SUPERSOLVABLE SIMPLICIAL ARRANGEMENTS

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Abstract. Simplicial arrangements are classical objects in discrete geometry. Their classification remains an open problem but there is a list conjectured to be complete at least for rank three. A further important class in the theory of hyperplane arrangements with particularly nice geometric, algebraic, topological, and combinatorial properties are the supersolvable arrangements. In this paper we give a complete classification of supersolvable simplicial arrangements (in all ranks). For each fixed rank, our classification already includes almost all known simplicial arrangements. Surprisingly, for irreducible simplicial arrangements of rank greater than three, our result shows that supersolvability imposes a strong integrality property; such an arrangement is called crystallographic. Furthermore we introduce Coxeter graphs for simplicial arrangements which serve as our main tool of investigation.

1. Introduction

A simplicial arrangement is a finite set of hyperplanes, i.e. codimension one subspaces, in a finite dimensional real vector space such that the ambient space is cut into simplicial cones by these hyperplanes. They were introduced by E. Melchior [Mel41] in 1941 by the means of triangulations of the projective plane by a finite set of projective lines.

B. Grünbaum [Grü09] gave a list of rank 3 simplicial arrangements, the slightly extended list [Cun12] is conjectured to be complete. But not much is known about simplicial arrangements of higher rank. In a series of papers I. Heckenberger and the first author investigate a class of objects called finite Weyl groupoids, a generalization of Weyl groups. Their work results in a complete classification of these objects, [CH15]. Since Weyl groupoids are in one to one correspondence with crystallographic arrangements [Cun11a], and these constitute a large subclass of the known simplicial arrangements, this explains a large subset of the arrangements in Grünbaum’s list.

The list given by Grünbaum contains two infinite series of irreducible simplicial arrangements of rank three parametrized by positive integers. They are denoted \( R(1) = \{ A(2n, 1) \mid n \geq 3 \} \) and \( R(2) = \{ A(4m + 1, 1) \mid m \geq 2 \} \). The irreducible simplicial 3-arrangements which do not belong to one of these infinite classes are called sporadic. One observes that each of the 94 sporadic arrangements in [Cun12] consists of no more than 37 hyperplanes. So the following is conjectured:

**Conjecture 1.1** (cf. [CG15 Conj. 1.6]). Let \( A \) be an irreducible simplicial arrangement of rank three. If \( |A| > 37 \) then \( A \in R(1) \cup R(2) \).

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D. Geis and the first author observed that simpliciality is a purely combinatorial property of the intersection lattice of an arrangement [CG15]. This combinatorial characterization suggests a connection of the class of simplicial arrangements with other classes of arrangements which can be defined combinatorially.

Supersolvable arrangements were first considered by R. Stanley [Sta72]. They are now a well studied class of arrangements. Supersolvable arrangements possess particularly nice algebraic, geometric, topological, and combinatorial properties, cf. [OT92, Theorems 2.63, 3.81, 4.58, 5.113]. Looking at the list of all known simplicial arrangements (including the known higher rank cases) one further observes that almost all of them belong to the class of supersolvable arrangements.

As the list (at least for rank 3) is conjectured to be complete and a conceptional approach towards a general classification is still missing, one might ask if there is an approach for a subclass with additional properties, e.g. supersolvable simplicial arrangements. This approach is chosen in the present article resulting in our following main theorem, a complete classification (for rank 3 up to lattice equivalence) of (irreducible) supersolvable simplicial arrangements:

**Theorem 1.2.** Let $A$ be an irreducible supersolvable simplicial $\ell$-arrangement, ($\ell \geq 3$). Then for $A$ one of the following cases holds:

1. $\ell = 3$ and $A$ is $L$-equivalent to exactly one of the arrangements in $\mathcal{R}(1) \cup \mathcal{R}(2)$, or
2. $\ell \geq 4$ and $A$ is linearly isomorphic to exactly one of the reflection arrangements $A(A_{\ell})$, $A(C_{\ell})$ or to $A_{\ell-1}^{\ell-1} = A(C_{\ell}) \setminus \{x_1 = 0\}$. In particular $A$ is crystallographic.

As a result of Part (1) of the above theorem we can reformulate Conjecture 1.1 in the following way:

**Conjecture 1.3.** Let $A$ be an irreducible simplicial 3-arrangement. If $|A| > 37$ then $A$ is supersolvable.

The article is organized as follows. Firstly we recall the basic notions from the theory of hyperplane arrangements and some properties of supersolvable and simplicial arrangements which we frequently need later on. In Subsection 2.2 we further comment on the more general notion of combinatorial simpliciality and its behavior with respect to some standard constructions for arrangements. In Section 3 we introduce Coxeter graphs, our main tool for a detailed investigation of simplicial arrangements. In the last three sections we prove our main theorem giving the aforementioned classification.

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2. Recollection and Preliminaries

We review the required notions and definitions, cf. [OT92]. Furthermore in Subsection 2.2 we prove some basic properties of simplicial arrangements.
2.1. Arrangements of hyperplanes. Let $\mathbb{K}$ be a field. An $\ell$-arrangement of hyperplanes is a pair $(\mathcal{A}, V)$, where $\mathcal{A}$ is a finite set of hyperplanes (codimension 1 subspaces) in the finite dimensional vector space $V \cong \mathbb{K}^\ell$. For $(\mathcal{A}, V)$ we simply write $\mathcal{A}$ if the vector space $V$ is unambiguous. We denote the empty $\ell$-arrangement by $\Phi_\ell$.

If $\alpha \in V^*$ is a linear form, we write $\alpha^+ = \ker(\alpha)$ and interpret $\alpha$ as a normal vector for the hyperplane $H = \alpha^\perp$. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $V = \mathbb{R}^\ell$. If we choose a basis $x_1, \ldots, x_\ell$ for $V^*$ and if $\alpha_j = \sum_{i=1}^\ell a_{ij} x_i \in V^*$ such that $H_j = \alpha_j^\perp$ then we say that $\mathcal{A}$ is given explicitly by the matrix $(a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq n} \in \mathbb{K}^{\ell \times n}$.

The intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$ is the set of all subspaces $X$ of $V$ of the form $X = H_1 \cap \ldots \cap H_r$, with $\{H_1, \ldots, H_r\} \subseteq \mathcal{A}$, partially ordered by reverse inclusion:

$$X \leq Y \iff Y \subseteq X,$$

for $X, Y \in L(\mathcal{A})$.

If $X \in L(\mathcal{A})$, then the rank $r(X)$ of $X$ is defined as $r(X) := \ell - \dim X$, i.e. the codimension of $X$ and the rank of the arrangement $\mathcal{A}$ is defined as $r(\mathcal{A}) := r(T(\mathcal{A}))$ where $T(\mathcal{A}) := \bigcap_{H \in \mathcal{A}} H$ is the center of $\mathcal{A}$. An $\ell$-arrangement $\mathcal{A}$ is called essential if $r(\mathcal{A}) = \ell$.

For $X \in L(\mathcal{A})$, we define the localization

$$A_X := \{H \in \mathcal{A} \mid X \subseteq H\}$$

of $\mathcal{A}$ at $X$, and the restriction of $\mathcal{A}$ to $X$, $(\mathcal{A}^X, X)$, where

$$A_X := \{X \cap H \mid H \in \mathcal{A} \setminus A_X\}.$$

For $X, Y \in L(\mathcal{A})$ with $X \leq Y$ we define the interval

$$[X, Y] = \{Z \in L(\mathcal{A}) \mid X \leq Z \leq Y\}.$$  

Note that we have $(\mathcal{A}_Y)^X = (\mathcal{A}^X)_Y$, and $L((\mathcal{A}_Y)^X) \cong [X, Y]$, i.e. intervals in $L(\mathcal{A})$ are again intersection lattices of restricted and localized arrangements.

For $0 \leq q \leq \ell$ we write $L_q(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid r(X) = q\}$. If $X$ is a subspace of $V$ and $X \subseteq H$ for all $H \in \mathcal{A}$ then $H/X$ is a hyperplane in $V/X$ for all $H \in \mathcal{A}$ and we can define the quotient arrangement $(\mathcal{A}/X, V/X)$ by $A/X := \{H/X \mid H \in \mathcal{A}\}$.

If $(\mathcal{A}, V)$ is not essential, i.e. $\dim(T(\mathcal{A})) > 0$, we sometimes identify it with the essential $r(\mathcal{A})$-arrangement $(\mathcal{A}/T(\mathcal{A}), V/T(\mathcal{A}))$.

Two arrangements $\mathcal{A}$ and $\mathcal{B}$ in $V$ are lattice equivalent or $L$-equivalent if $L(\mathcal{A}) \cong L(\mathcal{B})$ as lattices and in this case we write $\mathcal{A} \sim_L \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are arrangements in $V$ such that there is a $\varphi \in \text{GL}(V)$ with $\mathcal{B} = \varphi(\mathcal{A}) = \{\varphi(H) \mid H \in \mathcal{A}\}$ then we say that $\mathcal{A}$ is (linearly) isomorphic to $\mathcal{B}$.

The product $\mathcal{A} = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$ of two arrangements $(\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2)$ is defined by

$$\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus H_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\},$$

see [OT92 Def. 2.13]. In particular $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2|$. If an arrangement $\mathcal{A}$ can be written as a non-trivial product $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, i.e. $\mathcal{A}_i \neq \Phi_0$, then $\mathcal{A}$ is called reducible, otherwise irreducible.

**Proposition 2.1** ([OT92 Prop. 2.14]). Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product. Define a partial order on $L(\mathcal{A}_1) \times L(\mathcal{A}_2)$ by

$$(X_1, X_2) \leq (Y_1, Y_2) \iff X_1 \leq Y_1 \text{ and } X_2 \leq Y_2,$$
for \((X_1, X_2), (Y_1, Y_2) \in L(A_1) \times L(A_2)\). Then there is an isomorphism of lattices

\[
\pi : L(A_1) \times L(A_2) \rightarrow L(A_1 \times A_2)
\]

\[
(X_1, X_2) \mapsto X_1 \oplus X_2.
\]

**Corollary 2.2.** Let \(A = A_1 \times A_2\) be a product and \(X = X_1 \oplus X_2 \in L(A)\). Then there is an isomorphism of lattices

\[
\pi : L(A_1) \times L(A_2) \rightarrow L(A_1 \times A_2)
\]

\[
X \mapsto X_1 \oplus X_2.
\]

For an arrangement \(A\) the Möbius function \(\mu : L(A) \rightarrow \mathbb{Z}\) is defined by:

\[
\mu(X) = \begin{cases} 
1 & \text{if } X = V, \\
- \sum_{V \leq Y < X} \mu(Y) & \text{if } X \neq V.
\end{cases}
\]

We denote by \(\chi_A(t)\) the characteristic polynomial of \(A\) which is defined by:

\[
\chi_A(t) = \sum_{X \in L(A)} \mu(X) t^{\dim(X)}.
\]

**Remark 2.3.** If \(A\) is a 3-arrangement then the characteristic polynomial is given by

\[
\chi_A(t) = t^3 + \mu_1 t^2 + \mu_2 t + \mu_3,
\]

with

\[
\mu_1 = -|A|, \quad \mu_2 = \sum_{X \in L_2(A)} (|A_X| - 1), \quad \mu_3 = -1 - \mu_1 - \mu_2.
\]

**Lemma 2.4** ([OT92, Lem. 2.50]). Let \(A = A_1 \times A_2\) be a product of two arrangements. Then

\[
\chi_A(t) = \chi_{A_1}(t) \chi_{A_2}(t).
\]

We state the following geometric theorem generalizing the well known Sylvester-Gallai theorem in its dual version for real arrangements. It was first proved by Motzkin [Mot51] for \(\ell = 4\) and later by Hansen [Han65] for all \(\ell\).

**Theorem 2.5** (Hansen-Motzkin). Let \(A\) be an essential \(\ell\)-arrangement over \(\mathbb{R}\), \(\ell \geq 3\). Then there is an \(X \in L_{\ell-1}(A)\) and an \(H \in A\) such that \(X = H \cap Y\) for a \(Y \in L_{\ell-2}(A)\), and \(A_X = A_Y \cup \{H\}\). In particular \(A_X/X\) is reducible with \(A_X/X \cong A_Y/Y \times \{\{0\}\}\).

### 2.2. Simplicial arrangements.

Many of the notions in this subsection were introduced in the more general setting of simplicial arrangements on convex cones and Tits arrangements in [CMW17].

We firstly recall the definition of a simplicial arrangement.

**Definition 2.6.** Let \(A\) be an arrangement in a finite dimensional real vector space \(V\). Then \(A\) is called simplicial if every connected component of \(V \setminus \bigcup_{H \in A} H\) is an open simplicial cone. We denote by \(\mathcal{K}(A)\) the set of connected components of \(V \setminus \bigcup_{H \in A} H\); a \(K \in \mathcal{K}(A)\) is called a chamber.

Note that the only simplicial 1-arrangement is the arrangement \(A = \{\{0\}\}\), i.e. the non empty one, and every real 2-arrangement with more than one hyperplane is simplicial.

There are the following classical examples of simplicial arrangements.
**Example 2.7.** Let $W \leq \text{GL}(V)$ be a finite real reflection group acting on the real vector space $V$, i.e. a finite Coxeter group. Suppose that $W$ has full rank, i.e. $\text{rank}(W) = \dim(V)$. Then the reflection arrangement $(\mathcal{A}(W), V)$, (also called Coxeter arrangement), consisting of all the reflection hyperplanes of $W$ is a simplicial arrangement.

**Example 2.8.** For $0 \leq k \leq \ell$ let $\mathcal{A}^k_{\ell}$ be the $\ell$-arrangement defined as follows

$$\mathcal{A}^k_{\ell} := \{\ker(x_i \pm x_j) \mid 1 \leq i < j \leq \ell\} \cup \{\ker(x_i) \mid 1 \leq i \leq k\}.$$ 

The arrangements $\mathcal{A}^k_{\ell}$ are simplicial, cf. [CH15]. In particular $\mathcal{A}^0_{\ell} = \mathcal{A}(D_{\ell})$ and $\mathcal{A}^\ell_{\ell} = \mathcal{A}(C_{\ell})$ are the reflection arrangements of the finite reflection groups of type $D_{\ell}$ and $C_{\ell}$ respectively.

**Definition 2.9.** Let $\mathcal{A}$ be a simplicial $\ell$-arrangement in the real vector space $V$. For $\alpha \in V^*$ we write $\alpha^+ = \alpha^{-1}(\mathbb{R}_{>0})$ and $\alpha^- = (-\alpha)^+$ for the positive respectively negative open half-space defined by $\alpha$.

For $K \in \mathcal{K}(\mathcal{A})$ define the 	extit{walls} of $K$ as

$$W^K := \{H \in \mathcal{A} \mid \dim(H \cap \overline{K}) = \ell - 1\}.$$ 

If $R \subseteq V^*$ is a finite set such that $\mathcal{A} = \{\alpha^+ \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm \alpha\}$ for all $\alpha \in R$ then $R$ is called a (reduced) root system for $\mathcal{A}$.

If $B^K \subseteq V^*$ is such that $|B^K| = |W^K|$, $W^K = \{\alpha^+ \mid \alpha \in B^K\}$ and $K = \cap_{\alpha \in B^K} \alpha^+$ then $B^K$ is called a basis for $K$.

If $R$ is a root system for $\mathcal{A}$ we obtain a basis for $K$ as

$$B^K_R := \{\alpha \in R \mid \alpha^+ \in W^K \text{ and } \alpha^+ \cap K = K\}.$$ 

Furthermore for $\gamma \in B^K$ let $K_\gamma$ be the unique adjacent chamber in $\mathcal{K}(\mathcal{A})$, such that $\langle K \cap \overline{K_\gamma} \rangle = \gamma^+$ (the linear span of $K \cap \overline{K_\gamma}$). If there is a chosen numbering of $B^K = \{\alpha_1, \ldots, \alpha_\ell\}$ then we simply write $K_i = K_{\alpha_i}$.

**Remark 2.10.** The notions $W^K$, $R$ and $B^K$ also make sense for a not necessarily simplicial real arrangement $\mathcal{A}$. Since the normals of the facets of a cone constitute a basis if and only if the cone is simplicial, we observe that $B^K$ is indeed a basis of $V^*$ for all $K \in \mathcal{K}(\mathcal{A})$ if and only if $\mathcal{A}$ is simplicial.

