Spatial holeburning effects in the amplified spontaneous emission spectra of the non-lasing supermode in semiconductor laser arrays

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Abstract

The amplified spontaneous emission spectrum of the light field in the non-lasing supermode of two coupled semiconductor lasers is analyzed using linearized Langevin equations. It is shown that the interference between the laser mode and the fluctuating light field in the non-lasing mode causes spatial holeburning. This effect introduces a phase sensitive coupling between the laser field and the fluctuations of the non-lasing mode. For high laser fields, this coupling splits the spectrum of the non-lasing mode into a triplet consisting of two relaxation oscillation sidebands which are in phase with the laser light and a center line at the lasing frequency with a phase shift of $\pm \pi/2$ relative to the laser light. As the laser intensity is increased close to threshold, the spectrum shows a continuous transition from the single amplified spontaneous emission line at the frequency of the non-lasing mode to the triplet structure. An analytical expression for this transition is derived and typical features are discussed.
I. INTRODUCTION

The carrier density dynamics of semiconductor lasers couple not only to the lasing modes, but also modify the amplified spontaneous emission into the non-lasing modes. It is therefore possible to gain interesting insights into holeburning effects by studying the light field fluctuations in the non-lasing modes [1–4]. The most simple system which can exhibit holeburning effects is a laser array in which each of the individual lasers can be considered to operate in a single mode which is coupled to the local carrier density of that individual laser. A rate equation description for this type of device was developed by Winful and Wang [5,6] and has been discussed in a more general context by the authors [4]. In the following, this model will be referred to as the split density model since the characteristic feature of the model is the separation of the carrier density into distinct density pools associated with the individual lasers in the array. Originally, the split density model was intended to describe instabilities in semiconductor laser arrays. Indeed, the model for a two laser array exhibits a region of instability with a limit cycle [5]. However, the model can also describe the stable anti-phase locking between neighboring lasers in an array. This type of anti-phase locking is commonly observed in vertical cavity surface emitting laser arrays [7–10]. In particular, the presence of carrier diffusion leading to a stochastic exchange of carriers between the lasers was found to enhance the stability of the anti-phase locking considerably [4]. In the case of stable anti-phase locked operation, the special features of the model are still present in the dynamics of the quantum fluctuations causing amplified spontaneous emission in the non-lasing modes of the array. Specifically, the spatial holeburning represented by the carrier density difference between individual lasers is part of the fluctuation dynamics, modifying the amplified spontaneous emission spectrum in the non-lasing modes according to this coupling between the light field and the carrier density dynamics [11].

Experimentally, the symmetric non-lasing mode and the anti-symmetric lasing mode can be distinguished in the far field pattern. In particular, the spectrum of the non-lasing mode can be obtained in the center of the double-lobed far field where the anti-symmetric lasing mode vanishes due to destructive interference. At this very point, the non-lasing symmetric mode has an intensity maximum as a result of constructive interference. Assuming the lasing mode to be much stronger than the non-lasing mode, it should be possible to locate the center of the far field at the point where between the two maxima at the edges the total intensity is at its (local) minimum. If the far-field is observed off-center, a superposition of the laser spectrum and the spectrum of the non-lasing mode should be obtained. In this case it should be possible to distinguish the narrow laser line from the broad spectrum of the non-lasing mode. Spectral features of the fluctuations in the lasing mode such as relaxation oscillation sidebands should not be significant until the lasing mode intensity is much higher than the intensity of the non-lasing mode. Interference between the laser line and the fluctuations in the non-lasing mode are not to be expected since constructive and destructive interference are equally likely and consequently average out.

In this paper we derive a general expression for the spectrum of the non-lasing mode in a two laser array. In section II, we introduce the split density model for two lasers operating in the antisymmetric supermode. The Langevin equation for the light field in the non-lasing mode is derived and the effects of spatial holeburning on the fluctuation dynamics are discussed. In section III, the Langevin equation is solved and the analytic expression
of the spectrum is derived. General features of the spectrum are discussed. In section IV, the spectra of four typical cases are presented, illustrating the different spatial holeburning features which may arise in different types of semiconductor array devices.

II. SPLIT DENSITY MODEL OF A TWO LASER ARRAY

A. Rate equations

Generally a semiconductor laser array can exhibit a large range of dynamical effects involving far more than just two light field modes and two spatially separated carrier systems. However, the region of stable anti-phase locked operation relevant for at least some of the experimental realizations [7–10] can be described realistically by a split density model limited to only two light field modes and two carrier densities. The consistency of such a description with a more detailed model based on partial differential equations has been established in [12]. We will use a version of the model which includes both the possibility of carrier diffusion and of a difference in the cavity loss rate between the symmetric and the antisymmetric supermodes. The rate equations read

\[
\begin{align*}
\frac{d}{dt}E_1 &= \frac{w}{2}N_1(1 - i\alpha)E_1 - (\bar{\kappa} + i\bar{\omega})E_1 - \left(\frac{s}{2} - i\frac{\Omega}{2}\right)E_2 \\
\frac{d}{dt}N_1 &= \frac{\mu}{2} - \gamma N_1 - \Gamma(N_1 - N_2) - 2wE_1^*E_1N_1 \\
\frac{d}{dt}E_2 &= \frac{w}{2}N_2(1 - i\alpha)E_2 - (\bar{\kappa} + i\bar{\omega})E_2 - \left(\frac{s}{2} - i\frac{\Omega}{2}\right)E_1 \\
\frac{d}{dt}N_2 &= \frac{\mu}{2} - \gamma N_2 - \Gamma(N_2 - N_1) - 2wE_2^*E_2N_2,
\end{align*}
\]

where \(E_1, E_2\) are the field amplitudes of the individual lasers, normalized so that the intensities \(E_1^*E_1, E_2^*E_2\) correspond to photon numbers in the cavity and \(N_1, N_2\) are the carrier densities above transparency in the active media of laser one and two, respectively, each normalized to represent the actual number of carriers in the laser. The total carrier injection rate \(\mu\) is split equally between the two carrier pools. The linear gain coefficient \(w\) is then equal to the spontaneous emission rate into the laser mode. The rate of carrier losses due to electron-hole recombinations by spontaneous emission into modes not confined to the cavity is given by \(\gamma\). The properties of the individual laser cavities are given by the frequency at transparency \(\bar{\omega}\) and the cavity loss rate \(\bar{\kappa}\). The carrier density dependent frequency change is represented by the linewidth enhancement factor \(\alpha\).

