FINITE EQUAL NORM PARSEVAL WAVELET FRAMES OVER PRIME FIELDS

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Abstract. In the framework of wave packet analysis, finite wavelet systems are particular classes of finite wave packet systems. In this paper, using a scaling matrix on a permuted version of the discrete Fourier transform (DFT) of system generator, we derive a locally-scaled version of the DFT of system generator and obtain a finite equal-norm Parseval wavelet frame over prime fields. We also give a characterization of all multiplicative subgroups of the cyclic multiplicative group, for which the associated wavelet systems form frames. Finally, we present some concrete examples as applications of our results.

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1. Introduction

The theory of frames in finite dimensional Hilbert spaces has been recognized through its central role in signal representation methods [4, 9]. The best known frames in applications including data transmission such as packet communication networks [22], multiple antenna coding [25], perfect reconstruction filter banks (PRFBs) [27], quantization error problems in quantized frame expansions [23], and some areas of quantum communication theory [7] are finite equal-norm Parseval frames (ENPFs). The potential of these types of frames is due to the high capability of erasure-resilient, speedy implementation of reconstruction as well as the structure of equal energy frame vectors. The first comprehensive review of relevant studies on general equal-norm tight frames (ENTFs), and equal-norm Parseval frames (ENPFs) as the subclasses of ENTFs have been studied in [8]. Some classes of ENPFs such as (general) harmonic frames, Gabor frames (finite discrete Gabor frames) and ENPFs with single generator or more generators which have group structures, have been also introduced in [8]. Several other classes of ENPFs can be found in [10]. Moreover, a number of constructive methods to ENTFs and ENPFs from finite sets of vectors have been considered [8, 10]. Recently, some convergent constructive methods to ENPFs have been introduced [6]. Nevertheless, there is a lack of variety in classes of structured ENPFs, obtained from some generating vectors or with a simple structure which are important in the point of low rate of computation and
complexity. Furthermore, in practice, only one special class of ENPFs might not be guaranteed to be suitable for all applications.

Traditionally, the classical Gabor transforms and wavelet transforms have been used to perform time-frequency (resp. time-scale) analysis of a given function/signal in a Hilbert space, see [1–3]. In the last decades, generalized methods of Gabor transforms and wavelet transforms have been developed [14]. In the framework of wave packet analysis [11, 20], finite wavelet systems are particular classes of finite wave packet systems which have been recently introduced, see [15–18]. Extending the finite wavelet frames over prime fields in [19], the analytic structure, group theoretical and abstract aspects of the nature of such systems have been studied in [19, 21, 30].

In Theorem 4.2, using a scaling matrix (not necessarily uniform) on a permuted version of the discrete Fourier transform (DFT) of a given window function (system generator) under certain conditions, we derive a locally-scaled version of the DFT of the window function and obtain a finite equal-norm Parseval wavelet frame over prime fields. In [30], we presented a matrix representation of the DFT of the window function, which was based on a generator of the cyclic multiplicative group of integers modulo $p$, where $p$ is a prime number. This notion is a quite useful tool to determine whether a given system forms a frame. In this paper, we further apply this matrix notion. For any non-zero window function using this representation, we give a characterization of all multiplicative subgroups of the cyclic multiplicative group modulo $p$, for which the associated wavelet systems form frames. Although, this notion depends on a generator of the cyclic multiplicative group (not necessarily a specific generator) and there is not a general method to find its generator, however this is the key point in cryptosystems.

Construction of ENPFs in such systems relies on a local modifying of the frequency components of the system generator. For a prime integer $p$, the multiplicative group modulo $p$ is cyclic and based on that, Proposition 3.7 provides a convenient correspondence between the multiplicative subgroups and the main group. This leads us to simply manipulate the DFT of the system generator in order to obtain ENPFs of such systems.

This paper is organized as follows. Section 2 contains some basic definitions and results of Fourier transform on cyclic groups and finite frames. An overview for the notion and structure of finite wavelet groups appears in section 3. In section 4, we give main results of the current paper. The results will be accompanied by some concrete examples.

2. Preliminaries

Throughout this paper, $p$ is a prime positive integer. Here, we state a brief review of notations, basics and preliminaries of Fourier analysis and computational harmonic analysis over finite cyclic groups. For more details, we refer the readers to [16, 18, 19, 29]. Here we employ notations of the author in [13–21].
For a finite group $G$,
\[ C^G = \{ x : G \to \mathbb{C} \} \]
is a $|G|$-dimensional complex vector space. For any vector in $C^G$ the indices are taken to be elements in finite group $G$. This space is a Hilbert space under the inner product
\[ \langle x, y \rangle = \sum_{g \in G} x(g) \overline{y(g)} \]
for any $x, y \in C^G$. The induced norm is the $\| \cdot \|_2$-norm. Let $\mathbb{Z}_p$ denotes the cyclic group of $p$ elements $\{0, ..., p - 1\}$. Then for $C^{\mathbb{Z}_p}$, we write $C^p$. Also
\[ \| x \|_0 = |\{ k \in \mathbb{Z}_p : x(k) \neq 0 \}| \]
counts non-zero entries in $x \in C^p$.

For $k \in \mathbb{Z}_p$, the translation operator $T_k : C^p \to C^p$ is defined by
\[ T_k x(s) = x(s - k), \quad (x \in C^p, \ s \in \mathbb{Z}_p). \]
For $\ell \in \mathbb{Z}_p$, the modulation operator $M_\ell : C^p \to C^p$ is defined by
\[ M_\ell x(s) = e^{-2\pi i \ell s/p} x(s), \quad (x \in C^p, \ s \in \mathbb{Z}_p). \]

These operators are unitary operators in the $\| \cdot \|_2$-norm. For any $\ell, k \in \mathbb{Z}_p$, we have $(T_k)^* = T_{p-k}$ and $(M_\ell)^* = M_{p-\ell}$. The unitary discrete Fourier transform (DFT) of $x \in C^p$ is defined by $\hat{x}(\ell) = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} x(k) \overline{w_\ell(k)}$, for all $\ell \in \mathbb{Z}_p$ where for all $\ell, k \in \mathbb{Z}_p$ we have $w_\ell(k) = e^{2\pi i \ell k/p}$. Therefore, the DFT of $x \in C^p$ at $\ell \in \mathbb{Z}_p$ can be written as
\[ \hat{x}(\ell) = F_p(x)(\ell) = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} x(k) \overline{w_\ell(k)} = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} x(k) e^{-2\pi i \ell k/p}. \]

