ON SOME MEAN SQUARE ESTIMATES IN THE RANKIN-SELBERG PROBLEM

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An overview of the classical Rankin-Selberg problem involving the asymptotic formula for sums of coefficients of holomorphic cusp forms is given. We also study the function $\Delta(x; \xi)$ ($0 \leq \xi \leq 1$), the error term in the Rankin-Selberg problem weighted by $\xi$-th power of the logarithm. Mean square estimates for $\Delta(x; \xi)$ are proved.

1. THE RANKIN-SELBERG PROBLEM

The classical Rankin-Selberg problem consists of the estimation of the error term function

$$\Delta(x) := \sum_{n \leq x} c_n - Cx,$$

where the notation is as follows. Let $\varphi(z)$ be a holomorphic cusp form of weight $\kappa$ with respect to the full modular group $SL(2, \mathbb{Z})$, and denote by $a(n)$ the $n$-th Fourier coefficient of $\varphi(z)$ (see e.g., R.A. Rankin [15] for a comprehensive account). We suppose that $\varphi(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$, that is, $a(1) = 1$ and $T(n)\varphi = a(n)\varphi$ for every $n \in \mathbb{N}$. In (1.1) $C > 0$ is a suitable constant (see e.g., [9] for its explicit expression), and $c_n$ is the convolution function defined by

$$c_n = n^{1-\kappa} \sum_{m \mid n} m^{2(\kappa-1)} \left| a\left( \frac{n}{m^2} \right) \right|^2.$$

The classical Rankin-Selberg bound of 1939 is

$$\Delta(x) = O(x^{3/5}),$$

hitherto unimproved. In their works, done independently, R.A. Rankin [16] derives (1.2) from a general result of E. Landau [11], while A. Selberg [17] states the result

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with no proof. Although the exponent 3/5 in (1.2) represents one of the longest standing records in analytic number theory, recently there have been some developments in some other aspects of the Rankin-Selberg problem. In this paper we shall present an overview of some of these new results. In addition, we shall consider the weighted sum (the so-called Riesz logarithmic means of order $\xi$), namely

$$
\frac{1}{\Gamma(\xi + 1)} \sum_{n \leq x} c_n \log^n \left( \frac{x}{n} \right) := C x + \Delta(x; \xi) \quad (\xi \geq 0),
$$

where $C$ is as in (1.1), so that $\Delta(x) \equiv \Delta(x; 0)$. The effect of introducing weights such as the logarithmic weight in (1.3) is that the ensuing error term (in our case this is $\Delta(x; \xi)$) can be estimated better than the original error term (i.e., in our case $\Delta(x; 0)$). This was shown by Matsumoto, Tanigawa and the author in [9], where it was proved that

$$
\Delta(x; \xi) \ll \varepsilon x^{(3 - 2\xi)/5 + \varepsilon} \quad (0 \leq \xi \leq 3/2).
$$

Here and later $\varepsilon$ denotes arbitrarily small constants, not necessarily the same ones at each occurrence, while $a \ll \varepsilon b$ means that the constant implied by the $\ll$-symbol depends on $\varepsilon$. When $\xi = 0$ we recover (1.2) from (1.4), only with the extra $'\varepsilon'$ factor present. In this work we shall pursue the investigations concerning $\Delta(x; \xi)$, and deal with mean square bounds for this function.

### 2. THE FUNCTIONAL EQUATIONS

In view of (1.1) and (1.2) it follows that the generating Dirichlet series

$$
Z(s) := \sum_{n=1}^{\infty} c_n n^{-s} \quad (s = \sigma + it)
$$

converges absolutely for $\sigma > 1$. The arithmetic function $c_n$ is multiplicative and satisfies $c_n \ll \varepsilon n^\varepsilon$. Moreover, it is well known (see e.g., R.A. Rankin [14], [15]) that $Z(s)$ satisfies for all $s$ the functional equation

$$
\Gamma(s + \kappa - 1)\Gamma(s)Z(s) = (2\pi)^{\kappa - 2}\Gamma(\kappa - s)\Gamma(1 - s)Z(1 - s),
$$

