Abstract. Supersymmetry transformations are applied to the harmonic oscillator for generating potentials $V^j_k$ whose spectra have a gap with respect to the initial one. The extremal states are found and, as the reduction theorem conditions are satisfied, ensuring that the system has third order ladder operators and it is connected with Painlevé IV (PIV) equation, then solutions to this equation can be generated. An alternative transformation is applied, by adding the levels needed to recover the spectrum of $V^j_k$. The extremal states are found and, as the reduction theorem is met again, we get also solutions to the PIV equation which will be analysed.

1. Introduction

Supersymmetric quantum mechanics (SUSY QM) is a powerful tool for generating Hamiltonians with known spectra departing from a given initial one. In fact, through this technique it is possible to “add” or “delete” levels at will in the initial spectrum.

On the other hand, the Painlevé IV (PIV) equation has a direct connection with systems ruled by second-order polynomial Heisenberg algebras (PHA) [1]. Particular Hamiltonians having this kind of algebras arise by applying first-order SUSY QM to the harmonic oscillator. Note that the identification of the corresponding extremal states is the key to generate solutions to the PIV equation. Moreover, after a $k$-order SUSY transformations systems with $(2k + 1)$-th order ladder operators will be obtained. In addition, the SUSY partners of the harmonic oscillator that have also third order ladder operators have been recently identified [2]. Through them additional solutions to the PIV equation can be found.

In previous works, when $k$ new levels were created a pattern for the zeros of the non-singular PIV solution (the one arising from the extremal state with lowest energy) was recognized [2, 3]: the number of zeros becomes equal to $2k - 1$, where $k$ is the order of the transformation. Here, we will see that the order of the transformation is not what defines the number of zeros, but the number of steps of the finite ladder is indeed what characterizes this quantity.

In this paper, Hamiltonians with added or deleted levels whose spectra are the same (up to energy displacements), are generated through SUSY QM for the harmonic oscillator. The extremal states of those systems are found, and through them solutions to the PIV equation are generated. Finally, a comparison between such PIV solutions is made.
2. Supersymmetric quantum mechanics

In this work the names intertwining technique, factorization method and supersymmetric quantum mechanics are used as synonymous, since they are equivalent methods for generating exactly solvable quantum mechanical potentials, as we will show next.

2.1. Intertwining technique

In the $k$-th order intertwining technique $k+1$ Hamiltonians $H_j$ and $2k$ first-order operators $A^\pm_l$ of the form

$$H_j = -\frac{1}{2} \frac{d^2}{dx^2} + V_j(x), \quad A^\pm_l = \frac{1}{\sqrt{2}} \left( \frac{d}{dx} \pm \alpha_l(x, \epsilon_l) \right), \quad j = 0, \ldots, k, \quad l = 1, \ldots, k,$$

are intertwined in the way

$$H_i A^+_l = A^+_l H_{i-1}, \quad H_{i-1} A^-_l = A^-_l H_i, \quad i = 1, \ldots, k,$$

where the initial Hamiltonian $H_0$ has a known given spectrum. In order to fulfil (2) the superpotentials $\alpha_i(x, \epsilon_i)$, $i = 1, \ldots, k$ are determined by solutions to the initial Riccati equation \[2\]:

$$\alpha'_i(x, \epsilon_j) + \alpha_i^2(x, \epsilon_j) = 2(V_0 - \epsilon_j), \quad j = 1, \ldots, k.$$

Equivalently, the substitution $\alpha_1(x, \epsilon_j) = u'_j / u_j$ indicates that we require $k$ solutions to the corresponding stationary Schrödinger equation:

$$H_0 u_j = -\frac{1}{2} u''_j + V_0 u_j = \epsilon_j u_j, \quad j = 1, \ldots, k.$$

The potential $V_k$ is given in terms of them and the initial potential $V_0$ through \[2,3\]:

$$V_k(x) = V_0(x) - \sum_{i=1}^k \alpha'_i(x, \epsilon_i) = V_0(x) - (\ln \{W(u_1, u_2, \ldots, u_k)\})'',$$

where $W(u_1, \ldots, u_k)$ denotes the Wronskian of the $k$ seed solutions.

