Optimal Phase Description of Chaotic Oscillators

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We introduce an optimal phase description of chaotic oscillations by generalizing the concept of isochrones. On chaotic attractors possessing a general phase description, we define the optimal isophases as Poincaré surfaces showing return times as constant as possible. The dynamics of the resultant optimal phase is maximally decoupled of the amplitude dynamics, and provides a proper description of phase resetting of chaotic oscillations. The method is illustrated with the Rössler and Lorenz systems.

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I. INTRODUCTION

Phase description lies in the base of theory of self-sustained, autonomous oscillators\textsuperscript{1,2}. A prudently defined phase variable yields a one-dimensional description of the oscillator, allowing one to characterize important aspects of its dynamics such as regularity of oscillation, sensitivity to external forcing, etc. Moreover, the concept of phase is important for data analysis of oscillatory processes in physics, chemistry, biology, and technical applications, where various approaches exist for extracting different variants of phase variables from oscillatory scalar time series.

On a very basic level, every phase description starts with the identification of those states of the oscillator which are in the same phase. For a good phase description, the identification must be done in an invariant way – independent of the variables and observables used – in order to make statements about the oscillator’s phase dynamics non-arbitrary and comparable. The standard procedure of phase reduction is valid for periodic oscillators that possess a stable limit cycle. There, a certain family of Poincaré sections, called isochrones, is used for the identification of states: Each isochrone consists of those states which are mapped onto each other after one oscillation period $T$, and which converge to the corresponding state on the limit cycle\textsuperscript{3,4}.

Even though chaotic oscillators do not possess a stable limit cycle, a phase-like variable has been used for their description. In this sense, the phase dynamics of chaotic systems has been initially discussed in relation to diffusion properties of phase\textsuperscript{5,6} and to phase synchronization\textsuperscript{7,10}. However, to describe these features, one does not need a good microscopic, i. e. on the time scale of the order of a characteristic period $T$, definition of the phase because both diffusion and synchronization are defined macroscopically, i. e. for time scales much larger than $T$. On the other hand, in the theoretical description of phase synchronization a proper microscopic phase definition was pre-assumed\textsuperscript{11,12}, although no practical algorithm for construction of a phase variable with good properties has been presented. The reason is that the chaotic phase diffusion destroys a rigorous notion of time-coherent isochrones, because any two states of the chaotic oscillator which are thought to be initially at the same phase will diverge as their respective phases diffuse.

In this article, we suggest a numerical technique for phase description of chaotic oscillations. For this goal, we construct special Poincaré sections, which we call optimal isophases. The choice of isophases is based on the properties of their return times. In an application to chaotic oscillators, we demonstrate an intimate relation between optimal isophases, chaotic phase diffusion, and unstable periodic orbits. Specifically, we discuss the reduced phase dynamics of chaotic oscillations and the decoupling of the amplitudes from the phase dynamics. Next, we use the optimal phase to introduce a proper framework for the description of phase resetting of chaotic oscillators.

Starting with an outline of the standard phase definition for periodic oscillators via the isochrones, we introduce the generalized concept of isophases of chaotic oscillators in Sec.\textsuperscript{II}. In Sec.\textsuperscript{III} certain dynamical properties of the optimal phase are highlighted by the example of the Rössler oscillator. Thereafter, the relation between optimal isophases and unstable periodic orbits is presented (Sec.\textsuperscript{IV}). In Sec.\textsuperscript{V} certain aspects of the theory are presented for the Lorenz oscillator. In the last section we discuss our results.

II. ISOPHASES OF PERIODIC AND CHAOTIC OSCILLATORS

A. Periodic oscillators and their isochrones

Phase is a natural variable for the description of periodic motions in dynamical systems. It can be introduced in different ways, with different levels of mathematical rigor\textsuperscript{1,2,4,5}. Here we outline an approach that is mostly suited for a generalization to the case of chaotic
systems.

