Banach-Lie groupoids and generalized inversion

Daniel Beltită

Institute of Mathematics “S. Stoilow” of the Romanian Academy, 21 Calea Griviței Street, 010702 Bucharest, Romania

Tomasz Goliński, Grzegorz Jakimowicz

University in Białystok, Institute of Mathematics, Ciolkowskiego 1M, 15-245 Białystok, Poland

Fernand Pelletier

Unité Mixte de Recherche 5127 CNRS, Université de Savoie Mont Blanc, Laboratoire de Mathématiques (LAMA), Campus Scientifique, 73370 Le Bourget-du-Lac, France

Abstract

We study a few basic properties of Banach-Lie groupoids and algebroids, adapting some classical results on finite dimensional Lie groupoids. As an illustration of the general theory, we show that the notion of locally transitive Banach-Lie groupoid sheds fresh light on earlier research on some infinite-dimensional manifolds associated with Banach algebras.

Keywords: Banach manifold, Lie groupoid, Lie algebroid, Moore-Penrose pseudo-inverse

2010 MSC: 22A22, 58H05, 46L05

Contents

1 Introduction 2

2 Preliminaries and notation 3

3 n.n.H. Banach-Lie groupoids 7

4 Banach-Lie algebroids 14

5 Banach-Lie algebroids and s-simply connected groupoids 26
1. Introduction

The theory of Banach-Lie groups (i.e., infinite-dimensional Lie groups modeled on Banach spaces) is a rather old and well-developed research area that is interesting on its own and also for its applications to many problems in functional analysis, differential geometry, or mathematical physics. There exist however several topics whose natural background requires Banach manifolds endowed with an algebraic structure that is more general than the notion of group. We will briefly mention here very few references on some of these topics that can be better understood from the perspective of Banach-Lie groupoids, which is the main theme of our present paper:

• Moore-Penrose pseudo-inverses in $C^*$-algebras and their differentiability properties (cf. [ACM05], [Boa06], [ACG08], [LR12], [AM13]);
• Poisson structures on the predual of a $W^*$-algebra (cf. [OJS15] and [OS16]);
• Banach-Lie algebroids (cf. [An11], [CP12], [Pe12]).

Motivated by the above research directions, and also by the impressive development of the theory of finite-dimensional Lie groupoids (see for instance [Mac87], [MM03], [CF11]) we think it worthwhile to develop the basic theory of Banach-Lie groupoids and to illustrate it by a brief discussion on its relation to some differentiability questions in the theory of $C^*$-algebras. More specifically, the contents of this paper are as follows. Section 2 collects some basic notions on differential geometry and on topological groupoids that we need. In Section 3 we introduce the notion of (not-necessarily-Hausdorff) Banach-Lie groupoid along with several examples. The main result here (Theorem 3.3) concerns the differentiability properties of the orbits, extending the classical results on actions of Banach-Lie groups. In Section 4 we extend the Lie functor from Banach-Lie groups to Banach-Lie groupoids (Theorem 4.19) and we establish the link between the orbits of a split Banach-Lie groupoid and the orbits of its corresponding Banach-Lie algebroid (Theorem 4.24). In Section 5 we obtain a Banach-Lie groupoid version of the fact that for every Banach-Lie group there exists a simply connected Banach-Lie group with the same Lie algebra (Theorem 5.1). Section 6 is devoted to the study of locally transitive Banach-Lie groupoids, a class of groupoids that play a central role in the description of the differentiability properties of the Moore-Penrose pseudo-inverse in $C^*$-algebras. Among other things, we provide several equivalent characterizations of the groupoids of this type (Theorem 6.2) and we study the Atiyah bundles associated to Banach principal bundles (Proposition 6.6). Finally, in Section 7...
we briefly illustrate the general theory developed so far, by the locally transitive Banach-Lie groupoids associated to unital Banach algebras. As mentioned above, this provides a natural framework for some results that were established in the earlier literature in connection with the Moore-Penrose pseudo-inverse.

2. Preliminaries and notation

2.1. The context of not-necessarily-Hausdorff Banach manifolds

The terminology in the literature on infinite-dimensional manifolds is not uniform, so we mention that our general references are [Lan01, Ch. II–III] or [Bou71b]. For the sake of clarity and for later reference in this paper, we briefly recall here some classical notions on not-necessarily-Hausdorff manifolds modeled on Banach spaces.

A $C^\infty$-atlas on a set $M$ is a family $\{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ of subsets $U_\alpha$ of $M$ and maps $u_\alpha$ from $U_\alpha$ to a Banach space $M_\alpha$ such that:

- $u_\alpha$ is a bijection of $U_\alpha$ onto a open subset of $M_\alpha$ for all $\alpha \in A$;
- $M = \bigcup_{\alpha \in A} U_\alpha$;
- for any $\alpha$ and $\beta$ such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, then $u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1}: u_\beta(U_{\alpha\beta}) \subset M_\beta \rightarrow u_\alpha(U_{\alpha\beta}) \subset M_\alpha$ is a smooth map.

As usual, one has the notion of equivalent $C^\infty$-atlases on $M$. An equivalence class of $C^\infty$-atlases on $M$ defines a topology on $M$ which in general fails to have the Hausdorff property.

**Definition 2.1.** An equivalence class of $C^\infty$-atlases is called a not-necessarily-Hausdorff Banach manifold structure on $M$, for short a n.n.H. Banach manifold. This structure is called a Hausdorff Banach manifold structure on $M$ (for short a Banach manifold as in [Bou71b]) if the topology defined by this atlas is a Hausdorff topology.

It follows by the above definition that all Banach spaces $M_\alpha$ are topologically isomorphic on every connected component of $M$. If all connected components of $M$ are modelled on a fixed Banach space $M$ (up to an isomorphism) then we will say that $M$ is a pure Banach manifold. A pure Banach component of a Banach manifold $M$ is a pure Banach manifold which is a union

$$N = \bigcup_{\alpha \in A} M_\alpha$$

of connected components of $M$. We say that $N$ is a maximal pure (n.n.H.) Banach component if for any connected components $M_\lambda$ of $M$ such that $M_\lambda \cap N = \emptyset$ then $M_\lambda$ is modelled on a Banach space $M_\lambda$ which is not isomorphic to the model space of $N$.
Remark 2.2. If $M$ is a n.n.H. Banach manifold, then its corresponding topology is $T_1$ hence each finite subset of $M$ is closed. Moreover, since $M$ is locally Hausdorff, by Zorn’s Lemma, there exists a maximal open dense subset $M_0$ of $M$ which is an open Banach manifold in $M$. (See for instance [BG08, Lemma 4.2].)

It is clear that the classical construction of the tangent bundle $TM$ of a Banach manifold $M$ can be applied to a n.n.H. Banach manifold and so we get again a n.n.H. Banach manifold $TM$ which is a Banach manifold if and only if $M$ is.

A smooth map between n.n.H. Banach manifolds $\varphi: N \to M$ is called a weak immersion if its tangent map $T_x\varphi: T_xN \to T_{\varphi(x)}M$ is injective for every $x \in N$. If $\varphi$ is a weak immersion for which the range of its tangent map $T_x\varphi$ is a closed subspace of $T_{\varphi(x)}N$ for every $x \in N$, then $\varphi$ is called an immersion. If $\varphi$ is an immersion for which the range of $T_x\varphi$ is a split subspace of $T_{\varphi(x)}M$ (that is, there exists a closed linear subspace $V$ for which one has the direct sum decomposition $T_{\varphi(x)}M = (T_x\varphi)(T_xN) \oplus V$), then $\varphi$ is called a split immersion. We emphasize that this terminology is not generally used in the literature, since for instance the split immersions in the above sense are called immersions in [Bon71b, 5.7.1]. See [MO92] and [Gl15] for additional information.

An immersed (resp., weakly or split) n.n.H. Banach submanifold of a n.n.H. Banach manifold $M$ is an injective immersion (resp., weak or split immersion) $\iota: N \to M$. An immersed n.n.H. Banach manifold $\iota: N \to M$ is called a closed submanifold (resp. split submanifold) if $\iota(N)$ is a closed subset of $M$ (resp. $\iota$ is a split immersion).

A closed split submanifold of a n.n.H. Banach manifold $M$ will called simply a submanifold of $M$ and then the corresponding split immersion $\iota: N \to M$ is usually thought of as an inclusion map $N \hookrightarrow M$.

A submersion $p: N \to M$ between two n.n.H. Banach manifolds is a surjective smooth map such that $Tp(T_xN) = T_{p(x)}M$ and $\ker T_xp$ is a split subspace of $T_xN$ for each $x \in N$. A smooth map $f: N \to M$ between two n.n.H. Banach manifolds is a subimmersion if for each $x \in N$ there exist an open neighborhood $U$ of $x$, a Banach manifold $P$, a submersion $s: U \to P$, and an immersion $j: P \to M$ such that $f|_U = j \circ s$. If $f$ is a subimmersion, then $f^{-1}(y)$ is a submanifold of $N$ for each $y \in M$.

By locally trivial fibration we mean a submersion $p: N \to M$ such that for every $x \in M$ there exist an open neighborhood $U$ of $x$ and a diffeomorphism $\Phi: p^{-1}(U) \to U \times p^{-1}(x)$ such that $p_1 \circ \Phi = p$, where $p_1 = U \times p^{-1}(x) \to U$ is the canonical projection. If the basis $M$ is connected then all the fibers are diffeomorphic.

A n.n.H Banach bundle is a locally trivial fibration $\pi: \mathcal{A} \to M$ whose fiber is a Banach space. Of course $\mathcal{A}$ is Hausdorff if and only if $M$ is so. On each connected component $\mathcal{A}_\alpha$ of $\mathcal{A}$ the fibers are isomorphic to a common Banach space (called the typical fiber) but this space can change from one connected component to another. The (n.n.H) Banach bundle $\pi: \mathcal{A} \to M$ is called pure if its fibers have the same typical fiber and if $M$ is a pure Banach manifold. In
particular, if \( \pi : \mathcal{A} \to M \) is a Banach bundle, each connected component \( \mathcal{A}_\alpha \) of \( \mathcal{A} \) gives rise to a pure (n.n.H) Banach bundle \( \pi : \mathcal{A}_\alpha \to \pi(\mathcal{A}_\alpha) \). For example the tangent bundle \( TM \) of a (n.n.H) Banach manifold \( M \) is a Banach bundle and the tangent bundle \( TM_\alpha \to M_\alpha \) is a pure Banach bundle for each connected component \( M_\alpha \) of \( M \).

Notions like n.n.H. Banach bundle morphisms, n.n.H. Banach bundle isomorphisms, n.n.H. Banach subbundles etc., are defined as usual.

The algebra of smooth maps \( f : M \to \mathbb{R} \) will be denoted \( C^\infty(M) \), the \( C^\infty(M) \)-module of smooth sections of a bundle \( \pi : \mathcal{A} \to M \) will be denoted \( \Gamma(\mathcal{A}) \), and the \( C^\infty(M) \)-module of smooth vector fields on \( M \) will be denoted \( \Xi(M) \).

### 2.2. Topological groupoids

A topological groupoid \( \mathcal{G} \rightrightarrows M \) is a pair \((\mathcal{G}, M)\) of topological spaces such that \( \mathcal{G} \) may not be Hausdorff but \( M \) is Hausdorff, with the following structure maps:

1. **Source and Target Maps**
   - Two surjective open continuous maps \( s : \mathcal{G} \to M \) and \( t : \mathcal{G} \to M \) called **source** and **target** maps, respectively.

2. **Multiplication**
   - A continuous map \( m : \mathcal{G}^2 \to \mathcal{G} \), where \( \mathcal{G}^2 := \{(g, h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\} \) is provided with the induced topology from the product topology on \( \mathcal{G} \times \mathcal{G} \), called a **multiplication** denoted \( m(g, h) = gh \) and satisfying an associativity relation in the sense that the product \((gh)k\) is defined if and only if \( g(hk) \) is defined and in this case we must have \((gh)k = g(hk)\).

3. **Identity Section**
   - A continuous embedding \( \mathbf{1} : M \to \mathcal{G} \) called **identity section** which satisfies \( g\mathbf{1}_x = g \) for all \( g \in \mathcal{G}^{-1}(x) \), and \( \mathbf{1}_x \mathcal{G} = g \) for all \( g \in \mathbf{1}^{-1}(x) \) (what in particular implies \( s \circ \mathbf{1} = \text{id}_M = t \circ \mathbf{1} \)).

4. **Inversion**
   - A homeomorphism \( i : \mathcal{G} \to \mathcal{G} \), denoted \( i(g) = g^{-1} \) called **inversion**, which satisfies \( gg^{-1} = \mathbf{1}_{i(g)} \), \( g^{-1}g = \mathbf{1}_{s(g)} \) (what in particular implies \( s \circ i = t \), \( t \circ i = s \)).

The space \( M \) is called the **base** of the groupoid, and \( \mathcal{G} \) is called the **total space** of the groupoid. (See for instance [CF11, Def. 1.3] for more details.)

For any \( x, y \in M \), we denote \( \mathcal{G}(x, -) := \mathcal{G}^{-1}(x) \), \( \mathcal{G}(-, y) := \mathcal{G}^{-1}(y) \), and

\[
\mathcal{G}(x, y) := \{ g \in \mathcal{G} \mid s(g) = x, \ t(g) = y \} = \mathcal{G}(x, -) \cap \mathcal{G}(-, y).
\]

The **isotropy group** at \( x \in M \) is the set

\[
\mathcal{G}(x) = \{ g \in \mathcal{G} \mid s(g) = x = t(g) \} = \mathcal{G}(x, x) \subseteq \mathcal{G}
\]

and the **orbit** of \( x \in M \) is the set

\[
\mathcal{G}.x = \{ t(g) \mid g \in \mathcal{G}^{-1}(x) \} = t(\mathcal{G}(x, -)) \subseteq M.
\]

---

1. Each point \( g \in G \) can be regarded as an arrow \( g : s(g) \to t(g) \) which joins \( s(g) \) to \( t(g) \).
For any \( g \in G \), if \( s(g) = x \) and \( t(g) = y \), then we define its corresponding left translation
\[
L_g : G(-, x) \to G(-, y), \quad h \mapsto gh,
\]
and similarly the right translation \( R_g : G(x, -) \to G(y, -), \quad h \mapsto hg \). Each of these maps \( L_g \) and \( R_g \) is a homeomorphism.

The topological groupoid \( G \rightrightarrows M \) is called \( s \)-connected if its \( s \)-fibers \( G(x, -) \) are connected for all \( x \in M \), and \( s \)-simply connected if all its \( s \)-fibers are connected and simply connected.

The topological groupoid is called transitive if the map \((s, t) : G \to M \times M\) is surjective.

A topological morphism between topological groupoids \( G \rightrightarrows M \) and \( H \rightrightarrows N \) is given by a pair of continuous maps \( \Phi : G \to H \) and \( \phi : M \to N \) which are compatible with the structure maps, that is:

- \( \phi(s(g)) = s(\Phi(g)) \), \( \phi(t(g)) = t(\Phi(g)) \) and \( \Phi(g^{-1}) = \Phi(g)^{-1} \) for all \( g \in G \);
- if \( (g, g') \in G^{(2)} \) then \( \Phi(gg') = \Phi(g)\Phi(g') \);
- \( \Phi(1_x) = 1_{\phi(x)} \) for every \( x \in M \).

Note that \( \phi \) is uniquely determined by \( \Phi \) and therefore a morphism between the groupoids \( G \rightrightarrows M \) and \( H \rightrightarrows N \) is given only by a continuous map \( \Phi : G \to H \) which satisfies the above compatibility conditions.

A subgroupoid of \( G \rightrightarrows M \) is a groupoid \( H \rightrightarrows N \) such that \( H \subset G \) and the inclusion \( \iota : H \to G \) is a topological morphism of groupoids. A subgroupoid \( H \rightrightarrows N \) of \( G \rightrightarrows M \) is called a wide subgroupoid if \( N = M \).

### 2.3. Some technical lemmas

We will use the following results which are essentially extracted from the collection of Bourbaki’s books.

**Lemma 2.3.** A continuous left action of a n.n.H. topological group on a n.n.H. topological space \( G \times X \to X \), \( (g, x) \mapsto g.x \), is proper if and only if it satisfies the following condition: For every net \( \{ (g_j, x_j) \}_{j \in J} \) in \( G \times X \) for which there exists \( \lim_{j \in J} (g_j, x_j, x_j) = (b, a) \in X \times X \), there also exists \( \lim_{j \in J} g_j =: g \in G \) and \( g.a = b \).

**Proof.** See the comment after [Bou71a, Ch. III, §4, no. 1, Def. 1].

**Lemma 2.4.** Let \( G \times X \to X \), \( (g, x) \mapsto g.x \) be a continuous proper action of a n.n.H. topological group on a n.n.H. topological space. Then the quotient space \( G \setminus X \) is a Hausdorff topological space. Moreover, if \( G \) is Hausdorff, then \( X \) is also Hausdorff.

**Proof.** See [Bou71a, Ch. III, §4, no. 2, Prop. 3].
As noted in [Bou72, Ch. III, §1, no. 5, Prop. 10] in the case of finite-dimensional manifolds and in [Gl15, Th. I] for Banach manifolds acted on by Banach-Lie groups, the freeness hypothesis in (a) of the following lemma ensures that the tangent map of $\rho(x)$ is injective for every $x \in X$.

**Lemma 2.5.** Let $G \times X \to X$, $(g,x) \mapsto g.x$ be a smooth action of a Banach-Lie group on a n.n.H. Banach manifold, satisfying the following conditions:

(a) The action is free and proper.
(b) For every $x \in X$ the map $\rho(x): G \to X$, $g \mapsto g.x$ is a split immersion.

Then the quotient topological space $G \setminus X$ has the Hausdorff property and has the unique structure of a Banach manifold for which the quotient map $q: X \to G \setminus X$ is a submersion. Moreover $X$ is Hausdorff and $(X,G \setminus X,G,\pi)$ is a Banach principal $G$-bundle.

**Proof.** Any Banach-Lie group is Hausdorff by [Bou71a, Ch. III, §2, no. 6, Prop. 18(a)]. Then we can use Lemma 2.4 to obtain that both $G \setminus X$ and $X$ are Hausdorff. Finally, it follows by [Bou72, Ch. III, §1, no. 5, Prop. 10] that $G \setminus X$ has the unique structure of a Banach manifold for which the quotient map $q: X \to G \setminus X$ is a submersion.

**Lemma 2.6.** Let $Z \subseteq X \subseteq Y$ be n.n.H. Banach manifolds. If $Z \subseteq Y$ is a submanifold and $X \subseteq Y$ is a submanifold, then also $Z \subseteq X$ is a submanifold.

**Proof.** It follows by [Bou71b, 5.8.5] that the inclusion map $\iota: Z \hookrightarrow X$ is smooth. Then the map $\iota$ is an immersion as a direct consequence of the hypothesis. It remains to show that for every $z \in Z$ the subspace $T_z Z \subseteq T_z X$ is split. To this end, using that $Z \subseteq Y$ is a submanifold, we can find a closed linear subspace $V \subseteq T_z Y$ with $T_z Z \oplus V = T_z Y$. Since $T_z Z \subseteq T_z X \subseteq T_z Y$, it then follows that $T_z Z \oplus (V \cap T_z X) = T_z X$, and this completes the proof.

**Remark 2.7.** The proof of Lemma 2.6 uses only the fact that the subspace $T_z X \subseteq T_z Y$ is closed, but not that it is a split subspace, for $y \in Y$. However the splitting condition seems to be necessary in order to be able to use [Bou71b, 5.8.5].

3. n.n.H. Banach-Lie groupoids

3.1. Definition and basic properties

The definition of a n.n.H. Banach-Lie groupoid requires some basic facts established in the following proposition.

**Proposition 3.1.** Let $\mathcal{G} \rightrightarrows M$ be a topological groupoid satisfying the following conditions:

1. $\mathcal{G}$ is a n.n.H Banach manifold and $M$ is a Banach manifold;
2. the map $s: \mathcal{G} \to M$ is a submersion;
Then \( t \) is also a submersion and so for any \( x \in M \) each fiber \( G(x, -) \) and \( G(-, x) \) are n.n.H. Banach submanifolds of \( G \) and \( G(x, -) \) is Hausdorff if and only if \( G(-, x) \) is Hausdorff. Moreover we have:

(i) The topological space \( G^{(2)} \) is a n.n.H. Banach submanifold of \( G \times G \).

(ii) The map \( 1 : M \to G \) is smooth and \( 1(M) \) is a closed Banach submanifold of \( G \).

Moreover if \( G \) is a Banach manifold then for each \( x \in M \), each fiber \( G(x, -) \) and \( G(-, x) \) are Banach submanifolds of \( G \) and \( G^{(2)} \) is a Banach submanifold of \( G \times G \).

