Tridiagonal representation approach in quantum mechanics

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Received 19 February 2019, revised 22 May 2019
Accepted for publication 19 July 2019
Published 5 September 2019

Abstract

We present an algebraic approach for finding exact solutions of the wave equation. The approach, which is referred to as the tridiagonal representation approach, is inspired by the \(J\)-matrix method and based on the theory of orthogonal polynomials. The class of exactly solvable problems in this approach is larger than the conventional class. All properties of the physical system (energy spectrum of the bound states, phase shift of the scattering states, energy density of states, etc) are obtained in this approach directly and simply from the properties of the associated orthogonal polynomials.

Keywords: tridiagonal representation, orthogonal polynomials, recursion relation, asymptotics, energy spectrum, phase shift

(Some figures may appear in colour only in the online journal)

1. Introduction

A full understanding of the features and behavior of a physical submicroscopic system requires knowledge of the exact solution of the corresponding wave equation. This is especially evident when the system is in a critical state such as at phase transitions, singular limits, strong coupling, or when the physical parameters assume critical values, etc. In such circumstances, one may not be able to achieve full or correct understanding of the system by using numerical solutions. Now, since most of the quantum mechanical systems and physical processes are modeled by potential functions, then exact solutions of the wave equation with as many potential functions as possible remained as one of the prime challenges since the early inception of quantum mechanics. Additionally, exact solutions could also be used to test the accuracy and convergence of computational routines that were developed to obtain numerical solutions for complicated physical systems. Nonetheless, the class of exactly solvable potentials for a given wave equation (e.g. Schrödinger equation, Dirac equation, etc) is indeed very limited. In the physics and mathematics literature, the potential functions in this class are well known and their exact solutions are well established. For example, the class of exactly solvable potentials in the non-relativistic Schrödinger equation includes the Coulomb, harmonic oscillator, Morse, Pöschl–Teller, Scarf, Eckart, etc.

Several methods for obtaining exact solutions of the wave equation were developed over time. Most agree very well in their exact results but differ in accuracy and convergence when used to obtain approximations or numerical solutions of problems that are not exactly solvable. Among these methods, we mention supersymmetry [1], shape invariance [2], group theory [3–5], factorization [6, 7], asymptotic iteration [8, 9], point canonical transformation [10], path integral [11], Nikiforov–Uvarov [12], etc. Some of these are equivalent to each other but most (if not all) address the same class of exactly solvable potentials. Moreover, some are analytic methods while others are algebraic. The latter are usually preferred in numerical calculations especially if the former causes instabilities. The tridiagonal representation approach (TRA) was introduced in 2005 to obtain exact solutions of the wave equation [13]. It is an algebraic method based on the theory of orthogonal polynomials and was inspired by the \(J\)-matrix method [14]. It has been applied to relativistic as well as non-relativistic problems and in several spatial dimensions with separable potentials. The class of exactly solvable potentials in the TRA turns out to be larger than the conventional class. It includes new potential...
functions and generalizations of known ones. These potentials correspond to orthogonal polynomials that were not treated in the mathematics literature before. As an example of such potentials, we mention the infinite square potential well with sinusoidal rather than flat bottom. Another example, is a generalization of the hyperbolic Pöschl–Teller potential obtained by adding a term of the form $\tanh^2(x) / \cosh^2(x)$. A partial list of these new or generalized potentials is given in table 1 of [15]. Some of these new solutions lead to interesting applications in atomic, molecular and nuclear physics. As examples, we mention the anion problem where an electron becomes bound to a neutral molecule with an electric dipole moment [16, 17], the binding of a charged particle to an electric quadrupole in two-dimensions [18], energy density bands engineering [19], electric dipole and quadrupole contributions to valence electron binding in a charge-screening environment [20], etc. In applied mathematics, the TRA was also used to obtain series solutions of new types of ordinary differential equations of the second order with three and four singular points [21–23].

The TRA is now well established with proven success as an efficient analytical calculation tool and with rich mathematical underpinnings. Its foundation is built on the theory of spectral operators and the equivalence of special continuous systems (described by differential equations) to discrete systems (described by recursion relations). In the mathematics literature (e.g. the book by Atkinson [24]), the boundary value problem is cast as solutions of differential equation for continuous systems or solutions of difference equation (e.g. recursion relation) for discrete systems. Limited literature is devoted to the unification or equivalence of both. That is, to describe the system (whether continuous or discrete) by both solutions and establish the equivalence. Up to now, this has been done only for special boundary value problems most of which are of high significance in physics. The general theory of such unification/equivalence, which has its origin in the spectral theory of operators, is far from being well developed. In our approach (TRA), we studied some of these special physical systems that are described by differential equations (e.g. the Schrödinger wave equation for non-relativistic systems) and map them into an equivalent discrete system consisting of orthogonal polynomials that satisfy recursion relations. All features of the physical system (e.g. energy spectrum, phase shift, density of state, etc) are obtained from the properties of these polynomials (e.g. weight function, asymptotics, zeros, etc). Nonetheless, the TRA can handle only special systems but despite that, we were able to enlarge the conventional class of integrable physical systems and add few more with interesting applications. In the following section, we start by introducing and formulating the TRA and explain its two working modes.

2. Formulation of the TRA

The general spectrum of a quantum mechanical system is a mix of continuous and discrete energy states. The complete space–time wavefunction that represents such a system could be written in terms of its continuous and discrete Fourier components, in a standard notation, as follows

$$\Psi(x, t) = \int_{\Omega} e^{-iE_t} \psi(E, x) dE + \sum_{k} e^{-i\xi_k t} \psi_k(x),$$  \hspace{1cm} (1)

where $\Omega$ is the set of continuous scattering energy interval(s) and $\{E_k\}$ is the countably infinite or finite set of discrete bound state energies. The continuous and discrete wavefunction components $\psi(E, x)$ and $\psi_k(x)$ could be considered as elements in an infinite dimensional vector space spanned by the basis vectors $\{\phi_n(x)\}$. In physics, we are accustomed to writing vector quantities (e.g. force, velocity, electric field, etc) in terms of their components (i.e. projections on some conveniently chosen basis unit vectors in the space). For example, the force $\vec{F}$ is written in three-dimensional space with Cartesian coordinates as $\vec{F} = \hat{x}_f f_x + \hat{y}_f f_y + \hat{z}_f f_z$, where $\{\hat{x}_f, \hat{y}_f, \hat{z}_f\}$ are the projections of the force along the unit vectors $\{\hat{x}, \hat{y}, \hat{z}\}$. These projections contain all information about the physical quantity whereas the unit vectors (basis) are dummy, but must form a complete set to allow for a faithful representation of the vector quantity. In analogy, we write the wavefunction components $\psi(E, x)$ and $\psi_k(x)$ as a series expansion in terms of the complete set $\{\phi_n(x)\}$ of local unit vectors (square integrable basis functions in configuration space). That is, we write

