IN Variant Generalized Functions ON $\mathfrak{sl}(2, \mathbb{R})$ WITH VALUES IN A $\mathfrak{sl}(2, \mathbb{R})$-MODULE

P. Lavaud

Abstract. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Let $\rho : \mathfrak{g} \to \text{End}(V)$ be a representation of $\mathfrak{g}$ in a finite dimensional real vector space. Let $C_V = (\text{End}(V) \otimes S(\mathfrak{g}))^g$ be the algebra of End($V$)-valued invariant differential operators with constant coefficients on $\mathfrak{g}$. Let $U$ be an open subset of $\mathfrak{g}$. We consider the problem of determining the space of generalized functions $\phi$ on $U$ with values in $V$ which are locally invariant and such that $C_V \phi$ is finite dimensional.

In this article we consider the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{sl}(2, \mathbb{R})$. We prove that when $U$ is $SL(2, \mathbb{R})$-invariant, then $\phi$ is determined by its restriction to $U \setminus \mathcal{N}$ where $\phi$ is analytic (cf. Theorem 6.1). In general this is false when $U$ is not $SL(2, \mathbb{R})$-invariant and $V$ is not trivial. Moreover, when $V$ is not trivial, $\phi$ is not always locally $L^1$. Thus, this case is different and more complicated than the situation considered by Harish-Chandra (cf. [HC64, HC65]) where $\mathfrak{g}$ is reductive and $V$ is trivial.

To solve this problem we find all the locally invariant generalized functions supported in the nilpotent cone $\mathcal{N}$. We do this locally in a neighborhood of a nilpotent element $Z$ of $\mathfrak{g}$ (cf. Theorem 4.1) and on an $SL(2, \mathbb{R})$-invariant open subset $U \subset \mathfrak{sl}(2, \mathbb{R})$ (cf. Theorem 4.2). Finally, we also give an application of our main theorem to the Superpfaffian (cf. [Lav04]).

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. Let $\rho : \mathfrak{g} \to \text{End}(V)$ be a representation of $\mathfrak{g}$ in a finite dimensional real vector space. Let $C_V = (\text{End}(V) \otimes S(\mathfrak{g}))^g$ be the algebra of End($V$)-valued invariant differential operators with constant coefficients on $\mathfrak{g}$. It is the classical family algebra in the terminology of Kirillov (cf. [Kir00]). Let $U$ be an open subset of $\mathfrak{g}$. We consider the problem of determining the space of generalized functions $\phi$ on $U$ with values in $V$ which are locally invariant and such that $C_V \phi$ is finite dimensional.

When $V = \mathbb{R}$ is the trivial module and $\mathfrak{g}$ is reductive, the problem was solved by Harish-Chandra (cf. in particular [HC64, HC65]). Let $\phi$ be a locally invariant generalized function such that $S(\mathfrak{g})^0 \phi$ is finite dimensional. He proved that $\phi$ is locally $L^1$, $\phi$ is determined by its restriction $\phi|_{\mathfrak{g}'}$ to the open subset $\mathfrak{g}'$ of semi-simple regular elements of $\mathfrak{g}$ and $\phi|_{\mathfrak{g}'}$ is analytic.

In this article we consider the case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{sl}(2, \mathbb{R})$. In this case $\mathfrak{g}' = \mathfrak{sl}(2, \mathbb{R}) \setminus \mathcal{N}$. Let $\phi$ be a locally invariant generalized function on $U$ with values in $V$ such that $C_V \phi$ is finite dimensional. We prove that when $U$ is $SL(2, \mathbb{R})$-invariant, then $\phi$ is determined by its restriction to $U \setminus \mathcal{N}$ where $\phi$ is analytic (cf. Theorem 6.1). In general this is false when $U$ is not $SL(2, \mathbb{R})$-invariant and $V$ is not trivial. Moreover, when $V$ is not trivial, $\phi$ is not always locally $L^1$. Finally, we also give an application of our main theorem to the Superpfaffian (cf. [Lav04]).

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To solve the problem we find all the locally invariant generalized functions supported in the nilpotent cone $\mathcal{N}$. Let $V_n$ be the $n + 1$-dimensional irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathcal{U}$ be an open subset of $\mathfrak{sl}(2, \mathbb{R})$. We denote by $C^{-\infty}(\mathcal{U}, V_n)^{\mathfrak{sl}(2, \mathbb{R})}$ the set of locally invariant generalized functions on $\mathcal{U}$ with values in $V_n$. Let $\Box$ be the Casimir operator on $\mathfrak{g}$.

We denote by $\mathcal{N}^+$ (resp. $\mathcal{N}^-$) the “upper” (resp. “lower”) half nilpotent cone (cf. 4.1). We put:

1. $S^0_n(\mathcal{U}) = \{ \phi \in C^{-\infty}(\mathcal{U}, V_n)^{\mathfrak{sl}(2, \mathbb{R})} / \phi|_{\mathcal{U}\setminus\{0\}} = 0 \};$
2. $S^\pm_n(\mathcal{U}) = \{ \phi \in C^{-\infty}(\mathcal{U}, V_n)^{\mathfrak{sl}(2, \mathbb{R})} / \phi|_{\mathcal{U}\setminus(\mathcal{N}^\pm\cup\{0\})} = 0 \};$
3. $S_n(\mathcal{U}) = \{ \phi \in C^{-\infty}(\mathcal{U}, V_n)^{\mathfrak{sl}(2, \mathbb{R})} / \phi|_{\mathcal{U}\setminus\mathcal{N}} = 0 \}.$

Let $Z \in \mathcal{N}^+$. We assume that $\mathcal{U}$ is a suitable open neighborhood of $Z$ (cf. section 4.6). Let $\delta_{\mathcal{N}^\pm}$ be an invariant generalized function with support $\mathcal{N}^\pm \cup \{0\}$ (cf. section 4.4). We construct an invariant function $s_n$ on $\mathcal{N} \cap \mathcal{U}$ with values in $V_n$. We prove (cf. Theorem 4.1):

(i) When $n$ is even, $S_n(\mathcal{U})$ is an infinite dimensional vector space with basis:

(ii) When $n$ is odd, $\dim(S_n(\mathcal{U})) = \frac{n+1}{2}$ and a basis is given by:

We assume that $\mathcal{U}$ is an $SL(2, \mathbb{R})$-invariant open subset of $\mathfrak{sl}(2, \mathbb{R})$. If $\mathcal{U} \cap \mathcal{N} \neq \emptyset$, we have $\mathcal{N}^+ \subset \mathcal{U}$ or $\mathcal{N}^- \subset \mathcal{U}$. We prove (cf. Theorem 4.2):

(i) $S^0_n(\mathcal{U}) = \{0\}$ if $0 \notin \mathcal{U}$; $S^0_n(\mathcal{U}) \simeq (V_n \otimes S(\mathfrak{sl}(2, \mathbb{R})))^{\mathfrak{sl}(2, \mathbb{R})}$ if $0 \in \mathcal{U}$.

