Estimates for some convolution operators with singular measures on the Heisenberg group

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Abstract. We consider the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$. Let $\nu$ be the Borel measure on $\mathbb{H}^n$ defined by $\nu(E) = \int_{\mathbb{H}^n} \chi_E(w, \varphi(w)) \eta(w) dw$, where $\varphi(w) = \sum_{j=1}^{n} \eta_{j} |w_j|^2$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, $\eta_j \in \mathbb{R}$, and $\eta(w) = \eta_0 \left( |w|^2 \right)$ with $\eta_0 \in C_c^\infty(\mathbb{R})$. In this paper we characterize the set of pairs $(p, q)$ such that the convolution operator with $\nu$ is $L^p(\mathbb{H}^n) - L^q(\mathbb{H}^n)$ bounded. We also obtain $L^p$-improving properties of measures supported on the graph of the function $\varphi(w) = |w|^{2n}$.

1. Introduction

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with group law $(z, t) \cdot (w, s) = (z + w, t + s + (z, w))$ where $(z, w) = \frac{1}{2} \text{Im} \left( \sum_{j=1}^{n} z_j w_j \right)$. For $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, we write $x = (x', x'')$ with $x' \in \mathbb{R}^n$, $x'' \in \mathbb{R}^n$. So, $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n$ via the map $\Psi(x', x'') = x' + ix''$. In this setting the form $(z, w)$ agrees with the standard symplectic form on $\mathbb{R}^{2n}$. Thus $\mathbb{H}^n$ can be viewed as $\mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law

$$(x, t) \cdot (y, s) = \left( x + y, t + s + \frac{1}{2} W(x, y) \right)$$

where the symplectic form $W$ is given by $W(x, y) = \sum_{j=1}^{n} (y_{n+j} x_j - y_j x_{n+j})$, with $x = (x_1, \ldots, x_{2n})$ and $y = (y_1, \ldots, y_{2n})$, with neutral element $(0, 0)$, and with inverse $(x, t)^{-1} = (-x, -t)$.

Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$ be a measurable function and let $\nu$ be the Borel measure on $\mathbb{H}^n$ supported on the graph of $\varphi$, given by

$$\nu(E) = \int_{\mathbb{H}^n} \chi_E(w, \varphi(w)) \eta(w) dw,$$

with $\eta(w) = \prod_{j=1}^{n} \eta_{j} \left( |w_j|^2 \right)$, where for $j = 1, \ldots, n$, $\eta_j$ is a function in $C_c^\infty(\mathbb{R})$ such that $0 \leq \eta_j \leq 1$, $\eta_j(t) \equiv 1$ if $t \in [-1, 1]$ and $\text{supp}(\eta_j) \subset (-2, 2)$. Let $T_\nu$ be the right convolution operator by $\nu$, defined by

$$(2) \quad T_\nu f(x, t) = (f \ast \nu)(x, t) = \int_{\mathbb{R}^{2n}} f \left( (x, t) \cdot (w, \varphi(w))^{-1} \right) \eta(w) dw.$$

We are interested in studying the type set

$$E_\nu = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in [0, 1] \times [0, 1] : \|T_\nu\|_{pq} < \infty \right\}$$

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where the $L^p$-spaces are taken with respect to the Lebesgue measure on $\mathbb{R}^{2n+1}$. We say that the measure $\nu$ defined in (1) is $L^p$-improving if $E_\nu$ does not reduce to the diagonal $1/p = 1/q$.

This problem is well known if in (2) we replace the Heisenberg group convolution with the ordinary convolution in $\mathbb{R}^{2n+1}$. If the graph of $\varphi$ has non-zero Gaussian curvature at each point, a theorem of Littman (see [3]) implies that $E_\nu$ is the closed triangle with vertices $(0,0)$, $(1,1)$, and $\left(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\right)$ (see [4]). A very interesting survey of results concerning the type set for convolution operators with singular measures can be found in [5]. Returning to our setting $\mathbb{H}^n$, in [7] S. Secco obtains $L^p$-improving properties of measures supported on curves in $\mathbb{H}^1$, under the assumption that

$$\left| \begin{array}{cc} \phi_1^{(2)} & \phi_2^{(2)} \\ \phi_1^{(3)} & \phi_2^{(3)} \end{array} \right| (s) \neq \frac{(\phi_1^{(2)}(s))^2}{2}, \quad \forall s \in I$$

where $\Phi(s) = (s, \phi_1(s), \phi_2(s))$ is the curve on which the measure is supported. In [6] F. Ricci and E. Stein showed that the type set of the measure given by (1), for the case $\varphi(w) = 0$ and $n = 1$, is the triangle with vertices $(0,0)$, $(1,1)$, and $(1/2,1)$.

