Braided Yangians

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Abstract

Yangian-like algebras, associated with current $R$-matrices, different from the Yang ones, are introduced. These algebras are of two types. The so-called braided Yangians are close to the Reflection Equation algebras, arising from involutive or Hecke symmetries. The Yangians of RTT type are close to the corresponding RTT algebras. Some properties of these two classes of the Yangians are studied. Thus, evaluation morphisms for them are constructed, their bi-algebra structures are described, and quantum analogs of certain symmetric polynomials, in particular, quantum determinants, are introduced. It is shown that in any braided Yangian this determinant is always central, whereas in the Yangians of RTT type it is not in general so. Analogs of the Cayley-Hamilton-Newton identity in the braided Yangians are exhibited. A bozonization of the braided Yangians is performed.

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1 Introduction

The notion of the Yangians was introduced by V.Drinfeld in the framework of his Quantum Group (QG) theory [D]. The Yangian $\mathcal{Y}(gl(N))$ for the general linear Lie algebra $gl(N)$ is associated with the quantum $R$-matrix, due to Yang

$$R(u,v) = P - \frac{I}{u - v}. \quad (1.1)$$
Hereafter, $I$ is the identity matrix and $P$ stands for the usual flip or its matrix. The $R$-matrix \((1.1)\) meets the quantum Yang-Baxter equation
\[
R_{12}(u,v)R_{23}(u,w)R_{12}(v,w) = R_{23}(v,w)R_{12}(u,w)R_{23}(u,v),
\]
where $R_{12} = R \otimes I$ and $R_{23} = I \otimes R$. In general, the lower indices indicate the spaces where a given operator or a matrix is located.

Consider the following system
\[
R(u,v)L_1(u)L_2(v) - L_1(v)L_2(u)R(u,v) = 0,
\]
where $L(u) = \|l^j_i(u)\|$, $1 \leq i,j \leq N$, is a matrix-valued function depending on a formal parameter $u$, $L_1 = L \otimes I$, $L_2 = I \otimes L$.

We assume that the generating matrix $L(u)$ expands in a series
\[
L(u) = \sum_{k=0}^{\infty} L[k]u^{-k}, \quad L[0] = I,
\]
where the matrices $L[k] = \|l^j_i[k]\|$ are called the Laurent coefficients of the matrix $L(u)$.

By one of the equivalent definitions, the Yangian $\mathcal{Y}(gl(N))$ is the unital associative algebra, generated by the elements $l^j_i[k]$, $k \in \mathbb{Z}_+$, subject to the relations which arise from the system \((1.3)\). Thus, the current entries $l^j_i(u)$ of the generating matrix $L(u)$ are treated to be elements of $\mathcal{Y}(gl(N))[u^{-1}]$. Being presented in this form, the Yangian is generated by infinitely many generators $l^j_i[k]$, subject to infinitely many quadratic relations. However, each of these relations contains only a finite number of generators. In the sequel we deal with other quadratic algebras, which have the same property.

**Remark 1** Note that $R$-matrix $\mathcal{R}(u,v) = PR(u,v)$ is often used instead of $R(u,v)$. Being rewritten in terms of $\mathcal{R}$, the relation \((1.3)\) takes the form
\[
\mathcal{R}(u,v)L_1(u)L_2(v) - L_2(v)L_1(u)\mathcal{R}(u,v) = 0.
\]

Now, we list certain basic properties of the Yangians $\mathcal{Y}(gl(N))$.

