Instantons and Wormholes in $N = 2$ supergravity

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Abstract

In this paper, we construct Euclidean instanton and wormhole solutions in $d = 4$, $N = 2$ supergravity theories with hypermultiplets. The analytic continuation of the hypermultiplet action, involving pseudoscalar axions, is discussed using the approach originally developed by Coleman which determines the appearance of boundary terms. In particular, we investigate the conditions obtained by requiring the action to be positive-definite once the boundary terms are taken into account. The case of two hypermultiplets parameterizing the coset $G_{2,2}/SU(2) \times SU(2)$ is studied in detail. Orientifold projections which reduce the supersymmetry to $N = 1$ are also discussed.

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1 Introduction

Instantons and wormholes determine potentially important non-perturbative effects in string theory. Both can be obtained as saddle-points of the Euclidean action of the corresponding low-energy supergravity [1, 2, 3, 4, 5].

The literature discusses how wormholes can lead to several interesting effects. Some examples are the renormalization of coupling constants, a mechanism setting to zero the cosmological constant, quantum decoherence and creation of baby universes [6, 7, 8]. In [9, 10, 11, 12] wormholes in Anti-de Sitter spaces have been discussed. Recently, in [12] it was argued that wormholes in the AdS bulk can spoil locality and cluster decomposition in the context of the AdS/CFT correspondence.

In contrast to the non-local effects produced by wormholes, instantons produce local non-perturbative contributions to the low-energy effective action. In supergravity theories with extended supersymmetry there are BPS instanton solutions preserving half of the supersymmetries [13]. The broken supersymmetries in the instanton background generate fermionic zero modes which have to be soaked up by instanton-induced interaction terms in the path integral [14]. Note however that in some theories, such as $N = 2$ $d = 4$ supergravity theories with $n_H > 1$ hypermultiplets, extremal non-BPS instanton solutions can exist.

Instanton and wormhole solutions have been discussed for various theories and dimensions, in particular the axion/dilaton $SL(2, R)/U(1)$ coset [13, 14, 15, 16], the universal hypermultiplet in $N = 2, d = 4$ supergravity [17, 18, 19] and general hypermultiplets in $N = 2, d = 4$ theories [20, 21, 39].

The structure of the paper is as follows. In section 2 we review important properties of the hypermultiplet sector of $N = 2$ supergravity theories.

In section 3 we discuss the general properties of instantons and wormholes in $N = 2$ theories, in particular how the analytic continuation arises using an approach first employed in a paper by Coleman and Lee [4] (see also [12]). In general the instanton and wormhole solutions are constructed by complexifying the scalar fields and choosing a real section (i.e. the real section defines a particular analytic continuation of the original scalar fields). We propose a condition which leads to solutions which satisfy reality conditions and lead to a positive definite action. It is an interesting and open problem, whether for other real sections the instanton solutions are sensible saddle points dominating the path integral.

In section 4 we cover the universal hypermultiplet which is given by a $SU(2, 1)/U(2)$ coset nonlinear sigma model and has been the object of previous work (38).

In section 5 we study the case of two hypermultiplets parameterizing a $G_{2,2}/(SU(2) \times
SU(2)) coset. Explicit solutions are obtained using the conserved currents coming from the global symmetries of the coset sigma model. In particular, we study various consistent truncations and we present explicit solutions as well as their actions. We also discuss the existence of extremal non-BPS instanton solutions and the possibility to generate more general solutions using the $G_{2,2}$ global symmetry.

In section 6 the reduction of the supersymmetry due to orientifold projections is discussed and related to the consistent truncations of section 5 for the $G_{2,2}$ case. Finally, in section 7, we give a brief discussion of the open problems.

2 Hypermultiplets in $N = 2$ supergravity

$N=2$ supergravity theories are endowed with a very rich structure and stand between phenomenologically viable theories with $N = 1$ supersymmetry and theories with more than two supersymmetries which are almost completely fixed by their symmetries. Two recent examples of interest in these theories are the study of the attractor mechanism for extremal black holes [22, 23] and the discovery of the role that higher derivative corrections [24] and topological string amplitudes [25] play for the entropy of BPS black holes. In this section we will review the properties of the hypermultiplet sector of $N = 2$ theories.

2.1 Calabi-Yau compactification

The canonical example of obtaining four-dimensional $N = 2$ supergravity theories in string theory is the compactification of ten-dimensional type II (A or B) superstring theory on a six-dimensional Calabi-Yau manifold. The compactification breaks $N = 8$ supersymmetry down to $N = 2$. For length scales larger than the compactification scale (which in turn is larger than the string scale $l_s$) the theory is well approximated by the four-dimensional two-derivative effective supergravity action. The moduli space of scalars factorizes into vector and hypermultiplets, $\mathcal{M} = \mathcal{M}_{\text{vector}} \times \mathcal{M}_{\text{hyper}}$, where $\mathcal{M}_{\text{vector}}$ is given by a special Kähler manifold [26] and $\mathcal{M}_{\text{hyper}}$ is given by a quaternionic Kähler manifold [27]. The dimensionality of the respective moduli spaces depends on the Hodge numbers $h_{1,1}$ and $h_{2,1}$ of the Calabi-Yau manifold.

In terms of the conformal field theory, the compactification is encoded in a $c = 9, N = 2$ superconformal field theory [28]. The massless moduli come from the combination of chiral and anti-chiral primary states of the $N = 2$ SCFT.
Table 1: Dimensionality of moduli spaces

|               | \(\text{dim}(\mathcal{M}_{\text{vector}})\) | \(\text{dim}(\mathcal{M}_{\text{hyper}})\) |
|---------------|----------------------------------------|----------------------------------------|
| type IIA      | \(2h_{1,1}\)                           | \(4(h_{2,1} + 1)\)                     |
| type IIB      | \(2h_{2,1}\)                           | \(4(h_{1,1} + 1)\)                     |

### 2.2 Mirror symmetry and c-map

For type II string theories compactified on a circle, T-duality relates type IIA theory on a circle of radius \(R\) to type IIB on a circle of radius \(1/R\) [29]. There are two analogs of T-duality for Calabi-Yau compactifications.

First, Mirror symmetry has a simple realization in terms of the internal SCFT, where one changes the sign of the chiral \(U(1)\) current of the \(N = 2\) CFT. This transformation relates type IIA on a Calabi-Yau manifold \(\mathcal{M}\) to type IIB on a mirror Calabi-Yau manifold \(\tilde{\mathcal{M}}\). The two manifolds are topologically different since the Hodge numbers \(h_{1,1}\) and \(h_{2,1}\) are interchanged.

Second, the c-map is obtained [28] compactifying one of the four flat non-compact directions on a circle and performing a T-duality. This T-duality does not act on the internal \(N = 2\) SCFT and hence the c-map relates string theories compactified on the same Calabi-Yau manifold. It does however relate the gravity and vector multiplets of the type IIA theory to the hypermultiplets of type IIB and vice versa.

### 2.3 Hypermultiplet actions

The bosonic part of the hypermultiplet action given by a nonlinear sigma model which lives on a special quaternionic manifold. In the following, we will assume that the theory is obtained by compactifying type IIB string theory on a Calabi-Yau manifold. The quaternionic manifold is \(4n_H = 4(h_{1,1} + 1)\) dimensional [27]. It is parameterized by \(n_H - 1 = h_{1,1}\) complex scalars \(z^\alpha, \alpha = 1, 2, \cdots, n_H - 1\) together with \(2n_H\) real Ramond-Ramond scalars \(\zeta^I, \tilde{\zeta}_I, I = 0, 1, \cdots, n_H - 1\), the dilaton \(\phi\) and the NS-NS axion \(\sigma\). The explicit form of the action can be obtained by compactification [30] or applying the c-map on the gravity and vector multiplet action [27]. The resulting hypermultiplet action can be written as follows

\[
S = \int d^4x \sqrt{-g} \left\{ R - 2g_{\alpha\bar{\beta}} \partial_\mu z^\alpha \partial_\bar{\mu} \bar{z}^{\bar{\beta}} - \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} e^{-2\phi} (\partial_\mu \sigma + \frac{1}{2} \zeta^I \partial_\mu \tilde{\zeta}_I - \frac{1}{2} \tilde{\zeta}_I \partial_\mu \zeta^I)^2 - \frac{1}{2} e^{-\phi} I_{IJ} \partial_\mu \zeta^I \partial^\mu \zeta^J - \frac{1}{2} e^{-\phi} (\partial_\mu \tilde{\zeta}_I + R_{IK} \partial_\mu \zeta^K) (I^{-1})^{IJ} (\partial_\mu \zeta_I + R_{IL} \partial_\mu \zeta^L) \right\} \tag{2.1}
\]
The action is completely determined by a prepotential $F(X^I)$, where the projective coordinates $X^I, I = 0, 1, \ldots, h_{1,1}$ are related to the scalars $z^\alpha$ via $z^\alpha = X^\alpha/X^0$. The matrices $R_{IJ}$ and $I_{IJ}$ are determined in terms of the prepotential by the relations:

$$F_I = \frac{\partial F}{\partial X^I}, \quad F_{IJ} = \frac{\partial^2 F}{\partial X^I \partial X^J}, \quad N_{IJ} = \tilde{F}_{IJ} + 2i \frac{Im(F_{IL})Im(F_{JM})X^LX^M}{Im(F_{PQ})X^PX^Q}$$ (2.2)

and by:

$$R_{IJ} = Re(N_{IJ}) \quad I_{IJ} = Im(N_{IJ})$$ (2.3)

The scalars $z^\alpha = X^\alpha/X^0$ parameterize a special geometry with Kähler potential

$$K = -\ln \left( i(\bar{X}^IF_I - X^I\bar{F}_I) \right), \quad g_{\alpha\bar{\beta}} = \frac{\partial^2 K}{\partial z^\alpha \partial \bar{z}^{\bar{\beta}}}$$ (2.4)

In the following we will neglect worldsheet instanton corrections (alternatively one can work with a type IIA compactification where worldsheet instantons modify the vector multiplet moduli space). The prepotential, in the large volume limit, is then given by

$$F(X^I) = \frac{1}{6} C_{\alpha\beta\gamma} X^\alpha X^\beta X^\gamma X^0$$ (2.5)

where $C_{\alpha\beta\gamma}$ are the intersection numbers of the $H_{1,1}$ cycles on the Calabi-Yau manifold. For the prepotential (2.5) the matrices $R_{IJ}$ and $I_{IJ}$ are given by

$$R_{IJ} = \left( \begin{array}{cc} \frac{1}{3} C_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma & -\frac{1}{2} C_{\alpha\beta\gamma} x^\beta x^\gamma \\ -\frac{1}{2} C_{\alpha\beta\gamma} x^\beta x^\gamma & C_{\alpha\beta\gamma} x^\gamma \end{array} \right) \quad I_{IJ} = \left( \begin{array}{cc} \frac{k}{6} - C_{\alpha\beta\gamma} x^\alpha x^\beta y^\gamma + \frac{3C_{\alpha\beta\gamma} x^\alpha y^\beta C_{\delta\lambda\mu} x^\delta y^\gamma y^\lambda}{2k} & C_{\alpha\beta\gamma} x^\beta y^\gamma - \frac{3C_{\alpha\beta\gamma} y^\beta C_{\delta\lambda\mu} x^\delta y^\gamma y^\lambda}{2k} \\ -\frac{3C_{\alpha\beta\gamma} x^\gamma y^\alpha C_{\delta\lambda\mu} x^\delta y^\beta}{2k} & C_{\alpha\beta\gamma} y^\gamma + \frac{3C_{\alpha\gamma\delta} y^\gamma y^\delta C_{\beta\lambda\mu} y^\alpha y^\lambda}{2k} \end{array} \right)$$ (2.6)

where $k = C_{\alpha\beta\gamma} y^\alpha y^\beta y^\gamma$ and the complex scalars $z^\alpha$ have been split into real and imaginary part $z^\alpha = x^\alpha + iy^\alpha, \alpha = 1, 2, \cdots n_H - 1$. Note that with our conventions the matrix $I_{IJ}$ is positive-definite if $y^a > 0$ for $a = 1, 2, \cdots n_H - 1$.

### 2.4 Supersymmetry variations

The supersymmetry variation parameters $\epsilon^i, \ i = 1, 2$ and the hyperinos $\xi_a, \ a = 1, 2, \cdots 2n_H$ are complex Weyl spinors. The fermionic supersymmetry variations for the gravitino is given by:

$$\delta \psi_\mu^i = D_\mu \epsilon^i + (Q_\mu)^i_j \epsilon^j$$ (2.7)
where $D_\mu$ is the standard covariant derivative which includes the spin connection and $Q^i_j$ is a composite $SU(2)$ gauge connection defined by:

$$Q^i_j = \left( \frac{v - \bar{v}}{4} - \frac{X Im(F)dX - X Im(F)d\bar{X}}{4X Im(F)\bar{X}} \right) \left( \frac{-u}{4} + \frac{X Im(F)dX - X Im(F)d\bar{X}}{4X(ImF)\bar{X}} \right)$$ (2.8)

The hyperino variation is defined as:

$$\delta \xi_a = -iC_{ab}V^b_{\mu} \gamma^\mu \epsilon_i$$ (2.9)

Where $C_{ab}$ is the $Sp(2n_H)$ invariant tensor and $\epsilon_{ab}$ is the two-dimensional antisymmetric tensor. The quaternionic vielbein $V$ is a $2n_H \times 2$ dimensional matrix

$$V^a_i = \begin{pmatrix} u_\mu & v_\mu \\ \epsilon^A_\mu & E^A_\mu \\ -\bar{E}^A_\mu & \bar{\epsilon}^A_\mu \\ -\bar{v}_\mu & \bar{u}_\mu \end{pmatrix}$$ (2.10)

where the components appearing in (2.8) and (2.10) are given by

$$e^A_\mu = e^A_\alpha \partial_\mu \bar{z}^\alpha$$

$$E^A_\mu = -\frac{i}{\sqrt{2}} e^{-\frac{\phi}{2}} e^{A\alpha} \bar{f}^I_\alpha \left( N_{IJ} \partial_\mu \bar{\zeta}^J + \partial_\mu \bar{\zeta}^I \right)$$

$$u_\mu = \frac{i}{\sqrt{2}} e^{\frac{-\phi}{\sqrt{2}}} X^I \left( N_{IJ} \partial_\mu \bar{\zeta}^J + \partial_\mu \bar{\zeta}^I \right)$$

$$v_\mu = \frac{1}{2} \partial_\mu \phi + \frac{i}{2} e^{-\phi} \left( \partial_\mu \sigma + \frac{1}{2} \bar{\zeta}^I \partial_\mu \bar{\zeta}^I - \frac{1}{2} \bar{\zeta}^I \partial_\mu \bar{\zeta}^I \right)$$ (2.11)

where

$$X^I = \left( \begin{array}{c} 1 \\ \bar{z}^\alpha \end{array} \right), \quad f^I_\alpha = D_\alpha e^{K/2} X^I = \left( \partial_\alpha + \frac{\partial_\alpha K}{2} \right) e^{K/2} X^I$$ (2.12)

### 2.5 Shift symmetries

The hypermultiplet action (2.1) is invariant under $2n_H + 1$ shift symmetries. From the ten-dimensional point of view these symmetries arise because the scalars $\zeta^I, \bar{\zeta}^I, I = 0, 1, \ldots, n_H - 1$ are descending from RR tensors which only have derivative couplings. Similarly the axion
σ comes from the dualized NS-NS two-tensor field. The non-trivial Wess-Zumino term in the ten-dimensional IIB action leads to a mixing of σ and ζ^I, \tilde{ζ}^I shifts.

\[ \delta \zeta^I = \gamma^I, \quad \delta \tilde{\zeta}^I = \tilde{\gamma}^I, \quad \delta \sigma = \alpha + \frac{1}{2} \tilde{\gamma}_I \zeta^I - \frac{1}{2} \gamma^I \tilde{\zeta}^I \]  \hspace{1cm} (2.13)

where \gamma^I, \tilde{\gamma}^I, \alpha parameterize the 2n_H + 1 shift symmetries. The shift symmetries (2.13) have generators \Gamma^I, \tilde{\Gamma}^I and E, which satisfy a \( n_H \) dimensional Heisenberg algebra with central element E:

\[ [\Gamma_I, \tilde{\Gamma}^J] = \delta^J_I E, \quad [E, \Gamma_I] = 0, \quad [E, \tilde{\Gamma}^I] = 0 \]  \hspace{1cm} (2.14)

The action (2.1) also has a scaling symmetry

\[ \delta \phi = 2 \epsilon, \quad \delta \sigma = 2 \sigma \epsilon, \quad \delta \zeta^I = \zeta^I \epsilon, \quad \delta \tilde{\zeta}^I = \tilde{\zeta}^I \epsilon, \]  \hspace{1cm} (2.15)

which is generated by \( H \) and satisfies the following commutation relations with the generators of the shifts (2.19):

\[ [H, \Gamma_I] = \frac{1}{2} \Gamma_I, \quad [H, \tilde{\Gamma}^I] = \frac{1}{2} \tilde{\Gamma}^I, \quad [H, E] = E \]  \hspace{1cm} (2.16)

Finally, for a prepotential of the form (2.5), there are additional shift symmetries [31, 32] which involve the real parts of the NS-NS scalars \( z^a = x^a + iy^a \). We denote the generators of these shift symmetries \( B_\alpha \):

\[ \delta x^\alpha = \beta^\alpha, \quad \delta \zeta^\alpha = \beta^\alpha \zeta^0, \quad \delta \tilde{\zeta}_0 = -\beta^\alpha \tilde{\zeta}_0, \quad \delta \tilde{\zeta}_0 = -\beta^\alpha \tilde{\zeta}_0, \quad \delta \tilde{\zeta}_0 = -C_{\alpha\beta\gamma} \beta^\beta \zeta^\gamma, \quad \delta \sigma = 0 \]  \hspace{1cm} (2.17)

These extra axionic symmetries satisfy the commutation relations:

\[ [\Gamma_0, B_\alpha] = -\Gamma_\alpha, \quad [\Gamma_\alpha, B_\beta] = C_{\alpha\beta\gamma} \tilde{\Gamma}^\gamma, \quad [\tilde{\Gamma}^0, B_\beta] = 0, \quad [\tilde{\Gamma}^\alpha, B_\beta] = \delta^\alpha_\beta \tilde{\Gamma}^0, \quad [E, B_\beta] = 0 \]  \hspace{1cm} (2.18)

### 2.6 Quaternionic coset actions

The scalars in extended supergravities with more than eight supersymmetries are always described by sigma models with target spaces which are coset manifolds \( \mathcal{M}_{\text{scalar}} = G/H \). Here \( G \) is a noncompact group and \( H \) is a maximal compact subgroup. The simplest example is a \( SL(2,R)/U(1) \) coset sigma-model with Lagrangian

\[ \mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi \partial^\mu \phi + e^{\alpha\phi} \partial_\mu \chi \partial^\mu \chi \right) \]  \hspace{1cm} (2.19)
The value of the constant $a$ depends on the theory. For example, in ten dimensions for $a = 2$ one gets the action of the dilaton/axion scalars of IIB supergravity [33, 34]. In the case of $N = 2$ theories with hypermultiplets, the action (2.19) will appear as a subsector of the full hypermultiplet action. Instanton and wormhole solution for the action (2.19), in various dimensions and for various values of the parameter $a$, have been discussed in the literature [10, 12, 15, 16].

| coset manifold $G/H$ | dim($G$) | dim($H$) | $n_H$ |
|----------------------|---------|---------|------|
| $SU(n,2)/U(n) \times SU(2)$ | $(n+2)^2 - 1$ | $n^2 + 3$ | $n$ |
| $SO(n,4)/SO(n) \times SU(2) \times SU(2)$ | $(n+4)(n+3)/2$ | $(n-1)/2 + 6$ | $n$ |
| $Sp(n,1)/Sp(n) \times SU(2)$ | $(n+1)(2n+3)$ | $(2n+1)+3$ | $n$ |

Table 2: Infinite series of quaternionic coset spaces

There are cases, where the full hypemultiplet action (2.1) parameterizes a coset manifold. First, there are three infinite sets of cosets, which are non-compact versions of Wolf spaces [35, 36]. The first row of table 2 with $n = 1$ is the $S(2,1)/SU(2) \times U(1)$ coset which parameterizes the universal hypermultiplet and will be discussed briefly in section 4.

