Looking for good Hofmeister and Braunschadel bases

0 History

28-10-92 Document started
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24-01-93 New sections 6 and 7 added to record further results (for t=21 to 58), and
issues arising from them.
Summary table of the "best" bases added as Appendix 1.
14-03-93 Note on extended search (which confirms previous results) added.
29-09-14 Abstract and Reference sections added, and internal references updated, to
prepare the document for ArXiv publication. Some further explanatory
material added, as well as a new Appendix containing a transcript of
Selmer's "private communication" to me dated 1992/1993) (see Selmer, E. S.,
[8]).
Other than this additional explanatory material there are no changes to the
content of the paper as originally prepared in March 1993.

Abstract

A_k = (1, a_2, ... a_k) is an h-basis for n if every positive integer not exceeding n can be expressed as the
sum of no more than h values a_i. An "extremal" h-basis A_k is one for which n is as large as possible.
Computing extremal bases has become known as the "global" Postage Stamp Problem.

This paper describes the author's early attempts to identify extremal Hofmeister and Braunschadel
bases for large t (where t = 12h + r for some 0 <= r <= 11), and also includes a transcription of Prof.
Selmer's correspondence with the author about this work which has not been published before. See
(Selmer, E.S., Private Communication, [8], 1992/1993).

1 Early work

In August 1989 a colleague and personal friend Hector Prior suggested formulae of the following
kind for maximal d=4 sets:

\[ h = 12t + r \]
\[ a_4 = (k_{43}t + c_{43})a_3 + (k_{42}t + c_{42})a_2 + (k_{41}t + c_{41}) \]
\[ a_3 = (k_{32}t + c_{32})a_2 + (k_{31}t + c_{31}) \]
\[ a_2 = (k_{21}t + c_{21}) \]

with the k_{ij} independent of both t and r, and the c_{ij} dependent on r only. Using the computed maximal
sets for h <= 76, he suggested the following values for the k_{ij}:

(k_{21}, k_{31}, k_{32}, k_{41}, k_{42}, k_{43}) = (9, 4, 3, 6, 2, 2)

I then investigated formulae of this kind by computer, and it soon became apparent that the value for
k_{41} should be 7, rather than 6. I now concentrated on identifying the c_{ij}, and developed a program
(called exp21c) which took as input:

- a fixed value of r
- an initial value of $t$
- a range of values for each coefficient $c_{ij}$

and then, for successive values of $t$, considered each set in turn, printing out those whose cover was at least as good as that of some "good" set (as defined by a particular set of coefficients $c_{ij}$). This program was, in fact, the first of my programs to employ the "difficult target" idea (see Challis, M.F., [3]) and I was able to investigate ranges of $c_{ij}$ right up to $t=20$.

At this time, I believed that there must exist a "finite set of formulae" to describe the $d=4$ maximal sets that would be valid for all sufficiently large $h$, and also that we were close to identifying those formulae; clearly, the cycle length would be 12, and almost certainly we now had the correct values for the coefficients $k_{ij}$; the values I proposed for the $c_{ij}$ in September 1989 were:

Table 1

| $r$ | $c_{21}$ | $c_{31}$ | $c_{32}$ | $c_{41}$ | $c_{42}$ | $c_{43}$ |
|-----|---------|---------|---------|---------|---------|---------|
| 0   | 1       | 0       | 0       | 0       | 0       | 0       |
| 1   | 1       | 0       | 2       | 1       | 1       | 0       |
| 2   | 1       | 0       | 2       | 1       | 1       | 0       |
| 3   | 3       | 1       | 2       | 2       | 1       | 1       |
| 4   | 3       | 1       | 2       | 2       | 1       | 1       |
| 5   | 3       | 1       | 2       | 2       | 1       | 1       |
| 6   | 3       | 1       | 2       | 2       | 1       | 1       |
| 7   | 8       | 4       | 1       | 7       | 1       | 0       |
| 8   | 8       | 4       | 1       | 7       | 1       | 0       |
| 9   | 8       | 4       | 1       | 7       | 1       | 0       |
| 10  | 11      | 6       | 1       | 10      | 1       | 0       |
| 11  | 11      | 5       | 2       | 9       | 2       | 0       |

2 Braunschadel and Hofmeister bases

This is where the matter remained until late 1991 when I sent a copy of my results to Professor Ernst Selmer, recently retired from the University of Bergen. By this time, I had applied the "difficult target" technique to the $d=4$ case (in the program gen27c), and had calculated extremal sets as far as $h=122$. Selmer reviewed these results, and, in a letter dated January 2nd 1992, noted that two of them - for $h=107$ and 119 (ie $r=11$ for $t=8, 9$) - did not fit into the formula above; instead, he showed that they conformed to a very similar formula in which just one coefficient $k_{ij}$ was different: $k_{31} = 2$ instead of 4.

Selmer had recently been working on the concept of "dual" or "associate" bases (Selmer, E. S., [9]) and this new basis is the dual of the original one; it had first been discovered and investigated by Braunschadel (Braunschadel, R., [1]). It also turned out that formulae of the original kind had been discovered by Hofmeister and had been thoroughly investigated by Mossige in his thesis submitted in 1986 (Mossige, Svein, [6]), leading to the discovery of bases with an asymptotic coefficient of 2.008... - a most surprising result.

For these reasons, I now refer to bases for $h = 12t + r$ as follows:

$$(k_{21}, k_{31}, k_{32}, k_{41}, k_{42}, k_{43}) = (9, 4, 3, 7, 2, 2) \quad Hofmeister \ bases$$
Further results for $d=4$ (up to $h=158$ at the time of writing) suggest that all maximal sets are either of the Hofmeister or of the Braunschadel form, and so I returned to exp21 to see what coefficients $c_{ij}$ were best for larger values of $t$; in particular, there seemed to be some evidence that Braunschadel bases were beginning to take over from Hofmeister for large $t$.

The results are summarised in Tables 2 and 3 below, details of which I sent to Selmer on April 5th 1992. Although I shall talk of the best Hofmeister or Braunschadel basis for a given value of $t$, it is important to remember that I have not completed an exhaustive search: the search is only over those sets defined by coefficients $c_{ij}$ that lie within the ranges given in the tables.

### Table 2

Probably the best Hofmeister bases for $5 \leq t \leq 20$

| $r$   | $t_{\text{range}}$ | $c_{21}$ | $c_{31}$ | $c_{32}$ | $c_{41}$ | $c_{42}$ | $c_{43}$ | $c_{21}$ | $c_{31}$ | $c_{32}$ | $c_{41}$ | $c_{42}$ | $c_{43}$ |
|------|---------------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 0    | [4-5]               | 2        | 1        | 0        | 1        | 0        | 1        | [-9,11]  | [-5,5]   | [-3,3]   | [-10,10] | [-3,3]   | [-2,2]   |
|      | [6-20]              | 1        | 0        | 0        | 0        | 0        | 0        | [-9,11]  | [-5,5]   | [-1,5]   | [-9,11]  | [-2,4]   | [-2,2]   |
| 1    | [5-20]              | 1        | 0        | 2        | 1        | 1        | 0        | [-9,11]  | [-5,5]   | [-1,5]   | [-9,11]  | [-2,4]   | [-2,2]   |
| 2    | [5-6]               | 2        | 1        | 1        | 1        | 1        | 1        | [-9,11]  | [-5,5]   | [-1,5]   | [-9,11]  | [-2,4]   | [-2,2]   |
|      | [7-20]              | 1        | 0        | 2        | 1        | 1        | 0        | [-9,11]  | [-5,5]   | [-1,5]   | [-9,11]  | [-2,4]   | [-2,2]   |
| 3    | [1-20]              | 3        | 1        | 2        | 2        | 1        | 1        | [-7,13]  | [-4,6]   | [-1,5]   | [-8,12]  | [-2,4]   | [-1,3]   |
| 4    | [2-20]              | 3        | 1        | 2        | 2        | 1        | 1        | [-7,13]  | [-4,6]   | [-1,5]   | [-8,12]  | [-2,4]   | [-1,3]   |
| 5    | [4-20]              | 3        | 1        | 2        | 2        | 1        | 1        | [-7,13]  | [-4,6]   | [-1,5]   | [-8,12]  | [-2,4]   | [-1,3]   |
| 6    | [5-20]              | 3        | 1        | 2        | 2        | 1        | 1        | [-7,13]  | [-4,6]   | [-1,5]   | [-8,12]  | [-2,4]   | [-1,3]   |
| 7    | [2-11]              | 7        | 3        | 2        | 5        | 1        | 2        | [-2,18]  | [-1,9]   | [-2,4]   | [-3,17]  | [-2,4]   | [-2,2]   |
|      | [12-20]             | 8        | 4        | 1        | 7        | 1        | 0        | [-2,18]  | [-1,9]   | [-2,4]   | [-3,17]  | [-2,4]   | [-2,2]   |
| 8    | [1-16]              | 7        | 3        | 3        | 5        | 2        | 2        | [-3,17]  | [-2,8]   | [0,6]    | [-5,15]  | [-1,5]   | [0,4]    |
|      | [17-20]             | 8        | 4        | 1        | 7        | 1        | 0        | [-3,17]  | [-2,8]   | [0,6]    | [-5,15]  | [-1,5]   | [0,4]    |
| 9    | [1-20]              | 7        | 3        | 3        | 5        | 2        | 2        | [-3,17]  | [-2,8]   | [0,6]    | [-5,15]  | [-1,5]   | [0,4]    |
|      | [20]                | 11       | 6        | 1        | 10       | 1        | 0        | [-3,17]  | [-2,8]   | [0,6]    | [-5,15]  | [-1,5]   | [0,4]    |
| 10   | [2-17]              | 10       | 4        | 3        | 7        | 2        | 2        | [0,20]   | [-1,9]   | [0,6]    | [-3,17]  | [-1,5]   | [0,4]    |
|      | [18-20]             | 11       | 5        | 2        | 9        | 2        | 0        | [0,20]   | [-1,9]   | [0,6]    | [-3,17]  | [-1,5]   | [0,4]    |

Notes: This table is valid for all $t \geq 5$ ($h \geq 60$), and, in some cases, for smaller values of $t$; such cases are noted in the $t_{\text{range}}$ column.

