The simultaneous asymmetric perturbation method
for overdetermined free boundary problems

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Abstract

In this paper, we introduce a new method for applying the implicit function theorem to find nontrivial solutions to overdetermined problems with a fixed boundary (given) and a free boundary (to be determined). The novelty of this method lies in the kind of perturbations considered. Indeed, we work with perturbations that exhibit different levels of regularity on each boundary. This allows us to construct solutions that would have been out of reach otherwise. Another benefit of this method lies in the improvement of the regularity gap between the free boundary and the given one. Finally, some geometric properties of the solutions, such as symmetry and convexity, are also discussed.

Key words. two-phase, overdetermined problem, free boundary problem, shape derivatives, implicit function theorem.

AMS subject classifications. 35N25, 35J15, 35Q93

1 Introduction

1.1 Problem setting and known results

Let \( \Omega \) be a smooth bounded domain of \( \mathbb{R}^N \) \( (N \geq 2) \) and \( D \subset \overline{D} \subset \Omega \) be a subdomain with Lipschitz continuous boundary \( \partial D \). For simplicity, we will require that both boundaries \( \partial D \) and \( \partial \Omega \) are connected. Moreover, let \( n \) denote the outward unit normal vector to either \( \partial \Omega \) or \( \partial D \) depending on the context. In what follows, we consider various free boundary problems that satisfy the following assumptions:

- One of the two boundaries (say \( \partial D \)) is given, while the other (say \( \partial \Omega \)) is the free boundary to be determined.

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• The solution of a certain boundary value problem (depending on Ω and D) satisfies a given overdetermined condition on the free boundary.

• The pair \((\partial D_0, \partial \Omega_0)\) is some known solution to the free boundary problem (trivial solution).

Undoubtedly, one of the most famous examples of free boundary problems that satisfy the properties above is the following Bernoulli overdetermined problem.

**Problem 1.** Find a pair \((D, \Omega)\) such that the following overdetermined problem admits a solution for some real parameter \(c\).

\[
\begin{align*}
-\Delta u &= 0 \quad \text{in} \, \Omega \setminus \overline{D}, \\
u &= 1 \quad \text{on} \, \partial D, \\
u &= 0 \quad \text{on} \, \partial \Omega, \\
\partial_n u &= c \quad \text{on} \, \partial \Omega.
\end{align*}
\] (1.1)

Here \(\partial_n\) stands for the outward normal derivative at the boundary.

Clearly, (1.1) is solvable if \((D, \Omega)\) is a pair of concentric balls (trivial solution). Existence, regularity and qualitative properties of the solutions of (1.1) have been studied for a long time and a plethora of different approaches is known. As far as the study of the properties of single solutions is concerned, we refer to [Be, AC, HS] and the references therein. Furthermore, as far as the study of families of solutions is concerned, we refer to [Ac, HO] and the references therein.

Another example of a free boundary problem that fits the description above is given by the two-phase overdetermined problem of Serrin type. Given a positive constant \(\sigma_c \neq 1\), define the following piece-wise constant function:

\[
\sigma = \sigma_c \chi_D + \chi_{\Omega \setminus D},
\] (1.2)

where \(\chi_A\) is the characteristic function of the set \(A\) (i.e., \(\chi_A(x) = 1\) if \(x \in A\) and \(0\) otherwise).

**Problem 2.** Find a pair \((D, \Omega)\) such that the following overdetermined problem admits a solution for some real parameter \(c\).

\[
\begin{align*}
-\text{div} (\sigma \nabla u) &= 1 \quad \text{in} \, \Omega, \\
u &= 0 \quad \text{on} \, \partial \Omega, \\
\partial_n u &= c \quad \text{on} \, \partial \Omega.
\end{align*}
\] (1.3)
This problem was first studied by Serrin [Se] in the particular case $D = \emptyset$. He showed that, if $D = \emptyset$, Problem 2 admits a solution if and only if $\Omega$ is a ball. Just like Problem 1, (1.3) is solvable if $(D, \Omega)$ is a pair of concentric balls (trivial solution). Nontrivial solutions of Problem 2 have been studied only in recent years (see [CY1, CY2] for the local behavior of nontrivial solutions near the trivial ones) and still very little is known about them in the general case.

1.2 The simultaneous asymmetric perturbation method

In what follows, we will briefly describe the ideas behind the simultaneous asymmetric perturbation (SAP) method. For simplicity let $(D_0, \Omega_0)$ denote the pair of concentric balls centered at the origin with radii $R$ and 1 respectively ($0 < R < 1$). Now, let $\mathcal{F}$ and $\mathcal{G}$ be two suitable Banach spaces to be defined later. For sufficiently small $f \in \mathcal{F}$ and $g \in \mathcal{G}$, we introduce the perturbed domains $D_f$ and $\Omega_g$ (see (1.6) for the precise definition)

![Figure 1: Geometrical setting of the simultaneous asymmetric perturbation method.](image)

Figure 1: Geometrical setting of the simultaneous asymmetric perturbation method.

The first step to apply the SAP method consists in finding another Banach space $\mathcal{H}$ and a mapping

$$\Psi : \mathcal{F} \times \mathcal{G} \rightarrow \mathcal{H}$$

such that $\Psi(f, g) = 0$ if and only if the pair $(D_f, \Omega_g)$ solves the given overdetermined problem. The second step consists in applying the following version of the implicit function theorem ([AP]) for Banach spaces to $\Psi$.

**Theorem A** (Implicit function theorem). Let $\Psi \in \mathcal{C}^k(\Lambda \times W, \mathcal{H})$, $k \geq 1$, where $\mathcal{H}$ is a Banach space and $\Lambda$ (resp. $W$) is an open set of a Banach space $\mathcal{F}$ (resp. $\mathcal{G}$). Suppose
that \( \Psi(f^*, g^*) = 0 \) and that the partial derivative \( \partial_g \Psi(f^*, g^*) \) is a bounded invertible linear transformation from \( \mathcal{G} \) to \( \mathcal{H} \).

Then there exist neighborhoods \( \Theta \) of \( f^* \) in \( \mathcal{F} \) and \( W^* \) of \( g^* \) in \( \mathcal{G} \), and a map \( g \in \mathcal{O}^k(\Theta, \mathcal{G}) \) such that the following hold:

(i) \( \Psi(f, \tilde{g}(f)) = 0 \) for all \( f \in \Theta \),

(ii) If \( \Psi(f, g) = 0 \) for some \( (f, g) \in \Theta \times W^* \), then \( g = \tilde{g}(f) \),

(iii) \( \tilde{g}'(f) = -[\partial_y \Psi(p)]^{-1} \circ \partial_f \Psi(p) \), where \( p = (f, \tilde{g}(f)) \) and \( f \in \Theta \).

The essence of the SAP method relies upon the fine-tuned choice of the Banach spaces \( \mathcal{F}, \mathcal{G} \) and \( \mathcal{H} \) so that

- \( \mathcal{F} \) is as large as possible. That is, functions in \( \mathcal{F} \) enjoy the lowest regularity possible.
- \( \mathcal{G} \) is as small as possible. That is, functions in \( \mathcal{G} \) enjoy the highest regularity possible.
- The map \( \Psi : \mathcal{F} \times \mathcal{G} \to \mathcal{H} \) satisfies the hypotheses of the implicit function theorem in a neighborhood of \((0, 0) \in \mathcal{F} \times \mathcal{G} \). In particular \( \Psi \) has to be Fréchet differentiable jointly in the variables \( f \) and \( g \).

The most technical aspect of this method lies in showing the Fréchet differentiability of \( \Psi \) and in performing the resulting computations. This can be done by employing the theory of shape derivatives, some mathematical machinery for computing derivatives of shape functionals (such as \( \Psi \)) with respect to geometric perturbations of the boundary (see [DZ, HP]). The novelty of this paper lies in the kind of perturbations considered. Indeed, we will examine the case of perturbations that are both 

- simultaneous (in \( f \) and \( g \), that is, perturbing both \( \partial D_0 \) and \( \partial \Omega_0 \) at the same time) and 
- asymmetric (that is \( f \) and \( g \) enjoy different regularities). To the best of my knowledge, I am not aware of any other work in the literature where it is meaningful to consider shape derivatives with respect to simultaneous asymmetric perturbations of two boundaries.

