In this paper, Weyl’s procedure is used to investigate an alternative class of the q-metric that describes a deformed compact object in the presence of an external distribution of matter via exercising quadrupole moments and taking advantage of combining different approaches.

I. INTRODUCTION

In general relativity finding the exact and approximate solutions describing the real source is always of high interest. There exists quite a large number of the solutions of the Einstein field equations in the literature. However, when the solution supposed to describe a deviation from the spherical symmetry, this is a challenging task indeed.

Furthermore, within the scope of general relativity, Birkoff’s theorem states that Schwarzschild space-time is the unique static solution of the asymptotically flat vacuum Einstein field equation with a regular event horizon. However, by considering a static mass distribution with a quadrupole moment, characterizing the deviation from spherical symmetry, this theorem will no longer be valid.

Therefore it is possible to find the various vacuum solutions with mass and different quadrupole parameters. In this respect, the first static and axially symmetric solutions with arbitrary quadrupole moment are described by Weyl in 1917 [1]. Then Erez and Rosen discovered static solutions with arbitrary quadrupole in prolate spheroidal coordinates in 1959 [2]. Later Zipoy [3], and Voorhees [4] found a transformation that generates the simplest solution with quadrupole. This metric is known as \( \gamma \)-metric or \( \sigma \)-metric, and possesses interesting physical aspects with strange topologies. However, for exact spherical symmetry it reduces to the Schwarzschild metric. Later on, by representing this metric in terms of parameter q, it is known as q-metric [5].

The above procedure, however, leads to having Minkowski space as the limiting case. Indeed, this assumption is equivalent to restrict the solution to describing the isolated cases in the space. While this assumption is natural in the first place, the question of how the external distribution of mass may distort them might be of some interest. In this perspective, the second class of solutions describing non-asymptotically flat space-time, is obtained by assuming the existence of a static and axially symmetric external distribution of matter in the vicinity of the central source. In fact, this space-time is a local solution to the Einstein field equation by its construction [6]. In 1965, Doroshkevich and his colleagues considered an external gravitational field up to a quadrupole in the Schwarzschild space-time. They also showed that by adding quadrupole correction to the Schwarzschild space-time, the horizon remains regular [7]. However, a detailed analysis of the distorted Schwarzschild space-time’s global properties was introduced in 1982 by Geroch & Hartle [8]. Later the explicit form of this metric was presented in [9].

In this paper, the static q-metric has been generalized by considering an external matter distribution up to quadrupoles following the approach of Weyl and Eres-Rosen. Thus, the result is not asymptotically flat by its construction. Mathematically, it is possible to find various vacuum solutions with mass and different quadrupole parameters. Nevertheless, differences appear only at the higher multipoles [10]. This means that all the solutions can describe the exterior gravitational field of distorted mass distribution up to quadrupole moments, since they satisfy all the necessary conditions to describe the exterior gravitational field of realistic compact objects. In this paper, we also focus on exterior solutions. The first reason for choosing q-metric to work with is its simplicity. In fact, its mathematical structure is straightforward, which facilitates the study and can be handled analytically. From the physics perspective, this solution is similar to the q-metric solution with an additional external gravitational field, similar to the addition of a magnetic environment to the black hole solution.

Furthermore, aside from the simplicity of q-metric, there are several motives to consider this generalization. For instance, in the relativistic astrophysical study, it is assumed that astrophysical compact objects are described by the Schwarzschild or Kerr space-times. However, besides these relevant solutions, others can imitate a black hole’s properties, such as the electromagnetic signature. In this respect, this work aims at investigating how different backgrounds may describe an astrophysical phenomenon with departure from spherical geometry. Besides, one can take quadrupole moments as the additional physical degrees of freedom to the central compact object and its surroundings. This can facilitate searching for the link between observational phenomena and the compact object’s properties which is not isolated, since the identification of a compact object from initial observations is a challenging task. For example, the geometric configuration of an accretion disk located in this space-time depends explicitly on the value of both quadrupole parameters in a way that it is always possible to distinguish between a distorted Schwarzschild black hole and a distorted, deformed compact object. For example, testing quasi-periodic oscillations in this background, which is progressing, shows its ability to connect the data to the
model. Moreover, there is no doubt on the fundamental importance of gravitational waves in physics, where the experimental evidence finally supported the purely theoretical research in this area. In fact, this metric can also apply to the study of gravitational waves generated by a slightly non-isolated and self-gravitating axisymmetric distribution of mass. In fact, Ryan has shown [11] that we can extract the multipole moments of the central body from the gravitational wave signal. Thus, one can particularly apply this metric in the colliding gravitational waves where asymptotic flatness is not a requirement. As another example, the gravitational field of a very large plate of thin matter, where its gravity mainly focuses at a finite distance, and therefore it stops behaving well asymptotically [12].

