Switching-based Rejection of Multi-sinusoidal Disturbance in Uncertain Stable Linear Systems under Measurement Noise

Yang Wang, Gilberto Pin, Andrea Serrani and Thomas Parisini

Abstract—This paper addresses the problem of designing an Adaptive Feedforward Control (AFC) system for uncertain linear systems affected by a multi-sinusoidal disturbance with known frequencies. A novel State-Norm Estimator-based (SNE-based) switching mechanism is proposed to remove the long-standing assumption that either the sign of the real part or the imaginary part of the transfer function of the plant at the frequencies of excitation are needed to be known. The distinctive feature of the proposed mechanism in comparison to previous solutions from the authors is a lower order of the filters, allowing a decoupled design of the switching mechanism. Furthermore, the presence of the bounded noise is considered in the analysis. The effectiveness and robustness of the proposed method are illustrated by means of numerical examples.

I. INTRODUCTION AND PROBLEM FORMULATION

Consider the prototypical setup of the Adaptive Feedforward Control (AFC) problem for SISO LTI systems (see [1], [2]), where

\[
\dot{x} = A(\mu)x + B(\mu)\bar{d}(t) - d(t), \quad x(0) = x_0 \in \mathbb{R}^n_x
\]
\[
y = C(\mu)x, \quad y_d = y + \nu
\]  

(1)

is an \(n_x\)-dimensional realization of the internally stable interconnection of an uncertain plant model and a robust stabilizer. The variable \(y \in \mathbb{R}\) denotes the regulated output, whereas \(\nu \in \mathbb{R}\) is a bounded additive measurement noise, satisfying \(\|\nu(t)\|_\infty \leq \bar{\nu} < \infty\). System (1) is driven by the difference between the multi-sinusoidal disturbance (of known frequencies, \(\omega_i > 0, i = 1, 2, 3, \ldots, n_f\))

\[
d(t) = \sum_{i=1}^{n_f} a_i \sin(\omega_i t + \varphi_i) \tag{2}
\]

and an estimate of the disturbance, \(\hat{d}(t) \in \mathbb{R}\), generated by the AFC algorithm. It is assumed that the disturbance signal satisfies the following assumption:

Assumption 1: There exist constants \(\bar{a} > 0\) and \(\Delta \omega > 0\) such that the unknown amplitudes \(a_i\) and the known frequencies \(\omega_i\) are bounded by

\[
0 \leq a_i \leq \bar{a}, \quad |\omega_i - \omega_j| \geq \Delta \omega
\]

for all \(i, j \in \{1, 2, 3, \ldots, n_f\}\) and \(i \neq j\).

The vectors \(\mu \in \mathbb{R}^p\) collects the uncertain parameters of the plant model. It is assumed that \(\mu\) ranges on a given known compact set, \(\mathcal{P} \subset \mathbb{R}^p\). For future use, we let \(W(s) := C(\mu)(sI - A(\mu))^{-1}B(\mu)\) denote the transfer function of system (1). Furthermore, System (1) is assumed to be internally stable, robustly with respect to \(\mu \in \mathcal{P}\):

Assumption 2: There exist constants \(\alpha_0, \alpha_1, \alpha_2 > 0\) such that the parameterized family \(P_\nu : \mathbb{R}^p \to \mathbb{R}^{n_y \times n_x}\) of solutions of the Lyapunov equation \(P_\nu(\mu)A(\mu) + A^T(\mu)P_\nu(\mu) = -I\) satisfies \(\alpha_1 I \leq P_\nu(\mu) \leq \alpha_2 I\) for all \(\mu \in \mathcal{P}\). Moreover, \(-\text{Re}\{\lambda\} \geq \alpha_0\) for all \(\lambda \in \text{spec}(\mu), \forall \mu \in \mathcal{P}\).

The standard continuous-time AFC algorithm of [1], [3] requires that \(\beta_0 := \text{sign}\text{Re}\{W(j\omega_i)\}\) or \(\beta_0 := -\text{sign}\text{Im}\{W(j\omega_i)\}\) be known. In that case, the AFC module is comprised of the certainty-equivalence estimator

\[
\dot{\hat{\psi}}_i = -\varepsilon \beta_i \bar{a}_i e^{\mu_i T \tau} \psi_a \quad \text{or} \quad \dot{\hat{\psi}}_i = -\varepsilon \beta_i \bar{a}_i e^{\mu_i T \tau} \psi_a, \quad \hat{d}_i = \Gamma e^{\mu_i T \tau} \hat{\psi}_i, \quad \hat{d} = \sum_{i=1}^{n_f} \hat{d}_i, \quad i = 1, 2, 3, \ldots, n_f \tag{3}
\]

