Application of Bogolyubov’s approach to the derivation of kinetic equations for dissipative systems

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Abstract. The main goal of the present article is to extend the Bogolyubov method for deriving kinetic equations to dissipative many-body systems. The basic conjecture underlying the Bogolyubov approach is the functional hypothesis, according to which, the many-particle distribution functions are assumed to be functionals of the one-particle distribution function on kinetic time scales. Another ingredient in the Bogolyubov approach is the principle of the spatial weakening of correlations, which reflects statistical independence of physical values at distant spatial points. One can consider it as a reasonable mixing property of many-particle distribution functions. The motivation behind the generalization of Bogolyubov’s approach to (classical) many-body dissipative systems is the wish to describe the dynamics of granular systems, in particular granular fluids. To this end we first define a general dissipative fluid through a dissipation function, thereby generalizing the commonly employed models for granular fluids. Using the Bogolyubov functional hypothesis we show how a reduction of the pertinent BBGKY hierarchy can be achieved. The method is then employed to cases which can be treated perturbatively, such as those in which the interactions are weak or the dissipation is small or the particle density is small. Kinetic descriptions are obtained in all of these limiting cases. As a test case, we show that the Bogolyubov method begets the now standard inelastic Boltzmann equation for dilute monodisperse collections of spheres whose collisions are characterized by a fixed coefficient of normal restitution. Possible further applications and implications are discussed.

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1 Introduction

The dynamics of granular systems in general and granular gases in particular is of much current interest [1]. Except for the physical dimensions of typical macroscopic grains, the main difference between granular and molecular many-body systems is the dissipative nature of the interactions in the former. This property has far reaching consequences, many of which are a-priori counterintuitive.

While much of the theory of granular solids and quasi-static granular flows is of phenomenological nature, granular systems seem to be well described by kinetic theory [1], i.e. the Boltzmann [2,3], or Enskog-Boltzmann equations [4,5], with the possible exception of strongly inelastic systems. Attempts to go beyond the Boltzmann level of description are noted [6,7,8], but so far these directions have not been fully exploited.

The Bogolyubov method for deriving kinetic equations for many-body systems [10,11] is based on an assumption known as the functional hypothesis. According to it, for long times $t \gg \tau_0$, many-particle distribution functions become functionals of the corresponding one-particle distribution function. Characteristic time $\tau_0$ is of the order of the typical duration of a collision (for hard sphere collisions, $\tau_0$ is the time in which a particle traverses a distance that equals its diameter). The Bogolyubov functional hypothesis can be considered as a generalization of the Chapman-Enskog method for deriving hydrodynamic equations on the basis of kinetic equation. It should also be emphasized that the term “functional hypothesis” is not, in some measure, adequate because this statement was proved for some important cases (see, e.g., [12]). In fact, the first proof of the functional hypothesis was given by Gilbert in discussion of solutions of the Boltzmann equation.

An important component of the Bogolyubov method is the principle of spatial correlations weakening, which reflects statistical independence of physical values at distant spatial points. One can consider it as a reasonable mixing property of many-particle distribution functions. The
impact of the Bogolyubov approach on kinetic theory is described in [13], and a detailed exposition of his ideas in this field and some of their applications can be found in the monograph [12], whose techniques we generalize here to render them applicable to dissipative systems.

The present article has two main goals. The first is to present a (rather straightforward) generalization of the standard models of granular gas collisions by introducing a dissipation function in conjunction with a Hamiltonian formulation of classical mechanics for dissipative systems. The use of dissipation functions is of course not new, but their application to granular systems seems to be novel. On the basis of this formulation we develop a BBGKY hierarchy for dissipative systems. A BBGKY hierarchy for a system of hard inelastically colliding spheres, which is based on a pseudo-Liouville equation, is presented e.g., in [8,9]. However, in this approach there is a problem of adequacy of description of many-body dynamics by using a pseudo-Liouville equation.

The second goal of this article is to implement the Bogolyubov method to the derivation of kinetic equations to the case of dissipative many-body classical systems. As mentioned, this approach is based on the functional hypothesis. This conjecture seems to be borne out by all studied nonequilibrium systems we are aware of. In some non-trivial cases, such as the properties of non-equilibrium steady states, it was shown to successfully reproduce results obtained by other methods [14]. Moreover, the Bogolyubov method enables to study, for example, the problem of convergence and non-analyticity arising in a perturbation theory [15]. Whether in the realm of granular systems it will yield novel results, which the commonly used methods are incapable of producing, is at present unclear. However, given the difficulties one faces when studying granular systems we believe it is important to explore the possibilities afforded by an alternate formulation. Therefore, this article is devoted to the exposition of the Bogolyubov formulation of the kinetics of dissipative gases. As a simple test, it is shown that in the limit of a dilute collections of monodisperse spheres interacting by collisions characterized by a fixed coefficient of normal restitution, the present formulation reproduces the corresponding (inelastic) Boltzmann equation. Other cases, which can be treated perturbatively are presented below.

The structure of this paper is as follows. In section 2 we formulate a dissipative dynamics on the basis of Hamilton equations of motion and dissipation function. Then we derive the corresponding Liouville equation and BBGKY hierarchy and formulate the Bogolyubov principle of spatial correlation weakening for many-particle distribution functions. Section 3 introduces the Bogolyubov functional hypothesis and boundary condition as necessary concept for solving BBGKY hierarchy. It also formulates the basic equations for deriving kinetic descriptions in different limiting cases. In particular, it is shown how a kinetic equation in the limit of weak interactions can be derived. Section 4 deals with kinetic theory in the low density limit. Here a generalized Boltzmann equation for gases with dissipative interactions is obtained. Section 6 is devoted to a derivation of a generalized Boltzmann equation for the case of weak dissipation. At the end of this section a sketch of the theory of homogeneous cooling states is presented. Section 7 discusses the connection of the proposed kinetic theory of gases in the presence of dissipative interaction with the Boltzmann equation for inelastic rigid spheres. Finally, Section 8 comprises a brief summary and outlook.

2 Formulation, the Liouville equation and the BBGKY hierarchy

Consider a system composed of $N$ identical classical particles of mass $m$ each. Their reversible interactions are assumed to be derivable from a Hamiltonian, $H$, and their dissipative interactions are assumed to be determined by a dissipation function, $R$. Both $H$ and $R$ are assumed to depend on the spatial coordinates of the centers of mass of the particles, \{x$_i$ : 1 ≤ i ≤ N\} and the respective momenta \{p$_i$ : 1 ≤ i ≤ N\}. The generalized Hamilton equations are given by:

$$\dot{p}_i = -\frac{\partial H}{\partial x_i} - \frac{\partial R}{\partial p_i}, \quad \dot{x}_i = \frac{\partial H}{\partial p_i},$$

where we assume for simplicity that the particles experience only binary interactions:

$$H = H_0 + V = \sum_{1 \leq i < j \leq N} \frac{p_i^2}{2m} + \sum_{1 \leq i < j \leq N} V_{ij}, \quad V_{ij} \equiv V(x_{ij}),$$

and where $H_0$ denotes the kinetic part of the Hamiltonian. The dissipation function $R$ is taken to depend on the coordinate and momentum differences (to preserve Galilean invariance). For sake of simplicity it assumed to equal a sum of two-particle interactions:

$$R = \sum_{1 \leq i < j \leq N} R_{ij}, \quad R_{ij} \equiv R(x_{ij}, p_{ij}),$$

where $x_{ij} \equiv x_i - x_j$ and $p_{ij} \equiv p_i - p_j$. As the entity $R_{ij}$ is a scalar, it depends on $p_{ij}$, $x_{ij}$ and $p_{ij}$. The resulting equations of motion are invariant under Galilean transformations, as they should, and conserve momentum but not energy. The force exerted by particle $j$ on particle $i$ is defined as

$$F_{ij,n} \equiv F_n(x_{ij}, p_{ij}) = -\frac{\partial V_{ij}}{\partial x_{ij}} - \frac{\partial R_{ij}}{\partial p_{ij}}.$$  