In this article we consider first $\varphi(w) = \sum_{j=1}^{n} a_j |w_j|^2$, with $w_j \in \mathbb{R}^2$ and $a_j \in \mathbb{R}$. The Riesz-Thorin theorem implies that the type set $E_\nu$ is a convex subset of $[0,1] \times [0,1]$. In Lemmas 3 and 4 we obtain the following necessary conditions on the pairs $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\nu$,

$$\frac{1}{q} \leq \frac{1}{p}$$

$$\frac{1}{q} \geq \frac{2n+1}{p} - 2n$$

$$\frac{1}{q} \geq \frac{1}{(2n+1)p}$$

Thus $E_\nu$ is contained in the closed triangle with vertices $(0,0)$, $(1,1)$, and $\left(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\right)$. In Section 3 we prove that $E_\nu$ is exactly the closed triangle with these vertices. Indeed, we obtain the following

**Theorem 1.** If $\nu$ is the Borel measure defined by (1), supported on the graph of the function $\varphi(w) = \sum_{j=1}^{n} a_j |w_j|^2$, with $w_j \in \mathbb{R}^2$ and $a_j \in \mathbb{R}$, then the type set $E_\nu$ is the closed triangle with vertices

$$A = (0,0), \quad B = (1,1), \quad C = \left(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\right)$$

with $n \in \mathbb{N}$.

In a similar way we also obtain $L^p$-improving properties of the measure supported on the graph of the function $\varphi(w) = |w|^{2m}$. In fact we prove the following

**Theorem 2.** For $m, n \in \mathbb{N}_{\geq 2}$ let $\nu_{m,n}$ be the measure given by (7) with $\varphi(y) = |y|^{2m}$, $y \in \mathbb{R}^{2n}$. Then the type set $E_{\nu_{m,n}}$ contains the closed triangle with vertices $(0,0)$, $(1,1)$, $\left(\frac{2(n+m-1)m}{2(n+m)}, \frac{m}{2(n+m)}\right)$. Throughout this work, $c$ will denote a positive constant not necessarily the same at each occurrence.
2. Necessary conditions

We denote $B(r)$ the $2n+1$ dimensional ball centered at the origin with radius $r$.

**Lemma 3.** Let $\nu$ be the Borel measure defined by (1), where $\varphi$ is a bounded measurable function. If $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\nu$ then $p \leq q$.

**Proof.** For $(y, s) \in \mathbb{H}^n$ we define the operator $\tau_{(y, s)}$ by $\tau_{(y, s)} f(x, t) = f((y, s)^{-1} \cdot (x, t))$. Since $\tau_{(y, s)} T_\nu = T_\nu \tau_{(y, s)}$, it is easy to see that the $\mathbb{R}^n$ argument utilized in the proof of Theorem 1.1 in [2] works as well on $\mathbb{H}^n$. \hfill \Box

**Lemma 4.** Let $\nu$ be the Borel measure defined by (1), where $\varphi$ is a smooth function. Then $E_\nu$ is contained in the closed triangle with vertices 

$$(0, 0), (1, 1), \left(\frac{2n+1}{2n+2}, 1\right).$$

**Proof.** We will prove that if $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\nu$ then $\frac{1}{p} \geq \frac{2n+1}{p} - 2n$ and $\frac{1}{q} \geq \frac{1}{(2n+1)p}$. Then the lemma will follow by the Riesz-Thorin theorem. Let $f_\delta = \chi_{Q_\delta}$, where $Q_\delta = B(2\delta)$. Let $D = \{x \in \mathbb{R}^{2n} : \|x\| \leq 1\}$ and $A_\delta$ the set defined by

$$A_\delta = \left\{(x, t) \in \mathbb{R}^{2n} \times \mathbb{R} : x \in D, |t - \varphi(x)| \leq \frac{\delta}{4}\right\}.$$ 

For each $(x, t) \in A_\delta$ fixed, we define $F_{\delta, x}$ by

$$F_{\delta, x} = \left\{y \in D : \|x - y\|_{\mathbb{R}^{2n}} \leq \frac{\delta}{4n(1 + \|\nabla \varphi\|_{\text{supp}(\eta)}\|_\infty)}\right\}.$$