(ii) When $n$ is even, we have:

(iii) When $n$ is odd:

Finally, let $\mathcal{U}$ be an open subset of $\mathfrak{sl}(2, \mathbb{R})$. Let $V$ be the space of a real finite dimensional representation of $\mathfrak{g}$. Let $\phi$ be an invariant function defined on $\mathcal{U}$ such that $C_V \phi$ is finite dimensional. This last condition is equivalent to the existence of $r \in \mathbb{N}$ and $(a_0, \ldots, a_{r-1}) \in \mathbb{R}^r$ such that:

$\left(\Box^r + \sum_{k=0}^{r-1} a_k \Box^k\right) \phi = 0.$

Moreover, we assume that $\phi|_{\mathcal{U}\setminus\mathcal{N}} = 0$. We prove (cf. Theorem 5.3) that if $\mathcal{U}$ is $SL(2, \mathbb{R})$-invariant, then we have $\phi = 0.$
In general, when \( U \) is not \( SL(2, \mathbb{R}) \)-invariant, there exist non trivial solutions of the equation \( \Box^k \phi = 0 \) which are supported in the nilpotent cone (cf. Theorem 5.2).

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### 2. Notations

Let \( \mathfrak{g} \) be a finite dimensional real Lie algebra. Let \( \rho : \mathfrak{g} \to \text{End}(V) \) be a representation of \( \mathfrak{g} \) in a finite dimensional real vector space \( V \). Let \( U \) be an open subset of \( \mathfrak{g} \). We denote by \( D^\infty_c(U) \) the space of compactly supported smooth densities on \( U \). We put:

\[
\mathcal{C}^{-\infty}(U, V) = \mathcal{L}(D^\infty_c(U), V),
\]

where \( \mathcal{L} \) stands for continuous homomorphisms. It is the space of generalized functions on \( U \) with values in \( V \). We put \( \mathcal{C}^{-\infty}(U) = \mathcal{C}^{-\infty}(U, \mathbb{R}) \). For \( \phi \in \mathcal{C}^{-\infty}(U, V) \) and \( \mu \in D^\infty_c(U) \), we denote by:

\[
\int_U \phi(Z) d\mu(Z)
\]

the image of \( \mu \) by \( \phi \). We have:

\[
\mathcal{C}^{-\infty}(U, V) = \mathcal{C}^{-\infty}(U) \otimes V
\]

(we will also write \( \phi v \) for \( \phi \otimes v \)).
Let $Z \in \mathfrak{g}$. We denote by $\partial_Z$ the derivative in the direction $Z$. It acts on $C^{-\infty}(\mathcal{U})$ and on $C^{-\infty}(\mathcal{U}, V)$. We extend $\partial$ to a morphism of algebras from $S(\mathfrak{g})$ to the algebra of differential operators with constant coefficients on $\mathfrak{g}$. We denote by $L_Z$ the differential operator defined by:

\begin{equation}
(\mathcal{L}_Z \phi)(X) = \left. \frac{d}{dt} \phi(X - t[Z,X]) \right|_{t=0}.
\end{equation}

The map $Z \mapsto L_Z$ is a Lie algebra homomorphism from $\mathfrak{g}$ into the algebra of differential operators on $\mathfrak{g}$. Let $Z \in \mathfrak{g}$ and $\phi \otimes v \in C^{-\infty}(\mathcal{U}) \otimes V$, we put:

\begin{equation}
Z.(\phi \otimes v) = \phi \otimes \rho(Z)v + (L_Z \phi) \otimes v.
\end{equation}

In other words, if we extend $L_Z$ (resp. $\rho(Z)$) linearly to a representation of $\mathfrak{g}$ in $C^{-\infty}(\mathcal{U}, V)$, we have for $\phi \in C^{-\infty}(\mathcal{U}, V)$:

\begin{equation}
Z.\phi = (\rho(Z) + L_Z)\phi.
\end{equation}

We say that $\phi \in C^{-\infty}(\mathcal{U}, V)$ is locally invariant if for any $Z \in \mathfrak{g}$ we have $Z.\phi = 0$. We put:

\begin{equation}
C^{-\infty}(\mathcal{U}, V)^{\mathfrak{g}} = \{ \phi \in C^{-\infty}(\mathcal{U}, V) / \forall Z \in \mathfrak{g}, Z.\phi = 0 \}.
\end{equation}

### 3. Support $\{0\}$ distributions

In this section we assume that $\mathfrak{g}$ is unimodular. We choose an invariant measure $dZ$ on $\mathfrak{g}$. We define the Dirac function $\delta_0$ on $\mathfrak{g}$ with support $\{0\}$ (which depends on the choice of $dZ$) by the following. Let $C^\infty_c(\mathfrak{g})$ be the set of smooth compactly supported functions on $\mathfrak{g}$. Then:

\begin{equation}
\forall f \in C^\infty_c(\mathfrak{g}), \int_{\mathfrak{g}} \delta_0(Z)f(Z)dZ = f(0).
\end{equation}

We have the following well known theorem:

**Theorem 3.1.** Let $\mathfrak{g}$ be a finite dimensional unimodular real Lie algebra and $V$ be a finite dimensional $\mathfrak{g}$-module. Then:

\begin{equation}
\{ \phi \in C^{-\infty}(\mathfrak{g}, V)^{\mathfrak{g}} / \phi|_{\mathfrak{g}\{0\}} = 0 \} \simeq (V \otimes S(\mathfrak{g}))^{\mathfrak{g}}.
\end{equation}

The isomorphism (which depends on the choice of $dZ$) sends $\sum_i v_i \otimes D_i \in (V \otimes S(\mathfrak{g}))^{\mathfrak{g}}$ to $\sum_i (\partial_{D_i} \delta_0) v_i$.

### 4. Support in the nilpotent cone

From now on, we assume that $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. 

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4.1. Notations. We put:

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

We denote by \((h, x, y) \in (\mathfrak{sl}(2, \mathbb{R})^*)^3\) the dual basis of \((H, X, Y)\). Thus:

\[
\left( \begin{array}{cc} h & x \\ y & -h \end{array} \right) \in \mathfrak{sl}(2, \mathbb{R})^* \otimes \mathfrak{sl}(2, \mathbb{R})
\]

is the generic point of \(\mathfrak{sl}(2, \mathbb{R})\). Let \(N\) be the nilpotent cone of \(\mathfrak{sl}(2, \mathbb{R})\). It is the union of three orbits:

(i) \(\{0\}\).

(ii) the half cone \(N^+\) with equations \(h^2 + xy = 0; x - y > 0\).

(iii) the half cone \(N^-\) with equations \(h^2 + xy = 0; x - y < 0\).

We denote by \(\Box\) the Casimir operator of \(\mathfrak{sl}(2, \mathbb{R})\):

\[
\Box = \frac{1}{2}(\partial_H)^2 + 2\partial_Y \partial_X.
\]

It is an invariant differential operator with constant coefficients on \(\mathfrak{sl}(2, \mathbb{R})\).

Let \(V_1 = \mathbb{R}^2\) be the standard representation of \(\mathfrak{sl}(2, \mathbb{R})\). We denote by \((e = (1, 0), f = (0, 1))\) the standard basis of \(\mathbb{R}^2\). The symplectic form \(B\) such that \(B(e, f) = 1\) is \(\mathfrak{sl}(2, \mathbb{R})\)-invariant. For \(v \in V_1\), we define \(\mu_1(v) \in \mathfrak{sl}(2, \mathbb{R})\) as the unique element such that:

\[
\forall Z \in \mathfrak{sl}(2, \mathbb{R}), \tr(\mu_1(v)Z) = \frac{1}{2}B(v, Zv).
\]

It defines a (moment) map:

\[
\mu_1 : V_1 \to \mathfrak{sl}(2, \mathbb{R}).
\]

We have \(\mu_1(e) = \frac{1}{2}X\) and \(\mu_1(f) = -\frac{1}{2}Y\). The function \(\mu_1\) is a two-fold covering of \(N^+\) by \(V_1 \setminus \{0\}\).

