The self-consistent field model for Fermi systems with account of three-body interactions

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The self-consistent field equations are obtained and the thermodynamics is built of the many-particle Fermi system at finite temperatures with account of three-body interactions. It is shown that the delta-like three-body interaction gives no contribution into the self-consistent field and nonlocality of three-body interactions should be taken into account for their description within the framework of the self-consistent field model. The case of a spatially uniform system is considered and a formula for the fermion’s effective mass with account of contribution from three-body forces is derived. Dependencies of the effective mass and pressure on density are obtained for the potential of “semi-transparent sphere” type at zero temperature.

Key words: self-consistent field, three-body interactions, effective mass, fermion, equation of state

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I. INTRODUCTION

The self-consistent field model is an effective approach for describing systems of large number of particles, even in the case when the interaction between particles cannot be considered to be weak and the density of a system to be low. This model is applicable both for systems with a finite number of particles, such as atomic nuclei [1, 2] or electronic shells of atoms and molecules [3, 4], as well as for many-particle systems, when describing them by methods of the statistical physics. The method of the self-consistent field is especially useful in studying spatially non-uniform systems and phase transitions [4, 5]. The self-consistent field model is also the efficient main approximation in constructing a perturbation theory, including such theory for spatially non-uniform systems [6] and many-particle systems with broken symmetries [7, 8].

Systems of large number of particles, obeying Fermi statistics, within a phenomenological approach at large enough densities are usually described in the language of the Fermi liquid theory [9, 10], which at first was developed for normal limitless systems. Afterwards, the Fermi-liquid approach was generalized both for systems of finite dimensions [11, 12] and for many-particle Fermi systems with broken phase symmetry, possessing superfluid and superconducting properties [13]. This approach is, in essence, a phenomenological variant of the self-consistent field theory [14, 15].

Most frequently, the interaction between particles is described by means of pair potentials, on the assumption that the presence of other particles does not influence the interaction between the two selected particles. Meanwhile, for particles possessing an internal structure the interaction between a pair of particles is changed due to the presence of a third particle, that can be taken into account by introducing potentials depending on the coordinates of three particles. Such a representation follows from consideration of the exchange and multipole interactions of more than two particles in different orders of the perturbation theory [16, 17]. Essential is the fact that contribution from three-body forces not only gives quantitative corrections to characteristics of a system calculated with account of only pair interactions, but can prove to be necessary for qualitative understanding of some effects.

It was found that taking into account of three-body interactions is important in the theory of nuclear forces [3, 14, 21], because they model dependence of the nucleon-nucleon interaction potential on density. In particular, in the interaction potential proposed by Skyrme [22, 23], three-body forces are described by a simple delta-like potential.

Effects of three-body interactions should manifest themselves as well in the interaction of structureless particles situated in a polarizable medium, for example electrons in a lattice. This issue, however, is so far completely unexplored.

In this paper, the self-consistent field equations are obtained for normal (non-superfluid) Fermi systems at finite temperatures with account of three-body forces within an approach developed earlier for Fermi systems with pair interactions [3, 10, 16, 17]. It is shown that the delta-like three-body interaction gives no contribution into the self-consistent field, so that theoretical description of such interactions requires accounting for the effect of their nonlocality. In more detail the spatially uniform system with its thermodynamic properties is studied and the formula is derived...
II. HAMILTONIAN OF THE FERMI SYSTEM WITH ACCOUNT OF THREE-BODY INTERACTIONS

Potential energy of a system of $N$ particles possessing an internal structure can be represented as a sum of pair, three-body, etc. interactions

$$U(r_1, r_2, \ldots, r_N) = \sum_{i<j} U(r_i, r_j) + \sum_{i<j<k} U(r_i, r_j, r_k) + \ldots,$$

where $U(r_i, r_j) = U(r_j, r_i)$, $U(r_i, r_j, r_k)$ is a symmetric in all permutations of its coordinates function. In the second quantization representation Hamiltonian of the many-particle system with account of pair and three-body interactions has the form

$$H = T + V_2 + V_3.$$

Here

$$T = \int dq dq' \Psi^+(q) H_0(q, q') \Psi(q')$$

is the kinetic energy and the energy in external field $U_0(q)$, and

$$H_0(q, q') = -\frac{\hbar^2}{2m} \Delta \delta(q - q') + U_0(q) \delta(q - q').$$

The energies of pair $V_2$ and three-body $V_3$ interactions can be written in the form

$$V_2 = \frac{1}{2!} \int dq dq' \Psi^+(q) \Psi^+(q') U(q, q') \Psi(q') \Psi(q),$$

$$V_3 = \frac{1}{3!} \int dq dq' dq'' \Psi^+(q) \Psi^+(q') \Psi^+(q'') U(q, q', q'') \Psi(q'') \Psi(q') \Psi(q).$$

In the formulas (3) - (6) and below the symbol $q = (r, \sigma)$ designates the space coordinate $r$ and the spin projection $\sigma$. We assume $s = 1/2$. The field operators obey the known anticommutation relations [24]. In the following we will consider the case when all the potentials depend only on the space coordinates: $U(q, q') = U(r, r')$, $U(q, q', q'') = U(r, r', r'')$. Besides that, we assume that external potential does not depend on the spin projection, so that $U_0(q) = U_0(r)$. In order not to take into account the condition of conservation of total number of particles in all computations, we assume that the considered system can exchange particles with a thermostat. To account for it, a term with the chemical potential $\mu$ is introduced into Hamiltonian, so that $H_0(q, q')$ in (3) is replaced by one-particle Hamiltonian

$$H(q, q') = H_0(q, q') - \mu \delta(q - q').$$

In what follows, when talking about Hamiltonian, we imply that it includes the term with the chemical potential.

III. THE SELF-CONSISTENT FIELD MODEL WITH ACCOUNT OF THREE-BODY INTERACTIONS

In order to proceed to description of the many-particle Fermi system within the self-consistent field model, let us represent the total Hamiltonian [22] as a sum of two terms, the self-consistent Hamiltonian $H_0$ and the correlation Hamiltonian $H_C$:

$$H = H_0 + H_C.$$

The self-consistent Hamiltonian is defined by the relation

$$H_0 = \int dq dq' \Psi^+(q) [H(q, q') + W(q, q')] \Psi(q') + E_0,$$
which is quadratic in the field operators of creation and annihilation. Equation (9) includes the self-consistent potential \( W(q, q') \), representing the mean field acting on a single particle, as well as the non-operator term \( E_0 \), both still indeterminate. Owing to hermiticity of the Hamiltonian, the next property holds \( W(q, q') = W^*(q', q) \). Note that taking account of the non-operator term in (9) is essential for consistent description of the thermodynamics of the system within the considered approach. The correlation Hamiltonian accounts for all the effects, not accounted for in the self-consistent field model:

\[
H_C = \frac{1}{2i} \int dq dq' \Psi^+(q) \Psi^+(q') U(r, r') \Psi(q') \Psi(q) + \\
+ \frac{1}{3i} \int dq dq' dq'' \Psi^+(q) \Psi^+(q') \Psi^+(q'') U(r, r', r'') \Psi(q'') \Psi(q') \Psi(q) - \\
- \int dq dq' \Psi^+(q) W(q, q') \Psi(q') - E_0.
\]

Obviously, the total Hamiltonian \( H \) have not changed in consequence of the performed decomposition. Transition to the self-consistent field model consists in that, instead of the exact Hamiltonian (8), we will describe the system using the approximate quadratic Hamiltonian (9). Entering into (9), so far indeterminate, quantities \( W(q, q') \) and \( E_0 \) should be chosen in an optimal manner, as it will be done below. Effects conditioned by the correlation Hamiltonian can be accounted for by the perturbation theory [1, 9, 10]. In this paper we confine ourselves to consideration of only the main approximation. Note also that in constructing the self-consistent field theory here, we do not take into account the effects connected with breaking of phase symmetry and leading to the properties of superfluidity and superconductivity [9, 10].

