GIBBS MEASURE FOR THE FOCUSING FRACTIONAL NLS ON THE TORUS

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ABSTRACT. We study the construction of the Gibbs measures for the focusing mass-critical fractional nonlinear Schrödinger equation on the multi-dimensional torus. We identify the sharp mass threshold for normalizability and non-normalizability of the focusing Gibbs measures, which generalizes the influential works of Lebowitz-Rose-Speer (1988), Bourgain (1994), and Oh-Sosoe-Tolomeo (2021) on the one-dimensional nonlinear Schrödinger equations. To this purpose, we establish an almost sharp fractional Gagliardo-Nirenberg-Sobolev inequality on the torus, which is of independent interest.

1. Introduction

1.1. Focusing Gibbs measures. In this paper, we consider the focusing Gibbs measure $\rho_{s,p}$ on the $d$-dimensional torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$, formally given by

$$d\rho_{s,p}(u) = Z_{s,p}^{-1} \exp \left( \frac{1}{p} \int_{\mathbb{T}^d} |u|^p dx \right) d\mu_s(u)$$

for $p > 2$, where $Z_{s,p}$ is a normalization constant. Here, $\mu_s$ is the Gaussian probability measure with the density:

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}^d} |D_s u|^2 dx} du = Z_s^{-1} \prod_{n \neq 0} e^{-\frac{1}{2}(2\pi |n|)^2 \tilde{u}(n)^2} d\tilde{u}(n),$$

where $D = \sqrt{-\Delta}$ and $\tilde{u}(n)$ denotes the Fourier coefficient of $u$. When $s = 1$, the measure $\mu_s$ corresponds to the massless Gaussian free field on $\mathbb{T}^d$. A typical function $u$ in the support of $\mu_s$ is given by the random Fourier series:

$$u^\omega(x) = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \frac{g_n(\omega)}{(2\pi |n|)^s} e^{2\pi i n \cdot x},$$

where $\{g_n\}_{n \in \mathbb{Z}^d \setminus \{0\}}$ denotes a sequence of independent standard complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$. When $d = 1$, the expression (1.3) corresponds to the mean-zero Brownian loop for $s = 1$ (namely $u(0) = u(1)$) and to the mean-zero fractional Brownian loop for $\frac{1}{2} < s < \frac{3}{2}$. See [26, Section 5]. A standard computation shows that $u$ in (1.3) belongs to $\dot{W}^{s,p}(\mathbb{T}^d) \setminus \dot{W}^{s-d/2,p}(\mathbb{T}^d)$ for any $\sigma < s - \frac{d}{2}$ and $1 \leq p \leq \infty$ almost surely, where $\dot{W}^{s,p}(\mathbb{T}^d)$ denotes the homogeneous Sobolev space (= the Riesz potential space) defined by the norm:

$$\|u\|_{\dot{W}^{s,p}(\mathbb{T}^d)} = \|D^\sigma u\|_{L^p(\mathbb{T}^d)} = \|F^{-1}((2\pi |n|)^{2\sigma} \tilde{u}(n))\|_{L^p(\mathbb{T}^d)}.$$ 

In particular, when $s > \frac{d}{2}$, $u$ in the support of $\mu_s$ is almost surely a function, while it is merely a distribution when $s \leq \frac{d}{2}$. In the latter case, the potential energy $\frac{1}{p} \int_{\mathbb{T}^d} |u|^p dx$ in (1.1) does not make sense as it is and thus one needs to introduce renormalization. See Remark 1.5 below.

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In this paper, we focus on the case $s > \frac{d}{2}$ such that a typical element in the support of $\mu_s$ is a function.

The main difficulty in the construction of the focusing Gibbs measure $\mu_{s,p}$ in (1.1) comes from the unboundedness of the potential energy. In fact, with (1.3) and $L^2(\mathbb{Z}^d) \subset L^p(\mathbb{Z}^d)$, we immediately see that
\[
\mathbb{E}_{\mu_s} \left[ e^\frac{1}{p} \left( \int_{\mathbb{T}^d} |u|^2 \, dx \right)^\frac{p}{2} \right] \geq \mathbb{E}_{\mu_s} \left[ e^\frac{1}{p} \|u\|_{FLP(\mathbb{T}^d)}^p \right] \geq \prod_{n \in \mathbb{Z}^d} \mathbb{E} \left[ e^\frac{1}{p} \|\hat{u}(n)|^p \right] = \infty
\]
for $p > 2$, where $FLP(\mathbb{T}^d)$ denotes Fourier Lebesgue space defined by the norm
\[
\|u\|_{FLP(\mathbb{T}^d)} = \left( \sum_{n \in \mathbb{Z}^d} |\hat{u}(n)|^p \right)^\frac{1}{p}.
\]

In a seminal work [18], Lebowitz-Rose-Speer proposed to consider the focusing Gibbs measure of the following form (when $d = s = 1$):
\[
d\mu_{s,p}(u) = Z_{1,p}^{-1} \mathbb{1}([\|u\|_{L^2(\mathbb{T})} \leq K] \exp \left( \frac{1}{p} \int_{\mathbb{T}} |u|^p \, dx \right) d\mu_s(u). \tag{1.4}
\]

In [18, 6, 25], Lebowitz-Rose-Speer, Bourgain, and Oh-Sosoe-Tolomeo showed that the Gibbs measure $\mu_{1,p}$ on the one-dimensional torus $\mathbb{T}$ is indeed normalizable for (i) $2 < p < 6$ and any finite $K > 0$ and (ii) $p = 6$ and any $0 < K \leq \|Q\|_{L^2(\mathbb{R})}$, where $Q$ is the (unique) minimizer of the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}$ with $\|Q\|_{L^6(\mathbb{R})}^6 = 3 \|DQ\|_{L^2(\mathbb{R})}$, while it is not normalizable for (iii) $p > 6$ and (iv) $p = 6$ and $K > \|Q\|_{L^2(\mathbb{R})}$.

The main purpose of this paper is to study the focusing Gibbs measure $\mu_{s,p}$ in (1.4) with a mass cutoff for $s > \frac{d}{2}$ and to identify sharp conditions for its normalizability. More precisely, in the subcritical case (i.e. $2 < p < \frac{4s}{d} + 2$), we prove that the focusing Gibbs measure $\mu_{s,p}$ in (1.4) is normalizable for any $K > 0$, while we prove its non-normalizability for any $K > 0$ in the supercritical case (i.e. $p > \frac{4s}{d} + 2$). In the critical case (i.e. $p = \frac{4s}{d} + 2$), we then show that there is a critical mass threshold, characterized by an optimizer for the Gagliardo-Nirenberg-Sobolev inequality on $\mathbb{R}^d$ (see Section 2) such that the focusing Gibbs measure $\mu_{s,p}$ is normalizable below this critical mass threshold, while it is not above this threshold. See Theorem 1.1. Previously, these results were known only for $d = s = 1$ ([18, 6, 25]) and our aim is to extend these results for any $d \geq 1$, $s > 0$, and $p > 2$, provided that $s > \frac{d}{2}$ (such that the random Fourier series $u$ in (1.4) defines a function). See [24] for the case $s = \frac{d}{2}$. We refer the reader to [25] and the references therein for further background.

The energy functional associated with the Gibbs measure $\mu_{s,p}$ in (1.4) is given by
\[
H_{\mathbb{T}^d}(u) = \frac{1}{2} \int_{\mathbb{T}^d} |D^s u|^2 \, dx - \frac{1}{p} \int_{\mathbb{T}^d} |u|^p \, dx. \tag{1.5}
\]

We point out that the construction of the Gibbs measure is not only of interest in the area of mathematical physics such as constructive Euclidean quantum field theory, but is also crucial in the study of Hamiltonian PDEs [18, 6, 7, 22, 23, 20, 24, 25]. An important example of Hamiltonian PDEs corresponding to the energy functional (1.5) is the following fractional nonlinear Schrödinger equation:
\[
i \partial_t u + D^{2s} u = |u|^{p-2} u. \tag{1.6}
\]

The equation (1.6) corresponds to the nonlinear Schrödinger equation (NLS) when $s = 1$ ([6, 7]), to the biharmonic NLS when $s = 2$ ([27, 26, 28]), and to the nonlinear half-wave equation.
when \( s = \frac{1}{2} \) \((20)\). In the seminal work \([6,7]\), Bourgain showed that we can extend the local-in-time dynamics of \( (1.6) \) globally in time by using the Gibbs measure\(^1\) as a replacement of a conservation law. Over the last decade, we have seen a tremendous progress in the study of this subject. See \([14]\) for a survey on the subject and for the references therein.

We now state our main result.

**Theorem 1.1.** Let \( d \geq 1, s > \frac{d}{2}, \) and \( p > 2 \). Given \( K > 0 \), define the partition function \( Z_{s,p,K} \) by

\[
Z_{s,p,K} = E_{\mu_s} \left[ e^{\frac{1}{p} \int_{\mathbb{R}^d} |u|^p \, dx} 1\{\|u\|_{L^2(\mathbb{R}^d)} \leq K\} \right],
\]

where \( E_{\mu_s} \) denotes an expectation with respect to the law \( \mu_s \) of the random Fourier series in \( (1.3) \). Then, the following statements hold:

(i) (subcritical case) If \( 2 < p < \frac{4s}{d} + 2 \), then \( Z_{s,p,K} < \infty \) for any \( K > 0 \).

(ii) (critical case) Let \( p = \frac{4s}{d} + 2 \). Then, \( Z_{s,p,K} < \infty \) if \( K < \|Q\|_{L^2(\mathbb{R}^d)} \), and \( Z_{s,p,K} = \infty \) if \( K > \|Q\|_{L^2(\mathbb{R}^d)} \). Here, \( Q \) is the optimizer for the Gagliardo-Nirenberg-Sobolev inequality on \( \mathbb{R}^d \) such that \( \|Q\|_{L^p(\mathbb{R}^d)} = \frac{p}{2} \|D^s Q\|_{L^2(\mathbb{R}^d)}^2 \).

(iii) (supercritical case) If \( p > \frac{4s}{d} + 2 \), then \( Z_{s,p,K} = \infty \) for any \( K > 0 \).

As mentioned above, Theorem 1.1 extends the results in \([18,6,25]\) to \( d \geq 1 \) and \( s > \frac{d}{2} \). When \( d = s = 1 \), Bourgain \([6]\) proved Theorem 1.1(i) and also (ii) (but with sufficiently small \( K \ll 1 \)), using the dyadic pigeon hole principle and the Sobolev embedding theorem. In \([18]\), Lebowitz-Rose-Speer proved the non-normalizability in Theorem 1.1 (i.e. for \( K \gg \|Q\|_{L^2(\mathbb{R})} \) when \( p = 6 \) and for any \( K > 0 \) when \( p > 6 \)). Their argument was based on a Cameron-Martin type argument and the sharp Gagliardo-Nirenberg-Sobolev (GNS) inequality:

\[
\|u\|_{L^p(\mathbb{R})}^p \leq C_{\text{GNS}} \|u\|_{H^s(\mathbb{R})}^{\frac{p}{2} - 1} \|u\|_{L^2(\mathbb{R})}^{\frac{p}{2} + 1},
\]

where \( C_{\text{GNS}} \) is the optimal constant. In \([25]\), Oh-Sosoe-Tolomeo refined Bourgain’s argument and used the sharp GNS inequality to prove the normalizability part in Theorem 1.1(ii), thus identifying the optimal mass threshold at the critical mass nonlinearity. In the same paper, Oh-Sosoe-Tolomeo also proved the normalizability of the focusing Gibbs measure at the critical mass threshold \( K = \|Q\|_{L^2(\mathbb{R})} \) when \( d = s = 1 \). See Remark 1.4.