1.3 Main results

Let \( B \) denote an open ball centered at the origin such that \( D_0 \subset B \subset \overline{B} \subset \Omega_0 \). Moreover, for an integer \( m \geq 2 \) and a real number \( \alpha \in (0, 1) \), consider the following Banach spaces...
endowed with the usual norms (that will be simply denoted by \( \| \cdot \| \)):

\[
\mathcal{F} := \left\{ f \in C^{0,1}(B, \mathbb{R}^N) : f \equiv 0 \text{ on } \partial B \right\}, \quad \mathcal{G} := \left\{ g \in C^{m,\alpha}(\partial \Omega_0, \mathbb{R}) : \int_{\partial \Omega_0} g = 0 \right\},
\]

\[
\mathcal{H} := \left\{ h \in C^{m-1,\alpha}(\partial \Omega_0, \mathbb{R}) : \int_{\partial \Omega_0} h = 0 \right\}.
\]

(1.4)

Now, for small \((f, g) \in \mathcal{F} \times \mathcal{G}\) let \(\varphi_{f,g} \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)\) be a map such that \(\text{Id} + \varphi_{f,g} : \mathbb{R}^N \to \mathbb{R}^N\) is a diffeomorphism (here \(\text{Id}\) denotes the identity mapping of \(\mathbb{R}^N\)) and

\[
\varphi_{f,g} = f \text{ in } B, \quad \varphi_{f,g} = gn \text{ on } \partial \Omega_0.
\]

(1.5)

Finally, for \((f, g) \in \mathcal{F} \times \mathcal{G}\) sufficiently small, set

\[
D_f := (\text{Id} + \varphi_{f,g})(D_0), \quad \Omega_g := (\text{Id} + \varphi_{f,g})(\Omega_0).
\]

(1.6)

**Theorem I.** There exists a threshold \(\varepsilon > 0\) such that, for all \(f \in \mathcal{F}\) satisfying \(\|f\| < \varepsilon\) there exists a function \(g = g(f) \in \mathcal{G}\) such that the pair \((D_f, \Omega_g)\) is a solution to problem \(P\) for some \(c \in \mathbb{R}\). Moreover, this solution is unique in a small enough neighborhood of \((0, 0) \in \mathcal{F} \times \mathcal{G}\).

Moreover, the asymptotic behavior of the function \(g(f)\) above as \(\|f\| \to 0\) is given by Corollary 3.5.

Let us define

\[
s(k) = \frac{k(N+k-1) - (N+k-2)(k-1)R^{2-N-2k}}{k(N+k-1) + k(k-1)R^{2-N-2k}} \quad \text{for } k = 1, 2, \ldots,
\]

\[
\Sigma = \left\{ s \in (0, \infty) : s = s(k) \text{ for some } k = 1, 2, \ldots \right\}.
\]

**Theorem II.** Let \(\sigma_c \in (0, \infty) \setminus \Sigma\). Then, there exists a threshold \(\varepsilon > 0\) such that, for all \(f \in \mathcal{F}\) satisfying \(\|f\|_{C^{0,1}} < \varepsilon\), there exists a function \(g = g(f, \sigma_c) \in \mathcal{G}\) such that the pair \((D_f, \Omega_g)\) is a solution to problem \(Q\) for some \(c \in \mathbb{R}\). Moreover, this solution is unique in a small enough neighborhood of \((0, 0) \in \mathcal{F} \times \mathcal{G}\).

Moreover, the asymptotic behavior of the function \(g(f)\) as \(\|f\|_{C^{0,1}} \to 0\) is given by Corollary 4.1.

The local behavior of the solutions of Problem 2 when \(\sigma_c \in \Sigma\) has been carried out in [CY2].

This paper is organized as follows. In Section 2 we give the basic definitions concerning shape derivatives. In Section 3 we prove Theorem I through the implicit function theorem.
(Theorem A). Similarly, Section 4 is devoted to the proof of Theorem II. In Section 5 we give some general remarks on some geometric properties of the solutions \((D_f, \Omega_g)\) (namely regularity, symmetry and convexity). Finally, in the Appendix, we prove a technical lemma that is crucial for the SAP method.

2 Preliminaries on shape derivatives

In this section, we will introduce the concept of shape derivatives. Let us first introduce some basic notation. Let \(\omega \subset \mathbb{R}^N\) be a smooth domain at which we will compute the derivative of a shape functional \(J\) (to this end, we will require \(J(\bar{\omega})\) to be defined at least for all domains \(\bar{\omega}\) “sufficiently close” to the reference domain \(\omega\)). Let \(\varphi_0 : \bar{\omega} \to \mathbb{R}^N\) be a sufficiently “smooth” vector field. Let \(\omega_t := (\text{Id} + t\varphi_0)(\omega)\). For \(t > 0\) small enough the perturbation of the identity \(\text{Id} + t\varphi_0 : \bar{\omega} \to \bar{\omega}\) is a diffeomorphism. The shape derivative of \(J\) at \(\omega\) with respect to the perturbation field \(\varphi_0\) is then defined as

\[
J'(\omega)(\varphi_0) = \lim_{t \to 0} \frac{J(\omega_t) - J(\omega)}{t}.
\]

Of course, the definition above can be extended to functionals that take several domains as input as well.

The concept of shape derivative can be applied to shape functionals that take values in a general Banach space too. A fairly common example is given by a smoothly varying family of sufficiently “smooth” real-valued functions \(w_t\) defined on the set \(\omega_t\) (in many practical applications \(w_t\) is the solution to some boundary value problem defined on the perturbed domain \(\omega_t\)). Since each \(w_t\) lives in a different domain \(\omega_t\), the shape derivative \(w'\) has to be defined in an indirect way (see [DZ]), that is

\[
w'(x) = \dot{w}(x) - \nabla w \cdot \varphi_0(x),
\]

where \(\dot{w}\) is the so-called material derivative of \(w_t\), defined as

\[
\dot{w} = \frac{d}{dt} \bigg|_{t=0} w_t \circ (\text{Id} + t\varphi_0).
\]

We remark that, under suitable regularity conditions, the value of the shape derivative \(w'(x)\) at any point \(x \in \omega\) simply coincides with the derivative \(\frac{d}{dt} \bigg|_{t=0} w_t(x)\) (indeed, notice that \(x \in \omega_t\) for \(t\) small enough).
3 Proof of Theorem

3.1 Preliminaries

Let $\mathcal{F}$, $\mathcal{G}$ be the Banach spaces defined in (1.4). For $(f, g) \in \mathcal{F} \times \mathcal{G}$, let $u_{f,g}$ denote the solution to the following boundary value problem:

\[
\begin{cases}
-\Delta u = 0 \text{ in } \Omega \setminus D_f, \\
u = 1 \text{ on } \partial D_f, \\
u = 0 \text{ on } \partial \Omega_g.
\end{cases}
\]  

(3.8)

An elementary calculation yields that, for $(f, g) = (0, 0)$, the solution of (3.8) is given by the following radial function:

\[
u(r) = \begin{cases}
\frac{r^{2-N} - 1}{R^{2-N} - 1} & \text{for } N \geq 3, \\
\log \frac{r}{\log R} & \text{for } N = 2.
\end{cases}
\]

(3.9)

Moreover, by the above, we have

\[
\partial_n u \equiv \frac{2 - N}{R^{2-N} - 1}, \quad \partial_{nn} u \equiv \frac{2 - N}{R^{2-N} - 1} (1 - N) \text{ on } \partial \Omega_0,
\]

(3.10)

where we employed the following convention:

\[
\frac{2 - N}{R^{2-N} - 1} := \frac{1}{\log R} \text{ for } N = 2.
\]

(3.11)

