ASYMPTOTIC BEHAVIOR FOR A SCHRÖDINGER EQUATION WITH NONLINEAR SUBCRITICAL DISSIPATION

THIERRY CAZENAVE
Sorbonne Université & CNRS, Laboratoire Jacques-Louis Lions, B.C. 187
4 place Jussieu, 75252 Paris Cedex 05, France

ZHENG HAN∗
Department of Mathematics, Hangzhou Normal University
2318 Yuhangtang Road, 311121 Hangzhou, China

(Communicated by Joachim Krieger)

Abstract. We study the time-asymptotic behavior of solutions of the Schrödinger equation with nonlinear dissipation
\[ \partial_t u = i \Delta u + \lambda |u|^{\alpha} u \]
in \( \mathbb{R}^N \), \( N \geq 1 \), where \( \lambda \in \mathbb{C} \), \( \Re \lambda < 0 \) and \( 0 < \alpha < \frac{2}{N} \). We give a precise description of the behavior of the solutions (including decay rates in \( L^2 \) and \( L^\infty \), and asymptotic profile), for a class of arbitrarily large initial data, under the additional assumption that \( \alpha \) is sufficiently close to \( \frac{2}{N} \).

1. Introduction. In this paper, we consider the Schrödinger equation with nonlinear dissipation
\[ \begin{cases} \partial_t u = i \Delta u + \lambda |u|^{\alpha} u \\ u(0, x) = u_0, \end{cases} \tag{1.1} \]

where \( \lambda \in \mathbb{C} \) with \( \Re \lambda < 0 \) \( \tag{1.2} \)
and \( 0 < \alpha < \frac{2}{N} \).

Equation (1.1) is itself a particular case of the more general complex Ginzburg-Landau equation on \( \mathbb{R}^N \): \( u_t = e^{i\theta} \Delta u + z|u|^{\alpha} u \), where \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), \( z \in \mathbb{C} \) and \( \alpha > 0 \), which is a generic modulation equation describing the nonlinear evolution of patterns at near-critical conditions (see e.g. [20, 7, 16]).

Equation (1.1) is mass-subcritical, and is globally well-posed in \( L^2(\mathbb{R}^N) \) and \( H^1(\mathbb{R}^N) \). See Proposition 2.1 below.

Concerning the large-time asymptotic behavior of the solutions of (1.1) under assumption (1.2), \( \alpha = \frac{2}{N} \) is a limiting case. Indeed, if \( \alpha > \frac{2}{N} \), \( \lambda \in \mathbb{C} \), then a large set of initial values produces solutions that scatter as \( t \to \infty \), i.e. that are asymptotic to a solution of the free Schrödinger equation. (See [21, 9, 10, 6, 8, 17, 1, 4].)
If \( \alpha \leq \frac{2}{N} \), then in many cases solutions are known to decay faster than the solutions of the free Schrödinger equation. If \( \alpha = \frac{2}{N} \), then for a large class of initial values, the solutions of (1.1) can be described by an asymptotic formula, and have the decay rate \((t \log t)^{-\frac{N}{2}}\). See [19, 15, 5]. In addition, for some solutions, 
\[
(t \log t)^{\frac{n}{2}} \|u(t)\|_{L^\infty} \xrightarrow{t \to \infty} (\alpha |\Re \lambda|)^{-\frac{N}{2}}.
\]

See [5].

In the one-dimensional case \( N = 1 \), if \( \alpha < 2 \) is sufficiently close to 2 and
\[
\frac{\alpha}{2\sqrt{\alpha + 1}} |3\lambda| \leq |\Re \lambda|, \tag{1.3}
\]
then the large-time asymptotic behavior of solutions can be described for any initial data in \( H^1(\mathbb{R}) \cap L^2(\mathbb{R}, |x|^2 dx) \), and the solutions satisfy
\[
\|u(t)\|_{L^\infty} \lesssim t^{-\frac{1}{2}}, \tag{1.4}
\]
see [15]. In addition, in any space dimension \( N \geq 1 \), under assumption (1.3) and for \( \alpha < \frac{2}{N} \) sufficiently close to \( \frac{2}{N} \), all solutions with initial value in \( H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 dx) \) satisfy \( \|u(t)\|_{L^\infty} \lesssim t^{-\left(\frac{1}{2} - \frac{2}{N}\right)q} \) for all \( q < \frac{2}{N+1} \). See [11].

In space dimensions \( N \geq 1 \), if \( \alpha < \frac{2}{N} \) sufficiently close to \( \frac{2}{N} \), the upper estimate (1.4), as well as lower estimates, is established for sufficiently small initial data in a certain space. See [12].

Our purpose in this article is to complete the previous results for (1.1), and describe the large-time asymptotic behavior of the solutions for a class of arbitrarily large initial data. In order to state our result, we recall the definition of the space \( \mathcal{X} \) introduced [4], which we use in a essential way. We consider three integers \( k, m, n \) such that
\[
k > \frac{N}{2}, \quad n > \max\left\{ \frac{N}{2} + 1, \frac{N(N+1)}{4} \right\}, \quad 2m \geq k + n + 1 \tag{1.5}
\]
and we let
\[
J = 2m + 2 + k + n. \tag{1.6}
\]
We define the space \( \mathcal{X} \) by
\[
\mathcal{X} = \{ u \in H^J(\mathbb{R}^N); (x)^n D^\beta u \in L^\infty(\mathbb{R}^N) \text{ for } 0 \leq |\beta| \leq 2m, \quad (x)^n D^\beta u \in L^2(\mathbb{R}^N) \text{ for } 2m + 1 \leq |\beta| \leq 2m + 2 + k, \quad (x)^{J-|\beta|} D^\beta u \in L^2(\mathbb{R}^N) \text{ for } 2m + 2 + k < |\beta| \leq J \tag{1.7}
\}
\]
and we equip \( \mathcal{X} \) with the norm
\[
\|u\|_\mathcal{X} = \sum_{j=0}^{2m} \sup_{|\beta| = j} \| (x)^n D^\beta u \|_{L^\infty} + \sum_{\nu=0}^{k+1} \sum_{\mu=0}^{n} \sum_{|\beta|=\nu+\mu+2m+1} \| (x)^{n-\mu} D^\beta u \|_{L^2}. \tag{1.8}
\]
where
\[
(x) = (1 + |x|^2)^{\frac{1}{2}}.
\]
In particular, \( (\mathcal{X}, \| \cdot \|_\mathcal{X}) \) is a Banach space and \( \mathcal{X} \hookrightarrow H^J(\mathbb{R}^N) \).

Our main result is the following.

**Theorem 1.1.** Let \( \lambda \in \mathbb{C} \) satisfy (1.2), assume (1.5)-(1.6) and let \( \mathcal{X} \) be defined by (1.7)-(1.8). Given any \( K > 1 \), there exist \( \frac{2}{N+1} < \alpha_1 < \frac{2}{N} \) and \( b_1 > 0 \) with the
following property. Let $\alpha_1 \leq \alpha < \frac{2}{N}$. Suppose $u_0(x) = e^{i\frac{b|x|^2}{4}}v_0(x)$, where $b \geq b_1$ and $v_0 \in \mathcal{X}$ satisfies

$$\inf_{x \in \mathbb{R}^N} \langle x \rangle^\alpha |v_0(x)| > 0. \quad (1.9)$$

and

$$\|v_0\|_{\mathcal{X}} + \left( \inf_{x \in \mathbb{R}^N} \langle x \rangle^\alpha |v_0(x)| \right)^{-1} \leq K. \quad (1.10)$$

It follows that the corresponding solution $u \in C([0, \infty), H^1(\mathbb{R}^N))$ of (1.1) belongs to $L^\infty((0, \infty) \times \mathbb{R}^N)$. Moreover, there exist $f_0, \omega_0 \in L^\infty(\mathbb{R}^N)$, with $f_0$ real-valued and $\|f_0\|_{L^\infty} \leq \frac{1}{2}$, and $\langle \cdot \rangle^\alpha \omega_0 \in L^\infty(\mathbb{R}^N)$, such that

$$t^{\frac{1}{2} - \frac{\alpha}{N}} \|u(t) - z(t)\|_{L^2} + t^{\frac{1}{2}} \|u(t) - z(t)\|_{L^\infty} \leq Ct^{-\left(\frac{N+1}{2} - \frac{\alpha}{N}\right)}, \quad (1.11)$$

where

$$z(t, x) = (1 + bt)^{-\frac{N}{2}} e^{i\Theta(t, x)} \Psi \left( t, \frac{x}{1 + bt} \right) \omega_0 \left( \frac{x}{1 + bt} \right) \quad (1.12)$$

with

$$\Theta(t, x) = \frac{b|x|^2}{4(1 + bt)} - \frac{3\lambda}{\Re \lambda} \log \left( \Psi \left( t, \frac{x}{1 + bt} \right) \right) \quad (1.13)$$

and

$$\Psi(t, y) = \left( \frac{1 + f_0(y)}{1 + f_0(y) + \frac{2\alpha|\Re \lambda|}{b(2 - N\alpha)} |v_0(y)|^\alpha \left[ (1 + bt)^{\frac{2 - N\alpha}{2}} - 1 \right] } \right)^{\frac{1}{2}}. \quad (1.14)$$

Moreover,

$$|\omega_0|^\alpha = \frac{|v_0|^\alpha}{1 + f_0}, \quad (1.15)$$

so that $\frac{2}{N}|v_0|^\alpha \leq |\omega_0|^\alpha \leq 2|v_0|^\alpha$. Furthermore,

$$t\|u(t)\|_{L^\infty} \xrightarrow{t \to \infty} \frac{2 - N\alpha}{2\alpha|\Re \lambda|}, \quad (1.16)$$

and

$$t^{-\left(\frac{1}{2} - \frac{\alpha}{N}\right)(1 - \frac{2}{N})} \leq \|u(t)\|_{L^2} \leq t^{-\left(\frac{1}{2} - \frac{\alpha}{N}\right)(1 - \frac{2\alpha}{N})} \quad (1.17)$$

as $t \to \infty$, where $n$ is given by (1.5).

