Research Article

Hermite–Hadamard and Jensen-Type Inequalities via Riemann Integral Operator for a Generalized Class of Godunova–Levin Functions

Xiaoju Zhang,1 Khurram Shabbir2, Waqar Afzal,2 He Xiao,1 and Dong Lin3

1Xi’an Traffic Engineering Institute, Xi’an, Shaanxi 710300, China
2Department of Mathematics, GC University, Lahore, Pakistan
3Scientific Research Department, Xijing University, Xi’an, Shaanxi 710123, China

Correspondence should be addressed to Dong Lin; lindong@xijing.edu.cn

Received 14 June 2022; Accepted 20 July 2022; Published 31 August 2022

Academic Editor: Xiaolong Qin

Copyright © 2022 Xiaoju Zhang et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The generalization of Godunova–Levin interval-valued functions has been drastically studied in last few decades, as it has a remarkable applications in both pure and applied mathematics. The goal of this study is to introduce the notion of h-Godunova–Levin interval-valued functions. We establish Hermite–Hadamard and Jensen-type inequalities via Riemann integral operator.

1. Introduction

In last few years, the inequality theory gained the attention of many researchers working in analysis and other branches of mathematics [1–3]. Most of the real world problems may be viewed as integral equations. So, the generalization of integral inequalities is always appreciable and is more closer to applied problems[4, 5].

In the renowned celebrated book by Moore, the interactive analysis of numerical data starts the introduction into interval analysis in numerical analysis, see [6]. A tremendous number of applications have been developed over the past 50 years in areas including computer graphics [7], aeroelasticity [8], interval differential equations [9], and neural network optimization [10]. Numerous various integral inequalities have been investigated recently by different authors in the context of interval-valued functions (see [11, 12]).

It is well known that the convexity of functions plays an extremely important role in mathematics and other scientific fields such as economics, probability theory, and optimal control theory; moreover, several inequalities have been recorded in the literature (see [13–17]). In equality, the following inequality is called a classical Hermite–Hadamard inequality:

\[
φ\left(\frac{v + w}{2}\right) \leq \frac{1}{v - w} \int_v^w φ(μ) dμ \leq \frac{φ(v) + φ(w)}{2},
\]

where \(φ: S \subseteq R \rightarrow R\) is a convex on interval \(S\) and \(v, w \in S\) with \(v < w\). In the context of different generalizations and extensions of this inequality (see [16, 18]), the notion of \(h\)-convex was originally developed by Varošanec in 2007 (see [19]). Several authors have developed more sophisticated Hermite–Hadamard inequalities that include \(h\)-convex functions (see [20, 21]). Furthermore, Costa presented an inequality of the Jensen type for fuzzy interval-valued functions (see [22]). As well, Zhao et al. provide a new Hermite–Hadamard inequality for \(h\)-convex functions in the context of interval-valued functions (see [23]).

The following inequality was proved in 2019 by Almutairi and Kiliman using the \(h\)-Godunova-Levin function (see [24]).

**Theorem 1.** Let \(φ: [v, w] \rightarrow R\). If \(φ\) is \(h\)-Godunova–Levin convex function and \(h(1/2) \neq 0\). Then,
\[
\frac{h(1/2)}{2} \phi\left(\frac{\nu + \omega}{2}\right) \leq \frac{1}{w - \nu} \int_{\nu}^{w} \phi(\mu) d\mu
\leq [\phi(\nu) + \phi(\omega)] \int_{0}^{1} \frac{dx}{h(x)}
\]  
(2)

Motivated by Costa [22], Zhao et al. [23], Dragomir [25], and Almutairi and Kiliman [24], we introduce and explore the notion of \(h\)-Godunova–Levin interval-valued functions. Our new concept allowed us to develop fractional version of Hermite–Hadamard and Jensen-type inequalities via Riemann integral operator.

### 2. Preliminaries

In this study, we will review some fundamental definitions, properties, and notations. Let us say \(I\) is the collection of all intervals of \(R\), \([v, w]\) \ \(\in I\) is defined as follows:

\([v, w] = [x \in R | v \leq x \leq w]\), where real interval \([v, w]\) is closed and bounded subset of \(R\). The interval \([v, w]\) is said to be degenerated when \(v = w\) and the set of all negative and positive intervals are represented as \(R^+_\) and \(R^-\), respectively. The inclusion \(\subseteq\) is defined as

\([v, w] \subseteq [w, \overline{w}] \iff w \leq v, v \leq w\).

For any random real number \(\mu\) and \([v, w]\), the interval \(\mu [v, w]\) is given as

\[\mu \cdot [v, w] = \begin{cases} 
\mu [v, w], & \text{if } \mu > 0, \\
[v, w], & \text{if } \mu = 0, \\
[w, v], & \text{if } \mu < 0.
\end{cases}
\]

For \([v, w] = [y, \overline{y}]\) and \([w, w] = [w, \overline{w}]\), algebraic operations are defined as

\([v] + [w] = [y + w, v + w], \\
[v] - [w] = [y - w, v - w], \\
[v] - [w] = [v - w, v - w], \\
[v] \cdot [w] = [\min\{vw, yw, v\overline{w}, \overline{v}w\}], \max\{vw, yw, v\overline{w}, \overline{v}w\}],
\]

\[\frac{[v]}{[w]} = \left\{ \begin{array}{ll}
\min\left\{\frac{y}{w}, \frac{y}{w}, \frac{v}{w}, \frac{v}{w}\right\}, & \text{if } \frac{v}{w} > 0, \\
\frac{\overline{v}}{\overline{w}} & \text{if } \frac{v}{w} = 0, \\
\frac{v}{w}, & \text{if } \frac{v}{w} < 0.
\end{array} \right.
\]

where

\[0 \notin [y, \overline{y}].\]

For intervals \([y, \overline{y}], [y, \overline{w}]\) the Hausdorff–Pompeiu distance is defined as

\[d\left([y, \overline{y}], [w, \overline{w}]\right) = \max\{|v - w|, |\overline{v} - \overline{w}|\}.
\]

It is well known that the entire \((R_I, d)\) is complete metric space.