The coupling between the two lasers is described in terms of the parameters \(\Gamma, s\) and \(\Omega\). The carrier diffusion rate \(\Gamma\) represents diffusive carrier exchange between the two lasers which tends to equilibrate the carrier densities in the two regions of the gain medium. \(\Gamma\) may be derived from the ambipolar diffusion constant of the carriers, \(D_{\text{diff}}\), and the distance \(r\) between the two lasers, resulting in

\[
\Gamma = 4\pi^2 \frac{D_{\text{diff}}}{r^2}.
\]

Note that the consideration of carrier diffusion effects represents an extension of the original model of Winful and Wang [13]. The dissipative optical coupling term \(s\) represents
additional optical losses in the region between the two lasers, where the two field modes interfere. Constructive interference \( E_1 \) has the same phase as \( E_2 \) thus leads to increased losses, while destructive interference (phase difference of \( \pi \) and, therefore, opposite sign) leads to reduced losses. This coupling mechanism is itself sufficient to explain anti-phase locking [2,7]. The particular physical explanation associated with this term in a real laser device depends on the type of waveguiding mechanisms used to separate the two lasers. In gain guided twin stripe arrays, the additional losses are due to internal absorption in the region of negative gain between the lasers. In vertical cavity surface emitting laser arrays fabricated by destroying the Bragg mirrors between the lasers [7], this term represents the increased light emission from the cavity in that region. The coherent coupling of the two localized field modes by optical diffraction is represented by \( \Omega \), the frequency difference between the symmetric and the anti-symmetric supermodes in the cavity at \( N_1 = N_2 \).

Since this model represents a perfectly symmetric array of two lasers, it is useful to express the dynamics in terms of the symmetric and the anti-symmetric supermode, \( E_+ = 1/\sqrt{2}(E_1 + E_2) \) and \( E_- = 1/\sqrt{2}(E_1 - E_2) \) respectively. Also, the carrier densities can be expressed in terms of the total carrier density \( N = N_1 + N_2 \) and the carrier density difference \( \Delta = N_1 - N_2 \). The equations then read

\[
\begin{align*}
\frac{d}{dt} E_+ &= \frac{w}{2} N (1 - i\alpha) E_+ - (\kappa_+ + i\omega_+) E_+ + \frac{w}{2} \Delta (1 - i\alpha) E_- \\
\frac{d}{dt} E_- &= \frac{w}{2} N (1 - i\alpha) E_- - (\kappa_- + i\omega_-) E_- + \frac{w}{2} \Delta (1 - i\alpha) E_+ \\
\frac{d}{dt} N &= \mu - \gamma N - w(E_+^* E_+ + E_-^* E_-) N - w(E_+^* E_- + E_-^* E_+) \Delta \\
\frac{d}{dt} \Delta &= -(\gamma + 2\Gamma) \Delta - w(E_+^* E_+ + E_-^* E_-) \Delta - w(E_+^* E_- + E_-^* E_+) N,
\end{align*}
\]

where \( \kappa_\pm = \bar{\kappa} \pm s/2 \) and \( \omega_\pm = \bar{\omega} \pm \Omega/2 \). In this formulation, the rate equations of the two laser array appear as rate equations of a single two mode laser with the additional variable \( \Delta \) describing the spatial holeburning.

\[\text{B. Fluctuation dynamics}\]

If the effects of spatial holeburning are neglected by omitting all terms which depend on \( \Delta \), the stable solution is given by \( E_+ = 0 \), since \( \kappa_+ > \kappa_- \). Therefore, only the anti-symmetric supermode will show laser activity. In the presence of a non-vanishing spatial holeburning variable \( \Delta \), the stability analysis becomes more involved, as will be discussed below. However, there is always a stationary solution with

\[
\begin{align*}
\Delta &= 0 \\
N &= 2\kappa_-/w \\
E_+ &= 0 \\
E_- &= \sqrt{\frac{\mu}{4\kappa_-} - \frac{\gamma}{2w}} \exp[-i(\omega_- + \alpha\kappa_-)t] \\
&= \sqrt{I_0} \exp[-i(\omega_- + \alpha\kappa_-)t],
\end{align*}
\]
where $I_0$ is the total laser intensity in terms of the photon number in the anti-symmetric supermode. The linearized dynamics of small fluctuations around this stationary solution may be separated into the relaxation oscillations of the laser mode and the fluctuations in the non-lasing mode and the holeburning parameter. This independence of the linear dynamics in $E_+$ and $\Delta$ from fluctuations in $E_-$ and $N$ is a consequence of the fact that $E_- = \Delta = 0$ is stationary for all values of $E_+$ and $N$. Only the relaxation and oscillation rates of the small fluctuations in $E_+$ and $\Delta$ are altered by $E_+$ and $N$. Therefore, the fluctuations in $E_+$ and $N$ appear only in the quadratic terms of the fluctuation dynamics, as products with $\Delta$ and $N$, respectively.

While the linearized fluctuations in the lasing mode describe the unaltered relaxation oscillations of a single mode laser, the fluctuations in the non-lasing mode are described by

$$\frac{d}{dt}E_+ = -(\kappa_+ - \kappa_- + i\omega_+ + i\alpha\kappa_-)E_+ + \frac{w}{2} \Delta (1 - i\alpha) \sqrt{I_0} e^{-i(\omega_- + \alpha\kappa_-)t}$$

(5a)

$$\frac{d}{dt}\Delta = - (\gamma + 2\Gamma + wI_0)\Delta - 2\kappa_- \sqrt{I_0}(e^{-i(\omega_- + \alpha\kappa_-)t}E_+^* + e^{+i(\omega_- + \alpha\kappa_-)t}E_+).$$

(5b)

Note that the fluctuation dynamics can be written using $\delta E_+ = E_+$ and $\delta \Delta = \Delta$ because the averages of the non-lasing field mode $E_+$ and the holeburning parameter $\Delta$ are zero.