The following notion was first introduced in [Cun11a, Def. 2.3].

**Definition 2.11.** Let $\mathcal{A}$ be a simplicial arrangement. If there exists a root system $R \subseteq V^*$ for $\mathcal{A}$ such that for all $K \in \mathcal{K}(\mathcal{A})$ we have

$$R \subseteq \sum_{\alpha \in B^K_R} \mathbb{Z}\alpha,$$

then $\mathcal{A}$ is called 	extit{crystallographic} and in this case we call $R$ a 	extit{crystallographic root system} for $\mathcal{A}$.

**Example 2.12.** Let $W$ be a Weyl group, i.e. a crystallographic finite real reflection group with (reduced) root system $\Phi(W)$. Then the Weyl arrangement $\mathcal{A}(W) = \{\alpha^+ \mid \alpha \in \Phi(W)\}$ is a crystallographic arrangement with crystallographic root system $R = \Phi(W)$.
A complete classification of crystallographic arrangements by finite Weyl groupoids was obtained in \cite{CH15}, see also \cite{Cun11a}. It is worth mentioning that the class of crystallographic arrangements is much bigger than the class of Weyl arrangements with many more (74) sporadic cases. However, it turns out that irreducible crystallographic arrangements of rank greater or equal to 4 are all restrictions of (irreducible) Weyl arrangements (see for example \cite[Thm. 3.7]{CL17}).

**Theorem 2.13.** Let $A$ be an irreducible simplicial $\ell$-arrangement with $\ell \geq 4$. Then it is crystallographic if and only if it is a restriction of some (irreducible) Weyl arrangement.

We recall the following combinatorial characterization of simplicial 3-arrangements.

**Lemma 2.14.** \cite[Cor. 2.7]{CG15} Let $A$ be a 3-arrangement. Then $A$ is simplicial if and only if

$$
\mu_2 := \sum_{X \in L_2(A)} (|A_X| - 1) = 2|L_2(A)| - 3.
$$

More generally real simplicial $\ell$-arrangements are characterized by the next combinatorial property.

**Lemma 2.15.** \cite[Cor. 2.4]{CG15} Let $A$ be an $\ell$-arrangement. Then $A$ is simplicial if and only if

$$
\ell|\chi_A(-1)| - 2 \sum_{H \in A} |\chi_A H(-1)| = 0.
$$

**Definition 2.16.** Let $K$ be any field and $A$ an arrangement in $V = K^\ell$. Define

$$
s(A) := \ell|\chi_A(-1)| - 2 \sum_{H \in A} |\chi_A H(-1)|.
$$

If $A$ satisfies $s(A) = 0$ then $A$ is called *combinatorially simplicial*, see \cite{CG15}.

Simpliciality, at least geometrically for real arrangements, is compatible with taking localizations and restrictions, compare with the more general statements in \cite{CMW17}.

**Lemma 2.17.** Let $A$ be a simplicial arrangement over $\mathbb{R}$ and $X \in L(A)$. Then we have

1. $(A_X/X, V/X)$ is simplicial,
2. $(A^X, X)$ is simplicial.

**Proof.** Let $H_1, \ldots, H_{r(X)}$ be the walls of a chamber $K_X$ in $A_X$. They are a subset of the walls of a chamber $K \in K(A)$. If $\alpha_1, \ldots, \alpha_{r(X)}$ are corresponding normals of these walls pointing to the inside of $K$ and also $K_X$ then they are linearly independent, hence $K_X/X$ is a simplicial cone by Remark 2.10 and $A_X/X$ is simplicial.

Since every face of a simplicial cone is a simplicial cone, Statement (2) follows directly.

**Example 2.18.** Let $A = A(W)$ be the Coxeter arrangement of the finite real reflection group $W$ in $V$ and let $X \in L(A)$. Then $A_X/X$ is a reflection arrangement, namely the Coxeter arrangement of a parabolic subgroup of $W$. The arrangement $A_X/X$ is simplicial in accordance with Lemma 2.17(1).
In the next example we see that the bigger class of combinatorially simplicial arrangements defined over arbitrary fields is neither closed under taking localizations nor closed under taking restrictions.

**Example 2.19.** Let \( V = \mathbb{C}^4 \), \( \zeta = -\frac{1}{2}(1 - \sqrt{3}i) \) be a primitive third root of unity and \((A, V)\) the complex 4-arrangement containing 18 hyperplanes and defined by
\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & -\zeta & -\zeta^2 & -1 & -\zeta & -\zeta^2 \\
1 & 0 & 0 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & -\zeta & -\zeta^2 & -1 & -\zeta & -\zeta^2
\end{pmatrix}.
\]

Note that \( A \) is a subarrangement of the reflection arrangement of the complex reflection group \( G(3, 1, 4) \), see [OT92, Ch. 6.4] for a definition of these reflection arrangements. This is to say if \( B := A(G(3, 1, 4)) \)
\[
= \{ \ker(x_i - \zeta^kx_j) \mid 1 \leq i < j \leq 4, 0 \leq k \leq 2 \}
\cup \{ \ker(x_i) \mid 1 \leq i \leq 4 \},
\]
then we obtain \( A \) by removing 4 hyperplanes,
\[
A = B \setminus \{ \ker(x_1), \ker(x_2), \ker(x_3), \ker(x_3 - x_4) \}.
\]

A quick calculation shows that \( A \) satisfies \( s(A) = 0 \) so it is combinatorially simplicial. While for the reflection arrangement \( B \) all localizations and restrictions are again combinatorially simplicial, localizing \( A \) at the rank 3 intersection \( X = H_1 \cap H_2 \cap H_3 \in L(A) \), where the hyperplane \( H_i \) corresponds to the \( i \)-th column of the defining matrix above, yields the 3-arrangement \( C = A_X/X \). It contains 10 hyperplanes and is given by
\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & -\zeta & -\zeta^2 & -1
\end{pmatrix}.
\]

For \( C \) we have \( s(C) = 4 \), so it is not combinatorially simplicial.

Now let \( H = H_8 = (1, 0, 0, -\zeta) \perp A \). Then \( \mathcal{D} := A^H \) contains 10 hyperplanes and may be defined by
\[
\begin{pmatrix}
1 & \zeta & 1 & 0 & 0 & 0 & 0 & -1 & \zeta & 1 \\
0 & 0 & 0 & 1 & \zeta & 1 & 0 & \zeta & -1 & -1 \\
0 & 1 & -1 & 0 & 1 & -1 & 1 & 1 & 1 & 0
\end{pmatrix}.
\]

For \( \mathcal{D} \) we have \( s(\mathcal{D}) = 4 \), thus it is also not combinatorially simplicial.

The product construction described above is compatible with simplicity.

**Proposition 2.20.** Let \( A_1, A_2 \) be combinatorially simplicial arrangements in \( \mathbb{K}^{\ell_1} \) and \( \mathbb{K}^{\ell_2} \) respectively. Then the product \( A = A_1 \times A_2 \) is combinatorially simplicial.

**Proof.** Let \( A_1 \) and \( A_2 \) be combinatorially simplicial. Then we have
\[
s(A_1) = \ell_1|\chi_{A_1}(-1)| - 2 \sum_{H \in A_1} |\chi_{A_1^H}(-1)| = 0,
\]
and
\[ s(A_2) = \ell_2|\chi_{A_2}(-1)| - 2 \sum_{H \in A_2} |\chi_{A_2^H}(-1)| = 0. \]

By Lemma 2.4 we have \( \chi_A(t) = \chi_{A_1}(t)\chi_{A_2}(t) \). By Corollary 2.2 we get
\[
s(A) = \ell|\chi_A(-1)| - 2 \sum_{H \in A} |\chi_{A^H}(-1)|
\]
\[
= (\ell_1 + \ell_2)|\chi_{A_1}(-1)|\chi_{A_2}(-1)|
- 2 \sum_{H \in A_1} |\chi_{A_1^H}(-1)| - 2 \sum_{H \in A_2} |\chi_{A_2^H}(-1)|
\]
\[
= |\chi_{A_2}(-1)|(|\chi_{A_1}(-1)| - 2 \sum_{H \in A_1} |\chi_{A_1^H}(-1)|)
+ |\chi_{A_1}(-1)|(|\chi_{A_2}(-1)| - 2 \sum_{H \in A_2} |\chi_{A_2^H}(-1)|)
\]
\[
= |\chi_{A_2}(-1)|s(A_1) + |\chi_{A_1}(-1)|s(A_2) = 0.
\]

Hence \( A \) is combinatorially simplicial.

**Proposition 2.21.** Let \((A_1, V_1)\) and \((A_2, V_2)\) be two real arrangements. Then the product \((A_1 \times A_2, V)\) with \(V = V_1 \oplus V_2\) is simplicial if and only if \(A_1\) and \(A_2\) are both simplicial.

**Proof.** If \(A_1\) and \(A_2\) are simplicial, then \(A = A_1 \times A_2\) is simplicial by Proposition 2.20. Conversely, let \(A = A_1 \times A_2\) be simplicial. Then \(A_i\) is isomorphic to \(A_{X_i}/X_i\) for \(i = 1, 2\) as \(r(X_i)\)-arrangements in \(V/X_i\), where \(X_i = \{0\} \oplus V_{3-i}\). But these localizations regarded as essential arrangements in quotient spaces are simplicial by Lemma 2.17.

Combinatorial simpliciality of \(A_1 \times A_2\) does not imply combinatorial simpliciality of \(A_1\) and \(A_2\) in general:

**Example 2.22.** Let \(\zeta\), \(A\) and \(D\) be as in Example 2.19. Let \(A_1 = D\) and \(A_2 = A^H\) where \(H = H_5 = (1, 0, -\zeta, 0)^\perp\) as in Example 2.19. Define \(\omega := \frac{1}{3}(1 - \zeta)\). Then \(A_2\) is given by
\[
\begin{pmatrix}
1 & 0 & \omega & \omega & \omega & \omega & 0 & 0 \\
0 & 0 & 1 & \zeta & \zeta^2 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & 1 & 1
\end{pmatrix}.
\]

Recall that for the non combinatorially simplicial arrangement \(A_1\) we have \(s(A_1) = 4\). Furthermore \(\chi_{A_1}(t) = (t - 1)(t - 4)(t - 5) = \chi_{A_2}(t)\), and \(s(A_2) = -4\). So \(A_2\) is also not combinatorially simplicial. But, similar to the proof of Proposition 2.21 for \(A_1 \times A_2\) we have
\[
s(A_1 \times A_2) = |\chi_{A_2}(-1)|s(A_1) + |\chi_{A_1}(-1)|s(A_2)
= |\chi_{A_1}(-1)|s(A_1) + |\chi_{A_1}(-1)|s(A_2)
= |\chi_{A_1}(-1)|(s(A_1) + s(A_2))
= |\chi_{A_1}(-1)|(4 - 4)
= 0.
\]
So the product $A_1 \times A_2$ is combinatorially simplicial.

The following is true for all real simplicial arrangements, cf. [CMW17, Lem. 3.29].

**Lemma 2.23.** Let $(A, V)$ be a real simplicial arrangement, $K \in \mathcal{K}(A)$ a chamber with basis $B^K = \{\alpha_1, \ldots, \alpha_l\}$. Then for $\beta \in V^*$ with $\beta^\perp \in W^K_i$ and $K_i \subseteq \beta^+$ we have

1. $\beta \in \mathbb{R}_{\leq 0} \alpha_i$ if $\beta^\perp = \alpha_i^\perp$, or
2. $\beta \in \sum_{k=1}^{\ell} \mathbb{R}_{\geq 0} \alpha_k$ if $\beta^\perp \in W^K_i \setminus \{\alpha_i^\perp\}$.

**Lemma 2.24.** Let $A$ be a real simplicial $\ell$-arrangement and $K \in \mathcal{K}(A)$ with basis $B^K = \{\alpha_1, \ldots, \alpha_l\}$. Then for $1 \leq i, j \leq \ell$ there are $c_{ij}^K \in \mathbb{R}$ such that

$$\{\beta_j^i = \alpha_j - c_{ij}^K \alpha_i \mid j = 1, \ldots, \ell\}$$

is a basis for $K_i$. If $i \neq j$ then $c_{ij}^K \leq 0$ and $c_{ij}^K$ is uniquely determined by $B^K$. If $i = j$ then $c_{ij}^K > 1$.

**Proof.** Let $\beta \in V^*$ such that $\beta^\perp \in W^K_i$ and $K_i \subseteq \beta^+$. Suppose that $\beta^\perp \neq \alpha_i^\perp$. Then $\beta^\perp \in A_{\alpha_i^+ \cap \alpha_j^+}$ for some $1 \leq j \leq \ell$, $j \neq i$ and by Lemma 2.23 there are $a_i, a_j \in \mathbb{R}_{\geq 0}$ such that $\beta = a_i \alpha_i + a_j \alpha_j$. Since $B^K$ is a basis for $V^*$, and $\beta \notin \langle \alpha_i \rangle$ we further have $a_j > 0$. Setting $c_{ij}^K := -\frac{a_i}{a_j}$ and $\beta_j^i = \alpha_j - c_{ij}^K \alpha_i$ we have $\beta_j^i = \beta^\perp$ and $\beta_j^i = \beta^+$. Since $B^K$ is a basis for $V^*$, for $i \neq j$ the $c_{ij}^K$ are uniquely determined. Hence again by Lemma 2.23 for some $c_{ii}^K > 1$ we have that $\{\beta_j^i = \alpha_j - c_{ij}^K \alpha_i \mid j = 1, \ldots, \ell\}$ is a basis for $K_i$. 

**Definition 2.25.** Let $A$ be a real simplicial $\ell$-arrangement, $K \in \mathcal{K}(A)$, and $B^K = \{\alpha_1, \ldots, \alpha_l\}$ a basis for $K$. For $i \neq j$ let $c_{ij}^K$ be the uniquely determined coefficients from Lemma 2.24. For $1 \leq i \leq \ell$ we set $c_{ii}^K = 2$ and define the linear map $\sigma^K_i := \sigma^K_i$ by

$$\sigma^K_i(\alpha_j) := \alpha_j - c_{ij}^K \alpha_i$$

for $1 \leq j \leq \ell$. With respect to the basis $B^K$ this map is represented by the matrix

$$S^K_i := \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \cdots \\
-1 & \cdots & -c_{(i-1)i}^K & -c_{(i+1)i}^K & \cdots & -c_{\ell i}^K \\
0 & \cdots & 1 & 0 & 1
\end{pmatrix}.$$ 

**Remark 2.26.** We observe that $\sigma^K_i$ is a reflection at the hyperplane $\alpha_i^\perp$. In particular $\det(S^K_i) = -1$. Furthermore, $c_{ij}^K \neq 0$ if and only if $c_{ij}^K \neq 0$ since $c_{ii}^K = 0$ if and only if $|A_{\alpha_i^+ \cap \alpha_j^+}| = 2$ if and only $c_{ij}^K = 0$ by Lemma 2.23 (cf. [Cum11a]).

**Definition 2.27.** Let $A$ be a real arrangement with chambers $\mathcal{K}(A)$. A sequence $(K^0, K^1, \ldots, K^{n-1}, K^n)$ of distinct chambers in $\mathcal{K}(A)$ is called a *gallery* if for all $1 \leq i \leq n$ we have $\langle K^{i-1} \cap K^{i}\rangle = H \in A_i$, i.e. if $K^i$ and $K^{i-1}$ are adjacent with common wall $H$. We denote by $\mathcal{G}(A)$ the set of all galleries of $A$. 
We say that \( G \in \mathcal{G}(\mathcal{A}) \) has length \( n \) if it is a sequence of \( n + 1 \) chambers. For \( G = (K^0, \ldots, K^n) \in \mathcal{G}(\mathcal{A}) \) we denote by \( b(G) = K^0 \) the first chamber and by \( e(G) = K^n \) the last chamber in \( G \).

**Definition 2.28.** Let \( \mathcal{A} \) be a real simplicial \( \ell \)-arrangement. We fix a chamber \( K^0 \in \mathcal{K}(\mathcal{A}) \). Let \( \mathcal{G}(K^0, \mathcal{A}) = \{ G \in \mathcal{G}(\mathcal{A}) \mid b(G) = K^0 \} \) be the set of galleries starting with \( K^0 \).

