ANALYSIS OF A LYMPHATIC FILARIASIS-SCHISTOSOMIASIS COINFECTION WITH PUBLIC HEALTH DYNAMICS: MODEL OBTAINED THROUGH MITTAG-LEFFLER FUNCTION

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Abstract. In this paper, Lymphatic filariasis-schistosomiasis coinfected model is studied within the context of fractional derivative order based on Mittag-Leffler function of ABC in the Caputo sense. The existence and uniqueness of system model solution is derived by employing a well-known Banach fixed point theorem. The numerical solution based on the Mittag-Leffler function suggests that the dynamics of the coinfected model is well explored using fractional derivative order because of non-singularity.

1. Introduction. In recent years many researchers have found out that several real life problems can be mathematically formulated using power law [15, 8, 5, 6]. These applications can be found in areas such as engineering, life sciences, mathematics, physics etc. The study of mathematical biology is to transform biological processes into mathematical concept that can be represented in the form of derivatives. These derivatives have influence on the dynamics of model systems. Most of the researchers in the past have devoted substantial amount of time studying problems in the context of integer order [13, 14]. In current times, more attention have been shifted to the mechanism that seeks to provide better information which could have been hidden if the integer order is used [9, 10, 11, 12, 16]. In epidemiology studies presently, many researchers are making serious attempt to model infectious and non-infectious diseases in the light of non-integer order [9, 10, 11].

Basically, there are two kinds of differentiation in the existing literature: the rate of change and the one which hinges on convolution of some particular functions among them is the power law $\chi^{-\lambda}$ and Mittag-Leffler law [17, 1]. Naturally most of the physical problems appear not to follow the power law $\chi^{-\lambda}$ but rather Mittag-Leffler law which has the ability to vividly account for the fading memory effect.

Several authors in this field have applied the power law to model a lot of physical problems in areas such as engineering, biology, finance, economics etc. It has also

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been shown recently however, by expert in this area that this derivative does not have a non-local kernel since the associated integral appears not to be a fractional. A new dimension which seeks to solve such a problem has been proposed by Atangana and Beleanu. In this new operator the so called exponential decay kernel is substituted with the generalized Mittage-Leffer kernel [2, 3]. For instance, many researchers have shown that Caputo and Fabrizio operator is a mere filter which has a fractional regulator. This is because the kernel employed on this operator is local and also the integral is characterized by averaging the given function and its integral [17]. Many authors have successfully used the Atangana-Beneanu operator in Caputo sense (ABC) which hinges on generalized Mittag-Leffler function to model countless number of physical problems [4].

Lymphatic filariasis-schistosomiasis coinfection was recently studied by authors in integer order deterministic model with time dependent controls. They observed that incorporating five controls at the same time was the best option in controlling the spread of the co-infected disease. To the best of our knowledge there is no a single fractional order mathematical model on Lymphatic filariasis-schistosomiasis coinfection model and this study would help predict accurately because of the non-singularity and non-local kernel of the ABC operator.

In this paper, an attempt is made to apply the newly developed ABC fractional operator which is non-singular and non-local kernel to investigate the dynamics of Lymphatic filariasis-schistosomiasis coinfection which will avoid over prediction.

The paper is organized as follows: The mathematical definitions are presented in Section 2. In Section 3, the existence of solution of the Lymphatic filariasis-schistosomiasis co-infected model using the method of Picard-Lindelof is studied. Section 4 deals with special solution based on Atangana-Balenau derivative in Caputo sense. Numerical simulations on different values of alpha are discussed to show the effectiveness of the is operator is presented in Section 5. existence of solutions for the spread of rubella disease model via Picard-Lindelof method

2. New fractional differentiation based on Mittag-Leffler function. We present in this section, the recent fractional differentiation operator rooted from the famous Mittag-Leffler function. This novelty in Caputo sense is known as Atangana-Baleanu (ABC) and in Riemann-Liouville sense also known as Atangana-Baleanu Riemann-Liouville in sense (ABR). The various definitions can be found in many works of Atanaga et al., [2, 17, 11].

**Definition 2.1.** Let \( f \in H^1(a,b), b > a \) and \( \alpha \in [0,1] \) then the definition of the new fractional derivative (ABC) is given as

\[
\frac{ABC}{a} D_t^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \int_a^t f'(x)E_{\alpha} \left[ -\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx
\]  

(1)

The authors [11, 12, 16] in their previous work, made a classification that the function B possesses the same characteristics as in the case of Caputo and Fabrizio’s definition.

**Definition 2.2.** Let \( f \in H^1(a,b), b > a \) and \( \alpha \in [0,1] \) and not per say differentiable, then the definition of the new fractional derivative (ABR) is given as

\[
\frac{ABR}{a} D_t^\alpha (f(t)) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_a^t f(x)E_{\alpha} \left[ -\alpha \frac{(t-x)^\alpha}{1-\alpha} \right] dx
\]  

(2)
Definition 2.3. The fractional integral characterized with the new fractional derivative which is non-local kernel is expressed as

\[ A_{B}^{\alpha} I_{a}^{\alpha} \{ f(t) \} = \frac{1 - \alpha}{B(\alpha)} f(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_{a}^{t} f(y) (t - y)^{\alpha - 1} dy \]  

(3)

It is clearly seen from the above definition that when \( \alpha \) becomes zero, the initial function is arrived at, and also having being alpha equal to 1, the classical integral is achieved \([2, 17]\).

Theorem 2.4. The subsequent time fractional ordinary differential equation

\[ _{0}^{ABC} D_{t}^{\alpha} (f(t)) = u(t) - u(0) \]  

Possesses a singular solution which takes into consideration the inverse Laplace transform and make use of the convolution theorem as below \([2, 17]\).

\[ f(t) - f(0) = \frac{1 - \alpha}{B(\alpha)} u(t) + \frac{\alpha}{B(\alpha) \Gamma(\alpha)} \int_{a}^{t} u(y) (t - y)^{\alpha - 1} dy \]  

(5)

3. Analysis of existence and unicity of the new system. In this section, we shall reorganize the deterministic model proposed by Bonyah et al.,\([7]\) and substituting the time derivative with Atanagann Beneanu operator \((ABC)\). The application of local derivative to vividly describe many real life challenge problem has some serious set back. In this regard, the method of fractional differentiation hinged on convolution of \( \chi^{-\alpha} \) also could not adequately addressed the same problem because of convergence. Mittag- Leffler function, appears to have a generalized character in nature and likely to converge in the given time and can be employed to take care of many challenging physical problems.

\[ 0_{ABC} D_{t}^{\alpha} S_{h}(t) = (1 - r) \pi_{h} + \gamma R_{sc} + \sigma R_{l} + \omega R_{sl} + pV_{h} - \beta_{1} I_{v} S_{h} - \beta_{2} I_{sv} S_{h} - \mu_{h} S_{h} \]  

\[ 0_{ABC} D_{t}^{\alpha} I_{l}(t) = \beta_{1} I_{v} S_{h} + \beta_{1} I_{v} V_{h} - \beta_{2} I_{sv} I_{l} - (\alpha + \mu_{h} + m) I_{l} \]  

\[ 0_{ABC} D_{t}^{\alpha} I_{sc}(t) = \beta_{2} I_{sv} S_{h} + \beta_{2} I_{sv} V_{h} - \beta_{1} I_{v} I_{sc} - (\delta + \mu_{h} + n) I_{sc} \]  

\[ 0_{ABC} D_{t}^{\alpha} C_{ls}(t) = \beta_{1} I_{v} I_{sc} + \beta_{2} I_{sv} I_{l} - (\phi + \mu_{h} + \rho + \theta) C_{ls} \]  

