We formulate Dirac fermions on a (1+1)-dimensional lattice based on a Hamiltonian formalism. The species doubling problem of the lattice fermion is resolved by introducing hopping interactions that mix left- and right-handed fermions around the momentum boundary. Approximate chiral symmetry is realized on the lattice. The deviation of the fermion propagator from the continuum one is small.

In contrast with the great success of lattice gauge theory, lattice fermions remain a long-standing problem. Naive discretization causes the well-known species doubling problem [1]. The problem originates in the fact that the kinetic term of the fermion is proportional to the first-order derivative in real space. This means that the Fourier transform of the kinetic term is proportional to the momentum. Since an odd function cannot be periodic and have one zero, the fermion propagator is forced to have additional pole(s) on the lattice. The situation does not change regardless of how the lattice spacing is reduced as long as the space-time derivative is modeled as a naive difference.

Many attempts have been made to fix the doubling problem [1–7]. Wilson removed doublers at low energy by introducing an interaction that mixes left- and right-handed fermions [1]. The interaction is not a result of naive discretization and therefore has no counterpart in continuum theory. However, unwanted degeneracy persists at high energy and chiral symmetry is explicitly broken. To fix these problems, Kaplan modified Wilson’s fermion by introducing an extra dimension [4]. Kaplan’s fermion has an approximate chiral symmetry if the lattice size of the extra dimension is large. The fermion is useful for calculating physical quantities related to dynamical breaking of chiral symmetry. (See Ref. [8], for example.) However, the cost of numerical calculations based on Kaplan’s fermion is not cheap. If we find a method to perform such calculations without the extra dimension, calculation time decreases largely and a deeper understanding of quantum field theory becomes possible.

In addition to the doubling problem, the lattice fermion has another serious problem. The fermion propagator defined on a lattice deviates from the continuum one even if the doublers are removed with the existing techniques such as Kaplan’s fermion [5]. As a result, it may cause errors in numerical calculations. To remove this uncertainty, we need to modify the discretized propagator somehow so that it is close to the continuum one as far as possible. Such deviation of propagators becomes critical especially when supersymmetry is considered on a lattice because deviation of fermion and boson propagators gives wrong values for loop integrals (sums). For example, the zero-point energy does not cancel between fermions and bosons if the propagators deviate from the continuum ones. Accurate discretization of the propagators is a necessary condition for maintaining lattice supersymmetry.

In general, the extra dimension can be expressed as hopping interactions in a lower-dimensional system. Kaplan’s fermion is a formulation with an extra dimension, so there must be a corresponding Hamiltonian with no extra dimension. Also, the shape of the fermion propagator can be improved with hopping interactions. The Runge-Kutta method for differential equations is an example of such an improvement.

In this paper, based on a Hamiltonian formalism, we introduce ultralocal hopping interactions to remove doublers and improve momentum dependence of fermion energy. (The word “ultralocal” means that fermion hopping is restricted to a finite range on a real-space lattice [9].) From knowledge of the continuum theory, we know the correct momentum dependence of the energy. We start from momentum space and go back to real space by way of discrete Fourier transform. A real-space Hamiltonian is necessary to construct gauge theory. The method is a hybrid of the Wilson [1] and SLAC [3] approaches.

First, let us consider a free Dirac fermion on a (1+1)-dimensional Hamiltonian lattice. Time is continuous and space is discrete. The length of the spatial lattice is \( N \delta_l \), where \( a \) is a lattice spacing and the number of sites \( N \) is assumed to be even. According to the continuum theory, the Dirac fermion should be described by the following Hamiltonian in momentum space

\[
H = \frac{1}{a} \sum_{l=-N/2+1}^{N/2} p_l \bar{\gamma}^1 \gamma_l,
\]

where \( l \) is an index for momentum. \( \gamma_l \) and \( \bar{\gamma}_l \) are discrete Fourier transform of real-space two-component fermion operators

\[
\psi_n = \begin{pmatrix} \xi_n \\ \eta_n \end{pmatrix} = \frac{1}{\sqrt{N}} \sum_{l=-N/2+1}^{N/2} e^{i2\pi ln/N} \xi_l,
\]

and \( \bar{\psi}_n \equiv \psi_n^\dagger \gamma^0 \) for \( n = 1, 2, \ldots, N \), respectively. \( \xi_n \) and \( \eta_n \) satisfy

\[
\{\xi_m, \xi_n^\dagger\} = \{\eta_m, \eta_n^\dagger\} = \delta_{mn}.
\]
and other anticommutators are zero. The gamma matrices are
\[ \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]
Periodic boundary conditions are assumed in real space,
\[ p_l = \frac{2\pi l}{N}. \]
We try to create a real-space Hamiltonian with no doubler that reproduces Eq. (1). To find real-space representation of \( p_l \), let us consider the following function
\[ s(p) = \sum_{\alpha=1}^{M} S_\alpha \sin \alpha p, \]
In the limit \( M \to \infty \), the function (3) goes to \( p \). The function (3) necessarily has a node at the boundary \( p = \pm \pi \) because it has a periodicity of \( 2\pi \) (see Fig. 1). The node is the cause of the doubling problem. The doubler remains as a singularity at the boundary even if the limit \( M \to \infty \) is taken (by “doubler” we mean unwanted energy degeneracy that is not contained in the continuum theory). Anyway, the parameter \( M \) needs to be small for practical formulation because \( M \) corresponds to the maximum distance of fermion hopping in real space. The limit \( M \to \infty \) needs the infinite lattice.

