GENERALIZED SUPERELLIPTIC RIEMANN SURFACES

RUBÉN A. HIDALGO, SAÚL QUISPE, AND TONY SHASKA

Abstract. A conformal automorphism $\tau$, of order $n \geq 2$, of a closed Riemann surface $X$, of genus $g \geq 2$, which is central in $\text{Aut}(X)$ and such that $X/\langle \tau \rangle$ has genus zero, is called a superelliptic automorphism of level $n$. If $n = 2$, then $\tau$ is called hyperelliptic involution and it is known to be unique. In this paper, for the case $n \geq 3$, we investigate the uniqueness of the cyclic group $\langle \tau \rangle$. Let $\tau_1$ and $\tau_2$ be two superelliptic automorphisms of level $n$ of $X$. If $n \geq 3$ is odd, then $\langle \tau_1 \rangle = \langle \tau_2 \rangle$. In the case that $n \geq 2$ is even, then the same uniqueness result holds, up to some explicit exceptional cases.

1. Introduction

Let $X$ be a closed Riemann surface of genus $g \geq 2$ and let $G = \text{Aut}(X)$ be its group of conformal automorphisms. It is well known that $\text{Aut}(X)$ is finite [23] of order at most $84(g-1)$ [16], and that the order of any conformal automorphism is bounded above by $4g + 2$. This paper considers certain cyclic subgroups of $\text{Aut}(X)$ which behave similarly to the hyperelliptic involution.

Let $\tau \in G$ be an $n$-gonal automorphism, that is, it has order $n \geq 2$ and $X/\langle \tau \rangle$ has genus zero. In this case, $H = \langle \tau \rangle \cong C_n$ is called an $n$-gonal group and $X$ a cyclic $n$-gonal Riemann surface. Let $N$ be the normalizer of $\langle \tau \rangle$ in $G$. It follows from the results in [10, 27], that generically $N = G$.

If $n = 2$, then $\tau$ is the hyperelliptic involution and it is known to be unique and central in $G$; in particular, it is central in $G = N$.

If $n \geq 3$ is a prime integer and $s \geq 3$ is the number of fixed points of $\tau$, then $H$ is known to be the unique $n$-Sylow subgroup of $G$ if either (i) $2n < s$ [7] or (ii) $n \geq 5s - 7$ [12]. So, in this case, $N = G$; but it might be that $\tau$ is non-central.

If $n \geq 3$, not necessarily prime, such that: (i) every fixed point of a non-trivial power of $\tau$ is also a fixed point of $\tau$, and (ii) the rotation number of $\tau$ at each of its fixed points is the same, (some authors call $\tau$ a superelliptic automorphism and $X$ a superelliptic surface), then $\tau$ is central in $N$ (see Corollary 1), but in general $N \neq G$. In this case, under the extra condition that $g > (n-1)^2$, it is known that $N = G$ [18] (as a consequence of results in [1]). The computation of $G$ has been done in [22]. Superelliptic Riemann surfaces have been studied in [4, 19, 20] and those with many conformal automorphisms and with CM structures have been considered in [21].

If $\tau$ is central in $G$ (respectively, central in $N$), then we call it a superelliptic automorphism of level $n$ (respectively, generalized superelliptic automorphism of level $n$); we also say that $H = \langle \tau \rangle$ is a superelliptic group of level $n$.

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\( n \) (respectively, \textbf{generalized superelliptic group of level} \( n \)), and that \( \mathcal{X} \) is a \textbf{superelliptic curve of level} \( n \) (respectively, \textbf{generalized superelliptic curve of level} \( n \)). A superelliptic automorphism of level \( n \) is automatically a generalized one; the converse is in general false (but generically true). Also, as previously noted, a superelliptic automorphism of order \( n \) is a generalized superelliptic of level \( n \) (but it might not be a superelliptic automorphism of level \( n \)).

In this paper, (i) we provide necessary and sufficient conditions for an \( n \)-gonal automorphism to be a generalized superelliptic automorphism of level \( n \) (Theorem 1) and (ii) we provide conditions for a superelliptic curve of level \( n \) to have a unique superelliptic group of level \( n \) (Theorem 2 and Corollary 2).

Before stating the above two results, let us recall some facts on \( n \)-gonal automorphisms. Let us consider a pair \( (\mathcal{X}, \tau) \), where \( \tau \) is a \( n \)-gonal automorphism of \( \mathcal{X} \). Set \( H = \langle \tau \rangle \cong C_n \). Let \( \pi : \mathcal{X} \rightarrow \hat{\mathbb{C}} \) be a Galois branched covering, whose deck covering group is \( H = \langle \tau \rangle \), and let \( p_1, \ldots, p_s \in \hat{\mathbb{C}} \) be its branch values. Then there are integers \( l_1, \ldots, l_s \in \{1, \ldots, n-1\} \) satisfying that \( l_1 + \cdots + l_s \) is a multiple of \( n \) and \( \gcd(n, l_1, \ldots, l_s) = 1 \), such that \( \mathcal{X} \) can be described by an affine irreducible algebraic curve (which might have singularities) of the following form (called a \textbf{cyclic} \( n \)-gonal curve)

\[
y^n = \prod_{j=1}^{n}(x - p_j)^{l_j}.
\]

If one of the branch values is \( \infty \), say \( p_s = \infty \), then we need to delete the factor \( (x - p_s)^{l_s} \) from the above equation. In this algebraic model, \( \tau \) and \( \pi \) are given respectively by \( \tau(x, y) = (x, \omega_n y) \), where \( \omega_n = e^{2\pi i/n} \), and \( \pi(x, y) = x \).

**Theorem 1.** Let \( \mathcal{X} \) be a cyclic \( n \)-gonal Riemann surface, described by the cyclic \( n \)-gonal curve Eq. (1), and \( N \) be the normalizer of \( H = \langle \tau(x, y) = (x, \omega_n y) \rangle \) in \( \text{Aut}(\mathcal{X}) \). Let \( \theta : N \rightarrow \overline{\mathbb{N}} = N/H \) be the canonical projection homomorphism. Then \( \tau \) is a generalized superelliptic automorphism of level \( n \) if and only if for all \( p_j \) and \( \pi_i \) in the same \( \theta(N) \)-orbit it holds that \( l_j = l_i \).

**Corollary 1.** Let \( \mathcal{X} \) be a cyclic \( n \)-gonal Riemann surface, described by the cyclic \( n \)-gonal curve Eq. (1). If \( l_j = l, \) for every \( j, \) where \( \gcd(n, l) = 1, \) then \( \tau(x, y) = (x, \omega_n y) \in \text{Aut}(\mathcal{X}) \) is a generalized superelliptic automorphism of level \( n \).

**Remark 1.** The above corollary states that a superelliptic automorphism is always a generalized superelliptic automorphism of level \( n \) (but not necessarily a superelliptic automorphism of level \( n \) as the normalizer \( N \) might be smaller than the full group of automorphisms).

**Theorem 2.** Let \( \mathcal{X} \) be a cyclic \( n \)-gonal Riemann surface, admitting two superelliptic automorphisms \( \tau \) and \( \eta \), both of level \( n \), such that \( \langle \tau \rangle \neq \langle \eta \rangle \). Then

(I) \( \text{Aut}(\mathcal{X})/\langle \tau \rangle \) is either a non-trivial cyclic group of even order or a dihedral group of order a multiple of four;

(II) there is an integer \( d \geq 2 \) such that \( n = 2d \) and \( \mathcal{X} \) can be represented by a cyclic \( n \)-gonal curve of the form

\[
\mathcal{X} : \quad y^{2d} = x^2 (x^2 - 1)^{l_1} (x^2 - a_2)^{l_2} \prod_{j=3}^{L}(x^2 - a_j^{2})^{2l_j},
\]

where \( l_1, l_2, 2\widehat{l}_3, \ldots, 2\widehat{l}_L \in \{1, \ldots, 2d - 1\}, \) \( l_1 \) is odd, and either one of the two conditions (a) or (b) below holds for \( l_2 \).

(a) If \( l_2 = 2\widehat{l}_2 \), then \( \gcd(2d, l_1, \widehat{l}_2, \ldots, \widehat{l}_L) = 1 \).
the finite subgroups of the group $PSL$ of Möbius transformations. 2.1.

In these cases, $\tau(x, y) = (x, \omega_{2d}y)$, $\eta(x, y) = (-x, \omega_{2d}y)$ and $\langle \tau, \eta \rangle \cong C_{2d} \times C_2$.

Those superelliptic Riemann surfaces of level $n = 2d$, described by the cyclic $2d$-gonal curves in Theorem 2, will be called exceptional.

