1. INTRODUCTION

Compelling evidence has been accumulated that individual shell-type supernova remnant (SNR) shocks accelerate charged particles, i.e., electrons and probably ions, up to energies at least of the order of $10^{13} - 10^{14}$ eV (see, e.g., Cassiopeia A: Aharonian et al. 2001; RX J1713.7–3946: Aharonian et al. 2004; Tycho’s SNR: Acciari et al. 2011). Charged particles are likely to be accelerated by two simultaneous mechanisms: the so-called Fermi first-order, i.e., repeated shock crossing of the particle (Axford et al. 1977; Bell 1978a, 1978b; Blandford & Ostriker 1978; Krymskii 1977) and drift along the shock surface (Jokipii 1982, 1987). A magnetic field at the shock far exceeding the theoretically predicted shock-compressed field has been inferred from the detection of non-thermal X-ray rims (Vink & Laming 2003; Bamba et al. 2004), rapid timescale variability of X-ray hot spots (Uchiyama et al. 2007), and γ-ray emission in extended regions (Acciari et al. 2011). The X-ray rims could be due to damping of the magnetic field (Pohl et al. 2005) rather than to synchrotron emission, although corresponding narrow filaments in the radio emission have not been observed (Rothenflug et al. 2004; see, however, high-resolution radio images in Dyer et al. 2009). Magnetic field amplification might also be relevant to in situ measurements of the plasma downstream of the solar-wind termination shock (Burlaga et al. 2007), where fluctuations have been measured of the same order as the mean, or to radio observations of megaparsec-scale shocks at the edge of galaxy clusters (Brüggen et al. 2012). Whether or not such a magnetic field amplification in SNRs is to be associated with energetic particles at the shock is still the subject of controversy.

So far, SNR observations have not been able to rule out either of the following two mechanisms of field amplification: (1) microscopic plasma instability generated by the cosmic-ray current flowing upstream of the shock and therein exciting non-resonant magnetic modes (Bell 2004) and (2) macroscopic turbulent fluid motion downstream of the shock seeded by inhomogeneities of the upstream medium triggering vortical eddies and tangling the magnetic field lines hence amplifying the turbulent component (Giacalone & Jokipii 2007) (the cosmic-ray pressure gradient has also been proposed as the driver of the amplification in Drury & Falle 1986). The former mechanism was also discussed in the kinetic theory approach (Amato & Blasi 2009); however, numerical simulations (see, e.g., Riquelme & Spitkovsky 2009) could find only a moderate amplification ($B/B_0 \sim O(10)$), and the unfolding of its nonlinear extension and observational implications is currently active.

Collisionless shocks propagate in turbulent and inhomogeneous media undergoing rapid corrugation of their ideal planar surface. Numerical simulations have shown that the picture of a planar shock is inappropriate for describing the secular evolution of a downstream medium. The unshocked medium might be strongly inhomogeneous on several scales and the cold interstellar clumps strongly deform the shock surface. The passage of an oblique non-relativistic shock through an inhomogeneous medium has long been known to generate vorticity in the downstream flow (Ishizuka et al. 1964); in a conducting fluid the turbulent motion at scale $l$ with fluid velocity $v_l$ and local density $\rho$ leads to an exponentially amplified magnetic field $B^2 = 4\pi\rho v_l^2$ (Landau & Lifshitz 1960). Such a dynamo action amounts to a systematic conversion of the fluid kinetic energy into magnetic energy at each scale separately, possibly until equipartition is reached (Kulsrud 2005). Recent numerical 2D-MHD simulations have shown that such an amplification can be very efficient (Giacalone & Jokipii 2007). Further numerical studies using the thermal instability of the interstellar medium (ISM) turbulence, i.e., condensation of the interstellar gas due to catastrophic radiative cooling (Field 1965), confirmed the efficient magnetic field growth (Inoue et al. 2012, and references therein). 2D simulations of relativistic shocks (Mizuno et al. 2011) show that a small-scale dynamo can also operate downstream of the shocks of gamma-ray burst outflows, suggesting that the dynamo action downstream of shocks might shed light on the energy equipartition at magnetized shocks.

In this paper, we provide an analytic derivation of the vorticity generated by a clumpy unshocked medium, hence of the magnetic field amplification, downstream of a non-relativistic rippled collisionless shock. We apply the Rankine–Hugoniot jump conditions locally downstream of an MHD shock to compute the vorticity generated downstream. For the sake of

---

1. Associated Member of LUTH, Observatoire de Paris, CNRS-UMR8102 and Université Paris VII, 5 Place Jules Janssen, F-92195 Meudon Cédex, France.
simplicity, a 2D shock is considered, i.e., observables depend only upon two space coordinates. The downstream vorticity depends on the magnitude of the tangential component of the velocity (shear), although here an interpretation different from that of Kevlahan (1997) is given in terms of the density gradient at the boundary of clumps, and the curvature of the shock surface as previously found for purely hydrodynamic shocks (Truesdell 1952; Kevlahan 1997). We also compute the back-reaction to vortical motion of the seed magnetic field advected downstream, so far accounted for only numerically (e.g., Giacalone & Jokipii 2007; Inoue et al. 2012). Using the small-scale dynamo theory we determine the time evolution of the turbulent magnetic field in the downstream fluid until the saturation epoch.

