The Superfluid State of a Bose Liquid as a Superposition of a Single-Particle and Pair Coherent Condensates

E.A. Pashitskij\textsuperscript{1}, S.V. Mashkevich\textsuperscript{2}, S.I. Vilchynskyy\textsuperscript{3}

\textsuperscript{1}Institute for Physics, NAS of Ukraine, Kiev 03022, Ukraine
pashitsk@iop.kiev.ua

\textsuperscript{2}Institute for Theoretical Physics, Kiev 03143, Ukraine\textsuperscript{*}
mash@mashke.org

\textsuperscript{3}T. Shevchenko Kiev University, Kiev 03022, Ukraine
sivil@ap3.bitp.kiev.ua

November 23, 2018

Abstract

One considers the superfluid (SF) state of a Bose liquid with a strong repulsion between bosons, in which at $T = 0$, along with a weak single-particle Bose-Einstein condensate (BEC), there exists an intensive pair coherent condensate (PCC), analogous to the Cooper condensate in a Fermi liquid with an attraction between the fermions. Such a PCC emerges in a system of bosons due to an oscillating sign-changing momentum dependence of the Fourier component of the pair interaction potential, which is characteristic of a certain family of repulsion potentials, for example, for a regularized “hard spheres” model or for potentials with finite jumps or inflection points with an infinite derivative. In such cases, the Fourier component is negative in some domain of nonzero momentum transfer, which corresponds to an effective attraction of a quantum mechanical (diffraction) nature. The collective effects of renormalization (“screening”) of the initial interaction, which are described by the bosonic polarization operator, due to its negative sign on the “mass shell”, lead to a suppression of the repulsion and an enhancement of the effective attraction in the respective domains of momentum space. In the process of building a self-consistent model of the SF state, it is the ratio of the BEC density to the full density of the liquid $n_0/n \ll 1$ that is used as a small parameter—unlike in the Bogolyubov theory for a quasi-ideal Bose gas, in which the small parameter is the ratio of the number of supracondensate excitations to the number of particles in an intensive BEC, $(n-n_0)/n_0 \ll 1$. A closed system of nonlinear integral equations for the normal $\tilde{\Sigma}_1$ and anomalous $\tilde{\Sigma}_{12}$ self-energy parts is obtained, in the framework of a renormalized perturbation theory built on combined hydrodynamic (at $p \to 0$) and field (at $p \neq 0$) variables, whose usage eliminates infrared divergencies and ensures that the functions $\tilde{\Sigma}_{ij}(p, \epsilon)$ are analytic at $p \to 0$ and $\epsilon \to 0$ and that the SF order parameter $\tilde{\Sigma}_{12}(0, 0) \neq 0$ at $T = 0$.

In the framework of the hard-spheres model, a spectrum of quasiparticles is obtained, which is in good accordance with the experimental spectrum of elementary excitations in superfluid $^4$He. It is shown that the roton minimum in the quasiparticle spectrum is directly associated with the first negative minimum of the Fourier component of the renormalized potential of pair interaction between the bosons. Finally, the question of applicability of the Landau criterion to the description of the SF state of $^4$He in the absence of quantum vortices is discussed.

PACS: 67.57.-z

*On leave of absence.
1 Introduction

Despite the big progress achieved in the theory of superfluidity since the pioneer works by Landau [1], Bogolyubov [2], Feynman [3], and others [4]–[10], the task of constructing a microscopic theory of a superfluid (SF) state of a $^4$He Bose liquid cannot be considered complete. In fact, such questions of principle as: (i) the origin of the roton minimum in the spectrum of elementary excitations; (ii) applicability of the Landau criterion of superfluidity for the determination of the critical velocity of dissipationless flow in SF helium (He II); (iii) the quantum mechanical structure of the SF component of the $^4$He Bose liquid below the $\lambda$ point, at $T < T_\lambda = 2.17$ K, etc., remain unsolved. In particular, according to the latest results in quantum evaporation of $^4$He atoms [11], the maximal density $\rho_0$ of the single-particle Bose-Einstein condensate (BEC) in the $^4$He Bose liquid even at very low temperatures $T \ll T_\lambda$ does not exceed 10% of the total density $\rho$ of liquid $^4$He, whereas the density of the SF component $\rho_s \to \rho$ at $T \to 0$. Such a low density of the BEC is implied by a strong interaction between $^4$He atoms and is an indication of the fact that such an “exhausted” BEC cannot by itself form the microscopic basis of the SF component $\rho_s$. Therefore, the quantum structure of the SF condensate in He II with the “excess” density $(\rho_s - \rho_0) \gg \rho_0$ calls for a more thorough investigation [12]–[14].

On the other hand, numerous precise experiments on the restoration of the dynamic structure factor $S(\mathbf{p}, \varepsilon)$ in liquid $^4$He, involving inelastic neutron scattering [15]–[18], show that the temperature dependence of the spectrum of elementary excitations $E(\mathbf{p})$, associated with collective density oscillations in the $^4$He Bose liquid, is very weak all the way up to the $\lambda$ point, at all momenta, including the phonon, maxon, and roton bands. This means that the critical velocity, determined according to the Landau superfluidity criterion, $v_c = \min \{E(\mathbf{p})/p\}$, hardly changes as $T$ goes up and does not tend to zero as $T \to T_\lambda$. At the same time, the breakdown of superfluidity in macroscopic He II flows is known [19] to be implied by the processes of creation of Onsager-Feynman quantum vortices or Anderson closed vortex rings. As a result, the threshold velocity $v_c^*$ of breakdown of dissipationless flow as observed in He II, may be two orders of magnitude less than the critical velocity $v_c \approx [\Delta_r/p_r] \approx 60$ m/s associated with the roton gap $\Delta_r \approx 8.6$ K in the quasiparticle spectrum $E(\mathbf{p})$ at the point $p = p_r \approx 1.9$ Å$^{-1}$.

However, under the conditions where creation and motion of vortices (or vortex rings) is hindered, much higher values of threshold velocity can be achieved. For example, in ultrathin films and capillaries at $T < 1$ K, maximal values of $v_c^* \approx (2–3)$ m/s were observed [13], and for the passage of He II through narrow apertures in thin partitions, critical velocities $v_c^* \approx (8–10)$ m/s were registered [20], [21]. Moreover, in experiments on acceleration of ions in He II [22] at pressures $P \approx (15–20)$ bar, threshold velocities of more than 50 m/s were achieved, close to the roton limit $v_c \approx \Delta_r/p_r$. In this connection, the problem of ab initio theoretical description of the roton minimum in the quasiparticle spectrum $E(\mathbf{p})$ in liquid helium, remains topical.

For the first time, such a problem was considered in Ref. [23] (see also [24]) within the “hard-spheres” model, in which the Fourier component of the regularized pair
interaction for the $S$ scattering is a sign-changing function of momentum transfer. Substituting such a potential into the Bogolyubov quasiparticle spectrum of a weakly nonideal Bose gas [2] leads, under certain conditions, to the appearance of a minimum, analogous to the roton minimum in the empirical spectrum of liquid $^4$He. A similar problem was considered in Ref. [25] for a “semitransparent spheres” model with a finite potential jump. However, for a Bose liquid with strong interaction between particles and a suppressed BEC ($n_0 \ll n$) the Bogolyubov approximation is not applicable.

The questions discussed in this paper are both those of the quantum structure of the SF state of a Bose liquid and the calculation of the spectrum of elementary excitations based on a specific form of pair interaction between the bosons.

Our approach is based on the microscopic model [26] of superfluidity of a Bose liquid with a suppressed BEC and an intensive pair coherent condensate (PCC), which can arise from a sufficiently strong effective attraction between bosons in some domains of momentum space (see below) and is analogous to the Cooper condensate in a Fermi liquid with attraction between fermions near the Fermi surface [27]. As a small parameter, one uses the ratio of the BEC density to the total Bose liquid density ($n_0/n \ll 1$, unlike in the Bogolyubov theory [2] for a quasi-ideal Bose gas, in which the small parameter is the ratio of the number of supracondensate excitations to the number of density in the intensive BEC, $(n - n_0)/n_0 \ll 1$.

Because of this, the SF state within the model at hand can be described by a “truncated” self-consistent system of Dyson-Belyaev equations for the normal and anomalous single-particle Green functions $G_{ij}(k, \omega)$ and the self-energy parts $\tilde{\Sigma}_{ij}(k, \omega)$ without account for the diagrams of second and higher orders in the BEC density. The renormalized field perturbation theory [12]–[14] is used; it is built on combined field variables [28], [29], which in the long-wave limit ($p \rightarrow 0$) reduce to the hydrodynamic variables of macroscopic quantum (at $T = 0$) or two-liquid (at $T \neq 0$) hydrodynamics, whereas in the short-wave band they correspond to the bosonic quasiparticle creation and annihilation operators. In this case, the SF component $\rho_s$ is a superposition of the “exhausted” single-particle BEC and an intensive “Cooperlike” PCC with coinciding phases (signs) of the corresponding order parameters.

The pair interaction between bosons was chosen in the form of a regularized repulsion potential in the “hard spheres” model [23], [24], whose Fourier component $V(p)$ is an oscillating sign-changing function of momentum transfer $p$ due to mutual quantum diffraction of particles. The same oscillating sign-changing behavior is characteristic of Fourier components of potentials with finite jumps or inflection points with an infinite derivative.

As a result of renormalization (“screening”) of the initial interaction $V(p)$ due to multiparticle collective correlations, which are described by the boson polarization operator $\Pi(p, \omega)$, the interaction gets suppressed in the domains of momentum space where $V(p) > 0$ and enhanced where $V(p) < 0$. Such a suppression of repulsion and enhancement of attraction is implied by the negative sign of the real part of $\Pi(p, \omega)$ on the “mass shell” $\omega = E(p)$ for a decayless quasiparticle spectrum. It is shown that the integral contribution of the domains of effective attraction in the
renormalized sign-changing interaction

\[ \tilde{V}(p) = V(p) [1 - V(p) \text{Re}(\Pi(p, E(p)))]^{-1} \]

can be sufficient for the formation of an intensive bosonic PCC in momentum space (although not for the formation of bound boson pairs in real space).