$$|\psi(E, x)\rangle = \sum_n f_n(E) |\phi_n(x)\rangle, \hspace{1cm} (2a)$$

$$|\psi_k(x)\rangle = \sum_n g_n(k) |\phi_n(x)\rangle, \hspace{1cm} (2b)$$

where $f_n(E)$ and $g_n(k)$ are the projections of the corresponding wavefunction component on the basis unit vector $\phi_n(x)$. That is, $f_n(E) = \langle \phi_n(x) | \psi(E, x) \rangle$ and $g_n(k) = \langle \phi_n(x) | \psi_k (x) \rangle$, where $\phi_n(x)$ is the conjugate of $\psi_k(x)$ defined so that $\langle \phi_n(x) | \phi_m(x) \rangle = \delta_{nm}$. All physical information about the system (structure and dynamics) are contained in these projections. An alternative approach that leads to the same expansion of the wavefunction (2) is given in appendix A. It also shows how to choose the proper basis set $\{\phi_n(x)\}$.

Faithfulness of the representation of the system by the total wavefunction $\Psi(x, t)$ means that its space correlation over all times is as follows

$$\int \Psi(x, t) \overline{\Psi}(y, t) dt = \delta(x - y),$$  \hspace{1cm} (3)

where $\overline{\Psi}(y, t)$ is the conjugate wavefunction, which is obtained from (1) and (2) by $i \rightarrow -i$ and $\phi_n(x) \rightarrow \overline{\phi}_n(x)$. Using the fact that the discrete energy spectrum and the continuous energy interval(s) are distinct (do not overlap), we conclude that

$$\int e^{i\alpha(E - E')} dE \equiv 0. \hspace{1cm}$$

Thus, the overlap integral between the discrete and continuous energy components in equation (3) vanishes and we obtain

$$\sum_{n,m} \phi_n(x) \overline{\phi}_m(x) \left[ \int f_n^2(E) P_n(E) P_m(E) dE \right] + \sum_k g^2_n(k) Q_n(k) Q_m(k) = \delta(x - y), \hspace{1cm} (4)$$

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where we have written, in anticipation of future convenience, 
\( f_0(E) = f_0(E)P_0(E) \) and \( g_0(k) = g_0(k)Q_0(k) \) making \( P_0(E) = Q_0(k) = 1 \). Consequently, the completeness of the basis, which reads \( \sum_n \phi_n(x) \phi_n(y) = \delta(x - y) \) dictates that
\[
\int_{\Omega} \rho(E) P_n(E) P_m(E) dE + \sum_k \omega(k) Q_n(k) Q_m(k) = \delta_{nm}, \tag{5}
\]
where \( \rho(E) = f_0^2(E) \) and \( \omega(k) = g_0^2(k) \) are positive definite entire functions. This represents the completeness formula for the set \( \{ P_n(E), Q_n(k) \} \). Therefore, we can write the series expansion (2) as follows
\[
|\psi(E, x)| = \sqrt{\rho(E)} \sum_n P_n(E) |\phi_n(x)|, \tag{6a}
\]
\[
|\psi_k(x)| = \sqrt{\omega(k)} \sum_n Q_n(k) |\phi_n(x)|. \tag{6b}
\]

All physical information (both structural and dynamical) about the system are obtained from the properties of the set \( \{ P_n(E), Q_n(k) \} \). The initial values \( P_0(E) = Q_0(k) = 1 \) and orthogonality relation (5) suggest that this set represents a complete set of orthogonal polynomials with continuous and discrete elements \( \{ P_n(E) \} \) and \( \{ Q_n(k) \} \) whose corresponding weight functions are \( \rho(E) \) and \( \omega(k) \), respectively. It turns out that the type of orthogonal polynomials representing the physical system depends on the structure of its energy spectrum: purely continuous, purely discrete or a mix of both and on whether the discrete energy spectrum is infinite or finite. Table 1 shows this relationship with examples. These premonitions will be confirmed by the ensuing development below.

To simplify the presentation of the TRA to a general audience of quantum mechanics including undergraduate students, we limit the discussion in the following two sections (sections 3 and 4) to physical systems with either pure discrete energy bound states or pure continuous energy scattering states. For the more advanced readers, however, we give in section 5 an example of a system with mixed energy spectrum. Now, let the Hamiltonian operator of the system in configuration space be \( H(x) = T(x) + V(x) \), where \( T \) is the kinetic energy operator and \( V \) is the potential function. The wave equation is \( i\hbar \frac{\partial}{\partial t} \Psi(x, t) = H(x) \Psi(x, t) \) and thus the continuous or discrete Fourier components of the wave equation read as follows
\[
H(x)|\psi(x)| = E|\psi(x)|, \tag{7}
\]
where \( E \) stands for either continuous energies corresponding to the scattering states \( \psi(E, x) \) or discrete energies \( E_k \) corresponding to the bound states \( \{ \phi_n(x) \} \). Inserting the series expansion (6) in the wave equation (7) and projecting from left by \( \langle \phi_n(x) | \) result in the following equivalent matrix equation for the orthogonal energy polynomials
\[
\sum_m \{ \langle \phi_m | H |\phi_n \rangle - E \langle \phi_n |\phi_m \rangle \} P_{m}(E) = 0, \tag{8}
\]
where \( \mu \) stands for a set of physical parameters contained in the Hamiltonian. This equation could be rewritten as \( \sum_m J_{n,m}(E) P_m(E) = 0 \), where \( J_{n,m}(E) \) is the matrix representation of the wave operator \( J(x) = H(x) - E \) in the \( L^2 \) basis \( \{ \phi_n(x) \} \). Now, we come to the most crucial step in the TRA and that is to impose the condition on the basis that it must produce a tridiagonal matrix representation for the wave operator \( J(E) \). Moreover, since the Hamiltonian is Hermitian then the matrix \( J(E) \) should also be symmetric. That is, the matrix wave equation (8) must read
\[
J_{n,n-1}(E) P_{n-1}^\mu(E) + J_{n,n+1}(E) P_{n+1}^\mu(E) + J_{n,n}(E) P_{n}^\mu(E) = 0. \tag{9}
\]
This could always be reparametrized and rewritten as follows:
\[
z c_n P_n^{\mu}(z) = a_n P_n^{\mu}(z) + b_{n-1} P_{n-1}^{\mu}(z) + b_n P_{n+1}^{\mu}(z), \tag{10}
\]
where \( n = 1, 2, 3, \ldots \) and \( z \) is some proper function of the energy \( E \) and the set of physical parameters \( \mu \). If the basis elements are orthonormal (i.e., \( \langle \phi_n |\phi_m \rangle = \delta_{nm} \)) then \( z = E \) and \( c_n = 1 \) but this is not always the case. The coefficients \( a_n, b_n, c_n \) depend on \( \mu \) and \( n \) but are independent of \( z \) and such that \( b_n^2 > 0 \) for all \( n \). Equation (10) is, in fact, a recursion relation that makes \( P_n^{\mu}(z) \) a polynomial of degree \( n \) in \( z \) and determines all of them explicitly to any degree starting with the initial values \( P_0^{\mu}(z) = 1 \) and \( P_1^{\mu}(z) = (z c_0 - a_0) / b_0 \). Therefore, using the theory of orthogonal polynomials we can assert the following properties of the set \( \{ P_n(E), Q_n(k) \} \) introduced above in equations (4) and (5)