For $x \in C^p$, the inverse discrete Fourier transform (IDFT) is defined by
\[ x(k) = \frac{1}{\sqrt{p}} \sum_{\ell=0}^{p-1} \hat{x}(\ell) w_\ell(k) = \frac{1}{\sqrt{p}} \sum_{\ell=0}^{p-1} \hat{x}(\ell) e^{2\pi i \ell k/p}, \quad 0 \leq k \leq p - 1. \]

By $\| \cdot \|_2$-norm, DFT is a unitary transform. Thus, for all $x \in C^p$ Parseval formula $\| F_p(x) \|_2 = \| x \|_2$ satisfies. The Polarization identity implies $\langle x, y \rangle = \langle \hat{x}, \hat{y} \rangle$ for $x, y \in C^p$ which is known the Plancherel formula. The unitary DFT (2.1) satisfies
\[ \hat{T}_k \hat{x} = M_k \hat{x}, \ M_\ell \hat{x} = T_{p-\ell} \hat{x} \quad (x \in C^p, \ (k, \ell) \in \mathbb{Z}_p). \]

A finite sequence $A = \{ y_j : 0 \leq j \leq M - 1 \} \subset C^p$ is called a frame (or finite frame) for $C^p$, if there exists a positive constant $0 < A \leq B < \infty$ such that
\[ (2.2) \quad A \| x \|_2^2 \leq \sum_{j=0}^{M-1} |\langle x, y_j \rangle|^2 \leq B \| x \|_2^2, \quad (x \in C^p). \]
If (2.2) satisfies only the upper bound then $\mathfrak{A}$ is a Bessel sequence. Any finite sequence in $\mathbb{C}^p$ is a Bessel sequence, so that the condition in (2.2) can be reduced to

$$A\|x\|^2 \leq \sum_{j=0}^{M-1} |\langle x, y_j \rangle|^2, \quad (x \in \mathbb{C}^p).$$

for $0 < A < \infty$. If $A = B$, then $\mathfrak{A}$ is a $A$-tight frame and if $A = B = 1$, it is called a Parseval frame. Moreover, if all the frame elements have the same norm it is called equal-norm frame and if all the elements have norm 1 it is called unit-norm frame.

If $\mathfrak{A} = \{ y_j : 0 \leq j \leq M - 1 \}$ is a frame for $\mathbb{C}^p$, the synthesis operator $F : \mathbb{C}^M \to \mathbb{C}^p$ is defined by

$$F\{c_j\}^{M-1}_{j=0} = \sum_{j=0}^{M-1} c_j y_j, \quad (\{c_j\}^{M-1}_{j=0} \in \mathbb{C}^M).$$

The adjoint operator $F^* : \mathbb{C}^p \to \mathbb{C}^M$ which is known as analysis operator is defined by

$$F^*x = \{\langle x, y_j \rangle\}^{M-1}_{j=0}, \quad (x \in \mathbb{C}^p).$$

The frame operator $S : \mathbb{C}^p \to \mathbb{C}^p$ is defined by

$$x \mapsto Sx = FF^*x = \sum_{j=0}^{M-1} \langle x, y_j \rangle y_j, \quad (x \in \mathbb{C}^p),$$

and $x = \sum_{j=0}^{M-1} \langle x, S^{-1}y_j \rangle y_j = \sum_{j=0}^{M-1} \langle x, y_j \rangle S^{-1}y_j$. The redundancy of the finite frame $\mathfrak{A}$ is defined by $\text{red}_\mathfrak{A} = M/p$ where $M = |\mathfrak{A}|$.

3. Construction of Wavelet Frames over Prime Fields

In this section, we briefly state structure and basic properties of cyclic dilation operators (cf. [12, 13, 19, 26]). We shall present an overview for the notion and structure of finite wavelet groups over prime fields.

3.1. Structure of Finite Wavelet Group over Prime Fields. The set

$$\mathbb{U}_p := \mathbb{Z}_p - \{0\} = \{1, ..., p-1\},$$

is a finite multiplicative Abelian group of order $p - 1$ with respect to the multiplication module $p$ with the identity element 1. The multiplicative inverse for $m \in \mathbb{U}_p$ is $m_p$ satisfying $m_p m + np = 1$ for some $n \in \mathbb{Z}$ (cf. [24]).

For $m \in \mathbb{U}_p$, the cyclic dilation operator $D_m : \mathbb{C}^p \to \mathbb{C}^p$ is defined by

$$D_m x(k) := x(m_p k)$$

for $x \in \mathbb{C}^p$ and $k \in \mathbb{Z}_p$, where $m_p$ is the multiplicative inverse of $m$ in $\mathbb{U}_p$.

In the following propositions, we state some basic properties of the cyclic dilations.

**Proposition 3.1.** Let $p$ be a positive prime integer. Then
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\( (i) \) For \((m, k) \in \mathbb{U}_p \times \mathbb{Z}_p \), we have \( D_m T_k = T_{mk} D_m \).

\( (ii) \) For \( m, m' \in \mathbb{U}_p \), we have \( D_m m' = D_m D_{m'} \).

\( (iii) \) For \((m, k), (m', k') \in \mathbb{U}_p \times \mathbb{Z}_p \), we have \( T_{k+mk'} D_{mn'} = T_k D_m T_{k'} D_{m'} \).

\( (iv) \) For \((m, \ell) \in \mathbb{U}_p \times \mathbb{Z}_p \), we have \( D_m M_\ell = M_{mp\ell} D_m \).

The next result also states some properties of the cyclic dilations.

**Proposition 3.2.** Let \( p \) be a positive prime integer and \( m \in \mathbb{U}_p \). Then

\( (i) \) The dilation operator \( D_m : \mathbb{C}^p \to \mathbb{C}^p \) is a \( * \)-homomorphism.

\( (ii) \) The dilation operator \( D_m : \mathbb{C}^p \to \mathbb{C}^p \) is unitary in \( \| \cdot \|_2 \)-norm and satisfies

\[ (D_m)^* = (D_m)^{-1} = D_{mp}. \]

\( (iii) \) For \( x \in \mathbb{C}^p \), we have \( \overline{D_m x} = D_{mp} \overline{x} \).

