Weak field limit and gravitational waves in higher-order gravity

Salvatore Capozziello\textsuperscript{1,2,3,4}, Maurizio Capriolo\textsuperscript{5}, and Loredana Caso\textsuperscript{5}

\textsuperscript{1}Dipartimento di Fisica "E. Pancini", Università di Napoli “Federico II”, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy,
\textsuperscript{2}INFN Sezione di Napoli, Compl. Univ. di Monte S. Angelo, Edificio G, Via Cinthia, I-80126, Napoli, Italy,
\textsuperscript{4}Laboratory for Theoretical Cosmology, Tomsk State University of Control Systems and Radioelectronics (TUSUR), 634050 Tomsk, Russia,
\textsuperscript{4}Tomsk State Pedagogical University, ul. Kievskaya, 60, 634061 Tomsk, Russia,
\textsuperscript{5}Dipartimento di Matematica Università di Salerno, via Giovanni Paolo II, 132, Fisciano, SA I 84084, Italy.

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We wish to dedicate this paper to the beloved memory of Maria Transirico who passed away while this paper was being completed. We mourn the loss of a mentor and lament the loss of a dear friend. Thanks Mariella!

Abstract

We derive the weak field limit for a gravitational Lagrangian density \( L_{g} = (R + a_{0}R^{2} + \sum_{k=1}^{p} a_{k}R\Box^{k}R)\sqrt{-g} \) where higher-order derivative terms in the Ricci scalar \( R \) are taken into account. The interest for this kind of effective theories comes out from the consideration of the infrared and ultraviolet behaviors of gravitational field and, in general, from the formulation of quantum field theory in curved spacetimes. Here, we obtain solutions in weak field regime both in vacuum and in the presence of matter and derive gravitational waves considering the contribution of \( R\Box^{k}R \) terms. By using a suitable set of coefficients \( a_{k} \), it is possible to find up to \((p + 2)\) normal modes of oscillation with six polarization states with helicity 0 or 2. Here \( p \) is the higher order term in the \( \Box \) operator appearing in the gravitational Lagrangian. More specifically: the mode \( \omega_{1} \), with \( k^{2} = 0 \), has transverse polarizations \( \epsilon_{\mu\nu}^{(T)} \) and \( \epsilon_{\mu\nu}^{(S)} \) with helicity 2; the \((p + 1)\) modes \( \omega_{m} \), with \( k^{2} \neq 0 \), have transverse polarizations \( \epsilon_{\mu\nu}^{(T)} \) and non-transverse ones \( \epsilon_{\mu\nu}^{(TT)}, \epsilon_{\mu\nu}^{(TS)}, \epsilon_{\mu\nu}^{(L)} \) with helicity 0.

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1 Introduction

The need of extending General Relativity arises from astrophysical and cosmological reasons like the impossibility of explaining the early and late time accelerated expansion of the Universe and from the necessity to unify gravitation with the others fundamentals interactions under the standard of quantum field theory. Indeed, quantum effects intervene at ultraviolet regimes and Standard Cosmological Model, based on General Relativity, does not work. On the other hand, also at infrared scales, the Einstein theory is not explaining galactic, extra-galactic and cosmological structures, except by introducing exotic ingredients like dark matter and energy, not probed, until now, at fundamental scales. It is worth pointing out that dark energy and dark matter may be also unified under the same barotropic fluid, without necessarily invoking extensions of General Relativity as discussed in literature [1, 2, 3, 4, 5, 6]. Approaches toward quantum cosmology as a source for dark energy have been proposed in view of observations in some model-independent pictures [7, 8, 9, 10, 11, 12].

Besides the above approaches, Extended Theories of Gravity (ETGs) [13, 14, 15, 16] act on the geometry considering higher order curvature invariants, like $R_{\mu\nu}R_{\mu\nu}$, $R_{\mu\nu\lambda\sigma}R_{\mu\nu\lambda\sigma}$, or higher order derivative terms into the Hilbert-Einstein Lagrangian, linear in the Ricci scalar $\frac{1}{2} R$. In particular, invariant terms like $\Box^{k} R$, where $\Box$ is the D’Alembert operator, give rise to field equations with derivative orders higher than 2, namely of $(2k + 4)$ order. Specifically, this kind of contributions emerge in quantum field theory formulated on curved spacetimes [19] and in effective actions adopted for quantum gravity [20]. Physical phenomena related to these terms, specifically the light bending, the weak-field limit and ghost-free behavior are discussed in literature [21, 22, 23].

Considering possible effects at infrared scales, the idea is that further degrees of freedom coming from geometrical invariants allow to avoid the introduction of ad hoc fields (dark matter, dark energy, quintessence, etc.), explaining in a natural way cosmological dynamics and astrophysical structures like the inflation, the today observed acceleration of universe, the flat rotation curves of galaxies, the structure of galaxy clusters [24]. However, the possibility to really test theories of gravity is recently emerged with the discovery of gravitational waves. They, early predicted by Einstein’s General Relativity, were indirectly observed in 1982 thanks to the decrease of the orbital period observed for the binary pulsar PRS B1913+16, discovered in 1974 by Hulse and Taylor [25, 26]. On September 14th, 2015 at 09:50:45 UTC, the LIGO Hanford, WA, and Livingston, LA, observatories detected the gravitational-wave transient generated by the merge of a black hole binary system [27, 28]: this was the first direct detection. Several detections immediately followed opening the doors to the prova regina for any theory of gravity: features like gravitational wave polarization, amplitude, spin and so on can single out the underlying theory of gravity.

Also ETGs predict the existence of gravitational waves by solving the associated linearized field equations. However different properties with respect to General Relativity can emerge. While the Einstein gravitational radiation is quadrupolar, transverse and with helicity 2 - carried by massless spin 2 graviton - the one related to ETGs can have transverse and longitudinal polarizations with 0 and 2 helicity, carried by massive and massless gravitons with spin 0 and 2. As pointed out in literature [29], also ghost modes can emerge into dynamics. Here, we are going to investigate the weak field limit of higher order gravity and the related gravitational radiation putting in evidence

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1 Such invariants can be constructed starting from the Ricci tensor $R_{\mu\nu}$, the Riemann tensor $R_{\mu\nu\lambda\sigma}$, the Weyl tensor $\mathcal{W}_{\mu\nu\lambda\sigma}$, or the Gauss-Bonnet topological invariant [17] which is $\mathcal{G} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$. In general, any Kretschmann curvature invariant can give rise to ETGs [18].

2 It is worth saying that $f(R)$ gravity, if $f(R) \neq R$, is a fourth-order theory in metric representation.
the main differences with respect to General Relativity.

The paper is organized in the following way: in Sec. 2 we derive the field equations for higher-order Lagrangian including terms like $R \Box^k R$ by using a variational principle and then we derive the weak field limit. In Sec. 3, we recover the Newtonian limit reproducing the Poisson equation as a consistency check. Sec. 4 is devoted to the resolution of the linear field equations by using the Fourier transformation and the Green functions. These propagators give rise to the gravitational waves. In Sec. 5, the gravitational waves are analyzed by the polarization and helicity of the massive and non-massive modes. Conclusions are drawn in Sec. 6.