The consideration starts with a general dissipative dynamical system showing stable periodic oscillations; the system’s state $x(t)$ is, thus, attracted to a limit cycle $x_0(t)$ having period $T$. In a vicinity of this periodic attractor the state space can be foliated by a non-intersecting family of Poincaré sections $J(\varphi)$, parametrized by a phase variable $\varphi$ with period $2\pi$. With $J(\varphi)$, a phase variable $\varphi(t)$ can be assigned to each state of the trajectory $x(t) \in J(\varphi(t))$. Therefore, the family of isochrones $J(\varphi)$ provides a precise definition of what is meant by an oscillation: The system completes one oscillation if the variable $\varphi$ grows by $2\pi$, i.e. if the trajectory returns to a chosen isochrone, consequently passing through all sections in $J(\varphi)$. In order to simplify nomenclature and to distinguish this variable from the genuine phase, introduced below, we term $\varphi$ as protophase. Introducing coordinates on the sections $J(\varphi)$, one can parametrize each point by a vector of amplitudes $a$ and the protophase $\varphi$.

There are various equivalent ways to foliate the state space in such a way that $\varphi$ grows monotonically: for periodic oscillators with a period $T$, the optimal foliation does exist. It can be introduced by considering the stroboscopic map $x(t) \to x(t + T)$. Clearly, all points on the limit cycle are stable fixed points of this map. Hence, for each fixed point $x_0$ there exist a stable manifold which converges to $x_0$ under the action of the stroboscopic map. These stable manifolds, called isochrones, constitute a special foliation of the neighborhood of the limit cycle, for which by construction the Poincaré map is the same as the stroboscopic map.

In this way one introduces the phase of oscillation so that its time evolution does not depend on the amplitudes $a$. By virtue of a trivial reparametrization $\varphi \to \theta = \frac{2\pi}{T} \int \frac{d\varphi}{\dot{\varphi}}$ of this foliation, one can introduce the genuine phase $\theta$ which grows strictly uniformly in time, with a constant instantaneous frequency $\dot{\theta} = \omega = 2\pi/T$. This phase, defined in the whole basin of attraction of the limit cycle, serves as a basis for a theoretical description of perturbed periodic oscillations. In particular, one can easily formulate phase-resetting properties in terms of this phase: If a state on the limit cycle $x'$ is instantly perturbed to some other state (even outside of the limit cycle), $x' \to x''$, then the phase is reset by a value $\Delta \theta = \theta(x'') - \theta(x')$, which remains constant in the course of further evolution (see also Sec. IIIID below).

Noteworthy, the extension of the phase to a vicinity of a periodic orbit can be defined either for a stable or unstable limit cycle. In the latter case, instead of using the stable manifold, one constructs the isophases by using the unstable manifolds of the fixed points of the stroboscopic map. However, for saddle limit cycles having both stable and unstable directions, this construction fails. Here one can construct isochrones on the stable and unstable manifolds separately, but not in the whole vicinity of the cycle. With this in mind, we use below for the chaotic case, where isochrones do not exist, the term “isophases” instead of the usual “isochrones”.

### B. Protophase for chaotic oscillators

We start the generalization of phase description to chaotic oscillators by discussing the construction of the protophase. For this purpose, we need the chaotic attractor to show the same property as a limit cycle, namely that there exists a family of non-intersecting Poincaré sections $J(\varphi)$, monotonically parametrized by a protophase $\varphi$. The requirement includes periodicity, $J(\varphi + 2\pi) = J(\varphi)$, and that any trajectory on the attractor successively crosses each Poincaré section $J(\varphi)$ transversally. Of course, not all chaotic attractors possess such a family, but those which have such a foliation can be described in terms of phases and are the subject of further consideration here.

Let us consider as an example the Rössler oscillator

$$\begin{align*}
\dot{x} &= -y - z, \\
\dot{y} &= x + 0.15y, \\
\dot{z} &= 0.2 + z(x - 10),
\end{align*}$$

and take a family of Poincaré sections $J(\varphi_1)$ defined via the cylindrical coordinates

$$\varphi_1 = \tan^{-1} \frac{y}{x}; \quad a = (r, h) = \left(\sqrt{x^2 + y^2}, z\right).$$

This family of Poincaré sections with constant protophase $\varphi_1$ is shown in Fig. I(a)). However, other families can be defined as well; an example of another foliation based on the protophase $\varphi_2 = \varphi_1 + 0.7 \ln r$ is superposed in Fig. I(b).

