The symplectic reduction of the linearized Hamiltonian systems at elliptic relative equilibria of four-body problem

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Abstract

In this paper, we consider the elliptic relative equilibria of four-body problem. Here we prove that the corresponding linearized Hamiltonian system at such an elliptic relative equilibria of 4-bodies splits into two independent linear Hamiltonian systems, the first one is the linearized Hamiltonian system of the Kepler 2-body problem at Kepler elliptic orbit, and the other system is the essential part of the linearized Hamiltonian system, which is given implicitly. The reduction can be applied to the stability problem of such elliptic relative equilibria of four-body problem.

Keywords: four-body problem, elliptic relative equilibria, linear stability, reduction.

AMS Subject Classification: 58E05, 37J45, 34C25

1 Introduction and main results

For $n$ particles of mass $m_1, m_2, \ldots, m_n > 0$, let $q_1, q_2, \ldots, q_n \in \mathbb{R}^2$ the position vectors respectively. Then the system of equations for $n$-body problem is

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad \text{for} \quad i = 1, 2, \ldots, n,$$

(1.1)

where $U(q) = U(q_1, q_2, \ldots, q_n) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{\|q_i - q_j\|}$ is the potential or force function by using the standard norm $\| \cdot \|$ of vector in $\mathbb{R}^2$.

Note that $2\pi$-periodic solutions of this problem correspond to critical points of the action functional

$$\mathcal{A}(q) = \int_0^{2\pi} \left[ \sum_{i=1}^n \frac{m_i \|\dot{q}_i(t)\|^2}{2} + U(q(t)) \right] dt$$

defined on the loop space $W^{1,2}(\mathbb{R}/2\pi \mathbb{Z}, \hat{\mathcal{X}})$, where

$$\hat{\mathcal{X}} := \left\{ q = (q_1, q_2, \ldots, q_n) \in (\mathbb{R}^2)^n \left| \sum_{i=1}^n m_i q_i = 0, \ q_i \neq q_j, \ \forall i \neq j \right. \right\}$$

is the configuration space of the planar three-body problem.

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Letting \( p_i = m_i \dot{q}_i \in \mathbb{R}^2 \) for \( 1 \leq i \leq n \), then (1.1) is transformed to a Hamiltonian system

\[
p_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \text{for} \quad i = 1, 2, \ldots, n, \tag{1.2}
\]

with Hamiltonian function

\[
H(p, q) = H(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n) = \sum_{i=1}^{n} \frac{\|p_i\|^2}{2m_i} - U(q_1, q_2, \ldots, q_n). \tag{1.3}
\]

A central configuration is a solution \((q_1, q_2, \ldots, q_n) = (a_1, a_2, \ldots, a_n)\) of

\[
-\lambda m_i q_i = \frac{\partial U}{\partial q_i}(q_1, q_2, \ldots, q_n) \tag{1.4}
\]

for some constant \( \lambda \). An easy computation shows that \( \lambda = \frac{U(a)}{2I(a)} > 0 \), where \( I(a) = \frac{1}{2} \sum m_i ||a_i||^2 \) is the moment of inertia. Please refer [11] and [10] for the properties of central configuration.

It is well known that a planar central configuration of the \( n \)-body problem gives rise to solutions where each particle moves on a specific Keplerian orbit while the total of the particles move on a homographic motion. Following Meyer and Schmidt [9], we call these solutions as elliptic relative equilibria and in shorthand notation, simply ERE. Specially when \( e = 0 \), the Keplerian elliptic motion becomes circular motion and then all the bodies move around the center of masses along circular orbits with the same frequency, which are called relative equilibria traditionally.

In the current paper, we given the precise reduction of the linearized Hamiltonian system at each elliptic relative equilibria of 4-bodies. To describe our main reduction result more precisely, given positive masses \( m = (m_1, m_2, m_3, m_4) \in (\mathbb{R}^+)^4 \), let \( a = (a_1, a_2, a_3, a_4) \) be a central configuration of \( m \) with \( a_i = (a_{ix}, a_{iy}) \) for \( 1 \leq i \leq 4 \). For convenience, we define four corresponding complex numbers:

\[
z_{ai} = a_{ix} + \sqrt{-1}a_{iy}, \quad i = 1, 2, 3, 4. \tag{1.5}
\]

Without lose of generality, we normalize the three masses by

\[
\sum_{i=1}^{4} m_i = 1, \tag{1.6}
\]

and normalize the positions \( a_i, 1 \leq i \leq 4 \) by

\[
\sum_{i=1}^{4} m_i a_i = 0, \tag{1.7}
\]

\[
\sum_{i=1}^{4} m_i ||a_i||^2 = 2I(a) = 1. \tag{1.8}
\]

Using the notations in (1.5), (1.7) and (1.8) are equivalent to

\[
\sum_{i=1}^{4} m_i z_{ai} = 0, \tag{1.9}
\]

\[
\sum_{i=1}^{4} m_i ||z_{ai}||^2 = 2I(a) = 1. \tag{1.10}
\]
Moreover, we define
\[ \mu = U(a) = \sum_{1 \leq i < j \leq 4} \frac{m_im_j}{|a_i - a_j|} = \sum_{1 \leq i < j \leq 4} \frac{m_im_j}{|z_{a_i} - z_{a_j}|}, \quad \sigma = (\mu \rho)^{1/4}, \tag{1.11} \]
and
\[ \tilde{M} = \text{diag}(m_1, m_2, m_3, m_4), \quad M = \text{diag}(m_1, m_2, m_3, m_4). \tag{1.12} \]