The main difficulties in proving Theorem 1.1 come from the non-local nature of the fractional derivatives \( D^s = (-\Delta)^{\frac{s}{2}} \) and the non-integer critical exponents \( p = \frac{4s}{d} + 2 \). In particular, the non-local derivative poses extra difficulty in localizing the GNS inequality, initially on \( \mathbb{R}^d \), to the torus \( \mathbb{T}^d \). Inspired by \([3]\), we exploit a characterization of the \( \dot{H}^s(\mathbb{R}^d) \)-norm in terms of high order difference operators \((2.14)\). We then establish an almost sharp GNS inequality on \( \mathbb{T}^d \) (with the sharp constant \( C_{\text{GNS}} \) in \((1.8)\)) by using this new characterization. See Proposition 2.3.

With the sharp GNS inequality on \( \mathbb{T}^d \), our proof of Theorem 1.1 is based on the variational approach due to Barashkov-Gubinelli \([1]\) and is quite different from those in \([18,6,25]\), thus providing an alternative proof of the results when \( d = s = 1 \). We first express the partition function \( Z_{s,p,K} \) in \((1.7)\) in a stochastic optimization problem, using the Boué-Dupuis variational formula (Lemma 3.1). We then prove the normalizability part of Theorem 1.1 by using the almost sharp GNS inequality on \( \mathbb{T}^d \). As for the non-normalizability part, the main task is to

\(^1\)Strictly speaking, we need to modify the massless fractional Gaussian free field \( \mu_s \) in \((1.2)\) by the massive one to avoid an issue at the zeroth frequency. See Remark 1.3.
construct a sequence of drift terms which achieves the divergence of the partition function. Our construction of such drift terms is based on a scaling argument, analogous to that in [18, 25]. We point out that our proof of Theorem 1.1, based on the variational approach, is essentially a physical space approach, instead of the Fourier side approach in [6, 25]. It is thus expected that our approach is more flexible in geometric settings. See also Appendix B in [24], where the variational approach was used to prove Theorem 1.1 (i) with $s > \frac{d}{2}$ and $p = 4$.

Remark 1.2. The key idea in proving Theorem 1.1 lies in controlling the potential energy $\frac{1}{p} \|u\|_{L^p(T^d)}^p$ by the kinetic energy $\frac{1}{2} \|u\|_{H^s(T^d)}^2$ under the constraint $\|u\|_{L^2(T^d)} \leq K$. From Gagliardo-Nirenberg-Sobolev inequality Proposition 2.3, we see that the subcritical case $2 < p < \frac{d}{2} + 2$ corresponds to weaker potential energy. The critical exponent $p = \frac{4s}{d} + 2$ leads to the equivalence of potential and kinetic energy, where a restriction on the size $K$ is needed to guarantee the normalizability. For the supercritical case $p > \frac{4s}{d} + 2$, however, the kinetic energy losses control of the potential energy no matter how small the mass is.

Remark 1.3. As in [25], Theorem 1.1 also applies when we replace the mean-zero fractional Brownian loop in (1.3) by the fractional Ornstein-Uhlenbeck loop:

$$u(x) = \sum_{n \in \mathbb{Z}^d} g_n(\omega) \langle n \rangle^s e^{2\pi i n x}, \quad (1.9)$$

where $\langle n \rangle = (1 + 4\pi^2 |n|^2)^{\frac{1}{2}}$ and $\{g_n\}_{n \in \mathbb{Z}^d}$ is a sequence of independent standard complex-valued Gaussian random variables. See Remark 4.1 in [25]. The law $\tilde{\mu}_s$ of the fractional Ornstein-Uhlenbeck loop in (1.9) has the formal density

$$d\tilde{\mu}_s = \tilde{Z}_s^{-1} e^{-\frac{1}{2} \|u\|_{H^s(T^d)}^2} du.$$

As seen in [4], the measure $\tilde{\mu}_s$ is a more natural base Gaussian measure to consider for the (fractional) nonlinear Schrödinger equation (1.6) due to the lack of the conservation of the spatial mean under the dynamics.

Note that Theorem 1.1 also holds in the real-valued setting (i.e. with an extra assumption that $g_{-n} = \overline{g}_n$ in (1.3)). For example, this is relevant to the study of the dispersion generalized KdV equation on $\mathbb{T}$:

$$\partial_t u + D^{2s} \partial_x u = \partial_x (u^{p-1}).$$

Remark 1.4. We point out that Oh-Sosoe-Tolomeo [25] also showed the normalizability of the Gibbs measure (1.11) at the critical mass threshold when $s = d = 1$ and $p = 6$. This result is quite striking in view of the presence of the minimal mass blowup solution (at this critical mass) for the focusing quintic NLS on $\mathbb{T}$. We will not pursue this question for the fractional focusing Gibbs measure (1.4), as their argument is beyond the scope of the framework developed in this paper.

Remark 1.5. Since $s > \frac{d}{2}$, Theorem 1.1 only considers the non-singular case, namely, the measures $\mu_s$ and $\rho_{s,p}$ are supported on functions. One of the reasons for only considering the non-singular case is that the bifurcation phenomena at the critical mass (Theorem 1.1 (ii)) are only possible when $s > \frac{d}{2}$. As soon as $s \leq \frac{d}{2}$, we need to introduce a proper renormalization to define the potential energy $\frac{1}{p} \int_{T^d} |u|^p dx$, which necessitates $p$ to be an integer. When $s = \frac{d}{2}$, it was shown in [10, 24] that the renormalized focusing Gibbs measure $\rho_{\frac{d}{2},4}$ (with $p = 4$, critical), endowed with a (renormalized) mass cutoff, is not normalizable. It was also shown
in [24] that with the cubic interaction \((p = 3, \text{ subcritical})\), the renormalized focusing Gibbs measure \(\rho_{d,3}\) endowed with a renormalized mass cutoff is indeed normalizable. When \(d = 2\), this normalizability in the case of the cubic interaction was first observed by Bourgain [8]. When \(d = 3\), it has recently been shown that the cubic interaction \((p = 3)\) exhibits phase transition between weakly and strongly nonlinear regimes. See [21] for more details.

**Remark 1.6.** While the construction of the defocusing Gibbs measures has been extensively studied and well understood due to the strong interest in constructive Euclidean quantum field theory (see [31] [16] [32]), the (non-)normalizability issue of the focusing Gibbs measures, going back to the work of Lebowitz-Rose-Speer [18] and Brydges-Slade [10], is not fully explored. See related works [30] [9] [12] [20] [25] [24] [21] [33] on the non-normalizability (and other issues) for focusing Gibbs measures. In particular, recent works such as [20] [21] employ the variational approach developed in [1] and establish certain phase transition phenomena in the singular setting.

### 2. Sharp Gagliardo-Nirenberg-Sobolev inequality

In order to prove Theorem 1.1, we need the sharp Gagliardo-Nirenberg-Sobolev inequality on \(\mathbb{T}^d\). We first recall the definition of the homogeneous Sobolev space \(\dot{H}^s(\mathbb{R}^d)\) defined by the norm:

\[
\|u\|_{\dot{H}^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} (2\pi|\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi. \tag{2.1}
\]

As mentioned in Section 1, the optimizer for the Gagliardo-Nirenberg-Sobolev inequality with the optimal constant:

\[
\|u\|_{L^p(\mathbb{R}^d)}^p \leq C_{\text{GNS}}(d, p, s) \|u\|_{\dot{H}^s(\mathbb{R}^d)}^{(p-2d)/(2s)} \|u\|_{L^2(\mathbb{R}^d)}^{2 + \frac{p-2}{2s}(2s-d)} \tag{2.2}
\]

plays an important role in the study of the focusing Gibbs measures. We recall the following result,

**Theorem 2.1** (Theorem 2.1, [2]). Let \(d \geq 1\) and let (i) \(p > 2\) if \(d < 2s\), and (ii) \(2 < p \leq \frac{2d}{d-2s}\) if \(d \geq 2s\). Consider the functional

\[
J^{d,p,s}(u) = \frac{\|u\|_{\dot{H}^s(\mathbb{R}^d)}^{(p-2d)/(2s)} \|u\|_{L^2(\mathbb{R}^d)}^{2 + \frac{p-2}{2s}(2s-d)}}{\|u\|_{L^p(\mathbb{R}^d)}} \tag{2.3}
\]

on \(H^s(\mathbb{R}^d)\). Then, the minimum

\[
C_{\text{GNS}}^{-1} = C_{\text{GNS}}(d, p, s)^{-1} := \inf_{u \in H^s(\mathbb{R}^d), u \neq 0} J^{d,p,s}(u) \tag{2.4}
\]

is attained at a function \(Q \in H^s(\mathbb{R}^d)\).

**Remark 2.2.** It is easy to see that functions \(u(x) := cQ(b(x - a))\) for all \(c \in \mathbb{R}\setminus\{0\}\), \(b > 0\), and \(a \in \mathbb{R}^d\), are minimizers of the functional (2.3). Therefore, we may assume that

\[
\|Q\|_{L^2(\mathbb{R}^d)} = \|Q\|_{H^s(\mathbb{R}^d)}, \quad \|Q\|_{H^s(\mathbb{R}^d)}^2 = \frac{2}{p} \|Q\|_{L^p(\mathbb{R}^d)}. \tag{2.5}
\]
Under this specified scaling, we have $H_{\mathbb{R}^d}(Q) = 0$, where $H_{\mathbb{R}^d}$ is the Hamiltonian functional given in (1.5) with $\mathbb{T}^d$ being replaced by $\mathbb{R}^d$. Furthermore, this $Q$ solves the following semilinear elliptic equation on $\mathbb{R}^d$:

$$(p - 2)dD^2Q + (4s + (p - 2)(2s - d))Q - 4sQ^{p - 1} = 0.$$  

(2.6)

In the following, we restrict ourselves to (2.5) unless specified otherwise. In particular, we have

$$C_{\text{GNS}} = \frac{p}{2}||Q||_{L^2(\mathbb{R}^d)}^{2-p}.$$  

(2.7)

The uniqueness (in some sense) of this $Q$ for fractional value $s$ is a very challenging problem, which is only proved for some special cases, for instance when $d = 1$ and $s \in (0, 1)$. See [14].

