Differential geometry

A remark on the Bismut–Ricci form on 2-step nilmanifolds

Une remarque sur la forme de Bismut–Ricci des espaces homogènes sous l'action d'un groupe nilpotent de classe ≤ 2

Mattia Pujia, Luigi Vezzoni

Dipartimento di Matematica G. Peano, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

ARTICLE INFO

Article history:
Received 4 August 2017
Accepted after revision 3 January 2018
Available online 12 January 2018
Presented by the Editorial Board

ABSTRACT

In this note, we observe that, on a 2-step nilpotent Lie group equipped with a left-invariant SKT structure, the (1,1)-part of the Bismut–Ricci form is semi-negative definite. As an application, we give a simplified proof of the non-existence of invariant SKT static metrics on 2-step nilmanifolds and of the existence of a long-time solution to the pluriclosed flow in 2-step nilmanifolds.

© 2018 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

RÉSUMÉ

Nous observons que, sur un groupe de Lie nilpotent de classe ≤ 2, équipé d'une structure de Kähler forte avec torsion (SKT), invariante à gauche, la partie (1,1) de la forme de Bismut–Ricci est définie semi-négative. Comme application, nous donnons une démonstration simplifiée de la non-existence d'une métrique statique SKT sur un espace homogène sous l'action d'un groupe nilpotent de classe ≤ 2. Nous montrons également l'existence d'une solution à long terme du flot pluriclos dans ces mêmes espaces.

© 2018 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. The Bismut–Ricci form on 2-step SKT nilmanifolds

An Hermitian manifold is called SKT if its fundamental form is $\bar{\partial}\partial$-closed. The SKT condition can be described in terms of the Bismut connection by requiring that the torsion form is closed. Indeed, on any Hermitian manifold $(M, g)$, there is a unique Hermitian connection $\nabla$ such that the tensor $c := g(T(\cdot, \cdot), \cdot)$ is skew-symmetric in its entries [2], where $T$ is the torsion of $\nabla$. The metric $g$ is SKT if and only if $dc = 0$. In this note, we focus on the Ricci form of $\nabla$. In analogy to the Kähler case, the form is defined by
\[ \rho^B(X, Y) = \text{tr}_g R^B(X, Y, \cdot, \cdot), \]
\(\omega\) being the fundamental form of \(g\) and \(R^B\) the curvature tensor of \(\nabla\).

We consider as a manifold \(M\) a 2-step nilpotent Lie group \(G\) equipped with an invariant Hermitian structure \((J, g)\). Under these assumptions, the form \(\rho^B\) takes the following expression

\[ \rho^B(X, Y) = \text{tr}_g R^B(X, Y, \cdot, \cdot), \quad \text{for every } X, Y \in \mathfrak{g}, \tag{1} \]

where \(\{Z_r\}\) is an arbitrary \(g\)-unitary frame of the Lie algebra \(\mathfrak{g}\) of \(G\) (see [6,12], taking into account that here we adopt the convection \(\omega(\cdot, \cdot) = g(J \cdot, \cdot)\) in contrast to the one adopted in [6]). More generally, if \(G\) is just a Lie group with an invariant Hermitian structure, \(\rho^B\) takes the following expression

\[ \rho^B(X, Y) = -i \sum_{r=1}^n \left\{ g((X, Y)^{1,0}, Z_r) - g((X, Y)^{0,1}, \bar{Z}_r), Z_r \right\} - g((X, Y), [Z_r, \bar{Z}_r]) \tag{2} \]

(see [6,12], again). We have the following proposition.

**Proposition 1.1.** Let \(G\) be a 2n-dimensional 2-step nilpotent Lie group with a left-invariant SKT structure \((J, g)\). Then

\[ \rho^B(Z, \bar{Z}) = -i \sum_{r=1}^n \|[Z, \bar{Z}]_r\|^2 \]

for every \(Z \in \mathfrak{g}^{1,0}\), where \(\{Z_r\}\) is an arbitrary unitary frame. In particular,

\[ \rho^B(X, JX) \leq 0 \]

for every \(X \in \mathfrak{g}\).

