Affine Geometric Crystal of $A_n^{(1)}$ and Limit of Kirillov-Reshetikhin Perfect Crystals

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Abstract

Let $\mathfrak{g}$ be an affine Lie algebra with index set $I = \{0,1,2,\cdots,n\}$ and $\mathfrak{g}^\vee$ be its Langlands dual. It is conjectured in [10] that for each $k \in I \setminus \{0\}$ the affine Lie algebra $\mathfrak{g}$ has a positive geometric crystal whose ultra-discretization is isomorphic to the limit of certain coherent family of perfect crystals for $\mathfrak{g}^\vee$. Motivated by this conjecture we construct a positive geometric crystal for the affine Lie algebra $\mathfrak{g} = A_n^{(1)}$ for each Dynkin index $k \in I \setminus \{0\}$ and show that its ultra-discretization is isomorphic to the limit of a coherent family of perfect crystals for $A_n^{(1)}$ given in [29]. In the process we develop and use some lattice-path combinatorics.

1 Introduction

Let $\mathfrak{g} = \mathfrak{g}(A)$ be an affine Lie algebra with Cartan matrix $A = (a_{ij})_{i,j \in I}, I = \{0,1,\cdots,n\}$ and Cartan datum $(A, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$. Let $U_q(\mathfrak{g})$ denote the quantum affine algebra associated with $\mathfrak{g}$. We denote $\mathfrak{g}_0$ to be the subalgebra of $\mathfrak{g}$ with index set $I \setminus \{i\}$. Note that both $\mathfrak{g}_0$ and $\mathfrak{g}_n$ are isomorphic to the simple Lie algebra $A_n$. We denote the affine weight lattice and the dual affine weight lattice of $\mathfrak{g}$ by $P = \mathbb{Z} \alpha_0 \oplus \mathbb{Z} \alpha_1 \oplus \cdots \oplus \mathbb{Z} \alpha_n \oplus \mathbb{Z} \delta$ and $P^\vee = \mathbb{Z} \alpha_0^\vee \oplus \mathbb{Z} \alpha_1^\vee \oplus \cdots \oplus \mathbb{Z} \alpha_n^\vee \oplus \mathbb{Z} \delta$ respectively, where $\delta$ is the null root. For a dominant weight $\lambda \in P^+$ let $\lambda(c) = \mu \in P$ such that $\mu(h_i) \geq 0$ for all $i \in I)$ of level $l = \lambda(c)$ ($c = \text{canonical central element of } \mathfrak{g}$). Kashiwara defined the crystal base $(L(\lambda), B(\lambda))$ [11] for the integrable highest weight $U_q(\mathfrak{g})$-module $V(\lambda)$. The crystal $B(\lambda)$ is the $q = 0$ limit of the canonical basis [21] or the global crystal basis [12]. There are many known explicit realizations of the affine crystal $B(\lambda)$. One such realization is the path realization [8] using perfect crystals. A perfect crystal is a crystal for certain finite dimensional module called Kirillov-Reshetikhin module (KR-module for short) of the quantum affine algebra $U_q(\mathfrak{g})$ ([19], [4, 5]). The KR-modules are parametrized by two integers $(k, l)$, where $k \in I \setminus \{0\}$ and $l$ any positive integer. Let $\{\varpi_k\}_{k \in I \setminus \{0\}}$ be the set of level 0 fundamental weights [13]. Hatayama et al [4, 5] conjectured that any KR-module $W(l\varpi_k)$ admit a crystal base $B^{k,l}$ in the sense of Kashiwara and furthermore $B^{k,l}$ is perfect if $l$ is a multiple of $e_0^\vee := \max(1, 2, \frac{2}{\alpha_1^\vee, \alpha_2^\vee}).$ This conjecture has been proved for quantum affine algebras $U_q(\mathfrak{g})$ of classical types ([28], [2, 3]). When $B^{k,l}$ is a coherent family of perfect crystals [10] we denote its limit by $B^{k,\infty}(\varpi_k)$ (or just $B^{k,\infty}$).

In 2000, Berenstein and Kazhdan [1] introduced the notion of geometric crystal for reductive algebraic groups which was extended to Kac-Moody groups in [24]. For a given Cartan datum

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In this paper we have constructed positive geometric crystals associated with each Dynkin index \( k \in \mathbb{I} \setminus 0 \) for the affine Lie algebra \( A_{n}^{(1)} \) and have proved that its ultra-discretization is isomorphic to the limit \( B^{k,\infty} \) of the coherent family of perfect crystals \( \{B^{k,l}\}_{l \geq 1} \) given in \([8, 29]\). To construct the \( A_{n}^{(1)} \) positive geometric crystal we proceed as follows. We construct \( g_{0} \) and \( g_{n} \) positive geometric crystals \( \mathcal{V}_{1} = \{\mathcal{V}_{1}(x), \{\epsilon_{i}\}, \{\gamma_{i}\}, \{\varepsilon_{i}\} \mid 1 \leq i \leq n \} \) and \( \mathcal{V}_{2} = \{\mathcal{V}_{2}(y), \{\sigma_{i}\}, \{\tau_{i}\}, \{\tau_{i}\} \mid 0 \leq i \leq n - 1 \} \) respectively, in the fundamental representation \( W(\varpi_{k}) \). Then we define a birational isomorphism \( \sigma \) between \( \mathcal{V}_{1} \) and \( \mathcal{V}_{2} \), and using this isomorphism we patch them together to obtain an affine geometric crystal \( \mathcal{V}(A_{n}^{(1)}) \). In this case we conjecture that \( \gamma = \sigma(x) \) is the unique solution of \( \mathcal{V}_{2}(y) = a(x)\mathcal{V}_{1}(x) \) which would prove Conjecture 1.2 in \([16]\) completely.

This paper is organized as follows. In Section 2, we recall necessary definitions and facts about geometric crystals and ultra-discretization. In Section 3 we give \( g_{0} \) (resp. \( g_{n} \)) positive geometric crystal \( \mathcal{V}_{1} \) (resp. \( \mathcal{V}_{2} \)) explicitly. In Section 4 we develop some lattice-path combinatorics which is used in the sequel. In Section 5 we give the birational bi-positive map \( \tau \) between \( \mathcal{V}_{1} \) and \( \mathcal{V}_{2} \). In Section 6 we construct the positive affine geometric crystal \( \mathcal{V}(A_{n}^{(1)}) \). In Section 7, we show that the ultra-discretization of the affine geometric crystal \( \mathcal{V}(A_{n}^{(1)}) \) is isomorphic to the crystal \( B^{k,\infty} \) given in \([29]\). Finally in Section 8 as an application we give the explicit actions of the affine Weyl group on the geometric crystal \( \mathcal{V}(A_{n}^{(1)}) \) and its ultra-discretization \( B^{k,\infty} \).
2 Geometric crystals

We review Kac-Moody groups and geometric crystals following [1, 20, 24, 30].

2.1 Kac-Moody algebras and Kac-Moody groups

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix with a finite index set $I$, and $(t, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I})$ the associated root data, where $t$ is a vector space over $\mathbb{C}$ and $\{\alpha_i\}_{i \in I} \subset t^*$ and $\{\alpha_i^\vee\}_{i \in I} \subset t$ are linearly independent satisfying $\alpha_i(\alpha_i^\vee) = a_{ij}$.

The Kac-Moody Lie algebra $\mathfrak{g} = \mathfrak{g}(A)$ associated with $A$ is the Lie algebra over $\mathbb{C}$ generated by $t$, the Chevalley generators $e_i$ and $f_i$ ($i \in I$) with the usual defining relations ([15, 20]). There is the root space decomposition $\mathfrak{g} = \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in t^*| \alpha \neq 0, \mathfrak{g}_\alpha \neq (0)\}$. Set $Q = \sum_i Z\alpha_i$, $Q_+ = \sum_i Z_{\geq 0}\alpha_i$, $Q^\vee := \sum_i Z\alpha_i^\vee$ and $\Delta_+ := \Delta \cap Q_+$. An element of $\Delta_+$ is called a positive root. Let $P \subset t^*$ be a weight lattice such that $\mathbb{C} \otimes P = t^*$, whose element is called a weight.

Define simple reflections $s_i \in \text{Aut}(t)$ ($i \in I$) by $s_i(h) := h - \alpha_i(h)\alpha_i^\vee$, which generate the Weyl group $W$. It induces the action of $W$ on $t^*$ by $s_i(\lambda) := \lambda - \lambda(\alpha_i^\vee)\alpha_i$. Set $\Delta^\text{re} := \{w(\alpha_i)| w \in W, \; i \in I\}$, whose element is called a real root.

Let $\mathfrak{g}'$ be the derived Lie algebra of $\mathfrak{g}$ and let $G$ be the Kac-Moody group associated with $\mathfrak{g}'$ ([20]). Let $U_\alpha := \exp \mathfrak{g}_\alpha$ ($\alpha \in \Delta^\text{re}$) be the one-parameter subgroup of $G$. The group $G$ is generated by $U_\alpha$ ($\alpha \in \Delta^\text{re}$). Let $U^\pm$ be the subgroup generated by $U_{\pm \alpha}$ ($\alpha \in \Delta^\text{re}$), i.e., $U^\pm := \langle U_{\pm \alpha}| \alpha \in \Delta^\text{re}\rangle$.

For any $i \in I$, there exists a unique homomorphism: $\phi_i : SL_2(\mathbb{C}) \rightarrow G$ such that

$$
\phi_i \left( \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \right) = e^{\alpha_i^\vee}, \; \phi_i \left( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp(te_i), \; \phi_i \left( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp(tf_i).
$$

where $c \in \mathbb{C}^\times$ and $t \in \mathbb{C}$. Set $\alpha_i^\vee(c) := e^{\alpha_i^\vee}, \; x_i(t) := \exp(te_i), \; y_i(t) := \exp(tf_i), \; G_i := \phi_i(SL_2(\mathbb{C})), \; T_i := \phi_i(\text{diag}(c, c^{-1})| c \in \mathbb{C}^\times))$ and $N_i := N_G(T_i)$. Let $T$ (resp. $N$) be the subgroup of $G$ with the Lie algebra $t$ (resp. generated by the $N_i$’s), which is called a maximal torus in $G$, and let $B^\pm = U^\pm T$ be the Borel subgroup of $G$. We have the isomorphism $\phi : W \rightarrow N/T$ defined by $\phi(s_i) = N_i T/T$. An element $\pi_i := x_i(-1)y_i(1)x_i(-1) = \phi_i \left( \begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix} \right)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$.

2.2 Geometric crystals

Let $X$ be an ind-variety, $\gamma_i : X \rightarrow \mathbb{C}$ and $\varepsilon_i : X \rightarrow \mathbb{C}$ ($i \in I$) rational functions on $X$, and $e_i : \mathbb{C}^\times \times X \rightarrow X ((c, x) \mapsto e_i^c(x))$ a rational $\mathbb{C}^\times$-action.

Definition 2.1. A quadruple $(X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a $G$ (or $\mathfrak{g}$)-geometric crystal if

(i) $\{1\} \times X \subset \text{dom}(e_i)$ for any $i \in I$.

(ii) $\gamma_j(e_i^c(x)) = e^{\alpha_{ij}}\gamma_j(x)$. 


(iii) \( \{e_i\}_{i \in I} \) satisfy the following relations.

\[
\begin{align*}
e_i^1 e_j^2 &= e_j^2 e_i^1 & \text{if } a_{ij} = a_{ji} = 0, \\
e_i^1 e_j^2 e_i^1 &= e_j^2 e_i^1 e_j^2 e_i^1 & \text{if } a_{ij} = a_{ji} = -1, \\
e_i^1 e_j^2 e_i^1 e_j^2 &= e_j^2 e_i^1 e_j^2 e_i^1 & \text{if } a_{ij} = -2, a_{ji} = -1, \\
e_i^1 e_j^2 e_i^1 e_j^2 e_i^1 &= e_j^2 e_i^1 e_j^2 e_i^1 & \text{if } a_{ij} = -3, a_{ji} = -1,
\end{align*}
\]

(iv) \( \varepsilon_i(e_j^l(x)) = c^{-1} \varepsilon_i(x) \) and \( \varepsilon_i(e_j^l(x)) = \varepsilon_i(x) \) if \( a_{ij} = a_{ji} = 0 \).

The condition (iv) is slightly modified from the one in [6, 26, 27].

Let \( W \) be the Weyl group associated with \( w \in W \). For \( w \in W \) define \( R(w) \) by

\[
R(w) := \{ (i_1, i_2, \ldots, i_l) \in I^l \mid w = s_{i_l} s_{i_{l-1}} \cdots s_{i_1} \},
\]

where \( l \) is the length of \( w \). Then \( R(w) \) is the set of reduced words of \( w \). For a word \( i = (i_1, \ldots, i_l) \in R(w) \) \( (w \in W) \), set \( \alpha^{(j)} := s_{i_l} \cdots s_{i_{j+1}}(\alpha_{i_j}) \) \( (1 \leq j \leq l) \) and

\[
e_i : T \times X \to X
\]

\[
(t, x) \mapsto e_i^l(x) := e_{i_1}^{\alpha_1(t)} e_{i_2}^{\alpha_2(t)} \cdots e_{i_l}^{\alpha_l(t)}(x).
\]

Note that the condition (iii) above is equivalent to the following: \( e_i = e_i \) for any \( w \in W, i \in I \).

\[.\]

2.3 Geometric crystal on Schubert cell

Let \( w \in W \) be a Weyl group element and take a reduced expression \( w = s_{i_l} \cdots s_{i_1} \). Let \( X := G/B \) be the flag variety, which is an ind-variety and \( X_w \subset X \) the Schubert cell associated with \( w \), which satisfies \( X = \sqcup_{w \in W} X_w \) and has a natural geometric crystal structure ([1] [24]). For a reduced word \( i := (i_1, \ldots, i_k) \) of \( w \in W \), set

\[
B_i^w := \{ Y_i(c_1, \ldots, c_k) := Y_{i_1}(c_1) \cdots Y_{i_k}(c_k) \mid c_1, \ldots, c_k \in \mathbb{C}^\times \} \subset B^-,
\]

where \( Y_i(c) := y_i(\frac{1}{2}) a_i^w(c) \). If \( I = \{ i_1, \ldots, i_k \} \), this has a geometric crystal structure([24]) isomorphic to \( X_w \). The explicit forms of the action \( e_i^l \), the rational function \( \varepsilon_i \) and \( \gamma_i \) on \( B_i^w \) are given by

\[
e_i^l(Y_i(c_1, \ldots, c_k)) = Y_i(C_1, \ldots, C_k),
\]

where

\[
C_j := c_j \sum_{1 \leq m \leq j, i_m = i} c_1^{a_{11},i} \cdots c_{m-1}^{a_{m-1,i}} c_m + \sum_{1 \leq j < m, i_j = i} c_1^{a_{11},i} \cdots c_{m-1}^{a_{m-1,i}} c_m
\]

\[
(2.2)
\]

\[
\varepsilon_i(Y_i(c_1, \ldots, c_k)) = \sum_{1 \leq m \leq k, i_m = i} c_1^{a_{11},i} \cdots c_{m-1}^{a_{m-1,i}} c_m
\]

\[
(2.3)
\]

\[
\gamma_i(Y_i(c_1, \ldots, c_k)) = c_1^{a_{11},i} \cdots c_k^{a_{k,k-i}}.
\]

(2.4)
Remark. As in [23], the above setting requires the condition \( I = \{i_1, \cdots, i_k\} \). Otherwise, set \( J := \{i_1, \cdots, i_k\} \subset I \) and let \( g_J \subseteq g \) be the corresponding subalgebra. Then, by arguing similarly to [23, 4.3], we can define the \( g_J \)-geometric crystal structure on \( B_I^- \).

## 2.4 Positive structure and Ultra-discretizations

Let us recall the notions of positive structure and ultra-discretization following [1, 2, 5]. Let \( T = (\mathbb{C}^\times)^l \) be an algebraic torus over \( \mathbb{C} \) and \( X^\times(T) := \text{Hom}(T, \mathbb{C}^\times) \cong \mathbb{Z}^l \) (resp. \( X_+(T) := \text{Hom}(\mathbb{C}^\times, T) \cong \mathbb{Z}^l \)) be the lattice of characters (resp. co-characters) of \( T \). Define

\[
\eta : C(c) \setminus \{0\} \longrightarrow \mathbb{Z}
\]

where \( \text{deg} \) is the degree of poles at \( c = \infty \). Here note that for \( f_1, f_2 \in C(c) \setminus \{0\} \), we have

\[
v(f_1f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2)
\]

(2.5)

A non-zero rational function on an algebraic torus \( T \) is called \textit{positive} if it can be written as \( g/h \) where \( g \) and \( h \) are positive rational morphisms of algebraic tori.

**Definition 2.2.** Let \( f : T \to T' \) be a rational morphism between two algebraic tori \( T \) and \( T' \). We say that \( f \) is \textit{positive}, if \( \eta \circ f \) is positive for any character \( \eta : T' \to \mathbb{C} \).

Denote by \( \text{Mor}^+(T, T') \) the set of positive rational morphisms from \( T \) to \( T' \).

**Lemma 2.3 ([1]).** For any \( f \in \text{Mor}^+(T_1, T_2) \) and \( g \in \text{Mor}^+(T_2, T_3) \), the composition \( g \circ f \) is well-defined and belongs to \( \text{Mor}^+(T_1, T_3) \).

By this lemma, we can define a category \( \mathcal{T}_+ \) whose objects are algebraic tori over \( \mathbb{C} \) and arrows are positive rational morphisms.

Let \( f : T \to T' \) be a positive rational morphism of algebraic tori \( T \) and \( T' \). We define a map \( \hat{f} : X_+(T) \to X_+(T') \) by

\[
\langle \eta, \hat{f}(\xi) \rangle = v(\eta \circ f \circ \xi),
\]

where \( \eta \in X^\times(T') \) and \( \xi \in X_+(T) \).

**Lemma 2.4 ([1]).** For any algebraic tori \( T_1, T_2, T_3 \), and positive rational morphisms \( f \in \text{Mor}^+(T_1, T_2), g \in \text{Mor}^+(T_2, T_3) \), we have \( g \circ f = \hat{g} \circ \hat{f} \).

Let \( \mathsf{Set} \) denote the category of sets with the morphisms being set maps. By the above lemma, we obtain a functor:

\[
\mathcal{U}\mathcal{D} : \mathcal{T}_+ \times X_+(T) \longrightarrow \mathsf{Set}
\]

\[
(f : T \to T', \langle \eta, \xi \rangle) \mapsto (\hat{f} : X_+(T) \to X_+(T'))
\]

**Definition 2.5 ([1]).** Let \( \chi = \langle X, \{e_i\}_{i \in I}, \{\omega_i\}_{i \in I}, \{e_i\}_{i \in I} \rangle \) be a finite dimensional geometric crystal, \( T' \) an algebraic torus and \( \theta : T' \to X \) a birational isomorphism. The isomorphism \( \theta \) is called \textit{positive structure} on \( \chi \) if it satisfies
(i) for any $i \in I$ the rational functions $\gamma_i \circ \theta : T' \to \mathbb{C}$ and $\varepsilon_i \circ \theta : T' \to \mathbb{C}$ are positive.

(ii) For any $i \in I$, the rational morphism $e_{i, \theta} : \mathbb{C}^\times \times T' \to T'$ defined by $e_{i, \theta}(c, t) := \theta^{-1} \circ e_i^\times \circ \theta(t)$ is positive.