For a function $u$ defined on $\mathbb{T}^d$, we define the $\dot{H}^s(\mathbb{T}^d)$ norm via

$$||u||^2_{\dot{H}^s(\mathbb{T}^d)} = \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |n|^{2s} |\hat{u}(n)|^2.$$  

(2.8)

Due to the scaling invariance of the minimization problem (2.3), it is expected that the GNS inequality (2.2) also holds on the finite domains $\mathbb{T}^d$ with the same optimal constants.

**Proposition 2.3.** Let $d \geq 1$ and let (i) $p > 2$ if $d < 2s$, and (ii) $2 < p \leq \frac{2d}{d - 2s}$ if $d \geq 2s$. Then, given small $\delta > 0$, there is a constant $C = C(\delta) > 0$ such that

$$\|u\|_{L^p(\mathbb{T}^d)}^p \leq (C_{\text{GNS}}(d, p, s) + \delta)\|u\|^{(p - 2)d}_{\dot{H}^s(\mathbb{T}^d)}\|u\|_{L^2(\mathbb{T}^d)}^{2s} + C(\delta)\|u\|_{L^2(\mathbb{T}^d)}^p$$  

(2.9)

for $u \in H^s(\mathbb{T}^d)$, where $C_{\text{GNS}}$ is the constant defined in (2.4) and (2.7).

The main difficulty in showing Proposition 2.3 is due to the non-local nature of the fractional derivatives. To circumvent this difficulty, we recall the characterization of the $\dot{H}^s(\mathbb{R}^d)$ norm (2.1) based on the $L^2$-modulus of continuity. When $0 < s < 1$, one has

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d + 2s}} dxdy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x + y) - u(x)|^2}{|y|^{d + 2s}} dxdy$$

$$= \int_{\mathbb{R}^d} \left| \xi \right|^{-2s} \int_{\mathbb{R}^d} \frac{\left| e^{2\pi i y \xi} - 1 \right|^2}{\left| y \right|^{d + 2s}} dy \left| \xi \right|^{2s} |\hat{u}(\xi)|^2 d\xi.$$  

(2.10)

Denote the inner integral, a convergent improper integral for $0 < s < 1$, by

$$c_1(d, s) = \left| \xi \right|^{-2s} \int_{\mathbb{R}^d} \frac{\left| e^{2\pi i y \xi} - 1 \right|^2}{\left| y \right|^{d + 2s}} dy = \int_{\mathbb{R}^d} \frac{\left| e^{2\pi i x_1} - 1 \right|^2}{\left| x \right|^{d + 2s}} dx.$$  

(2.11)

which is a constant, i.e. independent of $\xi$. From (2.10) and (2.11), we have the following characterization of the $\dot{H}^s(\mathbb{R}^d)$ norm in (2.1) (see for example [3]),

$$\|u\|_{\dot{H}^s(\mathbb{R}^d)}^2 = c_1(d, s)^{-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d + 2s}} dxdy.$$  

(2.12)

We remark that on the torus $\mathbb{T}^d$ the $H^s(\mathbb{T}^d)$ norm defined in (2.8) has a similar equivalent characterization. See [3] Proposition 1.3. However, the identity as (2.12) fails for the torus case due to the lack of rotational invariance.

By using high order difference operators, we may generalize (2.12) to the cases $s \geq 1$. In particular, we have

$$\|u\|_{\dot{H}^s(\mathbb{R}^d)}^2 = c_k(d, s)^{-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\left| \Delta^k_y u(x) \right|^2}{|y|^{d + 2s}} dxdy.$$  

(2.13)
where $\Delta_y^k$ is the $k$-th forward difference operator with spacing $y$ defined by

$$
\Delta_y^k u(x) = \sum_{j=0}^{k} (-1)^{k-j} C_k^j u(x + jy),
$$

(2.14)

where $C_k^j$ are binomial coefficients, and

$$
c_k(d, s) = \int_{\mathbb{R}^d} \frac{|e^{2\pi i x \cdot y} - 1|^{2k}}{|x|^{d+2s}} \, dx.
$$

(2.15)

The proof of (2.13) is similar to that of (2.12). We thus omit the details.

Now we are ready to prove Proposition 2.3.

**Proof of Proposition 2.3.** For the pedagogical purpose, we present the proof for the case $0 < s < 1$ before demonstrating the general case $s > 0$, as the former is less complex in terms of notation.

We first consider $0 < s < 1$. Let $\psi \in C_0^\infty(B(0, \frac{1}{2}))$ be a bump function with $\|\psi\|_{L^1} = 1$ and $\psi_\delta(x) = \delta^{-d} \psi(\frac{x}{\delta})$. Define $\phi_\delta(x) = 1_{[-\frac{1}{2} + 3\delta, \frac{1}{2} - 3\delta]^d} * \psi_\delta(x)$. Then the following properties hold

(i) $\phi_\delta \in C_0^\infty(\mathbb{T}^d)$,
(ii) $\phi_\delta(x) = 1$ for $x \in [-\frac{1}{2} + 3\delta, \frac{1}{2} - 3\delta]^d$,
(iii) $\phi_\delta(x) = 0$ for $x \in ([-\frac{1}{2} + \delta, \frac{1}{2} - \delta]^d)^c$,
(iv) $|D^s \phi_\delta(x)| \lesssim \delta^{-s}$ for all $x \in \mathbb{R}^d$.

Let

$$
u_\delta(x) = \begin{cases} 
\phi_\delta(x) u(x), & x \in [-\frac{1}{2}, \frac{1}{2}]^d; \\
0, & \text{otherwise}. 
\end{cases}
$$

(2.16)

First we claim there exists $C(d) > 0$ such that for any $u \in L^p(\mathbb{T}^d)$ there exists $x_0 \in \mathbb{T}^d$ satisfying the following

$$
\|u\|_{L^p(\mathbb{T}^d)}^p \leq (1 + C(d)\delta) \|\phi_\delta(\cdot) u(\cdot + x_0)\|_{L^p(\mathbb{R}^d)}^p.
$$

(2.17)

From the definition of $\phi_\delta$, it suffices to show

$$
\|u\|_{L^p(\mathbb{T}^d)}^p \leq (1 + C(d)\delta) \|u(\cdot + x_0)\|_{L^p([-\frac{1}{2} + 3\delta, \frac{1}{2} - 3\delta]^d)}^p.
$$

(2.18)

We show (2.18) inductively. Recall that $\delta \ll 1$. When $d = 1$, we may split the interval $[-\frac{1}{2}, \frac{1}{2}]$ into $k = \lfloor \frac{1}{3\delta} \rfloor$ many equal subintervals. Then, from the pigeonhole principle, there must be a subinterval, say the $j$-th subinterval $[-\frac{1}{2} + \frac{j-1}{k}, -\frac{1}{2} + \frac{j}{k}]$, such that

$$
\int_{[-\frac{1}{2} + \frac{j-1}{k}, -\frac{1}{2} + \frac{j}{k}]} |u(x)|^p \, dx \leq \frac{1}{k} \int_{\mathbb{T}} |u(x)|^p \, dx,
$$

which implies

$$
\int_{\mathbb{T}} |u(x)|^p \, dx \leq (1 + \frac{1}{k}) \int_{[-\frac{1}{2} + \frac{j-1}{k}, -\frac{1}{2} + \frac{j}{k}]} |u(x + \frac{2j-1}{2k})|^p \, dx
\leq (1 + 12\delta) \int_{[-\frac{1}{2} + 3\delta]} |u(x + \frac{2j-1}{2k})|^p \, dx,
$$

where $\Delta_y^k$ is the $k$-th forward difference operator with spacing $y$ defined by

$$
\Delta_y^k u(x) = \sum_{j=0}^{k} (-1)^{k-j} C_k^j u(x + jy),
$$

(2.14)

where $C_k^j$ are binomial coefficients, and

$$
c_k(d, s) = \int_{\mathbb{R}^d} \frac{|e^{2\pi i x \cdot y} - 1|^{2k}}{|x|^{d+2s}} \, dx.
$$

(2.15)
provided \(\delta\) is sufficiently small. Thus we conclude (2.18) for \(d = 1\). Let us assume (2.18) holds for all \(1, 2, \ldots, d - 1\) dimensions. Then for \(x \in \mathbb{T}^d\), we may write \(x = (x', x_d)\) such that \(x' \in \mathbb{T}^{d-1}\) and \(x_d \in \mathbb{T}\). Then, from our assumption, there exist \(x_0^d \in \mathbb{T}\) and \(x_0' \in \mathbb{T}^{d-1}\) such that

\[
\int_{\mathbb{T}^d} |u(x)|^p dx = \int_{\mathbb{T}} \left( \int_{\mathbb{T}^{d-1}} |u(x', x_d)|^p dx' \right) dx_d \\
\leq (1 + C\delta) \int_{\mathbb{T}^{d-1}} \left( \int_{\mathbb{T}} |u(x', x_d + x_0^d)|^p dx' \right) dx_d \\
\leq (1 + C\delta) \int_{\mathbb{T}^{d-1}} \left( \int_{\mathbb{T}} |u(x', x_d + x_0^d)|^p dx' \right) dx' \\
\leq (1 + C(1)\delta) (1 + C(d - 1)\delta) \int_{[-\frac{1}{2} + 3\delta, \frac{1}{2} - 3\delta]^{d-1}} \left( \int_{[-\frac{1}{2} + 3\delta, \frac{1}{2} - 3\delta]} |u(x' + x_0', x_d + x_0^d)|^p dx_d dx' \right),
\]

where we used the assumption in the second and fourth steps. Thus we finish the proof of (2.18) for \(d\) dimension by taking \(x_0 = (x_0', x_0^d)\) and \(C(d) = 1 + (C(1) + 2C(d - 1))\) provided \(C(1)\delta < 1\). From (2.17), for the translated \(u(\cdot + x_0)\), still denoting by \(\mathcal{G}\), we have

\[
\|u\|_{L^p(\mathbb{T}^d)}^p \leq (1 + C\delta) \|u_\delta\|_{L^p(\mathbb{R}^d)}^p \\
\leq (1 + C\delta) C_{\text{GNS}}(d, p, s) \|u_\delta\|_{H^s(\mathbb{R}^d)} \|u_\delta\|_{L^2(\mathbb{R}^d)}^{2 + \frac{p - 2}{2d} (2s - d)} \\
\leq (1 + C\delta) C_{\text{GNS}}(d, p, s) \left( c_1(d, s) \right)^{-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_\delta(x) - u_\delta(y)|^2}{|x - y|^{d + 2s}} dx dy \|u\|_{L^2(\mathbb{T}^d)}^{2 + \frac{p - 2}{2d} (2s - d)}.
\]

To prove (2.9), it only needs to show

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|u_\delta(x) - u_\delta(y)|^2}{|x - y|^{d + 2s}} dx dy \leq (1 + C\delta) c_1(d, s) \|u\|_{H^s(\mathbb{T}^d)}^2 + C(\delta) \|u\|_{L^2(\mathbb{T}^d)}^2.
\]