### 3. Main Results

Now, we are ready to introduce the notion of interval-valued \(h\)-Godunova–Levin convex functions.
Definition 6. Let \( h: (0, 1) \subseteq M \rightarrow R \) be a nonnegative function. We say \( \varphi: M \rightarrow R \) is the interval \( h \)-Godunova–Levin convex function, or \( \varphi \in SGX \left( \left( 1/1, h \right), M, R_1^* \right) \); if, for all \( v, w \in M \) and \( \mu \in (0, 1) \), we have

\[
\frac{\varphi(v)}{h(\mu)} + \frac{\varphi(w)}{h(1-\mu)} \leq \varphi(\mu v + (1-\mu)w). \tag{13}
\]

If the above inequality is inverted, it is referred to interval \( h \)-Godunova–Levin concave functions are denoted by \( \varphi \in SGX \left( \left( 1/1, h \right), M, R_1^* \right) \) and \( \varphi \in SGX \left( \left( 1/1, h \right), M, R_1^* \right) \), respectively.

Proposition 1. Let \( \varphi: [v, w] \rightarrow R_1^* \) be \( h \)-Godunova–Levin convex interval-valued function such that \( \varphi(\mu) = [\varphi(\mu), \overline{\varphi}(\mu)] \). Then, if \( \varphi \in SGX \left( \left( 1/1, h \right), [v, w], R_1^* \right) \) and only if \( \varphi \in SGX \left( \left( 1/1, h \right), [v, w], R_1^* \right) \), then, from above definition and set inclusion, we have \( \varphi \in SGX \left( \left( 1/1, h \right), [v, w], R_1^* \right) \) This completes the proof.

Proof. Let \( \varphi \) be \( h \)-Godunova–Levin convex interval-valued function, and suppose that \( x, y \in [v, w], \mu \in (0, 1) \); then,

\[
\frac{\varphi(x)}{h(\mu)} + \frac{\varphi(y)}{h(1-\mu)} \leq \varphi(\mu x + (1-\mu)y), \tag{14}
\]

that is,

\[
\left[ \frac{\varphi(x)}{h(\mu)} + \frac{\varphi(y)}{h(1-\mu)} \right] \leq \left[ \frac{\varphi(x)}{h(\mu)} + \frac{\varphi(y)}{h(1-\mu)} \right]. \tag{15}
\]

It follows that we have

\[
\frac{\varphi(x)}{h(\mu)} + \frac{\varphi(y)}{h(1-\mu)} \geq \frac{\varphi(x)}{h(\mu)} \] \tag{16}

and

\[
\int_0^1 \varphi(xm + (1-x)m)dx + \int_0^1 \varphi((1-x)m + xn)dx \leq h\left( \frac{1}{2} \right) \int_0^1 \varphi\left( \frac{v+w}{2} \right) dx, \tag{20}
\]

\[
\int_0^1 \overline{\varphi}(xv + (1-x)w)dx + \int_0^1 \overline{\varphi}((1-x)v + xw)dx \leq h\left( \frac{1}{2} \right) \int_0^1 \overline{\varphi}\left( \frac{v+w}{2} \right) dx. \tag{21}
\]

It follows that we have

\[
\int v(\mu) d\mu \geq h\left( \frac{1}{2} \right) \int_0^1 \varphi\left( \frac{v+w}{2} \right) dx, \tag{22}
\]

Similarly,

\[
\int \overline{\varphi}(\mu) d\mu \leq h\left( \frac{1}{2} \right) \int_0^1 \varphi\left( \frac{v+w}{2} \right) dx. \tag{23}
\]

This implies that

\[
\frac{\varphi(x)}{h(\mu)} + \frac{\varphi(y)}{h(1-\mu)} \leq \varphi(\mu x + (1-\mu)y). \tag{17}
\]

This shows that \( \varphi \in SGX \left( \left( 1/1, h \right), [v, w], R_1^* \right) \) and \( \overline{\varphi} \in SGV \left( \left( 1/1, h \right), [v, w], R_1^* \right) \). Conversely, suppose that if \( \varphi \in SGX \left( \left( 1/1, h \right), [v, w], R_1^* \right) \) and \( \overline{\varphi} \in SGV \left( \left( 1/1, h \right), [v, w], R_1^* \right) \), then, from above definition and set inclusion, we have \( \varphi \in SGX \left( \left( 1/1, h \right), [v, w], R_1^* \right) \).
\[
\frac{h(1/2)}{2} \left[ \frac{\varphi(v + w)}{2}, \frac{\varphi(v + w)}{2} \right] \leq \frac{1}{w - v} \int _v ^w \varphi(\mu) d\mu.
\] (24)

As a result of applying the interval $h$-Godunova–Levin convex function, we have
\[
\frac{\varphi(v)}{h(x)} + \frac{\varphi(w)}{h(1 - x)} \leq \varphi(xv + (1 - x)w).
\] (25)

Integrate w.r.t “\(x\)” over (0, 1); we have
\[
\varphi(v) \int _0 ^1 dx \frac{1}{h(x)} + \varphi(w) \int _0 ^1 dx \frac{1}{h(1 - x)} \leq \int _0 ^1 \varphi(xv + (1 - x)w) dx.
\] (26)

Accordingly,
\[
[\varphi(v) + \varphi(w)] \int _0 ^1 dx \frac{1}{h(x)} \leq \int _0 ^w \varphi(\mu) d\mu \subseteq [\varphi(v) + \varphi(w)] \int _0 ^1 dx \frac{1}{h(x)}.
\] (27)