As announced in the introduction, we will apply the implicit function theorem to the function:

\[
\Psi : \mathcal{F} \times \mathcal{G} \rightarrow \mathcal{H} \\
(f, g) \mapsto \Pi_0 \left( \left( \partial_n u_{f,g} \right) \bigg|_{\partial \Omega_0} \right),
\]

(3.12)

where $\Pi_0 : C^{m-1, \alpha} (\partial \Omega_0) \rightarrow \mathcal{H}$ is the projection operator defined by

\[
\Pi_0(\varphi) = \varphi - \frac{1}{|\partial \Omega_0|} \int_{\partial \Omega_0} \varphi
\]

and $u_{f,g}$ is the solution of (3.8). Moreover, by a slight abuse of notation, here $\left( \partial_n u_{f,g} \right) \bigg|_{\partial \Omega_0}$ denotes the function of value

\[
\nabla u_{f,g}(x + g(x) n(x)) \cdot n_g(x + g(x) n(x)) \quad \text{for } x \in \partial \Omega_0.
\]

(3.13)

Finally, notice that, by the Hopf lemma and the boundary condition in (1.1), we can rewrite (3.13) as

\[
\left( \partial_n u_{f,g} \right) \bigg|_{\partial \Omega_0} (x) = - \left| \nabla u_{f,g}(x + g(x) n(x)) \right| \quad \text{for } x \in \partial \Omega_0.
\]

(3.14)
3.2 Computing the shape derivative of $u_{f,g}$

In this subsection, we will compute the explicit expression of the shape derivative of $u_{f,g}$ (the question of the shape differentiability of $u_{f,g}$ will be addressed by Remark 6.3 in the Appendix). Let $\varphi_0 : \Omega_0 \rightarrow \mathbb{R}^N$ satisfy (1.5). Then, the shape derivative $u'$ can be characterized as the unique solution of the following boundary value problem (the proof is analogous to that of [HP, Theorem 5.3.1]).

$$\begin{cases}
-\Delta u' = 0 & \text{in } \Omega_0 \setminus D_0, \\
u' = -\partial_n u f_0 \cdot n & \text{on } \partial D_0, \\
u' = -\partial_n u g_0 & \text{on } \partial \Omega_0.
\end{cases} \quad (3.15)$$

We remark that (3.15) depends on the perturbation field $\varphi_0$ only through its normal component $\varphi_0 \cdot n$ on $\partial D_0 \cup \partial \Omega_0$ (this fact holds in general and is known as the structure theorem for shape derivatives in the literature [NP]). In what follows we will also make use of the following notation for partial shape derivatives. Let $u'_{-}$ and $u'_{+}$ denote the solution to (3.15) corresponding to the pairs $(f_0,0)$ and $(0,g_0)$ respectively. Notice that, by linearity, we have $u' = u'_{-} + u'_{+}$.

Let $\{Y_{k,i}\}_{k,i} (k \in \{0,1,\ldots\}, i \in \{1,2,\ldots,d_k\})$ denote a maximal family of linearly independent solutions to the eigenvalue problem

$$-\Delta_\tau Y_{k,i} = \lambda_k Y_{k,i} \quad \text{on } S^{N-1}, \quad (3.16)$$

where $\Delta_\tau$ stands for the Laplace–Beltrami operator on the unit sphere $S^{N-1}$. The $k$-th eigenvalue $\lambda_k = k(N+k-2)$ has multiplicity $d_k$. Moreover, we consider the normalization $\|Y_{k,i}\|_{L^2(S^{N-1})} = 1$. The solutions to the eigenvalue problem above, usually referred to as spherical harmonics, form a complete orthonormal system of $L^2(S^{N-1})$. Finally, notice that the eigenspace corresponding to the eigenvalue $\lambda_0 = 0$ is the 1-dimensional space of constant functions on $S^{N-1}$.

Proposition 3.1. Let $(f_0,g_0) \in \mathcal{F} \times \mathcal{G}$ and assume that, for some real coefficients $\alpha_{k,i}^\pm$, the following expansions hold true in $L^2(S^{N-1})$:

$$f_0(R\theta) \cdot \theta = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k} \alpha_{k,i}^- Y_{k,i}(\theta), \quad g_0(\theta) = \sum_{k=1}^{\infty} \sum_{i=1}^{d_k} \alpha_{k,i}^+ Y_{k,i}(\theta), \quad \theta \in S^{N-1}. \quad (3.17)$$

Then, the function $u' = u'_{-} + u'_{+}$, solution to (3.15), admits the following explicit expression for $\theta \in S^{N-1}$ and $r \in [R,1]$:

$$u'_{\pm}(r,\theta) = \sum_{k=0}^{\infty} \sum_{i=1}^{d_k} \left(A_{k}^\pm r^{2-N-k} + B_{k}^\pm r^k \right) \alpha_{k,i}^\pm Y_{k,i}(\theta).$$
The values of the coefficients $A_k^\pm$ and $B_k^\pm$ are given by
\[
A_k^- = -B_k^- = \frac{2 - N}{R^{2-N} - 1} \frac{-R^2 - N}{(R^{2-N} - k) - R^2},
\]
\[
A_k^+ = -B_k^+ = \frac{2 - N}{R^{2-N} - 1} \frac{1}{(R^{2-N} - 2k) - 1},
\]
where we made use of the convention (3.11).

**Proof.** The proof is virtually identical to that of [Ca1, Section 4]. We will compute here the expression for $u_+''$ only, since the case of $u_-''$ is completely analogous. Let us pick arbitrary $k \in \{1, 2, \ldots\}$ and $i \in \{1, \ldots, d_k\}$. We will use the method of separation of variables to find the solution of problem (3.15) in the particular case when $f_0 = 0$ on $\partial D_0$ and $g_0 = Y_{k,i}$ on $\partial \Omega_0$ and then the general case will be recovered by linearity. We will be searching for solutions of the form $u_+'' = u_-''(r, \theta) = S(r) Y_{k,i}(\theta)$ (where $r := |x|$ and $\theta := x/|x|$ for $x \neq 0$).

Using the well known decomposition formula for the Laplace operator into its radial and angular components (see for instance [HP, Proposition 5.4.12]), the equation $\Delta u_+'' = 0$ in $\Omega_0 \setminus \overline{D_0}$ can be rewritten as
\[
\partial_{rr} S(r) Y_{k,i}(\theta) + \frac{N-1}{r} \partial_r S(r) Y_{k,i}(\theta) + \frac{1}{r^2} S(r) \Delta_r Y_{k,i}(\theta) = 0 \quad \text{for } r \in (R, 1), \theta \in S^{N-1}.
\]

By (3.16), we get the following equation for $S$:
\[
\partial_{rr} S + \frac{N-1}{r} \partial_r S - \frac{\lambda_k}{r^2} S = 0 \quad \text{in } (R, 1). \tag{3.18}
\]

Since we know that $\lambda_k = k(k + N - 2)$, it can be easily checked that any solution to the above consists of a linear combination of the following two independent solutions:
\[
S_{\text{sing}}(r) := r^{2-N-k} \quad \text{and} \quad S_{\text{reg}}(r) := r^k. \tag{3.19}
\]

Then, for some real constants $A_k^+, B_k^+$ we have
\[
S(r) = A_k^+ r^{2-N-k} + B_k^+ r^k \quad \text{for } r \in (R, 1).
\]

The coefficients $A_k^+$ and $B_k^+$ can then be obtained by the boundary conditions of problem (3.15) by setting $f_0 = 0$. We get the following system:
\[
\begin{cases}
A_k^+ R^{2-N-2k} + B_k^+ = 0,
A_k^+ + B_k^+ = \frac{(2-N)}{R^{2-N-1}},
\end{cases}
\]
where we make use of the convention (3.11) for $N = 2$. By solving it we obtain the coefficients of the series representation of $u_+''$. \qed
3.3 Computing the Fréchet derivative of $\Psi$

**Lemma 3.2.** The map $\Psi : F \times G \to H$ is Fréchet differentiable in a neighborhood of $(0, 0) \in F \times G$.

The proof of Lemma 3.2 is quite technical and will be postponed to the Appendix.