Self-consistent numerical calculations of the boson self-energy and polarization operator, pair order parameter, and quasiparticle spectrum at \( T = 0 \), involving an iteration scheme, have allowed us to find conditions for the theoretical spectrum \( E(p) \) to coincide with the experimentally observed elementary excitation spectrum in \(^4\text{He}\). At the same time it is shown that the roton minimum in the quasiparticle spectrum \( E(p) \) of a Bose liquid is directly associated with the first negative minimum of the Fourier component of the renormalized potential of pair interaction between the bosons (akin to the minimum in the Bogolyubov spectrum \([2]\) of a weakly nonideal dilute Bose gas \([23]–[26]\)).

Finally, the question of applicability of the Landau criterion to the description of the SF state of \(^4\text{He}\) in the absence of quantum vortices is discussed.

2 Green functions and equations for the self-energy parts in the model of a Bose liquid with a suppressed BEC in the renormalized perturbation theory

The main difficulty of the microscopic description of the SF state of a Bose liquid with a nonzero BEC is the fact that applying the renormalized perturbation theory directly \([4]\) leads, as was shown in \([12]–[14]\), to a whole number of divergences at small energies \( \epsilon \to 0 \) and momenta \( k \to 0 \) and, as a consequence, to erroneous results in the calculations of different physical quantities.

Thus, for example, for a Bose system with weak interaction, when the ratio of the mean potential energy \( V(k_0)k_0^3 \) (\( k_0 \) being a typical momentum transfer) to the corresponding kinetic energy \( k_0^2/2m \) of the bosons is small, the zeroth-approximation polarization operator \( \Pi(k, \omega) \) and the density-density response function \( \tilde{\Pi}(k_0, \omega) \) calculated to the first order in the small parameter of interaction \( \xi = mk_0V(k_0) \ll 1 \), are logarithmically divergent at \( k \to 0, \omega \to 0 \), whereas the exact values \( \Pi(0,0) \) and \( \tilde{\Pi}(0,0) \) are finite \([13]\):

\[
\Pi(0,0) = -\frac{\partial n}{\partial \mu} = -\frac{n}{mc^2}; \quad \tilde{\Pi}(0,0) = \frac{n}{m}(c_B^2 - c^2),
\]

where \( n \) is the total concentration of bosons, \( \mu \) the chemical potential, \( c_B = \sqrt{nV_0/m} \) the velocity of sound in the Bogolyubov approximation for a weakly nonideal Bose gas \([2]\), \( V_0 \equiv V(0) \) the zeroth Fourier component of the potential, and \( c \) the speed of sound in the \( k \to 0 \) limit for the spectrum of elementary excitations \( \epsilon(k) \simeq |k|c \) in the Belyaev theory \([4]\):

\[
c = \sqrt{\Sigma_{12}(0)/m^*}.
\]

Here \( \Sigma_{12}(0) \) is the anomalous self-energy part of bosons at zero 4-momentum \( p \equiv (k, \epsilon) = 0 \), and \( m^* \) is the effective mass of quasiparticles, which is determined by
the relation \[ \frac{1}{m^*} = \frac{2B}{2m} \left[ \frac{1}{2m} + \frac{\partial \Sigma_{11}(0)}{\partial |k|^2} - \frac{\partial \Sigma_{12}(0)}{\partial |k|^2} \right], \] (3)

where \( \Sigma_{11}(0) \) and \( \Sigma_{12}(0) \) are, respectively, the normal and anomalous self-energy parts (at \( k \to 0, \epsilon \to 0 \)), and

\[
B = \left[ 1 - \frac{\partial \Sigma_{11}(0)}{\partial \epsilon} \right]^2 - \Sigma_{11}(0) \frac{\partial^2 \Sigma_{12}(0)}{\partial \epsilon^2} + \frac{1}{2} \frac{\partial^2}{\partial \epsilon^2} [\Sigma_{12}(0)]^2.
\] (4)

The model of a dilute Bose system of hard spheres with a small parameter \( \beta = \sqrt{n/k_0^2} \ll 1 \), considered in Ref. [4], in which, by means of a summation of the “ladder” diagrams with single-particle Green functions \( G_0(p) \) in the zeroth approximation in \( \beta \), it is possible to exclude the infinite repulsion, leads to a finite value of \( \Sigma_{12}(0) \),

\[
\Sigma_{12}(0) = \frac{4\pi a_0}{m} n_0,
\] (5)
a_0 being the vacuum scattering amplitude of the particles and \( n_0 \) the concentration of bosons in the BEC \( (\rho_0 = mn_0) \).

At the same time, in Ref. [5], taking into account an exact thermodynamic equation

\[
\frac{\partial \Sigma_{11}(0)}{\partial \epsilon} = - \left( \frac{\partial n_1}{\partial n_0} \right)_\mu = 1 - \frac{1}{n_0} \Sigma_{12}(0) \frac{dn_0}{d\mu},
\] (6)

where \( n_1 = n - n_0 \) is the concentration of supracondensate bosons, exact asymptotic relations

\[
G_{11}(p \to 0) = -G_{12}(p \to 0) = \frac{n_0 mc^2}{n(\epsilon^2 - c^2k^2 + i\delta)}; \quad c^2 = \frac{n}{m} \frac{d\mu}{dn},
\] (7)

were obtained.

However, it was shown in Refs. [12], [13] (see also [29]) that at \( p \equiv (k, \epsilon) = 0 \) the anomalous self-energy part is precisely equal to zero, \( \Sigma_{12}(0) \equiv 0 \). Problems then emerge with the determination of the velocity of sound \( \tau \sim \Sigma_{12}^{-1}(0) \) [20], as well as with the asymptotic formulas for the normal \( G_{11}(p) \) and anomalous \( G_{12}(p) \) Green functions at \( p \to 0 \) [4]:

\[
G_{11}(p \to 0) = -G_{12}(p \to 0) = \frac{\Sigma_{12}(0)}{B(\epsilon^2 - c^2k^2 + i\delta)},
\] (8)

because at \( \Sigma_{12}(0) = 0 \) the relations (5) and (4) reduce to the identities

\[
\frac{\partial \Sigma_{11}(0)}{\partial \epsilon} \equiv 1, \quad B \equiv 0,
\] (9)

so that Eqs. (3) and (4) with account for (3) contain uncertainties of the 0/0 type.

With the purpose of fixing these controversies, as well as the infrared divergences and nonanalyticities at \( p \to 0 \) emerging in the nonrenormalized theory, a renormalization procedure for the field perturbation theory was worked out in Ref. [14].
employing the method of “combined variables” [28], which in the long-wave domain \(|k| < k_0\) are in fact the hydrodynamic variables in the spirit of Landau quantum hydrodynamics [4], while in the short-wave domain \(|k| > k_0\) they reduce to the bosonic creation and annihilation operators. As was shown in Refs. [14], [29], the perturbation theory built on such “adequate” field variables, does not suffer from infrared divergences at \(p \to 0\), whose source at \(T = 0\) is the divergence of long-wave quantum fluctuations (acoustic Goldstone oscillations), associated with a spontaneous breakdown of continuous gauge and translational symmetries in the SF state of a Bose system with a uniform coherent condensate. Such oscillations are essentially the hydrodynamic first sound in liquid \(^4\)He, propagating with the velocity of \(c_1 \simeq 236\) m/s.

The choice of combined variables [28], [29] leads to the renormalized anomalous self-energy part \(\tilde{\Sigma}_{12}(p)\) which does not vanish at \(p = 0\). Then one can formally restore all the results of the renormalized field theory [4], [9], but this time in terms of the renormalized quantities \(\tilde{G}_{ik}(p)\) and \(\tilde{\Sigma}_{ik}(p)\), which do not contain singularities at \(p \to 0\) (save for the pole part \(\tilde{G}_{ik}(p) \sim |p|^{-2}\)). In particular, the square of velocity of first sound \(c_1\), in accordance with Ref. (2), at \(T \to 0\) must be equal to

\[
c_1^2 \equiv \left(\frac{\partial P}{\partial \rho}\right)_\sigma = \frac{\tilde{\Sigma}_{12}(0)}{\tilde{m}^*}, \tag{10}
\]

where the derivative of the pressure \(P\) with respect to the total density \(\rho\) is taken at constant entropy \(\sigma\), and the renormalized effective mass \(\tilde{m}^*\) is determined by the relations (3) and (4) with \(\tilde{\Sigma}_{ik}(0)\) substituted for \(\Sigma_{ik}(0)\).

In view of the aforesaid, we will work with the combined variables [28], [29],

\[
\tilde{\Psi}(x) = \tilde{\Psi}_L(x) + \tilde{\Psi}_{sh}(x), \tag{11}
\]

where

\[
\tilde{\Psi}_L(x) = \sqrt{\langle \tilde{n}_L \rangle} \left[ 1 + \frac{\tilde{n}_L - \langle \tilde{n}_L \rangle}{2 \langle \tilde{n}_L \rangle} + i \tilde{\phi}_L \right]; \quad \tilde{\Psi}_{sh} = \psi_{sh} e^{i \tilde{\phi}_L}; \quad \psi_{sh} = \psi - \psi_L; \quad \psi_L(r) = \frac{1}{\sqrt{V}} \sum_{|k|<k_0} a_k e^{ikr} = \sqrt{\langle \tilde{n}_L \rangle} e^{i \tilde{\phi}_L}. \tag{12}
\]

Such an approach means that the separation of the Bose system into a macroscopic coherent condensate and a gas of supracondensate excitations is made not on the statistical level, like in the case of a weakly nonideal Bose gas [2], [4], but on the level of ab initio field operators, which are used to construct a microscopic theory of the Bose liquid. Note that the approximate expression for the long-wave part \(\tilde{\Psi}_L\) of the boson field operator \(\tilde{\Psi}\) is given with account only for first-order terms in the expansions over the slowly changing (hydrodynamic) phase \(\tilde{\phi}_L\) and the small deviation of the density \(\tilde{n}_L\) from its mean value \(\langle \tilde{n}_L \rangle\). In Refs. [28], [29] it was assumed that at low \(T\), because of rather weak interaction \((mk_0 V(k_0) \ll 1)\), almost all the particles are in the Bose condensate, and therefore the value of momentum \(k_0\) in Ref. [29] was chosen in such a way that the approximate equation \(\langle \tilde{n}_L \rangle \approx n_0\) \((n_0\) being the concentration of particles in the BEC) take place. However, in a Bose liquid with strong interaction, when the single-particle BEC is strongly suppressed
\((n_0 \ll n)\), the value \(\langle \tilde{n}_1 \rangle\) should be normalized to the density \(n_s = \rho_s/m\) of the SF component.

