The computational asymptotics of Gaussian variational inference and the Laplace approximation

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Abstract Gaussian variational inference and the Laplace approximation are popular alternatives to Markov chain Monte Carlo that formulate Bayesian posterior inference as an optimization problem, enabling the use of simple and scalable stochastic optimization algorithms. However, a key limitation of both methods is that the solution to the optimization problem is typically not tractable to compute; even in simple settings the problem is nonconvex. Thus, recently developed statistical guarantees—which all involve the (data) asymptotic properties of the global optimum—are not reliably obtained in practice. In this work, we provide two major contributions: a theoretical analysis of the asymptotic convexity properties of variational inference with a Gaussian family and the maximum a posteriori (MAP) problem required by the Laplace approximation; and two algorithms—consistent Laplace approximation (CLA) and consistent stochastic variational inference (CSVI)—that exploit these properties to find the optimal approximation in the asymptotic regime. Both CLA and CSVI involve a tractable initialization procedure that finds the local basin of the optimum, and CSVI further includes a scaled gradient descent algorithm that provably stays locally confined to that basin. Experiments on nonconvex synthetic and real-data examples show that compared with standard variational and Laplace approximations, both CSVI and CLA improve the likelihood of obtaining the global optimum of their respective optimization problems.

Keywords Bayesian statistics · variational inference · Laplace approximation · computational asymptotics · Bernstein-von Mises

1 Introduction

Bayesian statistical models are powerful tools for learning from data, with the ability to encode complex hierarchical dependence and domain expertise, as well as coherently quantify uncertainty in latent parameters. For many modern Bayesian models, exact computation of the posterior is intractable (Blei et al., 2017, Section 2.1) and statisticians must resort to approximate inference algorithms. Currently, the most popular type of Bayesian inference algorithm in statistics is Markov Chain Monte Carlo (MCMC) (Gelfand and Smith, 1990; Hastings, 1970; Robert and Casella, 2013), which provides approximate samples from the posterior distribution and is supported by a comprehensive literature of theoretical guarantees (Meyn and Tweedie, 2012; Roberts and Rosenthal, 2004). An alternative—variational inference (Blei et al., 2017; Jordan et al., 1998; Wainwright and Jordan, 2008)—approximates the intractable posterior with a distribution chosen from a pre-specified family, e.g., the family of Gaussian distributions parametrized by mean and covariance. The approximating distribution is chosen by minimizing a discrepancy (e.g., Kullback-Leibler (KL) (Murphy, 2012, Section 2.8) or Rényi divergence (Van Erven and Harremos, 2014)) to the posterior distribution over the family. Another alternative—the Laplace approximation (Hall et al., 2011; Shun and McCullagh, 1995)—involves first finding the maximum of the posterior density, and then fitting a Gaussian using a second-order Taylor expansion. Both methods convert Bayesian inference into an optimization problem, enabling the use of simple, scalable stochastic optimization algorithms (Bottou, 2004; Robbins and Monro, 1951) that require only a
subsample of the data at each iteration and avoid computation on the entire dataset.

But despite their scalability, both variational inference and the Laplace approximation have key limitations. First, one is forced to use an approximation of the posterior from a preselected parametric family of distributions. In particular, the Laplace approximation involves the family of Gaussians, while in variational inference the choice of family is left to the practitioner. In either case, it is in general difficult to know how limited the family is before actually optimizing; and not only that, it is also often difficult to estimate the approximation error once the optimization is complete (Huggins et al., 2020). For example, if one uses a mean-field Gaussian family with a diagonal covariance, the resulting posterior approximation will typically underestimate posterior variances and cannot capture its covariances (Murphy, 2012, Section 21.2.2), two quantities of particular interest to statisticians. The second key limitation is that even if the Laplace approximation or optimal variational approximation are known to have low error, the optimization problems required by both methods are typically nonconvex, and the global optimum cannot be found reliably.

The key to addressing the first limitation is to understand the optimal approximation error within the chosen family. Aside from nonparametric mixtures (Campbell and Li, 2019; Guo et al., 2016; Locatello et al., 2018; Miller et al., 2017)—which can be designed to achieve arbitrary approximation quality—available results in the finite-data setting are quite limited. For example, Han and Yang (2019) provides a non-asymptotic analysis of optimal mean-field variational approximations, but extending these results to more general distribution families is not straightforward. On the other hand, multiple threads of research have explored the statistical properties of parametric posterior approximations in the data-asymptotic regime by taking advantage of the limiting behavior of the Bayesian posterior. The Laplace approximation is well studied in statistical literature in this regard (Barber et al., 2016; Bassett and Deride, 2019; Hall et al., 2011; Miller, 2021; Shun and McCullagh, 1995), while research on variational inference is ongoing. Wang and Blei (2019) exploits the asymptotic normality of the distribution in a parametric Bayesian setting to show that the variational KL minimizer asymptotically converges to the KL minimizer to the limiting normal posterior distribution. Alquier and Ridgway (2020) analyze the rate of convergence of the variational approximation to a fractional posterior—a posterior with a tempered likelihood—in a high dimensional setting where the posterior itself may not have the ideal asymptotic behavior. Zhang and Gao (2020) studies the contraction rate of the variational distribution for non-parametric Bayesian inference and provides general conditions on the Bayesian model that characterizes the rate. Yang et al. (2020) and Jaiswal et al. (2019) build a framework for analyzing the statistical properties of α-Rényi variational inference, and provide sufficient conditions that guarantee an optimal convergence rate of the obtained point estimate. But while the literature has built a comprehensive understanding of the asymptotic guarantees of both the Laplace and optimal variational approximations, the nonconvexity of the optimization problems involved makes these guarantees difficult to obtain reliably in practice. In fact, Proposition 12 of the present work demonstrates that both the Laplace approximation and Gaussian variational inference involve nonconvex optimization, even in simple cases with ideal asymptotic posterior behaviour.

In this work, we address the nonconvexity of Gaussian variational inference and the maximum a posteriori (MAP) problem in the data-asymptotic regime when the posterior distribution admits asymptotic normality. Rather than focusing on the statistical properties of the optimum, we investigate the asymptotic properties of the optimization problems themselves (Section 4), and use these to design procedures (Section 3) which involve only tractable optimization and hence make theoretical results regarding global optima applicable. In particular, we develop consistent stochastic variational inference (CSVI) and consistent Laplace approximation (CLA), two algorithms for Gaussian posterior approximation. CSVI is guaranteed to find the optimal variational approximation, and CLA the maximum a posteriori (MAP), with probability that converges to 1 in the limit of observed data. The first key innovation in both CSVI and CLA is that we initialize the optimization at the mode of a smoothed posterior—the posterior distribution convolved with Gaussian noise. We prove that, with enough data, the smoothed MAP falls in a local region in which the optimization problem is locally convex and contains the global optimum, and that finding the smoothed MAP is a convex optimization problem and hence tractable (Section 4.3). The second innovation, which pertains only to CSVI, is a gradient scaling during stochastic optimization (Section 3.3) that ensures that the optimization remains inside the aforementioned local region and converges to the global optimum. Experiments on synthetic examples in Section 5 show that CSVI and CLA provide numerically stable and asymptotically consistent posterior approximations.

## 2 Variational and Laplace posterior approximations

In the setting of Bayesian inference considered in this paper, we are given a sequence of posterior distributions \( \Pi_n \), \( n \in \mathbb{N} \) each with full support on \( \mathbb{R}^d \). The index \( n \) represents the amount of observed data; denote \( \Pi_0 \) to be the prior. We also assume that each posterior \( \Pi_n \) has density \( \pi_n \) with respect to the Lebesgue measure.
2.1 Gaussian variational inference

Gaussian variational inference aims to find a Gaussian approximation to the posterior distribution by solving the optimization problem

$$\arg\min_{\mu \in \mathbb{R}^d, \Sigma \in \mathbb{R}^{d \times d}} D_{\text{KL}}(\mathcal{N}(\mu, \Sigma) || \mathcal{N}(0, I)) \quad \text{s.t.} \quad \Sigma > 0,$$

where the Kullback-Leibler divergence (Murphy, 2012, Section 2.8) is defined as

$$D_{\text{KL}}(Q || P) := \int \log \frac{dQ}{dP} dQ$$

for any pair of probability distributions $P, Q$ such that $Q \ll P$, and $\frac{dQ}{dP}$ is the Radon-Nikodym derivative of $Q$ with respect to $P$ (Folland, 1999, Section 3.2). We use the standard reparametrization of $\Sigma$ using the Cholesky factorization $\Sigma = n^{-1}LL^T$ to arrive at the common formulation of Gaussian variational inference (Kucukelbir et al., 2017) that is the focus of the present work:

$$\mu_n^*, L_n^* = \arg\min_{\mu \in \mathbb{R}^d, L \in \mathbb{R}^{d \times d}} -n^{-1} \log \det L + F_n(\mu, L)$$

$$\text{s.t.} \quad L \text{ lower triangular with positive diagonal},$$

where

$$f_n(x) := -n^{-1} \log \pi(x)$$

$$F_n(\mu, L) := \mathbb{E}[f_n(\mu + n^{-1/2}LZ)], \quad Z \sim \mathcal{N}(0, I).$$

Denote the optimal Gaussian distribution

$$\mathcal{N}_{\text{KL}} := \mathcal{N}(\mu_n^*, L_n^*).$$

Intuitively, this optimization problem encodes a tradeoff between maximizing the expected posterior density under the variational approximation—which tries to make $L$ small and move $\mu$ close to the maximum point of $\pi$—and maximizing the entropy of the variational approximation—which prevents $L$ from becoming too small. It crucially does not depend on the (typically unknown) normalization of $\pi$, which appears as an additive constant in Eq. (1); it is common to drop this constant and instead equivalently maximize the expectation lower bound (ELBO) (Blei et al., 2017). Note that there are a number of unconstrained parametrizations of the covariance matrix variable $\Sigma$ (Pinheiro and Bates, 1996). We select the (unique) positive-diagonal Cholesky factor $L$ as it makes the optimization problem Eq. (1) more amenable to both theoretical analysis and computational optimization.

One typically attempts to solve Eq. (1) using an iterative local descent optimization algorithm. As the expectation is intractable in most scenarios, this involves stochastic optimization (Hoffman et al., 2013; Kingma and Welling, 2014; Kucukelbir et al., 2017; Ranganath, 2014). In particular, assuming one can interchange expectation and differentiation (see Section 4.1 for details), the quantities

$$\hat{\nabla}_{\mu,n}(\mu, L, Z) := \nabla f_n(\mu + n^{-1/2}LZ)$$

$$\hat{\nabla}_{L,n}(\mu, L, Z) := -n^{-1} (\text{diag} L)^{-1} + n^{-1/2} \text{tril} \nabla f_n(\mu + n^{-1/2}LZ) Z^T,$$

are unbiased estimates of the $\mu$- and $L$-gradients of the objective in Eq. (1) given $Z \sim \mathcal{N}(0, I)$, where the functions $\text{diag} : \mathbb{R}^{d \times d} \to \mathbb{R}^d$ and $\text{tril} : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d}$ set the off-diagonal and upper triangular elements of their arguments to 0, respectively. These unbiased gradient estimates may be used in a wide variety of stochastic optimization algorithms (Bottou, 2004; Robbins and Monro, 1951) applied to Eq. (1).

In this paper, we will focus on projected stochastic gradient descent (SGD) (Bubeck, 2015, Section 3.) due to its simplicity: we expect that the mathematical theory in this work extends to other related methods. In general, Gaussian variational inference is a nonconvex optimization problem and standard iterative methods such as SGD are not guaranteed to produce a sequence of iterates that converge to $\mu_n^*, L_n^*$.

2.2 Laplace approximation

The Laplace approximation (Bishop and Nasrabadi, 2006, Section 4.4) constructs a Gaussian approximation to the posterior centered at the maximum a posteriori (MAP) point $\theta_n^*$, and with covariance based on a second-order Taylor expansion of $\log \pi_n$ at the MAP, i.e.,

$$\mathcal{N}_{\text{Lap}} := \mathcal{N}(\theta_n^*, \Sigma_{\text{Lap},n})$$

$$\theta_n^* = \arg\min_{\theta \in \mathbb{R}^d} f_n(\theta)$$

$$\Sigma_{\text{Lap},n} := (-\nabla^2 \log \pi_n(\theta_n^*))^{-1}.$$

Under certain regularity conditions, the total variation error of the Laplace approximation diminishes as the number of samples increases (Kass, 1990; Miller, 2021; Schillings et al., 2020); we present the precise statement in Theorem 5. However, since $\log \pi_n$ is typically not concave, obtaining the MAP point—and hence computing the Laplace approximation—is generally intractable.

3 Consistent variational and Laplace approximations

In this section, we provide two methods—Consistent Stochastic Variational Inference (CSVII) and Consistent Laplace (CLA)—that asymptotically solve the Gaussian variational inference and MAP problems in Eqs. (1) and (5) in the sense that the probability that the iterates converge to the global optimum converges to 1 in the asymptotic limit of observed data.
(see Definition 1). The development of the algorithms and results in this section depend heavily on asymptotic convexity and smoothness analysis later in Section 4. While the interested reader can refer to that section for a precise treatment, the key intuitive points in the development are that, very informally: (1) \( f_\theta \) is generally nonconvex, even asymptotically in well-behaved models, and this renders both the MAP and variational inference problems nonconvex; however, (2) both optimization problems are asymptotically locally strongly convex around a fixed location \( \theta_0 \), and \( \mu_\theta^n \) and \( \theta^n \) both converge to \( \theta_0 \); and (3) the posterior density smoothed by a Gaussian kernel is asymptotically log concave, and the smoothed MAP \( \theta^n \) also converges to \( \theta_0 \). Therefore in CSVVI and CLA, we first use the smoothed posterior MAP \( \hat{\theta}^n \) as a tractable initialization near the global optimum, and carefully scale gradient steps while optimizing so as to remain in the local basin around the optimum. In the remainder of this section, we provide the details of the smoothed MAP optimization, CSVVI, and CLA; theoretical details are deferred to Section 4.

Similar two-stage designs involving initialization followed by local optimization have been employed for nonconvex problems in statistics, e.g., Balakrishnan et al. (2017). However, this work assumes the existence of the initialization—which generally requires intractable nonconvex optimization itself—and requires a model-specific theoretical analysis to tune the local optimization method. In contrast, our work provides practical, model-agnostic, and asymptotically tractable optimization methods.

### 3.1 Smoothed MAP initialization

Given the \( n \)-th posterior distribution \( \Pi_n \), we define the smoothed posterior \( \Pi_n^\text{smooth} \) with smoothing variance \( \alpha_n \) to be the \( \theta \)-marginal of the generative process

\[
W \sim \Pi_n, \quad \theta \sim \mathcal{N}(W, \alpha_n I).
\]

The probability density function \( \hat{\pi}_n \) of \( \Pi_n^\text{smooth} \) is given by the convolution of \( \pi_n \) with a multivariate normal density,

\[
\hat{\pi}_n(\theta) = \mathbb{E} \left[ \frac{1}{(2\pi)^{d/2} \alpha_n^{d/2}} \exp \left( -\frac{1}{2\alpha_n} \| \theta - W \|^2 \right) \right]. \quad (6)
\]

Given these definitions, the smoothed MAP problem is the MAP inference problem for the smoothed posterior distribution, i.e.,

\[
\hat{\theta}_n = \arg \min_{\theta \in \mathbb{R}^d} -\log \mathbb{E} \left[ \exp \left( -\frac{1}{2\alpha_n} \| \theta - W \|^2 \right) \right]. \quad (7)
\]

Gaussian smoothing is commonly used in image and signal processing (Forsyth and Ponce, 2002; Haddad and Akansu, 1991; Lindeberg, 1990; Nixon and Aguado, 2012), and has previously been applied to reduce the presence of spurious local optima in nonconvex optimization problems, making them easier to solve with local gradient-based methods (Addis et al., 2005; Mobahi, 2013). This effect is demonstrated in Fig. 1, where we construct a synthetic Bayesian model where the posterior is asymptotically normal but has multiple modes even given a large sample size. The details of this synthetic model and the setting of the experiment can be found in Supp. A.1. The variance \( \alpha_n \) controls the degree of smoothing: larger values create a smoother density \( \hat{\pi}_n \), at the cost of making \( \hat{\pi}_n \) a poorer approximation of the original function \( \pi_n \). Fig. 2 demonstrates how increasing \( \alpha_n \) increases the smoothing effect, resulting in fewer and flatter local optima in the objective. In practice with a fixed finite data set, one needs to tune the smoothing constant \( \alpha_n \). However, we show later in Theorem 7 that if \( \alpha_n \) satisfies \( n\alpha_n^2 \to \infty \), then eventually Eq. (7) becomes a convex problem, and \( \hat{\theta}_n \) converges in probability to the original posterior mode \( \theta^\ast \).

We use SGD to solve the smoothed MAP problem. By change of variables and reparametrization, the gradient of the smoothed MAP objective function in Eq. (7) is

\[
\nabla (-\log \hat{\pi}_n(\theta)) = \alpha_n^{-1/2} \mathbb{E} \left[ W \pi_n \left( \theta - \alpha_n^{-1/2} W \right) \right] \mathbb{E} \left[ \pi_n \left( \theta - \alpha_n^{-1/2} W \right) \right]^{-1},
\]

where \( W \sim \mathcal{N}(0, I) \). Note that the unknown normalization constant in \( \pi_n \) cancels in the numerator and denominator. We obtain stochastic estimates of the gradient using a Monte Carlo approximation of the numerator and denominator using the same samples, i.e., self-normalized importance sampling (Robert and Casella, 2013, p. 95). It is known that the variance of this gradient estimate may be quite large or even infinite; although techniques such as truncation (Ionides, 2008) and smoothing (Vehtari et al., 2015) exist to address it, we leave this issue as an open problem for future work. The resulting SGD procedure with explicit gradient estimates are shown in Algorithm 1.

### 3.2 Consistent Laplace approximation (CLA)

The consistent Laplace approximation (CLA) involves first initializing \( \theta \) to the smoothed MAP \( \hat{\theta}_n \) (which we estimate...
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Fig. 1: Top: A synthetic Bayesian model with posterior density for increasing dataset sizes ($n = 10, 100, 1000, 10000$). The posterior density displayed in the figure are normalized, fixing the maximum at 100. The details of this synthetic model and experimental setup can be found in Supp. A.1. Bottom: The smoothed Bayesian posterior densities corresponding to the top row. Black curves show the smoothed posterior (with smoothing constant of $\alpha_n = 10n^{-0.3}$), the blue histogram shows the counts (over 100 trials) of the output of the smoothed MAP initialization, and the red dots denote the last iteration of the smoothed MAP optimization.

Fig. 2: Plots of the smoothed posterior density $\hat{\pi}_n$ with increasing smoothing variance $\alpha$. Note that $\alpha = 0$ denotes the original posterior.

with Algorithm 1), and then proceeding as in the standard Laplace approximation: we use gradient descent to find the posterior mode, and then construct $\mathcal{N}_{\text{Lap}, n}$ per Eq. (4). The only concern is that the iterates produced by gradient descent need to stay confined to the local basin of $f_n$ around the optimum point; we can ensure this by using a small enough step size, or by using a line-search to set the step size adaptively (Boyd and Vandenberghe, 2004, Page 466; Armijo, 1966; Bertsekas and Tsitsiklis, 2000). We present CLA with backtracking line search in Algorithm 2.

We provide a convergence result for CLA in Theorem 2. In particular, under mild regularity conditions on the Bayesian model, Theorem 2 shows that CLA asymptotically solves the MAP optimization problem Eq. (5), and hence enables reliable computation of the Laplace approximation. Definition 1 clarifies what it means for an algorithm to solve an optimization problem in the data-asymptotic limit. We use the usual notation $o_{P_n}$ and $O_{P_n}$ to denote stochastic order (van der Vaart, 2000, Section 2.2).

**Algorithm 2 Consistent Laplace approximation**

```
procedure CLA($f_n, \beta \in (0, 1)$, $K$)
    $\theta \leftarrow \text{SmoothedMAP (Algorithm 1)}$
    for $k = 1, \ldots, K$
        $t \leftarrow 1$
        while $f_n(\theta - t \nabla f_n(\theta)) > f_n(\theta) - \frac{t}{2} \|\nabla f_n(\theta)\|^2$
            $t \leftarrow \beta t$
        end while
        $\theta \leftarrow \theta - t \nabla f_n(\theta)$
    end for
    return $\theta, \frac{1}{n} \left(\nabla^2 f_n(\theta)\right)^{-1}$
end procedure
```

**Definition 1.** An iterative algorithm asymptotically solves a (random) sequence of optimization problems indexed by $n \in \mathbb{N}$, each with a single global optimum point $x_n^* \in \mathbb{R}^d$, if the sequence of iterates $(x_k, n)_{k \in \mathbb{N}}$ produced by the algorithm satisfies

$$\lim_{n \to \infty} P\left(\|x_k, n - x_n^*\|^2 = o_{P_n}(1)\right) = 1,$$

where $P$ denotes the law of the sequence of optimization problems and $P_n$ denotes the law of the iterates of the $n$th optimization problem. Further, we say that it asymptotically solves the sequence of problems at a rate $t_k$, if

$$\lim_{n \to \infty} P\left(\|x_k, n - x_n^*\|^2 = O_{P_n}(t_k)\right) = 1, \quad \lim_{k \to \infty} t_k = 0.$$
Theorem 2. Suppose Assumption 1 holds. Then there exist \( r, \ell > 0 \) and \( 0 < \eta < 1 \) such that if we initialize \( \theta \) such that \( \| \theta - \tilde{\theta}_n^* \| \leq \frac{r}{4(r+1)} \), then CLA asymptotically solves the MAP problem Eq. (5) at a rate of \( \eta^k \).

