Strong magnetic field induces superconductivity in Weyl semi-metal.

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Microscopic theory of the normal-to-superconductor coexistence line of a multi-band Weyl superconductor subjected to magnetic field is constructed. It is shown that Weyl semi-metal that is nonsuperconducting or having a small critical temperature $T_c$ at zero field, might become superconductor at higher temperature, when the magnetic field is tuned to a series of quantized values $H_n$. The pairing occurs on Landau levels. It is argued that the phenomenon is much easier detectable in Weyl semi-metals than in parabolic band metals since the quantum limit already has been approaches in several Weyl materials. The effect of Zeeman coupling leading to splitting of the reentrant superconducting regions on the magnetic phase diagram is considered. An experimental signature of the superconductivity on Landau levels is reduction of magnetoresistivity. This has already been observed in $Cd_3As_2$ and several other compounds. The novel kind of quantum oscillations of magnetoresistance detected in $ZrTe_5$ is discussed along these lines.

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I. INTRODUCTION

Conventional superconductivity arises from pairing of electrons in the vicinity of the Fermi surface, since the phonon mediated attraction is effective only when the electron’s energy is within a shell of the Debye energy width, $\hbar\Omega$ of order several hundreds of kelvin, see Fig.1. Within the BCS theory (in the adiabatic limit) the order parameter, $\Delta \sim T_c$, depends exponentially on the density of states (DOS) at Fermi level $D(\mu)$, so that in order to enhance the tendency for superconductivity, one should use any means to boost the density of states within this narrow shell. In quantum systems there is an obvious way to boost locally the DOS - quantization. Thus a natural mean to concentrate the spectral weight is a strong magnetic field that causes Landau quantization. The best known example of this phenomenon is 2D the electron gas in magnetic field, where DOS can be tuned to “infinity” at certain values of magnetic fields and the quantum Hall effect became visible.

In principle, one can imagine that strong magnetic field can enhance superconductivity as well, if the quantum limit (when the Fermi surface crosses the lowest Landau levels) is reached. At first glance there are two immediate problems with this scenario. First the magnetic field generally breaks the Cooper pairs due to the orbital instability that lead to suppression of superconductivity at $H_{c2}$. Second, the direct (Zeeman) coupling of the magnetic field to the electron’s spin also leads (for the singlet pairing) to the Chandrasekhar - Klogston\(^2\) pair breaking at $H_p$. However it was predicted in eighties of the last century (see\(\ref{1}^{3,5}\) and references therein) that paradoxically superconductivity can reappear on the Landau levels (LL) at fields far above $H_{c2}$. While the superconductivity enhancement can occur at any LL, it is stable against perturbations only near the ”quantum limit”, in which the lowest LL level crosses the Fermi energy $\mu$. The condition for that, $\mu \sim \hbar\omega_c$, however restricts the choice of material to those with extremely small electron density. Even for 100$T$ the Fermi level should be just 10$meV$.

In conventional metallic superconductors, even at $H_{c2} = \Phi_0/2\pi\xi^2$ (where $\xi$ is the coherence length at zero temperature and $\Phi_0$ is the flux quantum), the effect of the Landau quantization of the electron motion is negligible. For a metal with effective mass $m^*$, the separation between (equidistant) Landau levels is $\hbar\omega_c = \hbar eH/m^*c$. For typical values of the field $H_{c2} = 3T$ and effective mass $m^* \sim m_e$, the level spacing is $4K$, much smaller than $2\hbar\Omega$. Therefore, to take advantage of the Landau quantization effect on superconductivity, one should consider a super strong magnetic field of thousands Tesla. The estimate however is based on the assumption of the parabolic dispersion relation of the normal electrons (or holes).

Recently a new class of 2D and 3D multi-band materials with qualitatively different band structure near the Fermi level was discovered\(\ref{2,11}\) - Weyl (Dirac) semi-metals (WSM). Unlike in conventional semi-metals with several quasiparticle and hole bands, in WSM Dirac points occur due to the band inversion near the Fermi level. WSM are characterized by linear dispersion relation, $\varepsilon = vp$, and in many of them the chemical potential is tunable and small. Even a more important fact for pairing is that their inter - band tunneling is dominant. In some of this novel materials conventional phonon mediated superconductivity with $T_c$ up to 20$K$ (under pressure) with $H_{c2}$ of several $T$ was achieved\(\ref{12}\). Although mechanism of superconductivity is these materials does not differ much from the low $T_c$ metal\(\ref{12,13}\), the position of the Landau levels (LL) does. The notion of the effective mass does not apply for this essentially non-parabolic dispersion relation and LL are generally no longer equidistant\(\ref{14}\), see Fig.1. This raises a possibility that the Landau quantume quant limit is easier achievable in this case\(\ref{14}\). The first LL appears at $\hbar\omega_c = v\sqrt{2}\hbar eH/c$ should be equal to $\mu$ counted from the Dirac point. For a typical values of $v = 10^6cm/s$ and $H = 100T$, now one obtains $\mu \sim 0.4eV$, that favorably compares with the previous estimate of 10$meV$ in a ”conventional” parabolic band. The condition for the superconductivity enhancement in WSM is thus qualitatively different and quantum limit condition becomes $\omega_c \hbar \sim 2\hbar\Omega$. A more quantitative estimates and comparison between the conventional materials and the WSM is made below. Therefore it is important to extend the BCS type theory to the case of multi - band semi - metals like the WSM. The extension of conventional Gor’kov- Eliashberg approach in strong magnetic field\(\ref{4,5}\) to a multi - band semi - metals by no means trivial. For two parabolic (one quasi-particle and one hole) bands it was done in ref\(\ref{14}\). Since in WSM ratio $\mu/\hbar\Omega$ is relatively small, an important additional issue is the role of the retardation effects of the phonon mediated pairing in order remain within the bounds of the adiabatic approximation.

In this paper the effect of the phonon - mediated pairing in strong magnetic fields (including the quantum limit) in Weyl semi-metals is developed in wide range of temperatures and magnetic fields. The simplest model necessarily contains four (sub) bands (two Weyl subbands and two magnetically split spin subbands due to Zeeman coupling). The magnetic phase diagram consist of a series of superconducting domes in addition to the conventional $H_{c2}(T)$ line. Recent experiment\(\ref{6}\) on $Cd_3As_2$ in fields up to 52$T$ are reinterpreted as possible candidate of re-entrant superconductivity at $N = 2, 3$ Landau levels at 25$T$ and 46$T$. It is interesting to note that the upper bound on superconductivity at zero field in this material is 3$K$. Retardation effects of the phonon mediated pairing is discussed and taken into account phenomenologically.

The paper is organized as follows. The effect of re-entrant superconductivity at very high magnetic fields is more pronounced in two dimensions, so a sufficiently general 2D WSM model is defined in Section II. The superconductor-normal phase transition line in 2D WSM in high magnetic fields is derived in Section III. The phase diagram of
FIG. 1. Set of Landau levels in Weyl semimetals. Pairing due to phonons occurs in the energy shell of Debye energy width, $\hbar\Omega$, around the Fermi level $\mu$.

superconductivity on Landau levels is extended to Zeeman coupling and to the anisotropic 3D WSM in Section IV. Comparison with recent experiments, discussion and conclusions is the subject of Section V.

II. PHONON MEDIATED SUPERCONDUCTIVITY IN WSM IN STRONG MAGNETIC FIELD.

A. Pairing in 2D WSM under magnetic field

A Weyl material typically possesses several sublattices. We exemplify the effect of the WSM band structure on superconductivity using the simplest model with just two sublattices denoted by $\alpha = 1, 2$. The effective electron-electron attraction due to the electron-phonon coupling overcomes the Coulomb repulsion and induces pairing. Typically in WSM there are numerous bands. We assume that different valleys are paired independently and drop all the valley indices (including chirality, multiplying the density of states by $2N_f$). To simplify notations, we therefore consider just one spinor (left, for definiteness), the following Weyl Hamiltonian:

$$
K = \int r \psi_{\alpha}^\dagger(r) \left\{ -i\hbar v \left( D_x \sigma_{x\alpha\beta} + D_y \sigma_{y\alpha\beta} \right) - \mu \delta_{\alpha\beta} \right\} \psi_{\beta}(r) .
$$

Here $v$ is Fermi velocity assumed isotropic in the plane $x - y$ perpendicular to the applied magnetic field (assumed isotropic, generalized later to anisotropic 3D WSM). Chemical potential is denoted by $\mu$ - chemical potential. Pauli matrices $\sigma$ operate in the sublattice space (the indices $\alpha, \beta$ will be termed the pseudo-spin projections) and $s$ is spin projection. Magnetic field appears in the covariant derivatives via the vector potential, $D_i = \nabla_i - \frac{i}{\hbar c} A_i$. Here $A$ is the vector potential.

Further we assume the local density - density interaction Hamiltonian:

$$
V = \frac{g^2}{2} \int r \psi_{\alpha}^\dagger(r) \psi_{\alpha}^\dagger(r) \psi_{\beta}^\dagger(r) \psi_{\beta}(r) ,
$$

ignoring the Coulomb repulsion (that as usual is accounted for by a pseudopotential, so that $g$ is the electron-phonon coupling). It is important that the interaction has a cutoff Debye frequency $\Omega$, so that it is active in an energy shell of width $2\hbar\Omega$ around the Fermi level. We will discuss a more realistic dependence on frequency in Section III.
B. Matsubara Green’s functions and Gor’kov equations.