2.2. Factorization

Due to the intertwining relations (2), the $k+1$ Hamiltonians can be factorized as:

$$H_0 = A^-_1 A^+_1 + \epsilon_1, \quad H_k = A^+_k A^-_k + \epsilon_k,$$

$$H_i = A^-_i A^+_i + \epsilon_i = A^-_{i+1} A^+_{i+1} + \epsilon_{i+1}, \quad i = 1, \ldots, k - 1.$$

Another intertwining relation between $H_0$ and $H_k$ can be given, involving a $k$-th order differential intertwining operator which is factorized in terms of the first order ones as follows:

$$H_k B^+_k = B^+_k H_0, \quad B^+_k = A^+_k A^+_k \ldots A^+_1.$$

In addition, the two different products of $B^+_k$ can be factorized in the following way:

$$B^-_k B^+_k = \prod_{i=1}^k (H_0 - \epsilon_i), \quad B^+_k B^-_k = \prod_{i=1}^k (H_k - \epsilon_i).$$


2.3. Eigenfunctions

The formal eigenfunction of \( H_k \) for the initial eigenvalue \( E_n \) is expressed in terms of Wronskians involving the seed solutions \( u_1, \ldots, u_k \) and the eigenfunction \( \psi^{(0)}_n \) of \( H_0 \) which is transformed, and similar relations for those associated to the factorization energies \( \epsilon_i \) [1]:

\[
\psi^{(k)}_n \propto \frac{W(u_1, \ldots, u_k, \psi^{(0)}_n)}{W(u_1, \ldots, u_k)}, \quad \psi^{(k)}_{\epsilon_i} \propto \frac{W(u_1, \ldots, \hat{u}_i, \ldots, u_k)}{W(u_1, \ldots, u_k)}, \quad n = 0, 1, \ldots, \quad i = 1, \ldots, k, \tag{9}
\]

where \( \hat{u}_i \) indicates that the seed solution \( u_i \) is excluded of the corresponding Wronskian.

2.4. Supersymmetric quantum mechanics

The SUSY QM introduced by Witten in 1981 is realized through the following operators choice:

\[
Q_1 = \frac{Q^+ + Q^-}{\sqrt{2}}, \quad Q_2 = \frac{Q^+ - Q^-}{i\sqrt{2}}, \quad Q^+ = \begin{pmatrix} 0 & B^+_k \\ 0 & 0 \end{pmatrix}, \quad Q^- = \begin{pmatrix} 0 & 0 \\ B^-_k & 0 \end{pmatrix}. \tag{10}
\]

Since the operators \( B^\pm_k \) satisfy equations (7-8), it turns out that

\[
H_{ss} = \{Q^-, Q^+\} = \left( B^+_k B^-_k \begin{pmatrix} 0 & 0 \\ 0 & B^-_k B^+_k \end{pmatrix} \right) = \prod_{i=1}^k (H^p_k - \epsilon_i \mathbb{1}), \quad H^p_s = \begin{pmatrix} H_k & 0 \\ 0 & H_0 \end{pmatrix}, \tag{11}
\]

where \( H_k \) and \( H_0 \) in \( H^p_s \) are intertwined as in equations (7).

3. Second order polynomial Heisenberg algebra

In order to determine the general potential ruled by a second order PHA, a nonlinear second-order differential equation, the Painlevé IV equation, needs to be solved. In fact, let us take two Schrödinger Hamiltonians \( H \) and \( H_a \) such that:

\[
HL_1^+ = L_1^+(H_a + 1), \quad H_aL_2^+ = L_2^+H \quad \Rightarrow \quad [H, L^+] = L^+ , \tag{12}
\]

where the ladder operator \( L^+ \) is given by [1,2]:

\[
L^+ = L_1^+ L_2^+, \quad L_1^+ = \frac{1}{\sqrt{2}} \left[ -\frac{d}{dx} + f(x) \right], \quad L_2^+ = \frac{1}{2} \left[ \frac{d^2}{dx^2} + g(x) \frac{d}{dx} + h(x) \right]. \tag{13}
\]

