Aspects of Cubical Higher Category Theory
Camell Kachour
May 29, 2017

Abstract
In this article we show how to build aspects of articles \[24, 25, 34\] but with the cubical geometry. Thus we define a monad on the category $\mathcal{C}$Sets of cubical sets which algebras are models of cubical weak $\infty$-categories. Also for each $n \in \mathbb{N}$ we define a monad on $\mathcal{C}$Sets which algebras are models of cubical weak $(\infty, n)$-categories. And finally we define a monad on the category\(^2 \mathcal{C}$Sets\(^2\) which algebras are models of cubical weak $\infty$-functors, and a monad on the category $\mathcal{C}$Sets\(^4\) which algebras are models of cubical weak natural $\infty$-transformations.

Keywords. cubical weak $(\infty, n)$-categories, cubical weak $\infty$-groupoids, computer sciences.

Mathematics Subject Classification (2010). 18B40, 18C15, 18C20, 18G55, 20L99, 55U35, 55P15.

Contents
1 Cubical sets 3
   1.1 The cubical category .................................................. 3
   1.2 Reflexive cubical sets ............................................... 4
2 The category of strict cubical $\infty$-categories 6
   2.1 Definition ............................................................. 7
   2.2 The monad of cubical strict $\infty$-categories ..................... 8
3 The category of cubical weak $\infty$-categories 10
   3.1 The category of cubical categorical stretchings .................. 10
   3.2 Magmatic properties of cubical weak $\infty$-categories ........... 21
   3.3 Computations for low dimensions ................................ 22
4 Cubical $(\infty, m)$-sets 22
5 Cubical weak $(\infty, m)$-categories, $m \in \mathbb{N}$ 24
   5.1 Cubical strict $(\infty, m)$-categories, $m \in \mathbb{N}$ ............. 24
   5.2 The category of cubical weak $(\infty, m)$-categories, $m \in \mathbb{N}$ 25

\(^1\)In this "v2 arxived version" we have changed only $p = 0$ by $p = m + 1$ just before the section 5.2.
\(^2\)$\mathcal{C}$Sets\(^2\) means the cartesian product $\mathcal{C}$Sets $\times$ $\mathcal{C}$Sets, and $\mathcal{C}$Sets\(^n\) means the $n$-fold cartesian product of $\mathcal{C}$Sets with itself.
Introduction

In this article we explain how to build algebraic models of

- cubical weak $\infty$-categories (see 3)
- cubical weak $(\infty, m)$-categories (see 5.2)
- cubical weak $\infty$-functors (see 6)
- cubical weak natural $\infty$-transformations (see 7)

In particular cubical weak $(\infty, 0)$-categories known as cubical weak $\infty$-groupoids are very important for us because other models of cubical weak $\infty$-groupoids exist but are defined in an non-algebraic way [6, 7, 9, 10, 13, 17, 37], but more by considering kind of cubical Kan complexes.

As a matter of fact a very important feature of cubical higher category theory is their flexible possibility to have models of higher structures build by mimic simplicial methods for presheaves on the classical category $\Delta$, to presheaves on the reflexive cubical category $\mathcal{C}_r$ of cubical sets with connections (see 1.2), and in an other hand to have also models of higher structures build by mimic algebraic methods of the globular setting (see [24, 25, 34]).

For this last point it is important to notice that cubical strict $\infty$-categories (see 2) are very close in nature to their globular analogue : first datas of it are given by countable family of sets $(C_n)_{n\in\mathbb{N}}$, equipped both with kind of sources and targets, and partial operations, and two kinds of reflexions on each set $C_n$, subject to axioms. See [5, 12, 38].

Cubical sets have richer structure than globular sets, analogue to simplicial sets, and this richness allows to translate many definitions of simplicial higher category to cubical higher category (see [1, 17, 36]). But as we shall see, cubical higher category theory has the algebraic flexibility of globular higher category theory, which is a feature we have difficulty to see for simplicial higher category theory. This important aspect of cubical higher category theory push to see them as a bridge between simplicial higher category theory and globular higher category theory.

We believe that our models of cubical weak $\infty$-groupoids should opens new perspective to the Grothendieck conjecture on homotopy types of spaces, which is stated in the globular setting, and is as follow :

**Conjecture (Grothendieck)** The category of (some) models of globular weak $\infty$-groupoids is equipped with a Quillen model structure which is Quillen equivalent to the category of spaces equipped with its usual Quillen model structure, i.e those which weak equivalences are given by the homotopy groups, and which fibrations are Serre fibrations.

Finally it is important to notice that cubical strict higher structures have already applications and impacts in homology [2, 8, 9, 10] and in algebraic topology [11, 21, 36]. The use of connections with simplicial method can be found in [1, 17, 36].

This article is devoted to several work which main steps are as follow :
We define our own terminology in order to be as close as possible to the notation of the globular environment, and in it we define the monad of cubical strict ∞-categories on the category of cubical sets.

We define the category of cubical categorical stretchings which is the cubical analogue of the category of globular categorical stretchings of [34]. The key ingredient is a cubical analogue of the globular contractions build in [34]. Then we give a monad on the category of cubical sets which algebras are our models of cubical weak ∞-categories. This monad is the cubical analogue of the monad in [34], which algebras are the globular weak ∞-categories of Penon.

We define cubical (∞, n)-sets, which main tools are the cubical reversors. These are the cubical analogue of the globular (∞, n)-sets which has been defined in [25]. More precisely they are build by using cubical analogue of minimal (∞, n)-structures in the sense of [25].

We define the category of cubical reflexive ∞-magmas and then the category of cubical (∞, n)-categorical stretchings, which is the cubical analogue of the globular (∞, n)-categorical stretchings which has been defined in [25]. This allow us to build a monad on the category of cubical sets which algebras are our models of cubical weak (∞, n)-categories. This monad is the cubical analogue of the monad on globular sets build in [25] and which algebras are globular models of weak (∞, n)-categories.

In the sections 6 and 7 we extend globular weak ∞-functors and globular weak natural ∞-transformations to the cubical setting. In particular we shall see that the monad of cubical weak ∞-functors act on the category CSets × CSets and the monad of cubical weak natural ∞-transformations act on the category CSets × CSets × CSets. In these last sections some interesting internal 2-cubes appears in ∞-CCAT which actually are all easy cubical strict 2-categories.

Acknowledgement. I thank mathematicians of the IRIF, and the good ambience provided in the lab during my postdoctoral position in this team, especially I want to mention Paul-André Mélies, Mai Gehrke, Thomas Ehrhard, Pierre-Louis Curien, Yves Guiraud, Jordi Lopez-Abad, Maxime Lucas, Albert Burroni and François Métayer. I dedicate this work to Ronnie Brown.

1 Cubical sets

See also [1, 22] for more references on cubical sets.

1.1 The cubical category

Consider the small category C with integers n ∈ N as objects. Generators for C are, for all n ∈ N given by sources \( n \) \( \xrightarrow{s_{n-1,i,j}} n-1 \) for each \( j \in \{1, \ldots, n\} \) and targets \( n \) \( \xrightarrow{t_{n-1,i,j}} n-1 \) for each \( j \in \{1, \ldots, n\} \) such that for \( 1 \leq i < j \leq n \) we have the following cubical relations

\begin{align*}
(i) \quad s_{n-2,i} \circ s_{n-1,i,j} &= s_{n-2,j-1} \circ s_{n-1,i,j}, \\
(ii) \quad s_{n-2,i} \circ t_{n-1,i,j} &= t_{n-2,j-1} \circ s_{n-1,i,j}, \\
(iii) \quad t_{n-2,i} \circ s_{n-1,i,j} &= s_{n-2,j-1} \circ t_{n-1,i,j}, \\
(iv) \quad t_{n-2,i} \circ t_{n-1,i,j} &= t_{n-2,j-1} \circ t_{n-1,i,j},
\end{align*}

These generators plus these relations give the small category \( C \) called the cubical category that we may represent schematically with the low dimensional diagram :

---

3 Cubical sets in our terminology are the precubical sets of Richard Steiner
4 Also called (∞, n)-graphs in [25]
5 Three months have been financially supported (December 2016 until February 2017) under an European Research Council Project called Duall : https://www.irif.fr/~mgehrke/DualLL.htm.
and this category $C$ gives also the sketch $\mathcal{E}_S$ of cubical sets used especially in 2.2, 3 and 5.2 to produce the monads $S = (S, \lambda, \mu)$, $\mathcal{W} = (W, \eta, \nu)$ and $\mathcal{W}^m = (W^m, \eta^m, \nu^m)$ on $C_{Sets}$, which algebras are respectively cubical strict $\infty$-categories, cubical weak $\infty$-categories and cubical weak $(\infty, m)$-categories.

**Definition 1** The category of cubical sets $C_{Sets}$ is the category of presheaves $[C; Sets]$. The terminal cubical set is denoted $1$.

Occasionally a cubical set shall be denoted with the notation

$$C = (C_n, s^n_{n-1,j}, t^n_{n-1,j})_{1 \leq j \leq n, n \in \mathbb{N}}$$

in case we want to point out its underlying structures.

### 1.2 Reflexive cubical sets

Reflexivity for cubical sets are of two sorts: one is "classical" in the sense that they are very similar to their globular analogue; thus we shall use the notation $\Gamma$ for all maps $C(n) \xrightarrow{\Gamma} C(n + 1)$ which formally behave like globular reflexivity ([25]); the others are called *connections* and are given by maps $C(n) \xrightarrow{\gamma} C(n + 1)$ where the notation using the greek letter "Gamma" seems to be the usual notation. However we do prefer to use instead the notation $C(n) \xrightarrow{1^\alpha_{n+1,j}} C(n + 1)$ $(\gamma \in \{+, \ldots, n\})$ in order to point out the reflexive nature of connections.

Consider the cubical category $C$. For all $n \in \mathbb{N}$ we add in it generators $\mathbf{n} \xrightarrow{n^{-1}_{n,n,j}} \mathbf{n}$ for each $j \in \{1, \ldots, n\}$ subject to the relations:

| Relation | Description |
|----------|-------------|
| (i) $1^n_{n+1,i} \circ 1^n_{n,j} = 1^n_{n+1,j+1} \circ 1^n_{n,i}$ if $1 \leq i, j \leq n$; |
| (ii) $s^n_{n-1,i} \circ 1^n_{n,j} = 1^{n-2}_{n-1,j-1} \circ s^n_{n-2,i}$ if $1 \leq i < j \leq n$; |
| (iii) $s^n_{n-1,i} \circ 1^n_{n,j} = 1^{n-2}_{n-1,j} \circ s^n_{n-2,i-1}$ if $1 \leq j < i \leq n$; |
| (iv) $s^n_{n-1,i} \circ 1^n_{n,j} = id(n-1)$ if $i = j$. |

| Relation | Description |
|----------|-------------|
| (i) $1^n_{n+1,i} \circ 1^n_{n,j} = 1^n_{n+1,j+1} \circ 1^n_{n,i}$ if $1 \leq i, j \leq n$; |
| (ii) $t^n_{n-1,i} \circ 1^n_{n,j} = 1^{n-2}_{n-1,j-1} \circ t^n_{n-2,i}$ if $1 \leq i < j \leq n$; |
| (iii) $t^n_{n-1,i} \circ 1^n_{n,j} = 1^{n-2}_{n-1,j} \circ t^n_{n-2,i-1}$ if $1 \leq j < i \leq n$; |
| (iv) $t^n_{n-1,i} \circ 1^n_{n,j} = id(n-1)$ if $i = j$. |

These generators and relations give the small category $C_n$ called the *semireflexive cubical category* where a quick look at its underlying semireflexive structure is given by the following diagram:
Definition 2 The category of semireflexive cubical sets $\mathbf{C}_{\text{sr}}$ is the category of presheaves $[\mathbf{C}_{\text{sr}}; \text{Sets}]$. The terminal semireflexive cubical set is denoted $1_{\text{sr}}$.