![FIG. 1. The solid line plots the correct energy \( p \) of one-particle states from the continuum theory. The dot-dashed line plots the function (3) for \( M = 5 \). The dashed line plots the function \( s(p) \) modified with the Lanczos factor for \( M = 5 \). The oscillation of Eq. (3) has almost been removed.](image)

In addition to the doubler modes around the boundary, the function (3) has another degeneracy. The function oscillates around \( p \) and has local minima if the summation is truncated with a small \( M \). In Fourier analysis, it is called the Gibbs phenomenon, which occurs if a function to be expanded has a singularity [10]. The oscillation can be removed by replacing Eq. (3) with the function
\[ s(p) = \sum_{\alpha=1}^{M} S_\alpha \sin \alpha p, \]
where
\[ S_\alpha \equiv F_\alpha \frac{2(-1)^{\alpha-1}}{\alpha}, \quad F_\alpha \equiv \frac{M+1}{\pi\alpha} \sin \left( \frac{\pi\alpha}{M+1} \right). \]
\( F_\alpha \) is called the Lanczos factor [10]. As shown in Fig. 1, the factor almost removes the oscillation of Eq. (3). However, the doubler modes around the boundary still remain. We are going to remove them by a trick with hopping interactions. Let us consider the following momentum-space Hamiltonian:
\[ H = \sum_{l=-N/2+1}^{N/2} \left( s_l \tilde{\zeta}_l^1 \gamma^1 \zeta_l + m \tilde{\zeta}_l \gamma^5 \zeta_l \right), \]
where \( s_l \equiv s(p_l)/a \) and \( m \) is fermion mass. We write the Hamiltonian in the matrix form:
\[ H = \sum_{l=-N/2+1}^{N/2} \zeta_l^\dagger \left( \begin{array}{cc} s_l & m \\ m & -s_l \end{array} \right) \zeta_l. \]
We introduce an interaction \( c_l \) that mixes left- and right-handed fermions
\[ H = \sum_{l=-N/2+1}^{N/2} \zeta_l^\dagger \left( \begin{array}{cc} s_l & m + c_l \\ m + c_l & -s_l \end{array} \right) \zeta_l, \]
where \( c_l \) are assumed to be nonzero only for \( |l| \sim N/2 \). As shown later, \( s_l \) and \( c_l \) are expressed as ultralocal hopping interactions in real space. The Hamiltonian (7) can be diagonalized for each \( l \).
\[ H = \sum_{l=-N/2+1}^{N/2} \zeta_l^\dagger \left( \begin{array}{cc} k_l & 0 \\ 0 & -k_l \end{array} \right) \zeta_l, \]
where \( k_l \equiv \sqrt{s_l^2 + (m + c_l)^2} \) are energies of one particle states and \( \zeta_l^\dagger \) are transformed variables. Although better solutions may be found than \( c_l \) shown here, we give precedence to simplicity over accuracy in this paper. We understand that our strategy is successful if the properties of the continuum Dirac fermion are approximately reproduced.
Consider the function
\[-u(p - \pi)^2 + v = \frac{C_0}{2} + \lim_{M \to \infty} \sum_{\alpha=1}^{M} C_\alpha \cos(\alpha p), \quad (9)\]

where \(u\) and \(v\) are some positive real numbers. The Fourier coefficients \(C_\alpha\) are

\[C_\alpha = \frac{F_\alpha (-1)^\alpha}{\pi \alpha^2} \left[ -\sqrt{\frac{v}{u}} \cos \left( \alpha \sqrt{\frac{v}{u}} \right) + \frac{1}{\alpha} \sin \left( \alpha \sqrt{\frac{v}{u}} \right) \right].\]

The function (9) has a peak at \(p = \pm \pi\) and zeros at \(p = \pm \pi \mp \sqrt{v/u}\). As before, the infinite \(M\) cannot be realized on a finite lattice. For a finite \(M\), we define the function

\[c(p) = \frac{C_0}{2} + \sum_{\alpha=1}^{M} C_\alpha \cos(\alpha p). \quad (10)\]

When \(M\) is finite, \(c(p)\) does not reproduce the function (9) correctly. However, it does not matter in this consideration. We just want to use \(c(p)\) to remove the doubler modes. We do not need to care about the original shape of the function (9). The parameters \(u\) and \(v\) are adjusted so that the unnecessary degenerate modes around the boundary become non-degenerate normal modes. Figure 2 shows how the doubler modes of \(s(p)\) are removed using the function \(c(p)\). The function \(s^2(p)\) has two hemlines near the boundary because it has to vanish at \(p = \pm \pi\). We want to raise the both ends by adding a packet function \(c^2(p)\) to \(s^2(p)\). Let us consider a function \(k(p) \equiv \sqrt{s^2(p) + c^2(p)}\). If we choose \(M = 5, u = 130,\) and \(v = 8.4\), the function \(k^2(p)\) agrees well with \(p^2\) in the fundamental region \(|p| \leq \pi\) except for a small deviation around momentum \(|p| \sim 2.3\).

