Integrability of the Zakharov-Shabat Systems by Quadrature

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Abstract: We consider the general two-dimensional Zakharov-Shabat systems, which appear in application of the inverse scattering transform (IST) to an important class of nonlinear partial differential equations (PDEs) called integrable systems. We study their integrability in the meaning of differential Galois theory, i.e., their solvability by quadrature. It becomes a key for obtaining analytical expressions for solutions to the PDEs by using the IST. For a wide class of potentials, we prove that they are integrable in that meaning if and only if the potentials are reflectionless. It is well known that for such potentials particular solutions called n-solitons in the original PDEs are yielded by the IST.

1. Introduction

The inverse scattering transform (IST) is a powerful tool to solve the initial value problems for an important class of nonlinear partial equations (PDEs) called integrable systems such as the Korteweg-de Vries (KdV) equation and nonlinear Schrödinger (NLS) equation [1,3–6,19,24,35]. In application of the technique, eigenvalue problems for linear systems of ordinary differential equations (ODEs) called Zakharov-Shabat (ZS) systems need to be solved. Here we are interested in the question whether their solutions can be obtained by quadrature. Such solvability of linear ODEs can be determined by differential Galois theory [13,27], which is an extension of classical Galois theory for algebraic equations to linear ODEs. A linear ODE is said to be integrable in the meaning of differential Galois theory if its all solutions can be obtained by quadrature. The differential Galois theory was also utilized to develop a useful tool called the Morales-Ramis theory [8,21,23] for determining the nonintegrability of nonlinear ODEs. Some relations between nonintegrability and chaotic dynamics in two-degree-of-freedom Hamiltonian systems were described in [22,31,34] based on the Morales-Ramis theory. Moreover, the differential Galois theory was used to discuss bifurcations of homoclinic orbits in
four-dimensional ODEs [10,32] and a Sturm-Liouville problem of second-order ODEs on the infinite interval [11].

In this paper we study the integrability of the two-dimensional ZS systems,

\[ v_x = \left( \begin{array}{c}
-ik q(x) \\
-1
\end{array} \right) v, \quad v \in \mathbb{C}^2, \quad (1.1) \]

and

\[ v_x = \left( \begin{array}{c}
-ik q(x) \\
r(x) ik
\end{array} \right) v, \quad (1.2) \]

in the meaning of differential Galois theory, i.e., their solvability by quadrature, where the subscript \( x \) represents differentiation with respect to the variable \( x \), and \( k \in \mathbb{C} \) is a constant. Here the independent variable \( x \) is originally defined in \( \mathbb{R} \) but its domain is a little extended later. Moreover, the potentials \( q(x), r(x) \) which may be complex on \( \mathbb{R} \) are assumed to satisfy the following condition:

(A) The potentials \( q(x), r(x) \) are holomorphic in a neighborhood \( U \) of \( \mathbb{R} \) in \( \mathbb{C} \). Moreover, there exist holomorphic functions \( q_\pm, r_\pm : U_0 \to \mathbb{C} \) such that \( q_\pm(0), r_\pm(0) = 0 \) and

\[ q(x) = q_\pm(e^{\mp \lambda_\pm x}), \quad r(x) = r_\pm(e^{\mp \lambda_\pm x}) \]

for \( |\text{Re} \, x| \) sufficiently large, where \( U_0 \) is a neighborhood of the origin in \( \mathbb{C} \), \( \lambda_\pm \in \mathbb{C} \) are some constants with \( \text{Re} \, \lambda_\pm > 0 \), and the upper or lower signs are taken simultaneously depending on whether \( \text{Re} \, x > 0 \) or \( \text{Re} \, x < 0 \).

For the ZS system (1.1) condition (A) has a meaning only for \( q(x) \). In particular, \( q(x), r(x) \) tend to zero exponentially as \( x \to \pm \infty \) on \( \mathbb{R} \), so that \( q, r \in L^1(\mathbb{R}) \), if they satisfy condition (A). Condition (A) is a little restrictive, but it is satisfied by several wide classes of functions. For example, if \( q(x), r(x) \) are rational functions of \( e^{\lambda x} \) for some \( \lambda \in \mathbb{C} \) with \( \text{Re} \, \lambda > 0 \), have no singularity on \( \mathbb{R} \), and \( q(x), r(x) \to 0 \) as \( x \to \pm \infty \), then condition (A) holds (see also Sect. 5). We have another class of functions satisfying condition (A) as follows.

Remark 1.1. Let \( f(\xi) \) be a second- or higher-order polynomial of \( \xi \in \mathbb{R} \) such that for some \( \xi_- < \xi_+ \) \( f(\xi_\pm) = 0, \ f(\xi) > 0 \) on \( (\xi_-, \xi_+) \) and

\[ f_\xi(\xi_+) < 0 < f_\xi(\xi_-). \]

Then there exists a heteroclinic solution \( \xi^h(x) \) to the one-dimensional ODE

\[ \xi_x = f(\xi) \quad (1.3) \]

such that \( \lim_{x \to \pm \infty} \xi^h(x) = \xi_\pm \), where the upper or lower signs are taken simultaneously. Since the complexification of (1.3) is holomorphically equivalent to the linearized ODE

\[ \xi_x = f_\xi(\xi_\pm) \xi \]

near neighborhoods of \( \xi = \xi_\pm \) in \( \mathbb{C} \) (e.g., Theorem 5.5 in Chapter I of [16]), we see that \( q(x) = \xi^h_x(x) \) satisfies condition (A) with \( \lambda_\pm = \mp f_\xi(\xi_\pm) \).
It is well known that the ZS systems (1.1) and (1.2) appear in application of the IST for the following fundamental and important nonlinear PDEs (see, e.g., [3,4] or Sect. 1.2 of [6]):

- The KdV equation
  \[ q_t + 6qq_x + q_{xxx} = 0; \]  
  \[ u_{xt} = \sin u \]  
  \[ u_{xt} = \sinh u \]

Here \( q, r \) and \( u \) are assumed to depend on the time variable \( t \) as well as \( x \), and the superscript ‘*’ represents complex conjugate. The ZS system of the form (1.1) appears only for the KdV equation (1.4), and \( q, r \) and \( u \) are also usually assumed to be real on \( \mathbb{R} \) except for the NLS equation (1.5). When the minus sign is taken in (1.5) and (1.6), the ZS system (1.2) is typically considered under nonvanishing boundary conditions, in which \( \lim_{x \to \pm\infty} q(x), r(x) = 0 \) does not also hold. See [25,36] for more details. So these two cases may have to be excluded in the following discussions. On the other hand, under the transformation \((x + t, x - t) \mapsto (x, t)\), the sine- and sinh-Gordon equations (1.7) and (1.8) are changed to

\[ u_{tt} - u_{xx} + \sin u = 0 \]  
\[ u_{tt} - u_{xx} + \sinh u = 0, \]  

respectively, in the physical coordinate system. Throughout this paper we assume the symmetric relations between \( q(x) \) and \( r(x) \) in (1.5)–(1.8) for the ZS system (1.2).

Here we prove that the ZS system (1.1) (resp. (1.2)) is integrable in the meaning of differential Galois theory if and only if the potential \( q(x) \) is (resp. the potentials \( q(x), r(x) \) are) reflectionless. See Sect. 2 for the precise statement of the result along with the definition of reflectionless potentials. As stated above, the integrability of (1.1) and (1.2) in that meaning implies that they are solved by quadrature. See Sect. 3 for its more precise definition. It is also well known for the above five examples that when the potentials are reflectionless, the ZS systems (1.1) and (1.2) are solved by quadrature and particular solutions called \( n \)-solitons in the original PDEs are yielded from the potentials by the IST [1,5] (see also Sect. 4). This property is guaranteed for more general integrable PDEs by the recent result of Jiménez et al. [18], who considered a more general case including (1.1) and (1.2) and proved that the transformed, differential Galois group (see
or Sect. 3) are isomorphic to subgroups of the original one under the Darboux transformations, which produce sequences of \( n \)-solitons from the trivial solution \( u = 0 \) (see, e.g., Sect. 1.3 of [15]). Our result means that the ZS systems (1.1) and (1.2) are solved by quadrature only in such a case under condition (A). The ZS system (1.1) is transformed to a linear Schrödinger equation, as stated in Sect. 2. Its integrability in the meaning of differential Galois theory was also discussed in [7] for several classes of potentials which do not necessarily satisfy condition (A) and in [11] for a special potential which satisfies condition (A).

The outline of this paper is as follows: In Sect. 2 we state our main results along with necessary terminologies and setting, and give two examples to illustrate the results. We provide necessary information on differential Galois theory in Sect. 3 and on the IST theory for reflectionless potentials in Sect. 4. We also need some relations on scattering and reflection coefficients between the ZS system (1.1) and the corresponding linear Schrödinger equation, which are given in Appendix A. We prove the main theorems in Sects. 5 and 6.

2. Main Results

In this section we give our main results. Following the standard theory of the IST (e.g., [1,5]) with slight modifications, we first define some necessary terminologies for their statements.

Assume that \( k \neq 0 \). Taking \( x \to \pm \infty \) in (1.1) and (1.2), we have

\[
v_x = \begin{pmatrix} -ik & 0 \\ r_0 & ik \end{pmatrix} v, \tag{2.1}
\]

where \( r_0 = -1 \) in (1.1) and \( r_0 = 0 \) in (1.2). Equation (2.1) has

\[
\Phi(x; k) = T \begin{pmatrix} e^{-ikx} & 0 \\ 0 & e^{ikx} \end{pmatrix} T^{-1} \tag{2.2}
\]

as a fundamental matrix such that \( \Phi(0) = \text{id}_2 \), where \( \text{id}_2 \) denotes the \( 2 \times 2 \) identity matrix and

\[
T = \begin{pmatrix} 1 & 0 \\ ir_0 & 2k \end{pmatrix}. \tag{2.3}
\]

Let \( v = \phi(x; k), \tilde{\phi}(x; k), \psi(x; k), \tilde{\psi}(x; k) \) be solutions to (1.1) or (1.2) such that

\[
\phi(x; k) \sim T \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \quad \tilde{\phi}(x; k) \sim T \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} \quad \text{as} \quad x \to -\infty, \tag{2.4}
\]

\[
\psi(x; k) \sim T \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \quad \tilde{\psi}(x; k) \sim T \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \quad \text{as} \quad x \to +\infty.
\]

These solutions are called the Jost solutions and their existence for \( k \in \mathbb{C} \setminus \{0\} \) is guaranteed by a standard result on asymptotic behavior of linear ODEs (e.g., Theorem 8.1 in Sect. 3.8 of [12]). See also Sect. 6. Since \( v = \psi(x; k), \tilde{\psi}(x; k) \) are linearly independent solutions to (1.1) or (1.2), there exist constants \( a(k), \tilde{a}(k), b(k), \tilde{b}(k) \) such that

\[
\phi(x; k) = b(k) \psi(x; k) + a(k) \tilde{\psi}(x; k), \quad \tilde{\phi}(x; k) = \tilde{a}(k) \psi(x; k) + \tilde{b}(k) \tilde{\psi}(x; k). \tag{2.5}
\]
We refer to the constants \( a(k), \bar{a}(k), b(k), \bar{b}(k) \) as scattering coefficients. When \( a(k), \bar{a}(k) \neq 0 \), the constants

\[
\rho(k) = b(k)/a(k), \quad \bar{\rho}(k) = \bar{b}(k)/\bar{a}(k)
\]

are defined and called the reflection coefficients for (1.1) and (1.2). If \( \rho(k), \bar{\rho}(k) = 0 \) for any \( k \in \mathbb{R} \setminus \{0\} \), then \( q(x), r(x) \) are called reflectionless potentials.

