Fuzzy equilibrium existence for Bayesian abstract fuzzy economies and applications to random quasi-variational inequalities with random fuzzy mappings

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Abstract In this paper, we introduce a Bayesian abstract fuzzy economy model and we prove the Bayesian fuzzy equilibrium existence. As applications, we prove the existence of the solutions for two types of random quasi-variational inequalities with random fuzzy mappings and we also obtain random fixed point theorems.

Key words Bayesian abstract fuzzy economy, Bayesian fuzzy equilibrium, incomplete information, random fixed point, random quasi-variational inequalities, random fuzzy mapping.

AMS Subject Classification: 58E35, 47H10, 91B50, 91A44.

1 INTRODUCTION

The study of fuzzy games has begun with the paper written by Kim and Lee in 1998 [16]. This type of games is a generalization of classical abstract economies. For an overview of results concerning this topic, the reader is referred to [24]. Though, the existence of random fuzzy equilibrium has not been studied by now. We introduce the new model of Bayesian abstract fuzzy economy and explore the existence of the Bayesian fuzzy equilibrium. Our model is characterized by a private information set, an action (strategy) fuzzy mapping, a random fuzzy constraint one and a random fuzzy preference mapping. The Bayesian fuzzy equilibrium concept is an extension of
the deterministic equilibrium. We generalize the former deterministic models introduced by Debreu [8], Shafer and Sonnenschein [25], Yannelis and Prabhakar [29] or Patriche [24] and search for applications.

Since Fichera and Stampacchia introduced the variational inequalities (in 1960s), this domain has been extensively studied. For recent results we refer the reader to [1]-[4], [6],[7], [12], [17], [20]-[22], [26], [28] and the bibliography therein. Noor and Elsanousi [19] introduced the notion of a random variational inequality. Existence of solutions of the random variational inequality and random quasi-variational inequality problems has been proved, for instance, in [12], [13], [18], [27], [33].

In this paper, we first define the model of the Bayesian abstract fuzzy economy and we prove a theorem of Bayesian fuzzy equilibrium existence. Then we apply it in order to prove the existence of solutions for two types of random quasi-variational inequalities with random fuzzy mappings. We generalize some results obtained by Yuan in [27]. As a consequence, we obtain random fixed point theorems.

The paper is organized as follows. In the next section, some notational and terminological conventions are given. We also present, for the reader’s convenience, some results on Bochner integration. In Section 3, the model of differential information abstract fuzzy economy is introduced and the main result is stated. Section 4 contains existence results for solutions of random quasi-variational inequalities with random fuzzy mappings.

2 NOTATION AND DEFINITION

Throughout this paper, we shall use the following notation:

1. $\mathbb{R}_{++}$ denotes the set of strictly positive reals. $\text{co}D$ denotes the convex hull of the set $D$. $\text{co}D$ denotes the closed convex hull of the set $D$. $2^D$ denotes the set of all non-empty subsets of the set $D$. If $D \subset Y$, where $Y$ is a topological space, $\text{cl}D$ denotes the closure of $D$.

For the reader’s convenience, we review a few basic definitions and results from continuity and measurability of correspondences and Bochner integrable functions.

Let $Z$ and $Y$ be sets.

**Definition 1** The graph of the correspondence $P: Z \to 2^Y$ is the set $G_P = \{(z,y) \in Z \times Y : y \in P(z)\}$.

Let $Z$, $Y$ be topological spaces and $P : Z \to 2^Y$ be a correspondence.

1. $P$ is said to be upper semicontinuous if for each $z \in Z$ and each open set $V$ in $Y$ with $P(z) \subset V$, there exists an open neighborhood $U$ of $z$ in $Z$ such that $P(y) \subset V$ for each $y \in U$.

2. $P$ is said to be lower semicontinuous if for each $z \in Z$ and each open set $V$ in $Y$ with $P(z) \cap V \neq \emptyset$, there exists an open neighborhood $U$ of $z$ in $Z$ such that $P(y) \cap V \neq \emptyset$ for each $y \in U$. 
Lemma 1 (see [32]). Let $Z$ and $Y$ be two topological spaces and let $D$ be an open subset of $Z$. Suppose $P_1 : Z \to 2^Y$, $P_2 : Z \to 2^Y$ are upper semicontinuous correspondences such that $P_2(z) \subset P_1(z)$ for all $z \in D$. Then the correspondence $P : Z \to 2^Y$ defined by

$$P(z) = \begin{cases} P_1(z), & \text{if } z \notin D, \\ P_2(z), & \text{if } z \in D \end{cases}$$

is also upper semicontinuous.

Definition 2 Let $Y$ be a metric space and $Y'$ be its dual. $P : Y \to 2^{Y'}$ is said to be monotone if $\text{Re}\langle u - v, y - x \rangle \geq 0$ for all $u \in P(y)$ and $v \in P(x)$ and $x, y \in Y$.

Let now $(\Omega, F, \mu)$ be a complete, finite measure space, and $Y$ be a topological space.

1. The correspondence $P : \Omega \to 2^Y$ is said to have a measurable graph if $G_P \in F \otimes \beta(Y)$, where $\beta(Y)$ denotes the Borel $\sigma$-algebra on $Y$ and $\otimes$ denotes the product $\sigma$-algebra.

2. The correspondence $T : \Omega \to 2^Y$ is said to be lower measurable if for every open subset $V$ of $Y$, the set $\{\omega \in \Omega : T(\omega) \cap V \neq \emptyset\}$ is an element of $F$.

Recall (see Debreu [9], p. 359) that if $T : \Omega \to 2^Y$ has a measurable graph, then $T$ is lower measurable. Furthermore, if $T(\cdot)$ is closed valued and lower measurable then $T : \Omega \to 2^Y$ has a measurable graph.

Lemma 2 (see [15]). Let $P_n : \Omega \to 2^Y$, $n = 1, 2\ldots$ be a sequence of correspondences with measurable graphs. Then the correspondences $\bigcup_n P_n$, $\cap_n P_n$ and $Y \setminus P_n$ have measurable graphs.

Let $(\Omega, F, \mu)$ be a measure space and $Y$ be a Banach space.

It is known (see [15], Theorem 2, p. 45) that, if $x : \Omega \to Y$ is a $\mu$-measurable function, then $x$ is Bochner integrable if only if $\int_{\Omega} \|x(\omega)\|d\mu(\omega) < \infty$.

It is denoted by $L_1(\mu, Y)$ the space of equivalence classes of $Y$-valued Bochner integrable functions $x : \Omega \to Y$ normed by $\|x\| = \int_{\Omega} \|x(\omega)\|d\mu(\omega)$.

Also it is known (see [9], p. 50) that $L_1(\mu, Y)$ is a Banach space.

Definition 3 The correspondence $P : \Omega \to 2^Y$ is said to be integrably bounded if there exists a map $h \in L_1(\mu, R)$ such that $\sup\{||x|| : x \in P(\omega)\} \leq h(\omega)$ $\mu$-a.e.

We denote by $S^1_P$ the set of all selections of the correspondence $P : \Omega \to 2^Y$ that belong to the space $L_1(\mu, Y)$, i.e.

$$S^1_P = \{x \in L_1(\mu, Y) : x(\omega) \in P(\omega) \mu\text{-a.e.}\}.$$
Further, we will see the conditions under which $S_1^1$ is nonempty and weakly compact in $L_1(\mu,Y)$. Aumann measurable selection theorem (see Appendix) and Diestel’s Theorem (see Appendix) are necessary.

Let $F(Y)$ be a collection of all fuzzy sets over $Y$.

**Definition 4** A mapping $P : \Omega \to F(Y)$ is called a fuzzy mapping. If $P$ is a fuzzy mapping from $\Omega$, $P(\omega)$ is a fuzzy set on $Y$ and $P(\omega)(y)$ is the membership function of $y$ in $P(\omega)$.

Let $A \in F(Y)$, $a \in [0,1]$, then the set $(A)_a = \{y \in Y : A(y) \geq a\}$ is called an $a$–cut set of fuzzy set $A$.

**Definition 5** A fuzzy mapping $P : \Omega \to F(Y)$ is said to be measurable if for any given $a \in [0,1]$, $(P(\cdot))_a : \Omega \to 2^Y$ is a measurable set-valued mapping.

**Definition 6** We say that a fuzzy mapping $P : \Omega \to F(Y)$ is said to have a measurable graph if for any given $a \in [0,1]$, the set-valued mapping $(P(\cdot))_a : \Omega \to 2^Y$ has a measurable graph.

**Definition 7** A fuzzy mapping $P : \Omega \times X \to F(Y)$ is called a random fuzzy mapping if for any given $x \in X$, $P(\cdot, x) : \Omega \to F(Y)$ is a measurable fuzzy mapping.

3 BAYESIAN FUZZY EQUILIBRIUM EXISTENCE FOR BAYESIAN ABSTRACT FUZZY ECONOMIES

3.1 THE MODEL OF A BAYESIAN ABSTRACT FUZZY ECONOMY

We now define the next model of the Bayesian abstract fuzzy economy which generalizes the model in [23].

Let $(\Omega, F, \mu)$ be a complete finite measure space, where $\Omega$ denotes the set of states of nature of the world and the $\sigma$–algebra $F$, denotes the set of events. Let $Y$ denote the strategy or commodity space, where $Y$ is a separable Banach space.

Let $I$ be a countable or uncountable set (the set of agents). Let $X_i : \Omega \to F(Y)$ a fuzzy mapping and $z \in [0,1]$.

Let $L_{X_i} = \{x_i \in S_{(X_i(\cdot))_z} : x_i$ is $F_i$–measurable $\}$. Denote by $L_X = \prod_{i \in I} L_{X_i}$ and by $L_{X_{\neq i}}$, the set $\prod_{j \neq i} L_{X_j}$. An element $x_i$ of $L_{X_i}$ is called a strategy for agent $i$. The typical element of $L_X$ is denoted by $\tilde{x}_i$ and that of $X_i(\omega)$ by $x_i(\omega)$ (or $x_i$).

**Definition 8** A general Bayesian abstract fuzzy economy is a family $G = \{((\Omega, F, \mu), (X_i, F_i, A_i, P_i, a_i, b_i, z_i)_{i \in I})\}$, where
(1) $X_i: \Omega \rightarrow \mathcal{F}(Y)$ is the action (strategy) fuzzy mapping of agent $i$.
(2) $F_i$ is a sub-$\sigma$-algebra of $F$ which denotes the private information of agent $i$.
(3) for each $\omega \in \Omega$, $A_i(\omega, \cdot) : L_X \rightarrow \mathcal{F}(Y)$ is the random fuzzy constraint mapping of agent $i$;
(4) for each $\omega \in \Omega$, $P_i(\omega, \cdot) : L_X \rightarrow \mathcal{F}(Y)$ is the random fuzzy preference mapping of agent $i$;
(5) $z \in (0,1]$ is such that for all $(\omega, x) \in \Omega \times L_X$, $(A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} \subseteq (X_i(\omega))_{z}$ and $(P_i(\omega, \tilde{x}))_{p_i(\tilde{x})} \subseteq X_i(\omega)_{z}$;
(6) $a : S^1_X \rightarrow (0,1]$ is a random fuzzy constraint function and $p : S^1_X \rightarrow (0,1]$ is a random fuzzy preference function.

