Weak quasitriangular Quasi-Hopf algebra structure of minimal models

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ABSTRACT. The chiral vertex operators for the minimal models are constructed and used to define a fusion product of representations. The existence of commutativity and associativity operations is proved. The matrix elements of the associativity operations are shown to be given in terms of the 6-j symbols of the weak quasitriangular quasi-Hopf algebra obtained by truncating $\mathcal{U}_q(sl(2))$ at roots of unity.

1. Introduction

Structures related to quantum groups encode important information on conformal field theories. Whereas the chiral algebra (Virasoro, Kac-Moody, $W$ etc.) may be considered to describe the local properties of the theory, the relation to quantum groups nicely describes its global properties such as monodromies of correlation functions and exchange relations of operators. A nice picture emerges that puts (rational) conformal in analogy to classical group theory [MS][FFK]. This analogy may be formulated more precisely in the language of braided tensor categories, as has been worked out for (negative level) WZNW-models in [KL]. A similar presentation has not yet been rigorously worked out in the case of the minimal models. However, various investigations have produced a reasonable expectation of what the quantum group relevant for the minimal models should be: On one hand side, in [CGR] the operator algebra for Liouville theory at irrational central charges was determined. It was shown to be given in terms of the 6-j symbols of the quantum group $\mathcal{U}_q(sl(2))$. However, the truncation of the operator algebra due to the reducibility of the relevant Virasoro representations was not discussed there. In [MaS1] on the other hand side the quantum group structure that is compatible with the additional truncation at rational central charge was worked out. It was shown to be given by a structure called weak quasitriangular quasi-Hopf algebra, in which co-associativity has to be modified to account for the truncation. This structure was shown to be relevant in the simplest nontrivial example of the Ising model in [MaS2].

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In a sense, what remains to be done is to establish the foundations to apply the methods of [CGR][MaS1] to general minimal models. This is one of the aims of the present work. The main other objective is to introduce a formalism, based on the notion of a fusion product, which makes abovementioned analogy to classical group theory (resp. relation to braided tensor categories) more explicit.

One of the technical problems to deal with comes from the fact that there are no explicit expressions for the differential equations following from decoupling of general nullvectors. One therefore has to find indirect ways to obtain the required results.

The content of the present paper may be summarised as follows: After a brief review of results on the relevant Virasoro representations the third section describes the construction of chiral vertex operators, in particular a rigorous proof of the fusion rules. Although this has been done already in [FF3], it may be worthwhile to have an alternative more elementary approach. A concise description for the Virasoro transformation behaviour of general descendants is obtained.

In the next section the concept of a fusion product of representations is defined from that of chiral vertex operators. The conformal properties of the fusion product may be expressed in a very concise way.

The fifth section discusses the composition of fusion products as well as definition of correlation functions. The contact to the formalism of [BPZ] is established. Global properties of the fusion product, which are its commutativity and associativity, are considered in the following section. The main point is to show that an associativity law really exists. Given that, it is not difficult to show that the matrices which represent commutativity and associativity satisfy the pentagon an hexagon identities of [MS]. Finally, it is shown how the strategy of [CGR] can be applied to obtain the explicit expression for the matrix representing associativity, which is given in terms of the truncated q-6j symbols $U_h(sl(2))$ as defined in [KR].

### 2. Verma modules vs. irreducible highest weight modules

A Verma module $V_h$ is defined as the highest weight representation for which the states

$$ L_{-n_1} L_{-n_2} \ldots L_{-n_k} v_h \quad \text{with} \quad k \in \mathbb{N} \quad \text{and} \quad n_i \geq n_j \quad \text{for all} \quad i < j \quad i, j = 1 \ldots k $$

form a basis. $V_h$ may be decomposed into $L_0$ (energy) eigenstates.

$$ V_h = \bigoplus_{n \in \mathbb{N}} V_{h}^{(n)} \quad \quad L_0 V_{h}^{(n)} = (h + n) V_{h}^{(n)}.$$  

The subspace $V_{h}^{(n)}$ with energy $h + n$ is spanned by all vectors of the form (1) such that $n = \sum_{i=1}^{k} n_i$. The number $n$ will be called level in the following. There is a unique bilinear form $\langle \cdot , \cdot \rangle$ on $V_h$ such that $\langle v_h, v_h \rangle = 1$ and $\langle L_n \xi, \zeta \rangle = \langle \xi, L_{-n} \zeta \rangle$. One may prove [KaRa] that its kernel is the maximal proper submodule $S_h$ contained in $V_h$. $\langle \cdot , \cdot \rangle$ becomes nondegenerate on the irreducible representation $\mathcal{H}_h := V_h/S_h$. Reducibility is equivalent to the existence of vectors $n_h$ besides $v_h$ that obey $L_n n_h = 0, n > 0$, called null vectors. The nullvectors in $V_h$ generate the singular submodule $S_h$.

The case to be considered will be that of rational central charge $c = c_{p'p} = 1 - 6 \frac{(p' - p)^2}{p' p}$, with $p \in \mathbb{N}, p' \in \mathbb{N}$ coprime, and representations with highest weights

$$ h \in \mathcal{G} := \left\{ h_{j',j} = j' (j' + 1) \frac{p'}{p} + j (j + 1) \frac{p}{p'} - 2j'j - j' - j; 0 \leq j' \leq \frac{p - 2}{2} \wedge 0 \leq p \leq \frac{p' - 2}{2} \right\},$$
The tuple \((j', j)\) will be abbreviated \(J\). Correspondingly the spaces \(V_{h, j'}, \mathcal{H}_{h, j'}, \mathcal{S}_{h, j'}\) will be denoted \(V_J, \mathcal{H}_J, \mathcal{S}_J\).

In this case the structure of Verma-modules may be described as follows [FF1][FF2]: The singular submodule \(\mathcal{S}_J\) is generated by two nullvectors in \(V_J\): One at level \((2j' + 1)(2j + 1)\), the other at level \((p' - 2j' - 1)(p - 2j - 1)\). However, \(\mathcal{S}_J\) itself is reducible. One ends up with an infinite nested inclusion of Verma modules, one being generated by the nullvectors of the other. The structure of embeddings of Verma modules may be summarized by the following diagram:

```
  s_0  s_1  s_3  s_3  s_3  ...  
     \| \| \| \| \|     
  s_2  s_4  s_4  s_4  s_4  ...  
```

There is an arrow from a vector \(s_i\) to another vector \(s_j\) whenever \(s_j\) is a generator of the submodule \(\mathcal{S}_{h_1}\) in the Verma module generated from the null vector \(s_i\).

3. CHIRAL VERTEX OPERATORS

3.1. Conditions for existence. The first aim is to construct operators \(\psi_{\mathbb{H}}(z), \mathbb{H} = (h_2)\) such that:

(a) \(\psi_{\mathbb{H}}(z) : \mathcal{H}_{h_1} \to \mathcal{H}_{h_3}\),

(b) \([L_n, \psi_{\mathbb{H}}(z)] = z^n(z\partial + h_2(n + 1))\psi_{\mathbb{H}}(z)\)

Start by defining linear forms \(t_\mathbb{H}^{(n)}(\zeta)\) on \(\mathcal{V}_{h_3}^{(n)}\) by

- \(t_\mathbb{H}^{(0)}(v_{h_3}) = \mathcal{N}_{\mathbb{H}}, \) the number \(\mathcal{N}_{\mathbb{H}}\) being called the normalization of \(\psi_{\mathbb{H}}(z)\). This is extended to arbitrary \(n\) by

- \(t_\mathbb{H}^{(n)}(L_n\zeta^{(n-k)}) = (\Delta + n - k + h_2(k + 1))t_\mathbb{H}^{(n-k)}(\zeta^{(n-k)})\) for any \(\zeta^{(n-k)} \in \mathcal{V}_{h_3}^{(n-k)}\), where \(\Delta = h_3 - h_2 - h_1\).

**Proposition 3.1.** Necessary conditions for existence of chiral vertex operators \(\psi_{\mathbb{H}}(z)\) are \(t_\mathbb{H}^{(n)}(v_{h_3}) = 0\) for any nullvector \(n \in \mathcal{V}_{h_3}^{(n)}\).

