An optimal nonconforming finite element method for the Stokes equations

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Abstract

In this paper, we propose and develop an optimal nonconforming finite element method for the Stokes equations approximated by the Crouzix-Raviart element for velocity and the continuous linear element for pressure. Previous result in using the stabilization method for this finite element pair is improved and then proven to be stable. Then, optimal order error estimate is obtained and numerical results show the accuracy and robustness of the method.

Key words: Stokes equations, Crouzeix-Raviart element, linear element, finite element method, nonconforming finite element method, \textit{inf-sup} condition, stability, optimal estimate, numerical experiments

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1 Introduction

Nowadays, finite element methods have become an important and powerful tool in many scientific and technological fields. In particular, stable mixed finite element methods are a fundamental component in search for efficient numerical methods for solving the Stokes and Navier-Stokes equations governing incompressible flows [2, 3, 10, 5, 13]. For the incompressible flows, more researches have been directed toward the compatibility of the component approximations of velocity and pressure by satisfying the \textit{inf-sup} condition in the past decades, i.e., the stability condition. Some popular finite element pairs have been constructed for the incompressible Stokes and Navier-Stokes flows. However, the search for simpler and more efficient pairs for velocity and pressure approximations is still attractive and valuable.

Recently, a class of local stabilized mixed finite element methods have been developed and analyzed for the Stokes and Navier-Stokes equations approximated by the lower order finite element pairs. One of

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them uses the pressure projection method to stabilize the lower equal-order finite elements (i.e., $P_1 - P_1$ or $Q_1 - Q_1$). In practice, this method can also efficiently stabilize the equal-order conforming finite element pairs $P_r - P_r$, $r = 1, 2$ for the Stokes equations [14, 16] and Darcy equations [7]. Also, it can be easily promoted for solving the problems in elasticity, coupling free fluid and porous media system, fluid-fluid interaction in different media, etc. In [15], a stabilized finite element method is established for the Stokes equations approximated by the Crouzix-Raviart element for velocity and the continuous linear element for pressure (i.e., $NCP_1 - P_1$).

Compared with conforming finite element methods, nonconforming finite element methods for incompressible flows are more popular due to their simplicity and small support sets of basis functions. Furthermore, they seem much easier to fulfill the discrete inf-sup condition and can easily relax the high-order continuity requirement for conforming finite elements. Therefore, in practice, the nonconforming finite element methods seem superior to the conforming finite element methods. Based on the above heuristics and some existing result [11], we try to optimize the previous method [15] and furthermore establish the weak coercivity, well posedness and optimal estimates of the corresponding system.

As an example, this paper concentrates on a nonconforming finite element method for the Stokes equations, which uses the nonconforming and conforming piecewise linear polynomial approximations for velocity and pressure, respectively. This method is here defined in such a way that it can be easily generalized to the corresponding nonlinear problem. The present pair is different from the Crouzeix-Raviart pair [12], the $P_2 - P_0$ pair [10], the MINI-element $P_1 b - P_1$ pair [9], the Taylor-Hood $P_2 - P_1$ pair [8], and the conforming $P_1 - P_1$ pair and nonconforming $NP_1 - P_1$ pair with stabilization based on local Gauss integrations [16, 15]. Also, a better approximation for the pressure is obtained with the continuous piecewise linear element. In this paper, it is shown to be stable and optimal match using the nonconforming and conforming piecewise linear polynomial approximations for velocity and pressure without any stabilization treatment. It seems more computationally efficient without a loss of accuracy.

The rest of the paper is organized as follows: In the next section, an abstract functional setting for the stationary Stokes problem is described, along with some useful statements. Then, in the third section, the weak formulation, stability and well-posedness are established. Error estimates of optimal order for the method are derived in section 4. Finally, numerical experiments are given to show superiority of the present method for the Stokes equations.

## 2 Preliminaries

This section focus on the stationary Stokes equations with homogeneous Dirichlet boundary condition. Let the domain $\Omega$ be a bounded, convex and open subset of $\mathbb{R}^d, d = 2, 3$ with Lipschitz continuous boundary $\partial \Omega$. The Stokes equations are presented as follows

\[
\begin{align*}
-\nu \Delta u + \nabla p &= f & \text{in } \Omega, \\
\text{div } u &= 0 & \text{in } \Omega, \\
u u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

\(1\) \hspace{2cm} \(2\) \hspace{2cm} \(3\)
where $u = (u_1, u_2, u_d)$ represents the velocity vector, $p$ the pressure, $f$ the prescribed body force, and $v > 0$ the viscosity.