**Proof.** Let \(Z\) and \(W\) be vector fields of type \((1,0)\) on \(\mathfrak{g} \otimes \mathbb{C}\), and let \(\omega\) be the fundamental form of \(g\). Then we directly compute

\[
\begin{align*}
\partial \bar{\partial} \omega(Z, \bar{Z}, W, \bar{W}) &= -\partial \bar{\omega}(\{Z, \bar{Z}\}, [W, \bar{W}]) + \partial \omega(\{Z, W\}, \bar{Z}, \bar{W}) - \bar{\partial} \omega(\{Z, \bar{W}\}, \bar{Z}, W) \\
&\quad - \partial \omega(\{Z, \bar{W}\}, W, \bar{W}) + \bar{\partial} \omega(\{\bar{Z}, W\}, Z, \bar{W}) - \bar{\partial} \omega(\{\bar{Z}, W\}, Z, W) \\
&\quad - \partial \omega(\{Z, W\}, [W, \bar{W}]) + \bar{\partial} \omega(\{\bar{Z}, W\}, [Z, \bar{W}]) - \bar{\partial} \omega(\{\bar{Z}, W\}, [Z, W]) \\
&\quad - \partial \omega(\{Z, W\}, [W, \bar{W}]) + \bar{\partial} \omega(\{\bar{Z}, W\}, [Z, \bar{W}]) - \bar{\partial} \omega(\{\bar{Z}, W\}, [Z, W]) \\
&\quad - \partial \omega(\{Z, \bar{W}\}, [W, \bar{W}]) + \bar{\partial} \omega(\{\bar{Z}, \bar{W}\}, [Z, \bar{W}]) - \bar{\partial} \omega(\{\bar{Z}, \bar{W}\}, [Z, \bar{W}]) \\
&\quad + i g(\{Z, \bar{Z}\}, [W, \bar{W}]) + i g(\{Z, W\}, [ar{Z}, \bar{W}]) + i g(\{Z, W\}, [\bar{Z}, W]) + i g(\{\bar{Z}, \bar{W}\}, [Z, \bar{W}]) + i g(\{\bar{Z}, \bar{W}\}, [Z, W]) \\
&\quad + i g(\{Z, \bar{Z}\}, [W, \bar{W}]) + i g(\{Z, \bar{W}\}, [W, \bar{W}]).
\end{align*}
\]

The SKT assumption \(\partial \bar{\partial} \omega = 0\) implies

\[ g(\{Z, \bar{Z}\}, [W, \bar{W}]) = -g(\{Z, \bar{W}\}, [\bar{Z}, W]) \]

Therefore, in view of (1), we get

\[ \rho^B(Z, \bar{Z}) = i \sum_{r=1}^n g(\{Z, \bar{Z}\}, [Z_r, \bar{Z}_r]) = -i \sum_{r=1}^n g(\{Z, \bar{Z}_r\}, [\bar{Z}, Z_r]) \]

being \(\{Z_r\}\) an arbitrary unitary frame, and the claim follows.

**Remark 1.2.** Another description of the Bismut–Ricci form on 2-step nilmanifolds can be found in [1].

Next we observe that in general the form \(\rho^B\) is not seminegative definite if we drop the assumption on \(G\) to be nilpotent or on the metric to be SKT.
Example 1.3. Let \( g \) be the solvable unimodular Lie algebra with structure equations
\[
de^1 = 0, \quad de^2 = -e^{13}, \quad de^3 = e^{12}, \quad de^4 = -e^{23},
\]
equipped with the complex structure \( Je_1 = e_4 \) and \( Je_2 = e_3 \) and the SKT metric
\[
g = \sum_{i=1}^{4} e^i \otimes e^i + \frac{1}{2} (e^1 \otimes e^3 + e^3 \otimes e^1) - \frac{1}{2} (e^2 \otimes e^4 + e^4 \otimes e^2).
\]
By using (2) with respect to a unitary frame \( \{Z_i\} \), we easily get
\[
\rho^B = \frac{2}{3} e^{12} - \frac{2}{3} e^{13} + \frac{4}{3} e^{23}.
\]
In particular,
\[
\rho^B(e_2, Je_2) = \frac{4}{3} \quad \text{and} \quad \rho^B(4e_1 + e_2, J(4e_1 + e_2)) = -\frac{4}{3}
\]
which implies that \( \rho^B \) is not seminegative definite as \((1,1)\)-form.