**Remark 2.** The cyclic $2d$-gonal curves $X$, defined by Eq. (2), are cyclic $2d$-gonal curves $X$ admitting two commuting cyclic $2d$-gonal automorphisms, $\tau, \eta$, such that $\langle \tau \rangle \neq \langle \eta \rangle$. We should note that not all of them need to be hyperelliptic of level $n$; the theorem only asserts that the exceptional ones are some of them. For instance, in the case (b) with $d = 2$, $l_1 = 1$ and $l_2 = 3$, the genus five curve $\bar{\tau}, \eta$-gonal automorphisms, $\rho\tau\rho^{-1} = \tau^3$.

**Corollary 2.** Let $X$ be a Riemann surface admitting a superelliptic group $H$ of level $n$. Then $H$ is the unique superelliptic group of level $n$ of $X$ if either: (1) $n = 2$, or (2) $n \geq 3$ is odd , or (3) $n \geq 4$ is even and $X/H$ has no cone point of order $n/2$.

Finally, in the last section, we provide some discussion on the field of moduli of these superelliptic Riemann surfaces (see Theorem 5).

**Notations.** We denote by $C_n$ the cyclic group of order $n$, by $D_n$ the dihedral group of order $2n$, by $A_n$ the alternating group, and by $S_n$ the symmetric group.

### 2. Preliminaries

#### 2.1. The finite groups of Möbius transformations.

Up to $PSL_2(\mathbb{C})$-conjugation, the finite subgroups of the group $PSL_2(\mathbb{C})$ of Möbius transformations are given by (see, for instance, [3])

$$C_m := \langle a(x) = \omega_m x \rangle, \quad D_m := \langle a(x) = \omega_m x, b(x) = \frac{1}{x} \rangle, \quad A_4 := \langle a(x) = -x, b(x) = \frac{1}{x} \rangle, \quad S_4 := \langle a(x) = i x, b(x) = \frac{1}{x} \rangle, \quad A_5 := \langle a(x) = \omega_5 x, b(x) = \frac{(1-\omega^3_5)x + (\omega^3_5-1)}{(1-\omega^3_5)x + (\omega^3_5-1)} \rangle,$$

where $\omega_m$ is a primitive $m$-th root of unity. For each of the above finite groups $A$, a Galois branched covering $f_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, with deck group $A$, is given as follows

$$f_{C_m}(x) = x^m; \quad f_{D_m}(x) = x^m + x^{-m}; \quad \text{branching:} \quad (m, m).$$

$$f_{A_4}(x) = \frac{(x^4 - 2i\sqrt{3}x^2 + 1)^3}{-12i\sqrt{3}x^2(x^4 - 1)^2}; \quad f_{S_4}(x) = \frac{108x^4(x^4 - 1)^3}{108x^4(x^4 - 1)^4}; \quad \text{branching:} \quad (2, 3, 4).$$

$$f_{A_5}(x) = \frac{(-x^{20} + 228x^{15} - 494x^{10} - 228x^5 - 1)^3}{1728x^5(x^{10} + 11x^5 - 1)^5}; \quad \text{branching:} \quad (2, 3, 5).$$

see [14]. In the above, the branching corresponds to the tuple of branch orders of the cone points of the orbifold $\hat{\mathbb{C}}/A$. 
2.2. Fuchsian groups. A Fuchsian group is a discrete subgroup $\Delta$ of $\text{PSL}_2(\mathbb{R})$, the group orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$. It is called co-compact if the quotient orbifold $\mathbb{H}/\Delta$ is compact; its signature is the tuple $(g; n_1, \ldots, n_s)$, where $g$ is the genus of the quotient orbifold $\mathbb{H}/\Delta$, $s$ is the number of its cone points they having branch orders $n_1, \ldots, n_s$. The group $\Delta$ has a presentation as follows:

$$
\Gamma = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_s : c_1^{n_1} = \cdots = c_s^{n_s} = 1, c_1 \cdots c_s [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle,
$$

where $[a, b] = aba^{-1}b^{-1}$.

If a co-compact Fuchsian group $\Gamma$ has no torsion, then $X = \mathbb{H}/\Gamma$ is a closed Riemann surface of genus $g \geq 2$ and its signature is $(g; -)$. Conversely, by the uniformization theorem, every closed Riemann surface of genus $g \geq 2$ can be represented as above. In this case, by Riemann’s existence theorem, a finite group $G$ acts faithfully as a group of conformal automorphisms of $X$ if and only if there is a co-compact Fuchsian group $\Delta$ and a surjective homomorphism $\theta : \Delta \to G$ whose kernel is $\Gamma$.

2.3. Cyclic $n$-gonal Riemann surfaces. Let $X$ be a cyclic $n$-gonal Riemann surface of genus $g \geq 2$, $\tau \in \text{Aut}(X)$ be a $n$-gonal automorphism and $\pi : X \to \hat{\mathbb{C}}$ be a Galois branched cover whose deck group is the $n$-gonal group $H = \langle \tau \rangle \cong C_n$. Let $p_1, \ldots, p_s \in \hat{\mathbb{C}}$ be the branch values of $\pi$ and let us denote by $n_j \geq 2$ (which is a divisor of $n$) the branch order of $\pi$ at $p_j$.

Let $\Delta$ be a Fuchsian group such that (up to biholomorphisms) $\mathbb{H}/\Delta = X/\langle \tau \rangle$. Then $\Delta$ has signature $(0; n_1, \ldots, n_s)$ and a presentation

$$
\Delta = \langle c_1, \ldots, c_s : c_1^{n_1} = \cdots = c_s^{n_s} = 1, c_1 \cdots c_s = 1 \rangle.
$$

The branched Galois covering $\pi$ is determined by a surjective homomorphism $\rho : \Delta \to C_n = \langle \tau \rangle$ with a torsion-free kernel $\Gamma$ such that $X = \mathbb{H}/\Gamma$. (The homomorphism $\rho$ is uniquely determined up to post-composition by automorphisms of $C_n$ and pre-composition by an automorphism of $K$.) Let $\rho(c_j) = \tau^{l_j}$, where $c_j$ is as in Eq. (4), for $l_1, \ldots, l_s \in \{1, \ldots, n-1\}$. As a consequence of Harvey’s criterion [11],

(a) $n = \text{lcm}(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_s)$ for all $j$;

(b) if $n$ is even, then $\# \{ j \in \{ 1, \ldots, s \} : n/n_j \text{ is odd} \}$ is even.

The equality $c_1 \cdots c_s = 1$ is equivalent to have $l_1 + \cdots + l_s \equiv 0 \text{ mod}(n)$, and the condition for $\Gamma = \ker(\rho)$ to be torsion-free is equivalent to have $\gcd(n, l_j) = n/n_j$, for $j = 1, \ldots, s$. The surjectivity of $\rho$ is equivalent to have $\gcd(n, l_1, \ldots, l_s) = 1$, which in our case is equivalent to condition (a). Condition (b) is equivalent to saying that for $n$ even the number of $l_j$’s being odd is even, which trivially holds. Summarizing all the above,

\begin{align*}
(1) \quad & l_1, \ldots, l_s \in \{ 1, \ldots, n-1 \}, \quad & (3) \quad & \gcd(n, l_j) = n/n_j, \text{ for all } j, \\
(2) \quad & l_1 + \cdots + l_s \equiv 0 \text{ mod}(n), \quad & (4) \quad & \gcd(n, l_1, \ldots, l_s) = 1.
\end{align*}

The Riemann surface $X$ can be described by the affine curve

$$
X : \quad y^n = \prod_{j=1}^{s} (x - p_j)^{l_j},
$$

where, if one of the branched values is infinity, say $p_s = \infty$, then we need to delete the factor $(x - p_s)^{l_s}$ in the above equation.
In such an algebraic model, $\tau(x, y) = (x, \omega_n y)$, where $\omega_n = e^{2\pi i/n}$, and $\pi(x, y) = x$. The branch order of $\pi$ at $p_j$ is $n_j = n/\gcd(n, l_j)$ and, by the Riemann-Hurwitz formula, the genus $g$ of $X$ is given by

$$g = 1 + \frac{1}{2} \left( (s-2)n - \sum_{j=1}^{s} \gcd(n, l_j) \right).$$

3. PROOF OF THEOREM 1

Let $X$ be a curve given by equation Eq. (1), $\pi(x, y) = x$, and $\tau \in G = \text{Aut}(X)$. Let $N$ be the normalizer of $H = \langle \tau \rangle$ in $G$. There is a short exact sequence

$$1 \to H = \langle \tau \rangle \to N \xrightarrow{\theta} \frac{N}{H} \to 1,$$

where $\theta(\eta) \circ \pi = \pi \circ \eta$, for every $\eta \in N$.

The reduced group of automorphisms $\overline{N} = N/H < \text{PSL}_2(\mathbb{C})$ is a finite group keeping invariant the set $\{p_1, \ldots, p_s\}$.

