Abstract. In a previous paper, it was shown that a soluble model can be constructed for the description of a decaying system in analogy to the Lee-Friedrichs model of complex quantum theory. It is shown here that this model also provides a soluble scattering theory, and therefore constitutes a model for a decay scattering system. Generalized second resolvent equations are obtained for quaternionic scattering theory. It is shown explicitly for this model, in accordance with a general theorem of Adler, that the scattering matrix is complex subalgebra valued. It is also shown that the method of Adler, using an effective optical potential in the complex sector to describe the effect of the quaternionic interactions, is equivalent to the general method of Green’s functions described here.
1. Introduction

Quaternionic quantum mechanics$^1$ has recently become of interest as a generalization of the usual complex quantum theory with additional intrinsic degrees of freedom. Essentially new models for quantum field theory have emerged from this generalization$^1$. It therefore is potentially applicable to particle theory. These additional degrees of freedom, with the symmetry of the automorphisms of the quaternion algebra, moreover, imply that the anti-self-adjoint operators which are generators of groups have symmetric effective spectra. The generator of time evolution, if it is absolutely continuous on the half line, has an effective absolutely continuous spectrum on the whole line (this effective spectrum is evident from the presence of contributions from both right and left hand cuts in the scattering formalism developed by Adler$^2$), so that a conjugate symmetric “time” operator exists$^3$. The theory may, consequently, be useful for the description of unstable systems for which there exists a Lyapunov function$^4$.

In a previous work$^5$, it was shown that a soluble model can be constructed for the description of an unstable quaternionic quantum mechanical system, in analogy to the Lee-Friedrichs model$^6$ for complex quantum theory. Following the procedure of Wigner and Weisskopf$^7$ to define the decay of an unstable system, the probability amplitude for the system to remain in its initial (unstable) state is (we shall use the round bracket for proper kets, i.e., normalized vectors in the Hilbert space, and the angle bracket for the generalized states (labels of continuous spectral representation)

$$A(t) = (\psi_0|e^{-\hat{H}t}|\psi_0), \quad (1.1)$$

where

$$\hat{H} = \hat{H}_0 + \hat{V}, \quad (1.2)$$

The vector $\psi_0$ is an eigenstate of $\hat{H}_0$, i.e.,

$$\hat{H}_0|\psi_0) = |\psi_0)\imath E_0, \quad (1.3)$$

and we define the (generalized) continuum eigenstates by

$$\langle E|\hat{H}_0|f) = \imath E\langle E|f). \quad (1.4)$$

The Laplace transform of the amplitude $A(t)$ is defined as

$$\hat{A}(z) = \int_0^\infty dt \ e^{\imath z t} A(t), \quad (1.5)$$

where $z \in \mathbb{C}(1, i)$, the complex subalgebra of the quaternion algebra $\mathbb{H}$. One finds

$$H(z)\hat{A}(z) = \imath G(z), \quad (1.6)$$

where $H(z)$ is a complex-valued function of $z$ analytic in the upper half plane, containing both left and right hand cuts on the real $E$ axis;

$$G(z) = G_\alpha(z) + \imath G_\beta(z), \quad (1.7)$$
where \( G_\alpha(z) \) is a complex-valued analytic function of \( z \) and \( G_\beta(z) \) is a complex-valued analytic function of \( z^* \), as required by the quaternionic Cauchy-Riemann relations\(^5\) for left analytic \( G(z) \). The functions \( H(z) \) and \( G(z) \) depend only on the matrix elements \( \langle E|\bar{V}|\psi_0 \rangle \) of the potential, and are explicitly given in ref. 5; some of the analytic properties of \( H(z) \) are discussed there as well, making use of the full effective spectrum \( (-\infty, \infty) \) of the anti-self-adjoint operator \( \bar{H}_0 \). It was assumed, to obtain the result (1.7), that \( \bar{V} \) has no continuum-continuum matrix elements, i.e.,

\[
\langle E|\bar{V}|E' \rangle = 0. \tag{1.8}
\]

This restriction corresponds to that of the Lee-Friedrichs model of complex quantum theory.

