Lattice modulation spectroscopy of strongly interacting bosons in disordered and quasi-periodic optical lattices

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We compute the absorption spectrum of strongly repulsive one-dimensional bosons in a disordered or quasi-periodic optical lattice. At commensurate filling, the particle-hole resonances of the Mott insulator are broadened as the disorder strength is increased. In the non-commensurate case, mapping the problem to the Anderson model allows us to study the Bose-glass phase. Surprisingly we find that a perturbative treatment in both cases, weak and strong disorder, gives a good description at all frequencies. In particular we find that the infrared absorption rate in the thermodynamic limit is quadratic in frequency. This result is unexpected, since for other quantities like the conductivity in one-dimensional systems, perturbation theory is only applicable at high frequencies. We discuss applications to recent experiments on optical lattice systems, and in particular the effect of the harmonic trap.

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I. INTRODUCTION

In recent years, developments in the field of ultracold atomic gases have considerably enlarged the possibilities for exploring the physics of strongly correlated systems [1]. For instance, the study of quantum phase transitions, a subject of continuous theoretical interest, has been strongly stimulated by the experimental observation of the superfluid to Mott insulator transition using optical lattices [2]. Indeed, ultracold atom systems offer us an unprecedented control over the system parameters, which can allow us to ultimately understand the physics of very hard problems such as the phase diagram of the Hubbard model in two-dimensions [3].

A particularly interesting and fertile arena is the study of disordered ultra-cold atoms. Random potentials can be introduced in a controlled way by laser beams generating speckle patterns [4] or by loading in an optical lattice a mixture of two kinds of atoms, one heavy and one light. When the heavy atoms become randomly localized in the lattice, they will act as impurities for the lighter atoms [10]. Another available technique is to superimpose two optical lattices with incommensurate periodicities, thus generating a quasi-periodic potential [11]. The quasi-periodic lattices as well as the speckle patterns have been recently used in the experimental efforts to observe the effects of Anderson localization in dilute Bose gases expanding in highly elongated traps [12] [13]. Since Anderson localization is a single-particle effect, the next logical step is to study the interplay of disorder and interactions in strongly interacting ultra-cold atomic systems [14] [15] [16] [17]. The latter may be accessible by tuning interparticle interactions using Feshbach resonances [18], or by loading the atoms in sufficiently deep optical lattices [9] and/or strongly confining them to low-dimensions in tight traps [1].

In the context of the efforts described above, one of the experimental challenges is to observe in ultra-cold gases clear signatures of the theoretically-predicted transition from a superfluid to the Bose-glass phase [19] [20] [21]. A pioneering step in this direction was recently taken by the Florence group by using lattice modulation spectroscopy [11]. This technique consists in heating an ultracold gas loaded in a optical lattice by periodically modulating the depth of the lattice [22] [23] [24]. When perturbed in this way, the gas is driven out of equilibrium and absorbs energy. When the perturbation is switched off, and after re-thermalization, the broadening of the momentum distribution around zero momentum is taken as a measure of the energy absorbed by the system during the lattice modulation [22]. On the theory side, for non-disordered lattices, the calculation of the energy absorption rate due to the lattice modulation was first performed analytically within linear response theory by some of the present authors [23]. These results were confirmed and extended beyond linear response using time-dependent density-matrix renormalization group methods, both for the case of bosons [24] and fermions [25]. More recently, for disordered optical lattices, the energy absorption rate has been numerically calculated using full diagonalization in small systems [26] [27]. Other methods that have also been discussed in the literature for detecting signatures of the effect of disorder (or quasi-periodicity) on interacting boson systems in one dimen-
sion focus on the momentum distribution \cite{16,28} in disordered systems as well as quasi-periodic systems or the expansion dynamics in quasi-periodic systems \cite{15}.

Let us consider a one dimensional disordered Bose gas described by the following Hamiltonian:

\[
H = -J \sum_{j} (b_{j+1}^\dagger b_j + \text{h.c.}) + \frac{U}{2} \sum_{j} n_j (n_j - 1) + \sum_{j} (\epsilon_j + V_j^{\text{ho}}) n_j,
\]

where $b_j$ denotes the boson annihilation operator at sites $j$, $n_j = b_j^\dagger b_j$ being the local density. Here $J$ and $U$ are the usual parameters of the Bose-Hubbard model corresponding to the tunneling rate and the onsite repulsion ($U > 0$). The last term in the rhs of Eq. (1) accounts for the presence of both the harmonic trap, $V_j^{\text{ho}}$, and the disorder potential $\epsilon_j$.

The distribution of the on-site energies $\epsilon_j$ in Eq. (1) depends on the specific choice of the random (or pseudo-random) potential. In this work we extensively compare two cases:

1. $\epsilon_j$ uniformly distributed in $[-\Delta, \Delta]$ (2)
2. $\epsilon_j = \Delta \cos(2\pi j \sigma)$, $\sigma$ irrational, (3)

where $\Delta$ measures the strength of the disorder. From the experimental point of view, speckle patterns and lattice containing heavy atom impurities can be modeled by case a) whereas quasi-periodic potentials obtained by superimposing two optical potentials with incommensurate periodicities $d_1/d_2 = \sigma$ are described by case b).