Since Hamiltonian (9) is quadratic in the field operators, it can be represented in the form of Hamiltonian of an ideal gas of quasiparticles. The field operators and the operators of creation and annihilation of free particles \( a_j^+, a_j \) are connected by the relations

\[
\Psi(q) = \sum_j \phi_j(0)(q) a_j, \quad \Psi^+(q) = \sum_j \phi_j^*(0)(q) a_j^+, \tag{11}
\]

where the functions \( \phi_j(0)(q) \) are solutions of the Schrödinger equation for free particles

\[
\int dq' H_0(q, q') \phi_j(0)(q') = \varepsilon_j(0) \phi_j(0)(q). \tag{12}
\]

Here \( j \equiv (\nu, \sigma) \), where \( \nu \) is a full set of quantum numbers describing the state of a particle except the spin projection \( \sigma \). To represent Hamiltonian (9) in the form analogous to that for free particles, let us introduce the quasiparticle operators \( \gamma_j^+, \gamma_j \) connected with the field operators (11) by the relations

\[
\Psi(q) = \sum_j \phi_j(q) \gamma_j, \quad \Psi^+(q) = \sum_j \phi_j^*(q) \gamma_j^+, \tag{13}
\]

where the functions \( \phi_j(q) \) are now solutions of the self-consistent equation

\[
\int dq' [H(q, q') + W(q, q')] \phi_j(q') = \varepsilon_j \phi_j(q). \tag{14}
\]

Note that for \( \phi_j(q) \) functions, the same as for \( \phi_j(0)(q) \), the conditions of orthonormality and completeness hold

\[
\int dq \phi_j^*(q) \phi_j(q') = \delta_{jj'}, \quad \sum_j \phi_j^*(q) \phi_j(q') = \delta(q - q'). \tag{15}
\]

As a result, the self-consistent Hamiltonian (9) acquires the form of Hamiltonian of an ideal gas of quasiparticles

\[
H_0 = \sum_j \varepsilon_j \gamma_j^+ \gamma_j + E_0, \tag{16}
\]

where \( \varepsilon_j \) has the meaning of the quasiparticle energy and \( E_0 \) of the energy of the background on which the quasiparticles exist. In the self-consistent field model the reference of the energy cannot be arbitrary but, as will be shown
below, should be chosen in a way such that the thermodynamic relations were satisfied. From (11) and (13) it follows that the quasiparticle operators are explicitly expressed through the operators of free particles

\[ \gamma_j = \sum_{j'} a_{j'} \int dq \phi^*_j(q) \phi^{(0)}_{j'}(q), \quad \gamma^+_j = \sum_{j'} a_{j'}^* \int dq \phi^*_j(q) \phi^{(0)*}_{j'}(q). \]  

(17)

In a spatially uniform non-magnetic state the operators of particles and quasiparticles coincide, that corresponds to the known reasoning in the Fermi liquid theory about the invariance of classification of states during adiabatic “switching-on” of the interaction [11, 12].

IV. DERIVATION OF THE SELF-CONSISTENT POTENTIAL

Let us define the statistical operator

\[ \hat{\rho}_0 = \exp \beta (\Omega - H_0), \]  

(18)

where \( \beta = 1/T \) is the inverse temperature, the constant \( \Omega = -T \ln [\text{Sp} \exp(-\beta H_0)] \) is determined from the normality condition \( \text{Sp} \hat{\rho}_0 = 1 \) and, as will be seen below, has the meaning of the thermodynamic potential of the system in the self-consistent field model. The average of an arbitrary operator \( A \) in this approximation is defined by the relation

\[ \langle A \rangle \equiv \text{Sp}(\hat{\rho}_0 A). \]  

(19)

Since Hamiltonian (16) is quadratic, then for the averages with the statistical operator (18) the Bloch-de Dominicis (Wick) theorem [25] holds, in particular

\[ \langle \psi^+_1 \psi^+_2 \psi_3 \psi_4 \rangle = \langle \psi^+_1 \psi_4 \rangle \langle \psi^+_2 \psi_3 \rangle - \langle \psi^+_1 \psi_3 \rangle \langle \psi^+_2 \psi_4 \rangle, \]  

(20)

\[ \langle \psi^+_1 \psi^+_2 \psi^+_3 \psi_4 \psi_5 \psi_6 \rangle = \langle \psi^+_1 \psi_4 \rangle \langle \psi^+_2 \psi_5 \rangle \langle \psi^+_3 \psi_6 \rangle - \langle \psi^+_1 \psi_5 \rangle \langle \psi^+_2 \psi_4 \rangle \langle \psi^+_3 \psi_6 \rangle - \langle \psi^+_1 \psi_6 \rangle \langle \psi^+_2 \psi_5 \rangle \langle \psi^+_3 \psi_4 \rangle + \langle \psi^+_1 \psi_4 \rangle \langle \psi^+_2 \psi_5 \rangle \langle \psi^+_3 \psi_6 \rangle - \langle \psi^+_1 \psi_5 \rangle \langle \psi^+_2 \psi_4 \rangle \langle \psi^+_3 \psi_6 \rangle - \langle \psi^+_1 \psi_6 \rangle \langle \psi^+_2 \psi_5 \rangle \langle \psi^+_3 \psi_4 \rangle. \]  

(21)

The ground state energy in the self-consistent field model is determined from the requirement of equality of the averages calculated with the statistical operator (18) for the exact [8] and the approximating [9] Hamiltonians:

\[ \langle H \rangle = \langle H_0 \rangle. \]  

(22)

Hence we have

\[ E_0 = \frac{1}{2!} \int dq dq' U(r, r') \langle \Psi^+(q) \Psi^+(q') \Psi(q') \Psi(q) \rangle + \frac{1}{3!} \int dq dq' dq'' U(r, r', r'') \langle \Psi^+(q) \Psi^+(q') \Psi^+(q'') \Psi(q') \Psi(q'') \Psi(q) \rangle - \int dq dq' W(q, q') \langle \Psi^+(q) \Psi(q') \rangle. \]  

(23)

Let us define the one-particle density matrix by the relation

\[ \rho(q, q') = \langle \Psi^+(q') \Psi(q) \rangle = \sum_i \phi_i(q) \phi^*_i(q') f_i. \]  

(24)

Here the quasiparticle distribution function is defined by the expression \( \langle \gamma^+_j \gamma_j \rangle = f_i \delta_{ij} \). Based on the form of Hamiltonian (16), a straightforward calculation gives the Fermi-type distribution function

\[ f_i = f(\varepsilon_i) = (\exp \beta \varepsilon_i + 1)^{-1}. \]  

(25)

Since the quasiparticle energy \( \varepsilon_i \) is the functional of \( f_i \), the formula (25) represents a complicated nonlinear equation for the distribution function, being similar to that which takes place in the Landau phenomenological theory of a Fermi liquid [11]. Using (20), (21), we express the energy \( E_0 \) through \( \rho(q, q') \):

\[ E_0 = \frac{1}{2!} \int dq dq' U(r, r') \langle \Psi^+(q) \Psi^+(q') \Psi(q') \Psi(q) \rangle + \frac{1}{3!} \int dq dq' dq'' U(r, r', r'') \langle \Psi^+(q) \Psi^+(q') \Psi^+(q'') \Psi(q') \Psi(q'') \Psi(q) \rangle - \int dq dq' W(q, q') \langle \Psi^+(q) \Psi(q') \rangle. \]  

(26)
Hamiltonian (9):

\[ H = \frac{1}{2m} \int dq dq' U(r, r') \rho(q, q') \rho(q', q) - \rho(q', q) \rho(q, q') + \int dq dq' dq'' U(r, r', r'') \rho(q, q') \rho(q', q) - 3 \rho(q, q') \rho(q', q'') \rho(q''', q) + \rho(q, q') \rho(q', q'') \rho(q'', q''') - \int dq dq' W(q, q') \rho(q', q). \]

The variation of the thermodynamic potential \( \Omega = -T \ln \left[ \sum_n \langle n | e^{-\beta H_0} | n \rangle \right] \) is equal to the averaged variation of Hamiltonian (9):

\[ \delta \Omega = \frac{\sum_n \langle n | e^{-\beta H_0} | \delta H_0 | n \rangle}{\sum_n \langle n | e^{-\beta H_0} | n \rangle} = \langle \delta H_0 \rangle. \]

