A Note on Coloring Digraphs of Large Girth

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Abstract

The digirth of a digraph is the length of a shortest directed cycle. The dichromatic number \( \chi(D) \) of a digraph \( D \) is the smallest size of a partition of the vertex-set into subsets inducing acyclic subgraphs. A conjecture by Harutyunyan and Mohar [7] states that \( \chi(D) \leq \lceil \frac{\Delta}{4} \rceil + 1 \) for every digraph \( D \) of digirth at least 3 and maximum degree \( \Delta \). The best known partial result by Golowich [5] shows that \( \chi(D) \leq 2\frac{5}{\Delta} + O(1) \). In this short note we prove for every \( g \geq 2 \) that if \( D \) is a digraph of digirth at least \( g \) and maximum degree \( \Delta \), then \( \chi(D) \leq \left( \frac{1}{3} + \frac{1}{3g} \right)\Delta + O(1) \). This improves the bound of Golowich for digraphs without directed cycles of length at most 10.

1 Introduction

Preliminaries. All digraphs in this note are finite and do not contain loops or parallel arcs. Given a digraph \( D \), we denote by \( V(D) \) its vertex-set and by \( A(D) \) the arc-set. A digraph is called acyclic if it does not contain directed cycles. By \( \Delta(D), \Delta^+(D), \Delta^-(D), \delta^+(D), \delta^-(D) \) we denote, respectively, the maximum degree in (the underlying graph of) \( D \), and the extremal out- and in-degrees in \( D \). We furthermore denote by \( \Delta(D) = \max\{\sqrt{d^+(v)d^-(v)}|v \in V(D)\} \) the maximum geometric mean of the in- and out-degree of a vertex in \( D \). Note that in case \( D \) has no cycles of length 2, the inequality of geometric and arithmetic mean shows that \( \Delta(D) \leq \frac{\Delta}{2} \). Given a vertex set \( X \subseteq V(D) \), we denote by \( D[X] \) the induced subdigraph of \( D \) with vertex-set \( X \) and call \( X \) acyclic if \( D[X] \) is acyclic. By \( \bar{g}(D) \) we denote the digirth of \( D \), that is, the shortest length of a directed cycle in \( D \) (\( \bar{g}(D) := \infty \) if \( D \) is acyclic). Given a family \( A_1, \ldots, A_m \) of finite sets, a system of representatives of this family is a set \( X \subseteq \bigcup_{i=1}^{m} A_i \) such that \( X \cap A_i \neq \emptyset \) for all \( i \in [m] \).

We deal with a notion of coloring for directed graphs introduced in 1982 by Neumann-Lara [13]. Given a digraph \( D \), an acyclic coloring of \( D \) is a vertex-coloring in which all color classes are acyclic. The smallest number of colors sufficient for an acyclic coloring of \( D \) is denoted by \( \chi(D) \) and called dichromatic number of \( D \). This notion has received a fair amount of attention in the past two decades, see [1, 3, 4, 6, 9, 12] for some recent results. As for undirected graphs, there is a Brooks-type upper bound on the dichromatic number of a digraph, see [11, 13], which implies \( \chi(D) \leq \left[ \frac{\Delta(D)}{2} \right] + 1 \) for every digraph of girth at least 3 and maximum degree \( \Delta \geq 3 \). In this note, we are motivated by the following conjecture from [7], which claims that this Brook’s type bound can be improved by a factor of 2 if we forbid directed cycles of length 2 in the digraph.

Conjecture 1 (cf. [7], Conjecture 1.5). Let \( D \) be a digraph of digirth at least 3 and maximum degree \( \Delta \). Then \( \chi(D) \leq \left[ \frac{\Delta(D)}{4} \right] + 1 \leq \left[ \frac{\Delta}{4} \right] + 1 \).

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Approaching their conjecture, in [7] Harutyunyan and Mohar proved that there is a small absolute constant \( \varepsilon > 0 \) such that \( \bar{\chi}(D) \leq (1 - \varepsilon)\Delta(D) \leq \left( \frac{1}{2} - \frac{\varepsilon}{2} \right)\Delta(D) \) for every digraph \( D \) of digirth at least 3 and \( \Delta \) sufficiently large. Subsequently Golowich [5] improved the multiplicative constant in the upper bound, by showing that every digraph \( D \) of digirth at least 3 satisfies \( \bar{\chi}(D) \leq \frac{\Delta}{2}\Delta(D) + O(1) \). Our contribution is to further improve the multiplicative constant in this upper bound for digraphs without short directed cycles.