Let \( B \) be a \( 4 \times 4 \) symmetric matrix such that
\[
B_{ij} = \begin{cases} 
\frac{m_im_j}{|z_{a_i} - z_{a_j}|^r} & \text{if } i \neq j, 1 \leq i, j \leq 4, \\
-\sum_{j=1, j \neq i}^{4} \frac{m_im_j}{|z_{a_i} - z_{a_j}|} & \text{if } i = j, 1 \leq i \leq 4,
\end{cases} \tag{1.13}
\]
and
\[
D = \mu I_4 + \tilde{M}^{-1}B, \\
\tilde{D} = \mu I_4 + \tilde{M}^{-1/2}B\tilde{M}^{-1/2} = \tilde{M}^{1/2}D\tilde{M}^{-1/2}. \tag{1.14, 1.15}
\]
where \( \mu \) is given by (1.11).

For convenience, we define two linear maps \( \Phi, \Psi : \mathbb{C} \rightarrow \mathbb{R}^{2 \times 2} \) by
\[
\Phi(z) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}, \quad \forall z = x + \sqrt{-1}y \in \mathbb{C}, \quad x, y \in \mathbb{R}, \tag{1.16}
\]
\[
\Psi(z) = \begin{pmatrix} x & y \\ y & -x \end{pmatrix}, \quad \forall z = x + \sqrt{-1}y \in \mathbb{C}, \quad x, y \in \mathbb{R}. \tag{1.17}
\]

Our main result is the following:

**Theorem 1.1** In the planar 4-body problem with given masses \( m = (m_1, m_2, m_3, m_4) \in (\mathbb{R}^+)^4 \), denote the ERE with eccentricity \( e \in [0, 1) \) for \( m \) by \( q_{m,e}(t) = (q_1(t), q_2(t), q_3(t), q_4(t)) \). The linearized Hamiltonian system at \( q_{m,e} \) is reduced to the summation of two independent Hamiltonian systems, the first one is the linearized system of the Kepler 2-body problem at the corresponding Kepler orbit, the second one is the essential part of the linearized Hamiltonian system which is given by
\[
z' = J \begin{pmatrix} I & O & -J & 0 \\ O & I & 0 & -J \\ J & O & I_2 - \frac{\mu}{p} \Psi(\beta_{12}) & 0 \\ O & J & 0 & I_2 - \frac{\mu}{p} \Psi(\beta_{12}) \end{pmatrix} z, \tag{1.18}
\]
where \( \beta_2 = 1 - \frac{\nu(D)}{\mu} \) and \( \beta_{11}, \beta_{12}, \beta_{22} \) are given by (2.34) - (2.39) below.

In Section 2 of this paper we focus on the proof of Theorem 1.1. In Appendix, we give some properties of \( \Phi \) and \( \Psi \).

## 2 The symplectic reduction of the linearized Hamiltonian systems at elliptic relative equilibria

In [9] (cf. p.275), Meyer and Schmidt gave the essential part of the fundamental solution of the elliptic Lagrangian orbit. Their method is explained in [4] too. Our study on ERE is based upon their method.
As in Section 1, for the given masses \( m = (m_1, m_2, m_3, m_4) \) satisfying \((1.6)\), suppose the four particles are located at \( a_1 = (a_{1x}, a_{1y}), a_2 = (a_{2x}, a_{2y}), a_3 = (a_{3x}, a_{3y}), a_4 = (a_{4x}, a_{4y}) \), form a central configuration. Using notations \((1.5)-(1.12)\), since \( a_1, a_2, a_3, a_4 \) form a central configuration, we have

\[
\sum_{j=1, j\neq i}^{4} \frac{m_j(z_{a_j} - z_{a_i})}{|z_{a_j} - z_{a_i}|^3} = \frac{U(a)}{2I(a)} z_{a_i} = \mu z_{a_i}, \quad (2.1)
\]

Based on matrix \( B \) of \((1.13)\), \( D \) has two simple eigenvalues: \( \lambda_1 = \mu \) with \( v_1 = (1, 1, \ldots, 1)^T \), and \( \lambda_2 = 0 \) with \( v_2 = (z_{a_1}, z_{a_2}, z_{a_3}, z_{a_4})^T \). Exactly, we have

\[
(Dv_1)_i = \mu - \sum_{j=1, j\neq i}^{4} \frac{m_j}{|z_{a_j} - z_{a_i}|^3} + \sum_{j=1, j\neq i}^{4} \frac{m_j}{|z_{a_j} - z_{a_i}|^3} = \mu, \quad (2.2)
\]

\[
(Dv_2)_i = (\mu - \sum_{j=1, j\neq i}^{4} \frac{m_j}{|z_{a_j} - z_{a_i}|^3})z_{a_i} + \sum_{j=1, j\neq i}^{4} \frac{m_j z_{a_j}}{|z_{a_j} - z_{a_i}|^3}
\]

\[
= \mu z_{a_i} + \sum_{j=1, j\neq i}^{4} \frac{m_j z_{a_j}}{|z_{a_j} - z_{a_i}|^3}
= \mu z_{a_i} - \mu z_{a_i}
= 0, \quad (2.3)
\]

where in the second last equality, we used \((2.1)\). Moreover by \((1.6)-(1.8)\), we have

\[
\left< v_1, \tilde{M}v_1 \right> = \sum_{i=1}^{n} m_i = 1, \quad (2.4)
\]

\[
\left< v_1, \tilde{M}v_2 \right> = \sum_{i=1}^{n} m_i z_{a_i} = 0, \quad (2.5)
\]

\[
\left< v_2, \tilde{M}v_1 \right> = \sum_{i=1}^{n} m_i \tilde{z}_{a_i} = 0, \quad (2.6)
\]

\[
\left< v_2, \tilde{M}v_2 \right> = \sum_{i=1}^{n} m_i |z_{a_i}|^2 = 1. \quad (2.7)
\]