3.1. The following describes the form of those elements of $N$.

**Lemma 1.** Let $n \in N$ and $l \in \{1, \ldots, n-1\}$ (necessarily relatively prime to $n$) such that $\eta \tau \eta^{-1} = \tau^l$. If $b = \theta(\eta)$, then $\eta(x, y) = (b(x), y^l Q(x))$, where $Q(x)$ is a suitable rational map.

**Proof.** Let us note that $\eta(x, y) = (b(x), L(x, y))$, where $L(x, y)$ is a suitable rational map. As $\eta(\tau(x, y)) = \eta(x, \omega_n y) = (b(x), L(x, \omega_n y))$ and $\tau(\eta(x, y)) = \tau^l(b(x), L(x, y)) = (b(x), \omega^l_n L(x, y))$, the condition $\eta \tau \eta^{-1} = \tau^l$ holds if and only if $L(x, \omega_n y) = \omega^l_n L(x, y)$, that is, $L(x, y) = Q(x)^l$, for a suitable rational map $Q(x) \in \mathbb{C}(x)$.

**Remark 3.** (1) Lemma 1 asserts that those $\eta \in N$ commuting with $\tau$ have the form $\eta(x, y) = (b(x), Q(x)y)$. (2) If $t \in \text{PSL}_2(\mathbb{C})$, then replacing $\pi$ by $t \circ \pi$ only exchanges the set of branch points $\{p_1, \ldots, p_s\}$ for $\{t(p_1), \ldots, t(p_s)\}$ but keeps invariant the set of exponents $l_1, \ldots, l_s$.

3.2. Let $n \in N$ and assume $\theta(\eta)$ has order $m \geq 2$. As there is a suitable $t \in \text{PSL}_2(\mathbb{C})$ such that $t \theta(\eta)^{-1}(x) = \omega_m x$, we may assume (by post-composing $\pi$ with $t$) that $\theta(\eta)(x) = \omega_m x$. So the cyclic $n$-gonal curve Eq. (6) can be written as

$$y^n = x^\alpha \prod_{j=1}^{L} (x - q_j)^{i_j, 1} (x - \omega_m q_j)^{i_j, 2} \cdots (x - \omega_m^{m-1} q_j)^{i_j, m},$$

where

(a) the factor $x^\alpha$ only appears if one of the branch values $t(p_j)$ is equal to zero, and

(b) there is the following equality of sets of exponents in Eq. (9) and Eq. (6) (where $a$ needs to be deleted if none of the $t(p_j)$’s is equal to zero)

$$\{\alpha, l_1, \ldots, l_{1,m}, l_{2,1}, \ldots, l_{2,m}, \ldots, l_{L,1}, \ldots, l_{L,m}\} = \{l_1, \ldots, l_s\}.$$
and \( r_j = l_{j,1} + \cdots + l_{j,m} \), that is,

\[
Q(x)^n y^n = \frac{(\alpha + \sum_{j=1}^{l} r_j)x^n}{\prod_{j=1}^{L}(x-q_j)^{r_j} (x-\omega_{m} q_j)^{r_j}}.
\]

(11)

In particular,

\[
Q(x)^n y^{(l-1)n} = \frac{(\alpha + \sum_{j=1}^{l} r_j)x^n}{\prod_{j=1}^{L}(x-q_j)^{r_j} (x-\omega_{m} q_j)^{r_j}}.
\]

(12)

3.3. Let us assume \( \eta \) commutes with \( \tau \), that is, \( l = 1 \). We proceed to prove that the exponents \( l_{j,i} \) are the same for every \( i = 1, \ldots, m \). As \( \theta(\eta)^m = 1 \), it follows that \( \eta^m \in \langle \tau \rangle \), from which we must have that

\[
\left( \prod_{j=0}^{m-1} Q(\omega_{m}^j) \right)^n = 1.
\]

(13)

**Claim 1.** Equation Eq. (13) asserts that \( Q(x) \) is either an \( nm \)-root of unity or it has the form

\[
Q(x) = \lambda \prod_{u=1}^{A} \frac{x-\alpha_u}{x-\omega_{m}^{u} \alpha_u},
\]

where \( \lambda^{nm} = 1 \) and \( q_u \in \{1, \ldots, m-1\} \).

**Proof.** If we write

\[
Q(x) = \lambda \prod_{u=1}^{A} \frac{x-\alpha_u}{x-\omega_{m}^{u} \alpha_u},
\]

then

\[
\prod_{j=0}^{m-1} Q(\omega_{m}^j x) = \lambda^m \prod_{j=0}^{m-1} \omega_{m}^{(A-B)j} \prod_{u=1}^{A} \frac{\omega_{m}^{u-1}(x-\omega_{m}^{u} \alpha_u)}{\prod_{v=1}^{B} (x-\omega_{m}^{v} \beta_v)} =
\]

\[
= \lambda^m \omega_{m}^{(A-B)m(m-1)/2} \prod_{u=1}^{A} (x-\alpha_u^m) \prod_{v=1}^{B} (x-\beta_v^m).
\]

Equation Eq. (13) asserts that

\[
A = B, \quad \prod_{v=1}^{B} (x-\beta_v^m) = \lambda^{nm} \prod_{u=1}^{A} (x-\alpha_u^m).
\]

So, \( \lambda^{nm} = 1 \) and, up to a permutation of indices, we may assume \( \alpha_u^m = \beta_v^m \), for \( u = 1, \ldots, A \). \( \square \)

By Claim 1, either \( l_{j,i} - l_{j,i+1} = 0 \) or \( \omega_{m}^{i-1} q_j \) must be either a zero or a pole of order \( n \) of the left side of Eq. (12), that is, each \( l_{j,i} - l_{j,i+1} \in \{0, \pm n\} \). As \( l_{j,i} \in \{1, \ldots, n-1\} \), it follows that \( l_{j,1} = \cdots = l_{j,m} \).

3.4. In the other direction, let us assume that \( l_{j,1} = \cdots = l_{j,m} = l_j \), for every \( j = 1, \ldots, L \). In this case, \( X \) has equation

\[
y^n = x^\alpha \prod_{j=1}^{L} (x-q_j^m)^{l_j}.
\]

(14)

A lifting of \( \theta(\eta) \) under \( \pi(x, y) = x \) is of the form \( \tilde{\eta}(x, y) = (\omega_{m} x, \omega_{m}^n y) \). This asserts that \( \eta = \tilde{\eta}^k \), for some \( k \in \{0, \ldots, n-1\} \), i.e., \( \eta(x, y) = (\omega_{m} x, \omega_{m}^k \omega_{m}^\alpha y) \), that is \( l = 1 \).
3.5. **A consequence.** The above permits us to observe that, if \( \tau \) is a generalized superelliptic automorphism of level \( n \), and the reduced group \( \overline{G} \) admits an element of order \( m \), then \( X \) can be represented by a cyclic \( n \)-gonal curve of the form

\[
X : \quad y^n = x^{l_0}(x^m - 1)^{l_1} \prod_{j=2}^{L} (x^m - a_j^n)^{l_j},
\]

where any one of the following Harvey’s conditions is satisfied:

1. if \( l_0 = 0 \), then \( m(l_1 + \cdots + l_L) \equiv 0 \mod(n) \) and \( \gcd(n, l_1, \ldots, l_L) = 1 \); or
2. if \( l_0 \neq 0 \), then \( \gcd(n, l_0, l_1, \ldots, l_L) = 1 \).

Note that, in (2) above, either: (2.1) \( l_0 + m(l_1 + \cdots + l_L) \equiv 0 \mod(n) \) in case \( \infty \) is not a branch value, or (2.2) \( l_0 + m(l_1 + \cdots + l_L) \neq 0 \mod(n) \) in case \( \infty \) is a branch value.

4. **Proof of Theorem 2 and Corollary 2**

Let us assume \( X \) admits two superelliptic automorphisms \( \tau \) and \( \eta \), both of level \( n \), that is, each one being central in \( G = \text{Aut}(X) \). Let \( H = \langle \tau \rangle \) and the reduced group \( \overline{G} = G/H \). We proceed to investigate when it is possible to have that \( \eta \notin H \).

4.1. **Proof of Theorem 2.** As the case \( n = 2 \) corresponds to the hyperelliptic situation, and the hyperelliptic involution is unique, necessarily \( n \geq 3 \).

**Proposition 1.** If \( \overline{G} \) is either trivial, a dihedral group of order not divisible by 4 or \( A_4 \) or \( S_4 \) or \( A_5 \), then \( \eta \in H \).

**Proof.** Assume, to the contrary, that \( \eta \notin H \). Then \( \eta \) induces a non-trivial central element of the reduced group \( \overline{G} \). As the Platonic groups and the dihedral groups of order not divisible by 4, have no nontrivial central element, this is a contradiction.

