Flow Equations for Gluodynamics in the Coulomb Gauge

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Abstract

A systematic procedure to consistently formulate a field theoretical, QCD bound state problem with a fixed number of constituents is outlined. The approach entails applying the Hamiltonian flow equations, which are a set of continuous unitary transformations, to a QCD motivated Hamiltonian with a confining interaction. The method is developed in detail for gluodynamics in the Coulomb gauge to obtain an effective block-diagonal Hamiltonian appropriate to a reduced Fock space with fixed number of dynamical gluons. Standard many-body techniques are used to numerically diagonalize this Hamiltonian in a constituent two gluon Fock space. The calculated gluon condensates and glueball masses are in good agreement with QCD sum rule and lattice results.

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I. INTRODUCTION

Quantum Chromodynamics (QCD) is generally regarded as the theory governing hadronic structure and interactions. This broad acceptance follows, in part, from the success of high energy perturbative calculations where asymptotic freedom prevails. However, a significant gap still exists in implementing QCD in the non-perturbative, low energy regime where physical observables have only been described by lattice calculations, QCD sum rules and phenomenological models. Our study addresses this shortcoming and details a new approach which directly incorporates canonical QCD at high energies and phenomenology, including confinement, from low energy, to produce a realistic effective Hamiltonian appropriate for bound state investigations.

Because the relevant hadronic degrees of freedom do not clearly follow from the canonical QCD Hamiltonian, we look to phenomenological models, which are reasonably developed, for initial guidance. Of particular interest is gluonic structure, the focus of this work, which has been described by vastly alternative approaches: strings [1], flux tubes [2], stochastic gluon configurations [3] and constituents [4] confined by a bag [5] or potential [6,7]. In this paper we continue to build from our previous work [7–9] within the constituent picture, again viewing both quarks and gluons as dressed partons subject to an effective interaction. We assume this interaction is dominated by a confining potential whose explicit form is not essential, provided it is long ranged and generates bound states. Our effective Hamiltonian also contains perturbative interactions, including radiative corrections, governing the ultraviolet (UV) region. These terms are directly and unambiguously obtained from the leading order coupling constant expansion of the canonical Hamiltonian.

With these preliminaries we now address the essence of this paper, a systematic method for consistently solving a generic, field theoretical QCD motivated Hamiltonian. The method we present is based upon the flow equations suggested by Wegner [10] and applied to QED within the light-cone framework [11,12]. Because a field theoretical solution involves diagonalization in an infinite Fock space with arbitrary number of quanta, an exact solution
is not possible. Instead we seek a finite Fock space formulation and begin by dividing the complete Fock space into two components, a tractable $P$ subspace spanned by states with a small number of quanta and the remainder $Q = 1 - P$. This leads to a Hamiltonian matrix to be diagonalized of the form

$$H = \begin{pmatrix} PHP & PHQ \\ QHP & QHQ \end{pmatrix}. \quad (1.1)$$

Next we use the flow equation method to block diagonalize this matrix yielding the effective Hamiltonian

$$H_{eff} = \begin{pmatrix} P H_{eff} P & 0 \\ 0 & Q H_{eff} Q \end{pmatrix}. \quad (1.2)$$

One can then separately diagonalize the two blocks which are now uncoupled. It is important to note that the flow equation approach is significantly different than the Tamm Dancoff approximation (TDA) even though they both may use the same $P$ space. In the TDA the off-diagonal blocks $PHQ$ and $QHP$ are eliminated simply by truncation, whereas in the flow equation method they are eliminated by transformation to $PH_{eff}P$ and $QH_{eff}Q$.

In the flow equation approach the Hamiltonian is unitary transformed continuously

$$H(l) = U(l)H(0)U^\dagger(l), \quad (1.3)$$

$$U(l) = T_l \exp \left( \int_0^l \eta(l')dl' \right), \quad (1.4)$$

where $l$ is the continuous (flow) parameter ($l = 0$ corresponds to the initial Hamiltonian, Eq. (1.1)), $T_l$ the $l$ ordering operator and $\eta$ the generator of the transformation. Wegner’s flow equations are then an expression of this unitary transformation in differential form

$$\frac{dH(l)}{dl} = [\eta(l), H(l)], \quad (1.5)$$

with the specific generator choice

$$\eta(l) = [H_d(l), H(l)], \quad (1.6)$$
where $H_d$ is the diagonal part of $H$ and $H_{\text{eff}} = H_d(l \to \infty)$. Using this generator, it is shown in section II how the off-diagonal terms $QHP$, which connect Fock states with different particle number, can be eliminated as $l \to \infty$ and included in the block diagonal effective interaction. Hence coupling to the $Q$ space is incorporated into the $P$ space and represented by new terms in the effective Hamiltonian. Once $PH_{\text{eff}}P$ is determined, the bound state problem can be investigated in detail by direct diagonalization. In general, $PH_{\text{eff}}P$ will become more complicated as the $P$ space becomes smaller which may necessitate additional approximations in obtaining the effective Hamiltonian. Hence the strategy to find a balance between size of the $P$ space, which is determined by computational effort and resources, and level of approximations. In the current work we will apply this approach to gluonia and restrict the $P$ space to Fock states with at most two dynamical (dressed) gluons.

In the next section we review the flow equation approach for a general Hamiltonian. It is important to note, however, that the solution of these equations is quite involved. This is because the equations are nonlinear and coupled and also because the choice of generator needed to decouple the $P$ and $Q$ spaces generates complicated many-body interactions. For most realistic applications, therefore, the flow equations will be implemented iteratively. This has been demonstrated in a variety of investigations such as condense matter problems \cite{13}, \cite{14}, electron-phonon \cite{15} and positronium \cite{11} treatments, the latter two involving perturbative expansions in coupling constant. Here we adopt a slightly different perturbative scheme and invoke the weak coupling assumption commonly used in nuclear structure many-body calculations. There one introduces a mean field, or Hartree-Fock component of the interaction, and expresses the Hamiltonian interaction as a sum of this term and a residual force, the difference between the original potential energy and the mean field. An approximate effective interaction is then generated by treating the residual interaction perturbatively. The physical nuclear many-body states are finally obtained by diagonalizing this effective interaction in the $P$ space spanned by eigenfunctions of the mean field Hamiltonian. The success of the method is determined by choice of model space $P$ and phenomenological interactions. Although our problem is not a central field situation, we nevertheless can uti-
lize this concept since a dominant mean field is present in the form of the phenomenological confining interaction. This directly governs the quasi-particle (dressed) gluon basis functions which in turn form the many-body basis spanning the $P$ space. We identify our perturbative interaction as the difference between the original QCD interaction and the phenomenological confining potential and approximate this residual interaction at high energies by the leading order canonical Hamiltonian interaction. Again the overall utility of this method will depend significantly upon choice of $P$ space and phenomenological input. For example in our pure gluon application we assume that three and higher gluon Fock states only couple weakly to the $P$ space. Because the dressed gluon mass is of order $1 \text{ GeV}$, which sets the scale for a large energy denominator in a perturbative expansion, this approximation should be reasonable. However, it may develop in quark sector applications that the weak coupling approximation is not sufficient for low Fock space components such as exotic 4 quark ($qq\bar{q}\bar{q}$) and hybrid ($q\bar{q}g$) states. In such cases the $P$ space must be expanded to explicitly include these higher components and this is a subject for future investigations.

Finally we mention the issue of renormalization which can also be accommodated by the flow equations. Because the complete Fock state expansion contains states of arbitrarily large energies, UV divergences occur for certain matrix elements. However, since QCD is a renormalizable theory, these UV diverging terms can be isolated, regularized and then by introducing counterterms, renormalization can be achieved perturbatively order by order in the coupling constant $\lambda$. As demonstrated in the following sections, the flow parameter, actually $t^{-1/2}$, plays an additional role as an UV cut-off in the flow equation elimination of the off-diagonal terms. Hence, the flow equations embody the renormalization concept of Wilson wherein the high energy modes are integrated out in the path integral representation yielding a low energy effective action. Our method is also analogous to the Glazek and Wilson similarity transformation scheme \cite{16} with a continuous cut-off function $\lambda$. In particular, as shown in section IV.A, we obtain a well behaved gap equation with canonical counterterms cancelling the leading UV divergence. This is one reason why we do not incorporate the phenomenological confining terms in the implementation of the flow equations.
since they are not renormalizable. Further and related, including the confining potential in
the flow generator would produce additional effective interaction terms that are still phe-
omenological. Since an initial phenomenological interaction is already present, the model
framework remains unchanged and we therefore bypass this step.

This paper is divided into five sections and three appendices. In the next section we
summarize the general flow equation method, highlighting the key issues. In section III we
apply the flow equations and specify the explicit form of the model QCD Hamiltonian. In
addition to the weak coupling motivation discussed above, we also wish to disentangle all
complexities from chiral symmetry breaking and thus focus on pure gluodynamics. This
section also develops and presents our final effective Hamiltonian. Numerical results are
then given in section IV, with the gap equation, gluon constituent mass and condensate
analyzed in section IV.A and glueball bound states and masses in section IV.B. Finally,
results and future work are discussed in the concluding section along with an extensive
appendix containing full details of our comprehensive treatment.

II. THE FLOW EQUATION METHOD

We now summarize the important elements of flow equations for a general Hamiltonian.
We follow Wegner [10] who first formulated this approach to construct an effective bound
state Hamiltonian appropriate for field theories. Since this method does not depend on the
specific system or detailed nature of the Hamiltonian, it is beneficial to first conceptually
outline the general idea before our application in section III to a complicated field theory.
See ref. [12] for complete details.

Continuing with Eqs. (1.3-1.6), the first step is to divide the Hamiltonian, \( H(l) \), into
diagonal and "rest", or off-diagonal, components, \( H_d(l) \) and \( H_r(l) \), respectively

\[
H_d(l) = \begin{pmatrix}
PH(l)P & 0 \\
0 & QH(l)Q
\end{pmatrix},
\]

(2.1)
\[ H_r(l) = H(l) - H_d(l) = \begin{pmatrix} 0 & PH(l)Q \\ QH(l)P & 0 \end{pmatrix}. \] (2.2)

It is precisely the off-diagonal part, which connects Fock states with different particle number, that we wish to remove. Next, and most important, is the choice of generator. As shown by Wegner our block diagonalization goal can be achieved by choosing

\[ \eta(l) = [H_d(l), H(l)], \] (2.3)

which is off-diagonal by direct evaluation

\[ \eta(l) = \begin{pmatrix} 0 & P\eta(l)Q \\ Q\eta(l)P & 0 \end{pmatrix}, \] (2.4)

with \( P\eta Q = PHPHQ - PHQHQ \) and similar result for \( Q\eta P \) upon interchange of \( P \) and \( Q \). The flow equation, Eq. (1.5), can then be written for the important sector Hamiltonians \( PHP \) and \( PHQ \)

\[ \frac{d}{dl} PHP = P\eta QHP - PHQ\eta P, \] (2.5)

\[ \frac{d}{dl} PHQ = P\eta QHQ - PHP\eta Q. \] (2.6)

The essence of the flow equation approach is that with Wegner’s generator choice the absolute value of the off-diagonal terms, \( PH(l)Q, QH(l)P \), monotonically decreases to zero as the flow parameter tends to infinity. This can be directly shown by taking the trace of their product and computing the change with \( l \) yielding, using EQ. (2.6), a negative derivative

\[ \frac{d}{dl} \text{Tr}PHQHP = \text{Tr}\left(P\eta QQP\eta Q - PHQHPH\right) \]

\[ + \text{Tr}\left((PHPHQ - PHQHQ)\eta P\right) \]

\[ = 2\text{Tr}(P\eta Q\eta P) \]

\[ = -2\text{Tr}(P\eta Q(\eta Q)\dagger) \leq 0, \] (2.7)

where the last step follows from the antihermiticity of \( P\eta Q \). Since the trace of \( PHQHP \) is positive definite, \( P\eta(l)Q \rightarrow 0 \) in the limit \( l \rightarrow \infty \). Hence, \( H_d \) commutes with \( H \) and the
effective Hamiltonian, $H_{\text{eff}}$, approaches complete block diagonal form with increasing flow parameter.

