Effects of the Variational Consistency and Variational Inconsistency on Free Flexural Vibration Frequencies of Simply-supported Rectangular Isotropic Shear Deformable Beams

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Abstract. In order to derive governing differential equations of a displacement-based shear deformation beam theory, one can either utilise the variationally consistent approach (VCA) or the variationally inconsistent approach (VIA). In the VCA, variational principles are utilised to derive beam governing equations. Whereas in the VIA, beam gross / integrated two-dimensional equilibrium equations are utilised to derive beam governing equations. Shimpi et al. [2] have proposed the single variable shear deformation beam theory (SVSDBT) which is based on the VIA. Whereas, Shimpi et al. [1] have proposed the two variable shear deformation beam theory (TVSDBT) which is based on the VCA. In the SVSDBT and TVSDBT, the beam axial displacement and beam transverse displacement consist of bending and shearing components. In both theories, bending components do not take part in the cross-sectional shearing force, and shearing components do not take part in the cross-sectional bending moment. Both theories utilise linear strain-displacement relations. Displacement functions of both theories give rise to the beam transverse shear strain (and hence to the beam transverse shear stress) which varies quadratically through the beam thickness and maintains transverse shear stress-free beam surface conditions. For the case of dynamics, the SVSDBT has only one governing equation, and the TVSDBT has two governing equations which are inertially coupled. The aim of this paper is to present the comparison study on free flexural vibration frequencies of simply-supported rectangular isotropic shear deformable beams obtained by utilising the SVSDBT and TVSDBT. These frequencies are obtained for various values of the beam thickness-to-length ratio and beam vibration mode parameter. This comparison shows that for both theories, the VCA and VIA have a minimal effect on resulting beam free flexural vibration frequencies.

1. Introduction
It is well established that transverse shear deformation effects for microscopic as well as macroscopic shear deformable beams have a significant effect on their flexural vibration frequencies. Because of the effects of shear in the beam deformation, non-dimensional vibration frequencies of shear deformable beams decrease in comparison to corresponding non-dimensional frequencies of slender beams (Shimpi et al. [1]).

Due to features associated with the assumed displacement field of the Bernoulli-Euler beam theory (BEBT), it is unable to capture effects of shear on the beam deformation behaviour. Various displacement-based first-order (FOBT) and higher-order beam theories (HOBT) have been reported in the literature in order to account for beam transverse shear deformation effects. FOBT result in the constant transverse shear strain and HOBT result in more realistic non-linear transverse shear strain throughout the thickness of the beam.
In order to derive governing equations of FOBT or HOBT, the variationally consistent approach (VCA) or the variationally inconsistent approach (VIA) can be used. In the VCA, variational principles are utilised to minimise the beam potential energy (for the case of statics) or to minimise the Lagrangian (for the case of dynamics), to derive beam governing equations. Whereas in the VIA, beam gross/integrated two-dimensional equilibrium equations are utilised to derive beam governing equations. It should be noted that as the number of primary unknowns associated with the beam theory increases, it becomes increasingly challenging to derive beam governing equations using the VIA. Moreover, beam boundary conditions must be specified explicitly based on the physical understanding of the beam deformation behaviour in case of the VIA. Aforementioned challenges are not encountered when the VCA is utilised for deriving beam governing equations. Recently Shimpi et al. [1] have proposed the two variable shear deformation beam theory (TVSDBT) and Shimpi et al. [2] have proposed the single variable shear deformation beam theory (SVSDBT). These theories are applicable for beams undergoing small deformations. In both theories, the beam axial and transverse displacements have been split into bending components and shearing components. This splitting is such that bending components do not take part in the cross-sectional shearing force, and shearing components do not take part in the cross-sectional bending moment. Both theories result in the parabolic distribution of the beam transverse shear strain (and hence the beam transverse shear stress) through the beam thickness, maintaining transverse shear stress-free beam surface conditions, and therefore do not require a shear correction factor. Governing equations of TVSDBT have been obtained based on the VCA. Whereas, governing equation of SVSDBT has been obtained based on the VIA. For the case of dynamics, the TVSDBT has two governing equations which are inertially coupled, and the SVSDBT has only one governing equation.

In this paper, free flexural vibration frequencies of simply-supported rectangular isotropic shear deformable beams are presented which are obtained using the SVSDBT and TVSDBT for various values of the beam thickness-to-length ratio and beam vibration mode parameter. This comparison study aims to show the very small to none effect that the VCA and VIA have for a given shear deformation theory on the beam free flexural vibration frequencies. Corresponding results obtained from the BEBT and the theory of Touratier [3] adapted for beams by Sayyad [4] are also presented for the comparison purpose.