Where no ranges are defined, this is because the given set is known to be the extremal basis for those values of $t$. 

$(k_{21}, k_{31}, k_{32}, k_{41}, k_{42}, k_{43}) = (9, 2, 3, 7, 2, 2) \quad \text{Braunschadel bases}$
Table 3

Probably the best Braunschadel bases for 12<=t<=20

| r   | t_range | c_{21} | c_{31} | c_{32} | c_{41} | c_{42} | c_{43} |
|-----|---------|--------|--------|--------|--------|--------|--------|
| 0   | [12-20] | 2      | 2      | -1     | 3      | -1     | 0      |
|     |         | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
| 1   | [12-15] | -1     | 2      | -1     | 0      | 1      | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
| 16-20| 2      | 2      | -1     | 3      | -1     | 0      | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
| 2   | [12]    | 0      | -1     | 2      | -1     | 0      | 1      | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
|     |         |         |        |        |        |        |        | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
|     | [13-20] | 5      | 3      | -1     | 6      | -1     | 0      | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
| 3   | [12-20] | 5      | 3      | 0      | 6      | 0      | 0      | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
|     |         |         |        |        |        |        |        | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
| 4   | [12-17] | 3      | 0      | 3      | 2      | 1      | 1      | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
|     |         |         |        |        |        |        |        | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
|     | [18-20] | 5      | 3      | 0      | 6      | 0      | 0      | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
| 5   | [12-20] | 3      | 0      | 3      | 2      | 1      | 1      | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
| 6   | [12-20] | 3      | 0      | 3      | 2      | 1      | 1      | [-5,15]| [-10,10]| [-5,5]| [-5,15]| [-5,5]| [-2,2] |
| 7   | [12-20] | 8      | 4      | 1      | 9      | 1      | 0      | [-5,15]| [-10,10]| [-2,8]| [-5,15]| [-5,5]| [-2,2] |
| 8   | [12-20] | 9      | 4      | 1      | 9      | 1      | 1      | [0,20] | [-10,10]| [-2,8]| [-5,15]| [-5,5]| [-2,2] |
| 9   | [12-20] | 9      | 3      | 2      | 8      | 1      | 1      | [0,20] | [-10,10]| [-2,8]| [-5,15]| [-5,5]| [-2,2] |
| 10  | [12-14] | 7      | 1      | 4      | 5      | 2      | 2      | [0,20] | [-10,10]| [-2,8]| [-5,15]| [-5,5]| [-2,2] |
|     |         |         |        |        |        |        |        | [-3,17]| [-4,6]  | [1,7] | [-5,15]| [-1,5]| [0,4]  |
|     | [15-20] | 9      | 3      | 2      | 8      | 1      | 1      | [0,20] | [-10,10]| [-2,8]| [-5,15]| [-5,5]| [-2,2] |
|     |         |         |        |        |        |        |        | [-3,17]| [-4,6]  | [1,7] | [-5,15]| [-1,5]| [0,4]  |
| 11  | [12-20] | 11     | 4      | 2      | 10     | 1      | 2      | [0,20] | [-10,10]| [-2,8]| [-5,15]| [-5,5]| [-2,2] |
|     |         |         |        |        |        |        |        | [1,21] | [-1,9]  | [-1,5] | [0,20] | [-2,4] | [0,4]  |

* These are the only values of \( r \) where the best Braunschadel basis betters the best Hofmeister basis for some values of \( t \).

3 Discussion

We see that for Hofmeister sets the results for large \( t \) (eg \( t=20 \)) agree with the earlier results given in Table 1 above, with the exception of the \( r=9 \) case. The reason for this is that I noticed earlier in 1989 that the cover for the set with \( c_{ij} = (8,4,1,7,1,0) \) - given in Table 1 - is "catching up" the cover for the set with \( c_{ij} = (7,3,3,5,2,2) \) - given in Table 2: so although the latter is still best for \( t=20 \), it will not always remain so.

It turns out that for \( t<=20 \), the only cases where the best Braunschadel base has a greater cover than the best Hofmeister base are:

\[
\begin{align*}
  r &= 0 \quad \text{for} \quad t >= 12 \\
  r &= 11 \quad \text{for} \quad t >= 8 
\end{align*}
\]

However, examination of the full results suggested that some Braunschadel bases were catching up on their Hofmeister counterparts, and I showed in April 1992, for example, that for \( r=2 \), the \( t=20 \) Braunschadel basis has a greater cover than the \( t=20 \) Hofmeister basis for \( t>=21 \).
4 Coefficients of powers of \( t \)

I returned to look more closely at the "catching up" concept in October 1992, when I came to write up these notes for the first time.

We know that the cover of any Hofmeister or Braunschadel set can be expressed in the regular form as:

\[
C = (k_{54}t + c_{54})a_4 + (k_{53}t + c_{53})a_3 + (k_{52}t + c_{52})a_2 + (k_{51}t + c_{51})
\]

and, given a particular set for a particular (but reasonably large) value of \( t \), it turns out to be very easy to "guess" what these coefficients are.

As an example, consider the best Braunschadel basis for \( r=0, t=20 \):

\[
\begin{align*}
k_{ij} &= (9, 2, 3, 7, 2, 2), & c_{ij} &= (2, 2, -1, 3, -1, 0) \\
\text{gives} & & a_i &= \{1, 182, 10780, 438441\} \\
\text{with} & & C &= 28491279
\end{align*}
\]

We can easily work out that:

\[
28491279 = 64a_4 + 39a_3 + 58a_2 + 79
\]

from which we can make the educated guess that:

\[
C = (3t + 4)a_4 + (2t - 1)a_3 + (3t - 2)a_2 + (4t - 1)
\]

thus suggesting coefficients as follows:

\[
(k_{51}, k_{52}, k_{53}, k_{54}) = (4, 3, 2, 3) \\
\text{and} & & (c_{51}, c_{52}, c_{53}, c_{54}) &= (-1, -2, -1, 4)
\]

We can check out these values by calculating the cover using this formula for other values of \( t \), and checking that these match with the computed values.

The final step is to expand the formula for the cover into a polynomial in \( t \); we find that in this case:

\[
C = 162t^4 + 318t^3 + 68t^2 + 4t - 1
\]

We can compare this with the corresponding polynomial for the best Hofmeister basis for \( r=0 \), which turns out to be:

\[
C' = 162t^4 + 312t^3 + 137t^2 + 19t - 2
\]

From this it's immediately obvious that \( C > C' \) for large enough \( t \), although we know that \( C < C' \) for small \( t \).

Some results are given in Tables 4, 5, 6 and 7.
### Table 4
Coefficients for the best Hofmeister bases

| r | c_{21} | c_{43} | k_{51}-k_{54} | c_{51}-c_{54} | coeffs of powers of t |
|---|--------|--------|---------------|---------------|-----------------------|
| 0 | 2      | 1      | 1             | 0             | 1                     | 5 3 2 3 -1 -1 0 1 | 162 303 181 33 -1 |
| 1 | 0      | 0      | 0             | 0             | 0                     | 5 3 2 3 -1 -1 -1 3 | 162 312 137 19 -2 |
| 1 | 1      | 0      | 2             | 1             | 1                     | 5 3 2 3 -1 1 -1 2 | 162 366 252 46 2 |
| 2 | 2      | 1      | 1             | 1             | 1                     | 5 3 2 3 -1 0 0 2  | 162 411 366 125 11 |
| 1 | 0      | 2      | 1             | 1             | 0                     | 5 3 2 3 -1 1 -1 3 | 162 420 320 68 4  |
| 3 | 3      | 1      | 2             | 1             | 1                     | 5 3 2 3 0 1 0 2   | 162 483 504 207 27 |
| 4 | 3      | 1      | 2             | 1             | 1                     | 5 3 2 3 0 1 0 3   | 162 537 611 274 39 |
| 5 | 3      | 1      | 2             | 1             | 1                     | 5 3 2 3 0 1 0 4   | 162 591 718 341 51 |
| 6 | 3      | 1      | 2             | 1             | 1                     | 5 3 2 3 0 1 0 5   | 162 645 825 408 63 |
| 7 | 7      | 3      | 2             | 1             | 2                     | 5 3 2 3 2 1 1 3   | 162 690 1064 700 164 |
| 8 | 8      | 4      | 1             | 7             | 1 0                    | 5 3 2 3 2 0 -1 7  | 162 708 886 453 95 |
| 9 | 7      | 3      | 3             | 5             | 2 2                    | 5 3 2 3 2 2 1 4   | 162 798 1435 1109 308 |
| 10 | 7      | 3      | 3             | 5             | 2 2                    | 5 3 2 3 2 0 -1 9  | 162 816 1070 565 125 |
| 11 | 6      | 1      | 1 10          | 1             | 0                     | 9 3 2 3 8 0 -1 9 | 162 870 1298 741 180 |
| 11 | 10     | 4      | 3 7 2 2      | 5             | 3 2 3 4 2 1 5         | 162 906 1851 1642 533 |
| 11 | 11     | 5      | 2 9 2 0      | 5             | 3 2 3 4 1 -1 9        | 162 924 1565 1048 267 |