The constant \( r \) in the statement of Theorem 2 intuitively represents the radius of the local convex basin around the optimum point of \( f_n \), and \( \ell \) is the local Lipschitz smoothness constant, as defined in Lemma 9. In practice, since the constants \( r \) and \( \ell \) are not known, one should run the smoothed MAP optimization Algorithm 1 until convergence to \( \tilde{\theta}_n^* \) according to some diagnostic, e.g., a small gradient norm.

3.3 Consistent stochastic variational inference (CSVI)

Consistent stochastic variational inference (CSVI) begins by initializing \( L = I \) and \( \mu \) to the smoothed MAP \( \tilde{\theta}_n^* \) (which we estimate with Algorithm 1), and then optimizes the variational objective Eq. (1) using projected stochastic gradient descent (SGD) (Bubeck, 2015, Section 3). Much like in CLA, the remaining concern is that the iterates of SGD stay within the local region around the optimum \( \mu_n^*, L_n^* \) in which the variational objective is strongly convex. The major issue is that the regularization term \(-r^{-1} \log dL \) in the objective of Eq. (1) is not Lipschitz smooth, which both makes theoretical guarantees on convergence difficult to obtain and in practice results in instability in \( L \) during optimization. We address this issue by applying a novel scaling matrix to the gradient steps; in particular, define the scaled \( L \) gradient matrix \( \hat{\nabla}_{\mu, L, Z} \in \mathbb{R}^{d \times d} \) via

\[
[\hat{\nabla}_{\mu, L, Z}]_{ij} = \begin{cases} 
[\nabla_{\mu, L, Z}]_{ij} & j \neq i \\
\frac{1}{1+(nL_0)^{\ell/4}} [\nabla_{\mu, L, Z}]_{ii} & j = i, L_0 > 0 \\
-1 & j = i, L_0 = 0.
\end{cases}
\]

(8)

This scaling prevents the gradient of \( L \) from diverging when diagonal elements of \( L \to 0 \), and also creates a well-defined gradient for \( L \) at the boundary of the feasible region. Then given a sequence of step sizes \( \gamma_k \), Monte Carlo samples \( Z_k \) and initialization \( \mu_0 = \tilde{\theta}_n^*, L_0 = I \), the standard stochastic gradient update applied to the Gaussian variational inference problem is

\[
\mu_{k+1} = \mu_k - \gamma_k \hat{\nabla}_{\mu, L, Z} \mu_k, L_k, Z_k) \\
L_{k+1} = L_k - \gamma_k \hat{\nabla}_{L, Z} \mu_k, L_k, Z_k).
\]

After making this scaled gradient update, we ensure that the diagonal of \( L \) remains nonnegative by employing a simple projection step after each update: we set any negative diagonal entry in the current iterate \( L_k \) to 0. CSVI based on SGD with these two simple modifications is presented in Algorithm 3. The convergence of CSVI in the sense of Definition 1 is provided by Theorem 3.

Algorithm 3 Consistent stochastic variational inference

\[
\text{procedure CSVI} - \frac{1}{2} \log \pi_n, \mu_0, (\gamma_k)_{k \in \mathbb{N}}, K) \\
\mu \leftarrow \text{Smoothed MAP (Algorithm 1)} \\
L \leftarrow I \\
\text{for } k = 1, \ldots, K \text{ do} \\
\text{Sample } Z \sim \mathcal{N}(0, I) \\
g \leftarrow \hat{\nabla}_{\mu, L, Z} (\mu, L, Z) \text{ and } G \leftarrow \hat{\nabla}_{L, Z} (\mu, L, Z) \\
\text{for } i = 1, \ldots, d \text{ do} \\
\text{if } L_0 > 0 \text{ then} \\
G_i \leftarrow \frac{1}{1+(nL_0)^{\ell/4}} G_i \\
\text{else} \\
G_i \leftarrow -1 \\
\text{end if} \\
\text{end for} \\
\mu \leftarrow \mu - \gamma_k g \text{ and } L \leftarrow L - \gamma_k G \\
\text{for } i = 1, \ldots, d \text{ do} \\
L_0 \leftarrow \max \{0, L_0\} \\
\text{end for} \\
\text{end for} \\
\text{return } \mu, L \\
\text{end procedure}
\]

Theorem 3. Suppose Assumptions 1 and 2 hold. There exist constants \( r, C > 0 \) such that if we initialize \( \mu \) such that \( \| \mu - \tilde{\theta}_n^* \| \leq \frac{r}{2^2} \), and

\[
\gamma_k = \Theta(\kappa^\rho) \text{ for some } \rho \in (0.5, 1], \text{ and} \\
\forall k \in \mathbb{N}, \quad 0 < \kappa < C,
\]

then for any \( \rho' \in (0, \rho - 0.5) \), CSVI asymptotically solves Gaussian variational inference at rate \( k^{-\rho'} \).

Note that both constants \( r \) and \( C \) in the statement of Theorem 3 are not known in practice. The constant \( C \) is a step-size parameter that is typical in the analysis of stochastic gradient methods, and must be tuned. The constant \( r \), defined in Corollary 14, intuitively represents the radius of the convex local basin around the optimal variational parameters. In practice, one should run the smoothed MAP optimization Algorithm 1 until convergence to \( \tilde{\theta}_n^* \) according to some diagnostic, e.g., a small gradient norm. Further, note that one could use other stochastic gradient-based optimization schemes in CSVI and the smoothed MAP optimization, such as the Nesterov accelerated gradient algorithm (Nesterov, 1983), AdaGrad (Duchi et al., 2011), and Adam (Kingma and Ba, 2015); in our experiments we use Adam. Although our theoretical results do not cover those variants, we expect that they could be extended as long as one uses the gradient scaling on \( L \) in Eq. (8).

4 Computational asymptotic theory

In this section, we provide a detailed investigation of the MAP and variational inference optimization problems, which underpins the convergence of CLA (Theorem 2) and CSVI (Theorem 3). We take advantage of the theory of statistical
asymptotics to show that as we obtain more data, the optimum solutions of Eqs. (5) and (1) each converge to a fixed value, the objective functions become locally strongly convex around that fixed value, and the smoothed MAP initialization lies within that local region.

4.1 Statistical model and assumptions

As is common in past work (Ghosal et al., 2000; Kleijn and van der Vaart, 2012; Shen and Wasserman, 2001), we take a frequentist approach to analyzing Bayesian inference. We assume that the sequence of observations are independent and identically distributed \((X_i)_{i=1}^n \sim P_\theta\) from a distribution \(P_\theta\) with parameter \(\theta \in \mathbb{R}^d\) selected from a parametric family \(\{P_\theta : \theta \in \mathbb{R}^d\}\). We further assume that for each \(\theta \in \mathbb{R}^d, P_\theta\) has common support, has density \(p_\theta\) with respect to some common base measure, and that \(p_\theta(x)\) is a Lebesgue measurable function of \(\theta\) for all \(x\). Finally, we assume the prior distribution \(P_\theta\) has full support on \(\mathbb{R}^d\) with density \(\pi_0\) with respect to the Lebesgue measure. Thus by Bayes’ rule, the posterior distribution \(\Pi_n\) has density proportional to the prior density times the likelihood, i.e.,

\[
\pi_n(\theta) \propto \pi_0(\theta) \prod_{i=1}^n p_\theta(X_i).
\]

In order to develop the theory in this work, we require a set of additional technical assumptions on \(\pi_0\) and \(p_\theta\) given by Assumption 1. These are a collection of regularity conditions that are standard for parametric models, which guarantee that the maximum likelihood estimate (MLE) \(\theta_{MLE,n} := \arg\max_{\theta \in \mathbb{R}^d} \sum_{i=1}^n \log p_\theta(X_i)\) is well-defined and asymptotically \(\sqrt{n}\)-consistent for \(\theta_0\) (van der Vaart, 2000, Theorem 5.39), and that the Bayesian posterior distribution of \(\sqrt{n}(\theta - \theta_0)\) converges in total variation to a Gaussian distribution; this is known as the Bernstein-von Mises theorem (van der Vaart, 2000, Theorem 10.1). Also, Assumption 1 is sufficient to guarantee the consistency of MAP estimator \(\theta_n^*\) to \(\theta_0\) (Grendár and Judge, 2009, Lemma 2.1), and the asymptotic exactness of the Laplace approximation Eq. (4) (Miller, 2021, Theorem 4).

**Assumption 1. (Regularity Conditions)**

1. \(\{P_\theta : \theta \in \mathbb{R}^d\}\) is an identifiable family of distributions;
2. For all \(x, \theta\), the densities \(\pi_0, p_\theta\) are positive and twice continuously differentiable in \(\theta\);
3. There exists a measurable function \(L(x)\) such that for \(\theta, \theta'\) in a neighborhood of \(\theta_0\) and for all \(x\),

\[
|\log p_\theta(x) - \log p_{\theta'}(x)| \leq L(x)||\theta - \theta'||,
\]

\[
\mathbb{E}_{\theta_0}[L(X)^2] < \infty;
\]

4. For all \(\theta\),

\[
H_\theta := - \mathbb{E}_\theta [\nabla^2 \log p_\theta(X)] = \mathbb{E}_\theta [\nabla \log p_\theta(X) \nabla \log p_\theta(X)^T],
\]

and \(H_{\theta_0} \geq \epsilon I\) for some \(\epsilon > 0\). Further, for \(\theta, \theta'\) in a neighborhood of \(\theta_0\),

\[
(\theta, \theta') \rightarrow \mathbb{E}_{\theta'}[-\nabla^2 \log p_\theta(X)]
\]

is continuous in spectral norm;
5. There exists a measurable function \(g(x)\) such that for \(\theta\) in a neighborhood of \(\theta_0\) and for all \(x\),

\[
\max_{i,j \in [d]} |\nabla^2 \log p_\theta(x)|_{i,j} < g(x), \quad \mathbb{E}_{\theta_0}[g(X)] < \infty.
\]

**Theorem 4** (Bernstein-von Mises & MLE consistency (van der Vaart, 2000, Theorems 5.39, 10.1)).

**Under Assumption 1,**

\[
\sqrt{n}(\theta_{MLE,n} - \theta_0) \xrightarrow{d} \mathcal{N}(0, H_{\theta_0}^{-1}), \text{ and } D_{TV}\left(\Pi_n, \mathcal{N}\left(n^{-1/2}\Delta_n, \theta_0, n^{-1}H_{\theta_0}^{-1}\right)\right) \xrightarrow{p} 0,
\]

where \(\Delta_n, \theta_0 = n^{-1/2} \sum_{i=1}^n H_{\theta_0}^{-1/2} \nabla \log p_\theta(X_i)\).

**Theorem 5** (Laplace & MAP consistency (Grendár and Judge, 2009, Lemma 2.1; Miller, 2021, Theorem 4)).

**Under Assumption 1,**

\[
\theta_n \xrightarrow{p} \theta_0, \text{ and } D_{TV}\left(\Pi_n, \mathcal{N}\left(\theta_n^*, \Sigma_{Lap,n}\right)\right) \xrightarrow{p} 0.
\]

Note that the above conditions in Assumption 1 are stronger (van der Vaart, 2000, Lemmas 7.6 and 10.6) than the usual local asymptotic normality (van der Vaart, 2000, Section 7) and testability (van der Vaart, 2000, p. 141) conditions required for asymptotic posterior concentration and Gaussianity in the Bayesian asymptotics literature. Many of the results in this work would still hold with these weaker conditions, but we prefer Assumption 1 for the present work as these conditions are simpler to state and check.

The regularity conditions in Assumption 1—which essentially all pertain to a local neighborhood of \(\theta_0\)—are sufficient for the analysis of the smoothed MAP optimization and Laplace approximation. For variational inference, however, we additionally require asymptotic control on the global smoothness of the negative log posterior density \(f_n\). This is essentially because \(f_n\) in the variational objective Eq. (1) is an expectation of \(f_n\) under a normal distribution; if the tails of \(f_n\) are nonconvex and grow quickly, they can influence the convexity of \(f_n\) even locally around \(\theta_0\). In this work, we impose a bound on the second derivative, but we conjecture that bounds on higher-order derivatives would also suffice; see Section 4.5 for details.
Assumption 2. (Asymptotic Smoothness) There exists an $\ell > 0$ such that

$$\mathbb{P}\left( \sup_{\theta} \left| \frac{1}{n} \nabla^2 \log \pi_n(\theta) \right|_2 > \ell \right) \to 0,$$

where $\| \cdot \|_2$ denotes the matrix spectral norm.

4.2 Global optimum consistency

The first important property of both Gaussian variational inference and the MAP problem is that the optimum point converges to a fixed location; this substantially simplifies the convergence analysis of both CLA and CSVI. In particular, for the Laplace approximation, $\hat{\theta}_n^*$ converges in probability to the data-generating parameter $\theta_0$. For Gaussian variational inference, $\mu_n^*, L_n^*$ converges in probability to $\theta_0, L_0$, where $L_0$ is the unique positive-diagonal Cholesky factor of the inverse Fisher information matrix $H_0^{-1} = L_0 L_0^T$. The precise statement for the Laplace approximation was given earlier in Theorem 5, while Theorem 6 provides the precise statement of the optimal variational distribution (Wang and Blei, 2019) and the continuity of the positive-diagonal Cholesky decomposition (Schatzman, 2002, p. 295).

**Theorem 6.** Under Assumption 1,

$$\forall \varepsilon > 0, \lim_{n \to \infty} \mathbb{P}(\| \mu_n^* - \theta_0 \| < \varepsilon, \| L_n^* - L_0 \| < \varepsilon) = 1.$$

4.3 Convexity of smoothed MAP and consistency of $\hat{\theta}_n^*$

Next, we analyze the properties of the smoothed MAP problem and its optimum $\hat{\theta}_n^*$. Although intuitively reasonable, Gaussian smoothing typically does not typically come with strong practical theoretical guarantees, essentially because a good choice of the smoothing variance $\alpha_n$ is not known. Mobahi (2013) shows for a continuous integrable function with quickly decaying tails (at rate $\|x\|^{-d-3}$ as $\|x\| \to \infty$), the smoothed function is strictly convex given a large enough selection of $\alpha_n$. Addis et al. (2005) studies the smoothing effect of a log-concave kernel on a special type of piecewise constant function, and proves that the smoothed function is either monotonic or unimodal. To the best of our knowledge, previous analyses of smoothed optimization do not provide guidance regarding the choice of $\alpha_n$ or bounds on the error of the smoothed optimum point versus the original.

In contrast to past work, we leverage the asymptotic concentration of the statistical model as $n \to \infty$ to obtain error bounds as well as guidance on choosing $\alpha_n$. In particular, Theorem 7 shows that if the sequence $\alpha_n$ is chosen to decrease slower than $n^{-1/3}$, the smoothed MAP problem is eventually strictly convex within any arbitrary compact domain, and that the solution of the smoothed MAP problem $\hat{\theta}_n^*$ is asymptotically consistent for $\theta_0$ at a $\sqrt{n}$ rate. Therefore, we can tractably estimate $\hat{\theta}_n^*$, and—combined with the result of the previous section—use it as an initialization for $\theta$ in CLA and $\mu$ in CSVI that is guaranteed to be close to $\theta_n^*$ and $\mu_n^*$, respectively.

**Theorem 7.** Suppose Assumption 1 holds and $n \alpha_n^2 \to \infty$.

Then for all $M > 0$, the probability that the smoothed MAP optimization problem

$$\min_{\| \theta - \theta_0 \| \leq M} - \log \tilde{\pi}_n(\theta)$$

is strictly convex converges to 1 as $n \to \infty$ under the data generating distribution. Further, the optimal for the smoothed MAP problem is asymptotically $\sqrt{n}$-consistent, that is

$$\| \hat{\theta}_n^* - \theta_0 \| = O_{\mathbb{P}_{\theta_0}}(\sqrt{n}).$$

4.4 Asymptotic local convexity and smoothness of $f_n$

Note that the statistical consistency of the optima $\mu_n^*, L_n^*$, $\theta_n^*$ and smoothed MAP $\hat{\theta}_n^*$ alone do not provide a complete analysis of the asymptotics; in order to make use of these results, we require that the variational inference objective Eq. (1) and the MAP objective Eq. (5) are well-behaved in some sense. Since we have access only to (stochastic estimates of) the gradient of the objective function in Eqs. (1) and (5), and stochastic gradient descent is known to solve optimization problems with strongly convex and Lipschitz smooth objectives (Bottou, 2004; Rakshit et al., 2012), this amounts to analyzing the convexity and smoothness of the objective functions.¹ We begin with a generalization of the typical definitions of strong convexity and Lipschitz smoothness found in the literature (Boyd and Vandenberghe, 2004) in Definition 8.

**Definition 8** (Convexity and Smoothness). Let $g : \mathcal{X} \to \mathbb{R}$ be a twice differentiable function on a convex set $\mathcal{X} \subseteq \mathbb{R}^d$, and let $D : \mathcal{X} \to \mathbb{R}^{d \times d}$ be a positive definite matrix depending on $x$. Then $g$ is $D$-strongly convex if

$$\forall x \in \mathcal{X}, \nabla^2 g(x) \succeq D(x),$$

and $g$ is $D$-Lipschitz smooth if

$$\forall x \in \mathcal{X}, \nabla^2 g(x) \preceq D(x).$$

If either property holds only within a convex subset $\mathcal{Y} \subset \mathcal{X}$, we say it holds locally within $\mathcal{Y}$.

¹ There are many other properties one might require of a tractable optimization problem, e.g., pseudoconvexity (Crouzeix and Ferland, 1982), quasiconvexity (Arrow and Enthoven, 1961), or invexity (Ben-Israel and Mond, 1986). We focus on convexity as it does not impose overly stringent assumptions on our theory and has stronger implications than each of the aforementioned conditions.
Fig. 3: Plots of the function $f_n(y)$ from Example 10. Each row of figures represents a single realization of the sequence $(f_n)_{n \in \mathbb{N}}$ for increasing sample sizes 5, 50, 100, and 1000. Each column includes three repetitions of $f_n$ under a single $n$. As $n$ increases, the function $f_n(y)$ is more likely to be strongly convex and Lipschitz smooth with constants approaching 2.

In general, the MAP objective function in Eq. (5)—i.e., the scaled negative log posterior density $f_n$—need not be strongly convex or smooth for any particular $n$, and as such it is difficult to make any claim regarding convergence. This is where statistical asymptotics provides a major benefit in optimization: by Lemma 9, as $n \rightarrow \infty$, the probability that $f_n$ becomes locally strongly convex and Lipschitz smooth around $\theta_0$ converges to 1 (see Example 10 for an illustration of this effect). The convergence of CLA (Theorem 2) is then essentially a consequence of the fact that we initialize near $\theta_n$, which converges to $\theta_0$ by Theorem 7, and that the local convexity and smoothness in Lemma 9 ensures that gradient descent will contract towards $\theta_n^*$.

**Lemma 9.** Under Assumption 1, there exist $r, \epsilon, \ell > 0$ such that

$$\mathbb{P}(f_n \text{ is } \epsilon\ell\text{-strongly convex and } \ell\text{-Lipschitz smooth in the set } B_r(\theta_0)) \rightarrow 1,$$

as $n \rightarrow \infty$, where $B_r(\theta_0) := \{ \theta \in \mathbb{R}^d : \| \theta - \theta_0 \| \leq r \}$.

**Example 10.** Let $f_n(y) = y^2 + (\frac{1}{n} \sum_{i=1}^n X_i) \cos 5y$, where $X_i \sim \mathcal{N}(0, 1)$. Then

$$\frac{d^2 f_n}{dy^2} - 2 = 25 \cos(5y)\cdot n^{-1} \sum_{i=1}^n X_i.$$

Therefore by the law of large numbers and the fact that $|\cos(5y)| \leq 1$, for any $\epsilon > 0$, the sequence $(f_n)_{n \in \mathbb{N}}$ is asymptotically $(2 - \epsilon)$-strongly convex and $(2 + \epsilon)$-Lipschitz smooth. Fig. 3 visualizes the asymptotic behaviour of $f_n$ as $n$ increases.

4.5 Asymptotic local convexity and smoothness of $F_n$

The variational objective function Eq. (1) contains two terms: a regularization $-n^{-1} \log \det L$, and an expectation of the negative log posterior density under Gaussian noise, $F_n(\mu, L)$. The regularization term is known to be convex (Boyd and Vandenberghe, 2004, p.73) and Lipschitz smooth in any compact subset of the domain (the gradient scaling Eq. (8) essentially makes the smoothness uniform on the whole domain; see the proof of Theorem 3 in Supp. B). Therefore in this section we focus on the analysis of the data-dependent term, $F_n(\mu, L)$. The first main result is that $F_n(\mu, L)$ inherits the global convexity and smoothness behaviour of $f_n$.