**Remark 1.2.** Here are some comments on Theorem 1.1.

(i) We have $v_0 \in \mathcal{X} \hookrightarrow L^2(\mathbb{R}^N)$, $\mathcal{X} \hookrightarrow H^1(\mathbb{R}^N)$, and $\mathcal{X} \hookrightarrow L^2(\mathbb{R}^N, |x|^2 dx)$ (because $n > \frac{N}{2} + 1$), so that $u_0 \in H^1(\mathbb{R}^N)$. Therefore, the solution $u \in C([0, \infty), H^1(\mathbb{R}^N))$ of (1.1) is well defined, see Proposition 2.1. Moreover, $u$ is smoother than stated. Indeed, $u$ is given by the pseudo-conformal transformation (5.1) in terms of a solution $v \in C([0, \frac{1}{2}], \mathcal{X})$ of equation (1.19). In particular, $u$ is a classical solution of (1.1) ($C^1$ in $t$ and $C^2$ in $x$).

(ii) Theorem 1.1 is valid in any dimension $N \geq 1$ and for any $\lambda \in \mathbb{C}$ with $\Re \lambda < 0$. In particular, we do not require assumption (1.3). The main restrictions are that $\alpha$ must be sufficiently close to $\frac{2}{N}$ and that the initial value must be bounded from below in the sense (1.9) and sufficiently oscillatory in the sense that $b$ must be sufficiently large. Moreover, how close $\alpha$ must be to $\frac{2}{N}$ depends on a certain bound on the initial value through (1.10). On the other hand, there is no restriction on the size of $u_0$. 
A typical initial value which is admissible in Theorem 1.1 is

\[ v_0 = c(\cdot)^{-\alpha} + \varphi \]

with \( c \in \mathbb{C}, \ c \neq 0, \) and \( \varphi \in \mathcal{S}(\mathbb{R}^N), \ ||\varphi|| \leq ||c - \varepsilon||(\cdot)^{-\alpha}, \ \varepsilon > 0. \) Indeed, it is easy to check that \( v_0 \in X \) and \( v_0 \) satisfies (1.9). Then \( K \) must be chosen sufficiently large so that (1.10) holds and \( \alpha \) sufficiently close to \( \frac{N}{2}. \) Note that any value of \( n \) sufficiently large so that the second condition in (1.5) is satisfied, is admissible.

(vi) It follows from (1.16) and (1.17) that

\[ \|u(t)\|_{L^\infty} \rightarrow 0, \quad t \rightarrow \infty. \]

Thus we see that the asymptotic behavior of \( u(t) \) as \( t \rightarrow \infty \) is described by the function \( z(t) \) via the estimate (1.11). Note that the functions \( f_0 \) and \( \Psi \) are both real-valued, and that \( \frac{1}{2} \leq 1 + f_0 \leq \frac{3}{2} \) and \( 0 < \Psi \leq 1. \) The function \( \Theta \) is also real-valued. If \( 3\lambda \leq 0, \) then \( \Theta > 0. \) If \( 3\lambda > 0, \) then \( \Theta \) takes both positive and negative values.

Remark 1.3. If \( R\lambda > 0, \) then finite-time blowup occurs for equation (1.1), at least for \( H^1 \)-subcritical powers \((N-2)\alpha < 4. \) See [3, 2]. Moreover, if \( \alpha < \frac{2}{N}, \) then all nontrivial solutions blow up in finite or infinite time, see [1]. Finite-time blowup also occurs if \( R\lambda = 0, \) \( 3\lambda > 0, \) and \( \alpha \geq \frac{4}{N}, \) since in this case (1.1) is the focusing NLS. If \( R\lambda < 0, \) \( \alpha > \frac{4}{N} \) and condition (1.3) is not satisfied, then whether or not some solutions of (1.1) blow up in finite time seems to be an open question.

We apply the strategy of [4, 5] to prove Theorem 1.1. We require the nonvanishing condition (1.9), as well as strong decay and regularity of the initial data to overcome the difficulty of non-smooth nonlinearity and derivative loss in their estimates. This is why the various conditions in the definition of the space \( X \) arise. The other main ingredient is the application of the pseudo-conformal transformation. Given any \( b > 0, \) by the pseudo-conformal transformation

\[ v(t, x) = (1 - bt)^{-\frac{N}{2}} \left( \frac{t}{1 - bt} \right)^{\frac{x}{1 - bt}} e^{-\frac{bt|x|^2}{(1 - bt)^2}} \quad t \geq 0, \ x \in \mathbb{R}^N, \]

equation (1.1) is equivalent to the nonautonomous equation

\[
\begin{cases}
\partial_t v = i\Delta v + \lambda (1 - bt)^{-\frac{4 - N\alpha}{2}} |v|^\alpha v, \\
v(0, x) = v_0.
\end{cases}
\]

Note that the assumption \( \alpha \leq \frac{2}{N} \) implies that \( (1 - bt)^{-\frac{4 - N\alpha}{2}} \) is not integrable at \( 1/b. \) As in [5], we estimate the solution \( v(t, x) \) by allowing a certain growth of the various components of the \( X \)-norm of the solution, see (3.7)-(3.10). Using
Duhamel’s formula for (1.19), i.e.,
\[ v(t) = e^{it\Delta}v_0 + \lambda \int_0^t (1 - bs)^{-\frac{2-N\alpha}{2}} e^{i(t-s)\Delta} |v(s)|^\alpha v(s) \, ds \]  
(2.20)
and the elementary calculation
\[ \int_0^t (1 - bs)^{-1-\nu} \, ds = \frac{1}{b\nu}[(1 - bt)^{-\nu} - 1] \leq \frac{1}{b\nu} (1 - bt)^{-\nu}, \]  
(2.21)
we see that if \( e^{i(t-s)\Delta} |v(s)|^\alpha v(s) \) is estimated in a certain norm by \((1 - bs)^{-\mu}\), then \( v(t) \) can be controlled in that norm by \((1 - bs)^{-\mu + \frac{2-N\alpha}{2}}\). In the case \( \alpha = \frac{2}{N} \), one obtains the same power \((1 - bs)^{-\mu}\), and this can be used to close appropriate estimates. This is the strategy employed in [5]. In the present case \( \alpha < \frac{2}{N} \), we observe that if \( e^{i(t-s)\Delta} |v(s)|^\alpha v(s) \) is estimated in a certain norm by \((1 - bs)^{\mu - \frac{2-N\alpha}{2}}\), then \( v(t) \) can be controlled in that norm by \((1 - bs)^{-\mu}\). We obtain the extra decay by monitoring the decay of \(|v(s)|\) (see Lemma 3.1). The price to be paid is that the constants that appear in the calculations not only depend on \(1/b\), but also on \(\frac{2-N\alpha}{2}\). Therefore, in order to close the estimates, we are led to require not only that \(b\) is large, but also that \(\alpha\) is close to \(\frac{2}{N}\).

The rest of the paper is organized as follows. In section 2, we recall some estimates and a local well-posedness result in the space \(X\) for equation (1.19), taken from [4, 5]. The crucial estimate of the solutions is carried out in Section 3. Using these estimates, we describe in Section 4 the asymptotic behavior of the corresponding solutions of (1.19). Finally, we complete the proof of Theorem 1.1 in section 5, by applying the pseudo-conformal transformation.

2. Preliminary. We recall some properties of equation (1.1) which will be useful in the next sections. We begin with a global well-posedness result.

Proposition 2.1. Let \(0 < \alpha < \frac{4}{N}\) and let \(\lambda \in \mathbb{C}\) satisfy \(\Re\lambda \leq 0\). It follows that the Cauchy problem (1.1) is globally well-posed in \(L^2(\mathbb{R}^N)\) and in \(H^1(\mathbb{R}^N)\). More precisely, given any \(u_0 \in L^2(\mathbb{R}^N)\) there exists a solution \(u \in C([0, \infty), L^2(\mathbb{R}^N)) \cap L^{\alpha+2}(0, \infty), L^{\alpha+2}(\mathbb{R}^N))\) of (1.1). The solution is unique and depends continuously on \(u_0\) in \(C([0, T), L^2(\mathbb{R}^N)) \cap L^{\alpha+2}(0, T), L^{\alpha+2}(\mathbb{R}^N))\) for every \(T > 0\). If, in addition, \(u_0 \in H^1(\mathbb{R}^N)\), then \(u \in C([0, \infty), H^1(\mathbb{R}^N))\).

Proof. For the local theory (local existence, uniqueness, continuous dependence, regularity), see e.g. [13, 14]. For global existence, it is sufficient to estimate the \(L^2\) norm of \(u\). Multiplying (1.1) by \(\bar{u}\), taking the real part and integrating by parts, we obtain
\[ \|u(t)\|^2_{L^2} + (-\Re\lambda) \int_0^t \|u(s)\|_{L^{\alpha+2}}^{\alpha+2} = \|u_0\|^2_{L^2}. \]  
(2.21)
(This argument is formal, but (2.1) can be proved by standard approximation arguments, see for instance [18].) It follows that \(u\) is bounded in \(L^2(\mathbb{R}^N)\). \(\square\)

Next, we recall some estimates for the Schrödinger equation in the space \(X\).

Proposition 2.2 ([5, Proposition 2.1]). Assume (1.5)-(1.6) and let \(X\) be defined by (1.7)-(1.8). There exists \(C_1 \geq 1\) such that if \(T \geq 0, v_0 \in X\) and \(f \in C([0, T], X)\), then for all \(0 \leq t \leq T\), the solution \(v\) of
\[ \begin{cases} \partial_t v = i\Delta v + f, \\ v(0, x) = v_0, \end{cases} \]
satisfies the following estimates.