As the next step of this work, one can add rotation into this setup to describe more realistic scenarios. However, this is worth mentioning that it is feasible to explain some of the observational data with a static setup. For example, it has been shown that the possible resonant oscillations of the thick accretion disk, can be observed even if the source of radiation is steady and perfectly axisymmetric.

The paper’s organization is as follows: q-metric is briefly explained in Section II. Considering external source is presented in Section III. The conclusion is summarised in Section V.

In this paper, the geometrized units where $c = 1$ and $G = 1$, is used.

## II. q-METRIC

The q-metric describes a static, axially symmetric, and asymptotically flat solution to the Einstein equation with arbitrary quadrupole moment. The metric represents the exterior field of an isolated static axisymmetric mass distribution. It can be used to investigate the exterior fields of slightly deformed astrophysical objects in the strong-field regime [13]. In fact, the presence of quadrupole, independent of its value, changes the geometric properties of space-time drastically. The metric is presented as follows

$$ds^2 = \left(1 - \frac{2M}{r}\right)^{1+\alpha} dt^2 - \left(1 - \frac{2M}{r}\right)^{-\alpha} \left[1 + \frac{M^2 \sin^2 \theta}{r^2 - 2Mr} \right]^{-\alpha(2+\alpha)} \left(\frac{dr^2}{1 - \frac{2Mr}{r}} + r^2 d\theta^2\right) + r^2 \sin^2 \theta d\phi^2.$$  \hspace{1cm} (1)

However, in 1959, Erez and Rosen pointed out that the multipole structure of a static axially symmetric solution has a simpler form in the prolate spheroidal coordinates [14] rather than the cylindrical coordinates of Weyl [2, 10, 15]

$$ds^2 = \left(\frac{x-1}{x+1}\right)^{(1+\alpha)} dt^2 + M^2 (x^2 - 1) \left(\frac{x+1}{x-1}\right)^{(1+\alpha)} \left[\left(\frac{x^2 - 1}{x^2 - y^2}\right)^{\alpha(2+\alpha)} \left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2}\right) + (1 - y^2) d\phi^2\right],$$  \hspace{1cm} (2)

with the following transformation law

$$x = \frac{r}{M} - 1, \quad y = \cos \theta.$$  \hspace{1cm} (3)

This metric has the central curvature singularity at $x = -1$, where it occurs for all real values of $\alpha$. Moreover, an additional singularity appears at $x = 1$, and the norm of the time-like Killing vector at this radius vanishes. Also, outside this hypersurface, there exists no additional horizon. However, considering relatively small quadrupole moment, this hypersurface is located very close to the origin of coordinates so that a physically reasonable interior solution could be used to cover them [13]. Besides, out of this region there is no more singularity, and the metric is asymptotically flat. Although there are different definitions of multipole moments, at the lower order all are equivalent and link to parameter $\alpha$, where $\alpha \in (-1, \infty)$. Further, all odd multipole moments vanish because of the reflection symmetry with respect to the equatorial plane.

