SUPERSYMMETRIC GAUGE THEORIES IN THE EXACT RENORMALIZATION GROUP APPROACH

F. Vian

Università degli Studi di Parma
and
I.N.F.N., Gruppo collegato di Parma,
viale delle Scienze, 43100 Parma, Italy

Abstract

In these notes the exact renormalization group formulation of the scalar theory is briefly reviewed. This regularization scheme is then applied to supersymmetric theories. In case of a supersymmetric gauge theory it is also shown how to recover gauge invariance, broken by the introduction of the infrared cutoff.

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1 Introduction

The aim of this talk is to discuss qualitatively how the Wilson renormalization group (RG) can be implemented in supersymmetric gauge theories.

The RG formulation provides the most physical framework to study general properties of renormalized quantum field theories and, in particular, to deal with ultraviolet (UV) divergences. Furthermore, it is in this context that effective theories naturally arise. Recently supersymmetric gauge theories have been focussed and the definition of a low-energy supersymmetric Wilsonian effective action has become urgent. In the RG approach the bare action $S_{\text{int}}(\Lambda_0)$ can be viewed as the result of integrating out all degrees of freedom with frequencies larger than the UV cutoff $\Lambda_0$ of a more elementary underlying theory. By further integrating the fields in the path integral with frequencies larger than some scale $\Lambda_2 < \Lambda_0$, one obtains the so-called Wilsonian effective action $S_{\text{eff}}$. Such an operation can be carried out [2] by multiplying the quadratic part of the classical action by a cutoff function $K_{\Lambda \Lambda_0}(p)$ which falls off sufficiently rapid for $p^2$ outside the region $\Lambda^2 < p^2 < \Lambda_0^2$.

Despite the loss of gauge invariance at the scale $\Lambda$, when all cutoffs are removed the Slavnov-Taylor (ST) identity can be recovered (at least in perturbation theory) by properly fixing the boundary conditions of the RG equation.

The way the cutoffs are introduced in the RG formalism proves particularly suitable in the supersymmetric case. If the classical action is written in terms of superfields, the regularization procedure described above preserves supersymmetry (in components this corresponds to use the same cutoff function for all fields). [3] As our formulation works in $d = 4$, we can exploit the superspace technique, which is unambiguous and simplifies perturbative calculations.

2 Renormalization group flow

To start with, we briefly recall how the RG method works in the scalar case. [2] According to Wilson one integrates over the fields with frequencies $\Lambda^2 < p^2 < \Lambda_0^2$ in the path integral and obtains

$$Z[j] = N[j; \Lambda] \int D\phi \exp i\{ \frac{1}{2}(\phi D^{-1}\phi)_{0\Lambda} + (j\phi)_{0\Lambda} + S_{\text{eff}}[\phi; \Lambda] \} ,$$  

where $N[j; \Lambda]$ contributes to the quadratic part of $Z[j]$. The functional $S_{\text{eff}}$ is the Wilsonian effective action and is the generator of the connected amputated cutoff Green functions (except the tree-level two-point function). It contains a cutoff propagator $D_{\Lambda \Lambda_0}$, that is to say the free propagator $D(p)$ is multiplied by a cutoff function $K_{\Lambda \Lambda_0}$ which is one for $\Lambda^2 < p^2 < \Lambda_0^2$ and rapidly vanishes outside. Notice that at $\Lambda = \Lambda_0$ the Wilsonian effective action coincides with the bare action. The requirement that $Z[j]$ is independent of the IR cutoff $\Lambda$ can be translated into a flow equation for $S_{\text{eff}}$, referred to as the exact RG equation. For our purposes it is convenient to perform a Legendre transform on $S_{\text{eff}}$ in order to obtain the so-called “cutoff effective action” $\Gamma[\phi; \Lambda]$, which is the generator of 1PI cutoff vertex functions and reduces to the physical quantum effective action in the limits
$\Lambda \to 0$ and $\Lambda_0 \to \infty$. The next step consists in deriving a flow equation for $\Gamma[\phi; \Lambda]$, which is straightforward once the RG equation for $S_{\text{eff}}$ is given. It reads

$$
\Lambda \partial_\Lambda \left\{ \Gamma[\phi; \Lambda] - \frac{1}{2} \int \frac{d^4p}{(2\pi)^4} D_{\Lambda \Lambda_0}^{-1}(p) \phi(p)\phi(-p) \right\} 
= -\frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \Lambda \partial_\Lambda D_{\Lambda \Lambda_0}^{-1}(q) \frac{1}{\Gamma_2(q; \Lambda)} \bar{\Gamma}[q, -q; \phi; \Lambda] \frac{1}{\bar{\Gamma}_2(q; \Lambda)}.
$$

Equation (2), together with a set of suitable boundary conditions, can be thought as an alternative definition of the theory which in principle is non-perturbative. As far as one is concerned with its perturbative solution, the usual loop expansion is recovered by solving iteratively (2).