We can formally solve the flow equations to provide further detailed insight and also the starting point for our gluodynamics application in the next section. The Hamiltonian, $H = H^{\text{free}} + H^I$, consists of free (non-interacting), $H^{\text{free}}$, and interacting, $H^I$, components. We then work in a fixed basis spanned by eigenfunctions of the free sector Hamiltonians $PH^{\text{free}}P$ and $QH^{\text{free}}Q$ with eigenvalues $E_p$ and $E_q$, respectively. Here the indices $p$ and $q$ denote all states (both Fock and energy) in the respective $P$ and $Q$ spaces. Representing the matrix elements of transformed Hamiltonians $PH(l)P$, $PH(l)Q$, etc. in this basis by $h_{pp'}(l)$, $h_{pq}(l)$, ... the flow equations take the form

\[
\frac{dh_{pp'}}{dl} = \sum_q \left( \eta_{pp'} h_{qp'} - h_{pq} \eta_{qp'} \right),
\]

\[
\frac{dh_{pq}}{dl} = \sum_{q'} \eta_{pq'} h_{qp} - \sum_{p'} h_{pp'} \eta_{q'q},
\]

\[
\eta_{pq} = \sum_{p'} h_{pp'} \eta_{p'q} - \sum_{q'} h_{pq'} \eta_{q'q},
\]

where all matrix elements are functions of $l$. The initial conditions (flow parameter $l = 0$) for the matrix elements of the sector Hamiltonian $PH(0)P$ are, $h_{pp'}(0) = E_p(0) \delta_{pp'} + \delta E_p(0) \delta_{pp'} + h^I_{pp'}(1 - \delta_{pp'}) \equiv E_p^I(0) \delta_{pp'} + h^I_{pp'}|_{p\neq p'}$, and for the Hamiltonian $PH(0)Q$, $h_{pq}(0)$. Here $\delta E_p^I$ and $h^I_{pp'}|_{p\neq p'}$ are the diagonal and off-diagonal matrix elements of the interaction Hamiltonian $H^I$ in the $P$ space. We now separate the diagonal and off-diagonal matrix elements in Eqs. (2.8-2.10)

\[
\frac{dE_p^I}{dl} = \sum_q \left( \eta_{pq} h_{qp} - h_{pq} \eta_{qp} \right),
\]

\[
\frac{dh^I_{pp'}}{dl} = \sum_q \left( \eta_{pp'} h_{qp} - h_{pq} \eta_{qp} \right),
\]

\[
\frac{dh_{pq}}{dl} = -(E_p^I - E_q^I) \eta_{pq} + \sum_{q' \neq q} \eta_{pq'} h^I_{qp'} - \sum_{p' \neq p} h^I_{pp'} \eta_{q'q},
\]

\[
\eta_{pq} = (E_p^I - E_q^I) h_{pq} + \sum_{p' \neq p} h^I_{pp'} h_{p'q} - \sum_{q' \neq q} h_{pq'} h^I_{q'q}.
\]

This system of coupled equations can be approximately decoupled by assuming that the off-diagonal $PHQ$ sector matrix elements are small as compared to diagonal $PHP$ elements,
i.e. $|h_{pq}(0)| \ll |h_{pp'}(0)|$ (from the flow equations this is also valid for any finite $l$). This approximation is consistent with our weak coupling concept discussed earlier and is valid for our gluonic application which entails a nonperturbative confining potential in the diagonal sector Hamiltonian and a weak, perturbative residual interaction in the off-diagonal.

We now solve the flow equations perturbatively, iterating $h_{pq}$. We do not treat $h_{pp'}$ perturbatively, but note it depends very weakly upon $l$, being suppressed to second order in $h_{pq}$ since from Eqs. (2.11,2.12), $dE_p^I/dl \sim O(h_{pq}^2)$ and $dh_{pp'}^I/dl \sim O(h_{pq}^2)$. Hence, our leading iteration in Eqs. (2.13,2.14) uses $E_p^I(l) \sim E_p^I(0) = E_p(0) + \delta E_p^I(0)$ and $h_{pp'}^I(l) \sim h_{pp'}^I(0)$. We could systematically iterate Eqs. (2.11-2.14) carrying all terms, however, it is simpler and more direct to further approximate Eqs. (2.13,2.14) by recognizing that the two off-diagonal sums over $p'$ and $q'$ will tend to be suppressed by cancellation. To appreciate this point, consider the leading order iteration of Eq. (2.14) in Eq. (2.13) which generates terms of the type, $\sum_{p'} h_{pp'}^I(0)\eta_{p'q}(l) \sim \sum_{p'} (E_p^I(0) - E_q^I(0))h_{pp'}^I(0)h_{p'q}(l)$, with a similar expression for the $Q$-space sum, $\sum_{q'} \eta_{pq'}(l)h_{q'q}(0)$. Here the sum is over Fock state components and all momentum (energy) for each Fock state. Next we make the key observation that although the Q space has in general an infinite number of Fock states, the Fock sum over $q'$ is restricted to at most two adjacent sectors since $h_{qq'}$ is a two-body interaction which can change particle number at most by two. Hence the $P$ and $Q$ space ”volumes” in particle number (and energy for a given particle number sector) are the same. The two sums are therefore equivalent to the product of an averaged matrix element for each space with a common volume for both spaces. Because the matrix elements have random phases, in the limit of large numbers of energy states, the two sums will tend to cancel and thus be suppressed relative to the leading diagonal terms ($p = p', q = q'$).

Therefore in leading order the generator is

$$\eta_{pq}(l) = -\frac{1}{E_p^I(0) - E_q^I(0)} \frac{dh_{pq}(l)}{dl},$$

(2.15)

and the effective Hamiltonian $PH_{eff}P$ is given by $h_{pp'}^{eff} = h_{pp'}(\infty)$.
\[ h_{pp'}^{\text{eff}} = E_p'(0) \delta_{pp'} + h_{pp'}(0) \big|_{p \neq p'} - \int_0^\infty dl \sum_{q} \left( \frac{dh_{pq}(l)}{dl} h_{qp'}(l) + \frac{h_{pq}(l) dh_{qp'}(l)}{dl} \right) , \] (2.16)

having off-diagonal elements

\[ h_{pq}(l) = h_{pq}(0) \exp \left( -l(E_p(l) - E_q(l))^2 \right) . \] (2.17)

which can be used for the next iteration. The above process can then be repeated, replacing \( E_p', h_{pp'}(0) \) on the right hand side of the flow equations with the diagonal and off-diagonal matrix elements of \( h_{eff}^{pp'} \), respectively, and continued until convergence. Diagonalizing \( h_{eff}^{pp'} \) then produces the final eigenvalues \( E_p^{eff} \). We now apply these results to gluodynamics.

### III. EFFECTIVE HAMILTONIAN FOR GLUODYNAMICS

In constructing our QCD motivated Hamiltonian, \( H_{QCD} \), we have incorporated phenomenological elements of previous, related constituent studies [7,8]. Starting with the exact Coulomb gauge QCD Hamiltonian (pure gluodynamics) \( \hat{H}_{QCD} = H_{\text{free}} + H_{\text{QCD}}^{I} \), containing free and QCD interacting terms, we introduce a confining phenomenological interaction, \( H_{\text{phen}} \), to permit isolation of the nonperturbative and perturbative Hamiltonians, \( H_{NP} \) and \( H_{PT} \), respectively

\[ \hat{H}_{QCD} = H_{NP} + H_{PT} \cong H_{NP} + H_{\text{can}}^{I} \equiv H_{QCD} , \] (3.1)

\[ H_{NP} = H_{\text{free}} + H_{\text{phen}} , \] (3.2)

\[ H_{PT} = H_{QCD}^{I} - H_{\text{phen}} \cong H_{\text{can}}^{I} . \] (3.3)

The solvable, phenomenological interaction \( H_{\text{phen}} \) is introduced to provide a weaker, residual interaction \( H_{PT} \) at all energy scales. Following quark models, we assume that, in the constituent (quasiparticle) basis, \( H_{NP} \) predominantly governs the bound state bulk properties and should be treated nonperturbatively. We further assume that the residual interaction can be approximated at high energies by the canonical QCD interaction Hamiltonian
\[ H_{PT} = H_{can}^{I}(\Lambda_{UV} \to \infty) \] and is amenable to perturbation theory. Note that the residual interaction describes quantum fluctuations and may contribute UV divergent terms requiring regularization with cut-off parameter \( \Lambda_{UV} \).

The goal is to use the flow equations to scale the residual interaction to low energies to generate an effective interaction, which when added to \( H_{NP} \), can be diagonalized non-perturbatively for bound states. In this process renormalization will also be achieved by introducing second order \( O(g^2) \) canonical counterterms which remove the leading UV divergences. With an assumed weak residual interaction, the entire procedure can be conducted perturbatively. There is no over representation or double counting, since, as shown below, the phenomenological interactions determine the infrared (IR) behavior while the residual interactions, which are free from IR divergences, describe the UV part.

We now further detail the Hamiltonian. In the Coulomb gauge the physical degrees of freedom are the transverse gauge fields \( A^a T^a \) (with color matrices \( T^a \) and sum over index \( a \)) and their conjugate transverse momenta \( \Pi \). The free Hamiltonian is

\[
H_{\text{free}} = \text{Tr} \int d\mathbf{x} \left( \Pi^2(\mathbf{x}) + B^2_{\text{A}}(\mathbf{x}) \right),
\]

where \( B_{\text{A}} = \nabla \times A \) is the abelian component of the color magnetic field. The nonabelian contribution, \( B^2(\mathbf{x}) - B^2_{\text{A}}(\mathbf{x}) \), is included in \( H_{PT} \) (see below). Here \( B = B^a T^a \) is the complete magnetic field having components

\[
B^a_i = \epsilon_{ijk} \nabla_j A^a_k + \frac{g}{2} \epsilon_{ijk} f^{abc} A^b_j A^c_k,
\]

with bare coupling constant \( g \). As in [7] the phenomenological Hamiltonian is taken to be

\[
H_{\text{phen}} = -\frac{1}{2} \int d\mathbf{x} d\mathbf{y} \rho^a(\mathbf{x}) V_L(|\mathbf{x} - \mathbf{y}|) \rho^a(\mathbf{y}),
\]

with color charge density (only gluon component) \( \rho^a(\mathbf{x}) = f^{abc} A^b(\mathbf{x}) \Pi^c(\mathbf{x}) \) and linear confining potential

\[
V_L(|\mathbf{x} - \mathbf{y}|) = \sigma |\mathbf{x} - \mathbf{y}|,
\]

with string tension, \( \sigma = 0.18 \text{ GeV}^2 \), in accordance with lattice calculations and Regge phenomenology. The approximate residual interaction is given by the sum, \( H_{PT} = H_{NA} + H_{C} \),
with the nonabelian part including three-, $H_{3g}$, and four-gluon, $H_{4g}$, interactions (see Appendix A)

$$H_{NA} = \text{Tr} \int d\mathbf{x} \left( \mathbf{B}^2(\mathbf{x}) - \mathbf{B}_A^2(\mathbf{x}) \right) = H_{3g} + H_{4g}, \quad (3.7)$$

and the Coulomb interaction

$$H_C = -\frac{1}{2} \int d\mathbf{x} d\mathbf{y} \rho^a(\mathbf{x}) V_C(|\mathbf{x} - \mathbf{y}|) \rho^a(\mathbf{y}), \quad (3.8)$$

$$V_C(|\mathbf{x} - \mathbf{y}|) = -C_{adj} \frac{\alpha_s}{|\mathbf{x} - \mathbf{y}|}, \quad (3.9)$$

having $\alpha_s = \frac{g^2}{4\pi}$ and adjoint Casimir operator $C_{adj} = (N_c^2 - 1)/2N_c$ (here $N_c = 3$ colors). An advantage of the Coulomb gauge is the straightforward implementation of phenomenological potentials. We define the instantaneous interaction, $H_{L+C}$, by the combination

$$H_{L+C} = H_{\text{phen}} + H_C, \quad (3.10)$$

containing the sum, $V_L + V_C$, of linear and Coulomb potentials.

