PARTIALLY REGULAR WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^4 \times [0, \infty[$

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Abstract. We show that for any given initial data and any external force, there exist partially regular weak solutions of the Navier-Stokes equations in $\mathbb{R}^4$ which satisfy certain local energy inequalities and whose singular sets have locally finite 2-dimensional parabolic Hausdorff measure. With the help of a parabolic concentration-compactness theorem we are able to capture the lack of compactness arising in the spatially 4-dimensional setting by using defect measures, which we then incorporate into the partial regularity theory.

1. The Navier-Stokes Equations

1.1. Introduction. The nonstationary Navier-Stokes equations governing the motion of an incompressible viscous fluid in $\mathbb{R}^n \times [0, T]$ are given by

$$\begin{align*}
\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p &= f \quad (x, t) \in \mathbb{R}^n \times [0, T] \\
\text{div } u &= 0
\end{align*}$$

with initial condition $u(x, 0) = u_0(x), u_0 \in L^2(\mathbb{R}^n)$. Note that is suffices to consider weakly solenoidal forces $f$. Indeed, by Helmholtz-Weyl decomposition, for any external force $f \in L^q(\mathbb{R}^n), q > 1$, we have a decomposition $f = f_s + f_p'$ such that $\text{div } f_s = 0$ and $f_p' = \nabla p'$ for some $p' \in W^{1,q}(\mathbb{R}^n)$. We can insert the component $f_p'$ into the pressure term $\nabla p$.

The existence and regularity problem of the Navier-Stokes equations is one of the most significant open questions in the field of partial differential equations. The case $n = 2$ has been settled by Ladyzhenskaya [12] in 1959. The case $n = 3$ is one of the millennium problems and is still open. However, remarkable progress has been made since the pioneering work by Leray in 1930s. Leray [13] and Hopf [11] proved the existence of weak solutions of these equations in dimensions $n \geq 2$ in the whole space and on bounded open domains with smooth boundary in 1934 and 1950, respectively. These weak solutions, called Leray-Hopf weak solutions, satisfy (1.1) in the distributional sense and belong to $L^\infty_t L^2_x \cap L^2_t H^1_x(\mathbb{R}^n \times [0, T])$. Leray-Hopf weak solutions also satisfy the following global energy inequality

$$\frac{1}{2} \| u(\cdot, t) \|_{L^2(\mathbb{R}^n)}^2 + \int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 dx dt \leq \frac{1}{2} \| u_0 \|_{L^2(\mathbb{R}^n)}^2.$$

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Interestingly, because of the supercritical nature of the Navier-Stokes equations in dimensions $n \geq 3$, this inequality cannot be obtained rigorously by testing $\mathbf{(1.1)}$ with $u$. It is inherited from a Galerkin approximation with the help of weak convergence results. Indeed, we only have $(u \cdot \nabla)u \in L^{(n+2)/(n+1)}$ because of the embedding $L_\infty^\infty L_t^2 \cap L_t^2 H_x^1 \subset L_t^{2+4/n}$, and the product of this term with $u \in L_\infty^\infty L_t^2 \cap L_t^2 H_x^1$ is not necessarily integrable.

An important step towards a better understanding of the Navier-Stokes equations in dimension $n = 3$ was made by Scheffer [18, 19, 20] and Caffarelli, Kohn and Nirenberg [2]. In [20] Scheffer pioneered the partial regularity theory by introducing the notion of suitable weak solutions and proving their existence in dimension $n = 3$ when $f = 0$. Moreover, he proved that the singular sets of these suitable weak solutions have finite $\frac{3}{2}$-dimensional Hausdorff measure in space-time. In [19], Scheffer showed that in dimension $n = 4$, there exist weak solutions whose singular sets have finite 3-dimensional Hausdorff measure in space-time. Caffarelli, Kohn and Nirenberg made remarkable improvements and generalizations in dimension $n = 3$ by showing local partial regularity results for a general force and by proving that the 1-dimensional parabolic Hausdorff measure of the singular sets of suitable weak solutions is zero.

The suitable weak solutions are distributional solutions in the class $L_\infty^\infty L_t^2 \cap L_t^2 H_x^1$ which satisfy a local energy inequality, i.e., for any $-r_0^2 < t < 0$ and any scalar function $0 \leq \phi \in C^\infty(Q_{r_0})$ with $\phi = 0$ in $Q_{r_0} \setminus Q_{r_1}$ and $\phi = 1$ in $Q_{r_2}$ for any $0 < r_2 < r_1 < r_0$, the following inequality holds,

$$
(1.3) \quad \int_{Q_{r_0} \times \{t\}} |u|^2 \phi dx + 2 \int_{-r_0^2}^0 \int_{Q_{r_0}} |\nabla u|^2 \phi dx ds \\
\quad \quad \leq \int_{-r_0^2}^0 \int_{Q_{r_0}} \left( |u|^2 (\partial_t \phi + \Delta \phi) + (|u|^2 + 2p) u \cdot \nabla \phi + 2f \cdot u \right) dx ds.
$$

Note that it is unknown if Leray-Hopf weak solutions satisfy the local energy inequality, since $u \phi$ is not an admissible test function. Caffarelli, Kohn and Nirenberg [2] proved their existence in a general setting by discretizing the regularized equations in time and showing that the approximation sequence $\{u_k\}_{k \in \mathbb{N}}$ is relatively compact in $L_{t,x}^3$-topology. Because we have $L_\infty^\infty L_t^2 \cap L_t^2 H_x^1 \subset L_t^{2+4/n}$, the compactness in $L_{t,x}^3$ in dimension $n = 3$ can be obtained from boundedness of $\{u_k\}_{k \in \mathbb{N}}$ in $L_{t,x}^{10/3}$ and compactness in $L_{t,x}^2$. However, in dimension $n = 4$ we only have that the approximation sequence is relatively compact in $L_{t,x}^\alpha$ for $\alpha < 3$, which is not enough for the local energy inequality to hold in the limit. As the local energy inequality is the most important ingredient for partial regularity theory, the following natural question arises, mentioned by Dong and Du in Remark 1.1 of [4].

**Open question:** Do there exist partially regular weak solutions of the Navier-Stokes equations in 4D which satisfy certain local energy estimates?
This problem has not been answered for quite a long time; however, there have been many results in this direction. Early in 1978, Scheffer [19] constructed weak solutions $u$ in $\mathbb{R}^4 \times [0, +\infty)$ which are continuous outside a closed set of finite 3-dimensional Hausdorff measure. We remark that Scheffer’s weak solutions do not necessarily satisfy the local energy inequality, because of the loss of compactness mentioned above. Dong and Du [4] showed that the 2-dimensional Hausdorff measure of the singular sets of local-in-time regular weak solutions at the first blow-up time is zero. Under the assumption on the existence of the suitable weak solutions, Dong, Gu [5] and Wang, Wu [24] independently proved that the 2-dimensional parabolic Hausdorff measure of the singular sets is zero. A similar study of partial regularity has also been carried out for the magneto-hydrodynamic equations by Choe and Yang [3]. In the direction of the local energy inequality, Biryuk, Craig and Ibrahim [1] discussed the difficulty of validating the local energy inequality in higher dimensions $n \geq 4$. Taniuchi [23] proved the local energy inequality in the dimensions $3 \leq n \leq 10$, given some conditional regularity on distributional solutions.

1.2. New observations, main result, and the organization of this paper.

The aim of this paper is to answer the open question stated above, by constructing weak solutions of the Navier-Stokes equations in 4D satisfying the local energy inequalities (2.19) and (2.20) below and showing that these solutions are global-in-time partially regular with singular sets of finite 2-dimensional parabolic Hausdorff measure. Thus, we improve Scheffer’s result in [19] by refining the estimate of the Hausdorff dimension of singular sets from 3 to 2 and allowing general forces. We remark that the local energy inequalities (2.19) and (2.20) are slightly weaker than the local energy inequality (1.3). Nevertheless, they suffice to give all the partial regularity criteria which Caffarelli, Kohn and Nirenberg have obtained for 3D case.

As we have discussed before, the $L^3_{t,x}$-norm is critical for the local energy inequality (1.3) in dimension 4, in the sense that we need to deal with a possible loss of compactness of smooth approximating sequences in this norm. For this reason, we develop a parabolic concentration-compactness method to study concentration phenomena in a space-time topology. Our two new observations are as follows.

1. The measures $|u_k|^3dxdt$ induced by the solutions $u_k$ of the regularized equations (2.1) below are compact in the sense of measures.
2. The limit measures have the same scaling properties as classical solutions of (1.1) and satisfy the local energy inequalities (2.19) and (2.20).

With these ingredients, we are able to estimate the concentration locally and construct solutions satisfying local energy estimates involving concentration measures. To couple these measures and our weak solutions, we introduce a new notion of generalized solution, namely the notion of weak solution set in Definition 2.11. Finally, we use the iteration scheme as Caffarelli, Kohn and Nirenberg [2] to show that the functions $(u, p)$ in the weak solution set that we construct are partially regular.
Our main theorem is stated for $\mathbb{R}^4$ as follows. However, the method we use is robust and it also applies to more general open domains, for instance, bounded open domains with smooth boundary.

**Theorem 1.1.** Given a weakly solenoidal force $f \in L^q_{\text{loc}}(\mathbb{R}^4 \times [0,T]) \cap L^{3/2}(\mathbb{R}^4 \times [0,T]), q > 3$, there exists a weak solution set $(u,p,\lambda,\omega)$ for the nonstationary Navier-Stokes equations (1.1) in $\mathbb{R}^4$ which satisfies the local energy inequalities (2.19) and (2.20). Moreover, $(u,p)$ is a weak solution of the Navier-Stokes equations with $u \in L_1^\infty L_2^2 \cap L_1^2 H_x^1(\mathbb{R}^4 \times [0,T])$ and $p \in L^{3/2}(\mathbb{R}^4 \times [0,T])$, and the singular set $S$ of $u$ as defined in Definition 3.10 satisfies $P^2(S) < \infty$.

1.3. Connection with the stationary Navier-Stokes equations. Note that in the sense of energy estimates, the nonstationary Navier-Stokes equations in $\mathbb{R}^n \times [0,\infty]$ are similar to the stationary equations in $\mathbb{R}^{n+2}$. The stationary Navier-Stokes equations are given by

$$
-\Delta u + (u \cdot \nabla)u + \nabla p = f \quad x \in \mathbb{R}^{n+2}
$$

$$
div u = 0.
$$

The analogue of the 3D nonstationary case is the 5D stationary case. In 1988, Struwe proved partial regularity for the stationary case in $\mathbb{R}^5$ in [22]. Later, a similar approach was adapted for the stationary case in $\mathbb{R}^6$ by Dong and Strain [6]. Note that for general open subdomains of $\mathbb{R}^6$ or for an unbounded force $f$, the existence of regular solutions or suitable weak solutions to (1.4) for $n \geq 6$ is still open.

The strategy that we use to show Theorem 1.1 can also be applied to the stationary Navier-Stokes equations in dimension 6, since the measures $|u_k|^3dx$ induced by the solutions $u_k$ of the regularized equations are compact, modulo mass vanishing at infinity. However, the stationary 6D case and the nonstationary 4D case also differ in many interesting features. Although one time dimension counts for two space dimensions, the weak solutions are less regular in time than in space. Consequently, we may say the nonstationary case in space dimension $n = 4$ is less regular than the stationary case in $\mathbb{R}^6$. This slight regularity gap actually leads to a manifest difference in the concentration of $L_x^3$ mass. For instance, we only have point concentration in the stationary case, but line concentration might occur in the nonstationary case. The detailed discussion of the stationary case in 6D will be given in a separate paper.

1.4. Connection with variational problems. The concentration-compactness principle developed by Lions in [15] has been shown to be an effective tool for dealing with elliptic PDEs and variational problems. Basically, this tool may help us understand the process of passing to a weak limit in many cases. By lower semi-continuity, if $x_n \rightharpoonup x$ in a Banach space $X$, we have

$$
\|x\|_X \leq \liminf_{k \to +\infty} \|x_n\|_X.
$$

Usually, one would like to know if equality holds and if not, why equality fails.
A classical example is the existence of extremal functions for Sobolev embeddings. This amounts to find a function such that the following embedding inequality holds with equality,

\[ S\|u\|_{L^{n/(n-kl)}(\mathbb{R}^n)} \leq \|u\|_{W^{k,l}(\mathbb{R}^n)} \quad k \in \mathbb{N}, l \geq 1. \]  

After suitable translations and dilations, a minimizing sequence in this problem incurs no concentration, thus is relatively compact in \( L^{n/(n-kl)}(\mathbb{R}^n) \).

However, the stationary and the nonstationary Navier-Stokes equations are not known to admit a similar variational structure, and translations or dilations will change the external forces and boundary conditions, so it is not possible to normalize solutions using either of these tools. However, Gallagher, Koch and Planchon \cite{8} used profile decomposition to show that a local-in-time solution of the Navier-Stokes equations which develops a singularity at finite time must blow up in scale-invariant norms at this time, where they also study the residual term in weak convergence. Nevertheless, this approach does not work for our purpose because of the issue concerning controlling external forces, initial conditions and boundary conditions. Despite these technical difficulties that naturally arise in fluid dynamics equations, we shall see that possible concentration loss of \( L^3_{t,x} \) mass either can be controlled or causes no harm to the regularity theory we aim to pursue.

1.5. Connection with other PDEs. Apart from the stationary Navier-Stokes equations in \( \mathbb{R}^6 \), our strategy may be applied in a wide class of PDEs. For instance, the following incompressible magneto-hydrodynamic equations have a structure similar to the Navier-Stokes equations,

\[
\begin{align*}
\partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p &= (h \cdot \nabla) h \\
\text{div } u &= 0 \\
\partial_t h - \Delta h + (u \cdot \nabla) h - (h \cdot \nabla) u &= 0 \\
\text{div } h &= 0.
\end{align*}
\]  

Gu \cite{9} obtained some partial regularity criteria for suitable weak solutions to \( (1.7) \) in space dimension 4 assuming that these solutions exist. It is likely that one can construct partially regular weak solutions of the incompressible magneto-hydrodynamic equations with our strategy, but we do not pursue it here.

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2. Weak solution sets and local energy inequalities

To construct weak solutions for the Navier-Stokes equations in the space-time domain $D := \mathbb{R}^4 \times [0, T]$, we consider the regularized Navier-Stokes equations, namely

$$\begin{align*}
\partial_t u_k - \Delta u_k + \left[ (\chi_k * u_k) \cdot \nabla \right] u_k + \nabla p_k &= f \\
\text{div} u_k &= 0 \\
u(\cdot, 0) &= u_0,
\end{align*}$$

(2.1)

where $\{\chi_k\}_{k \in \mathbb{N}} \subset C_\infty^\infty(\mathbb{R}^4)$ are standard mollifiers. This regularization was used by Leray in [13] to show the existence of weak solutions in whole space, and the obtained weak solutions are suitable weak solutions in space dimension $n = 3$. The first step is to use a Galerkin method to construct stronger solutions for the regularized equations with uniform energy estimates. In this section, we also prove that the measures induced by $|u_k|^3 dx dt$ in the approximation sequence are compact in the weak sense.

Next, we set up a parabolic concentration-compactness framework. With all these ingredients and careful estimates on the pressure $p$, we can construct weak solution sets satisfying certain local energy estimates.

