Shift-plethystic trees and Rogers–Ramanujan identities

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Received: 15 October 2019 / Accepted: 8 May 2020 / Published online: 26 July 2020
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Abstract
By studying non-commutative series in an infinite alphabet, we introduce shift-plethystic trees and a class of integer compositions as new combinatorial models for the Rogers–Ramanujan identities. We prove that the language associated to shift-plethystic trees can be expressed as a non-commutative generalization of the Rogers–Ramanujan continued fraction. By specializing the non-commutative series to $q$-series, we obtain new combinatorial interpretations of the Rogers-Ramanujan identities in terms of signed integer compositions. We introduce the operation of shift-plethysm on non-commutative series and use this to obtain interesting enumerative identities involving compositions and partitions related to Rogers–Ramanujan identities.

Keywords Rogers–Ramanujan identities · Integer compositions · Non-commutative series

Mathematics Subject Classification Primary 05A17 · 11P84 · Secondary 05A15 · 05A19

1 Introduction
The Rogers–Ramanujan identities, equations (1) and (2), have had a fructiferous influence in many, some of them unexpected, subjects in Mathematics and Physics.

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^3) \cdots (1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+1})(1-q^{5k+4})}, \quad (1)$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^3) \cdots (1-q^n)} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{5k+2})(1-q^{5k+3})}. \quad (2)$$
They were discovered and proved by Rogers in 1894 [18], rediscovered by Ramanujan (without proof) in 1913, and again by Schur in 1917 [20]. It is impossible to summarize in a few lines the enormous amount of contributions related to the Rogers–Ramanujan identities and their generalizations. The reader is referred to the recent book of Sills [21], for further references and a nice introduction to the subject in its historical context. In [18] Rogers presented what is now known as the Rogers–Ramanujan continued fraction $\mathcal{R}(q)$,

$$\mathcal{R}(q) = q^{\frac{1}{5}} \frac{1}{1 + \frac{q}{1 + \frac{q^2}{\ddots}}}.$$

and proved that

$$\mathcal{R}(q) = q^{\frac{1}{5}} \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

In what follows we shall drop the factor $q^{\frac{1}{5}}$ from $\mathcal{R}(q)$, since our main concern here is about the combinatorial meaning of the Rogers–Ramanujan continued fraction and identities. MacMahon [13] and Schur [20] were the first in reporting the combinatorial meaning of the Rogers–Ramanujan identities. The left-hand side of (1) is the generating function for the number of partitions of positive parts with a difference of at least two among adjacent parts (2-distinct partitions in the terminology of [2]). Its right-hand side is the generating function of the partitions with each part congruent either with one or four module five. Similarly, the left-hand side of (2) is the generating function for the number of 2-distinct partitions, but having each part strictly greater than one. Its right-hand side counts the number of partitions with each part congruent either with two or three module five. Hence, each of them establish an equipotence between two different sets of partitions. Garsia and Milne gave a bijective proof of the Rogers–Ramanujan identities by establishing a complicated bijection between these two kinds of partitions [6]. For that end they created what now is called the Garsia–Milne involution principle. The Garsia–Milne proof was later simplified in [4].

We introduce here a non-commutative version of $\mathcal{R}(-q)$ in an infinite number of variables

$$X_0, X_1, X_2, X_3, \ldots,$$

and prove that its expansion is the language of words associated to a combinatorial structure we call shift-plethystic trees. Our model based on shift-plethystic trees lead us to consider compositions (instead of partitions) whose risings are at most one, and express the non-commutative version of $\mathcal{R}(-q)$ as a quotient of two generating functions on this kind of compositions. We call a $q$-umbral evaluation on a non-commutative series the procedure of substituting each variable $X_k$ by $zq^k$ or simply
by \( q^k \). By \( q \)-umbral evaluation of those generating functions we obtain an alternative (dual) combinatorial interpretation of Rogers–Ramanujan identities in terms of signed compositions (Sect. 5). A combinatorial understanding of the cancellations taking place in the signed compositions that we obtain would provide an elegant and, hopefully, simple proof of the Rogers–Ramanujan identities. In Sect. 6 we introduce shift-plethysm of non-commutative series. It generalizes the classical substitution of \( q \)-series. By \( q \)-umbral evaluating shift-plethysm on a particular class of non-commutative series we obtain the classical substitution of \( q \)-series. By means of elementary computation of inverses on generalized shift-plethystic trees we recover some classical identities in Sect. 6.1, and prove in Sect. 7 new ones relating Rogers–Ramanujan identities, compositions, partitions, and shift-plethystic trees.

Previous work on non-commutative versions of the Rogers–Ramanujan continued fractions can be found in [3,15]. Although their approach does not rely on an infinite number of variables, a coupling of both approaches would lead to novel identities involving signed compositions.

2 Formal power series in non-commuting variables

Let \( \mathbb{A} \) be an alphabet (a totally ordered set) with at most a countable number of elements (letters). Let \( \mathbb{A}^* \) be the free monoid generated by \( \mathbb{A} \). It consists of words or finite strings of letters in \( \mathbb{A} \), \( \omega = \omega_1 \omega_2 \ldots \omega_n \), including the empty string represented as 1. We denote by \( \ell(\omega) \) the length of \( \omega \). Let \( \mathbb{K} \) be a field of characteristic zero. A non-commutative formal power series in \( \mathbb{A} \) over \( \mathbb{K} \) is a function \( R : \mathbb{A}^* \rightarrow \mathbb{K} \). We denote \( R(\omega) \) by \( \langle R, \omega \rangle \) and represent \( R \) as a formal series

\[
R = \sum_{\omega \in \mathbb{A}^*} \langle R, \omega \rangle \omega, \quad \langle R, \omega \rangle \in \mathbb{K}.
\]

The sum and product of two formal power series \( R \) and \( S \) are, respectively, given by

\[
R + S = \sum_{\omega \in \mathbb{A}^*} \left( \langle R, \omega \rangle + \langle S, \omega \rangle \right) \omega,
\]

\[
R.S = \sum_{\omega \in \mathbb{A}^*} \left( \sum_{\omega_1 \omega_2 = \omega} \langle R, \omega_1 \rangle \langle S, \omega_2 \rangle \right) \omega.
\]

The algebra of non-commutative formal power series is denoted by \( \mathbb{K}(\langle \mathbb{A} \rangle) \). There is a notion of convergence on \( \mathbb{K}(\langle \mathbb{A} \rangle) \). We say that \( R_1, R_2, R_3, \ldots \) converges to \( R \) if for all \( \omega \in \mathbb{A}^* \), \( \langle R_n, \omega \rangle = \langle R, \omega \rangle \) for \( n \) big enough. If \( \langle R, 1 \rangle = \alpha \neq 0 \), then \( R \) has an inverse given by (see Stanley [22])

\[
R^{-1} = \frac{1}{\alpha} \sum_{n=0}^{\infty} \left( 1 - \frac{R}{\alpha} \right)^n.
\]
Let $B$ be a series having constant term equal to zero, $\langle B, 1 \rangle = 0$. We denote by $\frac{1}{1-B}$, the inverse of the series $1 - B$,

$$\frac{1}{1-B} := (1 - B)^{-1} = \sum_{n=0}^{\infty} B^n.$$ 

A language (on $\mathbb{A}$) is a subset of $\mathbb{A}^\ast$. We identify a language $L$ with its generating function, the formal power series

$$L = \sum_{\omega \in L} \omega.$$ 

We consider now a special kind of languages obtained from a given set of ‘links’ $B \subseteq \mathbb{A} \times \mathbb{A}$. Define

$$L_B = \{ \omega | (\omega_i, \omega_{i+1}) \in B, \text{ for every } i = 1, 2, \ldots, \ell(\omega) - 1 \},$$

and the language $L$ associated to $B$ by,

$$L = 1 + \mathbb{A} + L_B.$$ 

(3)