The encounter of a shock surface with a density clump, also called Richtmyer–Meshkov (RM) instability (Brouillette 2002), has been extensively investigated in plasma laboratory experiments (see Dimonte & Ramaprabhu 2010 and references therein). Numerical simulations of magnetic shocks proved that RM instability drives transient events in several regions of the Earth’s magnetosphere (Wu & Roberts 1999). Recent plasma laboratory experiments (Kuramitsu et al. 2011) made use of lasers to test the magnetic field amplification using density inhomogeneities at the shocks of SNRs (see Sano et al. 2012, and references therein).

The small-scale fluid vortices close behind the shock grow on a timescale smaller than the particle acceleration timescale, which depends on the seed magnetic field orientation, the isotropy of the turbulent component of the magnetic field, and the dependence of the spatial diffusion coefficients (parallel and perpendicular) on the particle energy. Therefore, the vortical field growth is unaffected by the presence of cosmic rays at the shock. The magnetic field may also be enhanced by field line stretching due to Rayleigh–Taylor (RT) instability (Jun et al. 1995) at the interface between the ejecta and the ISM, i.e., far downstream of the shock. In contrast with the vortical turbulence, late-time RT turbulence might be affected by the highest energy particle gyrating in the downstream fluid far from the shock (Frascetti et al. 2010). However, RT structures are unlikely to reach out to the blast wave (Frascetti et al. 2010 and references therein) and therefore to interact with vortical turbulence. Thus, the local dynamo amplification behind the shock can be temporally and spatially disentangled from the field line stretching due to RT instability.

This paper is organized as follows. In Section 2, we introduce the constitutive equations and describe the features of a rippled shock. In Section 3, we define the vorticity in the local rotated frame, compute the vorticity downstream of the shock, and interpret the result in terms of vorticity growth and field back-reaction. In Section 4, we apply the small-scale dynamo theory to determine the time evolution of the turbulent magnetic field. In Section 5, we identify the dependence of field growth and nonlinear field back-reaction on the physical shock parameters and discuss the implications for recent X-ray and γ-ray observations of non-relativistic SNR shocks. In Section 6, we summarize our findings.

2. RIPPLED SHOCK

We consider the propagation of a 2D non-relativistic shock front in an inhomogeneous medium. The time evolution of the two driving independent observables in an ideal MHD approximation, i.e., the fluid velocity \( \mathbf{v} = (v_x, v_y) \) and the magnetic field \( \mathbf{B} = B(x, y) \), is given, with no viscosity or heat conduction and for an infinitely conductive fluid, by

\[
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla P}{\rho} + \frac{1}{4\pi\rho} [\mathbf{B} \times (\nabla \times \mathbf{B})] = 0 \tag{1}
\]

\[
\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{2}
\]

where \( \rho \) and \( P \) are, respectively, density and hydrodynamic pressure of the fluid (here \( \partial_t = \partial / \partial t \)). Equation (2) does not include any field-generating term, such as Biermann’s battery (Kulsrud 2005), as the fluid on both sides of the shock is embedded in a pre-existing magnetic field. Note that the current density carried by cosmic rays is neglected here. We aim to identify the growth of the magnetic energy as generated by the vortical motion of the background fluid only. Thermal dissipation reduces the energy deposited in the magnetic turbulence and will be considered in a forthcoming publication.

The shock is a dynamic surface due to its interaction with the upstream clumps. The kinematics of a 2D hydrodynamic rippled shock propagating in an inhomogeneous medium (Ravindran & Prasad 1993) comprises a sequence of unstable configurations of the shock surface. The conservation of energy requires that sections of the corrugated shock surface that are ahead or lagging behind readjust to smooth out growing corrugations. The time evolution of the shock surface is accounted for by the rate change along the moving surface (Prasad 2001) of \( \partial (t, x, y) \), i.e., the local angle between the average direction of the shock motion and the local normal to the shock surface (see Figure 1). We assume here that such a self-deformation process of the shock surface occurs on a timescale much greater than the turnover time of the smallest eddies in the downstream flow of the shock. Thus, the shock profile is “frozen” during the exponentially fast amplification which proceeds as the field is advected downstream.

3. VORTICITY DOWNSTREAM OF AN MHD SHOCK

In the inviscid approximation used here the conservation of the vorticity flux applies: the shock-generated vorticity is transported along the flow “frozen” into the fluid as a
consequence of the Helmholtz–Kelvin theorem. The medium upstream of the shock has zero vorticity. The vorticity is calculated downstream at a distance from the shock large enough that the shock is infinitely thin, i.e., the thickness of the shock is much smaller than the local curvature radius at every point of the shock surface.