Now, combining (24) and (26), we get the required result:
\[
\frac{h(1/2)}{2} \varphi\left(\frac{v + w}{2}\right) \leq \frac{1}{w - v} \int _v ^w \varphi(\mu) d\mu \subseteq [\varphi(v) + \varphi(w)] \int _0 ^1 dx \frac{1}{h(x)}.
\] (28)

Remark 2

\[
\frac{h(1/2)}{2} \varphi\left(\frac{v + w}{2}\right) = \varphi(0) = [0, 3],
\]

\[
\frac{1}{w - v} \int _v ^w \varphi(\mu) d\mu = \frac{1}{2} \left[ \int _{1 - v} ^1 \mu^2 d\mu \int _0 ^1 (4 - \varepsilon^2) d\mu \right] = \frac{1}{2} \left[ \int _0 ^1 \mu^2 d\mu \int _0 ^1 (4 - \varepsilon^2) d\mu \right] = \left[ \frac{1}{3}, 4 - \frac{e}{\varepsilon} \right] \leq \frac{1, 4 + \frac{e}{\varepsilon}}{2}.
\] (32)

As a result,
\[
[0, 3] \supseteq \left[ \frac{1}{3}, 4 - \frac{e}{\varepsilon} \right] \supseteq \left[ 1, 4 + \frac{e}{\varepsilon} \right].
\] (33)

This proves the above theorem.

Corollary 1. Let \( \varphi : [v, w] \rightarrow R^+_1, h : (0, 1) \rightarrow R^+ \) and \( h(1/2) \neq 0 \) if \( \varphi \in SGV((1/h), [v, w], R^+_1) \) and \( \varphi \in IR_{[v, w]} \); then, we have
\[
\frac{h(1/2)}{2} \varphi\left(\frac{v + w}{2}\right) \leq \frac{1}{w - v} \int _v ^w \varphi(\mu) d\mu \subseteq [\varphi(v) + \varphi(w)] \int _0 ^1 dx \frac{1}{h(x)}.
\] (34)

Theorem 3. Let \( \varphi : [v, w] \rightarrow R^+_1, h : (0, 1) \rightarrow R^+ \) and \( h(1/2) \neq 0 \) if \( \varphi \in SGX((1/h), [v, w], R^+) \) and \( \varphi \in IR_{[v, w]} \), then we have

(i) If \( h(x) = 1 \), then Theorem 2 has the following result for interval P functions:
\[
\frac{1}{2} \varphi\left(\frac{v + w}{2}\right) \leq \frac{1}{w - v} \int _v ^w \varphi(\mu) d\mu \subseteq [\varphi(v) + \varphi(w)].
\] (29)

(ii) If \( h(x) = (1/x) \), then Theorem 2 has the following result for interval convex functions:
\[
\varphi\left(\frac{v + w}{2}\right) \leq \frac{1}{w - v} \int _v ^w \varphi(\mu) d\mu \subseteq [\varphi(v) + \varphi(w)].
\] (30)

(iii) If \( h(x) = (1/x^s) \), then Theorem 2 has the following result for interval s-convex function:
\[
\frac{2^{s-1} \varphi\left(\frac{v + w}{2}\right)}{2} \leq \frac{1}{w - v} \int _v ^w \varphi(\mu) d\mu \subseteq [\varphi(v) + \varphi(w)].
\] (31)

(iv) If \( \varphi = \varphi \), then Theorem 2 has the following result of
\( \text{Ohud Almutairi and Adem Kiliman} \) (see [24, Theorem 1]).

Example 1. Suppose that \( h : (0, 1) \rightarrow R^+_1 \) is defined as \( h(x) = \frac{1}{x} \) for \( x \in (0, 1) \), \([v, w] = [-1, 1]\), and \( \varphi : [v, w] \rightarrow R^+_1 \) is defined as \( \varphi(\mu) = [\mu^2, 4 - e^\mu] \), where
\[
\begin{align*}
\frac{\varphi(xv + (1-x)((v+w)/2))}{h(1/2)} + \frac{\varphi((1-x)v + x((v+w)/2))}{h(1/2)} & \\
\leq \varphi \left( \frac{xv + (1-x)((v+w)/2) + x((v+w)/2) + (1-x)v}{2} \right) \\
= \varphi \left( \frac{v + ((v+w)/2)}{2} \right) = \varphi \left( \frac{3v + w}{2} \right).
\end{align*}
\] (37)

Integrate the above inequality over \((0,1)\) w.r.t \(x\):

\[
\frac{1}{h(1/2)} \left[ \int_0^1 \varphi \left( xv + (1-x)\frac{v+w}{2} \right) dx + \int_0^1 \varphi \left( x\frac{v+w}{2} + (1-x)v \right) dx \right] \leq \varphi \left( \frac{3v + w}{2} \right). \quad (38)
\]

Then, the above inequality becomes as

\[
\frac{1}{h(1/2)} \left[ \int_{v-\frac{w}{2}}^v \varphi(\mu) d\mu + \int_{\frac{v+w}{2}-v}^{\frac{v+w}{2}} \varphi(\mu) d\mu \right] \leq \varphi \left( \frac{3v + w}{2} \right). \quad (39)
\]

Now, by using the property of integral, the above inequality becomes as

\[
\frac{1}{h(1/2)} \left[ \int_{v-\frac{w}{2}}^v \varphi(\mu) d\mu + \int_{\frac{v+w}{2}-v}^{\frac{v+w}{2}} \varphi(\mu) d\mu \right] \leq \varphi \left( \frac{3v + w}{2} \right). \quad (40)
\]

\[
\frac{4}{h(1/2)} \left[ \int_{v-\frac{w}{2}}^v \varphi(\mu) d\mu \right] \leq \varphi \left( \frac{3v + w}{2} \right). \quad (41)
\]

