Notes on Loewy series of centers of $p$-blocks

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Abstract

The present paper describes some results on the Loewy series of the center of a modular group algebra in order to solve a problem on the number of irreducible ordinary characters. For instance, we prove that a $p$-block of a finite group has at least $p+2$ characters if its defect group contains non-elementary center.

1 Introduction

In this paper we consider a problem on the number of irreducible ordinary characters of a finite group through some studies of the center of a modular group algebra.

Let $G$ be a finite group and $F$ an algebraically closed field of characteristic $p > 0$. For a $p$-block $B$ of the group algebra $FG$ with defect $d$, we denote by $k(B)$ the number of associated irreducible ordinary characters. In [4] Héthelyi and Külshammer conjectured that:

$$2\sqrt{p - 1} \leq k(B) \text{ unless } d = 0?$$

This is a refinement of a result by Maróti [10] (see also [9]) and is still unsolved in general. In modular representation theory it is known that $k(B)$ is equal to the dimension of the center $ZB$ of $B$. We accordingly examine the algebraic structure of $ZB$ in order to solve the problem above.

In the next section we prove the following results.

We first determine the Loewy structure of $ZB$ for cyclic defect groups. More precisely, it is shown that each codimension of the Loewy series of $ZB$ is equal to 1 or the inertial index of $B$. Moreover we refer to a remark on $ZB$ related to the problem above.

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The second subsection deals with bounds for the Loewy length \( LL(ZB) \), that is, the nilpotency index of the Jacobson radical of \( ZB \). Some papers [7, 8, 11, 12] have discussed its upper bounds: for example, Okuyama has stated \( LL(ZB) \leq p^d \). The main purpose of this section is to give a lower bound for \( LL(ZB) \). As an immediate corollary to this bound, we conclude that \( B \) has at least \( p + 2 \) characters provided its defect group contains non-elementary center.

Throughout this paper we use the following notations.

We fix a defect group \( D \) of order \( p^d \). Let \( N_{G}(D, b) \) be the inertial subgroup of a root \( b \) of \( B \) in \( N_{G}(D) \). Then the inertial index of \( B \) is \( e(B) = |N_{G}(D, b) : DC_{G}(D)| \). As usual \( l(B) \) denotes the number of irreducible Brauer characters associated to \( B \). For a finite-dimensional algebra \( A \) over \( F \) with Jacobson radical \( J(A) \), its Loewy length \( LL(A) \) is defined to be the smallest positive integer \( t \) such that \( J(A)^t = 0 \). For an integer \( n \geq 1 \) we define \( c(n) \) as the codimension of \( J(A)^n \) in \( J(A)^{n-1} \). Then \( c(n) \neq 0 \) for any \( 1 \leq n \leq LL(A) \).

2 Results

2.1 Cyclic defect groups

In this section we study \( c(n) \) for \( ZB \). It always holds \( c(1) = 1 \) as \( ZB \) is local. Moreover it is clear that \( c(2) = 0 \) if and only if \( d = 0 \). We next see the case of \( c(2) = 1 \) in the proposition below. This is a corollary to [7, Proposition 8] since it is easy to check \( c(n) \leq 1 \) for all \( 3 \leq n \).

**Proposition 1.** If \( d > 0 \), then the following are equivalent:

1. \( c(2) = 1 \);
2. \( ZB \) is uniserial;
3. \( B \) is nilpotent and \( D \) is cyclic.

In the next theorem we determine all the codimensions \( c(n) \) for cyclic defect groups. In the proof below we use the algebra of fixed points associated to \( B \). As \( N_{G}(D, b) \) acts on \( Z(D) \) we define

\[
F[Z(D)]^{N_{G}(D, b)} = \{ a \in F[Z(D)] : t^{-1}at = a \text{ for } t \in N_{G}(D, b) \}.
\]