Because the difference of any two protophases is bounded, the asymptotic properties of their phase dynamics, such as the mean frequency and the diffusion constant of the phase rotations, do not depend on the definition of the protophase. However, local, microscopic properties of the dynamics for two protophases are different as it becomes apparent through the irregularly fluctuating phase difference $\varphi_1(t) - \varphi_2(t)$ shown in Fig. I(c). The fluctuations show a bounded and irregular pattern that is specific to arbitrarily chosen variants of the Poincaré sections. In order to define a “genuine” phase, such as that of periodic oscillators, we need to define the “isophases” of chaotic attractors. (Notice that for chaotic oscillators the isochrones generally do not exist.) Because the phase of a chaotic system is in fact not as “genuine” and unique as in the periodic case (see discussion below), we will refer to it as optimal phase, in the sense that it represents oscillating properties of chaos in an optimal way.

### C. Optimal isophases for chaotic oscillators

The genuine phase of periodic oscillators is defined by the basic property that there exist Poincaré sections
with this period, we define a family of stroboscopic sets.

where all return times are exactly equal to the period of oscillations; i.e. the corresponding Poincaré maps are stroboscopic maps as well. Naturally, such a situation does not generally occur for chaotic oscillators. On the one hand, this is plausible because different periodic orbits embedded in chaos usually have different basic periods (total period divided by the number of crossings with a Poincaré surface, see Eq. (7) below). On the other hand, a coincidence of Poincaré and stroboscopic maps as well. Naturally, such a situation would also imply the absence of phase diffusion what, however, is a degenerate, sparsely observed situation [14].

Since isophases of chaotic oscillators defined as sections with constant return times do not exist in the strict sense, we introduce optimal isophases that approximate the property above with some accuracy. Practically, we construct the optimal isophases as a smooth Poincaré section with a minimal (bounded by the smoothness) variation of return times. As this condition is not unambiguous, we describe below an algorithm that we practically use.

The starting point of our construction is a suitable vector time series $\mathbf{x}(t), 0 \leq t \leq t_{end}$, of chaotic dynamics, which can be obtained by numerical simulation or by embedding observed oscillations [13]. The first step is to introduce an arbitrary protophase $\varphi$ as described above. Using it, we can estimate the average period of oscillations as

$$T = \frac{2\pi t_{end}}{\varphi(t_{end}) - \varphi(0)}.$$  

With this period, we define a family of stroboscopic sets for the trajectory $\mathbf{x}(t)$ as

$$\mathbf{x}_k(\hat{\varphi}) = \mathbf{x}\left(\frac{\hat{\varphi}}{2\pi} T + kT\right), \quad k = 1, 2, \ldots, K_{end}. \tag{3}$$  

Here $\hat{\varphi} \in [0, 2\pi)$ serves as a (still preliminary) phase parametrizing stroboscopic sets, and each set consists of $K_{end}$ points. These sets are invariant under the stroboscopic map with time interval $T$, but cannot serve as Poincaré maps as they are not smooth curves, because the rotation in chaotic systems is non-uniform. The larger the total time interval $t_{end}$, the stronger is the spreading of the points of the stroboscopic set. We illustrate this in Fig. 2. We note, that only in a degenerate case where the phase diffusion of the chaotic oscillator vanishes, these stroboscopic sets would be smooth lines which can be used as Poincaré sections; such degenerate chaotic attractors (see an example in [16]) possess the same rigorous attractors as periodic oscillators.

In order to obtain a proper smooth Poincaré section, we fit the stroboscopic set, in the sense of least squares, by a polynomial $\varphi(r)$ of order four are shown with black bold lines.

The coefficients can be computed by applying a linear least squares fit [17] to the stroboscopic sets. In this case the phase correction is represented using a set of coefficients $c_{mnl}$:

$$\Delta(\varphi, r, h) = \sum_{m=0}^{N_r} \sum_{n=0}^{N_n} \sum_{l=0}^{N_{\varphi}} c_{mnl} r^m h^n e^{i\varphi l}.$$  

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FIG. 1. (Color online) (a,b) Two different families of Poincaré sections or the Rössler system (red bold lines). Both families yield a proper definition of oscillation. The corresponding protophases $\varphi_{1,2}$ are, however, different, so that $\varphi_2(t) - \varphi_1(t)$ shows irregular bounded fluctuations (c), specific to the particular shapes of the Poincaré surfaces.