Since the integrand \(\frac{|u_\delta(x) - u_\delta(y)|^2}{|x - y|^{d + 2s}}\) in (2.19) is supported on \((x, y) \in (\mathbb{T}^d \times \mathbb{R}^d) \cup (\mathbb{R}^d \times \mathbb{T}^d)\), we have

LHS of (2.19) \(\leq \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{|u_\delta(x) - u_\delta(y)|^2}{|x - y|^{d + 2s}} dx dy + C \int_{(\mathbb{T}^d)^c} \int_{(\mathbb{T}^d)^c} \frac{|u_\delta(x)|^2}{|x - y|^{d + 2s}} dx dy. \)

For the second term in (2.20), since \(|x - y| > \delta\) in the integrand, we have

\[
\int_{(\mathbb{T}^d)^c} \int_{(\mathbb{T}^d)^c} \frac{|u_\delta(x)|^2}{|x - y|^{d + 2s}} dx dy \lesssim \left( \int_{|y| > \delta} \frac{1}{|y|^{d + 2s}} dy \right) \|u_\delta\|_{L^2(\mathbb{T}^d)}^2 \lesssim \delta^{-2s} \|u\|_{L^2(\mathbb{T}^d)}^2,
\]

which is sufficient for (2.19). Now we turn to the first term in (2.20). We note

\[
|u_\delta(x) - u_\delta(y)|^2 = |\phi_\delta(x)(u(x) - u(y)) + (\phi_\delta(x) - \phi_\delta(y))u(y)|^2 \\
= |\phi_\delta(x)(u(x) - u(y))|^2 + |(\phi_\delta(x) - \phi_\delta(y))u(y)|^2 \\
+ 2\phi_\delta(x)(\phi_\delta(x) - \phi_\delta(y))(u(x) - u(y))u(y).
\]

\(^2\text{We note that (2.9) is invariant under translation.}\)
Thus we have
\[
\begin{align*}
\int_{T^d} \int_{T^d} \frac{|u_\delta(x) - u_\delta(y)|^2}{|x - y|^{d+2s}} dxdy &\leq \int_{T^d} \int_{T^d} \frac{|u(x) - u(y)|^2}{|x - y|^{d+2s}} dxdy, \\
&+ \int_{T^d} \int_{T^d} \frac{|\phi_\delta(x)(\phi_\delta(x) - \phi_\delta(y))(u(x) - u(y))u(y)|}{|x - y|^{d+2s}} dxdy, \\
&+ \int_{T^d} \int_{T^d} \frac{|(\phi_\delta(x) - \phi_\delta(y))u(y)|^2}{|x - y|^{d+2s}} dxdy, \tag{2.22}
\end{align*}
\]
\[= A_1 + A_2 + A_3.\]

For the term $A_1$, we have
\[
A_1 \leq \int_{T^d} \int_{B(0,2)} \frac{|u(x) - u(x + z)|^2}{|z|^{d+2s}} dzdx
= \sum_{n \in \mathbb{Z}^d \setminus \{0\}} |n|^{-2s} \int_{B(0,2)} \frac{|e^{2\pi i x \cdot n} - 1|^2}{|x|^{d+2s}} dx \int_{R^d} |n|^{-2s} |\hat{u}(n)|^2,
\tag{2.23}
\]
where $B(0, 2) \subset \mathbb{R}^d$ is the ball centered at 0 with radius 2. It is easy to see that
\[
|n|^{-2s} \int_{B(0,2)} \frac{|e^{2\pi i x \cdot n} - 1|^2}{|x|^{d+2s}} dx \leq |n|^{-2s} \int_{R^d} \frac{|e^{2\pi i x \cdot n} - 1|^2}{|x|^{d+2s}} dx = c_1(d, s),
\]
which together with (2.23) shows the contribution from $A_1$ is bounded by the right hand side of (2.19). For the term $A_3$, we have
\[
A_3 \lesssim \delta^{-2} \int_{T^d} \int_{T^d} \frac{|u(y)|^2}{|x - y|^{d+2s-2}} dxdy \lesssim \delta^{-2} \|u\|_{L^2(T^d)}^2,
\tag{2.24}
\]
which is sufficient for our purpose. For $A_2$, by Young’s inequality we have
\[
A_2 \leq \delta A_1 + \frac{1}{\delta} A_3,
\tag{2.25}
\]
which is again acceptable. By collecting (2.22), (2.23), (2.25), and (2.24), we finish the proof of (2.19) and thus (2.9) when $0 < s < 1$.

In the following we consider the case $s \geq 1$. Assume $s \in [k - 1, k)$ for some $k \in \mathbb{Z}_+$. Similarly to (2.19) in the case $0 < s < 1$, it only needs to show
\[
\int_{R^d} \int_{R^d} \frac{|\Delta_y^k u_\delta(x)|^2}{|y|^{d+2s}} dydx \leq (1 + C\delta) c_k(d, s) \|u\|_{H^s(T^d)}^2 + C(\delta) \|u\|_{L^2(T^d)}^2.
\tag{2.26}
\]
Similarly to (2.21) and (2.23), we may reduce (2.26) to
\[
\int_{T^d} \int_{B(0,k)} \frac{|\Delta_y^k u_\delta(x)|^2}{|y|^{d+2s}} dydx \leq (1 + C\delta) c_k(d, s) \|u\|_{H^s(T^d)}^2 + C(\delta) \|u\|_{L^2(T^d)}^2.
\tag{2.27}
\]
In the following, we prove (2.27). First note that
\[
\Delta_y^k u_\delta(x) = \Delta_y^k (\psi_\delta(x)u(x)) = \sum_{j=0}^k C_j^k \Delta_y^{k-j} \psi_\delta(x) \Delta_y^j u(x + (k - j)y).
\]
Therefore, we have
\[
\int_{\mathbb{T}^d} \int_{B(0,k)} \frac{|\Delta_y^k u_\delta(x)|^2}{|y|^{d+2s}} dydx
= \int_{\mathbb{T}^d} \int_{B(0,k)} \frac{|\sum_{j=0}^{k-1} C^j_k \Delta_y^{k-j} \psi_\delta(x) \Delta_y^j u(x + (k - j)y)|^2}{|y|^{d+2s}} dydx
= \int_{\mathbb{T}^d} \int_{B(0,k)} \frac{|\psi_\delta(x) \Delta_y^k u(x)|^2}{|y|^{d+2s}} dydx
+ \sum_{j=0}^{k-1} \int_{\mathbb{T}^d} \int_{B(0,k)} \frac{|C^j_k \Delta_y^{k-j} \psi_\delta(x) \Delta_y^j u(x + (k - j)y)|^2}{|y|^{d+2s}} dydx
\]
\[
+ \sum_{\ell \neq k} \int_{\mathbb{T}^d} \int_{B(0,k)} \frac{C^j_k \Delta_y^{k-j} \psi_\delta(x) \Delta_y^j u(x + (k - j)y) C^\ell_k \Delta_y^{k-\ell} \psi_\delta(x) \Delta_y^\ell u(x + (k - \ell)y)}{|y|^{d+2s}} dydx
= B_1 + B_2 + B_3.
\]

For the term $B_1$ in (2.28), we have
\[
B_1 \leq \int_{\mathbb{T}^d} \int_{B(0,k)} \frac{|\Delta_y^k u(x)|^2}{|y|^{d+2s}} dydx
= \sum_{n \in \mathbb{Z}^d} \int_{B(0,k)} \frac{|e^{2\pi i y \cdot n} - 1|^{2k}}{|y|^{d+2s}} dy |\hat{u}(n)|^2
\leq c_k(d,s) \sum_{n \in \mathbb{Z}^d} |n|^{2s} |\hat{u}(n)|^2,
\]
where $c_k(d,s)$ is defined in (2.15). Thus the contribution of $B_1$ is bounded by the right hand side of (2.27). Similarly, we can control $B_2$ in (2.28) as
\[
B_2 \leq \sum_{j=0}^{k-1} \int_{\mathbb{T}^d} \int_{B(0,k)} \frac{|\Delta_y^{k-j} \psi_\delta(x) \Delta_y^j u(x + (k - j)y)|^2}{|y|^{d+2s}} dydx
\leq \sum_{j=0}^{k-1} \delta^{-2(k-j)} \int_{B(0,k)} \int_{\mathbb{T}^d} \frac{|\Delta_y^j u(x + (k - j)y)|^2}{|y|^{d+2s-2(k-j)}} dx dy
\leq \sum_{j=0}^{k-1} \delta^{-2(k-j)} \int_{\mathbb{T}^d} \int_{B(0,k)} \frac{|\Delta_y^j u(x)|^2}{|y|^{d+2s-2(k-j)}} dydx
\leq \sum_{j=0}^{k-1} \delta^{-2(k-j)} \|u\|_{\dot{H}^{s-k+j}(\mathbb{T}^d)}^2
\leq \delta \|u\|_{\dot{H}^s(\mathbb{T}^d)}^2 + C(\delta) \|u\|_{L^2(\mathbb{T}^d)}^2,
\]
where in the last step we used the interpolation between $L^2(\mathbb{T}^d)$ and $\dot{H}^s(\mathbb{T}^d)$. This shows that the contribution of $B_2$ is acceptable.

Finally, we turn to $B_3$ in (2.28). When $j < k$ and $\ell < k$, by H"{o}lder’s inequality we have
\[
\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{C^j_k \Delta_y^{k-j} \psi_\delta(x) \Delta_y^j u(x + (k - j)y) C^\ell_k \Delta_y^{k-\ell} \psi_\delta(x) \Delta_y^\ell u(x + (k - \ell)y)}{|y|^{d+2s}} dxdy \leq B_2,
\]
which is bounded by (2.30). Without loss of generality, we only consider the case \( j = k \). Then we have \( \ell < k \). By Young’s inequality we have

\[
\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \psi_\delta(x) \Delta^k u(x) C_k \Delta^{k-\ell} \psi_\delta(x) \Delta^\ell u(x + (k - \ell) y) \, dx \, dy \lesssim \delta B_1 + C(\delta) B_2,
\]

which is again sufficient for our purpose in view of (2.29) and (2.30).

We finish the proof of (2.27), and thus the proposition. \( \Box \)

**Remark 2.4.** Let \( u \) be a function defined on \( \mathbb{R}^d \). With a slight abuse of notation, we also use \( u \) to denote its restriction onto \( \mathbb{T}^d \). It follows from (2.13) and (2.15) that

\[
\|u\|_{\dot{H}^s(\mathbb{T}^d)} = c_k^{-1} \sum_{n \in \mathbb{Z}^d \{0\}} \left( \int_{\mathbb{R}^d} \frac{|e^{2\pi i y \cdot n} - 1|^{2k}}{|y|^{d+2s}} \, dy \right) |\hat{u}(n)|^2
\]

\[
= c_k^{-1} \int_{\mathbb{T}^d} \int_{\mathbb{R}^d} \frac{|\Delta^k u(x)|^2}{|y|^{d+2s}} \, dy \, dx
\]

\[
\leq c_k^{-1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Delta^k u(x)|^2}{|y|^{d+2s}} \, dy \, dx = \|u\|_{\dot{H}^s(\mathbb{R}^d)}^2,
\]

where \( k = \lfloor s \rfloor \) is the largest integer less than \( s \).