Let \( B^{K^0} = \{ \alpha_0^0, \ldots, \alpha_{\ell}^0 \} \) be a basis for \( K^0 \). For \( (K^0, \ldots, K^n) = G \in \mathcal{G}(K^0, \mathcal{A}) \) we denote by \( B^{K_{n}} = B_{G} \) the basis for \( K^n \) induced by \( G \) and \( B^{K^n} \), i.e. such that

\[
B^{K^{i+1}} = \sigma_{i}^{K_{i}}(B^{K^{i}}) = \{ \alpha_{j}^{i+1} = \sigma_{i}^{K_{i}}(\alpha_{j}^i) = \alpha_{j}^i - c_{i,j}^{K_{i}} \alpha_{i} \mid 1 \leq j \leq \ell \},
\]

where \( K^{i+1} = K_{i}^{i} \), \( \mu_i \in \{1, \ldots, \ell \} \), and \( 0 \leq i \leq n - 1 \).

**Definition 2.29.** Let \( \mathcal{A} \) be a real simplicial \( \ell \)-arrangement, \( K \in \mathcal{K}(\mathcal{A}) \). We call a basis \( B^{K} = \{ \alpha_{1}, \ldots, \alpha_{\ell} \} \) *locally crystallographic* if the \( c_{ij}^{K} \) are all integers.

If \( B^{K} \) is a locally crystallographic basis then we call the matrix \( C^{K} = (c_{ij}^{K})_{i,j=1,\ldots,\ell} \) the Cartan matrix of \( B^{K} \).

**Example 2.30.** Let \( \mathcal{A} = A_{\ell}^{x} \). Then \( \mathcal{A} \) is crystallographic with crystallographic root system \( R \). In particular for \( K \in \mathcal{K}(\mathcal{A}) \) the basis \( B_{R}^{K} \) is a locally crystallographic basis for \( K \) and the corresponding Cartan matrix is (up to simultaneous permutation of columns and rows) one of the matrices displayed in Table I see \[CH15\], Prop. 3.8).

**Definition 2.31.** Let \( B^{K} \) be a locally crystallographic basis with Cartan matrix \( C^{K} \). If (up to simultaneous permutation of columns and rows) \( C^{K} \) is one of the matrices shown in the left column of Table I then we say \( C^{K} \) is of type \( A, C, D \), or \( D' \) respectively.

If \( B^{K} \) is a locally crystallographic basis with Cartan matrix of type \( A, C, D \), or \( D' \) then the corresponding Coxeter graph \( \Gamma(K) \) (see Section III) is displayed in the right column of Table I

**Lemma 2.32.** Let \( \mathcal{A} \) be a simplicial \( \ell \)-arrangement, \( K \in \mathcal{K}(\mathcal{A}) \) with basis \( B^{K} = \{ \alpha_{1}, \ldots, \alpha_{\ell} \} \), and \( K_{i} \) an adjacent chamber. Then for all \( 1 \leq j \leq \ell \) we have \( c_{ij}^{K_{i}} = c_{ij}^{K} \) and in particular \( \sigma_{i}^{K_{i}} \circ \sigma_{i}^{K} = \sigma_{i}^{K_{i}} \circ \sigma_{i}^{K} = \text{id} \).

**Proof.** We have \( \sigma_{i}^{K}(\alpha_{j}) = \beta_{j}^{\perp} = (\alpha_{j} - c_{i,j}^{K} \alpha_{i}) \perp \in W_{K_{i}} \) but similarly \( \sigma_{i}^{K_{i}}(\beta_{j}^{\perp}) = \alpha_{j}^{\perp} = (\beta_{j}^{\perp} - c_{i,j}^{K_{i}} \alpha_{i} - c_{i,j}^{K_{i}}(-\alpha_{i})) \perp \in W_{K} \). Thus \( c_{ij}^{K_{i}} = c_{ij}^{K} \). \( \square \)

Similarly to the crystallographic case we have the following.

**Lemma 2.33** (cf. \[CH09\], Lem. 4.5). Let \( \mathcal{A} \) be a simplicial \( \ell \)-arrangement, \( K, B^{K}, \) and \( K_{i} \) as before. Let \( i \neq j \) and suppose \( c_{ij}^{K_{i}} = 0 \). Then \( c_{jk}^{K_{i}} = c_{jk}^{K} \) for all \( k \in \{1, \ldots, \ell \} \).

**Proof.** The proof is the same as in \[CH09\].

If \( k = i \) then by Lemma 2.32 \( c_{jk}^{K} = c_{jk}^{K_{i}} = 0 = c_{k,j}^{K_{i}} = c_{k,j}^{K} \). And if \( k = j \) then all the coefficients are equal to 2. So let \( k \in \{1, \ldots, \ell \} \setminus \{i,j\} \). Since \( c_{ij}^{K} = 0 \) we have \( |A_{\alpha_{i}^{\perp} \alpha_{j}^{\perp}}| = 2 \). So application of \( \sigma_{j}^{K_{i}} \circ \sigma_{i}^{K} \) and \( \sigma_{i}^{K_{i}} \circ \sigma_{j}^{K} \) on \( \alpha_{k} \) should yield a normal of the same wall of the
chamber $K\alpha_i\sigma^K_i(\alpha_j) = K\alpha_j\sigma^K_j(\alpha_i)$. Now

$$\sigma^K_j(\sigma^K_i(\alpha_k)) = \sigma^K_i(\alpha_k) - c_{jk}^K \sigma^K_i(\alpha_j)$$

$$= \alpha_k - c_{ik}^K \alpha_i - c_{jk}^K (\alpha_j - c_{ij}^K \alpha_i)$$

$$= \alpha_k - c_{ik}^K \alpha_i - c_{jk}^K \alpha_j,$$

and similarly

$$\sigma^K_i(\sigma^K_j(\alpha_k)) = \alpha_k - c_{jk}^K \alpha_j - c_{ik}^K \alpha_i.$$

Table 1. Cartan matrices and Coxeter graphs.
Since $i, j, k$ are pairwise different and $\{\alpha_1, \ldots, \alpha_\ell\}$ are linearly independent, comparing the coefficients of $\alpha_j$ in both terms gives $c_{jk}^K = c_{jk}^K$. □

2.3. Supersolvable arrangements. An element $X \in L(A)$ is called modular if $X + Y \in L(A)$ for all $Y \in L(A)$. An arrangement $A$ with $r(A) = \ell$ is called supersolvable if the intersection lattice $L(A)$ is supersolvable, i.e. there is a maximal chain of modular elements $V = X_0 < X_1 < \ldots < X_\ell = T(A)$.

$X_i \in L(A)$ modular. For example an essential 3-arrangement $A$ is supersolvable if there exists an $X \in L_2(A)$ which is connected to all other $Y \in L_2(A)$ by a suitable hyperplane $H \in A$, (i.e. $X + Y \in A$).

Lemma 2.34 ([OT92, Lem. 2.27]). Let $A$ be an essential $\ell$-arrangement, $X \in L_{\ell-1}(A)$ a modular element and $H \in A \setminus A_X$. Then $|A_H| = |A_X|$.

Supersolvability is preserved by taking localizations and restrictions, see [AHR14, Lem. 2.6], and [Sta72, Prop. 3.2]:

Lemma 2.35. Let $A$ be an arrangement, $X \in L(A)$ and $Y \in L(A)$ a modular element with $X \subseteq Y$. Then $Y$ is modular in $L(A_X)$. Moreover if $A$ is supersolvable so is $A_X$ for all $X \in L(A)$.

Lemma 2.36. Let $A$ be an arrangement, $X \in L(A)$ and $Y \in L(A)$ a modular element. Then $Y \cap X$ is modular in $L(A_X)$. In particular if $A$ is supersolvable so is $A_X$ for all $X \in L(A)$.

Combining the previous two lemmas with Lemma 2.17 we obtain the following.

Lemma 2.37. Let $A$ be a real supersolvable simplicial arrangement and $X \in L(A)$. Then we have

1. $(A_X/X, V/X)$ is supersolvable and simplicial,
2. $(A^X, X)$ is supersolvable and simplicial.

Furthermore, as (geometric) simpliciality, supersolvability is compatible with products.

Lemma 2.38 ([HR14 Prop. 2.5]). Let $A = A_1 \times A_2$ be a product. Then $A$ is supersolvable if and only if $A_1$ and $A_2$ are both supersolvable.

So together with Proposition 2.21 we get the following.

Proposition 2.39. Let $A_1$ and $A_2$ be real arrangements and $A = A_1 \times A_2$ their product. Then $A$ is supersolvable and simplicial if and only if $A_1$ and $A_2$ are both supersolvable and simplicial.

Because of the previous proposition, to classify supersolvable and simplicial arrangements, it suffices to classify the irreducible ones.

The following property of the characteristic polynomial of a supersolvable arrangement is due to Stanley [Sta72], cf. [OT92 Thm. 2.63].
Theorem 2.40. Let $A$ be a supersolvable $\ell$-arrangement with

$$V = X_0 < X_1 < \ldots < X_\ell = T(A)$$

a maximal chain of modular elements. Let $b_i := |A_{X_i} \setminus A_{X_{i-1}}|$ for $1 \leq i \leq \ell$. Then

$$\chi_A(t) = \prod_{i=1}^\ell (t - b_i).$$

A helpful result is due to Amend, Hoge, and Röhrle. They checked which restrictions of irreducible reflection arrangements are supersolvable, [AHR14 Thm. 1.3]. Here we only need the following weaker version for real reflection arrangements of rank greater or equal to 4.

Theorem 2.41. Let $A = A(W)$ be an irreducible real reflection arrangement of rank $\ell \geq 4$ associated to the finite reflection group $W$ and $X \in L(A)$ with $m := \dim(X) \geq 4$. Then $A^X$ is supersolvable if and only if one of the following holds:

1. $W = A_\ell$ and then $A^X = A(A_m)$,
2. $A^X = A_m^k$ with $k \in \{m, m-1\}$.

Together with Theorem [2.13] this gives us the following classification of irreducible supersolvable crystallographic arrangements of rank $\geq 4$.

Theorem 2.42. Let $A$ be an irreducible supersolvable crystallographic $\ell$-arrangement with $\ell \geq 4$. Then $A$ is isomorphic to exactly one of the reflection arrangements $A(A_\ell)$, $A(C_\ell)$ or $A_{\ell-1} = A(C_\ell) \setminus \{\{x_1 = 0\}\}$.

3. Coxeter graphs for simplicial arrangements

From now on until the end of this article we always assume arrangements to be real.

We introduce Coxeter graphs of chambers of simplicial arrangements and use the results from Subsection 2.2 to derive their properties.

Definition 3.1. Let $K \in K(A)$ be a chamber of the simplicial $\ell$-arrangement $A$ and $B^K$ some basis for $K$. We define a labeled non directed simple graph $\Gamma(K) = (V, E)$ with vertices $V = B^K$ and edges $E = \{\{\alpha, \beta\} \mid |A_{\alpha + \beta}| \geq 3\}$. An edge $e = \{\alpha, \beta\} \in E$ is labeled with $m^K(e) = m^K(\alpha, \beta) = |A_{\alpha + \beta}|$. Since the label $m(\alpha, \beta) = 3$ appears more often we omit it in drawing the graph. We call $\Gamma(K)$ the Coxeter graph of $K$. If we have chosen a numbering $B^K = \{\alpha_1, \ldots, \alpha_\ell\}$ then $\{\alpha_i, \alpha_j\} \in E$ is simply denoted by $\{i, j\}$ and $V = \{1, \ldots, \ell\}$, see Figure 1.

If $K_1$ is an adjacent chamber for some $1 \leq i \leq \ell$ and $\Gamma(K_i) = (V_i, E_i)$ its Coxeter graph with $V_i = B^{K_i} = \sigma_i^K(B^K) = \{\sigma_i^K(\alpha_1), \ldots, \sigma_i^K(\alpha_\ell)\}$, we similarly identify $\{\sigma_i^K(\alpha_k), \sigma_i^K(\alpha_j)\} \in E_i$ with $\{k, j\}$ and $V_i$ with $\{1, \ldots, \ell\}$.

Example 3.2. Let $A(W)$ be the Coxeter arrangement of the Coxeter group $W$. Then $A$ is a simplicial arrangement (c.f. Example 2.7) and for all $K \in K(A)$ the Coxeter graph $\Gamma(K)$ is indeed the Coxeter graph of $W$, see for example [Hum90 Ch. 2].
Lemma 3.3. Let \( \mathcal{A} \) be a simplicial \( \ell \)-arrangement, \( K \in \mathcal{K}(\mathcal{A}) \) with basis \( B^K = \{ \alpha_1, \ldots, \alpha_\ell \} \), \( K_i \) an adjacent chamber. Then for \( j \neq i \) we have \( c^K_{ij} = c^K_{ji} = 0 \) if and only if \( m^K(i, j) = 2 \). Furthermore if \( m^K(i, j) = 3 \) then for \( i \neq j \) we have \( c^K_{ij} = -1 \). In particular if \( c^K_{ij} = -1 \) then \( c^K_{ji} = c^K_{ij} \).

Proof. For \( j \neq i \) we have \( c^K_{ij} = c^K_{ji} = 0 \) if and only if \( \alpha_j \perp B^K \) if and only if \( m^K(i, j) = 2 \).

Lemma 3.4. Let \( \mathcal{A} \) be a simplicial \( \ell \)-arrangement, \( K \in \mathcal{K}(\mathcal{A}) \) a chamber, \( B^K = \{ \alpha_1, \ldots, \alpha_\ell \} \), \( \Gamma(K) = (\mathcal{V}, \mathcal{E}) \), and \( K_i \) an adjacent chamber with \( B^{K_i} = \{ \sigma^K_i(\alpha_1), \ldots, \sigma^K_i(\alpha_\ell) \} \) and \( \Gamma(K_i) = (\mathcal{V}_i, \mathcal{E}_i) \). Then if \( \{i, j\} \notin \mathcal{E} \) (\( i \neq j \)) but \( \{j, k\} \in \mathcal{E} \) then \( \{j, k\} \in \mathcal{E}_i \) (disregarding the labels).

Proof. Since \( \{j, k\} \in \mathcal{E} \) and \( m^K(j, k) \geq 3 \) by Lemma 3.3 we have \( c^K_{jk} \neq 0 \). Hence \( c^K_{ji} \neq 0 \) by Lemma 2.33 and so again by Lemma 3.3 \( m^{K_i}(j, k) \geq 3 \) and \( \{j, k\} \in \mathcal{E}_i \).

The next lemma is a direct generalization of [CH09, Prop. 4.6] from crystallographic arrangements to general simplicial arrangements. It may be proved completely analogously but here we give a more geometric proof.

Lemma 3.5. Let \( \mathcal{A} \) be a simplicial \( \ell \)-arrangement with chambers \( \mathcal{K}(\mathcal{A}) \). Then the following are equivalent.

1. \( \mathcal{A} \) is an irreducible arrangement.
2. \( \Gamma(K) \) is connected for all \( K \in \mathcal{K}(\mathcal{A}) \).
3. \( \Gamma(K) \) is connected for some \( K \in \mathcal{K}(\mathcal{A}) \).

Proof. We may assume that \( \ell \) is at least 2 since otherwise the statement of the theorem is trivial.