### 2.1 Boundary conditions

In order to set the boundary conditions we distinguish between relevant couplings and irrelevant vertices according to their mass dimension. Relevant couplings have non-negative mass dimension and are defined as the value of some vertices and their derivatives at a given normalization point. For the four-dimensional massless scalar field theory we are considering the relevant couplings $\sigma_i$’s are defined through

$$
\sigma_1(\Lambda) = \frac{d\Gamma_2(p; \Lambda)}{dp^2} \bigg|_{p^2 = \mu^2}, \quad \sigma_2(\Lambda) = \Gamma_2(p; \Lambda)|_{p^2 = 0}, \quad \sigma_3(\Lambda) = \Gamma_4(\bar{p}_i; \Lambda),
$$

with $\bar{p}_i$’s the momenta at the symmetric point $\bar{p}_i\bar{p}_j = \mu^2(\delta_{ij} - \frac{1}{4})$. Notice the introduction of the subtraction point $\mu$ to get rid of IR divergences.

We eventually assume the following boundary conditions:

(i) at the UV scale $\Lambda = \Lambda_0$ the simplest boundary condition for all the irrelevant vertices is that they vanish. As a matter of fact $\Gamma[\phi; \Lambda = \Lambda_0]$ reduces to the bare action, which must contain only renormalizable interactions in order to guarantee perturbative renormalizability;

(ii) the relevant couplings are fixed at the physical point $\Lambda = 0$ in terms of the physical couplings, such as the wave function normalization, the three-point coupling and the mass.

Figure 1: Graphical representation of the auxiliary functional $\bar{\Gamma}$. The box and the blob represent the functionals $\bar{\Gamma}$ and $\Gamma_{\text{int}}$, respectively. The dot indicates a cutoff full propagator.
Hence the boundary conditions to be imposed on the relevant couplings are
\[ \sigma_1(\Lambda = 0) = 1, \quad \sigma_2(\Lambda = 0) = 0, \quad \sigma_3(\Lambda = 0) = \lambda. \]

3 The Wess-Zumino model

Now that we have been acquainted with the RG formalism in the scalar case the extension to a supersymmetric theory is straightforward. In the Wess-Zumino model the classical Lagrangian reads
\[ S_{cl} = \frac{1}{16} \int \bar{\phi} \phi + \frac{\lambda}{48} \int d^4x \ d^2\theta \phi^3 + \text{h.c.}. \]
Integration over \( z \) means integration over the full superspace \( (d^4x \ d^4\theta) \) and \( \bar{\phi} \) is a chiral (anti-chiral) superfield satisfying \( \bar{D}_{\dot{\alpha}} \phi = 0 \) \( (D^\alpha \bar{\phi} = 0) \). Linearity of the supersymmetry transformation when acting on \( \phi \) \( (\bar{\phi}) \) \[4\] is a key ingredient of our formulation. As we previously did in the scalar case, we follow Wilson and integrate over the frequencies \( \Lambda^2 < p^2 < \Lambda_0^2 \). By collecting the fields and the sources in \( \Phi_i = (\phi, \bar{\phi}) \) and \( J_i = (J, \bar{J}) \) respectively, and introducing the general cutoff scalar products between fields and sources
\[ \frac{1}{2}(\Phi, D^{-1}\Phi)_{\Lambda_0} \equiv \frac{1}{16} \int_p K_{\Lambda_0}^{-1}(p) \bar{\phi}(-p, \theta) \phi(p, \theta), \quad \int_p \equiv \int \frac{d^4p}{(2\pi)^4} d^2\theta d^2\bar{\theta} \]
\[ (J, \Phi)_{\Lambda_0} \equiv \frac{1}{16} \int_p K_{\Lambda_0}^{-1}(p) \left\{ J(-p, \theta) \frac{D^2}{p^2} \phi(p, \theta) + \bar{J}(-p, \theta) \frac{\bar{D}^2}{p^2} \bar{\phi}(p, \theta) \right\} \]
(recall that \( \bar{D}^2 D^2 \phi = \phi \)), the generating functional can now be written in terms of the Wilsonian effective action \( S_{\text{eff}}[\Phi; \Lambda] \) as in \([1]\). Then we can derive the cutoff effective action \( \Gamma[\Phi; \Lambda] \) and it turns out it satisfies an evolution equation very similar to the one we met in the scalar case, slightly modified by the presence of covariant derivatives. \[3\] The auxiliary functional \( \bar{\Gamma} \) has exactly the same expansion in terms of the vertices of \( \Gamma \).