From the requirement that the variation of the thermodynamic potential with respect to the density matrix vanishes \( \delta \Omega / \delta \rho(q, q') = 0 \), we obtain the expression for the self-consistent potential which consists of the contributions from pair and three-body interactions \( W(q, q') = W^{(2)}(q, q') + W^{(3)}(q, q') \), where

\[
W^{(2)}(q, q') = -U(r, r') \rho(q, q') + \delta(q - q') \int dq'' U(r, r') \rho(q'', q'), \]

\[
W^{(3)}(q, q') = \int dq'' U(r, r', r'') \left[ \rho(q, q'') \rho(q'', q') - \rho(q', q'') \rho(q'', q') \right] + \left[ \int dq'' U(r, r', r'') \left[ \rho(q, q''') \rho(q''', q') - \rho(q', q''') \rho(q''', q') \right] \right].
\]

The self-consistent field can be also derived by an equivalent method from the requirement that the variation of the functional \( I \equiv \langle H - H_0 \rangle \) with respect to the density matrix vanishes, at that \( W(q, q') \) and the parameter \( E_0 \) are not varied. Such variation rule is conditioned by the fact that, as may be checked, the variations of the functional \( \delta I \) with respect to \( \delta W(q, q') \) and \( \delta E_0 \) are mutually compensated. Equation (14), together with the derived potentials (28), enables to obtain the quasiparticle wave functions \( \phi_i(q) \) and the quasiparticle energies \( \varepsilon_i \):

\[
\varepsilon_i = -\frac{\hbar^2}{2m} \int dq \phi_i^*(q) \Delta \phi_i(q) + \int dq U_0(q) |\phi_i(q)|^2 + \int dq dq' W(q, q') \phi_i^*(q) \phi_i(q') - \mu.
\]

Substitution of the potentials (28) into equation (14) leads to the integro-differential equation

\[
-\frac{\hbar^2}{2m} \Delta \phi_i(q) + \left[ U_0(q) - \mu - \varepsilon_i \right] \phi_i(q) + \left\{ \int dq' U(r, r') \rho(q', q') + \frac{1}{2} \int dq dq'' U(r, r', r'') \left[ \rho(q, q') \rho(q'', q'') - \rho(q', q'') \rho(q''', q') \right] \right\} \phi_i(q) + \int dq dq'' U(r, r', r'') \left[ \rho(q, q'') \rho(q'', q') - \rho(q', q'') \rho(q''', q) \right] \phi_i(q) - \int dq' U(r, r') \rho(q, q') \phi_i(q') = 0.
\]

The chemical potential \( \mu \) is associated with the average number of particles \( N \) by the relation

\[ N = \int dr \mathcal{N}(r), \quad \mathcal{N}(r) = \sum_q \rho(q, q). \]

In many cases, finding the equilibrium characteristics of the researched system does not require calculation of the quasiparticle wave functions, but it is sufficient to know the one-particle density matrix. From equations (24) and (25), the equation for the one-particle density matrix follows

\[
\frac{\hbar^2}{2m} \left[ \Delta \rho(q, q') - \Delta' \rho(q, q') \right] - \left[ U_0(q) - U_0(q') \right] \rho(q, q') + \int dq'' U(r, r') \left[ \rho(q, q'') \rho(q'', q') - \rho(q', q'') \rho(q''', q) \right] + \int dq dq'' U(r, r', r'') \left[ \rho(q, q''') \rho(q''', q') - \rho(q', q''') \rho(q''', q') \right] + \frac{1}{2} \rho(q, q') \int dq dq'' \left[ U(r, r', r'') - U(r', r'', r''') \right] \left[ \rho(q, q''') \rho(q'', q') - \rho(q', q''') \rho(q''', q') \right] = 0.
\]
In the absence of magnetic effects \( \rho(q, q') = \rho(r, r') \delta_{\sigma \sigma'} \) and the self-consistent potential is diagonal in spin indices as well \( W(q, q') = [W^{(2)}(r, r') + W^{(3)}(r, r')] \delta_{\sigma \sigma'} \), where

\[
W^{(2)}(r, r') = -U(r, r')\rho(r, r') + 2(\mathbf{r} - \mathbf{r}') \int d\mathbf{r}'' U(r, r'')\rho(r'', r'),
\]

\[
W^{(3)}(r, r') = \int d\mathbf{r}'' U(r, r'', r'')\rho(r'', r') - 2\rho(r, r') \int d\mathbf{r}'' U(r, r', r'')\rho(r'', r') +
\]

\[+ \delta(\mathbf{r} - \mathbf{r}') \int d\mathbf{r}'' d\mathbf{r}''' U(r, r'', r''') \left[ 2\rho(r'', r''')\rho(r''', r'') - \rho(r'', r''')\rho(r''', r') \right].
\]

In the following we will consider the system without account of magnetic effects.

V. THERMODYNAMIC RELATIONS

Entering into the definition of the statistical operator quantity \( \Omega \) has the meaning of the grand thermodynamic potential in the self-consistent field model. The entropy in the self-consistent field model is defined through the statistical operator \( \text{(18)} \) by the known expression

\[
S = -\text{Sp}(\hat{\rho}_0 \ln \hat{\rho}_0).
\]

With the help of \( \text{(36)} \) it is easy to verify that the usual thermodynamic definition of the grand thermodynamic potential holds \( \Omega = E - TS - \mu N \), where \( E \) is the total energy of the system. This potential is a function of the temperature \( T \) and the chemical potential \( \mu \), and also a functional of the one-particle density matrix \( \rho(q, q') = \rho(q, q'; T, \mu) \), which depends on these quantities as well. However, by virtue of the fact that the self-consistent potential was derived from the condition \( \delta \Omega / \delta \rho(q, q') = 0 \), when finding the derivatives of \( \Omega \) with respect to temperature and chemical potential one should account for only the explicit dependence of the thermodynamic potential on these quantities. As a consequence, at a fixed volume of the system the usual thermodynamic relation proves to be fulfilled in the self-consistent field model

\[
d\Omega = -SdT - Nd\mu.
\]

Calculation of the entropy and the number of particles either by means of averaging with the density matrix \( \text{(18)} \) or with the help of the thermodynamic relations \( S = -(\partial \Omega / \partial T)_\mu, N = -(\partial \Omega / \partial \mu)_T \) gives the same result. With the correct choice of the energy \( E_0 \), any kind of inconsistency in statistical description of systems within the self-consistent field model, mentioned in \( \text{(11)} \), does not appear.

The energy \( \text{(26)} \), with account of the obtained self-consistent potential \( \text{(28), (29)} \), acquires the form

\[
E_0 = \frac{1}{2} \int dq dq' U(r, r') \left[ \rho(q, q')\rho(q', q) - \rho(q, q)\rho(q', q') \right] +
\]

\[+ \int dq dq' dq'' U(r, r', r'') \left[ \rho(q, q)\rho(q', q'')\rho(q'', q') - \frac{1}{3} \rho(q, q)\rho(q', q')\rho(q''', q'') - \frac{2}{3} \rho(q, q')\rho(q', q'')\rho(q'', q) \right],
\]

and the thermodynamic potential is determined by the formula

\[
\Omega = E_0 - T \sum \ln(1 + e^{-\beta\varepsilon_i}).
\]

The total energy can be calculated either by averaging the Hamiltonian operator or by the use of the thermodynamic potential \( \text{(39)} \)

\[
E = \Omega - \mu \frac{\partial \Omega}{\partial \mu} - T \frac{\partial \Omega}{\partial T}.
\]

and has the form

\[
E = \sum \varepsilon_i f_i + \mu N +
\]

\[+ \frac{1}{2} \int dq dq' U(r, r') \left[ \rho(q, q')\rho(q', q) - \rho(q, q)\rho(q', q') \right] +
\]

\[+ \int dq dq' dq'' U(r, r', r'') \left[ \rho(q, q)\rho(q', q'')\rho(q'', q') - \frac{1}{3} \rho(q, q)\rho(q', q')\rho(q''', q'') - \frac{2}{3} \rho(q, q')\rho(q', q'')\rho(q'', q) \right].
\]
The entropy in this model is expressed through the distribution function formally in the same way as in the model of an ideal gas

\[ S = - \sum_i [f_i \ln f_i + (1 - f_i) \ln(1 - f_i)], \tag{42} \]

but, as remarked previously, the distribution function itself is derived from the complicated nonlinear equation (25) and includes effects conditioned by both pair and three-body interactions.