In this work we assume that the hopping amplitude $J$ in Eq. (1) is modulated periodically in time according to $J(t) = J + \delta J \cos \omega t$, where $\omega$ is the modulation frequency. We calculate the energy absorption rate within linear response theory, which is valid for weak lattice modulations $\delta J \ll J$.

We shall restrict ourselves to the strongly repulsive regime of Eq. (1), where the on-site repulsion is large compared to both the hopping amplitude and the disorder strength ($U \gg J, \Delta$). We first discuss systems in the thermodynamic limit by setting $V_j^{\text{ho}} = 0$. Effects of the harmonic trapping potential will be discussed later, in Sect. [V].

In the thermodynamic limit both the number $N$ of bosons and the length $M$ of the chain diverge. The ratio $\nu = N/M$ is instead finite and corresponds to the filling factor, i.e. the average number of bosons per lattice site. We must distinguish between two physically different situations. For incommensurate fillings $\nu < 1$, the system has gapless excitations. Thus, a good approximation to the absorption rate at frequencies $\omega \ll U$ can be obtained by formally taking $U \rightarrow +\infty$ and mapping the resulting hard-core bosons to non-interacting fermions using a Jordan-Wigner transformation (see e.g. Ref. [29]). The resulting single particle problem can be easily solved for a given choice of $\epsilon_i$ and $V_i^{\text{ho}}$ and the energy absorption obtained. On the other hand, for unit filling ($\nu = 1$) the ground state is a Mott insulator with a gap. In this case, the elementary excitations are particles and holes \cite{24} and in a homogeneous system the absorption can only occur at frequencies $\omega \sim U$. In the absence of disorder, the response of the system to the lattice modulation has been computed using degenerate perturbation theory within the subspace of particle and hole excitations \cite{23}. The same methods can be generalized to deal with disordered case, as shown below.

The resulting general picture of the energy absorption is depicted in Fig. 1. The low frequency $0 \lesssim \omega \lesssim W$, where $W = \max(4J, 2\Delta)$ is the effective bandwidth, is only present for incommensurate fillings, $\nu < 1$. A second distribution is centered at frequency $\omega = U$ and comes from particle-hole excitations generating doubly occupied sites. The width of this absorption line is also given by $W$.

The paper is organized as follows. In Section II we present the general formalism needed to calculate the absorption rate for incommensurate and unit filling. In Section III and Section IV we present our analytical and numerical results obtained for the random and the quasi-periodic potential, respectively. In Section V we discuss the effects of a trapping potential. Finally in Section VI we provide our conclusions. A derivation of the general formula [see Eq. (9)] for the absorption rate valid for weak disorder is given in the Appendix VI.

\section{II. ENERGY ABSORPTION: LINEAR RESPONSE THEORY}

For weak perturbations, corresponding to $\delta J \ll J$, the energy absorption rate $\dot{E}_\omega$ can be calculated using linear response theory. The general formula has been first derived in Ref. [23] and in the presence of disorder it takes
the form
\[ \hat{E}_\omega = \frac{1}{2} \delta J_0^2 \omega \text{Im} [- \chi_K(\omega)], \]
where \( \chi_K(\omega) \) is the Fourier transform of the \textit{retarded} correlation function \( \chi_K(t) = -i\Theta(t)[K(t),K(0)] \) of the hopping operator \( K = -\sum_j (b_j^\dagger b_j + \text{h.c.}) \), being \( \Theta(t) \) the step function. In Eq. (4) the bars mean average over different disorder (Sect. IV) or quasi-periodic (Sect. III) realizations.

In general, the calculation of the correlation function in Eq. (4) is a complicated many-body problem. However, in the limit of strong repulsion where \( U \gg J, \Delta \), calculations are considerably simplified by the fact that we can accurately restrict ourselves to work within a subspace of the total Hilbert space, whose detailed structure depends on the filling. In the case of an \textit{incommensurate} filling (i.e. \( \nu < 1 \)) this subspace can be described in terms of non-interacting fermion states (that is, Slater determinants). For commensurate filling (i.e. \textit{unit} filling, \( \nu = 1 \)) we can restrict ourselves to the subspace with one particle and one hole excitation, as described below.

