RECENT PROGRESS ON THE POSITIVE ENERGY THEOREM

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ABSTRACT. We give a short review of recent progress on the positive energy theorem in general relativity, especially for spacetimes with nonzero cosmological constant.

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1. INTRODUCTION

In general relativity, a spacetime is a 4-dimensional Lorentzian manifold $(\mathbb{L}^{3,1}, g)$ which satisfies the Einstein field equations

$$\mathbf{R}_{\alpha\beta} - \frac{\mathbf{R}}{2} g_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta},$$

where $\mathbf{R}_{\alpha\beta}$, $\mathbf{R}$ are Ricci and scalar curvatures of $g$ respectively, $\Lambda$ is the cosmological constant and $T_{\alpha\beta}$ is the energy-momentum tensor of matter. Indeed, $\Lambda$ is the Ricci curvature of vacuum spacetime $T_{\alpha\beta} = 0$ where the Einstein field equations reduce to $\mathbf{R}_{\alpha\beta} = \Lambda g_{\alpha\beta}$.

The maximally symmetric vacuum solutions are as follows

(i) $\Lambda = 0$, Minkowski spacetime $\mathbb{R}^{3,1}$. The symmetric group is (3+1)-Poincaré group generated by four translational Killing vectors $L_{\alpha}$ and six rotational Killing vectors $U_{\alpha\beta}$.
(ii) $\Lambda > 0$, de Sitter spacetime which is the hypersurface in $\mathbb{R}^{4,1}$

$$- (X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = \frac{3}{\Lambda}.$$ The symmetric group is (4+1)-Lorentzian group generated by ten rotational Killing vectors $U_{\alpha\beta}$.
(iii) $\Lambda < 0$, anti-de Sitter spacetime which is the hypersurface in $\mathbb{R}^{3,2}$

$$- (X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = \frac{3}{\Lambda}.$$ The symmetric group is (3+2)-Lorentzian group generated by ten rotational Killing vectors $U_{\alpha\beta}$. 

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Choose a frame in spacetime $\mathbb{L}^{3,1}$ with $e_0$ timelike and $e_i$ spacelike, $T_{00}$ is interpreted as the local mass density and $T_{0i}$ is interpreted as the local momentum density. The dominant energy condition asserts that

$$T_{00} \geq \sqrt{T_{01}^2 + T_{02}^2 + T_{03}^2}, \quad T_{00} \geq T_{\alpha\beta}.$$ 

Let $(M, g, h)$ be an initial data set, where $M$ is 3-dimensional spacelike hypersurface in $\mathbb{L}^{3,1}$, $g$ is the Riemannian metric of $M$ and $h$ is symmetric 2-tensor which is the second fundamental form of $M$. Let $\nabla$, $R$ be the Levi-Civita connection, the scalar curvature of $M$ respectively. Then $g$ and $h$ are constrained by the Gauss and Codazzi equations which give

$$T_{00} = \frac{1}{2}(R + (h^i_j)^2 - h_{ij}h^{ij}), \quad T_{0i} = \nabla^j h_{ij} - \nabla_i h^j_j.$$ 

For spacetimes which are asymptotic to the these maximal symmetric solutions, the famous Noether theorem in the framework of a Lagrangian theory asserts that the Killing fields of maximal symmetric spacetimes provide conserved quantities [2, 19]. Physically, the dominant energy condition for the energy-momentum tensor should indicate these conserved quantities are future timelike or null in suitable sense, and the above maximal symmetric spacetimes have the lowest energy and serve as the ground states. But this physical predication is far from obvious due to the well-known pseudo-tensor characteristic of the Lagrangian for gravity, e.g. [18]. It was therefore conjectured to be true and refereed as the positive energy conjecture. The conjecture plays a fundamental role and provides a self-consistent check for general relativity. However, what energy of the ground states is needs much deeper physical theory.

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2. $\Lambda = 0$

In this section, we assume the cosmological constant is zero. We shall review the main total energy-momentum inequalities in this case.

2.1. Positive energy theorem. When the cosmological constant is zero and spacetimes are asymptotically flat, the time translation $L_0$, the space translations $L_i$, the space rotations $U_{ij}$ and the time-space rotations $U_{i0}$ give rise to the total energy, the total linear momentum, the total angular momentum and the center of mass respectively.

An initial data set $(M, g, h)$ is asymptotically flat if there is a compact set $K \subset M$ such that $M \setminus K$ is the union of finite $M_c$, where each $M_c \cong \mathbb{R}^3 \setminus B_r$ is called an “end” of $M$, $B_r$ is the closed ball of radius $r$ with center at the
coordinate origin. In each end, let \( \{x^i\} \) be the Euclidean coordinates of \( \mathbb{R}^3 \), 
\( g_{ij} = \delta_{ij} + a_{ij} \), and \( a_{ij} \) satisfy
\[
a_{ij} = O\left(\frac{1}{r}\right), \quad \partial_r a_{ij} = O\left(\frac{1}{r^2}\right), \quad \partial_\theta a_{ij} = O\left(\frac{1}{r^3}\right), \quad h_{ij} = O\left(\frac{1}{r^3}\right),
\]

The total energy \( E \), the total linear momentum \( P_i \) for asymptotically flat initial data set \( (M, g, h) \) were defined by Arnowitt, Deser and Misner [2, 3]
\[
E = \frac{1}{16\pi} \int_{S_\infty} (\partial_j g_{ij} - \partial_i g_{jj}) * dx^i, \quad P_k = \frac{1}{8\pi} \int_{S_\infty} (h_{ki} - g_{ki} tr_g(h)) * dx^i
\]
where \( S_\infty \) is the sphere at infinity on end \( M_e \), \( 1 \leq k \leq 3 \). Furthermore, we assume the absolute values of scalar curvature \( |R| \) and \( |T_{0i}| \) are integrable over \( M \). These yield that the total energy-momentum are geometric quantities which are independent on the choice of coordinates \( \{x^i\} \) [9, 19].

The positive energy theorem for the ADM total energy-momentum was first proved by Schoen and Yau [33, 44, 45], then by Witten [31, 44]. Schoen-Yau’s proof included black hole’s case and Witten’s proof was extended to black hole’s case in [32]. Some higher dimensional cases were provided in [30, 9, 29, 54] for instance.

**The positive energy theorem:** If \( (M, g, h) \) is asymptotically flat, with possibly a finite number of apparent horizons which are inner boundaries of \( M \), each of them is topological \( S^2 \) whose mean curvature \( H \) satisfies \( H \leq tr_g(h|_{S^2}) = 0 \). Suppose the dominant energy condition holds, then
(i) \( E \geq \sqrt{P_1^2 + P_2^2 + P_3^2} \) for each end;
(ii) That \( E = 0 \) for some end implies \( M \) has only one end, and \( \mathbb{L}^{3,1} \) is flat along \( M \). If \( h = 0 \), then \( (M, g) \) is \( \mathbb{R}^3 \).

The total mass is \( \sqrt{E^2 - P_1^2 - P_2^2 - P_3^2} \) which is a Lorentzian invariant. When the positive energy theorem holds, the total mass is well-defined.

The positive energy theorem does not hold for spacetimes with naked singularity, e.g., Schwarzschild spacetime with \( m < 0 \). So \( E > 0 \) may be refereed as the Schwarzschild constraint. The dominant energy condition provides sufficient conditions for the Schwarzschild constraint.

### 2.2. Kerr Constraint

Kerr spacetime also has naked singularity for \( m < |a| \). So for general asymptotically flat spacetimes with the total energy \( E \), the total angular momentum \( J \) and the total mass \( m \), \( E > |J| \) or \( E > |J/m| \) for \( m \neq 0 \) may forbid naked singularities. Such an inequality is called the **Kerr constraint** by Schoen recently.