The system of Dyson-Belyaev equations \([4, 9]\), which allows one to express the normal \(\hat{G}_{11}\) and anomalous \(\hat{G}_{12}\) renormalized single-particle boson Green functions in terms of the respective self-energy parts \(\tilde{\Sigma}_{11}\) and \(\tilde{\Sigma}_{12}\), has the form

\[
\hat{G}_{11}(\mathbf{p}, \epsilon) = \left[ G_0^{-1}(-\mathbf{p}, -\epsilon) - \tilde{\Sigma}_{11}(-\mathbf{p}, -\epsilon) \right]/Z(\mathbf{p}, \epsilon) ;
\]

\[
\hat{G}_{12}(\mathbf{p}, \epsilon) = \tilde{\Sigma}_{12}(\mathbf{p}, \epsilon)/Z(\mathbf{p}, \epsilon) .
\]

Here

\[
Z(\mathbf{p}, \epsilon) = \left[ G_0^{-1}(-\mathbf{p}, -\epsilon) - \tilde{\Sigma}_{11}(-\mathbf{p}, -\epsilon) \right] \left[ G_0^{-1}(\mathbf{p}, \epsilon) - \tilde{\Sigma}_{11}(\mathbf{p}, \epsilon) \right] - |\tilde{\Sigma}_{12}(\mathbf{p}, \epsilon)|^2 ;
\]

\[
G_0^{-1}(\mathbf{p}, \epsilon) = \left[ \epsilon - \frac{\mathbf{p}^2}{2m} + \mu + i\delta \right] ; \quad (\delta \to +0) ,
\]

where \(\mu\) is the chemical potential of the quasiparticles, which satisfies the Hugengoltz-Peins relation \([3]\):

\[
\mu = \tilde{\Sigma}_{11}(0, 0) - \tilde{\Sigma}_{12}(0, 0) .
\]

Due to a strong hybridization of the single-particle and collective branches of elementary excitations in the Bose liquid with a finite BEC \((n_0 \neq 0)\), the poles of the two-particle and all many-particle Green functions, as well as the full four-pole function of the pair interaction of bosons \(\hat{\Gamma}(p_1, p_2, p_3, p_4)\), coincide with the poles of the single-particle Green functions \(\hat{G}_{ik}(\mathbf{p}, \epsilon)\) \([4, 7]\).

Therefore the spectrum of all elementary excitations with zero spirality is determined by the zeros of the function \(Z(\mathbf{p}, \epsilon)\):

\[
E(\mathbf{p}) = \left\{ \left[ \frac{\mathbf{p}^2}{2m} + \tilde{\Sigma}_{11}^a(\mathbf{p}, E(\mathbf{p})) - \mu \right]^2 - |\tilde{\Sigma}_{12}(\mathbf{p}, E(\mathbf{p}))|^2 \right\}^{1/2} + \tilde{\Sigma}_{11}^a(\mathbf{p}, E(\mathbf{p})) ,
\]

where

\[
\tilde{\Sigma}_{11}^a(\mathbf{p}, \epsilon) = \frac{1}{2} \left[ \tilde{\Sigma}_{11}(\mathbf{p}, \epsilon) \pm \tilde{\Sigma}_{11}(-\mathbf{p}, -\epsilon) \right] ,
\]

the (+) sign corresponding to the symmetric part of \(\tilde{\Sigma}_{11}^a\), the (−) sign—to the antisymmetric part \(\tilde{\Sigma}_{11}^a\). In the sequel, we will assume that \(\tilde{\Sigma}_{11}\) is an even function of \(\mathbf{p}\) and \(\epsilon\), so that \(\tilde{\Sigma}_{11}^a = 0\) and \(\tilde{\Sigma}_{11}^s = \tilde{\Sigma}_{11}\). Then, relation \([7]\) ensures the acoustic dispersion law for the quasiparticles at \(\mathbf{p} \to 0\), and the first derivative of \(\tilde{\Sigma}_{11}\) over \(\epsilon\) in Eq. \([4]\) at \(\epsilon \to 0\) and \(p \to 0\) vanishes: \(\frac{\partial \tilde{\Sigma}_{11}^a(0, 0)}{\partial \epsilon} = 0\), so that \(B \neq 0\), while \(\tilde{\Sigma}_{12}(0) = n_0 \frac{\partial \mu}{\partial n_0}\), in accordance with Eq. \([8]\).

As was shown in Ref. \([24]\), for a Bose liquid with strong enough interaction between particles, when the BEC is strongly suppressed, one can, when defining \(\tilde{\Sigma}_{ik}(\mathbf{p}, \epsilon)\) in the form of a sequence of irreducible diagrams containing condensate lines \([4]\), restrict oneself, with good precision, to the first (lowest) terms in the expansion over the small BEC density \((n_0 \ll n)\). Such an approximation is exactly opposite to the Bogolyubov approximation \([2]\) for a weakly nonideal Bose gas with an intensive BEC, when \(n_0 \simeq n\).
As a result, up to terms of first order in the small parameter $n_0/n \ll 1$, for a Bose liquid one gets the “trimmed” system of equations for $\tilde{\Sigma}_{ik}$ [26]:

$$\tilde{\Sigma}_{11}(p, \epsilon) = n_0 \Lambda(p, \epsilon) \tilde{V}(p, \epsilon) + n_1 V(0) + \tilde{\Psi}_{11}(p, \epsilon) ;$$

$$\tilde{\Sigma}_{12}(p, \epsilon) = n_0 \Lambda(p, \epsilon) \tilde{V}(p, \epsilon) + \tilde{\Psi}_{12}(p, \epsilon) ,$$

where

$$\tilde{\Psi}_{ij}(p, \epsilon) = i \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} G_{ij}(k) \tilde{V}(p-k, \epsilon - \omega) \Gamma(p, \epsilon, k, \omega) ;$$

$$\tilde{V}(p, \epsilon) = V(p) [1 - V(p) \Pi(p, \epsilon)]^{-1} .$$

Here $V(p)$ is the Fourier component of the input potential of pair interaction of bosons, $\tilde{V}(p, \epsilon)$ is the renormalized (“screened” due to multiparticle collective effects) Fourier component of the retarded (nonlocal) interaction; $\Pi(p, \epsilon)$ is the boson polarization operator:

$$\Pi(p, \epsilon) = i \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \Gamma(p, \epsilon, k, \omega)$$

$$\times \{ G_{11}(k, \omega) G_{11}(k + p, \epsilon + \omega) + G_{12}(k, \omega) G_{12}(k + p, \epsilon + \omega) \} ;$$

$$\Gamma(p, \epsilon; k, \omega)$$

is the vertex part (three-pole function), which describes multiparticle correlations; $\Lambda(p, \epsilon) = \Gamma(p, \epsilon, 0, 0) = \Gamma(0, 0, p, \epsilon)$, and $n_1$ is the number of supracondensate particles $(n_1 \gg n_0)$, which is determined from the condition of conservation of the total number of particles:

$$n = n_0 + n_1 = n_0 + i \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} G_{11}(k, \omega) .$$

In the sequel, as well as in Ref. [26], in the integral relations (21) and (23) we will only take into account the residues at the poles of single-particle Green functions $\tilde{G}_{ij}(p, \epsilon)$, neglecting the contributions of eventual poles of the functions $\Gamma(p, \epsilon, k, \omega)$ and $\tilde{V}(p, \epsilon)$, which do not coincide with the poles of $\tilde{G}_{ij}(p, \epsilon)$. As a result, taking into account relations (13), (16), (18), (19) and (22), Eqs. (21) on the mass shell $\epsilon = E(p)$ assume the following form (at $T = 0$):

$$\tilde{\Psi}_{11}(p) \equiv \tilde{\Psi}_{11}(p, E(p)) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \Gamma(p, E(p); k, E(k))$$

$$\times \tilde{V}(p-k, E(p) - E(k)) \left[ \frac{A(k)}{E(k)} - 1 \right] ;$$

$$\tilde{\Psi}_{12}(p) \equiv \tilde{\Psi}_{12}(p, E(p)) = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \Gamma(p, E(p); k, E(k)) \tilde{V}(p-k, E(p) - E(k))$$

$$\times \frac{n_0 \Lambda(k, E(k)) \tilde{V}(k, E(k)) + \tilde{\Psi}_{12}(k)}{E(k)} ,$$

where

$$A(p) = n_0 \Lambda(p, E(p)) \tilde{V}(p, E(p)) + n_1 V(0) + \tilde{\Psi}_{11}(p) + \frac{P^2}{2m} - \mu .$$
Then the nonlinear equation (18) for the quasiparticle spectrum \( E(p) \), according to Eqs. (19), (20), takes the form

\[
E(p) = \sqrt{A^2(p) - \left[n_0\Lambda(p, E(p))\bar{V}(p, E(p)) + \bar{\Psi}_{12}(p)\right]^2},
\]

(28)

and the total quasiparticle concentration in the Bose liquid is determined by the relation

\[
n = n_0 + \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \left[ \frac{A(k)}{E(k)} - 1 \right].
\]

(29)

The Hugengoltz-Peins relation (17), according to Eqs. (19) and (20), can be represented as

\[
\mu = n_1V(0) + \bar{\Psi}_{11}(0) - \bar{\Psi}_{12}(0),
\]

(30)

as a result of which Eq. (27) takes the form

\[
A(p) = n_0\Lambda(p, E(p))\bar{V}(p, E(p)) + \left[\bar{\Psi}_{11}(p) - \bar{\Psi}_{11}(0)\right] + \bar{\Psi}_{12}(0) + \frac{p^2}{2m}.
\]

(31)

From Eqs. (28) and (31) it follows that the quasiparticle spectrum, because of the analyticity of the functions \( \bar{\Psi}_{ij}(p, \epsilon) \), is acoustic at \( p \to 0 \), and its structure at \( p \neq 0 \) depends essentially on the character of the renormalized pair interaction of bosons.

