L-FUNCTIONS OF GL_{2n}:
P-ADIC PROPERTIES AND NON-VANISHING OF TWISTS

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Abstract. The principal aim of this article is to attach and study p-adic L-functions to cohomological cuspidal automorphic representations \( \Pi \) of \( GL_{2n} \) over a totally real field \( F \) admitting a Shalika model. We use a modular symbol approach, along the global lines of the work of Ash and Ginzburg, but our results are more definitive since we draw heavily upon the methods used in the recent and separate works of all the three authors. By construction our p-adic L-functions are distributions on the Galois group of the maximal abelian extension of \( F \) unramified outside \( p \). Moreover, we prove the so-called Manin relations between the p-adic L-functions at all critical points. This has the striking consequence that, given a unitary \( \Pi \) whose standard L-function admits at least two critical points, and given a prime \( p \) such that \( \Pi_p \) is ordinary, the central critical value \( L\left(\frac{1}{2}, \Pi \otimes \chi\right) \) is non-zero for all except finitely many Dirichlet characters \( \chi \) of \( p \)-power conductor.

Introduction

A crucial result in Shimura’s work on the special values of L-functions of modular forms concerns the existence of a twisting character to ensure that a twisted L-value is non-zero at the center of symmetry (see [Sh, Thm. 2]). Since then it has been a very important problem in the analytic theory of automorphic L-functions to find characters to render a twisted L-value non-zero. Rohrlich [Ro] proved such a non-vanishing result in the context of cuspidal automorphic representations of \( GL_2 \) over any number field. This was then generalized to \( GL_n \) over any number field by Barthel-Ramakrishnan [BR] and further refined by Luo [L]. However, neither [BR] nor [L] can prove this at the center of symmetry if \( n \geq 4 \). (For us the functional equation will be normalized so that \( s = 1/2 \) is the center of symmetry.) There have been other analytic machinery that has been brought to bear on this problem, for example, see Chinta-Friedberg-Hoffstein [CFH]. Even for simple situations involving L-functions of higher degree this problem is open. For example, suppose \( \pi \) is the unitary cuspidal automorphic representation associated to a primitive holomorphic cusp form for \( GL_2/\mathbb{Q} \), then it has been an open problem to find a Dirichlet character \( \chi \) so that the twisted symmetric cube L-function \( L\left(\frac{1}{2}, (\text{Sym}^3 \pi) \otimes \chi\right) \) is non-zero at the center. In this article, we prove the following result.

Theorem A. Let \( F \) be a totally real field and \( \Sigma_\infty \) the set of all its real places. Let \( \Pi \) be a unitary cuspidal automorphic representation of \( GL_{2n}(\mathbb{A}_F) \) admitting a Shalika model and such that \( \Pi_\infty \) is cohomological with respect to a pure dominant integral weight \( \mu \) such that

\[
\mu_{\sigma,n} > \mu_{\sigma,n+1}, \quad \text{for all } \sigma \in \Sigma_\infty.
\]

Assume that for all primes \( p \) above a given prime number \( p \), \( \Pi_p \) is unramified and \( U_p \)-ordinary (see [GM]). Then, for all but finitely many Dirichlet characters \( \chi \) of \( p \)-power conductor we have:

\[
L\left(\frac{1}{2}, \Pi \otimes (\chi \circ N_{F/\mathbb{Q}})\right) \neq 0.
\]

2010 Mathematics Subject Classification. Primary: 11F67; Secondary: 11S40, 11F55, 11F70.
For notions and notations that are not defined in the introduction, the reader will have to consult the main body of the paper. A more general statement is proven in Theorem 4.10. Furthermore, we can prove a stronger non-vanishing result covering the nearly-ordinary case (see Corollary 4.11) as well as a simultaneous non-vanishing result (see Corollary 4.11). Namely, suppose for 1 ≤ j ≤ r, Πj is a representation of GL2n_j(𝔸_F) as above, and p is ordinary for all of them, then

\[ L \left( \frac{1}{2}, \Pi_1 \otimes \chi \right) \cdots L \left( \frac{1}{2}, \Pi_r \otimes \chi \right) \neq 0 \]

for all but finitely many Hecke characters χ of p-power conductor. For example, with a classical normalization of L-functions, it follows from our results that there are infinitely many Dirichlet characters χ such that \( L(6, \Delta \otimes \chi)L(17, \text{Sym}^3(\Delta) \otimes \chi) \neq 0 \) for the Ramanujan \( \Delta \)-function. Our methods are purely arithmetic and involve studying \( p \)-adic distributions on the Galois group of the maximal abelian extension of \( F \) unramified outside \( p \) and \( \infty \) that are attached to suitable eigen-classes in the cohomology of \( GL_{2n} \).

Let’s now describe our methods and results in greater detail. We begin with a purely cohomological statement, without any reference to automorphic forms or L-functions. Let \( \mathcal{O}_F \) be the ring of integers of \( F \) and \( \mathfrak{d} \) its different. Take a pure dominant integral weight \( \mu \) for \( G = \text{Res}_{\mathcal{O}_F/\mathbb{Z}} GL_{2n} \), and let \( \mathcal{V}^\mu_E \) be the algebraic irreducible representation of \( G(E) \), for some ‘large enough’ \( p \)-adic field \( E \). If \( \mathcal{O} \) is the ring of integers of \( E \), then we also consider an \( \mathcal{O} \)-lattice \( \mathcal{V}^\mu_{\mathcal{O}} \) stabilized by \( G(\mathcal{O}) \). For any open compact subgroup \( K \subset G(\mathbb{A}_F) \) of the group of finite adeles, let \( \mathcal{V}^\mu_{\mathcal{O}} \) be the associated sheaf on the locally symmetric space \( S^G_K \) of \( G \) with level structure \( K \) and let’s consider the compactly supported cohomology \( H^q_G \left( S^G_K, \mathcal{V}^\mu_{\mathcal{O}} \right) \). As usual there is a Hecke action on \( H^q_G \left( S^G_K, \mathcal{V}^\mu_{\mathcal{O}} \right) \), and when \( K_p \) is the parahoric subgroup corresponding to the parabolic \( Q \) of type \((n,n)\) of \( G \) we consider an eigenclass \( \phi \in H^q_{c}(S^G_K, \mathcal{V}^\mu_{\mathcal{O}}) \) with eigenvalue \( \alpha_p^\phi \) for a particular Hecke operator \( U_p^\phi \), where the cohomology-degree \( q_0 \) is the dimension of a locally symmetric space for the Levi subgroup \( H = \text{Res}_{\mathcal{O}_F/\mathbb{Z}}(GL_n \times GL_n) \) of \( Q \). The weight \( \mu \) determines a contiguous string of integers \( \text{Crit}(\mu) \) which would correspond to the set of critical points for an \( L \)-function. For each \( j \in \text{Crit}(\mu) \) we attach an \( E \)-valued distribution \( \mu^j_{\phi} \) of growth at most \( v_p(\alpha_p^\phi) \) on \( \mathcal{O}_F^+(p^\infty) \). In particular, when \( \phi \) is \( U_p^\phi \)-ordinary then \( \mu^j_{\phi} \) is \( \mathcal{O} \)-valued, i.e., it is a measure. In \( \mathcal{O}_F \) we construct and study these distributions (see diagram 5.1) to get a quick overview of the sheaf-theoretic maps that are needed in their definition. Most importantly we prove a Manin type relation, namely

\[ \varepsilon_{\text{cyc}}^{j-j'}(\mu^j_{\phi}) = \mu^{j'}_{\phi}, \]

for all \( j, j' \in \text{Crit}(\mu) \), where \( \varepsilon_{\text{cyc}} \) denotes the universal cyclotomic character (see Theorem 2.4), allowing us to define a distribution \( \mu_{\phi} = \varepsilon_{\text{cyc}}(\mu_{\phi}) \) which is independent of \( j \).

Next we apply the above considerations to the situation when \( \phi \) is related to a cuspidal automorphic representation \( \Pi \) of \( G(\mathbb{A}) \) such that \( \Pi_\infty \) is cohomological with respect to the weight \( \mu \) (see 4.1.3). Friedberg and Jacquet related the period integral of cusp forms in \( \Pi \) over the subgroup \( H \) to the standard L-function \( L(s, \Pi) \), and for the unfolding of this integral to see the Eulerian property the representation is assumed to have a Shalika model (see 4.1.1). Such a cohomological interpretation was used in [GR2] to deduce algebraicity results for the critical values of \( L(s, \Pi \otimes \chi) \). The following result further investigates their \( p \)-adic integrality properties, under the assumption of \( Q \)-regularity at \( p \) (see Definition 3.6) which is shown to be always fulfilled when \( \Pi_p \) is \( U_p \)-ordinary for \( p \mid p \) (see Lemma 4.3).
Theorem B. Let $\Pi$ be a cuspidal automorphic representation of $GL_{2n}(\mathbb{A}_F)$ admitting a Shalika model and such that $\Pi_{\infty}$ is cohomological of weight $\mu$. Assume that for all primes $p$ above a given prime number $p$, $\Pi_p$ is spherical and admits a $Q$-regular refinement $\tilde{\Pi}_p$. There exists a distribution $\mu_{\Pi}$ on $\mathcal{O}_F^+(p^\infty)$ of growth at most $v_p(\alpha^0_p)$. Under the assumption:

$$v_p(\alpha^0_p) < \min_{\sigma \in \Sigma_{\infty}} (\mu_{\sigma,n} - \mu_{\sigma,n+1} + 1),$$

the push-forward of $\mu_{\Pi}$ by the norm $N_{F/Q} : \mathcal{O}_F^+(p^\infty) \to \mathcal{O}_Q^+(p^\infty)$ is uniquely determined by the following interpolation property: for any $j \in \text{Crit}(\mu)$ and for any finite order character $\chi$ of $\mathcal{O}_F^+(p^\infty)$ of conductor $\beta_p \geq 1$ at all $p \mid p$, letting $G(\chi_f)$ denote its Gauss sum one has

$$\int_{\mathcal{O}_F^+(p^\infty)} e^x(x) d\mu_{\Pi}(x) =$$

$$\gamma \cdot N_{F/Q}^{-n}(0) \prod_{p \mid p} \left( \alpha_p^{-1}q_p^{-n(j+1)} \right)^{\beta_p} \cdot \frac{G(\chi_f)^n \cdot L(j + \frac{1}{2}; \Pi \otimes \chi_f)}{\zeta_{\infty}^{-1}(j + \frac{1}{2}; \Omega_{\Pi_{\infty},j}) \cdot \chi_{\infty} \cdot \Omega_{\Pi}(\epsilon \chi_{\infty})},$$

where $\gamma \in \mathbb{Q}_\infty$ and the periods in the denominator are defined in Theorem 1.4.

Let us hint on how we deduce Theorem A. Theorem B whose formulation implicitly uses the earlier established Manin relations gives congruence relations between successive critical values, while (1) translates into a similar context, but his methods are entirely different from ours.

Let’s mention some relevant papers in the literature. First of all, Ash and Ginzburg [AG] started the study of $p$-adic $L$-functions for $GL_{2n}$ over a totally real field by considering the analytic theory developed by Friedberg and Jacquet [FJ]. However to quote the authors of [AG], their results are definitive only for $GL_4$ over $\mathbb{Q}$ and for cohomology with constant coefficients. Furthermore, they construct their distributions on local units while only suggesting that one should really work, as we do in this paper, on $\mathcal{O}_F^+(p^\infty)$ (see also the issues raised in [J]). This article uses the more recent results and techniques developed in independent papers by all the three authors; namely, [D], [BDJ], [GR2], and [J2, J3, J5]. Finally, we mention Gehrmann’s thesis [G] which also constructs $p$-adic $L$-functions in essentially a similar context, but his methods are entirely different from ours.

To conclude the introduction, our emphasis is on the purely sheaf-theoretic nature of the construction of the distributions attached to eigenclasses in cohomology which leads to a purely algebraic proof of Manin relations in a very general context. When specialized to a cohomology class related to a representation $\Pi$ of $GL_{2n}$, we get $p$-adic interpolation of the critical values of the standard $L$-function $L(s, \Pi)$, and Manin relations give non-vanishing of twists $L(s, \Pi \otimes \chi)$ at the center of symmetry. A non-vanishing theorem in the realms of analytic number theory admitting a decidedly algebraic proof is philosophically piquant.

Acknowledgements: This project started when the three of us met at a conference in July 2014 at IISER Pune on $p$-adic aspects of modular forms. Any subset of two of the authors is grateful to the host institute or university of the third author during various stages of this work. MD and AR are grateful to an Indo-French research grant from CEFIPRA that has facilitated visits by each to the work-place of the other. AR is grateful to Charles Simonyi Endowment that funded his stay at the Institute for Advanced Study, Princeton.
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References

1. Automorphic cohomology

Recall that \( F \) is a totally real number field with ring of integers \( \mathcal{O}_F \) and set of infinite places \( \Sigma_\infty \). For a set of places \( \Sigma \), we denote by \( \Lambda^{(\Sigma)} \) the topological ring of adeles of \( \mathbb{Q} \) outside \( \Sigma \). Let \( \Lambda_F = \Lambda \otimes_{\mathbb{Q}} F \) (resp. \( \Lambda_{F,f} \)) be the group of adeles (resp. finite adeles) of \( F \).

We consider \( G := \text{Res}_{\mathcal{O}_F/\mathbb{Z}}(GL_{2n}) \) as a reductive group scheme over \( \mathbb{Z} \), quasi-split over \( \mathbb{Q} \) and let \( Z := \text{Res}_{\mathcal{O}_F/\mathbb{Z}}(GL_1) \) be the center of \( G \). The standard Borel subgroup \( B \subseteq G \) is defined as the restriction of scalars of the standard Borel subgroup of all upper triangular matrices in \( GL_{2n}/\mathcal{O}_F \). We have \( B = TN \), where \( N \) is the unipotent radical of \( B \) and \( T \) is the standard torus of all diagonal matrices. Let \( H := \text{Res}_{\mathcal{O}_F/\mathbb{Z}}(GL_n \times GL_n) \), and \( \iota : H \hookrightarrow G \) be the map that sends \( (h_1, h_2) \) to \( \left( \begin{array}{cc} h_1 & 0 \\ 0 & h_2 \end{array} \right) \). Let \( Q = HU \) be the standard parabolic subgroup of type \( (n, n) \) whose Levi subgroup is \( H \) and unipotent radical is \( U \). Finally, the Shalika subgroup \( S \) of \( G \) is defined as \( S = \left\{ \left( \begin{array}{cc} h & 0 \\ 0 & h \end{array} \right) \left( \begin{array}{cc} 1 & X \\ 0 & 1 \end{array} \right) | h \in GL_n, X \in M_n \right\} \).

For any commutative ring \( A \), we let \( g_A, b_A, q_A, t_A, \mathfrak{h}_A, n_A \) and \( u_A \) stand for the Lie algebras of \( G, B, Q, T, H, N \) and \( U \) over \( A \), respectively. For \( a_A \) any amongst these, we let \( \mathcal{U}(a_A) \) stand for the enveloping algebra over \( A \). In the particular case \( A = \mathbb{R} \), let \( g_\infty := \mathfrak{g}_R \otimes \mathbb{R} \mathbb{C} \) denote the complexification and likewise for the other groups.

For any real reductive Lie group \( G \) we let \( G^0 \) denote the connected component of the identity. Let \( G_\infty = G(\mathbb{R}) \), and similarly \( Z_\infty = Z(\mathbb{R}) \).

1.1. Pure weights. We identify integral weights \( \mu \) of \( T \) with tuples of weights \( \mu = (\mu_\sigma)_{\sigma \in \Sigma_\infty} \) where \( \mu_\sigma = (\mu_{\sigma,1}, \ldots, \mu_{\sigma,2n}) \in \mathbb{Z}^{2n} \). The \( \mathbb{Q} \)-structure on \( G \) induces a natural
pure dominant integral weights of $T$. A weight $\mu$ is $B$-dominant if
\begin{equation}
\mu_{\sigma,1} \geq \cdots \geq \mu_{\sigma,2n}, \text{ for all } \sigma \in \Sigma_\infty.
\end{equation}
Let $X_+^\sigma(T)$ be the set of all such dominant integral weights. For $\mu \in X_+^\sigma(T)$ denote by $V^\mu$ the unique algebraic irreducible rational representation of $G$ of highest weight $\mu$. It is defined over the field of rationality $\Q(\mu)$ of $\mu$, and for any extension $E/\Q(\mu)$ we denote by $V^\mu_E = V^\mu_{\Q(\mu)} \otimes_{\Q(\mu)} E$ its $E$-valued points. Denote by $\mu^\vee$ the highest weight of the contragredient $(V^\mu)^\vee$ of $V^\mu$ which we consider as a rational character of $B$.

We call $\mu$ pure if there exists $w \in \Z$, called the purity weight of $\mu$, such that
\begin{equation}
V^\mu = V^{\mu^\vee} \otimes (N_{F/Q} \circ \det)^w,
\end{equation}
where $N_{F/Q} : \text{Res}_{F/Q}(GL_1) \to GL_1$ denotes the norm homomorphism. If $\mu$ is pure then
\begin{equation}
\mu_{\sigma,i} + \mu_{\sigma,2n-i+1} = w, \text{ for all } \sigma \in \Sigma_\infty.
\end{equation}
In particular $\sum_{i=1}^{2n} \mu_{\sigma,i} = wn$ is independent of $\sigma$. We let $X^\sigma_0(T) \subset X^\sigma_+(T)$ stand for the pure dominant integral weights of $T$. Given any $\mu \in X^\sigma_0(T)$, define the set
\begin{equation}
\text{Crit}(\mu) := \{ j \in \Z \mid \mu_{\sigma,n} \geq j \geq \mu_{\sigma,n+1}, \forall \sigma \in \Sigma_\infty \}.
\end{equation}
It is well known that only pure weights support cuspidal cohomology, and the motivation for this definition comes from the fact proved in [GR2, Prop.6.1] that if $\Pi$ is a cuspidal automorphic representation of $G(\A)$ having cohomology with respect to $\mu$ (see [LL1,3]) then $\frac{1}{2} + j$ with $j \in \Z$ is critical for the standard $L$-function $L(s, \Pi \otimes \chi)$ for any finite order character $\chi$ if and only if $j \in \text{Crit}(\mu)$. Note that the central point $\frac{w+1}{2}$ of $L(s, \Pi \otimes \chi)$ is critical, (i.e., $\frac{w}{2} \in \text{Crit}(\mu)$) if and only if $w$ is even.