(i) If $|\beta| \leq 2m$, then
\[ \|\langle x \rangle^n D^\beta v(t)\|_{L^\infty} \leq \|v_0\|_X + C_1 \int_0^t (\|v(s)\|_X + \|\langle x \rangle^n D^\beta f(s)\|_{L^\infty}) \, ds. \tag{2.2} \]

(ii) If $|\beta| = \nu + \mu + 2m + 1$ with $0 \leq \nu \leq k + 1$ and $0 \leq \mu \leq n$, then
\[ \|\langle x \rangle^{n-\nu} D^\beta v(t)\|_{L^\infty} \leq \|v_0\|_X + C_1 \int_0^t (\|v(s)\|_X + \|\langle x \rangle^{n-\nu} D^\beta f(s)\|_{L^2}) \, ds. \tag{2.3} \]

We now recall several estimates of the nonlinearity $|v|^{\alpha} v$. Given $\ell \in \mathbb{N}$, we set
\[ \|u\|_{1,\ell} = \sup_{0 \leq |\beta| \leq \ell} \|\langle \cdot \rangle^n D^\beta u\|_{L^\infty} \tag{2.4} \]
\[ \|u\|_{2,\ell} = \begin{cases} \sup_{2m+1 \leq |\beta| \leq \ell} \|\langle \cdot \rangle^n D^\beta u\|_{L^2} & \ell \geq 2m + 1 \\ 0 & \ell \leq 2m \end{cases} \tag{2.5} \]
and
\[ \|u\|_{3,\ell} = \begin{cases} \sup_{2m+3+k \leq |\beta| \leq \ell} \|\langle \cdot \rangle^{j-\ell} D^\beta u\|_{L^2} & \ell \geq 2m + 3 + k \\ 0 & \ell \leq 2m + 2 + k \end{cases} \tag{2.6} \]

We have the following estimates of the nonlinearity.

**Proposition 2.3** ([5], Proposition 3.1). Assume (1.5)-(1.6) and let $X$ be defined by (1.7)-(1.8). Let $\alpha > 0$ and suppose that, in addition to (1.5), $n \geq \frac{N}{2\alpha}$. It follows that there exists a constant $C_2 \geq 1$ such that if $v \in X$ satisfies
\[ \eta \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v(x)| \geq 1, \tag{2.7} \]
for some $\eta > 0$, then the following estimates hold.

(i) If $0 \leq |\beta| \leq 1$, then
\[ \|\langle \cdot \rangle^n D^\beta (|v|^{\alpha} v)\|_{L^\infty} \leq C_2 \|v\|_{L^\infty} \|\langle \cdot \rangle^n D^\beta v\|_{L^\infty}. \tag{2.8} \]

(ii) If $2 \leq |\beta| \leq 2m$, then
\[ \|\langle \cdot \rangle^n D^\beta (|v|^{\alpha} v)\|_{L^\infty} \leq C_2 \|v\|_{L^\infty} \|\langle \cdot \rangle^n D^\beta v\|_{L^\infty} + C_2 \|v\|_{L^\infty} \|\eta\|_{1,|\beta|-1} 2^{[|\beta|]} \|v\|_{1,|\beta|-1}. \tag{2.9} \]

(iii) If $2m + 1 \leq |\beta| \leq 2m + 2 + k$, then
\[ \|\langle \cdot \rangle^n D^\beta (|v|^{\alpha} v)\|_{L^2} \leq C_2 \|v\|_{L^2} \|\langle \cdot \rangle^n D^\beta v\|_{L^2} + C_2 \|\eta\|_{1,2m} 2^{J+\alpha} \|v\|_{1,2m} \tag{2.10} \]
\[ + C_2 \|v\|_{L^2} \|\eta\|_{1,2m} 2^{J} \|v\|_{2,|\beta|-1} \]

(iv) If $2m + 3 + k \leq |\beta| \leq J$, then
\[ \|\langle \cdot \rangle^{J-|\beta|} D^\beta (|v|^{\alpha} v)\|_{L^2} \leq C_2 \|v\|_{L^2} \|\langle \cdot \rangle^{J-|\beta|} D^\beta v\|_{L^2} + C_2 \|\eta\|_{1,2m} 2^{J+\alpha} \|v\|_{1,2m} \tag{2.11} \]
\[ + C_2 \|v\|_{L^2} \|\eta\|_{1,2m} 2^{J} \|v\|_{2,2m+2+k} + \|v\|_{3,|\beta|-1}. \]
Remark 2.4. Estimates (2.9)–(2.11) are not exactly the estimates of [5, Proposition 3.1]. First, 1 + \eta∥v∥_{1,\ell} is replaced by \eta∥v∥_{1,\ell} (with \ell = |\beta| - 1 in (2.9) and \ell = 2m in (2.10) and (2.11)). The two quantities are indeed equivalent, since by (2.7), \eta∥v∥_{1,\ell} ≥ 1. Next, the term \eta∥v∥_{1,2m}^{2J+\alpha}∥v∥_{2,|\beta| - 1} in [5, formula (3.9)] is replaced in formula (2.10) here by \eta∥v∥_{1,2m}^{2J}∥v∥_{2,|\beta| - 1}. This is in fact what the proof in [5] shows, see in particular [5, formulas (3.24) and (3.25)]. Finally, the term \eta∥v∥_{1,2m}^{2J+\alpha}∥v∥_{2,|\beta|} in [5, formula (3.10)] is replaced in formula (2.11) here by \eta∥v∥_{L_\infty}^{2J}∥v∥_{2,2m+2+\alpha} + ∥v∥_{3,|\beta| - 1}. Again, this is what the proof in [5] shows, see in particular [5, formulas (3.24) and (3.25)]. The term \eta∥v∥_{L_\infty}^{2J} in these estimates is important in our proof of Proposition 3.2 below.

Finally, we recall the local well-posedness of (1.19) in the space \mathcal{X}, see [4, Theorem 1] and [5, Proposition 4.1].

Proposition 2.5. Assume (1.5)-(1.6) and let \mathcal{X} be defined by (1.7)-(1.8). Let \alpha > 0 and suppose that, in addition to (1.5), \eta ≥ \frac{N}{2\alpha}. Let \lambda \in \mathbb{C} and \sigma \geq 0. If \nu_0 \in \mathcal{X} satisfies

\inf_{x \in \mathbb{R}^N} \left(\langle x \rangle^n \nu_0(x)\right) > 0, \tag{2.12}

then there exist 0 < T < \frac{1}{\lambda} and a unique solution \nu \in C([0, T], \mathcal{X}) of (1.19) satisfying

\inf_{0 \leq t \leq T} \inf_{x \in \mathbb{R}^N} \left(\langle x \rangle^n \nu(t, x)\right) > 0. \tag{2.13}

Moreover, \nu can be extended on a maximal existence interval [0, T_{\text{max}}) with 0 < T_{\text{max}} ≤ \frac{1}{\lambda} to a solution \nu \in C([0, T_{\text{max}}), \mathcal{X}) satisfying (2.13) for all 0 < T < T_{\text{max}} and if T_{\text{max}} < \frac{1}{\lambda}, then

\|\nu(t)\|_{\mathcal{X}} + \left(\inf_{x \in \mathbb{R}^N} \left(\langle x \rangle^n \nu(t, x)\right)\right)^{-1} \xrightarrow{t \uparrow T_{\text{max}}} \infty. \tag{2.14}

3. Estimates for (1.19). Throughout this section, we assume (1.5)-(1.6) and we let \mathcal{X} be defined by (1.7)-(1.8). We derive estimates for certain solutions of (1.19).

We first introduce several indices and seminorms. Let

\begin{align*}
\sigma_0 &= 0, \\
\sigma_1 &= \frac{1}{4[4J(J - 2m - 1) + 4J + (4/N) + 1](8m + 1)^{2m}}, \tag{3.1}
\end{align*}

and set

\begin{align*}
\sigma_j &= \begin{cases} 
(8m + 1)^j \sigma_1, & 2 \leq j \leq 2m \\
\frac{2 - N\alpha}{2} + (4J + 2\alpha + 1) \sigma_{2m}, & j = 2m + 1 \\
4J \sigma_{2m}(j - 2m - 1) + \sigma_{2m+1}, & 2m + 2 \leq j \leq J.
\end{cases} \tag{3.2}
\end{align*}

It follows that

\begin{align*}
0 = \sigma_0 < \sigma_1 \leq \sigma_j < \sigma_k \leq \sigma_J, & \quad 1 \leq j < k \leq J. \tag{3.3}
\end{align*}

Moreover, it follows from (3.3) that

\begin{align*}
\sigma_J = [4J(J - 2m - 1) + 4J + 2\alpha + 1](8m + 1)^{2m} \sigma_1 + \frac{2 - N\alpha}{2}. \tag{3.4}
\end{align*}

We deduce from (3.5) and (3.2) that

\begin{align*}
\sigma_J \leq \frac{1}{2} & \quad \text{for } \alpha \in \left[\frac{3}{2N}, \frac{2}{N}\right]. \tag{3.6}
\end{align*}
Given $0 < T < \frac{1}{b}$ and $v \in C([0, T], \mathcal{X})$ satisfying (2.13), we define

\begin{align*}
\Phi_{1,T} &= \sup_{0 \leq t \leq T} \sup_{0 \leq J \leq 2^m} (1 - bt)^{\sigma_j} \|v\|_{1,j} \\
\Phi_{2,T} &= \sup_{0 \leq t \leq T} \sup_{2^m+1 \leq J \leq 2^{m+1}+k} (1 - bt)^{\sigma_j} \|v\|_{2,j} \\
\Phi_{3,T} &= \sup_{0 \leq t \leq T} \sup_{2^m+3+k \leq J \leq J} (1 - bt)^{\sigma_j} \|v\|_{3,j} \\
\Phi_{4,T} &= \sup_{0 \leq t \leq T} \inf_{x \in \mathbb{R}^N} \frac{\Im\langle x \rangle^n |v(t, x)|}{(1 - bt)^{\sigma_1}} \\
\end{align*}

where the norms $\| \cdot \|_{l,j}$ are given by (2.4)–(2.6), and we denote

\begin{align*}
\Phi_T &= \max\{\Phi_{1,T}, \Phi_{2,T}, \Phi_{3,T}\} \\
\Psi_T &= \max\{\Phi_T, \Phi_{4,T}\}. \\
\end{align*}

From these definitions, it is easy to verify that

\begin{align*}
\|v\|_{L^\infty(0, T), \mathcal{X}} &\leq C_3 (1 - bt)^{-\sigma_J} \Phi_T, \\
\|v(t)\|_{\ell,j} &\leq (1 - bt)^{-\sigma_j} \Psi_T, \quad \ell = 1, 2, 3 \\
\frac{1}{|\langle x \rangle^n|} v(t, x) &\leq (1 - bt)^{-\sigma_1} \Psi_T, \\
\end{align*}

where the constant $C_3 \geq 1$ is independent of $t \in [0, T]$.