At a rippled shock, the MHD Rankine–Hugoniot jump conditions cannot be applied globally as the normal and tangential directions vary along the shock surface. In a 2D shock propagating on average in the $x$ direction (Figure 1) from the velocity field of the flow $v = (v_x, v_y, 0)$, the vorticity is given by $\omega = \nabla \times v = (0, 0, \omega_z)$ and the component in the $\hat{z}$ direction is given by $\omega_z = \partial_x v_y - \partial_y v_x$ (all quantities are independent of $z$ and we used $\hat{n} = \partial/\partial\hat{n}$). We use a local natural coordinate system $(\hat{n}, \hat{s})$, where $\hat{n} = (\cos(\theta(t, s), \sin(\theta(t, s)))$ is the coordinate along the normal to the shock surface, and $\hat{s} = (\sin(\theta(t, s), -\cos(\theta(t, s)))$ is the coordinate parallel to the shock surface (Figure 1). Note that $\theta = \theta(t, s)$ varies only along the shock surface and not along the orthogonal direction $n$, thus is independent of $n$ (also, the time evolution of $\theta$ in Ravindran & Prasad 1993, Equations (2.21)–(2.23), does not depend on $n$). In the local frame $(\hat{n}, \hat{s})$, the $z$ component of the vorticity becomes

$$\omega_z = \partial_z v_y - \partial_y v_x + v_z \partial_z \theta,$$

(3)

where $v_y = v_x \cos \theta + v_z \sin \theta$ and $v_z = v_x \sin \theta - v_y \cos \theta$ are, respectively, the local component of the velocity flow normal and tangential to the shock (see Figure 1). The local derivatives here can be expressed in terms of derivatives in directions $(\hat{x}, \hat{y})$ as $\partial_z = \hat{n} \cdot \nabla = \cos \theta \partial_x + \sin \theta \partial_y$, and $\partial_x = \hat{s} \cdot \nabla = \sin \theta \partial_x - \cos \theta \partial_y$. The vorticity generated downstream of the shock in the presence of a magnetic field $B = (B_n, B_s, 0)$, for $\omega_z = 0$ upstream (a detailed computation can be found in the Appendix), results in

$$\delta \omega_z = -\frac{1}{\rho C_r} \left\{ \frac{r - 1}{r} \left[ \frac{C_r}{\rho} \partial_x \delta \right] - \frac{\partial_s \delta B^2}{8\pi} + \frac{B_n \partial_s B_s}{4\pi} \right\},$$

(4)

where $C_r = C - v_n$ is the shock speed in the fluid frame ($C$ is the shock speed in the normal direction $n$), $r = \rho_d/\rho_u$ is the compression ratio at the shock, and $\delta f = [f]_{d} - [f]_{u}$ indicates the jump of $f$ from upstream ($u$) to downstream ($d$).

In the present paper, we deal with the hypothesis that the turbulent magnetic component is only generated in the downstream flow through a dynamo mechanism, i.e., turbulence is not required on both sides of the shock as in the diffusive shock acceleration. The contribution from far-upstream fluctuations of magnetic and flow velocity fields on several scales to strong dynamo amplification in the downstream flow will be included in a forthcoming work. If the magnetic field is quasi-perpendicular to the average direction of shock motion, charged particles can still be efficiently accelerated to high energy by the motional electric field by drifting along the shock, until they are advected in the downstream flow or escape upstream, provided they are not scattered back to the shock for further acceleration. In an oblique or quasi-parallel configuration, if the field is amplified only in the downstream flow as described here, other upstream turbulence or pre-existing magnetic instabilities are needed to scatter energetic particles back across the shock, increase the residence time in the upstream region, and release larger-energy accelerated particles. Since no upstream magnetic turbulence is included here, at the present stage this model reconciles dynamo amplification with particle acceleration occurring in the same region of the shock only for a quasi-perpendicular magnetic field. The turbulence of the upstream medium allows an extension to oblique and quasi-parallel cases.

We consider a seed-magnetic field upstream uniform and normal to the average direction of motion ($B_0 = (0, B_0^\parallel, 0)$, or $B_0 = B_0 \sin \theta$ and $B_z = -B_0 \cos \theta$); see Figure 1). For the first term in the second line of Equation (4), we can write $\delta \delta B^2 = \delta B \delta B \sim -2B_0^2 \delta (\sin^2 \theta \partial_x \theta) = 0$, as $\delta (\sin \theta) = \delta (\theta / \partial x) = 0$. The last term in Equation (4) vanishes: $\delta \{B_0 \cos \theta (\partial_x \theta) \} = 0$, and $\delta \partial_x B = 0$.

Assuming that amplification is efficient at the smallest scales (see Section 4), $B_n$ and $\delta B_s$ are drowned out by the turbulent components: $\delta B_s \sim -r B_0 - B_0^\text{turb} + B_0 \sim -B_0^\text{turb}$ and $B_n \sim B_n^\text{turb}$ for a perpendicular field (see also last paragraph in the present section). Therefore, the factors $\delta B_s$ and $B_n$ in Equation (4) include both the impulsive shock compression and the turbulence amplification. We can conclude that the vorticity produced downstream of a 2D shock propagating in an inhomogeneous medium with a uniform perpendicular upstream magnetic field (the same as for a parallel shock as shown later) can be recast, neglecting obliqueness, in a simple form:

$$|\delta \omega_z| = \frac{1}{r} \left( \frac{C_r}{\rho} \right) \partial_x \rho + \frac{B_n}{4\pi \rho C_r} \partial_x \theta,$$