Let us assume that \( \eta \notin H \). So, by the above, \( n \geq 3 \) and \( \overline{G} \) is either a non-trivial cyclic group or a dihedral group of order a multiple of 4. Let us consider, as before, the canonical quotient homomorphism \( \theta : G \to \overline{G} \), and let \( \pi : X \to \overline{C} \) be a Galois branched cover with deck group \( H \). As \( \tau \) is central, \( K = \langle \tau, \eta \rangle < G \) is an abelian group and \( \overline{K} = K/H = \langle \theta(\eta) \rangle \approx C_m \), where \( n = md \) and \( m \geq 2 \). Since \( \theta(\eta) \) has order \( m \), \( \eta^m \in H \) and it has order \( d \). So, replacing \( \tau \) by a suitable power (still being a generator of \( H \)) we may assume that \( \eta^m = \tau^m \). Now, as noted in Section 3.5, we may assume \( X \) to be represented by a cyclic \( n \)-gonal curve of the form

\[
X : \quad y^n = x^{l_0}(x^m - 1)^{l_1} \prod_{j=2}^{L} (x^m - a_j^n)^{l_j},
\]

where one of the following Harvey’s conditions is satisfied:

1. \( l_0 = 0, m(l_1 + \cdots + l_L) \equiv 0 \mod(n) \) and \( \gcd(n, l_1, \ldots, l_L) = 1 \); or
2. \( l_0 \neq 0 \) and \( \gcd(n, l_0, l_1, \ldots, l_L) = 1 \).

In this algebraic model, \( \tau(x, y) = (x, \omega_n y), \pi(x, y) = x \) and \( \theta(\eta)(x) = \omega_m x \) (where \( \omega_i = e^{2\pi i/i} \)). In this way, \( \eta(x, y) = (\omega_m x, \omega_m^{l_0/n} y) \). Since \( \eta^m = \tau^m \) and \( \eta \) has order \( n \), we may assume the following

\[
\begin{cases} 
\text{if } l_0 \neq 0 : & \eta(x, y) = (\omega_m x, \omega_m y) \quad \text{and} \quad l_0 = m, \\
\text{if } l_0 = 0 : & \eta(x, y) = (\omega_m x, y) \quad \text{and} \quad n = m.
\end{cases}
\]

**i): Case** \( l_0 = m \). In this case, \( \eta(x, y) = (\omega_m x, \omega_m y) \) and we are in case (C2) above. The \( \eta \)-invariant algebra \( \mathbb{C}[x, y]^{\langle \eta \rangle} \) is generated by the monomials \( u = x^m, v = y^n \) together those of the form \( x^a y^b \), where \( a \in \{0, 1, \ldots, m-1\} \) and \( b \in \{0, 1, \ldots, n-1\} \).
we must have a branch set of \( \pi \) as its deck group. Let us observe that the values 0, \( y \) (18) curve be the trivial group nor can it be isomorphic to the Klein group \( \mathbb{Z}/2\mathbb{Z} \). In particular, (19) coordinates. The group generated by all of these automorphisms is \( \mathcal{U} = \langle T_1, \ldots, T_{m-1} \rangle \cong C_m \). The curve \( \mathcal{Y} \) admits the automorphisms \( T_1, \ldots, T_{m-1} \), where \( T_j \) is just an amplification of the \( t_j \)-coordinate by \( \omega_m \) and acts as the identity on all the other coordinates. The group generated by all of these automorphisms is

\[
\mathcal{U} = \langle T_1, \ldots, T_{m-1} \rangle \cong C_m.
\]

The Galois branched cover map \( \pi_U : \mathcal{Y} \to \hat{\mathbb{C}} : (u, v, t_1, \ldots, t_{m-1}) \mapsto u \) has \( \mathcal{U} \) as its deck group. Let us observe that the values 0, \( a_1^m, \ldots, a_n^m \) belongs to the branch set of \( \pi_U \). Since \( \mathcal{Y} = \mathcal{X}/(\eta) \) has genus zero and the finite abelian groups of automorphisms of the Riemann sphere are either the trivial group, a cyclic group or \( V_4 = \mathbb{C}_2 \), the group \( \mathcal{U} \) is one of these three types. As \( m \geq 2 \), the group \( \mathcal{U} \) cannot be the trivial group nor can it be isomorphic to the Klein group \( V_4 = \mathbb{C}_2 \). It follows that \( \mathcal{U} \) is a cyclic group; so \( m = 2 \) and, in particular, \( n = 2d \), where \( d \geq 2 \), and

\[
y^{2d} = x^2(x^2 - 1)^{l_1} \prod_{j=2}^{L}(x^2 - a_j^2)^{l_j}.
\]

Harvey’s condition (a) is equivalent to have \( \gcd(2d, l_1, \ldots, l_L) = 1 \), which is satisfied if some of the exponents \( l_j \) is odd. Without loss of generality, we may assume that \( l_1 \) is odd. In this case the curve \( \mathcal{Y} \) is given by

\[
\mathcal{Y} : \begin{cases}
t_1^2 = uv, \\
v = (u-1)^{l_1} \prod_{j=2}^{L}(u - a_j^2)^{l_j},
\end{cases}
\]

which is isomorphic to the curve

\[
w^2 = (u-1)^{l_1} \prod_{j=2}^{L}(u - a_j^2)^{l_j}.
\]

As this curve must have genus zero, and \( l_1 \) is odd, the number of indices \( j \in \{2, \ldots, L\} \) for which \( l_j \) is odd must be at most one.

(i) If \( l_1 \) is the only odd exponent and \( l_j = 2\widehat{l}_j \), for \( j = 2, \ldots, L \), then the condition \( \gcd(2d, l_1, \widehat{l}_2, \ldots, 2\widehat{l}_L) = 1 \) is equivalent to \( \gcd(d, \widehat{l}_1, \widehat{l}_2, \ldots, \widehat{l}_L) = 1 \).

(ii) If there are exactly two of the exponents being odd, then we may assume, without loss of generality, that \( l_1 \) and \( l_2 \) are the only odd exponents. This means that the curve in Eq. (22) is isomorphic to \( \hat{w}^2 = (u-1)^{l_1} \prod_{j=3}^{L}(u - a_j^2)^{l_j/2} \), where \( \hat{w} = w/\prod_{j=3}^{L}(u - a_j^2)^{l_j/2} \). If we write \( l_j = 2\widehat{l}_j \), for \( j = 3, \ldots, L \), then the condition \( \gcd(2d, 2l_1, l_2, 2\widehat{l}_3, \ldots, 2\widehat{l}_L) = 1 \) is equivalent to \( \gcd(d, l_1, l_2, \widehat{l}_3, \ldots, \widehat{l}_L) = 1 \).
ii): Case \( l_0 = 0 \). In this case, \( m = n \), \( \eta(x, y) = (\omega_n x, y) \) and we are in case (C1) above. The \( \eta \)-invariants algebra \( \mathbb{C}[x, y]_{(\eta)} \) is generated by the monomials \( u = x^n, v = y \). As a consequence of the invariant theory, the quotient curve \( \mathcal{X}/(\eta) \) corresponds to one of the following algebraic curves

\[
\mathcal{Y}_1 : \begin{cases} 
    v^n = (u - 1)^{l_1} 
\end{cases}
\]

or

\[
\mathcal{Y}_2 : \begin{cases} 
    v^n = (u - 1)^{l_1} \prod_{j=2}^{L} (u - a_j^n)^{l_j}.
\end{cases}
\]

As \( \mathcal{Y} \) must have genus zero and \( n \geq 3 \), we should have either \( \mathcal{Y}_1 \) or \( \mathcal{Y}_2 \) with \( L = 2 \) and \( l_1 + l_2 \equiv 0 \mod (n) \). In particular, we have one of the two cases below for \( \mathcal{X} \):

\[
\begin{align*}
\text{(1)} & \quad \mathcal{X} : \quad y^n = (x^n - 1)^{l_1}, \\
\text{(2)} & \quad \mathcal{X} : \quad y^n = (x^n - 1)^{l_1}(x^n - a_2^n)^{l_2}, \quad l_1 + l_2 \equiv 0 \mod (n).
\end{align*}
\]

Note that, for situation (1) above, we may assume \( l_1 = 1 \) (this is the classical Fermat curve of degree \( n \)). As the group of automorphisms of the classical Fermat curve of degree \( n \) is \( C_n^2 \times S_3 \), we may see that \( \tau \) is not central; that is, it is not a generalized superelliptic Riemann surface of level \( n \). In case (2), Harvey’s conditions hold exactly when \( \gcd(n, l_1, l_2) = 1 \). As \( l_1 + l_2 \equiv 0 \mod (n) \) and \( l_1, l_2 \in \{1, \ldots, n - 1\} \), we have that \( l_1 + l_2 = n \). If we write \( l_2 = n - l_1 \), then

\[
\left( \frac{x^n - 1}{x^n - a_2^n} \right)^{l_1} = \frac{y^n}{(x^n - a_2^n)^{l_1}},
\]

and by writing \( l_1 = n - l_2 \) we also have that

\[
\left( \frac{x^n - a_2^n}{x^n - 1} \right)^{l_2} = \frac{y^n}{(x^n - 1)^{l_2}}.
\]

Then the Möbius transformation \( M(x) = a_2/x \) induces the automorphism

\[
\alpha(x, y) = \left( \omega_n \frac{a_2}{x}, \frac{-a_2^b(x^n - 1)(x^n - a_2^n)}{x^n y} \right),
\]

which does not commute with \( \eta(x, y) = (\omega_n x, y) \) since \( n \geq 3 \), a contradiction.