which provides then the analytic continuation of $Z(s)$. In modern terminology $Z(s)$ belongs to the Selberg class $\mathcal{S}$ of $L$-functions of degree four (see A. Selberg [18] and the survey paper of Kaczorowski–Perelli [10]). An important feature, proved by G. Shimura [19] (see also A. Sankaranarayanan [16]) is

$$
Z(s) = \zeta(s) \sum_{n=1}^{\infty} b_n n^{-s} = \zeta(s) B(s),
$$

where $b_n = c_n / \zeta(n)$.
where $B(s)$ is holomorphic for $\sigma > 0$, $b_n \ll n^\varepsilon$ (in fact $\sum_{n \leq x} b_n^2 \leq x \log^A x$ holds, too). It also satisfies the functional equation

$$B(s)\Delta_1(s) = B(1-s)\Delta_1(1-s),$$

$$\Delta_1(s) = \pi^{-3s/2} \Gamma(\frac{1}{2} (s + \kappa)) \Gamma(\frac{1}{2} (s + \kappa + 1)),$$

and actually $B(s) \in \mathcal{S}$ with degree three. The decomposition (2.3) (the so-called ‘Shimura lift’) allows one to use, at least to some extent, results from the theory of $\zeta(s)$ in connection with $Z(s)$, and hence to derive results on $\Delta(x)$.

### 3. THE COMPLEX INTEGRATION APPROACH

A natural approach to the estimation of $\Delta(x)$, used by the author in [8], is to apply the classical complex integration technique. We shall briefly present this approach now. On using Perron’s inversion formula (see e.g., the Appendix of [3]), the residue theorem and the convexity bound $Z(s) \ll |t|^{2-2\sigma+\varepsilon}$ ($0 \leq \sigma \leq 1$, $|t| \geq 1$), it follows that

$$\Delta(x) = \frac{1}{2\pi i} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \frac{Z(s)}{s} x^s \, ds + O_x \left( x^{\varepsilon \left( \frac{1}{2} + \frac{x}{T} \right)} \right) \quad (1 \ll T \ll x).$$

If we suppose that

$$\int_X^{2X} |B(\frac{1}{2} + it)|^2 \, dt \ll_x X^{\theta+\varepsilon} \quad (\theta \geq 1),$$

and use the elementary fact (see [3] for the results on the moments of $|\zeta(\frac{1}{2} + it)|$) that

$$\int_X^{2X} |\zeta(\frac{1}{2} + it)|^2 \, dt \ll X \log X,$$

then from (2.3),(3.2),(3.3) and the Cauchy-Schwarz inequality for integrals we obtain

$$\int_X^{2X} |Z(\frac{1}{2} + it)| \, dt \ll_x X^{(1+\theta)/2+\varepsilon}.$$ 

Therefore (3.1) gives

$$\Delta(x) \ll_x x^\varepsilon x^{\theta/2 - 1/2 + xT^{-1}} \ll_x x^{\frac{\theta}{T}+\varepsilon}$$

with $T = x^{1/(\theta+1)}$. This was formulated in [8] as

**THEOREM A.** If $\theta$ is given by (3.2), then

$$\Delta(x) \ll_x x^{\frac{\theta}{T}+\varepsilon}.$$
To obtain a value for $\theta$, note that $B(s)$ belongs to the Selberg class of degree three, hence $B(\frac{1}{2} + it)$ in (3.2) can be written as a sum of two Dirichlet polynomials (e.g., by the reflection principle discussed in [3, Chapter 4]), each of length $\ll X^{3/2}$. Thus by the mean value theorem for Dirichlet polynomials (op. cit.) we have $
exists \theta \leq 3/2$ in (3.2). Hence (3.5) gives (with unimportant $\varepsilon$) the Rankin-Selberg bound $\Delta(x) \ll x^{3/5+\varepsilon}$. Clearly improvement will come from better values of $\theta$. Note that the best possible value of $\theta$ in (3.2) is $\theta = 1$, which follows from general results on Dirichlet series (see e.g., [3, Chapter 9]). It gives $1/2 + \varepsilon$ as the exponent in the Rankin-Selberg problem, which is the limit of the method (the conjectural exponent $3/8 + \varepsilon$, which is best possible, is out of reach; see the author’s work [4]). To attain this improvement one faces essentially the same problem as in proving the sixth moment for $|\zeta(\frac{1}{2} + it)|$, namely