**Definition 9** A Bayesian fuzzy equilibrium for $G$ is a strategy profile $\tilde{x}^* \in L_X$ such that for all $i \in I$,

(j) $\tilde{x}^*_i(\omega) \in \text{cl}(A_i(\omega, \tilde{x}^*))_{a_i(\tilde{x}^*)} \mu - a.e.$
(j) $\tilde{x}^*_i(\omega) \in \text{cl}(A_i(\omega, \tilde{x}^*))_{a_i(\tilde{x}^*)} \cap (P_i(\omega, \tilde{x}^*))_{p_i(\tilde{x}^*)} = \emptyset \mu - a.e.$

**Remark 1** Now we assume that for each $i \in I$, $X_i$ is a compact convex nonempty subset of $Y$ and for each $\omega \in \Omega$, we set $(X_i(\omega))_{z} = X_i$. Then we obtain the deterministic classical model of Yannelis-Prabhakar in [29] for an abstract economy with any set of players.

**Remark 2** The interpretation of the preference fuzzy mapping $P_i$ is that $y_i \in (P_i(\omega, \tilde{x}))_{p_i(\tilde{x})}$ means that at the state $\omega$ of the nature, agent $i$ strictly prefers $y_i$ to $\tilde{x}_i(\omega)$ if the given strategy of other agents is fixed. The preference do not need to be reprezentable by utility functions. However, it will be assumed that $\tilde{x}_i(\omega) \notin (P_i(\omega, \tilde{x}))_{p_i(\tilde{x})} \mu - a.e.$

### 3.2 EXISTENCE OF THE BAYESIAN FUZZY EQUILIBRIUM

This is our first theorem. The constraint and preference correspondences derived from the constraint and preference fuzzy mappings verify the assumptions of measurable graph and weakly open lower sections. Our results is a generalization of Theorem 3 in [23].

**Theorem 1** Let $I$ be a countable or uncountable set. Let the family $G = \{(\Omega, F, \mu), (X_i, F_i, A_i, P_i, a_i, b_i, z_i)_{i \in I}, \}$ be a general Bayesian abstract economy satisfying $A.1)$-A.4). Then there exists a Bayesian fuzzy equilibrium for $G$.

For each $i \in I$ :

A.1) $X_i : \Omega \rightarrow \mathcal{F}(Y)$ is such that $\omega \rightarrow X_i(\omega)_z : \Omega \rightarrow 2^Y$ is a nonempty, convex, weakly compact-valued and integrably bounded correspondence.
(b) $X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X_i(\omega))_z : \Omega \to 2^Y$ is $F_i$-lower measurable;

A.2

(a) For each $\omega, \tilde{x} \in \Omega \times L_X$, $(A_i(\omega, \tilde{x}))_{a(\tilde{x})} : \Omega \to 2^Y$ is integrably bounded and has convex weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set \((A_i(\omega, \tilde{x}))_{a(\tilde{x})}^{-1}(\omega, y) = \{ \tilde{x} \in L_X : y \in (A_i(\omega, \tilde{x}))_{a(\tilde{x})} \}\) is weakly open in $L_X$;

(b) the correspondence $(\omega, \tilde{x}) \to (A_i(\omega, \tilde{x}))_{a(\tilde{x})} : \Omega \times L_X \to 2^Y$ has measurable graph i.e. $\{(\omega, \tilde{x}, y) \in \Omega \times L_X \times Y : y \in (A_i(\omega, \tilde{x}))_{a(\tilde{x})} \} \in F \otimes \beta_w(L_X) \otimes \beta(Y)$ where $\beta_w(L_X)$ is the Borel $\sigma$–algebra for the weak topology on $L_X$ and $\beta(Y)$ is the Borel $\sigma$–algebra for the norm topology on $Y$.

(c) the correspondence $(\omega, \tilde{x}) \to (A_i(\omega, \tilde{x}))_{a(\tilde{x})} : \Omega \times L_X \to 2^Y$ is weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set \((A_i(\omega, \tilde{x}))_{a(\tilde{x})}^{-1}(\omega, y) = \{ \tilde{x} \in L_X : y \in (A_i(\omega, \tilde{x}))_{a(\tilde{x})} \}\) is weakly open in $L_X$ for every norm open subset $V$ of $Y$.

A.3

(a) the correspondence $(\omega, \tilde{x}) \to (P_i(\omega, \tilde{x}))_{p(\tilde{x})} : \Omega \times L_X \to 2^Y$ has nonempty open convex values such that $(P_i(\omega, \tilde{x}))_{p(\tilde{x})} \subset (X(\omega))_z$ for each $(\omega, \tilde{x}) \in \Omega \times L_X$;

(b) the correspondence $(\omega, \tilde{x}) \to (P_i(\omega, \tilde{x}))_{p(\tilde{x})} : \Omega \times L_X \to 2^Y$ has measurable graph

(c) the correspondence $(\omega, \tilde{x}) \to (P_i(\omega, \tilde{x}))_{p(\tilde{x})} : \Omega \times L_X \to 2^Y$ has weakly open lower sections, i.e., for each $\omega \in \Omega$ and for each $y \in Y$, the set \((P_i(\omega, \tilde{x}))_{p(\tilde{x})}^{-1}(\omega, y) = \{ \tilde{x} \in L_X : y \in (P_i(\omega, \tilde{x}))_{p(\tilde{x})} \}\) is weakly open in $L_X$;

A.4

(a) For each $\tilde{x}, \tilde{x} \in L_{X_i}$, for each $\omega \in \Omega$, $\tilde{x}(\omega) \notin (A_i(\omega, \tilde{x}))_{a(\tilde{x})} \cap (P_i(\omega, \tilde{x}))_{p(\tilde{x})}$. We prove first that $L_X$ is a non-empty, convex, weakly compact subset in $L_1(\mu, Y)$.

Since $(\Omega, F, \mu)$ is a complete finite measure space, $Y$ is a separable Banach space and $X_i : \Omega \to 2^Y$ has measurable graph, by Aumann’s selection theorem (see Appendix) it follows that there exists a $F_1$-measurable function $f_i : \Omega \to Y$ such that $f_i(\omega) \in X_i(\omega)$ $\mu$ – a.e. Since $X_i$ is integrably bounded, we have that $f_i \in L_1(\mu, Y)$, hence $L_{X_i}$ is non-empty and $L_X = \prod_{i \in I} L_{X_i}$ is non-empty. Obviously $L_{X_i}$ is convex and $L_X$ is also convex. Since $X_i : \Omega \to 2^Y$ is integrably bounded and has convex weakly compact values, by Diestel’s Theorem (see Appendix) it follows that $L_{X_i}$ is a weakly compact subset of $L_1(\mu, Y)$. More over, $L_X$ is weakly compact. $L_1(\mu, Y)$ equipped with the weak topology is a locally convex topological vector space.

The correspondence $\Phi_i$ is convex valued, by Lemma 2 it has a measurable graph and for each $\omega \in \Omega$, $\Phi_i(\omega, \cdot)$ has weakly open lower sections. Let
U_i = \{ (\omega, \bar{x}) \in \Omega \times L_X : \Phi_i(\omega, \bar{x}) \neq \emptyset \}. For each \bar{x} \in L_X, let \( U_i^x = \{ \omega \in \Omega : \Phi_i(\omega, \bar{x}) \neq \emptyset \} \) and for each \( \omega \in \Omega \), let \( U_i^\omega = \{ \bar{x} \in L_X : \Phi_i(\omega, \bar{x}) \neq \emptyset \} \). The values of \( \Phi_i/U_i \) have non-empty interiors in the relative norm topology of \( X_i(\omega) \). By the Caratheodory-type selection theorem (see Appendix), there exists a function \( f_i : U_i \rightarrow Y \) such that \( f_i(\omega, \bar{x}) \in \Phi_i(\omega, \bar{x}) \) for all \((\omega, \bar{x}) \in U_i \).

We shall prove that \( G \) is an upper semicontinuous correspondence with respect to the weakly topology of \( L_X \) and has non-empty convex closed values. By applying Fan-Glicksberg’s fixed-point theorem [11] to \( G' \), we obtain a fixed point which is the equilibrium point for the abstract economy.

It follows by Theorem III.40 in [5] and the projection theorem that for each \( \bar{x} \in L_X \), the correspondence \( \bar{x} \rightarrow \text{cl} A_i(\cdot, \bar{x}) \) is jointly measurable on \( \Omega \times L_X \). By Lemma 1, for each \( \omega \in \Omega \), \( G_i(\omega, \cdot) \) is an upper semicontinuous correspondence and has non-empty interiors in the relative norm topology of \( X_i(\omega) \). By u. s. c. Lifting Theorem (see Appendix) it follows that \( G_i \) is weakly upper semicontinuous.

\( G' \) is an upper semicontinuous correspondence and has also non-empty closed values.

The set \( L_X \) is weakly compact and convex, and then, by Fan-Glicksberg’s fixed-point theorem in [11], there exists \( \bar{x}^* \in L_X \) such that \( \bar{x}^* \in G'(\bar{x}^*), \)

Then, \( \bar{x}^* \in L_X \) and \( \bar{x}^*_i(\omega) \in G_i(\omega, \bar{x}^*) \mu-a.e. \) Since \( \bar{x}_i^*(\omega) \notin A_i(\omega, \bar{x}^*) \), \( P_i(\omega, \bar{x}^*) \mu - a.e. \) it follows that \( (\omega, \bar{x}^*) \notin U_i \) for each \( i \in I \) and \( \bar{x}^*_i \in \text{cl} A_i(\omega, \bar{x}^*) \mu - a.e. \). We have also that \( (A_i(\omega, \bar{x}^*))_{a_i(\bar{x}^*)} \cap (P_i(\omega, \bar{x}^*))_{p_i(\bar{x}^*)} = \emptyset. \)

**Theorem 2** Let \( I \) be a countable or uncountable set. Let the family \( G = \{ (\Omega, F, \mu), (X_i, F_i, A_i, P_i, a_i, b_i, z_i)_{i \in I} \} \) be a general Bayesian abstract econo-
For each $i \in I$:

A.1)

(a) $X_i : \Omega \to F(Y)$ is such that $\omega \mapsto (X_i(\omega))_{z_i} : \Omega \to 2^Y$ is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;

(b) $X_i : \Omega \to F(Y)$ is such that $\omega \mapsto (X_i(\omega))_{z_i} : \Omega \to 2^Y$ is $F_i$-measurable and compact.

A.2)

(a) For each $(\omega, \tilde{x}) \in \Omega \times L_X$, $(A_i(\omega, \tilde{x}))_{a_i(\tilde{x})}$ is nonempty convex and compact.