**Proof:** Consider \(\psi_{\mathbb{H}}(z)v_{h_1}\): First observe that

\[\psi_{\mathbb{H}}(z)v_{h_1} := \sum_{n=0}^{\infty} z^{\Delta + n} \xi^{(n)}_{\mathbb{H}}\]

is necessary for (b) to hold in the case of \(n=0\). Next it is easy to see that

\[L_k \xi^{(n)}_{\mathbb{H}} = (\Delta + n - k + h_2(k + 1))\xi^{(n-k)}_{\mathbb{H}} \quad \text{for} \quad k \leq n.\]

is necessary for (b),\(n > 0\). One therefore needs to find necessary and sufficient conditions for the system of equations (3) to be solvable. It suffices to consider \(k = 1, 2\) since all other \(L_k, k > 2\) are generated from these. Suppose a solution exists for any \(m < n\). Then consider the linear map

\[A^{(n)} : \mathcal{H}_{h_3}^{(n)} \to \mathcal{K}_{h_3}^{(n)} \quad A^{(n)}(\xi^{(n)}) = L_1 \xi^{(n)} + L_2 \xi^{(n)},\]

where \(\mathcal{K}_{h_3}^{(n)} := \mathcal{H}_{h_3}^{(n-1)} \oplus \mathcal{H}_{h_3}^{(n-2)}\). If there is no nullvector in \(\mathcal{V}_{h_3}^{(n)}\) then \(A^{(n)}\) is invertible and one has a unique solution \(\zeta^{(n)}_{\mathbb{H}}\). If there are nullvectors then \(\mathcal{K}_{h_3}^{(n)} = \text{Im}A^{(n)} \oplus \mathcal{C}\) with a nontrivial \(\mathcal{C}\) and a solution will exist only if the vector \(\sum_{k=1,2}(\Delta + n - k + h_2(k + 1))\xi^{(n-k)}_{\mathbb{H}}\) has no components in \(\mathcal{C}\). This will be the case if it is annihilated by any linear form \(t_\mathbb{H}^{(n)}\) that
Another useful form of this expression is obtained by introducing and parametrizing \( h(t) \) and replacing \( c(n-1) + c(n-2) \) by \( n(n-1) + n(n-2) \), where \( n(n-1), n(n-2) \) are vectors in \( \mathcal{V}_{h_3} \) such that \( n = L_{-1}n(n-1) + L_{-2}n(n-2) \) is a null vector in \( \mathcal{V}_{h_3} \). By definition of \( t_{\xi_{\ell \ell}^{(n)}} \) this is just equivalent to \( t_{\xi_{\ell \ell}^{(n)}}(n) = 0. \square \)

**Proposition 3.2.** Fusion rules: Let \( h_i = h_{J_i}, J_i = (j_i^1, j_i^2), i = 1, 2, 3. \) Then a complete set of solutions to the condition that \( t_{\xi_{\ell \ell}^{(n)}}(n) = 0 \) for any nullvector \( n \in \mathcal{V}_{h_3}^{(n)} \) is provided by the triples \((J_1, J_2, J_3)\) such that

\[
\begin{align*}
|j_2^i - j_3^i| &\leq j_1^i = \min(j_2^i + j_3^i, p - 2 - j_2^i - j_3^i) \\
|j_2 - j_3| &\leq j_1 = \min(j_2 + j_3, p - 2 - j_2 - j_3).
\end{align*}
\]

**Proof:** Introduce the notation \( PJ = (\ell^2 - \ell, \ell^2 - j) \). The singular subspace \( \mathcal{D}_{h_3} \) of \( \mathcal{V}_{h_3} \) is generated by nullvectors \( n_{j_3}, n_{Pj_3} \) at level \((2j + 1)(2j^* + 1) \) and \((p - 2j - 1)(p^* - 2j^* - 1) \) respectively. Write them as

\[
J = J_3 \quad \text{or} \quad J = PJ_3,
\]

where the summation is performed over all vectors \( \pi = (n_k, n_{k-1}, \ldots, n_1) \) of integers such that \( n_{i+1} \geq n_i \) and \( \sum_{k=1}^{k} n_i = (2j^* + 1)(2j + 1) \), whereas \( L_{-\pi} = L_{-n_k}L_{-n_{k-1}} \ldots L_{-n_1} \). One has

\[
t_{\xi_{\ell \ell}^{(n)}}(L_{-n_k}L_{-n_{k-1}} \ldots L_{n_1}, v_J) = \prod_{i=1}^{k} \left( \Delta - \sum_{j=1}^{i-1} n_j + h_2(1-n_i) \right).
\]

\( t_{\xi_{\ell \ell}^{(n)}}(n_j) \) is of the form

\[
t_{\xi_{\ell \ell}^{(n)}}(n_j) = P_J(h_1, h_2; c),
\]

where \( P_J \) is some polynomial in \( h_1, h_2 \). A nice expression for this polynomial has been found by Feigin and Fuchs in [FF1]: They consider the following problem: There is a simple Virasoro representation on a vector space with basis \( \{f_n : n \in \mathbb{Z}\} \) defined by \( L_{-k}f_n := (\mu + n - \lambda(k + 1))f_{n+k}, \ cf_n := 0. \) What Feigin and Fuchs calculated is \( n_J f_0 = (\mu, \lambda; J; t) f_{(2j+1)(2j+1)}, \) where \( t = -p'/p \). It is easy to see that \( P_J(h_1, h_2; c) \) is proportional to \( Q(\Delta, -h_2; J, t) \). The expression given in [FF1] may be written as \( (\alpha_{\pm} := \pm(p'/p)^{1/2})\):

\[
(P_J(h_1, h_2; c))^2 = \prod_{-j \leq m \leq j'} \left\{ (h_2 - h_1)^2 + 2(h_2 + h_1)(m'\alpha_+ + ma_+) + h(m', m)h(-m', -m) \right\}.
\]

Another useful form of this expression is obtained by introducing \( \tau_{m'm} = m'\alpha_+ + ma_+ \), and parametrizing \( h_1 = \alpha_1^2 - \alpha_2^2, h_2 = (\alpha_1 + \delta)^2 - \alpha_0^2, \) \( 2\alpha_0 = \alpha_+ + \alpha_- \):

\[
P_J(h_1, h_2; c) = \prod_{-j \leq m \leq j'} (\delta - r_{m'm})(\delta + 2\alpha_1 + r_{m'm}).
\]
The necessary condition for existence of $\psi_3(z)$ is equivalent to the vanishing of $P_{J_3}(h_1, h_2; c)$ and $P_{J_5}(h_1, h_2; c)$. By writing $\alpha_1 = j'_1\alpha_- + j_1\alpha_+$ one easily reads off that the set of solutions to $P_{J_3}(h_1, h_2; c) = 0$ is given by

$$S_1 := \left\{ h_2 = h_{J_2}; J_2 = (j'_2, j_2) \left| \begin{array}{c} j'_2 = j'_2, j'_3 - j'_2 + 1, \ldots, j'_1 + j'_2 \\ j_2 = j_1, j_2, j_3 + 1, \ldots, j_1 + j_2 \end{array} \right. \right\}$$

The set of solutions to $P_{J_5}(h_1, h_2; c) = 0$ may similarly be read off by parametrizing $\alpha_1 = (E - 2)(E - 2)\alpha_+ - (E - 2)J_1\alpha_+.$

$$S_1 := \left\{ h_2 = h_{J_2}; J_2 = (j'_2, j_2) \left| \begin{array}{c} j'_2 = j'_2, j'_3 - j'_2 + 1, \ldots, p - 2 - j'_1 - j'_2 \\ j_2 = j_1, j_2, j_3 + 1, \ldots, p - 2 - j_1 - j_2 \end{array} \right. \right\}$$

The intersection of these sets may be parametrized as in the Proposition.

**Proposition 3.3.** The fusion rules are also sufficient for a chiral vertex operator $\psi_3(z)$ to exist.

**Proof:** From the proof of Proposition 3.1 one finds a unique definition of $\psi_3(z)\nu_{h_1}$. One may recursively extend the definition of $\psi_3(z)$ to arbitrary $\zeta \in \mathcal{V}_{h_1}$ by defining

$$\psi_3(z)L_{-k}\zeta = -z^k(\zeta \partial_z + h(1 - k))\psi_3(z)\zeta + L_{-k}\psi_3(z)\zeta.$$

Now condition (b) holds for $n < 0$ by definition. The case $n = 0$ is trivial, so it remains to consider $n > 0$. Validity of $[L_k, \psi_3(z)]\zeta_1 = z^k(\zeta \partial_z + h_k(k + 1))\psi_3(z)\zeta_1$ was shown for $\zeta_1 = \nu_{h_1}$ in the proof of proposition 3.1. It remains to show that validity for $\zeta_1 = \zeta$ implies validity for $\zeta_1 = L_{-m}\zeta$. One only needs to use (6) to move $L_{-m}$ to the left on both hand sides of $[L_k, \psi_3(z)]L_{-m}\zeta = z^k(\zeta \partial_z + h_k(k + 1))\psi_3(z)L_{-m}\zeta$ in order to get terms on which the assumption may be applied. It is then easy to see that the resulting terms cancel.