### Table 5
Coefficients for the best Braunschadel bases

| r | c_{21} | c_{43} | k_{51}-k_{54} | c_{51}-c_{54} | coeffs of powers of t |
|---|--------|--------|---------------|---------------|-----------------------|
| 0 | 2      | 2      | -1            | -1            | 0                     | 4 3 2 3 -1 -2 -1 4   | 162 318 68 4 -1 |
| 1 | 0      | 2      | -1            | -1            | 0                     | 5 3 2 3 2 1 0 1 162 363 218 35 4 |
| 2 | 2      | 2      | 1             | 0             | 0                     | 4 3 2 3 -1 -2 -1 5   | 162 372 84 6 0 |
| 2 | 0      | 2      | -1            | -1            | 0                     | 4 3 2 3 -2 1 0 2 162 417 303 60 -6 |
| 5 | 3      | 3      | 0             | 6             | 0                     | 4 3 2 3 0 -1 1 5 162 455 198 12 3 |
| 3 | 5      | 3      | 0             | 6             | 0                     | 4 3 2 3 0 1 0 2 162 525 570 240 32 |
| 6 | 3      | 3      | 0             | 6             | 0                     | 4 3 2 3 0 -1 1 6 162 543 415 155 28 |
| 5 | 3      | 3      | 2             | 1             | 1                     | 9 2 2 3 1 1 0 3 162 579 691 318 46 |
| 6 | 3      | 3      | 2             | 1             | 1                     | 9 2 2 3 1 1 0 4 162 633 812 396 60 |
| 7 | 8      | 4      | 1             | 9             | 1                     | 4 3 2 3 1 0 -1 7 162 696 854 460 108 |
| 8 | 9      | 4      | 1             | 9             | 1                     | 4 3 2 3 2 0 0 6 162 741 1123 738 188 |
| 9 | 9      | 3      | 2             | 8             | 1                     | 8 3 2 3 6 1 0 6 162 795 1324 938 243 |
| 10 | 7     | 1      | 4             | 5             | 2                     | 7 3 2 3 4 1 4 162 840 1577 1263 362 |
| 11 | 11     | 4      | 2             | 10            | 1                     | 4 3 2 3 3 1 1 6 162 912 1816 1546 478 |

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Table 6
Coefficients for some other good Braunschadel bases for r=11

The sets considered are all those with cover greater than or equal to that of the Braunschadel basis with $c_{ij} = (7, 1, 4, 5, 2, 2)$ in the coefficient range $[0,20], [-10,10], [-2,8], [-5,15], [-5,5], [-2,2])$.

| $c_{21}$ | ... | $c_{43}$ | $k_{51}-k_{54}$ | $c_{51}-c_{54}$ | coeffs of powers of $t$ |
|----------|-----|----------|-----------------|-----------------|--------------------------|
| 7        | 1   | 4        | 2               | 2               | 7 3 2 3 4 3 1 5 162 894 1767 1478 439 |
| 9        | 3   | 2        | 8               | 1               | 1 8 3 2 6 1 0 8 162 903 1602 1184 319 |
| 10       | 3   | 3        | 9               | 1               | 2 9 2 3 8 1 1 5 162 894 1773 1518 476 |
| 11       | 4   | 2        | 10              | 1               | 2 4 3 2 3 1 1 6 162 912 1816 1546 478 |
| 11       | 5   | 2        | 12              | 0               | 0 4 3 2 3 2 1 -1 9 162 912 1525 1058 292 |
| 12       | 5   | 1        | 12              | 1               | 0 4 3 2 3 3 0 -1 10 162 930 1436 839 226 |
| 12       | 6   | 1        | 13              | 1               | 1 4 3 2 3 3 0 0 8 162 903 1602 1189 347 |

Table 7
Coefficients for some other good Hofmeister bases for r=0

The sets considered are all those whose cover for $t=18$ is greater than 18634000 in the (narrow) coefficient range $[-2,2], [-3,1], [0,2], [-6,1], [0,2], [0,1])$.

| $c_{21}$ | ... | $c_{43}$ | $k_{51}-k_{54}$ | $c_{51}-c_{54}$ | coeffs of powers of $t$ |
|----------|-----|----------|-----------------|-----------------|--------------------------|
| 1        | 0   | 0        | 0               | 0               | 5 3 2 3 -1 -1 -1 3 162 312 137 19 -2 |
| -2       | -1  | 0        | -2              | 0               | 5 3 2 3 -3 -1 -1 4 162 312 55 -12 -8 |
| 2        | 1   | 0        | 1               | 0               | 5 3 2 3 -1 -1 0 1 162 303 181 33 -1 |
| 0        | 0   | 1        | 0               | 1               | 5 3 2 3 -2 0 -1 2 162 294 162 24 -2 |
| 0        | -1  | 2        | -1              | 1               | 6 3 2 3 -2 1 -1 1 162 294 148 2 -2 |
| 0        | 0   | 1        | 0               | 0               | 6 3 2 3 -2 0 -1 2 162 294 135 7 -2 |
| 0        | 0   | 0        | 0               | 0               | 5 3 2 3 -2 -1 -1 3 162 294 107 13 -2 |
| 1        | 1   | 0        | 1               | 1               | 5 3 2 3 -2 -1 0 1 162 285 181 34 -1 |
| 1        | 0   | 1        | 0               | 0               | 8 1 2 3 0 -1 1 0 162 285 149 21 0 |
| -2       | -1  | 1        | -2              | 0               | 6 3 2 3 -3 0 0 1 162 285 112 -17 -8 |
| -2       | -2  | 2        | -3              | 1               | 5 3 2 3 -2 1 0 0 162 285 107 -37 -4 |
| -2       | -2  | 1        | -3              | 0               | 5 3 2 3 -2 0 0 1 162 285 106 -28 -9 |
| -2       | -2  | 2        | -2              | 0               | 5 2 2 3 -2 2 0 0 162 285 71 -17 -6 |
| 2        | 1   | 0        | 1               | 0               | 8 1 2 3 0 -1 -1 2 162 276 162 47 2 |
| 2        | 1   | 0        | 1               | 0               | 8 1 2 3 1 -2 0 2 162 276 144 23 -1 |
| -1       | -2  | 2        | -3              | 2               | 5 3 2 3 -1 1 -1 1 162 276 139 -16 -3 |
| 2        | 0   | 0        | 0               | 0               | 8 1 2 3 2 -2 0 2 162 276 138 14 -2 |
| -1       | -1  | 1        | -2              | 1               | 6 3 2 3 -2 0 -1 2 162 276 126 0 -6 |
| -1       | -2  | 1        | -3              | 1               | 5 3 2 3 -1 0 -1 2 162 276 120 -10 -6 |
| -1       | -1  | 2        | -1              | 1               | 6 3 2 3 -3 1 -1 1 162 276 118 -11 -3 |
| -1       | -1  | 1        | -1              | 0               | 5 2 2 3 -2 0 0 2 162 276 117 -2 -4 |
| -1       | -2  | 2        | -2              | 1               | 5 3 2 3 -1 1 -1 1 162 276 112 -19 -1 |
| -1       | 0   | 0        | 0               | 0               | 6 3 2 3 -3 0 -1 2 162 276 105 -3 -2 |
| -1       | 0   | 0        | -1              | 1               | 5 3 2 3 -3 -1 -1 3 162 276 104 28 -8 |
| -1       | -1  | 2        | 0               | 0               | 5 2 2 3 -2 2 -1 1 162 276 82 -5 -1 |
| -1       | -1  | 2        | -1              | 0               | 5 2 2 3 -2 2 -1 1 162 276 82 -8 -2 |
| -1       | -1  | 0        | -2              | 0               | 8 1 2 3 -1 -2 0 3 162 276 80 -10 -5 |
| -1       | 0   | 0        | -1              | 0               | 6 3 2 3 -2 -1 -1 3 162 276 77 5 -4 |
| -1       | -2  | 0        | -3              | 0               | 5 3 2 3 -1 -1 -1 3 162 276 65 -18 -7 |
From these tables, we see that the best Braunschadel basis for \( t=20 \) will eventually better the best Hofmeister basis for \( t=20 \) only in the case of \( r = 0, 1 \) or 2. Of course, this does not mean that Hofmeister is always best in all other cases - simply that we have not yet found a Braunschadel basis with better coefficients.

The case of \( r=11 \) is particularly interesting, since we know that a Braunschadel basis is best for \( t = 8 \) to 20, whereas Tables 4 and 5 show that the best Hofmeister basis for \( t=20 \) must eventually dominate. But Table 6 reveals another Braunschadel basis with yet a higher coefficient of \( t^3 \), and we find that the optimal basis for \( r=11 \) is likely to switch from Hofmeister to Braunschadel and back again as follows:

| \( t \) range | type | \( c_{21} - c_{43} \) | coeffs of powers of \( t \) |
|---------------|------|----------------------|-----------------------------|
| [2-7]         | H    | (10, 4, 3, 7, 2, 2)  | 162 906 1851 1642 533       |
| [8-22]        | B    | (11, 4, 2, 10, 1, 2) | 162 912 1816 1546 748       |
| 23            | H    | (11, 5, 2, 9, 2, 0)  | 162 924 1565 1048 267       |
| [24-...]      | B    | (12, 5, 1, 12, 1, 0) | 162 930 1436 839 226        |

[ Later results confirm this behaviour - see section 6 and Appendix 1 ]

According to Mossige's thesis (see Mossige, Svein, [6], p38), the formula for the cover of any Hofmeister basis must conform to one of seven possibilities \( n_i(x,P) \). The coefficients \( c_{5j} \) in these possibilities depend on the coefficients \( c_{ij} \) for \( i \leq 4 \), but the coefficients \( k_{5j} \) are explicit, as follows:

\[
\begin{align*}
n_1 &= (8, 1, 0, 3) \\
n_2 &= (5, 3, 2, 3) \\
n_3 &= (9, 3, 2, 3) \\
n_4 &= (9, 3, 2, 3) \\
n_5 &= (8, 1, 2, 3) \\
n_6 &= (9, 1, 2, 3) \\
n_7 &= (5, 2, 2, 3)
\end{align*}
\]

Examining Tables 4 and 7 above show that the bases in these tables conform to these requirements with the exception of those with coefficients \( (6, 3, 2, 3) \). This was at first a mystery - even after I had taken into account Selmer's correction to Mossige's thesis (Selmer, E.S., [8]; see also Appendix 3) - but closer examination of my own notes after first reading the thesis in March 1992 (see particularly Mossige, Svein, [6] pp43-45) show that a further case omitted by Mossige and not noticed by Selmer can generate coefficients \( (6, 3, 2, 3) \).

5 The future

Future work should really be based on "Proposition 6.1" (and its analogue for Braunschadel bases - probably very similar) from Mossige's thesis: this gives conditions which, if met by a Hofmeister basis, determine that it is admissible, and also give a formula for its cover. It is possible (although by no means certain) that using these conditions would speed up \( \text{exp21} \) so that wider ranges of \( c_{ij} \) might be investigated (or, at least, easily rejected), and larger values of \( t \) investigated.