Let $\theta : T \to X$ be a positive structure on a geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$. Applying the functor $UD$ to positive rational morphisms $e_{i, \theta} : \mathbb{C}^\times \times T \to T$ and $\gamma_i \circ \theta, \varepsilon_i \circ \theta : T \to \mathbb{C}$ (the notations are as above), we obtain

$$\tilde{e}_i := UD(e_{i, \theta}) : X_\ast(T) \to X_\ast(T)$$
$$\text{wt}_i := UD(\gamma_i \circ \theta) : X_\ast(T) \to \mathbb{Z},$$
$$\varepsilon_i := UD(\varepsilon_i \circ \theta) : X_\ast(T) \to \mathbb{Z}.$$ 

Now, for given positive structure $\theta : T' \to X$ on a geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$, we associate the quadruple $(X_\ast(T'), \{\tilde{e}_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ with a free pre-crystal structure (see [1 Sect.7]) and denote it by $UD_{\theta, T'}(\chi)$. We have the following theorem:

**Theorem 2.6 ([1][24]).** For any geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ and positive structure $\theta : T' \to X$, the associated pre-crystal $UD_{\theta, T'}(\chi) = (X_\ast(T'), \{\tilde{e}_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ is a crystal (see [1 Sect.7]).

Now, let $GC^+$ be a category whose object is a triplet $(\chi, T', \theta)$ where $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ is a geometric crystal and $\theta : T' \to X$ is a positive structure on $\chi$, and morphism $f : (\chi_1, T'_1, \theta_1) \to (\chi_2, T'_2, \theta_2)$ is given by a rational map $\varphi : X_1 \to X_2 (\chi_i = (X_i, \cdots))$ such that

$$\varphi \circ e_{i_1}^X = e_{i_2}^X \circ \varphi, \quad \gamma_i^{X_1} \circ \varphi = \gamma_i^{X_2}, \quad \varepsilon_i^{X_1} \circ \varphi = \varepsilon_i^{X_2},$$

and $f := \theta_2^{-1} \circ \varphi \circ \theta_1 : T'_1 \to T'_2$,

is a positive rational morphism. Let $CR$ be the category of crystals. Then by the theorem above, we have

**Corollary 2.7.** The map $UD = UD_{\theta, T'}$, defined above is a functor

$$UD : GC^+ \to CR,$$

$$(\chi, T', \theta) \mapsto X_\ast(T'),$$

$$(f : (\chi_1, T'_1, \theta_1) \to (\chi_2, T'_2, \theta_2)) \mapsto (\tilde{f} : X_\ast(T'_1) \to X_\ast(T'_2)).$$

We call the functor $UD$ “ultra-discretization” as in ([24][25]) instead of “tropicalization” as in [1]. And for a crystal $B$, if there exists a geometric crystal $\chi$ and a positive structure $\theta : T' \to X$ on $\chi$ such that $UD(\chi, T', \theta) \cong B$ as crystals, we call an object $(\chi, T', \theta)$ in $GC^+$ a tropicalization of $B$, which is not standard but we use such a terminology as before.
3 $A_n$-Geometric Crystals $V_1$ and $V_2$ in $W(\varpi_k)$

3.1 Affine Lie Algebra $A_n^{(1)}$

In the sequel, we assume $\mathfrak{g}$ to be the affine Lie algebra $A_n^{(1)}$ ($n \geq 2$). The Cartan matrix $A = (a_{ij})_{i,j \in I}, I = \{0, 1, \cdots, n\}$ is given by:

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \equiv (j \pm 1) \mod (n + 1), \\ 0 & \text{otherwise}, \end{cases}$$

and its Dynkin diagram is as follows.

For the affine Lie algebra $A_n^{(1)}$, let $\{\alpha_0, \alpha_1, \cdots, \alpha_n\}, \{\alpha_0^\vee, \alpha_1^\vee, \cdots, \alpha_n^\vee\}$ and $\{\Lambda_0, \Lambda_1, \cdots, \Lambda_n\}$ be the set of simple roots, simple coroots and fundamental weights, respectively.

The standard null root $\delta$ and the canonical central element $c$ are given by

$$\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_n \quad \text{and} \quad c = \alpha_0^\vee + \alpha_1^\vee + \cdots + \alpha_n^\vee,$$

where $\alpha_0 = 2\Lambda_0 - \Lambda_1 - \Lambda_n + \delta, \alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1}, 1 \leq i \leq n-1, \alpha_n = -\Lambda_0 - \Lambda_{n-1} + 2\Lambda_n$.

Let $\sigma$ denote the Dynkin diagram automorphism such that $\sigma(\alpha_i) = \alpha_{i+1}$, where $i+1 = (i + 1) \mod (n + 1)$. Consider the level 0 fundamental weight $\varpi_k := \Lambda_k - \Lambda_0$ ($1 \leq k \leq n$). Let $I_0 = I \setminus \{0\}, I_n = I \setminus \{n\}$, and $\mathfrak{g}_i$ denote the subalgebra of $\mathfrak{g}$ associated with the index sets $I_i (i = 0, n)$. Then $\mathfrak{g}_0$ as well as $\mathfrak{g}_n$ is isomorphic to the simple Lie algebra of type $A_n$.

Let $W(\varpi_k)$ be the fundamental representation of $\mathcal{U}_q'(\mathfrak{g})$ associated with $\varpi_k$ ([13]). By [13 Theorem 5.17], $W(\varpi_k)$ is a finite-dimensional irreducible integrable $\mathcal{U}_q'(\mathfrak{g})$-module and has a global basis with a simple crystal. Thus, we can consider the specialization $q = 1$ and obtain the finite-dimensional $A_n^{(1)}$-module $W(\varpi_k)$, which we call a fundamental representation of $A_n^{(1)}$ and use the same notation as above. We shall present the explicit form of $W(\varpi_k)$ below.

3.2 Fundamental representation $W(\varpi_k)$ for $A_n^{(1)}$

The $A_n^{(1)}$-module $W(\varpi_k)$ is an $n+1C_k = \frac{(n+1)!}{k!(n+1-k)!}$-dimensional module with the basis:

$$\{(i_1, i_2, \cdots, i_k) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n + 1\},$$

where $(i_1, i_2, \cdots, i_k)$ denotes the one-column Young tableaux with entries $i_1, i_2, \cdots, i_k$. The explicit actions of $e_i, f_i$ and $\alpha_i^\vee(c) = c^{h_i} (c \in \mathbb{C}^\times)$ on these basis vectors are given as follows:
For $1 \leq i \leq n$, we have

\[
\begin{align*}
    f_i(i_1, \ldots, i_j, i_{j+1}, \ldots, i_k) &= \begin{cases} (i_1, \ldots, i + 1, i_{j+1}, \ldots, i_k), & \exists j: i_j = i < i_{j+1} - 1, \\ 0, & \text{otherwise}. \end{cases} \\
    e_i(i_1, \ldots, i_{j-1}, i_j, \ldots, i_k) &= \begin{cases} (i_1, \ldots, i_{j-1}, i, \ldots, i_k), & \exists j: i_{j-1} + 1 < i = i_{j+1}, \\ 0, & \text{otherwise}. \end{cases} \\
    \alpha_i(c)(i_1, \ldots, i_k) &= \begin{cases} c(i_1, \ldots, i_k), & \exists j: i_j = i < i_{j+1} - 1, \\ c^{-1}(i_1, \ldots, i_k), & \exists j: i_j + 1 < i_{j+1} = i + 1, \\ (i_1, \ldots, i_k), & \text{otherwise}, \end{cases}
\end{align*}
\]

\[
\begin{align*}
    f_0(i_1, \ldots, i_k) &= \begin{cases} (1, i_1, \ldots, i_{k-1}), & i_1 \neq 1, \ i_k = n + 1, \\ 0, & \text{otherwise}. \end{cases} \\
    e_0(i_1, \ldots, i_k) &= \begin{cases} (i_2, \ldots, i_k, n + 1), & i_1 = 1, \ i_k \neq n + 1, \\ 0, & \text{otherwise}. \end{cases} \\
    \alpha_0(c)(i_1, \ldots, i_k) &= \begin{cases} c(i_1, \ldots, i_k), & i_1 \neq 1, \ i_k = n + 1, \\ c^{-1}(i_1, \ldots, i_k), & i_1 = 1, \ i_k \neq n + 1, \\ (i_1, \ldots, i_k), & \text{otherwise}. \end{cases}
\end{align*}
\]

Note that in $W(\varpi_k)$ the vector $(1, 2, \ldots, k-1, k) =: u_1$ (resp. $(1, 2, \ldots, k-1, n+1) =: u_2$) is the $\varpi_0$ (resp. $\varpi_n$) highest weight vector with weight $\varpi_k = \Lambda_k - \Lambda_0$ (resp. $\sigma^{-1}\varpi_k = \Lambda_k - \Lambda_n$). We also find that $\tilde{c}_i^2 = \tilde{f}_i^2 = 0$ on $W(\varpi_k)$ and then $x_i(c) = 1 + c e_i$ and $y_i(c) = 1 + c f_i$ on $W(\varpi_k)$.

### 3.3 Geometric Crystals $\mathcal{V}_1$ and $\mathcal{V}_2$ in $W(\varpi_k)$

To construct the affine geometric crystal $\mathcal{V}(\mathcal{A}_n^{(1)})$ in $W(\varpi_k)$ $(1 \leq k \leq n)$ explicitly, we shall introduce two $\mathcal{A}_n$-geometric crystals $\mathcal{V}_1, \mathcal{V}_2 \subset W(\varpi_k)$.

For $\xi \in (t_n^0)_0$, let $t(\xi)$ be the translation as in [13 Sect 4] and $\varpi_k := \max(1, \frac{2}{(\alpha_2, \alpha_2)}) \varpi_k$ as in [14]. Indeed, $\varpi_k = \varpi_k$ in our case. Then we have

\[
\begin{align*}
    t(\varpi_k) &= \sigma^k(s_k s_{k'} \cdots s_1)(s_k s_{k'} s_{s_1} s_{s_2} \cdots s_{s_{n-1}} \cdots s_k), \\
    t(\sigma^{-1}\varpi_k) &= \sigma^k(s_{s_{k'} s_{s_1} s_{s_2} \cdots s_{s_{n-1}} \cdots s_k} s_k s_{k'} \cdots s_1),
\end{align*}
\]

where $k' = n+1-k$ and $\sigma$ is the Dynkin diagram automorphism as above. Set $w_1 := \sigma^{-k} t(\varpi_k) \in W$ and $w_2 := \sigma^{-k} t(\sigma^{-1}\varpi_k) \in W$. Associated with these Weyl group elements $w_1, w_2 \in W$, we shall define algebraic varieties $\mathcal{V}_1, \mathcal{V}_2 \subset W(\varpi_k)$. Take two variables $x, y$ of $(\mathbb{C}^\times)^{kk'}$ as follows:

\[
\left(\begin{array}{ccccccc}
    x_{(1)}^{(k)} & x_{(2)}^{(k)} & \cdots & x_{(k')}^{(k)} \\
    \vdots & \vdots & & \vdots \\
    x_{(1)}^{(2)} & x_{(2)}^{(2)} & \cdots & x_{(2)}^{(2)} \\
    x_{(1)}^{(1)} & x_{(1)}^{(1)} & \cdots & x_{(1)}^{(1)} \\
    x_{k+1}^{(1)} & x_{k+1}^{(1)} & \cdots & x_{n}^{(1)}
\end{array}\right), \quad
\left(\begin{array}{cccccc}
    y_{(1)}^{(k)} & y_{(2)}^{(k)} & \cdots & y_{(k')}^{(k)} \\
    \vdots & \vdots & & \vdots \\
    y_{(1)}^{(2)} & y_{(2)}^{(2)} & \cdots & y_{(2)}^{(2)} \\
    y_{(1)}^{(1)} & y_{(1)}^{(1)} & \cdots & y_{(1)}^{(1)} \\
    y_{k+1}^{(1)} & y_{k+1}^{(1)} & \cdots & y_{n}^{(1)}
\end{array}\right) \in (\mathbb{C}^\times)^{kk'},
\]
and set
\[
Y_{w_1}(x) = Y_k(x_k^{(k)}) \cdots Y_1(x_1^{(k)}) \cdots Y_n(x_n^{(1)}) \cdots Y_k(x_k^{(1)}),
\]
\[
Y_{w_2}(y) = Y_{k-1}(y_{k-1}^{(k)}) \cdots Y_0(y_0^{(k)}) \cdots Y_{n-1}(y_{n-1}^{(1)}) \cdots Y_{k-1}(y_{k-1}^{(1)}).
\]

Now, we define \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) as
\[
\mathcal{V}_1 := \{ v_1(x) := Y_{w_1}(x)u_1 | x \in (\mathbb{C}^\times)^{kk'} \} \subset W(\mathcal{W}_k), \quad (3.1)
\]
\[
\mathcal{V}_2 := \{ v_2(y) := Y_{w_2}(y)u_2 | y \in (\mathbb{C}^\times)^{kk'} \} \subset W(\mathcal{W}_k). \quad (3.2)
\]

Note that the dimensions of \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are \( kk' \).

Let us see the explicit \( g_0 \) (resp. \( g_n \)) -geometric crystal structure on \( \mathcal{V}_1 \) (resp. \( \mathcal{V}_2 \)) according to [23]. Indeed, the geometric crystal structure on \( \mathcal{V}_j \) \( (j = 1, 2) \) coincides with those on \( B_{w_j} \) \( (j = 1, 2) \), that is,
\[
\gamma_i(Y_{w_j}(x)u_j) = \gamma_i(Y_{w_j}(x)), \quad \varepsilon_i(Y_{w_j}(x)u_j) = \varepsilon_i(Y_{w_j}(x)),
\]
\[
e_i^j((Y_{w_j}(x)u_j) = Y_{w_j}(x')u_j, \quad (j = 1, 2).
\]

Set \( a := \max(k-i+1,1) \) and \( b = \min(k,n-i+1) \). Note that \( b \geq a \).

**Proposition 3.1.** The \( g_0 \)-geometric crystal structure of \( \mathcal{V}_1 = (\mathcal{V}_1, \{ \gamma_i \}, \{ \varepsilon_i \})_{i=1,2,\ldots,n} \) is described as follows: for \( v_1(x) = Y_{w_1}(x)^{(i)}u_1 \) and \( i = 1, 2, \ldots, n \), set
\[
D_i^{(l)}(x) := \frac{x_i^{(l)}(x_i^{(l+1)} \cdots x_i^{(b-1)} x_i^{(b)})^2}{x_i^{(l-1)} \cdots x_i^{(b+1)} x_i^{(b+2)} \cdots x_i^{(b-1)} x_i^{(b)} x_i^{(b+1)} \cdots x_i^{(b-1)} x_i^{(b)}}. \quad (3.3)
\]
\[
\gamma_i(v_1(x)) := D_i^{(a)} \frac{x_i^{(a)}}{x_i^{(a-1)}}, \quad \varepsilon_i(v_1(x)) := \sum_{l=1}^{b} \frac{1}{D_i^{(l)}}. \quad (3.4)
\]

Set \( e_i^j(v_1(x)) = v_1(x') \) \( (x = (x_m^{(l)}), x' = (x_m^{(l')})) \), then \( x_m^{(l')} \) is given by
\[
x_m^{(l')} = \begin{cases} 
\frac{\sum_{l=1}^{b} \frac{1}{D_i^{(l)}} + \sum_{p=1}^{c} \frac{1}{D_i^{(p)}}}{x_i^{(l)}}, & \text{if } m = i, \\
\sum_{l=1}^{b} \frac{1}{D_i^{(l)}} \sum_{p=1}^{c} \frac{1}{D_i^{(p)}}, & \text{otherwise.} 
\end{cases} \quad (3.5)
\]

For \( \mathcal{V}_2 \), we also obtain the similar description as follows: Set \( c := \max(k-i,1) \) and \( d = \min(k,n-i) \). Note that \( d \geq c \).

**Proposition 3.2.** The \( g_n \)-geometric crystal structure of \( \mathcal{V}_2 = (\mathcal{V}_2, \{ \gamma_i \}, \{ \varepsilon_i \})_{i=0,1,\ldots,n-1} \) is described as follows: for \( v_2(y) = Y_{w_2}(y)^{(i)}u_2 \) and \( i = 0, 1, 2, \ldots, n-1 \), set
\[
E_i^{(l)}(y) := \frac{y_i^{(l+1)} \cdots y_i^{(d-1)} y_i^{(d)} y_i^{(l)}}{y_i y_i^{(l+1)} \cdots y_i y_i^{(d-1)} y_i^{(d)} y_i^{(l+1)} \cdots y_i y_i^{(d-1)} y_i^{(d)}}. \quad (3.6)
\]
\( \tau_i(v_2(y)) := E_i^{(c)} \cdot \frac{y_i^{(c)}}{y_{i-1} y_{i+1}}, \quad \tau_i(v_2(y)) := \sum_{l=c}^d \frac{1}{E_i^{(c)}}. \) (3.7)

Set \( \tau'_i(v_2(y)) = v_2(y') (y = (y_m^{(i)}), y'(y_m^{(i)}), \text{then } y_m^{(i)'} \text{ is given by} \)

\[
y_m^{(i)'} = \begin{cases} 
\sum_{p=c}^{i-1} \frac{1}{E_i^{(p)}} + \sum_{p=i}^c \frac{c}{E_i^{(p)}} , & \text{if } m = i, \\
\sum_{p=c}^l \frac{1}{E_i^{(p)}} + \sum_{p=l+1}^c \frac{c}{E_i^{(p)}} , & \text{otherwise.} 
\end{cases} \] (3.8)

In the following sections, we shall define a birational isomorphism between \( V_1 \) and \( V_2 \), and using the isomorphism we shall patch them together to obtain an affine geometric crystal \( \mathcal{V}(A_n^{(1)}) \).

### 4. Lattice-Path Combinatorics

In this section, we shall treat rectangular lattices with certain weight on each edge. Here \( m \times n \)-lattice means a lattice whose vertical length is \( m \) and horizontal length is \( n \). Thus, for example, \( 2 \times 3 \)-rectangular lattice is visualized as

4.1 Weight of Paths

Now, we fix two positive integers \( n, k \) with \( k \leq n \) and take a \((k - 1) \times (n - k)\)-rectangular lattice. We associate a coordinate \((i, j) (1 \leq i \leq k < i + j \leq n + 1)\) to each lattice point on the \( i \)-th horizontal line from the bottom and \((i + j - k)\)-th line from the left and put a weight \( x_j^{(i)} \), which is denoted by \( L_1[n, k] = L_1[n, k](x_j^{(i)}) \) and called \( (x_j^{(i)}) \)-weighted lattice. Similarly, we define the \((y_j^{(i)})\)-weighted lattice \( L_2[n, k] = L_2[n, k](y_j^{(i)}) \) by associating a coordinate \((i, j) \) and a weight \( y_j^{(i)} \) \((1 \leq i \leq k \leq i + j \leq n)\) to each lattice point:
Indeed, those lattices are modelled after coordinates \(x = (x_j^{(i)})\), \(y = (y_j^{(i)}) \in (\mathbb{C}^x)^{kk'}\) as in Sect.4.

For the \(x\)-weighted lattice \(L_1[n,k](x)\), suppose that each horizontal strip \(s = (i,j)\rightarrow (i,j+1)\) has a weight \(wt(s) = \frac{x_i^{(j)}}{x_{j+1}^{(i)}}\) and each vertical strip has a weight 1. Similarly, for the \(y\)-weighted lattice \(L_2[n,k](y)\), suppose that each horizontal strip has a weight 1 and each vertical strip \(s = (i,j)\rightarrow (i-1,j+1)\) has a weight \(wt(s) = \frac{y_{i-1}^{(j)}}{y_j^{(i)}}\).