4.2. Proof of Corollary 2. Let \( \mathcal{X} \) be a cyclic \( n \)-gonal Riemann surface admitting superelliptic automorphisms \( \tau, \eta \), both of level \( n \), such that \( \langle \tau \rangle \neq \langle \eta \rangle \). By Theorem 2, \( \mathcal{X} \) has an equation of the form as in Eq. (20), where \( n = 2d \geq 4 \). The factor \( x^2 \) in such an equation asserts that \( 0 \) is a branch value of order \( d = n/2 \).

5. A REMARK ON THE FIELD OF MODULI OF SUPERELLIPTIC CURVES

5.1. Field of definitions and the field of moduli. As a consequence of the Riemann-Roch theorem, every closed Riemann surface \( \mathcal{X} \) can be described as a complex projective irreducible algebraic curve, say defined as the common zeros of the homogeneous polynomials \( P_1, \ldots, P_r \). If \( \sigma \in \text{Gal}(\mathbb{C}) \), the group of field automorphisms of \( \mathbb{C} \), then \( \mathcal{X}^\sigma \) will denote the curve defined as the common zeros of the polynomials \( P_1^\sigma, \ldots, P_r^\sigma \), where \( P_j^\sigma \) is obtained from \( P_j \) by applying \( \sigma \) to its coefficients. The new algebraic curve \( \mathcal{X}^\sigma \) is again a closed Riemann surface of the same genus. Let us observe that, if \( \sigma, \tau \in \text{Gal}(\mathbb{C}) \), then \( \mathcal{X}^{\sigma \tau} = (\mathcal{X}^\sigma)^\tau \) (we multiply the permutations from left to right). A subfield \( L \) of \( \mathbb{C} \) is called a field of definition of \( \mathcal{X} \) if there is a curve \( \mathcal{Y} \), defined over \( L \), which is isomorphic to \( \mathcal{X} \) over \( \mathbb{C} \). Weil’s descent theorem [28] provides sufficient conditions for a given subfield of
to be a field of definition of $X$. These conditions hold if $X$ has no non-trivial automorphisms (a generic situation for $g \geq 3$).

If $G_X$ is the subgroup of $\text{Gal}(\mathbb{C})$ consisting of those $\sigma$ so that $X^\sigma$ is isomorphic to $X$, then the fixed field $M_X$ of $G_X$ is called the field of moduli of $X$. The notion of the field of moduli was originally introduced by Shimura [26] for the case of abelian varieties and later extended to more general algebraic varieties by Koizumi [17]. In that same paper, Koizumi observed that: (i) $M_X$ is the intersection of all the fields of definition of $X$, and (ii) $X$ has a field of definition being a finite extension of $M_X$.

There are examples for which the field of moduli is not a field of definition [8, 26].

In [13] the following sufficient condition for a surface to be definable over its field of moduli was obtained.

**Theorem 3.** Let $X$ be a Riemann surface of genus $g \geq 2$ admitting a subgroup $L < \text{Aut}(X)$ so that $X/L$ has genus zero. If $L$ is unique in $\text{Aut}(X)$ and the reduced group $\text{Aut}(X)/L$ is different from trivial or cyclic, then $X$ is definable over its field of moduli.

If $X$ is hyperelliptic and $L$ is the cyclic group generated by the hyperelliptic involution, then the above result is due to Huggins [15].

Another sufficient condition on a curve $X$ to be definable over its field of moduli, which in particular contains the case of quasiplatonic curves, was provided in [2]. We say that $X$ has odd signature if $X/\text{Aut}(X)$ has genus zero and in its signature one of the cone orders appears an odd number of times.

**Theorem 4.** Let $X$ be a Riemann surface of genus $g \geq 2$. If $X$ has an odd signature, then $X$ can be defined over its field of moduli.

### 5.2. Minimal fields of definition of superelliptic curves

Let $X$ be a superelliptic curve of level $n$ and $H = \langle \tau \rangle \leq \text{Aut}(X)$ be a superelliptic group of level $n$. If $X$ is non-exceptional, then $H$ is unique (Corollary 2). So, if $\text{Aut}(X)/H$ is different from trivial or cyclic, then $X$ is definable over its field of moduli by Theorem 3. By Theorem 4, the same result holds if $X$ has odd signature.

At the level of the exceptional ones, we have seen that $H$ is not unique. But, if $\eta$ is another superelliptic automorphism of level $n$, then there is a power of $\eta$ inside $H$. In this case, we have seen that the quotient of $X$ by the abelian group $K = \langle \tau, \eta \rangle$ has an odd signature. If $\text{Aut}(X) = K$, then again $X$ is definable over its field of moduli.

**Theorem 5.** Let $H \cong C_n$ be a superelliptic group of a superelliptic curve $X$. Then $X$ is definable over its field of moduli if $X$ is non-exceptional with either (i) $\text{Aut}(X)/H$ different from trivial or cyclic or (ii) $\text{Aut}(X)/H$ either trivial or cyclic and $X$ has an odd signature.

### 6. Appendix A: Algebraic equations for generalized superelliptic curves

Let $X$ be a generalized superelliptic curve of level $n$ and $\tau \in G = \text{Aut}(X)$ be a generalized superelliptic automorphism of level $n$ (so, it is central in its normalizer $N$). We proceed to describe explicit algebraic equations for $X$ and also explicit generators for $N$, by making a subtle modification of the classical method done by Horiuchi in [14] for the hyperelliptic situation.
Let \( \pi : \mathcal{X} \to \hat{\mathbb{C}} \) be a Galois branched cover with deck group \( H = \langle \tau \rangle \) and let \( \mathcal{B}_\pi = \{p_1, \ldots, p_\alpha\} \subset \hat{\mathbb{C}} \) be its set of branch values. Let \( \theta : N \to \mathbb{N} \) be the surjective homomorphism satisfying \( \theta(\eta) \circ \pi = \pi \circ \eta \), for every \( \eta \in N \). Recall that \( \mathbb{N} \) is one of the finite subgroups of \( \text{PSL}_2(\mathbb{C}) \) (as described in Section 2.1) keeping the set \( \mathcal{B}_\pi \) invariant.

Let us consider the Galois branched cover \( f = f_\mathbb{N} : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) with \( \mathbb{N} \) as its deck group (as described in Section 2.1). Let \( P(x), Q(x) \in \mathbb{C}[x] \) be relatively prime polynomials such that \( f(x) = \frac{P(x)}{Q(x)} \).

The collection \( \mathcal{B}_\pi \) is \( \mathbb{N} \)-invariant and, by Theorem 1, if for \( t \in \mathbb{N} \) it holds that \( t(p_i) = p_j \), then \( l_i = l_j \). In particular, we may consider the partition \( \mathcal{B}_\pi = \mathcal{B}_\pi^{\text{crit}} \cup \mathcal{B}_\pi^s \), where \( \mathcal{B}_\pi^{\text{crit}} \) consists of those branch values with non-trivial \( \mathbb{N} \)-stabilizer. For simplicity, we assume \( \infty \notin \mathcal{B}_\pi^s \) (but it might happen that \( \infty \in \mathcal{B}_\pi^{\text{crit}} \)).

6.1. Horiuchi’s general process.

6.1.1. Computing algebraic models. There is at most \( T \leq 3 \) disjoint \( \mathbb{N} \)-orbits of the points in \( \mathcal{B}_\pi^{\text{crit}} \).

If \( \mathbb{N} \cong C_m \), then \( T \leq 2 \); each such orbit has cardinality one.

If \( \mathbb{N} \cong D_m \), then \( T \leq 3 \); at most one orbit of cardinality 2 and at most two others, each of cardinality \( m \).

If \( \mathbb{N} \cong A_4 \), then \( T \leq 3 \); at most one orbit of cardinality 6 and at most two others, each of cardinality 4.

If \( \mathbb{N} \cong S_4 \), then \( T \leq 3 \); at most one orbit of cardinality 8, one of cardinality 6 and another of cardinality 12.

