Linear orbital stability analysis of the pendulum-type motions of a Kovalevskaya top with a suspension point vibrating horizontally

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Abstract. The motion of a heavy rigid body is considered, with a point (suspension point) performing high-frequency horizontal vibration. The center of mass of the body is on the principal axis of inertia for this point. Within the framework of an approximate autonomous system, pendulum-type motions of the body are studied, they occur in a vertical plane containing the axis of vibration or in a plane perpendicular to it. For a body with a mass geometry corresponding to S. V. Kovalevskaya case, a linear orbital stability analysis of these motions is performed.

1. Introduction
The presence of high-frequency vibrations of the rigid body suspension has a significant influence on its motion. Vibrations can lead to the emergence of new types of body motions impossible for a body with a fixed point, or to a change in the stability of known particular cases of motion.

At the beginning of the 20th century, the phenomenon of dynamic stabilization of the inverted position of a mathematical pendulum due to vertical vibrations of the suspension point was first described [1]. Later, the mathematical pendulum and the pendulum systems were studied.

Recently, the influence of high-frequency vibrations on the dynamics of a heavy rigid body with a more complicated mass geometry, as well as more complex vibrations of its suspension point, have been investigated. In [2], an approximate autonomous system of Euler-Poisson type equations is obtained, which describes the motion of a heavy rigid body with an arbitrary mass geometry, with suspension point performing arbitrary periodic or conditionally periodic fast vibrations of small amplitude in three-dimensional space. Within the framework of this approximate system, a number of problems on dynamics of a body with a vibrating suspension has been solved. In particular, the orbital stability of the pendulum-type motions of the Lagrange top with a vibrating suspension has been studied [3].

Pendulum-type motions of a rigid body with a fixed point were first described in [4]. The orbital stability of the pendulum-type motions of a Kovalevskaya top with a fixed point was studied in [5] and [6].

This paper studies the influence of high-frequency horizontal periodic vibrations on the existence and stability of the pendulum-type motion of a rigid body with the center of mass on the principal axes of inertia for the suspension point. The problem of linear orbital stability of the pendulum-type motions for the Kovalevskaya top in the presence of such vibrations is considered in detail.
2. Statement of the problem. Equations of motion

We consider the motion of a heavy rigid body in a uniform gravitational field, assuming that one of the points of the body $O$ (the suspension point) makes a periodic, with frequency $\Omega$, motion along the horizontal line according to the law $O'O = \xi(t)$ relative to a fixed point $O'$. The average value of $\langle \xi(t) \rangle$ for the period is assumed to be zero.

We introduce a coordinate system $OXYZ$ moving translationally, with the axis $OY$ directed vertically upwards, and $OX$ along the direction of vibrations. The second coordinate system $Oxyz$ is rigidly attached to the body, with axes directed along the body principal axes of inertia for the point $O$. Let the center of mass $G$ of the body lie on the principal axis $Oz$, and $OG = z_G$. The principal moments of inertia for the point $O$ are $A, B$ and $C$, and the mass of the body is $m$. The orientation of the system $Oxyz$ relative to $OXYZ$ is defined by Euler angles $\psi, \theta$ and $\phi$.

Let the maximum deviation $h_\ast$ of the suspension point from the point $O'$ is small compared with the reduced body length $l = B / (mz_G)$, and the vibration frequency $\Omega$ of this point is large relative to the characteristic frequency $\omega_\ast = \sqrt{g/l}$. We introduce a small parameter $\varepsilon = \sqrt{h_\ast / l}$ and assume that $h_\ast \Omega \gg 1$.

We describe the motion of the body using the canonical Hamiltonian equations, choosing the Euler angles $\psi, \theta$ and $\phi$ for the generalized coordinates. Using the methods of the perturbation theory, we can obtain a near-identity canonical change of variables that reduces the Hamiltonian function of the system to a form, in which the main (approximate) part is autonomous.