\[ 0_{ABC} D_{t}^{\alpha} V_{h}(t) = r \Lambda_{h} - (p + \mu_{h}) V_{h} - \beta_{1} I_{v} V_{h} - \beta_{2} I_{sv} V_{h} - \beta_{2} I_{sv} V_{h} \]  

\[ 0_{ABC} D_{t}^{\alpha} R_{l}(t) = \alpha I_{l} - (\sigma + \mu_{h}) R_{l} + (1 - \phi) \eta C_{ls} \]  

\[ 0_{ABC} D_{t}^{\alpha} R_{sc}(t) = \delta I_{sc} - (\gamma + \mu_{h}) R_{sc} + (1 - \phi)(1 - \eta) C_{ls} \]  

\[ 0_{ABC} D_{t}^{\alpha} R_{ls}(t) = \phi C_{ls} - (\omega + \mu_{h}) R_{ls} \]  

\[ 0_{ABC} D_{t}^{\alpha} S_{v}(t) = \Lambda_{v} - \beta_{v} (I_{l} + C_{ls}) S_{v} - \mu_{v} S_{v} \]  

\[ 0_{ABC} D_{t}^{\alpha} I_{v}(t) = \beta_{v} (I_{l} + C_{ls}) S_{v} - \mu_{v} I_{v} \]  

\[ 0_{ABC} D_{t}^{\alpha} S_{sv}(t) = \Lambda_{sv} - \beta_{sv} (I_{sc} + C_{ls}) S_{sv} - \mu_{sv} S_{sv} \]  

\[ 0_{ABC} D_{t}^{\alpha} I_{sv}(t) = \beta_{sv} (I_{sc} + C_{ls}) S_{sv} - \mu_{sv} I_{sv} \]  

(6)

It is an indispensable fact in the concept of differential fractional calculus for one to prove the existence and the uniqueness solution of the model being examined. In this respect, we shall devote this part to prove the existence of solution of this system. The system model comprises the following compartments.
The interpretations of the various compartments and parameters meaning are the same as in Bonyah et al [7]. The system 6 is equivalent to that of Volterra type in which the integral is represented by Atangana-Baleanu fractional integral. It can be inferred that the Atangana-Baleanu fractional integral of a given function \( f(t) \) is just equivalent to average of the function \( f(t) \) and the Riemann-Liouville fractional integral. The detailed of the proof can be found in theorem 1. The proof is shown in theorem 1.

\[
\begin{align*}
S_h(t) - a_1(t) &= \frac{1}{\alpha} \int_0^t \{ (1 - r)\pi_h + \gamma R_{sv}(t) + \sigma R_l(t) + \omega R_{sl}(t) + pV_h(t) \\
&- \beta_1 I_s(t) S_h(t) - \beta_2 I_{sv}(t) S_h(t) - \mu_h S_h(t) \} dt \\
+ \frac{1-\alpha}{\Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} \{ (1 - r)\pi_h + \gamma R_{sv}(y) + \sigma R_l(y) + \omega R_{sl}(y) + pV_h(y) \\
&- \beta_1 I_s(y) S_h(y) - \beta_2 I_{sv}(y) S_h(y) - \mu_h S_h(y) \} dy \\
+ \frac{1-\alpha}{\Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} \{ \beta_1 I_s(t) S_h(t) + \beta_2 I_{sv}(t) V_h(t) - \beta_1 I_s(t) I_{sv}(t) \\
- (\delta + \mu_h + n) I_{sv}(t) \} dt \\
+ \frac{1-\alpha}{\Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} \{ \beta_1 I_s(y) S_h(y) + \beta_2 I_{sv}(y) V_h(y) - \beta_1 I_s(y) I_{sv}(y) \\
- (\delta + \mu_h + n) I_{sv}(y) \} dy \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} \{ \beta_1 I_s(y) I_{sv}(y) + \beta_2 I_{sv}(y) I_{sv}(y) - (\delta + \mu_h + n) I_{sv}(y) \} dy \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t (t - y)^{\alpha - 1} \{ \beta_1 I_s(t) I_{sv}(t) + \beta_2 I_{sv}(t) I_{sv}(t) - (\delta + \mu_h + n) I_{sv}(t) \} dt \end{align*}
\]

(7)
A possibility of converting the above system to iterative routine is given below

\[
S_{h(n+1)}(t) = \frac{1}{\beta_0}\left\{(1-r)\pi_h + \gamma R_{sc}(t) + \sigma R_l(t) + \omega R_{st}(t) + p V_h(t) \right. \\
- \beta_1 I_s S_h(t) - \beta_2 I_{sv} S_h(t) - \mu_h S_h(t) \bigg\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{(1-r)\pi_h + \gamma R_{sc}(y) + \sigma R_l(y) + \omega R_{st}(y) + p V_h(y) \right. \\
- \beta_1 I_s S_h(y) - \beta_2 I_{sv} S_h(y) - \mu_h S_h(y) \bigg\} dy \\
I_{l(n+1)}(t) = \frac{1}{\beta_0} \left\{\beta_1 I_v S_h(t) + \beta_1 q I_v V_h(t) - \beta_2 I_{sv} I_l(t) - (\alpha + \mu_h + m) I_l(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\beta_1 I_v S_h(y) + \beta_1 q I_v V_h(y) - \beta_2 I_{sv} I_l(y) \right. \\
- (\alpha + \mu_h + m) I_l(y) \bigg\} dy \\
I_{sc(n+1)}(t) = \frac{1}{\beta_0} \left\{\beta_2 I_{sv}(t) S_h(t) + \beta_2 g I_{sv}(t) V_h(t) - \beta_1 I_v(t) I_{sc}(t) - (\delta + \mu_h + n) I_{sc}(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\beta_2 I_{sv}(y) S_h(t) + \beta_2 g I_{sv}(y) V_h(t) - \beta_1 I_v(t) I_{sc}(t) \right. \\
- (\delta + \mu_h + n) I_{sc}(y) \bigg\} dy \\
C_{ls(n+1)}(t) = \frac{1}{\beta_0} \left\{\beta_1 I_v(t) I_{sc}(t) + \beta_2 I_{sv}(t) I_l(t) - (\phi + \mu_h + \rho + \theta) C_{ls}(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\beta_1 I_v(y) I_{sc}(y) + \beta_2 I_{sv}(y) I_l(y) - (\phi + \mu_h + \rho + \theta) C_{ls}(y) \bigg\} dy \\
V_{h(n+1)}(t) = \frac{1}{\beta_0} \left\{r \Lambda_h - (p + \mu_h) V_h(t) - \beta_1 q I_v(t) V_h(t) - \beta_2 g I_{sv}(t) V_h(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{r \Lambda_h - (p + \mu_h) V_h(y) - \beta_1 q I_v(y) V_h(y) - \beta_2 g I_{sv}(y) V_h(y) \bigg\} dy \\
R_{l(n+1)}(t) = \frac{1}{\beta_0} \left\{\alpha I_l(t) - (\sigma + \mu_h) R_l(t) + (1 - \phi) \eta C_{ls}(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\alpha I_l(y) - (\sigma + \mu_h) R_l(y) + (1 - \phi) \eta C_{ls}(y) \bigg\} dy \\
R_{sc(n+1)}(t) = \frac{1}{\beta_0} \left\{\delta I_{sc}(t) - (\gamma + \mu_h) R_{sc}(t) + (1 - \phi) (1 - \eta) C_{ls}(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\delta I_{sc}(y) - (\gamma + \mu_h) R_{sc}(y) + (1 - \phi) (1 - \eta) C_{ls}(y) \bigg\} dy \\
R_{ls(n+1)}(t) = \frac{1}{\beta_0} \left\{\phi C_{ls}(t) - (\omega + \mu_h) R_{ls}(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\phi C_{ls}(y) - (\omega + \mu_h) R_{ls}(y) \bigg\} dy \\
S_{l(n+1)}(t) = a_0(t) = \frac{1}{\beta_0} \left\{\Lambda_s - \beta_s I_l(t) + C_{ls}(t) S_v(t) - \mu_v S_v(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\Lambda_s - \beta_s I_l(y) + C_{ls}(y) S_v(y) - \mu_v S_v(y) \bigg\} dy \\
I_{l(n+1)}(t) = \frac{1}{\beta_0} \left\{\beta_s (I_l(t) + C_{ls}(t)) S_v(t) - \mu_v I_v(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\beta_s (I_l(y) + C_{ls}(y)) S_v(y) - \mu_v I_v(y) \bigg\} dy \\
S_{sv(n+1)}(t) = \frac{1}{\beta_0} \left\{\Lambda_{sv} - \beta_{sv} (I_{sc}(t) + C_{ls}(t)) S_{sv}(t) - \mu_{sv} S_{sv}(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\Lambda_{sv} - \beta_{sv} (I_{sc}(y) + C_{ls}(y)) S_{sv}(y) - \mu_{sv} S_{sv}(y) \bigg\} dy \\
I_{sv(n+1)}(t) = \frac{1}{\beta_0} \left\{\beta_{sv} (I_{sc}(t) + C_{ls}(t)) S_{sv}(t) - \mu_{sv} I_{sv}(t) \right\} \\
+ \frac{1}{\beta_0} \int_0^t (t-y)^{\alpha-1} \left\{\beta_{sv} (I_{sc}(y) + C_{ls}(y)) S_{sv}(y) - \mu_{sv} I_{sv}(y) \bigg\} dy.
\]
Where the initial conditions are given as