The total number of particles (32) is expressed through the distribution function by the formula

\[ N = \sum_i f_i. \tag{43} \]

Since, owing to interaction the energy of a single particle (30) differs from the energy of a free particle and includes collective effects, then in this case a particle should be treated as a quasiparticle and the function \( f_i \) as the distribution function of quasiparticles. According to (32), (43) the number of initial free particles and the number of quasiparticles coincide, as it takes place in the Landau theory of a normal Fermi liquid (11). In the presence of pair correlations, leading to the property of superfluidity, the number of quasiparticles is less than the number of initial free particles (30), because some fraction of particles participates in the formation of the pair condensate.

VI. INTERACTION POTENTIALS IN THE SELF-CONSISTENT FIELD MODEL

As a simplest model potential of the interparticle interaction, a delta-like potential is often chosen. Such choice of the pair interaction potential \( U(r', r'') = t_2 \delta(r' - r'') \) results in the following form of the self-consistent field

\[ W^{(2)}(r, r') = \frac{t_2}{2} \delta(r - r') n(r), \tag{44} \]

where \( n(r) = 2\rho(r, r) \) is the particle number density. In this case, the contribution into the self-consistent field (44) of the first (exchange) term proves to be two times less by absolute value than the contribution of the second term, describing the direct interaction, and has an opposite sign.

As can be immediately verified, when choosing three-body forces in the delta-like form

\[ U(r', r'', r''') = t_3 \delta(r' - r'') \delta(r'' - r'''), \tag{45} \]

which is used, for example, in (22), the self-consistent potential \( W^{(3)}(r, r') \) vanishes. Therefore, in order to obtain nonvanishing contribution of three-body forces into the self-consistent field, we have to take into account their nonlocality, that is their finite radius of action. In the case of an arbitrary three-body potential of the interparticle interaction, its contribution into the self-consistent potential (35) can be represented in the form

\[ W^{(3)}(r, r') = a_1(r, r') - 2\rho(r, r') a_2(r, r') + \delta(r - r') \left[ 2b_1(r) - b_2(r) \right], \tag{46} \]

where

\[ a_1(r, r') \equiv \int dr'' U(r, r', r'') \rho(r, r'') \rho(r'', r'), \quad a_2(r, r') \equiv \int dr'' U(r, r', r'') \rho(r'', r''), \]

\[ b_1(r) \equiv \int dr' dr'' U(r, r', r'') \rho(r', r'') \rho(r', r''), \quad b_2(r) \equiv \int dr' dr'' U(r, r', r'') \rho(r', r'') \rho(r'', r'). \tag{47} \]

Let us discuss the question of choosing the potentials of interparticle interactions, which are suitable for the self-consistent field model. In a spatially uniform state the interaction potentials have to satisfy two conditions. They must not change under the replacement \( r \rightarrow r + \mathbf{a} \) (\( \mathbf{a} \) is an arbitrary vector) of all vectors they depend on, and must be symmetric relative to any permutation of their arguments. The pair potential satisfying these conditions has the form

\[ U(r, r') = \frac{1}{2} \left[ U_2(r - r') + U_2(r' - r) \right], \tag{48} \]

where \( U_2(r) \) is a function of a vector argument. The three-body potential, depending on pairs of vector differences and satisfying the similar conditions, looks like

\[ U(r, r', r'') = \frac{1}{6} \left[ U_3(r - r', r - r'') + U_3(r - r'', r - r') + U_3(r' - r, r' - r'') + U_3(r' - r, r'' - r') + U_3(r'' - r, r' - r) + U_3(r'' - r, r'' - r) \right]. \tag{49} \]
Here \( U_3(\mathbf{r}, \mathbf{r}') \) is a function of two vector arguments. If there is dependence only on distances between particles, and the function in \([19]\) is symmetric \( U_3(\mathbf{r}, \mathbf{r}') = U_3(\mathbf{r}', \mathbf{r}) \), then \( U(\mathbf{r}, \mathbf{r}') = U_2(|\mathbf{r} - \mathbf{r}'|) \) and

\[
U(\mathbf{r}, \mathbf{r}', \mathbf{r}'') = \frac{1}{3} \left[ U_3(|\mathbf{r} - \mathbf{r}'|, |\mathbf{r} - \mathbf{r}''|) + U_3(|\mathbf{r}' - \mathbf{r}|, |\mathbf{r}' - \mathbf{r}''|) + U_3(|\mathbf{r}'' - \mathbf{r}|, |\mathbf{r}'' - \mathbf{r}'|) \right].
\]

(50)

In particular, the three-body potential can be chosen in the form proposed in \([20]\):

\[
U(|\mathbf{r} - \mathbf{r}'|, |\mathbf{r} - \mathbf{r}''|) = u_0 \exp \left[ - \frac{|\mathbf{r} - \mathbf{r}'| + |\mathbf{r} - \mathbf{r}''|}{r_0} \right].
\]

(51)

There is also a possibility of choosing the potential in the form of the Gauss function:

\[
U(|\mathbf{r} - \mathbf{r}'|, |\mathbf{r} - \mathbf{r}''|) = \frac{u_0}{\pi^{3/2} r_0^3} \exp \left[ - \frac{(\mathbf{r} - \mathbf{r}')^2 + (\mathbf{r} - \mathbf{r}'')^2}{r_0^2} \right].
\]

(52)

Such choice is characteristic in that in the limit \( r_0 \to 0 \) the potential \([52]\) turns into the potential of zero radius \([15]\), and its contribution into the self-consistent field vanishes. In principle, the model three-body potential can be chosen to depend on the three distances between three particles

\[
U(\mathbf{r}, \mathbf{r}', \mathbf{r}'') = U_3(|\mathbf{r} - \mathbf{r}'|, |\mathbf{r} - \mathbf{r}''|, |\mathbf{r}' - \mathbf{r}''|).
\]

(53)

Note that the derivation from first principles of the potential of interaction of three atoms as structureless entities presents a complex quantum mechanical problem \([20]\).

For describing the interaction between particles, the potentials are often used which rapidly tend to infinity at small distances. An example of such potential is given by the Lennard-Jones potential

\[
U_{LJ}(r) = 4\varepsilon \left[ \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^{6} \right],
\]

(54)

containing two parameters: the distance \( \sigma \) and the energy \( \varepsilon \). The use of the potentials with the hard core leads to considerable difficulties, especially in the quantum mechanical description \([27]\). Such potentials do not have the Fourier image and when using such potentials the self-consistent field becomes infinity. Sometimes, this fact is used as justification of non-applicability of the self-consistent field model in one or another case, for example for describing the liquid \([28]\). The use of the potential which rapidly tends to infinity at small distances means that an atom or other compound particle retains its identity at arbitrary high pressures. Meanwhile, it is clear that the critical pressure must exist at which atoms approach each other so closely that they should be “crushed” and lose their identity. Therefore, the requirement of absolute impermeability of particles at arbitrary high pressures is excessively rigorous and unphysical and, in our opinion, it is more reasonable to use the potentials which tend to a finite value at small distances. An example of such potential is given by the known Morse potential \( U_M(r) = \varepsilon \left\{ \exp[-2(r - r_0)/a] - 2 \exp[-(r - r_0)/a] \right\} \), \( \varepsilon \) is the energy parameter and \( r_0, a \) are specific distances. In addition, for the potentials with the hard core it is impossible to calculate such observable quantity as the scattering length by which the cross section is expressed at low energies. Such a problem is encountered, for example, when deriving the known Gross-Pitaevskii equation that is widely used presently in describing atomic Bose-Einstein condensates \([29]\). Here also, the integral \( \int U(r)d\mathbf{r} \) diverging for the potentials of type \([51]\) is replaced by a finite value, that in essence means the use of the potential not diverging to infinity at small distances. It should also be noted, that quantum mechanical calculations lead to the potentials having a finite value of energy at zero \([30, 31]\).

