THE GRADIENT FLOW OF A GENERALIZED FISHER INFORMATION FUNCTIONAL WITH RESPECT TO MODIFIED WASSERSTEIN DISTANCES

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Abstract. This article is concerned with the existence of nonnegative weak solutions to a particular fourth-order partial differential equation: it is a formal gradient flow with respect to a generalized Wasserstein transportation distance with nonlinear mobility. The corresponding free energy functional is referred to as generalized Fisher information functional since it is obtained by autodissipation of another energy functional which generates the heat flow as its gradient flow with respect to the aforementioned distance. Our main results are twofold: For mobility functions satisfying a certain regularity condition, we show the existence of weak solutions by construction with the well-known minimizing movement scheme for gradient flows. Furthermore, we extend these results to a more general class of mobility functions: a weak solution can be obtained by approximation with weak solutions of the problem with regularized mobility.

1. Introduction. This work is concerned with the existence of nonnegative weak solutions $u : [0, \infty) \times \Omega \to [0, \infty)$ to the partial differential equation

$$\partial_t u(t,x) = \text{div} \left( m(u(t,x)) \nabla \frac{\delta F}{\delta u}(u(t,x)) \right)$$  (1)

for $(t,x) \in (0, \infty) \times \Omega$, where $\Omega \subset \mathbb{R}^d \ (d \geq 1)$ is a bounded and convex domain with smooth boundary $\partial \Omega$ and exterior unit normal vector field $\nu$. The assumption on the mobility function $m$ and the free energy functional $F$ are specified below. Above, $\frac{\delta F}{\delta u}$ denotes the first variation of $F$ in $L^2$. Additionally, the sought-for solution $u$ to (1) is subject to the no-flux and homogeneous Neumann boundary conditions

$$m(u(t,x)) \partial_n \frac{\delta F}{\delta u}(u(t,x)) = 0 = \partial_n u(t,x)$$  (2)

for $t > 0$ and $x \in \partial \Omega$, and to the initial condition

$$u(0, \cdot) = u_0 \in L^1(\Omega), \text{ with } u_0 \geq 0 \text{ and } \int_{\Omega} u_0(x) \, dx = U,$$  (3)

for some fixed $U > 0$. 

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Formally, (1) possesses a gradient flow structure with respect to a modified version $W_m$ of the $L^2$-Wasserstein distance on the space of probability measures, reading as

$$W_m(u_0, u_1) = \left( \inf_{(u_s, w_s) \in \mathcal{C}} \int_0^1 \int_{\Omega} \frac{|w_s|^2}{m(u_s)} \, dx \, dt \right)^{1/2}, \quad (4)$$

where the admissible set $\mathcal{C}$ is a suitable subclass of curves $(u_s, w_s)_{s \in [0,1]}$ satisfying the continuity equation $\partial_s u_s = -\text{div} w_s$ on $[0,1] \times \Omega$ and the initial and terminal conditions $u_s|_{s=0} = u_0$, $u_s|_{s=1} = u_1$. We refer to [5, 9] (see also [4, 17]) for more details concerning the definition and properties of $W_m$.

Before stating our assumptions on the mobility function $m$, let us consider the linear case $m = \text{id}$. Then, it is well-known [2] that $W_m$ coincides (up to a scaling factor depending on the value of $U$) with the classical $L^2$-Wasserstein distance $W_2$ for probability measures on $\Omega$. Using techniques from optimal transportation theory (see, for instance, [16]), various equations of the form (1) for $m = \text{id}$ have been interpreted as gradient systems with respect to $W_2$. The specific gradient flow structure often allows for the analysis of the well-posedness of the underlying evolution equation as well as for the study of the qualitative behaviour of the associated gradient flow solutions [7, 14]. A powerful tool for those investigations is provided by the so-called minimizing movement scheme (cf. (6) below), a metric version of the implicit Euler scheme, which is used for the construction of a time-discrete approximative gradient flow. Even in very general situations, this approach can be of use (see, for example, [1] for an abstract description or [3, 17] in the context of coupled systems). A certain class of equations of the form (1) already has successfully been proved to be well-posed by Lisini, Matthes and Savaré [10] using its variational structure in spaces w.r.t. the distance $W_m$. This work aims at a further extension.

In their seminal paper [7] on the variational formulation of the Fokker-Planck equation, Jordan, Kinderlehrer and Otto rigorously interpreted the heat equation

$$\partial_t u = \Delta u$$

as $W_2$-gradient flow of Boltzmann's entropy

$$\mathcal{H}(u) = \int_{\Omega} u \log u \, dx.$$ 

Boltzmann’s entropy $\mathcal{H}$ shares an important property [13]: it is 0-convex along (generalized) geodesics with respect to the Wasserstein distance $W_2$ and thus generates a 0-contractive gradient flow (in the sense of Ambrosio, Gigli and Savaré [1]) along which $\mathcal{H}$ decreases most. The corresponding dissipation of $\mathcal{H}$ over time is given by the so-called Fisher information functional

$$\mathcal{F}(u) = \int_{\Omega} 4|\nabla \sqrt{u}|^2 \, dx.$$ 

Formally, $\mathcal{F}$ induces the (fourth-order) Derrida-Lebowitz-Speer-Spohn equation

$$\partial_t u = \text{div} \left( u \nabla \left( \frac{\Delta \sqrt{u}}{\sqrt{u}} \right) \right)$$

as $W_2$-gradient flow. A thorough analysis of the relationship between $\mathcal{H}$, $\mathcal{F}$ and their corresponding evolution equations has been done by Gianazza, Savaré and Toscani [6], later generalized by Matthes, McCann and Savaré [12] to more general energy functionals in the Wasserstein framework. Even if the Fisher information
functional $F$ does not admit a convexity condition along geodesics w.r.t. $W_2$, results on existence and convergence to equilibrium can be deduced using the fact that $F = |\partial H|^2$, i.e., being the squared Wasserstein slope of Boltzmann’s entropy $H$. The main aim of this work is to extend this specific connection to nonlinear mobility functions $m$ and the generalized Wasserstein distance $W_m$. 

However, in this case, the structure of the slope w.r.t. $W_m$ is not known explicitly. Moreover, convexity along geodesics in this space is a very rare property (cf. [4, 17]), with the following exception: It is known that the heat entropy functional $H(u) = \int_{\Omega} h(u(x)) \, dx$, with $h''(z)m(z) = 1$ for all $z$, is 0-convex along geodesics w.r.t. $W_m$ and generates the heat flow as its 0-contractive gradient flow. Formally, the dissipation of $H$ along its own gradient flow, i.e., its autodissipation, reads as

$$-\frac{d}{dt} H(u(t)) = \int_{\Omega} m(u(t))|\nabla h'(u(t))|^2 \, dx = \int_{\Omega} \frac{|\nabla u(t)|^2}{m(u(t))} \, dx$$

This motivates our specific definition for the free energy functional $F$ to be considered in (the consequently fourth-order equation) (1):

$$F(u) := \int_{\Omega} \frac{1}{2} |\nabla f(u(x))|^2 \, dx, \quad \text{with } f(z) := \int_0^z \sqrt{\frac{2}{m(r)}} \, dr,$$

if $f(u) \in H^1(\Omega)$, and $F(u) := +\infty$ otherwise. We call $F$ the generalized Fisher information functional associated to the mobility $m$. For linear mobility, functionals of the form (5) for other choices of $f$ have already been studied in [11].