Set

\[P_i[n,k] := \{\text{shortest paths on } L_i[n,k] \text{ from } (k,2-i) \text{ to } (1,n+1-i)\} \quad (i = 1, 2).\]

For a path \(p \in P_1[n,k]\) we use the expression \(p = (s_1, s_2, \ldots, s_k)\) where each \(s_i\) is consecutive horizontal strips on the \(i\)-th horizontal line from the top, that is, \(s_i\) with length \(m\) is in the form:

\[s_i = (k-i+1,j)\rightarrow (k-i+1,j+1) \rightarrow \cdots \rightarrow (k-i+1,j+m).\]

(4.1)

Note that we allow that some strips have length 0. For a path \(p \in P_2[n,k]\), we also use the similar expression by considering vertical strips.

To each path \(p \in P_1[n,k](x)\) (resp.\(P_2[n,k](y)\)) we associate the weight

\[x(p)\text{(resp. } y(p)) := \prod_{s: \text{strip in } p} wt(s).\]

(4.2)

For \(x\text{(resp. } y\text{-weighted lattice } L_1[n,k](x)\text{(resp. } L_2[n,k](y)), define } \varepsilon = \varepsilon(x) \text{ (resp. } E_2 = E_2(y)\text{)} as a total sum of weights of all shortest paths, that is, \(\varepsilon = \varepsilon(x) := \sum_{p \in P_1[n,k](x)} x(p)\) (resp. \(E_2 = E_2(y) := \sum_{p \in P_2[n,k](y)} y(p)\)). For \(i = 1, 2\), define the following set of partial paths in \(L_i[n,k]\):

\[P_i[n,k]_{l,m}^{(i)} := \{\text{shortest (partial) paths from } (l,m) \text{ to } (1,n+1-i)\}\]

\[P_i[n,k]_{m}^{(i)} := \{\text{shortest (partial) paths from } (k,2-i) \text{ to } (l,m)\}\]

We also define the weight \(x(p)\) (resp. \(y(p)\)) of a path \(p \in P_i[n,k]_{l,m}^{(i)}, P_i[n,k]_{m}^{(i)}\) by the similar way as in (4.2) and their total summations:

\[X_{m}^{(i)} = X_{m}^{(i)}(x) := \sum_{p \in P_i[n,k]_{m}^{(i)}} x(p), \quad X_{m}^{*(i)} = X_{m}^{*(i)}(x) := \sum_{p \in P_i[n,k]_{m}^{(i)}} x(p),\]

(4.3)

\[Y_{m}^{(i)} = Y_{m}^{(i)}(y) := \sum_{p \in P_i[n,k]_{m}^{(i)}} y(p), \quad Y_{m}^{*(i)} = Y_{m}^{*(i)}(y) := \sum_{p \in P_i[n,k]_{m}^{(i)}} y(p).\]

(4.4)
Lemma 4.1. We have the formulae:

\[ X_m^{(l)} = X_m^{(l-1)} + \frac{x_m^{(l)}}{x_{m+1}^{(l)}} X_m^{(l)}, \quad (4.5) \]
\[ Y_m^{(l)} = Y_m^{(l-1)} + \frac{y_{l+1}^{(l-1)}}{y_{l}^{(l)}} Y_m^{l+1}, \quad (4.6) \]
\[ X_{m+1}^{(l)} = X_m^{(l+1)} + \frac{x_m^{(l)}}{x_{m+1}^{(l)}} X_m^{(l+1)}, \quad (4.7) \]
\[ Y_{m+1}^{(l)} = Y_m^{(l+1)} + \frac{y_{l}^{(l)}}{y_{l+1}^{(l)}} Y_m^{(l+1)}. \quad (4.8) \]

Proof. We observe that the set of paths \( P_1[n,k]_m^{(l)} \) are divided into the following two sets: one is the set of paths through \( (l-1, m+1) \) and the other is the set of paths through \( (l, m+1) \). Thus, \( \text{(4.3)} \) is immediate consequence of the observation, and also \( \text{(4.7)} \) and \( \text{(4.8)} \) are obtained similarly.

Now, we define rational maps \( \Sigma : (\mathbb{C}^*)^{kk'} \to (\mathbb{C}^*)^{kk'} \) \( (x \mapsto y) \) and \( \Xi : (\mathbb{C}^*)^{kk'} \to (\mathbb{C}^*)^{kk'} \) \( (y \mapsto x) \) by

\[ y_m^{(l)} = \Sigma(x)_m^{(l)} := x_m^{(l+1)} \frac{X_m^{(l)}(x)}{X_m^{(l+1)}(x)} \quad (1 \leq l \leq k \leq l+m \leq n), \quad (4.9) \]
\[ x_m^{(l)} = \Xi(y)_m^{(l)} := y_m^{(l)} \frac{Y_m^{(l)}(y)}{Y_m^{(l+1)}(y)} \quad (1 \leq l \leq k < l+m \leq n+1), \quad (4.10) \]

where \( X_m^{(l)} = \frac{1}{x_m^{(l)}} \) if \( l+m = k \) and \( Y_m^{(l)} = \frac{1}{y_m^{(l)}} \). Note that \( y_m^{(k)} = Y_m^{(k)} \) and then \( y_m^{(k)} = \frac{1}{x_m^{(k)}} \).

Theorem 4.2. The rational maps \( \Sigma \) and \( \Xi \) are both birational and bi-positive. Furthermore, they are inverse to each other, that is, \( \Sigma = \Xi^{-1} \) and \( \Xi = \Sigma^{-1} \).

Proof. Bi-positivity of \( \Sigma \) and \( \Xi \) follows from their explicit forms. Thus, to prove the theorem, it suffices to show the birationality. Since

\[ (\Xi \circ \Sigma(x))_m^{(l)} = x_m^{(l+1)} \frac{X_m^{(l)}(x)}{X_m^{(l+1)}(x)} \frac{Y_m^{(l-1)}(\Sigma(x))}{Y_m^{(l)}(\Sigma(x))}, \quad (4.11) \]
\[ (\Sigma \circ \Xi(y))_m^{(l)} = y_m^{(l+1)} \frac{Y_m^{(l)}(y)}{Y_m^{(l+1)}(y)} \frac{X_m^{(l)}(\Xi(y))}{X_m^{(l+1)}(\Xi(y))}, \quad (4.12) \]

we shall show

\[ \frac{x_m^{(l)}}{x_m^{(l+1)}} = \frac{X_m^{(l)}(x)}{X_m^{(l+1)}(x)} \frac{Y_m^{(l-1)}(\Sigma(x))}{Y_m^{(l)}(\Sigma(x))}, \quad (4.13) \]
\[ \frac{y_m^{(l)}}{y_m^{(l+1)}} = \frac{Y_m^{(l)}(y)}{Y_m^{(l+1)}(y)} \frac{X_m^{(l)}(\Xi(y))}{X_m^{(l+1)}(\Xi(y))}. \quad (4.14) \]

The formulæ \( \text{(4.13)} \) and \( \text{(4.14)} \) are immediate from the following lemma.
Lemma 4.3. We have the formula

\[ x_m^{(l)} = X_m^{(l)}(x)Y_m^{(l-1)}(\Sigma(x)) \quad (1 \leq l \leq k < l + m \leq n + 1), \quad (4.15) \]

\[ y_m^{(l)} = Y_m^{(l)}(y)X_m^{(l)}(\Xi(y)) \quad (1 \leq l \leq k \leq l + m \leq n) \quad (4.16) \]

Proof of Lemma 4.3. Since the proof of (4.16) is similar to (4.15), we shall show (4.15) only. Let \( S \) be the set of the coordinates of the lattice points in \( L_1[n, k](x) \), that is, \( S := \{(l, m)|1 \leq l < l + m \leq n + 1\} \). We define a total order on \( S \) by setting: \((l, m) < (l', m')\) if \( l > l'\), or \( l = l'\) and \( m < m'\). The proof of (4.15) is proceed by induction on this order.

First, let us see the case \( l = k \), which is shown by induction on \( m \). The case \( m = 1 \) is clear from that

\[ Y_1^{(k-1)}(\Sigma(x)) = \frac{y_1^{(k-1)}}{x_1^{(k)}}(\Sigma(x)) = \frac{y_1^{(k-1)}}{x_1^{(k)}}X_0^{(k)}X_1^{(k)} = \frac{x_1^{(k)}}{X_1^{(k)}} \]

Thus, for \( m > 1 \) we obtain from (4.4), (4.8) and the hypothesis of the induction that

\[ Y_m^{(k-1)}(\Sigma(x)) = Y_{m-1}^{(k-1)}(\Sigma(x)) + \frac{y_m^{(k-1)}}{x_m^{(k-1)}}Y_{m-1}^{(k)}(\Sigma(x)) = \frac{x_m^{(k)}}{X_m^{(k)}}X_{m-1}^{(k-1)}x_m^{(k)}X_m^{(k)} \]

\[ = \frac{x_m^{(k)}}{X_m^{(k)}}X_{m-1}^{(k)}x_m^{(k)}X_m^{(k)} \]

Therefore, we have (4.15) for \( l = k \).

Let us show \( Y_m^{(l-1)}(\Sigma(x)) = x_m^{(l)}X_m^{(l)} \) for \( l < k \). By the induction hypothesis, we may assume

\[ Y_m^{(l-1)}(\Sigma(x)) = \frac{x_m^{(l)}}{X_m^{(l)}}, \quad Y_m^{(l-1)}(\Sigma(x)) = \frac{x_m^{(l)}}{X_m^{(l)}}. \]

Here, by (4.5), (4.8) and the induction hypothesis (4.17) we have

\[ Y_m^{(l-1)}(\Sigma(x)) = Y_m^{(l-1)}(\Sigma(x)) + \frac{y_m^{(l-1)}}{x_m^{(l-1)}}Y_m^{(l-1)}(\Sigma(x)) = \frac{x_m^{(l-1)}}{X_m^{(l-1)}}x_m^{(l)}X_m^{(l)} \]

\[ = \frac{x_m^{(l-1)}}{X_m^{(l-1)}}x_m^{(l)}X_m^{(l)} \]

Now, we have completed the proof of Lemma 4.3.

The lemma shows (4.13) and (4.14), and then (4.9) and (4.10), which completes the proof of the theorem.

4.2 Triangular decomposition of paths

Here we consider the \( x \)-weighted lattice \( L_1[n, k](x) (x = (x_j^{(i)})) \) and define the triangular decomposition of the set of paths on \( L_1[n, k] \).
A path \( p \) in \( P_1[n, k] \) is called a path above (resp. path below) \((l, m)\) if for any point \((l, j)\) on \( p \), we have \( j > m \) (resp. \( j < m \)). Set 
\[
LP[n, k](l, m) := \{ p \in P_1[n, k] \mid p \text{ is a path through } (l, m) \},
\]
\[
LA[n, k](l, m) := \{ p \in P_1[n, k] \mid p \text{ is a path above } (l, m) \},
\]
\[
LB[n, k](l, m) := \{ p \in P_1[n, k] \mid p \text{ is a path below } (l, m) \},
\]
and define
\[
R_l^{(l)}(x) := \sum_{p \in LP[n, k](l, m)} x(p), \quad U_l^{(l)}(x) := \sum_{p \in LA[n, k](l, m)} x(p), \quad V_l^{(l)}(x) := \sum_{p \in LB[n, k](l, m)} x(p).
\]

It is obvious from the definitions that 
\[
\varepsilon(x) = R_l^{(l)}(x) + U_l^{(l)}(x) + V_l^{(l)}(x), \quad U_l^{(l-1)} = U_l^{(l)} + R_l^{(l)}(x), \quad V_l^{(l+1)} = V_l^{(l)} + R_l^{(l)}(x).
\]  

\textbf{Lemma 4.4.} We obtain the following formula:
\[
U_{l+1}^{(l)} = U_l^{(l+1)} + X_{m+1}^{(l+1)} \cdot \frac{x_{m+1}^{(l+1)}}{x_{m+1}}, \quad (4.19)
\]
\[
U_l^{(l-1)} = U_{l+1}^{(l)} + X_{m+1}^{(l)} \cdot \frac{x_{m+1}^{(l-1)}}{x_{m+1}}, \quad (4.20)
\]
\[
V_{m+1}^{(l+1)} = V_m^{(l+1)} + X_m^{(l+1)} \cdot \frac{x_m^{(l)}}{x_{m+1}}, \quad (4.21)
\]

\textbf{Proof.} By the definition of \( LA[n, k](l, m) \) we find that \( LA[n, k](l+1) \) is a subset of \( LA[n, k](l+1) \) and their difference \( LA[n, k](l+1) \setminus LA[n, k](l+1) \) coincides with the set of paths which go through \((l+1, m)\) and \((l, m+1)\), which shows \((4.19)\). Considering similarly, the difference \( LA[n, k](l+1) \setminus LA[n, k](l+1) \) is just the set of paths going through \((l, m)\) and \((l-1, m+1)\), which implies \((4.20)\). Similarly, we obtain that \( LB[n, k](l+1) \) is a subset of \( LB[n, k](l+1) \) and their difference is same as the set of paths which go through \((l+1, m)\) and \((l, m+1)\).

\[]

\section{Birational map \( \sigma \)}

\subsection{Birational and Bi-positive isomorphism between \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \)}

We shall define a birational and bi-positive isomorphism between \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) by virtue of the birational maps \( \Sigma \) and \( \Xi \) introduced in the previous section (Theorem \ref{thm:biholomorphic}.)

\textbf{Proposition 5.1.} Let \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) be geometric crystals introduced in \ref{sec:biholomorphic} and \( \Sigma \) and \( \Xi \) the birational maps as in the previous section. Define a rational map \( \sigma : \mathcal{V}_1 \to \mathcal{V}_2 \) by \( \sigma(v_1(x)) := v_2(\Sigma(x)) \). Then, the rational map \( \sigma \) is a birational and bi-positive map and its inverse is given as \( \sigma^{-1}(v_2(y)) = v_1(\Xi(y)) \), where \( v_1(x) \) and \( v_2(y) \) are as in \ref{sec:biholomorphic} and \ref{sec:biholomorphic} respectively.

\textbf{Proof.} Since we have the following commutative diagram:
\[
\begin{array}{ccc}
(C^\times)^{kk'} & \longrightarrow & (C^\times)^{kk'} \\
\Sigma \downarrow & & \downarrow \sigma \\
\mathcal{V}_1 & \longrightarrow & \mathcal{V}_2
\end{array}
\]

(5.1)

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and \( v_1, v_2 \) and \( \Sigma \) are birational, it is clear that \( \bar{\sigma} \) is birational. By the definition of positive structure of geometric crystal, bi-positivity of \( \Sigma \) means that of \( \bar{\sigma} \).

\[ \Box \]

5.2 Properties of \( \bar{\sigma} \)

To induce an \( A_n^{(1)} \)-geometric crystal structure on \( \mathcal{V}_1 \), we shall see some crucial properties of \( \bar{\sigma} \). Let \( \mathcal{V}_1 = (\mathcal{V}_1, \{c_i\}, \{\gamma_i\}, \{\varepsilon_i\})_{i \in I_0} \) and \( \mathcal{V}_2 = (\mathcal{V}_1, \{\bar{c}_i\}, \{\bar{\gamma}_i\}, \{\bar{\varepsilon}_i\})_{i \in I_n} \) be the geometric crystals as above.

**Proposition 5.2.** For \( i \in \{1, 2, \ldots, n-1\} = I_0 \cap I_n \), we have

\[
\bar{\sigma} \circ c_i^c = \bar{c}_i^c \circ \bar{\sigma}, \quad \gamma_i = \bar{\gamma} \circ \bar{\sigma}, \quad \varepsilon_i = \bar{\varepsilon}_i \circ \bar{\sigma} \tag{5.2}
\]

**Proof.** Let \( y = \Sigma(x) \). Then, we have

\[
y_m^{(c)} y_m^{(m+1)} \cdots y_m^{(d)} = x_m^{(c-1)} x_m^{(c+2)} \cdots x_m^{(d+1)} \frac{X_m^{(c)}}{X_m^{(d+1)}} \tag{5.3}
\]

where if \( (l, m) \) is out of the lattice \( L_1[n, k] \), we understand \( X_m^{(l)} = 1 \). Therefore, applying (5.3) to (3.7) we have

\[
\gamma_i(v_1(x)) = \left( \frac{x_i^{(c+1)} x_i^{(c+2)} \cdots x_i^{(d+1)} X_i^{(c)}}{x_i^{(d+1)} x_i^{(c+1)} x_i^{(c+2)} \cdots x_i^{(d+1)} X_i^{(d+1)}} \right)^2.
\]

Here we set

\[
x_i(p, q) := x_i^{(p)} x_i^{(p+1)} \cdots x_i^{(q)} \quad \gamma_i(p, q) := \gamma_i x_i^{(p)} x_i^{(p+1)} \cdots x_i^{(q)}.
\]

Let us consider the following four cases (a)–(d):

(a) \( k - i \geq 1, n - i \geq k \). (b) \( k - i < 1, n - i \geq k \). (c) \( k - i < 1, n - i < k \). (d) \( k - i \geq 1, n - i < k \).

(a) In this case, \( a = k - i + 1, b = k, c = k - i \) and \( d = k \). Then we have

\[
\gamma_i(v_1(x)) = \left( \frac{x_i(k - i + 1, k) X_i^{(k-i)}}{x_i(k - i + 1) X_i^{(k-i) - 1} x_i(k - i, k) X_i^{(k-i) - 1}} \right)^2
\]

\[
= \left( \frac{x_i(k - i + 1, k)}{x_i(k - i + 1) x_i(k - i) x_i^{(k-i) - 1}} \right)^2
\]

\[
= \left( \frac{x_i(k - i + 1, k)}{x_i(k - i + 1) x_i(k - i, k) X_i^{(k-i) - 1}} \right)^2
\]

\[
= \frac{x_i(k - i + 1, k)^2}{x_i(k - i + 1, k) x_i(k - i, k) x_i^{(k-i) - 1}} = \gamma_i(v_1(x)).
\]
where note that $X^{(k+1)}_m = x^{(k+1)}_m = 1$ for any $m$ and $X^{(l)}_m = \frac{1}{x^{(l)}_m}$ if $l + m = k$.

(b) In this case, $a = 1$, $b = k$, $c = 1$ and $d = k$. Then, we find that

$$\tau_i(\sigma(v_1(x))) = \frac{\left( x_i(2, k) X_i^{(1)} \right)^2}{x_{i-1}(2, k) X_i^{(1)} \cdot x_{i+1}(2, k) X_i^{(1)}}$$

$$= \frac{\left( x_i(2, k) x_i^{(2)}(x_i^{(2)}) \right)^2}{x_{i-1}(2, k) x_i^{(2)}(x_i^{(2)})} = \frac{x_i(1, k)^2}{x_{i-1}(1, k) x_{i+1}(1, k)} = \gamma_i(v_1(x)),$$

where note that $x^{(k+1)}_{i-1} = 1$.

(c) In this case, $a = k - i + 1$, $b = n - i + 1$, $c = k - i$ and $d = n - i$. Then, we find that

$$\tau_i(\sigma(v_1(x))) = \frac{\left( x_i(k - i + 1, n - i + 1) X_i^{(k - i + 1)} \right)^2}{x_{i-1}(k - i + 1, n - i + 2) X_i^{(k - i + 1)} x_{i+1}(k - i, n - i + 1) X_i^{(k - i - 1)}}$$

$$= \frac{x_i(k - i + 1, n - i + 1)^2}{x_{i-1}(k - i + 1, n - i + 2) x_{i+1}(k - i, n - i + 1)} = \gamma_i(v_1(x)),$$

where note that $x^{(n-i+1)}_{i+1} = 1$ and $X^{(l)}_m = 1$ if $l + m = n + 1$.