Finite temperature properties of the superconducting condensate are described by the normal and the anomalous Matsubara Green’s functions (GF),

\[ G_{\alpha\beta}^{\text{ts}} (r, r' \tau') = - \left\langle T \psi_{\alpha}^i (r \tau) \psi_{\beta}^j (r' \tau') \right\rangle ; F_{\alpha\beta}^{\text{ts}} (r, r' \tau') = \left\langle T \psi_{\alpha}^i (r \tau) \psi_{\beta}^j (r' \tau') \right\rangle ; \]

with the spin Ansatz

\[ G_{\alpha\beta}^{\text{ts}} (r, r' \tau) = \delta_{\alpha\beta} G_{\alpha\beta} (r, r', \tau - \tau') ; F_{\alpha\beta}^{\text{ts}} (r, r' \tau) = -\varepsilon_{\alpha\beta} F_{\alpha\beta}^{\text{ts}} (r, r', \tau - \tau') ; \]

Here the normal GF is obtained from, \[ \sum_{\tau} - i \omega \sigma \cdot \mathbf{D} \cdot \mathbf{F} + \mu \sigma \cdot \mathbf{G} + \Delta \sigma, \] obeys the Pauli principle. The gap function consequently reads: \[ \Delta = \frac{1}{2} T \mathbf{r} \left[ \sigma^\gamma \Delta \right] \]. Notice, that in contrast to conventional metals with parabolic dispersion law, in the case of the Weyl semi-metals the second Gor’kov equation, Eq. (6), contains transposed Pauli matrices for isospins.

III. THE TRANSITION LINE

In this Section the superconductor-normal phase transition line in high magnetic fields is determined. The line breaks into a set of disconnected segments since in certain cases the superconductivity reappears when a Landau level crosses Fermi surface.

A. Linearization of the Gor’kov equations near the transition line

Near the normal-to-superconducting transition line the gap \( \Delta \) is small and the set of the Gor’kov equations can be linearized. In this case the gap equation describing the critical curve \( H_{c2} (T) \) has the form, see ref. [17] for details,

\[ \Delta (r) = \frac{g^2}{2} T \left( \int_{r'} \Delta^* (r') \sigma_{\alpha\beta}^\gamma G_{\beta\gamma}^{\text{ts}} (r', r) \right) \frac{\sigma_{\alpha\beta} G_{\alpha\beta}^1 (r, r')}{} \]

Here the normal GF is obtained from,

\[ \left[ iv \mathbf{D} \cdot \sigma \cdot \mathbf{F} + (i \omega + \mu) \delta_{\gamma\beta} \right] G_{\beta\gamma}^1 (r, r') = \delta_{\gamma\beta} (r - r') , \]

while a quantity \( \tilde{G}_{\beta\gamma} \) (an auxiliary function associated with \( G \) via a product of an axis reflection and time reversal) obeys a different equations:

\[ \left[ -iv \mathbf{D} \cdot \sigma_{\gamma\beta}^t + (-i \omega + \mu) \delta_{\gamma\beta} \right] \tilde{G}_{\beta\gamma}^2 (r', r) = \delta_{\gamma\beta} (r - r') . \]

Here \( \sigma^t \) is the transposed Pauli matrix that replaces \( \sigma \) in the customary normal state equation Eq. (5).
In the uniform magnetic field the GF can be written (in the symmetric gauge, \( A = \frac{1}{2} \mathbf{H} \times \mathbf{r} \)) in the following form:

\[
G^{1}_{\beta \kappa}(\mathbf{r}, \mathbf{r}') = \exp \left[ -i \frac{xy' - yx'}{2l^2} \right] g^{1}_{\beta \kappa}(\mathbf{r} - \mathbf{r}');
\]

\[
G^{2}_{\beta \kappa}(\mathbf{r}', \mathbf{r}) = \exp \left[ -i \frac{xy' - yx'}{2l^2} \right] g^{2}_{\beta \kappa}(\mathbf{r}' - \mathbf{r}).
\]

Here \( l^2 = c/eH \) is the magnetic length. This phase Ansatz indeed works. Substituting it into Eq.\((8)\) and Eq.\((9)\) respectively, the variables separate:

\[
\{(i\omega + \mu) \delta_{\gamma \beta} - v\mathbf{\Pi} \cdot \sigma_{\gamma \beta}\} g^{1}_{\beta \kappa}(\mathbf{r} - \mathbf{r}') = \delta^{\gamma \alpha} \delta (\mathbf{r} - \mathbf{r}');
\]

\[
\{-i\omega + \mu\} \delta_{\gamma \beta} + v\mathbf{\Pi} \cdot \sigma_{\gamma \beta} \} g^{2}_{\beta \kappa}(\mathbf{r}' - \mathbf{r}) = \delta^{\gamma \alpha} \delta (\mathbf{r} - \mathbf{r}').
\]

Here the ladder operators here are defined as

\[
\Pi_x = -i \frac{\partial}{\partial \rho_x} + \frac{1}{2l^2} \rho_y, \quad \Pi_y = -i \frac{\partial}{\partial \rho_y} - \frac{1}{2l^2} \rho_x,
\]

with relative distance denoted by \( \rho = \mathbf{r} - \mathbf{r}' \).

These equations are solved by expansion in the basis of eigenfunctions of harmonic oscillator in Appendix A. The resulting normal GF in terms of generalized Laguerre polynomials are:

\[
g^{1}_{11}(\rho) = \frac{(i\omega + \mu)}{2 \pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=0}^{\infty} \frac{L_n \left[ \rho^2/2l^2 \right]}{(i\omega + \mu)^2 - \omega^2_n (n + 1)};
\]

\[
g^{1}_{21}(\rho) = -i v \xi e^{-i\theta} \frac{2 \pi l^2}{4} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=1}^{\infty} \frac{L_{n-1} \left[ \rho^2/2l^2 \right]}{(i\omega + \mu)^2 - \omega^2_n (n + 1)};
\]

\[
g^{1}_{22}(\rho) = \frac{(i\omega + \mu)}{2 \pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=0}^{\infty} \frac{L_n \left[ \rho^2/2l^2 \right]}{(i\omega + \mu)^2 - \omega^2_n (n + 1)};
\]

\[
g^{1}_{12}(\rho) = -i v \xi e^{-i\theta} \frac{2 \pi l^2}{4} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=1}^{\infty} \frac{L_{n-1} \left[ \rho^2/2l^2 \right]}{(i\omega + \mu)^2 - \omega^2_n (n + 1)};
\]

Here the cyclotron frequency in WSM is denoted by \( \omega_c^2 = 2v^2/l^2 \) and \( \theta \) is the polar angle of \( \rho \). Similarly the associate GF are:

\[
g^{2}_{11}(-\rho) = \frac{-i \omega + \mu}{2 \pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=0}^{\infty} \frac{L_n \left[ \rho^2/2l^2 \right]}{(i\omega + \mu)^2 - \omega^2_n (n + 1)};
\]

\[
g^{2}_{12}(-\rho) = i v \xi e^{i\theta} \frac{2 \pi l^2}{4} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=1}^{\infty} \frac{L_{n-1} \left[ \rho^2/2l^2 \right]}{(i\omega + \mu)^2 - \omega^2_n (n + 1)};
\]

\[
g^{2}_{21}(-\rho) = i v \xi e^{-i\theta} \frac{2 \pi l^2}{4} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=1}^{\infty} \frac{L_{n-1} \left[ \rho^2/2l^2 \right]}{(i\omega + \mu)^2 - \omega^2_n (n + 1)};
\]

\[
g^{2}_{22}(-\rho) = \frac{-i \omega + \mu}{2 \pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=0}^{\infty} \frac{L_n \left[ \rho^2/2l^2 \right]}{(i\omega + \mu)^2 - \omega^2_n (n + 1)}.
\]

Now we are ready to return to the gap equation at criticality.

### B. Ansatz for the gap function and the angle integration

Substituting the phase factors of GF from Eq.\((10)\) into the gap equation, Eq.\((7)\), one obtains:

\[
\Delta (\mathbf{r}) = \frac{g^{2T}}{2} \sum_{\lambda} \int_{\mathbf{r}'} \exp \left[ -i \frac{xy' - yx'}{l^2} \right] \Delta^* (\mathbf{r}') \left( g_{22}^2 (-\rho) g_{11}^1 (\rho) + g_{11}^1 (-\rho) g_{22}^2 (\rho) + g_{12}^2 (-\rho) g_{21}^1 (\rho) + g_{21}^1 (-\rho) g_{12}^2 (\rho) \right).
\]
Adopting the gaussian Ansatz for the gap function,

$$\Delta (\mathbf{r}) = \exp \left[ -r^2 / 2\ell^2 \right],$$

(17)

used extensively in calculations since the seminal work, and substituting the above explicit expressions for the GF, one obtains,

$$1 = \frac{g^2 T}{\pi^2} \sum_\omega \int_0^\infty \rho \exp \left[ \frac{\rho}{4T} e^{i\theta} \right] \exp [ -2u] S (u, \omega) ,$$