1.2. Integral lattices. Let $E$ be a finite extension of $\Q_p$ and let $\mathcal{O}$ be its ring of integers. Given $\mu \in X^\sigma_+(T)$, we consider $V^\mu_\mathcal{O}$ as a representation of $G(E)$.

Let $v_0 \in V^\mu_\mathcal{O}$ be a non-zero lowest weight vector. Then the unipotent radical $N^-(E)$ of the Borel subgroup $B^-(E)$ of lower triangular matrices fixes $v_0$, while $T(E)$ acts on $v_0$ via the character $-\mu^\vee = w_{2n}(\mu)$ where $w_{2n}$ is the Weyl group element of longest length.

Observe that
\begin{equation}
V^\mu_\mathcal{O} := \mathcal{U}(\mathfrak{n}_\mathcal{O})v_0
\end{equation}
is an $\mathcal{O}$-lattice $V^\mu_\mathcal{O}$ endowed with a natural action of $G(\mathcal{O})$.

We fix once and for all for uniformizers $\varpi_p \in F_p$ and put $t_p = \iota(\varpi_p \cdot 1_n, 1_n) \in GL_{2n}(F_p)$. Define for any integral multi-exponent $\beta = (\beta_p)_{p|\ell}$ the element
\begin{equation}
t^\beta_p := \prod_{p|\ell} t^\beta_p \in T(\mathbb{Q}_p),
\end{equation}
and consider the semi-group
\begin{equation}
\Delta_p^+ := \{ t^\beta_p \mid \beta_p \in \Z_{\geq 0}, \forall p \mid \ell \}.
\end{equation}
Then by our choice of dominance condition, we have for any $t \in \Delta_p^+$:
\begin{equation}
\text{Ad}(t)Q(\mathcal{O}) = tQ(\mathcal{O})t^{-1} \subseteq Q(\mathcal{O}) \text{ and } \text{Ad}(t^{-1})U^-(\mathcal{O}) = t^{-1}U^-(\mathcal{O})t \subseteq U^-(\mathcal{O}).
\end{equation}
Consider the standard maximal parahoric subgroup $J_p = \prod_{p|\ell} J_p \subseteq G(\Z_p)$, where
\begin{equation}
J_p = t_p^{-1}GL_{2n}(O_{F,p})t_p \cap GL_{2n}(O_{F,p}).
\end{equation}
Since $J_p \supset Q(\mathbb{Z}_p)$ the parahoric decomposition is given by
\begin{equation}
J_p = (J_p \cap U^-(\mathbb{Z}_p))Q(\mathbb{Z}_p) = Q(\mathbb{Z}_p)(J_p \cap U^-(\mathbb{Z}_p)).
\end{equation}
Using (9) and (11) one sees that
\begin{equation}
\Lambda_p := \text{Ind}_G K \Delta_p K = Q(\mathbb{Z}_p)\Delta_p (J_p \cap U^-(\mathbb{Z}_p))
\end{equation}
is a semi-group. Moreover since $U^-(\mathbb{Z}_p) \subset N^-(\mathbb{Z}_p)$ acts trivially on $v_0$, the $J_p$-action on $V^\mu$ extends uniquely to an action $\bullet$ of the semi-group $\Lambda_p$ by letting $\Delta_p$ act trivially on the lowest weight vector $v_0$. Then for all $t \in \Delta_p$ and $v \in V^\mu_0$ one has:
\begin{equation}
t \bullet v = \mu^\nu(t)(t \cdot v)
\end{equation}
In fact by (6) one can write $v = m \cdot v_0$ for some $m \in U(n,\mathbb{Q})$ and using (9) one finds:
\[ t \bullet v = t \bullet (m \bullet v_0) = \text{Ad}(t)(m) \bullet (t \cdot v_0) = \mu^\nu(t)\text{Ad}(t)(m)(t \cdot v_0) = \mu^\nu(t)(t \cdot v). \]

1.3. Local systems on locally symmetric spaces for $\text{GL}_{2n}$. The standard maximal compact subgroup of $G_{\infty}$ is denoted $C^\infty = \prod_{\sigma \in \Sigma} C_{\sigma}$, where $C_{\sigma} \simeq O_{2n}(\mathbb{R})$. The determinant identifies the group of connected components $C^\infty / C^\infty_0$ with $\{ \pm 1 \}^{\Sigma \infty}$. Let $K_{\infty} = C^\infty Z_{\infty}$ and for any open compact subgroup $K$ of $G(\mathbb{A})$, we consider the locally symmetric space:
\begin{equation}
S^G_\infty := G(\mathbb{Q}) \backslash G(\mathbb{A}) / K K^\infty = G(\mathbb{Q}) \backslash (G(\mathbb{R}) / K_{\infty}) \times G(\mathbb{A}) / K.
\end{equation}
Note that $K^\infty = C^\infty Z^\infty = C^\infty_{\infty} Z_{\infty}$ since $2n$ is even. In general, $S^G_\infty$ is only a real orbifold. In the sequel we assume that $K$ is sufficiently small in the sense that for all $g \in G(\mathbb{A})$,
\begin{equation}
G(\mathbb{Q}) \cap g K K^\infty g^{-1} = Z(\mathbb{Q}) \cap K K^\infty,
\end{equation}
which implies in particular that $S^G_\infty$ is a real manifold.

Given a left $G(\mathbb{Q})$-module $V$ one can define $\mathcal{V}_K$ as the sheaf of locally constant sections of the local system:
\[ G(\mathbb{Q}) \backslash (G(\mathbb{A}) \times V) / K K^\infty \to S^G_\infty, \]
where $\gamma (g, v) k = (\gamma g k, \gamma \cdot v)$ for all $\gamma \in G(\mathbb{Q})$, $k \in K K^\infty$. Consider the canonical fibration $\pi : (G(\mathbb{R}) / K_{\infty}) \times G(\mathbb{A}) / K \to S^G_\infty$ given by going modulo the left action of $G(\mathbb{Q})$. Then for any open $U \subset S^G_\infty$ one has the sections $\mathcal{V}_K(U)$ over $U$ to be the set of all locally constant $s : \pi^{-1}(U) \to V$ such that $s(\gamma \cdot x) = \gamma \cdot s(x)$ for all $\gamma \in G(\mathbb{Q})$, $x \in \pi^{-1}(U)$. We denote by $\mathcal{V}_{K,E}$ the sheaf associated to $V^\mu_0$. The sheaf $\mathcal{V}_{K,E}$ is non-trivial if and only if
\begin{equation}
\mu(Z(\mathbb{Q}) \cap K K^\infty) = \{ 1 \}.
\end{equation}
Condition (10) is always satisfied if $\mu$ is pure, since $\det(F^\infty \cap K K^\infty_{\infty}) \subset \mathcal{O}_{F,+}^\times$.

In order to attach a sheaf to $V^\mu_0$ we need a slightly different construction. Given a left $K$-module $V$ satisfying (10) define $\mathcal{V}_K$ instead as the sheaf of locally constant sections of:
\[ G(\mathbb{Q}) \backslash (G(\mathbb{A}) \times V) / K K^\infty \to Y, \]
with left $G(\mathbb{Q})$-action and right $K K^\infty$-action given by $\gamma (g, v) k = (\gamma g k, k^{-1} \cdot v)$. Since $K$ acts on $V^\mu_0$ through its $p$-component $K_p \subset G(\mathbb{Z}_p) \subset G(\mathbb{O})$ we obtain a sheaf $\mathcal{V}^\mu_0$ on $S^G_\infty$.

When the actions of $G(\mathbb{Q})$ and $K$ on $V$ extend compatibly into a left action of $G(\mathbb{A})$, the two resulting local systems are isomorphic by $(g, v) \mapsto (g, g^{-1} \cdot v)$, justifying the abuse of notations.
1.4. Hecke operators. For any open compact subgroups $K' \subseteq K$ of $G(\mathbb{A}_f)$ the natural map $p_{K', K} : S^G_{K'} \to S^G_K$ induces an isomorphism of sheaves $p_{K', K}^* \sim \mathcal{V}_K \to \mathcal{V}_{K'}$.

When the $K$-action on $V$ extends to an action of a semi-group containing $K$ and $\gamma$, then one can define a Hecke operator $[K\gamma K]$ as a composition of three maps:

$$[K\gamma K] = \text{Tr}(p_{\gamma K\gamma^{-1} K,K}) \circ [\gamma] \circ p_{K\gamma^{-1} K,K}^* : H^0_c(S^G_{K\gamma^{-1} K}, \mathcal{V}_K) \to H^0_c(S^G_K, \mathcal{V}_K),$$

where $p_{K\gamma^{-1} K,K}^*$ is the pull-back, $\text{Tr}(p_{\gamma K\gamma^{-1} K,K})$ is the finite flat trace and

$$[\gamma] : H^0_c(S^G_{K\gamma^{-1} K\gamma}, \mathcal{V}_K \otimes \mathcal{V}_K) \to H^0_c(S^G_{K\gamma^{-1} K}, \mathcal{V}_K \otimes \mathcal{V}_K_{K\gamma^{-1} K})$$

is induced by the morphism of local systems given by $(g, v) \mapsto (g\gamma^{-1}, \gamma \cdot v)$ in the case of a right $K$-action.

When $K_p \subset J_p$, the above construction applies to $V^\mu_O$ on which the semi-group $\Lambda_p$ acts by the $\bullet$-action (see (12)) yielding for each $t \in \Delta_p^+$ a Hecke operator $[KtK]$ on $H^0_c(S^G_{K}, \mathcal{V}^\mu_{K, O})$. Note that while the natural inclusion $V^\mu_{\mathcal{O}} \subseteq V^\mu_K$ is $K_p$-equivariant, it is not $\Lambda_p$-equivariant (see (13)). As a consequence the natural map $H^0_c(S^G_{K}, \mathcal{V}^\mu_{K, O}) \to H^0_c(S^G_{K}, \mathcal{V}^\mu_{F})$ is equivariant for the $\bullet$-action of $[KtK]$ on the source and the action of optimally integral Hecke operator $[KtK]^{\circ} := \mu^\vee(t)[KtK]$ on the target. To ensure compatibility with extension of scalars we will also denote $[KtK]^{\circ}$ the Hecke operator $[KtK]$ acting (via the $\bullet$-action) on $H^0_c(S^G_{K}, \mathcal{V}^\mu_{K, O})$.

For any prime $p \mid p$ of $F$ the following Hecke operators will play an important role:

$$U_p := [Kt_p K] \quad \text{and} \quad U_{p, \delta}^\circ = \mu^\vee(t_p) U_p.$$  \hspace{1cm} (17)

For $\beta = (\beta_p)_{p\mid p}$ with $\beta_p \in \mathbb{Z}_{>0}$ we let $U_{p, \beta} := [Kt_p \beta K]$ and $U_{p, \beta}^\circ = \mu^\vee(t_p) U_{p, \beta}$.

Since $\text{im}(H^0_c(S^G_{K}, \mathcal{V}^\mu_{K, O}) \to H^0_c(S^G_{K}, \mathcal{V}^\mu_{F}))$ is a finitely generated $\mathcal{O}$-module, we may assume that $E$ is large enough so that all $U_{p, \beta}^\circ$-eigenvalues belong to $\mathcal{O}$.

2. Distributions attached to cohomology classes for $GL_{2n}$

Let $E = F \otimes \mathbb{Q}_p \equiv \prod_{p \mid p} F_p$. For a prime $p \mid p$ of $F$ we denote by $J_p$ (resp., $J_p$) the standard Iwahori (resp., parahoric) subgroup of $K_p := GL_{2n}(\mathcal{O}_{F_p})$ consisting of elements whose reduction modulo the $p$ belongs to $B(\mathcal{O}_F/p)$ (resp., to $Q(\mathcal{O}_F/p)$).

We let $K = K^{(p)} \times \prod_{p \mid p} K_p$ be an open compact subgroup of $G(\mathbb{A})$ such that:

(K1) $K^{(p)}$ is the principal congruence subgroup of modulus $m$, an ideal of $\mathcal{O}_F$ which is relatively prime to $p$, and $K^{(p)}G(\mathbb{Z}_p)$ satisfies (13),

(K2) $\left( \begin{array}{cc} T_n(\mathcal{O}_{F_{p, n}}) & M_n(\mathcal{O}_{F, p}) \\ 0_n & T_n(\mathcal{O}_{F_{p, n}}) \end{array} \right) \leq K_p \leq J_p$ for all $p \mid p$.

An important role will be played by the matrix $\xi_p \in GL_{2n}(\mathbb{A}_F)$, where

$$\xi_p := \left( \begin{array}{cc} 1_n & w_n \\ 0_n & w_n \end{array} \right) \in GL_{2n}(\mathcal{O}_{F, p}), \quad \text{for all } p \mid p, \quad \text{and } \xi_v = 1_{2n}, \quad \text{for all } v \mid p.$$

where $1_n$ and $0_n$ are the $n \times n$ identity and zero matrices, respectively, and $w_n$ is the longest length element in the Weyl group of $GL_n$, whose $(i, j)$-entry is $w_n(i, j) = \delta_{i, n-j+1}$.
We have \( \xi_p^{-1} = (\frac{1}{h_n} - \frac{1}{w_n}) \). Once and for all we record the identities

\[
\xi_p^{-1} \cdot \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \xi_p = \left( \begin{array}{cc} A - C & (A - D + B - C)w_n \\ w_nC & (C + D)w_n \end{array} \right), \quad \text{and}
\]

\[
\xi_p \cdot \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \xi_p^{-1} = \left( \begin{array}{cc} A + w_nC & Dw_n - A + Bw_n - w_nC \\ w_nC & Dw_n - w_nC \end{array} \right).
\]

2.1. Automorphic cycles. For any open-compact subgroup \( L \subset H(\mathbb{A}_f) \) we consider the locally symmetric space:

\[
\tilde{S}_L^H := H(\mathbb{Q}) \backslash H(\mathbb{A}) / LL_\infty, \quad \text{where} \quad L_\infty = H_\infty \cap K_\infty.
\]

Note that for each \( \sigma \in \Sigma_\infty \) one has \( L^0\sigma \simeq \left( \begin{array}{cc} SO_n(\mathbb{R}) & 0 \\ 0 & SO_2(\mathbb{R}) \end{array} \right) \mathbb{R}^{\infty} \). As in (15), \( \tilde{S}_L^H \) is a real manifold when \( L \) is sufficiently small in the sense that for all \( h \in H(\mathbb{A}) \),

\[
H(\mathbb{Q}) \cap hLL_\infty h^{-1} = Z(\mathbb{Q}) \cap LL_\infty.
\]

Recall the notation \( t_p = i(\varpi_p \cdot 1_n, 1_n) \) where \( \varpi_p \) is an uniformizer at \( p \mid p \). Recall also that for \( \beta = (\beta_p)_p \) with \( \beta_p \in \mathbb{Z}_{\geq 0} \) we let \( p^\beta = \prod_p \varpi_p^{\beta_p} \) and \( t_p^\beta = \prod_p t_p^{\beta_p} \in G(\mathbb{Q}_p) \).

For any ideal \( m \) of \( \mathcal{O}_F \), we denote by \( I(m) \) the open-compact subgroup of \( \mathbb{A}_{K,F} \) of modulus \( m \), and we consider the strict idele class group:

\[
\mathcal{O}_F^+(m) := F^\times \backslash \mathbb{A}_F^\times / I(m) F^\times.
\]

We let \( L_\beta = L^{(p)} \prod_p L_p^\beta \) be an open compact subgroup of \( H(\mathbb{A}_f) \) such that:

- (L1) \( L^{(p)} = K^{(p)} \cap H \) is the principal congruence subgroup of modulus \( m \), and
- (L2) \( L_p^\beta = H(F_p) \cap K_p \cap \xi_p^{\beta_p} K_p t_p^{-\beta_p} \xi^{-1} \) for all \( p \mid p \).

Note that conditions (11) and (L1) imply (22), in particular \( \tilde{S}_L^H \) is a real manifold.