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^6 \, dt \ll \varepsilon T^{1+\varepsilon},
$$

only this problem is even more difficult, because the arithmetic properties of the coefficients $b_n$ are even less known than the properties of the divisor coefficients $d_3(n) = \sum_{abc=\nu, a, b, c \in \mathbb{N}} 1$, generated by $\zeta^3(s)$. If we knew the analogue of the strongest sixth moment bound

$$
\int_0^T |\zeta(\frac{1}{2} + it)|^6 \, dt \ll T^{5/4} \log C T \quad (C > 0),
$$

namely the bound (3.2) with $\theta = 5/4$, then (3.1) would yield $\Delta(x) \ll x^{5/9+\varepsilon}$, improving substantially (1.2).

The essential difficulty in this problem may be seen indirectly by comparing it with the estimation of $\Delta_4(x)$, the error term in the asymptotic formula for the summatory function of $d_4(n) = \sum_{abcd=\nu, a, b, c, d \in \mathbb{N}} 1$. The generating function in this case is $\zeta^4(s)$. The problem analogous to the estimation of $\Delta(x)$ is to estimate $\Delta_4(x)$, given the product representation

$$
(3.6) \quad \sum_{n=1}^{\infty} d_4(n)n^{-s} = \zeta(s)G(s) = \zeta(s) \sum_{n=1}^{\infty} g(n)n^{-s} \quad (\sigma > 1)
$$

with $g(n) \ll n^\varepsilon$ and $G(s)$ of degree three in the Selberg class (with a pole of order three at $s = 1$). By the complex integration method one gets $\Delta_4(x) \ll x^{1/2+\varepsilon}$ (here ‘$\varepsilon$’ may be replaced by a log-factor) using the classical elementary bound $\int_0^T |\zeta(\frac{1}{2} + it)|^4 \, dt \ll T \log^4 T$. Curiously, this bound for $\Delta_4(x)$ has never been improved; exponential sum techniques seem to give a poor result here. However, if one knows only (3.6), then the situation is quite analogous to the Rankin–Selberg problem, and nothing better than the exponent $3/5$ seems obtainable. The bound $\Delta(x) \ll x^{1/2+\varepsilon}$ follows also directly from (3.1) if the Lindelöf hypothesis for $\zeta(s)$ (that $\zeta(\frac{1}{2} + it) \ll |t|^{\varepsilon}$) is assumed.
4. MEAN SQUARE OF THE RANKIN–SELBERG ZETA–FUNCTION

Let, for a given $\sigma \in \mathbb{R}$,

\[(4.1) \quad \mu(\sigma) = \limsup_{t \to \infty} \frac{\log |\zeta(\sigma + it)|}{\log t}\]

denote the Lindelöf function (the famous, hitherto unproved, Lindelöf conjecture for $\zeta(s)$ is that $\mu(\sigma) = 0$ for $\sigma \geq \frac{1}{2}$, or equivalently that $\zeta(\frac{1}{2} + it) \ll \varepsilon |t|^\varepsilon$). In [8] the author proved the following

**Theorem B.** If $\beta = 2/(5 - \mu(1/2))$, then for fixed $\sigma$ satisfying $\frac{1}{2} < \sigma \leq 1$ we have

\[(4.2) \quad \int_{1}^{T} |Z(\sigma + it)|^2 \, dt = T \sum_{n=1}^{\infty} c_n^2 n^{-2\sigma} + O_\varepsilon (T^{(2-2\sigma)/(1-\beta) + \varepsilon}).\]

This result is the sharpest one yet when $\sigma$ is close to 1. For $\sigma$ close to $\frac{1}{2}$ one cannot obtain an asymptotic formula, but only the upper bound (this is [7, eq. (9.27)])

\[(4.3) \quad \int_{T}^{2T} |Z(\sigma + it)|^2 \, dt \ll_\varepsilon T^{2\mu(1/2)(1-\sigma)+\varepsilon} (T + T^{3(1-\sigma)}) \quad (\frac{1}{2} \leq \sigma \leq 1).\]

