Research Article

New Iteration Scheme for Approximating a Common Fixed Point of a Finite Family of Mappings

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We introduce a new algorithm (horizontal algorithm) in a real Hilbert space, for approximating a common fixed point of a finite family of mappings, without imposing on the finite family of the control sequences \( \varsigma \).

Let \( Y \) be a nonempty set and \( S: Y \rightarrow Y \) be a mapping. A point \( y \in Y \) is called a fixed point of \( S \) if \( y = Sy \). If \( S: Y \rightarrow 2^Y \) is a multivalued mapping, then \( y \) is a fixed point of \( S \) if \( y \in Sy \). \( y \) is called a strict fixed point of \( S \) if \( Sy = \{ y \} \).

The set \( F(S) = \{ y \in D(S): y \in Sy \} \) (respectively, \( F(S) = \{ y \in D(S): y = Sy \} \)) is called the set of fixed points of the multivalued (respectively, single-valued) mapping \( S \), while the set \( F_S(S) = \{ y \in D(S): Sy = \{ y \} \} \) is called the set of strict fixed points of \( S \).

Let \( Y \) be a normed space. A subset \( K \) of \( Y \) is called proximinal if for each \( y \in Y \), there exists \( k \in K \) such that
\[
\| y - k \| = \inf \{ \| y - w \|: w \in K \} = d(y, K).
\]

It is known that every convex closed subset of a uniformly convex Banach space is proximinal. We shall denote the family of all nonempty closed and bounded subsets of \( Y \) by \( CB(Y) \), the family of all nonempty subsets of \( Y \) by \( 2^Y \), and the family of all proximinal subsets of \( Y \) by \( P(Y) \), for a nonempty set \( Y \).

Let \( \mathcal{D} \) denote the Hausdorff metric induced by the metric \( d \) on \( Y \), that is, for every \( A, B \in CB(Y) \),
\[
\mathcal{D}(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.
\]

Let \( Y \) be a normed space and \( S: D(S) \subseteq Y \rightarrow 2^Y \) be a multivalued mapping on \( Y \). \( S \) is called \( L \)–Lipschitzian if there exists \( L \geq 0 \) such that, for all \( x, y \in D(S) \),
\[
\mathcal{D}(Sx, Sy) \leq L\| x - y \|.
\]

In (3), if \( L \in [0, 1) \), then \( S \) is a contraction, while \( S \) is nonexpansive if \( L = 1 \). \( S \) is called quasi-nonexpansive if \( F(S) \neq \emptyset \) and for all \( p \in F(S) \),
\[
\mathcal{D}(Sx, Sp) \leq \| x - p \|.
\]

Clearly, every nonexpansive mapping with the nonempty fixed point set is quasi-nonexpansive. The multivalued mapping \( S \) is \( k \)-strictly pseudo-contractive-type of Isiogugu [1] using the terminology of Browder and Petryshen [2] for single-valued pseudo-contractive mapping and Markin [3] for the monotone operator if there exists \( k \in [0, 1) \) such that given any pair \( x, y \in D(S) \) and \( u \in Sx \), there exists \( v \in Sy \) satisfying \( \| u - v \| \leq \mathcal{D}(Sx, Sy) \) and
\[
\mathcal{D}^2(Sx, Sy) \leq \| x - y \|^2 + k\| x - u - (y - v) \|^2.
\]
If \( k = 1 \) in (5), then \( S \) is pseudo-contractive-type, while \( S \) is nonexpansive if \( k = 0 \). Every multivalued nonexpansive mapping \( S: D(S) \subseteq Y \to P(Y) \) is nonexpansive-type. \( S \) is of type-one in the sense of Isiogugu et al. \cite{4} if given any pair \( x, y \in D(S) \), then
\[
\|u - v\| \leq \varnothing(Sx, Sy), \quad \text{for all } u \in P_3x, v \in P_3y,
\]
where \( P_3S = \{u \in Sx: \|u - x\| = \varnothing(x, Sx)\} \). \( S \) is called a multivalued demicontractive in the sense of Isiogugu and Osilike \cite{5} using the terminology of Hicks and Kubic\v{e}k \cite{6} for single-valued demicontractive if \( F(S) \neq \emptyset \) and for all \( p \in F(S) \) and \( x \in D(S) \), there exists \( k \in [0, 1) \) such that
\[
\varnothing(Sx, Sp) \leq \|x - p\|^2 + kd^2(x, Sx).
\]
(7)

If \( k = 1 \) in (7), \( S \) is hemicontractive in the terminology of Naimpally and Singh \cite{7} for single-valued hemicontractive, while \( S \) is quasi-nonexpansive if \( k = 0 \).

Furthermore, every multivalued \( k \)-strictly pseudo-contractive-type in the sense of \cite{1} with the nonempty set of strict fixed points is demicontractive with respect to its set of strict fixed points.

A single-valued mapping \( S: D(S) \subseteq H \to H \) is called nonspreading in the sense of Kohsaka and Takahashi \cite{8, 9} if
\[
\|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\langle x - Sx, y - Sy \rangle, \quad \forall x, y \in C.
\]
(8)

Observe that if \( S \) is nonspreading and \( F(S) \neq \emptyset \), then \( S \) is quasi-nonexpansive. \( S \) is \( k \)-strictly pseudo-nonspreading in the sense of Osilike and Isiogugu \cite{10} if there exists \( k \in [0, 1) \) such that
\[
\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|x - Sx - (y - Sy)\|^2 + 2\langle x - Sx, y - Sy \rangle,
\]
(9)

for all \( x, y \in D(S) \). Clearly, every nonspreading mapping is \( k \)-strictly pseudo-nonspreading. If \( S \) is \( k \)-strictly pseudo-nonspreading and \( F(S) \neq \emptyset \), then \( S \) is demicontractive in the sense of \cite{6} (see also \cite{11}).