### 2.1. Solving regularized equations and weak compactness of measures.

Before we prove the existence of weak solutions of the regularized Navier-Stokes equations (2.1), we recall the following definition of distributional solutions of the Navier-Stokes equations (1.1), which also avoids any possible ambiguity of realizing the initial data $u_0$.

**Definition 2.1.** A pair of functions $(u, p) \in L^2_t H^1_{x,\text{loc}}(D) \times L^{1+2/n}_{\text{loc}}(D)$ are distributional solutions of (1.1) if $u$ is weakly divergence-free and for any $\varphi \in C_\infty^\infty(\mathbb{R}^4 \times [0, T])$ and any $t \in [0, T)$, we have

$$\begin{align*}
- \int_0^t \int_{\mathbb{R}^n} u^i \partial_i \varphi dx dt + \int_0^t \int_{\mathbb{R}^n} \partial_j u^i \partial_j \varphi_i dx dt + \int_0^t \int_{\mathbb{R}^n} u^i \partial_j u^j \varphi_i dx dt \\
- \int_0^t \int_{\mathbb{R}^n} p \text{div} \varphi dx dt - \int_0^t \int_{\mathbb{R}^n} f_i \varphi_i dx dt &= \int_{\mathbb{R}^n} \left( u_0 \cdot \varphi(0) - u(t) \cdot \varphi(t) \right) dx.
\end{align*}$$

(2.2)

Remark 2.2. If we restrict the test functions to divergence-free functions, then we have a weak formulation of (1.1) without $p$. If we test with $\nabla \phi$ where $\phi$ is a scalar function, then we obtain the following well-known elliptic equation for the pressure

$$- \Delta p = \partial_i \partial_j (u^i u^j).$$

(2.3)

Distributional solutions for the regularized Navier-Stokes equations (2.1) can be defined in a similar way. We now show that there exist weak solutions for (2.1) with uniform energy bounds and satisfying local energy inequality.

**Lemma 2.3.** Let $\{\chi_k\}_{k \in \mathbb{N}}$ be a sequence of standard mollifiers and $f \in L^{3/2}(D)$, then we have a sequence $\{(u_k, p_k)\}_{k \in \mathbb{N}} \subset L_t^\infty L_x^2 \cap L_t^2 H^1_x(D) \times L^{3/2}(D)$ such that
\((u_k, p_k)\) is a distributional solution to the regularized nonstationary Navier-Stokes equations \((2.1)\). Moreover,

1. \(\{u_k\}_{k \in \mathbb{N}}\) is uniformly bounded in \(L_t^\infty L_x^2 \cap L_t^2 H_x^1(D)\),
2. \(\{p_k\}_{k \in \mathbb{N}}\) is uniformly bounded in \(L^{3/2}(D)\),
3. \(\{\partial_t u_k\}_{k \in \mathbb{N}}\) is uniformly bounded in \(L_t^1 H_x^1(D)\),

where \(\mathbb{H}(\mathbb{R}^4) := \{\varphi \in H^1(\mathbb{R}^4) | \text{div } \varphi = 0\}\). Thereby we can pass to the weak limit,

\[
\begin{aligned}
&\quad u_k \to u \text{ weakly in } L_t^2 H_x^1, \\
&\quad u_k \to u \text{ weakly } \ast \text{ in } L_t^\infty L_x^2, \\
&\quad p_k \to p \text{ weakly in } L^{3/2}.
\end{aligned}
\]

This sequence satisfies the local energy inequality, i.e. for any bounded smooth function \(\phi\) with bounded derivatives,

\[
\begin{aligned}
\int \mathbb{R}^4 |u_k(t)|^2 \phi(t) dx - \int \mathbb{R}^4 |u_0|_0^2 \phi(0) dx + \int_0^t \int \mathbb{R}^4 |\nabla u_k|^2 \phi dx dt \\
\quad \leq \int_0^t \int \mathbb{R}^4 |u_k|^2 (\partial_t \phi + \Delta \phi) dx dt + \int_0^t \int \mathbb{R}^4 |u_k|^2 (\tilde{u}_k \cdot \nabla \phi) dx dt \\
\quad \quad + \int_0^t \int \mathbb{R}^4 2p_k (u_k \cdot \nabla \phi) dx + \int_0^t \int \mathbb{R}^4 f : u_k \phi dx dt,
\end{aligned}
\]

where \(\tilde{u}_k := \chi_k \ast u_k\).

**Remark 2.4.** We do not need \(\phi\) to have compact support. \(\chi_k \ast u_k\) is bounded and smooth for every \(k\). For any bounded function \(\phi\) with bounded derivatives, \(u_k \phi\) is in \(L_t^\infty L_x^2 \cap L_t^2 H_x^1(D) \subset L^3(D)\); thus, \(u_k \phi\) is an admissible test function.

**Proof.** The existence of \(u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(D)\) can be proved by a standard Galerkin method. We refer to Theorem 4.4 and Theorem 14.1 in [17] for an exposition. The existence of \(p \in L^{3/2}(D)\) is obtained by \(L^p\)-theory of elliptic operators and Calderon-Zygmund theory.

For the rest, note that in the regularized equations \(u_k\) and \(u_k \phi\) are admissible test functions. Testing with \(u_k\) and \(u_k \phi\) yields the uniform boundedness of \(\{u_k\}_{k \in \mathbb{N}}\) and \(\{p_k\}_{k \in \mathbb{N}}\) and the local energy inequality \((2.5)\).

For the uniform boundedness of \(\{\partial_t u_k\}_{k \in \mathbb{N}}\), we remark that the weak formulation \((2.2)\) for \(u_k\) is equivalent to

\[
\forall \xi \in \mathbb{H}_x, \quad \langle \partial_t u_k, \xi \rangle_{H^1_x} = - \int \mathbb{R}^4 \left( \partial_j u_k^i \partial_j \xi_i + u_k^i \xi_j \partial_j u_k^i - f_i \xi_i \right) dx
\]

for almost every \(t \in [0, T]\).

For every \(\xi \in C_0^\infty(\Omega), \Omega \subset \subset \mathbb{R}^4\) and almost every \(t \in [0, T]\), we can estimate

\[
\left| \int \mathbb{R}^4 \left( \partial_j u_k^i \partial_j \xi_i + u_k^i \xi_j \partial_j u_k^i - f_i \xi_i \right) dx \right| \leq \|\nabla u_k\|_{L_x^2} \|\nabla \xi\|_{L_x^2} + \|\xi\|_{L_x^2} \|f\|_{L_x^{3/2}}
\]

\[
\leq \left(C \left( \|u_k\|_{H_x^1}^2 + \|f\|_{L_x^{3/2}} \right) \|\xi\|_{H_x^1} \right).
\]
Then integrating in time yields that \( \{\partial_t u_k\}_{k \in \mathbb{N}} \) is uniformly bounded in \( L^1_t H^s_{x,loc}(D) \).

Next, we prove that certain measures in the limit are weakly compact, which yields a crucial requirement in our parabolic concentration-compactness framework.

**Lemma 2.5.** Let the assumptions be as in Lemma 2.3, then \( \{\nabla u_k \|^2 dx dt\}_{k \in \mathbb{N}}, \{u_k^2 dx dt\}_{k \in \mathbb{N}} \) and \( \{u_k^3 dx dt\}_{k \in \mathbb{N}} \) are tight in the sense of measures.

**Proof.** We define a cut-off function \( \xi \in C^\infty(\mathbb{R}^4) \) with bounded derivatives by

\[
0 \leq \xi \leq 1, \quad \xi|_{B_\rho} = 0, \quad \xi|_{\mathbb{R}^4 \setminus B_\rho} = 1, \quad |\nabla \xi| \leq C\rho^{-1}, \quad |\nabla^2 \xi| \leq C\rho^{-2}.
\]

and let \( \rho > 0 \) to be determined. From Remark 2.3 we know \( u_k \xi \) is an admissible test function. Testing the regularized Navier-Stokes equations (2.1) with \( u \xi \) yields

\[
\sup_t \int_{\mathbb{R}^4} |u_k(t)|^2 \xi dx - \int_{\mathbb{R}^4} |u_0|^2 \xi dx + \int_D |\nabla u_k|^2 \xi dx \leq \int_D |u_k|^2 |\Delta \xi| dx + \int_D |\nabla^2 u_k|^2 \xi dx
\]

The bounds for \( \xi \) gives

\[
\sup_t \int_{\mathbb{R}^4 \setminus B_\rho} |u_k(t)|^2 dx - \int_{\mathbb{R}^4 \setminus B_\rho} |u_0|^2 dx + \int_{D_{\alpha,2\rho}} |\nabla u_k|^2 dx \leq C\rho^{-2} \int_{D_{\alpha,2\rho}} |u_k|^2 dx + C\rho^{-1} \int_{D_{\alpha,2\rho}} |\nabla u_k|^2 dx + \int_{D_{\alpha,2\rho}} |\Delta \xi| dx
\]

where \( D_{\alpha,2\rho} := (\mathbb{R}^4 \setminus B_{2\rho}) \times [0, T] \), and \( D_{\alpha,2\rho} := (B_{2\rho} \setminus B_\rho) \times [0, T] \). The second inequality follows from Hölder inequality and the fact that \( \{u_k\}_{k \in \mathbb{N}} \) is uniformly bounded in \( L^3(D) \). Finally, letting \( \rho \) be arbitrarily large concludes the tightness of \( \{\nabla u_k \|^2 dx dt\}_{k \in \mathbb{N}} \). This also implies that \( \{u_k(t)|^2 dx\}_{k \in \mathbb{N}} \) is tight uniformly in \( t \), which leads to the tightness of \( \{u_k^3 dx dt\}_{k \in \mathbb{N}} \).

For the tightness of the measures \( \{u_k^3 dx dt\}_{k \in \mathbb{N}} \), we use Sobolev inequality and the same cutoff function,

\[
\int_D |u_k\xi|^3 dx dt \leq \|u_k\xi\|_{L^\infty_t L^2_x} \int_D |\nabla (u_k\xi)|^2 dx dt
\]

\[
\leq 2\|u_k\xi\|_{L^\infty_t L^2_x} \left( \int_D |\nabla u_k|^2 \xi^2 dx dt + \int_D |u_k|^2 |\nabla \xi|^2 dx dt \right)
\]

\[
\leq 2\|u_k\xi\|_{L^\infty_t L^2_x} \left[ \int_D |\nabla u_k|^2 \xi^2 dx dt + C\rho^{-2/3} T^{1/3} \left( \int_D |u_k|^3 dx dt \right)^{2/3} \right].
\]

Given the tightness of \( |\nabla u_k|^2 dx dt \) and uniform boundedness of \( u_k \) in the natural energy space, arbitrarily large \( \rho \) yields the tightness of \( |u_k|^3 dx dt \).

With the tightness of the measures, we obtain convergence of \( \{u_k\}_{k \in \mathbb{N}} \) in \( L^2(D) \).
Lemma 2.6. Let the assumptions be as in Lemma 2.3. The sequence \( \{ u_k \}_{k \in \mathbb{N}} \) is relatively compact in \( L^2(D) \). Consequently, the weak limit \((u, p)\) are distributional solutions of the Navier-Stokes equations (1.1).

Lemma 2.6 is a direct consequence of the bounds in Lemma 2.3 and the following compactness result.

Lemma 2.7 (Corollary 6, Simon, [21]). Let \( X, Y, B \) be Banach spaces and we have the embeddings

\[
X \xrightarrow{\text{compact}} B \hookrightarrow Y.
\]

If a sequence \( \{ u_k \}_{k \in \mathbb{N}} \) is bounded in \( L^\alpha(0, T, B) \cap L^1_{\text{loc}}(0, T, X), \alpha \in (1, +\infty] \) and \( \partial_t u_n \) is bounded in \( L^1_{\text{loc}}(0, T, Y), p \geq 1 \), then there exists a subsequence of \( \{ u_k \}_{k \in \mathbb{N}} \) which converges strongly in \( L^\beta(0, T, B) \) for any \( \beta \in [1, \alpha) \).

Proof of Lemma 2.6. Note that, to get a compact Sobolev embedding, we first restrict to \( B_l \subset \mathbb{R}^4 \) with \( l \in \mathbb{N}^* \), then letting \( \alpha = +\infty \) and \( X = H^1(B_l), \quad B = L^2(B_l), \quad Y = H^s(B_l). \)

give the strong convergence of a subsequence of \( \{ u_k \}_{k \in \mathbb{N}} \) in \( L^2(B_l \times [0, T]) \), then by enlarging \( r \) to infinity, a diagonal argument gives a subsequence which converges in \( L^2(\Omega \times [0, T]) \) for any compact subset \( \Omega \) of \( \mathbb{R}^4 \). Note that Lemma 2.5 yields the tightness of \( \{ |u_k|^2 dxdt \}_{k \in \mathbb{N}} \), then it is easy to show the subsequence converges in \( L^2(D) \).

With the strong convergence of \( u_k \) in \( L^2(D) \) and the weak convergence criteria in (2.4), it is easy to verify the weak limit \((u, p)\) solves (1.1) in the distributional sense. \( \square \)

2.2. Parabolic concentration-compactness. To obtain local energy inequalities for the weak limit \((u, p)\), one would like to pass to the limit \( k \to \infty \) in the local energy inequalities (2.3) for the approximation solutions. As we discussed in the introduction, this scenario is critical. In critical variational problems, concentration phenomena may occur. This motivates to look for an analogue of Lions’s [15] concentration-compactness principle in parabolic setting.

Note that concentration-compactness in elliptic setting may not be applicable to the parabolic setting, since it is hopeless to get \( \{ \nabla u_k(t) \}_{k \in \mathbb{N}} \) is bounded in \( L^2 \) for almost every \( t \), even for a subsequence. A relevant example in [16] by Lopes Filho and Nussenzveig Lopes shows that a bounded sequence in \( L^1 \) might blow up at almost every point up to any subsequence.

Lemma 2.8. Given a bounded sequence \( \{ u_k \}_{k \in \mathbb{N}} \subset L^\infty_t L^2_x \cap L^1_t H^1_x(D), \) let \( u \) be given by the limit in (2.4). Suppose \( u_k \) converges to \( u \) in \( L^1_{\text{loc}}(D) \). Assume that \( \mu_k = |\nabla u_k|^2 dxdt \to \mu, \nu_k = |u_k|^3 dxdt \to \nu \) weakly in the sense of measures, where \( \mu \)
and \( \nu \) are bounded nonnegative measures on \( \mathbb{R}^4 \times [0, T] \). Then there exist nonnegative finite measures \( \omega \) and \( \lambda \) on \( \mathbb{R}^4 \times [0, T] \), such that for any \( \varphi \in C_c^\infty(D) \),

\[
\begin{align*}
(2.6) & \quad \int \int \varphi \, d\mu = \int \int \varphi |\nabla u|^2 \, dx \, dt + \int \int \varphi \, d\lambda, \\
(2.7) & \quad \int \int \varphi \, d\nu = \int \int \varphi |u|^3 \, dx \, dt + \int \int \varphi \, d\omega.
\end{align*}
\]

Moreover, \( \omega \ll \lambda \), and we have for any open subdomain \( Q \) of \( D \),

\[
(2.8) \quad \int_Q d\omega \leq C \liminf_{k \to \infty} \|u_k - u\|_{L^\infty L^2(Q)} \int_Q d\lambda.
\]

In particular, the Radon-Nikodym derivative satisfies

\[
(2.9) \quad \frac{d\omega}{d\lambda} \leq C \lim_{r \to 0} \liminf_{k \to \infty} \|u_k - u\|_{L^\infty L^2(Q^* (x_0, t_0))},
\]

where \( Q^*_r(x_0, t_0) := B_r(x_0) \times (t_0 - \frac{r^2}{2}, t_0 + \frac{r^2}{2}) \).

