Content

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Realization of Biquadratic Impedances as Five-Element Bridge Networks

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Abstract

This paper studies the passive network synthesis problem of biquadratic impedances as five-element bridge networks. By investigating the realizability conditions of the configurations that can cover all the possible cases, a necessary and sufficient condition is obtained for biquadratic impedances to be realizable as two-reactive five-element bridge networks. Based on the types of the two reactive elements, the discussion is divided into two parts, and a canonical form for biquadratic impedances is utilized to simplify and combine the conditions. Moreover, the realizability result for the biquadratic impedance is extended to the general five-element bridge networks. Some numerical examples are presented for illustration.

Keywords: Network synthesis, passivity, biquadratic impedance, bridge network.

I. INTRODUCTION

Passive network synthesis has been one of the most important subjects in circuit and system theories. This field has experienced a “golden era” from the 1930s to the 1970s [1], [2], [13], [34]. The most classical transformerless realization method, the Bott-Duffin procedure [2], shows
that any positive-real impedance (resp. admittance) is realizable with a finite number of resistors, capacitors, and inductors. However, the resulting networks typically contain a large number of redundant elements, and the minimal realization problem is far from being solved even today for the biquadratic impedance. Recently, interest in investigation on passive networks has revived [7], [5], [6], [7], [10], [16], [17], [32], [26], [27], [29], [36], due to its connection with passive mechanical control using a new mechanical element named inerter [13], [18], [32]. In parallel, there are also some new results on the negative imaginary systems [35]. In addition to the control of mechanical systems, synthesis of passive networks can also be applied to a series of other fields, such as the microwave antenna circuit design [21], filter design [23], passivity-preserving balanced truncation [25], and biometric image processing [26]. Noticeably, the need for a renewed attempt to passive network synthesis and its contribution to systems theory has been highlighted in [18].

The realization problem of biquadratic impedances has been a focal topic in the theory of passive network synthesis [19], [20], [24], [26], [27], yet its minimal realization problem has not been completely solved to date. In fact, investigation on synthesis of biquadratic impedances can provide significant guidance on realization of more general functions. By the Bott-Duffin procedure [2] and Pantell’s simplification [22], one needs at most eight elements to realize a general positive-real biquadratic impedance. In [19], Ladenheim listed 108 configurations containing at most five elements and at most two reactive elements that can realize the biquadratic impedance. For each of them, values of the elements are explicitly expressed in terms of the coefficients of the biquadratic function, without any derivation given. Realizability conditions of series-parallel networks are listed, but those of bridge networks are not. Furthermore, the realizability problem of biquadratic impedances as five-element networks containing three reactive elements [20] has been investigated. Recently, a new concept named regularity is introduced and applied to investigate the realization problem of the biquadratic impedances as five-element networks in [16], in which a necessary and sufficient condition is derived for a biquadratic impedance to be realizable as such a network. It is noted that only the realizability conditions for five-element bridge networks that are not necessarily equivalent to the corresponding series-parallel ones are investigated in [16]. Hence, necessary and sufficient conditions for the biquadratic impedances to be realizable as five-element bridge networks are still unknown today.

The present paper is concerned with the realization of biquadratic impedances as five-element
bridge networks. As discussed above, this problem remains unsolved today. Pantell’s simplification [22] shows that non-series-parallel networks may often contain less redundancy. Besides, the non-series-parallel structure sometimes has its own advantages in practice [12]. It is essential to construct a five-element bridge network, the simplest non-series-parallel network to solve the minimal realization problem of biquadratic impedances. This paper focuses on deriving some realizability conditions of biquadratic impedances as two-reactive five-element bridge networks. Based on these and some previous results in [20], the realization result of five-element bridge networks without limiting the number of reactive elements will follow. The discussion on realization of two-reactive five-element bridge networks is divided into two parts, based on whether the two reactive elements are of the same type or not. Through investigating realizability conditions for configurations that can cover all the possible cases, a necessary and sufficient condition is obtained for a biquadratic impedance to be realizable as a two-reactive five-element bridge network. A canonical form for biquadratic impedances is utilized to simplify and combine the conditions. Furthermore, the corresponding result of general five-element bridge networks is further obtained. Throughout, it is assumed that the given biquadratic impedance is realizable with at least five elements. A part of this paper has appeared as a conference paper in Chinese [8] (Section V-C and a part of contents in Sections III and IV).

II. PRELIMINARIES

A real-rational function $H(s)$ is positive-real if $H(s)$ is analytic and $\mathbb{R}(H(s)) \geq 0$ for $\mathbb{R}(s) > 0$ [13]. An impedance $Z(s)$ is defined as $Z(s) = V(s)/I(s)$, and an admittance is $Z^{-1}(s)$, where $V(s)$ and $I(s)$ denote the voltage and current, respectively. A linear one-port time-invariant network is passive if and only if its impedance (resp. admittance) is positive-real, and any positive-real function is realizable as the impedance (resp. admittance) of a one-port network consisting of a finite number of resistors, capacitors, and inductors [2, 13], thus the network realizes (or being a realization of) its impedance (resp. admittance). A regular function $H(s)$ is a class of positive-real functions with the smallest value of $\mathbb{R}(H(j\omega))$ or $\mathbb{R}(H^{-1}(j\omega))$ being at $\omega = 0 \cup \infty$ [16]. The capacitors and inductors are called reactive elements, and resistors are called resistive elements. Moreover, the concept of the network duality is presented in [11].
III. Problem Formulation

The general form of a biquadratic impedance is
\[
Z(s) = \frac{a_2 s^2 + a_1 s + a_0}{b_2 s^2 + b_1 s + b_0},
\]
where \(a_2, a_1, a_0, b_2, b_1, b_0 \geq 0\). It is known from [6] that its positive-realness is equivalent to
\[
(\sqrt{a_0 b_2} - \sqrt{a_2 b_0})^2 \leq a_1 b_1.
\]
For brevity, the following notations are introduced:
\[
A = a_0 b_1 - a_1 b_0,
B = a_0 b_2 - a_2 b_0,
C = a_1 b_2 - a_2 b_1,
D_a := a_1 A - a_0 B,
D_b := -b_1 A + b_0 B,
E_a := a_2 B - a_1 C,
E_b := -b_2 B + b_1 C,
M := a_0 b_2 + a_2 b_0,
\Delta_a := a_1^2 - 4 a_0 a_2,
\Delta_b := b_1^2 - 4 b_0 b_2,
\Delta_{ab} := a_1 b_1 - 2 M,
R := A C - B^2,
\Gamma_a := R + b_0 b_2 \Delta_a,
\Gamma_b := R + a_0 a_2 \Delta_b.
\]

As shown in [16], if at least one of \(a_2, a_1, a_0, b_2, b_1, b_0\) is zero, then \(Z(s)\) is realizable with at most two reactive elements and two resistors. In [27], a necessary and sufficient condition for a biquadratic impedance with positive coefficients to be realizable with at most four elements was established, as below.

**Lemma 1:** [27] A biquadratic impedance \(Z(s)\) in the form of (III.1), where \(a_2, a_1, a_0, b_2, b_1, b_0 > 0\), can be realized with at most four elements if and only if at least one of the following conditions holds: 1) \(R = 0\); 2) \(B = 0\); 3) \(B > 0\) and \(D_a E_b = 0\); 4) \(B < 0\) and \(D_b E_a = 0\); 5) \(\Gamma_a \Gamma_b = 0\).

Therefore, when investigating the realizability problem of five-element networks, it suffices to assume that \(a_2, a_1, a_0, b_2, b_1, b_0 > 0\) but the condition of Lemma [1] does not hold in the consideration of minimal realizations. For brevity, the set of all such biquadratic functions (with positive coefficients and with the condition of Lemma [1] not being satisfied) is denoted by \(Z_b\) in this paper.

The present paper aims to derive a necessary and sufficient condition for a biquadratic impedance \(Z(s) \in Z_b\) to be realizable as a two-reactive five-element bridge network (Theorem [7]) and the corresponding result for a general five-element bridge network (Theorem [8]). Figs. [1]–[5] and [7] are the corresponding realizations. The configurations are assumed to be passive one-port time-invariant transformerless networks containing at most three kinds of passive elements, which are resistors, capacitors, and inductors, and the values of the elements are all positive and finite.

DRAFT
IV. A Canonical Biquadratic Form

A canonical form $Z_c(s)$ for biquadratic impedances stated in is expressed as

$$Z_c(s) = \frac{s^2 + 2U\sqrt{W} s + W}{s^2 + (2V/\sqrt{W})s + 1/W},$$

where

$$W = \sqrt{\frac{a_0b_2}{a_2b_0}}, \quad U = \frac{a_1}{2\sqrt{a_0a_2}}, \quad V = \frac{b_1}{2\sqrt{b_0b_2}}.$$  \hspace{1cm} (IV.2)

It is not difficult to verify that $Z_c(s)$ can be obtained from $Z(s)$ through $Z_c(s) = \alpha Z(\beta s)$, where $\alpha = b_2/a_2$ and $\beta = \sqrt[4]{a_0b_0/(a_2b_2)}$. If $Z(s)$ is realizable as a network $N$, then the corresponding $Z_c(s)$ must be realizable as another network $N_c$ with the same one-terminal-pair labeled graph by a proper transformation of the element values, and vice versa. Therefore, the realizability condition for $Z_c(s)$ as a network whose one-terminal-pair labeled graph is $N$ in terms of $U$, $V$, $W > 0$ can be determined from that of $Z(s)$ in terms of $a_2$, $a_1$, $a_0$, $b_2$, $b_1$, $b_0 > 0$, via transformation

$$a_2 = 1, \quad a_1 = 2U\sqrt{W}, \quad a_0 = W,$$

$$b_2 = 1, \quad b_1 = 2V/\sqrt{W}, \quad b_0 = 1/W.$$  \hspace{1cm} (IV.3)

Conversely, the realizability condition for $Z(s)$ as a network with one-terminal-pair labeled graph $N$ in terms of $a_2$, $a_1$, $a_0$, $b_2$, $b_1$, $b_0 > 0$ can be determined from that for $Z_c(s)$ in terms of $U$, $V$, $W > 0$, via transformation (IV.2). Furthermore, through (IV.3), one concludes that $Z_c(s)$ is positive-real if and only if $\sigma_c := 4UV + 2 - (W + W^{-1}) \geq 0$, as stated in [16]. Notations $\Delta_{ab}$, $R_c$, and $\Gamma_c$, as defined in Section [III] are respectively converted to $\Delta_{ab} := 4UV - 2(W + W^{-1})$, $R_c := -4U^2 - 4V^2 + 4UV(W + W^{-1}) - (W - W^{-1})^2$, $\Gamma_a := -4V^2 + 4UV(W + W^{-1}) - (W + W^{-1})^2$, and $\Gamma_b := -4U^2 + 4UV(W + W^{-1}) - (W + W^{-1})^2$. Also, $MR + 2a_0a_2b_0b_2\Delta_{ab}$ is converted to $-(W + W^{-1})^3 + 4UV(W + W^{-1})^2 - 4(U^2 + V^2)(W + W^{-1}) + 8UV$. Moreover, for brevity, denote $\lambda_c := 4UV - 4V^2W + (W - W^{-1})$. Defining $\rho^*(U, V, W) = \rho(U, V, W^{-1})$ and $\rho^+(U, V, W) = \rho(V, U, W)$ for any rational function $\rho(U, V, W)$, one can see that $\lambda_{c^*}W$, $\lambda_c/W$, $\lambda_c^*$, and $\lambda^*_c$ correspond to $D_a$, $D_b$, $E_a$, $E_b$, respectively, through (IV.3). Besides, by denoting $\eta_c := 4U^2 + 4V^2 + 4UV(3W - W^{-1}) + (W - W^{-1})(9W - W^{-1})$ and $\zeta_c := -4U^2 - 4V^2 + 4UV(W + W^{-1}) - (W - W^{-1})(3W - W^{-1})$, respectively, one has $\eta^*_c = \eta_c^*$ and $\zeta^*_c = \zeta_c^*$ corresponding to $(-R + 4a_0b_2(a_1b_1 + 2B))$ and $(R - 2a_0b_2B)$, respectively.
Denote $Z_{bc}$ as the set of biquadratic functions in the form of (IV.1), where the coefficients $U$, $V$, $W > 0$ and they do not satisfy the condition of Lemma III transformed through (IV.3). It is clear that $Z(s) \in Z_b$ if and only if $Z_c(s) \in Z_{bc}$.

In this paper, the canonical biquadratic form as in (IV.1) is introduced so as to further simplify the realizability conditions of (III.1) (in the proof of Theorems 2, 4, and 6).

V. MAIN RESULTS

Section V-A presents some basic lemmas that will be used in the following discussions. Section V-B investigates the realization of biquadratic impedances as a five-element bridge network containing two reactive elements of the same type. In Section V-C, the realization problem of biquadratic impedances as a five-element bridge network containing one inductor and one capacitor is investigated. Section V-D presents the final results (Theorems 7 and 8).

A. Basic Lemmas

Lemma 2: [14] If a biquadratic impedance $Z(s) \in Z_b$ is realizable with two reactive elements of different types and an arbitrary number of resistors, then $R < 0$. If a biquadratic impedance $Z(s) \in Z_b$ is realizable with two reactive elements of the same type and an arbitrary number of resistors, then $R > 0$.

Let $P(a, a')$ denote the path (see [34, pg. 14]) whose terminal vertices (see [34, pg. 14]) are $a$ and $a'$ [7]; let $\mathcal{C}(a, a')$ denote the cut-set (see [34, pg. 28]) that separates $\mathcal{N}$ into two connected subgraphs $\mathcal{N}_1$ and $\mathcal{N}_2$ containing $a$ and $a'$, respectively [7].

Lemma 3: [27] For a network with two terminals $a$ and $a'$ that realizes a biquadratic impedance $Z(s) \in Z_b$, its network graph can neither contain the path $P(a, a')$ nor contain the cut-set $\mathcal{C}(a, a')$ whose edges correspond to only one kind of reactive elements.