1. \( \{ P_n(E) \} \) constitutes a set of orthogonal polynomials with continuous spectrum whose discrete version is \( \{ Q_n(k) \} \).
2. Equation (5) is the generalized orthogonality relation with $\rho(E)$ and $\omega(k)$ being the continuous and discrete weight functions associated with $\{P_n(E)\}$ and $\{Q_n(k)\}$, respectively.

An example of orthogonal polynomial with purely continuous spectrum is the Meixner–Pollaczek polynomial whose discrete version is the Meixner polynomial (or Krawtchouk polynomial) with infinite (or finite) discrete spectrum [25]. Moreover, an orthogonal polynomial with a mix of continuous and discrete spectra whose generalized orthogonality relation has the same structure as equation (5) is the Wilson polynomial [25]. It is evident that if the system represented by the complete wavefunction $\psi(x, t)$ possesses only continuous energy scattering states, then the sum part of the orthogonality relation (5) disappears. However, if the system consists only of discrete energy bound states then the integral part of the orthogonality relation (5) disappears. These findings are supported by studies that established the connection between scattering theory and orthogonal polynomials [26–28] and by the recent extensive work on the formulation of quantum mechanics based on the theory of orthogonal polynomials (see, for example, [15, 29–31]).

The well-established connection between scattering and orthogonal polynomials dictates that out of all orthogonal polynomials that satisfy the above requirements, the physi-

cally relevant ones that enter in the expansion of the wave-
function (6a) are those with the following asymptotic ($n \to \infty$) behavior

$$P_n^\mu(z) \approx n^{-\tau}A_n^\mu(z)\{\cos[n^2 \theta(z)] + \varphi(z)\log n + \delta^\mu(z)] + O(n^{-1})\},$$

(11)

where $\tau$ and $\xi$ are real positive constants that depend on the particular energy polynomial. In most cases, either $\varphi(z) = 0$ or $\theta(z) = 0$. The studies in [15, 26–31] show that $A_n^\mu(z)$ is the scattering amplitude and $\delta^\mu(z)$ is the phase shift. Bound states, if they exist, occur at discrete real energies $E(z_0)$ that make the scattering amplitude vanish, $A_n^\mu(z_0) = 0$. The number of these bound states is either finite or infinite.

The main task in the TRA is to find the energy pol-

nomials with continuous and/or discrete spectrum associated with the given physical system because (as noted above and restated here for emphasis) all physical information about the system are obtained from the properties of these polynomials. There are two working modes of the TRA. In the first one, a potential function that models the given system is provided and the TRA is used to obtain the corresponding exact solution (i.e. the orthogonal energy polynomial). In the second mode, a general physical configuration is given and the TRA is used to obtain the class of all exactly solvable potential functions that are compatible with the configuration. For the first mode, we search for a complete square integrable basis set $\{\phi_j\} \text{ that satisfies the boundary conditions and supports a symmetric tridiagonal matrix representation for}$

the wave operator with the given potential. There is no prior guarantee that the task will be successful, but if it is, then we derive or identify the orthogonal polynomial that satisfies the corresponding three-term recursion relation. On the other hand, for the second mode we start by choosing a proper general basis that is compatible with the given physical configuration then compute $\langle \phi_i | \{T \phi_j\} \rangle$. Afterwards, we find all possible potential functions, $V(x)$, that leave the matrix representation of the wave operator, $\langle \phi_i | \{T \phi_j\} \rangle + \langle \phi_i | V | \phi_j \rangle = E \langle \phi_i | \phi_j \rangle$, triagonal. Finally, for each potential function in the class we derive or identify the orthogonal polynomial that satisfies the resulting three-term recursion relation. Subsequently, all properties of the physical system corresponding to a given potential function in the class are derived from the properties of the associated orthogonal polynomial. For example, we can use the asymptotics formula (11) to obtain the scattering amplitude and phase shift for the continuum scattering states as well as the energy spectrum for the discrete bound states. For clarity and simplicity of the presentation, we leave out details of the calculation but interested readers can find such details in [21, 32].

We should note that all new and generalized potential functions that are solved exactly using the TRA but have no exact solution using the conventional methods correspond to orthogonal polynomials that are either new or modified/ generalized versions of known polynomials. Most of these polynomials were not studied in the mathematics literature and some still constitute an open problem in orthogonal polynomials [33, 34]. In such cases and due to the absence of the analytic properties of these polynomials (such as the weight function, asymptotic formula, etc), we are forced to resort to numerical means to obtain the physical properties of the system.

In section 3, we present an example of how to use the TRA in the first mode of its application and obtain an exact solution for a conventional potential function (the Coulomb). For simplicity, we consider only scattering where the energy spectrum is purely continuous. In section 4, we obtain a solution of a non-conventional potential with purely discrete energy spectrum: the infinite square well with sinusoidal rather than flat bottom. In section 5 and for the more advanced readers, we present a system with mixed spectrum consisting of continuous scattering states and discrete bound states. In appendix B, we present an example of how to use the TRA in the second mode of its application and obtain the class of all exactly solvable potentials corresponding to a given physical configuration. Some of these systems are exactly solvable only by using the TRA.

