EXT ALGEBRA OF NICHOLS ALGEBRAS OF TYPE $A_2$

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Abstract. We give the full structure of the Ext algebra of a Nichols algebra of type $A_2$ by using the Hochschild-Serre spectral sequence. As an application, we show that the pointed Hopf algebras $u(D, \lambda, \mu)$ with Dynkin diagrams of type $A$, $D$, or $E$, except for $A_1$ and $A_1 \times A_1$ with the order $N_J > 2$ for at least one component $J$, are wild.

Introduction

For an algebra $R$ over a field $k$, its homological properties, such as the Calabi-Yau property [14], AS-regularity [16], support varieties [21], etc. rely exclusively on the structure of its Ext algebra $\text{Ext}^*_R(k, k)$.

Nichols algebras play an important role in the classification of pointed Hopf algebras [3, 4, 5, 11]. They are braided Hopf algebras in certain braided monoidal categories. In [5], the authors showed that if $H$ is a finite dimensional pointed Hopf algebra such that its coradical is an abelian group with order not divisible by primes less than 11, then $H$ is isomorphic to a deformation of the bosonization of a Nichols algebra of finite Cartan type. Thus the study of Nichols algebras not only helps us to classify pointed Hopf algebras, but also helps us to understand more about the properties of pointed Hopf algebras. In two recent papers [12, 18] support varieties of modules over Hopf algebras are introduced. It turns out that support varieties are useful tools to study homological properties and representations of (braided) Hopf algebras. To define and to compute support varieties over a (braided) Hopf algebra we need first to understand the Ext algebra of the (braided) Hopf algebra. In [11], the author raised the question of when the Ext algebra of a Nichols algebra is still a Nichols algebra. These facts motivate us to study the structure of the Ext algebra of a Nichols algebra in this paper.

As a first attempt to explore the structure of the Ext algebras for further study, we will give the full structure of the Ext algebra of a Nichols algebra of type $A_2$. 

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First we use the Hochschild-Serre spectral sequence to get a basis of the Ext algebra. We then construct the first segment of the minimal projective resolution of $k$ and give the relations that hold in the Ext algebra. We calculate the dimensions to verify that these relations are complete (see Theorems 2.12 and 2.13). The relations are braided commutative, which coincides with what have been proved in [19], where the authors also showed that the cohomology ring of a finite dimensional pointed Hopf algebra of finite Cartan type is finitely generated. Having the generators and relations of the Ext algebra, we can show that the Ext algebra of a Nichols algebra is not a Nichols algebra in general (see Proposition 2.15). However, the quotient algebra of the Ext algebra modulo the ideal generated the nilpotents can be a Nichols algebra (see Proposition 2.16). This partially answers one of the questions raised in [1, Sec. 2.1].

Finite dimensional pointed Hopf algebras with abelian group coradicals have support varieties [12, 19]. For a pointed Hopf algebra $A$ of type $A_2$, the support variety of $k$ over $A$ is isomorphic to the variety of $k$ over the associated graded algebra with respect to a certain filtration of $A$. This can be showed by using the full structure of the Ext algebra of the Nichols algebra of type $A_2$. Finally, we apply our main results to show that in many cases, the pointed Hopf algebras $u(D, \lambda, \mu)$ constructed in [5] are wild (Proposition 2.18).

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1. Preliminaries and Notations

Throughout the paper, we fix an algebraically closed field $k$ with $\text{char} k \neq 2$. All algebras are assumed to be finite dimensional and all modules are assumed to be finitely generated unless otherwise stated.

1.1. Nichols Algebras and pointed Hopf algebra of Cartan type. In [5], the authors classified finite dimensional pointed Hopf algebras whose coradicals are abelian groups. We need the following terminology:

- a finite abelian group $\Gamma$;
- a Cartan matrix $(a_{ij}) \in \mathbb{Z}^{\theta \times \theta}$ of finite type, where $\theta \in \mathbb{N}$;
- a set $\mathcal{X}$ of connected components of the Dynkin diagram corresponding to the Cartan matrix $(a_{ij})$. If $1 \leq i, j \leq \theta$, then $i \sim j$ means that they belong to the same connected component;
• elements \( g_1, \cdots, g_\theta \in \Gamma \) and characters \( \chi_1, \cdots, \chi_\theta \in \hat{\Gamma} \) such that

\[
(1) \quad \chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{ij}}, \quad \chi_i(g_i) \neq 1, \text{ for all } 1 \leq i, j \leq \theta.
\]

The collection \( D(\Gamma, (g_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, (a_{ij})_{1 \leq i, j \leq \theta}) \) is called a \textit{datum of finite Cartan type} for \( \Gamma \).

For simplicity, we define \( q_{ij} = \chi_j(g_i) \). Then equation (1) reads as

\[
(2) \quad q_{ij}q_{ji} = q^{a_{ij}}_{ii}, \quad q_{ii} \neq 1, \text{ for all } 1 \leq i, j \leq \theta.
\]

From now on, we assume that for all \( 1 \leq i \leq \theta \),

\[
(3) \quad q_{ii} \text{ has odd order, and}
\]

the order of \( q_{ii} \) is prime to 3, if \( i \) lies in a component \( G_2 \).

Since \( q_{ij}q_{ji} = q^{a_{ij}}_{ii}, \ 1 \leq i, j \leq \theta \), the order of \( q_{ii} \) is constant in each component \( J \in \mathcal{X} \) of the Dynkin diagram. Let \( N_J \) denote this common order.

Given a datum \( D \), we define a braided vector space as follows. Let \( V \) be a Yetter-Drinfeld module over the group algebra \( k\Gamma \) with basis \( x_i \chi_i g_i, \ 1 \leq i \leq \theta \). Then \( V \) is a braided vector space of diagonal type whose braiding is given by

\[
(4) \quad c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \quad 1 \leq i, j \leq \theta.
\]

Let \( \lambda = (\lambda_{ij})_{1 \leq i < j \leq \theta} \) be a set of scalars, such that

\[
\lambda_{ij} = 0 \text{ if } g_i g_j = 1 \text{ or } \chi_i \chi_j \neq \varepsilon,
\]

where \( \varepsilon \) is the identity in \( \hat{\Gamma} \). The set of scalars \( \lambda = (\lambda_{ij})_{1 \leq i, j \leq \theta} \) are called \textit{linking parameters}. The algebra \( U(D, \lambda) \) is defined to be the quotient Hopf algebra of the smash product \( k\langle x_1, \cdots, x_\theta \rangle \# k\Gamma \) modulo the ideal generated by the following relations

\[
\text{(Serre relations) } \quad (\text{ad}_c x_i)^{1-a_{ij}}(x_j) = 0, \quad 1 \leq i, j \leq \theta, \quad i \neq j, \quad i \sim j,
\]

\[
\text{(linking relations) } \quad x_i x_j - \chi_j(g_i) x_j x_i = \lambda_{ij}(1 - g_i g_j), \quad 1 \leq i < j \leq \theta, \quad i \sim j,
\]

where \( \text{ad}_c \) is the braided adjoint representation defined in [11, Sec. 1.4].

Let \( \Phi \) be the root system corresponding to the Cartan matrix \( (a_{ij}) \) with \( \Pi = \{ \alpha_1, \cdots, \alpha_\theta \} \) a set of fixed simple roots. Let \( \Phi_J, J \in \mathcal{X}, \) be the root system of the component \( J \). Assume that \( \mathcal{W} \) is the Weyl group of the root system \( \Phi \).

We fix a reduced decomposition of the longest element

\[
w_0 = s_{i_1} \cdots s_{i_p}
\]
of $W$ as a product of simple reflections. Then the positive roots $\Phi^+$ are precisely the followings

$$\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1}(\alpha_{i_2}), \ldots, \beta_p = s_{i_1}\cdots s_{i_{p-1}}(\alpha_{i_p}).$$

If $\beta_i = \sum_{j=1}^q m_j \alpha_j$, then we define

$$g_{\beta_i} = g_{\alpha_1}^{m_1} \cdots g_{\alpha_q}^{m_q} \text{ and } \chi_{\beta_i} = \chi_{\alpha_1}^{m_1} \cdots \chi_{\alpha_q}^{m_q}.$$

Similarly, we write $q_{\beta_j \beta_i} = \chi_{\alpha_1}^{\theta_1} \cdots \chi_{\alpha_q}^{\theta_q}$.

Let $x_{\beta_i}, 1 \leq j \leq p$, be the root vectors as defined in [5, Sec. 2.1]. Let $(\mu_\alpha)_{\alpha \in \Phi^+}$ be a set of scalars, such that

$$\mu_\alpha = 0 \text{ if } g_{\alpha_1}^{N_j} = 1 \text{ or } \chi_{\alpha_1}^{N_j} \neq \varepsilon, \alpha \in \Phi^+_J, J \in X.$$

This set of scalars are called root vector parameters. The finite dimensional Hopf algebra $u(D, \lambda, \mu)$ is the quotient of $U(D, \lambda)$ modulo the ideal generated by

$$(\text{root vector relations}) \quad x_{\alpha_1}^{N_j} - u_{\alpha}(\mu), \quad \alpha \in \Phi^+_J, J \in X,$$

where $u_{\alpha}(\mu) \in k\Gamma$ is defined inductively on $\Phi^+$ as in [5, Sec 4.2].