Both the amplitude $\sqrt{I_0}$ and the phase $-(\omega_- + \alpha\kappa_-)t$ of the average laser field appear in the fluctuation dynamics, representing the hole burning effects of the interference term between the weak non-lasing mode and the laser field. Thus the fluctuation dynamics are not independent of the phase of $E_+$. The phase sensitivity can be represented in the fluctuation dynamics by expressing the complex field amplitude $E_+$ in terms of the component $f_\parallel$ in phase with the laser field and the component $f_\perp$ which is $\pi/2$ out of phase with the laser field, i.e.

$$E_+ = (f_\parallel - if_\perp) \exp[-i(\omega_- + \alpha\kappa_-)t].$$

(6)

The fluctuation dynamics can now be rewritten as a linear matrix equation in a real three dimensional space. Adding the quantum noise terms $Q_\perp$ and $Q_\parallel$, one obtains the linearized Langevin equation of the fluctuations in the non-lasing mode,

$$\frac{d}{dt} \begin{pmatrix} f_\parallel \\ f_\perp \\ \Delta \end{pmatrix} = \begin{pmatrix} -s & +\Omega & w\sqrt{I_0}/2 \\ -\Omega & -s & \alpha w\sqrt{I_0}/2 \\ -4\kappa_- \sqrt{I_0} & 0 & -\gamma - 2\Gamma - wI_0 \end{pmatrix} \begin{pmatrix} f_\parallel \\ f_\perp \\ \Delta \end{pmatrix} + \begin{pmatrix} Q_\parallel \\ Q_\perp \\ 0 \end{pmatrix},$$

(7)

The quantum noise terms include quantum fluctuations of the light field vacuum entering the cavity and the dipole fluctuations of the gain medium. Here we have neglected the shot noise terms acting on the carrier density difference $\Delta$. This assumption can be well justified in the case of $\bar{\kappa} > \gamma$, where within an order of magnitude of the threshold current the total number of carriers in the gain medium is much larger than the number of photons in the cavity. The relative shot noise in the carrier system will therefore be much lower than the photonic shot noise in the light field system. Note that the presence of relaxation oscillations is an indicator that this assumption is valid. The limit of its validity is reached when the stimulated emission overdamps the relaxation oscillations.

The quantitative expression for the optical quantum noise terms $Q_\parallel$ and $Q_\perp$ may be derived by comparing the corresponding intensity noise with the expected shot noise for
the quantized processes. The vacuum noise entering the cavity thus corresponds to the quantization of photons emitted from the cavity and the dipole noise in the gain medium corresponds to the quantization of photon emission from the gain medium into the cavity. Note that reabsorption of photons due to incomplete inversion may add extra noise. In the following, however, we assume perfect inversion in the gain medium by using the minimum noise terms required by quantum mechanics,

\[ \langle Q_\parallel(t)Q_\parallel(t+\tau) \rangle = \langle Q_\perp(t)Q_\perp(t+\tau) \rangle = \kappa_\perp \delta(\tau). \] (8)

The fluctuation dynamics described by equation (8) depends on two relaxation times and two oscillation frequencies. In the case of weak laser light \( (I_0 \approx 0) \), the light field dynamics are governed by the frequency \( \Omega \) representing the frequency difference between the supermodes and the damping coefficient \( s \) representing the loss rate difference between the supermodes. The interference effects between the laser field and the light field fluctuations in the non-lasing mode couple the holeburning parameter \( \Delta \) to the field dynamics of the non-lasing mode. The total coupling strength is given by the relaxation oscillation frequency \( \sqrt{2\kappa_\perp w_0} \). The damping of the holeburning parameter \( \Delta \) is given by the sum of the spontaneous emission rate \( \gamma \), the Diffusion rate \( 2\Gamma \) and the rate of stimulated emission \( w_0 \).

To illustrate the physical processes involved in the coupling between the spatial holeburning parameter \( \Delta \) and the field fluctuations in the non-lasing mode, it is useful to examine the real time dynamics of the fluctuations in \( f_\parallel \) and \( f_\perp \) for large intensities \( I_0 \). If \( \sqrt{2\kappa_\perp w_0} \gg \Omega \), relaxation oscillations dominate the dynamics of the fluctuations. However, since the holeburning is only caused by the field fluctuation \( f_\parallel \) which are in phase with the laser field, any fluctuations in \( f_\perp \) will not cause oscillations. Instead, \( f_\perp \) relaxes exponentially with

\[ \Delta \approx f_\parallel \approx 0 \]

\[ f_\perp \approx f_0 e^{-(s+\alpha\Omega)t}. \] (9)

The effect of the frequency difference \( \Omega \) is suppressed by the fast relaxation oscillations and would not have any influence on the dynamics if it were not for the phenomena expressed by the linewidth enhancement factor \( \alpha \). The \( \alpha \) factor describes waveguiding properties induced by spatial holeburning. If a carrier density difference \( \Delta \) exists between the two cavities, \( \alpha \) induces a change in the relative phase between the local fields \( E_1 \) and \( E_2 \). In terms of the supermodes, this means that the components of \( E_- \) and \( E_+ \) which are out of phase by \( \pi/2 \) are coupled. Therefore, \( \Delta \) causes a change in \( f_\perp \) via \( \alpha \) by coherently converting part of the laser light amplitude into the non-lasing mode. In the presence of a strong laser field, even the negligibly small holeburning caused as \( \Omega \) rotates \( f_\perp \) into \( f_\parallel \) is sufficient to diminish \( f_\perp \) by destructive interference with the light coherently transferred from the laser mode.