Mossige's 2.008... result shows that there is no "finite set of formulae" for \( d=4 \), and I now conjecture instead that all maximal sets for sufficiently large \( h \) are either Hofmeister or Braunschadel - and that the coefficient sets \( c_{ij} \) will, for a fixed value of \( r \), change from time to time, slowly but surely moving away from \((0,0,0,0,0,0,0)\) without limit. Some interesting further questions can also be posed:
Will one or the other form eventually dominate?
Will one or the other form eventually dominate for a given value of r?
What is the "sufficiently large value" of h?

6 Afterword - further results

Following a request from Selmer in December 1992 for "good" bases for larger values of t, I looked again at the program I had been using to investigate these bases. I was able to incorporate some worthwhile improvements* to the algorithms used, allowing me to extend the searches for good bases to around t=58. Beyond this point the cover exceeds $2^{31}$, and a major coding change to double-length working will be needed to progress further.

*The first improvement was to check admissibility criteria: if a candidate set \{1, a_2, a_3, a_4\} proves to be inadmissible, then all other candidate sets \{1, a_2, a_3, a_4'\} are rejected without further ado. The second improvement was to the code which checks the cover of a "good" set. Advantage is taken of the fact that if $x$ has a generation:

$$x = c_4a_4 + c_3a_3 + c_2a_2 + c_1$$

then so do all $(x-ia_4)$ for $0 < i <= c_4$.

This significantly reduces the time taken to determine the full cover of a good set.

These new results are summarised in the following tables*, which should be considered as extensions of tables 2, 3, 4 and 5 above.

I have also calculated and compared the covers of the best Hofmeister and Braunschadel bases for each value of t and r, and the results are summarised in Appendix 1 which is, effectively, a table of the best bases found so far for $k=4$ and 144 <= h <= 707.

*These results have now (March 93) been further confirmed by extended searches. For Hofmeister bases, the searches covered the range:

$c_{21}$: [-8, 21] $c_{31}$: [-11, 18] $c_{32}$: [-15, 14] $c_{41}$: [-8, 21] $c_{42}$: [-14, 15] $c_{43}$: [-16, 13]

and for Braunschadel bases:

$c_{21}$: [-7, 22] $c_{31}$: [-10, 19] $c_{32}$: [-15, 14] $c_{41}$: [-5, 24] $c_{42}$: [-14, 15] $c_{43}$: [-16, 13]

Any basis whose cover for t=58 exceeded that of the best t=21 basis was printed out.

No improved bases for any t in the range 21 <= t <= 58 were found.
### Table 8
Probably the best Hofmeister bases for 21<=t<=58

| r   | t_range    | best coefficients | range of coefficients checked |
|-----|------------|-------------------|------------------------------|
|     |            | $c_{21}$ $c_{31}$ $c_{32}$ $c_{41}$ $c_{42}$ $c_{43}$ | $c_{21}$ $c_{31}$ $c_{32}$ $c_{41}$ $c_{42}$ $c_{43}$ |
| 0   | [21-22]    | 1 0 0 0 0 0       | [-2.9] [-1.6] [-6.1] [-2.10] [-2.1] [-6.1] |
|     | [23-39]    | 2 1 -1 2 0 -2     |                             |
|     | [40-58]    | 6 5 -4 8 -1 -5    |                             |
| 1   | [21-30]    | 1 0 2 1 1 0       | [-2.9] [-1.6] [-4.3] [-2.10] [-1.2] [-6.1] |
|     | [31-34]    | 2 1 -1 2 0 -2     |                             |
|     | [35-58]    | 6 4 -3 7 0 -5     |                             |
| 2   | [21-28]    | 1 0 2 1 1 0       | [-2.9] [-1.6] [-4.3] [-2.10] [-1.2] [-6.1] |
|     | [29-40]    | 5 3 -1 5 0 -2     |                             |
|     | [41-58]    | 6 4 -3 7 0 -5     |                             |
| 3   | [21-48]    | 3 1 2 2 1 1       | [-1.10] [-1.6] [-4.3] [-1.11] [-1.2] [-6.1] |
|     | [49-58]    | 6 4 -3 7 0 -5     |                             |
| 4   | [21-48]    | 3 1 2 2 1 1       | [1.12] [0.7] [-4.3] [0.12] [-1.2] [-6.1] |
|     | [49-58]    | 9 6 -3 10 0 -5    |                             |
| 5   | [21-45]    | 3 1 2 2 1 1       | [1.12] [0.7] [-4.3] [0.12] [0.3] [-6.1] |
|     | [46-58]    | 9 5 -2 9 1 -5     |                             |
| 6   | [21-49]    | 3 1 2 2 1 1       | [1.12] [0.7] [-4.3] [0.12] [0.3] [-6.1] |
|     | [50-58]    | 9 5 -2 9 1 -5     |                             |
| 7   | [21-36]    | 8 4 1 7 1 0       | [3.14] [1.8] [-3.4] [2.14] [0.3] [-5.2] |
|     | [37-58]    | 9 5 0 9 1 -2      |                             |
| 8   | [21-40]    | 8 4 1 7 1 0       | [3.14] [1.8] [-3.4] [2.14] [0.3] [-5.2] |
|     | [41-58]    | 9 5 0 9 1 -2      |                             |
| 9   | [21]       | 7 3 3 5 2 2       | [4.15] [2.9] [-2.5] [3.15] [0.3] [-4.3] |
|     | [22-29]    | 8 4 1 7 1 0       |                             |
|     | [30-58]    | 12 7 -1 12 1 -3   |                             |
| 10  | [21-34]    | 11 6 1 10 1 0     | [6.17] [3.10] [-3.4] [5.17] [0.3] [-5.2] |
|     | [35-58]    | 12 7 -1 12 1 -3   |                             |
| 11  | [21-40]    | 11 5 2 9 2 0     | [6.17] [3.10] [-3.4] [5.17] [0.3] [-5.2] |
|     | [41-58]    | 12 7 -1 12 1 -3   |                             |

Notes: The ranges given apply to all values of t for the given value of r. Although not as extensive as the ranges used when checking values of t<=20, some limited experiments with much wider ranges for certain values of r suggest that they are adequate [but see note on previous page: much more extensive searches now give greater confidence in these results (March 93)].
| r  | t_range   | c_{21} | c_{31} | c_{32} | c_{41} | c_{42} | c_{43} | c_{21} range | c_{31} range | c_{32} range | c_{41} range | c_{42} range | c_{43} range |
|----|-----------|--------|--------|--------|--------|--------|--------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0  | [21-27]   | 2      | 2      | -1     | 3      | -1     | 0      | [-3,10]     | [-1,7]      | [-6,2]      | [-2,12]     | [-3,1]      | [-6,2]      |
|    | [28-53]   | 3      | 3      | -2     | 5      | -1     | -2     |             |             |             |             |             |             |
|    | [54-58]   | 4      | 4      | -3     | 7      | -1     | -4     |             |             |             |             |             |             |
| 1  | [21-31]   | 2      | 2      | -1     | 3      | -1     | 0      | [-2,11]     | [0,8]       | [-7,1]      | [0,14]      | [-3,1]      | [-7,1]      |
|    | [32-37]   | 3      | 3      | -2     | 5      | -1     | -2     |             |             |             |             |             |             |
|    | [38-58]   | 7      | 6      | -4     | 11     | -1     | -5     |             |             |             |             |             |             |
| 2  | [21-31]   | 5      | 3      | -1     | 6      | -1     | 0      | [0,13]      | [1,9]       | [-7,1]      | [2,16]      | [-3,1]      | [-7,1]      |
|    | [32-43]   | 6      | 4      | -2     | 8      | -1     | -2     |             |             |             |             |             |             |
|    | [44-58]   | 7      | 6      | -4     | 11     | -1     | -5     |             |             |             |             |             |             |
| 3  | [21-30]   | 5      | 3      | 0      | 6      | 0      | 0      | [0,13]      | [1,9]       | [-6,2]      | [2,16]      | [-3,1]      | [-7,1]      |
|    | [31-50]   | 6      | 4      | -1     | 8      | 0      | -2     |             |             |             |             |             |             |
|    | [51-58]   | 7      | 6      | -4     | 11     | -1     | -5     |             |             |             |             |             |             |
| 4  | [21-34]   | 5      | 3      | 0      | 6      | 0      | 0      | [1,14]      | [1,9]       | [-6,2]      | [3,17]      | [-3,1]      | [-7,1]      |
|    | [35-47]   | 6      | 4      | -1     | 8      | 0      | -2     |             |             |             |             |             |             |
|    | [48-58]   | 10     | 7      | -4     | 14     | -1     | -5     |             |             |             |             |             |             |
| 5  | [21-25]   | 3      | 0      | 3      | 2      | 1      | 1      | [0,13]      | [0,8]       | [-4,4]      | [1,15]      | [-2,2]      | [-6,2]      |
|    | [26-37]   | 5      | 3      | 0      | 6      | 0      | 0      |             |             |             |             |             |             |
|    | [38-41]   | 6      | 4      | -1     | 8      | 0      | -2     |             |             |             |             |             |             |
|    | [42-58]   | 7      | 6      | -3     | 14     | 0      | -5     |             |             |             |             |             |             |
| 6  | [21-38]   | 8      | 4      | 0      | 9      | 0      | 0      | [3,16]      | [2,10]      | [-6,2]      | [5,19]      | [-2,2]      | [-7,1]      |
|    | [39-47]   | 9      | 5      | -1     | 11     | 0      | -2     |             |             |             |             |             |             |
|    | [48-58]   | 10     | 7      | -3     | 14     | 0      | -5     |             |             |             |             |             |             |
| 7  | [21-37]   | 8      | 4      | 1      | 9      | 1      | 0      | [3,16]      | [2,10]      | [-5,3]      | [5,19]      | [-2,2]      | [-7,1]      |
|    | [38-55]   | 9      | 5      | 0      | 11     | 1      | -2     |             |             |             |             |             |             |
|    | [56-58]   | 10     | 7      | -3     | 14     | 0      | -5     |             |             |             |             |             |             |
| 8  | [21]      | 9      | 4      | 1      | 9      | 1      | 1      | [5,18]      | [2,10]      | [-5,3]      | [6,20]      | [-2,2]      | [-6,2]      |
|    | [22-40]   | 8      | 4      | 1      | 9      | 1      | 0      |             |             |             |             |             |             |
|    | [41-52]   | 9      | 5      | 0      | 11     | 1      | -2     |             |             |             |             |             |             |
|    | [53-58]   | 13     | 8      | -3     | 17     | 0      | -5     |             |             |             |             |             |             |
| 9  | [21-27]   | 9      | 3      | 2      | 8      | 1      | 1      | [5,18]      | [2,10]      | [-4,4]      | [6,20]      | [-1,3]      | [-6,2]      |
|    | [28-49]   | 10     | 4      | 1      | 10     | 1      | -1     |             |             |             |             |             |             |
|    | [50-58]   | 13     | 8      | -2     | 17     | 1      | -5     |             |             |             |             |             |             |
| 10 | [21-24]   | 9      | 3      | 2      | 8      | 1      | 1      | [5,18]      | [2,10]      | [-4,4]      | [6,20]      | [-1,3]      | [-6,2]      |
|    | [25-37]   | 11     | 5      | 1      | 12     | 1      | 0      |             |             |             |             |             |             |
|    | [38-51]   | 10     | 4      | 1      | 10     | 1      | -1     |             |             |             |             |             |             |
|    | [52-58]   | 13     | 8      | -2     | 17     | 1      | -5     |             |             |             |             |             |             |
| 11 | [21-22]   | 11     | 4      | 2      | 10     | 1      | 2      | [6,19]      | [1,9]       | [-3,5]      | [5,19]      | [-1,3]      | [-4,4]      |
|    | [23-48]   | 12     | 5      | 1      | 12     | 1      | 0      |             |             |             |             |             |             |
|    | [49-58]   | 13     | 6      | 0      | 14     | 1      | -2     |             |             |             |             |             |             |