Consider the semireflexive cubical category $\mathbf{C}_{\text{sr}}$. For all integers $n \geq 1$ we add in it generators $n-1 \to n$ for each $j \in \{1, \ldots, n-1\}$ subject to the relations:

(i) for $1 \leq j < i \leq n$, $1^{n, \gamma}_{n+1, i} \circ 1^{n-1, \gamma}_{n, j} = 1^{n, \gamma}_{n+1, j+1} \circ 1^{n-1, \gamma}_{n, i}$;

(ii) for $1 \leq i \leq n-1$, $1^{n, \gamma}_{n+1, i} \circ 1^{n-1, \gamma}_{n, n} = 1^{n, \gamma}_{n+1, i+1} \circ 1^{n-1, \gamma}_{n, n}$;

(iii) for $1 \leq i, j \leq n$, $1^{n, \gamma}_{n+1, i} \circ 1^{n-1, \gamma}_{n, j} = 1^{n, \gamma}_{n+1, j+1} \circ 1^{n-1, \gamma}_{n, i}$ if $1 \leq j < i \leq n$;

(iv) for $1 \leq j \leq n$, $1^{n, \gamma}_{n+1, j} \circ 1^{n-1, \gamma}_{n, n} = 1^{n, \gamma}_{n+1, j} \circ 1^{n-1, \gamma}_{n, i}$;

(v) for $1 \leq i, j \leq n$,

\[
\begin{align*}
&\begin{cases}
  s^n_{n-1, i} \circ 1^{n-1, \gamma}_{n, j} = 1^{n-2, \gamma}_{n-1, i-1} \circ s^n_{n-2, j} & \text{if } 1 \leq i < j \leq n-1, \\
  = 1^{n-2, \gamma}_{n-1, j} \circ s^n_{n-2, i-1} & \text{if } 2 \leq j+1 < i \leq n
  \end{cases} \\
\text{and} \\
&\begin{cases}
  t^n_{n-1, i} \circ 1^{n-1, \gamma}_{n, j} = 1^{n-2, \gamma}_{n-1, j-1} \circ t^n_{n-2, j} & \text{if } 1 \leq i < j \leq n-1, \\
  = 1^{n-2, \gamma}_{n-1, j} \circ t^n_{n-2, i-1} & \text{if } 2 \leq j+1 < i \leq n
  \end{cases}
\end{align*}
\]

(vi) for $1 \leq j \leq n-1$, $s^n_{n-1, i} \circ 1^{n-1, \gamma}_{n, j} = s^n_{n-1, j+1} \circ 1^{n-1, \gamma}_{n, i} = 1^{n-1, \gamma}_{n-1}$ and $t^n_{n-1, j} \circ 1^{n-1, \gamma}_{n, j} = t^n_{n-1, j+1} \circ 1^{n-1, \gamma}_{n, j} = 1^{n-1, \gamma}_{n-1}$;

(vii) for $1 \leq j \leq n-1$, $s^n_{n-1, i} \circ 1^{n-1, \gamma}_{n, j} = s^n_{n-1, j+1} \circ 1^{n-1, \gamma}_{n, i} = 1^{n-2, \gamma}_{n-1, j} \circ s^n_{n-2, j}$;

(viii) for $1 \leq j \leq n-1$, $t^n_{n-1, i} \circ 1^{n-1, \gamma}_{n, j} = t^n_{n-1, j+1} \circ 1^{n-1, \gamma}_{n, i} = 1^{n-2, \gamma}_{n-1, j} \circ t^n_{n-2, j}$.

These generators and relations give the small category $\mathbf{C}_{\text{r}}$ called the reflexive cubical category and in it, connections have the following shape:

\[
\begin{align*}
&\begin{cases}
  s^n_{n-1, i} \circ 1^{n-1, \gamma}_{n, j} = 1^{n-2, \gamma}_{n-1, i-1} \circ s^n_{n-2, j} & \text{if } 1 \leq i < j \leq n-1, \\
  = 1^{n-2, \gamma}_{n-1, j} \circ s^n_{n-2, i-1} & \text{if } 2 \leq j+1 < i \leq n
  \end{cases} \\
\text{and} \\
&\begin{cases}
  t^n_{n-1, i} \circ 1^{n-1, \gamma}_{n, j} = 1^{n-2, \gamma}_{n-1, j-1} \circ t^n_{n-2, j} & \text{if } 1 \leq i < j \leq n-1, \\
  = 1^{n-2, \gamma}_{n-1, j} \circ t^n_{n-2, i-1} & \text{if } 2 \leq j+1 < i \leq n
  \end{cases}
\end{align*}
\]

Definition 3 The category of reflexive cubical sets $\mathbf{C}_{\text{r}}$ is the category of presheaves $[\mathbf{C}_{\text{r}}; \text{Sets}]$. The terminal reflexive cubical set is denoted $1_{\text{r}}$. □
2 The category of strict cubical $\infty$-categories

Cubical strict $\infty$-categories have been studied in [5, 12, 38].

In [3] the authors proved that the category of cubical strict $\infty$-categories with cubical strict $\infty$-functors as morphisms is equivalent to the category of globular strict $\infty$-categories with globular strict $\infty$-functors as morphisms. Consider a cubical reflexive set

$$(C, \{1^n_{n+1,j}\}_{n \in \mathbb{N}, j \in [1, n+1]}, \{1^n_{n+1,j}\}_{n \geq 1, j \in [1, n]})$$

equipped with partial operations $(\circ^n_j)_{n \geq 1, j \in [1, n]}$ where if $a, b \in C(n)$ then $a \circ^n_j b$ is defined for $j \in \{1, ..., n\}$ if $s^n_j(b) = t^n_j(a)$. We also require these operations to follow the following axioms of positions:

(i) For $1 \leq j \leq n$ we have: $s^n_{n-1,j}(a \circ^n_j b) = s^n_{n-1,j}(a)$ and $t^n_{n-1,j}(a \circ^n_j b) = t^n_{n-1,j}(a)$,

(ii) $s^n_{n-1,i}(a \circ^n_j b) = \begin{cases} s^n_{n-1,i}(a) \circ^n_{j-1} s^n_{n-1,i}(b) & \text{if } 1 \leq i < j \leq n \\ s^n_{n-1,i}(a) \circ^n_{j-1} s^n_{n-1,i}(b) & \text{if } 1 \leq j < i \leq n \end{cases}$

(iii) $t^n_{n-1,i}(a \circ^n_j b) = \begin{cases} t^n_{n-1,i}(a) \circ^n_{j-1} t^n_{n-1,i}(b) & \text{if } 1 \leq i < j \leq n \\ t^n_{n-1,i}(a) \circ^n_{j-1} t^n_{n-1,i}(b) & \text{if } 1 \leq j < i \leq n \end{cases}$

The following sketch $\mathcal{E}_M$ of axioms of positions as above shall be used in 2.2 to justify the existence of the monad on C$\mathcal{S}$ets of cubical strict $\infty$-categories. It is important to notice that the sketch just below has only one generation which means that diagrams and cones involved in it are not build with previous data of other diagrams and cones.

- For $1 \leq i < j \leq n$ we consider the following two cones:

$$
\begin{array}{ccc}
M_n \times_{M_{n-1,j}} M_n & \stackrel{\pi^n_{i,j}}{\longrightarrow} & M_n \\
\downarrow{s^n_{n-1,j}} & & \downarrow{t^n_{n-1,j}} \\
M_n & \rightarrow & M_n
\end{array}
\quad
\begin{array}{ccc}
M_{n-1} \times_{M_{n-2,j-1}} M_{n-1} & \stackrel{\pi^{n-1}_{i,j-1}}{\longrightarrow} & M_{n-1} \\
\downarrow{s^{n-1}_{n-1,i}} & & \downarrow{t^{n-1}_{n-2,j-1}} \\
M_{n-1} & \rightarrow & M_{n-2}
\end{array}
$$

and the following commutative diagram (definition of $s^n_{n-1,i} \times_{j-1} s^n_{n-1,i}$)

$$
\begin{array}{ccc}
M_n \times_{M_{n-1,j}} M_n & \stackrel{\pi^n_{i,j}}{\longrightarrow} & M_n \\
\downarrow{s^n_{n-1,i} \times_{j-1} s^n_{n-1,i}} & & \downarrow{\pi_{i,j-1}^{-1}} \\
M_{n-1} \times_{M_{n-2,j-1}} M_{n-1} & \stackrel{\pi^{n-1}_{i,j-1}}{\longrightarrow} & M_{n-1} \\
\downarrow{s^{n-1}_{n-1,i} \times_{j-1} s^{n-1}_{n-1,i}} & & \downarrow{t^{n-1}_{n-2,j-1}} \\
M_{n-1} & \rightarrow & M_{n-2}
\end{array}
$$

which gives the following commutative diagram

$$
\begin{array}{ccc}
M_n \times_{M_{n-1,j}} s^n_{n-1,i} \times_{j-1} s^n_{n-1,i} & \stackrel{s^n_{j}}{\longrightarrow} & M_{n-1} \times_{M_{n-2,j-1}} M_{n-1} \\
\downarrow{s^n_{j}} & & \downarrow{t^{n-1}_{j-1}} \\
M_n & \rightarrow & M_{n-1}
\end{array}
$$

- For $1 \leq j < i \leq n$ we consider the following two cones:
The previous data gives the following commutative diagram of axioms

\[
M_n \times M_n \longrightarrow M_n \quad M_{n-1} \times M_{n-1} \longrightarrow M_{n-1}
\]

\[
\pi_{0,j}^{n} \downarrow \quad \pi_{1,j}^{n} \downarrow
\]

\[
M_n \quad M_{n-1}
\]

\[
\pi_{0,j}^{n}^{-1} \quad \pi_{1,j}^{n}^{-1}
\]

\[
\pi_{n-1,j}^{n} \quad \pi_{n-1,j}^{n-1} \quad \pi_{n-2,j}^{n-1}
\]

\[
M_n \quad M_{n-1} \quad M_{n-2}
\]

and for \(1 \leq j \leq n\) we have the following commutative diagram of axioms

\[
M_n \times M_n \longrightarrow M_n \quad M_{n-1} \times M_{n-1} \longrightarrow M_{n-1}
\]

\[
\pi_{j}^{n} \downarrow \quad \pi_{j}^{n-1} \downarrow
\]

\[
M_n \quad M_{n-1}
\]

\[
\pi_{j}^{n-1} \quad \pi_{j}^{n-2}
\]

\[
M_n \quad M_{n-1} \quad M_{n-2}
\]

which actually complete the description of \(E_M\)

**Definition 4** Cubical reflexive \(\infty\)-magmas are cubical reflexive set equipped with partial operations like just above which follow axioms of positions. A morphism between two cubical reflexive \(\infty\)-magmas is a morphism of their underlying cubical reflexive sets. The category of cubical reflexive \(\infty\)-magmas is noted \(\infty\)-CMag.

**Remark 1** Cubical \(\infty\)-magmas are poorer structure: they are cubical sets equipped with partial operations like above with these axioms of positions. A morphism between two cubical \(\infty\)-magmas is a morphism of their underlying cubical sets. The category of cubical \(\infty\)-magmas is noted \(\infty\)-CMag.

\[\square\]

### 2.1 Definition

Strict cubical \(\infty\)-categories are cubical reflexive \(\infty\)-magmas such that partial operations are associative and also we require the following axioms:
(i) The interchange laws: \((a \circ^n_i b) \circ^n_j (c \circ^n_k d) = (a \circ^n_j c) \circ^n_i (b \circ^n_k d)\) whenever both sides are defined.

(ii) \(1^2_{n+1,i}(a \circ^n_j b) = 1^2_{n+1,j+1}(a) \circ^{n+1}_{j+1} 1^2_{n+1,i}(b)\) if \(1 \leq i \leq j \leq n\)
\(1^2_{n+1,i}(a \circ^n_j b) = 1^2_{n+1,j+1}(a) \circ^{n+1}_{j+1} 1^2_{n+1,i}(b)\) if \(1 \leq j < i \leq n + 1\).

(iii) \(1^{n,\gamma}_{n+1,i}(a \circ^n_j b) = 1^{n,\gamma}_{n,j+1}(a) \circ^{n+1}_{j+1} 1^{n,\gamma}_{n+1,i}(b)\) if \(1 \leq i \leq j \leq n\)
\(1^{n,\gamma}_{n+1,i}(a \circ^n_j b) = 1^{n,\gamma}_{n,j+1}(a) \circ^{n+1}_{j+1} 1^{n,\gamma}_{n+1,i}(b)\) if \(1 \leq j < i \leq n\).