**Theorem 11.** Suppose $f_n$ is $D$-strongly convex (Lipschitz smooth) for positive definite matrix $D \in \mathbb{R}^{d \times d}$. Then $F_n$ is $D'$-strongly convex (Lipschitz smooth), where

$$D' = \text{block}(D, n^{-1} D, \ldots, n^{-1} D) \in \mathbb{R}^{(d+1)d \times (d+1)d},$$

and blockd creates a block-diagonal matrix out of its arguments.

For example, if the posterior distribution $\Pi_n$ is a multivariate Gaussian distribution $\mathcal{N}(\mu_n, n^{-1} \Sigma_n)$ with mean $\mu_n$ and covariance $n^{-1} \Sigma_n$, then the expectation component of the Gaussian variational inference objective function becomes

$$F_n(\mu, L) = n^{-1} \text{tr} \Sigma_n^{-1} LL^T + (\mu - \mu_n)^T \Sigma_n^{-1} (\mu - \mu_n),$$

which is a jointly convex quadratic function in $\mu, L$ with Hessian matrix (for $\mu$ and columns of $L$ stacked together in a single vector) equal to

$$\text{blockd}(\Sigma_n^{-1}, \ldots, n^{-1} \Sigma_n^{-1}) \in \mathbb{R}^{(d+1)d \times (d+1)d}.$$

Combined with the convexity of the log determinant term $-n^{-1} \log \det L$ (Boyd and Vandenberghe, 2004, p.73), Gaussian variational inference for strongly convex and Lipschitz smooth negative log posterior density $-n^{-1} \log \pi_n$ is itself strongly convex and Lipschitz smooth in any compact set contained in the optimization domain.

However, in a typical statistical model, the posterior is typically neither Gaussian nor strongly convex. But when the Bernstein-von Mises theorem holds (van der Vaart, 2000), the posterior distribution (scaled and shifted appropriately) converges asymptotically to a Gaussian distribution. Thus, it may be tempting to think that the Bernstein von-Mises theorem implies that Gaussian variational inference should eventually become a convex optimization problem. This is unfortunately not true, essentially because Bernstein-von Mises only implies convergence to a Gaussian in total variation distance, but not necessarily in the log density function or its gradients. The second main result in this section—Proposition 12—is
a simple demonstration of the fact that the Bernstein-von Mises theorem is not sufficient to guarantee the convexity of Gaussian variational inference.

**Proposition 12.** Suppose $d = 1$, $f_n$ is differentiable to the third order for all $n$, that there exists an open interval $U \subseteq \mathbb{R}$ and $\varepsilon > 0$ such that

$$
\sup_{\theta \in U} \frac{d^2 f_n}{d\theta^2} \leq -\varepsilon,
$$

and that there exists $\eta > 0$ such that

$$
\sup_{\theta \in \mathbb{R}} \left| \frac{d^3 f_n}{d\theta^3} \right| \leq \eta.
$$

Then there exists a $\delta > 0$ such that

$$
\sup_{\alpha < \delta, \mu \in U} d^2 \text{KL}\left(\mathcal{N}(\mu, \sigma^2)\parallel \Pi_n\right) < 0.
$$

Although Proposition 12 is a negative result about the global convexity of $F_n$, it does hint at a very useful fact: the local convexity of $F_n$ matches that of $f_n$, assuming that we control the global growth of $f_n$ (e.g., in Proposition 12, we imposed a uniform bound on the 3rd derivative). This is due to the fact that $F_n(\mu, L) = \mathbb{E}_{f_n}(\mu + n^{-1/2}LZ)$, where $Z$ is a standard Gaussian random vector; intuitively, since the Gaussian distribution has very light tails, as $n$ grows the Taylor approximation

$$
F_n(\mu, L) \approx f_n(\mu) + \frac{1}{2} n^{-1} L^T \nabla^2 f_n(\mu) L + o(n^{-1})
$$

becomes accurate, assuming $f_n(x)$ does not grow too quickly as $||x|| \to \infty$. Therefore when $n$ is large, we expect $F_n$ to behave like $f_n$ as a function of $\mu$, and be roughly quadratic in $L$ with Hessian $n^{-1} \nabla \nabla^2 f_n(\mu)$. Thus, as long as $f_n$ is locally convex in $\mu$, we expect $F_n$ to be locally convex in $\mu$, as well. The third main result of this section specifies the general link between the local convexity behaviour of $f_n$ and $F_n$ assuming the global Lipschitz smoothness of $f_n$.

**Theorem 13.** Suppose there exist $\varepsilon, \ell, r > 0$ and $x \in \mathbb{R}^d$ such that $f_n$ is globally $\ell$-Lipschitz smooth and locally $\ell I$-strongly convex in the set $\{y : ||y - x|| \leq r\}$. Define

$$
D_n := \text{blockd}\left( I, n^{-1}I, \ldots, n^{-1}I \right) \in \mathbb{R}^{(d+1)d \times (d+1)d},
$$

$$
\tau_n(\mu, L) := 1 - \chi_d^2 + 2 \left( n \left( \frac{\varepsilon^2}{2} - 2 ||\mu - x||^2_2 \right) \right) / 2 \|L\|_F^2,
$$

where $\chi_d^2$ is the CDF of a chi-square random variable with $k$ degrees of freedom. Then $F_n$ reinterpreted as a function of $\mathbb{R}^{(d+1)d} \to \mathbb{R}$—by stacking $\mu$ and each column of $L$ into a single vector—is $D_n \ell$-Lipschitz smooth; and is $(\varepsilon - \tau_n(\mu, L)) D_n \ell$-strongly convex when $||\mu - x||^2 \leq \frac{\varepsilon^2}{2 \tau}$. The function $\tau_n(\mu, L)$ in Theorem 13 characterizes how much the tails of $f_n$ can influence the local strong convexity of $F_n$ around the point $x$. In particular, as long as $\mu$ is close to $x$ and $||L||_F$ (which modulates the effect of noise) is sufficiently small, then the argument of the $\chi_d^2$ CDF is large, so $\tau_n$ is small, so $(\varepsilon - \tau_n(\mu, L)) \approx \varepsilon$; thus we recover local strong convexity of the same magnitude as $f_n$.

A further note is that although Theorem 13 requires a uniform bound on the Hessian of $f_n$, we conjecture that a similar result would hold under the assumption of a uniform bound on the $k^{th}$ derivative. For simplicity of the result and ease of use later on in Section 3, we opted for the second derivative bound.

The last result of this section—Corollary 14—combines Theorems 13 and 6 and Lemma 9 to provide the key asymptotic convexity/smoothness result that we use in the development of the optimization algorithm in Section 3.

**Corollary 14.** Suppose Assumptions 1 and 2 hold, and define $D_n$ as in Theorem 13. Then there exist $\varepsilon, \ell, r > 0$ such that $F_n$ reinterpreted as a function of $\mathbb{R}^{(d+1)d} \to \mathbb{R}$—by stacking $\mu$ and each column of $L$ into a single vector—satisfies

$$
P \left( F_n \text{ is } \frac{\varepsilon}{2} D_n \ell \text{-strongly convex in } \mathcal{B}_{\mathcal{R}_n} \text{ and globally } (D_n, \ell \text{-Lipschitz smooth}) \to 1, \right.
$$

as $n \to \infty$, where

$$
\mathcal{B}_{\mathcal{R}_n} = \left\{ \mu \in \mathbb{R}^d, L \in \mathbb{R}^{d \times d} : ||\mu - \mu^*||^2 \leq \frac{\varepsilon^2}{4} \text{ and } ||L - L^*_n||_F^2 \leq 4 ||I - L^*_n||_F^2 \right\}.
$$

5 Experiments

In this section, we compare CSVI and CLA to standard Gaussian stochastic variational inference (SVI)$^2$ and the standard Laplace approximation on both synthetic and real data inference problems. By default we run both variational optimization algorithms for 100,000 iterations and the smoothed MAP optimization for 20,000 iterations. We base the gradients for the smoothed MAP mean initialization (Algorithm 1) on 100 samples, and the gradients for VI algorithms on a single sample. For both CLA and Laplace, we run backtracking line search for 20,000 iterations, with $\beta = 0.5$.

5.1 Synthetic Gaussian mixture

In the first experiment, we compare the reliability of CSVI and SVI, CLA and Laplace approximation on a simple target function under different initialization schemes, choices of

$^2$ Code for the experiments is available at https://github.com/zuhengxu/Consistent-Stochastic-Variational-Inference.
The computational asymptotics of Gaussian variational inference and the Laplace approximation

Fig. 4: The result of 20 trials of Gaussian approximation with the Gaussian mixture target (grey) given in Eq. (11). Each plot shows the target distribution and 20 Gaussian approximations obtained from one algorithm-initialization combination.

Fig. 5: The violin plots of the ELBOs of running 100 trials of Gaussian approximation with the Gaussian mixture target given in Eq. (11). The output of CSVI and CLA reliably finds the global optimum solution corresponding to the central mixture peak; SVI often provides solutions corresponding to local optima.

smoothing constant, and learning rates. The inferential goal is to approximate a Gaussian mixture distribution \( \Pi \).

\[
\Pi = 0.7 \mathcal{N}(0,4) + 0.15 \mathcal{N}(-30,9) + 0.15 \mathcal{N}(30,9).
\]  (11)

We set \( n = 1 \) in this example as there is no data likelihood. We use the smoothing constant \( \alpha_n = 100 \) in the implementation of CSVI, and initialize the smoothed MAP optimization, Laplace approximation and the mean of SVI uniformly in the range \((-50,50)\). The standard deviation \( \sigma \) for CSVI is initialized to be 1 and the log standard deviation \( \log \sigma \) for SVI is initialized uniformly in the range \((\log 0.1, \log 10)\). Unless otherwise indicated, we hand-tune the learning rates for both CSVI and SVI to optimize performance—we set \( \gamma_k = 5/(1+k) \) for CSVI, \( \gamma_k = 15/(1+k) \) for SVI, and \( t \) for CLA is set as 1.

In Section 5.1.1, we compare CLA, Laplace approximation, CSVI and SVI under various initialization schemes, aiming to dissect the contribution of each element of our methodology. Specifically, aside from the standard CSVI and SVI methods described above, we consider 4 additional combinations of VI algorithm and initialization: CSVI_RSD, SVI_Ind, SVI_SMAP and SVI_OPT. In particular, CSVI_RSD differs from CSVI by initializing \( \log \sigma \) uniformly in the range \((\log 0.5, \log 10)\), SVI_Ind denotes SVI with \( \sigma_0 = 1 \), SVI_SMAP is SVI using \( \mu_0 \) as the smoothed MAP, and SVI_OPT uses the optimal initialization (smoothed MAP for mean and \( \sigma = 1 \)) for SVI. The results demonstrate that both the smoothed MAP initialization and scaled gradient estimates are necessary to produce consistent and reliable VI approximations; and the smoothed MAP initialization also improve the reliability of Laplace method significantly.

In Section 5.1.2, we investigate the sensitivity of CSVI to the smoothing constant \( \alpha_n \). The results demonstrate that the performance of CSVI is very robust to the change of \( \alpha_n \). We also compare the reliability of CSVI and SVI across different optimization step schedules \( \gamma_k \), in which CSVI outperforms SVI in all settings and generally favours smaller learning rate.

5.1.1 Sensitivity to initialization

We first demonstrate the performance of CLA, Laplace, CSVI, SVI and their 4 variants under different initialization schemes—CSVI_RSD, SVI_Ind, SVI_SMAP and SVI_OPT. We run 100 trials for each variant. Fig. 4 visualizes 20 variational approximations and Laplace approximations that are randomly selected from the 100 trials. Note that the majority of the mass of the Gaussian mixture target distribution concentrates on the central mode with mean 0 and standard deviation 2; the optimal variational approximation has these same parameters. As shown in the plot, CSVI reliably learns the optimal variational distribution. CSVI_RSD—with randomly initialized standard deviation—occasionally becomes trapped in a local optimum that places the mean between the central mode and the adjacent peaks with a large standard deviation. The variants without gradient scaling (SVI, SVI_Ind, SVI_SMAP, and SVI_Opt) are significantly more likely to find this same local optimum; this is because the standard projected gradient has unstable behaviour for small \( \sigma \) due to the log-determinant regularization term. The comparison between CLA and standard Laplace method reinforce the im-
Fig. 6: The result of smoothed MAP of the Gaussian mixture target Eq. (11) across different values of $\alpha$. The black curve corresponds to the Gaussian mixture density and the red histogram shows the counts (over 100 trials) of the output of the smoothed MAP initialization.

Fig. 7: The result of running 100 trials of CSVI and CSVI_RSD with the Gaussian mixture target Eq. (11) using different smoothing values $\alpha = 20, 50, 200, 2000, 10000, 100000$. The scatter points mark ELBO outliers.

Fig. 8: The sensitivity results of CSVI and SVI with target distribution Eq. (11) across different $\gamma$. SVI often does not converge when $\gamma = 5$; the inset shows the results for these trials.

importance of a reliable initialization scheme for deterministic posterior approximation; Laplace approximation with random initialization is particularly sensitive to the local optima of posterior due to its lack of stochasticity.

These observations reveal two important facts. First, a good mean initialization is important and helps recover the global optimum. Second, the gradient scaling described in Eq. (8) aids the stability of the VI algorithm, which ensures that the algorithm stays in the region around the optimum, and hence converges to the optimal solution.

Fig. 5 presents a quantitative characterization of this result. In particular, we plot the final expectation lower bound (ELBO) (Blei et al., 2017) for each method, which is equivalent to the negative KL divergence between the posterior and variational distribution up to a constant; a larger ELBO value corresponds to a better approximation. We estimate the ELBO using 1000 Monte Carlo samples. As demonstrated in the violin plots, CSVI and CSVI_RSD find the global optimum significantly more reliably than SVI and its variants. Also, by comparing the distribution of the ELBO of the 100 trials of SVI and its variants, we find that the influence of the initial value alone is limited. This aligns well with our earlier theory in Corollary 14; in order to reliably find the global optimum of variational inference problem, one needs both a careful initialization and to stay in the basin of the global optimum during optimization. However, for the Laplace approximation, careful initialization determines its performance.

5.1.2 Sensitivity to smoothing and learning rate

Next, we study the sensitivity of the VI algorithms to the choice of the smoothing constant $\alpha$. Note that the smoothing constant $\alpha$ is the variance of the Gaussian smoothing kernel. A larger value corresponds to a more aggressive smoothing effect, and hence a lower likelihood of finding a spurious local peak. As shown in Fig. 6, as long as $\alpha$ is set large
enough, the smoothed MAP initialization has a reasonable chance to locate $\mu_0$ at the central mode of target distribution Eq. (11). As a result, CSVI and CSVI_RSD has a mean initialization close to that of the optimal variational distribution. Fig. 7 presents the distribution of the ELBOs over 100 trials of CSVI and CSVI_RSD. This figure demonstrates that CSVI and CSVI_RSD reliably find the optimal ELBO for a wide range of $\alpha_0$ ranging from about 100 to 100,000. In other words, the approach is not overly sensitive to the value of $\alpha_0$.

Finally, we illustrate the sensitivity of CSVI and SVI to the optimization step schedule. Both algorithms are run for 100 trials across different step schedules, i.e., $\gamma_k = C/(1 + k)$ for $C = 5, 10, 15, 20, 25, 30$. In Fig. 8, we display the spread of the ELBOs. In general, CSVI outperforms SVI for all choices of $\gamma_k$—it is more likely to find the optimum and the ELBO variation between trials is significantly smaller. This confirms that CSVI is less sensitive to the choice of learning rate. Further, SVI requires many more steps to converge than CSVI when $\gamma_k$ is small. As the step size gets larger, CSVI may overshoot its original local basin and converge to a suboptimal point.

5.2 Bayesian sparse linear regression

In this experiment, we compare the quality of CSVI, SVI, CLA and Laplace on a Bayesian sparse linear regression problem. As mentioned at the end of Section 3.3, we use Adam Kingma and Ba (2015) updates in both the smoothed MAP estimation and the variational inference algorithm to achieve faster convergence. The detailed implementation of the Adam version of CSVI is presented in Algorithm 4.

In the Bayesian sparse linear regression model, we are given a set of data points $(x_n, y_n)_{n=1}^N$ with feature $x_n \in \mathbb{R}^d$ and response $y_n \in \mathbb{R}$, we assume that the responses were generated from a Gaussian likelihood

$$y_n \mid x_n, \beta \sim \mathcal{N}(x_n^T \beta, \sigma^2),$$

and we assert that the feature coefficients each have a “spike and slab” prior distribution consisting of a mixture of two Gaussian distributions with different variances

$$\beta_i \sim \frac{1}{2} \mathcal{N}(0, \tau_1^2) + \frac{1}{2} \mathcal{N}(0, \tau_2^2),$$

where $\tau_1$ is set to a small value and $\tau_2$ is set to be large. Priors of this type are commonly used to encode variable selection (George and McCulloch, 1993). The goal of the inference is to approximate the posterior distribution of $(\beta_1, \ldots, \beta_d)$ with a full rank Gaussian distribution.

We run 100 trials of CSVI and SVI on two datasets—a synthetic dataset, and a dataset of measurements of 97 men with prostate cancer. For the synthetic example, we set $\sigma = 5, \tau_1 = 0.1, \tau_2 = 10$ and generate features $x_n \in \mathbb{R}^5$ i.i.d. from $\mathcal{N}(0, I)$ with $N = 10$. The response $y_n$ is generated from the following process,

$$y_n = [1 0 0 0 0]^T x_n + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, 0.25).$$

We use learning rates of $\gamma_k = 0.01$ for the optimization of SMAP, $t = 0.01$ for backtracking line search, and set the smoothing constant $\alpha_0 = 2$. Both the initial value of SMAP optimization, MAP optimization and the mean initialization of SVI are randomly sampled from the prior distribution. In terms of $L_0$, we consider two different initialization schemes—the identity matrix and a random diagonal matrix—for both CSVI and SVI, where for the random $L_0$, $\log |L_0|_d, i \in [d]$ is uniform in the range $(\log 0.1, \log 100)$. The learning rates for both VI algorithms are set to 0.001. Fig. 9 shows the ELBOs of VI inference on the synthetic data over 100 trials. Compared to SVI, CSVI reliably learns a better approximation to the posterior and rarely gets trapped at a local optimum. And CLA is much more robust to local minima compared to standard Laplace due to a good initialization. To visualize different Gaussian approximations, Fig. 10 shows contours of the optimal variational approximation (ELBO $= −1$) and three local approximations corresponding to ELBO $= −2, −6, −7$ respectively; it is clear that CSVI tends to find better local optima than SVI. Notice that in this example, the ELBO distribution of CLA is more consistent than CSVI’s. This is essentially due to the fact that CLA is a deterministic approximation method—the Hessian of log posterior evaluated at the mean determines the fitted covariance, while CSVI is able to produce multiple covariance fit given the Gaussian mean.

For the real dataset experiment, we subsample the original dataset to $N = 30$ data points. We apply the SMAP optimization for 200,000 iterations with the smoothing constant $\alpha_0 = 0.03$; and the learning rate for SMAP and VI algorithms are set to 0.002 and 0.0002 respectively. Other settings remain identical to the synthetic experiment. The results in Fig. 11 generally align with those from the previous synthetic experiment—CSVi and CLA outperform SVI and Laplace respectively with a more consistent and accurate Gaussian approximation. But an interesting observation is that CLA dominates all other methods and CSVI fails to learn those Gaussian distribution produced by CLA. It is mainly because that CLA fits into the dominating mode of posterior while CSVI prefers to fit a wider Gaussian distribution that covers the whole range of the posterior. A more detailed discussion, including several supporting visualizations is included in Supp. A.3.

5.3 Bayesian Gaussian mixture model

We finally compare the methods on a Bayesian Gaussian mixture model (GMM) applied to a synthetic dataset and the

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3 Available online at http://www.stat.cmu.edu/~ryantibs/statcomp/data/pros.dat.
we employ Adam for all optimization procedures involved in the inference and each experiment is repeated for 100 trials. The goal is to infer the posterior distribution of all the latent parameters.

Fig. 9: Sparse regression results on the synthetic dataset. The violin plots show the distribution of the ELBOs over 100 trials.

Shapley galaxy dataset\(^4\). Similar to the previous experiments, we employ Adam for all optimization procedures involved in the inference. The Shapley galaxy dataset is available by Michael Drinkwater, University of Queensland, which can be downloaded from https://astrostatistics.psu.edu/datasets/Shapley_galaxy.html.

Similar to the previous experiments, we pick a misspecified setting to make the posterior distributions of the redshifts for 4215 galaxies in the Shapley Concentration regions and was generously made available by Michael Drinkwater, University of Queensland, which can be downloaded from https://astrostatistics.psu.edu/datasets/Shapley_galaxy.html.

6 Conclusion

This work provides an extensive theoretical analysis of the computational aspects of Laplace approximation and Gaussian variational inference, and uses the theory to design a general procedure that addresses the nonconvexity of the problem in an asymptotic regime. We show that under mild conditions, the MAP estimation problem and Gaussian variational optimization are locally asymptotically convex. Based on this fact, we developed consistent stochastic variational inference (CSVI), a scheme that asymptotically solves Gaussian variational inference; and consistent Laplace approximation—its local approximation method which totally depends on the curvature of log posterior function; while variational inference is based on reducing KL divergence to the target, which is a global metric. The details of the real data experiment are deferred to Supp. A.4.2.

\[^4\] This dataset contains the measurements of the redshifts for 4215 galaxies in the Shapley Concentration regions and was generously made available by Michael Drinkwater, University of Queensland, which can be downloaded from https://astrostatistics.psu.edu/datasets/Shapley_galaxy.html.
The computational asymptotics of Gaussian variational inference and the Laplace approximation

Fig. 10: Visualization of different VI approximations to the posterior distribution on synthetic dataset ($\beta_1$ vs $\beta_2$; $\beta_3$ vs $\beta_2$). The grey area depicts the posterior density and four Gaussian approximations displayed in contour plots corresponding to different ELBOs.

