ON THE KERNEL OF THE ZERO-SURGERY HOMOMORPHISM FROM KNOT CONCORDANCE

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Abstract. Kawauchi defined a group structure on the set of homology $S^1 \times S^2$’s under an equivalence relation called $\tilde{H}$-cobordism. This group receives a homomorphism from the knot concordance group, given by the operation of zero-surgery. It is natural to ask whether the zero-surgery homomorphism is injective. We show that this question has a negative answer in the smooth category. Indeed, using knot concordance invariants derived from knot Floer homology we show that the kernel of the zero-surgery homomorphism contains a $\mathbb{Z}^\infty$-subgroup.

1. Introduction

In 1976, Kawauchi introduced an equivalence relation on 3-dimensional manifolds with the homology of $S^1 \times S^2$ [7]. This notion, which he refers to as $\tilde{H}$-cobordism, has the virtue of allowing a natural group structure induced by an operation $\ominus$ called the circle union. This group is denoted by $\Omega(S^1 \times S^2)$ and is called the $\tilde{H}$-cobordism group. An interesting feature of the $\tilde{H}$-cobordism group is that it receives a homomorphism from the knot concordance group $\mathcal{C}$ using the zero-surgery operation. It is natural to wonder how faithfully the concordance group is reflected in the $\tilde{H}$-cobordism group under this map.

Question 1. Is the zero-surgery homomorphism $\omega : \mathcal{C} \rightarrow \Omega(S^1 \times S^2)$ injective?

Closely related to $\tilde{H}$-cobordism is the more well-known notion of $\mathbb{Z}$-homology cobordism between 3-manifolds. A $\mathbb{Z}$-homology cobordism between $Y_0$ and $Y_1$ is a cobordism $W$ such that the inclusions $Y_i \hookrightarrow W$, $i = 0, 1$, induce isomorphisms on integral homology groups. In [3], the question of injectivity of the zero-surgery map from the knot concordance group to the set of all $\mathbb{Z}$-homology cobordism classes of 3-manifolds with the homology of $S^1 \times S^2$ was addressed by Cochran, Franklin, Hedden, and Horn. Inspired by their work, this paper investigates the kernel of $\omega$. We show that it is quite large.

Theorem 1. The kernel of the zero-surgery homomorphism $\omega : \mathcal{C} \rightarrow \Omega(S^1 \times S^2)$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$.

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There are a number of related questions that arise from our work. For instance, it is natural to wonder about the cokernel of the zero-surgery homomorphism.

**Question 2.** (c.f. [4]) Is $\omega : C \to \Omega(S^1 \times S^2)$ surjective? If not, how big is the coker($\omega$)?

It is also natural to ask if our result holds in the topological category.

**Question 3.** Is $\ker(C_{\text{top}} \to \Omega_{\text{top}}(S^1 \times S^2))$ non-trivial?

One might expect that in this latter category $\omega$ would be closer to an isomorphism.

Certainly the techniques we use are manifestly smooth.

In Section 2, we briefly review the $\tilde{H}$-cobordism group $\Omega(S^1 \times S^2)$ and the zero-surgery homomorphism $\omega : C \to \Omega(S^1 \times S^2)$. We also discuss several properties of the knot concordance invariants $\Upsilon, \tau$ and $\{V_i\}$ derived from knot Floer homology. In Section 3, we establish a relationship between satellite operations and $\tilde{H}$-cobordism. Using the aforementioned knot invariants, in Section 4 we show that there is a $\mathbb{Z}_\infty$-subgroup in $\ker(\omega)$.

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2. Preliminaries

2.1. An overview of the $\tilde{H}$-cobordism group $\Omega(S^1 \times S^2)$. In this subsection, we review the definitions and basic properties of Kawauchi’s $\tilde{H}$-cobordism group. We refer the reader to [7] for more details.

A 3-dimensional homology orientable handle is a compact, orientable 3-manifold whose integral homology groups are isomorphic to those of $S^1 \times S^2$. A distinguished homology handle is a pair $(Y, \alpha)$ consisting of an oriented homology handle $Y$ and a specified generator $\alpha$ of $H_1(Y; \mathbb{Z})$.

**Definition 2.1.** Two distinguished homology handles $(Y_0, \alpha_0)$ and $(Y_1, \alpha_1)$ are $\tilde{H}$-cobordant if there is a compact, connected, and oriented 4-dimensional manifold $W$ with $\partial W = -Y_0 \sqcup Y_1$ and a cohomology class $\varphi \in H^1(W; \mathbb{Z})$ such that

1. $\varphi|_{Y_i}$ are dual to $\alpha_i$ for $i = 0, 1$,
2. $H_*(\widetilde{W}_\varphi; \mathbb{Q})$ is finitely generated over $\mathbb{Q}$ for each $*$, where $\widetilde{W}_\varphi$ is the infinite cyclic covering of $W$ associated with $\varphi$.