(iv) First transport laws: for \(1 \leq j \leq n\)
\[1^{n,+}_{n+1,j}(a \circ^n_j b) = \begin{bmatrix} 1^2_{n+1,j}(a) & 1^2_{n,j}(a) \\ 1^2_{n+1,j+1}(a) & 1^2_{n,j+1}(b) \end{bmatrix}\]

(v) Second transport laws: for \(1 \leq j \leq n\)
\[1^{n,-}_{n+1,j}(a \circ^n_j b) = \begin{bmatrix} 1^2_{n+1,j}(a) & 1^2_{n,j+1}(a) \\ 1^2_{n+1,j+1}(b) & 1^2_{n,j+1}(b) \end{bmatrix}\]

(vi) for \(1 \leq j \leq n\), \(1^{n,+}_{n+1,i}(x) \circ^{n+1}_{i} 1^{n,-}_{n+1,i}(x) = 1^{n+1}_{n+1,i+1}(x)\) and \(1^{n,+}_{n+1,i}(x) \circ^{n+1}_{i+1} 1^{n,-}_{n+1,i}(x) = 1^n_{n+1,i}(x)\)

The category \(\infty\text{-CCAT}\) of strict cubical \(\infty\)-categories is the full subcategory of \(\infty\text{-C\text{-}Mag}\), spanned by strict cubical \(\infty\)-categories. A morphism in \(\infty\text{-CCAT}\) is called a strict cubical \(\infty\)-functor. We study it more specifically in \(\text{6}\) with the perspective to weakened it and to obtain cubical model of weak \(\infty\)-functors.

### 2.2 The monad of cubical strict \(\infty\)-categories

In this section we describe cubical strict \(\infty\)-categories as algebras for a monad on \(\mathbb{C}\text{Sets}\). We hope it to be a specific ingredient to compare globular strict \(\infty\)-categories with cubical strict \(\infty\)-categories.

Consider the forgetful functor: \(\infty\text{-CCAT} \xrightarrow{\text{U}} \mathbb{C}\text{Sets}\) which associate to any strict cubical \(\infty\)-category its underlying cubical set and which associate to any strict cubical \(\infty\)-functor its underlying morphism of cubical sets.

**Proposition 1** The functor \(\text{U}\) is right adjoint.

Its left adjoint is denoted \(F\).

**Proof** The proof is very similar to those in [34]: Actually it is not difficult to see that the category \(\infty\text{-CCAT}\) and the category \(\mathbb{C}\text{Sets}\) are both projectively sketchable. Let us denote by \(\mathcal{E}_C\) the sketch of \(\infty\text{-CCAT}\) and \(\mathcal{E}_S\) the sketch of \(\mathbb{C}\text{Sets}\). Main parts of \(\mathcal{E}_C\) are described just below and we see that \(\mathcal{E}_C\) contains \(\mathcal{E}_S\), and that this inclusion induces a forgetful functor \(\infty\text{-CCAT} \xrightarrow{\text{U}} \mathbb{C}\text{Sets}\) which has a left adjunction thanks to the sheafification theorem of Foltz [20]. Now we have the commutative diagram

\[
\begin{array}{ccc}
\text{Mod}(\mathcal{E}_C) & \xrightarrow{\text{iso}} & \text{Mod}(\mathcal{E}_S) \\
\text{iso} & & \text{iso} \\
\infty\text{-CCAT} & \xrightarrow{\text{U}} & \mathbb{C}\text{Sets}
\end{array}
\]

which shows that \(\text{U}\) is right adjoint.

The description of \(\mathcal{E}_C\) started with the description of \(\mathcal{E}_M\) in \(\text{2}\). We carry on to it in describing the sketch behind the interchange laws, which shall complete main parts of \(\mathcal{E}_C\):

- In the first generation of \(\mathcal{E}_C\) we start with three cones:
Then we consider the following commutative diagrams:
We consider then (still in the first generation) the following two commutative diagrams:

Finally we consider the following commutative diagram of interchange laws:

The monad of strict cubical ∞-categories on cubical sets is denoted \( S = (S, \lambda, \mu) \). Here \( \lambda \) is the unit map of \( S : \mathbf{1}_{\mathbf{S}ets} \to S \) and \( \mu \) is the multiplication of \( S : S^2 \to S \).

### 3 The category of cubical weak ∞-categories

#### 3.1 The category of cubical categorical stretchings

We defined the category \( \infty\text{-CMag}_r \) of cubical reflexive \( \infty \)-magmas in \( \mathbf{2} \). With objects of this category plus cubical strict \( \infty \)-categories, we are going to define the category \( \infty\text{-CtC} \) of cubical categorical stretchings. This category is the key to weakened cubical strict \( \infty \)-categories as it was done in \( \mathbf{34} \) for the globular setting. Our cubical weak \( \infty \)-categories are algebraic in the sense that they are algebras (5) for a monad on \( \mathbf{CSets} \) which is build by using the category of cubical categorical stretchings. Our way to build the category \( \infty\text{-CtC} \) allow to weakened the whole structure of cubical strict \( \infty \)-categories. As we shall see, the central notion of cubical contractions (see below) are more subtle than globular contractions of \( \mathbf{34} \): in particular they must be thought with an inductive definition on the dimension \( n \) of the \( n \)-cells \( (n \in \mathbb{N}) \).

The category \( \infty\text{-CtC} \) of cubical categorical stretchings has as objects quintuples

\[
E = (M, C, \pi, ([\cdot; -]_{n+1, j})_{n \in \mathbb{N}, j \in \{1, \ldots, n+1\}}, ([\cdot; -]_{n+1, j})_{n \geq 1, j \in \{1, \ldots, n\}, \gamma \in \{-, +\}})
\]

where \( M \) is a cubical \( \infty \)-magma, \( C \) is a cubical strict \( \infty \)-category, \( \pi \) is a morphism in \( \infty\text{-CMag}_r \)

\[
M \xrightarrow{\pi} C
\]
and \(([-;\leq]_{n+1,j}^n)\) are extra structures called the cubical bracketing structures, and which are the cubical analogue of the key structure of the Penon approach to weakening the axioms of strict \(\infty\)-categories; it is for us the key structures which are going to weakening the axioms of cubical strict \(\infty\)-categories. Let be more precise about it:

For \(n \geq 1\) and for all integer \(k \geq 1\), consider the following subsets of \(M_n \times M_n\)

- \(M_{n,k} = \{(\alpha, \beta) \in M_n \times M_n : \pi_n(\alpha) = \pi_n(\beta)\}\)
- \(M_{n,k}^+ = \{(\alpha, \beta) \in M_n \times M_n : s_{n-1,j}^n(\alpha) = s_{n-1,j}^n(\beta) \wedge \pi_n(\alpha) = \pi_n(\beta)\}\)
- \(M_{n,k}^- = \{(\alpha, \beta) \in M_n \times M_n : t_{n-1,j}^n(\alpha) = t_{n-1,j}^n(\beta) \wedge \pi_n(\alpha) = \pi_n(\beta)\}\)

and also we consider \(M_0 = \{(\alpha, \beta) \in M_0 \times M_0 : \alpha = \beta\}\)

These extra structures are given by maps

\(([-;\leq]_{n+1,j}^n : M_n \to M_{n+1})\) such that

- If \(1 \leq i < j \leq n + 1\), then
  \[s_{n,i}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = [s_{n-1,i-n}^n(\alpha) ; \pi_n(\alpha)]_{n-1,j-n}^{n-1} \land t_{n,i}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = [t_{n-1,i-n}^n(\alpha) ; \pi_n(\alpha)]_{n-1,j-n}^{n-1}\]

- If \(1 \leq j < i \leq n + 1\)
  \[s_{n,j}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = [s_{n-1,i-n}^n(\alpha) ; \pi_n(\alpha)]_{n-1,j-n}^{n-1} \land t_{n,j}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = [t_{n-1,i-n}^n(\alpha) ; \pi_n(\alpha)]_{n-1,j-n}^{n-1}\]

- If \(i = j\)
  \[s_{n,i}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = \alpha \land t_{n,i}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = \beta\]

- \(\pi_{n,j}^1(\alpha ; \beta)_{n+1,j}^{n+1} = 1_{n+1,j}^1(\pi_n(\alpha)) \land \pi_{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = 1_{n+1,j}^1(\pi_n(\beta))\)

and also are given by maps

\(([-;\leq]_{n+1,j}^{-n} : M_{n,j}^- \to M_{n+1})\) and \(([-;\leq]_{n+1,j}^{+n} : M_{n,j}^+ \to M_{n+1})\)

such that

- for \(1 \leq j \leq n\) we have:
  - \(s_{n,j}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = \alpha \land s_{n,j+1}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = \beta\)
  - \(t_{n,j}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = \alpha \land t_{n,j+1}^{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = \beta\)

  - \([s_{n,i-j}^n(\alpha) ; \pi_n(\alpha)]_{n-1,j-n}^{n-1} \land [t_{n-i-j}^n(\alpha) ; \pi_n(\alpha)]_{n-1,j-n}^{n-1}\)

  - \(\pi_{n,j}^1(\alpha ; \beta)_{n+1,j}^{n+1} = 1_{n+1,j}^1(\pi_n(\alpha)) \land \pi_{n+1}(\alpha ; \beta)_{n+1,j}^{n+1} = 1_{n+1,j}^1(\pi_n(\beta))\)

  - \(\forall \alpha \in M_n, [\alpha, \alpha]_{n+1,j}^{n+1} = 1_{n+1,j}^1(\pi_n(\alpha))\).
A morphism of cubical categorical stretchings 

\[(m, c) : E \to E'\]
is given by the following commutative square in \(\infty\text{-CMag}_c\),

\[
\begin{array}{ccc}
M & \xrightarrow{m} & M' \\
\pi & \downarrow & \pi' \\
C & \xrightarrow{c} & C'
\end{array}
\]
such that for all \(n \in \mathbb{N}\), and for all \((\alpha, \beta) \in \widetilde{M}_n\),

\[m_{n+1}(\alpha, \beta)_{n+1,j} = [m_n(\alpha), m_n(\beta)]_{n+1,j} \quad (j \in \{1, ..., n + 1\})\]

and

\[m_{n+1}(\alpha, \beta)_{n+1,j} = [m_n(\alpha), m_n(\beta)]_{n+1,j} \quad (j \in \{1, ..., n\}, \gamma \in \{1, ..., n\})\]

The category of cubical categorical stretchings is denoted \(\infty\text{-CtC}\).

Now consider the forgetful functor: \(\infty\text{-CtC} \xrightarrow{U} \text{CSets}\) given by:

\[
(M, C, \pi, ([\vdash ; \vdash]_n_{n+1,j})_{n \in \mathbb{N}, j \in \{1, ..., n\}}; ([\vdash ; \vdash]_n_{n+1,j})_{n \in \mathbb{N}, j \in \{1, ..., n\}}, \gamma \in \{-, +\}) \xrightarrow{U} M ,
\]

**Proposition 2** The functor \(U\) just above has a left adjoint which produces a monad \(\mathcal{W} = (W, \eta, \nu)\) on the category of cubical sets.

**Proof** The proof is very similar to those in [24, 34]: Actually it is not difficult to see that the category \(\infty\text{-CtC}\) and the category \(\text{CSets}\) are both projectively sketchable. The sketch of cubical sets is denoted by \(\mathcal{E}_S\) (see 1.1) and the sketch of the cubical categorical stretchings is denoted by \(\mathcal{E}_E\). Main parts of this sketch is described just below, and we see that \(\mathcal{E}_E\) contains \(\mathcal{E}_S\), and is such that it induces a forgetful functor \(\infty\text{-CtC} \xrightarrow{U} \text{CSets}\) which has a left adjunction thanks to the sheafification theorem of Foltz [20]. Now we have the commutative diagram

\[
\begin{array}{ccc}
\text{Mod}(\mathcal{E}_E) & \xrightarrow{\text{Mod}(\mathcal{E}_S)_{\text{iso}}} & \text{Mod}(\mathcal{E}_S)_{\text{iso}} \\
\downarrow & & \downarrow \\
\infty\text{-CtC} & \xrightarrow{U} & \text{CSets}
\end{array}
\]

which shows that \(U\) is right adjoint.