\textbf{Incommensurate filling}

For filling \( \nu < 1 \) and large on-site repulsion \( J, \Delta \ll U \), we take the hard core limit \( U \to +\infty \). Bosons are then mapped onto \textit{non interacting} spinless fermions via the Jordan-Wigner transformation:
\[ c_j = \exp \left[ \frac{i\pi}{\hbar} \sum_{k=1}^{j-1} n_j \right] b_j, \]
where \( c_j \) satisfy fermionic commutation relations \( \{c_j, c_j^\dagger\} = 0 \) and \( \{c_j, c_j^\dagger\} = 1 \). Under the above transformation, the Hamiltonian (1) is mapped onto the single particle Hamiltonian:
\[ H' = -J \sum_j (c_j^\dagger c_j + \text{h.c.}) + \sum_j c_j^\dagger c_j, \]
where the on-site repulsion \( U \) has disappeared and the hopping operator \( K \) in Eq. (4) is now given by \( K' = -\sum_j (c_j^\dagger c_j + \text{h.c.}) \). Notice that the mapping of observables that are non local in space is far less trivial, an example is the momentum distribution studied in Ref. [28] for disordered hard-core bosons in one dimension.

After some algebra, the absorption rate (4) becomes
\[ \hat{E}_\omega = \frac{\delta J_0^2 \pi \omega}{2} \sum_{\alpha,\beta} \mathcal{K}_{\alpha\beta} \left[ f(\varepsilon_\alpha) - f(\varepsilon_\beta) \right] \delta(\omega + \varepsilon_\alpha - \varepsilon_\beta), \]
where the matrix \( \mathcal{K} \) is defined as:
\[ \mathcal{K}_{\alpha\beta} = \left| \sum_j [\psi_\alpha(j+1)\psi_\beta(j) + \psi_\alpha(j)\psi_\beta(j+1)] \right|^2. \]
In the previous expressions \( \varepsilon_\alpha \) and \( \psi_\alpha(j) \) are the eigenvalues and eigenfunctions of \( H' \), respectively. In Eq. (7) \( f(\varepsilon) = (\exp[(\varepsilon - \mu)/T] + 1)^{-1} \) is the Fermi-Dirac distribution function at a temperature \( T \) and chemical potential \( \mu \). The latter is fixed by the normalization condition \( \nu = \sum_\alpha f(\varepsilon_\alpha) \). At zero temperature the only relevant processes in Eq. (7) correspond to transitions from an occupied level (with energy \( \varepsilon_\alpha < \mu(T = 0) \)) to an unoccupied level (\( \varepsilon_j > \mu(T = 0) \)). In particular, for unit filling the absorption (7) vanishes, consistently with the fact that a Mott insulator can only absorb at much higher frequencies \( \omega \sim U \).

In the absence of disorder (i.e. for \( \epsilon_i = 0 \) in Eq. (6), the hopping modulation commutes with the Hamiltonian \( H' \) and therefore the absorption rate vanishes to all orders, even beyond linear response. In the above expression, this is reflected in that, for \( \epsilon_i = 0 \), the eigenstates of \( H' \) become plane waves, \( \psi_k(j) \propto e^{i k j} \), with energy dispersion \( \varepsilon_k = -2J \cos k \) (\( k \) being the lattice momentum). Thus the matrix (8) is diagonal, i.e. \( K_{kk'} = 4\delta_{kk'} \cos^2 k \), which, together with the factor \( f(\varepsilon_k) - f(\varepsilon_{k'}) \) in Eq. (7) makes \( \hat{E}_\omega \) vanish.

At weak disorder (i.e. \( J > \Delta \)), the absorption rate (7) can be evaluated using perturbation theory (the details can be found in the Appendix), which yields:
\[ \hat{E}_\omega = \frac{\delta J_0^2 \pi \omega}{2} \sum_{k,k'} \left| V_{k-k'} \right|^2 \frac{f(\varepsilon_k) - f(\varepsilon_{k'})}{J^2} \delta(\omega + \varepsilon_k - \varepsilon_{k'}), \]
where \( V_k = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} e^{ikj} \varepsilon_j \) is the Fourier transform of the disorder potential. Equation (9) shows that the perturbation expansion in disorder is well defined provided the Fourier transform, \( V_k \), is finite.

Let us finally consider the so-called atomic limit, which corresponds to \( J < \Delta \) (yet \( \Delta < U \)). In this limit, tunneling can be neglected and the eigenstates are given by \( \psi_\alpha(j) = \delta_{jm} \). From Eq. (6) we find \( K_{jj'} = 1 \) if \( j \) and \( j' \) are nearest neighbor and zero otherwise. The absorption rate, Eq. (7), thus reduces to
\[ \hat{E}_\omega = \frac{\delta J_0^2 \pi \omega}{2} \sum_{r=\pm 1} \sum_{j=0}^{L-1} \left[ f(\varepsilon_j) - f(\varepsilon_{j+r}) \right] \delta(\omega + \varepsilon_j - \varepsilon_{j+r}). \]
In Sect. III we shall explicitly compare the results obtained using exact numerical diagonalization with the above results obtained both in the limit of weak (6) and strong (10) disorder. Finally, it is important to emphasize that Eq. (7) does not account for particle-hole excitations which are relevant at much higher frequencies, \( \omega \sim U \). These excitations become particularly important at unit filling (\( \nu = 1 \)), when the strongly repulsive Bose gas becomes a Mott insulator and the absorption at low frequency \( \omega \ll U \) predicted by Eq. (7) vanishes because there are no empty sites (i.e. holes) in the ground state. The contribution to the absorption from particle-hole excitations at unit filling will be discussed next.
Unit filling

We next turn our attention to the commensurate case with \( \nu = 1 \). For large enough \( U/J \) and \( U/\Delta \), the system becomes a bosonic Mott insulator. In Ref. 23 it was shown that for clean systems the absorption rate is zero at low frequencies and exhibits a narrow peak of width \( \sim J \) centered about \( \omega = U \). In this section, we consider the modifications of such peak due to a disorder or quasi-periodic potential, \( \epsilon_i \).