The upper bound in (4.3) follows easily from (2.3) and the fact that, as already mentioned, $B(s) \in \mathcal{S}$ with degree three, so that $B(\frac{1}{2} + it)$ can be approximated by Dirichlet polynomials of length $\ll t^{3/2}$, and the mean value theorem for Dirichlet polynomials yields

\[
\int_{T}^{2T} |B(\sigma + it)|^2 \, dt \ll_\varepsilon T^{\varepsilon} (T + T^{3(1-\sigma)}) \quad (\frac{1}{2} \leq \sigma \leq 1).
\]

Note that with the sharpest known result (see M.N. Huxley [2]) $\mu(1/2) \leq 32/205$ we obtain $\beta = 410/961 = 0.426638917\ldots$. The limit is the value $\beta = 2/5$ if the Lindelöf hypothesis (that $\mu(\frac{1}{2}) = 0$) is true. Thus (4.2) provides a true asymptotic formula for

\[
\sigma > \frac{1 + \beta}{2} = \frac{1371}{1922} = 0.7133194\ldots.
\]

The proof of (4.2), given in [8], is based on the general method of the author’s paper [6], which contains a historic discussion on the formulas for the left-hand side of (4.2) (see also K. Matsumoto [12]).

We are able to improve (4.2) in the case when $\sigma = 1$. The result is contained in
THEOREM 1. We have

\begin{equation}
\int_1^T |Z(1+it)|^2 \, dt = T \sum_{n=1}^{\infty} c_n^2 n^{-2} + O_{\epsilon}(\log T)^{2+\epsilon}.
\end{equation}

**Proof.** For \( \sigma = \Re s > 1 \) and \( X \geq 2 \) we have

\begin{equation}
Z(s) = \sum_{n \leq X} c_n n^{-s} + \int_{X}^{\infty} x^{-s} \, d\left( \sum_{n \leq x} c_n \right) + \int_{X}^{\infty} \frac{CX^{1-s}}{s-1} \, dx - \int_{X}^{\infty} \Delta(x) x^{-s-1} \, dx.
\end{equation}

By using (1.2) it is seen that the last integral converges absolutely for \( \sigma = \Re s > \frac{3}{5} \), so that (4.5) provides the analytic continuation of \( Z(s) \) to this region.

Taking \( s = 1 + it, 1 \leq t \leq T, X = T^{10} \), it follows that

\begin{equation}
\int_1^T |Z(1+it)|^2 \, dt = \int_1^T \left\{ \left| \sum_{n \leq X} c_n n^{-1-it} \right|^2 - 2 C \Im \left( \sum_{n \leq X} \frac{c_n}{nt} \left( \frac{X}{n} \right)^{it} \right) \right\} \, dt + O(1).
\end{equation}

By the mean value theorem for Dirichlet polynomials we have

\begin{equation}
\int_1^T \left| \sum_{n \leq X} c_n n^{-1-it} \right|^2 \, dt = T \sum_{n \leq X} c_n^2 + O \left( \sum_{n \leq X} c_n^2 n^{-1} \right)
= T \sum_{n=1}^{\infty} c_n^2 + O_{\epsilon}(\log T)^{2+\epsilon},
\end{equation}

where we used the bound (see K. Matsumoto [12])

\begin{equation}
\sum_{n \leq x} c_n^2 \ll_{\epsilon} x(\log x)^{1+\epsilon}
\end{equation}

and partial summation. Finally we have

\begin{equation}
\sum_{n \leq X} \frac{c_n}{n} \int_1^{T} \frac{1}{t} \left( \frac{X}{n} \right)^{it} \, dt \ll \log \log T.
\end{equation}