Fig. 11: Sparse regression results on the prostate cancer dataset. The violin plots show the distribution of the ELBOs over 100 trials.

There are many avenues of further exploration for the present work. For example, we limit consideration to the case of Gaussian variational families due to their popularity; but aside from the mathematical details, nothing about the overall strategy necessarily relied on this choice. It would be worth examining other popular variational families, such as mean-field exponential families (Xing et al., 2002).

Furthermore, the current work is limited to posterior distributions with full support on $\mathbb{R}^d$—otherwise, the KL divergence variational objective is degenerate. It would be of interest to study whether variational inference using a Gaussian variational family truncated to the support of the posterior possesses the same beneficial asymptotic properties and asymptotically consistent optimization algorithm as developed in the present work.

Another interesting potential line of future work is to investigate other probability measure divergences as variational objectives. For example, the chi-square divergence (Csiszár, 1967; Liese and Vajda, 1987, p. 51), Rényi $\alpha$-divergence (Van Erven and Harremos, 2014), Stein discrepancy (Stein, 1972), and more (Gibbs and Su, 2002) have all been used as variational objectives. Along a similar vein, we studied the convergence properties of only a relatively simple stochastic gradient descent algorithm; other base algorithms with better convergence properties exist (Duchi et al., 2011; Kingma and Ba, 2015; Nesterov, 1983), and it may be fruitful to see if they have similar asymptotic consistency properties.

A final future direction is to investigate the asymptotic behaviour of variational inference with respect to other measures of optimization tractability. In particular, (local) pseudoconvexity (Crouzeix and Ferland, 1982), quasiconvexity (Arrow and Enthoven, 1961), and invexity (Ben-Israel and Mond, 1986; Craven and Glover, 1985) are all weaker than (local) convexity, but provide similar guarantees for stochas-
Fig. 13: GMM results on the synthetic dataset Fig. 13a and real dataset Fig. 13b. The violin plots show the distribution of the ELBOs over 100 trials.

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A Details of experiments

A.1 Details of the toy example for smoothed MAP

The underlying synthetic model for Fig. 1 is as follows,

\[ \theta \sim \frac{1}{5} N(0,0.15^2) + \frac{1}{5} N(1,0.1^2) + \frac{1}{5} N(-4,0.3^2) + \frac{1}{5} N(4,0.3^2) + \frac{1}{5} \mathcal{N}(-8,0.1^2) \]

\[ X_i | \theta \sim \mathcal{N}(\theta, 0.1) \]

where the data are truly generated from \( (X_i)_{i=1}^n \sim \mathcal{N}(3, 10) \). For smoothed MAP optimization, we use a smoothing constant of \( \alpha_0 = 10n^{-0.3} \), and set the initial value uniformly within the range \((-50, 50)\). The learning rate for the SGD is chosen as \( \gamma_k = 15/(1 + k^{0.3}) \).

A.2 Algorithm: CSVI (Adam)

**Algorithm 4 CSVI with adaptive moment estimation**

procedure CSVI(f, \( \nu, \gamma, K \))

\( \mu_0 \leftarrow \text{SmoothedMAP (Algorithm 1)}, \ L_0 \leftarrow I \)

\( \epsilon \leftarrow 10^{-8}, \hat{\beta}_1 \leftarrow 0.9, \hat{\beta}_2 \leftarrow 0.9999 \)

\( m_k \leftarrow 0, \nu_k \leftarrow 0 \)

for \( k = 0, 1, \ldots, K - 1 \) do

- All operations on vector/matrix are elementwise
- Sample \( Z_k \sim \mathcal{N}(0, I) \)
- \( g_k \leftarrow (\hat{\nu}_{\mu, \nu}(\mu_k, L_k, Z_k), \hat{\nu}_{L, \nu}(\mu_k, L_k, Z_k)) \)
- \( \hat{\nu}_{\mu, \nu}, \hat{\nu}_{L, \nu} \) are same to Algorithm 3

\( m_{k+1} \leftarrow \hat{\beta}_1 m_k + (1 - \hat{\beta}_1) g_k \)

\( \nu_{k+1} \leftarrow \hat{\beta}_2 \nu_k + (1 - \hat{\beta}_2) g_k^2 \)

\( \tilde{\nu}_{k+1} \leftarrow \nu_k / (1 - \hat{\beta}_2) \)

\( \tilde{\nu}_{k+1} \leftarrow \nu_k / (1 - \hat{\beta}_2) \)

\( (\mu_{k+1}, L_{k+1}) \leftarrow (\mu_k, L_k) - \gamma_k / (\sqrt{\tilde{\nu}_k} + \epsilon) \)

for \( i = 1, \ldots, d \) do

\( L_{k+1,i} \leftarrow \max \{ 0, L_{k,i} \} \)

end for

end procedure

end procedure

A.3 Discussion of sparse regression experiment

In this section, we provide further discussion to the result presented in Fig. 11. Figs. 14 and 15 visualizes the Gaussian approximations produced by CLA and CSVI. Instead of fitting a single mode, CSVI covers the range of posterior and fit a Gaussian distribution with larger variance. Even though the performance of CSVI is consistent across runs, it does find the local optimum instead of the global solution. In this case, reverse KL—the objective function of Gaussian VI—can be limited. We compare the forward KL of these fitted Gaussians using 32000 posterior samples obtained from Stan, suggesting that CSVI find a solution that is better in forward KL.
A.4 Details of the GMM experiment

A.4.1 Variable transformations of Bayesian Gaussian mixture model

To transform \((\theta_1, \ldots, \theta_k) \in \Delta(K)\) and \(\sigma_{id} \in \mathbb{R}_+\) unconstrained space, we consider the change of random variables as below:

1. For \(\sigma_{id} \sim \text{LogNormal}(0, 1)\), we consider
   \[\tau_{id} = \log(\sigma_{id}) \sim \mathcal{N}(0, 1),\]
   which has a full support on \(\mathbb{R}\).

2. For \(\theta \sim \text{Dir}(\alpha_0)\), we consider using marginalized LogGamma random variables. Notice the relationship of Gamma distribution and Dirichlet distribution as follows,
   \[\frac{\exp(\lambda_1)}{\sum \exp(\lambda_i)} \sim \text{LogGamma}(\alpha_0, 1),\]
   then \(\lambda_1\) is supported on \(\mathbb{R}\).

Therefore, instead of inferring the original parameters, we perform Gaussian variational inference on the posterior distribution of \((\lambda_1, \mu_{1:k}, \tau_{1:kd})\).

A.4.2 Detailed settings for real data experiment

We subsample the Shapley galaxy dataset to \(N = 500\) and the goal is to cluster the distribution of galaxies in the Shapley Concentration region. In our experiment, we fix the number of component \(K = 3\) and set \(\alpha_0 = 1\). During inference, we initialize the MAP with random samples from the prior distribution, which are also used as the mean initialization for SVI. In SMAP, we perform the smoothed MAP estimation to a tempered posterior distribution \(\pi^\tau\) with \(\tau = 1/2\) and set the smoothing constant \(\tau_0 = 3\). The learning rate for SMAP and VI algorithms are chosen as 0.005 and 0.001 respectively. And similar to the synthetic experiment, CSVI and SVI-Ind use the identity matrix for \(L_0\) and SVI use random diagonal matrix for \(L_0\), whose log diagonal indices are uniformly in the range \((\log 0.1, \log 100)\).

B Proofs

B.1 Proof of Theorem 6

Proof. We consider the KL cost for the scaled and shifted posterior distribution. Let \(\Pi_{0}\) be the Bayesian posterior distribution of \(\sqrt{n}(\theta - \theta_0)\). The KL divergence measures the difference between the distributions of two random variables and is invariant when an invertible transformation is applied to both random variables (Qi and Minnetti, 2010, Theorem 1). Note that \(\Pi_{0}\) is shifted and scaled from \(\Pi_{0}\), and that this linear transformation is invertible, so

\[\text{DKL} \left( \mathcal{N}(\mu, \Sigma) || \Pi_{0} \right) = \text{DKL} \left( \mathcal{N}(\sqrt{n}(\mu_0 - \theta_0), n\Sigma) || \Pi_{0} \right).\]

Let \(\mu_{0}^*, \Sigma_{0}^*\) be the optimal Gaussian variational approximation to \(\Pi_{0}\), i.e.,

\[\mu_{0}^*, \Sigma_{0}^* = \arg \min_{\mu \in \mathbb{R}^d, \Sigma \in \mathbb{R}^{d \times d}} \text{DKL} \left( \mathcal{N}(\mu, \Sigma) || \Pi_{0} \right) \text{ s.t. } \Sigma > 0,\]

and let

\[\mathcal{N}_*^\tau := \mathcal{N}(\mu_{0}^*, \Sigma_{0}^*), \mathcal{N}_*^\tau \left( \sqrt{n}(\mu_{0}^* - \theta_0), n\Sigma_{0}^* L_{0}^* L_{0}^{*\top} \right).\]

Wang and Blei (2019, Corollary 7) shows that under Assumption 1,

\[\text{D}_{TV} \left( \mathcal{N}_*^\tau, \mathcal{N}(\theta_0, H_0^{-1}) \right) \overset{p}{\to} 0.\]

Convergence in total variation implies weak convergence, which then implies pointwise convergence of the characteristic function. Denote \(\tilde{f}_n(t)\) and \(f_0(t)\) to be the characteristic functions of \(\mathcal{N}_*^\tau\) and \(\mathcal{N}(\theta_0, H_0^{-1})\). Therefore

\[\forall t \in \mathbb{R}^d, \quad \tilde{f}_n(t) = \exp \left( i \sqrt{n} (\mu_{0}^* - \theta_0) - \text{Diag} (L_{0}^* L_{0}^{*\top} - H_0^{-1}) \right) \overset{p}{\to} 1,\]

which implies

\[\mu_{0}^* \overset{p}{\to} \theta_0, \quad \text{ and } L_{0}^* L_{0}^{*\top} \overset{p}{\to} H_0^{-1} = L_{0} L_{0}^{\top}.\]

Under Assumption 1, van der Vaart (2000, Theorem 8.14) states that

\[\| \theta_0 - \text{Diag}(\theta_{0\text{MLE}, n} - \theta_0) \|_{\mathbb{F}} \overset{p}{\to} 0,\]

and \(\theta_{0\text{MLE}, n} \overset{p}{\to} \theta_0\) according to Eq. (9), yielding \(\mu_{0}^* \overset{p}{\to} \theta_0\).

Finally since the Cholesky decomposition defines a continuous mapping from the set of positive definite Hermitian matrices to the set of lower triangular matrices with positive diagonals (both sets are equipped with the spectral norm) (Schatman, 2002, p. 295), we have

\[L_{0}^* \overset{p}{\to} L_{0}.\]

\(\square\)

B.2 Proof of Theorem 11

Proof. We provide a proof of the result for strong convexity; the result for Lipschitz smoothness follows the exact same proof technique. Note that if \(D'\) does not depend on \(x\), \(F_{0}(x) = D'\)-strongly convex if and only if \(F_{0}(x) - \frac{1}{2}x^TD'x\) is convex. We use this equivalent characterization of strong convexity in this proof.

Note that for \(Z \sim \mathcal{N}(0, I)\),

\[\mathbb{E} \left[ \frac{1}{2}(\mu + n^{-1/2}LZ)^TD(\mu + n^{-1/2}LZ) \right] = \frac{1}{2} \mu^T D \mu + \frac{1}{2} tr(L^T(n^{-1}D)L).\]

Define \(\lambda \in [0, 1]\), vectors \(\mu, \mu' \in \mathbb{R}^d\), positive-diagonal lower triangular matrices \(L, L' \in \mathbb{R}^{d \times d}\), and vectors \(x, x' \in \mathbb{R}^{(d+1)d}\) by stacking \(\mu\) and the columns of \(L\) and likewise \(\mu'\) and the columns of \(L'\). Define \(s(\lambda) = \lambda x + (1 - \lambda)x', \mu(\lambda) = \lambda \mu + (1 - \lambda)\mu', \text{ and } L(\lambda) = \lambda L + (1 - \lambda)L'\). Then

\[F_{0}(s(\lambda)) - \frac{1}{2} s(\lambda)^T \text{Diag}(D, n^{-1}D, \ldots, n^{-1}D)s(\lambda)\]

\[= F_{0}(\mu(\lambda), L(\lambda)) - \left( \frac{1}{2} \mu(\lambda)^T D \mu(\lambda) + \frac{1}{2} tr(L(\lambda)^T(n^{-1}D)L(\lambda)) \right)\]

\[= \mathbb{E} \left[ n^{-1} \log \sigma_{0}(\mu(\lambda) + n^{-1/2}L(\lambda)Z) - \frac{1}{2}(\mu(\lambda) + n^{-1/2}L(\lambda)Z)^T D(\mu(\lambda) + n^{-1/2}L(\lambda)Z) \right].\]
By the $D$-strong convexity of $n^{-1} \log \pi_n$,
\[
\begin{aligned}
&\leq \lambda \left( F_n(\mu, L) - \frac{1}{2} \mu^T D \mu - \frac{1}{2} \mu^T L^T (n^{-1} D)L \right) \\
&+ (1 - \lambda) \left( F_n(\mu^*, L^*) - \frac{1}{2} \mu^T D \mu^* - \frac{1}{2} \mu^T L^T (n^{-1} D)L^* \right) \\
= & \lambda \left( F_n(x) - \frac{1}{2} x^T \text{diag}(D,n^{-1} D,\ldots,n^{-1} D)x \right) \\
&+ (1 - \lambda) \left( F_n(x') - \frac{1}{2} x'^T \text{diag}(D,n^{-1} D,\ldots,n^{-1} D)x' \right).
\end{aligned}
\]

\section*{B.3 Proof of Proposition 12}
\textbf{Proof.} Note that by reparameterization,
\[
\arg \min_{\mu} \mathbb{D}_{KL}(\mathcal{N}(\mu, \sigma^2) || F_n) = \arg \min_{\mu} \mathbb{E} \left[ -n^{-1} \log \pi_n(\mu + \sigma Z) \right],
\]
where $Z \sim \mathcal{N}(0, 1)$. Using a Taylor expansion,
\[
-\mathbb{E} \left[ \frac{d^2}{d\mu^2} \left( -n^{-1} \log \pi_n(\mu + \sigma Z) \right) \right]
= -\mathbb{E} \left[ -n^{-1} \log \pi_n^{(2)}(\mu) - n^{-1} \log \pi_n^{(3)}(\mu^*) \cdot \sigma Z \right],
\]
for some $\mu^*$ between $\mu$ and $\mu + \sigma Z$. By the uniform bound on the third derivative and local bound on the second derivative, for any $\mu \in U$,
\[
-\mathbb{E} \left[ -n^{-1} \log \pi_n^{(2)}(\mu) - n^{-1} \log \pi_n^{(3)}(\mu^*) \cdot \sigma Z \right] \leq -\epsilon + \eta \sigma \mathbb{E} |Z| \leq -\epsilon + \eta \sigma.
\]
The result follows for any $0 < \epsilon < \epsilon/\eta$.

\section*{B.4 Proof of Theorem 13}
\textbf{Proof.} Note that we can split $L$ into columns and express $LZ$ as
\[
LZ = \sum_{i=1}^d L_i Z_i,
\]
where $L_i \in \mathbb{R}^p$ is the $i\text{th}$ column of $L$, and $(Z_i)_{i \neq j} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$. Denoting $\nabla^2 f_n := \nabla^2 f_n(\mu + LZ)$ for brevity, the $2\text{nd}$ derivatives in both $\mu$ and $L$ are
\[
\begin{aligned}
\nabla^2_{\mu \mu} f_n &= \mathbb{E} \left[ \nabla^2 f_n \right] \\
\nabla^2_{\mu L} f_n &= -n^{-1} \mathbb{E} \left[ Z_i^2 \nabla^2 f_n \right] \\
\nabla^2_{L L} f_n &= n^{-1/2} \mathbb{E} \left[ Z_i^2 \nabla^2 f_n \right]
\end{aligned}
\]
where we can pass the gradient and Hessian through the expectation by dominated convergence because $Z$ has a normal distribution and $f_n$ has $\ell$-Lipschitz gradients. Stacking these together in block matrices yields the overall Hessian,
\[
A = \left[ I \ n^{-1/2} Z_i I \ldots n^{-1/2} Z_i I \right] \in \mathbb{R}^{d \times d(d+1)}
\]
\[
\nabla^2 f_n = \mathbb{E} \left[ A \nabla^2 f_n A \right] \in \mathbb{R}^{d(d+1) \times d(d+1)}.
\]
Since $f_n$ has $\ell$-Lipschitz gradients, for all $x \in \mathbb{R}^d$, $-\ell I \leq \nabla^2 f_n(x) \leq \ell I$. Applying the upper bound and evaluating the expectation yields the Hessian upper bound (and the same technique yields the corresponding lower bound):
\[
\nabla^2 f_n = \mathbb{E} \left[ A \nabla^2 f_n A \right] \leq \mathbb{E} \left[ A^2 \right] = \ell^2 D_n.
\]

To demonstrate local strong convexity, we split the expectation into two parts: one where $n^{-1/2} D Z - x \leq r^2$, and the complement. Define
\[
r^2(\mu, L) := n \left( \frac{\| \mu - x \|^2}{2 \| D Z \|^2} \right).
\]
Noting that when $\|Z\|^2 \leq r^2(\mu, L)$,
\[
\|\mu + \frac{1}{\sqrt{n}} L Z - x\|^2 \
\leq 2 \| \mu - x \|^2 + 2n^{-1} \| L Z \|^2 \
\leq 2 \| \mu - x \|^2 + 2n^{-1} \| D Z \|^2 \
\leq r^2.
\]

Then we may write
\[
\nabla^2 f_n = \mathbb{E} \left[ A^2 \right] = \mathbb{E} \left[ A \right] + \mathbb{E} \left[ \|Z\|^2 \right] > r^2(\mu, L) A^2\mathbb{A}^2
\]

Since $f_n$ has $\ell$-Lipschitz gradients and is locally $\epsilon$-strongly convex,
\[
\nabla^2 f_n \geq \epsilon \cdot \mathbb{E} \left[ \|Z\|^2 \right] > r^2(\mu, L) A^2\mathbb{A}^2
\]

Note that $A^2$ has entries $1$ and $n^{-1} D Z_i^2$ along the diagonal, as well as $n^{-1} Z_i Z_j$, $i \neq j$ and $n^{-1/2} Z_i$ on the off-diagonals. By symmetry, since $Z$ is an isotropic Gaussian, censoring by $1 \| Z \|^2 \leq \ldots \leq 1 \| Z \|^2 \ldots$ maintains that the off-diagonal expectations are 0. Therefore the quantity $\mathbb{E} \left[ 1 \| Z \|^2 \right] \| Z \|^2 \leq r^2(\mu, L) A^2\mathbb{A}^2$ is diagonal with coefficients $1 - \alpha_n(\mu, L)$ and $n^{-1} \tau_n(\mu, L)$, and $\mathbb{E} \left[ 1 \| Z \|^2 > r^2(\mu, L) A^2\mathbb{A}^2$ is diagonal with coefficients $\alpha_n(\mu, L)$ and $n^{-1} \tau_n(\mu, L)$ where
\[
\alpha_n(\mu, L) = \mathbb{P} \left[ \| Z \|^2 > r^2(\mu, L) \right]
\]
\[
\tau_n(\mu, L) = \mathbb{E} \left[ 1 \| Z \|^2 \leq r^2(\mu, L) \right]
\]

Note that $\|Z\|^2 \sim \chi^2_d$; so $\alpha_n(\mu, L) = 1 - \chi^2_d(r^2(\mu, L))$ and
\[
\tau_n(\mu, L) = \int_{r^2(\mu, L)}^\infty (1 - \xi)^{d/2-1} \xi^{d/2-1} d\xi
\]

Therefore,
\[
\nabla^2 f_n \geq \mathbb{E} \left[ \|Z\|^2 \right] \| Z \|^2 \leq r^2(\mu, L) A^2\mathbb{A}^2
\]

Since $f_n$ has $\ell$-Lipschitz gradients, for all $x \in \mathbb{R}^d$, $-\ell I \leq \nabla^2 f_n(x) \leq \ell I$. Applying the upper bound and evaluating the expectation yields the
B.5 Proof of Lemma 9

Proof. Given Assumption 1, we know $f_\theta$ is twice continuously differentiable. Thus, using the second order characterization of strong convexity, it is equivalent to show that for all $\epsilon > 0$ such that

$$\mathbb{P}(\forall \theta \in B_r(\theta_0), \quad \nabla^2 f_\theta(\theta) < \epsilon I) \to 1,$$

as $n \to \infty$. Note that by Weyl’s inequality

$$\nabla^2 f_\theta(\theta) - \nabla^2 f_\theta(\theta) + H_\theta \leq \lambda_{\min} (\nabla^2 f_\theta(\theta) - H_\theta) \mathbb{I} + \lambda_{\max}(H_\theta) \mathbb{I}. \quad (12)$$

Condition 4 of Assumption 1 guarantees that $H_\theta \succeq \epsilon \mathbb{I}$ and that there exists a $\kappa > 0$ such that $H_\theta$ is continuous in $B_r(\theta_0)$. Hence there exists $0 < \kappa' \leq \kappa$, such that $\forall \theta \in B_{\kappa'}(\theta_0), H_\theta \succeq \frac{\epsilon}{2} I$.