In Fig. 3, the functions \(\pm p, \pm s(p)\), and \(\pm k(p)\) are compared. The function \(\pm k(p)\) corresponds to one-particle states given by the Hamiltonian (7) with \(m = 0\). The function \(\pm k(p)\) agrees well with the correct energy \(\pm p\) from the continuum theory in the fundamental region \(|p| \leq \pi\). The Hamiltonian (7) approximately reproduces the continuum theory for the (1+1)-dimensional Dirac spinor without doubler modes if we identify \(c_l = c(p_l)/a\) and \(k_l = k(p_l)/a\).

FIG. 2. The solid line plots the correct energy squared \(p^2\). The dashed line plots the function \(s^2(p)\) for \(M = 5\) with the Lanczos factor. The dotted line plots the function \(c^2(p)\) for \(M = 5, u = 130,\) and \(v = 8.4\) with the Lanczos factor. The dot-dashed line plots the function \(k^2(p)\). The doubler modes around the boundary \(|p| = \pi\) has been removed with \(c^2(p)\).

FIG. 3. The solid line plots the correct energy \(\pm p\) from the continuum theory. The dashed line plots the function \(\pm s(p)\) for \(M = 5\) with the Lanczos factor. The dot-dashed line plots the function \(\pm k(p)\) for \(M = 5, u = 130,\) and \(v = 8.4\) with the Lanczos factor. The function \(\pm k(p)\) almost agrees with \(\pm p\) in the fundamental region \(|p| \leq \pi\) except for a small deviation around momentum \(|p| \sim 2.3\).

In the new basis that diagonalizes Eq. (7) with \(m = 0\), \(\gamma_5\) is transformed into

\[\gamma_5' = \frac{s_l + k_l}{k_l^2 + s_l k_l} \begin{pmatrix} s_l & -c_l \\ -c_l & -s_l \end{pmatrix}\]

for \(l > 0\) and

\[\gamma_5' = \frac{-s_l + k_l}{k_l^2 - s_l k_l} \begin{pmatrix} s_l & c_l \\ c_l & -s_l \end{pmatrix}\]

for \(l < 0\). For \(l = 0\), the Hamiltonian (7) is diagonal with degenerate zero energy, so \(\gamma_5\) is not transformed.
Figure 4 shows the diagonal and off-diagonal matrix elements of the transformed $\gamma'_\alpha$. The diagonal (1,1) element of Eq. (11) is almost unity at low and intermediate energy $p_l < 2.3$ and deviates from unity at $p_l > 2.3$. The off-diagonal (1,2) element of Eq. (11) oscillates around zero at $p_l < 2.3$ and becomes unity at $p_l = \pi$. At low energy, the deviation of the off-diagonal elements from zero is not large and becomes smaller as the parameter $M$ increases. At $p_l < 2.3$, the transformed left- and right-handed fermions have approximately the correct chiral charges 1 and $-1$, respectively. The low-energy Hamiltonian (7) has approximate chiral symmetry because the commutation relation between the Hamiltonian (7) and chiral charge defined with $\gamma'_5$ is almost zero for small $l$. The errors associated with chiral symmetry can be improved in a systematic way by increasing $M$. The value used here for the parameter $M$ is sufficiently small and does not deny application of the model to actual numerical analysis with a computer.

\[ S_E = \sum_n \left\{ \frac{1}{2a} \sum_{\alpha=1}^M \sum_{\mu=1}^2 \left[ S_\alpha \left( \bar{\psi}_n \gamma_\mu \psi_{n+\alpha} - \bar{\psi}_{n+\alpha} \gamma_\mu \psi_n \right) + C_\alpha \left( \bar{\psi}_n \psi_{n+\alpha} + \bar{\psi}_{n+\alpha} \psi_n \right) \right] + \left( m + \frac{C_0}{a} \right) \bar{\psi}_n \psi_n \right\}, \] (14)

where $n$ indicates a site on a two dimensional Euclidean lattice and $\mu$ is a unit one-site vector in the $\mu$ direction. The Euclidean gamma matrices satisfy $\gamma^\dagger_\mu = \gamma_\mu$ and \{ $\gamma_\mu, \gamma_\nu$ \} = 2$\delta_{\mu\nu}$ and $\bar{\psi} = \psi^\dagger \gamma_0$. The definition of a lattice gauge theory based on the action (14) is ambiguous. An exponentiated gauge field can take any path between two ends. The most natural choice for gauge field is the shortest path that links the two ends because fermion hopping is parallel to one of the directions $\mu$.

In this paper, we have constructed doubler-free Hamiltonian and Euclidean action for Dirac fermions on a (1+1)-dimensional lattice. To realize approximate chiral symmetry at low energy, explicit breaking of chiral symmetry has been compressed to around the momentum boundary. In future works, it should be precisely checked if insertion of gauge interactions affects chiral properties.

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