Because of the phenomenological interaction, the trivial, perturbative vacuum is not the minimum ground state for the QCD motivated Hamiltonian, $H_{QCD}$. Instead, we introduce a trial nonperturbative vacuum state $|0\rangle$ which can be determined variationally, analogous to BSC type vacuum studies. The Fock space is constructed from this vacuum using quasiparticle operators $a^a_i(\mathbf{k}), a^{a\dagger}_i(-\mathbf{k})$ which appear in the field operator expansions

$$A_i^a(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{k}}} [a^a_i(\mathbf{k}) + a^{a\dagger}_i(-\mathbf{k})] e^{i\mathbf{kx}}$$

$$\Pi_i^a(\mathbf{x}) = -i \int \frac{d\mathbf{k}}{(2\pi)^3} \sqrt{\frac{\omega_\mathbf{k}}{2}} [a^a_i(\mathbf{k}) - a^{a\dagger}_i(-\mathbf{k})] e^{i\mathbf{kx}}, \quad (3.11)$$

with $a^a_i(\mathbf{k})|0\rangle = 0$. Here $\omega_\mathbf{k}$ is the variational gluon energy from the gap equation (see below), obtained by minimizing the ground state (vacuum) energy. The canonical commutation relation is

$$[a^a_i(\mathbf{k}), a^{b\dagger}_j(\mathbf{k}')] = (2\pi)^3 \delta^{ab} \delta^{(3)}(\mathbf{k} - \mathbf{k}') D_{ij}(\mathbf{k}), \quad (3.12)$$
where the gluon operators \( a_i^a(k) = \sum_{\lambda=1,2} \epsilon_i(k, \lambda)a^a(k, \lambda) \) are transverse, i.e. \( k \cdot a^a(k) = k \cdot a^a(k) = 0 \), and \( D_{ij}(k) \) is a polarization sum

\[
D_{ij}(k) = \sum_{\lambda=1,2} \epsilon_i(k, \lambda)\epsilon_j(k, \lambda) = \delta_{ij} - \hat{k}_i\hat{k}_j,
\]

(3.13)

with unit vector component \( \hat{k}_i = k_i/k \) and \( \hat{k}_i \cdot D_{ij}(k) = 0 \).

The complete QCD motivated Hamiltonian, Eq. (3.1), can now be expressed in second quantized form and normal ordered with respect to the trial vacuum. In addition to two-body interactions, normal ordering leads to condensate (vacuum expectations) and one-body operators, both having contributions from the instantaneous and four-gluon terms. The final expressions are complicated and are summarized in Appendix A.

Finally the flow equations are applied to \( H_{QCD} = H_{free} + H_{L+C} + H_{3g} + H_{4g} \). Separating the diagonal, \( H_d \), particle number conserving and off-diagonal, \( H_r \), particle number changing parts yields

\[
H_d = \left( H_{free}^{(0)} \right)_{nn} + \left( H_{L+C}^{(0)+2} \right)_{nn} + \left( H_{4g}^{(2)} \right)_{nn},
\]

\[
H_r = \left( H_{3g}^{(1)} \right)_{nm} + \left( H_{free}^{(2)} \right)_{nm} + \left( H_{C}^{(2)} \right)_{nm} + \left( H_{4g}^{(2)} \right)_{nm},
\]

(3.14)

where \( nn \) (\( nm \)) indicates particle number conserving (changing) matrix elements. Here and throughout, the superscript denotes the order (power) in the bare coupling constant. Note that the triple gluon Hamiltonian, \( H_{3g}^{(1)} \), which is first order, only contributes off-diagonally, while the four-gluon Hamiltonian, \( H_{4g}^{(2)} \), which is second order, contributes to both \( H_d \) and \( H_r \). Also, as discussed previously, the linear confining potential, which is zeroth order, is restricted to diagonal contributions. All terms are detailed in Appendix A along with the free Hamiltonian operator, containing the kinetic \( (K, K^{(0)}) \) and condensate \( (O, O^{(0)}) \) terms, which contributes to the diagonal part in lowest order, \( H_{free}^{(0)} = K^{(0)} + O^{(0)} \), and to the off-diagonal part in second order, \( H_{free}^{(2)} = (K - K^{(0)}) + (O - O^{(0)}) \).

Having identified the diagonal and rest parts we now construct the generator, \( \eta \), and solve the flow equations for the first two leading orders. The leading order \( O(g) \) (in coupling constant) flow equation is
\[
\frac{dH_{3g}^{(1)}}{dl} = [\eta^{(1)}(l), K^{(0)}(l)],
\]
\[
\eta^{(1)}(l) = [K^{(0)}(l), H_{3g}^{(1)}(l)].
\]

(3.15)

Here the kinetic and triple-gluon vertex terms are given (see Appendix A) respectively by

\[
K^{(0)}(l) = \int \frac{dk}{(2\pi)^3} \omega(k;l)a_i^{a_l}(k)a_i^a(k),
\]

(3.16)

and in symmetrized form

\[
H_{3g}^{(1)}(l) = \frac{i}{2\sqrt{2}} f^{abc} \int \left( \prod_{n=1}^{3} \frac{dk_n}{(2\pi)^3} \right) (2\pi)^3 \delta^{(3)}(\sum m k_m) \frac{1}{\sqrt{\omega_1\omega_2\omega_3}} \Gamma_{ijk}(k_1, k_2, k_3)
\times: \left[ g_0(k_1, k_2, k_3; l)a_i^{a_l}(k_1)a_j^{b_l}(k_2)a_k^{a_l}(k_3) + 3g_1(k_1, k_2, k_3; l)a_i^{a_l}(k_1)a_j^{b_l}(k_2)a_k^{a_l}(k_3) + \text{h.c.} \right],
\]

(3.17)

with shorthand notation \(\omega_1 = \omega(k_1)\), etc. and \(\Gamma_{ijk}\) defined by

\[
\Gamma_{ijk}(k_1, k_2, k_3) = \frac{1}{6} \left[ (k_1 - k_3)_j \delta_{ik} + (k_2 - k_1)_k \delta_{ij} + (k_3 - k_2)_l \delta_{jk} \right],
\]

satisfying

\[
\Gamma_{ijk}(-k_1, -k_2, -k_3) = -\Gamma_{ijk}(k_1, k_2, k_3),
\]

\[
\Gamma_{jik}(k_2, k_1, k_3) = -\Gamma_{ijk}(k_1, k_2, k_3).
\]

(3.18)

Notice that in implementing the flow equations new, effective coupling constants are generated (see below), \(g_0(k_1, k_2, k_3; l)\), \(g_1(k_1, k_2, k_3; l)\), which like the energy, \(\omega(k;l)\), depend upon the flow parameter \(l\). The coupling constants also acquire an effective energy/momentum dependence corresponding to a given Fock sector (different Fock sector operators flow differently in energy, indicated by 0 and 1 subscripts here and below). In leading order we omit the dependence of gluon energy on \(l\) in the triple vertex. The leading order generator, Eq. (3.13), is

\[
\eta^{(1)}(l) = \frac{i}{2\sqrt{2}} f^{abc} \int \left( \prod_{n=1}^{3} \frac{dk_n}{(2\pi)^3} \right) (2\pi)^3 \delta^{(3)}(\sum m k_m) \frac{1}{\sqrt{\omega_1\omega_2\omega_3}} \Gamma_{ijk}(k_1, k_2, k_3)
\times: \left[ \eta_0(k_1, k_2, k_3; l)a_i^{a_l}(k_1)a_j^{b_l}(k_2)a_k^{a_l}(k_3) + 3\eta_1(k_1, k_2, k_3; l)a_i^{a_l}(k_1)a_j^{b_l}(k_2)a_k^{a_l}(k_3) + \text{h.c.} \right],
\]

(3.19)

where
\[ \eta_0(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; l) = D_0(\omega_1, \omega_2, \omega_3)g_0(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; l), \]
\[ \eta_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; l) = D_1(\omega_1, \omega_2, \omega_3)g_1(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3; l), \] (3.20)

with energy terms
\[ D_0(\omega_1, \omega_2, \omega_3) = -(\omega_1 + \omega_2 + \omega_3), \quad D_1(\omega_1, \omega_2, \omega_3) = -(-\omega_1 + \omega_2 + \omega_3). \] (3.21)

The solutions of the flow equations, Eq. (3.15), for the effective coupling constants are
\[ g_0(l) = g(0)\exp(-D_0^2l), \quad g_1(l) = g(0)\exp(-D_1^2l), \] (3.22)
which, as anticipated, eliminates the triple-gluon vertex, Eq. (3.17), for \( l \to \infty \). Correspondingly, new operators in the particle number conserving sectors are generated, found from the second order \( O(g^2) \) flow equation
\[ \frac{dH^{(2)}_d(l)}{dl} = [\eta^{(1)}(l), H^{(1)}_{3g}(l)], \] (3.23)
which contribute to the effective two-body, self-energy and condensate operators in the effective Hamiltonian.  

Solving the flow equations, Eq. (3.15) and Eq. (3.23), the block-diagonal effective Hamiltonian, \( H_{\text{eff}} \), renormalized to the second order, is obtained (details in Appendix B)
\[ H_{\text{eff}} = H_d(\Lambda) + H_{\text{gen}}(\Lambda) + \delta X_{\text{CT}}(\Lambda), \] (3.25)
where \( H_d(\Lambda) \) is the particle number conserving part of the original Hamiltonian, Eq. (3.14), and \( H_{\text{gen}}(\Lambda) \) includes new operators generated via perturbative elimination of \( H_r \). The cut-off parameter, \( \Lambda \), regulates UV divergences in loop integrals. Renormalization is achieved through second order by adding the mass counterterm, \( \delta X_{\text{CT}}(\Lambda) \),

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\(^1\) One can always eliminate the particle number changing terms, \( H_r^{(2)} \), appearing in second order, by choosing the generator \( \eta^{(2)} = [K^{(0)}, H_r^{(2)}] \). The flow equation reads
\[ \frac{dH_r^{(2)}}{dl} = [\eta^{(1)}(l), H^{(1)}_{3g}(l)] + [\eta^{(2)}, K^{(0)}], \] (3.24)
and, due to the second term in Eq. (3.24), \( H_r^{(2)} \) decays exponentially with flow parameter.
\[ \delta X_{CT}(\Lambda) = m_{CT}^2(\Lambda) \text{Tr} \int d\mathbf{x} \mathbf{A}^2(\mathbf{x}), \]
\[ m_{CT}^2(\Lambda) = -\frac{\alpha_s}{\pi} N_c \frac{11}{6} \Lambda^2, \tag{3.26} \]

which absorbs the leading UV divergences in \( H_{eff} \) as \( \Lambda \to \infty \). Therefore we omit the cut-off notation in \( H_{eff} \), Eq. (3.25), because, through perturbative renormalization, the cut-off sensitivity is weak (see below).

We summarize the matrix elements of \( H_{eff} \) in the sectors of interest (up to two gluon states) \(^2\)

\[
\langle 0 | H_{eff} | 0 \rangle = O_{\text{ren}} + O_{L+C}(\Lambda) + O_{4g}(\Lambda) + O_{\text{gen}}(\Lambda),
\]
\[
\langle 1 | H_{eff} | 1 \rangle = K_{\text{ren}} + \Pi_{L+C}(\Lambda) + \Pi_{4g}(\Lambda) + \Pi_{\text{gen}}(\Lambda),
\]
\[
\langle 2 | H_{eff} | 2 \rangle = V_{L+C} + V_{4g} + V_{\text{gen}}, \tag{3.27}
\]

where \( |0\rangle \) is a shorthand notation for the zero-gluon sector (also vacuum state), \( |1\rangle \) the one-gluon sector, etc. One should distinguish between two types of \( H_{eff} \) terms. The first one arises from normal ordering the original Hamiltonian: the instantaneous interaction with linear plus Coulomb potentials, labeled by \( L + C \), and the four-gluon vertex, labeled by \( 4g \). This leads to the polarization operators \( \Pi_{L+C} \) and \( \Pi_{4g} \) in the one-body sector, and condensate terms \( O_{L+C} \) and \( O_{4g} \) in the zero-body sector. The energy of the ground state \( O \) comes from normal ordering the free Hamiltonian \( H_{\text{free}} \) with respect to the vacuum \( |0\rangle \). As shown below, given the phenomenological potential \( V_L \), the energy \( O \) of the ground state reproduces the known result \( \omega_0 \) from sum rules for the nonperturbative gluon condensate \(^{[21]}\).