The time derivative of total energy of the system is given by

$$\frac{dH}{dt} = -\sum_{1 \leq i \leq N} \frac{\partial R}{\partial p_{in}} \frac{\partial H}{\partial p_{in}} = -\sum_{1 \leq i \leq N} p_{in} \frac{\partial R}{\partial p_{im}}.$$  

We shall assume that the nature of dissipation is associated with friction of macroscopic particles. Therefore, as a model dissipation function, we take $R_{ij}$ as follows [16]:

$$R_{ij} = \frac{1}{2} \gamma (x_{ij}) p_{ij}^2,$$
moreover, $\gamma(x_{ij}) = 0$ when $|x_{ij}| \geq r_0$, where $r_0$ is the radius of dissipative interaction, i.e. the dissipative force acts at the moment of contact of particles. According to (5), (6), one finds

$$\frac{dH}{dt} = -\frac{2}{m} \sum_{i < j \in N} R_{ij} < 0.$$ 

Since $\gamma(x_{ij}) > 0$ (see [16]), the energy dissipation occurs.

As a precursor to the derivation of the corresponding Liouville equation for the system described by Eqs. (1), we first present some rather well known results concerning general systems of ordinary differential equations of the form:

$$\dot{x}_i(t) = h_i(x_1(t), ..., x_N(t)), \quad 1 \leq i \leq N. \tag{7}$$

In particular, if $h_i$ is a random field, then Eqs. (7) can be used to derive the kinetic description of stochastic systems [17]. Denote by $X_i(t,x)$ the solution of the Cauchy problem of this equation with initial condition $x \equiv (x_1, x_2, ..., x_N)$ ($X_i(x,0) \equiv x_i$). It is well known that Eqs. (7) admit the following formal solution:

$$X_i(t,x) = e^{tA(x)}x_i, \tag{8}$$

where

$$A(x) = \sum_{1 \leq i \leq N} h_i(x) \frac{\partial}{\partial x_i}. \tag{9}$$

When the evolution operator $e^{tA(x)}$ acts on an arbitrary function $\varphi(x)$ the result is as follows:

$$e^{tA(x)}\varphi(x) = \varphi(e^{tA(x)}x). \tag{10}$$

Notice that since Eqs. (7) are autonomous, the solution (8) can be inverted:

$$x' \equiv X_i(t, x) \Rightarrow x = X(-t, x'). \tag{11}$$

Define $\mathcal{D}(x,0)$ to denote the probability distribution of the initial conditions $x$ (see Eqs. (7)). Normalization requires that

$$\int dx \mathcal{D}(x,0) = 1 \quad (dx \equiv dx_1...dx_N). \tag{12}$$

The distribution function at time $t$ is, therefore, given by

$$\mathcal{D}(x,t) = \int dx' \mathcal{D}(x',0) \prod_{1 \leq i \leq N} \delta(x_i - X_i(t,x')). \tag{13}$$

Changing the integration variables from $x'$ to $y = X(t,x')$ and using the relation (11), one obtains

$$\mathcal{D}(x,t) = I(x,t)\mathcal{D}(X(-t,x),0), \tag{14}$$

where

$$I(x,t) = \left| \frac{\partial X(-t,x)}{\partial x} \right| \tag{15}$$

is the Jacobian of the transformation $x \rightarrow X(-t,x)$.

Following (8)-(10) and expression (14), it can be shown that the distribution function $\mathcal{D}(x,t)$ satisfies the equation:

$$\frac{\partial \mathcal{D}}{\partial t} = \left( \frac{\partial I}{\partial t} - I \sum_{1 \leq i \leq N} h_i \frac{\partial}{\partial x_i} \right) I^{-1} \mathcal{D}. \tag{16}$$

The equation of motion for the Jacobian $I(x,t)$ has the form

$$\frac{\partial I(x,t)}{\partial t} + \tilde{A}(x)I(x,t) = 0, \quad I(x,0) = 1, \tag{17}$$

with operator $\tilde{A}(x)$ defined by

$$\tilde{A}(x)\varphi(x) = \sum_{1 \leq i \leq N} \frac{\partial}{\partial x_i} (h_i(x)\varphi(x)). \tag{18}$$

In deriving (17) we have employed the fact that $\int dx \partial \mathcal{D}/\partial t = 0$ (see (13)) and that this relation should hold for any allowed initial distribution function. Upon elimination of $\partial I/\partial t$ on the right-hand side of (16), one obtains

$$\frac{\partial \mathcal{D}}{\partial t} + \tilde{A}(x)\mathcal{D}(x,t) = 0. \tag{19}$$

A comparison of (14) with the solution of Eq. (19) gives the following operator relation:

$$e^{-t\tilde{A}} = I(x,t)e^{-tA(x)}. \tag{20}$$

At this stage we return to the dynamical model (1)–(3). In this case, the corresponding Liouville equation (19) and the evolution equation (17) for the Jacobian (where $x_i = (x_i, p_i)$) assume the form:

$$\frac{\partial \mathcal{D}}{\partial t} - \{H, \mathcal{D}\} = \sum_{1 \leq i \leq N} \frac{\partial}{\partial p_i} \left( \mathcal{D} \frac{\partial R}{\partial p_i} \right), \tag{21}$$

$$\frac{\partial I}{\partial t} - \{H, I\} = \sum_{1 \leq i \leq N} \frac{\partial}{\partial p_i} \left( I \frac{\partial R}{\partial p_i} \right), \tag{22}$$

where $\{A, B\}$ is a Poisson bracket,

$$\{A, B\} = \sum_{1 \leq i \leq N} \left( \frac{\partial A}{\partial x_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial x_i} \right).$$

The operators $A(x), \tilde{A}(x)$, which are defined by (9), (18), become for the case of Eqs. (1):

$$A(x) = \sum_{1 \leq i \leq N} \frac{p_i}{m} \frac{\partial}{\partial x_i} + \sum_{1 \leq i,j \leq N} F_{ij,n} \frac{\partial}{\partial p_i}, \tag{23}$$

$$\tilde{A}(x) = \sum_{1 \leq i \leq N} \frac{p_i}{m} \frac{\partial}{\partial x_i} + \sum_{1 \leq i,j \leq N} \frac{\partial}{\partial p_i} F_{ij,n}. \tag{24}$$