**Lemma 3.1.** Suppose $\Re \lambda < 0$ and $\frac{3}{2N} \leq \alpha \leq \frac{2}{N}$. Let $K \geq 1$ and set

\begin{equation}
b_0 = \frac{16}{N} (4K)^{\frac{4}{2N}}. \\
\end{equation}

Let $b > 0$, let $v_0 \in \mathcal{X}$ satisfy (2.12), and let $v \in C([0, T_{\text{max}}], \mathcal{X})$ be the solution of (1.19) given by Proposition 2.5. If $v$ satisfies

\begin{equation}
\sup_{0 \leq t \leq T} \Psi_t \leq 4K, \\
\end{equation}

for some $0 < T < T_{\text{max}}$ and if $b \geq b_0$, then

\begin{equation}
\|v(t)\|_{L^\infty} \leq \frac{b(2 - N\alpha)}{2\alpha |\Re \lambda|} \frac{(1 - bt)^{-\frac{4-N\alpha}{2}}}{1 - (1 - bt)^{-\frac{4-N\alpha}{2}}}, \\
\end{equation}

for all $0 < t \leq T$.

**Proof.** Multiplying (1.19) by $\pi$, taking the real part and using that $|v| > 0$ on $[0, T_{\text{max}}] \times \mathbb{R}^N$ by Proposition 2.5, we obtain

\begin{equation}
\partial_t |v| = L + \Re \lambda (1 - bt)^{-\frac{4-N\alpha}{2}} |v|^{\alpha+1} \\
\end{equation}

where

\begin{equation}
L(t, x) = -\frac{\Im (\pi \Delta v)}{|v|}; \\
\end{equation}

and so

\begin{equation}
-\frac{1}{\alpha} \partial_t (|v|^{-\alpha}) = |v|^{-\alpha-1} L + \Re \lambda (1 - bt)^{-\frac{4-N\alpha}{2}}. \\
\end{equation}
Integrating (3.21) in $t$, we obtain
\[
\frac{1}{|v(t,x)|^\alpha} = \frac{1}{|v_0(x)|^\alpha} + \frac{2\alpha|\Re \lambda|}{(2-N\alpha)b} \left[ (1-bt)^{-\frac{2-N\alpha}{2}} - 1 \right] - \alpha \int_0^t |v(s,x)|^{-\alpha-1} L(s,x) \, ds
\]
so that
\[
|v(t,x)|^\alpha = \frac{|v_0(x)|^\alpha}{1 + f(t,x) + \frac{2\alpha|\Re \lambda|}{(2-N\alpha)b}|v_0(x)|^\alpha[(1-bt)^{-\frac{2-N\alpha}{2}} - 1]}
\] (3.22)
where
\[
f(t,x) = -\alpha \int_0^t |v_0(x)|^\alpha |v(s,x)|^{-\alpha-1} L(s,x) \, ds.
\] (3.23)
It follows from the definitions of $\Psi_T$ and $L(t,x)$ that, for any $0 \leq s \leq t \leq T$
\[
|v_0(x)|^\alpha |v(s,x)|^{-\alpha-1} |L(s,x)| \leq (\langle x \rangle^\alpha |v_0|) (\langle x \rangle^n |v|)^{-\alpha-1} (\langle x \rangle^n |\Delta v|)
\]
\[
\leq (4K)^{2\alpha+2}(1-bs)^{-(\alpha+1)}\sigma_1 - \sigma_2
\] (3.24)
where in the last inequality we used
\[(\alpha + 1)\sigma_1 + \sigma_2 \leq (\alpha + 2)\sigma_2 \leq 4\sigma_2 \leq \sigma_3\]
by (3.3). Using $\sigma_3 \leq \sigma_j \leq \frac{1}{2}$ (see (3.6)), we obtain
\[
|f(t,x)| \leq \alpha \int_0^t (4K)^{2\alpha+2}(1-bs)^{-\frac{\alpha}{2}} \, ds \leq \frac{2\alpha(4K)^{2\alpha+2}}{b}.
\] (3.25)
We deduce from (3.25) and (3.16) that
\[
\frac{3}{2} \geq 1 + f(t,x) \geq \frac{1}{2},
\] (3.26)
for $b \geq b_0$, $0 \leq t \leq T$ and $x \in \mathbb{R}^N$. In particular, $1 + f \geq 0$ and estimate (3.18) follows.

**Proposition 3.2.** Suppose $\Re \lambda < 0$. Given $K \geq 1$, let $\alpha_1 \in (\max\{\frac{3}{2N}, \frac{2}{N+1}\}, \frac{2}{N})$ be given by
\[
\frac{12C_1C_2(4K)^{4J+1} |\lambda|}{\sigma_1 |\Re \lambda|} \left( \frac{2}{\alpha_1} - N \right) = 1,
\] (3.27)
and let
\[
b_1 = \max\left\{ \frac{16}{N}(4K)^{\frac{N}{2}+2}, 8C_3, \frac{32(4K)^{4J+4} |\lambda| C_1 C_2}{\sigma_1}, \frac{2^{\frac{2}{N}+3}\alpha(4K)^2}{3^{N+1}} \right\},
\] (3.28)
where $\sigma_1$ is given by (3.2), $C_1$ by Proposition 2.2, $C_2$ by Proposition 2.3, and $C_3$ by (3.13). If $v_0 \in \mathcal{X}$ satisfies (1.10), then for every $\alpha \in [\alpha_1, \frac{2}{N}]$ and $b \geq b_1$, the corresponding solution $v \in C([0,T_{\max}), \mathcal{X})$ of (1.19) given by Proposition 2.5 satisfies $T_{\max} = \frac{1}{b}$ and
\[
\sup_{0 \leq t < \frac{1}{b}} \Psi_T \leq 4K.
\] (3.29)
\textbf{Proof.} We set
\begin{equation}
T^* = \sup\{0 \leq T < T_{\text{max}}; \Psi_T \leq 4K\}. \tag{3.30}
\end{equation}
Since \(\Psi_0 \leq K\) and \(v \in C([0, T_{\text{max}}), \mathcal{X})\), we see that \(0 < T^* \leq T_{\text{max}}\). We claim that if \(\alpha \in \left(\frac{N}{2}, \frac{N}{2}\right)\) and \(b > b_1\), then
\begin{equation}
T^* = T_{\text{max}}. \tag{3.31}
\end{equation}
We note that, since \(\alpha \geq \alpha_1 \geq \frac{2}{N+\beta}\), the second condition in (1.5) implies that \(n > \max\{\frac{N}{\beta}, \frac{N}{b}\}\), so that we may apply Propositions 2.3 and 2.5. Assuming (3.31), it follows from (3.13), (3.15) and (3.30) that for any \(T \in [0, T_{\text{max}}]\)
\begin{equation}
\|v(t)\|_X \left(\inf_{x \in \mathbb{R}^N} \langle x \rangle^n \|v(t, x)\|\right)^{-1} \leq C K (1 - bt)^{-\sigma_j}. \tag{3.32}
\end{equation}
If \(T_{\text{max}} < \frac{1}{b}\), then we deduce from (3.32) that
\begin{equation}
\sup_{0 \leq t < T_{\text{max}}} \left(\|v(t)\|_X \left(\inf_{x \in \mathbb{R}^N} \langle x \rangle^n \|v(t, x)\|\right)^{-1}\right) < \infty,
\end{equation}
which contradicts the blowup alternative (2.14). Therefore, we have \(T^* = T_{\text{max}} = \frac{1}{b}\), and (3.29) follows.

Now we prove the claim (3.31). We assume by contradiction that
\begin{equation}
T^* < T_{\text{max}}, \tag{3.33}
\end{equation}
then by the definition of \(T^*\), we have
\begin{equation}
\Psi_{T^*} = 4K. \tag{3.34}
\end{equation}
It follows from (3.13), (3.34) and (3.6) that
\begin{equation}
\int_0^{T^*} \|v(s)\|_X \, ds \leq 4KC_3 \int_0^{T^*} (1 - bs)^{-\sigma_j} \, ds \leq \frac{8KC_3}{b}. \tag{3.35}
\end{equation}
Using also (1.10) and (3.28), we see that
\begin{equation}
\|v_0\|_X + \int_0^{T^*} \|v(s)\|_X \, ds \leq 2K. \tag{3.36}
\end{equation}
Next, we set
\begin{equation}
\eta(t) = 4K(1 - bt)^{-\sigma_1}, \tag{3.37}
\end{equation}
so that (by definition of \(\Phi_{4,T^*}\))
\begin{equation}
\inf_{0 \leq t \leq T^*} \left\{\eta(t) \inf_{x \in \mathbb{R}^N} \langle x \rangle^n \|v(t, x)\|\right\} \geq 1. \tag{3.38}
\end{equation}
If \(2 \leq |\beta| \leq 2m\), we deduce from (3.37), (3.14) and (3.34) that
\begin{equation}
(\eta \|v\|_{1,|\beta|-1})^{2|\beta|} \|v\|_{1,|\beta|-1} \leq (4K(1 - bt)^{-\sigma_1} 4K(1 - bt)^{-(\sigma_1|\beta|-1)} 2|\beta| 4K(1 - bt)^{-(\sigma_1|\beta|-1)} \tag{3.39}
\end{equation}
\begin{equation}
\leq (4K)^{4|\beta|+1} (1 - bt)^{-(\sigma_1+|\beta|-1)} 2|\beta|-|\sigma_1| \leq (4K)^{8m+1} (1 - bt)^{-|\sigma_1|},
\end{equation}
since \((\sigma_1+|\beta|-1)|\beta|+|\sigma_1|-1 \leq (8m+1)|\sigma_1| = |\sigma_1| \) by (3.3). Similarly, if \(2 \leq |\beta| \leq 2m\), we deduce from (3.37), (3.14) and (3.34) that
\begin{equation}
(\eta \|v\|_{1,2m})^{2J+\alpha} \|v\|_{1,2m} \leq (4K)^{4J+2\alpha+1} (1 - bt)^{-(2J+\alpha)(|\sigma_1|+2m)} \tag{3.40}
\end{equation}
\begin{equation}
\leq (4K)^{4J+2\alpha+1} (1 - bt)^{-(4J+2\alpha+1)2m} = (4K)^{4J+2\alpha+1} (1 - bt)^{\frac{N\alpha}{2} - \sigma_2m+1}.
\end{equation}
where the last equality follows from the definition of $\sigma_{2m+1}$ by (3.3). As well, if $2m + 2 \leq |\beta| \leq J$ and $\ell \in \{2, 3\}$, then
\[
(\eta\|v\|_{L^{2m}})^2 \|v\|_{L^{(\ell+1)|\beta|-1}} \leq (4K)^{4J+1}(1 - bt)^{-2J(\sigma_1 + \sigma_{2m}) - |\sigma| - 1}
\leq (4K)^{4J+1}(1 - bt)^{-4J\sigma_{2m} - |\sigma| - 1}
\leq (4K)^{4J+1}(1 - bt)^{-\sigma|\beta|}
\]
where we used $2J(\sigma_1 + \sigma_{2m}) + \sigma|\beta| - 1 \leq 4J\sigma_{2m} + |\sigma| = |\sigma|$ by (3.3).