In the relaxation oscillations caused by the spatial holeburning effect of interference between the laser light and in-phase fluctuations in the non-lasing modes, the \( \alpha \) factor produces an out-of-phase component of \( \alpha \) times the in-phase fluctuation, i.e. \( f_\perp = \alpha f_\parallel \). Also, some of the field in \( f_\perp \) is rotated into \( f_\parallel \) by the frequency difference \( \Omega \) to cause an effective undamping of the oscillations. Note that this is the exact counterpart of the additional damping in the exponential decay of \( f_\perp \). The real time evolution of the relaxation oscillations reads
f_\parallel \approx f_0 \cos(\sqrt{2\kappa_w}I_0 t)e^{-(\gamma + 2\Gamma + wI_0 + s - \alpha\Omega)t/2}
\frac{f_\perp}{f_\parallel} \approx \alpha f_0 \cos(\sqrt{2\kappa_w}I_0 t)e^{-(\gamma + 2\Gamma + wI_0 + s - \alpha\Omega)t/2}
\Delta \approx -2\sqrt{\frac{2\kappa_w}{w}} f_0 \sin(\sqrt{2\kappa_w}I_0 t)e^{-(\gamma + 2\Gamma + wI_0 + s - \alpha\Omega)t/2}.

(10)

If the dynamics are dominated by the relaxation oscillations, the exponentially damped dynamics of \( f_\perp \) should produce a Lorentzian line at the laser frequency with a width of \( 2\sqrt{s + 2\alpha\Omega} \) in the spectrum of the non-lasing mode, while the relaxation oscillation dynamics should form two sidebands shifted from the laser line by the relaxation oscillation frequency and having a width of \( \gamma + 2\Gamma + wI_0 + s - \alpha\Omega \). If the \( \alpha \) factor was zero, these sidebands would be fully symmetric. The main effect of \( \alpha \) is to correlate the fluctuations in \( f_\parallel \) with the fluctuations in \( f_\perp \) as expressed in equation (10). If the relaxation starts at \( f_\parallel = f_0 \) and \( f_\perp = 0 \), the total relaxation dynamics are given by

\[
f_\parallel(t) = \cos(\sqrt{2\kappa_w}I_0 t)e^{-(\gamma + 2\Gamma + wI_0 + s - \alpha\Omega)t/2}
f_\perp(t) = \alpha \left( \cos(\sqrt{2\kappa_w}I_0 t)e^{-(\gamma + 2\Gamma + wI_0 + s - \alpha\Omega)t/2} - e^{-(s + \alpha\Omega)} \right).
\]

(11)

Figure 1 shows this relaxation trajectory for a typical choice of laser parameters. The question of whether this trajectory contributes to a high frequency line or to a low frequency line depends on its curvature. The parts of the trajectory curved left rotate counterclockwise and contribute mainly to the low frequency line, while those parts curved right rotate clockwise and therefore contribute mainly to the high frequency line. Note that the sections of the trajectory contributing mainly to the low frequency sideband are significantly longer than the sections contributing mainly to the high frequency sideband. It is therefore to be expected that the high frequency sideband is lower than the low frequency sideband because of the \( \alpha \) factor correlation between the relaxation oscillations along \( f_\perp = \alpha f_\parallel \) and the exponential relaxation along \( f_\parallel = 0 \).

If \( \alpha\Omega \) is larger than \( \gamma + 2\Gamma + s \) the relaxation oscillations may become undamped for intensities \( wI_0 < \alpha\Omega - \gamma - 2\Gamma - s \). However, as the relaxation oscillations are slower at low intensities \( wI_0 \), this effect does not necessarily occur at \( wI_0 = \alpha\Omega - \gamma - 2\Gamma - s \) as predicted by equation (11). To find the correct stability boundaries, we will in the following examine the stability condition of the complete fluctuation dynamics given by \( S \).

C. Stability Analysis

The stability boundaries associated with equation (11) can be determined from the matrix

\[
S = \begin{pmatrix}
s & -\Omega & -w\sqrt{I_0}/2 \\
\Omega & s & -\alpha w\sqrt{I_0}/2 \\
4\kappa_w\sqrt{I_0} & 0 & \gamma + 2\Gamma + wI_0
\end{pmatrix}
\]

(12)

describing the linearized dynamics. At the stability boundaries, \( S \) must have at least one eigenvalue with a real part of zero. This condition is satisfied if there exists a frequency \( \omega \) with
\[
\det\{S - i\omega\} = 0. \tag{13}
\]

The stability boundary for non-zero \(\omega\) corresponding to the undamping of relaxation oscillations by \(\alpha\Omega\) can be found by separating the real and imaginary part of equation (13) and eliminating \(\omega\). The condition for stability is then found to be

\[
\alpha\Omega < s + \gamma + 2\Gamma + wI_0 + \frac{s}{\kappa_- wI_0} \left(\Omega^2 + (s + \gamma + 2\Gamma + wI_0)^2\right). \tag{14}
\]

For the case of \(s = 0\) equation (14) reduces to the expressions derived in previous studies [4,5]. Since an instability is most likely to occur at low intensities \(I_0\), however, a closer examination of the stability boundary is of interest in the present context. In particular, it is possible to determine the condition for no instability at any laser intensity,

\[
\gamma + 2\Gamma + s > \alpha\Omega\left(1 + 2\frac{s}{\kappa_-} - 2\sqrt{\frac{s}{\kappa_-}(1 + \frac{s}{\kappa_-})(1 + \frac{1}{\alpha^2})}\right). \tag{15}
\]

If the device parameters satisfy this condition, the anti-phase locked operation of the two laser array is stable over the whole operating range described by the two density model.