Proof. At first, since \( s \circ i = t \), from (2) and (3), it follows that \( t \) is a submersion. It follows by [Lan01, Ch. II, Prop. 2.2] that the fibers \( G(x, -) \) and \( G(-, x) \) are n.n.H. Banach submanifolds of \( G \) for any \( x \in M \). Now assume that \( G(x, -) \) is Hausdorff. But \( G(-, x) = i(G(x, -)) \). Since \( i \) is a diffeomorphism of \( G \) this implies that \( G(-, x) \) is also Hausdorff. The same argument can be applied for the converse.

Since both \( s \) and \( t \) are submersions, the map \( (s, t) : G \times G \to M \times M \) is a submersion as well. Therefore, since the diagonal \( D \subseteq M \times M \) is a submanifold, it follows that the set \( G^{(2)} = (s, t)^{-1}(D) \) is a n.n.H. Banach submanifold of \( G \times G \). If \( G \) is Hausdorff then so are \( G^{(2)} \), \( G(x, -) \) and \( G(-, x) \) for all \( x \in M \).

We first assume that both \( G \) and \( M \) are pure Banach manifolds. This is the case for instance if \( G \) is connected, hence so is \( M \). Let \( G \) and \( M \) be the Banach spaces on which \( G \) and \( M \) are modeled, respectively. Fix some \( x \in M \) and set \( g = 1_x \) and so \( s(g) = x \). Since \( s \) is a submersion, we have a decomposition \( G \equiv T_g G = \ker T_g s \oplus F \) and \( T_g s(F) = T_x M \equiv M \). Thus we may assume that \( M \) is a split closed subspace of \( G \) and we write \( G = K \oplus M \equiv K \times M \). With these conventions and notations, there exists also a chart \((U, \phi)\) of \( G \) around \( g \) and \((U_0, \Phi_0)\) of \( M \) around \( s(g) \) such that \( s(U) = U_0 \), \( \phi(U) = V \times W \) where \( V \) and \( W \) are open sets of \( M \), \( \phi_0(U_0) = W \) and \( \phi_0 \circ s \circ \phi \) is the canonical projection on \( W \). It follows that the restriction \( s_0 \) of \( s \) to \( \phi^{-1}(\{\phi(g)\} \times W) \) is a diffeomorphism onto \( U_0 \). Since \( s \circ 1 = 1 \), it follows that \( 1_{U_0} : U_0 \to U \) is smooth and \( T_x 1 \) is injective. In particular, \( \phi^{-1}(\{\phi(g)\} \times W) = 1(U_0) \). This shows that \( 1(M) \) is a submanifold of \( G \). But since \( M \) is Hausdorff and the map \( 1 : M \to G \) is continuous and injective it follows that its range \( 1(M) \) is also Hausdorff. The general case is obtained by application of this result for each connected component of \( G \).

Definition 3.2. A n.n.H. Banach-Lie groupoid is a topological groupoid \( G \rightrightarrows M \) satisfying the following conditions:

(BLG1) \( G \) is a n.n.H. Banach manifold and \( M \) is a Banach manifold.

(BLG2) The map \( s : G \to M \) is a submersion.

(BLG3) The map \( i : G \to G \) is smooth.
The multiplication \( m : \mathcal{G}^2 \to \mathcal{G} \) is smooth.

A **Banach-Lie groupoid** is a n.n.H. Banach-Lie groupoid whose total space has the Hausdorff property, i.e., is a Banach manifold.

A n.n.H. Banach-Lie groupoid \( \mathcal{G} \rightrightarrows M \) is called **pure** if \( \mathcal{G} \) is a pure n.n.H. Banach manifold and \( M \) is a pure Banach manifold.

A n.n.H. Banach-Lie groupoid \( \mathcal{G} \rightrightarrows M \) is called **split** if for every \( x, y \in M \) the set \( \mathcal{G}(x, y) \) is submanifold of \( \mathcal{G} \).

A **Banach-Lie morphism** between the n.n.H Banach-Lie groupoids \( \mathcal{G} \rightrightarrows M \) and \( \mathcal{H} \rightrightarrows N \) is a topological morphism \( \Phi : \mathcal{G} \to \mathcal{H} \) which is a smooth map.

### 3.2. Examples

We will adapt to our context some classical examples of finite-dimensional Lie groupoids.

#### 3.2.1. Banach-Lie groups

Any Banach-Lie group \( G \) is a Banach-Lie groupoid: the set of arrows \( \mathcal{G} \) is the set \( G \) and the set of objects \( M \) is reduced to the singleton \( \{1\} \), where \( 1 \in G \) is the unit element.

#### 3.2.2. Banach-Lie pair groupoid

Given a Banach manifold \( M \), let \( \mathcal{G} := M \times M \) and let \( s \) and \( t \) be the Cartesian projections of \( M \times M \) on the first and the second factor, respectively. The multiplication map \( m \) and the inverse \( i \) are respectively \( m((x, y), (y, z)) = (x, z) \) and \( i(x, y) = (y, x) \). Finally the map \( 1 \) is \( 1(x) = (x, x) \). We thus obtain a Banach-Lie groupoid \( M \times M \rightrightarrows M \).

#### 3.2.3. General linear Banach-Lie groupoids

Let \( \pi : A \to M \) be a Banach vector bundle. The general linear Banach groupoid \( \mathcal{GL}(A) \rightrightarrows M \) is the Banach groupoid such that \( \mathcal{GL}(A) \) is the set of linear isomorphisms \( g : A_x \to A_y \) between each pair of fibers \( (A_x, A_y) \). The source map and the target map are obvious and the multiplication is the composition of linear isomorphisms and \( 1_x = Id_{A_x} \).

#### 3.2.4. Disjoint union of n.n.H. Banach-Lie groupoids

Let \( \{\mathcal{G}_\lambda \rightrightarrows M_\lambda\}_{\lambda \in \Lambda} \) be a family of n.n.H. Banach-Lie groupoids. We denote by \( \mathcal{G} := \bigsqcup_{\lambda \in \Lambda} \mathcal{G}_\lambda \) and \( M := \bigsqcup_{\lambda \in \Lambda} M_\lambda \). Since we consider here disjoint unions, the structure of the n.n.H. Banach-Lie groupoid \( \mathcal{G} \rightrightarrows M \) is clearly defined by the collection of structure maps for each particular \( \mathcal{G}_\lambda \rightrightarrows M_\lambda \) for \( \lambda \in \Lambda \).

For example if we consider a finite family of Banach bundles \( A_i \to M_i \), for \( i = 1, \ldots, n \), then we have the natural structure of a n.n.H. Banach-Lie groupoid \( \bigsqcup_{i=1}^n \mathcal{GL}(A_i) \rightrightarrows \bigsqcup_{i=1}^n M_i \). More generally, given any Banach-Lie n.n.H. groupoid \( \mathcal{G} \rightrightarrows M \), if \( M = \bigsqcup_{\lambda \in \Lambda} M_\lambda \) is a partition of \( M \) into open \( \mathcal{G} \)-invariant submanifolds, then the corresponding groupoids \( \mathcal{G}_\lambda \rightrightarrows M_\lambda \) obtained by restriction are n.n.H.
Banach-Lie groupoids and $\mathcal{G} \Rightarrow M$ is the disjoint union of the family $\{\mathcal{G}_\lambda \Rightarrow M\}_{\lambda \in \Lambda}$.

3.2.5. Action of a Banach-Lie group

To each smooth action $A : G \times M \to M$, $(g,x) \mapsto g.x$, of a Banach-Lie group on a Banach manifold there is associated a Banach-Lie groupoid $\mathcal{G} \Rightarrow M$ defined in the following way:

- $\mathcal{G} := M \times G$;
- $s(x,g) := x$ and $t(x,g) := A(g,x) = g.x$;
- if $y = g.x$ then $m((y,h),(x,g)) := (x,hg)$;
- $i(x,g) := (g.x,g^{-1})$;
- $1_x = (x,1)$.

It is easily seen that for any $x_0, y_0 \in M$ one has

$$\mathcal{G}(x_0, y_0) = \{x_0\} \times \{g \in G \mid g.x_0 = y_0\}$$

and in particular the isotropy group at $x_0$ is

$$\mathcal{G}(x_0) = \{x_0\} \times G(x_0)$$

where $G(x_0) := \{g \in G \mid g.x_0 = x_0\}$.

It is worth pointing out that the above groupoid needs not be split. To obtain a specific example in this connection, let $\mathcal{X}$ be any real Banach space with a closed linear subspace $\mathcal{X}_0$ which fails to be a split subspace of $\mathcal{X}$. (For instance the space of all bounded sequences of real numbers $\mathcal{X} := \ell^\infty(\mathbb{N})$ with its subspace $\mathcal{X}_0$ consisting of the sequences that converge to 0.) Then the abelian Banach-Lie group $G := (\mathcal{X}, \cdot \cdot)$ acts smoothly transitively on the Banach manifold $M := \mathcal{X}/\mathcal{X}_0$ by $A(g, x + \mathcal{X}_0) := g + x + \mathcal{X}_0$ for all $g, x \in \mathcal{X}$. If we define the groupoid $\mathcal{G} := M \times G \Rightarrow M$ as above, then the isotropy group at the point $\mathcal{X}_0 \in M$, that is,

$$\mathcal{G}(\mathcal{X}_0) = \{\mathcal{X}_0\} \times \mathcal{X}_0 \subseteq \mathcal{G},$$

is not a submanifold of $\mathcal{G} = M \times \mathcal{X}$ since $\mathcal{X}_0$ fails to be a split subspace of $\mathcal{X}$.

3.2.6. The gauge groupoid of a principal bundle

A principal Banach bundle is a locally trivial fibration $\pi : P \to M$ over a connected Banach manifold whose typical fiber is a Banach-Lie group $G$. We have then a right action of $G$ on $P$ whose corresponding quotient $P/G$ is canonically diffeomorphic to $M$. We get a right diagonal action of $G$ on $P \times P$ in an evident way. The gauge groupoid is the set of orbits of this action, that is, the quotient $\mathcal{G} := (P \times P)/G$ provided with the quotient topology. The source (resp. target) of the equivalence class of $(v, u)$ is $\pi(u)$ (resp. $\pi(v)$). The composition of the class of $(v, u)$ and of $(v', u')$ is the equivalence class of $(w', u)$ and the inverse of the equivalence class of $(v, u)$ is the equivalence class of $(u,v)$. (See [Mac87] for more details.)
3.2.7. Fundamental groupoid of a Banach manifold

Recall that for any topological space $M$ there is its corresponding fundamental groupoid $\Pi(M) \rightrightarrows M$ where $\Pi(M)$ is the set of homotopy classes of continuous paths with fixed end points. The source map (resp. target map) is the map which to a homotopy class $[\gamma]$ of a path $\gamma$ associates its origin $s(\gamma)$ (resp. its end $t(\gamma)$). The multiplication is obtained by the concatenation of paths: if $\gamma$ and $\gamma'$ are two paths defined on $[0,1]$ the concatenation $\gamma \ast \gamma'$ is the path defined by $\gamma \ast \gamma'(t) = \gamma(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $\gamma \ast \gamma'(t) = \gamma'(2t - 1)$ for $\frac{1}{2} \leq t \leq 1$. It is compatible with homotopy equivalence. The inverse of a homotopy class of a path $\gamma : [0,1] \to M$ is the homotopy class of the path $\gamma^{-1} : t \mapsto \gamma(1-t)$.

When $M$ is a connected Banach manifold, each source fiber is a universal covering of $M$. In particular such a fiber is a Banach manifold. Moreover it is also a principal bundle over $M$ whose structural group is the fundamental group $\pi_1(M)$. Then the gauge groupoid of this principal bundle can be identified with $\Pi(M)$ and so we get a structure of n.n.H. Banach-Lie groupoid structure on $\Pi(M)$.

3.3. Properties of orbits

The purpose of this subsection is to show the following results for Banach-Lie groupoids which are an adaptation of similar classical results in the finite dimensional case.

**Theorem 3.3.** Let $\mathcal{G} \rightrightarrows M$ be a n.n.H. Banach-Lie groupoid and define

$$(\forall x \in M) \quad t_x := t|_{\mathcal{G}(x,-)} : \mathcal{G}(x,-) \to M.$$ 

The following assertions hold.

(i) Let $N$ be a connected component of $M$. Then $\mathcal{G}^N = s^{-1}(N)$ and $\mathcal{G}^t_N = t^{-1}(N)$ are pure n.n.H. Banach manifolds and submanifolds of $\mathcal{G}$.

(ii) For all $x \in M$ and $y \in \mathcal{G}.x$ the set $\mathcal{G}(x,y)$ is a closed submanifold of $\mathcal{G}$. In particular the isotropy group $\mathcal{G}(x)$ is a Banach-Lie group and $T_{x_0}(\mathcal{G}(x)) = \ker T_{1_x} \cap \ker T_{1_x} t$.

(iii) For every $x \in M$ for which $\mathcal{G}(x,y)$ is a split submanifold of $\mathcal{G}$ for all $y \in \mathcal{G}.x = t_x(\mathcal{G}(x,-))$, the orbit $\mathcal{G}.x$ is a pure Banach manifold whose inclusion map $\mathcal{G}.x \hookrightarrow M$ is a weak immersion and $t_x : \mathcal{G}(x,-) \to \mathcal{G}.x$ is a Banach principal $\mathcal{G}(x)$-bundle.

**Proof.** We first prove Assertion (i). For any connected component $N$ of $M$, the set $\mathcal{G}^N$ is open and closed in $\mathcal{G}$ hence it is the union of some connected components of $\mathcal{G}$. Consider any connected component $\mathcal{G}_\alpha$ of $\mathcal{G}$ which is contained in $\mathcal{G}^N$. One clearly has $s(\mathcal{G}_\alpha) = N$. Now for $x \in N$, each fiber $\mathcal{G}_\alpha(x,-)$ of the restriction of $s$ to $\mathcal{G}_\alpha$ is an open submanifold of $\mathcal{G}(x,-)$. Denote by $N$ and $\mathcal{G}_\alpha$ the Banach spaces on which $N$ and $\mathcal{G}_\alpha$ are modeled respectively. Since $s$ is a submersion, $\mathcal{G}_\alpha$ is isomorphic to $T_x \mathcal{G}(x,-) \oplus N$ for any $g \in \mathcal{G}_\alpha(x,-)$ and $x \in N$. It follows easily that all connected components of $\mathcal{G}$ which are contained in $\mathcal{G}^N$. 


are modeled on the same Banach space. Thus $G_N^s$ is a pure n.n.H. Banach manifold and also a Banach submanifold of $G$. The same result for $G_N^t$ can be proved by similar arguments. This ends the proof of Assertion (i).

To prove Assertion (ii) we adapt the method of proof of [MM03, Th. 5.4]. To this end we will show that defining

$$\Delta_g = \ker T_g s \cap \ker T_g t$$

for all $g \in G(x,-)$, one obtains a Banach subbundle $\Delta := \bigcup_{g \in G(x,-)} \Delta_g \subseteq T G(x,-)$ which is integrable. Given any $g \in G(x,-)$ the left translation $L_g : G(x,-) \to G(-,t(g))$, $L_g(h) = gh$ is a diffeomorphism just as (4.1) and moreover $s \circ L_g = s|_{G(-,x)}$, hence

$$TL_g(\Delta_{1_x}) = \Delta_g.$$ 

Note that $\Delta_g$ is a closed subspace of $T_g G(-,x)$. The map

$$\Phi : G(x,-) \times \Delta_{1_x} \to TG(x,-), \quad \Phi(g,v) = TL_g(v)$$

is an injective morphism over $\text{id}_{G(-,x)}$ from the trivial bundle $G(x,-) \times \Delta_{1_x} \to G(x,-)$ to $TG(x,-) \to G(x,-)$, whose range is the distribution $\Delta$. Thus $\Delta$ defines a smooth trivial Banach bundle which is a Banach subbundle of $T G(x,-)$. For any $v \in \Delta_{1_x}$ denote by $X_v$ the vector field on $G(x,-)$ defined by $X_v(g) = TL_g(v)$. Then the set

$$\Gamma(\Delta) = \{ X_v \mid v \in \Delta_{1_x} \}$$

generates the distribution $\Delta$. Now any integral curve $\gamma : [0,\varepsilon] \to G(x,-)$ of $X \in \Gamma(\Delta)$ with $\gamma(0) = g$ is also contained in $G(-,t(g))$.

It follows that the Lie bracket $[X,Y](g)$ of vector fields $X$ and $Y$ in $\Gamma(\Delta)$ is tangent to $G(-,t(g))$ and so $\Gamma(\Delta)$ is stable under Lie bracket. From [Pe12, Th. 4.5], $\Delta$ is integrable, i.e., there exists a partition of $G(x,-)$ into immersed n.n.H. Banach manifolds and each one is modelled on the Banach space $\Delta_{1_x}$. Moreover since $\Delta_g = \ker T_g t_x$, the maximal leaf through $g \in G(x,-)$ is a connected component of $t^{-1}(t(g))$ and so is a closed subset of the Banach manifold $G(x,-)$. In particular the isotropy group $G(t(g))$ is an union of such leaves and so is a closed immersed n.n.H. Banach manifold. But from Remark 2.2 each point in $G(t(g))$ is closed thus it follows from [Boi71, Ch. III, §2, no. 6, Prop. 18(a)] that $G(t(g))$ is in fact Hausdorff. This implies that $G(t(g))$ has the structure of a Banach-Lie group.

Footnote: That theorem was proved in the context of (Hausdorff) Banach manifolds but all the arguments used in the proof of this result are based on local charts so the theorem also holds for n.n.H. Banach manifolds.
Finally, the left translation $L_g$ is a diffeomorphism from $G(-, x)$ to $G(-, t(g))$ and $TL_g(\Delta_x) = \Delta_g$, hence it follows that the restriction of $L_g$ to $G(x)$ is a diffeomorphism onto $G(t(g))$. We conclude that each fiber of $t_x$ is diffeomorphic to $G(x)$. This proves the first part of the assertion and $T_{1_x}(G(x)) = \ker T_{1_x} s \cap \ker T_{1_x} t$. This ends the proof of Assertion (iii).

For the proof of Assertion (iii) we consider the map

$$\Psi: G(x, -)/G(x) \to G.x, \quad g \cdot G(x) \mapsto t(g).$$

It is easy to prove that $\Psi$ is bijective. Therefore to prove the first part of Asser-

Since the map $s$ is a submersion, it follows by Proposition 3.1 that the set $G(x, -)$ is a n.n.H. submanifold of $G$. On the other hand, the quotient set $G(x, -)/G(x)$ is the set of orbits of the right group action

$$R: G(x, -) \times G(x) \to G(x, -)$$

defined by

$$(h, g) \mapsto R(h, g) = hg.$$

This is a smooth action of a Banach-Lie group on a n.n.H. Banach manifold since the groupoid multiplication is smooth. The action $R$ is free since every element of $G(x, -)$ has an inverse in $G$. Using Lemma 2.3 and continuity of multiplication and inversion maps of the groupoid $G$, it also follows that the action $R$ is proper. It then follows by Lemma 2.4 that both $G(x, -)/G(x)$ and $G(x, -)/G(x)$ are Hausdorff.

We now check that for every $h \in G(x, -)$ the map $r_h: G(x) \to G(x, -), \quad g \mapsto gh$, is an immersion. We have already seen that $r_h$ defines a diffeomorphism $G(x) \to G(x, y)$ where $y = t(h)$. One has

$$G(x, y) \subseteq G(x, -) \subseteq G$$

where we know that $G(x, y) \subseteq G$ is a submanifold and $G(x, -) \subseteq G$ is a submanifold, hence $G(x, y) \subseteq G(x, -)$ is a submanifold by Lemma 2.6. Thus the above map $r_h$ is a diffeomorphism of $G(x)$ onto the submanifold $G(x, y)$ of $G(x, -)$, and in particular $r_h$ is an immersion.

Thus we can apply Lemma 2.5 to the left action canonically associated to the right action $R$,

$$G(x) \times G(x, -) \to G(x, -), \quad (g, h) \mapsto R(h, g^{-1})$$

First, it follows that the set $G(x, -)/G(x)$ has the unique structure of a smooth manifold for which the quotient map $q: G(x, -) \to G(x, -)/G(x)$ is a submersion.

Moreover, one has $\Psi \circ q = t|_{G(x, -)}: G(x, -) \to G$, which is a smooth map. Since $q$ is a submersion, it then follows that $\Psi: G(x, -)/G(x) \to G$ is smooth. (See for instance [Bon71], 5.9.5.) This implies that the inclusion map $G.x \hookrightarrow M$ is smooth. Its tangent map at every point of $G.x$ is injective by [Gib13, Th. 1]. Finally, it follows by Lemma 2.5 that $G(x, -) t_x : G(x, -) \to G.x$ is a Banach principal $G(x)$-bundle.
Remark 3.4. Note that Theorem 3.3(i) ensures only that $G(x, y)$ is a closed immersed submanifold of $G$ but a priori not a submanifold as it is pointed out in 3.2.5. This problem is illustrated by the following examples.