Let \( \bar{v}_2 = (\bar{z}_{a_1}, \bar{z}_{a_2}, \bar{z}_{a_3}, \bar{z}_{a_4})^T \). Because, \( a_1, a_2, a_3, a_4 \) forms a nonlinear central configuration, \( \bar{v}_2 \) is independent with \( v_2 \). Moreover, \( \bar{v}_2 \) is also independent with \( v_1 \). So \( \bar{v}_2 \) is another eigenvector of \( D \) corresponding to eigenvalue \( \lambda_3 = 0 \).

Now, we construct \( v_3 \). We suppose

\[
v_3 = k \bar{v}_2 + lv_2 \quad (2.8)
\]

with \( k \in \mathbb{R}, l \in \mathbb{C} \) will be given later. If \( \bar{v}_2^T \tilde{M}v_2 = \sum_{i=1}^{n} m_i z_{a_i}^2 = 0 \), we set \( k = 1, l = 0 \), i.e., \( v_3 = \bar{v}_2 \). Then we have

\[
\bar{v}_1^T \tilde{M}v_3 = \sum_{i=1}^{n} m_i \tilde{z}_{a_i} = 0, \quad (2.9)
\]

\[
\bar{v}_2^T \tilde{M}v_3 = \sum_{i=1}^{n} m_i |z_{a_i}|^2 = 0, \quad (2.10)
\]
\[ \bar{v}_2^T \bar{M}v_3 = \sum_{i=1}^n m_i |z_{ai}|^2 = 1. \]  

(2.11)

In the other cases, we also hope (2.9)-(2.11) are satisfied. Thus we have

\[ 0 = \bar{v}_2^T \bar{M}v_3 = \bar{v}_2^T \bar{M} (k\bar{v}_2 + lv_2) = k \sum_{i=1}^4 m_i \bar{z}_{ai}^2 + l, \]

(2.12)

\[ 1 = \bar{v}_3^T \bar{M}v_3 = (kv_2 + l\bar{v}_2)^T \bar{M} (k\bar{v}_2 + lv_2) = k^2 + |l|^2 + kl \sum_{i=1}^4 m_i \bar{z}_{ai}^2 + kl \sum_{i=1}^4 m_i \bar{z}_{ai}. \]

(2.13)

Therefore, we have

\[ k = \frac{1}{\sqrt{1 - |\sum_{i=1}^4 m_i \bar{z}_{ai}|^2}} \]

(2.14)

\[ l = -\frac{\sum_{i=1}^4 m_i \bar{z}_{ai}}{\sqrt{1 - |\sum_{i=1}^4 m_i \bar{z}_{ai}|^2}}. \]

(2.15)

We now construct a unitary matrix \( \bar{A} \) based on \( v_1, v_2 \) and \( v_3 \). That is

\[ \bar{A} = \begin{pmatrix} 1 & z_{a_1} & b_1 & c_1 \\ 1 & z_{a_2} & b_2 & c_2 \\ 1 & z_{a_3} & b_3 & c_3 \\ 1 & z_{a_4} & b_4 & c_4 \end{pmatrix}. \]

(2.16)

where \((b_1, b_2, b_3, b_4) = v_3^T \), i.e., \( b_i = k\bar{z}_{ai} + lz_{ai}, 1 \leq i \leq 4 \). Then \( c_i = A_{i4} \), where \( A_{i4} \) is the algebraic cofactor of \( c_i \).

In the other hand, the signed area of the triangle formed by \( a_i, a_j \) and \( a_k \) is given by

\[ \Delta_{ijk} = \frac{\sqrt{-1}}{4} \det \begin{pmatrix} 1 & z_{a_1} & \bar{z}_{a_1} \\ 1 & z_{a_2} & \bar{z}_{a_2} \\ 1 & z_{a_3} & \bar{z}_{a_3} \end{pmatrix}. \]

(2.17)

Then \( c_1 = 4k \sqrt{-1} \Delta_{234} = -4k \sqrt{-1} \Delta_{234} \) and so on. Note that, for any \( \omega \in \mathbb{C}, |\omega| = 1 \), if \( c_i \) are replaced by \( \omega c_i, i = 1, 2, 3, 4, \) \( \bar{A} \) is also a unitary matrix. Thus we can let

\[ (c_1, c_2, c_3, c_4) = \left( \frac{4k \rho}{m_1} \Delta_{234}, -\frac{4k \rho}{m_2} \Delta_{134}, \frac{4k \rho}{m_3} \Delta_{124}, -\frac{4k \rho}{m_4} \Delta_{123} \right), \]

(2.18)

where

\[ \rho = \sqrt{m_1 m_2 m_3 m_4}. \]

(2.19)

For convenience, we also write \( v_4 \) as

\[ v_4 = (c_1, c_2, c_3, c_4)^T \in \mathbb{R}^4. \]

(2.20)

Now \( v_1, v_2, v_3, v_4 \) forms a unitary basis of \( \mathbb{C}^n \). Note that \( v_1, v_2, v_3 \) are eigenvectors of matrix \( D \), then \( v_4 \) is also an eigenvector of \( D \) with the corresponding eigenvalue

\[ \lambda_4 = tr(D) - \lambda_1 - \lambda_2 - \lambda_3 = tr(D) - \mu. \]