### 3. TRA solution for a conventional potential function

To ease the way into the TRA, we start by showing how the approach could be used in its first mode of application to obtain the exact solution of a well-known problem; the con-

tinuous energy scattering states of the Coulomb problem with $V(r) = Ze^{-r}$, where the electric charge number $Z$ is positive. In the atomic units $\hbar = m = \frac{e^2}{4\pi\epsilon_0} = 1$, the radial Schrödinger
wave equation for this problem reads as follows

$$\left[-\frac{1}{2} \frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} + \frac{Z}{r} - E\right] \psi(E, r) = 0,$$

(12)

where $\ell$ is the angular momentum quantum number. This equation has a regular singularity at $r = 0$ and an irregular (essential) singularity at infinity. Using this fact and that $r > 0$, the analysis in appendix A shows that a proper choice of a complete set of square integrable basis that satisfy the boundary conditions is

$$\phi_n(r) = A_n y^{\alpha} e^{-\gamma r^2/2} L_n^{\alpha}(y),$$

(13)

where $y = \lambda r$, $L_n^{\alpha}(y)$ is the associated Laguerre polynomial, $A_n$ is a normalization constant, $\lambda$ is a positive length scale parameter and $(\alpha, \nu)$ are real dimensionless parameters such that $\alpha > 0$ and $\nu > -1$. Using the differential equation and differential property of the Laguerre polynomial, we obtain the following action of the wave operator on the basis

$$J[\phi_n] = \frac{\lambda^2}{2} \left[ n + \nu + 1 - 2\alpha \right] + \frac{\ell(\ell + 1) - \alpha(\alpha - 1)}{y^2}
+ \frac{\alpha + 2\gamma}{y} - \varepsilon - \frac{1}{4} |\phi_n|,
+ \frac{\lambda^2}{2} \left[ (2\alpha - \nu - 1) \right] \frac{n + \nu}{y^2}
\frac{A_n}{A_{n-1}} |\phi_{n-1}|),
$$

(14)

where $\gamma = Z/\lambda$ and $\varepsilon = (\kappa/\lambda)^2$ with $\kappa$ being the linear momentum $\kappa^2 = 2E$. The recursion relation of the Laguerre polynomial shows that the matrix representation of the wave operator $\langle \phi_m | J[\phi_n] \rangle$ is tridiagonal if the terms with $y^{-2}$ in (14) vanish. Thus, we require that $2\alpha = \nu + 1$ and $\nu^2 = (2\ell + 1)^2$ giving $\nu = 2\ell + 1$ and $\alpha = \ell + 1$. Consequently, we obtain

$$\frac{2}{\lambda^2} \left( \langle \phi_m | J[\phi_n] \rangle = (n + \ell + 1 + 2\gamma) \langle \phi_m | y^{-1} |\phi_n \rangle
- \left( \varepsilon + \frac{1}{4} \right) \langle \phi_m | \phi_n \rangle. \right.
$$

(15)

Using the recursion relation of the Laguerre polynomial and its orthogonality along with $A_n = \sqrt{\Gamma(n + 1) / \Gamma(n + 2 + \ell)}$, we obtain

$$\frac{2}{\lambda^2} J_{m,n} = (n + \ell + 1 + 2\gamma) \delta_{m,n}
- \left( \varepsilon + \frac{1}{4} \right) \delta_{m,n}
- \sqrt{n(n + 2\ell + 1)} \delta_{m,n+1}
- \sqrt{(n + 1)(n + 2\ell + 2)} \delta_{m,n-1}.\right.$$

(16)

Therefore, the symmetric three-term recursion relation (9) becomes

$$-2\frac{\gamma}{\varepsilon + 1/4} \frac{P_{n+1}^{\nu}(E)}{2n + 2\ell + 1} = -2(n + \ell + 1) \varepsilon - \frac{1/4}{\varepsilon + 1/4} \frac{P_{n}^{\nu}(E)}{2n + 2\ell + 1} \sqrt{(n + 1)(n + 2\ell + 2)} \frac{P_{n-1}^{\nu}(E)}.\right.$$

(17)

Comparing this to the three-term recursion relation of the normalized version of the Meixner–Pollaczek polynomial $P_n^{\nu}(\zeta, \theta)$, which is given by equation (D2) of appendix D in [32], we can write $P_{n}^{\nu}(E) = P_{n}^{\nu+1}(\zeta, \theta)$, where

$$\cos \theta = \frac{-1/4}{\varepsilon + 1/4} \frac{(2\kappa/\lambda) - 1}{(2\kappa/\lambda) + 1},
\varepsilon = -\gamma - \frac{1/4}{\varepsilon + 1/4} \frac{(2\kappa/\lambda) - 1}{(2\kappa/\lambda) + 1}.\right.$$

(18)

The normalized version of the Meixner–Pollaczek polynomial is written in terms of the hyper-geometric function as given by equation (D1) in appendix D of [32]. Thus, the scattering wave function $(\psi_{0})$ is written as

$$\psi(E, r) = (\lambda r)^{\ell+1/2} e^{-\lambda r^2/2} \sqrt{\varepsilon + 1/4} \frac{(Z/\kappa, \theta)}{\varepsilon + 1/4} \frac{(2\kappa/\lambda) - 1}{(2\kappa/\lambda) + 1},\right.$$

(19)

where $\rho^2(\zeta, \theta)$ is the positive definite weight function associated with the Meixner–Pollaczek polynomial $P_n^{\nu}(\zeta, \theta)$ and given by equation (D3) in appendix D of [32]. In practice, the sum in (19) converges very quickly with the desired accuracy for the first few terms. The asymptotics of $P_n^{\nu}(\zeta, \theta)$ is given by equation (D4) in appendix D of [32]. It has exactly the same form as equation (11) from which the scattering phase shift (modulo $\pi/2$) is obtained directly and effortlessly as

$$\delta(E) = \arg[\Gamma(\ell + 1 - iZ/\kappa)],\right.$$

(20)

which is the well-known scattering phase shift for the Coulomb problem.