For convenience, set

$$
X = [H_0^1(\Omega)]^d, \quad Y = [L^2(\Omega)]^d, \quad M = \left\{ q \in L^2(\Omega) : \int_\Omega q \, dx = 0 \right\},
$$

$$
D(A) = [H^2(\Omega)]^d \cap X.
$$

The spaces $[L^2(\Omega)]^m, m = 1, 2, \text{or} 4$, are endowed with the $L^2$-scalar product $(\cdot, \cdot)$ and the $L^2$-norm $\| \cdot \|_0$, as appropriate. The space $X$ is equipped with the usual scalar product $(\nabla u, \nabla v)$ and the norm $\| \cdot \|_1$. Note that the norm equivalence between $\| \cdot \|_1$ and $\| \nabla u \|_0$ on $H_0^1(\Omega)$, we use the same notation for them. In fact, standard definitions are used for the Sobolev spaces $W^{m,r}(\Omega)$, with the norm $\| \cdot \|_{m,r}$ and the seminorm $| \cdot |_{m,r}$, $m, r \geq 0$. We will write $H^m(\Omega)$ for $W^{m,2}(\Omega)$ and $\| \cdot \|_m$ for $\| \cdot \|_{m,2}$.

Then, the weak formulation of (1)-(3) is to seek $(u, p) \in X \times M$ such that

$$
B((u, p); (v, q)) = (f, v) \quad \forall (v, q) \in X \times M, \quad (4)
$$

where

$$
B((u, p); (v, q)) = a(u, v) - d(v, p) - d(u, q)
$$

with

$$
a(u, v) = (-\Delta u, v) = (\nabla u, \nabla v),
$$

and

$$
d(v, p) = (\text{div} \ v, p).
$$

Clearly, the bilinear forms $a(\cdot, \cdot)$ and $d(\cdot, \cdot)$ are continuous on $X \times X$ and $X \times M$, respectively. Also, the bilinear form $d(\cdot, \cdot)$ satisfies the inf-sup condition [10, 13]

$$
\sup_{0 \neq q \in X} \frac{|d(v, q)|}{\| v \|_1} \geq \beta \| q \|_0, \quad (5)
$$

where $\beta$ is a positive constant depending only on $\Omega$.

The well-posedness of the model problem (1)-(3) follows from the results of the saddle-point problem. Assume that the domain $\Omega$ is so regular that ensures a $H^2$-regularity for the solution of (4), namely, the problem (1)-(3) has a unique solution $(u, p) \in D(A) \times H^1(\Omega)$ satisfying the following a priori estimate

$$
\| u \|_2 + \| p \|_1 \leq C \| f \|_0, \quad (6)
$$

where $C$ is a constant depending on $\Omega$. Subsequently, the constant $C > 0$ (with or without a subscript) will depend only on the data $(v, \Omega, f)$. 


3 The nonconforming finite element

Let $\mathcal{T}_h$ be a regular triangulation of $\Omega$ into elements $\{K_j\}$: $\Omega = \bigcup K_j$. Denote a boundary segment and an interior boundary by $\gamma_j = \partial \Omega \cap \partial K_j$ and $\gamma_{jk} = \partial K_j \cap \partial K_k$, respectively. The centers of $\gamma_j$ and $\gamma_{jk}$ are indicated by $\xi_j$ and $\xi_{jk}$, respectively. The finite element spaces are the following nonconforming and conforming finite elements for velocity and pressure:

$$\text{NCP}_1 = \{v \in Y : v|_K \in [P_1(K)]^d, v(\xi_{jk}) = v(\xi_j), v(\xi_{j}) = 0 \ \forall j, k, K \in \mathcal{T}_h\},$$

$$P_1 = \{q \in H^1(\Omega) \cap M : q|_K \in P_1(K) \ \forall K \in \mathcal{T}_h\}.$$

We will also use the piecewise constant spaces $P_0$:

$$P_0 = \{q \in M : q|_K \in P_0(K) \ \forall K \in \mathcal{T}_h\}.$$

Note that the nonconforming finite element space $\text{NCP}_1$ is not a subspace of $X$ any more.

Define the energy norm

$$\|v\|_{1,h} = \left(\sum_j |v|_{1,K_j}^2\right)^{1/2}, \ \forall v \in \text{NCP}_1.$$

The two finite element spaces $\text{NCP}_1$ and $P_1$ satisfy the approximation property: For $(v, q) \in [H^2(\Omega)]^d \times H^1(\Omega)$ there are two approximations $v_I \in \text{NCP}_1$ and $q_I \in P_1$ such that

$$\|v - v_I\|_0 + h(\|v - v_I\|_{1,h} + \|q - q_I\|_0) \leq C h^2 (\|v\|_2 + \|q\|_1). \quad (7)$$

Note that the following compatibility conditions hold for all $j$ and $k$:

$$\int_{\gamma_{jk}} [v] ds = 0 \ \forall v \in \text{NCP}_1 \quad (8)$$

and

$$\int_{\gamma_j} [v] ds = 0 \ \forall v \in \text{NCP}_1, \quad (9)$$

where $[v] = v|_{\gamma_{jk}} - v|_{\gamma_j}$ denotes the jump of the function $v$ across $\gamma_{jk}$. These conditions also hold for the rotated $Q_1$ space with the mean integral values as the degrees of freedom.