If \( \mathbb{N} \cong A_5 \), then \( T \leq 3 \); at most one orbit of cardinality 20, one of cardinality 30 and another of cardinality 12.

Let us denote these orbits (eliminating \( \infty \) from its orbit if it is a branch value of \( \pi \)) by

\[
\mathcal{O}_u^{\text{crit}} = \{q_{u,1}, \ldots, q_{u,s_u}\}, \ u = 1, \ldots, T,
\]

where \( s = s_1 + \cdots + s_T \) is the cardinality of \( \mathcal{B}_\pi^{\text{crit}} \) if \( \infty \notin \mathcal{B}_\pi^{\text{crit}} \) (otherwise, this cardinality is \( s + 1 \)).

Similarly, let the disjoint \( \mathbb{N} \)-orbits of the points in \( \mathcal{B}_\pi^s \) be given by

\[
\mathcal{O}_k^s = \{p_{k,1}, \ldots, p_{k,k/|\mathbb{N}|}\}, \ k = 1, \ldots, L,
\]

(so, \( L/|\mathbb{N}| \) is the cardinality of \( \mathcal{B}_\pi^s \)).

As, for \( k = 1, \ldots, L \),

\[
\prod_{j=1}^{\mathbb{N}} (x - p_{k,j}) = P(x) - f(p_{k,1})Q(x),
\]

our curve can be written as

\[
\mathcal{X} : \quad y^n = \prod_{u=1}^{T} R_u(x)^{\tilde{\ell}_u} \prod_{k=1}^{L} (P(x) - f(p_{k,1})Q(x))^{\tilde{\ell}_k},
\]

where

(1) \( R_u(x) = \prod_{j=1}^{s_u} (x - q_{u,j}) \)

(2) \( \tilde{\ell}_u \in \{0,1, \ldots, n-1\} \) and \( \tilde{\ell}_k \in \{1, \ldots, n-1\} \).

(3) \( \text{gcd}(n, \tilde{\ell}_1, \ldots, \tilde{\ell}_T, l_1, \ldots, l_L) = 1 \) (where we eliminate a zero if appears).

(4) if \( \infty \notin \mathcal{B}_\pi^s \), then \( \sum_{u=1}^{T} s_u \tilde{\ell}_u + \sum_{k=1}^{L} |\mathbb{N}| \tilde{\ell}_k \equiv 0 \mod n \).
(5) If $\infty \in \mathcal{O}_{\nu}^{\text{crit}}$, then
\[
(1 + s_v \hat{1}_v + \sum_{u=1,u\neq v}^T s_u \hat{1}_u + \sum_{k=1}^L |N| \hat{1}_k) \equiv 0 \mod n.
\]

6.1.2. Computing the elements of $N$. Let $\eta \in N$ and $b = \theta(\eta)$. As $\tau$ commutes with $\eta$, by Lemma 1, $\eta(x, y) = (b(x), F(x)y)$, where $F(x) \in \mathbb{C}(x)$. Below, we sketch how to compute such $F(x)$.

**Lemma 2.** Let $\mathcal{O} = \{a_1, \ldots, a_r\}$ a full $N$-orbit (in our case, this is one of the $\mathcal{O}_{\nu}^{\text{crit}}$ or $\mathcal{O}_k^*\$). If $b \in N$, then the following hold.

(1) If $\infty \notin \mathcal{O}$, then
\[
\prod_{j=1}^{r-1} (b(x) - a_j) = (b'(x))^{r/2} \left( \prod_{j=1}^{r-1} b'(a_j) \right)^{1/2} \prod_{j=1}^{r-1} (x - a_j).
\]

(2) If $a_r = \infty$ and $b(x) = \infty$, then
\[
\prod_{j=1}^{r-1} (b(x) - a_j) = (b'(x))^{(r-1)/2} \left( \prod_{j=1}^{r-1} b'(a_j) \right)^{1/2} \prod_{j=1}^{r-1} (x - a_j).
\]

(3) If $a_r = \infty$, $b(a_{r-1}) = \infty$ and $b(\infty) = a_s$, where $s \neq r - 1$, and $b(a_t) = a_{r-1}$, then
\[
\prod_{j=1}^{r-1} (b(x) - a_j) = \frac{(a_{r-1} - a_s)^{1/2}(a_{r-1} - a_t)^{1/2}}{x - a_{r-1}} (b'(x))^{(r-1)/2} \left( \prod_{j=1}^{r-2} b'(a_j) \right)^{1/2} \prod_{j=1}^{r-1} (x - a_j).
\]

(4) If $a_r = \infty$, $b(a_{r-1}) = \infty$ and $b(\infty) = a_{r-1}$, then
\[
\prod_{j=1}^{r-1} (b(x) - a_j) = -(b'(x))^{r/2} \left( \prod_{j=1}^{r-2} b'(a_j) \right)^{1/2} \prod_{j=1}^{r-1} (x - a_j).
\]

**Proof.** The equalities are consequence of the fact that, for $a, b(a) \in \mathbb{C},$
\[
b(x) - b(a) = b'(x)^{1/2}b'(a)^{1/2}(x - a).
\]

\[\square\]

If, in the above lemma, we replace $\mathcal{O}$ by $\mathcal{O}_{\nu}^{\text{crit}}$, then we obtain an equality
\[
R_u(b(x)) = \prod_{j=1}^{u-1} (b(x) - q_{u,j}) = Q_u(x) \prod_{j=1}^{u-1} (x - q_{u,j}) = Q_u(x) R_u(x).
\]

Similarly, if we replace $\mathcal{O}$ by $\mathcal{O}_k^*$ and set
\[
S_k(x) := P(b(x)) - f(p_{k,1}) Q(b(x)) = \prod_{j=1}^{[N]} (x - p_{k,j}),
\]
then we obtain an equality
\[
S_k(b(x)) = \prod_{j=1}^{[N]} (b(x) - p_{k,j}) = L_k(x) \prod_{j=1}^{[N]} (x - p_{k,j}) = L_k(x) S_k(x).
\]

It can be checked, by plugging directly into the equation for $X$, that
\[
F(x)^n = \prod_{u=1}^T Q_u(x)^{i_u} \prod_{k=1}^L L_k(x)^{i_k}.
\]
Table 1

| $N$  | Equation | Genus |
|------|----------|-------|
| $C_m$ | Eq. (33) | Eq. (34) |
| $D_m$ | Eq. (35) | Eq. (36) |
| $A_3$ | Eq. (37) | Eq. (38) |
| $S_4$ | Eq. (39) | Eq. (40) |
| $A_5$ | Eq. (41) | Eq. (42) |

6.2. *Explicit computations.* Below, for each of the possibilities for $\overline{N}$, we proceed to explicitly describe the above procedure. In the following, if $l_u > 0$, then we set $n_u = \gcd(n, l_u)$.

**Theorem 6.** Let $\mathcal{X}$ be a generalized superelliptic curve of level $n$, $\tau \in G = \Aut(\mathcal{X})$ be a generalized superelliptic automorphism of order $n$ and $N$ be the normalizer of $H = \langle \tau \rangle$ in $G$. Then, up to isomorphisms, $\mathcal{X}$, $\tau$ and $N$ are described as indicated in the above cases.

**Proof.** We will consider all cases one by one.

**Case $\overline{N} \cong C_m$:** In this case, $\overline{N} = \langle a(x) = \omega_m x \rangle$ and the curve $\mathcal{X}$ has the form

\begin{equation}
\mathcal{X} : \ y^n = x^l(x^{m-1})^t \prod_{j=2}^{s}(x^m - a_j^n)^{l_j},
\end{equation}

where (i) $a_2, \ldots, a_r \in C - \{0, 1\}$, $a_i^m \neq a_j^n$ and (ii) $\gcd(n, l_0, l_1, \ldots, l_r) = 1$. If $\alpha(x, y) = (\omega_m x, \omega_m^{l_0/n} y)$, then

\[ N = \langle \tau, \alpha : \tau^n = 1, \alpha^m = \tau^{l_0}, \tau \alpha = \alpha \tau \rangle. \]

