Borel–Cantelli lemmas and extreme value theory for geometric Lorenz models

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Abstract

We establish dynamical Borel–Cantelli lemmas for nested balls and rectangles centered at generic points in the setting of geometric Lorenz maps. We also establish the convergence of rare events point processes to the standard Poisson process, which implies extreme value laws for observations maximised at generic points for geometric Lorenz maps. Further, we extend our extreme value laws to the associated flows.

Keywords: Lorenz system, Borel–Cantelli lemmas, extreme value laws, REPP (rare events points process)

Mathematics Subject Classification numbers: 37, 62

(Some figures may appear in colour only in the online journal)

1. Introduction

In a chaotic system, the future behaviour of the system is very sensitive to the initial conditions, so a statistical description of the system’s behaviour is often the most appropriate object to study. We may investigate whether suitable versions of classical limit theorems from probability theory such as the law of large numbers, central limit theorem, Borel–Cantelli lemmas, extreme value theory and so on hold and use this knowledge to make predictions about the system’s behaviour. In this paper, we study a particular system, the Lorenz system, and establish strong Borel–Cantelli lemmas, Poisson laws for REPPs (rare events points processes) and the extreme value laws for it.

The equations defining the Lorenz system were first published in the Journal of Atmospheric Sciences ([14]) as a parametrised polynomial system of differential equations:

\[
\begin{align*}
\dot{x} &= \sigma (y - x) \\
\dot{y} &= x(r - z) - y \\
\dot{z} &= xy - \beta z
\end{align*}
\]
where $\sigma = 10$, $\rho = 28$, $\beta = 8/3$. The system was proposed as a simplified model for thermal fluid convection, motivated by a desire to understand weather systems. What is interesting is that the equations are deterministic but they produce chaotic behaviour, with trajectories spiralling around two attractors seemingly randomly. Figure 1 is the Lorenz attractor, the famous ‘butterfly’.

In order to achieve insights into this system, a very successful approach was taken by Afraimovich, Bykov and Shil’nikov [1], and Guckenheimer and Williams [9] independently: they constructed the so-called geometric Lorenz models. These models are flows in three dimensions which have properties very similar to the Lorenz systems and are easier to study. One can rigorously prove the existence of an attractor that contains an equilibrium point of the flow, together with regular solutions. The original proof of the existence of a chaotic attractor was made by Warwick Tucker in the year 2000, with the help of computer (see [19, 20]).

A brief version of the construction of the geometric Lorenz model is given in appendix A, and a more detailed version can be found in [8, section 2.1]. As we can see in the construction, the Lorenz map $F$ has a skew product form $F(x, y) = (T(x), G(x, y))$, where $F$ does not preserve the two-dimensional Lebesgue measure $m_2$ but the one-dimensional map $T$ preserves a measure absolutely continuously with respect to the one-dimensional Lebesgue measure $m$, with a Lipschitz density.

Much recent work has focused on the ergodic and statistical properties of Lorenz-like maps including rates of mixing, extreme value theory and return time statistics. Galatolo and Pacifico [8] proved that the Poincaré map, i.e. our Lorenz map $F$, associated with a Lorenz-like flow has an exponential decay of correlations with respect to Lipschitz observables and the hitting time statistics satisfies a logarithm law.
In the next few sections, Borel–Cantelli lemmas, extreme value laws and the rare event point process will be introduced. Readers who are familiar with this knowledge may skip to section 1.4 for the main results of this paper at a glance.

1.1. Local dimension

Let \((M, d)\) be a metric space and assume that \(\mu\) is a Borel probability measure on \(M\). Given \(x \in M\), let \(B(x, r) = \{y \in M : d(x, y) \leq r\}\) be the ball centered at \(x\) with radius \(r\). The local dimension of \(\mu\) at \(x \in M\) is defined by
\[
d_{\mu}(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r},
\]
if this limit exists. In this case, for any \(\epsilon > 0\), there exists small \(r_0\) such that for \(0 < r \leq r_0\),
\[
r d_{\mu}(x) - \epsilon \leq \mu(B(x, r)) \leq n d_{\mu}(x) + \epsilon.
\]

A result of Afraimovich and Pesin [2, theorem 9] ensures that for the Lorenz system, the local dimension exists and is constant for \(\mu\) a.e. point.

1.2. Borel–Cantelli lemmas

The classical Borel–Cantelli lemmas are as follows: suppose \((\Omega, \mathcal{B}, \mu)\) is a probability space. Let \(\mathbf{1}_A\) be the characteristic function of \(A\), given \(A\) is a measurable set of \(\Omega\). Then

1. if \((A_n)_{n=0}^{\infty}\) is a sequence of measurable sets in \(\Omega\) and \(\sum_{n=0}^{\infty} \mu(A_n) < \infty\), then \(\mu(\bigcap_{n=0}^{\infty} A_n) = 0\)
2. if \((A_n)_{n=0}^{\infty}\) is a sequence of independent sets in \(\Omega\) and \(\sum_{n=0}^{\infty} \mu(A_n) = \infty\), then for \(\mu\) a.e. \(x \in \Omega\)
   \[
   \frac{S_n(x)}{E_n} \to 1,
   \]
   where \(S_n(x) = \sum_{j=0}^{n-1} \mathbf{1}_{A_j}(x)\) and \(E_n = \sum_{j=0}^{n-1} \mu(A_j)\).

In the dynamical systems setting, \(T : \Omega \to \Omega\) is usually considered to be a measure-preserving transformation of the probability space \((\Omega, \mathcal{B}, \mu)\). Suppose that \((A_n)_{n=0}^{\infty}\) is a sequence of sets in \(\mathcal{B}\) such that \(\sum_{n=0}^{\infty} \mu(A_n) = \infty\). Let \(E_n = \sum_{j=0}^{n-1} \mu(A_j)\) and \(S_n(x) = \sum_{j=0}^{n-1} \mathbf{1}_{A_j} \circ T^j(x)\). Then we call the sequence \((A_n)\):

1. A Borel–Cantelli sequence (BC) if \(\mu(\{x : T^n x \in A_n i.o.\}) = 1\), i.e. \(S_n(x)\) is unbounded.
2. A strong Borel–Cantelli sequence (SBC) if \(\lim_{n \to \infty} \frac{S_n(x)}{E_n} = 1\), a.s.

Remark 1.0.1 If the sequence \((A_n)_{n=0}^{\infty}\) is nested balls of radius \(r_n\) about a point \(p\) in the dynamical system \((T, \Omega, \mu)\), then the question of whether \(T^n x \in A_i\) infinitely often for \(\mu\) a.e. \(x\) is called the shrinking target problem. In this paper, we establish strong Borel–Cantelli lemmas and extreme value laws for the Lorenz maps \(F\) with a shrinking target property.

1.3. Extreme value laws (EVLs)

We consider a dynamical system \((\Omega, \mathcal{B}, \mu, F)\), where \(F\) preserves an invariant measure \(\mu\). Consider the time series \(X_0, X_1, X_2 \cdots\) arising from such a system simply by evaluating a given random variable (r.v.) \(\varphi : \Omega \to \mathbb{R} \cup \{\pm \infty\}\) along the orbits of the system, that is to say, we define
for each \( n \in \mathbb{N} \). Clearly, \( X_0, X_1, \ldots \) defined in this way is not an independent sequence, but \( F \)-invariance of \( \mu \) guarantees that this stochastic process is stationary.

Here we suppose that \( \varphi \) has one global maximum at \( \zeta \in \Omega \) (\( \varphi(\zeta) = +\infty \) is allowed), and let \( u_F := \varphi(\zeta) \). By assuming that \( \varphi \) and \( \mu \) are sufficiently regular, the event

\[
U(u) := \{ x \in \Omega : \varphi(x) > u \} = [X_0 > u]
\]

corresponds to a topological ball centered at \( \zeta \) for \( u \) sufficiently close to \( u_F \).

**Definition 1.1 (Logarithmic singularity).** Consider a function \( \varphi : \Omega \to \mathbb{R} \) and a point \( x_0 \in \Omega \). Let \( d \) be a distance function on \( \Omega \). We say that \( \varphi \) has a logarithmic singularity at the point \( x_0 \) if \( x_0 \) has a neighbourhood where

\[
\varphi(x) = -C \log d(x, x_0) + g(x) \quad \text{with} \quad C > 0,
\]

where \( g \) is bounded and has a finite limit as \( x \to x_0 \).

We are always interested in studying the extremal behaviour of the stochastic process \( X_0, X_1, \ldots \), and it is associated with the occurrence of exceedances of high levels \( u \). The occurrence of an exceedance at time \( j \in \mathbb{N}_0 \) means that the event \( \{ X_j > u \} \) occurs, where \( u \) is close to \( u_F \). This is equivalent to saying that the orbit of the point \( x \) hits the ball \( U(u) \) at time \( j \), i.e.

\[
F^j(x) \in U(u).
\]

In order to consider the extremal behaviour of the system for which we define a new sequence of random variables \( M_0, M_1, \ldots \) given by

\[
M_n = \max\{X_0, \ldots, X_{n-1}\}.
\]

**Definition 1.2 (Extreme value laws (EVLs)).** We say that we have an EVL for \( M_n \) if there is a non-degenerate distribution function (d.f.) \( G : \mathbb{R} \to [0, 1] \) with \( G(0) = 0 \) and, for every \( \nu > 0 \), there exists a sequence of levels \( \nu_n = u_{\nu_n}, \ldots \) such that

\[
\lim_{n \to \infty} n\mu(X_0 > u_n) = \nu,
\]

and for which the following holds:

\[
\mu(M_n \leq u_n) \to G(\nu) \quad \text{as} \quad n \to \infty,
\]

where \( \bar{G} = 1 - G \).