We then consider $\lambda_{\min}(\nabla^2 f_\theta(\theta) - H_\theta)$. We aim to find a $0 < r \leq \kappa'$ such that $\lambda_{\min}(\nabla^2 f_\theta(\theta) - H_\theta)$ is sufficiently small. Note that for any fixed $r > 0$,

$$\sup_{\theta \in B_r(\theta_0)} \left| \lambda_{\min}(\nabla^2 f_\theta(\theta) - H_\theta) \right| \leq \sup_{\theta \in B_r(\theta_0)} \left\| \nabla^2 f_\theta(\theta) - H_\theta \right\|_2,$$

and

$$= \sup_{\theta \in B_r(\theta_0)} \left\| \nabla^2 f_\theta(\theta) - \mathbb{E}_\theta \left[ -\nabla^2 \log p_\theta(X) \right] + \mathbb{E}_\theta \left[ -\nabla^2 \log p_\theta(X) \right] - H_\theta \right\|_2 \leq \sup_{\theta \in B_r(\theta_0)} \left\| \nabla^2 f_\theta(\theta) - \mathbb{E}_\theta \left[ -\nabla^2 \log p_\theta(X) \right] \right\|_2 + \sup_{\theta \in B_r(\theta_0)} \left\| \mathbb{E}_\theta \left[ -\nabla^2 \log p_\theta(X) \right] - H_\theta \right\|_2.$$

Now we split $f_\theta$ into prior and likelihood, yielding that

$$\sup_{\theta \in B_r(\theta_0)} \left\| \nabla^2 f_\theta(\theta) - H_\theta \right\|_2 \leq \sup_{\theta \in B_r(\theta_0)} \left\| \nabla^2 \log p_\theta(X) \right\|_2 \quad (13)$$

Given Condition 2 of Assumption 1, for all $\theta$, $\pi_\theta(\theta)$ is positive and $\nabla^2 \pi_\theta(\theta)$ is continuous; and further due to the compactness of $B_r(\theta_0)$, we have that

$$\forall r > 0, \quad \sup_{\theta \in B_r(\theta_0)} \left\| -n^{-1} \nabla^2 \log \pi_\theta(\theta) \right\|_2 \to 0, \quad \text{as } n \to \infty. \quad (14)$$

Then, it remains to bound the first term and the last term of Eq. (13). For the first term, we aim to use the uniform weak law of large numbers to show its convergence to 0. By Condition 5 of Assumption 1, there exists a $0 < r_1 \leq \kappa'$ and a measurable function $g$ such that for all $\theta \in B_r(\theta_0)$ and for all $x$,

$$\max_{i \in [d]} \left( \nabla^2 \log p_\theta(x) \right)_{i,i} < g(x), \quad \mathbb{E}_\theta[g(X)] < \infty.$$

Then, by the compactness of $B_r(\theta_0)$, we can apply the uniform weak law of large numbers (Jennrich, 1969, Theorem 2), yielding that for all $i, j \in [d],$

$$\sup_{\theta \in B_r(\theta_0)} \left\| -n^{-1} \sum_{i=1}^n \nabla^2 \log p_\theta(X)_{i,j} \right\|_{l_2} \to 0.$$

Since the entrywise convergence of matrices implies the convergence in spectral norm,

$$\sup_{\theta \in B_r(\theta_0)} \left\| -n^{-1} \sum_{i=1}^n \nabla^2 \log p_\theta(X) - \mathbb{E}_\theta \left[ -\nabla^2 \log p_\theta(X) \right] \right\|_2 \to 0. \quad (15)$$

For the last term of Eq. (13), by Condition 4 of Assumption 1,

$$\lim_{r \to 0} \sup_{\theta \in B_r(\theta_0)} \left\| \mathbb{E}_\theta \left[ -\nabla^2 \log p_\theta(X) \right] - \mathbb{H}_\theta \right\|_2 \to 0.$$

Then, we can combine Eqs. (14) to (16) and pick $r' = \min(r_1, r_2) \leq \kappa'$, yielding that

$$\mathbb{P} \left( \sup_{\theta \in B_{r'}(\theta_0)} \left\| \mathbb{E}_\theta \left[ -\nabla^2 f_\theta(\theta) \right] - H_\theta \right\|_2 \leq \frac{\epsilon}{4} \right) \to 1, \quad (17)$$

as $n \to \infty$. Then the local strong convexity is established. Note that we have already shown for all $\theta \in B_{r'}(\theta_0), H_\theta \succeq \frac{\epsilon}{2} I$. By Eqs. (17) and (12), we conclude that for all $\epsilon \leq \frac{\epsilon}{4},$

$$\lim_{n \to \infty} \mathbb{P}(\forall \theta \in B_{r'}(\theta_0), \quad \nabla^2 f_\theta(\theta) \succeq \epsilon I) = 1.$$

The smoothness argument follows from the same strategy. Weyl’s inequality implies that

$$\nabla^2 f_\theta(\theta) - \nabla^2 f_\theta(\theta) + H_\theta \leq \lambda_{\max}(\nabla^2 f_\theta(\theta) - H_\theta) \mathbb{I} + \lambda_{\max}(H_\theta) \mathbb{I}.$$

By repeating the proof for local smoothness, we obtain that there exists a sufficiently small $0 < r''$, such that $\forall \theta > 0,

$$\sup_{\theta \in B_{r''}(\theta_0)} \left\| \mathbb{E}_\theta \left[ -\nabla^2 \log \pi_\theta(\theta) \right] \right\|_2 \to 0.$$

Therefore, there exists a $\epsilon > 0$ such that

$$\lim_{n \to \infty} \mathbb{P}(\forall \theta \in B_{r''}(\theta_0), \quad \nabla^2 f_\theta(\theta) \succeq \epsilon I) = 1.$$

Then the proof is complete by defining $r' := \min\{r', r''\}$. \qed

B.6 Proof of Corollary 14

Proof. We begin by verifying the conditions of Theorem 13 for $f_\theta$. By Assumption 1 we know that $f_\theta$ is twice differentiable. We also know that by Lemma 9, under Assumptions 1 and 2, there exist $\ell', r', \epsilon > 0$ such that

$$\mathbb{P} \left( \sup_{\theta} \left\| -n^{-1} \nabla^2 \log \pi_\theta(\theta) \right\|_2 > \ell' \right) \to 0$$

and

$$\mathbb{P} \left( \inf_{\theta \in [0,1]} \lambda_{\min} \left( -n^{-1} \nabla^2 \log \pi_\theta(\theta) \right) < \epsilon \right) \to 0.$$

By Theorem 6 we know that $\mu_{\theta_0}^{r_0} \to \theta_0$, so there exists an $r' > r > 0$ such that

$$\mathbb{P} \left( \inf_{\theta \in [0,1]} \lambda_{\min} \left( -n^{-1} \nabla^2 \log \pi_\theta(\theta) \right) < \epsilon \right) \to 0.$$
Therefore by Theorem 13, the probability that
\[ \forall \mu, \ell, \quad -tD_n \leq n^{-1} \sqrt{2\pi} E[\log \hat{p}_n(\mu + 1/\sqrt{7nLZ})] \leq tD_n, \quad (18) \]
and
\[ n^{-1} \sqrt{2\pi} E[\log \hat{p}_n(\mu + n^{-1/2}LZ)] \geq D_n(\varepsilon - \tau_n(\mu, L) \cdot (e + \ell)), \quad (19) \]
hold converges to 1 as \( n \to \infty \), where \( D_n \) and \( \tau_n(\mu, L) \) are as defined in Eq. (10) and \( x = \mu^* \). Note that the gradient and Hessian in the above expression are taken with respect to a vector in \( \mathbb{R}^{d(d+1)} \) that stacks \( \mu \) and each column of \( L \) into a single vector.

Then for all \( (\mu, L) \in \mathcal{B}_{r,n} \), we have
\[ \left\| \mu - \mu^*_n \right\|^2 \leq r^2/4. \]
\[ \left\| L - L^*_n \right\|^2 \leq 4\left\| L - L^*_n \right\|^2 \implies \left\| L \right\| \leq 2\left\| L - L^*_n \right\| + \left\| L^*_n \right\|. \]
yielding
\[ \frac{r^2 - 2\left\| \mu - \mu^*_n \right\|^2}{n^{-1}\left\| L^*_n \right\|^2} \geq \frac{2n^2}{(2\left\| L - L^*_n \right\| + \left\| L^*_n \right\|)^2}. \]
Hence for all \( (\mu, L) \in \mathcal{B}_{r,n} \), \( \tau_n(\mu, L) \to 0 \) as \( n \to \infty \), yielding that under sufficiently large \( n \),
\[ e - \tau_n(\mu, L) \cdot (e + \ell) > e/2. \]
Therefore, the probability that for all \( (\mu, L) \in \mathcal{B}_{r,n} \),
\[ \frac{1}{n} \sqrt{2\pi} E[\log \hat{p}_n(\mu + 1/\sqrt{7nLZ})] \geq \frac{e}{2} D_n \quad (20) \]
converges in \( P_{\hat{p}_n} \) to 1 as \( n \to \infty \).

Combining Eqs. (18) to (20), the proof is completed.\hfill\Box

### B.7 Proof of Theorem 7

#### B.7.1 Gradient and Hessian derivation

The gradient for smoothed posterior is as follows,
\[ \nabla \log \hat{p}_n(\theta) = \nabla \log \left\{ \mathbb{E} \left[ \exp \left( -\frac{1}{2\alpha_0} \|\theta - W\|^2 \right) \right] \right\} = \mathbb{E} \left[ \exp \left( -\frac{1}{2\alpha_0} \|\theta - W\|^2 \right) \left( -\frac{1}{\alpha_0} \right) (\theta - W) \right], \]
and the Hessian matrix is given by
\[ \nabla^2 \log \hat{p}_n(\theta) = \frac{1}{\alpha_0^2} \mathbb{E} \left[ \exp \left( -\frac{1}{2\alpha_0} \|\theta - W\|^2 \right) W(W-W)^T \right] - \frac{1}{\alpha_0} I, \quad (21) \]
where \( W, W^T \equiv I_{d^2} \).

#### B.7.2 Proof of 1st statement of Theorem 7

Proof of 1st statement of Theorem 7. To show the MAP estimation for smoothed posterior is asymptotically strictly convex, we will show that
\[ \lim_{n \to \infty} P \left( \sup_{\|\theta - \theta_0\| \leq M} \lambda_{\max} \left( \nabla^2 \log \hat{p}_n(\theta) \right) < 0 \right) = 1. \]

We focus on the first term of Eq. (21), and show that asymptotically it is uniformly smaller than \( c_0^{-1} \) so that the overall Hessian is negative definite. For the denominator of Eq. (21), define \( B_n := \{ W, W^T : \max(\|W' - \theta_0\|, \|W - \theta_0\|) \leq \delta_n \} \) for any sequence \( \delta_n = o(c_0) \).

Then we have
\[ \mathbb{E} \left[ \exp \left( -\frac{1}{2\alpha_0} \|\theta - W\|^2 \right) W(W-W)^T \right] \]
\[ = \frac{1}{2} \mathbb{E} \left[ \exp \left( -\frac{1}{2\alpha_0} \|\theta - W\|^2 \right) W(W-W)^T \right]. \]
and since \( \lambda_{\max} \left( (W-W')(W-W)^T \right) = \|W-W'\|^2 \),
\[ \lambda_{\max} \left( e^{-\frac{1}{2\alpha_0} \|\theta - W\|^2} W(W-W)^T \right) \]
\[ \leq \frac{1}{2} \mathbb{E} \left[ \exp \left( -\frac{1}{2\alpha_0} \|\theta - W\|^2 \right) \|W-W'\|^2 \right]. \]

With Eqs. (22) and (23), we can therefore bound the maximal eigenvalue of the Hessian matrix,
\[ \lambda_{\max} \left( \nabla^2 \log \hat{p}_n(\theta) \right) \]
\[ \leq \frac{1}{2\alpha_0^2} P(B_n) \mathbb{E} \left[ \exp \left( -\frac{1}{2\alpha_0} \|\theta - W\|^2 \right) \|W-W'\|^2 \right] - \frac{1}{\alpha_0^2}. \]

We now bound the supremum of this expression over \( \{\theta \in \mathbb{R}^d : \|\theta - \theta_0\| \leq M\} \). Focusing on the exponent within the expectation,
\[ \frac{1}{\alpha_0^2} \left( 2\|\theta - \theta_0\| + 2\alpha_0^2 \right) \mathbb{E} \left[ \frac{\|\theta - W\|^2}{2\alpha_0} \right] \]
\[ = \frac{1}{\alpha_0^2} \left( 2\|\theta - \theta_0\| + 2\alpha_0^2 \right) \mathbb{E} \left[ \frac{\|\theta - \theta_0 + \theta_0 - W\|^2}{2\alpha_0} \right] \]
\[ = \frac{1}{\alpha_0^2} \left( 2\beta_n^2 + 4M\beta_n \right) \left( \|\theta_0 - W\|^2 + 2\|\theta_0 - W\|^2 \right) \]
\[ + 2M \left( \|\theta_0 - W\| + \|\theta_0 - W\| \right), \]

where \( \beta_n \) is the support of the distribution. Combining these results, we have

\[ \frac{1}{\alpha_0^2} \left( 2\|\theta - \theta_0\| + 2\alpha_0^2 \right) \mathbb{E} \left[ \frac{\|\theta - W\|^2}{2\alpha_0} \right] \]
\[ \leq \frac{1}{\alpha_0^2} \left( 2\beta_n^2 + 4M\beta_n \right) \left( \|\theta_0 - W\|^2 + 2\|\theta_0 - W\|^2 \right) \]
\[ + 2M \left( \|\theta_0 - W\| + \|\theta_0 - W\| \right). \]

This completes the proof of the 1st statement of Theorem 7.
where the inequality is obtained by expanding the quadratic terms and bounding \(\|\theta - \theta_0\|\) with \(M\). We combine the above bound with Eq. (24) to show that \(\alpha_n^2 \Delta_{\text{high}}(V^2 \log \beta_n(\theta)) + \alpha_n\) is bounded above by

\[
\frac{\beta_n}{2\pi F(B_n)} e^{\frac{2 \pi^2 + \alpha_n}{m} \mathbb{E}} \left[ e^{\frac{2 \pi^2}{m} (W_\| - \|W\|)} \right].
\]

By multiplying and dividing by \(\beta_n\), one notices that

\[
\frac{\|W - W\|}{\beta_n} = \exp \left( \frac{\|W - W\|}{\beta_n} \right) \left( \left\| W - W \right\| \right) \left( \left\| W - W \right\| \right) \leq 4e^{-2} \exp \left( \frac{\|W - \theta_0\| + \|W - \theta_0\|}{\beta_n} \right),
\]

where the inequality is by the fact that \(x^2e^{-x}\) maximized at \(x = 2\) with value \(4e^{-2}\) and \(\|W - W\| \leq \|W\| + \|W\|\). If we combine this bound with Eq. (25) and note that \(W, W\) are iid, Eq. (25) is bounded above by

\[
\frac{2e^{-2} \beta_n}{2\pi F(B_n)} e^{\frac{2 \pi^2 + \alpha_n}{m} \mathbb{E}} \left[ e^{\frac{4 \pi^2 (M + \beta_n^{-1/2})}{m} \left( \frac{1}{\beta_n} \right) \left( \left\| W - \theta_0 \right\| \right) \right].
\]

To show that the Hessian is asymptotically negative definite, it suffices to show that Eq. (26) is \(o_p(\alpha_n)\). For the terms inside the expectation, \(\beta_n = o(\alpha_n)\) implies that \(2e^{-2} \beta_n e^{\frac{2 \pi^2 + \alpha_n}{m} \mathbb{E}} = o(\alpha_n)\), and Assumption 1 and Lemma 15 together imply that

\[
P(B_n) = \Pi_n \left( \left\{ W : \|W - \theta_0\| \leq \beta_n \right\} \right) \to 1,
\]

so

\[
2e^{-2} \beta_n \frac{2 \pi^2 + \alpha_n}{m} \mathbb{E} = o_p(\alpha_n).
\]

Therefore, in order to show Eq. (26) is \(o_p(\alpha_n)\), it is sufficient to show that

\[
\mathbb{E} \left[ e^{\left( \frac{4 \pi^2 (M + \beta_n^{-1/2})}{m} \left( \frac{1}{\beta_n} \right) \left( \left\| W - \theta_0 \right\| \right) \right)} \right] = O_p(1).
\]

The next step is to split the expectation into two regions---\(\|W - \theta_0\| \leq \beta_n\) and \(\|W - \theta_0\| > \beta_n\)—and bound its value within them separately.

1. When \(\|W - \theta_0\| \leq \beta_n\), the exponent inside the expectation is shrinking uniformly since \(\beta_n = o(\alpha_n)\):

\[
\mathbb{E} \left[ \left( \frac{4 \pi^2 (M + \beta_n^{-1/2})}{m} \left( \frac{1}{\beta_n} \right) \left( \left\| W - \theta_0 \right\| \right) \right) \right] = O_p(1).
\]

2. When \(\|W - \theta_0\| > \beta_n\), we take the supremum over the exponent (a quadratic function), yielding \(\|W - \theta_0\| = M + \alpha_n \beta_n^{-1/2}\) and the following bound,

\[
\left( \frac{1}{\alpha_n} M + \beta_n^{-1/2} \right) \leq \sup_{\|v - \theta_0\| \leq \beta_n} \left( \left( \frac{1}{\alpha_n} M + \beta_n^{-1/2} \right) \left( \left| v - \theta_0 \right| \right) \right) \leq \frac{1}{2\alpha_n} \left( \left| v - \theta_0 \right| \right)^2
\]

\[
= \frac{1}{\alpha_n} \left( M + \beta_n^{-1/2} \right) \left( M + \alpha_n \beta_n^{-1/2} \right) - \frac{1}{2\alpha_n} \left( M + \alpha_n \beta_n^{-1/2} \right)^2
\]

\[
= \frac{M^2}{2\alpha_n} + \frac{M}{\beta_n} + \frac{\alpha_n}{2\beta_n}.
\]

This yields

\[
\mathbb{E} \left[ I_{\left\{ \|W - \theta_0\| > \beta_n \right\}} e^{\left( \frac{4 \pi^2 (M + \beta_n^{-1/2})}{m} \left( \frac{1}{\beta_n} \right) \left( \left\| W - \theta_0 \right\| \right) \right)} \right] \leq \Pi_n \left( \left\{ W : \|W - \theta_0\| > \beta_n \right\} \right) \exp \left( \frac{M^2}{2\alpha_n} + \frac{M}{\beta_n} + \frac{\alpha_n}{2\beta_n} \right).
\]

Note that it is always possible to choose \(\beta_n = o(\alpha_n)\) with \(\beta_n = \omega(\alpha_n^2)\). With this choice of \(\beta_n\), the dominating term among the three of Eq. (27) is \(\frac{M^2}{2\alpha_n}\).

Then by Lemma 15, there exists a sequence \(\beta_n = o(\alpha_n)\) with \(\beta_n = \omega(\alpha_n^2)\) such that the following holds,

\[
\Pi_n \left( \left\{ W : \|W - \theta_0\| > \beta_n \right\} \right) = O_p(\alpha_n \left( \frac{M^2}{2\alpha_n} \right)),
\]

which implies

\[
\mathbb{E} \left[ I_{\left\{ \|W - \theta_0\| > \beta_n \right\}} e^{\left( \frac{4 \pi^2 (M + \beta_n^{-1/2})}{m} \left( \frac{1}{\beta_n} \right) \left( \left\| W - \theta_0 \right\| \right) \right)} \right] = O_p(1).
\]

This finishes the proof. □

In the last step of the above proof, we require an exponential tail bound for the posterior \(\Pi_n\). We provide this in the following lemma, following the general proof strategy of van der Vaart (2000, Thm 10.3). The proof of Lemma 15 involves many probability distributions; thus, for mathematical convenience and explicitness, in the proof of Lemma 15 we use square bracket—\(\mathbb{P} \left[ x \right]\)—to denote the expectation of random variable \(X\) with respect to a probability distribution \(P\). When taking expectation to a function of a data point \(f(X_1, \ldots, X_n)\), where \(X_i\) are iid \(\mathcal{P}\), we write \(\mathbb{P} \left[ f \right]\); and \(\mathbb{P}\) here represents the product measure.