(18)

where the integral have been shifted to $\rho = \mathbf{r} - \mathbf{r}'$. The scalar function $S$ depends on absolute value of $u$ only, so that the dimensionless variable $u = \rho^2 / 2\ell^2$ is used instead. It is a double sum over Landau levels:

$$S (u, \omega) = \left( \omega_c^2 + \mu^2 \right) \sum_{n,m=0}^{\infty} \left\{ \frac{L_n[u]L_m[u]}{\left[ (\omega - \omega_c)^2 - \omega_c^2 (n+1) \left( (\omega - \omega_c)^2 - \omega_c^2 (1+n) \right) + \left( (\omega - \omega_c)^2 - \omega_c^2 (n+1) \right) \right]} + \frac{L_n[u]L_m[u]}{\left[ (\omega - \omega_c)^2 - \omega_c^2 (n+1) \left( (\omega - \omega_c)^2 - \omega_c^2 (1+n) \right) \right]} \right\} ,$$

(19)

The integral over $\theta$ is just $2\pi$, so that the gap equation at criticality takes a form

$$1 = \frac{g^2 T}{4\pi^2} \sum_\omega \int_0^\infty \exp [ -2u] S (u, \omega) .$$

(20)

In what follows the integral over $u$ and the sum over the Matsubara frequencies is explicitly performed and the equation used to investigate the effect of Landau quantization of superconductivity in a WSM. Using the integrals over product of generalized Laguerre polynomial, the gap equation takes a form,

$$\int_0^\infty \frac{du}{\omega} \exp [ -2u] L_n (u) L_m (u) = \frac{(m+n)!}{2^{m+n+1}m!n!} ,$$

(22)

and

$$\int_0^\infty \frac{udu}{\omega} \exp [ -2u] L_{n-1}^1 (u) L_m (u) = \frac{(m+n)!}{2^{m+n+1}m! (n-1)!} ,$$

(23)

where the effective dimensionless electron - electron coupling $\lambda = g^2 \mu / 4\pi \omega_c^2$. It is also convenient to scale $\mu$ and $\omega_c$ by the temperature, $\overline{\omega} = \mu / T, \overline{\omega}_c = \omega_c / T$. After summation over the Matsubara frequency, one obtains, separating the zero LL ($n = 0$) from the rest,

$$\frac{1}{\lambda} = \frac{\omega_c^2}{4\pi^2} \sum_{n,m=1}^{\infty} \frac{(m+n)!}{2^{m+n+1}m!n!} \left\{ \sum_{n,m=0}^{\infty} \frac{f[n]f[m]}{2^{m+n+1}m!n!} s_{nm} + \sum_{n} \frac{f[n]f[0]}{2^{n}} s_n + \frac{f[0]^2}{2} s \right\} ,$$

(24)

where $f(n)$ will be discussed in the next subsection. The separation is required since the expressions in Appendix B are ambiguous for $n = 0$ and should be defined using L'Hopital’s rule. The $m, n > 0$ part (free of the ”ambiguous” terms) is:

$$s_{nm} = A \left[ \omega_c^2 (n+1) , \omega_c^2 (m+1) \right] + A \left[ \omega_c^2 n , \omega_c^2 m \right] +$$

$$= \frac{1}{\overline{\lambda}^2} \left[ \overline{\omega}_c^2 \left[ \omega_c^2 (n+1) , \omega_c^2 (m+1) \right] + \overline{\omega}_c^2 \left[ \omega_c^2 n , \omega_c^2 m \right] \right] .$$

(25)

The mixed zero-nonzero LL ($n = 0, m > 0$) part is

$$s_n = A \left[ \omega_c^2 (n+1) , \omega_c^2 \right] + A \left[ \omega_c^2 n , 0 \right] + \overline{\omega}_c^2 \left[ \omega_c^2 (n+1) , \omega_c^2 \right] + \overline{\omega}_c^2 \left[ \omega_c^2 n , 0 \right] ,$$

while the purely zero LL contribution

$$s = A \left[ \omega_c^2 , \omega_c^2 \right] + A [0, 0] + \overline{\omega}_c^2 \left[ \omega_c^2 , \omega_c^2 \right] + \overline{\omega}_c^2 [0, 0] .$$

(26)

Explicit form of functions $A$ and $B$ is given in Appendix B. It is shown there that the functions are finite for any value of magnetic field and temperature $T > 0$. The sum is computed numerically.
FIG. 2. The inverse effective electron coupling is presented for three temperatures $\hbar \Omega/200, \hbar \Omega/50, \hbar \Omega/20$, in a wide range of magnetic field up to $15\hbar c^2/ev^2$. The value of chemical potential is chosen as at $\mu = 5\hbar \Omega$.

C. Phonon retardation effects

Usually within the BCS approach, the interaction is approximated not just by a contact in space and a step function - like cutoff,

$$\mu - \hbar \Omega < \hbar \omega_c \sqrt{n} < \mu + \hbar \Omega,$$

(27)

see Fig.1. Therefore the sums over Landau levels in Eq. (23) is restricted. The approximation is not good enough for our purposes, since, when crossing a Landau level by increasing the field infinitesimally, the result of summation in the quantum regime jumps by a finite amount like Hall conductivity in 2DEG. This is unphysical since the step function dependence is just an approximation of a more realistic second order effective electron interaction due to phonon exchange.

Neglecting the dispersion of the optical phonon, the sharp cutoff will be replaced by the Lorentzian function of $\omega_s = \pi T (2s + 1) / \hbar$:

$$V(s, p) = \frac{g^2 \Omega^2}{\Omega^2 + \omega_s^2}. \tag{28}$$

In our scaled units the summation over Landau levels comes with a weight function,

$$f(n) = \frac{\Omega^2}{\Omega^2 + \left(\omega_c \sqrt{n} - \mu / \hbar\right)^2}. \tag{29}$$

The remaining sums over Landau levels in Eq. (24) were performed numerically to determine the normal - superconductor transition line.

D. The fragmented transition line

Magnetic phase diagram is the main result of the present paper. Although in experiments the material parameter $\lambda$ is fixed, while temperature and magnetic field (or both) are external parameters, it is more convenient to calculate the critical value of $\lambda$ as a function of temperature and magnetic field. In Fig. 2 the inverse effective electron - electron coupling $\lambda^{-1}$ is plotted as a function of magnetic field. Curves correspond to three temperatures $\hbar \Omega/200, \hbar \Omega/50, \hbar \Omega/20$, while the wide range of magnetic fields extends up to $25\hbar c^2/ev^2$. The value of chemical potential is chosen to be $\mu = 5\hbar \Omega$. To concreteness (and to facilitate a discussion of an experiment on $Cd_3As_2$) we use typical values of
FIG. 3. The fragmented $H-T$ phase diagram of the 2D Weyl semi-metal. Cross-sections (in gray) outline the superconducting "domed". Three values of the effective electron-electron coupling are given. a. $\lambda = 1$. b. $\lambda = 0.33$. c. $\lambda = 0.2$.

the Debye frequency $\Omega = 400K$ and the Fermi velocity $v = 10^8 cm/s$, so that temperatures and fields in Fig. 2 are given in kelvins and tesla respectively. Dashed lines mark the cases of a weak, $\lambda = 0.2$, an intermediate, $\lambda = 0.33$, and a relatively strong coupling $\lambda = 1$.

For the weak coupling the conventional $H_{c2}$ does not appear in the figure, since the critical temperature is below $2K$. The only superconducting "dome" appears at the quantum limit with Cooper pairs made on the lowest LL only. At the intermediate coupling the conventional $H_{c2} = 2T$ does appear (around $4K$), but now there are four additional superconducting domes at Landau levels $N = 1 - 4$. At the strong coupling regular $H_{c2}$ around $12T$ is clearly the dominant feature with numerous domes appearing at $T = 2K$. The problematic issue of rigorously defining the semi-classical notion of $H_{c2}$ from the microscopic calculation is the same as for the conventional superconductor (parabolic band). Of course at yet lower temperatures more domes appear.

In Fig. 3 the phase diagram in the $H-T$ is presented for the same three values of the effective electron-electron couplings.

The superconducting domes on Landau levels are clearly seem as gray areas. Generally they become very narrow as the LL index $N$ grows, at low temperatures and at weak couplings. The WSM, in which we suspect that the high magnetic field superconducting domes were observed (see Section IV), are anisotropic 3D WSM. In addition at fields as large as $50-60T$ applied in recent experiments, the Zeeman coupling to spin cannot be ignored. Therefore the next section is devoted to generalizations to the direct coupling to the electron spin and to 3D WSM.

IV. GENERALIZATIONS: ZEEMAN COUPLING AND 3D WSM.

A. Zeeman coupling, the paramagnetic limit

Along with the orbital effect of magnetic field on electrons and their pairing, at very high fields the direct (Zeeman) coupling of the magnetic field to spin becomes significant. A textbook example is the Chandrasekhar - Klogston pair
breaking phenomenon in conventional metallic (parabolic single band) superconductors.

To investigate the Zeeman coupling effect on superconductivity in (2D) WSM, let us consider the following Hamiltonian

$$ H = K + K_Z + V. $$

Here the kinetic energy term and the phonon mediated effective interaction are still defined in Eq. (1) and Eq. (2) respectively. The Zeeman coupling term is

$$ K_Z = -g_L \mu_B H \int_\mathbb{R} \psi_{\alpha}^\dagger (\mathbf{r}) \sigma_{st}^{\alpha} \delta_{\alpha\beta} \psi_{\beta}^\dagger (\mathbf{r}), \quad (31) $$

where $\sigma_{st}^{\alpha}$ is the Pauli matrix in spin space, $g_L$ and $\mu_B$ are the Lande factor and the Bohr magneton respectively.