\[
\begin{aligned}
S_{h0}(t) &= a_1(t) \\
I_{00}(t) &= a_2(t) \\
I_{sc0}(t) &= a_3(t) \\
C_{t=0}(t) &= a_4(t) \\
V_{h0}(t) &= a_5(t) \\
R_{t0}(t) &= a_6(t) \\
R_{sc0}(t) &= a_7(t) \\
R_{t=0}(t) &= a_8(t) \\
S_{v0}(t) &= a_9(t) \\
S_{sv0}(t) &= a_{10}(t) \\
I_{sv0}(t) &= a_{11}(t) \\
I_{scv0}(t) &= a_{12}(t)
\end{aligned}
\] (9)

Taking the limit for a large value of \( n \), we expect to obtain the exact solution.

### 3.1. Existence of solution using Pichard- Lindelof method

The proof is derived if one takes into consideration the following operator

\[
\begin{aligned}
g_1(t, x) &= (1 - r)\pi_h + \gamma R_{sc}(t) + \sigma R_i(t) + \omega R_{d1}(t) + pV_h(t) - \beta_1 I_v S_h(t) - \beta_2 I_{sv} S_h(t) - \mu_h S_h(t) \\
g_2(t, x) &= \beta_1 I_v S_h(t) + \beta_1 q I_s V_h(t) - \beta_2 I_{sv} I_i(t) - (\alpha + \mu_h + m) I_i(t) \\
g_3(t, x) &= \beta_2 I_{sv}(t) S_h(t) + \beta_2 g I_{sv}(t) V_h(t) - \beta_1 I_v(t) I_{sc}(t) - (\delta + \mu_h + n) I_{sc}(t) \\
g_4(t, x) &= \beta_1 I_v(t) I_{sc}(t) + \beta_2 I_{sv}(t) I_i(t) - (\phi + \mu_h + \rho + \theta) C_{ls}(t) \\
g_5(t, x) &= r\Lambda_h - (p + \mu_h) V_h(t) - \beta_1 q I_s(t) V_h(t) - \beta_2 g I_{sv}(t) V_h(t) \\
g_6(t, x) &= \alpha I_i(t) - (\sigma + \mu_h) R_i(t) - (1 - \phi) \eta C_{ls}(t) \\
g_7(t, x) &= \delta I_{sc}(t) - (\gamma + \mu_h) R_{sc}(t) + (1 - \phi)(1 - \eta) C_{ls}(t) \\
g_8(t, x) &= \phi C_{ls}(t) - (\omega + \mu_h) R_{ls}(t) \\
g_9(t, x) &= \Lambda_{ad} - \beta_v(I_i(t) + C_{ls}(t)) S_v(t) - \mu_v S_v(t) \\
g_{10}(t, x) &= \beta_v(I_i(t) + C_{ls}(t)) S_v(t) - \mu_v I_v(t) \\
g_{11}(t, x) &= \Lambda_{sv} - \beta_{sv}(I_{sc}(t) + C_{ls}(t)) S_{sv}(t) - \mu_{sv} S_{sv}(t) \\
g_{12}(t, x) &= \beta_{sv}(I_{sc}(t) + C_{ls}(t)) S_{sv}(t) - \mu_{sv} I_{sv}(t)
\end{aligned}
\] (10)

It is obvious that \( g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9, g_{10}, g_{11}, g_{12} \) are contraction with regard to \( x \) for the initial function, \( y \) for the second function, \( h \) represents the third function and the rest are \( k, r, q, s, u, j, m, n, p \) up to twelfth functions respectively.

We take into consideration the following

\[
\begin{aligned}
M_1 &= \sup \|g_1(t, x)\|, & M_2 &= \sup \|g_2(t, y)\| \\
M_3 &= \sup \|g_1(t, h)\|, & M_4 &= \sup \|g_2(t, k)\|
\end{aligned}
\]
\[ M_5 = \sup \| g_1(t, r) \|, \quad M_6 = \sup \| g_2(t, q) \| \]
\[ M_7 = \sup \| g_1(t, s) \|, \quad M_8 = \sup \| g_2(t, u) \| \]
\[ M_9 = \sup \| g_1(t, j) \|, \quad M_{10} = \sup \| g_2(t, m) \| \]
\[ M_{11} = \sup \| g_1(t, n) \|, \quad M_{12} = \sup \| g_2(t, p) \| \]  

where
\[
D_{b,c_1} = [t - b, t + b] \times [t - c_1, t + c_1] = B_1 \times C_1 \\
D_{b,c_2} = [t - b, t + b] \times [t - c_2, t + c_2] = B_1 \times C_2 \\
D_{b,c_3} = [t - b, t + b] \times [t - c_3, t + c_3] = B_1 \times C_3 \\
D_{b,c_4} = [t - b, t + b] \times [t - c_4, t + c_4] = B_1 \times C_4 \\
D_{b,c_5} = [t - b, t + b] \times [t - c_5, t + c_5] = B_1 \times C_5 \\
D_{b,c_6} = [t - b, t + b] \times [t - c_6, t + c_6] = B_1 \times C_6 \\
D_{b,c_7} = [t - b, t + b] \times [t - c_7, t + c_7] = B_1 \times C_7 \\
D_{b,c_8} = [t - b, t + b] \times [t - c_8, t + c_8] = B_1 \times C_8 \\
D_{b,c_9} = [t - b, t + b] \times [t - c_9, t + c_9] = B_1 \times C_9 \\
D_{b,c_{10}} = [t - b, t + b] \times [t - c_{10}, t + c_{10}] = B_1 \times C_{10} \\
D_{b,c_{11}} = [t - b, t + b] \times [t - c_{11}, t + c_{11}] = B_1 \times C_{11} \\
D_{b,c_{12}} = [t - b, t + b] \times [t - c_{12}, t + c_{12}] = B_1 \times C_{12} 
\]