3.1 Boundary conditions

We know that at the UV scale \( \Lambda = \Lambda_0 \) the cutoff effective action \( \Gamma[\Phi; \Lambda = \Lambda_0] \) reduces to the bare action, which contains only renormalizable supersymmetric interactions since in our formulation supersymmetry is manifest. \[3\] On the other hand, the relevant couplings are fixed at the physical point \( \Lambda = 0 \) in terms of the physical couplings, such as the wave function normalization, the three-point coupling and the mass. Let us consider as an example the massless chiral multiplet two-point function \( (i.e. \phi \bar{\phi}-\text{coefficient of the cutoff effective action}) \) \( \Gamma_2(p; \Lambda) = D^{-1}K_{\Lambda_0}^{-1}(p) + \Sigma(p; \Lambda) \), which contains the relevant coupling \( Z(\Lambda) = \Sigma(p; \Lambda) \mid_{p^2 = \mu^2} \). The boundary conditions we assume are \( Z(\Lambda = 0) = 0 \) and \( \Sigma_{\text{irr}}(p; \Lambda) \mid_{\Lambda = \Lambda_0} = (\Sigma(p; \Lambda) - Z(\Lambda)) \mid_{\Lambda = \Lambda_0} = 0 \).

\(^2\)The introduction of the cutoff does not spoil global symmetries as long as they are linearly realized.
3.2 Perturbative expansion

The iterative solution of the RG equation for $\Gamma$ automatically provides a renormalized perturbation theory in $\lambda$. As an example we compute the one-loop two-point function. The evolution equation for this vertex is given by

$$ \int_p \bar{\phi}(-p, \theta) \Lambda \partial_\Lambda \Sigma^{(1)}(p; \Lambda) \phi(p, \theta) = \frac{i}{64} \lambda^2 \int_{pq} \frac{K_{\Lambda \Lambda_0} (p + q) \Lambda \partial_\Lambda K_{\Lambda \Lambda_0} (q)}{q^2(p + q)^2} \times \bar{\phi}(-p, \theta_1) \phi(p, \theta_2) \delta^4(\theta_1 - \theta_2) \tilde{D}^2 \tilde{D}^2(q, \theta_2) \delta^4(\theta_1 - \theta_2). $$

By carrying out some standard $D$-algebra manipulations and implementing the boundary conditions, the solution of $\Sigma$ at the physical point $\Lambda = 0$ and in the $\Lambda_0 \rightarrow \infty$ limit is

$$ \Sigma^{(1)}(p; \Lambda = 0) = \frac{i}{128} \lambda^2 \int \frac{d^4q}{(2\pi)^4} \left( \frac{1}{q^2(p + q)^2} - \frac{1}{q^2(p + q)^2} \bigg|_{p^2 = \mu^2} \right). $$

Notice the crucial role of the boundary condition for $Z$, i.e. $Z^{(1)}(0) = 0$, which naturally provides the necessary subtraction to make the vertex function $\Sigma$ finite for $\Lambda_0 \rightarrow \infty$.

One immediately realizes that only the vertices with an equal number of $\phi$ and $\bar{\phi}$ are generated at this order. Anyway the coefficients of $(\phi \bar{\phi})^n$ with $n > 1$ are obviously irrelevant and power counting tells us they are finite, so that no subtraction is needed.

4 Supersymmetric gauge theories

When we have to face with (supersymmetric) gauge theories, the key issue are the boundary conditions, which in addition to ensuring (perturbative) renormalizability and providing the physical couplings $g(\mu)$ have to guarantee symmetry. In fact the cutoff $\Lambda$ explicitly breaks gauge invariance and we have to show that at the physical point $\Lambda = 0$ (in the $\Lambda_0 \rightarrow \infty$ limit) the Slavnov-Taylor identity can be recovered by properly fixing the boundary conditions, at least in perturbation theory.