Lemma 2.1. \( L_p^\beta \) consists of elements \( (h_1, h_2) \in GL_n(\mathcal{O}_{F,p}) \times GL_n(\mathcal{O}_{F,p}) \) such that

\[
i(h_1, h_2) \in K_p \cap \left( \begin{array}{cc} 1_n & w_n \\ w_n & 1_n \end{array} \right) K_p \left( \begin{array}{cc} 1_n & w_n \\ w_n & 1_n \end{array} \right), \quad \text{and} \quad h_1 h_2^{-1} \in 1 + \varpi_p^{\beta_p} M_n(\mathcal{O}_{F,p}).
\]

Proof. By (15) for all \( (h_1, h_2) \in H(F_p) \cap K_p = GL_n(\mathcal{O}_{F,p}) \times GL_n(\mathcal{O}_{F,p}) \) one has

\[
t_p^{-\beta_p} \xi^{-1} \left( \begin{array}{cc} h_1 \\ h_2 \end{array} \right) \xi_p^{\beta_p} = \left( \begin{array}{cc} h_1 \varpi_p^{-\beta_p(h_1 - h_2)w_n} \\ h_2 \varpi^{-\beta_p(h_1 - h_2)w_n-n} \end{array} \right).
\]

Hence \( h_1 - h_2 \in \varpi_p^{\beta_p} M_n(\mathcal{O}_{F,p}) \), and as \( \left( \begin{array}{cc} 1_n \varpi^{-\beta_p(\mathcal{O}_{F,p})} \\ 0_n \varpi^{-\beta_p(\mathcal{O}_{F,p})} \end{array} \right) \) we obtain \( (h_1, h_2^{-1}) \in K_p \). \( \square \)

Lemma 2.1 implies that the map \( \left( 1 + \varpi_p^{\beta_p} M_n(\mathcal{O}_{F,p}) \right) \times \mathcal{O}_{F,p}^\times \rightarrow \det(L_p^\beta) \) sending \( (x, y) \) to \( (xy, y) \) is an isomorphism. By the strong approximation theorem for \( SL_n(\mathbb{A}_F) \) the map

\[
(h_1, h_2) \mapsto \left( \frac{\det(h_1)}{\det(h_2)}, \det(h_2) \right)
\]

identifies the set of connected components of \( \tilde{S}_L^H \) with a product of two idele class groups:

\[
\pi_0(\tilde{S}_L^H) \xrightarrow{\sim} \mathcal{O}_F^+(p^n \mathfrak{m}) \times \mathcal{O}_F^+(\mathfrak{m}).
\]

It is easy to see that the fibre \( \tilde{S}_L^H[\delta] \) of \( [\delta] \in \pi_0(\tilde{S}_L^H) \) is connected of dimension

\[
q_0 := [F : \mathbb{Q}](n^2 + n - 1).
\]
If we consider a cohomology class on $S^G_K$ in degree $q_0$, and pull it back to $\tilde{S}^H_{L_\beta}$, then we end up with a top-degree class. The degree $q_0$ happens to be the top-most degree with non-vanishing cuspidal cohomology of $S^G_K$. This magical numerology is at the heart of what ultimately permits us to give a cohomological interpretation to an integral representing an $L$-value (see [GR3]) and allows us to study its $p$-adic properties.

2.2. Evaluation maps.

2.2.1. Automorphic symbols. By [L2] the map
\[ \iota_\beta : \tilde{S}^H_{L_\beta} \rightarrow S^G_K, \quad [h] \mapsto [\iota(h)\xi^\beta], \]
is well-defined. Since $\iota_\beta$ is proper by a well-known result of Borel and Prasad (see, for example, [A] Lem.2.7) one can consider the pull-back:
\[ \iota^*_\beta : H^j_c(S^G_K, V^\mu) \rightarrow H^j_c(\tilde{S}^H_{L_\beta}, \iota^*_\beta V^\mu). \]

2.2.2. Twisting. By [L1] the map $\iota : \tilde{S}^H_{L_\beta} \rightarrow S^G_K$, $[h] \mapsto [\iota(h)]$ is well-defined and proper. Since $\iota^*_\beta \in \Lambda_p$, using the $\bullet$-action from [M] one can consider the map
\[ H(\mathbb{A}) \times V^\mu_\mathbb{O} \rightarrow H(\mathbb{A}) \times V^\mu_\mathbb{O}, \quad (h, v) \mapsto (h, (\xi^\beta) \bullet v) \]
inducing a homomorphism of sheaves $\tau^\beta_\lambda : \iota^* V^\mu_\mathbb{O} \rightarrow \iota^* V^\mu_\mathbb{O}$ hence a map in cohomology
\[ \tau^\beta_\lambda : H^j_c(\tilde{S}^H_{L_\beta}, \iota^*_\beta V^\mu_\mathbb{O}) \rightarrow H^j_c(\tilde{S}^H_{L_\beta}, \iota^* V^\mu_\mathbb{O}). \]

Similarly using the natural action of $G(E)$ on $V^\mu_E$ instead of the $\bullet$-action one defines a map
\[ \tau_\beta : H^j_c(\tilde{S}^H_{L_\beta}, \iota^*_\beta V^\mu_E) \rightarrow H^j_c(\tilde{S}^H_{L_\beta}, \iota^* V^\mu_E), \]
and $\tau_\beta = \mu^\vee(t_p^{-\beta}) \tau^\beta_\lambda$, since by [L3] one has $(\xi^\beta) \bullet v = \mu^\vee(t_p^{-\beta})(\xi^\beta \bullet v)$ for all $v \in V^\mu_E$.

2.2.3. Critical maps. For $j_1, j_2 \in \mathbb{Z}$ let $V^{(j_1 j_2)}$ be the 1-dimensional $H$-representation
\[ (h_1, h_2) \mapsto N_{F/\mathbb{Q}}(\det(h_1))^{j_1} \det(h_2)^{j_2}. \]

Let $V^{(j_1 j_2)}$ be a free rank one $\mathcal{O}$-module on which the above defined natural $H(\mathbb{Z}_p)$-action is extended to a $H(\mathbb{Q}_p)$-action by letting $p \in \mathbb{Q}_p^\times$ act trivially. That this action is similar to the $\Lambda_p$-action on $V^\mu_\mathbb{O}$ defined in [L2].

It follows from [GR2] Prop.6.3 that $j \in \text{Crit}(\mu)$ (see (5)) if and only if
\[ \dim \left( \text{Hom}_H(V^\mu, V^{(j, w-j)}) \right) = 1. \]

Fix a non-zero $\kappa_j \in \text{Hom}_H(V^\mu, V^{(j, w-j)})$ normalized so as to get an integral map:
\[ \kappa_j : V^\mu_\mathbb{O} \rightarrow V^{(j, w-j)}_\mathbb{O}. \]

Denoting $V^{(j, w-j)}_\mathbb{O}$ the sheaf on $\tilde{S}^H_{L_\beta}$ attached to $V^{(j, w-j)}$ by the construction described in [L3] one obtains a homomorphism:
\[ \kappa_j : H^j_c(\tilde{S}^H_{L_\beta}, \iota^* V^\mu_\mathbb{O}) \rightarrow H^j_c(\tilde{S}^H_{L_\beta}, V^{(j, w-j)}_\mathbb{O}). \]

Putting (26), (27) and (30) together, for or each $j \in \text{Crit}(\mu)$, we get a map:
\[ \kappa_j \circ \tau^\beta_\lambda \circ \iota^*_\beta : H^j_c(S^G_K, V^\mu_\mathbb{O}) \rightarrow H^j_c(\tilde{S}^H_{L_\beta}, V^{(j, w-j)}_\mathbb{O}). \]
2.2.4. **Trivializations.** Given any \( \delta \in H(\mathbb{A}) \) the map

\[
\text{triv}_\delta : H(\mathbb{Q})\delta L_\beta H_\infty^0 \times V_{\mathbb{Q}}^{(j,w-j)} \to H(\mathbb{Q})\delta L_\beta H_\infty^0 \times V_{\mathbb{Q}}^{(j,w-j)}, \quad (\gamma\delta l_{h_\infty}, v) \mapsto (\gamma\delta l_{h_\infty}, l_p^{-1} \cdot v)
\]

is well-defined since \( H(\mathbb{Q}) \cap L_\beta H_\infty^0 \subset \ker(N_{F/p} \circ \det) \) acts trivially on \( V_{\mathbb{Q}}^{(j,w-j)} \). An easy check shows that \( \text{triv}_\delta \) induces a homomorphism of local systems

\[
\tilde{S}_L^H[\delta] \times V_{\mathbb{Q}}^{(j,w-j)} \to \left(V_{\mathbb{Q}}^{(j,w-j)}\right)|_{\tilde{S}_L^H[\delta]}
\]

where \([\delta]\) denotes the image of \( \delta \) in \( \pi_0(\tilde{S}_L^H) \), hence yields a homomorphism:

\[
\text{triv}^*_\delta : H^q_c(\tilde{S}_L^H[\delta], V_{\mathbb{Q}}^{(j,w-j)}) \to H^q_c(\tilde{S}_L^H[\delta], \mathbb{Z}) \otimes V_{\mathbb{Q}}^{(j,w-j)}.
\]

We will now render the trivializations independent of the choice of \( \delta \in [\delta] \in \pi_0(\tilde{S}_L^H) \).

By definition, for any \( \delta' \in H(\mathbb{Q})\delta l_{H_\infty} \) one has

\[
(32) \quad \text{triv}^*_{\delta'} = (\text{id} \otimes l_p^{-1}) \cdot \text{triv}^*_\delta = N_{F/p/q_p}^{-1} (\det(l_{1,p})^j \det(l_{2,p})^{w-j}) \text{triv}^*_\delta.
\]

The \( p \)-adic cyclotomic character \( \varepsilon \) seen as idele class character \( F^\times \backslash \mathbb{A}_F^\times \to \mathbb{Z}_p^\times \) sends \( y \) to \( N_{F/p}(y_p) \sqrt{\eta} / F \), hence is trivial on \( F^\times F_\infty^0 \) and given by \( N_{F/p/q_p} \) on \( (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \). Hence

\[
(33) \quad \text{triv}^*_{[\varepsilon]} = \varepsilon \left( \det(\delta_1^j \delta_2^{w-j}) \right) \text{triv}^*_\delta : H^q_c(\tilde{S}_L^H[\delta], V_{\mathbb{Q}}^{(j,w-j)}) \to H^q_c(\tilde{S}_L^H[\delta], \mathbb{Z}) \otimes V_{\mathbb{Q}}^{(j,w-j)}
\]

is independent of the particular choice of \( \delta \in [\delta] \in \pi_0(\tilde{S}_L^H) \).

2.2.5. **Connected components and fundamental classes.** Recall that for each \([\delta] \in \pi_0(\tilde{S}_L^H), \tilde{S}_L^H[\delta] \) is a \( q_0 \)-dimensional connected orientable real manifold and that choosing an orientation amounts to choosing a fundamental class, i.e., a basis \( \theta_{[\delta]} \) of its Borel-Moore homology \( H_{BM}^{q_0}(\tilde{S}_L^H[\delta]) \simeq \mathbb{Z} \). We choose such orientations in a consistent manner when \( \beta \) and \([\delta]\) vary as follows. First, we fix, once and for all, an ordered basis on the tangent space of the symmetric space \( H_\infty^0/L_\infty^0 \) yielding fundamental classes \( \theta_{[\delta]} \) of the connected components of identity \( \tilde{S}_L^H[1] \), when \( \beta \) varies. Then for each \([\delta] \in \pi_0(\tilde{S}_L^H) \) we consider the isomorphism \( \tilde{S}_L^H[1] \sim_{\delta} \tilde{S}_L^H[\delta] \) and define \( \theta_{[\delta]} = \delta, \theta_{\beta} \), which is clearly independent of the particular choice of \( \delta \in [\delta] \). Capping with \( \theta_{[\delta]} \) and fixing a basis of \( V_{\mathbb{Q}}^{(j,w-j)} \) (later in \((\text{II})\)) we will fix a particular basis in order to compare evaluations at different \( j \)'s yields an isomorphism:

\[
(H^q_c(\tilde{S}_L^H[\delta], \mathbb{Z}) \otimes V_{\mathbb{Q}}^{(j,w-j)} \sim \to V_{\mathbb{Q}}^{(j,w-j)} \sim \to \mathcal{O}).
\]

Combining this with \((\text{II})\) and \((\text{II})\) gives homomorphisms:

\[
(34) \quad \mathcal{E}_{\beta,\delta}^{j,w} = (- \cap \theta_{[\delta]} \cap \kappa_j \circ \tau_{\beta}^j \circ \iota_{\beta}^j : H^q_c(\tilde{S}_L^H, V_{\mathbb{Q}}^j) \to \mathcal{O},
\]

\[
\mathcal{E}_{\beta,\delta}^{j,w} = \varepsilon \left( \det(\delta_1^j \delta_2^{w-j}) \right) \cdot \mathcal{E}_{\beta,\delta}^{j,w} = (- \cap \theta_{[\delta]} \cap \kappa_j \circ \tau_{\beta}^j \circ \iota_{\beta}^j : H^q_c(\tilde{S}_L^H, V_{\mathbb{Q}}^j) \to \mathcal{O}.
\]
2.2.6. Summing over the second component. Consider a finite order $\mathcal{O}$-valued idele class character $\eta_0$ of $F$ which is trivial on $I(m)$, in particular unramified at all places above $p$. The character $\eta = \eta_0 \cdot |F|^s$ will later play a role when we discuss Shalika models for automorphic representations of $G$. The following map provides a section of $\eta$:

\[(35) \quad \delta(x,y) := (\text{diag}(xy,1,\ldots,1),\text{diag}(y,1,\ldots,1)) \in H.\]

When $(x,y) \in (\mathbb{A}_F^2)^2$ runs over a set of representatives of $\mathcal{O}_F^+(p^\beta \mathfrak{m}) \times \mathcal{O}_F^+(\mathfrak{m})$, $\bar{S}_{L_\beta}^H [\delta(x,y)]$ runs over the set of connected components of $\bar{S}_{L_\beta}^H$. Define the level $\beta$ evaluation:

\[(36) \quad \mathcal{E}_{\beta}^{j,\eta} = \sum_{[x] \in \mathcal{O}_F^+(p^\beta)} \mathcal{E}_{\beta,[x]}^{j,\eta} [x],\]

where the last sum runs over all $[x] \in \mathcal{O}_F^+(p^\beta)$ mapping to $[x]$ under the natural projection.

The following diagram recapitulates the steps in the construction of $\mathcal{E}_{\beta}^{j,\eta}$:

\[(37) \quad \begin{array}{ccc}
\mathcal{E}_{\beta}^{j,\eta} & \xrightarrow{\kappa_{j,\text{or}_{\beta} \circ \eta_0}} & \mathcal{E}_{\beta}^{j,\eta}(\bar{S}_{L_\beta}^H, \mathcal{V}_O^{j,\text{w},-j}) \\
\mathcal{E}_{\beta}^{j,\eta} & \xrightarrow{\sum_{[x] \in \mathcal{O}[\bar{S}_{L_\beta}^H]} (-\text{triv}_{[x]}) \circ \text{or}_{\beta}} & \mathcal{O}[\mathcal{O}_F^+(p^\beta) \times \mathcal{O}_F^+(\mathfrak{m})] \\
\mathcal{E}_{\beta}^{j,\eta} & \xrightarrow{\mathcal{O} \circ \mathcal{O}[\bar{S}_{L_\beta}^H]} & \mathcal{O}[\bar{S}_{L_\beta}^H] \end{array} \]

2.3. Distributions on $\mathcal{O}_F^+(p^\infty)$. The object of this section is to relate when $\beta$ varies the evaluation maps $\mathcal{E}_{\beta}^{j,\eta}$ whose definition is summarized in $[37]$.

2.3.1. The distributive property. Fix a $\beta = (\beta_p)_{\mid p}$ with $\beta_p \in \mathbb{Z}_{>0}$ for all $p \mid p$.

**Theorem 2.2.** Given a prime $p \mid p$ we let $p^{3\prime} = p^\beta p$ and consider the canonical projection $\text{pr}_{3\prime,\beta} : \mathcal{O}_F^+(p^{3\prime}) \rightarrow \mathcal{O}_F^+(p^\beta)$. For all $[x] \in \mathcal{O}_F^+(p^{3\prime})$ we have $\mathcal{E}_{\beta}^{j,\eta} \circ U_p^o = \text{pr}_{3\prime,\beta} \circ \mathcal{E}_{3\prime}^{j,\eta}$, i.e.,

\[\mathcal{E}_{\beta}^{j,\eta} \circ U_p^o = \sum_{[x] \in \text{pr}_{3\prime,\beta}^{-1}([x])} \mathcal{E}_{\beta,\delta(x,y)}^{j,\eta} [x].\]

**Proof.** Using $[32]$, $[35]$ and $[36]$ one has to show that for all $[x] \in \mathcal{O}_F^+(p^\beta \mathfrak{m}), [y] \in \mathcal{O}_F^+(\mathfrak{m})$:

\[\mathcal{E}_{\beta,\delta(x,y)}^{j,\eta} \circ U_p^o = \sum_{[x] \in \text{pr}_{3\prime,\beta}^{-1}([x])} N_{F_p/q_p}^{j,\eta} (u_{x'}) : \mathcal{E}_{3\prime,\delta(x',y)}^{j,\eta},\]

where $u_{x'} \in I(p^\beta)$ is such that $x' \in F^\times Xu_{x'}F_{x'}^{\infty}$. We proceed as in the proof of $[BDJ]$, Prop.3.5. Pulling back the definition of the Hecke operator $U_p^o$ (see $[1]$) by the automorphic symbols (see $[2.2.1]$) and the twisting operators (see $[2.2.2]$) yields a commutative diagram (we use...
implicitly that $p_{K_0(p), K}$ and $\pr_{\beta', \beta}$ have the same degree as $L_\beta/L_{\beta'} \simeq M_\mu(O/p)$:

$$
\begin{align*}
H^0_c(S^G_K, \psi^\mu_K) & \xrightarrow{\rho_{K_0(p), K}} H^0_c(S^G_{K_0(p)}, \psi^\mu_{K_0(p)}) \xrightarrow{[t_p]} H^0_c(S^G_{K_0(p)}, \psi^\mu_{K_0(p)}) \xrightarrow{\Tr(p_{K_0(p), K})} H^0_c(S^G_K, \psi^\mu_K) \\
H^0_c(\tilde{S}^{H}_{\beta'}, \psi^\mu_{K_\beta}) & \xrightarrow{\tau^\circ_{\beta'}} H^0_c(\tilde{S}^{H}_{\beta'}, \psi^\mu_{K_\beta}) \\
H^0_c(\tilde{S}^{H}_{\beta'}, \psi^\mu_{K_\beta}) & \xrightarrow{\tau^\circ_{\beta'}} H^0_c(\tilde{S}^{H}_{\beta'}, \psi^\mu_{K_\beta}) \xrightarrow{\Tr(\pr_{\beta', \beta})} H^0_c(\tilde{S}^{H}_{\beta'}, \psi^\mu_{K_\beta}) \\
H^0_c(\tilde{S}^{H}_{\beta'}, \psi^\mu_{K_\beta}) & \xrightarrow{\tau^\circ_{\beta'}} H^0_c(\tilde{S}^{H}_{\beta'}, \psi^\mu_{K_\beta}) \xrightarrow{\Tr(\pr_{\beta', \beta})} H^0_c(\tilde{S}^{H}_{\beta'}, \psi^\mu_{K_\beta}),
\end{align*}
$$

where the upper $[t_p]$ is induced by the morphism $(g, v) \mapsto (g \cdot t_p^{-1}, t_p \cdot v)$ of local systems, whereas the lower $[t_p]$ is induced by the morphism $(h, v) \mapsto (h, t_p \cdot v)$. Then

$$
\begin{align*}
H^0_c(S^H_{\beta'}, [\delta(x', y)], V^j_{\O_{\O}}) & \xrightarrow{\Tr(\pr_{\beta', \beta})} H^0_c(S^H_{\beta'}, [\delta(x, y)], V^j_{\O_{\O}}) \\
H^0_c(S^H_{\beta'}, [\delta(x', y)], \Z) \otimes V^j_{\O_{\O}} & \xrightarrow{\Tr(\pr_{\beta', \beta})} H^0_c(S^H_{\beta'}, [\delta(x, y)], \Z) \otimes V^j_{\O_{\O}},
\end{align*}
$$

is another commutative diagram by (32), hence the claim. 