3.1 The weak formulation

Set $(\cdot, \cdot)_j = (\cdot, \cdot)_{K_j}$, $(\cdot, \cdot)_{\partial K_j}$, and $|\cdot|_{m,j} = |\cdot|_{m,K_j}$. Then the discrete bilinear forms are given as follows:

$$a_h(u,v) = \sum_j (\nabla u, \nabla v)_j, \quad d_h(v,q) = \sum_j (\text{div} v, q)_j,$$

$$u|_j, v|_j \in [H^1(K_j)]^d, \ q \in L^2(\Omega). \quad (20.10)$$

Below we mention a few pairs of mixed finite element spaces for the Stokes equations. Earlier, the lowest-order Crouzeix-Raviart element using a nonconforming piecewise linear velocity and a piecewise
constant pressure was constructed [12], which was extended to a nonconforming piecewise bilinear velocity
and a piecewise constant pressure in [4]. Recently, we proposed the pressure projection method of noncon-
forming finite element method for the Stokes equations approximated by the Crouzeix-Raviart element and
piecewise linear element for velocity and pressure, respectively.

In general, the number of degrees of freedom for velocity should be larger than that for pressure. The
NCP $- P_0$ pair is shown to be stable. Is optimal mixed finite element space for the possible choice NCP $- 
P_1$ stable? It is still under develop.

As noted, we can not recognize the stable of the finite element pair until [11] have given us a hint related
to the choice NCP $- P_1$. Obviously, the bilinear form $a_h(\cdot, \cdot)$ and $d_h(\cdot, \cdot)$ are continuous and coercive with
respect to broken-norm:

$$|a_h(u_h, v_h)| \leq C \|u_h\|_{1,h} \|v_h\|_{1,h},$$  \hspace{1cm} (10)

$$a_h(v_h, v_h) \geq C \|v_h\|_{1,h}^2.$$  \hspace{1cm} (11)

For completeness, we will prove the inf-sup property which is important for the incompressible flow.

Finally, the discrete weak formulation of the Stokes equations (2.1)–(2.3) is to find $(u_h, p_h) \in \text{NCP}_1 \times \mathbb{P}_1$
such that

$$\mathcal{N}_h((u_h, p_h); (v_h, q_h)) = (f, v_h) \quad \forall (v_h, q_h) \in \text{NCP}_1 \times \mathbb{P}_1,$$  \hspace{1cm} (12)

where

$$\mathcal{N}_h((u_h, p_h); (v_h, q_h)) = a_h(u_h, v_h) - d_h(v_h, p_h) - d_h(u_h, q_h)$$
is the bilinear form on $(\text{NCP}_1, \mathbb{P}_1) \times (\text{NCP}_1, \mathbb{P}_1)$. In the next two sections, we will study (19) in terms of
stability, and existence and uniqueness.

### 3.2 Stability

In this section, we will study the stability of the nonconforming finite element method (19) for the Stokes
equations. The main result of this subsection is the existence and uniqueness of the nonconforming finite
element solution. The proof of this theorem is based on the inf-sup property that is proven in the following
lemma by adapting to a classical argument [11]. For completeness, we provide a detailed proof as follows.

**Theorem 3.1.** There exists a strictly positive constant $\beta > 0$ independent of $h$ such that for every
$q_h \in \mathbb{P}_1$, there exists a vector $v_h \in \text{NCP}_1$ such that

$$\sup_{v_h \in \mathcal{N}_h} \frac{d_h(v_h, q_h)}{\|v_h\|_{1,h}} \geq \beta \|q_h\|_0.$$  \hspace{1cm} (13)

**Proof.** First, we set a auxiliary space $\mathcal{R}_h = \{q_h \in Q_h : q_h = \sum_{i=1}^{N} q_i \chi_i \}$ where $\chi_i$
is a characteristic function of the support $S_i$ of the standard linear basis function $\varphi_i \in \mathbb{P}_1$ associated with the vertex $x_i, i = 1, 2, \cdots N$ and
$Q_h = \{q_h \in M : q_h|_K \in P_0(K), \int_{\Omega} q_h dx = 0 \}$. The relevant finite element space is also defined as follows

$$M_h \equiv \{q_h \in M : q_h|_K \in P_1(K), K \in \mathcal{T}_h \}.$$
Furthermore, we define an interpolation operator $I_h : M_1^1 \rightarrow R_h$ by

$$I_h q_h = \sum_{i=1}^{N} q_i \chi_i$$

with $q_h = \sum_{i=1}^{N} q_i \phi_i \in P_1$. Then, setting $\text{div}_h$ the divergence operator on each element, we observe that

$$\int_{\Omega} \text{div}_h I_h q_h dx = \int_{\Omega} \text{div}_h q_h dx = \sum_{i=1}^{N} q_i \int_{S_i} \text{div}_h v_h dx = \sum_{i=1}^{N} q_i \sum_{K \subset S_i} |K| \text{div}_h v_h.$$  \hfill (14)