Several algorithms have been introduced by different authors for the approximation of common fixed points of finite family of mappings \( \{S_i\}_{i=1}^N \), where \( N \in \mathbb{N} \) (the set of nonnegative integers) (see, for example, \cite{12–18} and references therein). One of the motivations for this aspect of research is the well-known convex feasibility problem which is reducible to the problem of finding a point in the intersection of the set of fixed points of a family of nonexpansive mappings (see, for example, \cite{19, 20}). The earliest of such algorithms was the cyclic algorithm introduced by Bauschke \cite{12} using a Halpern-type iterative process considered in \cite{21} for the approximation of a common fixed point of a finite family of nonexpansive self-mappings. He proved the following theorem.

**Theorem 1** (see \cite{12}, Theorem 3.1). Let \( K \) be a nonempty convex closed subset of a real Hilbert space \( H \) and \( S_1, S_2, \ldots, S_N \) be a finite family of nonexpansive mappings of \( K \) into itself with \( F = \cap_{i=1}^N F(S_i) \neq \emptyset \) with \( F = F(S_N S_{N-1} \ldots S_1) = F(S_1 S_N \ldots S_2) = F(S_{N-1} S_N \ldots S_2 S_1) \). Given points \( u, x_0 \in K \), let \( \{x_n\} \) be generated by
\[
x_{n+1} = c_n x_n + (1 - c_n) S_{n+1} x_{n+1}, \quad n \geq 0,
\]
(10)

where \( S_n = S_{n \mod N} \) and \( c_n \in (0, 1) \) satisfies \( \sum_{n=1}^\infty |c_{n+1} - c_n| < \infty \). Then, \( \{x_n\} \) converges strongly to \( P_Fu \), where \( P_F: H \to F \) is the metric projection.

The above algorithm of Bauschke was extended to approximate the family of more general class of strictly pseudo-contractive mappings (see, for example, \cite{22, 23}). Suantai et al. also considered similar algorithms (see, for example, \cite{24}) and references therein.

In 2008, Zhang and Guo \cite{25} considered a parallel iteration for approximating the common fixed points of a finite family of strictly pseudo-contractive mapping. They obtained the following theorem.

**Theorem 2** (see \cite{25}, Theorem 4.3). Let \( E \) be a real \( q \)-uniformly smooth Banach space which is also uniformly convex and \( K \) be a nonempty convex closed subset of \( E \). Let \( N \geq 1 \) be an integer, and for each \( 1 \leq i \leq N \), let \( S_i: K \to K \) be a \( k_i \)-strictly pseudo-contractive mapping for some \( 0 \leq k_i < 1 \). Let \( k = \min\{k_i: 1 \leq i \leq N\} \). Assume the common fixed point set \( \cap_{i=1}^N F(S_i) \) is nonempty. Assume also for each \( n, \{\lambda^i_n\}_{i=1}^N = \{\lambda^0_n\}_{i=1}^\infty \) is a finite sequence of positive numbers such that \( \sum_{i=1}^N \lambda^i_n \leq 1 \) for all \( n \) and \( \inf_{n \geq 1} \lambda^i_n > 0 \) for all \( 1 \leq i \leq N \). Given \( x_0 \in K \), let \( \{x_n\}_{n=1}^\infty \) be the sequence generated by the algorithm:
\[
x_{n+1} = c_n x_n + (1 - c_n) \sum_{i=1}^N \lambda^i_n S_i x_n, \quad n \geq 0.
\]
(11)

Let \( \{c_n\}_{n=1}^\infty \) be a real sequence satisfying the conditions
\[
\sum_{n=0}^\infty \sum_{i=1}^N |\lambda^i_{n+1} - \lambda^i_n| < \infty;
\]
(12)

\[
\sum_{n=0}^\infty (1 - c_n) \lambda^i_n \geq C q(1 - c_n)^{\infty - 1} = \infty.
\]

Then, \( \{x_n\} \) converges weakly to a common fixed point of \( \{S_i\}_{i=1}^N \).

Motivated by the parallel algorithm above, many authors have considered in a real Hilbert space, another form of parallel algorithm for a finite family \( \{S_i\}_{i=1}^N \) of \( k_i \)-strictly pseudo-contractive mappings defined by
\[
x_{n+1} = c_n x_n + \sum_{i=1}^N \lambda^i_n S_i x_n, \quad n \geq 1,
\]
(13)

where \( \{\lambda^i_n\}_{n=1}^\infty \subseteq (0, 1) \) for each \( i \) and \( \sum_{i=1}^N \lambda^i_n = 1 \) for each \( n \) (see, for example, \cite{13} and references therein).

In [14], Iemoto and Takahashi studied the approximation of common fixed points of a nonexpansive self-mapping \( T \) and a nonspreading self-mapping \( S \) in a real Hilbert space. If \( T, S: C \to C \) are, respectively, nonexpansive and nonspreading mappings, they considered the iterative scheme \( \{x_n\}_{n=1}^\infty \) generated from arbitrary \( x_1 \in C \) by
with a nonempty fixed point set $T$ of a nonexpansive mapping of $C$ into itself and $T$ a nonexpansive mapping of $C$ into itself such that $F(T) \cap F(S) \neq \emptyset$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in $C$ as follows:

$$
\begin{align*}
 x_1 & \in C, \\
 x_{n+1} &= (1 - \zeta_n)x_n + \zeta_n[\beta_nSx_n + (1 - \beta_n)Tx_n], \\
 & \quad n \geq 1,
\end{align*}
$$

(14)

where $\{\zeta_n\}$ and $\{\beta_n\}$ are suitable sequences in $[0, 1]$. They proved the following main theorem:

**Theorem 3** (see [14], Theorem 4.1). Let $H$ be a real Hilbert space. Let $C$ be a nonempty convex and closed subset of $H$. Let $S$ be a nonspreading mapping of $C$ into itself and $T$ a nonexpansive mapping of $C$ into itself such that $F(T) \cap F(S) \neq \emptyset$. Then the following hold:

(i) If $\liminf_{n \to \infty} \zeta_n (1 - \zeta_n) > 0$, then $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(S)$.

(ii) If $\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(T)$.