2 The field equations and the weak field limit

Let us consider the Lagrangian density $L$ and its variational derivative with respect to $g_{\mu\nu}$

$$L = L_g + L_m,$$

$$L_g = (R + a_0 R^2 + \sum_{k=1}^{p} a_k R \Box^k R) \sqrt{-g},$$

$$L_m = 2 \chi \sqrt{-g} L_m,$$

$$P_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta L_g}{\delta g_{\mu\nu}},$$

with $\chi = 8\pi G/c^4$ and $\Box = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu}$ the covariant D’Alembert operator. Here $P_{\mu\nu}$ is the tensor as defined in [30] relative to the gravitational Lagrangian and $T_{\mu\nu}$ is the matter energy-momentum tensor. Applying the principle of minimal action with the stationarity condition $\delta L/\delta g_{\mu\nu} = 0$, we can define [31, 32]:

$$\frac{\delta L}{\delta g_{\mu\nu}} = \sum_{m=0}^{2p+2} (-1)^m \left( \frac{\partial L}{\partial g_{\mu\nu,i_1 \cdots i_m}} \right)_{i_1 \cdots i_m}.$$

We then obtain the field equations related to the above Lagrangian density $L$:

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + 2 a_0(R R_{\mu\nu} - \frac{1}{4} R^2 g_{\mu\nu} - R_{\mu\nu} + g_{\mu\nu} \Box R)$$

$$+ 2 \sum_{j=1}^{p} a_j (R_{\mu\nu} \Box^j R - \frac{1}{4} g_{\mu\nu} R^j \Box R - (\Box^j R)_{\mu\nu} + g_{\mu\nu} \Box^{j+1} R)$$

$$+ \sum_{A=1}^{p} \sum_{j=A}^{p} a_j \frac{1}{2} \left\{ g_{\mu\nu} \left[ (\Box^{j-A} R) \lambda (\Box^{A-1} R)_{\lambda \sigma} + (\Box^{j-A} R) (\Box^A R) \right] - 2(\Box^{j-A} R)_{\mu} (\Box^{A-1} R)_{\nu} \right\} = \chi T_{\mu\nu}.$$

In the weak field limit, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$, the field equations became [33, 34, 35]:

$$\frac{1}{2} (h_{\mu,\nu,\rho} + h_{\rho,\nu,\mu} - \Box h_{\mu\nu} - h_{\mu\nu}) - (\eta_{\mu\nu} \eta_{\lambda\sigma} + 2 \eta_{\mu\lambda} \eta_{\nu\sigma}) \sum_{k=0}^{p} a_k \Box^k (h^{\tau\kappa}_{\tau\kappa} - \Box h)_{\lambda\sigma} = 4\pi (2 T_{\mu\nu} - T \eta_{\mu\nu}),$$

3The signature of the metric $g_{\mu\nu}$ is $(+, -, -, -)$, the Ricci tensor is defined as $R_{\mu\nu} = R^\rho_{\mu\nu\rho}$ and the Riemann tensor as $R^\alpha_{\beta\mu\nu} = \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \ldots$. 3
with $\Box = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}$. To verify the contracted Bianchi identities $\nabla_{\nu}G^{\mu\nu} = 0$, the energy-momentum tensor $T^{\mu\nu}$ must be unperturbed that is approximated to zero-order in $h$, so that the conservation law is not violated, $\partial_{\nu}T^{\mu\nu} = 0$. In fact the ordinary divergence of left- and right-hand sides of (2.4) must be the same, that is:

$$\frac{1}{2}(h^{\rho,\mu}_{\rho,\mu} + h^{\rho,\nu}_{\rho,\nu} - \Box h^{\mu\nu} - h^{\mu\nu}) - (\eta_{\mu\nu}\eta_{\lambda\sigma} + 2\eta_{\mu\lambda}\eta_{\nu\sigma}) \sum_{k=0}^{p} a_{k} \Box^{k}(h^{\tau\kappa}_{\tau\kappa} - \Box h)^{\lambda\sigma\mu} = 4\pi (2T^{\mu\nu} - T^{\nu\mu}\eta_{\mu\nu}) . \quad (2.5)$$

The trace of (2.4) gives

$$6 \sum_{k=0}^{p} a_{k} \Box^{k}(h^{\tau\kappa}_{\tau\kappa} - \Box h) = h^{\tau\kappa}_{\tau\kappa} - \Box h + 8\pi T , \quad \text{(2.6)}$$

and, from Eqs. (2.5) and (2.4), we have:

$$\frac{1}{2}(h^{\tau\kappa}_{\tau\kappa} - \Box h)_{,\nu} - \frac{1}{6}(\eta_{\mu\nu}\eta_{\lambda\sigma} + 2\eta_{\mu\lambda}\eta_{\nu\sigma})(h^{\tau\kappa}_{\tau\kappa} - \Box h + 8\pi T)^{\lambda\sigma\mu} = 4\pi (2T^{\mu\nu} - T^{\nu\mu}\eta_{\mu\nu}) , \quad \text{(2.7)}$$

namely:

$$\frac{1}{2}(h^{\tau\kappa}_{\tau\kappa} - \Box h)_{,\nu} - \frac{1}{6}(3\Box)(h^{\tau\kappa}_{\tau\kappa} - \Box h + 8\pi T)_{,\nu} = 4\pi (2T^{\mu\nu} - T^{\nu\mu}) , \quad \text{(2.8)}$$

verified for $\partial_{\nu}T^{\mu\nu} = 0$.

Eq. (2.4) is gauge-invariant that is a gauge transformation, such that the harmonic gauge is always verified, exists. It is

$$(h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}h)_{,\mu} = 0 \Rightarrow \begin{cases} h^{\tau\kappa}_{\tau\kappa} = \frac{1}{2}\Box h \\ h^{\rho,\mu}_{\rho,\mu} + h^{\rho,\nu}_{\rho,\nu} - h^{\mu\nu} = 0 \end{cases} . \quad \text{(2.9)}$$

The simplified equations gives:

$$\Box h_{\mu\nu} - (\eta_{\mu\nu}\eta_{\lambda\sigma} + 2\eta_{\mu\lambda}\eta_{\nu\sigma}) \sum_{k=0}^{p} a_{k} \Box^{k+1}h^{,\lambda\sigma} = -8\pi (2T^{(0)}_{\mu\nu} - T^{(0)}_{\nu\mu}\eta_{\mu\nu}) , \quad \text{(2.10)}$$

namely ten linear 2$(p + 2)$-order partial differential equations where six of them are independent because of the contracted Bianchi identities. The trace of Eq. (2.10) is:

$$\Box h - 6 \sum_{k=0}^{p} a_{k} \Box^{k+2}h = 16\pi T . \quad \text{(2.11)}$$

In more compact form, defining

$$\begin{cases} c_{1} = 1 \\ c_{l} = -6a_{l-2} \text{ if } l > 1 \end{cases} , \quad \text{(2.12)}$$

4
we get then
\[ \sum_{l=1}^{p+2} c_l \Box^l h = 16\pi T^{(0)} \] (2.13)

which is one of the main results of the present paper. It means that higher-order terms in the $\Box$ operator contribute to the gravitational wave equation as further derivatives. As we will see below, this fact is important in the Fourier analysis of the waves.