In order to obtain the energy absorption rate within linear response, we first use the spectral decomposition of the correlation function \( \chi_{\delta K}(\omega) \) in terms of the exact eigenstates of the unperturbed Hamiltonian. This yields the following expression for the energy absorption:

\[
\hat{E}_\omega = \delta J^2 \omega \frac{\pi}{2} \sum_n |\langle \Psi_n | K | \Psi_0 \rangle|^2 \delta (\omega + E_n - E_\alpha),
\]

where \( |\Psi_n \rangle \) with energies \( E_n \), and \( |\Psi_0 \rangle = |1,1,\ldots,1\rangle \) is the ground state in the Fock representation corresponding to one boson per lattice site. The low energy states are particle-hole excitations \( |\phi(m,j)\rangle = \frac{1}{\sqrt{2}} b_{m}^\dagger b_{m+j} |\Psi_0 \rangle \) corresponding to double occupation at site \( m \) and an empty site \( m+j \). These excitations are all degenerate with energy \( U \) in the absence of tunneling and \( \nu = 0 \).

For finite values of \( \Delta \) and \( J \), the eigenstates \( \Psi_n \) can be calculated using degenerate perturbation theory by writing \( |\Psi_n \rangle = \sum_{m,j} f_{m,j} |\phi(m,j)\rangle \), where the coefficients satisfy:

\[
\sum_{m',j'} \langle \phi(m,j) | H | \phi(m',j') \rangle f_{m',j'} = E f_{m,j}.
\]

The above matrix element and the energy absorption can be computed by taking into account that \( \langle \phi(m,j) | K | \Psi_0 \rangle = -\sqrt{2} \delta_{j,\pm 1} \), within the subspace containing just one particle and one hole. Hence, the matrix elements in Eq. (11) correspond to \( \langle \Psi_n | K | \Psi_0 \rangle = -\sum_{m,r=\pm 1} \sqrt{2} f_{m,r} \).

Again, let us note that, in the ‘atomic limit’ where the tunneling can be neglected, Eq. (11) simplifies considerably. Localized particle-hole excitations become the exact eigenstates \( |\Psi_n \rangle = |\phi(m,j)\rangle \) with energy \( E_n = U + \epsilon_m - \epsilon_{m+j} \), where \( E_n = \sum_{i=1}^M \epsilon_i \) is the energy of the ground state for a given realization of \( \epsilon_i \).

The absorption rate (11) then takes the form:

\[
\hat{E}_\omega = \delta j^2 \pi \omega \sum_{r=\pm 1} \sum_{m=1}^M \delta (\omega - U + \epsilon_m - \epsilon_{m+r}),
\]

which can be readily evaluated numerically once the disorder potential is known.

III. RESULTS FOR A DISORDER POTENTIAL

In this section we assume that the on-site energy \( \epsilon_i \) in Eq. (1) are random numbers uniformly distributed within the interval \([-\Delta, \Delta]\). The absorption rate (11) thus can be conveniently recast as:

\[
\hat{E}_\omega = M 6 j^2 F,
\]

where \( F \) is a dimensionless function that can be numerically evaluated. The results for incommensurate and commensurate cases are described below.

Incommensurate filling

As stated above, the absorption rate is calculated numerically at zero temperature starting from Eqs. (7) and (8). In Fig. 2 we plot the frequency dependence of the response function \( F \) for different fillings and increasing values of disorder \( \Delta = 0.1 J \) (upper panel), \( \Delta = J \) (central panel) and \( \Delta = 5 J \) (lower panel). Since, in the fermionic representation, the Hamiltonian of Eq. (6) is particle-hole symmetric, the absorption rate in Eq. (7) is unchanged under the transformation \( \nu \to 1 - \nu \), so we restrict our discussion to fillings \( \nu \leq 1/2 \).

A noticeable feature of Fig. 2 is that the response at high frequencies is independent of the filling factor. In this limit, the relevant processes contributing to the absorption mainly involve transitions from states far below the Fermi level \((\epsilon_\alpha \ll \mu)\) into empty states far above it \((\epsilon_\beta \gg \mu)\). As a result, the Fermi-Dirac distributions in Eq. (7) become irrelevant, and thus any dependence on the value chemical potential \( \mu \) disappears.