In the standard IST for the KdV equation (1.4), the linear Schrödinger equation

\[
w_{xx} + (k^2 + q)w = 0 \quad (2.6)
\]

is used instead of (1.1) and the scattering and reflection coefficients are defined for (2.6). See Appendix A for the relations on these coefficients between (1.1) and (2.6). Note that the first and second component of \( v \) in (1.1) are given by

\[
v_1 = -w_x + ikw, \quad v_2 = w. \quad (2.7)
\]

Actually, it follows from (2.6) and (2.7) that

\[
v_{2x} = w_x = -v_1 + ikv_2
\]

and

\[
v_{1x} = -w_{xx} + ikw_x = (k^2 + q)w + ikw_x = (k^2 + q)v_2 + ik(-v_1 + ikv_2) = -ikv_1 + qv_2.
\]

We now state the first of our main results.

**Theorem 2.1.** Suppose that \( q(x) \) is (resp. \( q(x), r(x) \) are) reflectionless and satisfies (resp. satisfy) condition (A). Then \( q(x) \) is a rational function (resp. \( q(x), r(x) \) are rational functions) of \( e^{\lambda x} \), where \( \lambda \in \mathbb{C} \) is some constant with \( \text{Re}\lambda > 0 \). Moreover, the ZS system (1.1) (resp. (1.2)), which is regarded as a linear system of differential equations over \( \mathbb{C}(e^{\lambda x}) \), is integrable in the meaning of differential Galois theory, i.e., it is solved by quadrature, for any \( k \in \mathbb{C} \setminus \{0\} \).

Theorem 2.1 is proved in Sect. 5.

**Remark 2.2.** For (1.1) we can take \( \lambda > 0 \) in Theorem 2.1 if \( q(x) \) is real on \( \mathbb{R} \). See Remark 5.4(ii).

Suppose that \( q(x), r(x) \) satisfy condition (A) such that \( q_{\pm}(s) = O(s^{\ell_{\pm}}) \) but \( q_{\pm}(s) \neq o(s^{\ell_{\pm}}) \) for some \( \ell_{\pm} \in \mathbb{N} \). Let \( \xi = p(x) \) be the solution to

\[
\xi_x = -(\xi + 1)(\xi - 1)(\xi - \alpha_+)(\xi - \alpha_-) \quad (2.8)
\]

satisfying \( \lim_{x \to \pm \infty} \xi(x) = \pm 1 \), where \( \alpha = \alpha_\pm \) are roots of

\[
\alpha^2 + \frac{1}{4}m(\ell_+\lambda_+ - \ell_-\lambda_-)\alpha + \frac{1}{4}m(\ell_+\lambda_+ + \ell_-\lambda_-) = 1
\]

with an integer \( m > 1 \). As in Remark 1.1, we see that there exist holomorphic functions \( p_{\pm}(s) : U_0 \to \mathbb{C} \) such that \( p(x) = p_{\pm}(e^{\mp m \ell_{\pm} x}) \) for \( |\text{Re}x| \) sufficiently large, where \( U_0 \) is a neighborhood of the origin in \( \mathbb{C} \) and the upper or lower signs are taken simultaneously depending on whether \( \text{Re}x > 0 \) or \( \text{Re}x < 0 \). Note that the derivative of the right hand side of (2.8) with respect to \( \xi \) is \( \mp m \ell_{\pm} \lambda_{\pm} \) at \( \xi = \pm 1 \) and \( \alpha_\pm \not\in [-1, 1] \) if \( m \) is appropriately chosen even when \( \alpha_\pm \in \mathbb{R} \). Let \( \hat{\Gamma}_R = \{ \hat{q}(x) = (q(x), p(x)) \mid x \in \mathbb{R} \} \).
By definition the curve $\hat{\Gamma}_R$ is analytic. Let $U_\pm$ be neighborhoods of $O_\pm = (0, \pm 1)$ in $\mathbb{C}^2$ and let $V_\pm \subset U_\pm$ be one-dimensional analytic complex manifolds with boundaries such that $V_\pm \supseteq U_\pm \cap \hat{q}(U_0)$ and $O_\pm \in V_\pm \setminus \partial V_\pm$. Let $R > 0$ be sufficiently large and let $U_R \subset U$ be a neighborhood of the open interval $(-R, R)$ in $\mathbb{C}$ such that $\hat{q}(U_R)$ does not contain $O_\pm$ but intersects $V_\pm$. Thus, we define a Riemann surface $\hat{\Gamma}_R$ that consists of $V_\pm$ and $\hat{q}(U_R)$. See Fig. 1.

We use $s_\pm = e^{\mp \lambda_\pm x}$ as the coordinates in $V_\pm$, and the original complex variable $x \in U$ as the coordinate in $\hat{q}(U_R)$. Note that $\hat{\Gamma}_R \supset \hat{\Gamma}_R$.

Let $A(x)$ be the coefficient matrix in (1.1) or (1.2), i.e.,

$$A(x) = \begin{pmatrix} -ikq(x) & -1 \\ 1 & ik \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -ikr(x) & -1 \\ 1 & ik \end{pmatrix}.$$

We express the ZS systems (1.1) and (1.2) as

$$\frac{d\eta}{dx} = A(x)\eta \quad (2.9)$$

in $\hat{q}(U_R)$, and

$$\frac{d\eta}{ds_\pm} = \mp \frac{1}{\lambda_\pm s_\pm} A_\pm(s_\pm)\eta, \quad (2.10)$$

in $V_\pm$, where

$$A_\pm(s_\pm) = \begin{pmatrix} -ikq_\pm(s_\pm) & -1 \\ 1 & ik \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -ikr_\pm(s_\pm) & -1 \\ 1 & ik \end{pmatrix}$$

for (1.1) or (1.2). Note that $s_\pm = 0$ at $O_\pm$ and $d/dx = \mp \lambda_\pm s_\pm d/ds_\pm$ in $V_\pm$. Thus, we can regard them as a linear system of differential equations on the Riemann surface $\hat{\Gamma}$.

**Theorem 2.3.** Suppose that $q(x)$ satisfies (resp. $q(x)$, $r(x)$ satisfy) condition (A). If the ZS system (1.1) (resp. (1.2)) is integrable in the meaning of differential Galois theory for all $k \in \mathbb{R} \setminus \{0\}$ when it is regarded as a linear system of differential equations on the Riemann surface $\hat{\Gamma}$, then $q(x)$ is (resp. $q(x)$, $r(x)$ are) reflectionless.

Theorem 2.3 is proved in Sect. 6. This theorem means that if the ZS systems (1.1) and (1.2) are integrable in the meaning of differential Galois theory, then the potentials are reflectionless. We also remark that the ZS systems (1.1) and (1.2) may be integrable on
If the ZS system (1.1) or (1.2) over $\mathbb{C}$ we immediately obtain the following result as a corollary of Theorem 2.3.

**Corollary 2.4.** Suppose that $q(x)$ is a rational function (resp. $q(x)$, $r(x)$ are rational functions) of $e^{\lambda x}$ for some $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > 0$, has (resp. have) no singularity on $\mathbb{R}$, and $q(x) \to 0$ (resp. $q(x)$, $r(x) \to 0$) as $x \to \pm \infty$. If the ZS system (1.1) (resp. (1.2)) over $\mathbb{C}(e^{\lambda x})$ is integrable in the meaning of differential Galois theory for all $k \in \mathbb{R} \setminus \{0\}$, then $q(x)$ is (resp. $q(x)$, $r(x)$ are) reflectionless.

**Proof.** If the ZS system (1.1) or (1.2) over $\mathbb{C}(e^{\lambda x})$ is integrable in the meaning of differential Galois theory for $k \in \mathbb{R} \setminus \{0\}$, then so is it as a linear system of differential equations on the Riemann surface $\hat{\Gamma}$. This yields the desired result. $\square$

In closing this section, we give two examples for the ZS system (1.1). They are immediately modified as those for (1.2).

**Example 2.5.** Let $q(x) = \alpha \text{sech}^2 x$ for some $\alpha > 0$. Obviously, $q(x)$ satisfies condition (A) with $\lambda_{-} = 2$. As shown in Sect. 2.5 of [19], it is a reflectionless potential in the linear Schrödinger equation (2.6) and consequently in the ZS system (1.1) (see Appendix A, especially Eq. (A.6)) if and only if $\alpha = n(n + 1)$ for some $n \in \mathbb{N}$. Using Theorem 2.1 and Corollary 2.4, we see that the ZS system (1.1) over $\mathbb{C}(e^{2x})$ is integrable in the meaning of differential Galois theory for $k \in \mathbb{R} \setminus \{0\}$ if and only if $\alpha = n(n + 1)$ for some $n \in \mathbb{N}$.

**Example 2.6.** Let $\alpha > 1$ and let $\xi^{h}(x)$ be a heteroclinic orbit in

$$\xi_{\pm} = \xi(\xi - 1)(\xi - \alpha) \quad (2.11)$$

and connect $\xi = 0$ to $\xi = 1$. We see that $\xi = \xi^{h}(x)$ satisfies

$$\frac{\xi^{\alpha - 1}(\alpha - \xi)}{(1 - \xi)^{\alpha}} = \frac{\xi^{h}(0)^{\alpha - 1}(\alpha - \xi^{h}(0))}{(1 - \xi^{h}(0))^{\alpha}} e^{x},$$

but it is difficult to obtain its closed expression. Let $q(x) = \xi^{h}_{\pm}(x)$, as in Remark 1.1. Then $q(x)$ satisfies condition (A) with $\lambda_{-} = \alpha$ and $\lambda_{+} = \alpha - 1$. Assume that $\alpha$ and $\alpha - 1$ are rationally independent. Then it follows from Theorem 2.3 that the ZS system (1.1) is not integrable in the meaning of differential Galois theory for all $k \in \mathbb{R} \setminus \{0\}$ since $q(x)$ is not a rational function of some exponential function and it is not reflectionless by Theorem 2.1.

3. Differential Galois Theory

In this and the next sections we give some prerequisites for our result. We begin with the differential Galois theory for linear differential equations, which is often referred to as the Picard-Vessiot theory, containing monodromy groups and Fuchsian equations. See the textbooks [13,27] for more details on the theory.
3.1. Picard-Vessiot extensions. Consider a linear system of differential equations

\[ y' = Ay, \quad A \in \text{gl}(n, \mathbb{K}), \]  

where \( \mathbb{K} \) is a differential field and \( \text{gl}(n, \mathbb{K}) \) denotes the ring of \( n \times n \) matrices with entries in \( \mathbb{K} \). We recall that a differential field is a field endowed with a derivation \( \partial \), which is an additive endomorphism satisfying the Leibniz rule. By abuse of notation we write \( y' \) instead of \( \partial y \). The set \( \mathbb{C}_\mathbb{K} \) of elements of \( \mathbb{K} \) for which \( \partial \) vanishes is a subfield of \( \mathbb{K} \) and called the field of constants of \( \mathbb{K} \). In our application of the theory in this paper, the differential field \( \mathbb{K} \) is the field of meromorphic functions on a Riemann surface \( \Gamma \), so that the field of constants is \( \mathbb{C} \).