**Remark 2.9.** We remark that this lemma only requires \( \{u_k\}_{k \in \mathbb{N}} \) to be bounded in \( L^\infty L^2 \cap L^2 H^1(D) \). \( \{u_k\}_{k \in \mathbb{N}} \) does not necessarily solve certain equations. Indeed, we only need \( u_k \to u \) converges in \( L^1_{loc}(D) \).

**Remark 2.10.** Although we only state this result for space dimension 4, one can easily see a trivial generalization to higher dimensions.

**Proof.** Let \( v_k = u_k - u \in L^\infty L^2 \cap L^2 H^1 \), then

\[
\begin{align*}
(2.10) & \quad v_k \to 0 \quad \text{strongly in } L^2_{t,x}, \text{ locally in space,} \\
(2.11) & \quad v_k \to 0 \quad \text{weakly in } L^2_t H^1_x, \\
(2.12) & \quad v_k \to 0 \quad \text{weakly-* in } L^\infty L^2_x.
\end{align*}
\]

Define \( \omega_k := |v_k|^3 \, dx \, dt \). It is easy to check \( \{\omega_k\}_{k \in \mathbb{N}} \) is tight. Indeed, for any compact subset \( \Omega \subset D \), denote \( \Omega^c := (\mathbb{R}^4 \setminus \Omega) \times [0, T] \), then

\[
\|v_k\|_{L^3(\Omega^c)} \leq \|u_k\|_{L^3(\Omega^c)} + \|u\|_{L^3(\Omega^c)}.
\]

Because of the weak convergence of \( \{v_k\}_{k \in \mathbb{N}} \), we know that \( \{v_k\}_{k \in \mathbb{N}} \) is tight and thus \( \|v_k\|_{L^3(\Omega^c)} \) is arbitrarily small given \( \Omega \) large enough. Thus we can extract a weakly convergent subsequence with a limit denoted by \( \omega \). For any \( \varphi \in C_c^\infty(D) \), we have

\[
\int \int \varphi \, d\nu = \lim_{k \to \infty} \int \int \varphi \, d\nu_k = \lim_{k \to \infty} \int \int \varphi |u_k|^3 \, dx \, dt
\]

\[
= \int \int \varphi |u|^3 \, dx \, dt + \lim_{k \to \infty} \int \int \varphi |v_k|^3 \, dx \, dt = \int \int \varphi |u|^3 \, dx \, dt + \int \int \varphi \, d\omega.
\]

The third equality follows from the fact that \( u_k \to u \) in \( L^\alpha \) locally in space for \( \alpha \in [1, 3] \), then all the interaction terms vanish. Let \( \lambda_k := |\nabla v_k|^2 \, dx \, dt \to \lambda \) weakly in the sense of measures. A similar argument verifies (2.7), and the interaction term vanishes there since \( u_k \to u \) weakly in \( L^2_t H^1_x(D) \).
Now we prove (2.8). For any \( \varphi \in C^\infty_c(D) \), we have
\[
\iint_D |\varphi|^3 d\omega = \lim_{k \to \infty} \iint_D |\varphi|^3 d\omega_k = \lim_{k \to \infty} \iint_D |v_k \varphi|^3 dx dt
\leq \liminf_{k \to \infty} \sup_{0 < t < T} \|v_k \varphi\|_{L^2_x}^2 \int_0^T \|v_k \varphi\|_{L^4_x}^2 dt
\]
(2.13)
\[
\leq C \liminf_{k \to \infty} \sup_{0 < t < T} \|v_k \varphi\|_{L^2_x} \iint_D |\nabla (v_k \varphi)|^2 dx dt
\leq C \liminf_{k \to \infty} \sup_{0 < t < T} \|v_k \varphi\|_{L^2_x} \iint_D |\varphi|^2 |\nabla v_k|^2 dx dt
\leq C \liminf_{k \to \infty} \sup_{0 < t < T} \|v_k \varphi\|_{L^2_x} \iint_D |\varphi|^2 d\lambda.
\]
The first inequality follows from the interpolation between \( L^2 \) and \( L^4 \). The second inequality follows from Sobolev embedding. For the third inequality, note that the terms converge to zero when at least one derivative hits \( \varphi \). Using smooth functions to approximate the indicator function of \( Q \) yields the inequality (2.8).

Therefore, \( \omega \) is absolutely continuous with respect to \( \lambda \), and by Radon-Nikodym theorem, we have
\[
\frac{d\omega}{d\lambda} \in L^1(D; \lambda)
\]
with
\[
\frac{d\omega}{d\lambda}(x_0, t_0) \leq C \lim_{r \to 0} \liminf_{k \to \infty} \|v_k\|_{L^4_x L^2_t(Q^*_r(x_0, t_0))}
\]
for any \((x_0, t_0) \in \mathbb{R}^4 \times (0, T)\).

Using the parabolic concentration-compactness framework in Lemma 2.8 and the tightness results in Lemma 2.5, we now can define the notion of weak solution sets involving concentration measures.

**Definition 2.11.** The quadruple \((u, p, \lambda, \omega)\) is a weak solution set of the Navier-Stokes equations (1.1) if

1. \( u \) and \( p \) are obtained as weak limits of the weak solutions \( \{(u_k, p_k)\}_{k \in \mathbb{N}} \) of the regularized Navier-Stokes equations (2.1), as in Lemma 2.3.
2. \( \lambda \) and \( \omega \) are obtained as weak limits of the measures in Lemma 2.8.

One can see that every weak solution set comes with a sequence of approximation solutions. However, this is in a sense necessary because a single \( L^p \) function is not able to represent concentration of any form. As we shall see, this is effective for analytical purposes in certain critical cases.

2.3. Local energy inequalities. In this subsection, we show two energy inequalities with purely local nature for weak solution sets. Although these inequalities are weaker than the local energy inequality (1.3) in a sense, they suffice to establish partial regularity of the distributional solutions \((u, p)\). For technical reasons only, in (2.19) and (2.20), we present two distinct forms of these estimates.
From the elliptic equation \((2.3)\) for the pressure \(p\), one may guess \(p\) has the same regularity as \(|u|^2\), so the pressure term in the local energy estimates \((2.5)\) may also exhibit concentration of mass. As a preparation for our main goal in this section, we show the concentration in \(|up|dxdt\) are localizable and comparable to the concentration in \(|u|^3dxdt\).

**Lemma 2.12.** Suppose \(\{(u_k,p_k)\}_{k \in \mathbb{N}}\) are the solutions of the regularized equations \((2.1)\) and \((u,p,\lambda,\omega)\) is the corresponding weak solution set, then

\[
\limsup_{k \to \infty} \int_D \zeta |u_k(p_k - \gamma) - u(p - \gamma)| dxdt \leq \int_D \zeta d\omega
\]

for any \(\zeta \in C_c^\infty(D)\) and any \(\gamma \in \mathbb{R}\) with \(\zeta \geq 0\).

**Proof.** To prove this result, we need an interpolation inequality. For any \(\alpha \in (3, +\infty), \beta \in (2, 3), \vartheta \in (0, 1)\) with \(\frac{1}{\alpha} + \frac{2}{\beta} = 1\) and \(\frac{1}{3} = \frac{\vartheta}{2} + \frac{1-\vartheta}{3}\), we have for any \(w \in L^\infty_t L^2_x \cap L^2_t H^1_x\),

\[
\int \|w(t)\|_{L^2_x}^3 dt = \int \|w(t)\|_{L^\beta_x} \cdot \|w(t)\|_{L^2_x}^2 dt \\
\leq \|w\|_{L^\alpha_t L^\beta_x} \|w\|_{L^2_t L^\vartheta_x} \|w\|_{L^2_t L^\vartheta_x} \|w\|_{L^2_t L^{(1-\vartheta)}_x} \\
= \|w\|_{L^2_t L^\vartheta_x} \|w\|_{L^2_t L^{(1-\vartheta)}_x} \left( \int \|w(t)\|_{L^2_t L^{\vartheta}_x}^\alpha dt \right)^{1/\alpha} \\
\leq \|w\|_{L^2_t L^\vartheta_x} \|w\|_{L^2_t L^{(1-\vartheta)}_x} \left( \int \|w(t)\|_{L^2_t L^{\alpha(4-\beta)/\beta}_x}^{\alpha(2\beta-4)/\beta} \|w(t)\|_{L^\infty_t L^\vartheta_x}^{\alpha(2\beta-4)/\beta} \|w(t)\|_{L^\infty_t L^2_x}^{\alpha(2\beta-4)/\beta} \|w(t)\|_{L^2_t L^{(4-\beta)/\beta}_x} \right)^{1/\alpha} \\
\leq \|w\|_{L^2_t L^\vartheta_x} \|w\|_{L^2_t L^{(1-\vartheta)}_x} \|w\|_{L^\infty_t L^2_x} \|
abla w\|_{L^2_t L^\vartheta_x}^{2/\alpha}.
\]

The first inequality follows from Hölder inequality. The second and the third inequalities follow from Lebesgue interpolation inequality. The fourth inequality follows from the Sobolev inequality.

Now we analyze the concentration phenomena of the measures involving the pressure \(p_k\). Note the following Poisson equation

\[-\Delta p_k = \partial_i \partial_j (\tilde{u}^i_k u^j_k),\]

where \(\tilde{u}_k := \chi_k * u_k\). From Remark \(2.2\) we know this equation holds in the sense of distributions for almost every \(t\), then we localize this equation with an arbitrary Lipschitz function \(\xi \in C^{0,1}(\mathbb{R}^4)\), i.e.

\[-\Delta(p_k \xi) = \xi \partial_i \partial_j (\tilde{u}^i_k u^j_k) - \text{div}(p_k \nabla \xi) - \nabla p_k \cdot \nabla \xi \]

\[-\Delta(p_k \xi) = \partial_i \partial_j (\tilde{u}^i_k u^j_k) - \text{div}(\tilde{u}_k u^i_k \partial_j \xi - \partial_j (\tilde{u}_k u^i_k) \partial_i \xi - \text{div}(p_k \nabla \xi) - \nabla p_k \cdot \nabla \xi \]

\[-\Delta(p_k \xi) = \partial_i \partial_j (\tilde{u}^i_k u^j_k) - \text{div}(\tilde{u}_k u^i_k \partial_j \xi + p_k \nabla \xi) - \partial_j (\tilde{u}_k u^i_k) \partial_i \xi - \nabla p_k \cdot \nabla \xi.\]
Next, we decompose the pressure \( p_k \xi = p_k^1 + p_k^2 + p_k^3 \) with

\[
\begin{align*}
-\Delta p_k^1 &= \partial_i \partial_j (\xi \tilde{u}^i_k u^j_k), \\
-\Delta p_k^2 &= - \text{div}(\tilde{u}^i_k u^j_k \partial_j \xi + p_k \nabla \xi), \\
-\Delta p_k^3 &= - \partial_j (\tilde{u}^i_k u^j_k) \partial_i \xi - \nabla p_k \cdot \nabla \xi,
\end{align*}
\]

and \( p \xi \) in a similar way. Intuitively, the concentration takes place in the component \( p_k^1 \), since at least one differentiation hits the cutoff function \( \xi \) in other components.

Now we do rigorous estimates term by term. \( p_k^1 \xi \) can be obtained by the Riesz transformation and Calderon-Zygmund theory yields

\[
\|p_k^1(t) - p^1(t)\|_{L^{3/2}_y} \leq \|\xi \tilde{u}^i_k(t) u^j_k(t) - \xi \tilde{u}^i(t) u^j(t)\|_{L^{3/2}_y} \\
\leq \|\xi (\tilde{u}^i_k(t) - \tilde{u}^i(t)) (u^j_k(t) - u^j(t))\|_{L^2_y} \\
+ \|\xi \tilde{u}^i(t) (u^j_k(t) - u^j(t))\|_{L^2_y} \\
+ \|\xi u^j(t) (\tilde{u}^i_k(t) - \tilde{u}^i(t))\|_{L^2_y} \tag{2.15}
\]

Since \( \omega_k \to \omega \) weakly, we have

\[
\limsup_{k \to \infty} \int_D |p_k^1 - p^1|^{3/2} \, dx \, dt \leq \limsup_{k \to \infty} \int_D |\xi^{3/2}| u_k - u|^3 \, dx \, dt \\
+ \limsup_{k \to \infty} \int_D |\xi^{3/2}| \tilde{u}^i (u^j_k - u^j)|^{3/2} \, dx \, dt \\
+ \limsup_{k \to \infty} \int_D |\xi^{3/2}| u^j (\tilde{u}^i_k - \tilde{u}^i)|^{3/2} \, dx \, dt \\
\leq \limsup_{k \to \infty} \int_D |\xi|^{3/2} \, d\omega.
\]

By Vitali’s convergence theorem, the second and third lines converge to zero, because \( |u_k^j - u^j|^3/2 \) is uniformly integrable with respect to \( |\xi|^{3/2}| \tilde{u}^i|^{3/2} \, dx \, dt \).