### 3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 which provides sharp criteria for the normalizability of the Gibbs measure (1.1) with focusing interaction.

#### 3.1. Variational formulation

In order to prove Theorem 1.1, we recall a variational formula for the partition functional \( Z_{s,p,K} \) as in [25]. Let \( W(t) \) denote a mean zero cylindrical Brownian motion in \( L^2(\mathbb{T}^d) \)

\[
W(t) = \sum_{n \in \mathbb{Z}^d \{0\}} B_n(t) e_n
\]

where \( \{B_n\}_{n \in \mathbb{Z}^d \{0\}} \) is a sequence of mutually independent complex-valued Brownian motions. Then define a centered Gaussian process \( Y_s(t) \) by

\[
Y_s(t) = D^{-s} W(t) = \sum_{n \in \mathbb{Z}^d \{0\}} \frac{B_n(t)}{|n|^s} e_n.
\]

(3.1)

We note that \( Y_s(t) \) is well-defined and

\[
\mathbb{E}[|Y_s(1)|^2] = \sum_{n \in \mathbb{Z}^d \{0\}} \frac{\mathbb{E}[|B_n(1)|^2]}{|n|^s} = \sum_{n \in \mathbb{Z}^d \{0\}} \frac{2}{|n|^s} < \infty,
\]

provided \( s > \frac{d}{2} \). In particular, we have

\[
\text{Law}(Y_s(1)) = \mu_s,
\]

(3.2)

where \( \mu_s \) is the massless Gaussian free field given in (1.2).

Let \( \mathcal{H}_s \) be the space of drifts, which consists of mean zero progressively measurable processes belonging to \( L^2([0,1]; L^2(\mathbb{T}^d)) \), \( \mathbf{P} \)-almost surely. One of the key tools in this paper is the following Boué-Dupuis variational formula [5] [34] [5]. See also [11] for the infinite dimensional setting.
Lemma 3.1. Let $Y_s$ be as in (3.1) with $s > \frac{d}{2}$. Suppose that $F : H^{s - \frac{d}{2}}(\mathbb{T}^d) \to \mathbb{R}$ is measurable and bounded from above. Then, we have
\begin{equation}
- \log \mathbb{E}\left[ e^{-F(Y_s(1))} \right] = \inf_{\theta \in \mathbb{H}_a} \mathbb{E}\left[ F(Y_s(1) + I_s(\theta)(1)) + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L_2}^2 dt \right],
\end{equation}
where $I_s(\theta)$ is defined by
\begin{equation}
I_s(\theta)(t) = \int_0^t D^{-s}P_{\neq 0}\theta(\tau) d\tau.
\end{equation}
and the expectation $\mathbb{E} = \mathbb{E}_p$ is with respect to the underlying probability measure $P$.

Since we only consider the non-singular case $s > \frac{d}{2}$, then $Y_s(t)$ and $I_s(\theta)(1)$ enjoy the following pathwise regularity bounds.

Lemma 3.2. (i) Given any $s > \frac{d}{2}$ and any finite $p, q \geq 1$, there exists $C_{s,p} > 0$ such that
\begin{equation}
\mathbb{E}[\|Y_s(1)\|_{L_p(\mathbb{T}^d)}^p] \leq \mathbb{E}[\|Y_s(1)\|_{L_\infty(\mathbb{T}^d)}^p] \leq C_{s,p} < \infty.
\end{equation}
(ii) For any $\theta \in \mathbb{H}_a$, we have
\begin{equation}
\|I_s(\theta)(1)\|_{H^s(\mathbb{T}^d)}^2 \leq \int_0^1 \|\theta(t)\|_{L_2}^2 dt.
\end{equation}

Proof. Part (i) follows from Hölder’s inequality, Sobolev embedding, Minkowski’s inequality, and Wiener chaos estimate [4, Lemma 2.4] with $k = 1$. As for Part (ii), the estimate (3.5) follows from Minkowski’s inequality and Cauchy-Schwarz’ inequalities. \qed

We conclude this subsection by recalling the following simple corollary of Fernique’s theorem [15]. See also Theorem 2.7 in [13] and Lemma 4.2 in [25].

Lemma 3.3. There exists a constant $c > 0$ such that if $X$ is a mean-zero Gaussian process with values in a separable Banach space $B$ with $\mathbb{E}[\|X\|_B] < \infty$, then
\begin{equation}
\int e^{\frac{||X||_B^2}{(\mathbb{E}[||X||_B])^2}} d\mathbb{P} < \infty.
\end{equation}
In particular, we have
\begin{equation}
\mathbb{P}(\|X\|_B \geq t) \lesssim \exp\left[ - \frac{ct^2}{(\mathbb{E}[\|X\|_B])^2} \right]
\end{equation}
for any $t > 1$.

3.2. Integrability. In this subsection, we demonstrate the proof of the integrability part of Theorem 1.1. Namely, we prove the boundedness of $Z_{s,p,K}$ (i) for all $K > 0$ when $2 < p < \frac{4s}{d} + 2$ and (ii) for all $K < \|Q\|_{L^2(\mathbb{R}^d)}$ when $p = \frac{4s}{d} + 2$, where $Q$ is the optimizer for the GNS inequality on $\mathbb{R}^d$.

Theorem 1.1 - (i) and the first half of (ii). It suffices to show the following bound
\begin{equation}
Z_{s,p,K} = \mathbb{E}_{\mu_s}\left[ \exp(R_p(u)) \cdot 1_{\{\|u\|_{L_2(\mathbb{T}^d)} \leq K\}} \right] < \infty,
\end{equation}
where $R_p(u)$ is the potential energy denoted by
\begin{equation}
R_p(u) := \frac{1}{p} \int_{\mathbb{T}^d} |u|^p dx.
\end{equation}
Observing that
\[ \mathbb{E}_{\mu_s} \left[ \exp(R_p(u)) \cdot 1_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}} \right] \leq \mathbb{E}_{\mu_s} \left[ \exp \left( R_p(u) \cdot 1_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}} \right) \right], \]
then the bound (3.6) follows once we have
\[ \mathbb{E}_{\mu_s} \left[ \exp \left( R_p(u) \cdot 1_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}} \right) \right] < \infty. \] (3.8)

From (3.2) and the Boué-Dupuis variation formula Lemma 3.1 it follows that
\[
- \log \mathbb{E}_{\mu_s} \left[ \exp \left( R_p(u) \cdot 1_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}} \right) \right] \\
= - \log \mathbb{E} \left[ \exp \left( R_p(Y_s(1)) \cdot 1_{\{\|Y_s(1)\|_{L^2(\mathbb{T}^d)} \leq K\}} \right) \right] \\
= \inf_{\theta \in \mathbb{R}} \mathbb{E} \left[ - R_p(Y_s(1) + I_s(\theta)(1)) \cdot 1_{\{\|Y_s(1) + I_s(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K\}} + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2}^2 \, dt \right],
\] (3.9)
where \( Y(1) \) is given in (3.1). Here, \( \mathbb{E}_{\mu_s} \) and \( \mathbb{E} \) denote expectations with respect to the Gaussian field \( \mu_s \) and the underlying probability measure \( \mathbb{P} \) respectively. In the following, we show that the right hand side of (3.9) has a finite lower bound. The key observation is that (i) in the subcritical setting, we view \( Y_s(1) \) as a perturbation with finite \( L^2(\mathbb{T}^d) \) norm; (ii) in the critical setting, we have \( Y_s(1) = P_{\leq N}Y_s(1) + \text{a perturbation} \) for large \( N \gg 1 \), where the perturbation term is small under \( L^2(\mathbb{T}^d) \) norm with large probability. We, therefore, distinguish two cases depending on subcritical/critical interactions.

**Case 1: subcritical** \( p < \frac{4s}{d} + 2 \). In this case, we prove (3.8) with a mass cut-off of any finite size \( K \). We first recall an elementary inequality, which is a direct consequence of the mean value theorem and the Young’s inequality. Given \( p > 2 \) and \( \varepsilon > 0 \), there exists \( C_\varepsilon \) such that
\[ |z_1 + z_2|^p \leq (1 + \varepsilon)|z_1|^p + C_\varepsilon|z_2|^p \] (3.10)
holds uniformly in \( z_1, z_2 \in \mathbb{C} \). From (3.7), (3.10), Proposition 2.3 and the fact
\[ \{\|Y_s(1) + I_s(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K\} \subset \{\|I_s(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K + \|Y_s(1)\|_{L^2(\mathbb{T}^d)}\}, \]
we obtain
\[
R_p(Y_s(1) + I_s(\theta)(1)) \cdot 1_{\{\|Y_s(1) + I_s(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K\}} \\
\leq (1 + \varepsilon)R_p(I_s(\theta)(1)) \cdot 1_{\{\|I_s(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K + \|Y_s(1)\|_{L^2(\mathbb{T}^d)}\}} + C_\varepsilon R_p(Y_s(1)) \\
\leq \frac{1 + \varepsilon}{p} \left( C_{\text{GNS}} + \delta(K + \|Y_s(1)\|_{L^2(\mathbb{T}^d)})^{2 + \frac{2p - 2}{d}} \|I_s(\theta)(1)\|_{H^s(\mathbb{T}^d)}^{\frac{(p-2)d}{2s}} \right) \\
+ C_\delta(K + \|Y_s(1)\|_{L^2(\mathbb{T}^d)})^p + C_\varepsilon R_p(Y_s(1)).
\]
Noting that \( \frac{(p-2)d}{2s} < 2 \) in this case, we apply Young’s inequality to continue with
\[
\leq C + C\|Y_s(1)\|_{L^2(\mathbb{T}^d)}^{2 + \frac{4s(p-2)}{4s(p-2)d}} + \frac{1}{4}\|I_s(\theta)(1)\|_{H^s(\mathbb{T}^d)}^2 + C\|Y_s(1)\|_{L^2(\mathbb{T}^d)}^p + CR_p(Y_s(1)) \] (3.11)
where $C$ is a constant depending on $\varepsilon, \delta, p, d, s, \|Q\|_{L^2}$, and $K$. By collecting (3.9), (3.11) and Lemma 3.2, we arrive at

$$-\log \mathbb{E}_{\mu_s} \left[ \exp \left( R_p(u) \cdot 1_{\{\|u\|_{L^2(\mathbb{T}^d)} \leq K\}} \right) \right]$$

$$\geq \inf_{\theta \in H_a} \mathbb{E} \left[ -C - C\|Y_s(1)\|_{L^2(\mathbb{T}^d)}^{2 + \frac{4s(p-2)}{4s(p-2)\eta}} - C\|Y_s(1)\|_{L^2(\mathbb{T}^d)}^p - CR_p(Y_s(1)) \right]$$

$$\geq \frac{1}{4} \|I_s(\theta)(1)\|_{H^s(\mathbb{T}^d)}^2 + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2}^2 dt$$

$$\geq \inf_{\theta \in H_a} \mathbb{E} \left[ -C - C\|Y_s(1)\|_{L^2(\mathbb{T}^d)}^{2 + \frac{4s(p-2)}{4s(p-2)\eta}} - C\|Y_s(1)\|_{L^2(\mathbb{T}^d)}^p - C\|Y_s(1)\|_{L^2(\mathbb{T}^d)}^p \right]$$

$$\geq -C - 2CC_{s,p} - CC_{s,2 + \frac{4s(p-2)}{4s(p-2)\eta}} > -\infty,$$

where $C_{s,r}$ is defined in Lemma 3.2 (i). Thus we finish the proof of (3.8) in the subcritical case.

