Single eta production in heavy quarkonia transitions

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Abstract

The $\eta$ production in the $(n,n')$ bottomonium transitions $\Upsilon(n) \rightarrow \Upsilon(n')\eta$, is studied in the method used before for dipion heavy quarkonia transitions. The widths $\Gamma_\eta(n,n')$ are calculated without fitting parameters for $n = 2, 3, 4, 5, n' = 1$. Resulting $\Gamma_\eta(4, 1)$ is found to be large in agreement with recent data.

1 Introduction

The $\eta$ and $\pi^0$ production in heavy quarkonia transitions is attracting attention of experimentalists for a long time [1]. The first result refers to the $\psi(2S) \rightarrow J/\psi(1S)\eta$ process (to be denoted as $\psi(2, 1)\eta$ in what follows, similarly for $\Upsilon$) with $\frac{\Gamma_\eta}{\Gamma_{\text{tot}}} = (3.09 \pm 0.08)\%$ [1], $\Gamma_{\text{tot}} = 337 \pm 13 \text{ keV}$.

For the $\Upsilon(2, 1)\eta$ and $\Upsilon(3, 1)\eta$ transitions only upper limits $B < 2 \cdot 10^{-3}$ and $B < 2.2 \cdot 10^{-3}$ were obtained in [2] and [3] correspondingly and preliminary results appeared recently in [4], $B(\Upsilon(2, 1)\eta) = (2.5 \pm 0.7 \pm 0.5)10^{-4}$ and $B(\Upsilon(2, 1)\pi^0) < 2.1 \cdot 10^{-4}$ (90\% c.l.). On theoretical side in [5] small ratios of widths

\[
\frac{\Gamma(\Upsilon(2, 1)\eta)}{\Gamma(\psi(2, 1)\eta)} \simeq 2.5 \cdot 10^{-3} \quad \text{and} \quad \frac{\Gamma(\Upsilon(3, 1)\eta)}{\Gamma(\psi(2, 1)\eta)} = 1.3 \cdot 10^{-3}
\] (1)
have been predicted, with the model property that the bottomonium yields of \( \eta \) would be smaller than those of charmonium; specifically in the method of [6], the width ratio is proportional to \( O \left( \left( \frac{m_c}{m_b} \right)^2 \right) \approx 0.1 \), for a discussion see also [6, 7].

However recently [8] new BaBar data have been published on \( \Upsilon(4,1)\eta \) with the branching ratio

\[
B(\Upsilon(4,1)\eta) = (1.96 \pm 0.06 \pm 0.09) \times 10^{-4}
\]

and

\[
\frac{\Gamma(\Upsilon(4,1)\eta)}{\Gamma(\Upsilon(4,1)\pi^+\pi^-)} = 2.41 \pm 0.40 \pm 0.12.
\]

This latter result is very large, indeed the corresponding ratio for \( \psi(2,1)\eta \) transition is \( \approx 0.2 \) and theoretical estimates [11] from [5] for a similar ratio of \( \Upsilon(3,1)\eta/\pi\pi \) yield 0.015. All this suggests that another mechanism can be at work in single \( \eta \) production and below we exploit the approach based on the Field Correlator Method (FCM) [9] recently applied to \( \Upsilon(n,n')\pi\pi \) transitions with \( n \leq 3 \) in [9, 10], \( n \leq 4 \) in [11] and \( n = 5 \) in [12, 13].

The method essentially exploits the mechanism of Internal Loop Radiation (ILR) with light quark loop inside heavy quarkonium and has two fundamental parameters – mass vertices in chiral light quark pair \( q\bar{q} \) creation \( M_{br} \approx f_\pi \) and pair creation vertex without pseudoscalars \( M_\omega \approx 2\omega \), where \( \omega (\omega_s) \) is the average energy of the light (strange) quark in the \( B(B_s) \) meson. Those are calculated with relativistic Hamiltonian [14] and considered as fixed for all types of transitions \( \omega = 0.587 \text{ GeV}, \omega_s = 0.639 \text{ GeV} \).

Any process of heavy quarkonium transition with emission of any number of Nambu-Goldstone (NG) mesons is considered in ILR as proceeding via intermediate states of \( BB, BB^* + c.c., B^\pm B^\mp \) etc. (or equivalently \( DD \) etc.) with NG mesons emitted at vertices.

