On relations between extreme value statistics, extreme random matrices and Peak-Over-Threshold method

Jacek Grela
LPTMS, CNRS, Univ. Paris-Sud, Université Paris-Saclay, 91405 Orsay, France

Maciej A. Nowak
M. Smoluchowski Institute of Physics and Mark Kac Complex Systems Research Centre, Jagiellonian University, Lojasiewicza 11, 30–348 Kraków, Poland

(Dated: November 20, 2017)

Using the thinning method, we explain the link between classical Fisher-Tippett-Gnedenko classification of extreme events and their free analogue obtained by Ben Arous and Voiculescu in the context of free probability calculus. In particular, we present explicit examples of large random matrix ensembles, realizing free Weibull, free Fréchet and free Gumbel limiting laws, respectively. We also explain, why these free laws are identical to Balkema-de Haan-Pickands limiting distribution for exceedances, i.e. why they have the form of generalized Pareto distributions and derive a simple exponential relation between classical and free extreme laws.

I. INTRODUCTION

Random matrix theory is one of the most universal probabilistic tools in physics and in several multidisciplinary applications. In the limit when the size of the matrix tends to infinity, random matrix theory bridges to free probability theory, which can be viewed as an operator valued (i.e. non-commutative) analogue of the classical theory of probability. Both calculi exhibit striking similarities. Wigner’s semicircle law can be viewed as an analogue of normal distribution, Marčenko-Pastur spectral distribution for Wishart matrices is an analogue of Poisson distribution in classical probability calculus, and Bercovici-Pata bijection is an analogue of Lévy stable processes classification for heavy-tailed distributions. It is therefore tempting to ask the question, how far can we extend the analogies between these two formalisms?

Extreme value theory in classical probability is the prominent application of probability calculus for several problems seeking extreme values for large number of random events. Its power comes from the fact, that it has universal probabilistic laws - according to Fisher-Tippett-Gnedenko classification of extreme values, only three generic statistics of extremes saturate all possibilities - Gumbel distribution, Fréchet distribution and Weibull distribution. Beyond applications of extreme value theory in physics in the theory of disordered systems, seminal applications include insurance, finances, hydrology, mechanics, biology, computer science, and several others. This is precisely also the domain of applications of random matrix theory. This coincidence raises a natural question - do we have an analogue of extreme values limiting distributions for the spectra of very large random matrices, i.e does Fisher-Tippett-Gnedenko classification exists in free probability? The positive answer to this crucial question was provided more than a decade ago by Ben Arous and Voiculescu. Using operator techniques, they have proven that free probability theory has also three limiting extreme distributions - free Gumbel, free Fréchet and free Weibull distribution. Although the domains of attraction are the same as in their classical counterparts, the functional form of these limiting distributions was however different. In all three cases, these limiting distributions are represented by certain generalized Pareto distribution. Surprisingly, these were identical to the threefold classification of exceedances in extreme value theory, i.e. they were represented by Balkema-de Haan-Pickands classification in classical probability theory.

In this paper, we shed some light on these unexpected links. In the first section, we introduce a thinning approach to order statistics, which allows us to re-express extreme values calculus in an intuitive way. This reformulation permits us to understand the operator-valued approach of in terms of the concepts of classical probability. Then we show how this reformulation is equivalent to Peak-Over-Threshold method, which explains, why the free probability extreme laws are governed by generalized Pareto distributions. We stress that extreme laws in both calculi have striking similarities - they admit not only the same domains of attraction, but even the same form of centering and

*jacek.grela@lptms.u-psud.fr
nowak@th.if.uj.edu.pl
II. ORDER AND EXTREME VALUE STATISTICS

Extremes of random numbers are described by order statistics. Given a set of \( m \) random variables \( \{x_1, \ldots, x_m\} \), we rearrange them in an ascending order \( \{x_{(1)}, \ldots, x_{(m)}\} \). As an example, for a set of such variables the following inequalities holds true

\[
\begin{align*}
x_{(1)} &\leq x_{(2)} &\leq &\cdots &\leq &x_{(m-1)} &\leq &x_{(m)} \\
x_3 &\leq &x_{m-1} &\leq &\cdots &\leq &x_6 &\leq &x_2 .
\end{align*}
\]

Typically one is interested in the extreme events and studies a particular element in the ordered set \( \{x_{(1)}, \ldots, x_{(m)}\} \) – either the largest \( x_{(m)} \) or the smallest one \( x_{(1)} \). One can study also the distributions of a subset of the ordered set – the \( n \) largest or smallest values.

In all these cases, the cumulative distribution function (or CDF) for the \( k \)-th order statistic \( x_{(k)} \) of a sample of \( m \) variables is given by:

\[
\mathcal{P}^{(m)}(x_{(k)} < x) = \mathcal{P}(x - x_{(k)}) = \sum_{i=k}^{m} \sum_{\sigma} \left[ \prod_{i=1}^{m} \int_{x}^{-\infty} dx_{\sigma(i)} \right] P(x_1, \ldots, x_m),
\]

where \( P(x_1, \ldots, x_m) \) is the joint probability distribution function (PDF) of the not-ordered set of variables \( \{x_1, \ldots, x_m\} \), \( \sum_{\sigma} \) is the summation over \( i \) permutations of \( m \) indices and \( \sum_{\delta} \) is \( m-i \) permutations of remaining \( m-i \) elements. Derivation of the above formula is straightforward when one realizes that in order for the variable to be a \( k \)-th order statistics \( x_{(k)} \), its CDF \( \mathcal{P}^{(m)}(x_{(k)} < x) \) comprises of all cases where at most \( m-k \) variables (i.e. 0, 1, 2 up to \( m-k \))
are larger than $x$ (the group of $\int_x^\infty$ integrals) and the remaining $k$ numbers are smaller than $x$ (forming the group of $\int_{-\infty}^x$ integrals).

In the simplest case of identically and independently distributed (or i.i.d.) variables $P(x_1, \ldots, x_m) = \prod_{i=1}^m p(x_i)$ we find $\int_{-\infty}^x p(t)dt = F(x)$ and $\int_{-\infty}^\infty p(t)dt = 1 - F(x)$ which produces a well-known CDF:

$$\mathcal{P}^{(m)}(x(k) < x) = \sum_{i=k}^m \binom{m}{i} [F(x)]^i [1 - F(x)]^{m-i}$$

(2)

since $\sum_{i=1}^m = \binom{m}{m}$ and $\sum_{i=k} = \binom{m-k}{k}$ = 1. In particular, the distribution function of the largest value for $k = m$ is just $\mathcal{P}^{(m)}(x(m) < x) = [F(x)]^m$.