The second type of terms are dynamical operators, generated by flow equations and labeled \( \text{gen} \). In Eq. (3.27) the renormalized condensate \( O_{\text{ren}} \) and kinetic \( K_{\text{ren}} \) terms are

\[
O_{\text{ren}} = O + \delta X_{CT}^{0\text{body}} = \frac{1}{2} (N_c^2 - 1) V \int \frac{dk}{(2\pi)^3} \left( \frac{k^2 + m_{CT}^2(\Lambda)}{\omega_k} + \omega_k \right),
\]

\(^2\)Since calculations are done through second order in the coupling constant, new terms are generated only up to the two-body sector (no three-body or higher interactions).
\[ K^{\text{ren}} = K + \delta X_{CT}^{\text{1body}} = \int \frac{dk}{(2\pi)^3} \omega_k a_i^\dagger(k)a_i(k) + \frac{1}{2} \int \frac{dk}{(2\pi)^3} \frac{(k^2 + m^2_{CT}(\Lambda)) - \omega_k}{\omega_k} \delta X + \frac{1}{2} \int \frac{dk}{(2\pi)^3} \frac{(k^2 + m^2_{CT}(\Lambda)) - \omega_k}{\omega_k}, \]  

where \( O \) and \( K \) are defined by Eqs. (A1) and (A3), respectively, and \( \delta X_{CT}^{\text{1body}} \) is the mass counterterm, given by Eq. (3.26) in the zero-body sector (analogous for \( \delta X_{CT}^{\text{0body}} \)). The mass counterterms \( \delta X_{CT} \) cancel the leading \( \Lambda^2 \) contribution from radiative corrections to the vacuum and kinetic terms. The subleading logarithmic dependence remains, producing a slow cut-off dependence of eigenvalues (see below). In Eq. (3.27) the corrections to \( O^{\text{ren}} \) and \( K^{\text{ren}} \), regulated by the exponential cut-off function, include the condensate terms (Appendix B)

\[ O_{L+C}(\Lambda) = \frac{1}{16} N_c (N_c^2 - 1) V \int \frac{dk \ dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q} \tilde{V}_{L+C}(k - q)(\omega_k - \omega_q)^2 \left(1 + (\hat{\omega}_k q)^2\right) e^{-(|q| + |k|)^2/\Lambda^2}, \]

\[ O_{4g}(\Lambda) = \frac{\alpha_s \pi}{4} N_c (N_c^2 - 1) V \int \frac{dk \ dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q} \left(3 - (\hat{\omega}_k q)^2\right) e^{-(|q| + |k|)^2/\Lambda^2}, \]

\[ O_{\text{gen}}(\Lambda) = -\alpha_s \pi N_c (N_c^2 - 1) V \int \frac{dk \ dq}{(2\pi)^6} \frac{1}{3 \omega_k \omega_q \omega_{k - q} \ (\omega_k + \omega_q + \omega_{k - q})} e^{-(|q| + |k| + |k - q|)^2/\Lambda^2}, \]

and the polarization operators (Appendix B)

\[ \Pi_{L+C}(\Lambda) = \frac{1}{8} N_c \int \frac{dk \ dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q} \tilde{V}_{L+C}(k - q)(1 + (\hat{\omega}_k q)^2) e^{-q^2/\Lambda^2} \]

\[ \times \left( (\omega_k^2 + \omega_q^2) a_i^\dagger(k) a_i(q) + (\omega_k^2 - \omega_q^2) \frac{1}{2} (a_i^\dagger(k) a_i(q)(-k) + \text{h.c.}) \right), \]

\[ \Pi_{4g}(\Lambda) = \frac{\alpha_s \pi}{2} N_c \int \frac{dk \ dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q} (3 - (\hat{\omega}_k q)^2) e^{-q^2/\Lambda^2} \]

\[ \times \left( a_i^\dagger(k) a_i(q) + \frac{1}{2} (a_i^\dagger(k) a_i(q)(-k) + \text{h.c.}) \right), \]

\[ \Pi_{\text{gen}}(\Lambda) = -\alpha_s \pi N_c \int \frac{dk \ dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q \omega_{k - q}} \frac{G(k, q)}{\omega_k + \omega_{k - q}} e^{-q^2/\Lambda^2} \]

\[ \times \left( a_i^\dagger(k) a_i(q) + \frac{1}{2} (a_i^\dagger(k) a_i(q)(-k) + \text{h.c.}) \right), \]

containing a tensor structure in the generated term

\[ G(k, q) = 2(1 - (\hat{\omega}_k q)^2) \left(k^2 + q^2 + \frac{k^2 q^2}{2(k - q)^2}(1 + (k \hat{q})^2)\right), \]
and $\tilde{V}_{L+C}$ is the momentum space representation of the linear plus Coulomb potentials (see Appendix A). Also, here and throughout $\hat{k}\hat{q} = \hat{k} \cdot \hat{q}$. The effective gluon interaction in the color singlet channel is

$$V_{\text{eff}} = \int \frac{d^4k d^4q}{(2\pi)^6} \frac{1}{\omega_k \omega_q} V_{ii'jj'}(k, q) :a_{j}^{b\dagger}(q)a_{j'}^{b\dagger}(-q)a_{i}^{a}(k)a_{i'}^{a}(\tilde{k}) :,$$

(3.33)

which describes the bound state of a glueball at rest. It includes the three interactions, $V_{L+C} + V_{4g} + V_{\text{gen}}$, that define the effective Hamiltonian in the two-body sector $\langle 2|H_{\text{eff}}|2 \rangle$, Eq. (3.27). Explicitly (see Appendix B)

$$V_{ii'jj'}^{L+C} = -\frac{1}{8} \tilde{V}_{L+C}(k - q)(\omega_k + \omega_q)^2 \delta_{ij} \delta_{i'j'} ,$$

$$V_{ii'jj'}^{4g} = \frac{\alpha_s \pi}{2} (\delta_{ii'} \delta_{jj'} - \delta_{ij} \delta_{i'j}) ,$$

$$V_{ii'jj'}^{\text{gen}} = -\alpha_s \frac{2\pi}{\omega_{k-q}^2} T_{ij,i'j'}(k, q) \left( 1 - \frac{(\omega_k - \omega_q)^2}{(\omega_k - \omega_q)^2 + \omega_{k-q}^2} \right) ,$$

(3.34)

where the tensor term in the generated interaction is

$$T_{ij,i'j'}(k, q) = (k - q)^2 \left( n_i n_{i'} \delta_{jj'} + n_j n_{i'} \delta_{ii'} + n_i n_j \delta_{i'j'} + n_{i'} n_j \delta_{ij} - n_i n_j \delta_{i'j'} + n_{i'} n_j \delta_{ij} - n_i n_j \delta_{i'j'} \right) + \frac{k^2 q^2}{(k - q)^4} (1 - (\hat{k}\hat{q})^2) \delta_{ij} \delta_{i'j'} ,$$

(3.35)

and $n_i$ are the components of the unit vector $\mathbf{n} = (k - q)/|k - q|$. All other possible two-body interactions that change particle number can be eliminated by the flow equations through second order and only contribute in higher orders. In the next section we utilize this effective, gluodynamical Hamiltonian, with matrix elements given by Eq. (3.27), to obtain and solve the gluon gap and glueball bound state equations.

### IV. GAP EQUATION AND GLUEBALLS

Now that we have the block diagonal, effective Hamiltonian, $H_{\text{eff}}$, valid through second order in the coupling constant, we can nonperturbatively diagonalize the $PH_{\text{eff}}P$ sector

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3 We do not consider color octet states.
Hamiltonian. Further, because $H_{\text{eff}}$ is also renormalized, equations for physical observables are free from the leading UV divergences. In subsection A we first investigate the vacuum by formulating and solving the gluon gap equation and also calculate the gluon condensate. Then we address the glueball spectrum in subsection B.

### A. Gap Equation

The gap equation allows determination of a nontrivial vacuum with gluon condensates and propagating quasiparticles, here gluons with a dynamical mass. There are several ways to obtain this equation, the most common based upon a variational principle to minimize the vacuum (ground state) energy. In the Bogoliubov-Valatin or BCS approach, the variational parameter is the angle of transformation from undressed to dressed particle (quasiparticle) operators. In this work the variational parameter is the quasiparticle energy, $\omega_k$, which defines a quasiparticle basis, Eq. (3.11), where the effective Hamiltonian is block-diagonal. Accordingly, minimizing the vacuum energy of the effective Hamiltonian

\[
\frac{\delta \langle 0 | H_{\text{eff}} | 0 \rangle}{\delta \omega_k} = 0,
\]

generates the gap equation for the unknown $\omega_k$. Using Eq. (3.30) for the condensate terms the following gap equation is obtained

\[
\omega_k^2 = k^2 + m_{CT}^2(\Lambda) + \frac{1}{4} N_c \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\omega_q^2} V_{L+C}(\mathbf{k} - \mathbf{q}) \left(1 + (\hat{k} \hat{q})^2\right) \left(\omega_q^2 - \omega_k^2\right) e^{-q^2/\Lambda^2} + \alpha_s \pi N_c \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\omega_q^2} \left(3 - (\hat{k} \hat{q})^2\right) e^{-q^2/\Lambda^2} - 2\alpha_s \pi N_c \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{\omega_q^2 \omega_{k-q} \omega_q + \omega_{k-q}} e^{-4q^2/\Lambda^2},
\]

where the mass counterterm $m_{CT}(\Lambda)$ is given by Eq. (3.26) and $G(k, q)$ by Eq. (3.32). Note that a momentum regulating function has been introduced in the first two integrals, while the third is naturally regulated from the flow equations.

The gap equation can also be obtained by using a renormalization group condition, namely the invariance of Hamiltonian form under the renormalization group transformation. Since flow equations are in this class of transformations, the renormalized effective Hamiltonian should not contain off-diagonal one-body terms of the type $a(\mathbf{k})a(-\mathbf{k})$ or $a^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})$. 

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Imposing this condition on the matrix elements of the effective Hamiltonian in the one-gluon sector $\langle 1|H_{\text{eff}}|1 \rangle$, Eq. (3.27) and Eq. (3.31), yields the same gap equation as above.

There are two types of UV divergences in the gap equation: the leading, $\Lambda^2$, and the subleading, $\ln \Lambda$, corresponding to relevant and marginal operators in the context of the renormalization group, respectively. Terms from the canonical Hamiltonian $H_{\text{PT}}$ generate both types, while the confining interaction $i\lambda$ from $H_{\text{NP}}$ only generates a $\ln \Lambda$ dependence. Renormalization is achieved by finding the counterterms associated with the canonical terms, since only the canonical part is renormalizable. We do not renormalize the confining interaction as this would only lead to ”noncanonical” divergences in each order of perturbation theory. The mass counterterm, Eq. (3.26), cancels the leading UV divergency in the gap equation, Eq. (4.2), while the uncanceled part leads to a gap energy with a logarithmic cut-off dependence, $\omega(k, \Lambda)$.

We have numerically solved the linearized gap equation retaining only the dominant, instantaneous term, with Coulomb and confining potentials, which leads to a different corresponding Coulomb mass counterterm

$$\tilde{m}^2_{\text{CT}}(\Lambda) = -\frac{\alpha_s}{\pi} \frac{2N_c}{3} \Lambda^2,$$

instead of $m^2_{\text{CT}}$ which is only appropriate for the full gap equation. For fixed cut-off $\Lambda = 4 \text{ GeV}$, the energy dispersion, $\omega(k)$, is displayed in Fig. (1). The free behavior, $\omega(k) = k$, is recovered at high energies while for low energies a constituent gluon mass, roughly $0.9 \text{ GeV}$ is obtained. As shown in Fig. (2), the analytic form, $\omega(k) = k + m(k)$, with a dynamical gluon mass, $m(k)$, given by

$$m(k) = m(0) \exp\left(-\frac{k}{\kappa}\right),$$

reproduces the numerical solution using parameters $m(0) = 0.9 \text{ GeV}$ and $\kappa = 0.95 \text{ GeV}$. The sensitivity of the effective gluon mass at zero momentum, $m(0)$, to cut-off $\Lambda$ is displayed in Fig. (3).

The gluon condensate is another nonperturbative quantity calculated within this approach. The condensate is given by the vacuum expectation value $O = \langle 0|\Pi^2 + B_A^2|0 \rangle$, Eq.
(3.28), but not renormalized by a counterterm. We regulate this value by subtracting the perturbative contribution with \( \omega(k) = k \) yielding

\[
\langle \frac{\alpha_s}{\pi} F_{\mu\nu}^a F_{\mu\nu}^a \rangle = \frac{N_c^2 - 1}{\pi^3} \int_0^\infty dk k^2 \alpha_s \frac{(\omega(k) - k)^2}{2\omega(k)}.
\] (4.5)

Using the above dispersion relation, \( \omega(k) \), and a cut-off \( \Lambda = 4 \text{ GeV} \) that we used in our previous work \cite{7}, the computed gluon condensate is \( 0.013 \text{ GeV}^4 \) in agreement with sum rule results \cite{21}. The condensate cut-off sensitivity is shown in Fig. \( (\text{I}) \). It is interesting that using the same cut-off as in \cite{7}, which calculated a condensate of \( 0.012 \text{ GeV} \), produces a similar result even though the two calculations are quite different. The logarithmic dependence of the effective gluon mass and condensate can both be absorbed by the running coupling constant \( \alpha_s(\Lambda) \).