In 1999, Zhang proved the positive energy theorem for *generalized asymptotically flat initial data set* \( (M, g, p) \) where \( p \) is general 2-tensor which is not necessary symmetric, and satisfies the following conditions
\[
a \in C^{2,\alpha}_{\tau}, \quad p \in C^{0,\alpha}_{-\tau-1}, \quad tr_g(p) \in W^{-1,\frac{1}{2}}_{-\tau-1}, \quad \{d\theta, d^\ast\theta\} \in L_{\frac{1}{2},-\tau-2}
\]

where \( \theta = (p_{ij} - p_{ji})e^i \wedge e^j \) is the associated 2-form of \( p \), \( \frac{1}{2} < \tau \leq 1 \) [55].
Geometrically, the second fundamental form $p$ is nonsymmetric when spacetimes equip with affine connections with torsion. In this case matter translates, meanwhile, it rotates. The idea using connection with torsion was initially due to E. Cartan \[14, 15, 16, 17\]. In \[55\], Zhang defined the following generalized linear momentum counting both translation and rotation

$$P_k = \frac{1}{8\pi} \int_{S_\infty} (p_{ki} - g_{ki}tr_g(p)) * dx^i,$$

and proved the following theorem.

The generalized positive energy theorem: If $(M, g, p)$ is generalized asymptotically flat, with a finite number of generalized apparent horizons which are inner boundaries of $M$, each of them is topological $S^2$ whose mean curvature $H$ satisfies $H \pm tr_g(p|_{S^2}) = 0$. Suppose the generalized dominant energy condition

$$\frac{1}{2} (R + (tr_g(p))^2 - |p|^2) \geq \max \{|\omega|, |\omega + \chi|\},$$

holds, $\omega_j = \nabla^i p_{ji} - \nabla^j p_{ij}$ (the term in left hand side can be interpreted as $T_{00}$ and the two terms in right hand side can be interpreted as $|T_{0i}|, |T_{i0}|$ in suitable sense), then

(i) $E \geq \sqrt{P_1^2 + P_2^2 + P_3^2}$ for each end;
(ii) That $E = 0$ for some end implies $M$ has only one end, and

$$R_{ijkl} + p_{ki}p_{jl} - p_{ij}p_{kl} = 0, \quad \nabla_i p_{jk} - \nabla_j p_{ik} = 0, \quad \nabla^i (p_{ij} - p_{ji}) = 0.$$

In Newton mechanics, let $\{x_1^0, x_2^0, x_3^0\}$ be the system’s center of mass, $T_v$ be the momentum density of the system, the total angular momentum of rigid body $V$ is

$$J_k = \int_{V} \epsilon_{kuv}(x^u - x_0^u)T^v * 1 = \int_{S_{\infty}} \epsilon_{kuv}(x^u - x_0^u)(h^v_i - \delta_{vi}tr(h)) * dx^i.$$

The second equality holds if, furthermore, $V = \mathbb{R}^3$, $T_v = \partial_i h^i_v - \partial_v tr(h)$. Usually, the total angular momentum is a set of anti-symmetric two tensors $J_{ij} = -J_{ji}$. In $\mathbb{R}^3$, the rotation along a plane is equivalent to the translation along an axis which is perpendicular to the plane, $J_k = \epsilon_{kij}J_{ij}$. So we can define them as co-vectors in 3-dimensional space. But the above definition of $J_k$ does not make sense in higher dimension.

In 1974, Regge-Teitelboim generalized it to general relativity and defined the total angular momentum for asymptotically flat initial data sets \[42\],

$$J_k(x_0) = \frac{1}{8\pi} \int_{S_{\infty}} \epsilon_{kuv}(x^u - x_0^u)\pi^v_i * dx^i, \quad \pi^v_i = h^v_i - g^v_i tr_g(h).$$

In general, the integrand is $O(\frac{1}{r})$ which may not be integrable. This ambiguity resolution requires stronger “Regge-Teitelboim” conditions on ends

$$g(x) - g(-x) = O(r^{-3}), \quad \pi(x) + \pi(-x) = O(r^{-3}).$$
However, as the integrand is not tensor, it is not possible to relate the
local density to Regge-Teitelboim’s total angular momentum. To resolve this
difficulty, Zhang defined trace free, non-symmetric tensor of local angular
momentum density \[ h_{ij} = \frac{1}{2} \epsilon^{uv} (\nabla_u \rho^2) (h_{vj} - g_{vj} tr_g(h)) \]
where \( \rho_z \) is the distance function w.r.t some \( z \in M \). If \( (M, g, h_{ij}) \) is gen-
eralized asymptotically flat, Zhang defined the total angular m omentum \[ \text{J}_k = \frac{1}{8\pi} \int_{S^\infty} \tilde{h}_{ik} \star dx \]
J is also a geometric quantity which is independent on the choice of coor-
dinates \( \{x^i\} \) if \( |\nabla h_{ij}| \) is integrable. In Kerr spacetime, \( h_{ij} = O(\frac{1}{r^4}) \), so the
total angular momentum is well-defined and it was found \( J = (0, 0, ma) \)
[55]. By taking \( p_{ij} = C h_{ij} \) for certain constant \( C > 0 \), Zhang proved the
Kerr constraint under the generalized dominant energy condition [55]. The
higher dimensional Kerr constraint is interesting in mathematics and it is
still open.

The dominant energy condition does not yield to the Kerr constraint. In
[34], Huang, Schoen and Wang showed that it is possible to perturb ar-
bitrarily vacuum asymptotically flat initial data sets to new vacuum ones
having exactly the same total energy, but with the arbitrary large total an-
gular momentum. However, the Kerr constraint holds for simply connected,
asymptotically flat, maximal, axisymmetric, vacuum black hole initial data
sets \( (M, g, h) \) defined on \( \mathbb{R}^3 \setminus \{ \rho = 0 \} \)
\[ g = e^{2\alpha-2U}(dr + d\rho^2) + e^{-2U} \rho^2 (d\phi + \rho B d\rho + A dz)^2, \quad tr_g(h) = 0, \]
where \( (\rho, \phi, z) \) are cylindrical coordinates, and all functions are \( \phi \) indepen-
dent [27, 20, 24, 46].

2.3. Bondi energy-momentum. Bondi-Sachs’ radiating metrics for grav-
itational waves are wave-like, vacuum solutions of the Einstein field equa-
tions, and take the following asymptotical forms
\[ g_{BS} = - \left(1 - \frac{2M}{r}\right)du^2 - 2du dr + 2Idud\theta + 2I sin \theta du d\psi \]
\[ + r^2 \left[ (1 + \frac{2c}{r})d\theta^2 + \frac{4I}{r} sin \theta d\theta d\psi + (1 - \frac{2c}{r}) sin^2 \theta d\psi^2 \right] \]
\[ + \text{lower order terms} \]
where \( u \) is retarded coordinate (\( u \)-slices are null hypersurfaces), \( r = x^1 \), \( \theta \)
and \( \psi \) are polar coordinates, \( M, c, d \) are smooth functions of \( u, \theta, \psi \) defined
on \( \mathbb{R} \times S^2 \) with regularity condition \( \int_0^{2\pi} c(u, \theta, \psi) d\psi = 0 \) for \( \theta = 0, \pi \) and all
\( u, l = c, \theta + 2c\cot \theta d, \psi \csc \theta, \bar{l} = d, \theta + 2d \cot \theta - c, \psi \csc \theta. \)
Throughout the paper, we denote $n^0 = 1, n^1 = \sin \theta \cos \psi, n^2 = \sin \theta \sin \psi, n^3 = \cos \theta$. At null infinity, the Bondi energy-momentum of $u_0$-slice are defined as

$$m_\nu(u_0) = \frac{1}{4\pi} \int_{S^2} M(u_0, \theta, \psi) n_\nu dS$$

for $\nu = 0, 1, 2, 3$. The famous Bondi energy loss formula asserts

$$\frac{d}{du} m_0(u) = -\frac{1}{4\pi} \int_{S^2} (c_u)^2 + (d_u)^2 \leq 0.$$ 

Let $|m(u)| = \sqrt{(m_1^2(u) + m_2^2(u) + m_3^2(u))}$. If $|m(u)| \neq 0$, it was generalized to the Bondi energy-momentum loss:

$$\frac{d}{du} \left(m_0(u) - |m(u)| \right) = -\frac{1}{4\pi} \int_{S^2} [(c_u)^2 + (d_u)^2] \left(1 - \frac{m_in_i}{m} \right) \leq 0.$$ 

This asserts that the Bondi energy-momentum can be viewed as the total energy-momentum measured after the loss due to the gravitational radiation up to that time.