The generalized Liouville equation (21) represents a basis for studying kinetics of systems with dissipative interaction. The next step is to derive the corresponding generalized BBGKY hierarchy, which is a starting point.
for derivation of kinetic equations. The \( s \)-particle distribution function is defined by

\[
  f_s(x_1, ..., x_s; t) = V^s \int dx_{s+1}...dx_N D(x_1, ..., x_N, t),
\]

(25)

where \( V \) is the volume of the system and \( x_i = (x_i, p_i) \). Using the Liouville equation (21), one now straightforwardly obtains the desired hierarchy:

\[
  \frac{\partial f_s}{\partial t} = -\dot{A}_s f_s - \sum_{1 \leq i \leq s} \frac{1}{\partial p_i} \int dx_{s+1} f_{s+1} F_{i,s+1,n},
\]

(26)

where \( \dot{A}_s \) is given by Eq.(24) with \( N = s \) and \( 1/v = N/V \) is the particle number density. The generalized BBGKY hierarchy (26) reduces to the standard BBGKY hierarchy when the dissipation function vanishes.

Our next goal is to solve the hierarchy (26) employing a perturbative approach. To this end we shall take the Bogolyubov principle of spatial correlation weakening [11] as a basis of our consideration. In terms of the many-particle distribution functions \( f_s(x_1, ..., x_s; t) \), this principle states that

\[
  f_s(x_1, ..., x_s, t) \xrightarrow{\tau \to \infty} f_s'(x'_1, ..., x'_s, t) f_s''(x''_1, ..., x''_s),
\]

(27)

where two groups of \( s' \) and \( s'' \) (\( s = s' + s'' \)) particles are formed from \( x_1, ..., x_s \), and \( r \) is the minimal distance between the particles from different groups. The relation (27) has a simple physical meaning: the phase variables of particles are statistically independent at large distance between particles. The property (27) of \( f_s(x_1, ..., x_s; t) \) holds in the thermodynamic limit and it specifies a set of functions, in terms of which one should seek a solution of the BBGKY hierarchy.

3 Kinetic stage of evolution

The present section is devoted to a description of a kinetic stage of evolution for the dissipative system under consideration. Following Bogolyubov, we assume his functional hypothesis as a basis of our investigation [12]. According to this hypothesis, for sufficiently large times, many-particle distribution functions depend on time and initial distribution functions only through one-particle distribution function [10,11]:

\[
  f_s(x_1, ..., x_s, t) \xrightarrow{t \to t_0} f_s(x_1, ..., x_s; f(t)),
\]

(28)

where

\[
  f_1(x_1, t) \xrightarrow{t \to t_0} f(x_1, t).
\]

Here \( f_s(x_1, ..., x_s; f) \) are functionals of the one-particle distribution function, \( t_0 \) is a microscopic time, usually estimated as a collision time. One can consider the above functional hypothesis (28) as a generalization of the Chapman-Enskog approach to the derivation of hydrodynamic equations proceeding from the Boltzmann equation.

According to the functional hypothesis (28), the functionals \( f_s(x_1, ..., x_s; f) \) are universal because they do not depend on initial conditions for the many-particle distribution functions \( f_s(x_1, ..., x_s; t = 0) \). These functionals can be calculated in a perturbative approach. To illustrate the subsequent steps in this direction, we first study a more simple perturbative approach for the case of weak interactions. The next section deals with the low density limit, which is adequate to the situation of granular systems, in particular granular fluids.

Following Eqs. (26), the single-particle distribution function satisfies the kinetic equation of the form

\[
  \frac{\partial f(x_1, t)}{\partial t} + \frac{p_i}{m} \frac{\partial f(x_1, t)}{\partial x_1} = L(x_1; f(t)),
\]

(29)

where the functional \( L(x_1, f) \) represents the generalized collision integral,

\[
  L(x_1; f) = -\frac{1}{\partial p_i} \int dx_2 f_2(x_1, x_2; f) F_{1,2,n}.
\]

(30)

The Bogolyubov functional hypothesis (28), in self-evident shorthand notation, gives:

\[
  \frac{\partial f_s(f(t))}{\partial t} = \int dx \frac{\delta f_s(f)}{\delta f(x)} \frac{\partial f(x, t)}{\partial t},
\]

(31)

where \( \delta f_s/\delta f \) denotes a functional derivative. This relation and Eqs.(26) yield the following equation for the functional \( f_s(f) \):

\[
  -\int dx \frac{\delta f_s(f)}{\delta f(x)} \frac{p_i}{m} \frac{\partial f(x)}{\partial x_n} + \sum_{1 \leq i \leq s} \frac{p_i}{m} \frac{\partial f_s(f)}{\partial x_i} = K_s(f),
\]

(32)

where we have introduced an auxiliary functional \( K_s(f) \),

\[
  K_s(f) = -\sum_{1 \leq i \leq j \leq s} \frac{\partial}{\partial p_i} (f_s(f) F_{i,j,n}) = -\sum_{1 \leq i \leq j \leq s} \frac{1}{\partial p_i} \int dx_{s+1} f_{s+1} F_{i,s+1,n} - \int dx \frac{\delta f_s(f)}{\delta f(x)} L(x; f).
\]

Next, following Bogolyubov again [10,11], in order to obtain an unambiguous solution to Eq. (32) we need to formulate a asymptotical condition ("boundary condition") for the functionals \( f_s(f) \). This condition should reflect the principle of spatial correlation weakening (27) and it should be written taking into account the evolution of the system in physical direction of time [10,11]. To this end, we introduce an auxiliary parameter \( \tau \), which has dimensions of time but does not represent physical time, and we use it in the following manner:

\[
  e^{-\tau A_0} f_s(x_1, ..., x_s; f) = f_s \left( \mathbf{x}_1 - \frac{p_1}{m} \tau, ..., \mathbf{x}_s - \frac{p_s}{m} \tau, \mathbf{p}; f \right) \xrightarrow{\tau \to \infty}
\]
Here \( e^{-\lambda t} \) is the evolution operator of free particles and \( A^0_s \) is given by the first term in (23) (see also (24)):

\[
A^0_s = \sum_{1 \leq i \leq s} p_m \frac{\partial}{\partial x_i}.
\]

The asymptotical condition (33) can be written in a more compact form,

\[
\lim_{\tau \to +\infty} e^{-\tau A^0} f_s(x_1, \ldots, x_s; e^{\tau A^0} f) = f_{s}^{0}(x_1, \ldots, x_s; f),
\]

where

\[
f_{s}^{0}(x_1, \ldots, x_s; f) \equiv \prod_{1 \leq i \leq s} f(x_i). \tag{36}
\]

In order to solve Eqs. (32) we recast them in the form

\[
\frac{\partial}{\partial \tau} e^{-\tau A^0} f_s(e^{\tau A^0} f) = -e^{-\tau A^0} K_s(e^{\tau A^0} f). \tag{37}
\]

