Scaling in Steiner Random Surfaces

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Abstract

It has been suggested that the modified Steiner action functional has desirable properties for a random surface action. In this paper we investigate the scaling of the string tension and massgap in a variant of this action on dynamically triangulated random surfaces and compare the results with the gaussian plus extrinsic curvature actions that have been used previously.

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1 Introduction

The issue of whether a non-trivial continuum limit exists for gaussian plus extrinsic curvature (GPEC) lattice actions of the form

\[ S = \sum_{<ij>} (\vec{X}_i - \vec{X}_j)^2 + \lambda \sum_{\Delta_i, \Delta_j} (1 - \vec{n}_i \cdot \vec{n}_j) \]  

on dynamically triangulated random surfaces, is of interest for the construction of well-defined lattice versions of string theory \[1, 2, 3, 4, 5\] as well as for constructing models of membranes in biophysics and chemistry. The second term in equ.(1), where the \( \vec{n}_i \) are the unit normals on neighbouring triangles, is a discretization of the extrinsic curvature and acts as a “stiffness” term. If this term is absent one has a gaussian discretization of the basic Polyakov action \[6\] which gives rise to pathologically crumpled surfaces due to the failure of the string tension to scale \[7\]. The dynamical nature of the triangulation is manifested as a sum over triangulations, \( \sum_T \), in the canonical (fixed number of points) partition function

\[ Z_N(\lambda) = \sum_T \int \prod_{i=1}^N d\vec{X}_i \delta(\sum_i \vec{X}_i) \exp(-S), \]  

where the delta function is inserted to kill the translational zero-mode, and \( N \) is the number of points. This means that we have in effect a fluid surface. The GPEC model of equs.(1,2) apparently has a low \( \lambda \) crumpled phase and a large \( \lambda \) smooth phase similar to those displayed by identical models on fixed triangulation surfaces \[8\] where the sum over triangulations in equ.(2) is dropped. The initial work in \[8\] found a pseudo-second order transition on small lattices, but later work \[2, 4, 5\] with larger lattices and better statistics suggested rather that the transition was higher order, or a crossover phenomenon \[5, 9\].

The strongest evidence so far that there is, indeed, a transition comes from the measurements of the scaling of the string tension and mass gap carried out in \[2\]. An earlier measurement of the string tension also found results that were consistent with scaling, but in this the points on the boundary, which constituted a large proportion of the total number of points, were physically pinned down \[3\] and the vanishing of the lattice string tension at the critical point was assumed. Although analytical calculations suggested \[10\] that the extrinsic curvature coupling \( \lambda \) in equ.(1) was asymptotically free and hence that there was no non-trivial theory for finite \( \lambda \) in the lattice action, the measurements in \[2\] were strongly indicative of scaling and hence a finite string tension in physical units. This implies a non-trivial continuum limit at finite \( \lambda \).

Pending clarification of the behaviour of the GPEC action on dynamical triangulations, we thought it a worthwhile exercise to investigate possible alternative lattice random surface actions in order to see if their behaviour was more (or less!) clear-cut. We have already conducted some preliminary simulations \[11, 12\] of actions containing terms of the form suggested by Savvidy et.al. \[13, 14\] that incorporate the modified Steiner functional \[15\]. The basic “Steiner” action is just

\[ S_{\text{Steiner}} = \frac{1}{2} \sum_{<ij>} |\vec{X}_i - \vec{X}_j| \theta(\alpha_{ij}), \]  

where \( \theta(\alpha_{ij}) = |\pi - \alpha_{ij}| \) and \( \alpha_{ij} \) is the angle between the embedded neighbouring triangles with common link \( <ij> \). This is essentially a coarse discretization of the absolute value of the trace of the second fundamental form of the surface, rather than its square which appears in the GPEC action. It was observed in \[16\] that an action containing only this term ran into problems with the entropy of vertices in smooth configurations and failed to give a well-defined grand canonical (varying number of vertices) partition function. It is a relatively simple matter however to concoct variations on this theme that constrain the errant planar vertices somewhat such as

\[ S_1 = \frac{1}{2} \sum_{<ij>} |\vec{X}_i - \vec{X}_j| + \frac{\lambda}{2} \sum_{<ij>} |\vec{X}_i - \vec{X}_j| \theta(\alpha_{ij}) \]  

or even

\[ S_2 = \sum_\Delta |\Delta| + \frac{\lambda}{2} \sum_{<ij>} |\vec{X}_i - \vec{X}_j| \theta(\alpha_{ij}), \]  