Quite often, the model of “semi-transparent sphere” potential is used as a simplest form of the pair potential where the noted problems with divergences are absent:

\[
U_2(r) = \begin{cases} 
U_{2m}, & r < r_2, \\
0, & r > r_2.
\end{cases}
\]

(55)

This potential is determined by two parameters, one of which \( U_{2m} \) has the energy dimension and defines the interaction strength, and the second one \( r_2 \) of the length dimension defines the radius of interaction. Such observable quantity as the scattering length can be expressed through the parameters of the potential \([53]\): \( a_0 = (m/4\pi\hbar^2) \int U_2(r)d\mathbf{r} = mU_{2m}r_2^3/3\hbar^2 \). A similar model can be used as well in the case of three-body forces. For the potential \([50]\) depending on pairs of distances between particles, we have

\[
U_3(r, r') = \begin{cases} 
U_{3m}, & r < r_3, r' < r_3, \\
0, & \text{else.}
\end{cases}
\]

(56)
Here, there are also two parameters: the strength $U_{3m}$ and the radius $r_3$. For this choice, the total potential $U(r, r', r'') = U_{3m}$, if distances between each pair of three particles are less than $r_3$. But if two distances between particles are less than $r_3$, and third one is greater than $r_3$ and less than $2r_3$, then $U(r, r', r'') = U_{3m}/3$. In other cases the potential vanishes.

For the potential (53), depending on three distances between particles, the model of “semi-transparent sphere” potential is defined by the formula

$$U_3(r, r', r'') = \begin{cases} U_{3m}, & r < r_3, r' < r_3, r'' < r_3, \\ 0, & \text{else}. \end{cases}$$  \hspace{1cm} (57)

In this case the potential is nonzero only under the condition that distances between each pair of three particles are less than $r_3$. Let us consider in more detail the spatially uniform system of Fermi particles in the absence of an external field and with account of three-body forces.

**VII. THE SPATIALLY UNIFORM SYSTEM**

In the spatially uniform state the one-particle density matrix is a function of absolute value of the coordinate difference $\rho(r, r') = \rho(|r-r'|)$, and the particle number density $n = 2\rho(0)$ is constant. For the potential (50) depending on pairs of distances between particles, the coefficients (47) which determine the contribution of three-body forces into the self-consistent potential acquire the form

$$a_1(r-r') = \frac{2}{3} \int d\tau'' U_3(|r-r'|, r'') \rho(r'') \rho(|r-r'| + r'') + \frac{1}{3} \int d\tau'' U_3(|r-r'| + r'', r'') \rho(r'') \rho(|r-r'|) \rho(|r-r'| + r''),$$

$$a_2(r-r') = \frac{2\rho(0)}{3} \int d\tau'' U_3(|r-r'|, r'') + \frac{\rho(0)}{3} \int d\tau'' U_3(|r-r'| + r'', r''),$$

$$b_1 = \rho^2(0) \int dr dr' U_3(r, r'),$$

$$b_2 = \frac{1}{3} \int dr dr' U_3(|r-r'|, r') \rho^2(r) + \frac{2}{3} \int dr dr' U_3(r, r') \rho^2(r').$$

Expansions of the three-body potential and the one-particle density matrix in Legendre polynomials can be used:

$$U_3(|r-r'|, r'') = \sum_{l=0}^{\infty} U_{3l}(r, r'; r'') P_l(\cos \theta),$$

$$\rho(|r-r'|) = \sum_{l=0}^{\infty} \rho_l(r, r') P_l(\cos \theta),$$  \hspace{1cm} (59)

where

$$U_{3l}(r, r'; r'') = \frac{2l+1}{2} \int_{-1}^{1} U_3(\sqrt{r'^2 + r'^2 - 2rr'x}, \cos \theta) P_l(x) \, dx,$$

$$\rho_l(r, r') = \frac{2l+1}{2} \int_{-1}^{1} \rho(\sqrt{r'^2 + r'^2 - 2rr'x}) P_l(x) \, dx = \frac{2l+1}{2\pi^2} \int_{0}^{\infty} f(\xi) j_l(kr) j_l(kr') k^2 dk,$$  \hspace{1cm} (60)

where $j_l(x)$ is the spherical Bessel function. With these expansions taken into account:

$$a_1(r) = \frac{8\pi}{3} \int_0^{\infty} U_3(r, r') \rho(r') \rho_0(r, r') r^2 \, dr' + \frac{4\pi}{3} \sum_{l=0}^{\infty} \frac{1}{2l+1} \int_0^{\infty} U_{3l}(r, r'; r') \rho(r') \rho_l(r, r') r^2 \, dr',$$

$$a_2(r) = \frac{8\pi\rho(0)}{3} \int_0^{\infty} U_3(r, r') r^2 \, dr' + \frac{4\pi\rho(0)}{3} \int_0^{\infty} U_{30}(r, r'; r') r^2 \, dr',$$

$$b_1 = 16\pi^2 \rho^2(0) \int_0^{\infty} r^2 \, dr \int_0^{\infty} U_3(r, r') r^2 \, dr',$$

$$b_2 = \frac{32\pi^2}{3} \int_0^{\infty} \rho^2(r) r^2 \, dr \int_0^{\infty} U_3(r, r') r^2 \, dr' + \frac{16\pi^2}{3} \int_0^{\infty} \rho^2(r) r^2 \, dr \int_0^{\infty} U_{30}(r, r'; r') r^2 \, dr'.$$  \hspace{1cm} (61)
In the spatially uniform case the state of a single particle can be characterized by its momentum, and equation (114) admits solutions in the form of plane waves:

\[ \phi_j(q) = \frac{\delta_{\sigma\sigma'}}{\sqrt{W}} e^{ikr}, \]

where \( q \equiv (r, \sigma), \quad j \equiv (k, \sigma'). \) As was noted, in the absence of magnetic effects \( \rho(q, q') = \delta_{\sigma\sigma'} \rho(r - r'), \) \( W(q, q') = \delta_{\sigma\sigma'} W(r - r'). \) Equation (30) gives the quasiparticle energy not depending on the spin projection:

\[ \varepsilon_k = \frac{\hbar^2 k^2}{2m} - \mu + W_k, \]

where \( W_k = \int dr W(r) e^{ikr}. \) The self-consistent potential can be represented as a sum of the direct and exchange terms

\[ W(r) = W_0 \delta(r) + W_C(r), \]

and, if the interaction depends only on a distance between particles, then also \( W_C(r) = W_C(r). \) In this case

\[ \varepsilon_k = \frac{\hbar^2 k^2}{2m} - \mu + W_0 + \frac{4\pi}{k} \int_0^{\infty} dr rW_C(r) \sin kr. \]

Considering that both pair and three-body interactions give contribution into the self-consistent potential \( W_0 = W_0^{(2)} + W_0^{(3)}, \) \( W_C(r) = W_C^{(2)}(r) + W_C^{(3)}(r), \) where

\[ W_0^{(2)} = nU_{20}, \quad W_C^{(2)}(r) = -U_2(r)\rho(r), \]

\[ W_0^{(3)} = 2b_1 - b_2, \quad W_C^{(3)}(r) = a_2(r) - 2\rho(r)a_2(r). \]