2. Theoretical formulation details of the SVSDBT and TVSDBT

In this section, brief details about the theoretical formulation of the SVSDBT and TVSDBT are presented. The beam geometry, co-ordinate system considered and symbols appearing in subsequent equations are the same as those of Shimpi et al. [1].

2.1. Expressions for the beam axial and transverse displacements

For the SVSDBT and TVSDBT, the displacement field was defined as follows

\[
\begin{align*}
  u &= u_h + u_s = -z \frac{\partial w_h}{\partial x} + h \left[ -\frac{5}{3} \left( \frac{z}{h} \right) + \frac{1}{4} \left( \frac{z}{h} \right)^2 \right] \frac{\partial w_s}{\partial x} \\
  w &= w_h + w_s
\end{align*}
\]  

(1)

2.2. Expressions for strains

As the SVSDBT and TVSDBT are valid for beams undergoing small deformations, expressions for the beam axial and transverse shear strain by utilising linear strain displacement relations were as follows

\[
\begin{align*}
  \dot{\varepsilon}_s &= -z \frac{\partial^2 w_h}{\partial x^2} + h \left[ -\frac{5}{3} \left( \frac{z}{h} \right) + \frac{1}{4} \left( \frac{z}{h} \right)^2 \right] \frac{\partial^2 w_s}{\partial x^2} \\
  \gamma_{sz} &= -\frac{5}{2} \left( \frac{z}{h} \right) + \frac{5}{4} \frac{\partial w_s}{\partial x}
\end{align*}
\]  

(3)

(4)

2.3. Expressions for stresses

For the SVSDBT and TVSDBT, expressions for the beam axial and transverse shear stress were as follows
2.4. Expressions for governing equations

Governing equations of the TVSDBT obtained using the VCA were as follows:

\[
E I \frac{\partial^4 w_b}{\partial x^4} - \rho I \left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w_b}{\partial x^2} \right) \right] + \rho A \frac{\partial^2 w_b}{\partial t^2} + \rho A \frac{\partial^2 w_x}{\partial t^2} = q \tag{7}
\]

\[
\frac{E I}{84} \frac{\partial^4 w_x}{\partial x^4} - \rho I \left[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w_x}{\partial x^2} \right) \right] - \frac{5 E A}{12 (1 + \mu)} \frac{\partial w_x}{\partial x} + \rho A \frac{\partial^2 w_b}{\partial t^2} + \rho A \frac{\partial^2 w_x}{\partial t^2} = q \tag{8}
\]

Governing equation of the SVSDBT obtained using the VIA was as follows:

\[
E I \frac{\partial^4 w_b}{\partial x^4} - \rho I \left[ 1 + \frac{12 (1 + \mu)}{5} \right] \left( \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2 w_b}{\partial x^2} \right) \right] + \rho A \frac{\partial^2 w_b}{\partial t^2} + \rho I \left[ \frac{12 (1 + \mu)}{5} \right] \frac{\partial^4 w_b}{\partial t^4} = q \tag{9}
\]

3. Free flexural vibrations of simply-supported isotropic shear deformable beams

In this section, the SVSDBT and TVSDBT are utilised to obtain respective free vibration characteristic equations of the simply-supported isotropic shear deformable beams.

3.1. Details with regard to SVSDBT

For the SVSDBT, boundary conditions for simply-supported ends were prescribed as follows:

\[
\left[ w_b - \frac{h^2 (1 + \mu)}{5} \frac{\partial^2 w_b}{\partial x^2} \right]_x = 0, L = 0 \tag{10}
\]

Following solution for \( w_b \) was assumed which satisfies boundary conditions (10):

\[
w_b = W_{bn} \sin \left( \omega_n t \right) \sin \left( \frac{n \pi x}{L} \right) \tag{11}
\]

Substituting this assumed expression for \( w_b \) in governing equation (9), following characteristic equation was obtained:

\[
\left( \omega_n^2 \frac{L^4}{EI} \rho A \frac{h^4}{60} \left( \frac{h}{L} \right) \right)^2 - \left( \omega_n^2 \frac{L^4}{EI} \rho A \frac{1}{5} \right) \left[ 1 + \left( \frac{n \pi h}{L} \right)^2 \left( \frac{1}{12} + \frac{1 + \mu}{5} \right) \right] + (n \pi)^4 = 0 \tag{12}
\]