On the other hand, letting $P_i$ be the barycentre of the element $K$,

$$d_h(v_h, q_h) = \int_{\Omega} \text{div}_h \left( \sum_{i=1}^{N} q_i \phi_i \right) dx = \sum_{i=1}^{N} q_i \text{div}_h v_h \sum_{K \subset S_i} \int_{K} \phi_i dx = \frac{1}{d+1} \sum_{i=1}^{N} q_i \text{div}_h v_h \sum_{K \subset S_i} |K|,$$  \hfill (15)

which together with (14) yields

$$d_h(v_h, q_h) = \frac{1}{d+1} \int_{\Omega} \text{div}_h v_h I_h q_h dx = \frac{1}{d+1} d_h(v_h, I_h q_h).$$  \hfill (16)

Then, noting that the lower order Crouzeix-Raviart element is stable, namely,

$$\sup_{v_h \in \mathbb{NCP}_1} \frac{d_h(v_h, I_h q_h)}{\|v_h\|_{1,h}} \geq \beta_1 \|q_h\|_0,$$  \hfill (17)

where $\beta_1 > 0$ only depends on $\Omega$, we can obtain that

$$\sup_{v_h \in \mathbb{NCP}_1} \frac{d_h(v_h, q_h)}{\|v_h\|_{1,h}} = \sup_{v_h \in \mathbb{NCP}_1} \frac{1}{d+1} \frac{d_h(v_h, I_h q_h)}{\|v_h\|_{1,h}} \geq \frac{\beta_1}{d+1} \|q_h\|_0 \geq \beta \|q_h\|_0,$$  \hfill (18)

where $\beta = \frac{\beta_1}{d+1}$. #

Furthermore, we can obtain the following result.
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Theorem 3.2. The bilinear form $\mathcal{B}_h((\cdot,\cdot);(\cdot,\cdot))$ satisfies the continuous property

$$\left|\mathcal{B}_h((u_h,p_h);(v_h,q_h))\right| \leq C(\|u_h\|_{1,h} + \|p_h\|_0)(\|v_h\|_{1,h} + \|q_h\|_0), \quad (u_h,p_h), (v_h,q_h) \in \text{NCP}_1 \times \mathbb{P}_1 \quad (19)$$

and the coercive property

$$\sup_{0 \neq (v_h,q_h) \in \text{NCP}_1 \times \mathbb{P}_1} \frac{\left|\mathcal{B}_h((u_h,p_h);(v_h,q_h))\right|}{\|v_h\|_{1,h} + \|q_h\|_0} \geq \beta^* (\|u_h\|_{1,h} + \|p_h\|_0), \quad (u_h,p_h) \in \text{NCP}_1 \times \mathbb{P}_1, \quad (20)$$

where $\beta > 0$ only depends on $\Omega$.

Proof. Using the continuous property of the bilinear forms $a_h(\cdot,\cdot)$ and $d_h(\cdot,\cdot)$, we can easily obtain the continuous property of $\mathcal{B}_h((\cdot,\cdot);(\cdot,\cdot))$.

As for the weakly coercivity of $\mathcal{B}_h((\cdot,\cdot);(\cdot,\cdot))$, there exists a positive constant $C_0$ and $w \in X$ for all $p_h \in \mathbb{P}_1 \subset M$, such that

$$(\text{div}w, p_h)_j = \|p_h\|_{0,j}^2, \quad \|w\|_{1,j} \leq C_0 \|p_h\|_{0,j}. \quad (21)$$