**Lemma 15.** Under Assumption 1, \(\alpha_n^n \to \infty\), there exists a sequence \(\beta_n\) satisfying \(\beta_n = o(\alpha_n), \beta_n = \omega(\alpha_n^2)\) and \(\beta_n = \omega(n^{-1/2})\) such that for any fixed constant \(M\),

\[
\Pi_n \left( \left\{ W : \|W - \theta_0\| > \beta_n \right\} \right) = o_p \left( \exp \left( \frac{M^2}{2\alpha_n} \right) \right).
\]

**Proof of Lemma 15.** In order to show that \(\beta_n\) satisfies the tail probability bound, it suffices to prove that

\[
\mathbb{E} \left[ I_{\left\{ \|W - \theta_0\| > \beta_n \right\}} \right] \to 0
\]

due to Markov’s inequality (we absorb the \(M^2/2\) constant into \(\alpha_n\) because it does not affect the proof). To achieve this, we take advantage of the existence of a test sequence applied from Assumption 1. By van der Vaart (2000, Lemma 10.6), given the 1st and the 2nd conditions of Assumption 1 and the fact that the parameter space \(\mathcal{X}\) is \(\sigma\)-compact, there exists a sequence of tests \(\Phi_n : \mathcal{X}^n \to [0, 1]\), where \(\mathcal{X}^n\) is the space of \((X_1, \ldots, X_n)\), such that as \(n \to \infty,

\[
\mathbb{P} \left[ \Phi_n(\theta) \right] \to 0, \quad \sup_{\|\theta - \theta_0\| \leq \epsilon} \mathbb{P} \left[ \|W - \theta_0\| > \beta_n \right] \to 0
\]

Further, by Kleijn (2004, Lemma 1.2) and van der Vaart (2000, Lemma 10.3), under Assumption 1 and the existence of the above test sequence \(\Phi_n\), for every \(M_n \to \infty\), there exists a constant \(C > 0\) and another sequence of tests \(\psi_n : \mathcal{X}^n \to [0, 1]\) such that for all \(\|\theta - \theta_0\| > M_n/\sqrt{n}\) and sufficiently large \(n\),

\[
\mathbb{P} \left[ \psi_n(\theta) \right] \leq \exp \left( -Cn \right), \quad \mathbb{P} \left[ \psi_n \right] \leq \exp \left( -Cn(\|\theta - \theta_0\|^2 \wedge 1) \right).
\]
Using $\psi_n$, we split the expectation as following,

$$e^{\frac{1}{n} \cdot P_0[\mathcal{I}_n(W : ||W - \theta_0|| > \beta_n)]} \leq e^{\frac{1}{n} \cdot P_0[\mathcal{I}_n(W : ||W - \theta_0|| > \beta_n)]} \cdot \left(1 - \psi_n \right)$$

and we aim to show both parts converging to 0.

For term (i), the first statement of Eq. (28) implies that $\exists C > 0$ such that

$$e^{\frac{1}{n} \cdot P_0[\mathcal{I}_n(W : ||W - \theta_0|| > \beta_n)]} \cdot \left(1 - \psi_n \right) \leq e^{\frac{1}{n} \cdot P_0[\psi_n]} \leq e^{\frac{1}{n} \cdot C} e^{-nC},$$

where the first inequality follows by $\mathcal{I}_n(W : ||W - \theta_0|| > \beta_n) \leq 1$. Since $n\alpha_n \to \infty$, the last bound in the above expression converges to 0.

For term (ii), we work with the shifted and scaled posterior distribution. Define $Z_n = \sqrt{n} (W - \theta_0)$ and $B_n = \{Z_n : ||Z_n|| > \sqrt{\beta_n} \}$, and let $\pi_n$ be the corresponding prior distribution on $Z_n$ and $\pi_n$ be the shifted and scaled posterior distribution, which yields

$$e^{\frac{1}{n} \cdot P_0[\mathcal{I}_n(W : ||W - \theta_0|| > \beta_n)]} \cdot \left(1 - \psi_n \right) = e^{\frac{1}{n} \cdot P_0[\mathcal{I}_n(B_n) \cdot \left(1 - \psi_n \right)]}.$$

Let $U$ be a closed ball around 0 with a fixed radius $r$, then restricting $\pi_n$ to $U$ defines a probability measure $\pi_n(U)$, i.e., for all measurable set $B$, $\pi_n(U) = \pi_n(B \cap U) / \pi_n(U)$. Write $P_n$ for the joint distribution of $n$ data points $(x_1, \ldots, x_n)$ parameterized under $\theta_0 + z_0 \sqrt{\beta_n}$ and hence write the marginal distribution of $(X_1, \ldots, X_n)$ under $\pi_n(U)$ for $P_n(U) = \int P_n(x) d\pi_n(U)$. The densities of these distributions will be represented using lower case, e.g., $p_{n,U}(x) = \int p_{n,U}(x) d\pi_n(U) dz$ is the PDF of $P_n(U)$. Here we abuse the notation that $x$ represents $(x_1, \ldots, x_n)$.

We replace $P_0$ in Eq. (29) with $P_{n,U}$. Under Assumption 1, by van der Vaart (2000, P. 141), $P_{n,U}$ is mutually contiguous to $P_0$ (LeCam, 1960), that is, for any statistics $T_n$ (a Borel function of $X^n$), $T_n \to 0$ iff $T_n \to 0$. Thus, considering $\pi_n(B_n) \cdot \left(1 - \psi_n \right)$ as the statistics $T_n$, the convergence to 0 of the expression in Eq. (29) is equivalent to

$$e^{\frac{1}{n} \cdot P_{n,U}[\mathcal{I}_n(B_n) \cdot \left(1 - \psi_n \right)]} \to 0.$$

Manipulating the expression of $P_{n,U}$ and $\pi_n(B_n)$ (we write $\pi_n(B_n,x)$ in the integral and write $\pi_n(B_n,x_{-i})$ in the expectation to make the dependence of posterior on the data explicit),

$$P_{n,U}[\pi_n(B_n(x_{-i}),x_{-i})] = \int \pi_n(B_n(x_{-i}),x_{-i}) dP_{n,U}(x) = \pi_n(B_n(x_{-i}),x_{-i}) d\pi_n(x_{-i}) dx.$$

Note that $p_{n,U}(x) = \int p_{n,U}(x) d\pi_n(U) dz$.

Recall that for all measurable set $B$, $\pi_n(U) = \pi_n(B \cap U) / \pi_n(U)$, thus

$$\frac{1}{\pi_n(U)} \int \pi_n(B_n(x_{-i}),x_{-i}) d\pi_n(x_{-i}) dx.$$

By using Bayes rule, we expand $\pi_n(B_n(x_{-i}),x_{-i}) = \frac{1}{\pi_n(U)} \int p_{n,U}(x_{-i}) d\pi_n(U)$.

Note that $\pi_n(U,x) = \frac{k}{\pi_n(U)} \int p_{n,U}(x_{-i}) d\pi_n(U)$.

By Fubini theorem and $\pi_n(U,x) \leq 1$,

$$\int \mathcal{I}_n(B_n(x_{-i}),x_{-i}) d\pi_n(U) \leq \int \mathcal{I}_n(B_n(x_{-i}),x_{-i}) d\pi_n(U) \cdot \left(1 - \psi_n \right) \pi_n(U,x) dx.$$

Note that $\pi_n[1 - \psi_n] = \pi_n[1 - \psi_n]$ for $\theta = \theta_0 + z_0 \sqrt{\beta_n}$ and that $\psi_n(\theta) \to \infty$ due to $\beta_n = O(n^{-1/2})$. Thus, we can use the second statement of Eq. (28) to bound $P_n[1 - \psi_n]$, yielding

$$\int \mathcal{I}_n(B_n(x_{-i}),x_{-i}) d\pi_n(U,x) \leq \int \mathcal{I}_n(B_n(x_{-i}),x_{-i}) d\pi_n(U,x) \cdot \left(1 - \psi_n \right) \pi_n(U,x) dx.$$
where the equality is by change of variable back to \( z = \sqrt{n}(w - \theta_0) \). Then, consider the integral on RHS. Using spherical coordinates, there exists a fixed constant \( D > 0 \) such that
\[
\int_{\|z\| > \sqrt{n}z_0} \exp\left\{-C\|z\|^2 \right\} \, dz = D \int_{\|r\| > \sqrt{n}s} e^{-Cr^2}r^{-1} \, dr = DC^{-d/2} \int_{\|z\| > Cn^{1/2}} e^{-s^2z_0^{-1}} \, ds,
\]
where the second equality is by setting \( s = Cr^2 \). Note that the integrand of RHS is proportional to the PDF of \( \Gamma_2^2(1,1) \). Using the tail properties of the Gamma random variable (Boucheron et al., 2013, p. 28), we have that for some generic constant \( D > 0 \),
\[
\int_{\|z\| > \sqrt{n}z_0} \exp\left\{-C\|z\|^2 \right\} \, dz \leq De^{-Cn^{1/2}}.
\]
Therefore, for some generic constants \( C, D > 0 \),
\[
\int_{\|z\| > \sqrt{n}z_0} \exp\left\{-C\|z\|^2 \wedge n \right\} \, dz \leq Dn^{-d/2}e^{-Cn^{1/2}} + e^{-Cn^{1/2}}.
\]
Then we combine Eqs. (30) and (31), yielding that for some constants \( C, D > 0 \) independent to \( n \),
\[
e^{\frac{1}{\sigma^2}} \frac{1}{P_{U}(f)} \left[ P_{U}(f)(B_0)(1 - \gamma_n) \right] \leq \frac{e^{\frac{1}{\sigma^2}}}{P_{U}(\theta)} \int_{\|z\| > \sqrt{n}z_0} \exp\left\{-C\|z\|^2 \wedge n \right\} \, dz \leq \frac{e^{\frac{1}{\sigma^2}}Cn}{\sqrt{\pi}}e^{-Cn^{1/2}} + e^{\frac{1}{\sigma^2}}De^{-Cn^{1/2}}.
\]
Lastly, it remains to show that there exists a positive sequence \( \beta_n \) satisfying both \( \beta_n = o(\alpha_n) \) and \( \beta_n = o(\alpha_n^2) \) such that the RHS converges to 0. The first term always converges to 0 due to \( \alpha_n^2 \rightarrow \infty \). For the second term, we consider two different cases. If \( \alpha_n = o(n^{-1/6}) \), we pick \( \beta_n = n^{-1/3} \), which is both \( o(\alpha_n) \) and \( o(\alpha_n^2) \). Then
\[
e^{\frac{1}{\sigma^2}}De^{-Cn^{1/2}} = D\exp\left\{\alpha_n^1 - Cn^{1/3}\right\} \rightarrow 0,
\]
where the convergence in the last line is by \( n\theta_0^2 \rightarrow \infty \Rightarrow \frac{1}{\theta_0^2} = o(n^{1/3}) \). If \( \alpha_n = o(n^{-1/6}) \), we pick \( \beta_n = \alpha_n^2 \). Then
\[
e^{\frac{1}{\sigma^2}}De^{-Cn^{1/2}} = D\exp\left\{\alpha_n^1 - Cn\alpha_n^2\right\}.
\]
Since \( \alpha_n = o(n^{-1/6}) \), \( \frac{1}{\theta_0^2} = o(n^{1/6}) \) and \( n\theta_0^2 = o(n^{1/3}) \), yielding that the above converges to 0 as \( n \rightarrow \infty \).

This completes the proof.

**B.7.3 Proof of 2nd statement of Theorem 7**

In this section, we show that the smoothed MAP estimator (\( \hat{\theta}_n^\star \)) is also a consistent estimator of \( \theta_0 \), but with a convergence rate that is slower than the traditional \( 1/\sqrt{n} \). This is the case because the variance of the smoothing kernel satisfies \( \alpha_n = o(n^{-1/6}) \), and the convergence rate of \( \hat{\theta}_n^\star \) is determined by \( \alpha_n \) via
\[
\|\hat{\theta}_n^\star - \theta_0\| = O_P(\sqrt{\alpha_n}).
\]
Recall that \( \theta_{MLE,n} \) is a \( \sqrt{n} \)-consistent estimate of \( \theta_0 \). Thus, it is sufficient to show that
\[
\|\hat{\theta}_n^\star - \theta_{MLE,n}\| = O_P(\sqrt{\alpha_n}).
\]
Note that \( \hat{\theta}_n^\star \) and \( \theta_{MLE,n} \) are maximal of stochastic process \( \pi_n(\theta) \) and \( \pi_n(\theta) \) respectively, which can be studied in the framework of M-estimator (van der Vaart, 2000; van der Vaart and Wellner, 2013). A useful tool in establishing the asymptotics of M-estimators is the Argmax Continuous Mapping theorem (van der Vaart and Wellner, 2013, Lemma 3.2.1), which is introduced as follows.

**Lemma 16 (Argmax Continuous Mapping (van der Vaart and Wellner, 2013)).** Let \( \{f_n(\theta)\} \) and \( f(\theta) \) be stochastic processes indexed by \( \theta \), where \( \theta \in \Theta \). Let \( \hat{\theta} \) be a random element such that almost surely, for every open sets \( G \) containing \( \theta \),
\[
f(\hat{\theta}) > sup_{\theta \in G} f(\theta),
\]
and \( \hat{\theta_n} \) be a random sequence such that almost surely
\[
f_n(\hat{\theta}_n) = sup_{\theta \in \Theta} f_n(\theta).
\]
If \( sup_{\theta \in \Theta} |f_n(\theta) - f(\theta)| = o_P(1) \) as \( n \rightarrow \infty \), then
\[
\hat{\theta}_n \overset{d}{\rightarrow} \hat{\theta}.
\]

The proof strategy of the 2nd statement of Theorem 7 is to apply Lemma 16 in a setting where \( f_n \) is \( \pi_n(\theta) \) and \( f \) is a Gaussian density. Using the Bernstein-von Mises Theorem 4, we show that \( \pi_n(\theta) \) converges uniformly to this Gaussian density, which implies that the MAP of \( \pi_n(\theta) \) converges in distribution to the MAP of this Gaussian distribution by the Argmax Continuous Mapping theorem. The detailed proof is as follows.

**Proof of 2nd statement of Theorem 7.** Note that
\[
\|\hat{\theta}_n^\star - \theta_0\| \leq \|\hat{\theta}_n^\star - \theta_{MLE,n}\| + \|\theta_{MLE,n} - \theta_0\|.
\]
By Theorem 4, we have \( \|\theta_{MLE,n} - \theta_0\| = O_P(1/\sqrt{n}) \). And in addition, given that \( \sqrt{\alpha_n} = o(1/\sqrt{n}) \), in order to get Eq. (32), it suffices to show
\[
\|\hat{\theta}_n - \theta_{MLE,n}\| = O_P(\sqrt{\alpha_n}).
\]
Thus, in this proof, we aim to show \( \|\hat{\theta}_n^\star - \theta_{MLE,n}\| = O_P(\sqrt{\alpha_n}) \) and it is sufficient to prove
\[
\frac{1}{\sqrt{\alpha_n}} (\hat{\theta}_n - \theta_{MLE,n}) = o_P(0).
\]
Let \( \xi = \frac{1}{\sqrt{\alpha_n}} (\theta - \theta_{MLE,n}) \), \( \xi_n^\star = \frac{1}{\sqrt{\alpha_n}} (\hat{\theta}_n - \theta_{MLE,n}) \) and \( t = \frac{1}{\sqrt{\alpha_n}} (w - \theta_{MLE,n}) \).

By expressing \( \pi_n(\theta) \), which is defined in Eq. (6),
\[
\xi_n^\star = \arg\max_{\xi} (\sqrt{\alpha_n} \xi + \theta_{MLE,n})
\]
\[
= \arg\max_{\xi} \int \pi_n(\sqrt{\alpha_n} \xi + \theta_{MLE,n}) \exp \left( -\frac{1}{2\alpha_n} \|\sqrt{\alpha_n} \xi + \theta_{MLE,n}\|^2 \right) \, dr
\]
\[
= \arg\max_{\xi} \int \alpha_n^{d/2} \pi_n(\sqrt{\alpha_n} \xi + \theta_{MLE,n}) \exp \left( -\frac{1}{2} \|\xi - t\|^2 \right) \, dr.
\]
Define
\[
f_n(\xi) = \int \alpha_n^{d/2} \pi_n(\sqrt{\alpha_n} \xi + \theta_{MLE,n}) \exp \left( -\frac{1}{2} \|\xi - t\|^2 \right) \, dr,
\]
\[
g_n(\xi) = \int \phi(t, 0, \frac{1}{\alpha_n}H_{\theta_0}^{-1}) \exp \left( -\frac{1}{2} \|\xi - t\|^2 \right) \, dr,
\]
\[
f(\xi) = (2\pi)^{d/2} \phi(\xi; 0, I),
\]
where \( \phi(\cdot; \mu, \Sigma) \) denotes the PDF of \( \mathcal{N}(\mu, \Sigma) \).

By adding and subtracting \( f(\xi) \),
\[
\xi_n^\star = \arg\max_{\xi} f_n(\xi)
\]
\[
= \arg\max_{\xi} \left\{ f_n(\xi) - f(\xi) + f(\xi) \right\}.
\]
We then apply Lemma 16 to show $\zeta^* \overset{d}{\to} \arg\max f(\xi)$. We start by verifying a condition of the argmax continuous mapping theorem that
\begin{equation}
\lim_{n \to \infty} \sup_{\xi} |f_n(\xi) - f(\xi)| = 0.
\end{equation}
By triangle inequality, for all $n$,
\begin{equation}
\sup_{\xi} |f_n(\xi) - f(\xi)| \leq \sup_{\xi} |f_n(\xi) - g_n(\xi)| + \sup_{\xi} |g_n(\xi) - f(\xi)|.
\end{equation}
Later we show both two terms on the RHS converging to 0.

For the first term, note that $\pi_n/\pi(\xi_0) + \theta_{\text{MLE},n}$ is the probability density function of $\pi_{\text{MLE},n}$, which is the posterior distribution parameterized on $\theta$. Thus, for all $n$,
\begin{equation}
\sup_{\xi} |f_n(\xi) - g_n(\xi)| = \sup_{\xi} \left\{ \int \pi_n(0, 1, n \theta_0) \cdot f(\xi) \right\} \leq D_{\text{TV}} \left( \pi_n, \pi(0, 1, n \theta_0) \right),
\end{equation}
where the inequality is by sup$_{\xi}$ $\exp(-\frac{1}{2} ||\xi - ||^2)$ $\leq 1$. Under Assumption 1, the posterior distribution admits Bernstein-von Mises theorem (Theorem 4) that
\begin{equation}
D_{\text{TV}} \left( \pi_n, \pi(0, 1, n \theta_0) \right) \to o(n^{-1}(1)).
\end{equation}
With the invariance of total variation under reparametrization, we have
\begin{equation}
D_{\text{TV}} \left( \pi_n, \pi(\theta_{\text{MLE},n}) \right) \to o(n^{-1}(1)).
\end{equation}
This shows the uniform convergence from $f_n(\xi)$ to $g_n(\xi)$.

For the second term in Eq. (34), Note that we can evaluate $g_n(\xi)$ since it is a convolution of two Gaussian PDFs, that is $g_n(\xi) = (2\pi)^{d/2} \phi(\xi, 0, 1, n \theta_0)^{-1/2} \cdot f(\xi)$. Comparing this to $f(\xi) = (2\pi)^{d/2} \phi(\xi, 0, 1, I)$, one notices that $1/n \theta_0 H_0^{-1} + I \to I$ due to $a_2n \to \infty$. And further for Gaussian distributions, the convergence of parameters implies the uniform convergence of PDFs, yielding that
\begin{equation}
\lim_{n \to \infty} \sup_{\xi} |g_n(\xi) - f(\xi)| = 0.
\end{equation}
Thus, we have Eq. (34) converging to 0 as $n \to \infty$.

Now we look at $f(\xi)$ with the goal to apply Lemma 16 and to obtain $\zeta^* \overset{d}{\to} \arg\max f(\xi)$, Note that
\begin{equation}
\arg\max f(\xi) = 0 \quad \text{and} \quad \sup_{\xi} f(\xi) = \det(I)^{-1/2} = 1.
\end{equation}
To apply Lemma 16, we need to ensure that for any open set $G$ that contains 0,
\begin{equation}
0 < \sup_{\xi \in G} f(\xi).
\end{equation}
This holds by the unimodality of standard Gaussian distribution.

Therefore, with both conditions Eq. (33) and Eq. (36), we can apply Lemma 16 to conclude that
\begin{equation}
\frac{1}{\sqrt{n}} (\theta^*_n - \theta_{\text{MLE},n}) \to 0.
\end{equation}
This completes the proof.

B.8 Proof of Theorem 2
Proof. Lemma 9 implies the existence of a locally asymptotically convex and smooth region $B^*_n(\theta_0)$; and Theorems 5 and 7 show the consistency of $\theta^*_n$ and $\theta^*$ to $\theta_0$ in data-asymptotics. Let $\theta_{\text{init}}$ be the initial value of Algorithm 2 satisfying $\|\theta_{\text{init}} - \theta^*\| \leq 1/2$. If
\begin{equation}
\quad \|\theta_{\text{init}} - \theta^*\| \leq r/4 \quad \text{and} \quad \|\theta^*_n - \theta_0\| \leq r/4,
\end{equation}
\begin{equation}
\quad \|\theta_{\text{init}} - \theta^*\| \leq 1/2 + \frac{1}{4(\ell + 1)^2},
\end{equation}
and $B_{2\epsilon^2}(\theta^*_n) \subseteq B^*_n(\theta_0)$. Combining above shows that there exists $0 < \epsilon' := \frac{1}{\sqrt{16}}$ such that
\begin{equation}
\lim_{n \to \infty} \mathbb{P}(\|\theta \in B^*_n(\theta_0), \quad \epsilon I \leq \nabla^2 f_n(\theta) \leq (I - \epsilon') = 1.
\end{equation}
Then, it remains to show that the iterates produced by Algorithm 2 will be confined in $B^*_n(\theta_0)$ as we get more and more data. Since $\theta^*_n$ is the global optimum of $f_n$, as long as backtracking line search ensures the decay of objective value within the locally strongly convex $B^*_n(\theta_0)$ (Boyd and Vandenberghe, 2004, Page 465), the gradient norm will decay in each iteration and hence the iterates converge to the optimum. Therefore, it is sufficient to show the first iteration stays inside.

