Fractals in Small-World Networks With Time Delay

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Abstract

The small-world networks recently introduced by Watts and Strogatz [Nature 393, 440 (1998)] has attracted much interests in studying the interesting properties of the networks without time-delay. However, a signal or influence travelling on the small-world networks often associated with time-delay features which are very common in biological and physical networks. We develop an analytical approach as well as numerical simulations to try to characterize the effect of time-delay on the properties of small-world networks. An analytical expression of the fractal dimension of the small-world networks is given and thus compared with the results from numerical simulations. Analysis shows that small-world networks with time-delay generally have the multifractals property.

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1 Introduction

The properties of complicated networks such as internet servers, power grids, forest fires and porous media are mainly determined by the structure of connections between the vertices or occupied sites. These networks can be generally represented as a graph network. Obviously, one extreme of such a graph is the regular network which has a high degree of local clustering and the average distance between the vertices is quite large, while the other extreme is the random network which shows negligible local clustering and the average distance is relative small. Recently, Watts and Strogatz [1] have presented a model for small-world networks with a high degree of local clustering and a small average distance. Such a small-world phenomenon can be obtained by adding randomly only a small fraction of the connections, and some common networks such as power grids, film stars networks and neural networks behave like small-world networks [2-6].

Recently, Moukarzel [3] studied the spreading and shortest paths in system with sparse long-range connections by using the small-world model. The spreading of some influence such as a forest fire, an infectious disease or a particle in percolating media is studied using a simple rule: at each time step, the influence propagates from the infected site to all uninfected sites connected to it via a link although this link is not necessary physical. A long-range connection or shortcut can simulate the spark that starts a new fire spot, the infect site (say person with flu) suddenly travels to a new place (site), or a portable computer with virus that start to connect to the network a new place. The Newman-Watts model [2] and Moukarzel model [3] were mainly focusing the immediate response of the network by adding randomly some long-distance shortcuts; there was no time delay in the network systems. However, in reality, a spark or an infection can not start a new fire spot or new infection immediately, it usually takes some time \( \delta \), called ignition time or waiting time, to start a new fire or infection. Thus the existing models are no longer able to predict the response in the networks or systems with time delay.

The purpose of the present work is to investigate the small-world networks with time delay by using or extending the existing the small-world theoretical framework developed by Watts and Strogatz (1998) so as to characterise the fractal dimension of the small-world network with time-delay. The present model will generally lead to a delay differential equation, whose solution is usually very difficult to obtain if it is not impossible. Thus the numerical simulation becomes essential [1]. We will take the analytical analysis as far as possible and compare with the results from numerical
simulations. In addition, we shall focus mainly on the 1-D and 2-D networks to study the effect of time-delay on fractal dimension and other properties of the small-world networks.

2 Delay Differential Equation for the Small-World Networks

Now considering a randomly connected network with n-dimensional lattice \([7,8,9,10]\) and \(n = 1, 2, 3, \ldots\) Assuming an influence spreads with a constant velocity \(u\) in all directions and a new infected spot in the other end of a shortcut will start but with a time delay \(\Delta\). Following the method developed by Newman and Watts [2] and Moukarzel [3], the total infected volume \(V(t)\) comes from two contributions: one is the primary influenced volume, and the other contribution is the secondary volume by shortcuts.

The primary volume at time \(t\) is the influenced part inside a (hyper) sphere of radius \(ut\) and the surface \(\Gamma_d(ut)^{n-1}\), so the primary volume is \(\Gamma_d u^n \int_0^t \zeta^{n-1} d\zeta\). When a primary sphere meets a shortcut end (with a density \(2p\) of shortcuts on the network where \(p\) is the probability of the long-range shortcuts), a new secondary sphere starts randomly at the other end of the shortcut. This secondary volume with a time-delay \(\Delta\) is \(\Gamma_d u^n \int_0^t [2pV(t - \zeta - \Delta)]\zeta^{n-1} d\zeta\). By using a continuum approach to the network, the total volume satisfies the time-delay equation of Newman-Watts-Moukarzel type