1
where the \( |\Delta| \) is just the area of triangle \( \Delta \) as seen in the space in which the surface is embedded. In [4] another alternative was suggested in which \( \theta \) was modified to \( \theta(\alpha_{ij}) = |\pi - \alpha_{ij}| \xi \) with \( \xi < 1 \) which also appeared to improve the convergence of the grand-canonical partition function.

In [10] we carried out simulations of \( S_1, S_2 \) along with a further permutation combining a gaussian term with the Steiner part

\[
S_3 = \frac{1}{2} \sum_{<ij>} (\vec{X}_i - \vec{X}_j)^2 + \frac{\lambda}{2} \sum_{<ij>} |\vec{X}_i - \vec{X}_j| \theta(\alpha_{ij})
\]  

and found rather similar behaviour to that seen for the GPEC action on small (72 and 144 nodes) meshes - namely peaks in the specific heats for the respective actions. For \( S_1 \) we have

\[
C = \frac{\lambda^2}{N} (\langle S_{Steiner}^2 \rangle - \langle S_{Steiner} \rangle^2).
\]

We also see, by visual inspection of the surfaces, a low \( \lambda \) crumpled phase and a large \( \lambda \) smooth phase. Although the gyration radius, a measure of the size of the embedded surfaces,

\[
X^2 = \frac{1}{9N(N-1)} \sum_{ij} (\vec{X}_i - \vec{X}_j)^2 q_i q_j
\]

where the \( q_i \) are the number of neighbours of point \( i \), was not monotone increasing with \( \lambda \) as for the GPEC action this could be explained by noting that the Steiner term, unlike the extrinsic curvature, was dimensionful.

2 Expected Scaling Properties

Our simulations described above were carried out on boundaryless surfaces with spherical topology. The simulations of [2], which were the most convincing demonstration to date of a non-trivial continuum limit, required surfaces with boundaries in order to carry out the scaling measurements of the string tension and mass-gap. These are constructed in an ingenious manner using twisted boundary conditions on a torus, which we now outline, in order to avoid pinning down a disproportionately large amount of lattice at the boundary loops or points. It was observed in [3] that on a torus the sum of vectors \( \vec{X}_{ij} \) along the edges of the triangulation on a closed path could take the values

\[
\vec{E}(n_1, n_2) = n_1 \vec{E}_1 + n_2 \vec{E}_2
\]

where the vectors \( \vec{E}_1, \vec{E}_2 \) are constant and the integers \( n_1, n_2 \) denote how many times the path winds round the two respective periods of the torus. For non-zero \( E_i \) this means that

\[
\vec{X}_i(k_1, k_2) = \vec{X}_i + k_1 \vec{E}_1 + k_2 \vec{E}_2
\]

where the \( k_i \) labelled the particular “copy” of the surface at a given point. The partition function in equ.(2) is now dependent on the choice of \( \vec{E}_1, Z_N(\lambda) \rightarrow Z_N(\lambda, \vec{E}_1, \vec{E}_2) \). Non-zero values of \( \vec{E}_1, \vec{E}_2 \) correspond to simulating the surface on a frame \( \vec{E}_1 \times \vec{E}_2 \). The important point to note is that it is not necessary to designate any of the points as boundary points in this procedure. It is thus possible to avoid potential problems with too many points on the boundary for small surfaces.

A canonical string tension \( \sigma(\lambda, N, y^2) \) for the system described above is defined by taking \( \vec{E}_1 = (y, 0, 0), \vec{E}_2 = (0, y, 0) \), \( F_N(\lambda, y^2) = -\log Z_N(\lambda, y^2) \) and

\[
\sigma(\lambda, N, y^2) = \frac{\partial F_N(\lambda, y^2)}{\partial y^2}
\]

where the translational invariance of \( Z_N \) means it depends on only the projected area \( y^2 \). Similarly a canonical massgap is defined by choosing \( \vec{E}_1 = (y, 0, 0), \vec{E}_2 = (0, 0, 0) \) and

\[
m(\lambda, N, y) = \frac{\partial F_N(\lambda, y)}{\partial y}.
\]
It is expected that the $N$ and $y^2$ dependence in the string tension appears as the ratio $r = y^2/N$ and in the massgap as the ratio $t = y/N$.