Second, there are exceptional cosets which are given in table 3. The coset $G_{2,2}/SU(2) \times SU(2)$ is an eight dimensional quaternionic manifold which will be discussed in detail in section 5. Note that an hypermultiplet sigma model coming from a generic Calabi-Yau compactification will in general not be a coset manifold. The increased symmetry of the coset manifolds makes it easier to find explicit instanton and wormhole solutions.

| coset manifold $G/H$ | dim($G$) | dim($H$) | $n_H$ |
|----------------------|---------|---------|------|
| $G_{2,2}/SU(2) \times SU(2)$ | 14 | 6 | 2 |
| $F_{4,4}/Sp(3) \times SU(2)$ | 52 | 24 | 7 |
| $E_6(+2)/SU(6) \times SU(2)$ | 78 | 38 | 10 |
| $E_7(-5)/SO(12) \times SU(2)$ | 133 | 69 | 16 |
| $E_8(-24)/E_7 \times SU(2)$ | 248 | 136 | 28 |

Table 3: Exceptional quaternionic coset spaces

3 Euclidean instantons and wormholes

In a semiclassical approximation, instantons and wormholes are viewed as saddle-points of the Euclidean action, i.e. they are solutions to the classical Euclidean equations of motion.
Instantons with finite action can provide an important contribution in the path integral calculation of some processes.

In a theory with pseudo-scalars fields that posses shift symmetries (the so-called axionic scalars), the analytic continuation from Minkowskian to Euclidean signature is non-trivial. In particular, regular instanton and wormhole solutions which carry charges associated with axionic scalars only exist if the sign of the kinetic terms for the axionic scalars are flipped as the theory is continued from Minkowskian to Euclidean signature. Since axionic scalars are ubiquitous in supergravity and string theory, it is important to have a sensible prescription for the analytic continuation in order to study non-perturbative effects in string theory.

A first approach is to dualize (in four dimensions) axions to rank-three antisymmetric tensor fields \([3, 14]\) and rewrite the hypermultiplet action as a tensor multiplet action \([19]\). For this theory the analytic continuation to Euclidean signature poses no problems and one obtains a positive-definite action. Dualization and analytic continuation, however, do not commute and one has to pick the order above to obtain a sensible result.

A more formal approach is to replace the Minkowskian quaternionic geometry by a para-quaternionic geometry in Euclidean space. \([13, 37]\) and to define the theory in Euclidean spacetime from the beginning.

In the following we discuss a third approach, originally formulated in a paper by Coleman and Lee \([4]\). This method was applied to the axion of the \(SL(2, R)/U(1)\) coset in \([12]\) and to the universal hypermultiplet in \([38]\). Here we want to apply the method to a general hypermultiplet action.

### 3.1 The Coleman approach

In this section, we consider imaginary-time transition amplitudes between initial and final states with constant values of the hypermultiplet fields:

\[
|I\rangle = |z_0^\alpha, \phi_0, \zeta_0^I, \bar{\zeta}_0^I, \sigma_0\rangle \\
|F\rangle = |z_F^\alpha, \phi_F, \zeta_F^I, \bar{\zeta}_F^I, \sigma_F\rangle
\] (3.1)

Following the approach of \([4]\), we can project initial and final states into eigenspaces of the shift symmetry charge densities inserting delta function projectors of the form:

\[
\delta(\rho_0,F - j_S\bar{x})|I,F\rangle \propto \int \mathcal{D}\alpha \exp \left\{ -i \int d^3\bar{x} \alpha(\bar{x})[\rho_0,F(\bar{x}) - j_S\bar{x}(\bar{x})] \right\}|I,F\rangle
\] (3.2)

Here \(j_S(\bar{x})\) is the Noether charge density corresponding to the shift of some field \(\chi\).

\[
j_S(\bar{x}) = \frac{\delta S}{\delta \partial_\tau \chi(\bar{x})}
\] (3.3)
\( \rho_0(\vec{x}) \) and \( \rho_F(\vec{x}) \) are the charge density eigenvalues. The overall amplitude can be obtained summing over the charge density eigenspaces. As we shall see, each term of this sum can be expressed through a path integral dominated by a single saddle-point of the Euclidean action. These saddle-points are exactly the instantons and wormholes which we will analyze in this paper.

The hypermultiplet action shift charge densities obey to commutation relations of the form:

\[
[j^I_a(\vec{x}, \tau), j^J_{n_H} + J(\vec{y}, \tau)] = \delta_{IJ} \delta^{3}(\vec{x} - \vec{y}) \delta^I_E(\vec{x}, \tau) \tag{3.4}
\]

where \( j^I_a \), \( j^J_{n_H} \), and \( j^E \) are the charge densities associated to the symmetries \( \Gamma_I \), \( \tilde{\Gamma}^J \) and \( E \). Because of the non-trivial commutation relation (3.4), initial and final states cannot be projected into eigenspaces of all the \( 2n_H + 1 \) shift densities. Instead, \( |I\rangle \) and \( |F\rangle \) can be decomposed into irreducible representations of the \( n_H \)-dimensional Heisenberg group \( H_n \) generated by the shift symmetries. Elements of such representations can be labeled by the charge density eigenvalues of a properly chosen set of commuting generators.

According to the Stone-Von Neumann theorem, there is a unique unitary irreducible representation of the Heisenberg group \( H_n \) for each value the central element \( E \). There are two qualitatively different cases:

- **Vanishing \( E \) charge density**: The central element in the algebra (3.4) is zero and it follows from the Stone-Von Neumann theorem that we can project initial and final states into eigenspaces of all the shift densities \( j^a_a, a = 0, 1, \cdots, 2n_H - 1 \). Saddle points of this kind correspond to instantons and wormholes charged only under the RR scalars shift symmetries (pure RR-charged instantons and wormholes).

- **Nonzero \( E \) charge density**: If we project initial and final states into eigenspaces of non-zero \( E \) charge density, elements within the same eigenspace belong to a unique irreducible representation of \( H_n \) with non-zero value of the central element. In the analogy with standard quantum mechanics, \( j^a_I, I = 0, 1, \cdots n_H - 1 \) play the role of position operators and \( j^J_{n_H + J}, J = 0, 1, \cdots n_H - 1 \) play the role of momentum operators. \( j^E \) is a central element and can be identified with \( \hbar \). These saddle-points correspond to instantons and wormholes charged under the NS-NS shift symmetry (NS-charged instantons and wormholes).

### 3.2 Classification of analytic continuations

In this section we will classify the possible analytic continuations corresponding to the two cases discussed above. The analytic continuation for the pure RR-charged and the NS-charged cases are different. Morover, in case of a prepotential of the form (2.5), we need
to take into account the extra shift symmetries $B_\alpha$. We will see in the next section that extra conditions need to be satisfied in order for the action to have a real saddle-point after analytic continuation.

### 3.2.1 Pure RR-charged case

For analyzing pure RR-charged instanton and wormhole solutions it is convenient to define:

$$\chi^I = \zeta^I, \quad \chi^{n_H + I} = \tilde{\zeta}^I, \quad \tilde{\sigma} = \sigma + \frac{1}{2} \chi^I \chi^{n_H + I}$$  \hspace{1cm} (3.5)

After analytic continuation to Euclidean time $t \to -i\tau$, the action (2.1) can be rewritten as follows:

$$S_E = \int d^4x \sqrt{g} \left\{ -R + 2g_{\alpha\bar{\beta}} \partial_\mu z^\alpha \partial^\mu \bar{z}^\beta + \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} e^{2\phi} (j_{E\mu})^2 + \frac{1}{2} e^{-\phi} M_{ab} \partial_\mu \chi^a \partial^\mu \chi^b \right\}$$  \hspace{1cm} (3.6)

With $a = 0 \ldots 2n_H - 1$. The matrix $M$ is positive-definite and given by:

$$M = \begin{pmatrix} I^{-1} & I^{-1}R \\ RI^{-1} & RI^{-1}R + I \end{pmatrix}$$  \hspace{1cm} (3.7)

with the matrices $R$ and $I$ from equation (2.3). We project initial and final state into charge density eigenspaces of the commuting $2n_H$ shift charges:

$$P_I \mathcal{I} = \delta(j_0^E) \prod_a \delta(\rho_{a0} - j_0^a) |I\rangle$$

$$\propto \int \mathcal{D} \alpha \prod_a \mathcal{D} \gamma a e^{i \int d^3\vec{x}\alpha j_0^E e^{-i \int d^3\vec{x}\gamma a (\rho_{a0} - j_0^a) x_0^a, \phi_0, \chi_0^a, \tilde{\sigma}_0}}$$

$$= \int \mathcal{D} \alpha \prod_a \mathcal{D} \gamma a e^{-i \int d^3\vec{x}\gamma a \rho_{a0} x_0^a, \phi_0, \chi_0^a + \gamma a, \tilde{\sigma}_0 + \gamma^{n_H + I} \chi^I + \alpha}$$  \hspace{1cm} (3.8)

Redefining:

$$\chi^a \to \chi^a - \gamma^a, \quad \tilde{\sigma} \to \tilde{\sigma} - \gamma^{n_H + I} \chi^I - \alpha$$  \hspace{1cm} (3.9)

The transition amplitude becomes:

$$\langle F| P_F e^{-H(\tau_F - \tau_0)} \delta(j_0^E) P_I |I\rangle = e^{i \int (\rho_{a0} \chi_0^a - \rho_{aF} \chi_F^a)} \int \mathcal{D} \Phi e^{-(S_E + \Sigma)}$$  \hspace{1cm} (3.10)

Where $\Sigma$ is a surface term given by:

$$\Sigma = i \int d^3\vec{x} [\rho_{a0}(\vec{x}) \chi^a(\vec{x}, \tau_0) - \rho_{aF}(\vec{x}) \chi^a(\vec{x}, \tau_F)]$$  \hspace{1cm} (3.11)
As an effect of the redefinition (3.9), the functional integration of the fields $\chi^a$ and $\sigma$ goes over configurations without fixed initial and final values. In particular $\chi^a(\vec{x}, \tau_{0,F})$ and $\tilde{\sigma}(\vec{x}, \tau_{0,F})$ do not equal $\chi^a_{0,F}(\vec{x})$ and $\tilde{\sigma}_{0,F}(\vec{x})$. Varying the action with respect to $\chi^a$ and $\tilde{\sigma}$ on the boundary leads to:

$$\frac{\delta S_E}{\delta \partial_T \chi^a} \bigg|_{\tau_{0,F}} = j_a^T = i\rho_{a,F}$$

$$\frac{\delta S_E}{\delta \partial_T \tilde{\sigma}} \bigg|_{\tau_{0,F}} = \tilde{j}_E = 0 \quad (3.12)$$

We can see that because of the surface term $\Sigma$, the path integral is dominated by a complex saddle-point. In order to evaluate the path integral with the semiclassical approximation, we have to analytically continue:

$$\chi^a \rightarrow i\chi^{'a}, \quad \tilde{\sigma} \rightarrow \tilde{\sigma}' \quad (3.13)$$

while the currents are continued as:

$$j_a^T \rightarrow ij_a'^T \quad (3.14)$$

After this analytic continuation, the action has a real saddle-point and the analytically continued currents obey the following relation

$$j_a'^T = \rho_{a,F} \quad \tilde{j}_E = 0 \quad (3.15)$$

### 3.2.2 NS-charged case

In case of NS-charged instanton and wormholes solutions, the choice of projectors is not unique. For every $Sp(2n_H, R)$ matrix $S$, we can define:

$$\left( \begin{array}{c} \chi^I \\ \chi^{n_H+1} \end{array} \right) = S \left( \begin{array}{c} \xi^I \\ \xi_I \end{array} \right) \quad \tilde{\sigma} = \sigma + \frac{1}{2} \chi^I \chi^{n_H+1} \quad (3.16)$$

and use the shift symmetries of the $\chi^I (I = 0 \ldots n_H - 1)$ as the commuting generators. The Euclidean hypermultiplet action can then be rewritten as:

$$S_E = \int d^4x \sqrt{g} \left\{ -R + 2g_{\alpha\beta}\partial_\mu z^\alpha \partial^\mu z^\beta + \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}e^{-2\phi}(\partial_\mu \tilde{\sigma} - \chi^{n_H+1} \partial_\mu \chi^I)^2 \\
+ \frac{1}{2}e^{-\phi} M_{ab} \partial_\mu \chi^a \partial^\mu \chi^b \right\} \quad (3.17)$$
With this notation $\tilde{M} = S^T M S$, $M$ given by (3.7) and $\tilde{M}$ is positive-definite. Now we can project initial and final states into charge density eigenspaces corresponding to the shifts of $\tilde{\sigma}$ and $\chi^I$:

$$P_I | I \rangle = \delta(\rho_E - j^E_I) \prod_I \delta(\rho_{I0} - j^E_I) | I \rangle$$

$$\propto \int D\alpha \prod_I D\gamma_I e^{-i \int d^3 \bar{x}(\alpha \rho_{E0} + \gamma^I \rho_{I0}) | z_0^\alpha, \phi_0, \chi^I_0 + \gamma^I, \chi^{n_H+I}_0, \tilde{\sigma} + \alpha \rangle}$$

The transition amplitude becomes:

$$\langle F | P_F e^{-H(\tau_F - \tau_0)} P_I | I \rangle = e^{i \int (\rho_{I0} \chi^I_0 + \rho_{E0} \tilde{\sigma} - \rho_{IF} \chi^I_F - \rho_{EF} \tilde{\sigma}_F)} \int D\Phi e^{-(S_E + \Sigma)}$$

The surface term $\Sigma$ is given by:

$$\Sigma = i \int d^3 \bar{x} [\rho_{I0}(\bar{x}) \chi^I(\bar{x}, \tau_0) + \rho_{E0}(\bar{x}) \tilde{\sigma}(\bar{x}, \tau_0) - \rho_{IF}(\bar{x}) \chi^I_F(\bar{x}, \tau_F) - \rho_{EF}(\bar{x}) \tilde{\sigma}_F]$$

As in the pure RR-charged case, the functional integration of the fields $\chi^I$ and $\tilde{\sigma}$ goes over configurations which do not have fixed initial and final values. In order to evaluate the path integral with the semiclassical approximation, we have to use a different kind of analytic continuation:

$$\chi^I \rightarrow i\chi'^I$$
$$\chi^{n_H+I} \rightarrow \chi^{n_H+I}$$
$$\tilde{\sigma} \rightarrow i\tilde{\sigma}'$$

Varying the action with respect to $\chi^I$ and $\tilde{\sigma}$ on the boundary leads to:

$$j^I_{iF}(\bar{x}, \tau_{0,F}) = \rho_{I0,F}$$
$$j^E_I(\bar{x}, \tau_{0,F}) = \rho_{E0,F}$$

**3.2.3 Large Calabi-Yau Manifolds**

In case of a prepotential of the form (2.5), there are extra shift symmetries corresponding to the shift of the $n_H - 1$ NS-NS axions $x^\alpha$. In the general case, the number of commuting generators is still $n_H$, but there are extra possible choices for the analytic continuation.

In particular, the generators $\tilde{\Gamma}^\alpha$, $B_\alpha$ and $\tilde{\Gamma}^0$ form a $n_H - 1$ dimensional Heisenberg algebra with central element $\tilde{\Gamma}^0$. We can re-define some of the axions so that $n_H$ of above symmetries
become simple shift symmetries:

\[
\chi^0 = \tilde{\zeta}_0 + \frac{1}{2} x^\alpha \zeta_0 + \frac{1}{2} \chi^{n_H+\alpha} + \frac{1}{12} C_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \zeta^0
\]

\[
\left( \begin{array}{c}
\chi^\alpha \\
\chi^{n_H+\alpha}
\end{array} \right) = S \left( \begin{array}{c}
\tilde{\zeta}_\alpha + C_{\alpha\beta\gamma} x^\beta \zeta^\gamma - \frac{1}{2} C_{\alpha\beta\gamma} x^\beta x^\gamma \zeta^0
\end{array} \right)
\]

\[
\chi^{n_H} = \sigma - \frac{1}{2} \tilde{\zeta}^I \tilde{\zeta}_I - \frac{1}{2} C_{\alpha\beta\gamma} x^\alpha \zeta^\beta \zeta^\gamma + \frac{1}{2} C_{\alpha\beta\gamma} x^\alpha x^\beta \zeta^\gamma \zeta^0 - \frac{1}{6} C_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \zeta^0^2
\]

\[
\chi^{2n_H} = \zeta^0
\]

\[
\chi^{2n_H+\alpha} = \zeta^\alpha - x^\alpha \zeta^0
\]

(3.23)

Here \( S \) is a \( Sp(2n_H - 2, R) \) matrix. We can then project initial and final states into charge density eigenstates corresponding to the shifts of \( \chi^I \) with \( I = 0 \ldots n_H \) as done in the NS-charged case. We obtain a third class of analytic continuations:

\[
\chi^I \rightarrow i\chi^{I'}
\]

\[
\chi^{n_H+\alpha} \rightarrow \chi^{n_H+\alpha}
\]

\[
\chi^{2n_H+I} \rightarrow \chi^{2n_H+I}
\]

(3.24)

Similarly, the case of vanishing \( \tilde{\Gamma}^0 \) leads to \( 2n_H - 2 \) commuting generators and is analogous to the pure RR-charged case.

### 3.3 Positive definite action for instantons and wormholes

In this section we will study the conditions which need to be satisfied in order for the actions \( (3.6) \) and \( (3.17) \) to be positive-definite for instanton and wormhole solutions. As we shall see, the boundary term introduced in section 3.2 is essential to obtain a positive-definite action. As we shall see, this condition restricts the possible analytic continuations in some cases.