(d) In this case, $a = 1$, $b = n - i + 1$, $c = 1$ and $d = n - i$. Then, we find that

$$\tau_i(\sigma(v_1(x))) = \frac{\left( x_i(2, n - i + 1) X_i^{(1)} \right)^2}{x_{i-1}(2, n - i + 2) X_i^{(1)} x_{i+1}(2, n - i) X_i^{(1)}}$$

$$= \frac{\left( x_i(2, n - i + 2) x_i^{(1)} \right)^2}{x_{i-1}(2, n - i + 2) x_i^{(1)} x_{i+1}(2, n - i + 1) x_i^{(1)}} = \frac{x_i(1, n - i + 1)^2}{x_{i-1}(1, n - i + 2) x_{i+1}(1, n - i + 1)} = \gamma_i(v_1(x)),$$

where note that $x^{(n-i+1)}_{i+1} = 1$. Thus, we obtain $\tau_i(\sigma(v_1(x))) = \gamma_i(v_1(x))$ ($i = 1, 2, \cdots, n - 1$).

Next, we shall see $\tau_i(\sigma(x)) = \varepsilon_i(x)$ for $i = 1, 2, \cdots, n - 1$. It follows from the explicit forms $[3.3]$ and $[3.4]$ that

$$E_i^{(l)}(\sigma(v_1(x))) = D_i^{(l+1)} \frac{X_i^{(l)} X_i^{(l+1)}}{X_i^{(1)} X_i^{(l+1)}}, \quad \text{for } l = 0, 1, \cdots, n - 1,$$

where $D_i^{(l)} = D_i^{(l)}(v_1(x))$, $X_i^{(l)} = X_i^{(l)}(v_1(x))$, etc. We get $X_i^{(l+1)} = X_i^{(l)} + \frac{x_i^{(l+1)}}{x_i^{(l+1)}} X_i^{(l+1)}$ by $[3.5]$.

Thus, substituting this to $[5.4]$, we have

$$\frac{1}{E_i^{(l)}(\sigma(v_1(x)))} = \frac{X_i^{(l+1)}}{D_i^{(l+1)} X_i^{(l+1)}} \left( X_i^{(l)} + \frac{x_i^{(l+1)}}{x_i^{(l+1)}} X_i^{(l+1)} \right).$$
As above we consider the cases (a)–(d).

(a) In this case, \( a = k - i + 1 \), \( b = k \), \( c = k - i \) and \( d = k \).

\[
\varepsilon_i(\mathcal{B}(v(x))) = \sum_{l=k-i}^{k} \frac{1}{E_i(l)}(\mathcal{B}(v(x)))
\]

\[
= \frac{X_{i+1}^{(k-i)}}{D_i^{(k-i+1)}} X_i^{(k-i+1)} + \sum_{l=i+1}^{k-1} \left( \frac{X_i^{(l)}}{D_i^{(l+1)}} X_i^{(l+1)} + \frac{X_i^{(l)} X_i^{(l+1)}}{D_i^{(l+1)} X_i^{(l+1)}} \right) + \frac{X_i^{(k)}}{X_i^{(k)}}
\]

\[
= \left( \frac{X_{i+1}^{(k-i)}}{D_i^{(k-i+1)}} X_i^{(k-i+1)} + \frac{X_i^{(k-i+1)} X_i^{(k-i+2)}}{D_i^{(k-i+2)} X_i^{(k-i+2)}} \right) + \left( \frac{X_i^{(k)} X_i^{(k+1)}}{D_i^{(k+1)} X_i^{(k+1)}} \right)
\]

\[
+ \sum_{l=i+2}^{k-1} \left( \frac{X_i^{(l-1)}}{D_i^{(l)}} X_i^{(l)} + \frac{X_i^{(l+1)}}{D_i^{(l+1)} X_i^{(l+1)}} \right) + \frac{1}{D_i^{(k+1)}} + \sum_{l=k+2}^{k-1} \frac{1}{D_i^{(l)}}
\]

\[
= \frac{1}{D_i^{(k)}} + \sum_{l=2}^{k-1} \frac{1}{D_i^{(l)}} = \varepsilon_i(v(x)),
\]

where for the 4th equality we use \( \varepsilon \), \( D_i^{(k)} = \frac{x_i^{(k)}}{x_i+i} \) and \( D_i^{(l+1)} = \frac{x_i^{(l+1)} x_i^{(l)}}{x_i+i x_i+i+1} D_i^{(l)} \).

(b) \( a = 1 \), \( b = k \), \( c = 1 \) and \( d = k \).

\[
\varepsilon_i(\mathcal{B}(v(x))) = \sum_{l=2}^{k} \frac{1}{E_i(l)}(\mathcal{B}(v(x))) = \sum_{l=1}^{k-1} \left( \frac{X_i^{(l)}}{D_i^{(l+1)}} X_i^{(l+1)} + \frac{X_i^{(l)} X_i^{(l+1)}}{D_i^{(l+1)} X_i^{(l+1)}} \right) + \frac{X_i^{(k)}}{X_i^{(k)}}
\]

\[
= \frac{x_i^{(k)}}{x_i^{(k)}} X_i^{(k)} + \frac{X_i^{(k)} X_i^{(k+1)}}{D_i^{(k+1)} X_i^{(k+1)}} + \sum_{l=2}^{k-1} \left( \frac{X_i^{(l)}}{D_i^{(l)}} X_i^{(l+1)} + \frac{X_i^{(l)} X_i^{(l+1)}}{D_i^{(l+1)} X_i^{(l+1)}} \right)
\]

\[
= \frac{1}{D_i^{(k)}} + \sum_{l=2}^{k-1} \frac{1}{D_i^{(l)}} = \varepsilon_i(v(x)),
\]

where for the 4th equality we also use \( \varepsilon \), \( D_i^{(k)} = \frac{x_i^{(k)}}{x_i+i} \) and \( D_i^{(l+1)} = \frac{x_i^{(l+1)} x_i^{(l)}}{x_i+i x_i+i+1} D_i^{(l)} \).

For the cases (c) and (d), we can show \( \varepsilon_i(\mathcal{B}(v(x))) = \varepsilon_i(v(x)) \) similarly.

Finally, let us show that \( \sigma \circ e_i^a = \varepsilon \circ \sigma \) \((i = 1, 2, \ldots, n - 1)\). Set \( v_2'(y') := \varepsilon_m(\mathcal{B}(v(x))) \) \((m = 1, 2, \ldots, n - 1)\) and show \( y_j^{m(i)} = y_j^{n(i)} \) for any \( i \in \{1, 2, \ldots, k\} \) and \( j = c, c+1, \ldots, d \).

We shall see the case \( j \neq m \). In this case, since \( y_j^{l(i)} = x_j^{(l+1)} X_j^{l(i)}(x) x_j^{(l+1)}(x) \) and \( y_j^{n(i)} = y_j^{l(i)} = x_j^{(l+1)} X_j^{l(i)}(x) X_j^{l(i)}(x) \), it suffices to show \( X_j^{(l)}(x) = X_j^{(l)}(x') \) for \( j \neq m \). Here, suppose \( j > m \). In this case, \( X_j^{(l)} \) does not depend on \( x_m \)'s by its definition and then we find that \( X_j^{(l)}(x) = X_j^{(l)}(x') \) for \( j > m \).

Next, suppose \( j = m - 1 \). We set

\[
A_i^{(l)} := \sum_{p=c}^{l-1} \frac{1}{E_i^{(p)}} + \sum_{p=l}^{d} \frac{c}{E_i^{(p)}}, \quad B_i^{(l)} := \sum_{p=a}^{l-1} \frac{1}{D_i^{(p)}} + \sum_{p=l}^{b} \frac{c}{D_i^{(p)}}. \tag{5.6}
\]

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Let us denote $X_j^{(l)}(x')$ by $X_j^{(l)}$. Applying the formula (4.5), we get

$$X_m^{(l)} = X_m^{(l-1)} + \frac{x_m^{(l)}}{x_m^{(l-1)}} X_m^{(l)} = \left( X_m^{(l-2)} + \frac{x_m^{(l-1)}}{x_m^{(l-1)}} X_m^{(l-1)} \right) + \frac{x_m^{(l)}}{x_m^{(l-1)}} \left( X_m^{(l-1)} + \frac{x_m^{(l-1)}}{x_m^{(l-1)}} X_m^{(l)} \right),$$

where

$$\frac{x_m^{(l)}}{x_m^{(l-1)}} + \frac{x_m^{(l-1)}}{x_m^{(l)}} = \frac{1}{B_m^{(l)}} \left( \frac{x_m^{(l-1)}}{x_m^{(l-1)}} B_m^{(l)} + \frac{x_m^{(l)}}{x_m^{(l)}} B_m^{(l+1)} \right)$$

$$= \frac{1}{B_m^{(l)}} \left( \frac{x_m^{(l-1)}}{x_m^{(l-1)}} B_m^{(l)} + \frac{\alpha}{D_m^{(l-1)}} - \frac{1}{D_m^{(l)}} \right) + \frac{x_m^{(l-1)}}{x_m^{(l)}} \left( \frac{B_m^{(l)}}{D_m^{(l)}} - \frac{1}{D_m^{(l)}} \right) = \frac{x_m^{(l-1)}}{x_m^{(l-1)}} \frac{D_m^{(l-1)}}{D_m^{(l)}} + \frac{x_m^{(l-1)}}{x_m^{(l)}} ,$$

so that the last equality we use the relation $D_m^{(l-1)} = \frac{x_m^{(l-1)}}{x_m^{(l-1)}} D_m^{(l)}$. Thus, substituting this to (5.7), we obtain

$$X_m^{(l)} = \left( X_m^{(l-2)} + \frac{x_m^{(l-1)}}{x_m^{(l-1)}} X_m^{(l-1)} \right) + \frac{x_m^{(l-1)}}{x_m^{(l-1)}} \left( X_m^{(l-1)} + \frac{x_m^{(l-1)}}{x_m^{(l-1)}} X_m^{(l)} \right) = X_m^{(l-1)}.$$

Let us show the cases $j < m - 1$ using descending induction on $j$. Applying (4.5) and the induction hypothesis, we have

$$X_j^{(l)} = X_j^{(l-1)} + \frac{x_j^{(l)}}{x_j^{(l-1)}} X_j^{(l-1)} = X_j^{(l-1)} + \frac{x_j^{(l)}}{x_j^{(l-1)}} X_j^{(l-1)} = X_j^{(l)} ,$$

which completes the proof of $X_j^{(l)} = X_j^{(l)}$ for $j \neq m$ and then $y_j^{(l)} = \frac{x_j^{(l)}}{x_j^{(l-1)}}$ for $j \neq m$.

Finally, let us show the case $m = j$, that is, $y_m^{(l)} = \frac{x_m^{(l)}}{x_m^{(l-1)}}$ for $l = 1, 2, \cdots, k$. Let $A_i^{(l)}$ and $B_i^{(l)}$ be as in (5.0). By the explicit forms of $e_m^{(l)}, \tau_m$ and $\sigma$, we have

$$y_m^{(l)} = \frac{x_m^{(l+1)}}{B_m^{(l+2)}} X_m^{(l+1)}, \quad y_m^{(l)} = \frac{x_m^{(l+1)}}{B_m^{(l+1)}} \left( \sigma(x) X_m^{(l)} \right)$$

Thus, we may show

$$B_m^{(l+1)} X_m^{(l)} = A_m^{(l)}(\sigma(x)) X_m^{(l)}$$

Suppose $\alpha = 1$. Since we have $B_m^{(l+1)}(x) |_{\alpha=1} = \tau_m(x)$, $A_m^{(l)}(\sigma(x)) |_{\alpha=1} = \tau_m(\sigma(x))$, $X_m^{(l)} = X_m^{(l)}$ and $\tau_m(x) = \tau_m(\sigma(x))$, we know that (5.9) holds for the case $\alpha = 1$. 

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Suppose that \( \alpha \) is generic. Since for \( j > m \) we have \( X_j^{(l)} = X_j^{(l)} \), we get

\[
B_m^{(l+1)} \sum_{m+1} X_m^{(l)} = B_m^{(l+1)} \left( \sum_{m+1} X_m^{(l)} \right) = B_m^{(l+1)} \sum_{m+1} X_m^{(l)} + B_m^{(l+1)} X_m^{(l)} + B_m^{(l+1)} X_m^{(l)} = B_m^{(l+1)} \sum_{m+1} X_m^{(l)} + B_m^{(l+1)} X_m^{(l)}.
\]

Thus, for \( l = 0 \), we have

\[
\sum_{m+1} X_m^{(l)} = 0.
\]

Therefore, since we obtained (5.9) for \( \alpha = 1 \), we may show the coefficients of \( \alpha \) in (5.10) and (5.11) coincide each other, that is,

\[
\sum_{m+1} X_m^{(l)} = 0, \quad \text{for } l = 0.
\]

We shall show (5.12) by descending induction on \( l \). Suppose \( l = d \). If \( k \leq n - m \), we have \( l = d = \min(k, n - m) = k \) and \( b = \min(k, n - m + 1) = k \), and then

\[
\text{L.H.S. of (5.12)} = \sum_{m+1} X_m^{(l)} D_m^{(l+1)} = X_m^{(k)}, \quad \text{R.H.S. of (5.12)} = X_m^{(k)} D_m^{(k+1)} X_m^{(k)} = X_m^{(k)}.
\]

In the case \( k > n - m \), we have \( l = d = n - m \) and \( b = n - m + 1 \), and then

\[
\text{L.H.S. of (5.12)} = \sum_{m+1} X_m^{(l)} D_m^{(l+1)} = X_m^{(n-m)} \left( \sum_{m+1} D_m^{(n-m)} \right) + X_m^{(n-m)} \left( \sum_{m+1} D_m^{(n-m+1)} \right), \quad \text{R.H.S. of (5.12)} = X_m^{(n-m)} \left( \sum_{m+1} D_m^{(n-m)} \right) X_m^{(n-m)} = X_m^{(n-m)}.
\]

where note that \( X_m^{(n-m)} = X_m^{(n-m)} = 1 \). Thus, we obtain (5.12) for \( l = d \). Here, suppose \( l < d \).
By the induction hypothesis, we have

\[ \text{R.H.S. of (5.12)} = X^{(l)}_m \frac{X^{(l+1)}_{m+1}}{D^{(l+1)}_m X^{(l+1)}_m} + X^{(l)}_m \times \text{L.H.S. of (5.12)} \text{ for } l+1 \]

\[ = \frac{X^{(l)}_{m+1}X^{(l+1)}_{m-1}}{D^{(l+1)}_m X^{(l+1)}_m} + X^{(l)}_m \left( \frac{x^{(l+1)}_{m+1}X^{(l+1)}_{m}}{x^{(l+1)}_{m+1}D^{(l+1)}_m X^{(l+1)}_m} + \sum_{p=2}^{b} \frac{1}{D^{(p)}_m} \right) \]

\[ = \frac{x^{(l)}_m X^{(l)}_{m+1}}{x^{(l)}_{m+1}D^{(l)}_m} + X^{(l)}_m \sum_{p=2}^{b} \frac{1}{D^{(p)}_m} = \text{L.H.S. of (5.12)}, \]

which shows (5.12) and then we obtain \( y^{(l)}_m = y^{(l)}_m \), that is, \( \sigma \circ e_i^a = e_i^a \circ \sigma \) for \( i = 1, 2, \cdots, n - 1 \).

Now, we have completed the proof of Proposition 5.2.

6 Affine geometric crystal \( V(A^{(1)}_n) \)

6.1 0-structures on \( V_1 \)

In this subsection, we induce the 0-structures on \( g_0 \)-geometric crystal \( V_1 \). Since \( V_2 \) is a \( g_n \)-geometric crystal, it is equipped with the 0-structure, on the other hand, \( V_1 \) does not hold 0-structure. We shall, however, define 0-structure on \( V_1 \) through the birational isomorphism \( \sigma \) by:

\[ \gamma_0 := \gamma_0 \circ \sigma, \quad \varepsilon_0 := \varepsilon_0 \circ \sigma, \quad e_0^c := \sigma^{-1} \circ \varepsilon_i \circ \sigma \]  

(6.1)

The following is one of the main theorems of this paper, which will be shown in the later subsections.

**Theorem 6.1.** \( g_0 \)-geometric crystal \( V_1 \) becomes an \( A^{(1)}_n \)-geometric crystal if it is equipped with 0-structures as in (6.1), which will be denoted by \( V(A^{(1)}_n) \). Furthermore, \( v_1 : (\mathbb{C}^\times)^{kk'} \rightarrow V(A^{(1)}_n) \) gives a positive structure.

Before showing this theorem, let us see the explicit forms of \( \gamma_0, \varepsilon_0 \) and \( e_0^c \).

**Proposition 6.2.** Let us set \( v_1(x') = e_0^c(v_1(x)) \) \( (c \in \mathbb{C}^\times) \). Then we have the following:

\[ \gamma_0(v_1(x)) = \frac{1}{x^{(1)}_n x^{(l)}_1}, \quad \varepsilon_0(v_1(x)) = x^{(1)}_n x^{(l)}_1, \quad (6.2) \]

\[ x^{(l)}_m = \begin{cases} x^{(l)}_m & \text{if } (l, m) \neq (1, n), \\ \alpha^{(l, m)}_m & \text{if } (l, m) = (1, n) \end{cases} \]  

(6.3)

where \( \alpha^{(l, m)}_m = \alpha^{(l, m)}(c) = U^{(l-1)}_m + cV^{(l)}_m \), and \( U^{(l)}_m \) and \( V^{(l)}_m \) are as in (4.2).
Proof. First, let us show \( \gamma_0(v_1(x)) = \tau_0(\sigma(v_1(x))) \). Set \( v_2(y) = v_2((y')^i) = \sigma(v_1(x)) \). By the explicit form of \( \sigma \) we know that \( y^{(l)}_m = x^{(l+1)}_m X^{(l)}_{m+1} \), and then have

\[
\tau_0(\sigma(v_1(x))) = \tau_0(v_2(y)) = \frac{y^{(k)}_0}{y^{(k-1)}_0 y^{(k)}_1} = \frac{1}{x^{(k)}_1 X^{(k)}_1} = \frac{1}{x^{(1)}_1 x^{(1)}_1} = \gamma_0(v_1(x)).
\]

Next, let us show \( \varepsilon_0(v_1(x)) = \tau_0(\sigma(v_1(x))) \). By (3.7), we get \( \tau_0(v_2(y)) = \frac{y^{(k)}_0}{y^{(k)}_0} \), and then

\[
\tau_0(v_2(y)) = \frac{y^{(k)}_1}{y^{(k)}_0} = \frac{x^{(1)}_1 X^{(1)}_1}{x^{(1)}_1} = x^{(1)}_1 \varepsilon(x) = \varepsilon_0(v_1(x)).
\]

Finally, it is sufficient to show \( \varepsilon_0 \circ \sigma = \sigma \circ \varepsilon_0 \). Set \( v_2(y') = \varepsilon_0 \circ \sigma(x) = \varepsilon_0(y') \) and \( v_2(y'') = \sigma \circ \varepsilon_0(x) = \sigma(x') \). We have \( \gamma^{(0)}_0 = c y^{(0)}_0, c = x^{(0)}_n \), and \( \gamma^{(1)}_0 = \frac{1}{x^{(1)}_1} \), which shows the case \( (l, m) = (k, 0) \).

