GRACEFUL NUMBERS

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We construct a labeled graph $D(n)$ that reflects the structure of divisors of a given natural number $n$. We define the concept of graceful numbers in terms of this associated graph and find the general form of such a number. As a consequence, we determine which graceful numbers are perfect.

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1. Introduction. In [2], Gallian presented a detailed survey of various types of graph labeling, the two best known being graceful and harmonious. Recall that a graph $G$ with $q$ edges is called graceful if one can label its vertices with distinct numbers from the set $\{0, 1, \ldots, q\}$ and mark the edges with differences of the labels of the end vertices in such a way that the resulting edge labels are distinct. A number of interesting results on graceful and graceful-like labelings are obtained in [1, 3, 4] and some other works. In this note, we give a description of natural numbers whose associated graph of divisors satisfies certain graceful-like conditions. For any natural number $n$, we construct a labeled graph $D(n)$ that reflects the structure of divisors of $n$. We define the concept of graceful number in terms of this associated graph and find the general form of such a number. As a consequence, we determine which graceful numbers are perfect.

2. Main results. Given a natural number $n$ one can generate a graph $D(n)$ that reflects the structure of divisors of $n$ as follows. The vertices of the graph represent all the divisors of the number $n$, each vertex is labeled by a certain divisor. (In what follows, we refer to the vertex of the graph $D(n)$ with label $k$ as the “vertex $k$.”) If $r$ and $s$ are two divisors of $n$ and $r > s$, then there is an edge between the vertices $s$ and $r$ if and only if $s$ divides $r$ and the ratio $r/s$ is a prime number. As in the theory of graceful graphs, we label such an edge by the difference $r - s$ of the labels of its vertices. In what follows, the sum of the labels of all edges of the graph $D(n)$ is denoted by $SD(n)$ while $SD(n)$ denotes the sum of labels of all edges of $D(n)$ except the edges terminating at $n$. (Clearly, if $n = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$ is the prime factorization of a natural number $n$, then $SD(n) = \sum_{i=1}^{r} (p^i - p^{i-1})$.)

Example 2.1. It is easy to see that if $n = p^r$, where $p$ is a prime number and $r$ is any positive integer, then $SD(n) = \sum_{i=1}^{r} (p^i - p^{i-1}) = p^r - 1$ and $SD(n) = \sum_{i=1}^{r-1} (p^i - p^{i-1}) = p^{r-1} - 1$, so that $SD(n) < n$. The graph $D(n)$ is shown in Figure 2.1.
The following example shows that there are numbers \( n \) such that \( SD(n) > n \), as well as numbers that satisfy the condition \( SD(n) = n \).

**Example 2.2.** Let \( n = 24 \) and \( m = 12 \). Then \( SD(n) = (12 - 6) + (12 - 4) + (8 - 4) + (6 - 3) + (6 - 2) + (4 - 2) + (3 - 1) + (2 - 1) = 30 > n \) and \( SD(m) = (6 - 3) + (6 - 2) + (4 - 2) + (3 - 1) + (2 - 1) = 12 = m \).

**Definition 2.3.** A natural number \( n \) is called **graceful** if \( SD(n) = n \).

In order to obtain the description of graceful numbers, we first find the value of \( SD(n) \) when \( n \) is a product of powers of two different prime numbers.

**Example 2.4.** Let \( n = p^r q^s \) where \( p \) and \( q \) are different prime numbers, \( r \geq 1 \), and \( s \geq 1 \). In this case the graph \( D(n) \) is of the form

\[
\begin{align*}
q^s & \quad q' \quad p q^s \quad \ldots \quad (q-1) \quad p (q-1) \\
p & \quad \ldots \quad p (q-1) \\
1 & \quad \ldots \quad p (q-1) \\
p q & \quad \ldots \quad p^r \\
p q^s & \quad \ldots \quad p^r (q-1) \\
p^r & \quad \ldots \quad p^r (q-1) \\
p^r q & \quad \ldots \quad p^r (q-1) \\
p^r q^s & \quad \ldots \quad p^r (q-1)
\end{align*}
\]

and \( \overline{SD}(n) = \sum_{i=0}^{r} \sum_{j=1}^{s} (p^i q^j - p^i q^{j-1}) + \sum_{i=1}^{r} \sum_{j=0}^{s} (p^i q^j - p^{i-1} q^j) = \sum_{i=0}^{r} p^i (q^s - 1) + \sum_{j=0}^{s} q^j (p^r - 1) \) (the first sum corresponds to the differences of the consecutive divisors of \( n \) when the exponent of \( q \) decreases, and the second sum takes care about the differences of consecutive divisors of \( n \) when the exponent of \( p \) decreases). Thus,

\[
\overline{SD}(n) = (q^s - 1) \sum_{i=0}^{r} p^i + (p^r - 1) \sum_{j=0}^{s} q^j = (q^s - 1) \frac{p^{r+1} - 1}{p - 1} + (p^r - 1) \frac{q^{s+1} - 1}{q - 1},
\]