Similarly, for interval \([ (v+w)/2, w ], \) we have

\[
\frac{1}{w-\frac{w}{2}} \int_{(v+w)/2}^w \varphi(\mu) d\mu \leq \frac{[h(1/2)]}{4} \varphi \left( \frac{3w + v}{2} \right). \quad (42)
\]

Adding inequality (41) and (42), we obtain

\[
\Delta_1 = \frac{[h(1/2)]}{4} \left[ \varphi \left( \frac{3v + w}{4} \right) + \varphi \left( \frac{3w + v}{4} \right) \right] \geq \frac{1}{w-\frac{w}{2}} \int_{(v+w)/2}^w \varphi(\mu) d\mu \]
\[
= \frac{1}{2} \left[ \varphi(\mu) \right]_{v-\frac{w}{2}}^v + \frac{2}{w-\frac{w}{2}} \int_{(v+w)/2}^w \varphi(\mu) d\mu \]
\[
\geq \frac{1}{2} \left[ \varphi(v) + \varphi \left( \frac{v + w}{2} \right) \right] \int_0^{1/h(x)} dx + \frac{1}{2} \left[ \varphi(\mu) + \varphi \left( \frac{v + w}{2} \right) \right] \int_0^{1/h(x)} dx \]
\[
= \frac{1}{2} \left[ \varphi(v) + \varphi(\mu) + 2\varphi \left( \frac{v + w}{2} \right) \right] \int_0^{1/h(x)} dx \]
\[
= \varphi \left( \frac{v}{2} + \frac{\varphi(\mu)}{2} + \varphi \left( \frac{v + w}{2} \right) \right] \int_0^{1/h(x)} dx = \Delta_2. \quad (43)
\]

Now,
\[
\frac{[h(1/2)]^2}{4} \varphi \left(\frac{v + w}{2}\right) = \frac{[h(1/2)]^2}{4} \varphi \left(\frac{1}{2} \left(\frac{3v + w}{4} + \frac{1}{2} \left(\frac{3w + v}{4}\right)\right)\right)
\]

\[
\geq \frac{[h(1/2)]^2}{4} \left[ \frac{\varphi ((3w + v)/4)}{h(1/2)} + \varphi ((3w + v)/4) \right]
\]

\[
= \frac{[h(1/2)]^2}{4} \left[ \frac{3v + w}{4} + \varphi \left(\frac{3v + w}{4}\right) \right]
\]

\[
= \frac{[h(1/2)]}{4} \left[ \frac{3v + w}{4} + \varphi \left(\frac{3v + w}{4}\right) \right]
\]

\[
= \frac{[h(1/2)]}{4} \left[ \varphi (v) + \varphi \left(\frac{v + w}{2}\right) \right] + \frac{1}{h(1/2)} \left[ \varphi (w) - \varphi \left(\frac{v + w}{2}\right) \right]
\]

\[
= \frac{[h(1/2)]}{4} \left[ \frac{1}{h(1/2)} \left[ \varphi (v) + \varphi \left(\frac{v + w}{2}\right) \right] + \frac{1}{h(1/2)} \left[ \varphi (w) + \varphi \left(\frac{v + w}{2}\right) \right] \right]
\]

\[
= \frac{1}{4} \left\{ \varphi (v) + \varphi (w) + 2\varphi \left(\frac{v + w}{2}\right) \right\}
\]

\[
= \frac{1}{2} \left[ \frac{\varphi (v) + \varphi (w)}{2} + \varphi \left(\frac{v + w}{2}\right) \right]
\]

\[
\geq \left[ \frac{\varphi (v) + \varphi (w)}{2} + \varphi \left(\frac{v + w}{2}\right) \right] \int_{\hspace{-1cm}0}^{1} \frac{dx}{h(x)}
\]

\[
\geq \left[ \frac{\varphi (v) + \varphi (w)}{h(1/2)} + \varphi \left(\frac{v}{h(1/2)} + \varphi \left(\frac{w}{h(1/2)} \right) \right) \int_{\hspace{-1cm}0}^{1} \frac{dx}{h(x)}
\]

\[
= \left[ \frac{\varphi (v) + \varphi (w)}{2} + \frac{1}{h(1/2)} \left[ \varphi (v) + \varphi (w) \right] \right] \int_{\hspace{-1cm}0}^{1} \frac{dx}{h(x)}
\]

\[
= \left\{ \left[ \varphi (v) + \varphi (w) \right] \left[ \frac{1}{2} + \frac{1}{h(1/2)} \right] \right\} \int_{\hspace{-1cm}0}^{1} \frac{dx}{h(x)}
\]

\[
\text{Example 2. Let } h: (0,1) \longrightarrow R^*_+ \text{ be defined as } h_i(x) = (1/x), \text{and for } x \in (0,1), [v,w] = [-1,1]. \varphi, \psi: [v,w] \longrightarrow R^*_+ \text{ is defined as } \varphi (\mu) = [\mu^2, 4 - e^\mu], \text{ where}
\]

\[
\frac{[h(1/2)]^2}{4} \varphi \left(\frac{v + w}{2}\right) \geq \frac{1}{w - v} \int_{\hspace{-1cm}v}^{w} \varphi (\mu) d\mu \geq \left[ \varphi (v) + \varphi (w) \right] \left[ \frac{1}{2} + \frac{1}{h(1/2)} \right] \int_{\hspace{-1cm}0}^{1} \frac{dx}{h(x)}
\]