FIG. 2. (Color online) The stroboscopic sets [Eq. (3)] for the Rössler attractor for two lengths of the trajectory (a) $t_{end} = 10^3$ and (b) $t_{end} = 5 \cdot 10^3$ are shown by squares. The trajectories are shown with gray lines, the optimal isophases obtained by fitting the by a polynomial $\varphi(r)$ of order four are shown with black bold lines.
FIG. 3. (Color online) A global approximation of optimal isophases (blue dots which look like bold lines), obtained for the Rössler attractor (gray) using the approximation (5) with $N_{\phi} = 4$, $N_r = 3$ and $N_h = 1$.

FIG. 4. (Color online) Return times $T_n$ for the Rössler oscillator [Eq. (1)]. Filled black squares correspond to an arbitrary Poincaré section $y = 0, x < 0$, here the spreading of the return times is large. Local (blue open circles) and global (red crosses) approximations (nearly coinciding on the figure) of the optimal isophases yield a strongly reduced spreading of the return times.

way it is easy to find an optimal phase globally, as a function of the state space coordinates $x$. We illustrate the isophases obtained in this way in Fig. 3.

In Fig. 4 we compare the quality of the optimal isophases obtained via representation (5) with the results of the local fitting of stroboscopic sets as in Fig. 2. We compare the return times for these isophases with the return times of the Poincaré section $y = 0, x < 0$. One can see that globally defined smooth isophases in the form (5) give a quite good minimization of the variability of the return times.

III. DYNAMICS OF THE OPTIMAL PHASE

In this section we discuss dynamical properties of the optimal phase introduced with help of optimal isophases.

A. Return time map

A natural way to characterize the time intervals $T_n$ between successive crossing of a Poincaré surface is to construct the return time map

$$T_{n+1} = M(T_n).$$

In fact, because $T_n$ is a function of the Poincaré map coordinate, it is just a scalar observable, and $M(T_n)$ is not a function but rather a one-dimensional projection of a Cantor set. Nevertheless, for nearly two-dimensional strange attractors the Poincaré map is nearly one-dimensional, and (6) looks like a curve (see Fig. 5a). In Fig. 5a we demonstrate, how this return time map changes if one uses an optimal isophase as a Poincaré surface. First, the range of variations of $T_n$ drastically shrinks. Second, one can hardly recognize the one-dimensional structure of the map: because now $T_n$ is a “bad” observable, it does not reproduce the nearly one-dimensional nature of the Poincaré map $a_n \to a_{n+1}$. This means, that with the dynamics of the new optimal phase looks like a random process even on a microscopic time scale, that is of the order of the period $T$.

B. Uniformity of phase rotations

The basic property of the phase for a periodic oscillator is that it rotates uniformly. For the optimal phase of a chaotic oscillator we cannot expect pure uniformity, but nevertheless it should be considerably increased compared to an arbitrary protophase. We illustrate this in Fig. 6. Here we show the velocities of the protophase $\dot{\phi}$ defined according to Eq. (2) and that of the optimal phase $\dot{\theta}$ defined according to isophases shown in Fig. 3. While fluctuations in the protophase velocity $\dot{\phi}$ heavily depend on $\phi$, the fluctuations of $\dot{\theta}$ are almost uniformly distributed and, notably, in some regions are larger than those of the protophase. Similar results are reported in Ref. [18]. We conclude that optimal isophases not only eliminate the amplitude dependence of the phase velocity, but they also flatten the phase dependence of its velocity fluctuations.
C. Decoupling of amplitude and phase dynamics