A differential field extension \( \mathbb{L} \supset \mathbb{K} \) is a field extension such that \( \mathbb{L} \) is also a differential field and the derivations on \( \mathbb{L} \) and \( \mathbb{K} \) coincide on \( \mathbb{K} \). A differential field extension \( \mathbb{L} \supset \mathbb{K} \) satisfying the following two conditions is called a Picard-Vessiot extension for (3.1):

(PV1) The field \( \mathbb{L} \) is generated by \( \mathbb{K} \) and entries of its fundamental matrix;
(PV2) The fields of constants for \( \mathbb{L} \) and \( \mathbb{K} \) coincide.

The system (3.1) admits a Picard-Vessiot extension which is unique up to isomorphism.

We give some notions on differential field extensions.

**Definition 3.1.** A differential field extension \( \mathbb{L} \supset \mathbb{K} \) is called

(i) an integral extension if there exists \( a \in \mathbb{L} \) such that \( a' \in \mathbb{K} \) and \( \mathbb{L} = \mathbb{K}(a) \), where \( \mathbb{K}(a) \) is the smallest extension of \( \mathbb{K} \) containing \( a \);

(ii) an exponential extension if there exists \( a \in \mathbb{L} \) such that \( a'/a \in \mathbb{K} \) and \( \mathbb{L} = \mathbb{K}(a) \);

(iii) an algebraic extension if there exists \( a \in \mathbb{L} \) such that it is algebraic over \( \mathbb{K} \) and \( \mathbb{L} = \mathbb{K}(a) \).

**Definition 3.2.** A differential field extension \( \mathbb{L} \supset \mathbb{K} \) is called a Liouvillian extension if it can be decomposed as a tower of extensions,

\[ \mathbb{L} = \mathbb{K}_n \supset \ldots \supset \mathbb{K}_1 \supset \mathbb{K}_0 = \mathbb{K}, \]

such that each extension \( \mathbb{K}_{j+1} \supset \mathbb{K}_j \) is either integral, exponential or algebraic.

Thus, if the Picard-Vessiot extension \( \mathbb{L} \supset \mathbb{K} \) is Liouvillian, then Eq. (3.1) is solved by quadrature.

We now fix a Picard-Vessiot extension \( \mathbb{L} \supset \mathbb{K} \) and fundamental matrix \( \Xi \) with entries in \( \mathbb{L} \) for (3.1). Let \( \sigma \) be a \( \mathbb{K} \)-automorphism of \( \mathbb{L} \), which is a field automorphism of \( \mathbb{L} \) that commutes with the derivation of \( \mathbb{L} \) and leaves \( \mathbb{K} \) pointwise fixed. Obviously, \( \sigma (\Xi) \) is also a fundamental matrix of (3.1) and consequently there is a matrix \( M_\sigma \) with constant entries such that \( \sigma (\Xi) = \Xi M_\sigma \). This relation gives a faithful representation of the group of \( \mathbb{K} \)-automorphisms of \( \mathbb{L} \) on the general linear group as

\[ R: \text{Aut}_{\mathbb{K}}(\mathbb{L}) \to \text{GL}(n, \mathbb{C}_\mathbb{L}), \quad \sigma \mapsto M_\sigma, \]

where \( \text{GL}(n, \mathbb{C}_\mathbb{L}) \) is the group of \( n \times n \) invertible matrices with entries in \( \mathbb{C}_\mathbb{L} \). The image of \( R \) is a linear algebraic subgroup of \( \text{GL}(n, \mathbb{C}_\mathbb{L}) \), which is called the differential Galois group of (3.1) and denoted by \( \text{Gal}(\mathbb{L}/\mathbb{K}) \). This representation is not unique and depends on the choice of the fundamental matrix \( \Xi \), but a different fundamental matrix only gives rise to a conjugated representation. Thus, the differential Galois group is unique up to conjugation as an algebraic subgroup of the general linear group.
Let $G \subset \text{GL}(n, \mathbb{C}_L)$ be an algebraic group. Then it contains a unique maximal connected algebraic subgroup $G^0$, which is called the connected component of the identity or connected identity component. The connected identity component $G^0 \subset G$ is a normal algebraic subgroup and the smallest subgroup of finite index, i.e., the quotient group $G/G^0$ is finite. By the Lie-Kolchin Theorem [13,27], a connected solvable linear algebraic group is triangularizable. Here a subgroup of $\text{GL}(n, \mathbb{C}_L)$ is said to be triangularizable if it is conjugated to a subgroup of the group of (lower) triangular matrices. The following theorem relates the solvability of the differential Galois group with a Liouvillian Picard-Vessiot extension (see [13,27] for the proof).

**Theorem 3.3.** Let $\mathbb{L} \supset \mathbb{K}$ be a Picard-Vessiot extension of (3.1). The connected identity component of the differential Galois group $\text{Gal}(\mathbb{L}/\mathbb{K})$ is solvable if and only if the extension $\mathbb{L} \supset \mathbb{K}$ is Liouvillian.

Thus, if the connected identity component of the differential Galois group $\text{Gal}(\mathbb{L}/\mathbb{K})$ is solvable, then Eq. (3.1) is solved by quadrature and called integrable in the meaning of differential Galois theory.

### 3.2. Monodromy groups and Fuchsian equations.

Let $\mathbb{K}$ be the field of meromorphic functions on a Riemann surface $\Gamma$. So the set of singularities in the entries of $A = A(x)$ is a discrete subset of $\Gamma$, which is denoted by $S$. We also refer to a singularity of the entries of $A(x)$ as that of (3.1). Let $x_0 \in \Gamma \setminus S$. We prolong the fundamental matrix $\Xi(x)$ analytically along any loop $\gamma$ based at $x_0$ and containing no singular point, and obtain another fundamental matrix $\gamma \ast \Xi(x)$. So there exists a constant nonsingular matrix $M_{[\gamma]}$ such that

$$\gamma \ast \Xi(x) = \Xi(x)M_{[\gamma]}.$$ \hfill (3.2)

The matrix $M_{[\gamma]}$ depends on the homotopy class $[\gamma]$ of the loop $\gamma$ and it is called the monodromy matrix of $[\gamma]$.

Let $\pi_1(\Gamma \setminus S, x_0)$ be the fundamental group of homotopy classes of loops based at $x_0$. We have a representation

$$\tilde{R}: \pi_1(\Gamma \setminus S, x_0) \to \text{GL}(n, \mathbb{C}), \quad [\gamma] \mapsto M_{[\gamma]}.$$  

The image of $\tilde{R}$ is called the monodromy group of (3.1). As in the differential Galois group, the representation $\tilde{R}$ depends on the choice of the fundamental matrix, but the monodromy group is defined as a group of matrices up to conjugation. In general, a monodromy transformation defines an automorphism of the corresponding Picard-Vessiot extension. We also just write $M_{\gamma}$ for $M_{[\gamma]}$ below.

A singular point $x = \bar{x}$ of (3.1) is called regular if for any sector $a < \arg(x - \bar{x}) < b$ with $a < b$ there exists a fundamental matrix $\Xi(x) = (\Xi_{ij}(x))$ such that for some $c > 0$ and integer $N$, $|\Xi_{ij}(x)| < c|x - \bar{x}|^N$ as $x \to \bar{x}$ in the sector; otherwise it is called irregular. In particular, if $A(x) = B(x)/x$, where $B(x)$ is holomorphic at $x = 0$, then Eq. (3.1) has a regular singularity at $x = 0$ (see, e.g., Sect. 2.4 of [9]). We have the following result, which plays an essential role in the proof of Theorem 2.3 in Sect. 6 (see, e.g., Theorem 5.8 in [27] for the proof).

**Theorem 3.4** (Schlesinger). Suppose that Eq. (3.1) is Fuchsian. Then the differential Galois group of (3.1) is the Zariski closure of the monodromy group.
Assume that Eq. (3.1) is Fuchsian and \( \text{tr} A(x) = 0 \). Then we have

\[
(\det \Xi(x))' = \text{tr} A(x) \det \Xi(x) = 0.
\]

Hence, by (3.2), \( \det \Xi(x) = \det \Xi(x) \det M_\gamma \), which yields

\[
\det M_\gamma = 1
\]

since \( \det \Xi(x) \neq 0 \). This means by Theorem 3.4 that \( \text{Gal}(\mathbb{L}/\mathbb{K}) \subset \text{SL}(n, \mathbb{C}) \). For \( n = 2 \) we can classify such algebraic groups as follows (see Sect. 2.1 of [21] for a proof).

**Proposition 3.5.** Any algebraic group \( G \subset \text{SL}(2, \mathbb{C}) \) is similar to one of the following types:

(i) \( G \) is finite and \( G^0 = \{ \text{id}_2 \} \);

(ii) \( G = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix} \Bigg| \lambda \text{ is a root of } 1, \mu \in \mathbb{C} \right\} \) and \( G^0 = \left\{ \begin{pmatrix} 1 & 0 \\ \mu & 1 \end{pmatrix} \Bigg| \mu \in \mathbb{C} \right\} \);

(iii) \( G = G^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \Bigg| \lambda \in \mathbb{C}^* \right\} \);

(iv) \( G = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -\beta^{-1} \\ \beta & 0 \end{pmatrix} \Bigg| \lambda, \beta \in \mathbb{C}^* \right\} \) and \( G^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \Bigg| \lambda \in \mathbb{C}^* \right\} \);

(v) \( G = G^0 = \left\{ \begin{pmatrix} \lambda & 0 \\ \mu & \lambda^{-1} \end{pmatrix} \Bigg| \lambda \in \mathbb{C}^*, \mu \in \mathbb{C} \right\} \);

(vi) \( G = G^0 = \text{SL}(2, \mathbb{C}) \).

The system (3.1) with \( n = 2 \) is integrable in the meaning of differential Galois theory unless its differential Galois group satisfies condition (vi). This proposition plays a key role in the proof of Theorem 2.3 in Sect. 6.

### 4. IST Theory for Reflectionless Potentials

We next give necessary information on the IST theory for reflectionless potentials. See the textbooks [1,5] for more details on the standard results. Many materials of Sect. 4.2 are not standard but immediate extensions of the standard ones.

#### 4.1. Scattering coefficients.

We first present some properties of the scattering coefficients. Noting that the trace of the coefficient matrices in (1.1) and (1.2) are zero, we see by (2.4) that the Wronskian of \( \phi(x) \) and \( \dot{\phi}(x) \) (resp. of \( \psi(x) \) and \( \dot{\psi}(x) \)) is one, i.e.,

\[
\det(\phi(x; k), \dot{\phi}(x; k)) = \det(\psi(x; k), \psi(x; k)) = 1. \tag{4.1}
\]

Hence, it follows from (2.5) that

\[
a(k)\dot{a}(k) - b(k)\dot{b}(k) = 1. \tag{4.2}
\]

Moreover, under the transformation \( x \mapsto kx \), the ZS systems (1.1) and (1.2) are rewritten as

\[
v_x = \begin{pmatrix} -i & \varepsilon q(\varepsilon x) \\ \varepsilon & i \end{pmatrix} v
\]
and
\[ v_x = \begin{pmatrix} -i & \varepsilon q(\varepsilon x) \\ \varepsilon r(\varepsilon x) & i \end{pmatrix} v, \]
respectively, where \( \varepsilon = 1/k \). This means that
\[ a(k), \bar{a}(k) \to 1, \quad b(k), \bar{b}(k) \to 0 \quad \text{as} \ |k| \to \infty. \quad (4.3) \]

We also have the following analyticity of the scattering coefficients.