It is actually more natural to define the physical string tension in a grand canonical ensemble (with a varying number of points) [17], which can be done by taking the Legendre transform of $F_N(\lambda, y^2)$

$$G(\mu, \lambda, y^2) = N\mu + F_N(\lambda, y^2)$$

where $\mu$ is the cosmological constant. For large $y^2$ one expects $G(\mu, \lambda, y^2) \simeq \bar{\sigma}(\lambda, \mu)y^2$ with

$$\bar{\sigma}(\lambda, \mu) = \frac{\partial F_N}{\partial y^2} = \sigma(\lambda, r).$$

The grand canonical $\bar{\sigma}(\lambda, \mu)$ is expected to scale as

$$\bar{\sigma}(\lambda, \mu) \simeq \sigma_0(\lambda) + d(\lambda)\mu_R^{2\nu}$$

where the exponent $\nu$ governs the scaling of the physical area $A_{\text{phys}} \simeq \mu_R^{2\nu}y^2$, with $\mu_R = (\mu - \mu_{\text{crit}})$. It is then possible to deduce the expected scaling of the canonical $\sigma(\lambda, r)$:

$$\sigma(\lambda, r) \simeq \sigma_0(\lambda) + \sigma_1(\lambda)r^{2\nu/(1-2\nu)}.$$  

The physical string tension $\sigma_{\text{phys}} = \bar{\sigma}(\lambda, \mu)/\mu_R^{2\nu}$ will be infinite unless $\sigma_0(\lambda) \simeq (\lambda - \lambda_{\text{crit}})^\alpha$ as we approach a critical point at some $\lambda_{\text{crit}}$ and this $\sigma_0$ is accessible in a canonical simulation.

It is also possible to play a similar game with the massgap, defining

$$G(\mu, \lambda, y) = N\mu + F_N(\lambda, y)$$

which is expected to behave as $G(\mu, \lambda, y) \simeq \bar{m}(\lambda, \mu)y$ for large $y$. In this case we have $\bar{m}(\lambda, \mu) \simeq \mu_R^{\nu}$ and

$$m(\lambda, t) \simeq D(\lambda)t^{\nu/(1-\nu)}$$

which is again accessible to a canonical simulation.

### 3 Numerical Simulations and Results

In this paper we apply the methods of [2] to analyse the scaling of the massgap and string tension for one of the variant Steiner actions, $S_1$. This was chosen because the two terms in the action $\frac{1}{2}\sum_{<ij>} |X_i^\mu - X_j^\mu|^{16}$ and $\sum_{<ij>} |X_i^\mu - X_j^\mu|\theta(\alpha_{ij})$ have the same scaling dimensions which simplifies somewhat the choice of observables. If we rescale the coordinates $\vec{X}_i \rightarrow y\vec{X}_i$ in $S_1$ we find $S_1(\vec{X}, y) \rightarrow yS_1(\vec{X}, 1)$, which means that with the appropriate boundary conditions for the string tension measurements

$$\sigma(\lambda, r) = \frac{\partial F_N(\lambda, r)}{\partial y^2} = <S_1 > - 3(N-1)$$

for a surface with $N$ points embedded in 3 dimensions. If we use the boundary conditions that are appropriate for the massgap measurements we find

$$m(\lambda, r) = \frac{\partial F_N(\lambda, t)}{\partial y} = <S_1 > - 3(N-1).$$

We thus simply measure the expectation value of the action with the appropriate choice of frame in order to access information about the string tension and massgap scaling.

In addition, we measure the specific heat as defined in equ.[3] and histogram the output data at the various $\lambda$ simulated in order to use the multi-histogram method of Ferrenberg and Swendsen [18], which allows one to estimate the density of states and hence the specific heat for arbitrary $\lambda$. We also measure the gyration radius, as defined in equ.[8], but a certain amount of care is needed with this because of the
twisted boundary conditions. We choose to measure the X2 using only the component transverse to the frame in the string tension measurements to avoid confusion, and the two components transverse to the line separating the pinned points in the case of the massgap measurements. The autocorrelation times for the various observables are calculated in order to ensure that we have reasonable statistics. We also measure the various acceptances for the lattice and X moves to check that the Monte-Carlo algorithm, which we now describe, is behaving reasonably.