In the next section, we discuss the basic structure of formal scattering theory in quaternionic quantum mechanics, and define the resolvent (Green’s function) in terms of an operator-valued Laplace transform which exists on a certain domain. We then derive, in Section 3, formulas analogous to those of the second resolvent equation (Lippmann-Schwinger equations), and in Section 4, use these to show that the soluble model discussed in ref. 5 provides a soluble model for scattering as well, and hence corresponds to a decay-scattering system\(^8\). We show, explicitly, in this soluble model, that the \( T \)-matrix is purely complex-valued, in accordance with a general theorem of Adler\(^1\), and furthermore show that for the general potential problem, Adler’s method for describing the solution of the problem in the complex sector with the use of an “optical potential,” describing the effect of the quaternionic interactions, is equivalent to the method of Green’s functions developed here.

2. Formal Scattering Theory

In this section we study the formal structure of quaternionic scattering theory. The resolvent techniques used in scattering theory for complex quantum mechanics do not apply directly, since the anti-self-adjoint generator of motion in time is not associated, in general, with any particular complex subalgebra. However, as in scattering theory in the complex case, the spectral representation of the unperturbed Hamiltonian may be used to construct the integral equations for the Green’s functions and the scattering operator. The methods we develop here can be used to define the spectrum of an operator, as the complement of the resolvent set\(\S\), and for many parallel applications of resolvent theory.

The fundamental condition for a scattering process is the existence of the strong limit

\[
\lim_{t \to \pm \infty} \|e^{-\bar{H}_t}\psi - e^{-\bar{H}_0 t}\phi\| = 0, \tag{2.1}
\]

where \( \psi \) is the Heisenberg state of the system, and \( \phi \) is an asymptotic free state. In this norm, we may bring the factor \( e^{-\bar{H}_t} \) to the second term. If the limit

\[
\lim_{t \to \pm \infty} e^{\bar{H}_t} e^{-\bar{H}_0 t}\phi
\]

\(\S\) In the complex Hilbert space, the resolvent set for an operator \( B \) is defined as the domain of analyticity of the operator \( (z - B)^{-1} \); this is an open set. Its complement, a closed set, is called the spectrum\(^9\).
exists on a dense set of the quaternionic Hilbert space, it defines the wave operators

$$
\Omega_\pm = \lim_{t \to \pm \infty} e^{\tilde{H}t} e^{-\tilde{H}_0 t}
$$

(2.2)

with the properties

$$
\tilde{H} \Omega_\pm = \Omega_\pm \tilde{H}_0,
$$

(2.3)

as given in ref. 1, Chap. 8. A sufficient condition for its existence is, as for the complex Hilbert space

$$
\lim_{t \to \pm \infty} \| \tilde{V} e^{-\tilde{H}_0 t} \phi \| = 0
$$

(2.4)
on a dense set.

We now define the Green’s function (the functions we define here are the Laplace transforms of the Green’s functions defined on $t$ in ref. 1)

$$
G(Z) = I \int_0^\infty dt \ e^{IZt} e^{-\tilde{H}t},
$$

(2.5)

where $Z, I \in \mathbb{C}(1, I)$, the complex subalgebra of the left operator valued algebra isomorphic to $\mathbb{H}$, constructed from the spectral family of $\tilde{H}_0$. The Green’s function (2.5) cannot be given in closed quaternionic operator algebraic form, since $I$ and $\tilde{H}$ do not, in general, commute. The unperturbed Green’s function, however, has the form

$$
G_0(Z) = I \int_0^\infty dt \ e^{IZt} e^{-\tilde{H}_0 t}
$$

(2.6)

$$
= -\frac{1}{Z + IH_0}
$$

for $Z$ formally in the upper half plane (positive coefficient of $I$ in $Z$).