\hfill \Box

2.3.2. Distribution for finite slope eigenvectors. Let $\phi \in H^0_c(S^G_K, \psi^\mu_K)$ be an eigenvector for $U^p_{\lambda}$ with eigenvalue $\alpha^\mu_{\lambda}$ for all $p \mid p$. After multiplying with a power of the uniformizer $\varpi$ we can and do assume that $\phi \in H^0_c(S^G_K, \psi^\mu_{\O})$. It is then an eigenvector for $U^\mu_{\lambda}$ with eigenvalue $\alpha^\mu_{\lambda} = \prod_{p \mid p} (\alpha^\mu_{\lambda})^b_p$ for all $\beta$.

**Definition 2.3.** We say that $\phi$ is of finite slope if $\alpha^\mu_{\lambda} \neq 0$, and in which case we define its slope as $v_\lambda(\alpha^\mu_{\lambda})$. A eigenvector $\phi$ of slope 0 is called $U^p_{\lambda}$-ordinary.

$U^p_{\lambda}$-ordinarity is equivalent to saying that the $U^p_{\lambda}$-eigenvalue $\alpha^\mu_{\lambda}$ satisfies $|\alpha^\mu_{\lambda}|_p = |\mu\beta(t_p)|^{-1}_p$ for all $p \mid p$ (the notion of $p$-ordinarity will be revisited in §1.2).

Given any $U^p_{\lambda}$-eigenvector $\phi$ of finite slope and any $j \in \Crit(\mu)$ by Theorem 2.2 one has a well-defined element

$$
(38) \quad \mu^j_{\phi} := \left( (\alpha^\mu_{\lambda})^{-1} \mathcal{E}^j_{\lambda}(\phi) \right)_{\beta} \in \lim_{\lambda} E[O^\nu_{\hat{\lambda}}(p^\lambda)] = E[[O^\nu_{\hat{\lambda}}(p^\lambda)]],
$$

which can be reinterpreted as an $E$-valued distribution on $O^\nu_{\hat{\lambda}}(p^\lambda)$ of growth at most $v_\lambda(\alpha^\mu_{\lambda})$.

We write $H^0_c(S^G_K, \psi^\mu_{\O})_{\ord}$ for the maximal $O$-submodule of $H^0_c(S^G_K, \psi^\mu_{\O})$ on which the operators $U^p_{\lambda}$ are invertible for all $p \mid p$ (it is a direct $O$-factor). Given any (not necessarily $U^\mu_{\lambda}$-eigen) non-torsion element $\phi \in H^0_c(S^G_K, \psi^\mu_{\O})_{\ord}$ one defines

$$
(39) \quad \mu^j_{\phi} := \left( \mathcal{E}^j_{\lambda}( (U^\mu_{\lambda})^{-1}(\phi) ) \right)_{\beta} \in \lim_{\lambda} O[O^\nu_{\hat{\lambda}}(p^\lambda)] = O[[O^\nu_{\hat{\lambda}}(p^\lambda)]],
$$

which can be reinterpreted as a measure \textit{(i.e., a bounded distribution)} on $O^\nu_{\hat{\lambda}}(p^\lambda)$. 

2.4. Manin relations. Consider the $p$-adic cyclotomic character $\varepsilon : \mathcal{O}_F^+(p^\infty) \to \Z_p^\times$ which is defined by composing the norm $N_{F/Q} : \mathcal{O}_F^+(p^\infty) \to \mathcal{O}_E^+(p^\infty)$ with the $p$-adic cyclotomic character over $\Q$. In this section we will prove the following result.

**Theorem 2.4.** Let $\mu \in X_0^0(T)$ and let $\phi \in H^0_c(S_K^G, V_\phi^0)$ be either a finite slope $U_p$-eigenvector, or else an ordinary vector.

If $j$ and $j+1$ both belong to Crit($\mu$) then the following equality holds in $E[[\mathcal{O}_F^+(p^\infty)]]$:

$$\varepsilon_{cyc}(\mu_{\phi}^{j+1,\eta}) = \mu_{\phi}^{j+1,\eta},$$

where $\varepsilon_{cyc}$ denotes the automorphism of $E[[\mathcal{O}_F^+(p^\infty)]]$ sending $[x]$ to $\varepsilon([x])[x]$. Hence

\begin{equation}
\mu_{\phi}^{j} := \varepsilon_{cyc}^{-j}(\mu_{\phi}^{j,\eta}) \in E[[\mathcal{O}_F^+(p^\infty)]],
\end{equation}

is independent of $j \in \text{Crit}(\mu)$.

This theorem has the following important consequences. First, by a well known result of Vishik and Amice-Vélu, it allows to uniquely determine $\mu_{\phi}^{v_0}$ in the positive slope case by evaluating $\mu_{\phi}^{v_0}$ on finite order characters when $v_p(\alpha_{\phi}^0) < \# \text{Crit}(\mu)$. An even more striking consequence is the overdetermination of $\mu_{\phi}^{v_0}$ evaluated at finite order characters in the ordinary case, which would lead to the main result of this paper. Before embarking on the proof of this theorem, we begin with some technical preparation.

2.4.1. Lie theoretic considerations. By the distributive property (see Theorem 2.2) we may reduce to strict $p$-power level with integral exponents $\beta \in \Z_{>0}$, ignoring the finer components $p \mid p$ for simplicity of notation. Recall that $b = t \oplus n$ and $q = b \oplus u$. With the notation $t_p = \iota(p, 1, 1)$, we observe for any $\beta \geq 0$ the relations

$$t_p^\beta n \varnothing t_p^{-\beta} \subseteq n \varnothing, \quad t_p^\beta u \varnothing t_p^{-\beta} = p^\beta u \varnothing.$$

Recall the matrix $\xi = (1_{w_n} w_n)$. A superscript $\xi(-)$ denotes left conjugation action by $\xi$.

**Proposition 2.5.** We have the relations

(i) $\varnothing \varnothing = b \varnothing + \xi b \varnothing$, and

(ii) $\xi(n \varnothing \cap b \varnothing) \subseteq [b, b \varnothing] + \xi n \varnothing$.

**Proof.** (i) Since $\xi \in G(\varnothing)$, it suffices to verify it over $E$, where it amounts to show that $\dim_E(\mathfrak{h}_E \cap \xi \mathfrak{b}_E^\infty) = n$. To this end, let $l_1, l_2$ be lower triangular matrices in $M_n(E)$ and $u \in M_n(E)$. Then

$$\xi \cdot \begin{pmatrix} l_1 & u \\ l_2 & \end{pmatrix} \cdot \xi^{-1} = \begin{pmatrix} l_1 + w_n u & l_2 w_n - l_1 - w_n u \\ w_n u & l_2 w_n - w_n u \end{pmatrix}$$

lies in $\mathfrak{h}_E$ if and only if $u = 0$ and $l_1 = l_2 w_n u$. Therefore, $l_1$ and $l_2$ are diagonal matrices determining each other uniquely.

(ii) Conjugation by $\xi^{-1}$ reduces the claim to the problem of solving

$$\begin{pmatrix} n_1 & \\
_2 & \end{pmatrix} = \begin{pmatrix} h_1 & (h_1 - h_2)w_n \\ h_2 w_n & \end{pmatrix} + \begin{pmatrix} \varnothing_1 & \\
_2 & \end{pmatrix}$$

for given $\iota(n_1, n_2) \in \mathfrak{h} \cap n \varnothing$ and unknowns $\iota(h_1, h_2) \in [\mathfrak{h}_\varnothing, \mathfrak{b}_\varnothing]$ and $\iota(\varnothing_1, \varnothing_2) \in n^{-}$. The choice

$$h_1 = h_2 = n_1 + n_2 w_n, \quad \varnothing_1 = -n_2 w_n, \quad \varnothing_2 = -n_1 w_n,$$

is a solution with the desired properties. \qed
Corollary 2.6. For any $\beta \geq 0$, the following relations hold inside $U(\mathfrak{g}_\mathcal{O})$:

(i) $U(\mathfrak{g}_\mathcal{O}) = U(\mathfrak{h}_\mathcal{O}) \cdot U(\xi \mathfrak{b}_\mathcal{O})$, and

(ii) $U(\xi \mathfrak{t}_p \mathfrak{n}_\mathcal{O}) \subseteq U(\mathfrak{h} \mathcal{O} + p^\beta \mathfrak{h}_\mathcal{O}) \cdot U(\xi \mathfrak{n}_\mathcal{O} + p^\beta \xi \mathfrak{b}_\mathcal{O})$.

Proof. (i) This is a consequence of Prop. 2.4(i) and the Poincaré-Birkhoff-Witt Theorem.

(ii) The decomposition $\mathfrak{n}_\mathcal{O} = (\mathfrak{h}_\mathcal{O} \cap \mathfrak{n}_\mathcal{O}) \oplus \mathfrak{u}_\mathcal{O}$, gives $\xi \mathfrak{t}_p \mathfrak{n}_\mathcal{O} \subseteq (\mathfrak{h}_\mathcal{O} \cap \mathfrak{n}_\mathcal{O}) \oplus p^\beta \mathfrak{u}_\mathcal{O}$. Conjugating by $\xi$ we get

$$\xi \mathfrak{t}_p \mathfrak{n}_\mathcal{O} \subseteq (\mathfrak{h} \mathcal{O} \cap \mathfrak{n}_\mathcal{O}) \oplus p^\beta \xi \mathfrak{u}_\mathcal{O}.$$  

Applying Proposition 2.4(ii) to the first summand and Prop. 2.4(i) to the second we get

$$\xi \mathfrak{t}_p \mathfrak{n}_\mathcal{O} \subseteq (\mathfrak{h} \mathcal{O} \cap \mathfrak{n}_\mathcal{O}) \oplus (\xi \mathfrak{n}_\mathcal{O} + p^\beta \xi \mathfrak{b}_\mathcal{O}).$$

One concludes again by the Poincaré-Birkhoff-Witt Theorem, because the sums within the parentheses on the right hand side are Lie $\mathcal{O}$-algebras.

2.4.2. Lattices and the projection formula. Recall from (1) the lowest weight vector $v_0 \in V^\mu_E$ and the $G(\mathcal{O})$-lattice $V^\mu_\mathcal{O} = U(\mathfrak{g}_\mathcal{O}) \cdot v_0 = U(\mathfrak{n}_\mathcal{O}) \cdot v_0$. Recall also the $\bullet$-action of the semi-group $\Lambda_p$ on $V^\mu_\mathcal{O}$ as in (13).

Given $j \in \text{Crit}(\mu)$ recall from (2.2.2.3) the map $\kappa_j : V^\mu_\mathcal{O} \rightarrow V^{(j \cdot w_- j)}_\mathcal{O}$. By Corollary 2.6(i)

$$V^\mu_\mathcal{O} = U(\mathfrak{h}_\mathcal{O}) \cdot \xi v_0,$$

which implies that $\kappa_j(\xi v_0)$ is an $\mathcal{O}$-basis of $V^{(j \cdot w_- j)}_\mathcal{O}$ yielding a surjective $\mathcal{O}$-linear map

$$\iota_j : V^\mu_\mathcal{O} \rightarrow \mathcal{O}, \quad \text{defined by} \quad \kappa_j(v) = \iota_j(v) \kappa_j(\xi v_0).$$

It is independent from the choice of $\kappa_j$ because of (29) and $\iota_j(\xi v_0) = 1$. We now come to the main technical result that is at the heart of our proof of the Manin relations.

Proposition 2.7. For any $\beta \geq 0$, $v \in (\xi \mathfrak{t}_p^\beta) \bullet V^\mu_\mathcal{O} \subset V^\mu_\mathcal{O}$ and for all $j_1, j_2 \in \text{Crit}(\mu)$ we have

$$\iota_{j_1}(v) \equiv \iota_{j_2}(v) \pmod{p^\beta \mathcal{O}}.$$

Proof. By (13) for $v \in (\xi \mathfrak{t}_p^\beta) \bullet V^\mu_\mathcal{O}$ there exists $m \in U(\mathfrak{n}_\mathcal{O})$ with

$$v = \xi \cdot \xi \mathfrak{t}_p^\beta \bullet (m v_0) = \xi \mathfrak{t}_p^\beta m \cdot \xi \mathfrak{t}_p^\beta v_0 = \xi \mathfrak{t}_p^\beta m \cdot \xi v_0 \in U(\xi \mathfrak{t}_p^\beta \mathfrak{n}_\mathcal{O}) \cdot \xi v_0.$$

By Corollary 2.6(ii) write

$$\xi \mathfrak{t}_p^\beta m = x y, \text{ with } x \in U([\mathfrak{h}, \mathfrak{h}]_\mathcal{O} + p^\beta \mathfrak{h}) \text{ and } y \in U(\xi \mathfrak{n}_\mathcal{O} + p^\beta \xi \mathfrak{b}_\mathcal{O}).$$

Let $x_0, y_0 \in \mathcal{O}$ be the degree zero terms of $x$ and $y$, respectively, and let

$$x_1 := x - x_0 \in ([\mathfrak{h}, \mathfrak{h}]_\mathcal{O} + p^\beta \mathfrak{h}) \cdot U([\mathfrak{h}, \mathfrak{h}]_\mathcal{O} + p^\beta \mathfrak{h}),$$

$$y_1 := y - y_0 \in (\xi \mathfrak{n}_\mathcal{O} + p^\beta \xi \mathfrak{b}_\mathcal{O}) \cdot U(\xi \mathfrak{n}_\mathcal{O} + p^\beta \xi \mathfrak{b}_\mathcal{O}),$$

be the higher degree terms in their respective enveloping algebras. Then

$$\iota_j(v) = \iota_j(xy \cdot \xi v_0) = x \cdot \iota_j(y \cdot \xi v_0) \quad \text{(since $\iota_j$ is $H$-equivariant)}$$

$$\equiv x \cdot \iota_j(y_0 \cdot \xi v_0) \pmod{p^\beta \mathcal{O}} \quad \text{(since $\xi \mathfrak{n}_\mathcal{O}$ acts trivially on $\xi v_0$)}$$

$$\equiv x_0 \cdot \iota_j(y_0 \cdot \xi v_0) \pmod{p^\beta \mathcal{O}} \quad \text{(since $[\mathfrak{h}, \mathfrak{h}]_\mathcal{O}$ acts trivially on a line)}$$

$$= x_0 y_0 \cdot \iota_j(\xi v_0) = x_0 y_0,$$

which does not depend on $j$ as claimed.

\qed
2.4.3. **Proof of Theorem 2.4** Since \( \mathcal{O}(\Omega_F^+(p^{\infty})) = \lim_{\beta} (\mathcal{O}/p^{\beta}\mathcal{O})[\Omega_F^+(p^{\beta})] \) and since \( \varepsilon \) (mod \( p^{\beta} \)) factors through \( \Omega_F^+(p^{\beta}) \), it is enough to check that given \( \beta \geq 1 \) and \( [x] \in \Omega_F^+(p^{\beta}) \) one has

\[
\varepsilon([x])E_{\beta,[x]}^j(\phi) \equiv E_{\beta,[x]}^{j+1}(\phi) \pmod{p^{\beta}}.
\]

Since by (33) one has \( \text{triv}_{\beta(x,y)}^* = \varepsilon(x^j y^m) \text{triv}_{\beta(x,y)}^* \), it suffices to show that (see (37))

\[
E_{\beta,\delta}^\tau = (- \cap \theta_{\beta\delta}) \circ \text{triv}_{\beta}^* \circ \kappa_j \circ \tau_{\beta}^* (\phi) \equiv (- \cap \theta_{\beta\delta}) \circ \text{triv}_{\beta}^* \circ \kappa_j^{*+1} \circ \tau_{\beta}^* \circ \iota_{\beta}^* (\phi) \pmod{p^{\beta}}.
\]

Now, by definition the homomorphism of sheaves \( \tau_{\beta}^* \) defined in (22.2) factors as:

\[
\tau_{\beta}^* V_{\mu}^O \longrightarrow \iota^* (\varepsilon t_{\beta}^2) \bullet V_{\mu}^O \longrightarrow \iota^* V_{\mu}^O
\]

in view of which Proposition 2.7 translates to the statement that (for the choice of basis of \( V_{\mu}^O \)) as in (41) \((- \cap \theta_{\beta\delta}) \circ \text{triv}_{\beta}^* \circ \kappa_j \circ \tau_{\beta}^* \circ \iota_{\beta}^* (\phi) \) is independent of \( j \in \text{Crit}(\mu) \). \( \square \)

3. **Local considerations**

We delineate some local calculations that will be needed in the global considerations of the next section. For only this section, \( F \) denotes a finite extension of \( \mathbb{Q}_p \), \( O \) its ring of integers, \( \mathcal{P} \) the maximal ideal, \( \varpi \in \mathcal{P} \) a uniformizer, \( q = \#(O/\mathcal{P}) \) and \( \delta \) the valuation of the different. We use local notations corresponding to the global notations introduced at the beginning of \( \S 1 \). For example \( G = \text{GL}_{2n}(F) \supset H = \text{GL}_n(F) \times \text{GL}_n(F) \), etc.