The motivation for using such normalising sequences (1.2) comes from the case when \( X_0, X_1, \ldots \) are independent and identically distributed (i.i.d.). In this i.i.d. setting, it is clear that

\[
P(M_n \leq u) = (Z(u))^n,
\]

where \( Z \) is the d.f. of \( X_0 \), i.e. \( Z(x) := P(X_0 \leq x) \). Hence, condition (1.2) implies that

\[
\mathbb{P}(M_n \leq u) = (1 - P(X_0 > u_n))^n \sim \left(1 - \frac{\nu}{n}\right)^n \to e^{-\nu},
\]

as \( n \to \infty \). This means that the waiting times between the exceedances of \( u_n \) are approximately exponentially distributed. Moreover, the reciprocal is also true. Note that in this case \( G(\nu) = 1 - e^{-\nu} \) is the standard exponential d.f.

**Remark 1.2.1.** We will give results on Lorenz flows \( f_t \) as well, in which case we consider the continuous time stochastic process \( \{ X_t \} \) such that \( X_t = \varphi \circ f_t \), and define the process of successive maxima \( \varphi_t := \sup_{0 \leq s \leq t}\{X_s\} \).

**Remark 1.2.2.** For independent and identically distributed (i.i.d.) processes, there are only three possible types of non-degenerate extremal distributions (subject to linear scaling):
• Type I

\[ G(x) = e^{-e^{-x}}, \quad -\infty < x < \infty \]

• Type II

\[ G(x) = \begin{cases} 0 & \text{if } x < 0; \\ e^{-\alpha x} & \text{for some } \alpha \text{ if } x > 0. \end{cases} \]

• Type III

\[ G(x) = \begin{cases} e^{-(1-\alpha)x} & \text{for some } \alpha > 0 \text{ if } x < 0; \\ 1 & \text{if } x > 0. \end{cases} \]

For dependent stationary processes \( \{X_n\} \), Leadbetter [13] gives two conditions called \( D(u_n) \) and \( D'(u_n) \) for sequence \( \{u_n\} \) satisfying (1.2), in which case EVL will hold and the limiting distribution is in a specific form, that is, \( \eta u(X_0 > u_0) \to \nu \) is equivalent to \( \mu(M_n \leq u_n) \to e^{-\nu} \), which implies \( G(\nu) = 1 - e^{-\nu} \) is a standard exponential d.f. In fact, the sequence \( \{u_n\} \) may not simply depend on any parameter \( \nu \) (e.g. a linear function of \( \nu \)) or may be more complicated functions than the linear one. However, most of recent work has focused on linear scaling.

There are no general techniques for proving these two conditions \( D(u_n) \) and \( D'(u_n) \). In Collet’s paper [4] he used the rate of decay of correlations of Hölder observations to establish \( D(u_n) \) for certain one-dimensional non-uniformly expanding maps. Freitas et al [5], based on Collet’s work, gave a condition \( D_2(u_n) \) which has the full force of \( D(u_n) \) and is relatively easier to establish in the dynamical setting by estimating the rate of decay of correlations of Hölder continuous observations or those of bounded variations. In this paper, we use \( D_2(u_n) \) and \( D'(u_n) \) to establish the results, and the definitions of \( D_2(u_n) \) and \( D'(u_n) \) are given in the section 1.3.1.

For \( x_0 \in \Omega \), if we consider

\[ \varphi(x) = -\log d(x, x_0), \quad (1.3) \]

where \( d(\cdot, \cdot) \) is the local metric on \( \Omega \) and \( x_0 \) is a generic point (we fix one point \( x_0 \)). Let \( X_n = \varphi \circ F^n \) and define \( U_n = \{X_0 > u_n\} \), where \( u_n = u_n(\nu) \) such that \( \mu(U_n) = e^{-\nu/n} \) (in this case, we will see later in this paper that \( u_n \) is roughly a linear function of this \( \nu \)). Here \( u_n \) is an increasing sequence going to \( \varphi(x_0) \) (which is \( +\infty \)) and assume \( U_n \) corresponds to a topological ball centered at \( x_0 \) with radius \( e^{-\nu} \). Then the corresponding processes \( \{X_n\} \) will satisfy Type I extremal distribution.

To be consistent, if not specified, we will use the definitions and assumptions above for the section on EVLs.

**Remark 1.2.3.** We define a function that we refer to as first hitting time function to a set \( A \in \mathcal{B} \), and denote by \( r_A : \Omega \to \mathbb{N} \cup \{\infty\} \) where

\[ r_A(x) = \min\{ j \in \mathbb{N} \cup \{\infty\} : F^j(x) \in A \}. \]

The restriction of \( r_A \) to \( A \) is called the first return time function to \( A \), denoted by \( R(A) \), as the minimum of the return time function to \( A \), i.e.

\[ R(A) = \min_{x \in A} r_A(x). \]
In [6], the equivalence between the EVL and the hitting time statistics (HTS)/returning time statistics (RTS) (for balls) of stochastic processes defined by (1.1) was obtained for dynamical systems $(\Omega, F, \mu)$ admitting an absolutely continuous invariant probability measure $\mu$, i.e. if such processes have an EVL $G$ then the system has HTS $G$ as well for balls ‘centered’ at $\zeta$, and vice versa. So, it is natural to use the observable (1.3), it gives balls centered at $x_0$, and we have:

$$\{M_n \leq u_n\} = \{r_{(x_0 > u_n)} > n\} = U_n^c$$

We may also consider the statistics of multiple returns, which we discuss in the next section.

1.3.1 Rare events points processes (REPPs) and the respective convergence. Let us introduce some formalism first. Let $\mathcal{S} = \{[a, b] \mid a, b \in \mathbb{R}^+\}$. Let $\mathcal{R}$ denote the ring generated by $\mathcal{S}$, i.e. $\mathcal{R} = \{J \mid \exists k \text{ such that } J = \bigcup_{j=1}^k I_j, I_j \in \mathcal{S}, j = 1, \ldots, k\}$. For $J = [a, b) \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, we define $\alpha J := [\alpha a, \alpha b)$ and $I + \alpha := [a + \alpha, b + \alpha)$. Similarly, for $J \in \mathcal{R}$ define $J + \alpha := (I_1 + \alpha) \cup \cdots \cup (I_k + \alpha)$.

**Definition 1.3.** For stationary processes $X_0, X_1, \ldots$ and sequences $(u_n)_{n \in \mathbb{N}}$ satisfying (1.2), we define the REPP by counting the number of exceedances (or hits to $U(u_n)$) during the re-scaled time period $a_n J \in \mathcal{R}$, where $J \in \mathcal{R}$ and $a_n := 1/\mu(X_0 > u_n)$ is, according to Kac’s theorem, the expected waiting time before the occurrence of one exceedance. To be more precise, for every $J \in \mathcal{R}$, set

$$N_0(J) := \sum_{j \in a_n J \cap [0, \infty)} 1_{\{X_j \geq u_n\}}$$

**Definition 1.4 (Poisson process).** Let $T_1, T_2, \ldots$ be an i.i.d. sequence of random variables with a common exponential distribution of mean $1/\theta$. Given this sequence of r.v., for $J \in \mathcal{R}$, set

$$N(J) = \left\{ i \in \mathbb{N} : \sum_{j=1}^i T_j \in J \right\}$$

where $|\cdot|$ denotes the cardinality of a set. We say that $N$ defined in this way is a Poisson process of intensity $\theta$. In a special case, when $J = [0, t)$, we also denote $N([0, t))$ by $N(t)$.

**Remark 1.4.1.** If $\theta = 1$ then we say that $N$ is a standard Poisson process and, for every $t > 0$, the random variable $N(t)$ has a Poisson distribution of mean $t$. In general, the random variable $N(J)$ has distribution

$$\mathbb{P}(N(J) = k) = e^{-m(J)}\frac{(m(J))^k}{k!}$$

where $m(J)$ is the Lebesgue measure of $J$.

**Definition 1.5.** Suppose that $(N_k)_{k \in \mathbb{N}}$ is a sequence of point processes defined on $\mathcal{S}$ and $N$ is a standard Poisson process defined on $\mathcal{S}$. We say that $N_k$ converges in distribution to $N$ if the sequence of vector random variables $(N_k(J_1), N_k(J_2), \ldots, N_k(J_k))$ converges in distribution to $(N(J_1), N(J_2), \ldots, N(J_k))$, for every $k \in \mathbb{N}$ and all $J_1, J_2, \ldots, J_k \in \mathcal{S}$ such that $N(\partial J_i) = 0$ a.s., for $i = 1, \ldots, k$.

**Remark 1.5.1.** The convergence of the REPP to the Poisson process is stronger than the existence of an EVL for $X_0, X_1, \ldots, X_n$. In particular, not only can we recover the distributional
limit for the maxima \( \{ M_n \} \) since \( \{ M_n \leq u_n \} = \{ N_0(n/a_n) = 0 \} \), we can also obtain the distributional limit of the order statistics, that is, if \( X_{1,n} \leq X_{2,n} \leq \cdots \leq X_{n,n} \) denote the order statistics of the first \( n \) random variables of the process, then \( \{ X_{n-L,n} \leq u_n \} = \{ N_0(n/a_n) \leq k \} \).