The underlying set \( \mathbb{U}_p \times \mathbb{Z}_p = \{(m, k) : m \in \mathbb{U}_p, k \in \mathbb{Z}_p\} \), equipped with the following group operations

\[ (m, k) \times (m', k') := (mm', k + mk'), \]

\[ (m, k)^{-1} := (m_p, m_p(p - k)), \]

denoted by \( \mathbb{W}_p \), is a finite non-Abelian group of order \( p(p - 1) \) and it is called as finite affine group on \( p \) integers or the finite wavelet group associated to the integer \( p \) or simply as \( p \)-wavelet group.

The next proposition states basic properties of the finite wavelet group \( \mathbb{W}_p \).

**Proposition 3.3.** Let \( p > 2 \) be a positive prime integer. Then \( \mathbb{W}_p \) is a non-Abelian group of order \( p(p - 1) \) which contains a copy of \( \mathbb{Z}_p \) as a normal Abelian subgroup and a copy of \( \mathbb{U}_p \) as a non-normal Abelian subgroup.

### 3.2. Wavelet Frames over Prime Fields

A finite wavelet system for the complex Hilbert space \( \mathbb{C}^p \) is a family or system of the form

\[ \mathcal{W}(y, \Delta) := \{ \sigma(m, k)y = T_k D_m y : (m, k) \in \Delta \subseteq \mathbb{W}_p \}, \]

for some window signal \( y \in \mathbb{C}^p \) and a subset \( \Delta \) of \( \mathbb{W}_p \).

If \( \Delta = \mathbb{W}_p \), we put \( \mathcal{W}(y) := \mathcal{W}(y, \mathbb{W}_p) \) and it is called a full finite wavelet system over \( \mathbb{Z}_p \). A finite wavelet system which spans \( \mathbb{C}^p \) is a frame and is referred to as a finite wavelet frame over the prime field \( \mathbb{Z}_p \).

If \( y \in \mathbb{C}^p \) is a window signal then for \( x \in \mathbb{C}^p \), we have

\[ \langle x, \sigma(m, k)y \rangle = \langle x, T_k D_m y \rangle = \langle T_{p-k}x, D_m y \rangle, \quad ((m, k) \in \mathbb{W}_p). \]

The following proposition states a formulation for wavelet coefficients via Fourier transform.

**Proposition 3.4.** Let \( x, y \in \mathbb{C}^p \) and \( (m, k) \in \mathbb{W}_p \). Then,

\[ \langle x, \sigma(m, k)y \rangle = \sqrt{p} \mathcal{F}_p(\overline{\hat{x}} D_m y)(p - k). \]

**Proof.** See Proposition 4.1 of [19].
Using an analytic approach, the author [19] has presented a concrete formulation for the \( \| \cdot \|_2 \)-norm of wavelet coefficients the formula of which is just stated hereby.

**Theorem 3.5.** Let \( p \) be a positive prime integer, \( M \) be a divisor of \( p - 1 \), and let \( M \) be a multiplicative subgroup of \( \mathbb{U}_p \) of order \( M \). Let \( y \in \mathbb{C}^p \) be a window signal and \( x \in \mathbb{C}^p \). Then,

\[
\sum_{m \in M} \sum_{k \in \mathbb{Z}_p} |\langle x, \sigma(m, k)y \rangle|^2
= p \left( M|\hat{y}(0)|^2|\hat{x}(0)|^2 + \left( \sum_{m \in M} |\hat{y}(m)|^2 \right) \left( \sum_{\ell \in \mathbb{M}} |\hat{x}(\ell)|^2 \right) + \sum_{\ell \in \mathbb{U}_p - M} \gamma_{\ell}(y, M)|\hat{x}(\ell)|^2 \right),
\]

where

\[
\gamma_{\ell}(y, M) := \sum_{m \in M} |\hat{y}(m\ell)|^2, \quad (\ell \in \mathbb{U}_p - M).
\]

**Proof.** See Theorem 4.2 of [19].

**Remark 3.6.** The formulation presented in Theorem 3.5 is an analytic formulation of wavelet coefficients associated to the subgroup \( M \). In detail, that formulation originated from an analytic approach which was based on direct computations of cyclic dilations in the subgroup \( M \).

Next proposition provides a particular partition of the cyclic multiplicative group \( \mathbb{U}_p \). Applying this, a constructive formulation for the \( \| \cdot \|_2 \)-norm of wavelet coefficients has been achieved in the following theorem.

**Proposition 3.7.** Let \( p \) be a positive prime integer, \( M \) be a divisor of \( p - 1 \), and let \( M \) be a multiplicative subgroup of \( \mathbb{U}_p \) of order \( M \). Let \( \epsilon \) be a generator of the cyclic group \( \mathbb{U}_p \) and \( a := \frac{p-1}{M} \). Then

(i) For \( 0 \leq r, s \leq a - 1 \), we have \( \epsilon^rM = \epsilon^sM \) iff \( r = s \).

(ii) \( \mathbb{U}_p/M = \{ \epsilon^tM : 0 \leq t \leq a - 1 \} \).

**Proof.** See Proposition 3.7 of [30].

The following theorem presents a constructive formulation for the \( \| \cdot \|_2 \)-norm of wavelet coefficients.

**Theorem 3.8.** Let \( p \) be a positive prime integer, \( M \) be a divisor of \( p - 1 \), and let \( M \) be a multiplicative subgroup of \( \mathbb{U}_p \) of order \( M \). Let \( \epsilon \) be a generator of the cyclic group \( \mathbb{U}_p \) and \( a := \frac{p-1}{M} \). Let \( y \in \mathbb{C}^p \) be a window signal and \( x \in \mathbb{C}^p \). Then,

\[
\sum_{m \in M} \sum_{k=0}^{a-1} |\langle x, \frac{T_k}{M}D_my \rangle|^2
= p \left( M|\hat{x}(0)|^2|\hat{y}(0)|^2 + \sum_{\ell \in H_t} \left( \sum_{w \in H_t} |\hat{x}(w)|^2 \right) \left( \sum_{w \in H_t} |\hat{y}(w)|^2 \right) \right),
\]
where \( H_t := \epsilon^t \mathbb{M} \) for all \( 0 \leq t \leq a - 1 \).

Proof. See Theorem 3.8 \[30\].

For a given multiplicative subgroup of \( \mathbb{U}_p \), the next theorem gives necessary and sufficient conditions for a finite wavelet system over prime field to be a frame.