We shall call an $L$ of this form a linked language. Define the K-dual $L^!$ to be the language associated with the complement set of links

$$L^! = 1 + \mathbb{A} + L_{B^c}$$

For linked languages we define a second formal power series,

$$L^g = \sum_{\omega \in L} (-1)^{\ell(\omega)} \omega.$$ 

We call it the graded generating function of $L$. We have the following inversion formula for linked languages. It is a non-commutative version of Theorem 4.1. in Gessel PhD thesis, [8], from where we borrow the terminology of linked sets. Propositions 1 and 2 are indeed particular instances of inversion formulas on generating functions for Koszul algebras and Koszul modules over Koszul algebras. Koszul algebras were introduced in [17], see also [16] for more details on Koszul algebras and the inversion formulas for generating functions of Koszul algebras and modules.

**Proposition 1** Let $L$ be a linked language, and $L^!$ its K-dual. Then we have

$$L^! = (L^g)^{-1}.$$ 

(4)

**Proof** The product $L^g . L^!$ is equal to

$$L^g . L^! = \sum_{(\omega, \omega') \in L \times L^!} (-1)^{\ell(\omega')} \omega . \omega'.$$ 

(5)
Define the function $\phi : L \times L^! \rightarrow L \times L^!$, 
\[
\phi \left( \omega_1 \omega_2 \ldots \omega_k, \omega_1' \omega_2' \ldots \omega_j' \right) = \left( \omega_1 \omega_2 \ldots \omega_k \omega_1', \omega_2' \ldots \omega_j' \right)
\]
when $(\omega_k, \omega_1') \in B$ or if $\omega = 1$ and $\omega' \neq 1$. Make 
\[
\phi \left( \omega_1 \omega_2 \ldots \omega_k, \omega_1' \omega_2' \ldots \omega_j' \right) = \left( \omega_1 \omega_2 \ldots \omega_k \omega_1', \omega_2' \ldots \omega_j' \right)
\]
if $(\omega_k, \omega_1') \in B^c$ or if $\omega' = 1$ and $\omega \neq 1$. Finally, make 
\[
\phi(1, 1) = (1, 1).
\]
Hence $L^! L^g = 1.1 = 1$. \qed

**Example 1** Let $X_+ = \{X_1, X_2, X_3, \ldots\}$. Denote by $\mathcal{P}$ the set of partitions $\lambda$ written in weak increasing order, $\lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$. The set $\mathcal{P}$ is represented as a language with letters in $X_+$, 
\[
\mathcal{P} = \sum_\lambda X_\lambda = \lim_{m \to \infty} \prod_{n=1}^m \frac{1}{1 - X_n} = \prod_{n=1}^\infty \frac{1}{1 - X_n}.
\]
It is a linked language, with set of links $B = \{(X_i, X_j) | i \leq j\}$. The complement is the set $B^c = \{(X_i, X_j) | i > j\}$, and hence the $K$-dual language $\mathcal{P}^!$ is the generating functions of the set of partitions in decreasing order with distinct parts. The graded generating function of $\mathcal{P}$ is equal to 
\[
\mathcal{P}^g = \sum_{\lambda \in \mathcal{P}} (-1)^{\ell(\lambda)} X_\lambda = \lim_{m \to \infty} \prod_{n=1}^m \frac{1}{1 + X_n} = \prod_{n=1}^\infty \frac{1}{1 + X_n}.
\]
By Proposition 1, since taking inverses is a continuous operation, 
\[
\mathcal{P}^! = (\mathcal{P}^g)^{-1} = \lim_{m \to \infty} (1 + X_m)(1 + X_{m-1}) \ldots (1 + X_1).
\]
This limit can be (symbolically) written as the product $\prod_{n=1}^\infty (1 + X_n)$. Since 
\[
(1 + X_m)(1 + X_{m-1}) \ldots (1 + X_1)
\]
\[
= 1 + \sum_{n=1}^m X_n(1 + X_{n-1})(1 + X_{n-2}) \ldots (1 + X_1),
\]
$\mathcal{P}^!$ is then equal to the series 
\[
\mathcal{P}^! = \prod_{n=1}^\infty (1 + X_n) = 1 + \sum_{n=1}^m X_n(1 + X_{n-1})(1 + X_{n-2}) \ldots (1 + X_1).
\]
Analogously, the $K$-dual of the language of partitions written in decreasing order, is the language of partitions with different parts, written in increasing order

$$\left(\prod_{n=\infty}^{1} \frac{1}{1-X_n}\right)!' = \left(1 + \sum_{n=1}^{\infty} X_n \prod_{j=n}^{1} \frac{1}{1-X_n}\right)!' = \prod_{n=1}^{\infty}(1 + X_n)$$ (6)

Let $L$ be a linked language $L = 1 + \mathbb{A} + L_B$. Given a subset $\mathbb{A}_1$ of $\mathbb{A}$, and another set of links $C \subseteq \mathbb{A}_1 \times \mathbb{A}$, let $N \subseteq \mathbb{A}_1 \times \mathbb{A}^*$, be the language defined by

$$N = \mathbb{A}_1 + L_{C,B},$$

where

$$L_{C,B} = \{\omega | (\omega_1, \omega_2) \in C \text{ and } (\omega_i, \omega_{i+1}) \in B, \ i = 2, 3, \ldots, \ell(\omega) - 1\}.$$

The language $N$ will be called a right (linked) $L$-module. Denote by $N!'$ (called the $K$-dual of $N$) the $L'$-module defined by

$$N'! = \mathbb{A}_1 + L_{C',B'},$$

where the complement of $C$ is taken over the set $\mathbb{A}_1 \times \mathbb{A}$, $C' = \mathbb{A}_1 \times \mathbb{A} - C$.

**Proposition 2** The generating function for the language $N'$ defined as above is given by the formula

$$N'! = N^g L'! = N^g (L^g)'^{-1},$$ (7)

where the graded generating function $N^g$ is defined as

$$N^g = \sum_{\omega \in N} (-1)^{\ell(\omega)-1} \omega.$$ (8)

**Proof** We have to prove that $N^g L'! = \sum_{\omega \in N'} \omega$. We have

$$N^g L'! = \sum_{(\omega, \omega') \in N \times L'} (-1)^{\ell(\omega)-1} \omega \omega'.$$

Define the function $\psi : N \times L' \rightarrow N \times L'$ by considering the following cases. Assume first that $\ell(\omega) \geq 2$. If $(\omega_k, \omega'_j) \in B$ we make

$$\psi \left(\omega_1 \omega_2 \ldots \omega_k, \omega_1' \omega_2' \ldots \omega'_j\right) = \left(\omega_1 \omega_2 \ldots \omega_k \omega_1', \omega_2' \ldots \omega'_j\right).$$