Let \(Z_0 \in N \setminus \{0\}\). Let \(U\) be a “small” neighborhood of \(Z_0\). In this section we determine:

\[
\{ \phi \in C^{-\infty}(U, V)^{\mathfrak{sl}(2, \mathbb{R})} / \phi|_{U \setminus N} = 0 \}.
\]

We can assume that \(Z_0 = X \in N^+\).

4.2. Restriction to \(X + \mathbb{R}Y\). We define a map:

\[
\pi : SL(2, \mathbb{R}) \times (X + \mathbb{R}Y) \to \mathfrak{sl}(2, \mathbb{R})
\]

\[
(g, Z) \mapsto Ad(g)(Z).
\]

This map is submersive. Let \(I_2\) be the identity matrix in \(SL(2, \mathbb{R})\). Let \(\Delta_X \subset X + \mathbb{R}Y\) be an open interval containing \(X\). We choose a connected open subset \(V \subset SL(2, \mathbb{R})\) such that \(I_2 \in V\). We put:

\[
U = \pi(V \times \Delta_X).
\]

It is an open neighborhood of \(X\) in \(\mathfrak{g}\).
Lemma 4.1. There is an injective (restriction) map:
\[
\mathcal{I}_X : C^{-\infty}(U, V)_{\mathfrak{sl}(2, \mathbb{R})} \to C^{-\infty}(\Delta_X, V) \\
\phi \mapsto \phi_X.
\]

Proof. The map
\[
\pi_U = \pi|_{V \times \Delta_X} : V \times \Delta_X \to U
\]
is a submersion. Thus if \( \phi \in C^{-\infty}(U, V) \), then \( \pi_U^*(\phi) \) is a well defined generalized function on \( V \times \Delta_X \) with values in \( V \). Moreover,
\[
\phi = 0 \iff \pi_U^*(\phi) = 0.
\]

Now, we assume that \( \phi \) is locally invariant. Then, \( \pi_U^*(\phi) \) is also locally invariant and
\[
\pi_U^*(\phi) \in C^\infty(V) \otimes C^{-\infty}(\Delta_X).
\]
(Where \( \otimes \) is a completed tensor product.) Thus \( \pi_U^*(\phi) \) can be restricted to \( \{I_2\} \times \Delta_X \subset V \times \Delta_X \) (cf. [HC64]). We identify \( \Delta_X \) and \( \{I_2\} \times \Delta_X \). We put:
\[
\phi_X \overset{\text{def}}{=} \pi_U^*(\phi)|_{\Delta_X}.
\]
Since \( V \) is connected and \( \phi \) is locally invariant, we have:
\[
\pi_U^*(\phi)(g, Z) = \rho(g)\phi_X(Z).
\]
Thus
\[
\phi_X = 0 \iff \pi_U^*(\phi) = 0.
\]

We have for \( Z \in \mathfrak{sl}(2, \mathbb{R}) \):
\[
\mathcal{L}_Z = -h\partial_{[Z, H]} - x\partial_{[Z, X]} - y\partial_{[Z, Y]}.
\]
In particular:
\[
\mathcal{L}_H = -2x\partial_X + 2y\partial_Y;
\]
\[
\mathcal{L}_X = 2h\partial_X - y\partial_H;
\]
\[
\mathcal{L}_Y = x\partial_H - 2h\partial_Y.
\]
If \( V \) is sufficiently small, we have \( x \neq 0 \) on \( U \). We assume that this condition is realized. It follows that on \( U \) we have:
\[
\partial_X = -\frac{1}{2x}\mathcal{L}_H + \frac{y}{x}\partial_Y;
\]
\[
\partial_H = \frac{1}{x}\mathcal{L}_Y + \frac{2h}{x}\partial_Y.
\]

We have \( \Delta_X \subset \{X + yY / y \in \mathbb{R}\} \). We use the coordinate \( y|_{\Delta_X} \), still denoted by \( y \), on \( \Delta_X \). Let \( \psi \in C^{-\infty}(\Delta_X, V_n) \). We put \( \psi(y) = \psi(X + yY) \).
Lemma 4.2. We have:

\[ \mathfrak{I}_X \left( C^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2, \mathbb{R})} \right) = \{ \psi \in C^{-\infty}(\Delta_X, V) / (\rho(X) + y\rho(Y)) \psi(y) = 0 \}. \]  

Thus:

\[ \mathfrak{I}_X : C^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2, \mathbb{R})} \to \{ \psi \in C^{-\infty}(\Delta_X, V) / (\rho(X) + y\rho(Y)) \psi(y) = 0 \} \]

is an isomorphism.

Proof. Since \( x|_{\Delta_X} = 1 \) and \( h|_{\Delta_X} = 0 \) we have for \( \phi \in C^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2, \mathbb{R})} \):

\[
(\mathcal{L}_X \phi)(Y)(y) = -y(\partial_H \phi)_X(y);
\]

and

\[
(\mathcal{L}_Y \phi)(X)(y) = (\partial_H \phi)_X(y).
\]

It follows that we have:

\[ (\mathcal{L}_X \phi)_X(y) + y(\mathcal{L}_Y \phi)_X(y) = 0. \]

Let \( \psi \in \mathfrak{I}_X \left( C^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2, \mathbb{R})} \right) \). Then, there is \( \phi \in C^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2, \mathbb{R})} \) such that \( \psi = \phi_X \).

We have:

\[
(\rho(X) + y\rho(Y))\psi(y) = (\rho(X) + y\rho(Y))\phi_X(y) = (\rho(X)\phi)_X(y) + y(\rho(Y)\phi)_X(y) + (\mathcal{L}_X \phi)_X(y) + y(\mathcal{L}_Y \phi)_X(y) = (\rho(X) + \mathcal{L}_X \phi)_X(y) + y(\rho(Y) + \mathcal{L}_Y \phi)_X(y) = 0.
\]

Let \( \psi \in C^{-\infty}(\Delta_X, V) \) such that \( (\rho(X) + y\rho(Y))\psi(y) = 0 \). We define \( \tilde{\psi} \in C^{-\infty}(\mathcal{V} \times \Delta_X) \) by the formula:

\[ \tilde{\psi}(g, y) = \rho(g)\psi(y). \]

Since \( \rho \) is a smooth function on \( SL(2, \mathbb{R}) \) with values in \( GL(V) \), this is a well defined generalized function on \( \mathcal{V} \times \Delta_X \) with values in \( V \).

Let \( (g, Z) \in \mathcal{V} \times \Delta_X \). Let \( (g', Z') \in \mathcal{V} \times \Delta_X \) such that \( \text{Ad}(g)(Z) = \text{Ad}(g')(Z') \). Then, \( \text{Ad}((g')^{-1}g)Z = Z' \). We put \( G^Z = \{ g'' \in SL(2, \mathbb{R}) / \text{Ad}(g'')(Z) = Z \} \). For \( g'' \in SL(2, \mathbb{R}) \), we have \( \text{Ad}(g'')(Z) \in \Delta_X \iff g'' \in G^Z \). Then, the fiber of \( \pi_{\mathcal{U}} \) at \( (g, Z) \) is included in \( \{ (g', Z) / g^{-1}g' \in G^Z \} \). Moreover, for \( Z' \in \mathfrak{sl}(2, \mathbb{R}) \), \( [Z, Z'] = 0 \iff Z' \in \mathbb{R}Z \).