Let $V$ be the braided vector space defined as in (4). The Nichols algebra $B(V)$ associated to $V$ is a braided Hopf algebra in the category of Yetter-Drinfeld modules over $k\Gamma$. By [5, Thm. 5.1], it is generated by $x_1, \ldots, x_\theta$ subject to relations

$$(\text{ad}_c x_i)^{1-a_{ij}} (x_j) = 0, \quad 1 \leq i, j \leq \theta, \quad i \neq j,$$

$$x_{\alpha_1}^{N_j} = 0, \quad \alpha \in \Phi^+_J, J \in X.$$

The details about Nichols algebras can be found in [4].

Corollary 5.2 in [5] showed that the associated graded Hopf algebra $Gr_u(D, \lambda, \mu)$ of the algebra $u(D, \lambda, \mu)$ with respect to the coradical filtration is $u(D, 0, 0)$. Moreover, we have that $B(V) \# k\Gamma \cong u(D, 0, 0)$. More detailed discussion about the algebras $U(D, \lambda)$ and $u(D, \lambda, \mu)$ can be found in [5].

The following set

$$\{x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} | 1 \leq a_i < N_j, \beta_i \in \Phi^+_J, 1 \leq i \leq p\}$$

forms a PBW basis of the Nichols algebra $B(V)$ [5]. As in [19, Sec. 2], define a degree on each element as

$$\deg x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p} = (\sum a_i \text{ht}(\beta_i), a_p, \ldots, a_1) \in \mathbb{N}^{p+1},$$
where \(ht(\beta_i)\) is the height of the positive root \(\beta_i\). That is, if \(\beta_i = \sum_{j=1}^{\theta} m_j \alpha_j\), then \(ht(\beta_i) = \sum_{j=1}^{\theta} m_j\). Order the elements in \(\mathbb{N}^{p+1}\) as follows

\[
(a_{p+1}, a_p, \cdots, a_1) < (b_{p+1}, b_p, \cdots, b_1)
\]

if and only if there is some \(1 \leq k \leq p+1\), such that \(a_i = b_i\) for \(i \geq k\) and \(a_{k+1} < b_{k+1}\).

By [10, Thm. 9.3], similar to the proof of Lemma 2.4 in [19], we obtain the following lemma.

**Lemma 1.1.** In the Nichols algebra \(\mathcal{B}(V)\), for \(j > i\), we have

\[
[x_{\beta_i}, x_{\beta_j}]_{c} = \sum_{a \in \mathbb{N}^{p}} \rho_a x_{\beta_{i,1}}^{a_1} \cdots x_{\beta_{p}}^{a_p},
\]

where \(\rho_a \in k\) and \(\rho_a \neq 0\) only when \(a = (a_1, \cdots, a_p)\) satisfies that \(a_k = 0\) for \(k \leq i\) and \(k \geq j\).

Therefore, if we order PBW basis elements by degree as in [5], we obtain a filtration on the Nichols algebra \(\mathcal{B}(V)\). The associated graded algebra \(\text{Gr}\mathcal{B}(V)\) is generated by the root vectors \(x_{\beta_i}\), \(1 \leq i \leq p\), subject to the relations

\[
[x_{\beta_i}, x_{\beta_j}]_{c} = 0, \text{ for all } i < j;
\]

\[
x_{\beta_i}^{N_j} = 0, \quad \beta_i \in \Phi^+_j, \quad 1 \leq i \leq p.
\]

### 1.2. Complexity and varieties.

We follow the definitions and the notations in [12]. Let \(A\) be a finite dimensional Hopf algebra and \(H^*(A, k) := \text{Ext}_A^*(k, k)\). The vector space \(H^*(A, k)\) is an associative graded algebra under the Yoneda product. The subalgebra \(H_{ev}(A, k)\) of \(H^*(A, k)\) is defined as

\[
H_{ev}(A, k) = \bigoplus_{n=0}^{\infty} H^{2n}(A, k).
\]

The algebra \(H_{ev}(A, k)\) is commutative, since \(H^*(A, k)\) is graded commutative. In the following, we say that a Hopf algebra \(A\) satisfies the assumption \((fg)\) if the following conditions hold:

\(fg1\) The algebra \(H_{ev}(A, k)\) is finitely generated.

\(fg2\) The \(H_{ev}(A, k)\)-module \(\text{Ext}_A^*(M, N)\) is finitely generated for any two finite dimensional \(A\)-modules \(M\) and \(N\).

Under the assumption \((fg)\), the variety \(V_A(M, N)\) for \(A\)-modules \(M\) and \(N\) is defined as

\[
V_A(M, N) := \text{MaxSpec}(H_{ev}(A, k)/I(M, N)),
\]

where \(I(M, N)\) is the annihilator of the action of \(H_{ev}(A, k)\) on \(\text{Ext}_A^*(M, N)\). It is a homogeneous ideal of \(H_{ev}(A, k)\). The support variety of \(M\) is defined

as \( \mathcal{V}_A(M) = \mathcal{V}_A(M, M) \). By [19, Thm 6.3], a finite dimensional pointed Hopf algebra of the form \( u(\mathcal{D}, \lambda, \mu) \) satisfies the assumption \( (\text{fg}) \).

For a graded vector space \( V^\bullet = \oplus_{n \in \mathbb{Z} \geq 0} V^n \), the growth rate \( \gamma(V^\bullet) \) is defined as
\[
\gamma(V^\bullet) = \min \{ c \in \mathbb{Z}, c \geq 0 \mid \exists b \in \mathbb{R}, \text{such that } \dim V^n \leq bn^{c-1}, \text{for all } n \geq 0 \}.
\]

Let \( M \) be an \( A \)-module and \( P_\ast : \cdots \to P_1 \to P_0 \to M \to 0 \) a minimal projective resolution of \( M \). Then the growth rate \( \gamma(P_\ast) \) is defined to be the complexity \( cx_A(M) \) of \( M \).

2. Ext algebras

It is clear that each Nichols algebra can be written as a twisted tensor product of a set of Nichols algebras, such that each of them satisfies that the Dynkin diagram associated to the Cartan matrix is connected. In [6], the authors showed that the Ext algebra of a twisted tensor algebra is essentially the twisted tensor algebra of the Ext algebras. Therefore, we only need to discuss the case where the Dynkin diagram is connected. Now we calculate the Ext algebra of a Nichols algebra of type \( A_2 \).

Let \( N \) be an integer, and let \( \bar{q} \) be a primitive root of 1 of order \( N \). Let \( q_{ij}, 1 \leq i, j \leq 2 \) be roots of 1, such that
\[
q_{11} = q_{22} = \bar{q}, \quad q_{12}q_{21} = \bar{q}^{-1}.
\]

Let \( V \) be a 2-dimensional vector space with basis \( x_1 \) and \( x_2 \), whose braiding is given by
\[
c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i, \quad 1 \leq i, j \leq 2.
\]

Then \( V \) is a braided vector space of type \( A_2 \).

2.1. Case \( N = 2 \). As discussed in [2], the Nichols algebra \( R = \mathcal{B}(V) \) is isomorphic to the algebra generated by \( x_1 \) and \( x_2 \), with relations
\[
x_1x_2x_1x_2 + x_2x_1x_2x_1 = 0, \quad x_1^2 = x_2^2 = 0.
\]

The dimension of \( R \) is 8.

Its Ext algebra can be calculated directly via the minimal projective resolution of \( k \).

Throughout, for an algebra \( R \), we write elements in the free module \( R^n, n \geq 1 \), as row vectors. A morphism \( f : R^m \to R^n \) is described by an \( m \times n \) matrix.
Proposition 2.1. Let \( R = \mathcal{B}(V) \) be the algebra mentioned before, then the algebra \( \text{Ext}^*_R(k, k) \) is generated by \( a_1, a_2 \) and \( b \) with \( \deg a_1 = \deg a_2 = 1 \) and \( \deg b = 2 \), subject to the relations

\[
\begin{align*}
    a_2 a_1 &= a_1 a_2 = 0, \quad a_1 b = b a_1, \quad a_2 b = b a_2.
\end{align*}
\]

Proof. We claim that the following complex is the minimal projective resolution of \( k \).

\[
\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to k,
\]

where \( P_n = \mathbb{R}^{n+1} \) and \( d_n \) is defined as

\[
d_n = \begin{pmatrix}
    x_1 & x_2 & x_1 & x_2 & \cdots & x_2 & x_1 \\
    x_2 x_1 x_2 & x_1 & x_2 & x_1 & \cdots & x_2 & x_1
\end{pmatrix},
\]

when \( n \) is odd and

\[
d_n = \begin{pmatrix}
    x_1 & x_2 & x_1 & x_2 & \cdots & x_2 & x_1 \\
    x_2 x_1 x_2 & x_1 & x_2 & x_1 & \cdots & x_2 & x_1
\end{pmatrix},
\]

when \( n \) is even. Especially, \( d_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). It is routine to check that \((7)\) is indeed a complex. Now we use induction to prove the exactness. It is clear that the minimal projective resolution starts as

\[
\mathbb{R}^3 \xrightarrow{d_2} \mathbb{R}^2 \xrightarrow{d_1} \mathbb{R} \to k \to 0,
\]

where \( d_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \) and \( d_2 = \begin{pmatrix} x_1 & x_2 x_1 x_2 \\ x_2 x_1 x_2 & x_1 x_2 x_1 \end{pmatrix} \). Assume that the complex \((7)\) is exact up to \( P_n \). If \( n \) is odd, then

\[
\dim(\ker d_n) = (1 + \dim P_1 + \dim P_3 + \cdots + \dim P_n) - (\dim P_0 + \dim P_2 + \cdots + \dim P_{n-1}) = 4n + 5.
\]