The straightforward differentiation of (37) gives Eqs. (32). Upon integrating Eq. (37) over \( \tau \) from 0 to \( +\infty \), and using the above boundary condition (35), one obtains the following chain of integral equations for the distribution functions:

\[
f_s(f) = f_{s}^{0}(f) + \int_0^{+\infty} d\tau e^{-\tau A^0} K_s(e^{\tau A^0} f). \tag{38}
\]

Equations (38) are solvable in a perturbative theory in weak interaction. In the leading approximation in small parameter \( \lambda \) (\( F_{ij,n} \sim \lambda \)), one obtains:

\[
f_{s}^{(0)}(f) = f_{s}^{0}(f),
\]

\[
L^{(1)}(x_1; f) = \frac{1}{v} \frac{\partial}{\partial p_1} f(x_1) \int dx_2 f(x_2) F_{12,n}.
\]

This yields the following kinetic equation (see (29)), correct to linear order in the interaction strength:

\[
\frac{\partial f(x_1)}{\partial t} + \frac{p_{1n}}{m} \frac{\partial f(x_1)}{\partial x_1} = \frac{1}{v} \frac{\partial}{\partial p_1} f(x_1) \left( \int dx_2 f(x_2) \frac{\partial V_{12}}{\partial x_{1n}} + \int dx_2 f(x_2) \left( \frac{\partial R_{12}}{\partial p_{1n}} \right) \right). \tag{39}
\]

In the absence of dissipative forces (i.e., in the case \( R = 0 \)) this kinetic equation reduces to a kinetic equation of Vlasov type with a self-consistent field \( U(x_1) \) given by:

\[
U(x_1) = \int dx_2 V_{12} \int dx_2 f_2(x_2, p_2).
\]

Another simple case is that of spatial homogeneity (but in the presence of dissipation). Then, Eq. (39) transforms to:

\[
\frac{\partial f(p_1)}{\partial t} = \frac{1}{v} \frac{\partial}{\partial p_1} f(p_1) \frac{\partial}{\partial p_1} \int dp_2 f(p_2) R_0(p_1), \tag{40}
\]

where

\[
R_0(p) \equiv \int dx R(x; p).
\]

When the dissipation function is given by (6), one obtains from (40), (41) the following equations for the densities of energy, momentum, and particle number:

\[
\frac{\partial}{\partial t} \int dp \frac{p^2}{2m} f(p) = -\frac{\gamma_0}{2vm} \int dp_1 dp_2 f(p_1) f(p_2) p_2^2 < 0,
\]

\[
\frac{\partial}{\partial t} \int dp p f(p) = 0, \quad \frac{\partial}{\partial t} \int dp f(p) = 0,
\]

where

\[
\gamma_0 \equiv \int dx \gamma(x).
\]

We see that the system becomes cool (the kinetic energy decreases) during its time evolution as it should be. The chain of integral equations (37) allows of studying the higher order approximations in interaction without any principal difficulties.

4 Kinetic equation for dilute gases with dissipative interaction

The present section is devoted to the case of small density with arbitrary in strength short-range interaction. In addition, we do not allow for the possibility of formation of complexes of particles, so as to avoid the necessity to introduce additional distribution functions. However, the Bogolyubov method can be applied here as well.

In principle, we can start from the chain of integral equations (38), as in the previous section. However, this is not convenient as the density expansion would require the use of nontrivial resummation techniques applied to the pertinent virial expansion (see, for example, [12]). Therefore, we choose to employ here an alternate approach, similar to those, developed by Bogolyubov to derive the Boltzmann equation.

Clearly, the expansion of the distribution functions \( f_s(x_1, \ldots, x_s; f) \) in Taylor functional series in the one-particle distribution, \( f(x) \), is equivalent to a density expansion (in powers of \( 1/v \)). Moreover, it is easy to see, on the basis of the structure of Eqs. (32), that the leading contribution to \( f_s(f) \) is proportional to \( f^s \). In accordance with this, it is convenient to rewrite Eqs. (32) in the form:

\[
- \int dx \frac{\delta f_s(f)}{\delta f(x)} \frac{p_n}{m} \frac{\partial f(x)}{\partial x_n} + \tilde{A}_s f_s(f) = Q_s(f), \tag{42}
\]

where we have introduced a new auxiliary functional \( Q_s(f) \),

\[
Q_s(f) \equiv - \sum_{1 \leq i \leq s} \frac{1}{v} \frac{\partial}{\partial p_{1n}} \int dx_{s+1} f_{s+1}(f) F_{i,s+1,n} - \int dx \frac{\delta f_s(f)}{\delta f(x)} L(x; f).
\]
The differential operator $\tilde{\mathcal{A}}_s$ is defined by (24) with $N = s$. The expansion of the left-hand side of Eqs. (42) includes terms that are proportional to $f^s$ and higher order contributions, whereas the right-hand side is, at least, of order $f^{s+1}$.

The next step is to formulate a boundary condition for (42) taking into account the evolution of the system in physical direction of time. This boundary condition reflects the principle of spatial correlation weakening. From (20), (10), (8) we have

$$e^{-\tau \tilde{\mathcal{A}}_s} f_s(f) = I_s(x, \tau) e^{-\tau \tilde{\mathcal{A}}_s} f_s(f) = I_s(x, \tau) f_s(X_1(\tau, t), \ldots, X_s(\tau, t); f),$$

where $I_s(x, \tau)$ denotes the Jacobian (15) for $s$-particle dynamics. The application of the principle of spatial correlation weakening (27) now yields

$$e^{-\tau \tilde{\mathcal{A}}_s} f_s(f) \bigg|_{\tau \to -\infty} = I_s(x, \tau) \prod_{1 \leq i \leq s} f_s(X_i^*(x) - \frac{\tau}{m} P_i^*(x), P_i^*(x)).$$

Here, following [10,11,12], we have introduced the asymptotic coordinates and momenta $X_i^*(x) = (X_i^*(x), P_i^*(x)) (i = 1, \ldots, s)$,

$$X_i(t, x) \bigg|_{t \to -\infty} \to X_i^*(x) + \frac{t}{m} P_i^*(x),$$

$$P_i(t, x) \bigg|_{t \to -\infty} \to P_i^*(x)$$

(X$_i(t, x) \equiv (X_i(t, x), P_i(t, x))$). The asymptotic coordinates and momenta $X_i^*(x)$ do exist, because, for long times in the past, the particles of the system with interaction under consideration are in a state of free motion. Now, according to (43), we need to find the limiting value of Jacobian, $I_s(x, t)$ as $t \to +\infty$. Making use of the definition (15), one obtains

$$I_s(x, t) = \frac{\partial X(\tau, t)}{\partial x} \bigg|_{\tau \to \infty} = \frac{\partial X(\tau, t)}{\partial X^*(x)} \frac{\partial X^*(x)}{\partial x},$$

whence, exploiting (44), we have

$$I_s(x, t) \bigg|_{t \to +\infty} \to I_s^*(x) = \frac{\partial X^*(x)}{\partial x}$$