(2.21)
Moreover, we define
\[ \beta_1 = -\frac{\lambda_3}{\mu} = 0, \]  
\[ \beta_2 = \frac{\lambda_4}{\mu} = 1 - \frac{\text{tr}(D)}{\mu}. \]  

In the following, if there is no confusion, we will use \( a_i \) to represent \( z_{ai} \), \( 1 \leq i \leq 4 \). By the definition of \((2.8)\) and \((2.20)\), \( Dv_k = \lambda_k v_k \), \( k = 3, 4 \) reads
\[ \mu b_i - \sum_{j=1,j\neq i}^{4} \frac{m_j(b_j - b_i)}{|a_i - a_j|^3} = \lambda_3 b_i, \quad 1 \leq i \leq 4, \]  
\[ \mu c_i - \sum_{j=1,j\neq i}^{4} \frac{m_j(c_j - c_i)}{|a_i - a_j|^3} = \lambda_4 c_i, \quad 1 \leq i \leq 4, \]  

Let
\[ F_i = \frac{4}{3} \sum_{j=1,j\neq i}^{4} \frac{m_j(a_j - a_i)}{|a_i - a_j|^3}, \quad G_i = \frac{4}{3} \sum_{j=1,j\neq i}^{4} \frac{m_j(a_j - a_i)}{|a_i - a_j|^3}, \quad 1 \leq i \leq 4, \]  
then we have
\[ F_i = (\mu - \lambda_3)m_i b_i = \mu(1 + \beta_1)m_i b_i, \quad G_i = (\mu - \lambda_4)m_i c_i = \mu(1 + \beta_2)m_i c_i. \]  

Now as p.263 of [9], Section 11.2 of [4], we define
\[ P = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}, \quad Q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix}, \quad Y = \begin{pmatrix} G \\ Z \\ W_1 \\ W_2 \end{pmatrix}, \quad X = \begin{pmatrix} g \\ z \\ w_1 \\ w_2 \end{pmatrix}, \]  
where \( p_i, q_i, i = 1, 2, 3, 4 \) and \( G, Z, W_1, W_2, g, z, w_1, w_2 \) are all column vectors in \( \mathbb{R}^2 \). We make the symplectic coordinate change
\[ P = A^T Y, \quad Q = AX, \]  
where the matrix \( A \) is constructed as in the proof of Proposition 2.1 in [9]. Concretely, the matrix \( A \in \text{GL}(\mathbb{R}^8) \) is given by
\[ A = \begin{pmatrix} I & A_1 & B_1 & C_1 \\ I & A_2 & B_2 & C_2 \\ I & A_3 & B_3 & C_3 \\ I & A_4 & B_4 & C_4 \end{pmatrix}, \]  
where each \( A_i \) is a \( 2 \times 2 \) matrix given by
\[ A_i = (a_i, J a_i) = \Phi(a_i), \]  
\[ B_i = (b_i, J b_i) = \Phi(b_i), \]  
\[ C_i = (c_i, J c_i) = \Phi(c_i) = c_i I_2, \]  
where \( \Phi \) is given by \((1.16)\). Moreover, by the definition of \( v_i \), \( 1 \leq i \leq 4 \), we obtain
\[ \bar{A}^T \bar{M} \bar{A} = (\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4)^T \bar{M} (v_1, v_2, v_3, v_4) = I_4 \]  

where
By (2.28), we have \( A^TMA = \Phi(\hat{A})^T\Phi(\hat{M})\Phi(\hat{A}) = \Phi(\hat{A}^T\hat{M}\hat{A}) = \Phi(I_4) = I_8 \) is fulfilled (cf. (13) in p.263 of [2]).

Now we consider the Hamiltonian function of the four-body problem. Under the coordinate change (1.11), we get the kinetic energy
\[
K = \frac{1}{2}(|G|^2 + |Z|^2 + |W_1|^2 + |W_2|^2),
\]
(2.35)
and the potential function
\[
U_{ij}(z, w_1, w_2) = \frac{m_im_j}{d_{ij}(z, w_1, w_2)},
\]
(2.36)
\[
U(z, w_1, w_2) = \sum_{1 \leq i < j \leq 4} U_{ij}(z, w_1, w_2),
\]
(2.37)
with
\[
d_{ij}(z, w_1, w_2) = |(A_i - A_j)z + (B_i - B_j)w_1 + (C_i - C_j)w_2|
= |\Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2|,
\]
(2.38)
where we used (2.31)-(2.33).

Let \( \theta \) be the true anomaly. Then under the same steps of symplectic transformation in the proof of Theorem 11.10 (p. 100 of [3]), the resulting Hamiltonian function of the 3-body problem is given by
\[
H(\theta, \bar{Z}, \bar{W}_1, \bar{W}_2, \bar{z}, \bar{w}_1, \bar{w}_2) = \frac{1}{2}(|\bar{Z}|^2 + |\bar{W}_1|^2 + |\bar{W}_2|^2) + (\bar{z} \cdot J\bar{Z} + \bar{w}_1 \cdot J\bar{W}_1 + \bar{w}_2 \cdot J\bar{W}_2)
+ \frac{p - \frac{r(\theta)}{2}}{2p}(|\bar{z}|^2 + |\bar{w}_1|^2 + |\bar{w}_2|^2) - \frac{r(\theta)}{\sigma}U(\bar{z}, \bar{w}_1, \bar{w}_2),
\]
(2.39)
where \( \mu \) is given by (1.11) and
\[
r(\theta) = \frac{p}{1 + e \cos \theta}.
\]
(2.40)

We now derived the linearized Hamiltonian system at the elliptic relative equilibrium.