In [7, p 4], two conditions \( D_3(u_n) \) and \( D'(u_n) \) (the same \( D'(u_n) \) we mentioned before) are given on the dependence structure of a general stationary stochastic process to ensure that the REPP \( N_0 \) converges in distribution to a standard Poisson process. \( D_3(u_n) \) is closely related to the condition \( D_2(u_n) \) and their proofs are similar, since they can be easily checked by using the decay of correlations of the Lorenz systems. We will prove \( D_3(u_n) \) and \( D'(u_n) \) for the REPP. The formulation of these three conditions are given as follows.

For every \( B \in \mathcal{R} \), let
\[
M(B) := \max \{ X_i : i \in B \cap \mathbb{Z} \}
\]
In particular, when \( B = [0, n) \), we have \( M(B) = \mu([0, n)) = M_0 \). Note that \( \{ M(B) \leq u_n \} = \{ N_0(a_n^{-1} B) = 0 \} \).

**Condition** \( D_3(u_n) \): We say that \( D_3(u_n) \) holds for the sequence \( X_0, X_1, X_2, \cdots \) if for all \( B \in \mathcal{R} \) and \( r \in \mathbb{N} \),
\[
\mu((X_0 > u_n) \cap (B + t) \leq u_n)) - \mu((X_0 > u_n) \mu((M(B) \leq u_n)) \leq \gamma(n, t),
\]
where \( \gamma(n, t) \) is non-increasing in \( t \) for each \( n \) and \( n\gamma(n, t_n) \to 0 \) as \( n \to \infty \) for some sequence \( t_n = o(n), t_n \to \infty \).

**Condition** \( D_2(u_n) \): We say that \( D_2(u_n) \) holds for the sequence \( X_0, X_1, X_2, \cdots \) if for any integers \( l, t \) and \( n \)
\[
\mu((X_0 > u_n) \cap (M_l \leq u_n)) - \mu((X_0 > u_n) \mu((M_l \leq u_n)) \leq \gamma(n, t),
\]
where \( M_l = \max \{ X_i, X_{i+1}, \cdots, X_{i+l-1} \} \), and \( \gamma(n, t) \) is non-increasing in \( t \) for each \( n \) and \( n\gamma(n, t_n) \to 0 \) as \( n \to \infty \) for some sequence \( t_n = o(n), t_n \to \infty \).

**Condition** \( D'(u_n) \): The condition \( D'(u_n) \) is said to hold for the stationary sequence \( \{ X_i \} \) and the sequence \( \{ u_n \} \) if
\[
\limsup_{n \to \infty} n^{-1} \sum_{j=1}^{[nk]} \mu(X_0 > u_n, X_j > u_n) \to 0,
\]
as \( k \to \infty \).

Condition \( D_3(u_n) \) is a sort of mixing requirement specially adapted to the problem of counting exceedances. While \( D_2(u_n) \) is a condition on the long-range dependence structure of the stochastic process \( X_0, X_1, \cdots \), condition \( D'(u_n) \) is a non-clustering condition, which states that if a large reading is observed (say the level \( u_n \)) at some time \( j < n \), then one must wait for a large time \( o(n) \to \infty \) before another reading larger than or equal to \( u_n \) is observed.

**Assumption A:** For \( \mu \) a.e. \( p \in \Omega \) there exists \( \tilde{d} = d(p) > 0 \) such that if \( A_{r, \epsilon}(p) = \{ y \in \Omega : r \leq d(p, y) \leq r + \epsilon \} \) is a shell of inner radius \( r \) and outer radius \( r + \epsilon \) about the point \( p \), and if \( r \) is sufficiently small and \( 0 < \epsilon \ll r < 1 \), then \( \mu(A_{r, \epsilon}) < \epsilon \tilde{d} \).

**Remark 1.5.2.** For systems satisfying Assumption A, condition \( D_2(u_n)(D_3(u_n)) \) often follows easily if there are good enough estimates on the decay of correlation for observations in a suitable Banach space.

Freitas, Haydn and Nicol [7] show that the REPP \( N_0 \) converges in distribution to a standard Poisson process for functions maximised at generic points in a variety of billiard systems. They prove this by verifying that the conditions \( D_3(u_n) \) and \( D'(u_n) \) hold for such systems.
1.4. Main results

In section 2, to introduce the main ideas of our analysis of the Lorenz system in a simpler setting, we establish the results of independent interest, namely shrinking target properties for general skew product maps which preserve the two-dimensional Lebesgue measure. We have the following theorem:

**Theorem 1.6.** Suppose \((Ω, B, m_2)\) is a probability space, where \(Ω = I \times I\), with \(I = \left[ -\frac{1}{2}, \frac{1}{2} \right] \) and \(m_2\) is the two-dimensional Lebesgue measure. Let \(m\) denote the one-dimensional Lebesgue measure. Suppose \(\Omega \mapsto \Omega\) is a map in the form \(F(x, y) = (T(x), G(x, y))\), where \(T : I \mapsto I\). Here \(F\) preserves the two-dimensional Lebesgue measure and \(T\) preserves the one-dimensional Lebesgue measure. In addition, \(T\) satisfies the exponential decay of correlation with observables in \(BV\) norm versus \(L^1\) norm (or \(L^1\) v.s. \(L^1\)), i.e.

\[
| \int \phi \circ T^n dm - \int \phi dm \int \psi dm | \leq C \theta^n \| \phi \|_{BV} \| \psi \|,
\]

or

\[
| \int \phi \circ T^n dm - \int \phi dm \int \psi dm | \leq C \theta^n \| \phi \| \| \psi \|,
\]

and \(F\) satisfies the exponential decay of correlation with observables in the Lipschitz norm versus the Lipschitz norm (or Lipschitz norm versus \(L^\infty\) norm):

\[
| \int \phi \circ F^n dm_2 - \int \phi dm_2 \int \psi dm_2 | \leq C \alpha^n \| \phi \|_{Lip} \| \psi \|_{Lip},
\]

or

\[
| \int \phi \circ F^n dm_2 - \int \phi dm_2 \int \psi dm_2 | \leq C \alpha^n \| \phi \|_{Lip} \| \psi \|_{L^\infty}.
\]

Consider nested balls \((B_i(p))\), centered at some \(p \in Ω\), with \(m_2(B_i(p)) \geq \frac{C}{i^r}\) for some \(γ_i > 0\), and \(\lim \sup (\log i)(m_2(B_i))^{\frac{1}{r}} \leq C\). Then we have the strong Borel–Cantelli property

\[
\frac{S_n(x, y)}{E_n} \rightarrow 1 \text{ a.s.}
\]

where \(S_n(x, y) = \sum_{j=0}^{n-1} 1_{B_j} \circ F^j(x, y), E_n = \sum_{j=0}^{n-1} m_2(B_j)\).

Then, in section 3 we establish the shrinking target property for the two-dimensional Lorenz map \(F\) and extreme value laws for the system with the Lorenz map. When we consider shrinking target balls, they have different shapes according to different metrics. Technically, balls of different shapes are equivalent to each other, but we are able to deal with rectangles and circular balls. While rectangles consist of local unstable manifold of same length, circular balls do not. For nested rectangles, we assume they centered at a same point and have same ratio between their lengths and widths. One special case is when they are squares, we prove the following theorem:

**Theorem 1.7.** Consider a sequence of nested squares \(\{A_i\}\) centered at a point \(p\), of side length \(2r(i)\) such that \(μ(A_i) \geq \frac{C}{i^r}\) with \(γ_i > 0\). Assume that \(p\) has a local product structure for sufficiently small neighbourhoods. Also, we assume \((\log i)m(π(A_i))\) is bounded, where \(γ\) is any local unstable manifold of \(A_i\) and \(π\) is the projection map onto the unstable manifold.
Then if $\sum_i \mu(A_i)$ diverges we have the strong Borel–Cantelli property for the squares $\{A_i\}$ of side length $2r(i)$.

**Remark 1.7.1.** Theorem 1.7 holds for rectangles of bounded aspect ratio as well.

Also, for circular balls, we show:

**Theorem 1.8.** Consider a sequence of nested circle balls $\{A_i\}$ centered at a point $p$ of radius $r(i)$ such that $\mu(A_i) \geq \frac{C_i}{\gamma_i}$, with $\gamma_i > 0$. Also, assume that $p$ has a local product structure for sufficiently small neighbourhoods. Then if $\sum_i \mu(A_i)$ diverges we have the strong Borel property for the balls $\{A_i\}$.

We use techniques from section 3.2 (about EVLs) to prove theorem 1.8, therefore we will prove it after we establish EVLs for the system with the Lorenz map $F$. We also show:

**Theorem 1.9.** Consider the dynamical systems $(\Omega, B, \mu, F)$, where $F$ is the Lorenz map and $F$ preserves the measure $\mu$, whose decomposition on the unstable leaves is absolutely continuous with respect to the one-dimensional Lebesgue measure. Let $\varphi$ be defined as (1.3), $X_0 = \varphi \circ F^\alpha$. For $\mu$ a.e. $x_0$ (we in particular assume $x_0$ is not periodic under $F$), we assume the sequence $\{u_n\}$ satisfies $\eta_\mu(X_0 > u_n) \to e^{-r}$. Then $\text{REPP} N_\mu$ given in definition 1.3 converges in distribution to the standard Poisson process.

Then the next corollary immediately follows:

**Corollary 1.10.** Under the same setting as above $X_n$ satisfies a type I EVL, i.e.

$$\lim_{n \to \infty} \mu(M_n \leq u_n) = e^{-e^r}.$$  

In section 4, we extend our results to the Lorenz flow.

