Vacuum effects in a vibrating cavity:  
time refraction, dynamical Casimir effect, and effective Unruh acceleration

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Two different quantum processes are considered in a perturbed vacuum cavity: time refraction and dynamical Casimir effect. They are shown to be physically equivalent, and are predicted to be unstable, leading to an exponential growth in the number of photons created in the cavity. The concept of an effective Unruh acceleration for these processes is also introduced, in order to make a comparison in terms of radiation efficiency, with the Unruh radiation associated with an accelerated frame in unbounded vacuum.

I. INTRODUCTION

Vacuum has always been an essential ingredient of our knowledge of the physical world, from Aristotle to the present days. In the low energy limit of the vacuum fluctuation spectrum, as described by quantum electrodynamics (or qed), vacuum effects predict the emission of pairs of photons, induced by some external perturbation. For higher energy fluctuations, qed also predicts the occurrence of vacuum nonlinearities, which are associated with virtual electron-positron pairs [1]. At even higher energies, electron-positron pairs will eventually be emitted from vacuum and become real. Other real and virtual particle-antiparticle pairs also have to be considered.

Here we restrict our analysis to the low energy range of quantum electrodynamics, where the influence of electron-positron pairs can be neglected. This is the range of quantum optics, which only deals with photon vacuum effects. It should be noted that the effects to be considered here could also occur at higher energies, involving other particles and other fields.

Three different effects have been discussed in the frame of photon qed or quantum optics: i) dynamical Casimir effect [2, 3, 4, 5, 6], ii) Unruh-Davies radiation [7, 8], and iii) time refraction [9, 10, 11]. Dynamical Casimir effect is a direct extension of the famous double plate geometry of the Casimir effect [12], which revealed the energy difference between different vacua. The dynamical Casimir setup considers one of the plates as periodically oscillating in time, due to some applied force. Unruh-Davies radiation (also called Unruh radiation) demonstrates the existence of thermal radiation, as seen from an accelerated reference frame in unbounded vacuum. At first sight this could be a purely kinematic effect, with no physical consequences. Its real existence has actually been seriously questioned [13]. However, any physical detector (such as a charged particle or an atom) moving with the accelerated frame, would be able to interact with such thermal radiation and thus respond accordingly. An important aspect of the Unruh effect is that it explores the equivalence between gravitation and acceleration, and is intimately related to Hawking radiation [14]. The relations between the dynamical Casimir effect and the Unruh effect have been explored in, e.g., Refs. [13, 16], and various views as to how close the relationship are have been presented in the literature [17]. The interplay between these two vacuum effects have important consequences for how experiments should be interpreted, see for example the recent debate [18, 19, 20].

In the present paper we will attempt to shed further light on these connections by tying the dynamical Casimir effect to the concept of time refraction. Time refraction is the temporal version of the well known concept of refraction. It is a low order effect, which is perceived by any photon in a time varying medium, and can be seen as the most basic mechanism leading to photon acceleration [21]. As a result of time refraction, superluminal frames with constant velocity can also observe a radiation spectrum resembling the Unruh radiation [22].

In a recent work [11], we were able to show that the concept of time refraction, when considered in the specific case of an optical cavity, is very similar to the dynamical Casimir effect. Time refraction always involves the presence of an optical medium, and is more general than the dynamical Casimir effect, in the sense that it is independent of boundary conditions, and can occur in unbounded media. On the other hand, the Unruh effect is quite often related with the dynamical Casimir (see e.g. [15, 16]), but the nature of this connection has been debated [13, 16, 17, 18, 19, 20].

One important aspect is that these three mechanisms can create radiation from vacuum with a finite energy, but with zero momentum. Time refraction, in a cavity or in an unbounded medium, creates pairs of photons propagating in opposite directions. The dynamical Casimir effect in a cavity creates a standing wave mode, which is equivalent to two counter-propagating photons. And, it was recently shown that the Unruh emission by accelerated electrons is also made of pairs of photons [23], with zero momentum in the instantaneous rest frame.
II. TIME REFRACTION IN A CAVITY

We first consider an empty cavity with a moving mirror. This is equivalent to an optical cavity with a fixed length, but filled with a time varying dielectric medium. By changing the refractive index, with the help for instance of an applied external field, we change the optical length of the cavity. Therefore, these two models of a varying cavity, with a variable length or with a varying dielectric medium, are equivalent from the point of view of the optical length.