Note: The ranges given apply to all values of t for the given value of r.
Table 10
Coefficients for the best Hofmeister bases for 21<=t<=58

| r  | c_{21} | c_{43} | k_{51}-k_{54} | c_{51}-c_{54} | coeffs of powers of t |
|----|--------|--------|----------------|----------------|----------------------|
| 0  | 1      | 0      | 0              | 0              | 5 3 2 3 -1 -1 -1 3  | 162 312 137 19 -2    |
| 2  | 1      | -1     | 2 0 -2         | 5 3 2 3 -1 -2 -3 7 | 162 330 -269 49 26  |
| 6  | 5      | -4     | 8 -1 -5        | 9 3 2 3 1 -5 -6 14 | 162 375 -2067 907 1443 |
| 1  | 1      | 0      | 2 1 1 0       | 5 3 2 3 -1 1 -1 2 | 162 366 252 46 2    |
| 2  | 1      | -1     | 2 0 -2         | 5 3 2 3 -1 -2 -3 8 | 162 384 -303 56 30  |
| 6  | 4      | -3     | 7 0 -5        | 5 3 2 3 0 -4 -6 14 | 162 429 -1875 444 1138 |
| 2  | 1      | 0      | 2 1 1 0       | 5 3 2 3 -1 1 -1 3 | 162 420 320 68 4    |
| 5  | 3      | -1     | 5 0 -2        | 9 3 2 3 2 -2 -3 8 | 162 438 -183 -57 70  |
| 6  | 4      | -3     | 7 0 -5        | 5 3 2 3 0 -4 -6 15 | 162 483 -2002 460 1215 |
| 3  | 3      | 1      | 2 1 1        | 5 3 2 3 0 1 0 2 | 162 483 504 207 27  |
| 6  | 4      | -3     | 7 0 -5        | 5 3 2 3 0 -4 -6 16 | 162 537 -2129 476 1292 |
| 4  | 3      | 1      | 2 1 1        | 5 3 2 3 0 1 0 3 | 162 537 611 274 39  |
| 9  | 6      | -3     | 10 0 -5       | 9 3 2 3 4 -4 -6 16 | 162 591 -1982 -313 1934 |
| 5  | 3      | 1      | 2 1 1        | 5 3 2 3 0 1 0 4 | 162 591 718 341 51  |
| 9  | 5      | -2     | 9 1 -5       | 5 3 2 3 2 -3 -6 16 | 162 645 -1736 -762 1381 |
| 6  | 3      | 1      | 2 1 1        | 5 3 2 3 0 1 0 5 | 162 645 825 408 63  |
| 9  | 5      | -2     | 9 1 -5       | 5 3 2 3 2 -3 -6 17 | 162 699 -1827 -819 1464 |
| 7  | 8      | 4      | 1 7 1        | 5 3 2 3 2 0 -1 7 | 162 708 886 453 95  |
| 9  | 5      | 0      | 9 1 -2      | 5 3 2 3 2 -1 -3 11 | 162 726 240 -234 66  |
| 8  | 8      | 4      | 1 7 1        | 5 3 2 3 2 0 -1 8 | 162 762 978 509 110  |
| 9  | 5      | 0      | 9 1 -2      | 5 3 2 3 2 -1 -3 12 | 162 780 266 -252 74  |
| 9  | 7      | 3      | 3 5 2        | 5 3 2 3 2 2 1 4 | 162 798 1435 1109 308 |
| 8  | 4      | 1      | 7 1 0       | 5 3 2 3 2 0 -1 9 | 162 816 1070 565 125 |
| 12 | 7      | -1     | 12 1 -3    | 5 3 2 3 3 -2 -4 15 | 162 861 -223 -939 584 |
| 10 | 11     | 6      | 1 10 1      | 9 3 2 3 8 0 -1 9 | 162 870 1298 741 180 |
| 12 | 7      | -1     | 12 1 -3    | 5 3 2 3 3 -2 -4 16 | 162 915 -224 -1002 623 |
| 11 | 11     | 5      | 2 9 2      | 5 3 2 3 4 1 -1 9 | 162 924 1565 1048 267 |
| 12 | 7      | -1     | 12 1 -3    | 5 3 2 3 3 -2 -4 17 | 162 969 -225 -1065 662 |
Table 11
Coefficients for the best Braunschadel bases for 21<=t<=58

| r  | c_{21} ... c_{43} | k_{51}-k_{54} | c_{51}-c_{54} | coeffs of powers of t |
|----|-------------------|---------------|---------------|-----------------------|
| 0  | 2 2 -1 3 -1 0     | 4 3 2 3      | -1 -2 -1 4    | 162 318 68 4 -1      |
| 1  | 3 -3 2 -1 -2 -2   | 4 3 2 3      | -1 -3 -3 8    | 162 354 -1400 642 444 |
| 2  | 3 -3 2 -1 1 -1    | 4 3 2 3      | -1 -2 -1 5    | 162 372 84 6 0       |
| 3  | 3 -3 2 -1 1 -1 7  | 4 3 2 3      | -1 -3 -3 9    | 162 390 -482 133 71  |
| 7  | 6 -4 11 -1 -5 7   | 4 3 2 3      | 0 -5 -6 15    | 162 435 -2207 851 1807 |
| 2  | 5 3 -1 6 -1 0     | 4 3 2 3      | 0 -2 -1 5    | 162 426 198 12 -3    |
| 6  | 4 -2 8 -1 -2 4    | 4 3 2 3      | 0 -3 -3 9    | 162 444 -368 -63 168 |
| 7  | 6 -4 11 -1 -5 4    | 4 3 2 3      | 0 -5 -6 17    | 162 543 -2493 917 2035 |
| 4  | 5 3 0 6 0 0       | 4 3 2 3      | 0 -1 -1 6    | 162 534 415 155 28   |
| 6  | 4 -1 8 0 -2 4    | 4 3 2 3      | 0 -2 -3 9    | 162 498 -179 -60 102 |
| 7  | 6 -4 11 -1 -5 7   | 4 3 2 3      | 0 -5 -6 17    | 162 543 -2493 917 2035 |
| 5  | 3 0 3 2 1 1 9     | 2 3 3 1      | 1 -1 -1 7    | 162 642 635 245 52   |
| 6  | 4 -1 8 0 -2 4    | 4 3 2 3      | 0 -2 -3 11   | 162 606 -207 -74 126 |
| 7  | 6 -4 11 -1 -5 7   | 4 3 2 3      | 0 -5 -6 17    | 162 543 -2493 917 2035 |
| 8  | 4 1 9 1 0 4      | 3 2 3 1      | 1 -2 -1 11   | 162 694 854 460 108  |
| 9  | 5 0 1 1 2 4      | 3 2 3 1      | 1 -1 -1 11   | 162 714 204 -179 87  |
| 10 | 7 -3 14 0 -5 4    | 3 2 3 1      | 1 -4 -6 18   | 162 705 -2183 -483 2421 |
| 7  | 8 4 0 9 0 0      | 4 3 2 3      | 1 -1 -1 7    | 162 642 635 245 52   |
| 9  | 5 -1 1 1 0 4      | 3 2 3 1      | 1 -2 -3 11   | 162 660 -39 -251 204 |
| 10 | 7 -3 14 0 -5 4    | 3 2 3 1      | 1 -4 -6 18   | 162 705 -2183 -483 2421 |
| 8  | 4 1 9 1 0 4      | 3 2 3 1      | 1 -2 -1 11   | 162 694 854 460 108  |
| 9  | 5 0 1 1 2 4      | 3 2 3 1      | 1 -1 -1 11   | 162 714 204 -179 87  |
| 10 | 7 -3 14 0 -5 4    | 3 2 3 1      | 1 -4 -6 18   | 162 705 -2183 -483 2421 |
| 8  | 4 1 9 1 0 4      | 3 2 3 1      | 1 -2 -1 11   | 162 694 854 460 108  |
| 9  | 5 0 1 1 2 4      | 3 2 3 1      | 1 -1 -1 11   | 162 714 204 -179 87  |
| 10 | 7 -3 14 0 -5 4    | 3 2 3 1      | 1 -4 -6 18   | 162 705 -2183 -483 2421 |
| 9  | 4 1 9 1 0 4      | 3 2 3 1      | 1 -2 -1 11   | 162 694 854 460 108  |
| 10 | 7 -3 14 0 -5 4    | 3 2 3 1      | 1 -4 -6 18   | 162 705 -2183 -483 2421 |
| 11 | 4 2 10 1 0 4     | 3 2 3 1      | 1 -2 -1 11   | 162 694 854 460 108  |
| 12 | 5 1 12 1 0 4     | 3 2 3 1      | 1 -2 -1 11   | 162 694 854 460 108  |
| 13 | 6 0 14 1 -2 8    | 3 2 3 1      | 8 -1 -3 14   | 162 948 588 -420 187 |
7 Sequences of bases

