Environment Induced Bipartite Entanglement

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Abstract

Recently, a sufficient condition on the structure of the Kossakowski-Lindblad master equation has been given such that the generated reduced dynamics of two qubits results entangling for at least one among their initial separable pure states. In this paper we study to which extent this condition is also necessary. Further, we find sufficient conditions for bath-mediated entanglement generation in higher dimensional bipartite open quantum systems.

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In standard quantum mechanics the focus is mainly upon closed physical systems, i.e. systems which can be considered isolated from the external environment and whose reversible time-evolution is described by one-parameter groups of unitary operators. On the other hand, when a system $S$ interacts with an environment $E$, it must be considered as an open quantum system whose time-evolution is irreversible and exhibits dissipative and noisy effects. A standard way of obtaining a manageable dissipative time-evolution of the density matrix $\rho_t$ describing the state of $S$ at time $t$ is to construct it as the solution of a Liouville-type Master equation $\partial_t \rho_t = \mathbf{L}[\rho_t]$. This can be done by tracing away the environment degrees of freedom [1 2] and by performing a Markovian approximation [3 4], i.e. by studying the evolution on a slow time-scale and neglecting fast decaying memory effects. Then the irreversible reduced dynamics of $S$ is described by one-parameter semigroups of linear
maps obtained by exponentiating the generator $L$ of Lindblad type \[5, 6\]: $\gamma_t = e^{tL}$, $t \geq 0$, such that $\varrho_t \equiv \gamma_t[\varrho]$.

The typical effect of noise and dissipation on a system $S$ immersed in a large environment $E$ is decoherence; however, in certain specific situations, the environment $E$ may even build quantum correlations between the subsystems which compose $S$. This possibility depends on the form of the Kossakowski matrix that characterizes the dissipative part of the generator $L$. In \[7\] an inequality was found, involving the entries of such a matrix which, if fulfilled, is sufficient to ensure that a specific initial separable pure state of two qubits gets entangled.

This inequality is basically derived by looking at first derivatives of evolving mean values that involve the generator only and not its powers. In this paper we show that, apart from marginal cases whose control needs second or higher powers of the generator, this inequality is also necessary for entangling two qubits via immersion within a common environment. Further, we consider higher dimensional bipartite systems composed of two $d$-level subsystems embedded in a common environment and provide sets of inequalities involving the entries of a higher rank Kossakowski matrix. It turns out that if at least one of these inequalities is fulfilled, the two parties get entangled by their reduced dynamics.

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In this section we will consider a system $S$ composed of two initially separable qubits immersed in a common external bath $E$ with which they weakly interact, but not directly interacting between each other (\[4\], pages 3124-3126); thus the total Hamiltonian is $H_T = H_1 + H_2 + H_B + \lambda H_I$, where $H_1, H_2$ and $H_B$ are Hamiltonians pertaining to the first and second qubit, respectively the bath within which they are immersed, while the interaction Hamiltonian is given by $H_I = \sum_{i=1}^{3} \left((\sigma_i \otimes I) \otimes B_i^{(1)} + (I \otimes \sigma_i) \otimes B_i^{(2)}\right)$ with $I$ the identity $2 \times 2$ matrix and $B_i^{(a)}$, $a = 1, 2$, $i = 1, 2, 3$, bath operators that describe the interaction with the two qubits. In the following, we shall use the convenient notation $\sigma_i^{(1)} := \sigma_i \otimes I$ and $\sigma_i^{(2)} := I \otimes \sigma_i$. By means of standard weak coupling limit techniques, the reduced dynamics of $S$ is given by the Master equation \[4\]

$$\frac{\partial \varrho_t}{\partial t} = L_H[\varrho_t] + D[\varrho_t] = -i[H_{eff}, \varrho_t] + D[\varrho_t] \tag{1}$$

where $H_{eff} = H^{(1)} + H^{(2)} + H^{(12)}$, with $H^{(a)} = \sum_{i=1}^{3} h_i^{(a)} \sigma_i^{(a)}$, $h_i^{(a)} \in \mathbb{R}$, $a = 1, 2$, Hamiltonians of the two qubits independently,

$$H^{(12)} = \sum_{i,j=1}^{3} h_{ij}^{(12)} (\sigma_i \otimes \sigma_j), \quad h_{ij}^{(12)} \in \mathbb{R} \tag{2}$$

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a Hamiltonian term describing a bath-mediated interaction between the qubits, while

\[ D[\rho(t)] = \sum_{i,j=1}^{3} \left( A_{ij} \left[ \sigma_j^{(1)} \rho \sigma_i^{(1)} - \frac{1}{2} \{ \sigma_j^{(1)} \sigma_i^{(1)}, \rho \} \right] + C_{ij} \left[ \sigma_j^{(2)} \rho \sigma_i^{(2)} - \frac{1}{2} \{ \sigma_j^{(2)} \sigma_i^{(2)}, \rho \} \right] + B_{ij} \left[ \sigma_j^{(1)} \rho \sigma_i^{(2)} - \frac{1}{2} \{ \sigma_j^{(1)} \sigma_i^{(2)}, \rho \} \right] + B^*_{ji} \left[ \sigma_j^{(2)} \rho \sigma_i^{(1)} - \frac{1}{2} \{ \sigma_j^{(2)} \sigma_i^{(1)}, \rho \} \right] \right) \]

(3)

is a Kossakowski-Lindblad contribution describing dissipation and noise. The \(3 \times 3\) matrices \(A = A^\dagger, C = C^\dagger\) and \(B\) form the so-called Kossakowski matrix

\[ K = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \]

(4)

whose coefficients come from the Fourier transform of the bath correlation functions (see [4]). In order to guarantee full physical consistency, namely that id \(\otimes\gamma_t\) be positivity preserving on all states of the compound system \(S + S_d\) for any inert ancilla \(S_d\), \(\gamma_t\) must be completely positive and this is equivalent to \(K\) being non-negative [8, 9].