Thus, since \( \mathcal{V} \) is connected, the condition \( (\rho(X) + y\rho(Y))\psi(y) = 0 \) on \( \Delta_X \) ensures that \( \tilde{\psi} \) is constant along the fibers of \( \pi_{\mathcal{U}} \). Thus there is a well defined generalized function \( \tilde{\psi} \) on \( \mathcal{U} \) such that:

\[ \pi_{\mathcal{U}}^*(\tilde{\psi}) = \tilde{\psi}. \]

It follows from the construction that \( \left( \tilde{\psi} \right)_X = \psi \).

The hypothesis \( \phi|_{\mathcal{U}\setminus \mathcal{N}} = 0 \) means that \( \phi_X \) is supported in \( \{ X \} \subset \Delta_X \).
4.3. **Radial part of** \( \Box \). In the neighborhood \( \mathcal{U} \) of \( X \) defined in section 4.2:

\[
\Box = \frac{1}{2} (\partial_H)^2 + 2\partial_Y \partial_X
\]

(46)

\[
= \frac{1}{2} \left( \frac{1}{x} \mathcal{L}_Y + \frac{2h}{x} \partial_Y \right)^2 + 2\partial_Y \left( -\frac{1}{2x} \mathcal{L}_H + \frac{y}{x} \partial_Y \right).
\]

We define the radial part of \( \Box \) as the differential operator \( \Box_X \) on \( C^{-\infty}(\Delta_X, V) \):

\[
\Box_X = \left( 3 + \rho(H) + 2y \frac{\partial}{\partial y} \right) \frac{\partial}{\partial y} + \frac{1}{2} \rho(Y)^2.
\]

(47)

This definition is justified by the following lemma:

**Lemma 4.3.** Let \( \phi \in C^{-\infty}(\mathcal{U}, V)^{sl(2,\mathbb{R})} \), then we have:

\[
(\Box \phi)_X = \Box_X \phi_X.
\]

(48)

**Proof.** Since \( x|_{\Delta_X} = 1 \) and \( h|_{\Delta_X} = 0 \), we have:

\[
(\Box \phi)_X = \frac{1}{2} \left( (\mathcal{L}_Y^2 + 2\mathcal{L}_Y \frac{h}{x} \frac{\partial}{\partial y}) \phi \right)_X + 2 \left( -\frac{1}{2} \left( \frac{\partial}{\partial y} \mathcal{L}_H \phi \right)_X + \left( \frac{\partial}{\partial y} \frac{\partial}{\partial y} \phi \right)_X \right)
\]

\[
= \frac{1}{2} \left( (\rho(Y)^2 + 2(x\partial_H - 2h\partial_Y) \frac{h}{x} \frac{\partial}{\partial y}) \phi \right)_X
\]

\[
+ 2 \left( -\frac{1}{2} \left( -\rho(H) \frac{\partial}{\partial y} \phi_X \right) + \frac{\partial}{\partial y} \phi_X + y \left( \frac{\partial}{\partial y} \right)^2 \phi_X \right)
\]

\[
= \frac{1}{2} \left( \rho(Y)^2 + 2 \frac{\partial}{\partial y} \right) \phi_X + \left( \rho(H) + 2 + 2y \frac{\partial}{\partial y} \right) \frac{\partial}{\partial y} \phi_X
\]

\[
= \left( 3 + \rho(H) + 2y \frac{\partial}{\partial y} \right) \frac{\partial}{\partial y} \phi_X + \frac{1}{2} \rho(Y)^2 \phi_X = \Box_X \phi_X.
\]

\( \Box \)

4.4. **The Dirac function** \( \delta_{\mathcal{N}^+} \) (resp. \( \delta_{\mathcal{N}^-} \)). Let \( dZ = dx\,dy\,dh \) be the Lebesgue measure on \( sl(2,\mathbb{R}) \). Let \( (e^*, f^*) \in (V_1^*)^2 \) be the dual basis of \( (e, f) \). The Lebesgue measure \( dv = -2de^*\,df^* \) on \( V_1 \) is \( sl(2,\mathbb{R}) \)-invariant. We define an invariant generalized function \( \delta_{\mathcal{N}^+} \) (resp. \( \delta_{\mathcal{N}^-} \)) on \( sl(2,\mathbb{R}) \) and supported in \( \mathcal{N}^+ \cup \{0\} \) (resp. \( \mathcal{N}^- \cup \{0\} \)) by:

\[
\forall g \in C_c^{\infty}(sl(2,\mathbb{R})), \quad \int_{sl(2,\mathbb{R})} \delta_{\mathcal{N}^+}(Z) g(Z)dZ \overset{\text{def}}{=} \int_{V_1} g \circ \mu_1(v)dv
\]

(50)

\[
\left( \text{resp.} \forall g \in C_c^{\infty}(sl(2,\mathbb{R})), \int_{sl(2,\mathbb{R})} \delta_{\mathcal{N}^-}(Z) g(Z)dZ \overset{\text{def}}{=} \int_{V_1} g \circ (-\mu_1)(v)dv \right).
\]

We put:

\[
\delta_X = (\delta_{\mathcal{N}^+})_X \in C^{-\infty}(\Delta_X).
\]

(51)

We still denote by \( dy \) the Lebesgue measure on \( \Delta_X \). It is invariant. Let \( g \in C_c^{\infty}(\Delta_X) \). Then we have:

\[
\int_{\Delta_X} \delta_X(y) g(y)dy = g(0).
\]

(52)
4.5. Irreducible representations. If $V = V^1 \oplus \cdots \oplus V^n$ where $V^i$ is an irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$, then we have:

\begin{equation}
C^{-\infty}(\mathcal{U}, V) = \bigoplus_{i=1}^{n} C^{-\infty}(\mathcal{U}, V^i),
\end{equation}

every subspace being stable for $\mathfrak{sl}(2, \mathbb{R})$. Thus we can assume from now on that the representation of $\mathfrak{sl}(2, \mathbb{R})$ in $V$ is irreducible.

We fix the Cartan subalgebra $\mathfrak{h} = \mathbb{R}H$ and the positive root $2h$ (we still denote by $h$ its restriction to $\mathfrak{h}$). Let $n \in \mathbb{N}$. We denote by $V_n$ the irreducible representation of $\mathfrak{sl}(2, \mathbb{R})$ with highest weight $nh$. We have $\dim(V_n) = n + 1$. We decompose $V_n$ under the action of $\mathbb{R}H$. We fix $v_0 \in V_n \setminus \{0\}$ a vector of weight $-nh$:

\begin{equation}
\rho(H)v_0 = -nv_0.
\end{equation}

We put for $0 \leq i \leq n$: $v_i = \rho(X)^i v_0$. We have $\rho(X)v_n = 0$ and $\rho(H)v_i = (-n + 2i)v_i$. On the other hand, $\rho(Y)v_0 = 0$ and for $1 \leq i \leq n$: $\rho(Y)v_i = (n - i + 1)iv_{i-1}$.

4.6. A basic function on $\mathcal{N}^+$. We construct a function $s_n : \mathcal{U} \cap \mathcal{N}^+ \rightarrow V_n$ which is the basic tool to generate all the generalized functions we are looking for.