In this instance, however, we shall explore the Banach fixed-point theorem employing the metric on \( D_{b,c_1}, (i = 1, 2, 3, 4, 5, 6, 7, \ldots, 12) \) characterized by a uniform norm as shown below
\[ \| g(t) \|_{\infty} = \sup_{t \in b, t + a} |g(t)| \]  

In this regard, the next operator is obtained by defining between two functional spaces which is of continuous functions and Pi- card’s operator as follows:
\[
O : D, (B_1, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12})
\rightarrow D, (B_1, C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12})
\]

For the purpose of simplicity we can state that \( g_i(x, t) = X(t), g_i(x, 0) = X_0(t), (i = 1, 2, 3, 4, \ldots, 12) \). Therefore, the model system can be reformulated as the following:
\[
OX(t) = X_0(t) + G(t, X(t)) \frac{1 - \alpha}{B(\alpha)}
\]
\[ + \frac{\alpha}{B(\alpha) t(\alpha)} \int_0^t (t - y)^{\alpha - 1} G(y, X(y)) dy \]  

where \( X \) denotes the matrix expressed as
\[
X(t) = \left\{ \begin{array}{c} S_h(t) \\
I_i(t) \\
I_{sc}(t) \\
C_{ls}(t) \\
V_h(t) \\
R_i(t) \\
R_{sc}(t) \\
R_{ls}(t) \\
S_v(t) \\
I_v(t) \\
S_{sv}(t) \\
I_{sv}(t) \end{array} \right. \\
X_0(t) = \left\{ \begin{array}{c} S_{h0}(t) \\
I_{i0}(t) \\
I_{sc0}(t) \\
C_{ls0}(t) \\
V_{h0}(t) \\
R_{i0}(t) \\
R_{sc0}(t) \\
R_{ls0}(t) \\
S_{v0}(t) \\
I_{v0}(t) \\
S_{sv0}(t) \\
I_{sv0}(t) \end{array} \right. \\
G(t, X(t)) = \left\{ \begin{array}{c} S_h(t) \\
I_i(t) \\
I_{sc}(t) \\
C_{ls}(t) \\
V_h(t) \\
R_i(t) \\
R_{sc}(t) \\
R_{ls}(t) \\
S_v(t) \\
I_v(t) \\
S_{sv}(t) \\
I_{sv}(t) \end{array} \right. 
\]

In most instances, there are hardly some diseases that can wipe out the entire world population and also target population is finite. It can therefore stated that
all the solutions are bound given a period of time.

\[
\|x(t)\|_\infty \leq \max\{c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8, c_9, c_{10}, c_{11}, c_{12}\} \quad (17)
\]

\[
\|OX(t) - X_0(t)\| = \left\| G(t, X(t)) \left\{ \frac{1-\alpha}{B(\alpha)} \int_0^t (t-y)^{\alpha-1} G(y, X(y)) dy \right\} \right\|
\]

\[
\leq \frac{1-\alpha}{B(\alpha)} \|G(t, X(t))\| + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \|G(y, X(y))\| dy
\]

\[
\leq \frac{1-\alpha}{B(\alpha)} M = \max\{M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{11}, M_{12}\}
\]

\[
\frac{\alpha}{B(\alpha)\Gamma(\alpha)} M^\alpha < Mb \leq c = \max\{b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}\}
\]

We ensure that

\[
b < \frac{c}{M}
\]

Again, the following equality is examined

\[
\|OX_1 - OX_2\|_\infty = \sup_{t \in B} |OX_1 - OX_2|
\]

(19)

Based on the defined operator being examined we derive the following

\[
\|OX_1 - OX_2\| = \left\| G(t, X_1(t)) - G(t, X_2(t)) \left\{ \frac{1-\alpha}{B(\alpha)} \int_0^t (t-l)^{\alpha-1} \left\{ \frac{G(l, X_1(l))}{-G(l, X_2(l))} \right\} dl \right\} \right\|
\]

\[
\leq \frac{1-\alpha}{B(\alpha)} \|G(t, X_1(t)) - G(t, X_2(t))\|
\]

\[
+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \|G(l, X_1(y)) - G(l, X_2(y))\| dy
\]

\[
\leq \frac{1-\alpha}{B(\alpha)} \alpha z \|X_1(t) - X_2(t)\|
\]

\[
+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \|X_1(y) - X_2(y)\| dy
\]

\[
\leq \left\{ \frac{1-\alpha}{B(\alpha)} \alpha z + \frac{\alpha z^\alpha}{B(\alpha)\Gamma(\alpha)} \right\} \|X_1(t) - X_2(t)\|
\]

\[
\leq bz \|X_1(t) - X_2(t)\|
\]

(20)

where \(z < 1\). Since \(G\) denotes a contraction then it can be stated that \(bz < 1\), in addition, the defined operator \(O\) is also considered as contraction. This indicates that the system being examined is characterized with a singular set of solutions.

### 3.2. Iteration solution approach.

By the fact that this coinfected model is highly nonlinear, it is usually difficult to obtain solution analytically therefore, an iterative techniques is preferred. In this regard, an iterative method which is hinged on integral transform of a well-known Sumudu transform operator will be used for this study. This operator possesses properties that are capable of keeping the parity of the function. The following subsequent theorem is crucial in order to explore the system in details and detailed discussion of the theorem can be found in [1].

**Theorem 3.1.** Let \(f \in H^1(b, c), c > b \) and \(\alpha \in [0, 1]\), the Sumudu transform of \(ABC\) is expressed as

\[
ST \left\{ \frac{ABC}{0} D_t^\alpha (f(t)) \right\} = \frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1) E_\alpha \left( \frac{1}{1-\alpha \gamma^\alpha} \right) \right) (ST(f(t)) - f(0)).
\]

(22)
Proof. For the Proof of the theorem readers are encouraged to consult the study by Atangana and Koca [1]. To obtain solution to the system 7, we employ the Sumudu transform of the Atangana-Baleanu fractional derivative of the system of both sides. The system below is then arrived at

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(S_h(t)) - S_h(0) \right) &= ST\{(1-r)\pi_h + \gamma R_{se}(t) + \\
&\sigma R_i(t) + \omega R_{st}(t) + pV_h(t) - \beta_1 I_v S_h(t) - \beta_2 I_v S_h(t) - \mu h S_h(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(I_t(t)) - I_t(0) \right) &= ST\{\beta_1 I_v S_h(t) + \beta_3 q I_v V_h(t) - \\
&\beta_2 I_v I_t(t) - (\alpha + \mu_h + m) I_t(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(I_{sc}(t)) - I_{sc}(0) \right) &= ST\{\beta_2 I_v S_h(t) + \\
&\beta_3 g I_v S_h(t) - \beta_1 I_v (t - (\delta + \mu_h + n)) I_{sc}(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(C_{ls}(t)) - I_{ls}(0) \right) &= ST\{\beta_1 I_v I_{sc}(t) + \\
&\beta_2 I_v I_{sc}(t) I_{ls}(t) - (\phi + \mu_h + \rho + \theta) C_{ls}(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(V_h(t)) - V_h(0) \right) &= ST\{r \Lambda_h - (p + \mu_h) V_h(t) - \\
&\beta_1 q I_v V_h(t) - \beta_2 g I_v V_h(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(R_i(t)) - R_i(0) \right) &= ST\{\alpha I_i(t) - (\sigma + \mu_h) R_i(t) + \\
&(1 - \phi)\eta C_{ls}(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(R_{ls}(t)) - R_{ls}(0) \right) &= ST\{\phi C_{ls}(t) - (\omega + \mu_h) R_{ls}(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(S_v(t)) - S_v(0) \right) &= ST\{\Lambda_{st} - \beta_v (I_t(t) + C_{ls}(t)) S_v(t) - \mu_v S_v(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(I_v(t)) - I_v(0) \right) &= ST\{\beta_v (I_v(t) + C_{ls}(t)) S_v(t) - \mu_v I_v(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(S_{sv}(t)) - S_{sv}(0) \right) &= ST\{\Lambda_{sv} - \beta_{sv} (I_{sc}(t) + C_{ls}(t)) S_{sv}(t) - \mu_{sv} S_{sv}(t)\}
\end{aligned}
\]

\[
\begin{aligned}
\frac{B(\alpha)}{1-\alpha} \left( \alpha \Gamma(\alpha + 1)E_\alpha \left(-\frac{1}{1-s^p}\right) \right) \left( ST(I_{sv}(t)) - I_{sv}(0) \right) &= ST\{\beta_{sv} (I_{sc}(t) + C_{ls}(t)) S_{sv}(t) - \\
&\mu_{sv} I_{sv}(t)\}.
\end{aligned}
\]