We introduce the dimensionless parameters, pulses and time by the formulas

$$
\alpha = \frac{(\xi_0^2) m^2 l^2}{A'\omega_\ast^4}, \quad \mu = \frac{B}{A}, \eta = \frac{B}{C}, \quad P_\psi = B\omega_\ast P_1, \quad P_\theta = B\omega_\ast P_2, \quad P_\phi = B\omega_\ast P_3, \quad \tau = \omega_\ast t
$$

Leaving the original notation for the Euler angles, we can write the approximate Hamiltonian in the form (here $\hat{\Pi}_v$ is the vibration potential [2])

$$
\hat{H} = \frac{\eta P_1^2}{2} + \left( \frac{\sin^2 \phi + \mu \cos^2 \psi}{2} \right) P_2^2 + \frac{(\mu - 1) \sin \phi \cos \phi (P_1 - P_2 \cos \theta) P_2}{\sin \theta} \\
+ \frac{(\cos^2 \phi + \mu \sin^2 \psi)(P_1 - P_2 \cos \theta)^2}{2 \sin^2 \theta} - \cos \psi \sin \theta + \hat{\Pi}_v
$$

(1)

$$
\hat{\Pi}_v = \frac{\alpha}{2} \left( \mu (\cos \psi \sin \phi + \sin \psi \cos \phi \cos \theta)^2 + (\cos \psi \cos \phi - \sin \psi \sin \phi \cos \theta)^2 \right)
$$

(2)

Further research is carried out in the framework of the approximate system with Hamiltonian (1), (2). The errors of the solutions of this system with respect to ones of the complete system as well as the time interval, on which the approximate solutions are studied, are given in [2].

The system with the Hamiltonian (1), (2) admits particular solutions that correspond to pendulum-like body motions, where one of its principal axes of inertia not containing the center of mass $G$, takes a fixed horizontal position, and the radius vector $OG$ moves in the vertical plane containing the axis of vibration (plane $OXY$) or perpendicular to it (plane $OYZ$).

In the first case $\theta = \pi / 2, \phi = 0, P_1 = 0, P_2 = 0$, and the change in the quantities $\psi$ and $P_1$ is described by canonical equations with the Hamiltonian function
\[ H_\alpha = \frac{P^2}{2} - \cos \psi + \frac{\alpha}{2} \cos^2 \psi \]  

(3)

To describe motions in the plane perpendicular to the axis of vibrations, we redirect the axes of the coordinate system \(OXYZ\) so that the vibrations of the suspension point occur along \(OZ\). Part of the Hamiltonian (1) without the last term remains unchanged, and the vibration potential takes the form

\[ \hat{\Pi}_v = \frac{\alpha}{2} \left( \mu \cos^2 \varphi + \sin^2 \varphi \right) \sin^2 \theta \]

(4)

The motions at hand are given by the relations \(\theta = \pi / 2, \varphi = \pi / 2\), \(P_2 = 0, P_1 = 0\), while the quantities \(\psi\) and \(P_1\) satisfy the canonical equations of the mathematical pendulum, defined by the Hamiltonian

\[ H_{01} = \frac{\mu P^2_1}{2} - \cos \psi \]

(5)

Note that in the case of the absence of vibrations of the body suspension point, the pendulum-like motions of the radius vector center \(OG\) can occur in any fixed vertical plane. However, there is a cyclic coordinate in the system if we choose the other coordinates axes and these motions are unstable with respect to spatial perturbations.

Further, we will assume that the body mass geometry corresponds to S. V. Kovalevskaya case, which is realized for \(\mu = 2, \eta = 1\).

The aim of the work is to describe and study the linear orbital stability of the pendulum-like motions of the Kovalevskaya top with a horizontally vibrating suspension point that correspond to the model systems with Hamiltonians (3) and (5).