In order to achieve a reasonable amount of vectorization in the code 64 systems were simulated in parallel, with measurements being taken after a sufficient number of sweeps were made to allow them to decorrelate. It proved to be convenient to store the link variables $\vec{X}_{ij}$ rather than the site variables $X_i$ which allows the incorporation of the twisted boundary conditions as $\vec{X}_{ij} = \vec{X}_i - \vec{X}_j + \vec{E}_{ij}$, where $\vec{E}_{ij} = n_{ij}^1 \vec{E}_1 + n_{ij}^2 \vec{E}_2$. The integers $n_{ij}$ are non-zero when the link $<ij>$ passes from one of the elementary cells in the parameter space (a plane for the torus) to another. Rounding errors during the simulation can be kept under control by using the transformations

$$
\begin{align*}
\vec{X}_i &\rightarrow \vec{X}_i + l_i^1 \vec{E}_1 + l_i^2 \vec{E}_2 \\
\vec{E}_{ij} &\rightarrow \vec{E}_{ij} + l_i^1 \vec{E}_1 + l_j^2 \vec{E}_2 - l_j^1 \vec{E}_1 - l_i^2 \vec{E}_2
\end{align*}
$$

where the $l$'s are arbitrary integers to keep the $E_{ij}$'s from straying.

The sum over lattices is effected by carrying out local flip moves on adjacent triangles, forbidding flips that lead to degenerate triangulations with less than 3 neighbours per point or with 2-loops. With non-trivial boundary conditions the $\vec{E}_{ij}$ for affected edges must be changed, whereas the $\vec{X}_i$ are left untouched. The coordinates $\vec{X}_i$ are updated with a simple Metropolis scheme, which does not affect the non-trivial boundary conditions. In this paper we report on simulations carried out on relatively small surfaces of size 64 and 144 nodes. We have not proceeded to larger surfaces in the current batch of simulations because there is a hidden penalty built into the direct transcription of the Steiner action we have used in $S_1$, compared with the GPEC action. Namely, the calculation of $\theta(\alpha_{ij})$ requires an inverse trigonometric operation, rather than the simple multiplications involved in calculating $\vec{a} \cdot \vec{n}$ in the GPEC action. It might be possible to avoid this in further simulations by using some trigonometric function with the requisite properties for $\theta(\theta(2\pi - \alpha) = \theta(\alpha), \theta(\pi) = 0, \theta(\alpha) \geq 0)$, but this begs the question of universality.

If we now move on to discuss the measurements made for the string tension and massgap with the choice of frames $\vec{E}_1 = (y, 0, 0)$, $\vec{E}_2 = (0, y, 0)$ and $\vec{E}_1 = (y, 0, 0)$, $\vec{E}_2 = (0, 0, 0)$ respectively, comparison of Fig.1 for the string tension and Fig.2 for the massgap with Figs.6,7 in the first of [19] reveal striking similarities. Looking at Fig.1 for the string tension first we see that the data points, as expected, fall on universal curves as a function of $r$ for a given $\lambda$ until finite size effects set in at small $r$. Lines are drawn to guide the eye through the points coming from the $N = 64 (8 \times 8)$ surface. For large $r$, just as for the GPEC action, we would expect a $\lambda$ independent limit which is what is observed. For small $r$ the $\lambda$ dependence becomes more marked and as the $\lambda = 3, 4$ values straddle zero as $r \rightarrow 0$ we can infer there is some $\lambda$ value intermediate between these, say $\lambda_C$ where $\sigma(\lambda, r) \rightarrow 0$ as $\lambda \rightarrow \lambda_C$ and $r \rightarrow 0$. This is one of the prerequisites for a finite physical string tension. For $\lambda = 4$ we see negative $\sigma(\lambda, r)$ at small $r$, which is due to the repulsion of the vertices, and the value where it is zero corresponds to the equilibrium configuration. This is again similar to the behaviour observed in the GPEC action, as are the very long autocorrelation times observed in this phase.