If they are $\tilde{H}$-cobordant, we write $(Y_0, \alpha_0) \sim (Y_1, \alpha_1)$ and call $(W, \varphi)$ (or simply $W$) an $\tilde{H}$-cobordism between $(Y_0, \alpha_0)$ and $(Y_1, \alpha_1)$. If a distinguished homology handle is $\tilde{H}$-cobordant to $(S^1 \times S^2, \alpha)$, where $\alpha$ is the homology class of $S^1 \times *$, then it is called null $\tilde{H}$-cobordant.
It can be easily checked that \((Y, \alpha)\) is null \(\tilde{H}\)-cobordant if and only if there is a compact, connected, and oriented 4-manifold \(\tilde{W}^+\) with \(\partial \tilde{W}^+ = Y\), and a class \(\varphi \in H^1(\tilde{W}^+; \mathbb{Z})\) such that \(\varphi|_Y = \alpha^*\) and \(H_* (\tilde{W}^+_\varphi; \mathbb{Q})\) is finitely generated.

**Lemma 2.2.** [7] \(\tilde{H}\)-cobordism is an equivalence relation.

*Proof.* The symmetry of the relation is trivial and the transitivity can be checked using the Mayer-Vietoris sequence. We verify reflexivity by showing that \(H_i(\tilde{Y}; \mathbb{Q})\) is finitely generated, where \(Y\) is an oriented homology handle and \(\tilde{Y}\) is the covering space associated with a cohomology class in \(H^1(Y; \mathbb{Z})\) dual to a generator of \(H_1(Y; \mathbb{Z})\). In [12] Proof of Assertion 5, it is shown that if \(H_1(Y; \mathbb{Q}) \cong \mathbb{Q}\), then \(H_1(\tilde{Y}; \mathbb{Q})\) is finitely generated by using the Milnor exact sequence for the cover \(\tilde{Y} \to Y\). By the partial Poincaré duality theorem, see [8, Theorem 2.3], \(H^0(\tilde{Y}; \mathbb{Q}) \cong H_2(\tilde{Y}; \mathbb{Q})\) since \(H_i(\tilde{Y}; \mathbb{Q})\) is finitely generated for \(i = 0, 1\). So, \(H_2(\tilde{Y}; \mathbb{Q}) \cong \mathbb{Q}\). □

**Lemma 2.3.** If there is an orientation-preserving diffeomorphism \(f : (Y_0, \alpha_0) \to (Y_1, \alpha_1)\) with \(f_*(\alpha_0) = \alpha_1\), then \((Y_0, \alpha_0) \sim (Y_1, \alpha_1)\).

*Proof.* Let \(W_0 = Y_0 \times [0, 1]\) and \(W_1 = Y_1 \times [0, 1]\). In the proof of Lemma 2.2, we checked that \(W_0\) and \(W_1\) are \(\tilde{H}\)-cobordisms. Let \(N_0\) and \(N_1\) be the collar neighborhoods of \(Y_0 \times 1\) and \(Y_1 \times 0\), respectively. Then \(N_0 \simeq Y_0 \times (1 - \epsilon, 1]\) and \(N_1 \simeq Y_1 \times [0, \epsilon]\). Define

\[ W = \frac{(W_0 \setminus (Y_0 \times 1)) \cup (W_1 \setminus (Y_0 \times 0))}{(x, 1 - \theta) \sim (f(x), \theta)} \]

for \(0 < \theta < \epsilon\). It is clear that \(W\) is a smooth 4-manifold with \(\partial W = -Y_0 \sqcup Y_1\). Moreover, the infinite cyclic covering \(\tilde{W}\) of \(W\) associated with the dual of \(\alpha_0\) (or \(\alpha_1\)) is the union of \(\tilde{W}_{0, \alpha_0^*}\) and \(\tilde{W}_{1, \alpha_1^*}\), where their intersection is \(\tilde{Y}_0 \times (1 - \epsilon, 1)\) (or \(\tilde{Y}_1 \times (0, \epsilon)\)). From the Mayer-Vietoris sequence, the homology groups of \(\tilde{W}\) over \(\mathbb{Q}\) are finitely generated since those of \(\tilde{W}_{0, \alpha_0^*}\) and \(\tilde{W}_{1, \alpha_1^*}\) are finitely generated, so \(W\) is an \(\tilde{H}\)-cobordism between \((Y_0, \alpha_0)\) and \((Y_1, \alpha_1)\). □

Using the above lemma, we see that \((S^1 \times S^2, \alpha), (-S^1 \times S^2, \alpha), (S^1 \times S^2, -\alpha),\) and \((-S^1 \times S^2, -\alpha)\) are all \(\tilde{H}\)-cobordant. Indeed, there are obvious orientation-preserving diffeomorphisms between them.