Since $\|v\|_{L^\infty} \leq 4K$ by (3.14), it follows from (1.21) that, given any $\sigma > 0$ and $0 \leq t < \frac{1}{4}$,
\[
\int_0^t (1 - bs)^{-\frac{4 - N\alpha}{2} - \sigma}\|v(s)\|_{L^\infty}^\sigma ds \leq \frac{2(4K)^\alpha}{b(2 - N\alpha + 2\sigma)}(1 - bt)^{-\frac{4 - N\alpha}{2} - \sigma}
\leq \frac{2(4K)^\alpha}{b\sigma}(1 - bt)^{-\frac{4 - N\alpha}{2} - \sigma}.
\]

Let $0 < t' < \frac{1}{4}$ be defined by $(1 - bt')^\frac{4 - N\alpha}{2} = \frac{1}{2}$, i.e. $(1 - bt') = 2^{-\frac{4 - N\alpha}{4}}$. It follows from the above inequality that if $0 \leq t \leq t'$, then
\[
\int_0^t (1 - bs)^{-\frac{4 - N\alpha}{2} - \sigma}\|v(s)\|_{L^\infty}^\sigma ds \leq \frac{2(4K)^\alpha}{b\sigma}(1 - bt)^{-\sigma}.
(3.42)
\]
Moreover, if $b \geq b_0$ and $t' \leq t < T^*$, then it follows from (3.18) and (1.21) that
\[
\int_{t'}^t (1 - bs)^{-\frac{4 - N\alpha}{2} - \sigma}\|v(s)\|_{L^\infty}^\sigma ds \leq \frac{b(2 - N\alpha)}{2\sigma|\Re\lambda|} \int_{t'}^t (1 - bs)^{-\sigma} ds \leq \frac{2 - N\alpha}{\sigma\alpha|\Re\lambda|}(1 - bt)^{-\sigma}.
(3.43)
\]
Using $\int_0^t = \int_0^{t'} + \int_{t'}^t$ if $t' < t < T^*$, we deduce from (3.42) and (3.43) that for all $b \geq b_0$ and all $0 \leq t < T^*$,
\[
\int_0^t (1 - bs)^{-\frac{4 - N\alpha}{2} - \sigma}\|v(s)\|_{L^\infty}^\sigma ds \leq \left(\frac{2(4K)^\alpha}{b\sigma} + \frac{2 - N\alpha}{\sigma\alpha|\Re\lambda|}\right)(1 - bt)^{-\sigma}.
(3.44)
\]
Now, we are ready to estimate $\Psi_{T^*}$ and the process is divided into four steps. We first estimate $\|\langle x \rangle^n v\|_{L^\infty}$. Since $\Re\lambda < 0$, it follows from (3.19) and (3.20) that
\[
\partial_t|v| \leq |L| \leq |\Delta v|,
\]
so that
\[
\langle x \rangle^n|v(t)| - \langle x \rangle^n|v_0| \leq \int_0^t \langle x \rangle^n|\Delta v| ds \leq \int_0^t \|v\|_{L^\infty} ds \leq \frac{8KC_3}{b},
\]
where we used (3.35) in the last inequality. Since $\langle x \rangle^n|v_0| \leq K$, we deduce that if $b \geq b_1$ with $b_1$ given by (3.28), then
\[
\|\langle x \rangle^n v\|_{L^\infty} \leq 2K.
(3.45)
\]
We next estimate $\|\langle x \rangle^n D^\beta v\|_{L^\infty}$ for $1 \leq |\beta| \leq 2m$. Applying (2.2) and (3.36), we obtain
\[
\|\langle x \rangle^n D^\beta v\|_{L^\infty} \leq 2K + |\lambda|C_1 \int_0^t (1 - bs)^{-\frac{4 - N\alpha}{2}}\|\langle x \rangle^n D^\beta (\langle v \rangle^\alpha v)\|_{L^\infty} ds.
(3.46)
\]
Using (3.38), (2.8)-(2.9), (3.14) and (3.39) and setting $\kappa = 0$ if $|\beta| = 1$ and $\kappa(\beta) = 1$ if $|\beta| \geq 2$, we see that
\[
\| \langle \cdot \rangle^n D^\beta (|v|^\alpha v) \|_{L^\infty} \leq C_2 \| |v|^\alpha \|_{L^\infty} \| \langle \cdot \rangle^n D^\beta v \|_{L^\infty} + \kappa C_2 \| v \|_{L^\infty} (\eta \| v \|_{L^1(1,|\beta|^{-1})})^{|2|\beta|} \| v \|_{L^1(|\beta|^{-1})} \leq 2C_2 (4K)^{2|m+1|} \| v \|_{L^\infty} (1 - bs)^{-\sigma(\beta)}. \tag{3.47}
\]
Applying now (3.44) and using $\sigma_{|\beta|} \geq \sigma_1$, we deduce from (3.46)-(3.47) that
\[
\| \langle x \rangle^n D^\beta v \|_{L^\infty} \leq 2K + 2(4K)^{2|m+1|} |\lambda| C_1 C_2 \left( \frac{2(4K)^{\alpha}}{b\sigma_1} + \frac{2 - N\alpha}{\sigma_1 \alpha |R\lambda|} \right) (1 - bt)^{-\sigma_{|\beta|}}.
\]
It follows, using also (3.27), (3.28) and (3.45), that
\[
\Phi_{1,T^*} \leq 3K. \tag{3.48}
\]
We next estimate similarly $\| \langle x \rangle^n D^\beta v \|_{L^2}$ for $2m + 1 \leq |\beta| \leq 2m + 2 + k$. It follows from (2.3) (with $\mu = 0$) and (3.36) that
\[
\| \langle x \rangle^n D^\beta (|v|^\alpha v) \|_{L^2} \leq 2K + C_1 |\lambda| \int_0^t (1 - bs)^{-\frac{4-N\alpha}{2}} \| \langle x \rangle^n D^\beta (|v|^\alpha v) \|_{L^\infty} ds.
\]
Using (2.10), (3.34), (3.40) and (3.41), we see that
\[
\| \langle x \rangle^n D^\beta (|v|^\alpha v) \|_{L^2} \leq C_2 (4K)^2 (1 - bs)^{-\sigma_{|\beta|}} \| v \|_{L^\infty}^2 + C_2 (4K)^2 J^{2 + 2\alpha + 1} (1 - bs)^{2 - \sigma_{2m+1}} + C_2 (4K)^2 J^{1 + 2\alpha + 1} (1 - bs)^{-\sigma_{|\beta|}} \| v \|_{L^\infty}^2,
\]
so that
\[
\| \langle x \rangle^n D^\beta v \|_{L^2} \leq 2K + C_1 C_2 (4K)^2 J^{2 + 2\alpha + 1} |\lambda| \int_0^t (1 - bs)^{-\sigma_{2m+1}} ds + C_1 C_2 (4K)^2 J^{1 + 2\alpha + 1} |\lambda| \int_0^t (1 - bs)^{-\sigma_{|\beta|}} \| v \|_{L^\infty}^2 ds.
\]
Applying (1.21) and (3.44) to estimate the integrals, we obtain
\[
\| \langle x \rangle^n D^\beta v \|_{L^2} \leq 2K + C_1 C_2 (4K)^2 J^{2 + 2\alpha + 1} |\lambda| (1 - bt)^{-\sigma_{2m+1}} + C_1 C_2 (4K)^2 J^{1 + 2\alpha + 1} |\lambda| \left( \frac{2(4K)^{\alpha}}{b\sigma_1} + \frac{2 - N\alpha}{\sigma_1 \alpha |R\lambda|} \right) (1 - bt)^{-\sigma_{|\beta|}}. \tag{3.49}
\]
Using $\sigma_1 \leq \sigma_{2m+1} \leq \sigma_{|\beta|}$, it follows that
\[
\| \langle x \rangle^n D^\beta v \|_{L^2} \leq 2K + C_1 C_2 (4K)^2 J^{2 + 2\alpha + 1} |\lambda| (1 - bt)^{-\sigma_{|\beta|}} + C_1 C_2 (4K)^2 J^{1 + 2\alpha + 1} |\lambda| \left( \frac{2(4K)^{\alpha}}{b\sigma_1} + \frac{2 - N\alpha}{\sigma_1 \alpha |R\lambda|} \right) (1 - bt)^{-\sigma_{|\beta|}}.
\]
Using also (3.27) and (3.28), we conclude that
\[
\Phi_{2,T^*} \leq 3K. \tag{3.50}
\]
Now we estimate $\| \langle x \rangle^{J-|\beta|} D^\beta v \|_{L^2}$ for $2m + k + 3 \leq |\beta| \leq J$. It follows from (2.3) (with $\mu = n + |\beta| - J, \nu = k + 1$) and (3.36) that
\[
\| \langle x \rangle^{J-|\beta|} D^\beta v \|_{L^2} \leq 2K + C_1 |\lambda| \int_0^t \| \langle x \rangle^{J-|\beta|} D^\beta (|v|^\alpha v) \|_{L^2} ds.
\]
Using (2.11), (3.34), (3.40) and (3.41), we see that
\[
\|\langle x \rangle^{J - |\beta|} D^\beta (|v|^\alpha v)\|_{L^2} \leq C_2 (4K) (1 - bs)^{-\sigma_1} \|v\|_{L^\infty}^2 + C_2 (4K)^{4J + 2m + 1} (1 - bs) \frac{2N_\alpha}{2 - N_\alpha} + 2C_2 (4K)^{4J + 1} (1 - bs)^{-\sigma_1} \|v\|_{L^\infty}^2,
\]
so that
\[
\|\langle x \rangle^{J - |\beta|} D^\beta v\|_{L^2} \leq 2K + C_1 C_2 (4K)^{4J + 2m + 1} |\lambda| \int_0^t (1 - bs)^{-1 - \sigma_2} ds
\]
\[
+ C_1 C_2 (4K)^{4J + 2} |\lambda| \int_0^t (1 - bs)^{-\frac{2N_\alpha}{2 - N_\alpha} - \sigma_1} \|v\|_{L^\infty}^2 ds.
\]
(3.51)
The right-hand side of (3.51) is similar to the right-hand side of (3.49), and we conclude as above that
\[
\Phi_{3,T*} \leq 3K.
\]
(3.52)
Finally, we estimate \(\Phi_{4,T*}\), we set
\[
w(t, x) = \langle x \rangle^n |v(t, x)|.
\]
Multiplying (3.21) by \(\langle x \rangle^{-n_\alpha}\) and integrating in \(t\), we obtain
\[
\frac{1}{|w(t, x)|^\alpha} \geq \frac{1}{|w(0, x)|^\alpha} + \alpha |\Re \lambda| \int_0^t \langle x \rangle^{-n_\alpha} (1 - bs)^{-\frac{2N_\alpha}{2 - N_\alpha} - \sigma_1} ds
\]
\[
- \alpha \int_0^t |w(s, x)|^{-\alpha - 1} \langle x \rangle^n L(s, x) ds.
\]
Applying (3.34), we see that \(\langle x \rangle^n |L| \leq (\langle x \rangle^n |\Delta v| \leq 4K (1 - bs)^{-\sigma_2}\), and \(|w|^{-\alpha - 1} \leq (4K)^{\alpha + 1} (1 - bs)^{-(\alpha - 1)\sigma_1}\). Since \((\alpha + 1)\sigma_1 + \sigma_2 \leq \sigma_3\) by (3.3), we deduce that
\[
\frac{1}{|w(t, x)|^\alpha} \leq K^\alpha + \alpha |\Re \lambda| \int_0^t (1 - bs)^{-\frac{2N_\alpha}{2 - N_\alpha} - \sigma_1} ds + \alpha (4K)^{\alpha + 2} \int_0^t (1 - bs)^{-\sigma_3} ds.
\]
Since \(-\frac{2N_\alpha}{2 - N_\alpha} = -1 - \alpha \sigma_1 + (\alpha \sigma_1 - \frac{2N_\alpha}{2 - N_\alpha})\), and since by (3.27) \(\alpha \sigma_1 \geq \frac{2N_\alpha}{2 - N_\alpha}\), we see that \(-\frac{2N_\alpha}{2 - N_\alpha} \geq -1 - \alpha \sigma_1\); and so, using (1.21) and \(\sigma_3 \leq \sigma_J \leq \frac{1}{2}\),
\[
\frac{1}{|w(t, x)|^\alpha} \leq K^\alpha + |\Re \lambda| \int_0^t (1 - bs)^{-1 - \alpha \sigma_1} ds + \alpha (4K)^{\alpha + 2} \int_0^t (1 - bs)^{-\sigma_3} ds
\]
\[
\leq K^\alpha + \frac{|\Re \lambda|}{b\sigma_1} (1 - bt)^{-\alpha \sigma_1} + \frac{2\alpha (4K)^{\alpha + 2}}{b}.
\]
It follows that
\[
\Phi_{4,T*}^0 \leq K^\alpha + \frac{|\Re \lambda|}{b\sigma_1} + \frac{2\alpha (4K)^{\alpha + 2}}{b}.
\]
Using (3.28), we deduce that for \(b \geq b_1\),
\[
\Phi_{4,T*}^0 \leq 3^\frac{\alpha}{\sigma} K^\alpha \leq (3K)^\alpha,
\]
(3.53)
since \(\alpha \geq \frac{1}{\sigma}\). Estimates (3.48), (3.50), (3.52) and (3.53) yield \(\Psi_{T*} \leq 3K\), which leads to a contradiction with (3.34). This completes the proof. \(\square\)
4. Asymptotics for (1.19). Throughout this section, we assume (1.5)-(1.6) and we let \( X \) be defined by (1.7)-(1.8). We describe the asymptotic behavior as \( t \to \frac{1}{b} \) of the solutions of (1.19) given by Proposition 3.2. More precisely, we have the following result.