**Theorem 3.3.** The Fréchet derivative $\Psi'(0, 0)$ defines a mapping from $F \times G$ to $H$ by the formula

$$\Psi'(0, 0)[f_0, g_0] = \partial_n u' - \partial_{nn} u g_0,$$

where $\partial_{nn} u = n \cdot (D^2 u n)$. In particular, following the definition of $u'_{\pm}$ given right after (3.15), we have the following expression for the partial Fréchet derivatives as well:

$$\partial_f \Psi(0, 0)[f_0] = \partial_n u' - \partial_{nn} u g_0,$$

$$\partial_g \Psi(0, 0)[g_0] = \partial_n u' + \partial_{nn} u g_0.$$

**Proof.** Fix $(f_0, g_0) \in F \times G$. For simplicity, set $u_t := u_{tf_0, tg_0}$ and $n_t := n_{g_0}$. Since $\Psi$ is Fréchet differentiable by Lemma 3.2, we can compute its Fréchet derivative as the following Gâteaux derivative:

$$\Psi'(0, 0)[f_0, g_0] = \frac{d}{dt} \bigg|_{t=0} \Psi(t f_0, t g_0) = \frac{d}{dt} \bigg|_{t=0} \Pi_0 \left( (\nabla u_t \cdot n_t) \circ (\text{Id} + t g_0 n) \right).$$

Now, since the projection operator $\Pi_0$ commutes with differentiation, we have

$$\Psi'(0, 0)[f_0, g_0] = \Pi_0 \left( \frac{d}{dt} \bigg|_{t=0} (\nabla u_t \cdot n_t) \circ (\text{Id} + t g_0 n) \right).$$

By (3.14), $\nabla u_t \cdot n_t = -|\nabla u_t| < 0$ on $\partial \Omega_0$. Therefore, we can write

$$\Psi'(0, 0)[f_0, g_0] = \Pi_0 \left( \frac{d}{dt} \bigg|_{t=0} |\nabla u_t| \circ (\text{Id} + t g_0 n) \right) = -\Pi_0 \left( \frac{1}{|\nabla u|} (\nabla u \cdot \nabla u' + (D^2 u \nabla u) \cdot g_0 n) \right) = \Pi_0 \left( \partial_n u' \right) + \partial_{nn} u \Pi_0 g_0,$$

where in the last equality we used the fact that $n = -\nabla u/|\nabla u|$ and that $\partial_{nn} u$ is constant on $\partial \Omega_0$. Now, since both maps $f \mapsto \partial_n u'_{\pm} |_{\partial \Omega_0}$ and $g \mapsto \partial_n u'_{\pm} |_{\partial \Omega_0}$ preserve the eigenspaces of the Laplace Beltrami operator in the sense of Proposition 3.1 and that $\Pi_0 f_0 = f_0$ and $\Pi_0 g_0 = g_0$ by construction, we obtain that

$$\Psi'(0, 0)[f_0, g_0] = \Pi_0 \left( \partial_n u'_{\pm} \right) + \Pi_0 \left( \partial_n u'_{\pm} \right) + \partial_{nn} u \Pi_0 g_0 = \partial_n u'_{\pm} + \partial_n u'_{\pm} + \partial_{nn} u g_0$$

as claimed. The representation formulas for the partial Fréchet derivatives $\partial_f \Psi(0, 0)$ and $\partial_g \Psi(0, 0)$ follow immediately by the definitions of $u'_{\pm}$ and $u'_{\pm}$. □
Combining Propositions 3.1 and 3.3 yields the following.

**Corollary 3.4.** Assume (3.17). Then the following hold true under the convention (3.11).

\[
\partial f \Psi(0,0)[f_0] = \frac{2 - N}{R^{2-N} - 1} \sum_{k=0}^{\infty} \sum_{i=1}^{d_k} \alpha_{k,i}^{-} Y_{k,i}(\theta), \\
\partial g \Psi(0,0)[g_0] = \frac{2 - N}{R^{2-N} - 1} \sum_{k=1}^{\infty} \sum_{i=1}^{d_k} \left(1 - k\right) R^k + \left(1 - N - k\right) R^{2-N-k} \frac{1}{R^{2-N-k} - R^k} \beta_{k,i} Y_{k,i}(\theta).
\]

### 3.4 Applying the implicit function theorem

**Proof of Theorem A**. In what follows, let us assume the result of Lemma 3.2 (see the Appendix for a proof). In order to apply the implicit function theorem (Theorem A of page 3) to \(\Psi\), we just need to ensure that the mapping (3.21) (or, equivalently, the one defined by the second formula of Corollary 3.4) is a bounded invertible linear transformation from \(G\) to \(H\). Linearity and boundedness ensue from the properties of the boundary value problem (3.15). We are left to show that \(\partial g \Phi(0,0) : G \to H\) is a bijection. First of all, by Corollary 3.4 we know that \(\partial g \Psi(0,0) : G \to H\) is given by the map

\[
\sum_{k=1}^{\infty} \sum_{i=0}^{d_k} \alpha_{k,i}^{-} Y_{k,i} \mapsto \sum_{k=1}^{\infty} \sum_{i=0}^{d_k} \beta_{k,i} \alpha_{k,i}^{+} Y_{k,i},
\]

where \(\beta_{k,i}\) is defined by

\[
\beta_{k,i} = 2 - N - (1 - k) \frac{R^k + (1 - N - k) R^{2-N-k}}{R^{2-N-k} - R^k},
\]

under the convention (3.11). Now, the injectivity of the map (3.24) is an immediate consequence of the fact that, for all \(k \geq 1\), the coefficient \(\beta_{k,i}\) in the above never vanishes.

Let us now show surjectivity. Take an arbitrary function \(h_0 \in H\). Since, in particular, \(h_0\) is continuous on \(\partial \Omega_0\), it admits a spherical harmonic expansion, say

\[
h_0 = \sum_{k=1}^{\infty} \sum_{i=0}^{d_k} \gamma_{k,i} Y_{k,i}.
\]

Set now

\[
g_0 = \sum_{k=1}^{\infty} \sum_{i=0}^{d_k} \gamma_{k,i} \beta_{k,i} Y_{k,i}.
\]

First of all, notice that, since the sequence \(1/\beta_{k,i}\) is bounded, the function \(g_0\) above is a well defined element of \(L^2(\partial \Omega_0)\). Moreover, the integral of \(g_0\) over \(\partial \Omega_0\) vanishes because the summation in (3.26) starts from \(k = 1\). Finally, if we let \(L\) denote the continuous
extension to $L^2(\partial \Omega_0) \to L^2(\partial \Omega_0)$ of the map defined by $\mathbf{[3.24]}$, it is clear that $g_0 = L^{-1}(h_0)$ by construction. Therefore, in order to prove the surjectivity of the original map $\partial \Psi(0, 0) : G \to \mathcal{H}$, we just need to show that the function $g_0$, defined by $\mathbf{[3.26]}$, is of class $C^{m, \alpha}$ whenever $h_0 \in C^{m-1, \alpha}$. To this end, we will proceed as in the proof of [KS, Proposition 5.2]. First of all, we recall that functions in the Sobolev space $H^s(\partial \Omega_0)$ can be characterized by the decay of the coefficients of their spherical harmonic expansion as follows:

$$\sum_{k=1}^{\infty} \sum_{i=0}^{d_k} (1 + k^2)^s \alpha_{k,i}^2 < \infty \iff \sum_{k=1}^{\infty} \sum_{i=0}^{d_k} \alpha_{k,i}^2 Y_{k,i} \in H^s(\partial \Omega_0).$$

Since, in particular, $h_0 \in C^{m-1, \alpha}(\partial \Omega_0) \subset H^{m-1}(\partial \Omega_0)$, the asymptotic behavior of the coefficients $\beta_k$ given in $\mathbf{[3.25]}$ yields that $g_0 \in H^m(\partial \Omega_0)$. Now, let $u'_+ \denote$ the solution to $\mathbf{[3.15]}$ where $f_0 = 0$ and $g_0$ is given by $\mathbf{[3.26]}$. By construction, $u'_+$ also satisfies:

$$\begin{cases}
-\Delta u'_+ = 0 & \text{in } \Omega_0 \setminus D_0, \\
u'_+ = 0 & \text{on } \partial D_0, \\
\partial_n u'_+ = h_0 - \partial_{mn} g_0 & \text{on } \partial \Omega_0.
\end{cases}$$