where \(\hat{\psi}_i(0) = \hat{\psi}_0 \in \mathbb{R}^2\), \(\Gamma = \begin{pmatrix} 1 & 0 \end{pmatrix}\),

\[
T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad g^b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4}
\]

and \(\varepsilon > 0\) is a gain parameter. Exponential stability of the interconnection (1)–(3) can be proved to hold for sufficiently small values of \(\varepsilon\) by way of averaging arguments [4], [5]. Persistence of the sign of \(\text{Re}\{W(j\omega_i)\}\) and \(\text{Im}\{W(j\omega_i)\}\) over the frequencies of interest is referred to as a SPR-like condition in the literature. Clearly, in the absence of the crucial information of either the sign of the real and the imaginary parts of \(W(j\omega_i)\), the standard strategy cannot be implemented.

An approach relying on Harmonic Steady State (HSS) response, also known as higher-harmonic control, was developed in [6]–[9]. In the cited references, the analysis assumes that the plant model is in steady-state, hence the dynamic interaction between the AFC algorithm and the plant dynamics is neglected. An adaptive solution that estimates \(\beta_1^c\) and \(\beta_2^c\) alongside \(\psi\) was pursued in [10], but its analysis is restricted to the averaged closed-loop system.

In this paper, we propose a switching-based AFC strategy that disposes of the knowledge of both \(\beta_1^c\) and \(\beta_2^c\), hence eliminates the need for the aforementioned SPR-like conditions, which are replaced by a weaker observability condition.
for the disturbance. We define the unknown parameter vector
\[ \vartheta_i^T(\mu) := (\Re \{W(j\omega_i)\}, \Im \{W(j\omega_i)\}), \]
for all \( i = 1, 2, \ldots, n_f \) and let the compact set \( \Theta \subset \mathbb{R}^2 \) be the annular region defined, for given real numbers \( 0 < \delta_1 < \delta_2 \), as
\[ \Theta := \{ \vartheta \in \mathbb{R}^2 | \delta_1^2 \leq \vartheta_1^2 + \vartheta_2^2 \leq \delta_2^2 \}. \]

Then, the typical SPR-like is replaced by the following assumption:

**Assumption 3:** The unknown parameter vector \( \vartheta(\mu) \) in (5) satisfies \( \vartheta(\mu) \in \text{int } \Theta \) for all \( \mu \in \mathcal{P} \).

The control problem is stated formally as follows:

**Problem 1.1:** Under Assumptions 1-3, for System (1), design a dynamic output-feedback controller of the form
\[
\begin{align*}
\dot{\xi} &= f_\mu(\xi, y_d) , \quad \xi(0) = \xi_0 \in \mathbb{R}^{n_0} \\
d_i &= h_\mu(\xi, y_d)
\end{align*}
\]
such that, for all \( \mu \in \mathcal{P} \), trajectories of the closed-loop system (1), (2) and (7) originating from all initial conditions 
\( x_0 \in \mathcal{X}_0 := \{ x_0 \in \mathbb{R}^{n_0} : |x_0| \leq r_0 \} \) and \( \xi_0 \in \mathcal{X} \), where \( r_0 > 0 \) is known and \( \mathcal{X} \subset \mathbb{R}^{n_0} \) is set to be determined, are bounded and satisfy \( \lim \sup_{t \to \infty} y_d(t) \leq r(\rho) \), where \( r(\cdot) \) is a class-\( \mathcal{K} \) function that depends on the tuning parameters of the controller and the switching mechanism.

**Remark 1.1:** Considering the definition of \( \mathcal{X}_0 \), the forthcoming results will be valid in a semi-global sense, that is, on the basis of an arbitrary but fixed choice of compact sets for the initial conditions of the plant, the controller and disturbance generator.

The solution is inspired by known results on state-norm estimators (SNE) [11] and logic-based switching control [12]-[16], and makes use of baseline controllers found in standard AFC schemes. In [17], the authors have proposed a switching strategy that avoids the necessity of relying on SPR-like conditions; however, the controller in [17] suffers from large dimensionality of the overall controller when extended to the case of multi-sinusoidal disturbance, which in turn will significantly increase the convergence time and computation cost. Furthermore, in the presence of sensor noise, robustness of the scheme depends on the choice of the switching strategy and may prevent the switching sequence from terminating.

To address these shortcomings, we present a new SNE-based switching logic. The proposed scheme provides robustness with respect to sensor noise and guarantees that the switching sequence terminates in a finite number of steps. Decoupling of the switching signal for each distinct frequency of excitation is attained by processing the output through the notch filter centered on the corresponding frequency. In this way, explosion of dimension of the controller with the number of frequency is avoided.