3 The Newtonian limit

Let us now consider the Newtonian approximation assuming that, beside the weak field, also the involved velocities are less than the speed of light. We will consider two approaches to deal with the Newtonian limit.

3.1 First approach

In the Newtonian approximation we impose that:

1. the field is weak $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$;
2. the field is static $g_{\mu\nu,0} = 0$ and $g_{00} = 0$;
3. the velocities are small compared to $c$ that is $v/c \ll 1$ (slow motion).

Using the matter energy-momentum tensor $T_{\mu\nu}$ at the zero order, $T_{\mu\nu} = \delta^0_\mu \delta^0_\nu \rho$, because we are in the slow motion approximation, from (2.10) we get the following field equations [36, 37, 38]:

\[ \Delta h_{\mu\nu} + (\eta_{\mu\nu} \eta_{\lambda\sigma} + 2\eta_{\mu\lambda} \eta_{\nu\sigma}) \sum_{k=0}^{p} a_k (-1)^{k+1} \Delta^{k+1} h_{\lambda\sigma} = 8\pi (2\delta^0_\mu \delta^0_\nu - \eta_{\mu\nu}) \rho , \]

whose trace is:

\[ \Delta h + 6 \sum_{k=0}^{p} a_k (-1)^{k+2} \Delta^{k+2} h = -16\pi \rho . \] (3.2)

Here $\rho$ is the perfect-fluid matter density. From (3.1), the equation of time – time component is:

\[ \Delta h_{00} + \sum_{k=0}^{p} a_k (-1)^{k+2} \Delta^{k+2} h_{00} = +8\pi \rho . \] (3.3)

From (3.1) the equations of the space components are:

\[ \Delta h_{ij} + \eta_{ij} \sum_{k=0}^{p} a_k (-1)^{k+2} \Delta^{k+2} h + 2 \sum_{k=0}^{p} (-1)^{k+1} \Delta^{k+1} h_{i,j} = 8\pi \rho \delta_{ij} . \] (3.4)

From (3.2) and (3.3) we obtain:

\[ 6\Delta h_{00} - \Delta h = 64\pi \rho . \] (3.5)
The metric perturbation $h_{\mu\nu}$ in harmonic gauge is

$$h^{ij} - \frac{1}{2} h^{ii} = 0 \, ,$$

which is:

$$\begin{cases} h^{11} + h^{12} + h^{13} + \frac{1}{2} h_{11} = 0 \\ h^{21} + h^{22} + h^{23} + \frac{1}{2} h_{22} = 0 \\ h^{31} + h^{32} + h^{33} + \frac{1}{2} h_{33} = 0 \, , \end{cases}$$

verified if:

$$h_{00} = h_{11} = h_{22} = h_{33} \, ,$$

and the remaining ones equal to zero. From (3.5) and (3.8), knowing that $h = -2h_{00}$, we get:

$$\Delta h_{00} = 8\pi \rho \, .$$

Setting $h_{00} = 2\phi$ we have:

$$h_{\mu\nu} = \text{diag}(2\phi, 2\phi, 2\phi, 2\phi) \, .$$

Substituting in (3.9) we obtain the standard Poisson equation

$$\Delta \phi = 4\pi \rho \, .$$

This means that the Newtonian limit of the theory is consistent with the one of General Relativity.

### 3.2 Second approach

The Poisson equation can be obtained in an alternative way by adding to the metric the hypotheses of asymptotic flatness and asymptotic behavior. We can relax the hypothesis of the static spacetime that becomes stationary [39, 40, 41]. Considering all the hypotheses, we have

1. the weak field limit;
2. the stationarity ($g_{\mu\nu,0} = 0$);
3. the slow motion;
4. the asymptotic flatness;
5. the asymptotic behavior.

Performing the following replacement:

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2} h^{\mu\nu} h \, ,$$

the harmonic gauge becomes $(\bar{h}^{\mu\nu})_{,\mu} = 0$. If we impose that the metric is:

1. stationarity

$$g_{\mu\nu,0} = 0 \, ,$$
2. asymptotically flat
\[ \lim_{r \to \infty} g_{\mu \nu} = \eta_{\mu \nu} , \]

3. asymptotic behavior
\[ \lim_{r \to \infty} r^M \left[ \tilde{h}^{\mu \nu} - \sum_{n=0}^M A^{\mu \nu}_n \right] = 0 \quad \forall M > 0 , \]

we can asymptotically develop the tensor \( \tilde{h}^{\mu \nu} \) in the following way:
\[ \tilde{h}^{\mu \nu} = \frac{A^{\mu \nu}}{r} + O \left( \frac{1}{r^2} \right) . \] (3.13)

Imposing the gauge condition to (3.13):
\[ 0 = \tilde{h}^{\mu \nu} \epsilon_{\mu}^{\nu} = \frac{A^{\mu i} n_i}{r^2} + O \left( \frac{1}{r^3} \right) \quad \text{with} \quad n_i = \frac{x_i}{r} , \] (3.14)

we get:
\[ A^{\mu i} = 0 , \quad |\tilde{h}^{00}| \gg |\tilde{h}^{0 i}| , \quad |\tilde{h}^{00}| \gg |\tilde{h}^{k i}| , \]
that is, only \( \tilde{h}^{00} \) survives to the order \( \frac{1}{r} \) because \( \tilde{h}^{\mu i} \approx \frac{1}{r} \). In general, we have:
\[ \Delta^{k+1} \tilde{h}^{\lambda \sigma} \approx \frac{1}{r^{2k+3}} \quad \text{and} \quad \Delta h_{\mu \nu} \approx \frac{1}{r^3} \Rightarrow \Delta^{k+1} \tilde{h}^{\lambda \sigma} \ll \Delta h_{\mu \nu} . \] (3.16)

So Eq. (3.1), in these hypotheses (weak field, asymptotic flatness and asymptotic behavior), becomes
\[ \Delta \tilde{h}^{00} = 16 \pi \left( 2 T_{00} - T \right) , \] (3.17)
because \( \tilde{h}^{00} = 2 h_{00} \).

Considering the further hypothesis that velocities are much lower than \( c \), i.e. the slow motion, we get:
\[ \left| \frac{v}{c} \right| \ll 1 \Rightarrow |T^{00}| \gg |T^{0 i}| \gg |T^{ij}| \Rightarrow T = T_{00} = \rho , \] (3.18)
and then
\[ \Delta h_{00} = 8 \pi \rho . \]

Setting \( h_{00} = 2 \phi \) the Poisson equation is again obtained
\[ \Delta \phi = 4 \pi \rho . \]

It is worth saying that the above approaches are equivalent from a practical point of view. However, the energetic behavior of the gravitational field is better stressed in the second than in the first one due to the considerations on the asymptotic behavior. In the first case, the structure of the weak filed metric is better defined.
4 The solution of the field equations

Let us solve the differential equations with the Fourier transform method on an unlimited domain $\mathbb{R}^4$ with the hypothesis of sufficient regularity for $T_{\mu\nu} \in L^2(\mathbb{R}^4)$. In order to integrate Eqs. (2.10), we first solve its trace by calculating $h(x)$ from (2.13), replace it in the field equations which, at this point, become a non-homogeneous wave-like equation and finally, we obtain the metric perturbation $h_{\mu\nu}$. We write the general integral as the sum of the homogeneous solution plus a particular solution due to the linearity of Eqs. (2.10) that is

$$h_{\mu\nu} = h_{\mu\nu}^{\text{homog}} + h_{\mu\nu}^{\text{part}}, \quad (4.1)$$

where we indicate $h_{\mu\nu}$ and $h_{\mu\nu}$ as the homogeneous and the particular solution respectively.