**Definition 2.4.** \(\Omega(S^1 \times S^2)\) is defined to be the set of all distinguished homology handles modulo the \(\tilde{H}\)-cobordism relation. We will denote elements of \(\Omega(S^1 \times S^2)\) by \([\langle Y, \alpha \rangle]\) and \([\langle S^1 \times S^2, \alpha \rangle]\) by 0.

Now, we introduce a group operation on \(\Omega(S^1 \times S^2)\). This operation is defined by round 1-handle attachment along curves representing the specified generators of \(H_1\).
In more detail, let \((Y_0, \alpha_0)\) and \((Y_1, \alpha_1)\) be distinguished homology handles. For each \(i = 0, 1\), choose a smoothly embedded simple closed oriented curve \(\gamma_i\) in \(Y_i\) such that \([\gamma_i] = \alpha_i\) in \(H_1(Y_i; \mathbb{Z})\). Then there exists a closed connected orientable surface \(F_i\) in \(Y_i\) which intersects \(\gamma_i\) in a single point. Let \(\nu(\gamma_i)\) be a tubular neighborhood of \(\gamma_i\). Then \(\nu(\gamma_i)\) is diffeomorphic to \(S^1 \times B^2\). Choose smooth embeddings
\[
\begin{align*}
h_0 & : S^1 \times B^2 \times 0 \to Y_0, \\
h_1 & : S^1 \times B^2 \times 1 \to Y_1
\end{align*}
\]
for \(\nu(\gamma_i)\) such that
1. there exist points \(s \in S^1\) and \(b \in \text{Int}(B^2)\) with \(h_i(s \times B^2 \times i) \subset F_i\) and \(h_i(S^1 \times b \times i) = \gamma_i\),
2. \(h_i\) is orientation reversing with respect to the orientation of \(S^1 \times B^2 \times i\) induced from an orientation of \(S^1 \times B^2 \times [0, 1]\).

Let \(Y_i' = Y_i \setminus \text{Int}(\nu(\gamma_i))\) and \(h_i' = h|_{S^1 \times \partial B^2 \times i}\). Now, define
\[
Y_0 \circ Y_1 := Y_0' \cup_{h_0'} (S^1 \times \partial B^2 \times [0, 1]) \cup_{h_1'} Y_1'.
\]
We claim \(Y_0 \circ Y_1\) is an oriented homology handle. To see this, let \(b' \in \partial B^2\) and \(\gamma'_i = h_i(S^1 \times b' \times i) \subset Y_i\). We give an orientation to \(\gamma'_i\) so that \([\gamma'_i] = [\gamma_i]\) in \(H_1(Y_i; \mathbb{Z})\). Let \(\mu_i = h_i(s \times \partial B^2 \times i) \subset Y_i\). We can check that \([\gamma'_i]\) is a generator of \(H_1(Y_i'; \mathbb{Z}) \cong \mathbb{Z}\) and \([\mu_i]\) is null in \(H_1(Y_i'; \mathbb{Z})\) since \(\mu_i\) bounds an orientable surface \(F_i \setminus h_i(s \times \text{Int}(B^2) \times i)\) in \(Y_i'\). From the Mayer-Vietoris sequence, we conclude that \(H_1(Y_0 \circ Y_1; \mathbb{Z}) \cong \mathbb{Z}\). Since \(Y_0 \circ Y_1\) is orientable, its homology groups are isomorphic to those of \(S^1 \times S^2\) by Poincaré duality.

From the above construction, we give an orientation to \(Y_0 \circ Y_1\) induced by the orientation of \(Y_0\) and \(Y_1\) and the generator \(\alpha\) of \(H_1(Y_0 \circ Y_1; \mathbb{Z})\) can be specified by the homology class of \(\gamma'_0\) or \(\gamma'_1\), which are homologous in \(Y_0 \circ Y_1\).

**Definition 2.5.** For two distinguished homology handles \((Y_0, \alpha_0)\) and \((Y_1, \alpha_1)\), we define \((Y_0, \alpha_0) \circ (Y_1, \alpha_1)\) to be the distinguished homology handle \((Y_0 \circ Y_1, \alpha)\) constructed as above and call it a **circle union** of \((Y_0, \alpha_0)\) and \((Y_1, \alpha_1)\).

The circle union operation satisfies the following properties.

**Proposition 2.6.**
1. \((Y_0, \alpha_0) \circ (Y_1, \alpha_1) \sim (Y_0, \alpha_0) \circ' (Y_1, \alpha_1)\), where \(\circ, \circ'\) are circle unions with different choices of \(\gamma\)’s and \(h\)’s used above, i.e., \([[(Y_0, \alpha_0) \circ (Y_1, \alpha_1)]\]
2. is well-defined.

