Inferential Induction: Joint Bayesian Estimation of MDPs and Value Functions

Christos Dimitrakakis∗†‡ Hannes Eriksson∗† Emilio Jorge∗†
Divya Grover† Debabrota Basu†

February 11, 2020

Abstract

Bayesian reinforcement learning (BRL) offers a decision-theoretic solution to the problem of reinforcement learning. However, typical model-based BRL algorithms have focused either on maintaining a posterior distribution on models or value functions and combining this with approximate dynamic programming or tree search. This paper describes a novel backwards induction principle for performing joint Bayesian estimation of models and value functions, from which many new BRL algorithms can be obtained. We demonstrate this idea with algorithms and experiments in discrete state spaces.

1 Introduction

Many reinforcement learning (RL) algorithms are grounded on the application of dynamic programming to a Markov decision process (MDP). When the underlying MDP $\mu$ is known, efficient algorithms for finding an optimal policy exist that exploit the Markov property. In reinforcement learning (RL), the learning agent simultaneously acts in and learns about the MDP. Model-based RL methods estimate an MDP model directly, while model-free methods estimate the expected utility of a policy. The latter category includes approximate and stochastic dynamic programming algorithms, which estimate a value function.

Here we focus on Bayesian reinforcement learning (BRL). There the underlying MDP $\mu$ is not known, but we assume some prior distribution $\beta$ over a set $\mathcal{M}$ of possible MDPs $\mu$. While Bayesian inference over MDPs is straightforward for the discrete case, it is not always clear how to define an appropriate prior in other cases. For that reason, some past BRL work has focused on maintaining distributions on value functions instead. However, as we show in this paper, this is not straightforward without strong assumptions.

We formulate a novel Bayesian paradigm for reinforcement learning that extends backwards induction$^\dagger\ddagger$ to the Bayesian setting. This allows us to perform joint Bayesian inference on MDPs and value function simultaneously, as well
as optimise the agent’s policy. This leads to a novel family of algorithms for Bayesian reinforcement learning with potential applications in domains where a natural prior on MDPs does not exist. We instantiate and experimentally analyse only one of the many possible algorithms in this family and show it is competitive with the current state of the art, even when such a natural prior exists.

1.1 Setting and Notation

In the following, we use $\mathbb{P}$ and $\mathbb{E}$ to refer to probability (measures) and expectations generally, with some abuse of notation for compactness.

Reinforcement learning is a sequential learning problem faced by agents acting in an unknown environment $\mu$, which is typically modelled as a Markov decision process [c.f. Puterman, 2005].

 Definition 1.1 (Markov Decision Process). An MDP $\mu$ with state space $S$ and action space $A$ is equipped with a reward distribution $P_\mu(s')|s,a)$ for states $s,s' \in S$ and actions $a \in A$.

At time $t$, the agent interacts with the environment as follows: At time $t$ an agent observes the environment state $s_t$, and then selects action $a_t$. It then receives, and observes the reward $r_t$ as well as the next state $s_{t+1}$. The RL agent is interested in the utility, i.e. the sum of future rewards $r_t$:

$$U_t \triangleq \sum_{k=t}^{T} \gamma^{k-t} r_k,$$

where $\gamma \in (0,1]$ is a discount factor and $T \in [1, \infty]$ is the problem horizon. Typically, the agent wishes to maximise the expected utility, but other objectives are possible.

The agent acts in the environment using a policy $\pi = (\pi_1, \ldots, \pi_t, \ldots)$ which takes action $a_t$ at time $t$ with probability $\pi_t(a_t | s_t, r_{t-1}, a_{t-1}, s_{t-1}, \ldots, r_1, a_1, s_1)$. The dependence on the complete observation history is necessary, if the agent is learning from experience. However, when $\mu$ is known, the policy $\pi_\star$ maximising expected utility is Markovian of the form $\pi_t(a_t | s_t)$ and can be calculated through dynamic programming. A useful algorithmic tool for achieving this is the value function, i.e. the expected utility of a policy $\pi$ from different starting states and action:

 Definition 1.2 (Value function). The state value function of policy $\pi$ in MDP $\mu$ is

$$V_{\pi,\mu}^\tau(s) \triangleq \mathbb{E}_\pi^\tau(U_t | s_t = s),$$

where we use $P_\pi^\tau$ and $\mathbb{E}_\pi^\tau$ to denote probabilities and expectations under the process jointly defined by $\pi, \mu$. The state-action value function of policy $\pi$ in MDP $\mu$ is

$$Q_{\pi,\mu}^\tau(s, a) \triangleq \mathbb{E}_\pi^\tau(U_t | s_t = s, a_t = a).$$

\footnote{In the partially-observable setting it instead observes another variable that is dependent on the state.}

\footnote{For finite horizon problems this policy is still Markovian, but can be non-stationary.}
We now define the Bellman operator:
\[ \mathcal{B}_\pi \mu V(s) \triangleq \rho_\mu(s) + \gamma \sum_{s' \in S} P_\pi(s' | s)V(s'), \]
which allows us to compute the value function recursively through
\[ V_{\mu,t+1} = \mathcal{B}_\pi \mu V_{\mu,t}. \]