4.1 The particular solution

Let us calculate the Green function for the linear differential operator

$$D = \sum_{l=1}^{p+2} c_l \Box^l,$$  \quad (4.2)

namely

$$\sum_{l=1}^{p+2} c_l \Box^l G_D(x, x') = \delta^4(x - x'). \quad (4.3)$$

Fourier integrals for the $G_D$ and $\delta$ are

$$G_D(x, x') = \int \frac{d^4k}{(2\pi)^4} \tilde{G}_D(k) e^{i k^\alpha (x_\alpha - x'_\alpha)}, \quad \delta(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{i k^\alpha (x_\alpha - x'_\alpha)}, \quad (4.4)$$

where $x = x^\alpha = (t, \mathbf{x}), k = k^\alpha = (\omega, \mathbf{k}), k^2 = k^\alpha k_\alpha, \mathbf{k} \cdot \mathbf{k} = q^2$ and $q = |\mathbf{k}|$.

By means of the formula

$$\Box^l G_D(x, x') = \int \frac{d^4k}{(2\pi)^4} (-1)^l k^{2l} \tilde{G}_D(k) e^{i k^\alpha (x_\alpha - x'_\alpha)}, \quad (4.5)$$

writing Eq. (4.3) in the space of $k$, we get the transformed Green function

$$\tilde{G}_D(k) = \frac{1}{\sum_{l=1}^{p+2} c_l (-1)^l k^{2l}}; \quad (4.6)$$

that is

$$G_D(x, x') = \int \frac{d^4k}{(2\pi)^4} \frac{1}{\sum_{l=1}^{p+2} c_l (-1)^l k^{2l}} e^{i k^\alpha (x_\alpha - x'_\alpha)}. \quad (4.7)$$

So the particular solution of (2.13) is:

$$h(x) = 16\pi \int d^4x' G_D(x, x') T(x'), \quad (4.8)$$
namely
\[ h(x) = \int \frac{d^3k}{(2\pi)^3} \left[ \int d^4x' \frac{16\pi T(x') e^{-ik^\alpha x_\alpha}}{\sum_{l=1}^{p+2} c_l (-1)^l k^{2l}} \right] e^{ik^\alpha x_\alpha}. \] (4.9)

Each derivative of the metric perturbation \( h_{\mu\nu} \), in the coordinate space, adds one \( ik_\mu \) in the Fourier space, that is \( h(x)_\mu \rightarrow ik_\mu \hat{h}(k) \). Replacing Eq. (4.9) into (2.10), we get
\[ \square h_{\mu\nu} - \int \frac{d^4k}{(2\pi)^4} \left\{ \hat{h}(k) \left[ (\eta_{\mu\nu} k^2 + 2k_\mu k_\nu) \sum_{l=0}^{p} a_l (-1)^{l+2} k^{2(l+1)} \right] e^{ik^\alpha x_\alpha} \right\} = -8\pi (2T_{\mu\nu} - T\eta_{\mu\nu}), \] (4.10)

and thus we have
\[ \square h_{\mu\nu} = -8\pi (2T_{\mu\nu} - T\eta_{\mu\nu}) + F_{\mu\nu}(x), \] (4.11)

\[ h_{\mu\nu}(x) = \int d^4x'' G_\square(x, x'') \left[ -8\pi (2T_{\mu\nu}(x'') - \eta_{\mu\nu}T(x'')) + F_{\mu\nu}(x'') \right] \] (4.12)

namely the particular solution of the field equations, where \( G_\square \) is the Green function for the \( \square \) operator:
\[ \square G_\square(x, x'') = \delta^4(x - x''), \] (4.13)
\[ G_\square(x, x'') = \int \frac{d^4k}{(2\pi)^4} \left( \frac{1}{k^2 - i\epsilon} \right) e^{ik^\alpha(x_\alpha - x'^\alpha')} \] (4.14)

From a physical point of view, this solution is interesting because, due to the presence of \( T_{\mu\nu} \) in it, it is strictly related to the distribution of matter-energy which determines the propagation of the gravitational interaction.

4.2 The homogeneous solution

In order to calculate the homogeneous solution of Eqs. (2.10), we perform the Fourier transform only of the spatial coordinates of the trace:
\[ h(t, x) = \int \frac{d^3k}{(2\pi)^3} h(t, k) e^{-ik^\alpha x_\alpha}, \] (4.15)

by means of the homogeneous Eq. (2.13)
\[ \sum_{l=1}^{p+2} c_l (\partial^2_0 - \Delta)^l h(x) = 0, \] (4.16)

Then we obtain:
\[ \sum_{l=1}^{p+2} c_l \left[ \frac{\partial^2}{\partial t^2} + q^2 \right]^l h(t, k) = 0, \] (4.17)
where \( q^2 = k \cdot k \). Let \( \tilde{D}_t \) be the following linear differential operator
\[
\tilde{D}_t = \partial_t^2 + q^2 ,
\]
we rewrite the equation in the following way
\[
\sum_{l=1}^{p+2} c_l \tilde{D}_t^l \mathfrak{h}(t, k) = 0 ,
\]
representing a homogeneous ordinary differential equation in \( t \) of \( 2(p+2) \)-degree, with constant coefficients. Its characteristic polynomial is:
\[
\sum_{l=1}^{p+2} c_l (\lambda^2 + q^2)^l = 0 ,
\]
which admits \( 2(p+2) \) solutions in \( \lambda \), for the fundamental theorem of the algebra. If we set
\[
\lambda^\pm = \pm i\omega_m ,
\]
we obtain \( p+2 \) polynomial solutions for \( \omega_m \). We define the four-vector \( k^\mu_m = (\omega_m, k) \) with
\[
k^2_m = \omega^2_m - q^2 ,
\]
and we have \( p+2 \) solutions \( k^2_m \) of the equation \( \sum_{l=1}^{p+2} c_l (-1)^l (k^2)^l = 0 \). A solution is definitely \( k^2_1 = 0 \), i.e. \( \omega_1 = q \), that give the gravitational waves predicted by General Relativity. In general \( \omega_m \) is a complex number and therefore there are also damped waves, but, considering only real \( \omega_m \), one can impose conditions on \( c_l \), that is on \( a_l \). So we will limit to the case of real, distinct \( \omega_m \): this means that they are exactly \( p+2 \) different real numbers from the imposed conditions. The solutions of Eq. (4.18) are 39, 42:
\[
\mathfrak{h}(t, k) = \sum_{m=1}^{p+2} \left[ A'_m(k) e^{i\omega_m t} + A''_m(k) e^{-i\omega_m t} \right] ,
\]
which, in coordinate space, becomes:
\[
\mathfrak{h}(t, x) = \sum_{m=1}^{p+2} \int \frac{d^3k}{(2\pi)^3} \left[ A'^+_m(k) e^{i(\omega_m t - k \cdot x)} + A''_m(k) e^{-i(\omega_m t - k \cdot x)} \right] ,
\]
where \( A'^+_m(k) = A'_m(k) \) and \( A''_m(k) = A''_m(-k) \). Since we want a real \( \mathfrak{h}(t, x) \) so that \( A_m^+(k) = A_m^-(k) \), we take the real part of the following integral:
\[
\Re \left\{ \sum_{m=1}^{p+2} \int \frac{d^3k}{(2\pi)^3} A_m(k) e^{i(\omega_m t - k \cdot x)} \right\} ,
\]
where \( A_m \) is a complex function. Later, even if not indicated, we will consider only the real part or equivalently the c.c.. Let us represent \( \mathfrak{h}_{\mu\nu}(t, x) \) by means of the Fourier transform with respect to the spatial coordinates only \( x \)
\[
\mathfrak{h}_{\mu\nu}(t, x) = \int \frac{d^3k}{(2\pi)^3} \mathfrak{h}_{\mu\nu}(t, k) e^{-ik \cdot x} ,
\]
where \( k_m^\mu = (\omega_m, k) \) as already defined. From the homogeneous Eq. (2.10), from the (4.24) and (4.25) in the Fourier space, we have:

\[
(\partial_0^2 + q^2) h_{\mu\nu} (t, k) = \sum_{m=1}^{p+2} \left[ \eta_{\mu\nu} k_m^2 + 2 (k_m)_{\mu} (k_m)_{\nu} \right] \sum_{l=0}^{p} a_l (-1)^{l+2} k_m^{2(l+1)} A_m (k) e^{i\omega_m t},
\]

(4.26)

where \( k_m^2 = (k_m)^\alpha (k_m)_\alpha = \omega_m^2 - q^2 \).

For \( m = 1 \), being \( k_1^2 = 0 \), the first term of the sum in \( m \) in Eq. (4.26) is cancelled and then the sum starts from \( m = 2 \). For \( 2 \leq m \leq p + 2 \), where \( k_m^2 \neq 0 \), from the following identity:

\[
\sum_{l=1}^{p+2} a_l (-1)^l k_m^{2l} = -k_m^2 \left( 1 + 6 \sum_{l=0}^{p} a_l (-1)^{l+2} k_m^{2(l+1)} \right) = 0 \Rightarrow \sum_{l=0}^{p} a_l (-1)^{l+2} k_m^{2(l+1)} = -\frac{1}{6},
\]

(4.27)

we get:

\[
(\partial_0^2 + q^2) h_{\mu\nu} (t, k) = \sum_{m=2}^{p+2} \rho_{\mu\nu} (k; m) e^{i\omega_m t},
\]

(4.28)

where

\[
\rho_{\mu\nu} (k; m) = \left\{ -\frac{k_m^2}{3} \left[ \eta_{\mu\nu} \frac{2}{m} + \frac{(k_m)_{\mu} (k_m)_{\nu} k_m^2}{m} \right] \right\} A_m (k) .
\]

(4.29)

The Fourier transform in the \( x \) coordinates of the function \( \rho_{\mu\nu} (t, x) \) is

\[
\rho_{\mu\nu} (x) = \sum_{m=2}^{p+2} \int \frac{d^3k}{(2\pi)^3} \rho_{\mu\nu} (k; m) e^{ik_m^\alpha x_\alpha} = \sum_{m=2}^{p+2} \rho_{\mu\nu} (x; m) ,
\]

(4.30)

where \( k_m^\alpha x_\alpha = \omega_m t - k \cdot x \). In the space of \( x \), we obtain

\[
\square h_{\mu\nu} (x) = \sum_{m=2}^{p+2} \rho_{\mu\nu} (x; m),
\]

(4.31)

that we solve as the sum of the homogeneous solution plus the particular one:

\[
h_{\mu\nu} (x) = \int \frac{d^3k}{(2\pi)^3} C_{\mu\nu} (k) e^{ik_m^\alpha x_\alpha} + \sum_{m=2}^{p+2} \int \frac{d^4x'}{(2\pi)^3} G_{\square} (x, x') \rho_{\mu\nu} (x'; m) ,
\]

(4.32)

that is the solution of the field equations in the vacuum. Equivalently:

\[
h_{\mu\nu} (x) = \int \frac{d^3k}{(2\pi)^3} C_{\mu\nu} (k) e^{ik_m^\alpha x_\alpha} + \sum_{m=2}^{p+2} \int \frac{d^3k}{(2\pi)^3} G_{\square} (k_m) \rho_{\mu\nu} (k; m) e^{ik_m^\alpha x_\alpha} ,
\]

(4.33)

that we can also write more explicitly

\[
h_{\mu\nu} (x) = \int \frac{d^3k}{(2\pi)^3} C_{\mu\nu} (k) e^{ik_m^\alpha x_\alpha} + \sum_{m=2}^{p+2} \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{1}{3} \left[ \eta_{\mu\nu} \frac{2}{m} + \frac{(k_m)_{\mu} (k_m)_{\nu} k_m^2}{m} \right] \right\} A_m (k) e^{ik_m^\alpha x_\alpha}.
\]

(4.34)
So the general integral of the field equations in presence of matter is:

\[
\begin{aligned}
\hat{h}_{\mu\nu}(x) &= \int d^4x' G^{\Box}(x, x') \left[-8\pi(2 T_{\mu\nu}(x') - \eta_{\mu\nu} T(x'))\right] + \int \frac{d^3k}{(2\pi)^3} C_{\mu\nu}(k) e^{i k^\alpha x^\alpha} \\
&\quad + \int d^4x' G^{\Box}(x, x') \left[F_{\mu\nu}(x') + \sum_{m=2}^{p+2} \rho_{\mu\nu}(x', m)\right] \\
&\quad \text{gravitational waves in matter, correction terms}
\end{aligned}
\]

This is the most general gravitational wave solution coming from higher-order gravitational theories.