(5)

where $\delta B_s$ is the jump across the shock of the magnetic field in the direction locally tangential to the shock surface including the Rankine–Hugoniot compressed seed field and the turbulently amplified field and $B_0$ is the component in the direction locally normal to the shock surface including the unchanged Rankine–Hugoniot and the turbulent component. The factor $(r - 1)/r$ shows the dependence of $\delta \omega_z$ on the shock compression. The first term in Equation (5) contains the density gradient $(\partial_x \rho)$, hence having the role of the baroclinic generation of vorticity, i.e., the term $\nabla P \times \nabla \rho$ appearing in the equation for $\omega$ time evolution: when an upstream clump (clump size is assumed greater than shock thickness) crosses the shock the clump is locally heated with a maximum compression at the front of the cloud and smaller along the side of the cloud, whereas the density gradient is directed across the fluid–clump interface (Klein et al. 1994): thus, the gradients of density and pressure are no longer parallel and vorticity is generated in the transition layer $(\nabla \rho \neq 0)$. The thermal instability model for ISM predicts a broad range (Field 1965; Begelman & McKee 1990) for the transition layer between the cloud and the ISM shocked gas as a function of the thermal conduction (see Section 5). The second, or corrugation, term in Equation (5) results from the finite curvature radius of the shock, i.e., for a planar shock $\partial_x C_r = 0$ at every point of the shock surface. As shown in the following section, the purely hydrodynamic terms (baroclinic and corrugation) drive the small–scale magnetic field growth.

The back-reaction of the small-scale turbulent field is represented by the last term: vortical eddies shear and whirl field lines around enhancing the turbulent field as long as the entailed magnetic tension increases to a strength large enough to halt such a growth. Similar dependence of the field back-reaction on the turbulence of the upstream medium allows an extension to oblique and quasi-parallel cases.
and back-reaction of the amplified field on the vorticity becomes negligible. Also, the vorticity downstream of a shock wave in Equation (5) holds regardless of the speed of the shock wave (Alfvén or fast or slow magnetosonic speed). We recall that in contrast with 3D turbulence, the shear of the fluid velocity along the vorticity, i.e., \((\omega \cdot \nabla)v\), vanishes in a 2D flow and the vorticity generation in a 2D fluid might be underestimated.

Note that if the magnetic field upstream is uniform and parallel to the average direction of motion (parallel to the average direction of motion in a 2D fluid might be underestimated), the vorticity downstream at the shock curvature scale as a consequence of the vorticity of the flow velocity field (Kulsrud 2005). We note that the ISM clump size is much larger than the shock thickness and the shock crossing is not impulsive on the vorticity generation timescale; thus, \(\varepsilon\) in the bracket of Equation (6) is continuously generated during the shock crossing and is not constant in time.

Neglecting the time dependence on \(\tau\) (the magnetic modes grow slowly for initially weak field; Kulsrud 2005), the solution is readily found:

\[
\frac{\varepsilon}{\varepsilon_0}(t) = \left(\frac{B}{B_0}\right)^2(t) = \frac{e^{2t/\tau}}{1 - \alpha \tau (1 - e^{2t/\tau})},
\]

for a uniform average interstellar matter density. Equation (7) provides the first analytic time evolution of a self-controlled amplification of magnetic energy through fluid vortical motion generated downstream of a 2D rippled shock.

For \(t \ll \tau\) the term proportional to \(v_\lambda^2\) in the denominator is negligible and \(B\) grows exponentially \((B/B_0 \sim e^{t/\tau})\). For times \(t \gg \tau\) the term proportional to \(v_\lambda^2\) dominates over the first term and the saturation is attained:

\[
B/B_0 \simeq \sqrt{2/\left(\alpha \tau v_\lambda^2\right)},
\]

corresponding to an increase in \(B/B_0 \sim M_A\), where \(M_A = C_A\sqrt{4\pi \rho}/B_0\) is the seed-field Alfvén Mach number (see Section 5). Note that this result applies only if the assumption that the turbulence \(B^{\text{th}}\) downstream dominates the compressed seed field \(rB_0\). For interplanetary shocks (\(M_A \lesssim 10\)), the measured downstream turbulence is rarely amplified to values much greater than the mean field (Burlaga et al. 2007), and the term \(rB_0\) is no longer negligible with respect to \(B^{\text{th}}\) and a different analysis is needed.

5. DISCUSSION

Whether the magnetic field is amplified by wave excitation in the shock precursor or downstream through the coupling of the motion of eddies with magnetic field lines, a synergy of the two mechanisms, or other processes, is not observationally settled yet. Only the former has been largely explored in recent years. This paper presents for the first time an analytic approach to the latter mechanism.

In what follows, we estimate \(\tau\) and \(\alpha\) showing that Equation (7) applies to the time evolution of magnetic energy in young and middle-aged SNRs. We do not aim here at predicting a typical length scale of the X-ray rims due to the large variety of SNR taxonomy, but rather we show that reasonable values of the thickness of the clump/fluid transition layer, shock curvature radius, and shock speed are capable of producing the inferred efficient magnetic field amplification.