A simple singlet Ansatz, Eq. (4), no longer solves the set of the Gor’kov equations. Therefore they should be explicitly solved. The number of the Greens functions in this case is doubled compared to the case considered in Section III. However the phase Ansatz for GF in magnetic field, Eq. (10), still holds. Substituting Eq. (10) into the gap equation (see Eq. (23) of Appendix C, where derivations also can be found), and using a pseudospin singlet Ansatz for the gap function, $\Delta_{st\alpha}^{\gamma} (\mathbf{r}) = \sigma_{st}^{\alpha} \exp (-\nu^2/21^2)$, one obtains equation for critical curve in $H - T$ plane:

$$ \frac{1}{\pi g_Z^2} = T \sum_n \int_\rho e^{-\nu^2/21^2} \left( g_{21}^{-1} (\rho) + g_{22}^{-1} (\rho) + g_{12}^{-1} (\rho) + g_{21}^{-1} (\rho) \right). $$

The set of the spin dependent GF is calculated in Appendix C (Eqs. (C7), (C8)). Substituting them into Eq. (32) performing integration over $\rho$, and summation on Matsubara frequencies one obtain relation for critical curve at the $\lambda^{-1} - H$ plane:

$$ \frac{1}{\lambda} = \frac{\omega_{\mu}}{4T} \sum_{n=0, m=0}^{\infty} \frac{(m+n)! f [n] f [m]}{m! n!} s_{nm}. \quad (33) $$

Here functions $s_{nm}$ are,

$$ s_{nm} = A_p \left[ \omega_{\mu}^2 (n+1), \omega_{\mu}^2 (m+1), \mu + \epsilon, \mu - \epsilon \right] + A_p \left[ \omega_{\mu}^2 (n+1), \omega_{\mu}^2 (m+1), \mu - \epsilon, \mu + \epsilon \right] + \left( \mu^2 - \pi^2 \right) \left( B_p \left[ \omega_{\mu}^2 (n+1), \omega_{\mu}^2 (m+1), \mu + \epsilon, \mu - \epsilon \right] + B_p \left[ \omega_{\mu}^2 (n+1), \omega_{\mu}^2 (m+1), \mu - \epsilon, \mu + \epsilon \right] \right) + nB_p \left[ \omega_{\mu}^2 (n+1), \omega_{\mu}^2 (m+1), \mu + \epsilon, \mu - \epsilon \right] + mB_p \left[ \omega_{\mu}^2 (m+1), \mu + \epsilon, \mu - \epsilon \right], $$

where the dimensionless ratio of the Zeeman energy and temperature, $\lambda = 2g_L \mu_B H/T$, is used. In the spin non-degenerate case the separation of the zero LL is not required due to the difference in chemical potentials of the spin projections. The Matsubara sums read:

$$ A_p [a, b, \mu_1, \mu_2] = \frac{1}{4\sqrt{a}} \left\{ \left( \frac{\sqrt{a} - \mu_1}{\sqrt{a} + \mu_1} \right)^2 + \left( \frac{\sqrt{a} + \mu_1}{\sqrt{a} - \mu_1} \right)^2 \right\} \left( \frac{a}{\mu_1} \leftrightarrow \frac{b}{\mu_2} \right); \quad (35) $$

$$ B_p [a, b, \mu_1, \mu_2] = \frac{1}{4\sqrt{a}} \left\{ \tanh \left( \frac{\sqrt{a} - \mu_1}{\sqrt{a} + \mu_1} \right) + \tanh \left( \frac{\sqrt{a} + \mu_1}{\sqrt{a} - \mu_1} \right) \right\} \left( \frac{a}{\mu_1} \leftrightarrow \frac{b}{\mu_2} \right). $$

The results of numerical calculations are presented in Fig. 4. The inverse effective coupling $\lambda^{-1}$ as function of magnetic field for six values of the material parameter characterizing the strength of the Zeeman coupling on superconductivity, $\alpha_p = g_L \mu_B e\Omega/e\nu^2$, $\alpha_p = 2 \cdot 10^{-4}, 5 \cdot 10^{-4}, 1.5 \cdot 10^{-3}, 3.5 \cdot 10^{-3}, 3.5 \cdot 10^{-3}, 1.7 \cdot 10^{-2}$, are plotted. Temperature is fixed at $T = 0.005 K$ (as above, we take $k_B = 400 K$ for concreteness this amounts to $2K$), $\mu = 5k_B$, while the range of magnetic fields is between $5\mu_B\Omega^2/e\nu^2$ to $30\mu_B\Omega^2/e\nu^2$. For a typical value of the Fermi velocity
FIG. 4. Superconductor - normal critical curve in the $\lambda^{-1} - H$ plane. Zeeman interaction splits the superconducting domes suppressing superconductivity at large values of the dimensionless of the paramagnetic coefficient $\alpha_p = g_L \mu_B c \Omega / ev^2$.

$c = 10^8 cm/s$ this corresponds to $25 - 150 T$. The magnetic phase $(H - T)$ diagram is obtained, as in the previous section, as a set of fields for a fixed $\lambda$.

One observes that while for the smallest Zeeman coupling (blue curve) there is no difference with the zero Zeeman splitting case (blue line in Fig.2), for the largest value the superconductivity is quenched due Chandrasekhar- Klongstone (paramagnetic) limit. For the intermediate values of $\alpha_p$ splitting of the superconducting domes of the fractured critical line is well pronounced.

Band structure calculations of one of the most promising WSM $Cd_3As_2$ show\cite{10} that the Dirac point in this system is formed by the spin mixed with the sublattice index. In this case the Zeeman interaction with the external magnetic field is more complicated than considered in our two band model. It is reasonable to expect however that qualitative features of the Zeeman coupling are similar. Another important characteristics of superconducting WSM is that many of them are three dimensional.

**B. Generalization to 3D WSM**

In this subsection the calculation of the magnetic phase diagram is generalized to 3D WSM with (typically several) Dirac points. The band structure of an asymmetric 3D WSM near such a point is captured by the Hamiltonian

$$K = \int r \psi_{s}^\dagger (r,z) \left\{ -ihv \left( D_x \sigma_{x\beta} + D_y \sigma_{y\beta} \right) - hv_z \partial_z \sigma_{z\beta} - \mu \delta_{\alpha\beta} \right\} \psi_{s} (r,z).$$

Here $v$ is Fermi velocity (assumed isotropic) in the $x - y$ plane perpendicular to magnetic field and $v_z$ the Fermi velocity along the field and the gauge in the covariant derivatives is chosen to be $A = H (-y/2, x/2, 0)$. The momentum $p_z$ in this gauge is a conserved quantum number.

The calculation is analogous to the 2D one, since magnetic field enters the dependence Greens functions on lateral dimensions only. The Fourier transform is defined now by

$$G_{\gamma\kappa} (r, z, \tau) = T \sum_s \exp \left[ -i\omega_s \tau + ip_z s \right] G_{\gamma\kappa} (\omega, r, p_z).$$
It is important to distinguish between the thin film and the "bulk" cases. For a film of thickness $d$, the field component of the "momentum" is discretized as:

$$ p_z = \frac{\pi \hbar}{d} M, M = \pm 1, 2... $$

The equations for two normal GF (see Eqs.(7)) in the 3D case read:

$$ [ivD_x^i + \sigma^i_{\gamma\beta} - v_z p_z \sigma^i_{\gamma\beta} + (i\omega + \mu) \delta_{\gamma\beta}] G^1_{\beta\alpha} (r, r', p_z) = \delta^{\alpha\gamma} \delta (r - r'), $$

$$ [-ivD_x^i + \sigma^i_{\gamma\beta} + v_z p_z \sigma^i_{\gamma\beta} + (-i\omega + \mu) \delta_{\gamma\beta}] G^2_{\beta\alpha} (r, r', p_z) = \delta^{\alpha\gamma} \delta (r - r'). $$

The magnetic phase Ansatz Eq.(10) still solves the 3D gap equation Eq.(7) (see Appendix D). Moreover the gaussian form of the gap function (independent of $z$), Eq.(17) is not changed. The equation determining the critical curve in the $H - T$ plane is now:

$$ \frac{1}{\lambda} = \frac{\zeta \omega^2}{4\mu^2} \sum_{M=0} \left\{ \sum_{n,m} \frac{(m + n)! f [m] f [n]}{2^{m+n+1} n!} s_{nmM} + \sum_n \frac{f [n] f [0]}{2^n} s_{nM} + \frac{f [0]^2}{2} s_M \right\}. $$

Here the 3D effective attraction strength (see Appendix D for the relevant DOS) is $\lambda = g^2 \mu^2 / 2\pi^2 v_z v^2$ and the dimensionless parameter inversely proportional to the thickness is defined by $\zeta = \pi v_z / dT$. The functions $s_{nmM}, s_{nM}, s_M$ depending on the new quantum number $M$, defined in Eq.38, and details of derivation (including the relevant GF in this case) are given in Appendix D, while the function $f$ containing the frequency dependence of the effective phonon mediated interaction remains as in 2D, see Eq.(29).

The result for films of two values of the film thickness corresponding to values $\zeta = 0.021$ and $\zeta = 0.11$ and fixed temperature $T = 0.005 \hbar \omega$ (for $\Omega = 400K$ it amount to $T = 2K$) are presented in Fig. 5a and 5b respectively. They demonstrate essential transformation of the superconducting - normal fractured critical line compared to the 2D case. The smaller value of $\zeta$ practically corresponds to the bulk, while the larger represents a thin film. In the bulk the domes become asymmetric due to the dispersion along the field. Generally larger coupling $\lambda$ is required to create the superconducting state on the Landau levels. The phenomenon of the re - entrant superconductivity itself however is clearly present due to enhancement of the DOS despite the fact that in 3D DOS does not vanishes between the LL.