The paper is organized as follows: the structure of the multi-controller employed in this paper is given in Section II, where we show that there exist candidate controllers that are capable of solving Problem 1.1. The switching logic is illustrated in Section III. Finally, a case study is presented in Section IV.

**Notation:** Throughout the paper, \( \lambda_i(M) \) denotes an eigenvalue of the matrix \( M \). \( \lambda_{\max}(M) \) and \( \lambda_{\min}(M) \) denote the maximum and minimum eigenvalues of \( M \), respectively. We denote with \( \| \cdot \| \) both the Euclidean vector norm and the corresponding induced matrix norm.

**II. CERTAINTY-EQUIVALENCE CONTROLLER DESIGN**

The overall control architecture follows the general paradigm of the switching control architecture proposed in [18], where a controller is selected among the members of a given family by a supervisor. For simplicity of notation and to enhance readability of the paper, we consider a disturbance comprised of 2 distinct excitation frequencies, i.e., \( n_f = 2 \). The design and analysis of the controller can be easily extended to higher dimensional disturbance models. The sinusoidal disturbance \( d(t) \) is regarded as the sum of the outputs of LTI exosystems of the form
\[
\begin{align*}
\hat{w}_i(t) &= \omega_i T \hat{w}_i, \quad w_{i0}(0) = w_{i0} \in \mathbb{R}^2, \\
d_i &= \Gamma w_i, \quad i = 1, 2
\end{align*}
\]

The family of candidate controllers \( \{ C_i^\gamma \} \) is selected as
\[
C_i^\gamma = \left\{ \begin{array}{ll}
\hat{w}_i^\gamma = \omega_i T \hat{w}_i^\gamma - kb^i e_i, & \hat{w}_i^\gamma(0) = \hat{w}_{i0} \in \mathbb{R}^2 \\
e_i = \omega_i F(\gamma) i + \gamma \omega_i G y_d, & t_i(0) = t_0 \in \mathbb{R}^2 \\
d_i^\gamma = \Gamma \hat{w}_i^\gamma \\
\end{array} \right., \quad i = 1, 2, j \in \mathcal{J} := \{ 1, 2, 3, 4 \}
\]
where \( F(\gamma) := \left( \begin{array}{cc}
-\gamma & i \\
\gamma & 1
\end{array} \right) \) is Hurwitz for all \( \gamma \in (0, 2) \), \( G = F^\top \), and \( b_i \in \mathbb{R}^2 \) are constant vectors given by
\[
b^1 = \left( \begin{array}{c}
1 \\
0
\end{array} \right), \quad b^2 = \left( \begin{array}{c}
0 \\
-1
\end{array} \right), \quad b^3 = \left( \begin{array}{c}
-1 \\
0
\end{array} \right), \quad b^4 = \left( \begin{array}{c}
0 \\
1
\end{array} \right)
\]
In (9), the signals \( e_i, i = 1, 2 \), are the output of notch filters
\[
e_i(s) = \gamma \omega_i \Gamma (sI - \omega_i F)^{-1} G y_d(s)
\]
whose Bode plots for several values of the gain \( \gamma \) are shown in Fig.1. The measured output \( y_d \) of the plant is processed by two notch filters, each driving a family of candidate controllers, where each candidate generates an estimate of the component \( d_i \). The control signal applied to the plant is
\[
\hat{d}(t) := \hat{d}_1^\gamma(t) + \hat{d}_2^\gamma(t)
\]
where \( \sigma_i : [0, \infty) \to \mathcal{J}, i = 1, 2 \) are piecewise-constant switching signals taking values in the index set of the family of the candidate controllers, \( \mathcal{J} \). The system that generates the switching signal is referred to as the supervisory system and will be described in the next section.

In this section, we show that, with proper selection of the tuning parameters, at least one member of each group of candidate controllers defined in (9) is capable of stabilizing the closed-loop system and solving Problem 1.1.

Assume that \( \sigma_i(t) \equiv s_i, i = 1, 2 \) for all \( t \in [t_0, \infty) \), with \( s_i \in \mathcal{J} \). Letting \( \Pi_i(\mu) \) be the unique solution of the Sylvester
Fig. 1. Bode diagram of the notch filter (11) with \( \omega_i = 1 \).