5.1. Field Growth

From the discussion below Equation (5), the magnitude of the vorticity downstream \(|\delta \omega_z|\) is expected to depend on the thickness of the layer of the interface clump/fluid and not on the size of the clump. Thermal instability model of two-phase
fluid suggests that such a thickness is given by the Field length $\ell_F$ (Field 1965; Inoue et al. 2006). However, $\ell_F$ depends on heat conduction (Begelman & McKee 1990) and is thus uncertain as the thermal conductivity might be sensitive to the magnetic field. The previous history of formation and evolution of the supernova progenitor might also affect the properties of the surrounding medium, i.e., uniform or stellar wind density profile.

The growth timescale can be recast as $\tau^{-1} = (\pi/3)(r - 1/r) [(C_r/\rho_0)\partial_\rho + \partial_\vartheta C_r] \sim (r - 1/r)C_r(R_c^{-1} + \ell_F^{-1})$. Neither of the terms $R_c^{-1}$ and $\ell_F^{-1}$ can be neglected because no particular assumption has been made on $R_c$, expected indeed to be comparable to the size of the clump that corrugated the shock, with respect to $\ell_F$.

From the previous argument it follows:

$$\tau \sim \frac{r - 1}{r - 1} \frac{R_c\ell_F}{R_c + \ell_F}. \quad (9)$$

The growth rate $\tau^{-1}$ of the turbulent field, and of $\delta w_{\alpha}$, increases with shock speed and it depends only on hydrodynamic quantities, except a possible unknown dependence on $B$ in $\ell_F$ is disregarded here. If $\ell_F \ll R_c$, from Equation (9) holds $\tau \sim \ell_F/C_r$ (see also Figure 4). Thus, the field amplification saturates faster in a region of smaller $\ell_F$ providing a constraint testable by multiwavelength observations. This justifies the description in Section 3 that the vorticity and the strong turbulent field are generated in the transition layer separating the cold clump from warm ISM and does not depend on the clump size.

Note that $\tau$ is smaller than the travel time of the shock across the interior of the ISM clump, i.e., cloud-crushing time $t_{cc}$, resulting from the ram-pressure equilibrium at the clump boundary (Klein et al. 1994): $t_{cc} = \sqrt{\rho_c/\rho_0} L/C_r$, where $\rho_c$ and $\rho_0$ are respectively mass--density in the clump and in the surrounding fluid. Structures of scale $L$ stable under external heating or radiative cooling condense if $L \gg \ell_F$; thus, $t_{cc} = \sqrt{\rho_c/\rho_0} L/C_r \gg \sqrt{\rho_c/\rho_0} \ell_F/C_r \sim \sqrt{\rho_c/\rho_0} \tau > \tau$. Thus, for any $\rho_c/\rho_0$ the ISM clump survives across the shock provided that the density gradient at the clump boundary is steep enough.

5.2. Nonlinear Back-reaction

As the magnetic field strengthens, it reacts to field lines whirling halting the turbulence growth. In more general terms, as the field increases by dynamo effect it also releases its tension by unwinding at a rate of the order of Alfvén speed: the back-reaction grows with the turbulent field Alfvén speed (Kulsrud 2005). The local back-reaction of the field $a \sim \partial_\vartheta \vartheta/C_r$ can be estimated by

$$a \sim \vartheta/(R_c C_r). \quad (10)$$

Clearly $\alpha$ is enhanced by a small curvature radius ($\alpha \sim 1/R_c$) as the more corrugated the shock is, and the shorter the eddy’s turnover time is, the more efficiently the turbulent field is created downstream and back-reacts to fluid motion. From Equation (6), it is clear that as $\epsilon$ increases, the back-reaction term grows faster than the driving term, eventually becoming dominant: the strongest field has the smallest growth (cf. the back-reaction dependence on $B$ in the nonlinear Landau damping).

5.3. Secular Evolution of the Turbulent Field

Figure 2 depicts the growth of the turbulent field for various shock speeds, assumed to be constant in time$^5$: given an ISM field of the order of $B_0 \sim 3 \mu G$, the turbulent field saturates at $B \sim 1.2$–$3$ mG for $C_r = 1500$–$5000$ km $s^{-1}$ on the year timescale. Such a rapid growth of magnetic energy is compatible with X-ray observations of SNRs RX J1713.7–3946 ($C_r < 4500$ km s$^{-1}$; Uchiyama et al. 2007) and Cas A (Patnaude & Fesen 2009) brightness variations detected on the year timescale in small-scale hot spot structures, attributed to synchrotron electron cooling. Using $R_c = 1500$ km s$^{-1}$ and $\ell_F = 10^6$ cm, we find an amplification to $B \sim 3$ mG within 3 yr. Such a value of $\ell_F$ is to be compared with the spatial scale of the Chandra RX J1713.7–3946 bright spots, estimated as $\lesssim 0.03$ pc. Similar length ($\sim 10^{10}$–$10^{18}$ cm) and time ($\sim 1$ yr) scales are found in simulations of the effects of magnetic field turbulence on the synchrotron emission images and spectra in SNRs (Bykov et al. 2008). Thus, the magnetic energy increase and the X-ray variability might have a timescale (1 yr) much lower than the SNR hydrodynamic timescale and might occur in middle-aged SNRs, not necessarily young SNRs (RX J1713.7–3946 age is estimated as 1600 yr; Stephenson & Green 2002). The high shock speed $C_r \sim 15,000$ km s$^{-1}$ in Figure 2 is comparable to observations of the youngest SNR in our galaxy, i.e., 100 yr old G1.9+0.3 (Reynolds et al. 2008). Thus, a rapid field saturation even up to $B \sim 10$ mG is predicted at SNR shocks within a few months.