Now, for each $(x, t) \in A_\delta$ fixed, we have

$$(3) \quad (x, t) \cdot (y, \varphi(y))^{-1} \in Q_\delta, \quad \forall y \in F_{\delta, x},$$

indeed

$$\| (x, t) \cdot (y, \varphi(y))^{-1} \|_{\mathbb{R}^{2n+1}} \leq \| x - y \|_{\mathbb{R}^n \times \mathbb{R}^{2n}}$$

$$+ |t - \varphi(x)| + |\varphi(x) - \varphi(y)| + \frac{1}{2} |W(x, y)|,$$

since

$$\frac{1}{2} |W(x, y)| \leq n \|x\|_{\mathbb{R}^{2n}} \|x - y\|_{\mathbb{R}^{2n}},$$

(3) follows. Then for $(x, t) \in A_\delta$ we obtain

$$T_\nu f_\delta(x, t) \geq \int_{F_{\delta, x}} \eta(y) dy \geq c \delta^{2n},$$

where $c$ not depends on $\delta$, $x$ and $t$. If $\left(\frac{1}{p}, \frac{1}{q}\right) \in E_\nu$ implies

$$c \delta^{\frac{n+1}{q} + 2n} = c \delta^{2n} |A_\delta|^{\frac{1}{q}} \leq \left(\int_{A_\delta} |T_\nu f_\delta(x, t)|^q\right)^{\frac{1}{q}} \leq \|T_\nu f_\delta\|_q \leq c \|f_\delta\|_p = c \delta^{\frac{n+1}{q}},$$

thus $\delta^{2n + \frac{1}{q}} \leq C \delta^{\frac{2n+1}{p}}$ for all $0 < \delta < 1$ small enough. This implies that

$$\frac{1}{q} \geq \frac{2n+1}{p} - 2n.$$

Now, the adjoint operator of $T_\nu$ is given by

$$T_\nu^* g(x, t) = \int_{\mathbb{R}^{2n}} g((x, t) \cdot (y, \varphi(y))) \eta(y) dy$$

and let $E^*_p$ be the type set corresponding. Since $T_p = (T_p^*)^*$, by duality it follows that $\left( \frac{1}{p}, \frac{1}{p'} \right) \in E_p$ if and only if $\left( \frac{1}{p'}, \frac{1}{p} \right) \in E^*_p$, thus if $\left( \frac{1}{p}, \frac{1}{p'} \right) \in E^*_p$ then $\frac{1}{q} \geq \frac{2n+1}{p} - 2n$. Finally, by duality it is also necessary that

$$\frac{1}{q} \geq \frac{1}{(2n+1)p}.$$ 

Therefore $E_p$ is contained in the region determined by these two conditions and by the condition $p \leq q$, i.e.: the closed triangle with vertices $(0, 0)$, $(1, 1)$, $(\frac{2n+1}{2n+2}, \frac{1}{2n+2})$.

**Remark** Lemma 4 holds if we replace the smoothness condition with a Lipschitz condition.

3. **The Main Results**

We consider for each $N \in \mathbb{N}$ fixed, an auxiliary operator $T_N$ which will be embedded in an analytic family of operators $\{T_{N,z}\}$ on the strip $-n \leq \text{Re}(z) \leq 1$ such that

$$\begin{cases} 
\|T_{N,z}(f)\|_{L^\infty(\mathbb{H}^n)} \leq c_z \|f\|_{L^1(\mathbb{H}^n)} & \text{Re}(z) = 1 \\
\|T_{N,z}(f)\|_{L^2(\mathbb{H}^n)} \leq c_z \|f\|_{L^2(\mathbb{H}^n)} & \text{Re}(z) = -n 
\end{cases}$$

where $c_z$ will depend admissibly on the variable $z$ and it will not depend on $N$. We denote $T_N = T_{N,0}$. By Stein’s theorem of complex interpolation, it will follow that the operator $T_N$ will be bounded from $L^{\frac{2n+2}{n+1}}(\mathbb{H}^n)$ in $L^{2n+2}(\mathbb{H}^n)$ uniformly on $N$, if we see that $T_N f(x, t) \to T_p f(x, t)$ as $N \to \infty$, a.e $(x, t) \in \mathbb{R}^{2n+1}$. Theorem 1 will then follow from Fatou’s lemma and the lemmas 3 and 4. To prove the second inequality in (4) we will see that such family will admit the following expression

$$T_{N,z}(f)(x, t) = (f * K_{N,z})(x, t),$$

where $K_{N,z} \in L^1(\mathbb{H}^n)$, moreover it is a poliradial function (i.e. the values of $K_{N,z}$ depend on $|w_1|, \ldots, |w_n|$ and $t$). Now our operator $T_{N,z}$ can be realized as a multiplication of operators via the group Fourier transform, i.e.