**Proposition 4.1.** Suppose \( \Re \lambda < 0 \). Let \( K \geq 1 \), let \( \alpha_1 \in (0, \frac{2}{N}) \) be given by (3.27) and let \( b_1 > 0 \) be given by (3.28). Let \( v_0 \in X \) satisfy (1.10), and let \( v \in C([0, \frac{1}{b}], X) \) be the solution of (1.19) given by Proposition 3.2. There exist \( f_0, \omega_0 \in L^\infty(\mathbb{R}^N) \), with \( f_0 \) real-valued, \( \|f_0\|_{L^\infty} \leq \frac{2}{b} \), and \( \langle \rangle^\alpha \omega_0 \in L^\infty(\mathbb{R}^N) \), such that

\[
\|\langle \rangle^\alpha (v(t) - \omega_0 \psi(t) e^{i\theta(t)})\|_{L^\infty} \leq C(1 - bt)^{\frac{1}{2}}
\]

for all \( t \in [0, \frac{1}{b}] \), where

\[
\psi(t, x) = \left( \frac{1 + f_0(x)}{1 + f_0(x) + \frac{2\alpha_i|\Re \lambda|}{b(2 - N\alpha)} |v_0(x)|^\alpha ((1 - bt)^{-\frac{2-N\alpha}{2}} - 1) } \right) ^{\frac{1}{2}}
\]

and

\[
\theta(t, x) = \frac{\Re \lambda}{\Re \lambda} \log(\psi(t, x)).
\]

Moreover,

\[
|\omega_0|^\alpha = \frac{|v_0|^\alpha}{1 + f_0},
\]

so that \( \frac{2}{b} |v_0|^\alpha \leq |\omega_0|^\alpha \leq 2 |v_0|^\alpha \). In addition,

\[
(1 - bt)^{-\frac{b - N\alpha}{2}} \|v(t)\|_{L^\infty} \to \frac{b(2 - N\alpha)}{2\alpha_i|\Re \lambda|} \text{ as } t \to \frac{1}{b},
\]

where \( n \) is given by (1.5).

**Proof.** We let \( f \) be defined by (3.23). It follows from (3.24) that \( f(t, \cdot) \) is convergent in \( L^\infty(\mathbb{R}^N) \) as \( t \uparrow \frac{1}{b} \). Then \( f \) can be extended to a continuous function \([0, \frac{1}{b}] \to L^\infty(\mathbb{R}^N) \) and we set

\[
f_0(x) = f \left( \frac{1}{b} x \right) = -\alpha \int_0^{\frac{1}{b}} |v_0(x)|^\alpha |v(s, x)|^{-\alpha - 1} L(s, x) ds.
\]

By using (3.24), (3.25), (3.16) and \( \sigma_3 \leq \sigma_j \leq \frac{1}{2} \) (see (3.6)), we have for all \( 0 \leq t \leq \frac{1}{b} \)

\[
\|f(t) - f_0\|_{L^\infty} \leq \frac{1}{4} (1 - bt)^{1 - \sigma_3},
\]

\[
\|f(t)\|_{L^\infty} \leq \frac{1}{4}.
\]

In particular, \( 1 + f_0 > 0 \), so that by (4.2),

\[
0 < \psi(t, x) \leq 1
\]

for all \( 0 \leq t < \frac{1}{b} \) and \( x \in \mathbb{R}^N \). Moreover, it follows from (4.8) that

\[
\left\| \frac{1}{1 + f_0(x) + \frac{2\alpha_i|\Re \lambda|}{b(2 - N\alpha)} |v_0(x)|^\alpha ((1 - bt)^{-\frac{2-N\alpha}{2}} - 1) \right\|_{L^\infty} \leq 2
\]
for all $0 \leq t < \frac{1}{b}$. We set

$$\vec{v}(t, x) = \left(1 + f_0(x) + \frac{2\alpha|RL|}{b(2 - N\alpha)}|v_0(x)|^\alpha[(1 - bt)^{-\frac{2 - N\alpha}{2}} - 1]\right)^{\frac{1}{\sigma}}.$$ \hspace{1cm} (4.11)

It follows from (1.10) and (4.10) that

$$\|\langle x \rangle^{\alpha} \vec{v}(t)\|_{L^\infty} \leq 2^{\frac{1}{\sigma}} K,$$

and we deduce from (3.22), (4.7) and (4.10) that

$$\|\langle x \rangle^{\alpha} (|v(t, \cdot)|^\alpha - \vec{v}(t, \cdot)|^\alpha)\|_{L^\infty} \leq \|\langle x \rangle^{\alpha} v_0\|_{L^\infty}(1 - bt)^{1 - \sigma_3} \leq K^\alpha(1 - bt)^{1 - \sigma_3}$$ \hspace{1cm} (4.12)

for all $0 \leq t < \frac{1}{b}$. Next, we introduce the decomposition

$$v(t, x) = \omega(t, x)\psi(t, x)e^{i\theta(t, x)},$$ \hspace{1cm} (4.13)

where $\psi(t, x)$ and $\theta(t, x)$ are defined by (4.2) and (4.3) respectively. Differentiating (4.13) with respect to $t$, we obtain

$$\partial_t \omega = \frac{e^{-i\theta}}{\psi} \partial_t v - \omega \frac{\partial_t \psi}{\psi} - i\omega \partial_t \theta.$$ \hspace{1cm} (4.14)