(b) For each $\tilde{x} \in L_X$, the correspondence $\omega \mapsto (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} : \Omega \to 2^Y$ has measurable graph;

(c) For each $\omega \in \Omega$, $\tilde{x} \mapsto (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} : L_X \to 2^Y$ is upper semicontinuous in the sense that the set \( \{ \tilde{x} \in L_X : (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} \} \subset V \) is weakly open in $L_X$ for every norm open subset $V$ of $Y$;

A.3)

(a) For each $(\omega, \tilde{x}) \in \Omega \times L_X$, the correspondence $\omega \mapsto (P_i(\omega, \tilde{x}))_{p_i(\tilde{x})} : \Omega \to 2^Y$ has nonempty convex compact values such that $(P_i(\omega, \tilde{x}))_{p_i(\tilde{x})} \subset (X(\omega))_{z_i}$ for each $(\omega, \tilde{x}) \in \Omega \times L_X$;

(b) For each $\tilde{x} \in L_X$, the correspondence $\omega \mapsto (P_i(\omega, \tilde{x}))_{p_i(\tilde{x})} : \Omega \rightarrow 2^Y$ has a measurable graph;

(c) For each $\omega \in \Omega$, $\tilde{x} \mapsto (P_i(\omega, \tilde{x}))_{p_i(\tilde{x})} : L_X \to 2^Y$ is upper semicontinuous.

A.4)

(a) For each $\tilde{x} \in L_X$, for each $\omega \in \Omega$, \( \tilde{x}_i(\omega) \notin (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} \cap (P_i(\omega, \tilde{x}))_{p_i(\tilde{x})} \);

(b) For each $\omega \in \Omega$, the set $U_{\omega} = \{ \tilde{x} \in L_X : (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} \cap (P_i(\omega, \tilde{x}))_{p_i(\tilde{x})} \neq \emptyset \}$ is weakly open in $L_X$.

Proof. For each $i \in I$, define $\Phi_i : \Omega \times L_X \to 2^Y$ by $\Phi_i(\omega, \tilde{x}) = (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} \cap (P_i(\omega, \tilde{x}))_{p_i(\tilde{x})}$.

We prove first that $L_X$ is a non-empty, convex, weakly compact subset in $L_1(\mu, Y)$.

Since $(\Omega, F, \mu)$ is a complete finite measure space, $Y$ is a separable Banach space and $(X_i(\cdot))_{z_i} : \Omega \to 2^Y$ has measurable graph, by Aumann’s selection theorem (see Appendix) it follows that there exists a $F_i$-measurable function $f_i : \Omega \to Y$ such that $f_i(\omega) \in X_i(\omega)$ $\mu$-a.e. Since $(X_i(\cdot))_{z_i}$ is integrably bounded, we have that $f_i \in L_1(\mu, Y)$, hence $L_X$ is non-empty and $L_X = \prod_{i \in I} L_X$ is non-empty. Obviously $L_X$ is convex and $L_X$ is also convex. Since $X_i : \Omega \to 2^Y$ is integrably bounded and has convex weakly compact values, by Diestel’s Theorem (see Appendix) it follows that $L_X$ is a weakly compact subset of $L_1(\mu, Y)$. Moreover, $L_X$ is weakly compact.
$L_1(\mu, Y)$ equipped with the weak topology is a locally convex topological vector space.

Define $G_i : \Omega \times L_X \to 2^Y$ by $G_i(\omega, \tilde{x}) = \{ \Phi_i(\omega, \tilde{x}) \text{ if } (\omega, \tilde{x}) \in U_i; \} (A_i(\omega, \tilde{x}))_{\alpha_i(\tilde{z})} \text{ if } (\omega, \tilde{x}) \notin U_i.$

For each $\tilde{x} \in L_X$, the correspondence $\omega \to (A_i(\omega, \tilde{x}))_{\alpha_i(\tilde{z})} : \Omega \to 2^Y$ has a measurable graph. Hence for each $\tilde{x} \in L_X$, the correspondence $G_i(\cdot, \tilde{x})$ has a measurable graph. By the assumption A4) (b), we have that $U_\omega^\sigma$ is weakly compact-valued and integrably bounded correspondence.

Theorem 3 Let $I$ be a countable or uncountable set. Let $(\Omega, F, \mu)$ be a complete finite separable measure space and let $Y$ be a separable Banach space. Suppose that the following conditions are satisfied:

For each $i \in I$:

(A.1)

(a) $X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to X_i(\omega)$ is a nonempty, convex, weakly compact-valued and integrably bounded correspondence.

(b) $X_i : \Omega \to \mathcal{F}(Y)$ is such that $\omega \to (X_i(\omega))_z : \Omega \to 2^Y$ is $F_i$-lower measurable;
(a) For each \((\omega, x) \in \Omega \times L_X\), \(\left( A_1(\omega, x) \right)_{a(\xi)} \) is convex and has a non-empty interior in the relative norm topology of \( \left( X(\omega) \right)_{a(\xi)} \).

(b) the correspondence \( \omega \), \( x \) \( \rightarrow \left( A_1(\omega, x) \right)_{a(\xi)} : \Omega \times L_X \rightarrow 2^Y \) has measurable graph i.e. \( \left\{ (\omega, x, y) \in \Omega \times L_X \times Y : y \in \left( A_1(\omega, x) \right)_{a(\xi)} \right\} \in F \otimes \beta_\omega(L_X) \otimes \beta(Y) \) where \( \beta_\omega(L_X) \) is the Borel \( \sigma \)-algebra for the weak topology on \( L_X \) and \( \beta(Y) \) is the Borel \( \sigma \)-algebra for the norm topology on \( Y \).

(c) the correspondence \( \omega \), \( x \) \( \rightarrow \left( A_1(\omega, x) \right)_{a(\xi)} \) has weakly open lower sections, i.e., for each \( \omega \in \Omega \) and for each \( y \in Y \), the set \( \left( \left( A_1(\omega, x) \right)_{a(\xi)} \right)^{-1}(y, \omega) = \{ x \in L_X : y \in \left( A_1(\omega, x) \right)_{a(\xi)} \} \) is weakly open in \( L_X \).

(d) For each \( \omega \in \Omega \), \( x \rightarrow \left( A_1(\omega, x) \right)_{a(\xi)} : L_X \rightarrow 2^Y \) is upper semicontinuous in the sense that the set \( \{ x \in L_X : \left( A_1(\omega, x) \right)_{a(\xi)} \subseteq V \} \) is weakly open in \( L_X \) for every norm open subset \( V \) of \( Y \).

A.3

\( \psi_1 : \Omega \times L_X \times Y \rightarrow R \cup \{-\infty, +\infty\} = \) such that:

(a) \( \tilde{x} \rightarrow \psi_i(\omega, \tilde{x}, y) \) is lower semicontinuous on \( L_X \) for each fixed \( \omega, y \in \Omega \times Y \);

(b) \( \tilde{x}(\omega) \notin \psi_i(\omega, \tilde{x}, y) > 0 \) for each fixed \( \omega, \tilde{x} \in \Omega \times L_X \);

(c) the correspondence \( \psi_i(\omega, \tilde{x}, y) \) is concave;

(d) \( \psi_i(\omega, \tilde{x}, y) \) is concave;

(e) \( \{(\omega, x) : \alpha_i(\omega, x) > 0\} \in F \cap GL(L_X) \).

Then, there exists \( \tilde{x}^* \in L_X \) such that for every \( i \in I \),

(i) \( \tilde{x}^*_i(\omega) \in cl \left( A_i(\omega, \tilde{x}^*) \right)_{a(\xi)} \);

(ii) \( sup_{y \in \left( A_i(\omega, \tilde{x}^*) \right)_{a(\xi)}} \psi_i(\omega, \tilde{x}^*, y) \leq 0 \).

Proof. For every \( i \in I \), let \( p_i : \Omega \times S^1_X \rightarrow F(Y) \) and \( p_i : S^1_X \rightarrow (0, 1) \) such that \( (p_i(\omega, \tilde{x}))_{p_i(\xi)} = \{ y \in Y : \psi_i(\omega, \tilde{x}, y) > 0 \} \) for each \( \omega, \tilde{x} \in \Omega \times L_X \).

We shall show that the abstract economy \( G = \{ (\Omega, F, \mu), (X_i, F_i, A_i, P_i, a_i, p_i, z_i)_{i \in I} \} \) satisfies all hypotheses of Theorem 1.

Suppose \( \omega \in \Omega \).

According to A3 a), we have that \( \tilde{x} \rightarrow (P_i(\omega, \tilde{x}))_{p_i(\xi)} : \Omega \rightarrow 2^Y \) has open lower sections with nonempty compact values and according to A3 b), \( \tilde{x}(\omega) \notin \left( P_i(\omega, \tilde{x}) \right)_{p_i(\xi)} \) for each \( \tilde{x} \in L_X \). Assumption A3 c) implies that \( \tilde{x} \rightarrow (P_i(\omega, \tilde{x}))_{p_i(\xi)} : \Omega \rightarrow 2^Y \) has convex values.

By the definition of \( \alpha_i \), we note that \( \{ \tilde{x} \in L_X : \alpha_i(\omega, \tilde{x}) \neq 0 \} \Rightarrow \{ \tilde{x} \in L_X : \alpha_i(\omega, \tilde{x}) > 0 \} \) so that \( \tilde{x} \in L_X : (A_1(\omega, \tilde{x}))_{a(\xi)} \cap (P_i(\omega, \tilde{x}))_{p_i(\xi)} \neq \emptyset \) is weakly open in \( L_X \) by A3 c).

According to A2 b) and A3 c), it follows that the correspondences \( (\omega, \tilde{x}) \rightarrow \left( A_1(\omega, \tilde{x}) \right)_{a(\xi)} : \Omega \times L_X \rightarrow 2^Y \) and \( (\omega, \tilde{x}) \rightarrow \left( P_i(\omega, \tilde{x}) \right)_{p_i(\xi)} : \Omega \times L_X \rightarrow 2^Y \) have measurable graphs.

Thus the Bayesian abstract fuzzy economy \( G = \{ (\Omega, F, \mu), (X_i, F_i, A_i, P_i, a_i, b_i, z_i)_{i \in I} \} \) satisfies all hypotheses of Theorem 1. Therefore, there exists \( \tilde{x}^* \in L_X \) such that for every \( i \in I \):

(\( \tilde{x}^*_i(\omega) \in cl \left( A_i(\omega, \tilde{x}^*) \right)_{a(\xi)} - a.e \) and
(\( A_i(\omega, \tilde{x}^*) \)\( a(\xi) \cap \left( P_i(\omega, \tilde{x}) \right)_{p_i(\xi)} = 0 \) \( - a.e \);
that is, there exists \( \tilde{x}^* \in L_X \) such that for every \( i \in I \):

i) \( \tilde{x}^*_i(\omega) \in \text{cl}(A_i(\omega, \tilde{x}^*))_{a(\tilde{x}^*)} \);

ii) \( \sup_{y \in (A_i(\omega, \tilde{x}^*))_{a(\tilde{x}^*)}} \psi_i(\omega, \tilde{x}^*, y) \leq 0. \)

If \( |I| = 1 \), we obtain the following corollary.

