A meshless solution of a of lid-driven cavity containing a heterogeneous porous medium

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Abstract. The main objective of the present paper is to define and evaluate a benchmark test for solving the Navier-Stokes equations in heterogeneous porous medium. The benchmark is of high relevance for proper numerical simulation of flow pattern in the liquid-solid phase-change related phenomena. The test case represents a lid-driven square cavity which consists of a solid, a liquid, and a mushy part. Three different variations of the problem are solved, in which the ratio between the liquid and the mushy part is modified. The Reynolds and the Darcy number of this case are equal to 400 and $2.5 \times 10^{-10}$, respectively. The Navier-Stokes equation, with an additional source term to take into account the Darcy effect, is solved with the local meshless diffuse approximate method. The method is structured by using the second-order polynomial basis vector and the Gaussian function to evaluate the weighted least squares approximation. The explicit Euler scheme is used to perform the time discretisation. The non-incremental pressure correction scheme is used to couple the pressure and velocity fields. Results are presented in terms of stream function, velocity magnitude, and mid-plane velocity profiles in the steady state. The accuracy of the numerical method is evaluated by a comparison with the finite volume solution obtained with a commercial solver and an investigation of the node arrangement effect.

1. Introduction

Solidification represents a key phenomenon in a spectra of classical casting and welding processes as well as in the contemporary technologies such as additive manufacturing [1]. The material structure and properties are direct results of this phenomenon. Understanding the solidification is therefore highly important when the defects, such as cracking, hot tearing, porosity, macrosegregation, surface roughness, etc., are tackled. Computer simulations are an extremely powerful and a very cost-effective tool used to perform numerical experiments and to investigate the related phenomena [2, 3]. Simulations on the macroscopic scale are considered in this paper. One of the suitable frameworks for simulation of solidification is the volume-averaged formulation of the two-phase liquid-solid system [4, 5]. The conservation equations for the mass, momentum, heat, and species, along with the adjacent constitutive relations, form the phase change model [6]. Obtaining a numerical solution of this numerical model is not trivial, since it is highly non-linear and strongly coupled [7]. The numerical models are therefore prone to errors, which can be identified and reduced with verification tests. The test described in this paper is focused on the mass and momentum conservation part of the solidifying systems with a special attention to the proper fluid flow evaluation in the mushy zone. The main purpose is to investigate the fluid flow in the situation where the liquid phase fraction decreases from one to zero. This is a typical situation, encountered in solidification systems. In order to focus on proper evaluation of fluid flow in
such a system, the model is decoupled from the energy and species conservation equations. The position of the mushy zone is thus constant and prescribed in advance. Such test enables the verification of the fluid flow in the mushy zone, which has a significant impact on the macrosegregation and other casting defects. This is the first benchmark of this type to the best of author’s knowledge. Other benchmarks with porous media also exist and are available in the literature, but they mainly consider homogenous porous media and the Darcy’s law.

Meshless numerical methods [8] represent one of the most rapidly developing fields of computational mechanics. They are praised for their flexibility in the choice of computational nodes. They do not require the mesh construction, which is beneficial in solidification problems. Most noticeably in cases where very fine node spacing is required in the mushy area and a sparse one in the solid phase, where time adaptive computational node arrangements are used to reduce the computational time, and where complex shaped geometries are considered (e.g., pressure die casting). Several successful applications of different types of meshless methods to multiphysics and multiscale phase-change related phenomena as well as other manufacturing phenomena have already been published in recent years [9–17]. The results presented in this paper are computed with the meshless diffuse approximation method (DAM) [18–20] and compared with the solution obtained with the reference finite volume model (FVM).

2. Model formulation

The volume averaged mass and momentum conservation equations for incompressible flow are solved at the macroscopic scale. The mushy zone is described as a porous medium with permeability $K$ which is a function of position in the proposed benchmark

$$\nabla \cdot \mathbf{v} = 0,$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \left[ \nabla p + \mu \nabla^2 \mathbf{v} - \frac{\mu g_l}{K} \right] = 0,$$

where $\mathbf{v}$ is the volume averaged velocity, $t$ is the time, $\rho$ is the density, $\mu$ is the dynamic viscosity, $p$ is the pressure, and $g_l$ is the liquid fraction ($= \text{porosity})$. The volume averaged velocity is directly linked to the liquid velocity $\mathbf{v}_l$ with the liquid fraction: $\mathbf{v} = g_l \mathbf{v}_l$. The pressure-velocity coupling is implemented with the non-incremental pressure-correction scheme (see [13] for more details).

3. Numerical benchmark definition

3.1. Dimensionless numbers analysis

The dimensionless numbers that effect the Navier-Stokes equation with the Darcy source term are the Reynolds ($\text{Re} = \rho v L / \mu$) and the Darcy number ($\text{Da} = K_0 / L^2$). These two numbers are set to $\text{Re} = 400$ and $\text{Da} = 2.5 \times 10^{-10}$ in all cases considered in this paper. The value of these numbers is consistent with the conditions, encountered in direct-chill casting of aluminium alloys [21].