3. Pendulum-type motions

Let us consider equilibrium positions of the model systems with Hamiltonians (3) and (5). The system with Hamiltonian (3) with \(\alpha < 1\) has a lower stable equilibrium at \(\psi = 0\) and upper unstable equilibrium at \(\psi = \pi\). If \(\alpha > 1\), then there are also stable inclined equilibrium positions at \(\psi = \pm \arccos(\alpha^{-1})\), and the lower and upper equilibria are unstable. The motions of the system with Hamiltonian (5) do not depend on the vibration intensity, and there are stable lower and unstable upper equilibria.

The model systems under study have the first integrals (energy integrals) of the form \(H_\alpha = h = \text{const}\), \(H_{01} = h_1 = \text{const}\). Depending on the value of the energy constant, the pendulum-like motions are oscillations around stable equilibrium positions, rotations, or asymptotic motions. Figure 1 a and 1 b show bifurcation diagrams on the parameter plane \(\alpha, h\) and \(\alpha, h_1\) for systems with Hamiltonians (3) and (5), respectively.

![Bifurcation diagrams](image)
In the diagram in figure 1 a the boundaries of regions with different types of motions are straight lines \( h = \pm 1 + \alpha / 2 \) and part of the hyperbola \( 2\alpha h = -1, \alpha > 1 \). The regions in the diagram in figure 1 b separated by straight lines \( h_0 = \pm 1 \). In the regions \( \Gamma_0 \) and \( \Gamma_0' \) motion is impossible. In the region \( \Gamma_1 \), the radius vector of the center of mass of the body oscillates around the inclined equilibrium. The regions \( \Gamma_2, \Gamma_2' \) correspond to the oscillations of this vector near the lower position, and the regions \( \Gamma_3 \) and \( \Gamma_3' \), to the rotations.

Let us integrate the equations of motion of the system with Hamiltonian (3) in each of the regions in figure 1 a. We solve the energy integral

\[
H_h = 0
\]

with respect to \( P_1 \) and, given that \( \psi' = P_1 \), we write the differential equation of the pendulum-type motion in the form

\[
\frac{d\psi}{d\tau} = \pm \left( 2h + 2\cos\psi - \alpha \cos^2\psi \right)^{1/2}
\]

Assuming that \( u = \cos\psi \) (\( |u| \leq 1 \)), we rewrite this equation as

\[
\tau = \pm \int_0^u \frac{du}{f(u)}, \quad f(u) = -\alpha (1-u^2)(u-u_1)(u-u_2)
\]

Here \( u_1 = \left( 1 - \sqrt{1 + 2\alpha h} \right) \alpha^{-1}, \ u_2 = \left( 1 + \sqrt{1 + 2\alpha h} \right) \alpha^{-1} \).

The integral on the right side of (6) is expressed in terms of elliptic functions. Further, we use the notation \( \text{sn} \) for elliptic sine, \( F(u,k) \) for elliptic integral of the first kind, \( K(k) \) for complete elliptic integral of the first kind, and \( k \) for module of the elliptic integral.

If the values of the parameters \( \alpha \) and \( h \) correspond to the region \( \Gamma_1 \), then the roots of the polynomial \( f(u) \) from (6) satisfy the condition \(-1 < u_1 < u_2 < 1 \). For oscillations in the vicinity of the inclined position, condition \( u_1 \leq u \leq u_2 \) satisfies. Assuming that \( u |_{\tau=0} = u_2 \) and using some formulas from [7], we obtain

\[
\tau = -\frac{F(u_2,k_1)}{\xi_1} u_2 = \arcsin \left( \frac{(1-u_1)(u_2-u)}{(1-u)(u_2-u_1)} \right)^{1/2}, \ \xi_1 = \frac{1}{2} \left( \alpha (1-u_1)(u_2+1) \right)^{1/2}, \ k_1 = \left( \frac{2(u_2-u_1)}{(1-u_1)(u_2+1)} \right)^{1/2}
\]

Reversing the function \( \tau(u) \), we have

\[
u = \frac{(u_2-u_1)\text{sn}^2(\xi_1,\xi_2,\xi_2) - u_2(1-u_1)}{(u_2-u_1)\text{sn}^2(\xi_1,\xi_2,\xi_2) - 1+u_1}
\]

The resulting solution is periodic, with frequency \( \omega_1 = \pi \xi_1 / (2K(k_1)) \).