**Proposition 2.1** Using notations in (2.28), elliptic relative equilibria \((P(t), Q(t))^T\) of the system (1.2) with
\[
Q(t) = (r(t)R(\theta(t))a_1, r(t)R(\theta(t))a_2, r(t)R(\theta(t))a_3, r(t)R(\theta(t))a_4)^T, \quad P(t) = M\dot{Q}(t)
\]
(2.41)
in time \( t \) with the matrix \( M = \text{diag}(m_1, m_2, m_2, m_3, m_3, m_4, m_4) \), is transformed to the new solution \((Y(\theta), X(\theta))^T\) in the variable true anomaly \( \theta \) with \( G = g = 0 \) with respect to the original Hamiltonian function \( H \) of (2.39), which is given by
\[
Y(\theta) = \begin{pmatrix} \hat{Z}(\theta) \\ \bar{W}_1(\theta) \\ \bar{W}_2(\theta) \end{pmatrix} = \begin{pmatrix} 0 \\ \sigma \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad X(\theta) = \begin{pmatrix} \bar{z}(\theta) \\ \bar{w}_1(\theta) \\ \bar{w}_2(\theta) \end{pmatrix} = \begin{pmatrix} \sigma \\ 0 \\ 0 \\ 0 \end{pmatrix},
\]
(2.42)

Moreover, the linearized Hamiltonian system at the elliptic relative equilibrium \( \xi_0 = (Y(\theta), X(\theta))^T = (0, \sigma, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T \in \mathbb{R}^{12} \) depending on the true anomaly \( \theta \) with respect to the Hamiltonian function \( H \) of (2.39) is given by
\[
\dot{\xi}(\theta) = J\xi(\theta),
\]
(2.43)
with

\[ B(\theta) = H''(\theta, \bar{Z}, \bar{W}_1, \bar{W}_2, \bar{z}, \bar{w}_1, \bar{w}_2) \big| \theta = \xi_0 \]

\[
= \begin{pmatrix}
I & O & O & -J & O & O \\
O & I & O & O & -J & O \\
O & O & I & O & O & -J \\
J & O & O & H_{zz}(\theta, \xi_0) & O & O \\
O & J & O & O & H_{\bar{w}_1 \bar{w}_1}(\theta, \xi_0) & H_{\bar{w}_1 \bar{w}_2}(\theta, \xi_0) \\
O & O & J & O & H_{\bar{w}_2 \bar{w}_1}(\theta, \xi_0) & H_{\bar{w}_2 \bar{w}_2}(\theta, \xi_0)
\end{pmatrix},
\]

(2.44)

and

\[
H_{zz}(\theta, \xi_0) = \begin{pmatrix}
\frac{2 - e^{\cos \theta}}{1 + e^{\cos \theta}} & 0 \\
0 & 1
\end{pmatrix},
\]

(2.45)

\[
H_{\bar{w}_i \bar{w}_j}(\theta, \xi_0) = I_2 - \frac{r}{p} \left[ \frac{3 + \beta_i}{2} I_2 + \Psi(\beta_i) \right], \quad i, j = 1, 2,
\]

(2.46)

\[
H_{\bar{w}_1 \bar{w}_2}(\theta, \xi_0) = -\frac{r}{p} \Psi(\beta_{12}),
\]

(2.47)

where \( \beta_1 = 0 \) and \( \beta_2 \) are given by (2.23), and \( \beta_{11}, \beta_{12}, \beta_{22} \) are given by

\[
\beta_{11} = \frac{3}{2\mu} \sum_{1 \leq i < j \leq 4} \frac{m_i m_j (a_i - a_j)^2 (\bar{b}_i - \bar{b}_j)^2}{|a_i - a_j|^5},
\]

(2.48)

\[
\beta_{12} = \frac{3}{2\mu} \sum_{1 \leq i < j \leq 4} \frac{m_i m_j (a_i - a_j)^2 (\bar{b}_i - \bar{b}_j)(\bar{c}_i - \bar{c}_j)}{|a_i - a_j|^5},
\]

(2.49)

\[
\beta_{22} = \frac{3}{2\mu} \sum_{1 \leq i < j \leq 4} \frac{m_i m_j (a_i - a_j)^2 (\bar{c}_i - \bar{c}_j)^2}{|a_i - a_j|^5},
\]

(2.50)

and \( H'' \) is the Hessian Matrix of \( H \) with respect to its variable \( \bar{Z}, \bar{W}_1, \bar{W}_2, \bar{z}, \bar{w}_1, \bar{w}_2 \). The corresponding quadratic Hamiltonian function is given by

\[
H_2(\theta, \bar{Z}, \bar{W}_1, \bar{W}_2, \bar{z}, \bar{w}_1, \bar{w}_2) = \frac{1}{2} |\bar{Z}|^2 + \bar{Z} \cdot J\bar{z} + \frac{1}{2} H_{zz}(\theta, \xi_0) |\bar{z}|^2 + H_{\bar{w}_1 \bar{w}_1}(\theta, \xi_0) |\bar{w}_1|^2 + H_{\bar{w}_1 \bar{w}_2}(\theta, \xi_0) \bar{w}_1 \cdot \bar{w}_2
\]

\[
+ \left( \frac{1}{2} |\bar{W}_1|^2 + \bar{W}_1 \cdot J\bar{w}_1 + \frac{1}{2} H_{\bar{w}_1 \bar{w}_1}(\theta, \xi_0) |\bar{w}_1|^2 \right)
\]

\[
+ \left( \frac{1}{2} |\bar{W}_2|^2 + \bar{W}_2 \cdot J\bar{w}_2 + \frac{1}{2} H_{\bar{w}_2 \bar{w}_2}(\theta, \xi_0) |\bar{w}_2|^2 \right).
\]

(2.51)

**Proof.** The proof is similar to those of Proposition 11.11 and Proposition 11.13 of [4]. We just need to compute \( H_{zz}(\theta, \xi_0), H_{\bar{w}_i \bar{w}_j}(\theta, \xi_0) \) and \( H_{\bar{w}_i \bar{w}_j}(\theta, \xi_0) \) for \( i, j = 1, 2 \).