(iii) If $\liminf_{n \to \infty} \zeta_n (1 - \zeta_n) > 0$ and $\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(T) \cap F(S)$.

Motivated by the above result, Osilike and Isiogugu obtained the following result.

**Theorem 4** (see [10], Theorem 3.1.1). Let $C$ be a nonempty convex closed subset of a real Hilbert space, and let $\{c_n\}_{n=1}^{\infty}$ be a sequence for $\{\beta_n\}_{n=1}^{\infty}$ be a nonspreading mapping of $C$ into itself and $T$ a nonexpansive mapping of $C$ into itself such that $F(T) \cap F(S) \neq \emptyset$. Then the following hold:

(i) If $\liminf_{n \to \infty} c_n (1 - c_n) > 0$, then $\sum_{n=1}^{\infty} (1 - \beta_n) < \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(S)$.

(ii) If $\sum_{n=1}^{\infty} \beta_n (1 - \beta_n) = \infty$ and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(T)$.

(iii) If $\liminf_{n \to \infty} c_n (1 - c_n) > 0$ and $\liminf_{n \to \infty} \beta_n (1 - \beta_n) > 0$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(T) \cap F(S)$.

2. **Preliminaries**

In the sequel, we shall need the following definitions and lemmas.

**Definition 1** (see, e.g., [26–27]). Let $Y$ be a Banach space and $S: D(S) \subseteq Y \to 2^Y$ be a multivalued mapping. $I - S$ is weakly demiclosed at zero if for any sequence $\{x_n\}_{n=1}^{\infty} \subseteq D(S)$ such that $\{x_n\}$ converges weakly to $p$ and a sequence $\{y_n\}$ with $y_n \in Sx_n$ for all $n \in \mathbb{N}$ such that $\limsup_{n \to \infty} d(y_n, y) = 0$, then $p \in Sp$ (i.e., $0 \in (I - S)p$).

**Definition 2**. A Banach space $Y$ is said to satisfy Opial’s condition [28] if whenever a sequence $\{x_n\}$ weakly converges to $x \in Y$, then it is the case that

$$
\liminf_{n \to \infty} \|x_n - y\| < \liminf_{n \to \infty} \|x_n - x\|,
$$

(17)

for all $y \in Y$, $y \neq x$.

**Definition 3** (see [29]). A multivalued mapping $S: C \to P(C)$ is said to satisfy condition (1) (see, for example, [29]) if there exists a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$
d(x, Sx) \geq f(d(x, F(S))), \quad \forall x \in C.
$$

(18)

**Definition 4** (see [4]). Let $Y$ be a normed space and $S: D(S) \subseteq Y \to 2^Y$ be a multivalued map. $S$ is of type-one if given any pair $x, y \in D(S)$, then

$$
\|u - v\| \leq d(Sx, Sy), \quad \text{for all } u \in P_x, v \in P_y.
$$

(19)
Lemma 1 (see [30]). Let \( \{a_n\} \) and \( \{y_n\} \) be sequences of nonnegative real numbers satisfying the following relation:
\[
a_{n+1} \leq a_n + y_n, \quad \forall n \in \mathbb{N}. \tag{20}
\]
If \( \sum y_n < \infty \), then \( \lim_{n \to \infty} a_n = \exists \).

3. Main Results

Let \( K \) be a nonempty convex and closed subset of a real Hilbert space \( H \). Suppose that \( \{S_i\}_{i=1}^N, N \geq 2 \) is a countable finite family of mappings \( S_i : K \to K \), and we consider the horizontal iteration process generated from arbitrary \( x_1 \) for the finite family of mappings \( \{S_i\}_{i=1}^N \), using a finite family of the control sequences \( \{\{c_{ij}\}_{j=1}^N\}_{i=1}^\infty \) as follows:

For \( N = 2 \),
\[
x_{n+1} = c_{11}x_n + (1 - c_{11})[c_{21}S_1x_n + (1 - c_{21})S_2x_n]. \tag{21}
\]
For \( N = 3 \),
\[
x_{n+1} = c_{11}x_n + (1 - c_{11})[c_{21}S_1x_n + (1 - c_{21})] \cdot [c_{31}S_2x_n + (1 - c_{31})S_3x_n]. \tag{22}
\]
For arbitrary but finite \( N \geq 2 \),
\[
x_{n+1} = c_{11}x_n + (1 - c_{11})[c_{21}S_1x_n + (1 - c_{21})c_{31}S_2x_n + (1 - c_{31})] \cdot \{ \ldots \{c_{N-1}S_{N-1}x_n + (1 - c_{N-1})S_Nx_n\} \ldots \} \]
\[
= c_{11}x_n + \sum_{i=2}^N c_{i1} \prod_{j=1}^{i-1} (1 - c_{ij})S_{j-1}x_n + \prod_{j=1}^{N} (1 - c_{ij})S_Nx_n, \quad n \geq 1. \tag{23}
\]

We now present the following results which are very useful in establishing our convergence theorems.

Proposition 1. Let \( \{\varsigma_i\}_{i=1}^N \subseteq \mathbb{R} \) be a countable subset of the set of real numbers \( \mathbb{R} \), where \( N \geq 2 \) is an arbitrary integer. Then, the following holds:
\[
\varsigma_1 + \sum_{i=2}^N \varsigma_i \prod_{j=1}^{i-1} (1 - \varsigma_j) + \prod_{j=1}^{N} (1 - \varsigma_j) = 1. \tag{24}
\]

Proof. For \( N = 2 \),
\[
\varsigma_1 + \sum_{i=2}^3 \varsigma_i \prod_{j=1}^{i-1} (1 - \varsigma_j) + \prod_{j=1}^{2} (1 - \varsigma_j)
= \varsigma_1 + \varsigma_2(1 - \varsigma_1)(1 - \varsigma_2)
= \varsigma_1 + (1 - \varsigma_1)[\varsigma_2 + (1 - \varsigma_2)]
= \varsigma_1 + (1 - \varsigma_1) = 1. \tag{25}
\]
We assume it is true for \( N \) and prove for \( N+1 \).