(23)

Reorganizing the above system the following inequalities are obtained where \(\psi = \frac{1}{1-\alpha}\)

\[
ST(S_h(t)) = S_h(0) + \frac{(1-\alpha)}{B(\alpha)\alpha \Gamma(\alpha + 1)E_\alpha(\psi r^p)} \times ST\{(1-r)\pi_h + \gamma R_{se}(t) + \\
\sigma R_i(t) + \omega R_{st}(t) + pV_h(t) - \beta_1 I_v S_h(t) - \beta_2 I_v S_h(t) - \mu h S_h(t)\}
\]
$ST(I(t)) = I_s(0) + \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \beta_1 I_c S_h(t) + \beta_1 q I_v V_h(t) - \beta_2 I_v I(t) - (\alpha + \mu_h + m) I(t) \}$

$ST(I_{sc}(t)) = I_{sc}(0) + \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \beta_2 I_{sv} L_{sc}(t) + \beta_2 q I_{sv} V_h(t) - \beta_1 I_v I_{sc}(t) - (\delta + \mu_h + n) I_{sv}(t) \}$

$ST(I_s(t)) = I_s(0) + \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \beta_1 I_c I_{sc}(t) + \beta_2 I_{sv}(t) I(t) - (\phi + \mu_h + \rho + \theta) C_{is}(t) \}$

$ST(V_h(t)) = V_h(0) + \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \phi \Lambda_h - (\mu + \rho) V_h(t) - \beta_1 q I_v V_h(t) - \beta_2 g I_{sv}(t) V_h(t) \}$

$ST(R_h(t)) = R_h(0) + \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \alpha I_t(t) - (\sigma + \mu_h) R_h(t) + (1 - \phi)(1 - \eta) C_{is}(t) \}$

$ST(I_{sc}(t)) = R_{sc}(0) + \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \delta I_{sc}(t) - (\gamma + \mu_h) R_{sc}(t) + (1 - \phi)(1 - \eta) C_{is}(t) \}$

$ST(S_v(t)) = S_v(0) + \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \Lambda_{sl} - \beta_v (I_t(t) + C_{is}(t)) S_v(t) - \mu_v S_v(t) \}$

$ST(I_v(t)) = I_v(0) + \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \beta_v (I_t(t) + C_{is}(t)) S_v(t) - \mu_v I_v(t) \}$

$ST(I_{sv}(t)) = I_{sv}(0) + \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \beta_v (I_{sc}(t) + C_{is}(t)) S_v(t) - \mu_v I_{sv}(t) \}.$

The recursive formula is obtained as

$S_{h(n+1)}(t) = S_h(n)(0) + ST^{-1} \left\{ \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ (1 - \tau) \pi_h + \gamma R_{sc}(n)(t) + \sigma R_t(n)(t) + \omega R_{al}(n)(t) + p V_h(n)(t) - \beta_1 I_{sv(n)} S_{h(n)}(t) - \beta_2 I_{sv(n)} S_h(n)(t) - \mu_h S_{h(n)}(t) \} \right\}$

$I_{t(n+1)} = I_t(n)(0) + ST^{-1} \left\{ \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \beta_1 I_{sv(n)} S_{h(n)}(t) + \beta_1 q I_{sv(n)} V_h(n)(t) - \beta_2 I_{sv(n)} I_t(n)(t) - (\alpha + \mu_h + m) I_t(n)(t) \} \right\}$

$I_{sc(n+1)}(t) = I_{sc(n)}(0) + ST^{-1} \left\{ \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \beta_2 I_{sv(n)}(t) S_{h(n)}(t) + \beta_2 I_{sv(n)} V_h(n)(t) - \beta_1 I_{sv(n)} I_{sc(n)}(t) - (\delta + \mu_h + n) I_{sc(n)}(t) \} \right\}$

$I_{sv(n+1)}(t) = I_{sv(n)}(0) + ST^{-1} \left\{ \frac{(1 - \alpha)}{B(\alpha)\Gamma(\alpha + 1)E_{\alpha}(\psi \rho^\alpha)} \times ST \{ \beta_1 I_{sv(n)}(t) I_{sc(n)}(t) + \beta_2 I_{sv(n)}(t) I_t(n)(t) - (\phi + \mu_h + \rho + \theta) C_{is(n)}(t) \} \right\}$
\[ V_{h(n+1)}(t) = V_{h(n)}(0) + ST^{-1} \left\{ \frac{1}{\mu} \left( \frac{(1-\alpha)}{\mu} \right) \right\} \times \]
\[ ST \{ rA_h - (p + \mu_h)\alpha V_{h(n)}(t) - \beta_1 qI_{v(n)}(t) + \beta_2 g I_{s(v(n))} + I_{h(n)}(t) \} \]
\[ R_{l(n+1)}(t) = R_{l(n)}(0) + ST^{-1} \left\{ \frac{1}{\mu} \left( \frac{(1-\alpha)}{\mu} \right) \right\} \times \]
\[ ST \{ \alpha I_{l(n)}(t) - (\sigma + \mu_h)R_{l(n)}(t) + (1 - \phi)\eta C_{l(n)}(t) \} \]
\[ R_{s(n+1)}(t) = R_{s(n)}(0) + ST^{-1} \left\{ \frac{1}{\mu} \left( \frac{(1-\alpha)}{\mu} \right) \right\} \times \]
\[ ST \{ \beta_1 qI_{v(n)}(t) - (\gamma + \mu_h)R_{s(n)}(t) + (1 - \phi)(1 - \eta)C_{l(n)}(t) \} \]
\[ R_{s(n+1)}(t) = R_{s(n)}(0) + ST^{-1} \left\{ \frac{1}{\mu} \left( \frac{(1-\alpha)}{\mu} \right) \right\} \times \]
\[ ST \{ \phi C_{l(n)}(t) - (\omega + \mu_h)R_{l(n)}(t) \} \]
\[ S_{v(n+1)}(t) = S_{v(n)}(0) + ST^{-1} \left\{ \frac{1}{\mu} \left( \frac{(1-\alpha)}{\mu} \right) \right\} \times \]
\[ ST \{ \Lambda_{l(n)}(t) - \beta_1 (I_{l(n)}(t) + C_{l(n)}(t))S_{v(n)}(t) - \mu_v S_{v(n)}(t) \} \]
\[ I_{v(n+1)}(t) = I_{v(n)}(0) + ST^{-1} \left\{ \frac{1}{\mu} \left( \frac{(1-\alpha)}{\mu} \right) \right\} \times \]
\[ ST \{ \beta_1 (I_{l(n)}(t) + C_{l(n)}(t))S_{v(n)}(t) - \mu_v I_{v(n)}(t) \} \]
\[ S_{sv(n+1)}(t) = S_{sv(n)}(0) + ST^{-1} \left\{ \frac{1}{\mu} \left( \frac{(1-\alpha)}{\mu} \right) \right\} \times \]
\[ ST \{ \Lambda_{sv(n)}(t) - \beta_{sv} (I_{sv(n)}(t) + C_{l(n)}(t))S_{sv(n)}(t) - \mu_{sv} S_{sv(n)}(t) \} \]
\[ I_{sv(n+1)}(t) = I_{sv(n)}(0) + ST^{-1} \left\{ \frac{1}{\mu} \left( \frac{(1-\alpha)}{\mu} \right) \right\} \times \]
\[ ST \{ \beta_{sv} (I_{sv(n)}(t) + C_{l(n)}(t))S_{sv(n)}(t) - \mu_{sv} I_{sv(n)}(t) \} \]