However, at low frequencies, the absorption rate vanishes quadratically with frequency for any strength \( \Delta \) of the disorder. To understand this, let us expand in Eq. (7) the distribution functions, \( f(\epsilon_\alpha) - f(\epsilon_\beta) \approx (\epsilon_\alpha - \epsilon_\beta) \partial_\epsilon f(\epsilon_\alpha). \) Taking into account that, at zero temperature, \( \partial_\epsilon f(\epsilon) = -\delta(\mu - \epsilon) \), we find that \( F \approx C \omega^2 \), where the constant

\[
C = \frac{\pi}{2} \sum_{\alpha \neq \beta} K_{\alpha \beta} \delta(\mu - \epsilon_\alpha) \delta(\mu - \epsilon_\beta)
\]

can be evaluated numerically. From Eq. (15) it can be seen that the constant \( C \) is non zero provided there are non-vanishing matrix elements \( K_{\alpha \beta} \) connecting two states \( \alpha \) and \( \beta \) at the Fermi level, \( \epsilon_\alpha = \epsilon_\beta = \mu \).

In the limit of weak disorder corresponding to \( \Delta \ll J \), the absorption can be evaluated using Eq. (9), where \( |V_k|^2 = \Delta^2 / 3 M \), as follows from the expression for the Fourier transform of the disorder potential, \( V_k \). Going to the thermodynamic limit and introducing the density of states \( \rho(\epsilon) = (2 \pi J \sqrt{1 - \epsilon^2 / 4 J^2})^{-1} \), we find:

\[
F = \frac{\omega \Delta^2}{24 \pi J^2} \int_{-b}^{b} \frac{d\epsilon}{\sqrt{1 - \epsilon^2 / 4 J^2} \sqrt{1 - (\epsilon + \omega)^2 / 4 J^2}}.
\]
where \( a = \max(-2J, \mu - \omega) \) and \( b = \min(\mu, 2J - \omega) \). Here the chemical potential \( \mu \) is related to the filling factor by \( \mu = -2J \cos \pi \nu \). By expanding Eq. (16) at low frequencies, we again obtain that the quadratic behavior discussed above:

\[
F = \frac{\Delta^2 \pi}{6J^2} \rho(\mu)^2 \omega^2 - \frac{\Delta^2}{24\pi J^4} \frac{\omega^2}{\sin^2(\pi \nu)}.
\]

It should be noticed that the right hand-side of Eq. (17) diverges in the limit of vanishing lattice filling \( \nu \to 0 \) because the density of states \( \rho(\epsilon) \) has a van Hove singularity at zero energy in one dimension. Thus the low filling limit, the quadratic behavior of Eq. (17) is only recovered at increasingly low frequencies, as shown in Fig. 2 (upper panel).

In Fig. 3 we show a comparison, in the limit of weak disorder \( (\Delta = 0.01 J) \), of the numerical results (open symbols) obtained using exact diagonalization with the analytical expression of Eq. (16) (continuous lines). The agreement is indeed very good over the entire frequency range. In the inset it is demonstrated that numerical results are consistent with the quadratic behavior of Eq. (17), expected at low frequencies.

On the other hand, in the opposite limit of strong disorder \( (\Delta \gg J) \), the tunneling can be neglected. In this limit, the absorption rate can be evaluated directly from Eq. (17), expected at low frequencies.

In Fig. 3 we show a comparison, in the limit of weak disorder \( (\Delta = 0.01 J) \), of the numerical results (open symbols) obtained using exact diagonalization with the analytical expression of Eq. (16) (continuous lines). The agreement is indeed very good over the entire frequency range. In the inset it is demonstrated that numerical results are consistent with the quadratic behavior of Eq. (17), expected at low frequencies.

On the other hand, in the opposite limit of strong disorder \( (\Delta \gg J) \), the tunneling can be neglected. In this limit, the absorption rate can be evaluated directly from Eq. (17). Taking into account that the on-site energies \( \epsilon_j \) at different sites are completely uncorrelated, we find

\[
F = \frac{\Delta^2 \pi}{6J^2} \rho(\mu)^2 \omega^2 - \frac{\Delta^2}{24\pi J^4} \frac{\omega^2}{\sin^2(\pi \nu)}.
\]

where \( a = \max(-2J, \mu - \omega) \) and \( b = \min(\mu, 2J - \omega) \). Here the chemical potential \( \mu \) is related to the filling factor by \( \mu = -2J \cos \pi \nu \). By expanding Eq. (16) at low frequencies, we again obtain that the quadratic behavior
FIG. 4: Comparison between numerics [symbols] and analyt-
ics [solid line, Eq.19] for strong disorder. The value of the
disorder is \( \Delta/J = 100 \) and the filling factors are \( \nu = 0.3 \) and
\( \nu = 0.5 \). The length of the chain is \( M = 2000 \) and the number
of disorder realizations is \( N_r = 2000 \). In the inset we compare
our numerics with the quadratic expansion (20) expected at
low frequency. In both cases the agreement is quite good.

and therefore \( \mu = (2\nu - 1)\Delta \). Using Eq. (18), the follow-
ing low-frequency behavior is obtained:

\[
F = \frac{\pi}{4\Delta^2} \omega \left[ \min(\mu, \Delta - \omega) - \max(-\Delta, \mu - \omega) \right].
\]  (19)