**Proposition 4.1.** (i) \( a(k), \bar{a}(k), b(k), \bar{b}(k) \) are analytic in \( \mathbb{R} \setminus \{0\} \).
(ii) \( a(k) \) and \( \bar{a}(k) \) can be analytically continued in the upper and lower \( k \)-planes, respectively.

**Proof.** By (2.5) and (4.1) we have
\[
\begin{align*}
 a(k) &= \det(\phi(x; k), \psi(x; k)), \quad \bar{a}(k) = \det(\bar{\psi}(x; k), \bar{\phi}(x; k)), \\
 b(k) &= \det(\bar{\psi}(x; k), \phi(x; k)), \quad \bar{b}(k) = \det(\bar{\phi}(x; k), \psi(x; k)).
\end{align*}
\]

(4.4)

We also see that \( \phi(x; k), \bar{\phi}(x; k), \psi(x; k), \bar{\psi}(x; k) \) are bounded and analytic in \( k \in \mathbb{R} \setminus \{0\} \). Actually, for instance, the derivative of \( \phi(x; k) \) with respect to \( k \) is given by the solution to the two-dimensional linear system
\[
w_x = \begin{pmatrix} -ik & q(x) \\ r_0 & ik \end{pmatrix} w + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \phi(x; k)
\]
with
\[
w(x) \sim -ixT \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \quad \text{as} \ x \to -\infty,
\]
and such a solution clearly exists for any \( k \in \mathbb{R} \setminus \{0\} \) (cf. Sect. 8 in Chapter 1 of [12]).

Regarding \( k \) as a complex-valued parameter, we show that \( \phi(x; k) \) is holomorphic in \( k \).
Using (4.4), we obtain part (i). Moreover, similar arguments show that \( \phi(x; k), \psi(x; k) \) (resp. \( \bar{\phi}(x; k), \bar{\psi}(x; k) \)) are bounded and analytic in the upper (resp. lower) \( k \)-plane near \( x = 0 \in \mathbb{R} \). This yields part (ii) along with (4.4).

**Remark 4.2.** (i) The ZS system (1.2) has the Jost solutions satisfying (2.4) for \( k = 0 \), so that the scattering coefficients are still defined and analytic at \( k = 0 \). So it follows by the identity theorem (e.g., Theorem 3.2.6 of [2]) from Proposition 4.1 and (4.3) that zeros of \( a(k), \bar{a}(k) \) are isolated and their numbers are finite.
(ii) For the Schrödinger equation (2.6), we can show that the corresponding scattering coefficient \( \hat{a}(k) \) (see Eq. (A.3) for its definition) has finitely many isolated zeros at most in the upper half plane, as shown for real potentials in [14]. By the relation (A.5) we see that the scattering coefficients \( a(k), \bar{a}(k) \) have finitely many isolated zeros at most for the ZS system (1.1).
(iii) The statements of Proposition 4.1 hold even if \( q(x), r(x) \) are not analytic. See, e.g., Sects. 9.1 and 9.2 of [1] and Sect. 2.2.2 of [5].

#### 4.2. Reflectionless potentials

We now assume that \( b(k), \bar{b}(k) = 0 \) for \( k \in \mathbb{R} \setminus \{0\} \), i.e., \( q(x) \) and \( q(x), r(x) \) are reflectionless potentials in (1.1) and (1.2), respectively. Note that \( a(k), \bar{a}(k) \neq 0 \) for \( k \in \mathbb{R} \setminus \{0\} \), by (4.2). For (1.1) and (1.2) separately, we provide some formulas for reflectionless potentials and Jost solutions.
4.2.1. ZS system (1.1) We begin with the ZS system (1.1). Basically following the standard IST theory for the KdV equation (1.4) (e.g., Chapter 9 of [1]), we discuss the linear Schrödinger equation (2.6) instead of (1.1). Let \( \hat{a}(k), \hat{b}(k) \) be the scattering coefficients for (2.6), as in Appendix A (see Eq. (A.3)). Note that \( \hat{b}(k) = 0 \) for \( k \in \mathbb{R} \setminus \{0\} \) by (A.5). Unlike the case of real potentials [14], the scattering coefficient \( \hat{a}(k) \) may have a non-simple zero when \( q(x) \) is not real. See Appendix B for the details. Moreover, it is well known that \( \hat{a}(k) \) can have a multiple zero if \( q(x) \) has a singularity, whether it locates in \( \mathbb{C} \setminus \mathbb{R} \) or not [20]. See also [17] for related results.

Suppose that \( \hat{a}(k) \) has \( n \) zeros \( \{k_j\}_{j=1}^n \) and the zero \( k_j \) is of multiplicity \( \nu_j \) with \( \text{Im} \ k_j > 0 \) for \( j = 1, \ldots, n \). Let

\[
\hat{M}(x; k) = \hat{\phi}(x; k)e^{ikx}, \quad \hat{N}(x; k) = \hat{\psi}(x; k)e^{ikx},
\]

(4.5)

where \( \hat{\phi}(x; k), \hat{\psi}(x; k) \) are the Jost solutions to (2.6) satisfying (A.1). Hence,

\[
\hat{M}(x; k) \sim 1 \quad \text{as} \quad x \to -\infty, \quad \hat{N}(x; k) \sim e^{2ikx} \quad \text{as} \quad x \to +\infty.
\]

(4.6)

Moreover, it follows from (A.3) that

\[
\hat{M}(x; k) = \hat{a}(k)\hat{N}(x; -k)e^{2ikx} + \hat{b}(k)\hat{N}(x; k).
\]

(4.7)

Let

\[
\hat{N}_j^r(x) := \frac{\partial^r}{\partial k^r}(x; k_j), \quad r = 0, \ldots, \nu_j - 1, \quad j = 1, \ldots, n.
\]

Differentiating (4.7) with respect \( k \) and substituting \( k = k_j \), we obtain

\[
\frac{\partial^r}{\partial k^r}(x; k_j) = \left. \frac{\partial^r}{\partial k^r}(\hat{b}(k_j)\hat{N}(x; k)) \right|_{k=k_j}, \quad r = 0, \ldots, \nu_j - 1
\]

(4.8)

for \( j = 1, \ldots, n \) since \( \hat{a}(k) \) has a zero of multiplicity \( \nu_j \) at \( k = k_j \). In particular, the right hand sides of (4.8) are represented by linear combinations of \( \hat{N}_j^r(x), r = 0, \ldots, \nu_j - 1 \). For example, when \( \nu_j > 1 \), we have

\[
\hat{M}(x; k_j) = \hat{b}(k_j)\hat{N}_j^0(x), \quad \frac{\partial \hat{M}}{\partial k}(x; k_j) = \hat{b}(k_j)\hat{N}_j^1(x) + \hat{b}_k(k_j)\hat{N}_j^0(x), \quad \ldots.
\]

We define the projection operators

\[
\mathcal{P}^\pm(f)(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(\kappa)}{\kappa - (k \pm i0)} \, d\kappa
\]

(4.9)

for \( f \in L^1(\mathbb{R}) \), where \( \text{Im} \ k > 0 \) or \( < 0 \) depending on whether the upper or lower signs are taken simultaneously. If \( f_+ \) (resp. \( f_- \)) is analytic in the upper (resp. lower) \( k \)-plane and \( f_\pm(k) \to 0 \) as \( |k| \to \infty \), then

\[
\mathcal{P}^\pm(f_\pm) = \pm f_\pm, \quad \mathcal{P}^\pm(f_\mp) = 0
\]

(see, e.g., Sect. 2.2.3 of [5]). Dividing both sides of (4.7) by \( \hat{a}(k) \), we apply \( \mathcal{P}^- \) to the resulting equation to obtain

\[
\hat{N}(x; k)e^{-2ikx} = 1 - \frac{1}{2\pi i} \sum_{j=1}^n \int_{|k - k_j| = \delta} \hat{M}(x; \kappa) \, d\kappa
\]

(4.10)
since \( \hat{b}(k) = 0 \) for \( k \in \mathbb{R} \setminus \{0\} \), \( \hat{a}(k) \to 1 \) as \( |k| \to \infty \) like (4.3), and \( \hat{M}(x; k) \) and \( \hat{N}(x; -k) \) are, respectively, analytic and tend to unity as \( |k| \to \infty \) in the upper and lower \( k \)-planes (see Lemma 9.1 of [1]), where \( \delta > 0 \) is sufficiently small. Using the method of residues, we see that the integrals in (4.10) are represented by linear combinations of \( N^r_j(x) \), \( r = 0, \ldots, \nu_j - 1 \). For example, we have

\[
\frac{1}{2\pi i} \int_{|k-k_j|=\delta} \hat{M}(x; \kappa) \frac{\hat{M}(x; \kappa)}{(k+\kappa)\hat{a}(\kappa)} d\kappa = \frac{\hat{b}(k_j)}{(k+k_j)\hat{a}_k(k_j)} \hat{N}^0_j(x),
\]

(4.11)

when \( \nu_j = 1, j = 1, \ldots, n \), and

\[
\frac{1}{2\pi i} \int_{|k-k_j|=\delta} \hat{M}(x; \kappa) \frac{\hat{M}(x; \kappa)}{(k+\kappa)\hat{a}(\kappa)} d\kappa = \frac{2}{(k+k_j)\hat{a}_{kk}(k_j)} \left( \hat{b}(k_j) \hat{N}^1_j(x) + \left( \hat{b}(k_j) - \frac{\hat{a}_{kk}(k_j)\hat{b}(k_j)}{3\hat{a}_{kk}(k_j)} \right) \hat{N}^0_j(x) \right)
\]

when \( \nu_j = 2, j = 1, \ldots, n \).