On the other hand, it follows easily from (4.2), (4.3) and (4.11) that

$$\frac{\partial_t \psi}{\psi} = \Re\lambda(1 - bt)^{-\frac{4 - N\alpha}{2}} \bar{v}^\alpha,$$

$$\partial_t \theta = \Im\lambda(1 - bt)^{-\frac{4 - N\alpha}{2}} \bar{v}^\alpha.$$

Therefore, we deduce from (4.14), (4.13) and (1.19) that

$$\partial_t \omega = \frac{e^{-i\theta}}{\psi} \partial_t v - \omega(1 - bt)^{-\frac{4 - N\alpha}{2}} \lambda \bar{v}^\alpha$$

$$= \frac{e^{-i\theta}}{\psi} (\partial_t v - \lambda(1 - bt)^{-\frac{4 - N\alpha}{2}} \bar{v}^\alpha v)$$ \hspace{1cm} (4.15)

$$= \frac{e^{-i\theta}}{\psi} (i\Delta v + \lambda(1 - bt)^{-\frac{4 - N\alpha}{2}} (|v|^\alpha - \bar{v}^\alpha) v).$$

Next, it follows from (4.2) and the property $1 + f_0 \geq \frac{1}{2}$ that

$$\psi(t, x)^{-\alpha} \leq 1 + \frac{2\alpha|RL|}{b(2 - N\alpha)} |v_0(x)|^\alpha(1 - bt)^{-\frac{2 - N\alpha}{2}}.$$ \hspace{1cm} (4.16)

Moreover, we deduce from (3.18) that if $t' \leq t < \frac{1}{b}$ where $t' \in (0, \frac{1}{b})$ is defined by $(1 - bt')^{\frac{2 - N\alpha}{2}} = \frac{1}{2}$, then

$$|v(t, x)|^\alpha \leq \frac{b(2 - N\alpha)}{\alpha|RL|}(1 - bt)^{\frac{2 - N\alpha}{2}},$$

hence

$$(1 - bt)^{-\frac{2 - N\alpha}{2}} \leq \frac{b(2 - N\alpha)}{\alpha|RL|} |v(t, x)|^{-\alpha}.$$ \hspace{1cm} (4.16)

Therefore, it follows from (4.16) that $\psi(t, x)^{-\alpha} \leq 1 + 2|v_0(x)|^\alpha |v(t, x)|^{-\alpha}$. Applying (3.15), (1.10) and (3.29), we conclude that

$$\psi(t, x)^{-\alpha} \leq C(1 - bt)^{-\alpha \sigma_1}.$$ \hspace{1cm} (4.17)
for $t' \leq t < \frac{1}{5}$. It follows from (4.15) and (4.17) that
\[
\|\langle \cdot \rangle^n \partial_t \omega \|_{L^\infty} \leq C(1-bt)^{-\sigma_1} \|\|\langle \cdot \rangle^n \Delta v\|_{L^\infty} + (1-bt)^{-\frac{2-N\alpha}{2}} \|\|v\|^{\alpha} - \bar{v}\|_{L^\infty} \|\langle \cdot \rangle^n v\|_{L^\infty}\|.
\]
Since $\|\langle \cdot \rangle^n \Delta v\|_{L^\infty} \leq 4K(1-bt)^{-\sigma_2}$ and $\|\langle \cdot \rangle^n v\|_{L^\infty} \leq 4K$ by (3.29), we deduce using (4.12) that
\[
\|\langle \cdot \rangle^n \partial_t \omega \|_{L^\infty} \leq C(1-bt)^{-\sigma_1} \left[(1-bt)^{-\sigma_2} + (1-bt)^{-\frac{2-N\alpha}{2}}\right] \leq C(1-bt)^{-\frac{1}{2}},
\]
(4.18) where we used $\sigma_1 + \sigma_2 \leq \sigma_1 + \frac{2-N\alpha}{2} + \sigma_3 \leq \sigma_j \leq \frac{1}{2}$ by (3.3) and (3.6). It follows from (4.18) that if $t' \leq t < \frac{1}{5}$, then
\[
\|\langle \cdot \rangle^n (\omega(t) - \omega(s))\|_{L^\infty} \leq C(1-bt)^{\frac{1}{2}},
\]
so that there exists $\omega_0$ such that $\langle x \rangle^n \omega_0 \in L^\infty(\mathbb{R}^N)$ and
\[
\|\langle \cdot \rangle^n (\omega(t) - \omega_0)\|_{L^\infty} \leq C(1-bt)^{\frac{1}{2}}
\]
(4.19) for all $t' \leq t < \frac{1}{5}$. Using (4.13), (4.9) and (4.19), we obtain
\[
\|\langle \cdot \rangle^n (v(t) - \omega_0 \psi(t) e^{i\theta(t)})\|_{L^\infty} \leq \|\langle \cdot \rangle^n (\omega(t) - \omega_0)\|_{L^\infty} \|\psi\|_{L^\infty} \leq C(1-bt)^{\frac{1}{2}},
\]
which proves (4.1).

Next, we prove (4.4). It follows from (4.1) (recall that $0 \leq \psi \leq 1$) that
\[
\| |v(t)| - |\omega_0 \psi(t)| \|_{L^\infty} \leq C(1-bt)^{\frac{1}{2}}.
\]
Using the elementary inequalities $|x^{\alpha} - y^{\alpha}| \leq |x - y|^\alpha$ if $\alpha \leq 1$ and $|x^{\alpha} - y^{\alpha}| \leq \alpha(x^{\alpha-1} + y^{\alpha-1})|x - y|$ if $\alpha \geq 1$, and the boundedness of $\|\langle \cdot \rangle^n v\|_{L^\infty}$, we deduce that
\[
\|v(t, t')\|^{\alpha} - (|\omega_0| \psi(t))^{\alpha}\|_{L^\infty} \leq C((1-bt)^{\frac{1}{2}} + (1-bt)^{\frac{3}{2}})
\]
Moreover, it follows from (4.12) and $\sigma_3 \leq \frac{1}{2}$ that
\[
\|v(t, t')\|^{\alpha} - \bar{v}(t, \cdot)^{\alpha}\|_{L^\infty} \leq K^\alpha(1-bt)^{\frac{1}{2}}.
\]
Thus we see that
\[
\|v(t, t')\|^{\alpha} - (|\omega_0| \psi(t))^{\alpha}\|_{L^\infty} \leq C(1-bt)^{\frac{1}{2}},
\]
where $\rho = \min\{\alpha, 1\}$. Using the explicit expressions (4.2) and (4.11), we obtain
\[
\frac{\|v_0(x)\|^{\alpha} - (1 + f_0(x))|\omega_0(x)|^{\alpha}|1 + f_0(x) + \frac{2\alpha |\mathcal{R}\lambda|}{b(2-N\alpha)}|v_0(x)|^{\alpha}|[(1-bt)^{-\frac{2-N\alpha}{2}} - 1]}{1 + f_0(x) + \frac{2\alpha |\mathcal{R}\lambda|}{b(2-N\alpha)}|v_0(x)|^{\alpha}|[(1-bt)^{-\frac{2-N\alpha}{2}} - 1]} \leq C(1-bt)^{\frac{1}{2}}.
\]
For $\frac{1}{2b} \leq t < \frac{1}{5}$, we have
\[
1 + f_0(x) + \frac{2\alpha |\mathcal{R}\lambda|}{b(2-N\alpha)}|v_0(x)|^{\alpha}|[(1-bt)^{-\frac{2-N\alpha}{2}} - 1] \leq C(1-bt)^{\frac{1}{2} - \frac{2-N\alpha}{2}},
\]
so that
\[
|v_0(x)|^{\alpha} - (1 + f_0(x))|\omega_0(x)|^{\alpha}| \leq C(1-bt)^{\frac{1}{2} - \frac{2-N\alpha}{2} - \frac{2-N\alpha}{2}}.
\]
since $\alpha > \min\{\frac{1}{N}, \frac{2}{N+1}\}$ by (3.27), we see that $\frac{1}{2} - \frac{2-N\alpha}{2} > 0$. Letting $t' \uparrow \frac{1}{5}$ in the above inequality, we obtain (4.4).

Now, we prove (4.5). Set
\[
Z(t, x) = (1-bt)^{-\frac{2-N\alpha}{2}} |\omega_0(x) \psi(t, w) e^{i\theta(t, w)}|^{\alpha} = (1-bt)^{-\frac{2-N\alpha}{2}} |\omega_0(x) \psi(t, w)|^{\alpha}.
\]
It follows from (4.2) and (4.4) that
\[
Z(t, x) = \frac{|v_0(x)|^\alpha (1 - bt)^{-\frac{2-N\alpha}{2}}}{1 + f_0(x) + \frac{2\alpha|\Re|}{b(2-N\alpha)}|v_0(x)|^\alpha [(1 - bt)^{-\frac{2-N\alpha}{2}} - 1]}.
\]
Since $1 + f_0 \geq 0$ by (4.8), we obtain
\[
Z(t, x) \leq \frac{b(2-N\alpha)}{2\alpha|\Re|} \frac{1}{1 - (1 - bt)^{-\frac{2-N\alpha}{2}}},
\]
so that
\[
\limsup_{t \uparrow \frac{1}{b}} \|Z(t)\|_{L^\infty} \leq \frac{b(2-N\alpha)}{2\alpha|\Re|}.
\]
Moreover, $1 + f_0 \leq 2$, so that
\[
Z(t, 0) \geq \frac{|v_0(0)|^\alpha (1 - bt)^{-\frac{2-N\alpha}{2}}}{2 + \frac{2\alpha|\Re|}{b(2-N\alpha)}|v_0(0)|^\alpha [(1 - bt)^{-\frac{2-N\alpha}{2}} - 1]}
\]
Since $|v_0(0)| > 0$ by (1.10), we deduce that
\[
\liminf_{t \uparrow \frac{1}{b}} \|Z(t)\|_{L^\infty} \geq \frac{b(2-N\alpha)}{2\alpha|\Re|}.
\]
Thus we see that $\|Z(t)\|_{L^\infty} \to \frac{b(2-N\alpha)}{2\alpha|\Re|}$ as $t \uparrow \frac{1}{b}$. On the other hand, it follows from (4.12) and (4.4) that
\[
|(1 - bt)^{-\frac{2-N\alpha}{2}} \|v(t)\|_{L^\infty}^\alpha - \|Z(t)\|_{L^\infty}| \leq K^\alpha (1 - bt)^{\frac{N\alpha}{2} - \sigma_3}.
\]
Since $\frac{N\alpha}{2} \geq \frac{1}{2} > \sigma_3$, (4.5) follows.