#### 3.3.1 Pure RR-charged case

In case of pure RR-charged instanton and wormhole solutions, the RR scalars shift symmetries have the following Noether currents:

\[
\mathcal{J}_{a\mu} = e^{-\phi'} M_{ab} \partial_\mu \chi^b
\]

(3.25)

Using the equations of motion \( (3.12) \) and the analytic continuation \( (3.13) \), the surface term can be rewritten:

\[
\Sigma = \int d^4 x \partial_\mu (\sqrt{g} j^a \chi^a) = \int d^4 x \sqrt{g} \partial_\mu \chi^a j^a_{\mu} = \int d^4 x \sqrt{g} (e^{-\phi'} M_{ab} \partial_\mu \chi^b \partial_\mu \chi^b)
\]

(3.26)
The analytic continuation \(^{(3.13)}\) flips the sign of the kinetic term for the \(\chi^a\) fields in the bulk part of the action \(^{(3.6)}\). As discussed before this flip of the sign in the action is essential for the existence of regular instanton and wormhole solutions. After the flip of the sign, the bulk part of the action is not positive-definite. However, adding the boundary term \(^{(3.26)}\) makes \(S_E + \Sigma\) manifestly positive-definite:

\[
S_E + \Sigma = \int d^4x \sqrt{g} \left\{ -R + 2g_{\alpha \beta} \partial_\mu z^\alpha \partial^\mu z^\beta + \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} e^{-\phi'} M_{ab} \partial_\mu \chi^a \partial^\mu \chi^b \right\} \tag{3.27}
\]

### 3.3.2 NS-charged case

For the NS-charged case we perform the analytic continuation \(^{(3.21)}\). After analytic continuation the relevant shift symmetries have Noether currents:

\[
j'^I_{\mu} = e^{-\phi' \tilde{M}_{IJ}} \partial_\mu \chi'^I - ie^{-\phi' \tilde{M}_{IJ}} \partial_\mu \chi'_{nH+j} \partial_\mu \chi'^{nH+j} - \delta_{IJ} \chi'\partial_\mu \chi'^{nH+j} \tag{3.28}
\]

\[
j'_{E\mu} = e^{-2\phi'} (\partial_\mu \tilde{\sigma}' - \chi'_{nH+j} \partial_\mu \chi'^{nH+j}) \tag{3.29}
\]

Because of the second term of \(^{(3.28)}\), the analytically continued action will not have a real saddle-point in the general case. Using the equations of motion \(^{(3.32)}\) and the analytic continuation \(^{(3.21)}\), the surface term can be rewritten as follows:

\[
\Sigma = \int d^4x \partial_\mu [\sqrt{g} (j'^I_{\mu} \chi'^I + j'_{E\mu} \tilde{\sigma}')] = \int d^4x \sqrt{g} (\partial_\mu \chi'^I j'^{\mu I} + \partial_\mu \tilde{\sigma}' j'^{\mu I})
\]

\[
= \int d^4x \sqrt{g} [e^{-\phi' \tilde{M}_{IJ}} \partial_\mu \chi'^I \partial_\mu \chi'^{nH+j} - ie^{-\phi' \tilde{M}_{IJ}} \partial_\mu \chi'^{nH+j} + (j'_{E\mu})^2] \tag{3.30}
\]

The action \(^{(3.17)}\) after the analytic continuation \(^{(3.21)}\) can be written as:

\[
S_E = \int d^4x \sqrt{g} \left\{ -R + 2g_{\alpha \beta} \partial_\mu z^\alpha \partial^\mu z^\beta + \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} e^{-2\phi'} (j'_{E\mu})^2 - \frac{1}{2} e^{-\phi'} \tilde{M}_{IJ} \partial_\mu \chi'^I \partial_\mu \chi'^{nH+j} + \frac{1}{2} e^{-\phi'} \tilde{M}_{IJ} \partial_\mu \chi'^{nH+j} \right\} \tag{3.31}
\]

\[
S_E + \Sigma \text{ becomes:}
\]

\[
S_E + \Sigma = \int d^4x \sqrt{g} \left\{ -R + 2g_{\alpha \beta} \partial_\mu z^\alpha \partial^\mu z^\beta + \frac{1}{2} e^{2\phi'} (j'_{E\mu})^2 + \frac{1}{2} e^{-\phi'} \tilde{M}_{IJ} \partial_\mu \chi'^{nH+j} \partial_\mu \chi'^{nH+j} + \frac{1}{2} e^{-\phi'} \tilde{M}_{IJ} \partial_\mu \chi'^{nH+j} \partial_\mu \chi'^{nH+j} \right\} \tag{3.32}
\]

Note the first term in the second line of the equation for the euclidean action \(^{(3.31)}\) is imaginary. The equations of motion derived from \(^{(3.31)}\) imply that the solution is complex.
A further analytic continuation \( \chi^{nH+I} \rightarrow i\chi^{nH+I}, \sigma \rightarrow i\sigma \) can be used to obtain real equations of motion. Note, however, that in this case the total action \( S_E + \Sigma \) is not positive definite anymore. Consequently, it is not guaranteed that after the further analytic continuation saddle point solution will give a dominant contribution to the path integral. For example, it might be possible that for some assignments of charges the instanton action could be negative and the multi instanton contribution in the dilute gas approximation would diverge.

One way to obtain real positive definite saddle point solutions, is to impose the following condition on the solution
\[
\tilde{M}_{I nH+J} \partial_{\mu} \chi^I \partial^\mu \chi^{mH+J} = 0 \tag{3.33}
\]
This eliminates the imaginary part in the equations of motion derived from (3.31). Consequently, the solution are real and the saddle point action including the boundary term (3.32) is positive definite.

Note, however, that the condition is not a constant of motion and it will lead to a truncation of the space of all solutions of the equations of motion. In the following we will impose the condition (3.33) and discuss the resulting truncations in section 5 using the concrete example of the \( G_{2,2}/SU(2) \times SU(2) \) coset.

It is an important open problem whether the more general solutions, where (3.33) is not imposed, make physical sense and contribute to the instanton induced effective action. In this paper we will not discuss the complexified solutions further.

### 3.4 SO(4)-invariant solutions

We will now focus on \( SO(4) \) invariant solutions of the equations of motion obtained by varying the actions (3.6) and (3.17). It is convenient to start with the following ansatz for the Euclidean metric:
\[
ds^2_E = \frac{e^{3U}}{4\tau^3} d\tau^2 + \frac{e^U}{\tau} d\Omega^2
\tag{3.34}
\]
Here \( d\Omega_3^2 \) is the metric of the unit three-sphere and \( \tau \) is a radial coordinate. The ansatz reduces to the flat metric in case \( U(\tau) \equiv 0 \), with the identification \( \tau = 1/r^2 \). Moreover, as a consequence of the \( SO(4) \) invariance, the scalar fields depend only on \( \tau \). With this choice for the metric, Einstein equation gives:
\[
G_{\tau \tau} = \frac{3}{4\tau^2} (1 - e^{2U} - 2\tau \dot{U} + \tau^2 \ddot{U}^2) = T_{\tau\tau}
\]
\[
G_{\alpha\beta} = e^{-2U} (1 - e^{2U} + 6\tau \dot{U} - 3\tau^2 \ddot{U}^2 + 4\tau^2 \dddot{U}) \eta_{\alpha\beta} = T_{\alpha\beta}
\tag{3.35}
\]
where the dot indicates a derivative with respect to \( \tau \) and \( \eta_{\alpha\beta} \) is the metric on the unit three-sphere. The radial and angular components of the energy-momentum tensor can be
obtained from (3.6) for the pure RR-charged case:

\[
T_{\tau\tau} = 2g_{\alpha\beta} \partial_{\tau} z^\alpha \partial_{\tau} z^\beta + \frac{1}{2} (\partial_{\tau} \phi')^2 - \frac{1}{2} e^{-\phi'} M_{ab} \partial_{\tau} \chi^a \partial_{\tau} \chi^b
\]

\[
T_{\alpha\beta} = -4\tau^2 e^{-2U} T_{\tau\tau} \eta_{\alpha\beta}
\]

(3.36)

For the NS-charged case one gets from (3.17)

\[
T_{\tau\tau} = 2g_{\alpha\beta} \partial_{\tau} z^\alpha \partial_{\tau} z^\beta + \frac{1}{2} (\partial_{\tau} \phi')^2 - \frac{1}{2} e^{-\phi'} (\partial_{\tau} \tilde{\phi}' - \chi^{nH+I} \partial_{\tau} \chi^I)^2 - \frac{1}{2} e^{-\phi'} \tilde{M}_{IJ} \partial_{\tau} \chi^I \partial_{\tau} \chi^J + ie^{-\phi'} \tilde{M}_{nH+J} \partial_{\tau} \chi^{nH+J} \partial_{\tau} \chi^{mH+J}
\]

(3.37)

A linear combination of the (3.36) depends only from the function \( U(\tau) \) and not on the energy momentum tensor. This gives a second order ordinary differential equation for the metric factor \( U(\tau) \).

\[
\partial^2 U = \frac{e^{2U} - 1}{\tau^2}
\]

(3.38)

All solutions can be brought in a form where \( U(\tau) \to 0 \) as \( \tau \to 0 \) with a simple rescaling of the radial coordinate \( \tau \). The equation (3.38) has two linearly independent solution. The first solution has the form:

\[
\begin{align*}
U(\tau) &= 4\sqrt{c} \tau \\
\end{align*}
\]

(3.39)

where \( c \) is a positive constant and \( \tau \) can assume any value from 0 to \( +\infty \). These solutions are always singular for \( \tau \to 0 \) and will not be studied in this paper. The second solution has the form:

\[
\begin{align*}
U(\tau) &= 4\sqrt{c} \frac{\tau}{\sin 4\sqrt{c} \tau} \\
\end{align*}
\]

(3.40)

The radial coordinate \( \tau \) can assume any value from 0 to \( \pi/4\sqrt{c} \). These solutions are regular and exhibit two flat asymptotic regions (\( \tau \to 0 \) and \( \tau \to \pi/4\sqrt{c} \)) connected by a wormhole. The neck of the wormhole is located at \( \tau = \pi/8\sqrt{c} \). The area of the three sphere at the neck is given by

\[
\text{Area}(S^3_{\text{neck}}) = 2\pi^2 R^3_{\text{neck}} = 16\pi^2 c^{3/4}
\]

(3.41)

Using equation (3.40) we can rewrite the first equation of (3.35) as follows for the pure RR-charged case

\[
-24 c = 2g_{\alpha\beta} \partial_{\tau} z^\alpha \partial_{\tau} z^\beta + \frac{1}{2} (\partial_{\tau} \phi')^2 - \frac{1}{2} e^{-\phi'} M_{ab} \partial_{\tau} \chi^a \partial_{\tau} \chi^b
\]

(3.42)
For the NS-charged case \((3.36)\) one obtains

\[
-24c = g_{\alpha \beta} \partial_\tau z^\alpha \partial_\tau z^{\bar{\beta}} + \frac{1}{2} (\partial_\tau \phi')^2 - \frac{1}{2} e^{-2\phi'} (\partial_\tau \delta' - \chi^{m_H+I} \partial_\tau \chi'^I)^2 - \frac{1}{2} e^{-\phi'} \tilde{M}_{IJ} \partial_\tau \chi'^I \partial_\tau \chi'^J + ie^{-\phi'} \tilde{M}_{n_H+J} \partial_\tau \chi'^I \partial_\tau \chi'^J \quad (3.43)
\]

These equations contain only the first derivatives of the fields. The limit \(c \to 0\) of \((3.40)\) gives the flat metric. Solutions of this kind are the extremal instantons. Note that for the \(SO(4)\) symmetric NS-charged solution the reality condition \((3.33)\) becomes

\[
e^{-\phi'} \tilde{M}_{n_H+J} \partial_\tau \chi'^I \partial_\tau \chi'^J = 0 \quad (3.44)
\]

If this condition is satisfied, the constraint \((3.43)\) can be satisfied by a real solution. As mentioned earlier, in general \((3.44)\) is not a conserved quantity, i.e. its time derivative does not vanish if the equations of motion are satisfied. This implies that \((3.44)\) imposes severe constraints on the solution since it has to be obeyed for all values of \(\tau\). As well shall see for the explicit example of \(G_{2,2}/SU(2) \times SU(2)\) coset only a few consistent truncations satisfy \((3.44)\).

### 3.5 BPS-condition, Extremality, non-extremality and attractors

In this section we limit ourselves to \(SO(4)\) invariant solutions. A bosonic solution preserves half of the supersymmetries if there exists a linear combination of the supersymmetry parameters \(\epsilon^1\) and \(\epsilon^2\) for which the gravitino \((2.7)\) and hyperino variation \((2.9)\) vanish. The gravitino variation determine the radial dependence of the unbroken supersymmetry, while the hyperino variation will determine which linear combination of the supersymmetries is unbroken. The condition that the hyperino variation \(\delta \xi_a = 0\) vanishes for the unbroken supersymmetry variation is equivalent to the statement that the the quaternionic vielbein \(V\) defined in \((2.10)\) is a \(2n_H \times 2\) dimensional matrix has non-maximal rank. This is the case if the \(2n_H\) dimensional columns are proportional.

\[
\begin{pmatrix}
  u_\mu \\
  e_\mu^A \\
  -\bar{E}_\mu^A \\
  -\bar{v}_\mu
\end{pmatrix} = c \begin{pmatrix}
  v_\mu \\
  E_\mu^A \\
  \bar{e}_\mu^A \\
  \bar{u}_\mu
\end{pmatrix} \quad (3.45)
\]

where the complex constant \(c\) determines the linear combination of the unbroken susy:
\[
\begin{pmatrix}
\epsilon^1 \\
\epsilon^2
\end{pmatrix} = \epsilon \begin{pmatrix}
1 \\
c
\end{pmatrix}
\]

(3.46)

Note that it follows from (3.45) that

\[u_\tau \bar{u}_\tau + v_\tau \bar{v}_\tau + e^A_\tau \bar{e}^A_\tau + E^A_\tau \bar{E}^A_\tau = 0\]  

(3.47)

Using the explicit form of the vielbein components (2.11) and the following identity of special geometry

\[f^I_\alpha \bar{f}^J_\beta g^{\alpha \beta} + e^K \bar{Z}^I Z^J = \frac{1}{2} (I^{-1})^{IJ}\]

(3.48)

One can show that the left hand side of (3.47) is proportional to \(T_{\tau \tau}\). It follows that all half-BPS solutions have \(c = 0\) and are therefore extremal instanton solutions. Note that the extremality condition (3.47) is single equation whereas the BPS conditions (3.45) are \(2n_H\) equations. It is therefore possible if \(n_H > 1\) to have an extremal solutions which break all supersymmetries. We will discuss such a case for the \(G_{2,2}/SU(2) \times SU(2)\) coset in section 5.

For a very similar system of hypermultiplets in five-dimensional supergravity it was shown in [20] (see also [39, 40, 41] for discussion for four-dimensional N=2 supergravity) that the BPS equations for pure RR-charged solutions are equivalent to the BPS attractor equations. The purely RR-charged instanton solution is related via the c-map to extremal BPS black holes. Similarly the extremal non-BPS instanton solutions are related to extremal non-BPS black hole solutions. For recent work on the attractor mechanism for extremal non-BPS black holes see e.g. [42, 43, 44, 45, 46].

Note also, that the \(SO(4)\) symmetric BPS-instanton solutions that can be mapped by the c-map to BPS black holes correspond to single center black holes. For black holes in \(N = 2\) supergravity there are however multicenter black hole solutions which are BPS invariant. If the c-map relates these solutions to instantons they would not be \(SO(4)\) invariant. Since multicenter black hole solutions are stationary instead of static they would necessarily be NS-charged. It is an interesting question whether such instantons exist and contribute to the path integral in the semiclassical approximation.

The extremal BPS instanton solutions break four of the eight real supersymmetries. The broken supersymmetries induce fermionic zero-modes and correlation function are non-zero only when the zero modes are soaked up by appropriate operator insertions [14, 51].

Four fermionic zero modes induce non-perturbative four-fermion terms which couple to the curvature tensor of the \(N = 2\) sigma model. By supersymmetry such terms are related to kinetic terms for the scalars in the sigma model. Hence instantons provide non-perturbative corrections to the geometry of the quaternionic hypermultiplet moduli space. For a recent discussion of such issues see [21, 52, 53, 54].
4 The $SU(2,1)/U(2)$ coset model

The universal hypermultiplet action can be derived from the general formulae given in section by setting $n_H = 1$ and $F = iX_0^2/2$. The action is given by:

$$S = \int d^4x \sqrt{g} \left\{ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} e^{2\phi} (\partial_\mu \phi + \tilde{\zeta} \partial_\mu \zeta)^2 + \frac{1}{2} e^\phi ((\partial_\mu \zeta)^2 + (\partial_\mu \tilde{\zeta})^2) \right\}$$

(4.1)

The possible analytic continuations and associated instanton and wormhole solutions for the universal hypermultiplet were discussed by the authors in a previous paper [38]. In this section we briefly review the results in the interest of completeness.

The action (4.1) is the sigma model action for the $SU(2,1)/SU(2) \times U(1)$ coset. The coset has eight global $SU(2,1)$ symmetries. The eight infinitesimal generators are given by

$$E \begin{cases} \delta \phi = 0 \\ \delta \sigma = 1 \\ \delta \zeta = 0 \\ \delta \tilde{\zeta} = 0 \end{cases} \quad E_q \begin{cases} \delta \phi = 0 \\ \delta \sigma = 0 \\ \delta \zeta = -\sqrt{2} \\ \delta \tilde{\zeta} = 0 \end{cases} \quad E_p \begin{cases} \delta \phi = 0 \\ \delta \sigma = \sqrt{2} \zeta \\ \delta \zeta = 0 \\ \delta \tilde{\zeta} = -\sqrt{2} \end{cases} \quad H \begin{cases} \delta \phi = 2 \\ \delta \sigma = 2 \sigma \\ \delta \zeta = \zeta \\ \delta \tilde{\zeta} = \tilde{\zeta} \end{cases}$$

$$F_p \begin{cases} \delta \phi = \sqrt{2} \zeta \\ \delta \sigma = \sqrt{2} (\zeta^3 - 3 \zeta \tilde{\zeta}^2) \\ \delta \zeta = \sqrt{2} \left( \sigma + \frac{3}{2} \zeta \tilde{\zeta} \right) \\ \delta \tilde{\zeta} = -\sqrt{2} \left( e^\phi + \frac{3}{4} \zeta^2 - \frac{1}{4} \tilde{\zeta}^2 \right) \end{cases} \quad F_q \begin{cases} \delta \phi = \sqrt{2} \zeta \\ \delta \sigma = \sqrt{2} \left( \sigma \zeta + e^\phi \tilde{\zeta} + \frac{1}{2} \zeta^3 \right) \\ \delta \zeta = \sqrt{2} \left( -e^\phi + \frac{1}{4} \zeta^2 - \frac{3}{4} \tilde{\zeta}^2 \right) \\ \delta \tilde{\zeta} = \sqrt{2} \left( -\sigma + \frac{1}{2} \zeta \tilde{\zeta} \right) \end{cases}$$

$$J \begin{cases} \delta \phi = 0 \\ \delta \sigma = \frac{1}{2} (\tilde{\zeta}^2 - \zeta^2) \\ \delta \zeta = -\zeta \\ \delta \tilde{\zeta} = \zeta \end{cases} \quad F \begin{cases} \delta \phi = -(2 \sigma + \zeta \tilde{\zeta}) \\ \delta \sigma = e^{2\phi} - \sigma^2 + e^\phi \tilde{\zeta}^2 - \frac{1}{16} \zeta^4 + \frac{3}{16} \zeta^4 + \frac{1}{8} \zeta^2 \tilde{\zeta}^2 \\ \delta \zeta = \sigma \zeta - e^\phi \zeta - \frac{3}{4} \zeta^2 \tilde{\zeta} - \frac{1}{4} \tilde{\zeta}^3 \\ \delta \tilde{\zeta} = -\sigma \tilde{\zeta} + e^\phi \zeta + \frac{1}{4} \zeta^3 - \frac{1}{4} \zeta \tilde{\zeta}^2 \end{cases}$$

(4.2)