The implication (2) \( \Rightarrow \) (3) is trivial.
(1)⇒(2). Suppose there is a \( K \in K(A) \) such that \( \Gamma(K) = (V, E) \) is not connected. Then there is a partition \( V = B^K = \Delta_1 \cup \Delta_2 \) such that \( |A_{\alpha+\gamma} | = 2 \) for \( \alpha \in \Delta_1 \) and \( \beta \in \Delta_2 \). Without loss of generality let \( \alpha \in \Delta_1 \). Then

\[
B^K = \{ -\alpha \} \cup \{ \alpha' + c_\alpha \alpha \mid \alpha' \in \Delta_1 \setminus \{ \alpha \} \} \cup \Delta_2
\]

is a basis for \( K_\alpha \) for certain \( c_{\alpha'} \geq 0 \), c.f. Lemma 2.24. Assume that there are \( \alpha' + \alpha \in B^K \) and \( \beta \in \Delta_2 \subseteq B^K_{\alpha} \) with \( |A_{(\alpha' + \alpha)_{\alpha+\gamma}}| \) and \( |A_{(\alpha' + \alpha)_{\alpha+\gamma}}| \) both \( \geq 3 \). Then there is a \( b > 0 \) such that \( \alpha' + \alpha + b \beta \in B^K \). Note that \( K_{\alpha}(\alpha) = K_{\beta} \) since \( |A_{\alpha+\gamma_{\alpha+\gamma}}| = 2 \). Then there is a \( d \geq 0 \) such that \( \alpha' + \alpha + b \beta + d(\alpha) = \alpha' + (c - d)\alpha + b \beta \in B^K \). But \( B^K_{\beta} = \Delta_1 \cup \{ -\beta \} \cup \{ \beta' + c_{\beta} \beta \mid \beta' \in \Delta_2 \setminus \{ \beta \} \} \) which gives a contradiction. So for all \( \alpha' + \alpha \in B^K_{\alpha} \) and \( \beta \in \Delta_1 \) we have \( |A_{(\alpha' + \alpha)_{\alpha+\gamma}}| = 2 \). We conclude that for all \( \gamma \in B^K \), for the corresponding adjacent chamber \( K \gamma \) there is a partition \( B^K \gamma = \Delta_1 \cup \Delta_2 \) with \( \Delta_i \subset \bigcup_{\lambda \in L_i} R \lambda \) and \( |A_{\alpha+\gamma_{\alpha+\gamma}}| = 2 \) for all \( \alpha \in \Delta_1 \), \( \beta \in \Delta_2 \). Hence for all \( H \in A \) by induction using a gallery from \( K \) to some chamber \( K' \) with \( H \in W^K \) we either have \( \gamma = (\sum_{\alpha \in \Delta_1} c_{\alpha} \alpha)^+ \) with \( c_{\alpha} \in R \), or \( H = (\sum_{\beta \in \Delta_2} c_{\beta} \beta)^+ \) with \( c_{\beta} \in R \) which means that \( A \) is reducible.

(3)⇒(1). Suppose that \( A \) is reducible. Then there exists a basis \( \{ x_1, \ldots, x_r \} \cup \{ y_1, \ldots, y_s \} \) of \( V^* \) with \( r, s \geq 1 \) such that for \( H \in A \) and \( H = \gamma^\perp \) for some \( \gamma \in V^* \) we either have \( \gamma \in \bigcup_{i=1}^{r} R x_i \) or \( \gamma \in \bigcup_{j=1}^{s} R y_j \). Let \( K \in K(A) \) be chamber of \( A \). Then \( B^K = \Delta_1 \cup \Delta_2 \) with \( \Delta_1 = B^K \cap \bigcap_{i=1}^{r} R x_i \) and \( \Delta_2 = B^K \cap \bigcap_{j=1}^{s} R y_j \). Since \( A \) is simplicial, \( B^K \) is a basis of \( V^* \) and we have \( \Delta_i \neq \emptyset \) for \( i = 1, 2 \). Furthermore \( A_{\alpha+\gamma_{\alpha+\gamma}} = \{ (\alpha^\perp, \beta^\perp) \} \) for \( \alpha \in \Delta_1 \), \( \beta \in \Delta_2 \) and hence \( \Gamma(K) \) is not connected.

**Lemma 3.6.** Let \( A \) be a simplicial \( \ell \)-arrangement, \( K \in K(A) \) with \( B^K = \{ \alpha_1, \ldots, \alpha_{\ell} \} \) and \( \Gamma(K) = (V, E) \) with vertices \( V = \{ 1, \ldots, \ell \} \). Suppose that \( \{ i, j \} \in E \) with label \( m^K(i, j) \) and there is a \( k \in V \setminus \{ i, j \} \) such that \( \{ k, i \} \notin E \) and \( \{ k, j \} \notin E \). Then \( \{ i, j \} \) is an edge in \( \Gamma(K_k) \) with the same label \( m^{K_k}(i, j) \).

**Proof.** That \( \{ i, j \} \) is an edge in \( \Gamma(K_k) \) is simply Lemma 3.4. The second statement holds because \( \sigma^K_k(\alpha_i) = \alpha_i \) and \( \sigma^K_k(\alpha_j) = \alpha_j \) and thus

\[
m^K_k(i, j) = |A_{\sigma^K_k(\alpha_i) \cap \sigma^K_k(\alpha_j) \cap K_k}| = |A_{\alpha_i \cap \alpha_j \cap K_k}| = m^K(i, j).
\]

**Lemma 3.7.** Let \( A \) be a simplicial \( \ell \)-arrangement, \( X \in L_q(A) \) for \( 1 \leq q \leq \ell \), and \( K_X \in K(A_X / X) \) be a chamber of the localization \( A_X / X \). Let \( K \in K(A) \) with \( B^K = \{ \alpha_1, \ldots, \alpha_{\ell} \} \) such that \( X = \bigcap_{i=1}^{q} \alpha_{\alpha_i}^+ \) and \( X = \bigcap_{i=1}^{q} \alpha_{\alpha_i}^+ / X \), and \( \Gamma(K) \) with corresponding vertices \( \mathcal{V} = \{ 1, \ldots, \ell \} \). Then \( \Gamma(K_X) \) is the induced subgraph on the q vertices \( \{ i_1, \ldots, i_q \} \subseteq \mathcal{V} \) of \( \Gamma(K) \) including the labels.

**Proof.** For \( q = 1 \) the statement is trivially true. For \( q \geq 2 \) this is easily seen as the intersection lattice \( L(A_X) \) is an interval in the intersection lattice \( L(A) \), i.e. \( L(A_X) = L(A_X)X = [V, X] = \{ Z \in L(A) \mid Z \subseteq X \} \).

With the correspondence from the previous lemma and Lemma 3.5 we obtain the following corollary for irreducible simplicial arrangements.
Corollary 3.8. Let \( \mathcal{A} \) be an irreducible simplicial \( \ell \)-arrangement and \( K \in \mathcal{K}(\mathcal{A}) \). Then there is an \( X \in L_{\ell-1}(W^K) \subseteq L(\mathcal{A}) \) such that \( (\mathcal{A}_X/X, V/X) \) is an irreducible simplicial \( (\ell - 1) \)-arrangement.

To describe the connection between restrictions of simplicial arrangements and Coxeter graphs we need a bit more notation.

Definition 3.9. Let \( \mathcal{A} \) be a simplicial arrangement, \( K \in \mathcal{K}(\mathcal{A}), \alpha \in B^K \) and \( H = \alpha^\perp \subseteq W^K \). Then we denote the induced chamber in the restriction \( \mathcal{A}^H \) by

\[
K^H = ( \bigcap_{\beta \in B^K \backslash \{\alpha\}} \beta^+ ) \cap H,
\]

and a basis for \( K^H \) is given by

\[
B^{K^H} = \{ \beta^H | \beta^H := \beta\mid_{H}, \text{ and } \beta \in B^K \backslash \{\alpha\} \}.
\]

Let \( \Gamma(K) = (\mathcal{V}, \mathcal{E}) \) be the Coxeter graph of \( K \) and suppose that there is an edge \( \{\alpha, \beta\} \in \mathcal{E} \) connecting the vertices \( \alpha \) and \( \beta \). Define \( \Gamma^{\alpha\beta} := (\mathcal{V}^{\alpha\beta}, \mathcal{E}^{\alpha\beta}) \) to be the (unlabeled) graph with vertices \( \mathcal{V}^{\alpha\beta} := \mathcal{V} \backslash \{\alpha, \beta\} \cup \{\alpha\beta\} \), and edges

\[
\mathcal{E}^{\alpha\beta} := \{ \{\gamma, \delta\} \in \mathcal{E} | \{\gamma, \delta\} \cap \{\alpha, \beta\} = \emptyset \} \cup \\
\{ \{\alpha\beta, \gamma\} | \{\alpha, \gamma\} \in \mathcal{E} \text{ or } \{\beta, \gamma\} \in \mathcal{E} \},
\]
i.e. the contraction of \( \Gamma(K) \) along the edge \( \{\alpha, \beta\} \).

It is convenient to use the following notation: If \( \Gamma(K) = (\mathcal{V}, \mathcal{E}) \) with \( \mathcal{V} = \{1, \ldots, \ell\} \) corresponding to \( B^K = \{\alpha_1, \ldots, \alpha_\ell\} \), \( I \subseteq \mathcal{V} \) with \( I = \{i_1, \ldots, i_r\} \) and \( X = \cap_{i \in I} \alpha_i^\perp \) then for the localization \( \mathcal{A}_X \) at the intersection adjacent to the chamber \( K \) we simply write \( \mathcal{A}^{I_1 \cap \cdots \cap I_r}_{K} \); e.g. for \( \mathcal{A}_{\alpha^1 \cap \alpha^2 \cap \alpha^3} \) we write \( \mathcal{A}^{K}_{123} \).

Lemma 3.10. Let \( \mathcal{A} \) be a simplicial \( \ell \)-arrangement and \( K \in \mathcal{K}(\mathcal{A}) \) with Coxeter graph \( \Gamma(K) = (\mathcal{V}, \mathcal{E}) \). Suppose \( \{\alpha, \beta\} \in \mathcal{E} \) is an edge. Let \( H \in \mathcal{A}_{\alpha^\perp \cap \beta^\perp} \) be the wall of \( K\alpha \) with \( H \neq \alpha^\perp \), i.e. \( H = \sigma^K_\alpha(\beta)^\perp \), and let \( \Gamma^H = (\mathcal{V}^H, \mathcal{E}^H) := \Gamma((K\alpha)^H) \) be the Coxeter graph of the chamber \( (K\alpha)^H \in \mathcal{K}(\mathcal{A}^H) \). Then we have the following:

1. The contracted graph \( \Gamma^{\alpha\beta} \) is isomorphic to a subgraph of \( \Gamma^H \) in the following way: Let \( \rho : \mathcal{V}^{\alpha\beta} \to \mathcal{V}^H \) be the bijective map defined by

\[
\rho(\gamma) := \begin{cases} 
\sigma^K_\alpha(\gamma)^H & \text{if } \gamma \neq \alpha\beta \\
\sigma^K_\alpha(\alpha)^H = (-\alpha)^H & \text{if } \gamma = \alpha\beta.
\end{cases}
\]

If \( \{\gamma, \delta\} \in \mathcal{E}^{\alpha\beta} \) then \( \{\rho(\gamma), \rho(\delta)\} \in \mathcal{E}^H \), i.e. \( \mathcal{E}^{\alpha\beta} \subseteq \mathcal{E}^H \) disregarding the labels.

2. If \( \{\alpha, \gamma\} \in \mathcal{E} \) (\( \gamma \neq \beta \)) is labeled with \( m(\alpha, \gamma) \) then for the corresponding label in \( \Gamma^H \) we have \( m^H(\rho(\alpha\beta), \rho(\gamma)) \geq m(\alpha, \gamma) \) (see Figure 2(a)).

3. If \( \{\alpha, \beta\}, \{\alpha, \gamma\} \), and \( \{\beta, \gamma\} \) are edges in \( \mathcal{E} \), then for the label of the edge \( \{\rho(\alpha\beta), \rho(\gamma)\} \) in \( \Gamma^H \) we have \( m^H(\rho(\alpha\beta), \rho(\gamma)) \geq m(\alpha, \gamma) + m(\beta, \gamma) - 2 \) (see Figure 2(b)).
It suffices to prove the statements for 3-arrangements (the statements are trivial for 2-arrangements). The general case then follows by taking localizations, the fact that $(\mathcal{A}^H)_X = (\mathcal{A}_X)^H$, and Lemma 3.7. Let $B^K = \{\alpha, \alpha_2, \alpha_3\}$ and denote the corresponding vertices of $\Gamma(K)$ by $\{1, 2, 3\}$.

If $\Gamma(K)$ is not connected, i.e. $A$ is reducible by Lemma 3.5, then either there is no edge in $\Gamma(K)$ and there is nothing to show, or $\Gamma(K)$ is the graph of Figure 3. In this case, all statements hold, since for all $H \in \mathcal{A}^K_{12}$ we then have $|A^H| = 2$. So $\mathcal{A}^H$ is reducible and the Coxeter graph of every chamber of $\mathcal{A}^H$ is the graph with 2 vertices which are not connected and which is exactly isomorphic to the contracted graph $\Gamma_{\alpha_1\alpha_2}$.

So assume $\Gamma(K)$ is connected. Without loss of generality let $H = \sigma^K_1(\alpha_2)^+ \in \mathcal{A}^K_{12}$. Since $(\sigma^K_1(\alpha_1)^H)^\perp = (\sigma^K_1(\alpha_1)^L)^\perp = (\alpha_2^\perp \cap \alpha_2^\perp \cap \alpha_2^\perp) \in \mathcal{A}^H$, and $(\sigma^K_1(\alpha_3)^H)^\perp = (\alpha_3^\perp \cap \alpha_3^\perp \cap \alpha_3^\perp) \in \mathcal{A}^H$ we have $(\sigma^K_1(\alpha_1)^L)^\perp \cap (\sigma^K_1(\alpha_3)^L)^\perp = \{0\}$. We have to show that $|A^H| \geq |B^H| + (|B^H| - 1)$ to obtain all three statements. Let $B = \mathcal{A}^K_{12} \cup \mathcal{A}^K_{23}$. Then $|A^H| \geq |B^H|$ and $|B| = |B^H| + |A^H| - 1$ (since $\mathcal{A}^K_{12} \cap \mathcal{A}^K_{23} = \{\alpha_3^\perp\}$). We now deduce that $|B^H| = |B| - 1$.

We have $W^K \subset B$ and $|(W^K)^H| = 2$. Now let $H_1, H_2 \in B \setminus W^K$ with $H_1 \neq H_2$. We first observe that $H \cap H_1 \neq H \cap H_2$ for any $H \in W^K$. But we also have $H_1 \cap H \neq H_2 \cap H$. Hence all $H' \in B \setminus W^K$ give different intersections with $H$. Thus we obtain

$$|A^H| \geq |B^H| = |(W^K)^H| + |(B \setminus W^K)^H|$$

$$= 2 + |(B \setminus W^K)| = 2 + |B| - 3$$

$$= |A^K_{12}| + |A^K_{23}| - 2.$$  

From this inequality by translating back to the corresponding Coxeter graphs (which are graphs with only two vertices) all statements directly follow.

**Lemma 3.11.** Let $\mathcal{A}$ be an irreducible simplicial $\ell$-arrangement and $X \in L_q(\mathcal{A})$. Then the restriction $\mathcal{A}^X$ is an irreducible simplicial $(\ell - q)$-arrangement.

**Proof.** It suffices to show the statement for $X = H \in \mathcal{A}$.

By Lemma 2.17 the restriction $\mathcal{A}^H$ is again simplicial. Since $\mathcal{A}$ is irreducible, there is an $X \in L_2(\mathcal{A})$ with $X \subseteq H$ and $|A_X| \geq 3$. So there is a chamber $K \in \mathcal{K}(\mathcal{A})$ with $\Gamma(K) = (V, E)$, $\{\alpha, \beta\} \in E$ such that $X = \alpha^\perp \cap \beta^\perp$, and $H$ the
wall of $K\alpha$ not equal to $\alpha^\perp$. Since $\mathcal{A}$ is irreducible, the Coxeter graph $\Gamma(K)$ is connected by Lemma 3.5 and by Lemma 3.10 the Coxeter graph $\Gamma((K\alpha)^H)$ of the chamber $(K\alpha)^H$ of $\mathcal{A}^H$ contains a subgraph on $\ell - 1$ vertices which is connected (as it is isomorphic to a contraction of the connected graph $\Gamma(K)$). So $\Gamma((K\alpha)^H)$ is also connected and hence again by Lemma 3.5 the restriction $\mathcal{A}^H$ is irreducible. □

4. The rank 3 case

We firstly collect some useful results about supersolvable simplicial 3-arrangements.

Lemma 4.1. Let $\mathcal{A}$ be a supersolvable simplicial 3-arrangement with two modular elements $X, Y \in L_2(\mathcal{A})$ such that $|\mathcal{A}_X| \neq |\mathcal{A}_Y|$. Then $\mathcal{A}$ is reducible.