(3.27)

Notice that, by assumption, the Neumann data in $\mathbf{[3.27]}$ belongs to $C^{m-1, \alpha}(\partial \Omega_0) + H^m(\partial \Omega_0)$. Therefore, by elliptic regularity for the Neumann problem, it must be that

$$u'_+ \in C^{m, \alpha}(\bar{\Omega}_0) + H^{m+1}(\Omega).$$

Let us argue by induction that

$$u'_+ \in C^{m, \alpha}(\bar{\Omega}_0) + H^{k/2}(\Omega_0) \text{ for all } k \geq 2m + 2.$$ 

Indeed, from the inductive assumption we see that the trace of $u'_+$ satisfies

$$u'_+\big|_{\partial \Omega_0} \in C^{m, \alpha}(\partial \Omega_0) + H^{(k-1)/2}(\partial \Omega_0),$$

which, in turn, implies that the Neumann data in $\mathbf{[3.27]}$ is in $C^{m-1, \alpha}(\partial \Omega_0) + H^{(k-1)/2}(\partial \Omega_0)$. Hence, by elliptic regularity for the Neumann problem, $u'_+ \in C^{m, \alpha}(\bar{\Omega}_0) + H^{(k+1)/2}(\Omega_0)$, which completes the inductive step. By Sobolev embedding, we now conclude that $u'_+ \in C^{m, \alpha}(\bar{\Omega}_0)$, so that its trace $u'_+\big|_{\partial \Omega_0} = h_0 - \partial_{mn} g_0$ belongs to $C^{m, \alpha}(\partial \Omega_0)$. In particular, this implies that $g_0 \in C^{m, \alpha}(\partial \Omega_0)$, as claimed. This concludes the proof of the invertibility of the map $\partial \Psi(0, 0) : G \to \mathcal{H}$. Finally, boundedness ensues by the Schauder boundary estimates and thus the proof of Theorem $\mathbf{i}$ is complete. \hfill $\square$
Moreover, item (iii) of Theorem A yields the following asymptotic behavior of $g(f)$.

**Corollary 3.5.** Suppose that $f \cdot n = \sum_{k=1}^{\infty} \sum_{i=1}^{d_k} \alpha_{k,i}^{-} Y_{k,i} \big|_{\partial D_0}$, then

$$g(f) = \sum_{k=1}^{\infty} \sum_{i=1}^{d_k} \frac{(2 - N - 2k)R^{2-N}}{(1-k)R^k + (1 - N - k)R^{2-N-k}} \alpha_{k,i}^{-} Y_{k,i} + o \left( \|f\|_{C^0,1} \right) \quad as \quad \|f\|_{C^0,1} \to 0.$$ 

4 Proof of Theorem II

The proof of Theorem II follows along the same lines as that of Theorem I, with some obvious modification. First of all, fix $\sigma_c \neq 1$ and let $\Psi : F \times G \to H$ denote the function defined by (3.12) in the sense of (3.13), with $u_{f,g}$ being the solution to the boundary value problem

$$\begin{cases}
-\text{div}(\sigma_{f,g} \nabla u) = 1 & \text{in } \Omega_g, \\
u = 0 & \text{on } \partial \Omega_g,
\end{cases} \quad (4.28)$$

where $\sigma_{f,g}$ denotes the piece-wise constant function defined by (1.2) with respect to the pair $(D_f, \Omega_g)$. Clearly, $\Psi(f,g) = 0$ if and only if the pair $(D_f, \Omega_g)$ solves Problem 2.

In what follows we will admit the shape differentiability of $u_{f,g}$ and the Fréchet differentiability of $\Psi$, while their proofs will be postponed to the Appendix. The actual explicit expressions for the shape derivative of $u_{f,g}$ and the Fréchet derivative of $\Psi$ can be obtained by following the proofs of [Ca1, Proposition 3.1, Proposition 3.2] and [CY1, Theorem 3.3] verbatim. As a result, we get the following expressions for the partial Fréchet derivatives of $\Psi$ under (3.17).

Assume (3.17). Then the following hold true.

$$\begin{align*}
\partial_f \Psi(0,0)[f_0] &= \sum_{k=0}^{d_k} \sum_{i=1}^{\infty} \frac{2 - N - 2k}{F} (\sigma_c - 1)k R^{1-k} \alpha_{k,i}^{-} Y_{k,i}(\theta), \\
\partial_g \Psi(0,0)[g_0] &= \sum_{k=1}^{d_k} \sum_{i=1}^{\infty} \frac{(N + k - 1)(\sigma_c - 1)k + (N - 2 + k + k\sigma_c)(k - 1)R^{2-N-2k}}{F} \alpha_{k,i}^{+} Y_{k,i}(\theta),
\end{align*}$$

where $F = N(N-2+k\sigma_c)R^{2-N-2k} + kN(1-\sigma_c)$. Now, we can proceed as in Subsection 3.4 and show that the mapping $\partial_g \Psi(0,0) : G \to H$ is a bounded bijection if and only if $\sigma_c \notin \Sigma$. This completes the proof of Theorem II. Furthermore, by item (iii) of Theorem A we get the following.
Corollary 4.1. Suppose that $f \cdot n = \sum_{k=1}^{\infty} \sum_{i=1}^{d_k} \alpha_{k,i} Y_{k,i}$ on $\partial D_0$. Then the following asymptotic behavior holds true as $\|f\|_{C^{0,1}} \to 0$.

$$g(f) = \sum_{k=1}^{\infty} \sum_{i=1}^{d_k} \frac{\alpha_{k,i}(N + 2k - 2)(\sigma_c - 1)k R^{1-k}}{(N + k - 1)(\sigma_c - 1)k + (N - 2 + k + k\sigma_c)(k - 1) R^2 - N - 2k} Y_{k,i} + o\|f\|_{C^{0,1}}.$$

5 Concluding remarks

5.1 On the optimal regularity of $\partial D_f$

To the best of our knowledge, there is no theory of shape derivatives that deals with perturbation fields that are not at least Lipschitz continuous. Indeed, it is worth noticing that many known results aim to generalize shape calculus in the other direction: that is, trying to define shape derivatives for non-regular sets (even just measurable sets) but with respect to “regular” perturbation fields (we refer the interested reader to [DZ, HP] and the references therein). So in this sense, we can state that our result yields the optimal (known) regularity for the fixed boundary $\partial D_f$.

Finally, we would like to remark that applying only the (classical) theory of shape derivatives with respect to Lipschitz continuous perturbations to both boundaries $\partial D_0$ and $\partial \Omega_0$ would not have been enough to show the existence of solutions as done in Theorems I and II. Indeed, the functional $\Psi(f, g)$ itself turns out to be not well-defined if the function $g$ is just Lipschitz continuous (since, for instance, one cannot define the trace of $\nabla u_{f,g}$ on $\partial \Omega_g$ if $\partial \Omega_g$ is not Lipschitz continuous).

5.2 On the optimal regularity of $\partial \Omega_g$

It can be shown (see [KN, Theorem 2]) that the free boundary $\partial \Omega_g$ given by Theorems I and II is indeed an analytic surface. Despite that, to our knowledge, it is not clear how this result could be obtained directly by the SAP method since the class of analytic functions is not naturally endowed with a Banach space structure.

In [Ca2], the author considered a variation of Problem 2 where the overdetermined condition on the normal derivative is been replaced by $\partial_n u(x) = cH(x)$ instead (here $H(x)$ denotes the mean curvature of $\partial \Omega$ at the point $x$). We remark that the SAP method can be applied in this case as well. As a consequence, one can prove existence of solutions of the form $(D_f, \Omega_g)$ where $\partial D_f$ is Lipschitz continuous and $\partial \Omega_g$ is of class $C^{m,\alpha}$ for arbitrarily large $m$. Notice that, unlike Problem 2, the machinery of [KN] cannot be applied to obtain
the analyticity of the solutions, since the overdetermined condition $\partial_n u(x) = cH(x)$ is not one of the types considered in [KN].