After several iterations the solution of system 25 is obtained as

\[
\begin{align*}
S_h(t) &= \lim_{n \to \infty} S_{h(n)}(t) \\
I_l(t) &= \lim_{n \to \infty} I_{l(n)}(t) \\
I_{sv}(t) &= \lim_{n \to \infty} I_{sv(n)}(t) \\
C_{ls}(t) &= \lim_{n \to \infty} C_{ls(n)}(t) \\
V_h(t) &= \lim_{n \to \infty} V_{h(n)}(t) \\
R_l(t) &= \lim_{n \to \infty} R_{l(n)}(t) \\
R_{sv}(t) &= \lim_{n \to \infty} R_{sv(n)}(t) \\
R_{ls}(t) &= \lim_{n \to \infty} R_{ls(n)}(t) \\
S_v(t) &= \lim_{n \to \infty} S_v(n(t)) \\
I_v(t) &= \lim_{n \to \infty} I_v(n(t)) \\
S_{sv}(t) &= \lim_{n \to \infty} S_{sv(n)}(t) \\
I_{sv}(t) &= \lim_{n \to \infty} I_{sv(n)}(t)
\end{align*}
\]

3.3 Fixed-point theorem for stability analysis of iteration method. Assume that \((X, ||\cdot||)\) is a Banach space and also \(H\) deemed to a self-map of \(X\). Furthermore, it is also considered that \(y_{n+1} = g(H, y_n)\) be specific recursive procedure. Assume that \(F(H)\) denotes the fixed-point set of \(H\) and contains at least a single element and that \(y^n\) converges to a point \(p \in F(H)\). Let \(\{x_n\} \subseteq X\) and define \(e_n = ||x_{n+1} - g(H, x_n)||\). If \(\lim_{n \to 0} e^n = 0\) it means that \(\lim_{n \to 0} x^n = p\) then the iteration
method \( y_{n+1} = g(H, y_n) \) is deemed to be \( H \) stable. It is also taken into considered that the sequence \( \{x_n\} \) possesses an upper boundary, else, it is not possible to anticipate the presence of convergence. If the entire conditions stated above are satisfied for \( \{x_n\} \) which is better known as Picard’s iteration, eventually, the iteration will be \( H \)-stable. The following theorem shall be stated.

**Theorem 3.2.** Let \((X, \|\cdot\|)\) be a Banach space and \( H \) a self-map of \( X \) satisfying \( \|H_x - H_y\| \leq C \|x - H_x\| + c \|x - y\| \) for all \( x, y \) in \( X \) where \( 0 \leq C, 0 \leq c < 1 \). Assume that \( H \) is Picard’s \( H \)-stable

The recursive formula 25 together with system equation 7 is taking into consideration as given below

\[
S_{h(n+1)}(t) = S_{h(n)}(0) + ST^{-1}\{\varphi ST\{(1 - r)\pi_h + \gamma R_{sc(n)}(t) + \sigma R_{l(n)}(t) + \omega R_{st(n)}(t) + pV_{h(n)}(t) - \beta_1 I_{v(n)}(t) - \beta_2 I_{sv(n)}(t) - \mu_h S_h(n)(t)\}\}
\]

\[
I_{l(n+1)} = I_{l(n)}(0) + ST^{-1}\{\varphi \beta_1 I_{v(n)}(t) + \beta_1 q I_{v(n)}(t) V_{h(n)}(t) - \beta_2 I_{sv(n)}(t) I_{l(n)}(t) - (\alpha + \mu_h + m) I_{l(n)}(t)\}
\]

\[
I_{sc(n+1)}(t) = I_{sc(n)}(0) + ST^{-1}\{\varphi ST\{(\beta_1 I_{v(n)}(t) I_{sc(n)}(t)) + \beta_2 I_{sv(n)}(t) I_{l(n)}(t) - (\phi + \mu_h + \rho + \theta) C_{ls(n)}(t)\}\}
\]

\[
V_{h(n+1)}(t) = V_{h(n)}(0) + ST^{-1}\{\varphi ST\{(r\Lambda_v - (p + \mu_h)V_{h(n)}(t) - \beta_1 q I_{v(n)}(t) V_{h(n)}(t) - \beta_2 g I_{sv(n)}(t) V_{h(n)}(t)\}\}
\]

\[
R_{l(n+1)}(t) = R_{l(n)}(0) + ST^{-1}\{\varphi ST\{(\alpha l(n)(t) - (\sigma + \mu_h) R_{l(n)}(t) + (1 - \phi) \eta C_{ls(n)}(t)\}\}
\]

\[
R_{sc(n+1)}(t) = R_{sc(n)}(0) + ST^{-1}\{\varphi ST\{(\gamma + \mu_h) R_{sc(n)}(t) + (1 - \phi)(1 - \eta) C_{ls(n)}(t)\}\}
\]

\[
S_{v(n+1)}(t) = S_{v(n)}(0) + ST^{-1}\{\varphi ST\{(\Lambda_{st} - \beta_v I_{v(n)}(t) + C_{ls(n)}(t)) S_{v(n)}(t) - \mu_v S_{v(n)}(t)\}\}
\]

\[
I_{v(n+1)}(t) = I_{v(n)}(0) + ST^{-1}\{\varphi ST\{(\beta_v I_{l(n)}(t) + C_{ls(n)}(t)) S_{v(n)}(t) - \mu_v I_{v(n)}(t)\}\}
\]

\[
S_{sv(n+1)}(t) = S_{sv(n)}(0) + ST^{-1}\{\varphi ST\{(\Lambda_{sv} - \beta_{sv} I_{sc(n)}(t) + C_{ls(n)}(t)) S_{sv(n)}(t) - \mu_{sv} S_{sv(n)}(t)\}\}
\]

\[
I_{sv(n+1)}(t) = I_{sv(n)}(0) + ST^{-1}\{\varphi ST\{(\beta_{sv} I_{sc(n)}(t) + C_{ls(n)}(t)) S_{sv(n)}(t) - \mu_{sv} I_{sv(n)}(t)\}\}
\]