Proof. By Theorem 2.40 two different roots of $\chi_{\mathcal{A}}(t)$ are given by $|\mathcal{A}_X| - 1$ and $|\mathcal{A}_Y| - 1$. So we have

$$\chi_{\mathcal{A}}(t) = (t - 1)(t - (|\mathcal{A}_X| - 1))(t - (|\mathcal{A}_Y| - 1)),$$

and by Remark 2.3 we further have

$$|\mathcal{A}| = -\mu_1 = |\mathcal{A}_X| + |\mathcal{A}_Y| - 1 \leq |\mathcal{A}_X \cup \mathcal{A}_Y|.$$ 

Then there is a hyperplane $H \in \mathcal{A}$ with $\mathcal{A}^H = \{X, Y\}$, i.e. $H = X + Y$. Hence $\mathcal{A}^H$ is reducible. By Lemma 3.11 the arrangement $\mathcal{A}$ is reducible. □

Lemma 4.2 ([Toh14, Lemma 2.1]). Let $\mathcal{A}$ be a supersolvable 3-arrangement. Then all elements $X \in L_2(\mathcal{A})$ with $|\mathcal{A}_X|$ maximal are modular.

Combining Lemma 4.1 and Lemma 4.2 we get the following lemma.

Lemma 4.3. Let $\mathcal{A}$ be an irreducible supersolvable simplicial 3-arrangement. Then $X \in L_2(\mathcal{A})$ is modular if and only if $|\mathcal{A}_X|$ is maximal among all localizations at intersections of rank two.

Corollary 4.4. Let $\mathcal{A}$ be an irreducible supersolvable simplicial 3-arrangement and $X \in L_2(\mathcal{A})$ a modular element. Then $|\mathcal{A}_X| \geq 3$.

Proof. Suppose $|\mathcal{A}_X| = 2$. Then $|\mathcal{A}_Z| = 2$ for all $Z \in L_2(\mathcal{A})$ by Lemma 4.3. Hence $\Gamma(K)$ is disconnected for all $K \in \mathcal{K}(\mathcal{A})$ and $\mathcal{A}$ is reducible by Lemma 3.5. A contradiction.

Definition 4.5. Let $n \in \mathbb{N}$ and $\zeta := \exp\left(\frac{2\pi i}{2n}\right)$ be a primitive 2n-th root of unity. We write

$$c_n(m) := \cos \frac{2\pi m}{2n} = \frac{1}{2}(\zeta^m + \zeta^{-m}),$$

and

$$s_n(m) := \sin \frac{2\pi m}{2n} = \frac{1}{2i}(\zeta^m - \zeta^{-m}).$$

For $n \geq 3$ the arrangements $\mathcal{A}(2n, 1)$ of the infinite series $\mathcal{R}(1)$ from [Gri09] may be defined by

$$
\begin{pmatrix}
-s_n(0) & -s_n(1) & \ldots & -s_n(n - 1) & c_n(1) & c_n(3) & \ldots & c_n(2n - 1) \\
c_n(0) & c_n(1) & \ldots & c_n(n - 1) & s_n(1) & s_n(3) & \ldots & s_n(2n - 1) \\
0 & 0 & \ldots & 0 & 1 & 1 & \ldots & 1
\end{pmatrix}$$
For $n \geq 2$ the arrangements $\mathcal{A}(4n+1,1)$ of the series $\mathcal{R}(2)$ are constructed as

$$\mathcal{A}(4n+1,1) = \mathcal{A}(4n,1) \cup \{(0,0,1)^{\perp}\}.$$ 

Some examples are displayed as projective pictures of the arrangements in Figure 4.

Remark 4.6. Let $\mathcal{A} = \mathcal{A}(2n,1) = \{H_1, \ldots, H_n, I_1, \ldots, I_n\}$ for an $n \geq 3$ where $H_i = (-s_n(i-1), c_n(i-1), 0)^{\perp}$ and $I_j = (c_n(2j-1), s_n(2j-1), 1)^{\perp}$, and let $X = \cap_{i=1}^{n} H_i \in L_2(\mathcal{A})$. The following facts are easily seen from the definition:

- The rank 2 part of the intersection lattice has the following form:

$$L_2(\mathcal{A}) = \{X\} \cup \{I_i \cap I_j \cap H_k \mid 1 \leq i, j, k \leq n, i \neq j \text{ and } i + j \equiv k \pmod{n}\}$$

$$\cup \{I_i \cap H_k \mid 1 \leq i, k \leq n \text{ and } k \equiv 2i \pmod{n}\}.$$ 

- The intersection $X$ is modular and hence $\mathcal{A}$ is supersolvable.

- We have the following multiset of invariants of $L(\mathcal{A})$:

$$\{|A_Z| \mid Z \in L_2(\mathcal{A})\} = \{2^n, 3^{L_2(\mathcal{A})-n-1}, n^1\}.$$
Lemma 4.7. Let $\mathcal{A} \in \mathcal{R}(1) \cup \mathcal{R}(2)$. Then $\mathcal{A}$ is irreducible, supersolvable and simplicial.

Lemma 4.8. Let $\mathcal{A}$ be a simplicial 3-arrangement such that $\chi_{\mathcal{A}} = (t - 1)(t - a)(t - b)$ factors over $\mathbb{N}$. If $|\mathcal{A}|$ is even, then exactly one of the numbers $a, b$ is even. If $|\mathcal{A}|$ is odd, then $a, b$ are also odd.

Proof. Compare the coefficient of $t$, i.e. by Remark 2.3 we have

$$ab + |\mathcal{A}| - 1 = ab + a + b = \mu_2.$$ 

Since $\mathcal{A}$ is simplicial we further have $\mu_2 = 2|L_2(\mathcal{A})| - 3$ by Lemma 2.14. So $\mu_2$ is always odd and we obtain $ab \equiv |\mathcal{A}| \pmod{2}$. Thus $|\mathcal{A}|$ is odd if and only if both $a$ and $b$ are odd. We further have $a + b + 1 = |\mathcal{A}|$. Hence if $|\mathcal{A}|$ is even then exactly one of the numbers $a, b$ is even.

Lemma 4.9. Let $\mathcal{A}$ be an irreducible simplicial 3-arrangement, $X \in L_2(\mathcal{A})$ a modular element, $n = |\mathcal{A}_X|$, and $K \in \mathcal{K}(\mathcal{A})$ a chamber with $\langle K \cap X \rangle = X$. Then the Coxeter graph $\Gamma(K)$ is the graph of Figure 5.

Proof. Let $B^K = \{\alpha_1, \alpha_2, \alpha_3\}$, and $\mathcal{V} = \{1, 2, 3\}$ the corresponding vertices of $\Gamma(K) = (\mathcal{V}, \mathcal{E})$. Since $\mathcal{A}$ is irreducible by Lemma 3.5 the graph $\Gamma(K)$ is connected. We may assume that $\{1, 2\}, \{2, 3\} \in \mathcal{E}$ and that $m(1, 2) = n$ by Corollary 1.3.

First suppose that $\{1, 3\} \in \mathcal{E}$ and let $H = \alpha_3^H$. Then in particular $H \in \mathcal{A} \setminus \mathcal{A}_X$ so $|\mathcal{A}_H| = n$ by Lemma 2.34. Let $\Gamma^H = (\mathcal{V}_H, \mathcal{E}_H) = \Gamma(K^H)$, $\mathcal{V}_H = \{\gamma, \delta\}$ and denote the
label by $m^H(\gamma, \delta) = |A^H|$. But by Lemma 3.10(3) we find that $n = m^H(\gamma, \delta) \geq m(1, 2) + m(2, 3) - 2 = n + 1$ which is absurd.

Now suppose that $m(2, 3) \geq 4$. Then $(\sigma_3^K(\alpha_2))^\perp$ (the blue line in Figure 6) intersects $(\sigma_2^K(\sigma_1^K(\alpha_3)))^\perp$ in $Z$. But $Z$ must lie in $(-\alpha_1)^+ \cap (\alpha_1^+ \cap \alpha_2^+)$ since otherwise $m(2, 3) \leq 3$, see Figure 6. This implies $Z + X \not\in L(A)$ which contradicts the modularity of $X$. \hfill \Box

**Proposition 4.10.** Let $A$ be an irreducible supersolvable simplicial 3-arrangement, and $X \in L_2(A)$ modular. Set $n := |A_X|$. Then there is a subarrangement $B \subseteq A$ with $A_X = B_X$ and $B \sim_L A(2n, 1)$.

**Proof.** Since $A$ is irreducible we have $n \geq 3$ by Corollary 4.4. We define

$$K_X := \{ K \in K(A) \mid (K \cap X) = X \},$$

i.e. the subset of chambers adjacent to $X$, and the subarrangement

$$B := \bigcup_{K \in K_X} W^K.$$  

Note that by the definition of $B$ and $K_X$ we have $A_X = B_X$ and $|K_X| = 2n$. Furthermore, for each $K \in K_X$ we have $|W^K \setminus B_X| = 1$, and for each $H \in B \setminus B_X$ there are exactly two adjacent chambers $K, K' \in K_X$ with $H \in W^K \cap W^{K'}$ by Lemma 4.9. Hence we have $|B \setminus B_X| = n$ and thus $|B| = 2n$. 
In the following we consider the projective picture of $A$ respectively $B$. Then in this picture the $n$ lines in $B \setminus B_X$ are the edge-lines of a convex $n$-gon. By Lemma 4.9 all chambers $K \in \mathcal{K}_X$ have the Coxeter graph of Figure 5 and for those we have $B_Y = A_Y$ for all $Y \in L_2(W^K)$, i.e. no line of $A \setminus B$ intersects the convex $n$-gon.

Now let $K_X(K^1, \ldots, K^{2n})$ such that $K^i$ and $K^j$ are adjacent for $1 \leq i, j \leq 2n$ with $j-i \equiv \pm 1 \pmod{2n}$. Let $B_X = \{H_1, \ldots, H_n\}$ and $B \setminus B_X = \{I_1, \ldots, I_n\}$ such that $H_{a_i}$ and $H_{b_i}$ are walls of $K^i$ with $a_i \equiv b_i - 1 \equiv i \pmod{n}$, $1 \leq a_i, b_i \leq n$, $1 \leq i \leq 2n$, and $I_k$ is a wall of both the chambers $K^{2k-1}$ and $K^{2k}$ for $1 \leq k \leq n$. Note that with this labeling for $1 \leq i, j \leq n$ we have $|B_{I_i \cap H_j}| = 2$ if $2i \equiv j \pmod{n}$ by Lemma 4.9 (since each edge of the $n$-gon contains one such point). The subarrangement $B$ is supersolvable with modular element $X$ because $A$ is supersolvable with modular element $X$ and $A_X = B_X$. Since exactly 2 edge-lines of the convex $n$-gon intersect in a common point we further have $|B_Y| \leq 3$ for all $Y \in L_2(B) \setminus \{X\}$. Suppose there is a $Y \in L_2(B)$ with $|B_Y| = 2$ and $Y \not\in L_2(W^K)$ for any chamber $K \in \mathcal{K}_X$, i.e. $Y$ is an intersection outside of the $n$-gon. By the modularity of $X$ in $L_2(B)$ we have $Y = I_i \cap H_j$ for some $1 \leq i, j \leq n$. But then $|B_Y| \geq n+1$ contradicting Lemma 2.34. Thus all intersections $Y$ outside the $n$-gon are of size 3, in particular $B_Y = \{I_i, I_j, H_k\}$ for some $1 \leq i < j \leq n$, and $1 \leq k \leq n$. We obtain the following multiset of invariants of the intersection lattice of $B$

\[
\{\{|B_Y| \mid Y \in L_2(B)\}\} = \{\{2^n, 3^{L_2(B)|-n-1}, n^1\}\}.
\]

In particular by Remark 4.6 we have

\[
\{\{|B_Y| \mid Y \in L_2(B)\}\} = \{\{|A(2n, 1)_Z| \mid Z \in L_2(A(2n, 1))\}\}.
\]

To finally see with Remark 4.6 that $B \sim_L A(2n, 1)$ we claim that

\[
\{I_i \cap I_j \cap H_k \mid 1 \leq i, j, k \leq n, i \neq j \text{ and } i + j \equiv k \pmod{n}\} \subseteq L_2(B).
\]

Let $1 \leq i < j \leq n$. Without loss of generality we may assume $i = 1$. We have $B^i = \{H_j \cap I_1 \mid 1 \leq j \leq n\}$ and there is exactly one simple intersection $H_2 \cap I_1$. But from the projective picture we see that the next intersection point $H_3 \cap I_1$ on $I_1$ is contained in the next edge-line $I_2$ of the $n$-gon. Continuing this way gives exactly $B^i = \{I_1 \cap H_2, I_1 \cap I_2 \cap H_3, \ldots, I_1 \cap I_{n-1} \cap H_n, I_1 \cap I_n \cap H_1\}$ and the claim follows.

\begin{remark}
Let $A \in \mathcal{R}(1) \cup \mathcal{R}(2)$. Then by [Cun11b] Thm. 3.6] there exists a minimal subfield $\mathbb{L} \leq \mathbb{R}$ such that there is an arrangement $B$ in $\mathbb{L}^3$ with $L(B) \cong L(A)$. Furthermore if $B'$ is another arrangement in $\mathbb{L}^3$ which is $L$-equivalent to $B$, then there is a collineation (projective semi-linear transformation) $\varphi \in \text{PGL}(\mathbb{L}^3)$ with $B' = \varphi(B) = \{\varphi(H) \mid H \in B\}$. Hence, by the fundamental theorem of projective geometry (see e.g. [Art88 Sec. II.9]) there is a field automorphism $\mu$ of $\mathbb{L}$ and a $\psi \in \text{GL}(\mathbb{R}^3)$ such that $\psi(\mu(\mathbb{L}) \otimes \mathbb{L}) = A$. So any (real) arrangement $A'$ which is $L$-equivalent to $A(2n, 1)$ or $A(4m+1, 1)$ is essentially this arrangement.

\end{remark}

\begin{proposition}
Let $A$ be an irreducible supersolvable simplicial 3-arrangement with modular element $X \in L_2(A)$, and $n := |A_X|$. Let $B$ be the subarrangement from Proposition 4.10 which is lattice equivalent to $A(2n, 1)$. Then
\begin{enumerate}
\item $|A \setminus B| \leq 1$, and
\item If $A = B \cup \{J\}$ then $n$ is even and $A \sim_L A(4n^2 + 1, 1)$.
\end{enumerate}
\end{proposition}
Proof. By the preceding remark we may assume that $B = A(2n, 1)$ and is given as in Definition 4.5.

Assume there is a $J \in A \setminus B$. Then for all $I' \in B \setminus A_X$, $Y = J \cap I'$ we have $|B_Y| = 3$ since otherwise $|A'| \geq n + 1$ contradicting Lemma 2.34. In particular if $K \in K(B)$ such that $K \cap J \neq \emptyset$, $W^K = \{H, I_1, I_2\}$ (since $K \cap X = \emptyset$) with $I \in B_X$, $H_1, H_2 \in B \setminus B_X$, then $I_1 \cap I_2 \subseteq J$. Furthermore for the adjacent chamber $K'$ with $\langle K \cap K' \rangle = H$, $W^{K'} = \{H, I_1', I_2'\}$ we also have $J \cap K' \neq \emptyset$ so similarly $I_1' \cap I_2' \subseteq J$. Since $I_1, I_2, I_1', I_2'$ are pairwise different, $J = I_1 \cap I_2 + I_1' \cap I_2'$, see Figure 7. Let $\tilde{J} \in A \setminus B$ be another hyperplane. Then there exists a chamber $K \in K(B)$ such that $J \cap K \neq \emptyset$ and $\tilde{J} \cap K \neq \emptyset$ (since otherwise there is an $I' \in B \setminus B_X$ such that $\tilde{J} \cap I' \notin L_2(B)$ which contradicts the modularity of $X$). Hence $J = \tilde{J}$. So there is only one possibility for such a $J$ and we obtain $|A \setminus B| \leq 1$.