It is an old question whether isolated gravitational systems can radiate away more energy than they have initially, i.e., whether Bondi’s Energy should be nonnegative. The proofs of this positivity were claimed by using both Schoen-Yau and Witten’s positive energy arguments (see [23] and references therein). However, extra conditions are required when two methods are worked out rigorously and completely [35].

Let $\mathcal{M}(u, \theta, \psi) = M(u, \theta, \psi) - \frac{1}{2}(l_\theta + l \cot \theta + l_\psi \csc \theta)$. In [35], it was derived that $\mathcal{M}_u = -c_u^2 - d_u^2$. Suppose $m_0(u) = |m(u)|$ on $[u_1, u_0]$. If $m_0(u) \neq 0$, then the Bondi energy-momentum loss formula implies $m_i(u) = \frac{d}{du}(m_i(u))$. This is impossible except $m_i = 0$ since $m_i$ are independent on $\theta, \psi$ but $n_i$ do depend on $\theta, \psi$. Thus $m_0(u) = 0$ on $[u_1, u_0]$. By the Bondi energy loss formula, we obtain that $c_u = d_u = \mathcal{M}_u = 0$ on $[u_1, u_0]$. Therefore,

(i) if $\mathcal{M}(u_0, \theta, \psi) = C$ for some constant $C$, then $\mathcal{M}(u, \theta, \psi) = C$ on $[u_1, u_0]$;
(ii) if $c(u_0, \theta, \psi) = d(u_0, \theta, \psi) = 0$, then $c(u, \theta, \psi) = d(u, \theta, \psi) = 0$ on $[u_1, u_0]$.

Now we discuss the positivity of Bondi energy-momentum. The following two theorems are proved in [35] and stated here more accurately. Firstly, Schoen-Yau’s method shows

\textbf{Positivity of Bondi energy (Schoen-Yau’s method):} Suppose there exists $u_0$ in vacuum Bondi’s radiating spacetime such that $\mathcal{M}(u_0, \theta, \psi)$ is constant.

(i) $m_0(u_0) \geq |m(u_0)|$, and the Bondi energy-momentum loss formula gives $m_0(u) \geq |m(u)|$ for all $u \leq u_0$;
(ii) If $m_0(u_0) = |m(u_0)|$ and there is $u_1 < u_0$ such that $m_0(u_1) = |m(u_1)|$, then $\mathcal{M}$ is constant on $[u_1, u_0]$. Thus the spacetime is flat in the region $(u_1, u_0)$.
The proof of the positivity using Witten’s method requires the positive
energy theorem for asymptotically null initial data sets. In Minkowski spacetime \( \mathbb{R}^{3,1} \), the null cone is \( \{ t = r \} \). The spacelike hypersurface \( t = \sqrt{1 + r^2} \) approaches to null infinity, and equips with the hyperbolic metric \( \tilde{g} \) and the nontrivial second fundamental form \( \tilde{h} \) in polar coordinates
\[
\tilde{g} = \tilde{h} = \frac{dr^2}{1 + r^2} + r^2(d\theta^2 + \sin^2 \theta d\psi^2).
\]
Denote \( \tilde{e}_i, \tilde{e}^i \) and \( \nabla \) the frame, coframe and the Levi-Civita connection respectively. The initial data set \((M, g, h)\) is asymptotically null if, on each end,
\[
\left\{ a_{ij}, \nabla_k a_{ij}, \nabla_l \nabla_k a_{ij}, b_{ij}, \nabla_k b_{ij} \right\} = O(r^{-\tau})
\]
for \( \tau > \frac{7}{4} \), where \( a_{ij} = g(\tilde{e}_i, \tilde{e}_j) - \tilde{g}(\tilde{e}_i, \tilde{e}_j), b_{ij} = h(\tilde{e}_i, \tilde{e}_j) - \tilde{h}(\tilde{e}_i, \tilde{e}_j) \). Let \( R, \nabla, \rho_z \) be the scalar curvature and the Levi-Civita connection of \( g \), and the distance function with respect to \( z \in M \) respectively. We further assume
\[
(R + 6)\rho_z \in L^1(M), \quad (\nabla^j g_{ij} - \nabla_i tr_g(h))\rho_z \in L^1(M).
\]
The asymptotic null total energy-momentum of the end \( M_i \) are
\[
E_\nu = \frac{1}{16\pi} \int_{S^\infty_\nu} \mathcal{E} n^\nu r \tilde{e}^2 \wedge \tilde{e}^3,
\]
where \( \nu = 0, 1, 2, 3, \mathcal{E} = \nabla^j g_{ij} - \nabla_i tr_g(h) + (a_{22} + a_{33}) + 2(b_{22} + b_{33}) \).

The positive energy theorem near null infinity \( [57] \): Let \((M, g, h)\) be an asymptotically null initial data set in spacetime \( \mathbb{L}^{3,1} \). Suppose \( \mathbb{L}^{3,1} \) satisfies the dominant energy condition, then
(i) \( E_0 \geq \sqrt{E_1^2 + E_2^2 + E_3^2} \) for any end;
(ii) That \( E_0 = 0 \) for some end implies \( M \) has only one end, and \( \mathbb{L}^{3,1} \) is flat along \( M \).

It ensures that the asymptotically null total mass \( \sqrt{E_0^2 - E_1^2 - E_2^2 - E_3^2} \) is well-defined.

In order to using this theorem to provide the positivity of Bondi energy, we use the spacelike hypersurface
\[
u = u_0 + \sqrt{1 + r^2} - r + \frac{c^2(u_0, \theta, \psi) + d^2(u_0, \theta, \psi)}{12r^3} + \frac{a_3(\theta, \psi)}{r^4} + o\left(\frac{1}{r^4}\right)
\]
to cut the Bondi-Sachs metrics and obtain asymptotically null initial data sets. Unfortunately, the metric and the second fundamental form of this spacelike hypersurface differ from the hyperbolic metric with the leading terms \( \frac{c(u_0)}{r} \) and \( \frac{c(u_0)}{r} \) \([35]\). This breaks the conditions required in the above positive energy theorem near null infinity. So we need to assume \( c(u_0, \theta, \psi) = d(u_0, \theta, \psi) = 0 \). Under these conditions
\[
E_\nu = \frac{1}{16\pi} \int_{S^\infty_\nu} \mathcal{E} n^\nu r \tilde{e}^2 \wedge \tilde{e}^3 = m_\nu(u_0)
\]
on the asymptotically null initial data sets. (The notations here are different from those in [35], $E_\nu$, $\mathcal{E}$ here are $E_\nu(X) - P_\nu(X)$, $\mathcal{E} - 2\mathcal{P}$ in [35] respectively. It is an error to integrate $\mathcal{E} - \mathcal{P}$ to get $E_\nu(X) - P_\nu(X) = \frac{5}{8}m_\nu(u_0)$ in [35] which should be corrected as above.) Then the positive energy theorem near null infinity shows [35].