It is natural to call the generated semigroup entangling if there exist at least two vector states \(|\psi\rangle\) and \(|\varphi\rangle\) of the two parties such that \(\gamma_t[Q]\) is entangled for some \(t > 0\), where \(Q := |\psi\rangle\langle\psi| \otimes |\varphi\rangle\langle\varphi|\).

Because of the Peres-Horodecki criterion [10], in the case of two qubits the semigroup \(\gamma_t\) is entangling if and only if there exist a separable initial projector \(Q \in M_4(\mathbb{C})\) and a vector \(\Phi \in \mathbb{C}^4\) such that:

\[ \langle \Phi | \tilde{Q} | \Phi \rangle = 0 \quad \text{and} \quad \langle \Phi | \tilde{L}[\tilde{Q}] | \Phi \rangle < 0 . \]

(6)

Notice that if \(\gamma_t\) cannot entangle initially separable pure states then it cannot entangle separable mixed states.
Vice versa, the semigroup $\gamma_t$ cannot be entangling if

$$\langle \Phi | \mathbf{L} [\tilde{Q}] | \Phi \rangle > 0$$

for all initial separable projectors $Q \in M_4(\mathbb{C})$ and vectors $\Phi \in \mathbb{C}^4$ such that $\langle \Phi | \tilde{Q} | \Phi \rangle = 0$.

**Remark 1** Notice that in case of an equality, the argument of Proposition 1 cannot be used to conclude that entanglement is or is not generated by the semigroup $\gamma_t$. A concrete instance of this fact will be given in Example 1.

**Proof:** As stated above, $\gamma_t$ results entangling if and only if there exist $Q$ and $t > 0$ such that $\tilde{\gamma}_t[\tilde{Q}] \not\geq 0$, i.e. if and only if there exist $Q$, $t > 0$ and $\Phi \in \mathbb{C}^4$ such that $\langle \Phi | \tilde{\gamma}_t[\tilde{Q}] | \Phi \rangle < 0$. Since $\langle \Phi | Q | \Phi \rangle \geq 0$, the latter condition is equivalent to the existence of a smallest $t^* \geq 0$ and $\epsilon_0 > 0$ such that

$$\langle \Phi | \tilde{\gamma}_{t^*} [\tilde{Q}] | \Phi \rangle = 0 \quad (a) \quad \text{and} \quad \langle \Phi | \tilde{\gamma}_{t^* - \epsilon} [\tilde{Q}] | \Phi \rangle < 0 \quad \forall \epsilon_0 \geq \epsilon > 0 \quad (b).$$

The assumption on $t^*$ means that $\tilde{\gamma}_{t^*} [\tilde{Q}]$ is still separable, whence it can be decomposed in a convex sum of pure separable projectors, $\tilde{\gamma}_{t^*} [\tilde{Q}] = \sum_{i,j} \lambda_{ij} Q_{ij}$, $0 \leq \lambda_{ij} \leq 1$. Then, condition (a) implies $\langle \Phi | Q_{ij} | \Phi \rangle = 0$ for all $Q_{ij}$, while, from the semigroup composition law (5) and condition (b) it follows that there exist $i$ and $j$ such that $\langle \Phi | \tilde{\gamma}_t [Q_{ij}] | \Phi \rangle < 0$ for all $0 < \epsilon \leq \epsilon_0$. The continuity of the semigroup formed by the $\tilde{\gamma}_t$ implies that (see for instance [11])

$$\lim_{\epsilon \to 0^+} \langle \Phi | \frac{\tilde{\gamma}_t [Q_{ij}] - Q_{ij}}{\epsilon} | \Phi \rangle = \langle \Phi | \mathbf{L} [Q_{ij}] | \Phi \rangle = \lim_{\epsilon \to 0^+} \frac{\langle \Phi | \tilde{\gamma}_t [Q_{ij}] | \Phi \rangle}{\epsilon}. $$

If $\gamma_t$ is entangling, then for at least one of the $Q_{ij}$ it must be true that $\langle \Phi | Q_{ij} | \Phi \rangle = 0$ and $\langle \Phi | \mathbf{L} [Q_{ij}] | \Phi \rangle \leq 0$, so that if (7) holds as stated, $\gamma_t$ cannot be entangling. Vice versa, if (6) holds then $\tilde{\gamma}_t[\tilde{Q}] \not\geq 0$ in a right neighborhood of $t = 0$ and the semigroup $\gamma_t$ is entangling.

$$\square$$

In order to concretely apply the previous result, we shall introduce the following notations. For given $|\psi\rangle$, $|\varphi\rangle \in \mathbb{C}^2$, let $|u\rangle$, $|v\rangle$ denote the vectors in $\mathbb{C}^3$ with components

$$u_i := \langle \psi | \sigma_i | \psi \rangle, \quad v_i := \epsilon_i \langle \varphi^* | \sigma_i | \varphi \rangle = \langle \varphi | \sigma_i | \varphi \rangle,$$

where $\sigma_i$, $i = 1, 2, 3$ are the Pauli matrices in the chosen standard representation, whence, under transposition, $\sigma_i^T = \epsilon_i \sigma_i$, with $\epsilon_i = +1$ when $i = 1, 3$ and $\epsilon_i = -1$ when $i = 2$. Moreover, $\{\psi, \psi_\perp\}$, $\{\varphi, \varphi_\perp\}$ are the orthonormal bases in $\mathbb{C}^2$ corresponding to $\psi$ and $\varphi$. Let $C^T$ denote the transposition of the $3 \times 3$ matrix $C$ in (4), $Re(B)$ the $3 \times 3$ matrix whose entries are $Re(B)_{ij} := \frac{B_{ij} + B^*_{ij}}{2}$ and $h^{(12)}$ is the $3 \times 3$ real matrix formed by the coefficients $h_{ij}^{(12)}$ of $H^{(12)}$ in (2).
From Proposition 1, it follows that we have to focus on \( \tilde{L} \). When \( L = L_H + D \) as in (2) and in (3), the action of the new generator explicitly reads

\[
\tilde{L}[\varrho] := T^{(2)} \circ L \circ T^{(2)}[\varrho] = -i \sum_{i=1}^{3} \left[ h_i^{(1)} \sigma_i^{(1)} + h_i^{(2)} \epsilon_i \sigma_i^{(2)} \right] \varrho
\]