Now suppose $n = |A_X|$ is odd. Since $A$ is supersolvable with modular element $X \in L_2(A)$ by Lemma 2.40 we have

$$\chi_A(t) = (t - 1)(t - a)(t - b),$$

with $a = n - 1$ and $b = |A| - n$. The first root $a$ is even so by Lemma 4.8 $b$ has to be odd, i.e. $|A|$ is even and hence $A = B$. If $n$ is even then either $A = B$ or there is one more hyperplane $J \in A \setminus B$ which has to be $J = (0, 0, 1)^\perp$ after a possible coordinate change and $A = A(4\frac{n}{2} + 1, 1)$.

We are now prepared to prove the main result of this section. Notice that if $A$ is not assumed to be finite, then one also obtains an infinite arrangement described in [CG17].

**Theorem 4.13.** Let $A$ be an irreducible supersolvable simplicial 3-arrangement. Then $A$ is lattice equivalent to exactly one of the arrangements in $R(1) \cup R(2)$.
Figure 8. Possible Coxeter graphs for an irreducible supersolvable simplicial 3-arrangement.

Proof. By Lemma 4.7 all arrangements in $\mathcal{R}(1) \cup \mathcal{R}(2)$ are irreducible, supersolvable, and simplicial.

Conversely by Proposition 4.10 and Proposition 4.12 we have $A \sim L A(2n, 1)$ if $n$ is odd, or $A \sim L A(2n, 1)$ or $A \sim L A(4n^2 + 1, 1)$ if $n$ is even. □

From the proof of Proposition 4.10 we obtain the following corollary.

Corollary 4.14. Let $A$ be an irreducible supersolvable simplicial 3-arrangement and $X \in L_2(A)$ a modular element. Then for all $X' \in L_2(A) \setminus \{X\}$ we have $|A_{X'}| \leq 4$.

From the proof of Proposition 4.12 we obtain:

Corollary 4.15. Let $A$ be an irreducible supersolvable simplicial 3-arrangement, $X \in L_2(A)$ modular with $n = |A_X|$, and $K \in \mathcal{K}(A)$. Then $\Gamma(K)$ is one of the Coxeter graphs of Figure 8. In particular, if $|A|$ is even or $n \leq 5$, then there is no chamber $K \in \mathcal{K}(A)$ such that $\Gamma(K) = \Gamma_3$ and if $n > 4$ and $|A|$ is even then there is also no chamber $K \in \mathcal{K}(A)$ such that $\Gamma(K) = \Gamma_3$.

Lemma 4.16. Let $A$ be an irreducible supersolvable simplicial 3-arrangement and $H \in A$. Then for all $H \in A$ we have

$$|A^H| \geq \left\lceil \frac{|A|}{4} \right\rceil + 1.$$ 

Proof. Let $X \in L_2(A)$ be modular, $n = |A_X|$, and $H \in A$. If $H \in A \setminus A_X$ then by Lemma 2.34 we have $|A^H| = n \geq \frac{|A|}{2} \geq \left\lceil \frac{|A|}{4} \right\rceil + 1$.

Let $t_r^H := |\{X \in A^H \mid |A_X| = r\}|$. Then we always have the identity $\sum_{r \geq 2}(r-1)t_r^H = |A| - 1$. By Corollary 4.14 for $H \in A_X$ we see that $t_r^H = 0$ for $r \notin \{2, 3, 4, n\}$, and $t_n^H = 1$. Furthermore by Theorem 4.13 we have $t_2^H \in \{0, 1, 2\}$ and $t_4^H = 1$ if and only if $|A| = 2n + 1$ and $n$ is even. So we obtain

$$t_3^H = \frac{|A| - 1 - t_2^H - 3t_4^H - (n - 1)t_n^H}{2} = \frac{|A| - n - t_2^H - 3t_4^H}{2},$$

and hence

$$|A^H| = t_2^H + t_3^H + t_4^H + t_n^H = \frac{n + t_2^H}{2} + 1 \geq \left\lceil \frac{|A|}{4} \right\rceil + 1.$$ □
5. The rank 4 case

The following proposition and its immediate corollary are the key for the classification of irreducible supersolvable simplicial arrangements of rank $\ell \geq 4$.

**Proposition 5.1.** Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement. Then for all $X \in L_2(\mathcal{A})$ we have $|\mathcal{A}_X| \leq 4$.

**Proof.** The proof is in three steps. First we show that if $X \in L_2(\mathcal{A})$ with $|\mathcal{A}_X| \geq 5$ then $X$ necessarily has to be the only rank 2 modular element in $L(\mathcal{A})$. From this we derive that $|\mathcal{A}_X| \leq 6$. Finally by some geometric arguments and using the classification in dimension 3 we exclude the cases $|\mathcal{A}_X| = 5, 6$.

Let $X \in L_2(\mathcal{A})$ be fixed and suppose $|\mathcal{A}_X| \geq 5$.

First assume that there is a modular $X' \in L_2(\mathcal{A}) \setminus \{X\}$. By the irreducibility of $\mathcal{A}$ there is an $H \in \mathcal{A}$ transversal to $X$ and $X'$, i.e. such that $X \not\subseteq H$, $X' \not\subseteq H$, and also $X \cap X' \not\subseteq H$ if $X \cap X' \in L_3(\mathcal{A})$. Let $Y = H \cap X$ and $Y' = H \cap X'$. By Lemma 2.36 and Lemma 3.11 the restriction $\mathcal{A}^H$ is an irreducible supersolvable simplicial 3-arrangement. Furthermore, $Y \neq Y'$ and $5 \leq |\mathcal{A}^H_Y| \leq |\mathcal{A}^H_{Y'}|$ for the 3-arrangement $\mathcal{A}^H$ by Lemma 1.3 since $Y'$ is a modular element in $L_2(\mathcal{A}^H)$ by Lemma 2.36. But this contradicts Corollary 4.14 the irreducible supersolvable simplicial 3-arrangement $\mathcal{A}^H$ cannot have two distinct rank 2 intersections of size greater or equal to 5, one of them modular. Hence $X$ is the only modular element in $L_2(\mathcal{A})$ and also the one single element in $L_2(\mathcal{A})$ with $|\mathcal{A}_X| \geq 5$.

From now on to the end of the proof let $Y \in L_3(\mathcal{A})$ be a fixed modular intersection of rank 3 with $Y > X$.

Suppose that $|\mathcal{A}_X| \geq 7$. Then since $\mathcal{A}$ is irreducible, by Lemma 2.35 the localization $\mathcal{A}_Y/Y$ regarded as an essential 3-arrangement in $V/Y$ is an irreducible supersolvable simplicial 3-arrangement with modular element $X/Y \in L_2(\mathcal{A}_Y/Y))$. So by Theorem 4.13 we have $|\mathcal{A}_Y| \geq 14$. Let $H \in \mathcal{A}_X$. By Lemma 3.11 the restriction $\mathcal{A}^H$ is irreducible and by Corollary 3.8 there is a $Y' \in L_2(\mathcal{A}^H) \setminus \{Y\}$ with $Y' \subseteq X$ such that $|(\mathcal{A}^H)_{Y'}| \geq 3$. Since $\mathcal{A}_Y/Y'$ is an irreducible supersolvable simplicial 3-arrangement with modular element $X/Y'$, as for $\mathcal{A}_Y$ we have $|\mathcal{A}_{Y'}| \geq 14$. By Lemma 2.36 the rank 3 intersection $Y \cap H = Y$ is modular in $L(\mathcal{A})$ for $H \in \mathcal{A}_X$. By Lemma 4.16 we further have $|(\mathcal{A}^H)_Y| = |(\mathcal{A}^H)_{Y'}| \geq 5$ and similarly $|(\mathcal{A}^H)_{Y'}| \geq 5$. Because of the choice of $Y' \in L_2(\mathcal{A}^H) \setminus \{Y\}$ the irreducible supersolvable simplicial 3-arrangement $\mathcal{A}^H$ has two distinct rank 2 intersections of size greater or equal to 5 which contradicts Corollary 4.14. Hence $|\mathcal{A}_X| \leq 6$.

To exclude the cases $|\mathcal{A}_X| \in \{5, 6\}$ first assume that $|\mathcal{A}_X| = 6$. We may assume that there is an $Y' \in L_3(\mathcal{A})$, $Y' \neq Y$, and $Y' > X$ such that $\mathcal{A}_{Y'/Y'}$ is an irreducible supersolvable simplicial 3-arrangement. So we have $\mathcal{A}_{Y'/Y'} \sim_L \mathcal{A}(12,1)$ or $\mathcal{A}_{Y'/Y'} \sim_L \mathcal{A}(13,1)$. But then there is an $H \in \mathcal{A}_X$ such that $|\mathcal{A}^H_{Y'}| \geq 5$ which is immediately clear by Figure 9. Since by Lemma 2.36 $Y = Y \cap H$ is a rank 2 modular element in $L(\mathcal{A}^H)$ different from $Y' \cap H = Y'' \in L_2(\mathcal{A}^H)$, with Corollary 4.14 we get a contradiction.

Finally suppose $|\mathcal{A}_X| = 5$. Then we have $\mathcal{A}_{Y'/Y'} \sim_L \mathcal{A}(10,1)$. Again we may assume that there is an $Y' \in L_3(\mathcal{A})$, $Y' \neq Y$, and $Y' > X$ such that $\mathcal{A}_{Y'/Y'}$ is an irreducible supersolvable simplicial 3-arrangement. So $\mathcal{A}_{Y'/Y'} \sim_L \mathcal{A}(10,1)$. Let $H \in \mathcal{A}_X$. Then $|\mathcal{A}^H_{Y'}| = |\mathcal{A}^H_Y| = 4$, see Figure 9. Since by Lemma 3.11 $\mathcal{A}^H$ is an irreducible supersolvable simplicial 3-arrangement with modular element $Y$ by Theorem 4.13 we have $\mathcal{A}^H \sim_L \mathcal{A}(9,1) \cong \mathcal{A}(B_3)$. For the
other restrictions $\mathcal{A}^{H'}$ with $H' \in \mathcal{A} \setminus \mathcal{A}_X$ we have $\mathcal{A}^{H'} \sim \mathcal{L}(10,1)$. The arrangement $\mathcal{A}$ is supersolvable and by Theorem 2.40 the characteristic polynomial factors as follows over the integers

$$\chi_{\mathcal{A}}(t) = (t - 1)(t - 4)(t - 5)(t - (|\mathcal{A}| - 10)).$$

Similarly for $H \in \mathcal{A}_X$ by Theorem 2.40 we have

$$\chi_{\mathcal{A}^H}(t) = (t - 1)(t - 3)(t - 5),$$

and for $H \in \mathcal{A} \setminus \mathcal{A}_X$

$$\chi_{\mathcal{A}^H}(t) = (t - 1)(t - 4)(t - 5).$$

Now we use Lemma 2.15 and insert the numbers,

$$0 = \ell|\chi_{\mathcal{A}}(-1)| - 2 \sum_{H \in \mathcal{A}} |\chi_{\mathcal{A}^H}(-1)|$$

$$= \ell|\chi_{\mathcal{A}}(-1)| - 2 \left( \sum_{H \in \mathcal{A}_X} |\chi_{\mathcal{A}^H}(-1)| + \sum_{H \in \mathcal{A} \setminus \mathcal{A}_X} |\chi_{\mathcal{A}^H}(-1)| \right)$$

$$= (4 \cdot 2 \cdot 5 \cdot 6)(|\mathcal{A}| - 9) - 2(5 \cdot 2 \cdot 4 \cdot 6 + (|\mathcal{A}| - 5 \cdot 2 \cdot 5 \cdot 6)$$

$$= 2|\mathcal{A}| - 18 - 4 - |\mathcal{A}| + 5$$

$$= |\mathcal{A}| - 17,$$

so $|\mathcal{A}| = 17$. Since $|\mathcal{A}^Y \cup \mathcal{A}^{Y'}| = 15$ there are exactly 2 other hyperplanes $H_1, H_2$ not contained in either $\mathcal{A}^Y$ or in $\mathcal{A}^{Y'}$. But then there is a $Z \in L_2(\mathcal{A})$, $Z \subseteq H_i$ for an $i = 1, 2$ such that $Z \notin \mathcal{A}^{H_i}$. This contradicts the modularity of $Y$ and finishes the proof. □

From the previous proposition, by taking localizations and Lemma 3.7 we immediately obtain the following theorem.

**Theorem 5.2.** Let $\mathcal{A}$ be an irreducible supersolvable simplicial $\ell$-arrangement with $\ell \geq 4$. Then for all $X \in L_2(\mathcal{A})$ we have $|\mathcal{A}_X| \leq 4$.

After establishing this strong constraint, in a sequence of lemmas we will decimate the number of possible Coxeter graphs for irreducible supersolvable simplicial 4-arrangements. We will use this to derive the crystallographic property at the end of this section.
Lemma 5.3. Let $A$ be an irreducible supersolvable simplicial 4-arrangement and let $K \in \mathcal{K}(A)$ be a chamber. Then $\Gamma(K)$ has no subgraph of the form shown in Figure 10.

Proof. Suppose there exists a chamber $K \in \mathcal{K}(A)$ with $B^K = \{\alpha_1, \ldots, \alpha_4\}$ such that $\Gamma(K)$ has a subgraph of this form. Then by Lemma 3.10 for $H = \sigma^{(\alpha_3)} K_1 \in A_{13}$ and the chamber $K_1^H \in \mathcal{K}(A^H)$ we find that the graph of Figure 11 is a subgraph of $\Gamma(K^H)$.

But this is a contradiction since by Lemma 2.36 and Lemma 3.11 the restricted arrangement $A^H$ is an irreducible supersolvable simplicial 3-arrangement and such a graph is not contained in the list of Corollary 4.15.

Lemma 5.4. Let $A$ be an irreducible supersolvable simplicial 4-arrangement and let $K \in \mathcal{K}(A)$ be a chamber. Then $\Gamma(K)$ has no subgraph of the form shown in Figure 12.

Proof. It is convenient to denote the graph of Figure 12 by $\tilde{\Gamma}$.

Suppose there is a chamber $K$ such that $\tilde{\Gamma}$ is a subgraph of $\Gamma(K)$ and let $K'$ be an adjacent chamber. By Lemma 5.3 the graph $\Gamma(K)$ cannot have a chord. But then by Lemma 3.4 the Coxeter graph $\Gamma(K')$ of the adjacent chamber also has a subgraph of the form shown in Figure 12 and hence, disregarding the labels, $\Gamma(K')$ is the same graph as $\Gamma(K)$. Thus by induction for all chambers $K \in \mathcal{K}(A)$ the graph $\tilde{\Gamma}$ is a subgraph of $\Gamma(K)$. Now let $X \in L_3(A)$ and $K \in \mathcal{K}(A)$ be some chamber adjacent to $X$, i.e. $X \in L_3(W^K)$. Then by Lemma 3.7 the Coxeter graph $\Gamma(K_X)$ for a chamber $K_X \in \mathcal{K}(A_X/X)$ contains an induced subgraph on 3 vertices of $\tilde{\Gamma}$ and thus is connected. So $A_X$ is irreducible for all $X \in L_3(A)$ by Lemma 3.5. This is a contradiction to Theorem 2.5.

To give a complete list of all possible Coxeter graphs of irreducible supersolvable simplicial 4-arrangements we need the explicit description of the change of Coxeter graphs for adjacent chambers in the three possible irreducible rank 3 localizations given by the next lemma.

Lemma 5.5. Let $A$ be one of the arrangements $A(A_3)$, $A(B_3)$ or $A_3^2$. Then Figure 13 gives a complete description of the change of the Coxeter graphs for adjacent chambers where an
\[ A(6, 1) \cong A(A_3): \begin{array}{ccc} 1 & 2 & 3 \\ \end{array} \cup \sigma_1, \sigma_2, \sigma_3 \]

\[ A(9, 1) \cong A(B_3): \begin{array}{ccc} 1 & 2 & 3 \\ \end{array} \cup \sigma_1, \sigma_2, \sigma_3 \]

\[ A(8, 1) \cong A_3^2: \begin{array}{ccc} 1 & 2 & 3 \\ \end{array} \cup \sigma_1, \sigma_2, \sigma_3 \]

\[ \begin{array}{ccc} 1 & 2 & 3 \\ \end{array} \cup \sigma_2, \sigma_3 \]

\[ \begin{array}{ccc} 1 & 2 & 3 \\ \end{array} \cup \sigma_3, \sigma_2 \]

\[ \begin{array}{ccc} 1 & 2 & 3 \\ \end{array} \cup \sigma_1, \sigma_3 \]

**Figure 13.** Diagrams for the change of Coxeter graphs of adjacent chambers in \( A(A_3), A(B_3), \) and \( A_3^2 \) respectively.