3.1. **Parahoric invariants.** Let \( K = \text{GL}_{2n}(O) \) be the standard maximal compact subgroup of \( G \). Define the parahoric (resp., Iwahori) subgroup \( J \) (resp., \( I \)) of \( K \) consisting of matrices whose reduction modulo \( \mathcal{P} \) belongs to \( Q(O/\mathcal{P}) \) (resp., to \( B(O/\mathcal{P}) \)). One has:

\[
J := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \mid a, d \in \text{GL}_n(O), c \in M_n(\mathcal{P}), b \in M_n(O) \right\}.
\]

Let \( \Pi \) be an algebraic unramified and generic representation of \( G \). Then there exists an unramified character \( \lambda = \otimes_{i=1}^{2n} \lambda_i : T \rightarrow \hat{Q} \xrightarrow{\lambda \circ \text{triv}} \hat{C} \) such that

\[
\Pi = \text{Ind}_B^G(|\cdot|^{-\frac{2n+1}{2}}\lambda),
\]

where the right hand side is the normalized parabolic induction, which differs from the usual induction by \( \delta_B^{-1/2} \) where

\[
\delta_B(t_1, t_2, \ldots, t_{2n}) = |t_1|^{2n-1}|t_2|^{2n-3} \cdots |t_{2n}|^{-2n}.
\]

Recall Jacquet’s exact functor sending an admissible \( G \)-representation \( V \) to the space of its co-invariants of \( U \) defined as \( V_U = V/\langle u \cdot v - v \mid u \in U, v \in V \rangle \) which is an admissible \( H \)-representation. The Weyl groups of \( G \supset H \) are given by \( \mathfrak{S}_{2n} \simeq W_G \supset W_H \simeq (\mathfrak{S}_n \times \mathfrak{S}_n) \). The group \( W_G \) acts on the right on characters of \( T \). There is a natural bijection:

\[
W_G/W_H \overset{\sim}{\rightarrow} \{ \tau \subset \{1, 2, \ldots, 2n\} \mid \#\tau = n \}, \ \rho \mapsto \{ \rho(1), \ldots, \rho(n) \}
\]
Lemma 3.1. The semi-simplification of the Jacquet module \( \Pi_U \) is isomorphic to:
\[
\bigoplus_{\tau \in W_G / W_H} \delta_Q^{1/2} \cdot \text{Ind}_{B \cap H}^G(\| \cdot \|^{2n-1} \lambda^\tau),
\]
where \( \delta_Q(t_1, t_2, \ldots, t_{2n}) = |t_1 \cdots t_n \cdot t_{n+1}^{-1} \cdots t_{2n}^{-1}|^n \). The semi-simplification can be omitted if \( \Pi \) is regular in the sense that \( \alpha_i = \lambda_i(\varpi) \) are pairwise distinct for \( 1 \leq i \leq 2n \).

The characteristic polynomial of the Hecke operator \( U_p = [J \ast \pi] \) acting on \( \Pi^J \) equals \( \prod_{\tau \in W_G / W_H} (X - q^{\frac{n(1-n)}{2}} \alpha^\tau) \), where \( \alpha^\tau := \prod_{i \in \tau} \alpha_i \).

Proof. The semi-simplification of the Jacquet module \( \Pi_N \) with respect to \( B \) is given by:
\[
\bigoplus_{\rho \in W_G} \delta_B^{1/2} \cdot \| \cdot \|^{2n-1} \lambda^\rho.
\]

Since \( \text{Ind}_B^G = \text{Ind}_B^G \text{Ind}_{B \cap H}^H \), Frobenius reciprocity implies that any irreducible sub-quotient of the Jacquet module of \( \Pi \) with respect to \( Q \) is isomorphic to one of the summands in (46).

The first claim then follows by a simple dimension count based on (47) and the transitivity of the Jacquet functors. By Bruhat decomposition:
\[
G = \bigoplus_{\rho \in W_G} B \rho I = \bigoplus_{\rho \in W_G / W_H} B \rho J,
\]
where the dimension of \( \Pi^J \) is \( \#(W_G / W_H) \). By Iwasawa decomposition \( H = (B \cap H) \cdot (H \cap J) \),
\[
\left( \text{Ind}_{B \cap H}^G(\delta_Q^{1/2} \cdot \| \cdot \|^{2n-1} \lambda^\rho) \right)^{H \cap J}
\]
is a line on which the central element \( \iota(1_n, \varpi 1_n) \) acts by \( q^{\frac{n(1-n)}{2}} \alpha^\tau \). Under the assumption that \( \Pi \) is regular, the image of \( \Pi^J \) by the Jacquet functor equals the direct sum of the above lines when \( \tau \) runs over \( W_G / W_H \), hence the second claim. The proof of the third claim is a standard double coset computation based (48) (see also [112]).

3.2. Twisted local Shalika integrals. We will review the theory of global Shalika models and \( L \)-functions in [41]. The computations in this section will be needed in \([43, 45]\) to evaluate the twisted local zeta integral.

Fix an additive character \( \psi : F \to \mathbb{C}^\times \) of conductor \( \varpi^{-\delta} \) and a multiplicative character \( \eta : F^\times \to \mathbb{C}^\times \).

Definition 3.2. We say that an admissible representation \( \Pi \) of \( G \) has a local \((\eta, \psi)\)-Shalika model if there is a non-trivial (and hence injective) intertwining of \( G = \text{GL}_{2n}(F) \)-modules
\[
\mathcal{S}_\psi^\eta : \Pi \hookrightarrow \text{Ind}_S^G(\eta \otimes \psi).
\]

For any \( W \in \text{Ind}_S^G(\eta \otimes \psi) \) and for any quasi-character \( \chi : F^\times \to \mathbb{C}^\times \) the zeta integral
\[
\zeta(s; W, \chi) := \int_{\text{GL}_{2n}(F)} W\left( \begin{pmatrix} h & 0 \\ 0 & 1_n \end{pmatrix} \right) \chi(\det(h))|\det(h)|^{s-1/2}dh
\]
is absolutely convergent for \( \Re(s) \gg 0 \). The following result is due to Friedberg and Jacquet.
Proposition 3.3. [FJ Prop. 3.1, 3.2] Assume that $\Pi$ has an $(\eta, \psi)$-Shalika model. Then for each $W \in S^0_\psi(\Pi)$ there is a holomorphic function $P(s; W, \chi)$ such that
\[
\zeta(s; W, \chi) = L(s, \Pi \otimes \chi)P(s; W, \chi).
\]
One may analytically continue $\zeta(s; W, \chi)$ by re-defining it as $L(s, \Pi \otimes \chi)P(s; W, \chi)$ for all $s \in \mathbb{C}$. Moreover, for every $s \in \mathbb{C}$ there exists a vector $W_\Pi \in S^0_\psi(\Pi)$ such that all unramified quasi-characters $\chi : F^* \to \mathbb{C}^*$ one has
\[
P(s; W_\Pi, \chi) = (q^{1/2-s}\chi(\varpi))^{-dn}.
\]
If $\Pi$ is spherical, then $W_\Pi$ can be taken to be the spherical vector $W^0_\Pi \in S^0_\psi(\Pi)$ normalized by the condition $W^0_\Pi(1_{2n}) = 1$.

Let’s recall a multiplicity one theorem for Shalika models due to Chen-Sun [CS] and Nien [N]:
\[
\dim \left( \text{Hom}_F(\Pi, \eta \otimes \psi) \right) \leq 1.
\]
A consequence is that we may and do assume that $W_\Pi$ is $\mathbb{Q}(\Pi)$-rational.

For ramified twists we need the following refinement of Proposition 3.3.

Proposition 3.4. Let $W \in S^0_\psi(\Pi)$ be a parahoric invariant vector, i.e.,
\[
(50)\quad W \left( \begin{pmatrix} h & X \\ 1_n \end{pmatrix} \right) = \eta(\det h)\psi(\varpi X)W(g),
\]
for all $h \in \text{GL}_n(F)$, $X \in M_n(F)$, $g \in G$ and $k \in J$. Then for every finite order character $\chi : F^* \to \mathbb{C}^*$ of conductor $\beta \geq 1$, and for all $s \in \mathbb{C}$ with $\Re(s) > 0$ one has
\[
\zeta(s; W(- \cdot \xi t_\varpi^\beta), \chi) = \mathcal{G}(\chi)^n \cdot q^\beta n(1-n) + (\beta + \delta)n(s - \frac{1}{2})W(t^{-\delta}).
\]

Proof. For any $h \in \text{GL}_n(F)$ and $X \in M_n(\mathcal{O})$ the Shalika property [50] implies that:
\[
W \left( \begin{pmatrix} h & X \\ 1_n \end{pmatrix} \xi t_\varpi^\beta \right) = W \left( \begin{pmatrix} h & X \\ 1_n \end{pmatrix} \right) \psi(\text{tr}(h \varpi^\beta X w_n)) \cdot W \left( \begin{pmatrix} h & X \\ 1_n \end{pmatrix} \xi t_\varpi^\beta \right),
\]
hence the zeta integral is supported over $\text{GL}_n(F) \cap \text{SL}_n(\mathcal{O})$. Also for $h \in \text{GL}_n(F)$:
\[
W \left( \begin{pmatrix} h & X \\ 1_n \end{pmatrix} \xi t_\varpi^\beta \right) = W \left( \begin{pmatrix} h & X \\ 1_n \end{pmatrix} \right) \psi(\text{tr}(h \varpi^\beta w_n)) \cdot W \left( \begin{pmatrix} h & X \\ 1_n \end{pmatrix} \xi t_\varpi^\beta \right).
\]
Using this and changing variable $h \mapsto h \varpi^{-\beta'}$ with $\beta' = \beta + \delta$ yields
\[
(51)\quad \zeta(s; W(- \cdot \xi t_\varpi^\beta), \chi) = \int_{\text{GL}_n(F) \cap M_n(\mathcal{O})} W \left( \begin{pmatrix} h^{-\delta} & 1_n \\ 1_n \end{pmatrix} \right) \psi(\text{tr}(h \varpi^{-\beta'})) (|\cdot|^s - \frac{1}{2})(\det(h \varpi^{-\beta'})) dh.
\]
Denote by $(e_{ij})_{1 \leq i,j \leq n}$ the standard basis of $M_n(\mathcal{O})$. Since $W$ is parahoric invariant, for any $i \neq j$ and $c \in \mathcal{O}$, right translation by $1_n + ce_{ij} \in \text{SL}_n(\mathcal{O})$ in [51] yields:
\[
\zeta(s; W(- \cdot \xi t_\varpi^\beta), \chi) = \int dh \ W \left( \begin{pmatrix} h^{-\delta} & 1_n \\ 1_n \end{pmatrix} \right) \psi(\text{tr}(h \varpi^{-\beta'})) (|\cdot|^s - \frac{1}{2})(\det(h \varpi^{-\beta'})) \int_{\mathcal{O}} dc \psi(ch_{ji} \varpi^{-\beta'}),
\]
and observe that $\int_{\mathcal{O}} \psi(ch_{ji} \varpi^{-\beta'}) dc = 0$ unless $h_{ji} \in \mathcal{P}$. 

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Similarly right translation by \( 1_n + (c - 1)e_i \) with \( c \in \mathcal{O}^\times \) shows that (51) equals:
\[
\int W \left( \left( \frac{h}{1_n} \right) \right) \psi(\text{tr}(h) - h_{ii})\omega^{-\beta}) (\chi(c) \cdot |c|^{n - \frac{1}{2}})(\det(\omega^{-\beta})) \left( \int_{\mathcal{O}^\times} \psi(ch_i \omega^{-\beta_i}) \chi(c) d^\times c \right) dh,
\]
and \( \left( \int_{\mathcal{O}^\times} \psi(ch_i \omega^{-\beta_i}) \chi(c) d^\times c \right) = 0 \) unless \( h_{ii} \in \mathcal{O}^\times \) as \( \beta \geq 1 \) equals the conductor of \( \chi \).

Therefore one can further restrict the domain of integration in (51) to the congruence subgroup \( \ker (\text{GL}_n(O) \rightarrow \text{GL}_n(O/P^\beta)) \cdot T_n(O) \), which by the Iwahori decomposition, may be identified to the product \( N_n^{-1}(P^\beta) \times T_n(O) \times N_n(P^\beta) \), where \( T_n \) denotes the diagonal subgroup of \( \text{GL}_n \) and \( N_n \) denotes the unipotent radical of the standard Borel subgroup \( B_n \). Hence
\[
\zeta(s; W(-, \omega^{-\beta}), \chi) = q^{\beta n} \left( \frac{1}{2} \right) W(t_{\omega}^{-\delta}) \int_{N_n^{-1}(P^\beta)T_n(O)N_n(P^\beta)} \psi(\text{tr}(\omega^{-\beta})) \chi(\det(\omega^{-\beta})) dk
\]
which can be simplified as
\[
q^{\beta n(1-n)+\beta n} \left( \frac{1}{2} \right) t_{\omega}^{-\delta} \prod_{1 \leq i < n} \int_{\mathcal{O}^\times} \psi(t_i \omega^{-\beta_i}) \chi(t_i \omega^{-\beta_i}) d^\times t_i =
\]
\[
q^{\beta n(1-n)+(\beta+\delta)n} \left( \frac{1}{2} \right) W(t_{\omega}^{-\delta}) \cdot G(\chi)^n,
\]
as desired. \( \square \)

### 3.3. Non-vanishing of a local twisted zeta integral.

In order to ensure the non-vanishing of the local twisted Shalika integral in Proposition 3.4, which is crucial for our applications, one has to exhibit a parahoric-spherical Shalika function \( W \) on \( G \) such that \( W(t_{\omega}^{-\delta}) \neq 0 \). Assume that \( \Pi \) is a spherical representation isomorphic to \( \text{Ind}^G_{\mathcal{O}}(|·|^\frac{2n-1}{2} \lambda) \) as in (45) and let \( \alpha_i = \lambda_i(\omega), 1 \leq i \leq 2n \). Consider an unramified character \( \eta \) of \( F^\times \).

**Definition 3.5.** Let \( \tau \in W_G/W_H \) thought of as an \( n \)-element subset of \( \{1, \ldots, 2n\} \) (see (49)). We say that \( \Pi = (\Pi, \tau) \) is \( Q \)-regular if is satisfies the following two conditions:

(i) \( q^{n(1-n)} \prod_{i \in \tau} \alpha_i \) is a simple eigenvalue for \( UP = \{J_{\tau} \omega J \} \) acting on \( \Pi^J \),

(ii) there exists \( \rho \in \mathfrak{S}_{2n} \) such that for all \( i \in \tau \), \( \rho(i) \notin \tau \) and \( \alpha_i \alpha_\rho(i) = q^{2n-1} \eta(\omega) \).

Assume that \( \Pi = (\Pi, \tau) \) is \( Q \)-regular. Then (i) together with Lemma 6.1 implies
\[
\prod_{i \in \tau, j \notin \tau} (\alpha_i - \alpha_j) \neq 0,
\]
while (ii) implies by [AG] Prop.1.3 that \( \Pi \) admits a \( (\eta, \psi) \)-Shalika model.

Without loss of generality assume from now on that \( \tau = \{n + 1, \ldots, 2n\} \) and that \( \rho \in \mathfrak{S}_{2n} \) is the order 2 element such that \( \rho(i) = n + i \) for all \( 1 \leq i \leq n \). In [AG] (1.3)] the authors construct an \( (\eta, \psi) \)-Shalika functional on \( \Pi \) sending \( f \in \text{Ind}^{\text{GL}_{2n}}_{\mathbb{B}_{2n}}(|·|^\frac{2n-1}{2} \lambda) \) to
\[
S(f)(g) := \int_{B_n \setminus \text{GL}_{2n}} \int_{M_n} f \left( \left( \begin{array}{cc} 1_n & 1_n \end{array} \right) \left( \begin{array}{c} h \end{array} \right) g \right) \eta^{-1}(\det(h))\psi(\text{tr}(X)) dX dh
\]

By [AG], Lem.1.5], this integral converges in a certain domain and, when multiplied by (52), can be analytically continued to \( \mathbb{C}^{2n} \), thus makes sense whenever (52) is non-zero. Let
Let \( W = S(f_0) \). Then \( W(t_{\infty}^{-\delta}) = 1 \). Moreover \( U_P \cdot f_0 = q^{-n(1-n)}(\prod_{i=0}^{2n} \alpha_i) f_0 \).

**Proof.** By Iwasawa decomposition \( GL_n = B_n K_n \) and as \( \iota(K_n, K_n) \subset J \) we see that

\[
W(t_{\infty}^{-\delta}) = S(f_0)(t_{\infty}^{-\delta}) = f_0 \left( \left( 1_n, 1_X^{-1} X \right) t_{\infty}^{-\delta} \right) \overline{\psi}(\text{tr}(X)) dX =
\]

\[
= \left( \left( 1_n, 1_X^{-1} X \right) \right) \overline{\psi}(\text{tr}(X)) dX =
\]

\[
= f_0 \left( \left( 1_n, 1_X^{-1} X \right) w_{2n} \right) q^{\delta n^2} \int_{M_n} f_0 \left( \left( 1_n, 1_X^{-1} X \right) \right) \overline{\psi}(\text{tr}(\omega X)) dX = 1.
\]

One checks that \( \left( 1_n, 1_X \right) \in B w_{2n} J \) if and only if \( X \in M_n(\mathcal{O}) \), in which case \( \overline{\psi}(\text{tr}(\omega X)) = 1 \). The parahoric decomposition of \( J = (J \cap U^-)(J \cap Q) = (J \cap U^-)(J \cap Q) \) implies

\[
J t_{\infty}^J = \bigsqcup_{m \in M_n(\mathcal{O}/\mathcal{P})} \left( \left( 1_n, m 1_1 \right) \right) t_{\infty} J.
\]

By (13) it suffices to compute \( (U_P \cdot f_0)(\rho) \) for all \( \rho \in W_G \). By the above decomposition

\[
(U_P \cdot f_0)(\rho) = \sum_{m \in M_n(\mathcal{O}/M_n(\mathcal{P}))} f_0 \left( \rho \left( \left( \omega, 1_n, m 1_1 \right) \right) \right).
\]

Note that \( \rho \left( \left( 1_n, m 1_1 \right) \right) t_{\infty} \) belongs to the support \( B w_{2n} J = B w_{2n} t_{\infty} J = B w_{2n} J - t_{\infty} \) of \( f_0 \) if and only if \( \rho \left( \left( 1_n, m 1_1 \right) \right) \in K \cap B w_{2n} J^- = (K \cap B) w_{2n} J^- = w_{2n} J^- \) (see (13), which implies \( \rho = w_{2n} \) and \( m \in M_n(\mathcal{P}) \)). Hence \( (U_P \cdot f_0)(\rho) = 0 \) for all \( \rho \neq w_{2n} \), while \( (U_P \cdot f_0)(w_{2n}) = f_0 \left( \left( 1_n, m 1_1 \right) \right) \right) = f_0 \left( \left( 1_n, 1_1 \right) \right) w_{2n} \left( \omega, 1_n \right) \right) = q^{-n(1-n)} (\prod_{i=0}^{2n} \alpha_i) f_0(w_{2n}). \)

One can check that using the \( \text{Aut}(\mathbb{C}) \)-action on local Shalika models as discussed in [GR2], §3.7 that \( S(f_0) \) defines a rational vector of the Shalika model.