If one assumes that the functions \( \Psi_{11} \) and \( \Psi_{12} \) depend weakly on the explicit form of the quasiparticle spectrum \( E(p) \), then by virtue of Eq. (4), one can, with good precision, assert \( B = 1 \), so that the effective mass, according to Eqs. (3), (19), (20), is determined by the expression

\[
\frac{1}{\tilde{m}} = \frac{1}{m} + \frac{\partial^2 \bar{\Psi}_{11}(0)}{\partial |p|^2} - \frac{\partial^2 \bar{\Psi}_{12}(0)}{\partial |p|^2},
\]

(32)

and the expression for the velocity of sound, according to Eqs. (28) and (31), can be cast in the form

\[
c = \sqrt{\Lambda(0, 0)\bar{V}(0, 0)\tilde{n}/\tilde{m}}; \quad \tilde{n} = n_0 + \frac{\bar{\Psi}_{12}(0)}{\Lambda(0, 0)\bar{V}(0, 0)},
\]

(33)

which is analogous to the expression for the Bogolyubov velocity of sound for a weakly nonideal Bose gas \( c_B = \sqrt{\bar{V}(0)n/m} \). Then the conditions \( \tilde{n} > 0 \) and \( c = c_1 \) imply severe constraints on the choice of the parameters of the model of interaction of the bosons (see below).

Indeed, at \( p \to 0 \) and \( E(p) = c_1|p| \to 0 \), Eq. (24), due to the the momentum dependence of the spectrum \( E(p) \) and the functions \( \bar{V}(p, E(p)) \equiv \bar{V}(p, E(p)) \), \( \Lambda(p) \equiv \Lambda(p, E(p)) \) and \( \bar{\Psi}_{12}(p) \) being isotropic, assumes the form

\[
\bar{\Psi}_{12}(0) = -\frac{1}{(2\pi)^2} \int_0^\infty \frac{dk}{E(k)} \left[n_0\Lambda^2(k)\bar{V}^2(k) + \Lambda(k)\bar{V}(k)\bar{\Psi}_{12}(k)\right].
\]

(34)

It follows that the first integral addend on the right-hand side of Eq. (34) is always negative, so that the value \( \bar{\Psi}_{12}(0) \), which plays the role of the pair order parameter
can also be negative at not too small values of \( n_0 \), regardless of the sign of the renormalized interaction \( \Lambda(k)\tilde{V}(k) \). The condition \( \Psi_{12}(0) < 0 \) means that the phase of the PCC is opposite to that of the BEC, because \( n_0 > 0 \). Moreover, in this case, due to the condition \( \Lambda(0)\tilde{V}(0) > 0 \), which ensures that the system is globally stable against a spontaneous collapse, at sufficiently small densities of the BEC, in accordance with Ref. \( \text{(33)} \), the values \( \tilde{n} \) and \( \tilde{c}_2 \) can become negative, which corresponds to an instability in the phonon spectrum, which occurs when \( |\Psi_{12}(0)| > n_0\Lambda(0)V(0) \).

However, if the pair interaction between bosons in a broad enough region of the momentum space has, due to some reason, the character of attraction, i.e., \( \Lambda(k)\tilde{V}(k) < 0 \) at \( k \neq 0 \), and if the magnitude of that attraction is large enough (see below) and the BEC density is small enough \((n_0 \ll n)\), the second (positive) addend on the right-hand side of Eq. \( \text{(34)} \) can outweigh the first (negative) one. Then \( \Psi_{12}(0) \) will be positive, and the phase of the PCC will coincide with that of the BEC, so that \( \tilde{n} > 0 \) and \( \tilde{c}_2 > 0 \).

Note that when the BEC is totally absent \((n_0 = 0)\), when the integral equation \( \text{(26)} \) is homogeneous, i.e., degenerate with respect to the phase \( \Psi_{12}(p) \), the condition of stability of the phonon spectrum \( \tilde{c}_2 = \tilde{\Psi}_{12}(0)/m^* > 0 \) is secured by the choice of the respective sign (phase) of the pair order parameter \( \tilde{\Psi}_{12}(0) > 0 \) (see \( \text{(24)} \)).

On the other hand, since at \( T = 0 \) the density of the SF component \( \rho_s \) coincides with the total density \( \rho = mn \) of the Bose liquid, assuming \( \tilde{n} = n \) and with account for \( \text{(20)} \), one gets the following relations:

\[
\rho_s = \rho_0 + \tilde{\rho}_s = m\frac{\tilde{\Sigma}_{12}(0)}{\Lambda(0)\tilde{V}(0)}; \\
\tilde{\rho}_s = mn_1 = m\frac{\tilde{\Psi}_{12}(0)}{\Lambda(0)\tilde{V}(0)},
\]

where \( \rho_0 = mn_0 \) is the single-particle BEC density, and \( \tilde{\rho}_s \) is the density of the “Cooper” PCC. The concentration \( n_1 = n - n_0 \) is then determined from the relation \( \text{(29)} \), and for liquid \( ^4\text{He} \) at \( T \to 0 \), in accordance with the experimental data \( \text{[11]} \), it should be not less than 90% of the full concentration of \( ^4\text{He} \) atoms. Thus, the SF component in this model is a superposition of a single-particle and a pair coherent condensates, and relations \( \text{(29)} \) and \( \text{(35)} \) impose additional constraints on the parameters of the microscopic theory of the SF Bose liquid.

3 Influence of a renormalized pair interaction on the spectrum of elementary excitations in a Bose liquid with a suppressed BEC

To describe interaction of helium atoms in real space, various phenomenological and semi-empirical potentials \( \text{[31]} - \text{[39]} \) are conventionally used, which describe strong repulsion at small distances and weak van der Waals attraction at large distances (see Table I). As was shown in \( \text{[40]} \) by means of a numerical solution of the Schrödinger
equation, some of those potentials give rise to bound states—discrete levels with very small binding energy ($\Delta E < 0.1$ K).

However, all these potentials are characterized by a divergence ($\pm \infty$) at $r \to 0$ and are not suitable for the description of pair interaction in momentum space, since the respective Fourier components are infinite. Moreover, in a condensed state one cannot directly use the free atomic potentials—it is rather necessary to address the problem of constructing adequate pseudopotentials [41]. Therefore, as the input potential of pair interaction, a regularized “hard-spheres” potential [23], [24] with the radius equal to the quantum radius of the helium atom, $r_0 = 1.22$ Å, will be employed.

As was shown in Refs. [23], [24], for spherically symmetric scattering ($S$-wave), taking into account that the radial wave function $u(r)$ of relative motion of particles vanishes at the infinite jump of the potential $V(r) \to \infty$ at $r = 2.44$ Å, as well as the boundary condition $V(r)u(r) = \lambda \delta(r - a)$, the Fourier component of the pair potential is a finite sign-changing function of momentum transfer:

$$V(p) = V_0 j_0(pa); \quad j_0(x) = \frac{\sin x}{x}.$$  \hspace{1cm} (37)

Here $j_0(x)$ is the spherical Bessel function of zeroth order, and $V_0$ is a positive constant which is determined in a self-consistent way from a nonlinear integral equation for the single-particle Green function at $p \to 0$ and depends on the dimensionless Bose liquid density $na^3$ (see Refs. [23], [24]).

The potential (37) is shown with a dashed line in Fig. 1. It corresponds to repulsion, $V(p) > 0$, in those regions of the momentum space where $\sin(pa) > 0$ (in particular, when $pa < \pi$), and to attraction, $V(p) < 0$, in those ones where $\sin(pa) < 0$ (for example, $\pi < pa < 2\pi$).

As a different example, consider the Fourier component of a model potential in the form of a “Fermi” function

$$V(r) = V_0 \left\{ \exp \left( \frac{r^2 - a^2}{b^2} \right) + 1 \right\}^{-1},$$  \hspace{1cm} (38)

which at $b = 0$ degenerates into a “step” of a finite height $V_0$ at $r < a$, corresponding to a model of semitransparent spheres [25]. In this case the Fourier component is expressed in terms of the first order spherical Bessel function:

$$V(p) = V_0 \frac{j_1(pa)}{pa}; \quad j_1(x) = \frac{\sin(x) - x \cos(x)}{x^2}.$$  \hspace{1cm} (39)

The Fourier component of a smooth potential $V(r)$ in the form of a Lindhardt function [27], whose derivative at the inflection point $r = a$ is $-\infty$, has the same shape (see Appendix A). The sign-changing oscillations of the Fourier component $V(p)$ in the momentum space are then formally analogous to the Ruderman-Kittel-Ksi-Yoshida oscillations [12] in exchange interaction of spins or to the Friedel oscillations [27] of the screened Coulomb potential with period $\pi/k_F$ in real space, which arise as a result of scattering of electrons (fermions) on a Fermi sphere, filled according to the Pauli principle, of diameter $2k_F$ ($k_F$ being the electron Fermi momentum).
It should be stressed that the sign-changing Friedel oscillations are not directly associated with the jump of the Fermi electron distribution function at \( T = 0 \), but rather with a relatively weak singularity (inflection point) in the momentum dependence of the static polarization operator \( \Pi_c(p, 0) \), which is characterized by a logarithmic divergence at \( p = 2k_F \) of the first derivative of \( \Pi_c(p, 0) \) with respect to \( p \). This means that the sign-changing oscillations of the Fourier component of the potential \( V(r) \) can arise not only in the “hard spheres” model with an infinite jump of \( V(r) \), but also with finite jumps of the potential, or for “smooth” potentials with weak fractures or inflection points with an infinite derivative with respect to \( r \).

Thus, the existence of negative values \( V(p) < 0 \), i.e., of an effective attraction in some regions of the momentum space, is a consequence of certain peculiarities (jumps, fractures, inflection points) in the radial dependence of the pair interaction potentials, and is of quantum mechanical nature.