3.2. Geometry

The test resembles the laminar lid-driven cavity benchmark case [22]. Global computational domain is a square shaped cavity with unit sides ($H = L = 1$), which is partially filled with a heterogeneous porous media (see figure 1). The permeability is infinite at the top of the cavity ($y > S + M$) and starts to decrease linearly at the vertical position $y > S + M$, where $S$ and $M$ are the thickness of the solid and the porous medium, respectively. It decreases to 0 at the bottom of the cavity ($y = S$). The permeability is defined through the liquid fraction and is more thoroughly elaborated in the next section. Three different geometrical cases are simulated, where the porous media thickness $M$ is varied (0.25, 0.50, and 0.75).
3.3. Heterogeneous porous media

The permeability $K$ in the porous media is modelled by using the Kozeny-Carman model

$$K = K_0 \frac{g_l + \varepsilon}{(1-g_l)^2},$$

(2)

where $K_0$ is permeability constant, which is set to $2.5 \times 10^{-10}$ m$^2$ in order to match the desired Darcy number. $\varepsilon$ is a small value (e.g., $1 \times 10^{-12}$), used to avoid the division by zero in the momentum conservation equation in case $g_l = 0$. The liquid fraction $g_l$ decreases linearly with the vertical position and is defined with the following relation

$$g_l(y) = \begin{cases} 
1 & \text{if } y > M + S \\
M^{-1}(y-S) & \text{if } S \leq y \leq M + S, \\
0 & \text{if } y < S
\end{cases}$$

(3)

where

$$S = 0.1H.$$  

(4)

3.4. Boundary conditions

The Dirichlet boundary conditions are used on all boundaries. The velocity in the $x$ direction is set to $v_x = 1$ m/s on the top boundary and to $v_x = 0$ m/s on the bottom and the side boundaries. The velocity in the $y$ direction is set to $v_y = 0$ m/s on all boundaries.

3.5. Initial conditions

The initial values for velocity in the $x$ and the $y$ direction are set to 0 m/s and the initial pressure is set to 0 Pa in the whole domain.
4. The numerical procedure

The partial differential equations are solved with the meshless-diffuse-approximate method. The DAM uses the weighted least squares formulation to determine a locally smooth approximation from a discrete set of data:

\[ \hat{f}(x) = p(x, x_0)\alpha = \sum_{j=1}^{J} p_j(x_j, x_0)\alpha, \]

where \( \alpha \) is the vector of coefficients and \( p(x, x_0) \) is the basis vector. The second-order polynomials are used as the basis vector. The Gaussian function is used to weight the approximation error in the minimization expression:

\[ \theta(x, x_0) = \exp\left(-c \frac{\|x-x_0\|}{h^2}\right), \]

where \( c \) and \( h \) are free-parameter and scaling factor, which are equal to 5 and to the Euclidian distance between \( x_0 \) and the farthest position from \( x_0 \) in the discrete set of data.

The method is localized by defining the smooth approximation for each computational node separately. This is performed by associating each node with a unique local neighbourhood, which is used for minimization. Either 9 or 14 nodes are included in the local subdomains for simulations. The stability of the advective term is attained with a shift of the Gaussian weight and evaluation location in the upwind direction. This approach is called the adaptive upwind weight function and it is a successful stabilization procedure [13]. Explicit Euler scheme is used for temporal discretization.

5. Time and spatial discretisation

Both structured and unstructured computational node arrangements are considered (see figure 2). Node arrangements with 50x50, 100x100, 150x150, 200x200, and 250x250 are used in order to observe the node spacing convergence. The time step is set to \( 1 \times 10^{-4} \) s in all simulations. The criterion for the steady state is that the relative change in the x mid-plane minimum horizontal velocity between two time steps is less than \( 1 \times 10^{-6} \). The reference (FVM) steady state solution is obtained only for the structured node arrangement with 250x250 nodes. The FVM model is calculated with the steady-state, pressure-based solver. It is solved in 2D with a double precision. The pressure-velocity coupling is performed with SIMPLE approach. Second order upwind stabilisation is used for the convective term. Residual limit of the steady-state solver for the continuity and velocity components are set to \( 1 \times 10^{-6} \).

![Figure 2](image-url)  
**Figure 2.** Comparison of (a) structured and (b) unstructured computational node arrangement (50x50).
6. Results
The final velocity field calculated with DAM is presented with the velocity magnitude contours and velocity direction vectors in figure 3. The results are additionally presented with figure 5, where the streamline contours and the mid-plane horizontal velocities obtained with DAM and FVM model are compared. The velocity magnitude decreases exponentially in the porous media. The horizontal velocity magnitude is therefore plotted with the logarithmic scale in the right part of figure 5.