For one \( \eta \) or \( \pi^0 \) emission one has diagrams shown in Fig.1, where dashed line is for the NG meson. As shown in [9, 10, 11], based on the chiral Lagrangian derived in [15], the meson emission vertex has the structure

\[
\mathcal{L}_{CDL} = -i \int d^4x \bar{\psi}(x) M_{br} \hat{U}(x) \psi(x)
\]

\[
\hat{U} = \exp \left( i\gamma_5 \frac{\varphi_a \lambda_a}{f_\pi} \right), \varphi_a \lambda_a = \sqrt{2} \begin{pmatrix} \frac{n}{\sqrt{6}} + \frac{\pi^0}{\sqrt{2}} & \pi^+, K^+ \\ \pi^-, K^- & \frac{n}{\sqrt{6}} - \frac{\pi^0}{\sqrt{2}} & K^0, K^0 \\ K^-, \bar{K}^0 & -\frac{2n}{\sqrt{6}} & \frac{n}{\sqrt{6}} - \frac{\pi^0}{\sqrt{2}} \end{pmatrix},
\]

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Fig.1 Single eta production (dashed line) from $\Upsilon(n)BB^*$ vertex (a), and $BB^*\Upsilon(n')$ vertex (b).

The lines (1,2,3) in the $\hat{U}$ matrix [2] refer to $u,d,s$ quarks and hence to the channels $B^+B^-, B^0\bar{B}^0, B^*_s\bar{B}^*_s$ (and to the corresponding channels with $B^*$ instead of $B$). Therefore the emission of a single $\eta$ in heavy quarkonia transitions requires the flavour $SU(3)$ violation and resides in our approach in the difference of channel contribution $B\bar{B}^*$ and $B_s\bar{B}_s^*$, while the $\pi^0$ emission is due the difference of $B^0\bar{B}^{0*}$ and $B^+B^{-*}$ channels (with $B \to D$ for charmonia).

The paper is devoted to the explicit calculation of single $\eta$ emission widths in bottomonium $\Upsilon(n,1)\eta$ transitions with $n = 2, 3, 4, 5$. Since theory has no fitting parameters (the only ones, $M_\omega$ and $M_{br}$ are fixed by dipion transitions) our predictions depend only on the overlap matrix elements, containing wave functions of $\Upsilon(nS), B, B_s, B^*, B^*_s$. The latter have been computed previously in relativistic Hamiltonian technic in [14] and used extensively in dipion transitions in [11, 12, 13].

The paper is organized as follows. In section 2 general expressions for process amplitudes are given; in section 3 results of calculations are presented and discussed and a short summary and prospectives are given.

## 2 General formalism

The process of single NG boson emission in bottomonium transition is described by two diagrams depicted in Fig.1, (a) and (b) which can be written according to the general formalism of FCM [9, 11, 12] as (we consider $\eta$ emission)

$$\mathcal{M} = \mathcal{M}^{(1)}_\eta + \mathcal{M}^{(2)}_\eta; \mathcal{M}^{(i)}_\eta = \mathcal{M}^{(i)}_{BB^*} - \mathcal{M}^{(i)}_{BB^*}, i = 1, 2$$

$$\mathcal{M}^{(1)}_\eta = \gamma \int \frac{J^{(1)}_n(p,k)J_{n'}(p)}{E - E(p)} \frac{d^3p}{(2\pi)^3}, \quad (7)$$
where $\mathcal{M}_{\eta}^{(2)}$ has the same form, but without NG boson energy in the denominator of (7). Here $\gamma = \frac{M_{bb}}{\sqrt{2\omega_{\eta}N_{s}f_{\eta}\sqrt{3}}}$ and the overlap integral of $\Upsilon(nS)$ and $BB^{*}$ wave functions is (for details see Appendix)

$$J_{n}^{(1)}(p,k) = \bar{\eta}_{n}^{(0)} I_{n,bb^{*}}(p)e^{-\frac{\kappa^{2}}{4\Delta_{n}^{2}}}$$

(8)

$$J_{n'}(p) = \bar{\eta}_{n'}^{(1)} I_{n',bb^{*}}(p)e^{-\frac{\kappa^{2}}{4\Delta_{n}}}$$