**Theorem 3.9.** Let \( p \) be a positive prime integer, \( \epsilon \) be a generator of \( \mathbb{U}_p \), \( M \) be a divisor of \( p - 1 \), \( \mathbb{M} \) be a multiplicative subgroup of \( \mathbb{U}_p \) of order \( M \), and let \( a := \frac{p - 1}{M} \). Let \( \Delta_M := \mathbb{M} \times \mathbb{Z}_p \) and \( y \in \mathbb{C}^p \) be a non-zero window signal. The finite wavelet system \( \mathcal{W}(y, \Delta_M) \) is a frame for \( \mathbb{C}^p \) if and only if the following conditions hold

(i) \( \widehat{y}(0) \neq 0 \)
(ii) For each \( t \in \{0, \ldots, a - 1\} \), there exists \( m_t \in \mathbb{M} \) such that \( \widehat{y}(\epsilon^t m_t) \neq 0 \).

Proof. See Theorem 3.9 \[30\].

The next result, in the matrix language also gives a constructive characterization for the frame conditions of finite wavelet systems over prime fields.

**Corollary 3.10.** Let \( p \) be a positive prime integer, \( \epsilon \) be a generator of \( \mathbb{U}_p \), \( M \) be a divisor of \( p - 1 \), \( \mathbb{M} \) be a multiplicative subgroup of \( \mathbb{U}_p \) of order \( M \), and let \( a := \frac{p - 1}{M} \). Let \( \Delta_M := \mathbb{M} \times \mathbb{Z}_p \) and \( y \in \mathbb{C}^p \) be a non-zero window signal. The finite wavelet system \( \mathcal{W}(y, \Delta_M) \) is a frame for \( \mathbb{C}^p \) if and only if \( \widehat{y}(0) \neq 0 \) and the matrix \( \mathbf{Y}(M, y) \) of size \( a \times M \) given by

\[
\mathbf{Y}(M, y) := \begin{bmatrix}
\widehat{y}(1) & \widehat{y}(\epsilon^a) & \cdots & \widehat{y}(\epsilon^{(M-1)a}) \\
\widehat{y}(\epsilon^a) & \widehat{y}(\epsilon^{a+1}) & \cdots & \widehat{y}(\epsilon^{(M-1)a+1}) \\
\vdots & \vdots & \ddots & \vdots \\
\widehat{y}(\epsilon^{a-1}) & \widehat{y}(\epsilon^{2a-1}) & \cdots & \widehat{y}(\epsilon^{Ma-1})
\end{bmatrix}_{a \times M}
\]

is a matrix such that each row has at least a non-zero entry.

Proof. See Corollary 3.10 \[30\].

4. **Finite equal norm Parseval wavelet frames over prime fields**

Weight frames, controlled frames \[5\], and scalable frames \[28\] have been already applied in order to tighten and also to produce Parseval frame of a given frame. However, in general the motivation of these methods in \[5, 28\] have not been aimed at the norm equality of the frame vectors. These methods are applied not only in finite dimensional Hilbert spaces but also in infinite dimensions. In fact, in the case of infinite dimensions, weighted frames algorithms are designed in such a way that decrease the condition number and provide approximately a tight frame. In \[28\], some equivalent conditions to scalable frames have been introduced. The principal significance of the constructive approach in the next theorem is that we apply a local scaling on the DFT of a given window function. The structure
of these systems allows one to simply manipulate the DFT of the system generator, in order to obtain an ENPF.

The following lemma provides an algebraic tool. This permutation depends on a multiplicative subgroup \( M \) of \( \mathbb{U}_p \) of a given size. Note that, for each divisor of the size of a finite cyclic group, there is exactly one subgroup of that size.

**Lemma 4.1.** Let \( p \) be a positive prime integer, \( \epsilon \) be a generator of \( \mathbb{U}_p \), \( M \) be a divisor of \( p - 1 \), and let \( a := \frac{p - 1}{M} \). Then

\[
\sigma(\ell) := \begin{cases} 
0, & \ell = 0, \\
\epsilon^\left\lfloor \frac{\ell - 1}{M} \right\rfloor + \left( \ell - \left\lfloor \frac{\ell - 1}{M} \right\rfloor M - 1 \right) a, & \ell \in \mathbb{U}_p,
\end{cases}
\]

is a permutation of \( \mathbb{Z}_p \) where \( \left\lfloor . \right\rfloor \) denotes the floor function.

**Proof.** Clearly \( \sigma \) is well-defined. In order to show that \( \sigma \) is bijective it suffices to prove that \( \sigma : \mathbb{U}_p \rightarrow \mathbb{U}_p \) is surjective. For any \( \ell \in \mathbb{U}_p \) there exists \( 0 \leq t \leq a - 1 \) such that \( tM + 1 \leq \ell \leq (t + 1)M \) and so it is clear that \( \left\lfloor \frac{\ell - 1}{M} \right\rfloor = t \). Thus the definition implies that \( \sigma(\ell) = \epsilon^{t + (\ell - tM - 1)a} \); in particular if \( 1 \leq \ell \leq M \), we have \( \sigma(\ell) = \epsilon^{(\ell - 1)a} \). We observe that \( \mathbb{M} = \left\langle \epsilon^a \right\rangle \) is a subgroup of \( \mathbb{U}_p \) of order \( M \) and \( \mathbb{U}_p = \bigcup_{t=0}^{a-1} \epsilon^t \mathbb{M} \), where \( \epsilon^t \mathbb{M} = \{ \epsilon^1, \ldots, \epsilon^{t(M-1)a} \} \) for any \( 0 \leq t \leq a - 1 \) and \( \epsilon^t \mathbb{M} \cap \epsilon^s \mathbb{M} = \emptyset \) for any \( t \neq s \). Given an arbitrary \( x \in \mathbb{U}_p \), there exists \( 0 \leq t \leq a - 1 \) such that \( x \in \epsilon^t \mathbb{M} \) and so there exists \( 0 \leq k \leq M - 1 \) such that \( x = \epsilon^{t+ka} \). One can easily check that \( \sigma(tM + k + 1) = x \). \( \square \)

Here, we apply lemma 4.1, in the following theorem. At first, the permutation defined in (4.1), is served to give a regular rearrangement to frequency components of the DFT of the window function and classify them in order to set cluster scales. Next again via this permutation, we are able to reverse the locally-scaled permuted version of the DFT of the window function to merely a locally-scaled version of the window function. Since the scales are defined positive thus, still this last version fulfills the frame conditions. Under the described procedure, we are thus led to construct a finite equal-norm Parseval frame of such systems for the Hilbert space \( \mathbb{C}^p \).