Also, for almost every \( t \), \( u_k(t) \in L^3(\mathbb{R}^4) \), then \( p_k^2 \xi \) can be obtained by convolution with singular kernels. Calderon-Zygmund theory yields

\[
\|p_k^2 - p^2\|_{L^{3/2}_y(D_r)} = \|(-\Delta)^{-1} \left[ - \text{div} \left( (u_k u^j_k - u^j) \partial_j \xi + (p_k - p) \nabla \xi \right) \right]\|_{L^{3/2}_y(D_r)} \\
\leq \|u_k u^j_k - u^j\|_{L^{3/2}_y L^{12/11}_x(D_r)} + \|p_k - p\|_{L^{3/2}_y L^{12/11}_x(D_r)} \\
\leq \|\nabla \xi\|_{L^{\infty}} \left( \|u_k u^j_k - u^j\|_{L^{3/2}_y L^{12/11}_x(D_r)} + \|p_k - p\|_{L^{3/2}_y L^{12/11}_x(D_r)} \right) \\
\leq \|\nabla \xi\|_{L^{\infty}} \left( \|u_k - u\|_{L^{3/2}_y L^{24/11}_x(D_r)} + \|p_k - p\|_{L^{3/2}_y L^{12/11}_x(D_r)} \right).
\]
Similarly, for $p_3^3 \xi$ we have

$$\|p_3^3 - p^3\|_{L^{3/2}(D)} \leq \|\nabla \xi\|_{L^\infty}(\|u_k - u\|_{L^3_t L^{24/11}(D)} + \|p_k - p\|_{L^{3/2}_t L^{12/11}(D)}).$$

Let $w = u_k - u$ and $\beta = \frac{24}{11}$, the interpolation inequality (2.14) yields

$$\text{(2.16)} \quad \limsup_{k \to \infty} \|u_k - u\|_{L^3_t L^{24/11}(D)} = 0$$

and Calderon-Zygmund theory yields

$$\text{(2.17)} \quad \limsup_{k \to \infty} \|p_k - p\|_{L^3_t L^{12/11}(D)} = 0.$$

Now we combine the estimates for $p_1^3$, $p_2^3$ and $p_3^3$. From (2.16) and (2.17), we know that $p_2^3$ and $p_3^3$ have no contribution to the concentration, then

$$\limsup_{k \to \infty} \|(p_k - p)\xi\|_{L^{3/2}_t L^{3/2}(D)}^{3/2} = \|(p_k - p)\xi\|_{L^{3/2}_t L^{3/2}(D)}^{3/2} \leq \sum_{l=1}^{3} \|p_k^l - p_l^l\|_{L^{3/2}_t L^{3/2}(D)}^{3/2}$$

$$\leq \int \int_{D} |\xi|^{3/2} d\omega. \quad \text{(2.18)}$$

Therefore, we can choose $\xi = \zeta^{2/3}$ and bound the concentration of the measure as follows,

$$\limsup_{k \to \infty} \int \int_{D} \zeta|u_k(p_k - \gamma) - u(p - \gamma)| dx dt$$

$$\leq \limsup_{k \to \infty} \int \int_{D} \zeta|p_k - p| dx dt + \limsup_{k \to \infty} \int \int_{D} \zeta|u_k - u||p - \gamma| dx dt$$

$$\leq \limsup_{k \to \infty} \int \int_{D} \zeta|u_k - u||p_k - p| dx dt + \limsup_{k \to \infty} \int \int_{D} \zeta|u||p_k - p| dx dt$$

$$\leq \int \int_{D} \zeta dx dt.$$

Due to Vitali’s convergence theorem, the second term in the second line and the second term in the third line converge to zero. Note that $\zeta$ is nonnegative and smooth. By Corollary A.2, $\xi$ is indeed a compactly supported Lipschitz continuous function. The last inequality follows from (2.18). \hfill \Box

Now we can prove the following local energy inequalities.
Proposition 2.13. Let the assumptions be as in Lemma 2.3. Then the following local energy inequalities hold,

\[
\limsup_{k \to \infty} \sup_t \int_{\mathbb{R}^4} |u_k(t)|^2 \varphi(t) dx + \int_{D} |\nabla u|^2 \varphi dx + \int_{D} \varphi d\lambda \\
\leq \int_{D} |u|^2 |\partial_t \varphi + \Delta \varphi| dxdt + 2 \sum_{i=1}^{n} \int_{D} |u|^3 |\nabla \varphi_i| dxdt + 3 \sum_{i=1}^{n} \int_{D} |\nabla \varphi_i| d\omega \\
+ 2 \sum_{i=1}^{n} \int_{D} |\nabla \varphi_i| |p - \gamma_i|^{3/2} dxdt + \int_{D} f \cdot u \varphi dxdt,
\]

(2.19)

\[
\limsup_{k \to \infty} \sup_t \int_{\mathbb{R}^4} |u_k(t)|^2 \varphi(t) dx + \int_{D} |\nabla u|^2 \varphi dx + \int_{D} \varphi d\lambda \\
\leq \int_{D} |u|^2 |\partial_t \varphi + \Delta \varphi| dxdt + \int_{D} |u|^2 (u \cdot \nabla) \varphi dxdt \\
+ 2 \int_{D} |u - \Pi \varphi|^3 |\nabla \varphi| dxdt + 3 \int_{D} |\nabla \varphi| d\omega \\
+ 2 \int_{D} p(u \cdot \nabla) \varphi dxdt + \int_{D} f \cdot u \varphi dxdt,
\]

(2.20)

for any \( n \in \mathbb{N} \), any \( \{\gamma_i\}_{1 \leq i \leq n} \subset \mathbb{R} \), any non-negative cut-off functions \( \varphi \in C_c^{\infty}(D) \) with \( \varphi(\cdot, 0) = 0 \) and \( \{\varphi_i\}_{1 \leq i \leq n} \subset C_c^{\infty}(D) \) with \( \varphi = \sum_{i=1}^{n} \varphi_i \), where

\[
\Pi \varphi(t) = \frac{1}{\mathcal{L}(\text{supp} \varphi(t))} \int_{\text{supp} \varphi(t)} u(x,t) dx.
\]

Proof. To prove local energy inequalities (2.19) and (2.20), we pass \( k \to \infty \) in the local energy inequality for approximation sequence \( u_k \). For the cutoff function \( \varphi \) defined above, the local energy inequality (2.5) reduces to

\[
\sup_t \int_{\mathbb{R}^4} |u_k(t)|^2 \varphi(t) dx + \int_{D} |\nabla u_k|^2 \varphi dx dt \leq \int_{D} |u_k|^2 |\partial_t \varphi + \Delta \varphi| dxdt \\
+ \int_{D} |u_k|^2 (\tilde{u}_k \cdot \nabla) \varphi dxdt + \int_{D} 2p_k(u_k \cdot \nabla) \varphi dx + \int_{D} f \cdot u_k \varphi dx dt.
\]

(2.21)

Since \( u_k \to u \) in \( L^2(D) \), the convergence of the third and the last terms is straightforward. The convergence of the second term is given by Lemma 2.5 and (2.6) in Lemma 2.8. The difference between the two inequalities and the technical difficulties come from the rest terms, namely the cubic term of \( u \) and the term involving \( p \).

For the cubic term of \( u \) in the local energy inequality (2.19), note that

\[
\int_{D} |\tilde{u}_k|^3 |\nabla \varphi| dxdt = \|h_1 + h_2\|^3_{L^1(D)}.
\]
where

\[ h_1(x, t) = \int_{\mathbb{R}^4} u_k(x - y, t) \chi_k(y) \left( |\nabla \varphi(x, t)|^{1/3} - |\nabla \varphi(x - y, t)|^{1/3} \right) dy, \]

\[ h_2(x, t) = \int_{\mathbb{R}^4} u_k(x - y, t) \chi_k(y) |\nabla \varphi(x - y, t)|^{1/3} dy. \]

For \( h_1 \), notice that \( d_k := \text{diam}(\text{supp} \, \chi_k) \to 0 \) and \( x \to |x|^{1/3} > 0 \) is \( 1/3 \)-Hölder continuous, then Young’s inequality for convolution yields

\[
\|h_1\|_{L^3} \leq \left\| \int_{\mathbb{R}^4} u_k(x - y, t) \chi_k(y) \frac{|\nabla \varphi(x, t)|^{1/3} - |\nabla \varphi(x - y, t)|^{1/3}}{|y|^{1/3}} d_k^{1/3} dy \right\|_{L^3(D)} \\
\leq C \text{diam}(\text{supp} \, \chi_k)^{1/3} \|\varphi\|_{C^2} \|\tilde{u}_k\|_{L^3(D)} \\
\leq C \text{diam}(\text{supp} \, \chi_k)^{1/3} \|\varphi\|_{C^2} \|u_k\|_{L^3(D)}.
\]

We can then deduce that \( h_1 \) part converges to zero in \( L^3 \) when \( k \) tends to infinity.

For \( h_2 \), Young’s inequality for convolution yields

\[
\|h_2\|_{L^3(D)} = \|(u_k|\nabla \varphi|^{1/3}) \ast \chi_k\|_{L^3(D)} \leq \|u_k|\nabla \varphi|^{1/3}\|_{L^3(D)}.
\]

Then these estimates for \( h_1 \) and \( h_2 \) yield

\[
\limsup_{k \to \infty} \int_D |u_k|^2 (\tilde{u}_k \cdot \nabla) \varphi dxd t \leq \frac{2}{3} \limsup_{k \to \infty} \int_D |u_k|^3 |\nabla \varphi| dxd t \\
+ \frac{1}{3} \limsup_{k \to \infty} \int_D |\tilde{u}_k|^3 |\nabla \varphi| dxd t \\
\leq \int_D |\nabla \varphi| |u|^3 dxd t + \int_D |\nabla \varphi| d\omega.
\]

(2.22)

For the term involving pressure in the local energy inequality (2.19), we use Lemma 2.12 and the fact that \( u_k \) is weakly divergence-free to bound

\[
\limsup_{k \to \infty} \int_D p_k (u_k \cdot \nabla) \varphi dxd t = \sum_{i=1}^n \limsup_{k \to \infty} \int_D p_k u_k \cdot \nabla \varphi_i dxd t \\
= \sum_{i=1}^n \limsup_{k \to \infty} \int_D (p_k - \gamma_i) u_k \cdot \nabla \varphi_i dxd t \\
\leq \frac{1}{3} \sum_{i=1}^n \int_D |u|^3 |\nabla \varphi_i| dxd t + \sum_{i=1}^n \int_D |\nabla \varphi_i| d\omega \\
+ \frac{2}{3} \sum_{i=1}^n \int_D |\nabla \varphi_i| |p - \gamma_i|^{3/2} dxd t.
\]
For the cubic term of $u$ in the local energy inequality (2.20), we use the fact that $u_k, \tilde{u}_k$ and $u$ are weakly divergence-free. Thus,

\begin{equation}
\int\int_D |u_k|^2 (\tilde{u}_k \cdot \nabla) \varphi \, dx \, dt = \int\int_D |u_k - \overline{u}_{k,\varphi} + \overline{u}_{k,\varphi}|^2 (\tilde{u}_k \cdot \nabla) \varphi \, dx \, dt \\
= \int\int_D [ |u_k - \overline{u}_{k,\varphi}|^2 + 2(u_k - \overline{u}_{k,\varphi}) \cdot \overline{u}_{k,\varphi}] (\tilde{u}_k \cdot \nabla) \varphi \, dx \, dt \\
= \int\int_D |u_k - \overline{u}_{k,\varphi}|^2 [(\tilde{u}_k - \overline{u}_{k,\varphi}) \cdot \nabla] \varphi \, dx \, dt \\
+ \int\int_D |u_k - \overline{u}_{k,\varphi}|^2 (\overline{u}_{k,\varphi} \cdot \nabla) \varphi \, dx \, dt \\
+ 2 \int\int_D [(u_k - \overline{u}_{k,\varphi}) \cdot \overline{u}_{k,\varphi}] (\tilde{u}_k \cdot \nabla) \varphi \, dx \, dt.
\end{equation}

Next, we argue that the individual terms above can be bounded by the weak limit $u$ and the concentration mass $\omega$. For the term in the third line of (2.23), since $\tilde{u}_k - \overline{u}_{k,\varphi} = (u_k - \overline{u}_{k,\varphi}) \ast \chi_k$, we can apply the same trick by replacing $u_k$ and $\tilde{u}_k$ with $u_k - \overline{u}_{k,\varphi}$ and $\tilde{u}_k - \overline{u}_{k,\varphi}$ and use Young’s inequality for convolution, therefore it is sufficient to look at the following term

\begin{align*}
\int\int_D |u_k - \overline{u}_{k,\varphi}|^2 [(u_k - \overline{u}_{k,\varphi}) \cdot \nabla] \varphi \, dx \, dt &
\leq \int\int_D |u_k - \overline{u}_{k,\varphi} - (u - \overline{u}_{\varphi})|^3 |\nabla \varphi| \, dx \, dt + \int\int_D |u - \overline{u}_{\varphi}|^3 |\nabla \varphi| \, dx \, dt \\
&+ \int\int_D 3|u_k - \overline{u}_{k,\varphi} - (u - \overline{u}_{\varphi})|^2 |u - \overline{u}_{\varphi}| |\nabla \varphi| \, dx \, dt \\
&+ \int\int_D 3|u_k - \overline{u}_{k,\varphi} - (u - \overline{u}_{\varphi})| |u - \overline{u}_{\varphi}|^2 |\nabla \varphi| \, dx \, dt \\
&\leq \int\int_D |u_k - u - (\overline{u}_{k,\varphi} - \overline{u}_{\varphi})|^3 |\nabla \varphi| \, dx \, dt + \int\int_D |u - \overline{u}_{\varphi}|^3 |\nabla \varphi| \, dx \, dt \\
&\rightarrow \int\int_D |\nabla \varphi| \, d\omega + \int\int_D |u - \overline{u}_{\varphi}|^3 |\nabla \varphi| \, dx \, dt \quad \text{as } k \rightarrow \infty.
\end{align*}

Since $u_k \rightarrow u$ in $L^2(D)$, $\overline{u}_{k,\varphi} - \overline{u}_{\varphi} \rightarrow 0$ in $L^2([0,T])$ as $k \rightarrow \infty$ for any compactly supported function $\varphi$. Thus, we can pass $k \rightarrow \infty$ in the remaining two terms in the
last line of (2.23). Hence we have
\[ \limsup_{k \to \infty} \int\int_{D} |u_k|^2(\bar{u}_k \cdot \nabla)\varphi \, dx \, dt \leq \int\int_{D} |\nabla \varphi| \, dw + \int\int_{D} |u - \bar{u}|^3|\nabla \varphi| \, dx \, dt \]
\[ + \int\int_{D} |u - \bar{u}|^2|\nabla \varphi| \, dx \, dt + 2 \int\int_{D} [(u - \bar{u}) \cdot \bar{u}] (u \cdot \nabla)\varphi \, dx \, dt \]
\[ = \int\int_{D} |\nabla \varphi| \, dw + \int\int_{D} |u - \bar{u}|^3|\nabla \varphi| \, dx \, dt \]
\[ + \int\int_{D} |u|^2(u \cdot \nabla)\varphi \, dx \, dt - \int\int_{D} |u - \bar{u}|^2 [(u - \bar{u}) \cdot \nabla] \varphi \, dx \, dt \]
\[ \leq \int\int_{D} |\nabla \varphi| \, dw + 2 \int\int_{D} |u - \bar{u}|^3|\nabla \varphi| \, dx \, dt + \int\int_{D} |u|^2(u \cdot \nabla)\varphi \, dx \, dt. \]

Finally, for the term involving pressure in the local energy inequality (2.20), Lemma 2.12 yields
\[ \limsup_{k \to \infty} \int\int_{D} p_k(u_k \cdot \nabla)\varphi \, dx \, dt \leq \int\int_{D} p(u \cdot \nabla)\varphi \, dx \, dt + \int\int_{D} |\nabla \varphi| \, dw. \]

3. Partial regularity theory

Partial regularity theory contains deep results of natural scaling and local energy estimates of the Navier-Stokes equations. In this section, we show that weak solution sets have the same scaling invariance as classical solutions, then we adapt Caffarelli, Kohn and Nirenberg’s argument to space dimension 4 with the presence of concentration measures.

As we mentioned in the introduction, Scheffer proved $\mathcal{H}^3(S) < \infty$. An interesting point is that Scheffer overcame the loss of compactness in $L^3_{t,x}$ by proving uniform local $L^3_{t,x}$ estimate for the approximate solutions $u^k$, then one can pass the local estimate to the weak limit without splitting the concentration measures and the weak limit $u \in L^3$. In Scheffer’s approach, local $L^3_{t,x}$ estimate gives the bound for $\mathcal{H}^3$ measure, while in our work, the $L^2_t H^1_x$ estimate gives refined bound for $\mathcal{H}^2$ measure.