(9)

$\bar{\eta}(BB^{*})$ and $\bar{\eta}_{2}(BB^{*})$ are Dirac traces of decay matrix elements $\Upsilon(nS) \rightarrow BB^{*}\eta$ and $BB^{*} \rightarrow \Upsilon(n')$, respectively they are defined in [9, 11] and below in Appendix. The special point in our case is that $\eta$ meson is emitted in $P$ wave, hence one must extract the corresponding term in the Dirac trace, for details see Appendix.

$$\bar{\eta}(BB^{*}) \cdot \bar{\eta}_{2}(BB^{*}) = \left( \frac{M_{bb}}{2\Omega} \right)^{2} \frac{4p^{2}u_{n}e_{i'i''}}{3\omega^{3}}k_{l}.$$ 

(10)

Here $u_{n} = \frac{2\Omega}{\Delta_{n}(\omega_{l}+\Omega)} \approx \frac{2\Omega}{\Delta_{n}}$, and $\Omega(\Omega_{s})$ is the average energy of the $b$ quark in $B(B_{s})$ meson; from Table IV in [9] one finds that $\Omega = 4.827$ GeV, $\Omega_{s} = 4.830$ GeV. In what follows we shall neglect the difference between $\Omega, \Omega_{s}$ and the mass of $b$ quark $M_{bb} = 4.8$ GeV. Note that these large masses cancel in all matrix elements and final expressions will depend only on energies $\omega$ and $\omega_{s}$ and differences of threshold positions: $\Delta M^{*} = M(B) + M(B^{*}) - M(\Upsilon(nS))$ and $\Delta M_{s}$ – the same for $B_{s}, B_{s}^{*}$ masses. Note, that the contribution of the $B^{*}B^{*}, B_{s}^{*}B_{s}^{*}$ channels vanish hence we shall consider only $BB^{*}$ and $B_{s}B_{s}^{*}$ channels.

Indices $i'j$ in $e_{i'j}$ in (10) refer to the $\Upsilon(n'S)$ and $\Upsilon(nS)$ polarizations respectively. Finally, coefficients $\beta_{2}, \beta_{1}$ and $\Delta_{n} = 2\beta_{1}^{2} + \beta_{2}^{2}$, refer to the expansion of realistic wave functions of $\Upsilon(nS), \Upsilon(n'S)$ and $B, B^{*}, B_{s}, B_{s}^{*}$ computed in [14] in series of oscillator functions and $\beta_{1}, \beta_{1}'$ and $\beta_{2}$ denote the $\chi^{2}$ fitted oscillator parameters for those functions respectively, see [11] for details.

Finally we define all quantities in the denominator of (7); in $\mathcal{M}_{BB^{*}}^{(1)}$, the denominator is

$$E - E(p) = M(\Upsilon(nS)) - (\omega_{\eta} + M_{B} + M_{B}^{*} + \frac{p^{2}}{2M_{B}} + (p - k)^{2}2M_{B}^{*}) \equiv -\Delta M^{*} - \omega_{\eta} - E(p,k).$$

(11)

For $\mathcal{M}_{\eta}^{(2)}$ one omits $\omega_{\eta}$ and $k$ in (11). Finally one can represent the matrix element $\mathcal{M}_{\eta}^{(i)}$ as follows:

$$\mathcal{M}_{\eta}^{(i)} = \gamma e_{i'i''}k \frac{\beta_{2}^{2}}{3\Delta_{n}} \left( \frac{1}{\omega_{s}^{3}}C_{s}^{(i)} - \frac{1}{\omega_{s}^{3}}C_{s}^{(1)} \right) e^{-\frac{k^{2}}{4\omega_{s}^{2}}}$$

(12)
\[ \mathcal{M}^{(2)}_\eta = \gamma e^{i\varphi k} \frac{\beta^2}{3\Delta n'} \left( \frac{1}{\omega_3} \mathcal{L}^{(2)} - \frac{1}{\omega_3} \mathcal{L}^{(2)} \right) e^{-\frac{i}{4\Delta n'}} \]  

with

\[ \mathcal{L}^{(1)} = \int \frac{d^3 p}{(2\pi)^3} \frac{I_{n_1}(p) I_{n_2}(p) e^{-\frac{p^2}{\eta^2}}}{\left( \Delta M^* + \omega + \frac{p^2}{2M_B} + \frac{(p-k)^2}{2M_B} \right)}. \]

\[ \mathcal{L}^{(2)} = \int \frac{d^3 p}{(2\pi)^3} \frac{I_{n_1}(p) I_{n_2}(p) e^{-\frac{p^2}{\eta^2}}}{\left( \Delta M^* + \frac{p^2}{2M_B} \right)}. \]