We impose the additional constraint $u(t, x) \leq S$, where $S > 0$ either is a fixed real number or equal to $+\infty$. The specific value of $S$ is determined by the structure of the mobility function $m$ which is assumed to satisfy the following.

$m \in C^2((0, S)); \quad m(z) > 0$ and $m''(z) \leq 0$ for all $z \in (0, S); \quad m(0) = 0$, and, if $S < \infty$, $m(S) = 0$. \hfill (M)

For mobilities satisfying (M), one can define the distance $W_m$ via formula (4) on the space

$$X := \left\{ u \in L^1(\Omega) : \ 0 \leq u \leq S \ a.e. \ in \ \Omega, \ \int_{\Omega} u \, dx = U \right\},$$

where $U > 0$ is a fixed given number, see [5, 9]. In this work, we use some of the topological properties of the metric space $(X, W_m)$. The respective statements are omitted here for the sake of brevity.

In the case $S = \infty$, we need an additional assumption on the growth of $m$ for large $z$:

There exist $\gamma_0, \gamma_1 \in [0, 1]$ with the additional requirement that

$$\frac{2 - \gamma_0}{2 - \gamma_1} < \frac{d}{d - 2}, \text{ if } d \geq 3, \text{ such that:} \quad (M-PG)$$

$$\lim_{z \to \infty} \frac{m(z)}{z^{\gamma_0}} \in (0, \infty) \quad \text{and} \quad \lim_{z \to \infty} \frac{m(z)}{z^{\gamma_1}} \in [0, \infty).$$
An important subclass of mobilities is the following: we say that $m$ satisfies the LSC condition (compare to [10]) if
$$\sup_{z \in (0,S)} |m'(z)| < \infty, \quad \text{and} \quad \sup_{z \in (0,S)} (-m''(z)m(z)) < \infty.$$
(M-LSC)

In particular, mobilities satisfying (M-LSC) are Lipschitz continuous. In order to also cover the framework of non-LSC mobilities, we will later require the following on the strength of the singularities of $m'$ at the boundary of $(0,S)$, if $m$ does not fulfill (M-LSC):
$$\lim_{z \searrow 0} m'(z)^2 f(z) = 0,$$
and, if $S < \infty$, additionally
$$\lim_{z \nearrow S} m'(z)^2 (f(S) - f(z)) = 0.$$
(M-S)

In particular, (M-S) is met by the paradigmatic examples $m(z) = z^\beta$ for $\beta \in (\frac{2}{3},1]$ and $S = \infty$ as well as by $m(z) = z^{\beta_1} (S-z)^{\beta_2}$ for $\beta_1, \beta_2 \in (\frac{2}{3},1]$ and $S < \infty$.

Compared to [10] where $\beta \in (\frac{1}{2},1]$ is allowed, we need a slightly stricter condition on $m$ here due to the appearance of $m$ in the definition of $f$. These non-Lipschitz mobilities especially come up in variants of the thin film equation with nonlinear mobility.

Our proof of existence of solutions to (1) with $\mathcal{F}$ defined as in (5) and mobilities satisfying (M) and (M-LSC) relies on the formal gradient structure of equation (1). We use the time-discrete minimizing movement scheme for the construction of weak solutions in the space $\mathbf{X}$: given a step size $\tau > 0$, define a sequence $(u_n^\tau)_{n \geq 0}$ in $\mathbf{X}$ recursively via
$$u_0^\tau := u_0, \quad u_n^\tau \in \arg\min_{u \in \mathbf{X}} \left( \frac{1}{2\tau} \mathbf{W}_m(u, u_{n-1}^\tau)^2 + \mathcal{F}(u) \right) \quad \text{for } n \in \mathbb{N}. \quad (6)$$

With the sequence $(u_n^\tau)_{n \geq 0}$ from (6), we define a time-discrete function $u_\tau : [0, \infty) \times \Omega \to [0, \infty]$ via piecewise constant interpolation:
$$u_\tau(t) := u_n^\tau \quad \text{if } t \in ((n-1)\tau, n\tau] \text{ for some } n \geq 1; \quad u_\tau(0) := u_0. \quad (7)$$

The resulting main theorem on the limit behaviour of $(u_\tau)_{\tau > 0}$ in the continuous time limit $\tau \searrow 0$ is as follows.

**Theorem 1.1** (Existence for Lipschitz mobilities). Assume that (M) and (M-LSC) hold, and if $S = \infty$, let (M-PG) be satisfied. Let an initial condition $u_0 \in \mathbf{X}$ with $\mathcal{F}(u_0) < \infty$ be given.

Then, for each step size $\tau > 0$, a time-discrete function $u_\tau : [0, \infty) \to \mathbf{X}$ can be constructed via the minimizing movement scheme (6)\&(7). Moreover, for each vanishing sequence $\tau_k \to 0$ of step sizes, there exists a (non-relabelled) subsequence and a limit function $u : [0, \infty) \to \mathbf{X}$ such that the following is true for arbitrary $T > 0$:

(a) $u \in C^{1/2}([0,T];(\mathbf{X}, \mathbf{W}_m)) \cap L^\infty([0,T];L^p(\Omega))$ for some $p > 1$; and $f(u) \in L^\infty([0,T];H^1(\Omega)) \cap L^2([0,T];H^2(\Omega))$.

(b) $u_{\tau_k}$ converges to $u$ as $k \to \infty$ strongly in $L^p([0,T];L^p(\Omega))$ and pointwise with respect to $t \in [0,T]$ in $(\mathbf{X}, \mathbf{W}_m)$. $f(u_{\tau_k})$ converges to $f(u)$ as $k \to \infty$ strongly in $L^2([0,T];H^1(\Omega))$ and weakly in $L^2([0,T];H^2(\Omega))$.

(c) For almost every $t \in [0,T]$, one has $\mathcal{F}(u_{\tau_k}(t)) \to \mathcal{F}(u(t))$ as $k \to \infty$; and the map $t \mapsto \mathcal{F}(u(t))$ is almost everywhere equal to a nonincreasing function.

(d) The map $u$ is a solution to (1)-(3) in the sense of distributions.
Notice that due to the non-convexity of the problem, we do not obtain uniqueness of solutions. The study of qualitative properties of \( u \) with respect to (large) time is postponed to future research.

Our second main theorem extends the result to mobilities \( m \) which are not Lipschitz continuous. We obtain a weak solution to (1) no longer by approximation via the minimizing movement scheme (6), but as a limit of solutions of (1) for mobilities \( m_\delta \) which are close to \( m \) and satisfy (M-LSC), as \( \delta \to 0 \). Specifically, we approximate \( m \) by \( m_\delta \) as in [10]:

Let \( \delta \in (0, \bar{\delta}) \) and \( \bar{\delta} > 0 \) sufficiently small.

If \( S < \infty \), define

\[
m_\delta(z) := m \left( \frac{z_{\delta,1} - z_{\delta,2}}{\delta} - z - z_{\delta,1} \right),
\]

where \( z_{\delta,1} < z_{\delta,2} \) are the two solutions of \( m(z) = \delta \).

If \( S = \infty \), define

\[
m_\delta(z) := m(z + \delta) - \delta,
\]

where \( z_\delta \) is the unique solution of \( m(z) = \delta \).