1. **Thinning approach to order statistics**

We describe a thinning procedure applied to order statistics. Consider the following problem – draw $m$ i.i.d. variables $\{x_1, \ldots, x_m\}$ from parent PDF $p(x)$ and CDF $F(x)$, pick out the $n$ largest ones $\{x_{(m)} \ldots x_{(m-n+1)}\}$ and look at their statistics. What will be the resulting probability density function (or PDF) and cumulative distribution function (or CDF)? We find the thinned CDF $F_k^{(m)}$ of the $n$ largest values selected out of $m$ values as a normalized sum of $n$ terms given by Eq. (2):

$$F_k^{(m)}(x) = \frac{1}{n} \sum_{i=m-n+1}^m \mathcal{P}^{(m)}(x(i) < x)$$

(3)

with $k = \frac{m}{n}$ and the corresponding thinned PDF is found by differentiation $p_k^{(m)}(x) = \frac{d}{dx} F_k^{(m)}(x)$:

$$p_k^{(m)}(x) = \frac{1}{n} \frac{d}{dx} \sum_{i=m-n+1}^m \mathcal{P}^{(m)}(x(i) < x).$$

(4)

By using Eq. (2) and $\frac{d}{dx} \mathcal{P}^{(m)}(x(i) < x) = m \binom{m-1}{i-1} p(x) [F(x)]^{i-1} [1 - F(x)]^{m-i}$, the PDF reads

$$p_k^{(m)}(x) = \frac{m}{n} p(x) \left[ 1 - \sum_{i=1}^{m-n} \binom{m-1}{i-1} [F(x)]^{i-1} [1 - F(x)]^{m-i} \right],$$

(5)

where the summation boundaries were changed by using an identity $\sum_{i=1}^m \binom{m-1}{i-1} [F(x)]^{i-1} [1 - F(x)]^{m-i} = 1$. Crucially, the presented approach is a generalization of the usual extreme value statistics (or EVS) as thinned CDF/PDF given by Eqs. (3) and (4) reduce to EVS counterparts for $n = 1$ (or $k = m$). Besides this special case, we consider limit $m, n \to \infty$ with fixed ratio $k = m/n$ where the sum in the thinned PDF of Eq. (5) is asymptotically given by

$$\sum_{i=1}^{m-n} \binom{m-1}{i-1} [F(x)]^{i-1} [1 - F(x)]^{m-i} \sim \begin{cases} 0, & F(x) > \alpha \\ 1, & F(x) < \alpha \end{cases}, \quad m, n \to \infty, \quad \text{and} \quad \frac{m}{n} \text{fixed},$$

(6)

with $\alpha = \frac{k}{k-1}$ and the details of this calculation are provided in App. A. We use the Heaviside theta function to re-express Eq. (5) and write down a simple asymptotic thinned PDF $p_k(x) = \lim_{m,n \to \infty} p_k^{(m/n)}(x)$ as:

$$p_k(x) = kp(x) \theta(F(x) - \alpha),$$

(7)

where $F(x)$ and $p(x)$ are the parent CDF and PDF respectively. The definition $F_k(x) = \int_{-\infty}^x dx' p_k(x')$ of the asymptotic thinned CDF gives:

$$F_k(x) = k(F(x) - \alpha) \theta(F(x) - \alpha).$$

(8)

Interpretation of both asymptotic thinned PDF $p_k$ and CDF $F_k$ is clear – picking $n$ largest values out of $m$ does not modify the shape of the parent distribution $p(x)$ but truncates it up to a point $x_\alpha$ such that $F(x_\alpha) = \alpha$. The point $x_\alpha$ is known in statistics as the last of the $k$-quantile and gives the point where the fraction of values smaller than $x_\alpha$ is $\alpha = \frac{k}{k-1}$. Importantly, since the large $n, m$ limit was taken the fraction $\alpha$ takes all real number between $(0, 1)$. In the following Sections we continue with relating the asymptotic thinned PDF and CDF to random matrices and Peak-Over-Threshold method.
III. EXTREME MATRIX STATISTICS

Extreme statistics of matrices were introduced in general operator language in Refs. [8, 9] and the special case of extreme matrices were discussed in detail in Ref. [10]. We first introduce these findings in what follows.

To quantify extremal features in a set of observables, a notion of order is indispensable – a characteristic of being either largest or smallest is needed. Although a natural inequality operator exists for real numbers (giving rise to standard EVS) for other objects it may not be the case. Already the complex plane lacks such an ordering; \( z_1 \geq z_2 \) for two complex numbers does not bear any natural interpretation. In particular, defining the order either by modulus or by comparing real and imaginary parts separately does not produce satisfactory results. To assess such problems in full generality, the order theory developed two notions of partially and totally ordered set. A partially ordered set is defined for Hermitian matrices

\[
H_a \leq H_b \quad \iff \quad E(H_a; [t, \infty)) \leq E(H_b; [t, \infty)),
\]

where the spectral projection for a matrix \( H_a \) is given by \( E(H_a; [t, \infty)) = \sum_{i=1}^{N} |\psi_i\rangle \langle \psi_i| \theta(\lambda_i - t) \) and \( A \leq B \) operation on the space of Hermitian projections is defined as \( \forall x : \ x^T (B - A) x \geq 0 \). With this ordering, the definitions of min (\( \wedge \)) and max (\( \lor \)) operations of Hermitian matrices are:

\[
H_a \wedge H_b \quad \iff \quad E(H_a \wedge H_b; [t, \infty)) = E(H_a; [t, \infty)) \wedge E(H_b; [t, \infty)),
\]

\[
H_a \lor H_b \quad \iff \quad E(H_a \lor H_b; [t, \infty)) = E(H_a; [t, \infty)) \lor E(H_b; [t, \infty)).
\]

If the matrices are random, the max operation becomes especially simple – given \( 2N \) eigenvalues of \( H_a, H_b \), we pick out of them the \( N \) largest eigenvalues and form the spectrum of \( H_a \lor H_b \). Since a random matrix is unitarily invariant, eigenvalues alone fully specify the matrix. We state the maximal law for asymptotically large matrices given in Refs. [8, 10].