Now, we define the total order \( \prec \) on the set of coordinates \( \mathcal{F} := \{ (l, n) \mid 1 \leq l \leq k \leq l + m \leq n \} \backslash \{ (k, 0) \} \) of the lattice \( L_2[n, k] \) by setting \( (l, m) \prec (l', m') \) if \( l < l' \) and \( m > m' \) if \( l = l' \). Thus, \( (1, n - 1) \) is the minimum and \( (k, 1) \) is the maximum. We shall show \( \gamma^{(0)}_m = y^{(m)}_m \) by the induction according to \( \prec \). Let us see the case \( (l, m) = (1, n - 1) \).

\[
y^{(1)}_{n-1} = x^{(1)}_{n-1} x^{(2)}_{n-1} X^{(1)}_{n-1} = x^{(1)}_{n-1} x^{(2)}_{n-1},
\]

\[
y^{(2)}_{n-1} = x^{(1)}_{n-1} x^{(2)}_{n-1} X^{(1)}_{n-1} = x^{(1)}_{n-1} x^{(2)}_{n-1},
\]

which shows \( \gamma^{(0)}_{n-1} = y^{(n-1)}_{n-1} \).

Let us see \( (l, m) = (1, n - 1) \). We assume \( \gamma^{(r)}_{q} = y^{(r)}_{q} \) for \( q < l \) and any \( r \). Thus, multiplying both sides for \( q = 1, 2, \ldots, l - 1 \) we get \( x^{(1)}_{r} x^{(2)}_{r} x^{(3)}_{r} \cdots x^{(r)}_{r} X^{(1)}_{r} = x^{(2)}_{r} x^{(3)}_{r} \cdots x^{(r)}_{r} X^{(1)}_{r} \) and then for \( r = m, m + 1 \)

\[
\frac{\alpha^{(2)}_{m} X^{(2)}_{m}}{\alpha^{(1)}_{m} X^{(1)}_{m}} = \frac{X^{(1)}_{m}}{X^{(1)}_{m}}, \quad \frac{\alpha^{(2)}_{m+1} X^{(2)}_{m+1}}{\alpha^{(1)}_{m+1} X^{(1)}_{m+1}} = \frac{X^{(1)}_{m+1}}{X^{(1)}_{m+1}}. \quad (6.4)
\]

Substituting \( X^{(1)}_{m+1} = \frac{x^{(1)}_{m+1}}{x^{(1)}_{m}} \), to (6.4), we obtain

\[
X^{(l)}_{m+1} = \frac{c \alpha^{(1)}_{m+1} X^{(1)}_{m+1}}{\alpha^{(1)}_{m+1} X^{(1)}_{m+1}}. \quad (6.5)
\]
Since we also assume that \( y^p_m = y_p^m \) for \( p > m \), we have \( x^\prime_m X^\prime_m X^\prime_{m+1} = x^\prime_{m+1} X^\prime_{m+1} \). Substituting this to \( X^\prime_m = X^\prime_m + \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} \), we find that

\[
X^\prime_m = X^\prime_m \left( 1 + \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} \right). \tag{6.6}
\]

Now, applying (6.4) to (6.6), we obtain

\[
X^\prime_m = \frac{\alpha_m^{(1)} X^\prime_m}{\alpha^{(1)}_{m+1}} \left( 1 + \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} \right). \tag{6.7}
\]

Using (6.4) and (6.7), we get

\[
y^p_m = x^p_m x^\prime_m = x^p_m \frac{\alpha_m^{(1)} X^\prime_m}{\alpha^{(1)}_{m+1}} \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} = \frac{\alpha_m^{(1)} X_m}{\alpha^{(1)}_{m+1}} \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} \tag{6.8}
\]

where for the last equality we use \( \alpha_m^{(1)} = \alpha^{(1)}_{m+1} = \varepsilon \). Here,

\[
\frac{\alpha_m^{(1)} X_m}{\alpha^{(1)}_{m+1}} \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} = (U^\prime + cR^\prime_m + eV^\prime_m)X^\prime_m + \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} (U^\prime + cR^\prime_m + eV^\prime_m)X^\prime_{m+1}
\]

\[
= (U^\prime + cR^\prime_m) \left( X^\prime_m + \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} X^\prime_{m+1} \right) + R^\prime_m \left( cX^\prime_m + \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} X^\prime_{m+1} \right)
\]

\[
= (U^\prime + cR^\prime_m) X^\prime_m + X^\prime_m X^\prime_{m+1} \left( cX^\prime_m + \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} X^\prime_{m+1} \right)
\]

\[
= X^\prime_m \left( U^\prime + \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} X^\prime_{m+1} \right) + c(R^\prime_m + X^\prime_m X^\prime_{m+1})
\]

\[
= X^\prime_m \left( U^\prime + \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} X^\prime_{m+1} \right) = X^\prime_m \left( \frac{\alpha_m^{(1)} X^\prime_m}{\alpha^{(1)}_{m+1}} \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} \right) \tag{6.9}
\]

where for the third equality we use the formula (4.20) and formula \( R^\prime_m = X^\prime_m X^\prime_{m+1} \) and, for the last equality we use the formula (4.19) and (4.21) in Lemma 4.11. Applying this to (6.8), we obtain

\[
y^p_m = x^p_m \frac{\alpha_m^{(1)} X^\prime_m}{\alpha^{(1)}_{m+1}} \frac{x^\prime_m X^\prime_{m+1}}{x^\prime_{m+1}} = x^p_m \frac{X_m}{X^\prime_m} = \frac{y^p_m}{y^p_m} = \frac{y^p_m}{y^p_m},
\]

which shows \( \sigma \circ e_0^p = e_0^p \circ \sigma \).
6.2 Proof of Theorem 6.1

In the rest of the section, for simplicity, let us denote $v_1(x)$ by $x$ and $v_2(y)$ by $y$, e.g., $\varepsilon_i(x)$ for $\varepsilon_i(v_1(x))$.

The positivity of $v_1$ on $V(A_n^{(1)})$ is trivial from the explicit forms of $\gamma_i$, $\varepsilon_i$ and $\tilde{e}_i^\varepsilon$ ($i \in I$) as in Proposition 5.2 and Proposition 6.1.

To prove Theorem 6.1 it suffices to show the relations of the geometric crystals related to 0-structures, that is,

$$
\begin{align*}
\gamma_0(e_i^0(x)) &= c^{\alpha_0} \gamma_0(x), \\
\varepsilon_i(e_i^0(x)) &= c^{-1} \varepsilon_i(x), \\
\varepsilon_i e_i^d &= e_i^c e_i^d, \quad (i = 2, 3, \cdots, n - 1), \\
e_i^c e_i^d e_0^0 &= e_i^d e_i^c e_0^0, \\
e_i^c e_i^d e_0^0 &= e_i^d e_i^c e_0^0.
\end{align*}
$$

As for (6.10), by Proposition 5.2 and 6.1, for $i = 0, 1, \cdots, n - 1$ we have

$$
\gamma_0(e_i^0(x)) = \gamma_0(\gamma_i(\varepsilon_i(x))) = \gamma_0(\gamma_i(\varepsilon_i(\varepsilon_i(x)))) = \gamma_0(\gamma_i(\varepsilon_i)) = \gamma_0(\varepsilon_i) = c^{\alpha_0} \gamma_0(x).
$$

Similarly, for $i = 1, \cdots, n - 1$ we obtain

$$
\gamma_i(e_i^0(x)) = \gamma_i(\varepsilon_i(\gamma_0(\varepsilon_i(x)))) = \gamma_i(\varepsilon_i(\gamma_0)) = c^{\alpha_0} \gamma_i(x).
$$

For $i = n$ case, by the explicit form of $\gamma_0$ we have

$$
\gamma_0(e_n^0(x)) = \frac{1}{ce_n^0} = c^{-1} \gamma_0(x).
$$

Similarly, by the explicit form of $\gamma_n$ we get

$$
\gamma_n(e_n^0(x)) = \frac{\gamma_n^2}{c_{e_n^0}} = \frac{x_n^{(1)}}{c_{e_n^0}} = c^{-1} \gamma_n(x),
$$

where note that $\gamma_n^2 = c_{e_n^0}$.

As for (6.11), due to (6.11) and the fact $\tilde{\varepsilon}_0(\tilde{\varepsilon}_0(v_2(y))) = c^{-1} \tilde{\varepsilon}_0(v_2(y))$ we have

$$
\varepsilon_i(e_i^0(x)) = \tilde{\varepsilon}_0(\tilde{\varepsilon}_0(e_i^0((x)))) = \tilde{\varepsilon}_0(\tilde{\varepsilon}_0) = c^{-1} \varepsilon_i(x).
$$

Since we have $\tilde{\varepsilon}_0 = c_{e_i^0} = c_{e_i^0}$ ($i = 2, \cdots, n - 1$) and $\tilde{\varepsilon}_0 = c_{e_i^0}$ on $V_2$, by using Proposition 5.2 and 6.1 we get (6.12) and (6.13). Thus, the remaining case to show is (6.14), which is the most difficult one since we have to show it by direct calculations.

6.3 Formula for Functions on Paths

In this subsection, for a function or a map $f$ on $V_1$, we shall denote $f(e_i^0(x))$ ($d \in \mathbb{C}$) by $f(x)$ for simplicity (except for $\gamma_i$, $\varepsilon_i$, $\tilde{e}_i^\varepsilon$). Then, e.g., $X_m^{(1)}(e_i^0(x))$ implies $X_m^{(1)}(e_i^0(x))$.

We shall give several formula for the functions related to the actions of $e_i^0$. 23
Lemma 6.3. We have the following formula:

\[
\begin{align*}
\overline{R}_{n-1}^{(1)} &= \frac{\varepsilon dR_{n-1}^{(1)}}{R_{n-1}^{(2)} + dR_{n-1}^{(1)}}, \\
\overline{R}_{n-1}^{(2)} &= \frac{\varepsilon R_{n-1}^{(2)}}{R_{n-1}^{(2)} + dR_{n-1}^{(1)}}, \\
\overline{U}_m^{(1)} &= \frac{\varepsilon U_m^{(1)}}{U_m^{(1)} + dR_m^{(1)}}, \\
\overline{R}_m^{(1)} &= \frac{\varepsilon dR_m^{(1)}}{U_m^{(1)} + dR_m^{(3)}} (m < n), \\
\overline{U}_m^{(l)} &= \frac{\varepsilon U_m^{(l)}}{\alpha_m^{(l+1)}(d)}, \\
\overline{V}_m^{(l)} &= \frac{d\varepsilon V_m^{(l)}}{\alpha_m^{(l)}(d)}, \\
\overline{R}_m^{(l)} &= \frac{d\varepsilon R_m^{(l)}}{\alpha_m^{(l)}(d)\alpha_m^{(l+1)}(d)} (l \geq 1), \\
\overline{X}_m^{(l)} &= \frac{d\varepsilon X_m^{(l)}}{\alpha_m^{(l)}(d)} (l, m) \neq (1, n).
\end{align*}
\]

Proof. Since we have \(\overline{c}_0^{d}(y) = \overline{c}_0^{d}(y)\), we get \(\varepsilon_0 e_0^{d}(x) = \varepsilon_0^{cd}(x)\). Thus, we have \((\varepsilon_0^{cd}(x))_{n-1}^{(1)} = (\varepsilon_0 e_0^{d}(x))_{n-1}^{(1)} + c\) and then

\[
\overline{R}_{n-1}^{(2)} + c\overline{R}_{n-1}^{(1)} = \frac{\varepsilon R_{n-1}^{(2)} + cdR_{n-1}^{(1)}}{R_{n-1}^{(2)} + dR_{n-1}^{(1)}}.
\]

By the formula \(\varepsilon_0(e_0^{d}(x)) = c^{-1}\varepsilon(x)\) in (6.11) and \(\varepsilon_0(x) = x_n^{(1)} \varepsilon(x)\) in (6.2), one has \(\varepsilon(e_0^{d}(x)) = \varepsilon(x)\). Applying this to (6.11), one gets

\[
\overline{R}_{n-1}^{(2)} + c\overline{R}_{n-1}^{(1)} = \varepsilon \frac{R_{n-1}^{(2)} + cdR_{n-1}^{(1)}}{R_{n-1}^{(2)} + dR_{n-1}^{(1)}}
\]

and comparing the terms with or without \(c\), one gets (6.15).

The formulae (6.15) are the special case of (6.17). Hence, it is sufficient to show (6.17). As considered above \((e_0^{cd}(x))_{m}^{(l)} = (e_0^{d}(x))_{m}^{(l)}\), we have

\[
\frac{U_m^{(l-1)} + cdV_m^{(l-1)}}{U_m^{(l)} + cdV_m^{(l+1)}} = \frac{U_m^{(l-1)} + dV_m^{(l)} U_m^{(l-1)} + cV_m^{(l)}}{U_m^{(l)} + dV_m^{(l+1)} U_m^{(l)} + cV_m^{(l+1)}}
\]

Multiplying the both sides of (6.20) for \(l = 1, 2, \ldots, l\), then one has

\[
\varepsilon \frac{U_m^{(l)} + cdV_m^{(l+1)}}{U_m^{(l)} + dV_m^{(l+1)} U_m^{(l)} + cV_m^{(l+1)}} = \varepsilon \frac{U_m^{(l)} + dV_m^{(l+1)} U_m^{(l)} + cV_m^{(l+1)}}{U_m^{(l)} + dV_m^{(l+1)} U_m^{(l)} + cV_m^{(l+1)}}.
\]

Since we have seen above \(\varepsilon(x) = \overline{x}(x)\), one gets

\[
\overline{U}_m^{(l)} + c\overline{V}_m^{(l+1)} = \varepsilon \cdot \frac{U_m^{(l)} + cdV_m^{(l+1)}}{U_m^{(l)} + dV_m^{(l+1)}}.
\]

Comparing the terms with or without \(c\), we have first two formulae in (6.17). The second formula in (6.17) is obtained from \(\overline{R}_m^{(l)} = \overline{V}_m^{(l+1)} - \overline{V}_m^{(l)}\). Here note that \(\alpha_m^{(l)}(d) = U_m^{(l-1)} + dV_m^{(l)}\).
Due to (6.20) and $R_m^{(l)} = X_m^{(l)} X_n^{(l)}$, one has
\[
X_m^{(l-1)} X_m = \frac{t_m^{(l-1)} - t_m^{(l-1)}}{R_m^{(l)}},
\]
and then sending $x$ to $e_0^{(l)}(x)$ and using the formula (6.17) one has
\[
\frac{X_m^{(l-1)}}{X_m^{(l)}} = \frac{U_m^{(l-1)} - U_m^{(l-1)}}{R_m^{(l)}} = \frac{\alpha_m^{(l+1)}(d)(U_m^{(l-1)} - U_m^{(l-1)})}{\alpha_m^{(l)}(d)R_m^{(l)}} = \frac{\alpha_m^{(l+1)}(d)X_m^{(l-1)}}{\alpha_m^{(l)}(d)X_m^{(l)}}.
\]
(6.22)
for the second equality we use
\[
U_m^{(l-1)} - U_m^{(l-1)} = \frac{V_m^{(l-1)}}{\alpha_m^{(l)}} = \frac{U_m^{(l-1)} - U_m^{(l-1)}}{\alpha_m^{(l)}(d)},
\]
which is obtained by using $V_m^{(l)} = \varepsilon - U_m^{(l-1)}$. Thus, one can find
\[
\frac{X_m^{(l)}}{X_m^{(l-1)}} = \frac{\alpha_m^{(l+1)}X_m^{(l-1)}}{\alpha_m^{(l+1)}X_m^{(l-1)}} = \cdots = \frac{\alpha_m^{(l+1)} \alpha_m^{(l-1)}}{\alpha_m^{(l+1)} \alpha_m^{(l-1)}} \cdots \frac{\alpha_m^{(2)} \alpha_m^{(1)}}{\alpha_m^{(2)} \alpha_m^{(1)}} \frac{X_m^{(1)}}{X_m^{(1)}} \frac{X_m^{(2)}}{X_m^{(2)}} \cdots \frac{X_m^{(n-1)}}{X_m^{(n-1)}} \frac{X_m^{(n)}}{X_m^{(n)}} \frac{X_m^{(n+1)}}{X_m^{(n+1)}}
\]
(6.23)
Here applying $X_m^{(l)} = \frac{x_m^{(l)}}{x_m^{(l-1)}}$, $X_m^{(l-1)} = \frac{x_m^{(l-1)}}{x_m^{(l-1)}}$, $\alpha_m^{(l)} = \frac{\alpha_m^{(l)}}{\alpha_m^{(l-1)}}$ and $\alpha_m^{(l-1)} = \frac{\alpha_m^{(l-1)}}{\alpha_m^{(l-1)}}$, we obtain the first one of (6.18). The second one of (6.18) is obtained by the similar method using the formula:
\[
\frac{X_m^{(l)}}{X_m^{(l-1)}} = \frac{U_m^{(l-1)} - U_m^{(l-1)}}{R_m^{(l-1)}}
\]
which follows from (6.20) and $R_m^{(l-1)} = X_m^{(l-1)} X_m^{(l-1)}$ also.

Here, we define the subset of $P_1[n,k]$: $^1P_1[n,k] := \{p \in P_1[n,k] | p$ is through $(1, n-1)\}$ and $^2P_1[n,k] := \{p \in P_1[n,k] | p$ is through $(2, n-1)\}$. Note that since all paths must go through $(1, n-1)$ or $(2, n-1)$, we have $P_1[n,k] = ^1P_1[n,k] \cup ^2P_1[n,k]$. For $j = 1, 2$, set
\[
^jW_m^{(l)} := LW_m^{(l)} \cap ^jP_1[n,k], \quad ^jP_1[n,k]^{(l)} := P_1[n,k]^{(l)} \cap ^jP_1[n,k],
\]
where $W = X, R, A, B$. For example, an element in $^1P_1[n,k]^{(l)}$ is a path through $(1, n-1)$ and above $(l, m)$. Here for $j = 1, 2$ we define:
\[
^jX_m^{(l)} := \sum_{p \in ^jP_1[n,k]^{(l)}} x(p), \quad ^jR_m^{(l)} := \sum_{p \in ^jP_1[n,k]^{(l)}} x(p),
\]
\[
^jU_m^{(l)} := \sum_{p \in ^jLW_m^{(l)}} x(p), \quad ^jV_m^{(l)} := \sum_{p \in ^jLB_m^{(l)}} x(p),
\]
where note that $W_m^{(l)} = ^1W_m^{(l)} + ^2W_m^{(l)} (W = X, R, U, V)$. Let us see the formulæ related to these functions.
 Lemma 6.4. For a function $f(x)$ on $V_1$ let us denote $\overline{f}(x) := f(e_0^d(x))$ as before. We get the following formula:

$$1\mathcal{M}_m^{(l)} = \frac{dz^2 \cdot 1U_m^{(l)}}{(R_n^{(2)} + dR_{n-1}^{(1)})\alpha_m^{(l+1)}(d)},$$

$$2\mathcal{M}_m^{(l)} = \frac{dz^2 \cdot 2Y_m^{(l)}}{(R_n^{(2)} + dR_{n-1}^{(1)})\alpha_m^{(l)}(d)},$$

$$1\mathcal{X}_m^{(l)} = \frac{d\varepsilon \alpha_m^{(l)}(d)}{(R_n^{(2)} + dR_{n-1}^{(1)})X_m^{(l)}(d)} \left( \frac{1U_m^{(l-1)}}{\alpha_m^{(l)}(d)} - \frac{1U_m^{(l)}}{\alpha_m^{(l+1)}(d)} \right),$$

$$1\mathcal{X}_m^{(l)} = \frac{d\varepsilon \alpha_m^{(l+1)}(d)}{(R_n^{(2)} + dR_{n-1}^{(1)})X_m^{(l+1)}(d)} \left( \frac{1U_m^{(l-1)}}{\alpha_m^{(l+1)}(d)} - \frac{1U_m^{(l)}}{\alpha_m^{(l+1)}(d)} \right),$$

where note that (6.27) is not immediate from (6.26).