(26)

where \( \varphi = \frac{1 - \alpha}{B(\alpha) \sigma (1 + \alpha) E_{\alpha}(\varphi)\alpha} \) denotes the fractional Lagrange multiplier.
Theorem 3.3. Let $H$ be a self-map defined expressed as in (27)

\[
H(S_h(n)(t)) = S_h(n-1)(t) + ST^{-1}\{\phi ST{\{\beta_1 I_v(n)S_h(n)(t) + \beta_2 I_v(n(t)S_h(n)(t) - \beta_1 I_v(n(t)S_h(n)(t) - \beta_2 I_v(n(t)S_h(n)(t) - \mu_h S_h(n)(t))}\}
\]

\[
H(I_{l(n)}(t)) = I_{l(n+1)} = I_{l(n)}(t) + ST^{-1}\{\phi ST{\{\beta_1 I_v(n)S_h(n)(t) + \beta_1 q I_v(n)S_h(n)(t) - \beta_2 I_v(n(t)S_h(n)(t) - \mu_h S_h(n)(t))}\}
\]

\[
H(I_{sc(n)}(t)) = I_{sc(n+1)} = I_{sc(n)}(t) + ST^{-1}\{\phi ST{\{\beta_1 I_v(n(t)S_h(n)(t) + \beta_2 g I_v(n(t)S_h(n)(t) - \beta_1 I_v(n(t)S_h(n)(t) - \mu_h S_h(n)(t))\}
\}
\]

\[
H(V_h(n)(t)) = V_{h(n+1)}(t) = V_{h(n)}(t) + ST^{-1}\{\phi ST{\{\lambda A_h - (p + \mu_h) V_{h(n)}(t) - \beta_1 q I_v(n(t)S_h(n)(t) - \beta_2 g I_v(n(t)S_h(n)(t))\}
\}
\]

\[
H(R_{sc(n)}(t)) = R_{sc(n+1)} = R_{sc(n)}(t) + ST^{-1}\{\phi ST{\{\lambda A_h - (p + \mu_h) R_{sc(n)}(t) + \lambda A_h - (p + \mu_h) R_{sc(n)}(t)\}
\}
\]

\[
H(S_v(n)(t)) = S_v(n+1)(t) = S_v(n)(t) + ST^{-1}\{\phi ST{\{\lambda A_h - (p + \mu_h) S_v(n)(t)\}
\}
\]

\[
H(I_{v(n+1)}(t)) = I_{v(n+1)}(t) = I_{v(n)}(t) + ST^{-1}\{\phi ST{\{\lambda A_h - (p + \mu_h) I_{v(n)}(t)\}
\}
\]

Thus, 27 is H- stable in $L^1(b,c)$ if the following statement can be arrived at below:

\[
(1 + (1 - r)\pi h + \gamma G(\varpi) + \sigma F(\varpi) + \omega J(\varpi) + pE(\varpi) - \beta_1 L_1 A(\varpi) - \beta_2 AN(\varpi) - \mu_h A(\varpi)) < 1
\]

\[
(1 + \beta_1 LA(\varpi) + \beta_1 q LE(\varpi) - \beta_2 NB(\varpi) - (\alpha + \mu_h + m) B(\varpi)) < 1
\]

\[
(1 + \beta_2 N(\varpi)A(\varpi) + \beta_2 g N(\varpi)E(\varpi) - \beta_1 L(\varpi)C(\varpi) - (\delta + \mu_h + n) C(\varpi)) < 1
\]

\[
(1 + \beta_1 L(\varpi)C(\varpi) + \beta_2 N(\varpi)B(\varpi) - (\delta + \mu_h + \rho + \theta) D(\varpi)) < 1
\]
Proof. We begin by making sure that $H$ possesses a fixed point. In order to establish this the following are examined for all $(n, m) \in \mathbb{N} \times \mathbb{N}$.

\[
H(S_{h(n)}(t)) - HS_{h(m)}(t) = S_{h(n)}(t) - S_{h(m)}(t)
\]

\[
+ST^{-1}\{\varphi \ast ST\{(1 - r)\pi_h + \gamma R_{sc(n)}(t) + \sigma R_{l(n)}(t) + \omega R_{sl(n)}(t) + pV_{h(n)}(t)\}
\]

\[
\left\{-\beta_1 I_{e(n)}S_{h(n)}(t) - \beta_2 I_{sv(n)}S_{h(n)}(t) - \mu_h S_{h(n)}(t)\right\}
\]

\[
\left\{-((1 - \tau)\pi_h + \gamma R_{sc(m)}(t) + \sigma R_{l(m)}(t) + \omega R_{sl(m)}(t) + pV_{h(m)}(t)\}
\]

\[
\left\{-\beta_1 I_{e(m)}S_{h(m)}(t) - \beta_2 I_{sv(m)}S_{h(m)}(t) - \mu_h S_{h(m)}(t)\right\}
\]

\[
\leq \left\| (S_{h(n)}(t) - S_{h(m)}(t) \right\|
\]

\[
\left\| +ST^{-1}\{\varphi \ast ST\{-[\beta_1 I_e + \beta_2 I_{sv} + \mu_h](S_{h(n)}(t) - S_{h(m)}(t)\})\} \right\|
\]

We now have

\[
\left\| H(S_{h(n)}(t)) - H(S_{h(m)}(t)) \right\| \leq \left\| S_{h(n)}(t) - S_{h(m)}(t) \right\|
\]

\[
\times (1 + \gamma G(\varpi) + \sigma F(\varpi) + \omega J(\varpi) + pE(\varpi) - [\beta_1 L(\varpi) + \beta_2 N(\varpi) + \mu_h] (A(\varpi)),
\]

\[
A(\varpi) \text{ denotes the } ST^{-1}\{\varphi \ast ST\} \text{ Because all the solutions depict same function the following can be stated}
\]

\[
\left\| H(I_{l(n)}(t)) - H(I_{l(m)}(t)) \right\| \leq \left\| I_{l(n)}(t) - I_{l(m)}(t) \right\|
\]

\[
\times (1 + \beta_1 L(\varpi) A(\varpi) + \beta_1 qL(\varpi) E(\varpi) - [\beta_2 N(\varpi) + (a + \mu_h + n)] (B(\varpi)),
\]

\[
\left\| H(I_{se(n)}(t)) - H(I_{se(m)}(t)) \right\| \leq \left\| I_{se(n)}(t) - I_{se(m)}(t) \right\|
\]

\[
\times (1 + \beta_2 N(\varpi) A(\varpi) + \beta_2 qN(\varpi) E(\varpi) - [\beta_1 L(\varpi) + (\delta + \mu_h + n)] (C(\varpi)),
\]

\[
\left\| H(I_{se(m)}(t)) - H(I_{se(m)}(t)) \right\| \leq \left\| I_{se(n)}(t) - I_{se(m)}(t) \right\|
\]

\[
\times (1 + \beta_1 L(\varpi) C(\varpi) + \beta_2 N(\varpi) B(\varpi) - [(\phi + \mu_h + \rho + \theta) C_{ls}(t)] (D(\varpi)),
\]

\[
\left\| H(V_{h(n)}(t)) - H(V_{h(m)}(t)) \right\| \leq \left\| V_{h(n)}(t) - V_{h(m)}(t) \right\|
\]

\[
\times (1 - [(p + \mu_h) + \beta_1 qL(\varpi) V_h(t) + \beta_2 gN(\varpi)] E(\varpi)),
\]
\[ \|H(R_{ll(m)}(t)) - H(R_{lm}(t))\| \leq \|R_{ll(m)}(t) - R_{lm}(t)\| \times (1 + (1 - \phi)\eta D(\omega) + \alpha B(\omega) - (\sigma + \mu h) F(\omega)), \]
\[ \|H(R_{sc(m)}(t)) - H(R_{sc}(t))\| \leq \|R_{sc(m)}(t) - R_{sc}(t)\| \times (1 + \delta C(\omega) + (1 - \phi)(1 - \eta)D(\omega) - (\sigma + \mu h) G(\omega)), \]
\[ \|H(R_{ls(m)}(t)) - H(R_{ls}(t))\| \leq \|R_{ls(m)}(t) - R_{ls}(t)\| \times (1 + \phi D(\omega) - (\omega + \mu h) J(\omega)), \]
\[ \|H(S_{ll(m)}(t)) - H(S_{lm}(t))\| \leq \|S_{ll(m)}(t) - S_{lm}(t)\| \times (1 - \beta_s(B(\omega) + D(\omega)) + \mu_c K(\omega)), \]
\[ \|H(I_{ll(m)}(t)) - H(I_{lm}(t))\| \leq \|I_{ll(m)}(t) - I_{lm}(t)\| \times (1 + \beta_s(B(\omega) + D(\omega))K(\omega) - \mu_N L(\omega)), \]
\[ \|H(S_{sc(m)}(t)) - H(S_{sc}(t))\| \leq \|S_{sc(m)}(t) - S_{sc}(t)\| \times (1 - \beta_s(C(\omega) + D(\omega)) + \mu_c M(\omega)), \]
\[ \|H(I_{sc(m)}(t)) - H(I_{sc}(t))\| \leq \|I_{sc(m)}(t) - I_{sc}(t)\| \times (1 + \beta_s(C(\omega) + D(\omega))M(\omega) - \mu_N N(\omega)). \]