**Corollary 1** Let \((\Omega, F, \mu)\) be a complete finite separable measure space and let \( Y \) be a separable Banach space. Suppose that the following conditions are satisfied:

\[ A.1 \]

a) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega))_{\sharp} : \Omega \to 2^Y \) is a nonempty, convex, weakly compact-valued and integrably bounded correspondence.

b) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega))_{\sharp} : \Omega \to 2^Y \) is \( F \)-lower measurable.

\[ A.2 \]

a) For each \( (\omega, \tilde{x}) \in \Omega \times L_X, (A(\omega, \tilde{x}))_{a(\tilde{x})} \) is convex and has a non-empty interior in the relative norm topology of \((X(\omega))_{\sharp}\).

b) the correspondence \( (\omega, \tilde{x}) \to (A(\omega, \tilde{x}))_{a(\tilde{x})} : \Omega \times L_X \to 2^Y \) has measurable graph i.e., \( \{ (\omega, \tilde{x}, y) : (A(\omega, \tilde{x}))_{a(\tilde{x})} = y \} \in F \otimes \beta_w(L_X)^{\otimes} \beta(Y) \) where \( \beta_w(L_X) \) is the Borel \( \sigma \)-algebra for the weak topology on \( L_X \) and \( \beta(Y) \) is the Borel \( \sigma \)-algebra for the norm topology on \( Y \).

c) the correspondence \( (\omega, \tilde{x}) \to (A(\omega, \tilde{x}))_{a(\tilde{x})} \) has weakly open lower sections, i.e., for each \( \omega \in \Omega \) and for each \( y \in Y \), the set \( ((A(\omega, \tilde{x}))_{a(\tilde{x})})^{-1}(\omega, y) = \{ \tilde{x} \in L_X : y \in (A(\omega, \tilde{x}))_{a(\tilde{x})} \} \) is weakly open in \( L_X \).

(d) For each \( \omega \in \Omega \), \( \tilde{x} \mapsto \text{cl}(A(\omega, \tilde{x}))_{a(\tilde{x})} : L_X \to 2^Y \) is upper semicontinuous in the sense that the set \( \{ \tilde{x} \in L_X : \text{cl}(A(\omega, \tilde{x}))_{a(\tilde{x})} \subseteq V \} \) is weakly open in \( L_X \) for every norm open subset \( V \) of \( Y \).

\[ A.3 \]

\( \psi_i : \Omega \times L_X \times Y \to R \cup \{-\infty, +\infty\} \) is such that:

a) \( \tilde{x} \mapsto \psi(\omega, \tilde{x}, y) \) is lower semicontinuous on \( L_X \) for each fixed \( (\omega, y) \in \Omega \times Y \);

b) \( \tilde{x}(\omega) \not\in \text{cl}\{ y \in Y : \psi(\omega, \tilde{x}, y) > 0 \} \) for each fixed \( (\omega, \tilde{x}) \in \Omega \times L_X \);

c) for each \( (\omega, \tilde{x}) \in \Omega \times L_X \), \( \psi(\omega, \tilde{x}, \cdot) \) is concave;

d) for each \( \omega \in \Omega \), \( \{ \tilde{x} \in L_X : a(\omega, \tilde{x}) > 0 \} \) is weakly open in \( L_X \), where \( a : \Omega \times L_X \to R \) is defined by \( a(\omega, \tilde{x}) = \sup_{y \in (A(\omega, \tilde{x}))_{a(\tilde{x})}} \psi(\omega, \tilde{x}, y) \) for each \( (\omega, \tilde{x}) \in \Omega \times L_X \);

(c) \( \{ (\omega, \tilde{x}) : a(\omega, \tilde{x}) > 0 \} \in F \otimes B(L_X) \).

Then, there exists \( \tilde{x}^* \in L_X \) such that:

i) \( \tilde{x}^*(\omega) \in \text{cl}(A(\omega, \tilde{x}^*))_{a(\tilde{x}^*)} \); 

ii) \( \sup_{y \in (A(\omega, \tilde{x}^*))_{a(\tilde{x}^*)}} \psi(\omega, \tilde{x}^*, y) \leq 0. \)

As a consequence of Theorem 2, we prove the following random generalized quasi-variational inequality with random fuzzy mappings. This theorem is comparable with Theorem 4.1 in [27], which is valid in a non-fuzzy framework and concerns upper-semicontinuous correspondences defined on metrizable spaces.
Theorem 4 Let \((\Omega, F, \mu)\) be a complete finite separable measure space and let \(Y\) be a separable Banach space. Suppose that the following conditions are satisfied:

For each \(i \in I\):

A.1) 
(a) \(X_i : \Omega \rightarrow F(Y)\) is such that \(\omega \rightarrow (X_i(\omega))_z : \Omega \rightarrow 2^Y\) is a nonempty, convex, weakly compact-valued and integrably bounded correspondence.

(b) \(X_i : \Omega \rightarrow F(Y)\) is such that \(\omega \rightarrow (X_i(\omega))_z : \Omega \rightarrow 2^Y\) is \(F_i\)-lower measurable;

A.2) 
(a) For each \((\omega, \tilde{x}) \in \Omega \times L_X\), \((A_i(\omega, \tilde{x}))_{a(i)}\) is convex and has a non-empty interior in the relative norm topology of \((X_i(\omega))_z\).

(b) the correspondence \((\omega, \tilde{x}) \rightarrow (A_i(\omega, \tilde{x}))_{a(i)} : \Omega \times L_X \rightarrow 2^Y\) has measurable graph i.e. \(\{\omega, \tilde{x}, y\} \in \Omega \times L_X \times Y : y \in \{A_i(\omega, \tilde{x}))_{a(i)}\} \subset F \otimes \beta_w(L_X) \otimes \beta(Y)\) where \(\beta_w(L_X)\) is the Borel \(\sigma\)-algebra for the weak topology on \(L_X\) and \(\beta(Y)\) is the Borel \(\sigma\)-algebra for the norm topology on \(Y\).

(c) the correspondence \((\omega, \tilde{x}) \rightarrow (A_i(\omega, \tilde{x}))_{a(i)}\) has weakly open lower sections, i.e., for each \(\omega \in \Omega\) and for each \(y \in Y\), the set \((A_i(\omega, \tilde{x}))_{a(i)}^{-1}(\omega, y) = \{\tilde{x} \in L_X : y \in (A_i(\omega, \tilde{x}))_{a(i)}\}\) is weakly open in \(L_X\);

(d) For each \(\omega \in \Omega\), \(\tilde{x} \rightarrow \text{cl}(A_i(\omega, \tilde{x}))_{a(i)} : L_X \rightarrow 2^Y\) is upper semicontinuous in the sense that the set \(\{\tilde{x} \in L_X : \text{cl}(A_i(\omega, \tilde{x}))_{a(i)} \subset V\}\) is weakly open in \(L_X\) for every norm open subset \(V\) of \(Y\);

A.3) 
\(G_i : \Omega \times Y \rightarrow F(Y')\) and \(g_i : Y \rightarrow (0, 1]\) are such that:

(a) for each \(\omega \in \Omega\), \(y \rightarrow (G_i(\omega, y))_{g_i(y)} : Y \rightarrow 2^{Y'}\) is monotone with non-empty values;

(b) for each \(\omega \in \Omega\), \(y \rightarrow (G_i(\omega, y))_{g_i(y)} : L \cap Y \rightarrow 2^{Y'}\) is lower semicontinuous from the relative topology of \(Y\) into the weak*-topology \(\sigma(Y', Y)\) of \(Y'\) for each one-dimensional flat \(L \subset \Omega\);

A.4) 
(a) \(f_i : \Omega \times L_X \times Y \rightarrow R \cup \{\infty\}\) is such that \(\tilde{x} \rightarrow f_i(\omega, \tilde{x}, y)\) is lower semicontinuous on \(\Omega \times L_X\) for each fixed \((\omega, y) \in \Omega \times Y\), \(f_i(\omega, \tilde{x}, \tilde{t}(\omega)) = 0\) for each \((\omega, \tilde{x}) \in \Omega \times L_X\) and \(y \rightarrow f_i(\omega, \tilde{x}, y)\) is concave on \(Y\) for each fixed \((\omega, \tilde{x}) \in \Omega \times L_X\);

(b) for each fixed \(\omega \in \Omega\), the set \(\{\tilde{x} \in S_X : \sup_{y \in (A_i(\omega, \tilde{x}))_{a(i)}}[\sup_{u \in (G_i(\omega, y))_{a_i(y)}} \Re\langle u, \tilde{x} - y \rangle + f_i(\omega, \tilde{x}, y) > 0\}\) is weakly open in \(L_X\);

(c) \(\{(\omega, \tilde{x}) : \sup_{u \in (G_i(\omega, \tilde{x}))_{a_i(y)}} \Re\langle u, \tilde{x} - y \rangle + f_i(\omega, \tilde{x}, y) > 0\} \in \mathcal{F} \otimes B(L_X)\).

Then, there exists \(\tilde{x}_* \in L_X\) such that for every \(i \in I\):

i) \(\tilde{x}_i^*(\omega) \in \text{cl}(A_i(\omega, \tilde{x}_i^*))_{a_i(\tilde{x}_i^*)}\);

ii) \(\sup_{u \in (G_i(\omega, \tilde{x}_i^*))_{a_i(\tilde{x}_i^*)}} \Re\langle u, \tilde{x}_i^*(\omega) - y \rangle + f_i(\omega, \tilde{x}_i^*, y) \leq 0\) for all \(y \in (A_i(\omega, \tilde{x}_i^*))_{a_i(\tilde{x}_i^*)}\).
Proof. Let us define \( \psi_i : \Omega \times L_X \times Y \to R \cup \{-\infty, +\infty\} \) by
\[
\psi_i(\omega, \bar{x}, y) = \sup_{u \in (G_i(\omega, y))_{\sigma_i(y)}} Re(u, \bar{x} - y) + f_i(t, \omega, \bar{x}, y)
\]
for each \( (\omega, \bar{x}, y) \in \Omega \times L_X \times Y \).

According to assumption A4 a), \( \bar{x} \to f_i(t, \omega, \bar{x}, y) \) is lower semicontinuous on \( L_X \) for each fixed \( (\omega, y) \in \Omega \times Y \) and \( f_i(\omega, \bar{x}, t(\omega)) = 0 \) for each \( (\omega, \bar{x}) \in \Omega \times L_X \) implies that \( \bar{x}(\omega) \notin \{ y \in Y : \psi_i(\omega, \bar{x}, y) > 0 \} \) for each fixed \( (\omega, \bar{x}) \in \Omega \times L_X \).

We also have that for each \( (\omega, \bar{x}) \in \Omega \times L_X, \psi_i(\omega, \bar{x}, \cdot) \) is concave. This fact is a consequence of assumption A3 b).