The superconducting "domes" become wider in slab geometry (Fig. 5a) and demonstrate in set of small secondary peaks (ripples) caused by the quantization of the momentum along the field ($p_z$) direction in a thin film. Higher LL disappear. To conclude in the bulk the third dimension "smooths" the effect on Landau quantization as it appears in 2D, but just slightly, while in thin films the shape is modified.
V. COMPARISON WITH EXPERIMENTS, DISCUSSION AND CONCLUSIONS

In this section experimental evidence for existence of the Cooper pairing in WSM above $H_{c2}$ is discussed. In addition we discuss the various tacit assumptions of our model and theoretical methods: speculate on possible transition to a triplet superconducting phase and a necessity to go beyond the adiabatic approximation used in the present paper. The conventional metals are explicitly contrasted with Weyl semi-metals.

A. Magnetoresistance as a signature of the superconducting state at Landau levels

A "smoking gun" revealing the existence of superconductivity on Landau levels would be the dependence of resistivity on magnetic field. In normal metal one observes the resistivity generally increase faster than $H$ superimposed with Shubnikov deHaas (SdH) oscillations around Landau levels. The picture is supported by detailed semi-classical theory valid for high Landau levels. In the present paper the superconductivity in the quantum limit was studied. How will it influence the magnetoresistance at previously unreachable fields of order 100T beyond the semiclassical regime?

Inside the "superconducting domes" (constituting a very tiny fraction of the magnetic phase diagram within the narrow range of fields) magnetoresistance does not vanish due to phenomenon of the "flux flow". Since 3D Weyl semi-metals can be made very clean, unpinned vortex liquid rather than pinned vortex glass is formed. When vortices are allowed to move, the dissipation inside the cores ensues, but the flux flow resistivity is much smaller than the normal state. In the vortex glass state the effect would be more dramatic: the resistivity drops (almost) to zero. It should be noted that "vortices" in the present context should be understood as an inhomogeneity of the order parameter, since the magnetic "envelop" (of the size of magnetic penetration depth) of multiple vortices strongly overlap at such fields. As a result magnetization is practically homogeneous. Damping of the amplitude of SdH oscillations in superconducting regions is not expected to be significant, as was already noted while analyzing the SdH oscillations in organic superconductor below the upper critical field of 3.6T. The physics of the superconducting state on the LL is still insufficiently studied (only the quantum limit for the parabolic band material was theoretically described in a series works).

In a remarkable experiments with magnetic fields up to 50T it was found that beyond several SdH oscillations at high LL riding on magnetoresistance quadratic in $H$ ($N = 6 - 15$ are clearly seen at $T = 3K$), upon approaching quantum limit at $N = 2 - 4$ the magnetoresistance levels off. The amplitude of the oscillations gradually increases. It is very difficult to explain why the fast increase of the magnetoresitivity is halted at 10 – 20 $T$. It is natural to interpret this as appearance of superconductivity as in Fig. 2 for moderate $\lambda$. Indeed the superconductivity (in the dynamic vortex liquid flux flow phase) would strongly reduce the magnetoresistance magnitude. Our calculation is 2D, however the effect of 3D in strong magnetic field is rather minor: the peaks in Fig. 2,3 will be broadened. In the experiment at $N = 2, 3$ a significant Zeeman and pseudospin splitting (with and accompanying Berry phase) are observed and these will be discussed below. The splitting is clearly seen in magnetoresistance data of ref. at fields above 25T.

Similar phenomenon (less pronounced since applied magnetic fields were up to 16T only) was observed in Weyl superconductor TaP above $H_{c2}$ (A quite conventional magnetic phase diagram was experimentally established in this materials with $H_{c2}(1K) = 3T$ and $T_c = 3.5K$). As before, the fast increase of magnetoresistance is leveled off at small $N$. Unfortunately it is difficult assign definite $N$ to SdH oscillations clearly seen at $T = 3K$. This would correspond to weak coupling case shown in Fig. 2,3. The same relates to the recent discovery of "logarithmic series" of oscillations in the same material at density of order 10$^{16}$. The quantum limit is reached and leveling of magnetoresistance is observed, but if superconductivity is formed at low Landau levels it is nonadiabatic (see below).

B. On the possibility of the triplet pairing

Our calculation was restricted to the singlet pairing. In some cases strong magnetic field might in principle favor triplet, however there is no experimental evidence in 3D Weyl semi-metals for a triplet state so far. One therefore can ask the following question: is the triplet state possible theoretically in models of WSM considered here. The question was addressed theoretically in a slightly different context of the 2D WSM surface state of topological insulator. In this system it was found that both the singlet and the triplet phases exist. However, although they are nearly degenerate in some cases (very small chemical potential $\mu$), the singlet always prevail energetically. It was also shown theoretically that magnetic impurities or proximity to the Stoner instability (local magnetic moment due to the exchange interaction) can favor the triplet state. In such case the triplet superconducting state in WSM must survive in extremely strong magnetic fields.
Another strong argument in 3D was put forward long ago by Rasolt and Tesanovic. They argued that the Chandrasekhar-Kalogstion breaking of the singlet state is ineffective due to spacial inhomogeneity of the order parameter in the field direction. This remains valid for WSM.

C. Comparison of the WSM superconductor to conventional parabolic band superconductor, adiabatic approximation

Let us complement the qualitative estimates made in the introduction on the comparison between the pairing on Landau levels in the parabolic band materials (including semi-metals) and WSM by contrasting the magnetic phase diagrams. In Fig. 6 the phase diagram of the 2D single parabolic band superconductor with the electron-phonon coupling $g$, Debye frequency $\Omega$ and the chemical potential $\mu = 5\hbar\Omega$ as for WSM is Section III (see blue curve in Fig. 2) is presented. The effective mass of the conventional metal is assumed to be equal to that of the free electron mass. The inverse effective coupling $\lambda^{-1}$ (calculated with pertinent density of states) is given as function of magnetic field at the same temperatures $T = 0.005, 0.0125, 0.05\hbar\Omega$ (corresponding to 2, 5, 20K, if $\hbar\Omega = 400K$). The range of magnetic field plotted is however much wider: 200 – 3000 T. The fields are necessarily super-high, if one were to attempt the quantum limit (low Landau level) for conventional metals, as follows from the qualitative estimate in Introduction. Inset shows a (slightly) more accessible fields.

One observes that although in quantum limit the coupling required is not large, field are inaccessible. On the other hand, even beyond 100T, one has superconducting "domes" at intermediate coupling at high LL $N > 10$ (so that the system enters the semi-classical regime with weak quantization effects). The effect therefore is smeared out by disorder other effects. Note that, as demonstrated in Fig.5, in 3D the peaks at higher LL is further broadened and become unobservable.

Very recently superconductivity in a two parabolic band semi-metal in strong magnetic field was considered. One of the bands is quasiparticle with distance of the band edge to the Fermi level $\mu_e >> \hbar\Omega$, well within the adiabatic approximation, while the second is hole with very small $\mu_h < \hbar\Omega$. The Landau quantization effect is most pronounced near the Lifshitz point, where superconducting "domes" in magnetic phase diagram are clearly seen.

It is important to note that assumptions of our calculation include the adiabatic pairing, namely that the Fermi level is larger than the Debye energy $\mu/\Omega > 1$. WSM like ZrTe$_5$ also can be tuned to small chemical potential, however to make use of the gaussian approximation (in the BCS form or the Eliashberg form) one typically relies on Migdal theorem. Here it is questionable. Therefore in the present paper only the adiabatic case $\mu/\hbar\Omega > 5$ was discussed. It would be interesting to investigate what will happen beyond this assumption since in many Dirac materials Fermi energy is very low. For example Fermi energy in ZrTe$_5$ grown on in experiment in large fields up to 100T no oscillations were observed at all. However in this experiment the density is below $10^{15} cm^{-3}$. 

FIG. 6. Magnetic phase $\lambda^{-1} - H$ for conventional one band metal.
D. Conclusions

Microscopic theory of phonon mediated superconductivity in Weyl semimetals at very high magnetic fields was constructed. Weak coupling was assumed, but the retardation effects were taken into account. It was shown that a Weyl semi-metal in 2D and 3D that is nonsuperconducting or having a small critical temperature $T_c$ at zero field becomes superconducting in narrow regions of the magnetic phase diagram around Landau levels, especially near the quantum limit. The Zeeman splitting sometimes becomes of significance at highest fields. Superconductivity has an effect on magneto-conductivity beyond conventional $H_{c2}$. Near the Landau levels the magneto-resistivity should diminish. This might explain the recent experiments on $Cd_3As_2$ and $TaP$ and perhaps other.

This enhancement is especially pronounced for the lowest Landau level. As a consequence, the reentrant superconducting regions in the temperature-field phase diagram emerge at low temperatures near the magnetic fields at which the chemical potential matches the Landau levels.