### 4. \( L \)-functions for \( GL_{2n} \)

#### 4.1. Global Shalika models and periods.

This subsection contains a brief review of the necessary ingredients from [GR2] and a discussion involving \( p \)-adically integrally refined Betti–Shalika periods. Henceforth, \( \Pi \) will stand for a (not necessarily unitary) cuspidal automorphic representation of \( G(\mathcal{A}) = GL_{2n}(\mathbb{A}_F) \). Keeping multiplicity one for \( GL_{2n} \) in mind, we will let \( \Pi \) also stand for its representation space within the space of cusp forms for \( G(\mathcal{A}) \). Fix the non-trivial additive unitary character \( \psi : \mathbb{A}_F/F \rightarrow \mathbb{A}_F/\mathbb{Q} \rightarrow \mathbb{C}^\times \) where the first map is the trace, whereas the second is the usual additive character \( \psi_0 \) on \( \mathbb{A}_F/\mathbb{Q} \) characterized by \( \ker(\psi_0|_{\mathbb{Z}_F}) = \mathbb{Z}_F \) for every prime number \( \ell \) and \( \psi_0|_{\mathbb{Z}_F}(x) = |x|_F \exp(2\pi i x). \) We remark that \( (\omega, \delta_0) \), where \( \delta_0 \) is the valuation at \( v \) of the different \( \mathfrak{d} \) of \( F \), is the largest ideal contained in \( \ker(\psi_v) \). The discriminant of \( F \) is \( N_{F/\mathbb{Q}}(\mathfrak{d}) \).
4.1.1. Global Shalika models. Let \( \eta : F^\times \backslash \mathbb{A}_F^\times \to \mathbb{C}^\times \) be a Hecke character such that \( \eta^n \) equals the central character of \( \Pi \). We get an automorphic character:

\[
\eta \otimes \psi : S(F) \backslash S(\mathbb{A}_F) \to \mathbb{C}^\times, \quad \left( \begin{array}{cc} h & hX \\ 0 & h \end{array} \right) \mapsto \eta(\det(h))\psi(Tr(X)).
\]

For a cusp form \( \varphi \in \Pi \) and \( g \in G(\mathbb{A}) \) consider the integral

\[
W^n_\varphi(g) := \int_{Z(\mathbb{A})S(F) \backslash S(\mathbb{A}_F)} \varphi(sg)(\eta \otimes \psi)^{-1}(s)ds,
\]

where Haar measures are normalized as in [GR2, §2.8]. It is well-defined by the cuspidality of the function \( \varphi \) (see [JS, §8.1]) and hence yields a function \( W^n_\eta : G(\mathbb{A}) \to \mathbb{C} \) such that

\[
W^n_\eta(sg) = (\eta \otimes \psi)(s) \cdot W^n_\varphi(g),
\]

for all \( g \in G(\mathbb{A}) \) and \( s \in S(\mathbb{A}) \). In particular, we obtain an intertwining of \( G(\mathbb{A}) \)-modules

\[
S^n_\psi : \Pi \to \text{Ind}_{S(\mathbb{A})}^{G(\mathbb{A})}(\eta \otimes \psi), \quad \varphi \mapsto W^n_\varphi.
\]

The following theorem, due to Jacquet and Shalika, gives a necessary and sufficient conditions for the existence of a non-zero intertwining as in [MK].

**Theorem 4.1.** ([JS Thm.1]) The following assertions are equivalent:

(i) There exists \( \varphi \in \Pi \) such that \( W^n_\varphi \neq 0 \).

(ii) There exists an injection of \( G(\mathbb{A}) \)-modules \( \Pi \hookrightarrow \text{Ind}_{S(\mathbb{A})}^{G(\mathbb{A})}(\eta \otimes \psi) \).

(iii) The twisted partial exterior square L-function \( \prod_{v \notin \Sigma_\Pi} L(s, \Pi_v, \wedge^2 \otimes \eta_v^{-1}) \) has a pole at \( s = 1 \), where \( \Sigma_\Pi \) is the set of places where \( \Pi \) is ramified.

This is proved in [JS] for unitary representations and its extension to the non-unitary case is easy. If \( \Pi \) satisfies any one, and hence all, of the equivalent conditions of Theorem 4.1 then we say that \( \Pi \) has a \((\eta, \psi)\)-Shalika model, and we call the isomorphic image \( S^n_\psi(\Pi) \) of \( \Pi \) under \([\mathbb{G}]\) a **global \((\eta, \psi)\)-Shalika model** of \( \Pi \). Then clearly \( \Pi \otimes \chi \) has an \((\eta \chi^2, \psi)\)-Shalika model for any Hecke character \( \chi \), by keeping the same model and only twisting the action.

The following proposition gives another equivalent condition for \( \Pi \) to have a global Shalika model (see [GR2] for more details).

**Proposition 4.2** (Asgari–Shahidi [AS]). Let \( \Pi \) be a cuspidal automorphic representation of \( \text{GL}_{2n}(\mathbb{A}_F) \) with central character \( \omega_{\Pi} \). Then the following assertions are equivalent:

(i) \( \Pi \) has a global \((\eta, \psi)\)-Shalika model for some character \( \eta \) satisfying \( \eta^n = \omega_{\Pi} \).

(ii) \( \Pi \) is the transfer of a globally generic cuspidal automorphic representation \( \pi \) of \( \text{GSpin}_{2n+1}(\mathbb{A}_F) \).

In particular, if any of the above equivalent conditions is satisfied, then \( \Pi \) is essentially self-dual. The character \( \eta \) may be taken to be the central character of \( \pi \).

4.1.2. Period integrals and L-functions. The following proposition, due to Friedberg and Jacquet, is crucial for much that will follow. It relates the period-integral over \( H \) of a cusp form \( \varphi \) of \( G \) to a certain zeta-integral of the function \( W^n_\varphi \) in the Shalika model corresponding to \( \varphi \) over one copy of \( \text{GL}_n \).
Proposition 4.3. [FJ Prop.2.3] Assume that $\Pi$ has an $(\eta, \psi)$-Shalika model. For $\varphi \in \Pi$

$$\Psi(s, \varphi, \chi, \eta) := \int_{Z(\mathbb{A})H(\mathbb{Q}) \backslash H(\mathbb{A})} \varphi \left( \begin{pmatrix} h & 0 \\ 0 & h_2 \end{pmatrix} \right) (\chi|\det(h)|^{-1/2} \frac{\det(h_1)}{\det(h_2)}) \eta^{-1}(\det(h_2)) dh_1 dh_2$$

converges absolutely for all $s \in \mathbb{C}$. For $\Re(s) \gg 0$ it is equal to

$$\zeta(s; W^\eta_\varphi, \chi) := \int_{GL_n(\mathbb{A}_F)} W^\eta_\varphi \left( \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \right) \chi(\det(h))|\det(h)|^{s-1/2} dh,$$

thus providing an analytic continuation of $\zeta(s; W^\eta_\varphi, \chi)$ to all of $\mathbb{C}$.

Suppose the representation $\Pi$ of $G(\mathbb{A}) = GL_2n(\mathbb{A}_F)$ decomposes as $\Pi = \otimes_v \Pi_v$, where $\Pi_v$ is an irreducible admissible representation of $GL_{2n}(F_v)$.

If $\Pi$ has a global Shalika model, then $\mathcal{S}^{\eta}_\psi$ defines local Shalika models at every place (see Definition [3.2]). The corresponding local intertwining operators are denoted by $\mathcal{S}^{\eta}_\psi$ and their images by $\mathcal{S}^{\eta}_\psi(\Pi_v)$, whence $\mathcal{S}^{\eta}_\psi(\Pi) = \otimes_v \mathcal{S}^{\eta}_\psi(\Pi_v)$. We can now consider cusp forms $\varphi$ such that the function $W^\eta_\varphi \in \mathcal{S}^{\eta}_\psi(\Pi)$ is factorizable as $W^\eta_\varphi = \otimes_v W^\eta_{\varphi_v}$, where

$$W^\eta_{\varphi_v} \in \mathcal{S}^{\eta}_\psi(\Pi_v) \subset \text{Ind}_{S(F_v)}^{GL_{2n}(F_v)}(\eta_v \otimes \psi_v).$$

Then the following factorisation holds for $\Re(s) \gg 0$:

$$(57) \quad \zeta(s; W^\eta_\varphi, \chi) = \prod_v \zeta_v(s; W^\eta_{\varphi_v}, \chi_v),$$

where the local zeta integrals $\zeta_v(s; W^\eta_{\varphi_v}, \chi_v)$ are related to $L$-functions in Proposition [3.3].

Proposition [4.3] relates this Shalika zeta integral to a period integral over $H$, and the main thrust of [AG], refined and generalized in [GR2], is that the period integral over $H$ admits a holomorphic interpretation, provided that $\Pi$ is of cohomological type.

4.1.3. An interlude on cuspidal cohomology. In this paragraph we recall some well-known facts from Clozel [C §3] (see also [GR2 §3.4] and [GR1 §5.5]). Assume from now on that the cuspidal automorphic representation $\Pi$ is cohomological with respect to a dominant integral weight $\mu \in X^+_c(T)$ (see [4]), i.e.,

$$H^q(\mathfrak{g}_\infty, K_\infty^\circ \otimes V_\xi) = H^q(\mathfrak{g}_\infty, K_\infty^\circ \otimes V_\xi) \otimes \Pi_f \neq 0$$

for some degree $q$. A necessary condition for the non-vanishing of this cohomology group is that the weight $\mu$ is pure, i.e., $\mu \in X^+_0(T)$. For each archimedean place $\sigma \in \Sigma_\infty$, $\Pi_\sigma$ can be explicitly described as follows. For any integer $\ell \geq 1$ consider the unitary discrete series representation $D(\ell)$ of $GL_2(\mathbb{R})$ of lowest non-negative $SO_2$-type $\ell + 1$ and central character $\text{sgn}^{\ell+1}$. Let $P$ be the parabolic subgroup of $GL_{2n}$ with Levi factor $\prod_{i=1}^{n} GL_2$. Then

$$\Pi_\sigma \simeq \text{Ind}_{P(\mathbb{R})}^{GL_{2n}(\mathbb{R})} \left( \bigotimes_{i=1}^{n} D(2(\mu_{\sigma,i} + n - i) + 1 - w) \otimes |\det|^{-w/2} \right), \quad \forall \sigma \in \Sigma_\infty,$$

in particular $\omega_{P_{\mu}} = |.|^{-n w}$. The highest degree supporting cuspidal cohomology of $G$ is

$$(58) \quad H^q(\mathfrak{g}_\infty, K_\infty^\circ \otimes V_\xi) = \text{Hom}_{K_\infty^\circ}(\Lambda^q(\mathfrak{g}_\infty/\mathfrak{t}_\infty), \Pi_\infty \otimes V_\xi),$$

is a line. The relative Lie algebra cohomology of $\Pi$ as above is a summand of the cuspidal cohomology which in turn injects into the cohomology with compact supports:

$$(59) \quad H^q(\mathfrak{g}_\infty, K_\infty^\circ \otimes V_\xi) \hookrightarrow H^q_{\text{cusp}}(S^G_K, \mathcal{V}_\xi) \hookrightarrow H^q_{\text{cusp}}(S^G_K, \mathcal{V}_\xi).$$
(see for example [GaR §2]). By $[\mathcal{C}]$ the cuspidal cohomology inherits a rational structure from sheaf-cohomology which allows one to deduce that the finite part $\Pi_f$ is defined over its rationality field that we denote $\overline{\mathbb{Q}}(\Pi)$.

If in addition $\Pi$ admits an $(\eta, \psi)$-Shalika model then $\eta$ is forced to be algebraic of the form $\eta = n|q^w$ with $n \in \mathbb{N}$ of finite order, $w$ is the purity weight of $\mu$ (see [GaR Thm.5.3]).

The reader should appreciate that the analytic condition on $\Pi$ of admitting a-Shalika model and the algebraic condition of contributing to the cuspidal cohomology of $G$ are of entirely different nature. One may construct examples of representations satisfying only one of these conditions and not the other (see [GR2 §3.5]).

4.1.4. Betti-Shalika periods. Let $\Pi$ be a cuspidal automorphic representation which is cohomological with respect to $\mu$. For any character $\epsilon : K_\infty / K_\infty^0 \to \{\pm 1\}$, we fix a basis $\Xi_\infty^\mu$ of the line $H^0(\mathfrak{g}_\infty, K_\infty^\mu; S_{\psi_i^\mu}^0(\Pi_\infty) \otimes V_2^\mu)[e]$ (see §2.5). We define exactly as in [GR2 §3.3] and [GR2 §4.2] an isomorphism $\Theta^\epsilon$ of $G(A_f)$-modules as the composition of the three isomorphisms:

\[
S_{\psi_i^\mu}(\Pi_f) \xrightarrow{\sim} S_{\psi_i^\mu}(\Pi_f) \otimes H^0(\mathfrak{g}_\infty, K_\infty^\mu; S_{\psi_i^\mu}^0(\Pi_\infty) \otimes V_2^\mu)[e] \xrightarrow{\sim} H^0(\mathfrak{g}_\infty, K_\infty^\mu; S_{\psi_i^\mu}^0(\Pi) \otimes V_2^\mu)[e] \xrightarrow{\sim} H^0(\mathfrak{g}_\infty, K_\infty^\mu; \Pi \otimes V_2^\mu)[e],
\]

where the first map is $W_f \mapsto W_f \otimes \Xi_\infty^\mu$, the second map is the natural one and the third map is the map induced in cohomology by $(S_{\psi_i^\mu})^{-1}$ from (55). For any field extension $E/\mathbb{Q}(\Pi)$, the Shalika model $S_{\psi_i^\mu}(\Pi_f)$ has an $E$-rational structure, and independently, the cohomology $H^0(\mathfrak{g}_\infty, K_\infty^\mu; \Pi \otimes V_2^\mu)[e]$ has an $E$-structure coming from (59). The Betti-Shalika period $\Omega_{\Pi}^\epsilon \in \mathbb{C}^\times$ is the homothety that is needed to modify $\Theta^\epsilon$ so that

\[
\Theta_{\mu^0}^\epsilon := (\Omega_{\Pi}^\epsilon)^{-1} \cdot \Theta^\epsilon
\]
preserves those $E$-structures (see [GR2 Prop.4.2.1]). The periods are well-defined up to $E^\times$ and will be refined in [3.3.1] when $E$ is a $p$-adic field, to be well-defined up to $\mathcal{O}^\times$.

4.2. Ordinarity and regularity. For $p$ dividing $p$, let $\mathcal{O}_p$ denote an uniformizer and let $q_p = |\mathcal{O}_p|^{-1}$ denote the cardinality of the residue field of $F_p$. Let $\Sigma_p = \prod_{p \mid \mathcal{O}_p} \Sigma_p$ be the partition induced by $i_p : \mathbb{Q} \hookrightarrow \mathcal{O}_p$, where $\Sigma_p = \{\sigma : F_p \hookrightarrow \mathcal{O}_p\}$. Since $p$ is unramified in $F$, we have $q_p = p^{f(\mathcal{O}_p)}$.

Recall from [1.1] that a weight $\mu \in X_+^\mu(T)$ yields a rational character $\mu : T = \text{Res}_{E/F} T_{2n} \to \text{GL}_1$, therefore induces a character $\mu_\mu = \otimes_{p \mid \mu} \mu_p$ of $T(\mathcal{Q}_p) = \prod_{p \mid \mu} T_{2n}(F_p)$, where

\[
\mu_\mu : T_{2n}(F_p) \to \mathcal{Q}_p^\times \text{ is given by } (\mu_\sigma)_{\sigma \in \Sigma_p} \text{ subject to the dominance condition }
\mu_{\sigma_1} \geq \mu_{\sigma_2} \geq \ldots \geq \mu_{\sigma_{2n}}, \text{ for all } \sigma \in \Sigma_p.
\]

Recall the maximal $(n,n)$-parabolic subgroup $Q \subseteq G$. Given a cuspidal automorphic representation $\Pi$ of $G(A)$ that is cohomological with respect to the weight $\mu \in X_+^\mu(T)$, we say that $\Pi_p$ is $U_p$-ordinary (resp. $p$-ordinary) if it is $Q$-ordinary (resp. $B$-ordinary) in the sense of Hida (see [1.1] [1.2]).