Proposition 2. Let \( \{\varsigma_i\}_{i=k}^N \subseteq \mathbb{R} \) be a countable subset of the set of real numbers \( \mathbb{R} \), where \( k \) is a fixed nonnegative integer and \( N \in \mathbb{N} \) is any integer with \( k + 1 \leq N \). Then, the following holds:
\[
\varsigma_k + \sum_{i=k+1}^N \varsigma_i \prod_{j=k}^{i-1} (1 - \varsigma_j) + \prod_{j=k}^{N} (1 - \varsigma_j) = 1. \tag{27}
\]

Proof. For \( k = 0 \), \( N = 1 \), and \( k = 1 \), \( N = 2 \), the proofs follow from Remark 1 and Proposition 1, respectively.
We assume it is true for \( k \) and \( N \). Now, for \( k \) and \( N + 1 \),
\[
\varsigma_k + \sum_{i=k+1}^{N+1} \varsigma_i \prod_{j=k}^{i-1} (1 - \varsigma_j) + \prod_{j=k}^{N+1} (1 - \varsigma_j)
= \varsigma_k + \sum_{i=k+1}^{N+1} \varsigma_i \prod_{j=k}^{i-1} (1 - \varsigma_j) + \prod_{j=k}^{N+1} (1 - \varsigma_j)
= \varsigma_k + \sum_{i=k+1}^{N+1} \varsigma_i \prod_{j=k}^{i-1} (1 - \varsigma_j) + \prod_{j=k}^{N+1} (1 - \varsigma_j)
= \varsigma_k + \sum_{i=k+1}^{N} \varsigma_i \prod_{j=k}^{i-1} (1 - \varsigma_j) + \prod_{j=k}^{N} (1 - \varsigma_j)
= \varsigma_k + \sum_{i=k}^{N} \varsigma_i \prod_{j=k}^{i-1} (1 - \varsigma_j) + \prod_{j=k}^{N} (1 - \varsigma_j)
= \varsigma_k + \sum_{i=k}^{N} \varsigma_i \prod_{j=k}^{i-1} (1 - \varsigma_j) + \prod_{j=k}^{N} (1 - \varsigma_j) = 1. \tag{28}
\]

Proposition 3. Let \( t, u, \) and \( v \) be arbitrary elements of a real Hilbert space \( H \). Let \( k \) be a fixed nonnegative integer and \( N \in \mathbb{N} \) be such that \( k + 1 \leq N \). Let \( \{w_i\}_{i=k}^N \subseteq \mathbb{H} \) and \( \{c_i\}_{i=k} \subseteq [0, 1] \) be a countable finite subset of \( H \) and \( \mathbb{R} \), respectively. Define
\[
y = c_0 t + \sum_{i=k}^N c_i \prod_{j=k}^{i-1} (1 - c_j) v_{i-1} + \prod_{j=k}^{N} (1 - c_j) v \tag{29}
\]
Then,
\[ \| y - u \|_2^2 = \gamma_k\| t - u \|_2^2 + \sum_{i=k+1}^N \gamma_i \prod_{j=k}^{i-1} (1 - \zeta_j) \| v_{i-1} - u \|_2^2 + \prod_{j=k}^N (1 - \zeta_j) \| v - u \|_2^2 \\
- \gamma_k \left[ \sum_{i=k+1}^N \gamma_i \prod_{j=k}^{i-1} (1 - \zeta_j) \| t - v_{i-1} \|_2^2 + \prod_{j=k}^N (1 - \zeta_j) \| v - v \|_2^2 \right] \\
- (1 - \gamma_k) \left[ \sum_{i=k+1}^N \gamma_i \prod_{j=k}^{i-1} (1 - \zeta_j) \| v_{i-1} - [\gamma_i v_i + \gamma_{i+1}] \|_2^2 \\
+ \gamma_N \prod_{j=k}^N (1 - \zeta_j) \| v - v_{N-1} \|_2^2 \right], \]

(30)

where \( \omega_k = \sum_{i=k+1}^N \gamma_i \prod_{j=k}^{i-1} (1 - \zeta_j) v_{i-1} + \prod_{j=k}^N (1 - \zeta_j) v, \ k = 1, 2, \ldots, N - 1, \) and \( \omega_N = (1 - \zeta_N) v. \)

**Proof.** Using the well-known identity,
\[ \| \tau x + (1 - \tau) y \|_2^2 = \tau \| x \|_2^2 + (1 - \tau) \| y \|_2^2 - \tau (1 - \tau) \| x - y \|_2^2, \]

(31)

which holds for all \( x, y \in H \) and for all \( \tau \in [0, 1], \) we prove by (i) direct computation and (ii) induction.