Setting the finite element approximation $w_h \in X_h$ of $w$, we have

$$\|w_h\|_{1,h} \leq C_1 \|p_h\|_0. \quad (22)$$

First, taking $(v_h,q_h) = (u_h - \alpha w_h,-p_h)$ for some positive constant $\frac{1}{\sqrt{C_0}} > \alpha$ yet to be determined in the bilinear term $\mathcal{B}_h((\cdot,\cdot);(\cdot,\cdot))$ to obtain

$$\mathcal{B}_h((u_h,p_h);(u_h - \alpha w_h,q_h)) = a_h(u_h,u_h - \alpha w_h) - d(u_h - \alpha w_h,p_h) + d(u_h,p_h)$$
$$= a(u_h,u_h) + \alpha d_h(w_h,p_h) - \alpha a_h(u_h,w_h)$$
$$= v\|u_h\|_{1,h}^2 + \alpha \|p_h\|_0^2 - \alpha a_h(u_h,w_h)$$
$$\geq \frac{v}{2} \|u_h\|_{1,h}^2 + \alpha \left(1 - \frac{vC_0^2\alpha}{2}\right) \|p_h\|_0^2$$
$$\geq \frac{v}{2} \|u_h\|_{1,h}^2 + \frac{\alpha}{2} \|p_h\|_0^2 \quad (23)$$

since

$$\alpha a_h(u_h,w_h) \leq v\alpha \|u_h\|_{1,h} \|w_h\|_{1,h} \leq v\alpha C_0 \|u_h\|_{1,h} \|p_h\|_0$$
$$\leq \frac{v}{2} \|u_h\|_{1,h}^2 + \frac{vC_0^2\alpha^2}{2} \|p_h\|_0^2. \quad (24)$$

Using a triangle inequality to obtain

$$\|u_h - \alpha w_h\|_{1,h} + \|p_h\|_0 \leq C_5 (\|u_h\|_{1,h} + \|p_h\|_0). \quad (25)$$

Setting $\beta^* = C_2C_3^{-1}$ and choosing $C_2 = \min\left\{\frac{v}{2}, \frac{\alpha}{2}\right\}$, we have

$$\mathcal{B}_h((u_h,p_h);(u_h - \alpha w_h,q_h)) \geq C_2 (\|u_h\|_{1,h}^2 + \|p_h\|_0^2), \quad (26)$$
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which together with (25) yields the following

\[
\sup_{(v_h, q_h) \in \mathbb{NCP}_1 \times P_1} \frac{\mathcal{R}_h((u_h, p_h); (v_h, q_h))}{\|v_h\|_{1,h} + \|q_h\|_0} \geq \frac{\mathcal{R}_h((u_h, p_h); (u_h - \alpha w_h, p_h))}{\|u_h - \alpha w_h\|_{1,h} + \|p_h\|_0} \geq C_2 C_3^{-1} (\|u_h\|_{1,h} + \|p_h\|_0) = \beta^* (\|u_h\|_{1,h} + \|p_h\|_0).
\]  

(27)

3.3 The well-posedness

Based on previous results, we can derive the existence and uniqueness of the nonconforming element solu-

tion for the Stokes equations.

**Theorem 5.2.** Under the assumptions of Theorem 3.2, the problem (19) admit s a unique solution.

4 Estimate in energy norm

In this section, we will derive optimal order error bounds for the Stokes equations. The non-conformity

error is controlled as in the following theorem, which is similar to Strange’s lemma in nonconforming finite

element version for the second order elliptic problem [6]. The proof of this lemma requires a bound on the

nonconforming error. For completeness, this bound is provided as follows.

**Lemma 4.1.** (Strange’s lemma for the Stokes equations) There exists a constant \( C > 0 \) depending

only on the coercivity and the continuity constants such that

\[
\sup_{w_h \in \mathbb{NCP}_1} \frac{|a(u, w_h) - d(w_h, p) - (f, w_h)|}{\|w_h\|_{1,h}} \leq C h (\|u\|_2 + \|p\|_1).
\]

(28)

**Proof.** By the definition of \( a(\cdot, \cdot) \) and \( d(\cdot, \cdot) \), it follows that

\[
a(u, w_h) = \sum_j (\nabla u, \nabla w_h)_j, \forall w_h \in \mathbb{NCP}_1
\]

\[
= \sum_j [-(\Delta u, w_h)_j + <\nabla u, [w_h] \cdot n>_j]
\]

(29)

and

\[
d(w_h, p) = \sum_j (\text{div} w_h, p)_j, \forall w_h \in \mathbb{NCP}_1
\]

\[
= \sum_j [-(\nabla p, w_h)_j + <p, [w_h] \cdot n>_j].
\]

(30)

Recalling that \( \bar{w}_h = \frac{1}{|\Omega|} \int_\Omega w_h ds \) defined above satisfying

\[
\int_{\partial K} (w_h - \bar{w}_h) ds = 0,
\]

\[
\|w_h - \bar{w}_h\|_{0,\Omega} \leq C h^{1/2} \|w_h\|_{1,K},
\]

(31) (32)
and noting that a constant and each interior edge appears twice in the sum of formulation, we can obtain that

\[ a(u, w_h) - d(w_h, p) - (f, w_h) = \sum_j <\nabla u + p, [w_h] \cdot n > j \]

\[ = \sum_j <\nabla u + p, ([w_h] - [\tilde{w}_h]) \cdot n > j. \]  \hspace{1cm} (33)