Finally, we prove (4.6). It follows from (4.2) and (4.4) that
\[
|\omega_0(x)^2 \psi(t, x)^2| = \left( \frac{|v_0(x)|^\alpha}{1 + f_0(x) + \frac{2\alpha|\Re|}{b(2-N\alpha)}|v_0(x)|^\alpha [(1 - bt)^{-\frac{2-N\alpha}{2}} - 1]} \right)^{\frac{1}{2}}.
\]
Recall that $\frac{1}{2} \leq 1 + f_0 \leq \frac{3}{2}$ and $\frac{1}{\alpha} \langle x \rangle^{-n} \leq |v_0(x)| \leq K \langle x \rangle^{-n}$. Therefore, for $\frac{1}{\alpha} \leq t < \frac{1}{b}$, we have
\[
a(x)^{-2n} \left( \frac{1}{1 + (1 - bt)^{-\frac{2-N\alpha}{2}} \langle x \rangle^{-n}} \right)^{\frac{1}{2}} \leq |\omega_0|^2 \psi^2 \leq \frac{A(x)^{-2n}}{(1 + (1 - bt)^{-\frac{2-N\alpha}{2}} \langle x \rangle^{-n})^{\frac{1}{2}}},
\]
for some constants $0 < a \leq A < \infty$. If $|x| \geq (1 - bt)^{-\frac{2-N\alpha}{2\alpha}}$, then $|\omega_0|^2 \psi^2 \geq |x|^{-2n}$ by the first inequality in (4.20). Since also $|\omega_0|^2 \psi^2 \lesssim |x|^{-2n}$ by the second inequality in (4.20), we deduce that
\[
a_1(1 - bt)^{\left(\frac{\alpha}{2} - N\right)(1 - \frac{2}{\alpha})} \leq \int_{|x|>(1 - bt)^{-\frac{2-N\alpha}{2\alpha}}} |\omega_0|^2 \psi^2 \leq A_1(1 - bt)^{\left(\frac{\alpha}{2} - N\right)(1 - \frac{2}{\alpha})},
\]
for some constants $0 < a_1 \leq A_1 < \infty$. If $|x| \leq (1 - bt)^{-\frac{2-N\alpha}{2\alpha}}$, then $|\omega_0|^2 \psi^2 \gtrsim (1 - bt)^{\left(\frac{2-N\alpha}{2\alpha}\right)^2}$ by the first inequality in (4.20). Since also $|\omega_0|^2 \psi^2 \lesssim (1 - bt)^{\left(\frac{2-N\alpha}{2\alpha}\right)^2}$ by the second inequality in (4.20), we deduce that
\[
a_2(1 - bt)^{\left(\frac{\alpha}{2} - N\right)(1 - \frac{2}{\alpha})} \leq \int_{|x|<(1 - bt)^{-\frac{2-N\alpha}{2\alpha}}} |\omega_0|^2 \psi^2 \leq A_2(1 - bt)^{\left(\frac{\alpha}{2} - N\right)(1 - \frac{2}{\alpha})},
\]
for some constants $0 < a_2 \leq A_2 < \infty$. It follows that
\[
a_3(1 - bt)^{\left(\frac{\alpha}{2} - N\right)(1 - \frac{2}{\alpha})} \leq \|\omega_0 \psi(t) e^{ibt(t)}\|_{L^2} \leq A_3(1 - bt)^{\left(\frac{\alpha}{2} - N\right)(1 - \frac{2}{\alpha})},
\]
for some constants $0 < a_3 \leq A_3 < \infty$. On the other hand, estimate (4.1) implies (since $n > \frac{N}{2}$)
\[ \|v(t) - \omega_0 \psi(t)e^{i\theta(t)}\|_{L^2} \leq C(1 - bt)^{\frac{1}{2}}. \] (4.22)
Since $\alpha > \frac{2}{N+1}$, we have
\[ \left(1 - \frac{N}{2}\right)\left(1 - \frac{N}{2n}\right) < 1 - \frac{N}{2} < \frac{1}{2} \]
and (4.6) follows from (4.21)-(4.22). \hfill \Box

5. Proof of Theorem 1.1. Let $\Re \lambda < 0$ and $K \geq 1$, and let $v_0 \in \mathcal{X}$ satisfy (1.10).
Let $\alpha_1$ and $b_1$ be given by Proposition 3.2. Given $\alpha_1 \leq \alpha < \frac{2}{N}$ and $b \geq b_1$, let $v \in C([0,1/b], \mathcal{X})$ be the corresponding solution of (1.19) given by Proposition 3.2. Let
\[ u(t, x) = (1 + bt)^{-\frac{N}{2}} e^{\frac{b|x|^2}{1+bt}} e^{\frac{t}{1+bt}(x)} e^{\frac{x}{1+bt}}, \quad t \geq 0, \ x \in \mathbb{R}^N. \] (5.1)
It follows that $u \in C([0, \infty), H^1(\mathbb{R}^N))$ is the solution of (1.1) with the initial condition $u_0(x) = e^{\frac{b|x|^2}{1+b}} v_0(x)$. Since $n > \frac{N}{2}$, we deduce from (4.1) in Proposition 4.1 that
\[ \|v(t, x) - \omega_0(x)\psi(t, x)e^{i\theta(t, x)}\|_{L^\infty L^2} \leq C(1 - bt)^{\frac{1}{2}}. \]
This proves (1.11), while (1.16) and (1.17) follow from (4.5) and (4.6), respectively. This completes the proof of Theorem 1.1.

REFERENCES

[1] T. Cazenave, S. Correia, F. Dickstein and F. B. Weissler, A Fujita-type blowup result and low energy scattering for a nonlinear Schrödinger equation, São Paulo J. Math. Sci., 9 (2015), 146–161.
[2] T. Cazenave, Z. Han and Y. Martel, Blowup on an arbitrary compact set for a Schrödinger equation with nonlinear source term, J. Dynam. Differential Equations (2020).
[3] T. Cazenave, Y. Martel and L. F. Zhao, Finite-time blowup for a Schrödinger equation with nonlinear source term, Discrete Contin. Dynam. Systems, 39 (2019), 1171–1183.
[4] T. Cazenave and I. Naumkin, Local existence, global existence, and scattering for the nonlinear Schrödinger equation, Commun. Contemp. Math., 19 (2017), 1650038, 20 pp.
[5] T. Cazenave and I. Naumkin, Modified scattering for the critical nonlinear Schrödinger equation, J. Funct. Anal., 274 (2018), 402–432.
[6] T. Cazenave and F. B. Weissler, Rapidly decaying solutions of the nonlinear Schrödinger equation, Comm. Math. Phys., 147 (1992), 75–100.
[7] M. C. Cross and P. C. Hohenberg, Pattern formation outside of equilibrium, Rev. Mod. Phys., 65 (1993), 851–1112.
[8] J. Ginibre, T. Ozawa and G. Velo, On the existence of the wave operators for a class of nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor., 60 (1994), 211–239.
[9] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations. I. Scattering theory, general case, J. Funct. Anal., 32 (1979), 33–71.
[10] J. Ginibre and G. Velo, On a class of nonlinear Schrödinger equations. III. Special theories in dimensions 1, 2 and 3, Ann. Inst. Henri Poincaré Sect. A (N.S.), 28 (1978), 287–316.
[11] N. Hayashi, C. H. Li and P. I. Naumkin, Time decay for nonlinear dissipative Schrödinger equations in optical fields, Adv. Math. Phys., (2016), Art. ID 3702378, 7 pp.
[12] N. Hayashi, C. H. Li and P. I. Naumkin, Upper and lower time decay bounds for solutions of dissipative nonlinear Schrödinger equations, Commun. Pure Appl. Anal., 16 (2017), 2089–2104.
[13] T. Kato, On nonlinear Schrödinger equations, Ann. Inst. H. Poincaré Phys. Théor., 46 (1987), 113–129.
[14] T. Kato, Nonlinear Schrödinger equations, Schrödinger Operators (Sonderborg, 1988), Lecture Notes in Phys., Springer, Berlin, 345 (1989), 218–263.
[15] N. Kita and A. Shimomura, Large time behavior of solutions to Schrödinger equations with a dissipative nonlinearity for arbitrarily large initial data, *J. Math. Soc. Japan*, 61 (2009), 39–64.

[16] A. Mielke, The Ginzburg-Landau equation in its role as a modulation equation, *Handbook of Dynamical Systems, North-Holland, Amsterdam*, 2 (2002), 759–834.

[17] K. Nakanishi and T. Ozawa, Remarks on scattering for nonlinear Schrödinger equations, *NoDEA Nonlinear Differential Equations Appl.*, 9 (2002), 45–68.

[18] T. Ozawa, Remarks on proofs of conservation laws for nonlinear Schrödinger equations, *Calc. Var. Partial Differential Equations*, 25 (2006), 403–408.

[19] A. Shimomura, Asymptotic behavior of solutions for Schrödinger equations with dissipative nonlinearities, *Comm. Partial Differential Equations*, 31 (2006), 1407–1423.

[20] K. Stewartson and J. T. Stuart, A non-linear instability theory for a wave system in plane Poiseuille flow, *J. Fluid Mech.*, 48 (1971), 529–545.

[21] W. A. Strauss, Nonlinear scattering theory at low energy: Sequel, *J. Funct. Anal.*, 43 (1981), 281–293.

Received for publication October 2019.

E-mail address: thierry.cazenave@sorbonne-universite.fr
E-mail address: hanzh_0102@hznu.edu.cn