Since \( m_\delta \) is constructed in such a way that (M-LSC) is fulfilled, there exists a weak solution \( u_\delta \) to (1) with initial condition \( u_0 \) (which is assumed to be independent of \( \delta \) and such that \( F_\delta(u_0) < \infty \), in the sense of Theorem 1.1, for each \( \delta \)). To indicate the dependence of \( \mathcal{F} \) on \( m \), we e.g. write \( f_\delta \) and \( F_\delta \) when \( m_\delta \) is considered in place of \( m \). Analysing the limit behaviour of the family \( (u_\delta)_{\delta>0} \) as \( \delta \searrow 0 \), we find that the properties of \( u_\delta \) carry over to the limit:

**Theorem 1.2 (Existence for non-Lipschitz mobilities).** Assume that \( m \) satisfies (M), (M-S), and if \( S = \infty \), also (M-PG), and define \( m_\delta \) for \( \delta \in (0, \bar{\delta}) \) and sufficiently small \( \bar{\delta} \) as in (8)/(9). Let an initial condition \( u_0 \in X \) with \( F_\delta(u_0) < \infty \) be given and denote by \( u_\delta \) a weak solution to (1) with \( m_\delta \) in place of \( m \) and initial condition \( u_0 \) in the sense of Theorem 1.1, for each \( \delta \in (0, \bar{\delta}) \). Then, there exists a vanishing sequence \( \delta_k \to 0 \) and a map \( u : [0, \infty) \to X \) such that for the sequence \( (u_{\delta_k})_{k\in\mathbb{N}} \) and the limit \( u \), one has

(a) \( u \in C^{1/2}([0,T]; (X, W_m)) \cap L^\infty([0,T]; L^p(\Omega)) \) for some \( p > 1 \); and \( f(u) \in L^\infty([0,T]; H^1(\Omega)) \cap L^2([0,T]; H^2(\Omega)) \).

(b) \( u_{\delta_k} \) converges to \( u \) as \( k \to \infty \) strongly in \( L^p([0,T]; L^p(\Omega)) \) and pointwise with respect to \( t \in [0,T] \) in \( (X, W_m) \). \( f_{\delta_k}(u_{\delta_k}) \) converges to \( f(u) \) as \( k \to \infty \) strongly in \( L^2([0,T]; H^1(\Omega)) \) and weakly in \( L^2([0,T]; H^2(\Omega)) \).

(c) For almost every \( t \in [0,T] \), one has \( F_{\delta_k}(u_{\delta_k}(t)) \to F(u(t)) \) as \( k \to \infty \); and the map \( t \to F(u(t)) \) is almost everywhere equal to a nonincreasing function.

(d) The map \( u \) is a solution to (1)–(3) in the sense of distributions.

The plan of the paper is as follows. Section 2 is concerned with the proof of Theorem 1.1. We first study the discrete curves obtained by the minimizing movement scheme in Section 2.1 before passing to the continuous time limit in Section 2.2. In Section 3, we show Theorem 1.2. Properties of the approximation with Lipschitz mobilities are investigated in Section 3.1, its convergence in Section 3.2

2. Lipschitz mobility functions. In this section, we prove Theorem 1.1. In advance of studying the properties of the scheme (6)&(7), we first prove an auxiliary
result on the relationship between $\mathcal{F}$ and $\mathcal{H}$. To this end, we make the following specific choice of the integrand $h$ of $\mathcal{H}$ (compare with [10]):

$$h(z) := \int_{s_0}^{z} \frac{z - r}{m(r)} \, dr, \quad \text{for } z \in (0, S). \quad (10)$$

Note that the following statement does not require condition (M-LSC).

**Lemma 2.1** (Estimate on $\mathcal{H}$). Assume that $m$ satisfies (M), and, if $S = \infty$, also (M-PG). Then, there exist $C > 0$ and $q \geq 1$ such that for all $u \in X$ with $f(u) \in H^1(\Omega)$, one has

$$0 \leq \mathcal{H}(u) \leq C(f(u)^q + 1). \quad (11)$$

**Proof.** Note that $h(z) \geq 0$ for all $z \in (0, S)$ and $h(s_0) = 0$. We first investigate the behaviour of $h$ as $z \downarrow 0$: for $z < s_0$, one has

$$h(z) = \int_{s_0}^{z} \frac{z - r}{m(r)} \, dr \leq \int_{z}^{s_0} \frac{(z - r)z}{m(z)r} \, dr,$$

where the last step follows from concavity of $m$, viz. $m(r) \geq \frac{m(z)}{m(z)r}$. One directly verifies that the limit $\lim_{z \downarrow 0} \int_{s_0}^{z} \frac{(z - r)z}{m(z)r} \, dr$ exists. Thanks to the monotonicity of $h$ for $z < s_0$ (clearly, $h'(z) < 0$ for $z < s_0$), also the limit $\lim_{z \downarrow 0} h(z)$ exists. If $S < \infty$, existence of the limit $\lim_{z \uparrow S} h(z)$ follows in analogy. Hence, the integrand $h$ can be continuously extended onto the boundary of $(0, S)$.

So, if $S < \infty$, $h$ is a bounded function. Hence, as $f$ is increasing, we have

$$h(z) \leq C(f(z)^2 + 1), \quad (12)$$

for some $C > 0$ and all $z \in [0, S]$, from which then (11) with $q = 1$ follows using Poincaré’s inequality.

Consider the case $S = \infty$. By assumption (M-PG), there exist $\bar{z} > \max(s_0, 1)$ and constants $C_0, C_1 > 0$ such that for all $z > \bar{z}$:

$$C_0 z^{7/2} \leq m(z) \leq C_1 z^{7/2} \quad (13)$$

In view of the previous arguments on bounded value spaces, we may restrict ourselves to the case $z > \bar{z}$ in the following. First, we have

$$h(z) = \int_{s_0}^{\bar{z}} \frac{z - r}{m(r)} \, dr + \int_{\bar{z}}^{z} \frac{z - r}{m(r)} \, dr,$$

by definition of $h$ (10). By similar considerations as above, one easily finds that

$$\int_{s_0}^{\bar{z}} \frac{z - r}{m(r)} \, dr \leq \bar{C} z,$$

for some $\bar{C} > 0$. Observe that

$$f(z) \geq C(\sqrt{z} - 1), \quad \text{for some } C > 0, \quad (14)$$

which can be verified by elementary calculations, using that $m(z) \leq \bar{C}(z + 1)$ as a consequence of assumption (M). With (14), we again arrive at

$$\int_{s_0}^{\bar{z}} \frac{z - r}{m(r)} \, dr \leq C(f(z)^2 + 1).$$
For the second term, we use (13) to obtain:
\[
\int_{\zeta}^{z} \frac{z - r}{m(r)} \, dr \leq \frac{1}{C_0} \int_{\zeta}^{z} (z r^{-\gamma_0} - r^{1-\gamma_0}) \, dr
\]
\[
= \left\{ \begin{array}{ll}
\frac{1}{C_0} \left[ z (\log z - \log \tilde{z}) - (z - \tilde{z}) \right] & \text{if } \gamma_0 = 1, \\
\frac{1}{C_0} \left[ \frac{z}{1-\gamma_0} (z^{1-\gamma_0} - \tilde{z}^{1-\gamma_0}) - \frac{1}{2-\gamma_0} (z^{2-\gamma_0} - \tilde{z}^{2-\gamma_0}) \right] & \text{if } \gamma_0 \in [0, 1).
\end{array} \right.
\]
All in all, we end up with
\[
h(z) \leq C(f(z)^2 + 1) + C'(1 + z \log z + z) \quad \text{if } \gamma_0 = 1, \quad \text{and}
\]
\[
h(z) \leq C(f(z)^2 + 1) + C'(1 + z^{2-\gamma_0}) \quad \text{if } \gamma_0 \in [0, 1).
\]
To estimate \( f \) from below, we use (13) again:
\[
f(z) = \int_{\zeta}^{z} \sqrt{\frac{2}{m(r)}} \, dr + \int_{\zeta}^{z} \sqrt{\frac{2}{m(r)}} \, dr
\]
\[
\geq C(\sqrt{z} - 1) + \int_{\zeta}^{z} \frac{2}{C_1} r^{-\frac{d}{2}} \, dr
\]
\[
\geq -\tilde{C}_1 + \tilde{C}_2 z^{1-\frac{d}{2}}.
\]
Consider the case \( \gamma_0 < 1 \). If \( d \geq 3 \), one has by (M-PG) that
\[
h(z) \leq C(f(z)^2 + 1) + C'(1 + z^{2-\gamma_0})
\]
\[
\leq C(f(z)^2 + 1) + C'(1 + z^{2\gamma_0} (1-\frac{2}{d}))
\]
\[
\leq C(f(z)^2 + 1) + \tilde{C}(1 + f(z)^{\frac{2}{d}}).
\]
Putting \( z = u(x) \) and integrating over \( x \in \Omega \), we obtain (11) for some \( q \geq 1 \) with the Gagliardo-Nirenberg-Sobolev inequality. The cases \( d \in \{1, 2\} \) and \( \gamma_0 = 1 \) can be treated by similar, but easier arguments. \( \square \)