In terms of the wave operators, the scattering $S$-matrix is

$$
S = \Omega_+^\dagger \Omega_-,
$$

(2.7)

and the corresponding $T$-matrix,

$$
T = S - 1
$$

(2.8)

can be expressed as

$$
T = \lim_{t \to -\infty} (\Omega_+^\dagger e^{\tilde{H}_0 t} - 1)
$$

(2.9)

where we have used the intertwining property (2.3) in the last step. Denoting by $\{ \langle E | f \rangle \}$ the part of the spectral representation of a vector $f$ in the absolutely continuous spectrum of $\tilde{H}_0$ ($\tilde{H}_0$ also has a discrete eigenfunction in the soluble model we are studying, which we shall call $\phi$; for example, in this case, $\int_0^\infty dE |E\rangle i(E) + |\phi\rangle i(\phi)$ represents the left algebraic operator $\mathbb{I}$), we have then that
\[ \langle E|T|f \rangle = \lim_{t \to -\infty} e^{iEt} \langle E|(\Omega^\dagger_+ - 1)e^{-\tilde{H}_0t}|f \rangle. \quad (2.10) \]

We now remark that
\[ -i \lim_{\epsilon \to 0^+} \epsilon \langle E|G_0(E + I\epsilon)|f \rangle = \langle E|f \rangle, \quad (2.11) \]
as can easily be seen from the integral representation
\[ -i\epsilon \langle E|G_0(E + I\epsilon)|f \rangle = \epsilon \int_0^\infty dt \ e^{i(E+i\epsilon)t} e^{-iEt} \langle E|f \rangle, \]
which, in the limit \( \epsilon \to 0 \), reduces to \( \langle E|f \rangle \). For the first term of (2.10), we use the fact that
\[ \langle E|\Omega^\dagger_+|f \rangle = -i \lim_{\epsilon \to 0} \epsilon \langle E|G(E + I\epsilon)|f \rangle. \quad (2.12) \]
This result follows from comparing
\[ \langle E|\Omega^\dagger_+|f \rangle = \langle E| \lim_{\epsilon \to 0} \epsilon \int_0^\infty dt e^{-\epsilon t} e^{\tilde{H}_0t} e^{-\tilde{H}t}|f \rangle \]
\[ = \lim_{\epsilon \to 0} \epsilon \int_0^\infty dt e^{i(E+i\epsilon)t} \langle E|e^{-\tilde{H}t}|f \rangle, \]
where we have used Abel’s formula* (assuming convergence), and
\[ \lim_{\epsilon \to 0} -i\epsilon \langle E|G(E + I\epsilon)|f \rangle = -i \lim_{\epsilon \to 0} \epsilon \langle E|I \int_0^\infty dt e^{i(E+i\epsilon)t} e^{-\tilde{H}t}|f \rangle, \]
which, with the properties of \( \langle E| \), coincides with the previous expression, and completes the proof of (2.12).

We therefore have the expression, familiar from complex scattering theory as well,
\[ \langle E|T|f \rangle = -i \lim_{t \to -\infty} e^{iEt} \epsilon \langle E|(G(E + I\epsilon) - G_0(E + I\epsilon))e^{-\tilde{H}_0t}|f \rangle. \quad (2.13) \]

Clearly, \( f \) cannot contain an eigenstate of \( \tilde{H}_0 \), or the limit \( t \to \infty \) will not converge. We must therefore take \( f \) entirely in the continuum of \( \tilde{H}_0 \).