If one substitutes the oscillating potential (37) (or (39)) into the Bogolyubov spectrum of a dilute quasi-ideal Bose gas \[ E_B(p) = \left\{ \frac{p^2}{2m} \left[ \frac{p^2}{2m} + 2nV(p) \right] \right\}^{1/2}, \] then, by choosing two parameters, \( V_0 \) and \( a \), independently, one can achieve a rather satisfactory coincidence of the spectrum \( E_B(p) \) with the elementary excitation spectrum \( E_{\text{exp}}(p) \) in liquid \(^4\)He derived from neutron scattering experiments (Fig. 2, solid curve). However, the self-consistent solution for \( a = 2.2 \, \text{Å} \) and \( na^3 \approx 0.23 \), which was obtained in Refs. [23], [24] in the framework of the regularized “hard spheres” model, differs considerably from \( E_{\text{exp}}(p) \), and for a more realistic parameter \( a = 2.44 \, \text{Å} \), characteristic of \(^4\)He, when \( na^3 = 0.315 \), the spectrum (40) with the potential (37) turns out to be unstable, because \( E_B^2(p) < 0 \) in some range of \( p \) (Fig. 2, dashed curve), which points out that the Bogolyubov theory [2] is unapplicable for the description of the \(^4\)He Bose liquid.

On the other hand, multiparticle collective effects in the Bose liquid, according to Ref. (22), lead to an essential renormalization of the pair interaction, which determines the normal and anomalous self-energy parts, Eqs. (19) and (20). Taking into account Eqs. (22) and (37), the vertex part (four-point function) of the retarded interaction between bosons assumes the form

\[
\tilde{V}(p, \omega) = \frac{V_0 \sin(pa)}{pa - V_0 \Pi(p, \omega) \sin(pa)},
\]

where \( \Pi(p, \omega) \) is the bosonic polarization operator (23) in the SF state, which, with account for the pole parts of the Green functions (13) and (14), is calculated in Appendix B.

An important feature of the renormalized interaction (11) is that in those regions of phase volume \( (p, \omega) \) in which \( \Pi(p, \omega) < 0 \), the repulsion (when \( \sin(pa) > 0 \)) gets suppressed while the attraction (when \( \sin(pa) < 0 \)) gets effectively enhanced.

As follows from Eqs. (28) and (31), the main influence on the quasiparticle spectrum \( E(p) \) comes from the shape of the interaction potential (11) on the “mass shell”, when \( \omega = E(p) \). The real part of the polarization operator then has the
form (see Appendix B):

\[
\text{Re} \Pi(p, E(p)) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{E(k) - E(k - p) - E(p)} \times \left\{ \frac{F_-(k, p)}{E(k)(E(k) + E(k - p) - E(p))} - \frac{F_+(k, p)}{E(k)(E(k) + E(k - p) + E(p))} \right\}
\]

(42)

where

\[
F_-(k, p) = \left[ E(k) + \frac{k^2}{2m} - \mu + \tilde{\Sigma}_{11}(k, E(k)) \right] \times \left[ E(k) - E(p) + \frac{(k - p)^2}{2m} - \mu + \tilde{\Sigma}_{11}(k - p, E(k) - E(p)) \right] + \tilde{\Sigma}_{12}(k, E(k))\tilde{\Sigma}_{12}(k - p, E(k) - E(p))
\]

(43)

\[
F_+(k, p) = \left[ E(k - p) + \frac{(k - p)^2}{2m} - \mu + \tilde{\Sigma}_{11}(k - p, E(k - p)) \right] \times \left[ E(k - p) + E(p) + \frac{k^2}{2m} - \mu + \tilde{\Sigma}_{11}(k, E(k - p) + E(p)) \right] + \tilde{\Sigma}_{12}(k - p, E(k - p))\tilde{\Sigma}_{12}(k, E(k - p) + E(p))
\]

(44)

The key feature of the screened potential \(\tilde{V}(p, E(p))\) is the fact that \(\text{Re} \Pi(p, E(p)) < 0\) for all \(p > 0\), if the quasiparticle spectrum \(E(p)\) is stable with respect to decays into a pair of quasiparticles \([5], [9]\), i.e., if for all \(p\) and \(k\) the conditions

\[
E(p) < E(k) + E(k - p) ; \quad E(k) < E(p) + E(k - p)
\]

(45)

are fulfilled.

Indeed, as follows from Eq. \([12]\), the common denominator in front of the curly braces is always negative,

\[
[E(k) - E(k - p) - E(p)] < 0 , \quad (46)
\]

whereas the denominator in the first term in the curly braces is always positive,

\[
[E(k) + E(k - p) - E(p)] > 0 \quad (47)
\]

and smaller than the positive denominator in the second term

\[
[E(k) + E(k - p) + E(p)] > 0 . \quad (48)
\]

At the same time, as numerical simulation has shown, the numerators \(F_-\) and \(F_+\) in both terms remain positive for any \(p\) and \(k\) (see Fig. 3a). Therefore, the common sign of integrands in Eq. \([12]\) is negative, so that \(\text{Re} \Pi(p, E(p)) < 0\). Figure 3b shows the \(p\) dependence of \(\text{Re} \Pi(p, E(p))\), obtained numerically by iterations.

Taking into account the negative sign and the relatively weak momentum dependence of \(\text{Re} \Pi(p, E(p))\) in a wide region \(p \neq 0\), one can approximate the renormalized potential \([11]\) at \(\omega = E(p)\) in the integrands of Eqs. \([23], [26]\) with a more simple one:

\[
\tilde{V}(p) = \frac{V_0 \sin(pa)}{pa + \alpha \sin(pa)} . \quad (49)
\]
where \( \alpha = V_0|\vec{\Pi}| \), and \( |\vec{\Pi}| = |\text{Re}(\Pi(p, \vec{E}(p))| \) is the mean absolute value of the real part of the polarization operator on the mass shell in the domain of existence of the spectrum \( E(p) \). In what follows, all the numerical simulations are based on the model potential (49), in which the quantities \( V_0 \) and \( \alpha \) are free fitting parameters. Figure 1 depicts the function \( V(p)/V_0 \) for different values of the dimensionless parameter \( \alpha \). As far as the vertices \( \Lambda \) and \( \Gamma \) are concerned, their relatively weak \( p \) and \( \omega \) dependence may be disregarded, putting \( \Lambda \simeq \Gamma \simeq \Lambda(0,0) = \text{const} \) and incorporating the constant value of \( \Lambda(0,0) \) into \( V_0 \).

As a result, equations (25) and (26) for the functions \( \tilde{\Psi}_{ij} \) boil down to a simpler form,

\[
\tilde{\Psi}_{11}(p) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \tilde{V}(p-k) \left[ A_0(k) \left( \frac{E(k)}{E(p)} - 1 \right) \right],
\]

\[
\tilde{\Psi}_{12}(p) = -\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \tilde{V}(p-k) \frac{n_0 \tilde{V}(k) + \Psi_{12}(k)}{E(k)},
\]

where

\[
E(p) = \sqrt{A_0^2(p) - \left[ n_0 \tilde{V}(p) + \Psi_{12}(p) \right]^2};
\]

\[
A_0(p) = n_0 \tilde{V}(p) + \left[ \tilde{\Psi}_{11}(p) - \tilde{\Psi}_{11}(0) \right] + \tilde{\Psi}_{12}(0) + \frac{p^2}{2m}.
\]

In this case, for \( p \rightarrow 0 \), from Eq. (52), taking into account (53) and (52), one gets

\[
E(p \rightarrow 0) = |p| \sqrt{\left[ n_0 \tilde{V}(0) + \Psi_{12}(0) \right]/m^*},
\]

where \( \tilde{V}(0) = V_0/\left[1 - V_0 \Pi(0,0)\right] \), and \( \Pi(0,0) \) is determined by the first relation in (1), so that for the velocity of sound one has the following relation (see (33)):

\[
c^2 = \frac{1}{m^*} \left[ \frac{n_0 V_0}{1 + n_0 V_0/mc^2} + \tilde{\Psi}_{12}(0) \right],
\]

which is equivalent to a biquadratic equation for \( c \). Under the conditions \( nV_0 \gg mc^2 \), \( n_0 \ll n \), from Eq. (53) there follows an approximate expression

\[
c^2 \simeq \frac{\tilde{\Psi}_{12}(0)}{m^*} \left( 1 + \frac{mn_0}{m^*n} \right).
\]

The parameters \( V_0 \) and \( \alpha \) were chosen in such a way that the phase velocity \( E(|p| \rightarrow 0)/|p| \) coincide with the hydrodynamic sound velocity \( c_1 \simeq 236 \text{ m/s} \) in liquid \( ^4\text{He} \). On the other hand, the choice of those parameters was to ensure the maximal coincidence of the spectrum \( E(p) \) with the experimental spectrum \( E_{\text{exp}}(p) \) in \( ^4\text{He} \) at all values of \( p \).

Figure 4 (a,b,c) depicts the momentum dependence of the functions \( \tilde{\Psi}_{11}(p) \), \( \tilde{\Psi}_{12}(p) \), and \( A_0(p) \), calculated according to Eqs. (50), (51) and (53) for the values of \( \alpha = 4.4 \), \( V_0/(4\pi^2a^3) = 17.7 \text{ K} \) at \( a = 2.44 \text{ Å} \), and Fig. 5a presents the quasiparticle spectrum \( E(p) \) obtained from Eq. (52) for those same values of the parameters. It can be seen that this spectrum is in qualitative agreement with the experimental spectrum \( E_{\text{exp}} \) in \( ^4\text{He} \), however the numerical correspondence of the positions and
values of the maximum and minimum of quasiparticle energy, $E_{\text{max}} = 15.9$ K at $p_{\text{max}} = 1.05$ Å$^{-1}$ and $E_{\text{min}} = 10.7$ K at $p_{\text{min}} = 1.72$ Å$^{-1}$, cannot be deemed satisfactory. Besides, the sound velocity calculated according to Eqs. (33) and (55) turns out to be too low, $c = 2.08 \times 10^4$ cm/s, while the total particle concentration calculated according to the formula (29) is too high, $n = 2.57 \times 10^{22}$ cm$^{-3}$, the concentration of the particles in the BEC being low, $n_0 = 0.03n$.