All hypotheses of Theorem 2 are satisfied. According to Theorem 2, there exists \( \bar{x}^\ast \in L_X \) such that \( \bar{x}^\ast_i(\omega) \in \text{cl}(A_i(\omega, \bar{x}^\ast))_{\sigma_i(\bar{x}^\ast)} \) for every \( i \in I \) and
\[
(1) \sup_{u \in A_i(\omega, \bar{x}^\ast)} \sup_{u \in G_i(\omega, y)} [Re(u, \bar{x}^\ast(\omega) - y) + f_i(\omega, \bar{x}^\ast, y)] \leq 0 \text{ for every } i \in I.
\]
Finally, we will prove that
\[
(2) \sup_{u \in G_i(\omega, z^\ast(\omega))} \sup_{u \in G_i(\omega, y)} [Re(u, \bar{x}^\ast(\omega) - y) + f_i(\omega, \bar{x}^\ast, y)] \leq 0 \text{ for every } i \in I.
\]
In order to do that, let us consider \( i \in I \) and the fixed point \( \omega \in \Omega \).
Let \( y \in (A_i(\omega, \bar{x}^\ast))_{\sigma_i(\bar{x}^\ast)}, \lambda \in [0, 1] \) and \( z^\ast_i(\omega) := \lambda \bar{x} + (1 - \lambda)\bar{x}^\ast_i(\omega) \).

According to assumption A2 b), \( z^\ast_i(\omega) \in A_i(\omega, \bar{x}^\ast) \).

According to (1), we have \( \sup_{u \in G_i(\omega, z^\ast(\omega))} \sup_{u \in G_i(\omega, y)} [Re(u, \bar{x}^\ast(\omega) - y) + f_i(\omega, \bar{x}^\ast, y)] \leq 0 \) for each \( \lambda \in [0, 1] \).

According to assumption A4 a), \( f_i(\omega, \bar{x}^\ast, \bar{x}^\ast(t, \omega)) = 0 \). For each \( y_1, y_2 \in Y \) and for each \( \lambda \in [0, 1] \), we also have that \( f_i(\omega, \bar{x}^\ast, \lambda y_1 + (1 - \lambda) y_2) \geq \lambda f_i(\omega, \bar{x}^\ast, y_1) + (1 - \lambda) f_i(\omega, \bar{x}^\ast, y_2) \).

Therefore, for each \( \lambda \in [0, 1] \), we have that
\[
t[\sup_{u \in G_i(\omega, z^\ast_i(\omega))} [Re(u, \bar{x}^\ast(\omega) - y) + f_i(\omega, \bar{x}^\ast, y)] \leq \sup_{u \in F_i(\omega, (G_i(\omega, z^\ast_i(\omega)))_{\sigma_i(z^\ast_i(\omega))})} t[Re(u, \bar{x}^\ast(\omega) - y) + f_i(\omega, \bar{x}^\ast, z^\ast_i(t, \omega)]] = \sup_{u \in G_i(\omega, z^\ast_i(\omega))} [Re(u, \bar{x}^\ast(\omega) - z^\ast_i(\omega)) + f_i(\omega, \bar{x}^\ast, z^\ast_i(\omega))] \leq 0.
\]
It follows that for each \( \lambda \in [0, 1] \),
\[
(2) \sup_{u \in G_i(\omega, z^\ast_i(\omega))} [Re(u, \bar{x}^\ast(\omega) - y) + f_i(\omega, \bar{x}^\ast, y)] \leq 0.
\]

Now, we are using the lower semicontinuity of \( y \to (G_i(\omega, y))_{\sigma_i(y)} : L \cap Y \to 2^Y \) in order to show the conclusion. For each \( z_0 \in (G_i(\omega, \bar{x}^\ast(\omega)))_{\sigma_i(z^\ast(\omega))} \) and \( e > 0 \) let us consider \( U_{z_0} \), the neighborhood of \( z_0 \) in the topology \( \sigma(Y', Y) \), defined by \( U_{z_0} := \{ z \in Y' : |Re(z_0 - z, \bar{x}^\ast(\omega) - y) | < e \} \).

As \( y \to (G_i(\omega, y))_{\sigma_i(y)} : L \cap Y \to 2^Y \) is lower semicontinuous, where \( L = \{ z_\lambda(\omega) : \lambda \in [0, 1] \} \) and \( U_{z_0} \cap (G_i(\omega, \bar{x}^\ast(\omega)))_{\sigma_i(z^\ast(\omega))} \neq \emptyset \), there exists a non-empty neighborhood \( N(\bar{x}^\ast(\omega)) \) of \( \bar{x}^\ast(\omega) \) in \( L \) such that for each \( z \in N(\bar{x}^\ast(\omega)) \), we have that \( U_{z_0} \cap (G_i(\omega, z))_{\sigma_i(z)} \neq \emptyset \). Then there exists \( \delta \in (0, 1], t \in (0, \delta) \) and \( u \in (G_i(\omega, z^\ast_i(\omega)))_{\sigma_i(z^\ast_i(\omega))} \cap U_{z_0} \neq \emptyset \) such that \( Re(z_0 - u, \bar{x}^\ast(\omega) - y) < e \). Therefore, \( Re(z_0, \bar{x}^\ast(\omega) - y) < Re(u, \bar{x}^\ast(\omega) - y) + e \). It follows that
\[
Re(z_0, \bar{x}^\ast(\omega) - y) + f_i(\omega, \bar{x}^\ast, y) < Re(u, \bar{x}^\ast(\omega) - y) + f_i(\omega, \bar{x}^\ast, y) + e < e.
\]
The last inequality comes from (2). Since \( \epsilon > 0 \) and \( z_0 \in (G_i(\omega, \bar{x}^*(\omega)))_{g_i(\bar{x}^*(\omega))} \) have been chosen arbitrarily, the next relation holds:
\[
\text{Re}(z_0, \bar{x}^*(\omega) - y) + f_i(\omega, \bar{x}^*, y) < 0.
\]
Hence, for each \( i \in I \), we have that \( \text{sup}_{u \in (G_i(\omega, \bar{x}^*(\omega)))_{g_i(\bar{x}^*(\omega))}} [\text{Re}(z_0, \bar{x}^*(\omega) - y) + f_i(\omega, \bar{x}^*, y)] \leq 0 \) for every \( y \in \text{cl}(A_i(\omega, \bar{x}^*)) \).

If \(|I|=1\), we obtain the following corollary.

Corollary 2 Let \((\Omega, F, \mu)\) be a complete finite separable measure space and let \( Y \) be a separable Banach space. Suppose that the following conditions are satisfied:

A.1)
(a) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega))_\omega : \Omega \to 2^Y \) is a nonempty, convex, weakly compact-valued and integrably bounded correspondence.

(b) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega))_\omega : \Omega \to 2^Y \) is \( F \)-lower measurable.

A.2)
(a) For each \((\omega, \bar{x}) \in \Omega \times L_X \), \((A(\omega, \bar{x}))_{a(\bar{x})}) \in \Omega \times L_X \to 2^Y \) is convex and has a non-empty interior in the relative norm topology of \((X(\omega))_\omega \).

(b) the correspondence \((\omega, \bar{x}) \to (A(\omega, \bar{x}))_{a(\bar{x})} : \Omega \times L_X \to 2^Y \) has measurable graph i.e. \( \{[(\omega, \bar{x}, y) \in \Omega \times L_X \times Y : y \in (A(\omega, \bar{x}))_{a(\bar{x})}] \in F \otimes \beta_\omega(L_X) \otimes \beta(Y) \) where \( \beta_\omega(L_X) \) is the Borel \( \sigma \)-algebra for the weak topology on \( L_X \) and \( \beta(Y) \) is the Borel \( \sigma \)-algebra for the norm topology on \( Y \).

(c) the correspondence \((\omega, \bar{x}) \to (A(\omega, \bar{x}))_{a(\bar{x})} \) has weakly open lower sections, i.e., for each \( \omega \in \Omega \) and for each \( y \in Y \), the set \( \{(A(\omega, \bar{x}))_{a(\bar{x})}^{-1}(\omega, y) = \{\bar{x} \in L_X : y \in (A(\omega, \bar{x}))_{a(\bar{x})}\} \) is weakly open in \( L_X \);

(d) For each \( \omega \in \Omega \), \( \bar{x} \to \text{cl}(A(\omega, \bar{x}))_{a(\bar{x})} : L_X \to 2^Y \) is upper semicontinuous in the sense that the set \( \{\bar{x} \in L_X : y \in (A(\omega, \bar{x}))_{a(\bar{x})}\} \subset V \) is weakly open in \( L_X \) for every norm open subset \( V \) of \( Y \);

A.3)
\[ G : \Omega \times Y \to F(Y') \] and \( g : Y \to (0,1] \) are such that:
(a) for each \( \omega \in \Omega \), \( y \to (G(\omega, y))_{g(y)} : Y \to 2^Y \) is monotone with non-empty values;

(b) for each \( \omega \in \Omega \), \( y \to (G(\omega, y))_{g(y)} : L \cap Y \to 2^Y \) is lower semicontinuous from the relative topology of \( Y \) into the weak*-topology \( \sigma(Y', Y) \) of \( Y' \) for each one-dimensional flat \( L \subset Y \);

A.4)
(a) \( f : \Omega \times L_X \times Y \to R \cup \{\infty, -\infty\} \) is such that \( \bar{x} \to f(\omega, \bar{x}, y) \) is lower semicontinuous on \( L_X \) for each fixed \( (\omega, y) \in \Omega \times Y \), \( f(\omega, \bar{x}, \bar{x}(t, \omega)) = 0 \) for each \( (\omega, \bar{x}) \in \Omega \times L_X \) and \( y \to f(\omega, \bar{x}, y) \) is concave on \( Y \) for each fixed \( (\omega, \bar{x}) \in \Omega \times L_X \);

(b) for each fixed \( \omega \in \Omega \), the set
\[ \{\bar{x} \in S_X : \sup_{y \in (A(\omega, \bar{x}))_{a(\bar{x})}} [\sup_{u \in (G(\omega, y))_{g(y)}} \text{Re}(u, \bar{x} - y) + f(\omega, \bar{x}, y)] > 0 \} \] is weakly open in \( L_X \).
Theorem 6

Let \( (\Omega, F, \mu) \) be a complete finite separable measure space and let \( Y \) be a separable Banach space. Suppose that the following conditions are satisfied:

(a) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega))_z : \Omega \to 2^Y \) is a nonempty, convex, weakly compact-valued and integrably bounded correspondence.

(b) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega))_z : \Omega \to 2^Y \) is \( F_i \)-lower measurable.

(c) For each \( (\omega, \tilde{x}) \in \Omega \times L_X \), \( (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} \) is convex and has a non-empty interior in the relative norm topology of \( (X(\omega))_z \).

(d) For each \( (\omega, \tilde{x}) \to (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} : \Omega \times L_X \to 2^Y \) has measurable graph i.e. \( \{ (\omega, \tilde{x}, y) : (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} \} \) is measurable in \( \Omega \times L_X \times Y \) where \( \beta(X) \) is the Borel algebra for the weak topology on \( L_X \) and \( \beta(Y) \) is the Borel algebra for the norm topology on \( Y \).