Assume from now on that $\Pi_p$ is unramified for all $p \mid p$. Since $\Pi$ is cohomological there exists an unramified algebraic character $\lambda_p : T_{2n}(F_p) \to \mathbb{Q}_p^\times \xrightarrow{\text{mod } \mathcal{O}_p} \mathbb{C}^\times$ such that (see [1.1]):

\[
\Pi_p = \text{Ind}_{B_{2n}(F_p)}^{G_{2n}(F_p)}(1 \cdot | \frac{2n-1}{2} \lambda_p).
\]
Using $i_p$ instead of $i_\infty$ allows us to see the Hecke parameters $\alpha_{p,i} = \lambda_{p,i}(\varpi_p)$, $1 \leq i \leq 2n$ as element of $\mathbb{Q}_p^\times$. Then $\Pi_p$ is $p$-ordinary relative to an ordering of its Hecke parameters $\alpha_{p,i}$ if and only if

$$|\mu_p^\vee(\varpi_p) \cdot q_p^{i-1} \alpha_{p,2n+1-i}|_p = 1, \quad \text{for all } 1 \leq i \leq 2n. \quad (64)$$

where $|\cdot|_p$ denotes the $p$-adic norm. The $B$-dominance condition (3) then implies that:

$$|\alpha_{p,1}|_p < |\alpha_{p,2}|_p < \cdots < |\alpha_{p,2n}|_p; \quad (65)$$

hence there exists at most one ordering of the Hecke parameters for which $\Pi_p$ is $p$-ordinary. Moreover this implies that a $p$-ordinary $\Pi_p$ is necessarily regular, i.e., the $\alpha_{p,i}$ are pairwise distinct.

Similarly, $\Pi_p$ is $U_p$-ordinary relative to $\tau \in W_G/W_H$ if and only if

$$\prod_{i \in \tau} |\alpha_{p,i}|_p = \left| q_p^{\frac{n(n-1)}{2}} \mu_p^\vee(i(\varpi_p^{-1}1_n, 1_n)) \right|_p. \quad (66)$$

We will make a key observation that $U_p$-ordinarity implies $Q$-regularity (see Definition 8).

**Lemma 4.4.** Assume that the cuspidal automorphic representation $\Pi$ of $G(\mathbb{A})$ is cohomological with respect to $\mu$ and admits an $(\eta, \psi)$-Shalika model. For $p$ dividing $p$, if $\Pi_p$ is spherical and $U_p$-ordinary, then

$$v_p(\alpha_{p,i}) < |\Sigma_p|^{w+2n-1} < v_p(\alpha_{p,j}), \quad \text{for all } i \in \tau, j \notin \tau. \quad (67)$$

In particular, $\Pi_p$ is $U_p$-ordinary only relative to $\tau$. Moreover, $\tilde{\Pi}_p = (\Pi_p, \tau)$ is $Q$-regular, i.e., $q_p^{\frac{n(n-1)}{2}} \alpha_p^\tau$ is a simple eigenvalue of $U_p$ acting on $\Pi_p^{\tau_p}$ and $\lambda_{p,i} \neq \lambda_{p,j}$ for all $i \in \tau, j \notin \tau$.

**Proof.** Consider the Hecke operators $U_{p,n-1} = [I_p t_{p,n-1} I_p]$ acting on $\Pi_{p,n}^{\tau_p}$, where $t_{p,n-1} = \text{diag}(\varpi_p 1_{n-1}, 1_{n+1}) \in \text{GL}_2n(F_p)$. Since $\mu_p^\vee(t_{p,n-1}) \cdot U_{p,n-1}$ preserves $p$-integrality its eigenvalues on $\Pi_{p,n}^{\tau_p}$ are $p$-integral, in particular for any $i \in \tau$ we have

$$|\alpha_{p,j}|_p \leq \left| \mu_p^\vee(i_{p,n-1}) \cdot q_p^{\frac{n(n-1)(n-2)}{2}} \right|_p. \quad (68)$$

Together with (67) this implies that $|\alpha_{p,i}|_p \leq |\mu_p^\vee(\varpi_p^{-1})q_p^{n-1}|_p = |\mu_{p,n+1}(\varpi)q_p^{n-1}|_p, \ i.e.,$

$$v_p(\alpha_{p,i}) \leq \sum_{\sigma \in \Sigma_p} (n - 1 + \mu_{\sigma,n+1}^p). \quad (68)$$

The existence of $(\eta_p, \psi_p)$-Shalika model for $\Pi_p$ gives by [AG, Prop.1.3] a $j = \rho(i)$ so that

$$v_p(\alpha_{p,i}) + v_p(\alpha_{p,j}) = |\Sigma_p|(2n - 1 + w). \quad (68)$$

The latter equality together with (67) and (68) yields for all $i \in \tau:

$$v_p(\alpha_{p,i}) \leq \sum_{\sigma \in \Sigma_p} (n - 1 + \mu_{\sigma,n+1}^p) < |\Sigma_p|^{w+2n-1} < \sum_{\sigma \in \Sigma_p} (n + \mu_{\sigma,n}) \leq v_p(\alpha_{p,\rho(i)}).$$

All claims follow then easily as clearly $\rho(\tau) \cap \tau = \emptyset$ as requested in Definition 8.5. □
4.3. **p-adic interpolation of critical values.** We suppose in the sequel that \( \Pi \) is a cuspidal automorphic representation of \( G(\mathbb{A}) \) which is cohomological with respect to a pure weight \( \mu \in X_0^+(T) \) and that \( \Pi \) admits an \((\eta, \psi)\)-Shalika model. Assume further that for all \( p \mid p \), \( \Pi_p \) is spherical and that \( \Pi_p = (\Pi_p, \tau) \) is \( Q\text{-regular} \) for \( \tau = \{n + 1, \ldots, 2n\} \) in the sense of Definition 3.5.

### 4.3.1. Choice of local Shalika vectors.

For \( v \mid p \infty \) we choose \( \mathbb{Q}(\Pi) \)-rational local vectors \( W_{\Pi_v} \in S^0_{v, \psi} ( \Pi_v ) \) as in Proposition 3.3.

For \( p \mid p \), \( \Pi_p \) is spherical and \( \Pi_p = (\Pi_p, \{n + 1, \ldots, 2n\}) \) is \( Q\text{-regular} \), in particular, \( \alpha_p := \prod_{n+1 \leq i \leq 2n} \alpha_{p,i} \) is a simple eigenvalue for the Hecke operator \( U_p \) acting on \( \Pi_p \). By Lemma 3.6 there exists an unique \( W_{\Pi_p} \) on the line \( S^0_{\psi_p} (\Pi_p)^{U_p} \) normalized so that

\[
W_{\Pi_p}(t_p^{-\delta}) = 1. \tag{70}
\]

In addition to the conditions \((K1)\) and \((K2)\) on \( K \) henceforth we assume that

\( (K3) \) \( K \) fixes \( W_{\Pi_f} \) and \( \eta \) is trivial on \( I(m) \) hence can be seen as a character of \( \mathcal{O}_E^\perp(\mathfrak{m}) \).

**Definition 4.5.** Fix a character \( \epsilon : K_\infty/K_\infty^0 \to \{\pm1\} \). Using \([55]\), the basis \( \Xi^\epsilon_\infty \) of the line \( H^{\eta}(g_\infty, K_\infty^0; S^0_{\psi, \infty} (\Pi_\infty) \otimes V^\mu_C)(\epsilon) \) from \([4.1.2]\) can be written as a \( K_\infty^0 \)-invariant element

\[
\Xi^\epsilon_\infty = \sum_{\epsilon=1}^r \omega^\epsilon \otimes W_{\infty, i}^\epsilon \otimes v^\epsilon_i \in \wedge^0 (g_\infty, K_\infty^0)^V \otimes S^0_{\psi, \infty} (\Pi_\infty) \otimes V^\mu_C. \tag{71}
\]

For \( 1 \leq i \leq r \), let \( \psi^\epsilon_i \in \Pi \) be the unique vector whose image under \([55]\) equals:

\[
W_{\psi^\epsilon_i} = W_{\Pi_f} \otimes W_{\infty, i}^\epsilon \in S^\eta_{\psi}(\Pi). \tag{72}
\]

For each character \( \epsilon \) of \( K_\infty/K_\infty^0 \) the isomorphism \( \Theta^\epsilon_0 = (\Omega_{\Pi_\infty})^{-1} \cdot \Theta^\epsilon \) from \([61]\) composed with the embedding of \([59]\) yields

\[
\Theta^\epsilon_0 : S^\eta_{\psi_f} (\Pi_f) \overset{\sim}{\longrightarrow} H^{\eta}(g_\infty, K_\infty^0; \Pi^K \otimes V^\mu_C)[\epsilon] \hookrightarrow H^{\eta}_{c}(S^0_{K, g_\infty^0}, V^\mu_C), \tag{72}
\]

Denote by \( \mathbb{Q}(\Pi)/\mathbb{Q}(\Pi) \) the extension obtained by adjoining the eigenvalues \( \alpha_p \) of \( U_p \) for \( p \mid p \). Recall that we fixed a finite extension \( E/\mathbb{Q}_p \) and embeddings \( \iota^\epsilon_p : \mathbb{Q} \subset \mathbb{Q}_p \) and \( \iota^\epsilon_\infty : \mathbb{Q} \to E \). We obtain a diagram

\[
\begin{array}{cccc}
H^{\eta}_{c}(S^0_{K, g_\infty^0}, V^\mu_C)[\epsilon] & \overset{i^{\epsilon}_\infty}{\longrightarrow} & H^{\eta}_{c}(S^0_{K, g_\infty^0}, V^\mu_C)[\epsilon] & \overset{i^{\epsilon}_p}{\longrightarrow} & H^{\eta}_{c}(S^0_{K, g_\infty^0}, V^\mu_C)[\epsilon] & \longrightarrow & H^{\eta}_{c}(S^0_{K, g_\infty^0}, V^\mu_C)[\epsilon] \\
\Theta^\epsilon_0 \downarrow & & \Theta^\epsilon_0 \downarrow & & \Theta^\epsilon_0 \downarrow & & \Theta^\epsilon_0 \downarrow \\
S^\eta_{\psi_f} (\Pi_f) & \overset{\phi^\epsilon_{\Pi_\infty}}{\longrightarrow} & H^{\eta}_{c}(S^0_{K, g_\infty^0}, V^\mu_C)[\epsilon] & & & & \end{array}
\]

Since \( W_{\Pi_f} \) is \( \mathbb{Q}(\Pi) \)-rational, the class \( \Theta^\epsilon_0( W_{\Pi_f} ) \) lies in the image of \( i^\epsilon_\infty \). The image of \( (i^\epsilon_p \circ (i^\epsilon_\infty)^{-1} \circ \Theta^\epsilon_0)( W_{\Pi_f} ) \) by the intertwining operator between the two sheaf construction described in \([1.3]\) yields, after possibly rescaling the periods \( \Omega_{\Pi_\infty} \) in order to render the image \( \mathcal{O} \)-integral, a cohomology class

\[
\phi^\epsilon_{\Pi_\infty} \in H^{\eta}_{c}(S^0_{K, g_\infty^0}, V^\mu_C)[\epsilon]. \tag{73}
\]
Recall that the Hecke operator $U_p^α$ defines an endomorphism of $H^0_c(S^1_K, V^μ_0)[c]$. Since $W_{Π_j}$ is an $U_{p^β}$-eigenvector with eigenvalue $α_{p^β} = \prod_{p|β} α_p^β$, it follows (after possibly rescaling by a power of $p$ killing the torsion in $H^0_c(S^1_K, V^μ_0)$ and modifying $Ω_{Π_i}$ accordingly) that one can assume $φ^c_{Π_i}$ is an $U^0_p$-eigenvalue with eigenvalue $α^c_{p^β} = μ^c(t_p^β)α_p^β$.

4.3.2. **Interpolation formula at critical points.** In this section we will relate the image of $φ^c_{Π_i}$ defined in (73) by the evaluation map $E^j_{β, [β]}$ from (34), to the Friedberg-Jacquet integral from Proposition 4.3.

**Proposition 4.6.** For any $ε ∈ \{±1\}^{Σ_∞}$ and any $[δ] ∈ \mathcal{O}_F^+(p^βm) × \mathcal{O}_F^+(m)$ we have:

$$μ^{j_{T_p}}(t_p^β) · E^j_{β, [β]}(φ^c_{Π_i}) = \frac{1}{Ω_{Π_i}} \int_{S^r_{L, [β], [δ], [m]}} \varphi^c_{Π_i}(hξt_p^β)|det(hξt_p^β)|_{Fdh},$$

where $φ^c_{Π_i} = \sum_{i=1}^r α^c_{i,j} · φ^c_{i}$ for suitable $α^c_{i,j} ∈ \mathbb{C}$.

**Proof.** We follow closely the proof of [BDJ Prop.4.6]. We will first prove that the left hand side belongs in fact to the number field $Q(Π)$. Consider the commutative diagram:

$$\begin{array}{cccc}
H^0_c(S^1_K, V^μ_{Q(Π)}) & (g,v) → (g, g^{-1}, v) & H^0_c(S^1_K, V^μ_{E}) \\
\downarrow \downarrow & & \downarrow \downarrow \\
H^0_c(S^l_{L, [β]}, V^{(j, w-j)}_{Π}) & (h,v) → (h, h^{-1}, v) & H^0_c(S^l_{L, [β]}, V^{(j, w-j)}_{E}) & (−\cap θ)οtriv^*οκ_{i,j} \\
\downarrow & N_{F_p/Q_p}(det(δ^j_{i, j}οδ^w_{i, j})) & \downarrow & (-\cap θ)οtriv^*οκ_{i,j} \\
Q(Π) & & E & \end{array}$$

where $τ_β = μ^{j_{T_p}}(t_p^β)$ is defined in (25), the horizontal maps are induced from the morphisms of local systems written above them, the map $T_β$ is induced from the morphisms of local systems $(h,v) → (hξt_p^β, v)$, and $triv^*_j$ is induced from the morphisms of local systems:

$$H(Q(Π))δ_{L_β}H^∞ \times V^{(j, w-j)}_{Q(Π)} → V^{(j, w-j)}_{Q(Π)|S^l_{L, [β], [m]}} ; (γδlh^∞, v) → (γδlh^∞, γ^{-1} · v).$$

Since $E^j_{β, [β]} = ε(\det(δ^j_{i, j}οδ^w_{i, j})) E^j_{β, [β]}$, where $ε = |_F · N_{F_p/Q}$ is the $p$-adic cyclotomic character, the above diagram shows that the proposition is equivalent to:

$$| \det(δ^j_{i, j}οδ^w_{i, j})|_F (−\cap θ)οtriv^*οκ_{i,j} o T_β) (Ω_{Π_i}) = \int_{S^r_{L, [β], [m]}} \varphi^c_{Π_i}(hξt_p^β)|det(hξt_p^β)|_{Fdh},$$

the left hand side being considered over $\mathbb{C}$ via the inclusion $i_∞ : \mathbb{Q} → \mathbb{C}$. By Definition 4.5

$$Ω_{Π_i} · φ^c_{Π_i} = (S^l_{l})^{−1} (W_{Π_j} ⊗ Ξ_∞) = \sum_{i=1}^r ω^c_i ⊗ φ^c_i ⊗ ν^c_i ∈ (Λ^{γ0}(g_∞, Φ_∞) ⊗ Π ⊗ V^{μ})^{K_{∞}}_c,$$

yielding a $V^μ_c$-valued de Rham differential $q_0$-form $\sum_{i=1}^r φ^c_i(g_∞ · ν^c_i)(g_∞^{-1})^*ω^c_i$ on $G^0_c/K_{∞}$.
Recall the basis $\kappa_j$ of the line $\text{Hom}_H(V^\mu, V^{(j,w-j)})$ from (29) and consider the map

$$
\kappa_j \circ \iota^* : H^{q_0}(g_{\infty}, K_{\infty}; \Pi_\infty \otimes V^H_C) \to H^{q_0}(\eta_{\infty}, L^0_{\infty}; \Pi_\infty \otimes V_C^{(j,w-j)})
$$

hence

$$(\kappa_j \circ T_{\beta})(\Omega_{\Pi}^C\phi_p^C) = \sum_{i=1}^r \iota^*\omega_i^C \otimes \varphi_i^C(- \cdot \xi t_p^\beta) \otimes \kappa_j(\nu_i^C) \in \left( \wedge^{q_0}(\eta_{\infty}/L_{\infty})^\vee \otimes \Pi \otimes V_C^{(j,w-j)} \right)^{L^0_{\infty}}.
$$

Let $\iota_j : V_C^{(j,w-j)} \overset{\sim}{\to} \mathbb{C}$ be the scalar extension of (111) and fix a basis of $\wedge^{q_0}(\eta_{\infty}/L_{\infty})^\vee$ given by a Haar measure $dh_{\infty}$. Then the restriction to $S^H_{\mu_{\beta}}[\delta]$ of $\kappa_j(T_{\beta}(\Omega_{\Pi}^C\phi_p^C))$ can be seen as $V_C^{(j,w-j)}$-valued de Rham differential $q_0$-form on $H^0_{\infty}/L^0_{\infty}$ given by

$$
\sum_{i=1}^r a_{i,j}^C \cdot \varphi_i^C(\delta_{h,\infty} t_p^\beta) \det(h^j_{1,\infty} h^{w,j}_{2,\infty}) dh_{\infty}
$$

for suitable $a_{i,j}^C \in \mathbb{C}$. Writing $h = \gamma \delta h^\infty_{\infty} \in H(Q) \delta L_\beta H^0_{\infty} \subset H(\mathbb{A})$, and using that $\text{triv}_\beta(\gamma \delta h^\infty_{\infty}, v) = (\gamma \delta h^\infty_{\infty}, \det(\gamma^{-1}_i J^j_{\infty} v))$ one obtains (14) and the Proposition from

$$
\big| \det(\delta^j_{1,f} \delta^{w,j}_{2,f}) \big|_F \big| \det(h^j_{1,\infty} h^{w,j}_{2,\infty}) \big|_F = \big| \det(\delta^j_{1,f} \delta^{w,j}_{2,f}) \big|_F \big| \det(h^j_{1,\infty} h^{w,j}_{2,\infty}) \big|_F.
$$

## 4.3.3. A distribution attached to $\tilde{\Pi}$

We recall that $\Pi$ is cuspidal automorphic representation of $G(\mathbb{A})$ admitting a global $(\psi, \eta)$-Shalika model, which is cohomological with respect to a pure dominant integral weight $\mu$. Recall also that $\Pi_p$ is spherical for all $p | p$ and that $\Pi_p = (\Pi_p, \{n + 1, \ldots, 2n\})$ is $Q$-regular, which by Lemma 4.3 is automatically fulfilled if $\Pi_p$ is $U_{\sigma}$-ordinary. In all cases $\Pi_p^{t_{\sigma}}$ contains a unique line on which $U_{\sigma}$ acts by $\alpha_{\sigma}$. Finally recall the $U_{\sigma}^{t_{\sigma}}$-eigenvectors $\phi_p^{t_{\sigma}}$ constructed in (73). Then

$$
\phi_{\Pi}^C := \sum_{\epsilon \in \{\pm 1\}^{\Sigma_{\infty}}} \phi_p^{t_{\sigma}}
$$

is an $U_{\sigma}^{t_{\sigma}}$-eigenvector with same eigenvalue. Consider the element

$$
\mu_{\Pi}^\sigma := \mu_{\phi_{\Pi}^C}^{t_{\sigma}} = \varepsilon_{\gamma_{\Pi}}^{t_{\sigma}}(\mu_{\phi_{\Pi}^C}^{t_{\sigma}}) \in E[[\mathcal{O}_F^+(p^{\infty})]],
$$

constructed in (38) and (10), which defines an $E$-valued distribution $d\mu_{\Pi}^\sigma$ on $\mathcal{O}_F^+(p^{\infty})$. When $\Pi$ is $U_{\sigma}$-ordinary, then $d\mu_{\phi_{\Pi}^C}$ is $O$-valued, hence $d\mu_{\phi_{\Pi}^C}$ is a measure on $\mathcal{O}_F^+(p^{\infty})$.