**Figure 14.** The arrangements \( A(8, 1) \) and \( A(9, 1) \)

arrow labeled with \( \sigma_i \) means crossing the \( i \)-th wall corresponding to the \( i \)-th vertex of the Coxeter graph.

**Proof.** The diagrams for \( A(A_3) \) and \( A(B_3) \) are obvious since they are reflection arrangements and hence for all chambers the Coxeter graph is the same.

The third diagram can be seen by looking at a projective picture of the arrangement (see Figure 14) or as a special case of [CH15, Prop. 3.8].

**Lemma 5.6.** Let \( A \) be an irreducible supersolvable simplicial 4-arrangement and let \( K \in \mathcal{K}(A) \) be a chamber. Then \( \Gamma(K) \) is not one of the graphs of Figure 13.

**Proof.** Let \( B^K = \{ \alpha_1, \ldots, \alpha_4 \} \) be a basis for \( K \).

First suppose that there is a \( K \in \mathcal{K}(A) \) such that \( \Gamma(K) = \Gamma_1 \). By Lemma 2.37(1) the arrangements \( A^K_{123} := A_X/X \) with \( X = \alpha_1^+ \cap \alpha_2^+ \cap \alpha_3^+ \) and \( A^K_{124} := A_{X'}/X' \) with \( X' = \alpha_1^+ \cap \alpha_2^+ \cap \alpha_4^+ \) are supersolvable and simplicial. By Lemma 3.7 both localizations
contain a chamber with Coxeter graph the induced subgraph on the vertices \{1, 2, 3\} or \{1, 2, 4\} of \(\Gamma_1\). Hence by Lemma 3.10 both localizations are irreducible and by Corollary 4.15 they are either lattice equivalent to \(A(8, 1)\) or \(A(9, 1)\). Since \(|A_{23}^{K}| = |A_{24}^{K}| = 3\) by Lemma 4.3 the intersection \(Y = \alpha_1^+ \cap \alpha_2^+\) is modular in \(A_X\) and \(A_Y\). Let \(H = \sigma_2^K(\alpha_1) \in A_Y\) then by Lemma 3.10 the Coxeter graph of \(K_2^H \in K(A^H)\) contains a subgraph of the form shown in Figure 16.

For the arrangements \(A(8, 1)\) and \(A(9, 1)\) in both cases we have \(|A_{123}^H| = |A_{234}^H| = 4\). So actually both labels of the Coxeter subgraph are equal to 4 and \(\Gamma(K^H)\) contains a subgraph as in Figure 11. This is a contradiction to Corollary 4.15 and we can exclude the graph \(\Gamma_1\) from the list of possible Coxeter graphs of irreducible supersolvable simplicial 4-arrangements.

Secondly suppose that \(\Gamma(K) = \Gamma_2\). Then by Lemma 3.10 there is a hyperplane \(H \in A_{23}^K\) and a chamber \(K^H \in K(A^H)\) such that the graph shown in Figure 11 is a subgraph of \(\Gamma(K^H)\) contradicting Corollary 4.15 again.

For the graphs \(\Gamma_3\) and \(\Gamma_4\) the localization \(A_{234}^K\) is an irreducible supersolvable simplicial 3-arrangement. By Theorem 5.2 it has rank 2 localizations of size at most 4. By Lemma 3.7 there is a chamber in \(A_{234}^K\) with Coxeter graph the induced subgraph on the vertices \{2, 3, 4\}. But this a contradiction to Corollary 4.15.

Finally suppose that there is a \(K \in K(A)\) such that \(\Gamma(K) = \Gamma_5\) and let \(B^K = \{\alpha_1, \ldots, \alpha_4\}\). Let \(X = \alpha_1^+ \cap \alpha_2^+ \cap \alpha_3^+, X' = \alpha_2^+ \cap \alpha_3^+ \cap \alpha_4^+, A_{123}^K = A_X/X\) and \(A_{234}^K = A_{X'}/X'\). By Lemma 2.37(1) these arrangements are supersolvable and simplicial, and by Lemma 3.7 Lemma 3.5 and Corollary 4.15 the two arrangements are either \(A(8, 1)\) or \(A(9, 1)\). If both arrangements are \(A(9, 1)\) then for all \(H \in A_Y\) with \(Y = \alpha_2^+ \cap \alpha_3^+\) we have \(|A_X^H| = |A_Y^H| = 4\) (see Figure 14) and similarly to the first part of this proof we can find an \(H'\) and a \(K^H' \in K(A^H)\) which contains the forbidden Coxeter subgraph of Figure 11. So assume without loss of generality that \(A_{123}\) is equal to \(A(8, 1)\). Using Lemma 3.7 Lemma 3.4 and Lemma 5.5 we obtain one of the sequences of Coxeter graphs for the corresponding sequence of chambers (A)–(D) of Figure 17 (depending on \(A_{234}^K\)). But in each sequence the last graph is (up to renumbering the vertices) one which we already excluded above. E.g. the last graph in sequence (A) is
(a) $A_{234}^{K_1} \sim_L A(6,1)$

(b) $A_{234}^{K_1} \sim_L A(8,1)$

(c) $A_{234}^{K_1} \sim_L A(9,1)$

(d) $A_{234}^{K_1} \sim_L A(8,1), A(9,1)$

Figure 17. Sequences of graphs of chambers in $A$ starting at $K$ and leading to a contradiction.

the graph $\Gamma_1$ which we already excluded. Hence $\Gamma_5$ is not the Coxeter graph of a chamber of an irreducible supersolvable simplicial 4-arrangement.

$\square$

**Proposition 5.7.** Let $A$ be an irreducible supersolvable simplicial 4-arrangement and $K \in \mathcal{K}(A)$. Then $\Gamma(K)$ is one of the Coxeter graphs displayed in Figure 18.

**Proof.** By Lemma 5.4 no big cycles are possible and by Proposition 5.1 all labels are at most 4. Furthermore with Lemma 3.7, Lemma 3.10 and Corollary 4.15 we see that the
graph cannot contain two edges labeled with 4 by looking at the appropriate restriction respectively localization not fitting into the classification of rank 3 arrangements and their Coxeter graphs (see Theorem 4.13 and Corollary 4.15). Now by Lemma 5.6 the only possible

Proposition 5.8. Let \( \mathcal{A} \) be an irreducible supersolvable simplicial 4-arrangement and \( K \in \mathcal{K}(\mathcal{A}) \).

1. There exists a locally crystallographic basis \( B^K \) for \( K \) such that the Cartan matrix \( C^K \) with respect to \( B^K \) is of type \( A, C, D', \) or \( D \).
2. If \( B^K = \{ \alpha_1, \ldots, \alpha_i \} \) is a locally crystallographic basis for \( K \) with \( C^K \) of type \( A, C, D', \) or \( D \), then for \( 1 \leq i \leq 4 \) the basis \( B^K_i = \sigma_i^K(B^K) = \{ \alpha_j + c_{ij} \alpha_i \mid 1 \leq j \leq 4 \} \) is a locally crystallographic basis with Cartan matrix \( C^K_i \) of type \( A, C, D', \) or \( D \).

Proof. Part (1). By Proposition 5.7 the Coxeter graph \( \Gamma(K) \) is one of the graphs of Figure 18. Let \( W^K = \{ H_1, \ldots, H_i \} \), and \( \Gamma(K) = (\mathcal{V}, \mathcal{E}) \) with numbering of the walls corresponding to the numbering of the vertices of the graphs in Figure 18.

Firstly suppose that \( \Gamma(K) = \Gamma_1^4 \). By Lemma 2.37 and Lemma 3.7 the localization \( \mathcal{A}_{123}^K \) adjacent to \( K \) is an irreducible supersolvable simplicial 3-arrangement with a modular rank 2 intersection of size at most 4 by Theorem 5.2. Hence by Theorem 4.13 and Corollary 4.15 it is the arrangement \( \mathcal{A}(6, 1) \) or \( \mathcal{A}(8, 1) \) and in particular crystallographic (see Example 2.30). By choosing a crystallographic root system for \( \mathcal{A}_{123}^K \) and taking the corresponding basis for the chamber in the localization by Example 2.30 there are \( \alpha_1, \alpha_2, \alpha_3 \in (\mathbb{R}^4)^* \) such that \( \alpha_1^\perp = H_i, \ K \subseteq \alpha_1^\perp \), \( (\alpha_1 + \alpha_2)^\perp \in W^{K_1}, W^{K_2} \), and \( (\alpha_2 + \alpha_3)^\perp \in W^{K_2}, W^{K_3} \). Let \( \alpha_4 \in (\mathbb{R}^4)^* \) such that \( \alpha_4^\perp = H_4 \) and \( \alpha_4^\perp \subseteq K \). Since \( \{3, 4\} \in \mathcal{E} \) with label \( m(K, 3, 4) = 3 \) there is a unique \( \lambda \in \mathbb{R}_{>0} \) such that \( (\alpha_3 + \lambda \alpha_4)^\perp \in W^{K_3}, W^{K_4} \). But then with \( \alpha_4 := \lambda \alpha_4 \) we have \( (\alpha_3 + \alpha_4)^\perp \in W^{K_3}, W^{K_4} \). Hence \( B^K := \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) is a locally crystallographic basis for \( K \) with Cartan matrix \( C^K = (\epsilon_{ij}^K) \) of type \( A \).

The same arguments work for the Coxeter graphs \( \Gamma_4^3 \) and \( \Gamma_4^4 \) since the vertex denoted as 4 is only connected with the vertex 3 and the localization \( \mathcal{A}_{123}^K \) is \( \mathcal{A}(6, 1) \) or \( \mathcal{A}(8, 1) \) by Theorem 4.13. So similarly there is a locally crystallographic basis \( B^K \) for \( K \) such that the Cartan matrix is of type \( D' \) if \( \Gamma(K) = \Gamma_3^3 \), or of type \( D \) if \( \Gamma(K) = \Gamma_4^4 \).

Now assume that \( \Gamma(K) = \Gamma_2^4 \). Then \( \mathcal{A}_{123}^K \) is \( \mathcal{A}(8, 1) = \mathcal{A}_3^2 \) or \( \mathcal{A}(9, 1) = \mathcal{A}(B_3) \). Then there are \( \alpha_1, \alpha_2, \alpha_3 \in (\mathbb{R}^4)^* \) such that \( \alpha_1^\perp = H_i, \ K \subseteq \alpha_1^\perp \), \( (2\alpha_1 + \alpha_2)^\perp \in W^{K_2}, (\alpha_1 + \alpha_2)^\perp \in W^{K_1} \), and \( (\alpha_2 + \alpha_3)^\perp \in W^{K_2}, W^{K_3} \) (by choosing a proper crystallographic root system for the

Figure 18. Remaining Coxeter graphs for an irreducible supersolvable simplicial 4-arrangement.
localization and taking the corresponding basis for the chamber in the localization). Again it is clear that we can find an \( \alpha_4 \in (\mathbb{R}^2)^* \), \( K \subseteq \alpha_4^\perp \) such that \( (\alpha_3 + \alpha_4)^\perp \in W^{K_3}, W^{K_3} \) and hence \( B^K := \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) is a locally crystallographic basis for \( K \) with Cartan matrix \( C^K = (c^K_{ij}) \) of type \( C \).

Part (2). For the second part we use Proposition 5.7, Lemma 2.32, Lemma 2.33, Lemma 3.3, and Lemma 5.5 to obtain the Coxeter graphs for the adjacent chamber \( K_i \) and deduce the claimed property of the induced basis \( B^{K_i} \) and the coefficients \( c^K_{ij} \).

We check the cases in turn. First assume that \( \Gamma(K) = \Gamma_4^1 \). \( B^K \) is locally crystallographic and \( C^K \) is of type \( A \). As we have seen in the proof of Part (1), the localization \( A^K_{123} \) is the arrangement \( A(6,1) \) or \( A_1(8,1) \).

Let \( i = 1 \). By Proposition 5.7, the Coxeter graph \( \Gamma(K_1) \) is one of the four graphs of Figure 18 and by Lemma 2.33 and Lemma 3.6 only \( \Gamma_1^4 \) is possible. Thus \( \Gamma(K_1) = \Gamma(K) \) and by Lemma 2.32, Lemma 2.33, Lemma 3.3, and Lemma 5.5 the basis \( B^{K_1} \) induced by \( C^K \) and \( B^K \) is locally crystallographic with Cartan matrix \( C^{K_1} = C^K \) of type \( A \). Note that from this case the statement also follows for \( \Gamma(K) = \Gamma_4^1 \) and \( i = 4 \) by symmetry.

Next let \( i = 2 \). If the localizations \( A^K_{123} \) and \( A^K_{234} \) are both isomorphic to \( A(6,1) \) then using the same lemmas from Subsection 2.2 as above, the basis \( B^{K_2} \) defined by \( C^K \) is locally crystallographic with Cartan matrix \( C^{K_2} = C^K \) of type \( A \). If \( A^K_{123} \) is the arrangement \( A(8,1) \) then \( A^K_{234} \) has to be the arrangement \( A(6,1) \) and \( \Gamma(K_2) = \Gamma_3^4 \). Otherwise by Lemma 5.5, we would get a forbidden Coxeter graph of Figure 15 for \( K_2 \). With the lemmas from Subsection 2.2 and Section 3 we again obtain all coefficients \( c^K_{K_2} \) except the ones with \( \{i,j\} = \{1,3\} \).

But \( A^K_{123} = A^K_{234} \) is the arrangement \( A(8,1) \) for which we know that with respect to the basis \( B^{K_3} = \{ \alpha_1, \alpha_2, \alpha_3 \} \subseteq B^K \) we have \( (\alpha_1 + 2\alpha_2 + \alpha_3)^\perp \in A^K_{123} \) and in particular \( (\alpha_1 + 2\alpha_2 + \alpha_3)^\perp = (\sigma_2^K(\alpha_1) + \sigma_2^K(\alpha_3))^\perp \in W^{(K_3)} \). So \( c^K_{K_2} = c^K_{31} = -1 \) and \( B^{K_2} \) is locally crystallographic with Cartan matrix \( C^{K_2} \) of type \( D' \). Note that from this case the statement also follows for \( \Gamma(K) = \Gamma_4^3 \) and \( i = 1, 2 \) by symmetry.

Now let \( i = 3 \). If the localizations \( A^K_{123} \) and \( A^K_{234} \) are both isomorphic to \( A(6,1) \) or if \( A^K_{234} \) is the arrangement \( A(8,1) \) then by symmetry we already handled these cases. So suppose that \( A^K_{123} \) is the arrangement \( A(8,1) \). Then by Lemma 5.5 and the lemmas from Subsection 2.2 and Section 3 we have \( \Gamma(K_3) = \Gamma_4^4 \). But we also obtain all \( c^K_{K_3} \) except \( c^K_{21} \).

But with respect to the basis \( B^{K_3} = \{ \alpha_1, \alpha_2, \alpha_3 \} \subseteq B^K \) we have \( (2\alpha_1 + \alpha_2 + \alpha_3)^\perp = \sigma^K_3(\alpha_1) + \sigma^K_3(\alpha_2)^\perp \in A^K_{123} \) so \( c^K_{21} = -2 \) and \( B^{K_3} \) is locally crystallographic with Cartan matrix \( C^{K_3} \) of type \( C \). Note that from this case the statement also follows for \( \Gamma(K) = \Gamma_4^2 \) and \( i = 3 \) by symmetry.

The other remaining cases, i.e.

- \( \Gamma(K) = \Gamma_2^4 \) and \( i \in \{1, 2, 4\} \),
- \( \Gamma(K) = \Gamma_3^4 \) and \( i \in \{3, 4\} \),
- \( \Gamma(K) = \Gamma_4^4 \) and \( i \in \{1, 2, 3, 4\} \),

can be handled completely analogously.