4. TRA solution for a non-convensional potential function

Now, we give an example that illustrates how the TRA could be used to solve a problem that is not exactly solvable using the conventional methods. One of the first problems that an under-graduate student is quantum mechanics is asked to solve is the infinite square potential well. Here, we add a twist on this problem and look for its solution when the bottom of the well is not flat but rather sinusoidal. Specifically, we want to solve the problem with the following potential function

$$V(x) = \begin{cases} V_0 \sin(\pi x/L), & -\frac{L}{2} \leq x \leq \frac{L}{2}, \\ \infty, & \text{otherwise}. \end{cases}$$

(21)

Since the system is totally confined in space, then it should be obvious that it has an infinite number of discrete energy bound states $\{\psi_n(x)\}$ and no continuous energy states. With modest efforts, it is not difficult to convince oneself that an exact solution for this potential is not achievable using any of the traditional methods (e.g. supersymmetry, factorization, asymptotic iteration, etc). Now, we employ the TRA and start by writing the corresponding Schrödinger equation in terms of the dimensionless coordinate $y = \sin(\lambda x)$ with $\lambda = \pi/L$ where we obtain

$$\frac{\hbar^2}{\left[ (1 - y^2) \frac{d^2}{dy^2} - y \frac{dy}{dy} - \gamma y + \varepsilon \right] \psi_n(y), \right.$$

(22)

with $\gamma = 2V_0/\lambda^2$, $\varepsilon = 2E/\lambda^2$ and $-1 \leq y \leq +1$. This equation has two regular singularities at $y = \pm 1$. Using these
Table 2. The lowest part of the energy spectrum (in units of $\frac{2}{\lambda^2}$) for the infinite potential well with sinusoidal bottom and for the chosen set of values of the potential parameter $\gamma$. We took the basis (23) with 50 elements.

| $n$ | $\gamma = 0$ | $\gamma = 2$ | $\gamma = 5$ | $\gamma = 10$ | $\gamma = 20$ |
|-----|--------------|--------------|--------------|--------------|--------------|
| 0   | 0.686 720 257 | $-0.595 539 559$ | $-3.622 765 814$ | $-10.838 068 721$ |              |
| 1   | 4.113 008 823 | $4.345 345 170$ | $3.873 494 394$ | $0.432 511 407$ |              |
| 2   | 9.057 352 856 | $9.354 964 694$ | $10.147 416 013$ | $10.358 251 307$ |              |
| 3   | 16.031 789 784 | $16.200 110 073$ | $16.813 060 198$ | $18.778 787 010$ |              |
| 4   | 25.020 212 925 | $25.126 692 366$ | $25.512 098 215$ | $27.111 504 117$ |              |
| 5   | 36.013 989 568 | $36.087 552 002$ | $36.351 914 438$ | $37.436 795 310$ |              |
| 6   | 49.010 257 797 | $49.064 156 865$ | $49.257 285 819$ | $50.040 106 169$ |              |
| 7   | 64.007 843 753 | $64.049 043 706$ | $64.196 465 710$ | $64.790 623 174$ |              |
| 8   | 81.006 192 252 | $81.038 711 488$ | $81.154 988 044$ | $81.622 257 081$ |              |
| 9   | 100.005 012 691 | $100.031 334 558$ | $100.125 413 252$ | $100.502 864 037$ |              |

It should be obvious that the infinite square potential well with flat bottom (where $\gamma = 0$) is a special case of (27) that results in a diagonal representation giving the well-known energy spectrum directly as $E_n = (n + 1)^2$. On the other hand, for $\gamma \neq 0$ we compare this recursion relation to that of the newly introduced orthogonal polynomial $H_{\alpha+\gamma}^{(\mu,\nu)}(z, \sigma)$, which is given as equation (E1) of appendix E in [32]. We then conclude that $P_{\alpha+\gamma}^{(\mu,\nu)}(E) = H_{\alpha+\gamma}^{(\mu,\nu)}(z, \sigma)$ and that the bound state wavefunction (66) is written as

$$
\psi_k(x) = \sqrt{\frac{2}{\lambda}} \cos(\lambda x) \sqrt{\omega_{(\mu,\nu)}^{(\frac{1}{2})}}(\epsilon, \gamma) \times \sum_{n=0}^{\infty} H_{\alpha+\gamma}^{(\mu,\nu)}(\epsilon_k, \gamma) U_n(\sin(\lambda x)),
$$

where $\omega_{(\mu,\nu)}^{(\frac{1}{2})}(z, \sigma)$ is the positive definite weight function associated with the new polynomial $H_{\alpha+\gamma}^{(\mu,\nu)}(z, \sigma)$. Unfortunately, the analytic properties of this polynomial (like its weight function, orthogonality, asymptotics, etc.) are not yet known. This is still an open problem in orthogonal polynomials [33, 34] despite the fact that this polynomial could be written explicitly to any desired degree (albeit not in closed form) using its recursion relation and initial value $H_0^{(\mu,\nu)}(z, \sigma) = 1$. Had the analytic properties of this polynomial been known, we would have easily and directly obtained the physical properties of the present system (e.g. the energy spectrum). Consequently, we are forced to resort to numerical means. In table 2, we give the lowest part of the energy spectrum (in units of $\frac{2}{\lambda^2}$) calculated for a chosen set of values of the potential parameter $\gamma$. Note that for $\gamma = 0$ we obtain the well-known result and for $\gamma \neq 0$ the higher excited states do not feel the sinusoidal structure at the bottom of the well. In figure 1, we plot the lowest un-normalized bound-state wavefunctions corresponding to the column $\gamma = 5$ of table 2. These were calculated using equation (28) where the sum converges quickly for the first 7–8 terms.

In appendix B, we use the second mode of application of the TRA to find the class of all potential functions associated with the physical configuration of the infinite square well and show that the potential well with sinusoidal bottom treated above is only a special case of a larger class (the generalized trigonometric Scarf potential).
In this section, we use the TRA to obtain the exact solution of the Schrödinger equation for a system with mixed spectrum consisting of discrete energy bound states and continuous energy scattering states. We give the condition \((s)\) for the existence of bound states and use the properties of the associated orthogonal energy polynomials to write down analytic expressions for the bound states energy spectrum and the scattering states phase shift. As an illustration, we choose the one-dimensional Morse potential \(V(x) = V_2 e^{2\lambda x} + V_1 e^{\lambda x}\), where \(-\infty < x < +\infty\) and \(\{\lambda, V_2, V_1\}\) are real parameters with \(\lambda\) being a positive length scale. In the atomic units \(\hbar = m = 1\), the corresponding Schrödinger equation in terms of the variable \(y = e^{-\lambda x}\) reads as follows

\[
-\frac{\hbar^2}{2} \left[ y^2 \frac{d^2}{dy^2} + y \frac{d}{dy} - U_2 y^2 - U_1 y + \varepsilon \right] \psi(E, y) = 0,
\]

(29)

where \(U_i = 2V_i / \hbar^2\). This equation has a regular singularity at \(y = 0\) and an irregular singularity at \(y \to +\infty\). Using this fact and that \(y \geq 0\), the analysis in appendix A shows that a proper choice of a complete set of square integrable basis that satisfy the boundary conditions is

\[
\phi_n(y) = A_n y^\alpha e^{-\gamma y/2} L_n^\nu(y).
\]