Examination of these results shows that from time to time sequences of good bases appear, whose coefficients differ one from the other by a constant amount. One sequence, particularly prominent in the Braunschadel results, has a coefficient difference set of (1, 1, -1, 2, 0, -2); examples include:

```
0  22-27  2  2  -1  3  -1  0
  [28-53]  3  3  -2  5  -1  -2
  [54-58]  4  4  -3  7  -1  -4

7  21-37  8  4  1  9  1  0
  [38-55]  9  5  0  11  1  -2

11  21-22  11  4  2  10  1  2
  [23-48]  12  5  1  12  1  0
  [49-58]  13  6  0  14  1  -2
```

and the sequence also appears in the Hofmeister results:

```
0  21-22  1  0  0  0  0  0
  [23-39]  2  1  -1  2  0  -2

7  21-36  8  4  1  7  1  0
  [37-58]  9  5  0  9  1  -2
```

In his thesis, Mossige gives criteria which the coefficients of a Hofmeister basis must satisfy in order to form a basis, and also a formula for the h-range when these criteria are satisfied. It is interesting to apply these criteria and formulae to the sequence of Hofmeister bases defined by the parametrised coefficient set:

\[(8+p, 4+p, 1-p, 7+2p, 1, -2p)\] for \(h = 12t + 7\)

Using the notation of (Mossige, Svein, [6] page 37) we have:

\[P = (d_1, d_2, d_3, d_4, d_5, d_6)\] where

\[
\begin{align*}
d_1 &= 8+p \\
d_2 &= 4+p \\
d_3 &= 1-p \\
d_4 &= 7+2p \\
d_5 &= 1 \\
d_6 &= -2p
\end{align*}
\]

We first check that both the "admissibility" conditions are always \(\geq 0\):

\[
\begin{align*}
f_8(P) &= 0 \\
f_9(P) &= 3p+4
\end{align*}
\]
Next, we check that there is no need to take Selmer's correction into account:

\[-d_1 - 3d_2 + 3d_4 = 2p + 1\]
\[-2d_3 + 3d_3 + 3 = 2p + 4\]

and now evaluate \(f_i(x, P)\) for \(i=1\) to \(7:\)

\[
\begin{align*}
  f_1(x, P) &\geq 0 \implies x \leq 4p + 7 \quad \text{[this is } f_{1A}, \text{ since the condition for } f_{1B} \text{ evaluates to } 2p + 3 < 0]\n  f_2(x, P) &\geq 0 \implies x \leq 4p + 6 \\
  f_3(x, P) &\geq 0 \implies x \leq 5p + 6 \\
  f_4(x, P) &\geq 0 \implies x \leq 5p + 8 \\
  f_5(x, P) &\geq 0 \implies x \leq 5p + 7 \\
  f_6(x, P) &\geq 0 \implies x \leq 4p + 6 \\
  f_7(x, P) &\geq 0 \implies x \leq 4p + 6
\end{align*}
\]

Clearly, the maximum permissible value of \(x\) is \(4p + 6\), and so \(n_2, n_6\) and \(n_7\) are candidate formulae for the cover. Of these, \(n_2\) has the lowest coefficient of \(a_3\) and so defines the value of the \(h\)-range:

\[
n_2(4p + 6, P) = (3t + (4p + 7))a_4 + (2t - (2p + 1))a_3 + (3t - p)a_2 + (5t + 2)
\]

Somewhat reassuringly, this means that:

\[
(k_{51}, k_{52}, k_{53}, k_{54}) = (5, 3, 2, 3)
\]
and

\[
(c_{51}, c_{52}, c_{53}, c_{54}) = (2, -p, -(2p + 1), (4p + 7))
\]

which is confirmed by the appropriate entries in Table 6 for \(r=7\).

Finally, we can calculate the coefficient \(X\) of \(t^3\) in the \(h\)-range; using the formula of Appendix 2 we have:

\[
X = \left( k_{54}k_{43}k_{32}c_{21} + k_{54}k_{43}c_{32}k_{21} + k_{54}c_{43}k_{32}k_{21} + c_{54}k_{43}k_{32}k_{21} \\
+ k_{54}k_{43}k_{31} + k_{54}k_{42}k_{21} + k_{53}k_{32}k_{21} \right)
= 18(8+p) + 54(1-p) - 162p + 54(4p+7)
= 18p + 708
\]

This shows that given any value \(X\) we can find a Hofmeister basis (valid for large enough \(t\)) whose cover is given by:

\[
162t^4 + X't^3 + O(t^2) \quad \text{for some } X' \geq X
\]

In other words, there is no limit to the value of the coefficient of \(t^3\) in the formula for the cover.
Appendix 1

This table summarises the best bases found so far for 144 <= h <= 707.

Probably the best Braunschadel or Hofmeister bases for 12<=t<=58

| r | t_range | type | c_21 | c_31 | c_32 | c_41 | c_42 | c_43 |
|---|---------|------|------|------|------|------|------|------|
| 0 | [12-27] | B    | 2    | 2    | -1   | 3    | -1   | 0    |
|   | [28-41] | B    | 3    | 3    | -2   | 5    | -1   | -2   |
|   | [42-58] | H    | 6    | 5    | -4   | 8    | -1   | -5   |
| 1 | [12-28] | H    | 1    | 0    | 2    | 1    | 1    | 0    |
|   | [29-31] | B    | 2    | 2    | -1   | 3    | -1   | 0    |
|   | [32-35] | B    | 3    | 3    | -2   | 5    | -1   | -2   |
|   | [36-54] | H    | 6    | 4    | -3   | 7    | 0    | -5   |
|   | [55-58] | B    | 7    | 6    | -4   | 11   | -1   | -5   |
| 2 | [12-20] | H    | 1    | 0    | 2    | 1    | 1    | 0    |
|   | [21-31] | B    | 5    | 3    | -1   | 6    | -1   | 0    |
|   | [32-41] | B    | 6    | 4    | -2   | 8    | -1   | -2   |
|   | [42-56] | H    | 6    | 4    | -3   | 7    | 0    | -5   |
|   | [57-58] | B    | 7    | 6    | -4   | 11   | -1   | -5   |
| 3 | [12-45] | H    | 3    | 1    | 2    | 2    | 1    | 1    |
|   | [46-49] | B    | 6    | 4    | -1   | 8    | 0    | -2   |
|   | [50-58] | H    | 6    | 4    | -3   | 7    | 0    | -5   |
| 4 | [12-48] | H    | 3    | 1    | 2    | 2    | 1    | 1    |
|   | [49-58] | H    | 9    | 6    | -3   | 10   | 0    | -5   |
| 5 | [12-45] | H    | 3    | 1    | 2    | 2    | 1    | 1    |
|   | [46-58] | H    | 9    | 5    | -2   | 9    | 1    | -5   |
| 6 | [12-49] | H    | 3    | 1    | 2    | 2    | 1    | 1    |
|   | [50-58] | H    | 9    | 5    | -2   | 9    | 1    | -5   |
| 7 | [12-36] | H    | 8    | 4    | 1    | 7    | 1    | 0    |
|   | [37-58] | H    | 9    | 5    | 0    | 9    | 1    | -2   |
| 8 | [12-16] | H    | 7    | 3    | 3    | 5    | 2    | 2    |
|   | [17-40] | H    | 8    | 4    | 1    | 7    | 1    | 0    |
|   | [41-58] | H    | 9    | 5    | 0    | 9    | 1    | -2   |
| 9 | [12-21] | H    | 7    | 3    | 3    | 5    | 2    | 2    |
|   | [22-29] | H    | 8    | 4    | 1    | 7    | 1    | 0    |
|   | [30-58] | H    | 12   | 7    | -1   | 12   | 1    | -3   |
| 10| [12-19] | H    | 7    | 3    | 3    | 5    | 2    | 2    |
|   | [20-34] | H    | 11   | 6    | 1    | 10   | 1    | 0    |
|   | [35-58] | H    | 12   | 7    | -1   | 12   | 1    | -3   |
| 11| [12-22] | B    | 11   | 4    | 2    | 10   | 1    | 2    |
|   | [23]    | H    | 11   | 5    | 2    | 9    | 2    | 0    |
|   | [24-43] | B    | 12   | 5    | 1    | 12   | 1    | 0    |
|   | [44-58] | H    | 12   | 7    | -1   | 12   | 1    | -3   |
Appendix 2

Here we give the expanded polynomial form of the formulae for \( a_i \) and the cover.

\[
\begin{align*}
    a_2 &= k_{21}t + c_{21} \\
    a_3 &= k_{32}k_{21}t^2 + (k_{32}c_{21} + c_{32}k_{21} + k_{31})t + c_{32}c_{21} + c_{31} \\
    a_4 &= k_{43}k_{32}k_{21}t^3 \\
        &+ (k_{43}k_{32}c_{21} + k_{43}c_{32}k_{21} + c_{43}k_{32}k_{21} + k_{43}k_{31} + k_{42}k_{21})t^2 \\
        &+ (k_{43}c_{32}c_{21} + c_{43}k_{32}c_{21} + c_{43}c_{32}k_{21} + k_{43}c_{31} + c_{43}k_{31} + k_{42}c_{21} + c_{42}k_{21} + k_{41})t \\
        &+ (c_{43}c_{32}c_{21} + c_{43}c_{31} + c_{42}c_{21} + c_{41}) \\
    C &= k_{54}k_{43}k_{32}k_{21}t^4 \\
        &+ (k_{54}k_{43}k_{32}c_{21} + k_{54}k_{43}c_{32}k_{21} + c_{54}k_{43}c_{32}k_{21})t^3 \\
        &+ (k_{54}k_{43}c_{32}c_{21} + k_{54}c_{43}c_{32}c_{21} + c_{54}k_{43}c_{32}c_{21})t^2 \\
        &+ (k_{54}c_{43}c_{32}c_{21} + c_{54}c_{43}c_{32}c_{21})t \\
        &+ (c_{54}c_{43}c_{32}c_{21} + c_{54}c_{43}c_{32}c_{21} + c_{54}c_{43}c_{32}c_{21})
\end{align*}
\]

Appendix 3

This is a transcript of Selmer's notes on "Good Hofmeister Bases" (Selmer, E.S., [8]).