### 2. Volume preserving skew products

As we mentioned in the introduction, the Lorenz map $F$ has a skew product form $F(x, y) = (T(x), G(x, y))$, and $F$ does not preserve the two-dimensional Lebesgue measure $m_2$, but the one-dimensional map $T$ preserves a measure absolutely continuously with respect to the one-dimensional Lebesgue measure $m$, with a Lipschitz density. In this section, we talk about general skew product maps $\tilde{F}(x, y) = (T(x), G(x, y))$, which preserve the two-dimensional Lebesgue measure, and $T$ preserves the one-dimensional Lebesgue measure. We stated our theorem 1.6 in the introduction, now let us prove it under the assumption that $T$ satisfies the exponential decay of correlation with observables in the $L^1$ norm versus the $L^1$ norm

$$| \int \phi \psi \circ T^a dm - \int \phi dm \int \psi dm | \leq C\theta \|\phi\|_{BV} \|\psi\|.$$

and $F$ satisfies the exponential decay of correlation with observables in the Lipschitz norm versus the Lipschitz norm:

$$| \int \phi \psi \circ F^a dm_2 - \int \phi dm_2 \int \psi dm_2 | \leq C\alpha \|\phi\|_{Lip} \|\psi\|_{Lip}.$$  

For other options of norms, the proof works in an analogous way with slight changes.

**Proof of theorem 1.6.** Let $f_k(x, y) = 1_{B_k} \circ \tilde{F}^k(x, y), E(f_k) = m_2(B_k)$ and let $a > \frac{\gamma_1}{\log \alpha}$, we will use $a$ later. To prove the strong Borel–Cantelli property, according to [11] it suffices to show the (SP) property (see appendix), i.e. for all $m < n$
\[
\sum_{i=m}^{n} \sum_{j=i+1}^{n} (E(f_jf_i) - E(f_i)E(f_j)) \leq C \sum_{i=m}^{n} E(f_i).
\]

We calculate
\[
E(f_jf_i) = \int 1_{B_i} \circ \tilde{F}^i(x,y) \cdot 1_{B_j} \circ \tilde{F}^j(x,y) \, dm_2
\]
\[
= \int 1_{B_i} \cdot 1_{B_j} \circ \tilde{F}^{j-i}(x,y) \, dm_2
\]
\[
= \frac{m_2(B_i \cap \tilde{F}^{j-i}(B_j))}{m_2(B_j \cap \tilde{F}^{j-i}(B_i))}
\]
\[
\leq C(m_2(B_i))^{\frac{1}{2}} \cdot m(\pi_\gamma B_i \cap T^{j-i}(\pi_\gamma B_j)) \quad (*)
\]

where $\pi_\gamma B_i$ is the projection of the maximal horizontal section of the ball $B_i$ onto the unstable manifold. If $\tilde{x} = (x,y) \in B_i$ and $\tilde{F}^{j-i}(\tilde{x}) \in B_j$, then their projection $x \in \pi_\gamma B_i$ and $T^{j-i}(x) \in \pi_\gamma B_j$ so
\[
m_2(B_i \cap \tilde{F}^{j-i}(B_j)) = \int m(B_i \cap \tilde{F}^{j-i}(B_j)) \, dm(y) = \int m([\tilde{x} = (x,y) ; \tilde{x} \in B_i, \tilde{F}^{j-i}(\tilde{x}) \in B_j]) \, dm(y) \leq \int m(x \in \pi_\gamma B_i, T^{j-i}(x) \in \pi_\gamma B_j) \, dm(y) \leq m(\pi_\gamma B_i) \cdot \int m(x \in \pi_\gamma B_i, T^{j-i}(x) \in \pi_\gamma B_j) \, dm(y),
\]
where $\pi_\gamma B_i$ is the projection of the maximal vertical section of $B_i$ onto the stable manifold, and
\[
m(\pi_\gamma B_i) = m(\pi_\gamma B_i) \leq C(m_2(B_i))^{\frac{1}{2}}.
\]
Thus, we continue calculating $E(f_jf_i)$:
\[
(*) = C(m_2(B_i))^{\frac{1}{2}} \cdot \int 1_{\alpha B_i} \cdot 1_{\alpha B_j} \circ T^{j-i}(x) \, dm
\]
\[
\leq C(m_2(B_i))^{\frac{1}{2}} \cdot \left( \int 1_{\alpha B_i} \int 1_{\alpha B_j} + C \theta^{j-i} \|1_{\alpha B_i}\|_{Lip} \|1_{\alpha B_j}\|_{Lip} \right)
\]
\[
\leq C(m_2(B_i))^{\frac{1}{2}} \left[ (m_2(B_i))^{\frac{1}{2}} \cdot (m_2(B_j))^{\frac{1}{2}} + C \theta^{j-i}(m_2(B_i))^{\frac{1}{2}} \right]
\]
\[
\leq C(m_2(B_i))^{\frac{3}{2}} + C \theta^{j-i}(m_2(B_i)).
\]

since $\int 1_{\alpha B_i} \, dm = m(\pi_\gamma B_i) = (m_2(B_i))^{\frac{1}{2}}$.

Recall $a \geq \frac{-\gamma}{\log \alpha}$, so that
\[
\sum_{j=i+1}^{n} (E(f_jf_i) - E(f_i)E(f_j))
\]
\[
\leq \left( \sum_{j=i+1}^{n} + \sum_{j > i + a \log i} \right) \left[ E(f_jf_i) - E(f_i)E(f_j) \right]
\]
\[
\leq C(\log i)(m_2(B_i))^{\frac{3}{2}} + Cm_2(B_i) \sum_{j > i + a \log i} C \alpha^{j-i} \|f_j\|_{Lip} \|f_i\|_{Lip} + O\left( \frac{1}{i^{7/2}} \right),
\]

where $\tilde{f}_i$ is a Lipschitz approximation to $f_i$, which is constructed as follows: $\tilde{f}_i = 1_B(x)$ if $x \in B_i, \tilde{f}_i = 0$ if $d(B_i, x) > 1/i^{2\gamma}, 0 \leq \tilde{f}_i \leq 1$ and $\|\tilde{f}_i\|_{Lip} \leq i^{2\gamma}$. 


Then
\[
\sum_{j > i + a \log i} (\sum_{j > i + a \log i} \alpha_{j+1} \|\tilde{f}_j\|_{L^p} \|\tilde{f}_j\|_{L^p}) \leq \sum_{j > i + a \log i} \alpha_{j} \beta^{\gamma j} \gamma^{j+1} \\
= \sum_{j=1}^{\infty} \alpha^{a \log j + \beta j} \gamma^j (i + a \log j + \beta)^{\gamma j} \\
\leq \frac{C}{j^{\gamma}} \leq C m_2(B_i).
\]

Since \((\log i)(m_2(B_i))^{\frac{1}{2}} \leq C\), the (SP) property is satisfied.

\begin{remark}
In fact, the (SP) property (the Sprindzuk property) is derived from the Gal–Koksma theorem, which is given in the appendix. Once we have the (SP) property, then the strong Borel–Cantelli property is established, because in the Gal–Koksma theorem, if we take \(f_k(x, y) = 1_{B_k} \circ F^k(x, y), h_k = g_k = E(f_k) = m_2(B_k)\), dividing both sides of the equation by \(\sum_{k=1}^{n} g_k\), we will have \(\frac{\text{dim}(x, y)}{\text{int}} \to 1\) a.s..
\end{remark}

\section{Lorenz system}

In this section, we present our results on the statistical properties of the Lorenz system, i.e. Borel–Cantelli lemmas and EVLs. Recall that the Lorenz map \(F\) does not preserve the two-dimensional Lebesgue measure \(m_2\), but preserves an invariant measure \(\mu\) which has absolutely continuous conditional measures on local unstable manifolds.

\subsection{Borel–Cantelli lemmas}

We let \(A_r(p)\) denote the square of side length \(2r\) centered at a point \(p\) in the two-dimensional space \(I \times I\). As a consequence of \([3, \text{proposition 2.4}]\), for \(\mu\) a.e. \(p\), there exists an \(r(p) > 0\) such that for all \(r < r(p), A_r(p)\) has a local product structure, and in particular \(\mu\) a.e. \(q \in A_r(p)\) has a local unstable manifold \(\gamma(q) := W_{\text{loc}}^u(q)\) which extends fully across \(A_r(p)\). The local stable manifolds are arbitrarily long for \(\mu\) a.e. \(q\). The set of local unstable manifolds \(\Gamma = \{\gamma(q)\}\) partition \(A_r(p)\) up to a set of zero \(\mu\) measure i.e. \(\mu(A_r(p)) = \mu(\cup_{q \in A_r(p)} \gamma(q) \cap A_r(p))\). We will drop the dependence on \(q\) and write \(\{\gamma\}\) for simplicity.

\begin{remark}[Young tower structure and local product structure]
A recent paper by Araujo, Melbourne and Varandas \([3]\) used a young tower construction to establish that a broad range of geometric Lorenz flows are rapidly mixing. Along the way they showed that \(\mu\) a.e. \(p \in M\) has a local product structure (this follows from their Proposition 2.4). More precisely, \(\mu\) a.e. \(p\) has the property that there exists a \(r(p) > 0\) such that for all \(r < r(p)\) if \(A(r)\) is a square of side length \(2r\), or a ball of radius \(r\), centered at \(p\), then \(\mu\) a.e. \(q \in A(r)\) has a local unstable manifold and a local stable manifold which fully crosses \(A(r)\). Moreover, if \(q_1, q_2\) are in \(A(r)\) then there is a unique point \(z = W_{\text{loc}}^u(q_1) \cap W_{\text{loc}}^s(q_2) \in A(r)\).
\end{remark}

\begin{remark}
As we mentioned before that theorem 1.7 holds for rectangles of bounded aspect ratio. We just need to change the setting above a little. Let \(A_r(p)\) denote the rectangle with side length \(2r\) of unstable manifold and side length \(2sr\) of stable manifold for some \(s\), where \(0 < s < \infty\). The proof will be the same as the proof of theorem 1.7 later in this section.
\end{remark}
Before we prove theorem 1.7, we introduce the notation. Given such a point \( p \) we let \( A \) be a square based at \( p \), with side length smaller than \( 2r(p) \).