As a result, (43) can be written in the final form

$$\lim_{\tau \to +\infty} e^{-\tau \tilde{\mathcal{A}}_s} f_s(x; e^{-\tau \tilde{\mathcal{A}}_s} f) = f_s^{(s)}(x; f), (46)$$

where

$$f_s^{(s)}(x; f) \equiv I_s^*(x) \prod_{1 \leq i \leq s} f_s(X_i^*(x)).$$

(above, in (35), (36), we have used a more detailed notation $(x_1, \ldots, x_s) \equiv x$). To solve Eqs. (42) taking into account the obtained boundary condition (46), we rewrite it as follows:

$$\frac{\partial}{\partial \tau} e^{-\tau \tilde{\mathcal{A}}_s} f_s(e^{-\tau \tilde{\mathcal{A}}_s} f) = -e^{-\tau \tilde{\mathcal{A}}_s} Q_s(e^{-\tau \tilde{\mathcal{A}}_s} f). \quad (48)$$

(The straightforward differentiation of (48) gives Eqs. (42)). Integration of this chain of equations over $\tau$ from 0 to $+\infty$ yields the following integral equations for the many-particle distribution functions:

$$f_s(f) = f_s^{(s)}(f) + \int_0^{+\infty} e^{-\tau \tilde{\mathcal{A}}_s} Q_s(e^{-\tau \tilde{\mathcal{A}}_s} f) d\tau.$$  \hspace{1cm} (49)

Equations (49) are solvable in a perturbative approach in density. Similar integral equations were obtained in [12] for Hamiltonian systems. The difference between both equations consists in the presence of the asymptotical value of Jacobian $I_s^*(x)$ in Eqs. (47).

In the leading order in density, Eqs. (49) give the two-particle distribution function, which is proportional to the squared density,

$$f_2^{(s)}(x_1, x_2; f) = I_2^*(x_1, x_2) \prod_{1 \leq i \leq 2} f(X_i^*(x_1, x_2), P_i^*(x_1, x_2)). \quad (50)$$

Next, using (29) and (30), one obtains the following kinetic equation:

$$\frac{\partial f(x_1, t)}{\partial t} + \frac{p_{1n}}{m} \frac{\partial f(x_1, t)}{\partial x_{1n}} = L_2^{(s)}(x_1; f(t)). \quad (51)$$

where the collision integral $L_2^{(s)}(x_1; f)$ is determined by

$$L_2^{(s)}(x_1; f) = -\frac{1}{v} \frac{\partial}{\partial p_{1n}} \int dx_2 f_2^{(s)}(x_1, x_2; f) F_{12n}. \quad (52)$$

The kinetic equation (51) is a generalization of the Boltzmann kinetic equation to non-Hamiltonian systems. The collision integral (52) is written in the Bogolyubov form and expressed through the asymptotic coordinates and momenta, and the Jacobian corresponding to two-particle dynamics.

In the case of weakly nonuniform states, when the gradients of the one-particle distribution function $f(x, p)$ are small, the collision integral (52) can be further simplified. For these states, the range $r_0$ of the interpartical forces is small compared to the characteristic scale of inhomogeneity $a$, $r_0 \ll a$, i.e. in comparison to those distances over which the one-particle distribution function $f(x, p)$ changes substantially. Also, we take into account that

$$|X^*_i(x_1, x_2) - x_i| \sim r_0, \hspace{1cm} P^*_i(x_1, x_2) \equiv P^*_i(x_{21}, p_1, p_2), \quad (53)$$

$(i = 1, 2)$. Following Eq. (50) these asymptotic coordinates and momenta of the two particles problem determine the two-particle distribution function $f_2^{(2)}(x_1, x_2; f)$ to second order in the particle density and zeroth order in the gradients. Using (42) one obtains:

$$\frac{\partial}{\partial p_{1n}} \frac{\partial f_2^{(2)}(x_1, x_2; f)}{\partial x_{1n}} + \frac{\partial}{\partial p_{1n}} (F_{2}^{(2)}(x_1, x_2; f) F_{12n}|_{n})$$
Integration of this equation over $x_2$ leads, using (52), to an expression for the collision integral $L^{(2,0)}(x; f)$,

$$L^{(2,0)}(x_1; f) = \frac{1}{v} \int dx_2 \frac{p_{21,n}}{m} \frac{\partial f^{(2,0)}(x_1, x_2; f)}{\partial x_{2n}},$$  \hspace{1cm} (54)

where

$$f^{(2,0)}(x_1, x_2; f) = I^*_2(x_1, x_2; f) = f(x_1, P^*_1(x_1, x_2))f(x_1, P^*_2(x_1, x_2)).$$  \hspace{1cm} (55)

(see (50), (53)).

Now, we evaluate integral over $x_2$ ($dx_2 = dx_0 dp_{21}$) in (54). The integration can be replaced by an integration over the difference $x_2$ (see (50), (53)). In performing the integral over $x_2$, we employ cylindrical coordinates $z$, $b$ and $\varphi$ with the origin at the point $x_1$ and the $z$-axis directed along the vector $p_{21}$:

$$L^{(2,0)}(x_1; f) = \frac{1}{v} \int dp_2 \int_0^{2\pi} \, d\varphi \int_0^\infty \, db \, \frac{|p_{21}|}{m} \times$$

$$\times f^{(2,0)}(x_1, x_2; f)|_{z=+\infty},$$  \hspace{1cm} (56)

where $f^{(2,0)}(x_1, x_2; f)$ is given by (55). The asymptotic momenta

$$P^*_i(x_1, z, p_1, p_2) \equiv P^*_i(x_{21}, p_1, p_2),$$

determine $f^{(2,0)}(x_1, x_2; f)$, have the following properties:

$$P^*_i(x_1, z, p_1, p_2)|_{z \to \infty} = P^*_i(b, \varphi, p_1, p_2),$$

(57)

where $x_{21} = (x_1)_{\perp} = (b, \varphi, z) = x_{21}P_{21}/|p_{21}|$. Indeed, according to (44), $P^*_i(x_1, x_2)$ are the momenta of two particles at the moment of time $t = -\infty$, if at $t = 0$ they have phase variables $x_1, p_1, x_2, p_2$. Then, the relationship

$$\left| (x_1 + \frac{p_1 t}{m}) - (x_2 + \frac{p_2 t}{m}) \right| = |x_{12}| + \frac{z}{t_0} t + O(t^2)$$

($t_0 = |x_{21}|/|p_{21}|$), which is valid for small $t$, shows the following: when $z > 0$, the collision of particles precedes the moment $t = 0$, whereas when $z < 0$, the collision occurs after $t = 0$. This reasoning explains the relations (57), where $P^*_i(b, \varphi, p_1, p_2)$ are the momenta of particles before the collision (precollisional momenta) after which the particles have momenta $p_1, p_2$.

With these observations we can now find the following expression for the generalized Boltzmann collision integral determined by (55) and (56):

$$L^{(2,0)}(x_1; f) = \frac{1}{v} \int dp_2 \int_0^{2\pi} \, d\varphi \int_0^\infty \, db \, \frac{|p_{21}|}{m} \times$$

$$\times \left[ I^*_2(x_1, p_1, p_2) f(x_1, P^*_1) f(x_1, P^*_2) - f(x_1, p_1) f(x_1, p_2) \right].$$  \hspace{1cm} (58)

where

$$I^*_2(x_1, p_1, p_2) = I^*(x_1, x_2)|_{z=+\infty}.$$  \hspace{1cm} (59)

The calculation of the collision integral $L^{(2,0)}(x_1; f)$ (much like the calculation of the two-particle distribution function) involves only the solution of the two-particle dynamics. This is elaborated in the subsection that appears immediately below.