Since the dimension of \( R \) is small, we can calculate the dimension of the submodule \( \text{Im} d_{n+1} \) of \( P_n \) directly, it is also \( 4n + 5 \). Then the complex is exact at \( P_{n+1} \). If \( n \) is even, by a similar discussion, we can also conclude that the
complex is exact at $P_{n+1}$. In this case $\dim(\ker d_n) = 4n + 7$. We have that $\text{Im} d_n \subseteq \text{rad} P_{n-1}$ for each $i \geq 0$. Therefore, the complex (7) is the minimal projective resolution of $k$. Since $k$ is a simple module, we have

$$\text{(8)} \quad \text{Hom}_R(P_n, k) \cong \text{Ext}^n_R(k, k)$$

as vector spaces for each $n \geq 0$. Let $a_1, a_2 \in \text{Hom}_R(P_1, k)$ be the functions dual to $(1, 0)$ and $(0, 1)$ respectively and $b \in \text{Hom}_R(P_2, k)$ be the function dual to $(0, 1, 0)$.

Let $f_1, g_1$ and $h_1$ be the morphisms described by the following matrices:

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x_1x_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$g_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$h_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then we have the following commutative diagrams:

$$P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow k,$$

$$P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow k,$$

$$P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow k.$$

These commutative diagrams show that the relation listed in the proposition hold.
Let $U$ be the algebra generated by $a_1$, $a_2$ and $b$ subject to the relations listed in the proposition. When $n$ is odd, $U_n$ has a basis
\[ \{a_1^n, a_1^{n-2}b, \cdots, a_1b^{n-1}, a_2b^{n-2}, \cdots, a_2^{n-2}b, a_2^n\} \]
and when $n$ is even, $U_n$ has a basis
\[ \{a_1^n, a_1^{n-2}b, \cdots, a_1b^{n-1}, b^{n-2}b, a_2b^{n-2}, \cdots, a_2^{n-2}b, a_2^n\}. \]
They are functions dual to $(1, 0 \cdots, 0), \cdots, (0, \cdots, 0, 1)$ respectively in the projective resolution $[\mathbf{7}]$. We have
\[
\dim U_n = n + 1 \\
= \dim \text{Hom}_R(P_n/(\text{rad } P_n), k) \\
= \dim \text{Hom}_R(P_n, k) \\
= \dim \text{Ext}^1_R(k, k),
\]
where the last equation follows from equation $[\mathbf{8}]$. So we have $\text{Ext}^1_R(k, k) = U$, which completes the proof of the proposition. \hspace{1cm} \square 

2.2. Case $N \geq 3$. In this case, the Nichols algebra $R = B(V)$ is the algebra generated by $x_1$ and $x_2$ subject to the relations
\[
x_1^2x_2 - (q_{12} + q_{12}q_{11})x_1x_2x_1 + q_{12}^2x_2x_1^2 = 0, \\
x_2^2x_1 - (q_{21} + q_{21}q_{22})x_2x_1x_2 + q_{21}^2x_2x_1^2 = 0, \\
x_1^N = x_2^N = (x_1x_2 - q_{12}x_2x_1)^N = 0. 
\]
The dimension of $R$ is $N^3$.

In the rest of the paper, we set $y = x_1x_2 - q_{12}x_2x_1$. From the above relations, we obtain that
\[
q_{21}x_1y - yx_1 = 0, \quad x_2y - q_{21}yx_2 = 0.
\]
Let $\alpha_1$ and $\alpha_2$ be the two simple roots. The element $\alpha_1\alpha_2\alpha_1$ is a reduced decomposition of the longest element in the Weyl group $W$ and $\{\alpha_1, \alpha_1+\alpha_2, \alpha_2\}$ are the positive roots. The corresponding root vectors are just $x_1$, $y$ and $x_2$.

So the set
\[
\{x_1^{a_1}y^{a_2}x_2^{a_3}, 0 \leq a_i < N, i = 1, 2, 3\}
\]
forms a PBW basis of $R$. The graded algebra $\text{Gr}R$ corresponding to $R$ is isomorphic to the algebra generated by $x_1$, $y$ and $x_2$ subject to the relations
\[
x_1y = q_{21}^{-1}yx_1, \quad x_1x_2 = qx_2x_1, \quad yx_2 = q_{21}^{-1}x_2y, \\
x_1^N = y^N = x_2^N = 0.
\]
We first show that the algebra $\text{Ext}^1_R(k, k)$ is generated in degree 1 and 2.
A connected graded algebra is called a $K_2$ algebra if the algebra $\text{EXT}^*_R(k, k)$ is generated by $\text{EXT}^*_R(k, k)$ and $\text{EXT}^*_R(k, k)$. Here $\text{EXT}^*_R(-, -)$ denotes the functor on graded category [8 Definition. 1.1].

**Remark 2.2.** For a finite dimensional connected algebra $R$, we have

$$\text{Ext}^*_R(k, k) \cong \text{EXT}^*_R(k, k).$$

In the following we just identify them.

Let $S$ be the subalgebra of $R$ generated by $x_1$ and $y$. To be more precise, it is isomorphic to the algebra generated by $x_1$ and $y$ subject to the relations

$$yx_1 = q_{21}x_1y, \quad x_1^N = y^N = 0.$$  

**Lemma 2.3.** The algebra $R = \mathcal{B}(V)$ is $K_2$.

**Proof.** The algebra $R$ is isomorphic to the graded Ore extension $R \cong S[x_2; \sigma, \delta]$, where $\sigma$ is the graded algebra automorphism of $S$ defined by $\sigma(x_1) = q_{12}^{-1}x_1$ and $\sigma(y) = q_{21}y$ and $\delta$ is the degree +1 graded $\sigma$-derivation of $S$ defined by $\delta(x_1) = -q_{12}^{-1}y$ and $\delta(y) = 0$. By [8] Thm 10.2, the $K_2$ property is preserved under graded Ore extension. From [19] Thm. 4.1, we can see that $S$ is $K_2$. Therefore, the algebra $R$ is $K_2$. □

The subalgebra $S$ is a normal subalgebra of $R$ (we refer to [15] Appendix for the definition of normal subalgebras). Now set $\overline{R} = R/(RS^+)$, where $S^+$ is the augmentation ideal of $S$. That is, $\overline{R} = k[x_2]/(x_2^N)$. We use the Hochschild-Serre spectral sequence (cf. [15])

$$E_2^{pq} = \text{Ext}^p_{\overline{R}}(k, \text{Ext}^q_S(k, k)) \implies \text{Ext}^{p+q}_R(k, k)$$

(9)

to calculate the Ext algebra of $R$. We show that $E_2 = E_\infty$.

The spectral sequence is constructed as follows. Let

$$\cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow k \rightarrow 0$$

and

$$\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow k \rightarrow 0$$

be free resolutions of $\overline{R}k$ and $Rk$ respectively. There is a natural $\overline{R}$-module action on $\text{Hom}_S(P_q, k)$ for $q \geq 0$. We form a double complex

$$E_0^{pq} = \text{Hom}_{\overline{R}}(Q_p, \text{Hom}_S(P_q, k)).$$

By taking the vertical homology and then the horizontal homology, we have

$$E_1^{pq} = \text{Hom}_{\overline{R}}(Q_p, \text{Ext}^q_S(k, k))$$
and

$$E_2^{pq} = \text{Ext}_R^p(k, \text{Ext}_S^q(k, k)).$$

Now we construct a free resolution of $k$ over $R$, which is a filtered complex. The corresponding graded complex is the minimal projective resolution of $k$ over $\mathfrak{g} \mathfrak{r} R$.

Let $\sigma, \tau : \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\sigma(a) = \begin{cases} 1, & \text{if } a \text{ is odd;} \\ N - 1, & \text{if } a \text{ is even} \end{cases}$$

and

$$\tau(a) = \begin{cases} \frac{a - 1}{2}N + 1, & \text{if } a \text{ is odd;} \\ \frac{a}{2}N, & \text{if } a \text{ is even}. \end{cases}$$

Let

$$P_* : \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \cdots \to P_1 \to P_0$$

be a complex of free $R$-modules constructed as follows. For each triple $(a_1, a_2, a_3)$, let $\Phi(a_1, a_2, a_3)$ be a free generator for $P_n$ with $n = a_1 + a_2 + a_3$. Set

$$P_n = \oplus_{a_1+a_2+a_3=n} R\Phi(a_1, a_2, a_3)(\tau(a_1) + 2\tau(a_2) + \tau(a_3), \tau(a_3), \tau(a_2), \tau(a_1)).$$

Here, $(\cdot, \cdot, \cdot, \cdot)$ denotes the degree shift. The differentials are defined by

$$\partial(\Phi(a_1, a_2, a_3)) = \begin{cases} (\delta_1 + \delta_2 + \delta_3)(\Phi(a_1, a_2, a_3)), & \text{if } a_2 \text{ is odd;} \\ (\delta_1 + \delta_2 + \tilde{\delta}_2)(\Phi(a_1, a_2, a_3)), & \text{if } a_2 \text{ is even}. \end{cases}$$

The maps $\delta_i$, $1 \leq i \leq 3$ and $\tilde{\delta}_2$ are defined as follows.