Let us first relate the temporal change in the refractive index $n(t)$, with the change in the empty cavity length $L(t)$. If we consider a given cavity mode with an integer number $m$ of wavelengths along the cavity axis, corresponding to the wavenumber $k_m = 2\pi m/L_0$, where $L_0$ is the cavity length, this mode frequency will vary in time according to

$$\omega_m(t) = \frac{k_m c}{n(t)} = \frac{2\pi mc}{L_0 n(t)} = \frac{2\pi mc}{L(t)}.$$  

By writing $n(t) = n_0 + \delta n(t)$, where $n_0$ is the unperturbed refractive index, we obtain for the variable cavity length $L(t) = L_0 + \delta L(t)$, where $\delta L(t) = L_0 \delta n(t)$ is the effective optical displacement. Keeping this equivalence in mind, we can now focus on the empty cavity case, and adapt previous results obtained for the variable refractive index case. For a single mode in vacuum, we have used an electric field operator of the form

$$\hat{E}_k(x,t) = i \sqrt{\frac{\hbar \omega(t)}{2\epsilon_0}} \left[ a_k(t)e^{ikx} - a_k^+(t)e^{-ikx} \right] \vec{e}_k,$$

where $\vec{e}_k$ is the unit polarization vector and the creation and destruction can be written as

$$a_k(t) = A_k(t)e^{-i\int \omega(t)dt}, \quad a_k^+(t) = A_k^+(t)e^{-i\int \omega(t)dt}.$$  

But for a cavity mode $m$, we need to associate the other momentum component $-k$, and it is more adequate to use the field mode operator

$$\hat{E}_m(x,t) = i \sqrt{\frac{\hbar \omega_m(t)}{2\epsilon_0}} \left[ a_m(t)\sin[k_m(t)x] + h.c. \right]\vec{e}_m,$$

where $k_m = 2\pi m/L(t)$. Using a non-perturbative field theory approach, it is then possible to show (2), that the mode operators will evolve in time according to the equations

$$\frac{da_m}{dt} = -i\omega_m a_m + \left( \frac{L'}{2L} \right) a_m^+,$$  

$$\frac{da_m^+}{dt} = i\omega_m a_m + \left( \frac{L'}{2L} \right) a_m^+,$$

where $L' \equiv dL/dt$. The creation and destruction operators can also be represented as

$$a_m(t) = A_m(t)e^{-\phi(t)}, \quad a_m^+(t) = A_m^+(t)e^{-\phi(t)},$$

with the phase

$$\phi(t) = \int_0^t \omega_m(t')dt'.$$

The evolution equations (3) can then be written in a simpler and more compact form, as

$$\frac{dA_m}{dt} = \nu(t)A_m^+, \quad \frac{dA_m^+}{dt} = \nu(t)^*A_m,$$

with the coupling function

$$\nu(t) = \left( \frac{L'}{2L} \right) \exp[2i\phi(t)].$$

This system of equations can easily be integrated, leading to the well known solutions

$$A_m(t) = \alpha(t)A_m(0) - \beta(t)A_m^+(0),$$

$$A_m^+(t) = \alpha(t)A_m^+(0) - \beta(t)A_m(0),$$

with

$$\alpha(t) = \cosh r(t), \quad \beta(t) = \sinh r(t),$$

where the squeezing function $r(t)$ is determined by

$$r(t) = \int_0^t \nu(t')dt' = \frac{1}{2} \int_0^t \left( \frac{d}{dt'} \ln L(t') \right) \exp[2i\phi(t')]dt'.$$
Notice that equations (10) are temporal Bogoliubov relations, obeying the usual hyperbolic condition $\alpha(t)^2 - \beta(t)^2 = 1$, which correspond to bosonic quantum states. These are the quantum laws of time refraction, adapted here to the case of an empty cavity with a variable length $L(t)$. Their physical implications, and their classical counterparts, were discussed in detail in reference [11].