**Positivity of Bondi energy (Witten’s method):** Suppose there exists $u_0$ in vacuum Bondi’s radiating spacetime such that $c(u_0) = d(u_0) = 0$.

(i) $m_0(u_0) \geq |m(u_0)|$, and the Bondi energy-momentum loss formula gives $m_0(u) \geq |m(u)|$ for all $u \leq u_0$;

(ii) If $m_0(u_0) = |m(u_0)|$ and there is $u_1 < u_0$ such that $m_0(u_1) = |m(u_1)|$, then $c(u_0) = d(u_0) = 0$ on $[u_1, u_0]$. Thus the spacetime is flat in the region $(u_1, u_0]$.

It ensures that the Bondi mass $\sqrt{m_0^2 - m_1^2 - m_2^2 - m_3^2}$ is well-defined.

It is an interesting question whether the total angular momentum can be detected near or at null infinity. As it is still open to find smooth Bondi-Sachs’ coordinates for Kerr spacetime, it is unclear what rotation means for gravitational systems traveling in the speed of light. On the other hand, B. Liu constructed certain asymptotically null initial data sets in Kerr spacetime on the dissertation of his Bachelor’s degree at the University of Science and Technology of China in 2013. Choose the spacelike hypersurface

$$t = r + 2m \ln r + \frac{c_1(\theta, \psi)}{r} + \frac{c_2(\theta, \psi)}{r^2} + \frac{c_3(\theta, \psi)}{r^3} + \cdots$$

in Kerr spacetime. Liu found the fall-offs of the metric and the second fundamental form by suitable choices of $c_i(\theta, \psi)$

$$g_{11} = 1 + O\left(\frac{1}{r^4}\right), \quad g_{12} = \frac{2m^2a^2 \sin 2\theta}{3r^3} + O\left(\frac{1}{r^4}\right), \quad g_{13} = -\frac{2m a \sin \theta}{r} - \frac{4m^2a \sin \theta}{r^2} - \frac{ma(8m^2 - 5a^2 \cos^2 \theta + a^2) \sin \theta}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$g_{22} = 1 + O\left(\frac{1}{r^5}\right), \quad g_{23} = O\left(\frac{1}{r^5}\right), \quad g_{33} = 1 + \frac{2ma^2 \sin^2 \theta}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$h_{11} = 1 - \frac{m^2a^2 \sin^2 \theta}{r^3} - \frac{m(16m^2a^2 - 3a^2) \sin^2 \theta + 2m}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$h_{12} = \frac{8m^2a^2 \sin 2\theta}{3r^3} + O\left(\frac{1}{r^4}\right), \quad h_{23} = O\left(\frac{1}{r^4}\right),$$

$$h_{13} = -\frac{2ma \sin \theta}{r} - \frac{4m^2a \sin \theta}{r^2} - \frac{ma(4a^2(4m^2 + 5) \sin^2 \theta + 16m^2 - 4a^2 + 3) \sin \theta}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$h_{22} = 1 + \frac{2m^2a^2 \sin^2 \theta}{r^2} - \frac{m(4m^2a^2 \sin^2 \theta - 1)}{r^3} + O\left(\frac{1}{r^4}\right),$$

$$h_{33} = 1 + \frac{2m^2a^2 \sin^2 \theta}{r^2} + \frac{m[(8m^2 - 1)a^2 \sin^2 \theta + 1]}{r^3} + O\left(\frac{1}{r^4}\right).$$
Let $\tilde{h}_{ij} = h_{ij} - g_{ij}$, $\tilde{\pi}_{ij} = \tilde{h}_{ij} - tr_g(\tilde{h})$. If $a \neq 0$, Liu found,

$$\tilde{\pi}_{12} = O\left(\frac{1}{r^3}\right), \quad \tilde{\pi}_{13} = O\left(\frac{1}{r^4}\right), \quad \text{other } \tilde{\pi}_{ij} = O\left(\frac{1}{r^4}\right).$$

The fall-offs of $g_{13}$, $\tilde{\pi}_{12}$ and $\tilde{\pi}_{13}$ are not sufficiently fast to define the total angular momentum. This indicates the rotation does not make sense also near null infinity.

3. $\Lambda > 0$

In this section, we assume the cosmological constant is positive. As recent cosmological observations indicated that our universe should have a positive cosmological constant, the positive energy theorem for asymptotically de Sitter spacetimes gains much more importance. There are a large number of papers contributing to issue of the total energy-momentum in spacetimes with the positive cosmological constant, e.g., [1], [8], [12], [31], [5], [6], but only few studying their positivity [47], [48], [36]. Even through, the formulations of the positivity are incomplete, and some of the proofs do not seem correct in mathematics. In [38], [37], the complete and rigorous study of the positive energy theorem were provided.

Denote $\lambda = \sqrt{\frac{3}{\Lambda}}$ throughout the section. De Sitter spacetime can be fully covered by the global coordinates equipped with the de Sitter metric

$$\tilde{g}_{dS} = -dT^2 + \lambda^2 \cosh^2 \frac{T}{\lambda} \left( dr^2 + \sin^2 r (d\theta^2 + \sin^2 \theta d\psi^2) \right).$$

In global coordinates, each time slice is a 3-sphere with constant curvature and has no spatial infinity. Therefore the ADM formulation of the energy-momentum is not available. There are essentially two ways to separate de Sitter spacetime into two parts which give two different spatial infinities. And the positive energy theorems can be established for spacetimes which are asymptotic to either half of the de Sitter spacetime under reasonable conditions.

Separating along the hypersurface $X^0 = X^4$, the half-de Sitter spacetime is covered by planar coordinates equipped with the de Sitter metric

$$\tilde{g}_{dS} = -dt^2 + e^{2T} g_\delta, \quad g_\delta = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

and the initial data set is ($\mathbb{R}^3$, $\tilde{g} = e^{2T} g_\delta$, $\tilde{K} = \frac{1}{2} \tilde{g}$). Separating along the hypersurface $X^4 = -\lambda$ ($X^4 = \lambda$), the half-de Sitter spacetime $X^4 < -\lambda$ ($X^4 > \lambda$) is covered by hyperbolic coordinates equipped with the de Sitter metric

$$\tilde{g}_{dS} = -dT^2 + \sinh^2 \frac{T}{\lambda} \tilde{g}_H, \quad \tilde{g}_H = dv^2 + \lambda^2 \sinh^2 \frac{T}{\lambda} (d\theta^2 + \sin^2 \theta d\psi^2)$$

and the initial data set is ($\mathbb{H}^3$, $\tilde{g} = \sinh^2 \frac{2T}{\lambda} \tilde{g}_H$, $\tilde{K} = \frac{1}{\lambda} \coth \frac{T}{\lambda} \tilde{g}$) where $\tilde{g}_H$ is the standard hyperbolic metric.
To define the total energy-momentum from the Hamiltonian point of view, an asymptotically de Sitter initial data set \((M, g, K)\) should satisfy
\[
g - \bar{g} = e^{2t_0} a = O\left(\frac{1}{r}\right), \quad K - \bar{K} = e^{t_0} b = O\left(\frac{1}{r^2}\right)
\]
for constant \(t_0\) on ends in planar coordinates. Then the ten Killing vectors \(U_{\alpha\beta}\) of \(\mathbb{R}^{4,1}\) and \(a\) and \(b\) are used to define the ADM-like total energy-momentum. But there is no energy-momentum inequalities for them in general. Alternatively, there is different approach to define the total energy-momentum from the initial data set point of view. We explain the main idea for this approach in planar coordinates for simplicity, and it is similar in hyperbolic coordinates.