Put

$$\begin{align*}
\delta_1(\Phi(a_1, a_2, a_3)) &= x_1^{\sigma(a_1)}(a_1 - 1, a_2, a_3), & \text{if } a_1 > 0; \\
\delta_2(\Phi(a_1, a_2, a_3)) &= (-1)^{a_1}q_{21}^{\sigma(a_2)\tau(a_1)}q_{12}^{\sigma(a_2)}\Phi(a_1, a_2 - 1, a_3), & \text{if } a_2 > 0; \\
\delta_3(\Phi(a_1, a_2, a_3)) &= (-1)^{a_1+a_2}q_{12}^{\sigma(a_3)\tau(a_1)}q_{21}^{\sigma(a_3)}\tau(a_2)x_2^{\sigma(a_2)}\Phi(a_1, a_2, a_3 - 1), & \text{if } a_3 > 0; \\
\tilde{\delta}_2(\Phi(a_1, a_2, a_3)) &= D\Phi(a_1 - 1, a_2 + 1, a_3 - 1), & \text{if } a_1, a_3 > 0, a_2 \text{ is even},
\end{align*}$$

where $D$ is an element in $R$ such that

$$Dy = -q_{21}^{\sigma(a_1)-1}q_{12}^{\sigma(a_3)-1}q_{21}^{-\sigma(a_3)}\tau(a_2)x_1^{\sigma(a_1)}x_2^{\sigma(a_3)}c.$$

The existence of such element $D$ will be explained in Lemma 2.4. For $i = 1, 2, 3$, if $a_i = 0$, set $\delta_i(\Phi(a_1, a_2, a_3)) = 0$. If $a_1 = 0$ or $a_3 = 0$, set $\tilde{\delta}_2(\Phi(a_1, a_2, a_3)) = 0$.

**Lemma 2.4.** The element $y$ is a right divisor of $[x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]$. 
(1) If $a_1, a_3 > 0$ are odd, then
\[
\hat{\delta}_2(\Phi(a_1, a_2, a_3)) = -\frac{a_2}{q_2} x_1 \Phi(a_1 - 1, a_2 + 1, a_3 - 1).
\]

(2) If $a_1 > 0$ is odd and $a_3 > 0$ is even, then
\[
\hat{\delta}_2(\Phi(a_1, a_2, a_3)) = q_2^{-1}(N-1) q_2 x_1 \Phi(a_1 - 1, a_2 + 1, a_3 - 1).
\]

(3) If $a_1 > 0$ is even and $a_3 > 0$ is odd, then
\[
\hat{\delta}_2(\Phi(a_1, a_2, a_3)) = q_2^{-1} x_1 \Phi(a_1 - 1, a_2 + 1, a_3 - 1).
\]

(4) If $a_1, a_3 > 0$ are even, then
\[
\hat{\delta}_2(\Phi(a_1, a_2, a_3)) = -q_2^{-1}(N-1) q_2 x_1 \Phi(a_1 - 1, a_2 + 1, a_3 - 1).
\]

**Proof.** (1) is easy to see. (2) and (3) follow from the following two equations,
\[
[x_1^{N-1}, x_2^{N-1}]_c = (1 + \bar{q}^{-1} + \ldots + \bar{q}^{-N+1})x_1^{N-2}y = -\bar{q}x_1^{N-2}y
\]
and
\[
x_2^{N-1}[x_1^{N-1}, x_2^{N-1}]_c = (1 + \bar{q}^{-1} + \ldots + \bar{q}^{-N+1})x_2^{N-2}y = -\bar{q}y_2^{N-2}y.
\]

For (4), by Lemma 2.5 below, both $\{x_1^{a_1}x_2^{a_2}x_3^{a_3}\}$ and $\{x_2^{a_1}y^{a_2}x_1^{a_3}\}$, $0 \leq a_i < N$, $i = 1, 2, 3$, are bases of $R$. Using an easy induction, we can see that $[x_1^{N-1}, x_2^{N-1}]_c$ can be expressed as
\[
[x_1^{N-1}, x_2^{N-1}]_c = x_1^{N-1}x_2^{N-1} - q_2^{-1}(N-1) x_1^{N-1}x_2^{N-1} - k_1x_1^{N-2}x_2^{N-2} + \ldots + k_{N-2}x_1^{N-2}x_2^{N-2} + k_{N-1}y^{N-1} - k_1x_1^{N-2}x_2^{N-2} + \ldots + k_{N-2}x_1^{N-2}x_2^{N-2} + k_{N-1}y^{N-1},
\]
with $k_i, l_i \in k, 1 \leq i \leq N - 1$. Observe that $y$ commutes with $x_1^tx_2^t$ and $x_2^tx_1^t$ for $t \geq 0$. Then the result follows. \qed
Lemma 2.5. Both the sets
\[ \{x_2^{a_1}y^{a_2}x_1^{a_3}\} \text{ and } \{x_1^{a_1}y^{a_2}x_2^{a_3}\}, \]
0 ≤ a_i < N, i = 1, 2, 3 form bases of the algebra R.

Proof. It is clear for the set \( \{x_1^{a_1}y^{a_2}x_2^{a_3}\} \). For the set \( \{x_2^{a_1}y^{a_2}x_1^{a_3}\} \), it is easy to see that
\[
x_2^{a_1}y^{a_2}x_1^{a_3} = q_1^{a_1+a_2+a_3}q_2^{-a_1}x_1^{a_1}y^{a_2}x_2^{a_3} + \sum_{i=1}^{\min\{a_1,a_2,a_3\}} k_ix_1^{a_1-i}y^{a_2-i}x_2^{a_3-i}
\]
and
\[
x_1^{a_1}y^{a_2}x_2^{a_3} = q_2^{a_1+a_2-a_3}q_1^{a_1}x_2^{a_3}y^{a_2}x_1^{a_1} + \sum_{i=1}^{\min\{a_1,a_2,a_3\}} l_ix_2^{a_3-i}y^{a_2-i}x_1^{a_1-i}
\]
with each \( k_i, l_i \in k \). So \( \{x_2^{a_1}y^{a_2}x_1^{a_3}\} \) also form a basis of R. □

Proposition 2.6. The complex (10) is a projective resolution of \( k \) over \( R \), the corresponding graded complex is the minimal projective resolution of \( k \) over \( \mathcal{G} \mathcal{R} \).

Proof. It is routine to check that (10) is indeed a complex. We see it in Appendix 3.1. The differentials preserve the filtration and the corresponding graded complex is just the minimal projective resolution of \( k \) over \( \mathcal{G} \mathcal{R} \) as constructed in [19, Sec. 4]. Since the filtration is finite, the complex \( P_\bullet \) is exact by [7, Chapter 2, Lemma 3.13]. Therefore, \( P_\bullet \) is a free resolution of \( k \) over \( R \). □

In the following, we will forget the shifting on the modules in the complex (10).

It is clear that it is still a projective resolution of \( k \) over \( R \). The only difference is that the differentials are not of degree 0. We denote this complex by \( P_\bullet \) as well.

It is well-known that the following complex is the minimal projective resolution of \( k \) over \( \mathcal{R} = k[x_2]/(x_2^N) \).

\[
Q_\bullet : \cdots \to \mathcal{R} \xrightarrow{x_2} \mathcal{R} \xrightarrow{x_2} \mathcal{R} \xrightarrow{x_2} \cdots \xrightarrow{x_2} \mathcal{R} \to k.
\]

Therefore, we have
\[
E_0^{pq} = \text{Hom}_{\mathcal{R}}(Q_p, \text{Hom}_S(P_q, k)) = \text{Hom}_S(\oplus_{a_1+a_2+a_3=q} R\Phi(a_1, a_2, a_3), k) = \oplus_{a_1+a_2+a_3=q} R\Phi(a_1, a_2, a_3),
\]
since \( \text{Hom}_S(R, k) \cong \mathcal{R} \). The double complex reads as follows
The vertical differentials are induced from the differentials of the complex (10).

By taking the vertical homology, we have $E_1^{pq} = \text{Hom}_{\mathcal{R}}(Q_p, \text{Ext}_S^q(k, k))$. Following from [19], the algebra $\text{Ext}_S^*(k, k)$ is generated by $u_1, u_y, w_1$ and $w_y$, where $\deg u_1 = \deg u_y = 2$ and $\deg w_1 = \deg w_y = 1$, subject to the relations

$$w_yw_1 = -q_{21}w_1w_y, \quad w_1^2 = w_y^2 = 0,$$

$$w_yu_1 = q_{21}^Ny_1w_y, \quad w_1u_1 = u_1w_1, \quad w_yu_y = u_yw_y, \quad w_1u_y = q_{21}^{-N}u_yw_1,$$

$$u_yu_1 = q_{21}^{-2}u_1u_y.$$

We use the notations $u_i$ and $w_i$ in place of the notations $\xi_i$ and $\eta_i$ used in [19]. Note that $w_1^2 = w_y^2 = 0$ holds since we assume that the characteristic of the field $k$ is 0. It should also be noticed that the Ext algebra in [19] is the opposite algebra here.