Differentiating (4.10) with respect to \( k \) up to \( \nu_j - 1 \) times and setting \( k = k_j \), we obtain a system of linear equations about \( N^r_j(x) \), \( r = 0, \ldots, \nu_j - 1, j = 1, \ldots, n \) or \( \tilde{n} \), and solve it to obtain them as rational functions of \( x \) and \( \{e^{2ik_jx}\}^n_{j=1} \), using basic arithmetic operations. Note that

\[
\frac{\partial}{\partial k} (\hat{N}(x; k)e^{-2ikx}) = \frac{\partial}{\partial k} (x; k)e^{-2ikx} - 2ix\hat{N}(x; k)e^{-2ikx}
\]

and so on. See Appendix C for further computation of \( \hat{N}_\ell(x) \), \( \ell = 1, \ldots, n \), when \( \nu_j = 1 \) for \( j = 1, \ldots, n \). Hence, we see by (4.5) that the Jost solutions \( \hat{\psi}(x; k) \), \( \tilde{\psi}(x; k) \) are also represented by rational functions of \( x \) and \( \{e^{ik_jx}\}^n_{j=1} \). Moreover, we show that

\[
q(x) = -\frac{1}{\pi} \frac{\partial}{\partial x} \left( \sum_{j=1}^{\tilde{n}} \left( \int_{|k-k_j|=\delta} \hat{M}(x; \kappa) \hat{a}(\kappa) d\kappa \right) \right)
\]

(4.12)

(see, e.g., Sect. 9.3 of [1]), so that \( q(x) \) is also represented by rational functions of \( x \) and \( \{e^{ik_jx}\}^n_{j=1} \). In particular, we have

\[
q(x) = -2i \sum_{j=1}^{\tilde{n}} \frac{\hat{b}(k_j)}{\hat{a}_k(k_j)} \frac{\partial}{\partial x} \hat{N}^0_j(x)
\]

(13.13)

when \( \nu_j = 1, j = 1, \ldots, n \), and

\[
q(x) = -4i \sum_{j=1}^{\tilde{n}} \frac{1}{\hat{a}_{kk}(k_j)} \left( \frac{\hat{N}^1_j(x)}{\hat{a}_k(k_j)} + \frac{3\hat{a}_{kk}(k_j)\hat{b}(k_j)}{\hat{N}^0_j(x)} \right) \frac{\partial}{\partial x}(x)
\]

when \( \nu_j = 2, j = 1, \ldots, n \). Since \( \text{Im } k_j > 0, j = 1, \ldots, n \), it follows from (4.6) and (4.12) that

\[
\lim_{x \to \pm \infty} q(x) = 0.
\]
In the KdV equation (1.4), Eq. (4.13) corresponds to an initial condition of an \( n \)-soliton when \( q(x) \) is real. See, e.g., Sect. 9.7 of [1] for more details. The two linearly independent solutions \( \psi(x; k), \psi(x; -k) \) to (1.4) are obtained via (A.2) from \( \hat{\psi}(x; k), \psi(x; -k) \) for \( k \neq 0 \).

### 4.2.2. ZS system (1.2)

We turn to the ZS system (1.2). Like (1.1), zeros of \( a(k), \tilde{a}(k) \) for (1.2) may be multiple (see, e.g., [26, 28, 29]). We slightly extend the standard IST theory for another class of integrable PDEs, which contains the examples of Sect. 1 except for the KdV equation (1.4), the NLS equation (1.5) with the minus sign, and the mKdV equation (1.6) with the minus sign.

Suppose that \( a(k) \) and \( \tilde{a}(k) \), respectively, have \( n \) and \( \tilde{n} \) zeros \( \{ k_j \}_{j=1}^n \) and \( \{ \tilde{k}_j \}_{j=1}^{\tilde{n}} \) and the zeros \( k_j \) and \( \tilde{k}_j \) are, respectively, of multiplicity \( \nu_j \) and \( \tilde{\nu}_j \) with \( \text{Im} k_j > 0 \) and \( \text{Im} \tilde{k}_j < 0 \) for \( j = 1, \ldots, n \) or \( \tilde{n} \). Let

\[
M(x; k) = \phi(x; k)e^{ikx}, \quad \tilde{M}(x; k) = \tilde{\phi}(x; k)e^{-ikx},
\]

\[
N(x; k) = \psi(x; k)e^{-ikx}, \quad \tilde{N}(x; k) = \tilde{\psi}(x; k)e^{ikx},
\]

where \( \phi(x; k), \tilde{\phi}(x; k), \psi(x; k), \tilde{\psi}(x; k) \) are the Jost solutions to (1.2) satisfying (2.4). Hence,

\[
M(x; k) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{M}(x; k) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{as} \quad x \to -\infty,
\]

\[
N(x; k) \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \tilde{N}(x; k) \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{as} \quad x \to +\infty.
\]

(4.14)

Moreover, it follows from (2.5) that

\[
M(x; k) = a(k)\tilde{N}(x; k) + b(k)N(x; k)e^{2ikx},
\]

\[
\tilde{M}(x; k) = \tilde{a}(k)N(x; k) + \tilde{b}(k)\tilde{N}(x; k)e^{-2ikx}.
\]

(4.15)

Let

\[
N'_j(x) := \frac{\partial^r N}{\partial k^r}(x; k_j), \quad r = 0, \ldots, \nu_j - 1, \quad j = 1, \ldots, n,
\]

\[
\tilde{N}'_j(x) := \frac{\partial^r \tilde{N}}{\partial \tilde{k}^r}(x; \tilde{k}_j), \quad r = 0, \ldots, \tilde{\nu}_j - 1, \quad j = 1, \ldots, \tilde{n}.
\]

Differentiating (4.15) with respect to \( k \) and substituting \( k = k_j \) or \( \tilde{k}_j \), we obtain

\[
\frac{\partial^r M}{\partial k^r}(x; k_j) = \frac{\partial^r}{\partial k^r} \left( b(k)N(x; k)e^{2ikx} \right) \bigg|_{k=k_j}, \quad r = 0, \ldots, \nu_j - 1,
\]

\[
\frac{\partial^r \tilde{M}}{\partial \tilde{k}^r}(x; \tilde{k}_j) = \frac{\partial^r}{\partial \tilde{k}^r} \left( \tilde{b}(k)\tilde{N}(x; k)e^{-2ikx} \right) \bigg|_{\tilde{k} = \tilde{k}_j}, \quad r = 0, \ldots, \tilde{\nu}_j - 1,
\]

(4.16)

for \( j = 1, \ldots, n \) or \( \tilde{n} \), since the zeros \( k_j \) and \( \tilde{k}_j \) of \( a(k) \) and \( \tilde{a}(k) \) are of multiplicity \( \nu_j \) and \( \tilde{\nu}_j \), respectively. In particular, the right hand sides of the first and second equations in (4.16) are, respectively, represented by linear combinations of \( N'_j(x), r = 0, \ldots, \nu_j - 1 \), and \( \tilde{N}'_j(x), r = 0, \ldots, \tilde{\nu}_j - 1 \), where the coefficients are given by polynomials of \( x \) and
exponential functions $\{e^{2ik_jx}\}_{j=1}^n$ and $\{e^{-2i\bar{k}_jx}\}_{j=1}^\bar{n}$. For example, when $\nu_j, \bar{\nu}_j > 1$, we have

$$M(x; k_j) = b(k_j)e^{2ik_jx}N_j^0(x), \quad \tilde{M}(x; k_j) = \tilde{b}(\bar{k}_j)e^{-2i\bar{k}_jx}\tilde{N}_j^0(x),$$

$$\frac{\partial M}{\partial k}(x; \bar{k}_j) = b(k_j)e^{2ik_jx}N_j^1(x) + (b_k(k_j) + 2ixb(k_j))e^{2ik_jx}N_j^0(x),$$

$$\frac{\partial \tilde{M}}{\partial k}(x; \bar{k}_j) = \tilde{b}(\bar{k}_j)e^{-2i\bar{k}_jx}\tilde{N}_j^1(x) + (\tilde{b}_k(\bar{k}_j) - 2ix\tilde{b}(\bar{k}_j))e^{-2i\bar{k}_jx}\tilde{N}_j^0(x).$$

Dividing both sides of the first and second equations of (4.15) by $a(k)$ and $\tilde{a}(k)$, we apply the projection operators $\mathcal{P}^-$ and $\mathcal{P}^+$ (see Eq. (4.9)) to the resulting equations and obtain

$$\tilde{N}(x; k) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \frac{1}{2\pi i} \sum_{j=1}^{n} \int_{|k - k_j| = \delta} M(x; \kappa) \frac{M(x; \kappa)}{(k - \kappa)a(\kappa)} d\kappa,$$

$$N(x; k) = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + \frac{1}{2\pi i} \sum_{j=1}^{\bar{n}} \int_{|\bar{k} - \bar{k}_j| = \delta} \tilde{M}(x; \kappa) \frac{\tilde{M}(x; \kappa)}{(k - \kappa)\tilde{a}(\kappa)} d\kappa,$$

like (4.10), where $\delta > 0$ is sufficiently small. The integrals in the first and second equations of (4.17) are, respectively, represented by linear combinations of $N_j^r(x)$ and $\tilde{N}_j^r(x)$, $r = 0, \ldots, \nu_j - 1$ or $\bar{\nu}_j - 1$, $j = 1, \ldots, n$ or $\bar{n}$, where the coefficients are given by polynomials of $x$, $\{e^{2ik_jx}\}_{j=1}^n$ and $\{e^{-2i\bar{k}_jx}\}_{j=1}^\bar{n}$, like (4.16). For example, we have

$$\frac{1}{2\pi i} \int_{|k - k_j| = \delta} M(x; \kappa) \frac{M(x; \kappa)}{(k - \kappa)a(\kappa)} d\kappa = e^{2ik_jx} b(k_j) \frac{b(k_j)}{k - k_j} N_j^0(x),$$

$$\frac{1}{2\pi i} \int_{|\bar{k} - \bar{k}_j| = \delta} \tilde{M}(x; \kappa) \frac{\tilde{M}(x; \kappa)}{(k - \kappa)\tilde{a}(\kappa)} d\kappa = e^{-2i\bar{k}_jx} \tilde{b}(\bar{k}_j) \frac{\tilde{b}(\bar{k}_j)}{k - \bar{k}_j} \tilde{N}_j^0(x)$$

when $\nu_j, \bar{\nu}_j = 1$, and

$$\frac{1}{2\pi i} \int_{|k - k_j| = \delta} M(x; \kappa) \frac{M(x; \kappa)}{(k - \kappa)a(\kappa)} d\kappa = 2e^{2ik_jx} \frac{b(k_j)}{(k - k_j)a_{kk}(k_j)} \left( b(k_j)N_j^1(x) + \left( b_k(k_j) + 2ixb(k_j) + \frac{b(k_j)}{k - k_j} - \frac{a_{kkk}(k_j)b(k_j)}{3a_{kk}(k_j)} \right) N_j^0(x) \right),$$

$$\frac{1}{2\pi i} \int_{|k - k_j| = \delta} \tilde{M}(x; \kappa) \frac{\tilde{M}(x; \kappa)}{(k - \kappa)\tilde{a}(\kappa)} d\kappa = 2e^{-2i\bar{k}_jx} \frac{\tilde{b}(\bar{k}_j)}{(k - \bar{k}_j)\tilde{a}_{kk}(\bar{k}_j)} \left( \tilde{b}(\bar{k}_j)\tilde{N}_j^1(x) + \left( \tilde{b}_k(\bar{k}_j) - 2ix\tilde{b}(\bar{k}_j) + \frac{\tilde{b}(\bar{k}_j)}{k - \bar{k}_j} - \frac{\tilde{a}_{kkk}(\bar{k}_j)\tilde{b}(\bar{k}_j)}{3\tilde{a}_{kk}(\bar{k}_j)} \right) \tilde{N}_j^0(x) \right).$$
when $\nu_j, \bar{v}_j = 2$. Differentiating (4.17) with respect to $k$ up to $\bar{v}_j - 1$ or $\nu_j - 1$ times and setting $k = \bar{k}_j$ or $k_j$, we obtain a system of linear equations about $N^r_j(x)$ and $\bar{N}^r_j(x)$, $r = 0, \ldots, \nu_j - 1$ or $\bar{v}_j - 1$, $j = 1, \ldots, n$ or $\bar{n}$, and solve it to obtain them as rational functions of $x$, $\{e^{2ik_jx}\}^n_{j=1}$ and $\{e^{2\bar{k}_jx}\}^\bar{n}_{j=1}$, using basic arithmetic operations, as in Sect. 4.2.1. Hence, we see that the Jost solutions $\psi(x; k)$, $\bar{\psi}(x; k)$ are also represented by rational functions of $x$, $\{e^{ik_jx}\}^n_{j=1}$ and $\{e^{i\bar{k}_jx}\}^\bar{n}_{j=1}$.