3.1. Dimensionless estimates in space dimension 4. The Navier-Stokes equations have a nice scaling property. If $(u, p)$ solves (1.1) with force $f$, then $u_r, p_r$ defined by
\[ u_r(x, t) = ru(rx, r^2t) \quad p_r(x, t) = r^2p(rx, r^2t) \]
solve (1.1) with force $f_r$ defined by
\[ f_r(x, t) = r^3 f(rx, r^2t). \]

The weak solution sets also have a similar scaling property.

\[ ^1 \text{One can see Lemma 2.6 in Scheffer [19] for details.} \]
Lemma 3.1. If \((u,p,\lambda,\omega)\) is a weak solution set of the Navier-Stokes equations \((1.1)\) with external force \(f\), then for any \(r > 0\), the scaled quadruple \((u_r,p_r,\lambda_r,\omega_r)\) is also a weak solution set of \((1.1)\) with external force \(f_r\), where \(u_r, p_r\) and \(f_r\) are defined as above and \(\lambda_r, \omega_r\) are defined as

\[
\int_E d\lambda_r := r^{-2} \int \int_{\{(rx,t^2) | (x,t) \in E\}} d\lambda
\]
\[
\int_E d\omega_r := r^{-3} \int \int_{\{(rx,t^2) | (x,t) \in E\}} d\omega
\]

for any \(E \subset \mathbb{R}^4 \times \mathbb{R}\).

Now we give short-hand notations for the following quantities are scale-invariant.

\[
A(x_0,t_0,r) = \limsup_{k \to +\infty} \sup_{t_0-r^2 < t < t_0} r^{-2} \int_{B_r(x_0)} |u_k|^2 dx
\]
\[
\delta(x_0,t_0,r) = r^{-2} \int_{Q_r(x_0,t_0)} |\nabla u|^2 dx dt
\]
\[
\delta_c(x_0,t_0,r) = r^{-2} \int_{Q_r(x_0,t_0)} d\lambda
\]
\[
G(x_0,t_0,r) = r^{-3} \int_{Q_r(x_0,t_0)} |u|^3 dx dt
\]
\[
G_c(x_0,t_0,r) = r^{-3} \int_{Q_r(x_0,t_0)} d\omega
\]
\[
H(x_0,t_0,r) = r^{-3} \int_{Q_r(x_0,t_0)} |u - \tilde{u}_{r,x_0}|^3 dx dt
\]
\[
K(x_0,t_0,r) = r^{-3} \int_{Q_r(x_0,t_0)} |p|^{3/2} dx dt
\]
\[
L(x_0,t_0,r) = r^{-3} \int_{Q_r(x_0,t_0)} |p - \tilde{p}_{r,x_0}|^{3/2} dx dt
\]
\[
F_1(x_0,t_0,r) = r^{3q-6} \int_{Q_r(x_0,t_0)} |f|^q dx dt
\]
\[
F_2(x_0,t_0,r) = \int_{Q_r(x_0,t_0)} |f|^2 dx dt
\]

where

\[
\tilde{u}_{r,x_0}(t) = \frac{1}{\mathcal{L}(B_r)} \int_{B_r(x_0)} u(x,t) dx
\]
\[
\tilde{p}_{r,x_0}(t) = \frac{1}{\mathcal{L}(B_r)} \int_{B_r(x_0)} p(x,t) dx
\]

and \(Q_r(x,t)\) is the parabolic cylinder centered at \((x,t)\) given by

\[Q_r(x,t) = B_r(x) \times (t - r^2, t).\]
When \((x_0, t_0) = (0, 0)\), we abbreviate \(A(0, 0, r)\) to \(A(r)\). This convention also applies to other quantities and parabolic cylinders. For technical reasons, we also need another quantity \(L'\) which is not scale-invariant.

\[
L'(x_0, t_0, r) = r^{-5/2} \int_{Q_r(x_0, t_0)} |p - \tilde{p}_{r, x_0}|^{3/2} dx dt = r^{1/2} L(x_0, t_0, r)
\]

Note that already in the work \([2]\) of Caffarelli, Kohn and Nirenberg, a quantity similar to \(L'\) that is not scale-invariant plays an important role.

One crucial component of proving partial regularity in space dimension 4 is interpolation inequalities. Next we introduce three interpolation inequalities based on the above dimensionless quantities.

**Lemma 3.2.** Suppose that \((u, p, \lambda, \omega)\) is a weak solution set of the Navier-Stokes equations (1.1) in space dimension 4 in \(Q_r(x_0, t_0)\). Then there exists an absolute constant \(C_1 > 0\), which is independent of \((x_0, t_0) \in \mathbb{R}^4 \times \mathbb{R}\) and \(r > 0\), such that

\[
G(x_0, t_0, r) \leq C_1 A^{3/2}(x_0, t_0, r) + C_1^{3/2}(x_0, t_0, r),
\]

\[
G_{\varepsilon}(x_0, t_0, r) \leq C_1 A^{1/2}(x_0, t_0, r) \delta(x_0, t_0, r),
\]

\[
H(x_0, t_0, r) \leq C_1 A^{1/2}(x_0, t_0, r) \delta(x_0, t_0, r).
\]

**Proof.** Since all quantities here are scale-invariant, it suffices to prove the inequalities for \(r = 1\). By Lebesgue interpolation inequality,

\[
\|u\|_{L^3(B_1(x_0))} \leq \|u\|_{L^4(B_1(x_0))}^{2/3} \|u\|_{L^2(B_1(x_0))}^{1/3},
\]

\[
\|u - \tilde{u}_{1,x_0}\|_{L^3(B_1(x_0))} \leq \|u - \tilde{u}_{1,x_0}\|_{L^4(B_1(x_0))}^{2/3} \|u - \tilde{u}_{1,x_0}\|_{L^2(B_1(x_0))}^{1/3}
\]

By Sobolev embedding and Sobolev-Poincaré inequality,

\[
\|u\|_{L^4(B_1(x_0))} \leq C_1 (\|u\|_{L^2(B_1(x_0))} + \|\nabla u\|_{L^2(B_1(x_0))})
\]

\[
\|u - \tilde{u}_{1,x_0}\|_{L^4(B_1(x_0))} \leq C_1 \|\nabla u\|_{L^2(B_1(x_0))}.
\]

Then we integrate in time and use Young’s inequality,

\[
\int_{Q_1(x_0, t_0)} |u|^3 dx dt \leq C_1 \int_{t_0}^{t_0} \left( A(x_0, t_0, 1) + \int_{B_1(x_0)} |\nabla u|^2 dx \right) A^{1/2}(x_0, t_0, 1) dt
\]

\[
= C_1 A^{3/2}(x_0, t_0, 1) + C_1 A^{1/2}(x_0, t_0, 1) \delta(x_0, t_0, 1)
\]

\[
\leq C_1 A^{3/2}(x_0, t_0, 1) + C_1 \delta^{3/2}(x_0, t_0, 1).
\]

In the first inequality, we use lower semi-continuity of the weak-* convergence to bound \(\|u\|_{L^\infty_t L^2_x}\) with \(\lim\sup\|u_k\|_{L^\infty_t L^2_x}\). Similarly, we also have

\[
\int_{Q_1(x_0, t_0)} |u - \tilde{u}_{r,x_0}|^3 dx dt \leq C_1 A^{1/2}(x_0, t_0, 1) \delta(x_0, t_0, 1).
\]

The second interpolation inequality follows directly from Lemma [28].
As we mentioned, these quantities are scale-invariant. Then we can obtain the inequalities via scaling when $r \neq 1$. □

The second key ingredient is the local energy inequality ($2.19$) and ($2.20$). To use this local energy inequality, we also need different types of estimates for the pressure term. We prove a 4-dimensional analogue of Lemma 3.2 in [2].

Lemma 3.3. Suppose that $(u, p, \lambda, \omega)$ is a weak solution set of the Navier-Stokes equations ($1.1$) in space dimension 4 in $Q_\rho(x_0, t_0)$. Then there exists an absolute constant $C_2 > 0$, which is independent of $(x_0, t_0) \in \mathbb{R}^4 \times \mathbb{R}$ and $\rho > 0$, such that

\begin{equation}
L'(x_0, t_0, r) \leq C_2 r^{-5/2} \int_{Q_\rho(x_0, t_0)} |u|^3 dx dt + C_2 r^5 \left( \sup_{t_0 - r^2 < t < t_0} \int_{2r < |y - x_0| < \rho} \frac{|u|^2}{|y - x_0|^5} dy \right)^{3/2} \tag{3.2}
\end{equation}

\begin{equation}
+ C_2 \frac{r^3}{\rho^{11/2}} \int_{Q_\rho(x_0, t_0)} (|u|^3 + |p|^{3/2}) dx dt,
\end{equation}

where $0 < r \leq \frac{\rho}{2}$.

Proof of Lemma 3.3. For simplicity, we prove the estimate when $(x_0, t_0) = (0, 0)$ at first. Choose a cutoff function $\psi \in C^\infty_c(\mathbb{R}^4)$ such that $0 \leq \psi \leq 1$ and

\begin{equation}
\psi \equiv 1 \text{ in } B_{3\rho/4}, \quad \psi \equiv 0 \text{ in } \mathbb{R}^4 \setminus B_{\rho}, \quad |\nabla \psi| \leq C_2 \rho^{-1}, \quad |\nabla^2 \psi| \leq C_2 \rho^{-2}. \tag{3.3}
\end{equation}

Then we localize the pressure equation and integrate by parts to move the differentiation from $u$ and $p$ to $\psi$,

\begin{equation}
p(x, t)\psi(x) = (-\Delta)^{-1}(-\Delta)(p(x, t)\psi(x)) \\
= \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \frac{1}{|x - y|^2} (\psi \partial_j \partial_j (u_i u_j) - 2\nabla \psi \cdot \nabla p - p \Delta \psi) dy \\
= \frac{1}{4\pi^2} \int_{\mathbb{R}^4} u_i u_j \psi \partial_j \left( \frac{1}{|x - y|^2} \right) dy \\
+ \frac{1}{4\pi^2} \int_{\mathbb{R}^4} u_i u_j \left( \frac{\partial_i \partial_j \psi}{|x - y|^2} + \partial_j \psi \frac{4(x_i - y_i)}{|x - y|^4} \right) dy \\
+ \frac{1}{4\pi^2} \int_{\mathbb{R}^4} p \left( \frac{\Delta \psi}{|x - y|^2} + \frac{4(x - y) \cdot \nabla \psi}{|x - y|^4} \right) dy \\
= p_1(x, t) + p_2(x, t) + p_3(x, t) + p_4(x, t), \tag{3.4}
\end{equation}
where by the fact that $\psi$ is supported in $B^\rho$,

\[
\begin{align*}
    p_1(x,t) &= \frac{1}{4\pi^2} \int_{B_{2r}} u_i u_j \psi \partial_i \partial_j \left( \frac{1}{|x-y|^2} \right) dy, \\
    p_2(x,t) &= \frac{1}{4\pi^2} \int_{B^\rho \setminus B_{2r}} u_i u_j \psi \partial_i \partial_j \left( \frac{1}{|x-y|^2} \right) dy, \\
    p_3(x,t) &= \frac{1}{4\pi^2} \int_{B^\rho} u_i \partial_i \psi \left( \frac{\partial_i \partial_j \psi}{|x-y|^2} + \partial_j \psi \frac{4(x_i - y_i)}{|x-y|^4} \right) dy, \\
    p_4(x,t) &= \frac{1}{4\pi^2} \int_{B^\rho} p \left( \frac{\Delta \psi}{|x-y|^2} + \frac{4(x-y) \cdot \nabla \psi}{|x-y|^4} \right) dy.
\end{align*}
\]

(3.5)

Now, we decompose $L'(x_0, t_0, r)$ into four terms involving $p_1, p_2, p_3$ and $p_4$ respectively and estimate them separately.

(3.6)

\[ L'(x_0, t_0, r) \leq \sum_{l=1}^{4} r^{-5/2} \int_{Q_r} |p_l - \tilde{p}_l, r|^{3/2} dx dt. \]

We interpret $p_1$ as $p_1 = T_{ij} (u_i u_j \psi)$, where $\{T_{ij}\}_{1 \leq i,j \leq 4}$ is given by

(3.7)

\[ T_{ij} \zeta = \left( \partial_i \partial_j \left( \frac{1}{|x|^2} \right) \right) \ast \zeta. \]

From Calderón-Zygmund theory we know $\{T_{ij}\}_{1 \leq i,j \leq 4}$ are bounded linear operators from $L^q(\mathbb{R}^4)$ to $L^q(\mathbb{R}^4)$ for any $1 < q < \infty$, hence let

\[ \zeta(y, t) = u_i(y, t) u_j(y, t) \psi(y) 1_{\{y \in B_{2r}\}} \]

and it yields

\[ \int_{B_r} |p_1|^{3/2} dx \leq C_2 \int_{B_{2r}} |u|^3 dx. \]

With a simple computation and integrate in time, we have

(3.8)

\[ \int_{Q_r} |p_1 - \tilde{p}_1, r|^{3/2} dx dt \leq C_2 \int_{Q_{2r}} |u|^3 dx dt. \]

We estimate the remaining terms by bounding the $L^\infty$-norm of the space derivatives of the pressure $p$.

For $p_2$, we can control its derivative as follows. When $(x, t) \in Q_r$,

\[ |\nabla p_2(x, t)| \leq C_2 \int_{2r < |y| < \rho} \frac{\psi |u|^2}{|x-y|^6} dx \leq C_2 \int_{2r < |y| < \rho} \frac{|u|^2}{|y|^5} dx. \]

The second inequality follows from $2|x-y| > |y|$ when $x \in B_r, y \in B^\rho_{2r}$ and we move the factor 2 into the absolute constant $C_2$. Then we can estimate the second term.
in (3.6) by mean value theorem as follows,
\[
\int \int_{Q_r} |p_2 - \tilde{p}_{2,r}|^{3/2} dx dt \leq \frac{\pi^2}{2} r^4 \int_{-r^2}^{0} \|p_2 - \tilde{p}_{2,r}\|^{3/2}_{L^\infty(B_r)} dt
\]
(3.9)
\[
\leq \frac{\pi^2}{2} r^{11/2} \int_{-r^2}^{0} \|\nabla p_2\|^{3/2}_{L^\infty(B_r)} dt
\]
\[
\leq C_2 r^{15/2} \left( \sup_{-r^2 < t < 0} \int_{2r < |y| < \rho} \frac{|u|^2}{|y|^3} dy \right)^{3/2}.
\]
Thus, for \(p_3\) and \(p_4\), note that \(\nabla \psi = 0\) and \(\nabla^2 \psi = 0\) in \(B_{3\rho/4}\). Moreover, when \(x \in B_r\) and \(y \in B_{\rho} \setminus B_{3\rho/4}, |x - y| > \frac{\rho}{4}\). Hence for \((x,t) \in Q_r\),
(3.10)
\[
|\nabla p_3 (x,t)| \leq C_2 \int_{B_{\rho} \setminus B_{3\rho/4}} |u|^2 \left( \frac{\nabla^2 \psi}{|x-y|^3} + \frac{\nabla \psi}{|x-y|^4} \right) dy \leq C_2 \rho^{-5} \int_{B_{\rho} \setminus B_{3\rho/4}} |u|^2 dy,
\]
\[
|\nabla p_4 (x,t)| \leq C_2 \int_{B_{\rho} \setminus B_{3\rho/4}} |p|^2 \left( \frac{\nabla^2 \psi}{|x-y|^3} + \frac{\nabla \psi}{|x-y|^4} \right) dy \leq C_2 \rho^{-5} \int_{B_{\rho} \setminus B_{3\rho/4}} |p|^2 dy.
\]
Thus,
(3.11)
\[
\sum_{l=3}^{4} \int \int_{Q_r} |p_l - \tilde{p}_{l,r}|^{3/2} dx dt \leq \sum_{l=3}^{4} \frac{\pi^2}{2} r^4 \int_{-r^2}^{0} \|p_l - \tilde{p}_{l,r}\|^{3/2}_{L^\infty(B_r)} dt
\]
\[
\leq \sum_{l=3}^{4} \frac{\pi^2}{2} r^{11/2} \int_{-r^2}^{0} \|\nabla p_l\|^{3/2}_{L^\infty(B_r)} dt
\]
\[
\leq C_2 \left( \frac{r}{\rho} \right)^{11/2} \int_{Q_{\rho}} |u|^3 + |p|^3 dx dt.
\]
The second inequality follows from mean value theorem and the last one follows from (3.10) and Hölder’s inequality.