5 Dissipative dynamics

5.1 Relative motion in two-particle dynamics

For the Hamiltonian systems the two-particle problem is reduced to study of relative motion of the particles. The same situation takes place in the presence of dissipative forces.

To obtain this result and some its consequences, let us consider the equations of motion for two particles in the presence of dissipative forces. According to Eqs. (1)-(4), these equations have the form

$$m\ddot{x}_1 = p_1, \quad m\ddot{x}_2 = p_2,$$  \hspace{1cm} (60)

$$\dot{p}_1 = F(x_{12}; p_{12}), \quad \dot{p}_2 = -F(x_{12}; p_{12}),$$  \hspace{1cm} (61)

where

$$F_n(x; p) = -\frac{\partial V(x)}{\partial x_n} - \frac{\partial R(x; p)}{\partial p_n}.$$  \hspace{1cm} (62)

Let us introduce the following new phase variables $x$, $p$, $x_c$, $p_c$ with clear meaning:

$$x = x_1 - x_2, \quad p = \frac{p_1 - p_2}{2},$$

$$x_c = \frac{x_1 + x_2}{2}, \quad p_c = p_1 + p_2.$$  \hspace{1cm} (63)

Then, the equations of motion (60), (61) are separated into equations for the center of mass and relative motion:

$$2m\ddot{x}_c = p_c, \quad \dot{p}_c = 0,$$  \hspace{1cm} (64)

$$m\ddot{x} = 2p, \quad \dot{p} = F(x, 2p).$$  \hspace{1cm} (65)

Let $x(t, x, p)$, $p(t, x, p)$ be a solution of Eqs. (64) with initial conditions $x$, $p$. According to (44), (62), the corresponding asymptotic coordinates and momenta are given by

$$p(t, x, p) \longrightarrow p^*(x, p),$$

$$x(t, x, p) \longrightarrow x^*(x, p) + \frac{2t}{m} p^*(x, p),$$  \hspace{1cm} (66)

where

$$p^*(x, p) \equiv \frac{1}{2} (P^*_1(x_1, x_2) - P^*_2(x_1, x_2))$$

$$x^*(x, p) \equiv X^*_1(x_1, x_2) - X^*_2(x_1, x_2).$$  \hspace{1cm} (67)

Integrating Eqs. (63), we can also come to the following identities:

$$p_1 + p_2 = P^*_1(x_1, x_2) + P^*_2(x_1, x_2).$$
\[ x_1 + x_2 = X^*_1(x_1, x_2) + X^*_2(x_1, x_2). \] (67)

The comparison of Eqs. (65)-(67) enables to express the asymptotic coordinates and momenta \( X^*_1(x_1, x_2) \) and \( P^*_1(x_1, x_2) \) through the functions \( x^*(x, p) \), \( p^*(x, p) \)

\[
P^*_1(x_1, x_2) = \frac{p_1}{2} + p^*(x, p), \quad P^*_2(x_1, x_2) = \frac{p_2}{2} - p^*(x, p),
\]

\[
X^*_1(x_1, x_2) = x_1 + \frac{1}{2} x^*(x, p), \quad X^*_2(x_1, x_2) = x_2 - \frac{1}{2} x^*(x, p).
\] (68)

We can see that the calculation of asymptotic phase variables for the two-particle dynamics is reduced to the calculation of asymptotic phase variables for the relative motion.

Next, consider the Jacobian \( I_2(x_1, x_2, t) \) that corresponds to the dynamics of two particles and determines the collision integral (58). According to Eq. (22), this Jacobian satisfies the following equation:

\[
\begin{aligned}
\frac{\partial I_2}{\partial t} - \{H_2, I_2\} &= \frac{\partial}{\partial p_{1n}} \left( I_2 \frac{\partial R}{\partial p_{1n}} \right) + \frac{\partial}{\partial p_{2n}} \left( I_2 \frac{\partial R}{\partial p_{2n}} \right),
\end{aligned}
\]

where \( H_2 \) is the two-particle Hamiltonian (see (2) for \( N = 2 \)). Changing the independent variables in this equation to \( x, p, x, p \) (see (62)), one finds:

\[
\begin{aligned}
\frac{\partial I_2}{\partial t} + p_{1n} \frac{\partial I_2}{\partial x_{1n}} + \frac{2p_n}{m} \frac{\partial I_2}{\partial x_n} - \frac{\partial V(x)}{\partial x_n} \frac{\partial I_2}{\partial p_n} &= \frac{1}{2} \frac{\partial}{\partial p_n} \left( I_2 \frac{\partial R}{\partial p_n} \right).
\end{aligned}
\]

Since \( I_2(x_1, x_2, 0) = 1 \), it follows from the latter equation that the Jacobian does not depend on \( x \) and \( p \), i.e. \( I_2(x_1, x_2, t) \equiv I_2(x, p, t) \). The Jacobian \( I_2(x, p, t) \) satisfies equation

\[
\begin{aligned}
\frac{\partial I_2}{\partial t} - \{h, I_2\} &= \frac{1}{2} \frac{\partial}{\partial p_n} \left( I_2 \frac{\partial R}{\partial p_n} \right), \quad I_2|_{t=0} = 1,
\end{aligned}
\] (69)

where

\[
\begin{aligned}
\{h, I_2\} &= \frac{\partial h}{\partial x_n} \frac{\partial I_2}{\partial p_n} - \frac{\partial h}{\partial p_n} \frac{\partial I_2}{\partial x_n}, \quad h \equiv \frac{p^2}{m} + V(x).
\end{aligned}
\] (70)

Next, introduce the Jacobian \( \tilde{I}_2(x, p, t) \) corresponding to the dynamics defined by (64). It can be easily seen, using Eqs. (22), (64), that this Jacobian satisfies the same equation and initial condition as \( I_2(x, p, t) \) (see Eqs. (69), (70)). Therefore, these two Jacobians are equal to each other,

\[
I_2(x, p, t) = \tilde{I}_2(x, p, t).
\] (71)

Taking into account this result and the definition (45) of the limiting Jacobian, we obtain

\[
I^*_2(x_1, x_2) = \frac{\partial (x^*, p)}{\partial (x, p)}.
\] (72)

### 5.2 Two-particle dynamics with weak dissipation

The present section is devoted to the study of the case of weak dissipation, namely the kinetics for which it is sufficient to consider only the linear order in an expansion of collision integral in powers of the dissipation function. This case is similar to the corresponding expansion in powers of the degree of inelasticity [2].