Example 1.4. Let \((g, J)\) be the 2-step nilpotent Lie algebra with structure equations
\[
de^1 = de^2 = de^3 = 0, \quad de^4 = e^{12}, \quad de^5 = -e^{23}, \quad de^6 = e^{13},
\]
equipped with the complex structure \( Je_1 = e_2, \) \( Je_3 = e_4 \) and \( Je_5 = e_6 \) and the non-SKT metric
\[
g = \sum_{i=1}^{6} e^i \otimes e^i + \frac{1}{2} (e^3 \otimes e^6 + e^6 \otimes e^3) - \frac{1}{2} (e^4 \otimes e^5 + e^5 \otimes e^4).
\]
Again by using (2) with respect to a unitary frame \( \{Z_i\} \), we easily get
\[
\rho^B = -e^{12} - \frac{1}{2} e^{23},
\]
which implies that \( \rho^B \) is not seminegative definite as \((1,1)\)-form.

2. Non-existence of invariant SKT metrics satisfying \((\rho^B)^{1,1} = \lambda \omega\)

In this section, we observe that our Proposition 1.1 easily implies that, on a 2-step nilpotent Lie group, there are no SKT invariant metrics such that
\[
(\rho^B)^{1,1} = \lambda \omega
\]
for some constant \( \lambda \). This result is already known: the case \( \lambda = 0 \) was studied in [4], while the case \( \lambda \neq 0 \) follows from [5].

Indeed, in the setting of Proposition 1.1, if we assume \( (\rho^B)^{1,1} = \lambda \omega \), then, taking into account that the center of \( G \) in not trivial, formula (1) implies \( \lambda = 0 \) and, from Proposition 1.1, it follows \([g^{1,0}, g^{0,1}] = 0\). Therefore, if \( \{\zeta^k\} \) is a unitary co-frame in \( g \), we have
\[
\delta \zeta^k = 0
\]
and we can write
\[
\delta \zeta^k = c^k_{rs} \zeta^r \wedge \zeta^s,
\]
for some \( c^k_{rs} \in \mathbb{C} \). Then
\[
\delta \delta \omega = i \delta (\sum_{k=1}^{n} \zeta^k \wedge \zeta^k) = -i \delta (\sum_{k=1}^{n} \zeta^k c^k_{rs} \zeta^r \wedge \zeta^s) = -i \sum_{k=1}^{n} \sum_{a,b} \bar{c}^k_{rs} \zeta^a \wedge \zeta^b \wedge \zeta^r \wedge \zeta^s
\]
and the SKT assumption implies that all the \( c^k_{rs} \)'s vanish in contrast to the assumption on \( G \) to be not abelian.
3. Long-time existence of the pluriclosed flow on 2-step nilmanifolds

The pluriclosed flow (PCF) is a parabolic flow of Hermitian metrics that preserves the SKT condition. The flow is defined on an SKT manifold \((M, \omega)\) as

\[ \partial_t \omega_t = - (\rho^B_\omega)^{1.1}, \quad \omega_{tt=0} = \omega, \]

where \(\rho^B_\omega\) is computed with respect to \(\omega_t\) and the superscript “1, 1” is the \((1, 1)\)-component with respect to \(J\). The flow was introduced in [8] and then investigated in [3,8–11], and it is a powerful tool in SKT geometry.

In [6], it is proved that on a 2-step nilpotent group, the flow has always a long-time solution for any initial invariant datum. The proof makes use of the bracket flow device introduce by Lauret in [7].