\[
+ \sum_{i,j=1}^{3} \left( i h_{ij}^{(12)} \epsilon_j \sigma_i^{(1)} \varrho \sigma_j^{(1)} - i h_{ij}^{(12)} \epsilon_j \sigma_i^{(1)} \varrho \sigma_j^{(2)} \right)
\]

\[
+ \sum_{i,j=1}^{3} A_{ij} \left( \sigma_j^{(1)} \varrho \sigma_i^{(1)} - \frac{1}{2} \left\{ \sigma_i^{(1)} , \sigma_j^{(1)} , \varrho \right\} \right)
\]

\[
+ \sum_{i,j=1}^{3} C_{ij} \epsilon_i \epsilon_j \left( \sigma_j^{(2)} \varrho \sigma_i^{(2)} - \frac{1}{2} \left\{ \sigma_i^{(2)} , \sigma_j^{(2)} , \varrho \right\} \right)
\]

\[
+ \sum_{i,j=1}^{3} B_{ij} \epsilon_i \left( \sigma_j^{(1)} \varrho \sigma_i^{(2)} - \frac{1}{2} \sigma_j^{(1)} \varrho \sigma_i^{(2)} - \frac{1}{2} \sigma_j^{(2)} \varrho \sigma_i^{(1)} \right)
\]

\[
+ \sum_{i,j=1}^{3} B_{ij}^* \epsilon_j \left( \sigma_j^{(2)} \varrho \sigma_i^{(1)} - \frac{1}{2} \sigma_j^{(2)} \varrho \sigma_i^{(1)} - \frac{1}{2} \sigma_j^{(1)} \varrho \sigma_i^{(2)} \right).
\]

By regrouping the terms in (9) as in (3), one sees that, with respect to (4), the Kossakowski matrix associated with \( \tilde{L} \) is now

\[
\tilde{K} = \begin{pmatrix} A & \tilde{B} \\ \tilde{B}^\dagger & \tilde{C} \end{pmatrix}, \quad \tilde{B}_{ij} := -\epsilon_i \left( \frac{B_{ij} + B_{ij}^*}{2} + i h_{ji}^{(12)} \right), \quad \tilde{C}_{ij} = \epsilon_i \epsilon_j C_{ij}.
\]

By regrouping the terms in (9) as in (3), one sees that, with respect to (4), the Kossakowski matrix associated with \( \tilde{L} \) is now

Notice that, in spite of the fact that (11) is positive semi-definite, \( \tilde{K} \) need not be so and therefore \( \tilde{\gamma}_t[\tilde{Q}] \) is not necessarily completely positive or even positive; this allows for the possibility that \( \tilde{\gamma}_t[\tilde{Q}] \) be not positive semi-definite.

Proposition 1 tells us that we can concentrate on the mean values of \( \tilde{L}[\tilde{Q}] \) with respect to \( \Phi \in \mathbb{C}^4 \) that belong to the subspace orthogonal to \( \tilde{Q} \). Therefore, we can restrict our attention upon the matrix \( \tilde{Q} \perp \tilde{L}[\tilde{Q}] \tilde{Q} \perp \), \( \tilde{Q} \perp := \mathbb{I} - \tilde{Q} \), that we will represent with respect to the following orthonormal basis

\[
|\Psi_1\rangle := |\psi\rangle \otimes |\varphi^*\rangle, \quad |\Psi_2\rangle := |\psi\rangle \otimes |\varphi^*\rangle,
\]

\[
|\Psi_3\rangle := |\psi\rangle \otimes |\varphi^*\rangle, \quad |\Psi_4\rangle := |\psi\rangle \otimes |\varphi^*\rangle,
\]

where \( |\psi\rangle, |\varphi\rangle \) are the 2-dimensional vectors which define \( Q \), and \( \tilde{Q} = |\Psi_1\rangle \langle \Psi_1| \).

In calculating the matrix elements \( \langle \Psi_i| \tilde{L}[\tilde{Q}] |\Psi_j\rangle \), one observes that only two scalar products contribute to them, either of the form \( \langle \Psi_i| \mathbb{I} \otimes \sigma_j |\Psi_1\rangle \) or of the form \( \langle \Psi_i| \sigma_j \otimes \mathbb{I} |\Psi_1\rangle \). So
$L^\perp := \tilde{Q}^\perp \tilde{L}[\tilde{Q}] \tilde{Q}^\perp = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, where $M_{ij} := \langle \Psi_i | \tilde{L}[\tilde{Q}] | \Psi_j \rangle$; explicitly,

\begin{align}
M_{22} &= \sum_{i,j} C_{ji} \epsilon_j \langle \varphi_\perp^* | \sigma_j | \varphi^* \rangle \langle \varphi^* | \sigma_i | \varphi_\perp \rangle = \langle v | C^T | v \rangle \tag{12} \\
M_{33} &= \sum_{i,j} A_{ij} \langle \psi_\perp | \sigma_j | \psi \rangle \langle \psi | \sigma_i | \psi_\perp \rangle = \langle u | A | u \rangle \tag{13} \\
M_{23} &= -\sum_{i,j} \left( i h_{ij}^{(12)} + \frac{B_{ji} + B_{ji}^*}{2} \right) \epsilon_j \langle \varphi_\perp^* | \sigma_j | \varphi^* \rangle \langle \psi | \sigma_i | \psi_\perp \rangle \\
&= -\langle v | (i h^{(12)})^T + \text{Re}(B) | u \rangle . \tag{14}
\end{align}

**Proposition 2** Given an initial projector $M_4(\mathbb{C}) \ni Q = |\psi\rangle \langle \psi| \otimes |\varphi\rangle \langle \varphi|$, with $|\psi\rangle$, $|\varphi\rangle \in \mathbb{C}^2$, consider the matrix $M = \begin{pmatrix} M_{22} & M_{23} \\ M_{23}^* & M_{33} \end{pmatrix}$ with entries as in (12)–(14). Then,

1. if $\text{Det}(M) \equiv \langle u | A | u \rangle \langle v | C^T | v \rangle - |\langle v | (Re(B) + i(h^{(12)})^T) | u \rangle|^2 < 0$, for at least one pair $|\psi\rangle$, $|\varphi\rangle \in \mathbb{C}^2$, the semigroup $\gamma_t$ with generator as in (1), (2), (3) entangles $Q$;

2. if $\text{Det}(M) \equiv \langle u | A | u \rangle \langle v | C^T | v \rangle - |\langle v | (Re(B) + i(h^{(12)})^T) | u \rangle|^2 > 0$, for all choices of $|\psi\rangle$, $|\varphi\rangle \in \mathbb{C}^2$, the semigroup $\gamma_t$ is not entangling.