As described in the appendix of [14], there is an action of $\overline{R}$ on $\text{Ext}_S^*(k, k)$ given by

$$x_2(u_y) = x_2(u_1) = 0, \quad x_2(w_y) = w_1, \text{ and } x_2(w_1) = 0.$$

This action is a derivation on $\text{Ext}_S^*(k, k)$. That is, $x_2(uw) = x_2(u)w + ux_2(w)$ for $u, w \in \text{Ext}_S^*(k, k)$.

The following lemma gives a basis of $\text{Ext}_S^*(k, \text{Ext}_S^q(k, k))$.

**Lemma 2.7.** As a vector space, $\text{Ext}_S^*(k, \text{Ext}_S^q(k, k))$ has a basis as follows

\[
\begin{align*}
&\begin{cases} 
  u_{i}^{1}u_{j}^{2}w_{1}, & \text{if } 2(i + j) + 1 = q, \quad q \text{ is odd and } p \text{ is even;} \\
  u_{i}^{1}u_{j}^{2}w_{y}, & \text{if } 2(i + j) + 1 = q, \quad q \text{ is odd and } p \text{ is odd;} \\
  u_{i}^{1}u_{j}^{2}(w_{1}w_{y})^{k}, & \text{if } k = 0, 1 \text{ and } 2(i + j) + 2k = q, \quad q \text{ is even.}
\end{cases}
\end{align*}
\]
Proof. Let $E = \text{Ext}_R^n(k, k)$. The lemma follows directly from the following facts:

(i) If $q$ is odd, then \{\(u_i^1 u_j^1 w_1, i, j \geq 0, 2(i + j) + 1 = q\) forms a basis of $x_2 E^q$ and \(\{e \in E^q | x_2 e = 0\} \}$.

(ii) If $q$ is even, then $x_2 E^q = 0$.

(iii) $x_2^{N-1} E = 0$. \(\square\)

Proposition 2.8. The spectral sequence

\[ E_2^{p,q} = \text{Ext}_R^p(k, \text{Ext}_R^q(k, k)) \implies \text{Ext}_R^{p+q}(k, k) \]

satisfies $E_2 = E_\infty$.

Proof. The elements $u_i^1 u_j^1 w_y$ and $u_i^1 u_j^1 w_1$ are represented by

\[ x_2^{N-2} \Phi(2i + 1, 2j, 0) + q_{i_2}^{-(j+1)} x_2^{N-1} \Phi(2i, 2j + 1, 0) \]

and

\[ x_2^{N-1} \Phi(2i + 1, 2j, 0), \]

while $u_i^1 u_j^1 w_y$ and $u_i^1 u_j^1 w_1 w_y$ are represented by

\[ x_2^{N-1} \Phi(2i, 2j, 0) \text{ and } x_2^{N-1} \Phi(2i + 1, 2j + 1, 0) \]

in $E_0$. In other words, all the elements in $E_0$ representing the elements in $E_2$ are mapped to 0 under the horizontal differentials. We conclude that $E_2 = E_\infty$. \(\square\)

We now can determine the dimension of $\text{Ext}_R^*_n(k, k)$. This dimension depends on the parity of $n$.

Corollary 2.9. We have

\[ \dim \text{Ext}_R^n(k, k) = \begin{cases} \frac{3n^2 + 8n + 5}{8}, & \text{if } n \text{ is odd;} \\ \frac{3n^2 + 10n + 8}{8}, & \text{if } n \text{ is even.} \end{cases} \]
Proof. Set $E^n = \oplus_{p+q=n} E^{pq}_2 = \oplus_{p+q=n} \text{Ext}^p_R(k, \text{Ext}^q_S(k, k))$. By Lemma 2.7 we can illustrate the dimensions of $E^{pq}_2$ with the following table:

| ... ... |
|--------|
| 4 4 4 4 4 4 ... |
| 7 7 7 7 7 7 ... |
| 3 3 3 3 3 3 ... |
| 5 5 5 5 5 5 ... |
| 2 2 2 2 2 2 ... |
| 3 3 3 3 3 3 ... |
| 1 1 1 1 1 1 ... |
| 1 1 1 1 1 1 ... |

Therefore, when $n$ is odd,

$$\dim E^n = (1 + 2 + \cdots + \frac{n+1}{2} + 1 + 3 + \cdots + n) = \frac{3n^2 + 8n + 5}{8};$$

When $n$ is even,

$$\dim E^n = (1 + 2 + \cdots + \frac{n}{2} + 1 + 3 + \cdots + n + 1) = \frac{3n^2 + 10n + 8}{8};$$

By Proposition 2.8 we have $E_2 = E_\infty$, so $\dim \text{Ext}^n_R(k, k) = \dim E^n$. This completes the proof. \qed

Now we give the first segment of the minimal projective resolution of a Nichols algebra of type $A_2$.

The algebra $R$ is a local algebra. Thus projective $R$-modules are free. Let

$$R^{n_4} \to R^{n_3} \to R^{n_2} \to R^{n_1} \to R^{n_0} \to k \to 0$$

be the first segment of the minimal projective resolution. Since $k$ is a simple module, we have

$$\dim \text{Ext}^i_R(k, k) = \dim \text{Hom}_R(R^{n_i}, k) = \dim \text{Hom}_R((R/(\text{rad} R))^{n_i}, k) = n_i.$$
From the computation of the dimensions of Ext$_R^*(k, k)$ in Corollary 2.9, we can see that the minimal projective resolution begins as
\[ R^{12} \rightarrow R^7 \rightarrow R^5 \rightarrow R^2 \rightarrow R \rightarrow k \rightarrow 0. \]
We give the differentials in the following proposition.

As in the construction of $\hat{\delta}_2$ in §2.2, let $D$ be the element in $R$ such that $Dy = [x_1^{N-1}, x_2^{N-1}]_c$.

**Proposition 2.10.** Let $R$ be a Nichols algebra of type $A_2$. The following sequence provides the first segment of the minimal projective resolution of $k$ over $R$,
\[(11) \quad R^{12} \xrightarrow{d_4} R^7 \xrightarrow{d_3} R^5 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \rightarrow k \rightarrow 0,\]
where the differentials are given by the following matrices:

\[
d_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
\]
\[
d_2 = \begin{pmatrix}
x_1^{N-1} & 0 \\
-(q_{12} + qq_{12})x_1x_2 + \bar{q}q_{12}^2x_2x_1 & x_2^2 \\
-q_{12}y^{N-1}x_2 & y^{N-1}x_1 \\
x_2^2 & \bar{q}q_{12}^2x_1x_2 - (q_{21} + \bar{q}q_{21})x_2x_1 \\
0 & x_2^{N-1} 
\end{pmatrix},
\]
\[
d_3 = \begin{pmatrix}
x_1 & 0 & 0 & 0 & 0 \\
q_{12}x_2 & x_1^{N-2} & 0 & 0 & 0 \\
0 & 0 & x_2 & q_{12}q_{21}y^{N-1}x_1 & 0 \\
0 & x_2 & 0 & x_1 & 0 \\
0 & -q_{21}y^{N-1}x_1 & 0 & 0 & 0 \\
0 & 0 & 0 & q_{12}x_2^{N-2} & x_1 \\
0 & 0 & 0 & 0 & x_2 
\end{pmatrix},
\]
\[
d_4 = \begin{pmatrix}
A_1 \\
A_2 \\
A_3 \\
A_4
\end{pmatrix},
\]
where
\[
A_1 = \begin{pmatrix}
x_1^{N-1} & 0 & 0 & 0 \\
-(q_{12}^{1+N} + \bar{q}q_{12}^{1+N})x_1x_2 + \bar{q}q_{12}^{2+N}x_2x_1 & x_2^2 & 0 & 0 \\
-q_{12}^{1+N}y^{N-1}x_2 & y^{N-1}x_1 & 0 & 0 \\
q_{12}x_2^2 & \bar{q}q_{12}^2x_1x_2 - (q_{21} + \bar{q}q_{21})x_2x_1 & 0 & 0 \\
0 & 0 & x_2^{N-1} & 0 \\
0 & 0 & 0 & -q_{12}^{N+2+N}y 
\end{pmatrix}.
\]
\[
A_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
q_{12}^{-N^2+N_2}x_1^{N-1} & 0 & 0 \\
0 & 0 & 0 \\
0 & q_{12}^{-N^2+N_2}x_1^{N-1} & 0
\end{pmatrix},
\]
\[
A_3 = \begin{pmatrix}
0 & 0 & x_2^2y_{N-1}^2x_1 & -q_{12}^{-1}y_{N-1}^2x_1 \\
0 & 0 & q_{21}^2x_1x_2 - (q_{21} + q_{21})x_2^2x_1 & q_{21}^{-1}y_{N-1}^2x_2 \\
0 & 0 & q_{21}^2x_2^{N-1} & q_{12}^{-1}y_{N-1}^2x_2 \\
0 & 0 & q_{21}^2x_1x_2 - (q_{21} + q_{21})x_2x_1 & 0
\end{pmatrix},
\]
\[
A_4 = \begin{pmatrix}
-(q_{12} + q_{12})x_1x_2 + q_{21}^2x_2x_1 & 0 & 0 \\
-q_{12}y_{N-1}^2x_2 & 0 & 0 \\
x_2^2y_{N-1}^2x_1 & 0 & 0 \\
0 & -(q_{12} + q_{12})x_1x_2 + q_{21}^2x_2x_1 & x_1^2y_{N-1}^2x_1 \\
0 & -q_{12}y_{N-1}^2x_2 & 0 & 0 \\
0 & q_{21}^2x_1x_2 - (q_{21} + q_{21})x_2x_1 & x_2^2y_{N-1}^2x_1
\end{pmatrix}.
\]