Here \( U_2(r) \) is the potential of the pair interaction, \( U_{20} = \int U_2(r) dr, \) \( n = 2\rho(0) \) is the particle number density, and quantities \( a_2(r), b_1, b_2 \) for the potential (54) are defined by the formulas (61). The one-particle density matrix has the form

\[ \rho(r) = \frac{1}{2\pi^2} \int_0^{\infty} dk k \sin(kr) f(\varepsilon_k), \]

where \( f(\varepsilon_k) = [\exp(\beta \varepsilon_k) + 1]^{-1}, \) and the expansion coefficients of the one-particle density matrix in Legendre polynomials (59) are defined by the formula

\[ p_l(r, r') = \frac{2l + 1}{4\pi^2} \int_0^{\infty} dk k f(\varepsilon_k) \int_{-1}^{1} \sin(k\sqrt{r^2 + r'^2 - 2rr'\cos x}) \frac{P_l(x)}{\sqrt{r^2 + r'^2 - 2rr'\cos x}} P_l(x) dx = \frac{2l + 1}{2\pi^2} \int_0^{\infty} f(\varepsilon_k) j_1(kr) j_1(k'r') k^2 dk. \]

At low temperatures in a degenerate system, its properties are determined by quasiparticles situated near the Fermi surface. In this case the notion of the effective mass of a quasiparticle \( m_* \) can be introduced. We define the Fermi wave number at finite temperatures by the relation

\[ \frac{\hbar^2 k_F^2}{2m} - \mu + W_0 + \frac{4\pi}{k_F} \int_0^{\infty} dr rW_C(r) \sin k_F r = 0. \]

Then near the Fermi surface \( k = k_F + \Delta k, \) so that the dispersion law of a quasiparticle can be represented in the form \( \varepsilon_k = (\hbar^2 k_F/m_*) \Delta k, \) where the effective mass is defined by the formula

\[ \frac{1}{m_*} = \frac{1}{m} - \frac{4\pi}{k_F^2} \int_0^{\infty} dr \frac{r^3 W_C(r) j_1(3k_Fr)}{j_1(k_Fr)}, \]

and \( j_1(x) = (\sin x - x \cos x)/x^2 \) is the first order spherical Bessel function. It is seen that the effective mass is determined by the exchange part of the self-consistent potential. The contribution of three-body interactions into the effective mass of a quasiparticle with different choices of the model potential will be studied in detail in a separate paper. In the approximation of the effective mass the distribution function acquires the form

\[ f_k = \frac{1}{\exp\left[ \frac{\hbar^2 k_F}{m_*} (k - k_F) \right] + 1} \approx \frac{1}{\exp\left[ \frac{\hbar^2}{2m_*} (k^2 - k_F^2) \right] + 1}. \]
For degenerate systems both representations of the distribution function (71) are equivalent. The only difference between the functions (71) and the distribution function of an ideal gas is dependence of the effective mass entering in (71) on temperature and density.

The effective mass can be introduced for the non-degenerate system as well, when the main contribution into the thermodynamics is given by particles with small momenta. Having used the expansion \( \sin kr \approx kr -(kr)^3/3! \), from (65) we find the dispersion law of quasiparticles in the non-degenerate case: 
\[
\varepsilon_k = \hbar^2 k^2 / 2m_c - \mu_s,
\]
where the effective mass \( m_c \) and the chemical potential \( \mu_s \) are now determined by the formulas:
\[
\frac{1}{m_c} = \frac{1}{m} - \frac{4\pi}{3\hbar^2} \int_0^\infty dr r^4 W_C(r),
\]
\[
\mu_s = \mu - W_0 - 4\pi \int_0^\infty dr r^4 W_C(r).
\]
(72)

Note that the formula (71) for the effective mass turns into the formula (72), if \( j_1(kFR) \approx kFR/3 \) is put in (71).

The thermodynamic potential per unit volume, with account of three-body interactions depending on pairs of distances between particles and in the approximation of the effective mass, according to (68), (69) has the form
\[
\Omega / V = -\frac{2T}{\Lambda^3} \Phi_{5/2} \left( \frac{\hbar^2 k_F^2}{2m_c} \right) + 4\pi \int_0^\infty U_2(r) \rho^2(r) r^2 dr - 2\rho^2(0) \int_0^\infty U_2(r) r^2 dr +
\]
\[
+ 16\pi^2 \left[ \frac{8}{3} \rho(0) \int_0^\infty r^2 dr \int_0^\infty U_3(r, r') \rho^2(r') r^2 dr' + \frac{4}{3} \rho(0) \int_0^\infty \rho^2(r) r^2 dr \int_0^\infty U_3(r, r'; r') r^2 dr' -
\]
\[
- \frac{8}{3} \rho^3(0) \int_0^\infty r^2 dr \int_0^\infty U_3(r, r') \rho(r') r^2 dr' - \frac{4}{3} \int_0^\infty \rho(r) r^2 dr \int_0^\infty U_3(r, r') \rho(r') \rho_0(r, r') r^2 dr' \right].
\]
(73)

Here \( \Lambda \equiv (2\pi\hbar^2 / \mu_s T)^{1/2} \) is the thermal wavelength. The first term contains one of the Stoner functions (32)
\[
\Phi_s(t) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{z^{s-1} dz}{e^{z/t} + 1},
\]
(74)
by which all the thermodynamical quantities of an ideal Fermi gas can be expressed. The first term in (73) gives the contribution of an ideal gas of non-interacting quasiparticles. Its difference from the thermodynamical potential of an ideal Fermi gas consists only in the replacement of mass by the effective mass. The rest terms in (73) are specific for the self-consistent field model and determined by pair and three-body interactions. Since pressure is associated with the thermodynamical potential by the known relation \( p = -\Omega / V \), then the formula (73) determines, except for sign, also pressure. The formula for the particle number density
\[
n = \frac{2}{\Lambda^3} \Phi_{3/2} \left( \frac{\hbar^2 k_F^2}{2m_c} \right),
\]
(75)
together with (73), determines pressure as function of density, that is the equation of state.

VIII. FERMI SYSTEM AT ZERO TEMPERATURE WITH THREE-BODY INTERACTIONS IN THE MODEL OF “SEMI-TRANSPARENT SPHERE”

The system of Fermi particles at zero temperature can be investigated in detail, if the three-body interaction is chosen in the form (68) which leads to somewhat more simple formulas than for the potential (60). For the potential (68), the parameters determining the self-consistent potential (71) are given by the formulas
\[
a_1(r) = 4\pi \sum_{l=0}^\infty \frac{1}{2l + 1} \int_0^\infty L_{3l}(r, r') \rho_l(r') r^2 dr', \quad a_2(r) = 4\pi \rho(0) \int_0^\infty L_{30}(r, r') r^2 dr',
\]
\[
b_1 = 16\pi^2 \rho^2(0) \int_0^\infty r^2 dr \int_0^\infty L_{30}(r, r') r^2 dr', \quad b_2 = 16\pi^2 \int_0^\infty r^2 dr \int_0^\infty L_{30}(r, r') \rho^2(r') r^2 dr'.
\]
(76)

Here
\[
U_3(r, r', |r - r'|) = \sum_{l=0}^\infty U_{3l}(r, r') P_l(\cos \theta),
\]
\[
U_{3l}(r, r') = \frac{2l + 1}{2} \int_{-1}^1 U_3 \left( r, r', \sqrt{r^2 + r'^2 - 2rr'x} \right) P_l(x) dx.
\]
(77)
In particular for the interaction potential of “semi-transparent sphere” type (57), we have:

\[ U_u(r, r') = U_{3m} \theta(r_3 - r)\theta(r_3 - r')(2l + 1) \left[ \theta(r_4 - r - r')\delta_{00} + \theta(r + r' - r_3) \right] \frac{1}{2} \int_{x_0}^{1} P_l(x) dx, \]

(78)

where \( x_0 = (r^2 + r'^2 - r_3^2)/2rr' \). At zero temperature \( f_k = \theta(k_F - k) \) and

\[ \rho(r) = \frac{k_F^2}{2\pi^2} j_1(k_F r), \quad \rho(0) = \frac{n}{2} = \frac{k_F^2}{6\pi^2}, \quad \rho(r, r') = \frac{2l + 1}{2\pi^2} \int_0^{k_F} dk k^2 j_i(kr)j_l(k'r'). \]

(79)