If $(\omega_k, \omega'_j) \in B^c$ or if $\omega' = 1$, define

$$\psi \left(\omega_1 \omega_2 \ldots \omega_k, \omega_1' \omega_2' \ldots \omega'_j\right) = \left(\omega_1 \omega_2 \ldots \omega_{k-1}, \omega_k \omega_1', \omega_2' \ldots \omega'_j\right).$$

\[\square\]
Assume now that $\ell(\omega) = 1$. If $\omega_1, \omega'_1 \in \mathcal{C}$ define
\[
\psi(\omega_1, \omega'_1, \omega'_2, \ldots, \omega'_j) = (\omega_1 \omega'_1, \omega'_2, \omega'_3, \ldots, \omega'_j).
\]
Otherwise, if $(\omega_1, \omega'_1) \in \mathcal{C}_c$ or if $\omega' = 1$ we make $\psi(\omega_1, \omega') = \psi(\omega_1, \omega'_1)$. The function $\psi$ is a sign reversing involution, its fixed points being of the form $(\omega_1, \omega'_1)$, if either $(\omega_1, \omega'_1) \in \mathcal{C}_c$ or $\omega' = 1$. Then
\[
N^g L^1 = \sum_{(\omega_1, 1) \in \mathcal{A}_1 \times 1} \omega_1 1 + \sum_{(\omega_1, \omega') : (\omega_1, \omega') \in \mathcal{C}_c, \omega' \in L^1 - \{1\}} \omega_1 \omega' = \mathcal{A}_1 + L_{C_c,B_c} = N^g.
\]

3 Shift and the shift-plethystic trees language

Consider the algebra $\mathbb{K}((\mathcal{X}))$, $\mathcal{X}$ being the alphabet
\[
\mathcal{X} = \{X_0, X_1, X_2, \ldots\}.
\]
Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_m)\in \mathbb{N}^m$ (a weak composition). We denote by $X_\kappa$ the word $X_{\kappa_1} X_{\kappa_2} \ldots X_{\kappa_m}$. As usual, the empty word will be denoted by 1. We denote by $|\kappa|$ the sum of its parts,
\[
|\kappa| = \kappa_1 + \kappa_2 + \cdots.
\]
Let $R$ be an element of $\mathbb{K}((\mathcal{X}))$. The series $R$ is written as
\[
R = \sum_{\kappa \in \mathbb{N}^*} \langle R, X_\kappa \rangle X_\kappa.
\]

Remark 1 Let $\mathcal{S}$ be set of weak compositions. In the rest of the article, when no risk of confusion, we identify $\mathcal{S}$ with the associated language $\{X_\kappa | \kappa \in \mathcal{S}\}$, and its generating series $\sum_{\kappa \in \mathcal{S}} X_\kappa$.

We shall call $\kappa$ a (strong) composition if $\kappa_i \neq 0$, for every $i$. In what follows, word ‘composition’ will mean by defect strong composition.

Definition 2 Define $\sigma : \mathbb{K}((\mathcal{X})) \rightarrow \mathbb{K}((\mathcal{X}))$ by extending the shift $\sigma X_i = X_{i+1}$, $i = 0, 1, 2, \ldots$, as a continuous algebra map. That is, by making it multiplicative and to commute with the series sum symbol.

Definition 3 A shift-plethystic (SP) tree is a plane rooted tree whose vertices are colored with colors in $\mathbb{N}$. The color of a given vertex indicates its height (length of the path from the root).
Let $T$ be a shift-plethystic tree. We associate to $T$ the word $\omega_T$ on $X$, obtained by reading the vertices of $T$ in preorder from left to right as follows. Assume that $T$ consists of $k \geq 0$ sub-trees, $T_1, T_2, \ldots, T_k$, attached to the root (of color 0). The preorder word of $T$ is then defined recursively by

$$\omega_T = \begin{cases} X_0 & \text{if } k = 0, \\ X_0 \sigma \omega_{T_1} \sigma \omega_{T_2} \cdots \sigma \omega_{T_k} & \text{if } k > 0. \end{cases} \quad (9)$$

We denote by $\mathcal{A}$ the language of shift-plethystic trees,

$$\mathcal{A} = \sum_T \omega_T. \quad (10)$$

It is easy to check that the tree $T$ is uniquely obtained from its word $\omega_T$. The series $\sigma \mathcal{A}$ gives us the language of shift-plethystic trees with the root colored with color 1, and every vertex colored with its height plus 1. Similarly, $\sigma^n \mathcal{A}$ is the language of shift-plethystic trees, the root colored $n$ and each vertex colored $n$ plus its height.

**Theorem 4** The language $\mathcal{A}$ can be expanded as the non-commutative continued fraction

$$\mathcal{A} = X_0 \frac{1}{1 - X_1 \frac{1}{1 - X_2 \frac{1}{\ddots}}} = \lim_{n \to \infty} X_0 \frac{1}{1 - X_1 \frac{1}{1 - X_2 \frac{1}{\ddots - X_{n-1} \frac{1}{1 - X_n}}}} \quad (11)$$

**Proof** Assume that the root of an SP tree has $k$ children, $k \geq 0$. By the definition of $\omega_T$ (Eq. 9), to read its colors we read first the root and then read in preorder from left to right the colors of each one (or none) of the $k$ trees above the root. Each of them will produce a word in $\sigma \mathcal{A}$. Hence we have the identity

$$\mathcal{A} = X_0 (1 + \sigma \mathcal{A} + (\sigma \mathcal{A})^2 + (\sigma \mathcal{A})^3 + \cdots) = X_0 \frac{1}{1 - \sigma \mathcal{A}}. \quad (12)$$

Applying $\sigma^{j-1}$, $j = 1, 2, \ldots$, to both sides of the above identity we get

$$\sigma^{j-1} \mathcal{A} = X_{j-1} \frac{1}{1 - \sigma^j \mathcal{A}}.$$
Recursively from Eq. (12), we obtain

\[ A = X_0 \frac{1}{1 - X_1} \frac{1}{1 - X_2} \cdots \frac{1}{1 - X_n} \frac{1}{1 - \sigma^n A}. \]  

(13)

Denote by \( A_n \) the language \( A \) restricted to the symbols \( \{ X_0, X_1, \ldots, X_n \} \) (the words of SP trees of height at most \( n \)). We have that \( \lim_{n \to \infty} A_n = A \) and since \( \sigma_n A_n = X_n \), from Eq. (13) we obtain the result.

\[ \square \]

**Proposition 3** The words coming from shift-plethystic trees are completely characterized by the following properties:

1. Its first letter is \( X_0 \).
2. If \( \ell(\omega_T) > 1 \) it is followed by a word of the form \( X_\kappa \), \( \kappa \) being a composition with first element equal to 1 and risings at most 1, \( \kappa_{i+1} - \kappa_i \leq 1 \).

**Proof** Easy from Eq. (9), by induction on the number of vertices.

\[ \square \]

**Definition 5** We denote by \( \mathcal{C} \) the language of compositions, and by \( \mathcal{C}^{(1)} \) language of compositions with risings at most 1. More generally we define \( \mathcal{C}^{(m)} \) to be the language of compositions with rising at most \( m \). Observe that \( \sigma \mathcal{C}^{(1)} \) consists of the compositions in \( \mathcal{C}^{(1)} \), but where every part is at least 2.

All the languages \( \mathcal{C} \) and \( \mathcal{C}^{(m)} \), \( m \geq 1 \), include the empty word. Proposition 3 can be now restated as follows, in terms of generating functions.