**Proof.** First, let us show (6.26) and (6.27) by descending induction on $m$. One can easily know that the formula (6.5) is valid for $1X_m^{(l)}$ and $2X_m^{(l)}$, namely,

$$jX_m^{(l)} = jX_{m+1}^{(l-1)} + \frac{x_m^{(l)}}{x_{m+1}^{(l-1)}} jX_{m+1}^{(l)} \quad (j = 1, 2).$$

Then, we also have

$$1X_m^{(l)} = 1X_{m+1}^{(l-1)} + \frac{x_m^{(l)}}{x_{m+1}^{(l-1)}} 1X_{m+1}^{(l)}.$$  

Applying the induction hypothesis on the right hand side of (6.29), more precisely, substituting (6.27) for $1X_{m+1}^{(l-1)}$ and (6.26) for $1X_{m+1}^{(l)}$, we obtain

$$1X_m^{(l)} = \frac{dz^{(l)}}{X_m^{(l)}X_{m+1}^{(l+1)}} (A - B + C - D),$$

where

$$A := \frac{x_m^{(l)} \alpha_m^{(l+1)} X_m^{(l)} U_m^{(l-1)}}{x_{m+1}^{(l)} \alpha_{m+1}^{(l)} \alpha_m^{(l+1)}}, \quad B := \frac{x_m^{(l)} X_m^{(l)} U_m^{(l-1)}}{x_{m+1}^{(l)} \alpha_m^{(l+1)}},$$

$$C := \frac{X_m^{(l)} U_m^{(l-1)}}{\alpha_m^{(l)}}, \quad D := \frac{X_m^{(l)} U_m^{(l-1)} \alpha_m^{(l+1)}}{\alpha_{m+1}^{(l+1)}}, \quad \xi := R_n^{(2)} + dR_{n-1}^{(1)}.$$  

By using (6.17), we get

$$A - D = \frac{1U_m^{(l-1)}}{\alpha_{m+1}^{(l+1)} \alpha_m^{(l+1)}} \left( X_m^{(l)} (\alpha_m^{(l+1)} - \alpha_{m+1}^{(l+1)}) - X_m^{(l+1)} \alpha_m^{(l+1)} \right).$$

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The following formulae are immediate from the definitions:

\[ \alpha^{(t+1)}_{m+1} - \alpha^{(t+1)}_m = (d - 1)X_m^{(t+1)}X_m^{(t+1)}, \quad (6.32) \]
\[ (d - 1)R_{m+1}^{(t+1)} - \alpha^{(t+1)}_{m+1} = \alpha^{(t)}_{m+1} - U^{(t+1)}_{m+1} - dV^{(t)}_{m+1}. \quad (6.33) \]

Applying these formulae to (6.31), we get

\[
A - D = \frac{1}{\alpha^{(t)}_{m+1} \alpha^{(t)}_m} \left( (d - 1)R_{m+1}^{(t+1)} - \alpha^{(t+1)}_{m+1} \right) = \frac{1}{\alpha^{(t)}_{m+1} \alpha^{(t)}_m} \left( -U^{(t+1)}_{m+1} - dV^{(t)}_{m+1} \right) \]
\[ = -\frac{1}{\alpha^{(t)}_{m+1} \alpha^{(t)}_m} \alpha^{(t)}_m \frac{1}{\alpha^{(t)}_{m+1}} = -\frac{1}{\alpha^{(t)}_m} \frac{1}{\alpha^{(t)}_m} X^{(t)}_{m+1} X^{(t)}_m. \quad (6.34) \]

Here applying (4.20) and \( R^{(t)}_{m+1} = X^{(t)}_{m+1} X^{(t)}_m \) to (6.34), we obtain

\[
(A - D)B = -\frac{1}{\alpha^{(t)}_m} \left( \frac{x_m^{(t)}}{x^{(t)}_{m+1}} X^{(t)}_{m+1} U^{(t)}_{m+1} + X^{(t)}_m U^{(t)}_m \right) = -\frac{X^{(t)}_{m+1}}{\alpha^{(t)}_m} X^{(t)}_m X^{(t+1)}_m + \frac{U^{(t)}_m}{\alpha^{(t)}_m} X^{(t+1)}_m. \quad (6.35) \]

Thus, finally applying this to (6.30) we get

\[
1X^{(t)}_m = \frac{\varepsilon do_m^{(t)} \alpha^{(t)}_m}{X^{(t)}_m} (A - B + C - D) = \frac{\varepsilon do_m^{(t)} \alpha^{(t)}_m}{X^{(t)}_m} \left( \frac{1}{\alpha^{(t)}_m} - \frac{1}{\alpha^{(t+1)}_m} \right), \quad (6.36) \]

which means (6.26). So, let us show (6.27) from (6.26). Let \( P \) (resp. \( Q \)) be the right-hand side of (6.26) (resp. (6.27)). Then, one has

\[
\frac{\xi}{d\varepsilon} (P - Q) = \frac{1}{\alpha^{(t+1)}_m X^{(t)}_m} - \frac{1}{\alpha^{(t)}_m X^{(t)}_m} \frac{\alpha^{(t+1)}_m U^{(t)}_m}{\alpha^{(t)}_m X^{(t)}_m} + \alpha^{(t)}_m \frac{U^{(t)}_m}{\alpha^{(t)}_m X^{(t)}_m} \frac{1}{\alpha^{(t+1)}_m X^{(t)}_m}. \quad (6.37) \]

**Lemma 6.5.** We get the following formula:

\[
\frac{1}{\alpha^{(t+1)}_m X^{(t)}_m} - \frac{1}{\alpha^{(t)}_m X^{(t)}_m} = \frac{1}{\alpha^{(t)}_m X^{(t)}_m} \left( \frac{1}{\alpha^{(t+1)}_m X^{(t)}_m} - \frac{1}{\alpha^{(t)}_m X^{(t)}_m} \right), \quad (6.38) \]
\[
\alpha^{(t+1)}_m X^{(t+1)}_m - \alpha^{(t)}_m X^{(t)}_m = \alpha^{(t+1)}_m X^{(t+1)}_m - \alpha^{(t)}_m X^{(t)}_m. \quad (6.39) \]

**Proof.** By considering similarly to (4.20), we get the formula

\[
1U^{(t)}_m = 1U^{(t+1)}_{m+1} X^{(t+1)}_m + 1X^{(t+1)}_{m+1} 1X^{(t+1)}_{m+1}. \quad (6.40) \]

It follows from (6.30) that we get (6.38).

Using the formula (4.20), \( U^{(t+1)}_{m+1} = U^{(t+1)} + R^{(t+1)}_{m+1} \) and \( \varepsilon = U^{(t+1)} + V^{(t+1)} \) we have

L.H.S of (6.39)
\[
= U^{(t+1)}_m X^{(t+1)}_m - U^{(t+1)}_m X^{(t+1)}_m + d(V^{(t+1)}_m X^{(t+1)}_m - V^{(t+1)}_m X^{(t+1)}_m)
= (1 - d)(U^{(t+1)}_m X^{(t+1)}_m - U^{(t+1)}_m X^{(t+1)}_m) + d(V^{(t+1)}_m X^{(t+1)}_m - X^{(t+1)}_m)
= ((1 - d)U^{(t+1)}_m + d\varepsilon)(X^{(t+1)}_m - X^{(t+1)}_m) = \alpha^{(t+1)}_m (X^{(t+1)}_m - X^{(t+1)}_m). \quad \Box
\]
Thus, applying (6.38) and (6.39) to (6.37) we obtain $P = Q$, which implies (6.27).

Next, let us show (6.24). By (6.18) and (6.26) one gets

$$1 \frac{R_m^{(l)}}{R_m} = x_m^{(l)} \cdot x_m^{(l)} = \frac{d \varepsilon^2}{R_{n-1}^{(l)} + d R_{n-1}^{(l)}} \left( \frac{1}{\alpha_m^{(l+1)}(d)} - \frac{1}{\alpha_m^{(l)}(d)} \right).$$

(6.41)

Applying (6.41) to $1 \frac{R_m^{(l)}}{R_m} = \sum_{j=\min(n+1-m,k)}^{l+1} \frac{R_j^{(l)}}{R_l}$, one gets (6.24).

Finally, let us show (6.25). One has

$$x_m^{(l)} = x_m^{(l)} - 1 x_m^{(l)} = \frac{d \varepsilon}{\xi \alpha_m^{(l)}(d)} \left( \xi V_m^{(l)} + \varepsilon^1 U_m^{(l-1)} - \alpha_m^{(l)}(d) R_m^{(l)} \right) = \frac{d \varepsilon}{\xi \alpha_m^{(l)}(d)} \cdot Z,$$

(6.42)

where $\xi = R_m^{(l)} + d R_m^{(l)}$. Then,

$$Z = V_m^{(l)} (\xi - d R_m^{(l)}) + \varepsilon^1 U_m^{(l)} - R_m^{(l)} U_m^{(l-1)}$$

$$= V_m^{(l)} R_m^{(l)} + (\varepsilon - R_m^{(l)} U_m) - R_m^{(l)} 2 U_m^{(l-1)}$$

$$= V_m^{(l)} R_m^{(l)} + R_m^{(l)} U_m^{(l)} - R_m^{(l)} 2 U_m^{(l-1)} = R_m^{(l)} (V_m^{(l)} + U_m^{(l-1)}) - R_m^{(l)} 2 U_m^{(l-1)}$$

$$= R_m^{(l)} (V_m^{(l)} + U_m^{(l-1)}) - R_m^{(l)} 2 U_m^{(l-1)} = \varepsilon \cdot 2 V_m^{(l)} ,$$

which shows (6.25).

\[ \square \]

### 6.4 Proof of (6.14)

Now, we shall show (6.14) by virtue of the formulae in the previous subsection. Set $\tilde{x} = e_0^{(n)} e_0^{(n)} e_0^{(n)}(x)$ and $x'' = e_0^{(n)} e_0^{(n)} e_0^{(n)}(x)$ where note that $x, \tilde{x}, x''$ mean $v_1(x), v_1(\tilde{x}), v_1(x'') \in \mathcal{V}_1$. It suffices to show that $x_m^{(l)} = x_m^{(l)}$ for any $(l, m)$, which will be done by using the induction on $(l, m) \in L_1[n, k]$ according to the total order $\prec(l, m) < (l', m') \Leftrightarrow l < l'$, or if $l = l'$, $m > m'$.

Since we have $e_0^{(n)}(x_m^{(l)}) = e_0^{(n)}(x)$ and $e_0^{(n)}(x_m^{(l)}) = \tilde{x}_m^{(l)}$, we obtain $\tilde{x}_m^{(l)} = x_m^{(l)} = x_m^{(l)}$, which shows the case $(l, m) = (1, n)$. Next, let us see the case $(l, m) = (1, n - 1)$. Let $\alpha_m^{(l)} = c R_m^{(l)}(c) = U_m^{(l)} + C V_m^{(l)}$ be as above. Thus, we have $\alpha_m^{(l)} = R_m^{(l)} + R_m^{(l)}$ and $\alpha_m^{(l)} = c R_m^{(l)} + R_m^{(l)}$, where note that $R_m^{(l)}$ depends on $x_m^{(l)}$ but not does $R_m^{(l)}$. Hence, we find that

$$\alpha_m^{(l)} = \frac{1}{cd} R_m^{(l)} + R_m^{(l)}, \quad \alpha_m^{(l)} = \frac{1}{cd} R_m^{(l)} + R_m^{(l)}.$$

(6.43)

Here note that

$$\tilde{x}_m^{(l)} = x_m^{(l)} \alpha_m^{(l)}(c) \left| x_m^{(l)} \right| x_m^{(l)} = x_m^{(l)} \frac{\alpha_m^{(l)}(d)}{\alpha_m^{(l)}(d)} \left| x_m^{(l)} \right| x_m^{(l)} \Rightarrow c \tilde{x}_m^{(l)}$$

(6.44)
Thus, it follows from (6.15), (6.43) and (6.44) that

\[
\hat{x}_n^{(1)} = \frac{1}{\alpha_n^{(1)}(d)} \cdot \frac{1}{dR_n^{(1)} + R_n^{(2)}} = \frac{1}{\alpha_n^{(1)}(d)} \cdot \frac{1}{dR_n^{(1)} + R_n^{(2)}}
\]

On the other hand, we have

\[
x_n^{(1)} = (e_n^d e_n^c e_n^c)_{n-1} = (e_n^d e_n^c)_{n-1} = x_n^{(1)} \left. \frac{R_n^{(1)} + R_n^{(2)}}{dR_n^{(1)} + R_n^{(2)}} \right|_{x_n^{(1)} \to x_n^{(1)}} = x_n^{(1)} \frac{1}{dR_n^{(1)} + R_n^{(2)}}.
\]

Then, by this and (6.46) we find that \( \hat{x}_n^{(1)} = x_n^{(1)} \).

Let us see the case \((1, m)\) with \(n < m - 1\). Here for a function \(f(x)\) we shall denote \(f(e_n^d e_n^c x)\) (resp. \(f(e_n^d e_n^c x)\)) by \(\hat{f}(x)\) (resp. \(\hat{f}(x)\)). The following are immediate from the explicit action of \(e_n^d e_n^c\):

\[
\hat{R}_m^{(1)} = \frac{1}{cd} R_m^{(1)}, \quad \hat{\xi} = \frac{1}{cd} R_m^{(1)} + R_m^{(2)}.
\]

It follows from (6.46) and (6.47) that

\[
\hat{x}_m^{(1)} = \frac{1}{\alpha_m^{(1)}(c)} \cdot \frac{1}{\alpha_m^{(2)}(c)} = x_m^{(1)} \left. \frac{U_m^{(1)} + R_m^{(1)}}{U_m^{(1)} + dR_m^{(1)}} \right|_{x_m^{(1)} \to x_m^{(1)}} = x_m^{(1)} \frac{1}{dR_m^{(1)} + R_m^{(2)}}.
\]

We also have the following:

\[
\hat{R}_m^{(1)} = \frac{1}{c} R_m^{(1)}, \quad \hat{\xi} = \frac{1}{c} R_m^{(1)} + R_m^{(2)}, \quad \hat{U}_m^{(1)} = \hat{\xi} - \hat{R}_m^{(1)}.
\]

Using these formulae, we get

\[
x_m^{(1)} = x_m^{(1)} \left. \frac{\alpha_m^{(1)}(cd)}{\alpha_m^{(2)}(cd)} \right|_{x_m^{(1)} \to x_m^{(1)}} = x_m^{(1)} \left. \frac{\alpha_m^{(1)}(d)}{U_m^{(1)} + dR_m^{(1)}} \right|_{x_m^{(1)} \to x_m^{(1)}} = x_m^{(1)} \left. \frac{1}{dR_m^{(1)} + R_m^{(2)}} \right|_{x_m^{(1)} \to x_m^{(1)}}.
\]

which coincides with (6.47) and then we have \( \hat{x}_m^{(1)} = x_m^{(1)} \) for \(m < n - 1\).

Now, let us see the general case \((l, m)\) with \(l > 1\). We have

\[
\hat{x}_m^{(l)} = x_m^{(l)} \left. \frac{\alpha_m^{(l)}(c)}{\alpha_m^{(l+1)}(c)} \right|_{x_m^{(l)} \to x_m^{(l)}} = x_m^{(l)} \left. \frac{\alpha_m^{(l)}(d)}{U_m^{(l)} + dR_m^{(l)}} \right|_{x_m^{(l)} \to x_m^{(l)}} = x_m^{(l)} \left. \frac{1}{dR_m^{(l)} + R_m^{(l+1)}} \right|_{x_m^{(l)} \to x_m^{(l)}}.
\]
Therefore, to obtain \( \hat{x}_m^{(l)} = x_m^{(l)} \) we shall show

\[
\frac{\alpha_m^{(l)}(d)}{\alpha_m^{(l+1)}(d)} \cdot \tilde{\alpha}_m^{(l)}(c) = \frac{\alpha_m^{(l)}(cd)}{\alpha_m^{(l+1)}(cd)}.
\]

(6.49)

To show (6.49), it suffices to show that there exists a function \( \Theta \) which does not depend on \((l, m)\) and possibly depends on \(x, c, d\) such that

\[
\alpha_m^{(l)}(d) \cdot \tilde{\alpha}_m^{(l)}(c) = \Theta \alpha_m^{(l)}(cd).
\]

(6.50)

**Lemma 6.6.** We have the following formulae:

\[
\hat{U}_m^{(l)} = \frac{1}{c} \hat{U}_m^{(l)} + 2 \hat{U}_m^{(l)},
\]

(6.51)

\[
\hat{V}_m^{(l)} = \frac{1}{c} \hat{V}_m^{(l)} + 2 \hat{V}_m^{(l)},
\]

(6.52)

\[
2^l \hat{U}_m^{(l)} = \frac{\varepsilon}{\alpha_m^{(l)}(d)} \left( U_m^{(l)} - \frac{d\varepsilon}{\xi} U_m^{(l)} \right),
\]

(6.53)

\[
1^l \hat{V}_m^{(l)} = \frac{d\varepsilon^2}{\xi} \left( \frac{R_{n-1}}{\varepsilon} - \frac{1}{\alpha_m^{(l)}(d)} \right).
\]

(6.54)

where \( \xi := \alpha_{n-1}^{(2)}(d) = R_{n-1}^{(2)} + dR_{n-1}^{(1)} \).

Proof. The formula (6.53) is an immediate consequence of (6.10), (6.24) and the fact \( U_m^{(l)} = 1^l U_m^{(l)} + 2^l U_m^{(l)} \). The formula (6.54) is obtained similarly.

We know that \( 1^l U_m^{(l)} \cdot x_m^{(1)} \) and \( 2^l U_m^{(l)} \) do not depend on \( x_m^{(1)} \), which induces (6.53). By considering similarly, we also obtain (6.52).

Due to this lemma and Lemma [6.4], we obtain

\[
\tilde{\alpha}_m^{(l)}(c) = \tilde{\alpha}_m^{(l)}(c) + c \tilde{\alpha}_m^{(l)}(c) + c(1^l \tilde{U}_m^{(l)} + 2^l \tilde{U}_m^{(l)}) = \frac{1}{cd} \tilde{U}_m^{(l-1)} + c \tilde{V}_m^{(l)} + c \tilde{U}_m^{(l)} + c \tilde{V}_m^{(l)}
\]

\[
= \frac{\varepsilon^2}{\xi \alpha_m^{(l)}(d)} \left( \frac{1}{c} \tilde{U}_m^{(l-1)} + 2 \tilde{U}_m^{(l-1)} + d \tilde{V}_m^{(l)} + cd \cdot 2 \tilde{V}_m^{(l)} \right)
\]

\[
= \frac{\varepsilon^2}{\xi \alpha_m^{(l)}(d)} \left( \tilde{U}_m^{(l-1)} + cd \tilde{V}_m^{(l)} \right) = \frac{\varepsilon^2}{\xi \alpha_m^{(l)}(d)} \tilde{\alpha}_m^{(l)}(cd),
\]

which shows (6.50) and \( \Theta = \frac{\varepsilon^2}{\xi} \) then (6.49). Hence, we completed to prove \( \hat{x}_m^{(l)} = x_m^{(l)} \) and then the relation (6.14).