Observe that, for \( k \leq N - 1, \ \omega_k = (1 - \zeta_k) [\gamma_k v_k + \omega_{k+1}]. \) Consequently, by the direct computation, we have
\[ q_k \| t - u \|^2 + (1 - c_k) \| s_{k+1} \| y_k - u \|^2 - c_k \| (1 - c_k) \| s_{k+1} \| y_k - \| t - y_k \|^2 \]
\[ - c_{k+1} (1 - c_k) \| (1 - c_k) \| y_k + [s_{k+1} + w_{k+2}] \|^2 \]
\[ + (1 - c_k) \| (1 - c_k) \| s_{k+2} + w_{k+2} - u \|^2 \]
\[ - c_k (1 - c_k) \| (1 - c_k) \| s_{k+1} \| y_k - \| t - y_k \|^2 \]
\[ = q_k \| t - u \|^2 + (1 - c_k) \| s_{k+1} \| y_k - u \|^2 - c_k \| (1 - c_k) \| s_{k+1} \| y_k - \| t - y_k \|^2 \]
\[ - c_{k+1} (1 - c_k) \| (1 - c_k) \| y_k + [s_{k+1} + w_{k+2}] \|^2 \]
\[ + (1 - c_k) \| (1 - c_k) \| s_{k+2} + w_{k+2} - u \|^2 \]
\[ - c_k (1 - c_k) \| (1 - c_k) \| s_{k+1} \| y_k - \| t - y_k \|^2 \]
\[ = q_k \| t - u \|^2 + (1 - c_k) \| s_{k+1} \| y_k - u \|^2 - c_k \| (1 - c_k) \| s_{k+1} \| y_k - \| t - y_k \|^2 \]
\[ + (1 - c_k) (1 - c_{k+1}) \| s_{k+2} \| y_k + \| w_k - t \|^2 \]
\[ - (1 - c_k) (1 - c_{k+1}) \| s_{k+2} \| y_k + \| w_k - t \|^2 \]
\[ - c_k (1 - c_k) \| (1 - c_k) \| s_{k+1} \| y_k - \| t - y_k \|^2 \]
\[ = q_k \| t - u \|^2 + (1 - c_k) \| s_{k+1} \| y_k - u \|^2 - c_k \| (1 - c_k) \| s_{k+1} \| y_k - \| t - y_k \|^2 \]
\[ + (1 - c_k) (1 - c_{k+1}) \| s_{k+2} \| y_k + \| w_k - t \|^2 \]
\[ - c_k (1 - c_k) \| (1 - c_k) \| s_{k+1} \| y_k - \| t - y_k \|^2 \]
\[ = q_k \| t - u \|^2 + \sum_{i=k+1}^{k+2} (1 - c_i) \| y_{k+1} - u \|^2 \]
\[ - c_k \| \sum_{i=k+1}^{k+2} (1 - c_i) \| y_{k+1} - u \|^2 \]
\[ - (1 - c_k) \| \sum_{i=k+1}^{k+2} (1 - c_i) \| y_{k+1} - [s_{i+1} + w_{i+1}] \|^2 \]
\[ + \prod_{j=k}^{k+2} \| (1 - c_j) \| y_{k+1} - \| s_{k+2} + w_{k+2} \|^2 \]
\[ - c_k \| \prod_{j=k}^{k+2} \| (1 - c_j) \| y_{k+1} - \| s_{k+2} + w_{k+2} \|^2 \]
prove by induction, we then assume that it is true for $k, N$, and prove for $k+1, N+1$.

Since induction holds for a fixed $k$ and each $N$ from direct computation, then it is true for $k, N = 1, 2, 3$. Thus, to prove by induction, we then assume that it is true for $k, N$ and prove for $k$ and $N + 1$. From

\[ y = \zeta_k t + \sum_{i=k+1}^{N+1} \zeta_i \prod_{j=k}^{i-1} (1 - \zeta_j) v_{i-1} + \prod_{j=k}^{N+1} (1 - \zeta_j)v \]

we have that

\[
\|y - u\|^2 = \zeta_k \|t - u\|^2 + \sum_{i=k+1}^{N+1} \zeta_i \prod_{j=k}^{i-1} (1 - \zeta_j) \|v_{i-1} - u\|^2 + \prod_{j=k}^{N+1} (1 - \zeta_j) \|v - u\|^2
\]

\[
- \zeta_k \left[ \sum_{i=k+1}^{N} \zeta_i \prod_{j=k}^{i-1} (1 - \zeta_j) \|t - v_{i-1}\|^2 + \prod_{j=k}^{N} (1 - \zeta_j) \|t - v\|^2 \right]
\]

\[
-(1 - \zeta_k) \left[ \sum_{i=k+1}^{N-1} \zeta_i \prod_{j=k}^{i} (1 - \zeta_j) \|v_{i-1} - [\zeta_{i+1} v_j + w_{i+1}]\|^2 \right]
\]

\[
+ \zeta_N \prod_{j=k}^{N} (1 - \zeta_j) \|v^* - v_{N-1}\|^2 \]

(32)

Therefore, it holds for $k, N$ from direct computation.

\[
\|v^* - v_{N-1}\|^2 = \|\zeta_N v_N + (1 - \zeta_{N+1}) v - v_{N-1}\|^2
\]

\[
= \zeta_N \|v_N - v\|^2 + (1 - \zeta_{N+1}) \|t - v\|^2
\]

\[
- \zeta_{N+1} \|v_{N+1} - v\|^2
\]

(35)

Also,

\[
\|v^* - v_{N-1}\|^2 = \|\zeta_N v_N + (1 - \zeta_{N+1}) v - v_{N-1}\|^2
\]

\[
= \|v_{N-1} - [\zeta_N v_N + (1 - \zeta_{N+1}) v]\|^2
\]

(36)

\[
= \|v_{N-1} - [\zeta_N v_N + w_{N+1}]\|^2.
\]
Furthermore,
\[
\| v - u \|^2 = \| \xi_{N+1} v_N + (1 - \xi_{N+1}) v - u \|^2 \\
= \xi_{N+1} \| u - v_N \|^2 + (1 - \xi_{N+1}) \| v - u \|^2 \\
- \xi_{N+1} (1 - \xi_{N+1}) \| v_N - v \|^2.
\]  
(37)

It then follows from (34–37) that
\[
\| y - u \|^2 = \xi_k \| t - u \|^2 + \sum_{i=k+1}^{N+1} \xi_i \| y_{i-1} - u \|^2 \\
+ \prod_{j=k}^{N+1} (1 - \xi_j) \| v - u \|^2 \\
- \xi_k \sum_{i=k+1}^{N+1} \xi_i \| y_{i-1} - u \|^2 + \prod_{j=k}^{N+1} (1 - \xi_j) \| t - v \|^2 \\
- (1 - \xi_k) \sum_{i=k+1}^{N+1} \xi_i \| y_{i-1} - u \|^2 - \xi_{N+1} \prod_{j=k}^{N+1} (1 - \xi_j) \| v_N - v \|^2.
\]  
(38)

We now apply Propositions 2 and 3 to prove the following weak and strong convergence theorems for type-one demicontractive mappings.

\[x_{n+1} = \xi_{N+1} x_n + \sum_{i=2}^{N} \xi_i \prod_{j=1}^{i-1} (1 - \xi_j) y_{ni-1} \]
\[+ \prod_{j=1}^{N} (1 - \xi_j) y_{nN}, \quad n \geq 1,
\]  
converges weakly to \( q \in \bigcap_{i=1}^{N} F(S_i) \), where \( y_{ni} \in S_i x_n \) for each \( i \) and \( \{\{\xi_{ni}\}_{i=1}^{N}\} \) is a countable finite family of real sequences in \([0, 1]\) satisfying the following:

(i) \( \xi_{n1} > \lambda > \max\{\lambda_i\}_{i=2}^{N}; \xi_{nj} < \zeta < 1 \), for each \( i \).