The motions in the region \( \Gamma_2 \) are oscillations that cover the lower and inclined equilibrium positions (if they exist), and \(-1 < u_1 \leq u \leq 1 < u_2 \). In this case, the solution with the initial condition \( u |_{\tau=0} = 1 \) is

\[
u = \frac{u_2(1-u_1)\text{sn}^2(\xi_2,\xi_2,\xi_2) + u_2 - u_1}{(1-u_1)\text{sn}^2(\xi_2,\xi_2,\xi_2) + u_2 - u_1}, \ \xi_2 = \frac{1}{2} \left( 2\alpha (u_2-u_1) \right)^{1/2}, \ k_2 = \left( \frac{(1-u_1)(u_2+1)}{2(u_2-u_1)} \right)^{1/2}
\]

and its frequency is \( \omega_2 = \pi \xi_2 / (2K(k_2)) \).

For the rotations in the region \( \Gamma_3 \) we have \( u_1 < -1 \leq u \leq 1 < u_2 \). The solution with the initial condition \( u |_{\tau=0} = 1 \) is represented as
\[ u = \frac{2u_2 \cdot \text{sn}^2(\xi_1, k_1) - u_2 - 1}{2 \text{sn}^2(\xi_1, k_1) - u_2 - 1}, \quad \xi_3 = \frac{1}{2} \left( \alpha (1 - u_1) (u_2 + 1) \right)^{1/2}, \quad k_3 = \left( \frac{2(u_2 - u_1)}{(1 - u_1)(u_2 + 1)} \right)^{1/2} \]

and its frequency is \( \omega_3 = \frac{\pi \xi_3}{2 K(k_3)} \).

At the boundary \( h = 1 + \alpha / 2 \), the function \( f(u) \) has a double root \( u = -1 \). Integrating, we find

\[ u = \frac{\mu^2 - (4\alpha - 6) \mu + 1}{-\mu^2 + (4\alpha - 2) \mu + 1}, \quad \mu = \exp \left( -2\sqrt{1 + \alpha} \right), \quad u|_{\alpha = 0} = 1 \]

This expression tends with time to the point \( u = -1 \) corresponding to the upper equilibrium position.

At the boundary of the regions \( \Gamma_1 \) and \( \Gamma_2 \), the function \( f(u) \) has a double root \( u = 1 \). Integration yields

\[ u = \frac{\mu^2 - (4\alpha - 6) \mu + 1}{\mu^2 + (4\alpha - 2) \mu + 1}, \quad \mu = \exp \left( 2\sqrt{\alpha - 1} \right), \quad u|_{\alpha = 0} = u_i \]

In this solution, the radius vector of the center of mass asymptotically approaches the lower equilibrium position.

The motion of the system with Hamiltonian (5) is described by the equation of the ordinary mathematical pendulum. The frequencies of the motions in the regions \( \Gamma_{21}, \Gamma_{31} \) are respectively

\[ \omega_{21} = \frac{\pi}{2\sqrt{2} K(k_{21})}, \quad \omega_{31} = \frac{\pi k_{21}}{\sqrt{2} K(k_{21}^{-1})}, \quad k_{21} = \left( \frac{h_1 + 1}{2} \right)^{1/2} \]

### 4. The study of orbital stability

To study the orbital stability of the motions considered, we introduce action-angle variables \( I, \omega \) in the regions \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_{31}, \Gamma_{21} \) of oscillations and rotations of the model systems. The corresponding particular solutions of the complete three-degree-of-freedom systems, in which the “pendulum part” is written in variables \( I, \omega \), will be taken as unperturbed motions.