Lemma 4: If $U, V, W > 0$ satisfy $W \neq 3$,

$$-4U^2 - 4V^2 + 4UV(W + W^{-1}) - (W - W^{-1})^2 > 0,$$

and

$$-4U^2 - 4V^2 + 4UV(W - 3W^{-1}) - (W - W^{-1})(W - 9W^{-1}) \geq 0,$$
then
\[ 4\zeta^* W^{-1} (UW - V)(UW - 3V) + 8\lambda^*(V^2 - U^2) > 0. \] (V.3)

**Proof:** See [11] for details.

\[ \Box \]

**B. Five-Element Bridge Networks with Two Reactive Elements of the Same Type**

**Lemma 5:** A biquadratic impedance \( Z(s) \in \mathbb{Z}_b \) is realizable as a five-element bridge network containing two reactive elements of the same type if and only if \( Z(s) \) is the impedance of one of configurations in Figs. 1 and 2.

**Proof:** The proof is straightforward based on Lemma 3 using the method of enumeration.

\[ \Box \]

![Fig. 1](image1.png)

Fig. 1. The two-reactive five-element bridge configurations containing the same type of reactive elements, which are respectively supported by two one-terminal-pair labeled graphs \( \mathcal{N}_1 \) and \( \text{Dual}(\mathcal{N}_1) \), where (a) is No. 85 configuration in [19] and (b) is No. 60 configuration in [19].

![Fig. 2](image2.png)

Fig. 2. The two-reactive five-element bridge configurations containing the same type of reactive elements, which are respectively supported by two one-terminal-pair labeled graphs \( \mathcal{N}_2 \) and \( \text{Dual}(\mathcal{N}_2) \), where (a) is No. 86 configuration in [19] and (b) is No. 61 configuration in [19].

**Theorem 1:** A biquadratic impedance \( Z(s) \in \mathbb{Z}_b \) is realizable as the configuration in Fig. 1 if and only if \( R - 4a_0a_2b_0b_2 \geq 0 \). Furthermore, if \( R - 4a_0a_2b_0b_2 \geq 0 \) and \( B > 0 \), then \( Z(s) \) is
realizable as the configuration in Fig. 1(a) with values of elements satisfying

\[ R_2 = \frac{a_0 - b_0 R_1}{b_0}, \quad (V.4a) \]

\[ R_3 = \frac{a_2 R_1}{b_2 R_1 - a_2}, \quad (V.4b) \]

\[ C_1 = \frac{(a_1 a_2 b_0 + a_0 C)R_1 - a_0 a_1 a_2}{(a_0 - b_0 R_1)R_1^2 M}, \quad (V.4c) \]

\[ C_2 = \frac{(a_2 b_0 b_1 + b_2 A) - b_0 b_1 b_2 R_1}{(a_0 - b_0 R_1)M}, \quad (V.4d) \]

and \( R_1 \) is the positive root of the following quadratic equation:

\[ b_0 b_2 \Gamma_a R_1^2 - (MR + 2a_0 a_2 b_0 b_2 \Delta_{ab}) R_1 + a_0 a_2 \Gamma_b = 0. \quad (V.5) \]

**Proof: Necessity.** The impedance of the configuration in Fig. 1(a) is

\[ Z(s) = \frac{a(s)}{b(s)}, \quad (V.6) \]

where \( a(s) = R_1 R_2 R_3 C_1 C_2 s^2 + ((R_1 R_2 + R_2 R_3 + R_1 R_3)C_1 + (R_1 + R_2)R_3 C_2)s + (R_1 + R_2) \)

and \( b(s) = (R_1 + R_3)R_2 C_1 C_2 s^2 + ((R_1 + R_3)C_1 + (R_1 + R_2 + R_3)C_2)s + 1. \)

Supposing that \( Z(s) \in \mathbb{Z}_0 \) is realizable as the configuration in Fig. 1(a), it follows that

\[ R_1 R_2 R_3 C_1 C_2 = k a_2, \quad (V.7a) \]

\[ (R_1 R_2 + R_3(R_1 + R_2))C_1 + (R_1 + R_2)R_3 C_2 = k a_1, \quad (V.7b) \]

\[ R_1 + R_2 = k a_0, \quad (V.7c) \]

\[ (R_1 + R_3)R_2 C_1 C_2 = k b_2, \quad (V.7d) \]

\[ (R_1 + R_3)C_1 + (R_1 + R_2 + R_3)C_2 = k b_1, \quad (V.7e) \]

\[ 1 = k b_0. \quad (V.7f) \]

From (V.7f), one obtains

\[ k = \frac{1}{b_0}. \quad (V.8) \]

From (V.7a) and (V.7d), it follows that \( 1/R_1 + 1/R_3 = b_2/a_2 \). The assumption that \( R_1 > 0 \) and \( R_3 > 0 \) implies

\[ b_2 R_1 - a_2 > 0. \quad (V.9) \]
Hence, \( R_3 \) is solved as \((\text{V.4b})\). Based on \((\text{V.7c})\) and \((\text{V.8})\), \( R_2 \) is solved as \((\text{V.4a})\), implying
\[
a_0 - b_0 R_1 > 0. \tag{V.10}
\]
Substituting \((\text{V.4a})\), \((\text{V.4b})\), and \((\text{V.8})\) into \((\text{V.7b})\) and \((\text{V.7e})\), \( C_1 \) and \( C_2 \) can be solved as \((\text{V.4c})\) and \((\text{V.4d})\), implying
\[
(a_1 a_2 b_0 + a_0 C) R_1 - a_0 a_1 a_2 > 0, \tag{V.11}
\]
\[
(a_2 b_0 b_1 \pm b_2 A) - b_0 b_1 b_2 R_1 > 0. \tag{V.12}
\]
Substituting \((\text{V.4a})-(\text{V.4d})\) and \((\text{V.8})\) into \((\text{V.7a})\) yields \((\text{V.5})\). The discriminant of \((\text{V.5})\) in \( R_1 \) is obtained as
\[
\delta = (-\mathrm{MR} - 2a_0 a_2 b_0 b_2 \Delta_{ab})^2 - 4b_0 b_2 \Gamma_a a_0 a_2 \Gamma_b
\]
\[
= M^2 R(\mathcal{R} - 4a_0 a_2 b_0 b_2), \tag{V.13}
\]
which must be nonnegative. Together with Lemma \( \text{[2]} \), it follows that \( \mathcal{R} - 4a_0 a_2 b_0 b_2 \geq 0 \).
Moreover, from \((\text{V.9})\) and \((\text{V.10})\), one obtains \( B > 0 \). Therefore, if \( Z(s) \in Z_b \) is realizable as the configuration in Fig. \( \text{IIb} \), then \( \mathcal{R} - 4a_0 a_2 b_0 b_2 \geq 0 \) and \( B < 0 \), which are obtained through the principle of duality \((a_2 \leftrightarrow b_2, a_1 \leftrightarrow b_1, \text{and } a_0 \leftrightarrow b_0) \) \( \text{(II)} \).

**Sufficiency.** By the principle of duality, it suffices to show that if \( \mathcal{R} - 4a_0 a_2 b_0 b_2 \geq 0 \) and \( B > 0 \), then \( Z(s) \in Z_b \) is realizable as the configuration in Fig. \( \text{IIa} \). Since \( \mathcal{R} - 4a_0 a_2 b_0 b_2 \geq 0 \), it follows that \( \Gamma_a \geq 4a_0 a_2 b_0 b_2 + b_0 b_2 \Delta_a = a_1^2 b_0 b_2 > 0 \), \( \Gamma_b \geq 4a_0 a_2 b_0 b_2 + a_0 a_2 \Delta_b = b^2 a_0 a_2 > 0 \), \( \mathrm{MR} + 2a_0 a_2 b_0 b_2(a_1 b_1 - 2M) \geq 4a_0 a_2 b_0 b_2 M + 2a_0 a_2 b_0 b_2(a_1 b_1 - 2M) = 2a_0 a_1 a_2 b_0 b_1 b_2 > 0 \), and the discriminant of \((\text{V.5})\) in \( R_1 \) as expressed in \((\text{V.13})\) is nonnegative. Hence, \((\text{V.5})\) in \( R_1 \) has one or two nonzero real roots, which must be positive.

Moreover, \( A C > 0 \) since \( \mathcal{R} = A C - B^2 > 0 \). Assume that \( A < 0 \), that is, \( a_0 b_1 < a_1 b_0 \).
Together with \( B > 0 \), that is, \( a_2 b_0 < a_0 b_2 \), one obtains that \((a_0 b_1)(a_2 b_0) < (a_1 b_0)(a_0 b_2)\), which is equivalent to \( C > 0 \). This contradicts the fact that \( A C > 0 \) as derived above. Therefore, it is only possible that \( A > 0 \) and \( C > 0 \), which implies that \( a_1 a_2 b_0 + a_0 C > 0 \) and \( a_2 b_0 b_1 + b_2 A > 0 \).
Replacing \( R_1 \) in \((\text{V.5})\) by \( a_0/b_0 \) and \( a_2/b_2 \) yields \( a_0^2 C^2 > 0 \) and \( a_2^2 A^2 > 0 \), respectively. Therefore, \( a_0 - b_0 R_1 \neq 0 \) and \( b_2 R_1 - a_2 \neq 0 \) provided that \( R_1 \) is the positive root of \((\text{V.5})\).

For the configuration in Fig. \( \text{IIa} \), let the values of the elements therein satisfy \((\text{V.4a})-(\text{V.4d})\), where \( R_1 \) is the positive root of \((\text{V.5})\). Let the value of \( k \) satisfy \((\text{V.8})\). It can be verified that \((\text{V.7a})-(\text{V.7b})\) hold. Hence, it follows that
\[
k^4 \mathcal{R} = ((R_1 + R_3)R_1 C_1 + (R_1 + R_2)R_3 C_2)^2 R_2^2 C_1 C_2,
\]
which implies that \( C_1 \) and \( C_2 \) must be simultaneously positive or negative. This means that 
\[
((a_1 a_2 b_0 + a_0 C) R_1 - a_0 a_1 a_2) \quad \text{and} \quad ((a_2 b_0 b_1 + b_2 A) - b_0 b_1 b_2 R_1)
\]
are simultaneously positive or negative. Assume that 
\[
(a_1 a_2 b_0 + a_0 C) R_1 - a_0 a_1 a_2 < 0 \quad \text{and} \quad (a_2 b_0 b_1 + b_2 A) - b_0 b_1 b_2 R_1 < 0.
\]
Then, one obtains 
\[
(a_1 a_2 b_0 + a_0 C)(a_2 b_0 b_1 + b_2 A) < a_0 a_1 a_2 b_0 b_2,
\]
which is equivalent to \( AC < 0 \). This contradicts \( AC > 0 \). Therefore, conditions (V.11) and (V.12) hold, which is equivalent to 
\[
(a_0 a_1 a_2)/(a_1 a_2 b_0 + a_0 C) < R_1 < (a_2 b_0 b_1 + b_2 A)/(b_0 b_1 b_2).
\]
Since \( (a_0 a_1 a_2)/(a_1 a_2 b_0 + a_0 C) > a_2/b_2 \) and \( (a_2 b_0 b_1 + b_2 A)/(b_0 b_1 b_2) < a_0/b_0 \) because of \( A > 0 \) and \( C > 0 \), it follows that conditions (V.10) and (V.9) must hold. Hence, the values of elements as expressed in (V.4a)–(V.5) must be positive and finite. As a conclusion, the given impedance \( Z(s) \) is realizable as the specified network.

**Lemma 6:** A biquadratic impedance \( Z(s) \in Z_b \) is realizable as the configuration in Fig. 2(a) with \( R_1 \neq R_2 \) if and only if there exists a positive root of
\[
b_0 E_a R_3^2 - (R + 2a_2 b_0 B) R_3 + a_2 D_a = 0 \tag{V.14}
\]
in \( R_3 \) such that
\[
b_2 R_3 - a_2 > 0, \tag{V.15a}
\]
\[
a_0 - b_0 R_3 > 0, \tag{V.15b}
\]
\[
b_0(C - 2a_2 b_1) CR_3 + a_2(a_0 a_2 b_1^2 - a_1^2 b_0 b_2) > 0. \tag{V.15c}
\]
Furthermore, if the condition is satisfied and if \( R_1 > R_2 \), then the values of the elements are expressed as
\[
R_1 = \frac{(a_0 - b_0 R_3)(1 + \sqrt{A})}{2b_0}, \tag{V.16a}
\]
\[
R_2 = \frac{(a_0 - b_0 R_3)(1 - \sqrt{A})}{2b_0}, \tag{V.16b}
\]
\[
C_1 = \frac{a_1 - b_1 R_2}{b_0(R_2 + R_3)(R_1 - R_2)}, \tag{V.16c}
\]
\[
C_2 = \frac{b_1 R_1 - a_1}{b_0(R_1 + R_3)(R_1 - R_2)}, \tag{V.16d}
\]
where
\[
\Lambda = 1 - \frac{4a_2 b_0 R_3}{(b_2 R_3 - a_2)(a_0 - b_0 R_3)}, \tag{V.17}
\]
and \( R_3 \) is the positive root of (V.14) satisfying (V.15a)–(V.15c).
\textbf{Proof: Necessity.} The impedance of the configuration in Fig. (a) is obtained as

\begin{equation}
Z(s) = \frac{a(s)}{b(s)},
\end{equation}

where

\begin{align*}
a(s) &= R_1 R_2 R_3 C_1 C_2 s^2 + ((R_2 + R_3) R_1 C_1 + (R_1 + R_3) R_2 C_2) s + (R_1 + R_2 + R_3) \\
b(s) &= (R_1 R_2 + R_2 R_3 + R_3 R_1) C_1 C_2 s^2 + ((R_2 + R_3) C_1 + (R_1 + R_3) C_2) s + 1.
\end{align*}