III. DYNAMICAL CASIMIR EFFECT

Let us now concentrate on the case where we have an oscillating mirror in the empty cavity, as described by

$$L(t) = L_0 + \epsilon \sin(\Omega t),$$

(13)

where we have assumed that the amplitude of the oscillations is much smaller than the cavity length $\epsilon \ll L_0$. From equation (11) we can see that the mode frequency will also oscillate in time according to

$$\omega_m(t) = \frac{2\pi mc}{L(t)} \approx \omega_{m0} \left[ 1 - \frac{\epsilon}{L_0} \sin(\Omega t) \right],$$

(14)

with $\omega_{m0} = k_{m0}c = 2\pi mc/L_0$. Let us calculate the corresponding values for the functions $\nu(t)$ and $r(t)$. From equation (10) we obtain

$$\nu(t) = \frac{\epsilon \Omega}{2L(t)} \cos(\Omega t) \exp[2i\omega_{m0}t + i\rho \cos(\Omega t)],$$

(15)

with

$$\rho = \frac{2\epsilon \omega_{m0}}{\Omega L_0}.$$  

(16)

Neglecting higher order corrections with respect to the small parameter $\epsilon/L_0$, we can write

$$\nu(t) = \frac{\rho \Omega^2}{4\omega_{m0}} \cos(\Omega t) e^{2i\omega_{m0}t} \sum_{n=-\infty}^{\infty} i^n J_n(\rho) e^{i n \Omega t},$$

(17)

where $J_n(\rho)$ are Bessel functions of the first kind. Replacing this in the definition of the squeezing function, we can easily see that only the constant terms of $\nu(t)$ will give a significant contribution to $r(t)$, while the others will average out to zero, for $t \gg 1/\Omega$. Constant terms of $\nu(t)$ only occur for

$$(n \pm 1)\Omega = 2\omega_{m0}.$$  

(18)

When such a condition is verified, the constant term of $\nu(t)$ is determined by

$$\nu_n = \frac{\rho \Omega^2}{2^{3/2} \omega_{m0}} J_a(\rho),$$

(19)

For $\rho \sim \epsilon/L_0$, the largest of these terms will correspond to $n = 0$, which leads us to the well known condition for an efficient dynamical Casimir effect, $\Omega = 2\omega_{m0}$. The corresponding expression for the squeezing function is simply determined by

$$r(t) = \nu_0 t = \frac{\epsilon}{L_0} \frac{\omega_{m0}}{2} J_0(\epsilon/L_0) t.$$  

(20)

Let us now calculate the number of photon pairs created from out of vacuum, due to the oscillations of the cavity mirror. To do this, we start from the usual definition of the photon number operator, for the cavity mode $m$, as given by $N_m(t) = A_m^+(t) A_m(t)$. The average number of photon pairs created at time $t$ in the cavity will then be determined by

$$\langle N_m(t) \rangle = \langle 0 | A_m^+(t) A_m(t) | 0 \rangle,$$  

(21)

where $| 0 \rangle$ is the vacuum state vector for the cavity mode $m$. Using the above solutions for the operators (10), we obtain

$$\langle N_m(t) \rangle = \sinh^2(\nu_0 t).$$  

(22)

For short times, such that $\nu_0 t \ll 1$, this equation predicts a linear growth, given by $\langle N_m(t) \rangle \approx \nu_0 t$. On the other hand, for very long times, such that $t \gg 1/\nu_0$, this leads to an exponential growth

$$\langle N_m(t) \rangle \approx \frac{1}{4} \exp(2\nu_0 t).$$  

(23)

This is the well known instability predicted for the dynamical vacuum cavity at the resonant excitation frequency $\Omega = 2\omega_{m0}$. Such an exponential growth is only possible if we neglect the cavity losses, due to absorption and diffraction at the cavity mirrors. These losses can be described by a linear damping rate $\gamma$, such that $Q_{m0} = \omega_{m0}/\gamma$ is the quality factor of the cavity. If we include losses, we can write the following balance equation for the average number of photon pairs

$$\frac{d}{dt} \langle N_m(t) \rangle = 2\nu_0 \sinh(\nu_0 t) \cosh(\nu_0 t) - \gamma \langle N_m(t) \rangle.$$  

(24)

For a small damping rate, this is approximately equal to

$$\frac{d}{dt} \langle N_m(t) \rangle = [2\nu_0 \coth(\nu_0 t) - \gamma] \langle N_m(t) \rangle.$$  