equation \( \omega_i \Pi_i(\mu)T = A(\mu)\Pi_i(\mu) + B(\mu)\Gamma \) and changing coordinates as \( \zeta_i := \tilde{w}_i^\top - \xi_i \) and \( z := x - \Pi_1(\mu)\zeta_1 - \Pi_2(\mu)\zeta_2 \), the aggregate system (1), (8) and (9) reads as

\[
\dot{z} = A(\mu)z + k\Pi_1(\mu)b^{s_1}e_1 + k\Pi_2(\mu)b^{s_2}e_2, \quad z(t_0) = z_{t_0}
\]

\[
\dot{\zeta}_i = \omega_i T \dot{\zeta}_i - k b^{s_i} e_i, \quad \zeta_i(t_0) = \zeta_{i,t_0}
\]

\[
y_d = C(\mu)z + \theta_1^\top(\mu)\zeta_1 + \theta_2^\top(\mu)\zeta_2 + \nu
\]

For the notch filters, we apply a further coordinate change, namely \( \xi_i := \xi_i - \Sigma_{i,1}\zeta_1 + \Sigma_{i,2}\zeta_2 \) where \( \Sigma_{i,l} \in \mathbb{R}^{2 \times 2}, i = 1, 2, l = 1, 2 \), is the solution of the Sylvester equation

\[
\omega_i \Sigma_{i,l} T = \omega_i F \Sigma_{i,l} + \gamma \omega_i G \theta_1^\top(\mu)
\]

Finally, defining \( \eta_i := \text{col} (\zeta_i, \xi_i) \in \mathbb{R}^4, i = 1, 2 \), the cascade connection of the filter and active controller becomes

\[
\dot{\eta}_i = E_i \eta_i + H_i \eta_j + \gamma \Xi_i (Cz + \nu)
\]

\[
e_i = L_i \eta_i + D_i \eta_j, \quad i, j = 1, 2, \quad j \neq i
\]

where

\[
E_i(\gamma) := \begin{pmatrix}
\omega_i T - k b^{s_1} \theta_1^\top & -k b^{s_1} \Gamma \\
(\Sigma_{i,1} b^{s_1} \theta_1^\top + \Sigma_{i,2} b^{s_2} \Sigma_{j,i}) & \omega_i F + \Sigma_{i,1} b^{s_1}
\end{pmatrix}
\]

\[
H_i := \begin{pmatrix}
k b^{s_1} \Sigma_{i,i} & 0 \\
(\Sigma_{i,1} b^{s_1} \Sigma_{j,i} + \Sigma_{i,2} b^{s_2} \Sigma_{j,j}) & k b^{s_2} \Sigma_{j,j}
\end{pmatrix}
\]

\[
\Xi_i^\top := \begin{pmatrix}
0 & \omega_i G \end{pmatrix}, \quad L_i := \begin{pmatrix}
\theta_1^\top & \Gamma
\end{pmatrix}, \quad D_i := \begin{pmatrix}
\Sigma_{i,i} & 0
\end{pmatrix}
\]

Note that \( E_i \) depends on the gain \( \gamma \) by way of the matrix \( F \). Let \( g_X \) denote the maximum value of the matrix norm of matrix \( X(\mu) \), that is \( g_X := \max_{\mu \in P} \| X(\mu) \| \). For future use, four subsets \( \mathcal{I}_i \) and \( \mathcal{I}_i^* \) are defined as follows:

\[
\mathcal{I}_i := \{ \sigma \in \mathcal{J} : \text{Re}\{\lambda_{\max}(E_i(\gamma))\} < 0 \}
\]

\[
\mathcal{I}_i^* := \{ \sigma \in \mathcal{J} : \text{Re}\{\lambda_{\max}(E_i(\gamma))\} \leq -\bar{\alpha}(k) \}
\]

where \( i = 1, 2 \) and \( \bar{\alpha}(k) \) is a class-\( K \) function that depends on the tuning parameters, to be determined. Clearly, \( \mathcal{I}_i \subseteq \mathcal{I}_i \subseteq \mathcal{I}_i \subseteq \mathcal{J} \) for all \( k > 0 \) and \( \gamma \in (0, 2) \).

Next, we establish properties and lemmas that are instrumental in the forthcoming analysis. All the proofs are omitted due to space limitation.

**Property 1:** Under Assumptions 1 and 3, the solutions \( \Sigma_{i,i} \) of the Sylvester equations (14) satisfy the properties

\[
\| \Sigma_{i,i} \| \leq \sqrt{2} \delta_i \quad \text{and} \quad \Gamma \Sigma_{i,i} = \theta_1^\top, \quad i = 1, 2
\]

\[
\| \Sigma_{i,j} \| \leq \gamma \delta_i \delta_j, \quad i, l = 1, 2, l \neq i
\]

where \( \delta_i > 0 \) depends only on \( \omega_i \), and \( \delta_2 \) is defined in (6).