Actually in 2 we described the sketch \(\mathcal{E}_M\) of cubical \(\infty\)-magmas, which where used to describe in 1 main part of the sketch \(\mathcal{E}_C\) of cubical strict \(\infty\)-categories. Thus we have already some part of the sketch \(\mathcal{E}_E\) that we complete by sketching operations \([\vdash ; \vdash]_n_{n+1,j}\) and \([\vdash ; \vdash]_{n+1,j}\) plus their axioms. With previous descriptions of sketches, and the one below, we shall see that we obtain the following inclusions of sketches:

\[
\mathcal{E}_S \xrightarrow{\mathcal{E}_M} \mathcal{E}_M \xrightarrow{\mathcal{E}_C} \mathcal{E}_C \xrightarrow{\mathcal{E}_E}
\]

**Description of \(\mathcal{E}_E\)**

- In the first generation we start with the following four cones:

\[
\begin{array}{ccc}
M_n & \xrightarrow{\pi^n_1} & M_n \\
\pi_0^n & \downarrow & \pi_n \\
M_n & \xrightarrow{\pi_n} & Z_n
\end{array}
\quad
\begin{array}{ccc}
M^n_{n+1,j} & \xrightarrow{\pi^n_{1,-j}} & M_{n} \\
\pi^n_0 & \downarrow & \pi^n_{0,-j} \\
M_n & \xrightarrow{\pi^n_{0,-j}} & Z_n
\end{array}
\quad
\begin{array}{ccc}
M^n_{n+1,j} & \xrightarrow{\pi^n_{1,-j}} & M_{n} \\
\pi^n_0 & \downarrow & \pi^n_{0,-j} \\
M_n & \xrightarrow{\pi^n_{0,-j}} & Z_n
\end{array}
\]
We consider the following commutative diagrams:

We have the following commutative diagrams which define the ”$s \times s$”:

- If $1 \leq i < j \leq n + 1$
  - for sources
    - $M_{n-1}$
    - $\pi_1^n$ 
    - $M_n$
  - for targets
    - $M_{n-1}$
    - $\pi_1^n$ 
    - $M_n$

If $1 \leq j < i \leq n + 1$
In the following diagrams we have \( \rho \in \{s, t\} \)

- If \( 1 \leq i < j \leq n \)
  - for sources
    
    \[
    \begin{array}{c}
    M_n \\
    \pi^n_0 \downarrow \\
    M_{n-1} \\
    \pi^{n-1}_0 \downarrow \\
    M_{n-2} \\
    \vdots \\
    \pi^{n-2}_{n-2} \downarrow \\
    M_1 \\
    \pi^{n-1}_{n-1} \downarrow \\
    M_0 \\
    \end{array}
    \]

- for targets
  
  \[
  \begin{array}{c}
  M_n \\
  \pi^n_0 \downarrow \\
  M_{n-1} \\
  \pi^{n-1}_0 \downarrow \\
  M_{n-2} \\
  \vdots \\
  \pi^{n-2}_{n-2} \downarrow \\
  M_1 \\
  \pi^{n-1}_{n-1} \downarrow \\
  M_0 \\
  \end{array}
  \]

- If \( 1 \leq j < i \leq n + 1 \)
  - for sources
    
    \[
    \begin{array}{c}
    M_n^\rho \\
    \pi^{n,\rho}_0 \downarrow \\
    M_{n-1, j-1}^\rho \\
    \pi^{n-1,\rho}_0 \downarrow \\
    M_{n-2, j-1} \\
    \vdots \\
    \pi^{n-2,\rho}_{n-2, j-1} \downarrow \\
    M_1^\rho \\
    \pi^{n-1,\rho}_{n-1, j-1} \downarrow \\
    M_0 \\
    \end{array}
    \]

- for targets
  
  \[
  \begin{array}{c}
  M_n^\rho \\
  \pi^{n,\rho}_0 \downarrow \\
  M_{n-1, j-1}^\rho \\
  \pi^{n-1,\rho}_0 \downarrow \\
  M_{n-2, j-1} \\
  \vdots \\
  \pi^{n-2,\rho}_{n-2, j-1} \downarrow \\
  M_1^\rho \\
  \pi^{n-1,\rho}_{n-1, j-1} \downarrow \\
  M_0 \\
  \end{array}
  \]
Second generation

- We consider the two cones:

\[ M_{\rho,n,j} \xrightarrow{\pi_{\rho,n,i-1}^{-1}} M_{\rho,n,i-1} \xrightarrow{\pi_{\rho,n,i}^{-1}} M_{\rho,n-1,j} \]

- for targets

\[ M_{\rho,n,j} \xrightarrow{\pi_{\rho,n,i-1}^{-1}} M_{\rho,n,i-1} \xrightarrow{\pi_{\rho,n,i}^{-1}} M_{\rho,n-1,j} \]

- for sources

\[ M_{\rho,n,j} \xrightarrow{\pi_{\rho,n,i-1}^{-1}} M_{\rho,n,i-1} \xrightarrow{\pi_{\rho,n,i}^{-1}} M_{\rho,n-1,j} \]

We denote \( k^- = j^n \circ \pi_{\rho,n}^- = j^n \circ \pi_{\rho,n}^- \) et \( k^+ = j^n \circ \pi_{\rho,n}^+ = j^n \circ \pi_{\rho,n}^+ \), and in this case we obtain the following commutative diagrams:

And we obtain the following commutative diagrams which give the definition of the \( s \times s \) for sets \( M_{\rho,n,j}^- \)

- If \( 1 \leq i < j \leq n \)
  - for sources
And we obtain the following commutative diagrams which give the definition of the $s \times s$ for sets $M_{n,j}^+$.

- If $1 \leq j < i \leq n$
  - for sources
    
  - for targets

- If $1 \leq i < j \leq n$
  - for sources
We consider the following commutative diagrams:

- If $1 \leq j < i \leq n + 1$
  - for sources
    
    \[
    \begin{array}{cccccc}
    M^n_{0,j} & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M_0 & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M_n & \xrightarrow{\pi^n_{i-1}} & M_{n-1} \\
    \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} \\
    M^n'_{0,j} & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M'_{0-1,j} & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M_{0-1} & \xrightarrow{\pi^n_{i-1}} & M_{n-1} \\
    \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} \\
    M^n''_{0,j} & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M''_{0-1,j} & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M''_{0-1,j} & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M_{n-1} \\
    \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} \\
    M^n'''_{0,j} & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M'''_{0-1,j} & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M'''_{0-1,j} & \xrightarrow{s^n_{0-1,i} \times s^n_{0-1,i}} & M_{n-1} \\
    \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} \\
    \end{array}
    \]

- for targets
    
    \[
    \begin{array}{cccccc}
    M^n_{+1,j} & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M_0 & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M_n & \xrightarrow{\pi^n_{i-1}} & M_{n-1} \\
    \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} \\
    M^n'_{+1,j} & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M'_{0-1,j} & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M_{0-1} & \xrightarrow{\pi^n_{i-1}} & M_{n-1} \\
    \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} \\
    M^n''_{+1,j} & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M''_{0-1,j} & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M''_{0-1,j} & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M_{n-1} \\
    \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} \\
    M^n'''_{+1,j} & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M'''_{0-1,j} & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M'''_{0-1,j} & \xrightarrow{t^n_{n-1,i} \times t^n_{n-1,i}} & M_{n-1} \\
    \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} & \downarrow & \pi^n_{0-1} \\
    \end{array}
    \]

- If $1 \leq i < j \leq n + 1$
• If $1 \leq j < i \leq n + 1$
  
  \[ M_{n-1} \xrightarrow{[-; ]_{n, j}^{n-1}} M_{n} \]
  
  \[ M_{n} \xrightarrow{[[-; ]_{n, j}^{n}} M_{n+1} \]

  \[ s_{n-1, i} \times s_{n-1, i}^{n-1} \]

  \[ M_{n, j}^{\gamma} \xrightarrow{[-; ]_{n, j}^{n-1}} M_{n+1} \]

• If $i = j$
  
  – for the operations $[[-; ]_{n+1, j}^{n}$

  \[ s_{n, j} \xrightarrow{[-; ]_{n+1, j}^{n}} M_{n+1} \]

  \[ M_{n} \xrightarrow{s_{0}} M_{n} \]

  \[ M_{n} \xrightarrow{s_{1}} M_{n+1} \]

  \[ \pi \]

  \[ \pi \]

  \[ e_{n, j}^{n+1} \]

  \[ e_{n, j}^{n+1} \]

  \[ \sigma_{n, j}^{n+1} \]

  \[ \sigma_{n, j}^{n+1} \]

  \[ q_{0}^{n} \]

  \[ q_{1}^{n} \]

  – Other possible diagram for this definition:
Commutative diagrams for axioms:

- For operations $[-; -]_{m+1,j}^{n}$

\[
\begin{array}{c}
M_n \\
\downarrow \pi_0 \\
M_n \\
\downarrow \pi_n \\
Z_n \xrightarrow{\iota_{n+1}} Z_{n+1}
\end{array}
\quad
\begin{array}{c}
M_{n+1} \\
\downarrow \pi_{n+1} \\
M_{n+1} \\
\downarrow \pi_n \\
Z_{n+1}
\end{array}
\]

- For operations $[-; -]_{m+1,j}^{n,\gamma}$

\[
\begin{array}{c}
M_n \\
\downarrow q_{0,\gamma}^{n,\gamma} \\
M_n \\
\downarrow \pi_n \\
Z_n \xrightarrow{\iota_{n+1}^{n,\gamma}} Z_{n+1}
\end{array}
\quad
\begin{array}{c}
M_{n+1} \\
\downarrow q_{1,\gamma}^{n,\gamma} \\
M_{n+1} \\
\downarrow \pi_{n+1} \\
Z_{n+1}
\end{array}
\]

Come back to the first generation

We build the diagonal with the following commutative diagrams:

\[
\begin{array}{c}
M_n \\
\downarrow \pi_0 \\
M_n \\
\downarrow \pi_n \\
Z_n
\end{array}
\quad
\begin{array}{c}
M_n \\
\downarrow \pi_1 \\
M_n \\
\downarrow \pi_n \\
Z_n
\end{array}
\]
Comeback to the second generation

The previous diagrams generate the following commutative diagrams:

Then we obtain the following commutative diagrams of first generation for axioms of reflexivity of the operations 

And the following commutative diagrams of second generation for axioms of reflexivity of the operations 

Definition 5 Cubical weak \(\infty\)-categories are algebras for the monad \(\mathcal{W}\) above.

Let us show with a simple example how cubical weak \(\infty\)-categories provide a richer weakened structure than the one of globular weak \(\infty\)-categories: for simplicity we show it inside an object \(\mathcal{E}\) of \(\infty\text{-}\text{CEtC}\):

\[
\mathcal{E} = (M, C, \pi, ([-; -]_{n+1,j})_{n \in \mathbb{N}, j \in \{1, \ldots, n+1\}}, ([{-}; {-}]_{n+1,j})_{n \in \mathbb{N}, j \in \{1, \ldots, n\}, \gamma \in \{-, +\}})
\]

Consider the following string in \(M(1)\)

and take the 1-cells \(x = (h \circ g) \circ f\) and \(y = h \circ (g \circ f)\). Because these cells belong to \(M_{1,0}^- \cap M_{1,0}^+\) we get the following 2-cells

\[
\begin{array}{cccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]
Remark 2 We could have defined cubical categorical stretchings slightly differently than those just above by means of using just the operations \([[-; -]_{n+1,j}]_{n \in N,j \in \{1, ... , n\}}\) to weaken the structure of cubical strict \(\infty\)-categories. Denote by \(\infty\text{-CETC}'\) the category of those slightly impoverished structures. We also have a forgetful functor \(\infty\text{-CETC}' \xrightarrow{U'} \text{CSets}\) which is right adjoint and which produce an other monad \(\mathcal{W}' = (W', \eta', \nu')\) which algebras could be also considered as enough good models of cubical weak \(\infty\)-categories. Also we have an evident forgetful functor \(\infty\text{-CETC} \xrightarrow{U} \infty\text{-CETC}'\) which is right adjoint and which produce a functor \(\mathcal{W}\text{-Alg} \xrightarrow{U} \mathcal{W}'\text{-Alg}'\) which shows that the models that we have chosen for our article are also models for these impoverished structures. And our choice to add operations \([[-; -]_{n+1,j}]_{n \in N,j \in \{1, ... , n\}, \gamma \in \{-, +\}}\) to get our models of cubical weak \(\infty\)-categories is similar to the one who choses cubical strict \(\infty\)-categories with connections instead of considering it without connections. We believe that our choice gives not only more refined models than those of the category \(\mathcal{W}'\text{-Alg}'\) but also is in fact really necessary for a good approach of cubical weak \(\infty\)-categories, where formalism of connections are implicit and used in our weakened structures.