**Proof:** The proof of the first statement is a simple application of Proposition 1. If $\text{Det}(M) < 0$, there exists a vector $|\Phi\rangle$ such that $\tilde{Q}|\Phi\rangle = 0$ and $\langle \tilde{Q}| \tilde{L}[\tilde{Q}] | \Phi \rangle < 0$. Then a first order expansion in $t \geq 0$ gives $\langle \tilde{Q}| \tilde{L}[\tilde{Q}] | \Phi \rangle \simeq t \langle \tilde{Q}| \tilde{L}[\tilde{Q}] | \Phi \rangle < 0$. This implies that $\gamma_t[\tilde{Q}]$ is not positive semi-definite in a right neighborhood of $t = 0$, whence $\gamma_t[\tilde{Q}]$ is entangled.

If $\text{Det}(M) > 0$, then (7) holds so, if $\tilde{Q}|\Phi\rangle = 0$ and $\Phi \neq \Psi_4$ (and $\Phi \neq \Psi_1$ since we only need to consider the subspace orthogonal to $\tilde{Q} = |\Psi_1\rangle \langle \Psi_1|)$, a first order expansion in $t \geq 0$ gives $\langle \tilde{Q}| \tilde{L}[\tilde{Q}] | \Phi \rangle \simeq t \langle \tilde{Q}| \tilde{L}[\tilde{Q}] | \Phi \rangle = t \langle \Phi| \tilde{L}^\perp | \Phi \rangle > 0$.

If $\Phi = \Psi_4$, then $\langle \Phi| \tilde{L}^\perp | \Phi \rangle = 0$ and the argument based on the first derivative seems to be not conclusive; however, $\Psi_4$ is separable and therefore $\langle \Psi_4| \tilde{L}[\tilde{Q}] | \Psi_4 \rangle \geq 0$ for all $t \geq 0$. Hence, if $\text{Det}(M) > 0$ for all choices of $|\psi\rangle$, $|\varphi\rangle \in \mathbb{C}^2$, $\gamma_t$ cannot be entangling.

Unlike the last part of the previous proof, when $M$ has an eigenvalue equal to zero, namely if $\langle u | A | u \rangle \langle v | C^T | v \rangle = |\langle v | (Re(B) + i(h^{(12)})^T) | u \rangle|^2$, then, in order to check whether

\footnote{This is essentially what was proved in [7].}
the semigroup $\gamma_t$ is entangling or not, one has to go to the second or higher order terms in the small $t \geq 0$ expansion of $\langle \Psi | \tilde{\gamma}_t(Q) | \Psi \rangle$. In fact, if $\text{Det}(M) = 0$, there exists $|\Psi^+_1\rangle$ such that $L^+|\Psi^+_1\rangle = 0$ so that $\langle \Psi^+_1 | \tilde{\gamma}_t(Q) | \Psi^+_1 \rangle \simeq t^2/2 \langle \Psi^+_1 | L^2[Q] | \Psi^+_1 \rangle$. As the following example shows, the non-negativity of the matrix $M$ does not fix the non-entangling character of $\gamma_t$: strict positivity as in point 2. of Proposition 2 is necessary for this to be true.

**Example 1** For sake of simplicity, we will set to zero the Hamiltonian terms in (1) and consider a Kossakowski matrix of the form

$$K = \begin{pmatrix} A & A \\ A & A \end{pmatrix} \quad \text{with} \quad A = \begin{pmatrix} 1 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & x \end{pmatrix},$$

with $x \geq 1$ so that $K \geq 0$. The purely dissipative generator

$$L[\rho] = \sum_{i,j=1}^{3} \sum_{p,q=1}^{2} A_{ij} \left( \sigma_j^{(p)} \rho \sigma_i^{(q)} - \frac{1}{2} \{ \sigma_i^{(q)} \sigma_j^{(p)} , \rho \} \right),$$

generates a continuous one-parameter semigroup of completely positive maps $\gamma_t = e^{tL}$. The Kossakowski matrix (10) associated with the generator $\tilde{L}$ of $\tilde{\gamma}_t := T^{(2)} \circ \gamma_t \circ T^{(2)}$ reads

$$\tilde{K} = \begin{pmatrix} A & -\text{Re}(A) \\ -\text{Re}(A)^\dagger & A^T \end{pmatrix} = \begin{pmatrix} 1 & 0 & i & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -i & 0 & x & 0 & 0 & -x \\ -1 & 0 & 1 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x & i & 0 & x \end{pmatrix}$$

and is not positive definite, its non-zero eigenvalues being $1 \pm \sqrt{2}$ and $x \pm \sqrt{1 + x^2}$. This ensures that $\tilde{\gamma}_t$ is not completely positive; moreover, it turns out that, for some values of $x$, it is not even positivity preserving, leaving a chance that for some initial separable projector $Q$ there exists $t > 0$ such that $\tilde{\gamma}_t[Q]$ might not be positive semi-definite.

Indeed, let $Q = |0\rangle\langle 0| \otimes |0\rangle\langle 0| = \tilde{Q}$ where $\sigma_3|0\rangle = |0\rangle$ and $\sigma_3|1\rangle = -|1\rangle$; then, $|u\rangle = |v\rangle = (1, -i, 0)$, and $\langle u|A|u\rangle = \langle u|A^T|u\rangle = \langle u|\text{Re}(A)|u\rangle = 1$. Therefore, $\text{Det}(M) = 0$ and the the vector $|\Psi^+_1\rangle = |\Psi_2\rangle + |\Psi_3\rangle$ is eigenvector of

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with eigenvalue 0.