**Property 2:** There exist constants \( k_0 > 0 \) and \( c_3 \geq c_2 > c_1 > 0 \) such that the solution \( P_o^i : (\vartheta, \varepsilon) \rightarrow \mathbb{R}^{2 \times 2} \) of the parametrized family of Lyapunov equations

\[
P_o^i \left[ \omega_i T - \varepsilon \theta_1^\top(\mu) \right] + \left[ \omega_i T - \varepsilon \theta_1^\top(\mu) \right]^\top P_o^i = -\varepsilon \theta_1^\top \theta_1 I
\]

satisfies \( c_1 I \leq P_o^i \leq c_2 I \) and \( \| P_o^i \| \leq c_3 \) for all \( (\vartheta, \varepsilon) \in \Theta \times (0, k_0) \).

**Property 3:** There exist constants \( c_6 \geq c_5 > c_4 > 0 \) such that the solution \( P_f : \gamma \rightarrow \mathbb{R}^{2 \times 2} \) of the Lyapunov equations

\[
P_f (\gamma) + F(\gamma) P_f^\top = -\gamma I
\]

satisfies \( c_4 I \leq P_f \leq c_5 I \) and \( \| P_f \| \leq c_6 \) \( \forall \gamma \in (0, 2) \).

The next lemma states that, for each \( \omega_i \), at least one candidate controller defined in (9) guarantees that system (15) is exponentially stable.

**Lemma 2.1:** If Assumptions 1-3 hold, there exist a constant \( \delta_3 > 0 \) and a class-\( K \) function \( k_3(\gamma) \) such that, letting \( \bar{\alpha}(k) = k \delta_3 \) in (17), the sets \( \mathcal{I}_i \) and \( \mathcal{I}_i^* \) defined in (16) and (17) respectively, are non-empty for all \( k \in (0, k_3(\gamma)) \) and all \( \gamma \in (0, 2) \). Moreover, there exist positive constants \( p_3 \geq p_2 > p_1 > 0 \) such that the solution \( P_c^i : (k, \gamma) \rightarrow \mathbb{R}^{4 \times 4} \) of the parametrized family of Lyapunov equations

\[
P_c^i E_i(\gamma) + E_i(\gamma) P_c^i = -k I, \quad i = 1, 2
\]

satisfies \( p_1 I \leq P_c^i \leq p_2 I \) and \( \| P_c^i \| \leq p_3 \) for all \( k \in (0, k_3(\gamma)) \) and all \( \gamma \in (0, 2) \).

Finally, we show that, with a suitable selection of the tuning parameters, any candidate controller that belongs to \( \mathcal{I}_i^* \) is capable of stabilizing the closed-loop system

\[
\dot{\eta}_i = E_i \eta_i + H_i \eta_j + \gamma \Xi_i (Cz + \nu), \quad i = 1, 2
\]

\[
z = A z + k \Pi_1 b^{s_1} (L_1 \eta_1 + D_1 \eta_2) + k \Pi_2 b^{s_2} (L_2 \eta_2 + D_2 \eta_1)
\]

\[
y_d = C z + \bar{L}_i \eta_1 + \bar{L}_i^\top \eta_2 + \nu
\]

with \( \bar{L}_i := \begin{pmatrix} \theta_1^\top & 0 \end{pmatrix} \), and the sinusoidal disturbance \( d(t) \) can be effectively attenuated in the presence of the sensor noise \( \nu \). Thanks to Property 1, there exists constants \( \delta_4, \delta_5 \) that only depends on the value of \( \omega_i \) and \( \vartheta_i \), such that

\[
\| H_i \| \leq k \gamma \delta_i, \quad \| D_i \| \leq \gamma \delta_i, \quad i = 1, 2
\]

**Theorem 2.1:** If Assumptions 1-3 hold, there exist positive constants \( k^* \) and \( \gamma^* \) such that, Problem 1.1 is solved with any \( \sigma_i \in \mathcal{I}_i^*, i = 1, 2, k \in (0, k^*) \) and \( \gamma \in (0, \gamma^*) \).
trajectories of system (21), yields, for all \((\vartheta_i, k, \gamma) \in \Theta \times (0, k_1(\gamma)] \times (0, 2),\)

\[
\dot{V} \leq -k\|\eta_i\|^2 + 2\eta_i^T P_{\alpha} H_1 \eta_j + 2\gamma \eta_i^T P_{\alpha} \Sigma_i (Cz + \nu) + 2kz^T P_x (\Pi_i b_{\sigma_i} L_i + \Pi_j b_{\sigma_j} D_j) \eta_i - \|z\|^2
\]

(23)

Thanks to Property 1 and inequalities (22), one obtains

\[
\|\Pi_i b_{\sigma_i} L_i + \Pi_j b_{\sigma_j} D_j\| \leq \delta_0, \quad i = 1, 2, \quad j \neq i
\]