\[
V(t) = \Gamma_d u^n \int_0^t \zeta^{n-1} [1 + \xi^{-n} V(t - \zeta - \Delta)] d\zeta, \quad n = 1, 2, 3, \ldots
\]  

where \(\Gamma_d\) is shape factor of a hypersphere in \(n\)-dimensions. The length scale of Newman-Watts type [2] can be defined as \(\xi = 1/(2p)^{1/n}\). However, by slightly rescaling the volume \(V\), a generalized definition is more appropriate. That is

\[
\xi = \frac{1}{(2pk^n)^{1/n}},
\]

where \(k\) being some fixed range. By proper rescaling \(V\) by \(\xi^{-n}\) and \(t\) by \(u(\xi^{-n}\Gamma_d(n-1))^{1/n}\), that is

\[
S = V\xi^{-n}, \quad \tau = tu(\xi^{-n}\Gamma_d(n-1))^{1/n}, \quad \delta = \Delta u(\xi^{-n}\Gamma_d(n-1))^{1/n}.
\]

Now we can rewrite (1) as

\[
S(t) = \frac{1}{(n-1)!} \int_0^\tau \zeta^{n-1} [1 + S(\tau - \zeta - \delta)] d\zeta, \quad n = 1, 2, 3, \ldots
\]

By differentiating the equation twice, we have the following time-delay equation

\[
\frac{d^n S}{d\tau^n} = 1 + S(\tau - \delta).
\]

This a delay differential equation, whose explicit solutions are not always possible depending on the initial conditions. For a proper initial condition \(S(\tau) = 0\) for \(-\delta \leq \tau \leq 0\), the solution can be explicitly obtained by using the time-forwarding method,

\[
S(\tau) = \sum_{j=1}^{\tau/\delta} \frac{(\tau - j\delta)^{nj}}{(nj)!}, \quad n = 1, 2, 3, \ldots
\]

Clearly, for \(\tau \neq 0\) and \(\delta \to 0\), the above solution degenerates into

\[
S(\tau) = \sum_{j=1}^{\infty} \frac{\tau^{nj}}{(nj)!},
\]

given by Moukarzel [3]. For \(n = 1\), \(S(\tau) = e^\tau - 1\). For \(n = 2\), \(S(\tau) = \cosh \tau - 1\). However, when \(\delta \neq 0\), there is no such simple expressions. It can be expected that the time-delay parameter \(\delta\) will have a strong effect on the evolution behaviour \(S(\tau)\) of an influence.


3 Fractal Dimension of Delay Small-World Networks

The evolutionary solution of $S(\tau)$ with time $\tau$ and time delay $\delta$ gives the areas affected by the influence such as fire, infectious diseases and computer virus. One can expect that fractal dimension of the small-world networks will give a measure of the network properties as first done by Newman and Watts (1999) which does not include the effect of time-delay. Since the velocity of the influence travels at a constant speed $u$, we can rewrite the solution (6) in terms of the distance $r = \tau = At/\xi$ where $A = u(\Gamma u(n-1))/n$, we have

$$S(r) = \sum_{j=1}^{n} \frac{(r-j\delta)^{nj}}{(nj)!}, \quad n = 1, 2, 3, ...$$

(8)

Since $S(r)$ increases as $r$ increases or $S(r) \sim r^D$, the fractal dimension $D$ can be calculated by differential formulae

$$D = \frac{d \log S(r)}{d \log r},$$

(9)

which can be calculated using (8). We now show how the time-delay affects the fractal dimension $D$ of the small-networks. For $\delta = 0$, we have

$$S(r)|_{\delta=0} = \sum_{j=1}^{\infty} \frac{r^{nj}}{(nj)!} = \frac{1}{n} \sum_{j=0}^{n-1} \exp [re^{i2\pi j/n}] - 1,$$