**Theorem 4.2.** Let \( p \) be a positive prime integer, \( \epsilon \) be a generator of \( \mathbb{U}_p \), \( \mathbb{M} \) be a multiplicative subgroup of \( \mathbb{U}_p \) of order \( M \), \( a := \frac{p - 1}{M} \) be the index of \( \mathbb{M} \) in \( \mathbb{U}_p \), and let \( \sigma \) is defined by

\[
(4.2) \quad \sigma := \begin{pmatrix} 
0 & 1 & 2 & \cdots & \ell & \cdots & p - 1 \\
0 & \epsilon^0 & \epsilon^a & \cdots & \epsilon^\left\lfloor \frac{\ell - 1}{M} \right\rfloor + \left( \ell - \left\lfloor \frac{\ell - 1}{M} \right\rfloor M - 1 \right) a & \cdots & \epsilon^{(Ma-1)}
\end{pmatrix}.
\]
Let \( y \in \mathbb{C}^p \) be a non-zero window signal and

\[
\hat{y}'(\ell) := \hat{y}(\sigma(\ell)) = \begin{cases}
\hat{y}(0), & \ell = 0, \\
\hat{y}(e^{\frac{(\ell - 1)}{M}} + (\ell - \frac{\ell - 1}{M})M^{-1}a), & \ell \in \mathbb{U}_p,
\end{cases}
\]

(4.3) \( \hat{y}'' := D(M, y)\hat{y}' \),

where

\[
D(M, y) := \begin{bmatrix}
[U']_{1 \times 1} & [U_0]_{M \times M} & [U_1]_{M \times M} & \cdots & [U_{a-1}]_{M \times M}
\end{bmatrix}_{p \times p}
\]

is a diagonal matrix with diagonal block matrices \([U']_{1 \times 1}\) and \([U_t]_{M \times M}\) for \( t \in \{0, \ldots, a - 1\} \), which are constant along diagonals by entries \( R' := \frac{1}{\sqrt{pM|\hat{y}(0)|^2}} \) and \( R_t := \frac{1}{\sqrt{\sum_{t=0}^{M-1} p|\hat{y}(e^{\tau_{t+a}})|^2}} \), respectively. If \( y \) satisfies the following conditions

(i) \( \hat{y}(0) \neq 0 \),
(ii) for each \( t \in \{0, \ldots, a - 1\} \), there exists \( r_t \in \{0, \ldots, M - 1\} \) such that \( \hat{y}(e^{\tau_{t+a+t}}) \neq 0 \),

then \( \mathcal{W}(y, \Delta_M) \) is an equal-norm Parseval frame for \( \mathbb{C}^p \), where \( \hat{y}(\sigma(\ell)) := \hat{y}''(\ell) \).

Proof. Let \( y \in \mathbb{C}^p \) be a window function which satisfies conditions (i) and (ii) and \( x \in \mathbb{C}^p \). Then \( |\hat{y}(0)|^2 \neq 0 \) and for any \( t \in \{0, \ldots, a - 1\} \), we have \( \sum_{\tau=0}^{M-1} |\hat{y}(e^{\tau_{t+a}})|^2 \neq 0 \). Thus, \( R' \) and \( R_t \) are well-defined for all \( t \in \{0, \ldots, a - 1\} \). By Lemma 4.1, \( \sigma \) presents a permutation of \( \mathbb{Z}_p \) and \( \sigma^{-1}(e^{\tau_{t+a}}) = r_t + tM + 1 \), for any \( t \in \{0, \ldots, a - 1\} \) and \( r_t \in \{0, \ldots, M - 1\} \). By Theorem
Now by (4.3), we obtain

\[
\sum_{m \in M} \sum_{k=0}^{p-1} |\langle x, T_k D_m y_\sigma \rangle|^2 = p M |\hat{x}(0)|^2 |\hat{y}_\sigma(0)|^2 + p \sum_{t=0}^{a-1} \left( \sum_{\ell \in H_t} |\hat{x}(\ell)|^2 \right) \left( \sum_{r_t=0}^{M-1} |\hat{y}_\sigma(\epsilon^{r_t a + t})|^2 \right)
\]

\[
= p M |\hat{x}(0)|^2 |\hat{y}_\sigma(0)|^2 + p \sum_{t=0}^{a-1} \left( \sum_{\ell \in H_t} |\hat{x}(\ell)|^2 \right) \left( \sum_{r_t=0}^{M-1} |\hat{y}_\sigma(\epsilon^{r_t a + t})|^2 \right)
\]

\[
= p M |\hat{x}(0)|^2 |\hat{y}_\sigma(0)|^2 + p \sum_{t=0}^{a-1} \left( \sum_{\ell \in H_t} |\hat{x}(\ell)|^2 \right) \left( \sum_{r_t=0}^{M-1} |\hat{y}_\sigma(\epsilon^{r_t a + t})|^2 \right)
\]

Now by (4.3), we get

\[
\hat{y}''(\ell) := D(M, y)\hat{y}'(\ell) = \begin{cases} R_1\hat{y}'(0), & \ell = 0, \\
R_{t+1} \hat{y}'(\ell), & \ell \in \mathbb{U}_p.
\end{cases}
\]

Thus, \( \hat{y}''(t M + r + 1) = R_t \hat{y}'(t M + r + 1) \) and we have

\[
\sum_{m \in M} \sum_{k=0}^{p-1} |\langle x, T_k D_m y_\sigma \rangle|^2
\]

\[
= |\hat{x}(0)|^2 + p \sum_{t=0}^{a-1} \left( \sum_{\ell \in H_t} |\hat{x}(\ell)|^2 \right) \left( \sum_{r_t=0}^{M-1} |R_t \hat{y}'(t M + r + 1)|^2 \right)
\]

\[
= |\hat{x}(0)|^2 + p \sum_{t=0}^{a-1} \left( \sum_{\ell \in H_t} |\hat{x}(\ell)|^2 \right) \left( \sum_{r_t=0}^{M-1} |R_t \hat{y}'(\sigma(t M + r + 1))|^2 \right)
\]

\[
= |\hat{x}(0)|^2 + p \sum_{t=0}^{a-1} \left( \sum_{\ell \in H_t} |\hat{x}(\ell)|^2 \right) \left( \sum_{r_t=0}^{M-1} |R_t \hat{y}'(\epsilon^{r_t a + t})|^2 \right)
\]

\[
= |\hat{x}(0)|^2 + p \sum_{t=0}^{a-1} \left( \sum_{\ell \in H_t} |\hat{x}(\ell)|^2 \right) \left( \sum_{r_t=0}^{M-1} \frac{1}{\sqrt{\sum_{r_t=0}^{M-1} |p|^{r_t a + t}}} |\hat{y}'(\epsilon^{r_t a + t})|^2 \right)
\]