In general, an initial data set \((M, g, K)\) in asymptotically de Sitter spacetimes should take the forms
\[
g = e^{2u(x)} \bar{g}(x), \quad h = K - \frac{1}{\lambda} \bar{g} = e^{u(x)} \bar{h}(x), \quad u(x) = \frac{ln}{\lambda} + o(1) \quad \text{and} \quad (M, \bar{g}, \bar{h}) \text{ is asymptotically flat.}
\]
In order to preserve the asymptotic flatness for \(g\) and \(h\), one must take \(u(x) = \frac{ln}{\lambda} + \frac{a}{r^2} + O\left(\frac{1}{r^4}\right)\). In this case it can be written that \(g = e^{2t_0} \bar{g}, \ h = e^{t_0} \bar{h}\) on ends, and the ADM total energy-momentum of \(\bar{g}, \bar{h}\) can serve as the total energy-momentum of \(g, h\) up to certain constant factors. As \(e^{\frac{ln}{\lambda}}\) is constant, it is equivalent to assume \(g = e^{2t_0} \bar{g}, \ h = e^{t_0} \bar{h}\) on whole \(M\), instead of on ends only.

Thus, an initial data set \((M, g, K)\) is \(\mathcal{P}\)-asymptotically de Sitter if \(g = \mathcal{P}\bar{g}, \ h = \mathcal{P}\bar{h}\) for certain constant \(\mathcal{P} > 0\), and \((M, \bar{g}, \bar{h})\) is asymptotically flat. Let \(E, \ P_k, \ J_k(z)\) be the total energy, the total linear momentum and the total angular momentum of the end \(M_l\) for \((M, \bar{g}, \bar{h})\) respectively. The corresponding quantities for \(\mathcal{P}\)-asymptotically de Sitter initial data set \((M, g, K)\) are
\[
E = \mathcal{P} \bar{E}, \quad P_k = \mathcal{P}^2 \bar{P}_k, \quad J_k(z) = \mathcal{P}^2 \bar{J}_k(z).
\]
and the positive energy theorem proved by Luo, Xie and Zhang [38] is as follows.

The positive energy theorem (planar coordinates): Let \((M, g, K)\) be a \(\mathcal{P}\)-asymptotically de Sitter initial data set in spacetime \(L^{3,1}\) with positive cosmological constant \(\Lambda > 0\). Suppose
\[
tr_g(K) \leq \sqrt{3}\Lambda
\]
If \(L^{3,1}\) satisfies the dominant energy condition, then
(i) \(E \geq \sqrt{P_1^2 + P_2^2 + P_3^2}\) for any end;
(ii) That \(E = 0\) for some end implies
\[
(M, g, K) \equiv \left(\mathbb{R}^3, \mathcal{P}^2 \bar{g}_8, \sqrt{\frac{\Lambda}{3}} \mathcal{P}^2 \bar{g}_8\right)
\]
and the spacetime \(L^{3,1}\) is de Sitter along \(M\).

It ensures that the total mass \(\sqrt{E^2 - \bar{P}_1^2 - \bar{P}_2^2 - \bar{P}_3^2}\) is well-defined.
The total angular momentum $J$ is computed to equal $(0, 0, \frac{ma}{(1 + \frac{\lambda}{2})^2})$ for Kerr-de Sitter spacetime [38]. It is interesting that $J_3(z)$ conjugates to $J_{23}$ of [33] by replacing the positive cosmological constant to the negative cosmological constant. The corresponding Kerr constraint was also proved in [38].

Let $(M, g, K)$ be $\mathcal{P}$-asymptotically de Sitter which has no apparent horizon in spacetime $(N^{1, 3}, \tilde{g})$ with positive cosmological constant $\Lambda > 0$. Denote $p = Ch^2$ where $h^2$ is local angular momentum density defined by $h$, and $C > 0$ is certain constant. Suppose that there exists a point $z \in M$ such that $(M, \tilde{g}, p)$ is generalized asymptotically flat. If $(M, \tilde{g}, p)$ satisfies the generalized dominant energy condition, then

(i) $E \geq C |J(z)|_{\tilde{g}p}$ for any end;
(ii) That $E = 0$ for some end implies that $M$ has only one end, and

$$R_{ijkl} + p_{ik}p_{jl} - p_{il}p_{jk} = 0, \quad \nabla_i p_{jk} - \nabla_j p_{ik} = 0, \quad \nabla^i(p_{ij} - p_{ji}) = 0.$$ 

In the Hamiltonian formulation, $g - \tilde{g} = O(r^{-1})$ and $K - \tilde{K} = O(r^{-2})$ are used to define the total energy-momentum. In the initial data set formulation, $g - \tilde{g} = O(r^{-1})$, $h = O(r^{-2})$ which give $K - \tilde{K} = O(r^{-1})$. Although the two approaches give the same total energy, the total momenta are completely different. While it is finite, defined in terms of $h$, the total linear momentum is infinite in general, defined in terms of $K - \tilde{K}$. So there should not be any energy-momentum inequality in the Hamiltonian formulation. But it is known that $E \geq 0$ if $\text{tr}_g(K) \leq \sqrt{3}\Lambda$ in this case.

In hyperbolic coordinates, denote $h = K - \frac{1}{2} \coth^2 \frac{T}{2}g$, $\mathcal{H} = \sinh \frac{T}{2}$ for certain constant $T$. Let $\tilde{e}_i$, $\tilde{e}_j$ and $\tilde{\nabla}^H$ be the frame, coframe and the Levi-Civita connection of the hyperbolic metric $\tilde{g}_H$. An initial data set $(M, g, K)$ is $\mathcal{H}$-asymptotically de Sitter if $g = \mathcal{H}^2 \tilde{g}$, $h = \mathcal{H} \tilde{h}$, and, on each end, $\tilde{g}$ and $h$ satisfy

$$\{a_{ij}, \tilde{\nabla}^H a_{ij}, \tilde{\nabla}^H \tilde{\nabla}^H a_{ij}, \tilde{h}_{ij}, \tilde{\nabla}^H \tilde{h}_{ij}\} = O(e^{-\frac{\tau}{2}})$$

for $\tau > \frac{3}{2}$, $a_{ij} = \tilde{g}(\tilde{e}_i, \tilde{e}_j) - \tilde{g}_H(\tilde{e}_i, \tilde{e}_j)$, $\tilde{h}_{ij} = \tilde{h}(\tilde{e}_i, \tilde{e}_j)$. Let $\tilde{R}$, $\tilde{\nabla}$, $\rho_z$ be the scalar curvature and the Levi-Civita connection of $\tilde{g}$, and the distance function with respect to $z \in M$ respectively. We further assume

$$(\tilde{R} + \frac{6}{\lambda^2})e^{\frac{\rho_z}{\lambda}}, \quad (\tilde{\nabla}^H \tilde{h}_{ij} - \tilde{\nabla}_j \text{tr}_\tilde{g}(\tilde{h}))e^{\frac{\rho_z}{\lambda}} \in L^1(M).$$