2. \((Y_0, \alpha_0) \sim (Y_1, \alpha_1)\) if and only if \((Y_0, \alpha_0) \circ (-Y_1, \alpha_1)\) is null \(\tilde{H}\)-cobordant.
3. If \((Y_0, \alpha_0)\) and \((Y_1, \alpha_1)\) are null \(\tilde{H}\)-cobordant, then \((Y_0, \alpha_0) \circ (Y_1, \alpha_1)\) is null \(\tilde{H}\)-cobordant.

Proposition 2.6 leads to the following theorem.
Lemma 2.8. \[7\] Lemma 2.4] The map \(\omega\) from the set of knots to the set of distinguished homology handles induces a homomorphism from the knot concordance group \(\mathcal{C}\) to \(\Omega(S^1 \times S^2)\), i.e.,

\[
(S^3_0(K_1 \# K_2), [m_0]) = (S^3_0(K_1), [m_0]) \circ (S^3_0(K_2), [m_1]),
\]

where \(m_i\) is the meridian of a knot \(K_i\) for each \(i = 0, 1\).

**Proof.** Let \(K_1\) and \(K_2\) be knots in \(S^3\). Then the exterior \(X(K_1 \# K_2)\) of the connected sum of \(K_1\) and \(K_2\) is the quotient space of the exteriors \(X(K_1)\) and \(X(K_2)\) of \(K_1\) and \(K_2\), respectively, formed by identifying annular neighborhoods of their meridians. So, \(S^3_0(K_1 \# K_2) = S^3_0(K_1) \circ S^3_0(K_2)\). Hence, it is sufficient to show that if \(K\) is a slice knot, then \((S^3_0(K), [m])\) is null \(\hat{H}\)-cobordant. Let \(B^4\) be a 4-ball with \(K\) in \(S^3 = \partial B^4\). Since \(K\) is slice, there is a smoothly embedded disk \(D^2\) in \(B^4\) such that \(\partial D^2 = K \subset \partial B^4\). Let \(W = B^4 \setminus \text{Int}(\nu(D^2))\), where \(\nu(D^2)\) is a closed tubular neighborhood of \(D^2\) in \(B^4\). Then \(W\) has the homology of a circle by Alexander duality. Moreover, \(\partial W = S^3_0(K)\). Since the map \(i_* : H_1(\partial W; \mathbb{Z}) \to H_1(W; \mathbb{Z})\) induced by inclusion is an isomorphism, we can choose a generator \(i_*([m])\) of \(H_1(W; \mathbb{Z})\), where \(m\) is a meridian of \(K\) with linking number +1 with \(K\). By \[12\] Assertion 5], the infinite cyclic covering \(\hat{W}\) of \(W\) associated with the dual of \(i_*([m])\) has finitely generated homology over \(\mathbb{Q}\) since the homology of \(W\) is of \(S^1\). Thus, \((S^3_0(K), [m])\) is null \(\hat{H}\)-cobordant. \(\square\)

2.2. Knot concordance invariants from knot Floer homology. We now briefly discuss the knot concordance invariants \(\Upsilon, \tau\) and \(\{\nu_i | i \in \mathbb{Z}\}\) without giving the definitions in detail. These are all derived from knot Floer homology. For introductions and details, see \[11, 2, 5, 6, 9, 10, 11, 12, 13, 15, 17\].

In \[13\], Ozsváth and Szabó defined the tau invariant \(\tau\), which is a group homomorphism from \(\mathcal{C}\) to \(\mathbb{Z}\), i.e., \(\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)\) and \(\tau(K) = 0\) for any slice knot \(K\).

Theorem 2.7. \[7\] Theorem 1.9] The set \(\Omega(S^1 \times S^2)\) is an abelian group under the sum \([([Y_0, \alpha_0]) + ([Y_1, \alpha_1])] = ([Y_0, \alpha_0]) \circ ([Y_1, \alpha_1])\), with identity \(0 = ([S^1 \times S^2, \alpha])\). The inverse \(-[([Y, \alpha])]\) of \([([Y, \alpha])]\) is \([([-Y, \alpha])]\).

Next, we define the zero-surgery homomorphism \(\omega\) from the knot concordance group \(\mathcal{C}\) to \(\Omega(S^1 \times S^2)\).

For any oriented knot \(K \subset S^3\), let \(S^3_0(K)\) be the closed 3-manifold obtained from 0-surgery along a knot \(K\). It is easily checked that \(S^3_0(K)\) is an oriented homology handle, i.e., the homology groups of \(S^3_0(K)\) are isomorphic to those of \(S^1 \times S^2\). We give an orientation to the meridian \(m\) so that the linking number with \(K\) is +1. Then the homology class \([m]\) represents a generator of \(H_1(S^3_0(K); \mathbb{Z})\). We define \(\omega(K)\) to be the distinguished homology handle \((S^3_0(K), [m])\). Sometimes, we write \(S^3_0(K)\) for \(\omega(K) = (S^3_0(K), [m])\) as the generator \([m]\) is well-understood.
Theorem 2.9. Let $P$ be the Mazur pattern shown in Figure 1. If $\tau(K) > 0$, then $\tau(P(K)) = \tau(K) + 1$.