2.1. The minimizing movement scheme. The next result shows the well-posedness of the scheme (6). Furthermore, the subsequent minimizers \( u^n \) gain in regularity, i.e., not only \( f(u^n) \in H^1(\Omega) \), but even \( f(u^n) \in H^2(\Omega) \) holds.

**Proposition 1** (Properties of the scheme). Assume that (M) and (M-LSC) hold, and if \( S = \infty \), let (M-PG) be satisfied. Let an initial condition \( u_0 \in X \) with \( \mathcal{F}(u_0) < \infty \) be given. Then, for each \( \tau > 0 \), the scheme (6) is well-defined and yields a sequence \( (u^n)_{n \geq 0} \) and a discrete solution \( u_\tau \) via (7).

Moreover, the following statements hold:

(a) For all \( n \in \mathbb{N}, s, t \geq 0 \), one has:
\[
\mathcal{F}(u^n_t) \leq \mathcal{F}(u^{n-1}_t) \leq \mathcal{F}(u^n_0) < \infty,
\]
\[
\sum_{n=1}^{\infty} W_m(u^n_t, u^{n-1}_t)^2 \leq 2\tau \mathcal{F}(u_0),
\]
\[
W_m(u_\tau(s), u_\tau(t)) \leq (2\mathcal{F}(u_0) \max(|s-t|, \tau))^{1/2}.
\]

(b) There exists a constant \( C > 0 \) such that for all \( \tau > 0 \) and all \( n \in \mathbb{N} \), one has:
\[
\int_{\Omega} \|
abla^2 f(u^n_\tau) \|^2 \, dx \leq \frac{C}{\tau} (\mathcal{H}(u^n_{\tau-1}) - \mathcal{H}(u^n_\tau)).
\]
There exists $p > 1$ such that for all $T > 0$, there exists a constant $C > 0$ such that for all $\tau > 0$, the following holds:

\[
\| f(u_\tau) \|_{L^\infty([0,T];H^1)} \leq C,
\]
\[
\| f(u_\tau) \|_{L^2([0,T];H^2)} \leq C,
\]
\[
\| u_\tau \|_{L^\infty([0,T];L^p)} \leq C.
\]

**Proof.** A straightforward application of the direct method from the calculus of variations and Poincaré’s inequality shows that, given that $\mathcal{F}(u_{\tau}^{n-1}) < \infty$, i.e., $f(u_{\tau}^{n-1}) \in H^1(\Omega)$, the Yosida penalization $u \mapsto \frac{1}{2\tau} \mathbf{W}_m(u, u_{\tau}^{n-1})^2 + \mathcal{F}(u)$ admits a minimizer $u_{\tau}^n$ on $X$ with $\mathcal{F}(u_{\tau}^n) < \infty$.

The properties in (a) are a direct consequence of the scheme (6)&(7) and are well-known (see, for instance, [1]).

To prove the additional regularity property (17), we apply the flow interchange technique from [12], using the heat entropy $\mathcal{H}$ as $0$-convex auxiliary functional. The relevant symbolic calculations are already contained in the proof of [11, Prop. 7.3] where a functional of similar form has been studied: here, it is sufficient to show that

\[
\frac{f'''(z)}{f''(z)^2} \geq 3 \quad \text{for all } z \in (0, S).
\]

(18)

Then, as in [11, Prop. 7.3], one obtains that (denoting by $u_s$ the solution to the heat equation with Neumann boundary conditions on $\partial \Omega$ and initial datum $u_{\tau}^n$)

\[
\liminf_{s \searrow 0} \left( -\frac{d}{ds} \mathcal{F}(u_s) \right) \leq \frac{1}{C} \| \nabla^2 u_{\tau}^n \|_{L^2},
\]

from which (17) follows immediately by the flow interchange lemma [12, Thm. 3.2].

Now, using the definition of $f$, one sees

\[
\frac{f'''(z)}{f''(z)^2} = 3 - 2m(z) \frac{m''(z)}{m'(z)^2},
\]

and (18) follows by assumption (M) on the mobility $m$.

The first estimate in (c) is an immediate consequence of the energy estimate (15) and Poincaré’s inequality. The last one is nontrivial only for $d \geq 3$ and $S = \infty$. Using the first estimate and the Gagliardo-Nirenberg-Sobolev inequality, the above estimate with $p = \frac{d}{d-2}$ follows from the inequality (14).

For the second statement in (c), integrate (17) over time and simplify the telescopic sum to see

\[
\int_0^T \int_{\Omega} \| \nabla^2 f(u_{\tau}) \|^2 \, dx \, dt \leq C \left( \mathcal{H}(u_0) - \mathcal{H}(u_{\tau}^n) \right).
\]

With (11) and the energy estimate (15), we have for some $C' > 0$:

\[
\int_0^T \int_{\Omega} \| \nabla^2 f(u_{\tau}) \|^2 \, dx \, dt \leq C' \left( \mathcal{F}(u_0)^q + \mathcal{F}(u_{\tau}^n)^q + 1 \right) \leq C' \left( 2\mathcal{F}(u_0)^q + 1 \right),
\]

which is a finite constant. □

With the techniques from [10, 11], one deduces an approximate weak formulation of equation (1) satisfied by the discrete curve $u_\tau$. The cornerstone of the derivation
again is the flow interchange lemma [12, Thm. 3.2]. This time, the auxiliary functional is the regularized potential energy
\[ \mathcal{V}(u) := \beta \mathcal{H}(u) + \int_{\Omega} u \phi \, dx, \]
where \( \beta > 0 \) and \( \phi \in C^\infty(\Omega) \) with \( \partial_\nu \phi = 0 \) on \( \partial \Omega \) are given: it is known [10, 17] under the assumptions (M) and (M-LSC) on \( \kappa \) that \( \mathcal{V} \) induces a \( \kappa \)-contractive gradient flow on \( (X, W) \), for \( \kappa_\beta = -\frac{C}{\beta} \) with a fixed constant \( C > 0 \). For the sake of brevity and since there is no conceptual novelty involved, we skip the calculations here and directly state the result:

**Lemma 2.2** (Discrte weak formulation). Let \( \tau > 0 \) and define the discrete solution \( u_\tau \) by (6) \& (7). Then, for all \( \beta > 0 \), all \( \phi \in C^\infty(\Omega) \) with \( \partial_\nu \phi = 0 \) on \( \partial \Omega \) and all \( \eta \in C^\infty_c((0, \infty) \cap C([0, \infty)) \), the following discrete weak formulation holds:

\[
\begin{align*}
\| \eta \|_{C^0[0, \infty)} & \mathcal{F}(u_0) + \beta \int_0^\infty \int_\Omega \eta \tau(t) - \eta \tau(t + \tau) h(u_\tau) \, dx \, dt \\
& \leq \int_0^\infty \int_\Omega \eta \tau(t) - \eta \tau(t + \tau) u_\tau \phi \, dx \, dt \\
& + \int_0^\infty \int_\Omega \eta \tau f'(u_\tau) \Delta f(u_\tau [\nabla m(u_\tau) \cdot \nabla \phi + m(u_\tau) \Delta \phi] \, dx \, dt \\
& \leq -\| \eta \|_{C^0[0, \infty)} \mathcal{F}(u_0) - \beta \int_0^\infty \int_\Omega \eta \tau(t) - \eta \tau(t + \tau) h(u_\tau) \, dx \, dt,
\end{align*}
\]

where \( \kappa_\beta < 0 \) is as above. For \( a : [0, \infty) \to \mathbb{R} \), we denote \( a_\tau(s) = a(\lfloor \frac{s}{\tau} \rfloor \tau) \).

Note that since \( f'(z) = \sqrt{2 \frac{m(z)}{m(z)}} \) holds, one can also rewrite the term involving \( f \) in (19) as
\[
\begin{align*}
\int_0^\infty \int_\Omega \eta \tau f'(u_\tau) \Delta f(u_\tau [\nabla m(u_\tau) \cdot \nabla \phi + m(u_\tau) \Delta \phi] \, dx \, dt \\
= \int_0^\infty \int_\Omega \eta \tau \sqrt{2 \Delta f(u_\tau) \left[ 2 \nabla \sqrt{m(u_\tau)} \cdot \nabla \phi + \sqrt{m(u_\tau)} \Delta \phi \right] \, dx \, dt.
\end{align*}
\]

### 2.2. Passage to the continuous time limit

The remainder of this section is concerned with the passage to the continuous time limit \( \tau \downarrow 0 \). In particular, we show that the passage to the limit inside (19) yields the time-continuous weak formulation of (1), completing the proof of Theorem 1.1. In order to obtain convergence in a strong sense, we make use of the following extension of the Aubin-Lions compactness lemma.

**Theorem 2.3** (Extension of the Aubin-Lions lemma [15, Thm. 2]). Let \( Y \) be a Banach space and \( A : Y \to [0, \infty) \) be lower semicontinuous and have relatively compact sublevels in \( Y \). Let furthermore \( W : X \times Y \to [0, \infty) \) be lower semicontinuous and such that \( W(u, \tilde{u}) = 0 \) for \( u, \tilde{u} \in \text{Dom}(A) \) implies \( u = \tilde{u} \).

Let \( (U_k)_{k \in \mathbb{N}} \) be a sequence of measurable functions \( U_k : (0, T) \to Y \). If
\[
\sup_{k \in \mathbb{N}} \int_0^T A(U_k(t)) \, dt < \infty,
\]
then
\[
\limsup_{k \downarrow \phi} \int_0^{T-h} W(U_k(t + h), U_k(t)) \, dt = 0,
\]
then there exists a subsequence that converges in measure w.r.t. \( t \in (0, T) \) to a limit \( U : (0, T) \to Y \).

**Proof of Theorem 1.1.** Let a vanishing sequence \((\tau_k)_{k \in \mathbb{N}}\) of step sizes be given and define the corresponding sequence of discrete solutions \((u_{\tau_k})\) via (6)\&(7). Thanks to the \textit{a priori} estimates from Proposition 1(a)-(c), there exists a map \( u \in C^{1/2}([0, T]; (X, W_m)) \cap L^\infty([0, T]; L^p(\Omega)) \) such that \( u_{\tau_k} \) converges (on a non-relaballed subsequence) to \( u \) both weakly in \( L^p([0, T]; L^p(\Omega)) \) (as a consequence of the Banach-Alaoglu theorem) as well as in \((X, W_m)\) pointwise w.r.t. \( t \in [0, T) \) (as a consequence of the topological properties of the distance \( W_m \) and a refined version of the Arzelà-Ascoli theorem [1, Prop. 3.3.1]).

Furthermore, Proposition 1(c) and the Banach-Alaoglu theorem yield the existence of \( v \in L^2([0, T]; H^2(\Omega)) \) such that \( f(u_{\tau_k}) \) converges weakly to \( v \) in \( L^2([0, T]; H^2(\Omega)) \), possibly extracting another subsequence. We now prove that \( f(u_{\tau_k}) \to f(u) \) strongly in \( L^2([0, T]; L^2(\Omega)) \) for \( u = f^{-1}v \). By a standard interpolation inequality, the desired strong convergence of \( f(u_{\tau_k}) \to f(u) \) in \( L^2([0, T]; H^1(\Omega)) \) then follows. Strong convergence of \( u_{\tau_k} \) to \( u \) in \( L^p([0, T]; L^p(\Omega)) \) (on a subsequence) is achieved by essentially the same technique as in [17, 11] applying Theorem 2.3 for the admissible choices \( Y := L^p(\Omega) \) (with \( p > 1 \) from Proposition 1(c)),

\[
A : Y \to [0, \infty] \quad \text{with} \quad A(\rho) := \begin{cases} \|\rho\|^2_{H^2} & \text{if } \rho \in H^2(\Omega), \\ +\infty & \text{otherwise}, \end{cases}
\]

and

\[
W : Y \times Y \to [0, \infty] \quad \text{with} \quad W(\rho, \bar{\rho}) := \begin{cases} W_m(\rho, \bar{\rho}) & \text{if } \rho, \bar{\rho} \in X, \\ +\infty & \text{otherwise}. \end{cases}
\]

We refer to [17, 11] for the details—again, this method is rather classical and provides no new insights into the theory. A straightforward application of Vitali’s convergence theorem subsequently yields the strong convergence of \( f(u_{\tau_k}) \to f(u) \) in \( L^2([0, T]; L^2(\Omega)) \).

Obviously, this convergence property shows in combination with (15) the claimed convergence and monotonicity of the energy \( F \) in Theorem 1.1(c). To prove that the limit \( u \) indeed is a weak solution to \( (1) \), we verify that for all \( \phi \in C^\infty(\bar{\Omega}) \) with \( \partial_\nu \phi = 0 \) on \( \partial \Omega \) and all \( \eta \in C^\infty_c((0, \infty)) \cap C([0, \infty)) \), the following \textit{continuous weak formulation} holds:

\[
0 = \int_0^\infty \int_\Omega \left( -\partial_t \eta \phi u + \eta f'(u) \Delta f(u) \left[ \nabla m(u) \cdot \nabla \phi + m(u) \Delta \phi \right] \right) \, dx \, dt. \tag{20}
\]

We set \( \beta_k := \sqrt{\tau_k} \) in the discrete weak formulation (19). Then, as \((H(u^n))_{n \in \mathbb{N}}\) is bounded (recall (11)\&(15)), it is immediate that one has

\[
\int_0^\infty \int_\Omega \partial_t \eta \phi u \, dx \, dt = \lim_{k \to \infty} \int_0^\infty \int_\Omega \eta_{\tau_k} \sqrt{2} \Delta f(u_{\tau_k}) \left[ 2 \nabla \sqrt{m(u_{\tau_k})} \cdot \nabla \phi + \sqrt{m(u_{\tau_k})} \Delta \phi \right] \, dx \, dt.
\]