(24)

with \(\delta_0 > 0\). Substituting (22) and (24) into (23), and applying Young's inequality to the sign-definite terms yield

\[
\dot{V} \leq - (1 - 8\gamma^2 p_0^2 (\omega_1^2 + \omega_2^2) + 8k\alpha_3\delta_0^2) \|z\|^2
\]

\[
- k \left( \frac{1}{4} - 4\gamma^2 p_0^2 \delta_0^2 \right) (\|\eta_i\|^2 + \|\eta_j\|^2) + 8\gamma^2 p_0^2 (\omega_1^2 + \omega_2^2)\|z\|^2
\]

Setting \(k^* := \min \left\{ \frac{1}{32\alpha_3\delta_0^2 k_1(\gamma)} \right\}\) and \(\gamma^* := \min \left\{ \frac{1}{4\sqrt{2}p_0\rho \gamma^2 \omega_1^2 + \omega_2^2}, \frac{1}{4\sqrt{2}p_0\rho \gamma^2}, \frac{1}{4} \right\}\), one obtains

\[
\dot{V} \leq -\frac{1}{2}\|z\|^2 - \frac{1}{4} k\|\eta_i\|^2 + \|\eta_j\|^2) + 8\gamma^2 p_0^2 (\omega_1^2 + \omega_2^2)\|z\|^2
\]

\[
\leq -\beta_0 V + \gamma^2 p_0^2 (\omega_1^2 + \omega_2^2)\|z\|^2
\]

where \(\beta_0 = \min\{1/2\alpha_2, k/8p_0\}\), thus ending the proof. 

III. STATE NORM ESTIMATOR-BASED SWITCHING

In this section, we introduce the switching mechanism. The strategy behind the method is to define a threshold based on an upper bound of \(e_i^2(t)\) under the assumption that \(\sigma_i \in T_i^*\). Then, if the performance index for the active controller violates the threshold \(\tilde{e}_i\), the assumption is invalid and \(\sigma_i(t) \notin T_i^*\). On the other hand, boundedness of the output suggests that the active controller is a stabilizing one.

A. Switching Logic

The proposed supervisor is a cascade connection of two subsystems: a scheduling logic \(J\) that generates the switching sequence \(m_i(\cdot) : [0, +\infty) \mapsto \mathbb{Z}^+\), and a routing function \(\beta(m_i) := \text{mod}(m_i + \sigma_i(0) - 1, 4) + 1\) that satisfies the revisitation property [18], with \(\sigma_i(0) \in J, i = 1, 2\) being the initial active controller of each candidate group.

The supervisory system keeps adjusting \(\sigma_i\) within the set \(J\) along a pre-specified path \(\beta(m_i)\) until the output \(e_i\) is small in a suitable sense. The flow chart of the switching logic \(J\) is given in Fig. 2, where \(\tilde{e}_i, \tilde{e}_{as}, T_{as}\) and \(T_f\) will be determined in the sequel.

B. Generation of the switching criteria

Let two sequences \(\{T_i^{m_i}\}_{i=1}^{2n+1}\), \(i = 1, 2\), denote the set of time instants at which the switching signal \(\sigma_i(t)\) changes. It is assumed that the same controller is kept active in the interval \([T_i^{m_i}, T_i^{m_{i+1}}]\). Without loss of generality, we only present the development of the threshold \(\tilde{e}_1\), as \(\tilde{e}_2\) can be established along the same lines.

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Referring to system (15), for \(t \in [T_i^{m_i}, T_i^{m_{i+1}}]\), the SNE for \(\eta_1(t)\) is designed as follows:

\[
\bar{J}_1 = -\frac{k}{2p_2} \bar{J}_1 + 4p_2^2k^2\gamma_1 \bar{Z}_1 \bar{Z}_2 + 8p_2^2\gamma_2 \bar{Z}_1 (\rho_0^2 \bar{z} + \bar{v}^2),
\]

\[
\bar{\eta}_1 = \bar{J}_1 \eta_1 + \bar{p}_2 \eta_2 (T_i^{m_{i+1}}) e^{- \frac{k}{\bar{p}_2}(t-T_i^{m_i})}, \quad \bar{J}_1 (T_i^{m_{i+1}}) = 0
\]

(25)

In the above equation, \(\delta_0\) is defined in (22), \(\bar{Z}_2\) is the upper bound of \(\|\eta_2\|^2\), \(\rho_1(\cdot)\) is the norm-bound of \(\eta_1(t)\), defined as follows:

\[
\|\eta_1(t)\| \leq \|\bar{Z}_1(\cdot)\| + \|\bar{Z}_2(\cdot)\| + ||\Sigma_1.1, \xi_1|| + ||\Sigma_1.2, \xi_2||
\]

\[
\leq (1 + \delta_2) (||\bar{Z}_1(\cdot)\| + \alpha) + ||\bar{Z}_2(\cdot)\| + \gamma \delta_0 \delta_2 (||\bar{Z}_2(\cdot)\| + \alpha) =: \rho_1(\cdot)
\]