Finally, combining the estimates (3.8), (3.9) and (3.11) and dividing them by \(r^{5/2}\) yield (3.12).

To obtain partial regularity theory in space dimension 4, we also need another estimate for the pressure \(p\).

**Lemma 3.4.** Suppose that \((u,p,\lambda,\omega)\) is a weak solution set of the Navier-Stokes equations (1.1) in space dimension 4 in \(Q_r(x_0,t_0)\). Then there exists an absolute constant \(C_3 > 0\), which is independent of \((x_0,t_0) \in \mathbb{R}^4 \times \mathbb{R}\) and \(r > 0\), such that
(3.12)
\[
K(x_0,t_0, \theta r) \leq C_1 C_3 \theta^{-3} A^{1/2}(x_0,t_0,r) \delta(x_0,t_0,r) + C_3 \theta K(x_0,t_0,r)
\]
for any \(\theta \in (0, \frac{1}{7}]\). The constant \(C_1 > 0\) is absolute and comes from Lemma 3.2.

**Proof.** Again it suffices to prove the estimate when \((x_0,t_0) = (0,0)\). Choose a cutoff function \(\psi \in C_c^\infty(\mathbb{R}^4)\) such that \(0 \leq \psi \leq 1\) and
(3.13)
\[
\psi \equiv 1 \text{ in } B_{3\rho/4}, \quad \psi \equiv 0 \text{ in } \mathbb{R}^4 \setminus B_{\rho}, \quad |\nabla \psi| \leq C_3 r^{-1}, \quad |\nabla^2 \psi| \leq C_3 r^{-2}.
\]
The pressure equation can be written as
\[-\Delta p = \partial_i \partial_j [(u_i - \tilde{u}_{i,r})(u_j - \tilde{u}_{j,r})].\]

We can localize this equation like (3.4) and (3.5) to obtain
\[p(x, t)\psi(x) = p_1(x, t) + p_2(x, t) + p_3(x, t),\]

\[p_1(x, t) = \frac{1}{4\pi^2} \int_{B_r} (u_i - \tilde{u}_{i,r})(u_j - \tilde{u}_{j,r}) \psi \partial_i \partial_j \left( \frac{1}{|x-y|^2} \right) dy,\]

\[p_2(x, t) = \frac{1}{4\pi^2} \int_{B_r} (u_i - \tilde{u}_{i,r})(u_j - \tilde{u}_{j,r}) \left( \frac{\partial_i \partial_j \psi}{|x-y|^2} + \partial_j \psi \frac{4(x_i - y_i)}{|x-y|^4} \right) dy,\]

\[p_3(x, t) = \frac{1}{4\pi^2} \int_{B_r} p \left( \frac{\Delta \psi}{|x-y|^2} + \frac{4(x - y) \cdot \nabla \psi}{|x-y|^4} \right) dy.\]

For \(p_1\), Calderón-Zygmund theory yields
\[\int_{B_{\theta r}} |p_1|^{3/2} dx \leq C_3 \int_{B_r} |(u_i - \tilde{u}_{i,r})(u_j - \tilde{u}_{j,r})|^{3/2} dx \leq C_3 \int_{B_r} |u - \tilde{u}_r|^3 dx.\]

Integrating in time gives
\[\iint_{Q_{\theta r}} |p_1|^{3/2} dx dt \leq C_3 \int_{Q_r} |u - \tilde{u}_r|^3 dx dt.\]

For \(p_2\), note that \(\nabla \psi\) is supported in \(B_r \setminus B_{3r/4}\). Then for \(x \in B_{\theta r}, |x - y| > \frac{r}{4}\) and the bounds in (3.13) give
\[|p_2| \leq C_3 r^{-4} \int_{B_r} |u - \tilde{u}_r|^2 dx.\]

Then integrate in \(Q_{\theta r}\) to obtain
\[\iint_{Q_{\theta r}} |p_2|^{3/2} dx dt \leq C_3 \int_{-(\theta r)^2}^{0} (\theta r)^4 \|p_2\|_{L^\infty(B_{\theta r})}^{3/2} dt \leq C_3 \theta^4 \int_{Q_r} |u - \tilde{u}_r|^3 dx dt.\]

For \(p_3\), likewise, we have
\[\iint_{Q_{\theta r}} |p_3|^{3/2} dx dt \leq C_3 \theta^4 \int_{Q_r} |p|^{3/2} dx dt.\]

Then combine the estimates for \(p_1, p_2\) and \(p_3\) and apply the interpolation inequality Lemma 3.2 to find
\[K(\theta r) \leq C_1 C_3 \theta^{-3} A^{1/2}(r) \delta(r) + C_3 \theta K(r),\]
as claimed. \(\square\)

For the proof of partial regularity results, two types of cutoff functions are introduced in following lemmas, respectively, for two local partial regularity results. Similar cutoff functions have been used by Scheffer [20] and Caffarelli, Kohn, and Nirenberg [2].
Lemma 3.5. Let \( r_n = 2^{-n} \) and \( Q_n = Q_{r_n} \). In space dimension 4, \( \{ \phi_n \}_{n \in \mathbb{N}} \) is a sequence of localized solutions of backward heat equations given by

\[
\phi_n(x,t) = \chi(x,t)\Psi_n(x,t) \quad (x,t) \in \mathbb{R}^4 \times (-\infty, 0),
\]

where \( \{ \Psi_n \}_{n \in \mathbb{N}} \) are the solutions of backward heat equations given by

\[
\Psi_n(x,t) = \frac{1}{(r_n^2 - t)^2} \exp \left( -\frac{|x|^2}{4(r_n^2 - t)} \right)
\]

and \( \chi \) is a cut-off function such that

\[
\chi \equiv 1 \text{ in } Q_{1/4}, \quad \chi \equiv 0 \text{ in } \mathbb{R}^4 \times (-\infty, 0) \setminus Q_{1/3},
\]

then the following statements hold for any integer \( n \in \mathbb{N} \):

1. \( \partial_t \phi_n + \Delta \phi_n = 0 \) in \( Q_{1/4} \);
2. \( |\partial_t \phi_n + \Delta \phi_n| \leq C_4 \) in \( \mathbb{R}^4 \times (-\infty, 0) \);
3. \( C_4^{-1} r_n^{-4} \leq \phi_n \leq C_4 r_n^{-4} \) and \( |\nabla \phi_n| \leq C_4 r_n^{-5} \) in \( Q_n \);
4. \( \phi_n \leq C_4 r_k^{-4} \) and \( |\nabla \phi_n| \leq C_4 r_k^{-5} \) in \( Q_{k-1} \setminus Q_k \) for any \( 2 \leq k \leq n \).

Note that the constant \( C_4 > 0 \) is absolute.

Proof. The first statement is obvious. For the second we compute

\[
\partial_t \phi_n + \Delta \phi_n = \Psi_n(\partial_t \chi + \Delta \chi) + \chi(\partial_t \Psi_n + \Delta \Psi_n) + 2\nabla \Psi_n \cdot \nabla \chi
\]

\[
= \Psi_n(\partial_t \chi + \Delta \chi) + 2\nabla \Psi_n \cdot \nabla \chi.
\]

Because any derivative of \( \chi \) vanishes in \( Q_{1/4} \) and \( \Psi_n, \nabla \Psi_n \) are uniformly bounded in \( (x,t) \in \mathbb{R}^4 \times (-\infty, 0) \setminus Q_{1/4} \), we can deduce that \( |\partial_t \phi_n + \Delta \phi_n| \) is bounded uniformly in \( (x,t) \in \mathbb{R}^4 \times (-\infty, 0) \) and in \( n \in \mathbb{N} \).

For the third, if \( (x,t) \in Q_n \), then \( r_n^2 \leq r_n^2 - t \leq 2r_n^2 \) and \( |x|^2 \leq r_n^2 \). We compute

\[
\nabla \phi_n(x,t) = \left( \frac{\nabla \chi(x,t)}{(r_n^2 - t)^2} - \frac{x \chi(x,t)}{2(r_n^2 - t)^3} \right) \exp \left( -\frac{|x|^2}{4(r_n^2 - t)} \right).
\]

The terms \( \chi, \nabla \chi \) and \( \exp \left( -\frac{|x|^2}{4(r_n^2 - t)} \right) \) are bounded from above and from below uniformly in \( (x,t) \in Q_n \) and in \( n \in \mathbb{N} \). Then the third statement follows from

\[
\frac{1}{(r_n^2 - t)^2} \leq r_n^{-4} \leq \frac{|x|^2}{(2r_n^2 - t)^3} \leq r_n^{-5}.
\]

For the fourth, if \( (x,t) \in Q_{k-1} \setminus Q_k \) and \( t \leq -r_k^2 \), we have \( |x|^2 \leq r_{k-1}^2 \) and \( r_n^2 - t \geq r_n^2 + r_k^2 \), then this statement follows from the argument for the third one. If \( (x,t) \in Q_{k-1} \setminus Q_k \) and \( t > -r_k^2 \), then \( r_k^2 \leq |x|^2 \leq r_{k-1}^2 \) and \( r_n^2 - t \leq r_n^2 + r_k^2 \), thus

\[
\phi_n(x,t) \leq \frac{\chi}{(r_n^2 - t)^2} \exp \left( -\frac{r_k^2}{2(4r_n^2 - t)} \right) \leq \chi r_k^{-4} e^{-\alpha^2/4},
\]

where \( \alpha = r_k(r_n^2 - t)^{-1/2} \) and the function \( \alpha^4 e^{-\alpha^2/4} \) is uniformly bounded. The bound for \( \nabla \phi_n \) follows similarly. \( \square \)
Lemma 3.6. In space dimension 4, fix $r > 0$. For any $0 < \theta \leq \frac{1}{2}$ we define
\[
\phi_{\theta}(x, t) = \frac{1}{[(\theta r)^2 - t]^2} \exp \left( - \frac{|x|^2}{4[(\theta r)^2 - t]} \right) \chi \left( \frac{x}{r}, \frac{t}{r^2} \right) \quad (x, t) \in \mathbb{R}^4 \times (-\infty, 0),
\]
where $\chi \in C_c^\infty(B_1 \times (-1, 1))$ is a cutoff function such that $\chi \equiv 1$ in $B_{1/2} \times (-\frac{1}{4}, \frac{1}{4})$. Then there exists an absolute constant $C_5 > 0$ such that
\begin{enumerate}[(1)]  
  \item $C_5^{-1}(\theta r)^{-4} \leq \phi_{\theta} \leq C_5(\theta r)^{-4}$ in $Q_{\theta r}$;
  \item In $Q_r$, we have following bounds,  
    \[
    \phi_{\theta} \leq C_5(\theta r)^{-4}, \\
    |\nabla \phi_{\theta}| \leq C_5(\theta r)^{-5}, \\
    |\partial_t \phi_{\theta} + \Delta \phi_{\theta}| \leq C_5 r^{-6}.
    \]
\end{enumerate}
\[\Box\]

Proof. This proof is analogue to the proof of Lemma 3.5.

These estimates will be fundamental for the local partial regularity results for the Navier-Stokes equations in space dimension 4. They involve some constants $C_1, C_2, C_3, C_4$ and $C_5$. All of these constants are absolute. This will be particularly important later.

3.2. Partial regularity results. The first partial regularity result states that $u$ is regular when $u, p, f$ and concentration measure $\omega$ satisfy a local smallness condition. This result is a version of Proposition 1 in [2] in space dimension 4 with concentration measures.

Proposition 3.7. There exist an absolute constant $\varepsilon > 0$ and, for any fixed $q > 3$, constants $\kappa = \kappa(\varepsilon, q)$ and $C = C(\varepsilon, q)$ depending on $\varepsilon$ and $q$ with the following property. If a weak solution set $(u, p, \lambda, \omega)$ of the Navier-Stokes equations (1.1) in $Q_1(0, 0)$ in space dimension 4 satisfies
\[\int_{Q_1} \left( |u|^3 + |p|^{3/2} \right) dx dt + \int_{Q_1} d\omega \leq \varepsilon \]
\[\int_{Q_1} |f|^q dx dt \leq \kappa,
\]
then $\|u\|_{L^\infty(Q_{1/2}(0, 0))} < C$.

Proof. Let $r_n = 2^{-n}$ and $Q_n = Q_{r_n}$, $n \in \mathbb{N}$. The strategy is to iteratively prove the following estimates
\[G(x_0, t_0, r_n) + G_c(x_0, t_0, r_n) + L'(x_0, t_0, r_n) \leq \varepsilon^{2/3} r_n^3
\]
\[A(x_0, t_0, r_n) + \delta(x_0, t_0, r_n) + \delta_c(x_0, t_0, r_n) \leq C_B \varepsilon^{2/3} r_n^2
\]
for all $n \in \mathbb{N}$. We use $\sum_{k=1}^n A(x_0, t_0, r_k)$, $\sum_{k=1}^n \delta(x_0, t_0, r_k)$ and $\sum_{k=1}^n \delta_c(x_0, t_0, r_k)$ to control $G(x_0, t_0, r_{n+1})$, $G_c(x_0, t_0, r_{n+1})$ and $L'(x_0, t_0, r_{n+1})$ by means of the interpolation inequalities in Lemma 3.2 and the regularity result for the pressure $p$ in
Claim 1: The inequality (3.15) holds.