This behavior is exhibited by the numerics, as shown in Fig. 2 (lower panel, see also discussion further below).
Furthermore, in the low frequency limit Eq. (19) we again recover the quadratic behavior described above on gen-
eral grounds:

\[
F = \frac{\pi}{4\Delta^2} \omega^2.
\]  (20)

Notice however that the proportionality constant is now independent of the filling factor. In Fig. 4 a more de-
tailed comparison of the numerics with the analytical result of Eq. (19) for the limit of strong disorder is shown.
In the inset we also compare our numerical results with the quadratic behavior (20) expected at low frequency.
In both cases the agreement is remarkably good.

Unit filling

The absorption rate for \( \nu = 1 \) is calculated numerically
starting from Eqs (11) and (12). The result is shown in
Fig. 5 as a function of frequency for different values of
the disorder strength \( \Delta \) and \( \Delta/U = 0 \) (dashed line), \( \Delta/U = 0.03 \), \( \Delta/U = 0.06 \), \( \Delta/U = 0.1 \).
In the inset we also compare our numerical results with
the quadratic behavior (20) expected at low frequency.
In both cases the agreement is remarkably good.

As the strength of disorder is increased, we see from
Fig. 5 that the absorption spectrum becomes broader and
progressively develops a triangle-like shape as the limit
\( \Delta \approx \Delta \approx \Delta \approx \Delta \approx \Delta \) is approached. This behavior can be obtained
analytically starting from Eq. (13). In the \( \Delta \ll \Delta \) limit,
since the on-site energies at different lattice sites are un-
correlated, we obtain:

\[
F \approx \frac{\pi \omega}{2\Delta^2} \int_{-\Delta}^{\Delta} de \int_{-\Delta}^{\Delta} de' \delta(\omega - U + \epsilon - \epsilon'),
\]  (22)

where \( \rho(\epsilon) = 1/2\Delta \) is the disorder-averaged density of
states in the atomic limit introduced above. Upon inte-
gration, Eq. (22) yields

\[
F \approx \frac{\pi \omega}{2\Delta^2} (2\Delta - |\omega - U|),
\]  (23)

showing that the lineshape of the absorption rate in the
atomic limit is approximately triangular and vanishes at
\( |\omega - U| = 2\Delta \) for \( \Delta \ll \Delta \).
IV. QUASI-PERIODIC POTENTIAL

In this section we assume that the external potential in Eq. (1) is $\epsilon_j = \Delta \cos(2\pi j \sigma)$ [14,16,17], being $\sigma$ an irrational number. This quasi-periodic potential distribution is realized experimentally by superimposing two different periodic potentials with incommensurate lattice periods [11]. Differently from the disorder potential considered in the previous section, where all states are localized (in the thermodynamic limit) for an arbitrarily small amount of disorder, in the quasi-periodic case there is a phase transition [8] at $\Delta_c = 2J$: for $\Delta < 2J$ all states are extended while for $\Delta > 2J$ all states are exponentially localized.

As described below, we find that the absorption spectra of bosons in the quasi-periodic potential are remarkably different from the spectra described in Sect. III for the bosons moving on a disorder potential.

Incommensurate filling

For lattice fillings $\nu < 1$ (i.e. less than a boson per site), we calculate the absorption rate numerically starting from Eqs (7) and (8). We restrict to zero temperature and fix $\sigma = 0.77145245$ which is relevant to the experiments carried out by the Florence group.

In Fig. 6 we plot the frequency dependence of the response function [14] for increasing values of the disorder strength $\Delta$. Compared to Fig. 2, we see that the absorption spectra in quasi-periodic potential exhibit a much richer structure. Interestingly, the response function becomes very large (and actually diverges) when $\omega$ matches special values. Moreover, for a given filling factor, the absorption rate is non-zero only within a certain range of frequencies.

These features can again be understood in the limits of weak and at strong disorder, where results can be obtained analytically. In the limit $\Delta \ll J$, the absorption rate can be calculated starting from Eq. (9) and taking the thermodynamic limit. This yields $\langle \varepsilon_k = -2J \cos k \rangle$:

$$F = \frac{\pi \omega}{2} \int \frac{dk}{2\pi} \frac{|V(k-k')|^2}{J^2} \delta(\omega + \varepsilon_k + \varepsilon_{k'}), \quad \text{(24)}$$

where the integration over momenta is restricted to $[0,2\pi]$. Using that the (modulus square of the) Fourier transform of the disorder potential is $|V(k)|^2 = \lim_{M \to \infty} |V_M(k)|^2$, where

$$|V_M(k)|^2 = \frac{\Delta^2}{4M} \left| \frac{e^{i(k+2\pi n)M}}{e^{i(k+2\pi n)M}} - 1 \right|^2,$$

and $k_0 = 2\pi \sigma$, for $M \to \infty$, the right hand-side of Eq. (25) becomes a series of delta functions:

$$|V(k)|^2 = \frac{\pi \Delta^2}{2} \sum_{n=-\infty}^{+\infty} \left[ \delta(k + k_0 + 2\pi n) + \delta(k - k_0 + 2\pi n) \right].$$