The total energy-momentum of the end $M_i$ are

$$E^H_{\nu} = \frac{\mathcal{H}^2}{16\pi} \int_{S_\infty} \mathcal{E}_\nu e^\frac{\rho_z}{\lambda} e^2 \wedge e^3,$$

where $\nu = 0, 1, 2, 3$, $\mathcal{E} = \tilde{\nabla}^H \tilde{g}_{ij} - \tilde{\nabla}^H_1 \text{tr}_\tilde{g}_H(\tilde{g}) + \frac{1}{2} (a_{22} + a_{33}) + 2(h_{22} + h_{33})$. The positive energy theorem in this case was also proved by Luo, Xie and Zhang [38].
The positive energy theorem (hyperbolic coordinates): Let \((M,g,K)\) be a \(\mathcal{H}\)-asymptotically de Sitter initial data set in spacetime \(L^{3,1}\) with positive cosmological constant \(\Lambda > 0\). Suppose 

\[
\text{tr}_g(K) \sinh \frac{T}{\Lambda} \leq \sqrt{3\Lambda} \cosh \frac{T}{\Lambda}.
\]

If \(L^{3,1}\) satisfies the dominant energy condition, then

(i) \(E^H_0 \geq \sqrt{(E^H_1)^2 + (E^H_2)^2 + (E^H_3)^2}\) for each end;

(ii) That \(E^H_0 = 0\) for some end implies

\[
(M,g,K) \equiv \left(\mathbb{H}^3, \sinh^2 \frac{T}{\Lambda} \tilde{g}_H, \sqrt{\frac{\Lambda}{3}} \sinh \frac{T}{\Lambda} \cosh \frac{T}{\Lambda} \tilde{g}_H \right)
\]

and the spacetime \(L^{3,1}\) is de Sitter along \(M\).

It ensures that the total mass \(\sqrt{(E^H_0)^2 - (E^H_1)^2 - (E^H_2)^2 - (E^H_3)^2}\) is well-defined.

The positive energy theorems hold also when \(M\) has a finite number of inner boundaries, and each \((\Sigma, \tilde{g}, \tilde{h})\) of them is topological \(S^2\) whose induced metric \(\tilde{g}\) and the second fundamental form \(\tilde{h}\) satisfy

\[
\pm \text{tr}_{\tilde{g}}(h|\Sigma) = \text{tr}_{g_\Sigma}(K|\Sigma) - 2\sqrt{\frac{\Lambda}{3}}
\]

in \(\mathcal{P}\)-asymptotically de Sitter initial data sets and

\[
\pm \text{tr}_{\tilde{g}}(h|\Sigma) = \text{tr}_{g_\Sigma}(K|\Sigma) - 2\sqrt{\frac{\Lambda}{3}} \tanh \frac{T}{2\Lambda}
\]

in \(\mathcal{H}\)-asymptotically de Sitter initial data sets. All these inner boundaries do not coincide with the future/past apparent horizon

\[
\pm \text{tr}_{\tilde{g}}(h|\Sigma) = \text{tr}_{g_\Sigma}(K|\Sigma)
\]

defined by the outward expansion of the future and past going light rays emanating from \(\Sigma\).

However, as pointed out by Witten \[52\], there is no positive conserved energy in de Sitter spacetime, and the corresponding Killing vector fields to the Lorentzian generators are timelike in some region of de Sitter spacetime and spacelike in some other region (see also \[1\]). This indicates that there should not have the positive energy theorem in the standard sense without extra mean curvature constraints.

The positive energy theorem indicates that any metric on \(\mathbb{R}^n\) \((n \geq 3)\) with scalar curvature \(\geq 0\) must be flat if it agrees with the Euclidean metric outside a compact set. In 1989, Min-Oo proved a theorem which implies any metric on \(\mathbb{H}^n\) with scalar curvature \(R \geq -n(n-1)\) \((n \geq 3)\) must be hyperbolic if it agrees with the hyperbolic metric outside a compact set \[40\]. In global coordinates, the de Sitter spacetime can be separated into two parts along the hypersurface \(X^3 = 0\). The time slices in this half-de Sitter spacetime are hemispheres. In 1995, Min-Oo conjectured that any metric
on the hemisphere $S^+_n$ ($n \geq 3$) with scalar curvature $R \geq n(n-1)$ must be the standard metric on $S^+_n$ if the boundary $\partial S^+_n$ is totally geodesic and the induced metric on $\partial S^+_n$ agrees with the standard metric on $\partial S^+_n$. In 2010, Brendle, Marques and Neves constructed some metrics on the hemisphere $S^+_n$ ($n \geq 3$) with scalar curvature $R \geq n(n-1)$ and $R > n(n-1)$ at some point, but agree with the standard metric in a neighborhood of $\partial S^+_n$ [11]. It therefore provides a counterexample to Min-Oo’s conjecture and provide one failure of the positive energy theorem in certain case for positive cosmological constant.

In 2012, Liang and Zhang constructed certain asymptotically de Sitter initial data sets with negative total energy in de Sitter spacetimes [37]. These initial data sets violate the mean curvature constraints in two positive energy theorems.

In planar coordinates, consider the graph $f(x) = t_0 + \varepsilon(1 + r^2)^{-\frac{1}{2}}$ for certain constant $t_0$ in de Sitter spacetime. The graph is $\mathcal{P}$-asymptotically de Sitter ($\mathcal{P} = e^\frac{T_0}{\lambda}$) for sufficiently small $\varepsilon$, and

$$g_{ij} = e^{\frac{2L}{\lambda}} \delta_{ij} - \varepsilon^2 \frac{x_i x_j}{(1 + r^2)^3},$$

$$tr_g(K) = \frac{3}{\lambda} + \frac{1}{2\lambda} \varepsilon^2 \frac{r^2}{(1 + r^2)^3} e^{-\frac{2L}{\lambda}} + O(r^{-5}).$$

Thus if $\varepsilon \neq 0$, then $tr_g(K) > \frac{3}{\lambda}$ for large $r$. Choose $\varepsilon < 0$, the total energy

$$E = \frac{\mathcal{P}}{16\pi} \lim_{r \to \infty} \int_{S_r} (\partial_j \bar{g}_{ij} - \partial_i \bar{g}_{jj}) * dx^j$$

$$= \frac{\mathcal{P}}{16\pi} \lim_{r \to \infty} \int_{S_r} \left( \frac{4\varepsilon}{\lambda} \frac{r}{(1 + r^2)^{\frac{3}{2}}} e^{\frac{2L - 3\lambda}{\lambda}} - 3\mathcal{P}^{-2} \varepsilon^2 \frac{r}{(1 + r^2)^3} \right)$$

$$= \frac{\varepsilon \lambda}{\lambda} e^{\frac{T_0}{\lambda}} < 0.$$

In hyperbolic coordinates, consider the graph $f(x) = T_0 + \varepsilon e^{-\frac{6r}{\lambda}}$ for certain constant $T_0 > 0$ in de Sitter spacetime. The graph is $H$-asymptotically de Sitter ($H = \sinh \frac{T_0}{\lambda}$) for sufficiently small $\varepsilon$, and

$$g = \left( \sinh^2 \frac{f}{\lambda} - \frac{9\varepsilon^2}{\lambda^2} e^{-\frac{6r}{\lambda}} \right) dr^2 + \lambda^2 \sinh^2 \frac{f}{\lambda} \sin^2 \frac{r}{\lambda} \left( d\theta^2 + \sin^2 \theta \psi^2 \right),$$

$$tr_g(K) \sinh \frac{T_0}{\lambda} - \frac{3}{\lambda} \cosh \frac{T_0}{\lambda} = -\frac{12\varepsilon}{\lambda^2 \sinh \frac{T_0}{\lambda}} e^{\frac{6r}{\lambda}} + O(e^{-\frac{6r}{\lambda}}).$$
Thus if ε < 0, then \( tr_g(K) \sinh \frac{T_0}{\lambda} > \frac{2}{\lambda} \cosh \frac{T_0}{\lambda} \) for large \( r \), and the total energy

\[
E_0^H = \frac{\mathcal{H}^2}{16\pi} \lim_{r \to \infty} \int_{S_r} \mathcal{E} n^0 \varepsilon^* e^2 \wedge e^3
\]

\[
= \sinh^2 \frac{T_0}{\lambda} \lim_{r \to \infty} \int_{S_r} \frac{16\epsilon}{\lambda^2} \tanh \frac{T_0}{2\lambda} e^{-\frac{2\epsilon}{\lambda^2} \sinh^2 \frac{r}{\lambda} \sin \theta d\theta d\psi}
\]

\[
= \epsilon \tanh \frac{T_0}{2\lambda} \sin^2 \frac{T_0}{\lambda} < 0.
\]