(ii) \( \liminf_{n \rightarrow \infty} \xi_{n1} \prod_{j=1}^{i-1} (1 - \xi_{nj}) (\xi_{nj} - \lambda_j) > 0 \), \( i = 2, 3, \ldots, N \).

(iii) \( \liminf_{n \rightarrow \infty} \prod_{j=1}^{N} (1 - \xi_{nj}) (\xi_{n1} - \lambda_N) > 0 \).

Also, if, in addition, \( S_i \) is \( L \)-Lipschitzian and satisfies condition (1) for each \( i \), then \( \{x_n\} \) converges strongly to \( q \in \bigcap_{i=1}^{N} F(S_i) \).

Proof. Setting \( x_{n+1} = y_n, \quad x_n = t_n, \quad p = u, \quad k = 1 \), and \( y_{nN} \in S_N x_n = v \) in Proposition 3, we obtain
\[
\| x_{n+1} - p \|^2 = \xi_{n1} \| x_n - p \|^2 + \sum_{i=2}^{N} \xi_i \prod_{j=1}^{i-1} (1 - \xi_j) \| y_{ni-1} - p \|^2 \\
+ \prod_{j=1}^{N} (1 - \xi_j) \| y_{nN} - p \|^2 \\
- \xi_{n1} \sum_{i=2}^{N} \xi_i \prod_{j=1}^{i-1} (1 - \xi_j) \| x_n - y_{ni-1} \|^2 \\
+ \prod_{j=1}^{N} (1 - \xi_j) \| x_n - y_{nN} \|^2.
\]  
(40)

Applying type-one demicontractive condition on each \( S_i \), we obtain

\[
\| x_{n+1} - p \|^2 \leq \xi_{n1} \| x_n - p \|^2 + \sum_{i=2}^{N} \xi_i \prod_{j=1}^{i-1} (1 - \xi_j) \| y_{ni-1} - p \|^2 \\
+ \prod_{j=1}^{N} (1 - \xi_j) \| y_{nN} - p \|^2 \\
- \xi_{n1} \sum_{i=2}^{N} \xi_i \prod_{j=1}^{i-1} (1 - \xi_j) \| x_n - y_{ni-1} \|^2 \\
+ \prod_{j=1}^{N} (1 - \xi_j) \| x_n - y_{nN} \|^2.
\]  
(41)
Consequently, if we set \( k = 1 \) in Proposition 2, we obtain
\[
\|x_n - p\|^2 - \|x_n - p\|^2 - \sum_{i=2}^{N} c_{n,i} \sum_{j=1}^{i-1} (1 - c_{n,j})(c_{n,1} - \lambda_{i-1}) \cdot \|x_n - y_{n,i-1}\|^2
+ \|x_n - y_{n} - y_{n,i}\|^2.
\]

Similarly, conditions (ii) and (iii) imply that the\(\lim\) construction of algorithms for approximating a common solution of an equilibrium problem and fixed point problem.

We now present the application of (Theorem 5 in the construction of algorithms for approximating a common solution of an equilibrium problem and fixed point problem.

**Lemma 2** (see [31]). Let \( C \) be a nonempty convex closed subset of a real Hilbert space \( H \) and \( F: C \times C \rightarrow \mathbb{R} \), a bifunction satisfying (A1)–(A4). Let \( r > 0 \) and \( g \in H \). Then, there exists \( z \in C \) such that
\[
F(z, h) + \frac{1}{r} \langle h - z, z - g \rangle \geq 0, \quad \forall h \in C.
\]

**Lemma 3** (see [32]). Let \( C \) be a nonempty convex closed subset of a real Hilbert space \( H \). Assume that \( F: C \times C \rightarrow \mathbb{R} \) satisfies (A1)–(A4). Let \( r > 0 \) and \( g \in H \). Define \( T_r: H \rightarrow 2^C \) by
\[
T_r(g) = \left\{ z \in C: F(z, h) + \frac{1}{r} \langle h - z, z - g \rangle \geq 0 \right\}, \quad \forall h \in C.
\]

Then, the following hold:
1. \( T_r \) is single valued.
2. \( T_r \) is firmly nonexpansive, that is, for any \( g, h, h' \in H \),
   \[
   \|T_r g - T_r h'\| \leq \langle T_r g - T_r h, h - h' \rangle.
   \]
3. \( F(T_r) = EP(F) \).
4. \( EP(F) \) is convex and closed.

**Lemma 4** (see [33]). Let \( C \) be a nonempty convex closed subset of a real Hilbert space \( H \) and \( F: C \times C \rightarrow \mathbb{R} \), a bifunction satisfying (A1)–(A4). Let \( r > 0 \) and \( g \in H \). Then, for all \( g \in H \) and \( p \in F(T_r) \),
\[
\|p - T_r g\|^2 + \|T_r g - p\|^2 \leq \|p - g\|^2.
\]

**Lemma 5.** Let \( H \) be a real Hilbert space, and let \( C \) be a nonempty convex closed subset of \( H \). Let \( P_C \) be the convex projection onto \( C \). Then, convex projection is characterized by the following relations:
1. \( g^* = P_C(g) \Leftrightarrow \langle g - g^*, h - g^* \rangle \leq 0, \quad \forall h \in C \).
2. \( \|g - P_C g\|^2 \leq \|g - h\|^2 - \|P_C g - P_C h\|^2 \).
3. \( \|g - P_C h\|^2 \leq \|g - h\|^2 - \|P_C h - h\|^2 \).