Recalling the definition of \( P^K_w h = \frac{1}{|K|} \int_{\partial K} w_k ds \) satisfying

\[ \int_{\partial K} (w_h - \tilde{P}^K_w h) ds = 0, \]

\[ ||w_h - \tilde{P}^K_w h||_{0,x} \leq Ch^{1/2} ||w_h||_1, \]  \hspace{1cm} (34)

we can obtain

\[ a(u, w_h) - d(w_h, p) - (f, w_h) = \sum_j <\nabla u + p, [w_h] \cdot n > j \]

\[ = \sum_j <(\nabla u + p) - \tilde{P}^K_w (\nabla u + p), ([w_h] - [\tilde{w}_h]) \cdot n > j \]

\[ \leq Ch(||u||_2 + ||p||_1). \]  \hspace{1cm} (36)

Thus, we can achieve the desired result.

**Theorem 4.2.** Under the assumption of Theorems 3.1-3.2, we can obtain that

\[ ||u_h - v_h||_{1,h} + ||p - p_h||_0 \leq Ch(||u||_2 + ||p||_1). \]  \hspace{1cm} (37)

**Proof.** First, multiplying (1) by \( v_h \in X_h \), integrating over \( \Omega \) and applying the Green formula we have

\[ a(u, v_h) - d(v_h, p) - \sum_K <\nabla u + p, [v_h] \cdot n >= (f, v_h). \]  \hspace{1cm} (38)

Using the same approach as for lemma 4.1 and setting \((e_h, \eta_h) = (u_l - u_h, p_1 - p_h)\), we find that

\[ \eta_h ||v_h||_{1,h} + ||q_h||_0 \]

\[ = \frac{\eta_h \text{Br}((e_h, \eta_h); (v_h, q_h))}{||v_h||_{1,h} + ||q_h||_0} \]

\[ = \frac{\text{Br}((u_l - u_h, p_l - p_h); (v_h, q_h))}{||v_h||_{1,h} + ||q_h||_0} + \sum_j <\nabla u + p, [v_h] \cdot n > j \]

\[ \leq C(||u - u_l||_{1,h} + ||p - p_l||_0) + \sum_j <\nabla u + p, [v_h] \cdot n > j \]

\[ \leq Ch(||u||_2 + ||p||_1). \]  \hspace{1cm} (39)

Then, we have

\[ ||e_h||_{1,h} + ||\eta_h||_0 \leq \frac{1}{\text{Br}((e_h, \eta_h); (v_h, q_h))} \]

\[ \leq Ch(||u||_2 + ||p||_1). \]  \hspace{1cm} (40)

Thus, using a triangle inequality and (7) to obtain (45). #
5 Estimate in $L^2$-norm

The velocity in $L^2$-norm for the nonconforming element method is here analyzed in the same way as it is done for the classical nonconforming methods. Firstly, we consider the dual problem: Find $(\Phi, \Psi) \in [H^2(\Omega) \cap X]^d \times L^2_0(\Omega)$ such that

\begin{align}
-\Delta \Phi + \nabla \Psi &= u - u_h \quad \text{in } \Omega, \\
\text{div } \Phi &= 0 \quad \text{in } \Omega, \\
\Phi|_{\partial \Omega} &= 0 \quad \text{on } \partial \Omega.
\end{align}

(41) \quad (42) \quad (43)

Because of the convexity of the domain $\Omega$, this problem has a unique solution that satisfies the regularity property

$$\|\Phi\|_2 + \|\Psi\|_1 \leq C\|u - u_h\|_0.$$  \hspace{1cm} (44)

**Theorem 5.1.** Under the assumption of Theorems 4.2, we can obtain that

$$\|u - u_h\|_0 \leq C h^2 (\|u\|_2 + \|p\|_1).$$  \hspace{1cm} (45)

**Proof.** Multiplying (41) and (42) by $e = u - u_h$ and $\eta = p - p_h$, respectively, integrating over $\Omega$, to obtain that

$$\|e\|_0^2 = -\nu \int_{\Omega} \Delta \Phi dx + \int_{\Omega} \nabla \Psi dx - \int_{\Omega} \text{div } \eta dx.$$  \hspace{1cm} (46)