The functions \( f(x), h(x), \) and \( V(x) \) at the end depend on just the function \( g(x) \),

\[
f = x + g, \quad h = -\frac{x^2}{2} + \frac{g'}{2} - \frac{g^2}{2} - 2xg + a, \quad V = \frac{x^2}{2} - \frac{g'}{2} + \frac{g^2}{2} + xg + \varepsilon_1 - \frac{1}{2}, \tag{14}
\]

which satisfies the PIV equation:

\[
g'' = \frac{(g')^2}{2g} + \frac{3}{2}g^3 + 4xg^2 + 2(x^2 - a)g + b \tag{15}
\]

with parameters \( a = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1 - 1 \) and \( b = -2\Delta^2, \Delta = \varepsilon_2 - \varepsilon_3 \). The generalized number operator is a polynomial of third degree in \( H \),

\[
N(H) = (H - \varepsilon_1)(H - \varepsilon_2)(H - \varepsilon_3). \tag{16}
\]
Thus, from an algebraic viewpoint the system will have at most three equidistant ladders, each one starting from $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$. The corresponding extremal states are given by:

$$\psi_{\epsilon_1} \propto e^{(-\frac{x^2}{2}-\int g dx)}, \quad \psi_{\epsilon_2} \propto \left( \frac{g'}{2g} - \frac{\Delta}{g} - x \right) e^{\int\left(\frac{g'}{2g} + \frac{\Delta}{g}\right)dx},$$ \hspace{1cm} (17)

$$\psi_{\epsilon_3} \propto \left( \frac{g'}{2g} - \frac{\Delta}{g} + x \right) e^{\int\left(\frac{g'}{2g} + \frac{\Delta}{g}\right)dx}.$$ \hspace{1cm} (18)

Therefore, if the extremal states for a system with third order ladder operators are given, then solutions to the PIV equation can be obtained through

$$g(x) = -x - \ln(\psi_{\epsilon_1})'.$$ \hspace{1cm} (19)

4. SUSY transformations for the harmonic oscillator

The $k$-th order SUSY partners of the harmonic oscillator are ruled by $2k$-th order PHA (with ladder operators of $(2k+1)$-th order). Moreover, if the seed solutions are connected by the standard annihilation and creation operators $a^\pm$ for $k > 1$, then the $(2k+1)$-th order ladder operators will be factorized, leading as well to third order ladder operators for our system. This is the so-called reduction theorem, which has been proven and analyzed in [2].

It has been shown recently that two different SUSY transformations producing a spectral gap of integer size can lead to the same final potential: in one transformation we delete an even number $k'$ of consecutive energy levels to produce a gap of size $k'+1$, while it appears a finite ladder of $k$ steps; in the second, we make a $k$-th order SUSY transformation which creates the finite ladder with $k$ steps and reproduce the same spectral gap. Thus, up to an energy displacement both spectra are equal, and by adjusting one final parameter both transformations lead indeed to the same final potential. In this paper we will illustrate this result by means of two particular examples; the general situation can be found elsewhere.

The eigenfunctions of the harmonic oscillator Hamiltonian are well known, and the general solution to the stationary Schrödinger equation for $\epsilon \neq E_n, n = 0, 1, \ldots$ is expressed in terms of confluent hypergeometric functions as follows [1, 2, 4]:

$$u(x) = e^{-\frac{x^2}{2}} \left[ F_1 \left( \frac{1-2\epsilon}{2}, \frac{1}{2}; x^2 \right) + 2\nu \frac{\Gamma(\frac{3-2\epsilon}{2})}{\Gamma(\frac{3-2\epsilon}{2})} F_1 \left( \frac{3-2\epsilon}{2}, \frac{3}{2}; x^2 \right) \right].$$ \hspace{1cm} (20)

Let us make a first-order transformation with factorization energy $\epsilon_1 = -\frac{5}{2}$, the final potential will be non singular for $\nu_1 \in (-1, 1)$, as proved in [2] (some examples of potentials of this family are shown in Figure 1). In particular, for $\nu_1 = 0$ we denote this potential as $V_{1}^{-3}$. On the other hand, a second-order transformation is implemented by choosing the first two excited states to generate the SUSY partner potential $V_2$. It turns out that:

$$V_2 = \frac{x^2}{2} + 2 - \frac{4}{2x^2 + 1} + \left( \frac{4x}{2x^2 + 1} \right)^2 = V_{1}^{-3} + 3.$$ \hspace{1cm} (21)