One of the goals of introducing the phase is to decouple its dynamics from that of the amplitude. For periodic oscillators this decoupling is perfect, whereas for chaotic oscillators, it is only approximate. To illustrate, how correlations of the phase dynamics with the amplitudes are reduced when the optimal phase is introduced, we performed a “mixing” experiment; the results are depicted in Fig. 7. We started an ensemble of initial conditions on a certain Poincaré surface and followed them for a time interval of length $5T$ (five average rotation periods). The trajectories starting at small, medium, and large amplitudes are marked separately in the plot. In the upper panels (a,b), where a “cylindrical” Poincaré surface $\phi = \text{const}$ is used, we see that the states started at small amplitudes are lagged behind while those started at large amplitudes are advanced. Contrary to this, using the optimal isophase as an initial condition, we see that after 5 rotations all points are mixed and one can hardly distinguish the points which had different amplitudes at the beginning. This is another illustration of the fact that the dynamics of the optimal phase is effectively decoupled from the amplitude.

D. Phase resetting of chaotic oscillators

A basic application of the phase description of periodic oscillators is quantification of the system response to pulse stimulation by means of phase response curves. Given a state on the limit cycle $x(\theta)$ one can determine the phase shift due to change of the state $x(\theta) \rightarrow x' = x(\theta) + k$ simply by calculating $\theta' = \theta(x')$. Because the phase rotates uniformly also outside of the limit cycle, the phase shift $\theta' - \theta$ remains invariant and characterizes the phase resetting (for noise-induced oscillations this notion can be also introduced in an optimal sense 19).

This approach has to be slightly modified when applied to chaotic oscillators. If both states $x$ and $x + k$ lie on the attractor, then their optimal phases are well-defined, and the phase shift can be simply calculated as $\theta(x + k) - \theta(x)$. However, generally the state $x + k$ lies outside of the attractor, and we have to generalize the definition of the optimal phases from the attractor to its vicinity. This is ambiguous, because the optimal isophases are not genuine isophases. They are not strictly invariant under time shifts and we cannot define the phase of the state $x + k$ by following its time evolution for arbitrarily large times. Instead, we have to fix the time interval after which the phase of the state $x + k$ is defined. For the Rössler model we choose the mean period $T$ as such an interval, as the relaxation time of approaching the attractor is typically smaller. So, we define $\theta(x + k) = \theta[\hat{T}(x + k)]$ where $\hat{T}$ is the operator of time evolution over the average period $T$ (see Fig. 8). Applying now the representation (5), we obtain the PRC of the Rössler attractor $\text{PRC}(k, x) = \theta[\hat{T}(x + k)] - \theta(x)$ as shown in Fig. 9.

IV. ISOPHASES AND UNSTABLE PERIODIC ORBITS

In this section we discuss a relation between optimal isophases of a chaotic system and unstable periodic orbits (UPOs) $x_0(t + \tau) = x_0(t)$ embedded in chaos. For each UPO one can define the phase on this orbit just from the
condition of uniform rotation. This approach is discussed in Sec. IV A Similar to the construction discussed in Sec. II A we can extend the notion of the phase for each periodic orbit to its stable or unstable manifold; these ideas are presented in Sec. IV B.

A. Approximation of orbit phase sets

For each UPO one can introduce a topological period (lap number) \( p \) as the number of intersections with a Poincaré section. With this number \( p \) and the total period \( \tau \), we define the oscillation period

\[
S = \frac{2\pi}{\nu} = \frac{\tau}{p},
\]

which is expected to be close, but not identical, to the mean period of chaotic oscillations (mean return time of the Poincaré map). Next, for the UPO we can introduce the phase \( \tilde{\theta} \) that rotates uniformly with frequency \( 2\pi/\nu \) so that \( \tilde{\theta}(\tau) = \tilde{\theta}(0) + 2\pi p \). With help of this phase, a family of point sets \( I(\tilde{\theta}) \), called orbit phase sets, can be defined as points which are attained at constant time intervals, equal to the oscillation period \( S \):

\[
I(\tilde{\theta}) = \{ x_0(nS) \mid n = 0, \ldots, p - 1 \},
\]

with some arbitrary choice of the zero phase.