\section*{III. SPECTRUM OF THE NON-LASING SUPERMODE}

\subsection*{A. General formula}

The Langevin equation (7) can be solved analytically in the frequency regime by determining the Greensfunction corresponding to the dynamical matrix \(S\) and applying it to the white noise input. The Greensfunction is obtained by inverting the matrix \(S + i\omega\). In the two dimensional subspace describing only the optical field components \(f_\parallel\) and \(f_\perp\), the Greensfunction of the two density model is

\[
G(\omega) = \frac{1}{(s + i\omega)^2 + \Omega^2 + M(\alpha\Omega + s + i\omega)} \begin{pmatrix} s + i\omega & \Omega \\ -\Omega - \alpha M & s + i\omega + M \end{pmatrix}
\]

with

\[
M = \frac{2\kappa_- wI_0}{\gamma + 2\Gamma + wI_0 + i\omega}. \tag{16}
\]

Note that \(M\) is a complex function of \(\omega\) and \(I_0\) which includes all parameters associated with the carrier dynamics. For \(M = 0\), the Greensfunction is that of a cavity mode with a frequency of \(-\Omega\) relative to the laser line and a damping rate of \(s\). \(M\) increases with increasing laser intensity \(I_0\), describing the transition from the cavity resonance to relaxation oscillations.

Since the noise input given by \(Q_\parallel\) and \(Q_\perp\) is white noise of intensity \(\kappa_-\) in both components, the resulting fluctuations are

\[
\begin{pmatrix} \langle |f_\parallel(\omega)|^2 \rangle & \langle f_\perp^*(\omega)f_\parallel(\omega) \rangle \\ \langle f_\parallel(\omega)^*f_\perp(\omega) \rangle & \langle |f_\perp(\omega)|^2 \rangle \end{pmatrix} = \frac{\kappa_-}{2\pi} G(\omega) G^\dagger(\omega). \tag{17}
\]
The spectrum is given by the average intensities of the field as a function of the frequency \( \omega \). It can therefore be determined from the fluctuations in \( f_\parallel \) and \( f_\perp \) using

\[
I_\pm(\omega) = \langle E_\pm^*(\omega)E_\pm(\omega) \rangle = \langle |f_\parallel(\omega)|^2 \rangle + \langle |f_\perp(\omega)|^2 \rangle + i\langle f_\parallel^*(\omega)f_\perp(\omega) \rangle - i\langle f_\perp^*(\omega)f_\parallel(\omega) \rangle.
\]

The spectrum of the non-lasing symmetric supermode \( I_+(\omega) \) for any given set of cavity parameters \( \Omega, s \) and any complex holeburning function \( M \) is then given by the general formula

\[
I_+(\omega) = \frac{\kappa_-}{2\pi} \left( \frac{1}{s^2 + (\omega - \Omega)^2} + (1 - i\alpha)(s - i(\omega - \Omega))M + (1 + i\alpha)(s + i(\omega - \Omega))M^* + (1 + \alpha^2)M^*M \right).
\]

### B. General properties of the spectrum

To understand the physical processes associated with the spectrum described by the general formula given in equation (19), it is helpful to discuss the two limiting cases given by \( M = 0 \) and \( M \to \infty \). For \( M = 0 \), equation (19) describes the amplified spontaneous emission spectrum expected without spatial holeburning at a total carrier density pinned to \( N = 2\kappa_-/w \),

\[
I_+(\omega)_{M=0} = \frac{\kappa_-}{\pi} \left( \frac{1}{s^2 + (\omega + \Omega)^2} \right).
\]

Here, the total photon number in the non-lasing cavity mode is given by \( \kappa_-/s \). In this case, no phase relation exists between the non-lasing mode and the lasing mode. In the opposite limit, i.e. \( M \to \infty \), only the component of amplified spontaneous emission out of phase with the laser mode remains undamped by the high stimulated emission rate and the spectrum shows a single line at the laser frequency \( \omega = 0 \),

\[
I_+(\omega)_{M\to\infty} = \frac{\kappa_-}{2\pi} \left( \frac{1 + \alpha^2}{(s + \alpha\Omega)^2 + \omega^2} \right).
\]

For \( M \to \infty \) the total photon number in the non-lasing supermode corresponding to this spectrum is \( \kappa_-(1 + \alpha^2)/2(s + \alpha\Omega) \). For \( \alpha = 0 \), this intensity is one half of the intensity at \( M = 0 \), reflecting the suppression of out-of-phase emissions. The linewidth enhancement factor \( \alpha \) changes this simple situation however. It enhances the noise effect by a factor of \( 1 + \alpha^2 \). This is the same mechanism which enhances the phase noise in the laser light and therefore increases the linewidth of the laser line. Moreover, \( \alpha \) introduces an additional damping effect of \( \alpha\Omega \), which makes this line wider than the one for \( M = 0 \) and reduces the total photon number accordingly.

The typical spectra pertaining to values of the complex rate \( M \) between these two extremes are triplet spectra. The case described by equations (10) features not only the centerline corresponding to \( M \to \infty \), but also a pair of relaxation oscillation sidebands.
corresponding to the in-phase component of amplified spontaneous emission. In the limit of fast relaxation oscillations assumed in equations (10), the sidebands are centered around the relaxation oscillation frequency \( \omega_R = \pm \sqrt{2} \kappa w_I_0 \). Therefore, the sidebands can be derived from equation (19) by approximating the spectrum for large \( \pm \omega_R \) in the vicinity of \( \omega = \omega_R \). The result of this approximation, including the lowest order asymmetry between the sidebands, reads

\[
I_+(\omega) \approx \frac{\kappa_-}{2\pi} \left( \frac{1 + \alpha^2}{(s + \alpha \Omega)^2 + \omega^2} \right) + \frac{\kappa_-}{2\pi} \left( \frac{1 + \alpha^2 + 2(\Omega + \alpha(s + \gamma + 2\Gamma + wI_0))/\sqrt{2\kappa - wI_0}}{(\gamma + 2\Gamma + wI_0 + s - \alpha \Omega)^2 + 4(\omega + \sqrt{2\kappa - wI_0})^2} \right) + \frac{\kappa_-}{2\pi} \left( \frac{1 + \alpha^2 - 2(\Omega + \alpha(s + \gamma + 2\Gamma + wI_0))/\sqrt{2\kappa - wI_0}}{(\gamma + 2\Gamma + wI_0 + s - \alpha \Omega)^2 + 4(\omega - \sqrt{2\kappa - wI_0})^2} \right).
\] (22)