\[
= |\hat{x}(0)|^2 + \sum_{t=0}^{a-1} \left( \sum_{\ell \in H_t} |\hat{x}(\ell)|^2 \right) = |\hat{x}(0)|^2 + \sum_{t=1}^{p-1} |\hat{x}(\ell)|^2 = \|\hat{x}\|_2^2 = |x|^2.
\]
Therefore, $\mathcal{W}(y_\sigma, \Delta_M)$ is a Parseval frame for $\mathbb{C}^p$. Note that, for any $k \in \mathbb{Z}_p$ and $m \in M$, the operators $T_k$ and $D_m$ are unitary operators and,
$$\|T_kD_my_\sigma\|_2 = \|y_\sigma\|_2.$$ Hence, $\mathcal{W}(y_\sigma, \Delta_M)$ is an equal-norm Parseval frame for $\mathbb{C}^p$. □

Next example demonstrates the design method of equal-norm Parseval finite wavelet system over prime fields, described in Theorem 4.2, in a numerical case.

**Example 4.3.** Let $p = 7$ and $U_7$ be the multiplicative group modulo 7. Then $|U_p| = 6$. One of its generators is 3. Consider $M$ be a subgroup of size 3 of $U_7$. So $M = \langle 3^{\frac{6}{3}} \rangle = \langle 2 \rangle = \{1, 2, 4\}$.

We use Theorem 4.2 to choose an appropriate window signal $y \in \mathbb{C}^7$ which satisfies the conditions (i) and (ii).

For simplicity in computation, we will consider a special case with least number of non-zero components of $\hat{y}$. Since the conditions (i) and (ii) rely on the DFT of $y$, so we use IDFT to get $y$. Now, let $\hat{y} = (1, 1, 0, 1, 0, 0, 0)$.

Then
$$y = (1.1339, 0.2731 + 0.4595i, 0.5295 + 0.0730i, -0.0467 + 0.5325i, -0.0467 - 0.5325i, 0.5295 - 0.0730i, 0.2731 - 0.4595i),$$

and $\mathcal{W}(y, \Delta_M)$ is a frame for $\mathbb{C}^7$. By Theorem 4.2, we have
$$\sigma := \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 3^0 & 3^2 & 3^4 & 3^1 & 3^3 & 3^5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 2 & 4 & 3 & 6 & 5 \end{pmatrix},$$

and
$$\hat{y}' = (\hat{y}(\sigma(0)), \hat{y}(\sigma(1)), \hat{y}(\sigma(2)), \hat{y}(\sigma(3)), \hat{y}(\sigma(4)), \hat{y}(\sigma(5)), \hat{y}(\sigma(6)))$$
$$= (\hat{y}(0), \hat{y}(1), \hat{y}(2), \hat{y}(4), \hat{y}(3), \hat{y}(6), \hat{y}(5)) = (1, 1, 0, 0, 1, 0, 0).$$

Also
$$\mathbf{D}(M, y) := \begin{bmatrix} \frac{1}{\sqrt{7}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{7}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{7}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{7}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{7}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{7}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{7}} \end{bmatrix}_{7 \times 7}.$$
By performing $D(M, y)$ on $\hat{y}'$, we get a locally-scaled version of $\hat{y}'$ which yields the following signals $\hat{y}''$ and $\hat{y}_\sigma$ given by

$$
\hat{y}'' = \left( \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{7}}, 0, 0, \frac{1}{\sqrt{7}}, 0, 0 \right) = (0.2182, 0.3780, 0, 0, 0.3780, 0, 0).
$$

and

$$
\hat{y}_\sigma = \left( \frac{1}{\sqrt{21}}, \frac{1}{\sqrt{7}}, 0, 0, \frac{1}{\sqrt{7}}, 0, 0 \right) = (0.2182, 0.3780, 0, 0.3780, 0, 0).
$$

Again using IDFT for $\hat{y}_\sigma$, we get

$$
y_\sigma = (0.3682, 0.0428 + 0.1737i, 0.1398 + 0.0276i, -0.0780 + 0.2013i,
-0.0780 - 0.2013i, 0.1398 - 0.0276i, 0.0428 - 0.1737i).
$$

Also

$$
H_0 = M = \{1, 2, 4\}, \quad H_1 = \{3, 6, 5\}.
$$

By Theorem 3.8 and a simple calculation for any $x \in \mathbb{C}^7$, we have

$$
\sum_{m \in M} \sum_{k=0}^{6} |\langle x, T_kD_m y_\sigma \rangle|^2 = 7 \left( 3|\tilde{x}(0)|^2|\tilde{y}_\sigma(0)|^2 + \sum_{t=0}^{1} \left( \sum_{\ell \in H_1} |\tilde{x}(\ell)|^2 \right) \left( \sum_{\ell \in H_1} |\tilde{y}_\sigma(\ell)|^2 \right) \right)
$$

$$
= 7 \left( 3|\tilde{x}(0)|^2|\tilde{y}_\sigma(0)|^2 + \left( |\tilde{x}(1)|^2 + |\tilde{x}(2)|^2 + |\tilde{x}(4)|^2 \right) \left( |\tilde{y}_\sigma(1)|^2 + |\tilde{y}_\sigma(2)|^2 + |\tilde{y}_\sigma(4)|^2 \right) + \left( |\tilde{x}(3)|^2 + |\tilde{x}(6)|^2 + |\tilde{x}(5)|^2 \right) \left( |\tilde{y}_\sigma(3)|^2 + |\tilde{y}_\sigma(6)|^2 + |\tilde{y}_\sigma(5)|^2 \right) \right)
$$

$$
= 7 \left( 3|\tilde{x}(0)|^2 \frac{1}{\sqrt{21}}^2 + \left( |\tilde{x}(1)|^2 + |\tilde{x}(2)|^2 + |\tilde{x}(4)|^2 \right) \left( \frac{1}{\sqrt{7}}^2 + |0|^2 + |0|^2 \right) + \left( |\tilde{x}(3)|^2 + |\tilde{x}(6)|^2 + |\tilde{x}(5)|^2 \right) \left( \frac{1}{\sqrt{7}}^2 + |0|^2 + |0|^2 \right) \right) = |x|^2
$$

Hence, $W(y_\sigma, \Delta_M)$ is an equal norm Parseval finite wavelet frame for $\mathbb{C}^7$.