4.6.1. Case $n$ even. In this case $V_n$ is isomorphic to the irreducible component of $S^{n \frac{h}{2}}(\mathfrak{sl}(2, \mathbb{R}))$ (under adjoint action of $\mathfrak{sl}(2, \mathbb{R})$) generated by $X^{n \frac{h}{2}}$. From now on we will identify $V_n$ with this component. We denote by $s_n : \mathcal{N} \rightarrow V_n$ the invariant map defined by:

\begin{equation}
s_n(Z) = Z^{n \frac{h}{2}}.
\end{equation}

4.6.2. Case $n = 1$. We recall that $\mu_1 : V_1 \setminus \{0\} \rightarrow \mathcal{N}^+$ is a two-fold covering with $\mu_1(e) = \frac{1}{2}X$. If $\mathcal{U}$ is a sufficiently small connected neighborhood of $X$, there exists a unique continuous section $s_1$ of $\mu_1$ in $\mathcal{U} \cap \mathcal{N}^+$ such that $s_1(\frac{1}{2}X) = e$. We have $s_1 : \mathcal{U} \cap \mathcal{N}^+ \rightarrow V_1$. It satisfies:

\begin{equation}
\forall Z \in \mathcal{U} \cap \mathcal{N}^+, \quad \mu_1(s_1(Z)) = Z.
\end{equation}

4.6.3. Case $n$ odd. More generally, when $n$ is odd, $V_n$ is isomorphic to the irreducible component of $V_1 \otimes S^{n \frac{h}{2} - 1}(\mathfrak{sl}(2, \mathbb{R}))$ generated by $e \otimes X^{n \frac{h}{2} - 1}$. From now on we will identify $V_n$ with this component. Let $\mathcal{U}$ be the above neighborhood of $X$. We define a function $s_n : \mathcal{U} \cap \mathcal{N}^+ \rightarrow V_n$ by:

\begin{equation}
\forall Z \in \mathcal{U} \cap \mathcal{N}^+, \quad s_n(Z) = s_1(Z) \otimes Z^{n \frac{h}{2} - 1} \in V_n.
\end{equation}

4.7. Basic theorem. Let $\mathcal{U}$ be an open subset of $\mathfrak{sl}(2, \mathbb{R})$. We put:

\begin{equation}
S_n(\mathcal{U}) = \{ \phi \in C^{-\infty}(\mathcal{U}, V_n)^{\mathfrak{sl}(2, \mathbb{R})}/ \phi|_{\mathcal{U} \cap \mathcal{N}} = 0 \}.
\end{equation}

**Theorem 4.1.** Let $n \in \mathbb{N}$. Let $\mathcal{U}$ be an open connected neighborhood of $X$ such that the function $s_n$ is well defined on $\mathcal{U} \cap \mathcal{N}$ (cf. section 4.6) and $\mathcal{J}_X$ is bijective (cf. section 4.2). Then:

(i) When $n$ is even, $S_n(\mathcal{U})$ is an infinite dimensional vector space with basis:

\begin{equation}
(\square^k(s_n^\delta_{\mathcal{N}^+}))_{k \in \mathbb{N}}.
\end{equation}

(ii) When $n$ is odd, $\dim(S_n(\mathcal{U})) = \frac{n+1}{2}$ and a basis is given by:

\begin{equation}
(\square^k(s_n^\delta_{\mathcal{N}^+}))_{0 \leq k \leq \frac{n-1}{2}}.
\end{equation}
Remark: Since $\delta_{\mathcal{N}^+}(Z)dZ$ is a measure on $\mathfrak{s}(2,\mathbb{R})$ with support $\mathcal{N}^+ \cup \{0\}$ and $s_n$ is a smooth function on $\mathcal{U} \cap \mathcal{N}$ with values in $V_n$, $s_n\delta_{\mathcal{N}^+}$ is a well defined generalized function on $\mathcal{U}$ with values in $V_n$.

Proof. Thanks to the isomorphism $\mathcal{J}_X$ we have to determine the space:

$$\{\psi \in C^{-\infty}(\Delta_X, V_n) / \psi|_{\Delta_X \setminus \{0\}} = 0 \text{ and } (\rho(X) + y\rho(Y))\psi(y) = 0\}.$$  

Let $\psi \in C^{-\infty}(\Delta_X, V_n)$. We write:

$$\psi(y) = \sum_{i=0}^{n} \psi_i(y)v_i,$$

where $\psi_i \in C^{-\infty}(\Delta_X)$ and $(v_i)_{0 \leq i \leq n}$ is the basis defined in section 4.5. We put:

$$\delta^k(y) = \left(\frac{\partial}{\partial y}\right)^k \delta_X(y).$$

Since $\psi$ is supported in $\mathcal{N}$ and $\Delta_X \cap \mathcal{N} = \{X\}$, there exists $a_{i,k} \in \mathbb{R}$, all equal to zero but for finite number, such that:

$$\psi_i(y) = \sum_{k \in \mathbb{N}} a_{i,k}\delta^k(y).$$

For $n = 0$, we have $\rho = 0$ and the condition $(\rho(X) + y\rho(Y))\psi(y) = 0$ is automatically satisfied.

For $n \geq 1$, we put $\alpha_i = (n - i + 1)i$. We have $y\delta^0(y) = 0$ and for $k \geq 1$, $y\delta^k(y) = -k\delta^{k-1}(y)$. Thus:

$$\sum_{0 \leq i \leq n-1, k \in \mathbb{N}} a_{i,k}\delta^k(y)v_{i+1} - \sum_{1 \leq i \leq n, k \geq 1} \alpha_i a_{i,k}k\delta^{k-1}(y)v_{i-1} = 0.$$

It follows:

$$\begin{cases} a_{n-1,k} = 0 & \text{for } k \geq 0; \\ a_{1,k} = 0 & \text{for } k \geq 1; \\ a_{n-1,k} = (k+1)(i+1)(n-i)a_{i+1,k+1} & \text{for } n \geq 2, 1 \leq i \leq n-1 \text{ and } k \geq 0. \end{cases}$$

It follows in particular

(i) from the first and the last relations that $\forall i, k \geq 0$ with $2i+1 \leq n$: $a_{n-(2i+1),k} = 0$;

(ii) from the last relation that $\forall i \geq 0$ with $2i \leq n$: $(a_{n-2i,k})_{k \geq 0}$ is completely determined by $(a_{n,k})_{k \geq 0}$.

We distinguish between the two cases according to the parity of $n$.

$n$ even: In this case, for $n \geq 2$, the second relation follows from (i). Hence the map:

$$\{\psi \in C^{-\infty}(\Delta_X, V_n) / \psi|_{\Delta_X \setminus \{0\}} = 0 \text{ and } (\rho(X) + y\rho(Y))\psi(y) = 0\} \rightarrow \mathbb{R}^N$$

$$\psi(y) = \sum_{0 \leq i \leq n, k \in \mathbb{N}} a_{i,k}\delta^k(y)v_i \mapsto (a_{n,k})_{k \in \mathbb{N}}$$

is bijective. This is also true for $n = 0$. 

\[ n \text{ odd}: \text{ It follows from the two last relations that for } k \geq i \geq 1 a_{2i-1,k} = 0. \text{ In particular, the map:} \]

\[
\psi \in C^{-\infty}(\Delta_X, V_n) / \psi|_{\Delta_X \setminus \{0\}} = 0 \text{ and } (\rho(X) + y\rho(Y))\psi(y) = 0 \rightarrow \mathbb{R}^{n+1/2} \\
\psi(y) = \sum_{0 \leq i \leq n, k \in \mathbb{N}} a_{i,k}\delta^k(y)v_i \mapsto (a_{n,0}, \ldots, a_{n, n+1})
\]

is bijective.