For simplicity, we omit all the upper bars on the variables of \( H \) in (2.39) in this proof. By (2.39), we have

\[
H_{\bar{z}} = JZ + \frac{p - r}{p} z - \frac{r}{\sigma} U_z(z, w_1, w_2),
\]

\[
H_{\bar{w}_i} = JW_i + \frac{p - r}{p} w_i - \frac{r}{\sigma} U_{w_i}(z, w_1, w_2), \quad i = 1, 2,
\]

and

\[
\begin{cases}
H_{zz} = \frac{p - r}{p} I - \frac{r}{\sigma} U_{zz}(z, w_1, w_2), \\
H_{\bar{w}_i \bar{w}_j} = \frac{p - r}{p} I - \frac{r}{\sigma} U_{\bar{w}_i \bar{w}_j}(z, w_1, w_2), \quad i, j = 1, 2, \\
H_{\bar{w}_1 \bar{w}_2} = \frac{p - r}{p} I - \frac{r}{\sigma} U_{\bar{w}_1 \bar{w}_2}(z, w_1, w_2),
\end{cases}
\]

(2.52)
where we write $H_z$ and $H_{w_l}$ etc to denote the derivative of $H$ with respect to $z$, and the second derivative of $H$ with respect to $z$ and then $w_i$ respectively. Note that all the items above are $2 \times 2$ matrices.

For $U_{ij}$ defined in (2.36) with $1 \leq i < j \leq n$, $1 \leq l \leq n - 2$, we have

\[
\frac{\partial U_{ij}}{\partial z}(z, w_1, w_2) = -\frac{m_1 m_j (a_i - a_j)^T}{|\Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2|^3} \left[ \Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2 \right],
\]

(2.53)

\[
\frac{\partial U_{ij}}{\partial w_1}(z, w_1, w_2) = -\frac{m_1 m_j (b_i - b_j)^T}{|\Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2|^3} \left[ \Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2 \right],
\]

(2.54)

\[
\frac{\partial U_{ij}}{\partial w_2}(z, w_1, w_2) = -\frac{m_1 m_j (c_i - c_j)^T}{|\Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2|^3} \left[ \Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2 \right],
\]

(2.55)

and

\[
\frac{\partial^2 U_{ij}}{\partial z^2}(z, w_1, w_2) = \frac{m_1 m_j |a_i - a_j|^2 I_2}{|\Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2|^5} \left[ \Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2 \right] \cdot \left[ \Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2 \right]^T \Phi(a_i - a_j),
\]

(2.56)

\[
\frac{\partial^2 U_{ij}}{\partial z \partial w_1}(z, w_1, w_2) = \frac{m_1 m_j |b_i - b_j|^2 I_2}{|\Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2|^5} \left[ \Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2 \right] \cdot \left[ \Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2 \right]^T \Phi(b_i - b_j),
\]

(2.57)

\[
\frac{\partial^2 U_{ij}}{\partial w_1^2}(z, w_1, w_2) = \frac{m_1 m_j |c_i - c_j|^2 I_2}{|\Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2|^5} \left[ \Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2 \right] \cdot \left[ \Phi(a_i - a_j)z + \Phi(b_i - b_j)w_1 + \Phi(c_i - c_j)w_2 \right]^T \Phi(b_i - b_j).
\]

(2.58)

Let

\[
K = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \Psi(1),
\]

where $\Psi$ is given by (1.17). Now evaluating these functions at the solution $\tilde{x}_0 = (0, \sigma, 0, 0, 0, 0, \sigma, 0, 0, 0, 0, 0)^T \in \mathbb{R}^8$ with $z = (\sigma, 0)^T$, $w_i = (0, 0)^T$, $1 \leq i \leq 2$, and summing them up, we obtain

\[
\frac{\partial^2 U}{\partial z^2}\bigg|_{\tilde{x}_0} = \sum_{1 \leq i < j \leq 4} \frac{\partial^2 U_{ij}}{\partial z^2}\bigg|_{\tilde{x}_0}
\]