**7 Ultra-discretization of \( \mathcal{V}(A_n^{(1)}) \)**

For basic notions of crystals, coherent family of perfect crystals and their limit we refer the reader to [10] (See also [8, 9]).
7.1 Crystal $B^{k,\infty}$

We review the $A_n^{(1)}$-crystal $B^{k,\infty}$ following [29]. We define the crystal $B^{k,\infty}$ by setting

$$B^{k,\infty} := \left\{ (b_{ji})_{1 \leq j \leq k, j \leq i \leq j + k'} \left| b_{ji} \in \mathbb{Z}, \sum_{i=j}^{j+k'} b_{ji} = 0 \text{ for any } j \right. \right\},$$

(7.1)

where $k' = n + 1 - k$. We also define $b_{\infty}$ in $B^{k,\infty}$ to be an element whose all entries are 0. For $b = (b_{ji}) \in B^{k,\infty}$, $\varepsilon_i(b)$, $\varphi_i(b)$, $\tilde{e}_i(b)$ and $\tilde{f}_i(b)$ for $i = 1, 2, \ldots, n$ are defined as follows:

For $i \in I \setminus \{0\}$, let us set $\beta = \max(0, i - k')$ and $\gamma = \min(k, i)$. Fix $b = (b_{ji}) \in B^{k,\infty}$. For $c \in \mathbb{Z}$ with $\beta < c \leq \gamma$ and $i \in I$, let us set

$$\Gamma_i(c) := \sum_{\beta < j < c} (b_{ji} - b_{j+1,i+1}), \quad \Gamma_i,\min := \min\{\Gamma_i(c) | \beta < c \leq \gamma\}. \quad \quad \quad (7.2)$$

Define

$$c_0 := \min\{c | \beta < c \leq \gamma, \Gamma_i(c) = \Gamma_i,\min\}, \quad c_1 := \max\{c | \beta < c \leq \gamma, \Gamma_i(c) = \Gamma_i,\min\}. \quad \quad \quad (7.3)$$

For $i \in I_0$ the functions $\varepsilon_i(b)$ and $\varphi_i(b)$ on $B^{k,\infty}$ are defined by

$$\varepsilon_i(b) := \sum_{\beta \leq j < c_0} (b_{j+1,i+1} - b_{ji}), \quad \varphi_i(b) := \sum_{c_1 \leq j \leq \gamma} (b_{ji} - b_{j+1,i+1}). \quad \quad \quad (7.4)$$

The Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ are given as follows: set $b' = (b'_{p,q}) = \tilde{e}_i(b)$ and $b'' = (b''_{p,q}) = \tilde{f}_i(b)$ ($i \in I_0$) and then

$$b'_{p,q} = b_{p,q} - \delta_{p,c_0}\delta_{q,i+1} + \delta_{p,c_0}\delta_{q,i}, \quad b''_{p,q} = b_{p,q} - \delta_{p,c_1}\delta_{q,i} + \delta_{p,c_1}\delta_{q,i+1}. \quad \quad \quad (7.5)$$

Note that $\tilde{f}_i^{-1} = \tilde{e}_i$. Next, let us define $\varepsilon_0$, $\varphi_0$, $\tilde{e}_0$ and $\tilde{f}_0$: Set

$$C := \{(c_0, c_1, \ldots, c_k) \in \mathbb{R}^{k+1} | 1 = c_0 < c_1 < \cdots < c_k \leq n + 1\} \quad \quad \quad (7.6)$$

and define a partial order $c \preceq c'$ on $C$ by $c_j \leq c'_j$ for any $j$. For fixed $b = (b_{ji}) \in B^{k,\infty}$, let $\Delta_b : C \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\Delta_b(c) = \sum_{j=1}^{k} \sum_{c_{j-1} < i \leq c_j} b_{ji},$$

and set $C_{\min}^b := \{c \in C | \Delta_b(c) \text{ is minimal} \}$. By [29] 5.3., we obtain the following lemma:

**Lemma 7.1.** Fix $b = (b_{ji}) \in B^{k,\infty}$.

(i) There exists unique $c^{(c)} \in C$ such that

$$\Delta_b(c^{(c)}) \leq \Delta_b(c) \quad \text{if} \quad c^{(c)} \preceq c,$$

$$\Delta_b(c^{(c)}) < \Delta_b(c) \quad \text{if} \quad c^{(c)} \not\preceq c,$$

for any $c \in C$. 31
(ii) There exists unique \( c^{(f)} \in C \) such that
\[
\Delta_b(c^{(f)}) \leq \Delta_b(c) \quad \text{if } c \preceq c^{(f)},
\]
\[
\Delta_b(c^{(f)}) < \Delta_b(c) \quad \text{if } c \prec c^{(f)},
\]
for any \( c \in C \).

Note that \( c^{(e)}, c^{(f)} \in C^{(b)} \) and then \( \Delta_b(c^{(e)}) = \Delta_b(c^{(f)}) \). Here, using these \( c^{(e)} \) and \( c^{(f)} \), for \( b = (b_{ji}) \in B^{k,\infty} \) we define
\[
\begin{align*}
\varepsilon_0(b) & := -b_{k,n+1} - \Delta_b(c^{(e)}), \\
b'_j & := b_j - \delta_{ic_j^{(e)}} + \delta_{ic_j^{(f)}}, \\
b''_j & := b_j - \delta_{ic_j^{(e)}} + \delta_{ic_j^{(f)}},
\end{align*}
\]
where \( b' = \tilde{\varepsilon}_0(b) \) and \( b'' = \tilde{f}_0(b) \). Note that \( \tilde{f}_0^{-1} = \tilde{\varepsilon}_0 \). Then, by these definitions the crystal \( B^{k,\infty} \) becomes a limit of coherent family of perfect crystals \( \{B^{k,l}\}_{l \geq l} \) \([10, 29]\). Note that \( w_0(b) = \varphi_0(b) - \tilde{\varepsilon}_0(b) = -b_{1,1} + b_{k,n+1} \) since \( \Delta_u(c^{(e)}) = \Delta_b(c^{(f)}) \).

### 7.2 Crystal structure on \( UD(\mathcal{V}(A_n^{(1)})) \)

Let us see the crystal structure on \( (UD(\mathcal{V}(A_n^{(1)})), UD(\gamma_i \circ v_1), UD(\varepsilon_i \circ v_1), UD(\varepsilon_i \circ v_1), UD(\gamma_i \circ v_1))_{i \in I} \), where \( v_1 : (C^\infty)^{kk'} \to \mathcal{V}(A_n^{(1)})(= V_1) \) is the positive structure on \( \mathcal{V}(A_n^{(1)}) \). For a horizontal strip \( s = (l, m) \) on \( L_1[n,k] \), let \( uw(s) := x_m^{(l)} - x_{m+1}^{(l)} \) and for a path \( p \in P_1[n,k] \) define \( ux(p) := \sum_{s \in \text{strip in } p} uw(s) \).

**Proposition 7.2.** The explicit forms of \( (UD(\mathcal{V}(A_n^{(1)})), UD(\varepsilon_i, v_1), UD(\gamma_i \circ v_1), UD(\varepsilon_i \circ v_1), UD(\gamma_i \circ v_1)) \) are as follows: \( UD(\mathcal{V}(A_n^{(1)})) = \mathbb{Z}^{kk'} \) where we denote the coordinate \( UD(x_j^{(i)}) \) also by \( x_j^{(i)} \).

\[
UD(\gamma_0 \circ v_1) = -x_n^{(1)} - x_1^{(k)},
\]
\[
UD(\gamma_i \circ v_1) = 2 \sum_{j=a}^{b} x_j^{(i)} - \sum_{j=a}^{b+1} x_j^{(i)} - \sum_{j=a-1}^{b} x_j^{(i)},
\]
\[
UD(\varepsilon_0 \circ v_1) = x_n^{(1)} + \max_{p \in P_1[n,k]} (ux(p)),
\]
\[
UD(\varepsilon_i \circ v_1) = \max_{a \leq p \leq b} (\overline{D}^{(i)}),
\]
\[
UD(\varepsilon_{0,v_1})^d x_m^{(l)} = x_m^{(l)} + \max_{p \in LA[n,k]^{(l-1)}} \left( \max_{a \leq p \leq l-1} (\overline{D}^{(i)}), \max_{l+1 \leq p \leq b} (d + \overline{D}^{(i)}) \right)
\]
\[
\quad + \max_{p \in LA[n,k]^{(l-1)}} \left( \max_{a \leq p \leq l-1} (\overline{D}^{(i)}), \max_{l \leq p \leq b} (d + \overline{D}^{(i)}) \right) \quad \text{if } m = i,
\]
\[
UD(\varepsilon_{i,v_1})^d x_m^{(l)} = \begin{cases} 
\begin{align*}
\max_{a \leq p \leq l-1} (\overline{D}^{(i)}), & \text{max} (d + \overline{D}^{(i)}) \\
\quad (d \in \mathbb{Z}),
\end{align*}
\end{cases}
\]
\[
\quad \begin{cases} 
\begin{align*}
\max_{a \leq p \leq l-1} (\overline{D}^{(i)}), & \text{max} (d + \overline{D}^{(i)}) \\
\quad (d \in \mathbb{Z}),
\end{align*}
\end{cases}
\]
where \( d \in \mathbb{Z}, 1 \leq i \leq n, (l, m) \in L_1[n, k] \), \( \mathbf{a} \) and \( \mathbf{b} \) are as in \((3.3)\) and
\[
\overline{D}_i^{(l)} = -\mathcal{U}D(D_i^{(l)}) = -x_i^{(l)} - 2 \sum_{j=+1}^{b+1} x_i^{(j)} + \sum_{j=l}^{b+1} x_i^{(j)} + \sum_{j=l}^{b} x_i^{(j)}.
\] (7.16)

**Proof.** All the formula are obtained trivially applying the formulae
\[
\mathcal{U}D(X \cdot Y^{\pm}) = \mathcal{U}D(X) \pm \mathcal{U}D(Y), \quad \mathcal{U}D(X + Y) = \max(X, Y),
\] to the explicit forms of \( e_i^d, \gamma_i, \epsilon_i \) \((i \in I)\) as in \((3.3), (3.21), (3.5), (6.2)\) and \((7.3)\). \( \square \)

In the sequel, we shall denote \( \mathcal{U}D(e_i, v_1), \mathcal{U}D(\gamma_i \circ v_1) \) and \( \mathcal{U}D(\epsilon_i \circ v_1) \) by \( e_i', w_i' \) and \( \epsilon_i' \) respectively.

## 7.3 Isomorphism between \( \mathcal{U}D(\mathcal{V}) \) and \( \mathcal{B}^{k, \infty} \)

We shall show that the crystal \( \mathcal{B}^{k, \infty} \) and \( \mathcal{Z}^{kk'} = \mathcal{U}D(A_n^{(1)}) \) \((k' = n - k + 1)\) are isomorphic to each other. Define the map \( \Omega : \mathcal{Z}^{kk'} = \mathcal{U}D(A_n^{(1)}) \to \mathcal{B}^{k, \infty} \) as follows: Set \((b_{ji}) = \Omega(x)\):
\[
b_{ji} := x_i^{(k-j+1)} - x_{i-1}^{(k-j+1)},
\] (7.17)
where we understand that \( x_m^{(l)} \) is 0 if \((l, m)\) is out of the lattice \( L_1[n, k] \). The following theorem gives an affirmative answer to the conjecture for \( A_n^{(1)} \) and generic Dynkin index \( k \).

**Theorem 7.3.** The map \( \Omega \) gives an isomorphism of crystals between \( \mathcal{B}^{k, \infty} \) and \( \mathcal{Z}^{kk'} \).

**Proof.** First, let us see the well-definedness of the map \( \Omega \), that is, we may show that
\[
\sum_{i=j}^{j+k'} \Omega(x)_{ji} = 0 \quad \text{for any } j.
\] (7.18)
The left hand-side of \((7.18)\) is
\[
(x_{j}(j) - x_{j-1}(j)) - (x_{j+1}(j) - x_{j}(j)) + \cdots + (x_{j+k'}(j) - x_{j+k'-1}(j)) = x_{j+k'}(j) - x_{j-1}(j) = 0 \quad (j'' = k - j + 1),
\] since \((j'', j + k') \) and \((j'', j - 1)\) are out of \( L_1[n, k] \). Indeed, \( j'' + (j + k') = n + 2 > n + 1 \) and \( j'' + j - 1 = k \).

Next, we shall see the bijectivity of \( \Omega \). It can be done easily by giving the inverse of \( \Omega \):
\[
\Omega^{-1}(b_{ji}) = b_{j'', j'' + b_{j'', j''+1} + \cdots + b_{j'', i-1} + b_{j'', i}}.
\] It is clear to show that this is an inverse of \( \Omega \). Indeed, denoting the right hand-side of the above formula by \( \Theta_i^{(j'' \circ \cdot)} \), we obtain
\[
\Theta_i^{(j'' \circ \cdot)} - \Theta_{i-1}^{(j'' \circ \cdot)} = b_{j,i}.
\]
Thus, the map \( \Omega \) is bijective.

Finally, let us show for any \( x \in \mathcal{Z}^{kk'} \) \((= \mathcal{U}D(\mathcal{V}(A_n^{(1)})))\) and \( i \in I \),
Lemma 7.4. There exists one-to-one correspondence between $P_1[n,k]$ and the set $C$ as in (7.6).

(i) $\text{wt}_i(x) = \text{wt}_i \circ \Omega(x)$.

(ii) $\varepsilon'_i(x) = \varepsilon_i \circ \Omega(x)$.

(iii) $\Delta_i(x) = \Omega^{-1} \circ \varepsilon_i \circ \Omega(x)$.

(i) First note that $\Delta_b(c^{(c)}) = \Delta_b(c^{(f)}) = \min \{ \Delta_b(c) | c \in C \}$ for $b \in B^{k,\infty}$. Thus, we get $\text{wt}_0(b) = \varepsilon_0(b) = -b_{1,1} + b_{k,n+1}$ and then

$$\text{wt}_0 \circ \Omega(x) = -x_n^{(k)} = \text{wt}_0'(x).$$

Suppose that $i \neq 0$. By the definition of $c_0$ and $c_1$ in (7.31), we have $\sum_{j=c_0}^{c_1-1}(b_{j,i} - b_{j+1,i+1}) = 0$. By using this, we obtain

$$\text{wt}_i(b) = \varphi_i(b) - \varepsilon_i(b) = \sum_{\beta \leq j < c_0} B_{ji} + \sum_{c_0 \leq j \leq \gamma} B_{ji} = \sum_{\beta \leq j < \gamma} B_{ji},$$

(7.19)

where $B_{ji} := b_{j,i} - b_{j+1,i+1}$. We have

$$\text{wt}_i \circ \Omega(x) = \sum_{\beta \leq j \leq \gamma} x_i^{(k-j+1)} - x_i^{(k-j)} - x_i^{(k-j+1)} + x_i^{(k-j)}.$$

Note that

$$\beta = \max(0, i - k') = \max(0, i - n + k - 1) = k + \max(-k, i - n - 1) = k - \min(k, n - i + 1) = k - b,$$

$$\gamma = \min(k, i) = k + 1 + \min(1, i - k - 1) = k + 1 - \max(1, k - i + 1) = k + 1 - a.$$ 

Hence, we get

$$\sum_{\beta \leq j \leq \gamma} x_i^{(k-j+1)} + x_i^{(k-j)} = \sum_{j=k-\gamma+1}^{k-\beta} x_i^{(j)} + \sum_{j=k-\gamma}^{k-\beta} x_i^{(j)} = \sum_{j=a}^{b} x_i^{(j)},$$

since $x_i^{(k-\gamma)} = x_i^{(a-1)} = 0$ and $x_i^{(k-\beta+1)} = x_i^{(b+1)} = 0$. Therefore, by (7.11)

$$\text{wt}_i \circ \Omega(x) = \sum_{\beta \leq j \leq \gamma} (x_i^{(k-j+1)} - x_i^{(k-j+1)} - x_i^{(k-j)} + x_i^{(k-j)})$$

$$= \sum_{j=a}^{b} x_i^{(j)} - \sum_{j=a}^{b} x_i^{(j)} - \sum_{j=a}^{b} x_i^{(j)} = \text{wt}_i'(x).$$

(ii) Let us see the relation between the set of paths $P_1[n,k]$ and the set $C$ as in (7.6).

Lemma 7.4. There exists one-to-one correspondence between $P_1[n,k]$ and $C$ defined as follows:

For $c = (1, c_1, \cdots, c_{k-1}, n+1) \in C$, let us define a path $p = (s_1, s_2, \cdots, s_k)$ by

$$s_j = (k - j + 1, c_{j-1}) \quad (k - j + 1, c_j - 1),$$

where $s_j$ is consecutive horizontal strips as in (4.1) with length $c_j - c_{j-1} - 1$. 

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The proof is immediate. Let us denote this correspondence by \( \pi : C \to P_1[n,k] \).

Using this correspondence, one has for \((b_{ji}) = \Omega(x)\) and \(c \in C\)

\[
\Delta_{\Omega(x)}(c) = \sum_{j=1}^{k} \sum_{c_{j-1} < c_j} b_{ji} = \sum_{j=1}^{k} \sum_{c_{j-1} < c_j} (x_i^{(k-j+1)} - x_i^{(k-j+1)}) = \sum_{j=1}^{k} (x_i^{(k-j+1)} - x_i^{(k-j+1)}) = -ux(\pi(c)), \tag{7.20}
\]

Then, for \((b_{ji}) = \Omega(x)\) by \((7.12)\) we get

\[
\varepsilon_0 \circ \Omega(x) = -b_{k,n+1} - \Delta_{\Omega(x)}(c^{(e)}) = x_n^{(1)} - \min_{c \in C} \{\Delta_{\Omega(x)}(c)\} = x_n^{(1)} - \max_{p \in P_1[n,k]} \{ux(p)\} = \varepsilon_0(x).
\]

Next, let us see the case \(i \neq 0\). By the definition of \(c_0\) as in \((7.3)\), for \(b = (b_{ji}) \in B^{k,\infty}\) one has

\[
\varepsilon_i(b) = \sum_{\beta \leq j < c_0} (b_{j+1,i+1} - b_{ji}) = b_{\beta+1,i+1} - b_{\beta,i} - \sum_{\beta < j < c_0} (b_{ji} - b_{j+1,i+1}) = b_{\beta+1,i+1} - b_{\beta,i} - \Gamma_i(c_0) = b_{\beta+1,i+1} - b_{\beta,i} - \min_{\beta < c \leq \gamma} \{\sum_{\beta < j < c} (b_{ji} - b_{j+1,i+1})\}
\]

Due to this formula, we get

\[
\varepsilon_i \circ \Omega(x) = x_i^{(k-\beta)} - x_i^{(k-\beta)} - x_i^{(k-\beta+1)} + x_i^{(k-\beta+1)} + \max_{\beta < c \leq \gamma} \{\sum_{j=\beta+1}^{c-1} x_i^{j} - x_i^{j} - x_i^{j+1} + x_i^{j+1}\}.
\]

Here one gets

\[
\max_{\beta < c \leq \gamma} \{\sum_{j=\beta+1}^{c-1} x_i^{j} - x_i^{j} - x_i^{j+1} + x_i^{j+1}\} = \max_{\beta < c \leq \gamma} \{\sum_{j=k-c+1}^{b-1} x_i^{j} - x_i^{j} - x_i^{j+1} + x_i^{j+1}\}
\]

where the last equality follows from that setting \(l = k - c + 1\) we have \(l = k - \gamma + 1 = a\) (resp. \(l = k - \beta = b\)) if \(c = \gamma\) (resp. \(c = \beta + 1\)). Therefore, by \((7.13)\) one has

\[
\varepsilon_i \circ \Omega(x) = x_i^{(b)} - x_i^{(b)} - x_i^{(b+1)} + x_i^{(b+1)} + \max_{a \leq i \leq b} \{\sum_{j=l+1}^{b} x_i^{j} - x_i^{j} - x_i^{j+1} + x_i^{j+1}\}
\]

\[
= \max_{a \leq i \leq b} \{-x_i^{(l)} - 2 \sum_{j=l+1}^{b} x_i^{j} + \sum_{j=l+1}^{b+1} x_i^{j} + \sum_{j=l+1}^{b} x_i^{j+1} \} = \max_{a \leq i \leq b} \{-D_i^{(l)} \} = \varepsilon_i(x).
\]

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Suppose $i = 0$. Let us show $\hat{c}_0 = \Omega \circ \hat{c}_0' \circ \Omega^{-1}$. The following lemma is immediate from the definitions of paths.