(2.1)
so that
\[ SD(n) = \overline{SD}(n) - \left( \left( n - \frac{n}{p} \right) + \left( n - \frac{n}{q} \right) \right). \]  
(2.2)

It follows from formulas (2.1) and (2.2) that a number \( n = p^r q^s \) (\( p \) and \( q \) are prime, \( r \geq 1 \), and \( s \geq 1 \)) is graceful if and only if \( p = 2 \) and \( s = 1 \), that is, \( n = 4q \) for some odd prime number \( q \).

Indeed, equality \( SD(n) = n \) can hold only for even numbers \( n \) (if \( n \) is odd, then (2.1) shows that \( SD(n) \) is even, whence \( SD(n) \neq n \). If \( n = 2^r q^s \), where \( r \geq 2 \), \( s \geq 2 \), then
\[
SD(n) - n = (2^r - 1) \sum_{i=0}^{s} q^i + (q^s - 1)(2^{r+1} - 1) - 2^{r+1} q^s + 2^{r-1} q^s + 2^{r-1} q^{s-1} - 2^r q^s \\
> (2^{r-1} - 2) q^s + (2^{r+1} q^{s-1} - q^{s-1} - 2^{r-1}) + (2^r - 1) \sum_{i=0}^{s-2} q^i \\
> 0,
\]
so that \( SD(n) > n \). Finally, if \( n = 2^r q \) (\( r \geq 1 \)), then \( SD(n) - n = (q - 1)(2^{r+1} - 1) + (2^r - 1)(q + 1) - 2^{r+1} q + 2^{r-1} q + 2^{r-2} q = q(2^{r-1} - 2) \), so that \( SD(2^r q) = 2^r q \) if and only if \( r = 2 \). Thus, for any two different prime numbers \( p \) and \( q \), and for any two nonnegative integers \( r \) and \( s \), the number \( p^r q^s \) is graceful if and only if \( p = 2 \), \( r = 2 \), and \( s = 1 \).

Now, we generalize formula (2.1) to the case of arbitrary number \( n \). More precisely, we show that if \( n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \) is a prime decomposition of a positive integer \( n \) (\( p_1, \ldots, p_k \) are different primes and \( r_1, \ldots, r_k \) are positive integers), then
\[
\overline{SD}(n) = \sum_{i=1}^{k} (p_i^{r_i} - 1) \prod_{1 \leq j \leq k, j \neq i} \left( \frac{p_j^{r_j+1} - 1}{p_j - 1} \right).
\]
(2.4)

We proceed by induction on \( n \). We have seen that the formula is true if \( n \) is a power of a prime number or a product of two powers of primes. In order to perform the step of induction, notice that
\[
\overline{SD}(n) = \overline{SD} \left( \frac{n}{p_1} \right) + (p_1^{r_1} - p_1^{r_1-1}) \sum_{i=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} p_2^{i_2} \cdots p_k^{i_k} + p_1^{r_1} \overline{SD} \left( \frac{n}{p_1} \right).
\]
(2.5)

Applying the inductive hypothesis and taking into account that
\[
\overline{SD}(n) = \sum_{i_2=0}^{r_2} \cdots \sum_{i_k=0}^{r_k} p_2^{i_2} \cdots p_k^{i_k} = \prod_{j=2}^{r_j} \sum_{i_1=0}^{r_1} p_1^{i_1} = \prod_{j=2}^{r_j} \left( \frac{p_j^{r_j+1} - 1}{p_j - 1} \right),
\]
(2.6)
we obtain that
\[
\overline{SD}(n) = \left( p_1^{r_1} - 1 \right) \prod_{j=2}^{r_j} \left( \frac{p_j^{r_j+1} - 1}{p_j - 1} \right) + \sum_{i=2}^{r_2} \cdots \sum_{i_k=2}^{r_k} \left( p_i^{r_i} - 1 \right) \prod_{1 \leq j \leq k, j \neq i} \left( \frac{p_j^{r_j+1} - 1}{p_j - 1} \right) \\
+ \left( p_1^{r_1} - p_1^{r_1-1} \right) \prod_{j=2}^{r_j} \left( \frac{p_j^{r_j+1} - 1}{p_j - 1} \right) + p_1^{r_1} \sum_{i=2}^{r_2} \cdots \sum_{i_k=2}^{r_k} \left( p_i^{r_i} - 1 \right) \prod_{1 \leq j \leq k, j \neq i} \left( \frac{p_j^{r_j+1} - 1}{p_j - 1} \right).
\]
\[ SD(n) = \sum_{i=1}^{k} (p_i^{r_i} - 1) \prod_{1 \leq j \leq k, j \neq i} \left( \frac{p_j^{r_j+1} - 1}{p_j - 1} \right) \]