In this transcript I have replaced Greek letters alpha, beta, gamma, delta by A, B, C and D, the summation symbol (Greek capital sigma) by \([\text{sigma}]\), the Greek capital "Phi" by P. The traditional printer's paragraph sign is replaced by \([\text{paragraph}]\), and umlauts are, in general, not mentioned: for example "Braunschadel" should really have umlauts on the "u" and second "a", but neither I nor, it seems, Selmer can be certain or consistent about this.

The first section - pages 1 to 8 - was written in December 1992, probably while Selmer was still at the University; I believe this was before he retired, after which he took a holiday before returning to his home to continue the document (pages 9 to 14) in March 1993 (as if nothing had happened in between!).

Page 17
The Hofmeister-Schell bases used by Mossige and Challis have the form

\[
\begin{align*}
    a_2 &= (9t + c_{21}) \\
    a_3 &= (4t + c_{31}) + (3t + c_{32}) a_2 \\
    a_4 &= (7t + c_{41}) + (2t + c_{42}) a_2 + (2t + c_{43}) a_3
\end{align*}
\]

The corresponding h-range, with

\[ h = 12t + r , \]

can be determined by Proposition 6.1 in Mossige's thesis, with my correction of Jan 92. For all bases (1) mentioned below, I have checked the formulas for \( n_h(A_4) \) given by Mossige and/or Challis, using Proposition 6.1.

By a "good" basis (1), we mean one with large h-range, often extremal. We have the following good bases available:

(I) Challis' extremal bases (A) in his paper.
(II) Mossige's bases in Table 4 of his thesis, for \( r = i = 2, 4, 5, 6, 9, 10 \) (not covered by (I)).
(III) Challis' good bases for \( 12 \leq t \leq 20 \) (his letter to me of April 25, 1992). Mostly covered by (I), and only two essentially new forms:

| \( r \) | \( c_{21} \) | \( c_{31} \) | \( c_{32} \) | \( c_{41} \) | \( c_{42} \) | \( c_{43} \) |
|------|-------|-------|-------|-------|-------|-------|
| (III.1) | 10 | 11 | 6 | 1 | 10 | 0 |
| (III.2) | 11 | 11 | 5 | 2 | 9 | 2 | 0 |

The bases at hand have two different forms of \( n_h(A_4) \), types "A" and "B":

(2A) \( n_h(A_4) = (2t + c_{51}) + (t + c_{52}) a_2 + (6t + c_{53}) a_3 + (3t + c_{54}) a_4 \)
(2B) \( n_h(A_4) = (3t + C_{51}) + (2t + C_{52}) a_2 + (4t + C_{53}) a_3 + (3t + C_{54}) a_4 \)

Both representations have coefficient sum \( h = 12t + r \), hence \( \sigma c = \sigma C = r \).

Type A dominates: All bases (I), bases (II) for \( r = 4, 5, 6, 9 \) (values of \( c_{5j} \) tabulated by Mossige), and basis (III.2), with

(III.2) \( c_{51} = 1, c_{52} = 2, c_{53} = 1, c_{54} = 7 \).
This gives a total of 14 bases - (9 different bases (I)).

In contrast to this, there are only 3 bases of type B at hand: r = 2, 10 for (II), and (III.1).

We list them all completely:

\[
\begin{array}{cccccccccccc}
\text{d}_1 & \text{d}_2 & \text{d}_3 & \text{d}_4 & \text{d}_5 - \text{d}_6 \\
\text{r} & \text{c}_{21} & \text{c}_{31} & \text{c}_{32} & \text{c}_{41} & \text{c}_{42} & \text{c}_{43} & \text{C}_{51} & \text{C}_{52} & \text{C}_{53} & \text{C}_{54} \\
\{ (II) \} & \{ 2 & 4 & 2 & 0 & 3 & 0 & -1 & 0 & 0 & 3 & \text{d}_1 \cdot 2 \} \\
\{ (II) \} & \{ 10 & 10 & 5 & 2 & 8 & 1 & 2 & 1 & 1 & 4 & 4 \} \\
\{ (III.1) \} & \{ 10 & 11 & 6 & 1 & 10 & 1 & 0 & 1 & 1 & 0 & 8 \} \\
\end{array}
\]

The \( c_{5j} \) of (III.2) and \( C_{5j} \) of (III.1) (not given by Challis) were calculated by me from Proposition 6.1.

So far, I have only systematized the "good" Hofmeister bases we have at hand. Now, I shall mention some very striking observations I have made for such bases. They resulted when I transformed the representations (2A), (2B) to regular form.

For (2A), the corresponding regular representation in all 14 cases is

\[
(4A) \quad n_{h}(A_4) = 5t + c_{51} + (c_{21} + 2c_{31} - 2c_{41}) \\
+ \{ 3t + c_{52} + (2c_{32} - 2c_{42} - 1) \} a_2 \\
+ \{ 2t + c_{53} - (2c_{43} + 2) \} a_3 \\
+ \{ 3t + c_{54} + 2 \} a_4
\]

The coefficient sum is \( 13t + r_1 \), where in fact \( r_1 = r \) in most cases (but 4 exceptions).

We note that when using Proposition 6.1 - which always gives the regular form - we must then end up with \( n_2(A_4) \) in (6.5).

Denoting \( n_{h}(A_4) \) of (4A) by

\[
n_{h}(A_4) = (5t + d_{51}) + (3t + d_{52}) a_2 + (2t + d_{53}) a_3 + (3t + d_{54}) a_4,
\]

there are two conditions for regularity:

\[
(5t + d_{51}) + (3t + d_{52}) a_2 < a_3 \\
(5t + d_{51}) + (3t + d_{52}) a_2 + (2t + d_{53}) a_3 < a_4.
\]

With the right hand sides given by (1), we
get the two necessary and sufficient conditions (at least for sufficiently large t):

\[ d_{52} < c_{32}, \ d_{53} < c_{43}, \]

which take the form

\[ c_{52} \leq 2c_{42} - c_{32}, \ c_{53} \leq 3c_{43} + 1. \]

When checking whether this was satisfied for the bases of type A, I discovered that in all 14 cases, these conditions were satisfied with equality! It was then natural to check if a similar linear relation exists between the \( c_{21} \), and in fact

\[ 3c_{31} - 2c_{41} + c_{51} + 2 = 0 \]

holds in all cases.

It thus seems that for our bases (1) of type A, the h-range (2A) can be immediately determined by

\[ (5A) \ c_{51} = 2c_{41} - 3c_{31} - 2, \ c_{52} = 2c_{42} - c_{32}, \ c_{53} = 3c_{43} + 1, \]

[ this line is strongly emphasised with ]
[ multiple vertical lines in the left hand ]
[ margin ]

together with the obvious relation [sigma] \( c_{5j} = r \):

\[ c_{54} = r - (c_{51} + c_{52} + c_{53}). \]

Substituting (5A) in (4A), we get

\[ (6A) \quad n_h(A_4) = (5t + c_{21} - c_{31} - 2) + (3t + c_{32} - 1) a_2 + (2t + c_{43} - 1) a_3 + (3t + c_{54} + 2) a_4. \]

This form is then equivalent to the observation (5A).

[Page 5]

Now we can prove (6A) and thus (5A), using Mossige's Proposition 6.1. His formulas (6.5) give the regular form of \( n_h(A_4) \) in the different cases, and should therefore be compared with (4A) above.

As already mentioned, Mossige's form \( n_2(X,P) \) is the only one giving the correct t-coefficients of (4A) (\( t = j \) in his notation). If we substitute in \( n_2(X,P) \) the coefficients

\[ d_1 = c_{21}, \ d_2 = c_{31}, \ d_3 = c_{32}, \ d_4 = c_{41}, \ d_5 = c_{42}, \ d_6 = c_{43}, \]

we just get the three first terms in (6A)
(the coefficients of \( a_4 \) cannot be compared).
This completes the promised proof.

If I had discovered directly
the above proof of (6A), I would in all
probability not have performed the calcu-
lation back to the interesting observation (5A).

One final remark: In my note of Jan. 92,
the corrections to Mossige's Proposition 6.1
did not affect the function \( f_2(X,P) \) of (6.3),
corresponding to \( n_2(X,P) \). But my condition for
a necessary correction:

\[
(7) \quad d_1 + 3d_2 - 3d_4 \geq 0 \quad \text{or} \quad 2d_3 - 3d_5 - 2 \geq 0 ,
\]

are satisfied in some of the 14 cases of type A,
and not in other cases.

[Page 6]

We now turn to the "good" Hofmeister
bases of type \( B \), with \( h \)-range given by
(2B). We only have the three bases (3) at hand.

The corresponding regular representation in
these cases is

\[
(4B) \quad n_h(A_4) = 9t + C_{51} + (c_{21} + c_{31} - c_{41})
+ \{ 3t + C_{52} + (c_{42} - c_{41} - 1) \} a_2
+ \{ 2t + C_{53} - (c_{43} + 1) \} a_3
+ \{ 3t + C_{54} + 1 \} a_4
= 9t + D_{51} + (3t + D_{52}) a_2 + (2t + D_{53}) a_3 + (3t + D_{54}) a_4 \quad (\text{say}).
\]

The necessary and sufficient conditions for
regularity are now

\[
D_{52} < c_{32} ,\; D_{53} < c_{43} ,
\]

which take the form

\[
C_{52} \leq c_{42} ,\; C_{53} \leq 2c_{43} .
\]

Again, these are satisfied with equality for
the three bases (3):

\[
(5B) \quad C_{52} = c_{42} ,\; C_{53} = 2c_{43} .
\]

But to find an expression for \( C_{51} \), in analogy
with \( c_{51} \) of (5A), is difficult from the
very few bases (3) at hand.