(25)

This can easily be integrated to give

$$\langle N_m(t) \rangle = \sinh^2(\nu_0 t) \exp(-\gamma t).$$  

(26)

Let us now consider nonlinear saturation effects, by assuming that the cavity oscillations are produced by an externally driven nonlinear element inserted in the cavity.
The existence of photons due to the dynamical Casimir instability will introduce a small change in the refractive index of the dielectric element, as described by the usual law \( n(t) = n_0(t) + n_1 I_m(t) \), where \( I_m(t) = \hbar \omega_{m0} c \langle N_m(t) \rangle \) is the intensity associated with the cavity mode \( m \). The nonlinear refractive index \( n_1 \) is proportional to the third order susceptibility of the dielectric medium. This nonlinear correction leads to a small detuning of the cavity mode, which can be approximately described by a nonlinear correction factor of the type \( (1 - \zeta \langle N_m(t) \rangle)^2 \), with \( \zeta \approx (\hbar \omega_{m0} c n_1)^2 \), multiplying the driving term of equation (24). This nonlinear effect will therefore reduce the rate of creation of photons inside the vibrating cavity.

**IV. EFFECTIVE UNRUH ACCELERATION**

We now compare the previous quantum effects associated with an oscillating cavity with those associated with Unruh radiation in unbounded vacuum. According to Unruh, an observer (or a physical object) moving in vacuum with acceleration \( a \), will perceive in its accelerated frame, a thermal spectrum with temperature defined by

\[
k_B T = \frac{\hbar}{2 \pi c} |a|.
\]

(27)

Noting the analogy between moving mirror radiation and the Unruh effect \[24\], a single moving cavity wall is treated as our accelerated observer. Furthermore, this accelerated observer will interact with a radiation spectrum having the following energy distribution per field mode \[25, 26\]

\[
W(\omega, t) = \left[ 1 + \frac{a^2(t)}{\omega^2 c^2} \right] W_T(\omega, t),
\]

(28)

where we define the thermal energy distribution

\[
W_T(\omega, t) = \frac{\hbar}{2} \omega \coth \left( \frac{\pi \omega c}{a(t)} \right)
= \hbar \omega \left\{ \frac{1}{2} + \frac{1}{\exp[2 \pi \omega c / |a(t)|] - 1} \right\}.
\]

(29)

This thermal spectrum includes both vacuum fluctuations and Planck spectrum with temperature defined by equation (27). In the low temperature limit \( \hbar \omega \gg k_B T \), or \( \omega \gg a(t)c \), it will reduce to the blackbody radiation. In alternative, we can use the photons number distribution \( N(\omega, t) = W(\omega, t) / \hbar \omega \). Neglecting the unphysical fractional number associated with the vacuum fluctuation term in equation (29), which represents in fact the average effect of virtual photons, we can write for the Unruh spectrum

\[
N(\omega) = \left[ 1 + \frac{a^2(t)}{\omega^2 c^2} \right] \frac{1}{\exp[2 \pi \omega c / |a|] - 1}.
\]

(30)

Let us now define an effective Unruh acceleration \( a_{\text{eff}} \), such that, for \( \omega = \omega_{m0} \) this nearly thermal spectrum gives the same amount of photons than those produced by the dynamical Casimir effect in a cavity. It will be determined by the equality

\[
\langle N_m(t) \rangle = \left[ 1 + \frac{a_{\text{eff}}(t)^2}{\omega_{m0}^2 c^2} \right] \frac{1}{\exp[2 \pi \omega_{m0} c / |a_{\text{eff}}(t)|] - 1},
\]

(31)

where \( \langle N_m(t) \rangle \) is determined by equation (22). This can also be written as

\[
[\exp(y(t) - 1) N_c(t) = 1 + \frac{4 \pi^2}{y(t)^2},
\]

(32)

with

\[
N_c(t) = \frac{\langle N_m(t) \rangle}{V_c}, \quad y(t) = 2 \pi \frac{\omega_{m0} c}{|a_{\text{eff}}(t)|}.
\]

(33)

For moderate values of the effective acceleration, such that \( y > 2\pi \), we simply obtain

\[
y(t) = \ln \left[ 1 + \frac{1}{N_c(t)} \right],
\]

(34)

or, in explicit form

\[
a_{\text{eff}}(t) = 2 \pi \frac{\omega_{m0} c}{\ln[1 + 1/N_c(t)]}.
\]

(35)

This effective Unruh acceleration associated with the dynamical Casimir effect can then be used to compare the Unruh radiation efficiency for unbounded vacuum with that of cavity vacuum effects. Such a comparison could help in the choice of the more appropriate configurations for quantum vacuum experiments.