Taking into account the latter relations, we find

\[ a_1(r) = U_{3m} \theta(r_3 - r) \frac{k_F^2}{2\pi^2} \frac{1}{k_F r} \left[ j_0^2(k_F r_3) - j_0(k_F r_3)j_0(k_F (r_3 - r)) + \int_{-k_F(r_3 - r)}^{k_F r} dy j_0(k_F r - y)j_1(y) \right], \]

\[ a_2(r) = \frac{U_{3m}}{r_3^6} (k_F r_3)^3 \theta(r_3 - r) \left[ \left( \frac{r}{r_3} \right)^3 - 12 \left( \frac{r}{r_3} \right) + 16 \right], \]

(80)

\[ b_1 = \frac{5\pi^2}{6} \rho^2(0) U_{3m} r_3^6 = \frac{5}{216\pi^2} U_{3m} (k_F r_3)^6 = 0.0023 U_{3m} (k_F r_3)^6, \]

\[ b_2 = \frac{U_{3m}}{12\pi^2} \left[ B_3(k_F r_3) - 12(k_F r_3)^2 B_1(k_F r_3) + 16(k_F r_3)^3 B_0(k_F r_3) \right], \]

where \( B_n(z) = \int_0^z y^n j_1^2(y) dy \). Note that when calculating \( a_1(r) \) the summation formula is used (see [32], p. 133)

\[ \sum_{l=0}^{\infty} (2l + 1) j_l(u) j_l(v) P_l(x) = j_0 \left( \sqrt{u^2 + v^2 - 2uvx} \right). \]

(81)

In the limit \( k_F r_3 \ll 1 \), we have more simple formulas

\[ b_2 \approx \frac{5}{216\pi^2} U_{3m} (k_F r_3)^6, \quad a_1(r) \approx \frac{U_{3m}}{432\pi^3 r_3^6} (k_F r_3)^6 \theta(r_3 - r) \left[ \left( \frac{r}{r_3} \right)^3 - 12 \left( \frac{r}{r_3} \right) + 16 \right]. \]

(82)

Note that even at \( k_F r_3 = 1 \), the formulas [82] give a mistake of the order and less than 10%. For the following it is convenient to define a characteristic value of density through the radius of the pair interaction:

\[ \frac{1}{n_*} \equiv \frac{4\pi}{3} r_3^3. \]

(83)

If, for example, \( r_2 = 3 \times 10^{-8} \text{ cm} \) is taken, then \( n_* = 0.88 \times 10^{22} \text{ cm}^{-3} \). This density is close to the particle number density in liquid. Since \( k_F r_3 = \left( \frac{9\pi}{4} \right)^{1/3} r_3 r_2 \left( \frac{n}{n_*} \right)^{1/3} \), then the condition \( k_F r_3 \ll 1 \) can be written in the form

\[ (r_3/r_2)(n/n_*)^{1/3} \ll 1. \]

(84)

Since it should be considered likely that \( r_3 \sim r_2 \), the noted condition is satisfied in the limit of low densities. However, as was noted, even at \( n \sim n_* \) the formulas [82] are valid with good accuracy. Let us introduce the designation, that will be used below, for the dimensionless density

\[ \tilde{n} \equiv \frac{n}{n_*} = \left( \frac{r_2}{r_3} \right)^3, \]

(85)

where \( r_3 = \left( \frac{3}{4\pi n} \right)^{1/3} \) is the radius of a sphere whose volume equals the volume per particle. In the following formulas we will use also the next designations:

\[ \lambda \equiv \frac{r_3}{r_2}, \quad \alpha \equiv \frac{a_0}{r_2}, \quad u \equiv \frac{U_{3m}}{U_{2m}}. \]

(86)
With account of (66), the relations (70), (80) determine the contribution of three-body interactions into the direct and exchange parts of the self-consistent potential

\[ W^{(3)}_0 = 2b_1 - b_2, \quad W^{(3)}_C(r) = a_3(r) - 3m \frac{j_1(k_Fr)}{k_Fr} a_2(r). \]  

(87)

The contribution of pair forces into these quantities for the potential of “semi-transparent sphere” type (55) is given by the formulas

\[ W^{(2)}_0 = U_{2m} \tilde{n}, \quad W^{(2)}_C(r) = \begin{cases} -U_{2m} \frac{3}{2} n \frac{j_1(k_Fr)}{k_Fr}, & r < r_2, \\ 0, & r > r_2. \end{cases} \]

(88)

The obtained relations (87), (88) for the self-consistent potential enable to determine by the formula (70) the contribution of pair and three-body interactions into the effective mass. With account of only pair forces, the effective mass is determined by the expression

\[ \frac{m}{m_*} = 1 + \frac{6}{\pi} \frac{\alpha}{(k_Fr_2)^2} B_2(k_Fr_2). \]

(89)

Hence it follows that for positive scattering length \( a_0 = mU_{2m}r_2^3/3\hbar^2 \) the effective mass proves to be less than mass of a free particle, and for negative \( a_0 \) it is greater. The effective mass with account of three-body interactions in this case has the form

\[ \frac{m}{m_*} = \frac{m}{m_{**}} - \frac{4\pi m}{k_F\hbar^2} \int_{r_0}^{r_3} a_1(r) j_1(k_Fr)r^3 dr + \frac{12\pi mn}{k_F^2\hbar^2} \int_{0}^{r_3} j_2^2(k_Fr)a_2(r)r^2 dr. \]

(90)

At low densities, when the condition (83) is satisfied, the formula for the total mass acquire the form

\[ \frac{m}{m_*} = 1 + \frac{3}{10} \alpha \tilde{n} - \frac{3}{70} \left( \frac{9\pi}{4} \right)^{2/3} \alpha \tilde{n}^{5/3} + \frac{159}{2560} \alpha u \lambda^3 \tilde{n}^2 = 1 + 0.3 \alpha \tilde{n} - 0.158 \alpha \tilde{n}^{5/3} + 0.0621 \alpha u \lambda^8 \tilde{n}^2. \]

(91)

In (69), (71) and below we use the designations (80). The second and third terms in (71) are determined by pair forces, and the fourth by three-body forces. As it is seen, with increasing density the contribution of three-body forces into the effective mass rises faster than that of pair forces, and the role of three-body forces rises at that. Also, attention is drawn to a strong dependence (as \( \lambda^8 \)) of the contribution of three-body forces on the ratio of radiiuses of three-body and pair interactions.

Dependencies of the effective mass on density, calculated by the formula (90), are shown in Figure 1. For negative scattering length and with neglect of three-body interactions, the effective mass proves to be greater than mass of a free particle and monotonically rises with increasing density, reaching maximum. With further increasing density the rise of the effective mass changes into its fall, although as before for all physically reasonable densities it remains greater than mass of a free particle (curve 1 in Figure 1). Accounting for the three-body interaction with a positive constant leads to reduction of the region of rise of the effective mass and to more faster its decreasing at high densities (curve 2 in Figure 1). For positive scattering length and in the absence of three-body forces, the effective mass for all reasonable densities is less than mass of a free particle (curve 3 in Figure 1). Accounting for the three-body interaction with a positive constant leads to more faster monotonic decrease of the effective mass (curve 4 in Figure 1).