Remark 2.10. Theorem 2.9 shows that for any knot $K$ with $\tau(K) > 0$, $K$ is not concordant to $P(K)$. If one merely wants to find examples of knots for which $\tau(P(K)) = \tau(K) + 1$, one can appeal to the slice-Bennequin inequality satisfied by $\tau$. For details, see [16] Theorem 3.1 and Corollary 3.2.

The definition of a satellite operation $P(K)$ will be given in Section 3.

In [14], Ozsváth, Stipsicz and Szabó introduced the $Upsilon$ invariant $\Upsilon$. This is a homomorphism $\Upsilon : \mathcal{C} \to PL([0, 2], \mathbb{R})$, $K \mapsto \Upsilon(K)(t)$, where $PL([0, 2], \mathbb{R})$ is the group of piecewise-linear functions on $[0, 2]$.

Theorem 2.11. [14] The invariants $\Upsilon_K(t)$ bound the slice genus of $K$, i.e., for $0 \leq t \leq 1$, $|\Upsilon_K(t)| \leq t g_s(K)$.

Theorem 2.12. [14] (c.f. [10]) The invariant $\Upsilon$ has the following properties:

1. $\Upsilon_K(2-t) = \Upsilon_K(t)$.
2. $\Upsilon_K(0) = 0$.
3. $\Upsilon'_K(0) = -\tau(K)$.
4. $\Upsilon_{K_1 \# K_2}(t) = \Upsilon_{K_1}(t) + \Upsilon_{K_2}(t)$.
5. $\Upsilon_{-K}(t) = -\Upsilon_K(t)$.
6. There are only finitely many singularities of $\Upsilon_K(t)$.
7. The derivative of $\Upsilon_K(t)$, where it exists, is an integer.

Note that Theorem 2.11 and Theorem 2.12 imply $\Upsilon : \mathcal{C} \to PL([0, 2], \mathbb{R})$ is a homomorphism.

In [17], Rasmussen introduced the local $h$-invariants, denoted $\{V_i | i \in \mathbb{Z}\}$ in [13], which are a family of integer-valued knot concordance invariants.

Theorem 2.13. [1] Let $K_1, K_2$ be two knots in $S^3$. Then for any non-negative integers $i_1, i_2$,

$$V_{i_1+i_2}(K_1 \# K_2) \leq V_{i_1}(K_1) + V_{i_2}(K_2).$$

The following results from Chen’s thesis [2] will be very useful.

Theorem 2.14. [2] For any knot $K$, $-2V_0(K) \leq \Upsilon_K(t) \leq 2V_0(-K)$.

Proposition 2.15. [2] Let $\{K_n | n \in \mathbb{Z}^+\}$ be a family of knots such that

$$\lim_{n \to \infty} \frac{\tau(K_n)}{V_0(K_n)} = \infty,$$

then there exists a subset of $\{K_n | n \in \mathbb{Z}^+\}$ which generates a $\mathbb{Z}^\infty$-subgroup in $\mathcal{C}$. 

Let $K$ be a knot in $S^3$. Let $P$ be a knot in a solid torus $S^1 \times D^2$. Let $p : S^1 \times D^2 \to S^3$ be an embedding which identifies a regular neighborhood of a knot $K$ with $S^1 \times D^2$ so that $p(S^1 \times pt)$ is the Seifert framing of $K$. Then the knot $P(K)$ is defined to be the image of $P$ in $S^1 \times D^2$ under the map $p$. $P_n(K)$ is defined to be $P(P_{n-1}(K))$ and $P_0(K) = K$. $P(K)$ is called a satellite knot with pattern $P$ and companion $K$. See Figure 1. The winding number of $P$ is the algebraic intersection number of $P$ with a meridian disk of the solid torus.

Theorem 3.1. Suppose that $P$ is a pattern knot with winding number $\pm 1$, and that $P(U)$ is a trivial knot in $S^3$, where $U$ is an unknot. Then for any knot $K$, $S^3_0(K) \simeq S^3_0(P(K))$.