**Case 2:** critical interaction $p = \frac{4s}{d} + 2$. We shall prove (3.8) below the critical mass threshold $K < \|Q\|_{L^2(\mathbb{T}^d)}$. To get the sharp mass threshold, we view $P_{\geq N}Y_s(1)$ as a perturbation instead. It turns out that as $N$ is getting larger, the probability of $P_{\geq N}Y_s(1)$ being large shrinks exponentially to zero. See (3.20).

Since $s > \frac{d}{4}$, it follows that

$$\lim_{N \to \infty} \|P_{\geq N}Y_s(1)\|_{L^2(\mathbb{T}^d)} = 0,$$

almost surely. Therefore, given small $\varepsilon > 0$, for $\omega \in \Omega$ almost sure, there exists an unique $N_\varepsilon := N_\varepsilon(\omega)$ such that

$$\|P_{\geq N_\varepsilon}Y_s(1)\|_{L^2(\mathbb{T}^d)} > \varepsilon \quad \text{and} \quad \|P_{> N_\varepsilon}Y_s(1)\|_{L^2(\mathbb{T}^d)} \leq \varepsilon. \quad (3.12)$$

Similar argument as before with (3.10), Proposition 2.3, and (3.12), yields that

$$R_p(Y_s(1) + I_s(\theta)(1)) \cdot 1_{\{\|Y_s(1) + I_s(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K\}}$$

$$\leq (1 + \varepsilon)R_p(P_{\leq N_\varepsilon}Y_s(1) + I_s(\theta)(1)) \cdot 1_{\{\|P_{\leq N_\varepsilon}Y_s(1) + I_s(\theta)(1)\|_{L^2(\mathbb{T}^d)} \leq K + \varepsilon\}} + C\varepsilon R_p(Y_s(1))$$

$$\leq \frac{1 + \varepsilon}{p} (C_{GNS} + \delta)(K + \varepsilon)^{p-2}(\|P_{\leq N_\varepsilon}Y_s(1)\|_{H^s(\mathbb{T}^d)}^2 + \|I_s(\theta)(1)\|_{H^s(\mathbb{T}^d)}^2)^{\frac{1}{2}}$$

$$+ C\varepsilon R_p(Y_s(1)) + C\delta(K + \varepsilon)^p$$

$$\leq \frac{(1 + \varepsilon)^2}{p} (C_{GNS} + \delta)(K + \varepsilon)^{p-2}(\|I_s(\theta)(1)\|_{H^s(\mathbb{T}^d)}^2 + C\|P_{\leq N_\varepsilon}Y_s(1)\|_{H^s(\mathbb{T}^d)}^2)$$

$$+ C\varepsilon R_p(Y_s(1)) + C\delta(K + \varepsilon)^p,$$

where $C$ is a constant depending on $\varepsilon, \delta, p, d, s, \|Q\|_{L^2}$, and $K$. Since $C_{GNS} = \frac{\eta}{4}\|Q\|_{L^2}^{2-p}$, $K < \|Q\|_{L^2(\mathbb{T}^d)}$ and $p > 2$, there exist $\eta, \varepsilon, \delta > 0$ such that

$$\frac{(1 + \varepsilon)^2}{p} (C_{GNS} + \delta)(K + \varepsilon)^{p-2} < \frac{1 - \eta}{2}. \quad (3.14)$$
By collecting (3.9), (3.13), (3.14), and Lemma 3.2 we arrive at

$$- \log E_{\mu_s} \left[ \exp \left( R_p(u) \cdot 1_{\{\|u\|_{L^2(T^d)} \leq K}\} \right) \right]$$

$$\geq \inf_{\theta \in \mathbb{B}_a} E \left[ - \frac{1 - \eta}{2} \left\| I_s(\theta)(1) \right\|_{H^s(T^d)}^2 - C_\varepsilon \left\| P_{\leq N_\varepsilon} Y_s(1) \right\|_{H^s(T^d)}^2 - C_\delta (K + \varepsilon)^p \right. \right.$$

$$\left. - C_\varepsilon R_p(Y_s(1)) + \frac{1}{2} \int_0^1 \left\| \theta(t) \right\|_{L^2_x}^2 dt \right]$$

$$\geq \inf_{\theta \in \mathbb{B}_a} E \left[ - C_\delta (K + \varepsilon)^p - C_\varepsilon R_p(Y_s(1)) - C_\varepsilon \left\| P_{\leq N_\varepsilon} Y_s(1) \right\|_{H^s(T^d)}^2 + \frac{\eta}{2} \int_0^1 \left\| \theta(t) \right\|_{L^2_x}^2 dt \right]$$

$$\geq E \left[ - C_\delta (K + \varepsilon)^p - C_\varepsilon \left( R_p(Y_s(1)) - C_\varepsilon \left\| P_{\leq N_\varepsilon} Y_s(1) \right\|_{H^s(T^d)}^2 \right) \right]$$

$$\geq - C_\delta (K + \varepsilon)^p - C_\varepsilon C_{s,p} - C_\varepsilon E \left[ \left\| P_{\leq N_\varepsilon} Y_s(1) \right\|_{H^s(T^d)}^2 \right],$$

where $C_{s,p}$ is given in (3.4). We remark that $Y_s(1) \notin \dot{H}^s(T^d)$ almost surely. Therefore, to prove (3.8), it still needs to show that

$$E \left[ \left\| P_{\leq N_\varepsilon} Y_s(1) \right\|_{H^s(T^d)}^2 \right] < \infty,$$

where $N_\varepsilon$ is a random variable given by (3.12).

Noting $Y_s(1)$ is a mean-zero random variable, we may decompose $\Omega$ (by ignoring a zero-measure set) as

$$\Omega = \bigcup_{N \geq 1} \Omega_N,$$

where

$$\Omega_N = \left\{ \omega \in \Omega : N_\varepsilon(\omega) \in \left[ \frac{N}{T^d}, N \right) \right\}.$$

By (3.16) and Hölder’s inequality, we have

$$E \left[ \left\| P_{\leq N_\varepsilon} Y_s(1) \right\|_{H^s(T^d)}^2 \right] \leq \sum_{N \geq 1} E \left[ \left\| P_{\leq N} Y_s(1) \right\|_{H^s(T^d)}^2 \cdot 1_{\Omega_N} \right]$$

$$\leq \sum_{N \geq 1} N^{2s} E \left[ \left\| P_{\leq N} Y_s(1) \right\|_{L^2(T^d)}^2 \cdot 1_{\Omega_N} \right]$$

$$\leq \sum_{N \geq 1} N^{2s} \left( E \left[ \left\| Y_s(1) \right\|_{L^2(T^d)}^4 \right] \right)^{\frac{1}{2}} \cdot P(\Omega_N)^{\frac{1}{2}}$$

$$\leq C_{s,4}^{\frac{1}{2}} \sum_{N \geq 1} N^{2s} P(\Omega_N)^{\frac{1}{2}},$$

where $C_{s,4}$ is given in (3.4). By a direct computation, we have

$$E \left[ \left\| P_{\geq \frac{N}{T^d}} Y_s(1) \right\|_{L^2(T^d)}^2 \right] \sim N^{d-2s}.$$
It then follows from (3.12), (3.17), Hölder’s inequality, Lemma 3.3, and (3.19), that
\[ P(\Omega_N) \leq P(\{\|P_{\geq N}Y_s(1)\|_{L^2} > \varepsilon\}) \]
\[ \lesssim \exp\left\{ -c\left(\frac{\varepsilon}{E(\|P_{\geq N}Y_s(1)\|_{L^2(T^d)}^2)}\right)^2\right\} \]
\[ \lesssim \exp\left\{ -\left(\frac{\varepsilon^2}{E(\|P_{\geq N}Y_s(1)\|_{L^2(T^d)}^2)}\right)\right\} \]
\[ \lesssim e^{-\tilde{c}\varepsilon^2N^{2s-d}}, \]
where \( c \) and \( \tilde{c} \) are constant. By collecting (3.18) and (3.20), we conclude that
\[ E[\|P_{\leq N}Y_s(1)\|_{H^s(T^d)}^2] \leq C_sA_1^2 \sum_{N \geq 1} N^{2s}e^{-\frac{\tilde{c}}{2}\varepsilon^2N^{2s-d}} < \infty, \]
which finishes the proof of (3.15), and thus (3.8) in the critical case.

Therefore, we finish the proof of Theorem 1.1 -(i) and the first half of (ii). □

3.3. Non-integrability. In this subsection, we prove the rest of Theorem 1.1, i.e. the non-integrability part of (ii) and (iii). In particular, we show that the partition function
\[ Z_{s,p,K} = E_{\mu_s}\left[ \exp(R_{Y_s}(u))1_{\{\|u\|_{L^2(T^d)} \leq K\}}\right] = \infty \]
under either of the following conditions
(i) critical nonlinearity: \( p = \frac{4s}{d} + 2 \) and \( K > \|Q\|_{L^2(\mathbb{R}^d)} \);
(ii) super-critical nonlinearity: \( p > \frac{4s}{d} + 2 \) and any \( K > 0 \).

Here \( Q \) is the optimizer of the GNS inequality given in Theorem 2.1 and Remark 2.2. To prove (3.21), we construct, within the ball \( \{\|u\|_{L^2(T^d)} \leq K\} \), a sequence of drift terms given by perturbed scaled “solitons”, along which the variational formula (3.3) diverges. The existence of such a sequence of scaled solitons is guaranteed by the following lemma;

Lemma 3.4. Assume (3.22) holds. Then, there exist a series of functions \( \{W_\rho\}_{\rho > 0} \subset H^s(T^d) \cap L^p(T^d) \) such that
\begin{align*}
(i) \quad & H_{T^d}(W_\rho) \leq -A_1\rho^{-\frac{dp}{2}+d}, \\
(ii) \quad & \|W_\rho\|^p_{L^p(T^d)} \leq A_2\rho^{-\frac{dp}{2}+d}, \\
(iii) \quad & \|W_\rho\|_{L^2(T^d)} \leq K - \eta, \\
(iv) \quad & P_0W_\rho \lesssim 1,
\end{align*}
where \( H_{T^d} \) is the Hamiltonian functional given in (1.5), and \( A_1, A_2, A_3, \eta > 0 \) are constant uniformly in sufficiently small \( \rho > 0 \).