Consider and put values:
\[
\frac{[h(1/2)]^2}{4} \varphi\left(\frac{v + \omega}{2}\right) = \varphi(0) = [0, 3],
\]

\[
\Delta_1 = \frac{[h(1/2)]}{4} \left[ \varphi\left(\frac{3v + \omega}{4}\right) + \varphi\left(\frac{3v + \omega}{4}\right) \right],
\]

\[
\Delta_1 = \frac{1}{2} \left[ \varphi\left(-\frac{1}{2}\right), \varphi(\frac{1}{2}) \right] = \left[ \frac{1}{4} \left(4 - \frac{\sqrt{3}}{\sqrt{3}} + (1/\sqrt{3}) \right) \right],
\]

\[
\frac{1}{w - v} \int_v^w \varphi(\mu) d\mu = \frac{1}{2} \left[ \int_{-1}^1 \mu^2 d\mu, \int_{-1}^1 (4 - e^\mu) d\mu \right]
\]

\[
= \frac{1}{2} \left[ \int_{-1}^1 \mu^2 d\mu, \int_{-1}^1 (4 - e^\mu) d\mu \right]
\]

\[
\Delta_2 = \left[ \frac{\varphi(v) + \varphi(w)}{2} + \varphi\left(\frac{v + w}{2}\right) \right] \int_0^1 h(x)
\]

\[
\Delta_2 = \frac{1}{2} \left[ 1, 4 - \frac{e + (1/e)}{2} \right] + [0, 3].
\]

\[
\Delta_2 = \left[ \frac{1}{2} \frac{7}{2} - \frac{e + (1/e)}{4} \right] \left[ \varphi(v) + \varphi(w) \right] \left[ \frac{1}{2} + \frac{1}{h(1/2)} \right] \int_0^1 h(x)
\]

\[
= \left[ 1, 4 - \frac{e + (1/e)}{2} \right].
\]

Thus, we obtain

\[
[0, 3] \supseteq \left[ \frac{1}{4} 4 - \frac{\sqrt{3}}{\sqrt{3}} + (1/\sqrt{3}) \right] \supseteq \left[ \frac{1}{3} 4 - \frac{e + (1/e)}{2} \right]
\]

\[
\supseteq \left[ \frac{1}{2} \frac{7}{2} - \frac{e + (1/e)}{4} \right] \supseteq \left[ 1, 4 - \frac{e + (1/e)}{2} \right].
\]

This proves the above theorem.

**Corollary 2.** Let \( \varphi: [v, w] \rightarrow R_+^e, h: (0, 1) \rightarrow R^+ \) and \( h(1/2) \neq 0 ; \) if \( \varphi \in SGV((1/h), I), [v, w], R^+ \) and \( \varphi \in IR_{[v,w]} \), then we have

\[
\frac{[h(1/2)]^2}{4} \varphi\left(\frac{v + \omega}{2}\right) \leq \Delta_1 \leq \int_v^w \varphi(\mu) d\mu \leq \Delta_2
\]

\[
\leq \left[ \varphi(v) + \varphi(w) \right] \left[ \frac{1}{2} + \frac{1}{h(1/2)} \right] \int_0^1 h(x)
\]

\[
\text{where} \quad M(v, w) = \varphi(v) \varphi(v) + \varphi(w) \varphi(w), \quad N(v, w) = \varphi(v) \varphi(w) + \varphi(w) \varphi(v).
\]

**Theorem 4.** Let \( \varphi: [v, w] \rightarrow R_+^e, h: (0, 1) \rightarrow R^+ \) be a continuous function. If \( \varphi \in SGX((1/h_1), [p, q], R_+^e) \), then,

\[
\phi(v) + \varphi(w) \leq \varphi(vx + (1-x)w),
\]

\[
\phi(v) h_1(x) + \varphi(w) h_1(1-x) \leq \varphi(vx + (1-x)w).
\]

\[
\phi(v) + \varphi(w) \leq \varphi(vx + (1-x)w).
\]

**Proof.** We assume that \( \varphi \in SGX((1/h_1), [v, w], R_+^e) \) and \( \phi \in SGX((1/h_2), [v, w], R_+^e) \); then, we have

\[
\varphi(v) + \varphi(w) \leq \varphi(vx + (1-x)w),
\]

\[
\phi(v) h_1(x) + \varphi(w) h_1(1-x) \leq \varphi(vx + (1-x)w).
\]
\[ \phi(vx + (1-x)w)\phi(vx + (1-x)w) = \frac{\phi(v)\phi(v)}{h_1(x)h_2(x)} + \frac{\phi(v)\phi(w)}{h_1(x)h_2(1-x)} + \frac{\phi(w)\phi(v)}{h_1(1-x)h_2(x)} + \frac{\phi(w)\phi(w)}{h_1(1-x)h_2(1-x)} \] (52)

Integrating both sides over \((0,1)\), w.r.t \(x\), we have

\[
\int_0^1 \phi(vx + (1-x)w)\phi(vx + (1-x)w)\,dx = \left[ \int_0^1 \phi(vx + (1-x)w)\phi(vx + (1-x)w)\,dx \right] 
\int_0^1 \frac{\phi(v)\phi(v)}{h_1(x)h_2(x)} + \frac{\phi(v)\phi(w)}{h_1(x)h_2(1-x)} + \frac{\phi(w)\phi(v)}{h_1(1-x)h_2(x)} + \frac{\phi(w)\phi(w)}{h_1(1-x)h_2(1-x)} \,dx \] (53)

It follows that

\[
\frac{1}{w-v} \int_v^w \phi(\mu)\phi(\mu)\,d\mu = M(v, w) \int_0^1 \frac{1}{h_1(x)h_2(x)} \,dx + N(v, w) \int_0^1 \frac{1}{h_1(x)h_2(1-x)} \,dx. \] (54)

Theorem is proved.