Then,

\begin{align*}
R_1 R_2 R_3 C_1 C_2 &= k a_2, \\
(R_2 + R_3) R_1 C_1 + (R_1 + R_3) R_2 C_2 &= k a_1, \\
R_1 + R_2 + R_3 &= k a_0, \\
(R_1 R_2 + R_2 R_3 + R_3 R_1) C_1 C_2 &= k b_2, \\
(R_2 + R_3) C_1 + (R_1 + R_3) C_2 &= k b_1, \\
1 &= k b_0.
\end{align*}

It is obvious that (V.19f) is equivalent to

\begin{equation}
k = \frac{1}{b_0}.
\end{equation}

Together with (V.19b) and (V.19c), $C_1$ and $C_2$ are solved as (V.16c) and (V.16d). From (V.19a) and (V.19d), one obtains

\begin{equation}
\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{b_2}{a_2}.
\end{equation}

As a result, condition (V.15a) is derived. Due to the symmetry of this configuration, one can assume that $R_1 > R_2$ without loss of generality. Therefore, from (V.19c), (V.20), and (V.21), $R_1$ and $R_2$ are solved as (V.16a) and (V.16b), which implies condition (V.15b). Substituting the expressions of (V.16a)–(V.16d) and (V.20) into $(R_1 R_2 R_3 C_1 C_2 - k a_2)$ gives

\begin{equation}
\frac{R_1 R_2 R_3 C_1 C_2 - k a_2}{b_0 (a_2 b_0 - \mathbb{B}) R_3 + a_2 D} = \frac{a_2}{b_0} - \frac{b_0 E_2 R_3^2 - (R + 2 a_2 b_0 \mathbb{B}) R_3 + a_2 D a}{b_0 (a_2 b_0 - \mathbb{B}) R_3 + a_2 D}.
\end{equation}

Then, one obtains (V.14). Since \( \Lambda \) must be nonnegative and $(b_0 b_2 R_3^2 + (2a_2 b_0 - \mathbb{B}) R_3 + a_0 a_2)$ cannot be zero, it follows that

\begin{equation}
b_0 b_2 R_3^2 + (2a_2 b_0 - \mathbb{B}) R_3 + a_0 a_2 < 0.
\end{equation}
Substituting the expressions of the roots of (V.14) into (V.23) yields \(( (R + 2a_2b_0B) + \sqrt{(R + 2a_2b_0B)^2 - 4a_2b_0D_a} \) \(2a_2b_1)C + 2a_2E_b(a_0a_2b_1^2 - a_1^2b_0b_2) > 0 \) or \(( (R + 2a_2b_0B) - \sqrt{(R + 2a_2b_0B)^2 - 4a_2b_0D_aE_b} ) (C - 2a_2b_1) + 2a_2E_b(a_0a_2b_1^2 - a_1^2b_0b_2) > 0 \), which is equivalent to condition (V.15c).

**Sufficiency.** Let the values of the elements satisfy (V.16a)–(V.16d), and \( R_3 \) be a positive root of (V.14) satisfying (V.15a) and (V.15b). Let \( k \) satisfy (V.20). Then, it can be verified that (V.19a)–(V.19f) are satisfied. Now, it suffices to prove that values of elements must be positive and finite. From the discussion in the necessity part, it is noted that condition (V.15c) yields \( \Lambda > 0 \), and conditions (V.15a) and (V.15b) imply \( R_1 > 0 \) and \( R_2 > 0 \). Besides, one has

\[
C_1C_2 = \frac{b_1^2(a_0 - b_0R_3)^2\Lambda - (A - a_1b_0 - b_0b_1R_3)^2}{(a_0 - b_0R_3)^2((a_0 + b_0R_3)^2 - (a_0 - b_0R_3)^2\Lambda)\Lambda}
= \frac{(b_2R_3 - a_2)(b_0b_1CR_3^2 - ACR_3 + a_1a_2\Lambda)}{(a_0 - b_0R_3)(b_0b_2R_3^2 + (2a_2b_0 - B)R_3 + a_0a_2)R_3^2B}
\]

(V.24)

where the third equality is (V.14). Since \( R_1 > R_2 \), it follows from (V.16c) and (V.16d) that \( C_1 > 0 \) and \( C_2 > 0 \).

The realizability condition for the configuration in Fig. 2a) with \( R_1 = R_2 \) is derived as follows.

**Lemma 7:** A biquadratic impedance \( Z(s) \in Z_b \) is realizable as the configuration in Fig. 2a) with \( R_1 = R_2 \) if and only if

\[
a_1b_1^2(2A - a_1b_0) - 4b_2A^2 \geq 0, \quad (V.25a)
\]

\[
a_1b_2(A - a_1b_0) = a_2b_1(2A - a_1b_0) > 0. \quad (V.25b)
\]

Furthermore, if the condition is satisfied, then the values of the elements are expressed as

\[
R_1 = \frac{a_1}{b_1}, \quad (V.26a)
\]

\[
R_2 = \frac{a_1}{b_1}, \quad (V.26b)
\]

\[
R_3 = \frac{A - a_1b_0}{b_0b_1}, \quad (V.26c)
\]

\[
C_1 = \frac{b_1^2b_2}{a_1(2A - a_1b_0)C_2}, \quad (V.26d)
\]

and \( C_2 \) is a positive root of

\[
a_1A(2A - a_1b_0)C_2^2 - a_1b_1^2(2A - a_1b_0)C_2 + b_1^2b_2A = 0. \quad (V.27)
\]
\textbf{Proof: Necessity.} Since it is assumed that \( R_1 = R_2 \), (V.19a)–(V.19f) become
\[
R_1^2 R_3 C_1 C_2 = k a_2, \quad (V.28a)
\]
\[
(R_1 + R_3) R_1 C_1 + (R_1 + R_3) R_1 C_2 = k a_1, \quad (V.28b)
\]
\[
2R_1 + R_3 = k a_0, \quad (V.28c)
\]
\[
(R_1 + 2R_3) R_1 C_1 C_2 = k b_2, \quad (V.28d)
\]
\[
(R_1 + R_3) C_1 + (R_1 + R_3) C_2 = k b_1, \quad (V.28e)
\]
\[
1 = k b_0. \quad (V.28f)
\]
It is obvious that (V.28f) is equivalent to
\[
k = \frac{1}{b_0}. \quad (V.29)
\]
From (V.28b) and (V.28c), it follows that \( R_1 \) satisfies (V.26a), implying that \( R_2 \) satisfies (V.26b). Then, substituting (V.26a) and (V.29) into (V.28c), one concludes that \( R_3 \) satisfies (V.26c), which implies \( A - a_1 b_0 > 0 \). Thus, it follows from (V.28d) and (V.28e) that \( C_1 \) satisfies (V.26d) and \( C_2 \) is a positive root of (V.27). Consequently, \( 2A - a_1 b_0 > 0 \). Since the discriminant of (V.27) should be nonnegative, one obtains condition (V.25a). Finally, substituting (V.26a)–(V.27) and (V.29) into (V.28a) yields
\[
a_1 b_2 (A - a_1 b_0) - a_2 b_1 (2A - a_1 b_0) = 0. \quad (V.25a)
\]
Since \( A - a_1 b_0 > 0 \), condition (V.25b) is obtained.

\textbf{Sufficiency.} Let the values of the elements satisfy (V.26a)–(V.27) and (V.29). Let \( k \) satisfy (V.29). \( A - a_1 b_0 > 0 \) and condition (V.25a) guarantee all the elements to be positive and finite. Since condition (V.25b) holds, it can be verified that (V.19a)–(V.19f) hold. Therefore, (V.18) is equivalent to (III.1).

Following Lemmas 6 and 7, one can derive the following theorem, where the realizability condition of the configuration in Fig. 2(b) follows from that of Fig. 2(a) based on the principle of duality [11].

\textbf{Theorem 2:} A biquadratic impedance \( Z(s) \in Z_0 \) is realizable as the configuration in Fig. 2(a) (resp. Fig. 2(b)) if and only if \( R > 0 \) and \( R - 4a_2 b_0 (a_1 b_1 - 2B) \geq 0 \) (resp. \( R > 0 \) and \( R - 4a_0 b_2 (a_1 b_1 + 2B) \geq 0 \)).
Proof: First, one can show that the condition of Lemma 6 is equivalent to

\begin{align*}
R > 0, & \quad B > 0, \quad R - 4a_2b_0(a_1b_1 - 2B) \geq 0, \quad (V.30) \\
2a_2b_0E_b < b_2(R + 2a_2b_0B) < 2a_0b_2E_b, \quad (V.31) \\
(R + 2a_2b_0B)(C - 2a_2b_1) + 2(a_0a_2b_0^2 - a_1^2b_0b_2)a_2E_b > 0. \quad (V.32)
\end{align*}

Suppose that the condition of Lemma 6 holds. The discriminant of (V.14) is obtained as \( \delta = R(R - 4a_2b_0(a_1b_1 - 2B)) \). By Lemma 2, one has \( R > 0 \). Together with \( \delta \geq 0 \), one concludes that \( R - 4a_2b_0(a_1b_1 - 2B) \geq 0 \) must hold. From (V.15a) and (V.15b), one obtains \( B > 0 \). Therefore, \( R > 0 \) indicates \( AC > B^2 > 0 \), which further implies that \( R + 2a_2b_0B > 0 \) and

\[
E_b = \frac{-b_2B^2 + b_1BC}{B} > \frac{-b_2AC + b_1BC}{B} = \frac{b_0C^2}{B} > 0. \quad (V.33)
\]

From [33] Ch.XV, Theorems 11 and 13, \( R > 0 \) yields \( \Delta_a > 0 \) and \( \Delta_b > 0 \). Substituting \( R_3 = a_2/b_2 \) and \( R_3 = a_0/b_0 \) into the left-hand side of (V.14) yields, respectively,

\[
\begin{align*}
b_0E_bR_3^2 - (R + 2a_2b_0B)R_3 + a_2D_a \big|_{R_3 = \frac{a_2}{b_2}} &= \frac{a^2_0\Delta_bB}{b^2_2} > 0, \quad (V.34) \\
b_0E_bR_3^2 - (R + 2a_2b_0B)R_3 + a_2D_a \big|_{R_3 = \frac{a_0}{b_0}} &= \Delta_aB > 0. \quad (V.35)
\end{align*}
\]

Thus, condition (V.31) holds. If \( C - 2a_2b_1 = 0 \), then condition (V.32) must hold because of (V.15c). Otherwise, substituting \( R_3 = -a_2(a_0a_2b_0^2 - a_1^2b_0b_2)/(b_0C(C - 2a_2b_1)) \) into the left-hand side of (V.14), one obtains

\[
\frac{a^2_0E_b(3a_1a_2b_0b_1 - 2a_0a_2b_2^2 + a_0a_1b_1b_2 - 2a_1^2b_0b_2)^2}{b_0C^2(C - 2a_2b_1)^2} \geq 0. \quad (V.36)
\]

Therefore, condition (V.32) is also satisfied. Conversely, following the above discussion, one can also prove that (V.30)–(V.32) yield the condition of Lemma 6.

By (IV.3), one converts (V.30)–(V.32) into \( W > 1 \) as well as

\[
\begin{align*}
R_c := -4U^2 - 4V^2 + 4UV(W + W^{-1}) - (W - W^{-1})^2 > 0, \quad (V.37) \\
- 4U^2 - 4V^2 + 4UV(W - 3W^{-1}) - (W - W^{-1})(W - 9W^{-1}) \geq 0, \quad (V.38) \\
- 4U^2 - 4V^2 + 4UV(W - W^{-1}) \\
- (W - W^{-1})(W - 5W^{-1}) + 8V^2W^{-2} > 0, \quad (V.39) \\
- 4V^2 + 4U^2 + 4UV(W - W^{-1}) - (W - W^{-1})(W + 3W^{-1}) > 0, \quad (V.40)
\end{align*}
\]
and
\[4W^{-1}\zeta(UW - V)(UW - 3V) + 8\lambda\varsigma(V^2 - U^2) > 0. \quad (V.41)\]

It is noted that condition (V.37) yields \(U > 1\) and \(V > 1\), and condition (V.38) yields \(W \geq 3\). If \(W = 3\), then \(U = V\) by (V.38), contradicting condition (V.41). Hence, \(W > 3\). Thus, Lemma 4 shows that conditions (V.37) and (V.38) with \(W \neq 3\) imply condition (V.41). If \(UV \leq (W - W^{-1})/2\), then conditions (V.39) and (V.40) hold: 
\[-4U^2 - 4V^2 + 4UV(W - W^{-1}) - (W - W^{-1})(W - 5W^{-1}) + 8V^22W^{-2} \geq (W - W^{-1})^2 - 8UVW^{-1} - (W - W^{-1})(W - 5W^{-1}) + 8V^22W^{-2} = 4W^{-1}(W - W^{-1}) - 8UVW^{-1} + 8V^22W^{-2} \geq 8V^22W^{-2} > 0\]
and 
\[-4V^2 + 4U^2 + 4UV(W - W^{-1}) - (W - W^{-1})(W + 3W^{-1}) \geq (W - W^{-1})^2 + 8U^2 - 8UVW^{-1} - (W - W^{-1})(W + 3W^{-1}) = -4W^{-1}(W - W^{-1}) - 8UVW^{-1} + 8U^2 \geq -8W^{-1}(W - W^{-1}) + 8U^2 > 8(U^2 - 1) > 0,\]
because of condition (V.37). Similarly, if \(UV > (W - W^{-1})/2\), then one can also show that conditions (V.39) and (V.40) hold because of condition (V.38). Therefore, the condition of \(W \neq 3\) and (V.37)–(V.38) together is equivalent to that of \(W > 1\) and (V.37)–(V.41). Moreover, through (IV.3), the condition of Lemma 7 is converted into
\[2WV - 3U > 0, \quad (V.42)\]
\[2WU^2 + 2WV^2 - UV(W^2 + 3) = 0, \quad (V.43)\]
\[U^2 + W^2V^2 + 3U^2V^2 - 2WUV^3 - 2WUV \leq 0. \quad (V.44)\]