(e) The correspondence \( (\omega, \tilde{x}) \to (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} \) has weakly open lower sections, i.e., for each \( \omega \in \Omega \) and each \( y \in Y \), the set \( \{ (\omega, \tilde{x})_{a_i(\tilde{x})} \} \) is weakly open in \( L_X \).

(f) For each \( \omega \in \Omega \), \( \tilde{x} \to \text{cl}(A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} : L_X \to 2^Y \) is upper semicontinuous in the sense that the set \( \{ \tilde{x} \in L_X : \text{cl}(A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} \subset V \} \) is weakly open in \( L_X \) for every norm open subset \( V \) of \( Y \).

Then, there exists \( \tilde{x}^* \in L_X \) such that for every \( i \in I, \tilde{x}^*_i(\omega) \in \text{cl}(A_i(\omega, \tilde{x}^*))_{a_i(\tilde{x}^*)} \).

If \( |I|=1 \), we obtain the following result.

Theorem 5

Let \( (\Omega, F, \mu) \) be a complete finite separable measure space and let \( Y \) be a separable Banach space. Suppose that the following conditions are satisfied:

(a) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega))_z : \Omega \to 2^Y \) is a nonempty, convex, weakly compact-valued and integrably bounded correspondence.

(b) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega))_z : \Omega \to 2^Y \) is \( F \)-lower measurable.

(c) \( \tilde{x}^* \in L_X \) is weakly open in \( L_X \) such that for every \( \omega \in \Omega \), \( \tilde{x}^*_i(\omega) \in \text{cl}(A_i(\omega, \tilde{x}^*))_{a_i(\tilde{x}^*)} \).
(a) For each \((\omega, \bar{x}) \in \Omega \times L_X\), \((A(\omega, \bar{x}))_{a(\bar{x})}\) is convex and has a non-empty interior in the relative norm topology of \((X(\omega))_2\).

(b) the correspondence \((\omega, \bar{x}) \mapsto (A(\omega, \bar{x}))_{a(\bar{x})} : \Omega \times L_X \to 2^Y\) has measurable graph i.e. \(\{(\omega, \bar{x}, y) \in \Omega \times L_X \times Y : y \in (A(\omega, \bar{x}))_{a(\bar{x})}\} \in F \otimes \beta_{w}(L_X) \otimes \beta(Y)\) where \(\beta_w(L_X)\) is the Borel \(\sigma\)-algebra for the weak topology on \(L_X\) and \(\beta(Y)\) is the Borel \(\sigma\)-algebra for the norm topology on \(Y\).

(c) the correspondence \((\omega, \bar{x}) \mapsto (A(\omega, \bar{x}))_{a(\bar{x})}\) has weakly open lower sections, i.e., for each \(\omega \in \Omega\) and for each \(y \in Y\), the set \(\{(A(\omega, \bar{x}))_{a(\bar{x})}^{-1}(\omega, y) = \{\bar{x} \in L_X : y \in (A(\omega, \bar{x}))_{a(\bar{x})}\}\}\) is weakly open in \(L_X\);

(d) For each \(\omega \in \Omega\), \(\bar{x} \mapsto \text{cl}(A(\omega, \bar{x}))_{a(\bar{x})} : L_X \to 2^Y\) is upper semicontinuous in the sense that the set \(\{\bar{x} \in L_X : \text{cl}(A(\omega, \bar{x}))_{a(\bar{x})} \subseteq V\}\) is weakly open in \(L_X\) for every norm open subset \(V\) of \(Y\);

Then, there exists \(\bar{x}^* \in L_X\) such that \(\bar{x}^*(\omega) \in \text{cl}(A(\omega, \bar{x}^*))_{a(\bar{x}^*)}\).

**Theorem 7** Let \(I\) be a countable or uncountable set. Let \((\Omega, F, \mu)\) be a complete finite separable measure space and let \(Y\) be a separable Banach space. Suppose that the following conditions are satisfied:

For each \(i \in I\):

(a) \(X_i : \Omega \to F(Y)\) is such that \(\omega \mapsto (X_i(\omega))_{\omega} : \Omega \to 2^Y\) is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;

(b) \(X_i : \Omega \to F(Y)\) is such that \(\omega \mapsto (X_i(\omega))_{\omega} : \Omega \to 2^Y\) is \(F_i\)-measurable;

(c) For each \(\omega \in \Omega\), \(\bar{x} \mapsto (A_i(\omega, \bar{x}))_{a(\bar{x})} : L_X \to 2^Y\) has a measurable graph;

(d) For each \(\omega \in \Omega\), \(\bar{x} \mapsto (A_i(\omega, \bar{x}))_{a(\bar{x})} : L_X \to 2^Y\) is upper semicontinuous;

\(A.3)\)

\[\psi_i : \Omega \times L_X \times Y \to R \cup \{-\infty, +\infty\}\] is such that:

(a) \(\bar{x} \mapsto \{y \in Y : \psi_i(\omega, \bar{x}, y) > 0\} : L_X \to 2^Y\) is upper semicontinuous with compact values on \(L_X\) for each fixed \(\omega \in \Omega\);

(b) \(\bar{x}(\omega) \notin \{y \in Y : \psi_i(\omega, \bar{x}, y) > 0\}\) for each fixed \(\omega \in \Omega \times L_X\);

(c) For each \(\omega \in \Omega \times L_X\), \(\psi_i(\omega, \bar{x}, \cdot)\) is concave;

(d) For each \(\omega \in \Omega\), \(\{\bar{x} \in L_X : \alpha_i(\omega, \bar{x}) > 0\}\) is weakly open in \(L_X\), where \(\alpha_i : \Omega \times L_X \to R\) is defined by \(\alpha_i(\omega, \bar{x}) = \sup_{y \in (A_i(\omega, \bar{x}))_{a(\bar{x})}} \psi_i(\omega, \bar{x}, y)\) for each \((\omega, \bar{x}) \in \Omega \times L_X\);

\(A.4)\)

\[\{\omega(\bar{x}) : \alpha_i(\omega, \bar{x}) > 0\} \in \mathcal{F}_i \otimes B(L_X).\]

Then, there exists \(\bar{x}^* \in L_X\) such that for every \(i \in I\),

1) \(\bar{x}^*_i(\omega) \in (A_i(\omega, \bar{x}^*))_{a(\bar{x}^*)}\);

2) \(\sup_{y \in (A_i(\omega, \bar{x}^*))_{a(\bar{x}^*)}} \psi_i(\omega, \bar{x}^*, y) \leq 0\).
Proof. For every \( i \in I \), let \( P_i : \Omega \times L_X \to \mathcal{F}(Y) \) and \( p_i : L_X \to (0, 1] \) such that \( (P_i(\omega, \tilde{x}))_{p_i}(\tilde{z}) = \{ y \in Y : \psi_i(\omega, \tilde{x}, y) > 0 \} \) for each \( (\omega, \tilde{x}) \in \Omega \times L_X \).

We shall show that the abstract economy \( G = \{ (\Omega, \mathcal{F}, \mu), (X_i, F_i, A_i, P_i, a_i, p_i, z_i)_{i \in I} \} \) satisfies all the hypotheses of Theorem 1.

Suppose \( \omega \in \Omega \).

According to A3 a), we have that \( \tilde{x} \to (P_i(\omega, \tilde{x}))_{p_i}(\tilde{z}) : L_X \to 2^Y \) is upper semicontinuous with nonempty values and according to A3 b), \( \tilde{x}(\omega) \notin (P_i(\omega, \tilde{x}))_{p_i}(\tilde{z}) \) for each \( \tilde{x} \in L_X \). Assumption A3 c) implies that \( \tilde{x} \to (P_i(\omega, \tilde{x}))_{p_i}(\tilde{z}) : L_X \to 2^Y \) has convex values.

By the definition of \( x_i \), we note that \( \{ \tilde{x} \in L_X : (A_i(\omega, \tilde{x}))_{a_i}(\tilde{z}) \cap (P_i(\omega, \tilde{x}))_{p_i}(\tilde{z}) \neq \emptyset \} = \{ \tilde{x} \in L_X : \alpha_i(\omega, \tilde{x}) > 0 \} \) so that \( \{ \tilde{x} \in L_X : (A_i(\omega, \tilde{x}))_{a_i}(\tilde{z}) \cap (P_i(\omega, \tilde{x}))_{p_i}(\tilde{z}) \neq \emptyset \} \) is weakly open in \( L_X \) by A3 d).

According to A2 b) and A3 e), it follows that the correspondences \( (\omega, \tilde{x}) \to (A_i(\omega, \tilde{x}))_{a_i}(\tilde{z}) : \Omega \times L_X \to 2^Y \) and \( (\omega, \tilde{x}) \to (P_i(\omega, \tilde{x}))_{p_i}(\tilde{z}) : \Omega \times L_X \to 2^Y \) have measurable graphs.

Thus the Bayesian abstract fuzzy economy \( G = \{ (\Omega, \mathcal{F}, \mu), (X_i, F_i, A_i, P_i, a_i, b_i, z_i)_{i \in I} \} \) satisfies all the hypotheses of Theorem 2. Therefore, there exists \( \tilde{x}^* \in L_X \) such that for every \( i \in I \):

- \( \tilde{x}^*_i(\omega) \in (A_i(\omega, \tilde{x}^*))_{a_i}(\tilde{z}) \mu - a.e \)
- \( (A_i(\omega, \tilde{x}^*))_{a_i}(\tilde{z}) \cap (P_i(\omega, \tilde{x}^*))_{p_i}(\tilde{z}) = \emptyset \) \( \mu - a.e \)

that is, there exists \( \tilde{x}^* \in L_X \) such that for every \( i \in I \):

1. \( \tilde{x}^*_i(\omega) \in (A_i(\omega, \tilde{x}^*))_{a_i}(\tilde{z}) \mu - a.e \)
2. \( \sup_{y \in (A_i(\omega, \tilde{x}^*))_{a_i}(\tilde{z})} \psi_i(\omega, \tilde{x}^*, y) \leq 0 \).

If \( |I|=1 \), we obtain the following corollary.

**Corollary 3** Let \( (\Omega, \mathcal{F}, \mu) \) be a complete finite measurable space and let \( Y \) be a separable Banach space. Suppose that the following conditions are satisfied:

1. \( A.1 \)

- \( a \) \( X : \Omega \to \mathcal{F}(Y) \) is such that \( \omega \to (X(\omega))_z : \Omega \to 2^Y \) is a nonempty, convex, weakly compact-valued and integrally bounded correspondence.