Let \( A_\gamma = A \cap \gamma \) for \( \gamma \in \Gamma \),

\[
\mu(A) = \int m_\gamma(A_\gamma) d\nu(\gamma),
\]

where \( m_\gamma \) is the induced measure of \( \mu \) on \( \gamma \) and \( \nu \) is the conditional measure in the decomposition of \( \mu \) with respect to the partition \{ \gamma \}.

Let \( \pi \) be the projection map onto the unstable manifold and note that \( m_\gamma(A_\gamma) \sim m(\pi A) \). Here \( \sim \) means the equivalence of measures on the various unstable manifolds.

For the Lorenz map \( F \) (see [8, theorem 4.7]), we have exponential decay in Lipschitz versus Lipschitz

\[
| \int \phi \psi \circ F^n d\mu - \int \phi d\mu \int \psi d\mu | \leq C \alpha^n \| \phi \|_{L^p} \| \psi \|_{L^p},
\]

and for the base map \( T \) (see [8, proposition 2.2]), we have exponential decay in \( L^1 \) versus \( BV \)

\[
| \int \phi \psi \circ T^n dm - \int \phi dm \int \psi dm | \leq C \theta^n \| \phi \|_{BV} \| \psi \|_1.
\]

By taking \( \phi = \psi = 1_{\pi A} = 1_{\pi A_i} \), we have

\[
m(\pi A_\gamma \cap T^{-n}(\pi A_\gamma)) - (m(\pi A_\gamma))^2 \leq C \theta^n \| 1_{\pi A_i} \|_{BV} \| 1_{\pi A_i} \|_1.
\]

That is

\[
m(\pi A_\gamma \cap T^{-n}(\pi A_\gamma)) \leq (m(\pi A_\gamma))^2 + C \theta^n m(\pi A_\gamma),
\]

since \( \| 1_{\pi A_i} \|_{BV} \) is bounded.

**Proof of theorem 1.7.** We will establish the (SP) property. Without loss of generality, we assume \( i < j \). We notice \( m(\pi A_\gamma) \) is equal for all \( \gamma \in \Gamma \) since they are square balls. Thus,

\[
\begin{align*}
\mu(A_i \cap F^{-j-i} A_j) &\leq \mu(A_i \cap F^{-j-i} A_j) \\
&\sim \int m(\xi) \in (A_i)_\gamma : F^{j-i}(\xi) \in A_j d\nu(\gamma) \\
&\leq \int m(\pi(A_i)_\gamma) \cap T^{-j-i}(\pi(A_j)_\gamma) d\nu(\gamma) \\
&\leq \int (m(\pi(A_i)_\gamma))^2 + C \theta^{j-i} m(\pi(A_j)_\gamma) d\nu(\gamma) \\
&= \int (m(\pi(A_i)_\gamma))^2 d\nu(\gamma) + C \theta^{j-i} \int m(\pi(A_i)_\gamma) d\nu(\gamma) \\
&= \int (m(\pi(A_i)_\gamma))^2 d\nu(\gamma) + C \theta^{j-i} \mu(A_i) \\
&\leq C m(\pi(A_i)_\gamma) \int m(\pi(A_j)_\gamma) d\nu(\gamma) + C \theta^{j-i} \mu(A_i) \\
&= C m(\pi(A_i)_\gamma) \mu(A_i) + C \theta^{j-i} \mu(A_i),
\end{align*}
\]

where \( \sim \) is defined the same as earlier, i.e. it means the equivalence of measures on the various unstable manifolds, and we use the notation \( m(\pi(A_i)_\gamma) \) for any \( \gamma \in \Gamma \) since they are all equal.
Since $1_{A_n}$ is not Lipschitz, we let $\phi_n$ be a Lipschitz approximation of $1_{A_n}$ such that

1. $\|1_{A_n} - \phi_n\| < (\mu(A_n))^{\frac{3}{2}}$,
2. $\|\phi_n\|_{L^p} < (\mu(A_n))^{-\frac{3}{2}}$.

Then

$$\left| \int 1_{A_n} \cdot 1_{A_n} \circ F^{-i} d\mu - \int 1_{A_n} d\mu \int 1_{A_n} d\mu \right|$$

$$= \left| \int ((1_{A_n} - \phi_n) + \phi_n) \cdot ((1_{A_n} - \phi_n) + \phi_n) \circ F^{-i} d\mu - \int ((1_{A_n} - \phi_n) + \phi_n) d\mu \int ((1_{A_n} - \phi_n) + \phi_n) d\mu \right|$$

$$\leq \left| \int \phi_n \circ F^{-i} d\mu - \int \phi_n d\mu \int \phi_n d\mu \right| + \left| \int (1_{A_n} - \phi_n) \phi_n \circ F^{-i} d\mu \right|$$

$$+ \left| \int (1_{A_n} - \phi_n) d\mu \int (1_{A_n} - \phi_n) d\mu \right| + \left| \int \phi_n d\mu \int (1_{A_n} - \phi_n) d\mu \right|$$

$$+ \left| \int (1_{A_n} - \phi_n) d\mu \int \phi_n d\mu \right|$$

$$\leq C \alpha^{i-\frac{3}{2}} \|\phi\|_{L^p} \|\phi\|_{L^p} + \tilde{C}(\mu(A_n))^{\frac{3}{2}}.$$

If we choose $a \geq \frac{\gamma}{\log k}$, then

$$\sum_{j=i+1}^{\infty} \alpha^{j-1} \|\phi\|_{L^p} \|\phi\|_{L^p} \leq \sum_{j=i+1}^{\infty} \alpha^{j-1} \gamma^{\frac{3}{2}}$$

$$= \sum_{j=1}^{\infty} \alpha^{j-1} \gamma^{\frac{3}{2}}(i + a \log i + \beta)^{\gamma}$$

$$\leq \alpha^{\frac{\gamma}{\log k}} C \gamma^{\frac{3}{2}}$$

$$\leq \frac{C}{\gamma}$$

$$\leq C \mu(A_n).$$

Thus, let $f_j = 1_{A_n} \circ F^j(x, y), E(f_j) = \mu(A_n)$, so that we have

$$\sum_{j=i+1}^{n} (E(f_j) - E(f_j)E(f_j))$$

$$= \sum_{j=i+1}^{n} \mu(A_n \cap F^{-j} A_j) - \mu(A_n) \mu(A_j)$$

$$= \sum_{j=i+1}^{n} \mu(A_n \cap F^{-j} A_j) - \mu(A_n) \mu(A_j) + \sum_{j=i+1}^{n} \left[ \mu(A_n \cap F^{-j} A_j) - \mu(A_n) \mu(A_j) \right]$$

$$\leq \sum_{j=i+1}^{n} \left[ m(\pi(A_n), \gamma) \mu(A_n) + C' \theta^{-j} \mu(A_n) \right] + C \mu(A_n)$$

$$\leq C(\log i) m(\pi(A_n), \gamma) \mu(A_n) + C' \mu(A_n) + C_2 \mu(A_n) \leq \tilde{C} \mu(A_n).$$
We have established the (SP) property and thus the strong Borel–Cantelli lemma for $\{A_i\}$.

\[\square\]

**Remark 3.0.4.** For general nested rectangles, the proof will be the same with a slight difference in the setting.

For circular balls, we have theorem 1.8. As we mentioned in the introduction, the proof of theorem 1.8 needs the techniques from section 3.2 so we will do the proof then.

### 3.2. REPPs and EVLs

In this section, we establish the convergence of the REPPs to the standard Poisson process (thus the EVL follows) for Lorenz maps by essentially showing that the two conditions $D_\delta(u_n)$ and $D'(u_n)$, which were introduced in section 1.3, are satisfied. Recall:

**Condition ($D_\delta(u_n)$):** We say that $D_\delta(u_n)$ holds for the sequence $\{X_0, X_1, X_2, \cdots\}$ if for all $B \in \mathcal{R}$ and $t \in \mathbb{N}$,

$$\mu(\{X_0 > u_n\} \cap \{M(B + t) \leq u_n\}) - \mu(\{X_0 > u_n\})\mu(\{M(B) \leq u_n\}) \leq \gamma(n, t),$$

where $\gamma(n, t)$ is non-increasing in $t$ for each $n$ and $n\gamma(n, t_n) \to 0$ as $n \to \infty$ for some sequence $t_n = o(n), t_n \to \infty$.

**Condition ($D'(u_n)$):** The condition $D'(u_n)$ is said to hold for the stationary sequence $\{X_i\}$ and the sequence $\{u_n\}$ if

$$\lim sup n \sum_{j=1}^{[n/k]} \mu(X_0 > u_n, X_j > u_n) \to 0,$$

as $k \to \infty$.

Before we prove theorem 1.9, let us prove the following two lemmas:

**Lemma 1.** Suppose we have a local product structure about a point $x_0$ and the local dimension exists, denoted by $d$. Then Assumption A is satisfied.