In the system with Hamiltonian (1), we introduce perturbations by the formulas

\[ r = I_0 - I, p_1 = P_2, p_2 = p_3, p_3 = q_2, p_q = \theta - \pi / 2, q_3 = q - q_0, \]

where \( I_0 \) and \( q_0 \) are values of the variable action and the angle \( \phi \) on the unperturbed motion. We consider the quadratic part, in the variables \( r^{1/2}, p_1, p_2, q_2, q_3, \) of the perturbed Hamiltonian, representing it in the form

\[ H = H_2 + \ldots, \quad H_2 = \omega_0 r + H_{2*}(p_2, q_2, p_3, q_3, \omega) \]  

(7)

Here \( \omega_0 \) is a frequency of unperturbed motion. If pendulum-like motions of the system with Hamiltonian (3) are taken for the unperturbed motion, then the function \( H_{2*} \) has the form

\[ H_{2*} = p_2^2 + P_2 q_2 p_2 + \frac{\cos \varphi + 2\alpha \sin^2 \varphi + P_1^2}{2} q_2^2 + P_1 p_2 q_3 - \frac{\alpha \sin 2\varphi}{2} q_3^2 + \frac{p_3^2}{2} + \frac{P_1^2 + \alpha \cos^2 \varphi}{2} q_3^2 \]  

(8)

If the unperturbed motion is described by a system with Hamiltonian (5), then we have

\[ H_{2*} = \frac{P_1^2}{2} - P_1 q_2 p_2 + p_2^2 + 2 P_1 q_2 p_3 + \frac{P_1^2 - \alpha + \cos \varphi}{2} q_2^2 + \frac{\alpha - \frac{P_1^2}{2}}{2} q_3^2 \]

(9)
In expressions (8) and (9), the functions \( w(\psi) \) and \( P(\psi) \) are calculated for the unperturbed motion.

Further, we will consider the linearized equations of perturbed motion described by the Hamiltonian \( H_2 \) from (7) with regard to relations (8) and (9). At the energy level \( H_2 = 0 \) that corresponds to the unperturbed motion we realize the isoenergetic reduction and, taking the quantity \( w \) as a new independent variable, we consider a non-autonomous reduced two-degree-of-freedom system with the Hamiltonian \( K = H_2, \alpha_0^{-1} \).

The stability criteria for the trivial equilibrium of a system with the Hamiltonian \( H_2 \) from (7) and the reduced system coincide [8]. The characteristic equation of the latter is

\[
\rho^4 - a_1 \rho^3 + a_2 \rho^2 - a_1 \rho + 1 = 0
\]

Here, \( a_1 \) is the trace of the monodromy matrix \( X(2\pi) \), and \( a_2 \) is the sum of its main second-order minors. Stability conditions are given by inequalities

\[-2 < a_2 < 6, \quad 4(a_2 - 2) < a_1^2 < (a_2 + 2)^2 / 4\]

5. Generating points

At energy levels of unperturbed motion corresponding to stable equilibrium positions, the system with the Hamiltonian \( K \) is autonomous. The eigenfrequencies \( \Omega_{i,j} \) \( (i \) is a region number, \( j = 1, 2 \) \) of the equations of small linear oscillations in the regions \( \Gamma_1, \Gamma_2, \Gamma_{21} \) have the form

\[
\Omega_{i,1} = \frac{\left(4\alpha^2 - 1 + 2\sqrt{1 - 16\alpha^2 + 16\alpha^4}\right)^{1/2}}{\sqrt{\alpha^2 - 1}}, \quad \Omega_{2,1} = \frac{\sqrt{2}}{\sqrt{1 - \alpha}}, \quad \Omega_{21,1} = \sqrt{2}\alpha,
\]

\[
\Omega_{i,2} = \frac{\left(4\alpha^2 - 1 - 2\sqrt{1 - 16\alpha^2 + 16\alpha^4}\right)^{1/2}}{\sqrt{\alpha^2 - 1}}, \quad \Omega_{2,2} = \frac{\sqrt{\alpha}}{\sqrt{1 - \alpha}}, \quad \Omega_{21,2} = \sqrt{2(1 - \alpha)}.
\]