Remark 3 In [5] the authors have proved that the category of cubical strict \(\infty\)-categories is equivalent to the category of globular strict \(\infty\)-categories. We suspect that such phenomena is still right in the world of weak models. Let us be more precise about what we are saying : denote by \(\mathcal{P}\) the Penon’s monad on the category of globular sets (see [34]) which algebras are particularly nice models of globular weak \(\infty\)-categories (see for example [4, 14]). It is suspected (see [39]) that its category of algebras \(\mathcal{P}\text{-Alg}\) can be equipped with a canonical Quillen model structure similar to the one build in [33] for strict globular \(\infty\)-categories, and we also suspect that \(\mathcal{W}\text{-Alg}\) can be equipped with such canonical Quillen model structure. Thus a weak version of the article [5] should be that the category \(\mathcal{P}\text{-Alg}\) is Quillen equivalent to \(\mathcal{W}\text{-Alg}\) when these categories are equipped with their canonical Quillen model structure.

3.2 Magmatic properties of cubical weak \(\infty\)-categories

Consider a cubical weak \(\infty\)-category \(W(C) \xrightarrow{\nu} C\). In this monadic presentation, \(W(C)\) has to be thought as the free cubical weak \(\infty\)-category representing the underlying syntax with which all algebras with underlying cubical set \(C\) are interpretations of it via their morphisms structural. For example here \(\nu\) is the morphism structural which plays the role of interpreting in \(C\) the "syntax" \(W(C)\), and thus put on \(C\) a structure of \(\mathcal{W}\)-algebra. We shall distinguished well notations of operations inside \(W(C)\) and inside \(C\) in order to separate the syntactic part from the model part of our algebras. For example the operations of compositions shall be denoted \(\circ^\gamma\) in the models, whereas we shall use the notation \(\star^\gamma\) instead when we work in the free models. The reflections are denoted \(\iota^\gamma_{n+1,j}\) in the models and \(\iota^\gamma_{n+1,j}\) in the free models. The connections are denoted \(\iota^\gamma_{n,\alpha}\) in the models and \(\iota^\gamma_{n,\alpha,j}\) in the free models. Definitions of operations for models use those for free models and the interpretative nature of \(W(C) \xrightarrow{\nu} C\) emerges then with the axiomatic of algebras for monads : for example we consider first the following definition of operations on \(C\):

(i) If \(a, b \in C(n)\) are such that \(s^\gamma_j(b) = \iota^\gamma_j(a)\) for \(j \in \{1, ..., n\}\) then we put \(a \circ^\gamma b = \nu_n(\eta(a) \star^\gamma \eta(b))\)
(iii) If \( a \in C(n) \) is an \( n \)-cell then we put \( \iota^n_{n+1,j}(a) = \nu_{n+1}(1^n_{n+1,j}(\eta(a))), n \in \mathbb{N}, j \in [1, n + 1] \)

(iii) If \( a \in C(n) \) is an \( n \)-cell then we put \( \iota^n_{n+1,j}(a) = \nu_{n+1}(1^n_{n+1,j}(\eta(a))), n \geq 1, j \in [1, n], \gamma \in \{-, +\} \)

Thus \( v \) puts on \( C \) a cubical \( \infty \)-magma structure and its interpretative nature is primarily expressed by the fact that it is a morphism of cubical \( \infty \)-magnas between the free cubical \( \infty \)-magma \( W(C) \) and this cubical \( \infty \)-magma on \( C \). It is the axioms of algebras which show us such important fact: actually we need to show that

\[
v(a \ast^n_b b) = v(a) \circ^n_b v(b), \quad (1^n_{n+1,j}(a)) = \iota^n_{n+1,j}(v(a)), \quad v(1^n_{n+1,j}(a)) = \iota^n_{n+1,j}(v(a)). \]

Let us show the first equality:

\[
v(a) \circ^n_b v(b) = v(\eta(v(a))) \circ^n_b \eta(v(b)) = v(W(\nu_1(W(C)(a)) \ast^n_b W(\nu_1(W(C)(b))))
\]

\[
= v(\nu_1(W(C)(a)) \ast^n_b \nu_1(W(C)(b))) \quad \text{for reflexion if its underlying reflexive set is an}\]

\[
\infty\text{-magma structure and its interpretative nature is primarily expressed by the}
\]

\[
\text{axioms of algebras which show us such important fact: actually we need to show that}
\]

\[
\text{\( v(a \ast^n_b b) = v(a) \circ^n_b v(b), \quad (1^n_{n+1,j}(a)) = \iota^n_{n+1,j}(v(a)), \quad v(1^n_{n+1,j}(a)) = \iota^n_{n+1,j}(v(a)). \) Let us show the first equality:}
\]

\[
v(a) \circ^n_b v(b) = v(\eta(v(a))) \circ^n_b \eta(v(b))
\]

\[
= v(W(\nu_1(W(C)(a)) \ast^n_b W(\nu_1(W(C)(b))))
\]

\[
= v(\nu_1(W(C)(a)) \ast^n_b \nu_1(W(C)(b)))
\]

\[
= v(\nu_1(W(C)(a)) \ast^n_b \nu_1(W(C)(b)))
\]

\[
= v(\nu_1(W(C)(a)) \ast^n_b \nu_1(W(C)(b)))
\]

\[
= v(a \ast^n_b b)
\]

Other equalities are shown similarly. In [34] J. Penon called magmatic such properties of algebras. In particular these shall be useful for concrete computations in any \( \mathcal{W} \)-algebras.

### 3.3 Computations for low dimensions

**Definition 6** Consider a reflexive cubical set \( C \in \mathcal{C}, \text{Sets} \). It has dimension \( p \in \mathbb{N} \) for reflexions if all its \( q \)-cells \( x \in C(q) \) for which \( q > p \) are of the form \( x = 1^{q-1}_{q,j}(y) \) and if there is at least one \( p \)-cell which is not of this form. It has dimension \( p \in \mathbb{N} \) for connections if all its \( q \)-cells \( x \in C(q) \) for which \( q > p \) are of the form \( x = 1^{q-1}_{q,j}(y) \) and if there is at least one \( p \)-cell which is not of this form. It has dimension \( p \in \mathbb{N} \), if it has dimension \( p \in \mathbb{N} \) for reflexions and connections.

**Definition 7** Consider a \( \mathcal{W} \)-algebra \((C, v)\). It has dimension \( p \in \mathbb{N} \) for reflexion if its underlying reflexive set produced by its underlying \( \infty \)-magma structure (see 3.2) has dimension \( p \in \mathbb{N} \) for reflexion. It has dimension \( p \in \mathbb{N} \) for connection if its underlying reflexive set produced by its underlying \( \infty \)-magma structure has dimension \( p \in \mathbb{N} \) for connections. It has dimension \( p \in \mathbb{N} \), if it has dimension \( p \in \mathbb{N} \) for reflexions and connections.

**Definition 8** A cubical bicategory is a 2-dimensional \( \mathcal{W} \)-algebra.

### 4 Cubical \((\infty, m)\)-sets

Globular \((\infty, m)\)-sets\(^6\) have been defined in [25, 26] and represent main parts of the underlying sketches for globular models of \((\infty, m)\)-categories: it algebraically formalize the idea of inverses, inverses of inverses, etc. that is, thanks to these structures, ideas of inverses are encoded with operations and thus give very elegant and tractable algebraic models of \((\infty, m)\)-categories (see [25]). For example we obtain models of \((\infty, m)\)-categories which are projectively sketchable, and this elegant categorical property is not clear for such models when it is build with simplicial methods. Also thanks to a result in [4] globular models that we obtain in [25] are algebras for the Batanin's operad (see [3, 25]) and these models of globular weak \( \infty \)-groupoids are probably very close to those proposed by Grothendieck (see [3, 32]).

Here we define cubical version of the formalism developed in [25]. This formalism of this cubical world is very similar to its globular world analogue however it is important to notice that other sketches are possible (see 5).

Consider a cubical set \( C = (C_n, s^n_{n-1,j}, t^n_{n-1,j}, 1 \leq j \leq n) \). If \( k \geq 1 \) and \( 1 \leq j \leq k \), then a \((k, j)\)-reversor on it is given by a map \( C_k \xrightarrow{j_k} C_k \) such that the following two diagrams commute:

---

\(^6\) Also called \((\infty, m)\)-graphs or \((\infty, m)\)-globular sets.
If for each \( k > m \) and for each \( 1 \leq j \leq k \), there are such \((k, j)\)-reversor \( j^k_j \) on \( C \), then we say that \( C \) is a cubical \((\infty, m)\)-set. The family of maps \((j^k_j)_{k>m, 1 \leq j \leq k}\) is called an \((\infty, m)\)-structure and in that case we shall say that \( C \) is equipped with the \((\infty, m)\)-structure \((j^k_j)_{k>m, 1 \leq j \leq k}\). Seen as cubical \((\infty, m)\)-set we denote it by \( C = ((C_n, s^n_{n-1,j}, f^n_{n-1,j})_{1 \leq j \leq n}, (j^k_j)_{k>m, 1 \leq j \leq k}) \). If \( C' = ((C'_n, s'^{n}_{n-1,j}, f'^{n}_{n-1,j})_{1 \leq j \leq n}, (j'^k_j)_{k>m, 1 \leq j \leq k}) \) is another \((\infty, m)\)-set, then a morphism of \((\infty, m)\)-sets

\[
 C \xrightarrow{f} C'
\]

is given by a morphism of cubical sets such that for each \( k > m \) and for each \( 1 \leq j \leq k \) we have the following commutative diagrams

![Diagram](image)

The category of cubical \((\infty, m)\)-sets is denoted \((\infty, m)\)-CSets

**Remark 4** This structural approach of inverses is much more powerful that the simplicial methods because with it we are able to build any kind of reversible higher structure. For example in our framework it is a simple exercise to build some exotic one which could be difficult to be build with simplicial method. For example

**Remark 5** The \((\infty, m)\)-structures that we used to define cubical \((\infty, m)\)-sets have globular analogues (see [25]) that we called the minimal \((\infty, m)\)-structures. The cubical analogue of the globular maximal \((\infty, m)\)-structures as defined in [25] is as follow : for all \( k > m \) and for each \( i_{m+1}, i_{m+2}, ..., i_k \) such that \( 1 \leq i_k \leq k, 1 \leq i_{k-1} \leq k-1, ..., 1 \leq i_{m+2} \leq m+2, 1 \leq i_{m+1} \leq m + 1 \), we have two diagrams in Sets each commuting serially :

![Diagram](image)

Such datas \((j^k_{i_k})_{k>m, 1 \leq i_k \leq k}\) is called a cubical maximal \((\infty, n)\)-structure.
More generally, an \((\infty, m)\)-structure is given by the following datas: For each \(k > m\), there exist \(p : m + 1 \leq p < k\) and diagrams:

with \(1 \leq i_j \leq j\) for all \(j \in \{p + 1, \ldots, k\}\). These datas \(((j^k_i)_{k>m,m+1 \leq p<k,l \in \{p+1, \ldots, k\}, 1 \leq i \leq l})\) are called an \((\infty, m)\)-structure. If we take \(p = m + 1\) and all \(i_j\) in the interval \([1; l]\) for all \(l \in \{m + 1, \ldots, k\}\) then we recover the maximal \((\infty, m)\)-structure \((j^k_i)_{k>m,m+1 \leq i \leq k}\) just above, and if for \(k > m\) we take \(p = k - 1\) we recover the minimal \((\infty, m)\)-structure. Actually we could define models of cubical weak \((\infty, m)\)-categories (5.2) with any of such cubical \((\infty, m)\)-structure, because as for the globular case, we have a strong suspicion that such models are equivalents through abstract homotopy theory.