Up to now the operator $\psi_3(z)$ was constructed to map from the Verma module $\mathcal{V}_{h_1}$. It remains to be shown that it vanishes on the singular subspace of $\mathcal{V}_{h_1}$. It is easy to see that necessary and sufficient for this is that $<\nu_{h_1}, \psi_3(z)\nu_{h_1}> = 0$ for any nullvector $\nu_{h_1} \in \mathcal{V}_{h_1}$.

As in the proof of Proposition 3.2 is found that $<\nu_{h_1}, \psi_3(z)\nu_{h_1}> = 0$ yields no new conditions.

**3.2. Descendant operators.** $\psi_3(z)$ is just one member of a whole class of operators $\psi_3(\xi_{h_2}|z)$ which may be labelled by vectors $\xi \in \mathcal{V}_{h_2}$. They are recursively defined by

$$\psi_3(\nu_{h_2}|z) := \psi_3(z) \quad \psi_3(L_{-}\xi|z) := \partial \psi_3(\xi|z)$$

$$\psi_3(L_{-n}\xi|z) := \frac{1}{n - 2!}(\partial^n - 2T_{<}(z)\psi_3(\xi|z) + \psi_{5}\partial^n - 2T_{>}(z))$$

where $n \geq 2$, and

$$T_{<}(z) := \sum_{n=0}^{\infty} z^n L_{-n-2} \quad T_{>}(z) := \sum_{n=1}^{\infty} z^{-n} L_{n-2}.$$

Instead of considering $\psi_3(\xi|z)$ as an operator that maps from $\mathcal{H}_{h_1}$ to $\mathcal{H}_{h_3}$, one may fix an element $\zeta \in \mathcal{H}_{h_1}$ and consider $\psi_3(\cdot | z)\zeta$ as an operator that maps from $\mathcal{V}_{h_2}$ to $\mathcal{H}_{h_3}$. In

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2Because of $\rho'\alpha_- + \rho_-\alpha_+ = 0$ the change $j_1 \rightarrow \frac{E - 2}{2} - j_1, j'_1 \rightarrow \frac{E - 2}{2} - j'_1$ doesn’t change $\alpha_1$, but only the labelling of solutions.
the next subsection, a formalism will be presented that puts both points of view on equal footing.

The following theorem gives a convenient characterization of the conformal properties of the descendants (I will omit the subscript $\mathbb{H}$ in the following):

**Theorem 3.4.**

\[
[L_n, \psi(\xi|z)] = \sum_{k=-1}^{l(n)} z^{n-k} \frac{n+1}{k+1} \psi(L_k \xi|z) \quad \text{where} \quad l(n) = \begin{cases} n & \text{for } n \geq -1 \\ \infty & \text{for } n < -1. \end{cases}
\]

Before giving the proof, I want to explain its implications: The content of the theorem may be summarized even more concise in the following rules:

1. \[ [T_>(u), \psi(\xi|z)] = \psi(T_>(u-z)\xi|z) \]
2. \[ [\psi(\xi|z), T_<(u)] = \psi(T_<(u-z)\xi|z) \]
3. \[ \psi(T_<(u-z)\xi|z) = T_<(u)\psi(\xi|z) + \psi(\xi|z)T_>(u) \]

These formulae allow to write down the complete operator product expansion of \( T(u)\psi(\xi|z) \):

\[
T(u)\psi(\xi|z) = [T_>(u), \psi(\xi|z)] + T_<(u)\psi(\xi|z) + \psi(\xi|z)T_>(u)
\]

\[
= \sum_{k=-\infty}^{\infty} (u-z)^{-k-2} \psi(L_k \xi|z)
\]

\[
= \psi(T(u-z)\xi|z).
\]

The sum is finite if \( \xi \) contains only finitely many \( L_{-n} \) generators. It is now possible to make contact with the more usual formulations of conformal field theories [BPZ]: One has

\[
\psi(L_{-n}\xi|z) = \text{Res}_{u-z}[(u-z)^{-n+1} T(u)\psi(\xi|z)]
\]

\[
= \oint \frac{du}{2\pi i} (u-z)^{-n+1} T(u)\psi(\xi|z).
\]

In [BPZ], these equations are used to **define** the formalism. In the present formalism they have been **derived** purely algebraically.

**Proof:** As a preliminary note that an alternative basis for \( \mathcal{V}_{h_2} \) may be written as follows: Let \( \vec{n} = (n_1, \ldots, n_k) \) be a vector of integers with \( n_1 \geq 0, n_k \geq 0 \) and \( n_i > 0 \) for \( i = 2 \ldots k - 1 \). Then a basis for \( \mathcal{V}_{h_2} \) is given by the set of all

\[
L_{n_1}^{-1} L_{-2}^{n_2} L_{-1}^{n_1} \cdots L_{n_k}^{-1} L_{-2}^{n_k} v_{h_2}
\]

It therefore suffices to use (7) for \( n = 1, 2 \) only. The theorem will be proved for vectors \( \xi \) of the form (15) by induction on the integer \( s \), defined as \( s := \sum_{i=1}^{k} n_i \). For \( s = 0 \) one easily recognizes the theorem as the covariant transformation law of \( \psi(z) \). Now assume that the theorem holds for \( \psi(\xi|z) \). Consider first \([L_n, \psi(L_{-1}\xi|z)]\): By using the definition of \( \psi(L_{-1}\xi|z) \) and the inductive assumption this is calculated as:

\[
[L_n, \psi(L_{-1}\xi|z)] = \sum_{k=-1}^{l(n)} z^{n-k} \frac{n+1}{k+1} \left( \frac{n-k}{z} \psi(L_k \xi|z) + \psi(L_{-1}L_k \xi|z) \right)
\]

The first sum may be rewritten as \( \sum_{k=0}^{l(n)} z^{n-k} \frac{n+1}{k+1} \psi([L_k, L_{-1}]\xi|z) \) which proves the theorem for \( \psi(L_{-1}\xi|z) \).

The key step for the computation of \([L_n, \psi(L_{-2}\xi|z)]\) is contained in the following:
Lemma 3.5.

\[ [L_n, T_< (z)] \psi(\xi | z) + \psi(\xi | z) [L_n, T_>(z)] = \sum_{k=1}^{l(n)} z^{n-k} \binom{n+1}{k+1} \psi([L_k, L_{-2}] \xi | z) \]

Proof: One has to distinguish two cases: \( n \geq -1 \) and \( n < -1 \). In the first case use

\[ [L_n, T_< (z)] = z^n (z \partial + 2(n+1)) T_< (z) + \sum_{m=1}^{n} z^{-m}(2n-m+2)L_{m-2} + \frac{c}{12} n(n^2-1)z^{n-2} \]

\[ [L_n, T_>(z)] = z^n (z \partial + 2(n+1)) T_>(z) - \sum_{m=1}^{n} z^{-m}(2n-m+2)L_{m-2}, \]

with the convention that \( \sum_{m=1}^{n} \ldots = 0 \) if \( n < 1 \) to find

\[ [L_n, T_< (z)] \psi(\xi | z) + \psi(\xi | z) [L_n, T_>(z)] = z^{n+1} \psi(L_{-3} \xi | z) + 2(n+1)z^n \psi(L_{-2} \xi | z) + \]

\[ + \frac{c}{12} n(n^2-1)z^{n-2} + \sum_{m=1}^{n} z^{-k}(2n-m+2)[L_{m-2}, \psi(\xi | z)], \]

where the sum on the right hand side is evaluated as

\[ S := \sum_{k=1}^{n} z^{-m}(2n-m+2) \sum_{k=1}^{m-2} z^{-2-k} \binom{m-1}{k+1} \psi(L_k \xi | z) \]

\[ = \sum_{k=1}^{n} z^{-k} \left( \sum_{m=k}^{n} (2n+2-m) \binom{m-1}{k-1} \right) \psi(L_{k-2} \xi | z). \]

One may prove by induction that the sum within the curly brackets equals \( (k+2) \binom{n+1}{k+1} \), so that

\[ S = \sum_{k=1}^{n} z^{-k} \binom{n+1}{k+1} \psi([L_k, L_{-2}] \xi | z) \frac{c}{12} n(n^2-1)z^{n-2} \]

Collecting the different terms one finds the claimed result. The case \( n < 1 \) proceeds analogously. \( \square \)

Given the lemma one only needs to apply the inductive assumption for the computation of \( T_< (z) [L_n, \psi(\xi | z)] + [L_n, \psi(\xi | z)] T_>(z) \). \( \square \)

3.3. Operator differential equations.

Proposition 3.6. The operators \( \psi_N(z) \) satisfy differential equations obtained by using (7) to evaluate in terms of \( \psi_H(z) \) the condition \( \psi_H(n|z) = 0 \) for \( n \) being any of the two nullvectors \( n_{J_2}, n_{P^J_{-2}} \) that generate the singular subspace of \( \mathcal{V}_{h_2} \).