Since \( \eta_{\tau_k} \to \eta \) uniformly and \( \Delta f(u_{\tau_k}) \to \Delta f(u) \) weakly in \( L^2([0, T]; L^2(\Omega)) \), it suffices to prove that \( \nabla \sqrt{m(u_{\tau_k})} \to \nabla \sqrt{m(u)} \) strongly in \( L^2([0, T]; L^2(\Omega)) \), in view of Poincaré’s inequality. Using the definition of \( f \) and writing \( g := f^{-1} \) (which exists by strict monotonicity, recall that \( m(z) > 0 \) for \( z \in (0, S) \)), this is equivalent to
proving that $\nabla g'(v_{\tau_k})$ converges to $\nabla g'(v)$ strongly in $L^2([0, T]; L^2(\Omega))$ for $v := f(u)$ and $v_{\tau_k} := f(u_{\tau_k})$, since one has

$$g'(w) = \frac{1}{f'(g(w))} = \sqrt{\frac{1}{2}m(g(w))} \text{ for all } w \in (0, f(S)). \quad (21)$$

Using the chain rule and (21), we obtain

$$g''(w) = \frac{1}{4} m'(g(w)).$$

Hence, for all $k \in \mathbb{N}$:

$$\nabla g'(v_{\tau_k}) = \frac{1}{4} m'(g(v_{\tau_k})) \nabla v_{\tau_k}. \quad (22)$$

Since $m$ satisfies (M) and (M-LSC), $m' \circ g$ is continuous and bounded. Due to the strong convergence of $v_{\tau_k}$ to $v$ in $L^2([0, T]; H^1(\Omega))$ and Vitali’s theorem, $(\nabla v_{\tau_k})_{k \in \mathbb{N}}$ is uniformly integrable in $L^2([0, T] \times \Omega)$. Without restriction, we may assume that $v_{\tau_k}$ to $v$ almost everywhere on $[0, T] \times \Omega$. Hence, another application of Vitali’s theorem yields the asserted strong convergence of $\frac{1}{4} m'(g(v_{\tau_k})) \nabla v_{\tau_k}$ to $\frac{1}{4} m'(g(v)) \nabla v$ in $L^2([0, T] \times \Omega)$, on a suitable subsequence.

\[ \square \]

3. Non-Lipschitz mobility functions. In this section, we consider mobility functions which do not satisfy (M-LSC), but can be approximated in a suitable way by LSC mobilities, see (8) and (9). Our strategy of proof for Theorem 1.2 is as follows: first, we demonstrate that the \textit{a priori} estimates from Proposition 1 are uniform w.r.t. the approximation parameter $\delta > 0$ when considering a family $(u_{\delta})_{\delta \in (0, \overline{\delta})}$ of weak solutions to (1) with initial condition $u_0$ (in the sense of Theorem 1.1). This will allow us to pass to the limit $\delta \searrow 0$ in the weak formulation (20) of (1) for $m_\delta$ to obtain the sought-for weak formulation of (1) for $m$.

3.1. \textbf{A priori estimates.} This section is devoted to the derivation of the necessary \textit{a priori} estimates on the family $(u_{\delta})_{\delta \in (0, \overline{\delta})}$ introduced above. For definiteness, notice that at each fixed $u \in \mathbf{X}$ and for all $0 < \delta_0 \leq \delta_1 \leq \overline{\delta}$, one has

$$\mathcal{F}_{\delta_0}(u) = \int_\Omega 1_{\{y \in \Omega: u(y) \in (0, S)\}}(x) f'_{\delta_0}(u(x))^2 |\nabla u(x)|^2 \, dx$$

$$= \int_\Omega 1_{\{y \in \Omega: u(y) \in (0, S)\}}(x) \frac{2}{m_{\delta_0}(u(x))^2} |\nabla u(x)|^2 \, dx$$

$$\leq \int_\Omega 1_{\{y \in \Omega: u(y) \in (0, S)\}}(x) \frac{2}{m_{\delta_1}(u(x))^2} |\nabla u(x)|^2 \, dx = \mathcal{F}_{\delta_1}(u),$$

since $m(z) \geq m_{\delta_0}(z) \geq m_{\delta_1}(z)$ for all $z \in (0, S)$ (see [10]). Hence,

$$\mathcal{F}_{\delta}(u_0) \leq \mathcal{F}_{\overline{\delta}}(u_0),$$

so the condition $\mathcal{F}_{\overline{\delta}}(u_0) < \infty$ and Theorem 1.1 provide the existence of $(u_{\delta})_{\delta \in (0, \overline{\delta})}$.

\textbf{Lemma 3.1 (A priori estimates).} Let a sufficiently small $\overline{\delta} > 0$ be given and let $(u_{\delta})_{\delta \in (0, \overline{\delta})}$ be a family of weak solutions to (1) for $m_\delta$ in place of $m$ with initial condition $u_0$ (in the sense of Theorem 1.1). Then, for all $\delta \in (0, \overline{\delta})$ and all $T > 0$:

(a) $\mathcal{F}_\delta(u_{\delta}(t)) \leq \mathcal{F}_\delta(u_0)$ for all $t \in [0, T]$,

(b) $\mathbb{W}_m(u_{\delta}(s), u_{\delta}(t)) \leq \sqrt{2 \mathcal{F}_{\delta}(u_0)}|s - t|$ for all $s, t \in [0, T],$

(c) $\|f_\delta(u_{\delta})\|_{L^\infty([0, T]; H^1)} \leq C,$
(d) \( \| f_\delta(u_\delta) \|_{L^2([0,T];H^2)} \leq C \),

(c) \( \| u_\delta \|_{L^\infty([0,T];L^p)} \leq C \),

for some \( p > 1 \) and \( C > 0 \) independent of \( \delta \).

Proof. For part (a), we obtain by Theorem 1.1(c):

\[
F_\delta(u_\delta(t)) \leq F_\delta(u_0) \leq F_\delta(w_0) < \infty \quad \text{for all } t \in [0,T].
\]

Consequently, (b) immediately follows from the Hölder estimate for \( u_\delta \) in \((X, W_m)\) (recall (16) and Theorem 1.1(b)) and the monotonicity \( W_m(u, \bar{u}) \leq W_m(u, \bar{u}) \) for each \( u, \bar{u} \in X \). The claims (c)–(e) are a consequence of (a) and the respective estimates of Proposition 1(c) the proof of which does not rely on condition (M-LSC).

\[ \square \]

### 3.2. Convergence

In this section, we prove Theorem 1.2. As a preparation, we show

**Lemma 3.2** (Local uniform convergence). Let a vanishing sequence \((\delta_k)_{k \in \mathbb{N}}\) in \((0, \delta)\) be given and denote, for each \( k \), \( g_{\delta_k} := f_{\delta_k}^{-1} \). The following statements hold:

(a) If \( S = \infty \), there exists a (non-relabelled) subsequence on which the sequence \((G_{\delta_k})_{k \in \mathbb{N}}\), defined by

\[ G_{\delta_k} : [0, \infty) \to \mathbb{R}, \quad G_{\delta_k}(w) := m_{\delta_k}'(g_{\delta_k}(w)) \sqrt{w}, \]

converges locally uniformly to the continuous map

\[ G : [0, \infty) \to \mathbb{R}, \quad G(w) := m'(g(w)) \sqrt{w} \quad \text{for } w > 0, \quad G(0) := 0. \]