(10)

so that fractal dimension becomes

$$D = \frac{r \left( \frac{1}{n} \sum_{j=0}^{n-1} \exp \left[ \frac{12\pi j}{n} - re^{i2\pi j/n} \right] \right)}{1/n \sum_{j=0}^{n-1} \exp \left[ \frac{12\pi j}{n} + re^{i2\pi j/n} \right] - 1}$$

(11)

For $n = 1$, this becomes $D = re^{r}/(e^r - 1)$. Clearly, when $r \to 0$, the limit of $D$ leads to one. One the other hand, when $r \to \infty$, the $r$-term dominates, so we have $D \to r = At/\xi$. For $\delta \neq 0$, there is no general explicit asymptotic expressions and numerical evaluations can be easily done. For small time delay and shorter length scale ($r = At/\xi \ll 1$), the fractal dimension remains nearly a constant. For $r \gg 1$, the fractal dimension increases essentially linearly with the radius $r$, thus the dimension of a small-world network depends on the length scale on which one looks at it. The dependence of the fractal dimension $D$ on the lengthscale $\xi$ through $r$ (since $r$ and $\tau$ is rescaled by the length scale $\xi$) and the time-delay $\delta$ suggests the multifractal features of the small-networks.

On the other hand, for a network of finite size, we can change the shortcut probability $p$ to modulate the effective connectivity and fractal dimension. For a very small $p \to 0$, the effective length scale $\xi \to \infty$, so any finite-sized network satifies the condition $r \ll 1$, and thus the fractal dimension $D \to 1$. As $p$ increases, the length scale $\xi$ reduces, so that the fractal dimension increases and the network is closely interconnected with increasing efficiency.

In order to compare the analytical solution with numerical simulations, we use the simulation method given by Watts and Strogatz (1998) and Newman and Watts (1999). The numerical simulations for the one-dimensional $n = 1$ and the comparison of fractal dimension with the analytical solution (8) is shown in Figure 1 where the network size $N = 500,000$, $\xi = 500$ (or $p = 0.002$ and $k = 2$) and $\delta = 0, 1, 5, 10$. We can see that numerical results (marked with $\circ, \circ$ etc) are in good agreement with the analytical solution (solid lines) by using (8). As expected, for $r \ll 1$, both numerical results and analytical fractal dimension approach $n = 1$. The time-delay has a very strong effect on the fractal dimension $D$ of the small-world networks. For $\delta \gg 1$, $D$ is substantially reduced compared with the one without time-delay $\delta = 0$, and $D$ is quite near $n$ for most of the region. The large time-delay actually makes the network become a larger world network, and the influence slowly spread in more localized areas.

The transition from a very small to higher value of $p$ or very long to moderate length scale $\xi^* = 1/(2pkn)^{1/n}$ suggests that the increasing shortcut probability $p$ will greatly increase the fractal dimension $D$ of the small-world network. This has very important implications for real world networks. For a small-world network such as the internet, road networks, and business partnerships,
the introduction of a small fraction of long-range shortcuts or connections, will essentially make the network behave more effectively since the effective scale is becoming smaller. Real networks have always some time delay that usually makes the network larger than networks without time delay. Thus there is a trade-off between these two competitive factors and the proper network design is very important to ensure the efficiency of real networks such as the internet as well as business partnerships.

In summary, numerical simulations and analytical analysis for small-world networks with time-delay show that the time-delay parameter $\delta$ has a very strong effect on the fractal dimension and other properties such as the speed and saturation time of the delay networks. A small-world network can become larger if sufficient time-delay in the system response is introduced. On the other hand, in order to make a large-world network transfer into a small-world, a slightly higher probability $p$ of long-range random shortcuts are necessary compared with the one without time-delay because this essentially reduces the characteristic length scale $\xi = O(1/p)$ quite significantly. Further studies of the dynamics of the delay small-world networks are now in progress.

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Figure 1: Fractal dimension of small-world networks with time-delay $\delta = 0, 1, 5, 10$ for a network size $N = 500,000$, $\xi = 500$ (or $p = 0.002$) and $n = 1$. Numerical results (marked with ◯, ⊕ etc) agree well with analytical express (solid) calculated from (9).