(30)

The real parameters \((\alpha, \nu)\) are dimensionless such that \(\alpha > 0\) and \(\nu > -1\). Using the differential equation and differential property of the Laguerre polynomial, we obtain the following action of the wave operator on the basis

\[
J|\phi_n\rangle = -\frac{\hbar^2}{2} \left[ y^2 \left( \frac{1}{4} - U_2 \right) - y \left( n + \alpha + \frac{1}{2} + U_1 \right) + n(2\alpha - \nu) + \alpha^2 + \varepsilon \right] |\phi_n\rangle
\]

\[
+ (2\alpha - \nu)(n + \nu) \frac{A_n}{A_{n-1}} |\phi_{n-1}\rangle.
\]

(31)

Using the recursion relation of the Laguerre polynomial, its orthogonality and that \(\langle \phi_m | F(y) | \phi_n \rangle\), for a given function \(F(y)\), is the following integral

\[
\langle \phi_m | F(y) | \phi_n \rangle = \lambda \int_0^{\infty} \phi_m(x) F(y) \phi_n(x) dx
\]

\[
= A_m A_n \int_0^{\infty} y^{2\alpha} e^{-y/2} L_m^\nu(y) F(y) L_n^\nu(y) dy / y,
\]

(32)

where \(A_n = \sqrt{\Gamma(n+1)/\Gamma(n+\nu+1)}\). We can show that equation (31) produces a symmetric tridiagonal matrix representation for \(\langle \phi_m | J | \phi_n \rangle\) in either one of the following two cases

(a) \(2\alpha = \nu, \quad \nu^2 = -4\varepsilon\),
(b) \(2\alpha = \nu + 1, \quad U_2 = \frac{1}{4}\).

(33a), (33b)

The first case is valid for negative energies, which corresponds to a spectrum consisting only of bound states. Therefore, we dismiss this case and consider case (33b) corresponding to the following two-parameter 1D Morse potential

\[
V(x) = \frac{\lambda^2}{8} e^{2\lambda x} + V_1 e^{\lambda x}.
\]

(34)

Putting all the above together enables us to calculate the elements of the tridiagonal matrix representation of the wave
Comparing this with the three-term recursion relation of the normalized version of the continuous dual Hahn polynomial \( S^*_n(z^2; a, b) \), which is given by equation (B4) of appendix B in [15], we can write \( P^*_n(E) = S^*_n(z^{\frac{1}{2}}; \frac{\nu + 1}{2}, \nu + 1) \). The normalized version of the continuous dual Hahn polynomial is written in terms of the hypergeometric function \( F_2 \) as shown by equation (B1) in appendix B of [15]. The properties of the polynomial \( S^*_n(z^2; a, b) \) shows that if \( \tau > 0 \) then it has only a continuous spectrum with \( z^2 > 0 \). However, if \( \tau < 0 \) then the spectrum is a mix of continuous and discrete parts with the discrete part being of finite size \( N \) where \( N \) is the largest integer less than or equal to \( - \tau \). Therefore, if \( V_I > -(\lambda/2)^2 \) (i.e. \( U_I > -\frac{1}{2} \)) then the potential cannot support bound states whereas if \( V_I < -(\lambda/2)^2 \) (i.e. \( U_I < -\frac{1}{2} \)) then it can and the system’s total wavefunction has a continuous energy component and discrete energy components of a finite size as given by equation (1) with the sum being for \( k = 0, 1, ..., N \) where \( N \) is the largest integer less than or equal to \( -U_I - \frac{1}{2} \). The continuous and discrete energy components of the wavefunction are written as

\[
\psi(E, x) = e^{\frac{-i}{2}E}\exp \left( -\frac{1}{2}e^{i\lambda} \right) \sqrt{\rho^*(z^2; \frac{\nu + 1}{2}, \frac{\nu + 1}{2})} \times \sum_{n=0}^{\infty} A_n S^*_n(z^2; \frac{\nu + 1}{2}, \frac{\nu + 1}{2}) L^*_n(\rho^*(z)) ,
\]

\[
\psi_k(x) = e^{\frac{-i}{2}E}\exp \left( -\frac{1}{2}e^{i\lambda} \right) \sqrt{\rho(\frac{\lambda}{2}; \frac{\nu + 1}{2}, \frac{\nu + 1}{2})} \times \sum_{n=0}^{\infty} A_n S^*_n(z^2; \frac{\nu + 1}{2}, \frac{\nu + 1}{2}) L^*_n(\rho(z)),
\]

where \( \rho(z^2; a, b) \) is the continuous part of the weight function associated with \( S^*_n(z^2; a, b) \) and given by equation (B2) in appendix B of [15]. On the other hand, \( \omega(z^2; a, b) \) is the discrete part of the weight function obtained from formula (B3) in appendix B of [15], which has exactly the same form as equation (3) above. The asymptotic of \( S^*_n(z^2; a, b) \) is given by equation (B5) in appendix B of [15], which has the same form as equation (11) above with \( \theta(z) = 0 \). The scattering amplitude and scattering phase shift obtained from the asymptotics, which are shown as equations (B6) and (B7) in [15], give

\[
E_k = -\frac{\lambda^2}{2} \left( k + \frac{1}{2} + \frac{2V_i}{\lambda^2} \right)^2.
\]
infinite potential well with sinusoidal bottom, which is a confined system with only discrete energy spectrum and is exactly solvable only by using the TRA. Then in appendix B, we showed that this potential well is only an element in a larger class of potentials represented by a three-parameter generalized trigonometric Scarf potential. Finally, using the unique features of orthogonal polynomials with mixed spectrum we presented, as an illustrative example, the two-parameter 1D Morse potential that supports both discrete energy bound states as well as continuous energy scattering states.

Acknowledgments

The support provided by the Saudi Center for Theoretical Physics (SCTP) is highly appreciated. We also acknowledge partial support by King Fahd University of Petroleum and Minerals (KFUPM).