Let us now take a Poincaré surface which passes through the orbit phase set \( I(\tilde{\theta}) \). (Of course, there are many possibilities to draw such a surface, e.g., one can use splines.) Then it will be an approximation to an optimal isophase, as at least on the orbit phase set all the return times will be equal to \( S \). We illustrate this with Fig. 10(a), where we show orbit phase sets of two UPOs, with topological periods \( p = 10 \) and \( p = 9 \), for the Rössler system (1). Since the orbits do not share any state, the zero phases can be chosen separately. Practically, the phase offsets have been chosen in a way that the orbit phase sets are mostly close to each other and approximate the same isophase, which is also drawn for comparison. One can see that the orbit phase states indeed can serve as approximations for the isophases.

This approximation is expected to work better for larger periods and for periodic orbits which are most “typical”. The probability for a trajectory to approach the orbit depends on the stability of the UPO, quantified by its unstable Floquet multiplier [20]. Therefore, it is expected that the correspondence between isophases and the orbit phase sets will be better for UPOs which are visited more often because they are less unstable. To check this for the Rössler system, we introduce a measure \( d \) of the distance of a \( p \)-orbit \( y \) to the optimal isophase.
where

\[ \text{of} \]

we also optimized the zero phase to achieve a minimum

\( (\text{dashed line}) \), the orbit phase set [Eq. (8)] can be extended

FIG. 11. (Color online) For a UPO of the Rössler oscillator

shown in Fig. 4 with a green line, as

\[ d = \sqrt{p^{-1} \sum_{k=0}^{p-1} ||y_k^l - y_k||^2}, \quad (9) \]

where \( y_k^l \) are coordinates of the orbit phase set (for which

we also optimized the zero phase to achieve a minimum

of \( d \)) and \( y_k \) are the crossings of the periodic orbits with

the isophase. This measure was calculated for the 80

available UPOs together with their Floquet multipliers.

It was found that orbits showing a larger distance had a tendency to be less stable (cf. Fig. 11(b)).

**B. Orbit isophase**

As described in Sec. II A, after the phase on a periodic

orbit is introduced, the isophases in its vicinity can be

defined separately on the stable and unstable manifolds

of the orbit, as the stable and the unstable manifolds of

the fixed points of the stroboscopic (with the period of

the orbit) map. This definition can be applied to the

UPOs in chaos, where the unstable manifolds are espe-

cially interesting as they lie in the attractor.

Let us consider the simplest UPO of the Rössler os-

cillator that has topological period \( p = 1 \). Its oscillation

period is \( S \approx 6.024 \) whereas the mean period of a

typical trajectory is \( T \approx 6.073 \). Numerically, we calcu-

lated the isophase on the unstable manifold of this orbit

using the oscillation period \( S \) for the stroboscopic map and

obtained the blue line in Fig. 11. This isophase be-

comes folded together with the unstable manifold, and is

not close to the optimal isophases obtained by another

methods.

It is instructive to try to construct the isophase on the

unstable manifold of the UPO using not its period \( S \), but

the mean period \( T \). It is clear that such an isophase can-

not exist, but trying approximate it (see appendix A for
details) we obtain a singular curve (Fig. 11). This is an-

other representation of non-smoothness of stroboscopic

sets that appears in the algorithm described in Sec. II C
due to non-existence of true isophases. In fact, when one

tries to construct an isophase, such a singularity will ap-

pear for every periodic orbit, and the procedure should be

constrained by the requirement that the isophase should

be sufficiently smooth.

**V. PHASE OF THE LORENZ SYSTEM**

In our presentation above we have used the Rössler model [Eq. (1)] as the basic example. Here we discuss how the approach works for the Lorenz system

\[ \begin{align*}
\dot{x} &= 10 \cdot (y - x), \\
\dot{y} &= 28 \cdot x - y - xz, \\
\dot{z} &= \frac{8}{3} \cdot z + xy.
\end{align*} \quad (10) \]

Chaotic phase diffusion of the Lorenz system is orders of magnitude stronger than that of the Rössler oscillator Eq. (1), thus introducing its phase is a more challenging task. The main difficulty lies in the unboundedness of the return times of the Poincaré map, due to presence of the saddle steady state at the origin \( (x = y = z = 0) \).