In our setting, let \(G\) be a 2-step nilpotent Lie group with a left-invariant complex structure \(J\) and consider the PCF equation starting form an invariant SKT form \(\omega\). The solution \(\omega_t\) holds invariant for every \(t\) and, therefore, the flow can be regarded as an ODE on \(\Lambda^2 g^* \otimes g\), where \(g\) is the Lie algebra of \(G\). The bracket flow device consists in evolving the Lie bracket on \(g\) instead of the form \(\omega\). For this purpose, one considers the bracket variety \(A\) consisting of the elements \(\lambda \in \Lambda^2 g^* \otimes g\) such that

\[
\begin{align*}
\lambda(\lambda(X, Y), V) &= 0, \\
\lambda(JX, JY) - J\lambda(JX, Y) - J\lambda(X, JY) - \lambda(X, Y) &= 0, \\
\partial_t \lambda_{\omega_t} &= 0,
\end{align*}
\]

for every \(X, Y, V \in g\), where the operators \(\partial_t\) and \(\tilde{\partial}_t\) are computed by using the bracket \(\lambda\). Any \(\lambda \in A\) gives a structure of 2-step nilpotent Lie algebra to \(g\) such that \((J, \omega)\) is an SKT structure. It turns out that the PCF is equivalent to a bracket flow-type equation, i.e. an ODE in \(A\). The equivalence between the two equations is obtained by evolving the initial bracket \(\mu\) of \(g\) as

\[
\mu_t(X, Y) = h_t\mu(h_t^{-1}X, h_t^{-1}Y), \quad X, Y \in g,
\]

being \(h_t\) the curve in \(\text{End}(g)\) solving

\[
\frac{d}{dt} h_t = -\frac{1}{2} h_t P_{\omega_t}, \quad h_{tt=0} = 1
\]

and \(P_{\omega_t} \in \text{End}(g)\) is defined by

\[
\omega_t(P_{\omega_t} X, Y) = \frac{1}{2} (\rho^B_{\omega_t}(X, Y) + \rho^B_{\omega_t}(JX, JY)).
\]

The form \(\omega_t\) reads in terms of \(h_t\) as

\[
\omega_t(X, Y) = \omega(h_t X, h_t Y).
\]

Now, in view of formula (1),

\[
\rho^B_{\omega_t}(X, \cdot) = 0 \quad \text{for every } X \in \xi
\]

and then \(\omega_t(X, \cdot) = \omega(X, \cdot)\) for every \(X \in \xi\), where \(\xi\) is the center of \(\mu\). Let \(\xi^\perp\) be the \(g\)-orthogonal complement of \(\xi\) in \(g\) and let \(g_t\) be the Hermitian metric corresponding to the solution to the PCF equation starting from \(\omega\). Then

\[
\frac{d}{dt} g_t(X, \cdot) = 0 \quad \text{for every } X \in \xi
\]

and \(g_t\) preserves the splitting \(g = \xi \oplus \xi^\perp\), and the flow evolves only the component of \(g\) in \(\xi^\perp \times \xi^\perp\). It follows that \(h_t\) preserves the splitting \(g = \xi \oplus \xi^\perp\) and

\[
h_{t|\xi} = 1_{\xi}.
\]

Since \((g, \mu)\) is 2-step nilpotent, then \(\mu(X, Y) \in \xi\) for every \(X, Y \in g\) and

\[
\mu_t(X, Y) = \mu(h_t^{-1}X, h_t^{-1}Y).
\]