(b) If \( S < \infty \), there exists a (non-relabelled) subsequence on which the sequence \((\tilde{G}_{\delta_k})_{k \in \mathbb{N}}\), defined by

\[ \tilde{G}_{\delta_k} : [0, f(S)] \to \mathbb{R}, \quad \tilde{G}_{\delta_k}(w) := m_{\delta_k}'(g_{\delta_k}(w)) \sqrt{w(f(\delta_k)(S) - w)}, \]

converges uniformly to the continuous map

\[ \tilde{G} : [0, f(S)] \to \mathbb{R}, \quad \tilde{G}(w) := m'(g(w)) \sqrt{w(f(S) - w)} \quad \text{for } w \in (0, f(S)), \]

\[ \tilde{G}(0) := 0, \quad \tilde{G}(f(S)) := 0. \]

Proof. At first, we prove—for arbitrary \( S \)—that \((g_{\delta_k})_{k \in \mathbb{N}}\) converges (on a suitable subsequence) locally uniformly on \([0, S]\) to \( g := f^{-1} \). Indeed, using the monotonicity \( m_{\delta_k} \leq m \) on \([0, S]\) and (21), we obtain the differential estimate

\[ 0 \leq g'_{\delta_k}(w) = \sqrt{\frac{1}{2} m_{\delta_k}(g_{\delta_k}(w))} \leq \sqrt{\frac{1}{2} m(g_{\delta_k}(w))} \leq C(g_{\delta_k}(w) + 1), \]

where the constant \( C > 0 \) does not depend on \( k \). Using Gronwall’s lemma, we deduce that \( g_{\delta_k} \), and consequently also \( g_{\delta_k}' \), is \( k \)-uniformly bounded on compact subsets of \([0, S]\). The application of the Arzelà-Ascoli theorem yields the desired local uniform convergence.

Consider the case \( S = \infty \). Using the monotonicity properties (see [10] again)

\[ m_{\delta_k}'(z) \leq m'(z) \quad \text{and} \quad g(w) \geq g_{\delta_k}(w) \]

in combination with the concavity of \( m \), we find that for all \( w > 0 \):

\[ m_{\delta_k}'(g(w)) \sqrt{w} \leq G_{\delta_k}(w) \leq m'(g_{\delta_k}(w)) \sqrt{w}. \]

Clearly, \( G_{\delta_k}(w) \to G(w) \) if \( w > 0 \). Moreover, this property also extends to \( w = 0 \) by existence of the limit as \( w \searrow 0 \), cf. condition (M-S), and \( G \) is continuous. Observe
that \((m'_k(g(w))\sqrt{w})_{k\in\mathbb{N}}\) and \((m'(g_{\delta_k}(w))\sqrt{w})_{k\in\mathbb{N}}\) are monotonic in \(k\) at each fixed \(w > 0\). Invoking Dini’s theorem and a diagonal argument, we deduce the claimed local uniform convergence of \(G_{\delta_k}\) to \(G\), extracting a certain subsequence.

Consider now the case \(S < \infty\). Fix \(0 < \tilde{z}_0 < \tilde{z}_1 < S\) such that \(m'(\tilde{z}_0) > 0\) and \(m'(\tilde{z}_1) < 0\), and define \(\tilde{w}_0 = f(\tilde{z}_0)\), \(\tilde{w}_1 = f(\tilde{z}_1)\). We distinguish the cases \(w \in [0, \tilde{w}_0)\), \(w \in [\tilde{w}_0, \tilde{w}_1]\) and \(w \in [\tilde{w}_1, f(S)]\). Clearly, by smoothness of \(m\) and \(m_{\delta_k}\), the claim is true for \(w \in [\tilde{w}_0, \tilde{w}_1]\). For the case \(w \in [0, \tilde{w}_0]\), we argue similarly as above; recall that \(f_{\delta_k}(S) \downarrow f(S)\) and observe that (23) also holds in the case at hand:

\[ m'_k(g(w))\sqrt{w} \leq \tilde{G}_{\delta_k}(w) \leq m'(g_{\delta_k}(w))\sqrt{w}(f_{\delta_k}(S) - w). \]

Again, \(\tilde{G}_{\delta_k}(w) \rightarrow \tilde{G}(w)\) if \(w \geq 0\), and \((m'_k(g(w))\sqrt{w}(f(S) - w))_{k\in\mathbb{N}}\) and \((m'(g_{\delta_k}(w))\sqrt{w}(f_{\delta_k}(S) - w))_{k\in\mathbb{N}}\) are monotonic, so proceed as above. For the remaining case \(w \in [\tilde{w}_1, f(S)]\), we have:

\[ |\tilde{G}_{\delta_k}(w)| \leq |m'(g_{\delta_k}(w))\sqrt{w}(f_{\delta_k}(S) - w)| \leq |\tilde{G}(w)| \sqrt{f_{\delta_k}(S) - w} \leq 2|\tilde{G}(w)|, \]

provided that \(k\) is sufficiently large. Consequently, by (M-S), there exists for each \(\varepsilon > 0\) a \(\bar{w} \geq \tilde{w}_1\) such that \(|\tilde{G}_{\delta_k}(w) - \tilde{G}(w)| \leq 2\tilde{G}(w) < \varepsilon\) for all \(w \in [\bar{w}, f(S)]\) and sufficiently large \(k \in \mathbb{N}\). Combining this with the—again obvious—uniform convergence of \(\tilde{G}_{\delta_k}(w)\) to \(\tilde{G}(w)\) on \([\bar{w}, \tilde{w}]\) yields the claim. \(\square\)

Now, we are in position to complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** In complete analogy to the proof of Theorem 1.1 from Section 2.2, one deduces the existence of a limit map \(u : [0, \infty) \rightarrow \mathbb{X}\) such that, on a suitable subsequence of the vanishing sequence \((\delta_k)_{k\in\mathbb{N}}\) in \((0, \delta)\), the claims (a)–(c) in Theorem 1.2 are true. It remains to show that the limit \(u\) satisfies the weak formulation (20), given that \(u_{\delta_k}\) satisfies (20) for \(m_{\delta_k}\) and \(f_{\delta_k}\) in place of \(m\) and \(f\), respectively. As before, the proof is complete if, extracting a further subsequence, \(\nabla\sqrt{m_{\delta_k}(u_{\delta_k})} \rightarrow \nabla\sqrt{m(u)}\) strongly in \(L^2([0, T]; L^2(\Omega))\). Again, introducing the inverse functions \(g_{\delta_k} := f_{\delta_k}^{-1}\) and \(g := f^{-1}\), we have to verify that \(m'(g_{\delta_k}(v_{\delta_k}))\nabla v_{\delta_k} \rightarrow m'(g(v))\nabla v\) strongly in \(L^2([0, T]; L^2(\Omega))\), for \(v := f(u)\) and \(v_{\delta_k} := f(u_{\delta_k})\) (recall (21) & (22)).