## 4.3.4. Main theorem on $p$-adic interpolation

For any character $\epsilon : \{\pm 1\}^{\Sigma_{\infty}} \to \{\pm 1\}$ and any $j \in \text{Crit}(\mu)$, by a careful inspection of proof of Proposition 4.3 one sees that the cohomological test vector $W^\sigma_{\Pi_{\infty}, j} = \sum_{i=1}^r a_{j,i}^\sigma W^\sigma_{\Pi_{\infty}, i}$ decomposes as pure tensor $\otimes_{\sigma \in \Sigma} W^\sigma_{\Pi_{\infty}, j}$. A crucial result of Sun [Su] asserts the following non-vanishing:

$$
\zeta_\infty(j + \frac{1}{2}; W^\sigma_{\Pi_{\infty}, j}, \chi_\sigma) = \prod_{\sigma \in \Sigma} \zeta_\sigma(j + \frac{1}{2}; W^\sigma_{\Pi_{\infty}, j}, \chi_\sigma) \in \mathbb{C}^\times.
$$

Recall the auxiliary ideal $m$ from (111) in (2.1) and, for brevity, let’s define

$$
\gamma = \# \mathcal{O}_F^+(m) \cdot \# \mathcal{GL}_n(O_F/m) \cdot \# \mathcal{PGL}_n(O_F/m) \cdot \prod_{p | p} \left( q_p^{-n^2} \cdot \# \mathcal{GL}_n(O_F/p) \right) \in \mathbb{Q}^\times.
$$
Theorem 4.7. For any finite order character $\chi$ of $\mathcal{O}_p^+(p^\infty)$ of conductor $\beta_p \geq 1$ at $p \mid p$:

$$\int_{\mathcal{O}_p^+(p^\infty)} \epsilon^j(x) \chi(x) d\mu^\beta_{\Pi}(x) =$$

$$\gamma \cdot N_{F/Q}(\mathfrak{d}) \cdot \prod_{p \mid p} \left( \alpha_p^{-1} q_p^{n+1} \right)^{\beta_p} \cdot \frac{G(\chi_f)^n \cdot L(j + \frac{1}{2}, \Pi_f \otimes \chi_f)}{\zeta_{\infty}^1(j + \frac{1}{2}; W^{(\epsilon \gamma)^\infty}_{\Pi_{\infty,j}}; \chi_{\infty}) \cdot \Omega_{\Pi_{\infty,j}}^{(\epsilon \gamma)^\infty}}.$$ 

Remark 4.8. Note first that while the archimedean zeta integral $\zeta_{\infty}(j + \frac{1}{2}; W^{(\epsilon \gamma)^\infty}_{\Pi_{\infty,j}}; \chi_{\infty})$ and the period $\Omega_{\Pi_{\infty,j}}$ both depend on the choice of the cohomological choice $\Xi_{\infty}$ in $[\text{4.1.A}]$, their quotient appearing in the denominator of the above equation is independent of that choice. Further note that the main result of [GR2] asserts that the quantity

$$L^{\text{alg}}(\frac{1}{2} + j, \Pi_f \otimes \chi_f) := \frac{G(\chi_f)^n \cdot L(j + \frac{1}{2}, \Pi_f \otimes \chi_f)}{\zeta_{\infty}^1(j + \frac{1}{2}; W^{(\epsilon \gamma)^\infty}_{\Pi_{\infty,j}}; \chi_{\infty}) \cdot \Omega_{\Pi_{\infty,j}}^{(\epsilon \gamma)^\infty}} \in \mathbb{Q},$$

and is $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant. Finally by [J4, Thm.A] we may choose $\Xi_{\infty}$ so that

$$\zeta_{\infty}^1(j + \frac{1}{2}; W^{(\epsilon \gamma)^\infty}_{\Pi_{\infty,j}}; \chi_{\infty}) \in (2\pi i)^{n+1} \mathbb{Q}, \text{ for all } j \in \text{Crit}(\mu).$$

Proof of Theorem 4.7. Using (40), (38) and (36) we find:

$$\int_{\mathcal{O}_p^+(p^\infty)} \epsilon^j(x) \chi(x) d\mu^\beta_{\Pi}(x) = \int_{\mathcal{O}_p^+(p^\infty)} \chi(x) d\mu^\beta_{\Pi}(x) = (\alpha_p^{-1})^{-1} \sum_{[x] \in \mathcal{O}_p^+(p^\infty)} \chi([x]) \xi_{\beta}(\phi_{\Pi})$$

$$= \alpha_p^{-1} \cdot \mu^\beta(t_{p}^{-\beta}) \sum_{[x] \in \mathcal{O}_p^+(p^\infty)} \chi([x]) \eta_0([y]) \xi_{\beta}(\phi_{\Pi}).$$

Since $\pi_0(S^H_{L^\beta}) \simeq \mathcal{O}^+_p(p^\infty) \times \mathcal{O}^+_p(m)$ by (73) and Proposition 4.6, the integral equals:

$$\alpha_p^{-1} \cdot \sum_{x \in (\pm 1)^2 \mathbb{Z}_{\Pi}} \frac{1}{\Lambda_{\Pi_{\beta}}} \int_{S^H_{L^\beta}} \phi_{\Pi_{\beta}}^\epsilon(h \xi_\beta) \chi \left( \frac{\det(h_1)}{\det(h_2)} \right) \left( \frac{\det(h_1)}{\det(h_2)} \right)^j \eta^{-1}(\det(h_2)) dh.$$ 

Note that the integrand is $L^\epsilon_{\infty}(\mathfrak{f})$-invariant and $L_{\infty}/L^\epsilon_{\infty}Z_{\infty}$ acts on it by $e^{\epsilon \gamma}_{\infty} \chi_{\infty}$, hence the integral vanishes unless $\epsilon = \epsilon \gamma_{\infty}$. Since $Z(\mathfrak{f}) \cap L_{\beta}$ is independent of $\beta$, after some volume computation, one further finds:

$$\int_{\mathcal{O}_p^+(p^\infty)} \epsilon^j(x) \chi(x) d\mu^\beta_{\Pi}(x) = \frac{\Psi \left( j + \frac{1}{2}, \phi_{\Pi_{\beta}}(\epsilon \gamma)^\infty (- \cdot \xi_{\beta}) \chi, \eta \right)}{\Omega_{\Pi_{\beta}}^{(\epsilon \gamma)^\infty}} \prod_{p | p} \left( \alpha_p^{-1} q_p^{n+1} \right)^{\beta_p}.$$ 

By Proposition 4.3, the Friedberg-Jacquet integral has an Euler product for $\Re(s) \gg 0$:

$$\Psi(s, \phi_{\Pi_{\beta}}^\epsilon(- \cdot \xi_{\beta}) \chi, \eta) = \prod_{v \notin \infty} \zeta_v(s; W_{\Pi_v}, \chi_v) \cdot \prod_{p | p} \zeta_p(s; W_{\Pi_p}(- \cdot \xi_{\beta}), \chi_p) \cdot \zeta_{\infty}(s; W_{\Pi_{\infty}}(\epsilon \gamma)^\infty, \chi_{\infty}).$$ 

Since $L(s, \Pi \otimes \chi)$ has trivial Euler factors at all places $p | p$ (as $\Pi_p$ is spherical while $\chi_p$ is ramified), Proposition 3.3 implies that:

$$\prod_{v \notin \infty} \zeta_v(j + \frac{1}{2}; W_{\Pi_v}, \chi_v) = N_{F/Q}^{j+1}(\mathfrak{d}) \chi(\mathfrak{d}^{-1})^n L(j + \frac{1}{2}, \Pi_f \otimes \chi_f).$$
Theorem 4.7, the measure
Proof. We will first show that
\[ 0 \leq C_F \] (79)
Let \( \Pi_m \) be a cuspidal automorphic representation which is cohomological with respect to the \( \sigma \) representation, we see that
\[ 1 + 2 \otimes \text{unramified} \] but finitely many Dirichlet characters \( \chi \) and \( \omega \), hence, Theorem 4.9. 4.4.1. The main theorem.

Non-vanishing of twists.

4.4. Non-vanishing of twists.

Theorem 4.9. Let \( \mu \) be a pure dominant integral weight such that
\[ \mu_{\sigma,n} > \mu_{\sigma,n+1}, \text{ for all } \sigma \in \Sigma_\infty. \]
Let \( \Pi \) be a cuspidal automorphic representation which is cohomological with respect to the weight \( \mu \) and admitting an \((\eta, \psi)\)-Shalika model. Assume that for all primes \( p \) above a prime number \( p \), \( \Pi_p \) is unramified and \( U_p \)-ordinary. Then for all \( j \in \text{Crit}(\mu) \) and for all but finitely many Dirichlet characters \( \chi \) of \( F \) of \( p \)-power conductor we have:
\[ L \left( \frac{1}{2} + j, \Pi \otimes (\chi \circ N_{F/Q}) \right) \neq 0. \]

We begin with a few comments. Since \( \Pi^o = \Pi \otimes |w|^{w/2} \) is a unitary cuspidal automorphic representation, we see that \( \frac{1 + w}{2} \) is the center of symmetry for the \( L \)-function of \( \Pi \):
\[ L \left( \frac{1 + w}{2}, \Pi \otimes \chi \right) = L \left( \frac{1}{2}, \Pi^o \otimes \chi \right). \]

By regularity one knows that \( \text{Crit}(\mu) \) is non-empty and condition \((29)\) is equivalent to assuming that \( \text{Crit}(\mu) \) has at least two elements. If \( \Pi \) is unitary then \( w = 0 \) and \( \frac{1}{2} \in \text{Crit}(\Pi \otimes \chi) = \text{Crit}(\mu) \), whence Theorem 4.9 is a particular case of Theorem 4.4.1.

If Leopoldt’s conjecture holds for \( F \) at \( p \) then one readily obtains a statement for all but finitely many \( p \)-power conductor Hecke characters, as opposed to Dirichlet characters.

We show non-vanishing of critical values of twisted \( L \)-functions by showing non-vanishing statement about distributions on the cyclotomic \( Z_p \)-extension of \( F \). Recall the \( p \)-adic cyclotomic character \( \varepsilon : \mathcal{O}^+_Q(p^\infty) \sim Z_p^\times = \mu_{2p} \times (1 + 2p\mathbb{Z}_p) \) the first component of which is given by the Teichmüller character \( \omega \), while the fixed field of the kernel of the second component \( \varepsilon^{-1} \omega \) is the cyclotomic \( Z_p \)-extension of \( Q \). Then by a well-known result due to Serre there is an isomorphism \( \mathcal{O}[1 + 2p\mathbb{Z}_p] \simeq \mathcal{O}[T] \) sending \( 1 + 2p \) to \( 1 + T \). Composing with the norm map \( N_{F/Q} : \mathcal{O}^+_F(p^\infty) \to \mathcal{O}^+_Q(p^\infty) \) allows us to lift Dirichlet characters to Hecke characters over \( F \), thus to push-forward of a measure on \( \mathcal{O}^+_F(p^\infty) \), such as \( \mu_{\Pi} \), to a measure on \( \mathcal{O}^+_Q(p^\infty) \). Further composing with \( \omega^m : \mu_{2p} \to \mathcal{O}^\times \) for \( 0 \leq m \leq p - 1 \) allows us to define a measure on \( 1 + 2p\mathbb{Z}_p \), \( \omega^m(\mu_{\Pi}) \in \mathcal{O}[T] \).

Proof. We will first show that \( \omega^m(\mu_{\Pi}) \neq 0 \) for all \( m \in \mathbb{Z} \). By the interpolation property in Theorem 4.7 the measure \( \omega^m(\mu_{\Pi}) \) interpolates the algebraic parts of \( L \left( \frac{1}{2} + j, \Pi \otimes \omega^{-m-j} \chi \right) \) for \( j \in \text{Crit}(\mu) \) and \( \chi \) runs over all Dirichlet characters of (non-trivial) \( p \)-power order and conductor. Our hypothesis \((29)\) implies that we find \( j \in \text{Crit}(\mu) \) satisfying \( j \geq \frac{1}{2} \), hence \( \frac{1}{2} + j \) lies outside the interior of the critical strip for \( L(s, \Pi) \) and thus \( L \left( \frac{1}{2} + j, \Pi \otimes \omega^{-m-j} \chi \right) \neq 0 \). Therefore \( \omega^m(\mu_{\Pi}) \neq 0 \) as claimed.

By the Weierstrass preparation theorem, a non-zero element of \( \mathcal{O}[T] \) admits only a finitely many zeros in \( \mathbb{Z}_p \). Again by Theorem 4.7 this means that, given any \( j \in \text{Crit}(\mu) \) and \( m \), there are at most finitely many Dirichlet characters \( \chi \) of \( p \)-power order and conductor
such that $L(\frac{1}{2} + j, \Pi \otimes \omega^{m-j} \chi) = 0$. Since any $p$-power conductor Dirichlet character is of that form for some $0 \leq m \leq p - 1$, the theorem follows. \hfill $\Box$

4.4.2. Variations.

**Corollary 4.10** (Nearly-ordinary case). Under the hypotheses of Theorem 4.9, let $\nu$ be a finite order character of $\mathcal{O}_F^+(p^\infty)$. Then for all but finitely many Dirichlet characters $\chi$ of finite order and with $p$-power conductor we have:

$$L(\frac{\nu+1}{2}, \Pi \otimes \nu \chi) \neq 0.$$  

Proof. Use the twisted norm map $[x] \mapsto \nu(x)[N_{F/Q}x]$ to push forward $\mu_{\Pi}$ to a measure on $\mathcal{O}_F^+(p^\infty)$. Then proceed mutatis mutandis as in the proof of Theorem 4.9. \hfill $\Box$

This result is slightly stronger because the representation $\Pi \otimes \nu$, even though of cohomological type and admitting a Shalika model, is no longer $U_p$-ordinary, nor spherical.

The following corollary of Theorem 4.9 follows from the fact that we have non-vanishing for all but finitely many Dirichlet characters $\chi$ of finite order and with $p$-power conductor.

**Corollary 4.11** (Simultaneous non-vanishing). For $1 \leq k \leq r$ fix $n_k \in \mathbb{Z}_{>0}$ and let $\mu_k$ be a pure dominant integral weight for $GL_{2n_k}$ over $F$. Suppose that each $\mu_k$ satisfies the regularity condition in (79) and that its the purity weight $w_k$ is even. Let $\Pi_k$ be a cuspidal automorphic representation of $GL_{2n_k}(\mathbb{A}_F)$ of cohomological weight $\mu_k$ admitting a Shalika model. For a rational prime $p$, suppose that each $\Pi_k$ is unramified at $p$ and $U_p$-ordinary. Then, for all but finitely many Dirichlet characters $\chi$ of $p$-power conductor, we have:

$$L(\frac{\mu_k+1}{2}, \Pi_1 \otimes \chi) L(\frac{\mu_k+1}{2}, \Pi_2 \otimes \chi) \cdots L(\frac{\mu_k+1}{2}, \Pi_r \otimes \chi) \neq 0.$$  

Let’s note that this is a simultaneous non-vanishing result at the central point. We will leave it to the reader to formulate the stronger version of simultaneous non-vanishing combining Corollaries 4.10 and 4.11.

As an example illustrating an application of simultaneous non-vanishing to algebraicity results, let’s consider the unitary cuspidal automorphic representation $\pi(\Delta)$ of $GL_2(\mathbb{A})$ associated to the Ramanujan $\Delta$-function. A particular case of Corollary 4.11 gives infinitely many Dirichlet characters $\chi$ such that

$$L(17, \text{Sym}^3(\Delta) \otimes \chi) L(6, \Delta \otimes \chi) = L(\frac{1}{2}, \text{Sym}^3(\pi(\Delta)) \otimes \chi) L(\frac{1}{2}, \pi(\Delta) \otimes \chi) \neq 0.$$  

By [R] Cor.5.2, Prop.5.4 we get an algebraicity result for the central critical value

$$L(28, \text{Sym}^5(\Delta) \otimes \chi) = L(\frac{1}{2}, \text{Sym}^5(\pi(\Delta)) \otimes \chi)$$  

exactly analogous to [R] (5.6): we leave the whimsical details to the interested reader.

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