## 5 Polarization and helicity states in vacuum

In order to study the polarization and the helicity of waves \[13\ [14\ [24\], let us consider separately the mode with \( k_1^2 = 0 \), that we indicate with the oscillation mode \( A_1 \), from the \((p + 1)\) modes with \( k_m^2 \neq 0 \) for \( 2 \leq m \leq p + 2 \), that we indicate with the \( A_m \) modes. All the modes are classified in Table 1:

| Mode | Wave number | Gauge condition |
|------|-------------|-----------------|
| \( A_1 \) | \( k^2 = k_1^2 = 0 \) & \( \bar{h} = C_p(\bar{h}) = 0 \) ⇒ \( \bar{h}_{\mu\nu} \) | \( \bar{h}_{\mu\nu} k^\mu = 0 \) |
| \( A_2 \) | \( k^2 = k_2^2 = \omega_2^2 - q^2 \) & \( \bar{h} = A_2(\bar{h}) \neq 0 \) ⇒ \( \bar{h}_{\mu\nu} = \frac{i}{4} \left( \frac{2m}{\omega_2} + \frac{(k_2)(k_2)}{k_2^2} \right) \bar{h} \) | \( \bar{h}_{\mu\nu} k^\mu - \frac{1}{2} \bar{h} k^\nu = 0 \) |
| \( \cdots \) | \( \cdots \) | \( \cdots \) |
| \( A_{p+2} \) | \( k^2 = k_{p+2}^2 = \omega_{p+2}^2 - q^2 \) & \( \bar{h} = A_m(\bar{h}) \neq 0 \) ⇒ \( \bar{h}_{\mu\nu} = \frac{i}{4} \left( \frac{2m}{\omega_{p+2}} + \frac{(k_{p+2})(k_{p+2})}{k_{p+2}^2} \right) \bar{h} \) | verified |

Table 1: Classification of waves in vacuum

Consider a wave propagating along the \( z \)-axis. From Eq. \ref{5.1}, we have:

\[
\begin{aligned}
\hat{h}_{\mu\nu}(t, z) &= \int \frac{d^3k}{(2\pi)^3} C_{\mu\nu}(k) e^{i \varepsilon_{\mu\nu}(t - z)} + \sum_{m=2}^{p+2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{3} \left( \frac{\eta_{\mu\nu}}{2} + \frac{(k_m)(k_m)}{k_m^2} \right) \right] A_m(k) e^{i(\omega_m t - k_m z)}
\end{aligned}
\]

(5.1)

where \( k_1^2 = (\omega_1, 0, 0, k_z) \) and \( k_m^2 = (\omega_m, 0, 0, k_z) \). The oscillation mode \( A_1 \) with \( k^2 = k_1^2 = 0 \), i.e. \( \omega_1 = q = k_z > 0 \), could have the trace of the metric perturbation not equal to zero, \( \bar{h} \neq 0 \). However, by exploiting the degrees left free, we can perform an infinitesimal transformation that makes the trace equal to zero. In fact, from Eq. \ref{5.1}, considering the plane wave associated with the \( \omega_1 \) mode to \( k \) constant that propagates along the \( z \)-axis, using the symmetry of the polarization tensor \( \epsilon_{\mu\nu}^A \), we have:

\[
\begin{aligned}
\hat{h}_{\mu\nu}^A(t, z) &= \epsilon_{\mu\nu}^A(k) e^{i \varepsilon_{\mu\nu}(t - z)} = \begin{pmatrix}
\varepsilon_{00} & \varepsilon_{01} & \varepsilon_{02} & \varepsilon_{03} \\
\varepsilon_{10} & \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
\varepsilon_{20} & \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
\varepsilon_{30} & \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{pmatrix} e^{i \omega_1(t - z)},
\end{aligned}
\]

(5.2)
where we have set \( C_{\mu\nu} (k) = \epsilon_{\mu\nu}^A (k) \). From the four gauge relations
\[
\epsilon_{\mu\nu} k^\mu - \frac{1}{2} \epsilon k^\nu = 0 ,
\] (5.3)
in the \( k \)-space, the ten unknowns become six and precisely: \( \epsilon_{01}, \epsilon_{02}, \epsilon_{03}, \epsilon_{11}, \epsilon_{12} \) and \( \epsilon \). Thus explicitly written:
\[
\begin{align*}
\epsilon_{00} + \epsilon_{30} &= \frac{1}{2} \epsilon \\
\epsilon_{01} + \epsilon_{31} &= 0 \\
\epsilon_{02} + \epsilon_{32} &= 0 \\
\epsilon_{03} + \epsilon_{33} &= -\frac{1}{2} \epsilon 
\end{align*}
\] (5.4)
If we perform the infinitesimal transformation \( x'^\mu = x^\mu + \xi^\mu \), the polarization tensor \( \epsilon_{\mu\nu}^A \), at first-order in \( |\xi| \), becomes
\[
\epsilon'_{\mu\nu} = \epsilon_{\mu\nu} + k_\mu \theta_\nu + k_\nu \theta_\mu ,
\] (5.5)
if
\[
\xi^\mu = i\theta^\mu e^{ik^\alpha x_\alpha} .
\] (5.6)
In the mode \( k_\alpha^2 = 0 \), we have \( \Box \xi^\mu = 0 \) that is, the gauge remains unchanged (gauge invariant) and therefore we can find a particular gauge transformation that makes the metric perturbation, transverse and traceless. In the Fourier space, it is equivalent to impose the two conditions \( k^\mu \epsilon'_{\mu\nu} = 0 \) and \( \epsilon' = 0 \), known as the TT gauge, where we have only two degrees of freedom. So for the system (5.4), we have:
\[
\begin{align*}
\epsilon'_{00} &= \epsilon_{00} + 2\omega_1 \theta_0 \\
\epsilon'_{11} &= \epsilon_{11} \\
\epsilon'_{22} &= \epsilon_{22} \\
\epsilon'_{33} &= \epsilon_{33} - 2\omega_1 \theta_3 \\
\epsilon'_{01} &= \epsilon_{01} + \omega_1 \theta_1 \\
\epsilon'_{02} &= \epsilon_{02} + \omega_1 \theta_2 \\
\epsilon'_{03} &= \epsilon_{03} + \omega_1 \theta_3 - \omega_1 \theta_0 \\
\epsilon'_{12} &= \epsilon_{12} \\
\epsilon'_{13} &= \epsilon_{13} - k_\omega \theta_1 \\
\epsilon'_{23} &= \epsilon_{23} - k_\omega \theta_2
\end{align*}
\] (5.7)
By choosing the 4-vector associated to the infinitesimal transformation as
\[
\theta_\mu = \left( \frac{\epsilon}{4\omega_1}, \frac{\epsilon_{30}}{2\omega_1}, -\frac{\epsilon_{01}}{\omega_1}, -\frac{\epsilon_{02}}{\omega_1}, -\frac{\epsilon}{4\omega_1} - \frac{\epsilon_{30}}{2\omega_1} \right) ,
\] (5.8)
the polarization tensor \( \epsilon'_{\mu\nu}^A \) becomes
\[
\epsilon'_{\mu\nu}^A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \epsilon_{11} & \epsilon_{12} & 0 \\
0 & \epsilon_{12} & -\epsilon_{11} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \epsilon_{11} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + \epsilon_{12} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} ,
\] (5.9)
namely for the mode $A_1$, we have the two transversal polarization states of General Relativity. So the plane wave $A_1$ can be written as:

$$h_{\mu\nu}^{A_1}(t, z) = \sqrt{2} \left[ \epsilon_{11}^{(+)\mu\nu} + \epsilon_{12}^{(\times)\mu\nu} \right] e^{i\omega_1(t-z)},$$

(5.10)

where we have indicated the two polarization states of General Relativity as

$$\epsilon^{(+)}_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \epsilon^{(\times)}_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (5.11)$$