**Proposition 4** The language \( A \) can be expanded as

\[ A = X_0 \left( 1 + \sum_{\kappa \in \mathcal{C}^{(1)}, \kappa_1 = 1} X_\kappa \right). \]  

(14)

Define the alphabets

\[ \mathbb{X}_+ = \{ X_1, X_2, X_3, \ldots \} \quad \text{and} \quad \mathbb{X}_{+2} = \{ X_2, X_3, \ldots \}. \]

**Definition 6** We denote by \( N \) the \( \mathcal{C}^{(1)} \)-module of compositions \( \kappa \) such that \( \kappa_1 \geq 2 \).

\[ N = \sum_{\kappa \in \mathcal{C}^{(1)}, \kappa_1 \geq 2} X_\kappa, \]  

(15)

The languages \( \mathcal{C}^{(1)} \) and \( \sigma \mathcal{C}^{(1)} \) are both linked. The language \( \mathcal{C}^{(1)} \subset \mathbb{X}_+^* \) with set of links

\[ B = \{(X_i, X_j)|j - i \leq 1\} \subset \mathbb{X}_+ \times \mathbb{X}_+, \]

\[ \square \]
and \( \sigma C^{(1)} \subset X_{2+}^* \) with the shifted set of links

\[
\sigma B = \{(X_i, X_j) | j - i \leq 1, \ i, j \geq 2\} \subset X_{2+} \times X_{2+}.
\]

The \( C^{(1)} \)-module \( N \) has as set of links \( C \subset X_2 \times X_2^+ \),

\[
C = \{(X_i, X_j) | j - i \leq 1, \ i \geq 2\} \subset X_{2+} \times X_{2+}.
\]

**Definition 7** We denote by \( \mathcal{P}_m \) the language of \( m \)-distinct partitions (in the terminology of [2]). Being more explicit, \( \mathcal{P}_m \) is the language of words of the form \( X_\lambda \), \( \lambda \) being a partition (written in increasing order), \( 1 \leq \lambda_1 < \lambda_2 < \lambda_3, \ldots \), satisfying

\[
\lambda_{i+1} - \lambda_i \geq m,
\]

(16)

(the empty and the singleton words being included in \( \mathcal{P}_m \)). In particular we have that \( \mathcal{P}_0 = \mathcal{P} \) is the language of partitions with repetitions, \( \mathcal{P}_1 \), that of partitions without repetitions, and finally \( \mathcal{P}_2 \) is the language of 2-distinct partitions, directly related to the combinatorics of the Rogers–Ramanujan identities. Observe that \( \sigma \mathcal{P}_2 \) is the language of 2-distinct partitions, where each part is at least 2.

**Proposition 5** We have that \( \mathcal{P}_2 \) is the \( K \)-dual of \( \mathcal{C}^{(1)} \), \( \sigma \mathcal{P}_2 \) is the \( K \)-dual of \( \sigma \mathcal{C}^{(1)} \). The \( K \)-dual of \( N \) is the \( \mathcal{P}_2 \)-module \( \mathcal{N} \) of 2-distinct partitions with first part greater than 2,

\[
\mathcal{P}_2 = (\mathcal{C}^{(1)})!, \quad \sigma \mathcal{P}_2 = (\sigma \mathcal{C}^{(1)})!, \quad \mathcal{N} = \mathcal{N}!. \]

**Proof** Easy, by simple inspection. \( \square \)

**Remark 2** Proposition 5 can be easily generalized to obtain the duality \( \mathcal{P}_m = (\mathcal{C}^{(m-1)})! \) for arbitrary \( m \geq 1 \). This duality leads to \( m \)-generalization of the the Rogers–Ramanujan continued fraction, related to \( m \)-distinct partitions. These results will appear in a forthcoming paper.

Observe that since \( \lambda \) is a partition, if \( \lambda_1 > 1 \), the rest of parts have also to be greater than 1, and the series \( \mathcal{N} \) equals the non-constant part of \( \sigma \mathcal{P}_2 \),

\[
\mathcal{N} = \sigma \mathcal{P}_2 - 1.
\]

Their graded generating functions are related as follows:

\[
\mathcal{P}_2^g = 1 + \sum_{\lambda_{i+1} - \lambda_i \geq 2} (-1)^{\ell(\lambda)} X_\lambda = 1 - \sum_{\lambda_{i+1} - \lambda_i \geq 2} (-1)^{\ell(\lambda)-1} X_\lambda = 1 - \mathcal{N}^g. \quad (17)
\]

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Theorem 8  The language $\mathcal{A}$ can be expressed as the product

$$\mathcal{A} = X_0 \left( \sigma \mathcal{P}_2^g \right) \left( \mathcal{P}_2^g \right)^{-1}. \quad (18)$$

Proof  By Eq. (14) we have

$$\mathcal{A} = X_0 \left( \mathcal{C}(1) - N \right).$$

Since the operation of taking duals is involutive, $\mathcal{P}_2^l = \mathcal{C}(1)$, $(\sigma \mathcal{P}_2^g)^! = \sigma \mathcal{C}(1)$, and $N^! = N$. By Eq. (17), Propositions 2 and 1,

$$\mathcal{C}(1) - N = \left( \mathcal{P}_2^g \right)^{-1} - \left( \mathcal{N}^g \right)^{-1} = (1 - \mathcal{N}^g) \left( \mathcal{P}_2^g \right)^{-1} = \left( \sigma \mathcal{P}_2^g \right) \left( \mathcal{P}_2^g \right)^{-1}. \quad \Box$$

Equation (18) can be written more explicitly as the product

$$\mathcal{A} = X_0 \left( \sigma \mathcal{P}_2^g \right) \left( \mathcal{P}_2^g \right)^{-1} = X_0 \left( 1 + \sum_{\lambda_{i+1} - \lambda_i \geq 2, \lambda_i \geq 2} (-1)^{\ell(\lambda)} X_\lambda \right)^{-1} \left( 1 + \sum_{\lambda_{i+1} - \lambda_i \geq 2, \lambda_i \geq 1} (-1)^{\ell(\lambda)} X_\lambda \right) \quad (19)$$

Since $(\mathcal{C}(1))^{-1} = \mathcal{P}_2^g$, and $(\sigma \mathcal{C}(1))^{-1} = \sigma \mathcal{P}_2^g$, Theorem 8 has the following dual form

Corollary 9  The language $\mathcal{A}$ can be written as the product

$$\mathcal{A} = X_0 (\sigma \mathcal{C}(1))^{-1} \mathcal{C}(1) = X_0 \left( 1 + \sum_{\kappa_{i+1} - \kappa_i \leq 1, \kappa_i \geq 1} X_\kappa \right)^{-1} \left( 1 + \sum_{\kappa_{i+1} - \kappa_i \leq 1, \kappa_i \geq 2} X_\kappa \right). \quad (20)$$

4 Path length and $q$-series

For a series $S$ on the alphabet $X$, making the $q$-umbral evaluation $X_k \to zq^k$ in the pair of commuting variables $z$ and $q$, we obtain a $q$-series that by abuse of language we denote with the same symbol $S$, $S(z, q)$. Observe that for every series $S$ in $K \langle \langle X \rangle \rangle$ we have

$$(\sigma S)(z, q) = S(zq, q). \quad (21)$$

Recall that the path length of a rooted tree is defined to be the sum of the heights of its vertices. When we make the substitution $X_k \to zq^k$ in the word $\omega_T$ associated to a tree $T$ we get $z^n q^{pl(T)}$, where $n$ is the number of vertices of $T$ and $pl(T)$ its path length. For example, in the tree of Fig. 1, the $q$-substitution in the word $X_0 X_1 X_2 X_2 X_1 X_2 X_2 X_2 X_3$ gives us