**B. Glueball Bound State Equation**

We model the glueball bound state as two valence constituent gluons. The glueball wave function in the rest frame is

\[
|\psi_n\rangle = \int \frac{d\mathbf{q}}{(2\pi)^3} \phi_n^{ij}(\mathbf{q}) a_i^{\alpha \dagger}(\mathbf{q}) a_j^{\alpha \dagger}(-\mathbf{q}) |0\rangle,
\] (4.6)

where repeated indices are summed and \( a_i^{\alpha \dagger}(\mathbf{q}) \) creates a quasiparticle from the nontrivial vacuum \( |0\rangle \) with energy \( \omega(\mathbf{q}) \). Since in this constituent basis our effective Hamiltonian \( H_{eff} \) is block-diagonal, mixing with three gluon and higher states is suppressed. Therefore we can directly diagonalize in the two gluon \( P \) space and our bound state equation follows by projecting the fundamental equation \( H_{eff} |\psi_n\rangle = E_n |\psi_n\rangle \) on the \( P \) space

\[
\langle \psi_n | [H_{eff}, a_i^{\alpha \dagger}(\mathbf{q}) a_j^{\alpha \dagger}(-\mathbf{q})] |0\rangle = (E_n - E_0) X_{ij'}^{ij}(\mathbf{q}) \phi_n^{i'j'}(\mathbf{q}) ,
\] (4.7)

where

\[
X_{ij'}^{ij}(\mathbf{q}) \phi_n^{i'j'}(\mathbf{q}) = D_{ii'}(\mathbf{q}) D_{jj'}(\mathbf{q}) \phi_n^{i'j'}(\mathbf{q}) + D_{ij'}(\mathbf{q}) D_{ji'}(\mathbf{q}) \phi_n^{i'j'}(-\mathbf{q}) .
\] (4.8)
The polarization term, $D_{ij}$, is given by Eq. (3.13) and $H_{eff}$ by Eq. (3.25), with matrix elements given in Eq. (3.27). Equation (4.7) determines the glueball mass, $M_n = E_n - E_0$, where $E_0$ is the vacuum energy defined by $H_{eff}|0\rangle = E_0|0\rangle$.

Using our $H_{eff}$ we have evaluated Eqs. (4.7,4.8) and summarize in Appendix C the complete form of the bound state equation with all possible terms through second order. As mentioned in the introduction and important to repeat, even though our analysis has the appearance of the TDA, we do not make this approximation (truncation to the two gluon sector). Quite the contrary, we include the truncation corrections which introduces complicated new terms in the bound state equation as detailed in Appendix C and below.

As a numerical application we consider here only the dominant, instantaneous part of the kernel, neglecting perturbative transverse gluon exchange and terms from the four-gluon vertex. The bound state equation for glueball states having total angular momentum, $J$, parity, $P$, and charge conjugation, $C$, is then

$$M_n \phi_n(q) = \left[ \frac{q^2 + \tilde{m}_{CT}^2(\Lambda)}{\omega_q} + \omega_q \right] \phi_n(q) + \frac{1}{4} N_c \int \frac{d^3 p}{(2\pi)^3} \tilde{V}_{L+}(p-q) \left( 1 + (\hat{p}\hat{q})^2 \right) \frac{\omega_p^2 + \omega_q^2 e^{-p^2/\Lambda^2}}{\omega_p \omega_q} \phi_n(p) - \frac{1}{8} N_c \int \frac{d^3 p}{(2\pi)^3} \tilde{V}_{L+C}(p-q) \frac{(\omega_p + \omega_q)^2}{\omega_p \omega_q} F^{JPC}(p, q) \phi_n(p).$$

(4.9)

The spectroscopic terms, $F^{JPC}$, for the scalar, $0^{++}$, and pseudoscalar, $0^{-+}$, states are

$$F^{0^{++}}(p, q) = 1 + (\hat{p}\hat{q})^2,$n explosives (4.10)

Here the Coulomb counterterm, $\tilde{m}_{CT}(\Lambda)$, is given by Eq. (1.3) and $\tilde{V}_{L+C}$ is the sum of linear and Coulomb potentials in momentum space.

Besides the UV divergences from the Coulomb interaction, the bound state equation, Eq. (4.9), also has an IR singularity from the confining potential. The UV divergences in the first integral are regulated by the exponential cut-off function and the leading divergent part is canceled by the mass counterterm $\tilde{m}_{CT}(\Lambda)$. The UV behavior of the second integral
is regulated by the wave function, which decreases with increasing momenta. Although the kinetic (self-energy term) and potential parts both contribute IR divergent pieces, the bound state equation is IR finite due to complete cancelation for small momenta. This cancelation only occurs for the color singlet state \[19\].

Numerical solutions of Eq. \([4.9]\) are obtained variationally with a set of gaussian test functions. Results for the ground state scalar and pseudoscalar glueball masses, and first excited states, are presented in Table \[\] and compared with recent quenched lattice data \[22\], generated with an improved SII action and anisotropic lattice. It is more consistent to compare with quenched lattice results, rather than improved unquenched treatments, since we also omit the quark sector. Notice agreement is better in the scalar channel and that the mass of this state is about twice the constituent gluon mass.

**V. CONCLUSIONS AND OUTLOOK**

In this work we have described in detail a systematic procedure for solving a bound state problem in QCD. Our study is motivated by the success of the constituent quark model. Central to our approach are the flow equations, which we have applied perturbatively to a QCD motivated Hamiltonian with confining interaction. The flow equations generated a block diagonal, effective Hamiltonian which provides an improved framework for vacuum and excited hadron state investigations. In the block diagonal process, the effective Hamiltonian was also renormalized through second order in the coupling constant. In particular, we determined the appropriate mass counterterm to cancel the leading UV divergences.

Applications to the gluon vacuum and excited glueball spectrum produced a nonlinear gap equation and a bound state equation that is superior to the TDA. Numerical solutions reproduced the QCD sum rule gluon condensate value and also provided reasonable agreement with quenched lattice glueball calculations. Our results support the constituent picture for gluonia.

While we have only considered pure gluodynamics in this work, the same approach can
be directly applied to the full QCD Hamiltonian, including dynamical quarks, where the
issue of chiral symmetry breaking is present. Of keen interest will be the structure of the
full gap and bound state equations in the combined quark and gluon sectors. In particular,
application to exotic four quark and hybrid systems will provide significant new information,
especially the degree of mixing between $q\bar{q}$, $qq\bar{q}$, $q\bar{q}g$, and $gg$ states. Much of this work is
in progress and will be reported in a future publication.

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APPENDIX A: COMPLETE QCD MOTIVATED HAMILTONIAN

Here we specify, in second quantized form, the full QCD motivated Hamiltonian of glu-
dynamics in the Coulomb gauge, decomposed in the basis Eq. (3.11). One-body operators
and condensate terms arise from normal ordering with respect to the trial vacuum state $|0\rangle$.
The upper index over a Hamiltonian operator (e.g. $K^{(0)}$, $H^{(1)}$) denotes the order (power)
in (bare) coupling constant.

**Free gluon part** : Eq. (3.4) includes the gluon kinetic energy

$$K = \frac{1}{2} \int \frac{dk}{(2\pi)^3} \left\{ \frac{k^2}{\omega_k} a_i^{a\dagger}(k) a_i^a(k) + \left( \frac{k^2}{\omega_k} - \omega_k \right) \frac{1}{2} (a_i^a(k) a_i^a(-k) + h.c.) \right\}$$

$$= \int \frac{dk}{(2\pi)^3} \omega_k a_i^{a\dagger}(k) a_i^a(k) + \frac{1}{2} \int \frac{dk}{(2\pi)^3} \left( \frac{k^2}{\omega_k} - \omega_k \right)$$

$$\times (a_i^{a\dagger}(k) a_i^a(k) + \frac{1}{2} (a_i^a(k) a_i^a(-k) + h.c.)) = K^{(0)} + (K - K^{(0)}) \, , \tag{A1}$$

with

$$K^{(0)} = \int \frac{dk}{(2\pi)^3} \omega_k a_i^{a\dagger}(k) a_i^a(k) \, , \tag{A2}$$
and condensate term

\[ O = \frac{1}{2}(N_c^2 - 1)V \int \frac{dk}{(2\pi)^3} \left( \frac{k^2}{\omega_k} + \omega_k \right) \]

\[ = (N_c^2 - 1)V \int \frac{dk}{(2\pi)^3} \omega_k + \frac{1}{2}(N_c^2 - 1)V \int \frac{dk}{(2\pi)^3} \frac{k^2}{\omega_k} - \omega_k = O^{(0)} + (O - O^{(0)}) , \quad (A3) \]

with

\[ O^{(0)} = (N_c^2 - 1)V \int \frac{dk}{(2\pi)^3} \omega_k , \quad (A4) \]

and volume \( V = (2\pi)^3 \delta(3)(0) \).

**Instantaneous interaction:** Eq. (3.10) includes the linear confining and Coulomb interactions

\[ H^{(2)}_{L+C} = -\frac{1}{8} f^{abc} f^{ade} \int \left( \prod_{n=1}^{4} \frac{dk_n}{(2\pi)^3} \right) (2\pi)^3 \delta^{(3)}(\sum m k_m) \left( \frac{\omega_2 \omega_4}{\omega_1 \omega_3} \right)^{1/2} \tilde{V}_{L+C}(k_1 + k_2) \]

\[ \times: [a_i^b(k_1) + a_i^{b\dagger}(-k_1)] [a_i^c(k_2) - a_i^{c\dagger}(-k_2)] [a_j^d(k_3) + a_j^{d\dagger}(-k_3)] [a_j^e(k_4) - a_j^{e\dagger}(-k_4)] : , \]

where the supercript \( (2) \) refers to the Coulomb term, which is second order, and \( \tilde{V}_{L+C}(k) \) is

\[ \tilde{V}_{L+C}(k) = 2\pi C_{adj} \frac{\alpha_s}{k^2} + 4\pi \frac{\sigma}{k^4} . \quad (A6) \]

Terms arising from normal ordering are the one-body operator

\[ \Pi_{L+C} = \frac{N_c}{4} \int \frac{dk dq}{(2\pi)^6} \tilde{V}_{L+C}(k + q) \left( \frac{\omega_q^2 + \omega_k^2}{\omega_k \omega_q} \right) D_{ij}(q) \left[ a_i^{a\dagger}(k) a_j^a(k) \right] \]

\[ + \frac{N_c}{8} \int \frac{dk dq}{(2\pi)^6} \tilde{V}_{L+C}(k + q) \left( \frac{\omega_q^2 - \omega_k^2}{\omega_k \omega_q} \right) D_{ij}(q) \left[ a_i^a(k) a_j^{a\dagger}(k) + h.c. \right] , \quad (A7) \]

and the condensate term

\[ O_{L+C} = \frac{1}{8} N_c (N_c^2 - 1)V \int \frac{dk dq}{(2\pi)^6} \tilde{V}_{L+C}(k + q) \left( \frac{\omega_q}{\omega_k} - 1 \right) \left( 1 + (k q)^2 \right) . \quad (A8) \]

**Nonabelian gluon part:** Eq. (3.7) includes to order \( O(g) \) the triple-gluon coupling

\[ H^{(1)}_{3g} = \frac{i g}{2\sqrt{2}} f^{abc} \int \left( \prod_{n=1}^{3} \frac{dk_n}{(2\pi)^3} \right) (2\pi)^3 \delta^{(3)}(\sum_m k_m) \frac{k_{1,j}}{\sqrt{\omega_1 \omega_2 \omega_3}} \]

\[ \times: [a_i^a(k_1) + a_i^{a\dagger}(-k_1)] [a_j^b(k_2) + a_j^{b\dagger}(-k_2)] [a_k^c(k_3) + a_k^{c\dagger}(-k_3)] : , \quad (A9) \]
where $\omega_1 \equiv \omega_{k1}$, etc. In order $O(g^2)$, the normal ordered four-gluon vertex is

$$H_{4g}^{(2)} = \frac{\alpha_s \pi}{4} f^{abc} f^{ade} \int \left( \prod_{n=1}^{4} \frac{dk_n}{(2\pi)^3} \right) (2\pi)^3 \delta^{(3)} \left( \sum_m k_m \right) \frac{1}{\sqrt{\omega_1 \omega_2 \omega_3 \omega_4}}$$  \tag{A10}

$\times \left[ a_i^{\dagger}(k_1) + a_i^b(-k_1) \right] \left[ a_j^{\dagger}(k_2) + a_j^c(-k_2) \right] \left[ a_i^{\dagger}(k_3) + a_i^d(-k_3) \right] \left[ a_j^{\dagger}(k_4) + a_j^e(-k_4) \right]$.