3. **Second Resolvent Equations**

The relation corresponding to (2.13) in the complex quantum theory may be analyzed with the help of the second resolvent equations
\[ G(z) = G_0(z) - G_0(z)VG(z) \]
\[ = G_0(z) - G(z)VG_0(z). \quad (3.1) \]

* That is, that \( \lim_{t \to \infty} g(t) = \lim_{\epsilon \to 0} \epsilon \int_0^\infty e^{-\epsilon t} g(t) dt \).
One can obtain equations analogous to (3.1) by taking the formal Laplace transform of the derivative of the unitary evolution operator,

\[ \frac{d}{dt} e^{-\tilde{H}t} = (\tilde{H}_0 + \tilde{V}) e^{-\tilde{H}t} = e^{-\tilde{H}t} (\tilde{H}_0 + \tilde{V}). \tag{3.2} \]

Multiplying from the left by \( e^{IZt} \) and integrating between \((0, \infty)\), one obtains from the first of (3.2) [the first term on the left hand side of these equations is due to integration by parts]

\[ 1 + ZG(Z) = -I\tilde{H}_0 G(Z) + \int_0^\infty dt \ e^{IZt} \tilde{V} e^{-\tilde{H}t} \tag{3.3} \]

and from the second of (3.2),

\[ 1 + ZG(Z) = -IG(Z)(\tilde{H}_0 + \tilde{V}). \tag{3.4} \]

We now make use of the symplectic decomposition available for quaternion linear operators\(^1\) to define

\[ \tilde{V} = V_\alpha + JV_\beta, \tag{3.5} \]

where \( V_\alpha \) and \( V_\beta \) are \( \mathbb{C}(1, I) \) valued operators, and \( J \) is the second generator of the algebra, isomorphic to \( \mathbb{H} \), spanned over the reals by the left multiplying operators \( \{1, I, J, K\} \). From (3.3), we then obtain

\[ 1 + ZG(Z) = -I\tilde{H}_0 G(Z) - IV_\alpha G(Z) - JIG(-Z^*). \tag{3.6} \]

Substituting the decomposition

\[ G(Z) = G_\alpha(Z) + JG_\beta(Z), \tag{3.7} \]

we obtain

\[ 1 + ZG_\alpha(Z) = -I\tilde{H}_0 G_\alpha(Z) - IV_\alpha G_\alpha(Z) - IV_\beta^* G_\beta(-Z^*) \tag{3.8} \]

and

\[ Z^* G_\beta(Z) = -I\tilde{H}_0 G_\beta(Z) + IV_\alpha^* G_\beta(Z) - IV_\beta G_\alpha(-Z^*); \tag{3.9} \]

we use the asterisk here to represent the involution of the left quaternionic algebra as well as its (isomorphic) use as quaternion conjugation of the right algebra. For formally complex-valued operators \( B_\alpha \), we have that \( B_\alpha^* \equiv JB_\alpha J^* \).

With the definition (2.6), one then obtains from (3.8) and (3.9) the two equations (see ref. 1, Chap. 7, Sec. 7.2 and Chap. 8, and ref. 2 for related results)

\[ G_\alpha(Z) = G_0(Z) + G_0(Z)IV_\alpha G_\alpha(Z) + G_0(Z)V_\beta^* IG_\beta(-Z^*) \tag{3.10} \]

and

\[ G_\beta(Z) = -G_0(Z)^* IV_\alpha^* G_\beta(Z) + G_0(Z)^* IV_\beta G_\alpha(-Z^*). \tag{3.11} \]
Similarly, from (3.4), substituting (3.5) and (3.7), we obtain

\[ 1 + ZG_\alpha(Z) = -IG_\alpha(Z)(\tilde{H}_0 + V_\alpha) + IG_\beta(Z)^*V_\beta \]

\[ Z^*G_\beta(Z) = IG_\beta(Z)(\tilde{H}_0 + V_\alpha) + IG_\alpha(Z)^*V_\beta \]

(3.12)

It then follows, again using (2.6), that

\[ G_\alpha(Z) = G_0(Z) + IG_\alpha(Z)V_\alpha G_0(Z) - IG_\beta(Z)^*V_\beta G_0(Z) \]

(3.13)

and

\[ G_\beta(Z) = IG_\alpha(Z)^*V_\beta G_0(-Z^*) + IG_\beta(Z) V_\alpha G_0(-Z^*) \].