In order to establish such a comparison, we can imagine an observer moving in unbounded vacuum with a constant acceleration \( a_0 = \delta \Omega^2 \), equal to the maximum acceleration value attained by the vibrating mirror of a cavity, or

\[
a_0 = 4 \epsilon \omega_{m0}^2 = (2 \pi \omega_{m0} c) \frac{4 \epsilon}{L_0} m.
\]

(36)

The corresponding Unruh spectrum observed in unbounded vacuum, would be given by equation (30) with \( a = a_0 \). The ratio between the two quantities \( a_{\text{eff}}(t) / a_0 \), corresponding to effective and real Unruh acceleration values, is given by

\[
R(t) = \frac{a_{\text{eff}}(t)}{a_0} = \frac{L_0}{4m \epsilon \ln[1 + 1/N_c(t)]}. \tag{37}
\]

The condition \( R(t) \gg 1 \) can be achieved for large effective values of \( a_{\text{eff}} \) such that

\[
\ln \left[ 1 + \frac{1}{N_c(t)} \right] > \frac{L_0}{4m \epsilon}. \tag{38}
\]
Such a condition can easily be achieved. For instance, assuming that $L_0 \approx 4\epsilon t$, which would correspond to $\epsilon = \frac{\lambda m}{4}$, where $\lambda$ is the wavelength of the cavity radiation mode $m$, the threshold for high effective acceleration regime, such that $R(t) \geq 1$, is attained for $N_c(t) = \frac{1}{(e-1)^{-1}}$. If we take the approximate expression for the number of photons created by the dynamical Casimir effect, equation (23), condition (38) is verified for

$$t \geq \frac{1}{2\nu_0} \ln \left( \frac{1}{\epsilon - 1} \right) \simeq \frac{1}{4\nu_0},$$

(39)

This clearly shows that, for $t \gg 1/\nu_0$, we are in a regime where the dynamical Casimir effect is much more favorable, for quantum vacuum observations, that the corresponding Unruh effect in unbounded vacuum, for an observer moving with the maximum mirror acceleration $a_0 = c\Omega^2$. This is due to the unstable character of the dynamical Casimir effect.

V. SUMMARY AND CONCLUSIONS

In this work we have considered vacuum quantum processes in an oscillating cavity. Two different quantum processes have been discussed in this specific configuration: time refraction and dynamical Casimir radiation. We have shown that time refraction in a cavity is physically identical to the dynamical Casimir effect, if we identify the varying optical path (due to the change of the refractive index of the medium inside a cavity with fixed boundaries) with the actual varying length of an empty cavity with a moving mirror. We have shown the possible occurrence of an exponential growth of the number of photons created from vacuum by both models, if the cavity length oscillates at a frequency $\Omega$ with is twice the frequency of a given cavity mode $\omega_{\nu_0}$. The instability growth rate is linear with respect to the mirror displacement $\epsilon$, or to the equivalent oscillation amplitude of the refractive index $\delta n = \epsilon/L_0$. Previous results obtained for these two models [11] were confirmed and refined. In particular the instability saturation mechanisms, due to cavity losses and nonlinear detuning has been included.

We have also introduced the concept of an effective Unruh acceleration for the dynamical Casimir effect, in order to create a bridge between the bounded and unbounded quantum vacuum effects. Using this new concept, we were able to compare the efficiency of the dynamical Casimir effect with that of an equivalent Unruh radiation, for observers moving in unbounded vacuum with the maximum value of mirror acceleration $a_0 = 2\epsilon\Omega$. We have shown that very efficient regimes for photon creation from vacuum can be attained in a vibrating cavity, with respect to those of unbounded vacuum. This suggests that dynamical cavity experiments appear to be better candidates for the observation of photon creation from a perturbed quantum vacuum than those in unbounded vacuum, for comparable values of acceleration.

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