**The Bayesian Setting.** In RL, the MDP is unknown. In the Bayesian setting, we can represent this with a subjective belief, in the form of a probability measure \( \beta \), over possible MDPs. This in turn, allows us to define the Bayesian value function:
\[ V_{\pi,\beta,t}(s) \triangleq \int_M V_{\pi,\mu,t}(s) \, d\beta(\mu). \]

The Bayesian value function is simply the expectation of the value function distribution. The Bayes-optimal policy achieves the Bayes-optimal value function \( V_{\pi,\beta,t}(s) = \sup_\pi V_{\pi,\beta,t}(s) \), calculating this value function, however, involves integrating \( V_{\pi,\mu,t} \) for all \( \mu \), while obtaining the optimal value function requires exponential time. Nevertheless, information about the value function distribution can be a useful tool for constructing near-optimal policies. In addition, distributional information may be important for finding risk-sensitive policies.

In this paper, we will refer to the initial belief \( \beta \) as the prior distribution. By interacting with the environment until time \( t \), the agent obtains data \( D = (s_1, a_1, r_1, \ldots, s_t) \). This can be used to calculate a posterior distribution \( \beta(\mu | D) \), which can be done in closed form when the MDP is discrete, in which case a Dirichlet-product prior can be used or when continuous and the dynamics are assumed linear, in which case a Gaussian-Wishart prior can be used [DeGroot [1970]].

**Distributions over Value Functions.** Consider the value function \( V \), with \( V = (V_1, \ldots, V_T) \) for finite-horizon problems, and some prior belief \( \beta \) over MDPs, and some previously collected data \( D = (s_1, a_1, r_1, \ldots, s_t) \) from some policy \( \pi \). Then the posterior value function distribution can be written in terms of the MDP posterior:
\[ P_{\beta}(V | D) = \int_M P_\mu(V) \, d\beta(\mu | D). \]

This empirical measure \( \hat{P}_{MC} \) defined below corresponds to the standard Monte-Carlo estimate
\[ \hat{P}_{MC}(B) \triangleq N^{-1} \sum_{k=1}^{K} \mathbb{I} \{ v^{(k)} \in B \}, \]
where \( \mathbb{I} \{ \} \) is the indicator function. In practice, this can be implemented through Algorithm 1.

---

4In the discounted setting the value function converges to \( V_{\mu} \equiv V_{\mu,1} \) as \( T \to \infty \).
5Even though Gaussian process inference can be expressed in a closed-form, in practice the inference must be approximate, as the computational complexity scales quadratically with time.
6This has \( O(N_\mu N_\beta N A T) \) complexity for policy evaluation, while policy optimisation can be performed through approximate dynamic programming [Dimitrakakis [2011]] or Bayesian gradient ascent [Ghavamzadeh and Engel [2006]].
Algorithm 1 Monte-Carlo Estimation of Value Function Distributions

1: Select a policy $\pi$.
2: for $k = 1, \ldots, N$ do
3: Sample an MDP $\mu(k) \sim \beta$.
4: Calculate $v(k) = V_{\mu(k)} \sim \beta$.
5: end for
6: return $\hat{P}_{MC}(\{v(k)\})$

1.2 Related Work

Bayesian value function distributions have been considered extensively in model-free Bayesian reinforcement learning. Engel et al. [2003] developed the Gaussian Process Temporal Difference (GPTD) algorithm, which has a Gaussian process (GP) prior $\beta(V)$ on value functions. It then combines this with the likelihood function

$$P(D | V) \propto \prod_{t} \exp\{-|V(s_t) - r_t - \gamma V(s_{t+1})|^2\}.$$ 

However, this makes the implicit assumption that the deterministic empirical MDP model is true. They relaxed this assumption in their follow up paper Engel et al. [2005] by allowing for correlation between sequentially visited states. Later, Deisenroth et al. [2009] developed a dynamic programming algorithm that uses a GP prior on value functions and obtains value function distributions given an explicit GP model of the MDP. Finally, Tang and Agrawal [2018] introduces the VDQN temporal difference algorithm that generalises such methods to Bayesian neural networks.

However, the assumptions these methods implicitly make about the MDP are difficult to interpret, and we find the use of an empirical model of the MDP independently of the value function distribution unsatisfactory. We argue that explicitly reasoning about the joint value function and MDP distribution is necessary to obtain a coherent Bayesian procedure.

Naturally, if a posterior over MDPs is available, then there is a simple way to calculate a distribution over value functions: First, sample from the MDP posterior and then calculate the value function of each MDP, as illustrated in (3). This approach was used in the discrete case by Dimitrakakis [2011] and in the continuous case by Osband et al. [2016]. We, however, focus on whether it is possible to compute the value function distribution through a backwards induction procedure.