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Appendix A: Calculation of the normal Green’s functions

In this Appendix the normal state Green’s functions are calculated. In the matrix form the equations [11], [12] read:

$$\hat{h}^a g^a (\rho) = \delta (\rho)$$  \hspace{1cm} (A1)

with 2D matrix operators $\hat{h}^1 = i \omega + \mu - \Pi \cdot \sigma$; $\hat{h}^2 = - i \omega + \mu + \Pi \cdot \sigma^t$, where $a = 1,2$ and $\Pi = \{ \Pi_x, \Pi_y \}$ are the ladder operators. In the symmetric gauge

$$\Pi_x = - i \frac{\partial}{\partial \rho_x} + \frac{1}{2l^2} \rho_y, \Pi_y = - i \frac{\partial}{\partial \rho_y} - \frac{1}{2l^2} \rho_x.$$  \hspace{1cm} (A2)

It is convenient to rewrite them via creation and annihilation operators for a bosonic field

$$a = \frac{l}{\sqrt{2}} (\Pi_x - i \Pi_y); a^\dagger = \frac{l}{\sqrt{2}} (\Pi_x + i \Pi_y)$$  \hspace{1cm} (A3)

with the commutation relations $[\Pi_x, \Pi_y] = - i / l^2$, $[a, a^\dagger] = 1$.

The matrix elements of the $2 \times 2$ matrices $h^a$ are defined by relations:

$$\hat{h}_{11} = \hat{h}_{22}^1 = i \omega + \mu; \hat{h}_{11}^2 = \hat{h}_{22}^2 = - i \omega + \mu;$$

$$\hat{h}_{12} = \hat{h}_{21} = - \omega_c a; \hat{h}_{12}^1 = \hat{h}_{21}^2 = - \omega_c a^\dagger.$$  \hspace{1cm} (A4)

Here $\omega_c = v \sqrt{2} / l$ is the Larmor frequency in Weyl semimetals. Equations for normal GF can be represented in the following form (suppressing the index $a$):

$$h_{11} g_{11} + \hat{h}_{12} g_{21} = \delta (\rho); \hat{h}_{21} g_{12} + h_{22} g_{22} = \delta (\rho);$$

$$h_{11} g_{12} + \hat{h}_{12} g_{22} = 0, \hat{h}_{21} g_{11} + h_{22} g_{21} = 0.$$  \hspace{1cm} (A5)

Since $h_{11}, h_{22}$ are just numbers (not operators acting on $\rho$), one first solves the second pair of equations for the off diagonal elements:

$$g_{21} = - \frac{1}{h_{22}} \hat{h}_{21} g_{11}; \quad g_{12} = - \frac{1}{h_{11}} \hat{h}_{12} g_{22}.$$  \hspace{1cm} (A6)

Substituting into the first pair, one obtains:

$$\left( h_{22} h_{11} - \hat{h}_{12} \hat{h}_{21} \right) g_{11} (\rho) = h_{22} \delta (\rho);$$

$$\left( h_{11} h_{22} - \hat{h}_{21} \hat{h}_{12} \right) g_{22} (\rho) = h_{11} \delta (\rho).$$  \hspace{1cm} (A7)

Substituting into the first pair, one obtains:

$$\left( h_{22} h_{11} - \hat{h}_{12} \hat{h}_{21} \right) g_{11} (\rho) = h_{22} \delta (\rho);$$

$$\left( h_{11} h_{22} - \hat{h}_{21} \hat{h}_{12} \right) g_{22} (\rho) = h_{11} \delta (\rho).$$  \hspace{1cm} (A8)
We present next a detailed calculation of the normal GF, while the associate GF are obtained similarly. For \( g_{11} \), after substitution of the matrix elements from Eq. \([A4]\), one obtains the following second order linear differential equation with a source:

\[
\left( (i \omega + \mu)^2 - \Pi^2 - i [\Pi_x, \Pi_y] \right) g_{11}^1 (\rho) = (i \omega + \mu) \delta (\rho) .
\]  

(A9)

This is written via Laplacian,

\[
\hat{L} = \frac{l^2}{2} \left\{ \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{i}{2l^2} \frac{\partial}{\partial \theta} + \frac{\rho^2}{4l^4} \right\} ,
\]  

(A10)
as represented into the form:

\[
\left( (i \omega + \mu)^2 - \frac{\omega_c^2}{2} - \omega_c^2 \hat{L} \right) g_{11}^1 (\rho) = (i \omega + \mu) \delta (\rho) .
\]  

(A11)

Since the operator \( \hat{L} \) in this equation is rotation invariant, \( g_{11}^1 (\rho) \) is a scalar (independent of the polar angle). The operator \( \hat{L} \) has the following eigenfunctions and eigenvalues:

\[
\epsilon_n^m = n + \frac{|m| + m + 1}{2} ,
\]  

(A12)
and eigenfunctions

\[
\varphi_n^m = \frac{1}{\sqrt{1+|m|}} \sqrt{\frac{nl}{2^{|m|} (|m| + n)!}} \exp \left[ -\frac{\rho^2}{4l^2} \right] \rho^{|m|} L_n^{|m|} \left( \frac{\rho^2}{2l^2} \right) e^{im \theta} .
\]  

(A13)

Here \( n \) and \( m \) are integers and \( L_n^m \) are the generalized Laguerre polynomials.

In specific case of a scalar the azimuthal number \( m = 0 \), and one obtains:

\[
\varphi_n^0 = \frac{1}{\sqrt{2\pi l}} \exp \left[ -\frac{\rho^2}{4l^2} \right] L_n \left( \frac{\rho^2}{2l^2} \right) .
\]  

(A14)

Expanding the GF \( g_{11}^1 (\rho) \) by series of the scalar eigenfunctions of the \( \hat{L} \) operator, \( g_{11}^1 (\rho) = \sum_n \epsilon_n^0 \varphi_n^0 \), and making the scalar product with \( \varphi_n^0 \), one obtains:

\[
\int \rho \varphi_n^0 \sum_n \left[ (i \omega + \mu)^2 - \omega_c^2 \right] \epsilon_n^0 \varphi_n^0 = (i \omega + \mu) \int \rho \varphi_n^0 (\rho) \delta (\rho) \]  

(A15)

Performing the integration, finally

\[
g_{11}^1 (\rho) = \frac{i \omega + \mu}{2\pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_n \frac{L_n \left( \rho^2/2l^2 \right)}{(i \omega + \mu)^2 - \omega_c^2 \left( 1 + n \right)} .
\]  

(A16)

Using the relation Eq. \([A6]\), the off diagonal matrix element \( g_{21}^1 (\rho) \) reads:

\[
g_{21}^1 (\rho) = \frac{\omega_c}{i \omega + \mu} a^\dagger g_{11}^1 (\rho) .
\]  

(A17)

Since

\[
a^\dagger = \frac{i}{\omega_c} e^{i \theta} \left( \frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \theta} + \frac{\rho}{2l^2} \right) ,
\]  

(A18)

using the relation between Laguerre polynomials \([A8]\), the result is:

\[
g_{21}^1 (\rho) = \frac{i \rho}{2\pi l^2} e^{i \theta} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=1}^{L_{n-1}} \frac{L_n \left( \rho^2/2l^2 \right)}{(i \omega + \mu)^2 - \omega_c^2 \left( 1 + n \right)} .
\]  

(A19)
In order to calculate the next pair of the GF matrix elements, \( g_{22}^1 \) and \( g_{12}^1 \), one has to solve the second Eq. (A7). The corresponding equation is similar,

\[
(-\omega^2 \hat{a} \hat{a} + (i\omega + \mu)^2) g_{22}^1 (\rho) = (i\omega + \mu) \delta (\rho), \quad (A20)
\]

\[
\left\{ (i\omega + \mu)^2 - \omega^2 \hat{L} \right\} g_{22}^1 (\rho) = (i\omega + \mu) \delta (\rho). \quad (A21)
\]

Repeating the procedure this results in

\[
g_{22}^1 (\rho) = \frac{i\omega + \mu}{2\pi l^2} \exp \left[ -\rho^2 / 4l^2 \right] \sum_{n=0}^{\infty} \frac{L_n [\rho^2 / 2l^2]}{(i\omega + \mu)^2 - \omega^2 n}. \quad (A22)
\]

Using the relation \( g_{12}^1 (\rho) = \frac{\omega}{i\omega + \mu} a g_{22}^1 (\rho) \), one obtains in view of

\[
a = -\frac{ie^{-i\theta}}{2\pi l^2} \left( -\frac{\partial}{\partial \rho} - \frac{i}{\rho} \frac{\partial}{\partial \theta} + \frac{\rho^2}{2l^2} \right),
\]

\[
g_{12}^1 (\rho) = \frac{\omega}{i\omega + \mu} \exp \left[ -\rho^2 / 4l^2 \right] \sum_{n=1}^{\infty} \frac{L_n [\rho^2 / 2l^2]}{(i\omega + \mu)^2 - \omega^2 n}. \quad (A23)
\]

The associated GF is calculated in the same manner, replacing matrix elements as it’s presented in Eq. (A4). All of the GF are presented in Eq. (14), (15).