5.3 Local symmetry of solutions

Let $(D_f, \Omega_g)$ be a solution of Problem 1 (resp. Problem 2) for small enough $(f, g)$ under the hypotheses of Theorem I (resp. Theorem II) in particular, assume $\sigma_c \notin \Sigma$. Then the functions $f$ and $g$ “share the same symmetries”. This can be made precise by the following:

**Proposition 5.1.** Let $\gamma$ be an element of the orthogonal group $O(N)$ such that $f$ is invariant with respect to $\gamma$ (i.e. $f \circ \gamma = f$). Then, $g$ is also invariant with respect to $\gamma$.

**Proof.** By assumption, $(D_f, \Omega_g)$ is a solution of Problem 1 (resp. Problem 2). Now, since $\gamma$ is a rigid motion, $(D_{f \circ \gamma}, \Omega_{g \circ \gamma})$ is also a solution. Furthermore, $D_f = D_{f \circ \gamma}$ by hypothesis. Since, by assumption $f$ and $g$ are small enough and $\sigma_c \notin \Sigma$, then we can apply item (ii) of Theorem A to conclude that $g = g \circ \gamma = g(f)$ as claimed.

We remark that, for Problem 2, this result holds only for sufficiently small $(f, g)$ and $\sigma_c \notin \Sigma$. Indeed, as shown in [CY2], Problem 2 admits solutions of the form $(D_0, \Omega_g)$ for $g \neq 0$ branching from the bifurcation point $\sigma_c = s_k \in \Sigma$. In this case, $g$ is invariant with respect to a strictly smaller (and non empty) subset of $O(N)$.

5.4 A counterexample concerning convexity

Let $(D, \Omega)$ be a solution of (1.1). It is known that, if $D$ is convex then $\Omega$ must be as well (see [HS]). In what follows we will show that the converse does not hold. To this end, we will make use of the SAP method and construct a counterexample showing that the convexity of $\Omega$ does not necessarily imply that of $D$. Let $(0, R) \in \mathbb{R}^{N-1} \times \mathbb{R}$ be the north pole of $\partial D_0$ and let $f_\vee : \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}^{N-1} \times \mathbb{R}$ be the following Lipschitz continuous function:

$$f_\vee(x, y) = \begin{cases} (0, |x| - \varepsilon) & \text{for } (x, y) \in \mathbb{R}^{N-1} \times \mathbb{R}, |x| \leq \varepsilon, |y - R| \leq \varepsilon, \\ (0, 0) & \text{otherwise.} \end{cases}$$

Here $\varepsilon$ is a positive parameter to be chosen such that $\text{supp} f_\vee \subset B$. Let $(D_t, \Omega_t)$ denote the solution of (1.1) given by Theorem I for $f = t f_\vee$ (see Figure 2). Notice that, by taking $t$ sufficiently small, $\partial \Omega_t$ can be made arbitrarily close to $\partial \Omega_0$ in the $C^2$ norm. As
a consequence, we can find some $t > 0$ such that $\partial\Omega_t$ has positive sectional curvature everywhere and thus is a convex set by [Sa]. On the other hand, by construction, the set $D_t$ is never convex, no matter how small $t$ is.

Clearly, the same considerations can be made for Problem 2 by applying Theorem II.

6 Appendix: proof of the Fréchet differentiability of $\Psi$

In this section, we give a proof of the Fréchet differentiability of the map $\Psi$. The procedures used are standard (see for instance [DZ, HP]) but some technicalities arise when we try to “glue together” perturbations with different regularities.

6.1 For the Problem 1

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{m+1,\alpha}$ and let $D \subset \overline{D} \subset \Omega$ be an open set with Lipschitz continuous boundary. Suppose that $D$ has “no holes” so that $\Omega \setminus \overline{D}$ is a domain. Moreover, for some sufficiently small constant $\delta > 0$, set

$$K := \left\{ x \in \overline{\Omega} : \text{dist}(x, \partial\Omega) \leq \delta \right\}, \quad K' := \left\{ x \in \overline{\Omega} : \text{dist}(x, \partial\Omega) \leq 2\delta \right\}, \quad (6.29)$$

$$B := \left\{ x \in \Omega : \text{dist}(x, \overline{D}) < \delta \right\}, \quad (6.30)$$

where $\text{dist}(x, \partial\Omega)$ denotes the distance between the point $x$ and the set $\partial\Omega$. Notice that, by taking $\delta$ small enough, we can assume that $K' \cap \overline{D} = \emptyset$. Finally, define the following Banach space

$$\mathcal{A} := \left\{ \varphi \in C^{0,1}(\overline{\Omega}, \mathbb{R}^N) : \varphi \equiv 0 \quad \text{in} \quad \Omega \setminus (K^c \cup B), \quad \varphi|_{K'} \in C^{m,\alpha}(K', \mathbb{R}^N) \right\} \quad (6.31)$$
endowed with the norm
\[ \|\varphi\| := \|\varphi\|_{W^{1,\infty}(\Omega, \mathbb{R}^N)} + \|\varphi\|_{C^{m,\alpha}(K', \mathbb{R}^N)}. \]

Notice that, for sufficiently small \( \varphi \in \mathcal{A} \), the sets
\[ D_\varphi := (\text{Id} + \varphi)(D), \quad \Omega_\varphi := (\text{Id} + \varphi)(\Omega) \quad (6.32) \]
are simply connected domains with boundaries of class \( C^{0,1} \) and \( C^{m,\alpha} \) respectively. Furthermore, notice that the points in \( \Omega \setminus (K' \cup B) \) are not moved by \( \text{Id} + \varphi \). Now, let \( u_\varphi \) denote the solution to the following boundary value problem.
\[
\begin{cases}
-\Delta u_\varphi = 0 & \text{in } \Omega_\varphi \setminus D_\varphi, \\
u_\varphi = 1 & \text{on } \partial D_\varphi, \\
u_\varphi = 0 & \text{on } \partial \Omega_\varphi.
\end{cases}
(6.33)
\]
By the standard Schauder theory for elliptic operators \([\text{GT}]\), \( u_\varphi \) belongs to
\[ H^1(\Omega_\varphi \setminus D_\varphi) \cap C^{m,\alpha}(\Omega_\varphi \cap W) \]
for any arbitrary open neighborhood \( W \) of \( \partial \Omega_\varphi \) that does not intersect \( \partial D_\varphi \). Finally, notice that, if \( \|\varphi\| \) is small enough, the function
\[ v_\varphi := u_\varphi \circ (\text{Id} + \varphi) \big|_{\Omega \setminus \overline{D}} \quad (6.34) \]
is a well defined element of \( H^1(\Omega \setminus \overline{D}) \cap C^{m,\alpha}(K') \). Then the following holds true.
Lemma 6.1. The map \( \varphi \mapsto v_\varphi \in H^1(\Omega \setminus D) \cap C^{m,\alpha}(K') \) is of class \( C^\infty \) in a neighborhood of \( \varphi = 0 \in A \).

Proof. The proof of this Lemma is quite technical but the overall strategy is simple: we just apply Theorem A of page 3 to some map \( F \) in order to show the smoothness of the auxiliary function \( w_\varphi := v_\varphi - v_0 \) in a neighborhood of \( \varphi = 0 \in A \) (here \( v_0 = u_0 \) denotes the function \( v_\varphi \) corresponding to \( \varphi = 0 \)).