This proves the first part of the theorem on the dimension of \( S_n(U) \). It remains to prove that the functions \( \square^k(s_n\delta_N) \) form a basis of \( S_n(U) \). We have for \( \psi(y) = \sum_{i=0}^n \sum_{k \in \mathbb{N}} a_{i,k}\delta^k(y)v_i \in C^{-\infty}(\Delta_X, V_n) \) such that \( \rho(X + yY)\psi(y) = 0: \)

\[
\square_X \psi(y) = (3 + \rho(H) + 2y\partial_Y)\sum_{k \in \mathbb{N}} a_{n,k}\delta^{k+1}(y)v_n + \sum_{i=0}^{n-1} \ldots v_i \\
= \sum_{k \in \mathbb{N}} (n - 2k - 1)a_{n,k}\delta^{k+1}(y)v_n + \sum_{i=0}^{n-1} \ldots v_i
\]

where \( \ldots \) are elements of \( C^{-\infty}(\Delta_X) \).

\[ n \text{ even}: \text{ Since } v_n \in X^2, \text{ we have } (s_n\delta_N)_X(y) = \delta_X(y)X^2. \text{ By induction on } k, \text{ it follows:} \]

\[
(\square^k(s_n\delta_N))_X(y) = (n - 2k + 1) \ldots (n - 1)\delta^k(y)X^2 + \text{ terms with } X^{2-i} \text{ for } i \geq 1.
\]

Since \( n \) is even \( n - 2k + 1 \neq 0 \). The result follows.

\[ n \text{ odd}: \text{ Since } v_n = e \otimes X^{n-1}, \text{ we have } (s_n\delta_N)_X(y) = \delta_X(y)(e \otimes X^{n-1}). \text{ By induction on } k, \text{ it follows:} \]

\[
(\square^k(s_n\delta_N))_X(y) = (n - 2k + 1) \ldots (n - 1)\delta^k(y)(e \otimes X^{n-1}) + \text{ terms with } e \otimes X^{n-1-i} \text{ for } i \geq 1.
\]

In this case for \( k = \frac{n+1}{2}, n - 2k + 1 = 0 \). Thus, since \( \square^k(s_n\delta_N) \) is invariant, it follows from the isomorphism (68) that for \( k \geq \frac{n+1}{2} \): \( \square^k(s_n\delta_N) = 0 \). The result follows.

\[ \square \]

4.8. **Global version.** Let \( U \) be an open subset of \( \mathfrak{sl}(2, \mathbb{R}) \). We put:

\[
S_n^0(U) = \{ \phi \in C^{-\infty}(\mathbb{U}, V_n)_{\mathfrak{sl}(2, \mathbb{R})} / \phi|_{U \setminus \{0\}} = 0 \}; \\
S_n^\pm(U) = \{ \phi \in C^{-\infty}(\mathbb{U}, V_n)_{\mathfrak{sl}(2, \mathbb{R})} / \phi|_{U \setminus (\mathcal{N}^\pm \cup \{0\})} = 0 \}.
\]

**Theorem 4.2.** Let \( U \) be an \( SL(2, \mathbb{R}) \)-invariant open subset of \( \mathfrak{sl}(2, \mathbb{R}) \). Then we have:

(i)

\[
\begin{cases}
S_n^0(U) = \{0\} & \text{if } 0 \notin U; \\
S_n^0(U) \simeq (V_n \otimes S(\mathfrak{sl}(2, \mathbb{R})))_{\mathfrak{sl}(2, \mathbb{R})} & \text{if } 0 \in U.
\end{cases}
\]
(ii) When \( n \) is even, we have:
\[
S_n(U) = S_n^+(U) = S_n^-(U) = S_n^0(U)
\]

Proof. (i) It follows from Theorem 3.1.

(ii) When \( n \) is even, the function \( \delta^+_n \) is defined on \( \mathfrak{sl}(2, \mathbb{R}) \), the function \( s_n \) is defined on \( \mathcal{N} \) and the product \( s_n \delta^+_n \) is well defined (cf. Remark of Theorem 4.1). Then the result follows from Theorem 4.1.

(iii) Let \( n \) be odd. We assume that \( \mathcal{U} \cap \mathcal{N} \neq \emptyset \). Since \( \mathcal{U} \) is \( SL(2, \mathbb{R}) \)-invariant, we have \( \mathcal{N}^+ \subset \mathcal{U} \) or \( \mathcal{N}^- \subset \mathcal{U} \). We assume that \( \mathcal{N}^+ \subset \mathcal{U} \) (the case \( \mathcal{U} \subset \mathcal{N}^- \) is similar).

Let \( \phi \in C^{-\infty}(\mathcal{U}, \mathcal{V})^{\mathfrak{sl}(2, \mathbb{R})} \). Let \( \mathcal{U}_0 \subset \mathcal{U} \) be a suitable neighborhood of \( X \) where \( s_1 \) (and thus \( s_n \)) is defined (cf. section 4.6). There exists \( (a_0, \ldots, a_{\frac{n-1}{2}}) \in \mathbb{R}^{\frac{n+1}{2}} \) such that on \( \mathcal{U}_0 \) (cf. Theorem 4.1):
\[
\phi(Z) = \sum_{k=0}^{\frac{n+1}{2}} a_k \Box^k (s_n(Z)\delta_{N^+}(Z)) = \sum_{k=0}^{\frac{n+1}{2}} a_k \Box^k ((s_1(Z) \otimes Z^\frac{n-1}{2})\delta_{N^+}(Z)).
\]

Since \( \mu_1 : V_1 \setminus \{0\} \to \mathcal{N}^+ \) is a non trivial two-fold covering, there is not any continuous section. In other words there is not any continuous \( SL(2, \mathbb{R}) \)-invariant map \( s : \mathcal{N}^+ \to V_1 \) such that for any \( Z \in V_0 \), \( s(Z) = s_1(Z) \). Thus \( a_0 = \cdots = a_{\frac{n-1}{2}} = 0 \). The result follows. \( \square \)

5. Invariant solutions of differential equations

5.1. Introduction. Let \( C_V = (\text{End}(V) \otimes S(\mathfrak{sl}(2, \mathbb{R})))^{\mathfrak{sl}(2, \mathbb{R})} \) be the algebra of \( \text{End}(V) \)-valued invariant differential operators with constant coefficients on \( \mathfrak{g} \). It is the classical family algebra in the terminology of Kirillov (cf. [Kir00]). When \( V = V_n \) is the \((n+1)\)-dimensional irreducible representation of \( \mathfrak{sl}(2, \mathbb{R}) \), we put \( C_n = C_{V_n} \).

Let \( \mathcal{U} \subset \mathfrak{sl}(2, \mathbb{R}) \) be an open subset. It is a natural and interesting problem to determine the generalized functions \( \phi \in C^{-\infty}(\mathcal{U}, \mathcal{V})^{\mathfrak{sl}(2, \mathbb{R})} \) such that \( C_V \phi \) is finite dimensional.