**Theorem 1.** Let \( g \geq 2 \) a natural number, and let \( D \) be a digraph with \( \bar{\chi}(D) \geq 2g - 1 \) and maximum degree \( \Delta \). Then \( \bar{\chi}(D) \leq \left( \frac{1}{3} + \frac{1}{3g} \right)\Delta + (g + 1) \).

## 2 Proof of Theorem 1

We need three auxiliary results by Neumann-Lara, by Aharoni, Berger and Kfir, and by Lovász.

**Lemma 2** (cf. [13, Theorem 5]). Let \( k \in \mathbb{N} \) and let \( D \) be a \((k + 1)\)-critical digraph, that is, \( \bar{\chi}(D) = k + 1 \) but \( \bar{\chi}(D') \leq k \) for every proper subdigraph \( D' \subseteq D \). Then \( \delta^+(D), \delta^-(D) \geq k \).

**Lemma 3** (cf. [2, Corollary II.13]). Let \( D \) be a digraph of digirth at least \( \gamma \geq 2 \) and let \( V_1, V_2, \ldots, V_m \) be a partition of \( V(D) \). If \( \vert V_i \vert \geq \frac{3}{2^m} \Delta^+(D) \) for all \( i \in [m] \), then there is a system \( X \) of representatives of \( V_1, \ldots, V_m \) which is acyclic in \( D \).

**Lemma 4** ([10]). Let \( G \) be an undirected graph, \( k \in \mathbb{N} \). Then \( V(G) \) admits a partition \( X_1, \ldots, X_k \) such that for every \( v \in X_i, i \in [k] \), we have \( \deg_{G[V]}(v) \leq \frac{1}{k} \deg(v) \).

The proof of Theorem 1 relies on the following bound on the dichromatic number for digraphs of large girth compared to their maximum out-degree.

**Lemma 5.** Let \( D \) be a digraph such that \( \bar{\chi}(D) > \Delta^+(D) \). Then

\[
\bar{\chi}(D) \leq \left\lfloor \frac{\Delta(D)}{3} \right\rfloor + 2.
\]

**Proof.** Abbreviate \( \Delta = \Delta(D) \) and \( \gamma = \bar{\chi}(D) \) and put \( k := \left\lfloor \frac{\Delta}{3} \right\rfloor + 1 > \frac{\Delta}{3} \). By Lemma 2 there is a partition \( X_1, \ldots, X_k \) of \( V(D) \) such that for every \( i \in [m] \) we have \( \Delta(D[X]) \leq \frac{\Delta}{3} < 3 \). Hence, \( D[X_i] \) is a disjoint union of oriented paths and oriented cycles. For every \( i \), let us denote by \( \bar{C}_i \) the set of all directed cycles in \( D[X_i] \) and put \( V' := \bigcup_{i \in [k], C \in \bar{C}_i} V(C) \). We claim that there is an acyclic set \( X \subseteq D \) such that \( X \cap V(C) \neq \emptyset \) for all \( C \in \bar{C}_i \) and \( i \in [k] \). To see this, note that \( \deg_{V'}(v) \geq \gamma \geq \frac{3}{2^m} \Delta^+(D) \geq \frac{3}{2^m} \Delta^+(D[V']) \) for every \( C \in \bar{C}_i \) and \( i \in [k] \). We can therefore apply Lemma 3 to the digraph \( D[V'] \) equipped with the partition \( (V'(C) \in \bar{C}_i, i \in [k]) \) to find a system of representatives \( X \) which is acyclic in \( D[V'] \) and thus in \( D \). Next we claim that each of the sets \( X_i \setminus X, i \in [k] \) is acyclic in \( D \). Indeed, the digraph \( D[X_i \setminus X] = D[X_i] - (X_i \cap X) \) is obtained from a disjoint union of oriented paths and cycles by removing at least one vertex from each directed cycle, and is therefore acyclic. Hence, \( X_1 \setminus X, X_2 \setminus X, \ldots, X_k \setminus X \) is a partition of \( V(D) \) into acyclic sets which certifies that \( \bar{\chi}(D) \leq k + 1 = \left\lfloor \frac{\Delta}{3} \right\rfloor + 2 \).