Proof: By using (7) one may find that \( <v_{h_3}, \psi_H(n|z) v_{h_1}> \) is proportional to \( P_J(h_1, h_3; c) \), \( J = J_2 \) or \( J = P^J_{-2} \), which vanishes whenever the fusion rules are satisfied. Furthermore, it is an immediate consequence of Theorem 3.4 that

\[ [L_n, \psi_H(n|z)] = z^n (z \partial + (h_2 + l)(n+1)) \psi_H(n|z), \]

where \( l \) denotes the level of \( n \). But this means that any matrix element of \( \psi_H(n|z) \) may be expressed as some differential operator acting on \( <v_{h_3}, \psi_H(n|z) v_{h_1}> \). Since \( P_J(h_1, h_3; c) = 0 \), the operator vanishes altogether. \( \square \)
In the simplest nontrivial case of \( J_2 = (0, 1/2) \) one gets
\[
\partial^2 \psi_{\mathbb{H}}(z) = \alpha^2_\pm (T_< (z) \psi_{\mathbb{H}} + \psi_{\mathbb{H}} T_>(z)),
\]
which provides the link of the present treatment to Liouville theory as treated in \[\text{[GN][CGR]}.\]

4. Fusion Product

4.1. Consider \( \psi_{\mathbb{H}}(\xi_2|z)\xi_1, \xi_i \in \mathcal{H}_{h_i}, \ i = 1, 2 \): Instead of viewing it as the action of an operator on some state one may view it as the result of taking some kind of product of two states:
\[
\psi_{\mathbb{H}}(\xi_2|z)\xi_1 := [\xi_2(z) \hat{\otimes} \xi_1(0)]_{h_3}
\]
The state \( \xi_2 \) is considered to be located at \( z \), \( \xi_1 \) at 0. In order to make this more precise I will now consider the action of the translation operator \( e^{z L_{-1}} \) on states:
\[
\xi(z) := e^{z L_{-1}} \xi.
\]
In fact, translated states are nothing new:

**Lemma 4.1.** For \( \mathbb{H} = \left( \frac{h}{h_0} \right) \) one has \( \xi(z) = \psi_{\mathbb{H}}(\xi|z)v_0 \).

**Proof:** This may be verified by noting that
- one has \( v_h(z) = \psi_{\mathbb{H}}(z)v_0 \) since \( v_h(z) \) satisfies \( L_k v_h(z) = z^k (z \partial + h (k + 1)) v_h(z), \ k \geq 0, \) which are the conditions used to define \( \psi_{\mathbb{H}}(z)v_0 \) in the proof of Proposition 3.1, and that
- \( (L_{-1} \xi)(z) = \partial \xi(z), \ (L_{-2} \xi)(z) = T_< (z) \xi(z), \) from which the Lemma may be inductively proved for arbitrary \( \xi \). \qed

The conformal properties of translated states may be conveniently summarized by
\[
T_> (u) \xi(z) = (T_>(u - z) \xi)(z) \quad (T_< (u - z) \xi)(z) = T_< (u) \xi(z).
\]

Let me also mention the following important special case of (20):
\[
T_>(u) v_h(z) = \left( \frac{h}{(u - z)^2} + \frac{1}{u - z} \partial \right) v_h(z).
\]

**Definition:** The fusion products \([\hat{\otimes}]_{h_3} \) of two representations \( \mathcal{H}_{h_1} \) and \( \mathcal{H}_{h_2} \) are defined as \( z_1, z_2 \) dependent bilinear maps
\[
[\hat{\otimes}]_{h_3}: \mathcal{H}_{h_1} \otimes \mathcal{H}_{h_2} \rightarrow [\xi_2(z_2) \hat{\otimes} \xi_1(z_1)]_{h_3} := \psi_{\mathbb{H}}(\xi_2|z_2)\xi_1(z_1).
\]

The concept of the fusion product is completely equivalent to that of the chiral vertex operator: Having defined a fusion product, one may for any fixed \( \xi_2 \) define
\[
\psi_{\mathbb{H}}(\xi_2|z_2) : \mathcal{H}_{h_1} \rightarrow [\xi_2(z_2) \hat{\otimes} \xi_1(0)]_{h_3}.
\]
4.2. Local properties of the fusion product. The conformal properties of \( \psi(\xi|z) \) derived above may now be restated as rules for moving \( T(u) \) within fusion products:

\[
\begin{align*}
[\xi(z) \otimes T_c(u) \xi'(z')] &= T_c(u)[\xi(z) \otimes \xi'(z')] + [T_c(u)\xi(z) \otimes \xi'(z')] \\
[T_c(u)\xi(z) \otimes \xi'(z')] &= T_c(u)[\xi(z) \otimes \xi'(z')] + [\xi(z) \otimes T_c(u)\xi'(z')] \\
T_c(u)[\xi(z) \otimes \xi'(z')] &= [T_c(u)\xi(z) \otimes \xi'(z')] + [\xi(z) \otimes T_c(u)\xi'(z')],
\end{align*}
\]

which have to be supplemented by (20). Since \( T_c(u)\xi(z) \) will always involve a derivative with respect to \( z \), one may view these relations as describing the response of the fusion product with respect to infinitesimal changes of the parameters \( z, z' \), i.e. the local properties of the fusion product.

5. Composition of fusion products; Conformal blocks

5.1. Triple products. On the level of formal power series it is possible to define repeated products such as

\[
[[\xi_3(z_3) \otimes \xi_2(z_2)]_{h_{32}} \otimes \xi_1(z_1)]_h = \xi^{(n)}_{32} \xi^{(m)}_{321}(z_1),
\]

where \( \xi_i \in \mathcal{H}_{h_i}, \ i = 1, 2, 3 \). Consider i.e. \([\xi_3(z_3) \otimes \xi_2(z_2)]_{h_{32}} \otimes \xi_1(z_1)]_h \), where \( \xi_i \) will be assumed to have definite level \( n_i \); By definition, the inner bracket \([\xi_3(z_3) \otimes \xi_2(z_2)]_{h_{32}} \) may be written as the formal series

\[
[\xi_3(z_3) \otimes \xi_2(z_2)]_{h_{32}} = \sum_{n=0}^{\infty} (z_3 - z_2)^{\Delta_{32} + n} \xi^{(n)}_{32}(z_2),
\]

where \( \Delta_{32} = h_{32} - h_2 - n_2 - h_3 - n_3 \) and \( \xi^{(n)}_{32} \in \mathcal{H}_{h_{32}}^{(n)} \). Then the iterated product is defined as

\[
[[\xi_3(z_3) \otimes \xi_2(z_2)]_{h_{32}} \otimes \xi_1(z_1)]_h :=
\]

\[
= \sum_{n=0}^{\infty} (z_3 - z_2)^{\Delta_{32} + n} \xi^{(n)}_{32} \otimes \xi_1(z_1)]_h
\]

\[
= \sum_{n=0}^{\infty} (z_3 - z_2)^{\Delta_{32} + n} \sum_{m=0}^{\infty} (z_2 - z_1)^{\Delta_{321} + m - n} \xi^{(n,m)}_{321}(z_1)
\]

where \( \Delta_{321} = h_{32} - h_2 - h_1 - n_1 \) and \( \xi^{(n,m)}_{321} \in \mathcal{H}_{h_{321}}^{(m)} \). The sum over \( m \) is a formal sum over the homogeneous components \( \mathcal{H}_{h}^{(m)} \). It may therefore be exchanged with the summation over \( n \) which then defines a vector in \( \mathcal{H}_{h}^{(m)} \) if it converges:

\[
\xi^{(m)}_{321}(z_3, z_2, z_1) := (z_3 - z_2)^{\Delta_{32}} \sum_{n=0}^{\infty} (z_3 - z_2)^n (z_2 - z_1)^m \xi^{(n,m)}_{321}(z_1)
\]

It may be useful to see what the different possible ways of iterating the fusion product correspond to in the language of chiral vertex operators:

\[
\begin{align*}
[[\xi_3(z_3) \otimes \xi_2(z_2)]_{h_{32}} \otimes \xi_1(z_1)]_h &= \sum_{n} (z_3 - z_2)^{\Delta_{32} + n} \psi^{(h_{321})}_{\xi_3(z_3)}(z_3) \psi^{(h_{21})}_{\xi_2(z_2)}(z_2) \xi_1(z_1)
\end{align*}
\]
The order \([A[BC]]\) of taking the fusion product therefore simply corresponds to the composition of chiral vertex operators, whereas the expression on the r.h.s of the second line has the form one expects the terms of the operator product expansion of\(\psi \left( h_{h_{21}} \right) \psi \left( h_{h_3} \right) (\xi_3 z_3)\psi \left( h_{h_2} \right) (\xi_2 z_2)\) to have \([MS]\).