When \(W = 3\) and conditions (V.37) and (V.38) hold, one obtains \(U = V = 2\sqrt{3}/3\), implying that (V.42)–(V.44) hold. The proof is completed if one can show that conditions (V.42)–(V.44) can imply conditions (V.37) and (V.38). Indeed, by the following transformation
\[U = x \cos \frac{\pi}{4} - y \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}(x - y), \quad (V.45)\]
\[V = x \sin \frac{\pi}{4} + y \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}(x + y), \quad (V.46)\]

conditions (V.42)–(V.44) are further converted into \((2W - 3)x + (2W + 3)y > 0\), \((W + 1)(W + 3)y^2 - (W - 1)(W - 3)x^2 = 0\), and \((3 - 2W)x^4 - 4Wx^3y - 6x^2y^2 + 4Wxy^3 + (2W + 3)y^4 + 2((W - 1)x + (W + 1)y)^2 \leq 0\) with \(x > 0\), which are in term equivalent to
\[y = \pm \sqrt{\frac{(W - 1)(W - 3)}{(W + 1)(W + 3)}}x, \quad (V.47)\]
\[x \geq \frac{W + 1}{2W} \sqrt{\frac{(W - 1)(W + 3)}{2}}. \quad (V.48)\]
Next, conditions (V.37) and (V.38) are converted into

\begin{align}
2W^{-1}(W - 1)^2x^2 - 2W^{-1}(W + 1)^2y^2 - W^{-2}(W^2 - 1)^2 > 0, \\
2W^{-1}(W+1)(W - 3)x^2 - 2W^{-1}(W + 3)(W - 1)y^2 - W^{-2}(W^2 - 9)(W^2 - 1) \geq 0.
\end{align}

Substituting (V.47) into (III.2) and (III.3) gives

\begin{align}
8(W - 1)x^2 - \frac{(W + 1)^2(W - 1)^2}{W^2} \geq 0, \\
8(W - 3)x^2 - \frac{(W^2 - 1)(W^2 - 9)}{W^2} \geq 0,
\end{align}

respectively. It is obvious that that conditions (V.47) and (V.48) can imply conditions (III.2) and (III.3).

Therefore, \(Z_c(s) \in \mathcal{Z}_b\) is realizable as the configuration in Fig. 2(a) if and only if conditions (V.37) and (V.38) hold. Through (IV.2), the condition for \(Z(s) \in \mathcal{Z}_b\) is obtained as stated in the theorem.

Theorem 3: A biquadratic impedance \(Z(s) \in \mathcal{Z}_b\) is realizable as a five-element bridge network containing two reactive elements of the same type if and only if \(R > 0\) and at least one of \((R - 4a_0b_2(a_1b_1 + 2B)), (R - 4a_2b_0(a_1b_1 - 2B)),\) and \((R - 4a_0a_2b_0b_2)\) is nonnegative.

Proof: Combining Lemma 5 and Theorems 1 and 2 yield the result.

C. Five-Element Bridge Networks with One Inductor and One Capacitor

Lemma 8: A biquadratic impedance \(Z(s) \in \mathcal{Z}_b\) is realizable as a five-element bridge network with two reactive elements of different types if and only if \(Z(s)\) is the impedance of one of configurations in Figs. 3-5.

Proof: This lemma is proved by a simple enumeration.

The realizability condition of Fig. 3 has already been established in [16], as follows.

Lemma 9: [16] A biquadratic impedance \(Z(s) \in \mathcal{Z}_b\) is realizable as the configuration in Fig. 3(a) (resp. Fig. 3(b)) if and only if \(B < 0, R - 4a_0b_2(a_1b_1 + 2B) \leq 0\) (resp. \(B > 0, R - 4a_2b_0(a_1b_1 - 2B) \leq 0\), and signs of \(D_b, E_a,\) and \((R - 2a_0b_2B)\) (resp. \(D_a, E_b,\) and \((R +..."}
Fig. 3. The two-reactive five-element bridge configurations containing different types of reactive elements, which are respectively supported by two one-terminal-pair labeled graphs $\mathcal{N}_3$ and Dual($\mathcal{N}_3$), where (a) is No. 70 configuration in [19], and (b) is No. 95 configuration in [19].

Fig. 4. The two-reactive five-element bridge configurations containing different types of reactive elements, which are respectively supported by two one-terminal-pair labeled graphs $\mathcal{N}_4$ and Dual($\mathcal{N}_4$), where (a) is No. 105 configuration in [19], and (b) is No. 107 configuration in [19].

$2a_2b_0B$) are not all the same. If $R - 2a_0b_2B = 0$ (resp. $R + 2a_2b_0B = 0$), then $D_bE_a < 0$ (resp. $D_aE_b < 0$).

By the star-mesh transformation [29], it can be verified that the configuration in Fig. 4(a) is equivalent to that in Fig. 6. The element values for configurations in Figs. 4–6 have been listed in [19], without any detail of derivation.

**Lemma 10:** A biquadratic impedance $Z(s) \in \mathcal{Z}_b$ is realizable as the configuration in Fig. 6 if and only if $R < 0$ and one of the following two conditions is satisfied:

Fig. 5. The two-reactive five-element bridge configuration containing different types of reactive elements, which is supported by a one-terminal-pair labeled graph $\mathcal{N}_5$ satisfying $\mathcal{N}_5 = \text{Dual}(\mathcal{N}_5)$, and it is No. 108 configuration in [19].
1. $\Gamma_a < 0$, either $D_b > 0$ for $B < 0$ or $E_b > 0$ for $B > 0$, and the signs of $\Delta_b$, $-\Delta_{ab}$, and $\Gamma_a$ are not all the same (when one of them is zero, the other two are nonzero and have different signs);
2. $\Gamma_a > 0$, $\Delta_b > 0$, $\Delta_{ab} > 0$, and either $D_b + b_0 B < 0$ for $B < 0$ or $E_b - b_2 B < 0$ for $B > 0$.

Furthermore, if the above condition is satisfied, then the values of the elements are expressed as

\[
R_1 = \frac{a_2 - b_2 R_3}{b_2}, \quad (V.53a)
\]
\[
R_2 = \frac{a_0 - b_0 R_3}{b_0}, \quad (V.53b)
\]
\[
L_1 = \frac{M - 2 b_0 b_2 R_3}{b_0 b_1}, \quad (V.53c)
\]
\[
C_1 = \frac{b_1 b_2}{M - 2 b_0 b_2 R_3}, \quad (V.53d)
\]

and $R_3$ is a positive root of

\[
b_0 b_2 \Delta_a R_3^2 - 2 b_0 b_2 \Delta_{ab} R_3 + \Gamma_a = 0, \quad (V.54)
\]
satisfying

\[
R_3 < \min\{a_2/b_2, a_0/b_0\}. \quad (V.55)
\]

**Proof: Necessity.** The impedance of the configuration in Fig. 6 is obtained as

\[
Z(s) = \frac{a(s)}{b(s)}, \quad (V.56)
\]

where $a(s) = (R_1 + R_3)L_1 C_1 s^2 + ((R_1 R_3 + R_2 R_3 + R_1 R_2)C_1 + L_1)s + (R_2 + R_3)$ and $b(s) =
\[ L_1 C_1 s^2 + (R_1 + R_2) C_1 s + 1. \] Thus,

\[ (R_1 + R_3) L_1 C_1 = k a_2, \quad \text{(V.57a)} \]
\[ (R_1 R_3 + R_2 R_3 + R_1 R_2) C_1 + L_1 = k a_1, \quad \text{(V.57b)} \]
\[ R_2 + R_3 = k a_0, \quad \text{(V.57c)} \]
\[ L_1 C_1 = k b_2, \quad \text{(V.57d)} \]
\[ (R_1 + R_2) C_1 = k b_1, \quad \text{(V.57e)} \]
\[ 1 = k b_0. \quad \text{(V.57f)} \]

From (V.57f), one obtains

\[ k = \frac{1}{b_0}. \quad \text{(V.58)} \]

Based on (V.57a) and (V.57d), one concludes that \( R_1 \) satisfies (V.53a). From (V.57c), it follows that \( R_2 \) satisfies (V.53b). Therefore, condition (V.55) must hold. Substituting (V.53a), (V.53b), and (V.58) into (V.57c) yields the value of \( C_1 \) as (V.53d). As a result, the value of \( L_1 \) is obtained from (V.57d) as (V.53c). Finally, substituting (V.53a)–(V.53d) into (V.57b) yields (V.54). It follows that the discriminant of (V.54) in \( R_3 \) is

\[ \delta = (2 b_0 b_2 \Delta_{ab})^2 - 4 b_0 b_2 \Delta_b \Gamma_a = -4 b_0 b_2^2 b_2 R_3. \quad \text{(V.59)} \]

Since (V.54) must have at least one positive root, one concludes that \( R < 0 \), and at most one of \( \Delta_b \), \( \Delta_{ab} \), and \( \Gamma_a \) is zero. Substituting \( R_3 = a_0/b_0 \), \( R_3 = a_2/b_2 \), and \( R_3 = (a_0/b_0 + a_2/b_2)/2 = M/(2b_0b_2) \) into the left-hand side of (V.54), one obtains, respectively,

\[ b_0 b_2 \Delta_b R_3^2 - 2 b_0 b_2 \Delta_{ab} R_3 + \Gamma_a \big|_{R_3 = a_0/b_0} = -B D b_0, \quad \text{(V.60)} \]
\[ b_0 b_2 \Delta_b R_3^2 - 2 b_0 b_2 \Delta_{ab} R_3 + \Gamma_a \big|_{R_3 = a_2/b_2} = B E b_2, \quad \text{(V.61)} \]
\[ b_0 b_2 \Delta_b R_3^2 - 2 b_0 b_2 \Delta_{ab} R_3 + \Gamma_a \big|_{R_3 = M/(2b_0b_2)} = \frac{b_2^2 B^2}{4b_0 b_2} > 0. \quad \text{(V.62)} \]

Since the condition of Lemma 1 does not hold, \( \Gamma_a \neq 0 \). When \( \Gamma_a < 0 \), based on (V.62) it follows that (V.54) has only one positive root in \( R_3 \) such that (V.55) holds. Therefore, the signs of \( \Delta_b \), \( -\Delta_{ab} \), and \( \Gamma_a \) are not all the same (when one of them is zero, the other two are nonzero and have different signs). Moreover, if \( B < 0 \), then \( a_0/b_0 < M/(2b_0b_2) < a_2/b_2 \), implying that
$D_b > 0$ to guarantee \((V.60)\) to be positive; if $B > 0$, then $a_0/b_0 > \mathcal{M}/(2b_0b_2) > a_2/b_2$, implying that $E_b > 0$ to guarantee \((V.61)\) to be positive.

When $\Gamma_a > 0$, based on \((V.62)\) it follows that $\Delta_b > 0$ and $-\Delta_{ab} < 0$. If $B < 0$, then $a_0/b_0 < \mathcal{M}/(2b_0b_2) < a_2/b_2$. Therefore, in either the case when \((V.60)\) is negative or the case when \((V.60)\) is nonnegative, one has $\Delta_{ab}/\Delta_b < a_0/b_0$ holds. The above two cases correspond to $-b_1A < -b_0B$ and $-b_0B \leq -b_1A < -2b_0B$, respectively. Hence, combining them yields $-b_1A < -2b_0B$, which is equivalent to $D_b + b_0B < 0$. Similarly, if $B > 0$, then $E_b - b_2B < 0$.

**Sufficiency.** Let the values of the elements in Fig. 6 satisfy \((V.53a)-(V.53d)\), and $R_3$ be a positive root of \((V.54)\) satisfying \((V.55)\). Then, $a_2 - b_2R_3 > 0$, $a_0 - b_0R_3 > 0$, and $\mathcal{M} - 2b_0b_2R_3 > 0$. Letting $k$ satisfy \((V.58)\), it can be verified that \((V.57a)-(V.57f)\) hold. $R < 0$ implies that the discriminant of \((V.54)\) as expressed in \((V.59)\) is positive.

If condition 1 is satisfied, then as discussed in the necessity part there exists a unique positive root of \((V.54)\) in terms of $R_3$ such that \((V.55)\) holds.

If condition 2 holds, and either $-b_1A < -b_0B$ for $B < 0$ or $-b_2B < -b_1C$ for $B > 0$, then it can be proved that there exists a unique positive root for \((V.54)\) in terms of $R_3$ such that \((V.55)\) holds.

If condition 2 holds, and either $-b_0B \leq -b_1A < -2b_0B$ for $B < 0$ or $-2b_2B < -b_1C \leq -b_2B$ for $B > 0$, then there are two positive roots for \((V.54)\) in terms of $R_3$ such that \((V.55)\) holds.

As a conclusion, the values of elements must be positive and finite. The given impedance $Z(s)$ is realizable as the specified network.

The values of elements in Fig. 4(a) can be obtained from those in Fig. 6 via the following transformation: $R_P/R_2 \rightarrow R_1$, $R_P/R_3 \rightarrow R_2$, $R_P/R_1 \rightarrow R_3$, $C_1 \rightarrow C_1$, and $L_1 \rightarrow L_1$, where $R_P = R_1R_2 + R_2R_3 + R_3R_1$.

Since the realizability condition of the configuration in Fig. 4(a) is equivalent to that of Lemma 10 a necessary and sufficient condition for the realizability of the configurations in Fig. 4 is obtained as follows.

**Theorem 4:** A biquadratic impedance $Z(s) \in \mathcal{Z}_b$ is realizable as one of the configurations in Fig. 4 if and only if $R < 0$ and one of the following three conditions is satisfied:

1. $\Gamma_a < 0$, either $D_b > 0$ for $B < 0$ or $E_b > 0$ for $B > 0$, and the signs of $\Delta_b$, $-\Delta_{ab}$, and $\Gamma_a$ are not all the same (when one of them is zero, the other two are nonzero and have
different signs);

2. \( \Gamma_b < 0 \), either \( D_a > 0 \) for \( E > 0 \) or \( E_a > 0 \) for \( E < 0 \), and the signs of \( \Delta_a, -\Delta_{ab} \), and \( \Gamma_b \) are not all the same (when one of them is zero, the other two are nonzero and have different signs);

3. \( \Gamma_a > 0 \), \( \Gamma_b > 0 \), and \( \Delta_{ab} > 0 \).