In the next lemma, we construct an approximation \( Z_M \) to \( Y_s(1) \) in (3.1) through solving a stochastic differential equation. These \( Z_M \) act as controllable stochastic perturbations in defining the drift terms. See (3.28) and (3.29) in the following. Similar approximation has appeared in [24].
Lemma 3.5. Given \( s > \frac{d}{2} \) and a dyadic number \( M \sim \rho^{-1} \gg 1 \), define the \( Z_M(t) \) by its Fourier coefficients: Let \( \hat{Z}_M(n,t) \) for \( 0 < |n| \leq M \) be as follows:

\[
\left\{
\begin{array}{l}
  d\hat{Z}_M(n,t) = |n|^{-s} M^d \hat{Y}_s(n,t) - \hat{Z}_M(n,t) dt \\
  \hat{Z}_M|_{t=0} = 0,
\end{array}
\right.
\]

and \( \hat{Z}_M(n,t) = 0 \) for \( n = 0 \) and \( |n| > M \). Then the following holds:

\[
\mathbb{E}[\|Z_M(1) - Y_s(1)\|_{L^p(T_d^\rho)}^p] \lesssim \max(M^{-s+\frac{d}{2}}, M^{-\frac{d}{2}+})^\frac{p}{2}, \text{ for } p \geq 1, \tag{3.25}
\]

\[
\mathbb{E}\left[\left\|D^s \frac{d}{dt} Z_M(t)\right\|_{L^2(T_d^\rho)}^2\right] \lesssim \max(M^{-\frac{d}{2}-s}, M^{\frac{d}{2}+}), \tag{3.26}
\]

for any \( M \gg 1 \).

The proofs of Lemma 3.4 and Lemma 3.5 will be postponed to the next subsection. Now we are ready to prove the rest of Theorem 1.1.

Proof of Theorem 1.1 - the second half of (ii) and (iii). We shall prove (3.21) under conditions (3.22). Observing that

\[
\mathbb{E}_{\mu_s}\left[\exp(R_p(u)) \cdot 1\{\|u\|_{L_2(T_d^\rho)} \leq K\}\right] \geq \mathbb{E}_{\mu_s}\left[\exp\left(R_p(u) \cdot 1\{\|u\|_{L_2(T_d^\rho)} \leq K\}\right)\right] - 1,
\]

then (3.21) follows from

\[
\mathbb{E}_{\mu_s}\left[\exp\left(R_p(u) \cdot 1\{\|u\|_{L_2(T_d^\rho)} \leq K\}\right)\right] = \infty. \tag{3.27}
\]

To apply Lemma 3.1 we construct the series of drift terms as follows. Let \( W_\rho \) be as in Lemma 3.4 and

\[
\theta(t) \in \left\{- D^s \frac{d}{dt} Z_M(t) + D^s W_\rho\right\}_{\rho > 0}, \tag{3.28}
\]

where \( \rho \ll 1 \) and \( M \sim \rho^{-1} \) is a dyadic number. From (3.28), we have

\[
I_s(\theta)(1) = \int_0^1 D^{-s} P_{\neq 0} \theta(t) dt = \int_0^1 \left(P_{\neq 0} W_\rho - \frac{d}{dt} Z_M(t)\right) dt = P_{\neq 0} W_\rho - Z_M(1). \tag{3.29}
\]
Thus, from Lemma 3.1, (3.28), and (3.29) we have

\[ -\log \mathbb{E}_{\mu_a} \left[ \exp \left( R_p(u) \cdot 1_{\{||u||_{L^2(T^d)} \leq K\}} \right) \right] \]

\[ = \inf_{\theta \in \Theta} \mathbb{E} \left[ \left( -R_p(Y_s(1) + I_s(\theta)(1)) \cdot 1_{\{||Y_s(1) + I_s(\theta)(1)||_{L^2(T^d)} \leq K\}} + \frac{1}{2} \int_0^1 \|\theta(t)\|_{L^2(T^d)}^2 dt \right) \right] \]

\[ \leq \inf_{0 < \rho \leq 1} \mathbb{E} \left[ \left( -R_p(Y_s(1) - Z_M(1) + P_{\neq 0} W_\rho) \cdot 1_{\{||Y_s(1) - Z_M(1) + P_{\neq 0} W_\rho||_{L^2(T^d)} \leq K\}} \right. \right. \]

\[ \left. + \frac{1}{2} \int_0^1 \left\| -\frac{d}{dt} Z_M(t) + P_{\neq 0} W_\rho \right\|_{L^2(T^d)}^2 dt \right] \]

\[ = \inf_{0 < \rho \leq 1} \left[ \left( -R_p(\rho W_\rho) + \frac{1}{2} \||W_\rho||_{H^s(T^d)}^2 \right) \right. \]

\[ \left. + \left( R_p(P_{\neq 0} W_\rho) - R_p(Y_s(1) - Z_M(1) + P_{\neq 0} W_\rho) \cdot 1_{\{||Y_s(1) - Z_M(1) + P_{\neq 0} W_\rho||_{L^2(T^d)} \leq K\}} \right) \right. \]

\[ \left. + R_p(P_{\neq 0} W_\rho) \cdot 1_{\{||Y_s(1) - Z_M(1) + P_{\neq 0} W_\rho||_{L^2(T^d)} > K\}} \right. \]

\[ \left. + \frac{1}{2} \int_0^1 \left\| -\frac{d}{dt} Z_M(t) \right\|_{H^s(T^d)}^2 dt - 2 \left\langle \frac{d}{dt} Z_M(t), W_\rho \right\rangle_{H^s(T^d)} dt \right] \]

\[ = \inf_{0 < \rho \leq 1} (A + B + C + D + E). \tag{3.30} \]

In what follows, we consider these terms one by one for $0 < \rho \leq 1$.

For term (A), from (3.23) - (i), we have

\[ A = -R_p(\rho W_\rho) + \frac{1}{2} \||W_\rho||_{H^s(T^d)}^2 = H_{T^d}(W_\rho) \lesssim -\rho^{-\frac{d}{2} + \frac{d(p-1)}{2}}. \tag{3.31} \]

For term (B), from (3.23) - (iv) and the mean value theorem, we have

\[ B = \frac{1}{p} \int_{T^d} \left| W_\rho \right|^p - \left| W_\rho - P_0 W_\rho \right|^p \] dx

\[ \lesssim \int_{T^d} \left( \left| P_0 W_\rho \right|^p + \left| P_0 W_\rho \right| \left| W_\rho \right|^{p-1} \right) \] dx

\[ \lesssim 1 + \left| W_\rho \right|^{p-1}_{L^p(T^d)}, \]

Then, by interpolating (3.23) - (ii) and (iii), we obtain

\[ B \lesssim \rho^{-\frac{d(p-1)}{2} + d}. \tag{3.32} \]

For term (C), by using the mean value theorem we see that

\[ \int_{T^d} \left( \left| P_{\neq 0} W_\rho \right|^p - \left| Y_s(1) - Z_M(1) + P_{\neq 0} W_\rho \right|^p \right) \] dx

\[ \lesssim \int_{T^d} \left( \left| Y_s(1) - Z_M(1) \right|^p + \left| Y_s(1) - Z_M(1) \right| \left| P_{\neq 0} W_\rho \right|^{p-1} \right) \] dx,
which together with Lemma 3.5 and Lemma 3.4 gives
\[
C = E \left[ \left( R_p(P_{\neq 0}W_\rho) - R_p(Y_s(1) - Z_M(1) + P_{\neq 0}W_\rho) \right) \cdot 1_{\{\|Y_s(1) - Z_M(1) + W_\rho\|_{L^2(\mathbb{T}^d)} \leq K\}} \right]
\]
\[
\lesssim \int_{\mathbb{T}^d} \left( E[|Y_s(1) - Z_M(1)|^2] + E[|Y_s(1) - Z_M(1)| |P_{\neq 0}W_\rho|^p] \right) dx
\]
\[
\lesssim \max(M^{-s+\frac{d}{2}}, M^{-\frac{d}{2}+}) + \max(M^{-s+\frac{d}{2}}, M^{-\frac{d}{2}+}) \|P_{\neq 0}W_\rho\|_{L^p(\mathbb{T}^d)}^{p-1}
\]
\[
\lesssim (\|P_{\neq 0}W_\rho\|_{L^p(\mathbb{T}^d)}^{p-1} - \|W_\rho\|_{L^p(\mathbb{T}^d)}) + \|W_\rho\|_{L^p(\mathbb{T}^d)}^{p-1}
\]
\[
\lesssim \rho^{-\frac{(p-1)}{2}+d},
\]
where in the last step, to bound (\|P_{\neq 0}W_\rho\|_{L^p(\mathbb{T}^d)}^{p-1} - \|W_\rho\|_{L^p(\mathbb{T}^d)}) , we used a similar argument as in estimating term (B). Now we turn to term (D), by using Chebyshev’s inequality, (3.23) - (iii), (3.25), and (3.32), we have
\[
D = E \left[ R_p(P_{\neq 0}W_\rho) \cdot 1_{\{\|Y_s(1) - Z_M(1) + P_{\neq 0}W_\rho\|_{L^2(\mathbb{T}^d)} > K\}} \right]
\]
\[
\leq R_p(P_{\neq 0}W_\rho) \cdot E \left[ 1_{\{\|Y_s(1) - Z_M(1)\|_{L^2(\mathbb{T}^d)} > K - \|W_\rho\|_{L^2(\mathbb{T}^d)}\}} \right]
\]
\[
\leq R_p(P_{\neq 0}W_\rho) \frac{E[|Y_s(1) - Z_M(1)|^2_{L^2(\mathbb{T}^d)}]}{(K - \|W_\rho\|_{L^2(\mathbb{T}^d)})^2}
\]
\[
\lesssim \rho^{-\frac{dp}{2}+d} \max(M^{-s+\frac{d}{2}}, M^{-\frac{d}{2}+})
\]
\[
\lesssim \max(\rho^{-\frac{(p-1)}{2}+s}, \rho^{-\frac{p-1}{2}+})
\]
where in the last step we use the relation $M \sim \rho^{-1}$. For term (E), from (3.24) and (3.26), we have
\[
E = \frac{1}{2} \int_0^t E \left[ \left\| -\frac{d}{dt} Z_M(t) \right\|^2_{L^2(\mathbb{T}^d)} \right] dt \lesssim \max(\rho^{-\frac{dp}{2}+s}, \rho^{-\frac{d}{2}+})
\]
where we used the fact that $Z_M$, removing the zero frequency, is a mean zero Gaussian random variable. By collecting estimates (3.31), (3.32), (3.33), (3.34), and (3.35), we conclude that
\[
A + B + C + D + E \lesssim -\rho^{-\frac{dp}{2}+d},
\]
where we used (3.22) and the assumption $s > \frac{d}{2}$. Finally, the desired estimate (3.27) follows from (3.30) and (3.36). We thus finish the proof of Theorem 1.1. □

3.4. **Proof of the auxiliary lemmas.** It remains to prove Lemmas 3.3 and 3.5, which is the main purpose of this subsection. We first present the proof of Lemma 3.3.