$$\widehat{T_{N,z}(f)}(\lambda) = \widehat{f}(\lambda) \widehat{K_{N,z}}(\lambda)$$

where, for each $\lambda \neq 0$, $\widehat{K_{N,z}}(\lambda)$ is an operator on the Hilbert space $L^2(\mathbb{R}^n)$ given by

$$\widehat{K_{N,z}}(\lambda)g(\xi) = \int_{\mathbb{H}^n} K_{N,z}(s, t)\pi_\lambda(s, t)g(\xi)dsdt.$$ 

It then follows from Plancherel’s theorem for the group Fourier transform that

$$\|T_{N,z}f\|_{L^2(\mathbb{H}^n)} \leq A_z \|f\|_{L^2(\mathbb{H}^n)}$$

if and only if

$$\left\| \widehat{K_{N,z}}(\lambda) \right\|_{op} \leq A_z$$

uniformly over $N$ and $\lambda \neq 0$. Since $K_{N,z}$ is a poliradial integrable function, then by a well known result of Geller (see Lemma 1.3, p. 213 in [H]), the operators $\widehat{K_{N,z}}(\lambda) : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$ are, for each $\lambda \neq 0$, diagonal with respect to a Hermite basis for $L^2(\mathbb{R}^n)$. This is

$$\widehat{K_{N,z}}(\lambda) = C_n (\delta_{\gamma, \alpha} \mu_{N,z}(\alpha, \lambda))_{\gamma, \alpha \in \mathbb{N}_0^\alpha}$$

where $C_n$ is a constant.
where \( C_n = (2\pi)^n \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \delta_\gamma, \alpha = 1 \) if \( \gamma = \alpha \) and \( \delta_\gamma, \alpha = 0 \) if \( \gamma \neq \alpha \), and the diagonal entries \( \mu_{N,z}(\alpha_1, \ldots, \alpha_n, \lambda) \) can be expressed explicitly in terms of the Laguerre transform. We have in fact

\[
\mu_{N,z}(\alpha_1, \ldots, \alpha_n, \lambda) = \int_0^\infty \cdots \int_0^\infty K_{N,z}^L(r_1, \ldots, r_n) \prod_{j=1}^n \left( r_j L_j^0 \left( \frac{1}{2} |r_j| e^{-\frac{1}{2} |r_j|^2} \right) \right) dr_1 \cdots dr_n
\]

where \( L_j^0(s) \) are the Laguerre polynomials, i.e. \( L_j^0(s) = \sum_{i=0}^k \left( \frac{4^i s^i}{(i-k)!} \right) \) and \( K_{N,z}(\varsigma) = \int \mathcal{K}_{N,z}(\varsigma, t) e^{i\lambda t} dt \). Now \([5]\) is equivalent to

\[
\|T_{N,z}f\|_{L^2(\mathbb{H}^n)} \leq A \|f\|_{L^2(\mathbb{H}^n)}
\]

if and only if

(6) \( |\mu_{N,z}(\alpha_1, \ldots, \alpha_n, \lambda)| \leq A \)

uniformly over \( N, \alpha_j \) and \( \lambda \neq 0 \). If \( \text{Re}(z) = -n \) we prove that \( |\mu_{N,z}(\alpha_1, \ldots, \alpha_n, \lambda)| \leq A \), with \( A \) independent of \( N, \lambda \), \( \alpha_j \), and then we obtain the boundedness on \( L^2(\mathbb{H}^n) \) that is stated in \([4]\).

We consider the family \( \{I_z\}_{z \in \mathbb{C}} \) of distributions on \( \mathbb{R} \) that arises by analytic continuation of the family \( \{I_z\} \) of functions, initially given when \( \text{Re}(z) > 0 \) and \( s \in \mathbb{R} \setminus \{0\} \) by

(7) \( I_z(s) = \frac{2^{-\frac{z}{2}}}{\Gamma \left( \frac{z}{2} \right)} |s|^{-\frac{z}{2}} \).

In particular, we have \( \hat{I}_z = I_{1-z} \), also \( I_0 = c \delta \) where \( \hat{}\) denotes the Fourier transform on \( \mathbb{R} \) and \( \delta \) is the Dirac distribution at the origin on \( \mathbb{R} \).

Let \( H \in S(\mathbb{R}) \) such that \( \text{supp}(\hat{H}) \subseteq (-1,1) \) and \( \int H(t) dt = 1 \). Now we put \( \phi_N(t) = H(\frac{t}{\sqrt{N}}) \) thus \( \hat{\phi}_N(\xi) = N \hat{H}(N \xi) \) and \( \hat{\phi}_N \to \delta \) in the sense of the distribution, as \( N \to \infty \).