Proof of Claim 1. Hölder’s inequality gives
\[G(x_0, t_0, r_1) + G_c(x_0, t_0, r_1) + L(x_0, t_0, r_1)\]
\[\leq 8 \int_{Q_{1/2}(x_0, t_0)} |u|^3 dxdt + 8 \int_{Q_{1/2}(x_0, t_0)} dω + 16 \int_{Q_{1/2}(x_0, t_0)} |p|^{3/2} dxdt.\]

Then we impose the first condition on \( ε > 0, \)
\[ε \leq 2^{-2} \]

Now we can invoke initial smallness condition and it yields
\[G(x_0, t_0, r_1) + G_c(x_0, t_0, r_1) + L(x_0, t_0, r_1) \leq 16ε \leq ε^{2/3} r_1^3.\]

Claim 2: \( \{3.15\}_k \) \( 1 \leq k \leq n \) implies \( 3.16 \) \( n+1. \)

Proof of Claim 2. Let \( φ_n \) be the localized solution of the backward heat equation
\[φ_n(x, t) = \frac{χ(x, t)}{(r^2 - t)} \exp \left(-\frac{|x|^2}{4(r^2 - t)}\right)\]
with \( χ \) as given in Lemma \( 3.5 \) Define smooth cutoff functions \( \{η_k\}_k \in \mathbb{N} \) such that
\[η_k \equiv 1 \text{ in } Q_{r_k/8}, \quad η_k \equiv 0 \text{ in } \mathbb{R}^4 \times (-\infty, 0)\setminus Q_k, \quad |∇η_k| \leq C'_r r_k^{-1}.\]

Then define \( ϕ_k := φ_n(η_k - η_{k+1}) \) for \( 1 \leq k \leq n - 1 \) and \( ϕ_n := φ_n η_n. \) It is easy to check the bound
\[|∇ϕ_k| = |φ_n ∇η_k + η_k ∇φ_n| \leq C_4 C'_r r_k^{-5} \text{ for any } k \leq n\]

and the fact \( φ_n = \sum_{k=1}^n ϕ_k. \)

We use \( φ_n \) as the cutoff function in the local energy inequality \( 2.19 \) and choose the constants \( γ_k = p r_k. \) This yields
\[\limsup_{k→∞} \sup_{t} \int_{B_{1/2}} |u_k|^2 φ_n dx + \int_{Q_{1/2}} φ_n (|∇u|^2 dxdt + dλ)\]
\[\leq I_1 + 3I_2 + 2I_3 + I_4,\]
where

\[ I_1 = \int\int_{Q_{1/2}} |u|^2 |\partial_t \phi_n + \Delta \phi_n| dt\ dx \]

\[ I_2 = \sum_{k=1}^{n} \int\int_{Q_{1/2}} |\nabla \varphi_k| (|u|^3 dt + d\omega) \]

\[ I_3 = \sum_{k=1}^{n} \int\int_{Q_{1/2}} |\nabla \varphi_k| |p - \tilde{p}_r|^3/2 dt\ dx \]

\[ I_4 = \int\int_{Q_{1/2}} |u||f||\phi_n| dt\ dx. \]

With the bounds in Lemma 3.5, we can deduce

\[ C_4^{-1} r_n^{2}(A(r_n+1) + \delta(r_n+1) + \delta_c(r_n+1)) \leq I_1 + I_2 + I_3 + I_4 \]

For \( I_1 \), we use the bounds in Lemma 3.5, Hölder’s inequality and the initial smallness condition (3.14),

\[ I_1 \leq C_4 \left( \int\int_{Q_{1/2}} |u|^2 dt\ dx \right)^{1/3} \left( \int\int_{Q_{1/2}} |u|^3 dt\ dx \right)^{2/3} \]

\[ \leq C_4 \left( \int\int_{Q_1} |u|^3 dt\ dx \right)^{2/3} \leq C_4 \epsilon^{2/3} \]

For \( I_2 \), we need to decompose the integral over \( Q_{1/2} \) into integrals over parabolic rings. Then for each subintegral we use the bounds in Lemma 3.5 and our induction hypothesis \([3.15]\) \(1 \leq k \leq n\) to obtain

\[ I_2 = \sum_{k=2}^{n} \int\int_{Q_{k-1} \setminus Q_k} |\nabla \varphi_k| (|u|^3 dt + d\omega) + \int\int_{Q_n} |\nabla \varphi_k| (|u|^3 dt + d\omega) \]

\[ \leq C_4 C' \sum_{k=2}^{n} r_k^{-5} \int\int_{Q_{k-1} \setminus Q_k} (|u|^3 dt + d\omega) + C_4 C' r_n^{-5} \int\int_{Q_n} (|u|^3 dt + d\omega) \]

\[ \leq C_4 C' 2^5 \sum_{k=2}^{n} r_k^{-5} \int\int_{Q_{k-1}} (|u|^3 dt + d\omega) + C_4 C' r_n^{-5} \int\int_{Q_n} (|u|^3 dt + d\omega) \]

\[ \leq C_4 C' 2^5 \sum_{k=1}^{n} r_k \epsilon^{2/3}. \]

We estimate \( I_4 \) and \( I_3 \) similarly to \( I_2 \), doing the decomposition and using the bounds in Lemma 3.5, Hölder’s inequality, the initial smallness condition (3.14) and
the induction hypothesis } (3.15)_{k} \text{ for } 1 \leq k \leq n,

\[ I_4 \leq 2C_4 \sum_{k=2}^{n} \int_{Q_{k-1} \setminus Q_k} |u||f||\phi_k|dxdt + \int_{Q_n} |u||f||\phi_k|dxdt \]

\[ \leq 2C_4 \sum_{k=1}^{n} r_k^{-4} \left( \int_{Q_k} |u|^3 dxdt \right)^{1/3} \left( \int_{Q_k} |f|^q dxdt \right)^{1/q} \left( \int_{Q_k} 1 dxdt \right)^{2/3 - 1/q} \]

\[ \leq 2C_4 \sum_{k=1}^{n} r_k^{2-6/q} \varepsilon^{2/9} \kappa^{1/q}, \]

\[ I_3 \leq C_4 C'' \sum_{k=2}^{n} r_k^{-5} \int_{Q_{k-1} \setminus Q_k} |p - \tilde{p}_r_k|^3/2 dxdt + C_4 C'_r n^{-5} \int_{Q_n} |p - \tilde{p}_r_n|^3/2 dxdt \]

\[ \leq C_4 C'' \sum_{k=1}^{n} r_k^{1/2} \varepsilon^{2/3} \]

\[ \leq C_4 C'' \varepsilon^{2/3}, \]

respectively.

Now we can combine the estimates for \( I_1, I_2, I_3, I_4 \) and the inequality \( (3.20) \),

\[ A(r_{n+1}) + \delta(r_{n+1}) \leq C_3^2 C_4 r_{n+1}^2 \left( \varepsilon^{2/3} + 96 C' \varepsilon^{2/3} + 2C' \sum_{k=1}^{n} r_k^{2/3} \varepsilon^{2/3} + \sum_{k=1}^{n} r_k^{2-6/q} \varepsilon^{2/9} \kappa^{1/q} \right). \]

Since \( q > 3 \), we can choose the constants \( C_B \) and \( \kappa \) as

\[ C_B = C_4^2 \left( 1 + 96 C' + 2C' \sum_{k=1}^{n} r_k^{2/3} + \sum_{k=1}^{\infty} r_k^{2-6/q} \right), \]

\[ \kappa = \varepsilon^{4q/9}. \]

Because \( C' \) and \( C_4 \) are absolute, note that \( C_B \) only depends on \( q \). Then it yields

\[ A(r_{n+1}) + \delta(r_{n+1}) + \delta_{c}(r_{n+1}) \leq C_B \varepsilon^{2/3} r_{n+1}^2. \]

Since this argument and the constants \( C_B, \kappa \) are uniform for all \( (x_0, t_0) \in Q_{1/2}(0, 0) \), we can deduce that \( (3.16) \) \(_{n+1} \) holds true. \( \Box \)

**Claim 3:** \( \{3.16\}_k \) \(_{2 \leq k \leq n} \) implies \( (3.15)_n \).

**Proof of Claim 3.** For simplicity, let \( (x_0, t_0) = (0, 0) \). The interpolation inequality Lemma \( 3.2 \) yields for any \( 2 \leq k \leq n \),

\[ (3.21) \quad G(r_k) + G_\varepsilon(r_k) \leq C_4 A^{3/2}(r_k) + C_1 \delta^{3/2}(r_k) + C_1 \delta^{3/2}_\varepsilon(r_k) \leq C_1 C_B \varepsilon r_k^3. \]
Let $\rho = r_1, r = r_n$, Lemma \[3.3\] yields

\[ L'(r_n) \leq C_2 r_n^{-5/2} \int_{Q_{n-1}} |u|^3 dx dt + C_2 r_n^{5} \left( \sup_{-r_n^2 < t < 0} \int_{2r_n < |y| < r_1} \frac{|u|^2}{|y|^5} dy \right)^{3/2} + C_2 \frac{r_n^3}{r_1^{1/2}} \int_{Q_2} (|u|^3 + |p|^{3/2}) dx dt \]

(3.22)

\[ \leq C_2 r_n^{1/2} G(r_{n-1}) + C_2 r_n^{5} \left( \sup_{-r_n^2 < t < 0} \int_{2r_n < |y| < r_1} \frac{|u|^2}{|y|^5} dy \right)^{3/2} + C_2 \varepsilon r_n^{3} \]

\[ \leq C_2 (8C_1 C_B r_n^{1/2} + 1) \varepsilon r_n^{3} + C_2 r_n^{5} \left( \sup_{-r_n^2 < t < 0} \int_{2r_n < |y| < r_1} \frac{|u|^2}{|y|^5} dy \right)^{3/2}. \]

In the first line of (3.22), for the first term, we use the interpolation inequality in Lemma \[3.2\] and our induction hypothesis \[3.16\] \[n-1\]. For the third term, we use the initial smallness condition \[3.14\]. For the second term, we decompose this integral into integrals over rings and estimate it using induction hypothesis \{3.16\} \[2 \leq k \leq n\].

\[ \sup_{-r_n^2 < t < 0} \int_{2r_n < |y| < r_1} \frac{|u|^2}{|y|^5} dy \leq \sum_{k=2}^{n-1} \sup_{-r_n^2 < t < 0} \int_{r_k < |y| < r_{k-1}} \frac{|u|^2}{|y|^5} dy \]

\[ \leq \sum_{k=2}^{n-1} r_k^{-3} A(r_{k-1}) \]

\[ \leq C_B \varepsilon^{2/3} \sum_{k=2}^{n-1} r_k^{-1} \]

\[ \leq C_B \varepsilon^{2/3} r_n^{-1}. \]

In the second inequality we use $|y|^{-5} \leq r_k^{-5}$ when $r_k < |y| < r_{k-1}$. The third inequality follows from our induction hypothesis \{3.16\} \[2 \leq k \leq n\].

Hence, from (3.21), (3.22) and (3.23), we can deduce that

\[ G(r_n) + G_C(r_n) + L'(r_n) \leq \left[ C_2 (8C_1 C_B r_n^{1/2} + 1 + C_3^{3/2} r_n^{1/2}) + C_1 C_B \right] \varepsilon r_n^{3}. \]

Now we impose the second condition on $\varepsilon > 0$,

\[ [C_2 (8C_1 C_B + 1 + C_3^{3/2}) + C_1 C_B] \varepsilon^{1/3} < 1. \]

It yields

\[ G(r_n) + G_C(r_n) + L'(r_n) \leq \varepsilon^{2/3} r_n^{3}. \]

Then we can argue for any $(x_0, t_0) \in Q_{1/2}(0, 0)$. Because $C_1$ and $C_2$ are absolute and $C_B$ only depends on $q$, the choice of $\varepsilon$ is uniform for any $(x_0, t_0) \in Q_{1/2}(0, 0)$. □

Now we can deduce that \[3.16\] \[k\] holds for any $k \geq 2$. This gives

\[ \sup_{t_0 - r_n^2 < t < t_0} \int_{B_{r_n}(x_0)} |u|^4 dx \leq \limsup_{k \to \infty} \sup_{t_0 - r_n^2 < t < t_0} \int_{B_{r_n}(x_0)} |u_k|^2 dx \leq C_B \varepsilon^{2/3} \]
for any \((x_0, t_0) \in Q_{1/2}(0, 0)\) and \(n \geq 2\). Hence
\[
|u(x, t)| \leq C_B^{1/2} \varepsilon^{1/3},
\]
given that \((x, t) \in Q_{1/2}(0, 0)\) is a Lebesgue point of \(u\).

\[\square\]

**Remark 3.8.** Observe that the above proof crucially uses the term \(L'\). If instead of \(L'\), we were to carry out the estimate (3.22) with \(L\), we would obtain \(\varepsilon r^{5/2}_n\) when estimating the term in the second line of (3.22), which cannot be bounded by \(\varepsilon^{2/3} r^{3}_n\) uniformly in \(n \in \mathbb{N}\).

The second partial regularity result corresponds to a version of Proposition 2 in [2] in space dimension 4 with concentration measures. We use an idea from Lin’s work [14] where he gave a simpler proof for the results in [2]. As a consequence, we are able improve Scheffer’s result in [19], to show that the 2-dimensional parabolic Hausdorff measure of the singular set of \(u\) in space dimension 4 is finite.

**Proposition 3.9.** There exists an absolute constant \(\tau > 0\) with the following property. Suppose that \((u, p, \lambda, \omega)\) is a weak solution set of the Navier-Stokes equations (1.1) on some cylinder \(Q_{\rho}(x_0, t_0)\) in space dimension 4 with \(f \in L^q_{\text{loc}}(Q_{\rho}(x_0, t_0))\) for some \(q > 3\). If
\[
\limsup_{r \to 0} \frac{1}{r^2} \int_{Q_r(x_0, t_0)} (|\nabla u|^2 dx dt + d\lambda) \leq \tau,
\]
then \(\|u\|_{L^\infty(Q_{r_0}(x_0, t_0))} < Cr_0^{-1}\) for some \(0 < r_0 < \rho\). Note that \(C = C(\varepsilon, q)\) depends on \(\varepsilon\) and \(q\) in Proposition 3.7.

**Proof.** Because \(q > 3\), there holds \(3q - 6 > 3\). Then there exists \(r_1 > 0\), such that for any \(0 < r \leq r_1\),
\[
F_1(r, x_0, t_0) = r^{3q-6} \int_{Q_r(x_0, t_0)} |f|^q dx dt \leq \kappa.
\]

**Claim 1:** For some \(0 < r_2 \leq r_1\),
\[
(3.24) \quad r_2^{-3} \int_{Q_{r_2}(x_0, t_0)} \left( |u|^3 + |p|^{3/2} \right) dx dt + r_2^{-3} \int_{Q_{r_2}} d\omega \leq \varepsilon.
\]

If this claim holds, Proposition 3.7 yields \(\|u\|_{L^\infty(Q_{r_2}(x_0, t_0))} < Cr_2^{-1}\) immediately.