Define the asymptotic eigenvalue PDF (or the spectral density) of the random matrix \( H \) as \( \rho_H(t) = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_{i=1}^{N} \delta(\lambda_i - t) \right\rangle \) where the average is taken wrt. a prescribed matrix PDF. Asymptotic eigenvalue CDF is in turn given by \( F_H(x) = \int_{-\infty}^{x} dt \rho_H(t) \) of \( H_a, H_b \). The asymptotic eigenvalue CDF of the maximum \( H_a \lor H_b \) of two random matrices \( H_a, H_b \) is given by:

\[
F_{H_a \lor H_b}(x) = \max(0, F_{H_a}(x) + F_{H_b}(x) - 1).
\]

As a special case, for a maximum of \( k \) i.i.d. matrices each with eigenvalue CDF \( F_H(x) \) we find:

\[
F_{H^\lor k}(x) = \max(0, kF_H(x) - (k - 1)),
\]

where \( H^\lor k = H \lor \ldots \lor H \).
1. GUE case

It is instructive to show the spectrum of extremal matrices as given by Eq. (10) on a classic example. We compute both eigenvalue PDF and CDF in the simplest case of Gaussian Unitary Ensemble where the matrix is drawn from a Gaussian distribution $P(H)dH \sim \exp\left(-\frac{N}{2} \text{Tr}H^2\right)dH$. The asymptotic eigenvalue PDF is given by the celebrated Wigner’s semicircle law:

$$\rho_H(t) = \frac{1}{2\pi} \sqrt{4 - t^2}.$$  \hspace{1cm} (11)

The corresponding eigenvalue CDF is given by

$$F_H(x) = \frac{1}{2} + \frac{1}{4\pi}x\sqrt{4 - x^2} + \frac{1}{\pi} \text{arcsin}\frac{x}{2}.$$  \hspace{1cm} (12)

![Graph showing analytical and numerical density of eigenvalues.](image)

FIG. 2. Analytical (lines) and numerical (points) density of eigenvalues $\rho_{H^\vee k}$ of the maximum matrix $H^\vee k = H_1 \vee H_2 \vee \ldots H_k$ where each $H_i$ is of size $500 \times 500$ and drawn from a Gaussian Unitary Ensemble and the ordering in the space of matrices is defined by Eq. (9).

Using Eq. (10) and the above eigenvalue CDF, we compute the $k$-maximal CDF and the corresponding PDF by differentiation $\rho_{H^\vee k}(x) = \frac{d}{dx} F_{H^\vee k}(x)$. The resulting formula is again a semicircle however truncated to an interval $(x_{-}, 2)$:

$$\rho_{H^\vee k}(x) = \frac{k}{2\pi} \sqrt{4 - x^2}, \quad x \in (x_{-}, 2),$$  \hspace{1cm} (13)

with an approximate boundary at $x_{-} \sim 2 - \left(\frac{3\pi}{2k}\right)^2$, obtained by imposing the normalization condition $\int_{x_{-}}^{2} \rho_{H^\vee k}(x)dx = 1$.

In Fig. 1 we present how the density behaves for different values of $k$ – the distribution retains its shape in all cases and becomes truncated into an interval $(x_{-}, 2)$. Moreover, for values of $k \sim 1000$ we observe a deviation from the formula (13) – numerical distribution seem to develop a tail for $x > 2$ and is smoothened near $x \sim x_{-}$.

2. Equivalence between extreme matrices and the thinning method

We show how the extremal law for random matrices given by Eq. (10) is expressible as the thinning procedure introduced in Sec. II 1. We rewrite the asymptotic thinning CDF given by Eq. (8):

$$F_k(x) = (kF(x) - (1-k))\theta\left(F(x) - \frac{1-k}{k}\right),$$

then, since $\theta(af) = \theta(f)$ for $a > 0$ and $\max(0, f) = f\theta(f)$, we have

$$F_k(x) = \max(0, kF(x) - (1-k)).$$
There is an exact correspondence with the CDF given by Eq. [10]. The parent CDF \( F(x) \) and asymptotic thinning CDF \( F_h(x) \) corresponds to the single random matrix CDF \( F_H(x) \) and the \( k \)-maximal CDF \( F_{H \land k}(x) \) respectively. This equivalence is given formally by the formula
\[
F_{H \land k}(x) = F_h(x).
\]

The matrix interpretation of the thinning method is understood by the following replacements:

- \( n, m \) (sample sizes) → \( N, M \) (matrix sizes)
- \( \) values → eigenvalues
- \( \) parent PDF \( p(x)/CDF F(x) \) → eigenvalue PDF \( \rho_H(x)/CDF F_H(x) \)

\textit{a. Equivalence in the GUE case.} We relate the extremal eigenvalue PDF to the asymptotic thinning PDF found in Eq. (7) in the previous example of Gaussian Unitary Ensemble. By making replacements as in the above list, the thinning procedure is picking \( N \) largest eigenvalues out of \( M = kN \) eigenvalues drawn from \( \rho_H(x) \) which readily coincides with the extremal law for matrices given by \([13]\). We see it also in the formulas itself, Eq. (7) gives the asymptotic thinned PDF:
\[
p_k(x) = k\rho_H(x)(H_F(x) - \alpha).
\]

which agrees with Eq. \([13]\) as \( k\rho_H(x) = \frac{k}{2\pi} \sqrt{4 - x^2} \). The Heaviside theta function gives the appropriate truncation at \( F_H(x_+) = \alpha \) equivalent to \( \int_{x_+}^{x_\ast} \rho_H(x)dx = \alpha = 1 - 1/k \). Since \( \int_{x_\ast}^{\infty} \rho_H(x)dx = 1 \), we find
\[
\int_{x_\ast}^{\infty} k\rho_H(x)dx = 1,
\]

which is the normalization condition for the PDF \( \rho_{H \land k} \) given by Eq. \([13]\). We conclude that \( x_- = x_\ast \) and consequently Eq. \([13]\) is recreated. The same calculation can be done to retrieve the CDF \( F_{H \land k} \) given by Eq. \([12]\).

All above considerations are true in the limit when the matrix sizes grow to infinity \( M, N \to \infty \). This corresponds to the free probability regime in which extreme matrix laws are valid. Importantly however, it is also possible to consider the maximum principle also for finite matrices. In this case, the thinning method as described in Sec. \([11,12]\) no longer applies as the eigenvalues are correlated and no longer drawn from a semicircle law. The correlations necessitates the use of the general formula given by Eq. \([11]\) with \( \mathcal{P} \) consisting of \( k \) eigenvalue PDFs and construct a \textit{correlated} thinning framework. Although the problem is harder, it is a promising direction through Coulomb gas techniques as proven useful in extracting extreme value statistics of highly correlated PDFs of eigenvalues (see Refs. \([11,12]\)).

\textbf{IV. LINKING THINNING APPROACH AND PEAK-OVER-THRESHOLD METHOD}

We turn to investigate an apparent connection between the thinning approach and Peak-Over-Threshold method as already signalled in the initial work \([8]\) on extreme matrices. The method is closely related to the notion of \textit{exceedances} which arise conditioned on the event that the random variable \( X \) is larger than some threshold \( u \). For \( t \geq u \), the exceedance distribution function \( F_{[u]}(t) \) is then
\[
F_{[u]}(t) = \mathcal{P}(X < t|X > u) = \frac{\mathcal{P}(X < t, X > u)}{\mathcal{P}(X > u)} = \frac{F(t) - F(u)}{1 - F(u)}.
\]

where we used the usual definition of conditional probability \( \mathcal{P}(A|B) = \mathcal{P}(A, B)/\mathcal{P}(B) \) and parent CDF \( F(x) \). The Peak-Over-Threshold method (or POT) developed in Refs. \([14,15]\) in turn looks at excess distribution functions of events \( X \) above some threshold \( u \):
\[
\mathcal{P}_{\text{POT}}(X < u + t|X > u) = \frac{F(u + t) - F(u)}{1 - F(u)}.
\]

An excess of \( t \) is therefore a variant of the exceedance shifted by the threshold \( u \) itself, i.e. \( F_{[u]}(t + u) \).