Moreover, we have the relations

$$q(x) = 2i \lim_{|k| \to \infty} k N_1(x; k), \quad r(x) = -2i \lim_{|k| \to \infty} k \bar{N}_2(x; k),$$

(4.18)

where $N_\ell(x; k)$ and $\bar{N}_\ell(x; k)$ are the $\ell$th components of $N(x; k)$ and $\bar{N}(x; k)$, respectively, for $\ell = 1, 2$ (see, e.g., Sect. 2.2.2 of [5]). Hence, we see that $q(x)$ and $r(x)$ are represented by rational functions of $x$, $\{e^{2ik_jx}\}^n_{j=1}$ and $\{e^{2i\bar{k}_jx}\}^\bar{n}_{j=1}$. In particular, we have

$$q(x) = 2i \sum_{j=1}^{n} \frac{\bar{b}(\bar{k}_j) \bar{N}^0_{j1}(x) e^{-2i\bar{k}_jx}}{\bar{a}_k(k_j)}, \quad r(x) = -2i \sum_{j=1}^{\bar{n}} \frac{b(k_j) N^0_{j2}(x) e^{2ik_jx}}{a_k(k_j)}$$

when $\nu_j, \bar{v}_j = 1$ for any $j = 1, \ldots, n$ or $\bar{n}$, and

$$q(x) = 4i \sum_{j=1}^{\bar{n}} \left( \frac{\bar{b}(\bar{k}_j) \bar{N}^1_{j1}(x) + (\bar{b}_k(\bar{k}_j) - 2ix\bar{b}(\bar{k}_j)) \bar{N}^0_{j1}(x)}{\bar{a}_k(k_j)} - \frac{\bar{a}_{kkk}(\bar{k}_j) \bar{b}(\bar{k}_j) \bar{N}^0_{j1}(x)}{3\bar{a}_k(k_j)^2} \right) e^{-2i\bar{k}_jx},$$

$$r(x) = -4i \sum_{j=1}^{n} \left( \frac{b(k_j) N^1_{j2}(x) + (b_k(k_j) + 2ixb(k_j)) N^0_{j2}(x)}{a_k(k_j)} - \frac{a_{kkk}(k_j) b(k_j)}{3a_k(k_j)^2} N^0_{j2}(x) \right) e^{2ik_jx}$$

when $\nu_j, \bar{v}_j = 2$ for any $j = 1, \ldots, n$ or $\bar{n}$, where $N^r_{j\ell}(x)$ and $\bar{N}^r_{j\ell}(x)$ are the $\ell$-th components of $N^r_j(x)$ and $\bar{N}^r_j(x)$, respectively, for $\ell = 1, 2$. It follows from (4.3), (4.14), (4.17) and (4.18) that

$$q(x), r(x) \to 0 \quad \text{as} \quad x \to \pm \infty.$$

In the four examples (1.5)-(1.8) except for the minus sign case in (1.5) and (1.6), Eq. (4.18) corresponds to an initial condition of an $n$-soliton (or antisoliton) when $r(x)$ is appropriately defined with $n = \bar{n}$. See [26,28,29] as well as Sect. 2.3 of [5], for more details.

5. Proof of Theorem 2.1

In this section we prove Theorem 2.1 for (1.1) and (1.2) separately.
5.1. ZS system (1.1). We begin with the ZS system (1.1). Henceforth we assume that the potential \(q(x)\) is reflectionless and satisfies condition (A). We first prove the following.

**Lemma 5.1.** \(q(x)\) is a rational function of \(e^{\lambda x}\) for some constant \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > 0\).

**Proof.** Since \(\{ \hat{N}_r^\ell(x) \}_{r=0}^\nu, \ell = 1, \ldots, n, \) are rational functions of \(x\) and/or \(\{ e^{2ik_jx} \}_{j=1}^n \) as shown in Sect. 4.2.1, we see that \(q(x)\) is a rational function of \(x\) and/or \(\{ e^{2ik_jx} \}_{j=1}^n \) after the order of \(\{ k_j \}_{j=1}^n \) are changed if necessary, where \(1 \leq n_0 \leq n\). If it is actually a rational function of \(x\) or there does not exist a constant \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > 0\) such that \(k_j = in_j \lambda\) with some integer \(n_j > 0\) for each \(j = 1, \ldots, n_0\), then \(q(x)\) does not satisfy condition (A) obviously. Thus, we obtain the result. \(\Box\)

Let \(\hat{\psi}(x; k)\) be the Jost solution to the linear Schrödinger equation (2.6) satisfying (A.1), as in Sect. 4.2.1 and Appendix A.

**Lemma 5.2.** \(\hat{\psi}(x; k)\) is a rational function of \(x\), \(e^{\lambda x}\) and \(e^{ikx}\) for any \(k \in \mathbb{C}\) and some constant \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > 0\).

**Proof.** Let \(q_0(s)\) be a rational function of \(s\) such that \(q(x) = q_0(e^{\lambda x})\) with \(q_0(0) = 0\) and \(\lim_{s \to \infty} q_0(s) = 0\). The existence of such a rational function is guaranteed by Lemma 5.1. Using the transformation \(s = e^{\lambda x}\), we rewrite (2.6) as

\[
s^2 w_{ss} + sw_s + \frac{k_j^2 + q_0(s)}{\lambda^2} w = 0 \tag{5.1}
\]

at \(k = k_j, j = 1, \ldots, n\). Equation (5.1) is a linear differential equation over \(\mathbb{C}(s)\) and has regular singularities at \(s = 0\) and \(\infty\). See, e.g., Section 7.1 of [13] for the definition of regular singularities in higher-order differential equations, which is similar to that in linear systems of first-order differential equations such as (3.1). The indicial equations (e.g., Sect. 7.1 of [13]) at \(s = 0\) and \(\infty\) coincide and are given by

\[
\rho^2 + \frac{k_j^2}{\lambda^2} = 0,
\]

which has two roots at

\[
\rho = \mp \frac{ik_j}{\lambda} := \pm \rho_j,
\]

for \(j = 1, \ldots, n\). Note that \(\text{Re} ik_j < 0\) and consequently \(\text{Re} \rho_j > 0\) for \(j = 1, \ldots, n\). If \(\rho_j\) is a rational number for some \(j = 1, \ldots, n\), then we change \(\lambda\) appropriately to take an integer for it. So \(\rho_j, j = 1, \ldots, n\), can be assumed to be positive integers, irrational or non-real numbers.

Assume that \(\rho_j\) is an irrational or non-real number. At \(k = k_j\), since \(\hat{a}(k_j) = 0\), the Jost solution \(w = \hat{\psi}(x; k_j) = \hat{N}_j^0(x)e^{-ik_jx}\) to (2.6) converges to \(w = 0\) as \(x \to \pm \infty\) by (A.3), and corresponds to a solution to (5.1) which has the forms

\[
w = s^{\rho_j}w_1(s) \tag{5.2}
\]

near \(s = 0\) and

\[
w = s^{-\rho_j}w_2(1/s) \tag{5.3}
\]
near \(s = \infty\), where \(w_\ell(s), \ell = 1, 2,\) are holomorphic functions of \(s\) (see, e.g., Sect. 7.1 of [13]). If it has the form (5.3) near \(s = \infty\), then \(\tilde{N}^0_j(x)\) is an analytic function of \(e^{-\lambda x}\), so that it does not have the form (5.2) near \(s = 0\). This yields a contradiction. Thus, for each \(j = 1, \ldots, n,\) \(\rho_j\) is a positive integer and \(k_j = i n_j \lambda\) with some \(n_j \in \mathbb{N}\). Hence, \(\{\tilde{N}^0_j(x)\}_{r=0}^{\nu_j}, j = 1, \ldots, n,\) are rational functions of \(x\) and \(e^{\lambda x}\). So we obtain the desired result from the arguments of Sect. 4.2.1.

\[\square\]

Remark 5.3. In Lemma 5.2, if all zeros of \(\hat{\alpha}(k)\) are simple, i.e., \(\nu_j = 1\) for all \(j = 1, \ldots, n\), then \(\hat{\psi}(x; k)\) is shown to be a rational function of \(e^{\lambda x}\) and \(e^{kx}\).

Proof of Theorem 2.1 for (1.1). The first part immediately follows from Lemma 5.1. We next regard the ZS system (1.1) as a linear system over \(\mathbb{C}(e^{\lambda x})\). Since \(\hat{\psi}(x; k)\) and \(\hat{\psi}(x; -k)\) are linearly independent solutions to the linear Schrödinger equation (2.6), we see via Lemma 5.2 and Eq. (A.2) that the Picard-Vessiot extension of (1.1) is Liouvillian. Thus, we obtain the desired result from Lemma 5.1 and 5.2 and consequently in that of Theorem 2.1 for (1.1).

5.2. ZS system (1.2). We turn to the ZS system (1.2). Henceforth we assume that the potentials \(q(x), r(x)\) are reflectionless and satisfy condition (A). We proceed as in Sect. 5.1. We first prove the following like Lemma 5.1.

Lemma 5.5. \(q(x), r(x)\) are rational functions of \(e^{\lambda x}\) for some constant \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > 0\).

Proof. Since \(\{N^\ell_r(x)\}_{r=0}^{\nu_\ell} \text{and} \{\tilde{N}^\ell_r(x)\}_{r=0}^{\nu_\ell}, \ell = 1, \ldots, n\) or \(\tilde{n}\), are rational functions of \(x\), \(\{e^{2ik_jx}\}_{j=1}^{n_1}\) and \(\{e^{2ik_jx}\}_{j=1}^{\tilde{n}}\), as shown in Sect. 4.2.2, we see that \(q(x), r(x)\) are rational functions of \(x, \{e^{2ik_jx}\}_{j=1}^{n_1}\) and \(\{e^{2ik_jx}\}_{j=1}^{\tilde{n}}\) after the orders of \(\{k_j\}_{j=1}^{n_1}\) and \(\{\tilde{k}_j\}_{j=1}^{\tilde{n}}\) are changed if necessarily, where \(1 \leq n_0 \leq n\) and \(1 \leq \tilde{n}_0 \leq \tilde{n}\). If they are actually rational functions of \(x\) or there does not exist a constant \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > 0\) such that \(k_j = in_j \lambda\) with some integer \(n_j > 0\) for \(j = 1, \ldots, n_0\) and \(\tilde{k}_j = -i\tilde{n}_j \lambda\) with some integer \(\tilde{n}_j > 0\) for \(j = 1, \ldots, \tilde{n}_0\), then \(q(x), r(x)\) do not satisfy condition (A) obviously. Thus, we obtain the result.