\section{Cubical weak \((\infty, m)\)-categories, \(m \in \mathbb{N}\)}

They are few works on cubical weak \(\infty\)-categories with kind of inverses involved. However in low dimensions, simplicial methods have been used in \([6, 7, 9, 10, 13, 17, 37]\) to study it. Some applications of it to homology have been considered in \([2, 8, 9, 10]\), and other applications in algebraic topology have also been carried out in \([11, 21, 36]\).

As we said in the beginning of this article our first scope is to provide these higher cubical notions with the perspective to carry on applications of cubical higher categories to homological algebra, algebraic topology and computer sciences.

\subsection{Cubical strict \((\infty, m)\)-categories, \(m \in \mathbb{N}\)}

Cubical strict \((\infty, m)\)-categories are just cubical strict \(\infty\)-categories such that all \(k\)-cells for \(k > m\) are \(\phi^k_i\)-isomorphisms for \(1 \leq i \leq k\). The studies of cubical strict structures using inverses has been done in \([6, 7, 9, 10, 13, 17, 30, 37]\), especially for the scope to generalize many known results involving low dimensional groupoids and algebraic topology. For example generalization of cubical strict fundamental groupoid to higher dimensions has been undertaken in \([8, 9]\) in order to obtain higher version of Van Kampen type Theorem.

In this article we use cubical strict \((\infty, m)\)-categories as an underlying part of the structure of the cubical \((\infty, m)\)-categorical stretchings (see 5.2) which are the adapted stretchings to weakened cubical strict \((\infty, m)\)-categories. Thus they are an important step for our approach of cubical weak \(\infty\)-categories.

Consider a cubical strict \(\infty\)-category \(\mathcal{C}\) as defined in 2. We say that it is a cubical strict \((\infty, m)\)-category if its underlying cubical set is equipped with an \((\infty, m)\)-structure \((j^k_i)_{k>m,1 \leq i \leq k}\) such that we require the following identities: \(\forall j, k\) such that \(1 \leq j \leq k\) and \(m \leq m < k\),

\[\forall \alpha \in \mathcal{C}_k, \ \alpha \circ j^k_i(\alpha) = 1^1_{k,j}(s^{k-1}_{k-1,j}(\alpha)) \text{ and } j^k_i(\alpha) \circ \alpha = 1^1_{k,j}(s^{k-1}_{k-1,j}(\alpha))\]

\textbf{Proposition 3} A cubical strict \((\infty, m)\)-category \(\mathcal{C}\) as above has a unique underlying \((\infty, m)\)-set.
Remark 6 Cubical strict \((\infty, m)\)-categories are richer that globular strict \((\infty, m)\)-categories. For example in \([30]\) the author has shown that they are others and equivalent ways (see also \([27]\)) to define inverses for strict \((\infty, m)\)-categories, and it seems then that we can imagine other \((\infty, m)\)-structures which could lead to other approach of algebraic models of cubical weak \((\infty, m)\)-categories as defined below 5.2.

A cubical strict \(\infty\)-functor preserve \((k, j)\)-reversors. Thus morphisms between cubical strict \((\infty, m)\)-categories are just cubical strict \(\infty\)-functors. The category of cubical strict \((\infty, m)\)-category is denoted \((\infty, m)\)-\text{CCat}. As in 2.2 it is not difficult to show the following proposition

**Proposition 4** The evident forgetful functor

\[
(\infty, m)\text{-CCat} \xrightarrow{U} \text{CSets}
\]

is right adjoint and monadic. \(\qed\)

The monad of cubical strict \((\infty, m)\)-category on cubical sets is denoted \(S^m = (S^m, \lambda^m, \mu^m)\). Here \(\lambda^m\) is the unit map of \(S^m : 1_{\text{CSets}} \xrightarrow{\lambda^m} S^m\) and \(\mu^m\) is the multiplication of \(S^m : (S^m)^2 \xrightarrow{\mu^m} S^m\)

### 5.2 The category of cubical weak \((\infty, m)\)-categories, \(m \in \mathbb{N}\)

A cubical reflexive \((\infty, m)\)-magma is an object of \(\infty\text{-Commag}\), such that its underlying cubical set is equipped with an \((\infty, m)\)-structure. Morphisms between cubical reflexive \((\infty, m)\)-magnas are those of \(\infty\text{-Commag}\) which are also morphisms of \((\infty, m)\)-\text{CSets}, i.e they preserve the underlying \((\infty, m)\)-structures. The category of cubical reflexive \((\infty, m)\)-magnas is denoted \((\infty, m)\)-\text{Commag}.

The category \((\infty, m)\)-\text{CtC} of cubical reflexive \((\infty, m)\)-categorical stretchings has as objects quintuples

\[
E = (M, C, \pi, ([-; -]^n_{n+1,j})_{n \in \mathbb{N}, j \in \{1, ..., n\}}, ([-; -]^n_{n+1,j})_{n \in \mathbb{N}, j \in \{1, ..., n\}, \gamma \in \{-, +\}})
\]

where \(M\) is a cubical reflexive \((\infty, m)\)-magma, \(C\) is a cubical strict \((\infty, m)\)-category, \(\pi\) is a morphism in \((\infty, m)\)-\text{Commag},

\[
M \xrightarrow{\pi} C
\]

and \(([[-; -]^n_{n+1,j})_{n \in \mathbb{N}, j \in \{1, ..., n\}}, ([[-; -]^n_{n+1,j})_{n \in \mathbb{N}, j \in \{1, ..., n\}, \gamma \in \{-, +\}}\) are the cubical bracketing structures which have already been defined in 3. A morphism of cubical \((\infty, m)\)-categorical stretchings

\[
E \xrightarrow{(m, c)} E'
\]

is given by the following commutative square in \((\infty, m)\)-\text{Commag},

\[
\begin{array}{ccc}
M & \xrightarrow{m} & M' \\
\pi & \downarrow & \pi' \\
C & \xrightarrow{c} & C'
\end{array}
\]

such that for all \(n \in \mathbb{N}\), and for all \((\alpha, \beta) \in \overline{M}_n\),

\[
m_{n+1}((\alpha, \beta)^n_{n+1,j}) = [m_n(\alpha), m_n(\beta)]^n_{n+1,j} \quad (j \in \{1, ..., n + 1\})
\]

and

\[
m_{n+1}((\alpha, \beta)^n\gamma_{n+1,j}) = [m_n(\alpha), m_n(\beta)]^{n\gamma}_{n+1,j} \quad (j \in \{1, ..., n\}, \gamma \in \{1, ..., n\})
\]

The category of cubical \((\infty, m)\)-categorical stretchings is denoted \((\infty, m)\)-\text{CtC}.

Now consider the forgetful functor:

\[
(\infty, m)\text{-CtC} \xrightarrow{U} \text{CSets}
\]

given by

\[
(M, C, \pi, ([-; -]^n_{n+1,j})_{n \in \mathbb{N}, j \in \{1, ..., n\}}, ([-; -]^n_{n+1,j})_{n \in \mathbb{N}, j \in \{1, ..., n\}, \gamma \in \{-, +\}}) \xrightarrow{U} M
\]

This functor has a left adjoint which produces a monad \(\mathcal{W}^m = (\mathcal{W}^m, \eta^m, \mu^m)\) on the category of cubical sets.
Definition 9 Cubical weak $(\infty, m)$-categories are algebras for the monad $W^m$ above.

Thus the category of our models of cubical weak $(\infty, m)$-categories is denoted $W^m$-$\Alg$.

If it is evident to see that we have "an embedded" of $(\infty, m)$-$\CCat$ in $\infty$-$\CCat$, and this also the case for the weak case, even it is more subtle: It comes from the forgetful functor

$$\begin{array}{ccc}
(\infty, m)$-$\CCat & \xrightarrow{U} & \infty$-$\CCat \\
\xrightarrow{F} & \xleftarrow{id} & \\
\text{Sets} & \xrightarrow{U^m} & \text{Sets}
\end{array}$$

which forgets the underlying $(\infty, m)$-structures. Also we have the following morphism of the category $\Adj$ of adjunctions:

$$\begin{array}{ccc}
(\infty, m)$-$\CCat & \xrightarrow{V^*} & (\text{Sets}^\text{op}, W^m) \\
\xrightarrow{F} & \xleftarrow{U} & \\
\text{Sets} & \xrightarrow{\text{id}} & \text{Sets}
\end{array}$$

because $U \circ V = U^m$ (see [24]), thus it produces a morphism $V^*$ in the category $\text{Alg}$ of monads

$$(\text{Sets}^\text{op}, W) \xrightarrow{V^*} (\text{Sets}^\text{op}, W^m)$$

and passing to algebras, gives the following functor $\text{Alg}(V^*)$ which is the "embedded" we were looking for

$$W^m$-$\Alg \xrightarrow{\text{Alg}(V^*)} W$-$\Alg$$

Also each $m > 0$ we have an evident forgetful functor $(\infty, m-1)$-$\CCat \xrightarrow{V^{m-1}} (\infty, m)$-$\CCat$ and by using the same technology as just above we obtain the following filtration in $\text{CAT}$

$$W^0$-$\Alg \xrightarrow{\cdots} W^m$-$\Alg$$

However it is important to notice that the filtered colimit of it in $\text{CAT}$ doesn’t give $W$-$\Alg$, even up to equivalence of categories. See [25] for a discussion of a similar phenomena for the globular approach.

Definition 10 Cubical weak $(\infty, 0)$-categories are our models of cubical weak $\infty$-groupoids. In particular our models are algebraic, in the sense that these cubical weak $\infty$-groupoids are algebras for the monad $W^0 = (W^0, \eta^0, \nu^0)$ defined above on the category $\text{Sets}$ of cubical sets.

Also the dimension of $W^m$-algebras is defined as for $W$-algebras (see 3.3). We now propose a definition of cubical bigroupoids.

Definition 11 A cubical bigroupoid is a 2-dimensional $W^0$-algebra.

6 The category of cubical weak $\infty$-functors

In [35] Jacques Penon had proposed algebraic models of globular weak $\infty$-functors which were extended to all kind of globular weak higher transformations in [24]. The methods used in [24, 35] has consisted to use different kind of stretchings used to weakened different kind of strict structure. For example in [35] he build a category of stretchings named in [24] the category of $(0, \infty)$-categorical stretchings which were adapted to weakened strict $\infty$-functors. And in [24] the author used the category of $(n, \infty)$-categorical stretchings to weakened all kind of globular strict $n$-transformations for all $n \geq 2$ (strict natural $\infty$-transformations correspond to $n = 2$ and strict $\infty$-modifications correspond to $n = 3$, etc.) As we are going to see, our models of cubical weak $\infty$-functors are build with similar technology: we are going to define cubical functorial stretchings which contains $(0, \infty)$-categorical stretchings must not be confused with $(\infty, 0)$-categorical stretchings used in 5.2 to weakened cubical strict $(\infty, 0)$-categories and which were used in [25] to weakened globular strict $(\infty, 0)$-categories.
all "informations" of the structure behind cubical weak $\infty$-functors. This structure produces a monad on the category $\mathbb{C}Sets \times \mathbb{C}Sets$ which algebras are our models of cubical weak $\infty$-functors. In 7 we shall investigate similar constructions but for cubical weak natural $\infty$-transformations.

Cubical strict $\infty$-functors have been defined in 2.1. A morphism between two cubical strict $\infty$-functors $C \xrightarrow{F} D$ and $C' \xrightarrow{F'} D'$ is given by a commutative 2-cube in $\mathbb{C}CAT$

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
C \\
F
\end{array}
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
D \\
F
\end{array}
\end{array}
\end{array}
\]

The category of cubical strict $\infty$-functors is denoted $\infty$-$\mathbb{C}Fun$. 