By $\ell$-smoothness of $f_n$ inside $B^*_n(\theta_0)$, we have $\|\nabla f_n(\theta_{\text{init}})\| \leq \frac{\|\nabla f_n(\theta_{\text{init}})\|}{\sqrt{\epsilon'}}$, and hence
\begin{equation}
\lim_{n \to \infty} \mathbb{P}(\|\theta_{\text{init}} - \nabla f_n(\theta_{\text{init}}) \leq \|\theta_{\text{init}} - \theta^*\| = 1).
\end{equation}
This shows the confinement. The convergence rate of iterates is $\eta^4$, where
\begin{equation}
0 \leq \eta = \max \left\{ \left| 1 - \frac{2r}{\ell + 1} \left| 1 - \frac{2r}{\ell + 1} \right| \right| \right\} < 1,
\end{equation}
which follows from the standard theory of gradient descent under $\epsilon$-strong convexity and $\ell$-Lipschitz smoothness (Ryu and Bond, 2016, Page 15). This completes the proof.

B.9 Proof of Theorem 3
Proof of Theorem 3. In this proof, we aim to apply Theorem 17 with $\mathcal{X} = \{ \mu \in \mathbb{R}^d, L \text{ lower triangular with non-negative diagonals} \}$, which is closed and convex. Note that in the notation of this theorem, $x = (\mu^T, l_{11}^T, \ldots, l_{d1}^T)^T \in \mathbb{R}^{(d+1)d}$ and $V \in \mathbb{R}^{(d+1)d \times (d+1)d}$ is set to be a diagonal matrix with entries 2 for the $\mu$ components and $r/(2\|I - L^*_n\|_F)$ for the $L$ components. Therefore
\begin{equation}
J(x) = J(\mu, L) = 4\|\mu - \mu_n^*\|^2 + \frac{r^2}{4\|I - L_n^*\|_F} \|L - L_n^*\|^2_2.
\end{equation}
This setting yields two important facts. First, by Theorems 7 and 6, $\theta^*_n \overset{p}{\to} \theta_0$ and $\mu_n^* \overset{p}{\to} \mu_0$, yielding that
\begin{equation}
\mathbb{P}(\|\theta_n^* - \theta_0\| + \|\mu_n^* - \theta_0\| \leq \frac{r}{4\sqrt{2}} \to 1, \quad \text{as} \ n \to \infty.
\end{equation}
For $\|\mu_0 - \mu^*_n\|^2 \leq \frac{r^2}{2}$, by triangle inequality, the probability that the following inequalities hold converges to 1 in $P_{\theta_0}$ as $n \to \infty$:

$$\|\mu_0 - \mu^*_n\| \leq \|\mu_0 - \hat{\mu}_n\| + \|\hat{\mu}_n - \mu^*_n\| \leq \frac{r}{2\sqrt{2}}.$$  

Further with $L_0 = 1$, $J(\mu_0, L_0) \leq \frac{L_0^2}{4} \leq r^2$. Hence, if we initialize $L_0 = 1$ and $\mu_0$ such that $\|\mu_0 - \hat{\mu}_n\| \leq \frac{r}{\sqrt{2}}$,

$$P(\exists_0 \in \{x : J(x) \leq r^2\}) \to 1, \text{ as } n \to \infty.$$ (37)

Second, if $J \leq r^2$ then $\mu$ is close to the optimal and $\|L\|_{\mathcal{F}}$ is not too large, i.e.,

$$J(\mu, L) \leq r^2 \implies \|\mu - \mu^*_n\|^2 \leq r^2/4$$

$$J(\mu, L) \leq r^2 \implies \|L - L^*_n\|^2 \leq 4|I - I^*_n|^2$$

yielding that $\{J(\mu, L) \leq r^2\} \subseteq \mathcal{B}_n$. Recall that

$$\mathcal{B}_n = \left\{ \mu \in \mathbb{R}^d, L \in \mathbb{R}^{d^2} : \|\mu - \mu^*_n\|^2 \leq r^2/4 \text{ and } \|L - L^*_n\|^2 \leq 4|I - I^*_n|^2 \right\}.$$ 

Then by Corollary 14, under Assumptions 1 and 2, the probability of the event that

$$\mathcal{B}_n$$

converges to 1 in $P_{\theta_0}$ as $n \to \infty$.

For brevity, we make the following definitions for the rest of this proof: recall the definition of $f_n, F_n$ in Eqs. (2) and (3) (we state here again):

$$L_n : \mathcal{X} \to \mathbb{R}, \quad L_n(x) := -\frac{1}{n} \log \det L$$

$$f_n : \mathbb{R}^d \to \mathbb{R}, \quad f_n(\theta) := -\frac{1}{n} \log \pi_n(\theta)$$

$$\tilde{f}_n : (\mathcal{X}, \mathcal{F}) \to \mathbb{R}, \quad \tilde{f}_n(x, Z) := -\frac{1}{n} \log \pi_n \left( \mu + \frac{1}{n} \mathbb{E}(\mathcal{Z}) \right)$$

$$F_n : \mathcal{X} \to \mathbb{R}, \quad F_n(x) = \mathbb{E} \left[ \frac{1}{n} \log \pi_n \left( \mu + \frac{1}{n} \mathbb{E}(\mathcal{Z}) \right) \right]$$

$$\Phi_n := L_n + \tilde{f}_n, \quad \Phi_n := L_n + \tilde{f}_n.$$ 

Here $\Phi_n(x, z)$ is the KL cost function with no expectation, and $\Phi_n(x)$ is the cost function with the expectation. To match the notation of Theorem 17, we reformulate the scaled gradient estimator defined in Eq. (8) as $g_n$,

$$g_n(x, z) = R_n(x) \nabla \Phi_n(x, z) \Psi_n(y, z), \quad \text{for } x \in \mathcal{X} \setminus \mathcal{Z}^0$$

$$g_n(x, z) = R_n(x) \nabla \Phi_n(x, z) \Psi_n(y, z), \quad \text{for } x \in \mathcal{X} \setminus \mathcal{Z}^0$$

for a diagonal scaling matrix $R_n(x) \in \mathbb{R}^{d(d+1) \times d(d+1)}$. Define that $R_n(x)$ has entries 1 for the $\mu$ components, 1 for the off-diagonal L components, and $L/(1 + (nL_0)^{-1})$ for the diagonal L components. Note that $x \to \mathcal{D}$ means that $L_0 \to 0$, ensuring that $g_n(x, z)$ has entries $-1$ for the $L_0$. Since $Z$ is a standard normal random variable, under the event that $-\frac{1}{n} \log \pi_n$ has Lipschitz gradient, the gradient can be passed through the expectation so that the true gradient is defined as below,

$$G_n(x) := \mathbb{E}[g_n(x, Z)] = R_n(x) \nabla \Phi_n(x).$$

Note that the projected stochastic iteration

$$x_{k+1} = \Pi_x (x_k - \gamma_k G_n(x_k, Z_0)), \quad k = \mathbb{N} \cup \{0\},$$

with $\Pi_x(\cdot) := \arg\min_{y \in \mathcal{X}} \|V(x - y)\|^2$ is equivalent to the iteration described in Algorithm 3. Note that the differentiability of $\Phi_n$ only holds for $x \in \mathcal{X} \setminus \mathcal{Z}$. For the case where $L_0 = 0$ for some $i \in [d]$, we can use continuation via the limit $\lim_{\delta \to 0^+} \pi_n(x) - \pi_n(x + \delta \mathcal{Z}) = -1$ to evaluate even though the gradient is not defined. For the following proof, we do not make special treatments to those boundary points when applying Taylor expansion and taking derivative.

Next we apply Theorem 17 to carry out the proof. The rest of the proof consists two parts: to show the confinement result (statement 2. of Theorem 17) and to show the convergence result (statement 3. of Theorem 17). We prove these two results under the event that Eqs. (37) and (38) hold; since the probability that these events hold converges in $P_{\theta_0}$ to 1 as $n \to \infty$, the final result holds with the same convergent probability.

We first show the confinement result by analyzing $\varepsilon_n(x), \tilde{\varepsilon}(x)$, and $\sigma^2(x)$, which are defined in Eqs. (44) and (45) respectively. We aim to obtain that

$$\alpha_0 = 1 + \mathbb{I} \left(J(x) \leq r^2 \right) (2 \varepsilon_n(x) + 2 \tilde{\varepsilon}(x)) \in (0, 1)$$

holds for all $x \in \mathcal{X}$, i.e.,

$$\forall x \in \mathcal{X} : J(x) \leq r^2, \quad 0 \leq 2 \varepsilon_n(x) - 2 \tilde{\varepsilon}(x) \leq 1. \quad (39)$$

ii. $\sigma^2(x) \to 0$ as $n \to \infty$ to guarantee the SGD iterations are eventually locally confined as $n \to \infty$ (based on Theorem 17).

To show the statement i., Eq. (39), we start by deriving a lower bound for $2 \varepsilon_n(x) - 2 \tilde{\varepsilon}(x)$. Examine the expression,

$$2 \varepsilon_n(x) - 2 \tilde{\varepsilon}(x)$$

$$= 2 \varepsilon(x)(x - x^*)^T V^T V R_n(x) (V \Phi_n(x) - V \Phi_n(x^*))$$

$$- 2 \varepsilon(x)(x - x^*)^T R^T(x)(x) V^T V R_n(x) (V \Phi_n(x) - V \Phi_n(x^*))$$

$$= 2 \varepsilon(x)(x - x^*)^T V^T V R_n(x) (\int \cdots (x) V^{-1} (V(x - x^*))$$

$$- 2 \varepsilon(x)(x - x^*)^T R(x) V(x - x^*) V^{-1} (V(x - x^*))$$

where $\{\cdots \} = \int_0^1 \nabla^2 \Phi_n((1-t)x^* + tx) dt$. By splitting $\Phi_n$ into the regularization $L_n(x)$ and the expectation $F_n(x)$; and defining

$$A(x) := V R_n(x) \left( \int_0^1 \nabla^2 L_n((1-t)x^* + tx) dt \right) V^{-1}$$

$$B(x) := V R_n(x) \left( \int_0^1 \nabla^2 F_n((1-t)x^* + tx) dt \right) V^{-1}$$

$$V(x) := V(x - x^*),$$
the above expression can be written as
\[
2\mathcal{H}(x) - 2\gamma^2 \ell^2(x) = 2\gamma \left( v(x)^T A(x) v(x) - \gamma \|A(x) v(x)\|^2 \right) \geq 2\gamma \left( v(x)^T A(x) v(x) + v(x)^T B(x) v(x) \right) - 2\gamma \|A(x) v(x)\|^2 \geq 2\gamma \left\{ \left( v(x)^T A(x) v(x) + v(x)^T B(x) v(x) \right) - 2\gamma \|A(x) v(x)\|^2 \right\} \geq 2\gamma \left[ \left( v(x)^T A(x) v(x) + v(x)^T B(x) v(x) \right) - 2\gamma \|A(x) v(x)\|^2 \right]
\]

Note that by Corollary 14 that \( \frac{1}{2} D(n) \preceq \nabla^2 F(n) \preceq D(n) \) and all the \( V, R(n) \) are positive diagonal matrices, leading to
\[
B(x) - VR(n) \geq \frac{1}{2} D(n) \nabla^2 F(n) \geq 0 \quad \text{and} \quad \left\|B(x) - VR(n)\right\| \leq \frac{1}{2} D(n).
\]
Thus, the above expression can be bounded below by
\[
2\gamma \left( v(x)^T A(x) v(x) + v(x)^T B(x) v(x) \right) - 2\gamma \|A(x) v(x)\|^2 \geq 2\gamma \left[ \left( v(x)^T A(x) v(x) + v(x)^T B(x) v(x) \right) - 2\gamma \|A(x) v(x)\|^2 \right]
\]

This is true if
\[
\sup_{x \in \mathcal{X}, J(x) \leq J} \epsilon(x) \leq \gamma\ell^{-1}.
\]
Since \( \gamma \to 0 \) as \( k \to \infty \), the above holds if \( \sup_{x \in \mathcal{X}, J(x) \leq J} \epsilon(x) \) is bounded above by a constant that is independent of \( n \). Now we consider the upper bound for \( \sup_{x \in \mathcal{X}, J(x) \leq J} \epsilon(x) \). Expanding \( \epsilon(x) \),
\[
\epsilon(x) = J(x)^{-1} (x - x^*)^T \nabla^2 F(n) (x - x^*) = v(x)^T (A(x) + B(x)) v(x) \leq \|v(x)\|^2 \leq \lambda_{\max} (A(x) + B(x)) \lambda_{\max} R(n)^{1/2}.\]

Split \( \Phi(n) \) into the regularization \( L(n) \) and the expectation \( F(n) \). For the expectation, by Corollary 14 that \( \nabla^2 F(n) \preceq D(n) \) and entries of \( R(n) \) are bounded by 1, we have
\[
R(n)^{1/2} \nabla^2 F(n) R(n)^{1/2} \preceq I,
\]
and for the regularization, note that \( \nabla^2 L(n) \) is a diagonal matrix with 0 for \( \mu \) and off-diagonals of \( L \) and \( L^{-1} / n \) for diagonals of \( L \), so
\[
\frac{1}{2} \left\| V^2 L(n) \left( (1 - t)x^* + tx \right) \right\| R(n)^{1/2} = \frac{1}{2} \left\| V^2 L(n) \right\| \left\| \left( (1 - t)x^* + tx \right) \right\| R(n)^{1/2} = \frac{1}{2} \left\| V^2 L(n) \right\| \left\| \left( (1 - t)x^* + tx \right) \right\| \leq \frac{1}{2} \min_{|d|=|l|} L^{-1}_{n}.
\]

Therefore, we have completed the proof for statement I, Eq. (39).

Then we show the statement II, by getting upper bound on \( \sigma^2(r) \). Recall that \( \sigma^2(r) \) is the upper bound of the fourth moment of
\[
\|V R(n) (V \Phi(n, x, z) - V \Phi(n, x))\|.
\]
Since the regularizer is cancelled in this expression, we only consider the expectation part. Note that \( V R(n) \) is a diagonal matrix with positive diagonals,
\[
\mathbb{E} \left[ \left\| V R(n) (V \Phi(n, x, z) - V \Phi(n, x)) \right\|^4 \right]^{1/4} \leq \max_{i \in [d]} \left\| V R(n) \right\| \mathbb{E} \left[ \left\| \Phi(n, x, z) - \Phi(n, x) \right\|^4 \right]^{1/4}.
\]
Let \( Z_1, Z_2 \) be independent copies, by tower property of conditional expectation,
\[
\max_{i \in [d]} \left( V R(n) \right)_{ii} \mathbb{E} \left[ \left\| \Phi(n, x, z) - \Phi(n, x, z) \right\|^4 \right]^{1/4}.
\]
By the convexity of \( \| \cdot \| \) and Jensen’s inequality,
\[
\mathbb{E} \left[ \left\| \Phi(n, x, z) - \Phi(n, x, z) \right\|^4 \right]^{1/4} \leq \max_{i \in [d]} \left( V R(n) \right)_{ii} \mathbb{E} \left[ \left\| \Phi(n, x, z) - \Phi(n, x, z) \right\|^4 \right]^{1/4}.
\]

It remains to show the second inequality of Eq. (39), i.e.,
\[
\sup_{x \in \mathcal{X}, J(x) \leq J} 2\gamma \epsilon(x) \leq 2\gamma \ell^2(x) \leq 1.
\]
By \( \| \nabla f_n(x, Z) - \nabla f_n(x, Z) \| \leq \| \nabla f_n(x, Z_1) \| + \| \nabla f_n(x, Z_2) \| \) and Minkowski for some constant C, we obtain

\[
\leq \max_{i \in [d]} \langle V R_n(x) \rangle \leq \left\{ E \left[ \| \nabla f_n(x, Z_1) \|^4 \right] / \sqrt{n} \right\}^{1/4} + \left\{ E \left[ \| \nabla f_n(x, Z_2) \|^4 \right] / \sqrt{n} \right\}^{1/4}.
\]

\[
= 2 \max_{i \in [d]} \langle V R_n(x) \rangle \leq \left\{ E \left[ \| \nabla f_n(x, Z) \|^4 \right] / \sqrt{n} \right\}^{1/4}.
\]

Now we focus on bounding \( E \left[ \| \nabla f_n(x, Z) \|^4 \right] / \sqrt{n} \). We examine \( \| \nabla f_n(x, Z) \|^4 \).

\[
\nabla f_n(x, Z) = \left( \begin{array}{c}
\nabla f_n(x, Z) \\
\nabla f_n(x, Z)
\end{array} \right) \in \mathbb{R}^{(d+1)},
\]

yielding

\[
\| \nabla f_n(x, Z) \|^4 = \left( \begin{array}{c}
\nabla f_n(x, Z) \\
\nabla f_n(x, Z)
\end{array} \right)^4 = \left( 1 + \frac{Z^T Z}{n} \right)^2.
\]

By Cauchy-Schwarz inequality,

\[
E \left[ \| \nabla f_n(x, Z) \|^4 \right] / \sqrt{n} \leq E \left[ \| \nabla f_n(x, Z) \|^4 \right] / \sqrt{n} \left( 1 + \frac{Z^T Z}{n} \right)^4 / \sqrt{n}.
\]

We then bounds these two terms on RHS separately. We use the sub-Gaussian property of \( \| \nabla f_n(x, Z) \| \) to bound its 8th moment. First notice that \( \| \nabla f_n(x, Z) \| \) is a max of \( L \mu \)-Lipschitz function of \( Z \).

\[
\| \nabla f_n(x, Z) \| \leq \max_{i \in [d]} \langle V R_n(x) \rangle \leq \left\{ E \left[ \| \nabla f_n(x, Z) \|^4 \right] / \sqrt{n} \right\}^{1/4} + \left\{ E \left[ \| \nabla f_n(x, Z) \|^4 \right] / \sqrt{n} \right\}^{1/4}.
\]

\[
= \max_{i \in [d]} \langle V R_n(x) \rangle \leq \left\{ E \left[ \| \nabla f_n(x, Z) \|^4 \right] / \sqrt{n} \right\}^{1/4}.
\]

Given that \( \| \nabla f_n \| \leq L_1 \), the above is bounded by

\[
\frac{e}{L_1} \sqrt{(Z_1 - Z_2)^2 / L_1} \leq \frac{e}{L_1} \max_L (L, E \| Z_1 - Z_2 \|) = \frac{e}{L_1} \max_L \| Z_1 - Z_2 \|.
\]

Since a Lipschitz function of Gaussian noise is sub-Gaussian (Kon-torovich, 2014, Thm 1), i.e., let \( Z \sim N(0, \Sigma) \), \( \psi : \mathbb{R}^d \to \mathbb{R} \) be \( L \)-Lipschitz, then

\[
\mathbb{P}(\| \psi(Z) - \mathbb{E}[\psi(Z)] \| > e) \leq 2 \exp \left( -\frac{e^2}{4L_1^2} \right).
\]

Thus, \( \| \nabla f_n(x, Z) \| = \| \nabla f_n(x, Z) \| ^4 \) is \( \max_{i \in [d]} \langle V R_n(x) \rangle \)-sub-Gaussian. Then note that for a \( \sigma^2 \)-sub-Gaussian random variable \( X \in \mathbb{R} \), for any positive integer \( k \geq 2 \), \( \mathbb{E} \| X \|^k \leq \sigma^k \mathbb{E} \| X \|^k \). Hence we obtain

\[
\mathbb{E} \left[ \| \nabla f_n(x, Z) \|^4 \right] / \sqrt{n} \leq \frac{2e^4}{L_1^4} \sqrt{n} \max_L \| Z_1 - Z_2 \|.
\]

Along with the fact that Gaussian random variable has arbitrary order moments,

\[
\mathbb{E} \left[ \left( 1 + \frac{Z^T Z}{n} \right)^4 \right] \leq C,
\]

\[
\mathbb{E} \left[ \left( 1 + \frac{Z^T Z}{n} \right)^{1/4} \right] \leq \frac{2e^4}{L_1^4} \sqrt{n} \max_L \| Z_1 - Z_2 \|,
\]

and hence

\[
\mathbb{E} \left[ \| \nabla f_n(x, Z) - \nabla f_n(x) \|^4 \right] / \sqrt{n} \leq \frac{2e^4}{L_1^4} \sqrt{n} \max_L \| Z_1 - Z_2 \|.
\]

Taking supremum over \( J(x) \leq r^2 \), the RHS is bounded above by a universal constant, we therefore conclude that

\[
\sigma^2(r) = \sup_{x \in X \cap J(x) \leq r^2} \mathbb{E} \left[ \| \nabla f_n(x, Z) - \nabla f_n(x) \|^4 \right] \rightarrow 0, \quad n \to \infty.
\]

Therefore, with \( \forall x \in \mathcal{X} : J(x) \leq r^2, 0 \leq 2 \mathbb{E} \| u(x) - 2 \mathbb{E} \| u(x) \|^2 \leq 1 \) and \( \sigma^2(r) \to 0 \) as \( n \to \infty \), applying Theorem 17 yields the confinement result, i.e.,

\[
\mathbb{P} \left( \sup_{x \in \mathcal{X}} (J(x) \leq r^2) \right) \rightarrow 1
\]

in \( P_{\text{th}} \) as \( n \to \infty \).