We can now complete the proof of Theorem 1 by applying Lemma 4 a second time.

**Proof of Theorem 1** Let \( \ell := \left\lfloor \frac{\Delta}{3g} \right\rfloor + 1 \). By Lemma 4 there exists a partition \( Y_1, \ldots, Y_\ell \) of \( V(D) \) such that \( \Delta(D[Y_i]) \leq \Delta < 3g \) for every \( i \in [\ell] \). We claim that for every \( i \in [\ell] \), we have \( \bar{\chi}(D[Y_i]) \geq g + 1 \). Suppose by way of a contradiction that \( \bar{\chi}(D[Y_i]) \geq g + 2 \) for some \( i \in [\ell] \).
Consider a subgraph $D_i$ of $D[Y_i]$ with $\chi(D_i) \geq g + 2$ minimizing $|V(D_i)| + |A(D_i)|$. Clearly, $D_i$ is $(g+2)$-critical, and thus $\delta^-(D_i) \geq g + 1$ by Lemma 2. Hence we have

$$\Delta^+(D_i) \leq \Delta(D_i) - \delta^-(D_i) \leq \Delta(D[Y_i]) - \delta^-(D_i) \leq (3g - 1) - (g + 1) = 2g - 2 < g(D) \leq g(D_i).$$

We can therefore apply Lemma 5 to obtain $\chi(D_i) \leq \left\lfloor \frac{3g - 1}{3g} \right\rfloor + 2 = g + 1$, which is the desired contradiction. This shows that indeed we have $\chi(D[Y_i]) \leq g + 1$ for all $i \in [\ell]$. The claim now follows from

$$\chi(D) \leq \sum_{i=1}^{\ell} \chi(D[Y_i]) \leq (g + 1) \left( \left\lfloor \frac{\Delta}{3g} \right\rfloor + 1 \right) \leq \left( \frac{1}{3} + \frac{1}{3g} \right) \Delta + (g + 1).$$

\[\square\]

References

[1] P. Aboulker, N. Cohen, F. Havet, W. Lochet, P. Moura, and S. Thomassé. Subdivisions in digraphs of large out-degree or large dichromatic number. Electronic Journal of Combinatorics, 26(3):P3.19, 2019.

[2] R. Aharoni, E. Berger, and O. Kfir. Acyclic systems of representatives and acyclic colorings of digraphs. Journal of Graph Theory, 59:177–198, 2008.

[3] S. D. Andres and W. Hochstättler. Perfect digraphs. Journal of Graph Theory, 79(1):21–29, 2015.

[4] J. Bensmail, A. Harutyunyan, and N. Khang Le. List coloring digraphs. Journal of Graph Theory, 87(4):492–508, 2018.

[5] N. Golowich. The m-degenerate chromatic number of a digraph. Discrete Mathematics, 339(6):1734–1743, 2016.

[6] A. Harutyunyan, T.-N. Le, S. Thomassé, and H. Wu. Coloring tournaments: From local to global. Journal of Combinatorial Theory, Series B, 2019.

[7] A. Harutyunyan and B. Mohar. Strengthened brooks’ theorem for digraphs of girth at least three. Electronic Journal of Combinatorics, 18(1), 2011. P195.

[8] A. Harutyunyan and B. Mohar. Planar digraphs of digirth five are 2-colorable. Journal of Graph Theory, 84(4):408–427, 2017.

[9] Z. Li and B. Mohar. Planar digraphs of digirth four are 2-colorable. SIAM J. Discrete Math., 31:2201–2205, 2017.

[10] L. Lovász. On decompositions of graphs. Studia Sci. Math. Hungar., 1:237–238, 1966.

[11] B. Mohar. Eigenvalues and colorings of digraphs. Linear Algebra and its Applications, 432:2273–2277, 2010.

[12] B. Mohar and H. Wu. Dichromatic number and fractional chromatic number. Forum of Mathematics, Sigma, 4, E32, 2016.

[13] V. Neumann-Lara. The dichromatic number of a digraph. Journal of Combinatorial Theory, Series B, 33(3):265–270, 1982.