For \( \mathcal{L}^{(1)}, \mathcal{L}^{(2)} \), one replaces \( \Delta M^* \) with \( \Delta M_3^* \) and \( M_B, M_B^* \) with \( M_{B_3}, M_{B^*_3} \). Here

\[ \beta_{\eta}^{-2} = \frac{\Delta_n}{\Delta_n + \Delta_{n'}.} \]

The width of the \( \Upsilon(n, n')\eta \) decay is obtained from \( |\mathcal{M}|^2 \) averaging over vector polarizations as

\[ \Gamma_\eta = \frac{1}{3} \sum_{i,i'} |\mathcal{M}|^2 d\Phi = \frac{2k^2}{27} \gamma e^{-\frac{k^2}{2\eta^2}} d\Phi \left| u_n \left( \frac{\mathcal{L}^{(1)}_{s_1}}{\omega_3} - \frac{\mathcal{L}^{(1)}_{s_3}}{\omega_3} \right) + u_{n'} \left( \frac{\mathcal{L}^{(2)}_{s_1}}{\omega_3} - \frac{\mathcal{L}^{(2)}_{s_3}}{\omega_3} \right) \right|^2 \]

where the phase space factor \( d\Phi = \frac{d^3 k}{(2\pi)^3} 2\pi \delta(M(\Upsilon(n)) - M(\Upsilon(n')) - \omega - \frac{k^2}{2M(\Upsilon(n'))}) \).

Introducing the average \( \bar{\omega} = \frac{1}{2}(\omega_3 + \omega) \), one can rewrite (16) as

\[ \Gamma_\eta = \left( \frac{M_{B_3}}{M_\eta} \right)^2 \left( \frac{M_\omega}{2\bar{\omega}} \right)^2 \zeta_\eta \frac{k^3}{\omega_4} e^{-\frac{k^2}{2\bar{\omega}^2}} \left| u_n \left[ \left( \frac{\bar{\omega}}{\omega_3} \right)^3 \mathcal{L}^{(1)}_{s_1} - \left( \frac{\bar{\omega}}{\omega_3} \right)^3 \mathcal{L}^{(1)}_{s_3} \right] + \right| u_{n'} \left[ \left( \frac{\bar{\omega}}{\omega_3} \right)^3 \mathcal{L}^{(2)}_{s_1} - \left( \frac{\bar{\omega}}{\omega_3} \right)^3 \mathcal{L}^{(2)}_{s_3} \right] \right|^2 \]

with \( \zeta_\eta = \frac{16}{\pi^3 N_\tau^3} \simeq 7 \cdot 10^{-3} \).

One can see from the general structure of \( \Gamma_\eta \), that the main effect comes from the difference \( \left| \left( \frac{\bar{\omega}}{\omega_3} \right)^3 - \left( \frac{\bar{\omega}}{\omega_3} \right)^3 \right| \approx |0.882 - 1.139| \approx 0.257 \), and from the difference of \( |\mathcal{L}^{(1)}_{s_1} - \mathcal{L}^{(1)}_{s_3}| \leq 0.05 \).

### 3 Results and discussion

We consider here the single \( \eta \) emission in bottomonium transitions \( \Upsilon(n, 1)\eta \) with \( n = 2, 3, 4, 5 \). The corresponding values of \( \Delta M^*, \Delta M_{s^*}^*, \omega, k \) are given in the Table 1.
Table 1.
Mass parameters of $\Upsilon(n,n')\eta$ transitions (all in GeV, $k$ in GeV/c) and matrix elements $L^{(i)}, L_s^{(i)}$ (in GeV).