**Proof.** It is routine to check that (11) is indeed a complex. But we need to mention that the following two equations hold
\[
\overline{D}x_1 - x_1^{N-1}x_2^{N-2} = 0,
\]
\[
x_2^{N-1}x_1^{N-2} - q_{12}^{-N^2+2N}Dx_2 = 0.
\]
These equations follow from Lemma [2.5] and the equations
\[
[x_1^{N-1}, x_2^{N-1}]c_{x_1} = yx_1^{N-1}x_2^{N-2},
\]
\[
[x_1^{N-1}, x_2^{N-1}]c_{x_2} = q_{12}^{-N^2+2N}yx_2^{N-1}x_1^{N-2}.
\]
The complex (11) is homotopically equivalent to the first segment of the resolution $P_\bullet$ (without shifting) constructed in Section 2. Therefore, it is exact. □

**Remark 2.11.** In [17, Theorem 6.1.3], the authors give a set of linearly independent 2-cocycles on $R$, indexed by the positive roots. In the resolution (11), the functions dual to $(1,0,0,0,0)$, $(0,0,1,0,0)$ and $(0,0,0,0,1)$ are just those 2-cocycles, corresponding to the positive roots $\alpha_1$, $\alpha_1 + \alpha_2$ and $\alpha_2$ respectively.

Now we give our main theorems about the structure of the Ext algebra of a Nichols algebra of type $A_2$. 
Theorem 2.12. Let $R$ be a Nichols algebra of type $A_2$ with $N = 3$, then $\text{Ext}_R^4(k, k)$ is generated by $a_i$, $b_i$, $c_i$, $i = 1, 2$ and $b_y$ with
\[
\deg a_i = 1, \quad \deg b_i = \deg b_y = \deg c_i = 2,
\]
subject to the relations
\[
\begin{align*}
 a_1^2 &= a_2^2 = 0, \quad a_1 a_2 = a_2 a_1 = 0, \\
 a_1 b_1 &= b_1 a_1, \quad a_1 b_y = q_{12}^3 b_y a_1, \quad a_1 b_2 = q_{12}^3 b_2 a_1, \\
 a_1 c_1 &= \bar{q}^2 q_{12} c_1 a_1, \quad a_1 c_2 = \bar{q} q_{12}^2 c_2 a_1, \\
 q_{12}^3 a_2 b_1 &= b_1 a_2, \quad q_{12}^3 a_2 b_y = b_y a_2, \quad a_2 b_2 = b_2 a_2, \\
 a_2 c_1 &= \bar{q} q_{12}^2 c_1 a_2, \quad \bar{q}^2 q_{12} a_2 c_2 = c_2 a_2, \\
 b_1 c_2 &= q_{12}^6 c_1^2, \quad q_{12}^6 b_2 c_1 = c_2^2, \quad b_1 b_2 = q_{12}^3 c_1 c_2, \quad c_1 c_2 = q_{12}^3 c_2 c_1, \\
 b_1 b_y &= q_{12}^9 b_y b_1, \quad b_1 b_2 = q_{12}^9 b_2 b_1, \quad b_y b_2 = q_{12}^9 b_2 b_y, \\
 q_{12}^3 c_1 b_1 &= b_1 c_1, \quad c_1 b_y = q_{12}^3 b_y c_1, \quad c_1 b_2 = q_{12}^6 b_2 c_1, \\
 q_{12}^6 c_2 b_1 &= b_1 c_2, \quad q_{12}^3 c_2 b_y = b_y c_2, \quad c_2 b_2 = q_{12}^3 c_2 c_2.
\end{align*}
\]

Theorem 2.13. Let $R$ be a Nichols algebra of type $A_2$ with $N > 3$, then $\text{Ext}_R^4(k, k)$ is generated by $a_i$, $b_i$ and $c_i$, $i = 1, 2$ and $b_y$ with
\[
\deg a_i = 1, \quad \deg b_i = \deg b_y = \deg c_i = 2,
\]
subject to the relations
\[
\begin{align*}
 a_1^2 &= a_2^2 = 0, \quad a_1 a_2 = a_2 a_1 = 0, \\
 a_1 b_1 &= b_1 a_1, \quad a_1 b_y = q_{12}^N b_y a_1, \quad a_1 b_2 = q_{12}^N b_2 a_1, \\
 q_{12}^N a_2 b_1 &= b_1 a_2, \quad q_{12}^N a_2 b_y = b_y a_2, \quad a_2 b_2 = b_2 a_2, \\
 a_1 c_2 &= \bar{q} q_{12}^2 c_2 a_1, \quad a_2 c_1 = \bar{q} q_{12}^2 c_1 a_2, \\
 a_1 c_1 &= c_1 a_1 = c_2 a_2 = a_2 c_2 = 0, \quad c_1 a_2 = c_2 a_1, \\
 c_1^2 &= c_2^2 = c_1 c_2 = c_2 c_1 = 0, \\
 b_1 b_y &= q_{12}^{N^2} b_y b_1, \quad b_1 b_2 = q_{12}^{N^2} b_2 b_1, \quad b_y b_2 = q_{12}^{N^2} b_2 b_y, \\
 q_{12}^N c_1 b_1 &= b_1 c_1, \quad c_1 b_y = q_{12}^N b_y c_1, \quad c_1 b_2 = q_{12}^{2N} b_2 c_1, \\
 q_{12}^{2N} c_2 b_1 &= b_1 c_2, \quad q_{12}^N c_2 b_y = b_y c_2, \quad c_2 b_2 = q_{12}^{2N} b_2 c_2.
\end{align*}
\]
Proof of Theorems 2.12 and 2.13. We prove Theorem 2.12. Theorem 2.13 can be proved similarly. Consider the minimal resolution (11) showed in Proposition 2.10 we have Ext$_R^1(k, k) = \text{Hom}_R(R^2, k)$ and Ext$_R^2(k, k) = \text{Hom}_R(R^5, k)$, since $k$ is a simple module. Let $a_1, a_2 \in \text{Ext}_R^1(k, k)$ be the functions dual to $(1, 0)$ and $(0, 1)$ respectively. Let $b_1, c_1, c_2, b_2 \in \text{Ext}_R^2(k, k)$ be the functions dual to $(1, 0, 0, 0), \cdots, (0, 0, 0, 1)$ respectively. The relations listed in the theorem can be verified by constructing suitable commutative diagrams, we do this in Appendix 3.2. Let $U$ be an algebra generated by $b_1, b_y, b_2$ and $a_1, c_i, i = 1, 2$, subject to the relations listed in the theorem. Then any element in $U$ can be written as a linear combination of elements of the form $b_1^{a_1} b_y^{b_2} b_2^{a_i}, b_1^{b_1} b_y^{b_2} b_2^{c_i}$ and $b_1^{b_1} b_y^{b_2} b_2^{c_1} a_2$, with $b_1, b_2, b_3 \geq 0, a_i, c_i \in \{0, 1\}, i = 1, 2$. By Lemma 2.3 the algebra $U$ is a quotient of Ext$_R^2(k, k)$. When $n$ is odd,

$$\dim U_n = \left(\frac{n-1}{2} + 2\right)\left(\frac{n-1}{2} + 1\right) + \frac{1}{2}\left(\frac{n-1}{2}\right)\left(\frac{n-1}{2} + 1\right) = \frac{3n^2 + 8n + 3}{8}.$$  

When $n$ is even,

$$\dim U_n = \left(\frac{n}{2}\right)\left(\frac{n}{2} + 1\right) + \frac{1}{2}\left(\frac{n}{2}\right)\left(\frac{n}{2} + 2\right) = \frac{3n^2 + 10n + 8}{8}.$$  

It follows from Corollary 2.9 that $\dim U_n = \dim \text{Ext}_R^2(k, k)$, for all $n \geq 0$, so $U = \text{Ext}_R^2(k, k)$, which completes the proof of the theorem. \qed

Remark 2.14. In [19] Thm 5.4, the authors showed that the Ext algebra of a Nichols algebra of finite Cartan type is braided commutative. This coincides with the results we obtain in Theorems 2.12 and 2.13.