Simplified to gives the following

$$\|e\|_0^2 = a_h(e, \Phi) - d_h(e, \Psi) - d_h(\Phi, \eta) - \sum_j < \frac{\partial \Phi}{\partial n}, e >_j + \sum_j < e \cdot n, \Psi >_j$$

$$= a_h(e, \Phi - \Phi_I) - d_h(e, \Psi - \Psi_I) - d_h(\Phi - \Phi_I, \eta) - \sum_j < \frac{\partial \Phi}{\partial n}, e >_j + \sum_j < e \cdot n, \Psi >_j$$

$$+ a_h(e, \Phi_I) - d_h(e, \Psi_I) - d_h(\Phi_I, \eta).$$

The difference of (1) and (19) tested against $v_h = \Phi_I$, implies that

$$a_h(e, \Phi_I) - d_h(e, \Psi_I) - d_h(\Phi_I, \eta) = \sum_j < \nabla u + p, [\Phi_I] \cdot n >_j.$$  \hspace{1cm} (49)

Thus,

$$\|e\|_0^2 = a_h(e, \Phi - \Phi_I) - d_h(e, \Psi - \Psi_I) - d_h(\Phi - \Phi_I, \eta)$$

$$- \sum_j < \frac{\partial \Phi}{\partial n}, e >_j + \sum_j < e \cdot n, \Psi >_j$$

$$+ \sum_j < \frac{\partial u}{\partial n}, \Phi_I >_j + \sum_j < u \cdot n, \Phi_I >_j$$

$$= E_1 + E_2 + E_3.$$  \hspace{1cm} (47)
Here,
\[
|E_1| \leq Ch(\|e\|_{1,h} + \|\eta\|_0)(\|\Phi\|_1 + \|\Psi\|_1) \\
\leq Ch^2(\|\Phi\|_2 + \|\Psi\|_1) \leq Ch^2\|e\|_0. \tag{48}
\]
Using the same approach as Lemma 4.1, yields
\[
\left| - \sum_j < \frac{\partial \Phi}{\partial n}, e >_j + \sum_j < e \cdot n, \Psi >_j \right| \leq Ch^2(\|\Phi\|_2 + \|\Psi\|_1) \leq Ch^2\|e\|_0. \tag{49}
\]
\[
\left| \sum_j < \frac{\partial u}{\partial n}, \Phi_I >_j + \sum_j < u \cdot n, \Psi_I >_j \right| \leq Ch^2(\|\Phi\|_2 + \|\Psi\|_1) \leq Ch^2\|e\|_0. \tag{50}
\]
Combining (47) with (48)-(50), and using (44) and a triangle inequality, yields (46)

6 Numerical analysis

This section concentrates on the performance of the nonconforming finite element method approximated by the Crouzeix-Raviart element and continuous linear element for the incompressible Stokes equations. We compare the present method with the stable Crouzeix-Raviart element/piecewise constant element [6, 12], the pressure projection stabilization finite element method approximated by the Crouzeix-Raviart element/continuous linear element and piecewise linear element/piecewise linear element for the incompressible Stokes equations [15, 16].

In order to illustrate the features of the present method, three test problems are considered to verify the performance of the present method including a nonphysical example with a known exact solutions, the driven cavity flow and a flow over a cylinder.

**Problem I (nonphysical example with analytical solution).** In this case, we consider a unit square with an exact flow solution given by
\[
\begin{align*}
\mathbf{u}(x) &= (u_1(x_1, x_2), u_2(x_1, x_2)), \quad p(x_1, x_2) = \cos(\pi x_1) \cos(\pi x_2), \\
\mathbf{u}_1(x_1, x_2) &= 2\pi \sin^2(\pi x_1) \sin(\pi x_2) \cos(\pi x_1), \quad u_2(x_1, x_2) = -2\pi \sin(\pi x_1) \sin(\pi x_2)^2 \cos(\pi x_1).
\end{align*}
\]
Then, the body force \(f(x, t)\) is deduced from the exact solution and (1). We here pay more attention to convergence rate of four different methods with the same mesh and the same UMFPACK code. The results in tables 1-4 suggest that there are no significant differences between three different nonconforming finite element methods in terms of the relative \(H^1\)- and \(L^2\)-norms for velocity. Obviously, the present method is more efficient than other methods by comparison. Especially, the \(P_{1nc} - P_1\) and the stabilized \(P_{1nc} - P_1\) schemes have almost achieve the same superconvergence rate \(O(h^2)\) for pressure. However, the latter did not improve on the accuracy of the stabilized schemes whilst being significantly more expensive.
Table 1. The standard Galerkin method for the Crouzeix-Raviart element.