Due to this, the stroboscopic sets are spread over the at-

tractor and cannot serve as a basis for construction of

isophases like described above. Therefore we applied the

following iterative procedure for obtaining smooth opti-

mal isophases. First, we use projections of the trajectory

onto the plane \( (u = \sqrt{x^2 + y^2}, z) \). On this plane the tra-

jectory rotates around a center approximately at \( (12, 27) \),

and the protophases can be easily defined (cf. [21]). We

choose a Poincaré surface and find the points of the tra-

jectory at the intersection with this surface, these are

\[ x(t_k), y(t_k), z(t_k), \quad k = 1, 2, \ldots \] Of course, the times \( t_k \)

are not equidistant because the Poincaré map is far from

the stroboscopic one. We adjust the times \( t_k \) trying to

make them equal, by introducing a parameter \( s \) on which

these times depend, and letting them evolve according to

\[ \frac{dt_k}{ds} = - \frac{\partial V(t_1, t_2, \ldots)}{\partial t_k}, \quad V = \frac{1}{2} \sum_k (t_{k+1} - t_k - T)^2 \quad (11) \]

where \( T \) is the average period. One can easily see that the “evolution” of \( t_k \) according to Eq. (11) leads to equalization of the intervals \( t_{k+1} - t_k \) because of minimization of the Lyapunov function \( V \). However, we “evolve” the times \( t_k \) only for a finite interval of \( s \), and obtain new times \( \tilde{t}_k = t_k(s) \). The new points \( \tilde{x}(\tilde{t}_k), \tilde{y}(\tilde{t}_k), \tilde{z}(\tilde{t}_k) \)
form a new, distorted and singular Poincaré section. We smoothen this set by applying a kernel technique \cite{22} and obtain a smooth new Poincaré section with more equidistant time intervals. We make several iterations of this procedure, and finally obtain the approximate smooth isophases as depicted in Fig. 12.

![Fig. 12.](image1)

**FIG. 12.** (Color online) Optimal isophases (different markers/colors depict different isophases) of the Lorenz attractor [Eq. (10)] (grey line).

To characterize the quality of the introduced isophases for the Lorenz system, we plot the return times for an initial arbitrary Poincaré section and for the obtained isophase in Fig. 13. We see that the variations of the return times decrease only slightly, and the singularity (corresponding to the stable manifold of the origin) remains.

![Fig. 13.](image2)

**FIG. 13.** (Color online) Return times for the Poincaré section $u = 12$, $z < 27$ (filled circles) and for the optimal isophases resulting from its iterations (shown in Fig. 12 with black filled squares) (open circles). The variations of the return times only slightly decrease.

In Fig. 14, we use orbit phase sets of UPOs of the Lorenz system to approximate isophases. Nine periodic orbits of the Lorenz system with topological period 6 are shown with grey line. By manually adjusting phase shifts of these orbits, it is possible to arrange the isophase sets (different markers) to build a set close to a curve (drawn manually as a black line) which can serve as an optimal isophase. The form of this curve is close to one of the optimal isophases presented in Fig. 12.

![Fig. 14.](image3)

**FIG. 14.** (Color online) Building an isophase using nine UPOs of the Lorenz system with $p = 6$ (see text for details).

**VI. CONCLUSION**

In summary, we have proposed a method of phase reduction of chaotic oscillators by generalizing the concept of standard isophases (isochrones). In the absence of a stable limit cycle, the definition of optimal isophases of chaotic oscillations is solely based on their return times. Because of non-vanishing chaotic diffusion and embedded unstable periodic orbits with different periods, isophases could only be obtained in an optimal, approximate way constrained by certain smoothness conditions. In the case of the Rössler attractor, where the phase diffusion is relatively small, we obtain the optimal isophases by smoothening the stroboscopic sets of a chaotic trajectory. For the Lorenz attractor, where phase diffusion is large, an iterative numerical scheme was proposed. Using the Rössler oscillator as an example, we have presented different aspects of the phase dynamics. Specifically, the decoupling of the phase dynamics from the amplitudes, as well as a way to describe phase resetting of chaotic oscillators have been outlined.