3.2. Details with regard to TVSDBT

For the TVSDBT, boundary conditions for simply-supported ends were prescribed as follows:

\[
\left[ w_b \right]_x = 0, L = 0 \quad \text{and} \quad \left[ E I \frac{\partial^2 w_b}{\partial x^2} \right]_x = 0, L = 0 \tag{13}
\]

\[
\left[ w_x \right]_x = 0, L = 0 \quad \text{and} \quad \left[ \frac{84 E I \partial^2 w_x}{\partial x^2} \right]_x = 0, L = 0 \tag{14}
\]
Following solutions for \( w_b \) and \( w_s \) were assumed which satisfy boundary conditions (13) and (14):

\[
\begin{bmatrix}
w_b \\
w_s \\
\end{bmatrix} = \begin{bmatrix} W_{bn} \\
W_{sn} \\
\end{bmatrix} \sin(\omega_n t) \sin\left(\frac{n \pi x}{L}\right)
\]  

(15)

Substituting these assumed expressions for \( w_b \) and \( w_s \) in governing equations (7) and (8), following equations were obtained which are of the standard eigenvalue form:

\[
\begin{bmatrix}
k_{11n} & k_{12n} \\
k_{21n} & k_{22n} \\
\end{bmatrix} - (\omega_n)^2 \begin{bmatrix} m_{11n} & m_{12n} \\
m_{21n} & m_{22n} \\
\end{bmatrix} \begin{bmatrix} W_{bn} \\
W_{sn} \\
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
\end{bmatrix}
\]

(16)

with

\[
k_{11n} = \frac{n^4 \pi^4 E I}{L^4}
\]

\[
m_{11n} = \frac{n^2 \pi^2 \rho I}{L^2} + \rho A
\]

\[
k_{12n} = k_{21n} = 0
\]

\[
m_{12n} = m_{21n} = \rho A
\]

\[
k_{22n} = \frac{n^4 \pi^4 E I}{84 L^4} + \frac{5 n^2 \pi^2 E A}{12 (1 + \mu) L^2}
\]

\[
m_{22n} = \frac{n^2 \pi^2 \rho I}{84 L^2} + \rho A
\]

4. Numerical results

In this section, numerical results pertaining to free flexural vibration frequencies of simply-supported rectangular isotropic shear deformable beams obtained by utilising the SVSDBT, TVSDBT, BEBT and theory of Touratier adapted for beams are presented in tabular form in tables 1 through 4 for various values of the beam thickness-to-length ratio \( h / L \) and beam vibration mode parameter \( n \). It is then followed by the relevant discussion.

The non-dimensional natural frequency of the beam vibration \( \bar{\omega}_n \) was defined as follows:

\[
\bar{\omega}_n = \sqrt{\frac{\rho A L^4}{E I}} \omega_n
\]

(17)

Following points were considered while obtaining results which are presented in tables 1 through 4:

- The Poisson’s ratio \( \mu \) was assumed to be 0.3.
- The beam vibration mode parameter \( n \) was assumed to have values 1, 2, 3 and 4.
- The beam thickness-to-length ratio \( h / L \) was assumed to have values 0.01, 0.05, 0.10, 0.15 and 0.20.
- Numerical results obtained by utilising the higher-order trigonometric theory of Touratier and BEBT have been calculated by the present authors.
  i. In case of the higher-order trigonometric theory of Touratier, eqs. (14), (30) through (36) of Sayyad [4] were utilised.
  ii. In case of the BEBT, eq. (12) of Timoshenko [5] (wherein effects of rotary inertia are taken into consideration) was utilised.
- In tables 1 through 4, the percentage difference quoted in parentheses against results reported by the particular theory (either by the SVSDBT, TVSDBT or BEBT, as the case may be) was calculated as follows:

\[
\% \text{ difference} = \left( \frac{\text{The result reported by the particular theory}}{\text{The corresponding result of the theory of Touratier}} - 1 \right) \times 100
\]

(18)
5. Discussion on numerical results

Free flexural vibration frequencies of simply-supported rectangular isotropic shear deformable beams in the non-dimensional form (Tables 1 through 4) calculated using the SVSDBT and TVSDBT are in quantitative agreement with each other and with corresponding values calculated using the higher-order trigonometric theory of Touratier [4] adapted for beams for all values of the beam thickness-to-length ratio and beam vibration mode parameters considered. E.g., Percentage difference in calculating $\bar{\omega}_n$ for $h / L = 0.20$ and $n = 4$ is $-0.54$ in case of the SVSDBT and $-0.13$ in case of the TVSDBT. Hence for both theories, the VCA and VIA have a minimal effect on resulting beam free flexural vibration frequencies.