so formula (2.4) is proved.

Now, formulas (2.2) and (2.4) imply that

\[ SD(n) = \sum_{i=1}^{k} (p_i^{r_i} - 1) \prod_{1 \leq j \leq k, j \neq i} \left( \frac{p_j^{r_j+1} - 1}{p_j - 1} \right) \]

\[ = \sum_{i=1}^{k} (p_i^{r_i} - 1) \prod_{1 \leq j \leq k, j \neq i} \left( \frac{p_j^{r_j+1} - 1}{p_j - 1} \right), \]

(2.7)

Formula (2.8) shows, in particular, that if a number \( n \) is odd, then \( SD(n) \) is even (it is easily seen that both sums in the right side of the formula are even if \( n \) is odd). Therefore, every graceful number must be even, that is,

\[ n = 2^r q_1^{s_1} \cdots q_m^{s_m} \quad (2.9) \]

for some odd primes \( q_1, \ldots, q_m \) \((m \geq 1, s_i \geq 1 \text{ for } i = 1, \ldots, m)\). As we have seen, if \( m = 1 \), then the number \( n \) is graceful if and only if \( s_1 = 1 \) and \( r = 2 \), that is, \( n = 4q_1 \).

We show that if \( m \geq 2 \), then \( SD(n) > n \), so the only graceful numbers are the numbers of the form \( 4q \) where \( q \) is an odd prime.

First of all, notice that \( SD(2^r q_1^{s_1}) \geq 2^r q_1^{s_1} \) for \( r \geq 1, \ s \geq 2 \) (see Example 2.4) and \( SD(2q_1 q_2) \geq 2q_1 q_2 \) for any two different primes \( q_1 \) and \( q_2 \) (applying formula (2.1)) we obtain that \( SD(2q_1 q_2) = (q_1 + 1)(q_2 + 1) + (q_1 - 1)(q_2 + 1) + (q_2 - 1)(q_1 + 1) - 6q_1 q_2 + 3q_1 + 2q_1 + 2q_2 = 2q_1 q_2 + 3(q_1 + q_2) - 5 > 2q_1 q_2 \). Therefore, in order to prove that \( SD(n) > n \) for any number \( n \) of the form (2.9) with \( m \geq 2 \), it is sufficient to prove that \( SD(n) > q_m^{s_m} SD(n/q_m^{s_m}) \). But the last inequality is a consequence of equality (2.5).

Indeed,

\[ SD(n) = SD(n) - n = SD\left( \frac{n}{q_m} \right) + q_m^{s_m} - q_m^{s_m-1} \sum_{i=0}^{s_1} \sum_{t_1=0}^{s_2} \cdots \sum_{t_{m-1}=0}^{s_m-1} 2^i q_1^{t_1} \cdots q_m^{t_{m-1}} \]

\[ + q_m^{s_m} SD\left( \frac{n}{q_m} \right) - n > q_m^{s_m} SD\left( \frac{n}{q_m} \right) - n = q_m^{s_m} SD\left( \frac{n}{q_m} \right). \]

(2.10)

We arrive at the following result.

**Theorem 2.5.** A natural number \( n \) is graceful if and only if \( n = 4q \) where \( q \) is an odd prime.

Recall that a positive integer \( m \) is called a perfect number if it is equal to the sum of all its proper divisors (i.e., of all divisors of \( m \) except of the number \( m \) itself). It is known (cf. [4], Theorem 5.10) that every even perfect number is of the form \( 2^{k-1}(2^k - 1) \), where the number \( 2^k - 1 \) is prime. Thus, our theorem implies the following result.

**Corollary 2.6.** The only perfect graceful number is 28.
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