Proof. We will construct an $\tilde{H}$-cobordism $W$ between $S^3_0(K)$ and $S^3_0(P(K))$ to show that $S^3_0(K) \sim S^3_0(P(K))$. Let $X^1$ be the 4-manifold by attaching a 1-handle to the outgoing boundary of the 4-manifold $X = S^3_0(K) \times [0,1]$. This boundary is depicted in Figure 2(a), where we replace the dotted circle $P(U)$ typically used to denote a 1-handle with a zero-framed curve since the resulting boundaries are diffeomorphic. Now let $W$ be the 4-manifold obtained by attaching a 0-framed 2-handle to $\partial^+ X^1$ along the red circle shown Figure 2(b). Because $P(U)$ is an unknotted knot, using an isotopy from $P(U)$ to a trivial unknotted, we have the following Figure 2(c). See Figure 3 for schematic pictures of $X$, $X^1$ and $W$. By handle slides, one can show that $\partial^+ W \simeq S^3_0(P(K))$, see [3, Theorem 2.1].

We now show that this cobordism is an $\tilde{H}$-cobordism, i.e., there is a cohomology class $\varphi$ for which the infinite cyclic covering $\tilde{W}_\varphi$ of $W$ has finitely generated rational homology groups. Let $\pi_1(X) = \langle x_1, x_2, \ldots, x_k | r_1, r_2, \ldots, r_l, \lambda \rangle$ be the fundamental group of $X$, where $\langle x_1, x_2, \ldots, x_k | r_1, r_2, \ldots, r_l \rangle$ is the knot group of $K$ and $\lambda$ is a relator coming from the disk bounded by a 0-framed longitude. The abelianization map $A_X$ from $\pi_1(X)$ to $H_1(X; \mathbb{Z})$ provides a generator $[x_1] = [x_2] = \cdots = [x_k]$ of $H_1(X; \mathbb{Z}) \cong \mathbb{Z}$, which corresponds to the meridian of the knot $K$. 

Figure 1. Satellite operation

3. $\tilde{H}$-cobordism and satellite knots
Corresponding to $\ker(A_X)$, we have the infinite cyclic covering $\tilde{X}$ of $X$ associated with $A_X$, and $\pi_1(\tilde{X}) \cong \ker(A_X)$. Indeed, $\ker(A_X)$ is the commutator subgroup $[\pi_1(X), \pi_1(X)]$ of $\pi_1(X)$ and the covering $\tilde{X}$ is the universal abelian covering space. Attaching the 1-handle to the boundary $\partial^+X$ adds one extra generator $b$ to the presentation of $\pi_1$, so $\pi_1(X^1) = \langle x_1, x_2, \ldots, x_k, b | r_1, r_2, \ldots, r_l, \lambda \rangle$, where we orient $b$ to be compatible with winding number of the pattern $P$. Let $A_{X^1}$ be the abelianization map from $\pi_1(X^1)$ to $H_1(X^1; \mathbb{Z}) \cong \mathbb{Z}[\langle x_1 \rangle = \langle x_2 \rangle = \cdots = \langle x_k \rangle] \oplus \mathbb{Z}[\langle b \rangle]$. When we attach the 0-framed 2-handle to $\partial^+X^1$ to get $W$, the attaching region is homologous to $[x_1] + [b]$. Thus, the homomorphism $\varphi_1 : H_1(X^1; \mathbb{Z}) \to \mathbb{Z}$ defined by $[x_i] + [b] \mapsto 0$ and $[x_i] \mapsto 1$ can be considered as a map from $H_1(X^1; \mathbb{Z})$ to $H_1(W; \mathbb{Z}) \cong \mathbb{Z}[\langle x_1 \rangle = \langle x_2 \rangle = \cdots = \langle x_k \rangle = -[b]]$. Let

$$
\psi_1 := \varphi_1 \circ A_{X^1} : \pi_1(X^1) \to H_1(X^1; \mathbb{Z}) \to H_1(W; \mathbb{Z}) \cong \mathbb{Z}.
$$

Note that the attaching region of the 1-handle in $\partial^+X$ is a disjoint union of two 3-balls, which are simply-connected. So, the attaching region can be lifted to
Thus, the infinite cyclic covering $\tilde{X}^1$ of $X^1$ associated with $\varphi_1$ is obtained by attaching infinitely many 1-handles to the infinite cyclic covering $\tilde{X}$ of $X$. See Figure 4.

It follows that

$$H_i(\tilde{X}^1; \mathbb{Q}) = \begin{cases} H_i(\tilde{X}; \mathbb{Q}), & i \neq 1 \\ H_i(\tilde{X}; \mathbb{Q}) \oplus \mathbb{Q}[t, t^{-1}], & i = 1 \end{cases},$$

where $t$ is a generator of the deck transformation group of the covering spaces.