Now we can answer the question whether the Ext algebra of a Nichols algebra is still a Nichols algebra. In general, the answer is negative.

Proposition 2.15. The Ext algebra of a Nichols algebra of type $A_2$ is not a Nichols algebra.

Proof. We consider the case $N = 2$ first. Denote the Ext algebra by $E$. From Proposition 2.1, $E$ is generated by $a_1, a_2$ and $b$ subject to the relations

$$a_2 a_1 = a_1 a_2 = 0, \quad a_1 b = b a_1, \quad a_2 b = b a_2.$$  

If $E$ is a Nichols algebra with respect to some braided vector space $V$, then $a_1, a_2$ and $b$ should form a basis of $V$. This is because as an algebra, a Nichols algebra $B(V)$ is generated by elements in $V$. With relation $a_2 a_1 = a_1 a_2, a_1 b = b a_1$ and $a_2 b = b a_2$, the vector space $V$ is of diagonal type. This contradicts to the relation $a_2 a_1 = a_1 a_2 = 0$. Therefore, $E$ is not a Nichols algebra. By a
similar argument, we can conclude that when $N \geq 3$, the Ext algebra is not a Nichols algebra either. □

However, we have the following positive result.

**Proposition 2.16.** Let $R$ be a Nichols algebra of type $A_2$ with $N > 3$. Then $\text{Ext}^*_R(k, k)/\mathcal{N}$ is a Nichols algebra of diagonal type, where $\mathcal{N}$ is the ideal generated by nilpotent elements.

**Proof.** From the proof of Theorem 2.13, the elements $b_1^{b_1} b_2^{b_2} a_i$, $b_1^{b_1} b_2^{b_2} c_i$ and $b_1^{b_1} b_2^{b_2} b_1 c_1 a_2$, with $b_1, b_2, b_3 \geq 0$, $a_i, c_i \in \{0, 1\}$, $i = 1, 2$ form a basis of $\text{Ext}^*_R(k, k)$. With the relation listed in that theorem, the elements $b_1^{b_1} b_2^{b_2} a_i$, $b_1^{b_1} b_2^{b_2} c_i$ and $b_1^{b_1} b_2^{b_2} c_1 a_2$ are nilpotent. However, linear combination of elements $b_1^{b_1} b_2^{b_2} b_2^{b_2}$ are not nilpotent. Then the algebra $\text{Ext}^*_R(k, k)/\mathcal{N}$ is generated by $b_1, b_2$ subject to the relations

\[ b_1 b_2 = q_{12}^{N_2} b_2 b_1, \quad b_1 b_2 = q_{12}^{N_2} b_2 b_1, \quad b_2 b_2 = q_{12}^{N_2} b_2 b_2. \]

It is obvious that it is a Nichols algebra of diagonal type with Cartan matrix of type $A_1 \times A_1 \times A_1$. □

The following corollary is a direct consequence from Theorem 2.12 and 2.13.

**Corollary 2.17.** Let $A = u(\mathcal{D}, 0, \mu)$ be a pointed Hopf algebra of type $A_2$ with $N \geq 3$ and $R = \mathcal{B}(V)$ the corresponding Nichols algebra. Then

\[ cx_R(k) = cx_A(k) = 3. \]

In addition, $V_A(k) \cong V_{(\mathcal{G}R)/\#k\Gamma}(k)$.

**Proof.** For the Nichols algebra $R$, the complexity

\[ cx_R(k) = \gamma(\text{Ext}^*_R(k, k)) = 3 \]

follows directly from Proposition 2.10 or Theorems 2.12 and 2.13. By 19, Lemma 6.1, we have

\[ H^*(u(\mathcal{D}, 0, \mu), k) \cong H^*(u(\mathcal{D}, 0, 0), k). \]

In addition, we also have

\[ \text{Ext}^*_u(\mathcal{D}, 0, 0)(k, k) \cong \text{Ext}^*_R(k, k)\Gamma. \]

Observe that for each positive root $\alpha$, some power of $b_\alpha$ is invariant under the group action. Indeed, from the discussion in Section 6 in 17, each $b_\alpha$ (denoted by $f_\alpha$ there) can be expressed as a function $R^+ \times R^+ \rightarrow k$. Then we see that $b_\alpha^{M_\alpha}$ is $\Gamma$-invariant, where $M_\alpha$ is the integer such that $\chi_\alpha^{M_\alpha} = \varepsilon$. Hence,
\[ \gamma(H^*(u(D,0,0),k)) = 3, \] which implies that \( \text{cx}_A(k) = 3. \] With the relations in Theorems 2.12 and 2.13 we see that
\[ V_A(k) \cong \text{MaxSpec}(k[b_1^{m_1}, b_y^{m_y}, b_2^{m_2}]), \]
where \( m_1, m_y \) and \( m_2 \) are the least integers such that \( b_1^{m_1}, b_y^{m_y}, b_2^{m_2} \in H^*(u(D,0,0),k). \) That is, \( V_A(k) \) is isomorphic to the maximal spectrum of the polynomial algebra \( k[y_1,y_2,y_3]. \) By \( [19, \text{Thm. 4.1}] \) \( V_{G_{R#\Gamma}}(k) \) is also isomorphic to the maximal spectrum of \( k[y_1,y_2,y_3]. \) So \( V_A(k) \cong V_{G_{R#\Gamma}}(k). \]\]

To end this section, we give an easy application of the main theorems. We show that a large class of finite dimensional pointed Hopf algebras of finite Cartan type are wild.

**Proposition 2.18.** Let \( A = u(D,\lambda,\mu) \) be a pointed Hopf algebra such that the components of the Dynkin diagram are of type \( A, D, \) or \( E, \) except for \( A_1 \) and \( A_1 \times A_1, \) and the order \( N_J > 2 \) for at least one component. Then \( A \) is wild.

**Proof.** In view of \([12, \text{Thm. 3.1}]\) we only need to prove that \( \text{cx}_A(k) \geq 3. \) Using \([19, \text{Lemma 6.1}]\) again, we have \( \text{cx}_A(k) = \text{cx}_{u(D,\lambda,0)}(k). \) However, \( u(D,\lambda,0) \) contains a Hopf subalgebra \( B \) which is of type \( A_2 \) with the order \( N \geq 3. \) Thus \( \text{cx}_{u(D,\lambda,0)}(k) \geq \text{cx}_B(k) \geq 3 \) by \([12, \text{Prop 2.1}]\). \( \square \)

We conjecture that the isomorphism \( V_A \cong V_{G_{R#\Gamma}} \) in Corollary 2.17 holds for general finite dimensional pointed Hopf algebra \( A = u(D,\lambda,\mu) \) of finite Cartan type.

### 3. Appendix

3.1. In this subsection, we verify that the complex \([11]\) in \([12, \text{Sec. 2}]\) is indeed a complex.

The following equations follow directly from Lemma 2.4

\[ Dy = \begin{cases} yD, & \text{if } a_1, a_3 \text{ are both even or both odd;} \\ q_{21}^{-N+2}yD, & \text{if } a_1 \text{ even and } a_3 \text{ is odd;} \\ q_{21}^{N-2}yD, & \text{if } a_1 \text{ odd and } a_3 \text{ is even.} \end{cases} \]

It is clear that \( \delta_i^2 = 0 \) for \( i = 1, 2, 3. \) So if \( a_2 \) is odd,
\[ \partial^2(\Phi(a_1,a_2,a_3)) = ((\delta_2\delta_1 + \delta_1\delta_3 + \delta_2\delta_3) + (\delta_2\delta_3 + \delta_3\delta_2) + (\delta_1\delta_2 + \delta_2\delta_1))\Phi(a_1,a_2,a_3). \]
Put
\[ A = (\delta_3 \delta_1 + \delta_1 \delta_3 + \delta_2 \delta_2) \Phi(a_1, a_2, a_3), \]
\[ B = (\delta_2 \delta_3 + \delta_3 \delta_2) \Phi(a_1, a_2, a_3), \]
\[ C = (\delta_1 \delta_2 + \delta_2 \delta_1) \Phi(a_1, a_2, a_3). \]

We show that \( A = B = C = 0. \)

\[ A = (\delta_3 \delta_1 + \delta_1 \delta_3 + \delta_2 \delta_2) \Phi(a_1, a_2, a_3) = ((-1)^{a_1-1+a_2} q_{12}^{\sigma(a_2) \tau(a_1-1)} q_{21}^{\sigma(a_3) \tau(a_2)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c + (-1)^{a_3} q_{12}^{\tau(a_1) y D} \Phi(a_1 - 1, a_2, a_3 - 1), \]

where \( D \) satisfies that
\[ Dy = -q_{12}^{\tau(a_1-1)} q_{21}^{\sigma(a_3) \tau(a_2-1)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c + q_{21}^{\tau(a_1-1) D y} = 0. \]

That is,
\[ q_{12}^{\sigma(a_3) \tau(a_1-1)} q_{21}^{\sigma(a_3) \tau(a_2-1)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c + q_{21}^{\sigma(a_3) \tau(a_1-1) D y} = 0. \]