In [30], the matrix notion presented in Corollary 3.10, have been applied as a useful tool to determine whether a finite wavelet system forms a frame for $\mathbb{C}^p$. As another application of this notion, the next result for any given window function derives a characterization of all multiplicative subgroups of $\mathbb{U}_p$, for which the associated wavelet system form frames for $\mathbb{C}^p$.

**Theorem 4.4.** Let $p$ be a positive prime integer, $y \in \mathbb{C}^p$ such that $\hat{y}(0) \neq 0$, and let

$$
\Phi(p) = p - 1 = \prod_{i=1}^{k} q_i^{p_i},
$$
be the factorization of $\Phi(p)$ into prime powers, where the prime factors $q_i$ are distinct such that $q_1 < q_2 < \ldots < q_k$, $k \geq 1$ and $\alpha_i \geq 0$. Let $\epsilon$ be a generator of $\mathbb{U}_p$ and

$$
\Lambda := \{(r_1, r_2, \ldots, r_k) : \prod_{i=1}^{k} q_i^{r_i} \leq \|y\|_0 - 1, 0 \leq r_i \leq \alpha_i, \quad y(< \epsilon \prod_{i=1}^{k} q_i^{r_i}, y)
$$

has $\prod_{i=1}^{k} q_i^{r_i}$ nonzero rows\}.

Then the set of all subgroups of $\mathbb{U}_p$ for which the associated finite wavelet systems are frames for $\mathbb{C}^p$ consists of exactly those subgroups $M$ of $\mathbb{U}_p$ of the form

$$
M = < \epsilon \prod_{i=1}^{k} q_i^{r_i - s_i} >, \quad ((r_1, ..., r_k) \in \Lambda, 0 \leq s_i \leq r_i, i = 1, ..., k).
$$

Hence, the order of any such $M$ is

$$
|M| = \prod_{i=1}^{k} q_i^{\alpha_i - r_i + s_i}
$$

Proof. Let $m := \|y\|_0 - 1$. Since $p - 1 = \Phi(p) = |\mathbb{U}_p|$, so the set of divisors of $|\mathbb{U}_p|$ is the set of numbers of the form $\prod_{i=1}^{k} q_i^{r_i}$, where $0 \leq r_i \leq \alpha_i$ for $i = 1, ..., k$.

Suppose $M$ be an arbitrary multiplicative subgroup of $\mathbb{U}_p$. Then, we have $M = < \epsilon \prod_{i=1}^{k} q_i^{r_i - s_i} >$. Let $a_M := \frac{p-1}{|M|}$. The order of $M$ is a divisor of $p - 1$, thus $a_M$ is of the form $\prod_{i=1}^{k} q_i^{r_i}$ for some $0 \leq r_i \leq \alpha_i, i = 1, ..., k$. For any such $M$, there are two cases to consider, $a_M > m$ or $a_M \leq m$.

Case $a_M = \prod_{i=1}^{k} q_i^{r_i} > m$. In this case, by Corollary 3.10, the frame conditions for the finite wavelet system $W(y, \Delta_M)$ can not be satisfied. In fact, we have some zero rows in $Y(M, y)$.

Case $a_M = \prod_{i=1}^{k} q_i^{r_i} \leq m$. Again applying Corollary 3.10, $W(y, \Delta_M)$ is a finite wavelet frame if and only if $Y(M, y)$ has $a_M$ non-zero rows. Also if $M'$ be a subgroup of $\mathbb{U}_p$ such that $|M| = \prod_{i=1}^{k} q_i^{\alpha_i - r_i} |M'|$, i.e., $M$ is a subgroup of $M'$ then $W(y, \Delta_M) \subseteq W(y, \Delta_{M'})$. By this, if $W(y, \Delta_M)$ is a frame for $\mathbb{C}^p$ then, $W(y, \Delta_{M'})$ is also a frame for $\mathbb{C}^p$. In other words, the remaining cases of subgroups $M'$ of $\mathbb{U}_p$ satisfying the frame conditions of the finite wavelet system $W(y, \Delta_{M'})$ for $\mathbb{C}^p$ are those of the form, $M = < \epsilon \prod_{i=1}^{k} q_i^{r_i - s_i} >$ or $|M'| = \prod_{i=1}^{k} q_i^{\alpha_i - r_i + s_i}$ for $(r_1, ..., r_k)$ $\in \Lambda$ and $0 \leq s_i \leq r_i$.

Next example gives a numerical illustration of Theorem 4.4.

**Example 4.5.** Let $p = 13$ and $y \in \mathbb{C}^{13}$, such that

$$
\tilde{y} = a\delta_0 + b\delta_2 + c\delta_3 + d\delta_8 + e\delta_{11} + f\delta_{12}, \quad (a, b, c, d, e, f \neq 0),
$$
where \( \{ \delta_i \}_{i=0}^{12} \) is an orthonormal basis for \( \mathbb{C}^{13} \). Then \( \| \hat{y} \|_0 - 1 = 5 \) and \( |U_{13}| = \Phi(13) = 12 = 2^2 \times 3^1 \). Also \( \epsilon = 2 \) is a generator of \( U_{13} \). The only divisors of 12 so that are equal or less than 5 are, 1,2,3,4. Thus by definition of \( \Lambda \), we have \( \Lambda \subseteq \{ (0,0), (1,0), (0,1), (2,0) \} \).

Let \( m_1 = (2,0) \), then \( M_{1} = \langle 2^1 \rangle = < 3 > \) and we get

\[
Y(M_{1}, y) := \begin{bmatrix}
\hat{y}(1) & \hat{y}(3) & \hat{y}(9) \\
\hat{y}(2) & \hat{y}(6) & \hat{y}(5) \\
\hat{y}(4) & \hat{y}(12) & \hat{y}(10) \\
\hat{y}(8) & \hat{y}(11) & \hat{y}(7)
\end{bmatrix} = \begin{bmatrix}
0 & c & 0 \\
b & 0 & 0 \\
0 & f & 0 \\
d & e & 0
\end{bmatrix}_{4 \times 3}
\]