**Proof:** Conditions 1 and 2 can be obtained from Lemma 10 based on the principle of duality [11]. To obtain condition 3, it suffices to show that

\[
\Gamma_{ac} > 0, \quad \Gamma_{bc} > 0, \quad \Delta_{abc} > 0
\]

is equivalent to the union of the following two conditions:

a. \( \Gamma_{ac} > 0 \), \( V > 1 \), \( \Delta_{abc} > 0 \), and either \( 2UV - 2WV^2 + (W - W^{-1}) < 0 \) for \( W < 1 \) or \( 2UV - 2W^{-1}V^2 - (W - W^{-1}) < 0 \) for \( W > 1 \);

b. \( \Gamma_{bc} > 0 \), \( U > 1 \), \( \Delta_{abc} > 0 \), and either \( 2UV - 2WU^2 + (W - W^{-1}) < 0 \) for \( W < 1 \) or \( 2UV - 2W^{-1}U^2 - (W - W^{-1}) < 0 \) for \( W > 1 \).

First, one can show that condition a or b implies (V.63). Without loss of generality, assume that \( W > 1 \). Then, one obtains \( W + W^{-1} < 2UV < (W - W^{-1}) + 2V^2W^{-1} \) from condition a. Hence, it follows from condition a that \( \Gamma_{bc} = -4U^2 + 4UV(W + W^{-1}) - (W + W^{-1})^2 = -V^{-2}(2UV - V(V - \sqrt{V^2 - 1})(W + W^{-1}))(2UV - V(V + \sqrt{V^2 - 1})(W + W^{-1})) > 0 \), since \( (W + W^{-1}) - V(V - \sqrt{V^2 - 1})(W + W^{-1}) = \sqrt{V^2 - 1}(V - \sqrt{V^2 - 1})(W + W^{-1}) > 0 \) and \( (W - W^{-1}) + 2V^2W^{-1} - V(V + \sqrt{V^2 - 1})(W + W^{-1}) = -(W - W^{-1})(V^2 - 1) - V\sqrt{V^2 - 1}(W + W^{-1}) < 0 \). Therefore, condition a yields condition (V.63). Similarly, condition b implies \( \Gamma_{ac} > 0 \), which also yields condition (V.63). In addition, the case of \( W < 1 \) can be similarly proved.

Now, it remains to show that condition (V.63) implies condition a or b. Assume that \( W > 1 \). Since \( \Gamma_{ac} > 0 \) and \( \Gamma_{bc} > 0 \) can yield respectively \( U > 1 \) and \( V > 1 \), if \( 2UV - 2W^{-1}V^2 - (W - W^{-1}) < 0 \) then condition a holds. Otherwise, one obtains \( U - \sqrt{U^2 - 2W^{-1}(W - W^{-1})} \leq 2VW^{-1} \leq U + \sqrt{U^2 - 2W^{-1}(W - W^{-1})} \). It can be verified that \( U(W + W^{-1}) + (W - W^{-1})\sqrt{U^2 - 1} - W(U + \sqrt{U^2 - 2W^{-1}(W - W^{-1})}) > 0 \). Together with \( R_c < 0 \), one has \( V < (U(W + W^{-1}) - (W - W^{-1})\sqrt{U^2 - 1})/2 \), which implies that \( 2UV - 2W^{-1}U^2 - (W - W^{-1}) < U^2(W + W^{-1}) - (W - W^{-1})U\sqrt{U^2 - 1} - 2W^{-1}U^2 - (W - W^{-1}) = (W - W^{-1})\sqrt{U^2 - 1} - 1 - U < 0 \). Hence, condition b is obtained. The case of \( W < 1 \) can be similarly proved.
Theorem 5: A biquadratic impedance $Z(s) \in \mathbb{Z}_b$ is realizable as the configuration in Fig. 5 if and only if $R < 0$ and the signs of $\Gamma_a$, $\Gamma_b$, and $(MR + 2a_0a_2b_0b_2\Delta_{ab})$ are not all the same (when $MR + 2a_0a_2b_0b_2\Delta_{ab} = 0$, $\Gamma_a\Gamma_b < 0$). Furthermore, if the above condition holds, then the values of the elements are expressed as

$$R_1 = \frac{a_0}{b_0}, \quad (V.64a)$$

$$R_3 = \frac{a_2}{b_2}, \quad (V.64b)$$

$$L_1 = \frac{(a_1a_2b_0 + a_0C)R_2 + a_0a_1a_2}{(b_0R_2 + a_0)M}, \quad (V.64c)$$

$$C_1 = \frac{b_0b_1b_2R_2 + (a_0b_1b_2 - b_0C)}{(b_0R_2 + a_0)M}, \quad (V.64d)$$

and $R_2$ is a positive root of

$$b_0b_2\Gamma_aR_2^2 + (MR + 2a_0a_2b_0b_2\Delta_{ab})R_2 + a_0a_2\Gamma_b = 0, \quad (V.65)$$

satisfying

$$b_0b_1b_2R_2 + (a_0b_1b_2 - b_0C) > 0, \quad (V.66a)$$

$$(a_1a_2b_0 + a_0C)R_2 + a_0a_1a_2 > 0. \quad (V.66b)$$

Proof: Necessity. The impedance of the configuration in Fig. 5 is given by

$$Z(s) = \frac{a(s)}{b(s)}, \quad (V.67)$$

where $a(s) = (R_1 + R_2)R_3L_1C_1s^2 + (R_1R_2R_3C_1 + (R_1 + R_2 + R_3)L_1)s + (R_2 + R_3)R_1$ and $b(s) = (R_1 + R_2)L_1C_1s^2 + ((R_1R_2 + R_2R_3 + R_1R_3)C_1 + L_1)s + R_2 + R_3$. Thus,

$$(R_1 + R_2)R_3L_1C_1 = ka_2, \quad (V.68a)$$

$$R_1R_2R_3C_1 + (R_1 + R_2 + R_3)L_1 = ka_1, \quad (V.68b)$$

$$(R_1 + R_2)L_1C_1 = ka_0, \quad (V.68c)$$

$$(R_2 + R_3)R_1 = kb_0, \quad (V.68d)$$

$$(R_1R_2 + R_2R_3 + R_1R_3)C_1 + L_1 = kb_1, \quad (V.68e)$$

$$R_2 + R_3 = kb_0. \quad (V.68f)$$
From (V.68a) and (V.68d), it follows that \( R_3 \) satisfies (V.64b). From (V.68c) and (V.68f), it follows that \( R_1 \) satisfies (V.64a). Substituting (V.64b) into (V.68f) yields
\[
k = \frac{b_2 R_2 + a_2}{b_0 b_2}. \tag{V.69}
\]

Therefore, \( L_1 \) and \( C_1 \) can be solved from (V.68b) and (V.68e) as (V.64c) and (V.64d). The assumption that all the values of the elements are positive and finite implies conditions (V.66a) and (V.66b). Substituting (V.64a)–(V.64d) into (V.68d) yields (V.65). The discriminant of (V.65) in terms of \( R_2 \) is obtained as
\[
\delta = M^2 R (R - 4a_0 a_2 b_0 b_2). \tag{V.70}
\]

Since the discriminant must be nonnegative to guarantee the existence of real roots, together with Lemma 2 one has \( R < 0 \), implying that \( \delta > 0 \) and at most one of \( \Gamma_a, \Gamma_b, \) and \( (MR + 2a_0 a_2 b_0 b_2 \Delta_{ab}) \) is zero. If one of them is zero, then it is only possible that \( MR + 2a_0 a_2 b_0 b_2 \Delta_{ab} = 0 \) and \( \Gamma_a \Gamma_b < 0 \). If none of them is zero, then it follows that the signs of them cannot be the same to guarantee the existence of the positive root.

**Sufficiency.** Let the values of the elements in Fig. 5 be (V.64a)–(V.64d), and \( R_2 \) be a positive root of (V.65) satisfying conditions (V.66a) and (V.66b). Let \( k \) satisfy (V.69). It can be verified that (V.68a)–(V.68f) hold, implying that (V.67) is equivalent to (III.1).

It suffices to show that (V.65) always has a positive root, such that \( R_1, R_3, L_1, C_1 \) expressed as (V.64a)–(V.64d) are positive. Since \( R_1 \) and \( R_3 \) expressed as (V.64a) and (V.64b) are obviously positive, one only needs to discuss \( L_1 \) and \( C_1 \).

It is not difficult to see that if the signs of \( \Gamma_a, \Gamma_b, \) and \( (MR + 2a_0 a_2 b_0 b_2 \Delta_{ab}) \) satisfy the given conditions, then (V.65) must have at least one positive root, since \( R < 0 \) implies that the discriminant of (V.65) shown in (V.70) is always positive. Furthermore, \(-k^4 R = (R_1 + R_2)^2(R_2 + R_3)^2(R_1 R_3 C_1 - L_1)^2 L_1 C_1.\) Together with \( R < 0 \), it follows that \( L_1 \) and \( C_1 \) are both positive or negative. Hence, \( \chi_1 \chi_2 > 0 \), where \( \chi_1 := b_0 b_1 b_2 R_2 + (a_0 b_1 b_2 - b_0 b_2) \) and \( \chi_2 := (a_1 a_2 b_0 + a_0 C) R_2 + a_0 a_1 a_2. \) Assume that \( \chi_1 < 0 \) and \( \chi_2 < 0 \). Then, by letting \( a_0 a_2 \chi_1 + b_0 b_2 \chi_2 \), one obtains \( a_0 a_2 \chi_1 + b_0 b_2 \chi_2 = (a_1 b_0 b_2 R_2 + a_0 a_2 b_1)M < 0. \) This contradicts the assumption that all the coefficients are positive. Hence, \( \chi_1 > 0 \) and \( \chi_2 > 0 \), suggesting that \( L_1 > 0 \) and \( C_1 > 0 \).

Combining Lemma 9, Theorems 4 and 5, one obtains the following result.
Theorem 6: A biquadratic impedance $Z(s) \in \mathcal{Z}_b$ is realizable as a five-element bridge network containing one inductor and one capacitor if and only if $R < 0$, and $Z(s)$ is regular or satisfies the condition of Lemma 9.

Proof: Necessity. By Lemma 2, $R < 0$. It is shown in [16] that biquadratic impedances that can realize configurations in Figs. 4 and 5 must be regular. Based on Lemma 8, the necessity part is proved.

Sufficiency. Based on Lemma 9, one only needs to consider the case when $R < 0$ and $Z(s)$ is regular, which means that the corresponding $Z_c(s)$ is regular. Assuming that the condition of Lemma 1 does not hold, $Z_c(s)$ is regular if and only if (1) $\lambda_c > 0$ or $\lambda_c^* > 0$ when $W < 1$; (2) $\lambda_c^* > 0$ or $\lambda_c^{**} > 0$ when $W > 1$. It suffices to show that if $R_c < 0$ and $Z_c(s) \in \mathcal{Z}_b$ is regular then $Z_c(s)$ is realizable as one of the configurations in Figs. 4 and 5.

Case 1: $\Gamma_{ac} < 0$ and $\Gamma_{bc} < 0$. If $U < 1$ and $V < 1$, then $\Delta_{ab_c} = 4UV - 2(W + W^{-1}) < 0$. Suppose that $U \geq 1$. Then, $\Gamma_{ac} < 0$ implies that $V < (W + W^{-1})(U - \sqrt{U^2 - 1})/2$ or $V > (W + W^{-1})(U + \sqrt{U^2 - 1})/2$, and $\Gamma_{bc} < 0$ implies $V < (4U^2 + (W + W^{-1})^2)/(4U(W + W^{-1}))$.

Since $2U(W + W^{-1})^2(U + \sqrt{U^2 - 1}) - ((W + W^{-1})^2 + 4U^2) > 4(2U(U + \sqrt{U^2 - 1}) - 1) - 4U^2 = 4(U^2 - 1) + 8U\sqrt{U^2 - 1} > 0$, it is only possible that $V < (W + W^{-1})(U - \sqrt{U^2 - 1})/2$.

Hence, $\Delta_{ab_c} = 4UV - 2(W + W^{-1}) < 2(W + W^{-1})(U^2 - U\sqrt{U^2 - 1}) - 2(W + W^{-1}) = 2(W + W^{-1})\sqrt{U^2 - 1}(\sqrt{U^2 - 1} - U) < 0$. Making use of Theorem 4 and (IV.3), $Z_c(s)$ is realizable as one of the configurations in Fig. 4.

Case 2: Only one of $\Gamma_{ac}$ and $\Gamma_{bc}$ is negative. By Theorem 5 and (IV.3), $Z_c(s)$ is realizable as the configuration in Fig. 5.

Case 3: $\Gamma_{ac} > 0$ and $\Gamma_{bc} > 0$. One obtains that $U > 1$ and $V > 1$. If $\Delta_{ab_c} > 0$, then $Z_c(s)$ is realizable as one of the configurations in Fig. 4 by Theorem 4 and (IV.3). If $\Delta_{ab_c} \leq 0$, then $R_c = -4U^2 - 4V^2 + 4UV(W + W^{-1}) - (W - W^{-1})^2 < 0$ yields $-(W + W^{-1})^3 + 4UV(W + W^{-1})^2 - 4(U^2 + V^2)(W + W^{-1}) + 8UV < -(W + W^{-1})^3 + 4UV(W + W^{-1})^2 + ((W - W^{-1})^2 - 4UV(W + W^{-1}))(W + W^{-1}) + 8UV = -4(W + W^{-1}) + 8UV = 2\Delta_{ab_c} \leq 0$. Therefore, $Z_c(s)$ is realizable as the configuration in Fig. 5 based on Theorem 5 and (IV.3).