The relation of the symmetry generators and the roots of $SU(2,1)$ are given in Figure 4. The eight global symmetries lead to eight Noether currents given by:

$$j_\mu^a = \sum_{i=1}^4 \frac{\delta L}{\delta (\partial_\mu \Phi_i)} \delta_a \Phi_i$$

(4.3)

The shift symmetries of the NS-NS axion $\sigma$ and the RR axions $\zeta$ and $\tilde{\zeta}$ are generated by $E, E_q$ and $E_p$ respectively and form a Heisenberg algebra. We have to distinguish two cases depending on whether the central element vanishes or not.
For a zero value of the charge density $j_E$, the initial and final states can be projected onto eigenstates of $j_{E_p}$ and $j_{E_q}$. This case is called "pure RR charged". Applying the Coleman approach the path integral is dominated by a complex saddle-point where both $\zeta$ and $\tilde{\zeta}$ are pure imaginary. We can make the saddle-point real by the analytic continuation

$$\zeta \rightarrow i\zeta', \quad \tilde{\zeta} \rightarrow i\tilde{\zeta}'$$

(4.4)

For a non-zero value of the charge density $\rho_E$, the initial and final states are projected onto eigenstates of fixed $\rho_E$ and $\rho_{E_q}$. This case is called "mixed NS-R charged". Applying the Coleman approach the path integral is dominated by a complex saddle-point where both $\zeta'$ and $\sigma'$ are pure imaginary. We can make the saddle-point real by the analytic continuation

$$\zeta \rightarrow i\zeta', \quad \sigma \rightarrow i\sigma'$$

(4.5)

In both cases the instanton and wormhole solutions can be expressed in terms of the conserved charges. See our previous paper [38] for more details.
The next simplest example is the quaternionic symmetric space $G_{2,2}/SU(2) \times SU(2)$. This model has $n_H = 2$ and corresponds to the prepotential $F = (X^1)^3/X^0$. The complex scalar $z$ can be split into real and imaginary part $z = x + iy$. The Euclidean action (before any analytic continuations are performed) is:

$$S_E = \int \sqrt{g} \left\{ \frac{3}{2} \frac{(\partial_{\mu} x)^2 + (\partial_{\mu} y)^2}{y^2} + \frac{1}{2} (\partial_{\mu} \phi)^2 + \frac{1}{2} e^{-2\phi} (\partial_{\mu} \sigma - \xi^I \partial_{\mu} \tilde{\xi}_I)^2 + \frac{1}{2} e^{-\phi} \partial_{\mu} \xi^T M \partial^\mu \xi \right\} \right.$$  \hspace{1cm} (5.1)

where the matrix $M_{ab}$ is given by

$$M = \frac{1}{y^3} \begin{pmatrix} 1 & x & -x^3 & 3x^2 \\ x & y^2/3 + x^2 & -x^4 - x^2y^2 & 3x^2 + 2xy^2 \\ -x^3 & -x^4 - x^2y^2 & (x^2 + y^2)^3 & -3(x^2 + y^2)^2 x \\ 3x^2 & 3x^3 + 2xy^2 & -3(x^2 + y^2)^2 x & (3x^2 + 2y^2)^2 - y^4 \end{pmatrix} \right.$$ \hspace{1cm} (5.2)

It is convenient to collect the four RR axions into a vector:

$$\zeta = (\tilde{\zeta}_0, \zeta_1, \zeta_0, \zeta_1) \right.$$ \hspace{1cm} (5.3)

The model has a period matrix:

$$N = \frac{1}{2} \begin{pmatrix} z^3 + 3z^2 \bar{z} & -3z^2 - 3z \bar{z} \\ -3z^2 - 3z \bar{z} & 9z + 3 \bar{z} \end{pmatrix} \right.$$ \hspace{1cm} (5.4)

Hence the $I$ and $R$ matrices are given by

$$I = \begin{pmatrix} 3x^2y + y^3 & -3xy \\ -3xy & 3y \end{pmatrix} \right.$$ \hspace{1cm} (5.5)

$$R = \begin{pmatrix} 2x^3 & -3x^2 \\ -3x^2 & 6x \end{pmatrix} \right.$$ \hspace{1cm} (5.5)

With our conventions, $y$ is positive and $I$ is positive-definite. The one forms $u, v, E, e$ are defined as:

$$u = \frac{i}{4} y^{-\frac{3}{2}} e^{-\phi} (-z^3 d\zeta^0 + 3z^2 d\zeta^1 + d\tilde{\zeta}_0 + zd\tilde{\zeta}_1) \right.$$ \hspace{1cm} (5.6)

$$v = \frac{1}{2} d\phi + \frac{i}{2} e^{-\phi} (d\sigma - \xi^I d\tilde{\xi}_I) \right.$$ \hspace{1cm} (5.7)

$$E = \frac{\sqrt{3}}{4} y^{-\frac{3}{2}} e^{-\phi} \left( z\bar{z}^2 d\zeta^0 - \bar{z}(2\bar{z} + z) d\zeta^1 - d\tilde{\zeta}_0 - \frac{2\bar{z} + z}{3} d\tilde{\zeta}_1 \right) \right.$$ \hspace{1cm} (5.8)

$$e = \frac{\sqrt{3}}{2} \frac{dz}{y} \right.$$ \hspace{1cm} (5.9)
5.1 Global symmetries and conserved charges

Since the action (5.1) is a sigma model for the $G_{2,2}/SU(2) \times SU(2)$ coset manifold, there are 14 global symmetries which form a $G_{2,2}$ group. Our definition of the generators agrees with [55, 56]. For completeness we give the explicit form of all generators of the group in appendix A. Here we display the root diagram in Figure 2. There are five generators which produce shift symmetries

$$
\begin{pmatrix}
\delta x = 0 \\
\delta y = 0 \\
\delta \phi = 0 \\
\delta \zeta^0 = \frac{1}{3} \\
\delta \zeta^1 = 0 \\
\delta \sigma = \frac{1}{3} \tilde{\zeta}^0 \\
\end{pmatrix}_{E_{p0}}
\begin{pmatrix}
\delta x = 0 \\
\delta y = 0 \\
\delta \phi = 0 \\
\delta \zeta^0 = 0 \\
\delta \zeta^1 = \frac{1}{3} \\
\delta \sigma = \frac{1}{3} \tilde{\zeta} \\
\end{pmatrix}_{E_{p1}}
\begin{pmatrix}
\delta x = 0 \\
\delta y = 0 \\
\delta \phi = 0 \\
\delta \zeta^0 = 0 \\
\delta \zeta^1 = 0 \\
\delta \sigma = 0 \\
\end{pmatrix}_{E_{q0}}
\begin{pmatrix}
\delta x = 0 \\
\delta y = 0 \\
\delta \phi = 0 \\
\delta \zeta^0 = 0 \\
\delta \zeta^1 = 0 \\
\delta \sigma = 0 \\
\end{pmatrix}_{E_{q1}}
\begin{pmatrix}
\delta x = 0 \\
\delta y = 0 \\
\delta \phi = 0 \\
\delta \zeta^0 = 0 \\
\delta \zeta^1 = 0 \\
\delta \sigma = -\frac{1}{2\sqrt{3}} \\
\end{pmatrix}_{E}
$$

Figure 2: The $G_{2,2}$ root diagram. $Y_0$ and $H$ are the Cartan generators.
These five generators obey a two-dimensional Heisenberg algebra where $E$ is the central
element.

$$[E_q, E_p] = 2\delta_{IJ}E, \quad I, J = 1, 2 \quad (5.11)$$

It is convenient to express the conserved Noether currents associated to symmetries other
than the shift generators (5.10) in terms of the shift currents (5.10) using the following
relations,

$$\frac{\delta L}{\delta \partial_\mu \zeta^I} = 3 j^\mu_{E_p I} + 2 \sqrt{3} \zeta_{I J} j^\mu_E$$

$$\frac{\delta L}{\delta \partial_\mu \zeta^I} = \frac{1}{\sqrt{3}} j^\mu_{E q I}$$

$$\frac{\delta L}{\delta \partial_\mu \zeta^I} = -2 \sqrt{3} j^\mu_E \quad (5.12)$$

For example we obtain the following expressions for the currents associated to the generators
$H$ and $Y_0$:

$$j^\mu_H = 3 \zeta^0 j^\mu_{E p 0} + 3 \zeta^1 j^\mu_{E p 1} + \frac{\zeta_0}{\sqrt{3}} j^\mu_{E q 0} + \frac{\zeta_1}{\sqrt{3}} j^\mu_{E q 1} - 2 \sqrt{3} (2 \sigma - \zeta^I \zeta_I) j^\mu_E + 2 \partial^\mu \phi$$

$$j^\mu_{Y_0} = \frac{3}{2} (3 \zeta^0 j^\mu_{E p 0} + \zeta^1 j^\mu_{E p 1} + \frac{\zeta_0 j^\mu_{E q 0}}{\sqrt{3}} - \frac{\zeta_1 j^\mu_{E q 1}}{3 \sqrt{3}}) + \sqrt{3} (3 \zeta^0 \zeta_0 + \zeta^1 \zeta_1) j^\mu_E - 3 \frac{\partial^\mu |z|^2}{2 y^2}$$

We have similar expressions for all the symmetry generators in the appendix. Like in the case
of the universal hypermultiplet, we focus our analysis on $SO(4)$ invariant solutions. With
the spherically symmetric ansatz, the angular components of the Noether currents vanish
and the $\tau$ components of the Noether charges are constants. Since the action (5.1) contains
eight scalar fields the equation of motion are eight second order differential equations. The
expressions for the fourteen conserved charges would allow us, in principle, to reduce the
problem to a system of two ODEs and to express the solutions in terms of fourteen charges
and two integration constants.

However, the problem is algebraically too complex for obtaining the general solution in this
way. In the general discussion about the analytic continuation in section 3.2, it became clear
that the instanton and wormhole action will be in real only in particular cases. For these rea-
sons we will consider truncations for which the conserved charges can be used to completely
solve the equations of motion and for which the action is real. The general solutions can be
obtained from the truncated solutions by acting with a group transformation, as outlined in
section 5.4.

Since $G_{2,2}$ has rank two, there are two linearly independent Casimir operators of degree two
and six. Like in the case of the $SU(2,1)/SU(2) \times U(1)$ coset, the quadratic Casimir is proportional to the non-extremality parameter $c$. However, the $G_{2,2}/SU(2) \times SU(2)$ solutions do not obey a constraint in terms of the degree six Casimir, which is not in general equal to zero.

5.2 Consistent truncations and instanton solutions

The problem greatly simplifies if we consider solutions where only a subset of the eight scalar fields have a non-trivial profile. In order to identically set some fields to constants, we need the truncation to be consistent with the equations of motion. It is easy to express the first derivatives of the RR scalars in terms of the conserved charges:

$$
\left( \dot{\zeta}^I, \ddot{\zeta}^I \right) = e^\phi M^{-1} \left( \begin{array}{c}
3q_{E_{P I}} + 2\sqrt{3}\bar{\zeta}^I q_E \\
q_{E_{Q I}} \sqrt{3} - 2\sqrt{3}\bar{\zeta}^I q_E
\end{array} \right)
$$

(5.13)

To be able to set to constants some of the RR axions without setting to zero all the four corresponding shift charges, we need the matrix $M$ which is defined in (5.2) to have vanishing nondiagonal terms. This requires the NS axion $x$ to vanish identically as all the non-diagonal terms are linear, quadratic or cubic in $x$. The equations on motion for $x$ give an extra condition for consistency: two RR axions cannot both have non-trivial profile if the corresponding off-diagonal term of (5.2) is linear in $x$. Moreover, if we want a solution with $q_E \neq 0$ we need both $\bar{\zeta}_I$ and $\zeta_I$ to be constant for some $I = 0, 1$. This leaves us with several possible consistent truncations, which we will list in the following.

5.2.1 The RR truncation I

To get a consistent truncation involving non-trivial RR fields we need to set the modulus $x$ to zero. We also set the shift charge corresponding to the NS axion $\sigma$ to zero obtaining a pure RR truncation.

$$
x \equiv 0, \quad \zeta^1 \equiv \text{const}, \quad \bar{\zeta}_0 \equiv \text{const}, \quad q_E \equiv 0
$$

(5.14)

This truncation is consistent as the $\partial_r \zeta^0 \partial_r \bar{\zeta}_1$ term in the action is quadratic in $x$. It is easy to see that the charges $q_{E_{P1}}$ and $q_{E_{Q0}}$ are equal to zero as well. Moreover, the equations for the conserved charges simplify. It is possible to solve the equations for $q_Y$, $q_{Y-}$, $q_{Y+}$ and $q_H$.
for the fields $\zeta^0, \zeta^1, \tilde{\zeta}_0$ and $\tilde{\zeta}_1$. We get:

$$\zeta^0 = \frac{q_H}{12q_{E_{p0}}} + \frac{q_{Y_0}}{6q_{E_{p0}}} - \frac{1}{6q_{E_{p0}}} \left( \partial_\tau \phi - 3 \frac{\partial_\tau y}{y} \right)$$

(5.15)

$$\zeta^1 = \frac{q_Y}{3\sqrt{2}q_{E_{q1}}}$$

(5.16)

$$\tilde{\zeta}_0 = \sqrt{2} \frac{q_{E_{p0}}q_{Y_0}}{q_{E_{q1}}}$$

(5.17)

$$\tilde{\zeta}_1 = \sqrt{3} \frac{q_{E_{p0}}q_{Y_0}}{2q_{E_{q1}}} - \sqrt{2} \frac{q_{E_{p0}}q_{Y_0}}{q_{E_{q1}}}$$

(5.18)

Substituting these expressions into equation (A.13) and equation (A.16) we get two decoupled ODEs:

$$\left( \frac{q_H^2}{4} + \frac{q_H q_{Y_0} + q_{Y_0}^2}{3} + \frac{2q_{E_{p0}}q_{E_{p0}}}{3\sqrt{3}q_{E_{q1}}} - \frac{8q_{E_{p0}}q_{Y_0}}{9} \right) - 4\frac{q_{E_{p0}}q_{Y_0}}{4} e^\phi y^3 - \left( 3 \frac{\partial_\tau y}{y} - \frac{\partial_\tau \phi}{y} \right)^2 = 0$$

(5.19)

And:

$$\left( \frac{q_H^2}{4} - \frac{q_H q_{Y_0}}{3} + \frac{q_{Y_0}^2}{9} + \frac{2q_{E_{p0}}q_{E_{p0}}}{3\sqrt{3}q_{E_{q1}}} - \frac{8q_{E_{p0}}q_{Y_0}}{9} \right) - \frac{4}{3} q_{E_{q1}} e^\phi y^3 - \left( \frac{\partial_\tau y}{y} + \frac{\partial_\tau \phi}{y} \right)^2 = 0$$

(5.20)

The appropriate analytic continuation for this case is:

$$\zeta^1 \rightarrow i\zeta'^1, \quad \tilde{\zeta}_1 \rightarrow i\tilde{\zeta}'_1$$

(5.21)

It is convenient to define:

$$\gamma_1 = \frac{q_{F_{p0}}q_{E_{p0}}}{2} + \frac{q_{E_{p0}}q_{E_{p0}}^2}{2\sqrt{3}q_{E_{q1}}} - \frac{q_{E_{p0}}^2}{16} - \frac{q_{Y_0}}{4}$$

$$\gamma_2 = \frac{q_{F_{p0}}q_{E_{p0}}^2}{6} - \frac{q_{E_{p0}}q_{E_{p0}}^2}{6\sqrt{3}q_{E_{q1}}} - \frac{2q_{E_{p0}}q_{Y_0}}{9} - \frac{q_{E_{p0}}^2}{16} + \frac{q_{Y_0}q_{E_{p0}}}{12} - \frac{q_{E_{p0}}^2}{36}$$

(5.22)

The solution is then given by:

$$e^{-\eta_1} = \frac{9q_{E_{p0}}^2}{\gamma_1} \cos^2 \left[ \sqrt{\gamma_1} (\tau + c_1) \right]$$

$$e^{-\eta_2} = \frac{q_{E_{p0}}^2}{3\gamma_2} \cos^2 \left[ \sqrt{\gamma_2} (\tau + c_2) \right]$$

(5.23)
And:

\[ \phi' = \frac{\eta_1' + 3\eta_2'}{4}, \quad y' = e^{\frac{\eta_1' - \eta_2'}{4}} \]  

(5.24)

Equation (A.15) and the equation for \( q_F \) simplify and reduce to two constraints on the charges:

\[ q_{E\alpha} q_{E\alpha} q_{F\nu} q_{F\nu} - \sqrt{2}(q_H - \frac{2}{3}q_{Y_0}) \frac{q_{E\alpha} q_{Y_+}}{q_{E\alpha} q_{Y_+}} - \sqrt{\frac{2}{3}}(2q_{Y_0} - q_H)q_{Y_+} = 0 \]  

(5.25)

It is instructive to consider the particular case in which the fields \( \zeta_1, \tilde{\zeta}_0 \) and \( \sigma \) are equal to zero. In this case, the charges \( q_{Y_\pm}, q_{F\nu}, q_{E\alpha} \) and \( q_F \) vanish while the constraints (5.26) are automatically satisfied and the non-vanishing charges correspond to two perpendicular \( SL(2, R) \) subalgebra in root space. Moreover, \( \gamma_1, \gamma_2 \) are proportional to the quadratic Casimirs of these subalgebras.

The action of the instanton solution is given by the surface term (3.11). Plugging in (5.18) and (5.23) we get:

\[ S = 3q_{E\alpha} \Delta \zeta_0 + \frac{q_{E\alpha}}{\sqrt{3}} \Delta \tilde{\zeta}_1 = -\frac{1}{2} \partial_{\tau} \eta_1 - \frac{3}{2} \partial_{\tau} \eta_2 \bigg|_{\tau} \]

\[ = \sqrt{\gamma_1} \tan \left[ \sqrt{\gamma_1}(\tau + c_1) \right] \bigg|_0^{\tau} + 3 \sqrt{\gamma_2} \tan \left[ \sqrt{\gamma_2}(\tau + c_2) \right] \bigg|_0^{\tau} \]  

(5.27)

The non-extremality parameter \( c \) can be expressed as:

\[ c = \frac{\gamma_1 + 3\gamma_2}{48} \]  

(5.28)

Imposing regularity leads to the conditions:

\[ \gamma_1, \gamma_2 \geq 0, \quad \sqrt{\gamma_1} c_1, \sqrt{\gamma_2} c_2 \geq -\frac{\pi}{2}, \quad \sqrt{\gamma_1} \left( \frac{\sqrt{3}\pi}{\sqrt{\gamma_1} + 3\gamma_2} + c_1, \right) \leq \frac{\pi}{2} \]  

(5.29)

In particular, there are non-singular solutions for some values of the integration constants provided that:

\[ \gamma_2 \geq \frac{2}{3} \gamma_1 \]  

(5.30)
Figure 3: Action of regular RR wormhole solutions as a function of $\gamma_1$ and $\gamma_2$ for fixed values of $\phi$ and $y$ at $\tau = 0$. The plot is obtained setting $e^{-\phi_0} = 0.001$, $y_0 = 10$ and $q'_{E_1} = q'_{E_0} = 1$. Extremal instantons correspond to an absolute minimum of the action for $\gamma_1 = \gamma_2 = 0$.