Also, after normal ordering one obtains a one-body operator

$$\Pi_{4g} = \alpha_s \pi N_c \int \frac{dk \, dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q} \left[ 2\delta_{ij} - D_{ij}(q) \right]$$

$$\times \left[ a_i^{\dagger}(k)a_j^{\dagger}(k) + \frac{1}{2} \left( a_i^a(k)a_j^a(-k) + \text{h.c.} \right) \right],$$  \tag{A11}

and a condensate term

$$O_{4g} = \frac{\alpha_s \pi}{4} N_c (N_c^2 - 1) V \int \frac{dk \, dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q} \left( 3 - \langle \hat{k} q \rangle^2 \right).$$  \tag{A12}

One-body and condensate terms diverge in the UV region and must be regulated.

**APPENDIX B: SECOND ORDER FLOW EQUATIONS**

Solving the second order flow equation for the particle number conserving part $H_d$, Eq. (3.23), generates three types of Hamiltonian operators: two-body effective interactions, one-body polarization terms and condensates. We consider each separately.

1. **Effective Interaction (two-body sector)**

We calculate an effective gluon interaction in the color singlet channel for glueball Fock states. The general form of an effective interaction is given by Eq. (3.33) and includes interactions from the original Hamiltonian and a new interaction, generated to second order by flow equations in the two-body sector. Consider first the generated terms. The $t$-channel diagrams, arising from the commutator $[a_1^a b_2^b c_3^c, a_1^{\dagger a} b_2^{\dagger b} c_3^{\dagger c}]$ (diagram without backward motion) and from the commutator $[a_1^a b_2^b c_3^c, a_1^{\dagger a} b_2^{\dagger b} c_3^{\dagger c}]$ (Z-graph in t-channel), do not
contribute to the color singlet state; only the s-channel diagram, coming from the commutator \([a_{i1}^a a_{j2}^b, a_{i3k}^c, a_{i'1}^{b'} a_{j'2}^{d'} a_{j''k'}^{e''}]\), needs to be calculated (here the notation is \(a_{i1}^a = a^a_i(k_i)\), etc.).

In the c.m. frame the flow equation for the effective interaction is

\[
\frac{dV_{\text{gen}}(l)}{dl} = \frac{1}{8} \int f_{abc} f_{a'b'c} \int \frac{dk dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q \omega_{k-q}}
\]

\[
\times \Gamma_{ijk}(k, -q, -(k - q)) \Gamma_{i'j'k'}(k, -q, -(k - q)) D_{kk'}(k - q) \cdot 4 \cdot 9
\]

\[
\times \left[ \eta_i(q, k, k - q; l) g_i(k, q, k - q; l) + \eta_i(k, q, k - q; l) g_i(q, k, k - q; l) \right]
\]

\[
\times :a_j^{b'}(q) a_{j'}^{b''}(q) a_i^a(\bar{q}) a_i^a(\bar{q}) :,
\]

where the factor 4 is the number of permutations and the property of \(\Gamma\)-factors, Eq. (3.18), is used. The generators are given in Eq. (3.20) and coupling constants in Eq. (3.22). The \(\eta g\) sum corresponds to two different time-ordered s-channel diagrams. We introduce the factor \(S_{ijk, i'j'k'}(k, q)\) by

\[
\Gamma_{ijk}(k, -q, -(k - q)) \Gamma_{i'j'k'}(k, -q, -(k - q)) = \frac{1}{6} 4 S_{ijk, i'j'k'}(k, q).
\]

which, due to transversality, can be expressed as

\[
S_{ijk, i'j'k'}(k, q) = (k_j \delta_{ik} - \frac{1}{2} (k + q)_k \delta_{ij} + q_i \delta_{jk})(k_{i'} \delta_{k'k} - \frac{1}{2} (k + q)_{k'} \delta_{i'j'} + q_{i'} \delta_{j'k'}),
\]

and is symmetric under interchange of \(k\) and \(q\). The tensor structure of the generated interaction is defined by the contraction \(S_{ijk, i'j'k'}(k, q) D_{kk'}(k - q) \equiv T_{ij, i'j'}(k, q)\), where

\[
T_{ij, i'j'}(k, q) = \left( q_i q_{i'} \delta_{jj'} + k_j k_{i'} \delta_{ii'} + q_i (k_{i'} \delta_{jj'} - k_j \delta_{i'j'}) + q_{i'} (k_j \delta_{ij'} - k_{i'} \delta_{ij}) \right)
\]

\[
+ \frac{k^2 q^2}{(k - q)^2} \left( (k - q)_j \delta_{i'j'} \right)
\]

\[
= (k - q)^2 \left( n_i n_{i'} \delta_{jj'} + n_j n_{i'} \delta_{ii'} + n_i n_j \delta_{i'j'} + n_{i'} n_j \delta_{ij} - n_i n_{i'} \delta_{jj'} - n_i n_{i'} \delta_{jj'} \right)
\]

\[
+ \frac{k^2 q^2}{(k - q)^4} \left( (k - q)_j \delta_{ij} \delta_{i'j'} \right),
\]

and \(n_i\) are the components of the unit vector \(n = (k - q)/|k - q|\). Integrating the flow equation, Eq. (3.31), we obtain the generated interaction \(V_{\text{gen}}(l \to \infty) \equiv V_{\text{gen}}\) (the initial value is \(V_{\text{gen}}(l = 0) = 0\).
\[ V_{\text{gen}} = \alpha_s 2\pi f^{abc} f^{a'b'c} \int \frac{d\mathbf{k} d\mathbf{q}}{(2\pi)^6} \frac{1}{\omega_k \omega_q \omega_{k-q}} T_{ij,i'j'}(\mathbf{k}, \mathbf{q}) \frac{D_1 + D_{1'}}{D_1^2 + D_{1'}^2} \times :a_j^b(\mathbf{q}) a_{j'}^{b'}(\mathbf{-q}) a_i^a(\mathbf{k}) a_{i'}^{a'}(\mathbf{-k}): \]

with

\[ D_1 = -(-\omega_k + \omega_q + \omega_{k-q}), \quad D_{1'} = -(-\omega_q + \omega_k + \omega_{k-q}). \]

This reduces to

\[ V_{\text{gen}} = -\alpha_s 2\pi f^{abc} f^{a'b'c} \int \frac{d\mathbf{k} d\mathbf{q}}{(2\pi)^6} \frac{1}{\omega_k \omega_q \omega_{k-q}} T_{ij,i'j'}(\mathbf{k}, \mathbf{q}) \left( 1 - \frac{(\omega_k - \omega_q)^2}{(\omega_k - \omega_q)^2 + \omega_{k-q}^2} \right) \times :a_j^b(\mathbf{q}) a_{j'}^{b'}(\mathbf{-q}) a_i^a(\mathbf{k}) a_{i'}^{a'}(\mathbf{-k}):. \]

Now consider the second order gluon interactions that conserve particle number. The generator \( \eta^{(1)} \), Eq. (B.19), produces second order interactions that change the number of quasiparticles, for example, an interaction \( W \) having form \( a_{i_1}^{a_1} a_{j_2}^{b_2} a_{k_3}^{c_3} a_{l_4}^{d_4} \). However, the generator choice, \( \eta = \eta^{(1)} + \eta^{(2)} \), eliminates \( W \), where the second order generator is calculated from \( \eta^{(2)} = [K^{(0)}, W] \).

To maintain gauge invariance we also include the normal ordered instantaneous, Eq. (A3), and the four-gluon, Eq. (A10), interactions of the original Hamiltonian in the color-singlet channel. Projecting these terms on the color singlet state yields

\[ V_{L+C} = -\frac{1}{8} f^{abc} f^{a'b'c} \int \frac{d\mathbf{k} d\mathbf{q}}{(2\pi)^6} \frac{(\omega_k + \omega_q)^2}{\omega_k \omega_q} \bar V_{L+C}(\mathbf{k} - \mathbf{q}) \delta_{ij} \delta_{i'j'} :a_j^b(\mathbf{q}) a_{j'}^{b'}(\mathbf{-q}) a_i^a(\mathbf{k}) a_{i'}^{a'}(\mathbf{-k}):. \]

\[ V_{4g} = \frac{\alpha_s}{4} f^{abc} f^{a'b'c} \int \frac{d\mathbf{k} d\mathbf{q}}{(2\pi)^6} \frac{2}{\omega_k \omega_q} (\delta_{i'j'} \delta_{ij} - \delta_{i'j} \delta_{ij'}) :a_j^b(\mathbf{q}) a_{j'}^{b'}(\mathbf{-q}) a_i^a(\mathbf{k}) a_{i'}^{a'}(\mathbf{-k}):. \]

The complete effective gluon interaction is given by

\[ V_{\text{eff}} = V_{L+C} + V_{4g} + V_{\text{gen}}, \]

with the generated interaction defined by Eq. (B7).

2. Polarization Operator (one-body sector)

Two contraction terms from the commutator \( [\eta^{(1)}, H_{3g}^{(1)}] \) contribute to one-body operators.
\[
\Pi = \int \frac{dk}{(2\pi)^3} \left( \Pi_{ij}^a(k) a_i^a(k) a_j^b(k) + M_{ij}^a(k) \frac{1}{2} \left( a_i^a(k) a_j^b(-k) + \text{h.c.} \right) \right), \quad (B10)
\]

with polarization amplitudes, \( \Pi_{ij}^a(k) \), \( M_{ij}^a(k) \), given by gluon loop integrals of the form

\[
\Pi_{ij}^a(k) = \int \frac{dq}{(2\pi)^3} \bar{\Pi}_{ij}^a(k, q). \quad (B11)
\]

The specific integrand structure is given below. Note, from rotational invariance, the one-body operators \( \Pi_{ij}^a(k) \) and \( M_{ij}^a(k) \) have tensor structure, \( A^{ab}(k)\delta_{ij} + B^{ab}(k)\hat{k}_i\hat{k}_j \), however, due to transversality, only the term proportional to \( \delta_{ij} \) survives after integrating Eq. (B10). Hence, we only need to extract \( A^{ab}(k) \) which can be achieved by choosing \( k = k_z \) and considering components \( i = j = x \) and \( y \) but \( \neq z \). Then in Eq. (B11), where \( \bar{\Pi}_{ij}^a(k, q) \) also has terms of the type \( \delta_{ij} \) and \( \hat{q}_i\hat{q}_j \) (there are no terms proportional to \( \hat{k}_i \) and \( \hat{k}_j \)), one can substitute

\[
\hat{q}_i\hat{q}_j \rightarrow \frac{1}{2}(1 - (\hat{k}_q)^2)\delta_{ij}, \quad (B12)
\]

to extract the \( \delta_{ij} \) component in Eq. (B11). From the second order flow equation, Eq. (B.23), the gluon polarization operator is

\[
\frac{d\Pi(l)}{dl} = \frac{1}{8} N_c \delta^{aa'} \int \frac{dk dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q \omega_{k-q}} \times \Gamma_{ijk}(k, -q, -(k - q)) \Gamma'_{j'k'}(k, -q, -(k - q)) D_{j'k'}(q) D_{kk'}(k - q) \cdot 2 \cdot 9 \cdot 2
\]
\[
\left[ (\eta_1(k, q, k - q; l) g_1(k, q, k - q; l) + \eta_0(k, q, k - q; l) g_0(k, q, k - q; l)) a_i^{a'}(k) a_j^{a'}(k) + (\eta_1(k, q, k - q; l) g_0(k, q, k - q; l) + \eta_0(k, q, k - q; l) g_1(k, q, k - q; l)) \right.
\]
\[
\times \frac{1}{2} \left( a_i^a(k) a_j^{a'}(-k) + \text{h.c.} \right) \right], \quad (B13)
\]

where \( \eta_1 g_1 \) corresponds to a gluon loop without backward motion, \( \eta_0 g_0 \), a \( Z \)-graph, and the last two terms \( \eta_1 g_0 \) and \( \eta_0 g_1 \), are gluon loop diagrams with two incoming and two outgoing gluon lines. In calculating Eq. (B13) we have also used the property, Eq. (B.18), for \( \Gamma \)-factors.