Example 3. Suppose that \(h: (0, 1) \rightarrow R_+^*\) defined as \(h_1(x) = (1/x)\) and \(h_2(x) = 1\) for \(x \in (0, 1)\), \([v, w] = [0, 1]\), and

\[
\frac{1}{w-v} \int_v^w \phi(\mu)\phi(\mu)\,d\mu = \left[ \int_0^1 \mu^2 \,d\mu, \int_0^1 (4-\epsilon^2)(3-\mu^2)\,d\mu \right] = \left[ \frac{1}{4}, \frac{35}{3}, 2\epsilon \right], \] \[
M(v, w) \int_0^1 \frac{1}{h_1(x)h_2(x)} \,dx = M(0, 1) \int_0^1 x\,dx = \left[ \frac{17}{2}, \frac{9}{2} - \epsilon \right], \]

\[
N(v, w) \int_0^1 \frac{1}{h_1(x)h_2(1-x)} \,dx = N(0, 1) \int_0^1 x\,dx = \left[ \frac{9}{2} - \frac{3\epsilon}{4} \right]. \]

It follows that

\[
\left[ \frac{1}{4} + \frac{35}{3} - 2\epsilon \right] + \left[ \frac{1}{2} + \frac{9}{2} - \epsilon \right] = \left[ \frac{1}{4} + \frac{7\epsilon}{4} \right]. \] (56)

Consequently, the above theorem is verified.

Corollary 3. Let \(\phi: [v, w] \rightarrow R_+^*\), \(h: (0, 1) \rightarrow R^*\) be a continuous function. If \(\phi \in SGV(((1/h_1), [v, w], R_+^*), \phi \in SGV(((1/h_2), [v, w], R_+^*), and \(\phi, \phi \in IR_{[v, w]}\), then we have

\[
\frac{1}{w-v} \int_v^w \phi(\mu)\phi(\mu)\,d\mu \leq M(v, w) \int_0^1 \frac{1}{h_1(x)h_2(x)} \,dx + N(v, w) \int_0^1 \frac{1}{h_1(x)h_2(1-x)} \,dx. \] (57)
Theorem 5. Let \( \varphi, \phi : [v, w] \rightarrow R^*_+; h : (0, 1) \rightarrow R^+ \) be a continuous function. If \( \varphi \in \text{SGX}((1/h_1), [v, w], R^*_e) \), \( \phi \in \text{SGX}((1/h_2), [v, w], R^*_2) \) and \( \varphi, \phi \in IR_{[v, w]} \), then we have

\[
\begin{align*}
\frac{h_1(1/2)h_2(1/2)}{2} & \phi\left(\frac{v + w}{2}\right) + \frac{1}{w - v} \int_v^w \varphi(\mu)\phi(\mu) d\mu + M(v, w) \int_0^1 \frac{1}{h_1(x)h_2(x)} dx \\
& + N(v, w) \int_0^1 \frac{1}{h_1(x)h_2(x)(1 - x)} dx.
\end{align*}
\]

Proof. By hypothesis, one has

\[
\begin{align*}
\phi\left(\frac{v + w}{2}\right) & \phi\left(\frac{v + w}{2}\right) \geq \frac{1}{h(1/2)h(1/2)} [\varphi(vx + (1-x)w)\phi(vx + (1-x)w) + \varphi(v(x-1) + xw)\phi(v(x-1) + xw)] \\
& + \frac{1}{h(1/2)h(1/2)} [\varphi(vx + (1-x)w)\phi(vx + (1-x)w) + \varphi(v(x-1) + xw)\phi(vx + (1-x)w)] \\
& + \frac{1}{h(1/2)h(1/2)} [\varphi(vx + (1-x)w)\phi(vx + (1-x)w) + \varphi(v(x-1) + xw)\phi(vx + (1-x)w)] \\
& + \frac{1}{h(1/2)h(1/2)} \left(\frac{\varphi(v)}{h_1(x)} + \frac{\varphi(w)}{h_1(1-x)}\right) \left(\frac{\phi(w)}{h_2(x)} + \frac{\phi(\phi_2(1-x))}{h_2(1-x)}\right) \\
& = \frac{1}{h(1/2)h(1/2)} [\varphi(vx + (1-x)w)\phi(vx + (1-x)w) + \varphi(v(x-1) + xw)\phi(v(x-1) + xw)] \\
& + \frac{1}{h(1/2)h(1/2)} \left[\left(\frac{\varphi(v)}{h_1(x)} + \frac{\varphi(w)}{h_1(1-x)}\right) \left(\frac{\phi(w)}{h_2(x)} + \frac{\phi(\phi_2(1-x))}{h_2(1-x)}\right)\right] M(v, w) \int_0^1 \frac{1}{h_1(x)h_2(x)(1 - x)} N(v, w).
\end{align*}
\]

Integrating above inequality over \((0, 1)\), w.r.t \( x \), we obtain

\[
\begin{align*}
\int_0^1 \phi\left(\frac{v + w}{2}\right) & \phi\left(\frac{v + w}{2}\right) dx = \int_0^1 \phi\left(\frac{v + w}{2}\right) \phi\left(\frac{v + w}{2}\right) dx, \int_0^1 \varphi(\frac{v + w}{2})\phi\left(\frac{v + w}{2}\right) dx \\
& = \frac{\varphi(v + w)}{1/2} \phi\left(\frac{v + w}{2}\right) dx \\
& \geq \frac{1}{h(1/2)h(1/2)} \left[\frac{1}{w - v} \int_v^w \varphi(\mu)\phi(\mu) d\mu\right] \\
& + \frac{2}{h(1/2)h(1/2)} \left[M(v, w) \int_0^1 \frac{1}{h_1(x)h_2(x)(1 - x)} dx + N(v, w) \int_0^1 \frac{1}{h_1(x)h_2(x)} dx\right].
\end{align*}
\]

Multiply both sides by \((h(1/2)h(1/2))/2\); from the above equation, we obtain
Theorem 6. (Jensen-type inequality). Let

\[
\frac{h_1(1/2)h_2(1/2)}{2} \phi \left( \frac{v + w}{2} \right) \phi \left( \frac{v + w}{2} \right) - \frac{1}{w - v} \int_{v}^{w} \varphi(\mu) \phi(\mu) d\mu + M(v, w) \int_{0}^{1} \frac{1}{h_1(x)h_2(1 - x)} dx + N(v, w) \int_{0}^{1} \frac{1}{h_1(x)h_2(x)} dx. 
\]

(62)

This completes the proof.