As it is well-known, the BEBT does not account for the beam transverse shear deformation effects, and hence it overestimates beam free flexural vibration frequencies. E.g., Percentage difference in calculating $\bar{\omega}_n$ for $h / L = 0.20$ and $n = 4$ in case of the BEBT is 36.32 %.
Non-dimensional natural frequencies of the beam vibration

$$\bar{\omega}_n = \sqrt{\frac{\mu A L^4}{EI}} \omega_n$$

(Values in parentheses indicate the percentage difference)\(^{b}\)

| Theory       | \(h / L =\) | \(h / L =\) | \(h / L =\) | \(h / L =\) | \(h / L =\) |
|--------------|--------------|--------------|--------------|--------------|--------------|
|              | 0.01         | 0.05         | 0.10         | 0.15         | 0.20         |
| SVSDBT       | 88.6914      | 85.6619      | 78.1547      | 69.5062      | 61.4581      |
|              | (0.00 \%)    | (-0.01 \%)   | (-0.04 \%)   | (-12 \%)     | (-26 \%)     |
| TVSDBT       | 88.6914      | 85.6634      | 78.1719      | 69.5629      | 61.5746      |
|              | (0.0 \%)     | (0.00 \%)    | (-0.02 \%)   | (-0.04 \%)   | (-0.07 \%)   |
| BEBT [5]     | 88.7936      | 88.0158      | 85.7108      | 82.2414      | 78.0234      |
|              | (0.12 \%)    | (2.74 \%)    | (9.63 \%)    | (18.18 \%)   | (26.62 \%)   |
| Touratier [4]| 88.6915      | 85.6671      | 78.1855      | 69.5908      | 61.6192      |

$^{b}$ % difference is calculated by utilising eq. (18).

### Table 4.
Free flexural vibration frequencies of simply-supported rectangular isotropic shear deformable beams in the non-dimensional form, \(n = 4\) and \(\mu = 0.3\).

Non-dimensional natural frequencies of the beam vibration

$$\bar{\omega}_n = \sqrt{\frac{\mu A L^4}{EI}} \omega_n$$

(Values in parentheses indicate the percentage difference)\(^{b}\)

| Theory       | \(h / L =\) | \(h / L =\) | \(h / L =\) | \(h / L =\) | \(h / L =\) |
|--------------|--------------|--------------|--------------|--------------|--------------|
|              | 0.01         | 0.05         | 0.10         | 0.15         | 0.20         |
| SVSDBT       | 157.4877     | 148.3846     | 128.6660     | 109.2588     | 93.2594      |
|              | (0.00 \%)    | (-0.01 \%)   | (-0.09 \%)   | (-26 \%)     | (-54 \%)     |
| TVSDBT       | 157.4878     | 148.3924     | 128.7389     | 109.4660     | 93.6436      |
|              | (0.00 \%)    | (-0.01 \%)   | (-0.03 \%)   | (-0.07 \%)   | (-13 \%)     |
| BEBT [5]     | 157.8099     | 155.3785     | 148.4480     | 138.7083     | 127.8170     |
|              | (0.20 \%)    | (4.70 \%)    | (15.27 \%)   | (26.62 \%)   | (36.32 \%)   |
| Touratier [4]| 157.4882     | 148.4036     | 128.7792     | 109.5453     | 93.7660      |

$^{b}$ % difference is calculated by utilising eq. (18).

6. **Concluding remarks**
In this paper, details regarding the variationally consistent two variable refined beam theory by Shimpi et al. [1] and variationally inconsistent single variable refined beam theory by Shimpi et al. [2] are presented. The main feature of these theories is that both of them have identical displacement fields. However, these theories differ in the approach utilised for deriving their respective governing equations. These two theories were used for calculating free flexural vibration frequencies of simply-supported rectangular isotropic shear deformable beams for various values of the beam thickness-to-length ratio and beam vibration mode parameters. It is observed that for these two theories, variationally consistent and variationally inconsistent approach utilised for deriving their respective governing equations has minimal effect on resulting beam free flexural vibration frequencies.

7. **References**
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