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\[
\begin{align*}
\frac{\partial^2 U}{\partial w_1^2} |_{\xi_0} &= \sum_{1 \leq i < j \leq 4} \left( -\frac{m_i m_j |a_i - a_j|^2}{|a_i - a_j| \sigma^3} I + \frac{3}{2} \frac{m_i m_j \sigma^2 |a_i - a_j|^2 K_1 |a_i - a_j|^2}{|a_i - a_j| \sigma^3} \right) \\
&= \frac{1}{\sigma^3} \sum_{1 \leq i < j \leq 4} \left( m_i m_j \right) K \\
&= \frac{\mu}{\sigma^3} K, \\
\frac{\partial^2 U}{\partial w_1^2} |_{\xi_0} &= \sum_{1 \leq i < j \leq 4} \frac{\partial^2 U_{ij}}{\partial w_1^2} |_{\xi_0} \\
&= \sum_{1 \leq i < j \leq 4} \left( -\frac{m_i m_j |b_i - b_j|^2}{|a_i - a_j| \sigma^3} I + \frac{3}{2} \frac{m_i m_j \sigma^2 |b_i - b_j|^2 \Phi(b_i - b_j) \Phi(a_i - a_j)^T \Phi(b_i - b_j)}{|a_i - a_j| \sigma^3} \right) \\
&= \sum_{1 \leq i < j \leq 4} \left( -\frac{m_i m_j |b_i - b_j|^2}{|a_i - a_j| \sigma^3} I + \frac{3}{2} \frac{m_i m_j \sigma^2 |b_i - b_j|^2 \Phi(b_i - b_j) \Phi(a_i - a_j)^T \Phi(b_i - b_j)}{|a_i - a_j| \sigma^3} \right) \\
&+ \sum_{1 \leq i < j \leq 4} \left( \frac{3}{2} \frac{m_i m_j \sigma^2 \Phi(b_i - b_j)^T \Phi(a_i - a_j) \Phi(1) \Phi(a_i - a_j)^T \Phi(b_i - b_j)}{|a_i - a_j| \sigma^3} \right) \\
&= \sum_{1 \leq i < j \leq 4} \left( -\frac{m_i m_j |b_i - b_j|^2}{|a_i - a_j| \sigma^3} I + \frac{3}{2} \frac{m_i m_j \sigma^2 |b_i - b_j|^2 |a_i - a_j|^2}{|a_i - a_j| \sigma^3} \right) \\
&+ \sum_{1 \leq i < j \leq 4} \left( \frac{3}{2} \frac{m_i m_j \sigma^2 |b_i - b_j|^2 |a_i - a_j|^2}{|a_i - a_j| \sigma^3} \right) \\
&= \frac{1}{2 \sigma^3} \sum_{1 \leq i < j \leq 4} \left( \frac{m_i m_j |b_i - b_j|^2}{|a_i - a_j|^3} \right) I_2 + \frac{1}{\sigma^3} \Psi \left( \frac{3}{2} \sum_{1 \leq i < j \leq 4} \frac{m_i m_j (a_i - a_j)^2 (\bar{b}_i - \bar{b}_j)^2}{|a_i - a_j|^5} \right) \\
&= \frac{1}{2 \sigma^3} \left( \sum_{i=1}^4 \sum_{j=1, j \neq i}^4 \frac{m_i m_j (b_i - b_j)}{|a_i - a_j|^3} \right) I_2 + \frac{1}{\sigma^3} \Psi \left( \frac{3}{2} \sum_{1 \leq i < j \leq 4} \frac{m_i m_j (a_i - a_j)^2 (\bar{b}_i - \bar{b}_j)^2}{|a_i - a_j|^5} \right) \\
&= \frac{1}{2 \sigma^3} \left( \sum_{i=1}^4 b_i F_i \right) I_2 + \frac{1}{\sigma^3} \Psi \left( \frac{3}{2} \sum_{1 \leq i < j \leq 4} \frac{m_i m_j (a_i - a_j)^2 (\bar{b}_i - \bar{b}_j)^2}{|a_i - a_j|^5} \right) \\
&= \frac{\mu(1 + \beta_1)}{2 \sigma^3} I_2 + \frac{1}{\sigma^3} \Psi \left( \frac{3}{2} \sum_{1 \leq i < j \leq 4} \frac{m_i m_j (a_i - a_j)^2 (\bar{b}_i - \bar{b}_j)^2}{|a_i - a_j|^5} \right) \\
&= \frac{\mu(1 + \beta_1)}{2 \sigma^3} I_2 + \frac{\mu}{\sigma^3} \Psi(\beta_{11}), \\
\end{align*}
\]

where in the third equality of the first formula, we used (2.26), and in the last equality of the second formula, we use the definition (2.27) and (2.48). Similarly, we have

\[
\frac{\partial^2 U}{\partial w_2^2} |_{\xi_0} = \frac{\mu(1 + \beta_2)}{2 \sigma^3} I_2 + \frac{1}{\sigma^3} \Psi \left( \frac{3}{2} \sum_{1 \leq i < j \leq 4} \frac{m_i m_j (a_i - a_j)^2 (\bar{c}_i - \bar{c}_j)^2}{|a_i - a_j|^5} \right) \\
= \frac{\mu(1 + \beta_2)}{2 \sigma^3} I_2 + \frac{\mu}{\sigma^3} \Psi(\beta_{22}),
\]
\[
\frac{\partial^2 U}{\partial w_1 \partial w_2} |_{\xi_0} = \frac{1}{\sigma^3} \Psi \left( \frac{3}{2} \sum_{1 \leq i < j \leq 4} \frac{m_i m_j (a_i - a_j)^2 (\vec{b}_i - \vec{b}_j)(\vec{a}_i - \vec{a}_j)}{|a_i - a_j|^3} \right) \\
= \frac{\mu}{\sigma^3} \Psi(\beta_{12}).
\]