(3.14)

The formulas (3.10), (3.11), (3.13) and (3.14) are the generalized second resolvent formulas for quaternionic Green’s functions.

4. Solubility of the Model

In this section we apply the generalized second resolvent equations to the soluble model discussed in ref. 5. The model is described by Eqs. (1.2) – (1.4), with the condition (1.8). The matrix element \( \langle \psi_0 | \tilde{V} | \psi_0 \rangle \), included in the analysis of the decay system carried out in ref. 5, does not play a direct role in the structure of the equations for the scattering amplitude, but the Green’s function \( G(Z) \) appearing in the formula (2.13) depends implicitly on this part of the potential as well.

To display explicitly the property that the continuum-continuum matrix elements of the interacting Green’s functions depend only on the discrete-discrete matrix elements (with transition amplitudes provided by the symplectic components of the potential) in the soluble model we are considering, we use (3.13) and (3.14) in (3.10) and (3.11) to obtain

\[ G_\alpha(Z) = G_0(Z) + G_0(Z)[IV_\alpha - V_\alpha G_\alpha(Z)V_\alpha + V_\alpha G_\beta(Z)^*V_\beta - V_\beta^*G_\alpha(-Z^*)V_\alpha - V_\beta^*G_\alpha(-Z^*)^*V_\beta]G_0(Z), \]

(4.1)

and

\[ G_\beta(Z) = G_0(Z)^*[IV_\beta + V_\alpha^*G_\beta(Z)V_\alpha + V_\alpha^*G_\alpha(Z)^*V_\beta - V_\beta G_\alpha(-Z^*)V_\alpha + V_\beta G_\beta(-Z^*)^*V_\beta]G_0(-Z^*). \]

(4.2)

We now return to Eq. (2.13) for the \( T \)-matrix; recalling that the vector \( f \) must be in the continuous spectrum of \( \tilde{H}_0 \) if the limits are to be well-defined, we see that we are interested in the continuum-continuum matrix elements of

\[ G(Z) - G_0(Z) = G_\alpha(Z) - G_0(Z) + JG_\beta(Z). \]

(4.3)

It is clear from the structure of the potential (1.8), and the form of (4.1) and (4.2) of the symplectic components of the Green’s function, that the continuum-continuum matrix elements entering the definition of the \( T \) matrix are, in fact, expressed in terms of the discrete-discrete matrix elements of the Green’s function only. Since a closed formula can
be obtained for the discrete-discrete matrix element of the Green’s function\(^\dagger\) in this model, this result completes the proof that the scattering theory for this model is exactly soluble as well.

In the following, we shall give the explicit form of the \(T\)-matrix,

\[
\langle E|T|f \rangle = -i \lim_{t \to -\infty} e^{iEt} \langle E|(G_\alpha(E + i\epsilon) - G_0(E + i\epsilon) + JG_\beta(E + i\epsilon))e^{-\tilde{H}_0t}|f \rangle. \tag{4.4}
\]

For the contribution of \(G_\alpha - G_0\) to (4.4), we see from (4.1) that the first operator valued factor (leftmost) in the matrix element is \(G_0(E + i\epsilon)\); under the limit \(\epsilon \to 0_+\), this factor, according to (2.11), with the pre-factor \(-i\epsilon\), has the effect of multiplication by unity. The contribution of the \(JG_\beta\) term contains, as seen from (4.2), for the first operator valued factor, \(JG_0(E + i\epsilon)^* = G_0(E + i\epsilon)J\). Hence, applying (2.11) again, the result (after the limit \(\epsilon \to 0_+\)) is

\[
\langle E|J|f' \rangle = j\langle E|f' \rangle, \tag{4.5}
\]

where \(f'\) is the vector generated from \(f\) by the action of the remaining operators.