Step 1: find a functional \( F \) such that \( F(\varphi, w_\varphi) = 0 \). First of all, notice that the function \( v_\varphi \) is characterized as the unique element of \( H^1(\Omega \setminus D) \) that satisfies

\[
\int_{\Omega \setminus D} A_\varphi \nabla v_\varphi \cdot \nabla \psi = 0 \quad \text{for all } \psi \in H^1_0(\Omega), \quad v_\varphi = 1 \text{ on } \partial D, \quad v_\varphi = 0 \text{ on } \partial \Omega. \tag{6.35}
\]

where \( J_\varphi \) is the Jacobian of the map \( \text{Id} + \varphi \) and

\[
A_\varphi := J_\varphi (I + D\varphi)^{-1} (I + D\varphi^T)^{-1}. \tag{6.36}
\]

This can be proved by explicitly computing the change of variable \( \text{Id} = \varphi + D\varphi \) in the weak formulation of \( u_\varphi \). Let now consider \( w_\varphi \). By the above, \( w_\varphi \) can be characterized as the unique solution of

\[
\int_{\Omega} A_\varphi \nabla w_\varphi \cdot \nabla \psi + \int_{\Omega} A_\varphi \nabla v_0 \cdot \nabla \psi \quad \text{for all } \psi \in H^1_0(\Omega), \quad w_\varphi \in X, \tag{6.37}
\]

where \( X \) denotes the Banach space

\[
X := \left\{ w \in H^1_0(\Omega \setminus D) : \Delta w \equiv 0 \text{ in } \Omega \setminus (K \cup B), \quad w \in C^{m,\alpha}(K') \right\},
\]

endowed with the norm \( \| \cdot \| := \| \cdot \|_{H^1_0(\Omega \setminus D)} + \| \cdot \|_{C^{m,\alpha}(K')} \).

Let us now consider the following mapping:

\[
F : A \times X \ni (\varphi, w) \mapsto -\text{div} (A_\varphi \nabla w) - \text{div} (A_\varphi \nabla v_0) \in Y, \tag{6.38}
\]

where \( Y \) denotes the Banach space

\[
Y := \left\{ h \in H^{-1}(\Omega \setminus D) : h \equiv 0 \text{ in } \Omega \setminus (K \cup B), \quad w \in C^{m-2,\alpha}(K') \right\},
\]

endowed with the norm \( \| \cdot \| := \| \cdot \|_{H^{-1}(\Omega \setminus D)} + \| \cdot \|_{C^{m-2,\alpha}(K')} \). By \( \text{[6.37]} \), we have \( F(\varphi, w_\varphi) = 0 \).

Step 2: show that \( F \) is smooth. First, we claim that \( F \) is differentiable infinitely many times in a neighborhood of \( (0, 0) \). As a matter of fact, the map \( A \ni \varphi \mapsto J_\varphi = \det (I + D\varphi) \in L^\infty(\Omega) \cap C^{m-1,\alpha}(K') \) is differentiable infinitely many times because also \( \varphi \mapsto I + D\varphi \in L^\infty(\Omega, \mathbb{R}^{N \times N}) \cap C^{m-1,\alpha}(K', \mathbb{R}^{N \times N}) \) is, and the application \( \det(\cdot) \) is a polynomial in
its entries and is therefore continuous. Similarly, the map \( \varphi \mapsto (I + D\varphi)^{-1} \) can be expressed as a Neumann series as \( (I + D\varphi)^{-1} = \sum_{k=0}^{\infty} (-1)^k (D\varphi)^k \) and thus it is \( C^\infty \) in a neighborhood of \( 0 \in A \). Therefore, the map \( A \ni \varphi \mapsto A\varphi \in L^\infty(\Omega, \mathbb{R}^{N \times N}) \cap C^{m-1,\alpha}(K', \mathbb{R}^{N \times N}) \) is also of class \( C^\infty \). Thus, the map \( \left( L^\infty(\mathbb{R}^N, \mathbb{R}^{N \times N}) \cap C^{m-1,\alpha}(K, \mathbb{R}^{N \times N}) \right) \times X \to H^{-1}(\Omega) \cap C^{m-2,\alpha}(K) \) defined by \( (A, v) \mapsto - \text{div}(A \nabla v) \) is also of class \( C^\infty \) because both bilinear and continuous. By composition, we conclude that the full map \( (\varphi, w) \mapsto F(\varphi, w) \) is of class \( C^\infty \).

**Step 3: show that \( \partial_w F(0, 0) \) is a bounded bijection.** It is easy to see that the partial Fréchet derivative with respect to the variable \( w: \partial_w F(0, 0) : X \to Y \) is given by the formula \( w \mapsto - \Delta w \). In what follows, let us show that the map \( X \to Y \) given by \( w \mapsto - \Delta w \) is indeed a bounded bijection as needed by the hypotheses of Theorem A. Fix \( h \in Y \). Let \( w \) denote the unique solution of \( - \Delta w = h \) in \( H^1_0(\Omega \setminus D). \) We will show that \( w \in X \). By assumption \( w \) is harmonic in \( \Omega \setminus (K \cup B) \). Take now another compact set \( K'' \) such that \( K \subset K'' \subset K' \). By the classical boundary Schauder estimates for the Poisson equation, we get \( w \in C^{m,\alpha}(K'') \). Moreover, since \( w \) is harmonic (and thus real analytic) in the whole \( \Omega \setminus (K \cup B) \), in particular \( w \in C^{m,\alpha}(K) \) also holds. Since \( h \in Y \) was arbitrary, we showed that \( \partial_w F(0, 0) : X \to Y \) is a bijection. Finally, boundedness ensues by the standard regularity theory for the Laplace operator.

**Step 4: apply the implicit function theorem.** As a consequence of the above, we can apply Theorem A to show the existence of a \( C^\infty \) branch \( \varphi \mapsto w(\varphi) \in X \) defined for sufficiently small \( \varphi \in A \) such that \( F(\varphi, w(\varphi)) = 0 \). Unique solvability for problem (6.37) yields that \( w(\varphi) = w_\varphi \). Therefore, we obtain the smoothness of the map \( \varphi \mapsto v_\varphi \), as claimed.

For sufficiently small \( \delta > 0 \) let \( B \) and \( K \) be the sets defined at the beginning of this subsection. Suppose that \( \delta > 0 \) is sufficiently small so that \( B \cap K = \emptyset \). Now, let \( \mathcal{F} \) and \( \mathcal{G} \) denote the Banach spaces defined in (1.4). Take any sufficiently small pair \( (f, g) \) in \( \mathcal{F} \times \mathcal{G} \). We will construct a map \( \varphi_{f,g} \in A \) that verifies (1.5). Let \( \Gamma' = \partial K' \setminus \partial \Omega_0 \) and consider the following two boundary value problems:

\[
\begin{align*}
- \Delta \tilde{d} &= 0 \quad \text{in} \ (K')^\circ, \\
\tilde{d} &= 0 \quad \text{on} \ \Gamma', \\
\tilde{d} &= 1 \quad \text{on} \ \partial \Omega.
\end{align*}
\]

\[
\begin{align*}
- \Delta \tilde{g} &= 0 \quad \text{in} \ (K')^\circ, \\
\tilde{g} &= 0 \quad \text{on} \ \Gamma', \\
\tilde{g} &= \frac{1}{|\nabla \tilde{d}|} g \quad \text{on} \ \partial \Omega.
\end{align*}
\]
Now, for \( x \in \overline{\Omega} \) set

\[
\tilde{\varphi}_{f,g}(x) := \begin{cases} 
  f(x) & x \in \overline{B}, \\
  0 & x \in \overline{\Omega} \setminus (\overline{B} \cup K'), \\
  \varrho(x) \nabla \tilde{d}(x) & x \in K'.
\end{cases}
\]

Let now \( \xi \in C^\infty([0, \infty), [0, 1]) \) be a cut off function that verifies

\[
\xi \equiv 1 \quad \text{in} \quad [0, \frac{1}{3} \delta], \quad \xi \equiv 0 \quad \text{in} \quad \left[\frac{2}{3} \delta, \infty\right)
\]

and set

\[
\varphi_{f,g}(x) := \tilde{\varphi}_{f,g}(x) \xi \left(\text{dist}(x, \partial \Omega)\right).
\]  

(6.40)

By construction, \( \varphi_{f,g} \) is a well-defined element of \( A \) that satisfies (1.5) (see also [F]). Let now \( v_{f,g} \) denote the function (6.34) corresponding to \( \varphi = \varphi_{f,g} \).