**Proof.** As before, the conditional measure $\mu_\gamma$ is equivalent to Lebesgue in the local unstable direction, and $r$ is small, i.e. $r < 1$. Let $\epsilon = r^w$, with $w > 1$. We need to prove that the measure of the annular region $S = A_{r^w}(x_0) \setminus A_r(x_0)$ is small.

We decompose $\mu$ in a neighbourhood of $x_0$ as follows

$$\mu(A) = \int_{\Gamma} m(\gamma \cap A) d\nu(\gamma)$$

where $\gamma$ is the foliation into local unstable manifolds. Since we have a local product structure at $x_0$, these extend all the way across a sufficiently small rectangular neighbourhood of $x_0$.

Now consider the equation of the circles $x^2 + y^2 = r^2$ and $x^2 + y^2 = (r + \epsilon)^2 = r^2 + 2r\epsilon + r^{2w}$. The larger circle contains some local unstable manifolds which are not in the smaller circle but the greatest length of these is found by setting $y^2 = r^2$ in the second equation and solving for $\delta x \leq r^{w+1}$. Their length is less than $r^{w+1}$, so that

$$\mu(S) \leq \int_{\Gamma} (S \cap \gamma) d\nu(\gamma) < r^{w+1} < \epsilon^{w+1} < \epsilon^{1/2}. \quad \square$$
Lemma 2.
(a) For μ a.e. x₀, for every ε > 0, there exists an N ∈ ℤ such that for all n ≥ N
\[
\frac{1}{d + \epsilon} (v + \log n) \leq u_\delta(v) \leq \frac{1}{d - \epsilon} (v + \log n),
\]
where d is the local dimension.

(b) Denote by S(n, x₀) = A_{e^{-u_\delta}(x₀)} \setminus A_{e^{-u_\delta} - \epsilon}(x₀), the annulus region between balls centered at x₀ of radius e^{-u_\delta} and e^{-u_\delta} - \epsilon. There exists δ = δ(x₀) ∈ (0, 1) such that for n large enough
\[
\mu(S(n, x₀)) \leq C \delta^{-2d\epsilon - \delta \log n}.
\]

Proof.
(a) By the definition of the local dimension, for any ε > 0, there is an N such that for all n ≥ N,
\[
(e^{-u_\delta})^{d + \epsilon} \leq \mu(U_n) \leq (e^{-u_\delta})^{d - \epsilon}, \quad \text{and} \quad \mu(U_n) = e^{-\epsilon/n}.
\]
We immediately get
\[
\frac{1}{d + \epsilon} (v + \log n) \leq u_\delta(v) \leq \frac{1}{d - \epsilon} (v + \log n).
\]
(b) According to section 1.1 and remark 3.0.2, the Lorenz system has a local dimension and local structure, and by lemma 1, Assumption A is satisfied for the Lorenz system. So, there exists a δ ∈ (0, 1) such that
\[
\mu(S(n, x₀)) \leq C(e^{-u_\delta})^{d\delta}
\]
\[
= C e^{-\delta u_\delta} \leq C \exp \left( \frac{-\delta}{(d + \epsilon)^2} (v + \log n)^2 \right) \leq C n^{-2d\epsilon - \delta \log n}.
\]

Proof of theorem 1.9. It suffices to show D₃(uₐ) and D₄'(uₐ). As mentioned in remark 1.5.2, D₃(uₐ) is easily proven if the system satisfies Assumption A and good enough estimates for the decay of correlations. The proof here adopts similar arguments to that in the proof of [7, theorem 3.1]. We already know Assumption A is satisfied for Lorenz systems, and we have the exponential decay of correlation for the Lorenz map
\[
| \int \phi \circ F^n d\mu - \int \phi \circ F d\mu | \leq C n^\theta \| \phi \|_{L^p} \| \psi \|_{L^p}.
\]
So, we prove D₃(uₐ) in the following paragraph.

By part (b) of lemma 2, we have μ(S(n, x₀)) ≤ C n^{-2d\epsilon - \delta \log n}, where δ = δ(x₀) ∈ (0, 1) for n large enough and υ could be any number. Take φₐ to be the Lipschitz approximation of Iₐ = I_{[x_{0}, u_{\delta}]} such that φₐ(x) = 1 if x inside A_{e^{-u_\delta} - \epsilon}(x₀), φₐ = 0 if x is outside U_n, and decays to 0 at a linear rate on S(n, x₀). So we have the estimate \|φₐ - I_{[x_{0}, u_{\delta}]}\| ≤ μ(S(n, x₀)) and \|φₐ\|_{L^p} ≤ e^{\delta \nu}. Also let ψₐ = I_{(M,B) ∈ \mathcal{L}} \phiₐ, where B ∈ R so that B = \cup_{j=1}^j [a_j, b_j]. By ([7], Lemma 3.1), we then have
\[ | \int \phi_n \psi^n_B \circ F^t \, d\mu - \int \phi_n \, d\mu | \leq O(1)(\| \phi_n \|_{\infty} \gamma^{t/2}) + \| \phi_n \|_{\text{Lip}} \gamma^{t/2}), \]

and

\[ |\mu(X_0 > u_n) \cap \{ M(B + t) \leq u_n \} - \mu(X_0 > u_n) \mu(M(B) \leq u_n) | \]

\[ = | \int 1_{u_n} \cdot \psi^n_B \circ F^t \, d\mu - \mu(U_n) \int \psi^n_B \, d\mu | \]

\[ = | \int 1_{u_n} \cdot \psi^n_B + [t/2] \circ F^{-t/2} \, d\mu - \mu(U_n) \int \psi^n_B + [t/2] \, d\mu | \]

\[ \leq | \int (1_{u_n} - \phi_n) \psi^n_B + [t/2] \circ F^{-t/2} \, d\mu | + | \int \phi_n \psi^n_B + [t/2] \circ F^{-t/2} \, d\mu - \int \phi_n \, d\mu | \int \psi^n_B + [t/2] \, d\mu | \]

\[ \leq 2\mu(S(n, x_0)) + O(1)(\| \phi_n \|_{\infty} \gamma^{t/4} + \| \phi_n \|_{\text{Lip}} \gamma^{t/4}) \]

\[ \leq O(1)(n^{-2\delta \log n} + \| \phi_n \|_{\infty} \gamma^{t/4} + \| \phi_n \|_{\text{Lip}} \gamma^{t/4}), \]

where \( \gamma \) is from [7, proposition 3.2]. Let \( \gamma(n, t) = O(1)(n^{-2\delta + \delta \log n} + \| \phi_n \|_{\text{Lip}} \gamma^{t/4}) \), where \( \gamma \) is the projection map onto the unstable manifold. For each leaf \( \gamma \in \Gamma \), define \( B_{n, \gamma} = (p_i, e^{-u_n}, \gamma(x_0)) \), where \( r_n = e^{-u_n}, 0 < q_n < 1. \)

Consider points where the local unstable manifold is less than \( r_{n, \gamma} ^3 \), so that the integral splits as follows

\[ \int_{\gamma \in \Gamma} m(\pi(U_{n, \gamma})) d\nu(\gamma) = \int_{\Gamma_1} m(\pi(U_{n, \gamma})) d\nu(\gamma) \]

\[ + \int_{\Gamma_2} m(\pi(U_{n, \gamma})) d\nu(\gamma), \]

where \( \Gamma_1 \) is the set of local unstable manifolds which has a length less than \( r_{n, \gamma} ^3 \), and \( \Gamma_2 = \Omega/\Gamma_1 \).

The reason for doing so is because if the local unstable manifold has a short length, the point in the projection probably has no short return.

**Remark 3.0.5 (Short returns for one-dimensional Lorenz-like maps).** Gupta, Holland and Nicol [10] established extreme value statistics for Lorenz-like maps. The proofs used a crucial estimate on the measure of points with short returns. In particular, they showed that for \( \mu \) a.e. \( p \), and for all sufficiently small \( r < r(p) \), if \( B_r(p) \) is a ball of radius \( r \) based at \( p \), then there are constants \( C > 0, 0 < \alpha < 1 \) such that for all \( 1 \leq j \leq (\log r)^{1/2}, \mu(B_r \cap T^{-j}B_r) \leq \mu(B_r) e^{-\alpha (\log r)^2}. \)

We will adopt similar arguments in the following part. \[ \Box \]
For $\gamma \in \Gamma_2$, define
\[
E_{k, \gamma} = \left\{ x \in B_{\gamma, \gamma} : d(T^j x, x) < \frac{1}{k^{1/3}}, \text{ for some } 1 \leq j \leq (\log k)^\beta \right\}.
\]

By [10, proposition 4.2], there exists $0 < a < 1$, $0 < \bar{\theta} < 1$ such that
\[
m(E_{k, \gamma}) < \bar{\theta}^{(\log k^{1/3})^\rho},
\]
We only need $a < 1/2$, so we take $a = 1/3$.

Let $0 < \beta \leq \frac{1}{2}$ and let $0 < \rho < 1$ such that $\rho \beta < \beta/3$.

Define the set
\[
F_{k, \gamma} := \left\{ m(B_{q_k, \exp(-k^{1/3})}, x) \cap E_{\exp(k^{1/3})} \geq m(B_{q_k, \exp(-k^{1/3})}, x) \exp(-k^{1/3}) \right\}.
\]

If $x \in F_{k, \gamma}$ then
\[
\frac{m(B_{q_k, \exp(-k^{1/3})}, x) \cap E_{\exp(k^{1/3})}}{m(B_{q_k, \exp(-k^{1/3})}, x)} \geq \exp(-k^{1/3}),
\]
If we define
\[
M_j(x) := \sup_{r > 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} 1_{E_k}(y) dm(y)
\]
we see immediately from the definition of $M_j(x)$ and (3.3) that for every $x \in F_{k, \gamma}$, $M_j(x) \geq e^{-k^{1/3}}$.