5.2. Correlation functions. Since any multiple product will have to be understood as a formal series over the homogeneous components \(H_{h_{m}}(z)\) of some \(H_{h_{m}}\), the definition of higher iterated products is equivalent to the definition of their matrix elements with arbitrary \(\xi^{(m)} \in H_{h_{m}}\). In the case of triple products one has the four point functions such as
\[
< \xi^{(m)}, \{[\xi_3 z_3] \hat{\otimes} \xi_2 z_2 \} \}_{\frac{1}{J_32}} \hat{\otimes} \xi_1 (z_1) > = (z_2 - z_1)^{A_{321} + m} < \xi^{(m)}(z_3), \xi^{(m)}(z_3, z_2, z_1) >
\]
In the five point case such as \(< \xi_3, [\xi_4(z_4) \hat{\otimes} \xi_3(z_3) \hat{\otimes} \xi_2(z_2) \hat{\otimes} \xi_1(z_1)] \}_{\frac{1}{J_321}} \hat{\otimes} \xi \_5 \) one finds an apparent ambiguity: it can be expressed as a sum over four point functions in two ways: either
\[
\sum_{n=0}^{\infty} (z_2 - z_1)^{A_{321} + n} < \xi_5, [\xi_4(z_4) \hat{\otimes} \xi_3(z_3) \hat{\otimes} \xi_2(z_2) \hat{\otimes} \xi_1(z_1)] \}_{\frac{1}{J_321}} \hat{\otimes} \xi \_5 \_2 \_3 \_4 \_5 > 
\]
In the first case one ends up with a power series of the form
\[
z^{A_{321} + A_{321} + A_{321}} \sum_{n=0}^{\infty} \left( \frac{z_{21}}{z_{31}} \right)^n \sum_{m=0}^{\infty} \left( \frac{z_{31}}{z_{41}} \right)^n < \xi_5, [\xi_4(z_4) \hat{\otimes} \xi_3^{(m)}(z_{31}) \hat{\otimes} \xi_5] \hat{\otimes} \xi_5 \_3 \_4 \_5 >
\]
where \(z_{ij} := z_i - z_j\), whereas in the second case one has a similar power series with exchanged summations over \(n\) and \(m\). There is no ambiguity if the power series converge to holomorphic functions, since then the summations can be freely interchanged. On the level of formal power series one may remove the ambiguity by adopting the convention that the series expansion is performed starting from the innermost brackets to the outermost ones.

5.3. Conformal Ward identities; Decoupling equations. It is now easy to make contact to the formulation of \([BPZ]\): Consider correlation functions such as
\[
< v_0, [\ldots [\xi_i(z_i) \hat{\otimes} \ldots [\xi_i(z_i) \hat{\otimes} \xi_{i+1}(z_{i+1})]_{h_{i}}] \ldots ]_0 >
\]
It is useful to have a more concise notation: The possible multiplications of states \(\xi_i\), \(i = 1, \ldots, n\) are characterized by the following data:

1. A permutation \(\sigma(i), i = 1, \ldots, n\),
2. a complete binary bracketing of \(\sigma(1) \ldots \sigma(n)\) such as \(((3, 5), (1, 4), 6), 2)\),
3. The set of tuples \((h_i, \xi_i, z_i)\), where \(\xi_i\) is a state in the Verma module of conformal weight \(h_i\) and \(z_i\) is the position where the state is supposed to be inserted, and finally
4. a set of real numbers \(h', r = 1, \ldots, n - 1\) associated with each pair of brackets which denote the weights of the "intermediate" representations appearing in the multiplication.

Let \(\mathfrak{B}_n\) denote the set of all collections of data (1)-(2), i.e. of all bracketings \(((1, 4), \ldots)\). The elements of \(\mathfrak{B}_n\) will be denoted by \(T, T'\) etc.. The tuples \((h^1, \ldots, h^{n-1} := 0)\) will be abbreviated as \(H\). One may distinguish the 'external' data \((h_i, \xi_i, z_i)\) from the 'internal' data \(\Gamma := (T, H)\), which parametrize the possible ways to form fusion products of \(\xi_i(z_i)\). Correlation functions such as (29) may then be abbreviated as \(< \xi_1(z_1) \ldots \xi_n(z_n) >_T\).
In order to derive the conformal Ward identities consider
\[ <\xi_1(z_1) \ldots (L_{-n}\xi_i)(z_i) \ldots \xi_n(z_n)>_\Gamma. \]

By using (20) one rewrites \((L_{-n}\xi_i)(z_i)\) as \(\oint_{2\pi i} (u-z_i)^{-n+1} T_{<}(u) \xi_i(z_i)\), where the integral is to be understood as the operation of taking the residue in the formal power series. By using (22)-(24) one moves \(T_{<}(u)\) until one gets
\[ \sum_{j \neq i} \oint_{2\pi i} (u-z_i)^{-n+1} <\xi_1(z_1) \ldots (T_{<}(u)\xi_j(z_j)) \ldots \xi_n(z_n)>_\Gamma. \]

Using (20) again one gets terms with \(\oint_{2\pi i} (u-z_i)^{-n+1} (T_{<}(u-z_j)\xi_j)(z_j)\). Expanding \(T_{<}(u-z_j)\) one finds precisely what the conformal Ward identities of [BPZ] amount to. Besides derivatives with respect to \(z_j\) this will only produce generators \(L_n, n \geq 0\) which preserve or lower the degree of the homogeneous components of \(\xi_j\). By repeated application of these operations one recovers the fact that any correlations function \(<\xi_1(z_1) \ldots \xi_n(z_n)>_\Gamma\) may be expressed in terms of meromorphic differential operators acting on \(<v_1(z_1) \ldots v_n(z_n)>_\Gamma\), which are usually called the conformal blocks.

One now also immediately gets the differential equations that the conformal blocks have to satisfy: First of all, from \(0 = <L_k v_0, \ldots >_\Gamma\) for \(k = -1, 0, 1\) one finds the equations expressing projective invariance of the conformal blocks:
\[ \sum_{i=1}^{n} z_i^k \left( z_i \frac{\partial}{\partial z_i} + (k+1)h_i \right) <v_1(z_1) \ldots v_n(z_n)>_\Gamma = 0; \quad k \in \{-1, 0, 1\}. \]

In addition one gets partial differential equations from decoupling of the null vectors in \(\mathcal{V}_{j_i}, i = 1, \ldots, n\). If the nullvector in \(\mathcal{V}_{j_i}\) is written as in the proof of Proposition 3.2, then the resulting differential equation will be
\[ \sum_{j} C_z L_{-\pi}^{(i)} <v_1(z_1) \ldots v_n(z_n)>_\Gamma = 0, \quad \text{where} \quad L_{-\pi}^{(i)} := L_{-n_1}^{(i)} \ldots L_{-n}^{(i)}, \quad L_{-n}^{(i)} := \sum_{j \neq i} \left( h_k^1 (1-n) \frac{1}{(z_i-z_j)^m} + \frac{1}{(z_i-z_j)^{m-1}} \right) \]

The conformal blocks as constructed from fusion products provide formal power series solutions to these differential equations. By using standard results on partial differential equations with regular singular points such as given in [Kn], Appendix B, it should be easy to prove that the formal power series in question actually converge and may be analytically continued to multivalued analytic functions on the complement of the hyperplanes \(z_i = z_j\) in \(\mathbb{C}^n\).