It turns out that this is possible to do for utility (rather than value) distributions through Bellman updates, in the context of distributional reinforcement learning. Essentially, this is the problem of estimating $P_{\mu}(U_i | U_{i+1})$ for some given MDP. Morimura et al. [2010] used such an approach to construct risk-sensitive policies, and Bellemare et al. [2017] showed that modelling the full utility distribution may also be useful for exploration. For use in RL, they relax the unknown MDP assumption by using either an empirical estimate of the MDP or with stochastic approximation theory to obtain a distribution on utilities. However, the utility distribution is due to the stochasticity of the transition kernel $P_{\mu}(s_{t+1} | s_t, a_t)$ rather than uncertainty about the MDP, and hence a completely different quantity from the value function distribution.
In this paper, we wish to estimate the value function distribution from data, which makes our problem much more challenging technically. In particular, there is no closed-form expression for calculating the value distribution at step $i$ from the value distribution at step $i+1$ when the MDP is unknown, even in simple scenarios.

1.3 Our Contribution

This paper introduces a Bayesian approach to value function estimation in reinforcement learning. In particular, we develop Bayesian backwards induction: a framework where the conditional value function distribution at step $t$ is calculated from the value function distribution at step $t+1$, similarly to backwards induction for the expectation. This can be combined with a policy optimisation step similarly to value iteration. This choice opens up a new family of inferential induction Bayesian reinforcement learning algorithms, specifying a joint distribution over value functions and models. This has potential advantages over previous approaches, even when a natural prior over MDP models exists. While we only instantiate one member of this family in this paper, our experimental results show that it performs similarly to the current state of the art in discrete domains.

Unlike previous approaches which make implicit model assumptions, in our formulation the model of the process is explicit. In fact, our derivations imply that it is impossible to be completely model-free in a probabilistic sense when we wish to estimate value functions.

2 Inferential Induction

The problem is calculating the distribution of value functions $V$ from the current time step $t$ to the horizon $T$, i.e. $P^\pi_\beta(V_t, \ldots, V_T \mid D)$ for a policy $\pi$. Our framework inductively calculates $P^\pi_\beta(V_{i+1} \mid D)$ from $P^\pi_\beta(V_i \mid D)$ for $i \geq t$:

$$P^\pi_\beta(V_{i} \mid D) = \int_{V} P^\pi_\beta(V_i \mid V_{i+1}, D) dP^\pi_\beta(V_{i+1} \mid D).$$

Let $\psi_{i+1}$ be a (possibly approximate) representation of $P^\pi_\beta(V_{i+1} \mid D)$. Then the remaining problem is to define the term $P^\pi_\beta(V_{i} \mid V_{i+1}, D)$ appropriately and finally calculate the complete distribution. This is what we focus on in the rest of this section.

2.1 The Link Distribution: $P(V_i \mid V_{i+1}, D)$

A simple idea for dealing with the term linking the two value functions is to marginalise over the MDP as follows:

$$P^\pi_\beta(V_{i} \mid V_{i+1}, D) = \int_{\mathcal{M}} P^\pi_{\mu}(V_i \mid V_{i+1}) dP^\pi_\beta(\mu \mid V_{i+1}, D).$$

This equality holds because given $\mu$, $V_i$ is uniquely determined by the policy $\pi$ and $V_{i+1}$ through the Bellman operator. However, it is crucial to note that
This section develops a Monte Carlo method for approximating (4). The core

The other important design decision concerns the distribution $P(\mu | V_{i+1}, D)$. By sampling $s$ from the MDP and the policy $\pi$, the quantity $P(\mu | V_{i+1}, D)$ may appear problematic, but we can use utility rollouts $U$ sampled from the MDP $\mu$ and the policy $\pi$ from step $t + 1$ onwards, averaged over states:

$$P^\pi_{\mu}(V_{i+1}) = \int_S dq(s) \int_{-\infty}^{\infty} P^\pi_{\mu}(V_{i+1} | U, s) P^\pi_{\mu}(U | s) dU$$

As a design choice, we define the density of $V_{i+1}$ given a sample $u_m$ from state $s_m$ to be a Gaussian with variance $\sigma^2$:

$$\frac{d}{dA} P(V_{i+1} | u_m, s_m) \propto \frac{1}{\sqrt{2\pi}} e^{-|V_{i+1}(s_m) - u_m|^2 / 2\sigma^2}.$$

In order to maintain a correct estimate of uncertainty, we must specify an appropriate conditional distribution $P^\pi_{\mu}(V_{i+1}, D)$. We focus on the idea of maintaining an approximation $\psi_i$ of value function distributions and combining this with the MDP posterior through an appropriate kernel, as detailed in Sec. 2.2.

2.2 The Conditional MDP Distribution: $P(\mu | V_{i+1}, D)$

The other important design decision concerns the distribution $P^\pi_{\beta}(\mu | V_{i+1}^{(k)})$. Expanding this term, we obtain, for any subset of MDPs $A \subseteq M$:

$$P^\pi_{\beta}(\mu \in A | V_{i+1}, D) = \frac{\int_A P^\pi_{\mu}(V_{i+1}) d\beta(\mu | D)}{\int_M P^\pi_{\mu}(V_{i+1}) d\beta(\mu | D)},$$

since $P^\pi_{\mu}(V_{i+1} | D) = P^\pi_{\mu}(V_{i+1})$, as $\mu, \pi$ are sufficient for calculating $V_{i+1}$. Let us now see how we can perform inference using Monte Carlo methods.