$$X_0 X_1 X_2 X_2 X_1 X_2 X_2 X_2 X_3 \mapsto z(zq)(zq^2)^2(zq)(zq^2)^3(zq^3) = z^9 q^{15}. \quad \square$$
Then, the $q$-series $\mathcal{A}(z, q)$ counts the number of plane rooted trees according with their path length. Observe that the path length of a plane rooted tree with $n$ vertices is bounded by the path length of the branchless tree, which is equal to $0 + 1 + 2 + 3 + \cdots + n - 1 = \binom{n}{2}$. From Eq. (13) we get

$$
\mathcal{A}(z, q) = \sum_{n=1}^{\infty} \left( \sum_{m=0}^{\left(\binom{n}{2}\right)} P(n, m) q^m \right) z^n = \frac{z}{1 - \frac{z q}{1 - \frac{z q^2}{1 - \frac{z q^3}{1 - \cdots}}}}, \tag{22}
$$

where $P(n, m)$ is the number of plane rooted tree on $n$ vertices having path length equal to $m$,

$$
\mathcal{A}(z, q) = z + qz^2 + (q^2 + q^3)z^3 + (q^3 + 2q^4 + q^5 + q^6)z^4
+ (q^4 + 3q^5 + 3q^6 + 3q^7 + 2q^8 + q^9 + q^{10})z^5
+ (q^5 + 4q^6 + 6q^7 + 7q^8 + 7q^9 + 5q^{10})z^6
+ 5q^{11} + 3q^{12} + 2q^{13} + q^{14} + q^{15})z^6 + \cdots.
$$

From Eq. (19),

$$
\mathcal{A}(z, q) = z \frac{P_2(zq, z)}{P_2(z, q)} = z \frac{1 + \sum_{\lambda \in \sigma} P_2(-z) q^{\ell(\lambda)} |\lambda|}{1 + \sum_{\lambda \in \sigma} P_2(-z) q^{\ell(\lambda)} |\lambda|}. \tag{23}
$$

From Eq. (20) we obtain the dual expression

$$
\mathcal{A}(z, q) = z \frac{C^{(1)}(z, q)}{C^{(1)}(zq, q)} = z \frac{1 + \sum_{\kappa \in \sigma} C^{(1)}(-z) q^{\ell(\kappa)} |\kappa|}{1 + \sum_{\kappa \in \sigma} C^{(1)}(-z) q^{\ell(\kappa)} |\kappa|}. \tag{24}
$$

### 5 Rogers–Ramanujan identities and compositions

**Theorem 10** We have the following identities:

$$(C^{(1)})(-1, q) = (C^{(1)})^g(1, q)$$

\[ \mathcal{A} \] Springer
\begin{equation}
1 + \sum_{k \in \mathcal{C}^{(-1)}} (-1)^{\ell(k)} q^{k|} = \prod_{k=0}^{\infty} (1 - q^{5k+1})(1 - q^{5k+4}), \quad (25)
\end{equation}

\begin{equation}
(\sigma \mathcal{C}^{(1)})^{-1}(1, q) = (\sigma \mathcal{C}^{(1)}) \mathcal{C}(1, q)
\end{equation}

\begin{equation}
1 + \sum_{k \in \mathcal{C}^{(2)}} (-1)^{\ell(k)} q^{k|} = \prod_{k=0}^{\infty} (1 - q^{5k+2})(1 - q^{5k+3}). \quad (26)
\end{equation}

**Proof** From Proposition 1, \((\mathcal{C}^{(1)}) \mathcal{C} = (\mathcal{P}_2)^{-1}\) and \((\sigma \mathcal{C}^{(1)}) \mathcal{C} = (\sigma \mathcal{P}_2)^{-1}\). Then, \(q\)-umbral evaluation gives us

\begin{equation}
(\mathcal{C}^{(1)}) \mathcal{C}(z, q) = \sum_{k \in \mathcal{C}^{(1)}} (-1)^{\ell(k)} z^{\ell(k)} q^{k|} = \frac{1}{1 + \sum_{\lambda \in \mathcal{P}_2} z^{\ell(\lambda)} q^{k|}},
\end{equation}

\begin{equation}
(\sigma \mathcal{C}^{(1)}) \mathcal{C}(z, q) = \sum_{k \in \sigma \mathcal{C}^{(1)}} (-1)^{\ell(k)} z^{\ell(k)} q^{k|} = \frac{1}{\sum_{\lambda \in \sigma \mathcal{P}_2} z^{\ell(\lambda)} q^{k|}}.
\end{equation}

By the well-known identities

\begin{equation}
\sum_{\lambda \in \mathcal{P}_2} z^{\ell(\lambda)} q^{k|} = \sum_{n=0}^{\infty} \frac{z^n q^{n^2}}{(1 - q)(1 - q^2) \cdots (1 - q^n)},
\end{equation}

\begin{equation}
\sum_{\lambda \in \sigma \mathcal{P}_2} z^{\ell(\lambda)} q^{k|} = \sum_{n=0}^{\infty} \frac{z^n q^{n(n+1)}}{(1 - q)(1 - q^2) \cdots (1 - q^n)},
\end{equation}

using the Rogers–Ramanujan identities (Eqs. 2 and 1) we get the result. \(\square\)

Let \(\mathcal{C}^{(1)}[n]\) and \(\mathcal{C}^{(1)}[n, k]\), respectively, be the set of compositions of \(n\) in \(\mathcal{C}^{(1)}\), and the set of compositions of \(n\) in \(\mathcal{C}^{(1)}\) having exactly \(k\) parts. Similarly, define \(\sigma \mathcal{C}^{(1)}[n]\) and \(\sigma \mathcal{C}^{(1)}[n, k]\). From Theorem 10, we get the identities

\begin{equation}
1 + \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} (-1)^k |\mathcal{C}^{(1)}[n, k]| \right) q^n = \prod_{k=0}^{\infty} (1 - q^{5k+1})(1 - q^{5k+4}), \quad (27)
\end{equation}

\begin{equation}
1 + \sum_{n=2}^{\infty} \left( \sum_{k=1}^{n} (-1)^k |\sigma \mathcal{C}^{(1)}[n, k]| \right) q^n = \prod_{k=0}^{\infty} (1 - q^{5k+2})(1 - q^{5k+3}). \quad (28)
\end{equation}

Observe that the series in the right-hand side of equation (27) gives us the partitions (in decreasing order) with distinct parts congruent with 1 or 4 module 5, signed by its number of parts. The right-hand side of (28) enumerates a similar kind of signed partitions, each part congruent with 2 or 3 module 5. For example 4 + 1 is the only partition of 5 enumerated by the right-hand side of Eq. (27), and \(7 + 3, 8 + 2\) are the only partitions of 10 enumerated by the right-hand side of Eq. (28). The compositions in \(\sigma \mathcal{C}^{(1)}(10)\) and in \(\sigma \mathcal{C}^{(1)}(11)\) are given, respectively, in Tables 1 and 2.
### Table 1  Compositions in $\sigma \mathcal{E}^{(1)}[10]$

| $k$ | Weight |
|-----|---------|
| 1   | 10      | $-1$   |
| 2   | 55      | 64  | $73$ | $82$ | 2 |
| 3   | 532     | 523  | 622  | 442  | 433  | 343  | 334  | 334  | 7 |
| 4   | 2233    | 2323  | 3223  | 2332  | 3232  | 3322  | 4222  | 4222  | 7 |
| 5   | 22222   |       |       |       |       |       |       |       | $-1$ |

### Table 2  Compositions in $\sigma \mathcal{E}^{(1)}[11]$

| $k$ | Weight |
|-----|---------|
| 1   | 11      | $-1$ |
| 2   | 56      | 65  | 74  | $83$ | 92  | 4   |
| 3   | 722     | 632  | 623  | 542  | 533  | 452  | 443  | $-9$ |
|     | 434     | 344  |       |       |       |       |       |       |       |
| 4   | 522     | 4322  | 4232  | 4223  | 3422  | 3332  | 3323  | 11  |
|     | 3233    | 2333  | 2342  | 2234   |       |       |       |       |
| 5   | 32222   | 23222  | 22322  | 22232  | 22223  |       |       | $-5$ |

Consider the set $\widehat{\mathcal{E}}^{(1)}[n, k]$ of restricted compositions. This set only excludes from $\mathcal{E}^{(1)}[n, k]$ the strictly decreasing compositions with all its parts congruent with 1 or 4 module 5. In a similar way, we define $\sigma \widehat{\mathcal{E}}^{(1)}[n, k]$.