This triplet spectrum can be characterized by the total photon numbers in the three lines,

\[
I_{\text{centerline}} = \frac{(1 + \alpha^2)\kappa_-}{2(s + \alpha \Omega)}
\]

\[
I_{\text{sidebands}} = \frac{(1 + \alpha^2)\kappa_-}{4(\gamma + 2\Gamma + wI_0 + s - \alpha \Omega)} \left( 1 \pm \frac{2\Omega + 2\alpha(s + \gamma + 2\Gamma + wI_0)}{(1 + \alpha^2)\sqrt{2\kappa - wI_0}} \right).
\] (23)

The sidebands are clearly damped by the stimulated emission into the laser line, \( wI_0 \), as well as by spontaneous recombinations and carrier diffusion expressed by \( \gamma + 2\Gamma \). The increase in the stimulated emission rate is responsible for the suppression of the sidebands with increasing laser intensity. Note that the undamping of relaxation oscillations discussed in section II also appears in the total intensity of the sidebands, possibly raising the sideband intensities far above the intensity of the centerline at low laser intensities.

The asymmetry of the sidebands can be traced back to two separate contributions, one proportional to the frequency difference between the modes \( \Omega \) and one proportional to \( \alpha(s + \gamma + 2\Gamma + wI_0) \). In both cases, the low frequency sideband intensity is greater than the high frequency sideband intensity. The first term can be understood as a remnant of the cavity resonance of the non-lasing supermode in the absence of spatial holeburning. The low frequency line continuously evolves from the line at \( \omega = -\Omega \) for \( M = 0 \) while the high frequency line only emerges as the laser intensity is increased. However, the low frequency line is also stronger for \( \Omega = 0 \). This is the effect explained in section II.B which is caused by the \( \alpha \) factor. Within the approximation assumed here, the sideband intensity ratio corresponding to the parameters given in Figure 1 is about two to one.

While the general considerations given above already cover the qualitative effects which can be expected in the amplified spontaneous emission spectrum, the quantitative changes in the spectrum as spatial holeburning effects get stronger with increasing laser intensity are best illustrated and discussed on spectra pertaining to typical examples of device parameters. In the following, we will therefore discuss four cases, with either strong centerlines (\( \Omega < s \)) or weak centerline (\( \Omega > s \)) and either low carrier diffusion (\( \gamma + 2\Gamma \ll \kappa_- \)) or high carrier diffusion (\( \gamma + 2\Gamma = \kappa_-/2 \)).
IV. EXAMPLES OF SPECTRA FOR TYPICAL DEVICE PARAMETERS

A. Diffusion rates and array size

In the following we will discuss spectra assuming diffusion rates of $\Gamma \approx 5$ GHz to $\Gamma \approx 100$ GHz. Based on a typical ambipolar diffusion constant $D_{\text{diff}}$ in semiconductor lasers of about $10 \text{ cm}^2\text{s}^{-1}$ the diffusion rate $\Gamma$ is related to the separation $r$ between the lasers according to equation (2) by

$$\left( \frac{r}{1\mu m} \right)^2 \approx \frac{40\text{GHz}}{\Gamma}.$$  

Thus for GaAs based semiconductor lasers, the diffusion rates discussed correspond to arrays with separations between $r = 0.6\mu m$ and $r = 3\mu m$.

In this context the separations between the lasers of an array of about $r = 3\mu m$ represent diffusion effects which are much weaker than the spatial hole burning effects. This situation is to be expected for most of the semiconductor laser arrays realized today. Note that the spectra for higher diffusion rates and larger separations between the lasers are qualitatively similar to the examples given by the diffusion rates $\gamma + 2\Gamma = 10$ GHz and $\gamma + 2\Gamma = 15$ GHz. Consequently, these results also apply to array structures with much larger separations between the lasers. Below a separation between the lasers of $r = 3\mu m$ the effects of diffusion drastically increase. In the following we investigate two representative cases of such strong diffusion effects with $\gamma + 2\Gamma = 50$ GHz and $\gamma + 2\Gamma = 200$ GHz. This corresponds to laser separations of about $r = 1.25\mu m$ and $r = 0.65\mu m$, respectively. These values were chosen in view of a future realization of highly integrated and/or short wavelength semiconductor laser arrays. The spectra for $\gamma + 2\Gamma = 10$ GHz and $\gamma + 2\Gamma = 15$ GHz may thus be considered typical for the type of arrays already realized in experiments while the spectra for $\gamma + 2\Gamma = 50$ GHz and $\gamma + 2\Gamma = 200$ GHz represent possibilities accessible only if further integration can be achieved.

B. Triplet structure

A typical triplet structure can be observed in devices where the frequency difference $\Omega$ is smaller than the dissipative optical coupling $s$. The intensity of the centerline is then comparable to the intensity of the single line at $M = 0$. The carrier relaxation and diffusion rate $\gamma + 2\Gamma$ will suppress the sidebands, so a low value for this rate produces the strongest possible sidebands. The evolution of the spectrum as a function of laser intensity for the device parameters $\kappa_- = 400$ GHz, $s = 3$ GHz, $\Omega = 1$ GHz, $\gamma + 2\Gamma = 10$ GHz and $\alpha = 3$ is shown in Figure 2 and 3. The laser intensity is given in terms of the stimulated emission rate $wI_0$ which can be converted to units of threshold current by dividing $wI_0$ by the spontaneous emission rate $\gamma$. Since $\gamma$ is usually close to one Gigahertz, realistic values for $wI_0$ range from about zero to ten Gigahertz.

Figure 2 shows the evolution of the triplet structure with increasing intensity. Equation (22) describes this evolution very well, except for the immediate vicinity of the threshold. Note that the asymmetry of the sidebands is extremely strong. At $wI_0 = 1$ GHz, the choice
of parameters corresponding to figure 4, the intensity ratio between the sidebands is about two to one, as predicted by the approximate equation (22).