In the following we provide a connection between POT method and the thinning approach. It is evident that both methods study extremes – POT method looks at values above some threshold whereas the thinning approch focuses
on a fraction $k$ of largest values drawn from the sample of $m$ observations. To establish the link, it suffices to relate the POT threshold $u$ to the thinning fraction $k$:

$$F(u) = 1 - \frac{1}{k},$$

(16)

with a known parent CDF $F(x)$. This one-to-one relation dictates where one should position the threshold in order to capture a fraction $k$ of values in the sample. This relation is strict in the limit of large samples as only then the inter-sample fluctuations vanish. By the same reason, this relation makes sense for any real value of $k$.

We show now that both asymptotic thinned CDF and PDF given by Eqs. (8) and (7) respectively are related to the POT excess distribution function $P_{\text{POT}}$ and its derivative:

$$\mathcal{P}_{\text{POT}}(X < u + t | X > u) = F_{k(u)}(u + t),$$

$$\frac{d}{dt}\mathcal{P}_{\text{POT}}(X < u + t | X > u) = p_{k(u)}(u + t),$$

(17)

where $k(u) = \frac{1}{F(u)}$ follows from the relation (16). By Eq. (8), the thinned CDF reads:

$$F_{k(u)}(u + t) = k(u) \left( F(u + t) - 1 + \frac{1}{k(u)} \right) \theta \left( F(u + t) - 1 + \frac{1}{k(u)} \right),$$

and by simply computing $F(u + t) - 1 + \frac{1}{k(u)} = F(u + t) - F(u)$ and $\theta(F(u + t) - F(u)) = \theta(t)$ is rewritten as

$$F_{k(u)}(u + t) = \frac{F(t + u) - F(u)}{1 - F(u)} \theta(t),$$

which recreates the POT excess distribution function given by Eq. (15) given an implicit assumption that $t > 0$. This expression is also exactly that of Def. 7.2 given in Ref. [8]. By similar computation one can show the PDF equivalence stated in Eq. (17) as

$$p_{k(u)}(u + t) = \frac{1}{1 - F(u)} p(t + u) \theta(t),$$

where $p(x)$ is the parent PDF. On the other hand, from the POT excess distribution function given by Eq. (15) we arrive at the same formula with the help of $\frac{d}{dt}F(u + t) = p(u + t)$.

V. EXTREME LAWS OF VALUES, MATRICES AND POTS

Up to now, a close relation between extreme matrix statistics, Peak-Over-Threshold method and the thinning approach was established. We investigate these links further from the point of view of extreme laws.

a. Classic extreme laws. We first revise the classic extreme laws arising when inspecting the distribution of the largest value $x_{(m)}$ in the sample of $m$ i.i.d. variables drawn from parent PDF $p(x)$. An $m$-independent form of the maximal CDF $F_{\text{max}}(x)$ exists:

$$\lim_{m \to \infty} \mathcal{P}^{(m)}(x_{(m)} < a_m + b_m x) = F_{\text{max}}(x),$$

(18)

with $m$ dependent constants $a_m$ and $b_m$ representing centering and scaling respectively. There exist three limiting forms of $F_{\text{max}}(x)$ depending on the properties of the parent PDF as summarized in Tab. [I] (see Ref. [16] for a pedagogical review).

The thinning approach encompasses these classic extreme laws as the special case $k = m$ of Eq. (3) is $\mathcal{P}(x_{(m)} < x) = F_{m}^{(m)}(x)$.

b. Free extreme laws. Highly similar free (or matrix) extreme laws exist for the CDF of noncommutative (or free) random variables defined as the limit of the formula given by Eq. (10):

$$\lim_{k \to \infty} F_{H^k}(a_k + b_k x) = F_{\text{free}}(x),$$

(19)

with some scaling and centering constants $a_k, b_k$. The classic and free extreme laws are highly similar – they admit the same domains of attraction, constants $a_k, b_k$ and properties of the parent distributions. The expressions for extreme CDF’s are however different and summarized in Tab. [II].
Gumbel

Weibull

Free Cauchy

Wigner’s semicircle

name | Gumbel | Fréchet | Weibull
---|---|---|---
properties of parent PDF $p(x)$ | tails falls off faster than any power of $x$ | $p(x)$ falls off as $x^{-(\gamma+1)}$ and is infinite | $p(x)$ is finite, $p(x) = 0$ for $x > x_+$ and $p(x) \sim (x-x_+)^{-\gamma-1}$
maximal CDF $F_{\text{max}}^I(x) = \exp \left( -e^{-x} \right)$, $x \in \mathbb{R}$ | $F_{\text{max}}^I(x) = \begin{cases} 0, & x < 0 \\ \exp \left( -x^{-\gamma} \right), & x > 0 \end{cases}$ | $F_{\text{max}}^I(x) = \begin{cases} \exp \left( -\left( -x \right)^\gamma \right), & x < 0 \\ 1, & x > 0 \end{cases}$
a
$F^{-1}\left(1 - \frac{a_n}{\gamma+1}\right)$ | $F^{-1}\left(1 - \frac{b_n}{\gamma+1}\right)$ | $x_+ - F^{-1}\left(1 - \frac{b_n}{\gamma+1}\right)$

TABLE I. Table summarizing three classic extreme laws of Gumbel, Fréchet and Weibull. Functional inverse of the parent CDF $F$ is denoted by $F^{-1}$.

c. POT extreme laws. Lastly, the Peak-Over-Threshold method produces a highly similar family of extreme law under the name of generalized Pareto distribution (see Ref. [13]). By definition (15), the limiting distribution reads:

$$\lim_{u \to \infty} P_{\text{POT}}(X < u + \tilde{a}_u + \tilde{b}_u x | X > u) = F^{\text{POT}}(x),$$

for some constants $\tilde{a}_u, \tilde{b}_u$ which although different than in the classic and free cases, we can fix them by demanding that extreme-matrix and POT limiting distributions are exactly the same $F^{\text{POT}}(x) = F^{\text{free}}(x)$. This is possible as the constants are not unique. Moreover, from the point of view of previously derived equivalencies this is hardly a surprise - both free and POT extreme laws ought to be related as both are expressed by the same extremal PDF and CDF given by Eqs. (7) and (8) respectively.