## III. GENERALIZATION OF q-METRIC

In this section, we explain briefly the Weyl procedure. The first static and axisymmetric solutions with arbitrary quadrupole moment are described via the Weyl metric

$$ds^2 = -e^{2\psi_0} dt^2 + e^{2(\gamma_0 - \psi_0)} (d\rho^2 + dz^2) + e^{-2\psi_0} \rho^2 d\phi^2,$$  \hspace{1cm} (4)

where $\psi_0 = \psi_0(\rho, z)$ and $\gamma_0 = \gamma_0(\rho, z)$ are the metric functions, and $\psi_0$ plays the role of gravitational potential. The relation between Schwarzschild coordinates and Weyl coordinates is given by

$$\rho = \sqrt{r(r - 2M)} \sin \theta, \quad z = (r - M) \cos \theta,$$  \hspace{1cm} (5)

where $M$ is a parameter that is considered as the mass of the compact object. If we consider three-dimension manifold $N$, orthogonal to the static Killing vector field, then this metric induces the flat metric on $N$. Besides, the metric function $\psi_0$ with respect to this flat metric obeys the Laplace equation

$$\psi_{0,pp} + \frac{1}{\rho} \psi_{0,p} + \psi_{0,zz} = 0.$$  \hspace{1cm} (6)
which is the key factor in this approach. Also, the field \( \gamma_0 \) is determined by the following relation \[1 \]

\[ \nabla \gamma_0 \nabla \rho = \rho (\nabla \psi_0)^2, \]  

where \( \nabla := \partial_\rho + i \partial_z \), or

\[ \gamma_{0,\rho} = \rho (\psi_{0,\rho} - \psi_{0,z}^2), \]  

\[ \gamma_{0,z} = 2\rho \psi_{0,\rho} \psi_{0,z}. \]  

Now, Weyl’s technique takes advantage of the linearity of Laplace’s equation for \( \psi_0 \) and modifies our original metric by replacing the field \( \psi_0 \) by

\[ \psi = \psi_0 + \dot{\psi}. \]  

Since equation (7) is not linear with respect to \( \gamma_0 \), therefore with the same modification as \( \psi \), it is not possible to find a new field \( \tilde{\gamma} \). In fact, it turns out that there will also be a contribution of another term, say \( \delta \gamma \), that satisfies

\[ \nabla (\delta \gamma) = C(\rho)(\nabla \psi_0), \]  

where \( C \) is a function of \( \rho \) [1, 2]. This relation determines \( \delta \gamma \) up to some constant. However, the requirement of elementary flatness [16], in the neighbourhood of the symmetry axis fixes the constant, and it should be set equal to zero. In what follows, we have chosen them via a standard procedure employing Legendre expansion.

However, following the Erez-Rosen approach, it is more appropriate to rewrite this metric in prolate spheroidal coordinates \((t, x, y, \phi)\), via this transformation relation

\[ x = \frac{1}{2M} \left( \sqrt{\rho^2 + (z + M)^2} + \sqrt{\rho^2 + (z - M)^2} \right), \]  

\[ y = \frac{1}{2M} \left( \sqrt{\rho^2 + (z + M)^2} - \sqrt{\rho^2 + (z - M)^2} \right). \]

Equivalently

\[ \rho = M \sqrt{(x^2 - 1)(1 - y^2)}, \]  

\[ z = Mxy. \]

Also, the equations (6) and (8) in the prolate spheroidal coordinates are written as

\[ (x^2 - 1)\psi_{0,xx} + 2x\psi_{0,x} + \psi_{0,yy} + \frac{y}{\sqrt{1 - y^2}} \psi_{0,y} = 0, \]  

\[ \gamma_{0,x} = \frac{y^2 - 1}{x^2 - y^2} [x (\psi_{0,y}^2 - (x^2 - 1)\psi_{0,x}^2) + 2y(1 - x^2)\psi_{0,x} \psi_{0,y}], \]  

\[ \gamma_{0,y} = \frac{1 - y^2}{x^2 - y^2} \frac{y}{\sqrt{1 - y^2}} (\psi_{0,y}^2 - (x^2 - 1)\psi_{0,x}^2) + 2x\psi_{0,y} \psi_{0,x}. \]

To sum up, all the process reduces to choosing two fields that provide a solution of the vacuum field equations and take them as the seed for introducing new fields that satisfy the above relations. In this approach, metric components are expressed through integrability conditions as functions of a single harmonic function and its partial derivatives in a surprisingly simple mathematical procedure.