Moreover, we have
\[
\frac{\partial^2 U}{\partial z \partial w_1} |_{\xi_0} = \sum_{1 \leq i < j \leq 4} \frac{\partial^2 U_{ij}}{\partial z \partial w_1} |_{\xi_0}
= \sum_{1 \leq i < j \leq 4} \left( -\frac{m_i m_j \Phi(a_i - a_j)^T \Phi(b_i - b_j)}{|a_i - a_j|^3} + 3 \frac{m_i m_j \sigma^2 |a_i - a_j|^2 K \Phi(a_i - a_j)^T \Phi(b_i - b_j)}{|a_i - a_j|^5} \right)
= \frac{K}{\sigma^3} \Phi \left( \sum_{1 \leq i < j \leq 4} \frac{m_i m_j \vec{a}_i (b_i - b_j)}{|a_i - a_j|^3} - \sum_{1 \leq i < j \leq 4} \frac{m_i m_j \vec{a}_j (b_i - b_j)}{|a_i - a_j|^3} \right)
= \frac{K}{\sigma^3} \Phi \left( \sum_{1 \leq i < j \leq 4} \frac{m_i m_j \vec{a}_i (b_i - b_j)}{|a_i - a_j|^3} - \sum_{1 \leq i < j \leq 4} \frac{m_j m_i \vec{a}_i (b_i - b_j)}{|a_i - a_j|^3} \right)
= \frac{K}{\sigma^3} \Phi \left( \sum_{i=1}^{4} \sum_{j \neq i} a_i \sum_{j=1}^{4} \frac{m_i m_j (b_i - b_j)}{|a_i - a_j|^3} \right)
= \frac{K}{\sigma^3} \Phi \left( \mu (1 + \beta_1) \sum_{i=1}^{4} \vec{a}_i b_i \right)
= 0.
\]

where in the second last equation, we used (2.1), and in the last equality, we used (2.27). Similarly, we have
\[
\frac{\partial^2 U}{\partial z \partial w_2} |_{\xi_0} = 0.
\]

By and (2.59)-(2.64), we have
\[
H_{z z} |_{\xi_0} = \frac{p-r}{p} I - \frac{r \mu}{\sigma^2} K = I - \frac{r \mu}{p \sigma^2} K = I - \frac{r}{p} (I + K) = \begin{pmatrix} \frac{2 \pi \cos \theta}{r} & 0 \\ 0 & 1 \end{pmatrix},
\]
\[
H_{z w_i} |_{\xi_0} = -\frac{r}{\sigma} \frac{\partial^2 U}{\partial z \partial w_i} |_{\xi_0} = O, \quad 1 \leq i \leq 2,
\]
\[
H_{w_1 w_1} |_{\xi_0} = \frac{p-r}{p} I - \frac{r}{\sigma} \left[ \mu (1 + \beta_1) I_2 + \frac{\mu}{\sigma^3} \Psi(\beta_{11}) \right] = \frac{p-r}{p} I - \frac{r}{p} \left[ \frac{1 + \beta_1}{2} I_2 + \Psi(\beta_{11}) \right]
\]
\[
\begin{align*}
H_{w_2 w_1 | \xi_0} &= H_{w_2 w_1 | \xi_0} = -r \frac{\partial^2 U}{\partial w_1 \partial w_2} |_{\xi_0} = -r \frac{\Psi}{p}. \tag{2.65}
\end{align*}
\]

Thus the proof is complete. \[ \square \]

Theorem 1.1 immediately follows from Theorem 2.1.

Remark 2.2 If \( \beta_{12} = 0 \), by (2.46), we have \( H_{w_1 w_2} = 0 \), and hence the linearized Hamiltonian system (2.43) can be separated into three independent Hamiltonian systems, the first one is the linearized Hamiltonian system of the Kepler two-body problem at Kepler elliptic orbit, and each of the other two systems can be written as

\[
\dot{\zeta}_i(\theta) = J B_{i,0}(\theta) \zeta_i(\theta), \tag{2.66}
\]

with

\[
B_{i,0} = \begin{pmatrix} I_2 & -J_2 \\ J_2 & J_2 - \frac{s}{p} \frac{3 + \beta_{11}}{2} I_2 + \Psi(\beta_{11}) \end{pmatrix}, \tag{2.67}
\]

for \( i = 1, 2 \). Thus the linear stability problem of the elliptic relative equilibrium of the four-body problem can be reduced to the linear stability problems of system (2.66) with \( i = 1, 2 \).

However in general, \( \beta_{12} = 0 \) does not hold. But in some special cases, such as the four-body system with two small masses, we precisely have \( \beta_{12} = 0 \), and we will study such system below.

3 Appendix: Properties on \( \Phi \) and \( \Psi \)

Direct computation shows that:

Lemma 3.1 (i) If \( z \in \mathbb{R} \), then

\[
\Phi(z) = z I_2, \quad \Psi(z) = z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{3.68}
\]

(ii) For any \( z \in \mathbb{C} \), we have

\[
\Phi(z)^T = \Phi(\bar{z}), \quad \Psi(z)^T = \Psi(z); \tag{3.69}
\]

(iii) For any \( z, w \in \mathbb{C} \), we have

\[
\Phi(z) \Phi(w) = \Phi(z w), \tag{3.71}
\]

\[
\Psi(z) \Psi(w) = \Phi(z \bar{w}), \tag{3.72}
\]

\[
\Phi(z) \Psi(w) = \Psi(z w), \tag{3.73}
\]

\[
\Psi(z) \Phi(w) = \Psi(z \bar{w}). \tag{3.74}
\]

Specially, we have

\[
\Phi(\bar{z}) \Phi(z) = \Phi(z) \Phi(\bar{z}) = \Phi(|z|^2) = |z|^2 I_2, \tag{3.75}
\]

\[
\Psi(z) \Psi(\bar{z}) = \Psi(\bar{z}) \Psi(z) = \Phi(|z|^2) = |z|^2 I_2. \tag{3.76}
\]
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