We recall that \( S(\mathfrak{sl}(2, \mathbb{R}))^{\mathfrak{sl}(2, \mathbb{R})} = \mathbb{R}[\Box] \). It is a subalgebra of \( C_V \). An other subalgebra of \( C_V \) is \( \text{End}(V)^{\mathfrak{sl}(2, \mathbb{R})} \). When \( V = V_n \), we put:
\[
M_n = \rho_n(X)Y + \rho_n(Y)X + \frac{1}{2} \rho_n(H)H \in C_n
\]
According N. Rozhkovskaya (cf. [Roz03]), \( C_n \) is a free \( S(\mathfrak{sl}(2, \mathbb{R}))^{\mathfrak{sl}(2, \mathbb{R})} \)-module with basis
\[
B_n = (1, M_n, \ldots, (M_n)^n).
\]

Lemma 5.1. Let \( \phi \in C^{-\infty}(\mathcal{U}, \mathcal{V})^{\mathfrak{sl}(2, \mathbb{R})} \). Then we have:
\[
\dim_{\mathbb{R}} (C_V \phi) < \infty \iff \dim_{\mathbb{R}} (\mathbb{R}[\Box] \phi) < \infty
\]
Proof. We argue as in [Roz03]. Let \( H \) be the set of harmonic polynomials in \( S(\mathfrak{sl}(2, \mathbb{R})) \). Then, \( S(\mathfrak{sl}(2, \mathbb{R})) = \mathbb{R}[\Box] \otimes H \) (cf. [Kos63]), and:
\[
C_V = \mathbb{R}[\Box] \otimes (H \otimes \text{End}(V))^{\mathfrak{sl}(2, \mathbb{R})}.
\]
Since \( \dim_{\mathbb{R}} (H \otimes \text{End}(V))^{\mathfrak{sl}(2, \mathbb{R})} < \infty \), the result follows.
Remark: Since \( \mathbb{R}[^{\square}] \subset \mathbb{R}[^{\square}] \otimes \text{End}(V)^{\mathfrak{sl}(2,\mathbb{R})} \subset \mathcal{C}_V \), the condition \( \dim(\mathcal{C}_V \varphi) < \infty \) is also equivalent to the existence of \( r \in \mathbb{N} \) and \((A_0, \ldots, A_{r-1}) \in (\text{End}(V)^{\mathfrak{sl}(2,\mathbb{R})})^r \) such that:

\[
(82) \quad (\square^r + A_{r-1} \square^{r-1} + \ldots + A_1 \square + A_0) \varphi = 0.
\]

Useful examples of (82) are \((\square - \lambda)^k \varphi = 0\) for \( \lambda \in \mathbb{C} \) and generalized functions with values in a complex representation. We give such an example below.

Definition 5.1. Let \( \varphi \in C^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2,\mathbb{R})} \). We say that \( \varphi \) is \( \square \)-finite if \( \dim_{\mathbb{R}}(\mathbb{R}[^{\square}] \varphi) < \infty \).

In other words, \( \varphi \) is \( \square \)-finite if there exists \( r \in \mathbb{N} \) and \((a_0, \ldots, a_{r-1}) \in \mathbb{R}^r \) such that

\[
(83) \quad (\square^r + a_{r-1} \square^{r-1} + \ldots + a_1 \square + a_0) \varphi = 0.
\]

Example: (This was our original motivation to study this problem.) Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra. We define the generalized functions on \( \mathfrak{g} \) as the generalized functions on \( \mathfrak{g}_0 \) with values in the exterior algebra \( \Lambda(\mathfrak{g}_1^*) \) of \( \mathfrak{g}_1^* \):

\[
(84) \quad C^{-\infty}(\mathfrak{g}) \overset{\text{def}}{=} C^{-\infty}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1^*) = C^{-\infty}(\mathfrak{g}_0, \Lambda(\mathfrak{g}_1^*)).
\]

We assume that \( \mathfrak{g} \) has a non degenerate invariant symmetric even bilinear form \( B \). Let \( \Omega \in S^2(\mathfrak{g}) \) be the Casimir operator associated with \( B \). We have \( \Omega = \Omega_0 + \Omega_1 \) with \( \Omega_0 \in S^2(\mathfrak{g}_0) \) and \( \Omega_1 \in \Lambda^2(\mathfrak{g}_1) \). We consider \( \Omega_1 \) as an element of \( \text{End}(\Lambda(\mathfrak{g}_1^*)) \) acting by interior product. When they can be evaluated (cf. for example [Lav98, Chapitre III.5]), the Fourier transforms of the coadjoint orbits in \( \mathfrak{g}^* \) are invariant generalized functions \( \varphi \) on \( \mathfrak{g} \) subject to equations of the form \((\Omega - \lambda) \varphi = 0\) with \( \lambda \in \mathbb{C} \). It can be written \((\Omega_0 + (\Omega_1 - \lambda)) \varphi = 0\) (for \( \mathfrak{g}_0 = \mathfrak{sl}(2,\mathbb{R}) \)) it is of the form (82) with \( \Omega_0 = \square \) and \( A_0 = \Omega_1 - \lambda \). We have:

\[
(85) \quad (\Omega_0 - \lambda)^k = \sum_{i=0}^{k} \binom{k}{i} (\Omega - \lambda)^i (-\Omega_1)^{k-i}.
\]

For \( k > \frac{\dim(\mathfrak{g}_1)}{2} \), we have \( \Omega_1^k = 0 \). It follows that for \( k > 1 + \frac{\dim(\mathfrak{g}_1)}{2} \) we have:

\[
(86) \quad (\Omega_0 - \lambda)^k \varphi = 0.
\]

this equation is of the form of (82).

5.2. Generalized functions with support \( \{0\} \). We immediately obtain from Theorem 3.1

Theorem 5.1. Let \( V \) be a representation of \( \mathfrak{sl}(2,\mathbb{R}) \). Let \( \varphi \in C^{-\infty}(\mathfrak{sl}(2,\mathbb{R}), V)^{\mathfrak{sl}(2,\mathbb{R})} \) such that \( \varphi|_{\mathfrak{sl}(2,\mathbb{R}) \setminus \{0\}} = 0 \) and \( \varphi \) is \( \square \)-finite. Then, we have \( \varphi = 0 \).
5.3. Support in the nilpotent cone: local version.

**Theorem 5.2.** Let $n \in \mathbb{N}$. Let $V_n$ be the irreducible $n + 1$-dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$. Let $W$ be a finite dimensional vector space with trivial action of $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathcal{U}$ be an open connected neighborhood of $X$ such that the function $s_n$ is well defined on $\mathcal{U} \cap \mathcal{N}$ (cf. section 4.6) and $\mathcal{J}_X$ is bijective (cf. section 4.2). Let $\phi \in C^{-\infty}(\mathcal{U}, W \otimes V_n)^{\mathfrak{sl}(2, \mathbb{R})}$ such that $\phi|_{\mathcal{U} \cap \mathcal{N}} = 0$. Let $r \in \mathbb{N}$ and $(a_0, \ldots, a_r) \in \mathbb{R}^r$ such that: \( \left( \Box^r + \sum_{k=0}^{r-1} a_k \Box^k \right) \phi = 0. \)

Then, we have $\phi = 0$ when at least one of the following conditions is satisfied:

(i) $n$ is even;

(ii) $n$ is odd and $a_0 \neq 0$.