Following Weyl’s procedure by using the q-metric fields as the seed, we can obtain the solution of a deformed source surrounding with the external mass distribution, which can be generalized to the whole class of q-metric. Thus, we introduce auxiliary fields \( \psi \) and \( \gamma \) such that

\[ e^{2\psi} := \left( \frac{x - 1}{x + 1} \right)^{(1 + \alpha)} e^{2\dot{\psi}}, \]  

\[ e^{2\gamma} := \left( \frac{x^2 - 1}{x^2 - y^2} \right)^{\alpha(1 + \alpha)} e^{2\dot{\gamma}}. \]

Notice that the exponential factors, distorted terms, represent the surrounding matter. As this process suggests, we preserve the q-metric fields by taking \( \dot{\gamma} = \dot{\psi} = 0 \). It can easily be checked that in the limits \( \dot{\psi} = 0, \dot{\gamma} = 0 \), and \( \alpha = 0 \) we recover the Schwarzschild fields, and in the case \( \alpha \neq 0 \), \( \dot{\psi} \neq 0 \) we obtain the distorted Schwarzschild fields written in the prolate spheroidal coordinates.

In this way, field functions preserve their properties as before by its construction. Ultimately, the metric is then given by

\[ ds^2 = - \left( \frac{x - 1}{x + 1} \right)^{(1 + \alpha)} e^{2\dot{\psi}} dt^2 + M^2 (x^2 - 1) e^{-2\dot{\psi}} \left( \frac{x + 1}{x - 1} \right)^{(1 + \alpha)} \left( \frac{x^2 - 1}{x^2 - y^2} \right)^{\alpha(2 + \alpha)} e^{2\dot{\gamma}} \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} + (1 - y^2) d\phi^2 \right), \]

where \( t \in (-\infty, +\infty), x \in (1, +\infty), y \in [-1, 1], \) and \( \phi \in [0, 2\pi) \). In this work for choosing the explicit form of these distortion functions we followed the standard procedure via the separation of variables method [17]. In this approach \( \dot{\psi} \) generally can be expressed in terms of Legendre polynomials of the first kind and second kind [9, equation 6-7]. This method was introduced first by Geroch and Hartle [8]. However, up to the quadrupole \( \beta \), these are expressed as follows [9] [18],

\[ \dot{\psi} = -\frac{\beta}{2} \left[ -3x^2 y^2 + x^2 + y^2 - 1 \right], \]  

\[ \dot{\gamma} = -2x\beta(1 - y^2) + \frac{\beta^2}{4} (x^2 - 1)(1 - y^2)(9x^2 y^2 + x^2 + y^2 - 1). \]
We may refer to quadrupole $\alpha$ as a deformation parameter and $\beta$ as a distortion parameter to be compatible with the proceeding related works.

Therefore, this metric contains three free parameters: the total mass, deformation parameter $\alpha$, and distortion $\beta$, which are taken to be relatively small and connected to the compact object deformation and external mass presence distribution, respectively. The result is also locally valid by its construction [6, 8], and may be considered as the local q-metric or distorted q-metric. From the physics perspective, this solution is similar to the q-metric solution with an additional external gravitational field, like adding a magnetic surrounding [19–21].

IV. EFFECTIVE POTENTIAL

Regarding the symmetries in the metric, there are two constants of geodesic motion $E$ and $L$,

\[
E = -g_{tt} \dot{t} = \left( \frac{x - 1}{x + 1} \right)^{(1+\alpha)} e^{2\hat{\phi}} \dot{t},
\]

\[
L = g_{\phi\phi} \dot{\phi} = M^2 (x^2 - 1) \left( \frac{x + 1}{x - 1} \right)^{(1+\alpha)} e^{-2\hat{\phi}}. \tag{23}
\]