**Appendix B: Matsubara summations**

The sums over reduced Matsubara frequency \( \overline{\omega} = \pi (2s + 1) \) in Eq. (22) read:

\[
A_1 [a, b] = \sum_{s=-\infty}^{\infty} \frac{\overline{\omega}^2 + \mu^2}{(-i\overline{\omega} + \mu)^2 - \overline{\omega}^2 (n + 1)} \left( (i\overline{\omega} + \mu)^2 - \overline{\omega}^2 (m + 1) \right)
\]

\[
= \frac{(\sqrt{a} - \mu)^2 \tanh \left( \frac{\sqrt{a} + \mu}{2} \right)}{(\sqrt{a} - \mu)(\sqrt{a} - 2\mu)} + \frac{(\sqrt{b} - \mu)^2 \tanh \left( \frac{\sqrt{b} + \mu}{2} \right)}{(\sqrt{b} - \mu)(\sqrt{b} - 2\mu)};
\]

\[
A_2 [a, b] = \sum_{s} \frac{\overline{\omega}^2 + \mu^2}{(-i\overline{\omega} + \mu)^2 - \overline{\omega}^2 n} \left( (i\overline{\omega} + \mu)^2 - \overline{\omega}^2 m \right)
\]

\[
= \frac{(\sqrt{a} + \mu)^2 \tanh \left( \frac{\sqrt{a} + \mu}{2} \right)}{4\sqrt{a}(-b + (\sqrt{a} - 2\mu)^2)} + \frac{(\sqrt{b} + \mu)^2 \tanh \left( \frac{\sqrt{b} + \mu}{2} \right)}{4\sqrt{b}(-a + (\sqrt{b} + 2\mu)^2)};
\]

\[
B_1 [a, b] = \sum_{s} \frac{\overline{\omega}^2 + \mu^2}{(-i\overline{\omega} + \mu)^2 - \overline{\omega}^2 (n + 1)} \left[ (i\overline{\omega} + \mu)^2 - \overline{\omega}^2 m \right]
\]

\[
= \frac{\tanh \left( \frac{\sqrt{a} + \mu}{2} \right)}{4\sqrt{a}(-b + (\sqrt{a} - 2\mu)^2)} - \frac{\tanh \left( \frac{\sqrt{b} + \mu}{2} \right)}{4\sqrt{b}(-a + (\sqrt{b} + 2\mu)^2)};
\]
and

\[ B_2[a, b] = \sum_s \left( \frac{-i\omega_a + p^2 - \omega_s^2}{2} \frac{m}{(i\omega_a + p^2 - \omega_s^2 (m + 1)} \right) \left( \frac{\tanh \left( \frac{\sqrt{a} + p}{2} \right)}{4\sqrt{a}(-b + (\sqrt{a} + 2p)^2)} - \frac{\tanh \left( \frac{\sqrt{b} + p}{2} \right)}{4\sqrt{b}(-a + (\sqrt{b} + 2p)^2)} \right). \]  

(B4)

Functions \( A[a, b] \) and \( B[a, b] \) in the Eq. (24) are subsequently composed as:

\[ A[a, b] = A_1[a, b] + A_2[a, b]; \quad B[a, b] = B_1[a, b] + B_2[a, b]. \]  

(B5)

Appendix C: Zeeman Effect

1. The Zeeman term in Gorkov equations

In the case of the WSM Hamiltonian containing the Zeeman term, Eq. (30), the Gor’kov equations for normal Green Function at criticality reads,

\[ \frac{\partial G_{\gamma\kappa}^{st}(X, X')}{\partial \tau} = i\sigma_{\gamma\beta}^i \partial_t G_{\beta\kappa}^{st}(X, X') + \mu G_{\gamma\kappa}^{st}(X, X') + g_L\mu_B H \tau_{\gamma\kappa} G_{\gamma\kappa}^{st}(X, X') - \delta^{\gamma\kappa}\delta^{s}\delta(X - X'), \]  

(C1)

while the equation for the anomalous average becomes:

\[ \frac{\partial F_{\gamma\kappa}^{st+}(X, X')}{\partial \tau} = i\nu_{\alpha\gamma}^i \nabla_r F_{\alpha\kappa}^{st+}(X, X') - \mu F_{\gamma\kappa}^{st+}(X, X') - \frac{\nu^2}{4} \varepsilon_{s_1 s_2} \varepsilon_{s_3 s_4} F_{\alpha\kappa}^{s_1 s_2} (X, X') F_{\alpha\kappa}^{s_3 s_4} (X, X') - g_L\mu_B H \tau_{\gamma\kappa} F_{\gamma\kappa}^{st+}(X, X'). \]  

(C2)

Number of GF in this case is doubled, although due to symmetry for singlet pairing solution one observes that 
\( G_{\gamma\kappa}^{s_1} = G_{\gamma\kappa}^{s_2} = F_{\gamma\kappa}^{s_1 s_2} = F_{\gamma\kappa}^{s_2 s_1} = 0. \)

Self-consistent equation for the gap function is,

\[ \Delta_{\alpha\kappa}^{\frac{s}{\gamma}}(r) = \frac{-\gamma^2}{4} \int_i \left( G_{\beta\gamma}^{\Omega s}(r, r') \Delta_{\alpha\kappa}^{s}(r') G_{\alpha\kappa}^{s}(r, r') + G_{\beta\gamma}^{ss}(r, r') \Delta_{\alpha\kappa}^{s}(r') G_{\alpha\kappa}^{ss}(r, r') \right), \]  

(C3)

while the GF in magnetic field are

\[ G_{\beta\kappa}^{ss_1}(r, r') = \exp \left[ -\frac{x y' - y x'}{2l^2} \right] \frac{s_{s_1}^{s_1}}{g_{s_1}^{s_1}}(r - r'); \]  

(C4)

\[ G_{\beta\kappa}^{ss_2}(r', r) = \exp \left[ -\frac{x y' - y x'}{2l^2} \right] \frac{s_{s_2}^{s_2}}{g_{s_2}^{s_2}}(r' - r'). \]

(C5)

here \( s = \uparrow, \downarrow \).

Substituting Eq. (C4) into Eq. (C3) and using the singlet assumption, \( \Delta_{\alpha\kappa}^{s}(r) = \Delta(r) \sigma_{\alpha\kappa}^s \), one obtains Eq. (18), and after the angle integration Eq. (20), with the only difference being the modified function \( S \):

\[ S_Z(r, \omega) = \left( \begin{array}{c|c} g_{21}^{11}(-\rho) & g_{21}^{11}(\rho) \\ \hline g_{22}^{11}(\rho) & g_{22}^{11}(-\rho) \end{array} \right) \left( \begin{array}{c|c} g_{21}^{21}(\rho) & g_{21}^{21}(-\rho) \\ \hline g_{22}^{21}(-\rho) & g_{22}^{21}(\rho) \end{array} \right) + \left( \begin{array}{c|c} g_{11}^{11}(\rho) & g_{11}^{11}(-\rho) \\ \hline g_{12}^{11}(-\rho) & g_{12}^{11}(\rho) \end{array} \right) \left( \begin{array}{c|c} g_{11}^{21}(\rho) & g_{11}^{21}(-\rho) \\ \hline g_{12}^{21}(-\rho) & g_{12}^{21}(\rho) \end{array} \right) + \left( \begin{array}{c|c} g_{21}^{11}(\rho) & g_{21}^{11}(-\rho) \\ \hline g_{22}^{11}(-\rho) & g_{22}^{11}(\rho) \end{array} \right) \left( \begin{array}{c|c} g_{21}^{21}(\rho) & g_{21}^{21}(-\rho) \\ \hline g_{22}^{21}(-\rho) & g_{22}^{21}(\rho) \end{array} \right) \right), \]  

(C5)

2. Calculation of the GF

Calculation of the GF is performed along the lines described in Appendix A. In this case however we get two separate equations for each GF with different spin projections. The equations for first GF are:
\[ i\omega G_{\gamma n}^{↓↓} (\mathbf{r}, \mathbf{r}') - i\sigma_\gamma^i \sigma_\beta \partial_n^i G_{\beta n}^{↓↓} (\mathbf{r}, \mathbf{r}') + (\mu + g_L \mu_B H) G_{\gamma n}^{↓↓} (\mathbf{r}, \mathbf{r}') = \delta^{\gamma n} \delta (\mathbf{r} - \mathbf{r}'); \quad (C6) \]
\[ i\omega G_{\gamma n}^{↑↑} (\mathbf{r}, \mathbf{r}') - i\sigma_\gamma^i \sigma_\beta \partial_n^i G_{\beta n}^{↑↑} (\mathbf{r}, \mathbf{r}') + (\mu - g_L \mu_B H) G_{\gamma n}^{↑↑} (\mathbf{r}, \mathbf{r}') = \delta^{\gamma n} \delta (\mathbf{r} - \mathbf{r}'). \]

Therefore the solution coincides with that of the GF Eq.\[ \text{[14]} \] for two different values of the chemical potential. The result is

\[ g_{11}^{↓↑,↓↓} (\rho) = \frac{(i\omega + \mu \pm g_L \mu_B H)}{2\pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \sum_{n=0} L_n \left[ \frac{\rho^2}{2l^2} \right] \right]; \quad (C7) \]
\[ g_{21}^{↓↑,↓↓} (\rho) = \frac{i\nu \rho e^{i\theta}}{2\pi l^4} \exp \left[ -\frac{\rho^2}{4l^2} \sum_{n=1} L_{n-1} \left[ \frac{\rho^2}{2l^2} \right] \right]; \]
\[ g_{22}^{↓↑,↓↓} (\rho) = \frac{(i\omega + \mu \pm g_L \mu_B H)}{2\pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \sum_{n=0} L_n \left[ \frac{\rho^2}{2l^2} \right] \right]; \]
\[ g_{12}^{↓↑,↓↓} (\rho) = \frac{i\nu \rho e^{-i\theta}}{2\pi l^4} \exp \left[ -\frac{\rho^2}{4l^2} \sum_{n=1} L_{n-1} \left[ \frac{\rho^2}{2l^2} \right] \right]. \]