In the same way Theorem 3.3(iii) ensures only that each orbit $G.x$ is an immersed submanifold of $M$ but a priori not a closed submanifold and in particular of course not a submanifold.

Example 3.5. If $G \rightarrow M$ is a finite-dimensional Lie groupoid, then it is well known that $G(x, y)$ is a submanifold of $G$ for every $x \in M$ and $y \in G.x$. See for instance [Mac87, Cor. 1.4.11] and [MM03, Th. 5.4]. Thus $G \rightarrow M$ is split.

Example 3.6. Let $G \rightarrow M$ be the Banach-Lie groupoid associated as in 3.2.5 to the smooth action $A : G \times M \rightarrow M$ of a Banach-Lie group on a Banach manifold. Let $x \in M$ with its isotropy group $G(x) := \{ g \in G \mid A(g, x) = x \}$ and with its orbit $O(x) := \{ A(g, x) \mid g \in G \}$. One has

$$G(x) = \{(h, x) \in G \times M \mid h \in G(x)\} = G(x) \times \{x\} \subseteq G \times M = G$$

hence $G(x)$ is a submanifold of $G$ if and only if $G(x)$ is a submanifold of $G$, that is, if and only if $G(x)$ is a Banach-Lie subgroup of $G$.

For any $y \in O(x)$ and $g_0 \in G$ with $A(g_0, x) = y$ one has

$$G(x, y) = \{(g, x) \in G \times M \mid A(g, x) = y\} = \{(g_0 h, x) \in G \times M \mid h \in G(x)\} = g_0 G(x) \times \{x\} \subseteq G \times M = G.$$

This shows that the condition that $G(x)$ is a Banach-Lie subgroup of $G$ for all $x \in M$ is equivalent to the assumption that $G$ is split. Under this assumption the conclusion of Theorem 3.3(iii) recovers the classical fact that if $G(x)$ is a Banach-Lie subgroup of the Banach-Lie group $G$, then $G/G(x)$ is a smooth homogeneous Banach manifold (cf. for instance [Be06, Th. 4.19]).

For the above Banach-Lie groupoid $G \rightarrow M$, Theorem 3.3(iii) says that $G(x, y)$ is a closed submanifold of $G$ if $y \in O(x)$. By the above description of $G(x, y)$, this property is equivalent to the fact that the isotropy group $G(x)$ is a closed submanifold of $G$, that is, $G(x)$ is a closed subset of $G$ and $G(x)$ has the structure of a Banach manifold for which the inclusion map $G(x) \hookrightarrow G$ is an immersion as in Subsection 2.1. For the sake of completeness, we note that this conclusion (which does not mean that $G(x)$ is a Banach-Lie subgroup of $G$ unless $\dim G < \infty$, cf. [Ho75, Ex. II.11]) can also be derived from an infinite-dimensional version of Cartan’s theorem on closed subgroups [Be06, Cor. 3.7], since $G(x)$ is a closed subgroup of the Banach-Lie group $G$.

4. Banach-Lie algebroids

4.1. Structure of Banach-Lie algebroid

Let $\pi : \mathcal{A} \rightarrow M$ be a Banach vector bundle on a Banach manifold.
**Definition 4.1.** A morphism of vector bundles $\rho : A \to TM$ covering identity is called an anchor. The structure $(A, \pi, M, \rho)$ is then called an anchored Banach bundle.

The above morphism $\rho$ induces a map $\Gamma(A) \to \Gamma(M)$, again denoted by $\rho$, defined on any section $s \in \Gamma(A)$ by $(\rho(s))(x) := (\rho \circ s)(x)$ for every $x \in M$.

**Remark 4.2.** If one has a local trivialization of the vector bundle $\pi : A \to M$ in which the manifold $M$ is modeled on a Banach space $\mathcal{M}$ and the typical fiber of $\pi$ is a Banach space $\mathcal{A}$, then the anchor $\rho$ has the local expression

$$\rho(x, u) \equiv (x, u) \mapsto (x, \rho(x)(u))$$

where $\rho : U \to L(\mathcal{A}, \mathcal{M})$ is a smooth map.

**Definition 4.3.** A Lie bracket on the anchored Banach bundle $(A, \pi, M, \rho)$ is a skew-symmetric $\mathbb{R}$-bilinear map $[, ]_A : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ satisfying the following conditions:

1. Leibniz property: $[\sigma_1, f \sigma_2]_A = f[\sigma_1, \sigma_2]_A + df(\rho(\sigma_1))\sigma_2$ for all $f \in C^\infty(M)$ and $\sigma_1, \sigma_2 \in \Gamma(A)$.
2. Jacobi identity: $[\sigma_1, [\sigma_2, \sigma_3]_A]_A + [\sigma_2, [\sigma_3, \sigma_1]_A]_A + [\sigma_3, [\sigma_1, \sigma_2]_A]_A = 0$ for all $\sigma_1, \sigma_2, \sigma_3 \in \Gamma(A)$.

**Remark 4.4.** An anchored Banach bundle provided with a Lie bracket $[, ]_A$ as above was sometimes called a Banach-Lie algebroid in the earlier literature (for instance in [An11]) and was denoted $(A, M, \rho, [, ]_A)$ if $\rho$ satisfies:

$$\rho([\sigma_1, \sigma_2]_A) = [\rho(\sigma_1), \rho(\sigma_2)] \text{ for all } \sigma_1, \sigma_2 \in \Gamma(A)$$

where $[, ]$ denotes the usual Lie bracket of vector fields on $M$.

If $(A, M, \rho, [, ]_A)$ is a Banach-Lie algebroid in this sense, then $(\Gamma(A), [, ]_A)$ is a Lie algebra and $\rho$ is a Lie algebra morphism from $(\Gamma(A), [, ]_A)$ to the Lie algebra $(\Gamma(M), [, ])$ of smooth vector fields on $M$.

In this paper we prefer the definition of a Banach-Lie algebroid (Definition 4.7) which, in addition to the Lie algebra morphism property of the anchor map from Remark 4.4, also involves the localizability property that we will discuss right now.

**Definition 4.5.** A Lie bracket $[, ]_A$ on an anchored bundle $(A, \pi, M, \rho)$ respects the sheaf of sections of $\pi : A \to M$ or, for short, is localizable (see for instance [Mar08]), if for every nonempty open subset $U \subseteq M$ one has a Lie bracket $[, ]_U$ on the anchored bundle $(A|_U, \pi|_U, U, \rho|_U : A|_U \to TU)$ satisfying the following conditions:

(i) For $U = M$ one has $[, ]_M = [, ]_A$.
(ii) For any open subsets $V \subseteq U \subseteq M$ and all $\sigma_1, \sigma_2 \in \Gamma(A|_U)$ one has

$$[\sigma_1|_V, \sigma_2|_V]_V = ([\sigma_1, \sigma_2]|_V |_V.$$

15
Remark 4.6. In finite dimensions it is well known that every Lie bracket \([\cdot, \cdot]_A\) on an anchored bundle \((A, \pi, M, \rho)\) is localizable. (See for instance [Mar08].) To extend this fact to infinite dimensions, one needs the following notion: A bump function on a Banach space \(X\) is a smooth function on \(\varphi: X \to \mathbb{R}\) whose support is a bounded nonempty subset of \(X\). See for instance [KM97, Ch. III, §14] for background information on this notion.

We claim that if the Banach space \(X\) admits bump functions, then for every \(x_0 \in X\) there exist a bump function \(\varphi_{x_0}: X \to \mathbb{R}\) and an open neighborhood \(V\) of \(x_0\) with \(\varphi_{x_0}(x) = 1\) for every \(x \in V\). Without loss of generality let us chose a bump function \(\varphi: X \to \mathbb{R}\) such that \(\varphi(x_0) = 1\). For convenience we will fix open neighborhoods \((1/2, 3/2) \subseteq (1/3, 5/3)\) of \(1 \in \mathbb{R}\). Then there exist open neighborhoods \(V \subseteq W\) of \(x_0 \in X\) with \(\varphi(V) \subseteq (1/3, 5/3)\) and \(\varphi(W) \subseteq (1/2, 3/2)\). If \(\psi: \mathbb{R} \to \mathbb{R}\) is any smooth function satisfying \(\psi(t) = 0\) if \(t \in \mathbb{R} \setminus (1/2, 3/2)\) and \(\psi(t) = 1\) for every \(t \in (1/3, 5/3)\), then \(\varphi_{x_0} := \psi \circ \varphi: X \to \mathbb{R}\) is a smooth function satisfying \(\varphi_{x_0}(x) = 1\) for every \(x \in V\). Moreover, using that \(0 \not\in (1/3, 3/4)\), it is straightforward to check that the support of \(\varphi_{x_0}\) is contained in the support of \(\varphi\), hence \(\varphi_{x_0}\) is a bump function as claimed.

A Banach manifold \(M\) is called smoothly regular if the model space of \(M\) at every point \(x \in M\) is a Banach space that admits bump functions. By the same arguments as in finite dimensions and using the above observation on bump functions that are constant on a neighborhood of any given point, one can prove that if the Banach manifold \(M\) is smoothly regular then any Lie bracket \([\cdot, \cdot]_A\) on an anchored bundle \((A, \pi, M, \rho)\) is localizable. (See [CP12, Prop. 3.6] and also [Pe12].)

If \(M\) is not smoothly regular, we cannot prove that any Lie bracket is localizable. Unfortunately in the Banach framework, we have no example of Lie algebroid whose Lie bracket is not localizable. Moreover we will see later that if a Banach-Lie algebroid is integrable then its Lie bracket is localizable. Therefore this condition is necessary in order to find conditions under which a Banach-Lie algebroid is integrable.

For these reasons we make the following definition (cf. [Pe12] and [CP12]).

Definition 4.7. The structure of an anchored Banach bundle endowed with a Lie bracket \((A, M, \rho, [\cdot, \cdot]_A)\) is called a Banach-Lie algebroid if \([\cdot, \cdot]_A\) is a localizable Lie bracket and \(\rho\) is a Lie algebra morphism.

When \(\pi: A \to M\) is a pure Banach bundle then \((A, M, \rho, [\cdot, \cdot]_A)\) will be called a pure Banach-Lie algebroid which corresponds to the definition of Banach-Lie algebroid given in [Pe12] and [CP12].

Consider two Lie algebroids \((A, M, \rho, [\cdot, \cdot]_A)\) and \((A', M', \rho', [\cdot, \cdot]_{A'})\). A Banach bundle morphism \(\Phi: A \to A'\) over a map \(\phi: M \to M'\) is called a Banach-Lie algebroid morphism if it satisfies the conditions

\[ T\phi \circ \rho = \rho' \circ \Phi \quad \text{and} \quad \Phi([\sigma_1, \sigma_2]_A) = [\Phi(\sigma_1), \Phi(\sigma_2)]_{A'} \quad \text{for all} \quad \sigma_1, \sigma_2 \in \Gamma(A). \]

The following notion of admissibility is used in the proof of Lemma 4.26 below, and it thus plays a key role for Theorem 4.24.
Definition 4.8. If \((A, M, \rho, [\cdot, \cdot]_A)\) is a Banach-Lie algebroid, then a piecewise smooth curve \(c: \mathbb{R} \supset [a, b] \to M\) is called an \(A\)-admissible curve if there exists piecewise smooth curve \(\alpha: [a, b] \to A\) with \(\pi(\alpha) = c\) and \(\rho(\alpha(t)) = c'(t)\) for each \(t \in [a, b]\) at which \(c\) is smooth. In these conditions \(\alpha\) is called an \(A\)-lift of \(c\) and more generally \(\alpha\) is called an \(A\)-path. We define a binary relation on \(M\) associated to \((A, M, \rho, [\cdot, \cdot]_A)\) in the following way:

\[
x \sim y \iff \text{there exists } \gamma: [a, b] \to M \text{ \(A\)-admissible with } \gamma(a) = x \text{ and } \gamma(b) = y.
\]

Since any concatenation of two piecewise smooth paths is in turn piecewise smooth, the above is an equivalence relation, and the equivalence class of \(x\) is called the \(A\)-orbit of \(x\) with respect to the Banach-Lie algebroid \((A, M, \rho, [\cdot, \cdot]_A)\).

Lemma 4.9. Let \(\alpha\) be an \(A\)-lift of the curve \(c\), both these curves being defined on some interval \(J\). Then for any smooth map \(\phi: J \to J\), the curve \((\alpha \circ \phi) \circ \dot{\phi}\) is an \(A\)-lift of \(c \circ \phi\).

Proof. Since \(\alpha\) is an \(A\)-lift of \(c\), one has \(c = \pi \circ \alpha\) and \(\rho \circ \alpha = \dot{c}\), and these equalities imply:

1. \(\pi \circ (\alpha \circ \phi) = c \circ \phi\);
2. \(\rho \circ (\alpha \circ \phi) = \dot{c} \circ \phi\).

Now one obtains \((c \circ \phi) = (\dot{c} \circ \phi) \circ \dot{\phi} = (\rho \circ (\alpha \circ \phi)) \circ \dot{\phi} = \rho \circ ((\alpha \circ \phi) \circ \dot{\phi})\) where the second equality follows by 2. above. To check the second equality needed for \((\alpha \circ \phi) \circ \dot{\phi}\) to be an \(A\)-lift of \(c \circ \phi\) one just has to note that \(\pi(\alpha \circ \phi \circ \dot{\phi}) = \pi(\alpha \circ \phi) = c \circ \phi\) where the first and last equality follow by 1. above, while the second equality follows by the fact that \(\pi(tv) = \pi(v)\) for any real \(t\) and any \(v \in A\).

Lemma 4.10. For each \(y\) in the \(A\)-orbit of \(x\) there exists an \(A\)-admissible smooth curve \(\gamma: [a, b] \to M\) with a smooth \(A\)-lift such that \(\gamma(a) = x\) and \(\gamma(b) = y\).

Proof. By the definition of an \(A\)-orbit there exists a piecewise smooth curve \(c: [a, b] \to A\) such that \(c(a) = x\) and \(c(b) = y\) with a piecewise smooth \(A\)-lift \(\alpha\). There exists a smooth bijection \(\phi: [a, b] \to [a, b]\) for which both \(c \circ \phi\) and \(\alpha \circ \phi\) are smooth. As \(\phi\) one may take any smooth homeomorphism whose derivatives of arbitrary order vanish at the finitely many points where \(c\) or \(\alpha\) fail to be smooth. Then from Lemma 4.10 it follows that \(\gamma = c \circ \phi\) is the required smooth curve with its smooth \(A\)-lift \((\alpha \circ \phi) \circ \dot{\phi}\).

Lemma 4.10 shows that the algebroid orbits introduced in Definition 4.8 can be equivalently defined using only smooth paths that are \(A\)-admissible in the sense defined in [CF03] for the case of finite-dimensional Lie algebroids.

Definition 4.11. A Banach-Lie algebroid \((A, M, \rho, [\cdot, \cdot]_A)\) is called split on a connected component \(N\) of \(M\) if for each \(x \in N\) the kernel of \(\rho_x = \rho|_{\pi^{-1}(x)}\) is a split subspace of \(\pi^{-1}(x)\). The algebroid is called split if it is split on all the connected components of \(M\).
If \((\mathcal{A}, M, \rho, [\cdot, \cdot]_\mathcal{A})\) is a finite-dimensional Lie algebroid, it is well known that each \(\mathcal{A}\)-orbit is a smooth immersed manifold of \(M\). In the Banach context from [Pe12 Th. 5] applied to each subbundle \(\pi: \mathcal{A}_\alpha \to \pi(\mathcal{A}_\alpha)\) we draw the following conclusion.

**Theorem 4.12.** If \((\mathcal{A}, M, \rho, [\cdot, \cdot]_\mathcal{A})\) is a split Banach-Lie algebroid and the distribution \(\rho(\mathcal{A})\) on \(M\) is closed, then each orbit is an immersed submanifold of \(M\).

### 4.2. The Banach-Lie algebroid of a n.n.H. Banach-Lie groupoid

The main result of this subsection is Theorem 4.19 on the construction of the Banach-Lie algebroid of a n.n.H. Banach-Lie groupoid \(G \rightrightarrows M\). This is formally the same as in the case of finite-dimensional Lie groupoids (cf. [Mac 87]) and, for infinite-dimensional Lie groupoids modelled on locally convex spaces it was considered in [SW15] and [SW16]. However, since the basis \(M\) may not be smoothly regular (see Remark 4.6), the localizability property of the Lie bracket must be proved separately. This is a technical aspect that occurs neither in the case of finite-dimensional Lie groupoids nor in the case of Banach-Lie groups.

In order to deal with this problem we use the following notion.

**Definition 4.13.** Let \(G \rightrightarrows M\) be a n.n.H. Banach-Lie groupoid, \(H\) be a n.n.H. Banach manifold, and \(\sigma: H \to M\) be a surjective submersion. Then the set \(H^* G := \{(h, g) \in H \times G \mid \sigma(h) = t(g)\}\) is the inverse image of the diagonal of \(M \times M\) through the submersion \((\sigma, t): H \times G \to M \times M\), hence \(H^* G\) is a closed split submanifold of \(H \times G\).

In the above framework, a **right action of \(G\) on \(H\)** is a smooth map \(H^* G \to H\), \((h, g) \mapsto h.g\), satisfying the following conditions:

- If \(g_1, g_2 \in G\) with \(s(g_1) = t(g_2)\), and \(h \in H\) with \(\sigma(h) = t(g_1)\), then \(h.(g_1 g_2) = (h.g_1).g_2\).
- If \((h, g) \in H^* G\) then \(\sigma(h.g) = \sigma(h)\).

We denote \(T^\sigma H := \ker(T\sigma) \subseteq TH\), hence \(T^\sigma H\) is an integrable distribution on \(H\), since \(\sigma: H \to M\) is a submersion. We denote by \(\Gamma(T^\sigma H)\) the vector space of smooth sections of \(T^\sigma H\) regarded as a sub-bundle of the tangent bundle \(TH \to H\).

For every \(g \in G\) one has the diffeomorphism

\[
R_g: \sigma^{-1}(t(g)) \to \sigma^{-1}(s(g)), \quad h \mapsto h.g.
\]  

(4.1)

We denote by \(\Gamma_{inv}(H)\) the set of all vector fields \(X \in \Gamma(T^\sigma H)\) satisfying the invariance condition

\[
X_{h,g} = (T_h(R_g))(X_h) \text{ for all } g \in G \text{ and } h \in \sigma^{-1}(t(g)).
\]  

(4.2)

**Remark 4.14.** Assume the setting of Definition 4.13. Since \(T^\sigma H\) is an integrable distribution, it follows that \(\Gamma(T^\sigma H)\) is a Lie algebra of smooth vector fields on \(H\). We also note that if \(X \in \Gamma(T^\sigma H)\) and \(h \in H\), then \(\sigma^{-1}(\sigma(h)) \subseteq H\) is a closed submanifold and one has \(X_h \in T_h(\sigma^{-1}(\sigma(h))) \subseteq T_hH\).
We prove the following lemma for the sake of completeness—its proof is similar to that of its counterpart from the construction of the Lie algebroid of a finite-dimensional Lie groupoid [Mac05].

Lemma 4.15. In the setting of Definition 4.13, the following assertions hold.

(i) Let $U \subseteq H$ be a subset such that $H = \{ u, g \mid (u, g) \in H \ast \mathcal{G} \text{ and } u \in U \}$. If $X$ is a smooth vector field on $H$, then one has $X \in \Gamma_{inv}(H)$ if and only if $X_u \in T_u(\sigma^{-1}(\sigma(u)))$ and $X_{u,g} = T_Rg(X_u)$ for all $(u, g) \in H \ast \mathcal{G}$ with $u \in U$. In particular, the value of a right invariant vector field is determined by its values at the points of $U$.

(ii) If $X, Y \in \Gamma_{inv}(H)$, then $[X, Y] \in \Gamma_{inv}(H)$.

(iii) $\Gamma_{inv}(H)$ is a subalgebra of the Lie algebra $\Gamma(T^\sigma H)$.

Proof. If $X \in \Gamma_{inv}(H)$ then it clearly satisfies the conditions from the statement. Conversely, let us assume that $X$ is a smooth vector field satisfying these conditions. For arbitrary $h \in H$ there exists $(u, g) \in H \ast \mathcal{G}$ with $u \in U$ and $h = u \cdot g$. Then $X_h = (T_u(R_g))(X_u)$ with $X_u \in T_u(\sigma^{-1}(\sigma(u))) = T_u(\sigma^{-1}(t(g)))$. Using the diffeomorphism (4.1) and the fact that $R_g(u) = u \cdot g = h$, we then obtain $X_h \in T_h(\sigma^{-1}(s(g))) \subseteq T^\sigma H$. Thus $X \in \Gamma(T^\sigma H)$. It is straightforward to check that $X$ also verifies the invariance condition (4.2), hence $X \in \Gamma_{inv}(H)$.