6. Global properties of the fusion product: Commutativity and associativity

6.1. Commutativity. The logarithm used to define \((z_2 - z_1)^{h_{21} - h_1 - h_2}\) is taken to be the principal value. One therefore has to distinguish two zones: \(\mathbb{C}_+^2 := \{(z_2, z_1) \in \mathbb{C}^2| \arg(z_2 - z_1) \in (0, \pi)\}\) and \(\mathbb{C}_-^2 := \{(z_2, z_1) \in \mathbb{C}^2| \arg(z_2 - z_1) \in (-\pi, 0)\}\).

\[ [\xi_2(z_2) \otimes \xi_1(z_1)]_{h_{21}} = \begin{cases} e^{i\pi(h_{21} - h_1 - h_2)} [\xi_1(z_1) \otimes \xi_2(z_2)]_{h_{21}} & \text{for } (z_2, z_1) \in \mathbb{C}_+^2 \quad \text{for } (z_2, z_1) \in \mathbb{C}_+^2 \quad \text{for } (z_2, z_1) \in \mathbb{C}_-^2 \end{cases} \]
The phase factor will in the following be abbreviated by

\[ \Omega\left(\frac{h_{21}}{h_{22}}\right) = e^{\pi i(h_{21}-h_{12})} \]

6.2. Associativity.

**Theorem 6.1.** To any four conformal dimensions \( h_i = h_j \), there exists an invertible matrix \( F_{J_21, J_22}^{J_3, J_2} \) such that

\[ [\xi_3(z_3) \hat{\otimes} \xi_2(z_2) \hat{\otimes} \xi_1(z_1)]_{J_21} = \sum_{J_{23}} F_{J_{21}, J_{22}}^{J_3, J_2} [[\xi_3(z_3) \hat{\otimes} \xi_2(z_2)]_{J_{23}} \hat{\otimes} \xi_1(z_1)]_{J_1}, \]

for any \( \xi_i \in \mathcal{H}_{h_i}, i = 1, 2, 3 \). The summation ranges over all \( J_{23} \) such that the triples \( (J_{23}, J_2, J_3) \) and \( (J_4, J_1, J_{23}) \) satisfy the fusion rules.

**Proof:** The Theorem will be deduced (Proposition 6.5) from the corresponding statement for the four-point conformal blocks

\begin{align*}
\mathcal{G}_{3(21)}^{J_2}(z_1, z_2, z_3, z_4) &:= <v_0, [v_1(z_1) \hat{\otimes} [v_2(z_2) \hat{\otimes} v_3(z_3)]_{J_2}]_{J_1}\rangle_0 > \\
\mathcal{G}_{3(22)}^{J_2}(z_1, z_2, z_3, z_4) &:= <v_0, [v_1(z_1) \hat{\otimes} [[[v_3(z_3) \hat{\otimes} v_2(z_2)]_{J_{22}} \hat{\otimes} v_1(z_1)]_{J_1}\rangle_0 >.
\end{align*}

As a preliminary for the proof one needs results on the differential equations that \( \mathcal{G}_{3(21)}^{J_2} \) and \( \mathcal{G}_{3(22)}^{J_2} \) satisfy. The three equations (30) determine \( \mathcal{G} \) to be of the form

\[ \mathcal{G}(z_1, z_2, z_3, z_4) = \prod_{i>j}(z_i - z_j)^{h_i-h_j} F \left( \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)} \right), \]

where \( h = \frac{1}{3} \sum_{i=1}^4 h_i \). In addition one has eight partial differential equations from null vector decoupling, cf. (31). The crucial result is then the following:

**Proposition 6.2.** \( \{\mathcal{G}_{3(21)}^{J_2}\} \) and \( \{\mathcal{G}_{3(22)}^{J_2}\} \) form complete sets of solutions of the 11 ordinary differential equations that follow from null vector decoupling and projective invariance.

**Proof:** As preparation I will need two Lemmas that are proved in the appendix.

**Lemma 6.3.** The ordinary differential equation that follows from the decoupling of a nullvector \( n_i \in \mathcal{V}_{J_i}, i = 1, 2, 3, 4 \) by using (36) are of fuchsian type (regular singular points at \( z_i = z_j \), \( i \neq j \in \{1, 2, 3, 4\} \)) and \( z_i = \infty \) only, see i.e. [CL], i.e. of the form

\[ \sum_{k=0}^l q^{(k)}(z_i \mid z_j; j \neq i) \delta_k^i \mathcal{G} = 0 \quad \text{where} \quad q^{(k)} = \prod_{j \neq i}(z_i - z_j)^{-(n-k)} p^{(k)}(z_i \mid z_j; j \neq i), \]

where \( l \) is the level at which the nullvector occurs and \( p^{(k)}(z_i \mid z_j; j \neq i) \) is a polynomial in \( z_i \) of order \( 2(n-k) \) (in particular \( q^{(n)} = \text{const.} \)).

**Lemma 6.4.** The indicial equation of the fuchsian differential equation from a nullvector in \( \mathcal{V}_{J_i} \) at the singular point \( z_j \) is equal to \( P_{J_i}(h_{J_{ij}}, h_{J_j}; c) = 0 \).

Consider the two differential equations from the nullvectors in \( \mathcal{V}_{J_i} \), which will be denoted \( (J_1) \) and \( (P_{J_1}) \). Let \( S_1 \) (\( S_2 \)) be the set of roots of indicial equation for \( (J_1) \) (resp. \( (P_{J_1}) \)), and denote \( \mu_1(s), (\mu_2(s)) \) the multiplicity of the root \( s \in S_1 \) (\( s \in S_2 \)). By the Frobenius method in the theory of ordinary differential equations of fuchsian type, see [CL], it is shown that there is a basis \( \{ f_{s,k}; s \in S_1, k = 0, \ldots, \mu_1(s) \} \) of solutions to equation \( (J_1) \) with leading asymptotics \( (z_2 - z_1)^s \log(z_2 - z_1)^k \). The number \( s \) is called exponent of the
solution $f_{s,k}$. The solutions $f_{s,k}$ are of the form $(z_2 - z_1)^s \log(z_2 - z_1)^k g_{s,k}(z_2 - z_1)$, where $g_{s,k}(z)$ may be expanded as

$$g_{s,k}(z) = 1 + \sum_{i=1}^{\infty} \sum_{j=0}^{M} a_{ij} z^j (\log(z))^i$$

with some $M \in \mathbb{Z}_{\geq 0}$. The logarithmic terms may appear (but must not) only if for given $s$ there is a $s' \in S_1$ such that $s - s'$ is a positive integer.

In the present case the roots of the indicial equations for $(J_1)$ and $(PJ_1)$ are parametrized by the index sets

$$I_1 := \left\{ J = (j', j)| \ j' = j_2' - j_3' - j_4' + 1 \ldots j_1' + j_2' - 1 \right\}$$

$$I_2 := \left\{ J = (j', j)| \ j' = j_2' - j_3' - j_4' + 1 \ldots j_1' - j_2' - 2 \right\}$$

The corresponding roots of the indicial equations are $s_J := h_J - h_{J_1} - h_{J_2}$. A basis for the space of common solutions to $(J_1)$ and $(PJ_1)$ must be contained in the set of solutions to $(J_1)$ or $(PJ_1)$ that have leading asymptotics $(z_2 - z_1)^s \log(z_2 - z_1)^k$ with $J \in I_{12} := I_1 \cap I_2$ for $z_1 \rightarrow z_2$. The crucial fact that may be established by direct calculation is that the roots $s_J$ with $J \in I_{12}$ are nondegenerate, i.e. there is no $J' \in I_1$ such that $s_{J'} = s_J$ and no $J'' \in I_2$ such that $s_{J''} = s_J$. It follows that the space of common solutions of $(J_1)$ and $(PJ_1)$ has dimension less or equal to the cardinality of the set of $J$ such that the triple $(J, J_2, J_1)$ satisfies the fusion rules. Similarly one finds that the dimension of the space of common solutions to $(J_1)$ and $(PJ_3)$ is bounded from above by the cardinality of the set of $J$ such that $(J_4, J_3, J)$ obeys the fusion rules. Consideration of the remaining equations $(J_2)$, $(PJ_2)$, $(J_3)$ and $(PJ_3)$ will not lower the bounds on the dimensionalities.