**Lemma 6.2** (Fréchet differentiability of \( v_{f,g} \)). The map \( \mathcal{F} \times \mathcal{G} \ni (f, g) \mapsto v_{f,g} \in X \) is Fréchet differentiable in a neighborhood of \( (f, g) = (0, 0) \).

**Proof.** First of all, by the linearity of the two boundary value problems in (6.39), we deduce that the map \( (f, g) \mapsto \varphi_{f,g} \in A \) is bilinear. Moreover, by the standard Schauder estimates for the Laplace equation, there exists some constant \( C > 0 \) (independent of \( f \) and \( g \)) such that

\[
\|\varphi_{f,g}\| \leq C \left( \|f\|_{W^{1,\infty}(B, \mathbb{R}^N)} + \|g\|_{C^{m,\alpha}(\partial \Omega_0, \mathbb{R})} \right).
\]

That is, \( (f, g) \mapsto \varphi_{f,g} \) is a continuous bilinear map from \( \mathcal{F} \times \mathcal{G} \) to \( A \), hence it is Fréchet differentiable. The claim now follows from Lemma 6.1 by composition. \( \square \)

**Remark 6.3.** By (2.7), the result above implies the shape differentiability of \( u_{f,g} \) at \((0, 0)\).

We finally have all the ingredients to prove Lemma 3.2.

**Proof of Lemma 3.2.** Notice that, by change of variables, we have

\[
\nabla u_{f,g} \circ (\text{Id} + gn)^{-1} = (\text{Id} + D\varphi_{f,g})^{-1} \nabla v_{f,g} \quad \text{on} \quad \partial \Omega_0
\]  

(6.41)

where \( \varphi_{f,g} \) is the map defined by (6.40). The statement of Lemma 3.2 follows by combining (3.12), (3.14), (6.41) and Lemma 6.2. \( \square \)
6.2 For Problem 2

As done in the previous subsection, the result will be given under a fairly general geometrical setting. The proofs will be omitted altogether since they follow almost verbatim from those in the previous subsection. Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain of class \( C^{m+1,\alpha} \) and let \( D \subset \overline{D} \subset \Omega \) be a measurable set. For small enough \( \delta > 0 \), define the sets \( K, K' \) and \( B \) as in (6.29) and define \( A \) as in (6.31). For \( \varphi \in A \) small enough, let \( D_\varphi \) and \( \Omega_\varphi \) be the sets defined in (6.32). Moreover, for \( \varphi \in A \) small enough, let \( u_\varphi \) denote the solution to the boundary value problem

\[
\begin{aligned}
- \text{div} (\sigma_\varphi \nabla u_\varphi) &= 1 \quad \text{in } \Omega_\varphi, \\
 u_\varphi &= 0 \quad \text{on } \partial \Omega_\varphi,
\end{aligned}
\]

with \( \sigma_\varphi := \sigma_\varphi X_{D_\varphi} + X_{\Omega_\varphi \setminus D_\varphi} \).

By the classical regularity theory for elliptic operators in divergence form and the Schauder boundary estimates for the Laplace operator ([GT]), \( u_\varphi \) belongs to

\[ H_0^1(\Omega_\varphi) \cap C^{m,\alpha}(\Omega_\varphi \cap W), \]

where \( W \) is any open neighborhood of \( \partial \Omega_\varphi \) that does not intersect \( \partial D_\varphi \). Moreover, if \( \|\varphi\| \) is small enough, the function

\[ v_\varphi := u_\varphi \circ (\text{Id} + \varphi) \mid_\Omega \]

is a well defined element of \( H_0^1(\Omega) \cap C^{m,\alpha}(K') \). As done in Lemma 6.1, we can obtain the smoothness of \( w_\varphi := v_\varphi - v_0 \) in the \( X \)-norm in a neighborhood of \( \varphi = 0 \in A \). The shape differentiability of \( u_\varphi \) and the Fréchet differentiability of \( \Psi \) (defined as in (3.12)) then follow immediately.

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References

[Ac]  A. ACKER, On the qualitative theory of parametrized families of free boundaries. J. Reine Angew. Math. 393 (1989), 134—167.
[AI] A.D. ALEXANDROV, *Uniqueness theorems for surfaces in the large*. V. Vestnik Leningrad Univ., 13 (1958): 5–8 (English translation: Trans. Amer. Math. Soc., 21 (1962), 412–415).

[AC] H.W. ALT, L.A. CAFFARELLI, *Existence and regularity for a minimum problem with free boundary*. J. reine angew. Math., 325 (1981): 105–144.

[AP] A. AMBROSETTI, G. PRODI, *A Primer of Nonlinear Analysis*, Cambridge Univ. Press (1983).

[ACM] L. AMBROSIO, A. CARLOTTO, A. MASSACCESI, *Lectures on Elliptic Partial Differential Equations*, Appunti. Sc. Norm. Super. Pisa (N. S.) 18, Edizioni della Normale, Pisa (2019).

[Be] A. BEURLING, *On free-boundary problems for the Laplace equation*. Sem. on Analytic Functions I, Inst. for Advanced Study Princeton (1957), 248–263.

[Ca1] L. CAVALLINA. *Stability analysis of the two-phase torsional rigidity near a radial configuration*. Published online in Applicable Analysis (2018). Available at https://www.tandfonline.com/doi/full/10.1080/00036811.2018.1478082

[Ca2] L. CAVALLINA. *Local analysis of a two phase free boundary problem concerning mean curvature*. To appear in Indiana University Mathematics Journal.

[CY1] L. CAVALLINA, T. YACHIMURA, *On a two-phase Serrin-type problem and its numerical computation*, ESAIM: Control, Optimisation and Calculus of Variations (2020). https://doi.org/10.1051/cocv/2019048

[CY2] L. CAVALLINA, T. YACHIMURA, *Symmetry breaking solutions for a two-phase overdetermined problem of Serrin-type*, to appear in the volume Trends in Mathematics, Research Perspectives. Birkhäuser. https://arxiv.org/abs/2001.10212

[DZ] M.C. DELFOUR, Z.P. ZOLÉSIO, *Shapes and Geometries: Analysis, Differential Calculus, and Optimization*. SIAM, Philadelphia (2001).

[Fo] R.L. FOOTE, *Regularity of the distance function*, Proc. Am. Math. Soc.,92, (1984): 153–155.

[GT] D. GILBARG, N.S. TRUDINGER. *Elliptic Partial Differential Equation of Second Order*, second edition. Springer.

[HO] A. HENROT, M. ONODERA, *Hyperbolic Solutions to Bernoulli’s Free Boundary Problem*. Arch Rational Mech Anal (2021). https://doi.org/10.1007/s00205-021-01620-z

[HP] A. HENROT, M. PIERRRE, *Shape variation and optimization (a geometrical analysis)*, EMS Tracts in Mathematics, Vol.28, European Mathematical Society (EMS), Zürich, (2018).

[HS] A. HENROT, H. SHAHRKHOLIAN, *Convexity of free boundaries with Bernoulli type boundary condition*, Nonlinear Analysis: Theory, Methods & Applications Vol 28 No 5 (1997), 815–823.
[KS] N. Kamburov, L. Sciaraffia, Nontrivial solutions to Serrin’s problem in annular domains, Annales de l’Institut Henri Poincaré C, Analyse non linéaire Vol 38 No 1 (2021), 1–22.

[KN] D. Kinderlehrer, L. Nirenberg, Regularity in free boundary problems, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, Série 4, Tome 4 no. 2, (1977), 373–391.

[NP] A. Novruzi, M. Pierre, Structure of shape derivatives. Journal of Evolution Equations 2 (2002): 365–382.

[Sa] R. Sacksteder, On Hypersurfaces with no Negative Sectional Curvatures, American Journal of Mathematics Vol. 82, No. 3 (July 1960), pp. 609-630.

[Se] J. Serrin, A symmetry problem in potential theory, Arch. Rat. Mech. Anal., 43 (1971), 304–318.

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