Thus, \( W(y, \Delta_{M_{1}}) \) is a frame for \( \mathbb{C}^{13} \). Since \( M_{1} \) is of order 3, hence the corresponding wavelet systems to subgroups of sizes 6 and 12 i.e., \( M_{2} = \langle 2^2 \rangle = < 4 > \) and \( M_{3} = \langle 2^1 \rangle = \langle U_{13} \rangle \) are also frames for \( \mathbb{C}^{13} \). Note that \( M_{2} \) and \( M_{3} \) are corresponding subgroups to \( m_{2} = (1,0) \) and \( m_{3} = (0,0) \) respectively. So, now we just examine in the case of \( m_{4} = (0,1) \). We have \( M_{4} = \langle 2^3 \rangle = < 8 > \), and

\[
Y(M_{4}, y) := \begin{bmatrix}
\hat{y}(1) & \hat{y}(8) & \hat{y}(12) & \hat{y}(5) \\
\hat{y}(2) & \hat{y}(3) & \hat{y}(11) & \hat{y}(10) \\
\hat{y}(4) & \hat{y}(6) & \hat{y}(9) & \hat{y}(7)
\end{bmatrix} = \begin{bmatrix}
0 & d & f & 0 \\
b & c & 0 & e \\
0 & 0 & 0 & 0
\end{bmatrix}_{3 \times 4}
\]

Hence, \( W(y, \Delta_{M_{4}}) \) is not a frame for \( \mathbb{C}^{13} \).

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References

[1] A.A. Arefijamaal, R.A. Kamyabi-Gol, On the square integrability of quasi regular representation on semidirect product groups, J. Geom. Anal. 19 (3) (2009) 541-552.
[2] A.A. Arefijamaal, E. Zekaee, Signal processing by alternate dual Gabor frames, Appl. Comput. Harmon. Anal. 35 (2013) 535-540.
[3] A.A. Arefijamaal, E. Zekaee, Image processing by alternate dual Gabor frames, Bull. Iranian Math. Soc. 42(6) (2016) 1305-1314.
[4] P. Balazs, Frames and Finite Dimensionality: Frame Transformation, Classification and Algorithms, Appl. Math. Sci. 2(43) (2008) 2131-2144.
[5] P. Balazs, J.P. Antonio, A. Grybos, Weighted and controlled frames, Int. J. Wavelets Multiresolut. Inf. Process. 8(1) (2010) 109-132.
[6] B.G. Bodmann, P.G. Casazza, The road to equal-norm Parseval frames, J. Funct. Anal. 258 (2010) 397-420.
[7] B.G. Bodmann, D.W. Kribs, V.I. Paulsen, Decoherence-insensitive quantum communications by optimal C*-encoding, IEEE Trans. Inform. Theory. 53 (2007) 4738-4749.
[8] P.G. Casazza, J. Kovačević, Equal-norm tight frames with erasures, Adv. Comput. Math. 18 (2003) 387-430.
[9] P.G. Casazza, G. Kutyniok. Finite Frames, Theory and Applications, Applied and Numerical Harmonic Analysis (Birkhäuser, Boston, 2013).
[10] P.G. Casazza, N. Leonhard, Classes of equal-norm Parseval frames with erasures, Contemp. Math. 451 (2008) 11-31.
[11] O. Christensen, A. Rahimi, Frame properties of wave packet systems in $L^2(\mathbb{R}^d)$, Adv. Comput. Math. 29(2) (2008) 101-111.
[12] K. Flornes, A. Grossmann, M. Holschneider and B. Torrésani, Wavelets on discrete fields, Appl. Comput. Harmon. Anal. 1 (1994) 137-146.
[13] A. Ghaani Farashahi, Cyclic wave packet transform on finite Abelian groups of prime order, Int. J. Wavelets Multiresolut. Inf. Process. 12(6) (2014). Article ID 1450041, 14 pages.
[14] A. Ghaani Farashahi, Classical wavelet transforms over finite fields, J. Linear Topol. Algebra, 4(4) (2015) 241-257.
[15] A. Ghaani Farashahi, Cyclic wavelet systems in prime dimensional linear vector spaces, Wavelets Linear Algebra 2(1) (2015) 11-24.
[16] A. Ghaani Farashahi, Wave packet transform over finite fields, Electron. J. Linear Algebra, 30 (2015) 507-529.
[17] A. Ghaani Farashahi, Classical wavelet systems over finite fields, Wavelets and Linear Algebra, 3(2) (2016) 1-18.
[18] A. Ghaani Farashahi, Wave packet transforms over finite cyclic groups, Linear Algebra Appl. 489 (2016) 75-92.
[19] A. Ghaani Farashahi, Structure of finite wavelet frames over prime fields, Bull. Iranian Math. Soc. 43(1) (2017) 109-120.
[20] A. Ghaani Farashahi, Theoretical frame properties of wave-packet matrices over prime fields, Linear and Multilinear Algebra (2017) to appear, http://dx.doi.org/10.1080/03081087.2016.1278196.
[21] A. Ghaani Farashahi and M. Mohammad-Pour, A unified theoretical harmonic analysis approach to the cyclic wavelet transform (CWT) for periodic signals of prime dimensions, Sahand Commun. Math. Anal. 1(2) (2014) 1-17.
[22] V. K. Goyal, Single and multiple description transform coding with bases and frames, (SIAM 2002).
[23] V. K. Goyal, J. J. Kovačević and M. Vetterli, Quantized frame expansions with erasures, J. Appl. Comput. harmon. Anal. 10(3) (2001) 203-233.
[24] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, (Oxford University Press, 1979).
[25] B. Hassibi, B. Hochwald, A. Shokrollahi and W. Sweldens, Representation theory for high rate multiple antenna code design, IEEE Trans. Inform. Theory 47(6) (2001) 2355-2367.
[26] C.P. Johnston, *On the pseudodilation representations of flornes, grossmann, holschneider, and torrésani*, J. Fourier Anal. Appl. 3(4) (1997) 377-385.

[27] J. Kovačević, P.L. Dragotti, V.K. Goyal, *Filter banks expansions with erasures*, IEEE Trans. Inform. Theory 48(6) (2002) 1439-1450.

[28] G. Kutyniok, K.A. Okoudjou, F. Philipp and E.K. Tuley, *scalable frames*, IEEE Trans. Inform. Theory 438(5) (2013) 2225-2238.

[29] G. Pfander. *Gabor Frames in Finite Dimensions*, in Finite Frames, G.E. Pfander (eds). P.G. Casazza, and G. Kutyniok, , Applied Numerical Harmonic Analysis (Birkhäuser/Springer, New York, 2013) pp. 193-239.

[30] A. Rahimi, N. Seddighi, *A constructive approach to the finite wavelet frames over prime fields*, Proc. Math. Sci. (2017) to appear.

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