Let us start with $N = 1$ Super Yang-Mills. The classical action reads

$$ S_{\text{SYM}} = -\frac{1}{128g^2} \text{Tr} \int d^4x \ d^2\theta \ W^a W_a + \text{h.c.}, \quad W_\alpha = \tilde{D}^2 (e^{-gV} D_\alpha e^{gV}), $$

where $V(x, \theta)$ is the $N = 1$ vector supermultiplet which belongs to the adjoint representation of the gauge group $G$ ($V = V^a \tau_a$). As the classical action is gauge invariant, in order to quantize the theory we have to fix the gauge and choose a regularization procedure. Instead of the familiar Wess-Zumino gauge—in which the supersymmetry transformation is not linear—we choose a supersymmetric gauge fixing. Hence we add to the action a gauge fixing term which is a supersymmetric extension of the Lorentz gauge and the corresponding Faddeev-Popov term. [4] The classical action $S_{\text{cl}} = S_{\text{SYM}} + S_{\text{gf}} + S_{\text{FP}}$ is invariant under the usual BRS transformation.
Eventually we take into account matter fields. Matter is described by a set of chiral superfields $\phi^I(x, \theta)$ which belong to some representation $R$ of the gauge group. The BRS action for the matter fields is

$$S_m = \frac{1}{16} \int_x \bar{\phi} e^{g V^a T_a} \phi$$

plus a possible superpotential $W(\phi)$.

Developing the RG formalism for supersymmetric theories is straightforward once one replaces the sets of fields and sources in (1) with

$$\Psi_i = (V, c_+, \bar{c}_-, c_-, \bar{c}_+ \phi, \bar{\phi}), \quad \gamma_i = (\gamma_V, \gamma_{c_+}, \gamma_{\bar{c}_-}, \gamma_{\phi}, \gamma_{\bar{\phi}}),$$

$$J_i = (J_V, \xi_+ + D^2 \gamma_V, -\xi_+, -\xi_-, \bar{\xi}_- - D^2 \gamma_V, J, \bar{J}),$$

where the ghost $c_+$ and the anti-ghost $c_-$ are chiral fields, like the gauge parameter. The sources $\gamma_i$ are associated to the BRS variations of the respective superfields, so that we are able to manage composite operators.

### 4.1 Boundary conditions

The relevant part of the cutoff effective action involves full superspace integrals of monomials in the fields, sources and derivatives local in $\theta$, with dimension not larger than two, invariant under Lorentz and global gauge transformations. Due to the dimensionless nature of the field $V$, $\Gamma_{\text{rel}}$ contains infinite terms which can be classified according to the number of gauge fields. The couplings $\sigma_i(\Lambda)$, i.e. the coefficients of those monomials, can be expressed in terms of the cutoff vertices at a given subtraction point, generalizing the procedure used in subsect. 3.1 to define the coupling $Z(\Lambda)$.

As usual the boundary condition we impose on the irrelevant part of the cutoff effective action is $\Gamma_{\text{irr}}[\Phi, \Psi; \Lambda = \Lambda_0] = 0$. The way in which the boundary conditions for the relevant couplings $\sigma_i(\Lambda)$ are determined is not straightforward. In the case of a gauge theory there are interactions in $\Gamma_{\text{rel}}$ which are not present in $S_{\text{cl}}$, so that only some of the relevant couplings are connected to the physical couplings (such as the wave function normalizations and the three-vector coupling $g$ at a subtraction point $\mu$). Therefore, in order to fix the boundary conditions for all the relevant couplings, one needs an additional fine-tuning procedure which implements the gauge symmetry at the physical point.