Figure 3 displays an enlarged part of figure 2 close to threshold. In this region, the triplet has not yet been formed. Between \( wI_0 = 0 \) and \( wI_0 = 0.03 \) GHz the maximum of the single spectral line rapidly shifts to lower frequencies, increasing in intensity in the process. Between \( wI_0 = 0.03 \) GHz and \( wI_0 = 0.15 \) GHz, the line widens and finally forms a plateau on the low frequency side. At about \( wI_0 = 0.25 \) GHz, the plateau on the low frequency side splits from the center line to form the lower sideband. The high frequency sideband emerges in a similar fashion between \( wI_0 = 0.1 \) GHz and \( wI_0 = 0.5 \) GHz.

C. Sideband suppression

Since the center line is only damped by the sum of rates \( s + \alpha \Omega \) while the sidebands are additionally damped by carrier diffusion, the sidebands will be suppressed in the presence of fast carrier diffusion. In this case, only a single line which gets frequency locked to the laser frequency as the laser intensity increases remains in the spectrum. This effect can indeed be seen in the spectra by choosing the same parameters as for the triplet structure (section IV B), \( \kappa_- = 400 \) GHz, \( s = 3 \) GHz, \( \Omega = 1 \) GHz and \( \alpha = 3 \), except for an increased carrier recombination and diffusion rate of \( \gamma + 2\Gamma = 200 \) GHz.

Figure 4 clearly shows the evolution of a single line. However, the peak frequency is not just reduced to zero as the amplified spontaneous emission is phase locked to the laser light by the increasingly strong spatial holeburning effects. Instead, the peak even shifts to larger negative frequencies just above threshold. Between \( wI_0 = 0.5 \) GHz and \( wI_0 = 1.0 \) GHz the peak frequency is about 2 GHz, twice as large as the frequency at \( wI_0 = 0 \). This shift to higher frequency differences between the laser light and the non-lasing mode is similar to the one observed in the case of low carrier diffusion discussed above. However, the shift is not as rapid in the high diffusion case and instead extends to much higher laser intensities \( wI_0 \). In the vicinity of \( wI_0 = 0 \), an analytical expression may be derived for the change in peak position by considering the coupling terms between the holeburning parameter \( \Delta \) and the field components \( f_\parallel \) and \( f_\perp \) in the dynamical matrix \( S \) as weak perturbations. The peak position \( \omega_p \) is given by the imaginary part of the eigenvalues of \( S \). Near \( wI_0 = 0 \) it shifts at a rate of

\[
\frac{d(\omega_p)}{d(wI_0)} = \frac{\alpha \kappa_-}{(\gamma + 2\Gamma)}.
\]

Note that this equation generally applies to all parameter sets. However, in the case of low carrier diffusion it is only valid extremely close to threshold. In particular, the shift of \( d(\omega_p)/d(wI_0) = 120 \) for the low diffusion case in figure 2 extends only to \( wI_0 = 0.03 \) GHz. In the low diffusion case displayed in figure 3 the shift of \( d(\omega_p)/d(wI_0) = 6 \) is the dominant feature of the spectrum up to intensities of \( wI_0 = 0.4 \) GHz, and even at \( wI_0 = 2 \) GHz the peak of the line is still 1 GHz below the laser frequency. These features in the evolution of the spectrum demonstrate how carrier diffusion suppresses the spatial holeburning effects. We note that a quantitative expression for the relative importance of spatial holeburning is given by the complex rate \( M \) introduced in equation (16). It is the denominator of \( M \) which
includes the carrier diffusion rate $2\Gamma$. Thus increased carrier diffusion reduces the absolute value of $M$, thereby suppressing spatial holeburning effects.

D. Undamped sidebands

If $\alpha \Omega$ is larger than $s + \gamma + 2\Gamma$, the undamping effect may cause an instability as discussed in section IV. If the instability is only narrowly avoided by choosing parameters close to the stability condition (15), the sidebands should grow large and become narrow as the laser intensity passes the region of instability in the phase diagram. Figure 5 shows the phase diagram for variable damping of the holeburning parameter, $\gamma + 2\Gamma$, and $\kappa^- = 100$ GHz, $s = 5$ GHz, $\Omega = 10$ GHz, and $\alpha = 3$. The horizontal line at $\gamma + 2\Gamma = 15$ GHz shows the section of this phase diagram corresponding to the choice of parameters in figure 4. The sidebands clearly have their maximum at a stimulated emission rate near $wI_0 = 5$ GHz, the point closest to the unstable region in the phase diagram. The linewidth of the sidebands in this region is close to 1 GHz, as compared to the linewidth given by $2s = 10$ GHz. The maximal value of the total photon number in the non-lasing mode is roughly 240 at $wI_0 = 5$ GHz. For comparison, the total photon number at $wI_0 = 0$ is 20 and the centerline photon number for $wI_0 \to \infty$ is 14.3. The asymmetry of the lines is given by an intensity ratio of about three to one, e.g. at $wI_0 = 5$ GHz the photon numbers in the sidebands are approximately 180 in the low frequency sideband and 60 in the high frequency sideband.

E. Sidebands in the presence of strong carrier diffusion effects

If the carrier diffusion is increased, the undamping of the sidebands is suppressed. However, the sidebands still remain the dominant feature of the spectrum if the line at $wI_0 = 0$ is well separated from the laser line by $\Omega > s$. This effect can be illustrated using the same parameters as those in the case of undamped sidebands (section IV), $\kappa^- = 100$ GHz, $s = 5$ GHz, $\Omega = 10$ GHz and $\alpha = 3$, except for an increased carrier damping and diffusion rate of $\gamma + 2\Gamma = 50$ GHz. As shown in figure 7, the low frequency sideband continuously evolves from the amplified spontaneous emission line at threshold as intensity increases. Meanwhile, the weak phase locking effects cause a low intensity mirror image line to appear at the corresponding position on the high frequency side. Note that the frequency separation from the laser line increases from $\Omega = 10$ GHz at $wI_0 = 0$ to almost 40 GHz at $wI_0 = 10$ GHz. The relaxation oscillation frequency at $wI_0 = 10$ GHz is about 45 GHz. This indicates that the line at $wI_0 = 10$ GHz is actually a relaxation oscillation line created by the carrier dynamics, while the suppression of the high frequency sideband is not only an effect of $\Omega$ but also of the $\alpha$ factor as described by equation (11). Indeed, the approximation given by equation (22) predicts an intensity ratio of twenty to one between the sidebands for the parameters used here.