Relation between the constants are found by

$$F^{\text{free}}(x) = \lim_{k \to \infty} F_{\text{Fréchet}}(a_k + b_k x) = \lim_{k \to \infty} F_k(a_k + b_k x) = \lim_{u \to \infty} F_{k(u)} (a_k(u) + b_k(u)x),$$

and using a well-known limit composition theorem: if $\lim f_k = c$, $\lim k(u) = \infty$ and $f_k$ is continuous then $\lim f_k(u) = c$. On the other hand, we find

$$F^{\text{POT}}(x) = \lim_{u \to \infty} P_{\text{POT}}(X < u + \tilde{a}_u + \tilde{b}_u x | X > u) = \lim_{u \to \infty} F_{k(u)} (u + \tilde{a}_u + \tilde{b}_u x),$$

which gives the relations between the constants:

$$F^{\text{POT}}(x) = F^{\text{free}}(x) \quad \longrightarrow \quad \begin{cases} u + \tilde{a}_u = a_k(u) \\ \tilde{b}_u = b_k(u) \end{cases} .$$

(20)

As was mentioned before, all three laws in the POT approach are typically expressed in terms of the generalized Pareto distribution $G_\beta(x)$:

$$G_{\beta \geq 0}(x) = \begin{cases} 0, & x < 0 \\ 1 - (1 + \beta x)^{-\frac{1}{\beta}}, & x > 0 \end{cases} \quad \text{and} \quad G_{\beta < 0}(x) = \begin{cases} 0, & x < 0 \\ 1 - (1 + \beta x)^{-\frac{1}{\beta}}, & x \in \left(0, -\frac{1}{\beta}\right) \\ 1, & x > -\frac{1}{\beta} \end{cases},$$

where $\lim_{\beta \to 0} \left( 1 - (1 + \beta x)^{-\frac{1}{\beta}} \right) = 1 - e^{-x}$. We find $F_I^{\text{POT}}(x) = G_0(x)$, $F_{II}^{\text{POT}}(x) = G_{1/\gamma}(\gamma(x-1))$ and $F_{III}^{\text{POT}}(x) = G_{-1/\gamma}(\gamma(x+1))$ as can be seen also in Tab. II.
A. Relating classical and free/POT extreme laws via exponentiation

Although classical and free/POT extreme CDFs have different functional forms, they seem to be related by a striking expression

$$F_{\text{free}}(x) \approx 1 + \ln F_{\text{max}}(x), \quad \text{or}$$
$$F_{\text{max}}(x) \approx \exp \left( F_{\text{free}}(x) - 1 \right), \quad (21)$$

Such relation between the POT extreme and classical EVS has been observed in classical probbaility, see e.g. [7]. Unfortunately, this relation is valid only for the functional forms (see Tabs. I and II) and not for whole functions as their domains do not simply match up. In what follows we derive a slightly modified formula [21] with valid treatment of domains by using the thinning method. This derivation highlights the bijection between free and classical extreme statistics.

The classic extreme laws $F_{\text{max}}$ are found from the extremal CDF given by Eq. (2)

$$\mathcal{P}_m^{(m)}(x) = [F(x)]^m \theta(F(x)) \quad (22)$$

with the parent CDF $F(x)$. The step function is added freely as the parent CDF is always a positive function. On the other hand, the free extreme laws $F_{\text{free}}$ arise from the asymptotic thinned CDF given by Eq. (19) and step function $T(x)$:

$$F_m(x) = m \left( F(x) - 1 + \frac{1}{m} \right) \theta \left( F(x) - 1 + \frac{1}{m} \right),$$

where we set $k \to m$ for clarity. We formally solve above equation for $F(x)$:

$$F(x) = 1 + \frac{1}{m} \left( \frac{F_m(x)}{T_m(x)} - 1 \right)$$

and let $T_m(x) = \theta \left( F(x) - 1 + \frac{1}{m} \right)$ be the step function. We plug it back to Eq. (22), set $x \to a_m + b_m x$ with constants given in Tab. I and consider the limit $m \to \infty$ to obtain the classic extreme law CDF according to Eq. (18):

$$F_{\text{max}}(x) = \lim_{m \to \infty} \mathcal{P}_m^{(m)}(a_m + b_m x) = \lim_{m \to \infty} \left( \theta(a_m + b_m x) \right) \left[ 1 + \frac{1}{m} \left( \frac{F_m(a_m + b_m x)}{T_m(a_m + b_m x)} - 1 \right) \right]^m. \quad (23)$$

The ratio $F_m/T_m$ in this formula is in turn expressed asymptotically by free extreme law CDF Eq. (19) and step function $T(x)$:

$$\lim_{m \to \infty} \frac{F_m(a_m + b_m x)}{T_m(a_m + b_m x)} = \frac{F_{\text{free}}(x)}{T(x)}. \quad (24)$$

Thus, both step functions $t(x), T(x)$ admit $m$-independent form given by

$$T(x) = \lim_{m \to \infty} \theta \left( F(a_m + b_m x) - 1 + \frac{1}{m} \right),$$
$$t(x) = \lim_{m \to \infty} \theta(F(a_m + b_m x)). \quad (25)$$

After plugging these definitions back into Eq. (23), we use finally the exponentiation formula $\lim_{m \to \infty} (1 + x/m)^m = e^x$ and the corrected formula (21) relating free and classic extreme laws reads

$$F_{\text{max}}(x) = t(x) \exp \left( \frac{F_{\text{free}}(x)}{T(x)} - 1 \right), \quad (26)$$

where step functions $t$ and $T$ are defined in Eq. (25). Based on explicitly calculated examples in Sec. V A 1 we propose the form for these functions:

- $t(x) = \theta(x)$ if the parent CDF belongs to the Fréchet domain and $t(x) = 1$ if parent CDF belongs to either Gumbel or Weibull domains whereas
- $T(x) = \theta(x + \alpha)$ with $\alpha = -1$ for Fréchet domain, $\alpha = 0$ for Gumbel domain and $\alpha = 1$ for Weibull domain.
To support the form of $T(x)$, we rewrite the free CDFs of Tab. II with the use of step functions:

- Gumbel domain gives $F^{\text{free}}_I(x) = \theta(x)(1 - e^{-x})$,
- Fréchet domain gives $F^{\text{free}}_II(x) = \theta(x - 1)(1 - x^{-\gamma})$,
- Weibull domain gives $F^{\text{free}}_{III}(x) = \theta(x + 1)[1 - \theta(-x)(-x)^\gamma]$.

and observe how in all three above formulas, there is a corresponding $T(x)$ step function present and cancelled in the ratio.

1. Examples

We turn to several examples of extreme laws relevant in the random matrix context and utilize the formulae in Tab. I and II to derive universal CDF’s and support the relation between classical and free/POT extreme laws.