2. \( b \) \( X : \Omega \to \mathcal{F}(Y) \) is such that \( \omega \to (X(\omega))_z : \Omega \to 2^Y \) is \( F \)-lower measurable;

2. \( A.2 \)

- \( a \) For each \( (\omega, \tilde{x}) \in \Omega \times L_X \), \( (A(\omega, \tilde{x}))_{a(z)} \) is non-empty convex and compact;

3. \( b \) For each \( \tilde{x} \in L_X \), the correspondence \( \omega \to (A(\omega, \tilde{x}))_{a(z)} : \Omega \to 2^Y \) has a measurable graph;

4. \( c \) For each \( \omega \in \Omega \), \( \tilde{x} \to (A(\omega, \tilde{x}))_{a(z)} : L_X \to 2^Y \) is upper semicontinuous;

3. \( A.3 \)

- \( \psi : \Omega \times L_X \times Y \to R \cup \{-\infty, +\infty\} \) is such that:

  - \( a \) \( \tilde{x} \to \{ y \in Y : \psi(\tilde{x}, \tilde{x}, y) > 0 \} : L_X \to 2^Y \) is upper semicontinuous with weakly compact values on \( L_X \) for each fixed \( \omega \in \Omega \).
(b) \( \tilde{x}(\omega) \notin \{ y \in Y : \psi(\omega, \tilde{x}, y) > 0 \} \) for each fixed \((\omega, \tilde{x}) \in \Omega \times L_X; \)
(c) for each \((\omega, \tilde{x}) \in \Omega \times L_X, \psi(\omega, \tilde{x}, \cdot)\) is concave;
(d) for each \(\omega \in \Omega, \{ \tilde{x} \in L_X : \alpha(\omega, \tilde{x}) > 0 \}\) is weakly open in \(L_X;\)

where \(\alpha : \Omega \times L_X \to \mathbb{R} \) is defined by \(\alpha(\omega, \tilde{x}) = \sup_{y \in (A(\omega, \tilde{x}))_{a(\tilde{x})}} \psi(\omega, \tilde{x}, y)\)
for each \((\omega, \tilde{x}) \in \Omega \times L_X;\)

(e) \(\{(\omega, \tilde{x}) : \alpha(\omega, \tilde{x}) > 0\} \in \mathcal{F} \otimes B(L_X).\)

Then, there exists \(\tilde{x}^* \in L_X\) such that:

(i) \(\tilde{x}^*(\omega) = (A(\omega, \tilde{x}^*))_{a(\tilde{x}^*)};\)
(ii) \(\sup_{y \in (A(\omega, \tilde{x}^*))_{a(\tilde{x}^*)}} \psi(\omega, \tilde{x}^*, y) \leq 0.\)

As a consequence of the Theorem 2, we prove the following Tan and Yuan’s type (1995) random quasi-variational inequality with random fuzzy mappings.

**Theorem 8** Let \((\Omega, F, \mu)\) be a complete finite separable measure space and let \(Y\) be a separable Banach space. Suppose that the following conditions are satisfied:

For each \(i \in I:\)

**A.1**

(a) \(X_i : \Omega \to F(Y)\) is such that \(\omega \to (X_i(\omega))_{z_i} : \Omega \to 2^Y\) is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;

(b) \(X_i : \Omega \to F(Y)\) is such that \(\omega \to (X_i(\omega))_{z_i} : \Omega \to 2^Y\) is \(F_i\)-lower measurable;

**A.2**

(a) For each \((\omega, \tilde{x}) \in \Omega \times L_X, (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})}\) is nonempty convex and weakly compact;

(b) For each \(\tilde{x} \in L_X, the correspondence \omega \to (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} : \Omega \to 2^Y\) has a measurable graph;

(c) For each \(\omega \in \Omega, \tilde{x} \to (A_i(\omega, \tilde{x}))_{a_i(\tilde{x})} : L_X \to 2^Y\) is upper semicontinuous;

**A.3**

\(G_i : \Omega \times Y \to F(Y')\) and \(g_i : Y \to (0, 1]\) are such that:

(a) For each fixed \((\omega, y) \in \Omega \times Y, \tilde{x} \to \{ y \in Y : \sup_{u \in (G_i(\omega, y))_{g_i(\omega)}} \text{Re}(u, \tilde{x}_i(\omega) - y) > 0\} : L_X \to 2^Y\) is upper semicontinuous with compact values;

(b) for each fixed \(\omega \in \Omega, the set\)
\[
\{ \tilde{x} \in L_X : \sup_{u \in (G_i(\omega, y))_{g_i(\omega)}} \text{Re}(u, \tilde{x}_i(\omega) - y) > 0 \} \text{ is weakly open in } L_X
\]

(c) \(\{ (\omega, \tilde{x}) : \sup_{u \in (G_i(\omega, y))_{g_i(\omega)}} \text{Re}(u, \tilde{x}_i(\omega) - y) > 0 \} \in \mathcal{F} \otimes B(L_X).\)

**A.4**

\(H_i : \Omega \times Y \to F(Y')\) and \(h_i : Y \to (0, 1]\) are such that:

(a) For each fixed \((\omega, y) \in \Omega \times Y, (H_i(\omega, y))_{h_i(y)} \subseteq (G_i(\omega, y))_{g_i(y)};\)

(b) for each \(\omega \in \Omega, y \to (H_i(\omega, y))_{h_i(y)} : Y \to 2^{Y'}\) is monotone with non-empty values;
(c) for each \( \omega \in \Omega \), \( y \to (H_1(\omega, y))_{h_1(y)} : L \cap Y \to 2^{Y'} \) is lower semicontinuous from the relative topology of \( Y \) into the weak*–topology \( \sigma(Y', Y) \) of \( Y' \) for each one-dimensional flat \( L \subset Y \).

Then, there exists \( \tilde{x}^* \in L_X \) such that for every \( i \in I \):

i) \( \tilde{x}^*_i(\omega) \in (A_i(\omega, \tilde{x}^*))_{a_i(\tilde{x}^*)} \);

ii) \( \sup_{u \in H_i(\omega, \tilde{x}^*_i(\omega))} Re(u, \tilde{x}^*_i(\omega) - y) \leq 0 \) for all \( y \in (A_i(\omega, \tilde{x}^*))_{a_i(\tilde{x}^*)} \).

Proof. Let us define \( \psi_i : \Omega \times L_X \times Y \to R \cup \{-\infty, +\infty\} \) by

\[ \psi_i(\omega, x, y) = \sup_{u \in (G_i(\omega, y))_{a_i(y)}} Re(u, x - y) \]

for each \( (\omega, x, y) \in \Omega \times L_X \times Y \).

We have that \( \tilde{x}^*_i(\omega) \notin \{ y \in \Omega : \psi_i(\omega, x, y) > 0 \} \) for each fixed \( (\omega, \tilde{x}^*_i) \in \Omega \times L_X \) and, as a consequence of assumption A3 b), it follows that for each \( (\omega, \tilde{x}^*_i) \in \Omega \times L_X \), \( \psi_i(\omega, x, y) \) is concave.

All the hypotheses of Theorem 2 are satisfied. According to Theorem 2, there exists \( \tilde{x}^*_i \in L_X \) such that \( \tilde{x}^*_i(\omega) \in \text{cl}(A_i(\omega, \tilde{x}^*))_{a_i(\tilde{x}^*)} \) for every \( i \in I \) and

\[ \sup_{u \in A_i(\omega, \tilde{x}^*)} Re(u, \tilde{x}^*_i(\omega) - y) \leq 0 \] for every \( i \in I \).

Finally, we prove that

\[ \sup_{u \in A_i(\omega, \tilde{x}^*)} Re(u, \tilde{x}^*_i(\omega) - y) \leq 0 \]

for every \( \lambda \in [0, 1] \).

For each \( \lambda \in [0, 1] \), we have that

\[ t(\sup_{u \in H_i(\omega, z^*_i(\omega))} Re(u, \tilde{x}^*_i(\omega) - y)) = \sup_{u \in H_i(\omega, z^*_i(\omega))} t Re(u, \tilde{x}^*_i(\omega) - y) = \sup_{u \in H_i(\omega, z^*_i(\omega))} Re(u, \tilde{x}^*_i(\omega) - z^*_i(\omega)) \leq 0. \]

It follows that for each \( \lambda \in [0, 1] \),

\[ \sup_{u \in H_i(\omega, z^*_i(\omega))} Re(u, \tilde{x}^*_i(\omega) - y) \leq 0. \]

Now, we are using the lower semicontinuity of \( y \to (H_i(\omega, y))_{h_i(y)} : L \cap Y \to 2^{Y'} \) in order to show the conclusion. For each \( z_0 \in (H_i(\omega, \tilde{x}^*_i(\omega)))_{h_i(\tilde{x}^*_i(\omega))} \) and \( e > 0 \) we let us consider \( U^i_{z_0} \), the neighborhood of \( z_0 \) in the topology \( \sigma(Y', Y) \), defined by \( U^i_{z_0} := \{ z \in Y' : |Re(z_0 - z, \tilde{x}^*_i(\omega) - y) | < e \} \).

As \( y \to (H_i(\omega, y))_{h_i(y)} : L \cap Y \to 2^{Y'} \) is lower semicontinuous, where \( L = \{ z^*_i(\omega) : \lambda \in [0, 1] \} \) and \( U^i_{z_0} \cap (H_i(\omega, \tilde{x}^*_i(\omega)))_{h_i(\tilde{x}^*_i(\omega))} \neq \emptyset \), there exists a non-empty neighborhood \( N(\tilde{x}^*_i(\omega)) \) of \( \tilde{x}^*_i(\omega) \) in \( L \) such that for each \( z \in N(\tilde{x}^*_i(\omega)) \), we have that \( U^i_{z_0} \cap (H_i(\omega, z))_{h_i(z)} \neq \emptyset \). Then there exists \( \delta \in (0, 1] \), \( t \in (0, \delta) \) and \( u \in (H_i(\omega, z^*_i(\omega)))_{h_i(z^*_i(\omega))} \cap U^i_{z_0} \neq \emptyset \) such that

\[ Re(z_0 - u, \tilde{x}^*_i(\omega) - y) < e. \]

Therefore, \( Re(z_0, \tilde{x}^*_i(\omega) - y) < Re(u, \tilde{x}^*_i(\omega) - y) + e. \)

It follows that
\[ \text{Re}(z_0, \tilde{x}_i^*(\omega) - y) < \text{Re}(a_i, \tilde{x}_i^*(\omega) - y) + e < e. \]

The last inequality comes from (2). Since \( e > 0 \) and \( z_0 \in (H_i(\omega, \tilde{x}^*(\omega)))_{h_i(\tilde{x}^*)} \) have been chosen arbitrarily, the next relation holds:

\[ \text{Re}(z_0, \tilde{x}_i^*(\omega) - y) < 0. \]

Hence, for each \( i \in I \), we have that \( \sup_{a_i \in (H_i(\omega, \tilde{x}^*(\omega)))_{h_i(\tilde{x}^*)}} \text{Re}(z_0, \tilde{x}_i^*(\omega) - y) \leq 0 \) for every \( y \in (A_i(\omega, \tilde{x}^*))_{a_i(\tilde{x}^*)} \).

If \( |I| = 1 \), we obtain the following corollary.

**Corollary 4** Let \((\Omega, F, \mu)\) be a complete finite separable measure space and let \( Y \) be a separable Banach space. Suppose that the following conditions are satisfied:

\( A.1) \)

\( a) \) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega)_z : \Omega \to 2^Y \) is a nonempty, convex, weakly compact-valued and integrably bounded correspondence.