The polarization operator tensor structure is given by a double contraction of the \( S(k, q) \) factor, Eq. (B3), with polarization terms in Eq. (B13). We then define
\[ F_{ii'}(k, q) \equiv S_{ijk,i'j'k'}(k, q)D_{ij'}(q)D_{kk'}(k-q) = \delta_{ii'} k^2 q^2 \left( \frac{1}{q^2} + \frac{1}{(k-q)^2} \right) (1 - \hat{k} \hat{q})^2 + 2q_i q'_i \left( 1 - \frac{k^2}{2(k-q)^2} (1 - \hat{k} \hat{q})^2 \right), \] (B14)

with repeated indices summed. Due to transversality, only the \( \delta_{ii'} \) component is retained, therefore from Eq. (B12)

\[ F_{ii'}(k, q) \rightarrow \delta_{ii'} G(k, q)/2, \]

\[ G(k, q) = 2(1 - \hat{k} \hat{q})^2 \left( k^2 + q^2 + \frac{k^2 q^2}{2(k-q)^2} (1 + \hat{k} \hat{q})^2 \right). \] (B15)

We integrate the flow equation, Eq. (B13), with the generators and coupling constants given by Eq. (3.20) and Eq. (3.22), respectively, producing the second order polarization operator correction, \( \delta \Pi = \Pi(l) - \Pi(l_0 = 0) \). The polarization operator, at a scale \( l = 1/\lambda^2 \), is then

\[ \Pi_{\text{gen}}(\lambda) = \alpha_s \pi N_c \int \frac{dk dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q \omega_{k-q}} G(k, q) \times \left[ \left( \frac{1}{2D_0} e^{-2D_0^2/\lambda^2} + \frac{1}{2D_1} e^{-2D_1^2/\lambda^2} \right) a_i^a(k) a_i^a(k) + \frac{D_0 + D_1}{D_0^2 + D_1^2} e^{-(D_0^2 + D_1^2)/\lambda^2} \frac{1}{2} (a_i^a(k) a_i^a(-k) + \text{h.c.}) \right], \] (B16)

with

\[ D_0 = -\left( \omega_k + \omega_q + \omega_{k-q} \right), \quad D_1 = -\left( -\omega_k + \omega_q + \omega_{k-q} \right). \] (B17)

For large momenta flowing in the loop, one has \( D_0 \sim D_1 \sim -\left( \omega_q + \omega_{k-q} \right) \sim -2\omega_q \). Finally, rescaling the cut-off, \( \lambda \rightarrow \sqrt{2} \lambda \), which does not change results, the polarization operator generated by the flow equations through second order is

\[ \Pi_{\text{gen}}(\lambda) = -\alpha_s \pi N_c \int \frac{dk dq}{(2\pi)^6} \frac{G(k, q)}{\omega_k \omega_q \omega_{k-q}} \frac{1}{\omega_q + \omega_{k-q}} e^{-4q^2/\lambda^2} \times \left( a_i^a(k) a_i^a(k) + \frac{1}{2} (a_i^a(k) a_i^a(-k) + \text{h.c.}) \right), \] (B18)

where in the exponential factor we have used the free dispersion relation, \( \omega_q = q \), valid for large cut-off values. We then calculate one-body operators, by normal ordering the instantaneous interactions and the four-gluon vertex, Eq. (A5) and Eq. (A10), respectively,
and again extract the $\delta_{ij}$ component using Eq. (B12). The regulated polarization operators are

$$\Pi_{L+C}(\lambda) = \frac{1}{8} N_c \int \frac{dkdq}{(2\pi)^6} \frac{1}{\omega_k \omega_q} \tilde{V}_{L+C}(k - q)(1 + (\hat{k}q)^2)e^{-q^2/\lambda^2}$$

$$\times \left((\omega_q^2 + \omega_k^2)a_i^a(k)a_i^a(k) + (\omega_q^2 - \omega_k^2)\frac{1}{2}(a_i^a(k)a_i^a(-k) + h.c.)\right)$$ (B19)

$$\Pi_{4g}(\lambda) = \frac{\alpha_s \pi}{2} N_c \int \frac{dkdq}{(2\pi)^6} \frac{1}{\omega_k \omega_q} (3 - (\hat{k}q)^2)e^{-q^2/\lambda^2}$$

$$\times \left(a_i^a(k)a_i^a(k) + \frac{1}{2}(a_i^a(k)a_i^a(-k) + h.c.)\right).$$

We choose the same regulating function to match energy denominators. Note that in the generated term the regulator emerges naturally from the flow equations, while the normal ordered terms require introducing a loop integral regulator with form consistent with the generated term. The complete polarization operator is then

$$\Pi(\lambda) = \Pi_{L+C}(\lambda) + \Pi_{4g}(\lambda) + \Pi_{\text{gen}}(\lambda).$$ (B20)

We conclude this subsection by specifying the counterterm for the leading UV divergences in the polarization operator. The instantaneous (Coulomb only), four-gluon and generated terms all contribute to the quadratically UV divergent part, $\Pi^{\text{div}}$, with respective weights, $1, 2, -1/4$

$$\Pi^{\text{div}}(\Lambda) = \frac{\alpha_s}{\pi} \frac{2N_c}{3} \Lambda^2 \left(1 + 2 - \frac{1}{4}\right)$$

$$\times \int \frac{dk}{(2\pi)^3} \frac{1}{2\omega_k} \left(a_i^a(k)a_i^a(k) + \frac{1}{2}(a_i^a(k)a_i^a(-k) + h.c.)\right).$$ (B21)

The appropriate counterterm is then

$$\delta X'_{CT}(\Lambda) = \int \frac{dk}{(2\pi)^3} \frac{m^2_{CT}(\Lambda)}{2\omega_k} \left(a_i^a(k)a_i^a(k) + \frac{1}{2}(a_i^a(k)a_i^a(-k) + h.c.)\right)$$

$$= \frac{m^2_{CT}(\Lambda)}{2} \int dx :A_i^a(x)A_i^a(x):,$$ (B22)

with mass

$$m^2_{CT}(\Lambda) = -\frac{\alpha_s}{\pi} N_c \frac{11}{6} \Lambda^2.$$ (B23)

When the quark sector is included using this same procedure, the counterterm coefficient reproduces the QCD $\beta$-function. This result was also obtained in our previous study \[9\].
3. Gluon Condensate (zero-body sector)

The second order flow equation, Eq. (3.23), for the condensate term is

\[
\frac{dO_{\text{gen}}(l)}{dl} = \frac{1}{8} N_c(N_c^2 - 1)V \int \frac{d\mathbf{k}d\mathbf{q}}{(2\pi)^6} \frac{1}{\omega_k \omega_q \omega_{k-q}} \times \Gamma_{ijk}(\mathbf{k}, -\mathbf{q}, -(\mathbf{k} - \mathbf{q})) \Gamma_{i'j'k'}(\mathbf{k}, -\mathbf{q}, -(\mathbf{k} - \mathbf{q})) D_{ii'}(\mathbf{k}) D_{jj'}(\mathbf{q}) D_{kk'}(\mathbf{k} - \mathbf{q}) \cdot 6 \cdot 2 \\
\times \eta_0(\mathbf{k}, \mathbf{q}, \mathbf{k} - \mathbf{q}; l) g_0(\mathbf{k}, \mathbf{q}, \mathbf{k} - \mathbf{q}; l),
\]

where only the commutator \( [a_1^a a_2^b a_3^c, a_1^{a'} a_2^{b'} a_3^{c'}] \) contributes to the vacuum expectation value. The factor of 6 is the number of permutations and the volume is \( V = (2\pi)^3 \delta^{(3)}(0) \).

The tensor structure of the condensate term is given by the triple contraction of the S-factor, Eq. (B3), with polarization terms, i.e. \( S_{ijk,i'j'k'}(\mathbf{k}, \mathbf{q}) D_{ii'}(\mathbf{k}) D_{jj'}(\mathbf{q}) D_{kk'}(\mathbf{k} - \mathbf{q}) \equiv G(\mathbf{k}, \mathbf{q}) \), where the function \( G(\mathbf{k}, \mathbf{q}) \) is given by Eq. (B13). Integrating yields the correction, \( \delta O = O(l) - O(l_0 = 0) \), where \( O(l) \) is a condensate for flow parameter \( l \), related to the energy scale \( \lambda \) by \( l = 1/\lambda^2 \). The resulting generated condensate term through second order is

\[
O_{\text{gen}}(\lambda) = \alpha_s \pi N_c(N_c^2 - 1)V \int \frac{d\mathbf{k}d\mathbf{q}}{(2\pi)^6} \frac{1}{\omega_k \omega_q \omega_{k-q}} \frac{G(\mathbf{k}, \mathbf{q})}{3D_0} e^{-2D_0^2/\lambda^2},
\]

where the energy difference \( D_0 \) is given by Eq. (B14). The three regulated condensate terms, instantaneous, Eq. (A8), four-gluon, Eq. (A12), and generated can be summarized

\[4 \text{ In the gap equation calculations we use the symmetrized form} \]

\[
O_{\text{gen}}(\lambda) = \alpha_s \pi N_c(N_c^2 - 1)V \int \frac{d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3}{(2\pi)^9} \frac{1}{\omega_1 \omega_2 \omega_3} \times (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \frac{\tilde{G}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)}{3D_0} e^{-2D_0^2/\lambda^2},
\]

where \( D_0 \) is given in Eq. (3.21), and

\[
\tilde{G}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) = \left( (k_1^2 + k_2^2 + k_3^2)(1 - 2(\hat{k}_1 \hat{k}_2)(\hat{k}_1 \hat{k}_3)(\hat{k}_2 \hat{k}_3)) + k_1(\hat{k}_1 \hat{k}_2)^2(\hat{k}_1 \hat{k}_3)^2 + k_2(\hat{k}_1 \hat{k}_2)^2(\hat{k}_2 \hat{k}_3)^2 + k_3(\hat{k}_1 \hat{k}_2)^2(\hat{k}_3 \hat{k}_3)^2 \right) \\
\times (2\pi)^3 \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) \to G(\mathbf{k}, \mathbf{q}) .
\]
\[ O_{L+C}(\lambda) = \frac{1}{8}N_c(N_c^2 - 1)V \int \frac{dk dq}{(2\pi)^6} \hat{V}(k - q) \left( \frac{\omega_q}{\omega_k} - 1 \right) \left( 1 + (\hat{k}q)^2 \right) e^{-|q|+|k|/\lambda^2} \]

\[ O_{4g}(\lambda) = \frac{\alpha_s \pi}{4} N_c(N_c^2 - 1)V \int \frac{dk dq}{(2\pi)^6} \frac{1}{\omega_k \omega_q} \left( 3 - (\hat{k}q)^2 \right) e^{-|q|+|k|/\lambda^2} \]  \hspace{1cm} (B26)

\[ O_{\text{gen}}(\lambda) = -\alpha_s \pi N_c(N_c^2 - 1)V \int \frac{dk dq}{(2\pi)^6} \frac{G(k, q)}{3\omega_k \omega_q \omega_{k-q}} (\omega_k + \omega_q + \omega_{k-q}) e^{-|q|+|k|+|k-q|/\lambda^2}. \]

The regulating procedure is the same as above. The complete radiative correction to the gluon condensate, Eq. (A3), is then

\[ O(\lambda) = O_{L+C}(\lambda) + O_{4g}(\lambda) + O_{\text{gen}}(\lambda). \]  \hspace{1cm} (B27)

Finally, we determine the counterterm for the leading UV divergences in the zero-body sector. Again, the instantaneous (only Coulomb part), four-gluon and generated terms contribute, respectively

\[ O^{\text{div}}(\Lambda) = \frac{\alpha_s}{\pi} \frac{2N_c(N_c^2 - 1)V}{3} \Lambda^2 \left( 1 + 2 - \frac{1}{4} \right) \int \frac{dk}{(2\pi)^3} \frac{1}{2\omega_k}. \]  \hspace{1cm} (B28)