Appendix A. Basis set for the expansion of the wavefunction

In this appendix, we give an alternative view for the origin of the wavefunction expansion given by equation (2) and show how to choose a proper square integrable basis \{\phi_n(x)\} for a given problem. Performing a general coordinate transformation \(x \rightarrow y(x)\) typically makes the wave equation a second order differential equation in \(y\). For such an equation with regular singular points and due to Fuchs’ theorem, we can use Frobenius method to write the following power series expansion for the solution

\[
\psi(E, x) = (y - y_1)^{a_1}(y - y_2)^{a_2} \cdots (y - y_r)^{a_r} \times \sum_{n=0}^{\infty} c_n (y - \hat{y})^n, \quad (A1)
\]

where \([y_i]\) are the singular points of the differential equation and \(\hat{y}\) is one of these points or any other regular point. The exponents \([a_n]\) are dimensionless parameter to be determined by the ‘indicial equation’. The expansion coefficients \([c_n]\) depend on the parameters of the wave equation and the energy. In the expansion (A1), we assumed that the singular points are finite. However, if one of them is irregular (essential) at \(+\infty\) then the factors multiplying the sum contains the term \(e^{-\lambda y}\) and if, on the other hand, two of them are irregular at \(\pm\infty\) then the factors include \(e^{-\lambda y}\). Substituting (A1) into the differential wave equation will result in the indicial equation and a recurrence relation for the expansion coefficients. It is easy to see that we can replace \((y - \hat{y})^n\) by a polynomial \(p_n(y)\) of degree \(n\) in \(y\) with different expansion coefficients as follows

\[
\psi(E, x) = R(y - \hat{y}) \sum_{n=0}^{\infty} f_n p_n(y) \equiv \sum_{n=0}^{\infty} f_n \phi_n(x), \quad (A2)
\]

where \(R(y)\) is an entire function of \(y\) and \([f_n]\) is a new set of expansion coefficients that depend on \(\hat{y}\), \([c_n]\) and the parameters of the polynomial \(p_n(y)\), which is a polynomial of degree \(n\) in \(y\) with \(n\) distinct real zeros. These polynomials in configuration space are not to be confused with the energy polynomials \([P_n(x)]\) defined in section 2. Therefore, equation (A2) allows us to write \(\phi_n(x) = R(y - \hat{y})p_n(y)\). Faithful representation of the wavefunction \(\psi(E, x)\) dictates that \(\{\phi_n(x)\}\) be a complete set and so is \([p_n(y)]\). That is

\[
\sum_n \phi_n(x) \phi_n(y) = \frac{\delta(x - y)}{\omega(x)}, \quad (A3)
\]

where \(\omega(x)\) is a positive definite entire function. This allows us to define the conjugate basis set \(\{\bar{\phi}_n(x)\}\) whose elements are

\[
\bar{\phi}_n(x) = \omega(x)^{1/2} \phi_n(x) \equiv R(y - \hat{y})p_n(y), \quad (A4)
\]

where \(R(y - \hat{y}) = \omega(x)R(y - \hat{y})\). Thus, we can write another series for the conjugate wave function as \(\bar{\psi}(E, x) = \sum_{n=0}^{\infty} f_n \bar{\phi}_n(x)\). The orthogonality \(\langle \phi_n(x)\bar{\phi}_m(x)\rangle = \langle \bar{\phi}_n(x)\bar{\phi}_m(x)\rangle = \delta_{nm}\) results in the following orthogonality relation for the polynomials \([p_n(y)]\).

\[
\int R(y - \hat{y})R(y - \hat{y})p_n(y)p_m(y) \frac{dy}{y'} = \delta_{nm}, \quad (A5)
\]

where \(y' = dy/dx\). Thus, the weight function \(W(y)\) associated with the configuration space polynomials \([p_n(y)]\) is

\[
W(y) = R(y - \hat{y})R(y - \hat{y})/y'. \quad (A6)
\]

In most cases, the requirements of square integrability of the basis and the fact that it carries a tridiagonal matrix representation for the wave operator dictate that \([R(y - \hat{y})]^2\) has the same form as the weight function \(W(y)\). Now, our choice of the polynomial \(p_n(y)\) is limited to those that satisfy the following two criteria:

1. \(p_n(y)\) must satisfy a differential equation whose singular points are either \([y_i]\) or a subset thereof.

2. The nature and range of the polynomial argument \(y\) must be compatible and consistent with that of the physical configuration space coordinate. For example, if \(y \geq 0\) then the Laguerre polynomial \(L_n^a(y)\) whose weight function is \(y^a e^{-y}\) becomes a proper choice and we can write \(R(y - \hat{y}) = y^a e^{-\beta y}\). Whereas, if \(-1 \leq y \leq +1\) then the Jacobi polynomial \(P_n^{(\mu, \nu)}(y)\) will be an appropriate choice with the corresponding weight function \((1 - y)^\mu (1 + y)^\nu\) and we can write \(R(y - \hat{y}) = (1 - y)^\mu (1 + y)^\nu\). On the other hand, if \(-\infty \leq y \leq +\infty\) then the Hermite polynomial \(H_n(y)\) with its weight function \(e^{-y^2}\) would be a proper choice to take for \(p_n(y)\) where we can also write \(R(y - \hat{y}) = e^{-\alpha y^2}\).

Based on our past experience, all known exactly solvable potentials correspond to coordinate transformations \(y(x)\) such that \(y \in [-\infty, +\infty]\), \(y \geq 0\), \(y \geq a\) or \(y \in [a, b]\). These problems are associated with the Hermite polynomial, Laguerre polynomial and Jacobi polynomial. Note that there are many special cases of the Jacobi polynomial like the Legendre, Chebyshev, Gegenbauer, etc.
Appendix B. TRA solution for a new class of potential functions

In this appendix, we use the second mode of application of the TRA to find the class of all potential functions corresponding to the physical configuration of section 4 and show that the potential well with sinusoidal bottom treated therein is only a special case of a larger class. Therefore, we start by defining the configuration space as the interval \(-\frac{L}{2} \leq x \leq +\frac{L}{2}\) and write the one-dimensional Schrödinger equation in terms of the dimensionless coordinate \(y = \sin(\lambda x)\), with \(\lambda = \pi / L\) and \(-1 \leq y \leq +1\), as follows

\[
-\frac{\hbar^2}{2} \left[ (1 - y^2) \frac{d^2}{dy^2} - y \frac{d}{dy} - U(y) + \varepsilon \right] \psi_\lambda(x) = 0, \quad (B1)
\]

where \(U(y) = 2V(y) / \lambda^2\). The analysis in Appendix A shows that the most general complete basis set appropriate for this problem has the following elements

\[
\phi_\alpha(x) = A_\alpha (1 - y)^\alpha (1 + y)^\beta P_n^{(\mu, \nu)}(y), \quad (B2)
\]

where \(P_n^{(\mu, \nu)}(y)\) is the Jacobi polynomial and the normalization constant is chosen as \(A_\alpha = \sqrt{\frac{2^n \mu + \nu + 1}{\Gamma(n + \mu + \nu + 1)}} \frac{\Gamma(n + \mu + 1)}{\Gamma(n + \mu + \nu + 1)} \frac{\Gamma(n + \mu + \nu + 2)}{\Gamma(n + \mu + \nu + 3)}\).