Motivated by Algorithm 19 of Isiogugu et al. [34], we obtain the following result using a selection of Algorithm 4.2 above in the sense of [34].

**Theorem 6.** Let \( C \) be a nonempty convex closed subset of a real Hilbert space \( H \), \( f_i: C \times C \rightarrow \mathbb{R} \), a bifunction satisfying (A1)–(A4) and \( \{T_i\}_{i=1}^{N} \) be such that \( T_i: C \rightarrow P(C) \) is type-one \( \lambda_i \)-strictly pseudo-contractive-type mappings, and \( (I - T_i) \) is weakly demiclosed at zero for each \( i = 1, 2, \ldots, N \). Suppose that \( F = \bigcap_{i=1}^{N} F(T_i) \cap EP(f) \neq \emptyset \). Let \( \{x_n\} \) be a sequence generated from arbitrary \( x_0 \in C \) as follows:

**Remark 2.** If \( N = 2 \) and we set \( c_{n,1} = c_n \) and \( c_{n,2} = \beta_n \) for all \( n, 1 \) (the identity mapping) \( S_1 \), and \( S = S_2 \), we obtain
\[
x_{n+1} = c_n x_n + \sum_{i=2}^{N} c_{n,i} \sum_{j=1}^{i-1} (1 - c_{n,j})S_{n,i-1} + \sum_{j=1}^{2} (1 - c_{n,j})S_{n,n}
= c_n x_n + (1 - c_n)\beta_n x_n + (1 - c_n)(1 - \beta_n)S x_n
= c_n x_n + (1 - c_n) \beta x_n + (1 - \beta)S x_n.
\]

which was considered by Osilike and Isiogugu [10].

**4. Applications**

We now present the application of Theorem 5 in the construction of algorithms for approximating a common solution of an equilibrium problem and fixed point problem.

For solving the equilibrium problems for a bifunction \( F: C \times C \rightarrow \mathbb{R} \), let us assume that \( F \) satisfies the following conditions:

(A1): \( F(g, g) = 0 \) for all \( g \in C \)
(A2): \( F \) is monotone, that is, \( F(g, h) + F(h, g) \leq 0 \), for all \( g, h \in C \)
(A3): for each \( g, h, z \in C \), \( \lim_{t \downarrow 0} F(tz + (1 - t)g, h) \leq F(g, h) \)
(A4): for each \( g \in C \), \( h \rightarrow F(g, h) \) is convex and lower semicontinuous
Algorithm 1.

\[
\begin{aligned}
\ y_n &= \varsigma_{n_1}x_n + \sum_{i=1}^{N} \varsigma_{n_i} \prod_{j=1}^{i-1} (1 - \varsigma_{n_j})y_{n_{j-1}} + \sum_{i=1}^{N} (1 - \varsigma_{n_i})y_{n,N}, \\
\ u_n &\in K \text{ such that } F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - y_n \rangle \geq 0, \quad \forall y \in K, \\
\ x_{n+1} &= \frac{1}{2}(u_n + x_n),
\end{aligned}
\]

where \( y_{n,i} \in T_i x_n \) for each \( i \) and \( \{\varsigma_{n_i}\}_{i=1}^{\infty} \) is a finite family of real sequences in \([0, 1]\) for each \( i \) satisfying

\[
\|x_{n+1} - p\|^2 = \frac{1}{2} (|x_n + u_n) - p\|^2
\]

\[
= \frac{1}{2} \|x_n - p\|^2 + \frac{1}{2} \|u_n - p\|^2 - \frac{1}{4} \|x_n - u_n\|^2
\]

\[
\leq \frac{1}{2} \|x_n - p\|^2 + \frac{1}{2} \|y_n - p\|^2 - \frac{1}{4} \|x_n - u_n\|^2
\]

\[
= \frac{1}{2} \|x_n - p\|^2 - \frac{1}{4} \|x_n - u_n\|^2 + \frac{1}{2} \varsigma_{n,i} x_n + \sum_{i=2}^{N} \varsigma_{n,i} \prod_{j=1}^{i-1} (1 - \varsigma_{n_j})y_{n_{j-1}} + \prod_{j=1}^{N} (1 - \varsigma_{n,j})y_{n,N} - p\|^2
\]

\[
\leq \frac{1}{2} \|x_n - p\|^2 - \frac{1}{4} \|x_n - u_n\|^2 + \frac{1}{2} \left\| \sum_{i=2}^{N} \varsigma_{n,i} \prod_{j=1}^{i-1} (1 - \varsigma_{n_j})\langle x_n - y_{n_{j-1}} \rangle \|x_n - y_{n_{j-1}}\|^2 \right\|
\]

\[
= \frac{1}{4} \|x_n - u_n\|^2 + \|x_n - p\|^2 - \frac{1}{2} \left\| \sum_{i=2}^{N} \varsigma_{n,i} \prod_{j=1}^{i-1} (1 - \varsigma_{n_j})\langle x_n - y_{n_{j-1}} \rangle \|x_n - y_{n_{j-1}}\|^2 \right\|
\]

\[
\left(1 - \varsigma_{n,i}\right)\left(\varsigma_{n,1} - \lambda_{N}\right)\|x_n - y_{n,N}\|^2.
\]

\[
\text{Thus, from (i), (ii), and (iii), we have that } \lim_{n \to \infty} \|x_n - p\| = 0, \text{ for all } i = 1, 2, \ldots, N. \text{ Furthermore, } \lim_{n \to \infty} \|x_n - u_n\| = 0. \text{ Consequently, } \lim_{n \to \infty} \|x_{n+1} - x_n\|^2 = \lim_{n \to \infty} \|x_n - u_n\|^2 = 0 \text{ which implies that } \{x_n\} \text{ is a Cauchy sequence in } K. \text{ Also, since } K \text{ is convex and closed, } \{x_n\} \text{ converges strongly to some } q \in K. \text{ From the Opial condition of } H \text{ and the demiclosedness property of } T_i, \text{ we have that } q \in T_i q, \text{ for all } i = 1, 2, \ldots, N.
\]

The remaining part of the proof is similar to the method of [34], Theorem 20. Therefore, it is omitted.
5. Examples

We present the numerical computation of the iteration scheme of Theorem 5.