Considering the second order term in the expansion of $\langle \Psi^+_1 | \tilde{\gamma}_t[\tilde{Q}] | \Psi^+_1 \rangle$, explicit calculations give $\langle \Psi^+_1 | L^2[\tilde{Q}] | \Psi^+_1 \rangle = 16x - 24$; thus, the semigroup $\tilde{\gamma}_t$ is not positivity preserving and $\gamma_t = e^{tL}$ entangles $Q$ for $1 \leq x < 3/2$, it does not do so for $x > 3/2$, while if $x = 3/2$ one has to consider the third order term of the expansion in $t \geq 0$.

**Remark 2** If $A = B = C = 0$ in (1) and $H^{(a)} = 0$ in (1), the time-evolution is purely reversible and governed by the interaction Hamiltonian $H^{(12)} = \sum_{i,j=1}^{3} h_{ij}^{(12)}(\sigma_i \otimes \sigma_j)$ in (2); then
the sufficient condition for entanglement in Proposition 2 reduces to (see in particular (14))

\[ |\langle u|h^{(12)}|v\rangle|^2 > 0 \quad (15) \]

for some \(|u\rangle, |v\rangle \in \mathbb{C}^3\) of the form (8).

In [12] it is shown that there can be found local unitary transformations

\[ \sigma^A_i := U_A^\dagger \sigma_i U_A = \sum_{k=1}^{3} O^A_{ik} \sigma_k, \quad \sigma^B_j := U_B^\dagger \sigma_j U_B = \sum_{l=1}^{3} O^B_{lj} \sigma_l, \quad (16) \]

such that \( H^{(12)} = \sum_{i,j=1}^{3} h^{(12)}_{ij} (\sigma_i \otimes \sigma_j) \) can always be recast as

\[ (U_A \otimes U_B) H^{(12)} (U_A^\dagger \otimes U_B^\dagger) := \hat{H}^\pm = \mu_1 \sigma_1^A \otimes \sigma_1^B \pm \mu_2 \sigma_2^A \otimes \sigma_2^B + \mu_3 \sigma_3^A \otimes \sigma_3^B \]

(\( \hat{H}^+ \) if \( Det(h^{(12)}) \geq 0 \), \( \hat{H}^- \) if \( Det(h^{(12)}) < 0 \)), where \( \mu_1 \geq \mu_2 \geq \mu_3 \geq 0 \) are the sorted eigenvalues of \( \sqrt{(h^{(12)})^\dagger h^{(12)}} \). Further, the maximal entangling capability \( \eta_{\text{max}} \) of an interaction Hamiltonian \( H^{(12)} \) is defined as

\[ \eta_{\text{max}} := \max |\langle \chi^1 \otimes \chi^2 | H^{(12)} | \chi^1_\perp \otimes \chi^2_\perp \rangle| = \max |\langle \tilde{\chi}^1 \otimes \tilde{\chi}^2 | \hat{H}^\pm | \tilde{\chi}^1_\perp \otimes \tilde{\chi}^2_\perp \rangle| = (\mu_1 + \mu_2)^2, \]

with \( |\tilde{\chi}^1 \rangle := U_A |\chi^1 \rangle \) and \( |\tilde{\chi}^2 \rangle := U_B |\chi^2 \rangle \), the maximum value being attained at \( |\tilde{\chi}^1 \rangle = |0\rangle_A \) eigenstate of \( \sigma_3^A \) and \( |\tilde{\chi}^2 \rangle = |1\rangle_B \) eigenstate of \( \sigma_3^B \). Now consider (15) and observe that

\[ |\langle u|h^{(12)}|v\rangle|^2 = |\sum_{i,j} h^{(12)}_{ij} \epsilon_j \langle \psi_\perp | \sigma_i \langle \varphi^* | \sigma_j | \varphi^*_\perp \rangle| = (\mu_1 + \mu_2)^2 \]

for \(|\psi\rangle = U_A |0\rangle_A \) and \(|\varphi^*\rangle = U_B |0\rangle_B \). Therefore, the maximal entanglement capability of a two qubit Hamiltonian coincides with the the largest possible value that fulfils the sufficient condition (15).

\textbf{Remark 3} Since partial transposition provides an exhaustive entanglement witness also in the case of a two-level system coupled to a three-level system [10], similar arguments as those developed above can be applied to derive necessary and sufficient conditions for entanglement generation even in this case. The proofs both of necessity and of sufficiency would be the direct generalization of that of the two-qubit case with Pauli matrices acting on the first subsystem and, for instance, Gell-Mann matrices acting on the second, as will be explained at the beginning of the next subsection. A concrete physical example of entanglement conditions for a spin-1/2 coupled to a spin-1 will be given in a subsequent paper. In the following, we shall rather discuss bipartite systems consisting of two \( d \)-level systems; this includes, for instance, bipartite systems of \( n \) qubits each, which are a natural generalization of the system previously considered, although positivity under partial transposition is not sufficient to exclude entanglement.
The argument of the proof of sufficiency in Proposition 2 can be extended to higher dimensional bipartite systems consisting of two \( d \)-dimensional subsystems. It must be noticed that when \( d \geq 3 \), no extension of condition (7) is possible for there can be entangled states which remain positive under partial transposition [13], that is, \( \gamma_t \) might result entangling despite \( \tilde{\gamma}_t \) being positive on initially separable states. On the other hand, though, the larger \( d \) gets, the more sufficient conditions we can obtain for the generation of entanglement.

Let \( \{F_k\}_{k=0}^{d^2-1} \), \( F_0 := I_d/\sqrt{d} \), be an orthonormal set of \( d \times d \) Hermitian matrices such that \( \text{Tr}(F_i F_j) = \delta_{ij} \) and, under transposition, \( F_k^T = \eta_k F_k \), where \( \eta_k = \pm 1 \). For instance, as \( F_k \)'s we can take the generalized Gell-Mann matrices [14] which satisfy this request.