2.3 Monte Carlo Inference

This section develops a Monte Carlo method for approximating (4). The core of the algorithm calculates an approximate belief $\psi_i$ on $V_i$ from $\psi_{i+1}$ on the next-step value function, $V_{i+1}$. We start by combining (4) and (6) and replacing $\psi_{i+1}$ for the next-step belief to obtain a conditional probability measure on value functions:

$$P^\pi_{\beta}(V_i \in B | D) = \int_V \int_M P^\pi_{\mu}(V_i \in B | V_{i+1}) dP^\pi_{\beta}(\mu | V_{i+1}, D) d\psi_{i+1}(V_{i+1})$$

By sampling $V_{i+1}^{(k)} \sim \psi_{i+1}$ we obtain the following Monte Carlo estimate of $\psi_i$

$$\frac{1}{N_V} \sum_{k=1}^{N_V} \int_M 1 \{ P^\pi_{\mu}(V_{i+1}^{(k)} \in B) \} dP^\pi_{\beta}(\mu | V_{i+1}^{(k)}, D).$$

Let us now move onto calculating (6). The quantity $P^\pi_{\mu}(V_{i+1})$ may appear problematic, but we can use utility rollouts $U$ sampled from the MDP $\mu$ and the policy $\pi$ from step $t + 1$ onwards, averaged over states:
For discrete MDPs, we can use a uniform distribution $q$ over states and sum over all of them, while we can sample from $q$ in the continuous case. Finally, we can write:

$$P^\pi_\mu(V_{i+1}) \approx \frac{1}{n} \sum_{m=1}^{n} P^\pi_\mu(V_{i+1} \mid u_m, s_m), \quad u_m \sim P^\pi_\mu(U)$$

This leads to the following approximation for (6):

$$P^\pi_\beta(\mu \in A \mid V_{i+1}, D) \approx \int_A \sum_{m=1}^{n} e^{-\frac{|V_{i+1}(s_m) - u_m|^2}{2\sigma^2}} d\beta(\mu \mid D) \cdot \int_M \sum_{m=1}^{n} e^{-\frac{|V_{i+1}(s_m) - u_m|^2}{2\sigma^2}} d\beta(\mu \mid D).$$

If we generate $\mu^{(j)} \sim \beta(\mu \mid D)$ and set:

$$w_{jk} \triangleq \frac{\sum_{m=1}^{n} e^{-\frac{|V_{i+1}(s_m) - u_m|^2}{2\sigma^2}}}{\sum_{j'=1}^{N_\mu} \sum_{m=1}^{n} e^{-\frac{|V_{i+1}(s_m) - u_m'|^2}{2\sigma^2}}}$$

then $E w_{jk} = P^\pi_\beta(\mu \in M \mid V_{i+1}, D)$. This allows us to obtain value function samples for step $i$,

$$V^{(j,k)} \triangleq \mathbb{P}^\pi_{\mu^{(j)}} V^{(k)},$$

each weighted by $w_{jk}$, leading to the following Monte Carlo estimate of the value function distribution at step $i$

$$\psi_i(B) = \frac{1}{N_\mu N_\nu \sum_{k=1}^{N_\mu} \sum_{j=1}^{N_\nu} 1 \{V^{(j,k)} \in B\} w_{jk}.}$$

This ends the general description of the Monte-Carlo method. The actual algorithmic details depend on the representation we use for $\psi_i$ and whether the MDP is discrete or continuous. We will describe specific instantiations of (10) in Section 3.

2.4 A Parenthesis on Mean-field Approximation

If we ignore the value function information by assuming that $P^\pi_\beta(\mu \mid V_{i+1}, D) = P_\beta(\mu \mid D)$, we obtain

$$P^\pi_\beta(V_i \mid D) = \int \int P^\pi_\mu(V_i \mid V_{i+1}) d\beta(\mu \mid D) dP^\pi_\beta(V_{i+1} \mid D).$$

Unfortunately, this corresponds to a mean-field approximation. This will eventually eliminate all our uncertainty about the value function as it is equivalent to assuming the mean MDP is true. For that reason, we do not consider this approximation any further.

3 Policy Evaluation: Inference Algorithms

Algorithm 2 below is a concise description of the Monte Carlo procedure we developed. At each time step $t$, the algorithm is called with the prior and data
D collected so far, and it looks ahead up to some lookahead factor $H$. When the horizon $T$ is small, we can set $H = T - t$. We then instantiate it below for the discrete case, with an algorithm for continuous MDPs relegated to the appendix.