In accordance with Eqs. (45), (57)-(59), in order to derive a kinetic equation in the case of weak dissipation, we have to calculate the asymptotic coordinates \( X^*_i(x_1, x_2) \) and momenta \( P^*_i(x_1, x_2) \) for the two-particle dynamics function \( R(x, p) \). However, in the previous sub-section, we have shown that it is sufficient to find the asymptotic coordinates \( x^*(x, p) \) and momenta \( p^*(x, p) \) for relative motion. This motion is described by the solution \( x(t, x), x(t, p), p(t, x), p(t, p) \) of Eqs. (64), which can be written in the form

\[
x(t, x) = e^{t(\lambda_0 + \lambda_1)} x, \quad p(t, x, p) = e^{t(\lambda_0 + \lambda_1)} p.
\] (73)

where the operators \( \lambda_0, \lambda_1 \) have the following structure:

\[
\lambda_0 = \frac{2p_n}{m} \frac{\partial}{\partial x_n} - \frac{\partial V(x)}{\partial x_n} \frac{\partial}{\partial p_n}, \quad \lambda_1 = \frac{1}{2} \frac{\partial R(x, 2p)}{\partial p_n} \frac{\partial}{\partial p_n}
\] (74)

(see Eqs. (7)-(9)).

In the sequel, while calculating \( x(t, x, p), p(t, x, p) \) we shall consider \( \lambda_1 \) as a small perturbation. The unperturbed relative motion is expressed as

\[
p^{(0)}(t, x, p) = e^{t\lambda_0} p, \quad x^{(0)}(t, x, p) = e^{t\lambda_0} x.
\] (75)

Since the operators \( \lambda_0 \) and \( \lambda_1 \) do not commute, we use the following well known expansion:

\[
e^{t(\lambda_0 + \lambda_1)} = e^{t\lambda_0} + \int_0^t dt' e^{t'\lambda_0} \lambda_1 e^{(t-t')\lambda_0} + ...
\] (76)

This formula, in conjunction with (73)-(75), gives

\[
p(t, x) = p^{(0)}(t, x) + \int_0^t dt' \lambda_1 p^{(0)}(t - t', x) \bigg|_{x=x^{(0)}(t', x)},
\]

\[
x(t, x) = x^{(0)}(t, x) + \int_0^t dt' \lambda_1 x^{(0)}(t - t', x) \bigg|_{x=x^{(0)}(t', x)},
\]

\[
(x^{(0)}(t, x), p^{(0)}(t, x)). \] Using (65), it is easy to find the asymptotic momentum \( p^*(x) \) and coordinate \( x^*(x) \) by letting \( t \to -\infty \):

\[
p^*(x) = p^{*(0)}(x) - \int_{-\infty}^0 dt \lambda_1 p^{*(0)}(x) \bigg|_{x=x^{(0)}(t, x)},
\]

\[
x^*(x) = x^{*(0)}(x) - \int_{-\infty}^0 dt \lambda_1 \left\{ x^{*(0)}(x) - \frac{2t}{m} x^{*(0)}(x) \bigg|_{x=x^{(0)}(t, x)} \right\}.\] (77)
Consider now the Jacobian $I_2(x, t)$ in the linear order in the dissipation function $R$. To this order in $R$, it follows from Eq. (69) (with $I_2(x, t) = 1$ at $R = 0$) that

$$\frac{\partial I_2}{\partial t} + \lambda_0 I_2 = \frac{1}{2} \frac{\partial^2 R}{\partial p_a \partial p_n}. $$

whence

$$I_2(x, t) = 1 + \frac{1}{2} \int_{-\infty}^{0} dt \frac{\partial^2 R(x, 2p)}{\partial p_a \partial p_n} \bigg|_{x \rightarrow x^{(0)}(t, x)}. $$

According to Eq. (45), the asymptotic value of the Jacobian $I_2(x, t)$ is given by the formula:

$$I_2^*(x) = 1 + \frac{1}{2} \int_{-\infty}^{0} dt \frac{\partial^2 R(x, 2p)}{\partial p_a \partial p_n} \bigg|_{x \rightarrow x^{(0)}(t, x)}. $$

$$ (78) $$

### 6 Kinetic equation in the weak dissipation approximation

In this section we study the kinetic equation (29) with the collision integral (58) in the weak dissipation approximation. In the spatially homogeneous case and in the linear approximation in $R$, the collision integral Eq. (58) assumes the form:

$$L^{(2)}(p_1, f) = L_0^{(2)}(p_1, f) + L_1^{(2)}(p_1, f), $$

$$ (79) $$

where $L_0^{(2)}(p_1, f)$ is the Boltzmann collision integral, which accounts only for the reversible (potential) interactions:

$$L_0^{(2)}(p_1, f) = \frac{1}{v} \int dp_2 \int_{0}^{2\pi} d\varphi \int_{0}^{\infty} db \frac{|p_{21}|}{m} \times $$

$$ \times \{f(p_{10})f(p_{20}) - f(p_1)f(p_2)\} $$

$$ (80) $$

with $|p_{21}| = \delta [I'(x, p_1, p_2)]_{f_2}$. The second term in (79) is a correction to the Boltzmann collision integral, which accounts for dissipation to linear order in $R$:

$$L_1^{(2)}(p_1, f) = \frac{1}{v} \int dp_2 \int_{0}^{2\pi} d\varphi \int_{0}^{\infty} db \frac{|p_{21}|}{m} \times $$

$$ \times \delta[(\delta_{x^{(0)}(t, x)}), \mathbf{p}^{(0)}]_{f_1^0}$$

(57)

(68)

and (77), (78) for the asymptotic values of the momentum and Jacobian, we find

$$L_1^{(2)}(p_1, f) = \frac{1}{v} \int dp_2 \int_{0}^{2\pi} d\varphi \int_{0}^{\infty} db \frac{|p_{21}|}{m} \times $$

$$ \times \left\{a(x, p) + b_n(x, p) \left(\frac{\partial}{\partial p_{1n}} - \frac{\partial}{\partial p_{2n}}\right)\right\} \times $$

$$ f(p_1^0)f(p_2^0)|_{p_1^0, p_2^0 \rightarrow p_1^0, p_2^0}, $$

$$ (81) $$

where

$$a(x, p) = \frac{1}{2} \int_{-\infty}^{0} dt \frac{\partial^2 R}{\partial p_n \partial p_a} |_{x \rightarrow x^{(0)}(t, x), z \rightarrow +\infty}, $$

$$b_n(x, p) = \frac{1}{2} \int_{-\infty}^{0} dt \frac{\partial R}{\partial p_n} |_{x \rightarrow x^{(0)}(t, x), z \rightarrow +\infty}. $$

$$ (82) $$

$$ (83) $$

Next we wish to evaluate (81) for the case in which the particle interactions vanish ($V = 0$). In this case the solution of the equations of motion assumes the form

$$x^{(0)}(t, x) = x + \frac{2t}{m} p $$

and, as expected, the asymptotic momentum and space coordinate coincide with their respective initial values, $p^{(0)} = p, x^{(0)} = x$. Noting that

$$\int_{-\infty}^{0} dt g(x + \frac{2t}{m} p) = \int_{-\infty}^{0} dt g(x, z + \frac{2t}{m} p) = \frac{m}{2p} \int_{-\infty}^{z} dz' g(x, z'), $$

is valid for an arbitrary function $g(x) = g(x, z)$ (the $z$-axis of the cylindrical coordinates we employ is chosen to coincide with the respective initial values, $p^{(0)} = p, x^{(0)} = x$). Noting that

$$a(x, p) = \frac{m}{4p} \int_{-\infty}^{\infty} dz \frac{\partial^2 R(x, z, 2p)}{\partial p_n \partial p_a}, $$

$$b_n(x, p) = \frac{m}{4p} \int_{-\infty}^{\infty} dz \frac{\partial R(x, z, 2p)}{\partial p_n} . $$

Finally, upon substituting these expression into (81) and noting that $p_{10} = p_1, p_{20} = p_2$ for $V = 0$, one obtains

$$L_1^{(2)}(p_1, f) = \frac{1}{v} \int dp_{1n} \frac{\partial}{\partial p_{1n}} f(p_1) \int dp_2 f(p_2) R_0(p_{12}) $$

$$ (84) $$

where $R_0(p)$ is defined by Eq. (41). Formula (84) coincides with (40) obtained within the weak interaction approximation.