1. Wigner’s semicircle law (free Weibull domain). Previously considered case of truncated semicircle law given by Eq. (13) is a case belonging to the Free Weibull domain. To show this, we choose scaling parameters $a_k = 2, b_k = a_k - 2^{-1}(\alpha)$, set $x = a_k + b_k \tilde{x}$ and find

$$\theta(F(x) - \alpha) = \theta(x - F^{-1}(\alpha)) = \theta ((2 - F^{-1}(\alpha))(\tilde{x} + 1)) = \theta(\tilde{x} + 1).$$

With the approximation $F^{-1}(\alpha) \sim 2 - (3\pi/2)^{2}$ we find:

$$\lim_{k \to \infty} p_k(a_k + b_k \tilde{x}) = \frac{3}{2}(-\tilde{x})^{1/2}\theta(\tilde{x} + 1)\theta(-\tilde{x})d\tilde{x},$$

where the second Heaviside theta function arises by truncating the semicircle as $\theta(2 - x) = \theta(-\tilde{x})$. The full CDF therefore reads:

$$F^{\text{GUE}}_{III}(x) = \begin{cases} 0 & , \ x < -1 \\ 1 - (-x)^{3/2} & , \ x \in (-1,0) \\ 1 & , \ x > 0 \end{cases},$$

and is an example of the free Weibull distribution of Tab. II with parameter $\gamma = 3/2$.

2. Marčenko-Pastur law (free Weibull domain). We check that Marčenko-Pastur law for Wishart matrices also belongs to the free Weibull domain. To this end, we write down both parent PDF and CDF for rectangularity $0 < r < 1$:

$$p_{MP}(x) = \frac{1}{2\pi r x} \sqrt{(x_+ - x)(x - x_-)}, \quad x_{\pm} = (1 \pm \sqrt{r})^2,$$

$$F_{MP}(x) = \frac{1}{2} + \frac{\sqrt{(x_+ - x)(x - x_-)}}{2\pi r} + \frac{1 - r}{2\pi r} \arctan \left( \frac{(1 - r)^2 - x(1 + r)}{(1 - r)\sqrt{(x_+ - x)(x - x_-)}} \right) +$$

$$- \frac{1 + r}{2\pi r} \arctan \left( \frac{1 + r - x}{\sqrt{(x_+ - x)(x - x_-)}} \right).$$

We expand CDF around the endpoint $F_{MP}(x_+ - \delta) \sim 1 - \frac{2}{3\pi} (1 + \sqrt{r})^{3/2} \delta^{3/2}$ and the inverse CDF around unity $F^{-1}_{MP} (1 - \frac{1}{k}) \sim x_+ - \delta$ to find an approximate formula to the latter

$$F^{-1}_{MP}(\alpha) \sim x_+ - \left(\frac{3\pi}{2k}\right)^{2/3} (1 + \sqrt{r})^{4/3} \sqrt{\delta}.$$ 

We compute the rescaled PDF for scaling parameters $a_k = x_+, b_k = x_+ - F^{-1}_{MP}(\alpha)$ and find that it belongs to the free Weibull domain with parameter $\gamma = 3/2$:

$$F^{\text{MP}}_{III}(x) = \begin{cases} 0 & , \ x < -1 \\ 1 - (-x)^{3/2} & , \ x \in (-1,0) \\ 1 & , \ x > 0 \end{cases}.$$
3. Free Arcsine (free Weibull distribution). Arcsine distribution is given by the PDF $p(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}$. In the classical probability, this is a special case of the beta distribution $Beta\left(\frac{1}{2}, \frac{1}{2}\right)$. In the free probability, such spectral measure corresponds to the free convolution of the identical, mutually free discrete measures concentrated on two points (Dirac deltas), i.e. $f(x) = \frac{1}{2}(\delta(x) + \delta(x-1/2))$ (see Ref. [2]). A rerun of arguments presented for semicircle (with $F_{\text{arcsine}}(x) = \frac{2}{\pi} \arcsin\sqrt{x}$), yields immediately to the conclusion, that extreme statistics for arcsine law belongs to the Weibull domains (both classical and free) with $\gamma = 1/2$.

4. Free Cauchy (free Fréchet domain). In free probability, there exists the whole class of spectral distributions, which are stable under the free convolution, modulo the affine transformation. They form exactly the analogue of Lévy heavy (fat) tail distributions in classical probability theory. This one-to-one analogy is called Bercovici-Pata bijection [4]. As the simplest example in the free probability context, the following PDF and CDF are considered:

$$\rho_C(x) = \frac{1}{\pi} \frac{1}{1 + x^2},$$
$$F_C(x) = \frac{1}{2} + \frac{1}{\pi} \arctan(x), \quad F_C^{-1}(x) = -\cot(x\pi).$$

This is the symmetric, spectral Cauchy distribution. The realization of such free heavy-tailed ensembles is non-trivial, e.g. the potential, which by the entropic argument yields Cauchy spectrum, reads explicitly [17]

$$V(\lambda) = \frac{1}{2} \ln(\lambda^2 + 1)$$

so it is non-polynomial - note, that for the Gaussian ensembles $V(\lambda) \sim \lambda^2$. However, to get the extreme law we do not need at any moment the form of the potential. According to Tab. [1] we choose $a_k = 0, b_k = F_C^{-1}(\alpha)$, compute $\theta(a_k + b_k \bar{x} - F_C^{-1}(\alpha)) = \theta(\bar{x} - 1)$ and find:

$$\lim_{k \to \infty} p_k(a_k + b_k \bar{x}) b_k d\bar{x} = \frac{1}{\bar{x}^2} \theta(\bar{x} - 1) d\bar{x}$$

which gives the CDF:

$$F_C^{II}(x) = \begin{cases} 0, & x < 1 \\ 1 - x^{-1}, & x > 1 \end{cases}$$

belonging to the free Fréchet class with $\gamma = 1$.

5. Lévy-Smirnov distribution (free Fréchet domain). Free Lévy-Smirnov distribution is another, simple example belonging to the free Fréchet domain. It is realized through an entropic argument with a confining potential $V(\lambda) = \frac{1}{2\pi} + \ln \lambda$ [17]. The PDF and CDF read respectively:

$$\rho_{LS}(\lambda) = \frac{1}{2\pi} \frac{\sqrt{4\lambda - 1}}{\lambda^2}, \quad \lambda \in \left(\frac{1}{4}, \infty\right),$$
$$F_{LS}(x) = \frac{2}{\pi} \arccos\left(\frac{1}{2\sqrt{x}}\right) - \frac{\sqrt{4x - 1}}{2\pi x}.$$

The inverse CDF is found by expanding CDF $F_{LS}(x) \sim 1 - \frac{2}{\pi} \sqrt{\frac{1}{2} - x}$ around $x \to \infty$ which results in

$$F_{LS}^{-1}(\alpha) \sim \left(\frac{2k}{\pi}\right)^2.$$

The scaling parameters read $a_k = 0, b_k = F_{LS}^{-1}(\alpha)$ and so the limiting CDF given by Eq. (8) reads

$$F_{LS}^{II}(x) = \begin{cases} 0, & x < 1 \\ 1 - x^{-\frac{1}{2}}, & x > 1 \end{cases}$$

belonging to the free Fréchet class with $\gamma = 1/2$. 