Corollary 1: A biquadratic impedance $Z(s) \in \mathcal{Z}_b$ with $R < 0$ is regular if and only if it is realizable as one of the configurations in Figs. 4 and 5.

1 A necessary and sufficient condition for a biquadratic impedance to be regular is presented in [16] Lemma 5.
Proof: This corollary is obtained based on the proof of Theorem 5.

Corollary 2: A biquadratic impedance, which can be realized as an irreducible five-element series-parallel network containing one inductor and one capacitor, can always be realized as a five-element bridge network containing one inductor and one capacitor.

Proof: It has been proved in [16] that if the biquadratic impedance \( Z(s) \) in the form of (III.1) is realizable as a five-element series-parallel network containing one inductor and one capacitor, then \( Z(s) \) must be regular. Since the network is irreducible, it follows that \( Z(s) \in \mathcal{Z}_b \) and \( R < 0 \). Hence, the conclusion directly follows from Theorem 5.

Corollary 3: If a biquadratic impedance \( Z(s) \in \mathcal{Z}_b \) can be realized as a network containing one inductor, one capacitor, and at least three resistors, then the network will always be equivalent to a five-element bridge network containing one inductor and one capacitor.

Proof: It can be proved by Theorem 6 and a theorem of Reichert [24].

D. Summary and Notes

Theorem 7: A biquadratic impedance \( Z(s) \in \mathcal{Z}_b \) is realizable as a two-reactive five-element bridge network if and only if one of the following two conditions holds:

1. \( R > 0 \), and at least one of \((R - 4a_0b_2(a_1b_1 + 2B)), (R - 4a_2b_0(a_1b_1 - 2B))) \) and \((R - 4a_0a_2b_0b_2)\) is nonnegative;
2. \( R < 0 \), and \( Z(s) \) is regular or satisfies the condition of Lemma 5.

Proof: Combining Theorems 3 and 5 leads to the conclusion.

Now, a corresponding result for general five-element bridge networks directly follows.

Theorem 8: A biquadratic impedance \( Z(s) \in \mathcal{Z}_b \) is realizable as a five-element bridge network if and only if \( Z(s) \) satisfies the condition of Theorem 7 or is realizable as a configuration in Fig. 7.

Proof: Sufficiency. The sufficiency part is obvious.

Necessity. Since \( R \neq 0 \), the McMillan degree (see [1] Chapter 3.6]) of \( Z(s) \in \mathcal{Z}_b \) satisfies \( \delta(Z(s)) = 2 \). Since the McMillan degree is equal to the minimal number of reactive elements for realizations of \( Z(s) \) [1 pg. 370], there must exist at least two reactive elements. Since

2 An irreducible network means that it can never become equivalent to the one containing fewer elements.

3 A necessary and sufficient condition for the realizability of Fig. 7 is presented in [16] Theorem 7.
$Z(s) \in \mathcal{Z}_b$ has no pole or zero on $j\mathbb{R} \cup \infty$, the number of reactive elements cannot be five. If the number of reactive elements is four (only one resistor), then $Z(0) = Z(\infty)$, which is equal to the value of the resistor. This means that $B = 0$, contradicting the assumption that the condition of Lemma 1 does not hold. Therefore, the number of reactive elements must be two or three. If the number of reactive elements is two, then the condition of Theorem 7 holds. If the number of reactive elements is three, then their types cannot be the same by Lemma 3. Furthermore, by the discussion in [20], the network is equivalent to either a five-element series-parallel structure containing one inductor and one capacitor or a configuration in Fig. 7. By Corollary 2, the theorem is proved.

The condition of Theorem 7 can be converted into a condition in terms of the canonical form $Z_c(s) \in \mathcal{Z}_{b_c}$ through (IV.3), which is shown on the $U$–$V$ plane in Fig. 8 when $W = 2$. If $(U, V)$ is within the shaded region (excluding the inside curves $\Gamma_{a_c} \Gamma_{b_c} = 0$ and $\lambda_c^* \lambda_{c^\dagger} = 0$), then $Z_c(s)$ is realizable as two-reactive five-element bridge networks. The hatched region ($\sigma_c < 0$) represents the non-positive-realness case, where $Z_c(s)$ cannot be realized as a passive network.

Some important notes are listed as follows.

Remark 1: Ladenheim [19] only listed element values for configurations in Figs. 1–5 without showing any detail of derivation. Neither explicit conditions in terms of the coefficients of $Z(s)$ only (like the condition of Theorem 1) nor a complete set of conditions are given in [19]. Besides, the special case when $R_1 = R_2$ for Fig. 2(a) (No. 86 configuration in [19]) is not discussed in [19].

Remark 2: In [16], necessary and sufficient conditions are derived only for the realizability of bridge networks that sometimes cannot realize regular biquadratic impedances. In the present paper, through discussing other two-reactive five-element bridge networks and by combining
Fig. 8. The \( U-V \) plane showing a necessary and sufficient condition for any \( Z_c(s) \in \mathcal{Z}_b \) to be realizable as the two-reactive five-element bridge network, when \( W = 2 \).

their conditions, a complete result is obtained (Theorem 7). Together with previous results in [20], the result has been further extended to the general five-element bridge case (Theorem 8).

Remark 3: Corollary 1 shows that the regular biquadratic impedance \( Z(s) \in \mathcal{Z}_b \) with \( R < 0 \) is always realizable as one of the three five-element bridge configurations in Figs. 4 and 5. It was shown in [16] that a group of four five-element series-parallel networks can be used to realize such a function. Therefore, the number of configurations covering the case of regularity with \( R < 0 \) is reduced by one in the present paper.

Remark 4: The logical path of lemmas and theorems in this paper is as follows. Theorem 8 follows from Theorem 7 together with some results in the existing literature. Theorem 7 is the combination of Theorems 3 and 6, where Theorem 3 follows from Lemma 5 and Theorems 1 and 2 and Theorem 6 is derived from Lemmas 2, 8, and 9 and Theorems 4 and 5. The proof of Lemma 5 makes use of Lemma 3, the proof of Theorem 1 makes use of Lemma 2, the proof of Theorem 2 makes use of Lemmas 2, 4, 6, and 7 and the proof of Theorem 4 makes use of Lemma 10.

Remark 5: The configurations of this paper can be connected to \( RC \) low-pass filters. By appropriately setting the values of components in the filters, the frequency responses of the resulting networks can be adjusted, in order to better reject high-frequency noises and guarantee the low-frequency responses to approximate to those of the original configurations. Applying passive
network synthesis to the circuits with filter implementations needs to be further investigated.

VI. NUMERICAL EXAMPLES

**Example 1:** As shown in [31], the function

\[
Z_{e,2}^{sy} = \frac{s^2 + 2.171 \times 10^8 s + 4.824 \times 10^9}{1.632s^2 + 1.575 \times 10^8s + 2.838 \times 10^8}
\]  

(VI.1)

is the impedance of an external circuit in the machatronic suspension system, which optimizes the settling time at a certain velocity range and is realizable as a five-element series-parallel configuration in [31, Fig. 18]. Since \( R > 0 \), \( R - 4a_0a_2b_0b_2 > 0 \), and \( R - 4a_2b_0(a_1b_1 - 2B) > 0 \), \( Z_{e,2}^{sy} \) satisfies the condition of Theorem 7, so is realizable as a five-element bridge network. Furthermore, \( Z_{e,2}^{sy} \) is realizable as Fig. 1(a) with \( R_1 = 16.232 \Omega, R_2 = 0.637 \Omega, R_3 = 0.766 \Omega, C_1 = 0.0329 \text{ F}, \) and \( C_1 = 1.411 \times 10^{-8} \text{ F}, \) and \( Z_{e,2}^{sy} \) is also realizable as Fig. 2(a) with \( R_1 = 14.425 \Omega, R_2 = 1.378 \Omega, R_3 = 1.194 \Omega, C_1 = 4.157 \times 10^{-9} \text{ F}, \) and \( C_2 = 0.0355 \text{ F}. \)

**Example 2:** As shown in [30], the function

\[
Z_{e,J_1}^{2nd} = \frac{1.665 \times 10^5s^2 + 5.776 \times 10^5s + 5.466 \times 10^7}{s^2 + 1.544 \times 10^6s + 0.342}
\]  

(VI.2)

is the impedance of an external circuit in the machatronic suspension system (LMIS3 layout), which optimizes \( J_1 \) (ride comfort) and is realizable as a five-element series-parallel configuration in [30, Fig. 2(c)]. It can be verified that \( R < 0 \) and \( Z_{e,J_1}^{2nd} \) is regular, implying that \( Z_{e,J_1}^{2nd} \) is realizable as a five-element bridge network by Theorem 7. Furthermore, \( Z_{e,J_1}^{2nd} \) is realizable as Fig. 5 with \( R_1 = 1.598 \times 10^8 \Omega, R_2 = 0.374 \Omega, R_3 = 1.665 \times 10^5 \Omega, C_1 = 0.0282 \text{ F}, \) and \( L_1 = 0.108 \text{ H}. \)

VII. CONCLUSION

This paper has investigated the realization problem of biquadratic impedances as five-element bridge networks, where the biquadratic impedance was assumed to be not realizable with fewer than five elements. Through investigating the realizability conditions of configurations covering all the possible cases, a necessary and sufficient condition was derived for a biquadratic impedance to be realizable as a two-reactive five-element bridge network, in terms of the coefficients only. Through the discussions, a canonical form for biquadratic impedances was utilized to simplify and combine the obtained conditions. Finally, a necessary and sufficient condition was obtained for the realizability of the biquadratic impedance as a general five-element bridge network.
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REFERENCES

[1] B. D. O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*. NJ: Prentice Hall, 1973.

[2] R. Bott and R. J. Duffin, “Impedance synthesis without use of transformers,” *Journal of Applied Physics*, vol. 20, no. 8, p. 816, 1949.

[3] M. Z. Q. Chen, C. Papageorgiou, F. Scheibe, F.-C. Wang, and M. C. Smith, “The missing mechanical circuit element,” *IEEE Circuits Syst. Mag.*, vol. 9, no. 1, pp. 10–26, 2009.

[4] M. Z. Q. Chen and M. C. Smith, “Restricted complexity network realizations for passive mechanical control,” *IEEE Trans. Automatic Control*, vol. 54, no. 10, pp. 2290–2301, 2009.

[5] M. Z. Q. Chen and M. C. Smith, “A note on tests for positive-real functions,” *IEEE Trans. Automatic Control*, vol. 54, no. 2, pp. 390–393, 2009.

[6] M. Z. Q. Chen, Y. Hu, and B. Du, “Suspension performance with one damper and one inerter,” in *Proceedings of 24th Chinese Control and Decision Conference (CCDC)*, 2012, pp. 3534–3539.

[7] M. Z. Q. Chen, K. Wang, Z. Shu, and C. Li, “Realizations of a special class of admittances with strictly lower complexity than canonical forms,” *IEEE Trans. Circuits and Systems I: Regular Papers*, vol 60, no. 9, pp. 2465–2473, 2013.

[8] M. Z. Q. Chen, K. Wang, C. Li, and G. Chen, “Realizations of biquadratic impedances as five-element bridge networks containing one inductor and one capacitor,” in *Proceedings of the 33rd Chinese Control Conference*, 2014, pp. 7498–7503.

[9] M. Z. Q. Chen, K. Wang, Y. Zou, and G. Chen, “Realization of three-port spring networks with inerter for effective mechanical control,” *IEEE Trans. Automatic Control*, vol. 60, no. 10, pp. 2722–2727, 2015.

[10] M. Z. Q. Chen, Y. Hu, L. Huang, and G. Chen, “Influence of inerter on natural frequencies of vibration systems,” *Journal of Sound and Vibration*, vol. 333, no. 7, pp. 1874–1887, 2014.

[11] M. Z. Q. Chen, K. Wang, C. Li, G. Chen, “Supplementary material to: Realizations of biquadratic impedances as five-element bridge networks,” Technical Report, Department of Mechanical Engineering, The University of Hong Kong, Hong Kong.

[12] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1980, vol. II.

[13] E. A. Guillemin, *Synthesis of Passive Networks*. New York: Wiley, 1957.

[14] T. H. Hughes and M. C. Smith, “Algebraic criteria for circuit realisations,” K. Hüper and J. Trumpf (Eds.), *Mathematical System Theory*, Charlotte: CreateSpace, 2012.

[15] Y. Hu, M. Z. Q. Chen, Z. Shu, and L. Huang, “Analysis and optimisation for inerter-based isolators via fixed-point theory and algebraic solution,” *Journal of Sound and Vibration*, vol. 346, pp. 17–36, 2015.

[16] J. Z. Jiang and M. C. Smith, “Regular positive-real functions and five-element network synthesis for electrical and mechanical networks,” *IEEE Trans. Automatic Control*, vol. 56, no. 6, pp. 1275–1290, 2011.