Let $H = (\mathbb{R}^m, \|\cdot\|, \leq) \to \mathbb{R}^m$ with the usual norm $\|\cdot\|$ on $\mathbb{R}^m$ and partial order $\leq$ on $\mathbb{R}$, $C = \{x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : x_1 = x_2 = \ldots = x_t = \ldots = x_m\}$. Observe that $(C, \|\cdot\|, \leq)$ is a convex closed linear totally ordered subset of $\mathbb{R}^m$ with $a \leq b$ if and only if $a_i \leq b_i$ for all $i = 1, 2, 3, \ldots, m$. Denote the order interval $[a, b] = [\min(a, b), \max(a, b)]$, and let $\{S_i\}_{i=1}^m$ be a countable infinite family of mappings and $S_i : C \to CB(C)$ define for each $i$ and $x \in C$ by

$$S_i x = \left\{ \begin{array}{ll}
\left\lfloor \frac{4i}{2i+1} \right\rfloor x, & x \geq \bar{b}, \\
\left\lfloor \frac{3i}{2i+1} \right\rfloor x, & x < \bar{b}.
\end{array} \right.$$

(50)

Clearly, for each $i$,

(1) $F(S_i) = \{\emptyset\}$.

(2) $P_S \bar{x} = \{-(3i/(2i+1))\bar{x}\}$.

$$\mathcal{D}(S, x, S, y) = \begin{cases}
\frac{4i}{2i+1} \|x - y\|, & x, y \geq \bar{b}, \\
\frac{4i}{2i+1} \|x - y\|, & x, y < \bar{b}, \\
\left\| \frac{3i}{2i+1} x - \frac{4i}{2i+1} \right\|, & x \geq \bar{b}, y < \bar{b}.
\end{cases}$$

(51)
(III) \( \| u - v \| = (3i/(2i + 1))\| x - y \| \leq \mathcal{O}(S_i, S_i, y) \), for all \( u \in P_{S_i} x, v \in P_{S_i} y \).

(IV) \( \cap_{i=1}^N F(S_i) = \emptyset \).

(V) \( d^2(x, S, y) = \| x - (-3i/(2i + 1))x \|^2 = \| x + (3i/(2i + 1))x \|^2 = (5i + 1)/(2i + 1))\| x \|^2 \).

(VI) \( d(x, F(S_i)) = d(x, \emptyset) = \| x \| \).

(VII) \( H^2(S_i, x, S_0) = \| ((4i/2i + 1))x \|^2 = \| x \|^2 + \left( ((4i/2i + 1))^2 - 1 \right)\| x \|^2 = \| x \|^2 + (12 (i)^2 - 4i - 1/(2i + 1)^2)\| x \|^2 \).

It then follows from (V) and (VII) that (VIII) \( H^2(S_i, x, S_0) = \| x - \emptyset \|^2 + (12 (i)^2 - 4i - 1/25 (i)^2 + 10i + 1)\| x \|^2 \).

Also, from (V) and (VI), we obtain that…

Figure 1: Errors vs. iteration numbers (n): case 1a (a); case 1b (b); case 2a (c); case 2b (d).
(IX) $d(\mathfrak{x}, S\mathfrak{x}) \geq f(\mathfrak{x}, F(S))$, where $f: [0, \infty) \rightarrow [0, \infty)$ is defined by $f(r) = r$.

In summary, for each $i$, we have from (III), (VIII), and (IX) that $S_i$ is type-one demicontractive mapping with contraction coefficient $\lambda_i = (12(i)^2 - 4i - 125) + 10i + 1$ and satisfies condition (1).

Observe that $\sup_{j} |\lambda_j| = 12(25) = \lim_{i \to \infty} \lambda_i$. Therefore, if we set $-3i/(2i + 1)\mathfrak{x}_n = \mathfrak{y}_{ni} \in S_i\mathfrak{x}_n$ and define \( \{c_{ni}\}_{i=1}^{\infty} \subseteq (0, 1) \) by
\[
\rho_{ni} = \frac{38(ni)^2 + 37}{50(ni)^2 + 1} \tag{52}
\]
then
(i) $\rho_{ni} > (37/50) > (12(25)) = \sup_{j} |\lambda_j|$, 
\[
\rho_{ni} < (37/50) \times 1.
\]
(ii) $\lim_{n \to \infty} \rho_{ni} \prod_{j=1}^{i-1} (1 - \rho_{nj}) (\rho_{nj} - 1) = \lim_{n \to \infty} \rho_{ni} \prod_{j=1}^{i-1} (1 - \rho_{nj}) (\rho_{nj} - 1) = (38/50) (1 - (38/50))^{-1} (38/50 - 1) > 0$, 
\[
2 \leq i < N - 1.
\]
(iii) $\lim_{n \to \infty} \prod_{j=1}^{N} (1 - \rho_{nj}) (\rho_{nj} - \lambda_{ij}) = \lim_{n \to \infty} \prod_{j=1}^{N} (1 - \rho_{nj}) (\rho_{nj} - \lambda_{ij}) = (1 - (38/50))^{N} (38/50 - 0) > 0$.

Table 1 and Figure 1 show the sequences for $N = 5$ and $N = 10$. The values are rounded up to 9 decimal places.

6. Conclusion

A horizontal iteration scheme for the approximation of a common fixed point of a finite family of mappings is introduced in a real Hilbert space. This algorithm does not require the imposition of sum = 1 on the control sequences. Its applicability in developing other algorithms is demonstrated in Algorithm 1. Furthermore, its computability is also exhibited in our numerical computations presented in Section 5.

Data Availability

All data generated or analyzed during this study are included in this published article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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