6. Free Gaussian (free Gumbel domain). To apply our procedure for this case, we have to choose the spectral distribution whose tails fall faster than any power of $x$. We can use the powerful result [18, 19], noticing that the normal distribution is freely infinitely divisible. This implies, that there exists a random $N \times N$ matrix ensemble, whose spectrum in the large $N$ limit approaches the normal distribution. Entropic argument can even help to find the shape of the confining potential yielding such distribution [20]

$$V(\lambda) = c + \frac{\lambda^2}{2} 2F_2 \left( 1, 1; \frac{3}{2}, 2; -\frac{\lambda^2}{2} \right), \quad c = -\gamma + \log \frac{2}{e},$$

which is a solution to $V(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \ln |x - \lambda| dx$. Luckily, we do not need the shape of the potential to find the free extreme laws. The resulting PDF, CDF, and inverse CDF (quantile) for the spectral normal distribution read, respectively:

$$\rho_G(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

$$F_G(x) = \frac{1}{2} \left( 1 + \operatorname{erf}(x/\sqrt{2}) \right), \quad F_G^{-1}(x) = \sqrt{2}\operatorname{erf}^{-1}(2x - 1).$$

According to the Table I, we have $a_k = F_G^{-1}(1/(1/k))$, $b_k = F_G^{-1}(1/(e/k)) - F_G^{-1}(1/(1/k))$ and so with $x = a_k + b_k \tilde{x}$ we have to perform the limit

$$\lim_{k \to \infty} a_k(b_k + b_k \tilde{x})b_k d\tilde{x} = \lim_{k \to \infty} \frac{k}{\sqrt{2\pi}} b_k e^{-[a_k + b_k \tilde{x}]^2/2} d\tilde{x}.$$  

The limit is subtle, since the inverse error function develops the singularity when its argument approaches unity

$$\operatorname{erf}^{-1}(z)|_{z \to 1} \sim \frac{1}{\sqrt{2}} \ln \left( \frac{2}{\pi(z - 1)^2} \right) - \ln \left( \ln \left( \frac{2}{\pi(z - 1)^2} \right) \right).$$

We set $k = \sqrt{2\pi} e^{u/2}$ and find asymptotic series for both scaling parameters $a_k \sim \sqrt{u - \ln u}$ and $b_k \sim \sqrt{2 + u - \ln(2 + u)} - \sqrt{u - \ln u}$. These asymptotic expansions result in $a_k^2 \sim u - \ln u$, $a_k b_k \sim 1$, $b_k^2 \sim \frac{1}{u}$ and $\ln b_k \sim -\frac{1}{2} \ln u$ which makes all divergent terms cancel out and only the $a_k b_k \sim 1$ survives, yielding

$$\lim_{k \to \infty} \frac{k}{\sqrt{2\pi}} b_k e^{-[a_k + b_k \tilde{x}]^2/2} d\tilde{x} = e^{\tilde{x}^2/2} \theta(\tilde{x}) d\tilde{x},$$

which in turn gives the CDF:

$$F_G^G(x) = \begin{cases} 
0 & x < 0 \\
1 - e^{-x} & x > 0
\end{cases}.$$

an instance of free Gumbel domain of Tab. II.

Other examples include e.g. several free infinite divisible gamma distributions [21], with the simplest $p(x) = e^{-x}$ for non-negative $x$. Since $F(x) = 1 - e^{-x}$, the application of scaling and centering formulae from Table I yields, $a_k = \ln k$ and $b_k = 1$, which trivially reproduces the free Gumbel PDF.

We would like to stress that the same functional form of the PDF may lead to either classical or free Extreme Value Statistics, depending if the PDF represents the one-dimensional classical probability or represents spectral PDF of the ensemble of asymptotically large matrices. Examples 3, 4 and 6 show it explicitly, for each domain: Weibull, Fréchet and Gumbel, respectively.

VI. CONCLUSIONS

We have reformulated laws for extreme random matrices and Peak-Over-Threshold method as a thinning approach to order statistics (inspecting distributions of a subset of ordered variables). This reformulation allowed us to explain several noticed similarities between these different areas. In particular, we have provided explicit examples of random matrix ensembles exhibiting universal laws of extreme statistics in free probability theory. The considered here extreme laws hold for the whole ensembles. They should be not confused with the issue of maximum single eigenvalue
distribution in Gaussian Unitary Ensemble or Wishart $N$ by $N$ ensembles, where universal Tracy-Widom distribution \cite{22} is separating weakly and strongly coupled phases corresponding to right and left shoulders of Tracy-Widom distribution, corresponding to different speeds ($N$ and $N^2$, respectively) \cite{23}. From the point of view of free extreme theory, both Gaussian Unitary Ensembles and Wishart ensembles belong to the free Weibull domain, since they have finite spectral support.

Free and classical probability calculi have several striking similarities and we hope that this work will help to understand further the subtle relations between both calculi. Last but not least, since our approach is operational and expressible in terms of large random matrix models, we anticipate practical applications of extreme random ensembles classification.

VII. ACKNOWLEDGMENTS

The authors appreciate discussions with M. Bożejko, S. Majumdar and D. V. Voiculescu, and comments from P. Warchol and W. Tarnowski. The research was supported by the MAESTRO DEC-2011/02/A/ST1/00119 grant of the Polish National Center of Science.

Appendix A: Proof of asymptotic Eq. (6)

We compute asymptotically the sum:

$$ S = \sum_{i=1}^{m-n} \binom{m-1}{i-1} F(x)^{i-1} (1 - F(x))^{m-i} $$

as $n, m \to \infty$, keeping $m/n = k$ constant and setting $\alpha = \frac{k-1}{k}$ for convenience. First we separate the trivial factors and we use the reparametrization

$$ S = \sum_{i=1}^{m-s} \frac{1}{\Gamma(m-i+1)\Gamma(i)} \left( \frac{F}{1-F} \right)^i. $$

(A2)

With $S = (m-1)!\frac{(1-F)^m}{F} S'$, use the Euler-Maclaurin formula on $S'$:

$$ S' = \sum_{i=1}^{m-s} \frac{1}{\Gamma(m-i+1)\Gamma(i)} \left( \frac{F}{1-F} \right)^i \sim m \int_0^\alpha \frac{dt}{\Gamma(m(1-t) + 1)\Gamma(mt)} \left( \frac{F}{1-F} \right)^mt. $$