Lastly, by statement 3. of Theorem 17, we prove the convergence result by checking

\[
\inf_{x \in \mathcal{X}, J(x) \leq r^2} \mathbb{E}[e(x)] > 0.
\]

We use the similar way to expand the expression,

\[
e(x) = J(x)^{-1}(u(x) - u(x)^T V R_n(x)) (V \Phi_n(x) - V \Phi(x)) = v(x)^T (A(x) + B(x)) v(x) / \| v(x) \|^2 \geq \lambda_{\text{max}} A(x) + B(x) = \lambda_{\text{max}} R_n(x)^{1/2} \left( \int_0^1 V^2 \Phi_n((1 - t)x^* + tx) \right) R_n(x)^{1/2}.
\]

By splitting \( \Phi_n \) into the regularization and the expectation, we have

\[
R_n(x)^{1/2} \left( \int_0^1 V^2 \Phi_n((1 - t)x^* + tx) \right) R_n(x)^{1/2} = \mathcal{O}(0, \ldots, (nL_0)^{-1} 1 / (1 + (nL_0)^{-1} L_0^2, 0, \ldots, 0) (42)
\]

and

\[
R_n(x)^{1/2} \left( \int_0^1 V^2 \Phi_n((1 - t)x^* + tx) \right) R_n(x)^{1/2} \geq \mathcal{O}(1/2) / 2 \geq 0.
\]

We then combine Eqs. (42) and (43) and use Weyl’s inequality to bound the minimal eigenvalue of the summation of two Hermitian matrices, yielding

\[
\inf_{x \in \mathcal{X}, J(x) \leq r^2} \mathbb{E}[e(x)] > \epsilon / n > 0.
\]

This gives the convergence result.

Then the proof is complete by applying Theorem 17. We know that \( \xi_k \) is strictly positive. Since \( \epsilon(r) > 0 \) and \( \epsilon(r) \) is bounded above,
there exists \( \gamma_k = \Theta(k^{-p}), p \in (0.5, 1) \) so that it satisfies the condition of the theorem. We have that \( \sigma \to 0 \), which makes 3. in the statement of Theorem 17 become

\[
P(\|V(x_k - x^*)\|^2 = O_P(k^{-p})) \xrightarrow{p} 1, p' \in (0, 0.5) \quad n \to \infty
\]

Even though \( D \) is a function of \( n, n \) is fixed as Algorithm 3 runs. Since \( D \) is invertible,

\[
P(\|x_k - x^*\|^2 = O_P(k^{-p})) \xrightarrow{p} 1, p' \in (0, 0.5) \quad n \to \infty
\]

which is exactly our desired result: as the number of data \( n \to \infty \), the probability that Algorithm 3 finds the optimum in a rate of \( k^{-p} \) (as we take more iterations, \( k \to \infty \)) converges to 1. In other words, variational inference gets solved asymptotically in a rate of \( k^{-p} \).

\[\square\]

**Theorem 17.** Let \( \mathcal{X} \subseteq \mathbb{R}^p \) be closed and convex, \( g : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^p \) be a function, \( \mathcal{G}(x) := \mathbb{E}[g(x,Z)] \) for a random element \( Z \in \mathcal{Z} \), \( x^* \in \mathcal{X} \) be a point in \( \mathcal{X} \) such that \( \mathcal{G}(x^*) = 0, V \in \mathbb{R}^{p \times p} \) be invertible, \( J(x) := \|V(x-y)\|^2 \), and \( r > 0 \). Consider the projected stochastic iteration

\[x_0 \in \mathcal{X}, \quad x_{k+1} = \Pi_{\mathcal{X}}(x_k - \gamma_k g(x_k, Z_k)), \quad k = \mathbb{N} \cup \{0\},\]

with independent copies \( Z_k \) of \( Z, \gamma_k \geq 0 \), and \( \Pi_{\mathcal{X}}(x) := \arg \min_{y \in \mathcal{X}} \|V(x-y)\|^2 \).

1. For all \( k \in \mathbb{N} \cup \{0\} \), the step sizes satisfy

\[
\forall x \in \mathcal{X} : J(x) \leq r^2, \quad 0 \leq 2\gamma_k \mathbb{E}(x) - 2\gamma_k^2 \ell(x) \leq 1 \tag*{(44)}
\]

\[
\mathbb{E}(x) := \frac{1}{J(x)} (x - x^*)^TV^TV(\mathcal{G}(x) - \mathcal{G}(x^*))
\]

\[
\ell(x) := \frac{1}{J(x)} \|V(\mathcal{G}(x) - \mathcal{G}(x^*))\|^2.
\]

2. For all \( x \in \mathcal{X}, \mathbb{E}[\|\mathcal{G}(x,Z) - \mathcal{G}(x)\|] / \sqrt{n} \leq \sigma(x) \) for \( \sigma : \mathcal{X} \to \mathbb{R}_{>0} \) and

\[
\sigma(r) := \sup_{x \in \mathcal{X}, J(x) \leq r^2} \sigma(x), \tag*{(45)}
\]

then

1. The iterate \( x_k \) is locally confined with high probability:

\[
P(J(x_k) \leq r^2) \geq \frac{\xi_k^2}{\xi_k^2 + 8\sigma(r)^2}\xi_k^2
\]

\[
\xi_k(r) := \max\{0, r^2 - J(x_k) - 2\sigma^2(r) \sum_{j=1}^k \ell_j\}
\]

\[
\xi_k(r) := r^2 \sum_{j=1}^k \ell_j^2 + \sigma^2 \sum_{j=1}^k \ell_j^2.
\]

2. The iterate \( x_k \) stays locally confined for all \( k \in \mathbb{N} \) with high probability:

\[
P\left(\sup_{x \in \mathcal{X}} J(x) \leq r^2 \right) \geq \frac{\xi_k^2}{\xi_k^2 + 8\sigma^2(r)^2}\xi_k^2
\]

\[
\xi_k(r) := \lim_{k \to \infty} \xi_k(r), \quad \xi(r) := \lim_{k \to \infty} \xi_k(r).
\]

3. If additionally

\[
\inf_{x \in \mathcal{X}}, J(x) \leq r^2 , \quad \mathbb{E}(x) > 0 \quad \text{and} \quad \gamma_k = \Theta(k^{-p}), p \in (0.5, 1],
\]

the iterate \( x_k \) converges to \( x^* \) with high probability:

\[
\lim_{k \to \infty} P(J(x_k) \leq k^{-p}) \geq P\left(\sup_{x \in \mathcal{X}} J(x) \leq r^2 \right), \quad \forall p' \in (0, 0.5).
\]

**Proof.** To begin, we show \( \Pi_{\mathcal{X}} \) is non-expansive,

\[
\|V(\Pi_{\mathcal{X}}(x) - \Pi_{\mathcal{X}}(y))\|^2 \leq \|V(x - y)\|^2.
\]

For all \( x, y \in \mathbb{R}^p, \) define \( (x,y) \mapsto x^TV^TV \). Since \( V \) is invertible, \( V^TV \) is symmetric and positive definite, and hence \( \mathbb{R}^p, (\cdot, \cdot) \) forms a Hilbert space. Any projection operator of a Hilbert space is non-expansive (Bauschke and Combettes, 2011, Prop. 4.4).

Note that \( x^* = \Pi_{\mathcal{X}}(x^*) \) and the projection operation is non-expansive, expanding the squared norm yields

\[
\|V(x_{k+1} - x^*)\|^2 \leq \|V(x_k - x^*)\|^2
\]

\[
-2\gamma_k (x_k - x^*)^TV^TV(\mathcal{G}(x_k, Z_k) - \mathcal{G}(x^*))
\]

\[
+ 2\gamma_k \|V(\mathcal{G}(x_k, Z_k) - \mathcal{G}(x^*))\|^2 - 2\gamma_k \mathbb{E}[\|\cdot\|^2]
\]

\[
\ell_k(x) := \frac{1}{J(x)} (x - x^*)^TV^TV(\mathcal{G}(x) - \mathcal{G}(x^*))
\]

\[
\ell_k(x) := \frac{1}{J(x)} \|V(\mathcal{G}(x) - \mathcal{G}(x^*))\|^2,
\]

we have that

\[
J(x_{k+1}) \leq J(x_k) (1 - 2\gamma_k \ell_k(x_k) + 2\gamma_k^2 \ell_k^2(x_k) + \beta_k(x_k) + 2\gamma_k^2 \sigma^2(x_k)).
\]

We now define the filtration of \( \sigma \)-algebras

\[
\mathcal{F}_k = \sigma(x_1, \ldots, x_k, z_1, \ldots, z_{k-1}),
\]

and the stopped process for \( r > 0 \),

\[
Y_0 = J(x_0), \quad Y_{k+1} = \begin{cases} Y_k & \text{if } J(x_{k+1}) > r^2, \\ J(x_{k+1}) & \text{otherwise} \end{cases}
\]

Note that \( Y_k \) is \( \mathcal{F}_k \)-measurable, and that \( Y_k \) "freezes in place" if \( J(x_k) \) ever jumps larger than \( r^2 \); so for all \( r \leq r_j \),

\[
P\left(J(x_k) > r^2 \right) = P(\{J(x_k) \geq r, Y_{k-1} > r^2\} + P(\{J(x_k) \geq r, Y_{k-1} \leq r^2\)}
\]

\[
= P(\{J(x_k) > r, Y_k > r, Y_{k-1} > r^2\} + P(\{Y_k > r, Y_{k-1} \leq r^2\})
\]

\[
\leq P(\{Y_k > r, Y_{k-1} > r^2\}) + P(\{Y_k > r, Y_{k-1} \leq r^2\})
\]

\[
= P(\{Y_k > r^2\}).
\]

Therefore if we obtain a tail bound on \( Y_k \), it provides the same bound on \( J(x_k) \). Now substituting the stopped process into the original recursion and collecting terms,

\[
Y_{k+1} \leq Y_k (1 + 1 \{Y_k \leq r^2\} (-2\gamma_k \ell_k(x_k) + 2\gamma_k^2 \ell_k^2(x_k))) + 1 \{Y_k \leq r^2\} [\beta_k(x_k) + 2\gamma_k^2 \sigma^2(x_k)]
\]

\[
\leq Y_k (1 + 1 \{Y_k \leq r^2\} (-2\gamma_k \ell_k(x_k) + 2\gamma_k^2 \ell_k^2(x_k))) + 1 \{Y_k \leq r^2\} \beta_k(x_k) + 2\gamma_k^2 \sigma^2(r).
\]

Using the notation of Lemma 18, set

\[
\alpha_k = 1 + 1 \{Y_k \leq r^2\} (-2\gamma_k \ell_k(x_k) + 2\gamma_k^2 \ell_k^2(x_k))
\]

\[
\alpha_k = 1
\]

\[
\beta_k = 1 \{Y_k \leq r^2\} \beta_k(x_k)
\]

\[
c_k = 2\gamma_k^2 \sigma^2(r).
\]
By the fourth moment assumption, \( \beta_k \) has variance bounded above by \( \sigma^2_k \) conditioned on \( \mathcal{F}_k \), where

\[
\sigma^2_k = 8 \mathbb{E} \left[ |Y_k |^2 \right] \| V(x_k - x^*) \|^2 \mathbb{E} \left[ \sigma^2(x_k) \right]^2 + 8 \mathbb{E} \left[ |Y_k |^2 \right] \sigma^4(x_k) \\
\leq 8 \mathbb{E} \sigma^2(r)^2 + 8 \mathbb{E} \sigma^4(r).
\]

Therefore, using the descent Lemma 18,

\[
P(Y_k > r^2) \leq \frac{\xi_k}{\max(r^2 - \xi_k, 0)^2 + \xi_k}
\]

\[
\xi_k = J(x_0) + 2\sigma^2(r) \sum_{j \in J_k} \xi_{j/k} = 8\sigma^2(r) \left( \sum_{j \in J_k} \gamma_{j/k}^2 + \sigma^2(r) \sum_{j \in J_k} \gamma_{j/k}^2 \right),
\]

yielding the first result. Now since \( Y_{k+1} \leq r^2 \implies Y_k \leq r^2 \) for all \( k \geq 0 \), the sequence of events \( \{ Y_k \leq r^2 \} \) is decreasing. Therefore the second result follows from

\[
P(\bigcap_{k=0}^{\infty} \{ Y_k \leq r^2 \}) = \lim_{k \to \infty} P(Y_k \leq r^2) \geq \lim_{k \to \infty} 1 - \frac{\xi_k}{\max(r^2 - \xi_k, 0)^2 + \xi_k} = \frac{\max(r^2 - \xi_k, 0)^2 + \xi_k}{\max(r^2 - \xi_k, 0)^2 + \xi_k},
\]

where \( \xi_k = \lim_{k \to \infty} \xi_k \) and \( \zeta_k = \lim_{k \to \infty} \zeta_k \). Finally, we analyze the conditional tail distribution of \( Y_k \) given that it stays confined, i.e., \( \forall k \geq 0, Y_k \leq r^2 \). In the notation of Lemma 18, redefine

\[
0 \leq \alpha_k := \sup_{x \in \mathcal{F}_k} \{ 1 - 2 \rho \mathbb{E}(x) + 2\rho^2 \mathbb{E}(x^2) \leq 1, \}
\]

i.e., \( \alpha_k \) is the largest possible value of \( \alpha_k \) when \( Y_k \leq r^2 \). So again applying Lemma 18,

\[
P(Y_k > t_k \forall k, Y_k \leq r^2) \leq \lim_{k \to \infty} P(Y_k \leq r^2) \leq \lim_{k \to \infty} P(Y_k > t_k \forall k, Y_k \leq r^2) \leq \frac{\xi_k}{\max(r^2 - \xi_k, 0)^2 + \xi_k} \leq \frac{\max(\xi_k, 0)^2 + \xi_k}{\max(r^2 - \xi_k, 0)^2 + \xi_k}
\]

\[
\xi_k = J(x_0) + 2\sigma^2(r) \sum_{j \in J_k} \gamma_{j/k}^2 + \sigma^2(r) \sum_{j \in J_k} \gamma_{j/k}^2.
\]

Here \( t_k = \Theta(k^{-p'}) \), \( p' \in (0, 0.5) \) is a decreasing sequence, whose shrinking rate—such that Eq. (46) still converges to 0—will determine the convergence rate of \( Y_k \).

To understand the rate of Eq. (46), the key is to characterizing the order of \( \prod_{j \in J_k} \alpha_j \) and \( \sum_{j \in J_k} \gamma_{j/k} \prod_{j \in J_k} \alpha_j \). Since \( \beta_k = \Theta(k^{-p'}) \), \( p \in [0.5, 1] \), and \( \epsilon' := \inf_{x \in \mathcal{F} \setminus \mathcal{J}(x) \subset \mathcal{J}_0} \mathbb{E}(x) > 0 \), we know that

\[
\prod_{j \in J_k} \alpha_j = \prod_{j \in J_k} \left\{ \sup_{x \in \mathcal{F} \setminus \mathcal{J}(x) \subset \mathcal{J}_0} \left( 1 - 2 \gamma_j \mathbb{E}(x) + 2 \gamma_j^2 \mathbb{E}(x^2) \right) \right\} \\
\leq \prod_{j \in J_k} \left( 1 - 2 \gamma_j \right) \inf_{x \in \mathcal{F} \setminus \mathcal{J}(x) \subset \mathcal{J}_0} \left( \mathbb{E}(x) - \gamma_j \mathbb{E}(x^2) \right) \\
\leq \prod_{j \in J_k} \left( 1 - 2 \gamma_j \right) \\
= \exp \left( \sum_{j \in J_k} \log \left( 1 - 2 \gamma_j \right) \right) \leq \exp \left( 2 \sum_{j \in J_k} \gamma_j \right)
\]

for some \( c > 0 \), yielding that \( \prod_{j \in J_k} \alpha_j = \Theta(\exp(-Ck^{-2p})) \) for some \( C > 0 \). For the second term, since \( \alpha_k \in (0, 1) \),

\[
\sum_{j=0}^{k} \sum_{i \in J_{i+1}} \alpha_j^2 \leq \sum_{i=0}^{k} \sum_{j=0}^{i} \alpha_j^2 = \Theta(k^{-2p}). \quad (47)
\]

Similarly, \( \sum_{j=0}^{k} \sum_{i \in J_{i+1}} \alpha_j^2 = \Theta(k^{-2p}) \). Therefore,

\[
\sum_{j=0}^{k} \sum_{i \in J_{i+1}} \alpha_j^2 = \Theta(k^{-2p}).
\]

Combined with the fact that \( t_k = \Theta(k^{-p'}) \), \( p' \in (0, 0.5) \), this implies that Eq. (46) is \( o(1) \) and hence for all \( \epsilon > 0, p' \in (0, 0.5) \),

\[
\lim_{k \to \infty} P(k^{\epsilon'} Y_k > \epsilon) \forall k, Y_k \leq r^2 = 0.
\]

Therefore, \( \forall p' \in (0, 0.5) \),

\[
\lim_{k \to \infty} P(Y_k \leq k^{-p'}) \geq \lim_{k \to \infty} \prod_{k \to \infty} P(Y_k \leq k^{-p'}) \forall Y_k \leq r^2 \leq P \forall Y_k \leq r^2,
\]

and the result follows.

\section*{Lemma 18 (Descent). Suppose we are given a filtration \( \mathcal{F}_k \subseteq \mathcal{F}_{k+1} \), \( k \geq 0 \). Let

\[
Y_{k+1} \leq \alpha_k Y_k + \beta_k + c_k, \quad k \geq 0,
\]

where \( Y_1 \geq 0 \) and \( 0 \leq \alpha_k \leq \alpha_k \) are \( \mathcal{F}_k \)-measurable, \( \beta_k \) is \( \mathcal{F}_{k+1} \)-measurable and has mean 0 and variance conditioned on \( \mathcal{F}_k \) bounded above by \( \sigma^2_k \), and \( Y_0, \sigma^2_k, c_k, \alpha_k \geq 0 \) are \( \mathcal{F}_0 \) measurable. Then

\[
P(Y_1 \geq t) \leq \frac{\xi_k}{\max(r^2 - \xi_k, 0)^2 + \xi_k}
\]

\[
\xi_k = Y_0 \sum_{i=0}^{k-1} \alpha_i + \sum_{i=0}^{k-1} c_i \left( \sum_{j=0}^{i} \alpha_j \right) - \sum_{i=0}^{k-1} \xi_j \left( \sum_{j=0}^{i} \alpha_j \right),
\]

\section*{Proof. Solving the recursion,

\[
Y_k \leq \alpha_k Y_{k-1} + \beta_{k-1} + c_{k-1}
\]

\[
\leq \alpha_k \left( \alpha_{k-1} Y_{k-2} + \beta_{k-2} + c_{k-2} \right) + \beta_{k-1} + c_{k-1}
\]

\[
\leq Y_k \sum_{i=0}^{k-1} \alpha_i + \sum_{i=0}^{k-1} \beta_i \sum_{j=0}^{i} \alpha_j + \sum_{i=0}^{k-1} c_i \sum_{j=0}^{i} \alpha_j
\]

\[
\leq Y_0 \sum_{i=0}^{k-1} \alpha_i + \sum_{i=0}^{k-1} \beta_i \sum_{j=0}^{i} \alpha_j + \sum_{i=0}^{k-1} c_i \sum_{j=0}^{i} \alpha_j.
\]

5 Although the bound in Eq. (47) is loose, the order described by the bound is actually tight. A more detailed analysis can be obtained by approximating the summation with an integral, which yields the same order.
So

\[ P(Y_k \geq t) \leq P \left( \sum_{j=0}^{k-1} \beta_j \prod_{j+1}^{k-1} \alpha_j \geq t - \xi_k \right). \]

By Cantelli’s inequality and the fact that the \( i \)th term in the sum is \( \mathcal{F}_{i+1} \)-measurable,

\[
P(Y_k \geq t) \leq \frac{\sum_{i=0}^{k-1} \mathbb{E} \left[ \beta_i^2 \prod_{j+1}^{k-1} \alpha_j^2 \right]}{\max \{ t - \xi_k, 0 \}^2 + \sum_{i=0}^{k-1} \mathbb{E} \left[ \beta_i^2 \prod_{j+1}^{k-1} \alpha_j^2 \right]}
\leq \frac{\sum_{j=0}^{k-1} \tau_j \prod_{j+1}^{k-1} \tilde{\alpha}_j^2}{\max \{ t - \xi_k, 0 \}^2 + \sum_{j=0}^{k-1} \tau_j \prod_{j+1}^{k-1} \tilde{\alpha}_j^2}
= \frac{\zeta_k}{\max \{ t - \xi_k, 0 \}^2 + \zeta_k}.
\]