The computation above indicates a linear increase of the saturation value with the shock speed (see Figure 2), or $M_A$. From Section 4 the saturation can be written as

$$\frac{B}{B_0} \lesssim \sqrt{\frac{2}{\alpha \tau v_A^2}} \sim \sqrt{\frac{C_r^2 + \ell_F}{v_A^2 R_c \ell_F \partial_\vartheta \vartheta \sim M_A \sqrt{\frac{2 R_c + \ell_F}{\ell_F \vartheta}}}} \quad (11)$$

The equipartition between the magnetic pressure of the turbulent component downstream of the shock, the thermal pressure, and the fluid ram pressure ($B^2/8\pi \sim \rho C_s^2$) implies $(B/B_0)^2 \sim (8\pi B_0^2/R_c^2 \rho C_s^2) \sim 2M_A^2$. Thus, the scaling $B/B_0 \sim M_A$ indicates that the amplified field pressure might locally become comparable to the ram pressure (and the gas pressure), according to the factor $(R_c + \ell_F)/(\ell_F \vartheta)$; the nonlinear back-reaction, found using jump conditions at rippled shocks in this paper, is no longer negligible (cf. Bell 2004) and the rates of growth and unwinding are equal.

Figure 3 shows the turbulent field growth for various values of $R_c$, with a shock speed $C_r = 4800$ km s$^{-1}$ (close to the highest
measured for the SNR1006 shock; Katsuda et al. 2013). From the Hubble Space Telescope observations of SN1006 (Raymond et al. 2007), $R_c$ is inferred to be less than 1/10 of the forward shock radius, estimated as $R_s \sim 9.1$ pc = $2.8 \times 10^{10}$ cm. However, the overall smoothness of SN1006 suggests that the ripple scale might be lower ($R_c \sim 10^{16}$ cm) in more clumpy-structured SNRs. This justifies the smaller value of $R_c$ used in Figure 2. Moreover, the absence in SN1006 of hot spots with rapid time variability, as detected in synchrotron cooling regions of RX J1713.7−3946 (Uchiyama et al. 2007), might result from the spatial location at high Galactic latitude where small-scale clumps are not present (see also Katsuda et al. 2010). Thus, a saturation $B/B_0 \sim 500$–800 for $R_c = 10^{15}$ cm is compatible with the optical constraints on SN1006 (Raymond et al. 2007), indicating that a region with a strong amplified field variable on a 10 yr timescale could still be observed in future high-spatial resolution efforts before the advected amplified field is drowned by projection effects.

Despite several uncertainties, a numerical estimated range for $\ell_F$ in the ISM is given by $[10^{16}–10^{18}]$ cm (see Equations (5.1) and (2.7) in Begelman & McKee 1990). The field growth is clearly more efficient at the small-scale $\ell_F$, as also shown in Figure 4. In Figure 3, an increase in $R_c$ reduces the field backreaction ($\alpha \sim 1/R_c$) resulting in a larger saturation value. By contrast (Figure 4) an increase in $\ell_F$ ($\ell_F \ll R_c$) reduces the growth rate ($\tau \sim \ell_F/R_c$) and therefore $B/B_0$.

ISM density turbulence has been known for long time through radio scintillation to obey the Kolmogorov scaling over several decades in length scale (Lee & Jokipii 1976; Armstrong et al. 1995) from the outer scale, or injection scale, $L_c \sim 3$–50 pc (Noutos 2012). If the turbulent fluid velocity downstream of the shock follows a 1D Kolmogorov power spectrum at a certain scale $k$ as numerical simulations seem to suggest (Inoue et al. 2012) ($P_v(k) \sim k^{-5/3}$), the vorticity spectrum at scale $k$ is given by $P_{\vartheta}(k) \sim k^2 P_v(k) \sim k^{1/3}$; in the inertial range the vortical energy on small scales is greater than on large scales. Also, since the field on a small scale grows and saturates faster, at that scale the magnetic energy coincides with the turbulent energy (Kulsrud 2005). Here we indicate such a scale to be $\lesssim 0.01$ pc. However, as a result of the uncertainties on the Field length $\ell_F$ (Field 1965), within the scenario presented here a unique length scale of magnetic growth cannot be identified. A range of scales consistent with the ISM fluctuations is shown here to account for optical and X-ray observations. The confirmation will be sought through kinetic analysis or 3D numerical simulations in forthcoming publications.