The last operator valued factor in the two types of terms in the matrix element, i.e., \(G_0(E + i\epsilon)\) and \(G_0(-E + i\epsilon)\), combine with the prefactor \(e^{iEt}\) and the unitary operator \(e^{-\tilde{H}_0t}\), in the limit \(t \to -\infty\) to form delta functions. We now study these contributions. Since \(f\) is in the continuous spectrum of \(\tilde{H}_0\), we may represent it by

\[
f = \int_0^\infty |E'\rangle\langle E'|f \rangle dE'. \tag{4.6}
\]

Consider, for the first of these types, the expression

\[
e^{iEt}G_0(E + i\epsilon)e^{-\tilde{H}_0t}|E'\rangle. \tag{4.7}
\]

Using the definition (2.6) for \(G_0\), one obtains

\[
\lim_{t \to -\infty} |E'\rangle \frac{e^{-i(E' - E)t}}{(E' - E) - i\epsilon} = |E'\rangle 2\pi i \delta(E - E'). \tag{4.8}
\]

For the second type, there is a factor \(j\), so that we must evaluate, in place of (4.7), the limit

\[
e^{-iEt}G_0(-E + i\epsilon)e^{-\tilde{H}_0t}|E'\rangle; \tag{4.7'}
\]

\(^\dagger\) A closed formula is given for \(\hat{A}(z)\) in ref. 5; from the definition (2.5) of the Green’s function and of the amplitude \(\hat{A}(z)\) in Eqs. (2.1) and (3.5) of ref. 5 it follows that

\[
\langle \psi_0|G(Z)|\psi_0 \rangle = i\hat{A}(z).
\]
the corresponding result is

\[
\lim_{t \to -\infty} |E'\rangle e^{-i(E+E')t} \frac{E}{E+E' - i\epsilon} = |E'\rangle 2\pi i\delta(E + E').
\]  

(4.9)

Since \(E, E' > 0\), this second term can give no contribution. The \(T\)-matrix is therefore entirely complex-valued, relative to the ray convention we have chosen, in accordance with a theorem of Adler\(^1\), stating that the \(S\)-matrix necessarily has values entirely in the complex subalgebra \(C(1, i)\).

Inserting these results into the formula (4.4) for the \(T\)-matrix, we have that

\[
\langle E|T|f \rangle = \int_0^\infty dE \; \langle E|T|E' \rangle \langle E'|f \rangle,  
\]  

(4.10)

where \(\langle E|T|E' \rangle\) is complex-valued:

\[
\langle E|T|E' \rangle = 2\pi i\delta(E - E')\langle E|\mathcal{O}|E' \rangle,  
\]  

(4.11)

where, from (4.1) and (4.2) (the \(IV_\alpha\) term of \(G_\alpha - G_0\) and the \(IV_\beta\) term of (4.2) do not contribute, in this model, to the continuum-continuum matrix elements),

\[
\mathcal{O} = -V_\alpha G_\alpha (E + I\epsilon)V_\alpha + V_\alpha G_\beta (E + I\epsilon)^*V_\beta - V_\beta^* G_\beta (-E + I\epsilon)V_\alpha - V_\alpha^* G_\alpha (-E + I\epsilon)^*V_\beta.
\]  

(4.12)

One could, alternatively use here (in place of (4.6)) the spectral representation for \(|f\rangle\) which makes explicit the symmetric spectrum\(^3\) of \(\tilde{H}_0\), i.e.,

\[
|f\rangle = \int_{-\infty}^{\infty} dE \; \langle E| \langle f|E\rangle,  
\]

where \(\langle E|\) admits both positive and negative values of \(E\) with the identification \(\langle -|E\rangle = |E\rangle j\). The subscript \(c\) on the bra \(\langle E|\) indicates that the complex scalar product must be taken\(^13\). The equivalence between (4.6) and this expression can be seen by noting\(^3\) that \(c\langle f|E\rangle = f_\alpha(E)\), and therefore

\[
\int_{-\infty}^{\infty} dE \; \langle E| \langle E|f\rangle = \int_0^\infty dE \left( |E\rangle \langle E|f\rangle_c + |E\rangle j|E\rangle f_\alpha(E) \right)  
\]

\[
= \int_0^\infty dE \left( |E\rangle f_\alpha(E) + |E\rangle j f_\beta(E) \right).
\]