**Proof.** Let $\phi \in C^{-\infty}(\mathcal{U}, W \otimes V_n)^{\mathfrak{sl}(2, \mathbb{R})}$ such that $\phi|_{\mathcal{U} \cap \mathcal{N}} = 0$. From Theorem 4.1 we obtain that there exist $p \in \mathbb{N}$, with $p = \frac{n-1}{2}$ if $n$ is odd and $(w_0, \ldots, w_p) \in W^{p+1}$, such that:

\[
(87) \quad \phi = \sum_{i=0}^{p} w_i \otimes \Box^i(s_n \delta_{\mathcal{N}^+}).
\]

Then:

(i) When $n$ is even, for $0 \leq j \leq p + r$, we have $\sum_{k+i=j} a_k w_i = 0$.

(ii) When $n$ is odd, for $0 \leq j \leq \frac{n-1}{2}$, we have $\sum_{k+i=j} a_k w_i = 0$.

The result follows. \( \square \)

**Remark:** When $n$ is odd, in contrast with the classical case ($V = V_0$ is the trivial representation) there exist (in a neighborhood of $X$) non trivial locally invariant solutions of the equation $\Box^k \phi = 0$ supported in the nilpotent cone! For example, if $k \geq \frac{n+1}{2}$ the functions $\phi = \Box^i(s_n \delta_{\mathcal{N}^+})$ for $0 \leq i \leq \frac{n-1}{2}$ are not trivial, supported in the nilpotent cone and satisfy the equation $\Box^k \phi = 0$.

When we consider the equation $(\Box - \lambda)^k \phi = 0$ for $\lambda \in \mathbb{C} \setminus \{0\}$, then the trivial solution is again the only one supported in the nilpotent cone.

5.4. Support in the nilpotent cone: global version.

**Theorem 5.3.** Let $V$ be a real finite dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathcal{U}$ be an $SL(2, \mathbb{R})$-invariant open subset of $\mathfrak{sl}(2, \mathbb{R})$. Let $\phi \in C^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2, \mathbb{R})}$ such that $\phi|_{\mathcal{U} \cap \mathcal{N}} = 0$ and $\phi$ is $\Box$-finite. Then we have $\phi = 0$.

**Proof.** It is enough to prove the theorem for $V$ irreducible. Then, the result follows from Theorem 4.2, Theorem 5.2 and Theorem 5.1. \( \square \)

6. **General invariant generalized functions**

6.1. **Main theorem.**

**Theorem 6.1.** Let $V$ be a real finite dimensional representation of $\mathfrak{sl}(2, \mathbb{R})$. Let $\mathcal{U}$ be an $SL(2, \mathbb{R})$-invariant open subset of $\mathfrak{sl}(2, \mathbb{R})$. Let $\phi \in C^{-\infty}(\mathcal{U}, V)^{\mathfrak{sl}(2, \mathbb{R})}$ such that $\phi$ is $\Box$-finite. Then $\phi$ is determined by $\phi|_{\mathcal{U} \cap \mathcal{N}}$ and $\phi|_{\mathcal{U} \cap \mathcal{N}}$ is an analytic function.
Proof. The fact that \( \phi \) is determined by \( \phi|_{U\setminus \mathcal{N}} \) follows from Theorem 5.3. The fact that \( \phi|_{U\setminus \mathcal{N}} \) is analytic can be proved exactly as in [HC65].

\( \square \)

Remark: In general \( \phi \) will not be locally \( L^1 \). Indeed, let \( \phi_0 \in C^{-\infty}(sl(2, \mathbb{R}))^{sl(2, \mathbb{R})} \) a non zero \( \Box \)-finite generalized function. Then \( \phi_0 \) is locally \( L^1 \), but for \( k \in \mathbb{N}^* \):

\[
M^k_n \phi_0 \in C^{-\infty}(sl(2, \mathbb{R}), \text{End}(V_n))^{sl(2, \mathbb{R})}
\]

is usually not locally \( L^1 \).

6.2. Application to the Superpfaffian. Let us consider the Lie superalgebra \( g = spo(2, 2n) \). Its even part is \( g_0 = sl(2, \mathbb{R}) \oplus so(2n, \mathbb{R}) \). Its odd part is \( g_1 = V_1 \otimes W \) where \( W \) is the standard \( 2n \)-dimensional representation of \( so(2n, \mathbb{R}) \).

In [Lav04] we constructed a particular invariant generalized function \( Spf \) on \( spo(2, 2n) \) called Superpfaffian. It generalizes the Pfaffian on \( so(2n, \mathbb{R}) \) and the inverse square root of the determinant on \( sl(2, \mathbb{R}) \). As it is a polynomial of degree \( n \) on \( so(2n, \mathbb{R}) \), we may consider that we have:

\[
\text{Spf} \in C^{-\infty}(sl(2, \mathbb{R}), \bigoplus_{k=0}^n S^k(so(2n, \mathbb{R})^*) \otimes \Lambda(g_1^*))^{sl(2, \mathbb{R})}.
\]

Let \( \Omega \) (resp. \( \Box \), \( \Omega_0 \), \( \Omega_1 \)) be the Casimir operator on \( spo(2, 2n) \) (resp. on \( sl(2, \mathbb{R}) \), \( so(2n, \mathbb{R}) \), \( g_1 \)). Then \( \Omega = \Box + \Omega_0 + \Omega_1 \) and

\[
\Omega_0' + \Omega_1 \in \text{End} \left( \bigoplus_{k=0}^n S^k(so(2n, \mathbb{R})^*) \otimes \Lambda(g_1^*) \right)^{sl(2, \mathbb{R})}
\]

is a nilpotent endomorphism. The superpfaffian satisfies:

\[
\left( \Box + (\Omega_0' + \Omega_1) \right) \text{Spf} = \Omega \text{Spf} = 0.
\]

The function \( \text{Spf} \) is analytic on \( sl(2, \mathbb{R}) \setminus \mathcal{N} \) and in [Lav04] an explicit formula is given for \( \text{Spf}(X) \in \bigoplus_{k=0}^n S^k(so(2n, \mathbb{R})^*) \otimes \Lambda(g_1^*) \) with \( X \in sl(2, \mathbb{R}) \setminus \mathcal{N} \). However, since \( \text{Spf} \) is not locally \( L^1 \) (cf. [Lav04]), it is not clear whether \( \text{Spf} \) is determined by its restriction to \( sl(2, \mathbb{R}) \setminus \mathcal{N} \) or not. In [Lav04] we proved that \( \text{Spf} \) is characterized, as an invariant generalized function on \( sl(2, \mathbb{R}) \), by its restriction to \( sl(2, \mathbb{R}) \setminus \mathcal{N} \) and its wave front set.

From the preceding results we obtain this new characterization of \( \text{Spf} \):

**Theorem 6.2.** Let \( \phi \in C^{-\infty}(sl(2, \mathbb{R}), \bigoplus_{k=0}^n S^k(so(2n, \mathbb{R})^*) \otimes \Lambda(g_1^*))^{sl(2, \mathbb{R})} \) such that:

(i) for \( X \in sl(2, \mathbb{R}) \setminus \mathcal{N} \), \( \phi(X) = \text{Spf}(X) \in \bigoplus_{k=0}^n S^k(so(2n, \mathbb{R})^*) \otimes \Lambda(g_1^*) \);

(ii) \( \Omega \phi = 0 \).

Then we have \( \phi = \text{Spf} \).

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