Therefore

\[
\frac{d}{dt} \mu_t(X, Y) = -\mu(h_t^{-1}h_t X, h_t^{-1}Y) - \mu(h_t^{-1}X, h_t^{-1}h_t^{-1}Y)
\]

\[
= -\mu_t(h_t^{-1}X, Y) - \mu_t(X, h_t^{-1}Y) = \frac{1}{2} \mu_t(P_{\mu_t} X, Y) + \frac{1}{2} \mu_t(X, P_{\mu_t} Y),
\]

and

\[
\frac{d}{dt} \mu_t(X, Y) = \mu_t(X, P_{\mu_t} Y).
\]
where for any $\lambda \in \mathcal{A}$ we set
\[
\omega_t(P_{\lambda}X, Y) = \frac{1}{2} \sum_{r=1}^{n} \left( g(\lambda(X, Y), \lambda(Z_r, \bar{Z}_r)) + g(\lambda(JX, JY), \lambda(Z_r, \bar{Z}_r)) \right)
\]
being $\{Z_r\}$ an arbitrary $g$-unitary frame and in the last step we have used
\[
h_t P_{\omega_t} = P_{\mu_t} h_t.
\]
Hence the bracket flow equations writes as
\[
\frac{d}{dt} \mu_t(X, Y) = \frac{1}{2} \mu_t(P_{\mu_t} X, Y) + \frac{1}{2} \mu_t(X, P_{\mu_t} Y), \quad \mu|_{t=0} = \mu
\]
and its solution satisfies
\[
\frac{d}{dt} g(\mu_t, \mu_t) = 2g(\mu_t, \mu_t) = 4 \sum_{r,s=1}^{2n} g(\mu_t(P_{\mu_t} e_r, e_s), \mu_t(e_r, e_s))
\]
being $\{e_r\}$ an arbitrary $g$-orthonormal frame. In view of Proposition 1.1, all the eigenvalues of any $P_{\mu_t}$ are nonpositive. Fixing $t$ and taking as $\{e_r\}$ an orthonormal basis of eigenvectors of $P_{\mu_t}$, we get
\[
\frac{d}{dt} g(\mu_t, \mu_t) = 4 \sum_{r,s=1}^{2n} a_r g(\mu_t(e_r, e_s), \mu_t(e_r, e_s)) \leq 0.
\]
It follows that solution $\mu_t$ to (6) will stay forever in a compact subset, which implies that $\mu_t$ is defined for every $t \in [0, \infty)$, and the claim follows.

Acknowledgement

The authors are grateful to Anna Fino for very useful conversations.

References

[1] R. Arroyo, R. Lafuente, The long-time behavior of the homogeneous pluriclosed flow, arXiv:1712.02075.
[2] J.-M. Bismut, A local index theorem for non-Kähler manifolds, Math. Ann. 284 (4) (1989) 681–699.
[3] J. Boling, Homogeneous solutions of pluriclosed flow on closed complex surfaces, J. Geom. Anal. 26 (3) (2016) 2130–2154.
[4] N. Enrietti, Static SKT metrics and strong Kähler with torsion metrics, J. Symplectic Geom. 10 (2) (2012) 203–223.
[5] N. Enrietti, A. Fino, L. Vezzoni, Tamed symplectic forms and strong Kähler with torsion metrics, J. Symplectic Geom. 10 (2) (2012) 203–223.
[6] N. Enrietti, A. Fino, L. Vezzoni, The pluriclosed flow on nilmanifolds and Tamed symplectic forms, J. Geom. Anal. 25 (2) (2015) 883–909.
[7] J. Laurent, The Ricci flow for simply connected nilmanifolds, Commun. Anal. Geom. 19 (5) (2011) 831–854.
[8] J. Streets, G. Tian, A parabolic flow of pluriconcaved metrics, Int. Math. Res. Not. (2010) 3101–3113.
[9] J. Streets, G. Tian, Regularity results for pluriclosed flow, Geom. Topol. 17 (4) (2013) 2389–2429.
[10] J. Streets, Pluriclosed flow, Born–Infeld geometry, and rigidity results for generalized Kähler manifolds, Commun. Partial Differ. Equ. 41 (2) (2016) 318–374.
[11] J. Streets, Pluriclosed flow on manifolds with globally generated bundles, Complex Manifolds 3 (2016) 222–230.
[12] L. Vezzoni, A note on canonical Ricci forms on 2-step nilmanifolds, Proc. Amer. Math. Soc. 141 (1) (2013) 325–333.