| $(n,n')$ | 2,1 | 3,1 | 4,1 | 5,1 |
|----------|-----|-----|-----|-----|
| $\Delta M^*$ | 0.582 | 0.26 | 0.026 | -0.26 |
| $\Delta M_s^*$ | 0.757 | 0.425 | 0.20 | -0.08 |
| $\omega_\eta$ | 0.562 | 0.87 | 1.075 | 1.325 |
| $k$ | 0.115 | 0.674 | 0.923 | 1.20 |
| $L^{(1)}$ | 0.263 | -0.188 | 0.29 | 1.48 $\cdot 10^{-3}$ |
| $L_s^{(1)}$ | 0.240 | -0.174 | 0.121 | 1.41 $\cdot 10^{-3}$ |
| $L^{(2)}$ | 0.390 | -0.340 | 0.255 | -0.347 |
| $L_s^{(2)}$ | 0.341 | -0.298 | 0.226 | +0.0584 |

The resulting values of $\Gamma_\eta(n,n')$ have been computed as in (17) with $\omega = 0.587$ GeV and $\omega_s = 0.639$ GeV, calculated earlier in [14], see Table 4 of [9], and $u_n = \frac{\beta_2^2}{\Delta_n}$ with $\beta_2 = 0.48$ GeV, and $\Delta_n$ both fitted to the realistic wave functions in [11], with $\Delta_n = 2.56; 1.54, 1.21, 1.05$ and 1.35 (all in GeV$^2$) for $n = 1, 2, 3, 4, 5$ respectively.

Results of calculations are given in Table 2.

Table 2.
Values of $\Gamma_\eta(n,n')$ (in keV) calculated using Eq. (17) vs experimental data $\Gamma_{\eta}^{\text{exp}}(n,n')$ (in keV).

| $(n,n')$ | 2,1 | 3,1 | 4,1 | 5,1 |
|----------|-----|-----|-----|-----|
| $\Gamma_\eta(n,n')$ | 5.0 $\cdot 10^{-2}$ | 2.9 | 1.81 | 7.04 |
| $\Gamma_{\eta}^{\text{exp}}(n,n')$ | $(0.8 \pm 0.3) \cdot 10^{-2}$ | - | 4.02 $\pm 0.6$ | - |

Looking at the Table 2, one can see, that there is an order of magnitude agreement with experiment. Indeed, the factor $\left(\frac{M_{\eta}}{M_{\pi}}\right)^2 \left(\frac{M_{\pi}}{2\omega}\right)^2$ can be estimated from $\Upsilon(n,n')\pi\pi$ transitions studied in [9]-[13] to be roughly in the range $[\frac{1}{2}, 2]$. At the same time $\Gamma_{\eta}^{\text{exp}}(2,1)$ differs from our calculated value several times, and more accurate measurements as well as theoretical calculations are highly welcome here.
Another point, not shown in Table 2, is the old upper limit [3] on $B_{\eta}(3,1)$, namely $B(\Upsilon(3,1)\eta) < 2.2 \cdot 10^{-3}$, which yields $\Gamma_{\eta}(3,1) < (4.5 \pm 0.4) \cdot 10^{-2}$ keV and is two orders of magnitude below our calculated value. Hopefully new measurements can resolve this disagreement. On theoretical side our formulas (14)-(16) automatically produce the width $\Gamma_{\eta}(n,n')$ of the order of $O(1 \text{ keV})$, for all $(n,1)$ transitions except for $(2,1)$, where a small phase space factor $k^3$ gives two orders of magnitude suppression of $\Gamma_{\eta}(2,1)$. For the $\Gamma_{\eta}(5,1)$ one obtains a 7 keV value, which is however small as compared with the $\Gamma_{\pi\pi}(5,1)$, the latter being $O(1 \text{ MeV})$. For $\Gamma_{\eta}(4,1)$ and $\Gamma_{\pi\pi}(4,1)$ from [11] the calculated ratio is $R_{\eta/\pi\pi} \equiv \frac{\Gamma_{\eta}(4,1)}{\Gamma_{\pi\pi}(4,1)} \approx 3 \left( \frac{M_{br}}{2\omega} \right)^2 \left( \frac{f_\pi}{M_{br}} \right)^2 \approx 3$ which roughly agrees with experimental value $R_{\eta/\pi\pi}^{exp} = 2.41 \pm 0.40 \pm 0.12$.