Oscillation modes $A_m$, with $2 \leq m \leq p + 2$ and $k^2 = k_m^2 \neq 0$, have the perturbation trace $h \neq 0$, otherwise one gets $A_m(k) = 0$, that is the trivial solution $h_{\mu\nu} = 0$. From Eq. (5.1), we consider the plane wave related to the mode $m$ for fixed $k$ that is:

$$h_{\mu\nu}^{A_m}(t, z) = \epsilon_{\mu\nu}^{A_m} e^{i\omega_m t - k_z z} = \frac{A_m(k)}{3} \begin{pmatrix} \frac{1}{2} + \frac{\omega^2}{k_m^2} & 0 & 0 & -\frac{\omega k_m}{k_m^2} \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ \frac{\omega k_m}{k_m^2} & 0 & 0 & -\frac{1}{2} + \frac{k^2}{k_m^2} \end{pmatrix} e^{i\omega_m t - k_z z},$$

(5.12)

where $\epsilon_{\mu\nu}^{A_m}(k) = \epsilon_{\mu\nu}(k) \frac{A_m(k)}{3}$ and the harmonic gauge is verified in the Fourier space

$$\epsilon_{\mu\nu}k^\mu - \frac{1}{2}ck^\nu = 0.$$  

(5.13)

The harmonic gauge, in general, is not an invariant gauge, and for a general gauge transformation $x'^\mu = x^\mu + \xi^\mu$, it turns out to be:

$$\partial_\mu \left( h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \right) = \partial_\mu \left( h^{\mu\nu} - \frac{1}{2} \eta^{\mu\nu} h \right) - \Box \xi^\nu. \quad (5.14)$$

Our transformation maps harmonic gauge into harmonic gauge if and only if $\xi^\nu$ is the solution of equation $\Box \xi^\nu = 0$. So we will have as a solution

$$\xi^\nu(x) = \int \frac{d^3k}{(2\pi)^3} \theta^\mu(k) e^{i k_m x_\alpha} \Leftrightarrow k_m^2 = 0.$$  

(5.15)

But, for massive modes $k_m^2 \neq 0$, we have:

$$\Box \xi^\nu(x) = \int \frac{d^3k}{(2\pi)^3} \theta^\mu(k) (-k_m^2) e^{i k_m x_\alpha} \neq 0,$$

(5.16)

unless $\theta^\nu = 0$, namely the identical transformation. For the massive modes there is no gauge transformation that leaves the harmonic gauge unchanged, that is the polarization tensor $\epsilon_{\mu\nu}^{A_m}$ cannot be modified. Hence it is not possible to make the polarization tensor for these massive
modes neither completely spatial nor traceless. Expressing it as a function of a suitable orthonormal polarization basis, we have:

\[
\epsilon_{\mu\nu}^m = \frac{A_m(k)}{3} \begin{pmatrix}
\frac{1}{2} + \frac{\omega^2}{k_m^2} & 0 & 0 & -\frac{\omega_m k_z}{k_m^2} \\
0 & -\frac{1}{2} + \frac{\omega_m k_z}{k_m^2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} + \frac{\omega_m k_z}{k_m^2} & 0 \\
-\frac{\omega_m k_z}{k_m^2} & 0 & 0 & -\frac{1}{2} + \frac{k^2}{k_m^2}
\end{pmatrix} = A_m(k) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
+ \left(-\frac{1}{2}\right) \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} + \left(-\frac{1}{2}\right) \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \right).
\]

Indicating the four polarization states as:

\[
\epsilon_{\mu\nu}^{(TT)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \epsilon_{\mu\nu}^{(TS)} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\epsilon_{\mu\nu}^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad \epsilon_{\mu\nu}^{(L)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

we can express \(A_m\) plane waves as:

\[
\mathbf{h}_m^{\mu\nu}(t, z) = \left[\frac{1}{3} \left(\frac{1}{2} + \frac{\omega_m^2}{k_m^2}\right) \epsilon_{\mu\nu}^{(TT)} - \frac{\sqrt{2} \omega_m k_z}{3 k_m^2} \epsilon_{\mu\nu}^{(TS)} + \frac{\sqrt{2}}{6} \epsilon_{\mu\nu}^{(1)} + \frac{1}{3} \left(-\frac{1}{2} + \frac{k^2}{k_m^2}\right) \epsilon_{\mu\nu}^{(L)}\right] A_m(k) e^{i(\omega_m t - k_z z)}.
\]

So the general solution for a wave that propagates along the z-axis considering the \((p+2)\) oscillation modes \(\omega_m\) becomes:

\[
\mathbf{h}_m^{\mu\nu}(t, z) = \int \frac{d^3k}{(2\pi)^3} \left[\sqrt{2} \epsilon_{\mu\nu}^{(x)} + \sqrt{2} \epsilon_{\mu\nu}^{(s)}\right] e^{i\omega_1(t-z)}
+ \sum_{m=2}^{p+2} \int \frac{d^3k}{(2\pi)^3} \left[\frac{1}{3} \left(\frac{1}{2} + \frac{\omega_m^2}{k_m^2}\right) \epsilon_{\mu\nu}^{(TT)} - \frac{\sqrt{2} \omega_m k_z}{3 k_m^2} \epsilon_{\mu\nu}^{(TS)} + \frac{\sqrt{2}}{6} \epsilon_{\mu\nu}^{(1)} + \frac{1}{3} \left(-\frac{1}{2} + \frac{k^2}{k_m^2}\right) \epsilon_{\mu\nu}^{(L)}\right] A_m(k) e^{i(\omega_m t - k_z z)}.
\]

expressed with respect to the polarization basis \(\epsilon_{\mu\nu}^{(+)}, \epsilon_{\mu\nu}^{(x)}, \epsilon_{\mu\nu}^{(TT)}, \epsilon_{\mu\nu}^{(TS)}, \epsilon_{\mu\nu}^{(1)}, \epsilon_{\mu\nu}^{(L)}\). In terms of
amplitudes, the solution can be written as:

\[
\begin{align*}
    h_{\mu\nu}(t, z) &= A^{(+)}(t-z) \epsilon^{(+)}_{\mu\nu} + A^{(x)}(t-z) \epsilon^{(x)}_{\mu\nu} \\
    &+ \sum_{m=2}^{p+2} \left[ A_{m}^{(TT)}(t-v_{G_{m}}z) \epsilon^{(TT)}_{\mu\nu} + A_{m}^{(TS)}(t-v_{G_{m}}z) \epsilon^{(TS)}_{\mu\nu} \\
    &+ A_{m}^{(L)}(t-v_{G_{m}}z) \epsilon^{(L)}_{\mu\nu} \right],
\end{align*}
\]  

(5.21)

where \( v_{G_{m}} \) is the group velocity, related to the massive mode \( m \), defined below in Eq. (5.29). The polarization tensors have been chosen to form an orthonormal basis, that is they have to verify the following relation:

\[
\text{Tr} \left\{ \epsilon^{(i)} \epsilon^{*(j)} \right\} \equiv \epsilon^{(i)}_{\mu\nu} \epsilon^{*(j)\mu\nu} = \delta^{ij} \quad \text{with} \quad i, j \in \{+, x, TT, TS, 1, L\} 
\]  

(5.22)

The three polarization states \( \epsilon^{(+)}_{\mu\nu}, \epsilon^{(x)}_{\mu\nu} \) and \( \epsilon^{(L)}_{\mu\nu} \) are transversal and verify the relation \( \epsilon_{\mu\nu} k^{\mu} = 0 \):

\[
\epsilon^{(+)}_{\mu\nu} k^{\mu} = 0, \quad \epsilon^{(x)}_{\mu\nu} k^{\mu} = 0, \quad \epsilon^{(L)}_{\mu\nu} k^{\mu} = 0, 
\]  