**Algorithm 2** Inferential Induction based Policy Evaluation

1: **Input:** Prior $\beta$, data $D$, lookahead $H$, discount $\gamma$, policy $\pi$, $N_\mu, N_V$.
2: Initialise $\psi_H$.
3: Sample $M \triangleq \{\mu(j) \mid j \in [N_\mu]\}$ from $\beta(\mu \mid D)$.
4: for $i = H - 1, \ldots, 1$ do
5: Sample $V^{(k)} \sim \psi_{i+1}(v)$ for $k \in [N_V]$.
6: Generate $n$ utility samples $u_m$.
7: Calculate $w_{jk}^i$ from (8) and $V^{(j,k)}_i$ from (9).
8: Calculate $\psi_i$ from (10) or (11).
9: end for
10: return $\{\psi_i \mid i = 1, \ldots, H\}$

### 3.1 Discrete MDPs

When the MDPs are discrete, the algorithm is straightforward. Then the belief $\beta(\mu \mid D)$ admits a conjugate prior in the form of a Dirichlet-product for the transitions. In that case, it is also possible to use a histogram representation for $\psi_i$, so that it can be calculated by simply adding weights to bins according to (10).

However, as a histogram representation is not convenient for a large number of states, we focus on using a Gaussian $\psi_i$. To do this, we can simply use the sample mean and covariance of the weighted value function samples $V^{(j,k)}_i$:

$$m_i = \frac{1}{N_V} \sum_{k=1}^{N_V} \sum_{j=1}^{N_\mu} V^{(j,k)}_i w_{jk}$$

$$\Sigma_i = \frac{1}{N_V} \sum_{k=1}^{N_V} \sum_{j=1}^{N_\mu} (V^{(j,k)}_i - m_i)(V^{(j,k)}_i - m_i)^T w_{jk}$$

so that $\psi$ is a multivariate normal distribution.

### 4 Bayesian Backwards Induction

We are now ready to describe an algorithm for policy optimisation. We will use a dynamic programming algorithm that looks ahead $T$ steps, and at each step $i$ calculates a policy maximising the Bayesian expected utility in the next $i + 1$ steps. Just as in standard backwards induction, at each step, we can calculate...
π_i by keeping π_{i+1}, . . . , π_H fixed:
\[ Q_i(s,a) \triangleq E_{\beta}(U | s_i = s, a_i = a, D) \]
\[ = \int_{\mathcal{M}} \rho_{\mu}(s,a) + \sum_{s'} p_{\mu}^{(j)}(s'|s,a) V_{\mu,j}^{\pi_{i+1},\ldots,\pi_H}(s') \]
\[ \approx \sum_{j,k} [\rho_{\mu}^{(j)}(s,a) + \sum_{s'} p_{\mu}^{(j)}(s'|s,a) V_{\mu,j}^{(k)}(s')] w_{jk} N_{\mu} N_V. \] (12)

We can then set π_i(a|s) = 1 for a = arg max Q_i(s,a) and calculate the value function distribution for the partial policy (π_i, π_{i+1}, . . . , π_H). Since the algorithm only calculates Markov policies, they cannot be Bayes-optimal. Nevertheless, we expect them to be competitive with other methods that calculate Markov policies, such as posterior sampling, even in cases where a natural MDP prior exists.

**Algorithm 3 Bayesian Backwards Induction**

1: **Input:** Prior β, data D, lookahead H, discount γ, N_μ, N_V.
2: Initialise ψ_i.
3: Sample \( \hat{M} \triangleq \{ \mu^{(j)} | j \in [N_\mu] \} \) from \( \beta(\mu | D) \).
4: for \( i = H - 1, \ldots, 1 \) do
5: Sample \( V^{(k)} \sim \psi_{i+1}(v) \) for \( k \in [N_V] \).
6: Generate n utility samples \( u_i \).
7: Calculate \( w_{jk} \) from (8).
8: Calculate \( D_i \) from (12).
9: Set π_i(a|s) = 1 for \( a \in \arg \max D_i(s,a) \).
10: Calculate \( w_{jk} \) from (8) and \( V_i^{(j,k)} \) from (9) using policy π_i in the Bellman operator, i.e.
\[ V_i^{(j,k)} \triangleq D_i^{\pi_{i}(j)} V_{i+1}^{(k)} \]
11: Calculate ψ_i from (10) or (11).
12: end for
13: return π = (π_1, . . . , π_H).

Risk-sensitive decision making. While many Bayesian algorithms only approximate the expected value function, inferential induction (and Bayesian backwards induction) approximates the value function distribution. This can be used to compute risk-sensitive version of the expected utility, with the risk being relative to our uncertainty about the MDP rather than the MDP stochasticity.

If we are risk-seeking, we will compute an optimistic value function and explore more. We use the estimate of the expectation of the \( \alpha \) fraction of the upper tail i.e. \( E[X | P(X \leq x) \geq \alpha] \) for estimating the optimistic value function. For Gaussian marginal of the value distribution, it is estimated as
\[ V_i^{OPT}(s) = \hat{\mu}_i(s) + \hat{\sigma}_i(s) \frac{\Phi(\Phi^{-1}(\alpha))}{\alpha} \]
where \( \hat{\mu}_i(s) \) is the empirical mean, \( \hat{\sigma}_i(s) \) is the empirical variance, \( \Phi^{-1} \) is the inverse CDF of standard normal, \( \phi \) the PDF of standard normal, and \( \alpha \in (0.5, 1] \).
If $\alpha \to 0$ we essentially maximize for the most optimistic V.F. If $\alpha = 1$, we maximize the regular expectation.