The total pressure \( p = -\Omega/V \), besides the contribution of an ideal gas with the effective mass

\[ p_0 = \frac{(3\pi^2/2)^2/3 \ h^2}{5 \ m_* \ n^{5/3}}, \]

(92)

contains the contributions of both pair and three-body interactions so that \( p = p_0 + p_2 + p_3 \), where in the model of “semi-transparent sphere”:

\[ p_2 = -\frac{U_{2m} n_*}{2} \left[ \frac{6}{\pi} \tilde{n} B_0(k_Fr_2) - \tilde{n}^2 \right], \]

(93)

\[ p_3 = -\int d\mathbf{r}d\mathbf{r}' U_3(r, r', |\mathbf{r} - \mathbf{r}'|) \left[ 4\rho(0)\rho^2(r') - \frac{8}{3}\rho^3(0) - \frac{4}{3}\rho(r)\rho(r')\rho(|\mathbf{r} - \mathbf{r}'|) \right]. \]

(94)
Using the general formulas (47) for the coefficients of the self-consistent potential, we find the following representation of the contribution (94) of three-body forces into pressure

\[ p_3 = 4\rho(0)\left(\frac{2}{3}b_1 - b_2\right) + \frac{16\pi}{3} \int_0^\infty \rho(r)a_1(r)r^2 dr. \]  

(95)

Here \( b_1, b_2, a_1(r) \) are defined by the formulas (80). In the case of densities \( n \leq n_* \), using the expansion

\[ B_n(z) \approx \frac{2^{n+3}}{9(n + 3)} - \frac{2^{n+5}}{45(n + 5)} + \frac{2^{n+7}}{525(n + 7)} \]

we find the dependence of pressure conditioned by the pair interaction on density

\[ \frac{p_2}{p_{0*}} = \frac{5\alpha}{(12\pi^2)^{1/3}} \left[ \tilde{n}^2 + \frac{3}{25} \left( \frac{9\pi}{4} \right)^{2/3} \tilde{n}^{8/3} - \frac{81\pi}{4900} \left( \frac{9\pi}{4} \right)^{1/3} \tilde{n}^{10/3} \right] = \frac{5\alpha}{(12\pi^2)^{1/3}} \left[ \tilde{n}^2 + 0.442 \tilde{n}^{8/3} - 0.0997 \tilde{n}^{10/3} \right]. \]  

(96)

Here pressure is related to pressure of an ideal gas of particles at the characteristic density (83):

\[ p_{0*} = \left( \frac{3\pi^2}{2} \right)^{2/3} \frac{\hbar^2}{m} \frac{n_*^{5/3}}{5}. \]

The sign of pressure (96) is determined by the sign of the pair interaction constant and at negative value of this constant the interaction contribution into pressure is negative. The main term in the expansion of pressure conditioned by the three-body interaction (95) has the form

\[ \frac{p_3}{p_{0*}} = \frac{477}{2560} \alpha u \lambda^8 \tilde{n}^{11/3} = 0.186 \alpha u \lambda^8 \tilde{n}^{11/3}. \]  

(97)

Let us pay attention, that the expansion of \( p_2 \) holds in even powers of the quantity \( \tilde{n}^{1/3} \) and the expansion of \( p_3 \) in odd powers, and this expansion begins with a high power.

Since the effective mass depends on density, then pressure of an ideal gas of quasiparticles (92), with account of (91), can also be represented in the form of expansion in powers of density:

\[ \frac{p_0}{p_{0*}} = \tilde{n}^{5/3} + \frac{3}{10} \alpha \tilde{n}^{8/3} - \frac{3}{70} \left( \frac{9\pi}{4} \right)^{2/3} \alpha \tilde{n}^{10/3} + \frac{159}{2560} \alpha u \lambda^8 \tilde{n}^{11/3} = \tilde{n}^{5/3} + 0.3 \alpha \tilde{n}^{8/3} - 0.158 \alpha \tilde{n}^{10/3} + 0.0621 \alpha u \lambda^8 \tilde{n}^{11/3}. \]  

(98)
Taking into account (96) – (98), we find the expansion of the total pressure in powers of density:

\[
\frac{p}{p_0^*} = \hat{n}^{5/3} + \frac{5}{(12\pi^2)^{1/3}} \alpha \hat{n}^2 + \frac{3}{4} \alpha \hat{n}^{8/3} - \frac{207}{1960} \left(\frac{3\pi^2}{2}\right)^{1/3} \alpha \hat{n}^{10/3} + \frac{159}{640} \alpha u \lambda^8 \hat{n}^{11/3} =
\]

\[
\hat{n}^{5/3} + 1.018 \alpha \hat{n}^2 + 0.75 \alpha \hat{n}^{8/3} - 0.259 \alpha \hat{n}^{10/3} + 0.248 \alpha u \lambda^8 \hat{n}^{11/3}.
\] (99)

Comparison of the accurate dependence of the total pressure on density, calculated by the formulas (92), (93), (95) and the approximate dependence, calculated by the formula (99), is given in Figure 2. It is seen, that even at \(\hat{n} \sim 1\) pressure, calculated by the approximate formula, differs weakly from its accurate value.

Figure 2: Comparison of the accurate dependence of pressure (1), calculated by the formulas (92), (93), (95) and the approximate dependence (2), calculated by the formula (99), at \(\alpha = -0.3, u = -1, \lambda = 1\) (3) an ideal Fermi gas.

Let us discuss the question of the thermodynamic stability of the Fermi system, assuming \(\hat{n} < 1\). For system to be stable, it is necessary that its compressibility or (that is the same) the squared speed of sound should be positive, so that \(\partial p/\partial n > 0\). Since in this case the two first terms give the main contribution into the total pressure (99) and taking into account qualitatively the new effect of three-body interactions, we find the condition of stability in the form

\[
1 + \frac{9}{2} \left(\frac{4}{9\pi}\right)^{2/3} \alpha \hat{n}^{1/3} + \frac{1749}{3200} \alpha u \lambda^8 \hat{n}^2 > 0.
\] (100)

For positive constants of pair and three-body interactions, the system is always stable. More interesting is the case when the constant of the pair interaction and, consequently, the scattering length are negative. Then, without account of three-body interactions, the condition of stability should be satisfied that can be represented in equivalent forms

\[
\hat{n}^{1/3} < \left(\frac{\pi^2}{18}\right)^{1/3} |\alpha|^{-1} \approx 0.82 |\alpha|^{-1}, \quad r_S > \left(\frac{18}{\pi^2}\right)^{1/3} |a_0| \approx 1.22 |a_0|.
\] (101)

Accounting for the three-body interaction, if it has the repulsion character, extends the region of stability and can lead to stabilization of the system with negative scattering length at arbitrary densities in case of fulfillment of the following from (100) condition

\[
|u| > \frac{200}{583 \pi^4 3^3} \frac{5^5 |\alpha|^5}{\lambda^8} \approx 0.408 |\alpha|^5 \lambda^8.
\] (102)

Some dependencies of pressure on density for different signs of the scattering length are shown in Figure 3. For negative scattering length and with neglect of three-body interactions, the spatially uniform state is stable only at low densities for which the condition (101) is satisfied. At high densities pressure decreases with increasing density
Figure 3: Dependencies of pressure on density: 1) $\alpha = -0.3, u = 0; 2) \alpha = -0.3, u = -0.5; 3) \text{an ideal Fermi gas}; 4) \alpha = 0.3, u = 1$. It is everywhere assumed $\lambda = 1$.

Figure 4: Influence of three-body repulsive forces ($U_{3m} > 0$) on stability of the Fermi system with negative scattering length: 1) $u = 0; 2) u = -0.085; 3) u = -0.098; 4) u = -0.11; 5) u = -0.13$. It is everywhere assumed $\alpha = -0.3$ and $\lambda = 1$.

(curve 1 in Figure 3) and the spatially uniform state ceases to be stable. Sufficiently intensive three-body repulsive forces lead to stabilization of the system with negative scattering length (curve 2 in Figure 3).

The influence of three-body repulsive forces on the stability of the Fermi system with negative scattering length is illustrated in more detail in Figure 4. With increasing strength of the three-body repulsive interaction, the regions of stability of such system extend (curve 2, 3 in Figure 4), and further growth of intensity of the three-body interaction leads to stabilization of the system for all physically reasonable values of density.
IX. CONCLUSION

The self-consistent field equations and the thermodynamic relations are obtained for the normal system of Fermi particles interacting via both pair and three-body forces. It is shown that three-body interactions of zero radius give no contribution into the self-consistent field and in order to account for the effects, conditioned by such interactions, it is necessary to account for their nonlocality. The case of the spatially uniform system is considered separately. The formulas for the equation of state and the effective mass with account of three-body forces are derived. Dependencies of the quasiparticle effective mass and pressure on density are obtained at zero temperature with account of pair and three-body interactions in the model of interaction potentials of “semi-transparent sphere” type. It is shown that the role of three-body forces rises with increasing density. The influence of three-body forces on the stability of the Fermi system is considered.

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