It is particularly interesting to consider the limit $g_S \ll 1/y_0 \ll 1$ with $g_S = e^{-\phi_0/2}$. In this case the action reduces to:

$$S = \frac{3|q'_{E_0}|}{g_S \sqrt{y_0}} \left[ 1 + \frac{\sqrt{3\gamma_1 y_0 g_S}}{|q'_{E_1}|} \cot \left( \frac{3\gamma_2}{\gamma_1 + 3\gamma_2} \pi \right) \right] + \frac{\sqrt{3}|q'_{E_1}| \sqrt{y_0}}{g_S} \left[ 1 + \frac{\sqrt{3\gamma_2 g_S}}{|q'_{E_0}|} \cot \left( \frac{3\gamma_1}{\gamma_1 + 3\gamma_2} \pi \right) \right]$$

This expression further simplifies if the value for $\gamma_2$ is not close to $\frac{2}{3}\gamma_1$. Note that the action is proportional to $1/g_S$ as expected for D-brane instantons.

### 5.2.2 The RR truncation II

A second consistent truncation has a different pair of non-trivial RR fields.

$$x \equiv 0, \quad \zeta^0 \equiv \text{const}, \quad \tilde{\zeta}_1 \equiv \text{const}, \quad q_E \equiv 0$$
In this case the solution is given by:

\[
\zeta^0 = -\sqrt{\frac{2}{27}} \frac{q_{Y_+}}{q_{E_+}} - \frac{q_{E_0} q_{Y_+}}{3\sqrt{2}q_{E_1}^2}
\]
\[
\zeta^1 = \frac{q_H}{4q_{E_1}} + \frac{q_{Y_0}}{6q_{E_1}} - \frac{1}{2q_{E_1}} \left( \partial_\tau \phi - \frac{\partial_\tau y}{y} \right)
\]
\[
\zeta_0 = \frac{\sqrt{3}q_H}{4q_{E_0}^2} - \frac{\sqrt{3}q_{Y_0}}{2q_{E_0}} - \frac{\sqrt{3}}{2q_{E_0}} \left( \partial_\tau \phi + 3\frac{\partial_\tau y}{y} \right)
\]
\[
\zeta_1 = -\sqrt{3} \frac{q_{Y_+}}{2q_{E_1}}
\]

And by:

\[
e^{-\eta'_1} = \frac{q_{E_0}^2}{3\gamma_1} \cos^2 \left[ \sqrt{\gamma_1} (\tau + c_1) \right]
\]
\[
e^{-\eta'_2} = \frac{q_{E_1}^2}{\gamma_2} \cos^2 \left[ \sqrt{\gamma_2} (\tau + c_2) \right]
\]

with:

\[
\phi' = \frac{3\eta'_2 + \eta'_1}{4}, \quad y' = e^{\eta'_1} - \eta'_2
\]
\[
\gamma_1 = \frac{q_{E_0}^2}{2} - \frac{q_{E_0}^2 q_{Y_+}^2}{2\sqrt{3}q_{E_1}^2} - \frac{q_H^2}{16} + \frac{q_H q_{Y_0}}{4} - \frac{q_{Y_0}^2}{4}
\]
\[
\gamma_2 = \frac{q_{F_0}}{6} + \frac{q_{E_0} q_{Y_+}^2}{6\sqrt{3}q_{E_1}^2} + \frac{2q_{Y_+} q_{Y_-}}{9} - \frac{q_H^2}{16} - \frac{q_H q_{Y_0}}{12} - \frac{q_{Y_0}^2}{36}
\]

The charges obey to the constraints:

\[
q_{E_0}^2 q_{F_0} + q_{F_0}^2 q_{E_0} + \sqrt{2} (q_H^2 + \frac{2}{3} q_{Y_0}^2) \frac{q_{E_0}^2 q_{Y_+}}{q_{E_1}^2} = 0
\]
\[
q_{F_0}^2 q_H^2 + \frac{1}{\sqrt{2}} q_{F_0} q_{Y_+}^2 - \sqrt{\frac{2}{3}} q_{F_0} q_{Y_-} + q_{F_0}^2 q_{E_1}^2 + \frac{2\sqrt{2}q_{E_0}^2 q_{Y_+}^2}{3\sqrt{3}q_{E_1}^2} - \left(q_{E_0}^2 q_{F_0} - \frac{2}{3} q_H q_{Y_0} - \frac{4}{3} q_{Y_+} q_{Y_-} q_{F_0}^2 \frac{q_{Y_+}}{\sqrt{2}q_{E_1}^2} \right) = 0
\]

Like in the case of the RR truncation I, these solutions are regular for some values of the integration constants if \( \gamma_2 \geq 2/3\gamma_1 \) and have actions proportional to \( 1/g_S \) as expected for D-instantons.
5.2.3 The NS-NS truncation

A different kind of truncation can be obtained by setting all RR axions to be constant.

\[ \zeta^I = \text{const}, \quad \tilde{\eta}_I = \text{const}, \quad I = 0, 1 \quad (5.41) \]

The definitions of the shift currents \[(5.12)\] give the following expressions for the RR axions in terms of the charges:

\[ \xi^0 = \frac{q_{E\rho}}{6q_E}, \quad \tilde{\xi}_0 = -\frac{\sqrt{3}q_{E\rho}}{2q_E} \quad (5.42) \]
\[ \xi^1 = \frac{q_{E\phi}}{6q_E}, \quad \tilde{\xi}_1 = -\frac{\sqrt{3}q_{E\phi}}{2q_E} \quad (5.43) \]

Substituting equations \[(A.9)\] into \[(A.12)\] into the expression for \(q_F\) and equations \[(A.10)\] and \[(A.11)\] into equation \[(A.12)\] we are left with two ODE:

\[ -\frac{(3q_{E\rho} q_{E\rho} + q_{E\phi} q_{E\phi} - 4q_E q_Y)^2}{144q_E^2} + \frac{2\tilde{q}_Y^2 q_Y}{9} + \frac{2}{27} q_Y^2 y^2 + \frac{(\partial_Y y)^2}{y^2} = 0 \quad (5.44) \]
\[ \frac{(q_{E\rho} q_{E\rho} + q_{E\phi} q_{E\phi})^2}{16q_E^2} - q_E \tilde{q}_F - \frac{q_H^2}{4} + 12q_E c^2 + \frac{(\partial_Y y)^2}{2} = 0 \quad (5.45) \]

In the above equations, we have re-defined the charges \(\tilde{q}_Y\) as:

\[ \tilde{q}_Y = q_Y - \frac{\sqrt{6}q_{E\rho} q_{E\rho} - \sqrt{2}q_{E\phi}}{4q_E} \quad (5.46) \]

and the charge \(\tilde{q}_F\) as:

\[ \tilde{q}_F = q_F + \left( \frac{q_{E\rho} q_{E\rho}}{2q_E^2} + \frac{q_{E\phi} q_{E\phi}}{6q_E^2} \right) q_{E\rho} - \frac{q_{E\phi}^2 q_{E\phi}}{12\sqrt{3}q_E^3} + \frac{3q_{E\rho}^2 q_{E\rho}}{8q_E^3} + \frac{q_{E\rho} q_{E\rho} q_{E\phi} q_{E\phi}}{4q_E^3} \]
\[ + \frac{q_{E\phi}^2 q_{E\phi}}{8q_E^3} + \left( \frac{q_{E\phi}^2 q_{E\phi}}{12\sqrt{3}q_E^3} + \frac{q_{E\rho}^2 q_{E\rho} q_{E\phi} q_{E\phi}}{\sqrt{6}q_E^3} \right) \tilde{q}_Y - \left( \frac{q_{E\phi}^2 q_{E\phi}}{3\sqrt{2}q_E^3} - \frac{q_{E\phi}^2 q_{E\phi}}{3\sqrt{2}q_E^3} \right) \tilde{q}_Y \quad (5.47) \]

The equations \[(A.13)\] lead to four constraints in the charges:

\[ \frac{q_{E\rho}^2 q_{E\rho}}{2q_E} - \frac{q_{E\phi}^3}{3\sqrt{3}q_E} - q_{E\rho} \left( \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\rho} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\rho} q_{E\phi} q_{E\phi}}{2q_E} \right) = q_{E\rho} q_{E\rho} \quad (5.48) \]
\[ \frac{q_{E\rho}^2 q_{E\rho}}{2q_E} + \sqrt{3}q_{E\phi} q_{E\phi} - q_{E\phi} \left( \frac{q_{E\phi}^3}{2q_E} + \sqrt{3}q_{E\phi} q_{E\phi} \right) - q_{E\rho} \left( \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} \right) = q_{E\rho} q_{E\rho} \quad (5.49) \]
\[ \frac{q_{E\rho}^2 q_{E\rho}}{3\sqrt{3}q_E} + q_{E\rho} \left( \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} \right) = q_{E\rho} q_{E\rho} \quad (5.50) \]
\[ \frac{q_{H}^2}{2} - \frac{q_{Y}^2}{3} - q_{E\rho} q_{E\rho} q_{E\rho} - q_{E\phi} q_{E\phi} q_{E\phi} + q_{E\phi} q_{E\phi} q_{E\phi} \left( \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \sqrt{3}q_{E\phi} q_{E\phi} \right) + \frac{q_{E\phi} q_{E\phi} q_{E\phi}}{2q_E} + \frac{q_{E\rho} q_{E\rho} q_{E\rho}}{3q_{E\rho} q_{E\rho}} = q_{E\rho} q_{E\rho} \quad (5.51) \]
Finally, the fields $x$ and $\sigma$ can be expressed as follows:

$$x = -\sqrt{\frac{3}{32}} \frac{3q_{E,p_0}q_{E,q_0} + q_{E,p_1}q_{E,q_1}}{q_{E}q_{\nu_-}} + \sqrt{\frac{2}{3q_{E}}} \frac{q_{Y_0} + 3\tilde{q}_{Y}}{q_{Y_-}} \quad (5.52)$$

$$\sigma = -\frac{q_{E,p_0}q_{E,q_0} + q_{E,p_1}q_{E,q_1}}{8\sqrt{3q_{E}^2}} - \frac{q_{H} - 2\tilde{q}_{\tau_1}q_{E}}{4\sqrt{3q_{E}}} \quad (5.53)$$

These solutions have non-zero $q_{E}$ charge and require an analytic continuation of the kind (3.21). As the condition (3.33) is satisfied, the action after analytic continuation has a real saddle-point and is definite-positive after the surface term is taken into account. If we choose the analytic continuation:

$$\zeta^0 \to i\zeta^0, \quad \zeta^1 \to i\zeta^1, \quad \sigma \to i\sigma' \quad (5.54)$$

the charges are continued as follows:

$$q_{E,p_1} \to iq_{E,p_1}', \quad q_{E} \to iq_{E}', \quad q_{Y_\pm} \to \tilde{q}_{Y_\pm}', \quad \tilde{q}_{F} \to i\tilde{q}_{F}', \quad q_{Y_0} \to q_{Y_0}' \quad (5.55)$$

Note that the charges $q_{Y_\pm}'$ and $q_{F}'$ are not real nor purely imaginary. In the particular case of the NS-NS truncated solution, the real and imaginary parts of these charges are separately conserved while the analytically continued fields are real. We then obtain the solution:

$$y^{-2} = \frac{2\tilde{q}_{Y}^2}{27\gamma_1} \cosh^2 \left[ \sqrt{\gamma_1}(\tau + c_1) \right]$$

$$e^{-2\phi} = \frac{12q_{E}^2}{\gamma_2} \cos^2 \left[ \sqrt{\gamma_2}(\tau + c_2) \right] \quad (5.56)$$

with:

$$\gamma_1 = \frac{(3q_{E,p_0}'q_{E,q_0}' + q_{E,p_1}'q_{E,q_1}')^2}{144q_{E}^2} - \frac{2\tilde{q}_{Y_\pm}^2}{9}$$

$$\gamma_2 = \frac{(q_{E,p_0}'q_{E,q_0}' + q_{E,p_1}'q_{E,q_1}')}{16q_{E}^2} + \frac{q_{E}q_{\nu}}{4} \quad (5.57)$$

Note that these instantons are charged under the shifts of the NS axions and correspond to worldsheet instantons. The non-extremality parameter can be expressed as:

$$c = \frac{\gamma_2 - 3\gamma_1}{48} \quad (5.58)$$
Figure 4: Action of regular NS-NS wormhole solutions as a function of $\gamma_1$ and $\gamma_2$ for fixed values of $\phi$ and $y$ at $\tau = 0$. The plot is obtained setting $e^{-\phi_0} = 0.001$, $y_0 = 10$ and $q_E' = \tilde{q}_Y' = 1$. Extremal instantons correspond to an absolute minimum of the action.

These solutions are always singular.

NS-NS truncated solutions also admit analytic continuations of the kind \[3.24\] \footnote{This analytic continuation is slightly different from \[3.24\] since the field $\tilde{\zeta}_1$ is continued as well. However, the lack of a surface term for the field $\tilde{\zeta}_1$ poses no problem in this case because $\tilde{\zeta}_1$ has constant profile.}

$$\tilde{\zeta}_0 \rightarrow i\tilde{\zeta}_0', \quad \tilde{\zeta}_1 \rightarrow i\tilde{\zeta}_1', \quad x \rightarrow ix', \quad \sigma \rightarrow i\sigma'$$ (5.59)

In this case, the charges are continued as:

$$q_{E_{q1}} \rightarrow i\tilde{q}'_{E_{q1}}, \quad q_E \rightarrow i\tilde{q}'_E, \quad \tilde{q}_Y \rightarrow i\tilde{q}'_Y$$

$$q_{E_{p1}} \rightarrow q'_{E_{p1}}, \quad \tilde{q}_F \rightarrow i\tilde{q}'_F$$ (5.60)

With this analytic continuation we obtain the non-extremality parameter:

$$c = \frac{3\gamma_1 + \gamma_2}{48}$$ (5.61)
With $\gamma_1$ and $\gamma_2$ given by:

$$
\gamma_1 = -\frac{(q_{E_{p1}}q_{E_{q1}} - 4q_{E}q_{Y_{0}})^2}{144q_E^2} - \frac{2q_Y q_{Y_{+}}}{9}
$$

$$
\gamma_2 = \frac{(q_{E_{p0}}q_{E_{q0}} + q_{E_{p1}}q_{E_{q1}})^2}{16q_E^2} + \frac{q_Y^2}{4}
$$

(5.62)

The solution for $\phi$ and $y$ is given by:

$$
y^{-2} = \frac{2q_Y^2 Y}{27\gamma_1} \cos^2 \left[ \sqrt{\gamma_1}(\tau + c_1) \right]
$$

$$
e^{-2\phi} = \frac{12q_E^2}{\gamma_2} \cos^2 \left[ \sqrt{\gamma_2}(\tau + c_2) \right]
$$

(5.63)

In this case, there exist non-singular solutions for some value of the integration constants provided that $\gamma_1 \geq \frac{2}{3}\gamma_2$. The action for these solutions is given by:

$$
S = 3\sqrt{\gamma_1} \tan \left[ \sqrt{\gamma_1}(\tau + c_1) \right] + \sqrt{\gamma_2} \tan \left[ \sqrt{\gamma_2}(\tau + c_2) \right] \Big|_0^{\pi \sigma}
$$

(5.64)

As done for the RR truncation, it is instructive to consider the $g_s \ll 1/y_0 \ll 1$ limit (corresponding to the weak coupling limit for a large Calabi-Yau manifold). In this limit, we get:

$$
S = \frac{\sqrt{\frac{2}{3}}|q_Y^2|y_0}{1 + \frac{\sqrt{3}\gamma_1}{\sqrt{2y_0}|q_Y^2|} \cot \left( \sqrt{\frac{27\gamma_1}{3\gamma_1 + \gamma_2}} \pi \right)} + \frac{\sqrt{12}|q_E|g_s}{1 + \frac{\sqrt{3}\gamma_2 g_s^2}{\sqrt{12}|q_E|} \cot \left( \sqrt{\frac{3\gamma_2}{3\gamma_1 + \gamma_2}} \pi \right)}
$$

(5.65)

The second term presents a $1/g_s^2$ dependence characteristic of a fivebrane instanton, while the first term is proportional to the volume of a two-cycle of the manifold as expected for a worldsheet instanton.

### 5.2.4 NS-R truncations

A fourth truncation in which the off-diagonal terms of (5.2) are at least quadratic in $x$ is given by setting $\zeta^1 = \tilde{\zeta}_1 = \text{const}$ and $x \equiv 0$. The resulting truncation is however more complicated, since the $y$ and $\phi$ equations do not decouple. This is related to the fact that the non-zero global symmetry charges are not associated with commuting subalgebras.
It is possible to generate NS-R truncated solutions acting on a particular solution with a global $G_{2,2}$ transformation. We first consider the case in which the modulus $y$ is constant. It is convenient to set:

$$q_E = q_{E_0} = q_{E_1} = q_{F_0} = q_{F_1} = 0, \quad y \equiv y_0 = \sqrt{3} \left( \frac{q_{E_0}}{q_{E_0}} \right)^{\frac{1}{3}}$$  \hspace{1cm} (5.66)

If $q_E = 0$, it is easy to see from the equations of motion of the truncated lagrangian that $y_0$ corresponds to a minimum of the potential for the field $y$. In this case, we can substitute (A.9), (A.10) and (A.13) into (A.15) obtaining:

$$- \frac{q_{E_0} q_{F_0}}{2} - \frac{q_{E_0} q_{F_0}}{2} - \frac{q_H^2}{4} - \frac{q_{E_0} q_{E_0}}{3} + 4 \sqrt{3} |q_{E_0} q_{E_0}| e^\phi + (\partial_\tau \phi)^2 = 0$$  \hspace{1cm} (5.67)

This ODE is solved by:

$$e^{-\phi} = \frac{2 \sqrt{3} |q_{E_0} q_{E_0}|}{C_2} \cosh \left( \frac{\sqrt{C_2}}{2} \tau + A_1 \right)$$  \hspace{1cm} (5.68)

With this notation, $C_2$ is the quadratic Casimir operator:

$$C_2 = \frac{q_H^2}{4} + \frac{q_{E_0}^2}{3} + \frac{q_{E_0} q_{F_0}}{2} + \frac{q_{E_0} q_{F_0}}{2} = -48c$$  \hspace{1cm} (5.69)

where $c$ is the non-extremality parameter. The expression for $q_F$ reduces to a constraint in the charges:

$$54 q_{E_0} q_{E_0} q_{F_0} - 9 q_{E_0} q_{F_0} (3q_H - 2q_{Y_0}) + (3q_H + 2q_{Y_0}) (4q_{Y_0}^2 - 6q_H q_{Y_0} + 9q_{E_0} q_{F_0}) = 0$$  \hspace{1cm} (5.70)

The non-trivial RR fields can be obtained from (A.9) and (A.10):

$$\zeta^0 = \frac{3q_H + 2q_{Y_0} + 6 \sqrt{C_2} \tanh \left( \frac{\sqrt{C_2}}{2} \tau + A_1 \right)}{18 q_{E_0}}$$  \hspace{1cm} (5.71)

$$\bar{\zeta}_0 = \frac{3q_H - 2q_{Y_0} + 6 \sqrt{C_2} \tanh \left( \frac{\sqrt{C_2}}{2} \tau + A_1 \right)}{2 \sqrt{3} q_{E_0}}$$  \hspace{1cm} (5.72)

If we consider a set of charges such that:

$$q_{Y_0} = q_{E_0} = q_{E_1} = q_{F_0} = q_{F_1} = 0$$  \hspace{1cm} (5.73)
together with a set of arbitrary values for the fields \( \phi \) and \( y \) at \( \tau = 0 \) and act with a group transformation of the form:

\[
g_{\alpha,\beta,\gamma} = e^{-\alpha F_{p0}} e^{-\beta E_{q0}} e^{-\gamma E_{q0}} \tag{5.74}
\]

we can solve numerically for the values of the transformation parameters \( \alpha, \beta, \gamma \) such that the transformed solution obeys to (5.66) and (5.70). The equations we obtain admit solutions only for some of the random values of charges and fields at \( \tau = 0 \). The interpretation of this fact is that the solutions obtained applying the inverse of (5.74) on the solution (5.68-5.72) constitute a patch of non-zero measure of the set of solutions obeying to (5.73).