Hence
\[
F_{k, \gamma} \subset \{ e^{k^{1/3}}(x) \geq e^{-k^{1/3}} \}.
\]

A theorem of Hardy and Littlewood [15, theorem 2.19] implies that
\[
m(|M_j| > c) \leq \frac{|1_E|}{c},
\]
$|\cdot|$ is with respect to the one-dimensional Lebesgue measure. As $m(E_{q_j, \gamma}) \leq O(1)\theta^{(\log k^{1/3})^\beta}$ (recall $a = 1/3$),
\[
m(F_{k, \gamma}) \leq O(1)m(E_{\exp(k^{1/3})})e^{-k^{1/3}} \leq O(1)(e^{(k^{1/3}+1)^\beta}),
\]
where $\alpha := \log \bar{\theta}$ and $k$ is large enough. Since $\beta/3 > \beta \rho$, $\sum_{k > 0} m(F_{k, \gamma}) < \infty$. By the Borel–Cantelli lemma, $m(\limsup F_{k, \gamma}) = 0$, and hence for $m$ almost every $x$ there exists an $N_x$ such that for all $k \geq N_x$, $x \notin F_{k, \gamma}$ for each $\gamma$.

Let $x_0$ be such a generic point, and let $N_{x_0}$ be the corresponding index beyond which $x_0$ does not belong to any $F_{k, \gamma}$. Since $\lim_{k \to \infty} \exp(k^{1/3})e^{-k^{1/3}} = 1$ the fact that we restricted to a subsequence is of no consequence, and we obtain the following estimate for all $n$ sufficiently large.
If \(1 \leq j \leq (\log n)^5\), then
\[
m(B_{\gamma, \gamma} \cap T^{-j}B_{\gamma, \gamma}) \leq m(B_{\gamma, \gamma}) \exp(-u_{\gamma}^d).
\] (3.5)

Summing over \(1 \leq j \leq (\log n)^5\) and taking limits as \(n \to \infty\) we obtain:
\[
n \sum_{1}^{(\log n)^5} \int_{\gamma \in \Gamma_1} m(\pi(U_n) \cap T^{-j}(\pi(U_n)))d\nu(\gamma)
\leq n \sum_{1}^{(\log n)^5} e^{-u_{\gamma}^d} \int_{\gamma \in \Gamma_1} m(\pi(U_n))d\nu(\gamma)
\leq n \sum_{1}^{(\log n)^5} e^{-u_{\gamma}^d} \mu(U_n)
= (\log n)^5 e^{-u_{\gamma}^d} \to 0,
\]

since \(u_{\gamma}\) has estimates in part (a) of lemma 2.

For \(\Gamma_1\),
\[
n \sum_{1}^{(\log n)^5} \int_{\gamma \in \Gamma_1} m(\pi(U_n) \cap T^{-j}(\pi(U_n)))d\nu(\gamma) \leq n(\log n)^5 e^{-3u_{\gamma}^d} \to 0.
\]

Consequently we have
\[
n \sum_{1}^{(\log n)^5} \mu(X_0 > u_{n, X_0} \circ F^j > u_{n}) \to 0.
\]

Finally, similar to the argument in the case of planar dispersing billiard maps in \([10,\ \text{section 4.1.3}]\), we use the exponential decay of correlations to show
\[
\lim_{n \to \infty} n \sum_{(\log n)^5} \mu(X_0 > u_{n, X_0} \circ F^j > u_{n}) = 0.
\]

Now we can prove theorem 1.8.

**Proof of theorem 1.8.** We follow the steps of proof of theorem 1.7, and use the techniques from the proof of theorem 1.9, we have:
\[
\mu(A_i \cap F^{-j-i}A_i) \leq \mu(A_i \cap F^{-j-i}A_i)
\sim \int_{\Gamma} \int_{\Gamma} m(\tilde{x} \in (A_i) \cap F^{-j-i}(\pi(A_i))d\nu(\gamma)
\leq \int_{\Gamma} m(\pi(A_i) \cap T^{-j-i}(\pi(A_i)))d\nu(\gamma)
= \int_{\Gamma} m(\pi(A_i) \cap T^{-j-i}(\pi(A_i)))d\nu(\gamma)
+ \int_{\Gamma} m(\pi(A_i) \cap T^{-j-i}(\pi(A_i)))d\nu(\gamma),
\]

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where $\sim$ is defined the same as before and $\Gamma_1$ is the set of local unstable manifolds which has a length less than $r_n^3$, and $\Gamma = \Omega \Gamma_1$. Thus,

$$
\int_{\Omega_1} m(\pi(A), \cap T^{-1}(\pi(A), \gamma)) d\nu(\gamma) \leq r_1^3 \int_{\Omega_1} d\nu(\gamma) \leq r_1^3 \mu(A_{\Gamma})^{\frac{3}{1+\epsilon}},
$$

where $\epsilon$ is such that $r_n^{d+\epsilon} \leq \mu(A_{\Gamma}) \leq r_n^{d-\epsilon}$ for $n$ large and $d$ is the local dimension. We let $u_i = -\log r_i$, that is, $r_i = e^{-u_i}$, where the $u_i$ is determined by the radius $r_i$, not necessarily satisfying (1.2). By (3.5)

$$
\int_{\Omega_1} m(\pi(A), \cap T^{-1}(\pi(A), \gamma)) d\nu(\gamma) \leq \mu(A_{\Gamma}) \exp(-u_i^3).
$$

Let $f_k = 1 \circ F_k(x, y)$ and $E(f_k) = \mu(A_k)$, so that

$$
\sum_{j=i+1}^n (E(f_j f_j) - E(f_j) E(f_j)),
$$

$$
= \sum_{j=i+1}^n \mu(A_j \cap F^{-1}(A_j)) - \mu(A_j) \mu(A_j)
$$

$$
= \sum_{j=i+1}^{(\log i)^\gamma} \mu(A_j \cap F^{-1}(A_j)) - \mu(A_j) \mu(A_j)
$$

$$
+ \sum_{j=i+1}^{(\log i)^\gamma} \mu(A_j \cap F^{-1}(A_j)) - \mu(A_j) \mu(A_j)
$$

$$
\leq \sum_{j=i+1}^{(\log i)^\gamma} \mu(A_j) \exp(-u_i^3) + C \mu(A_i)
$$

$$
\leq (\log i)^\gamma \mu(A_i) \exp(-u_i^3) + C \mu(A_i) + C \mu(A_i) \leq \bar{C} \mu(A_i).
$$

where $0 < \rho < 1/3$ and $(\log i)^\gamma \exp(-u_i^3)$ is bounded since $\exp(-u_i^3) \leq \exp(- \frac{1}{2} \log \mu(A)) \leq \exp(- \frac{1}{2} \log i)^\gamma$. Therefore, we have the (SP) property, and the strong Borel–Cantelli property then follows. $\square$

### 4. Lorenz flow

Let $M$ be the Riemannian manifold, associated with Lorenz flows, endowed with a metric $d_M$, and $f_0 : M \to M$ the Lorenz $C^1$-flow. $\Omega \subset M$ is a transverse cross-section of the flow, which is a $C^1$-submanifold with a boundary, as we stated in the previous sections. We know $F : \Omega^+ \to \Omega$ preserves a probability measure $\mu$, where $\Omega = [-1/2, 1/2] \times [-1/2, 1/2]$ and $\Omega^+ = \Omega \setminus \Omega_1$. Let $h : \Omega \to \mathbb{R}$, be the first return time of the flow to $\Omega$, and $h \in L^1(\mu)$. Consider the suspension space

$$
\Omega^h = \{(p, u) \in \Omega \times \mathbb{R} \mid 0 \leq u \leq h(p)\}/\sim, \quad \text{where} \ (p, h(p)) \sim (F(p), 0).
$$

We model the flow $f_0 : M \to M$ in the standard way by the suspension flow $\tilde{f}_0 : \Omega^h \to \Omega^h$, $\tilde{f}_0(p, u) = (p, u + 1)/\sim$. Denote the metric on $\Omega$ by $d_\Omega$, and we define a metric $d_{\Omega^h}$ on $\Omega^h$ by

$$
d_{\Omega^h}((p, u), (q, v)) = \sqrt{d_\Omega(p, q)^2 + |u - v|^2}.
$$
Then we can introduce a projection map \( \pi : \Omega^h \to M, (p, t) \mapsto f_t(p) \), which is a local \( C^1 \)-diffeomorphism. \( \mu \) is an invariant ergodic probability measure for the first return map, i.e. our Lorenz map, \( F : \Omega^r \to \Omega \). This induces (in the standard way) an invariant measure \( \mu^h \), on the suspension \( \Omega^h \), which is given by \( d\mu \times dh \). Then \( \mu^h \) determines a \( f_t \)-invariant measure \( \mu^h \) on \( M \) by \( \mu^h(A) = \mu^h(\pi^{-1}_M A) \) for measurable sets \( A \).