Proposition 5.8 immediately tells us that an irreducible supersolvable simplicial 4-arrangement defines a Weyl groupoid and thus a crystallographic arrangement (cf. Cun11a).
Thm. 1.1]). Since we did not introduce the notion of a Weyl groupoid, we repeat the argument without this terminology:

**Proposition 5.9.** Let $A$ be an irreducible supersolvable simplicial $4$-arrangement, and fix a chamber $K^0 \in \mathcal{K}(A)$. Then there exists a basis $B^{K^0}$ for $K^0$ such that

$$R := \bigcup_{G \in \mathcal{G}(K^0,A)} B_G$$

is a crystallographic root system for $A$.

**Proof.** By Proposition 5.8(1) for $K^0$ there exists a locally crystallographic basis $B^{K^0}$ with Cartan matrix of type $A, C, D'$ or $D$. Such a basis will have the desired property and from now on we fix it.

First we show that for $K \in \mathcal{K}(A)$ the basis $B^K_G$ does not depend on the chosen $G \in \mathcal{G}(K^0,A)$ with $e(G) = K$. Let $G, \tilde{G} \in \mathcal{G}(K^0,A)$ with $e(G) = e(\tilde{G}) = K$, say

$$G = (K^0, K^1, \ldots, K^{n-1}, K^n = K),$$

and

$$\tilde{G} = (K^0, \tilde{K}^1, \ldots, \tilde{K}^{m-1}, \tilde{K}^m = K).$$

Then

$$B_G = (\sigma_{\mu_{n-1}}^{K^{n-1}} \circ \ldots \circ \sigma_{\mu_0}^{K^0})(B^{K^0}),$$

where the linear map $\sigma_{\mu_{n-1}}^{K^{n-1}} \circ \ldots \circ \sigma_{\mu_0}^{K^0}$ is represented with respect to $B^{K^0}$ by a product of reflection matrices

$$S_{\mu_{n-1}}^{K^{n-1}} \cdots S_{\mu_0}^{K^0} =: S.$$ 

By Proposition 5.8(2) and an easy induction over the length $n$ of $G$ all reflection matrices $S^K_i$ are integral matrices with determinant $-1$. Hence the product $S$ has only entries in $\mathbb{Z}$ and has determinant $\pm 1$. Similarly for $\tilde{G}$ we have

$$B_{\tilde{G}} = (\sigma_{\tilde{\mu}_{m-1}}^{\tilde{K}^{m-1}} \circ \ldots \circ \sigma_{\tilde{\mu}_0}^{\tilde{K}^0})(B^{K^0}),$$

where the linear map is represented with respect to $B^{K^0}$ by a product of integral reflection matrices

$$S_{\tilde{\mu}_{m-1}}^{\tilde{K}^{m-1}} \cdots S_{\tilde{\mu}_0}^{\tilde{K}^0} =: \tilde{S},$$

and $\tilde{S}$ also has only entries in $\mathbb{Z}$ and determinant equal to $\pm 1$. Now $S\tilde{S}^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_4)P$ where $P$ is a permutation matrix and $\lambda_i \in \mathbb{R}_{>0}$ because $\{\alpha^+ \mid \alpha \in B_G\} = \{\beta^+ \mid \beta \in B_{\tilde{G}}\}$. But $S\tilde{S}^{-1}$ is an integer matrix entries and has determinant $\pm 1$. Hence $\text{diag}(\lambda_1, \ldots, \lambda_4)$ has determinant $1$ so $\lambda_1 = \ldots = \lambda_4 = 1$ and thus $B_G = B_{\tilde{G}}$ (up to a permutation of the basis elements).

From the above consideration we obtain

$$B_G \subseteq \sum_{\alpha \in B_G^t} \mathbb{Z}\alpha,$$
for $G, G' \in \mathcal{G}(K^0, \mathcal{A})$. Hence for $R$ we have
\[
R \subseteq \sum_{\alpha \in B^\mathcal{A}_K} \mathbb{Z}\alpha,
\]
for all $K \in \mathcal{K}(\mathcal{A})$ since $B^\mathcal{A}_K = B_G$ for some $G \in \mathcal{G}(K^0, \mathcal{A})$ with $e(G) = K$ and each $\beta \in R$ is contained in some $B_G, G' \in \mathcal{G}(K^0, \mathcal{A})$.

It remains to show that $R$ is reduced, i.e. that for $\beta \in R$ we have $R \cap R_\beta = \{\pm \beta\}$. So suppose that $\beta \in R$ and $\lambda \beta \in R$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Then there are $G, G' \in \mathcal{G}(K^0, \mathcal{A})$ such that $\beta \in B_G$ and $\lambda \beta \in B_{G'}$. But as above $\lambda \beta \in \mathbb{Z}\beta$ and $\beta \in \mathbb{Z}(\lambda \beta)$. Hence $\lambda \in \{\pm 1\}$. □

**Theorem 5.10.** Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement. Then $\mathcal{A}$ is isomorphic to exactly one of the reflection arrangements $\mathcal{A}(A_4), \mathcal{A}(C_4)$, or $\mathcal{A}_3^4 = \mathcal{A}(C_4) \setminus \{x_1 = 0\}$. In particular $\mathcal{A}$ is crystallographic.

**Proof.** By Proposition 5.9 there exists a crystallographic root system for $\mathcal{A}$, so the arrangement $\mathcal{A}$ is crystallographic. By Theorem 2.42 the only irreducible crystallographic 4-arrangements which are supersolvable are the three arrangements $\mathcal{A}(A_4), \mathcal{A}(C_4)$, and $\mathcal{A}_3^4 = \mathcal{A}(C_4) \setminus \{x_1 = 0\}$.

**Corollary 5.11.** Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement and $K \in \mathcal{K}(\mathcal{A})$. Then $\Gamma(K)$ is not the Coxeter graph $\Gamma_4^4$ of Figure 18.

6. **The rank $\geq 5$ case**

**Lemma 6.1.** Let $\mathcal{A}$ be an irreducible simplicial supersolvable $\ell$-arrangement and let $K \in \mathcal{K}(\mathcal{A})$ be a chamber. Then $\Gamma(K)$ has no cycles with more than 3 vertices.

**Proof.** Suppose there is a chamber $K \in \mathcal{K}(\mathcal{A})$ such that $\Gamma(K)$ has a cycle with more than three vertices. Then we localize at the intersection of the walls corresponding to these vertices and use Lemma 3.7 and Lemma 3.10 (possibly several times) to arrive at a 4-arrangement which is irreducible by Lemma 3.11 simplicial and supersolvable by Lemma 2.37 and contains a chamber $K'$ such that the Coxeter graph $\Gamma(K')$ contains a subgraph of the form displayed in Figure 12. This is a contradiction to Lemma 5.4. □

**Lemma 6.2.** Let $\mathcal{A}$ be an irreducible supersolvable simplicial $\ell$-arrangement, $\ell \geq 5$ and let $K \in \mathcal{K}(\mathcal{A})$ be a chamber. Then the Coxeter graph $\Gamma(K)$ does not contain a subgraph of the form shown in Figure 19.

**Proof.** Let $B^K = \{\alpha_1, \ldots, \alpha_\ell\}$, and suppose that $\Gamma(K)$ has a subgraph of this form containing the vertices $\{i_1, \ldots, i_5\} \subseteq \{1, \ldots, \ell\}$. By Lemma 3.7 and Lemma 3.10 localizing $\mathcal{A}_{i_1 \cdots i_5}^K$ and restricting to $H = \sigma_{i_2}^K(\alpha_{i_3})^\perp$ gives the irreducible supersolvable simplicial 4-arrangement...
Proposition 6.3. Let \( \mathcal{A} \) be an irreducible supersolvable simplicial \( \ell \)-arrangement, \( \ell \geq 4 \) and let \( K \in \mathcal{K}(\mathcal{A}) \) be a chamber. Then \( \Gamma(K) \) is one of the Coxeter graphs of Figure 20.

Proof. By Lemma \([5,2]\) the Coxeter graph \( \Gamma(K) \) cannot have a triangle somewhere in the middle.

The statement then follows by induction on \( \ell \), Lemma \([3,7]\), Proposition \([5,7]\), Corollary \([5,11]\) and Lemma \([6,1]\).

Proposition 6.4. Let \( \mathcal{A} \) be an irreducible supersolvable simplicial \( \ell \)-arrangement, \( \ell \geq 4 \) and \( K \in \mathcal{K}(\mathcal{A}) \).

(1) There exists a locally crystallographic basis \( B^K \) for \( K \) such that the Cartan matrix \( C^K \) is of type \( A,C \) or \( D' \).

(2) If \( B^K = \{\alpha_1, \ldots, \alpha_4\} \) is a locally crystallographic basis for \( K \) with \( C^K \) of type \( A,C \) or \( D' \), then for \( 1 \leq i \leq \ell \) the basis \( B^K_i = \sigma_i^K(B^K) = \{\alpha_j + c^K_{ij} \alpha_i \mid 1 \leq j \leq \ell\} \) is a locally crystallographic basis with Cartan matrix \( C^{K_i} \) of type \( A,C \) or \( D' \).

Proof. We argue by induction on \( \ell \geq 4 \). For \( \ell = 4 \) this is Proposition \([5,8]\). Let \( \ell \geq 5 \) and assume both statements are true for \( \ell - 1 \). By Proposition \([6,3]\) the Coxeter graph \( \Gamma(K) \) is one of the graphs of Figure 20. Let \( W^K = \{H_1, \ldots, H_\ell\} \) where the numbering of the walls corresponds to the numbering of the vertices in \( \Gamma^1_\ell, \Gamma^2_\ell, \Gamma^3_\ell \) respectively.

By the induction hypothesis for the localization \( A_{12,\ldots,(\ell-1)}^K \) there are \( \{\alpha_1, \ldots, \alpha_{\ell-1}\} \subseteq (\mathbb{R}^\ell)^* \) which form a locally crystallographic basis for the corresponding chamber in \( A_{12,\ldots,(\ell-1)}^K \). Furthermore \( \alpha_i^+ = H_i \) for \( 1 \leq i \leq \ell - 1 \), there are \( c^K_{ij} \in \mathbb{Z}, 1 \leq i, j \leq \ell - 1 \) such that \( (\alpha_j - c^K_{ij} \alpha_i)^\perp \in W^{K_i} \), and the matrix \( C'^K = (c^K_{ij})_{i,j,\ell} \) is a Cartan matrix of type \( A,C \), or \( D' \). But in \( \Gamma(K) \) the vertex \( \ell \) is only connected to \( \ell - 1 \) by an edge with label 3. Hence there is an \( \alpha_\ell \in (\mathbb{R}^\ell)^* \) such that \( \alpha_\ell^+ = H_\ell, K \subseteq \alpha_\ell^+, (\alpha_\ell + \alpha_\ell)^\perp \in W^{K_{\ell-1}}, W^{K_\ell} \). This is to say for \( B^K := \{\alpha_1, \ldots, \alpha_\ell\} \) we have \( c^K_{(\ell-1)\ell} = c^K_{\ell(\ell-1)} = -1, c^K_{ij} = c^K_{ji} = 0 \) for \( j \notin \{\ell - 1, \ell\} \), the other \( c^K_{ij} \) are given by the localization \( A_{12,\ldots,(\ell-1)}^K \), and hence \( B^K \) is a locally crystallographic basis for \( K \) with Cartan matrix of type \( A,C \), or \( D' \) if \( \Gamma(K) \) is \( \Gamma^1_\ell, \Gamma^2_\ell, \) or \( \Gamma^3_\ell \) respectively.

Now let \( B^K \) be a locally crystallographic basis with Cartan matrix of type \( A,C \), or \( D' \) and \( B^K_i \) the induced basis for \( K_i \).
If \( i = \ell \) then in each case \( \Gamma(K_i) = \Gamma(K) \) by Lemma 3.4, Lemma 3.6 and Proposition 6.3 where the vertex \( k \) in \( \Gamma(K_i) \) corresponds to the root \( \sigma^K_\ell(\alpha_k) \). In all graphs \( \Gamma^K_\ell \) the vertex \( \ell \) is not connected with the vertex \( j \) for \( j \leq \ell - 2 \), and \( m^K(i,j) = 3 \) for all \( i, j \in \{1, \ldots, \ell\} \) except possibly for \( \{i, j\} = \{1, 2\} \). So by Lemma 2.32 and Lemma 2.33 the induced basis \( B^K_\ell \) is locally crystallographic with Cartan matrix \( C^K_\ell = C^K \) of type \( A, C \), or \( D' \).

For \( i \in \{1, \ldots, \ell - 1\} \) we have \( A^K_{1,\ldots,(\ell-1)} = A^K_{1,\ldots,(\ell-1)} \). So at least \( C^{K_1} = (c^K_{ij})_{1\leq i,j \leq \ell-1} \) is a Cartan matrix of type \( A, C \), or \( D' \) by the induction hypothesis. If \( C^{K_1} \) is of type \( C \), or \( D \) then by Proposition 6.3 the Coxeter graph \( \Gamma(K_i) \) is \( \Gamma^C_\ell \), or \( \Gamma^D_\ell \) respectively with the numbering of the vertices corresponding to \( B^K_\ell = \{\sigma^K_1(\alpha_1), \ldots, \sigma^K_\ell(\alpha_\ell)\} \). If \( C^{K_1} \) is of type \( A \) we may also assume that \( \Gamma(K_i) \) is the Coxeter graph \( \Gamma^A_\ell \) since otherwise we can renumber the bases \( B^K \) and \( B^K_\ell \) respectively such that \( C^{K_1} \) is of type \( C \), or \( D' \) and we actually are in one of the above cases. We observe next that in \( \Gamma(K_i) \) the vertex \( \ell \) is not connected with the vertex \( j \) for \( j \leq \ell - 2 \). So \( c^K_{ij} = c^K_{ij} = 0 \) for \( 1 \leq j \leq \ell - 2 \).

If \( i \in \{1, \ldots, \ell - 2\} \) we have \( c^K_{ii} = 0 \) and then by Lemma 2.33 we get \( c^K_{i,\ell-1} = c^K_{i,\ell-1} \). But \( m^K_\ell(\ell-1,\ell) = 3 \) and by Lemma 3.3 for the last remaining coefficient we obtain \( c^K_{i,\ell-1} = -1 \) and \( B^K_\ell \) is a locally crystallographic basis with Cartan matrix of type \( A, C \), or \( D' \).

Finally for \( i = \ell - 1 \) by Lemma 2.32 we also have \( c^K_{(\ell-1),\ell} = c^K_{(\ell-1),\ell} = -1 \). Again since \( m^K_{\ell-1}(\ell-1,\ell) = 3 \), by Lemma 3.3 for the remaining coefficient we get \( c^K_{(\ell-1),\ell} = -1 \) and \( B^K_{\ell-1} \) is a locally crystallographic basis with Cartan matrix \( C^K_{\ell-1} \) of type \( A, C \), or \( D' \). This finishes the proof. \( \square \)

**Proposition 6.5.** Let \( A \) be an irreducible supersolvable simplicial \( \ell \)-arrangement, \( \ell \geq 4 \), and fix a chamber \( K^0 \in K(A) \). Then there exists a basis \( B^K_0 \) for \( K^0 \) such that

\[
R := \bigcup_{G \in G(K^0, A)} B_G
\]

is a crystallographic root system for \( A \).

**Proof.** This is exactly the same as the proof of Proposition 5.9 using Proposition 6.4 instead of Proposition 5.8. \( \square \)

**Theorem 6.6.** Let \( A \) be an irreducible simplicial supersolvable \( \ell \)-arrangement, \( \ell \geq 4 \). Then \( A \) is isomorphic to exactly one of the reflection arrangements \( A(A_\ell) \), \( A(C_\ell) \), or \( A^{\ell-1}_\ell = A(C_\ell) \setminus \{x_1 = 0\} \). In particular \( A \) is crystallographic.

**Proof.** By Proposition 6.5 there exists a crystallographic root system for \( A \), so the arrangement \( A \) is crystallographic. By Theorem 2.42 the only irreducible crystallographic \( \ell \)-arrangements, \( \ell \geq 4 \) which are supersolvable are the arrangements \( A(A_\ell) \), \( A(C_\ell) \), and \( A^{\ell-1}_\ell = A(C_\ell) \setminus \{x_1 = 0\} \). \( \square \)

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