We can also be risk-averse with a lower tail estimate. This ensures safe exploration while facing uncertainty. This is desired in many applications like autonomous driving. In contrast to the optimistic formulation, we use the estimate of the expectation of the $1 - \alpha$ fraction of the lower tail i.e, $E[X \mid \mathbb{P}(X \leq x) \leq 1 - \alpha]$ for estimating the safe value function. For Gaussian marginal of the value distribution, it is estimated as

$$V_{SAFE}^i(s) = \hat{\mu}_i(s) - \hat{\sigma}_i(s) \frac{\phi(\Phi^{-1}(\alpha))}{1 - \alpha}.$$  

![Figure 1: Comparison of value function posteriors obtained by inferential induction and Monte Carlo evaluations at different time steps for a fixed policy. We plot for five runs of inferential induction at each time step. The value of the mean MDP is shown by a vertical line.](image)

![Table 1: Wasserstein distance to the true distribution of the value function, for II and the mean MDP model, for NChain. For II the distance is averaged over five runs. The distances correspond to the plots in Figure 1](image)

| Time steps | II     | Mean MDP |
|------------|--------|----------|
|            |        |          |
| 10         | 22.80  | 30.69    |
| 100        | 16.41  | 17.90    |
| 1000       | 4.18   | 4.27     |

5 Experimental Analysis

We begin by evaluating whether inferential induction (II) based policy evaluation (Alg. 2) results in a good approximation of the actual value function posterior. We then evaluate the performance of Bayesian backward induction (BBI) (Alg. 3) in different environments. As a baseline, we compare with posterior sampling [PSRL, Strens 2000] and exploration by distributional reinforcement learning [VDQN, Tang and Agrawal 2018]. In Section 5.1, we describe the experimental setup and the priors used for implementation. In Section 5.2 we illustrate different environments used for empirical evaluation.
In Section 5.3 we analyse the results obtained for different environments in the context of estimation, efficiency, exploration and safety.

5.1 Experimental Setup

**Parameters.** We run the algorithms for the infinite-horizon formulation of value function with discount factor $\gamma = 0.99$. We evaluate their performance in terms of the evolution of average reward to $T = 10^6$ time-steps. Each algorithm updates its policy at steps $t = 1, 3, 6, 10, \ldots$. While implementing Alg. 2 we set the horizon to be 100 while approximating the policy evaluation. For the optimistic version of the algorithms, we set the scaling factor $\alpha = 0.25$ and $\alpha = 0.9$ for the safe version. More implementation details can be found in the supplementary material.

**Prior.** For discrete MDPs, we use Dirichlet $\text{Dir}(\alpha)$ priors over each of the transition probabilities $P(s' | s, a)$. The prior parameter $\alpha$ for each transition is set to 0.5. We use separate NormalGamma $\text{NG}(\mu, \kappa, \alpha, \beta)$ priors for each of the reward distributions $P(r | s, a)$. We set the prior parameters to $[\mu_0, \kappa_0, \alpha_0, \beta_0] = [0, 1, 1, 1]$. While we use the same prior parameters for all algorithms, we have not attempted to do an exhaustive unbiased evaluation by tuning their hyperparameters on a small set of runs, hence, our results should be considered preliminary.

5.2 Description of Environments

We evaluate the algorithms on four different environments that emphasize different aspects of performance such as estimation, efficiency, exploration, and safety. We use the OpenAI gym framework to implement and test the environments.

**NChain.** NChain is a discrete state, discrete action MDP environment. We implement the 5-state, 2-action chain environment (ref. Fig 1 [Strens 2000]).
Figure 3: Evolution of average reward for $5 \times 7$ and $10 \times 10$ LavaLake environments. The results are averaged over 20 and 30 runs respectively with a length of $10^6$ for each algorithm. The runs are exponentially smoothened with a half-life 1000 before averaging.

Taking the first action returns a reward 2 for all states and transitioning to the first state. Taking the second action returns 0 reward in the first four states (and the state increases by one) but returns 10 for the fifth state and the state remains unchained. There is a probability of slipping of 0.2 with which its action has the opposite effect. This environment requires both exploration and planning to be solved effectively and thus acts as an evaluator of posterior estimation, efficient performance and effective exploration.

**DoubleLoop.** DoubleLoop is another discrete MDP environment with two loops of states and two actions (ref. Fig 2 Strens [2000]). Taking the first action yields traversal of the right loop and a reward 1 for every 5 state traversal. Taking the second action yields traversal of the left loop and a reward 2 for every 5 state traversal. The transitions are deterministic. This environment acts as an evaluator of efficient performance and effective exploration.

**LavaLake.** Lava lake is a stochastic grid world Leike et al. [2017] where every state gives a reward of -1, unless you reach the goal, in which case you get 50, or fall into lava, when you get -50. This environment is a good evaluation of safety aspects of algorithms. We tested on the $5 \times 7$ and a $10 \times 10$ versions of the environment.