By using these relations and the normalization condition $g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\epsilon$, where $\epsilon$ can take values $-1$, $0$ and $1$, for the space-like, light-like and for the time-like trajectories, respectively, then the geodesic equation is reduced to

\[
\left( 1 + \frac{x^2 - 1}{x^2 - y^2} \right)^{\alpha(2+\alpha)} M^2 e^{2\hat{\phi}} \dot{x}^2 + V^2 = E^2, \tag{24}
\]

where

\[
V^2 = \frac{x^2 - 1}{1 - y^2} \left( \frac{x^2 - 1}{x^2 - y^2} \right)^{\alpha(2+\alpha)} M^2 e^{2\hat{\phi}} y^2
\]

\[+ \frac{L^2 e^{4\hat{\phi}}}{M^2 (1 - y^2)(x + 1)^2} \left( \frac{x - 1}{x + 1} \right)^{(\alpha+1)} e^{2\hat{\phi}} \epsilon. \tag{25}
\]

One can interpret this equation as the motion along the $x$ coordinate in terms of this potential, which is called the effective potential $V_{\text{eff}}$. However, this

Furthermore, by analysing the effective potential in the equatorial plane, one can investigate general properties of the particle dynamics in the equatorial plane. In fact, this motion is determined by the energy’s boundaries given by $E^2 = V_{\text{eff}}$. In general, possible types of orbits, dependent on the parameters $\epsilon$, $E$, $L$, $\alpha$ and $\beta$. And in an analytical exploration depending on the number of positive real zeros and the sign of $E^2 - \epsilon$, one obtains different types of trajectories. In Figure 1 the trajectories for some choices of the parameters have plotted. In fact, for each valid pair of parameters, one can distinguish different trajectories in this background. Depending on initial values and parameters, particles can fall onto the central object, escape from it, or trap in some region and forming a toroidal shape around the central object. As one can see in the plots, only by a slight change in the initial values one can obtain different trajectories for a chosen pair of parameters.

V. SUMMARY AND CONCLUSION

This paper has presented a class of q-metric for the relatively small quadrupole moment via Weyl’s procedure. This class explains the exterior of a deformed body locally in the presence of an external distribution of matter up to the quadrupole. It contains three free parameters: the total mass, deformation parameter $\alpha$, and the distortion parameter $\beta$, referring to the central object’s quadrupoles and its surrounding mass distribution.

Exercising the quadrupole moments may open the door to a broader range of applications via considering them as new degrees of freedom to both central object and its surrounding, to search for their observational fingerprints. For instance, in the study of the gravitational wave, or quasi-periodic oscillations generated by a non-isolated, self-gravitating axisymmetric distribution of mass.

We propose that this class has promising features, and it may serve to link the metric to the physical nature of the compact object via its parameters. Although its simplicity still this generalization worth considering to have more vacuum metric available, for example, to study a vacuum solution that is accelerating or embedded in an external field. Therefore, it may be of some interest to investigate how the external field affects the geometry and geodesics of the q-metric. Of course, real models will be inaccurate due to incomplete knowledge of the underlying physics, or perhaps a desire for simplicity.

In the present contribution we further explored the existence of different trajectories for a particle in the equatorial plane for correct choices of parameters. In general, test particles motion can be chaotic in this background for some combinations of parameters $\alpha$ and $\beta$. Also, collision may occur in this vicinity of central object. Further, the model can reproduce many of the relevant features of the numerical results. At this stage, such qualitative agreement is all that can be hoped for.

The next step of this work could be a study on off-equatorial time-like and light-like geodesics. Studying the topological implication of this background may be the next stage of study in this area. Also, the construction of accretion disks, the study of gravitational waves and quasi-periodic oscillation in this background are in progress. We expect future work in this field to be guided by questions of astrophysics, and other areas where strong-field gravitational theory applies.
FIG. 1. Timelike geodesic for pairs of ($\alpha = 0.1, \beta = 0.000001$). The trajectories in the ($r, \phi$) section and in the complete 3D are plotted. Configurations in the first column correspond to a bounded trajectory, in the second column to a falling trajectory, and in the third column to an escaping trajectory.

VI. ACKNOWLEDGEMENTS

The author gratefully acknowledges Prof. Hernando Quevedo, Prof. Domenico Giulini, and Dr. Eva Hackmann for their valuable comments on this work. Also, thanks to Dr. Audry Trova for valuable discussions. This work is supported by the research training group GRK 1620 ”Models of Gravity”, funded by the German Research Foundation (DFG).

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