Now one moreover requires the common solutions to $(J_1)$ and $(PJ_i)$, $i = 1, 2, 3, 4$ to also obey (30), i.e. to be of the form (36). These solutions will have asymptotics proportional to $(z_2 - z_1)^{h_{J_2} - h_{J_1} - k}$ for $z_2 \rightarrow z_1$ iff it has asymptotics $(z_4 - z_3)^{h_{J_4} - h_{J_3} - k}$ for $z_4 \rightarrow z_3$. Therefore only those $s_J$ which are such that both $(J, J_2, J_1)$ and $(J_3, J_4, J)$ satisfy the fusion rules can appear as exponents. The dimension of the space of solutions to all 11 equations is therefore bounded from above by the dimension of the space of conformal blocks. One thereby deduces completeness of the set of solutions to all 11 equations that is provided by the conformal blocks. \(\square\)

**Proposition 6.5.** The theorem is valid iff one has

$$\mathcal{G}^{J_2}_{3(21)}(z_1, z_2, z_3, z_4) = \sum_{J_3} F_{J_2, J_3} \left[ J_3 \ J_2 \right] \mathcal{G}^{J_2}_{3(21)}(z_1, z_2, z_3, z_4)$$

**Proof:** First of all note that the triple fusion products are defined as formal sums over vectors in the homogeneous components $\mathcal{H}_{J_4}^{(n)}$ of $\mathcal{H}_{J_4}$. By nondegeneracy of the form $\langle, \rangle$ on $\mathcal{H}_{J_4}^{(n)}$ the theorem is equivalent to the corresponding equation for the matrix elements with arbitrary vectors $\xi^{(n)}_i \in \mathcal{H}_{J_4}^{(n)}$. By repeated application of the conformal Ward identities it is straightforward to see that any such matrix element may be expressed as some meromorphic differential operator acting on

$$\mathcal{F}^{J_2}_{3(21)}(z_1, z_2, z_3) := \langle v_4, [v_3(z_3) \hat{\otimes} v_2(z_2) \hat{\otimes} v_1(z_1)]_{J_21} J_4 \rangle$$

$$\mathcal{F}^{J_2}_{3(22)}(z_1, z_2, z_3) := \langle v_4, [[v_3(z_3) \hat{\otimes} v_2(z_2)]_{J_21} \hat{\otimes} v_1(z_1)]_{J_4} \rangle.$$
However to each $F_{j_2}^{j_1}$ or $F_{j_2}^{j_1'}$ one has a $G_{j_2}^{j_1}$ resp. $G_{j_2}^{j_1'}$ such that (see (36)):

$$
\lim_{z_i \to \infty} z_i^{2h_i} G(z_1, z_2, z_3, z_4) = F(z_1, z_2, z_3).
$$

The proof of Theorem 6.1 is thereby completed. \( \square \)

7. POLYNOMIAL EQUATIONS

The data $F$ and $\Omega$ satisfy certain identities. To derive these, introduce the following conformal blocks to each permutation $(kji)$ of (321):

\begin{align}
G_{j_i}^{j_{ki}} & := \langle v_{j_i}^* [v_k(z_k) \otimes v_j(z_j) \otimes v_i(z_i)]_{j_i} \rangle_{J_i} \\
G_{(kji)} & := \langle v_{j_i}^* [v_k(z_k) \otimes v_j(z_j) \otimes v_i(z_i)]_{j_i} \rangle_{J_i}
\end{align}

These are analytic functions on the universal cover of $\Lambda := \mathbb{H}^3 / \mathbb{Z}^3$ where $j_i, (kji)$: $j_i = 1, 2, 3$ of the following form

\begin{align}
G_{j_i}^{j_{ki}} & = \Delta_{ji}(z_j - z_i) \Delta_{ki}(z_k - z_i) H_{j_i}(z_j - z_i) \\
G_{(kji)} & = \Delta_{kji}(z_k - z_j) \Delta_{ki}(z_k - z_i) H_{(kji)}(z_k - z_j)
\end{align}

where $\Delta_{ji} = h_{ji} - h_j - h_i$, $\Delta_{kji} = h_k - h_{ji} - h_j$. The functions $H(z)$ are holomorphic and single-valued in a neighborhood of 0. Consider the region in $\mathbb{C}^3$ where $(z_2, z_1), (z_3, z_1), (z_3, z_2)$ are all in $\mathbb{C}_+^3$. Then one has the following relations between the functions $G$:

\begin{align}
G_{3(21)} & = \Omega(J_2, J_1) G_{3(12)} G_{3(21)} \\
G_{3(21)} & = \Omega(J_4, J_3, J_2) G_{(21)} G_{3(21)} \\
G_{3(21)} & = \Omega(J_4, J_3, J_2) G_{(21)} G_{3(21)}
\end{align}

In addition one has the associativity relations

\begin{align}
G_{j_i}^{j_{ki}} & = \sum_{J_{ki}} F_{J_i J_k} [J_k J_i] G_{j_i}^{j_{ki}} \\
G_{(kji)} & = \sum_{J_{ki}} F_{-1}^{-1} [J_k J_i] G_{j_i}^{j_{ki}}
\end{align}

Now the expression of $G_{3(21)}$ in terms of $G_{(21)}$ may be computed in two ways: Either by using (42) or by a sequence of operations that may be symbolically written as $3(21) \to (32) \to (23) \to 2(31) \to 2(13) \to (21)3$. By the linear independence of the $G_{(21)}$, one gets the following identity (hexagon):

$$
\Omega(J_3, J_2, J_1) = \sum_{J_{23}, J_{32}, J_{21}} F_{J_2 J_1, J_{21}} [J_{21} J_{22}] \Omega(J_{22}, J_{21}) F_{-1}^{-1} [J_{21} J_{22}] \Omega(J_{33}, J_{31}) F_{J_3 J_1, J_{21}} [J_{21} J_{22}].
$$

Similarly one gets

$$
\Omega(J_{12}, J_3) = \sum_{J_{23}, J_{32}} F_{-1}^{-1} [J_{21} J_{22}] \Omega(J_{21}, J_{22}) F_{J_2 J_1, J_{32}} [J_{32} J_{31}] \Omega(J_{33}, J_{31}) F_{J_3 J_2, J_{32}} [J_{32} J_{31}].
$$

The inverse of $F$ may be calculated in terms of $\Omega, F$ by representing $(12)3 \to 1(23)$ as the sequence of moves $(12)3 \to (21)3 \to 3(21) \to (32)1 \to 1(32) \to (1)(23)$. The result is simply

$$
F_{J_2 J_1, J_{32}} [J_{32} J_{31}] = F_{J_2 J_1, J_{32}} [J_{11} J_{33}].$$
A further important identity may be derived by considering fusion products of four highest weight states, or equivalently five point conformal blocks: 

$$[v_4 \otimes [v_3 \otimes [v_2 \otimes v_1]]]$$

may be expressed in terms of $$[[v_4 \otimes v_3] \otimes [v_2 \otimes v_1]]$$ in two ways: Either by $$4(3(21)) \rightarrow (43)(21) \rightarrow ((43)21)$$ or by $$4(3(21)) \rightarrow 4((32)1) \rightarrow (4(32))1 \rightarrow ((43)21)$$. This leads to the identity (pentagon)

$$\tag{46} F_{J_{21},J_{22}}[J_{43} J_{42}] F_{J_{21},J_{22}}[J_{43} J_{41}] = \sum_{J_{22}} F_{J_{21},J_{22}}[J_{33} J_{32} J_{31}] F_{J_{21},J_{22}}[J_{33} J_{32} J_{31}] F_{J_{21},J_{22}}[J_{33} J_{32} J_{31}].$$

If one then considers eqn. (46) in the special cases $$J_1 = (0,1/2), J_{21} = (j_2', j_2 + 1/2)$$ and $$J_{1} = (1/2,0), J_{21} = (j_2', j_2, j_2),$$ one finds that it allows to express the F-matrices with $$J_1 = (j_1', j_1 + 1/2)$$ (or $$J_1 = (j_1', 1/2, j_1)$$) in terms of those with $$J_1 = (j_1', i_1), i_1 \leq j_1', i_1 \leq j_1$$. Equation (46) therefore uniquely determines $$F_{J_{21},J_{22}}[J_{33} J_{32} J_{31}]$$ in terms of $$F_{J_{21},J_{22}}[J_{33} J_{32} J_{31}], F_{J_{21},J_{22}}[J_{33} J_{32} J_{31}].$$

8. Conformal Bootstrap

Now one is in the position to apply the results of [CGR] to determine the matrices $$F$$ explicitly. The result is

**Theorem 8.1.** There is a number $$g(J_3 J_2 J_1)$$ for any triple $$(J_3, J_2, J_1)$$ that satisfies the fusion rules such that

$$\tag{47} F_{J_{21},J_{22}}[J_{33} J_{32} J_{31}] = \frac{g(J_3 J_2 J_1) g(J_3 J_2 J_1)}{g(J_{21} J_{22}) g(J_{33} J_{32} J_{31})} \{J_{33} J_{32} J_{31}\}_q,$$

where $$\{J_{33} J_{32} J_{31}\}_q$$ is defined as

$$\{J_{33} J_{32} J_{31}\}_q := (-)^{2j'_2(j_j + j_2 - j - j_4) + 2j_2(j'_2 + j_2 - j'_4 - j'_4)} \{J_{33} J_{32} J_{31}\}_q,$$

in terms of the restricted q-6j symbols defined in [KR].