Now assume that $X, Y \in \Gamma_{inv}(H)$. In particular, $X, Y \in \Gamma(T^\sigma H)$ hence, since the distribution $T^\sigma H \subseteq TH$ is integrable, we obtain $[X, Y] \in \Gamma(T^\sigma H)$. For any $g \in \mathcal{G}$, it follows by (4.2) that the restrictions of $X$ to the manifolds $\sigma^{-1}(s(g))$ and $\sigma^{-1}(t(g))$ are $R_g$-related, using the diffeomorphism (4.1). Similarly, the restrictions of $Y$ to the manifolds $\sigma^{-1}(s(g))$ and $\sigma^{-1}(t(g))$ are $R_g$-related. Since both $X$ and $Y$ are tangent to the manifolds $\sigma^{-1}(s(g))$ and $\sigma^{-1}(t(g))$, it then follows that the restrictions of $[X, Y]$ to the manifolds $\sigma^{-1}(s(g))$ and $\sigma^{-1}(t(g))$ are $R_g$-related. This is equivalent to the fact that $[X, Y]$ satisfies the invariance condition (4.2) with $X$ replaced by $[X, Y]$. Thus $[X, Y] \in \Gamma_{inv}(H)$.

The third assertion in the statement follows by the second assertion, and this concludes the proof.

Proposition 4.16. Let $\mathcal{G} \rightrightarrows M$ be a n.n.H. Banach-Lie groupoid, $U \subseteq M$ be a nonempty open subset, and define $\mathcal{G}^U := t^{-1}(U) \subseteq \mathcal{G}$.

Then the following assertions hold.

(i) For $H := \mathcal{G}^U$ and $\sigma := s|_{\mathcal{G}^U} : \mathcal{G}^U \rightarrow M$, the groupoid multiplication defines by restriction a right action $H \ast \mathcal{G} \rightarrow H$.

(ii) If $X \in \Gamma_{inv}(\mathcal{G}^U)$, then $Tt(X_g) = Tt(X_{1_{t(g)}})$ for all $g \in \mathcal{G}$.

(iii) $Tt$ induces a morphism of Lie algebras from $\Gamma_{inv}(\mathcal{G}^U)$ into the Lie algebra of vector fields on $U$.

Proof. For Assertion (i) we only need that $s, t : \mathcal{G} \rightarrow M$ are surjective submersions.

For Assertion (ii) consider $g \in \mathcal{G}^U$ and set $x = s(g)$ and $y = t(g)$. Then we have:

$$ Tt(X_g) = Tt(X_{1_{s(g)}}) = Tt \circ TR_g(X_{1_y}) = T(t \circ R_g)(X_{1_y}). $$
Since $t_x \circ R_g = t_y$ we get $T_t(X_g) = T_t(X_y)$.

Assertion (iii) follows by Assertions (i)–(ii) along with Lemma 4.15.

For any nonempty open subset $U \subseteq M$ we now define the Banach vector bundle $\pi_U: (AG)_U \to U$ as the pullback of the Banach bundle $T^*(G^U) \to G^U$ through the map $1: U \to G^U$. (If we identify $U$ with its image through the map $1: U \to G$, then we may say that $(AG)_U$ is the restriction of $T^*(G^U)$ to $U$.) Note that $(AG)_U$ is a Banach manifold since $M$ is Hausdorff.

One has a natural Banach bundle morphism $(1_U)^*: (AG)_U \to T^*(G^U)$ over the map $1: U \to G^U$, from the bundle $(AG)_U \to U$ to the bundle $T^*(G^U) \to G^U$, which is a bundle isomorphism onto the restriction of the bundle $T^*(G^U) \to G^U$ to $1(U) \subseteq G^U$. On the other hand, let $\Gamma_1(T^*(G^U))$ be the vector space of all smooth sections of the restriction of the bundle $T^*(G^U) \to G^U$ to $1(U) \subseteq G^U$. By Lemma 4.15 we obtain a Lie bracket on $\Gamma_1(T^*(G^U))$ with a Lie algebra isomorphism $\Gamma_1(T^*(G^U)) \to \Gamma_{inv}(G^U)$. Now the aforementioned bundle isomorphism $(1_U)^*$ gives rise to a linear isomorphism $\Gamma((AG)_U) \to \Gamma_1(T^*(G^U))$. We thus obtain a linear isomorphism

$$\Gamma((AG)_U) \to \Gamma_{inv}(G^U), \quad X \mapsto \tilde{X}$$

and we thus see that there exists a unique Lie bracket $[\cdot, \cdot]_U$ on the vector space $\Gamma((AG)_U)$ for which the above linear isomorphism $X \mapsto \tilde{X}$ is a Lie algebra isomorphism. For later reference we note that

$$\tilde{X}, \tilde{Y} = [\tilde{X}, \tilde{Y}]$$

for all $X, Y \in \Gamma((AG)_U)$

and

$$f \tilde{X} = (f \circ t)\tilde{X}$$

for all $X \in \Gamma((AG)_U)$ and $f \in C^\infty(M)$.

**Proposition 4.17.** For every open subset $U \subseteq M$, the map

$$\rho_U := T_t \circ (1_U)^*: (AG)_U \to TU$$

is a bundle morphism over $\text{id}_U$ whose corresponding map $\rho_U: \Gamma((AG)_U) \to \Gamma(TU)$ is a morphism of Lie algebras. Moreover for any $f \in C^\infty(M)$ and $X, Y \in \Gamma((AG)_U)$ we have the Leibniz property:

$$[X, fY]_U = f[X, Y]_U + df(\rho_U(X))Y.$$

**Proof.** The definition of $\rho_U$ implies clearly that it is a bundle morphism. From the construction of the isomorphism (4.3) and Proposition 4.16(iii) it follows that $\rho_U$ induces a morphism of Lie algebras as indicated in the statement. Using (4.4) and (4.5) we have

$$\tilde{X}, f\tilde{Y} = [\tilde{X}, f\tilde{Y}]$$

$$= [\tilde{X}, (f \circ t)\tilde{Y}]$$

$$= (f \circ t)[\tilde{X}, \tilde{Y}] + d(f \circ t)(\tilde{X})\tilde{Y}$$

20
\[ (f \circ t)[X,Y] + df(Tt(X))Y = (f \circ t)[X,Y] + df(\tilde{X})Y, \]

(see \[MM03\] page 152). Using the isomorphism (4.3) again, we are done.

**Notation 4.18.** In Proposition 4.16 if \( U = M \) then \( G^U = G \). Therefore, in the above constructions we drop \( U \) from the notation whenever \( U = M \). We thus denote \( AG \) instead of \( (AG)_M \), and we obtain a Lie bracket \([\cdot,\cdot]\) (rather than \([\cdot,\cdot]_M\)) on \( \Gamma(AG) \). Also, we use the notation \( \rho := \rho_M : AG \to TM \) for the map defined in Proposition 4.17.

We are now in a position to prove the following theorem, which lays the foundations of a Lie theory for n.n.H. Banach-Lie groupoids.

**Theorem 4.19.** For every n.n.H. Banach-Lie groupoid \( G \rightrightarrows M \), the structure \((AG, M, \rho, [\cdot, \cdot])\) is a Banach-Lie algebroid.

**Proof.** It follows by Proposition 4.17 that for every open subset \( U \subseteq M \) one has a Lie bracket \([\cdot,\cdot]_U\) on the Banach anchored bundle \((AG)_U, \pi_U, U, \rho_U\). It follows directly by the construction that \( \pi_U : (AG)_U \to U \) is the restriction of the bundle \( \pi = \pi_M : AG \to M \) to \( M \).

We still need to prove that the Lie bracket \([\cdot,\cdot]\) on \( \Gamma(AG) \) is localizable, and to this end we must check that for any open subsets \( V \subseteq U \subseteq M \) the following conditions are satisfied:

- The vector bundle \((AG)_V\) is the restriction of the vector bundle \((AG)_U\) to \( V \).
- The restriction map \( \Gamma((AG)_U) \to \Gamma((AG)_V) \), \( \sigma \mapsto \sigma|_V \), is a morphism of Lie algebras with respect to the Lie brackets \([\cdot,\cdot]_U\) and \([\cdot,\cdot]_V\) on \( \Gamma((AG)_U) \) and \( \Gamma((AG)_V) \), respectively.

For the first of these conditions we note that \( G^V \) is an open subset of \( G^U \), and the vector bundle \( T^s(G^V) \to G^V \) is equal to the vector bundle \( T^s(G^U) \to G^U \) restricted to \( G^V \). One then uses the definition of the vector bundles \((AG)_V\) and \((AG)_U\). For the second of the above conditions we note the commutative diagram

\[
\begin{array}{ccc}
\Gamma((AG)_U) & \xrightarrow{X \mapsto \tilde{X}} & \Gamma_{inv}(G^U) \\
\downarrow & & \downarrow \\
\Gamma((AG)_V) & \xrightarrow{X \mapsto \tilde{X}} & \Gamma_{inv}(G^V)
\end{array}
\]

where we denoted by \( \Xi(N) \) the Lie algebra of all smooth vector fields on any n.n.H. Banach manifold \( N \), the vertical arrows are restriction maps, and the maps \( X \mapsto \tilde{X} \) are Lie algebra isomorphisms as in (4.3). Since \( G^V \) is an open subset of \( G^U \), it is well known that the restriction map \( \Xi(G^U) \to \Xi(G^V) \) is a Lie algebra morphism, and then by the above commutative diagram we obtain that the restriction map \( \Gamma((AG)_U) \to \Gamma((AG)_V) \) is a Lie algebra morphism, too. This completes the proof.
Remark 4.20. Given a n.n.H. Banach-Lie groupoid \( G \rightarrow M \), the construction of the Banach-Lie algebroid \((AG, M, \rho, [\cdot, \cdot])\) in Theorem 4.19 can be summarized as follows:

- The Banach vector bundle \( \pi: AG \rightarrow M \) has its fiber \( (AG)_x = T_{1_x}(G(x, -)) \) at any \( x \in M \).
- The anchor \( \rho: AG \rightarrow TM \) is a morphism of vector bundles whose fiber at \( x \in M \) is the differential of the target map \( t_x = t|_{G(x, -)}: G(x, -) \rightarrow M \) at the point \( 1_x \in G(x, -) \).

Hence, just as in the special case of Banach-Lie groups, the underlying structure of the anchored bundle of a Banach-Lie algebroid of a n.n.H. Banach-Lie groupoid does not require invariant vector fields. These vector fields are only needed in order to define the Lie bracket \([\cdot, \cdot]\) on \( \Gamma(AG) \) for which the anchor \( \rho \) is a morphism of Lie algebras, which extends one of the methods to define the Lie bracket on the Lie algebra of a Lie group. We note however that there is no essential difference between the total space \( AG \) and the section space \( \Gamma(AG) \) in the case of Lie groups, where the base of the vector bundle \( AG \) is a singleton \( M = \{1\} \). See also Remark 4.22 below.

Using the above remark we now make the following definition.

Definition 4.21. If \( G \rightarrow M \) is a n.n.H. Banach-Lie groupoid, then the Banach-Lie algebroid \((AG, M, \rho, [\cdot, \cdot])\) constructed in Theorem 4.19 is said to be the Banach-Lie algebroid associated to the n.n.H. Banach-Lie groupoid \( G \rightarrow M \). For the sake of simplicity this algebroid will be simply denoted \( AG \).

Let \( H \rightarrow M \) be another n.n.H. Banach-Lie groupoid and \( \Phi: G \rightarrow H \) be a morphism of Banach-Lie groupoids over \( id_M \). Then \( s \circ \Phi = s \), hence for every \( x \in M \) one has \( \Phi(G(x, -)) \subseteq H(x, -) \), and we may define the bounded linear operator

\[
(A\Phi)_x: (AG)_x \rightarrow (AH)_x, \quad (A\Phi)_x := (T_{1_x}\Phi)|_{(AG)_x}: (AG)_x \rightarrow (AH)_x
\]

We then define \( A\Phi: AG \rightarrow AH \) as the morphism of Banach vector bundles over \( id_M \) whose restriction to the fiber \( (AG)_x \) is the above operator \( (A\Phi)_x \) for every \( x \in M \).

Remark 4.22. In Definition 4.21 for every \( x \in M \) one has the commutative diagram

\[
\begin{array}{ccc}
G(x, -) & \xrightarrow{\Phi|_{G(x, -)}} & H(x, -) \\
\downarrow t & & \downarrow t \\
M & \xrightarrow{id_M} & M
\end{array}
\]

from which, computing the tangent maps at \( 1_x \in G(x, -) \), one obtains

\[
\begin{array}{ccc}
(AG)_x & \xrightarrow{(A\Phi)_x} & (AH)_x \\
\downarrow \rho & & \downarrow \rho \\
T_{1_x}M & \xrightarrow{id_{T_{1_x}M}} & T_{1_x}M
\end{array}
\]
which is again a commutative diagram, where we denoted by \( \rho \) the anchor maps of both Banach-Lie algebroids \( \mathcal{A} \mathcal{G} \rightarrow M \) and \( \mathcal{A} \mathcal{H} \rightarrow M \).

In addition to the above, one can check that \( \mathcal{A} \Phi : \mathcal{A} \mathcal{G} \rightarrow \mathcal{A} \mathcal{H} \) is a morphism of Banach-Lie algebroids over \( \text{id}_M \), just as in finite dimensions. (See for instance [Mac05, Prop. 3.5.10].) Moreover, let \( \mathbf{GRPD}_M \) be the category of n.n.H. Banach-Lie groupoids with the base \( M \), in which the morphisms are the morphisms of Banach-Lie groupoids with the base \( M \), in which the morphisms are the morphisms of Banach-Lie algebroids over \( \text{id}_M \). Then one can also check that \( \mathcal{A} : \mathbf{GRPD}_M \rightarrow \mathbf{ALGBD}_M \) is a functor, which is equal to the Lie functor from Banach-Lie groups to Banach-Lie algebras in the special case when \( M = \{1\} \) is a singleton, with the Lie bracket defined via right-invariant vector fields on Banach-Lie groups. (See [SW15, §4] for a more complete discussion.)

In the sequel, we will need the following result:

**Proposition 4.23.** For any \( y \in M \) the kernel of the restriction of \( \rho \) to the fiber \( \mathcal{A} \mathcal{G}_y = \pi^{-1}(y) \) is exactly \( 1^{-1}(\ker T_{1_y}s \cap \ker T_{1_y}t) \). In particular \( \mathcal{A} \mathcal{G} \) is split on the connected component \( N \) of \( M \) if and only if \( \ker T_{1_y}s \cap \ker T_{1_y}t \) is a split subspace of \( \ker T_{1_y}s \) for all \( y \in N \). In particular \( \mathcal{G} \) is split if and only if \( \mathcal{A} \mathcal{G} \) is split.

**Proof.** Recall that by construction \( \rho \) is the composition of \( Tt \) restricted to \( \ker T_{1_y}s \) with \( 1_\ast \). Since \( 1_\ast \) is an isomorphism in restriction to \( \mathcal{A} \mathcal{G}_y \) we obtain directly the first part. The second part is a direct consequence of Definition 4.11 and the previous argument. The last assertion is then a direct consequence of Theorem 3.3.21 since \( s \) is a submersion.

**4.3. Link between Banach-Lie algebroids and n.n.H. Banach-Lie groupoid orbits**

**Theorem 4.24.** If \( \mathcal{G} \Rightarrow M \) is a split n.n.H. Banach-Lie groupoid, then for any \( x \in M \) its orbit \( \mathcal{G}.x \) is a weakly immersed submanifold of \( M \) whose connected components are orbits of the Banach-Lie algebroid \( \mathcal{A} \mathcal{G} \).

**Remark 4.25.**

1. According to Theorem 3.3.14, the previous Theorem 4.24 implies that the orbits of an integrable split Banach-Lie algebroid (cf. section 5.2) are always weakly immersed Banach submanifold of \( M \). On the other hand for a general Banach-Lie algebroid \( (\mathcal{A}, M, \rho, [\cdot, \cdot]) \) if this algebroid is split and the associated distribution \( \rho(\mathcal{A}) \) is closed, then Theorem 4.1.12 shows that its orbits are immersed submanifolds. In finite dimensions, any orbit of a Lie algebroid is a split immersed submanifold of \( M \). In the general context of Banach-Lie algebroids we think that the orbits may not be weakly immersed submanifolds but unfortunately, we have no specific example in this connection.

2. The Banach-Lie algebroid of any split n.n.H. Banach-Lie groupoid is split. Indeed for any \( x \in M \), we have \( \ker \rho_x = T_x(\mathcal{G}(x)) \), which is the Lie algebra of the isotropy group \( \mathcal{G}(x) \). As in the proof of Theorem 3.3.11, for the
sequence \( \mathcal{G}(x) \subseteq \mathcal{G}(x,-) \subseteq \mathcal{G} \) we know that \( \mathcal{G}(x) \subseteq \mathcal{G} \) and \( \mathcal{G}(x,-) \subseteq \mathcal{G} \) are submanifolds, and so by Lemma 2.6 also \( \mathcal{G}(x) \subseteq \mathcal{G}(x,-) \) is a submanifold. Therefore the inclusion of tangent spaces \( \ker \rho_x \subseteq (\mathcal{A} \mathcal{G})_x \) is split since \((\mathcal{A} \mathcal{G})_x = T_x(\mathcal{G}(x,-)) \) and \( t_x : \mathcal{G}(x,-) \to \mathcal{G}x \) is a submersion.

The proof of Theorem 4.24 is based on Lemma 4.26 below, which in turn needs some preparations.

For any connected component \( N \) of \( M \) we denote

\[
\mathcal{G}_N := \{ g \in \mathcal{G} \mid s(g) \in N, t(g) \in N \}
\]

and let \( \mathcal{G}_N^0 \) be the connected component of \( \mathcal{G}_N \) that contains \( 1(N) \). Just as for finite-dimensional Lie groupoids (see for instance [CF11, Prop. 1.30]), it is easy to show that \( \mathcal{G}_N \cong N \) (resp. \( \mathcal{G}_N^0 \cong N \)) provided with the restrictions \( s^0_N, t^0_N \) and \( \iota_N \) (resp. \( s^0_N, t^0_N \) and \( \iota_N \)) of \( s, t \) and \( \iota \) to \( \mathcal{G}_N \) (resp. \( \mathcal{G}_N^0 \)), and the restriction \( m^0_N \) (resp. \( m^0_N^0 \)) of \( m \) to \( \mathcal{G}_N^0 \) (resp. \( \mathcal{G}_N^0 \)) is a n.n.H. Banach-Lie groupoid (resp. \( s \)-connected n.n.H. Banach-Lie groupoid). Let \( \mathcal{A} \mathcal{G}_N \to N \) be the restriction of \( \mathcal{A} \mathcal{G} \) to \( N \). Then the restriction \( \rho_N \) of \( \rho \) to \( \mathcal{A} \mathcal{G}_N \) take values in \( TN \) and the restriction of the Lie bracket \( [\cdot, \cdot] \) on \( \mathcal{A} \mathcal{G} \) to \( \mathcal{A} \mathcal{G}_N \) is a Lie bracket on the anchored bundle \( (\mathcal{A} \mathcal{G}_N, N, \rho_N) \) is a Lie bracket again denoted by \( [\cdot, \cdot] \). Finally \((\mathcal{A} \mathcal{G}_N, N, \rho_N, [\cdot, \cdot]) \) (denoted as \( \mathcal{A} \mathcal{G}_N \) for short) is the algebroid associated to \( \mathcal{G}_N \cong N \). Note that \( \mathcal{A} \mathcal{G}_N \) is also the Banach-Lie algebroid of \( \mathcal{G}_N^0 \cong N \) (cf. Remark 4.20).

We define \( \mathcal{G}^0 \) as the set of all \( g \in \mathcal{G} \) that belong to the connected component of \( 1 \) in \( \mathcal{G}(s(g),-) \), and then \( \mathcal{G}^0 \) is an \( s \)-connected open wide subgroupoid of \( \mathcal{G} \) just as in [CF11, Prop. 1.30].

**Lemma 4.26.** Let \( N \) be a connected component of \( M \). For any \( x \in N \), its \( \mathcal{G}_N^0 \)-orbit \( \mathcal{G}_N^0 x \) coincides with the \( \mathcal{A} \mathcal{G}_N \)-orbit of \( x \) and is a weakly immersed submanifold of \( N \).

We postpone the proof of Lemma 4.26 until after the proof of Theorem 4.23.

**Lemma 4.27.** Let \( \mathcal{O} \subseteq M \) be an orbit of \( \mathcal{G} \), and assume that \( \mathcal{O} \) is endowed with a topology such that for every \( x \in \mathcal{O} \) the map \( t_{\mathcal{G}(x,-)} : \mathcal{G}(x,-) \to \mathcal{O} \) is continuous and open. Then \( \{ \mathcal{G}^0 \cdot x \mid x \in \mathcal{O} \} \) is the family of all connected components of \( \mathcal{O} \).

**Proof.** It suffices to prove that \( \{ \mathcal{G}^0 \cdot x \mid x \in \mathcal{O} \} \) is a family of open, connected, mutually disjoint subsets of \( \mathcal{O} \) whose union is equal to \( \mathcal{O} \).