For \( z \in \mathbb{C} \) and \( N \in \mathbb{N} \), we also define \( J_{N,z} \) as the distribution on \( \mathbb{H}^n \) given by the tensor products

(8) \( J_{N,z} = \delta \otimes \cdots \otimes \delta \otimes \left( I_z \ast_R \hat{\phi}_N \right) \)

where \( \ast_R \) denotes the usual convolution on \( \mathbb{R} \) and \( I_z \) is the fractional integration kernel given by \([4]\). Finally, for \( z \in \mathbb{C} \) and \( N \in \mathbb{N} \) fixed, we defined the operator \( T_{N,z} \) by

(9) \( T_{N,z}f(x,t) = (f \ast \nu \ast J_{N,z})(x,t) \)

We observe that \( T_{N,0}f(x,t) \to cI_t f(x,t) as \ N \to \infty a.e. (x,t) \in \mathbb{R}^{2n+1}, \) since \( J_{N,0} = \delta \otimes \cdots \otimes \delta \otimes c\phi_N \to \delta \otimes \cdots \otimes \delta \otimes c\delta \) in the sense of the distribution, as \( N \to \infty \).

Before proving Theorem 1 we need the following lemmas,

**Lemma 5.** If \( \text{Re}(z) \leq -1 \) then \( \nu \ast J_{N,z} \in L^p(\mathbb{H}^n), \forall p \geq 1. \)

**Proof.** For \( \text{Re}(z) \leq -1 \) and \( N \in \mathbb{N} \) fixed, a simple calculation gives

\[
(\nu \ast J_{N,z})(x, \sigma) = \eta(x) \left( I_z \ast_R \hat{\phi}_N \right)(\sigma - \varphi(x)).
\]

We see that is enough to prove that \( \left( I_z \ast \hat{\phi}_N \right)(s) \in L^p(\mathbb{R}), \) if \( \text{Re}(z) \leq -1. \) For them we observe that if \( g \in S(\mathbb{R}) \) with \( \text{supp}(g) \cap [-\epsilon, \epsilon] = \emptyset \) for some \( \epsilon > 0, \) then
for \( \text{Re}(z) \leq -1 \)

\[
I_z(g) = \frac{2^{-\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_{|t| \geq \epsilon} |t|^{z-1} g(t) dt, \quad \text{if } z \notin -2\mathbb{N}
\]

and

\[
I_z(g) = 0, \quad \text{if } z \in -2\mathbb{N}.
\]

From this observation and the fact that

\[
\text{supp} \left( \tau_s \left( \tilde{\phi}_N \right) \right) \subset \left[ s - \frac{1}{N}, s + \frac{1}{N} \right] \subset [-\infty, -1] \cup [1, +\infty] \quad \text{for } |s| \geq \frac{N+1}{N}
\]

(where \( \phi^v(x) = \phi(-x) \) and \( \tau_s \phi)(x) = \phi(x - s) \)), we obtain

\[
\left| \left( I_z \ast \tilde{\phi}_N \right)(s) \right| = \left| I_z \left( \tau_s \left( \tilde{\phi}_N \right) \right) \right| \leq c |s - \text{sign}(s)|^{-2}, \quad \text{if } |s| \geq \frac{N+1}{N}.
\]

Finally, since \( \left| \left( I_z \ast \tilde{\phi}_N \right)(s) \right| \leq c \) for all \( s \in [-2, 2] \), the lemma follows.

\[\square\]

**Lemma 6.** For \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) we put

\[
F_{n,k} \left( \sigma \right) := \chi_{(0,\infty)}(\sigma)L^{n-1}_k(\sigma) e^{-\frac{s}{2} \sigma^2 n^{-1}}
\]

then

\[
\widehat{F_{n,k}}(\xi) = \frac{(k+n-1)!}{k!} \left( -\frac{i}{2} + i\xi \right)^k \left( \frac{1}{2} + i\xi \right)^{k+n}.
\]