The attaching region of the 2-handle in $\partial^+ X^1$ is homotopic to $x_1 b$ and is contained in $\ker(\psi_1)$, which is the image of $\pi_1(\tilde{X}^1)$ under the covering map. Hence, the attaching region of the 2-handle can be lifted to the covering $\tilde{X}^1$. Attaching the 2-handle to $\partial^+ X^1$ adds a relator $x_1 b$ to the presentation of $\pi_1$, so we have

$$\pi_1(W) = \langle x_1, x_2, \ldots, x_k, b | r_1, r_2, \ldots, r_l, x_1 b \rangle$$

and

$$H_1(W) = \mathbb{Z} / \langle [x_1] = [x_2] = \cdots = [x_k] \rangle \oplus \mathbb{Z} / \langle [b] \rangle$$

$$= \mathbb{Z} / \langle [x_1] = [x_2] = \cdots = [x_k] = [-b] \rangle \cong \mathbb{Z}.$$

Let $\varphi$ be the dual cohomology class of $[x_1] = [x_2] = \cdots = [x_k] = [-b]$ in $H_1(W; \mathbb{Z})$. It is clear that $\varphi|_{\partial^\pm W}$ is dual to the generator $[x_i]$ of $H_1(\partial^\pm W; \mathbb{Z})$.

The infinite cyclic covering $\tilde{W}_\varphi$ of $W$ associated with $\varphi$ is obtained from $\tilde{X}^1$ by attaching infinitely many 0-framed 2-handles along curves homotopic to elements $t^n x_1 b$, $n \in \mathbb{Z}$, in $\pi_1(\tilde{X}^1)$. Thus, $H_i(\tilde{W}_\varphi; \mathbb{Q}) = H_i(\tilde{X}; \mathbb{Q})$, and they are all finitely generated over $\mathbb{Q}$. \qed

Remark 3.2. In [3], the cobordism $W$ constructed in the proof of Theorem 3.1 is used to show that $S^3_0(K)$ and $S^3_0(P(K))$ are $\mathbb{Z}$-homology cobordant rel meridians.
under the same assumption as Theorem 3.1. The latter means that the positively-oriented meridians of knots are homologous in \( H_1(W; \mathbb{Z}) \). So, the cobordism \( W \) is a non-trivial cobordism which is simultaneously a \( \mathbb{Z} \)-homology and \( \mathcal{H} \)-cobordism. In fact, we can easily find \( \mathcal{H} \)-cobordisms which are not \( \mathbb{Z} \)-homology cobordisms. But, we do not know whether every \( \mathbb{Z} \)-homology cobordism is an \( \mathcal{H} \)-cobordism.

4. \( \mathbb{Z}^\infty \)-SUBGROUP IN \( \ker(\omega) \)

In this section, using the properties of the knot concordance invariants reviewed in Section 2.2 and in conjunction with Theorem 3.1, we establish our theorem on the kernel of the zero-surgery homomorphism \( \omega \).

**Theorem 4.1.** The zero-surgery homomorphism \( \omega : \mathcal{C} \to \Omega(S^1 \times S^2) \) is not injective. Indeed, there is a \( \mathbb{Z}^\infty \)-subgroup in \( \ker(\omega) \).

**Proof.** Let \( P \) be the Mazur pattern shown in Figure 1. Let \( T_{2,3} \) be a \( (2,3) \)-torus knot. Note that \( V_0(T_{2,3}) = 1 \), \( V_0(-T_{2,3}) = 0 \), and \( \tau(T_{2,3}) = 1 \). Moreover, \( \Upsilon_{T_{2,3}}(t) = -t \) for \( t \in [0,1] \). Let \( K_n = P^n(T_{2,3})^{#} - T_{2,3} \).

We first claim that the family \( \{K_n| n \in \mathbb{Z}^+\} \) is mapped to 0 in \( \Omega(S^1 \times S^2) \) by \( \omega \), and that \( K_n \) is a not slice knot for each \( n \). By Theorem 3.1, \( S^3_0(T_{2,3}) \simeq S^3_0(P^n(T_{2,3})) \). Then

\[
0 = [S^3_0(P^n(T_{2,3})) \circ \omega(S^3_0(T_{2,3}))]
\]

Also, \( \tau(K_n) = \tau(P^n(T_{2,3})^{#} - T_{2,3}) = \tau(P^n(T_{2,3})) - \tau(T_{2,3}) = n + 1 - 1 = n \) by Theorem 2.9. This proves the claim, showing that the zero-surgery homomorphism \( \omega \) is not injective.

Next, we will show that there is a subset of \( \{K_n| n \in \mathbb{Z}^+\} \) which generates a \( \mathbb{Z}^\infty \)-subgroup in \( \ker(\omega) \). By Theorem 2.13

\[
V_0(K_n) = V_0(P^n(T_{2,3})^{#} - T_{2,3}) \leq V_0(P^n(T_{2,3})) + V_0(-T_{2,3}).
\]