To check stability of our results, we have used for the wave function of $B_s$ the realistic wave function different from that of $B$. As a result one obtains for $\frac{\Gamma_{\eta}(n,1)}{\Gamma_{\pi\pi}(n,1)}$ the values ($2.74 \cdot 10^{-2}$; 1.13; 0.44; 7.3) keV for $n = 2, 3, 4, 5$ respectively, which should be compared with numbers in the upper line of Table 2. The same type of sensitivity occurs for modifications of other wave functions, implying that our results strictly speaking yield the correct order of magnitude but not exact values of $\Gamma_{\eta}$.

Summarizing, we have calculated the single $\eta$ production width $\Gamma_{\eta}(n,n')$ for $\Upsilon(n,1)\eta$ transitions with $n = 2, 3, 4, 5$. We have found that $\Gamma_{\eta}(n,1)$ are of the order of and larger than $\Gamma_{\pi\pi}(n,1)$ for $n = 3, 4$. This fact is in agreement with the latest measurements in [3] of $\Gamma_{\eta}^{exp}(4,1)$ and disagrees with earlier experimental limit on $\Gamma_{\eta}^{exp}(3,1)$. Our calculations do not contain fitting parameters; the only two parameters $M_{br}, M_\omega$ are fixed by previous comparison with dipion data. One should stress that $\eta$ production in bottomonium is not suppressed in our approach as compared to $\eta$ production in charmonium transitions. This is in contrast with the results of method of [6].

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Appendix 1

Matrix element of single $\eta$ emission

According to the general theory in [9][11], the matrix elements $\mathcal{M}^{(1)}_\eta, \mathcal{M}^{(2)}_\eta$ for $\Upsilon(n,n')\eta$ corresponding to diagrams of Fig.1, (a) and (b) respectively, can be written as

$$
\mathcal{M}^{(1)}_\eta(n) = \frac{M_{br}M_{\omega}}{f_\eta N_c \sqrt{2\omega_\eta}} \int \frac{d^3p}{(2\pi)^3} \sum_{n_2,n_3} \frac{J_{n_2n_3}^{(p,k)}J_{n'n_2n_3}^{(p)}}{E - E_{n_2n_3}(p)}. \tag{A.1}
$$

Here $n_2, n_3$ are channels of intermediate state, with e.g. $n_2 = B, B^*, B_s, B_{s^*}, ...$, we omit indices $n_2, n_3$ and write

$$
J_n(p,k) = \int \frac{d^3q}{(2\pi)^3} \tilde{\psi}_n(cp - \frac{k}{2} + q)\tilde{\psi}_{n_2}(q)\tilde{\psi}_{n_3}(q - k). \tag{A.2}
$$
where $\tilde{\Psi}_n, \tilde{\psi}_n$ are momentum space wave functions of $\Upsilon(nS)$ and $B(B^*)$ mesons respectively.

The vertex factor $\bar{y}_1(y)$ is calculated in the same way as in [9], namely from the Dirac trace of the projection operators for the decay process, in our case this is $\Upsilon(nS) \rightarrow BB^*\eta$. Identifying the creation operators as $\bar{\psi}_b\gamma_i\psi_b$, $\bar{\psi}_b\gamma_5\psi_n$, $\bar{\psi}_b\gamma_k\psi_n$, $n = u, d, s$ and extracting vertex of $\eta$ creation from the Lagrangian $\Delta L = -\int \bar{\psi}_n \hat{U}_{br} \psi_n d^4x$ which gives $iM_{br}\bar{\psi}_n \gamma_5 \hat{\lambda} \psi_n$, with $\hat{\lambda} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & -2 \end{pmatrix}$, one has for the decay process (cf. Appendix 1 of [9])

$$G(\Upsilon \rightarrow BB^*\eta) = tr[\gamma_i S_b(u, w)\gamma_5 S_n(w, x)\gamma_5 S_n(x, w)\gamma_k S_b(w', v)] \quad (A.3)$$

As shown in [9], appendix 1 and 2, the quark Green’s functions can be split into two factors $S(x, y) = \Lambda^\pm G(x, y)$, with the projection operators $\Lambda^\pm_k = \frac{m_k + \omega_k \gamma_4 + \nu^{(k)}_\eta}{2\omega_k}$, $k = b, n$ and the scalar part $G(x, y)$, where spins are present only in spin-dependent interaction and treated as corrections. Here $\omega_k$ is the average energy of quark in given meson. Hence one is brought to the spin factor $Z$.