5.3 Experimental Results

**Estimation.** In order to evaluate the effectiveness of estimating the value function distribution using inferential induction (II) we compare it with the Monte Carlo distribution and the mean MDP. We compare this for posteriors after 10, 100 and 1000 time steps, obtained with a fixed policy in NChain that visits all the states, in Figure 1 for 5 runs of II. The fixed policy selects the first action with probability 0.8 and the second action with probability 0.2. The Monte Carlo estimate is done through 1000 samples of the value function vector
Table 2: Average number of times the goal and lava states are reached in $10^6$ steps for both $5 \times 7$ and $10 \times 10$ LavaLake environments (for 20 and 30 runs respectively).

| LavaLake Environments | Algorithms | BBI-Safety | BBI-Optimism | BBI | PSRL | VDQN |
|------------------------|------------|------------|--------------|-----|------|------|
|                        | Lava       | 496        | 31942        | 3637| 11143| 432* |
|                        | Goal       | 18810      | 49095        | 54325| 60621| 36277|
| $5 \times 7$           | Lava       | 65*        | 9546         | 1172| 2667 | 5445 |
|                        | Goal       | 403        | 31620        | 30569| 29636| 13243|
| $10 \times 10$         | Lava       |            |              |     |      |      |
|                        | Goal       |            |              |     |      |      |

($\gamma = 0.99$). This shows that the II estimate reasonably captures the uncertainty in the true distribution. For this data, we also compute the Wasserstein distance between the true and the estimated distributions at the different time steps as can be found in Table 1. There we can see that the distance to the true distribution decreases over time.

**Online experiments.** We compare BBI with other algorithms on four discrete MDP environments, namely NChain, DoubleLoop LavaLake $5 \times 7$ and LavaLake $10 \times 10$ (Sec. 5.2). Figures 2a and 2b show the average reward of BBI, BBI+Optimism, PSRL, and VDQN for NChain and DoubleLoop respectively. The results reported over 50 runs of each algorithm for $10^6$ time steps show BBI, BBI+Optimism and PSRL to have comparable performances while VDQN performs worse. Comparisons with MMBI Dimitrakakis [2011] and Bayesian Q-learning Dearden et al. [1998] can be found in the supplementary material.

**Exploration.** Figures 2 and 3 show that BBI generally manages to converge to the optimal solution quickly, hence has good exploration properties. Maintaining a distribution over value function allows us to use optimism to increase exploration, and we see that this has a positive effect in non-adverse environments.

**Safety.** Conversely, we see that optimism has a negative effect in environments like lava lake. For that reason we also used a conservative value function for safer exploration and compared the algorithms in lava lake in terms of the number of times they reached the goal or fell into the lava, as shown in Table 3. Comparisons with additional algorithms can be found in the supplementary material.

### 6 Discussion and Future Work

This paper offers a new perspective on Bayesian value function estimation. The central idea is to calculate the conditional value function distribution $P^*_\mu(V_t \mid V_{t+1}, D)$ on the data and apply it inductively to obtain the complete value function distribution $P^*_\mu(V_1, \ldots, V_t \mid D)$.

To place it in context, standard backwards induction (i.e. value iteration) calculates $V_t = \mathbb{E}_\mu V_{t+1}$ and distributional reinforcement learning methods calculate $P^*_\mu(U_t \mid U_{t+1})$, both for a given, underlying MDP. In an RL setting this

---

*A common distance metric between probability distributions*
can be replaced either through stochastic approximation (like in Q-learning) or through an explicit empirical model.

This is also unlike previous Bayesian value function methods, which essentially cast the problem into regression. For example, GPTD \cite{Engel2003} can be written as Bayesian inference with a GP prior over value functions and a data likelihood that uses a deterministic empirical model of the MDP. While this can be relaxed by using temporal correlations as in \cite{Engel2005}, the fundamental problem remains. Even though such methods have practical value, we show that Bayesian estimation of value functions requires us to explicitly think about the MDP distribution as well.

Even though we only developed one algorithm from this family, our experimental results appear promising. We see that BBI is comparable to state-of-the-art methods like PSRL, and it significantly outperforms methods relying on approximate inference such as VDQN. Thus, our proposed framework of inferential induction is a potentially useful basis for developing new Bayesian reinforcement learning algorithms.

References

Marc G Bellemare, Will Dabney, and Rémi Munos. A distributional perspective on reinforcement learning. In Proceedings of the 34th International Conference on Machine Learning-Volume 70, pages 449–458. JMLR. org, 2017.

Richard Dearden, Nir Friedman, and Stuart Russell. Bayesian q-learning. In Aaai/iaai, pages 761–768, 1998.

M. H. DeGroot. Optimal Statistical Decisions. John Wiley & Sons, 1970.

M.P. Deisenroth, C.E. Rasmussen, and J. Peters. Gaussian process dynamic programming. Neurocomputing, 72(7-9):1508–1524, 2009.

C. Dimitrakakis. Robust Bayesian reinforcement learning through tight lower bounds. In European Workshop on Reinforcement Learning (EWRL 2011), pages 177–188, 2011.