**Proof.** On \( \mathbb{R} \) we define the Fourier transform by \( \hat{g}(\xi) = \int_{\mathbb{R}} g(\sigma) e^{-i\sigma \xi} d\sigma \) thus

\[
\widehat{F_{n,k}}(\xi) = \int_{0}^{\infty} L^{n-1}_k(\sigma) \sigma^{n-1} e^{-\sigma \xi} d\sigma
\]

and since \( L^{n-1}_k(\sigma) \sigma^{n-1} = \frac{\sigma}{k!} \left( \frac{d}{d\sigma} \right)^k \left( e^{-\sigma} \sigma^{k+n-1} \right) \) for each \( n \in \mathbb{N} \) and each \( k \in \mathbb{N}_0 \), we obtain

\[
\widehat{F_{n,k}}(\xi) = \frac{1}{k!} \int_{0}^{\infty} \left( \frac{d}{d\sigma} \right)^k \left( e^{-\sigma} \sigma^{k+n-1} \right) d\sigma = \frac{1}{k!} \int_{0}^{\infty} \sigma^{k+n-1} e^{-\sigma} d\sigma
\]

the third equality follows from the rapid decay of the function \( e^{-z} \) on the region \( \{ z : \text{Re}(z) > 0 \} \). Then we apply the Cauchy’s theorem.

\[\square\]

**Proof of Theorem 1.** For \( \text{Re}(z) = 1 \) we have

\[
\|T_{N,z}f\|_{\infty} = \|(f \ast \nu \ast J_{N,z})\|_{\infty} \leq \|f\|_{1} \|\nu \ast J_{N,z}\|_{\infty}
\]

Since

\[
(\nu \ast J_{N,z})(x,\sigma) = \eta(x) \left( I_z \ast \tilde{\phi}_N \right) (\sigma - \varphi(x))
\]
it follows that \( \| \nu * J_{N,z} \|_\infty \leq c \left| \Gamma \left( \frac{z}{2} \right) \right|^{-1} \). Then, for \( \Re(z) = 1 \), we obtain
\[
\| T_{N,z} \|_{1,\infty} \leq c \left| \Gamma \left( \frac{z}{2} \right) \right|^{-1}.
\]

From Lemma 5, in particular, we have that \( \nu * J_{N,z} \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n) \). In addition \( \nu * J_{N,z} \) is a poliradial function. Thus the operator \((\nu * J_{N,z}) (\lambda)\) is diagonal with respect to a Hermite base for \( L^2(\mathbb{R}^n) \), and its diagonal entries \( \mu_{N,z}(\alpha, \lambda) \), with \( \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n \), are given by
\[
\mu_{N,z}(\alpha, \lambda) = \int_0^\infty \cdots \int_0^\infty (\nu * J_{N,z})(r_1, ..., r_n, -\lambda) \prod_{j=1}^{\infty} \left( r_j L_{k_j}^0 \left( \frac{1}{2} |\lambda| r_j^2 \right) e^{-\frac{i}{2} |\lambda| r_j^2} \right) dr_1 ... dr_n
\]

Thus, it is enough to study the integral \( \int_0^\infty \eta_1(r^2) L_{\alpha_1}^0 \left( \frac{|\lambda|^2}{2} \right) e^{-\frac{|\lambda|^2}{2} r^2} e^{i\lambda_1 r^2} r \, dr \), where \( \alpha_1 \in \mathbb{R} \) and \( \eta_1 \in C_c^\infty (\mathbb{R}) \). We make the change of variable \( \sigma = \frac{|\lambda|^2}{2} \) in such integral and we obtain
\[
\int_0^\infty \eta_1(\sigma) L_{\alpha_1}^0 \left( \frac{|\lambda|^2}{2} \right) e^{-\frac{|\lambda|^2}{2} r^2} e^{i\lambda_1 r^2} r \, dr
\]