Substituting (5B) in (4B), we get

\[ n_b(A_4) = (9t + C_{51} + c_{21} + c_{31} - c_{41}) + (3t + c_{32} - 1) a_2 
+ (2t + c_{43} - 1) a_3 + (3t + C_{54} + 1) a_4. \]

[Page 7]

For the corresponding (6A), we could identify this uniquely with the function \( n_3(X,P) \) of (6.5) in Mossige, by looking at the \( t \)-coefficients. For (6B), however, these coefficients leave both \( n_3(X,P) \) and \( n_4(X,P) \) as candidates. For the bases (3), only \( n_3(X,P) \) (possibly modified, cf. below) turns up. This was to be expected, since both coefficients of \( a_2 \) and \( a_3 \) in (6B) fit for \( n_3(X,P) \) but not for \( n_4(X,P) \).

But now another complication turns up, because of my correction of Mossige's Proposition 6.1. If the condition (7) is not satisfied, Prop. 6.1 goes unmodified, with the constant term

\[ 9t + 2d_1 + 3d_2 - 3d_4 - 2 \]

of \( n_3(X,P) \). A comparison with (6B) shows that we must then expect

\[ C_{51} = c_{21} + 2c_{31} - 2c_{41} - 2. \]

This holds for the last basis (3), which is the only one where (7) fails. The numerical material is not overwhelming!

If on the other hand (7) holds, we must correct Proposition 6.1 as in my note of Jan. 92. In particular, \( n_3(X,P) \) is replaced by \( n_{3B}(X,P) \), with a different constant term

\[ 9t + d_1 - 2. \]

[Page 8]

We must then expect

\[ C_{51} = c_{41} - c_{31} - 2. \]

and this holds for the first two bases (3) (which both satisfy (7)).
Because of these complications, it may safely be said that the (dominating) type A is the most interesting one!

I have asked earlier why the good Hofmeister bases seem to give larger h-ranges than the Braunshadel bases. Now a similar question turns up, why type A is usually "better" than type B.

It seems very difficult to answer any of the questions. But one thing would be interesting to look into: In his good bases (III), Challis list one (III.1) of type B as high up as for \( t > 19 \). I therefore have the following question to him: **Would it take much time to extend the list of "good" Hofmeister bases to \( t > 20 \)?**

So far, all examined (by Prop. 6.1) Hofmeister bases have led to \( n_2(X,P) \) or \( n_3(X,P)/n_{3B}(X,P) \). It would be interesting to give some bases where other functions \( n_l(X,P) \) turn up, that is, bases which are not of type A or B. Preferably, such bases should be "good".

Selmer, March 1993

On "good" Hofmeister bases (continued).

I have now received Challis' note "Looking for good Hofmeister and Braunschadel bases" (17 pp, but not paginated!), with very many interesting observations.

I asked about the occurrence of my "dominating" type A, with \( k_{51} - k_{54} = (5, 3, 2, 3) \) by (4A), versus type B with \( (9, 3, 2, 3) \) in (4B). In Challis' Table 4, my basis (III.1) is - correctly - the only case B (for \( r = 10 \)). More interesting is of course Challis' extension to \( t \leq 58 \) in Table 10 of "best" Hofmeister bases. It still contains only A and B, but now with four cases B. However, only two of these, for \( r = 0 \) and \( r = 4 \), apply...
for the largest \( t = 58 \), cf. Table 8. So it seems that type A is "strongly" dominating for "very good" Hofmeister bases for large \( t \). For the "not so good" bases for \( r = 0 \) in Table 7, only half (14 of 29) of the bases are of type A. Quite surprisingly, type B has disappeared completely, but there are three other types \((6, 3, 2, 3), (8, 1, 2, 3)\) and \((5, 2, 2, 3)\) instead. The first of these inspired Challis' correction of Mossige's Proposition 6.1.

For Braunschadel bases, a similar dominance appears for \( k_{51} - k_{54} = (4, 3, 2, 3) \), at least for larger \( t \) in Table 11. This now contains seven bases of other types, but again only two of these, for \( r = 0 \) and \( r = 11 \), apply for \( t = 58 \), cf. Table 9.

In [para] 5, "The future", Challis conjectures that all extremal bases for sufficiently large \( h \) are either Hofmeister or Braunschadel - and the coefficient sets \( c_{ij} \) for a fixed value of \( r \), will "move slowly" away from \((0, 0, 0, 0, 0, 0)\) without limit.

Again, I must return to the question: What is really (and "naturally") a Hofmeister or Braunschadel basis? In my "Comments to Challis' letter of 19.10.92" (Dec. 1992), I exemplify this question with the bases \((1)\) and \((2)\) page C, of which \((1)\) is not Hofm/Braunsch. A similar and much more striking example will appear below.

Challis also wonders whether one or the other "form" will eventually dominate. I assume that by form, he refers to Hofmeister versus Braunschadel. But we may also ask if, among for instance extremal Hofmeister bases, there will be a dominance of "types" (like A above), that is, of the choice \( n_i \) in Mossige's (6.5),

with Challis' latest addition.

For the determination of \( n_i \), we cannot use only (asymptotic) parameter bases, but need exact coefficients.
Outside Challis' range $t \leq 58$, we only have one such case at disposal, namely Mossige's "optimal" basis of Theorem 11.1 (in thesis, = Th. 6.1 in Math. Scand paper). Putting $bt = T$, and then replacing $At$ by $t$, this basis may be written as

\[
\begin{align*}
\{ & h = 12t \quad (r = 0) \\
\{ & a_2 = 9t + 15T \\
(8) \{ & a_3 = 4t + 14T + (3t - 15T + 2) a_2 \\
\{ & a_4 = 9t + 23T + (2t - 2T) a_2 + (2t - 20T) a_3.
\end{align*}
\]

From Mossige's condition $A \geq 25b$, we get $T \leq t/25$. Is this a Hofmeister basis or not? Judging from the $t$-coefficients $(9, 4, 3, 7, 2, 2)$, one would say "yes". On the other hand, Mossige's very good choice $T = t/206$ (see also his example p. 48) gives a non-Hofmeister basis with prefactor $P = 2.008$.

But under all circumstances, the $h$-range of the basis (8) can be calculated from Mossige's Proposition 6.1, which was specially designed to find the $h$-range of a Hofmeister basis. In (6.3), we must use $f_{1A}$, and $2d_3 - 2d_5 - 2 < 0$, so Challis' latest modification does not apply, nor does my earlier modification. In Prop. 6.1, we then find $Z = 45T - 1$, $L = \{1, 7\}$, and the $h$-range $\min \{n_1, n_7\} = n_1$ is just the one given by Theorem 11.1.

The "type" $k_51 - k_54 = (8, 1, 0, 3)$ is not the type A which dominates for smaller $t$. However we shall see that it is easy to construct alternative "optimal" bases of several different types, more specifically corresponding to $n_2$ (type A), $n_3$, $n_6$ and $n_7$.

Using Prop. 6.1 on the basis (8), we find

\[
\begin{align*}
f_{1A} & \{ -1 \\
f_2 & \{ 4 \quad f_4 = -X + 57T \\
f_3 & = -X + 45T + \{ 2 \quad f_5 = -X + 51T - 1 \\
f_6 & \{ 2 \\
f_7 & \{ -1
\end{align*}
\]
The constant addends to the left result from
the choice \( P = (0, 0, 2, 0, 0, 0) \) in Theorem 11.1.
The smallest addends -1 determine \( L = \{1, 7\} \)
in line 3 above. By changing \( P \), we can
get other minimal addends, for instance

\[
\begin{align*}
P &= (1, 0, 0, 0, 0, 0) : \text{min for } f_3 \\
P &= (-1, 0, 0, 0, 0, 0) : \text{min for } f_6 \\
P &= (0, 0, 0, 0, -1, 0) : \text{min for } f_7 \\
\end{align*}
\]

\[ Z = 45T + 1 \] in all cases

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So the types corresponding to \( n_3, n_6 \)
and \( n_7 \) all turn up.

But the most interesting - and
"good looking" - choice is all zeros ,
\( P = (0, 0, 0, 0, 0, 0) \). Then \( Z = 45T + 2, L = \{2, 3, 6\} \),
and the minimum at the bottom of Mossige's
p. 39 occurs for \( n_2 \) (type A), with

\[
(10) \quad \{ \quad n_5(A_4) = (3t + 45T + 3) a_4 + (2t - 20T - 1) a_3 \\
\quad \quad \quad + (3t - 15T + 1) a_2 + (5t + T - 2) .
\]

Of course, all such cases contain
the same term \( (3t + 45T) a_4 \), and thus
give the same asymptotic value of \( n_5(A_4) \).
The advantage of the last, "all zero" case
is that it is analogous to my asymptotic
parameter bases, where I drop all constant
terms. For instance, the addend +2 in
the \( a_2 \)-coefficient in Mossige's example p. 48
now disappears, resulting in the h-range

\[
n(h, C) = (663t + 3) a_4 + (392t - 1) a_3 \\
\quad + (603t + 1) a_2 + (1031t - 2)
\]
of (10).

When I asked Mossige why he chose
the vector \( P = (0, 0, 2, 0, 0, 0) \), he explained
that it was to get an \( h_0 \)-basis, since
his condition of (6.4),

\[
f_8(P) = - d_1 - d_3 + i + 2 >= 0 \quad (i = 0)
\]
is then sharp. With the zero vector,
\( f_8(P) = 2 \), meaning that \( h = h_0 + 2 \).

It follows from (9) that with the
same vector \( B \), the \( A_4 \) of Theorem 11.1
can never have a h-range of the form
("type") $n_4$ or $n_5$, whatever the choice of $P$. This raises an interesting (and certainly difficult) question: Can such a h-range, with the same asymptotic prefactor 2.008, be obtained from a different choice of the vector $B$?

Generally, is Mossige's B "God-given"? [this line is emphasised as described previously]

[The End]

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