(5.23)

while the remaining three polarization states \( \epsilon^{(TT)}_{\mu\nu}, \epsilon^{(TS)}_{\mu\nu}, \epsilon^{(1)}_{\mu\nu} \) are not transversal:

\[
\epsilon^{(TT)}_{\mu\nu} k^{\mu} \neq 0, \quad \epsilon^{(TS)}_{\mu\nu} k^{\mu} \neq 0, \quad \epsilon^{(1)}_{\mu\nu} k^{\mu} \neq 0. 
\]  

(5.24)

In summary, there are 6 polarization states. In order to study the helicity of such waves, we see how the polarization basis \( \{ \epsilon^{(+)}_{\mu\nu}, \epsilon^{(x)}_{\mu\nu}, \epsilon^{(TT)}_{\mu\nu}, \epsilon^{(TS)}_{\mu\nu}, \epsilon^{(1)}_{\mu\nu}, \epsilon^{(L)}_{\mu\nu} \} \) under a rotation of an angle \( \varphi \) around the \( z \)-axis:

\[
R_{\mu}^{\nu} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \varphi & \sin \varphi & 0 \\
0 & -\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]  

(5.25)

that is \( \tilde{\epsilon}_{\mu\nu} = R_{\mu}^{\sigma} R_{\nu}^{\rho} \epsilon_{\sigma\rho} \). The four polarizations \( \epsilon^{(TT)}_{\mu\nu}, \epsilon^{(TS)}_{\mu\nu}, \epsilon^{(1)}_{\mu\nu} \) and \( \epsilon^{(L)}_{\mu\nu} \) remain unchanged under rotations and therefore the waves \( A_{m} \) with \( 2 \leq m \leq p+2 \) have helicity equal to zero.

If we introduce two further polarization states, called circular,

\[
\epsilon^{(R)}_{\mu\nu} = \frac{1}{\sqrt{2}} \left( \epsilon^{(+)}_{\mu\nu} + i \epsilon^{(x)}_{\mu\nu} \right),
\]  

(5.26)

and

\[
\epsilon^{(L)}_{\mu\nu} = \frac{1}{\sqrt{2}} \left( \epsilon^{(+)}_{\mu\nu} - i \epsilon^{(x)}_{\mu\nu} \right),
\]  

(5.27)

we see that, under rotation, they transform as:

\[
\epsilon^{(R)}_{\mu\nu} = e^{2i\varphi} \epsilon^{(R)}_{\mu\nu},
\]  

(5.28)

that is, waves like \( A_{1} \) have two-helicity because they are the standard ones of General Relativity.

It is possible to prove that our Lagrangian (2.1) is conformally equivalent to Einstein’s theory with \( (p+1) \) appropriate scalar fields [32]. Setting \( k_{m}^{2} = M_{m}^{2} \) for the waves \( A_{m} \), the dispersion
relation becomes \( \omega_m(q) = \sqrt{M_m^2 + q^2} \). We can interpret the \((p + 1)\) oscillation modes as massive scalar fields of mass \( M \) with four polarization states, one transverse \( \epsilon^{(1)}_{\mu\nu} \) and three longitudinal \( \epsilon^{(TT)}_{\mu\nu}, \epsilon^{(TS)}_{\mu\nu}, \epsilon^{(L)}_{\mu\nu} \) with helicity equal to zero. We can associate a massless tensor field to the wave \( A_1 \), with two transverse polarization states with helicity two. The wave associated with massless mode \( A_1 \) has velocity \( c \), whereas \( A_m \) waves have a velocity other than \( c \), due to the dispersion law. That is, if we consider the wave packet associated with such modes, the group velocity is:

\[
v_{G_m} = \frac{d\omega_m(q)}{dq} = \frac{\sqrt{\omega_m^2 - M_m^2}}{\omega_m}, \tag{5.29}
\]

which allows us to associate a velocity to the wave \( A_m \) and therefore to the scalar field. A summary of polarizations and helicity states is reported in Table 2.

| Mode | Dispersion relation | Polarization | Helicity | Mass of the associated field |
|------|---------------------|--------------|----------|-----------------------------|
| \( A_1 \) | \( \omega_1 = q \) | \( \epsilon^{(+)\mu\nu}, \epsilon^{(\times)\mu\nu} \) | 2 | 0 |
| \( A_2 \) | \( \omega_2 = \sqrt{k_2^2 + q^2} \) | \( \epsilon^{(TT)}_{\mu\nu}, \epsilon^{(TS)}_{\mu\nu}, \epsilon^{(1)}_{\mu\nu}, \epsilon^{(L)}_{\mu\nu} \) | 0 | \( M_2 = k_2 \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( A_{p+2} \) | \( \omega_{p+2} = \sqrt{k_{p+2}^2 + q^2} \) | \( \epsilon^{(TT)}_{\mu\nu}, \epsilon^{(TS)}_{\mu\nu}, \epsilon^{(1)}_{\mu\nu}, \epsilon^{(L)}_{\mu\nu} \) | 0 | \( M_{p+2} = k_{p+2} \) |

Table 2: Polarizations and helicity states

6 Conclusions

Properties of gravitational waves provide fundamental information for any theory of gravity. In particular, they allow to set constraints on the gravitational Lagrangian considering the multipolar radiation, polarization and helicity states [45, 46] also considering, indirectly, astrophysical dynamics [47]. Here we have taken into account a generic higher-order gravitational Lagrangian density \( L_g = (R + a_0 R^2 + \sum_{k=1}^{p} a_k R^k R) \sqrt{-g} \). We perturbed the metric \( g_{\mu\nu} \) with respect to the flat Minkowski spacetime \( \eta_{\mu\nu} \) and thus, we obtained the linearized equations in the perturbed metric \( h_{\mu\nu} \). The solutions are gravitational waves with \((p + 2)\) normal modes of helicity 0 and 2 with 6 polarization states, three transverse and three longitudinal. Here \( p \) is the order of the theory so the result is completely general for theories of any order. It is important to stress the fact that, in four dimensions, the maximal allowed number of polarization state is always 6 for any theory of gravity. This fact has a deep intrinsic meaning that can be related to the fundamental structure of spacetime and the degrees of freedom of gravitational field (see also [29, 48]).

In principle, the emitted power from a gravitational radiating source can be related to the gravitational energy-momentum pseudo-tensor (see [30, 49, 50] for details in metric and teleparallel gravity). Then features of sources and further gravitational modes could be strictly related. In this sense, the so called multimessenger astrophysics is a powerful tool to discriminate among concurring gravitational theories (see [51, 52, 53]).
In particular, relating terrestrial laser interferometers like LIGO (Livingston and Hanford, USA), VIRGO (Cascina, Italy), GEO 600 (Hannover, Germany), Tama 300 (Mitsaka, Japan), KAGRA (Japan), LIGO-India (India) and, in principle, the space interferometer LISA, could be the best approach to detect or exclude possible further gravitational modes.

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