**Proof of Claim 1.** We use the local energy inequality (2.20) to derive the smallness condition (3.24). Fix \(r \in (0, \rho), \theta \in (0, \frac{1}{2}]\) and \((x_0, t_0) = (0, 0)\). We choose cutoff function \(\phi_\theta\) as stated in Lemma 3.6. Then from (2.20) we deduce
\[
(3.25) \quad \frac{1}{C_5(\theta r)^2} \limsup_{k \to \infty} \sup_{-(\theta r)^2 \leq t < 0} \int_{J_{B_{\theta r}}} |u_k|^2 dx + \frac{1}{C_5(\theta r)^2} \int_{Q_{\theta r}} (|\nabla u|^2 dx dt + d\lambda) \leq I_1 + I_2 + I_2' + I_3 + I_4,
\]
where

\[ I_1 = (\theta r)^2 \int_{Q_r} |u|^2 (\partial_t \phi_\theta + \Delta \phi_\theta) \, dx \, dt, \]
\[ I_2 = (\theta r)^2 \int_{Q_r} |u|^2 u \cdot \nabla \phi_\theta \, dx \, dt, \]
\[ I_2' = 3(\theta r)^2 \int_{Q_r} |\nabla \phi_\theta| (|u - \bar{\phi}_\theta|^3 \, dx \, dt + d\omega), \]
\[ I_3 = 2(\theta r)^2 \int_{Q_r} pu \cdot \nabla \phi_\theta \, dx \, dt, \]
\[ I_4 = (\theta r)^2 \int_{Q_r} f \cdot u \phi_\theta \, dx \, dt. \]

For \( I_1, I_3 \) and \( I_4 \), we simply use Hölder’s inequality and the bounds in Lemma 3.6. For \( I_2' \), we use the interpolation inequalities in Lemma 3.2. Thus, we obtain

\[
I_1 \leq C_5 \theta^2 G^{2/3}(r), \\
I_3 \leq C_5 \theta^{-3} K^{2/3}(r) G^{1/3}(r), \\
I_4 \leq C_5 \theta^{-2} F^{1/2}_2(r) G^{1/3}(r), \\
I_2' \leq C_5 \theta^{-3} [H(r) + G_c(r)].
\]

For \( I_2 \), we reduce the estimates on \( u \) to estimates on \( u - \bar{u}_r \), then we use the interpolation inequality in Lemma 3.2. Note that \( |u|^2 \in L_t^3 W_x^{1,1}(Q_r) \), because \( u \in L_t^2 H_x^1(Q_r) \cap L_t^\infty L_x^2(Q_r) \). We have

\[
I_2 = (\theta r)^2 \int_{Q_r} |u|^2 (u - \bar{u}_r) \cdot \nabla \phi_\theta \, dx \, dt + (\theta r)^2 \int_{Q_r} |u|^2 \bar{u}_r \cdot \nabla \phi_\theta \, dx \, dt \\
= (\theta r)^2 \int_{Q_r} |u - \bar{u}_r|^2 (u - \bar{u}_r) \cdot \nabla \phi_\theta \, dx \, dt + 2(\theta r)^2 \int_{Q_r} u \cdot \bar{u}_r (u - \bar{u}_r) \cdot \nabla \phi_\theta \, dx \, dt \\
+ (\theta r)^2 \int_{Q_r} |\bar{u}_r|^2 (u - \bar{u}_r) \cdot \nabla \phi_\theta \, dx \, dt + (\theta r)^2 \int_{Q_r} |u|^2 \bar{u}_r \cdot \nabla \phi_\theta \, dx \, dt \\
\leq C_5 (\theta r)^{-3} \int_{Q_r} |u - \bar{u}_r|^3 \, dx \, dt - 2(\theta r)^2 \int_{Q_r} \left[ ((u - \bar{u}_r) \cdot \nabla) u \right] \cdot \bar{u}_r \phi_\theta \, dx \, dt \\
- 2(\theta r)^2 \int_{Q_r} u \cdot ((\bar{u}_r \cdot \nabla) u) \phi_\theta \, dx \, dt.
\]
where we use the bounds in Lemma 3.6 and integration by parts. Notice that $u$ and $u - \tilde{u}_r$ are divergence-free. Furthermore, we have

$$(3.28)$$
\[
\iint_{Q_r} \left[ ((u - \tilde{u}_r) \cdot \nabla) u \right] \cdot \tilde{u}_r \phi dy dt \leq C_5 (\theta r)^{-4} \int_{-r^2}^0 r^{-4} \int_{B_r} |u| dx \int_{B_r} |u - \tilde{u}_r| |\nabla u| dx dt
\]
\[
\leq C_5 \theta^{-4} r^{-16/3} \int_{-r^2}^0 \left( \int_{B_r} |u|^3 dx \right)^{1/3} \left( \int_{B_r} |u - \tilde{u}_r|^2 dx \right)^{1/2} \left( \int_{B_r} |\nabla u|^2 dx \right)^{1/2} dt
\]
\[
\leq C_5 \theta^{-4} r^{-5} \left( \iint_{Q_r} |u|^3 dx dt \right)^{1/3} \left( \iint_{Q_r} |\nabla u|^2 dx dt \right)^{1/2} \left( \sup_{-r^2 < t < 0} \int_{B_r} |u|^2 dx \right)^{1/2}.
\]

The first inequality follows from the bounds in Lemma 3.6. The remaining inequalities follow from Hölder’s inequality. Analogously,

$$(3.29)$$
\[
\iint_{Q_r} u \cdot ((\tilde{u}_r \cdot \nabla) u) \phi dy dt \leq C_5 \theta^{-4} r^{-5} \left( \iint_{Q_r} |u|^3 dx dt \right)^{1/3} \left( \iint_{Q_r} |\nabla u|^2 dx dt \right)^{1/2} \left( \sup_{-r^2 < t < 0} \int_{B_r} |u|^2 dx \right)^{1/2}.
\]

Now, from (3.27), (3.28) and (3.29), we deduce

$$(3.30)$$
\[
I_2 \leq 2C_5 \theta^{-2} G^{1/3}(r) A^{1/2}(r) \delta^{1/2}(r).
\]

Now, we are in a position to plug (3.26) and (3.30) into (3.25) and to invoke the interpolation inequality in Lemma 3.2

$$(3.31)$$
\[
A(\theta r) + 2[\delta(\theta r) + \delta_c(\theta r)] \leq C_5^2 \left( \theta^2 G^{2/3}(r) + \theta^{-3} K^{2/3}(r) G^{1/3}(r) \right.
\]
\[
+ 2\theta^{-2} F_2^{1/2}(r) G^{1/3}(r) + \theta^{-3}[H(r) + G_c(r)]
\]
\[
+ 3\theta^{-2} G^{1/3}(r) A^{1/2}(r) \delta^{1/2}(r) \right)
\]
\[
\leq C_5^2 \left( 4\theta^2 G^{2/3}(r) + \theta^{-8} K^{4/3}(r) + \theta^{-6} F_2(r) \right.
\]
\[
+ C_1 \theta^{-3} A^{1/2}(r)[\delta(r) + \delta_c(r)] + \frac{9}{4} \theta^{-6} A(r) \delta(r) \right)
\]
\[
\leq C_5^2 \left[ 4C_1 \theta^2 A(r) + 5C_1 \theta^2 [\delta(r) + \delta_c(r)] + C_1 \theta^{-8} A(r)[\delta(r) + \delta_c(r)] \right.
\]
\[
+ \frac{9}{4} \theta^{-6} A(r) \delta(r) + \theta^{-8} K^{4/3}(r) + \theta^{-6} F_2(r) \right] .
\]

In the second inequality, we use Young’s inequality to move $\theta G^{1/3}(r)$ to the first term. In the third inequality, we use the interpolation inequality in Lemma 3.2. And Young’s inequality moves $\theta^{-3} A^{1/2}(r) \delta^{1/2}(r)$ to the term $4\theta^{-6} A(r) \delta(r)$.
On the other hand, with Lemma 3.4 we deduce
\begin{equation}
K^{4/3}(\theta r) \leq C_1^{4/3} C_3^{4/3} \theta^{-4} A^{2/3}(r) \delta^{5/3}(r) + C_3^{4/3} \theta^{4/3} K^{4/3}(r) \\
\leq C_1^{4/3} C_3^{4/3} \theta^{-12} A(r) \delta(r) + \theta^{12} \delta^2(r) + C_3^{4/3} \theta^{4/3} K^{4/3}(r).
\end{equation}
(3.32)

Taking the sum of (3.31) and \( \theta^{-9} \times (3.32) \) gives
\begin{equation}
A(\theta r) + \theta^{-9} K^{4/3}(\theta r) + 2[\delta(\theta r) + \delta_c(\theta r)] \\
\leq 4C_1 C_5^2 \theta^2 A(r) + 5C_1 C_5^2 \theta^2 [\delta(r) + \delta_c(r)] + \theta^3 \delta^2(r) \\
+ \left( \frac{9}{4} C_5^2 + C_1^2 C_3^2 \theta^{-15} + C_1 C_5^2 \theta^{-2} \right) \theta^{-9} A(r) [\delta(r) + \delta_c(r)] \\
+ (C_2^2 \theta + C_3^{4/3} \theta^{4/3}) \theta^{-9} K^{4/3}(r) + C_5^2 \theta^{-6} F_2(r).
\end{equation}
(3.33)

Since \( C_1, C_3 \) and \( C_5 \) are absolute positive constants, we can fix \( \theta \in (0, \frac{1}{2}] \) such that
\begin{equation}
4C_1 C_5^2 \theta^2 \leq \frac{1}{4}, \\
C_5^2 \theta + C_3^{4/3} \theta^{4/3} \leq \frac{1}{2}, \\
(5C_1 C_5^2 + 1) \theta^2 \leq \frac{1}{8}.
\end{equation}
(3.34)

Then we choose \( \tau \in (0, 1) \) such that
\begin{equation}
\left( \frac{9}{4} C_5^2 + C_1^2 C_3^2 \theta^{-10} + C_1 C_5^2 \theta^{-2} \right) \theta^{-9} \cdot 2 \tau \leq \frac{1}{4}.
\end{equation}
(3.35)

Because
\[ \limsup_{r \to 0} \delta(r) + \delta_c(r) \leq \tau, \quad \lim_{r \to 0} F_2(r) = 0, \]
we can choose \( r' > 0 \) such that for any \( 0 < r \leq r' \),
\[ \delta(r) + \delta_c(r) \leq 2 \tau, \quad C_5^2 \theta^{-6} F_2(r) \leq \frac{\tau}{4}. \]

Let \( E(r) = A(r) + \theta^{-9} K^{4/3}(r) + 2[\delta(\theta r) + \delta_c(\theta r)] \), then (3.33) yields for any \( 0 < r \leq r' \),
\begin{equation}
E(\theta r) \leq \frac{1}{2} E(r) + \frac{\tau}{2}.
\end{equation}
(3.36)

Iterating this inequality yields for any \( k \in \mathbb{N} \)
\[ A(\theta^k r') + \theta^{-9} K^{4/3}(\theta^k r') + 2[\delta(\theta^k r') + \delta_c(\theta^k r')] = E(\theta^k r') \leq \frac{1}{2k} E(r') + \tau. \]

Then there exists some \( r_2 > 0 \) such that
\[ A(r_2) + \delta(r_2) + \delta_c(r_2) + \theta^{-9} K^{4/3}(r_2) \leq 4 \tau. \]

Again by the interpolation inequality in Lemma 3.2 we can bound \( G(r_2) + G_c(r_2) \) with \( A(r_2) + \delta(r_2) + \delta_c(r_2) \). Then we can impose another condition on \( \tau \) to ensure (3.21). This additional condition on the choice of \( \tau \) depends on \( \varepsilon, C_1 \) and \( \theta \), so it does not produce any circular reasoning. This concludes the proof. \( \square \)
Now, we give definitions to singular set of weak solution set and parabolic Hausdorff measure of a space-time set.

**Definition 3.10.** Suppose that \((u, p, \lambda, \omega)\) is a weak solution set of the Navier-Stokes equation in \(\mathbb{R}^4 \times [0, T]\). A point \((x, t)\) is called a regular point if there exists \(r > 0\) such that \(u \in L^\infty(Q_r(x, t))\). Otherwise, \((x, t)\) is called a singular point. The singular set is the set of all singular points.

**Definition 3.11.** Given a set \(D \subset \mathbb{R}^4 \times \mathbb{R}\), for a fixed positive real number \(s\), the \(s\)-dimensional parabolic Hausdorff measure is defined as
\[
P_s^s(D) = \lim_{\delta \to 0^+} P_s^s(D),
\]
where
\[
P_s^s(D) = \inf \left\{ \sum_{i=1}^\infty r_i^s \left| D \subset \bigcup_{i \in \mathbb{N}^+} Q_{r_i}^s(x_0, t_0), 0 < r_i < \delta, (x_0, t_0) \in \mathbb{R}^4 \times \mathbb{R} \right. \right\}.
\]
Here, \(Q_{r_i}^s(x, t)\) is centered parabolic cylinder defined by
\[
Q_{r_i}^s(x, t) = B_r(x) \times \left( t - \frac{r^2}{2}, t + \frac{r^2}{2} \right).
\]

Theorem 1.1 follows from Lemma 2.3, Proposition 2.13, Proposition 3.9 and the following standard covering argument.

**Proof of Theorem 1.1.** We assume \(S\) is bounded and \(S \subset B_{\rho_0} \times [0, T]\) for some \(\rho_0 > 0\). Let \(D' := \overline{B}_{\rho_0+1}(\mathbb{R}^4) \times [0, T]\). Let \(V\) be a parabolic neighborhood (neighborhood given by parabolic cylinders) of \(S\) in \(D'\) and fix \(\delta > 0\). According to Proposition 3.9, for each \((x, t) \in S\), we choose \(Q_r(x, t) \subset V\) with \(r < \delta\) such that
\[
r^{-2} \int_{Q_r(x, t)} (|\nabla u|^2 dxdt + d\lambda) > \tau.
\]

Because \(S\) is bounded, we can use Vitali Covering lemma to obtain a family of disjoint parabolic cylinders \(\{Q_{r_i}(x_i, t_i)\}_{i \in \Lambda}\) such that
\[
S \subset \bigcup_{i \in \Lambda} Q_{5r_i}(x_i, t_i).
\]
Here \(\Lambda\) is a finite set. Then
\[
\sum_{i \in \Lambda} r_i^2 \leq \frac{1}{\tau} \sum_{i \in \Lambda} \int_{Q_{r_i}(x_i, t_i)} (|\nabla u|^2 dxdt + d\lambda) \leq \frac{1}{\tau} \int_V (|\nabla u|^2 dxdt + d\lambda).
\]
Since \(\delta\) is arbitrary, we know the Lebesgue measure of \(S\) is zero and
\[
(3.37) \quad P^s(S) \leq \frac{5}{\tau} \int_V (|\nabla u|^2 dxdt + d\lambda).
\]

In case that \(S\) is unbounded, we look at \(S \cap B_r \times [0, T]\) with \(r \to \infty\), then \(P^s(S \cap B_r \times [0, T])\) is bounded uniformly in \(r\), which concludes our proof. \(\Box\)
Remark 3.12. Dong, Gu [5] and Wang, Wu [24] proved that suitable weak solutions satisfy $P^2(S) = 0$, but they were not able to show that such solutions exist. Here, we can only prove $P^2(S) < \infty$, since the presence of the concentration measures leads to nontriviality of the Hausdorff measure of the singular set $S$ in this covering argument.

Appendix A. Fractional power of nonnegative smooth function

In this appendix, we prove that certain fractional power of any nonnegative smooth function is Lipschitz continuous, which is a direct consequence of the following lemma proved by Fefferman and Phong [7].

**Lemma A.1** (Fefferman and Phong [7]; Lemma 4, Guan [10]). If $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^{3,1}$ nonnegative function, with $\|f\|_{C^4} \leq A$, then there is $N \in \mathbb{N}$ (only depends on $n$) and functions $g_1, g_2, \ldots, g_N \in C^{1,1}$, with $\|g_j\|_{C^2} \leq C$, such that

$$f = \sum_{j=1}^{N} g_j^2,$$

where the constant $C$ depends on $n$ and $A$.

**Corollary A.2.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^{3,1}$ nonnegative function, then $h := f^\alpha$ is Lipschitz continuous for any $\alpha \in \left[\frac{1}{2}, 1\right]$.

**Proof.** This result follows from Lemma A.1 and the following bound.

$$|h'| = \frac{2\alpha \left| \sum_{j=1}^{N} g_j g_j' \right|}{\left( \sum_{j=1}^{N} g_j^2 \right)^{1-\alpha}} \leq \sum_{j=1}^{N} \frac{2\alpha \left| g_j^{2\alpha-1} g_j' \right|}{\left( \sum_{i=1}^{N} (g_i^2/g_j^2) \right)^{1-\alpha}} \leq \sum_{j=1}^{N} 2\alpha \left| g_j^{2\alpha-1} g_j' \right|.$$  

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