The constant \( \Lambda = \frac{3}{\lambda^2} \) in the above two positive energy theorems can be chosen arbitrarily which is not necessary the cosmological constant. Based on the experimental dates from Planck 2015\(^{1}\) Hubble constant \( H = \frac{1}{\lambda} \) is chosen and \( \tilde{g}_{ds} \) is taken as the FLRW metric with \( k = 0 \)

\[
\tilde{g}_{FLRW} = -dt^2 + e^{2Ht} g_\delta, \quad g_\delta = (dx^1)^2 + (dx^2)^2 + (dx^3)^2
\]

which is isotropic and homogeneous, where

\[ 3H^2 = \rho_m + \Lambda_c \]

and \( \rho_m \cong 0.3156 \times 3H^2 \) is the matter density containing dark matter, \( \Lambda_c \cong 0.6844 \times 3H^2 \) is the real value of cosmological constant representing dark energy. The universe takes asymptotically FLRW metrics and satisfies the dominant energy condition, and initial data sets are \( \mathcal{P} \)-asymptotically de Sitter. If universe’s volume expansion ratio \( tr_g(K) \) can be detected to be less than or equal to \( 3H \) from Planck 2015, then the universe has positive total energy counting dark matter and dark energy. More precisely, the universe has more energy than the metric \( \tilde{g}_{FLRW} \) has. Otherwise it may have negative total energy.

The case of \( \mathcal{H} \)-asymptotically de Sitter initial data sets corresponds to FLRW metrics with \( k < 0 \).

4. \( \Lambda < 0 \)

In this section, we assume the cosmological constant is negative. In this case spacetimes are asymptotically anti-de Sitter and initial data sets have asymptotically hyperbolic metrics and the asymptotically zero second fundamental forms. There are also a large number of papers to devote to define the total energy-momentum and prove its positivity in a physical manner, see, e.g. \([1, 33, 4]\) and references therein. (It seems the total energy was first defined in \([1]\), and which also contained the proof of its positivity via SUGRA, exactly as the proof for zero cosmological constant \([28]\).) However, the mathematical rigorous and complete proofs were given only in \([19, 22]\) for asymptotically anti-de Sitter initial data sets with the zero second fundamental form, and in \([39, 25]\) for the initial data sets with the nonzero

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\(^{1}\)Planck 2015 results I. Overview of Scientific Results. ArXiv: 1502.01582; XIII. Cosmological Parameters. ArXiv: 1502.01589.
second fundamental form where the energy-momentum matrix was proved to be positive semi-definite, and some energy-momentum inequalities were proved in certain specific coordinate systems. The positive energy theorem near null infinity in [53] gives a different energy-momentum inequality for asymptotically anti-de Sitter initial data sets with the nonzero second fundamental form whose trace is nonpositive [53]. In [50], the complete and rigorous study of the positive energy theorem was provided.

Denote \( \kappa = \sqrt{-\frac{\lambda}{3}} \) throughout the section. Under coordinate transformations

\[
X^0 = \frac{\cos(\kappa t)}{\kappa} \cosh(\kappa r), \quad X^i = \frac{\sinh(\kappa r)}{\kappa} n^i, \quad X^4 = \frac{\sin(\kappa t)}{\kappa} \cosh(\kappa r),
\]

the anti-de Sitter metric is

\[
\tilde{g}_{\text{AdS}} = -\cosh^2(\kappa r)dt^2 + dr^2 + \frac{\sinh^2(\kappa r)}{\kappa^2}(d\theta^2 + \sin^2 \theta d\psi^2)
\]

and the initial data set is \( (\mathbb{H}^3, \tilde{g} = g_{\mathcal{H}}, \tilde{h} = 0) \). The Killing vectors \( U_{\alpha \beta} \) depend on \( t \) restricting on anti-de Sitter spacetime.

Let \( \tilde{e}_i, \tilde{\xi}^i \) and \( \tilde{\nabla} \) be the frame, coframe and the Levi-Civita connection of the hyperbolic metric \( \tilde{g} \) respectively. The initial data set \( (M, g, h) \) is asymptotically anti-de Sitter if \( M \) has a finite number of ends,

\[
\left\{ a_{ij}, \tilde{\nabla}_k a_{ij}, \tilde{\nabla}_l \tilde{\nabla}_k a_{ij}, h_{ij}, \tilde{\nabla}_k h_{ij} \right\} = O(e^{-\kappa \tau t}), \quad \tau > \frac{3}{2},
\]

on each end, where \( a_{ij} = g(\tilde{e}_i, \tilde{e}_j) - \tilde{g}(\tilde{e}_i, \tilde{e}_j) \), \( h_{ij} = h(\tilde{e}_i, \tilde{e}_j) \). Moreover, let \( R, \nabla, \rho_z \) be the scalar curvature and the Levi-Civita connection of \( g \), and the distance function with respect to \( z \in M \) respectively,

\[
(R + 6\kappa^2)e^{\kappa r}z, \quad (\nabla^i h_{ij} - \nabla_i tr_g(h))e^{\kappa r}z \in L^1(M).
\]

In 1985, Henneaux and Teitelboim defined the total energy-momentum \( J_{\text{HT}}^{ab} \) for asymptotically AdS spacetimes associated to \( U_{ab} \) [33]. In our notation, these quantities are

\[
E_0 = \frac{\kappa}{16\pi} \int_{S_{\infty}} \mathcal{E} U_{40}^{(0)} \tilde{\omega},
\]

\[
c_i(t) = \frac{\kappa}{16\pi} \int_{S_{\infty}} \mathcal{E} U_{14}^{(0)} \tilde{\omega} + \frac{\kappa}{8\pi} \int_{S_{\infty}} \mathcal{P}_A U_{14}^{(A)} \tilde{\omega},
\]

\[
c'_i(t) = \frac{\kappa}{16\pi} \int_{S_{\infty}} \mathcal{E} U_{10}^{(0)} \tilde{\omega} + \frac{\kappa}{8\pi} \int_{S_{\infty}} \mathcal{P}_A U_{10}^{(A)} \tilde{\omega},
\]

\[
J_i = \frac{\kappa}{8\pi} \int_{S_{\infty}} \mathcal{P}_A V_i^{(A)} \tilde{\omega},
\]

where \( \mathcal{E} = \tilde{\nabla}^j g_{ij} - \tilde{\nabla}^j tr_g(h) + \kappa(a_{22} + a_{33}), \mathcal{P}_j = h_{ji} - g_{ji} tr_g(h), \tilde{\omega} = \tilde{\xi}^2 \wedge \tilde{\xi}^3, U_{\alpha \beta} = U_{\alpha \beta}^{(\gamma)} \tilde{\xi}_\gamma, V_i = \frac{1}{2} \varepsilon_{ijk} U_{jk} = V_i^{(A)} \tilde{\xi}_A, A = 2, 3. \)
Denote $c = (c_1, c_2, c_3)$, $c' = (c'_1, c'_2, c'_3)$, $J = (J_1, J_2, J_3)$ and

$$L = (|c|^2 + |c'|^2 + |J|^2)^{\frac{1}{2}},$$

$$A = (|c \times c'|^2 + |c \times J|^2 + |c' \times J|^2)^{\frac{1}{2}},$$

$$V = (\varepsilon_{ijk}c_i'c_j'J_k)^{\frac{1}{3}}.$$

The quantities $2L$, $2A^2$ and $V^3$ are the (normalized) length, surface area and volume of the parallelepiped spanned by $c$, $c'$ and $J$, which are independent on $t$. Clearly, $L^2 \geq 3V^2$, and $|c|^2 + |c'|^2$ is independent on $t$.