Then, the Rogers–Ramanujan identities have the following combinatorial form in terms of compositions.

**Theorem 11** The signed sets $\widehat{\mathcal{E}}^{(1)}[n]$ and $\sigma \widehat{\mathcal{E}}^{(1)}[n]$ have both zero total weight

\[
\sum_{k=1}^{n} (-1)^k |\widehat{\mathcal{E}}^{(1)}[n, k]| = 0, \quad \text{for } n \geq 1 \tag{29}
\]

\[
\sum_{k=1}^{n} (-1)^k |\sigma \widehat{\mathcal{E}}^{(1)}(n, k)| = 0, \quad \text{for } n \geq 2. \tag{30}
\]

### 6 Shift-plethysm, and general shift-plethystic trees

**Definition 12** Let $R$ be a series in $\mathbb{K}[\langle X \rangle]$ with zero constant term, $\langle R, 1 \rangle = 0$. We define the *shift-plethystic substitution* of $R$ in a word $X_k = X_{k_1}X_{k_2}X_{k_3} \cdots X_{k_l}$, as the substitution of the shift $\sigma^{k_1}R$ on each of the letters of $X_k$,

\[X_k \circ_s R := (\sigma^{k_1} R)(\sigma^{k_2} R) \cdots (\sigma^{k_l} R).\]
For a formal power series $T$, define the shift-plethysm $T \circ_s R$ by

$$T \circ_s R = \sum_{\kappa \in \mathbb{N}^*} \langle T, X_{\kappa} \rangle X_{\kappa} \circ_s R = \sum_{\kappa \in \mathbb{N}^*} \langle T, X_{\kappa} \rangle \sigma^{\kappa_1} R (\sigma^{\kappa_2} R) \cdots (\sigma^{\kappa_l} R).$$  \hspace{1cm} (31)

The series in the right-hand side of Eq. (31) is convergent. To see this, let us denote by $R_{\kappa}$ the product $(\sigma^{\kappa_1} R) (\sigma^{\kappa_2} R) \cdots (\sigma^{\kappa_l} R)$. We see that $\langle R_{\kappa}, X_{\tau} \rangle = 0$ whenever either $l = \ell(\kappa) > |\tau|$ or $|\kappa| > |\tau|$. Hence the set $\{ \tau | \langle R_{\kappa}, X_{\tau} \rangle \neq 0 \}$ is finite. Shift-plethysm is an associative operation having $X_0$ as identity.

**Proposition 6** Every series $R$ with zero constant term and such that $\langle R, X_0 \rangle \neq 0$ has a two-sided shift-plethystic inverse, denoted $R^{(\bot)}$, $R \circ_s R^{(\bot)} = R^{(\bot)} \circ_s R = X_0$.

**Proof** Let $\alpha \neq 0$ be the value of $R$ at $X_0$, $\langle R, X_0 \rangle = \alpha \neq 0$. Define $R_+ = R - \alpha X_0$, and the series $T$ by the implicit equation

$$T = \alpha^{-1} (X_0 - R_+ \circ_s T).$$

From here we get $\alpha T + R_+ \circ_s T = X_0$. Which can be written as $(\alpha X_0 + R_+) \circ_s T = R \circ_s T = X_0$. Then, $R^{(\bot)} = T$. \hfill $\square$

### 6.1 Shift-plethysm and $q$-composition of series

In this subsection we show how shift-plethysm generalizes the classical definition of $q$-composition of series. This is relevant due to the importance of the $q$-Lagrange inversions formulas for applications in proving identities in $q$-series (see [1,5,7,9,10,12]). A general shift-plethystic Lagrange inversion, not yet found, would lead to new forms of $q$-Lagrange inversion as well as to the reinterpretation in a general context of the known ones. Shift-plethysm also offers the advantage, in contrast to $q$-substitution, of being an associative operation. From that, the plethystic inverse is a bilateral one, also in contrast to the known forms of $q$-composition inverse.

Let $C$ be the series

$$C = \sum_{n=1}^{\infty} c_n X_0 X_1 X_2 \cdots X_{n-1}.$$  

Consider the shift-plethysm $H := C \circ_s R$, $R$ being an arbitrary series with zero constant term. We have

$$H = C \circ_s R = \sum_{n=1}^{\infty} c_n R (\sigma R) (\sigma^2 R) \cdots (\sigma^{n-1} R).$$  \hspace{1cm} (32)
Taking \( q \)-series, by Eq. (21) we recover the classical \( q \)-substitution,

\[
H(z, q) = (C \circ R)(z, q) = \sum_{n=1}^{\infty} c_n R(z, q) R(zq, q) R(zq^2, q) \cdots R(zq^{n-1}, q).
\]  

(33)

Now consider \( R \) to be a series in the variable \( X_0 \), and express it in the form \( R(X_0) = X_0 \phi^{-1}(X_0) \). Shift-plethysm with \( C \) will give us

\[
H = \sum_{n=1}^{\infty} c_n \frac{X_0}{\phi(X_0)} \frac{X_1}{\phi(X_1)} \cdots \frac{X_{n-1}}{\phi(X_{n-1})},
\]

(34)

which, by \( q \)-umbral evaluation, gives

\[
H(z, q) = \sum_{n=1}^{\infty} c_n \frac{q^{\binom{n}{2}} z^n}{\phi(z) \phi(zq) \cdots \phi(zq^{n-1})}.
\]

(35)

Obtaining \( c_n \) in terms of the \( h_n \) in the expansion of \( H(z, q) \) is similar to the \( q \)-Lagrange inversion problem in [1].

6.2 Enriched shift-plethystic trees

In this section we introduce the \( M \)-enriched shift-plethystic trees, \( M \) being a normalized invertible (non-commutative) series, based on the similar notion formalized by Joyal in [11] and its plethystic generalization in the commutative framework of colored species [14].

**Definition 13** Let \( M \) be a series with constant term equal to 1, \( \langle M, 1 \rangle = 1 \). We define the \( M \)-enriched trees series by the implicit equation

\[
\mathcal{A}_M = X_0 (M \circ_s \mathcal{A}_M).
\]

(36)

**Proposition 7** The shift-plethystic inverse of \( \mathcal{A}_M \) is given by the formula

\[
(\mathcal{A}_M)^{(-1)} = X_0 M^{-1}.
\]

**Proof** From Eq. (36) we have \( \mathcal{A}_M (M^{-1} \circ_s \mathcal{A}_M) = (X_0 M^{-1}) \circ_s \mathcal{A}_M = X_0. \)

The plethystic inversion for enriched trees, in the most elementary examples where the shift-plethystic inverse can be easily computed, leads by \( q \)-umbral evaluation to generalization of some classical formulas.