Putting Eq. (26) this result into Eq. (24) we obtain:

$$F = \frac{\omega \pi \Delta^2}{4J^2} \int_0^{2\pi} \frac{dk}{2\pi} \delta(\omega - \varepsilon_k - \varepsilon_{k+k_0}) \Theta(\mu + \varepsilon_k) \Theta(\omega - \varepsilon_{k+k_0}). \quad \text{(27)}$$

The integral in Eq. (27) can be evaluated using the iden-
tity $a \sin k + b \cos k = c \cos(k + \gamma)$, where $c = \sqrt{a^2 + b^2}$ and $\tan \gamma = -a/b$. Since $a = \sin k_0$ and $b = 1 - \cos k_0$ we obtain $c = 2 \sin(k_0/2)$ and $\tan \gamma = -1/\tan(k_0/2)$. The zeroes of the $\delta$-function in Eq. (27) occur at $k = k_{\pm} = -\gamma \pm \arccos(\omega/2Jc)$. Hence, we obtain:

$$F = \frac{\Delta^2}{4J^2} \frac{\omega/2}{\sqrt{(2Jc)^2 - \omega^2}} \sum_{r=\pm} \Theta(-\xi_r)\Theta(\omega + \xi_r), \quad (28)$$

where $\xi_r = -\epsilon_{k_r} - \mu = 2J \cos k_r - \mu$. Eq. (28) shows that the absorption is finite only in a range of frequencies given by the conditions $\xi_r < 0$ and $\xi_r + \omega > 0$, where $\xi_r$ itself depends on $\omega$ through the $\omega$ dependence of $k_r$. Moreover, the response diverges for $\omega = 2Jc$, provided this frequency value is allowed for a given filling factor. Since $c = 2 \sin \pi \sigma = 1.31576$, the divergence occurs at $\omega = 2.6315J$, as found numerically and shown in Fig. [6] (upper panel).

For a potential strength comparable to the tuneling rate, i.e. $\Delta \sim J$, the absorption spectrum develops very sharp peaks as shown in the central panel of Fig. [6]. These peaks gradually disappear as $\Delta$ increases and becomes much larger than $J$. In this regime, however, the energy absorption spectrum becomes indeed rather similar to the case of weak quasi-periodic potential, as can be seen by comparing the upper and lower panels of Fig. [6]. This peculiar effect can be explained analytically starting from Eq. (10) which applies in the atomic limit. Introducing the variable $y = 2\pi \sigma n$, in the thermodynamic limit, we obtain:

$$F = \omega \pi \int_0^{2\pi} \frac{dy}{2\pi} \delta(\omega + \Delta \cos y - \Delta \cos(y + 2\pi \sigma)) \Theta(\mu - \Delta \cos y)\Theta(\omega + \Delta \cos y - \mu) \quad (29)$$

The integral in Eq. (29) becomes the integral in Eq. (27) after a change of variable $k = y + \pi$. We thus obtain:

$$F = \frac{\omega/2}{\sqrt{(\Delta c)^2 - \omega^2}} \sum_{r=\pm} \Theta(\mu + \Delta \cos k_r)\Theta(\omega - \Delta \cos k_r - \mu), \quad (30)$$

showing that the behavior of the absorption rate at strong quasi-periodic potential can be obtained from Eq. (28) by simply replacing $2J$ with $\Delta$.

**Unit filling**

We have obtained the absorption spectrum at unit filling numerically using Eq. (11) and Eq. (12), for the same value of $\sigma = 0.77145245$. The result is plotted in Fig. [7] for fixed $J/U = 0.01$ and increasing values of the quasi-periodic potential strength, $\Delta$. The dashed line corresponds to the clean case, $\Delta = 0$, where the absorption rate is given by Eq. (21).

We see that the shape of the absorption spectrum changes considerably as $\Delta$ increases. For sufficiently weak quasi-periodic potential, $\Delta \lesssim 3J$, the spectrum does not become broad, but instead satellite peaks appear on the sides of the central absorption feature. For stronger quasi-periodic potentials, the central peak disappears and the spectrum develops a two hump structure. To understand these features, let us again focus on the ‘atomic limit’ where the absorption rate can be obtained analytically using Eq. (13). Introducing the variable $y = 2\pi \sigma n$ and passing to the continuum limit, we obtain the result:

$$F = 2\omega \pi \int_0^{2\pi} \frac{dy}{2\pi} \delta(\omega - U + \Delta \cos y - \Delta \cos(y + 2\pi \sigma)). \quad (31)$$

The integral (31) can be readily evaluated using the identity $a \sin y + b \cos y = c \cos(y + \gamma)$, where $\tan \gamma = -a/b$ and $c = \sqrt{a^2 + b^2}$. From Eq. (31) we have that $c = 2 \sin \pi \sigma$ and therefore,

$$F = \frac{2\omega}{\sqrt{(\Delta c)^2 - (\omega - U)^2}}, \quad (32)$$

showing that the absorption rate diverges at the edge where $(\omega - U) = \Delta c$. This means that the shape of the absorption spectrum changes completely going from weak to strong quasi-periodic potential, as obtained numerically and shown in Fig. [7]. Finally, upon comparing Eq. (28) and Eq. (32) we see that in a quasi-periodic potential the absorption spectrum is (at least in the atomic limit) narrower because $c < 2$. 