The theory of optimal isophases can possibly provide a refined understanding of emergent behavior of weakly coupled oscillating systems. For example, a theoretical phase description of weakly coupled limit cycle oscillators can be extended to ones of greater complexity, such as stochastic or chaotic oscillators (cf. \cite{11, 12}). In this way, one can treat more realistic models of natural systems. Furthermore, the theory can easily be utilized in the analysis of observed chaotic oscillations where the numerical scheme described above can be used to refine a preliminary phase description. This can help to reduce certain systematic errors which may be present in phase-related quantities such as coupling strengths.

The theory is easily adaptable for the analysis of non-linear oscillations with a random component (for theoretical approaches see, e.g. \cite{23, 25}). Here, the return times to optimal isophases have to be interpreted in an average sense. The corresponding results will be presented...
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Appendix A: Isophases of unstable Stuart-Landau oscillator

To give an analytically tractable example of isophases of UPOs on their unstable manifold, let us consider the unstable Stuart-Landau oscillator governed by

\[ \dot{r} = r(r^2 - 1); \ \dot{\varphi} = \alpha - \kappa r^2. \]  

(A1)

It is exactly solvable: For the initial conditions \( r(0) = R \) and \( \varphi(0) = \Phi \), it has the well-known solution

\[ r(t) = \left[ 1 + \frac{R^2}{2} e^{2t} \right]^{-1/2}, \]  

(A2)

\[ \varphi(t) = (\alpha - \kappa)t - \kappa \ln r(t) + \Phi + \frac{\kappa}{\lambda} \ln R. \]  

(A3)

Oscillator (A1) shows an unstable periodic orbit (UPO) with frequency \( \omega = \alpha - \kappa \). Depending on the initial conditions, it either performs decaying oscillations (for \( R < 1 \)), or diverges in finite time (for \( R > 1 \)).

As the characteristic period we first choose that of the UPO: \( S = \frac{2\pi}{\omega} \). In order to obtain a phase which rotates independently of \( r \), we set \( \theta = \omega t + \Phi + \frac{\kappa}{\lambda} \ln R \). Inserting \( \theta \) into Eq. (A3), we find that optimal isophases \( I(\theta) \) are solutions of the equation

\[ \theta = \varphi + \kappa \ln r. \]  

(A4)

For each \( (\Phi, R) \in I(\theta) \) the return time for \( \theta \to \theta + 2\pi \) is equal to \( S \) because \( \dot{\theta} = \omega \). This is the standard definition of the isophases.

Alternatively, one may think of the unstable Stuart-Landau oscillator as of a rarely visited part of the state space of a bigger chaotic system which has a different characteristic frequency \( \frac{\kappa}{\lambda} \omega = \omega + \Delta \omega \). It means, average period \( T \) is different from the period \( S \) of the UPO. Therefore, the condition that the return time for a Poincaré surface is equal to \( T \) cannot be fulfilled on the orbit. To fulfill the condition for states off the periodic orbit, we now seek a phase with the dynamics \( \dot{\theta} = \omega_0 \). Therefore, we rewrite Eq. (A3) in terms of \( \omega_0 t \):

\[ \varphi(t) = \omega_0 t + \Phi + \kappa \ln R - \kappa \ln r - \Delta \omega_0 r. \]  

(A5)

Here, we need to rewrite time as a function of radius, using (A2). We get

\[ t(r) = \frac{1}{2} \ln |r^2 - 1| - \ln r + \ln \frac{R}{\sqrt{1 - R^2}}. \]  

(A6)

After the substitution, a uniformly rotating phase is given by \( \theta = \omega_0 t + \Phi + \kappa \ln R + \Delta \omega_0 \ln(\sqrt{1 - R^2}/R) \). Comparing the result with Eq. (1), we obtain the phase correction as

\[ \Delta(r) = -\kappa \ln r - \frac{\Delta \omega}{2} \ln |r^2 - 1|. \]  

(A7)

While the return time is equal to \( T \) off the periodic orbit, the phase correction diverges as \( \ln |1 - r| \) in the limit \( r \to 1 \). Thus, the “isophase” is singular and winds itself infinitely often around the limit cycle.

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