In conclusion of this section we briefly concern the question of the evolution of the system described by the kinetic equation

$$\frac{\partial f(p, t)}{\partial t} = L_0^{(2)}(p, f(t)) + L_1^{(2)}(p, f(t)). $$

(85)
The description of this state can be based on the functional hypothesis of the form

\[ f(p, t) \xrightarrow{t \to \infty} f(p, \varepsilon(t)), \quad (86) \]

where the asymptotic value of the energy density \( \varepsilon(t) \) is defined by

\[ \int dp f(p, t) \frac{p^2}{2m} \xrightarrow{t \to \infty} \varepsilon(t). \]

This functional hypothesis results in the equation for \( \varepsilon(t) \),

\[ \frac{\partial \varepsilon(t)}{\partial t} = L(\varepsilon(t)) \]

with the following right-hand side:

\[ L(\varepsilon) \equiv \int dp \frac{p^2}{2m} L_1^{(2,0)}(p; f(\varepsilon)) \]

(the Boltzmann collision integral \( L_1^{(2,0)}(p; f) \) does not contribute to \( L(\varepsilon) \)).

The distribution function \( f(\varepsilon) \), according to Eq. (85) and the functional hypothesis (86), satisfies the equation

\[ \frac{\partial f(\varepsilon)}{\partial \varepsilon} L(\varepsilon) = L_0^{(2,0)}(p; f(\varepsilon)) + L_1^{(2,0)}(p; f(\varepsilon)). \quad (87) \]

This equation is solvable in a perturbative approach in powers of the dissipation function. We shall not discuss here the study of the homogeneous cooling state based on the obtained equations. This can be done similarly to those theories developed for spatially nonuniform states in [2,4]. Finally, we note that in another terminology, the sketched theory is the application of the Chapman-Enskog method to the solution of the kinetic equation (85).

7 Connection to the Boltzmann equation for inelastic rigid spheres

In this section we compare our kinetic equation (58) with that obtained by considering a system of rigid particles experiencing instantaneous inelastic collisions characterized by a fixed coefficient of normal restitution (see, e.g., [2,4]):

\[ L'(p_1; f) = \frac{d^2}{m^2} \int_{kp_{12} > 0} dp_2 \int d^2k(kp_{12}) \times \]

\[ \times \left\{ \frac{1}{\varepsilon^2} f(p'_1)f(p'_2) - f(p_1)f(p_2) \right\}, \quad (88) \]

where \( k \) is a unit vector pointing from the center of sphere 1 to that of sphere 2 at the moment of contact (\( d^2k = \sin \theta d\theta d\varphi \); the polar axis \( z \) is directed along \( p_{21} = p_2 - p_1 \)).

\( d \) is the diameter of a sphere, and \( \varepsilon \) is the coefficient of normal restitution. Using the identity

\[ \int_0^{2\pi} d\varphi \int_0^\infty b db |p_{12}| \cdots = d^2 \int_{kp_{21} > 0} d^2k(kp_{21}) \cdots \]

\( b = d \sin \theta, \ 0 \leq \theta \leq \pi/2 \), one obtains from (88):

\[ L'(p_1; f) = \frac{1}{v} \int dp_2 \int_{0}^{2\pi} d\varphi \int_{0}^{\infty} db \frac{|p_{21}|}{m} \times \]

\[ \times \left\{ \frac{1}{\varepsilon^2} f(p'_1)f(p'_2) - f(p_1)f(p_2) \right\}. \quad (89) \]

Here \( p'_1, p'_2 \) are the precollisional momenta determined by

\[ p'_1 = p_1 + \frac{1+\varepsilon}{\varepsilon}k(pk), \quad p'_2 = p_2 - \frac{1+\varepsilon}{\varepsilon}k(pk), \quad (90) \]

where \( p = (p_1 - p_2)/2 \). The collision integral (58) is determined by the asymptotic \( (t \to -\infty) \) values of the momenta, coordinates, and Jacobian, which specify the two-particle dynamics. In terms of the relative momenta \( p \), the collision law (90) can be written as

\[ p' = p + \frac{1+\varepsilon}{\varepsilon}k(pk). \quad (91) \]

Within the framework of the formalism developed in this article, one needs to know the relation between the asymptotic values of coordinate \( x' \) and the initial coordinate \( x \). We establish this relation in the terms of the relative coordinate \( x = x_1 - x_2 \) as follows:

\[ x' = x + \frac{1+\varepsilon}{\varepsilon}k(xk). \quad (92) \]

Using the above \( x' \) (we assume \( k = \text{const} \)) we obtain the Jacobian:

\[ \frac{\partial(x', p')}{\partial(x, p)} = \frac{1}{\varepsilon^2}. \]

Substitution of this Jacobian into the collision integral (58) gives the collision integral (89). Therefore, when (90), (91) are satisfied, the collision integrals (58), (88) coincide, as they should.

8 Conclusion

We have shown that the Bogolyubov method of derivation of kinetic equations can be applied to dissipative many-body systems with the corresponding modifications. In the case of inelastically colliding hard spheres we reproduce the inelastic Boltzmann equation. The reader may be justified in asking whether yet another formulation is needed to study dissipative systems. We believe that the answer is that given the difficulties encountered by other approaches, in particular the problems emanating from the lack of scale separation in granular systems, it is advantageous to consider a powerful approach such as that of Bogolyubov. The application of this approach to dense systems, for instance, would not only serve to complement the results obtained by using the Enskog corrected Boltzmann equation, but may also enable the study of systems (such as binary granular mixtures) where a naive application of the Enskog-Boltzmann equation has been
shown to be invalid even in the framework of elastically interacting particles [18]. Much like any other approach to many-body systems, the present one is not directly useful: perturbative expansions need to be implemented to obtain physically significant results. However, as the formulation is rather different from e.g., those directly based on the Boltzmann equation or its ring corrections, one may be able to study hitherto inaccessible cases (or limits), e.g. when gradients are large (typical of granular systems) or many-body contacts are of importance. Whether the present approach will indeed provide useful results for these and other cases of dissipative systems remains to be seen.

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