We first consider the case \(S = \infty\) and observe that

\[ m'(g_{\delta_k}(v_{\delta_k}))\nabla v_{\delta_k} = 2G_{\delta_k}(v_{\delta_k})\nabla \sqrt{v_{\delta_k}} \quad \text{a.e. on} \ [0, T] \times \Omega. \]

By Lemma 3.2 and since, without restriction, \(\nabla v_{\delta_k} \rightarrow \nabla v\) pointwise almost everywhere on \([0, T] \times \Omega\), we obtain that

\[ 2G_{\delta_k}(v_{\delta_k})\nabla \sqrt{v_{\delta_k}} \rightarrow 2G(v)\nabla \sqrt{v} = m'(g(v))\nabla v \]

pointwise almost everywhere on \([0, T] \times \Omega\). Furthermore, using (M-S) and (M) yields \(G_{\delta_k}(v_{\delta_k}) \leq C(\sqrt{w} + 1)\) for some \(k\)-independent constant \(C > 0\). Hence, for each measurable set \(A \subset [0, T] \times \Omega\), one has for sufficiently large \(k\) that

\[
\int_A [2G_{\delta_k}(v_{\delta_k})\nabla \sqrt{v_{\delta_k}}]^2 \, dx \, dt \leq 4 \left( \int_0^T \|\nabla \sqrt{v_{\delta_k}}\|_{L^4}^4 \, dt \right)^{1/2} \left( \int_A G_{\delta_k}(v_{\delta_k})^4 \, dx \, dt \right)^{1/2} \\
\leq C' \left( \int_0^T \|\nabla^2 v_{\delta_k}\|_{L^2}^2 \, dt \right)^{1/2} \left( \int_A (v_{\delta_k}^2 + 1) \, dx \, dt \right)^{1/2},
\]
where we used the well-known Lions-Villani estimate on square roots \([8]\) (see \([10,\) \(^{2}\) Lemma A.1\]) for a formulation in the framework at hand) in the last step. Since \((v_{\delta k})_{k \in \mathbb{N}}\) is \(L^{2}\)-uniformly integrable (by Vitali’s theorem), the former estimate also yields \(L^{2}\)-uniform integrability of \((2G_{\delta k}(v_{\delta k})\nabla \sqrt{v_{\delta k}})_{k \in \mathbb{N}}\) as \(v_{\delta k}\) is \(k\)-uniformly bounded in \(L^{2}([0,T];H^{2}(\Omega))\). Applying Vitali’s theorem once again gives \(m'(g_{\delta k}(v_{\delta k}))\nabla v_{\delta k} \to m'(g(v))\nabla v\) strongly in \(L^{2}([0,T];L^{2}(\Omega))\), extracting a subsequence if necessary.

Consider the remaining case \(S < \infty\) and notice that

\[
m'(g_{\delta k}(v_{\delta k}))\nabla v_{\delta k} = \frac{2}{f_{\delta k}(S)} \tilde{G}_{\delta k}(v_{\delta k}) \left[ \nabla \sqrt{v_{\delta k}} + \nabla \sqrt{f_{\delta k}(S) - v_{\delta k}} \right]
\]

almost everywhere on \([0, T] \times \Omega\). Thanks to Lemma 3.2, one has

\[
\frac{2}{f_{\delta k}(S)} \tilde{G}_{\delta k}(v_{\delta k}) \left[ \nabla \sqrt{v_{\delta k}} + \nabla \sqrt{f_{\delta k}(S) - v_{\delta k}} \right]
\]

\[
\to \frac{2}{f(S)} \tilde{G}(v) \left[ \nabla \sqrt{v} + \nabla \sqrt{f(S) - v} \right] = m'(g(v))\nabla v
\]

almost everywhere on \([0, T] \times \Omega\). Now, similarly as above, for each measurable set \(A \subset [0, T] \times \Omega\):

\[
\int_{A} \left( \frac{2}{f_{\delta k}(S)} \tilde{G}_{\delta k}(v_{\delta k}) \left[ \nabla \sqrt{v_{\delta k}} + \nabla \sqrt{f_{\delta k}(S) - v_{\delta k}} \right] \right)^{2} \, dx \, dt
\]

\[
\leq C' \left( \int_{0}^{T} \| \nabla^{2} v_{\delta k} \|_{L^{2}}^{2} \, dt \right)^{1/2} \left( \int_{A} (v_{\delta k}^{2} + 1) \, dx \, dt \right)^{1/2},
\]

for some \(C' > 0\) which does not depend on \(k\). Applying Vitali’s theorem as in the former case yields the asserted strong convergence \(m'(g_{\delta k}(v_{\delta k}))\nabla v_{\delta k} \to m'(g(v))\nabla v\) in \(L^{2}([0,T];L^{2}(\Omega))\).

All in all, we have proved that \(u = f(v)\) satisfies the weak formulation (20) of (1), so the proof of Theorem 1.2 is finished. \(\square\)

REFERENCES

[1] L. Ambrosio, N. Gigli and G. Savaré, Gradient Flows in Metric Spaces and in the Space of Probability Measures, 2nd edition, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2008.

[2] J.-D. Benamou and Y. Brenier, A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, Numer. Math., 84 (2000), 375–393.

[3] A. Blanchet and P. Laurençot, The parabolic-parabolic Keller-Segel system with critical diffusion as a gradient flow in \(\mathbb{R}^{d}\), \(d \geq 3\), Comm. Partial Differential Equations, 38 (2013), 658–686.

[4] J. A. Carrillo, S. Lisini, G. Savaré and D. Slepčev, Nonlinear mobility continuity equations and generalized displacement convexity, J. Funct. Anal., 258 (2010), 1273–1309.

[5] J. Dolbeault, B. Nazaret and G. Savaré, A new class of transport distances between measures, Calc. Var. Partial Differential Equations, 194 (2009), p133.

[6] U. Gianazza, G. Savaré and G. Toscani, The Wasserstein gradient flow of the Fisher information and the quantum drift-diffusion equation, Arch. Ration. Mech. Anal., 194 (2009), 133–220.

[7] R. Jordan, D. Kinderlehrer and F. Otto, The variational formulation of the Fokker-Planck equation, SIAM J. Math. Anal., 29 (1998), 1–17.

[8] P.-L. Lions and C. Villani, Régularité optimale de racines carrées, C. R. Acad. Sci. Paris Sér. I Math., 321 (1995), 1537–1541.

[9] S. Lisini and A. Marigonda, On a class of modified Wasserstein distances induced by concave mobility functions defined on bounded intervals, Manuscripta Math., 133 (2010), 197–224.
[10] S. Lisini, D. Matthes and G. Savaré, Cahn-Hilliard and thin film equations with nonlinear mobility as gradient flows in weighted-Wasserstein metrics, *J. Differential Equations*, 253 (2012), 814–850.
[11] D. Loibl, D. Matthes and J. Zinsl, Existence of weak solutions to a class of fourth order partial differential equations with Wasserstein gradient structure, *Potential Analysis*, 45 (2016), 755–776.
[12] D. Matthes, R. J. McCann and G. Savaré, A family of nonlinear fourth order equations of gradient flow type, *Comm. Partial Differential Equations*, 34 (2009), 1352–1397.
[13] R. J. McCann, A convexity principle for interacting gases, *Adv. Math.*, 128 (1997), 153–179.
[14] F. Otto, The geometry of dissipative evolution equations: The porous medium equation, *Comm. Partial Differential Equations*, 26 (2001), 101–174.
[15] R. Rossi and G. Savaré, Tightness, integral equicontinuity and compactness for evolution problems in Banach spaces, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 2 (2003), 395–431.
[16] C. Villani, *Topics in Optimal Transportation*, vol. 58 of Graduate Studies in Mathematics, American Mathematical Society, Providence, 2003.
[17] J. Zinsl and D. Matthes, Transport distances and geodesic convexity for systems of degenerate diffusion equations, *Calc. Var. Partial Differential Equations*, 54 (2015), 3397–3438.

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