By Remark 3.2, \( S^3_0(P^n(T_{2,3})) \) and \( S^3_0(T_{2,3}) \) are \( \mathbb{Z} \)-homology cobordant. Note that \( V_0 \) is an invariant of the \( \mathbb{Z} \)-homology cobordism class of the zero-surgery, see Theorem 3.1, i.e., if two knots have \( \mathbb{Z} \)-homology cobordant 0-surgeries, then they have the same \( V_0 \). So, \( V_0(P^n(T_{2,3})) = V_0(T_{2,3}) = 1 \), and hence \( V_0(K_n) \leq 1 \). By Theorem 2.14

\[
-2V_0(K_n) \leq \Upsilon_{K_n}(t) = \Upsilon_{P^n(T_{2,3})}(t) - \Upsilon_{T_{2,3}}(t).
\]

On \([0, \delta_n)\), with sufficiently small \( \delta_n \) having no singularity of \( \Upsilon_{P^n(T_{2,3})}(t) \),

\[
\Upsilon_{P^n(T_{2,3})}(t) - \Upsilon_{T_{2,3}}(t) = -(n + 1)t + t = -nt,
\]

since \( \Upsilon'(0) = -\tau \) by Theorem 2.12(3). This implies \( -2V_0(K_n) \leq -nt \), and hence \( V_0(K_n) > 0 \). So, \( 0 < V_0(K_n) \leq 1 \). Then \( \lim_{n \to \infty} \frac{\tau(K_n)}{V_0(K_n)} = \infty \) because \( \tau(K_n) = n \).
By Proposition 2.15, there exists a subset of \( \{ K_n | n \in \mathbb{Z}^+ \} \) which generates a \( \mathbb{Z}^\infty \)-subgroup in \( \ker(\epsilon) \).

□

Remark 4.2. In [18], Yasui showed that there exists a pair of non-concordant knots in \( S^3 \) with the same 0-surgery, and that there exist infinitely many distinct pairs of such knots. His result, therefore, can alternatively establish that the zero-surgery homomorphism \( \omega \) is not injective.

References

[1] J. Bodnár, D. Celoria, M. Golla, A note on cobordisms of algebraic knots, Algebr. Geom. Topol. 17 (2017), no. 4, 2543–2564.
[2] W. Chen, Some inequalities of Heegaard Floer concordance invariants of satellite knots, Michigan State University, ProQuest Dissertations Publishing, 2019. 13878056.
[3] T. D. Cochran, B. D. Franklin, M. Hedden and P. D. Horn, Knot concordance and homology cobordism, Proc. Amer. Math. Soc. 141 (2013), no. 6, 2193–2208.
[4] M. Hedden, M. H. Kim, T. Mark, K. Park, Irreducible 3-manifolds that cannot be obtained by 0-surgery on a knot, Trans. Amer. Math. Soc. 372 (2019), no. 11, 7619–7638.
[5] M. Hedden, L. Watson, On the geography and botany of knot Floer homology, Selecta Math. (N.S.) 24 (2018), no. 2, 997–1037.
[6] J. Hom, A survey on Heegaard Floer homology and concordance, J. Knot Theory Ramifications 26 (2017), no. 2, 1740015, 24 pp.
[7] A. Kawauchi, \( \tilde{H} \)-cobordism, I; The groups among three dimensional homology handles, Osaka Math. J. 13 (1976), no. 3, 567–590.
[8] A. Kawauchi, A partial Poincaré duality theorem for infinite cyclic coverings, Quart. J. Math. Oxford Ser. (2) 26 (1975), no. 104, 437–458.
[9] A. S. Levine, Nonsurjective satellite operators and piecewise-linear concordance, Forum Math. Sigma 4 (2016), e34, 47 pp.
[10] C. Livingston, Notes on the knot concordance invariant upsilon, Algebr. Geom. Topol. 17 (2017), no. 1, 111–130.
[11] C. Manolescu, An introduction to knot Floer homology. Physics and mathematics of link homology, 99–135, Contemp. Math., 680, Centre Rech. Math. Proc., Amer. Math. Soc., Providence, RI, 2016.
[12] J. Milnor, Infinite cyclic coverings, 1968 Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967) pp. 115–133 Prindle, Weber & Schmidt, Boston, Mass.
[13] Y. Ni, Z. Wu, Cosmetic surgeries on knots in \( S^3 \), J. Reine Angew. Math. 706 (2015), 1–17.
[14] P. Ozsváth, A. Stipsicz and Z. Szabó, Concordance homomorphisms from knot Floer homology, Adv. Math. 315 (2017), 366–426.
[15] P. Ozsváth, Z. Szabó, Knot Floer Homology and the four-ball genus, Geom. Topol. 7 (2003), 615–639.
[16] O. Plamenevskaya, Bounds for the Thurston-Bennequin number from Floer homology, Algebr. Geom. Topol. 4 (2004), 399–406.
[17] J. A. Rasmussen, Floer homology and knot complements, Harvard University, ProQuest Dissertations Publishing, 2003. 3091665.
[18] K. Yasui, Corks, exotic 4-manifolds and knot concordance, arXiv preprint arXiv:1505.02551 (2015).

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