There is a much better numerical agreement at the values of $\alpha = 4.52$, $V_0/(4\pi^2a^2) = 11.2$ K and $a = 3.0$ Å. The respective spectrum $E(p)$ is shown in Fig. 5b and is characterized by the following values: $E_{\text{max}} = 14.28$ K at $p_{\text{max}} = 1.13$ Å$^{-1}$, $E_{\text{min}} = 9.8$ K at $p_{\text{min}} = 1.95$ Å$^{-1}$; $c = 2.34 \times 10^4$ cm/s, the total concentration being $n = 2.18 \times 10^{22}$ cm$^{-3}$ and the BEC concentration $n_0 = 0.09n$ (which agrees with the experimental data [11]). Such an agreement should be deemed quite satisfactory, taking into account the simplified model of pair interaction employed.

It is seen from Fig. 4–5 that the nonmonotonous nature of the spectrum $E(p)$, including, in particular, the presence of the “roton” minimum, is determined mainly by the momentum dependences of the functions $\tilde{\Psi}_{11}(p)$ and $A_0(p)$, which have deep minima due to the oscillations of the sign-changing potential $\tilde{V}(p)$ in the domain $p < 2\pi/a$ (see Fig. 1). The theoretical spectra obtained are in good agreement with the experimental spectrum of $^4$He both in the positions and the absolute values of the maximum and minimum of $E(p)$.

One should stress that at comparatively small values of the parameter $\alpha$ it is impossible to render the quantity $\tilde{\Psi}_{12}(0)$ positive, because repulsion between bosons prevails over effective attraction when integrating over $p$ in Eq. (51). At the same time, because of the BEC density being small ($n_0 \ll n$), the quantity $\Sigma_{12}(0) = n_0\Lambda(0)\tilde{V}(0) - |\tilde{\Psi}_{12}(0)|$ becomes negative, which corresponds, according to (10) and (33), to a phonon instability of the spectrum ($c^2 < 0$).

On the other hand, at large enough values of $\alpha$, the effective attraction $\tilde{V}(p) < 0$ in the region $\pi < p/a < 2\pi$ (Fig. 1) turns out to be so strong that around the negative minimum of $\tilde{V}(p)$, the radicand in Eqs. (28) and (52) becomes negative, i.e., the quasiparticle spectrum $E(p)$ becomes absolutely unstable (imaginary) in the region in question, analogously to the Bogolyubov spectrum with a nonrenormalized potential in the “hard spheres” model (see Fig. 2). Therefore, the domain of parameters $V_0$ and $\alpha$ in which there exists a quasiparticle spectrum (52) which would be stable at all $p$ and be in agreement with experiment, is rather narrow. It is possible, in principle, to consider the inverse problem and find the effective “pseudopotential” of pair interaction of particles in the $^4$He Bose liquid, starting from the shape of the empirical quasiparticle spectrum $E_{\text{exp}}(p)$, however this is quite tedious.

4 The criterion of superfluidity and limiting critical velocities

We conclude with a brief discussion of the applicability of the Landau criterion of superfluidity to He II and of the value of limiting critical velocity in the absence of quantum vortices in the Bose liquid where the BEC and PCC coexist.
As was noted in Introduction, the elementary excitation spectrum \( E_{\text{exp}}(p) \) as observed in neutron scattering experiments \([15]–[18]\), leads to a rather overestimated value of the critical velocity, determined by the roton minimum (in accordance with the Landau criterion), as compared with the experimentally measured velocities of destruction of the SF flow. This has to do with the emergence of quantum vortices and vortex rings in He II \([19]\), however under conditions when creation and/or motion of those vortices is hindered, the critical velocities increase sharply \([20]\), \([21]\), and at low temperatures \( T_c < 1K \) they can assume values comparable with \( v_c = \min[\epsilon(p)/p] \) \([22]\).

One should stress that this situation is quite analogous to the one with the type I superconductors, in which the critical current \( j_c \) is determined by the condition of creation and pinning of Abrikosov quantum vortices on the surface of the superconductor or near various defects of the crystal lattice, whereas the true maximal value of \( j_c \)—the so-called decoupling critical current, which is determined by the process of decay of Cooper pairs \([27]\),—is much larger and can only be observed in thin wires, whose thickness is much less than the London depth of penetration of magnetic field into the superconductor, which hinders vortex creation.

In the \(^4\text{He} \) Bose liquid at finite temperatures, \( T \neq 0 \), together with the spectrum \( E_{\text{exp}}(p) \), which at \( p \to 0 \) corresponds to the first (hydrodynamic) sound with phase velocity \( c_1 \), there arises—due to the emergence of the normal component \( \rho_n \) in He II—the second sound, whose velocity \( c_2 \ll c_1 \) at \( T > 1 \text{ K} \). It is known \([10]\), \([43]\) that, due to heat expansion of liquid \(^4\text{He} \) being weak, the second sound branch reduces basically to oscillations of temperature (entropy) without any noticeable transfer of total mass of the normal (\( \rho_n \)) and SF (\( \rho_s \)) components, whose oscillations are in counterphase. Therefore, the excitations of second sound with energy \( \epsilon_2(p) = c_2 p \) cannot be observed in standard neutron scattering experiments, unlike the first sound \( \epsilon_1(p) = c_1 p \), which corresponds to cophase oscillations of the densities \( \rho_s \) and \( \rho_n \). At the same time, in the two-component Bose liquid there may coexist two different types of acoustic Goldstone excitations, associated, on the one hand, with spontaneous breakdown of gauge symmetry due to the degeneracy over the phase of the coherent SF condensate \( \rho_s \) at \( T \to 0 \), and the breakdown of continuous translational symmetry, i.e., of the homogeneity of the total density \( \rho = \rho_n + \rho_s \) (first sound); on the other hand, with spatially inhomogeneous deviations of temperature \( T \) from the uniform distribution due to the oscillations of the density of the gas of normal excitations (second sound). Nevertheless, the second sound excitations do carry over some energy and therefore have to be taken into account in the determination of the minimum critical velocity, according to the original concept of the Landau superfluidity criterion \([1]\), which includes all the types of excitations in the quantum liquid.

In this respect, one can assume that at those temperatures where \( c_2(T) < v_c \simeq 60 \text{ m/s} \), the maximal allowed critical velocity of the macroscopic SF flow in He II in the absence (or with strong pinning) of quantum vortices cannot exceed a number of the order of the second sound velocity \( c_2(T) \), which tends to zero as \( T \to T_\lambda \), together with the SF component density \( \rho_s(T) \). Precisely such a situation is characteristic of superconductors, in which the decoupling critical current turns into zero in the critical point \( T = T_c \) together with the energy gap \( \Delta \) in the quasiparticle spectrum.
Finally, it is worth noting that the coexistence of a weak BEC and an intensive PCC conserves the integer value of the quantum of circulation of the SF velocity in the vortices \( \kappa = \hbar / m \), due to the total mutual coherence of those condensates in the SF component \( \rho_s \). Indeed, a sufficiently strong effective attraction for the screened Fourier component of the singular “hard spheres” potential provides for the formation of a condensate of bound bosonic pairs with a positive sign of the pair order parameter \( \tilde{\Psi}_{12}(0) \), whose phase in that case coincides with the one of the BEC.

5 Conclusions

Thus, employing the renormalized field theory for the description of the SF state of the Bose liquid at \( T \to 0 \) with account for a small density of the single-particle BEC allows one to formulate a self-consistent model of superfluidity, in which the SF component at \( T \to 0 \) is a coherent superposition of the single-particle BEC, suppressed due to interaction, and an intensive PCC, which arises due to an effective attraction between the bosons in the momentum space. Such an approach lets one obtain, in the framework of the “hard spheres” model, an explicit form of the quasiparticle spectrum, which, with a suitable choice of parameters, coincides with good precision with the experimental spectrum of elementary excitations in \(^4\)He.

We are sincerely grateful to P.I. Fomin for numerous useful discussions.

6 Appendix A

In order to calculate the Fourier components of the empirical potentials presented in the Table, it is necessary, in order to avoid the divergence at \( r \to 0 \), to make a cutoff at some distance \( r_c \), representing the diameter of a “hard core”. For example, for the Lennard–Jones potential, up to the main terms, one has

\[
U(p) = 4\pi \varepsilon \int_{r_c}^{\infty} r dr \sin(pr) \left[ \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^{6} \right] \simeq 2\pi \varepsilon \sigma^3 \left[ \frac{1}{6} \left( \frac{\sigma}{r_c} \right)^{9} \left( \frac{\sin(pr_c)}{pr_c} + \frac{\cos(pr_c)}{9} - \frac{pr_c \sin(pr_c)}{7r_c} \right) \right. \\
- \left. \frac{1}{2} \left( \frac{\sigma}{r_c} \right)^{3} \left\{ \frac{\sin(pr_c)}{pr_c} + \frac{\cos(pr_c)}{3} - \frac{pr_c \sin(pr_c)}{6} - \frac{p^2 r_c^2 \cos(pr_c)}{6} - \frac{p^3 r_c^3}{6} \left( \frac{\pi}{2} - \text{Si}(pr_c) \right) \right\} \right]
\]

where \( \text{Si}(x) \) is the integral sine. This potential can be extended to the region \( r < r_c \) of the regularized Fourier component of the infinite repulsion potential in the “hard spheres” model \[23\], \[24\] or of the finite potential of “semi-transparent” spheres \[25\]. Such an approach is analogous to the method of pseudopotential in solid state theory \[41\]. Sign-changing oscillations of the Fourier component (A.1) are associated with a sharp fracture and a finite jump of the potential at the point \( r = r_c \). In particular,
Eq. (A.1) by itself corresponds, in coordinate space, to the model of pseudopotential with an “empty skeleton”, when the potential is $U(r) = 0$ at $r < r_c$, and adding the Fourier components of the potentials of “hard” or “semitransparent” spheres to that equation corresponds to the model of “filled skeleton” [11].

Oscillating sign-changing Fourier components are also characteristic of certain types of smooth potentials with inflection points. As an example, apart from the Fermi function [38], there is a potential in the form of a static Lindhart function [27] in real space:

$$W(r) = \frac{W_0}{2} \left[ 1 + \frac{1 - x^2}{2x} \log \left| \frac{1 + x}{1 - x} \right| \right]; \quad x = r/r_0,$$

which has an inflection point with an infinite negative derivative at $r = r_0$. The Fourier component of this function can be calculated exactly, and equals

$$W(p) = W_0 \frac{j_1(pr_0)}{pr_0},$$

i.e., its momentum dependence coincides with the one of the Fourier component of the potential of semi-transparent spheres with a finite jump at $r = r_0$ (see [39]). This is formally equivalent to the calculation of the Ruderman-Kittel-Ksui-Yoshida static oscillations [12] in real space for indirect exchange interaction of spin in metals with the Fermi distribution function at $T = 0$ and with the Fourier component of spin susceptibility of electrons in the form of Eq. (A.2) with $x = p/(2k_F)$ ($k_F$ being the Fourier momentum of the electrons).