6.1 The category of cubical $(0, \infty)$-magmas

A cubical $(0, \infty)$-magma is given by a morphism $M_0 \xrightarrow{F_M} M_1$ of $\mathbb{C}Sets$ such that $M_0$ and $M_1$ are objects of $\infty$-$\mathbb{C}Mag$. Such object is denoted $(M_0, F_M, M_1)$. A morphism between $(0, \infty)$-magmas $(M_0, F_M, M_1) \xrightarrow{m} (M_0', F_M', M_1')$ is given by two morphisms of $\infty$-$\mathbb{C}Mag$:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
M_0 \\
M_0'
\end{array}
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
M_1 \\
M_1'
\end{array}
\end{array}
\end{array}
\]

such that the following diagram commutes in $\mathbb{C}Sets$

The category of cubical $(0, \infty)$-magmas is denoted by $(0, \infty)$-$\mathbb{C}Mag$. 

6.2 The category of cubical $(0, \infty)$-categorical stretchings

A $(0, \infty)$-stretching is given by a triple $E = (E_0, E_1, F_M, F_C)$ such that $E_0$, $E_1$ are cubical categorical stretchings given by

\[
E_0 = (M_0, C_0, \pi_0, (0^-; -^{|n|} \mathbb{N}^+_{n+1,j})_{n \in \mathbb{N}, j \in \{1, \ldots, n\}}, (0^-; -^{|n|} \mathbb{N}^+_{n+1,j})_{n \in \mathbb{N}, j \in \{1, \ldots, n\}, \gamma \in \{-, +\}})
\]

and

\[
E_1 = (M_1, C_1, \pi_1, (1^-; -^{|n|} \mathbb{N}^+_{n+1,j})_{n \in \mathbb{N}, j \in \{1, \ldots, n\}}, (1^-; -^{|n|} \mathbb{N}^+_{n+1,j})_{n \in \mathbb{N}, j \in \{1, \ldots, n\}, \gamma \in \{-, +\}})
\]

$(M_0, F_M, M_1)$ is an object of $(0, \infty)$-$\mathbb{C}Mag$, $C_0 \xrightarrow{F_C} C_1$ is a strict cubical $\infty$-functor, such that the following square is commutative in $\mathbb{C}Sets$:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
M_0 \\
F
\end{array}
\end{array}
\end{array}
\]
\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
M_1 \\
F
\end{array}
\end{array}
\end{array}
\]
A morphism of $(0, \infty)$-stretchings

$$E = (E_0, E_1, F_M, F_C) \xrightarrow{U} E' = (E'_0, E'_1, F'_M, F'_C)$$

is given by the following commutative diagram in $\mathcal{C}Set$s:

\[
\begin{array}{cccc}
M_0 & \xrightarrow{F_M} & M_1 \\
\downarrow{\pi_0} & & \downarrow{\pi_1} \\
C_0 & \xrightarrow{F_C} & C_1
\end{array}
\]

such that $(m_0, m_1)$ is a morphism of $(0, \infty)$-Cat, $(m_0, c_0)$ and $(m_1, c_1)$ are morphisms of $\infty\mathcal{C}EtC$. The category of $(0, \infty)$-stretchings is denoted $(0, \infty)\mathcal{C}EtC$.

Now consider the forgetful functor:

\[
(0, \infty)\mathcal{C}EtC \xrightarrow{U} \mathcal{C}Set \times \mathcal{C}Set
\]

given by

$$E = (E_0, E_1, F_M, F_C) \xrightarrow{U} (M_0, M_1)$$

This functor has a left adjoint which produces a monad $\mathbb{T}^0 = (T^0, \lambda^0, \mu^0)$ on the category $\mathcal{C}Set \times \mathcal{C}Set$.

**Definition 12** Cubical weak $\infty$-functors are algebras for the monad $\mathbb{T}^0$ above. □

Thus a cubical weak $\infty$-functor is given by a quadruple $(C_0, C_1, v_0, v_1)$ such that if we note $T^0(C_0, C_1) = (T^0_0(C_0, C_1), T^0_1(C_0, C_1))$ then we get its underling morphisms of $\mathcal{C}Set$s

\[
\begin{array}{ccc}
T^0_0(C_0, C_1) & \xrightarrow{v_0} & C_0 \\
T^0_1(C_0, C_1) & \xrightarrow{v_1} & C_1
\end{array}
\]

and these morphisms of $\mathcal{C}Set$s put on $(C_0, C_1)$ a structure of cubical weak $\infty$-functor $C_0 \xrightarrow{F} C_1$ defined by

$$F = v_1 \circ F_M \circ \lambda^0_0(C_0, C_1)$$

with
7 The category of cubical weak $\infty$-natural transformations

We finish this article on cubical higher category theory by building a monad on the category $$(\mathcal{CSe})^4 = \mathcal{CSe} \times \mathcal{CSe} \times \mathcal{CSe} \times \mathcal{CSe}$$ which algebras are our models of cubical weak natural $\infty$-transformations. In [24] we defined globular natural $\infty$-transformations by using the structure given by an adapted category of stretchings, namely the category of (globular) $(1, \infty)$-stretchings. Here we use similar technology by defining first the category of cubical $(1, \infty)$-stretchings which contains the underlying structure needed to weakened cubical strict natural $\infty$-transformations. In particular it leads to a monad on the category $(\mathcal{CSe})^4$ which algebras are our models of cubical weak natural $\infty$-transformations.

7.1 The category of cubical strict $\infty$-natural transformations

Cubical strict natural transformations were introduced in [23]. Here we give the evident strict and higher version of it. A cubical strict $\infty$-natural transformation is given by a 2-cube in $\infty$-CCAT

$$
\begin{array}{ccc}
C_{0,0} & \xrightarrow{F} & C_{1,0} \\
\downarrow H & & \downarrow G \\
C_{0,1} & \xrightarrow{K} & C_{1,1}
\end{array}
$$

which 0-cells corresponds to four cubical strict $\infty$-categories $C_{0,0}$, $C_{0,1}$, $C_{1,0}$, $C_{1,1}$, which 1-cells corresponds to four cubical strict $\infty$-functors $F$, $G$, $H$, $K$, and which the only 2-cell $\tau$ corresponds, for all 0-cells $a$ in $C_{0,0}$, to a 1-cell $G(F(a)) \xrightarrow{\tau(a)} K(H(a))$ such that for all 1-cells $a \xrightarrow{f} b$ of $C_{0,0}$ we have the following commutative diagram :

$$
\begin{array}{ccc}
G(F(a)) & \xrightarrow{\tau(a)} & K(H(a)) \\
G(F(f)) & & K(H(f)) \\
G(F(b)) & \xrightarrow{\tau(b)} & K(H(b))
\end{array}
$$

A morphism between two cubical strict $\infty$-natural transformations $\tau$ and $\tau'$ is given by a 3-cube in $\infty$-CCAT
such that $c_{1,0}F = F'c_{0,0}, c_{1,1}G = G'c_{1,0}, c_{0,1}H = H'c_{0,0}$ and $c_{1,1}K = K'c_{0,1}$. The category of cubical strict $\infty$-natural transformations is denoted $(1, \infty)$-$\text{CTrans}$, and we obtain an internal 2-cube in $\text{CAT}$

\[
\begin{array}{ccc}
(1, \infty)$-$\text{CTrans}$ & \xrightarrow{\boldsymbol{\sigma}_{1,1}^2} & \infty$-$\text{CFunc}t \\
& \xrightarrow{\boldsymbol{\sigma}_{1,2}^2} & \xrightarrow{\boldsymbol{\tau}_{0,1}^i} CC\text{AT} \\
& \xrightarrow{\boldsymbol{\tau}_{1,1}^2} & \end{array}
\]

**Proposition 5** The internal 2-cube of $\text{CAT}$ just above can be structured in a strict cubical 2-category \(\square\)

**Proof** Consider the following object $\tau \in (1, \infty)$-$\text{CTrans}$:

\[
\begin{array}{ccc}
C_{0,0} & \xrightarrow{F} & C_{1,0} \\
\downarrow{H} & \tau \downarrow & \downarrow{G} \\
C_{0,1} & \xrightarrow{K} & C_{1,1}
\end{array}
\]

such that $\sigma_{1,1}^2(\tau) = F, \sigma_{1,2}^2(\tau) = H, \tau_{1,1}^2(\tau) = K$ and $\tau_{1,2}^2(\tau) = G$, and such that $\sigma_0^0$ and $\tau_0^i$ are clearly defined.

**Definition of the classical reflexivity**:

\[
\begin{array}{ccc}
(1, \infty)$-$\text{CTrans}$ & \xleftarrow{1_{1,1}^i} & \infty$-$\text{CFunc}t \xrightarrow{1_i^i} CC\text{AT} \\
& \xleftarrow{1_{1,2}^i} & \end{array}
\]

$1_{1,1}^i(F)$ is given by

\[
\begin{array}{ccc}
C_{0,0} & \xrightarrow{F} & C_{1,0} \\
\downarrow{}_{1_{c_{0,0}}} & \downarrow{1_{1,1}^i(F)} & \downarrow{}_{1_{c_{1,0}}} \\
C_{0,0} & \xrightarrow{F} & C_{1,0}
\end{array}
\]

and is such that $1_{1,1}^i(F)(a) = 1_0^0(F(a))$ for all 0-cells $a \in C_{0,0}(0)$, and also $1_{1,2}^i(F)$ is given by
and is such that $1_{2,2}(F)(a) = 1_0^0(F(a))$ for all 0-cells $a \in C_{0,0}(0)$.

**Definition of the connections:**

$1_{2,1}^{\pm}(F)$ is given by

and is such that $1_{2,1}^{\pm}(F)(a) = 1_1^0(F(a))$ for all 0-cells $a \in C_{0,0}(0)$, and $1_{2,1}^{\pm}(F)$ is given by

and is such that $1_{2,1}^{\pm}(F)(a) = 1_1^0(F(a))$ for all 0-cells $a \in C_{0,0}(0)$.

The following shape of 2-cells

allows to define the composition $\rho \circ_2^1 \tau$
by the formula:

$$(\rho \circ_{1,1} \tau)(a) = \rho(H(a)) \circ G'(\tau(a))$$

and the following shape of 2-cells

allows to define the composition $\rho \circ_{1,2} \tau$

by the formula:

$$(\rho \circ_{1,2} \tau)(a) = K'(\tau(a)) \circ \rho(F(a))$$

The proof that these data put a structure of cubical strict 2-categories on the internal 2-cube of the proposition is left to the reader.  

\[\blacksquare\]

### 7.2 The category of cubical $(1, \infty)$-magmas

A cubical $(1, \infty)$-magma is an object with shape

such that $(M_{0,0}, F_M, M_{1,0})$, $(M_{1,0}, G_M, M_{1,1})$, $(M_{0,0}, H_M, M_{0,1})$ and $(M_{0,1}, K_M, M_{1,1})$ are objects of $(0, \infty)$-$\text{CMag}_r$, and such that $\tau_M$ is a map $M_{0,0}(0) \xrightarrow{\tau_M} M_{1,1}(1)$ which sends each 0-cell $a$ of $M_{0,0}$ to an 1-cell $\tau_M(a) \in M_{1,1}(1)$ such that $s_0(\tau_M(a)) = G_M(F_M(a))$ and $t_0(\tau_M(a)) = K_M(H_M(a))$. We want to avoid heavy notations and shall denote usually just by $\tau_M$ such object of a category $(1, \infty)$-$\text{CMag}_r$, where we have to think this greek letter $\tau$ as the variable usually used for natural transformations and the subscript $M$ in it just means "Magmatic".
Given \( \tau_M \) and \( \tau'_M \) two objects of \((1, \infty)-\mathcal{CMag}_r\), a morphism between them is given by a commutative diagram in \( \infty\text{-CSets} \):

\[
\begin{array}{cccccc}
M_{0,0} & \xrightarrow{m_{0,0}} & M'_0,0 & \xrightarrow{F'_M} & M'_{1,0} & \xrightarrow{m_{1,0}} M_{1,0} \\
\downarrow{H_M} & & \downarrow{F_M} & & \downarrow{G'_M} & \downarrow{G_M} \\
M_{0,1} & \xrightarrow{\tau_M} & M'_0,1 & \xrightarrow{G_M} & M'_{1,1} \\
\downarrow{m_{0,1}} & & \downarrow{G_M} & & \downarrow{m'_{1,1}} & \\
M_{1,1} & & & & M_{1,1} \\
\end{array}
\]

such that \((m_{0,0}, m_{1,0}), (m_{1,0}, m_{1,1}), (m_{0,0}, m_{0,1}), (m_{0,1}, m_{1,1})\) are morphisms of \((0, \infty)-\mathcal{CMag}_r\). It is important to note that commutativity of this diagram means also the equality \(m_{1,1} \circ \tau_M = \tau'_M \circ m_{0,0}\).