Consider a measurable observation \( \varphi : \Omega^h \to \mathbb{R} \) such that \( \varphi(x) = -\log d\mu(x_0) \) where \( x_0 \) is any point in \( \Omega^h \), then \( \varphi \) has a logarithmic singularity at \( x_0 \). Define \( \Phi : \Omega \to \mathbb{R} \) by

\[
\Phi(p) := \max\{ \varphi(f_t(p, 0)) \mid 0 \leq s < h(p) \}, \ p \in \Omega
\]

Denote

\[
\varphi(x) := \max\{ \varphi(f_t(x)) \mid 0 \leq s < t \}, \ x \in \Omega^h;
\]

\[
\Phi_n(p) := \max\{ \varphi(F^k(p)) \mid 0 \leq k < n \}, \ p \in \Omega.
\]

Then we have our main theorem in the flow case:

**Theorem 4.1.** Assume that \( F \) is the Lorenz map and \( f_t \) is the corresponding Lorenz flow. Assume the levels \( \{ u_n \} \) satisfy

\[
n\mu(\Phi > u_n) \to e^{-h}, \quad (4.1)
\]

Then \( \Phi_n \) satisfies the extreme value law:

\[
\mu(\Phi_n \leq u_n) \to e^{-e^{-h}}, \quad (4.2)
\]

and \( \varphi_t \) also satisfies the extreme value law,

\[
\mu^h(\varphi_t \leq u_{n/h}) \to e^{-e^{-h}}. \quad (4.3)
\]

**Proof.** The flow \( f_t \) is modelled by the suspension flow \( \tilde{f}_t \) in the standard way. By \cite[lemma 2.10]{ref}, \( \Phi \) has the same type of maxima as \( \varphi \), i.e. logarithmic singularities, which implies that \( \{ u_n \} \) satisfies the lemma 2a. Then (4.2) follows immediately under the condition (4.1), since we have shown that for the Lorenz map. Equation (4.3) is a consequence of \cite[theorem 2.5]{ref}, as the proof is a straightforward modification of the proof of \cite[theorem 2.5]{ref}.

**Remark 4.1.1.** The sequence \( \{ u_n \} \) in theorem 4.1 is not necessarily a linear function of \( v \), and it is determined by condition (4.1).

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**Appendix A. Geometric Lorenz model**

Consider a linear system in \([-1, 1]^3\):

\[
(\dot{x}, \dot{y}, \dot{z}) = (\lambda x, \lambda y, \lambda z),
\]
with $\lambda_1, \lambda_2, \lambda_3$ satisfying
$$0 < \frac{\lambda_1}{2} \leq -\lambda_3 < \lambda_1 < -\lambda_2.$$ 
For any initial point $(a, b, c) \in \mathbb{R}^3$ near the equilibrium $(0, 0, 0)$, the trajectories are given by
$$\tilde{L}_t(a, b, c) = (ae^{\lambda_1 t}, be^{\lambda_2 t}, ce^{\lambda_3 t}),$$
where $\tilde{L}_t$ denotes the linear flow.

Consider $\Omega = \{(x, y, z) : |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\} = \Omega^- \cup \Omega^+ \cup \Omega^*$, where
$$\Omega^- = \{(x, y, 1) \in \Omega : x < 0\},$$
$$\Omega^+ = \{(x, y, 1) \in \Omega : x > 0\},$$
$$\Omega^* = \{(x, y, 1) \in \Omega : x = 0\}.$$
$\Omega$ is a transverse section to the linear flow $\tilde{L}_t$, and since $\lambda_3 < 0$, every trajectory that would cross $\Omega$, will cross in the direction of the negative $z$ axis. Let $\Omega^* = \Omega^- \cup \Omega^+$, and let $\tilde{\Omega} = \{(x, y, z) : |x| = 1\} = \tilde{\Omega}^- \cup \tilde{\Omega}^+$ with $\tilde{\Omega}^\pm = \{(x, y, z) : x = \pm 1\}$. For each $(a, b, 1) \in \Omega^*$, the time $t$ such that $\tilde{L}_t(a, b, 1) \in \tilde{\Omega}$ is given by
$$|ae^{\lambda_1 t}| = 1 \implies t(a) = -\frac{1}{\lambda_1} \log|a|,$$
the time only depends on the first component of the point in $\Omega^*$ and $t(a) \to \infty$ as $a \to 0$. Thus, we can express the point in $\tilde{\Omega}$ mapped from point $(a, b, 1) \in \Omega^*$ explicitly:
$$\tilde{L}_{t(a)}(a, b, 1) = (\text{sgn}(a), be^{\lambda_2 t(a)}, ce^{\lambda_3 t(a)}) = \left(\text{sgn}(a), b|a|^\beta, a|a|^\alpha\right),$$
where $\text{sgn}(a) = a/|a|$ for $a = 0$. In this way, we just defined a map $L : \Omega^* \to \tilde{\Omega}^\pm$ by
$$L(x, y, 1) = (\text{sgn}(x), y|x|^\beta, |x|^\alpha),$$
where $\beta = -\frac{\lambda_2}{\lambda_1}, \alpha = -\frac{\lambda_3}{\lambda_1}$ satisfying $\frac{1}{2} < \alpha < 1 < \beta$, since $0 < \frac{\lambda_3}{2} \leq -\lambda_3 < \lambda_1 < -\lambda_2$.

Then we should let the sets $L(\Omega^*)$ return to the cross section $\Omega$ through a flow defined by a suitable composition of a rotation $R_\pm$, an expansion $E_{\pm \theta}$ and a translation $T_{\pm}$. More precisely, for $(x, y, z) \in \tilde{\Omega}^\pm$,
$$R_\pm(x, y, z) = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix},$$
and for $(x, y, z) \in \Omega$,
$$E_{\pm \theta}(x, y, z) = \begin{pmatrix} \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
$\theta$ and $T_{\pm}$ will be chosen to satisfy certain conditions.

So the Poincaré first return map, i.e. our Lorenz map, $F : \Omega^* \to \Omega$, is defined as
$$F(x, y) = \begin{cases} T_+ \circ E_{+ \theta} \circ R_+ \circ L(x, y, 1) & \text{for } x > 0 \\ T_- \circ E_{- \theta} \circ R_- \circ L(x, y, 1) & \text{for } x < 0 \end{cases}.$$
Combining the effect of the rotation with expansion and translation, $F$ must have the form:

$$F(x, y) = (T(x), G(x, y)),$$

where $T : I \setminus \{0\} \to I$ and $G : (I \setminus \{0\}) \times I \to I$, where $I = \left[\frac{-1}{2}, \frac{1}{2}\right]$. Here $T$ is given by

$$T(x) = \begin{cases} f_1(x^\alpha), & x < 0 \\ f_0(x^\alpha), & x > 0 \end{cases}$$

with $f_i(x) = (-1)^i \theta \cdot x + b_i, i \in \{0, 1\}$ such that $\theta \cdot \left(\frac{1}{2}\right)^\alpha < 1$ and $\theta \cdot \alpha \cdot 2^{1-\alpha} > 1$. $T$ is the quotient map of $F$, usually referred to as the Lorenz-like map, see figure A1. It has the following properties:

1. $T$ is discontinuous at $x = 0$, and the lateral limits $T(0^\pm)$ do exist, $T(0^+) = \mp \frac{1}{2}$.
2. $T$ is $C^2$ on $I \setminus \{0\}$ and $T(x) > 1$ for all $x \in I \setminus \{0\}$.
3. $\lim_{x \to 0} T(x) = +\infty$.

$G$ is given by

$$G(x, y) = \begin{cases} g_1(x^\alpha, y \cdot x^\beta), & x < 0 \\ g_0(x^\alpha, y \cdot x^\beta), & x > 0 \end{cases}$$

where $g_1|I^- \times I \to I$ and $g_0|I^+ \times I \to I$ are suitable affine maps. Here $I^- = (-1/2, 0)$, $I^+ = (1/2, 0)$.

**Appendix B. Gal–Koksma theorem**

We recall the following result of Gal and Koksma as formulated by Schmidt [16, 17] and stated by Sprindzuk [18]:

![Figure A1. Lorenz like map $T$.](image-url)
Let \((\Omega, \mathcal{B}, \mu)\) be a probability space and let \(f_k(\omega), (k = 1, 2, \ldots)\) be a sequence of non-negative \(\mu\) measurable functions and \(g_k, h_k\) be sequences of real numbers such that \(0 \leq g_k \leq h_k \leq 1\), \((k = 1, 2, \ldots)\). Suppose there exists \(C > 0\) such that
\[
\int \left( \sum_{m<k \leq n} (f_k(\omega) - g_k) \right)^2 \, d\mu \leq C \sum_{m<k \leq n} h_k,  \tag{*}
\]
for arbitrary integers \(m < n\). Then for any \(\epsilon > 0\)
\[
\sum_{1 \leq k \leq n} f_k(\omega) = \sum_{1 \leq k \leq n} g_k + O(\Theta^{1/2}(n) \log^{3/2+\epsilon}(\Theta(n))),
\]
for \(\mu\) a.e. \(\omega \in \Omega\), where \(\Theta(n) = \sum_{1 \leq k \leq n} h_k\).

Appendix C. (SP) Property

Let \((\Omega, \mathcal{B}, \mu, F)\) be a dynamical system and \(\{B_n\}\) be a sequence of nested balls. Define \(f_k = 1_{B_k} \circ F^k\), then \(E(f_k) = \mu(B_k)\). We say the (SP) property is satisfied, if for all \(n > m\), we have
\[
\sum_{i=m}^{n} \sum_{j=i+1}^{n} (E(f_jf_i) - E(f_i)E(f_j)) \leq C \sum_{i=m}^{n} E(f_i),
\]
for some constant \(C\).

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