The points of the resonances of the first and second orders are generating points for parametric resonance regions and single resonant curves on the parameter plane \( \alpha, \hbar \) or \( \alpha, \hbar_i \). At resonances of the first order, the relations \( \Omega_{i,j} = N \) are satisfied, and at resonances of the second order, the relations \( \Omega_{i,j} = 2N - 1 \), \( \Omega_{i,1} - \Omega_{i,2} = N \), \( \Omega_{i,1} + \Omega_{i,2} = N \). Here \( N \) is an integer number. Let \( \alpha_{i,k} \) be resonant values of the parameter \( \alpha \) characterizing intensity of vibrations, where \( i \) is region number, and \( k \) resonant relation.

Consider the generating points of the resonances of the first and second orders for pendulum-type motions of the system with Hamiltonian (3). For all types of resonances at hand, the sequences of values of \( \alpha_{i,k} \) monotonically tend to \( \alpha = 1 \) as the value \( N \) increases. Further, we will limit ourselves to the following points:

\[
\alpha_{1,\Omega_{1,2,3,4}} = 1/2, \quad 7/9, \quad 7/8; \quad \alpha_{1,\Omega_{1,3,2,5,2,7,2}} = 1/9, \quad 17/25, \quad 41/49;
\]
\[
\alpha_{1,\Omega_{1,1,2}} = 1/2, \quad 4/5; \quad \alpha_{1,\Omega_{1,\Omega_{1,2,3,5,2}}} = 1/5, \quad 9/13, \quad 25/29;
\]
\[
\alpha_{1,\Omega_{2,1,2,3,4,5}} = 0.16808, \quad 0.68575, \quad 0.78804; \quad \alpha_{1,\Omega_{1,\Omega_{1,2,1}}} = 1/2.
\]
\[
\alpha_{2,\Omega_{1,1}} = 1.80892; \quad \alpha_{1,\Omega_{1,2,3,2,2}} = 3.05607, \quad 1.44949;
\]
\[
\alpha_{2,\Omega_{2,2}} = 1.43404; \quad \alpha_{3,\Omega_{1,2,1}} = 1.81363; \quad \alpha_{1,\Omega_{1,\Omega_{1,2,3,5,6,7}}} = 2.28032, \quad 1.57899, \quad 1.35168.
\]

For pendulum motions of the system with Hamiltonian (5), at the lower boundary of the region there is a finite number of resonance points:
\[ \alpha_{21,\Omega_1=1} = 1/2; \quad \alpha_{21,\Omega_1=1/2} = 7/8; \quad \alpha_{21,\Omega_1=1/2} = 1/2; \quad \alpha_{21,\Omega_1=1} = 1/8; \]
\[ \alpha_{21,\Omega_1=1/2} = 1/2; \quad \alpha_{21,\Omega_1=1/2} = 0.06698; \quad \alpha_{21,\Omega_1=1} = 0.93301 \]

Note that in the regions \( \Gamma_2 \) and \( \Gamma_{21} \) at \( \alpha = 1/2 \) the cases of multiple resonances \( \Omega_{2,1} = 2, \Omega_{2,2} = 1 \) and \( \Omega_{21,1} = 1, \Omega_{21,2} = 1 \) occur.

The study showed that for the pendulum-type motions of the first model system, the generating points for parametric resonance regions are only points \( \alpha = 1/2, 7/9, 0.78804, 4/5, 7/8 \) on the lower boundary of the region \( \Gamma_2 \), and points \( \alpha = 1.80892, 1.57899, 1.43404 \) on the lower boundary of the region \( \Gamma_1 \). For pendulum motions of the second model system, at the lower boundary of the region \( \Gamma_{21} \), only one point \( \alpha = 1/2 \) of multiple resonance generates a parametric resonance regions.

From each of the remaining generating points for both systems, the single resonant curve is arisen, and there are no instability regions.