To obtain an explicit form for these solutions we note that a finite transformation generated by \( F_{p0} \) acts on the truncated fields as:

\[
\begin{align*}
\frac{e^{\phi/2}}{y^{3/2}} & \rightarrow \frac{e^{\phi/2}}{y^{3/2}} \left( 1 + 6\alpha \zeta^0 \right)^2 + 36\alpha^2 \frac{e^\phi}{y^7}
\end{align*}
\]

\[
e^{\phi} y \rightarrow e^{\phi} y
\]

\[
\zeta^0 \rightarrow \zeta^0 + 6\alpha \zeta^0 \zeta^0 + 6\alpha \frac{e^\phi}{y^7}
\]

\[
(1 + 6\alpha \zeta^0)^2 + 36\alpha^2 \frac{e^\phi}{y^7} \tag{5.75}
\]

The finite transformations generated by \( F_{p0} \) leave the fields \( x \), \( \zeta_1 \) and \( \tilde{\zeta}_1 \) equal to zero and allows to obtain an expression for the fields \( \phi \) and \( y \) of a truncated solution obeying to (5.73).

The appropriate analytic continuation is the one corresponding to the NS-charged case:

\[
\begin{align*}
\zeta^0 & \rightarrow i\zeta^0 & \tilde{\zeta}_0 & \rightarrow \tilde{\zeta}_0 & \sigma & \rightarrow i\sigma' \\
q_{E_{q0}} & \rightarrow iq'_{E_{q0}} & q_{E_{q0}} & \rightarrow q'_{E_{q0}} & q_E & \rightarrow iq'_E \\
q_{F_{q0}} & \rightarrow iq'_{F_{q0}} & q_{F_{q0}} & \rightarrow q'_{F_{q0}} & q_F & \rightarrow iq'_F \\
\end{align*}
\]

we need to continue two of the transformation parameters as well:

\[
\alpha \rightarrow i\alpha \quad \gamma \rightarrow i\gamma \tag{5.79}
\]

The solution is:

\[
e^{\phi'} = \frac{12\sqrt{2c}|\alpha|}{|q_{E_{q0}}^2 q_{E_{q0}}'|^{\frac{1}{2}}} \sqrt{b_1 \sin^2 \left( 2\sqrt{3c\tau} + A_2 \right) - 1} \tag{5.80}
\]

\[
y' = \frac{\sqrt{2c}}{|\alpha q_{E_{q0}}'|} \sqrt{\frac{\sec \left( 2\sqrt{3c\tau} + A_1 \right)}{b_1 \sin^2 \left( 2\sqrt{3c\tau} + A_2 \right) - 1}} \tag{5.81}
\]

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The constants $b_1$ and $A_2$ are given by:

\[
b_1 = 2 + \frac{(q_{E_p0}^r + \alpha q_H^r + \frac{2}{3} \alpha q_{Y_0}^r)^2}{96 \alpha^2 c},
\]
\[
A_2 = A_1 - \tan^{-1}\left(\frac{q_{E_p0}^r + \alpha q_H^r + \frac{2}{3} \alpha q_{Y_0}^r}{8 \alpha \sqrt{3c}}\right).
\]

This solution is parametrized by eight charges, three transformation parameters and one integration constant. The actual Noether charges of the solution (5.81) can be obtained applying the inverse of (5.74) to the charges $q_H^r, q_{Y_0}^r \ldots q_F^r$ which transform in the adjoint representation. The expression for the two RR axions and the NS-NS axion can be obtained solving the equations (5.9-5.10) and (5.13) for the fields $\zeta^0, \tilde{\zeta}^0$ and $\sigma$. Finally, more general NS-R truncated solutions can be obtained by acting with the shifts of the the fields $\sigma, \zeta^1$ and $\tilde{\zeta}_1$ on the above solution.

One may wonder whether it is possible to recover the universal hypermultiplet as a truncation of the $G_2/(SU(2) \times SU(2))$ model. From a ten-dimensional perspective, the NS-R truncation has the same non-trivial axionic fields of the universal hypermultiplet. However, NS-R truncated solutions have a non-trivial profile of the modulus $y$ which corresponds to the volume of the compactification manifold and has no analogue in the $SU(2,1)/(SU(2) \times U(1))$ sigma model covered in section 4.

### 5.3 Extremal limit and supersymmetry analysis

In case of the RR truncated solutions, we have regular (extremal) instanton solutions for $\gamma_1 \to 0$ and $\gamma_2 \to 0$. For the RR truncation I, we get:

\[
e^{-\phi'} = \sqrt{\frac{|q_{E_p0}^r \tau + h_1||q_{E_p1}^r \tau + h_2|^3}{3}}, \quad y' = \sqrt{\frac{27}{|q_{E_p0}^r \tau + h_1|^2}}.
\]
Direct computation of the quaternionic vielbein (5.6-5.9) leads to:

\[ u'_\tau = -\frac{3}{4} i \left( q'_{E\tau_0} \frac{\eta_1'}{\sqrt{3}} + \frac{q'_{E\tau_1}}{\sqrt{3}} e^{\frac{\eta_2'}{2}} \right) \]  
(5.84)

\[ v'_\tau = -\frac{3}{4} \left( |q'_{E\tau_0}| \frac{\eta_1'}{\sqrt{3}} + \frac{|q'_{E\tau_1}|}{\sqrt{3}} e^{\frac{\eta_2'}{2}} \right) \]  
(5.85)

\[ E'_\tau = \frac{1}{4} \left( 3\sqrt{3}|q'_{E\tau_0}| e^{\frac{\eta_1'}{2}} - q'_{E\tau_1} e^{\frac{\eta_2'}{2}} \right) \]  
(5.86)

\[ e'_\tau = \frac{i}{4} \left( 3\sqrt{3}|q'_{E\tau_0}| e^{\frac{\eta_1'}{2}} - |q'_{E\tau_1}| e^{\frac{\eta_2'}{2}} \right) \]  
(5.87)

with \( \eta_1' \) and \( \eta_2' \) defined in (5.18). It is easy to see that the BPS condition (3.45) is satisfied only if \( q'_{E\tau_0}q'_{E\tau_1} \geq 0 \). In contrast, solutions with \( q'_{E\tau_0}q'_{E\tau_1} < 0 \) are extremal non-BPS instantons. Note that the extremal non-BPS instantons will be related to extremal non-BPS black holes by the c-map. The existence and properties of such black hole solutions and their relation to the attractor mechanism was discussed in several papers recently [42, 43, 44, 45, 71]. Note that an extremal non-BPS solutions is still an instanton, i.e. it has only one asymptotic region and induces a local operator insertion. However the fact that it breaks all supersymmetries implies that there are twice as many fermionic zero-modes and consequently the instanton will only contribute to higher derivative corrections to the \( N = 2 \) effective action.

Similarly, direct computation of the vielbein for the RR truncation II leads to:

\[ u'_\tau = -\frac{1}{4} \left( \frac{q'_{E\tau_0}}{\sqrt{3}} e^{\frac{\eta_1'}{2}} - 3q'_{E\tau_1} e^{\frac{\eta_2'}{2}} \right) \]  
(5.88)

\[ v'_\tau = -\frac{1}{4} \left( |q'_{E\tau_0}| e^{\frac{\eta_1'}{2}} + 3|q'_{E\tau_1}| e^{\frac{\eta_2'}{2}} \right) \]  
(5.89)

\[ E'_\tau = -\frac{i}{4} \left( q'_{E\tau_0} e^{\frac{\eta_1'}{2}} + \sqrt{3}q'_{E\tau_1} e^{\frac{\eta_2'}{2}} \right) \]  
(5.90)

\[ e'_\tau = -\frac{i}{4} \left( |q'_{E\tau_0}| e^{\frac{\eta_1'}{2}} - \sqrt{3}|q'_{E\tau_1}| e^{\frac{\eta_2'}{2}} \right) \]  
(5.91)

These solutions are supersymmetric only if \( q'_{E\tau_0}q'_{E\tau_1} \leq 0 \). We will have non-supersymmetric extremal solutions if \( q'_{E\tau_0}q'_{E\tau_1} \geq 0 \).

Finally, taking the extremal limit for the NS-NS truncation leads to:

\[ e^{-\phi'} = 2\sqrt{3} \left| q'_E \tau + h_2 \right|, \quad y' = \sqrt{\frac{2}{27}} \left| q'_{Y^{-}\tau} + h_1 \right|^{-1} \]  
(5.92)

It is easy to see the NS-NS truncated extremal solutions are BPS since \( u_\tau = E_\tau = 0 \).
5.4 General solution

The truncations allowed for the exact solution of the equations of motion using the conserved charges to solve for the radial dependence of all fields. Applying this method in the most general case leads to algebraic equations which cannot be solved explicitly. In the following we will describe using the method of solution generating transformations to obtain the general solution.

The RR truncated solutions obey to five constraints in the charges. It is in principle possible to obtain the general solution using a five parameter group transformation. The general solution will then be characterized by fourteen parameters: two integration constants, seven charges and five transformation parameters. We first integrate the $F_{p1}$ generator to obtain a finite group transformation. The result for the fields $x, y, \phi$ and $\zeta^0$ is:

\[
\begin{align*}
  x & \rightarrow x + 6a(\zeta^0 - \zeta^1) \\
  y & \rightarrow y\sqrt{(1 + 6a\zeta^0)^2 + 36a^2\frac{e^\phi}{y^3}} \\
  e^\phi & \rightarrow \frac{e^\phi}{\sqrt{(1 + 6a\zeta^0)^2 + 36a^2\frac{e^\phi}{y^3}}} \\
  \zeta^0 & \rightarrow \frac{\zeta^0 + 6a\zeta^0 + 6a\frac{e^\phi}{y^3}}{(1 + 6a\zeta^0)^2 + 36a^2\frac{e^\phi}{y^3}}
\end{align*}
\] (5.93)

It is easy to show that the other fields transform so that:

\[
\begin{align*}
  \zeta^1 - \zeta^0 x & = \text{const} \\
  3\zeta^1 x + \tilde{\zeta}_1 & = \text{const} \\
  \sigma + 2\zeta^1 x - \zeta^0 \zeta^1 x^2 & = \text{const} \\
  \delta F_{p0} \delta F_{p0} (\tilde{\zeta}_0 - \zeta^1 x^2) & = 0
\end{align*}
\] (5.94)

The finite transformation generated by $Y_+$ will be simpler as the $Y_+$ action on the RR axions

---

2The dimensional reduction to coset sigma models and the use of global symmetries to generate solutions has a long history for black holes in (super)gravity, see e.g. [57, 58, 59, 60].
is nilpotent:

\[
x \rightarrow \sqrt{6} \frac{(\sqrt{6} + ax)x + ay^2}{(\sqrt{6} + ax)^2 + a^2y^2}
\]

\[
y \rightarrow \frac{6y}{(\sqrt{6} + ax)^2 + a^2y^2}
\]

\[
\zeta^0 \rightarrow \zeta^0 + \sqrt{\frac{3}{2}} a \zeta^1 + \frac{a^2}{6} \zeta_1 - \frac{a^3}{6\sqrt{6}} \zeta_0
\]

\[
\zeta^1 \rightarrow \zeta_1 - \sqrt{\frac{2}{27}} a \zeta_1 + \frac{a^2}{6} \zeta_0
\]

\[
\tilde{\zeta}_1 \rightarrow \tilde{\zeta}_1 - \sqrt{\frac{3}{2}} a \tilde{\zeta}_0
\]

(5.95)

It is slightly more involved to obtain the finite transformation generated by \( F_{q_0} \). It is convenient to consider the transformation rules for the combination \( e^{\frac{\phi}{2}} \left( \frac{x^2 + y^2}{y} \right)^{\frac{3}{2}} \). We get:

\[
e^{\frac{\phi}{2}} \left( \frac{x^2 + y^2}{y} \right)^{\frac{3}{2}} \rightarrow \frac{e^{\frac{\phi}{2}} \left( \frac{x^2 + y^2}{y} \right)^{\frac{3}{2}}}{\left( 1 + \frac{2}{\sqrt{3}} a \tilde{\zeta}_0 \right)^2 + \frac{4}{3} a^2 e^{\phi} \left( \frac{x^2 + y^2}{y} \right)^{\frac{3}{2}}}
\]

\[
\tilde{\zeta}_0 \rightarrow \frac{\tilde{\zeta}_0 + \frac{2}{\sqrt{3}} a \left( \tilde{\zeta}^2_0 + e^{\phi} \left( \frac{x^2 + y^2}{y} \right)^{\frac{3}{2}} \right)}{\left( 1 + \frac{2}{\sqrt{3}} a \tilde{\zeta}_0 \right)^2 + \frac{4}{3} a^2 e^{\phi} \left( \frac{x^2 + y^2}{y} \right)^{\frac{3}{2}}}
\]

(5.96)

(5.97)

The transformations for the fields \( x, y \) and \( \phi \) can be obtained observing that:

\[
\delta_{F_{q_0}} \left( e^{-\phi} \frac{x^2 + y^2}{y} \right) = 0
\]

(5.98)

\[
\delta_{F_{q_0}} \left( \frac{x}{x^2 + y^2} \tilde{\zeta}_0 + \frac{\tilde{\zeta}_1}{3} \right) = 0
\]

(5.99)

and that:

\[
\frac{x}{x^2 + y^2} \rightarrow \frac{x}{x^2 + y^2} + \frac{2a}{\sqrt{3}} \left( \frac{x}{x^2 + y^2} \tilde{\zeta}_0 + \frac{\tilde{\zeta}_1}{3} \right)
\]

(5.100)

The general solution can be obtained by acting with a five parameter group transformation on one of the truncated solutions. In particular, we can act with the \( G_{2,2} \) element:

\[
g = e^{\alpha_5 F_{q_0}} e^{\alpha_4 Y} e^{\alpha_3 E_{q_1}} e^{\alpha_2 E_{q_1}} e^{\alpha_1 F_{q_0}}
\]

(5.101)

40
on the RR truncation II (5.33-5.38). It can be checked numerically that the inverse of the above transformation can map generic values of the fourteen $G_{2,2}$ charges into values obeying to the constraints characterizing the RR truncation (equations 5.39-5.40 and the vanishing of three shift charges). On the other side, some random assignments for the values of the fourteen charges cannot be mapped into a truncated solution. As in the case of the NS-R truncation, the interpretation of this fact is that the set of solutions obtained by acting with a transformation of the form (5.101) on a truncated solution represent a patch of non-zero measure in the space of general solutions.

Solutions with $q'_E = 0$ admit an analytic continuation of the form (3.13) and lead to a real positive-definite action. On the other hand, solutions with $q_E \neq 0$ need to satisfy the condition (3.33) in order to have a real positive-definite action. This condition poses a strong constraint on the solutions. Indeed, we suspect that a real positive-definite action can be obtained only with the truncations studied in section 5.2.

6 Orientifolding and $N = 1$ supergravities

Orientifolding of $N = 2$ supergravity theories can be used to obtain consistent truncations which reduces the supersymmetry of the theory to $N = 1$. The orientifolding can be understood purely from the perspective of the four-dimensional supergravity [61, 62] or microscopically from the Calabi-Yau compactification of type II string theory [63, 64], where the orientation reversal on the worldsheet is accompanied by an involution acting on the Calabi-Yau manifold.

6.1 Orientifolding $N = 2$ theories

The simplest orientifold $O_1$ projection (corresponding to an orientation reversal on the worldsheet with a trivial involution on the Calabi-Yau manifold)

\[
O_1 \phi = \phi, \quad O_1 \sigma = -\sigma, \quad O_1 \zeta^0 = \zeta^0, \quad O_1 \tilde{\zeta}_0 = -\tilde{\zeta}_0 \\
O_1 x^a = -x^a, \quad O_1 y^a = y^a, \quad O_1 \zeta^a = -\zeta^a, \quad O_1 \tilde{\zeta}_a = \tilde{\zeta}_a, \quad a = 1, 2, \ldots, h_{1,1} \tag{6.1}
\]

Projecting out the odd fields one obtains the action:

\[
S = \int d^4x \sqrt{-g}\left\{R - 2g_{ab}\partial_\mu y^a \partial_\nu y^b - \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}e^\phi \partial_\mu \zeta a (I^{-1})^{ab} \partial_\nu \tilde{\zeta}_b - \frac{1}{2}e^\phi \partial_\mu \zeta^0 I_{00} \partial_\nu \zeta^0 \right\} \tag{6.2}
\]

\textsuperscript{3}Solutions obtained by applying a finite transformation generated by $Y_-$ to these truncations will have positive-definite action as well.
The microscopic projection in this case is:

\[
O_1 = \Omega_p \sigma_i, \quad \sigma_i \Omega_{3,0} = \Omega_{3,0}, \quad \sigma_i \omega_a = \omega_a, \quad a = 1, 2, \ldots, h_{1,1}
\]  
(6.3)

and corresponds to the insertion of spacetime filling orientifold O5/O9 planes. Here \( \Omega_p \) corresponds to orientation reversal on the worldsheet. The map \( \sigma_i \) is an involution on the Calabi-Yau and is not to be confused with the spacetime field \( \sigma \), which is the NS-NS axion.