For
\begin{eqnarray*}
(1 + \gamma G(\omega) + \sigma F(\omega) + \omega J(\omega) + pE(\omega) - \beta_1 LA(\omega) - \beta_2 AN(\omega) - \mu h A(\omega) < 1, \\
(1 + \beta_1 LA(\omega) + \beta_1 q LE(\omega) - \beta_2 NB(\omega) - (\alpha + \mu h + m)B(\omega) < 1, \\
(1 + \beta_2 N(\omega)A(\omega) + \beta_2 q N(\omega)E(\omega) - \beta_1 L(\omega)C(\omega) - (\delta + \mu h + n)C(\omega) < 1, \\
(1 + \beta_1 L(\omega)C(\omega) + \beta_2 N(\omega)B(\omega) - (\phi + \mu h + \rho + \theta)D(\omega) < 1, \\
(1 + rA_h - (p + \mu h) E(\omega) - \beta_1 q L(\omega)E(\omega) - \beta_2 q N(\omega)E(\omega) < 1, \\
(1 + \alpha B(\omega) - (\sigma + \mu h) F(\omega) + (1 - \phi)\eta D(\omega) < 1, \\
(1 + \delta C(\omega) - (\gamma + \mu h)G(\omega) + (1 - \phi)(1 - \eta)D(\omega) < 1, \\
(1 + \phi D(\omega) - (\omega + \mu h) J(\omega)) < 1, \\
(1 - \beta_s(B(\omega) + D(\omega))K(\omega) - \mu_c K(\omega) < 1, \\
(1 + \beta_s(B(\omega) + D(\omega))K(\omega) - \mu_c L(\omega) < 1, \\
(1 + \Lambda_{sc} - \beta_{sc}(C(\omega) + D(\omega))M(\omega) - \mu_{sc} M(\omega)) < 1, \\
(1 + \beta_{sc}(C(\omega) + D(\omega))M(\omega) - \mu_{sc} N(\omega)) < 1, \\
\end{eqnarray*}

then \( H \) - self possesses mapping a fixed point. Again, non-linear mapping \( H \) must meet the condition below. We assume that

\[ k_1 = (0, 0, 0, 0, 0, 0, 0, 0, 0, 0) \]

\[
\begin{cases}
(1 + \gamma G(\omega) + \sigma F(\omega) + \omega J(\omega) + pE(\omega) - \beta_1 LA(\omega) - \beta_2 AN(\omega) - \mu h A(\omega) < 1, \\
(1 + \beta_1 LA(\omega) + \beta_1 q LE(\omega) - \beta_2 NB(\omega) - (\alpha + \mu h + m)B(\omega) < 1, \\
(1 + \beta_2 N(\omega)A(\omega) + \beta_2 q N(\omega)E(\omega) - \beta_1 L(\omega)C(\omega) - (\delta + \mu h + n)C(\omega) < 1, \\
(1 + \beta_1 L(\omega)C(\omega) + \beta_2 N(\omega)B(\omega) - (\phi + \mu h + \rho + \theta)D(\omega) < 1, \\
(1 + rA_h - (p + \mu h) E(\omega) - \beta_1 q L(\omega)E(\omega) - \beta_2 q N(\omega)E(\omega) < 1, \\
(1 + \alpha B(\omega) - (\sigma + \mu h) F(\omega) + (1 - \phi)\eta D(\omega) < 1, \\
(1 + \delta C(\omega) - (\gamma + \mu h)G(\omega) + (1 - \phi)(1 - \eta)D(\omega) < 1, \\
(1 + \phi D(\omega) - (\omega + \mu h) J(\omega)) < 1, \\
(1 - \beta_s(B(\omega) + D(\omega))K(\omega) - \mu_c K(\omega) < 1, \\
(1 + \beta_s(B(\omega) + D(\omega))K(\omega) - \mu_c L(\omega) < 1, \\
(1 + \Lambda_{sc} - \beta_{sc}(C(\omega) + D(\omega))M(\omega) - \mu_{sc} M(\omega)) < 1, \\
(1 + \beta_{sc}(C(\omega) + D(\omega))M(\omega) - \mu_{sc} N(\omega)) < 1, \\
\end{cases}
\]
3.4. **Numerical simulations.** This section presents the parameter values for numerical simulation based on the one used in [7], with various \( \alpha \) values using the numerical scheme in [3]. The parameter values employed are given as \( m = 0.2 \), \( \beta_1 = 0.034 \), \( \beta_v = 0.09 \), \( \beta_2 = 0.406 \), \( \beta_{sv} = 0.615 \), \( \mu_h = 0.00004 \), \( \mu_v = 0.143 \), \( \mu_{sv} = 0.000569 \), \( \sigma = 0.167 \), \( \gamma = 0.0013 \), \( \omega = 0.013 \), \( \Lambda_h = 800 \), \( \Lambda_v = 1000 \), \( \Lambda_s = 100 \), \( \delta = 0.35 \), \( \omega = 0.0181 \), \( \psi = 0.5 \), \( \tau = 0.1 \), \( \eta = 0.0039 \), \( \theta = 0.1 \) together with following initial conditions \( S_h(0) = 100 \), \( I_l(0) = 10 \), \( I_{sc}(0) = 0 \), \( C_{ls}(0) = 5 \), \( V_h(0) = 1 \), \( R_l(0) = 0 \), \( R_{sc}(0) = 0 \), \( R_{ls}(0) = 0 \), \( S_v(0) = 0 \), \( I_v(0) = 0 \), \( S_{sv}(0) = 0 \) and \( I_{sv}(0) = 0 \).

We show only the infected compartment because of the many compartments of system 6. The results shown in Figures 1-5 depict clearly that the dynamical behaviour of Lymphatic filariasis-schistosomiasis coinfected model hinges on the fractional derivative order. In fact, the fractional Lymphatic filariasis-schistosomiasis coinfected system 6 within the general Mittag-Leffler function (ABC fractional operator) shows some asymptotic characteristics which is as a result of memory effects which could not be observed when the model is examined at \( \alpha = 1 \).

![Figure 1. Approximate solution for \( \alpha = 0.3 \).](image-url)

4. **Conclusion.** This paper seeks to extend the Lymphatic filariasis-schistosomiasis coinfected models within the concept of fractional derivative in the form of Mittag-Leffler function. The fixed point theorem was used to study the existence of generalized model. The Sumudu transform technique of Atanagana-Balenau derivative in the light of Caputo sense was used to derive the solution of the model system 6. The H-stable analysis approach was used to validate the method used to derive the solution. The numerical simulation results also showed that the dynamical behaviour of the coinfected model depends on the fractional order derivative and revealed some hidden properties which could not have been seen when \( \alpha = 1 \).
Figure 2. Approximate solution for $\alpha = 0.5$.

Figure 3. Approximate solution for $\alpha = 0.65$. 
Figure 4. Approximate solution for $\alpha = 0.75$.

Figure 5. Approximate solution for $\alpha = 0.95$. 
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