![Figure 7: Energy absorption in the Mott insulator phase for bosons in a quasi-periodic potential with $\sigma = 0.77145245$.](image-url)
In conclusion we have investigated the energy absorbed by a disordered strongly interacting Bose gas in the presence of periodically modulated optical lattices. For filling factor less than one, the absorption rate has been calculated exactly in the hard-core limit via the Bose-Fermi mapping. For commensurate filling, corresponding to one boson per lattice site, the gas is a Mott insulator and can only absorb energy at much higher frequency (of the order of the repulsive interaction $U$). The disorder induced broadening of the absorption spectrum has been calculated by restricting to the subspace of particle-hole excitations.

We have performed extensive calculations comparing two different sources of disorder: a random potential, which is relevant for current experiments based on speckle patterns, and a quasiperiodic potential, which is obtained by superimposing two optical lattices with incommensurate periods. Our results indicate that the response of the gas to the lattice modulation significantly

VI. CONCLUSIONS

We have performed extensive calculations comparing two different sources of disorder: a random potential, which is relevant for current experiments based on speckle patterns, and a quasiperiodic potential, which is obtained by superimposing two optical lattices with incommensurate periods. Our results indicate that the response of the gas to the lattice modulation significantly
to introduce the matrix elements so that using new indices $k$ are plane waves $ψ_k$. This work was partly supported by ANR grant 08-BLAN-0165-01 and by the Swiss National Fund under MaNEP and Division II. AI gratefully acknowledges financial support from CONICET and UNLP. During the initial stage of this project, GO was also supported by the Marie Curie Fellowship under contract n. EDUG-03970. MAC was supported by MEC (Spain) Grant No. FIS2007-066711-C02-02 and CSIC (Spain) through Grant No. PIE 200760/007.

VII. APPENDIX

In this Appendix we shall derive the asymptotic formula of yielding the general expression

$$E_ω = \frac{δJ_0^2πω}{2} \sum_{k,k'} K_{kk'} [f(ε_k) - f(ε_{k'})] δ(ω + ε_k - ε_{k'})$$

(33)

using new indices $k$ and $k'$. Moreover we find convenient to introduce the matrix elements

$$A_{kk'} = \sum_j [ψ_0^k(j + 1)ψ_0^{k'}(j) + ψ_0^k(j)ψ_0^{k'}(j + 1)]$$

(34)

so that $K_{kk'} = |A_{kk'}|^2$.

In the absence of disorder $Δ = 0$, the eigenstates are plane waves $ψ_k(n) = e^{i kn}/√L$ with energy $ε_k = -2J cos k$. Therefore from Eq. (34) we find

$$A_{kk'}^0 = 2δ_{kk'} cos k,$$

(35)

showing that the matrix $A$ is diagonal in momentum space. Since the matrix $K_{kk'} = 4δ_{kk'} cos^2 k$ is also diagonal, the absorption rate vanishes.

For $Δ ≪ J$, we formally expand the rhs of Eq. (34) in powers of the disorder strength $A_{kk'} = A_{kk'}^0 + A_{kk'}^1 + A_{kk'}^2 + O(Δ^3)$, so the matrix $K$ takes the form

$$K_{kk'} = K_{kk'}^0 + A_{kk'}^1 + A_{kk'}^2 + A_{kk'}^3 + O(Δ^3).$$

(36)

(37)

Taking Eq. (35) into account, we see that the only non-diagonal term appearing in the expansion (37) is $|A_{kk'}^1|^2$, which is second order in $Δ$. This term can be readily evaluated from Eq. (34) by applying first order perturbation theory for the eigenstates

$$ψ_k = ψ_k^0 + \sum_{q≠k} (ψ_0^q | V | ψ_0^k) e^{iq}ψ_0^q.$$  

(38)

After a simple algebra we obtain

$$A_{kk'}^1 = \frac{(ψ_0^q | V | ψ_0^k)}{ε_k - ε_q} 2(cos k - cos k'),$$

(39)

which is valid up to linear order in $Δ$. Finally, by using the dispersion relation $ε_k = -2J cos k$, Eq. (39) further simplifies yielding

$$A_{kk'}^1 = \frac{(ψ_0^q | V | ψ_0^k)}{J}.$$  

(40)

Substituting Eq. (40) into Eq. (38) and replacing the eigenstates by their zero order values $ε_k = ε_k^0$, we recover the asymptotic formula (9).

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