The pseudo-Euclidean space $\mathbb{R}^{3,2}$ has two timelike Killing vectors and three spacelike Killing vectors. Physically, $E$ measures the rotation on the plane $(X^0, X^4)$, $c_i$ measures the rotation on the plane $(X^i, X^4)$, $c_i'$ measures the rotation on the plane $(X^0, X^i)$ and $J_i$ measures the rotation on the plane $(X^j, X^k)$ where $\{i, j, k\}$ is the even permutation of $\{1, 2, 3\}$. But these rotations are all observed from a curved hyperboloid, so they contain both translation and rotation of an asymptotically anti-de Sitter spacetime. This indicates that we can not simply refer them as the center of mass as well as the total angular momentum. The total effect of translation and rotation is given by the parallelepiped spanned by $c$, $c'$ and $J$ which can be measured from its length of the edges, surface area and the volume.

In 2001, Wang established the mathematically rigorous and complete proofs for asymptotically anti-de Sitter initial data sets with $a_{12} = a_{13} = 0$ and the zero second fundamental form [49]. And these extra conditions on $a$ were removed by Chrúsciel-Herzlich later [22]. Under the dominant energy condition, they proved, at $t = 0$,

(i) $E_0 \geq \sqrt{c_1^2 + c_2^2 + c_3^2}$ for each end;
(ii) That $E = 0$ for some end implies $M$ is hyperbolic.

In 2006, Maerten used the Lorentzian setting to treat $M$ as a spacelike hypersurface in spacetime $\mathbb{L}^{3,1}$. Using Witten’s approach and studying the restriction of spin geometry of $\mathbb{L}^{3,1}$ over $M$ with the nonzero second fundamental form, he proved that there is a positive semi-definite $4 \times 4$ Hermitian matrix $Q$ involving $E_0$, $c_i(0)$, $c_i'(0)$ and $J_i$ under the dominant energy condition [39]. Soon later, Chrúsciel, Maerten and Tod [25] proved, at $t=0$, if $E_0 > \sqrt{c_1^2 + c_2^2 + c_3^2}$, one can make $SO(3, 1)$ coordinate transformations such that

$$\sqrt{E_0^2 - c_1^2 - c_2^2 - c_3^2} \rightarrow E_0, \ c_i, c_i', J_1, J_2 \rightarrow 0, \ c_1' \rightarrow \tilde{c}_1', \ c_2' \rightarrow \tilde{c}_2', \ J_3 \rightarrow \tilde{J}_3.$$

In the new coordinates, which we refer to the “center of AdS mass” coordinates, they proved

(i) $E_0 \geq \sqrt{|\bar{c}|^2 + |\bar{J}|^2 + 2|\bar{c} \times \bar{J}|}$ for each end;
(ii) That $E = 0$ for some end implies $M$ has only one end, and $\mathbb{L}^{3,1}$ is anti-de Sitter along $M$. 
In [50], it was shown that the trace, sum of the second-order minors, sum of the third-order minors and the determinant of $Q$ are
\[
tr Q = 4E_0, \quad Q^{(2)} = 6E_0^2 - 2L^2, \quad Q^{(3)} = 4E_0(E_0^2 - L^2) + 8V^3,
\]
\[
det Q = (E_0^2 - L^2)^2 + 8E_0V^3 - 4A^4.
\]
But Witten’s argument indicates only that these quantities are nonnegative, which do not yield $E_0 > \sqrt{c_1^2 + c_2^2 + c_3^2}$ in general when the second fundamental form is nonzero, required for the center of AdS mass coordinate transformations. (e.g. $Q^{(2)} > 0 \implies E_0 > \sqrt{\frac{1}{2}(|c|^2 + |c'|^2 + |J|^2)} \not\implies E_0 > \sqrt{c_1^2 + c_2^2 + c_3^2}$.) On the other hand, the form of Chrusciel-Maerten-Tod’s energy-momentum inequality is not $SO(3,1)$ invariant. It changes when it is transformed back to the non-center of AdS mass coordinates. These problems are the motivation to establish the inequality for Henneaux and Teitelboim’s total energy-momentum in general non-center of AdS mass coordinates at general $t$, and the positive energy theorem in this case was proved by Wang, Xie and Zhang [50].

The positive energy theorem: If $(M, g, h)$ is an asymptotically anti-de Sitter initial data set in spacetime $\mathbb{L}^{3,1}$ with negative cosmological constant $\Lambda < 0$, with possibly a finite number of apparent horizons which are inner boundaries of $M$, each of them is topological $S^2$ whose mean curvature $H$ satisfies $H \pm tr_g(h|_{S^2}) = 0$. Suppose the dominant energy condition holds, then
(i) $E_0 \geq \sqrt{L^2 - 2V^2 + 2\left(\max\{A^4 - L^2V^2, 0\}\right)}$ for each end;
(ii) That $E = 0$ for some end implies $M$ has only one end, and $\mathbb{L}^{3,1}$ is anti-de Sitter along $M$.

If three vectors $c$, $c'$, $J$ are linearly dependent, i.e., $V = 0$, then the above energy-momentum inequality reduces to $E_0 \geq \sqrt{L^2 + 2A^2}$ which provides more general energy-momentum inequality than Chrusciel-Maerten-Tod’s.

In [50], Wang, Xie and Zhang also proved that $\det Q$ is invariant under the following admissible coordinate transformation on ends
\[
\dot{t} = t + o(e^{-\frac{2\pi}{r}}), \quad \dot{e}_0(t) = \tilde{e}_0(t) + o(e^{-\frac{2\pi}{r}}),
\]
\[
\dot{r} = r + o(e^{-\frac{2\pi}{r}}), \quad \dot{e}_1(r) = \tilde{e}_1(r) + o(e^{-\frac{2\pi}{r}}),
\]
\[
\dot{\theta}^A = \theta^A + o(e^{-\frac{2\pi}{r}}), \quad \dot{e}_B(\theta^A) = \tilde{e}_B(\theta^A) + o(e^{-\frac{2\pi}{r}}).
\]
Thus it serves as the geometric invariant of asymptotically anti-de Sitter spacetimes. Moreover, $\det Q = \left(\frac{\kappa}{16\pi}\right)^4 (I_1^2 + I_2)$, where
\[
I_1 = \frac{1}{2} J^HT_aJ^HT_a, \quad I_2 = \frac{1}{2} J^HT_aJ^HT_bJ^HT_cJ^HT_dJ^HT_a - \frac{1}{4}(J^HT_aJ^HT_bJ^HT_a)^2.
\]
are two $O(3,2)$ Casimir invariants [33]. Therefore $\sqrt{\det Q}$ is $O(3,2)$ invariance and serves as the total rest mass of asymptotically anti-de Sitter spacetimes.
Finally, we would like to remark that the total energy-momentum for negative cosmological constant may have some physical implications in the theory of strongly coupled superconductors based on the point of view of Anti-de Sitter/Conformal Field Theory correspondence. We refer to the recent introductory overview of relevant theory \[13\] and references therein.

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