An attractive feature of the microinstability mechanism in Bell (2004) is that magnetic field growth is associated with energetic particles at the shock and the field saturation is driven by the cosmic rays of highest energy, possibly $10^{15}$ eV protons. In contrast, the lack of direct association between particle acceleration and field amplification in the dynamo mechanism here does not disfavor it as an interpretation of the X-ray rims in SNRs: the particle acceleration might proceed through the motional electric field $E = |\mathbf{v} \times \mathbf{B}|/c$ along the rippled shock surface (such an acceleration is known to be faster than in the parallel field case; Jokipii 1987). We note that the scale of the smallest eddies, or highest turbulent field, inferred here ($\ell_F \sim 10^{16}$ cm) is comparable to the gyroscale $r_g$ of energetic protons with energy $E = 10^{15}$ eV in a field $B = 300$ G ($r_g = 10^{16}$ cm); hence, the mean free path $\lambda$ of the highest energy particles, except the Bohm scattering case, i.e., $\lambda \sim r_g$, is typically greater than $\ell_F$. If this happens, the turbulence downstream is able to enhance the scattering of the highest energy particles, thereby leading to an additional particle acceleration at the shock possibly beyond the knee of the cosmic-ray spectrum.

6. CONCLUSION

We have investigated the generation of vorticity downstream of a non-relativistic 2D rippled shock front typical of a shell-like SNR expanding in a turbulent ISM. By using only the jump condition at a rippled shock surface, we have derived the temporal evolution and the saturation of the turbulent magnetic field downstream of the shock, including the nonlinear field back-reaction. We conclude that the saturation of $B$ by small-scale dynamo action depends on the shock speed, on the thickness of the layer of the ISM clumps, and on the shock curvature radius. The saturation value is found counter-intuitively not to depend on the size of the ISM clumps, as is also the case for purely hydrodynamic shocks (Klein et al. 1994). Perpendicular and parallel field upstream cases lead to the same saturation values, if amplification is very strong, due to fast isotropization of the downstream turbulence.

Our finding shows that a small-scale dynamo might explain non-thermal X-ray and optical observations of young/middle-aged SNR shocks. The secular evolution of the turbulent magnetic field derived here might help to shed light on the
evolution of young SNRs to be discovered. The youngest SNRs in our galaxy or other "historical" SNRs might be in the growing phase of a magnetic field described here. The current and next generation of hard X-ray observatories (NuSTAR, Astro-H) will provide a helpful probe for the mechanism described here.

The author thanks J. Giacalone and J. R. Jokipii for helpful discussions, P. Prasad and N. Kevlahan for correspondence on moving surface theories, R. Kulsrud and J. Raymond for comments, Observatory Paris-Meudon, where part of this work was done, and the anonymous referee for helpful comments which significantly improved the manuscript. The support from NASA through Grants NNX10AF24G and NNX11AO64G is gratefully acknowledged.

## APPENDIX

### COMPUTATION OF THE VORTICITY

The \( \omega_z \) generated downstream of the shock is found, for convenience, by computing separately the jump across the shock of every term on the right-hand side of Equation (3), with the shock profile being frozen. Recalling that if \( \mathbf{B} \neq 0 \) the transverse flow velocity is not conserved across the shock (\( \delta v_n \neq 0 \)), we find

\[
\delta \omega_z = \partial_s (\delta v_n) - \partial_t (\delta v_s) + (\delta v_s) \partial_t \theta.
\]  

(A1)

We compute the term \( \delta (\delta v_s) \) by using the equations of motion, i.e., Equations (1) and (2), and following the algorithm used in Kevlahan (1997). The magnetic field \( \mathbf{B} = (B_n, B_s) \neq 0 \) accounts for the back-reaction to the vorticity growth (see Equation (5)). Rewriting the two components of Equation (1) in coordinates \((n, s)\), one finds two new equations:

\[
\begin{align*}
\cos \theta \mathbf{F} + \sin \theta \mathbf{G} &= 0, \\
\sin \theta \mathbf{F} - \cos \theta \mathbf{G} &= 0,
\end{align*}
\]

(A2)

where we have defined

\[
\mathbf{F} = \frac{d v_n}{d t} + v_n \frac{d \theta}{d t} + \frac{\partial_t P}{\rho} + \frac{B_s}{4\pi \rho} \left( \partial_n B_s - \partial_s B_n - B_s \partial_t \theta \right)
\]

and

\[
\mathbf{G} = \frac{d v_s}{d t} - v_s \frac{d \theta}{d t} + \frac{\partial_t P}{\rho} - \frac{B_n}{4\pi \rho} \left( \partial_n B_s + \partial_s B_n - B_s \partial_t \theta \right),
\]

and \( \frac{d}{d t} = \partial_t + v_n \partial_n + v_s \partial_s \).

Equation (A2) implies \( \mathbf{F} = \mathbf{G} = 0 \). We recast \( \mathbf{G} = 0 \) as (see also Section 2 in Ravindran & Prasad 1993)

\[
\frac{d \theta}{d t} - \frac{1}{\rho} \frac{d v_n}{d t} - \frac{\partial_t P}{\rho v_n} - \frac{B_n}{4\pi \rho v_n} \left( - \partial_n B_s + \partial_s B_n + B_s \partial_t \theta \right) = 0.
\]

(A3)

We now evaluate the jump of Equation (A3) across the shock (\( \delta (\delta v_s) \)). Multiplying Equation (A3) by \( \rho \nu_n \nu_s \) and making use of the mass conservation \( \delta \rho = -\delta \rho \rho_C \), we obtain

\[
(\rho \nu_n \nu_s) \delta (\delta v_s) = \delta (\rho \nu_n \nu_s + \rho v_n \partial_t v_s) - (\rho \nu_n \nu_s) \delta (\partial_t v_s) \partial_t \theta + \partial_t \delta P + C \delta \rho (\partial_n v_s) - C \delta \rho (\partial_s v_n) - C \delta \rho (\nu_s) \partial_t \theta + \frac{B_n}{4\pi} \delta (- \partial_n B_s + \partial_s B_n + B_s \partial_t \theta) = 0.
\]

(A4)

Note that if \( \mathbf{B} = 0 \) this expression is equivalent to Equations (2.8)–(2.10) in Kevlahan (1997) with \( \delta \theta = 0 \).