Let \(\psi(x; k), \tilde{\psi}(x; k)\) be the Jost solutions to the ZS system (1.2) satisfying (2.4), as in Sect. 4.2.2. We also prove the following like Lemma 5.2.

Lemma 5.6. \(\psi(x; k), \tilde{\psi}(x; k)\) are rational functions of \(x, e^{\lambda x}\) and \(e^{kx}\) for any \(k \in \mathbb{C}\) and some constant \(\lambda \in \mathbb{C}\) with \(\text{Re} \lambda > 0\).

Proof. Let \(q_0(s), r_0(s)\) be rational functions of \(s\) such that \(q(x) = q_0(e^{\lambda x})\) and \(r(x) = r_0(e^{kx})\) with \(q_0(0), r_0(0) = 0\) and \(q_0(s), r_0(s) \to 0\) as \(s \to \infty\). The existence of such rational functions is guaranteed by Lemma 5.5. Using the transformation \(s = e^{\lambda x}\), we rewrite (1.2) as

\[
v_s = \frac{1}{\lambda s} \begin{pmatrix} -ik & q_0(s) \\ r_0(s) & ik \end{pmatrix} v.
\]

(5.4)
Equation (5.4) is a linear system of differential equations over $\mathbb{C}(s)$ and has regular singularities at $s = 0$ and $\infty$.

Assume that $\rho_j = ik_j/\lambda$ is not an integer. Then we can also assume that $\rho_j$ is not even a rational number, as in the proof of Lemma 5.2. Noting that $\text{Im} \ k_j > 0$ and using Theorem 5 in Chapter 2 of [9], we see that at $k = k_j$ the Jost solution $v = \psi(x; k_j) = N_j^0(x)e^{ik_jx}$ to (1.2) converges to $w = 0$ as $x \to \pm \infty$ by (2.5), and corresponds to a solution to (5.4) which has the forms

$$v = s^{-\rho_j}v_1(s) \quad (5.5)$$

near $s = 0$ and

$$v = s^{\rho_j}v_2(1/s) \quad (5.6)$$

near $s = \infty$, where $v_\ell(s)$, $\ell = 1, 2$, are vectors whose components are holomorphic functions of $s$, as in the proof of Lemma 5.2. This yields a contradiction since if it has the form (5.5) near $s = 0$, then $N_j^0(x)$ is an analytic function of $e^{\lambda x}$, so that it does not have the form (5.6) near $s = \infty$. Thus, for each $j = 1, \ldots, n$, $\rho_j = ik_j/\lambda$ is an integer and $k_j = in_j\lambda$ for some $n_j \in \mathbb{N}$. Similarly, we can show that for each $j = 1, \ldots, n$, $ik_j/\lambda$ is an integer and $k_j = -i\tilde{n}_j\lambda$ for some $\tilde{n}_j \in \mathbb{N}$. This implies that $(N_j^r(x))_{r=0}^{v_j-1}$ and $\tilde{N}_j^r(x))_{r=1}^{\tilde{v}_j-1}$, $j = 1, \ldots, n$ or $\tilde{n}$, are rational functions of $x$ and $e^{\lambda x}$. So we obtain the desired result from the arguments of Sect. 4.2.2.

**Remark 5.7.** In Lemma 5.6, if all zeros of $a(k)$ and $\tilde{a}(k)$ are simple, i.e., $v_j = \tilde{v}_j = 1$ for all $j = 1, \ldots, n$ or $\tilde{n}$, then $\psi(x; k), \tilde{\psi}(x; k)$ are shown to be rational functions of $e^{\lambda x}$ and $e^{ikx}$, as in Remark 5.3.

**Proof of Theorem 2.1 for (1.2).** The first part immediately follows from Lemma 5.5. We next regard the ZS system (1.2) as a linear system over $\mathbb{C}(e^{\lambda x})$. Since the linearly independent solutions $\psi(x; k), \tilde{\psi}(x; k)$ are represented by rational functions of $x$, $e^{\lambda x}$ and $e^{ikx}$ by Lemma 5.6, we see that the Picard-Vessiot extension of (1.2) is Liouvillian. Thus, we obtain the second part by Theorem 3.3.

**Remark 5.8.** If the ZS system (1.2) is regarded as a linear system of differential equations over $\mathbb{C}(x, e^{2ik_1x}, \ldots, e^{2ik_nx}, e^{2i\tilde{k}_1x}, \ldots, e^{2i\tilde{k}_n})$, then it is always integrable in the meaning of differential Galois theory, even when condition (A) does not hold (cf. Remark 5.4(i)).

### 6. Proof of Theorem 2.3

In this section we finally prove Theorem 2.3. Similar approaches were previously used to discuss a relationship between nonintegrability and chaos for two-degree-of-freedom Hamiltonian systems in [31,33,34].

We first see that Eq. (2.10) has regular singularities at $s_{\pm} = 0$ since the matrices $A(s_{\pm})$ are holomorphic. Thus, we regard the ZS systems (1.1) and (1.2) as linear ODEs of Fuchs type on the Riemann surface $\hat{\Gamma}$. Let $M_{\pm}$ be monodromy matrices of (2.10) around $s_{\pm} = 0$. Note that there exists no singularity on $\hat{\Gamma}(U_R)$. Let $K = \{k \in \mathbb{R} \setminus \{0\} \mid ik(\lambda_{\pm}^{-1} - \lambda_{\mp}^{-1}) \notin \mathbb{Z}\}$. If $\lambda_{\pm}^{-1} - \lambda_{\mp}^{-1} \notin i\mathbb{R}$, then $K = \mathbb{R} \setminus \{0\}$.

**Lemma 6.1.** The monodromy matrices $M_{\pm}$ have eigenvalues $e^{2\pi k/\lambda_{\pm}}$ and $e^{-2\pi k/\lambda_{\pm}}$ for $k \in K$. 

Proof. Let $k \in \mathcal{K}$. Since $A_\pm(0)$ have eigenvalues $\pm ik$, the characteristic exponents of (2.10) are given by $\mp ik/\lambda_\pm$ and $\pm ik/\lambda_\pm$, the difference of which is not an integer. Hence, by Theorem 5 in Chapter 2 of [9], we compute the local monodromy matrices of (2.10) around $s_\pm = 0$ as

$$\exp \left( \mp \frac{2\pi i}{\lambda_\pm} A_\pm(0) \right),$$

which have eigenvalues $e^{2\pi k/\lambda_\pm}$ and $e^{-2\pi k/\lambda_\pm}$. This means the desired result. □

Let $\Psi(x; k)$ be a fundamental matrix to (1.1) or (1.2) for $k \in \mathbb{R} \setminus \{0\}$. Using Theorem 8.1 in Sect. 3.8 of [12], we show that the limits

$$B_\pm(k) = \lim_{x \to \pm \infty} \Phi(-x; k) \Psi(x; k)$$

exist and $B_\pm(k)$ are nonsingular (cf. Lemma 3.1 of [30]). Recall that $\Phi(x; k)$ is a fundamental matrix to (2.1) with $\Phi(0) = \text{id}_2$ and given by (2.2). Hence, we have

$$\Psi(x; k) \sim \Phi(x; k) B_\pm(k) \quad \text{as } x \to \pm \infty$$

since $\Phi(x; k)^{-1} = \Phi(-x; k)$. Letting

$$B_0(k) = B_+(k) B_-(k)^{-1} \quad \text{and} \quad \Psi_-(x; k) = \Psi(x; k) B_-(k)^{-1},$$

we have

$$\Psi_-(x; k) \sim \Phi(x; k) \quad \text{as } x \to -\infty,$$

$$\Psi_-(x; k) \sim \Phi(x; k) B_0(k) \quad \text{as } x \to +\infty. \quad (6.1)$$

So the first and second column vectors of $\Psi_-(x; k)$ give the Jost solutions $\phi(x; k)$ and $\bar{\phi}(x; k)$, respectively. Similarly, the first and second column vectors of

$$\Psi_+(x; k) = \Psi(x; k) B_+(k)^{-1}$$

give the Jost solutions $\bar{\psi}(x; k)$ and $\psi(x; k)$, respectively. From (2.5) and (6.1) we see that

$$B_0(k) = \begin{pmatrix} a(k) & \bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}. \quad (6.2)$$

In particular, $\det B_0(k) = 1$ by (4.2) for $k \in \mathbb{R} \setminus \{0\}$.

**Lemma 6.2.** Let $k \in \mathcal{K}$. The monodromy matrices can be expressed as

$$M_+ = B_0^{-1} \begin{pmatrix} e^{-2\pi k/\lambda_+} & 0 \\ 0 & e^{2\pi k/\lambda_+} \end{pmatrix} B_0, \quad M_- = \begin{pmatrix} e^{2\pi k/\lambda_-} & 0 \\ 0 & e^{-2\pi k/\lambda_-} \end{pmatrix} \quad (6.3)$$

for a common fundamental matrix.
Proof. Let $\tilde{\Psi}(x; k) = \Psi(x; k)B_-(k)^{-1}$. Then $\tilde{\Psi}(x; k)$ is also a fundamental matrix to (1.1) or (1.2) such that

$$\lim_{x \to -\infty} \Phi(-x; k)\tilde{\Psi}(x; k) = \text{id}_2, \quad \lim_{x \to +\infty} \Phi(-x; k)\tilde{\Psi}(x; k) = B_0.$$ 

Consider the transformed ZS system consisting of (2.9) and (2.10) on $\hat{\Gamma}$, and take a fundamental matrix corresponding to $\tilde{\Psi}(x; k)$. Its analytic continuation yields the (local) monodrmy matrices

$$\begin{pmatrix}
 e^{\mp 2\pi k/\lambda_{\pm}} & 0 \\
 0 & e^{\mp 2\pi k/\lambda_{\pm}}
\end{pmatrix},$$

along small loops around $O_{\pm}$, which are estimated from asymptotic expressions

$$T^{-1}\Phi\left(\mp \frac{1}{\lambda_{\pm}} \log s_{\pm}; k\right)T$$

of its fundamental matrices with $T$ given by (2.3). Hence, we choose the base point near $O_{-}$ to obtain the desired result. \hfill \Box

Proof of Theorem 2.3. Assume that the hypothesis of Theorem 2.3 holds and $k \in \mathcal{K}$. Then the connected identity component $\hat{G}^0$ of the differential Galois group $\hat{G}$ is triangularizable but $G$ may not. Let $\mathcal{M}$ denote the monodromy group generated by $M_{\pm}$. We have the following.