Y. Engel, S. Mannor, and R. Meir. Reinforcement learning with Gaussian process. In International Conference on Machine Learning, pages 201–208, 2005.

Yaakov Engel, Shie Mannor, and Ron Meir. Bayes meets Bellman: The Gaussian process approach to temporal difference learning. In Proceedings of the 20th International Conference on Machine Learning (ICML-03), pages 154–161, 2003.

M. Ghavamzadeh and Y. Engel. Bayesian policy gradient algorithms. In NIPS 2006, 2006.

Jan Leike, Miljan Martic, Victoria Krakovna, Pedro A Ortega, Tom Everitt, Andrew Lefrancq, Laurent Orseau, and Shane Legg. Ai safety gridworlds. arXiv preprint arXiv:1711.09883, 2017.

Tetsuro Morimura, Masashi Sugiyama, Hisashi Kashima, Hirotaka Hachiya, and Toshiyuki Tanaka. Nonparametric return distribution approximation for
reinforcement learning. In Proceedings of the 27th International Conference on Machine Learning (ICML-10), pages 799–806, 2010.

Ian Osband, Benjamin Van Roy, and Zheng Wen. Generalization and exploration via randomized value functions. In ICML, 2016.

M. L. Puterman. Markov Decision Processes: Discrete Stochastic Dynamic Programming. John Wiley & Sons, New Jersey, US, 2005.

Malcolm Strens. A Bayesian framework for reinforcement learning. In ICML 2000, pages 943–950, 2000.

Yunhao Tang and Shipra Agrawal. Exploration by distributional reinforcement learning. In Proceedings of the 27th International Joint Conference on Artificial Intelligence, pages 2710–2716. AAAI Press, 2018.

W.R. Thompson. On the Likelihood that One Unknown Probability Exceeds Another in View of the Evidence of two Samples. Biometrika, 25(3-4):285–294, 1933.
Table 3: Average number of times the goal and lava states are reached in $10^6$ steps for both $5 \times 7$ and $10 \times 10$ LavaLake environments (for 20 and 30 runs respectively).

| LavaLake Environments | Algorithms | BBI-Safety | BBI-Optimism | BBI | MMBI | PSRL | VDQN | BQL |
|------------------------|------------|------------|--------------|-----|------|------|------|-----|
| 5 × 7                  | Lava       | 496        | 31942        | 3637| 3371 | 11143| 432* | 445.6|
|                        | Goal       | 18810      | 49095        | 54325| 53058| 60621| 36277| 1268|
| 10 × 10                | Lava       | 65*        | 9546         | 1172| 1278 | 2667 | 5445 | 598 |
|                        | Goal       | 403        | 31620        | 30569| 30437| 29636| 13243| 1   |

**A Rollout computation**

To speed up the computation of rollouts, we have used three possible methods that essentially bootstrap previous rollouts or use value function samples:

$$u_{t}^{\mu,\pi}(s) = r(s, a) + \gamma u_{t+1}^{\mu,\pi}(s')$$  \hspace{1cm} (13)

$$u_{t}^{\mu,\pi}(s) = \sum_{s'} r(s, a) + \gamma P(s'|s, a) u_{t+1}^{\mu,\pi}(s')$$  \hspace{1cm} (14)

$$u_{t}^{\mu,\pi}(s) = \sum_{s'} r(s, a) + \gamma P(s'|s, a) V_{t+1}(s')$$  \hspace{1cm} (15)

where $V_{t+1} \sim \psi_{t+1}$. In experiments, we have found no significant difference between them.

**B Additional results**

**The Bayesian value function.** In this experiment, we evaluate the Bayesian (i.e., mean) value function of the proposed algorithm (BBI) by with the lower and upper bounds on the Bayes-optimal value function. These are calculated from $\int_{\mathcal{M}} \max_\pi V_\mu \pi d\beta(\mu \mid D)$ and $\int_{\mathcal{M}} V_\mu \hat{\pi} d\beta(\mu \mid D)$, respectively, where $\hat{\pi}$ is a heuristic stationary policy, that by definition is not Bayes-optimal. We estimate these bounds through 100 MDP samples for NChain. We plot the time evolution of our value function (BBI) and the simulated Bayes bounds in Figure 4 for $10^5$ steps. We observe that this becomes closer to both bounds as we obtain more data.

**Comparisons with further algorithms.** We have also run additional experiments with other Bayesian algorithms. In particular, here we show comparisons with MMBI and BQL.
Figure 4: Bayes bounds for NChain with the value functions for BBI and inferential induction. Upper and lower bounds are calculated from 100 MDPs and plotted for $10^5$ time steps.

Figure 5: Evolution of average reward for NChain and DoubleLoop environments, averaged over 50 runs of length $10^6$ for each algorithm. The runs are exponentially smoothened with a half-life 1000 before averaging.
Figure 6: Evolution of average reward for $5 \times 7$ and $10 \times 10$ LavaLake environments. The results are averaged over 20 and 30 runs respectively with a length of $10^6$ for each algorithm. The runs are exponentially smoothened with a half-life 1000 before averaging.