Hence the counterterm is

\[ \delta X''_{CT}(\Lambda) = (N_c^2 - 1)V \int \frac{dk}{(2\pi)^3} \frac{m_{CT}^2(\Lambda)}{2\omega_k} = \frac{m_{CT}^2(\Lambda)}{2} \int d\mathbf{x} \langle 0| A_i^a(x) A_i^a(x) |0 \rangle, \]  \hspace{1cm} (B29)

where the mass \( m_{CT}(\Lambda) \) is given by Eq. (B23). Combining the counterterms from one-body, Eq. (B22), and zero-body, Eq. (B29), sectors yields

\[ \delta X_C = \delta X'_C + \delta X''_{CT} = m_{CT}^2(\Lambda) \text{Tr} \int d\mathbf{x} A^2(\mathbf{x}), \]  \hspace{1cm} (B30)

which is the complete mass counterterm in the effective Hamiltonian, renormalized through second order, Eq. (B.23). Note, that the one- and zero-body counterterms have the same mass coefficient, \( m_{CT}(\Lambda) \). This suggests one can work directly in a field theoretical basis without a Fock operator decomposition.
APPENDIX C: COMPLETE GLUEBALL BOUND STATE EQUATION

In this appendix we present the complete bound state equation, including all second order terms. Projecting the effective Hamiltonian, Eq. (3.25), on the two-gluon Fock basis, Eq. (4.7), generates the bound state equation for a glueball at rest with mass $M = E - E_0$ (we suppress the excited state quantum index number)

\[
M \left( X_{in}^{ij}(q)\phi_{+}^{ln}(q) + Y_{in}^{ij}(q)\phi_{-}^{ln}(q) \right) = \left[ \frac{g^2 + m_{\mu\nu}^2\Lambda}{\omega_{q}} + \omega_{q} \right] \nonumber
\]

\[
+ \frac{1}{4}N_c \int \frac{dp}{(2\pi)^3} \tilde{V}_{L+C}(p - q) \left( (\hat{q}^2)^2 - \frac{\omega_{p}^2 + \omega_{q}^2}{\omega_{p}\omega_{q}} e^{-p^2/\Lambda^2} \right) \nonumber
\]

\[
+ \alpha_s \pi N_c \int \frac{d \mathbf{p}}{(2\pi)^3} \frac{1}{\omega_{p}\omega_{q}} \left( 3 - (\hat{k}^2)^2 \right) e^{-p^2/\Lambda^2} \nonumber
\]

\[
- \alpha_s \pi N_c \int \frac{d \mathbf{p}}{(2\pi)^3} \frac{1}{\omega_{p}\omega_{q}\omega_{p-\mathbf{q}}} G(p, q) e^{-4p^2/\Lambda^2} \left[ X_{in}^{ij}(q)\phi_{+}^{ln}(q) + Y_{in}^{ij}(q)\phi_{-}^{ln}(q) \right] \nonumber
\]

\[
+ \left[ -\frac{1}{8}N_c \int \frac{dp}{(2\pi)^3} \tilde{V}_{L+C}(p - q) \frac{(\omega_{p} + \omega_{q})^2}{\omega_{p}\omega_{q}} \left( X_{km}(q)X_{ln}(p)\phi_{+}^{ln}(p) + Y_{km}(q)Y_{ln}(p)\phi_{-}^{ln}(p) \right) \right. \nonumber
\]

\[
+ \alpha_s \pi N_c \int \frac{d \mathbf{p}}{(2\pi)^3} \frac{2}{\omega_{p}\omega_{q}} \left( D_{ij}(q)D_{ln}(p)\phi_{+}^{ln}(p) - \frac{1}{4}X_{km}(q) \left[ X_{ln}(p)\phi_{+}^{ln}(p) + Y_{ln}(p)\phi_{-}^{ln}(p) \right] \right) \nonumber
\]

\[
+ \alpha_s 2 \pi N_c \int \frac{d \mathbf{p}}{(2\pi)^3} \frac{1}{\omega_{p}\omega_{q}\omega_{p-\mathbf{q}}} \left( \omega_{p} - \omega_{q} \right) \left[ X_{kn'}(q)X_{ln'}(p)\phi_{+}^{ln}(p) + Y_{kn'}(q)Y_{ln'}(p)\phi_{-}^{ln}(p) \right] \right], \tag{C1}
\]

where the factor $T_{km,k'm'}$ is defined in Eq. (B4) and

\[
\phi_{+}^{ln}(p) = \frac{1}{2}(\phi^{ln}(p) \pm \phi^{ln}(-p)) \nonumber
\]

\[
X_{ij}(q) = D_{ij}(q)D_{jn}(q) + D_{in}(q)D_{jl}(q) \nonumber
\]

\[
Y_{ij}(q) = D_{ij}(q)D_{jn}(q) - D_{in}(q)D_{jl}(q), \tag{C2}
\]

involving the polarization sum $D_{ij}(q)$, Eq. (3.13), and glueball momentum wavefunction, $\phi_{+}^{ln}(p)$, introduced in Eq. (4.6).

The bound state equation can be further reduced by forming the appropriate tensor contractions for $\phi^{ij}(p)$ corresponding to a specific glueball state having quantum numbers $J$, total angular momentum, $P$, parity, and $C$, charge conjugation. For the applications in
this paper we considered the scalar \((J^{PC} = 0^{++})\) and pseudoscalar \((J^{PC} = 0^{-+})\) channels having tensor contractions \(\phi^{ij}(q) = \delta^{ij}\phi(q)\) and \(\phi^{ij}(q) = \epsilon^{ijk}\hat{q}^k\phi(q)\), respectively. The resulting equation describing both \(J = 0\) states is

\[
M\phi(q) = \left[ \left( \frac{q^2 + m^2_{\text{CT}}(\Lambda)}{\omega_q} + \omega_q \right) \right]
\]

+ \(\frac{1}{4} N_c \int \frac{dp}{(2\pi)^3} \tilde{V}_{L+C}(p - q) \left( 1 + (\hat{p} \hat{q})^2 \right) \frac{\omega_p + \omega_q}{\omega_p \omega_q} e^{-p^2/\Lambda^2}
\]

+ \(\alpha_s \pi N_c \int \frac{dp}{(2\pi)^3} \frac{1}{\omega_p \omega_q} \left( 3 - (\hat{k} \hat{q})^2 \right) e^{-p^2/\Lambda^2}
\]

- \(\alpha_s \pi N_c \int \frac{dp}{(2\pi)^3} \frac{1}{\omega_p \omega_q \omega_{p-q}} \frac{G(p, q)}{\omega_p + \omega_{p-q}} \phi(p)
\]

+ \(\left[ -\frac{1}{8} N_c \int \frac{dp}{(2\pi)^3} \tilde{V}_{L+C}(p - q) \left( \omega_p + \omega_q \right)^2 \frac{e^{-4p^2/\Lambda^2}}{\omega_p \omega_q} \right] \phi(p)
\]

+ \(\alpha_s 2\pi N_c \int \frac{dp}{(2\pi)^3} \frac{1}{\omega_p \omega_q} \frac{D^{J^{PC}}(p, q)}{\omega_{p-q}} \left( 1 - \frac{(\omega_p - \omega_q)^2}{(\omega_p - \omega_q)^2 + \omega_{p-q}^2} \right) \phi(p)
\]

(C3)

with spectroscopic terms \(D^{J^{PC}}, E^{J^{PC}}\) and \(F^{J^{PC}}\) for each channel given by

\[F^{0^{++}}(p, q) = 1 + (\hat{p} \hat{q})^2\]

\[F^{0^{-+}}(p, q) = 2(\hat{p} \hat{q})\]

\[E^{0^{++}}(p, q) = 1\]

\[E^{0^{-+}}(p, q) = 0\]

\[D^{0^{++}}(p, q) = G(p, q) = 2(1 - (\hat{p} \hat{q})^2) \left( p^2 + q^2 + \frac{p^2 q^2}{2(p-q)^2} (1 + (\hat{p} \hat{q})^2) \right)\]

\[D^{0^{-+}}(p, q) = 2p^2 q^2 (1 - (\hat{p} \hat{q})^2) \left( \frac{2}{pq} + \frac{1}{(p-q)^2}(\hat{p} \hat{q}) \right)\]

(C4)

An approximate form for Eq. (C3) is used (see Eq. (4.9)) in the main text for numerical calculations.
REFERENCES

[1] H. Nielson and P. Oleson, Nucl. Phys. B61, 45 (1973); Y. Nambu, Phys. Rev. D 10, 4262 (1974).

[2] N. Isgur and J. Paton, Phys. Rev. D 31, 2910 (1985).

[3] Y. A. Simonov, Nucl. Phys. B307, 512 (1988) and B324, 67 (1989); H. G. Dosch, Phys. Lett. 190B, 177 (1987); H. G. Dosch and Y. A. Simonov, Phys. Lett. 205B, 339 (1988).

[4] T. Barnes, Z. Phys. C 10, 275 (1981); P. Hasenfratz, R. R. Horgan, J. Kuti, and J. M. Richard, Phys. Lett. 95B, 229 (1981); T. Barnes, F. E. Close, and F. De Viron, Nucl. Phys. B224, 241 (1983); M. Chanowitz and S. Sharpe, Nucl. Phys. B222, 211 (1983).

[5] R. L. Jaffe and K. Johnson, Phys. Lett. 60B, 201 (1976).

[6] E. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane, and T. M. Yan, Phys. Rev. D 21, 203 (1980); E. Eichten and F. Feinberg, Phys. Rev. D 23, 2724 (1981).

[7] A. Szczepaniak, E. S. Swanson, C.-R. Ji, and S. R. Cotanch, Phys. Rev. Lett. 76, 2011 (1996).

[8] F. J. Llanes-Estrada and S. R. Cotanch, Phys. Rev. Lett. 84, 1102 (2000).

[9] D. G. Robertson, E. S. Swanson, A. P. Szczepaniak, C.-R. Ji, and S. R. Cotanch, Phys. Rev. D 59, 074019 (1999).

[10] F. Wegner, Ann. Physik 3, 77 (1994).

[11] E. Gubankova and F. Wegner, Phys. Rev. D 58, 025012 (1998), hep-th/9710233.

[12] E. Gubankova, H. C. Pauli, F. Wegner, and G. Papp, hep-th/9809143; E. Gubankova and G. Papp, hep-th/9904081.

[13] S. K. Kehrein and A. Mielke, J. Phys. A 27, 4259 (1994); Ann. Phys. (N.Y.) 252, 1 (1996).
[14] S. K. Kehrein, A. Mielke, and P. Neu, Z. Phys. B 99, 269 (1996); S. K. Kehrein and A. Mielke, Ann. Phys. (Leipzig) 6, 90 (1997); J. Stat. Phys. 90, 889 (1998).

[15] P. Lenz and F. Wegner, Nucl. Phys. B482, 693 (1996).

[16] St. D. G/ąszek and K. G. Wilson, Phys. Rev. D 48, 5863 (1993); ibid. 49, 4214 (1994).

[17] D. Zwanziger, Nucl. Phys. B485, 185 (1997); preprint NYU-TH PH97205.

[18] A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, Phys. Rev. D 29, 1233 (1984); A. Le Yaouanc, L. Oliver, S. Ono, O. Pène, and J.-C. Raynal, ibid. 31, 137 (1985).

[19] S. Adler and P. Davis, Nucl. Phys. B244, 469 (1984); J. R. Finger and J. E. Mandula, Nucl. Phys. B199, 168 (1982).

[20] P. Bicudo, N. Brambilla, E. Ribeiro and A. Vairo, Phys. Lett. 442A, 349 (1998) and references there in.

[21] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. B147, 385 (1980).

[22] C. Morningstar and M. Peardon, hep-lat/9901004.
TABLE I. Glueball spectrum for the first two scalar and pseudoscalar states 
($\alpha_s = \frac{g^2}{4\pi} = 0.4$, $\sigma = 0.18\, GeV^2$, $\Lambda = 4\, GeV$).

| $J^{PC}$       | 0$^{++}$ | 0*$^{++}$ | 0$^{-+}$ | 0*$^{-+}$ |
|----------------|----------|-----------|----------|-----------|
| calculated (MeV)| 1760     | 2697      | 2142     | 2895      |
| lattice data [22] (MeV) | 1730(80) | 2670(130) | 2590(130) | 3640(180) |
FIG. 1. One particle dispersion relation, $\omega(q)$. Dots are the numerical solution of the gap equation for parameters $\alpha_s = 0.4, \sigma = 0.18\, GeV^2$ and $\Lambda = 4\, GeV$. The solid line is the free dispersion, $\omega(q) = q$.

FIG. 2. Dynamical gluon mass. Dots represent the numerical solution for $m(q) = \omega(q) - q$ (same parameters as in Fig. 1). Solid line is $m(q) = 0.9\exp(-q/0.95)$ (parameters are in GeV).
FIG. 3. Cut-off dependence of the constituent gluon mass (same parameters as in Fig. 1).

FIG. 4. Gluon condensate cut-off dependence (same parameters as in Fig. 1).