Hence,
\[ q_{12}^{\sigma(a_3) \tau(a_1-1)} q_{21}^{\sigma(a_3) \tau(a_2-1)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c + q_{21}^{\tau(a_1) y D} = q_{21}^{-\sigma(a_3) \tau(a_1-1) D y}. \]

By equation (12), we have \( q_{21}^{-\sigma(a_3) \tau(a_1-1) D y} = q_{21}^{\tau(a_1)} y D. \) So
\[ A = ((-1)^{a_1-1+a_2} q_{12}^{\sigma(a_2) \tau(a_1-1)} q_{21}^{\sigma(a_3) \tau(a_2)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c + (-1)^{a_3} q_{12}^{\tau(a_1) y D} \Phi(a_1 - 1, a_2, a_3 - 1) = ((-1)^{a_1-1+a_2} q_{12}^{\sigma(a_2) \tau(a_1-1)} q_{21}^{\sigma(a_3) \tau(a_2)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c + (-1)^{a_3} q_{12}^{\tau(a_1) y D} \Phi(a_1 - 1, a_2, a_3 - 1) = 0. \]

The equations \( B = 0 \) and \( C = 0 \) can be verified directly. For example,
\[ B = (\delta_2 \delta_3 + \delta_3 \delta_2) (\Phi(a_1, a_2, a_3)) = ((-1)^{a_1-1+a_2} q_{21}^{\sigma(a_2) \tau(a_1-1)} q_{21}^{\sigma(a_3) \tau(a_2)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c + (-1)^{a_3} q_{21}^{\tau(a_1) y D} \Phi(a_1 - 1, a_2, a_3 - 1) = 0. \]

since \( \tau(a_2 - 1) + \sigma(a_2) = \tau(a_2). \)

If \( a_2 \) is even, then
\[ \partial^2(\Phi(a_1, a_2, a_3)) = ((\delta_1 \delta_3 + \delta_3 \delta_1 + \delta_2 \delta_2) + (\delta_1 \delta_2 + \delta_2 \delta_1) + (\delta_1 \delta_3 + \delta_3 \delta_2)) \Phi(a_1, a_2, a_3). \]

The equation \( (\delta_1 \delta_3 + \delta_3 \delta_1 + \delta_2 \delta_2) \Phi(a_1, a_2, a_3) = 0 \) follows directly from the definition of \( \delta_2. \) As in the case in which \( a_2 \) is odd,
\[ (\delta_2 \delta_3 + \delta_3 \delta_2) \Phi(a_1, a_2, a_3) = 0 \text{ and } (\delta_1 \delta_2 + \delta_2 \delta_1) \Phi(a_1, a_2, a_3) = 0. \]
can be also verified via a straightforward computation. Now, we show that

\[(\tilde{\delta}_2 \delta_1 + \delta_1 \tilde{\delta}_2)\Phi(a_1, a_2, a_3) = 0\] case by case, using Lemma 2.4

Case (i) $a_1$ and $a_3$ are both odd,

\[
(\tilde{\delta}_2 \delta_1 + \delta_1 \tilde{\delta}_2)\Phi(a_1, a_2, a_3) = (x_1(q_{21}^{-} q_{21}^{N-2}) - q_{21}^{N} q_{21}^{N-1})\Phi(a_1 - 2, a_2 + 1, a_3) = 0.
\]

Case (ii) $a_1$ is odd and $a_3$ is even,

\[
(\tilde{\delta}_2 \delta_1 + \delta_1 \tilde{\delta}_2)\Phi(a_1, a_2, a_3) = (x_1(-q_{21}^{-(N-1)q_{21}^{N-1} - (N-1)q_{21}^{N-2}q_{21}^{N+1}}) - q_{21}^{N} q_{21}^{N-1} (k_1 x_1^{N-2} x_2^{N-2} + \ldots + k_{N-2} y^{N-3} x_1 x_2 + k_{N-1} y^{N-2}) + x_2^{N-2} x_1^{N-1})\Phi(a_1 - 2, a_2 + 1, a_3) = 0,
\]

since $q_{12}^{-N^2+2N} x_1^{N-1} x_2^{N-1} = x_2^{N-2} x_1^{N-1} y$.

Case (iii) $a_1$ is even and $a_3$ is odd,

\[
(\tilde{\delta}_2 \delta_1 + \delta_1 \tilde{\delta}_2)\Phi(a_1, a_2, a_3) = -q_{21}^{N} x_1^{N-1} + q_{21}^{N} x_1^{N-1} \Phi(a_1 - 2, a_2 + 1, a_3) = 0.
\]

Case (iv) $a_1$ and $a_3$ are both even,

\[
(\tilde{\delta}_2 \delta_1 + \delta_1 \tilde{\delta}_2)\Phi(a_1, a_2, a_3) = x_1^{N-1} (q_{21}^{(N-1)q_{21}^{N} q_{21}^{N-1} - (N-1)q_{21}^{N-2}q_{21}^{N+1}}) + (-q_{21}^{N} q_{21}^{N-1} (k_1 x_1^{N-2} x_2^{N-2} + \ldots + k_{N-2} y^{N-3} x_1 x_2 + k_{N-1} y^{N-2}) x_1)\Phi(a_1 - 2, a_2 + 1, a_3) = 0,
\]

since $[x_1^{N-1} x_2^{N-1}] = y x_1^{N-1} x_2^{N-2}$.

Similarly, we can prove that $\tilde{\delta}_2 \delta_3 + \delta_3 \tilde{\delta}_2)\Phi(a_1, a_2, a_3) = 0$.

In conclusion, we have $\tilde{\partial}^2 = 0$. 

Set

\[ X_1 = q_{12}^{-(N-1)(N-3)}x_1^{N-3}x_2 + k_1 y x_1^{N-4} x_2^2 + \cdots + k_{N-3} y^{N-3}, \]

where \( k_i \in k, 1 \leq i \leq N - 3 \), such that \( x_2^{N-1} x_1^{N-3} = X_1 x_2^2 \), and

\[ X_2 = q_{12}^{(N-3)(N-1)}x_2^{N-3}x_1^{N-3} + l_1 y x_2^{N-4} x_1^2 + l_2 y^2 x_2^{N-5} x_1^{N-5} + \cdots + l_{N-3} y^{N-3}, \]

where \( l_i \in k, 1 \leq i \leq N - 3 \), such that \( x_1^{N-1} x_2^{N-3} = X_2 x_1^2 \).

Let \( f_1^1, f_2^1, f_3^1 \) and \( g_1^j, g_2^j, g_3^j, 1 \leq i \leq 5 \) and \( j = 1, 2 \) be the morphisms described by the following matrices:

\[ f_1^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q_{12}^N & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ x_1^{N-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{12} q_{12}^{1-N} y x_2 & -q_{12}^{1-N} y^{N-2} x_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_3^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ x_1^{N-3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{12}^N \end{pmatrix}, \]

\[ f_1^2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_2^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -q_{12}^{1-N} y x_2 & 0 & 0 & 0 & 0 \\ x_1^{N-3} & q_{12}^{1-N} y x_2 & 0 & 0 & 0 \end{pmatrix}, \quad f_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \]

\[ f_1^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q_{12}^N & 0 & 0 & 0 \\ 0 & 0 & q_{12}^N & 0 & 0 \\ 0 & 0 & 0 & q_{12}^N & 0 \\ 0 & 0 & 0 & 0 & q_{12}^N \end{pmatrix}, \quad f_2^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & q_{12}^{1-N} y x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{12}^{1-N} y x_2 & 0 \\ 0 & 0 & 0 & 0 & q_{12}^{1-N} y x_2 \\ 0 & 0 & 0 & 0 & q_{12}^{1-N} y x_2 \\ 0 & 0 & 0 & 0 & q_{12}^{1-N} y x_2 \end{pmatrix}, \quad f_3^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \]
Then we have the following commutative diagrams

\[
\begin{align*}
& R^{12} \xrightarrow{f_4} R^7 \xrightarrow{f_5} R^5 \xrightarrow{f_1} R^2 \xrightarrow{f_1} R, \\
& f_4 \downarrow \quad f_5 \downarrow \quad f_1 \downarrow \quad f_1 \downarrow \quad \downarrow k
\end{align*}
\]
It is also routine to check the commutativity of the diagrams (13) and (14). But we need to mention that the following equations hold

\[ X_1(q_{12}^2 x_1 x_2 - (q_{21} + \bar{q}q_{21}) x_2 x_1) = -q_{12}^{-N^2+2N} \mathcal{D}, \]
\[ X_2(q_{12}^2 x_2 x_1 - (q_{12} + \bar{q}q_{12}) x_1 x_2) = -\mathcal{D}, \]

which follow from Lemma 2.5 and the following two equations

\[ q_{12}^{-N^2+2N} \mathcal{D}_{x_2} = x_2^{-N} x_1^{-2} = X_1 x_2^2 x_1 = X_1(-\bar{q}q_{21}^2 x_1 x_2 + (q_{21} + \bar{q}q_{21}) x_2 x_1) x_2, \]
\[ \mathcal{D}_{x_1} = x_1^{-N} x_2^{-2} = X_2 x_1 x_2^2 = X_2(-\bar{q}q_{12}^2 x_2 x_1 + (q_{12} + \bar{q}q_{12}) x_1 x_2) x_1. \]

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