The variable \(\bar{z}\) is the upper bound of \(\|z\|^2\), given by

\[
\bar{z} = -\frac{1}{2\alpha_2} \bar{z} + 4\alpha_2^2 k^2 (\rho_1^2, e_1^2 + \rho_0^2 e_2^2), \quad \bar{z}(0) = 0
\]

(27)

with \(\bar{Z}_0\) is the bound on the initial condition \(z(0)\), computed as follows:

\[
\|z(0)\| \leq \|x_0\| + \rho_1, \|\xi_1(0)\| + \rho_2 \bar{Z}_2(\xi_2) \|\xi_2(0)\|
\]

\[
\leq \rho_0 + \rho_1, (\bar{a} + \|\bar{w}_1, (0)\|) + \rho_2 (\bar{a} + \|\bar{w}_2, (0)\|) =: \bar{Z}_0
\]

The initial conditions \(J_1(T_i^{m_i}), \bar{z}(0)\) are set to zero without any loss of generality (see [19]). The following lemma establishes the switching criterion for \(\sigma_1(t)\).

**Lemma 3.1:** If Assumptions 1-3 hold, then the inequality

\[
|e_1(t)|^2 \leq \epsilon_1 := (1 + \delta_2^2) \bar{Z}_1 + \gamma^2 \delta_2^2, \bar{Z}_2
\]

(28)

holds for all \(\sigma_1(t) \in T_i^*\) and \(k \in (0, k^*)\), \(\gamma \in (0, \gamma^*)\).

A detailed proof based on Lyapunov analysis can be found in [19]. Following similar arguments, one can show that

\[
|e_2(t)|^2 \leq \epsilon_2 := (1 + \delta_2^2) \bar{Z}_2 + \gamma^2 \delta_2, \bar{Z}_1
\]

(29)

also holds for all \(\sigma_2(t) \in T_i^*\) and \(k \in (0, k^*)\), \(\gamma \in (0, \gamma^*)\).

Inequalities (28) and (29) indicate that the switching sequence terminates in finite time and next lemma shows that, with a suitable choice of the gain parameters, the
condition $|e_i(t)|^2 \leq \bar{e}_i, i = 1, 2,$ holds for all $t \geq T_\ast := \max\{T_{\omega_1}^2, T_{\omega_2}^2\}$, thus proving that $e_i \in L_\infty$, and all closed-loop trajectories are bounded.

Lemma 3.2: If Assumptions 1-3 hold, there exists a positive constant $k^{**}$ such that, for all $k \in (0,k^{**})$ and $\gamma \in (0,\gamma^*)$, the active controllers selected by the conditions (28) and (29) guarantee that the states of the closed-loop system are bounded.

The proof of this result, omitted for reasons of space, can be obtained using arguments similar to those presented in [20]. At this point, boundedness of all internal variables still does not guarantee that Problem 1.1 is solved, as two cases are possible:

1) The closed-loop system is internally stable, i.e., Problem 1.1 is solved.

2) The closed-loop system is “neutrally stable”, which is the pathological case we mentioned before.

Case 2) is a pathological one, where the active controller introduces a pair of eigenvalues on the imaginary axis, while boundedness of the output signal is maintained. In that case, the output may still be bounded by the threshold, but the active controller is not a stabilizing one. Therefore, the last step is to check the steady state of $e_i$ to exclude case 2) above. To this end, we need to first estimate the time at which steady state is (approximately) reached. Referring to inequality (26) and Lemma 2.1, the system is deemed to have reached its steady state if $t > T_{ss}(t_\ast) := \log \frac{\hat{p}(t_\ast)}{\varepsilon_0 m_{\omega_i}}$ with $\hat{p}(t_\ast) = \max\{p_1(t_\ast), p_2(t_\ast)\}$, $t_\ast := \max\{T_{\omega_1}^1, T_{\omega_2}^1\}$ and $\varepsilon_0 > 0$ is a (sufficiently small) positive constant chosen by the designer.