For a bipartite system where the two parties consist of \( n \) qubits, one chooses the matrices \( F_k \) as tensor products of \( n \) Pauli matrices, whence \( \eta_k \) is the product of \( n \varepsilon_i \), where \( \sigma_i^T = \varepsilon_i \sigma_i \) as in the previous section.

If we consider a system \( S \) composed of two subsystems each of finite dimension \( d \) immersed in a common external bath \( E \) with which they weakly interact, but not directly interacting between each other, we can generalize the 2-dimensional Master equation (1) with the \( F_k \) matrices defined above. Thus the total Hamiltonian is \( H_T = H_1 + H_2 + H_B + \lambda H_I \), where \( H_1, H_2 \) and \( H_B \) are Hamiltonians pertaining to the first and second subsystem, respectively to the bath, while the interaction Hamiltonian is given by

\[
H_I = \sum_{i=1}^{d^2-1} \sum_{a=1}^2 F_i^{(a)} \otimes B_i^{(a)} , \quad F_i^{(1)} := F_i \otimes I , \quad F_i^{(2)} := I \otimes F_i ,
\]

with \( I \) the \( d \times d \) identity matrix and \( B_i^{(a)} \) bath operators. Again by means of standard weak coupling limit techniques, the reduced dynamics of \( S \) is given by the Master equation

\[
\frac{\partial \varrho_t}{\partial t} = L_H[\varrho_t] + D[\varrho_t] = -i[H_{\text{eff}}, \varrho_t] + D[\varrho_t]
\]

where \( H_{\text{eff}} = H^{(1)} + H^{(2)} + H^{(12)} \), with \( H^{(a)} = \sum_{i=1}^{d^2-1} h_i^{(a)} F_i^{(a)} \), \( h_i^{(a)} \in \mathbb{R} \), \( a = 1, 2 \), Hamiltonians of the two subsystems independently,

\[
H^{(12)} = \sum_{i,j=1}^{d^2-1} h_{ij}^{(12)} (F_i \otimes F_j) , \quad h_{ij}^{(12)} \in \mathbb{R} ,
\]
a Hamiltonian term describing a bath-mediated interaction between the subsystems, while

\[
D[\varrho(t)] = \sum_{i,j=1}^{d^2-1} \left( A_{ij} \left[ F_j^{(1)} \varrho F_i^{(1)} - \frac{1}{2} \{ F_j^{(1)} F_i^{(1)} , \varrho \} \right] \\
+ C_{ij} \left[ F_j^{(2)} \varrho F_i^{(2)} - \frac{1}{2} \{ F_j^{(2)} F_i^{(2)} , \varrho \} \right] \\
+ B_{ij} \left[ F_j^{(1)} \varrho F_i^{(2)} - \frac{1}{2} \{ F_j^{(1)} F_i^{(2)} , \varrho \} \right] \\
+ B_{ji}^* \left[ F_j^{(2)} \varrho F_i^{(1)} - \frac{1}{2} \{ F_j^{(2)} F_i^{(1)} , \varrho \} \right] \right),
\]

is a Kossakowski-Lindblad contribution describing dissipation and noise. \( A = A^\dagger, C = C^\dagger \) and \( B \) are \((d^2-1) \times (d^2-1)\) matrices which define a \(2(d^2-1) \times 2(d^2-1)\) Kossakowski matrix \( K = \begin{pmatrix} A & B \\ B^\dagger & C \end{pmatrix} \). As in the \(d = 2\) case, in order to guarantee the complete positivity of the dynamical map \( \gamma_t = e^{tL} \) and thus its full physical consistency against coupling with generic ancillas and the existence of entangled states, the Kossakowski matrix must be positive semi-definite, \( K \geq 0 \) \([8, 9]\).

As in Proposition 1 and Proposition 2, let \( Q = |\psi\rangle \langle \psi| \otimes |\varphi\rangle \langle \varphi|, \) with \(|\psi\rangle, |\varphi\rangle \in \mathbb{C}^d\), be an initial separable projector of the two \(d\)-level systems. We shall then consider the semigroup \( \tilde{\gamma}_t = e^{tL} \) and its generator \( \tilde{L} = T^{(2)} \circ L \circ T^{(2)} \), where \( T^{(2)} \) is the partial transposition operated on the second factor. The form of \( \tilde{L} \) is the same as in \([8, 9]\) and \([10]\) with the \( F_k \) matrices instead of the Pauli matrices and \( \eta_i \) in place of \( \varepsilon_i \).

According to Proposition 1 and the proof of point 1 in Proposition 2, in order to find sufficient conditions for \( \gamma_t \) to be entangling, we just have to study when \( \tilde{Q}^{\perp} \tilde{L} \tilde{Q} \tilde{Q}^{\perp} \) has a negative eigenvalue, where \( \tilde{Q} = T^{(2)}[Q] = |\psi\rangle \langle \psi| \otimes |\varphi^*\rangle \langle \varphi^*|, \) \( \tilde{Q}^{\perp} = \mathbb{I} - \tilde{Q} \).

Let \( \{|\psi_i\rangle\}_{i=1}^{d^2} \) and \( \{|\varphi_i\rangle\}_{i=1}^{d^2} \) be two orthonormal bases for the two parties, with \(|\psi_1\rangle = |\psi\rangle\) and \(|\varphi_1\rangle = |\varphi^*\rangle\). A convenient enumeration for the corresponding basis of the composite system is as follows: \(|\Psi_{d(k-1)+\ell}\rangle := |\psi_k \otimes \varphi_\ell\rangle\) for \(k, \ell = 1, 2, \ldots, d\). Set \(i = d(k-1) + \ell, i = 1, 2, \ldots, d^2\); then, \( \tilde{Q} := |\Psi_1\rangle \langle \Psi_1| \) and \( \tilde{Q}^{\perp} := \sum_{i=2}^{d^2} |\Psi_i\rangle \langle \Psi_i| \).