To see that (4.8) holds, note first that for \( X - X/\log T \leq n \leq X \) the integral over \( t \) is trivially estimated as \( \ll \log T \), and the total contribution of such \( n \) is

\[ \ll \log T \sum_{X - X/\log T \leq n \leq X} \frac{c_n}{n} \, dx \ll 1 \]
on using (1.1)–(1.2). For the remaining \( n \) we note that the integral over \( t \) equals
\[
\left. \frac{(X/n)^i}{it \log(X/n)} \right|_1^T + \frac{1}{i \log(X/n)} \int_1^T \left( \frac{X}{n} \right)^i dt / T^2.
\]
The contribution of those \( n \) is, using (1.1)–(1.2) again and making the change of variable \( X/u = v \),
\[
\ll \sum_{1 \leq n \leq X - X/\log T} \frac{c_n}{n \log(X/n)} = \int_1^{X - X/\log T} \frac{1}{u \log(X/u)} \frac{C + \Delta(u)}{u} \frac{\Delta(u)}{u \log(X/u)} du + O(1)
\]
\[
\ll \int_1^{X - X/\log T} \frac{1}{u \log(X/u)} du + 1 = \int_1^{X} \frac{1}{\log(v)} du + 1
\]
\[
= \log \log X - \log \log(1 - 1/\log T) + 1 = \ll \log \log T,
\]
and (4.8) follows.

One can improve the error term in (4.4) to \( O(\log^2 T) \), which is the limit of the method. I am very grateful to Prof. Alberto Perelli, who has kindly indicated this to me. The argument is very briefly as follows. Note that the coefficients \( c_n^2 \) are essentially the tensor product of the \( c_n \)'s, and the \( c_n \) are essentially the tensor product of the \( a(n) \)'s; “essentially” means in this case that the corresponding \( L \)-functions differ at most by a “fudge factor”, i.e., a Dirichlet series converging absolutely for \( \sigma > 1/2 \) and non-vanishing at \( s = 1 \). In terms of \( L \)-functions, the tensor product of the \( a(n) \) (the coefficients of the tensor square \( L \)-function) corresponds to the product of \( \zeta(s) \) and the \( L \)-function of \( Sym^2 \) (Shimura’s lift). Moreover, Gelbart–Jacquet [1] have shown that \( Sym^2 \) is a cuspidal automorphic representation, so one can apply to the above product the general Rankin-Selberg theory to obtain “good properties” of the corresponding \( L \)-function. Since \( Sym^2 \) is irreducible, the \( L \)-function corresponding to \( c_n^2 \) has a double pole at \( s = 1 \) and a functional equation of Riemann type. It follows that the sum in (4.7) is asymptotic to \( Dx \log x \) for some \( D > 0 \), and the assertion follows by following the preceding argument.

In concluding this section, let it be mentioned that, using (4.5), it easily follows that \( Z(1 + it) \ll \log |t| (t \geq 2) \).

### 5. MEAN SQUARE OF \( \Delta(x; \xi) \)

In this section we shall consider mean square estimates for \( \Delta(x; \xi) \), defined by (1.3). Although we could consider the range \( \xi > 1 \) as well, for technical reasons we shall restrict ourselves to the range \( 0 \leq \xi \leq 1 \), which is the condition that will be assumed henceforth to hold. Let
\[
(5.1) \quad \beta_\xi := \inf \left\{ \beta \geq 0 : \int_1^x \Delta^2(x; \xi) dx \ll X^{1+2\beta} \right\}.
\]
The definition of $\beta_\xi$ is the natural analogue of the classical constants in mean square estimates for the generalized Dirichlet divisor problem (see [3, Chapter 13]). Our first result in this direction is

**Theorem 2.** We have

\[
3 - \frac{2\xi}{8} \leq \beta_\xi \leq \max \left( \frac{1 - \xi}{2}, \frac{3 - 2\xi}{8} \right) \quad (0 \leq \xi \leq 1).
\]

**Proof.** First of all, note that (5.2) implies that $\beta_\xi = (3 - 2\xi)/8$ for $\frac{1}{2} \leq \xi \leq 1$, so that in this interval the precise value of $\beta_\xi$ is determined. The main tool in our investigations is the explicit Voronoï type formula for $\Delta(x; \xi)$. This is

\[
\Delta(x; \xi) = V_\xi(x, N) + R_\xi(x, N),
\]

where, for $N \gg 1$,

\[
V_\xi(x, N) = (2\pi)^{-1 - \xi} x^{(3 - 2\xi)/8} \sum_{n \leq N} c_n n^{-(5 + 2\xi)/8} \cos \left( 8\pi (xn)^{1/4} + \frac{1}{2} \left( \frac{1}{2} - \xi \right) \pi \right),
\]

\[
R_\xi(x, N) \ll (xN)^{\varepsilon} \left( 1 + x^{(3 - \xi)/4} N^{-1 + \xi}/4 + (xN)^{(1 - \xi)/4} + x^{(1 - 2\xi)/8} \right).
\]