We obtain an internal 2-cube in \(\mathcal{CAT}\):

\[
\begin{array}{cccccc}
(1, \infty)-\mathcal{CMag}_r & \xrightarrow{\sigma^1_2} & (0, \infty)-\mathcal{CMag}_r & \xrightarrow{\sigma^0_1} & \infty\text{-CMag}_r \\
\downarrow{\tau^1_2} & & \downarrow{\tau^0_1} & & \downarrow{\tau^0_0} & \\
(1, \infty)-\mathcal{CMag}_r & & & & \infty\text{-CMag}_r \\
\end{array}
\]

**Proposition 6** The internal 2-cube of \(\mathcal{CAT}\) just above can be structured in a cubical reflexive 2-magma.

**Proof** The proof is easy and basic data have been already defined in 5.

\[\blacksquare\]

### 7.3 The category of cubical \((1, \infty)\)-categorical stretchings

A cubical \((1, \infty)\)-categorical stretching is given by a commutative diagram in \(\infty\text{-CSets}\):

\[
\begin{array}{cccccc}
M_{0,0} & \xrightarrow{H_M} & M_{0,1} & \xrightarrow{\tau_M} & M_{1,1} & \xrightarrow{K_M} M_{1,1} \\
\downarrow{F_M} & & \downarrow{\pi_{0,1}} & & \downarrow{G_M} & \downarrow{\pi_{1,1}} \\
C_{0,0} & \xrightarrow{H_C} C_{0,1} & \xrightarrow{\tau_C} C_{1,1} & \xrightarrow{K_C} C_{1,1} \\
\end{array}
\]

such that \((\pi_{0,0}, F_M, F_C, \pi_{1,0}), (\pi_{1,0}, G_M, G_C, \pi_{1,1}), (\pi_{0,0}, H_M, H_C, \pi_{0,1})\) and \((\pi_{0,1}, K_M, K_C, \pi_{1,1})\) are objects of \((0, \infty)-\mathcal{C)tC\), and also \(\tau_M\) is an object of \((1, \infty)\)-\(\mathcal{CMag}_r\) and \(\tau_C\) is an object of \((1, \infty)\)-\(\mathcal{CTrans}\). It is important to note that commutativity of this diagram means also that the equality \(\pi_{1,1} \circ \tau_M = \tau'_C \circ \pi_{0,0}\) holds. Such cubical \((1, \infty)\)-categorical stretching can be denoted \((\tau_M, \tau_C)\). Given another cubical \((1, \infty)\)-categorical stretching \((\tau'_M, \tau'_C)\):

\[33\]
a morphism \((\tau_M, \tau_C) \to (\tau'_M, \tau'_C)\) of such cubical \((1, \infty)\)-categorical stretchings is given by

- a morphism of \((1, \infty)\)-C\text{Mag}_G underlied by \((m_{0,0}, m_{1,0}, m_{0,1}, m_{1,1})\), a morphism of \((1, \infty)\)-C\text{Trans} underlied by \((c_{0,0}, c_{1,0}, c_{0,1}, c_{1,1})\):

- the following morphisms: \(((m_{0,0}, c_{0,0}), (m_{1,0}, c_{1,0})), ((m_{0,1}, c_{1,0}), (m_{1,1}, c_{1,1})), ((m_{0,0}, c_{0,0}), (m_{0,1}, c_{0,1})), ((m_{0,1}, c_{0,1}), (m_{1,1}, c_{1,1})), ((0, \infty)\)-C\text{EtC}
We denote $(1, \infty)$-CEtC the category of cubical $(1, \infty)$-categorical stretchings. Now we have a forgetful functor:

$$(1, \infty)$-CEtC \xrightarrow{U} (\mathsf{CSets})^4$$

which sends the object $(\tau_M, \tau_C)$ to the object $(M_{0,0}, M_{0,1}, M_{0,1}, M_{1,1})$.

This functor has a left adjoint which produces a monad $T^1 = (T^1, \lambda^1, \mu^1)$ on the category $(\mathsf{CSets})^4$.

**Definition 13** Cubical weak natural $\infty$-transformations are algebras for the monad $T^1$ above.

Thus we obtain a 2-cube in the category $\mathcal{A}dj$ of pairs of adjunctions defined in [24]

which allow to obtain a 2-cocube in the category $\mathcal{M}nd$ of categories equipped with monads defined in [24]

And finally it gives the following 2-cube in $\mathcal{C}AT$

$$(\mathcal{T}^1, \mathcal{T}^0) \xrightarrow{\mathcal{W}} (\mathcal{W}, \mathcal{W})$$
Proposition 7 The internal 2-cube of $\text{CAT}$ just above can be structured in a cubical weak 2-category.

PROOF Detail of the proof is quite long but is not difficult. For example basic datas of such structure are similar to those build in 5.

References

[1] Antolini, R., Geometric realisations of cubical sets with connections, and classifying spaces of categories, Appl. Categ. Structures 10 (5) (2002) 481–494.

[2] Ashley, N., Simplicial $T$-complexes and crossed complexes: a nonabelian version of a theorem of Dold and Kan, Dissertationes Math. (Rozprawy Mat.) 265 (1988) 1–61. With a prefence by R. Brown. xviii, 396.

[3] Michael Batanin, Monoidal globular categories as a natural environment for the theory of weak-n-categories, Advances in Mathematics (1998), volume 136, pages 39–103.

[4] Michael Batanin, On the Penon method of weakening algebraic structures, Journal of Pure and Appl. Algebra (2002), volume 172-1, pages 1–23.

[5] Fahd Ali Al-Agl and Ronald Brown and Richard Steiner, Multiple Categories: The Equivalence of a Global and a Cubical Approach, Advances in Mathematics 170, 71–118 (2002).

[6] Brown, R., A new higher homotopy groupoid: the fundamental globular $\omega$-groupoid of a filtered space, Homology, Homotopy Appl. 10 (1) (2008) 327–343.

[7] Brown, R., double modules, double categories and groupoids, and a new homotopy double groupoid, arXiv Math (0903.2627) (2009) 8 pp.

[8] Brown, R. and Higgins, P. J. The equivalence of $\infty$-groupoids and crossed complexes, Cahiers Topologie Géom. Différentielle 22 (4) (1981) 371–386.

[9] Brown, R. and Higgins, P. J. The equivalence of $\omega$-groupoids and cubical $T$-complexes, Cahiers Topologie Géom. Différentielle 22 (4) (1981) 349–370.

[10] Brown, R. and Higgins, P. J. Tensor products and homotopies for $\omega$-groupoids and crossed complexes, J. Pure Appl. Algebra 47 (1) (1987) 1–33.

[11] Brown, R. and Loday, J.-L., Homotopical excision, and Hurewicz theorems for $n$-cubes of spaces, Proc. London Math. Soc. (3) 54 (1) (1987) 176–192.

[12] Brown, R. and Mosa, G.H., Double categories, 2-categories, thin structures and connections, Theory Appl. Categ. 5 (1999) No. 7, 163–175 (electronic).

[13] Brown, R. and Spencer, C. B., Double groupoids and crossed modules, Cahiers Topologie Géom. Différentielle 17 (4) (1976) 343–362.

[14] Cheng Eugenia and Makkai Michael, A note on Penon’s definition of weak n-category, Cahiers de Topologie et de Géométrie Différentielle Catégorique, volume 50, page 83–10, 2009.

[15] Denis-Charles Cisinski, Batanin higher groupoids and homotopy types, Contemporary Mathematics (2007) volume 431, pages 171–186.

[16] Laurent Coppey and Christian Lair, Leçons de théorie des esquisses, Université Paris VII, (1985).

[17] Dakin, M. K., Kan complexes and multiple groupoid structures, Esquisses Math. 32. With a preface by R. Brown.

[18] Eduardo Dubuc, Adjoint triangles, Lecture Notes in Mathematics (Springer-Verlag 1968) volume 61, pages 69–91.

[19] Evrard, M., Homotopie des complexes simpliciaux et cubiques, Preprint.
[20] F. Foltz, *Sur la catégorie des foncteurs dominés*, Cahiers de Topologie et de Géométrie Différentielle Catégorique (1969), volume 11(2), pages 101–130.

[21] Gilbert, N. D., *On the fundamental cat n-group of an n-cube of spaces*, In "Algebraic topology, Barcelona, 1986", Lecture Notes in Math., Volume 1298. Springer, Berlin (1987), 124–139. 2, 24

[22] Marco Grandis and Mauri, *Cubical sets and their site*, Theory Appl. Categories 11 (2003) 185–201. 3

[23] Marco Grandis and Paré Robert, *An introduction to multiple categories (On weak and lax multiple categories, I)*, Cahiers de Topologie et de Géométrie Différentielle Catégorique, fascicule 2, volume LVII (2016). 29

[24] Kamel Kachour, *Définition algébrique des cellules non-strictes*, Cahiers de Topologie et de Géométrie Différentielle Catégorique, volume 1 (2008), pages 1–68. 1, 2, 12, 26, 29, 35

[25] Camell Kachour, *Algebraic definition of weak (∞, n)-categories*, Theory and Applications of Categories (2015), Volume 30, No. 22, pages 775-807 1, 2, 3, 4, 22, 23, 26

[26] Camell Kachour, *An algebraic approach to weak ω-groupoids*, Australian Category Seminar, 14 September 2011. http://web.science.mq.edu.au/groups/coact/seminar/cgi-bin/speaker-info.cgi?name=Camell+Kachour 22

[27] Camell Kachour, * (∞, n)-ensembles cubiques*, CLE Seminar, 23 novembre 2016. https://sites.google.com/site/logiquecategorique/Contenus/201611-kachour 25

[28] G.M.Kelly, *A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on*, Bulletin of the Australian Mathematical Society (1980), volume 22, pages 1–83.

[29] Tom Leinster, *Higher Operads, Higher Categories*, London Mathematical Society Lecture Note Series, Cambridge University Press (2004), volume 298.

[30] Maxime Lucas, *Cubical (ω, p)-categories*, https://arxiv.org/pdf/1612.07050v1.pdf. 24, 25

[31] Georges Maltsiniotis, *La catégorie cubique avec connections est une catégorie test stricte*, (preprint) (2009) 1–16.

[32] Georges Maltsiniotis, *Grothendieck ∞-groupoids, and still another definition of ∞-categories*, available online: http://arxiv.org/pdf/1009.2331v1.pdf (2010). 22

[33] Yves Lafont, François Metayer, and Krzysztof Worytkiewicz, *A folk model structure on omega-cat*, Advances in Mathematics (2010), volume 224, pages 1183–1231. 21

[34] Jacques Penon, *Approche polygraphique des ∞-catégories non-strictes*, Cahiers de Topologie et de Géométrie Différentielle Catégorique (1999), pages 31–80. 1, 2, 3, 8, 10, 12, 21, 22

[35] Jacques Penon, *∞-catégorification de structures équationnelles*, Séminaires Itinérants de Catégories (S.I.C) à Amiens, Septembre 2005. 26

[36] Timothy Porter, *n-types of simplicial groups and crossed n-cubes*, Topology 32 (1) (1993) 5–24. 2, 24

[37] Razak Salleh, *Union theorems for double groupoids and groupoids; some generalisations and applications*, Ph.D. thesis, University of Wales (1976). 2, 24

[38] Richard Steiner, *Thin fillers in the cubical nerves of omega-categories*, Theory Appl. Categ. 16 (2006) No. 8, 144–173. 2, 6

[39] Tuyéras Rémy, *Sketches in higher category theory*, Ph.D thesis, Cambridge University Press, Volume 95, Issue 1, February 2017. 21

Camell KACHOUR
Institut de Recherche en Informatique Fondamentale CNRS (UMR 8243), Paris, France.
Phone: 00 336 458 473 25
Email: camell.kachour@gmail.com

37