Since \( F_i^{(1)} = F_i \otimes \mathbb{I} \) and \( F_i^{(2)} = \mathbb{I} \otimes F_i \), with respect to the chosen basis, only the entries \( M_{ij} := \langle \Psi_i | \tilde{Q}^{\perp} \tilde{L} \tilde{Q} \tilde{Q}^{\perp} | \Psi_j \rangle \) with either \(k = 1\) or \(\ell = 1\) in \(i = \ell + d(k-1)\) survive, while all those with \(k \neq 1\) and \(\ell \neq 1\) vanish. There are \(2(d-1)\) basis vectors with either \(k = 1\) or \(\ell = 1\):

\[
|\Psi_2\rangle = |\psi_1 \otimes \varphi_2\rangle, \hspace{1cm} |\Psi_3\rangle = |\psi_1 \otimes \varphi_3\rangle, \ldots \hspace{1cm} |\Psi_d\rangle = |\psi_1 \otimes \varphi_d\rangle \\
|\Psi_{d+1}\rangle = |\psi_2 \otimes \varphi_1\rangle, \hspace{1cm} |\Psi_{2d+1}\rangle = |\psi_3 \otimes \varphi_1\rangle, \ldots \hspace{1cm} |\Psi_{(d-1)d+1}\rangle = |\psi_d \otimes \varphi_1\rangle,
\]

and \((d-1)^2\) vectors with \(k \neq 1\) and \(\ell \neq 1\). Therefore we can focus on \( \tilde{Q}^{\perp} \tilde{L} \tilde{Q} \tilde{Q}^{\perp} \) restricted to the subspace spanned by the vectors in \((20)\), i.e. on \((2(d-1)) \times 2(d-1)\) non-zero submatrix that we will call \(M\). This matrix is composed of four \((d-1)\)-dimensional square blocks and
its entries can be written in analogy to the $d = 2$ case generalizing the vectors $|u\rangle$, $|v\rangle$ in (8). Explicitly, we define $(d-1)$ vectors $|u^{(n)}\rangle$ and $(d-1)$ vectors $|v^{(m)}\rangle$ with $(d^2-1)$ components each, given by

$$u_i^{(n)} := \langle \psi_1 | F_i | \psi_n \rangle, \quad n = d + 1, d + 2, \ldots, 2d - 1, \quad i = 1, \ldots, d^2 - 1,$$

$$v_i^{(m)} := \eta_i \langle \varphi_1 | F_i | \varphi_m \rangle, \quad m = 2, \ldots, d, \quad i = 1, \ldots, d^2 - 1. \quad (21)$$

Further, we introduce a Hermitian $2(d-1) \times 2(d-1)$ matrix $M = [M_{,\beta}]$ with entries

$$M_{\alpha,\beta} := \langle u^{(\alpha+1)} | C^T | v^{(\beta+1)} \rangle, \quad \alpha, \beta = 1, 2, \ldots, d-1$$

$$M_{\alpha,\beta} := \langle u^{(\alpha+1)} | A | v^{(\beta+1)} \rangle, \quad \alpha, \beta = d, d+1, \ldots, 2(d-1)$$

$$M_{\alpha,\beta} := -\langle u^{(\alpha+1)} | (i(h^{(12)}T + Re(B)) | v^{(\beta+1)} \rangle, \quad \alpha = 1, \ldots, d-1, \quad \beta = d, \ldots, 2(d-1)$$

$$M_{\alpha,\beta} := -\langle u^{(\beta+1)} | (i(h^{(12)}T + Re(B)) | u^{(\alpha+1)} \rangle, \quad \alpha = d, \ldots, 2(d-1), \quad \beta = 1, \ldots, d-1. \quad (22)$$

In Proposition 2, we saw that the negativity of the determinant of the matrix $M = [M_{\alpha,\beta}]$ with $d = 2$ is a sufficient condition for a bath-mediated entanglement of an initial separable projector. For a bipartite system composed of two $d$-level subsystems, the argument generalizes as follows.

**Proposition 3** If at least one of the principal minors of the $2(d-1) \times 2(d-1)$ matrix $M = [M_{\alpha,\beta}]$ is negative, then the semigroup $\gamma_t$ is entangling. Therefore there are $2d(2^{d-1} - 1) + 1$ conditions at the most, each one of them ensuring bipartite entanglement generation through immersion in a common environment.

**Proof:** Let $R$ be one of $M$’s principal sub-matrices. If $Det(R) < 0$, then there exists a vector $|\Phi\rangle$ in the support of $Q^{-1}L[Q]Q^{-1}$ such that $\langle \Phi | L[Q] | \Phi \rangle < 0$ and $\langle \Phi | Q | \Phi \rangle = 0$. Thus, an expansion at small times $t \geq 0$ yields

$$\langle \Phi | \tilde{\gamma}_t[\tilde{Q}] | \Phi \rangle \simeq t \langle \Phi | L[\tilde{Q}] | \Phi \rangle < 0,$$

which implies that $\tilde{\gamma}_t[\tilde{Q}]$ is not positive semi-definite in a right neighborhood of $t = 0$ and $\gamma_t(Q)$ becomes entangled in that time-interval.

Being a $2(d-1) \times 2(d-1)$ matrix, $M$ has $2^{2(d-1)} - 1$ principal sub-matrices; however, since $A$ and $C$ in (19) are positive matrices, all their $2^{(2d-1) - 1}$ principal minors cannot be negative: therefore, all the diagonal elements of $M$ are surely non-negative. Thus we are left with, at most, $(2^{2(d-1)} - 1) - 2(2^{d-1} - 1) = 4^{d-1} - 2^d + 1$ principal minors that are not necessarily positive. $\square$

---

3In the two qubit case, $d = 2$ and we get $4 - 2^2 + 1 = 1$ sufficient condition for entanglement, as found in Proposition 2.
Example 2. We will consider four qubits \((1, 2, 3, 4)\) immersed in a dissipative environment such that their states evolve in time according to a Master equation \((17)\) with a purely dissipative generator of the form

\[
L[\rho] = \sum_{p=1}^{4} \sum_{i,j=1}^{3} C_{ij}^{(1)} \left( \sigma_j^{(p)} \rho \sigma_i^{(p)} - \frac{1}{2} \{ \sigma_j^{(p)} , \sigma_i^{(p)} \} \right) + \sum_{p \neq q=1}^{4} \sum_{i,j=1}^{3} C_{ij}^{(2)} \left( \sigma_j^{(p)} \rho \sigma_i^{(q)} - \frac{1}{2} \{ \sigma_j^{(p)} , \sigma_i^{(q)} \} \right).
\]  