Thus, it is enough to estimate \( \int \eta_1(r^2) L_{\alpha_1}^0 (\sigma) e^{-\frac{|\lambda|^2}{2} r^2} e^{i\lambda_1 r^2} r \, dr \), where
\[
F_{\alpha_1}(\sigma) := \chi_{(0,\infty)}(\sigma) L_{\alpha_1}^0 (\sigma) e^{-\frac{|\lambda|^2}{2} r^2}
\]
and
\[
G_{\lambda}(\sigma) := \eta_1 \left( \frac{2\sigma}{|\lambda|^2} \right)
\]
Now
\[
\left| (\hat{F}_{\alpha_1} * \hat{G}_{\lambda})(-2sgn(\lambda)\alpha_1) \right| \leq \left\| \hat{F}_{\alpha_1} * \hat{G}_{\lambda} \right\|_\infty \leq \left\| \hat{F}_{\alpha_1} \right\|_\infty \left\| \hat{G}_{\lambda} \right\|_1 = \left\| \hat{F}_{\alpha_1} \right\|_\infty \left\| \hat{G}_{\lambda} \right\|_1.
\]
So it is enough to estimate \( \left\| \hat{F}_{\alpha_1} \right\|_\infty \). Now, from lemma 6, with \( n = 1 \) and \( k = \alpha_1 \), we obtain
\[
\left| \hat{F}_{\alpha_1}(\xi) \right| = \frac{1}{|\frac{1}{2} + i\xi|}
\]
Finally, for \( \Re(z) = -n \), we obtain
\[
|\mu_{N,z}(\alpha_1, ..., \alpha_n, \lambda)| \leq 2^n |I_{1-z}(-\lambda)\varphi_N(\lambda)| |\lambda|^{-n} \prod_{j=1}^{\infty} \left\| \hat{\eta}_j \right\|_1
\]
\[
\leq 2^n \left| \Gamma \left( \frac{1-z}{2} \right) \right|^{-1} \left| H \left( \frac{\lambda}{N} \right) \right| \prod_{j=1}^{\infty} \left\| \hat{\eta}_j \right\|_1
\]
\[
\leq 2^n \left| \Gamma \left( \frac{1-z}{2} \right) \right|^{-1} \left\| H \right\|_\infty \prod_{j=1}^{\infty} \left\| \hat{\eta}_j \right\|_1
\]
by (9) it follows, for $Re(z) = -n$, that

$$\|T_{N,z}f\|_{L^2(\mathbb{H}^n)} \leq c \left(\frac{(2\pi)^n 2^n}{\Gamma \left(\frac{1}{2} - z\right)}\right) \|f\|_{L^2(\mathbb{H}^n)}$$

It is easy to see, with the aid of the Stirling formula (see [10], p. 326), that the family $\{T_{N,z}\}$ satisfies, on the strip $-n \leq Re(z) \leq 1$, the hypothesis of the complex interpolation theorem (see [3], p. 205) and so $T_{N,0}$ is bounded from $L^{2n+2}(\mathbb{H}^n)$ into $L^{2n+2}(\mathbb{H}^n)$ uniformly on $N$, then doing $N$ tend to infinity, we obtain that the operator $T_u$ is bounded from $L^{2n+2}(\mathbb{H}^n)$ into $L^{2n+2}(\mathbb{H}^n)$ with $n \in \mathbb{N}$. 

**Proof of Theorem 2.** We consider for each $N \in \mathbb{N}$ fixed, the analytic family of operators $\{U_{N,z}\}$ on the strip $-\left(n + \frac{1}{2}m\right) \leq Re(z) \leq 1$, defined by $U_{N,z}f = f * \nu_m * J_{N,z}$, where $J_{N,z}$ is given by (8) and $U_{N,0}f \to U_{\nu_m}f = f * \nu_m$ as $N \to \infty$.

Proceeding as in proof of Theorem 1 it follows, for $Re(z) = 1$, that $\|U_{N,z}\|_{1,\infty} \leq c \left(\frac{1}{\Gamma \left(\frac{1}{2} - z\right)}\right)$. Also it is clear that, for $Re(z) = -\left(n + \frac{1}{2}m\right)$, the kernel $\nu_m * J_{N,z}$ is $L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$ and it is also a radial function. Now, our operator $(\nu_m * J_{N,z})\tau(\lambda)$ is diagonal, with diagonal entries $v_{N,z}(k,\lambda)$ given by

$$v_{N,z}(k,\lambda) = \frac{k!}{(k + n - 1)!} \int_0^\infty (\nu_m * J_{N,z})(s, -\lambda) L_k^{n-1} \left(\frac{\lambda s^2}{2}\right) e^{-\frac{\lambda s^2}{2}} s^{2n-1} ds$$

$$= \frac{k!}{(k + n - 1)!} L_{1-z}(-\lambda) \phi_N(\lambda) \int_0^\infty \eta_0(s^2) L_k^{n-1} \left(\frac{\lambda s^2}{2}\right) e^{-\frac{\lambda s^2}{2}} e^{i\lambda s^2} s^{2n-1} ds.$$

Now we study the integral

$$\int_0^\infty \eta_0(s^2) L_k^{n-1} \left(\frac{\lambda s^2}{2}\right) e^{-\frac{\lambda s^2}{2}} e^{i\lambda s^2} s^{2n-1} ds.$$