For special quaternionic geometries a more general version of the orientifold projection \( O_1 \) can be constructed. Defining the following split of the indices into a positive and a negative set

\[
A_+ = \{1, 2, \ldots, n_+\}, \quad A_- = \{n_+ + 1, n_+ + 2, \ldots, h_{1,1}\}
\]  
(6.4)

We can define the following projection

\[
\begin{align*}
O'_1 y_{A_+} &= y_{A_+}, \\
O'_1 x_{A_-} &= -x_{A_-}, \\
O'_1 x_{A_+} &= x_{A_-}, \\
O'_1 y_{A_-} &= -y_{A_-} \\
O'_1 \zeta_{A_+} &= -\zeta_{A_+}, \\
O'_1 \zeta_{A_-} &= \zeta_{A_-}, \\
O'_1 \tilde{\zeta}_{A_+} &= \tilde{\zeta}_{A_+}, \\
O'_1 \tilde{\zeta}_{A_-} &= -\tilde{\zeta}_{A_-}, \\
O'_1 \phi &= \phi, \\
O'_1 \sigma &= -\sigma, \\
O'_1 \zeta^0 &= \zeta^0, \\
O'_1 \tilde{\zeta}_0 &= -\tilde{\zeta}_0
\end{align*}
\]  
(6.5)

It can be shown that all terms in the hypermultiplet action (2.1) linear in the odd fields under \( O'_1 \) vanish if the intersection numbers \( C_{abc} \) satisfy the following condition

\[
C_{A_+A_+A_-} = C_{A_-A_-A_-} = 0
\]  
(6.6)

Hence projecting out the odd fields is a consistent truncation and the projected action is given by

\[
\begin{align*}
S &= \int d^4 x \sqrt{-g} \left\{ R - 2g_{++} \partial_\mu y^+ \partial y^+ - 2g_{--} \partial_\mu x^+ \partial x^+ - \frac{1}{2} (\partial_\mu \phi)^2 \\
&\quad - \frac{1}{2} e^\phi \partial_\mu \zeta^- \partial^\mu \zeta^- - \frac{1}{2} e^\phi \partial_\mu \tilde{\zeta}^+ \partial^\mu \tilde{\zeta}^- - \frac{1}{2} e^\phi \partial_\mu \tilde{\zeta}^+ (I^{-1})^{++} \partial^\mu \tilde{\zeta}^+ - e^\phi \partial_\mu \tilde{\zeta}^+ (I^{-1})^{+I} R_{IJ} \partial^\mu \tilde{\zeta}^- \right\}
\end{align*}
\]  
(6.7)

Where schematically the index + runs over \( A_+ \) and the index − runs over \( \{0\} \cup A_- \). Microscopically this orientifold projection is the same as \( O_1 \) where in addition the involution \( \sigma_i \) of the Calabi-Yau manifold acts non-trivially on the (1,1) forms splitting them into even and odd forms.

\[
\begin{align*}
O'_1 &= \Omega_p \sigma_i, \\
\sigma_i \Omega_{3,0} &= \Omega_{3,0} \\
\sigma_i \omega_a &= +\omega_a, \quad a \in A_+, \quad \sigma_i \omega_a &= -\omega_a, \quad a \in A_-
\end{align*}
\]  
(6.8)

(6.9)
Like the first orientifold projection $O'_1$ corresponds to the insertion of space filling O5/O9 planes, the non-trivial action of the involution $\sigma_i$ arises from the way the O5 plane is embedded in the Calabi-Yau manifold.

A second projection $O_2$ exchanges the role of $\zeta^i$ and $\tilde{\zeta}^i$. The same argument as above shows that the orientifold projection is consistent.

$$O_2 \phi = \phi, \quad O_2 \sigma = -\sigma, \quad O_2 \zeta^0 = -\zeta^0, \quad O_2 \tilde{\zeta}_0 = \tilde{\zeta}_0$$
$$O_2 x^a = -x^a, \quad O_2 y^a = y^a, \quad O_2 \zeta^a = \zeta^a, \quad O_2 \tilde{\zeta}_a = -\tilde{\zeta}_a, \quad a = 1, 2, \ldots, h_{1,1} \quad (6.10)$$

Since the action is even under the orientifold projection there are no linear terms involving odd fields. Hence setting to zero the odd fields under $O_2$ is a consistent truncation. The projected action is given by

$$S = \int d^4x \sqrt{-g} \left\{ R - 2g_{ab} \partial_\mu y^a \partial_\mu y^b - \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} e^\phi \partial_\mu \zeta^a I_{ab} \partial_\mu \zeta^b - \frac{1}{2} e^\phi \partial_\mu \tilde{\zeta}_0 (I^{-1})^{00} \partial_\mu \tilde{\zeta}_0 \right\} \quad (6.11)$$

Microscopically, the orientifold projections on a IIB Calabi-Yau compactification can be understood as combining a worldsheet parity reversal $\Omega_p$ with an involution $\sigma_i$ of the Calabi-Yau manifold.

$$O_2 = (-1)^{F_L} \Omega_p \sigma_i, \quad \sigma_i \Omega_{3,0} = -\Omega_{3,0}, \quad \sigma_i \omega_a = \omega_a, \quad a = 1, 2, \ldots, h_{1,1} \quad (6.12)$$

where $F_L$ is the left-moving spacetime fermion number. This projection corresponds to the insertion of spacetime filling orientifold O3/O7 planes.

There is a generalized orientifold projection $O'_2$ associated with $O_2$ which can be obtained from $O'_1$ by exchanging the roles of $\zeta^i$ and $\tilde{\zeta}^i$. Microscopically this projection is given by

$$O'_2 = \Omega_p (-1)^{F_L} \sigma_I, \quad \sigma_i \Omega_{3,0} = -\Omega_{3,0} \quad (6.13)$$
$$\sigma_i \omega_a = +\omega_a, \quad a \in A_+, \quad \sigma_i \omega_a = -\omega_a, \quad a \in A_- \quad (6.14)$$

and corresponds to the insertion of space filling O3/O7 planes. The form of the projected action can be easily worked out.

It is useful to compare the orientifold projections to the truncations of the $N = 2$ action for the $G_{2,2}/(SU(2) \times SU(2))$ coset space, which was explicitly constructed. The orientifold projections $O_1$ and $O_2$ set to zero the NS-NS fields $\sigma$ and $x$. Hence they correspond to the RR truncations discussed in section 5.2. For a general $N = 2$ action all four orientifold projections set to zero $\sigma$ and correspond to a pure RR-charged case. Whether the solution exists depends on whether the behavior of the $x^a$ and $\xi^a, \tilde{\xi}_a$ for $a = 1, 2, \ldots, h_{1,1}$ in the full
\( N = 2 \) instanton solution is consistent with the particular orientifold projection. This means that the fields which are to be projected out in the orientifold projection are zero for all values of Euclidean time in the full \( N = 2 \) solution.

In section 5 the NS-NS truncation was also discussed. One may wonder whether there is an orientifold projection associated with this truncation as in the case of the RR truncation above. From the perspective of four-dimensional \( N = 2 \) supergravity such a projection indeed exists and was called “heterotic” in [65], where all RR axions are projected out. Such a projection does not have an interpretation as an action on the ten-dimensional type II string theory and will not be discussed further here.

### 6.2 Supersymmetry

For all orientifold projections given above the supersymmetry is reduced from \( N = 2 \) to \( N = 1 \). The projection on the fermions can also be worked out either from the consistency of the supersymmetry transformations or microscopically from the Calabi-Yau compactification.

The reduction of the supersymmetry can be achieved by choosing a linear combination of the two \( N = 2 \) gravitinos \( \psi^\alpha_\mu \) as the single \( N = 1 \) gravitino. Also the half the degrees of freedom of the hyperino \( \xi^a \) is set to zero producing the fermionic components of the \( N = 1 \) chiral multiplet.

The supersymmetry is reduced by setting to zero a linear combination of the two infinitesimal supersymmetry transformation parameters \( \epsilon_1, \epsilon_2 \) which are Weyl spinors of positive chirality \( \gamma_5 \epsilon^{1,2} = +\epsilon^{1,2} \). The supersymmetry transformation parameters of negative chirality are labelled \( \epsilon_{1,2} \) related to the positive chirality by \( \epsilon_1 = (\epsilon^1)^*, \epsilon_2 = (\epsilon^2)^* \). The supersymmetries which are preserved by the O3/O7 and the O5/O9 orientifold can be derived by the consistency of the orientifold projection on the bosonic fields and the supersymmetry variations of the fermionic fields [65]. We remind the reader of the \( N = 2 \) gravitino variation

\[
\delta \psi^i_\mu = D_\mu \epsilon^i + (Q^i_\mu)^j \epsilon^j
\]  

(6.15)

where \( D_\mu \) is the standard covariant derivative which includes the spin connection, and \( Q^i_\mu \) is the composite \( SU(2) \) gauge connection, which for both orientifold projections \( O_1 \) and \( O_2 \) reduces to

\[
Q^i_\mu = \begin{pmatrix}
0 & -u \\
\bar{u} & 0
\end{pmatrix}
\]  

(6.16)
The components of the vielbein \( u_\mu \) and \( v_\mu \) after the orientifold projection are given by

\[
O_1: \quad v_\mu = \frac{1}{2} \partial_\mu \phi, \quad u_\mu = -\frac{1}{\sqrt{2}} e^{\frac{K-\phi}{2}} \left( \frac{1}{6} C_{abc} y^a y^b y^c \partial_\mu \zeta^0 + y^a \partial_\mu \bar{\zeta} \right)
\]
\[
\bar{u}_\mu = -\frac{1}{\sqrt{2}} e^{\frac{K-\phi}{2}} \left( \frac{1}{6} C_{abc} y^a y^b y^c \partial_\mu \bar{\zeta}^0 + y^a \partial_\mu \bar{\zeta} \right)
\]

\( (6.17) \)

\[
O_2: \quad v_\mu = \frac{1}{2} \partial_\mu \phi, \quad u_\mu = \frac{i}{\sqrt{2}} e^{\frac{K-\phi}{2}} \left( \partial_\mu \bar{\zeta}^0 - \frac{1}{2} C_{abc} y^a y^b \partial_\mu \zeta^a \right)
\]
\[
\bar{u}_\mu = \frac{i}{\sqrt{2}} e^{\frac{K-\phi}{2}} \left( \partial_\mu \bar{\zeta}^0 - \frac{1}{2} C_{abc} y^a y^b \partial_\mu \bar{\zeta} \right)
\]

\( (6.18) \)

The supersymmetry is reduced by setting to zero a linear combination of the two infinitesimal supersymmetry transformation parameters \( \epsilon_1, \epsilon_2 \) which are Weyl spinors of positive chirality \( \gamma_5 \epsilon_{1,2} = + \epsilon_{1,2} \). The supersymmetry transformation parameters of negative chirality are labelled \( \epsilon_{1,2} \) related to the positive chirality by \( \epsilon_1 = (\epsilon_1)^* \), \( \epsilon_2 = (\epsilon_2)^* \). The supersymmetries which are preserved by the O3/O7 and the O5/O9 orientifold can be derived by the consistency of the orientifold projection on the bosonic fields and the supersymmetry variations of the gravitino \([65]\). The following combination of supercharges is consistent with the orientifold projection

\[
O_1: \quad \epsilon_\alpha = \epsilon_\alpha^1 + i \epsilon_\alpha^2, \quad \epsilon_\dot{\alpha} = \epsilon_{1 \dot{\alpha}} - i \epsilon_{2 \dot{\alpha}}
\]

\[
O_2: \quad \epsilon_\alpha = \epsilon_\alpha^1 - i \epsilon_\alpha^2, \quad \epsilon_\dot{\alpha} = \epsilon_{1 \dot{\alpha}} - \epsilon_{2 \dot{\alpha}}
\]

\( (6.19) \)

Where for clarity we have written the un-dotted (positive chirality) and dotted (negative chirality) spinor indices. A second approach uses the microscopic definition of the orientifold projection and the world sheet definition of the supersymmetry generators \([66, 67]\) and leads to the same conditions.

Since the chirality of the surviving supersymmetries for the instanton solutions is very important, we repeat the hyperino variations for both chiralities for the special case of \( SO(4) \) symmetric solutions

\[
\delta \xi^a = -i V^{a \alpha} \gamma^\tau \epsilon^\alpha, \quad \delta \bar{\xi}_a = i V_{\tau a \alpha} \gamma^\tau \epsilon^\alpha
\]

\( (6.20) \)

Where \( (V^{a \alpha})^* = V_{\tau a \alpha} \). After multiplication by \( \gamma^\tau \) the condition that \( N = 1 \) supersymmetry is preserved for the hyperino variation for the negative chirality supersymmetry becomes

\[
\bar{u}_\tau \epsilon_1 + v_\tau \epsilon_2 = 0, \quad -\bar{v}_\tau \epsilon_1 + \bar{u}_\tau \epsilon_2 = 0, \quad \epsilon^A_\tau \epsilon_1 + E^A_\tau \epsilon_2 = 0, \quad -\bar{E}^A_\tau \epsilon_1 + \bar{E}^A_\tau \epsilon_2 = 0
\]

\( (6.21) \)

and for the positive chirality supersymmetry one gets

\[
- v_\tau \epsilon^1 + u_\tau \epsilon^2 = 0, \quad \bar{u}_\tau \epsilon^1 + \bar{v}_\tau \epsilon^2 = 0, \quad -\epsilon^A_\tau \epsilon_1 + E^A_\tau \epsilon_2 = 0, \quad \bar{E}^A_\tau \epsilon^1 + \bar{E}^A_\tau \epsilon^2 = 0
\]

\( (6.22) \)
Note that for all orientifold projections the NS-NS axion field $\sigma$ is projected out and the shift isometries of the remaining RR fields all commute. The analytic continuation à la Coleman always leads to a saddlepoint with a real action since (3.33) is projected out. The analytic continuation of the RR axion fields for the RR-charged instanton solution is

$$ O_1 : \zeta^0 \to i\zeta_0^\prime, \quad \bar{\zeta}_a \to i\bar{\zeta}_a^\prime $$

$$ O_2 : \bar{\zeta}_0 \to i\zeta_0^\prime, \quad \zeta^a \to i\zeta^\prime_a $$

(6.23)

and it follows that the vielbein component $u$ after analytic continuation is imaginary for $O_1$ and real for $O_2$. The same is true for the components $e^A$ and $E^A$. The consistency of the first two equations in (6.21) and (6.22) implies that the following linear combinations $\epsilon'$ parameterize the unbroken supersymmetries for an extremal BPS instanton solution.

$$ O_1 : \epsilon'^\alpha = \epsilon_1^\alpha \pm i\epsilon_2^\alpha, \quad \epsilon'^\dot{\alpha} = \epsilon_{1\dot{\alpha}} \mp i\epsilon_{2\dot{\alpha}} $$

$$ O_2 : \epsilon'^\alpha = \epsilon_1^\alpha \mp i\epsilon_2^\alpha, \quad \epsilon'^\dot{\alpha} = \epsilon_{1\dot{\alpha}} \mp i\epsilon_{2\dot{\alpha}} $$

(6.24)

The choice of sign corresponds to a choice of sign for the RR charge and gives an instanton or anti instanton solution. The comparison of (6.19) and (6.24) shows of the four real supersymmetries which survive the orientifold projection, two are identical to unbroken supersymmetries of the (anti)-instanton solution. Two of the four broken supersymmetries generate fermionic zero modes in the instanton background. In order to obtain non-zero correlation function the fermionic zero modes have to be soaked up by the appropriate operator insertions. The resulting terms are instanton induced F-terms

$$ \int d^4x \int d^2 \theta F(\Phi)e^{-S_{\text{inst}}} + \int d^4x \int d^2 \bar{\theta} F(\bar{\Phi})e^{-S_{\text{inst}}} $$

(6.25)

Such are potentially important in constructing phenomenologically viable superstring models as they can lift moduli and be responsible for supersymmetry breaking.

Such terms have been analyzed in several models such as intersecting D-branes in orientifolds [68, 69, 70]. We will leave the evaluation of such terms for the theories obtained by orientifold projections discussed in this section for future work.

7 Discussion

In this paper instanton and wormhole solutions in $d = 4 \ N = 2$ supergravity theories coming from large volume Calabi-Yau compactification of type II string theories were discussed using

4 Note however that it follows from (6.17) that after analytic continuation one still has the relations $u_\mu = \bar{u}_\mu$ for the $O_1$ truncation and $u_\mu = -\bar{u}_\mu$ for the $O_2$ truncation.
a method due to Coleman. It is an interesting question whether other prescriptions (e.g. the
dualization of axions to tensor fields) give the same results for solutions, boundary term and
saddlepoint action. For the case of the $SU(2,1)/SU(2) \times U(1)$ coset this is indeed the case
as shown in a previous paper [38]). It would be interesting to find a general proof for the
equivalence of the prescriptions in order to show that there is no arbitrariness in the analytic
continuation procedure.

The Coleman method allows for a classification of possible analytic continuation depend-
ing on the charge the Euclidean solution is carrying. Furthermore this prescription produces
the boundary terms which are necessary to get a non-zero action for the instanton. The
positive definiteness of the saddlepoint action is however not guaranteed. We proposed an
additional condition which guarantees the reality of the solution as well as the positive def-\niniteness of the action. This condition can only be satisfied for truncated solutions More
general real solutions exist after an additional analytic continuation is performed and the
action is not positive definite anymore. Whether these solutions give physical sensible saddle
point contributions is an open problem.

We discussed two cases: the $SU(2,1)/SU(2) \times U(1)$ coset (which was discussed in [38]) and the $G_{2,2}/SU(2) \times SU(2)$ coset. Instanton and wormhole solutions were constructed using
the conserved Noether charges associated with all the global symmetries of the coset. The
solutions are then explicitly obtained for some truncations which give a real saddle-point
action. The method of using the global symmetries to generate the most general solution
was discussed for the $G_{2,2}/SU(2) \times SU(2)$ coset.

For higher dimensional cosets the Noether-method can also be applied to reduce number
of independent equations of motion by using the conservation equations to replace fields and
their derivatives by conserved charges. The usefulness of this approach for more complicated
cases is however limited by the fact that for an exact solution it would be necessary to
solve algebraic equations in terms of the charges of high degree which can not be done
analytically in general. Generic Calabi-Yau compactifications which are not cosets, have
fewer symmetries and the Noether-method is less useful.

The various orientifold projections which reduce the four dimensional supersymmetry
from $N = 2$ to $N = 1$ provide truncations of the $N = 2$ theory. $N = 2$ instanton solutions
will lead to solutions of the truncated theory, as long as in the solutions all the fields which are
projected out are trivial. Such $N = 1$ instanton solution can lead to interesting contributions,
since the orientifold projection reduces the number of fermionic zero modes. Hence such
instantons can contribute to F-terms in the effective action.

In this paper we focussed on solutions which are $SO(4)$ invariant since the equations of
motion reduce to ordinary differential equations, it would be interesting to generalize the
solutions to situation with less symmetry. Via the c-map such solutions could be related to rotating or multi-center black holes.