Example 4. Suppose that \( h: (0, 1) \rightarrow R^+ \) is defined as \( h(x) = (1/x) \), \( h_2(x) = 2 \) for \( x \in (0, 1) \), \([v, w] = [0, 1] \),

\[
\frac{h_1(1/2)h_2(1/2)}{2} \phi \left( \frac{v + w}{2} \right) \phi \left( \frac{v + w}{2} \right) = 2 \phi \left( \frac{1}{2} \right) \phi \left( \frac{1}{2} \right) = \left[ \frac{1}{4} \right],
\]

\[
\frac{1}{w - v} \int_{v}^{w} \varphi(\mu) \phi(\mu) d\mu = \left[ \frac{1}{2} \right], 
\]

\[
M(v, w) \int_{0}^{1} \frac{1}{h_1(x)h_2(1 - x)} dx = \left[ \frac{1}{2} \right],
\]

\[
N(v, w) \int_{0}^{1} \frac{1}{h_1(x)h_2(x)} dx = \left[ \frac{1}{2} \right].
\]

(63)

It follows that

\[
\left[ \frac{1}{8} \right] \left[ \frac{11}{4} - \frac{11 \sqrt{e}}{4} \right] \left[ \frac{1}{4} \right] = \left[ \frac{1}{4} \right] + \left[ \frac{17}{4} - \frac{17 \sqrt{e}}{4} \right] + \left[ \frac{9}{4} - \frac{9 \sqrt{e}}{4} \right] = \left[ \frac{1}{2} \right].
\]

(64)

This proves the above theorem.

Corollary 4. Let \( \varphi, \phi: [v, w] \rightarrow R^+ \), \( h: (0, 1) \rightarrow R^+ \) be a continuous function. If \( \varphi \in SGV((1/h_1), [v, w], R^+), \phi \in SGX((1/h_2), [v, w], R^+) \), and \( \varphi, \phi \in IR_{[v, w]} \), then we have

\[
\phi \left( \sum_{i=1}^{k} \frac{s_i}{S_k} \right) \geq \frac{1}{k} \sum_{i=1}^{k} \left[ \frac{\varphi(z_i)}{h(S/S_k)} \right],
\]

(66)

where \( S_k = \sum_{i=1}^{k} s_i \).

Theorem 6. (Jensen-type inequality). Let \( s_1, s_2, s_3, \ldots, s_k \in R^+ \) with \( k \geq 2 \). If \( h \) is super multiplicative nonnegative function and \( \phi \in SGX((1/h), [v, w], R^+) \), \( z_1, z_2, z_3, \ldots, z_k \in I \), then the inequality become as
Proof. When \( k = 2 \), the above inequality is trivially true, i.e., reduce to official definition of \( h \)-Godunova–Levin interval-valued function. Now, we suppose that inequality is true for \( k - 1 \); then, consider
\[
\varphi \left( \frac{1}{S_k} \sum_{i=1}^{k} s_iz_i \right) = \varphi \left( \frac{s_k}{S_k} + \sum_{i=1}^{k-1} \frac{s_i}{S_k} z_i \right) \\
\geq \frac{\varphi(z_k)}{h(s_k/S_k)} + \frac{\varphi \left( \frac{1}{S_k} \sum_{i=1}^{k-1} s_iz_i \right)}{h(s_k/1)} \\
= \frac{\varphi(z_k)}{h(s_k/S_k)} + \frac{\varphi \left( \sum_{i=1}^{k-1} s_iz_i \right)}{h(s_k/1)} \\
= \varphi \left( \frac{\varphi(z_k)}{h(s_k/S_k)} + \frac{\varphi \left( \sum_{i=1}^{k-1} s_iz_i \right)}{h(s_k/1)} \right) > \varphi \left( \frac{\varphi(z_k)}{h(s_k/S_k)} + \frac{\varphi \left( \sum_{i=1}^{k-1} s_iz_i \right)}{h(s_k/1)} \right).
\]

Thus, the result is proved by using mathematical induction. \( \square \)

4. Conclusions

We introduced the notion of \( h \)-Godunova–Levin interval-valued functions and established Hermite–Hadamard and Jensen-type inequalities for the introduced class of functions. Our results are extension of many existing results. It is interesting for the researchers to establish fractional version of established inequalities for the \( h \)-Godunova–Levin interval-valued functions.

Data Availability

All data needed for this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors have equally contributed to this paper.