The last part of this theorem is due to [5, Corollary 2.8].
Theorem 2. If $D$ is cyclic, then

$$c(n) = \begin{cases} 1 & (n = 1) \\ e(B) & (n = 2) \\ 1 & (3 \leq n \leq LL(ZB)) \end{cases}$$

and

$$LL(ZB) = \frac{p^d - 1}{e(B)} + 1.$$ 

Proof. Put $I = N_G(D, b)/C_G(D)$. Since $B$ is perfectly isometric to its Brauer correspondent in $N_G(D)$ we may assume $B = FH$ where $H = D \times I$ by [6]. Let $\Gamma$ be the ideal of $ZF H$ spanned by all class sums of defect 0. Then $ZF H = F D N_G(D, b) \oplus \Gamma$ (cf. [3, Theorem 1.1]). Since $\Gamma$ is contained in the socle of $F H$ we have $\Gamma^2 = 0$. Thus $J(ZFH) = J(FD^{N_G(D, b)}) \oplus \Gamma$ and $J(ZFH)^2 = J(FD^{N_G(D, b)})^2$. As $FD^{N_G(D, b)}$ is uniserial (see the proof of [5, Corollary 2.8]), it follows that $c(n) = 1$ for $3 \leq n \leq LL(ZB)$. Therefore $c(2) = k(B) - 1 - \{LL(ZB) - 2\} = e(B)$ by Dade’s theory.

For cyclic defect groups $l(B) = e(B)$ and

$$k(B) = \frac{p^d - 1}{e(B)} + e(B) \geq 2\sqrt{p^d - 1}.$$ 

In general the relations between $c(n), l(B)$ and $e(B)$ are as yet unknown. However it is possible to answer the question in Introduction under the conditions below inspired by Theorem 2.

We now assume $d > 0$ and one of the following holds:

(i) $e(B) \leq c(n)$ for some $2 \leq n \leq LL(ZB)$;

(ii) $e(B) \leq l(B)$.

Then we obtain $2\sqrt{\lambda - 1} \leq k(B)$ where $\lambda = LL(F[Z(D)])$ as follows:

By [5, Corollary 2.7],

$$\frac{\lambda - 1}{e(B)} + 1 \leq LL(F[Z(D)])^{N_G(D, b)} \leq LL(ZB).$$

It is known that $l(B)$ is equal to the dimension of the Reynolds ideal of $ZB$. Since it is contained in the socle of $ZB$,

$$k(B) \geq 1 + e(B) + \{(\lambda - 1)/e(B) - 1\} \geq 2\sqrt{\lambda - 1}$$

in either case.

If $Z(D)$ has type $(p^{a_1}, \ldots, p^{a_r})$, then $\lambda = p^{a_1} + \cdots + p^{a_r} - r + 1$. Therefore we deduce $2\sqrt{p - 1} \leq k(B)$ in the assumptions.
2.2 Lower bounds

In this subsection we consider a lower bound for $LL(ZB)$. By a result of Broué [1, Proposition (III) 1.1] there exists an ideal $K$ of $ZB$ such that $ZB/K$ is isomorphic to $F[Z(D)]^{N_G(D,b)}$. Thereby $LL(F[Z(D)]^{N_G(D,b)}) \leq LL(ZB)$.

In the next theorem we give a different lower bound for the left side from [5, Corollary 2.7].

**Theorem 3.** Let $p^m$ be the exponent of $Z(D)$. Then

$$\frac{p^m + p - 2}{p - 1} \leq LL(F[Z(D)]^{N_G(D,b)}) \leq LL(ZB).$$

**Proof.** We fix an orbit $O$ of an element in $Z(D)$ with maximal order by the action of $N_G(D, b)$. Remark that $|O|$ divides $e(B)$ since $DC_G(D)$ acts trivially on $Z(D)$, namely $|O| ̸= 0$ in $F$. We put $a = |O| - \sum_{u \in O} u$ where 1 is the unit in $G$. As $a$ is contained in $J(F[Z(D)]^{N_G(D,b)})$ it suffices to prove that $a^t ̸= 0$ where

$$t = 1 + p + \cdots + p^{m-1}.$$  

For $0 \leq i \leq m-1$,

$$a^p = |O|^p 1 - \sum_{u \in O} u^p = |O| 1 - \sum_{u \in O} u^p$$

by Fermat’s theorem. Hence each term of $a^t = a \cdot a^p \cdots a^{p^{m-1}}$ has the form

$$(-1)^{|I||O|^{m-|I|}} \prod_{i \in I} (u_i)^{p^i}$$

where $I \subseteq \{0, 1, \ldots, m - 1\}$ and $u_i \in O$. Suppose now that $\prod_{i \in I} (u_i)^{p^i} = 1$ for some $I ̸= \emptyset$. Since the order of $u_i$ is $p^m$ for any $i \in I$, we may assume $|I| \geq 2$. If $r = \min\{I\}$ and $s = \min\{I - \{r\}\}$, we obtain

$$1 = (1)^{p^{m-s}} \cdot \prod_{i \in I} (u_i)^{p^i} = \prod_{i \in I} (u_i)^{p^{m-s+i}} = (u_r)^{p^{m-s+s}} ̸= 1,$$

a contradiction. Thus the coefficient of 1 in $a^t$ is $|O|^m ̸= 0$. Therefore $a^t ̸= 0$ as claimed.