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A Some details on $G_{2,2}$

In this appendix we give the explicit form of the fourteen infinitesimal transformations of the action which generate the Lie algebra of $G_{2,2}$. For completeness we recall the definition of the matrix $M_{ab}$ is

$$M = \frac{1}{y^2} \begin{pmatrix} 1 & x & -x^3 & 3x^2 \\ x & y^2/3 + x^2 & -x^4 - x^2y^2 & 3x^3 + 2xy^2 \\ -x^3 & -x^4 - x^2y^2 & (x^2 + y^2)^3 & -3(x^2 + y^2)^2x \\ 3x^2 & 3x^3 + 2xy^2 & -3(x^2 + y^2)^2x & (3x^2 + 2y^2)^2 - y^4 \end{pmatrix}$$ \hfill (A.1)

First, we repeat the infinitesimal shift generators, which produce a Heisenberg algebra.

$$E_{p0} \begin{cases} \delta x = 0 \\ \delta y = 0 \\ \delta \phi = 0 \\ \delta \xi^0 = 1/3 \\ \delta \xi^1 = 0 \\ \delta \sigma = -\frac{1}{3} \tilde{\xi}_0 \end{cases} \quad E_{p1} \begin{cases} \delta x = 0 \\ \delta y = 0 \\ \delta \phi = 0 \\ \delta \xi^0 = 0 \\ \delta \xi^1 = 0 \\ \delta \sigma = -\frac{1}{3} \tilde{\xi}_1 \end{cases} \quad E_{q0} \begin{cases} \delta x = 0 \\ \delta y = 0 \\ \delta \phi = 0 \\ \delta \zeta^0 = 0 \\ \delta \zeta^1 = 0 \\ \delta \sigma = 0 \end{cases} \quad E_{q1} \begin{cases} \delta x = 0 \\ \delta y = 0 \\ \delta \phi = 0 \\ \delta \zeta^0 = 0 \\ \delta \zeta^1 = 0 \\ \delta \sigma = 0 \end{cases} \quad E \begin{cases} \delta x = 0 \\ \delta y = 0 \\ \delta \phi = 0 \\ \delta \zeta^0 = 0 \\ \delta \zeta^1 = 0 \\ \delta \sigma = -\frac{1}{2\sqrt{3}} \end{cases}$$ \hfill (A.2)

In the next set of generator $H$ generates a scale transformation, whereas $Y_{0,+,−}$ generators a $SL(2, R)$ action on the moduli $x, y$.

$$H \begin{cases} \delta x = 0 \\ \delta y = 0 \\ \delta \phi = 0 \\ \delta \xi^0 = \xi^0 \\ \delta \xi^1 = \xi^1 \\ \delta \sigma = 2\sigma \end{cases} \quad Y_0 \begin{cases} \delta x = -x \\ \delta y = -y \\ \delta \phi = 0 \\ \delta \xi^0 = \frac{3}{2} \xi^0 \\ \delta \xi^1 = \frac{1}{2} \xi^1 \\ \delta \sigma = 0 \end{cases} \quad Y_+ \begin{cases} \delta x = \frac{1}{\sqrt{6}}(y^2 - x^2) \\ \delta y = -\sqrt{3/2} xy \\ \delta \phi = 0 \\ \delta \xi^0 = \frac{3}{2} \xi^1 \\ \delta \xi^1 = -\frac{3}{2\sqrt{3}} \tilde{\xi}_1 \\ \delta \sigma = 0 \end{cases} \quad Y_- \begin{cases} \delta x = -\sqrt{3/2} \\ \delta y = 0 \\ \delta \phi = 0 \\ \delta \xi^0 = 0 \\ \delta \xi^1 = -\sqrt{3/2} \tilde{\xi}_1 \\ \delta \sigma = 0 \end{cases}$$ \hfill (A.3)

The rest of the generators are quite complicated and complete the $G_{2,2}$ algebra. $F_{p0}$ and $F_{p1}$

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are given by:

\[
F_{p0} = \begin{cases} 
\delta x = 6(x\zeta^0 - \zeta^1) \\
\delta y = 6y\zeta^0 \\
\delta \phi = -6\zeta^0 \\
\delta \zeta^0 = 6 \left( \frac{e^\phi}{y^3} - \zeta^{02} \right) \\
\delta \zeta^1 = 6 \left( \frac{e^\phi}{y^3} - \zeta^{0}\zeta^1 \right) \\
\delta \zeta_0 = 6 \left( \frac{e^\phi}{y^3} + \sigma \right) \\
\delta \zeta_1 = 18 \left( \zeta^{12} - \frac{e^\phi}{y^3} \zeta^1 \right) \\
\delta \sigma = 6 \left( \frac{e^\phi x^3}{y^3} \zeta^0 - \frac{3e^\phi x^2}{y^3} \zeta^1 + 2\zeta^1 \right) 
\end{cases}
\]

\[
F_{p1} = \begin{cases} 
\delta x = 2(x^2 - y^2)\zeta^0 + 2x\zeta^1 + \frac{4}{3}\zeta_1 \\
\delta y = 2y(\zeta^1 + 2x\zeta^0) \\
\delta \phi = -6\zeta^1 \\
\delta \zeta^0 = 6 \left( \frac{e^\phi x}{y^3} - \zeta^{0}\zeta^1 \right) \\
\delta \zeta^1 = 2 \left[ \frac{e^\phi (3x^2 + 2xy^2)}{y^3} - \zeta^{12} + \frac{2}{3}\zeta^{0}\zeta_1 \right] \\
\delta \zeta_0 = 6 \left[ \frac{e^\phi (x^4 + x^2y^2)}{y^3} - \frac{\zeta^2}{9}\zeta_1 \right] \\
\delta \zeta_1 = -6 \left[ \frac{e^\phi (3x^3 + 2xy^2) - \sigma + \frac{4}{3}\zeta^{1}\zeta_1}{y^3} \right] \\
\delta \sigma = 6 \left[ \frac{e^\phi x^2}{y^3} \zeta^0 - \frac{e^\phi (3x^3 + 2xy^2)}{y^3} \zeta^1 + \frac{2}{3}\zeta^{12}\zeta_1 \right] 
\end{cases}
\]  

(A.4)

with \(|z|^2 = x^2 + y^2\). The \(F_{q0}\) generator is:

\[
F_{q0} = \begin{cases} 
\delta x = \frac{2}{\sqrt{3}} \left[ \frac{1}{3} (y^2 - x^2)\zeta_1 - x\zeta_0 \right] \\
\delta y = -\frac{2}{\sqrt{3}} (e^\phi \zeta_0 + \frac{2}{3}e\zeta_1) \\
\delta \phi = -\frac{2}{\sqrt{3}} \zeta_0 \\
\delta \zeta^0 = \frac{2}{\sqrt{3}} \left( \frac{e^\phi}{y^3} - \zeta^{0}\zeta_0 + \zeta^{1}\zeta_1 - \sigma \right) \\
\delta \zeta^1 = \frac{2}{\sqrt{3}} \left[ \frac{e^\phi x^2}{y^3} - \frac{1}{9}\zeta^2 \right] \\
\delta \zeta_0 = \frac{2}{\sqrt{3}} \left[ \frac{e^\phi}{y^3} - \zeta_0 \right] \\
\delta \zeta_1 = -\frac{2}{\sqrt{3}} \left[ \frac{3e^\phi x^2}{y^3} \zeta^0 - \frac{3e^\phi x^2}{y^3} \zeta^1 - \zeta_0 \sigma - \frac{1}{27}\zeta_1 \right] \\
\delta \sigma = \frac{2}{\sqrt{3}} \left[ \frac{e^\phi}{y^3} \zeta^0 - \frac{3e^\phi x^2}{y^3} \zeta^1 - \zeta_0 \sigma - \frac{1}{27}\zeta_1 \right] 
\end{cases}
\]  

(A.5)

and the \(F_{q1}\) generator is:

\[
F_{q1} = \begin{cases} 
\delta x = -\frac{2}{\sqrt{3}} \left[ 2(x^2 - y^2)\zeta^1 - \zeta_0 + \frac{1}{3}x\zeta_1 \right] \\
\delta y = -\frac{2}{\sqrt{3}} \left[ 4xy\zeta^1 + \frac{1}{3}y\zeta_1 \right] \\
\delta \phi = -\frac{2}{\sqrt{3}} \zeta_1 \\
\delta \zeta^0 = -2\sqrt{3} \left( \frac{e^\phi x^2}{y^3} - \zeta^{12} \right) \\
\delta \zeta^1 = -\frac{2}{\sqrt{3}} \left[ \frac{e^\phi (3x^3 + 2xy^2)}{y^3} + \sigma - \zeta^{0}\zeta_0 + \frac{1}{3}\zeta^{1}\zeta_1 \right] \\
\delta \zeta_0 = -\frac{2}{\sqrt{3}} \left[ \frac{3e^\phi x^2}{y^3} \zeta^0 + \zeta_0 \zeta_1 \right] \\
\delta \zeta_1 = 2\sqrt{3} \left[ \frac{e^\phi (3x^3 + 4x^2y^2 + y^4)}{y^3} - 2\zeta^1 \zeta_0 - \frac{1}{9}\zeta_1 \right] \\
\delta \sigma = -2\sqrt{3} \left[ \frac{e^\phi}{y^3} \zeta^0 - \frac{e^\phi (3x^3 + 4x^2y^2 + y^4)}{y^3} \zeta^1 + \frac{1}{9}\zeta_1 \sigma + \zeta^{12}\zeta_0 \right] 
\end{cases}
\]  

(A.6)

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Finally, the last generator is:

\[
F \left\{ \begin{array}{l}
\delta x = 2\sqrt{3}(x^2 - y^2)(\zeta^{12} + \frac{1}{3}\zeta^0\zeta_1) + \frac{1}{2}\zeta^2 - \zeta^1\zeta_0 + x(\zeta^0\zeta_0 + \frac{1}{3}\zeta^1\zeta_1) \\
\delta y = 2\sqrt{3}(2xy\zeta^{12} + \frac{1}{2}y^2\zeta^1 + y\zeta^0\zeta_0 + \frac{1}{2}xy\zeta^0\zeta_1) \\
\delta \phi = -4\sqrt{3}(\frac{1}{2}\zeta^0\zeta_0 + \frac{1}{2}\zeta^1\zeta_1 - \sigma) \\
\delta \zeta^0 = 2\sqrt{3}\left[ e^\phi(M\zeta)_1 - \zeta^{13} - \zeta^0\zeta_0 - \zeta^0\zeta_1 + \zeta^1 \right] \\
\delta \zeta^1 = 2\sqrt{3}\left[ e^\phi(M\zeta)_2 + \zeta^0\zeta_1 - \zeta^1\zeta_0 + \zeta^1 \right] \\
\delta \zeta_0 = 2\sqrt{3}\left[ -e^\phi(M\zeta)_3 - \frac{1}{4}\zeta^3 + \zeta_0 \sigma \right] \\
\delta \zeta_1 = 2\sqrt{3}\left[ -e^\phi(M\zeta)_4 + 3\zeta^1\zeta_0 - \frac{3}{2}\zeta^1\zeta_1 + \zeta_1 \right] \\
\delta \sigma = 2\sqrt{3}\left\{ -e^\phi[(M\zeta)_3\zeta^0 + (M\zeta)_4]\zeta^1\right\} - e^{2\phi} + 2\zeta^1\zeta_0 - \frac{2\zeta^2}{\sigma^2} + \sigma^2 \right\}
\right.
\]

Here \((M\zeta)_i\) is the \(i\)-th component of the vector \(M\zeta\) with \(M\) defined in (5.2) and the vector \(\zeta\) is defined in (5.3). The generators (A.2-A.7) form a \(G_{2,2}\) group of global symmetries. Some of the relevant nonvanishing commutation relations are:

\[
\begin{align*}
[E_{p_1}, E_{q_1}] &= -2\delta^{I J} E_I \\
[Y_{-}, Y_{+}] &= Y_0 \\
[E_{p_0}, F_{p_0}] &= H + 2Y_0 \\
[E_{p_1}, F_{p_1}] &= H + \frac{2}{3}Y_0 \\
[E_{p_1}, F_{q_1}] &= -\frac{4\sqrt{2}}{3}Y_+ \\
[Y_{+}, E_{p_1}] &= \frac{\sqrt{3}}{2}E_{p_0} \\
[Y_{+}, E_{q_1}] &= -\sqrt{2}E_{p_1} \\
[Y_{+}, E_{q_0}] &= -\sqrt{3}E_{q_1} \\
[Y_{+}, F_{p_1}] &= \frac{2}{3}F_{p_1} \\
[Y_{+}, F_{q_1}] &= -\frac{2}{3}F_{p_1} \\
[E, F_{q_1}] &= -E_{p_1} \\
\end{align*}
\]

Finally, we get the following expressions for the Noether current associated with the symmetries \(H\):

\[
j_H^\mu = 3\zeta^0 j_{E_{p_0}}^\mu + 3\zeta^I j_{E_{p_I}}^\mu + \frac{\zeta_0}{\sqrt{3}} j_{E_{q_0}}^\mu + \frac{\zeta_1}{\sqrt{3}} j_{E_{q_1}}^\mu - 2\sqrt{3}(2\sigma - \zeta^1\zeta_1) j_E^\mu + 2\partial^\mu \phi \tag{A.9}
\]

and the current associated with \(Y_0\):

\[
j_{Y_0}^\mu = \frac{3}{2}(3\zeta^0 j_{E_{p_0}}^\mu + \zeta^I j_{E_{p_I}}^\mu - \frac{\zeta_0}{\sqrt{3}} j_{E_{q_0}}^\mu - \frac{\zeta_1}{\sqrt{3}} j_{E_{q_1}}^\mu) + \sqrt{3}(3\zeta^0\zeta_0 + \zeta^I\zeta_1) j_E^\mu - \frac{3}{2y^2}(\zeta^2)^2 \tag{A.10}
\]

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and the current associated with \( Y_- \):
\[
j_{Y_-}^\mu = \frac{\bar{\zeta}}{\sqrt{2} E_{q_0}} - \frac{3\sqrt{3}}{\sqrt{2}} \zeta^0 j_{E_{p_1}}^\mu + 3\sqrt{2} \zeta^1 j_{E_{q_1}}^\mu - 3\sqrt{2}(3\zeta^{12} + \zeta^0 \bar{\zeta}_1) j_E^\mu - \frac{3\sqrt{3}}{\sqrt{2}} \partial^\mu y \tag{A.11}
\]
and the current associated with \( Y_+ \):
\[
j_{Y_+}^\mu = \frac{\sqrt{3}}{\sqrt{2}} \left( 3\zeta^1 j_{E_{q_0}}^\mu - \frac{\bar{\zeta}}{\sqrt{3}} j_{E_{q_1}}^\mu - \frac{2}{3} \bar{\zeta}_1 j_{E_{p_1}}^\mu - \frac{x^2 - y^2}{y^2} \partial^\mu x - 2 \frac{x \partial^\mu y}{y} \right) + \sqrt{2}(3\zeta^1 \bar{\zeta}_0 - \frac{\zeta^2}{3}) j_E^\mu \tag{A.12}
\]
The current associated with the \( F_{p_1} \) and \( F_{q_1} \) are more complicated. \( F_{q_0} \) gives the current:
\[
j_{F_{q_0}}^\mu = \left( \frac{e^\phi}{y^3} - \zeta^0 \zeta^1 \right) j_{E_{q_0}}^\mu + \left( \frac{e^\varphi x}{y^3} - \zeta^0 \zeta^1 \right) j_{E_{p_1}}^\mu + \frac{1}{3\sqrt{3}} \left( \frac{e^\varphi x^3}{y^3} + \sigma \right) j_{E_{q_0}}^\mu
\]
\[
- \frac{1}{\sqrt{3}} \left( \frac{e^\varphi x^2}{y^3} - \zeta^2 \right) j_{E_{q_1}}^\mu + \frac{2}{3} \frac{e^\varphi}{y^3} \left( 3\zeta^2 \zeta^1 - \zeta^3 \zeta^0 + \bar{\zeta}_0 + x \bar{\zeta}_1 \right) j_{E_{q_0}}^\mu
\]
\[
- \left( 2 \zeta^{13} + \zeta^0 \zeta^2 + \zeta^0 \zeta^1 \zeta_1 \right) j_E^\mu - \frac{1}{3} \zeta^0 \partial^\mu \phi + \frac{x \zeta^0 - \zeta^1}{y^2} \partial^\mu x + \frac{\partial^\mu y}{y} \tag{A.13}
\]
The current associated to \( F_{p_1} \) is:
\[
j_{F_{p_1}}^\mu = \left( \frac{e^\varphi x}{y^3} - \zeta^0 \zeta^1 \right) j_{E_{p_0}}^\mu + \left( \frac{e^\varphi x^3 + \frac{2}{3} xy^2}{y^3} - \sigma + \frac{4\zeta^1 \bar{\zeta}_1}{9} \right) j_{E_{q_0}}^\mu + \left( \frac{e^\varphi x^2 + \frac{y^2}{3}}{y^3} + \frac{2}{3} \zeta^1 \bar{\zeta}_1 - \frac{\zeta^2}{3} \right) j_{E_{p_1}}^\mu + \left( \frac{e^\varphi x^2 |z|^2}{3y^3} - \frac{\zeta^1}{27} \right) j_{E_{q_0}}^\mu
\]
\[
+ \frac{\zeta^1}{3} \partial^\mu \phi + \frac{2}{3} \frac{e^\varphi}{y^3} \left( (3x^3 + 2xy^2) \zeta^1 - x^2 |z|^2 \zeta^0 + x \bar{\zeta}_0 + (x^2 + \frac{y^2}{3}) \bar{\zeta}_1 \right) + \frac{2}{3} \frac{\zeta_1}{\sqrt{3}} \partial^\mu x + \frac{\zeta^1}{3y} \partial^\mu y \tag{A.14}
\]
and the current associated to \( F_{q_0} \) is:
\[
j_{F_{q_0}}^\mu = \left( \frac{e^\varphi x^3}{y^3} + \zeta^0 \bar{\zeta}_0 + \zeta^1 \bar{\zeta}_1 - \sigma \right) j_{E_{q_0}}^\mu + \left( \frac{e^\varphi x^2 |z|^2}{y^3} - \frac{\bar{\zeta}^2}{9} \right) j_{E_{p_1}}^\mu
\]
\[
+ \left( \frac{e^\varphi |z|^6}{y^3} - \bar{\zeta}_0 \right) j_{E_{q_0}}^\mu + \left( \frac{e^\varphi x |z|^4}{y^3} + \frac{\bar{\zeta}_0 \bar{\zeta}_1}{3} \right) j_{E_{q_1}}^\mu - \frac{\bar{\zeta}_0}{3} \partial^\mu \phi + \frac{2}{\sqrt{3}} \frac{e^\varphi}{y^3} \left( 3x |z|^4 \zeta^1 - |z|^6 \zeta^0 + x^3 \bar{\zeta}_0 + x^2 |z|^2 \bar{\zeta}_1 \right) j_E^\mu
\]
\[
+ \frac{2}{\sqrt{3}} \left( \zeta^0 \bar{\zeta}_0 + \zeta^1 \bar{\zeta}_0 \bar{\zeta}_1 - \frac{2 \bar{\zeta}_3}{27} \right) j_E^\mu - \frac{3x \bar{\zeta}_0 + (x^2 - y^2) \bar{\zeta}_1}{3y^2} \partial^\mu x + \frac{\bar{\zeta}_0 + \frac{2}{3} x \bar{\zeta}_1}{y} \partial^\mu y \tag{A.15}
\]
Finally, the current associated to $F_{q1}$ is:

$$\begin{align*}
\frac{j_{F_{q1}}^\mu}{2\sqrt{3}} &= 3 \left( \zeta^{12} - \frac{e^\phi x^2}{y^3} \right) j_{E_{\rho 0}}^\mu - \left( \frac{e^\phi(3x^3 + 2xy^2)}{y^3} + \sigma - \zeta^0 \zeta_0 + \frac{\zeta_1^1 \zeta_1}{3} \right) j_{E_{\rho 1}}^\mu + \frac{\zeta_1}{3} \partial^\mu \phi \\
&\quad - \left( \frac{e^\phi x |z|^4}{y^3} + \frac{\zeta_0 \zeta_1}{3} \right) j_{E_{\phi}}^\mu + \left( \frac{e^\phi(3x^4 + 4x^2y^2 + y^4)}{y^3} - 2\zeta^1 \zeta_0 - \frac{\zeta_2}{9} \right) j_{E_{q1}}^\mu \\
&\quad + \frac{2\sqrt{3}e^\phi}{y^3} \left( 3x |z|^4 \zeta^0 - (3x^4 + 4x^2y^2 + y^4) \zeta^1 - x^2 \zeta_0 - (x^3 + \frac{2xy^2}{3}) \zeta_1 \right) j_{E}^\mu \\
&\quad + \frac{2\zeta^0 \zeta_0 \zeta_1 + 12\zeta^{12} \zeta_0 - \frac{2}{3} \zeta^1 \zeta_1}{\sqrt{3}} j_{E}^\mu - \frac{2(x^2 - y^2) \zeta^1 - \zeta_0 + \frac{x}{3} \zeta_1}{y^2} \partial^\mu x - \frac{4x \zeta^1 + \frac{\zeta_1}{3}}{y} \partial^\mu y \tag{A.16}
\end{align*}$$

These expressions greatly simplify in the truncations considered in section 5.
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