At the same time, Fourier components of smooth continuous potentials with finite derivatives, nonsingular at $r = 0$, are characterized by weak oscillations. Consider for example the modified nonsingular Buckingham potential (see Table), in which the divergent term $(-1.49/r^6)$ is replaced, at $r < r_0 = 2.61$ Å, with an exponential $(-Be^{-\beta r})$, finite at $r = 0$. The parameters, $B = 0.57$ eV and $\beta = 1.84$ Å⁻¹, are chosen in such a way that at $r < r_0$, the expression

$$V_1(r) = Ae^{-\alpha r} - Be^{-\beta r}$$

with $A = 481$ eV, $\alpha = 4.6$ Å⁻¹, would vanish at $r = r_0$, and its derivative over $r$ at that point would be equal to that of the potential in the $r > r_0$ range:

$$V_2(r) = Ae^{-\gamma r} - \frac{D}{r^6} - \frac{F}{r^8},$$

at $C = 611$ eV, $D = 0.94$ eV Å⁶, $F = 1.87$ eV Å⁸, and $\gamma = \alpha$. The Fourier component of this potential is

$$V(p) = 4\pi \left\{ \frac{2\alpha A}{(\alpha^2 + p^2)^2} - \frac{(A - C)e^{-\alpha r_0}}{p(\alpha^2 + p^2)} \left[ \alpha r_0 \sin(pr_0) + pr_0 \cos(pr_0) \right] \\
- \frac{1}{(\alpha^2 + p^2)^2} \left( (\alpha^2 - p^2) \sin(pr_0) + 2\alpha p \cos(pr_0) \right) \right\} - \frac{2\beta B}{(\beta^2 + p^2)^2}$$
\[ + \frac{Be^{-\beta r_0}}{p (\beta^2 + p^2)} \left[ \beta r_0 \sin(pr_0) + pr_0 \cos(pr_0) \right] \] 
\[- \frac{1}{\beta^2 + p^2} \left( (\beta^2 - p^2) \sin(pr_0) + 2p\beta \cos(pr_0) \right) \right] \right] \right) \} - \frac{4\pi D}{p} I_5(p) - \frac{4\pi F}{p} I_7(p), \]

where
\[ I_5(p) = \int_{r_0}^{\infty} \sin(pr) \frac{dr}{r^5} = \frac{\sin(pr_0)}{4r_0^4} \left( 1 - \frac{p^2 r_0^2}{6} \right) \]
\[ + \frac{p \cos(pr_0)}{12r_0^3} \left( 1 - \frac{p^2 r_0^2}{3} \right) + \frac{p^4}{36} \left[ \frac{\pi}{2} - \text{Si}(pr_0) \right] ; \quad (A.7) \]
\[ I_7(p) = \int_{r_0}^{\infty} \sin(pr) \frac{dr}{r^7} = \frac{\sin(pr_0)}{6r_0^6} \left( 1 - \frac{p^2 r_0^2}{20} - \frac{p^4 r_0^4}{120} \right) \]
\[ + \frac{p \cos(pr_0)}{30r_0^5} \left( 1 - \frac{p^2 r_0^2}{12} - \frac{p^4 r_0^4}{24} \right) - \frac{p^6}{720} \left[ \frac{\pi}{2} - \text{Si}(pr_0) \right] . \quad (A.8) \]

Evidently, the oscillation amplitudes in this case are exponentially small as compared to the smooth part of the potential.

7 Appendix B

The polarization operator \( \Pi(p, \omega) \) can be calculated without account for the vertex part \( \Gamma \), making use of the expressions \( (13) \)–\( (16) \) in the form
\[ \Pi(p, \omega) = \int \frac{d^3k}{(2\pi)^3} \left[ I_{11}(p, k, \omega) + I_{12}(p, k, \omega) \right] , \quad (B.1) \]

where
\[ I_{ij}(p, k, \omega) = i \int \frac{dz}{2\pi} \tilde{G}_{ij}(k, z) \tilde{G}_{ij}(k - p, z - \omega) . \quad (B.2) \]

Assume that the Green functions \( \tilde{G}_{ij} \) have only one pole within the integration contour and are equal to
\[ \tilde{G}_{11}(k, \epsilon) = \frac{\epsilon + \frac{k^2}{2m} - \mu + \tilde{\Sigma}_{11}(-k, -\epsilon)}{\epsilon^2 - E^2(k) + i\delta} ; \quad (B.3) \]
\[ \tilde{G}_{12}(k, \epsilon) = \frac{\tilde{\Sigma}_{12}(k, \epsilon)}{\epsilon^2 - E^2(k) + i\delta} ; \quad (\delta \to 0) . \quad (B.4) \]

Calculating the integrals \( (B2) \) with account for the poles at the points \( \epsilon = E(k) \)
and \( \epsilon = E(k - p) + \omega \) yields

\[
I_{11}(p, k, \omega) = \frac{1}{2[E(k) - E(k - p) - \omega]} \left\{ \left[ E(k) + \frac{k^2}{2m} - \mu + \tilde{\Sigma}_{11}(k, E(k)) \right] \right.
\]
\[
\times \left[ E(k) - \omega + \frac{(k - p)^2}{2m} - \mu + \tilde{\Sigma}_{11}(k - p, E(k) - \omega) \right] \right. 
\]
\[
\left. \left[ E(k) + E(k - p) - \omega \right] \right. 
\]
\[
- \left[ E(k - p) + \frac{(k - p)^2}{2m} - \mu + \tilde{\Sigma}_{11}(k - p, E(k - p)) \right] \right]
\]
\[
\times \left[ E(k - p) + \omega + \frac{k^2}{2m} - \mu + \tilde{\Sigma}_{11}(k, E(k - p) + \omega) \right] \right. 
\]
\[
\left. \left[ E(k - p) + E(k - p) + \omega \right] \right. 
\]
\[
= \frac{1}{2[E(k) - E(k - p) - \omega]} \left\{ \frac{\tilde{\Sigma}_{12}(k, E(k))\tilde{\Sigma}_{12}(k - p, E(k) - \omega)}{E(k) + E(k - p) - \omega} \right. 
\]
\[
- \frac{\tilde{\Sigma}_{12}(k, E(k - p) + \omega)\tilde{\Sigma}_{12}(k - p, E(k - p))}{E(k - p) + E(k - p) + \omega} \right\} 
\]
\[
(B.5)
\]
\[
I_{12}(p, k, \omega) = \frac{1}{2[E(k) + E(k - p) + \omega]} \left\{ \frac{\tilde{\Sigma}_{12}(k, E(k))\tilde{\Sigma}_{12}(k - p, E(k) - \omega)}{E(k) + E(k - p) - \omega} \right. 
\]
\[
- \frac{\tilde{\Sigma}_{12}(k, E(k - p) + \omega)\tilde{\Sigma}_{12}(k - p, E(k - p))}{E(k - p) + E(k - p) + \omega} \right\} 
\]
\[
(B.6)
\]

In the statistical limit \((\omega \to 0, p \to 0)\), expression \((B.5)\) reduces to

\[
I_{11}(0, k, 0) = -\frac{1}{4} \left\{ \frac{1}{E^2(k)} \left[ \epsilon(k) + \frac{k^2}{2m} - \mu + \tilde{\Sigma}_{11}(k, \epsilon(k)) \right]^2 \right. 
\]
\[
+ \left. \left[ \frac{2}{\epsilon(k)} \left( 1 + \frac{\partial \tilde{\Sigma}_{11}(k)}{\partial \epsilon} \right) - \frac{k}{m} \frac{1}{\epsilon(k)} \frac{\partial \epsilon(k)}{\partial k} \right] \left[ \epsilon(k) + \frac{k^2}{2m} - \mu + \tilde{\Sigma}_{11}(k, \epsilon(k)) \right] \right\} 
\]
\[
(B.7)
\]

It follows that in a large region of momentum space, \( I_{11}(0, k, 0) < 0 \). The same result is obtained for the function \((B.6)\) at \( p = 0 \) and \( \omega = 0 \), i.e., \( I_{12}(0, k, 0) < 0 \), so that the static bosonic polarization operator \( \Pi(0, 0) \) is negative, which corresponds to a suppression of the “screened” repulsion at \( p \to 0 \). From Eqs. \((B.5)\) and \((B.6)\) it can also be seen that on the mass shell \( \omega = E(p) \), the integrals \( I_{11} \) and \( I_{12} \) remain negative in a wide region of momentum space because of the negative sign of the common denominator \([E(k) - E(k - p) - E(p)] < 0 \) and the positive sign of the denominator \([E(k) + E(k - p) - E(p)] > 0 \) due to the fact that the quasiparticle spectrum \( E(p) \) is decaying [conditions \((15)\)].

Thus, the polarization operator on the mass shell \( \epsilon = E(p) \) is negative on the whole range \( p < 2\pi/a \), where \( \sin(pa) < 0 \). It is necessary to stress that this feature of the polarization operator, \( \Pi(p, E(p)) < 0 \), is only characteristic of Bose systems, in which the single-particle and collective spectra coincide with each other and are counted from the common zero of energy, unlike the Fermi systems, in which the single-particle excitation spectrum begins at the Fermi energy, due to the Pauli principle. Therefore, in the Fermi liquid \((^3\text{He})\) there can be no corresponding effective enhancement of the negative values of the “input” interaction potential \( V(p) \),
so that the formation of Cooper pairs is only possible for nonzero orbital momenta, due to the true weak van der Waals attraction between fermions. Apparently, it is this fact that has to do with the critical temperatures of the SF transition in $^4$He i $^3$He differing by three orders of magnitude.