[17] J. Z. Jiang and M. C. Smith, “Series-parallel six-element synthesis of biquadratic impedances,” *IEEE Trans. Circuits and Systems I: Regular Papers*, vol. 59, no. 11, pp. 2543–2554, 2012.
[18] R. Kalman, “Old and new directions of research in system theory,” in Perspectives in Mathematical System Theory, Control, and Signal Processing, J. C. Willems, S. Hara, Y. Ohta, and H. Fujioka (Eds.), New York: Springer-Verlag, 2010, LNCIS, vol. 398, pp. 3–13.
[19] E. L. Ladenheim, “A synthesis of biquadratic impedances,” Master’s thesis, Polytechnic Inst. of Brooklyn, N.Y., 1948.
[20] E. L. Ladenheim, “Three-reactive five-element biquadratic structures,” IEEE Trans. Circuit Theory, vol. 11, no. 1, pp. 88–97, 1964.
[21] J. Lavaei, A. Babakhani, A. Hajimiri, and J. C. Doyle, “Solving largescale hybrid circuit-antenna problems,” IEEE Trans. Circuits and Systems I: Regular Papers, vol. 58, no. 2, pp. 374–387, 2011.
[22] R. H. Pantell, “A new method of driving-point impedance synthesis,” Proceedings of the IRE, vol. 42, no. 5, p. 861, 1954.
[23] N. Rajapaksha, A. Madanayake, and L. T. Bruton, “2D space–time wave-digital multi-fan filter banks for signals consisting of multiple plane waves,” Multidimensional Systems and Signal Processing, vol. 25, no. 1, pp. 17–39, 2014.
[24] M. Reichert, “Die kanonisch und übertragerfrei realisierbaren Zweipolfunktionen zweiten Grades (Transformerless and canonic realization of biquadratic immittance functions),” Arch. Elek. Übertragung, vol. 23, pp. 201–208, 1969.
[25] T. Reis and T. Stykel, “PABTEC: Passivity-preserving balanced truncation for electrical circuits,” IEEE Trans. Computer-Aided Design of Integrated Circuits and Systems, vol. 29, no. 9, pp. 1354–1367, 2010.
[26] K. Saeed, “Carathéodory-Toeplitz based mathematical methods and their algorithmic applications in biometric image processing,” Applied Numerical Mathematics, vol. 75, pp. 2–21, 2014.
[27] S. Seshu and M. B. Reed, Linear Graphs and Electrical Networks. Boston: Addison-Wesley, 1961.
[28] M. C. Smith, “Synthesis of mechanical networks: the inverter,” IEEE Trans. Automatic Control, vol. 47, no. 10, pp. 1648–1662, 2002.
[29] L. Versfeld, “Remarks on star-mesh transformation of electrical networks,” Electronics Letters, vol. 6, no. 19, pp. 597–599, 1970.
[30] F.-C. Wang, H.-A. Chan, J. Z. Jiang, and M. C. Smith, “Optimization and network synthesis for a mechatronic system,” in Proceedings of the 3rd International Conference on Mathematical Aspects of Computer and Information Sciences (MACIS 2009), Fukuoka, Japan, December, 2009, pp. 386–390.
[31] F.-C. Wang, M.-R. Hsieh, and H.-J. Chen, “Stability and performance analysis of a full-train system with inverters,” Vehicle System Dynamics, vol. 50, no. 4, pp. 545–571, 2012.
[32] K. Wang and M. Z. Q. Chen, “Generalized series-parallel RLC synthesis without minimization for biquadratic impedances,” IEEE Trans. Circuits and Systems II: Express Briefs, vol. 59, no. 11, pp. 766–770, 2012.
[33] K. Wang, M. Z. Q. Chen, and Y. Hu, “Synthesis of biquadratic impedances with at most four passive elements,” Journal of the Franklin Institute, vol. 351, no. 3, pp. 1251–1267, 2014.
[34] K. Wang and M. Z. Q. Chen, “Minimal realizations of three-port resistive networks,” IEEE Trans. Circuits and Systems I: Regular Papers, vol. 62, no. 4, pp. 986–994, 2015.
[35] J. Xiong, I. R. Petersen, and A. Lanzon, “On lossless negative imaginary systems,” Automatica, vol. 48, no. 6, pp. 1213–1217, 2012.
[36] B. S. Yarman, R. Kopru, N. Kumar, and C. Prakash, “High precision synthesis of a Richards immittance via parametric approach,” IEEE Trans. Circuits and Systems I: Regular Papers, vol. 61, no. 4, pp. 1055–1067, 2014.
Supplementary Material to: Realizations of Biquadratic Impedances as Five-Element Bridge Networks

I. INTRODUCTION

This report presents some supplementary material to the paper entitled as “Realizations of Biquadratic Impedances as Five-Element Bridge Networks” [1]. For more background information of this field, refer to [2]–[32] and references therein.

II. DEFINITIONS OF THE NETWORK DUALITY

Regardless of the values of the elements, any one-port passive RLC network $N$ can be regarded as a one-terminal-pair labeled graph $\mathcal{N}$ with two distinguished terminal vertices (see [34, pg. 14]), in which the labels designate passive circuit elements, namely resistors, capacitors, and inductors, which are labeled as $R_i$, $C_i$, and $L_i$, respectively.

Two natural maps acting on the labeled graph are defined as follows:

1) GDu := Graph duality, which takes the one-terminal-pair graph into its dual (see [34, Definition 3-12]) while preserving the labeling.

2) Inv := Inversion, which preserves the graph but interchanges the reactive elements, that is, capacitors to inductors and inductors to capacitors with their labels $C_i$ to $L_i$ and $L_i$ to $C_i$.

Consequently, one defines

$$\text{Dual} := \text{network duality of one terminal-pair labeled graph} := \text{GDu} \circ \text{Inv} = \text{Inv} \circ \text{GDu}.$$ 

Consider a network $N$ whose one-terminal-pair labeled graph is $\mathcal{N}$. Denote $N^i$ as the network whose one-terminal-pair labeled graph is $\text{Inv}(\mathcal{N})$, resistors are of the same values as those of $N$, and inductors (resp. capacitors) are replaced by capacitors (resp. inductors) with reciprocal values. Denote $N^{id}$ as the network whose one-terminal-pair labeled graph is $\text{GDu}(\mathcal{N})$ and elements are of the reciprocal values to those of $N$. Denote $N^d$ as the network whose one-terminal-pair labeled graph is $\text{Dual}(\mathcal{N})$, resistors are of reciprocal values to those of $N$, and inductors (resp. capacitors) are replaced by capacitors (resp. inductors) with same values. Based on the mesh current and node voltage method, it can be proved that $Z(s)$ (resp. $Y(s)$) is realizable as the impedance (resp. admittance) of a network $N$ whose one-terminal-pair labeled graph is $\mathcal{N}$, if
and only if \( Z(s^{-1}) \) (resp. \( Y(s^{-1}) \)) is realizable as the impedance (resp. admittance) of \( N^i \) whose one-terminal-pair labeled graph is \( \text{Inv}(\mathcal{N}) \), if and only if \( Z(s^{-1}) \) (resp. \( Y(s^{-1}) \)) is realizable as the admittance (resp. impedance) of \( N^{id} \) whose one-terminal-pair labeled graph is \( \text{GDu}(\mathcal{N}) \), and if and only if it is realizable as the admittance (resp. impedance) of \( N^d \) whose one-terminal-pair labeled graph is \( \text{Dual}(\mathcal{N}) \).

If a necessary and sufficient condition is derived for \( H(s) = \sum_{k=0}^{m} a_k s^k / \sum_{k=0}^{m} b_k s^k \) to be realizable as the impedance (resp. admittance) of the one-port network \( N \) whose one-terminal-pair labeled graph is \( \mathcal{N} \), then the corresponding condition for \( N^i \) whose one-terminal-pair labeled graph is \( \text{Inv}(\mathcal{N}) \) can be obtained from that for \( N \) through conversion \( a_k \leftrightarrow a_{m-k} \) and \( b_k \leftrightarrow b_{m-k} \), \( k \in \{0, 1, \ldots, \lfloor m/2 \rfloor \} \). Consequently, the corresponding condition for \( N^{id} \) whose one-terminal-pair labeled graph is \( \text{GDu}(\mathcal{N}) \) can be obtained from that for \( N \) through conversion \( a_k \leftrightarrow b_{m-k} \), \( k \in \{0, 1, \ldots, m \} \). Furthermore, the corresponding condition for \( N^d \) whose one-terminal-pair labeled graph is \( \text{Dual}(\mathcal{N}) \) can be obtained from that for \( N \) through conversion \( R_i \rightarrow R_{i-1} \), \( C_i \rightarrow L_i \), \( L_i \rightarrow C_i \), and \( a_k \leftrightarrow b_k \), \( k \in \{0, 1, \ldots, m \} \).

### III. Proof of Lemma 4

It follows from condition (6) and the assumption of \( W \neq 3 \) that \( W > 3 \). In order to simplify the proof of this lemma, one utilizes the coordinate transformation

\[
U = x \cos \frac{\pi}{4} - y \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} (x - y),
\]

\[
V = x \sin \frac{\pi}{4} + y \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} (x + y),
\]

which does not affect the nature and proof of the lemma. Thus, conditions (5)–(7) are converted into the following:

\[
\Psi_1(x, y) := 2W^{-1}(W - 1)^2 x^2 - 2W^{-1}(W + 1)^2 y^2 - W^{-2}(W^2 - 1)^2 > 0, \tag{III.2}
\]

\[
\Psi_2(x, y) := 2W^{-1}(W + 1)(W - 3)x^2 - 2W^{-1}(W + 3)(W - 1)y^2 - W^{-2}(W^2 - 9)(W^2 - 1) \geq 0, \tag{III.3}
\]
and
\[
\Sigma(x, y) := -4W^{-2}(W + 3)(W + 1)^3y^4 + 8W^{-2}(W + 3)(W + 1)(W^2 - 2W - 1)xy^3 \\
+ 16W^{-1}(W^2 - 5)x^2y^2 - 8W^{-2}(W - 1)(W - 3)(W^2 + 2W - 1)x^3y \\
+ 4W^{-2}(W - 3)(W - 1)^3x^4 - 2W^{-3}(W - 1)(W + 3)(W^2 - 3)(W + 1)^2y^2 \\
+ 4W^{-3}(W^2 - 9)(W^2 - 1)^2xy - 2W^{-3}(W - 3)(W + 1)(W^2 - 3)(W - 1)^2x^2 > 0, 
\]
where \( x > 0 \) and \( W > 3 \). Note that \( (\text{III.2}) \) and \( (\text{III.3}) \) are equivalent to
\[
x > \frac{W + 1}{W - 1} \sqrt{\frac{2Wy^2 + (W - 1)^2}{2W}} =: x_{\Psi_1}(y), 
\]
\[
x \geq \sqrt{\frac{(W + 3)(W - 1)(2Wy^2 + (W + 1)(W - 3))}{2W(W + 1)(W - 3)}} =: x_{\Psi_2}(y),
\]
respectively. Hence, it can be verified that condition \( (\text{III.5}) \) yields condition \( (\text{III.6}) \) when \( y \in [-y_1, y_1] \), and condition \( (\text{III.6}) \) yields condition \( (\text{III.5}) \) when \( y \in (-\infty, -y_1) \cup (y_1, +\infty) \), where
\[
y_1 = \frac{W - 1}{2W} \sqrt{\frac{(W + 1)(W - 3)}{2}}.
\]
Consider the case of \( y \in [-y_1, y_1] \). It suffices to show that condition \( (\text{III.5}) \) implies condition \( (\text{III.4}) \). It can be verified that
\[
\Sigma_x(x, y) := \frac{\partial}{\partial x} \Sigma(x, y) \\
= 16W^{-2}(W - 1)^3(W - 3)x^3 - 24W^{-2}(W - 1)(W - 3)(W^2 + 2W - 1)x^2y \\
+ 32W^{-1}(W^2 - 5)xy^2 - 4W^{-3}(W - 3)(W + 1)(W^2 - 3)(W - 1)^2x \\
+ 8W^{-2}(W + 3)(W + 1)(W^2 - 2W - 1)y^3 + 4W^{-3}(W^2 - 1)^2(W^2 - 9)y, 
\]
and
\[
\Sigma_{xx}(x, y) := \frac{\partial^2}{\partial x^2} \Sigma(x, y) \\
= 48W^{-2}(W - 1)^3(W - 3)x^2 - 48W^{-2}(W - 1)(W - 3)(W^2 + 2W - 1)xy 
\]
\[
+ 32W^{-1}(W^2 - 5)y^2 - 4W^{-3}(W + 1)(W - 1)^2(W^2 - 3)(W - 3).
\]
Regarding \( (\text{III.8}) \) as a quadratic function of \( x \), its symmetric axis \( x_s(y) = (W^2 + 2W - 1)y/(2(W - 1)^2) \) satisfies \( x_s(y) < x_{\Psi_1}(y) \), as \( 4W(W - 1)^4 ((x_s(y))^2 - (x_{\Psi_1}(y))^2) = W(W^2 +
$2W - 1)^2y^2 - 2(W + 1)^2(2Wy^2 + (W - 1)^2) = - W(3W^2 + 2W - 3)(W^2 - 2W - 1)y^2 - 2(W + 1)^2(W - 1)^4 < 0$. Furthermore, one obtains $\Sigma_{xx}(x_\Psi(y), y) = \Sigma_{xx1}(x_\Psi(y), y) - \Sigma_{xx2}(x_\Psi(y), y)$, where
\[
\begin{align*}
\Sigma_{xx1}(x_\Psi(y), y) &= \frac{16(3W^4 - 4W^3 - 12W^2 - 4W + 9)y^2}{W^2} + \frac{4(W + 1)(5W^2 - 3)(W - 1)^2(W - 3)}{W^3}, \\
\Sigma_{xx2}(x_\Psi(y), y) &= \frac{24\sqrt{2}(W + 1)(W - 3)(W^2 + 2W - 1)y}{W^2} \sqrt{2W^2y^2 + (W - 1)^2}.
\end{align*}
\]
It is obvious that $\Sigma_{xx1}(x_\Psi(y), y) > 0$. If $y \in [-y_1, 0]$, then $\Sigma_{xx2}(x_\Psi(y), y) \leq 0$, indicating that $\Sigma_{xx}(x_\Psi(y), y) > 0$. If $y \in (0, y_1]$, then $\Sigma_{xx2}(x_\Psi(y), y) > 0$. One further obtains $(\Sigma_{xx1}(x_\Psi(y), y))^2 - (\Sigma_{xx2}(x_\Psi(y), y))^2 = -2048W^{-3}(W^2 - 3W - 2)(3W^4 - 2W^3 - 18W^2 - 8W + 9)y^4 + 256W^{-4}(W + 1)(W - 3)(3W^5 - 19W^4 + 6W^3 + 86W^2 + 27W - 39)(W - 1)^2y^2 + 16W^{-6}(W - 3)^2(y^2 - 2W^2 - 3)^2(W - 1)^4$, which is positive for $y \in (0, y_1]$. Hence, one concludes that $\Sigma_{xx}(x, y) > 0$ for $y \in [-y_1, y_1]$ and $x > x_\Psi(y)$, indicating that $\Sigma_{xx}(x, y)$ increases monotonically in $x$ when $x > x_\Psi(y)$ and $y$ is fixed for $y \in [-y_1, y_1]$. For (III.7), one has $\Sigma_{x}(x_\Psi(y), y) = \Sigma_{x2}(x_\Psi(y), y) - \Sigma_{x1}(x_\Psi(y), y)$, where
\[
\begin{align*}
\Sigma_{x1}(x_\Psi(y), y) &= \frac{16(W + 1)(W^4 - 8W^2 - 8W + 3)y^3}{W^2(W - 1)} + \frac{8(W - 1)(W - 3)(W + 2)(W + 1)^2y}{W^2}, \\
\Sigma_{x2}(x_\Psi(y), y) &= \frac{2\sqrt{2}(W + 1)}{W^3(W - 1)} \sqrt{2W^2y^2 + (W - 1)^2} \\
&\times \left(4W(W^4 - 4W^2 - 8W + 3)y^2 + (W - 3)(W + 1)(W^2 + 1)(W - 1)^2\right).
\end{align*}
\] It can be verified that $\Sigma_{x2}(x_\Psi(y), y) > 0$. If $y \in [-y_1, 0]$, then $\Sigma_{x1}(x_\Psi(y), y) \leq 0$, implying that $\Sigma_{xx}(x_\Psi(y), y) > 0$. If $y \in (0, y_1]$, then $\Sigma_{x1}(x_\Psi(y), y) > 0$. One obtains
\[
f(Y) := (\Sigma_{x2}(x_\Psi(y), y))^2 - (\Sigma_{x1}(x_\Psi(y), y))^2 = \frac{2048(W + 1)^2S_1Y^3}{W^2(W - 1)^2} - \frac{256(W + 1)^2S_2Y^2}{W^4} + \frac{16W^{-6}(W + 1)^3(W - 1)^2(W - 3)S_3Y}{W^2} + \frac{8W^{-7}(W + 1)^4(W^2 + 1)^2(W - 1)^4(W - 3)^2}{W^2},
\] where $S_1 = W^4 - 6W^2 - 8W + 3 > 0$, $S_2 = W^6 - 8W^5 - 5W^4 + 36W^3 + 63W^2 - 4W - 3$, $S_3 = W^6 - 10W^5 + 15W^4 + 44W^3 + 39W^2 - 34W + 9$, and $Y = y^2$. The Sturm chain for (III.9) can be obtained through $f_0(Y) = f(Y)$, $f_1(Y) = f'(Y)$, $f_2(Y) = -\text{rem}(f_0, f_1)$,
and $f_3(Y) = -\text{rem}(f_1, f_2)$, where $\text{rem}(p_i, p_j)$ denotes the remainder of the polynomial long division of $p_i$ by $p_j$. The sign of this chain at $Y = 0$ and $Y = Y_1 = y_1^2$ is as shown in Table II (the special case when $f_3(0) = f_3(Y_1) = -\infty$ is excluded, which does not affect the result), where $T = W^{12} + 18W^{11} + 18W^{10} - 242W^9 - 685W^8 + 60W^7 + 3516W^6 + 6260W^5 + 2407W^4 - 1422W^3 + 1234W^2 - 258W - 27$ and $X = 3W^{10} + 4W^9 - 47W^8 - 104W^7 + 54W^6 + 552W^5 + 1178W^4 + 552W^3 - 1065W^2 - 1356W + 549$, and $V(Y)$ denote the number of sign variations in the Sturm chain for $(\text{III.9})$. By investigating the roots of $V(Y)$, it is also noted that $\text{rem}(p_1, p_2)$ division of $X$. Then, it has been checked that $f_3(0) = f_3(Y_1) = -\infty$. By investigating the roots of $V(Y)$, it is noted that $\text{rem}(p_1, p_2)$ division of $X$, as it has been checked that $f_3(0) = f_3(Y_1) = -\infty$. Moreover, it is observed that if $y \in (-y_1, 0)$, then $\Sigma_2(x_\Psi_1(y), y) \leq 0$, implying that $\Sigma(x_\Psi_1(y), y) \geq 0$. If $y \in [0, y_1]$, then one has