This follows from the work of U. Vorhauer [20] (for $\xi = 0$ this is also proved in [9]), specialized to the case when

\[
A = \frac{1}{(2\pi)^2}, \quad B = (2\pi)^4, \quad M = L = 2, \quad b_1 = b_2 = d_1 = d_2 = 1, \quad \beta_1 = \kappa - \frac{1}{2}, \quad b_1 = \frac{1}{2},
\]

\[
\delta_1 = \kappa - \frac{1}{4}, \quad \delta_2 = -\frac{1}{2}, \quad \gamma = 1, \quad p = B, \quad q = 4, \quad \lambda = 2, \quad \Lambda = -1, \quad C = (2\pi)^{-5/2}.
\]

In (5.3)–(5.4) we take $N = x$, so that $R_\xi(x, N) \ll x^{(1 - \xi)/2 + \varepsilon}$. Since $\frac{1 - \xi}{2} \leq \frac{3 - 2\xi}{8}$ for $\xi \geq \frac{1}{2}$, the lower bound in (5.2) follows by the method of [4]. For the upper bound we use $c_n \ll n^{\varepsilon}$ and note that ($e(z) = \exp(2\pi i z)$)

\[
\int_x^{2x} \left| \sum_{K < k \leq 2K} c_k k^{-(5 + 2\xi)/8} e(4(kx)^{1/4}) \right|^2 \, dx \ll X + \sum_{k_1 \neq k_2} c_{k_1} c_{k_2} (k_1 k_2)^{-(5 + 2\xi)/8} \int_x^{2x} e(4x^{1/4}(k_1^{1/4} - k_2^{1/4})) \, dx \ll \varepsilon X^{3/4 + \varepsilon} K^{-(5 + 2\xi)/4} \sum_{k_1 \neq k_2} \left| k_1^{1/4} - k_2^{1/4} \right|^{-1} \ll \varepsilon X^{3/4 + \varepsilon} K^{(1 - \xi)/2},
\]
where we used the first derivative test (cf. [3, Lemma 2.1]). Since $K \ll X$ and
\[
\int_X^{2X} \Delta^2(x; \xi) \, dx \ll \int_X^{2X} \lVert V_\xi(x, N) \rVert^2 \, dx + \int_X^{2X} \lVert R(x, N) \rVert^2 \, dx,
\]
it follows that
\[
\int_X^{2X} \Delta^2(x; \xi) \, dx \ll \varepsilon \int_X^{2X} \Delta^2(x; \xi) \, dx \ll \varepsilon X^{(7-2\xi)/4+\varepsilon} + X^{2-\xi+\varepsilon},
\]
which clearly proves the assertion.

Our last result is a bound for $\beta_\xi$, which improves on (5.2) when $\xi$ is small. This is

**Theorem 3.** We have
\[
(5.5) \quad \beta_\xi \leq \frac{2 - 2\xi}{5 - 2\mu(\frac{1}{2})} \left(0 \leq \xi \leq \frac{1}{6}(1 + 2\mu(\frac{1}{2}))\right).
\]

**Proof.** We start from
\[
(5.6) \quad \Delta(x; \xi) = \frac{1}{2\pi i} \lim_{t \to \infty} \int_{c-iT}^{c+iT} Z(s) \frac{x^s}{s+1} \, ds,
\]
where $0 < c = c(\xi) < 1$ is a suitable constant (see K. Matsumoto [13] for a detailed derivation of formulas analogous to (5.6)). By the Mellin inversion formula we have (see e.g., the Appendix of [3])
\[
Z(s)s^{-\xi-1} = \int_0^\infty \Delta(1/x; \xi)x^{s-1} \, dx \quad (\Re s = c).
\]