**Example 14** From Eq. (12), the SP trees \( \mathcal{A} \) satisfy the implicit equation

\[
\mathcal{A} = X_0 \left( \frac{1}{1 - X_1 \circ_s \mathcal{A}} \right).
\]

(37)

\( \square \) Springer
Hence, they are obtained by enriching with the series \( \frac{1}{1-X_1} \), \( \mathcal{A} = \mathcal{A} \frac{1}{1-X_1} \). From this we get

\[
\mathcal{A}(1 - \sigma \mathcal{A}) = \mathcal{A} - \mathcal{A} \sigma \mathcal{A} = (X_0 - X_0X_1) \circ_s \mathcal{A} = X_0,
\]

its shift-plethystic inverse

\[
\mathcal{A}^{-1} = X_0 - X_0X_1,
\]

and the implicit equation

\[
\mathcal{A} = X_0 + (X_0X_1) \circ_s \mathcal{A}.
\]

The \( q \)-series of the SP trees satisfies the implicit equations

\[
\begin{align*}
\mathcal{A}(z, q) &= z + \mathcal{A}(z, q) \mathcal{A}(zq, q), \\
\mathcal{A}(z, q) &= \frac{z}{1 - \mathcal{A}(zq, q)}.
\end{align*}
\]

Those equations were studied by Garsia in [5] in relation with his \( q \)-Lagrange inversion formulas, but without any combinatorial interpretation.

**Example 15** Let \( \mathbb{L} \) be the language

\[
\mathbb{L} = 1 + X_0 + X_0X_1 + X_0X_1X_2 + X_0X_1X_2X_3 + \cdots.
\]

The language of the branchless trees, enriched with \( 1 + X_1 \) is equal to \( \mathbb{L}_+ = \mathbb{L} - 1 \),

\[
\mathbb{L}_+ = \mathcal{A}_{1+X_1} = X_0(1 + \sigma \mathbb{L}_+).
\]

Its shift-plethystic inverse is equal to

\[
\mathbb{L}_+^{-1} = X_0 \frac{1}{1 + X_1}.
\]

Then,

\[
\mathbb{L}_+ \circ_s \left( X_0 \frac{1}{1 + X_1} \right) = \sum_{n=1}^{\infty} \prod_{j=1}^{n} X_{j-1} \frac{1}{1 + X_j} = X_0.
\]

The \( q \)-umbral evaluation gives us

\[
\sum_{n=1}^{\infty} \frac{q^{(n)} z^n}{(1 + zq)(1 + zq^2) \cdots (1 + zq^n)} = z.
\]

From that

\[
\sum_{n=0}^{\infty} \frac{q^{(n)} z^n}{(1 + zq)(1 + zq^2) \cdots (1 + zq^n)} = 1 + z.
\]
Making \( z = 1 \) and \( z = -1 \) we recover, respectively, the classical identities A.1 and A.4 in [21].

**Example 16** Let \( \mathbb{L}_+^{(e)} \) be the even form of \( \mathbb{L}_+ \),

\[
\mathbb{L}_+^{(e)} = \sum_{n=0}^{\infty} X_0 X_2 X_4 \ldots X_{2n-2} = X_0 \left( 1 + \sigma \mathbb{L}_+^{(e)} \right) = \mathcal{A}(1+X_2).
\]

Its shift-plethystic inverse is equal to

\[
(\mathbb{L}_+^{(e)})^{(-1)} = X_0 \frac{1}{1 + X_2}.
\]

Hence we have the identity

\[
\mathbb{L}_+^{(e)} \circ X_0 \frac{1}{1 + X_2} = \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} X_{2j+1} \frac{1}{1 + X_{2j+3}} = X_0.
\]  \( (38) \)

The odd version of \( \mathbb{L}_+ \), \( \mathbb{L}_+^{(o)} \) is equal to the shift \( \sigma \mathbb{L}_+^{(e)} \).

\[
\mathbb{L}_+^{(o)} = \sum_{n=1}^{\infty} X_1 X_3 X_5 \ldots X_{2n-1}.
\]

From Eq. (38), by shifting and adding 1 we obtain

\[
\mathbb{L}_+^{(o)} \circ X_0 \frac{1}{1 + X_2} = 1 + \mathbb{L}_+^{(o)} \circ X_0 \frac{1}{1 + X_2} = 1 + \sum_{n=1}^{\infty} \prod_{j=0}^{n-1} X_{2j+1} \frac{1}{1 + X_{2j+3}} = 1 + X_1.
\]

Multiplying by \( (1 + X_1)^{-1} \) the left of both sides of the rightmost equality

\[
\frac{1}{1 + X_1} + \sum_{n=1}^{\infty} \frac{1}{1 + X_1} \prod_{j=0}^{n-1} X_{2j+1} \frac{1}{1 + X_{2j+3}} = 1.
\]

By \( q \)-umbral evaluation we obtain

\[
\sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(1 + zq)(1 + zq^3) \ldots (1 + zq^{2n+1})} = 1. \quad (39)
\]

Equation (39) generalizes Rogers identity (C) 6 in [19, p. 333], obtained by specializing to \( z = -1 \). See also [21], Formula (A.2).
Example 17  Denote by $\Sigma_0 \mathbb{L}_+$ the language obtained from $\mathbb{L}_+$ by left shift-plethysm with the series $\Sigma_0 = \sum_{j=0}^{\infty} X_j$,

$$\Sigma_0 \mathbb{L}_+ = \sum_{j=0}^{\infty} X_j \circ \mathbb{L}_+ = \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} X_j X_{j+1} \ldots X_{n+j-1}. \quad (40)$$

Since $\Sigma_0 - \sigma \Sigma_0 = X_0$, the shift-plethystic inverse of $\Sigma_0$ is equal to $X_0 - X_1$. Hence

$$(\Sigma_0 \mathbb{L}_+)^{(-1)} = (\Sigma_0 \circ \mathbb{L}_+)^{(-1)} = (\mathbb{L}_+)^{(-1)} \circ_s (X_0 - X_1) = (X_0 - X_1) \frac{1}{1 + (X_0 - X_1)}.$$

By plethystic composition with $\Sigma_0 \mathbb{L}_+$ we obtain the identity

$$\sum_{j=0}^{\infty} \sum_{n=1}^{\infty} \frac{(X_j - X_{j+1})}{1 + (X_{j+1} - X_{j+2})} \ldots \frac{1}{1 + (X_{n+j-1} - X_{n+j})} = X_0.$$

Interchanging sums, by $q$-umbral evaluation,

$$\sum_{n=1}^{\infty} z^n q^{(n)} (1 - q)^n \sum_{j=0}^{\infty} \frac{q^{jn}}{\prod_{k=1}^{n} (1 - zq^{j+k}(1 - q))} = z. \quad (41)$$

Equivalently, making the change $z(1 - q) \mapsto z$,

$$\sum_{n=1}^{\infty} z^n q^{(n)} \sum_{j=0}^{\infty} \frac{q^{jn}}{\prod_{k=1}^{n} (1 - zq^{j+k})} = \frac{z}{1 - q}. \quad (42)$$

Example 18  The shift-plethystic trees enriched with the language $\sigma \mathbb{L}$, $\mathcal{A}_{\sigma \mathbb{L}}$, satisfy the equation

$$\mathcal{A}_{\sigma \mathbb{L}} = X_0 (1 + \sigma \mathcal{A}_{\sigma \mathbb{L}} + (\sigma \mathcal{A}_{\sigma \mathbb{L}})(\sigma^2 \mathcal{A}_{\sigma \mathbb{L}}) + \ldots).$$