$$Z = tr(\gamma_i \Lambda_b^+ \gamma_5 \Lambda_n^- \gamma_5 \Lambda_n^+ \gamma_k \Lambda_b^-) \quad (A.4)$$

which is equal to

$$Z = \frac{m_b^2 + \Omega^2}{4\Omega^2\omega^2}((\omega^2 - p_i^2 p_j^2)\delta_{ik} - p_i^2 p_k^2 + p_k^2 p_i^2). \quad (A.5)$$

Here $\Omega, \omega$ are average energies of $b$ and $n$ quark in $B$ or $B^*$. One can identify the momenta of $B$ and $B^*$ as $P_1 = p$ and $P_2 = -p - k$, then $q$ in (A.2) can be expressed as

$$p_q = -q + \frac{\omega}{\omega + \Omega}p, \quad p_q = q - \frac{\omega}{\omega + \Omega}p - k\frac{\Omega + 2\omega}{\Omega + \omega}, \quad (A.6)$$

and $Z$ is (we put $m_b \equiv \Omega$)

$$Z = \frac{1}{2\omega^2}(-kq\delta_{ik} + k_i q_k + k_k q_i). \quad (A.7)$$

It is important, that we are looking for the $P$-wave of emitted $\eta$, and hence for $P$ wave of relative $BB^*$ motion, hence the integral (A.2) should yield the term $pk$. This indeed happens, when one approximates $\Psi_n, \psi_n$ as series of oscillator wave functions and (A.2) has the form

$$J_n(p, k) = \tilde{y}_{n23}^\eta e^{-\frac{p^2}{2m_n}} \frac{k^2}{4\pi^2} (0) I_n(p). \quad (A.8)$$
In the process of $dq$ integration in (A.2) one changes the integration variable $q_i \rightarrow q_i' - u_n p_i + O(k_i)$ with $u_n = \beta^2_2 / \Delta_n$ are oscillator parameters, found by $\chi^2$ procedure. Thus result of $d^3q$ integration yields

$$\bar{y}_{123}^\eta = \frac{u_n}{2\omega^2} (-kp\delta_{ik} - k_ip_k + k_k p_i). \quad (A.9)$$

In an analogous way one obtains for $J_{n'}(p)$ in (A.1) the form

$$J_{n'}(p) = \frac{j_{n'}(p)}{2\omega^2} e^{-p^2/\Delta_{n'}} I_{n'}(p) \quad (A.10)$$

and $j_{n'}(p)$ is obtained from the Dirac trace for the process $BB^* \rightarrow \Upsilon(n'S),$

$$Z_2(BB^*) = \frac{1}{2\omega} e^{i' kl}(-2q_l + \frac{2\omega}{\omega + \Omega}p_l) \quad (A.11)$$

and the result of integration over $d^3q$ yields in (A.10)

$$\bar{y}_{n'23}^{(n)} = -e_{i' ki} \frac{p_l}{\omega}. \quad (A.12)$$

Here $i'$ is the polarization of $\Upsilon(n'S)$ (represented by $\bar{\psi}_b\gamma_i\psi_b$) and $k$ as in (A.9) is the polarization of $B^*$. Averaging over angles of $p$ one obtains

$$\langle \bar{y}_{n'23}^{(n)} \rangle_p = \frac{u_n}{3} \frac{p_l^2}{\omega^3} (e_{i' ki}) \quad (A.13)$$

and finally one writes as in (12)

$$\mathcal{M}^{(1)}(n,n') = \frac{M_{\omega}M_{br}u_n}{2\omega^3 \sqrt{3}} \left( \frac{L^{(1)} - L^{(1)}_{\omega^3}}{\omega^3} \right) e^{-\frac{p^2}{4\omega^2}}. \quad (A.14)$$

For $\mathcal{M}^{(2)}(n,n')$ one can use time inversion and interchange indices $i,i'$ and change sign of $k$, obtaining in this way Eqs. (14) and (15) of the main text. For the intermediate state of $B^*B^*$ the summation over polarizations of $B^*$ yields a net zero result, therefore we are left with only $(BB^* + B^*B)$ intermediate state.