,2} \cdots w_{2,b}$$

where $w_{i,j} \in V$, modulo the following relations:

- Interchanging the two entries of a column negates the vector $w$.
- Interchanging two columns of the same length does not change $w$.
- For each $1 \leq j \leq b$ and $j < k \leq a + b$, let $z_1$ (resp. $z_2$) be the vector obtained from $w$ by interchanging $w_{1,k}$ with $w_{1,j}$ (resp. $w_{2,j}$). Then $w = z_1 + z_2$.

These relations are the exchange conditions of [F], page 81, worked out in our situation.

The natural surjection from $\otimes^a V \to S_\lambda(V)$ induces the $g$–action on $S_\lambda(V)$. Note that the highest weight of $\Gamma_{a,b}$ is $(a\mu_1 + b\mu_2) = (a + b)(-\beta'' + b\beta')$. From Figure 2 recall that $v''_{\beta} = -ie_4 + e_5$ and $v'_{\beta} = ie_2 + e_6$. Let $w_\lambda \in S_\lambda(V)$ be the vector written in the form given in equation (1) whose first row entries are all equal to $-ie_4 + e_5$ and whose second row entries are all equal to $ie_2 + e_6$. Then we find that $w_\lambda$ has weight $(a\mu_1 + b\mu_2)$. So $\Gamma_{a,b} = U(g)w_\lambda$, and $w_\lambda$ is the highest weight vector of $\Gamma_{a,b}$.

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