| $1/h$ | $\|u - u_h\|_0$ | $\|u - u_h\|_1,h$ | $\|p - p_h\|_0$ | $L_2$rate | $H_1$rate | $pL_2$rate |
|-------|------------------|------------------|------------------|---------|---------|-----------|
| 10    | 0.151784         | 0.470627         | 0.129512         |         |         |           |
| 20    | 0.0426512        | 0.248179         | 0.0571999        | 1.831361482 | 0.923203046 | 1.179001248 |
| 30    | 0.0194144        | 0.167253         | 0.0366254        | 1.941080342 | 0.97330847 | 1.099503145 |
| 40    | 0.0110149        | 0.125928         | 0.0270155        | 1.981967072 | 0.99184018 | 1.037124587 |
| 50    | 0.00707796       | 0.100926         | 0.0214341        | 1.987917211 | 0.99457037 | 1.025685073 |
| 60    | 0.00492609       | 0.0841883        | 0.0177783        | 1.987917211 | 0.99457037 | 1.025685073 |

Table 2. The standard Galerkin method for the $P_{1nc} - P_{1}$ pair.

| $1/h$ | $\|u - u_h\|_0$ | $\|u - u_h\|_1,h$ | $\|p - p_h\|_0$ | $L_2$rate | $H_1$rate | $pL_2$rate |
|-------|-----------------|-----------------|-----------------|---------|---------|-----------|
| 10    | 0.152528        | 0.474482        | 0.0761601       |         |         |           |
| 20    | 0.0428488       | 0.250109        | 0.0224344       | 1.8317474 | 0.923796385 | 1.763322773 |
| 30    | 0.0195085       | 0.168554        | 0.010413        | 1.940555072 | 0.973303601 | 1.892987718 |
| 40    | 0.0110695       | 0.126911        | 0.00597363      | 1.969732367 | 0.98640182 | 1.931647106 |
| 50    | 0.00711356      | 0.101716        | 0.00386744      | 1.981642554 | 0.99174471 | 1.948351144 |
| 60    | 0.00495107      | 0.0848496       | 0.00270709      | 1.987692039 | 0.99442074 | 1.956355138 |

Table 3. The stabilized nonconforming finite method for the $P_{1nc} - P_{1}$ pair.

| $1/h$ | $\|u - u_h\|_0$ | $\|u - u_h\|_1,h$ | $\|p - p_h\|_0$ | $L_2$rate | $H_1$rate | $pL_2$rate |
|-------|-----------------|-----------------|-----------------|---------|---------|-----------|
| 10    | 0.153018        | 0.474483        | 0.0764451       |         |         |           |
| 20    | 0.0429933       | 0.250109        | 0.0225213       | 1.831517616 | 0.923799426 | 1.763133924 |
| 30    | 0.0195752       | 0.168554        | 0.0104535       | 1.940440287 | 0.973303601 | 1.89294877 |
| 40    | 0.0111076       | 0.126911        | 0.00599673      | 1.96965315 | 0.98640182 | 1.931724578 |
| 50    | 0.00713812      | 0.101716        | 0.00388229      | 1.981594896 | 0.99174471 | 1.948472776 |
| 60    | 0.0049682       | 0.0848496       | 0.00271736      | 1.987652153 | 0.99442074 | 1.95678522 |

Table 4. The stabilized finite method for the $P_{1} - P_{1}$ pair.

| $1/h$ | $\|u - u_h\|_0$ | $\|u - u_h\|_1,h$ | $\|p - p_h\|_0$ | $L_2$rate | $H_1$rate | $pL_2$rate |
|-------|-----------------|-----------------|-----------------|---------|---------|-----------|
| 10    | 0.0884909       | 0.269308        | 0.203893        |         |         |           |
| 20    | 0.0206359       | 0.130988        | 0.0961777       | 2.100372742 | 1.039822436 | 1.084037915 |
| 30    | 0.0087527       | 0.086592        | 0.0544149       | 2.080976891 | 1.020679148 | 1.40470588 |
| 40    | 0.00490909      | 0.0647394       | 0.03482142      | 2.0584535 | 1.011136883 | 1.54967603 |
| 50    | 0.00311108      | 0.051703        | 0.02443922      | 2.04405932 | 1.00766018 | 1.589281981 |
| 60    | 0.00214673      | 0.0430371       | 0.018301        | 2.034998961 | 1.006207251 | 1.586387822 |
Problem II (The driven cavity flow). The driven cavity is considered for the four different methods. It is a box full of liquid with its lid moving horizontally at speed one. The results for both velocity and pressure are given in Figures 1-2. Numerical result of the present method shows the same performance as that of other methods.

Problem III (The exterior of a 2d cylinder). We build a computation mesh the exterior of a 2d cylinder. A fluid recirculation zone produced by the hole must be captured correctly.

The geometry for the numerical model of the problem are given in Figure 3. Also, the Diriclet boundary conditions is designed for this model and $u_1, u_2$ and $p$ denote the velocity components in $x$ and $y$ direction and the pressure. Simulations have been performed with the given viscosity $\nu = 1$. Here, a set of sample results is given in Figure 3. In order to verify the correctness of the method, a comparison of the results with the standard Taylor-Hood element shows that the present method is creditable.

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