Hence by Parseval’s formula for Mellin transforms (op. cit.) we obtain, for $\beta_\xi < \sigma < 1$,
\[
(5.7) \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|Z(\sigma + it)|^2}{|\sigma + it|^{2\xi+2}} \, dt = \int_0^\infty \Delta^2(1/x; \xi)x^{2\sigma-1} \, dx
\]
\[
= \int_0^\infty \Delta^2(x; \xi)x^{-2\sigma-1} \, dx \gg X^{-2\sigma-1} \int_X^{2X} \Delta^2(x; \xi) \, dx.
\]

Therefore if the first integral converges for $\sigma = \sigma_0 + \varepsilon$, then (5.7) gives
\[
\int_X^{2X} \Delta^2(x; \xi) \, dx \ll X^{2\sigma+1},
\]
namely $\beta_\xi \leq \sigma_0$. The functional equation (2.2) and Stirling’s formula in the form
\[
|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-1/2}e^{-\pi|t|/2}(1 + O(|t|^{-1})) \quad (|t| \geq t_0 > 0)
\]
imply that

\[ Z(s) = \mathcal{X}(s)Z(1-s), \quad \mathcal{X}(\sigma + it) \approx |t|^{2-4\sigma} \quad (s = \sigma + it, 0 \leq \sigma \leq 1, |t| \geq 2). \]

Thus it follows on using (4.3) that

\[
\int_T^{2T} |Z(\sigma + it)|^2 \, dt \ll T^{4-8\sigma} \int_T^{2T} |Z(1-\sigma + it)|^2 \, dt \ll \varepsilon T^{4-8\sigma + 2\mu(\frac{1}{2})\sigma + \max(1,3\sigma) + \varepsilon}.
\]

But we have

\[
4 - 8\sigma + 2\mu(\frac{1}{2})\sigma + \max(1,3\sigma) = 4 - 5\sigma + 2\mu(\frac{1}{2})\sigma < 2\xi + 2
\]

for

\[ \sigma > \sigma_0 = \frac{2 - 2\xi}{5 - 2\mu(\frac{1}{2})}, \]

provided that \( \sigma_0 \geq 1/3 \), which occurs if \( 0 \leq \xi \leq \frac{1}{5}(1+2\mu(\frac{1}{2})) \). Thus the first integral in (5.7) converges if (5.9) holds, and Theorem 3 is proved. Note that this result is a generalization of Theorem 7 in [8], which says that

\[ \beta_0 \leq \frac{2 - 2\xi}{5 - 2\mu(\frac{1}{2})}. \]

In the case when \( \beta_\xi = (3 - 2\xi)/8 \) we could actually derive an asymptotic formula for the integral of the mean square of \( \Delta(x; \xi) \), much in the same way that this was done in [9] for the square of

\[
\Delta_1(x) := \int_0^x \Delta(u) \, du,
\]

where it was shown that

\[ \int_1^X \Delta_1^2(x) \, dx = DX^{13/4} + O_\varepsilon(X^{3+\varepsilon}) \]

with explicit \( D > 0 \) (in [12] the error term was improved to \( O_\varepsilon(X^3(\log X)^{3+\varepsilon}) \)). In the case of \( \Delta(x; 1) \) the formula (5.10) may be used directly, since

\[ \frac{1}{x} \Delta_1(x) = \frac{1}{x} \int_0^x \Delta(u) \, du = \Delta(x; 1) + O_\varepsilon(x^\varepsilon). \]

To see that (5.11) holds, note that with \( c = 1 - \varepsilon \) we have

\[
\Delta(x; 1) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(s) \frac{x^s}{s^2} \, ds
\]

\[
= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(s) \frac{x^s}{s(s+1)} \, ds + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(s) \frac{x^s}{s^2} \, ds
\]

\[
= \frac{1}{x} \Delta(u) \, du + \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(s) \frac{x^s}{s^2(s+1)} \, ds
\]

\[
= \frac{1}{x} \Delta_1(x) + O_\varepsilon(x^\varepsilon),
\]
on applying (5.8) to the last integral above.

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