The limit of (4.7) in the \(E' < 0\) interval contains a sign change (due to the factor \(j\)) in the eigenvalue of the operator \(I\) (with the parameter \(E'\) in the coefficients positive). Hence only the sign of \(i\) in both numerator and denominator of the left hand side of (4.8) is changed; the sign of the right hand side is therefore also changed, consistent with simply multiplying both sides of (4.8) by \(j\) on the right. Similarly, in this interval, (4.7') leads
to (4.9) with the sign of \( i \) changed on both left and right sides. The construction of the \( E' < 0 \) extension therefore commutes with the limiting process, and leads to the same conclusion.

5. The Optical Potential

Since the \( T \)-matrix is complex-valued and acts separately in the complex and quaternionic sectors, the formulation of Adler\(^1\) for scattering in the complex sector with the use of an effective potential, called the “optical potential,” to provide the effect of perturbation due to the interactions of the quaternionic sector should, in fact, coincide with the results we have obtained from the general theory of Green’s functions. To obtain this equivalence, we return to Eqs.\((3.8), (3.9)\) and \((3.12)\). We define the Green’s function associated with the complex part of the full Hamiltonian by

\[
G_c(Z) = -\frac{1}{Z + I\tilde{H}_0 + IV_\alpha}. \tag{5.1}
\]

Grouping terms in a different way, it then follows that, from \((3.8)\) and \((3.9)\),

\[
\begin{align*}
G_\alpha(Z) &= G_c(Z) + G_c(Z)V_\beta^*IG_\beta(-Z^*) \\
G_\beta(Z) &= G_c(Z)V_\beta G_\alpha(-Z^*)
\end{align*} \tag{5.2}
\]

and, from \((3.12)\), we obtain

\[
\begin{align*}
G_\alpha(Z) &= G_c(Z) - IG_\beta(Z)V_\beta G_c(Z) \\
G_\beta(Z) &= IG_\alpha(Z)V_\beta G_c(-Z^*).
\end{align*} \tag{5.3}
\]

Substituting the quaternionic part \( G_\beta(Z) \) of the Green’s function into the first of each of these equations, one obtains

\[
\begin{align*}
G_\alpha(Z) &= G_c(Z) - G_c(Z)V_\beta^*G_c(-Z^*)V_\beta G_\alpha(Z) \\
G_\beta(Z) &= G_c(Z)V_\beta G_c(-Z^*). \tag{5.4}
\end{align*}
\]

and

\[
\begin{align*}
G_\alpha(Z) &= G_c(Z) - G_\alpha(Z)V_\beta^*(Z)G_c(-Z^*)V_\beta G_c(Z). \\
G_\beta(Z) &= G_c(Z)V_\beta G_c(-Z^*). \tag{5.5}
\end{align*}
\]

Making the identifications with the notation of Adler\(^1\), for which the complex part of the Hamiltonian \( H_\alpha \), the unperturbed Hamiltonian \( \tilde{H}_0 \) and the complex part of the potential \( V_\alpha \) are related to Hermitian operators by

\[
\begin{align*}
H_\alpha &= IH_1, \\
\tilde{H}_0 &= IH_0, \\
V_\alpha &= IV_1,
\end{align*} \tag{5.6}
\]

we see that the conjugated Green’s function appearing in (5.4) and (5.5)

\[
G_c(-Z^*)^* = \frac{1}{Z - I\tilde{H}_0 - IV_\alpha}, \tag{5.7}
\]
an analytic function in the upper half plane, has the form

\[ G_c(-Z^*)^* = \frac{1}{Z + H_0 + V_1} \]

\[ = \frac{1}{Z + H_1}, \tag{5.8} \]

so that we recognize that

\[ V_{opt}(Z) = V_{opt}(E) \]

is the analytic function in the upper half plane with boundary value \( V_{opt}(E) \) on the real axis defined by Adler\(^1\). If \( H_1 \) has positive spectrum, there is no discontinuity as one approaches the real line.