Its shift-plethystic inverse is equal to

$$\mathcal{A}_{\sigma \mathbb{L}}^{(-1)} = X_0 (\sigma \mathbb{L})^{-1} = X_0 (1 + X_1 + X_1 X_2 + X_1 X_2 X_3 + \ldots)^{-1}.$$

Example 19  The series of shift-plethystic trees enriched with

$$M = (1 - \sigma \mathbb{L}_+)^{-1} = (1 - (X_1 + X_1 X_2 + X_1 X_2 X_3 + \ldots))^{-1}$$

satisfies the implicit equations

$$\mathcal{A}_{M} = X_0 \frac{1}{1 - \sigma \mathbb{L}_+ \circ_s \mathcal{A}_{M}},$$
$$\mathcal{A}_M = X_0 + (\mathcal{A}_M)(\sigma \mathcal{A}_M) + (\mathcal{A}_M)(\sigma^2 \mathcal{A}_M) + \cdots.$$ 

Its shift-plethystic inverse is equal to

$$\mathcal{A}_M^{-1} = X_0 - X_0\mathcal{L}.$$ 

Taking $q$-series we obtain the implicit equation

$$\mathcal{A}_M(z, q) = z + \mathcal{A}_M(z, q)\mathcal{A}_M(zq, q) + \mathcal{A}_M(z, q)\mathcal{A}_M(zq, q)\mathcal{A}_M(zq^2, q) + \cdots.$$ 

7 Some other shift-plethystic identities

In this section we establish some relations between the languages of partitions, compositions, and shifted plethystic trees. As a motivating example for Theorem 20, let us take the following composition in $C(1)_1$, 

$$\kappa = 56763454343342332.$$ 

Placing a bar before each local (non-strict) minimum of the sequence, 

$$\mid 5676 \mid 3454 \mid 34 \mid 3 \mid 34 \mid 233 \mid 2.$$ 

We see that the local minima form a partition in weakly decreasing form $\lambda = 5333322$. Each word between two bars is associated to a word of a shifted plethystic tree,

$$X_{5676}X_{3454}X_3X_{34}X_{233}X_2$$

is in the language $(\sigma^5 \mathcal{A})(\sigma^3 \mathcal{A})(\sigma^3 \mathcal{A})(\sigma^3 \mathcal{A})(\sigma^2 \mathcal{A})(\sigma^2 \mathcal{A}) = X_\lambda \circ_2 \mathcal{Z}.$

**Theorem 20** We have the following identities:

$$C(1)_1 = \prod_{n=\infty}^{1} \frac{1}{1 - X_n} \circ_5 \mathcal{A} = \prod_{n=\infty}^{1} \frac{1}{1 - \sigma^n \mathcal{A}}. \quad (43)$$

$$\sigma C(1)_1 = \prod_{n=\infty}^{2} \frac{1}{1 - X_n} \circ_5 \mathcal{A} = \prod_{n=\infty}^{2} \frac{1}{1 - \sigma^n \mathcal{A}}. \quad (44)$$

**Proof** Let $\kappa$ be a composition in $C(1)_1$. Define $i_1 = 1$ and $\lambda_1 = \kappa_1$, and while the set $A_{r-1} = \{i > i_{r-1} | \kappa_i \leq \kappa_{i_{r-1}} = \lambda_{r-1}\}$ is non-empty define recursively

$$i_r = \min A_{r-1} \quad \text{and} \quad \lambda_r = \kappa_{i_r}.$$ 

Let $\kappa^{(r)}$ be the segment of $\kappa$ after (and including) $\lambda_r = \kappa_{i_r}$ and before (and excluding) $\kappa_{i_{r+1}} = \lambda_{r+1}$. We claim that each word $X_{\kappa^{(r)}}$ is in the language $\sigma^{i_r} \mathcal{A}$. If $\ell(\kappa^{(r)}) = 1$ the statement is trivial. If $\ell(\kappa^{(r)}) > 1$, it follows since $\kappa^{(r)}$ is in $C(1)_1$ and all of its
parts after \( \lambda_r \) (the shifted height of the root) are greater than it. Hence, for each word \( X_\kappa \in \mathcal{C}(1) \), there exists a unique partition \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \), as defined above, such that \( X_\kappa \in X_\lambda \circ_s \mathcal{A} \). Conversely, every word in \( X_\lambda \circ_s \mathcal{A} \) is in \( \mathcal{C}(1) \). Then \( \mathcal{C}(1) \) can be expanded as follows, using the generating function of the weakly decreasing partitions (6)

\[
\mathcal{C}(1) = \sum_{\lambda} X_\lambda \circ_s \mathcal{A} = \prod_{n=\infty}^{1} \frac{1}{1 - X_n} \circ_s \mathcal{A}.
\]

Equation (44) follows immediately by shifting. \qed

From Eq. (43) we get

\[
(\mathcal{C}(1)^g = \left( \prod_{n=\infty}^{1} \frac{1}{1 - X_n} \right) \circ_s \mathcal{A}(-X),
\]

where \( \mathcal{A}(-X) \) is the graded generating function of \( \mathcal{A} \). Using the fact that the shift-plethystic inverses of \( \mathcal{A} \) and \( \mathcal{A}(-X) \) are, respectively, equal to \( X_0 - X_0X_1 \) and \( X_0X_1 - X_0 \), we get the identities

\[
\mathcal{C}(1) \circ_s (X_0 - X_0X_1) = \prod_{n=\infty}^{1} \frac{1}{1 - X_n}
\]

\[
\mathcal{P}_2 \circ_s (X_0X_1 - X_0) = ((\mathcal{C}(1)^g)^{-1} \circ_s (X_0X_1 - X_0) = \prod_{n=1}^{\infty} (1 - X_n). \quad (47)
\]

The left-hand side of Eqs. (46) and (47) are, respectively, equal to

\[
\mathcal{C}(1) \circ_s (X_0 - X_0X_1) = \sum_{\kappa \in \mathcal{C}(1)} \prod_{i=1}^{\ell(\kappa)} X_{k_i}(1 - X_{k_i+1}),
\]

\[
\mathcal{P}_2 \circ_s (X_0X_1 - X_0) = \sum_{\lambda \in \mathcal{P}_2} \prod_{i=1}^{\ell(\lambda)} X_{\lambda_i}(X_{\lambda_i+1} - 1).
\]

Substituting \( X_n \mapsto q^n \), we get the identities

\[
\sum_{n=0}^{\infty} q^n \sum_{\kappa \in \mathcal{C}(1)[n]} \prod_{i=1}^{\ell(\kappa)} (1 - q^{k_i+1}) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n},
\]

\[
\sum_{n=0}^{\infty} q^n \sum_{\lambda \in \mathcal{P}_2[n]} \prod_{i=1}^{\ell(\lambda)} (q^{\lambda_i+1} - 1) = \prod_{n=1}^{\infty} (1 - q^n).
\]
Acknowledgements The author is grateful to the referee for his careful reading and for calling our attention to references [3,15] which consider non-commutative versions of the Rogers–Ramanujan continued fraction. In particular, [15, Theorem 3.3.2] gives an explicit formula for (in our notation) the continued fraction \( \mathcal{A}(z, q) \) as a quotient of \( q \)-series. This gives a novel expression for the generating function \( \mathcal{C}^{(1)}(1, z, q) \). These methods could help in finding explicit formulas for \( \mathcal{C}^{(1)}(z, q) \), and more generally for \( \mathcal{C}^{(m)}(z, q) \), an interesting problem for future research.

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