The signature of $\mathcal{X}/H$ is

\[ \left\{ \begin{array}{ll}
0; \frac{n}{n_1}, \frac{m}{n_1}, \ldots, \frac{m}{n_r}, \frac{m}{n_r}, \frac{n}{n_r} & \text{if } l_0 = 0, \ m \sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\
0; \frac{n}{n_0}, \frac{m}{n_1}, \ldots, \frac{m}{n_r}, \frac{m}{n_r}, \frac{n}{n_r} & \text{if } l_0 \neq 0, \ l_0 + m \sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\
0; \frac{n}{n_0}, \frac{n}{n_{r+1}}, \frac{m}{n_1}, \frac{m}{n_2}, \ldots, \frac{m}{n_r} & \text{if } l_0 \neq 0, \ l_0 + m \sum_{j=1}^{r} l_j \not\equiv 0 \mod (n),
\end{array} \right. \]

where (in the last situation) $l_{r+1} \in \{1, \ldots, n-1\}$ is the class of $-(l_0 + m \sum_{j=1}^{r} l_j)$ module $n$. The signature of $\mathcal{X}/N$ is

\[ \left\{ \begin{array}{ll}
0; m, m, \frac{n}{n_1}, \frac{n}{n_2}, \ldots, \frac{n}{n_r} & \text{if } l_0 = 0, \ m \sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\
0; m, \frac{m}{n_0}, \frac{n}{n_1}, \frac{n}{n_2}, \ldots, \frac{n}{n_r} & \text{if } l_0 \neq 0, \ l_0 + m \sum_{j=1}^{r} l_j \equiv 0 \mod (n), \\
0; \frac{m}{n_0}, \frac{m}{n_{r+1}}, \frac{n}{n_1}, \frac{n}{n_2}, \ldots, \frac{n}{n_r} & \text{if } l_0 \neq 0, \ l_0 + m \sum_{j=1}^{r} l_j \not\equiv 0 \mod (n),
\end{array} \right. \]

The genus of $\mathcal{X}$ is

\begin{equation}
1 + \frac{1}{2} \left( (rm - 2)n - m \sum_{j=1}^{r} n_j \right), \quad \text{if } l_0 = 0, \ m \sum_{j=1}^{r} l_j \equiv 0 \mod (n),
\end{equation}

\begin{equation}
1 + \frac{1}{2} \left( (rm - 1)n - m \sum_{j=1}^{r} n_j \right), \quad \text{if } l_0 \neq 0, \ l_0 + m \sum_{j=1}^{r} l_j \equiv 0 \mod (n),
\end{equation}

\begin{equation}
1 + \frac{1}{2} \left( rnm - m \sum_{j=1}^{r} n_j \right), \quad \text{if } l_0 \neq 0, \ l_0 + m \sum_{j=1}^{r} l_j \not\equiv 0 \mod (n).\]
Case $\mathcal{N} \cong D_m$: In this case, $D_m := \langle a(x) = \omega_m x, b(x) = \frac{1}{x} \rangle$ and the curve $\mathcal{X}$ has the form

$$\mathcal{X} : \ y^n = x^h(x^m - 1)^{l_{r+1}}(x^m + 1)^{l_{r+2}} \prod_{j=1}^{n}(x^{2m} - (a_j^m + a_j^{-m})x^m + 1)^{l_j},$$

where (i) $\alpha_i^{\pm m} \neq a_j^{\pm m} \neq 0, \pm 1$, (ii) $2l_0 + m(l_{r+1} + l_{r+2}) + 2m(l_1 + \cdots + l_r) \equiv 0 \mod (n)$ and (iii) gcd$(n, l_0, l_1, \ldots, l_r) = 1$. If $\alpha$ and $\beta$ are as follows

$$\alpha(x, y) = (\omega_m x, \omega_m^{lo/n} y), \quad \beta(x, y) = \left(1, \frac{(-1)^{l_{r+1}/n}}{x^{l_{r+1}/n}(2l_0 + m(l_{r+1} + l_{r+2} + 2(l_1 + \cdots + l_r))/n)} y\right),$$

then

$$N = \langle \tau, \alpha, \beta : \tau^n = 1, \alpha^m = \tau^{l_0}, \beta^2 = \tau^{l_{r+1}}, \tau \alpha = \alpha \tau, \tau \beta = \beta \tau \rangle.$$

The signature of $\mathcal{X}/H$ is

$$\left(0; \frac{n}{n_0}, \frac{n}{n_1}, \frac{n}{n_2}, \ldots, \frac{n}{n_r}\right),$$

and the genus of $\mathcal{X}$ is

$$g = 1 + \frac{1}{2} \left(2m(r + 1)n - 2n_0 - m \left(n_{r+1} + n_{r+2} + 2 \sum_{j=1}^{r} n_j \right)\right).$$

Case $\mathcal{N} \cong A_4$: In this case, $A_4 := \langle a(x) = -x, b(x) = \frac{-x}{x + 1} \rangle$ and $\mathcal{X}$ has the form

$$\mathcal{X} : \ y^n = R_1(x)^{l_{r+1}} R_2(x)^{l_{r+2}} R_3(x)^{l_{r+3}} \prod_{j=1}^{n}(R_1(x)^3 + 12b_j \sqrt{3} R_3(x)^2)^{l_j},$$

where

$$R_1(x) = x^4 - 2i \sqrt{3} x^2 + 1,$n

$$R_2(x) = x^4 + 2i \sqrt{3} x^2 + 1,$n

$$R_3(x) = x^4 - 1,$n

$$f(x) = \frac{-12i \sqrt{3} R_3(x)^2}{R_1(x)^3},$$

such that (i) $b_j \neq b_i \in \mathbb{C}\setminus\{0, 1\}$, (ii) $(l_{r+1} + l_{r+2}) + 6l_{r+3} + 12(l_1 + \cdots + l_r) \equiv 0 \mod (n)$, and (iii) gcd$(n, l_1, \ldots, l_{r+3}) = 1$. If $\alpha(x, y) = (-x, (-1)^{l_{r+3}/n} y), \quad \beta(x, y) = (b(x), F(x)y),$ where

$$F(x) = \frac{12(l_{r+1} + l_{r+2})/n(1 - 1\sqrt{3} l_{r+1} + 1\sqrt{3} l_{r+2} + 8l_{r+3})/n(-64)^{l_1 + \cdots + l_r})/n}{(x + 1)^{l_{r+1} + l_{r+2} + 3l_{r+3} + 12(l_1 + \cdots + l_r)/n}}$$

then

$$N = \langle \tau, \alpha, \beta : \tau^n = 1, \alpha^2 = \tau^{l_{r+3}}, \beta^2 = \tau^{l_{r+1} + l_{r+2} + l_{r+3} + l_{r+4} + \cdots + l_r}, \alpha \beta = \alpha \tau, \tau \beta = \beta \tau \rangle.$$

The signature of $\mathcal{X}/H$ is

$$\left(0; \frac{n}{n_0}, \frac{n}{n_1}, \frac{n}{n_2}, \ldots, \frac{n}{n_r}\right),$$

and the genus of $\mathcal{X}$ is

$$g = 1 + \frac{1}{2} \left(2m(r + 1)n - 2n_0 - m \left(n_{r+1} + n_{r+2} + 2 \sum_{j=1}^{r} n_j \right)\right).$$
and the genus of $\mathcal{X}$ is
\begin{equation}
(38) \quad g = 1 - n + 2n_{r+1} + 2n_{r+2} + 3n_{r+3} - 6\sum_{j=1}^{\tau} n_j.
\end{equation}

**Case $\mathbb{N} \cong S_4$:** In this case, $S_4 := \langle a(x) = ix, b(x) = \bar{x}^{l_{r+2}} \rangle$ and $\mathcal{X}$ has the form
\begin{equation}
(39) \quad \mathcal{X} : \quad y^n = R_1(x)^l R_2(x)^l R_3(x)^{l+3} \prod_{j=1}^{\tau} (R_1(x)^3 - 108b_j R_3(x)^4)^{l_j},
\end{equation}
where
\begin{align*}
R_1(x) &= x^8 + 14x^4 + 1, \\
R_2(x) &= x^{12} - 33x^8 - 33x^4 + 1, \\
R_3(x) &= x(x^4 - 1), \\
f(x) &= \frac{R_1(x)^3}{\text{dim}_{\mathbb{C}}(x)},
\end{align*}
such that (i) $b_j \neq b_i \in \mathbb{C}\backslash\{0, 1\}$, (ii) $8l_{r+1} + 12l_{r+2} + 6l_{r+3} + 24(l_1 + \cdots + l_r) \equiv 0 \mod (n)$ and (iii) $\gcd(n, l_1, \ldots, l_{r+3}) = 1$. If
\begin{align*}
\alpha(x, y) &= (ix, i^{l_{r+3}/n}y), \\
\beta(x, y) &= (b(x), F(x)y), \\
F(x) &= \frac{16l_{r+1} + (1 - 44)^{l_{r+2} + n} (48)^{l_{r+3} + n} 496^{(l_1 + \cdots + l_r) + n}}{(x + 1)^{12l_{r+1} + 2l_{r+2} + 2l_{r+3} + 24(l_1 + \cdots + l_r) + n}},
\end{align*}
then
\begin{align*}
N &= \langle \tau, \alpha, \beta : \tau^n = 1, \alpha^4 = \tau^{l_{r+3}}, \beta^3 = \tau^{l_{r+1} + l_{r+2} + l_{r+3} + l_1 + \cdots + l_r}, \\
(\alpha \beta)^2 &= \tau, \tau \alpha = \alpha \tau, \tau \beta = \beta \tau\rangle.
\end{align*}
The signature of $\mathcal{X}/H$ is
\begin{align*}
(0; \frac{n}{l_{r+1}}, \ldots, \frac{n}{l_{r+1}}, \frac{n}{l_{r+2}}, \frac{n}{l_{r+2}}, \frac{n}{l_{r+3}}, \ldots, \frac{n}{l_{r+3}}),
\end{align*}
and the signature of $\mathcal{X}/N$ is
\begin{align*}
(0; \frac{3n}{l_{r+1}}, \frac{2n}{l_{r+2}} \frac{4n}{l_{r+3}}, \frac{n}{l_1}, \frac{n}{l_2}, \ldots, \frac{n}{l_r}),
\end{align*}
and the genus of $\mathcal{X}$ is
\begin{equation}
(40) \quad g = 1 + 2(1 + r)n - 4n_{r+1} - 6n_{r+2} - 3n_{r+3} - 12\sum_{j=1}^{\tau} n_j.
\end{equation}