In the light of (21), it is possible to show that the steady state trajectory of $e_i$ can be bounded as follows:

$$|e_{i,ss}| \leq \frac{\|L_i\eta_{i,ss} + D_i\eta_{j,ss}\|}{\gamma (1 + \bar{d}_2 + \gamma \bar{d}_2)(\omega_1 + \omega_2)\bar{\nu}} =: \bar{e}_{ss}$$

for $i = 1, 2$ and $j \neq i$. Therefore, as shown in Fig.2, if (30) is satisfied for another $T_{ss} := \max\{\frac{2s}{\omega_1}, \frac{2s}{\omega_2}\}$ seconds, one can claim that the aforementioned case 1) occurs and the disturbance $d(\cdot)$ has been successfully rejected. Otherwise, the active controller (or both) does not belong to $I_i^*$ and the switching sequence shall continue.

IV. ILLUSTRATIVE EXAMPLE

As a case study, we consider two stable non-minimum phase plant models described by

$$W_1(s) = \frac{2s - 2}{s^2 + 2s + \frac{5}{2}}, \quad W_2(s) = \frac{2s - 1}{s^2 + 2s + 2},$$

and disturbance signal given by $d(t) = 5 \sin(\omega_1 t) + 4 \sin(\omega_2 t + \pi)$ with $\omega_1 = 2, \omega_2 = 3$. The corresponding frequency response parameter together with and set of stabilizing controllers are listed in Table I. The Runge-Kutta integration method has been employed for all simulations with fixed sampling interval $T_s = 10^{-3}s$.

| Plant | Frequency | $\omega_i$ | $t_i^*$ | $I_i^*$ |
|-------|-----------|------------|---------|--------|
| $W_1(s)$ | $\omega_1 = 2$ [rad/s] | $(0.82, 0.7)$ | $(1)$ | $\{1, 4\}$ |
| $W_2(s)$ | $\omega_2 = 3$ [rad/s] | $(0.9, -0.23)$ | $(1)$ | $\{1, 2\}$ |

The additive noise $n(t)$ is a random noise with uniform distribution within the interval $[-0.5, 0.5]$. The tuning parameters selected as, $k = 0.05, \gamma = 0.08$. The thresholds are chosen as $\varepsilon_0 = 0.5$ and $\bar{\nu} = 0.5$. Other parameters used in the SNE are: $\alpha_1 = 1, p_3 = 2$.

First, for $t \in [0, 800]$, we consider $W_1(s)$ and the initial activating controllers are both destabilizing ones: $\sigma_1(0) = 4$ and $\sigma_2(0) = 3$, which the initial guess of estimates are located far from the true value. The results of the simulations are shown in Fig. 3 and Fig. 4. It is observed that the proposed multiple controller with the SNE-based switching mechanism succeeds in rejecting the multi-sinusoidal disturbance under the effect of the sensor noise. The second supervisory system deselects the initial candidate controller and finally selects the stabilizing one after 3 switches. The closed-loop system reaches steady state in about 400 seconds. From the magnified steady state behaviour, one can see that, the effect of the sensor noise has been heavily reduced due to the use of the notch filters.

Then, we consider a more complex case in which the plant model undergoes a step change, namely, at $t \geq 800[s]$, the transfer function of the plant becomes $W_2(s)$. To tackle this challenge, we choose more conservative parameters with $\alpha_2 = 2, p_2 = 8$. From Table I, we can see that controller $C_1^4$ is not a stabilizing controller for $W_2(s)$ with $\omega = 2$. As shown in Fig.4, the supervisor system automatically detects this change at $t \approx 1400[s]$ and immediately switches to the stabilizing one. It worth pointing out that the switching for the first family of candidate controllers and the transient behaviour of the output of the plant have negligible effect on the second supervisory system, such that the stabilizing controller $C_1^4$ remains active for both plants. After another 200 seconds, the output is brought back within a neighborhood of zero.
The main benefit of the proposed scheme versus existing adaptive and switching solutions is the relative low-dimensionality of the overall controller. As a matter of fact, the order of the controller grows linearly (instead of exponentially) with the number of distinct frequencies of excitation; specifically, $6n_f$ for $n_f$ sinusoidal components, which compares very favorably with the order of the multiple-model controller proposed in [18]. Furthermore, the presented method shows good robustness with respect to sensor noise and guarantees a finite number of switching. Conversely, obvious drawbacks of the approach are the complexity of the switching logic and potentially slow convergence speed, due to the conservative choice of the tuning parameters.

The extension of the methodology to the case in which the frequencies of disturbance are unknown is the subject of current investigation.

V. CONCLUDING REMARKS

A novel methodology has been proposed in this work to remove a longstanding requirement in the problem of rejecting multi-sinusoidal disturbances for an uncertain linear system, namely the necessity to impose SPR-like conditions on the transfer function at frequencies of interest. At the core of the proposed method is the selection of candidate controllers of fixed structure by way of a novel switching mechanism, which relies on state-norm estimators to remove possibly destabilizing controllers from the candidate set.

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