References

[1] T. H. Zhao, Z. Y. He, and Y. M. Chu, “Sharp bounds for the weighted hölder mean of the zero-balanced generalized complete elliptic integrals,” Computational Methods and Function Theory, vol. 21, no. 3, pp. 413–426, 2021.
[2] T. H. Zhao, M. K. Wang, and Y. M. Chu, “Concavity and bounds involving generalized elliptic integral of the first kind,” Journal of Mathematical Inequalities, vol. 15, no. 2, pp. 701–724, 2021.
[3] S. Furuta, Inequalities, MDPI-Multidisciplinary Digital Publishing Institute, Basel, Switzerland, 2020.
[4] L. Yang and R. Yang, “Some new hermite-hadamard type inequalities for \( h \)-convex functions via quantum integral on finite intervals,” The Journal of Mathematics and Computer Science, vol. 18, no. 01, pp. 74–86, 2018.
[5] M. Kamenskii, G. A. R. I. K. Petrosyan, and C. F. Wen, “An existence result for a periodic boundary value problem of fractional semilinear differential equations in a banach space,” Journal of Nonlinear and Variational Analysis, vol. 5, pp. 155–177, 2021.
[6] P. S. Dwyer, “Interval Analysis: By Ramon E. Moore. 145 Pages, Diagrams, 6 x 9 in. New Jersey, Englewood Cliffs, Prentice-Hall, 1966. Price, 9.00,” Journal of the Franklin Institute, vol. 284, no. 2, pp. 148-149, 1967.
[7] J. M. Snyder, “Interval analysis for computer graphics,” in Proceedings of the 19th Annual Conference on Computer Graphics and Interactive Techniques, pp. 121–130, New York, NY, USA, July 1992.
[8] Y. Li and T. Wang, “Interval analysis of the wing divergence,” Aerospace Science and Technology, vol. 74, pp. 17–21, 2018.
[9] N. A. Gasilov and S. Emrah Amrahov, “Solving a non-homogeneous linear system of interval differential equations,” Soft Computing, vol. 22, no. 12, pp. 3817–3828, 2018.
[10] E. de Weerd, Q. P. Chu, and J. A. Mulder, “Neural network output optimization using interval analysis,” IEEE Transactions on Neural Networks, vol. 20, no. 4, pp. 638–653, 2009.
[11] Z. Zhang, M. A. Ali, H. Budak, and M. Z. Sarikaya, “On Hermite-Hadamard type inequalities for interval-valued multiplicative integrals,” Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, vol. 69, no. 2, pp. 1428–1448, 2020.
[12] H. Roman-Flores, Y. Chalco-Cano, and W. A. Lodwick, “Some integral inequalities for interval-valued functions,” Computational and Applied Mathematics, vol. 37, no. 2, pp. 1306–1318, 2018.
[13] S. Faisal, M. A. Khan, and S. Iqbal, “Generalized hermite-hadamard-mercer type inequalities via majorization,” Filomat, vol. 36, no. 2, pp. 469–483, 2022.
[14] S. Faisal, M. Adil Khan, T. U. Khan, T. Saeed, A. M. Alshehri, and E. R. Nwaeze, “New “contricte” hermite–hadamard–jensen–mercer fractional inequalities,” Symmetry, vol. 14, no. 2, pp. 294, 2022.
[15] S. Faisal, M. Adil Khan, T. U. Khan, T. Saeed, and Z. M. Mohammad Mahdi Sayed, “Unifications of continuous and discrete fractional inequalities of the hermite–hadamard–jensen–mercer type via majorization,” Journal of Function Spaces, vol. 2022, Article ID 6964087, 24 pages, 2022.
[16] S. S. Dragomir, “Inequalities of hermite-hadamard type for functions of self-adjoint operators and matrices,” Journal of Mathematical Inequalities, vol. 11, no. 1, pp. 241–259, 2017.
[17] A. Almutairi and A. Kilicman, “New refinements of the hadamard inequality on coordinated convex function,” Journal of Inequalities and Applications, vol. 2019, no. 1, pp. 192–199, 2019.
[18] S. S. Dragomir, “Some inequalities of hadamard type,” Soschow Journal of Mathematics, vol. 21, no. 3, pp. 335–341, 1995.
[19] S. Varošanec, “On h-convexity,” Journal of Mathematical Analysis and Applications, vol. 326, no. 1, pp. 303–311, 2007.
[20] M. Bombardelli and S. Varošanec, “Properties of h-convex functions related to the hermite–hadamard–fejér
inequalities,” *Computers & Mathematics with Applications*, vol. 58, no. 9, pp. 1869–1877, 2009.

[21] S. S. Dragomir, “Inequalities of hermite-hadamard type for h-convex functions on linear spaces,” *Proyecciones (Antofagasta)*, vol. 34, no. 4, pp. 323–341, 2015.

[22] T. M. Costa, “Jensen’s inequality type integral for fuzzy-interval-valued functions,” *Fuzzy Sets and Systems*, vol. 327, pp. 31–47, 2017.

[23] D. Zhao, T. An, G. Ye, and W. Liu, "New jensen and hermite–hadamard type inequalities for h-convex interval-valued functions," *Journal of Inequalities and Applications*, vol. 2018, no. 1, pp. 302–314, 2018.

[24] O. Almutairi and A. Kilicman, "Some integral inequalities for h-godunova-levin preinvexity," *Symmetry*, vol. 11, no. 12, p. 1500, 2019.

[25] S. S. Dragomir, “Integral inequalities of Jensen type for convex functions,” *Matematicki Vesnik*, vol. 68, no. 1, pp. 45–57, 2016.

[26] R. E. Moore, *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1966.

[27] J. E. Peajcharaiaac and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Cambridge, MA, USA, 1992.

[28] E. K. Godunova and V. I. Levin, “Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii,” *Vycislitel. Mat. i. Fiz. Mez- vuzov. Sb. Nauc. Trudov, MGPI, Moskva*, vol. 9, pp. 138–142, 1985.

[29] W. W. Breckner, “Stetigkeitsaussagen fur eine klasse verallgemeinerter konvexer funktionen in topologischen linearen räumen,” *Publications de l’Institut Mathématique*, vol. 23, no. 37, pp. 13–20, 1978.

[30] M. A. Noor, K. I. Noor, M. U. Awan, and S. Khan, “Fractional hermite-hadamard inequalities for some new classes of godunova-levin functions,” *Applied Mathematics & Information Sciences*, vol. 8, no. 6, pp. 2865–2872, 2014.