The results show that the flow is restricted by the decreased permeability. The effect of porous media is very large even at very high liquid fraction, causing the main vortex to form almost exclusively in the liquid phase. The size of the main vortex is therefore controlled by parameter $M$. Recirculation is observed for $M = 0.25$ and 0.50, while only the main vortex is observed for 0.75.

![Figure 3](image)

Figure 3. Direction vectors of the velocity field. The DAM results for 250x250 structured node arrangement is shown.

6.1. Convergence
The convergence of the node spacing is evaluated for all three cases in figure 4. The minimum horizontal component of the velocity vector along the vertical line through the geometric centre of the cavity is compared. This value is simply referred as the minimum velocity in the further text. The results show
that the solution on the finest node spacing \((h = 0.004)\) is independent of node distribution. This observation is valid for both structured and unstructured node arrangement.

![Graph showing node spacing convergence for structured and unstructured node arrangement.](image)

**Figure 4.** Node spacing convergence for the structured and unstructured node arrangement.

### 6.2. Verification

The solution obtained with the presented meshless model is compared with the solution obtained with the reference FVM model. Comparisons are presented in figure 4 on the finest node density. The results have a very good match, especially at the top, liquid part of the cavity, where the streamlines overlap completely. The relative difference in the minimum velocity is calculated for the 250x250 (structured) node arrangements

\[
E = \frac{V_{\text{DAM}, \min} - V_{\text{FVM}, \min}}{V_{\text{FVM}, \min}},
\]

where superscripts DAM and FVM are used to denote which model is used to obtain the velocity value. The relative difference between the results is equal to \(-1.09 \times 10^{-3}\), \(-6.33 \times 10^{-3}\), and \(-7.43 \times 10^{-3}\) for \(M = 0.25, 0.50, \) and \(0.75\), respectively. The agreement between the results is good, especially in the upper part of the porous media, where permeability is relatively high \((g_l > 0.9)\). The difference between the results increases when the absolute value of minimum velocity is below \(1 \times 10^{-6}\). This is expected, because the FVM solution is calculated with a steady state solver, with the residual limit for the velocity components set to \(1 \times 10^{-6}\). It is observed that the velocity in the non-permeable (solid) phase is not zero for the FVM solution, which is probably a numerical error. This is not the case in DAM solution, where an additional condition is added to the solver so that the momentum conservation equation is not solved in the solid phase. Therefore, the comparison of the results is not valid in the solid phase.

### 6.3. Comparison of computational node arrangements

The solution of the problem is obtained on both structured and unstructured node arrangement (see figure 2). The visual comparison shows a slight difference between the results (see figure 4 and figure 5). The difference is evaluated by calculating the relative difference in the minimum velocities for the finest node arrangement

\[
E = \frac{V_{\text{Unstructured}, \min} - V_{\text{Structured}, \min}}{V_{\text{Unstructured}, \min}},
\]

where the superscripts are used to denote the type of node arrangement on which the solution is obtained. The relative difference between both node arrangements is equal to \(4.35 \times 10^{-4}\), \(-1.71 \times 10^{-3}\), and \(-1.90 \times 10^{-3}\) for \(M = 0.25, 0.50, \) and \(0.75\), respectively.
Figure 5. Comparison of DAM and FVM results. (a) DAM (solid black line) and FVM (dashed black line) stream function contours. The horizontal green lines are used to display the liquid fraction isolines. The vertical blue line is used to show the position of the vertical mid-plane cross section. (b) Horizontal component of the velocity magnitude in the logarithmic scale is compared at the vertical cross section. The dashed line is used to plot the FVM results. The red and the blue dots are used to plot the DAM results on the structured and unstructured computational node arrangements. The red line is used to show liquid fraction/porosity.
7. Conclusions
A new benchmark for fluid flow in heterogeneous porous medium is proposed. The test case is designed in such a manner that the conditions are similar to the ones during direct-chill casting of aluminium alloys. Three different cases are studied, where the thickness of the porous media is varied. The benchmark is solved by using the local meshless diffuse approximate method. The meshless solutions are obtained on the structured and the unstructured computational node arrangements with different node spacing. These results show that the solution is independent of the node distribution type. Furthermore, the solution is compared with the reference FVM model. Agreement with the reference results is very good, especially in the liquid and in the highly permeable part of the porous region. The results of this paper show that DAM is suitable for solving the Navier-Stokes equation in heterogeneous porous media. Furthermore, this means that the method is suitable for simulation of melt flow in solidification problems, including the industrial processes of direct-chill casting. The upgrade of the test will include the time dependent solution of the problem with variation of the mushy zone dynamics as well as the anisotropic behaviour of the permeability.

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