**Lemma 6.3.** The monodromy group $\mathcal{M}$ is triangularizable.

**Proof.** From Theorem 3.4 we first notice that $\mathcal{M}$ has the same classifications as stated in Proposition 3.5. So $\mathcal{M}$ is not an algebraic group of type (vi) in Proposition 3.5 obviously. On the other hand, by Lemma 6.1 the eigenvalues of $M_{\pm}$ are not roots of 1 since $\lambda_{\pm}$ are not purely imaginary. Hence, neither case (i), (ii) nor (iv) occurs for $\mathcal{M}$. Thus, the monodromy group $\mathcal{M}$ is of type (iii) or (v). \hfill \Box

Theorem 2.3 is now easily proved. Substituting (6.2) into the first equation of (6.3), we have

$$M_+ = \begin{pmatrix}
 a(k)\tilde{a}(k)e_- - b(k)\tilde{b}(k)e_+ & \tilde{a}(k)\tilde{b}(k)(e_- - e_+)
 \\
 a(k)b(k)(e_+ - e_-) & a(k)\tilde{a}(k)e_+ - b(k)\tilde{b}(k)e_-
\end{pmatrix},$$

where $e_\pm = e^{\pm 2\pi k/\lambda_\pm}$. Hence, if the monodromy group $\mathcal{M}$ is triangularizable, then

$$a(k)b(k) = 0 \quad \text{or} \quad \tilde{a}(k)\tilde{b}(k) = 0.$$

Since by (4.3) $a(k)$ and $\tilde{a}(k)$ only have discrete zeros, we have $b(k) = 0$ or $\tilde{b}(k) = 0$ for any $k \in \mathbb{R} \setminus \{0\}$ by the identity theorem (e.g., Theorem 3.2.6 of [2]). We also see that if $b(k) = 0$, then $\tilde{b}(k) = 0$ and vice versa for (1.4)–(1.9) (see Eq. (A.5) and Sect. 2.2.2 of [5]). This completes the proof. \hfill \Box

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Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.
Appendix A. Relations on Scattering and Reflection Coefficients between (1.1) and (2.6)

Following Sect. 3d of [24] basically, we define the scattering and reflection coefficients for (2.6) (see also Sect. 9.1 of [1]). Equation (2.6) has the Jost solutions
\[ \hat{\varphi}(x; k) \sim e^{-ikx} \quad \text{as} \quad x \to -\infty, \]
\[ \hat{\psi}(x; k) \sim e^{ikx} \quad \text{as} \quad x \to +\infty. \]

Since
\[ \hat{\varphi}(x; -k) \sim e^{ikx} \quad \text{as} \quad x \to -\infty, \]
\[ \hat{\psi}(x; -k) \sim e^{-ikx} \quad \text{as} \quad x \to +\infty, \]
we have the relations
\[
\phi(x; k) = -\frac{i}{2k} \left( -\hat{\varphi}_x(x; k) + ik\hat{\varphi}(x; k) \right),
\]
\[
\tilde{\phi}(x; k) = \left( -\hat{\varphi}_x(x; -k) + ik\hat{\varphi}(x; -k) \right),
\]
\[
\psi(x; k) = \left( -\hat{\psi}_x(x; k) + ik\hat{\psi}(x; k) \right),
\]
\[
\tilde{\psi}(x; k) = -\frac{i}{2k} \left( -\hat{\psi}_x(x; -k) + ik\hat{\psi}(x; -k) \right),
\]
between the Jost solutions to (1.1) and (2.6) by (2.7). Define the scattering coefficients \( \hat{a}(k) \) and \( \hat{b}(k) \) for (2.6) as
\[ \hat{\varphi}(x; k) = \hat{a}(k)\hat{\psi}(x; -k) + \hat{b}(k)\hat{\psi}(x; k) \]
like (2.5). Since
\[ \hat{\varphi}(x; -k) = \hat{a}(-k)\hat{\psi}(x; k) + \hat{b}(-k)\hat{\psi}(x; -k), \]
we have
\[ \hat{a}(k)\hat{a}(-k) - \hat{b}(k)\hat{b}(-k) = 1 \]
like (4.2). From (A.2) we obtain

$$
\phi(x; k) = \hat{a}(k)\tilde{\psi}(x; k) - \frac{i}{2k}\hat{b}(k)\psi(x; k),
$$

$$
\bar{\phi}(x; k) = \hat{a}(-k)\psi(x; k) + 2ik\hat{b}(-k)\tilde{\psi}(x; k),
$$

which are compared with (2.5) to yield

$$
a(k) = \hat{a}(k), \quad \bar{a}(k) = \hat{a}(-k), \quad b(k) = -\frac{i}{2k}\hat{b}(k), \quad \bar{b}(k) = 2ik\hat{b}(-k). \quad (A.5)
$$

Moreover, for the reflection coefficients we have

$$
\rho(k) = -\frac{i}{2k}\hat{\rho}(k), \quad \bar{\rho}(k) = 2ik\hat{\rho}(-k), \quad (A.6)
$$

where \(\hat{\rho}(k) = \hat{b}(k)/\hat{a}(k)\).

**Appendix B. Condition for the Simplicity of Zeros of \(\hat{a}(k)\)**

We consider the linear Schrödinger equation (2.6) and follow an approach in the proof of Theorem 1 of [14] to provide a necessary and sufficient condition for a zero of the scattering coefficient \(\hat{a}(k)\) to be simple.

Let \(\hat{\phi}(x; k), \hat{\psi}(x; k)\) be the Jost solutions satisfying (A.1). Since

$$
[\hat{\psi}(x; k), \hat{\psi}(x; -k)] := \hat{\psi}_x(x, k)\hat{\psi}(x; -k) - \hat{\psi}(x, k)\hat{\psi}_x(x; -k) = 2ik
$$

by (A.1), we have

$$
[\hat{\psi}(x; k), \hat{\phi}(x; k)] = [\hat{\psi}(x, k), \hat{a}(k)\hat{\psi}(x; -k) + \hat{b}(k)\hat{\psi}(x; k)] = 2ik\hat{a}(k)
$$

by (A.3). Hence,

$$
\frac{d}{dk}(2ik\hat{a}(k)) = 2i\hat{a}(k) + 2ik\hat{a}_k(k)
$$

$$
= [\hat{\psi}_k(x; k), \hat{\phi}(x; k)] + [\hat{\psi}(x; k), \hat{\phi}_k(x; k)]. \quad (B.1)
$$

On the other hand, since \(\hat{\psi}(x; k)\) is a solution to (2.6), we immediately have

$$
\hat{\psi}_{k,xx}(x; k) + q(x)\hat{\psi}_k(x; k) = -k^2\hat{\psi}_k(x; k) - 2k\hat{\psi}(x; k),
$$

so that
\[
\frac{d}{dx} \left[ \hat{\psi}_k(x; k), \hat{\phi}(x; k) \right] = \hat{\psi}_{k,xx}(x; k) \hat{\phi}(x; k) - \hat{\psi}_k(x; k) \hat{\phi}_{xx}(x; k) \\
= -2k \hat{\psi}(x; k) \hat{\phi}(x; k).
\]  
\tag{B.2}

Similarly, we have
\[
\frac{d}{dx} \left[ \hat{\psi}(x; k), \hat{\phi}_k(x; k) \right] = 2k \hat{\psi}(x; k) \hat{\phi}(x; k).
\]  
\tag{B.3}

Let \( k = k_0 \) be a zero of \( \hat{a}(k) \) with \( \text{Im} k_0 > 0 \). Then by (A.1) and (A.3) we have
\[
\lim_{x \to +\infty} \hat{\psi}_k(x; k_0) \hat{\phi}(x; k_0) = 0, \quad \lim_{x \to -\infty} \hat{\psi}(x; k_0) \hat{\phi}_k(x; k_0) = 0.
\]

Hence, it follows from (B.2) and (B.3) that
\[
\left[ \hat{\psi}_k(x; k_0), \hat{\phi}(x; k_0) \right] = 2k_0 \int_{-\infty}^{\infty} \hat{\psi}(x; k_0) \hat{\phi}(x; k_0) dx,
\]
\[
\left[ \hat{\psi}(x; k_0), \hat{\phi}_k(x; k_0) \right] = 2k_0 \int_{-\infty}^{\infty} \hat{\psi}(x; k_0) \hat{\phi}(x; k_0) dx,
\]

which yield
\[
i \hat{a}_k(k_0) = \int_{-\infty}^{\infty} \hat{\psi}(x; k_0) \hat{\phi}(x; k_0) dx
\]
along with (B.1). Using (A.3), we have the following proposition.

**Proposition B.4** The zero \( k = k_0 \) of \( \hat{a}(k) \) with \( \text{Im} k_0 > 0 \) is simple if and only if
\[
\int_{-\infty}^{\infty} \hat{\psi}(x; k_0)^2 dx \neq 0.
\]  
\tag{B.4}

**Remark B.5** Assume that \( q(x) \) is real on \( \mathbb{R} \) and let \( k = k_0 \) be a zero of \( \hat{a}(k) \). Then \( k_0 \) is purely imaginary as stated in Remark 5.4(ii) and we can take a real function on \( \mathbb{R} \) as \( \hat{\psi}(x; k_0) \), which is a solution to (2.6) with \( k = k_0 \) such that \( \lim_{x \to \pm 0} w(x) = 0 \). Hence, condition (B.4) holds. Thus, any zero of \( \hat{\psi}(x; k_0) \) is simple, as shown in [14].

**Appendix C. Computation of \( \hat{N}_\ell(x) \), \( \ell = 1, \ldots, n \), in a Simple Case**

We consider the linear Schrödinger equation (2.6) and compute \( \hat{N}_\ell(x) \), \( \ell = 1, \ldots, n \), which were defined in Sect. 4.2, when the zeros \( k = k_j, j = 1, \ldots, n, \) of \( \hat{a}(k) \) are all simple.

Substituting (4.11) into (4.10) and setting \( k = k_\ell \), we have
\[
\hat{N}_\ell(x) = e^{2ik_\ell x} \left( 1 - \sum_{j=1}^{n} \frac{\hat{C}_j \hat{N}_j(x)}{k_\ell + k_j} \right), \quad \ell = 1, \ldots, n,
\]  
\tag{C.1}

where
\[
\hat{C}_j = \frac{\hat{b}(k_j)}{\hat{a}_k(k_j)}, \quad j = 1, \ldots, n.
\]
Let
\[
\Delta(x) = \begin{vmatrix}
1 + (2k_1)^{-1} \hat{C}_1 e^{2ik_1x} (k_1 + k_2)^{-1} \hat{C}_2 e^{2ik_1x} & \cdots & (k_1 + k_n)^{-1} \hat{C}_n e^{2ik_1x} \\
(k_2 + k_1)^{-1} \hat{C}_1 e^{2ik_2x} & 1 + (2k_2)^{-1} \hat{C}_2 e^{2ik_2x} & \cdots & (k_2 + k_n)^{-1} \hat{C}_n e^{2ik_2x} \\
\vdots & \vdots & \ddots & \vdots \\
(k_n + k_1)^{-1} \hat{C}_1 e^{2ik_nx} & (k_n + k_2)^{-1} \hat{C}_2 e^{2ik_nx} & \cdots & 1 + (2k_n)^{-1} \hat{C}_n e^{2ik_nx}
\end{vmatrix}
\]
and let \(\Delta_\ell(x), \ell = 1, \ldots, n\), be the determinants of the matrices obtained by replacing the \(\ell\)th column of \(\Delta(x)\) with
\[
\begin{pmatrix}
e^{2ik_1x} \\
e^{2ik_2x} \\
\vdots \\
e^{2ik_nx}
\end{pmatrix}
\]
Using Cramer’s rule, we represent the solution to the system of linear algebraic equations \((C.1)\) as
\[
\hat{N}_\ell(x) = \Delta_\ell(x) / \Delta(x), \quad \ell = 1, \ldots, n.
\]
In particular, \(\hat{N}_\ell(x), \ell = 1, \ldots, n\), are rational functions of \(e^{2ik_jx}, j = 1, \ldots, n\).

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