The four dimensionless real parameters are such that \((\alpha, \beta)\) positive and \((\mu, \nu)\) greater than \(-1\). Using the differential equation and differential property of the Jacobi polynomial, we obtain the following matrix representation of the wave operator in this basis \([21, 32]\]

\[
\frac{\hbar^2}{2} \left[ (1 - y^2) \frac{d^2}{dy^2} - y \frac{d}{dy} - U(y) + \varepsilon \right] \psi_\lambda(x) = V(x),
\]

which is given as equation \((B4a)\) and \((B4b)\) and \((B4c)\)

where \(U_0\) and \(U_\pm\) are arbitrary dimensionless parameters. The relevant case to the present treatment is the first one corresponding to

\[
V(x) = \left\{
\begin{array}{ll}
V_\lambda - V_0 \sin((\pi x)/L) & , -L/2 \leq x \leq +L/2 \\
0 & , \text{otherwise}
\end{array}
\right.
\]

where \(V_\lambda\) and \(V_0\) are real potential parameters, \(\mu^2 \geq -\frac{1}{4} + \frac{2}{3} V_\lambda - V_0\), \(\nu^2 \geq \frac{1}{4} + \frac{2}{3} \left( V_\lambda - V_0 \right)\) and \(U_0 = 2V_\lambda / \lambda^2\). Reality dictates that \(V_\lambda \geq \left| V_0 \right| - \frac{1}{\lambda^2}\). It is obvious that the potential well with sinusoidal bottom \((21)\) is a special case with \(V_\lambda = 0\). Note that if \(V_\lambda = 0\) then the potential function \((B5)\) is the well-known trigonometric Scarf potential, which is exactly solvable using conventional methods.

Substituting \((B4a)\) in \((B3)\) and using the recursion relation of the Jacobi polynomial and its orthogonality, we obtain an infinite symmetric tridiagonal matrix representing the wave operator with the following elements

\[
\frac{\hbar^2}{2} \left[ (1 - y^2) \frac{d^2}{dy^2} - y \frac{d}{dy} - U(y) + \varepsilon \right] \psi_\lambda(x) = J_{n,m} \delta_{n,m}, \quad (B6)
\]

where \(J_{n,m} = \left[ (n + \frac{\mu + \nu + 1}{2})^2 + U_0 C_n - \varepsilon \right] \delta_{n,m} + U_0 (D_n - \delta_{n,m-1} + D_n \delta_{n,m+1})\).

Comparing this recursion relation to that of \(H_{n}^{(\mu, \nu)}(z, \sigma)\), which is given as equation \((E1)\) of appendix E in \([32]\), we conclude that \(P_n^{(\mu, \nu)}(E) = H_{n}^{(\mu, \nu)}(z, \sigma)\) and that the bound state wave function \((6b)\) is written as

\[
\psi_\lambda(x) = \sqrt{\cos(\lambda x)} \left[ 1 - \sin(\lambda x) \right] \left[ 1 + \sin(\lambda x) \right] \left[ 1 - \sin(\lambda x) \right] \times \left[ 1 + \sin(\lambda x) \right] \sum_{n=0}^{\infty} A_n H_n^{(\mu, \nu)}(z, \sigma) \left( D_n - \delta_{n-1} + D_n \delta_{n+1} \right) P_n^{(\mu, \nu)}(E).
\]

Again, due to the lack of knowledge of the analytic properties of the polynomial \(H_n^{(\mu, \nu)}(z, \sigma)\) we resort to numerical means to obtain the physical properties of the quantum system. In table 3, we give the lowest part of the energy spectrum (in
Table 3. The lowest part of the energy spectrum (in units of $\frac{1}{2}\lambda^2$) for the potential (B5) with the parameter values $\{V_0, V_1, V_2\} = \{7, 5, 3\}$ in units of $\frac{1}{2}\lambda^2$. Rapid convergence with an increase in the basis size $N$ is evident.

| $n$ | $N = 10$ | $N = 11$ | $N = 12$ | $N = 13$ | $N = 100$ |
|-----|----------|----------|----------|----------|-----------|
| 0   | 7.680 625 404 | 7.680 625 404 | 7.680 625 404 | 7.680 625 404 | 7.680 625 404 |
| 1   | 14.338 493 494 | 14.338 493 494 | 14.338 493 494 | 14.338 493 494 | 14.338 493 494 |
| 2   | 22.546 540 967 | 22.546 540 967 | 22.546 540 967 | 22.546 540 967 | 22.546 540 967 |
| 3   | 32.767 801 800 | 32.767 801 800 | 32.767 801 800 | 32.767 801 800 | 32.767 801 800 |
| 4   | 40.034 852 009 | 40.034 852 009 | 40.034 852 009 | 40.034 852 009 | 40.034 852 009 |
| 5   | 49.334 170 172 | 49.334 170 172 | 49.334 170 172 | 49.334 170 172 | 49.334 170 172 |
| 6   | 59.634 553 948 | 59.634 553 948 | 59.634 553 948 | 59.634 553 948 | 59.634 553 948 |
| 7   | 69.938 897 443 | 69.938 897 443 | 69.938 897 443 | 69.938 897 443 | 69.938 897 443 |
| 8   | 79.238 493 494 | 79.238 493 494 | 79.238 493 494 | 79.238 493 494 | 79.238 493 494 |
| 9   | 88.538 493 494 | 88.538 493 494 | 88.538 493 494 | 88.538 493 494 | 88.538 493 494 |

units of $\frac{1}{2}\lambda^2$ for a chosen set of values of the potential parameters $V_0$ and $V_2$. The table shows rapid convergence of the calculation with an increase in the basis size.

Finally, if we were to choose either case (B4b) or (B4c), then we would have obtained the class of potentials given by (B5) but with $V_0 = 0$, which is the well-known trigonometric Scarf potential that is exactly solvable using any of the conventional methods. Note, however, that in this case and in contrast to (B5) there is a constraint on the value of only one of the two potential parameters. That is, either $V_0 \geq -\frac{\lambda}{4}^2$ or $V_0 \leq 2\lambda^2$ while the other is arbitrary and does not have to obey $V_2 \geq 2\lambda^2$. Moreover, the orthogonal energy polynomial in this case becomes the discrete version of the Wilson polynomial whose analytic properties are well known (see, for example, the appendices in [31] and citations therein).

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