**Proof of Lemma 3.3** Define $W_\rho \in H^s(\mathbb{T}^d)$ by
\[
W_\rho(x) := \alpha \rho^{-\frac{d}{2}} \phi_\delta(x) Q(\rho^{-1} x),
\]
where $\phi_\delta$ is the same as in Proposition 2.3, $\alpha > 0$ is to be determined later and $Q$ is given in Theorem 2.1 and Remark 2.2. Then (iv) follows directly from $\|W_\rho\|_{L^1(\mathbb{T}^d)} \lesssim \|W_\rho\|_{L^2(\mathbb{T}^d)}$. We only consider (i) – (iii) in what follows. We distinguish two cases based on the conditions in (3.22):
**Case 1:** critical nonlinearity. In this case, we have \( p = \frac{4s}{d} + 2 > 2 \) and \( K > \|Q\|_{L^2(\mathbb{R}^d)} \). Fix \( \alpha > 1 \) such that

\[
\|\alpha Q\|_{L^2(\mathbb{R}^d)} = \alpha \|Q\|_{L^2(\mathbb{R}^d)} = K - \eta,
\]

where \( \eta \) is given in (3.38). Recall that \( H_d(Q) = 0 \) from Remark 2.2. We then have

\[
H_d(\alpha Q) = \frac{\alpha^2}{2} \int_{\mathbb{R}^d} |D^s Q|^2 dx - \frac{\alpha^p}{p} \int_{\mathbb{R}^d} |Q|^p dx < 0.
\]

Then, it follows from Remark 2.3 that

\[
H_d(W_\rho) = \frac{\alpha^2}{2} \int_{\mathbb{T}^d} |D^s (\phi_\delta Q_\rho)|^2 dx - \frac{\alpha^p}{p} \int_{\mathbb{T}^d} |\phi_\delta(x) Q_\rho(x)|^p dx
\]

\[
\leq \frac{\alpha^2}{2} \|D^s(\phi_\delta Q_\rho)\|_{L^2(\mathbb{R}^d)}^2 - \frac{\alpha^p}{p} \int_{\mathbb{T}^d} |\phi_\delta(x) Q_\rho(x)|^p dx,
\]

where \( Q_\rho = \rho^{-d} Q(p^{-1} x) \). By the fractional Leibniz rule [17, 19] and Sobolev embedding, we may continue with

\[
\leq \frac{\alpha^2 + \epsilon}{2} \|D^s Q_\rho\|_{L^2(\mathbb{R}^d)}^2 - \frac{\alpha^p}{p} \int_{\mathbb{T}^d} |\phi_\delta(x) Q_\rho(x)|^p dx + C \|Q_\rho\|_{W^{s,2}}^2,
\]

then by interpolation we can continue with

\[
\leq \frac{\alpha^2 + 2\epsilon}{2} \rho^{-2s} \|D^s Q\|_{L^2(\mathbb{R}^d)}^2 - \frac{\alpha^p}{p} \rho^{-\frac{dp}{2} + d} \int_{\mathbb{R}^d} \phi_\delta(\rho x) Q(x)|^p dx + C \|Q_\rho\|_{L^2(\mathbb{R}^d)}^2,
\]

we have \( \int_{\mathbb{R}^d} |\phi_\delta(x) Q(x)|^p dx > \frac{1}{\alpha^2} \int_{\mathbb{R}^d} |Q(x)|^p dx \), provided that \( \rho \) is sufficiently small. Thus, combining with (2.5) and the fact \( 2s = \frac{dp}{2} - d \) from (3.22) - (i), we may continue with

\[
\leq \left( \frac{\alpha^2 + 2\epsilon}{p} - \frac{\alpha^p}{p} \right) \rho^{-\frac{dp}{2} + d} \|Q\|_{L^p(\mathbb{R}^d)}^p + C \|Q\|_{L^2(\mathbb{R}^d)}^2,
\]

which finishes the proof of (3.23) - (i) by choosing \( \epsilon \) small enough and setting

\[
A_1 := \left( \frac{\alpha^p - \alpha^2}{p} + 2\epsilon \right) \|Q\|_{L^p(\mathbb{R}^d)}^p.
\]

As to (3.23) - (ii) and (iii), we note that

\[
\|W_\rho\|_{L^p(\mathbb{T}^d)}^p \leq \alpha^p \rho^{-\frac{dp}{2} + d} \|Q\|_{L^p(\mathbb{R}^d)}^p = A_2 \rho^{-\theta},
\]

\[
\|W_\rho\|_{L^2(\mathbb{T}^d)} \leq \alpha \|Q_\rho\|_{L^2(\mathbb{R}^d)} = \alpha \|Q\|_{L^2(\mathbb{R}^d)} = K - \eta,
\]

with \( A_2 := \alpha^p \|Q\|_{L^p(\mathbb{R}^d)}^p \) and \( \eta \) being the one in (3.38). Thus, we finish the proof.

**Case 2:** super-critical nonlinearity. In what follows, we assume \( p > \frac{4s}{d} + 2 \). It only needs to prove (3.23) - (i) and (iii), since (ii) follows the same way as that of Case 1. Given \( K > 0 \), we choose \( \alpha \ll 1 \) in (3.37) so that

\[
\|W_\rho\|_{L^2(\mathbb{T}^d)} \leq \|W_\rho\|_{L^2(\mathbb{R}^d)} = \alpha \|Q\|_{L^2(\mathbb{R}^d)} < K - \eta,
\]
which gives (3.23) - (iii). Similar computation as in the previous case, we have

\[ H_{T^d}(W_\rho) = \frac{\alpha^2}{2} \int_{T^d} |D^{s}(\phi_\delta Q_\rho)|^2 dx - \frac{\alpha^p}{p} \int_{T^d} |\phi_\delta Q_\rho|^p dx \]

\[ \leq \frac{\alpha^2 + 2\varepsilon}{2} \rho^{-2s} \|D^{s}Q\|_{L^2(T^d)}^2 - \frac{\alpha^p - \varepsilon}{p} \rho^{-\frac{dp}{2} + d} \|Q\|_{L^p(T^d)}^p + C\|Q_\rho\|_{L^2(T^d)}^2, \]

for sufficiently small \( \rho \) and \( \delta \). Also, note that \( p > \frac{4s}{d} + 2 \) implies \(-2s > -\frac{dp}{2} + d\). Thus, recalling (2.5), we may continue with

\[ \leq \left( \frac{\alpha^2 + 2\varepsilon}{p} \rho^{-2s} - \frac{\alpha^p - \varepsilon}{2p} \rho^{-\frac{dp}{2} + d} \right) \|Q\|_{L^p(T^d)}^p + C\|Q\|_{L^2(T^d)}^2 \]

\[ \leq -\tilde{A}_1 \rho^{-\frac{dp}{2} + d}, \]

for sufficiently small \( \rho > 0 \) and some constant \( \tilde{A}_1 > 0 \). Thus, we obtain (3.23) - (i). We finish the proof of Lemma 3.4. □

Next, we present the proof of Lemma 3.5.

**Proof of Lemma 3.5**

Let

\[ X_n(t) = \tilde{Y}_s(n, t) - \tilde{Z}_M(n, t), \quad 0 < |n| \leq M. \]

(3.39)

Then, from (3.1) and (3.24), we see that \( X_n(t) \) solves

\[
\begin{cases}
    dX_n(t) = -|n|^{-s} M^\frac{d}{2} X_n(t) dt + |n|^{-s} dB_n(t) \\
    X_n(0) = 0
\end{cases}
\]

for \( 0 < |n| \leq M \). Solving the above stochastic differential equation yields

\[ X_n(t) = |n|^{-s} \int_0^t e^{-|n|^{-s} M^\frac{d}{2}(t-t')} dB_n(t'). \]

(3.40)

Then, from (3.39) and (3.40), we have

\[ \tilde{Z}_M(t) = \tilde{Y}_s(n, t) - |n|^{-s} \int_0^t e^{-|n|^{-s} M^\frac{d}{2}(t-t')} dB_n(t'), \]

(3.41)

for \( 0 < |n| \leq M \). In what follows, we show that \( Z_M \) approximates to \( Y_s \) as \( M \sim \rho^{-1} \) tends to infinity. From (3.41), the independence of \( \{B_n\}_{n \in \mathbb{Z}^d} \), and Ito’s isometry, we have

\[
\mathbb{E} \left[ |Z_M(1) - Y_s(1)|^2 \right] = \sum_{0 < |n| \leq M} |n|^{-2s} \int_0^t e^{-2|n|^{-s} M^\frac{d}{2}(t-t')} dt' + \sum_{|n| > M} |n|^{-2s}
\]

\[ \lesssim \sum_{0 < |n| \leq M} |n|^{-s} M^{-\frac{d}{2}} + M^{-2s + d} \]

\[ \lesssim \max(M^{-s + \frac{d}{2}}, M^{-\frac{d}{2}}), \]

(3.42)

which is sufficient for (3.25) with \( p = 2 \).

When \( p = 1 \), (3.25) follows from (3.42) together with Hölder’s inequality

\[ \mathbb{E} \left[ |Z_M(1) - Y_s(1)| \right] \lesssim \left( \mathbb{E} \left[ |Z_M(1) - Y_s(1)|^2 \right] \right)^{\frac{1}{2}}. \]
Then the case for \( 1 < p < 2 \) follows from interpolation. When \( p > 2 \), we note that \( Z_M(1) - Y_s(1) \in H_1 \), homogeneous Wiener chaoses of order 1. Then, by using Wiener chaos estimate [31, Lemma I.22], we obtain

\[
\mathbb{E}[|Z_M(1) - Y_s(1)|^p] \lesssim \mathbb{E}[|Z_M(1) - Y_s(1)|^2] \lesssim \left( \mathbb{E}[|Z_M(1) - Y_s(1)|^2] \right)^{\frac{p}{2}},
\]

which together with (3.42) implies (3.25) for \( p > 2 \).

Finally, we turn to (3.26). From (3.24) and (3.39), we have

\[
\mathbb{E} \left[ \left\| D^s \frac{d}{dt} Z_M(t) \right\|_{L^2(T^d)}^2 \right] = M^d \sum_{0 < |n| \leq M} \mathbb{E}[|X_n(t)|^2]
\]
\[
= M^d \sum_{0 < |n| \leq M} |n|^{-2s} \int_0^t e^{-2|n|^{-s} M^{\frac{d}{2}} (t-t')} dt'
\]
\[
\lesssim M^d \sum_{0 < |n| \leq M} |n|^{-s} M^{-\frac{d}{2}}
\]
\[
\lesssim \max(M^{\frac{d}{2} - s}, M^{\frac{d}{2} + s}).
\]

We finish the proof of (3.26) and thus we conclude this lemma. \( \Box \)

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