In Theorem 3 it seems that $Z(D)$ is generally not replaced with $D$ itself. Suppose that $B$ is a $p$-block with defect group

$$M_{p^d} = \langle x, y ; x^{p^{d-1}} = y^p = 1, y^{-1}xy = x^{1+p^{d-2}} \rangle$$

where $d \geq 4$. As is well known, $M_{p^d}$ has exponent $p^{d-1}$ and has cyclic center of order $p^{d-2}$. By [8, Proposition 10],

$$LL(ZB) = \frac{p^{d-2} - 1}{l(B)} + 1 \leq p^{d-2} < \frac{p^{d-1} + p - 2}{p - 1}.  $$
Furthermore we note that Theorem 3 is clear in this case since $l(B) \leq p - 1$ (cf. [13, Theorem 8.1, 8.8]).

As a corollary to this bound and [12, Proposition 2.2], we deduce:

**Corollary 4.** Let $p^m$ be the exponent of $Z(D)$. Then

$$\frac{p^m + p - 2}{p - 1} \leq k(B) - l(B) + 1 \leq k(B).$$

In particular, $p + 2 \leq k(B)$ provided $m \geq 2$.

At the end of this section we reconsider the previous remark. Here assume $d > 0$ and one of the following holds:

(iii) $Z(D)$ is not elementary abelian;

(iv) $e(B) \leq mr \cdot c(n)$ for some $2 \leq n \leq LL(ZB)$;

(v) $e(B) \leq mr \cdot l(B)$

where $p^m$ and $r$ are the exponent and the rank of $Z(D)$, respectively. Then $2\sqrt{p - 1} \leq k(B)$ by Corollary 4 and the methods in (i) and (ii).

### 3 Examples

We introduce two examples by Brough-Schwabrow [2] and Héthelyi-Külshammer [4] in order to understand our theorems. In the following let $P$ be a Sylow $p$-subgroup of $G$.

1. Assume $p = 3$ and $G = 3^2G_2(q)$ is the simple Ree group where $q$ is a power of 3 at least 27. Then $FG$ decomposes into two 3-blocks: the principal block $B_0$ with $q + 7$ characters and simple block $B_1$ with the Steinberg character. By [2, Theorem 1.1], we have $LL(ZB_0) = 3$. In this case Theorem 3 indicates that $Z(P)$ has exponent 3. In fact

$$Z(P) = \{x(0,0,v) ; v \in \mathbb{F}_q\}$$

by the notations [2, page 59] and

$$x(0,0,v)^3 = x(0,0,3v) = x(0,0,0).$$

2. We denote by $k(G)$ and $l(G)$ the numbers of irreducible ordinary and Brauer characters of $G$, respectively and apply Corollary 4 to $P$ as a defect group of the principal block. Then it follows that $p + 2 \leq k(G)$
when \( Z(P) \) is non-elementary, otherwise it is likely that \( k(G) - l(G) + 1 \) is "small". Now assume \( p \equiv 23 \mod 264 \). Let \( X \) be a cyclic group of order \( x = (p - 1)/22 \) and \( H \) the unique 2-fold covering group of the four-degree symmetric group. Then \( H \times X \) acts freely on an elementary abelian group \( P \) of order \( p^2 \) and the Frobenius group \( G = P \rtimes (H \times X) \) satisfies

\[
k(G) = \frac{p^2 - 1}{48x} + 8x = \frac{217}{264}p + \frac{25}{264}
\]

as mentioned in [4, Remark (iii)]. Since \( P \) is a normal Sylow \( p \)-subgroup of \( G \),

\[
l(G) = k(H \times X) = 8x = \frac{4}{11}p - \frac{4}{11}
\]

and

\[
k(G) - l(G) + 1 = \frac{11}{24}p + \frac{35}{24}.
\]

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