In order to see that the results of [CGR] may be applied here, I will briefly review the overall strategy:

In the case of $$J_1 = (0,1/2)$$ or $$J_1 = (1/2,0)$$ one gets second order differential equations from null vector decoupling, which may be reduced to the hypergeometric differential equation. The matrix $$F$$ for this case is thereby found in terms of Gamma-functions. It is then shown in [CGR] how to determine $$g(J_3 J_2 J_1)$$ such that for irrational $$c$$ and the special cases $$J_1 = (0,1/2)$$ or $$J_1 = (1/2,0)$$ equation (47) holds (of course with unrestricted q-6j). However, one may explicitly check that in these cases the elements of F matrices and q-6j symbols do not vanish in the limit $$c \rightarrow c_{p'},$$ provided the triples $$(J_1, J_2, J_{12}), (J_{12}, J_3, J_{23}), (J_{23}, J_1, J_3)$$ satisfy the fusion rules. Equation (47) will therefore hold for $$c = c_{p'},$$ if the q-6j symbols are simply taken to be the restricted q-6j as defined in [KR]. Validity of equation (47) for general $$J_1$$ will then follow recursively from the pentagon equation.
9. Appendix

9.1. Proof of Lemma 6.3. Call differential operators $D$ fuchsian iff $Df(z) = 0$ is of fuchsian type.

It suffices to show that

$$\mathcal{L}_{-m_k} \ldots \mathcal{L}_{-m_1} = \mathcal{D}_{-n} = \mathcal{D}_{-\pi}, \quad \text{where } \mathcal{D}_{-\pi} \text{ is fuchsian.}$$

For notational simplicity consider the equations from the nullvectors on $V_{i}$, and let $z := z_1$, $\partial := \partial_1$. (48) may be proved by induction on $k$: It is easy to see that it holds for $k = 1$ by direct calculation using (36): One has

$$\mathcal{L}_{-m} = \mathcal{D}_{-m} = \left( \frac{c_2}{z_{21}^{m-1}} + \frac{1}{z_{21}^{m-1}} \mathcal{D}_{-m} \right) \mathcal{G},$$

with some constants $c_2, c_3, c_4$, which is easily checked to be fuchsian. An important property of $\mathcal{D}_{-m}$ is that its coefficients are functions of $z_{21}, z_{31}, z_{41}$ only.

Now assume that $\mathcal{D}_{-\pi}$ is in the set $\mathcal{F}(l)$ of differential operators of the form

$$\mathcal{D}_{-\pi} = \sum_{k=0}^{l} q^{(k)} \partial^k, \quad q^{(k)} = \frac{p^{(k)}(z_{21}, z_{31}, z_{41})}{(z_{21}z_{31}z_{41})^{n-k}},$$

where $p^{(k)}(z_{21}, z_{31}, z_{41})$ is polynomial of order $2(l - k)$ in $z_1$, and consider $\mathcal{L}_{-m} \mathcal{D}_{-\pi} = [\mathcal{L}_{-m}, \mathcal{D}_{-\pi}] \mathcal{G} + \mathcal{D}_{-\pi} \mathcal{D}_{-m} \mathcal{G}$. The differential operator $\mathcal{D}_{-\pi} \mathcal{D}_{-m}$ is easily seen to be in $\mathcal{F}(l+m)$, so it remains to consider the first term. Write $\mathcal{L}_{-m} = r_m + s_{m-1} \partial_i$, where $i$ is summed over $i = 2, 3, 4$.

$$[\mathcal{L}_{-m}, \mathcal{D}_{-\pi}] = \sum_{k=0}^{l} (\mathcal{L}_{-m} q^{(k)}) \partial^k + (\mathcal{D}_{-\pi} s_{m-1}^i) \partial_i + \mathcal{D}_{-\pi} r_m.$$ 

The last term is trivially fuchsian, for the second one note that $q^{(k)}(\partial^k s_{m-1}^i) \partial_i$ is proportional to $q^{(k)} s_{m+k-1}^i \partial_i$, which is again fuchsian when acting on $\mathcal{G}$, see (49). It remains to show that

$$\mathcal{L}_{-m} q^{(k)} := r_m q^{(k)} + s_{m-1}^i \partial_i q^{(k)} = \frac{r^{(n+m-k)}(z_{21}, z_{31}, z_{41})}{(z_{21}z_{31}z_{41})^{n+m-k}},$$

where $r^{(n+m-k)}(z_{21}, z_{31}, z_{41})$ is of order $2(l + m - k)$ in $z_1$. This reduces be the verification that $(z_{41}z_{31})^{m-1} \partial_2 + (z_{41}z_{21})^{m-1} \partial_3 + (z_{21}z_{31})^{m-1} \partial_4) p^{(k)}(z_{21}, z_{31}, z_{41})$ is (a) of order $2(l + m - k)$ and (b) annihilated by $\sum_{i=1}^{4} \partial_i$. For (a) note that because of $\sum_{i=1}^{4} \partial_i p^{(k)} = 0$, $(\partial_2 + \partial_3 + \partial_4) p^{(k)}$ is of order $2(l - k) - 1$ in $z_1$. (b) follows from the fact that $\sum_{i=1}^{4} \partial_i$ commutes with $\mathcal{L}_{-m}$.

9.2. Proof of Lemma 6.4. It suffices to show that the indicial equation for the differential operator $\mathcal{D}_{-\pi}$ is given by

$$\prod_{i=1}^{k} \left( \Delta - \sum_{j=1}^{i-1} n_j + h_2(1 - n_i) \right) = 0.$$ 

Use induction on $k$: From the previous lemma one has

$$\mathcal{D}_{-\pi} = \sum_{k=0}^{l} q^{(k)} \partial_i^{(k)}, \quad \text{with } q^{(l-k)} = z_i^{-(l-k)} a^{(k)} + O(z_i^{-(l-k)+1})$$
such that the indicial equation is $\sum_{k=0}^{l} a^{(k)}(s)_k = 0$, where $(s)_k := s(s-1)\ldots(s-k+1)$. Now write

$$\mathcal{L}_{-m} \mathcal{D}_- \mathcal{G} = \sum_{k=0}^{l} \left( (\mathcal{L}_{-m} q^{(k)}) \partial^k_i + q^{(k)}(s^j_{m-1} \partial^k_j) \right) \mathcal{G}$$

The first term contributes $(h_j(m-1))a^{(k)}$. To evaluate the second, use (36) and observe that because of ($\tilde{z} := \frac{z_1 z_2}{z_3 z_4}$)

$$\partial^2 F(\tilde{z}) = \frac{z_3 z_4}{z_2 z_3} \partial_1 F(\tilde{z}) \quad \partial^3 F(\tilde{z}) = \frac{z_3^2 z_4}{z_2^3 z_3^2} \partial_1 F(\tilde{z}) \quad \partial^4 F(\tilde{z}) = \frac{z_2 z_3^3}{z_2^4 z_3^4} \partial_1 F(\tilde{z}),$$

only the $j = 2$ term is relevant, so that the contribution to the indicial equation is $-a^{(k)}(s)_{k+1}$. Collecting terms one therefore finds the indicial equation to be

$$0 = \sum_{k=0}^{l} a^{(k)}((s)_k(k + h_j(m-1))(s)_k - a^{(k)}(s)_{k+1} = (h_j(m-1) - s + l) \sum_{k=0}^{l} a^{(k)}(s)_k.$$

The claim follows.

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