**Case $\mathbb{N} \cong A_5$:** In this case,
\begin{align*}
A_5 := \langle a(x) = \omega_5 x, b(x) = \frac{(1 - \omega_5^2) x + (\omega_5^2 - \omega_3^2)}{(\omega_5 - \omega_3^2) x + (\omega_3^2 - \omega_5^2)} \rangle
\end{align*}
and $\mathcal{X}$ has the form
\begin{equation}
(41) \quad \mathcal{X} : \quad y^n = R_1(x)^l R_2(x)^l R_3(x)^{l+3} \prod_{j=1}^{\tau} (R_1(x)^3 - 1728b_j R_3(x)^5)^{l_j},
\end{equation}
where
\begin{align*}
R_1(x) &= -x^{20} + 228x^{15} - 494x^{10} - 228x^5 - 1, \\
R_2(x) &= x^{30} + 522x^{25} - 10005x^{20} - 10005x^{10} - 522x^5 + 1, \\
R_3(x) &= x(x^{10} + 11x^5 - 1), \\
f(x) &= \frac{R_3(x)^3}{1728R_3(x)^5},
\end{align*}
such that
\begin{enumerate}
\item $b_j \neq b_i \in \mathbb{C}\backslash\{0, 1\}$,
\item $20l_{r+1} + 30l_{r+2} + 12l_{r+3} + 60(l_1 + \cdots + l_r) \equiv 0 \mod (n)$,
\item $\gcd(n, l_1, \ldots, l_{r+3}) = 1$.
\end{enumerate}
\[
N = \langle \tau, \alpha, \beta : \alpha^6 = \tau^{l+3}, \beta^3 = \tau^l, (\alpha \beta)^3 = \tau^l \rangle,
\]
such that \(\alpha(x, y) = (a(x), \omega_5^{l+3/n}y)\) and \(\beta(x, y) = (b(x), F(x)y)\), where \(F(x)\) is a rational map satisfying
\[
F(b^2(x)) = F(b(x)) \cdot F(x) = \omega_n^l,
\]
for a suitable \(l \in \{0, \ldots, n - 1\}\), and
\[
F(x)^n = T_{1}^{l+3(l_1 + \cdots + l_s)}(x)T_{2}^{l+2}(x)T_{3}^{l+3}(x),
\]
where \(T_j(x) = \frac{R_j(b(x))}{R_j(x)}\), for \(j = 1, 2, 3\).

The signature of \(\mathcal{X}/H\) is
\[
(0; \frac{n}{n}, \frac{20}{n}, \frac{n}{n}, \frac{30}{n}, \frac{\cdot}{\cdot}, \frac{12}{n}, \frac{n}{n}, \frac{\cdot}{\cdot}, \frac{n}{n}, \frac{n}{n}, \frac{\cdot}{\cdot}, \frac{n}{n}, \frac{n}{n}, \frac{\cdot}{\cdot}, \frac{n}{n})
\]
and the signature of \(\mathcal{X}/N\) is
\[
(0; \frac{3n}{n}, \frac{2n}{n}, \frac{5n}{n}, \frac{n}{n}, \frac{n}{n}, \frac{n}{n}, \frac{\cdot}{\cdot})
\]
and the genus of \(\mathcal{X}\) is
\[
g = 1 + 30(r+1)n - 10n_{r+1} - 15n_{r+2} - 6n_{r+3} - 30 \sum_{j=1}^{r} n_j.
\]

\[\square\]

\section{Appendix B: Computing cyclic \(n\)-gonal curves}

Consider the collection \(\mathcal{F}_g\) of all the tuples \((n, s; n_1, \ldots, n_s)\) satisfying the following Harvey’s conditions:
\begin{enumerate}
\item \(n \geq 2\), \(s \geq 3\), \(2 \leq n_1 \leq n_2 \leq \cdots \leq n_s \leq n\);
\item \(n_j\) is a divisor of \(n\), for each \(j = 1, \ldots, s\);
\item \(\text{lcm}(n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_s) = n\), for every \(j = 1, \ldots, s\);
\item if \(n\) is even, then \(#\{j \in \{1, \ldots, s\} : n/n_j\ \text{is odd}\}\) is even;
\item \(2(g-1) = n \left( s - 2 - \sum_{j=1}^{s} n_j^{-1} \right) \).
\end{enumerate}

For each tuple \((n, s; n_1, \ldots, n_s) \in \mathcal{F}_g\) we consider the collection \(\mathcal{F}_g(n, s; n_1, \ldots, n_s)\) of tuples \((l_1, \ldots, l_s)\) so that
\begin{enumerate}
\item \(l_1, \ldots, l_s \in \{1, \ldots, n - 1\}\);
\item \(l_1 + \cdots + l_s \equiv 0 \ \text{mod}(n)\);
\item \(\text{gcd}(n, l_j) = n/n_j\), for each \(j = 1, \ldots, s\).
\end{enumerate}

Now, for each such tuple \((l_1, \ldots, l_s) \in \mathcal{F}_g(n, s; n_1, \ldots, n_s)\) we may consider the epimorphism
\[
\theta : \Delta = \langle c_1, \ldots, c_s : c_1^{n_1} = \cdots = c_s^{n_s} = c_1 \cdots c_s = 1 \rangle \to C_n = \langle \tau \rangle : c_j \mapsto \tau^{l_j}.
\]

Our assumptions ensure that the kernel \(\Gamma = \ker(\theta)\) is a torsion free normal cocompact Fuchsian subgroup of \(\Delta\) with \(\mathcal{X} = \mathbb{H}/\Gamma\) a closed Riemann surface of genus \(g\) admitting a cyclic group \(H \cong C_n\) as a group of conformal automorphisms with quotient orbifold \(\mathcal{X}/H = \mathbb{H}/\Delta\); a genus zero orbifold with exactly \(s\) cone points of respective orders \(n_1, \ldots, n_s\). The surface \(\mathcal{X}\) corresponds to a cyclic \(n\)-gonal curve
\[
C(n, s; l_1, \ldots, l_s; p_1, \ldots, p_s) : \ y^n = \prod_{j=1}^{s} (x - p_j)^{l_j},
\]
for pairwise different values \(p_1, \ldots, p_s \in \mathbb{C}\), and \(H\) generated by \(\tau(x, y) = (x, \omega_n y)\).

Different tuples \((l_1, \ldots, l_s), (l'_1, \ldots, l'_s) \in \mathcal{F}_g(n, s; n_1, \ldots, n_s)\) might provide isomorphic pairs \((\mathcal{X}, H)\) and \((\mathcal{X}', H')\) (i.e., there is an isomorphism between the Riemann surfaces conjugating the cyclic groups). In general this is a difficult problem.
to determine if different tuples define isomorphic pairs. But, in the non-exceptional fully generalized superelliptic situation (see Theorem 2) the uniqueness of the superelliptic cyclic group of level \( n \) permits us to see that \((X, H)\) and \((X', H')\) are isomorphic pairs if and only if the corresponding curves \( C(n, s; l_1, \ldots, l_s; p_1, \ldots, p_s) \) and \( C(n, s; l'_1, \ldots, l'_s; p'_1, \ldots, p'_s) \) are isomorphic, this last being equivalent to the existence of Möbius transformation \( t \in PSL_2(\mathbb{C}) \), a permutation \( \eta \in S_n \) and an element \( u \in \{1, \ldots, n-1\} \) with \( \gcd(u, n) = 1 \), such that

(a) \( l'_j \equiv ul_{\eta(j)} \mod (n) \), for \( j = 1, \ldots, s \),
(b) \( p'_{\eta(j)} = t(p_j) \), for \( j = 1, \ldots, s \).

The above (together with Theorem 1) may be used to construct all the possible (generalized) superelliptic curves of lower genus in a similar fashion as done in [20] for the superelliptic case.

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