To this end we first note that for arbitrary \( x \in \mathcal{O} \) one has \( x \in \mathcal{G}^0 \cdot x \), hence \( \bigcup_{x \in \mathcal{O}} \mathcal{G}^0 \cdot x = \mathcal{O} \). We now check that if \( x, y \in \mathcal{O} \) and \( \mathcal{G}^0 \cdot x \cap \mathcal{G}^0 \cdot y \neq \emptyset \) then \( \mathcal{G}^0 \cdot x = \mathcal{G}^0 \cdot y \). In fact, if \( z \in \mathcal{G}^0 \cdot x \cap \mathcal{G}^0 \cdot y \), then there exist \( g \in \mathcal{G}^0(x,z) \) and \( h \in \mathcal{G}^0(y,z) \). For every \( w \in \mathcal{G}^0 \cdot x \) there exists \( k \in \mathcal{G}^0(x,w) \), and then \( k g^{-1} h \in \mathcal{G}^0(y,w) \), hence \( w \in \mathcal{G}^0 \cdot x \). This shows that \( \mathcal{G}^0 \cdot x \subseteq \mathcal{G}^0 \cdot y \), and the converse inclusion can be proved similarly.
We have recalled above that $\mathcal{G}^0$ is an $s$-connected groupoid. Since the image of a connected set by a continuous map is connected and one has $\mathcal{G}^0, x = t^0(\mathcal{G}^0(x, -))$, it then follows that $\mathcal{G}^0, x$ is a connected subset of $\mathcal{O}$. Moreover, as also recalled above, $\mathcal{G}^0$ is an open subset of $\mathcal{G}$, and this implies that $\mathcal{G}^0 \cap \mathcal{G}(x, -)$ is an open subset of $\mathcal{G}(x, -)$. The map $t|_{\mathcal{G}(x, -)}: \mathcal{G}(x, -) \to \mathcal{O}$ is open by hypothesis, hence the equality $\mathcal{G}^0, x = t^0(\mathcal{G}^0(x, -))$ implies that $\mathcal{G}^0, x$ is an open subset of $\mathcal{O}$, and this completes the proof.

**Proof (Proof of Theorem 4.24).** By Theorem 3.3(iii), $\mathcal{G}(x, -)$ is a pure Hausdorff Banach manifold and the map $t_x : \mathcal{G}(x, -) \to \mathcal{G}, x$ is a locally trivial fibration, hence in particular continuous and open. The assertion then follows at once by Lemmas 4.20 and 4.27.

**Proof (Proof of Lemma 4.26).** **Step 1:** The $\mathcal{G}^0_N$-orbit of $x$ is contained in its $\mathcal{A}\mathcal{G}_N$-orbit.

For every $y \in \mathcal{G}^0_N, x$ there exists $g \in \mathcal{G}^0_N$ with $s(g) = x$ and $t(g) = y$. Since the groupoid $\mathcal{G}^0_N \rightrightarrows N$ is $s$-connected, there exists a smooth curve $\gamma : [0, 1] \to \mathcal{G}^0_N(y, -)$ with $\gamma(0) = 1_x$ and $\gamma(1) = g$. For $c := t \circ \gamma : [0, 1] \to N$ one has $c(0) = x$ and $c(1) = y$. Hence in order to prove that $y$ belongs to the $\mathcal{A}\mathcal{G}_N$-orbit of $x$ it suffices to show that $c$ is an $\mathcal{A}\mathcal{G}^0_N$-admissible curve. To this end we set

$$\alpha(t) = TR_{\gamma(t)^{-1}}(\gamma(t)) \in T_{1_{t(\gamma(t))}}(\mathcal{G}^0_N(\gamma(t), -))$$

for every $t \in [0, 1]$ and we will check that $\alpha$ is an $\mathcal{A}\mathcal{G}^0_N$-lift of $c$.

We recall from Remark 4.20 that the fiber of $\mathcal{A}\mathcal{G}^0_N$ at any $x \in N$ is $T_{1_x}(\mathcal{G}^0_N(x, -))$. Hence, if we denote by $\pi : \mathcal{A}\mathcal{G}^0_N \to N$ the vector bundle projection, then one has $\pi(\alpha(t)) = t(\gamma(t)) = c(t)$. If $\rho : \mathcal{G}^0_N \to TN$ is the anchor, then we also have $\rho(\alpha(t)) = c(t)$ for the following reason. Since $t(\gamma(t)) = c(t)$ and $s(\gamma(t)) = y$, one has $t_y \circ R_{\gamma(t)} = t_y$ (cf. the equality $t_y \circ R_y = t_y$ in the proof of Proposition 3.16(ii)). Therefore $t_y = t_y \circ R_{\gamma(t)^{-1}} : \mathcal{G}^0_N(y, -) \to M$ with $R_{\gamma(t)^{-1}}(t_y) = 1_{t_y}$. Computing the differential of both sides of the above equality at the point $\gamma(t) \in \mathcal{G}^0_N(y, -)$ and using the definition of $\rho$ (cf. Remark 4.20) we obtain

$$T(t_y) = \rho(T_{\mathcal{G}^0_N(z, t)}(\gamma(t))R_{\gamma(t)^{-1}} : T_{\gamma(t)}(\mathcal{G}^0_N(y, -)) \to T_{\gamma(t)}M.$$

Evaluating this equality at $\gamma(t) \in T_{\gamma(t)}(\mathcal{G}^0_N(y, -))$ we obtain $c(t) = \rho(\alpha(t))$, and this completes the proof of the fact that $\alpha$ is an $\mathcal{A}\mathcal{G}^0_N$-lift of $c$.

**Step 2:** The $\mathcal{A}\mathcal{G}_N$-orbit of $x$ is contained in its $\mathcal{G}^0_N$-orbit.

To prove this, let $y$ be arbitrary in the $\mathcal{A}\mathcal{G}_N$-orbit of $x$. We will prove that there exists $g \in \mathcal{G}^0_N$ with $s(g) = 1_x$ and $t(g) = 1_y$, using an idea from the proof of [CF03, Prop. 1.1].

To this end, note that from Lemma 3.10 it follows that there exists a smooth curve $\alpha : [0, 1] \to \mathcal{A}\mathcal{G}_N$ whose projection $c := \pi \circ \alpha$ on $N$ is a smooth $\mathcal{A}\mathcal{G}_N$-admissible curve, satisfying $c(0) = x$ and $c(1) = y$, and for which there exists an open subset $U \subseteq N$ for which $c([0, 1]) \subseteq U$ and the restriction of $\mathcal{A}\mathcal{G}_N$ to $U$ is trivializable. If $E$ is the typical fiber of $\mathcal{A}\mathcal{G}_N$, without loss of generality, we can
identify this restriction with \( U \times E \). We can then write \( \alpha(t) = (c(t), u(t)) \in U \times E \) for all \( t \in [0, 1] \) and we denote by \( \{\alpha_t\}_{t \in [0, 1]} \) the smooth family of smooth sections of \( A_G N \) over \( U \) defined by

\[
(\forall t \in [0, 1])(\forall y \in U) \quad \alpha_t(y) = (y, u(t)).
\]

Now, for each \( t \in [0, 1] \), consider the right-invariant vector field \( \tilde{\alpha}_t \) on the restriction of \( G^0_N \) to \( U \), satisfying \( \tilde{\alpha}_t(1_y) = \alpha_t(y) \in (A_G N)_y = T_{1_y}(G^0_N(y, -)) \) for every \( y \in U \). (Recall Remark \[4.20\]) Let \( \Phi^t_s \) be the flow of the vector field \( \tilde{\alpha}_t \) with initial conditions \( \Phi^t_0(1_c(t)) = 1_c(t) \). Since \( c([0, 1]) \subseteq N \) is compact there exist \( \delta > 0 \) and an open subset \( W \subseteq G_N \) with \( \{1_z \mid z \in c([0, 1])\} \subseteq W \cap N \subseteq G^0_N \) such that \( \Phi^t_s : W \to G^0_N \) is defined for all \( s \in [0, \delta] \) and all \( t \in [0, 1] \). For any \( t \in [0, 1] \), using the fact that \( 1_y = 1_{c(a)} \in W \), we define

\[
\gamma(t) := \Phi^t_0(1_y) \in G^0_N.
\]

Since \( \tilde{\alpha}_t \) is right invariant it follows that \( \tilde{\alpha}_t \) is s-vertical and so \( s(\gamma(t)) = s(1_{c(t)}) = c(t) \) for all \( t \in [0, 1] \). Hence for \( g := \gamma(a) \) one has \( s(g) = c(0) = x \) and \( t(g) = t(1_{c(a)}) = c(1) = y \), as we have wished for.

Now in the most general situation assume that \( x, y \in N \) belong to the same \( A_G N \)-orbit. Then by the Lemma \[4.3\] there exists a smooth curve \( \alpha : [0, 1] \to A_G N \) whose projection \( c \) on \( N \) is an \( A_G N \)-admissible curve with \( c(0) = x \) and \( c(1) = y \).

Since \( c([0, 1]) \) is compact, there exists a family of open sets \( \{U_i\}_{i=1,...,n} \) with the following properties:

- \( A_G N \) is trivializable over each \( U_i \) for \( i = 1, \ldots, n \);
- if \( n > 1 \) then \( U_i \cap U_{i+1} \neq \emptyset \) for \( i = 1, \ldots, n-1 \) and \( U_i \cap U_j = \emptyset \) for \( 1 \leq i+1 < j \leq n \) and \( i = 1, \ldots, n-2 \) if \( n > 2 \);
- \( c([0, 1]) \subseteq \bigcup_{i=1}^n U_i \).

We fix points \( t_0 = 0 < t_1 \leq \cdots \leq t_i \leq \cdots \leq t_n = 1 \) in \([0, 1]\) with \( c(t_i) \in U_i \cap U_{i+1} \) for \( i = 1, \ldots, n-1 \) if \( n > 1 \). Set \( x_i = c(t_i) \) for \( i = 0, \ldots, n \). By the arguments above, there exists \( g_i \in G_N \) with \( s(g_i) = x_i \) and \( t(g_i) = x_{i+1} \) for \( i = 0, \cdots, n-1 \). Then the product \( g = g_{n-1} \cdots g_1 \cdots g_0 \) is well defined and, by construction, we have \( s(g) = x \) and \( t(g) = y \). Thus \( y \) belongs to the \( G_N \)-orbit of \( x \).

5. Banach-Lie algebroids and s-simply connected groupoids

The main result of this section is Theorem \[5.1\] and its proof requires the idea of monodromy group of a foliation.
5.1. Monodromy groupoid of a foliation

Consider an involutive split subbundle \( F \) of the tangent \( TM \) of a connected Banach manifold \( M \). If \( M \) is modeled on a Banach space \( \mathbb{M} \), we have a decomposition \( \mathbb{M} = \mathbb{F} \oplus \mathbb{T} \) and from the Frobenius theorem we have a "foliated atlas" \( \{(U_{\alpha},\varphi_{\alpha})\} \) on \( M \) with the following properties:

For each \( \alpha \), \( \varphi_{\alpha} : U_{\alpha} \to \mathbb{M} \equiv \mathbb{F} \times \mathbb{T} \) is a diffeomorphism with \( \varphi_{\alpha}(U_{\alpha}) = P_{\alpha} \times Z_{\alpha} \subset \mathbb{F} \times \mathbb{T} \), and \( \varphi_{\alpha}^{-1}(P_{\alpha} \times \{z\}) \) is contained in the leaf of \( F \) through \( \varphi_{\alpha}^{-1}(p,z) \) for any \( (p,z) \in P_{\alpha} \times Z_{\alpha} \). If we fix some point \( x \in U_{\alpha} \) then, up to a translation in \( \mathbb{T} \), we can always assume that \( \varphi_{\alpha}(x) = (p,0) \in \mathbb{F} \times \mathbb{T} \). Then \( S_{\alpha} = \varphi_{\alpha}^{-1}(\{p\} \times Z_{\alpha}) \) is called a transversal at \( x \) and for any \( y \in S_{\alpha} \) if \( \varphi_{\alpha}(y) = (p,z) \) then \( \varphi_{\alpha}^{-1}(P_{\alpha} \times \{z\}) \) is called the plaque through \( y \) and of course is contained in the leaf through \( y \).

If \( U_{\alpha} \cap U_{\beta} \neq \emptyset \) then the transition diffeomorphism \( \varphi_{\alpha} \circ \varphi_{\beta}^{-1}(p,z) = (f(p,z),h(z)) \) is a local diffeomorphism of \( \mathbb{F} \times \mathbb{T} \) which respect this product structure.

Let \( \Pi(F) \) be the set of homotopy classes with fixed end points of piecewise smooth paths contained in leaves of \( F \). One can define groupoid structure on \( \Pi(F) \) in the same way as in the construction of fundamental groupoid \( \Pi(M) \), see [3, 27]. The differentiable structure is obtained by a direct adaptation to the Banach framework, step by step, of such a structure as it is built in [Ph87, Sect. 2]. We will describe this construction in detail in our Banach context:

**Step 1:** Consider a leaf \( L \) of \( F \) and a piecewise smooth path \( \gamma : [0,1] \to L \), and denote \( x_0 := \gamma(0) \) and \( x_1 := \gamma(1) \).

1. There exists two charts \( (U_{i},\varphi_{i}) \) around \( x_{i}, \ i = 0, 1 \), with \( \varphi_{i}(U_{i}) = P_{i} \times Z \) where \( P_{i} \) (resp. \( T \)) and \( \varphi_{i}(x_{i}) = (p_{i},0) \in \mathbb{F} \times \mathbb{T} \).

2. For \( i = 0, 1 \) if \( S_{i} := \varphi_{i}^{-1}(\{p_{i}\} \times Z) \) then there exists a continuous map \( H_{\gamma} : [0,1] \times S_{0} \to M \) which is smooth in the second variable and \( H_{\gamma}(\cdot,y) \) is a piecewise smooth path contained in the leaf through \( y \in S_{0} \) with the following properties: \( H_{\gamma}(:,x_{0}) = \gamma; \ H_{\gamma}(0,y) = y \) and \( H_{\gamma}(1,y) \in S_{1} \) for all \( y \in S_{0} \); the mapping \( S_{0} \to S_{1}, y \mapsto H_{\gamma}(1,y) \), is a diffeomorphism.

**Proof.** Let \( \tilde{S} \subseteq M \) be an embedded submanifold with \( x_{0} \in \tilde{S} \) and \( T_{x} \tilde{S} \oplus F_{x} = T_{x}M \) for every \( x \in \tilde{S} \). There exist a finite set of charts \( (V_{i},\psi_{i}) \) for \( i = 0, \ldots, n \) of the foliated atlas and a partition \( t_{0} < t_{1} < \cdots < t_{n} < 1 = t_{n+1} \) satisfying the following conditions:

- \( \psi_{i}(V_{i}) = P_{i} \times Z_{i} \) where \( P_{i} \) (resp. \( Z_{i} \)) is a simply connected open neighborhood of \( 0 \in \mathbb{F} \) (resp. \( 0 \in \mathbb{T} \)) for \( i = 0, \ldots, n \);
- \( \gamma(t_{i}) \in V_{i-1} \cap V_{i} \) for \( i = 1, \ldots, n \), \( \gamma(0) = x_{0} \in V_{0} \), and \( \gamma(1) = x_{1} \in V_{n} \);
- \( \gamma([t_{i},t_{i+1}]) \subseteq P_{i} \times \{0\} \subseteq V_{i} \) for \( i = 0, \ldots, n \).

By suitable translations in \( \mathbb{T} \) and \( \mathbb{F} \) we may assume that there exist \( p_{i} \in P_{i} \) with \( \psi(x_{i}) = (p_{i},0) \in P_{i} \times Z_{i} \) for \( i = 0, 1 \), and we may also select the foliation chart \( (V_{0},\psi_{0}) \) such that \( S_{0} := \psi_{0}^{-1}(\{p_{0}\} \times Z_{0}) \) is an open neighborhood of \( x_{0} \in \tilde{S} \).
We will now prove by induction on \( n \) that if one has a family of local charts satisfying the above conditions then there exists a continuous mapping \( H_γ \) with the properties required in (2) above, with \((U_0, φ_0) := (V_0, ψ_0)\) and \((U_1, φ_1) := (V_{n+1}, ψ_{n+1})\).

**Case \( n = 0 \):** We denote \( ψ := ψ_0, V := V_0, P := P_0, \) and \( Z := Z_0 \) for simplicity. Using the path \( \hat{γ} := ψ \circ γ : [0, 1] \to P \equiv P \times \{0\}, \) define
\[
\tilde{H} : [0, 1] \times Z \to P \times Z, \quad \tilde{H}(t, z) := (\hat{γ}(t), z).
\]
Obviously \( \tilde{H} \) is continuous and piecewise smooth (resp. smooth) in \( t \) (resp. \( z \)) and the path \( t \mapsto \tilde{H}(t, z) \) is contained in \( P \times \{z\}. \)

The charts \((U_0, φ_0) = (U_1, φ_1) := (V, ψ)\) satisfy condition (1). For \( j = 0, 1 \) and \( S_j := \psi^{-1}(\{p_j\} \times Z), \) one has a bijection \( ψ_{p_j} : S_j \to Z \) satisfying \( ψ(y) = (p_j, ψ_{p_j}(y)) \) for all \( y \in S_j \), and then we can define
\[
H_γ : [0, 1] \times S_0 \to M, \quad H_γ(t, y) := \psi^{-1}(\tilde{H}(t, ψ_{p_0}(y))) = \psi^{-1}(\hat{γ}(t), ψ_{p_0}(y)).
\]
Hence \( H_γ(0, y) = \psi^{-1}(p_0, ψ_{p_0}(y)) = y \) for all \( y \in S_0 \). For every \( y \in S_0 \) one then has
\[
ψ(H_γ(1, y)) = (\hat{γ}(1), ψ_{p_0}(y)) = (ψ(x_1), ψ_{p_0}(y)) = (p_1, ψ_{p_0}(y))
\]
and
\[
H_γ(1, y) = ψ^{-1}(p_1, ψ_{p_0}(y)) = ψ_{p_1}^{-1}(ψ_{p_0}(y)).
\]
Since \( ψ_{p_j} : S_j \to Z \) is a global chart of \( S_j \) for \( j = 1, 2 \), the above equality shows that the mapping \( S_0 \to S_1, y \mapsto H_γ(1, y) \) is a diffeomorphism that maps \( p_0 \in S_0 \) to \( p_1 \in S_1 \). Condition (2) from Step 1 above is thus satisfied as well.

**Induction step:** Assume that \( n \geq 1 \) and the assertion has been proved for \( n - 1 \).

Then, on the one hand, by the induction hypothesis applied for the path \( γ' := γ|_{[0, t_n]} \), one obtains a continuous map \( H_{γ'} : [0, t_n] \times S_0 \to M \) which is smooth in the second variable and \( H_{γ'}(\cdot, y) \) is a piecewise smooth path contained in the leaf through \( y \in S_0 \) with the following properties: \( H_{γ'}(\cdot, x_0) = γ'; H_{γ'}(0, y) = y \) and \( H_{γ'}(t_n, y) \in S_{n-1} := \psi_{p_{n-1}}^{-1}(\{p_{n-1}\} \times Z) \) for all \( y \in S_0 \); the mapping
\[
Ψ : S_0 \to S_{n-1}, \quad y \mapsto H_{γ'}(t_n, y)
\]
is a diffeomorphism.

On the other hand, by the Case \( n = 0 \) applied for the path \( γ'' := γ|_{[t_n, 1]} \) and with the above embedded submanifold \( S_{n-1} \subseteq M \) in the role of \( S \), one obtains a continuous map \( H_{γ''} : [t_n, 1] \times S_{n-1} \to M \) which is smooth in the second variable and \( H_{γ''}(\cdot, y) \) is a piecewise smooth path contained in the leaf through \( y \in S_{n-1} \) with the following properties: \( H_{γ''}(\cdot, γ(t_{n-1})) = γ''; H_{γ''}(t_n, y) = y \) and \( H_{γ''}(1, y) \in S_n := \psi_{p_n}^{-1}(\{p_n\} \times Z) \) for all \( y \in S_{n-1} \); the mapping \( S_{n-1} \to S_n, y \mapsto H_{γ''}(1, y) \) is a diffeomorphism.

We now define \( H_γ : [0, 1] \times S_0 \to M \) by
\[
H_γ(t, y) = \begin{cases} 
H_{γ'}(t, y) & \text{if } t \in [0, t_n], \\
H_{γ''}(t, Ψ(y)) & \text{if } t \in [t_n, 1]
\end{cases}
\]
for all \( y \in S_0 \). This definition is correct since \( H_\gamma(t_n, y) = \Psi(y) = H_\gamma(t_n, \Psi(y)) \).

It is straightforward to check the other properties of \( H_\gamma \) required in the statemen-
tment of Step 1 with the charts \((U_0, \phi_0) := (V_0, \psi_0)\) and \((U_n, \phi_n) := (V_n, \psi_n)\), and this completes the proof.