(A1)

with $f(t) = -t \log \left( \frac{F}{1-F} \right) + \frac{1}{m} \log \left[ \Gamma(m(1-t) + 1)\Gamma(mt) \right]$. We find

$$ f(t) \sim -t \log \left( \frac{F}{1-F} \right) - 1 + \log m + (1-t) \log(1-t) + t \log t + \frac{1}{m} \log \left( 2\pi \sqrt{\frac{1-t}{t}} \right) $$

so that

$$ S' \sim \frac{m-m+1}{2\pi} \int_0^\alpha dt \sqrt{\frac{t}{1-t}} e^{-mf(t)}, \quad \tilde{f}(t) = -t \log \left( \frac{F}{1-F} \right) + (1-t) \log(1-t) + t \log t - 1. $$

We find saddle point $\tilde{f}'(t_s) = 0$ being equal to $t_s = F$ and consider $F > 0$. When $F < \alpha$, the saddle is inside the integral bounds $(0, \alpha)$ and moves outside for $F > \alpha$. Firstly we consider the $F < \alpha$ case where the saddle-point approximation holds. We compute the integral around $t = t_s + \frac{\delta}{\sqrt{m}}$, find $e^{-m\tilde{f}(t_s + \frac{\delta}{\sqrt{m}})} \sim (1-F)^{-m} e^{m} - \frac{m}{\sqrt{m}} \frac{F}{(1-F)^m}$. which results in

$$ S'_{F<\alpha} \sim \frac{m-m+1}{2\pi} \frac{1}{\sqrt{m}} \sqrt{\frac{F}{1-F}} \frac{e^m}{(1-F)^m} \int_0^\alpha \frac{F}{(1-F)^m} \sqrt{2\pi} = \frac{m-m+1/2m}{\sqrt{2\pi}} \frac{F}{(1-F)^m}. $$

(A3)

For $F > \alpha$, the main contribution to the integral comes from the endpoint at $t = \alpha$:

$$ S'_{F>\alpha} \sim \frac{m-m+1}{2\pi} \sqrt{\frac{\alpha}{1-\alpha}} e^{-mf(\alpha)} = \frac{m-m+1}{2\pi} \sqrt{\frac{\alpha}{1-\alpha}} e^m \left( \frac{F}{\alpha(1-F)} \right)^{ma} \frac{1}{(1-\alpha)^m}. $$

(A4)
Lastly we use Stirling formula \((m - 1)! \sim e^{-m} m^{m-1/2} \sqrt{2\pi}\) in (A2) and combine it with (A3) which readily gives \(S_{F<\alpha} = 1\). Putting together Eqs (A4) and (A2) results in the following formula in the \(F > \alpha\) case:

\[
S_{F>\alpha} \sim \sqrt{\frac{m}{\sqrt{2\pi}F}} \sqrt{\frac{\alpha}{1-\alpha}} \left[ \left(1 - F\right) \left(\frac{1 - \alpha}{1 - F\alpha}\right)^\alpha \right]^m \to 0, \quad \text{as} \quad m \to \infty,
\]

which vanishes when \(m \to \infty\) as the term in the square brackets is less than one: \(\left(1 - F\right) \left(\frac{1 - \alpha}{1 - F\alpha}\right)^\alpha < 1\) or \((1 - \alpha)^{\alpha-\alpha} < (1 - F)^{\alpha-\alpha}\). This inequality follows from the base condition \(F > \alpha \to f(F) > f(\alpha)\) for a function \(f(x) = x^{-\alpha} (1 - x)^{\alpha-1}\) which is monotonically growing given that \(x \in (\alpha, 1)\).

In conclusion, we finally obtain Eq. (6):

\[
S = \sum_{i=1}^{m-n} \left(\begin{array}{c} m - 1 \\ i - 1 \end{array}\right) F(x)^{i-1} (1 - F(x))^{m-i} \sim \begin{cases} 0, & F(x) > \alpha \\ 1, & F(x) < \alpha \end{cases}, \quad m, n \to \infty,
\]

with \(\frac{m}{n} = k, \alpha = \frac{k-1}{k}\. 

---

[1] G. Akemann, J. Baik and P. Di Francesco (Editors), "The Oxford Handbook of Random Matrix Theory", Oxford University Press (2011).

[2] D.V. Voiculescu, Invent. Math 104, 201 (1991); D.V. Voiculescu, K.J. Dykema and A. Nica, "Free Random Variables", Am. Math. Soc., Providence, RI (1992).

[3] A. Nica and R. Speicher, Amer. J. Math. 118, 799 (1996).

[4] H. Bercovici and D. V. Voiculescu, Ind. Univ. Math. J. 42, 733 (1993); H. Bercovici and V. Pata, Ann. of Mathematics 149, 1023 (1999), Appendix by P. Biane.

[5] R. A. Fisher and L. H. C. Tippett, Proc. Cambridge Phil. Soc. 24, 180 (1928); B.V. Gnedenko, Ann. of Mathematics 44, 423 (1943).

[6] J.-Y. Fortin and M. Clusel, J. Phys. A: Math. Theor. 48, 183001 (2015) and references therein; G. Biroli, J.-P. Bouchaud and M. Potters, J. Stat. Mech., P07019 (2016).

[7] R. D. Reiss and M. Thomas, "Statistical Analysis of Extreme Values", Second edition, Birkhäuser Verlag, Basel (2001).

[8] G. Ben Arous and D.V. Voiculescu, Ann. Probab. 34, 2037 (2006).

[9] G. Ben Arous, and V. Kargin, Probab. Theory Relat. Fields 147, 161 (2010).

[10] B. Davis and M. de la Peña, Probab. Theory Relat. Fields 131, 17 (2005).

[11] S. N. Majumdar, and A. Pal, lecture notes available at [arXiv:1406.6768](http://arxiv.org/abs/1406.6768) (2015).

[12] I. P. Castillo, J. Stat. Mech. Theor. Exp. 2016, 063207 (2016).

[13] A. A. Balkema and L. de Haan, Ann. Probab. 2, 792 (1974).

[14] J. I. Pickands, Ann. Statist. 3, 119 (1975).

[15] R. L. Smith, Ann. Statist. 15 1174 (1987).

[16] P. Vivo, Eur. J. Phys. 36, 055037 (2015).

[17] Z. Burda et al., Phys. Rev. E65, 021106 (2002).

[18] S. Belinschi, M. Boţeţko, F. Lehner and R. Speicher, Adv. Math. 226, 3677 (2011); M. Boţeţko and T. Hasebe, Prob. Math. Stat. 33(2013), 363 (2013).

[19] M. Anshelevich, S. T. Belinschi, M. Boţeţko and F. Lehner, Math. Res. Lett. 17 905 (2010).

[20] T. Tierz, "On stable random matrix ensembles", [arXiv:cond-mat/0106485](http://arxiv.org/abs/cond-mat/0106485).

[21] T. Hasebe, Electron. J. Probab. 19 (81), 1, 2014.

[22] C. A. Tracy and H. Widom, Commun. Math. Phys. 159, 151 (1994).

[23] S. Majumdar and G. Schehr, J. Stat. Mech., P01012 (2014), and references therein.