References

[1] L.D. Landau. ZHETF, 11, 592 (1941); 17, 91 (1947).
[2] N.N. Bogolyubov. Izv. AN SSSR, Seriya Fizika, 11, 77 (1947); Phys. 9, 23 (1947).
[3] R.P. Feynman, Phys.Rev. 94, 262 (1954).
[4] S.T. Belyaev, ZHETF, 34, 417, 433 (1958).
[5] L.P. Pitaevskij, ZHETF, 31, 536, (1956); ZHETF, 36, 1168, (1959).
[6] N. Hugengoltz, D. Pines. Phys.Rev. 116, 489 (1959).
[7] D.Pines, P.Nozières, ”Theory of Quantum Liquids”, Academic, New-York (1969).
[8] J. Ganoret, P. Nozières, Ann. of Phys. 28, 349 (1964).
[9] A.A. Abrikosov et al. “Methods of Quantum Field Theory in Statistical Physics”, Fizmatgiz, Moscow (1962).
[10] I.M. Khalatnikov. “Theory of Superfluidity”, Nauka, Moscow (1971).
[11] A.F.G. Wyatt. Nature, 391, No. 6662, p. 56 (1998).
[12] Yu.A. Nepomnyaschij, A.A. Nepomnyaschij. Pis’ma v ZHETF, 21, 3 (1976).
[13] Yu.A. Nepomnyaschij, A.A. Nepomnyaschij. ZHETF, 75, 976 (1978).
[14] Yu.A. Nepomnyaschij. ZHETF, 85, 1244 (1983); 89, 511 (1985).
[15] H.R. Glyde, E.C. Swenson, in: ”Neutron Scattering”. D.L. Price, K. Skold (eds). Methods of Experimental Physics, vol. 23, part B, Academic Press, New York (1987), p.303.
[16] E.F. Tabbot, H.R. Glyde, W.G. Stirling, E.C. Swenson, Phys.Rev. B38, 11229 (1988).
[17] H.R. Glyde, W.G. Stirling, Phys.Rev. B42, 4224 (1990).
[18] K.H. Andersen, W.G. Stirling, R. Scherm, A. Stanault, B. Fak, H. Godfrin, A.J. Dianoux, J.Phys.:Condensed Matter, 6, 821 (1994).
[19] S. Patterman. “Hydrodynamics of a Superfluid Liquid”, Mir, Moscow (1968).
[20] G.B. Gess. Low Temp.Phys. LPT-13, Plenum, NY (1974), p.302.
[21] J.R. Hulin, D’Humieres, B. Perrin, A. Lichaber, Phys.Rev. A9, 885 (1974).
[22] G. Rayfield, Phys.Rev.Lett. 20, 1467 (1968).
[23] K.A. Brueckner, K. Sawada, Phys.Rev. 106, 1117, 1128 (1957).
[24] K. Brueckner, “Theory of Nuclear Matter”, Mir, Moscow (1964).
[25] E.A. Pashitsky, Ukp.Fiz.Zh., 18, 1439 (1973).
[26] Yu.A. Nepomnyaschij, E.A. Pashitskij, ZHETF, 98, 178 (1990).
[27] J. Schrieffer. ”Theory of Superconductivity”, Nauka, Moscow, 1970.
[28] V.H. Popov. ”Path Integrals in Quantum Field Theory and Statistical Physics”, Nauka, Moscow, 1973.
[29] V.H. Popov, A.V. Serednyakov, ZHETF, 77, 377 (1979).
[30] N. Morii, K. Miyaka, T. Usui, Prog. Theor. Phys. 56, 1360 (1976).
[31] J.C. Slatter, J.G. Kirkwood, Phys.Rev. 37, 682 (1931).
[32] C.H. Page, Phys.Rev. 53, 426 (1938).
[33] H. Margenau, Phys.Rev. 56, 1000 (1939).
[34] R.A. Buckingham, J. Hamilton, H.S.W. Massey, Proc.Roy.Soc. A179, 103 (1941).
[35] P. Rosen, J.Chem.Phys. 18, 1182 (1950).
[36] E.A. Mason, J.Chem.Phys. 22, 1678 (1954).
[37] J.O. Hirschfelder, Ch.F. Curtiss, R.B. Bird, Molecular Theory of Gases and Liquids, New York (1954). The molecular interaction from the diatomic molecules to the polymer. translated from English; edited by B.Plushkin; Mir, Moscow (1981).
[38] I.G. Kaplan, Introduction to the Theory of Intermolecular Interaction, Moscow, Nauka (1982).
[39] Physical Encyclopedia, v.3, p. 88–91, Moscow, Bol’shaya Rossijskaya Enciklopediya (1995).
[40] S.V. Mashkevich, S.I. Vilchynskyy, Bound state of interacting helium atoms. Journal of Molecular Liquids, 2001 (in press).
[41] V. Heine, M. Koen, D. Wape, Theory of Psedupotential, Mir, Moscow (1973).
[42] R. White, Quantum Theory of Magnetism. Mir, Moscow (1985).
[43] B.H. Eselson, V.H. Grigor’ev et al. Solutions of Quantum Liquids ³He–⁴He. Nauka, Moscow (1973).
[44] V.L. Pokrovsky, A.Z. Patashinsky. The Fluctuation Theory of Phase Transitions. Nauka, Moscow (1965).
| Name of potential | Form of potential $\Phi(r)$ | Energy level |
|-------------------|-----------------------------|--------------|
| Rosen-Margenau    | $[925 \exp(-4.4r) - 560 \exp(-5.33r) - \frac{1.31}{r^6} - \frac{3}{r^8}] \cdot 10^{-12}$ erg | None |
| Slater-Kirkwood   | $[770 \exp(-4.6r) - \frac{1.24}{r^6}] \cdot 10^{-12}$ erg | None |
| Intem-Schneider   | $[1200 \exp(-4.72r) - \frac{1.24}{r^6} - \frac{1.89}{r^8}] \cdot 10^{-12}$ erg | None |
| Lennard-Jones     | $4\epsilon \left[ \left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6} \right]$ | Case 1 $\sigma = 2.556, \epsilon = 10.22$ K Case 2 $\sigma = 2.642, \epsilon = 10.80$ K. None |
| Buckingham        | $\begin{cases} [770 \exp(-4.6r) - \frac{1.24}{r^6}] \cdot 10^{-12}, & r \leq 2.61 \text{ Å}; \\ [977 \exp(-4.6r) - \frac{1.50}{r^6} - \frac{2.51}{r^8}] \cdot 10^{-12}, & r \geq 2.61 \text{ Å.} \end{cases}$ | $\epsilon = 0.00632$ K |
| Massey-Buckingham | $[1000 \exp(-4.6r) - \frac{1.51}{r^6}] \cdot 10^{-12}$ erg | $\epsilon = 0.0622$ K |
| Buckingham-Hamilton | $[977 \exp(-4.6r) - \frac{1.51}{r^6} - \frac{2.51}{r^8}] \cdot 10^{-12}$ erg | $\epsilon = 0.00229$ K |
Figure Captions

Fig. 1. The dependence of the ratio \( \frac{\tilde{V}(p\alpha)}{V_0} \) on the dimensionless momentum \( p \) for different values of the dimensionless parameter \( \alpha = V_0|\Pi| \): Dashed curve 1, the “hard spheres” input potential at \( \alpha=0 \); solid curve 2 corresponds to \( \alpha=2 \), curve 3—to \( \alpha=3 \); curve 4—to \( \alpha=3.5 \).

Fig. 2. The Bogolyubov spectrum (40) for a dilute quasi-ideal Bose gas, obtained by substituting the potential (37) with an independent choice of the two parameters \( \alpha = 2.5 \) and \( V_0/(4\pi a^3) = 169.9 \) K at \( a = 2.44 \) Å. The solid curve corresponds to the instability of the Bogolyubov spectrum obtained by substituting the potential (37) into (40) for \( n = 2.17 \times 10^{22} \) cm\(^{-3} \) i \( a = 2.44 \) Å, characteristic of \(^4\)He.

Fig. 3. (a) The numerators \( F_+ \) and \( F_- \) of both terms in expressions (43) and (44) remain positive at any \( p \) and \( k \). (b) The \( p \) dependence of the polarization operator \( \Pi(p, E(p)) \) (42) on the mass shell, obtained numerically by iterations.

Fig. 4. The momentum dependence of the functions (a) \( \Psi_{11}(p) \), (b) \( \Psi_{12}(p) \), (c) \( A_0(p) \), obtained according to Eqs. (50), (51) i (53), for \( V_0/(4\pi^2 a^3) = 17.7 \) K, \( \alpha = 4.4 \).

Fig. 5. (a) The elementary excitation spectrum \( E(p) \) obtained from Eq. (53), for the same set of parameters as on Fig. 4. The maximum energy in the quasiparticle spectrum (52) at \( p/\hbar = 1.05 \) Å\(^{-1} \) is \( E_{\text{max}} = 15.9 \) K, and the minimum one, at the roton minimum at \( p/\hbar = 1.72 \) Å\(^{-1} \), is \( E_{\text{min}} = 10.7 \) K. The hydrodynamic sound velocity \( c_1 = 2.08 \times 10^4 \) cm/s and the total quasiparticle concentration \( n = 2.57 \times 10^{22} \) 1/cm\(^3\) at the BEC concentration \( n_0 = 0.03n \) and the diameter \( a = 2.44 \) Å, found according to Eqs. (33) and (29); (b) The spectrum \( E(p) \) for \( V_0/(4\pi^2 a^3) = 11.2 \) K, \( \alpha = 4.52 \). The maximum energy of the spectrum (52) at \( p/\hbar = 1.23 \) Å\(^{-1} \) is \( E_{\text{max}} = 15.85 \) K, and the minimum one, at the roton minimum at \( p/\hbar = 2 \) Å\(^{-1} \) is \( E_{\text{min}} = 9.81 \) K. The hydrodynamic sound velocity \( c_1 = 2.34 \times 10^4 \) cm/s and the total quasiparticle concentration \( n = 2.18 \times 10^{22} \) 1/cm\(^3\) at the BEC concentration \( n_0 = 0.09n \) and the diameter \( a = 2.95 \) Å, found according to (33) and (29). The dashed line corresponds to the experimental curve.
Fig. 1
Fig. 3b
Fig. 4a
Fig. 5a
Fig. 5b