6. Stability diagrams

To solve the orbital stability problem of the pendulum-type motions considered, one needs to integrate the equations of motion of a non-autonomous system with the Hamiltonian \( K \) and calculate the coefficients of the characteristic equation (10). The integration of the equations of motion has been carried out numerically. For the system corresponding to the model system with Hamiltonian (3), the range of parameters \( \alpha \in (0;5] \) and \( h \in [-1;3.5] \) is considered, and for the system corresponding to the model system with Hamiltonian (5), the region \( \alpha \in (0;5] \) and \( h \in [-1;2] \). The results of the orbital stability study of the pendulum-type motions defined by Hamiltonians (3) and (5) are shown in figure 2 a and 2 b, respectively.

![Figure 2. Stability diagrams.](image)

Here, thin lines are the curves of resonances of the first and second orders. The instability regions are indicated in gray, the linear orbital stability regions are not painted over.

We note that in the vicinity of the point \( \alpha = 1 \) on the lower boundary curves in the regions \( \Gamma_1 \) and \( \Gamma_2 \), there is a countable set of points generating parametric resonance regions and single resonant curves. These regions and curves are located in the vicinity of the ray \( h = \alpha / 2 - 1, \alpha > 1 \) corresponding to the asymptotic motion of the radius vector of the body mass center. The domain, in which they are located in figure 2 b, is shaded. It is bounded by sections of two resonant curves obtained.
6.1. Orbital stability of pendulum-type motions of the system corresponding to model Hamiltonian (3)

For oscillations in the region $\Gamma_1$, there is a large instability region near the oscillation energy level, corresponding to the asymptotic motion. It is generated by three points indicated in the previous section, and on its upper border has two corner points with coordinates $(2.71815; 0.28091)$ and $(2.99157; 0.43130)$. The stability domains for oscillations near the inclined equilibrium position are found up to the upper boundary of the region $\Gamma_1$. With an increase in the intensity of vibrations, the stability regions are expanding.

For oscillations in the region $\Gamma_2$, three significant instability regions can be distinguished. One of these regions is adjacent to the upper boundary of the oscillation region and originates at the generating points $\alpha = 1/2$ and $\alpha = 7/9$. Parts of its borders have corner points $(0;0)$, $(0.49252; -0.34305)$, $(0.64130; -0.17655)$, $(1.22538; -0.05679)$, and $(2.77955; 1.25049)$. Two instability regions generated by dots $\alpha = 0.78804$ and $\alpha = 4/5$, with an increase in the intensity of vibrations, go along the lower boundary of the region. The instability region arising from the point $\alpha = 7/8$ is small and merges into the lower one of the large instability regions.

The significant stability regions for oscillations near the lower position are available at low and high vibration intensities. With an increase in the intensity of vibrations, the stability regions are expanding. For the rotation, of the stability regions are not detected.

6.2. Orbital stability of pendulum-type motions of the system corresponding to model Hamiltonian (5)

For oscillations around the lower equilibrium position, there is an instability region that occupies almost the entire region $\Gamma_2$. It is born from the point of multiple resonance $\alpha = 1/2$ and has angular points $(0; -0.37233)$, $(0.66882; 0.43844)$, $(1.15900; -0.37413)$ and $(1; -0.5)$. Two stability regions are adjacent to the lower boundary of the region $\Gamma_2$ at $\alpha < 1$. A significant stability region exists only for small values of the vibration intensity of the suspension point.

In the rotation region for the range of parameters considered, two stability regions are found. One of them, with a corner point $(2.72745; 2.68839)$, is expanding with increasing level of rotation energy, and the other one is enclosed between the corner points $(2.83023; 2.53740)$, $(2.67780; 2.40480)$ and $(2.70412; 2.41939)$.

Note that in the limiting case where the suspension point of the body is fixed the results on the orbital stability of the pendulum-type motions of the Kovalevskaya top obtained in the paper are consistent with the results obtained in [5, 6].

Acknowledgments
This work was carried out within the framework of the state assignment (project No. 3.3858.2017/4.6).

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