(23)

In terms of the matrices \(F_k\) in \((19)\), one has \(F_k := F_{i+3(p-1)} = \frac{1}{\sqrt{2}} \sigma_i^{(p)}\), \(i = 1, 2, 3\), \(p = 1, 2\) and a Kossakowski matrix \(K_4\)

\[
K_4 = \begin{pmatrix}
C^{(1)} & C^{(2)} & C^{(3)} & C^{(4)} \\
C^{(2)} & C^{(1)} & C^{(3)} & C^{(4)} \\
C^{(2)} & C^{(2)} & C^{(1)} & C^{(4)} \\
C^{(2)} & C^{(2)} & C^{(2)} & C^{(1)}
\end{pmatrix}.
\]

Taking \(C^{(1)} = \begin{pmatrix} 1 & iz & 0 \\ -iz & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\) and \(C^{(2)} = \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 0 \end{pmatrix}\), with \(z, x\) real numbers, in order for the semigroup generated by \((23)\) to be completely positive, the Kossakowski matrix \(K_4\) must be positive semi-definite; this implies \(z^2 + 9x^2 \leq 1\) as its non-zero eigenvalues are \(1 \pm \sqrt{x^2 + z^2}\) and \(1 \pm \sqrt{9x^2 + z^2}\).

Consider the fully separable state \(Q_4 := \left| 0 \right\rangle \left\langle 0 \right| \otimes \left| 0 \right\rangle \left\langle 0 \right| \otimes \left| 0 \right\rangle \left\langle 0 \right|\), \(\sigma_3 |0\rangle = |0\rangle\); the vectors defined in \((21)\) are

\[
|u^{(5)}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -i \end{pmatrix}, \quad |v^{(2)}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix}, \quad |u^{(6)}\rangle = \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |v^{(3)}\rangle = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]

while \(|u^{(7)}\rangle\) and \(|v^{(4)}\rangle\) are the null vector. Therefore, the \(2(d-1) \times 2(d-1) = 6 \times 6\) matrix \(M = [M_{\alpha\beta}]\) reduces to a \(4 \times 4\) matrix of the form

\[
M = 2 \begin{pmatrix}
1 + z & 0 & x & x \\
0 & 1 + z & x & x \\
x & x & 1 + z & 0 \\
x & x & 0 & 1 + z
\end{pmatrix}.
\]

Since \(z^2 \leq z^2 + 9x^2 \leq 1\), its principal minors of order 1, 2(1+z), are all positive; those of order 2 are the determinants of \(2 \begin{pmatrix} 1 + z & 0 \\ 0 & 1 + z \end{pmatrix}\), \(2 \begin{pmatrix} 1 + z & x \\ x & 1 + z \end{pmatrix}\), and are also positive if
Those of order 3, \( D(x, z) := 8(1 + z)((1 + z)^2 - 2x^2) \), namely the determinants of the matrices

\[
2 \begin{pmatrix}
1 + z & 0 & x \\
0 & 1 + z & x \\
x & x & 1 + z
\end{pmatrix},
2 \begin{pmatrix}
1 + z & x & x \\
x & 1 + z & 0 \\
x & 0 & 1 + z
\end{pmatrix},
\]

can nevertheless be negative. In fact, there is a region in the plane \((x, z)\) where \( z^2 + 9x^2 \leq 1 \) and \((1 + z)^2 > x^2 \), while \((1 + z)^2 < 2x^2 \). The corresponding values \((x, z)\) ensure that \( K_4 \) is positive semi-definite, whereas \( M \) is not; correspondingly, the 4-qubit dissipative dynamics entangles the two pairs \((1, 2)\) and \((3, 4)\) initially in the pure separable state \( Q_4 \).

The fact that the principal minors of order 2 are non-negative has the following physical interpretation. The form of the generator of the dissipative dynamics of the two pairs of qubits is such that if we eliminate any pair of qubits by taking the trace of (23) over their Hilbert spaces, a generator results for the remaining pair \((i, j)\) of qubits that amounts to keeping only the \(i\)-th and \(j\)-th row and column of \( K_4 \), thereby leading to a same dissipative time-evolution associated with the Kossakowski matrix \( K_2 = \begin{pmatrix} C^{(1)} & C^{(2)} \\ C^{(2)} & C^{(1)} \end{pmatrix} \) for any pair \((i, j)\) of qubits. Moreover, given \( Q_4 \), any pair of qubits is in the same state \( Q_2 = |0\rangle \langle 0| \otimes |0\rangle \langle 0| \) with the vectors \(|u\rangle\) and \(|v\rangle\) defined in (8) that read \(|u\rangle = (1, -i, 0)\), \(|v\rangle = (1, i, 0)\) and yield a matrix \( M = 2 \begin{pmatrix} 1 + z & x \\
x & 1 + z \end{pmatrix} \). By assumption \( \text{Det}(M) > 0 \), thus, according to Proposition 2, the two-qubit semigroup associated with the Kossakowski matrix \( K_2 \) cannot entangle \( Q_2 \); nonetheless, the four-qubit semigroup associated with \( K_4 \) entangles \( Q_4 \).

We have shown that the sufficient condition found in [7] for the creation of entanglement between two qubits immersed in a common environment by means of their reduced dissipative dynamics is, apart from marginal cases, also necessary. Moreover, we have extended to higher dimensional bipartite open quantum systems the basic argument in [7] thereby obtaining sufficient conditions for entangling a separable pure state in terms of the negativity of the principal minors of certain matrices that depend on the generator of the dynamics and on the given state to get entangled. Since the number of principal minors increases with the dimension of the parties, for more than two qubits a richer variety of noise-induced entanglement is available. As an example, we provided a purely dissipative time evolution that entangles two subsystems, each consisting of two qubits, without entangling any two single qubits.

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