We make the change of variable $\sigma = \frac{\lambda s^2}{2}$ to obtain

$$\int_0^\infty \eta_0(\sigma) L_k^{n-1} \left(\frac{\lambda \sigma}{2}\right) e^{-\frac{\lambda \sigma}{2}} e^{i\lambda \sigma} \sigma^{2n-1} d\sigma$$

$$= 2^{n-1} |\lambda|^{-n} \int_0^\infty \eta_0 \left(\frac{2\sigma}{|\lambda|}\right) L_k^{n-1} (\sigma) e^{-\frac{2\sigma}{|\lambda|}} e^{i2^n sgn(\lambda)|\lambda|^{1-n}\sigma^m} \sigma^{n-1} d\sigma$$

$$= 2^{n-1} |\lambda|^{-n} (F_{n,k} \hat{G}_\lambda R_\lambda)(0) = 2^{n-1} |\lambda|^{-n} (\hat{F}_{n,k} \ast \hat{G}_\lambda \ast \hat{R}_\lambda)(0)$$

$$= 2^{n-1} |\lambda|^{-n} (\hat{F}_{n,k} \ast \hat{G}_\lambda \ast \hat{R}_\lambda)(0)$$

where $F_{n,k}$ is the function defined in the lemma 6, $G_\lambda(\sigma) = \eta_0 (2\sigma/|\lambda|)$ and $R_\lambda(\sigma) = \chi_{(0,|\lambda|]}(\sigma)e^{i2^n sgn(\lambda)|\lambda|^{1-n}\sigma^m}$. If $n \geq 2$, from lemma 6 we get

$$\|\hat{F}_{n,k} \ast \hat{G}_\lambda \ast \hat{R}_\lambda\|_{\infty} \leq \|\hat{F}_{n,k}\|_1 \|\hat{G}_\lambda\|_1 \|\hat{R}_\lambda\|_\infty$$

$$= \frac{(k + n - 1)!}{k!} \left(\int_0^\infty \frac{d\xi}{(\frac{1}{2} + \xi^2)^\frac{1}{2}}\right) \|\hat{\eta}_0\|_1 \|\hat{R}_\lambda\|_\infty$$
Now, we estimate $\|\widetilde{R}_\lambda\|_\infty$. Taking account of Proposition 2 (p.332 in [9]), we note that

$$\widetilde{R}_\lambda(\xi) = \frac{\lambda}{|\lambda|} \int_0^1 e^{i(2^m s \eta_n(\lambda))|\lambda|^{1-m}\sigma^m - \xi \sigma} d\sigma \leq \frac{C_m}{|\lambda|^{1-m}}$$

where the constant $C_m$ does not depend on $\lambda$. Then for $Re(z) = -(n + \frac{1-m}{m})$, we have

$$|u_{N,z}(k,\lambda)| \leq \frac{k!}{(k + n - 1)!} |I_{1-z}(-\lambda)\phi_N(\lambda)| 2^{n-1} |\lambda|^{-n} \left\| F_{n,k} * \left( G_\lambda + \tilde{R}_\lambda \right) \right\|_\infty$$

$$\leq |I_{1-z}(-\lambda)| |\phi_N(\lambda)| 2^{n-1} |\lambda|^{-n} \left( \int_{\mathbb{R}} \frac{d\xi}{(\frac{n}{2} + |\xi|^2)^\frac{n}{m}} \right) \left\| \tilde{\phi} \right\|_1 \frac{C_m}{|\lambda|^{1-m}}$$

$$\leq C_m 2^{n-1} \left( \frac{1}{m} \right) \left\| H \right\|_\infty \left( \int_{\mathbb{R}} \frac{d\xi}{(\frac{n}{2} + |\xi|^2)^\frac{n}{m}} \right) \left\| \tilde{\gamma} \right\|_1$$

Finally, by (6) it follows that, for $Re(z) = -(n + \frac{1-m}{m})$

$$\|U_{N,z}f\|_{L^2(\mathbb{H}^n)} \leq \frac{C_{n,m}}{|\Gamma(\frac{1}{m})|} \|f\|_{L^2(\mathbb{H}^n)}$$

is clear that the family $\{U_{N,z}\}$ satisfies, on the strip $(n + \frac{1-m}{m}) \leq Re(z) \leq 1$, the hypothesis of the complex interpolation theorem. Thus $U_{N,0}$ is bounded from $L^{2(1+m)/m}(\mathbb{H}^n)$ into $L^{2(1+n)/m}(\mathbb{H}^n)$ uniformly on $N$, then doing $N$ tend to infinity we obtain that the operator $U_{n,m}$ is bounded from $L^{2(1+m)/m}(\mathbb{H}^n)$ into $L^{2(1+n)/m}(\mathbb{H}^n)$, for $m,n \in \mathbb{N}_{\geq 2}$.

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