\( b) \) \( X : \Omega \to F(Y) \) is such that \( \omega \to (X(\omega)_z : \Omega \to 2^Y \) is \( F \)-lower measurable.

\( A.2) \)

\( a) \) For each \( (\omega, \tilde{x}) \in \Omega \times L_X \), \((A(\omega, \tilde{x}))_{a(\tilde{x})} \) is non-empty convex and compact.

\( b) \) For each \( \tilde{x} \in L_X \), the correspondence \( \omega \to (A(\omega, \tilde{x}))_{a(\tilde{x})} : \Omega \to 2^Y \) has a measurable graph.

\( c) \) For each \( \omega \in \Omega \), \( \tilde{x} \rightarrow (A(\omega, \tilde{x}))_{a(\tilde{x})} : L_X \rightarrow 2^Y \) is upper semicontinuous.

\( A.3) \)

\( G : \Omega \times Y \to F(Y') \) and \( g : Y \to (0,1] \) are such that:

\( a) \) For each fixed \( (\omega, y) \in \Omega \times Y \), \( \tilde{x} \rightarrow \{ y \in Y : \sup_{u \in (G(\omega, y)))_{g(\omega)}} \text{Re}(u, \tilde{x}(\omega) - y) > 0 \} \) is upper semicontinuous with compact values;

\( b) \) For each fixed \( \omega \in \Omega \), the set \( \{ \tilde{x} \in L_X : \sup_{y \in (A(\omega, \tilde{x}))_{a(\tilde{x})}} \sup_{u \in (G(\omega, y)))_{g(\omega)}} \text{Re}(u, \tilde{x}(\omega) - y) > 0 \} \) is weakly open in \( L_X \);

\( c) \) \( \{ (\omega, \tilde{x}) : \sup_{u \in (G(\omega, y)))_{g(\omega)}} \text{Re}(u, \tilde{x}(\omega) - y) > 0 \} \in F \otimes B(L_X); \)

\( A.4) \)

\( H : \Omega \times Y \to F(Y') \) and \( h : Y \to (0,1] \) are such that:

\( a) \) For each fixed \( (\omega, y) \in \Omega \times Y \), \((H(\omega, y))_{h(y)} \) \( \subset \ (G(\omega, y)))_{g(\omega)} \);

\( b) \) for each \( \omega \in \Omega \), \( y \rightarrow (H(\omega, y))_{h(y)} : Y \rightarrow 2^{Y'} \) is monotone with non-empty values;

\( c) \) for each \( \omega \in \Omega \), \( y \rightarrow (H(\omega, y))_{h(y)} : L \cap Y \rightarrow 2^{Y'} \) is lower semicontinuous from the relative topology of \( Y \) into the weak \( ^* \)-topology \( \sigma(Y', Y) \) of \( Y \) for each one-dimensional flat \( L \subset Y \).

Then, there exists \( \tilde{x}^* \in L_X \) such that:

\( i) \) \( \tilde{x}^*(\omega) \in (A(\omega, \tilde{x}^*))_{a(\tilde{x}^*)} ; \)

\( ii) \) \( \sup_{u \in (H(\omega, \tilde{x}^*(\omega)))_{h(\tilde{x}^*)}} \text{Re}(u, \tilde{x}^*(\omega) - y) \leq 0 \) for all \( y \in (A(\omega, \tilde{x}^*))_{a(\tilde{x}^*)} . \)

We obtain the following random fixed point theorem as a corollary.
Theorem 9 Let $(\Omega, F, \mu)$ be a complete finite separable measure space and let $Y$ be a separable Banach space. Suppose that the following conditions are satisfied:

For each $i \in I$:
A.1) 
(a) $X_i : \Omega \to F(Y)$ is such that $\omega \to (X_i(\omega))_z : \Omega \to 2^Y$ is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;
(b) $X_i : \Omega \to F(Y)$ is such that $\omega \to (X_i(\omega))_z : \Omega \to 2^Y$ is $F_i$-lower measurable;
A.2) 
(a) For each $(\omega, \tilde{x}) \in \Omega \times L_X$, $(A_i(\omega, \tilde{x}))_{a(\tilde{x})} : \Omega \to 2^Y$ has a measurable graph;
(b) For each $\tilde{x} \in L_X$, the correspondence $\omega \to (A_i(\omega, \tilde{x}))_{a(\tilde{x})} : \Omega \to 2^Y$ is upper semicontinuous;
Then, there exists $\tilde{x}^* \in L_X$ such that for every $i \in I$, $\tilde{x}^*_i(\omega) \in (A_i(\omega, \tilde{x}^*))_{a(\tilde{x}^*)}$.

If $|I| = 1$, we obtain the following result.

Theorem 10 Let $(\Omega, F, \mu)$ be a complete finite separable measure space and let $Y$ be a separable Banach space. Suppose that the following conditions are satisfied:

A.1) 
(a) $X : \Omega \to F(Y)$ is such that $\omega \to (X(\omega))_z : \Omega \to 2^Y$ is a nonempty, convex, weakly compact-valued and integrably bounded correspondence;
(b) $X : \Omega \to F(Y)$ is such that $\omega \to (X(\omega))_z : \Omega \to 2^Y$ is $F$-lower measurable;
A.2) 
(a) For each $(\omega, \tilde{x}) \in \Omega \times L_X$, $(A(\omega, \tilde{x}))_{a(\tilde{x})} : \Omega \to 2^Y$ has a measurable graph;
(b) For each $\tilde{x} \in L_X$, the correspondence $\omega \to (A(\omega, \tilde{x}))_{a(\tilde{x})} : \Omega \to 2^Y$ is upper semicontinuous;
Then, there exists $\tilde{x}^* \in L_X$ such that for every $i \in I$, $\tilde{x}^*_i(\omega) \in (A(\omega, \tilde{x}^*))_{a(\tilde{x}^*)}$.

5 APPENDIX

The results below have been used in the proof of our theorems. For more details and further references see the paper quoted.
Theorem 11 (Projection theorem). Let \((\Omega, F, \mu)\) be a complete, finite measure space, and \(Y\) be a complete separable metric space. If \(H\) belongs to \(F \otimes \mathcal{B}(Y)\), its projection \(\text{Proj}_\Omega(H)\) belongs to \(F\).

Theorem 12 (Aumann measurable selection theorem [30]). Let \(F\) be a separable Banach space. Let \(\Omega\) be a measurable space, i.e., \(Y\) be a complete, separable metric space and \(G\) and \(\beta\) be a nonempty valued correspondence. Then there is a measurable function \(f : \Omega \to Y\) such that \(f(\omega) \in T(\omega)\) \(\mu\) - a.e.

Theorem 13 (Diestel's Theorem [30, Theorem 3.1]). Let \((\Omega, F, \mu)\) be a complete finite measure space, \(X\) be a separable Banach space and \(T : \Omega \to 2^Y\) be an integrably bounded, convex, weakly compact and nonempty valued correspondence. Then \(S_T = \{x \in L_1(\mu, Y) : x(\omega) \in T(\omega) \mu\text{-a.e.}\}\) is weakly compact in \(L_1(\mu, Y)\).

Theorem 14 (Carathéodory-type selection theorem [15]). Let \((\Omega, F, \mu)\) be a complete measure space, \(Z\) be a complete separable metric space and \(Y\) a separable Banach space. Let \(X : \Omega \to 2^Y\) be a correspondence with a measurable graph, i.e., \(G_X \in F \otimes \mathcal{B}(Y)\) and let \(T : \Omega \times Z \to 2^Y\) be a convex valued correspondence (possibly empty) with a measurable graph, i.e., \(G_T \in F \otimes \mathcal{B}(Z) \otimes \mathcal{B}(Y)\) where \(\mathcal{B}(Y)\) and \(\mathcal{B}(Z)\) are the Borel \(\sigma\) - algebras of \(Y\) and \(Z\), respectively.

Suppose that:
(a) for each \(\omega \in \Omega\), \(T(\omega, x) \subseteq X(\omega)\) for all \(x \in Z\).
(b) for each \(\omega \in \Omega\), \(T(\omega, \cdot)\) has open lower sections in \(Z\), i.e., for each \(\omega \in \Omega\) and \(y \in Y\), \(T^{-1}(\omega, y) = \{x \in Z : y \in T(\omega, x)\}\) is open in \(Z\).
(c) for each \((\omega, x) \in \Omega \times Z\), if \(T(\omega, x) \neq \emptyset\), then \(T(\omega, x)\) has a non-empty interior in \(X(\omega)\).

Let \(U = \{(\omega, x) \in \Omega \times Z : T(\omega, x) \neq \emptyset\}\) and for each \(x \in Z\), \(U_x^\omega = \{\omega \in \Omega : (\omega, x) \in U\}\) and for each \(\omega \in \Omega\), \(U_x^\omega = \{x \in Z : (\omega, x) \in U\}\). Then for each \(x \in Z\), \(U_x^\omega\) is a measurable set in \(\Omega\) and there exists a Carathéodory-type selection from \(T|_U\), i.e., there exists a function \(f : U \to Y\) such that \(f(\omega, x) \in T(\omega, x)\) for all \((\omega, x) \in U\), for each \(x \in Z\), \(f(\cdot, x)\) is measurable on \(U_x^\omega\) and for each \(\omega \in \Omega\), \(f(\omega, \cdot)\) is continuous on \(U_x^\omega\). Moreover, \(f(\cdot, \cdot)\) is jointly measurable.

Theorem 15 (U. s. c. Lifting Theorem. [30]). Let \(Y\) be a separable space, \((\Omega, F, \mu)\) be a complete finite measure space and \(X : \Omega \to 2^Y\) be an integrably bounded, nonempty, convex valued correspondence such that for all \(\omega \in \Omega\), \(X(\omega)\) is a weakly compact, convex subset of \(Y\). Denote by \(S_X\) the set \(\{x \in L_1(\mu, Y) : x(\omega) \in X(\omega) \mu\text{-a.e.}\}\). Let \(T : \Omega \times S_X \to 2^Y\) be a nonempty, closed, convex valued correspondence such that \(T(\omega, x) \subseteq X(\omega)\) for all \((\omega, x) \in \Omega \times S_X^\omega\). Assume that for each fixed \(x \in S_X\), \(T(\cdot, x)\) has a measurable graph and that for each fixed \(\omega \in \Omega\), \(T(\omega, \cdot) : S_X \to 2^Y\) is u.s.c.
in the sense that the set \( \{ x \in S_X : T(\omega, x) \subset V \} \) is weakly open in \( S_X \) for every norm open subset \( V \) of \( Y \). Define the correspondence \( \Phi : S_X \to 2^{S_X} \) by

\[
\Phi(x) = \{ y \in S_X : y(\omega) \in T(\omega, x) \mu - a.e. \}. 
\]

Then \( \Phi \) is weakly u.s.c., i.e., the set \( \{ x \in S_X : \Phi(x) \subset V \} \) is weakly open in \( S_X \) for every weakly open subset \( V \) of \( S_X \).

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