$$(\Sigma_1(x_\Psi_1(y), y))^2 - (\Sigma_2(x_\Psi_1(y), y))^2 = 4W^{-8}(W + 1)^4(8W^2y^2 - (W + 1)(W - 3)(W - 1)^2)^2 \geq 0.$$ 

Hence, $\Sigma(x_\Psi_1(y), y) \geq 0$ for $y \in [-y_1, y_1]$. As a result, $\Sigma(x, y) > 0$ for $y \in [-y_1, y_1]$ and $x > x_\Psi_1(y)$. 

Now, it remains to consider the case of $y \in (-\infty, -y_1) \cup (y_1, +\infty)$. In this case, one only needs to consider condition (\text{III.6}), that is, $x \geq x_\Psi_2(y)$, as it has been checked that $x_\Psi_2(y) > x_\Psi_1(y)$. It suffices to show that $\Sigma_x(x, y) > 0$ for $y \in (-\infty, -y_1) \cup (y_1, +\infty)$ and $x \geq x_\Psi_2(y)$, since it has been checked that $\Sigma(x_\Psi_2(y), y) > 0$ for $y \in (-\infty, -y_1) \cup (y_1, +\infty)$. These can be straightforwardly verified but the tedious detail is omitted for similarity of the presentation herein.
TABLE I

The sign of the Sturm chain for (III.9).

| $\text{sign}(f_i(Y))$ | $i = 0$ | $i = 1$ | $i = 2$ | $i = 3$ | Variations |
|------------------------|---------|---------|---------|---------|------------|
| $Y = 0$                | +       | +       | $\text{sign}(S_i)$ | $\text{sign}(-T)$ | $-$ | $V(0)$ |
| $Y = Y_1$              | +       | +       | $\text{sign}(-X)$ | $-$ | $V(Y_1)$ |

REFERENCES

[1] M. Z. Q. Chen, K. Wang, C. Li, and G. Chen, “Realization of biquadratic impedances as five-element bridge networks,” *IEEE Trans. Circuits and Systems I: Regular Papers*, vol. 64, no. 6, pp. 1599–1611, 2017.

[2] M. Z. Q. Chen, *Passive Network Synthesis of Restricted Complexity*, Ph.D. Thesis, Cambridge Univ. Eng. Dept., U.K., 2007.

[3] M. Z. Q. Chen and M. C. Smith, “Electrical and mechanical passive network synthesis,” in *Recent Advances in Learning and Control*, V. D. Blondel, S. P. Boyd, and H. Kimura (Eds.), New York: Springer-Verlag, 2008, LNCIS, vol. 371, pp. 35–50.

[4] M. Z. Q. Chen, “A note on PIN polynomials and PRIN rational functions,” *IEEE Trans. Circuits and Systems II: Express Briefs*, vol. 55, no. 5, pp. 462–463, 2008.

[5] M. Z. Q. Chen and M. C. Smith, “Restricted complexity network realizations for passive mechanical control,” *IEEE Trans. Automatic Control*, vol. 54, no. 10, pp. 2290–2301, 2009.

[6] M. Z. Q. Chen and M. C. Smith, “A note on tests for positive-real functions,” *IEEE Trans. Automatic Control*, vol. 54, no. 2, pp. 390–393, 2009.

[7] M. Z. Q. Chen, C. Papageorgiou, F. Scheibe, F.-C. Wang, and M. C. Smith, “The missing mechanical circuit element,” *IEEE Circuits Syst. Mag.*, vol. 9, no. 1, pp. 10–26, 2009.

[8] M. Z. Q. Chen, K. Wang, Y. Zou, and J. Lam, “Realization of a special class of admittances with one damper and one inerter for mechanical control,” *IEEE Trans. Automatic Control*, vol. 58, no. 7, pp. 1841–1846, 2013.

[9] M. Z. Q. Chen, K. Wang, M. Yin, C. Li, Z. Zuo, and G. Chen, “Synthesis of $n$-port resistive networks containing $2n$ terminals,” *International Journal of Circuit Theory and Applications*, vol. 43, no. 4, pp. 427–437, 2015.

[10] M.Z.Q. Chen, K. Wang, Y. Zou, and G. Chen, “Realization of three-port spring networks with inerter for effective mechanical control,” *IEEE Trans. Automatic Control*, vol. 60, no. 10, pp. 2722–2727, 2015.

[11] M. Z. Q. Chen, “The classical $n$-port resistive synthesis problem,” in *Workshop on “Dynamics and Control in Networks”*, Lund University, 2014 [http://www.lccc.lth.se/media/2014/malcolm3.pdf](http://www.lccc.lth.se/media/2014/malcolm3.pdf) last accessed on 19/01/2015).

[12] M. Z. Q. Chen, Y. Hu, and B. Du, “Suspension performance with one damper and one inerter,” in *Proceedings of the 24th Chinese Control and Decision Conference*, Taiyuan, China, 2012, pp. 3534–3539.

[13] M. Z. Q. Chen, Y. Hu, L. Huang, and G. Chen, “Influence of inerter on natural frequencies of vibration systems,” *Journal of Sound and Vibration*, vol. 333, no. 7, pp. 1874–1887, 2014.

[14] M. Z. Q. Chen, Y. Hu, C. Li, and G. Chen, “Performance benefits of using inerter in semiactive suspensions,” *IEEE Trans. Control Systems Technology*, vol. 23, no. 4, pp. 1571–1577, 2015.
[15] M. Z. Q. Chen, Y. Hu, F.-C. Wang, “Passive mechanical control with a special class of positive real controllers: application to passive vehicle suspensions,” *Journal of Dynamic Systems, Measurement, and Control*, vol. 137, no. 12, pp. 121013-1–121013-11, 2015.

[16] X. Dong, Y. Liu, and M. Z. Q. Chen, “Application of inerter to aircraft landing gear suspension systems,” in *Proceedings of the 34th Chinese Control Conference*, July 28–30, 2015, Hangzhou, China, pp. 2066–2071.

[17] Y. Hu, M. Z. Q. Chen, and Z. Shu, “Passive vehicle suspensions employing inerters with multiple performance requirements,” *Journal of Sound and Vibration*, vol. 333, no. 8, pp. 2212–2225, 2014.

[18] Y. Hu, M. Z. Q. Chen, Z. Shu, and L. Huang, “Analysis and optimisation for inerter-based isolators via fixed-point theory and algebraic solution,” *Journal of Sound and Vibration*, vol. 346, pp. 17–36, 2015.

[19] Y. Hu, K. Wang, and M. Z. Q. Chen, “Semi-active suspensions with low-order mechanical admittances incorporating inerters,” in *Proceedings of the 27th Chinese Control and Decision Conference*, 2015, pp. 79–84.

[20] Y. Hu and M. Z. Q. Chen, “Performance evaluation for inerter-based dynamic vibration absorbers,” *International Journal of Mechanical Sciences*, vol. 99, pp. 297–307, 2015.

[21] Y. Hu, H. Du, M. Z. Q. Chen, “An inerter-based electromagnetic device and its application in vehicle suspensions,” in *Proceedings of the 34th Chinese Control Conference*, 2015, pp. 2060–2065.

[22] Y. Hu, K. Wang, and M. Z. Q. Chen, “Performance optimization for passive suspensions with one damper one inerter and three springs,” in *Proceeding of the 2015 IEEE International Conference on Information and Automation*, 2015, pp. 1349–1354.

[23] Y. Hu, M. Z. Q. Chen, S. Xu, and Y. Liu, “Semi-active inerter and its application in adaptive tuned vibration absorbers,” *IEEE Trans. Control Systems Technology*, vol. 25, no. 1, pp. 294–300, 2017.

[24] Y. Liu, Y. Hu, and M. Z. Q. Chen, “Effect of play in inerter on vehicle suspension system,” in *Proceedings of the 27th Chinese Control and Decision Conference*, 2015, pp. 2497–2502.

[25] Y. Liu, M. Z. Q. Chen, Y. Tian, “Nonlinearities in landing gear model incorporating inerter,” in *Proceeding of the 2015 IEEE International Conference on Information and Automation*, 2015, pp. 696–701.

[26] K. Wang and M. Z. Q. Chen, “Generalized series-parallel RLC synthesis without minimization for biquadratic impedances,” *IEEE Trans. Circuits and Systems II: Express Briefs*, vol. 59, no. 11, pp. 766–770, 2012.

[27] K. Wang, M. Z. Q. Chen, and Y. Hu, “Synthesis of biquadratic impedances with at most four passive elements,” *Journal of the Franklin Institute*, vol. 351, no. 3, pp. 1251–1267, 2014.

[28] K. Wang and M. Z. Q. Chen, “Minimal realizations of three-port resistive networks,” *IEEE Trans. Circuits and Systems I: Regular Papers*, vol. 62, no. 4, pp. 986–994, 2015.

[29] K. Wang, M. Z. Q. Chen, and G. Chen, “Realization of a transfer function as a passive two-port RC laddernetwork with a specified gain,” *International Journal of Circuit Theory and Applications*, in press (DOI: 10.1002/cta.2328).

[30] F.-C. Wang, M.-K. Liao, B.-H. Liao, W.-J. Su, and H.-A. Chan, “The performance improvements of train suspension systems with mechanical networks employing inerters,” *Vehicle System Dynamics*, vol. 47, no. 7, pp. 805–830, 2009.

[31] F.-C. Wang and H.-A. Chan, “Vehicle suspensions with a mechatronic network strut,” *Vehicle System Dynamics*, vol. 49, no. 5, pp. 811–830, 2011.

[32] M. C. Smith, “Synthesis of mechanical networks: the inerter,” *IEEE Trans. Automatic Control*, vol. 47, no. 10, pp. 1648–1662, 2002.

[33] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1980, vol. II.

[34] S. Seshu and M. B. Reed, *Linear Graphs and Electrical Networks*. Boston: Addison-Wesley, 1961.