V. CONCLUSIONS

We have demonstrated that the spatial holeburning caused by interference between the laser light in the antisymmetric supermode of a two laser array and the amplified sponta-
neous emission in the non-lasing symmetric supermode can give rise to relaxation oscillations which phase-lock the amplified spontaneous emission to the laser light. In the limit of high laser intensities, this effect splits the spectrum into a center line at the laser frequency and two sidebands. The center line represents the out-of-phase component of the amplified spontaneous emission which does not interfere with the laser light. The two sidebands correspond to the relaxation oscillations between the in-phase component of amplified spontaneous emission and the depth of spatial holeburning in the carrier distribution.

The sidebands may become undamped by the linewidth enhancement factor $\alpha$, which describes the conversion of laser light from the antisymmetric mode to the symmetric mode. This effect can be countered by carrier diffusion, which dampens the sidebands and may actually suppress them. Depending on the device parameters, it is therefore possible to alternatively find in the spectrum only the sidebands, only a single center line, or the full triplet. The continuous transition from a single line at the frequency of the symmetric supermode to any of the three possibilities can be described analytically using equation (19).

In all cases, the amplified spontaneous emission in the non-lasing supermode is strongly modified by the spatial holeburning effects. The spectra of the non-lasing modes above laser threshold are therefore quite different from the spectra at or below threshold described by linear optics. It should be possible to observe such spectra experimentally by measuring the spectrum of anti-phase locked laser arrays in the center of the farfield, where contributions from the anti-symmetric laser mode is close to zero. The type of spectrum observed will then allow a determination of the coupling strength and the type of coupling in the respective device.

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FIGURES

FIG. 1. Relaxation of $f_\parallel(t = 0) = 1, f_\perp(t = 0) = 0$ for $\kappa_- = 400$ GHz, $s = 3$ GHz, $\Omega = 1$ GHz, $\gamma + 2\Gamma = 10$ GHz, $\alpha = 3$, and $wI_0 = 1$ GHz. The oscillations effectively correspond to a mainly counterclockwise rotation.

FIG. 2. Amplified spontaneous emission spectra for $\kappa_- = 400$ GHz, $s = 3$ GHz, $\Omega = 1$ GHz, $\gamma + 2\Gamma = 10$ GHz and $\alpha = 3$. (a) shows the contour plot of the spectrum as a function of laser intensity from $wI_0 = 0$ to $wI_0 = 2$ GHz, while (b) shows the spectra at $wI_0 = 0$ (no offset), $wI_0 = 0.5$ GHz (offset of 6/GHz), $wI_0 = 1.0$ GHz (offset of 12/GHz), and $wI_0 = 1.5$ GHz (offset of 18/GHz).

FIG. 3. Amplified spontaneous emission spectra close to threshold for the same parameters as in figure 1. (a) shows the contour plot of the spectrum as a function of laser intensity from $wI_0 = 0$ to $wI_0 = 0.3$ GHz, while (b) shows the spectra at $wI_0 = 0$ (no offset), $wI_0 = 0.1$ GHz (offset of 3/GHz), $wI_0 = 0.2$ GHz (offset of 6/GHz), and $wI_0 = 0.3$ GHz (offset of 9/GHz).

FIG. 4. Amplified spontaneous emission spectrum for $\kappa_- = 400$ GHz, $s = 3$ GHz, $\Omega = 1$ GHz, $\gamma + 2\Gamma = 200$ GHz and $\alpha = 3$. (a) shows the contour plot of the spectrum as a function of laser intensity from $wI_0 = 0$ to $wI_0 = 2$ GHz, while (b) shows the spectra at $wI_0 = 0$ (no offset), $wI_0 = 0.5$ GHz (offset of 6/GHz), $wI_0 = 1.0$ GHz (offset of 12/GHz), and $wI_0 = 1.5$ GHz (offset of 18/GHz).

FIG. 5. Stability boundary for $\kappa_- = 100$ GHz, $s = 5$ GHz, $\Omega = 10$ GHz, $\alpha = 3$ and variable carrier recombination and diffusion rates $\gamma + 2\Gamma$. The diagonal line is the approximated boundary for sideband undamping given by $\gamma + 2\Gamma + wI_0 + s - \alpha\Omega = 0$. The horizontal line at $\gamma + 2\Gamma = 15$ GHz represents the choice of parameters in section 4 C and in figure 6.

FIG. 6. Amplified spontaneous emission spectrum for $\kappa_- = 100$ GHz, $s = 5$ GHz, $\Omega = 10$ GHz, $\gamma + 2\Gamma = 15$ GHz and $\alpha = 3$. (a) shows the contour plot of the spectrum as a function of laser intensity from $wI_0 = 0$ to $wI_0 = 10$ GHz, while (b) shows the spectra at $wI_0 = 3$ GHz (no offset), $wI_0 = 4$ GHz (offset of 5/GHz), $wI_0 = 5$ GHz (offset of 10/GHz), $wI_0 = 6$ GHz (offset of 15/GHz), and $wI_0 = 7$ GHz (offset of 20/GHz).

FIG. 7. Amplified spontaneous emission spectrum for $\kappa_- = 100$ GHz, $s = 5$ GHz, $\Omega = 10$ GHz, $\gamma + 2\Gamma = 50$ GHz and $\alpha = 3$. (a) shows the contour plot of the spectrum as a function of laser intensity from $wI_0 = 0$ to $wI_0 = 10$ GHz, while (b) shows the spectra at $wI_0 = 0$ (no offset), $wI_0 = 2$ GHz (offset of 0.2/GHz), $wI_0 = 4$ GHz (offset of 0.4/GHz), $wI_0 = 6$ GHz (offset of 0.6/GHz), $wI_0 = 8$ GHz (offset of 0.8/GHz), and $wI_0 = 10$ GHz (offset of 1.0/GHz).