Rewriting (5.4) and (5.5), we obtain

\[ G_\alpha(Z) = G_c(Z) - G_c(Z)V_{opt}(Z)G_\alpha(Z) \tag{5.10} \]

and

\[ G_\alpha(Z) = G_c(Z) - G_\alpha(Z)V_{opt}(Z)G_c(Z). \tag{5.11} \]

These are the second resolvent equations for a complex quantum mechanical scattering problem for which the “unperturbed” Green’s function corresponds to the scattering problem with the complex part of the quaternionic potential; the complete problem, described by \( G_\alpha(Z) \)(since the part proportional to \( j \) in (4.4) vanishes in the scattering limit), carries the perturbation induced by the interactions of the quaternionic sector through the optical potential. This corresponds to the method used by Adler\(^1\) to solve the scattering problem in the complex sector, and therefore the solutions for the \( T \)-matrix coincide.

From (5.10) and (5.11) we see that

\[ G_\alpha(Z) = G_c(Z)(G_\alpha(Z)^{-1} - V_{opt}(Z))G_\alpha(Z) \tag{5.12} \]

and

\[ G_\alpha(Z) = G_\alpha(Z)(G_\alpha(Z)^{-1} - V_{opt}(Z))G_c(Z), \tag{5.13} \]

so that the effective Hamiltonian for the problem is the complex part plus the (energy dependent) optical potential.

### 6. Conclusions

In a previous work\(^5\) a soluble model for the description of a decaying system was discussed. In this model, the unperturbed Hamiltonian has an absolutely continuous spectrum on \([0, \infty)\) (and hence, an effective absolutely continuous spectrum \(^3\) on \((-\infty, \infty)\)) and a discrete eigenstate; the perturbing potential has no continuum-continuum matrix elements. This property makes it possible to obtain a closed formula for the exact computation of the diagonal matrix element of the full Green’s function in the discrete eigenstate of the unperturbed Hamiltonian (this result can be easily generalized to a finite set of discrete eigenstates, as for the complex case treated, e.g., in ref. \(^8\)). We have shown here that this model also provides an exactly soluble model in the scattering channel.

11
From the fundamental strong convergence relation of scattering theory, a formula for the scattering $T$-matrix was obtained in terms of the Green’s functions of the theory. The full quaternionic Green’s function does not have a simple quaternionic operator form, but from its definition as a formal integral (convergent on some domain), the quaternionic analog of the second resolvent equations were developed. It was then shown that the $T$-matrix, expressed in term of the operator $G - G_0 = G_{\alpha} - G_0 + JG_{\beta}$ reduces to a form in which the Green’s functions appear between two factors consisting of the complex or quaternionic parts of the potential. Since the potential, in this model, connects the unperturbed continuous spectrum only to the discrete, the $T$-matrix then becomes exactly solvable in terms of the known solutions for the discrete matrix element of the Green’s function. Hence, the model provides an exactly soluble scattering theory as well, and serves as a decay-scattering system.

The result for the $T$ matrix is purely complex-valued, in accordance with a theorem of Adler. We have shown, moreover, that in the framework of the general scattering theory, the Green’s function that generates the $T$-matrix is precisely the Green’s function for the complex part of the problem perturbed by the “optical potential” defined by Adler. In his formulation of the scattering problem in the complex sector with the effective perturbation due to the quaternionic interactions, the optical potential plays the same role. Hence the $T$-matrix defined by Adler with the optical potential in the complex sector coincides (as it must, since the $T$-matrix is purely complex valued, and acts in each sector separately) with the scattering matrix we have obtained in the framework of the general theory of quaternionic Green’s functions.

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