**Step 2:** For any piecewise smooth path \( \gamma: [0, 1] \to M \) contained in a leaf, use Step 1 to define

\[
\mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma) := \{ [\gamma'] \in \Pi(\mathcal{F}) \mid s(\gamma') \in S_0, \ t(\gamma') \in S_1, \ [\gamma'] = [\alpha_\gamma * H_\gamma(\cdot, y) * \beta_\gamma] \text{ for some } y \in S_0 \}
\]

where \( \alpha_\gamma \) is a path which joins \( s(\gamma') \) to \( H_\gamma(0, y) = y \) and lies in the plaque \( \phi_0^{-1}(P_0 \times \{z_0(y)\}) \) through \( y \in S_0 \), while and \( \beta_\gamma \) is a path which joins \( H_\gamma(1, y) \) to \( t(\gamma') \) and lies in the plaque \( \phi_1^{-1}(P_1 \times \{z_1(y)\}) \) through \( H_\gamma(1, y) \in S_1 \), where \( \phi_j(y) = (p_j, z_j(y)) \in P_j \times Z \) for \( j = 0, 1 \). Then there exists a unique topology on \( \Pi(\mathcal{F}) \) for which the set of all the above sets \( \mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma) \) is a neighborhood base for \( [\gamma] \in \Pi(\mathcal{F}) \).

**Proof.** For every \([\gamma] \in \Pi(\mathcal{F})\) let us denote by \( B([\gamma]) \) the set of all the above subsets \( \mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma) \) of \( \Pi(\mathcal{F}) \). Then the assertion will follow as soon as we will have checked that the following properties:

(BP1) For every \([\gamma] \in \Pi(\mathcal{F})\) one has \( B([\gamma]) \neq \emptyset \), and \([\gamma] \in \mathcal{V} \) for all \( \mathcal{V} \in B([\gamma]) \).

(BP2) If \([\gamma] \in \Pi(\mathcal{F})\), \( \mathcal{V} \in B([\gamma]) \) and \([\gamma'] \in \mathcal{V} \), then there exists \( \mathcal{V}' \in B([\gamma']) \) with \( \mathcal{V}' \subseteq \mathcal{V} \).

(BP3) If \([\gamma] \in \Pi(\mathcal{F})\) and \( \mathcal{V}, \mathcal{V}' \in B([\gamma]) \), then there exists \( \mathcal{V}'' \in B([\gamma]) \) with \( \mathcal{V}'' \subseteq \mathcal{V} \cap \mathcal{V}' \).

(See for instance [En89, Prop. 1.2.3].)

The property (BP1) directly follows by Step 1.

For (BP2), let \([\gamma'] \in \mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma) \). We can assume that \( \gamma' \) is the path \( \alpha_{\gamma'} \circ H_\gamma(\cdot, y) \circ \beta_{\gamma'} \) for some \( y \in S_0 \). Since \( \phi_0(U_i) = P_i \times Z \), if \( \phi_0(y) = (0, z) \) we have \( \phi_0(\gamma'(0)) = (p_0, z) \) and \( \phi_1(\gamma'(1)) = (p_1, H_\gamma(1, y)) \). By composition of \( \phi_1 \) by translation in \( \mathbb{F} \) and \( \mathbb{T} \) and after shrinking the open set \( U_i \) if necessary, we get a new foliated chart \((U_i', \phi_i')\) such that \( U_i' \subset U_i, \phi_i' \circ \gamma'(i) = (p_i', 0) \) and \( \phi_1(U_i') = P_i' \times Z_i \) with \( Z_i \subset Z \) for \( i = 0, 1 \). If \( S_i' = (\phi_i')^{-1}(\{p_i'\} \times Z_i) \) as in the proof of Step 1 we can build a map \( H_\gamma' : [0, 1] \times S_0' \to M \) with properties of Step 1(2) with respect to \((U_i', \phi_i')\) for \( i = 0, 1 \). We have

\[
[\gamma'] \in \mathcal{V}(\gamma', (U_0', \phi_0'), (U_1', \phi_1'), H_{\gamma'}) \subseteq \mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma).
\]

For (BP3), let

\[
\mathcal{V} := \mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma) \text{ and } \mathcal{V}' = \mathcal{V}(\gamma, (U_0', \phi_0'), (U_1', \phi_1'), H_{\gamma'}).
\]

By a method similar to the one used for (BP3) above, one can then find a local chart \((U_j'', \phi_j'')\) at \( \gamma(j) \in M \) with \( U_j'' \subseteq U_j \cap U_j' \) for \( j = 0, 1 \) and with

\[
\mathcal{V}(\gamma, (U_0'', \phi_0''), (U_1'', \phi_1''), H_{\gamma''}) \subseteq \mathcal{V} \cap \mathcal{V}'
\]

and this completes the proof of Step 2.
Step 3: The set of all the basic open sets \( \mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma) \) defines a smooth atlas on \( \Pi(F) \) modeled on \( F \times T \times F \). The source map \( s \) and the target \( t \) are submersions, and the inversion \( i \) and the multiplication \( m \) are also smooth maps.

Proof. We already noted that \( \mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma) \) depends only of the homotopy class \( [\gamma] \in \Pi(F) \). Consider the map

\[
\Phi_\gamma : \mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma) \to F \times T \times F,
\]

\[
\Phi_\gamma((\mu)) = (\phi_0(\mu(0)), \phi_1(\mu(1)), \phi_1(\mu(1))).
\]

It is clear that \( \Phi \) is well defined and injective and depends only of the homotopy class of \( \gamma \). In addition, since \( \phi_i U_i = P_i \times Z \) is a product of simply connected open sets, it follows from the proof of Step 1 that \( \Phi_\gamma \) is injective.

For \( i = 0, 1 \) if \( P'_i \subseteq P_i \) and \( Z' \subseteq Z \) are open sets then \( \Phi^{-1}_\gamma(P'_i \times Z' \times P'_i) \) is the basic set \( \mathcal{V}(\gamma, (U'_0, \phi'_0), (U'_1, \phi'_1), H'_\gamma) \) where \( U'_i = \phi_i^{-1}(P'_i \times Z') \), \( \phi'_i = \Phi_i|_{U'_i} \) for \( i = 0, 1 \) and \( H'_\gamma \) is the restriction of \( H_\gamma \) to \( [0, 1] \times S' \) with \( S' = S_0 \cap U'_0 \) and so \( \Phi_\gamma \) is continuous. It follows by the arguments of the previous proof that \( \Phi_\gamma \) is open and so \( \Phi_\gamma \) is a homeomorphism. It is obvious that each transition map \( \Phi_\gamma \circ \Phi^{-1}_\gamma \) between charts is smooth.

Now as the smoothness is a local property, it is obvious that \( s, t \) and \( i \) restricted to a chart \( (\mathcal{V}(\gamma, (U_0, \phi_0), (U_1, \phi_1), H_\gamma), \Phi_\gamma) \) are smooth. For the multiplication \( m \) it is also easy to verify the smoothness in local charts in \( \Pi(F) \times \Pi(F) \) and \( \Pi(F) \). The splitness properties of the tangent maps of \( s \) and \( t \) is also clear from the definition of \( \Phi_\gamma \) on a chart.

5.2. Integrable Banach-Lie algebroids and \( s \)-simply connected groupoids

As in finite dimension, we say that a Banach-Lie algebroid \( \mathcal{A} \) is integrable if there exists a n.n.H Banach-Lie groupoid \( \mathcal{G} \) such that \( \mathcal{A} \) is isomorphic to the algebroid \( \mathcal{A}\mathcal{G} \). If this is the case, then we say that the Banach-Lie algebroid \( \mathcal{A} \) integrates to the n.n.H. Banach-Lie groupoid \( \mathcal{G} \).

The purpose of this subsection is to prove the following theorem which generalizes a well known result in finite dimensions and shows that every integrable Banach-Lie algebroid integrates to certain \( s \)-simply connected n.n.H. Banach-Lie groupoid. (See [MM03, Prop. 6.6].)

Theorem 5.1. If \( \mathcal{G} \rightrightarrows M \) is a n.n.H. Banach-Lie groupoid, then there exist an \( s \)-simply connected n.n.H. Banach-Lie groupoid \( \mathcal{G} \rightrightarrows M \) and a morphism of Banach-Lie groupoids \( \Phi : \mathcal{G} \to \mathcal{G} \) over id\(_M\), which is a local diffeomorphism and for which \( \mathcal{A}\Phi : \mathcal{A}\mathcal{G} \to \mathcal{A}\mathcal{G} \) is an isomorphism of Banach-Lie algebroids over id\(_M\).

Proof. This is an adaptation to our context of the proof of [CF11, Th. 1.31]. At first note that we can prove the result for the sub-groupoid associated to each connected component of \( M \). Therefore from now, assume that \( \mathcal{G} \rightrightarrows M \) is a n.n.H. Banach-Lie groupoid, where \( M \) is a connected (hence pure) Banach manifold. From the construction of the algebroid \( \mathcal{A}\mathcal{G} \) and using the local normal
form of a submersion in the Banach framework (that is, submersion charts), with
the arguments of the proof of [CF11, Prop. 1.30] without loss of generality, we
can also assume that \( \mathcal{G} \) is \( s \)-connected.

For any \( x \in M \) denote by \( \tilde{\mathcal{G}}(x,-) \) the universal covering of \( \mathcal{G}(x,-) \) and we set
\[
\tilde{\mathcal{G}} = \bigcup_{x \in M} \tilde{\mathcal{G}}(x,-).
\]
In fact \( \tilde{\mathcal{G}} \) is the set of homotopy classes of paths in each \( \mathcal{G}(x,-) \) (with fixed end points) and starting at \( 1_x \).

Consider the foliation \( F_s \) of \( \mathcal{G} \) defined by the fibration \( s : \mathcal{G} \to M \). The
construction of the n.n.H. Banach-Lie Banach groupoid structure on the mon-
odromy groupoid of a (regular) foliation (cf. subsection [5.1]) used only local
arguments. Therefore the set \( \Pi(F_s) \) of homotopy classes with fixed end of path
contained in leaves of \( F_s \) has a structure of n.n.H. Banach manifold and more-
over, since the smoothness is a local property the source \( s(\gamma) = \gamma(0) \) and the
target \( t(\gamma) = \gamma(1) \) are submersions from \( \Pi(F_s) \) onto \( \mathcal{G} \).

We also define \( \tilde{m}(\gamma_1, \gamma_2) \) as the homotopy class of the concatenation of
\( \gamma_2 \) with \( R_{\gamma_1(1)} \circ \gamma_1 \). Just as in the finite dimensional context we thus obtain a
n.n.H. Banach-Lie groupoid structure on \( \tilde{\mathcal{G}} \).

Now, since \( s : \tilde{\mathcal{G}} \to M \) is a submersion and \( M \) is connected, from the normal
form of a submersion in Banach framework (i.e., submersion charts), it follows
that for any \( g \in \mathcal{G} \) there exists a neighborhood \( U \) which is diffeomorphic to a
product of open sets \( U \times V \subset M \times \mathbb{K} \) where the Banach space \( M \) (resp. \( \mathbb{K} \)) is the
model space of \( M \) (resp. of the typical fiber of \( s \)). In the same way, since \( \tilde{s} : \tilde{\mathcal{G}} \to M \) is a submersion, any \( \gamma \in \tilde{\mathcal{G}} \) has a neighborhood \( \tilde{U} \) diffeomorphic to a product
of open \( \tilde{U} \times \tilde{V} \subset M \times \tilde{\mathbb{K}} \) where \( \tilde{\mathbb{K}} \) is the typical Banach model of the typical fiber
of \( \tilde{s} \). But the typical fiber of \( \tilde{s} \) is the universal covering of the typical fiber of \( s \) so \( \tilde{\mathbb{K}} \) is isomorphic to \( \mathbb{K} \). But, from the construction of the smooth structure on
\( \Pi(F_s) \) and the characterization of \( \tilde{\mathcal{G}} \) as submanifold of \( \Pi(F_s) \), for each \( \gamma \in \tilde{\mathcal{G}} \) one has an open neighborhood of type \( \mathcal{V}(\gamma,(U_0,\phi_0),(U_1,\phi_1),H_\gamma) \cap \tilde{\mathcal{G}} \) and the
restriction of \( \Phi_\gamma \) to this open set is \( \Phi_\gamma(\mu) = (\phi_1(H_\gamma(1,\mu(0))),\mu(1)) \) which is a diffeomorphism on an open set of \( \tilde{U} \times \tilde{V} \subset M \times \tilde{\mathbb{K}} \). It follows that the
map \( \Phi : \tilde{\mathcal{G}} \to \mathcal{G} \) given by \( \Phi(\gamma) = \gamma(1) \) is a local diffeomorphism. From
the construction of the Banach-Lie algebroid this last property implies that
\( A\Phi : A\tilde{\mathcal{G}} \to A\mathcal{G} \) is an isomorphism of Banach-Lie algebroids, and this completes
the proof.

6. Locally transitive Banach-Lie groupoids and transitive algebroids

6.1. Locally transitive Banach-Lie groupoids

As usual for topological groupoids, we will say that a Banach-Lie groupoid
\( \mathcal{G} \rightrightarrows M \) is locally transitive if each orbit \( \mathcal{G}.x \) is open in \( M \). This condition is
equivalent to the following one:

For every \( x \in M \) the map \( t_x : \mathcal{G}(x,-) \to M \) is a submersion. (6.1)
In fact, we know that \( t_x : \mathcal{G}(x, -) \to \mathcal{G}.x \) is a submersion since it is a principal bundle by Theorem 3.3(iii). If \( \mathcal{G}.x \) is open in \( M \) so is a submanifold of \( M \) whose tangent space at any point coincides to the tangent space of \( M \) at that point and so \( t_x : \mathcal{G}(x, -) \to M \) is a submersion.

**Lemma 6.1.** If \( \mathcal{G} \to M \) is a locally transitive Banach-Lie groupoid, then the following assertions hold:

(i) Every orbit of \( \mathcal{G} \) is a union of some connected components of \( M \).

(ii) If an orbit of \( \mathcal{G} \) is connected as a topological subspace of \( M \) (for instance if it is the orbit of a point \( x \in M \) whose \( s \)-fiber \( \mathcal{G}(x, -) \) is connected), then that orbit coincides with some connected component of \( M \).

(iii) If some connected component of \( M \) is invariant under the action of \( \mathcal{G} \) on \( M \), then that connected component coincides with an orbit of \( \mathcal{G} \).

**Proof.** It suffices to prove Assertion (i) and then the other assertions follow directly.

Since for arbitrary \( x \in M \) its orbit \( \mathcal{G}.x := t(\mathcal{G}(x, -)) \) is open and \( M \) is a disjoint union of the orbits of \( \mathcal{G} \), it then follows that every orbit is also closed because its complement in \( M \) is the union of the other orbits, which is open. Thus every orbit of \( \mathcal{G} \) is simultaneously open and closed, and then it is a union of some connected components of \( M \).

Extending the finite-dimensional case, we have the following result.

**Theorem 6.2.** Let \( \mathcal{G} \to M \) be a Banach-Lie groupoid. Then the following properties are equivalent:

(i) The groupoid \( \mathcal{G} \to M \) is locally transitive.

(ii) The map \((s, t) : \mathcal{G} \to M \times M\) is a submersion.

(iii) The orbits of \( \mathcal{G} \) are unions of connected components of \( M \).

(iv) The anchor \( \rho : A\mathcal{G} \to TM \) is split and surjective.

If the above conditions are satisfied, then \( \mathcal{G} \) is a split Banach-Lie groupoid and the map \( t_x : \mathcal{G}(x, -) \to \mathcal{G}.x \) is a \( \mathcal{G}(x) \)-principal bundle for every \( x \in M \).

**Proof.** (i) \(\Rightarrow\) (ii): Denoting \( \alpha := (s, t) : \mathcal{G} \to M \times M \)

we must check that the map \( T_g\alpha : T_g\mathcal{G} \to T_xM \times T_yM \) is surjective and its kernel is a split subspace of \( T_g\mathcal{G} \) for arbitrary \( g \in \mathcal{G} \) with \( \alpha(g) =: (x, y) \).

To this end first note that the hypothesis implies via (6.1) that \( \mathcal{G}(u, v) = t^{-1}_u(v) \) is a submanifold of \( \mathcal{G} \) for all \( u, v \in M \) with \( \mathcal{G}(u, v) \neq \emptyset \). That is, the n.n.H. Banach-Lie groupoid \( \mathcal{G} \to M \) is split. Moreover,

\[
\ker T_g\alpha = T_g(\alpha^{-1}(\alpha(g))) = T_g(\alpha^{-1}(x, y)) = T_g(\mathcal{G}(x, y))
\]

hence this is a split subspace of \( T_g\mathcal{G} \) since we have just seen that \( \mathcal{G}(x, y) \) is a submanifold of \( \mathcal{G} \).
It remains to prove that the map $T_y\alpha: T_y\mathcal{G} \to T_x M \times T_y M$ is surjective. To this end it suffices to check that the range of $T_y\alpha$ contains both linear subspaces $T_x M \times \{0\}$ and $\{0\} \times T_y M$. Our present hypothesis ensures via (6.1) that the map $T|_{\mathcal{G}(x,-)}: \mathcal{G}(x,-) \to M$ is a submersion, hence $T_y t(T_y(\mathcal{G}(x,-))) = T_y M$. This further implies $T_y t(T_y \mathcal{G}) = T_y M$, hence $T_y \alpha(T_y \mathcal{G}) \supseteq \{0\} \times T_y M$.

On the other hand, since $T|_{\mathcal{G}(y,-)}: \mathcal{G}(y,-) \to M$ is a submersion and $s = t \circ i$, where $i: \mathcal{G} \to \mathcal{G}$ is a diffeomorphism, we obtain that $s|_{\mathcal{G}(y,-)}: \mathcal{G}(y,-) \to M$ is a submersion. Then, as above, we obtain $T_y \alpha(T_y \mathcal{G}) \supseteq T_x M \times \{0\}$, and we are done.

\( (\text{ii}) \Rightarrow (\text{iv}) \): Let $g \in \mathcal{G}$ be arbitrary and denote $\alpha(g) = (x, y) \in M \times M$. Since $\alpha$ is a submersion, there exist open sets $U, V \subseteq M$ with $x \in U$ and $y \in V$, for which there exists a smooth map $\sigma: U \times V \to \mathcal{G}$ with $\sigma(x, y) = g$ and $\alpha \circ \sigma = \text{id}_{U \times V \times V}$. In particular, for every $v \in V$ we obtain $s(\alpha(x, v)) = x$ and $t(\alpha(x, v)) = v$. Therefore we obtain the well-defined smooth map

$$
\tau: V \to \mathcal{G}(x,-), \quad \tau(\cdot) := \alpha(x, \cdot),
$$

which is defined on the neighborhood $V$ of $y \in M$ and satisfies $t \circ \tau = \text{id}_V$. Since $y \in \mathcal{G}, x$ is arbitrary, we thus see that the groupoid $\mathcal{G} \rightrightarrows M$ is locally transitive.

\( (\text{ii}) \Rightarrow (\text{iii}) \): This implication is exactly Lemma 6.1.4.

\( (\text{iii}) \Rightarrow (\text{ii}) \): This is clear.

\( (\text{ii}) \Leftrightarrow (\text{v}) \): For every $x \in M$ one has

$$
\rho|_{(\mathcal{G}x)_x} = T_{1_x}(t|_{\mathcal{G}(x,-)}): T_{1_x}(\mathcal{G}(x,-)) \to T_x M
$$

(see for instance Remark 4.20), and then it is clear that (\text{v}) is equivalent to (\text{ii}), which is further equivalent to (\text{iv}).

Now the last part is a direct application of Theorem 3.3.3 and the fact that each $\mathcal{G}_N \rightrightarrows N$ is a principal bundle for any connected component $N$ of $M$.

**Remark 6.3.** We note for later use that if $\mathcal{G} \rightrightarrows M$ is a topological groupoid that satisfies the condition (BLG1) of Definition 3.2 (i.e., $\mathcal{G}$ is a n.n.H. Banach manifold and $M$ is a Banach manifold) and for which the map $(s, t): \mathcal{G} \to M \times M$ is a submersion, then the condition (BLG2) follows automatically, that is, the map $s: \mathcal{G} \to M$ is a submersion.

**Remark 6.4.** Let $\mathcal{G} \rightrightarrows M$ be a Banach-Lie groupoid and denote by $M_0$ the set of all $x \in M$ whose corresponding map $t_x : \mathcal{G}(x,-) \to M$ is a submersion. It is clear that $M_0$ is an open subset of $M$. It follows by the discussion after (6.1) that $M_0$ is the set of all points in $M$ whose $\mathcal{G}$-orbits are open. In particular, $M_0$ is an open $\mathcal{G}$-invariant subset of $M$, and the groupoid $\mathcal{G} \rightrightarrows M$ is locally transitive if and only if $M_0 = M$.