**Lemma 7.5.** For $c = (c_0, c_1, \cdots, c_{k+1}) \in C$, set $\pi(c) \in P_1[n, k]$.

(i) Path $p$ is above $(l, m)$ iff $c_{k-l} > m$.

(ii) Path $p$ is below $(l, m)$ iff $c_{k-l+1} \leq m$.

(iii) Path $p$ is through $(l, m)$ iff $c_{k-l} \leq m < c_{k-l+1}$.

Since $U_m^{(l-1)}$ (resp. $V_m^{(l)}$) is the total sum of all weight of paths above $(l-1, m)$ (resp. below $(l, m)$), by using (7.20) and the above lemma, we obtain

$$UD(U_m^{(l-1)}(\Omega^{-1}(b))) = \max_{c_{k-l+1} > m} \{-\Delta_b(c)\}, \quad UD(V_m^{(l)}(\Omega^{-1}(b))) = \max_{c_{k-l+1} \leq m} \{-\Delta_b(c)\}.$$ 

Then we obtain

$$UD(\alpha_m^{(l)}(d))(\Omega^{-1}(b)) = \max \left( \max_{c_{k-l+1} > m} \{-\Delta_b(c)\}, d + \max_{c_{k-l+1} \leq m} \{-\Delta_b(c)\} \right) =: \Lambda_m^{(l)}(d). \quad (7.21)$$

Thus,

$$UD(e_0)^d(\Omega^{-1}(b))^{(l)} = x_m^{(l)}(\Omega^{-1}(b)) + \Lambda_m^{(l)}(d) - \Lambda_m^{(l+1)}(d), \quad (7.22)$$

Therefore, we get

$$(\Omega \circ UD(e_0)^d(\Omega^{-1}(b)))_{ji}^{(l)} = UD(e_0)^d(\Omega^{-1}(b))_{i}^{(k-j+1)} - UD(e_0)^d(\Omega^{-1}(b))_{i}^{(k-j+1)} = x_i^{(k-j+1)}(\Omega^{-1}(b)) + A_i^{(k-j+1)}(d) - A_i^{(k-j+1)}(d) - x_i^{(k-j+1)}(\Omega^{-1}(b)) - A_i^{(k-j+1)}(d) - A_i^{(k-j+1)}(d) + A_i^{(k-j+1)}(d) - A_i^{(k-j+1)}(d) + A_i^{(k-j+1)}(d), \quad (7.23)$$

Let us denote the last formula of (7.23) by $P(d)$. Now we know that for $b = (b_{ji}) \in B^{k, \infty}$ it suffices to show

$$\hat{c}_0(b)_{ji} = b_{ji} - \delta_i \tau_{j-1} + \delta_i \tau_j = P(1), \quad (7.24)$$

where set $e^{(c)} = (\tau_0, \tau_1, \cdots, \tau_{k+1})$. One has

$$A_i^{(k-j+1)}(1) = - \min_{c_{j+1} > 1} \{ \Delta_b(c) \}, -1 + \min_{c_{j+1} \leq 1} \{ \Delta_b(c) \}. \quad (7.25)$$

We shall see the cases: (I) $\tau_j = i$. (II) $\tau_j > i$. (III) $\tau_j < i$.

(I) Suppose $\tau_j = i$. One has $\min_{c_{j+1} \leq i} \{ \Delta_b(c) \} = \Omega(c)$. Let $c' \in C$ be an element such that $\min_{c_{j+1} > 1} \{ \Delta_b(c) \} = \Omega(c') c_j > i$. By the definition of $e^{(c)}$, we find that if $c' > c^{(c)}$, $\Delta_b(c') \geq \Delta_b(c^{(c)})$ and if $c' \not< c^{(c)}$, $\Delta_b(c') > \Delta_b(c^{(c)})$. Then, in any case we get $A_i^{(k-j+1)}(1) = 1 - \Delta_b(c^{(c)})$ and $A_i^{(k-j+1)}(1) = -\Delta_b(c^{(c)})$, and then

$$A_i^{(k-j+1)}(1) - A_i^{(k-j+1)}(1) = 1.$$
(II) Suppose \( \tau_j > i \). One has \( \min_{c \leq j} \{ \Delta_b(c) \} = \Delta_b(c(e)) \). Let \( c' \in C \) be an element such that \( \min_{c \leq j} \{ \Delta_b(c) \} = \Delta_b(c'(e)) \) and \( n' \leq i \), which means \( c' \neq c(e) \) and then \( \Delta_b(c') > \Delta_b(c(e)) \). Therefore, we obtain \( A_i^{(k-j+1)}(1) = -\Delta_b(c(e)) \). Next, we also have \( \min_{c \leq j} \{ \Delta_b(c) \} = \Delta_b(c'(e)) \) since \( \tau_j > i > i-1 \). Let \( c'' \in C \) be an element such that \( \min_{c \leq j} \{ \Delta_b(c) \} = \Delta_b(c'(e)) \) and \( n' \leq i - 1 \), which means \( c'' \neq c(e) \) and then \( \Delta_b(c'') > \Delta_b(c(e)) \). Thus, we have \( A_i^{(k-j+1)}(1) = -\Delta_b(c(e)) \). Therefore, we obtain \( A_i^{(k-j+1)}(1) - A_{i-1}^{(k-j+1)}(1) = 0 \).

(III) Suppose \( \tau_j < i \). One gets \( \min_{c \leq j} \{ \Delta_b(c) \} = \Delta_b(c(e)) \). Let \( c' \in C \) be an element such that \( \min_{c \leq j} \{ \Delta_b(c) \} = \Delta_b(c'(e)) \) and \( n' > i \). If \( c' = c(e) \), then \( \Delta_b(c') \geq \Delta_b(c(e)) \), and if \( c' \neq c(e) \), then \( \Delta_b(c') > \Delta_b(c(e)) \). In both cases, we have \( A_i^{(k-j)}(1) = 1 - \Delta_b(c(e)) \). Next, we also have \( \min_{c \leq j} \{ \Delta_b(c) \} = \Delta_b(c'(e)) \) since \( \tau_j < i \). It is easy to get \( \min_{c \leq j} \{ \Delta_b(c) \} \geq \Delta_b(c(e)) \). Thus, we have \( A_i^{(k-j)}(1) = 1 - \Delta_b(c(e)) \) and then we obtain \( A_i^{(k-j)}(1) - A_{i-1}^{(k-j)}(1) = 0 \). By the result from (I), (II) and (III), we find that

\[
A_i^{(k-j)}(1) - A_{i-1}^{(k-j)}(1) = \delta_{i, \tau_i}, \quad A_i^{(k-j+1)}(1) - A_{i-1}^{(k-j+1)}(1) = \delta_{i, \tau_{i-1}}.
\]

(7.26)

Substituting (7.20) in (7.23) and setting \( d = 1 \), we obtain

\[
(\Omega \circ e'_0 \circ \Omega^{-1}(b))_{ji} = (\Omega \circ UD(e_0) \circ \Omega^{-1}(b))_{ji} = b_{ji} + \delta_1 \pi_j - \delta_1 \pi_{j-1} = \tilde{e}_0(b)_{ji}.
\]

Here note that if we set \( d = -1 \), we obtain the action of \( \tilde{f}_0 \).

Finally, consider the case \( i \in \{1, 2, \ldots, n\} \). The action of Kashiwara operators \( \tilde{e}_i \) and \( \tilde{f}_i \) on \( B^{k, \infty} \) are given in (7.24). By the explicit form of \( D^{(i)}_m \) in (7.23), for \( x \in \mathbb{Z}^{kk'} \), one gets

\[
UD(D^{(i)}_m(x)) = x_i^{(p)} - x_{i+1}^{(p)} + \sum_{r=p+1}^{b} 2x_i^{(r)} - x_{i+1}^{(r)},
\]

and then using \( \Omega^{-1}(b)_i^{(i)} = \sum_{s=k-j+1}^{i} b_{k-j+1,s} \), for \( b = (b_{ji}) \in B^{k, \infty} \) one has

\[
UD(D^{(i)}_m) = \sum_{r=p+1}^{b+1} b_{k-r+1,i} - \sum_{r=p}^{b} b_{k-r+1,i+1} =: D_{p,i} \quad (7.27)
\]

Thus, it follows from (7.15) that for \( d \in \mathbb{Z} \),

\[
UD(e_i)^d(\Omega^{-1}(b))^{(i)}_m = \sum_{s=1}^{m} b_{k-s+1,i+1} + \delta_{i,m} \left( \max_{a \leq p \leq l-1} (-D_{p,i})_d + \max_{l \leq p \leq b} (-D_{p,i}) \right) - \max_{a \leq p \leq l} (-D_{p,i})_d + \max_{l+1 \leq p \leq b} (-D_{p,i})_d.
\]

And then,

\[
\Omega \circ UD(e_i)^d(\Omega^{-1}(b))_{l=m} = UD(e_i)^d(\Omega^{-1}(b))_{l=m} - UD(e_i)^d(\Omega^{-1}(b))_{l=i+1} \quad (7.28)
\]

\[
= b_{i,m} + \delta_{i,m} \left( \max_{a \leq p \leq l-1} (-D_{p,i})_d + \max_{l \leq p \leq b} (-D_{p,i})_d \right) - \max_{a \leq p \leq l} (-D_{p,i})_d + \max_{l+1 \leq p \leq b} (-D_{p,i})_d.
\]

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where \( l' = k - l + 1 \). Here, recalling \( \beta = k - b \) and \( \gamma = k + 1 - a \) as above, we get

\[
\max_{a \leq p \leq l' - 1} (-D_{p,i}) = - \min_{l + 1 \leq p \leq \gamma} (D_{p,i}) = - \min_{l + 1 \leq p \leq \gamma} \left( \sum_{r=\beta}^{p-1} (b_{r,i} - b_{r+1,i+1}) \right)
\]

\[
\max_{l' \leq p \leq b} (-D_{p,i}) = - \min_{\beta + 1 \leq p \leq l} (D_{p,i}) = - \min_{\beta + 1 \leq p \leq l} \left( \sum_{r=\beta}^{p-1} (b_{r,i} - b_{r+1,i+1}) \right).
\]

Since \( \sum_{r=\beta}^{p} (b_{r,i} - b_{r+1,i+1}) = \Gamma_i(p) + (b_{\beta,i} - b_{\beta+1,i+1}) \) and in general, \( \min(A + x, B + x) - \min(C + x, D + x) = \min(A, B) - \min(C, D) \), by setting \( d = 1 \) we obtain

\[
\Omega \circ \mathcal{U}D(e_i)(\Omega^{-1}(b))_{l,m} = b_{l,m} + (\delta_{i,m-1} - \delta_{i,m}) \times \left( \min_{l + 1 \leq p \leq \gamma} (\Gamma_i(p)), -1 + \min_{\beta + 1 \leq p \leq l} (\Gamma_i(p)) \right)
\]

\[
= \min_{l \leq p \leq \gamma} (\Gamma_i(p)), -1 + \min_{\beta + 1 \leq p \leq l - 1} (\Gamma_i(p)) \right) =: b_{l,m} + (\delta_{i,m-1} - \delta_{i,m}) \times G.
\]

Now, to finalize the proof, let us show \( G = -\delta_{l,c_0} \).

where \( c_0 \) is as in \( \text{[23]} \). By the definition of \( c_0 \) in \( \text{[23]} \), we find that

(i) If \( l > c_0 \), \( G = (-1 + \Gamma_i(c_0)) - (-1 + \Gamma_i(c_0)) = 0 \).

(ii) If \( l < c_0 \), \( G = \Gamma_i(c_0) - \Gamma_i(c_0) = 0 \).

(iii) If \( l = c_0 \), \( G = (-1 + \Gamma_i(c_0)) - \Gamma_i(c_0) = -1 \).

Thus, we have \( G = -\delta_{l,c_0} \). Therefore, for \( i \in \{1, 2, \ldots, n\} \) we obtain

\[
\Omega \circ \mathcal{U}D(e_i)(\Omega^{-1}(b))_{l,m} = b_{l,m} + \delta_{m,i} \delta_{l,c_0} - \delta_{m,i+1} \delta_{l,c_0} = \tilde{e}_i(b)_{l,m}.
\]

Here, we have completed the proof of Theorem \( \text{[23]} \).

8 Affine Weyl Group Action

In this section, as an application of the affine geometric crystal structure on \( \mathcal{V}(A_n^{(1)}) \) and its ultra-discretization, we shall describe a birational action of affine Weyl group and a piecewise-linear action of affine Weyl group explicitly.

8.1 Birational action of affine Weyl group \( W(A_n^{(1)}) \)

Let \( (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I}) \) be a geometric crystal of \( G \) or \( g \).

Theorem 8.1 \( \text{[19]} \). Define \( s_i(x) := e_i^{\gamma_i(x)^{-1}}(x) \) for \( x \in X \). Then, \( \langle s_i| i \in I \rangle \) gives a birational actions on \( X \) of Weyl group \( W \) associated with \( G \) (or \( g \)).
Indeed, we can easily find that $s_i^2 = \text{id}_X$, $s_is_j = s_js_i$ if $a_{ij} = a_{ji} = 0$, and the braid relation $s_is_j s_i = s_j s_is_j$ holds if $a_{ij} = a_{ji} = -1$. Since we have obtained the $A_n^{(1)}$-geometric crystal structure on $\mathcal{V}(A_n^{(1)})$, we get the birational action of the affine Weyl group $W(A_n^{(1)})$ on $\mathcal{V}(A_n^{(1)})$. Its explicit forms are as follows:

**Theorem 8.2.** Let us set

$$F_m^{(p)} := \frac{x_m^{(p)}(x_m^{(a)} \cdots x_m^{(p-1)})^2}{x_m^{(a+1)} \cdots x_m^{(a)} x_{m+1}^{(a)} \cdots x_{m+1}^{(p-1)}}. \quad (8.1)$$

Then, we obtain the action of $s_i$:

$$s_i(x)_m^{(l)} = \begin{cases} 
\sum_{p=0}^{l-1} F_i^{(p)} + \frac{1}{D_i^{(p)}}, & \text{if } m = i, \ (i = 1, \cdots, n), \\
\sum_{p=0}^{l-1} F_i^{(p)} + \frac{1}{D_i^{(p)}}, & \text{otherwise}, \\
x_m^{(l)}, & \text{for } (l, m) \neq (1, n), \\
x_m^{(l)}, & \text{for } (l, m) = (1, n). 
\end{cases} \quad (8.2)$$

$$s_0(x)_m^{(l)} = \frac{x_m^{(l)} U_m^{(l-1)} + x_m^{(1)} x_1^{(k)} Y_m^{(l)} + x_m^{(l-1)} U_m^{(l-1)} + x_m^{(1)} x_1^{(k)} Y_m^{(l-1)}}{x_1^{(k)}} \quad (8.3)$$

**Proof.** We know that $\gamma_i(x) = \frac{D_i^{(a-1)}}{x_i^{(a)}}$ for $i = 1, \cdots, n$ by (8.4), and $\gamma_0(x) = \frac{1}{x_n^{(a)} x_1^{(a)}}$ for $i = 0$ by (8.2). Then, due to Theorem 8.1 substituting these formulae to (8.5) and (8.6), we get (8.2) and (8.3). $\square$

### 8.2 Piecewise-linear action of the affine Weyl group $W(A_n^{(1)})$

We shall describe the action of $W(A_n^{(1)})$ on the crystal $B^{k, \infty}$.

As an application of Theorem 8.1 and ultra-discretization of positive geometric crystals, we obtain the following easily.

**Theorem 8.3.** Let $(X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ be a positive geometric crystal of $G$ (or $\mathfrak{g}$) and $(B, \{\tilde{e}_i\}_{i \in I}, \{\tilde{\varepsilon}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ its ultra-discretized crystal. Define $\tilde{s}_i(x) := \tilde{e}_i^{-\gamma_i(b)}(b)$ for $b \in B$. Then, $(\tilde{s}_i | i \in I)$ gives a piecewise-linear actions on $B$ of Weyl group $W$ associated with $G$ (or $\mathfrak{g}$).

**Theorem 8.4.** Let $\beta, \gamma$ and $\Delta_b(c)$ be as in (8.1) For $p \in \{\beta + 1, \cdots, \gamma\}$ and $b = (b_{j,i}) \in B^{k, \infty}$ set

$$\tilde{\Gamma}_i(p) := \sum_{p \leq j \leq \gamma} (b_{j,i} - b_{j+1,i+1}), \quad \Delta_b^{(1)}(c) := \Delta_b(c) + b_{11}, \quad \Delta_b^{(k)}(c) := \Delta_b(c) + b_{k,n+1}. \quad (8.4)$$
Then the action of $\tilde{s}_i$ is as follows:

$$
\tilde{s}_i(b)_{l,m} = b_{l,m} + (\delta_{i,m-1} - \delta_{i,m}) \times 
\left( \min_{l+1 \leq p \leq \gamma} \tilde{\Gamma}_i(p), \min_{\beta+1 \leq p \leq \gamma} \tilde{\Gamma}_i(p) \right) - \min \left( \min_{l \leq p \leq \gamma} \tilde{\Gamma}_i(p), \min_{\beta \leq p \leq \gamma} \tilde{\Gamma}_i(p) \right) 
\right) 
(i = 1, 2, \cdots, n),
\right)
\tilde{s}_0(b)_{l,m} = b_{l,m} + \min \left( \min_{c_{i-1} > m} \Delta_{b}^{(1)}(c), \min_{c_{i-1} \leq m} \Delta_{b}^{(k)}(c) \right) - \min \left( \min_{c_{i} > m} \Delta_{b}^{(1)}(c), \min_{c_{i} \leq m} \Delta_{b}^{(k)}(c) \right) 
+ \min \left( \min_{c_{i-1} \geq m} \Delta_{b}^{(1)}(c), \min_{c_{i-1} < m} \Delta_{b}^{(k)}(c) \right) - \min \left( \min_{c_{i-1} \geq m} \Delta_{b}^{(1)}(c), \min_{c_{i-1} < m} \Delta_{b}^{(k)}(c) \right).
\right)
\right)

Proof. As for the cases $i \neq 0$, substituting $d = -\text{wt}_i(b) = \sum_{\beta+1 \leq j \leq \gamma} (b_{ji} - b_{j+1,i+1})$ for (7.28) we obtain the formula (8.5).

As for the $i = 0$-case, substituting $d = -\text{wt}_0(b) = b_{11} - b_{k,n+1}$ for (7.28), we get the formula (8.6). □

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