We would expect the results for $m(\lambda, t)$ in Fig.2 to fall on universal curves for different $\lambda$ with $t = y/N$ until finite size effects set in at small $t$, and this is, indeed, what is seen. We have again drawn lines through the $N = 64 (8 \times 8)$ points to guide the eye. Only the crudest of fits to the scaling law in equ.(18) ($m \simeq D(\lambda) \nu^{\beta(\lambda)}$, where $\beta(\lambda) = \nu(\lambda)/(1 - \nu(\lambda))$ ) is possible with our data, and we find $\nu$ to be in the region of 0.3 for $\lambda = 3$ and 0.2 for $\lambda = 4$. This gives an estimate of $\nu$ at $\lambda_C$ which is lower than that for the GPEC action, 0.38 $< \nu < 0.42$. This difference could be due to finite size effects. It has recently been pointed out that these may be so strong that they completely mask the presence of the tachyon, and hence the real continuum physics, for multiple spin models on dynamical lattices up to very large sizes [19], though the effects may well be less severe here as they depend sensitively on the extrinsic Hausdorff dimension (of the order of 4 in this case) and probably non-universal constants. We note that our value
is consistent with an earlier estimate for the GPEC action in the second of [2], which was obtained using system sizes similar to those studied here. A fit to the scaling law in equ.17 for $\sigma(\lambda, r)$ gives a lower estimate for $\nu \leq 0.2$ for both $\lambda = 3, 4$ but the quality of the fits here is even poorer, so the discrepancy is not too disconcerting.

The results for the specific heat are shown in Fig.3 for some selected $r$ values, where we have not plotted the Ferrenberg-Swendsen interpolation between the measured points for simplicity. The Ferrenberg-Swendsen interpolation gives a value for $\lambda_C$ that is close to 4. The quality of this interpolation deteriorates as the relative size of the frame (ie $r$ or $t$) is increased, suggesting that finite size effects may be quite important in such cases. Finally the time evolution of $X_2$ for $r = 1$ is shown in Fig.4 for one of the square frames used in the string tension measurements, and is at first sight rather surprising as it decreases with time after a disordered start for large $\lambda$. However, it should be remembered that we measure transverse fluctuations to the frame, so if the surface becomes more rigid with increasing $\lambda$ this is what we would expect to see as the transverse fluctuations are suppressed. When the frame size is reduced sufficiently the customary behaviour of a larger $X_2$ in the large $\lambda$ phase reasserts itself.

To summarize the numerical results of this paper: for the action $S_1$, containing an edge length and a Steiner term, the scaling behaviour of both the string tension and the massgap are very similar to those seen in the GPEC action. There is some evidence for universal behaviour in the value of the exponent $\nu$, but our measurements of this are very preliminary. The behaviour of the specific heat peak is consistent with that seen in our earlier, smaller scale, simulations of $S_1$. Finally $X_2$, both with and without framing the surfaces, gives every indication of a smooth or rigid phase at larger $\lambda$. There is no sign, however, of sharper scaling behaviour than is seen with GPEC actions, for $S_1$ at any rate. For future work it is possible that a subdivision invariant action such as $S_2$ might offer a faster approach to the continuum limit $[10]$. A more judicious choice of $\theta$ from the numerical point of view for any of the Steiner actions $S_1, S_2, S_3$ might also offer the possibility of more efficient simulations. Nonetheless, the current batch of simulations has demonstrated that there is evidence for scaling and hence a non-trivial continuum theory for a particular Steiner action, just as with the GPEC action.

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Figure Captions

**Fig. 1.** The canonical string tension $\sigma(\lambda, r)$ is plotted for the various $\lambda$ values to show the scaling with $r = y^2/N$.

**Fig. 2.** The canonical massgap $m(\lambda, t)$ is plotted for the various $\lambda$ to show the scaling with $t = y/N$.

**Fig. 3.** The specific heat vs $\lambda$ for some selected $r$ values. Our original small scale simulations of $S_1$ [11] are also included for comparison.

**Fig. 4.** The decrease of the perpendicular $X2$ with time at large $\lambda$ is evident from these measurements taken with $r = 1.0$.  

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