The condition $M_0 = M$ cannot be weakened to the condition that $M_0$ intersects every connected component of $M$. That is, if for every connected component $N$ of $M$ there exists $x \in N$ whose corresponding map $t_x : \mathcal{G}(x,-) \to N$ is a submersion, then the groupoid $\mathcal{G} \rightrightarrows M$ need not be locally transitive. Examples in this connection are provided by any action of a Lie group on a connected
manifold $A: G \times M \to M$ with a dense open nontrivial orbit. For instance, the tautological action of the group of all invertible matrices $G = \text{GL}(n, \mathbb{R})$ on $M = \mathbb{R}^n$ has its orbits $\mathbb{R}^n \setminus \{0\}$ and $\{0\}$, and the corresponding groupoid $M \times G \Rightarrow M$ (see subsection 3.2.4) is not locally transitive. A wider perspective on this problem is offered by the following discussion on transitive groupoids.

6.2. Transitive Banach-Lie groupoids

A Banach-Lie groupoid $G \Rightarrow M$ is called transitive if the map $(s, t) : G \to M \times M$ is a surjective submersion.

It follows by Theorem 6.2 that if $G \Rightarrow M$ is a Banach-Lie groupoid then the following properties are equivalent:

(i) The groupoid $G \Rightarrow M$ is transitive.

(ii) $G$ is split and the maps $T(s, t) : TG \to TM \times TM$ and $(s, t) : G \to M \times M$ are surjective.

(iii) The groupoid $G$ is locally transitive and has only one orbit, namely $M$.

If moreover the base $M$ is connected, then it has only one connected component, hence the above conditions are equivalent to the fact that the groupoid is locally transitive, which is further equivalent to any of the following properties:

(iv) The anchor $\rho : AG \to TM$ is split and surjective.

(v) For every $x \in M$ the map $t_{x} : G(x, -) \to M$ is a surjective submersion.

If this is the case, then for every $x \in M$ the map $t_{x} : G(x, -) \to M$ is a $G(x)$-principal bundle.

Consider a locally transitive Banach-Lie groupoid $G \Rightarrow M$. Then the restriction $G_{N} \Rightarrow N$ to any connected component $N$ of $M$ is a Banach-Lie groupoid whose algebroid $AG_{N}$ is the restriction of $AG$ to $N$. Therefore $G_{N} \Rightarrow N$ is a transitive Banach-Lie groupoid which is the gauge groupoid of the principal bundle $t_{x} : G_{N}(x, -) \to N$ for any $x \in N$.

6.3. Banach-Lie algebra bundles and transitive Banach-Lie algebroids

Motivated by [Mac87, Def. 3.3.8] we define a Banach-Lie algebra bundle (for short an LAB) as a Lie algebroid $(L, M, \rho, [\cdot, \cdot])$ with anchor $\rho \equiv 0$, such that for each $x \in M$ there exists a local trivialization $\Psi : \pi^{-1}(U) \to U \times g$ with $x \in U$, where $g$ is a Banach-Lie algebra, and the restriction $\Psi_{y} := \Psi|_{\pi^{-1}(y)} : \pi^{-1}(y) \to \{y\} \times g$ is a Lie algebra isomorphism for all $y \in U$.

A morphism of LAB is a bundle morphism which is a morphism of Lie algebras between each fiber.

As in [Mac87, Prop. 3.3.9], any characteristic subalgebra $\mathfrak{h}$ of $g$ (i.e., $\varphi(\mathfrak{h}) = \mathfrak{h}$ for all automorphism $\varphi$ of $g$) generates a sub-LAB $(K, M, [\cdot, \cdot])$ of $(L, M, [\cdot, \cdot])$. In particular if $\mathfrak{h}$ is the center $Zg$ (resp. the derived ideal $[g, g]$) of $g$ we get an associated sub-LAB denoted $ZL$ (resp. $[L, L]$).

For any Banach bundle $E \to M$, we denote by $\text{End}(E)$ the bundle over $M$ of bundle morphisms of $E$. By same argument as in [Mac87, Sect. 3.3], $\text{End}(E)$ is a LAB with typical fiber $\text{End}(g)$ and provided with the classical
Since we have to build a local trivialization of $A$ on the other hand $T\phi$ that $KAG$ active. Moreover, the restriction of $\rho$ of $\rho$ on the connected component of $M$ that contains $x$. We simply denote by $K := \ker \rho_x$ and $F := F_x$. Therefore we can also identify $A$ with $K \times F$. Moreover for $x$ fixed, $F$ is isomorphic to the model space $M$ of $M$ at $x$, and then the trivialization can be viewed as a map $\Psi : A_U \to \phi(U) \times A$ such that $(U, \phi)$ is a local chart at $x$. We can identify $A_x$ with the typical fiber $A$ of $A$ on the connected component of $M$ that contains $x$.

We simply denote by $K := \ker \rho_x$ and $F := F_x$. Therefore we can also identify $A$ with $K \times F$. Moreover for $x$ fixed, $F$ is isomorphic to the model space $M$ of $M$ at $x$, and then the trivialization can be viewed as a map $\Psi : A_U \to U \times K \times M$. On the other hand $\phi : TM|_U \to \phi(U) \times M$ is a trivialization of $TM$ over $U$. Since we have to build a local trivialization of $A$ whose restriction induces a local trivialization for $K$, without loss of generality, we may assume that $U$ is an open subset of $M$, $TM|_U$ is the trivial bundle $U \times M$ and $A_U$ is the trivial bundle $U \times K \times M$. With this notation, $\rho$ can be written as a map $(y, u) \mapsto (y, \rho_y(u))$ where $y \mapsto \rho_y$ is a map from $U$ to $L(K \times M, M)$ and moreover the restriction of $\rho_x$ to $M$ belongs to $GL(M)$. Thus after shrinking $U$ if necessary, we may assume that the restriction of $\rho_y$ to $M$ belongs to $GL(M)$ for any $y \in U$. We set $\sigma_y = -(\rho_y|_M)^{-1} \circ \rho_y$. Then the map $y \mapsto \sigma_y$ is a smooth map from $U$ to $L(K \times M, M)$ and has the following properties:

- $\sigma_y|_M = -1dM$
- $\ker \sigma_y = \ker \rho_y = (\rho_y|_M)^{-1} \circ \rho_y(K)$.

It follows that $k_y : (k, m) \mapsto (k, \sigma_y(k, m))$ is an isomorphism from $K \times M$ to $(\ker \rho_y) \times M$ and $y \mapsto \kappa_y$ is a smooth map from $U$ to $GL(K)$. This implies that $(y, u) \mapsto (y, \kappa_y^{-1}(u))$ is a trivialization of $U \times K$ whose restriction to $K|_U = \cup_{y \in U} \ker \rho_y$ defines a trivialization of $K|_U$.

Motivated by Theorem 6.2 iv, a split Banach-Lie algebroid $(A, M, \rho, [, ,])$ is called transitive if its anchor $\rho$ is surjective. In this case, as in finite dimension we have the following result.

**Proposition 6.5.** If $(A, M, \rho, [, ,])$ is a transitive Banach-Lie algebroid, then $\ker \rho$ is a Banach subbundle of $A$.

**Proof.** Denote $K = \ker \rho$ and fix some $x \in M$. By hypothesis we have $A_x = \ker \rho_x \oplus F_x$ for a suitable closed subspace $F_x \subseteq A_x$, and $\rho_x$ is surjective. It follows that the restriction $\rho_x|_{F_x}$ is an isomorphism onto $T_xM$. Choose a trivialization $\Psi : A_U \to \phi(U) \times A$ such that $(U, \phi)$ is a local chart at $x$. We can identify $A_x$ with the typical fiber $A$ of $A$ on the connected component of $M$ that contains $x$. We simply denote by $K := \ker \rho_x$ and $F := F_x$. Therefore we can also identify $A$ with $K \times F$. Moreover for $x$ fixed, $F$ is isomorphic to the model space $M$ of $M$ at $x$, and then the trivialization can be viewed as a map $\Psi : A_U \to U \times K \times M$. On the other hand $\phi : TM|_U \to \phi(U) \times M$ is a trivialization of $TM$ over $U$.

Now given a transitive Banach-Lie algebroid $(A, M, \rho, [, ,])$ since the restriction of $\rho$ to $K$ is null and the bracket in restriction to global (resp. local) sections of $K$ is a characteristic algebra and so we get the following result. Therefore $\rho$ is a sub-LAB of $\rho$ called the adjoint LAB of $L$.

In the case of the algebroid $\mathcal{AG}$ of Banach-Lie groupoid $\mathcal{G}$ over $M$, according to section 6.2, the algebroid $\mathcal{AG}$ is transitive if and only if $\mathcal{G}$ is locally transitive. Moreover, the restriction $\mathcal{G}_N$ to any connected component $N$ of $M$ has a structure of principal bundle whose structural Banach-Lie group is the typical
model $G_N$ of the isotropy group (in $G_N$) of any point $x \in N$ and so $G_N$ is a

gauge groupoid of this principal bundle.

6.4. Atiyah exact sequence of a principal bundle

A particularly important transitive Banach-Lie groupoid is the gauge groupoid

of a Banach principal bundle. Therefore we will look for a famous short exact

sequence of Banach algebroids canonically associate to this context: the Atiyah

exact sequence.

Let $\pi : P \to M$ be a Banach principal bundle with structural Banach-Lie

group $G$ and $\tau : E \to P$ be a Banach bundle over $P$. Assume that there exists

a smooth right action $E \times G \to E$, $(\xi, g) \mapsto \xi g$ such that

1. $\xi \mapsto \xi g$ is a bundle isomorphism over the right translation $R_g : P \to P$.

2. $E$ is covered by equivariant trivializations in the sense that around each

   $u_0 \in P$ there is an open set of the form $\mathcal{U} = \pi^{-1}(U)$ where $U$ is a neigh-

   borhood of $\pi(u_0)$ in $M$ and a Banach bundle chart

   $\psi : \mathcal{U} \times E \to E_{\mathcal{U}}$

   which is equivariant in the sense that $\forall u \in \mathcal{U}, \xi \in E, g \in G$ then

   $\psi(ug, \xi) = \psi(u, \xi)g$.

Under these assumptions we have the following result.

**Proposition 6.6.** The quotient set $E/G$ has a canonical structure of Banach

bundle $\hat{\tau} : E/G \to M$ such that the natural projection $q : E \to E/G$ is a

surjective submersion and a bundle morphism over $\pi : P \to M$. Moreover

$\tau : E \to P$ can be identified with the pull back of $E/G \to M$ by $\pi : P \to M$.

The method of proof is the same as that of [Mac05, Prop. 3.1.1] and we only
give the key points which are essential in this Banach context.

**Proof.** We denote by $\hat{\xi}$ the $G$-orbit of any $\xi \in E$, and then we define

$\hat{\tau} : E/G \to M, \quad \hat{\tau}(\hat{\xi}) = (\pi \circ \tau)(\xi)$.

This map is well defined since for all $\xi \in E$ and $g \in G$ one has $\pi(\tau(\xi g)) =

\pi(\tau(\xi)g) = \pi(\tau(\xi))$, where the first equality follows from the above hypoth-

esis (1) while the second equality follows by the fact that $\pi : P \to M$ is a

apricental bundle with its structural group $G$.

We now define a Banach structure on each fiber $\hat{\tau}^{-1}(x)$. If $\hat{\xi}$ and $\hat{\eta}$ belongs to

$\hat{\tau}^{-1}(x)$ then there exists $g \in G$ such that $\xi g$ and $\eta$ such that $\tau(\xi g) = \tau(\eta)$. Thus

the sum $\xi g + \eta$ is well defined and we can define $\hat{\xi} + \hat{\eta} = \hat{\xi g + \eta}$ and $\lambda \hat{\xi} = \lambda \xi$.

It is easy to see that these operations are well defined and endow $\hat{\tau}^{-1}(x)$ with a

vector bundle structure. Now from assumption (2) preceding the statement of

Proposition 6.6, the action of $G$ on $E$ must be proper. Indeed, consider a net
Finally by construction we have \( \hat{\tau} \) as \( \hat{\tau} \in J \). Therefore the set of charts of type \( 2.3 \), the action of \( G \) by \( g \) on \( U \equiv U \) with \( \tau(g,\xi) \) is a trivialization of \( U \). Thus, restriction of \( \psi \) given by \( \psi \) is a trivialization of \( U \). Clearly implies assumption \( 2 \).

Identify \( U \) with \( \tau \) as \( \tau \in J \). For any \( u \in \pi^{-1}(x) \) the restriction \( q_u: E_u \to (E/G)_x \) is a linear map which is surjective. The assumption \( 2 \) implies that \( q_u \) injective and then \( q_u \) is an (algebraic) isomorphism. But \( E_u \) is topological subspace of \( E \) so \( q_u: E_u \to E/G \) is continuous. Therefore \( (E/G)_x \) can be provided with a Banach space structure isomorphic to \( E_u \). Moreover since \( q_u \) is an isomorphism the inclusion of \( (E/G)_x \) in \( E/G \) is continuous.

Next we must show that \( \hat{\tau}: E/G \to M \) is a Banach bundle. Fix some \( x_0 \in M \) and \( u_0 \in E_{x_0} \). According to assumption \( 2 \), consider an equivariant trivialization \( \psi: U \times E \to E_{x_0} \) so that around \( u_0 \) which can be chosen so that \( U \) is a trivialization of \( P \) which is isomorphic to \( U \times G \). For simplicity we can identify \( U \) with \( U \times G \). Then consider the map

\[
\psi^G: U \times E \to (E/G)\big|_U
\]
given by \( \psi^G(x,\xi) = \psi(x,e,\xi) \). Clearly \( \psi^G \) is an injective continuous map. The restriction of \( \psi^G \) to \( \{x\} \times E \) is nothing else but \( q_u \) according to the identification \( U \equiv U \times G \) and so is an isomorphism. From the assumption \( 2 \) and the fact that the action of \( G \) on \( P \) is proper, it follows easily that \( \psi^G \) is a homeomorphism and so we get a chart on \( E/G \). Now consider two such charts \( \psi^G_i: U_i \times E \to (E/G)\big|_{U_i} \) for \( i = 1, 2 \) with \( (E/G)\big|_{U_1} \cap (E/G)\big|_{U_2} \neq \emptyset \) then we have

\[
(\psi^G_1)^{-1} \circ \psi^G_2(x,\xi) = (\psi_1)^{-1} \circ \psi_2(x,e,\xi)
\]

Therefore the set of charts of type \( \psi^G: U \times E \to (E/G)\big|_U \) defines a Banach manifold structure on \( E/G \) which is also a Banach bundle structure on \( M \).

Finally by construction we have \( \hat{\tau} \circ q = \pi \circ \tau \) and since \( q_u \) is an isomorphism it follows that \( E \to P \) is the pullback of \( \hat{\tau}: E/G \to M \) over \( \pi \).

Application of Proposition \([6.6]\) for \( E = TP \): As in finite dimensions (see [Mae03, §3.2]), there is a natural right action of \( G \) on \( TP \) by \( (X,g) \in T_uP \times G \to TR_g(X) \), where \( R_g \) is the right translation by \( g \) on \( P \). This action satisfies the assumptions \( 1 \)–\( 2 \) above. Indeed \( 1 \) is obvious and for \( 2 \) we choose any open set \( U \) such that \( P_U \) can be identified with \( U \times G \). Then, if \( g \) is the Lie algebra of \( G \), the tangent bundle \( TP \) can be identified with \( (U \times M) \times (G \times \mathfrak{g}) \) over \( U \times G \). Then \( X \in T(x,\gamma)P \) can be written as \( (x,\gamma,\dot{x},\bar{\mathfrak{X}}) \), and \( R_g(x,\gamma) = (x,\gamma,g) \). Hence

\[
TR_g(X) \equiv (x,\gamma,g,\dot{x},\text{Ad}_g^{-1}\bar{\mathfrak{X}})
\]

which clearly implies assumption \( 2 \).
Therefore applying Proposition 6.6 to $TP$ we get the following diagram

$$
\begin{array}{ccc}
TP & \longrightarrow & TP/G \\
\downarrow q & & \downarrow p \\
P & \longrightarrow & M
\end{array}
$$

The vector bundle $TP/G$ is known as the **Atiyah bundle** and was firstly introduced by Atiyah in [At57]. Since $TP \to P$ is the pullback of $TP/G \to M$ each local or global section of this last bundle gives rise to a local or global pull-back section of $TP \to P$ which is $G$-invariant and conversely. Therefore the $G$ invariant local or global vector fields on $P$ can be identified with sections of $TP/G \to M$.

Now recall that the gauge groupoid of $P$ is the quotient set $(P \times P)/G$ (see Example 3.2.6). Since any invariant local vector fields on $P$ can be identified with local sections of $TP/G \to M$; from the construction of the algebroid of a groupoid, it follows that the Lie algebroid associated to the groupoid $(P \times P)/G$ is exactly $TP/G \to M$ and the associated anchor will denoted $\rho : TP/G \to TM$ in the sequel. In fact $\rho$ is induced from the canonical map $T\pi : TP \to TM$ in restriction to $G$-invariant vectors and so $\rho$ is surjective.

On the one hand consider the vertical subbundle $VP \to M$ of $TP \to M$ that is the kernel of $T\pi$. Since $\pi \circ R_g = \pi$ for any $g \in G$ it follows that $VP$ is $G$-invariant.

On the other hand as in finite dimension by formally same arguments we can show that $VP$ is isomorphic to $P \times g$. (See for instance the proof of [Mac05, Prop. 3.2.2].) Now according to (6.3) we can identify $VP$ with $P \times g$ with the action $(x, \gamma, X)g = (x, \gamma, g, \Ad_g^{-1}X)$ and so the assumption of Proposition 6.6 are satisfied for $E = P \times g$ we get a Banach bundle $(P \times g)/G \to M$ which is in fact a subbundle of $TP/G \to M$ which is the kernel of $\rho$. Note that $(P \times g)$ provided with the induced bracket $[\cdot, \cdot]$ of the Lie algebroid $TP/G$ is a LAB. We finally the following exact sequence of Banach bundles over $M$

$$
0 \longrightarrow (P \times g)/G \overset{j}{\longrightarrow} TP/G \overset{\rho}{\longrightarrow} TM \longrightarrow 0
$$

which is called the **Atiyah sequence** of $P$.

### 7. Perspectives on generalized inverses in Banach algebras

In this final section of the present paper we show that the preceding theory of Banach-Lie groupoids sheds fresh light on the generalized inverses in Banach algebras, in particular on Moore-Penrose inverses in $C^*$-algebras, a research area that has been rather active. From the extensive literature that is available, the most relevant references for our present paper include [HM92], [Ko01], [AC04], [ACM05], [Boa06], [ACG08], [LR12], [AM13].

We begin by a general construction of groupoids associated to semigroups, which we will afterwards specialize to the multiplicative semigroups underlying the associative Banach algebras.
7.1. Groupoids associated to semigroups

Lemma 7.1. Let \((A, \cdot)\) be an arbitrary semigroup and define

\[
Q(A) := \{ a \in A \mid a^2 = a \} \quad \text{and} \quad G(A) := \{ (a, b) \in A \times A \mid aba = a, \ bab = b \}.
\]

Then one has a groupoid \(G(A) \rightrightarrows Q(A)\) with its source/target maps

\[
s, t : G(A) \to Q(A), \quad s(a, b) := ba, \ t(a, b) := ab
\]

with its multiplication \((a_1, b_1) \cdot (a_2, b_2) := (a_1a_2, b_2b_1)\) if \(s(a_1, b_1) = t(a_2, b_2)\), and with its inversion map \((a, b) \mapsto (b, a)\).

Proof. We only need to check that the source/target maps and the multiplication indeed take values in \(Q(A)\) and \(G(A)\), respectively. For the source and target maps we note that if \((a, b) \in A \times A\) satisfy \(aba = a\) and \(bab = b\), then \((ab)^2 = ab\) and \((ba)^2 = ba\), hence \(ab, ba \in Q(A)\).

For the multiplication, the condition \(s(a_1, b_1) = t(a_2, b_2)\) is equivalent to \(b_1a_1 = a_2b_2\), and then one obtains \((a_1a_2, b_2b_1) \in G(A)\) since

\[
(a_1a_2)(b_2b_1)(a_1a_2) = a_1a_2b_2(2a_2b_2)a_2 = a_1(a_2b_2)^2a_2 = a_1(a_2b_2)a_2 = a_1a_2
\]

where the first equality follows by \(b_1a_1 = a_2b_2\), the third equality follows by the property \((a_2b_2)^2 = a_2b_2\) (which is a consequence of \((a_2, b_2) \in G(A)\) as we have already seen above), and the fourth equality follows by \((a_2, b_2) \in G(A)\).