Similarly for the second set of GF:

\[ g_{11}^{↑↑,↑↓} (-\rho) = \frac{(-i\omega + \mu \pm g_L \mu_B H)}{2\pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \sum_{n=0} L_n \left[ \frac{\rho^2}{2l^2} \right] \right]; \quad (C8) \]
\[ g_{12}^{↑↑,↑↓} (-\rho) = \frac{i\nu \rho e^{i\theta}}{2\pi l^4} \exp \left[ -\frac{\rho^2}{4l^2} \sum_{n=1} L_{n-1} \left[ \frac{\rho^2}{2l^2} \right] \right]; \]
\[ g_{21}^{↑↑,↑↓} (-\rho) = \frac{-i\nu \rho e^{-i\theta}}{2\pi l^4} \exp \left[ -\frac{\rho^2}{4l^2} \sum_{n=1} L_{n-1} \left[ \frac{\rho^2}{2l^2} \right] \right]; \]
\[ g_{22}^{↑↑,↑↓} (-\rho) = \frac{(-i\omega + \mu \pm g_L \mu_B H)}{2\pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \sum_{n=0} L_n \left[ \frac{\rho^2}{2l^2} \right] \right]. \]

Appendix D: Generalization to 3D

1. Density of states for a film in zero magnetic field

Using the dispersion law in the form,

\[ \varepsilon = \sqrt{v^2 (\rho_x^2 + \rho_y^2) + v_z^2 p_z^2}, \quad (D1) \]

one obtains for the density of electrons for the bulk anisotropic sample,

\[ n = \frac{1}{(2\pi)^3 \hbar^3} \int_p \Theta (\varepsilon [p] - \mu) = \frac{\mu^3}{6\pi^2 v_z c^2 \hbar^3}, \quad (D2) \]

while the density of electron states

\[ D (\mu) = \frac{\mu^2}{2\pi^2 v_z c^2 \hbar^3}. \quad (D3) \]
In films of thickness \( d \) the quantization of the momentum along axes \( z \) is important and the density of the electrons reads,

\[
\frac{n}{A d} = \frac{N}{(2\pi)^2 \hbar^2} \frac{1}{2d} \int \sum_M \Theta (\varepsilon [p, M] - \mu),
\]

where \( \varepsilon [p, M] = v^2 (p_x^2 + p_y^2) + v_z^2 (\pi \hbar M / d)^2 = v^2 p_x^2 + v_z^2 (\pi \hbar M / d)^2 \), and the chemical potential is \( \mu = \sqrt{v^2 u + v_z^2 (\pi \hbar M / d)^2} \).

The density of states in this case is

\[
D(\mu) = \frac{d n}{d \mu} = \frac{1}{8\pi \hbar^2 d} \sum_{M: \mu > \mu_M} \frac{2 \mu}{\nu^2} = \frac{\mu}{4\pi \hbar^2 d \nu^2} F[\mu].
\]

Here \( \mu_M \equiv \frac{\pi \hbar v_z}{d} | M |, M [\mu] = \frac{d\mu}{\pi \hbar v_z} \) and \( F[\mu] \) is the step-like function \( F = 2n \) in the interval \( n \pi \hbar v_z / d < \mu < (n + 1) \pi \hbar v_z / d, n = 1, 2, 3, ... \)

**2. Green’s functions in 3D**

In this Appendix the normal state Green’s functions for 3D are calculated. In the matrix form the equations (11), (12) read:

\[
\hat{h}^a g^a (\rho) = \delta (\rho);
\]

where \( a = 1, 2 \), with 3D matrix operators

\[
\hat{h}^1 = i \omega + \mu - \Pi \cdot \sigma - v_z p_z \sigma^z; \hat{h}^2 = -i \omega + \mu + \Pi \cdot \sigma^t + v_z p_z \sigma^z
\]

Substituting \( \hat{h}^1 \) and \( \hat{h}^2 \) into Eq. (C6) and solving set of eight equations in the manner similar to that described in Appendix A, one obtains the first set of GF

\[
g_{11}^1 (\rho, p_z) = \frac{v_z p_z + i \omega + \mu}{2\pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=0}^{L_n} \frac{L_n [\rho^2 / 2l^2]}{(i \omega + \mu)^2 - v_z^2 p_z^2 - \omega_c^2 (n + 1) ;}
\]

and the second set,

\[
g_{11}^2 (-\rho, -p_z) = \frac{-v_z p_z + i \omega + \mu}{2\pi l^2} \exp \left[ -\frac{\rho^2}{4l^2} \right] \sum_{n=0}^{L_n} \frac{L_n [\rho^2 / 2l^2]}{(i \omega + \mu)^2 - v_z^2 p_z^2 - \omega_c^2 (1 + n) ;}
\]

These functions allow to solve exactly the gap equation.
3. Solution of the gap equation in 3D

The gap equation in 3D takes a form:

\[
\Delta (\mathbf{r}) = \frac{g^2 T}{2} \sum \omega \int d\mathbf{r}' \exp \left[ -i \frac{xy' - yx'}{l^2} \right] \Delta^* (\mathbf{r}') \left[ g_{12}^2 (-\rho_z - p_z) g_{11}^1 (\rho, p_z) + g_{11}^1 (\rho, p_z) g_{12}^2 (-\rho_z - p_z) \right], \tag{D10}
\]

where \( \rho = \mathbf{r} - \mathbf{r}', \mathbf{r}, \mathbf{r}' \) are vectors in the \( x - y \) plane. Substituting the Ansatz for the gap function, Eq. (17), and GF, Eqs. (D8) and (D9), into Eq. (D10), and performing integration over the angle as in 2D case, one obtains the equation (using notation \( u = \rho^2/2l^2 \)):

\[
\frac{2}{g^2} = \frac{1}{2\pi l^2} \sum \omega, p_z \int u e^{-2u} S (u, p_z, \omega). \tag{D11}
\]

Here

\[
S (u, p_z, \omega) = \sum_{n, m=0} \left\{ \frac{\omega^2 + (\mu + v_z p_z)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 (n + 1)} + \frac{\omega^2 + (\mu + v_z p_z)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 m} \right\} \left( \frac{\omega^2 + (\mu + v_z p_z)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 (n + 1)} + \frac{\omega^2 + (\mu + v_z p_z)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 m} \right) \right\}.
\]

After integration over \( u \) it is written as a double sum:

\[
\frac{1}{\lambda} = \frac{\zeta \pi^2}{4 \mu^2} \sum_{s, M} \left\{ \sum_{n, m=1, s} \frac{(m + n)! f [n] f [m]}{2^{m + n + 1} m! n!} S_1 + \sum_{n=1, s} \frac{f [n] f [0]}{2^n} S_2 + \frac{f [0]^2}{2} S_3 \right\}, \tag{D13}
\]

where

\[
S_1 = \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 (n + 1)} \left( \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 m} \right) \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 (n + 1)} \left( \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 m} \right).
\]

\[
S_2 = \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 (n + 1)} \left( \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 m} \right) \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 (n + 1)} \left( \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 m} \right).
\]

\[
S_3 = \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 (n + 1)} \left( \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 m} \right) \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 (n + 1)} \left( \frac{\omega^2 + \mu^2 + (\zeta M)^2}{\left( -i \omega + \mu \right)^2 - \omega^2 m} \right).
\]

The abbreviations are as in 2D and in addition \( v_z \to v_z / T \). For 3D, after performing summation on Matsubara frequencies, one finally obtains,

\[
\frac{1}{\lambda} = \frac{\zeta \pi^2}{4 \mu^2} \sum_{M > 0} \left\{ \sum_{n, m} \frac{(m + n)! f [n] f [m]}{2^{m + n + 1} m! n!} s_{nmM} + \sum_{n} \frac{f [n] f [0]}{2^n} s_{nM} + \frac{f [0]^2}{2} s_M \right\}. \tag{D15}
\]

The summands are,
\[ s_{nmM} = A \left( \varpi^2_{c} (n+1) + (\zeta M)^2, \varpi^2_{c} (m+1) + (\zeta M)^2 \right) + \left( \mu^2 + (\zeta M)^2 \right) \left( B \left[ \varpi^2_{c} (n+1) + (\zeta M)^2, \varpi^2_{c} (m+1) + (\zeta M)^2 \right] \right. \\
+ \left. m \varpi^2_{c} B \left[ \varpi^2_{c} (n+1) + (\zeta M)^2, \varpi^2_{c} (m+1) + (\zeta M)^2 \right] \right) \]

\[ s_{nM} = A \left[ \varpi^2_{c} (n+1) + (\zeta M)^2, \varpi^2_{c} + (\zeta M)^2 \right] + A \left[ \varpi^2_{c} + (\zeta M)^2, (\zeta M)^2 \right] \\
+ \left( \mu^2 + (\zeta M)^2 \right) B \left[ \varpi^2_{c} (n+1) + (\zeta M)^2, \varpi^2_{c} + (\zeta M)^2 \right] + \left( \mu^2 + (\zeta M)^2 \right) G \left[ \varpi^2_{c} n + (\zeta M)^2, (\zeta M)^2 \right] \]

and

\[ s_{M} = A \left[ \varpi^2_{c} + (\zeta M)^2, \varpi^2_{c} + (\zeta M)^2 \right] + A \left[ (\zeta M)^2, (\zeta M)^2 \right] \\
+ \left( \mu^2 + (\zeta M)^2 \right) B \left[ \varpi^2_{c} + (\zeta M)^2, \varpi^2_{c} + (\zeta M)^2 \right] + \left( \mu^2 + (\zeta M)^2 \right) B \left[ (\zeta M)^2, (\zeta M)^2 \right] \]

with functions \( A \) and \( B \) given in Appendix B.
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