### 4.2 Gauge symmetry

The gauge symmetry requires that the physical effective action satisfies the ST identity. One can show that for the Wilson effective action $S_{\text{eff}}$ such identity can be rephrased as

$$S_J Z[J, \gamma] = N[J, \gamma; \Lambda] \int D\Psi e^{i \left[ \frac{1}{2} \{ (\Psi, D^{-1} \Psi)_{\alpha\Lambda} + (J, \Psi)_{\alpha\Lambda} + S_{\text{eff}}[\Psi; \Lambda] \} \right] \Delta_{\text{eff}}[\Psi, \gamma; \Lambda]},$$

where $S_J$ is the usual ST operator. Restoration of symmetry, $S_J Z[J, \gamma] = 0$, translates into

$$\Delta_{\text{eff}}[\Psi, \gamma; \Lambda] = 0 \quad \text{for any } \Lambda.$$
From a perturbative point of view, instead of studying $\Delta_{\text{eff}}$ it is convenient to introduce its Legendre transform $\Delta_\Gamma$, in which reducible contributions are absent. We refer to literature for the derivation of $\Delta_\Gamma$. For our purposes it suffices to observe that $\Delta_\Gamma$ is made up of two pieces, the first being essentially the standard ST operator applied to $\Gamma(\Lambda)$, the second, $\hat{\Delta}_\Gamma$, being $O(\hbar)$ and vanishing for $\Lambda \to 0$. The functional $\Delta_\Gamma$ at the UV scale is schematically represented in fig. 2. What we have to show to recover symmetry is we can set $\Delta_\Gamma = 0$ in perturbation theory. Hence we need the evolution equation for such a functional. It turns out the evolution of the vertices of $\Delta_\Gamma$ at the loop $\ell$ depends on vertices of $\Delta_\Gamma$ itself at lower loop order, so that if $\Delta_\Gamma^{(\ell')} = 0$ at any loop order $\ell' < \ell$, then $\Delta_\Gamma^{(\ell)}$ is constant. Thus one can analyse $\Delta_\Gamma$ at an arbitrary value of $\Lambda$. A natural choice is $\Lambda = \Lambda_0$, since at this scale $\Delta_\Gamma$ is local, or, more precisely, $\Delta_{\Gamma,\text{irr}}(\Lambda_0) = O(\frac{1}{\Lambda_0})$. Once the locality of $\Delta_\Gamma(\Lambda)$ is shown, the solvability of the equation $\Delta_\Gamma(\Lambda) = 0$ can be proven using cohomological methods. This is a consequence of the $\Lambda$-independence of $\Delta_\Gamma$ and the solvability of the same equation at $\Lambda = 0$, where the cohomological problem reduces to the standard one.

The requirement that the physical effective action satisfies the ST identity translates at $\Lambda = \Lambda_0$ into the fine-tuning equation

$$S_{\Gamma} \Gamma(\Lambda_0) = -\hat{\Delta}_\Gamma(\Lambda_0),$$

which allows to determine the couplings of the relevant, non-invariant interactions in $\Gamma(\Lambda_0)$. Once the field normalization and the gauge coupling are fixed at the physical point $\Lambda = 0$, the problem of assigning the boundary conditions of the RG equation is finally solved.

However, when the matter representation is such that the matching conditions for the anomaly cancellation are not fulfilled, we tailored a simple method to compute the chiral anomaly. In our framework a violation of the ST identity results in the impossibility of fixing the relevant couplings in $\Gamma(\Lambda_0)$ in such a way that (3) is satisfied. What happens is that part of the matter contribution to the vertices $c_+ V^n$ and $\bar{c}_+ V^n$ of $\hat{\Delta}_\Gamma$ yields a cutoff independent result with no match in the l.h.s. of (3), which is precisely the anomaly. It is known the anomaly can be expressed as an infinite series in the gauge field $V$.

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Figure 2: Graphical representation of $\hat{\Delta}_\Gamma(\Lambda_0)$. The box and the circle represent the functionals $\tilde{\Gamma}$ and $\Gamma$ respectively. The top line is the cutoff full propagator of the field $\Psi$; the bottom full line represents the field $\Psi$ while the double line is the corresponding BRS source $\gamma$. The cross denotes the product of the two functionals with the insertion of the cutoff function $K_{0\Lambda_0}(p)$. Integration over the loop momentum is understood.
Finally, though we restricted our analysis to the perturbative regime, the RG formula-
tion is in principle non-perturbative and provides a natural context in which to clarify the
connection between exact results and those obtained in perturbation theory. In particular,
the relation between the Novikov-Shifman-Vainshtein-Zakharov $\beta$ function for the gauge
coupling in $N = 1$ supersymmetric gauge theories and the exact 1-loop $\beta$ function for the
gauge coupling in a Wilsonian action is presently under investigation.

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