Lemma 7.2. Let \(A\) be a \(*\)-semigroup and define

\[
P(A) := \{ a \in Q(A) \mid a^\ast = a \} \quad \text{and} \quad J(A) := \{ a \in A \mid aa^* a = a \}.
\]

Then one has a groupoid \(J(A) \rightrightarrows P(A)\) with its source/target maps

\[
s, t : J(A) \to P(A), \quad s(a) := a^* a, \ t(a) := aa^*
\]

with its multiplication obtained as the restriction of the multiplication of \(A\) and with its inversion map \(a \mapsto a^\ast\). Moreover there is the injective morphism of groupoids \(J(A) \to G(A), \ a \mapsto (a, a^\ast)\).

Proof. The structure \(J(A) \rightrightarrows P(A)\) is a groupoid by [Law98, §4.2, Th. 3], and it is straightforward that the map \(a \mapsto (a, a^\ast)\) is an injective morphism of groupoids.

7.2. Generalized inverses and groupoids associated to Banach algebras

For any associative algebra \(A\), an element \(a \in A\) is called regular if \(a \in aAa\), and if this is the case then every element \(b \in A\) with \(a = aba\) is called a generalized inverse of \(a\). The generalized inverse of a regular element is not uniquely determined in general, and it is therefore difficult to extend the classical continuity and differentiability properties of the inversion mapping from invertible elements to regular elements.
A way out of the above difficulty is to regard $A$ as a multiplicative semigroup, and to consider the set $G(A)$ from Lemma 7.1, that is, the set of all pairs of regular elements $(a, b) \in A \times A$ for which $b$ is a generalized inverse of $a$ and $a$ is a generalized inverse of $b$. When $A$ is a Banach algebra, the differential geometry of the set $G(A)$ was investigated in [ACM05], and we will show below that the corresponding results have their natural place in the theory of Banach-Lie groupoids. To this end we specialize the construction of Lemma 7.1 for the multiplicative semigroups defined by associative Banach algebras. See for instance [Be06, App. A] and the references therein for real analytic mappings on Banach manifolds.

**Theorem 7.3.** If $A$ is a unital associative Banach algebra, then $G(A) \Rightarrow Q(A)$ is a real analytic Banach-Lie groupoid. Moreover, this groupoid is locally transitive and its isotropy group at $1 \in Q(A)$ is the Banach-Lie group $A^{\times}$ of invertible elements of $A$.

**Proof.** It follows by [ACM05, Cor. 1.3 and Th. 1.5] and [CPR90, Cor. 1.5] that both $G(A)$ and $Q(A)$ are real analytic submanifolds of $A$, and in particular their tangent spaces at any point are split closed linear subspaces of $A$. Since the multiplication map $A \times A \to A$, $(a, b) \mapsto ab$, is clearly real analytic, it then follows by [Bou71], 5.8.5 that the structure maps of the groupoid $G(A) \Rightarrow Q(A)$ are real analytic.

Moreover, the map $(s, t) : G(A) \to P(A) \times P(A)$ is a submersion by [ACM05, Prop. 1.12] and [CPR90, Th. 2.1], hence by Remark 6.3 we obtain that $G(A) \Rightarrow P(A)$ is a locally transitive Banach-Lie groupoid.

**Remark 7.4.** From the perspective of Theorem 7.3 it is natural to ask if, in the case when $A$ is endowed with a continuous involution, the corresponding groupoid $J(A) \Rightarrow P(A)$ given by Lemma 7.2 is a Banach-Lie groupoid and moreover if it is a Banach-Lie subgroupoid of $G(A) \Rightarrow Q(A)$. For general associative Banach $*$-algebras, it is not difficult to check that $P(A)$ is a submanifold of $Q(A)$, since it is the fixed-point set of the involutive diffeomorphism $a \mapsto a^*$ of $Q(A)$. However it is less clear how $J(A)$ should be given a manifold structure with respect to which the source/target maps of the groupoid $J(A) \Rightarrow P(A)$ would be submersions.

We will see below that the above questions can be satisfactorily answered in the important case of $C^*$-algebras, but it would be interesting to understand what happens for other important examples of associative Banach $*$-algebras, as for instance the restricted Banach algebra related to the restricted Grassmann manifold from [PS86, Sect. 6.2]. (See also [BTR07, Sect. 6] for the contrast between the restricted Banach algebra and the $C^*$-algebras.)

### 7.3. The special case of $C^*$-algebras

The following definition was suggested by [OS16] and is a specialization of the construction from Lemma 7.2. (We recall that in the special case of the matrix algebra $A = M_n(C)$ or bounded operators on Hilbert space $A = B(H)$)
there is a one-to-one correspondence from the orthogonal projections \( p = p^2 = p^* \in \mathfrak{A} \) onto the linear subspaces of \( \mathbb{C}^n \) or \( \mathcal{H} \), and this why the set of orthogonal projections defined below for any \( C^* \)-algebra \( \mathfrak{A} \) is called the Grassmann manifold associated with \( \mathfrak{A} \), see also e.g. [PR87]. We slightly change the notation of Lemma 7.2 in order to emphasize the importance of the class of \( C^* \)-algebras among the Banach +-algebras.

**Definition 7.5.** For any \( C^* \)-algebra \( \mathfrak{A} \) we introduce the following subsets:

- \( \mathcal{P}(\mathfrak{A}) := \{ p \in \mathfrak{A} \mid p = p^2 = p^* \} \) (the Grassmann manifold of \( \mathfrak{A} \));
- \( \mathcal{U}(\mathfrak{A}) := \{ a \in \mathfrak{A} \mid aa^*, a^*a \in \mathcal{P}(\mathfrak{A}) \} \) (the set of partial isometries in \( \mathfrak{A} \)).

The *groupoid associated to \( \mathfrak{A} \) is \( \mathcal{U}(\mathfrak{A}) \rightrightarrows \mathcal{P}(\mathfrak{A}) \) with the following structure maps:

- the target/source maps \( t, s : \mathcal{U}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{A}) \), \( t(a) = aa^*, s(a) = a^*a \);
- the inversion map \( i : \mathcal{U}(\mathfrak{A}) \to \mathcal{U}(\mathfrak{A}) \), \( i(a) = a^* \);
- the composition defined on \( \mathcal{U}(\mathfrak{A})^{(2)} := \{ (a, b) \in \mathcal{U}(\mathfrak{A}) \times \mathcal{U}(\mathfrak{A}) \mid s(a) = t(b) \} \) by \( \mu : \mathcal{U}(\mathfrak{A})^{(2)} \to \mathcal{U}(\mathfrak{A}) \), \( \mu(a, b) := ab \).

If \( \mathfrak{B} \) is another \( C^* \)-algebra and \( \varphi : \mathfrak{A} \to \mathfrak{B} \) is a \( * \)-morphism, then \( \mathcal{U}(\varphi) := \varphi|_{\mathcal{U}(\mathfrak{A})} \).

**Theorem 7.6.** For any unital \( C^* \)-algebra \( \mathfrak{A} \) its corresponding groupoid \( \mathcal{U}(\mathfrak{A}) \rightrightarrows \mathcal{P}(\mathfrak{A}) \) is a real analytic Banach-Lie groupoid. Moreover, this groupoid is locally transitive. The isotropy group of the above groupoid at \( 1 \in \mathcal{P}(\mathfrak{A}) \) is the Banach-Lie group of unitary elements of \( \mathfrak{A} \).

**Proof.** It follows by [ACM05, Prop. 3.3] and [PR87, (3)] that both \( \mathcal{U}(\mathfrak{A}) \) and \( \mathcal{P}(\mathfrak{A}) \) are real analytic submanifolds of \( \mathfrak{A} \), and in particular their tangent spaces at any point are split closed linear subspaces of \( \mathfrak{A} \). Since the multiplication map \( \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}, (a, b) \mapsto ab \), and the adjoint \( \mathfrak{A} \to \mathfrak{A}, a \mapsto a^* \), are clearly real analytic, it then follows by [Bou71, 5.8.5] that the structure maps of the groupoid \( \mathcal{U}(\mathfrak{A}) \rightrightarrows \mathcal{P}(\mathfrak{A}) \) are real analytic.

Moreover, the map \( (s, t) : \mathcal{U}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{A}) \times \mathcal{P}(\mathfrak{A}) \) is a submersion by [ACM05, Prop. 3.4], hence by Remark 5.3 we obtain that \( \mathcal{U}(\mathfrak{A}) \rightrightarrows \mathcal{P}(\mathfrak{A}) \) is a locally transitive Banach-Lie groupoid.

Finally, the isotropy group of the groupoid \( \mathcal{U}(\mathfrak{A}) \rightrightarrows \mathcal{P}(\mathfrak{A}) \) at \( 1 \in \mathcal{P}(\mathfrak{A}) \) is

\[
(\mathcal{U}(\mathfrak{A}))(1) = \{ a \in \mathcal{U}(\mathfrak{A}) \mid s(a) = t(a) = 1 \} = \{ a \in \mathfrak{A} \mid a^*a = aa^* = 1 \}
\]

which is exactly the unitary group of \( \mathfrak{A} \). Since \( \mathcal{U}(\mathfrak{A}) \rightrightarrows \mathcal{P}(\mathfrak{A}) \) is a Banach-Lie groupoid, it follows by Theorem 5.3.11 that all its isotropy groups are Banach-Lie groups. However, in the special case of the unitary group of a \( C^* \)-algebra \( \mathfrak{A} \), it is well known that this is a Banach-Lie group. This follows for instance from the fact that the unitary group of \( \mathfrak{A} \) is an algebraic subgroup (of degree \( \leq 2 \)) of the group of invertible elements of \( \mathfrak{A} \), hence one can use [Ped6 Th. 4.13 and Ex. 2.21]. This completes the proof.
Remark 7.7. Let \( \text{CSTAR} \) be the category of \( C^* \)-algebras and \( \text{GRPD} \) be the category of Banach-Lie groupoids. Then it is easily checked that the correspondence \( U: \text{CSTAR} \to \text{GRPD} \) is a functor.

Since the above functor takes values in the category of Banach-Lie groupoids, we can also compose it with the functor that associates to every Banach-Lie groupoid its Lie algebroid. In this connection we note that the differentiable structures of the source-fibers of the groupoid \( U(\mathfrak{A}) \) were discussed in [AC04].

7.4. Moore-Penrose inverse in \( C^* \)-algebras

The research on Moore-Penrose inverses in \( C^* \)-algebras and even in more general Banach algebras has been rather active. We will briefly discuss here the relation between some of these recent results and the theory of Banach-Lie groupoids that we developed in this paper. In particular, we show that a part of the operator theoretic research in this area can be cast in a natural way in the framework of groupoids.

For the sake of simplicity we will discuss here Moore-Penrose invertibility only in \( C^* \)-algebras. If \( \mathfrak{A} \) is a unital \( C^* \)-algebra, its set of regular elements is denoted by

\[
\mathfrak{A}^\dagger := \{ a \in \mathfrak{A} \mid a \in a\mathfrak{A}a \}.
\]

It follows by [HM92, Th. 6] that \( \mathfrak{A}^\dagger \) is exactly the set of all \( a \in \mathfrak{A} \) for which there exists a Moore-Penrose inverse, that is, a unique element \( a^\dagger \in \mathfrak{A} \) satisfying

\[
\begin{align*}
(aa^\dagger)a &= a, \\
(a^\dagger a)a^\dagger &= a^\dagger, \\
(a^\dagger a)^* &= a^\dagger a, \\
(aa^\dagger)^* &= aa^\dagger.
\end{align*}
\]

It then follows that for every \( a \in \mathfrak{A}^\dagger \) one has \( a^\dagger \in \mathfrak{A}^\dagger \) and \( (a^\dagger)^\dagger = a \). We recall from [Ko01, Ex. 1.1] that in general \( \mathfrak{A}^\dagger \) is not an open subset of \( \mathfrak{A} \). It is also known that although the mapping \( \mathfrak{A}^\dagger \to \mathfrak{A}^\dagger, a \mapsto a^\dagger \), is well-defined, in general it is not continuous (cf. [LR12, Sect. 3]). Also, if \( a, b \in \mathfrak{A}^\dagger \) then we may have \( ab \notin \mathfrak{A}^\dagger \). These observations show that both the algebraic and analytic structures of the set \( \mathfrak{A}^\dagger \) are pathological in some sense, unlike the group \( \mathfrak{A}^\times \) of all invertible elements, which is always an open subset of \( \mathfrak{A} \) and is a Banach-Lie group.

Despite the above aspects, it is clearly desirable to have a framework in which the Moore-Penrose inversion has better continuity and differentiability properties. One possible approach to that problem is to understand the relation between the Moore-Penrose inversion and the locally transitive Banach-Lie groupoids from Theorems 7.3 and 7.6. To conclude this paper, we take a very first step in that direction.

**Proposition 7.8.** For every unital \( C^* \)-algebra \( \mathfrak{A} \) the following assertions hold:

(i) The mappings \( \eta: \mathfrak{A}^\dagger \to \mathcal{G}(\mathfrak{A}), a \mapsto (a, a^\dagger), \) and \( \pi: \mathcal{G}(\mathfrak{A}) \to \mathfrak{A}^\dagger, (a, b) \mapsto a \), are well defined and \( \pi \circ \eta = \text{id}_{\mathfrak{A}^\dagger} \).

(ii) For any sequence \( \{a_n\}_{n \geq 1} \) in \( \mathfrak{A}^\dagger \setminus \{0\} \) and any \( a \in \mathfrak{A}^\dagger \setminus \{0\} \) with \( \lim_{n \to \infty} a_n = a \) one has \( \lim_{n \to \infty} \eta(a_n) = \eta(a) \) if and only if \( \lim_{n \to \infty} s(\eta(a_n)) = s(\eta(a)) \).
(iii) One has $\mathcal{U}(A) \subseteq A^\dagger$ and the map $\eta|_{\mathcal{U}(A)} : \mathcal{U}(A) \to G(A)$ is an injective morphism of Banach-Lie groupoids.

**Proof.** Assertion (i) is straightforward.

Assertion (ii) follows by [Ko01, Th. 1.6].

Assertion (iii) follows by the well-known fact that every partial isometry in $A$ is a regular element and, more exactly, one has $\mathcal{U}(A) = \{a \in A^\dagger \mid a^\dagger = a^*\}$. This completes the proof.

**Acknowledgment**

We wish to thank Anatol Odzijewicz and Aneta Sliżewska for their helpful suggestions at the beginning of this project. We extend our thanks to the Referee for numerous suggestions that greatly helped us improve the exposition and even some results of our paper. The authors D. Beltită and F. Pelletier also acknowledge financial support from the Centre Francophone en Mathématiques de Bucarest and the GDRI ECO-Math.

**References**

[An11] M. Anastasiei, *Banach-Lie algebroids*. An. Științ. Univ. Al. I. Cuza Iași. Mat. (N.S.) 57 (2011), no. 2, 409–416.

[At57] M.F. Atiyah, *Complex analytic connections in fibre bundles*. Trans. Amer. Math. Soc. 85 (1957), 181–207.

[AC04] E. Andruchow, G. Corach, *Differential geometry of partial isometries and partial unitaries*. Illinois J. Math. 48 (2004), no. 1, 97–120.

[ACM05] E. Andruchow, G. Corach, M. Mbekhta, *On the geometry of generalized inverses*. Math. Nachr. 278 (2005), no. 7–8, 756–770.

[ACG08] M.L. Arias, G. Corach, M.C. Gonzalez, *Generalized inverses and Douglas equations*. Proc. Amer. Math. Soc. 136 (2008), no. 9, 3177–3183.

[AM13] M.L. Arias, M. Mbekhta, *A-partial isometries and generalized inverses*. Linear Algebra Appl. 439 (2013), no. 5, 1286–1293.

[BG08] M. Baillif, A. Gabard *Manifolds: Hausdorffness versus homogeneity*. Proc. Amer. Math. Soc. 136 (2008), no. 3, 1105–1111.

[Be06] D. Beltită, “Smooth homogeneous structures in operator theory”. Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, 137. Chapman & Hall/CRC, Boca Raton, FL, 2006.

[BTR07] D. Beltită, T.S. Ratiu, A.B. Tumpach, *The restricted Grassmannian, Banach Lie-Poisson spaces, and coadjoint orbits*. J. Funct. Anal. 247 (2007), no. 1, 138–168.
[Boa06] E. Boasso, *On the Moore-Penrose inverse in $C^*$-algebras*. Extracta Math. **21** (2006), no. 2, 93–106.

[Bou71a] N. Bourbaki, “Topologie générale”. Chapitres 1 à 4. Hermann, Paris, 1971.

[Bou71b] N. Bourbaki, “Variétés différentielles et analytiques”. Fascicule de résultats. Springer, 1971.

[Bou72] N. Bourbaki, “Groupes et algèbres de Lie”. Chapitres II et III. Actualités Scientifiques et Industrielles, No. 1349. Hermann, Paris, 1972.

[CP12] P. Cabau, F. Pelletier, *Almost Lie structures on an anchored Banach bundle*. J. Geom. Phys. **62** (2012), no. 11, 2147–2169.

[CPR90] G. Corach, H. Porta, L. Recht, *Differential geometry of systems of projections in Banach algebras*. Pacific J. Math. **143** (1990), no. 2, 209–228.

[CF03] M. Crainic, R.L. Fernandes, *Integrability of Lie brackets*. Ann. of Math. (2) **157** (2003), no. 2, 575–620.

[CF11] M. Crainic, R.L. Fernandes, *Lectures on integrability of Lie brackets*. In: “Lectures on Poisson geometry”, Geom. Topol. Monogr., 17, Geom. Topol. Publ., Coventry, 2011, pp. 1–107.

[En89] R. Engelking, “General topology”. Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989.

[G15] H. Glöckner, *Fundamentals of submersions and immersions between infinite-dimensional manifolds*, Preprint arXiv:1502.05795v4 [math.DG].

[HM92] R. Harte, M. Mbekhta, *On generalized inverses in $C^*$-algebras*. Studia Math. **103** (1992), no. 1, 71–77.

[Ho75] K.H. Hofmann, *Théorie directe des groupes de Lie*. II. Séminaire P. Dubreil (27e année: 1973/74), Algèbre, Fasc. 1, Exp. No. 2. 16 pp. Secrétariat Mathématique, Paris, 1975.

[Ko01] J.J. Koliha, *Continuity and differentiability of the Moore-Penrose inverse in $C^*$-algebras*. Math. Scand. **88** (2001), no. 1, 154–160.

[KM97] A. Kriegl, P.W. Michor, “The convenient setting of global analysis”. Mathematical Surveys and Monographs, 53. American Mathematical Society, Providence, RI, 1997.

[Lan01] S. Lang, “Fundamentals of differential geometry”. Springer, 2001.

[Law98] M.V. Lawson, “Inverse semigroups”. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.
[LR12] J. Leiterer, L. Rodman, *Smoothness of generalized inverses*. Indag. Math. (N.S.) **23** (2012), no. 3, 487–521.

[Mac87] K. Mackenzie “Lie groupoids and Lie algebroids in differential geometry”. London Mathematical Society Lecture Note Series, 124. Cambridge University Press, Cambridge, 1987.

[Mac05] K.C.H. Mackenzie, “General theory of Lie groupoids and Lie algebroids.” London Mathematical Society Lecture Note Series, 213. Cambridge University Press, Cambridge, 2005.

[MO92] J. Margalef Roig, E. Outerelo Domínguez, “Differential topology.” North-Holland Mathematics Studies, 173. North-Holland Publishing Co., Amsterdam, 1992.

[Mar08] E. Martínez *Variational calculus on Lie algebroids*. ESAIM Control Optim. Calc. Var. **14** (2008), no. 2, 356–380.

[MM03] I. Moerdijk, J. Mrčun, “Introduction to foliations and Lie groupoids”. Cambridge University Press, 2003.

[OJS15] A. Odzijewicz, G. Jakimowicz, A. Sliżewska, *Banach-Lie algebroids associated to the groupoid of partially invertible elements of a W*-algebra*. J. Geom. Phys. **95** (2015), 108–126.

[OS16] A. Odzijewicz, A. Sliżewska, *Groupoids and inverse semigroups associated to W*-algebras*. J. Symplectic Geom. **14** (2016), no. 3, 687–736.

[Pe12] F. Pelletier, *Integrability of weak distributions on Banach manifolds*. Indag. Math. (N.S.) **23** (2012), no. 3, 214–242.

[Ph87] J. Philips, *The holonomic imperative and the homotopy groupoid of a foliated manifold*. Rocky Mountain J. Math. **17** (1987), no. 1, 151–165.

[PR87] H. Porta, L. Recht, *Minimality of geodesics in Grassmann manifolds*. Proc. Amer. Math. Soc. **100** (1987), no. 3, 464–466.

[PS86] A. Pressley, G. Segal, “Loop groups.” Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1986.

[SW15] A. Schmeding, C. Wockel, *The Lie group of bisections of a Lie groupoid*. Ann. Global Anal. Geom. **48** (2015), no. 1, 87–123.

[SW16] A. Schmeding, C. Wockel, *(Re)constructing Lie groupoids from their bisections and applications to prequantisation*. Differential Geom. Appl. **49** (2016), 227–276.