The eigenfunctions for both potentials turn out to be the same and, after identifying and sorting the extremal states, the corresponding PIV solutions become (see Figure 2):

$$g_{\epsilon_1}^{1,-3} = g_{\epsilon_1}^{2,1} = \frac{4x}{2x^2 + 1} \quad \text{for} \quad a = 3, b = -8,$$ \hspace{1cm} (22)

$$g_{\epsilon_2}^{1,-3} = g_{\epsilon_2}^{2,1} = -\frac{4x^4 + 3}{4x^5 + 8x^3 + 3x} \quad \text{for} \quad a = -6, b = -2,$$ \hspace{1cm} (23)

$$g_{\epsilon_3}^{1,-3} = g_{\epsilon_3}^{2,1} = \frac{8x^5 + 6x}{1 - 4x^4} \quad \text{for} \quad a = 0, b = -18.$$ \hspace{1cm} (24)
Note that the non-singular solution in Figure 2a has one node; by realizing that the finite ladder generated by adding a new level at \(-\frac{9}{2}\), for different values of \(\nu_1\) (for \(\nu_1 = 0\) it is called \(V^{-3}_1\)); (b) Spectrum associated to this family of potentials, which is also the one of \(V^4_1\) but displaced down by 3.

**Figure 1.** (a) Plot of three potentials belonging to the family of first-order SUSY partners of the oscillator generated by adding a new level at Figure 1. (b) Plot of both potentials is given in Figure 3:

\[
V^2_1 = V^{-6}_2 + 6
\]

\[
= \frac{x^2}{2} - \frac{64 (8x^3 + 16x) x^3 + 8 (24x^2 + 16) x^4 + 96 (2x^4 + 8x^2 + 15) x^2}{8 (2x^4 + 8x^2 + 15) x^4 + 45}
\]

\[
+ \frac{(8 (8x^3 + 16x) x^4 + 32 (2x^4 + 8x^2 + 15) x^2)^2}{(8 (2x^4 + 8x^2 + 15) x^4 + 45)^2}.
\]

The second example appears by doing first a fourth order transformation which deletes four excited state levels starting from the second one and then implementing a second order SUSY to create two levels at \(-\frac{9}{2}\) and \(-\frac{11}{2}\), leading to (a plot of both potentials is given in Figure 3):

\[
g_{\ell_1}^2 = g_{\ell_1}^{4,2} = \frac{8x (32x^{10} + 208x^8 + 432x^6 + 504x^4 + 90x^2 - 135)}{(4x^4 + 12x^2 + 3) (16x^8 + 64x^6 + 120x^4 + 45)}
\]

\[
\text{for } a = 7, b = -32,
\]

\[
g_{\ell_2}^2 = g_{\ell_2}^{4,2} = \frac{32x^3 (4 (x^2 + 3) x^2 + 15)}{8 (2x^4 + 8x^2 + 15) x^4 + 45}
\]

\[
- \frac{20x (8 (2x^6 + 12x^4 + 27x^2 + 15) x^2 + 45)}{2x^2 (4 (4x^8 + 30x^4 + 90x^2 + 75) x^2 + 225) - 225}
\]

\[
\text{for } a = -11, b = -8,
\]

\[
g_{\ell_3}^2 = g_{\ell_3}^{4,2} = -\frac{32 (4x^4 + 9) x^3}{16x^8 + 72x^4 - 135}
\]

\[
+ \frac{32 (4 (x^2 + 3) x^2 + 15) x^3}{8 (2x^4 + 8x^2 + 15) x^4 + 45} - 2x - \frac{1}{x}
\]

\[
\text{for } a = 1, b = -72.
\]
Let us note that, although $g_{4,2}^4 \varepsilon_1$ comes from a fourth order SUSY transformation, it has just three zeros which appear again from the same empirical formula $2k - 1$, where the number of steps of the finite ladder is now $k = 2$.

5. Conclusions

In this article we have derived some rational solutions to the PIV equation, from potentials whose spectra have a gap of integer size. Since such Hamiltonians appear from two SUSY transformations of different orders, we realized that the number of zeros associated to the non-singular PIV solution which comes from the ground state is equal to $2k - 1$, where $k$ is now the number of steps of the finite ladder. A proof of this conjecture is still an open question.

Acknowledgments
The authors acknowledge the support of Conacyt, project 152574.

References
[1] Carballo J M, Fernández D J, Negro J and Nieto L M 2004 Polynomial Heisenberg algebras J Phys A: Math Gen 37 10349
[2] Bermudez D and Fernández D J 2011 Supersymmetric Quantum Mechanics and Painlevé IV equation SIGMA 7 025
[3] Bermudez D and Fernández D J 2013 Solution Hierarchies for the Painlevé IV Equation Geometric Methods in Physics, Trends in Mathematics ed P Kielanowski et al (Heidelberg: Springer Basel) p 199
[4] Junker G and Roy P 1998 Conditionally exactly solvable potentials: a supersymmetric construction method Ann Phys 270 155177