In Equation (A4), the mass conservation readily gives

\[
\delta (\rho \nu_n \nu_s \delta \nu_s) = \delta (\partial_t \rho \nu_n \nu_s + \rho v_n \partial_t \nu_s) - (\rho \nu_n \nu_s) \delta (\partial_t \nu_s) \partial_t \theta + \partial_t (\rho \nu_s \delta \nu_s) + C \delta \rho (\partial_n \nu_s) - C \delta \rho (\partial_s \nu_n) - C \delta \rho (\nu_s) \partial_t \theta + \frac{B_n}{4\pi} \delta (- \partial_n B_s + \partial_s B_n + B_s \partial_t \theta) = 0.
\]

(A5)

We replace the expression for \( \delta \delta P \) from Equation (A5) into Equation (A4) and substitute the resulting expression for \( \delta (\delta v_s) \) into the vorticity downstream (Equation (A1)). By using the jump condition \( \delta v_n = (r - 1/r) [C \theta] \), one can write

\[
\delta \omega_z = -\frac{1}{\rho (C \theta)} \left[ \delta \left( \rho \nu_n \nu_s \frac{\Delta \nu_s}{\Delta t} \right) + \frac{1}{r} [C \theta] \nu_s \right],
\]

(A6)

where \( \frac{\Delta \nu_s}{\Delta t} = (\partial_t + \nu_n \partial_n + \nu_s \partial_s) \nu_s \). Note that, if \( \mathbf{B} = 0 \), the second line in Equation (A6) vanishes and the remaining terms correspond to the result in Kevlahan (1997).

In Equation (A6) we can approximate to first order in \( \partial_t \) the shear term: \( \nu_s \cdot \left( C \theta \Delta \nu_s / \Delta t \right) \sim C \theta [C \theta] \). The mass conservation \( \delta (\rho C \theta) = 0 \) for the first term in Equation (A6) gives \( \delta (\rho \nu_n \nu_s \Delta \nu_s / \Delta t) \sim \rho C \theta [C \theta] \). The time-derivative \( \partial_t \Delta \nu_s / \Delta t \sim C \theta / \Delta t \) for an MHD shock wave depends on the shock speed, magnetic field components, and density gradient in a complicated manner (Prasad 2001). Remarkably, in the shear term of Equation (A6) the only non-vanishing jump is \( \delta (\delta v_s) \partial_t \theta \) thus independent of the particular equation of motion for \( \partial_t \theta \).

Using the MHD Rankine–Hugoniot conditions for the transverse velocity, \( \delta v_s = -B_s \delta B_s / (4 \pi \rho C \theta) \), we can recast Equation (A6) as Equation (4) in the main text.

## REFERENCES

Acciari, V. A., Aliu, E., Arlen, T., et al. 2011, ApJL, 730, L20
Aharonian, F. A., Akhperjanian, A. G., Aye, K.-M., et al. 2004, Natur, 432, 75
Aharonian, F. A., Akhperjanian, A. G., Barrio, J., et al. 2001, A&A, 370, 112
Amato, A. E., & Blasi, P. 2009, MNRAS, 392, 1591
Armstrong, J. W., Rickett, B. J., & Spangler, S. R. 1995, ApJ, 443, 209
Axford, W. I., Leer, E., & Skadron, G. 1977, in Proc. 15th Int. Cosmic Ray Conf., Vol. 11 (Plovdiv: Central Research Institute), 132
Balsara, D. S., & Kim, J. 2005, ApJ, 643, 390
Bamba, A., Yamazaki, R., Ueno, M., & Koyama, K. 2004, AdSpR, 33, 376
Bell, A. R. 2004, MNRAS, 353, 550
Blandford, R. D., & Ostriker, J. P. 1978, ApJL, 221, L29
Bougie, L. F., Ness, N. J., & Acosta, M. H. 2007, ApJ, 678, 1246
Bykov, A. M., Uvarov, Y. A., & Ellison, D. C. 2008, ApJL, 689, L13
Dinmonte, G., & Ramaprabhu, P. 2010, PhFl, 22, 014104
Drury, L. O'C., & Falle, S. A. E. G. 1986, MNRAS, 223, 353
Dyer, K. K., Cornwell, T. J., & Middalena, R. J. 2009, AJ, 137, 2956
Field, G. B. 1965, ApJ, 142, 531
Fraschetti, F., & Jokipii, J. R. 2011, ApJ, 734, 83
Fraschetti, F., Teyssier, R., Ballet, J., & Decourchelle, A. 2010, A&A, 515, A104