1. The Yangian has a bi-algebraic structure: the corresponding coproduct is defined on the generating matrix by the rule
\[
\Delta(L(u)) = L(u) \otimes L(u).
\]
Hereafter the notation $A \otimes B$ stands for the matrix with entries $(A \otimes B)_{ij}^{kl} = \sum_k A_k^i \otimes B_k^j$, where $A$ and $B$ are square matrices of the same size. In terms of the Laurent coefficients the coproduct reads
\[
\Delta(L[m]) = \sum_{k=0}^{m} L[k] \otimes L[m-k].
\]
The counit $\varepsilon$ is as follows
\[
\varepsilon(L(u)) = I \quad \Leftrightarrow \quad \varepsilon(l^j_i[m]) = \delta_{m,0} \delta^j_i.
\]
2. In the Yangian $\mathcal{Y}(gl(N))$ there are well-defined quantum analogs of some symmetric polynomials, namely, the elementary symmetric polynomials and power sums. The highest quantum elementary symmetric polynomial is called the (quantum) determinant. This determinant (more precisely, its Laurent coefficients) generate the center of the Yangian $\mathcal{Y}(gl(N))$. The quantum elementary symmetric polynomials generate a commutative subalgebra in the Yangian $\mathcal{Y}(gl(N))$. The power sums are related with elementary polynomials by a quantum analog of Newton relations and generate the same commutative subalgebra.
3. There exists the so-called evaluation morphism, which sends the Yangian to the enveloping algebra $U(gl(N))$ enabling one to endow any $U(gl(N))$-module with a $Y(gl(N))$-module structure.

For these and other properties of the Yangians the reader is referred to [Mo].

A big interest in the Yangians is motivated in the first turn by their significance in the integrable system theory. In particular, the Yangians play the role of a symmetry group for the Nonlinear Schrödinger hierarchies [MRSZ]. Also, by using the Yangian $Y(gl(N))$, D.Talalaev [T] succeeded in finding higher Hamiltonians of the rational Gaudin model.

There are known numerous attempts to generalize the Yangians. Thus, there exist the so-called twisted Yangians which are associated with simple Lie algebras of the series different from $A_N$ (see [Mo] and the references therein) and the super-Yangians (see [Ni, Ga]), associated with super-flips $P_m|n$.

**Remark 2** The super-flip $P_m|n$ acts on the tensor square of the super-space $V^\otimes 2$ by the rule $P_m|n(x \otimes y) = (-1)^{\pi \bar{\pi}} y \otimes x$. Here $V = V_0 \oplus V_1$, $V_0$ (resp., $V_1$) is even (resp., odd) subspace, dim$V_0 = m$, dim$V_1 = n$, elements $x$ and $y$ are assumed to be homogenous, and $\pi, \bar{\pi} \in \mathbb{Z}_2$ are their parities. We want to emphasize that the super-Yangians from the cited papers are defined via the relation (1.5) with $R = I + \frac{P_m|n}{u-v}$. Consequently, they differ from our super-Yangians of both types (see below), which are particular cases of our general definitions, provided $R = P_m|n$.

The main purpose of the current paper is to introduce some new Yangian-like algebras, associated with a large class of $R$-matrices and to study their properties. These algebras are of two types. Algebras of the first type are similar to the Reflection Equation (RE) algebras\footnote{There are known numerous forms of the RE algebras. In the present paper we use this term for the algebras associated with constant braidings.} and we call them braided Yangians. They are the main objects of the current paper. Algebras of the second type are similar to RTT algebras, we call them Yangians of RTT-type. These two types of the Yangian-like algebras have different properties. This difference stems from a similar difference between the RE algebras and RTT ones.

Recall that the RTT and RE algebras are respectively defined by the following systems of relations

$$R_{12}T_1T_2 - T_1T_2R_{12} = 0, \quad T = \|t^j_{ij}\|_{1 \leq i,j \leq N}. \quad (1.6)$$

$$R_{12}L_1R_{12}L_1 - L_1R_{12}L_1R_{12} = 0, \quad L = \|l^j_{ij}\|_{1 \leq i,j \leq N}. \quad (1.7)$$

Here, $R = R^V : V^\otimes 2 \rightarrow V^\otimes 2$, is a braiding. By a braiding we mean a solution of the constant (i.e. without parameters) quantum Yang-Baxter equation

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}. \quad (1.8)$$

and $V$ is a finite-dimensional vector space $V$ over the ground field $\mathbb{K}$ (we put $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$), dim$_{\mathbb{K}} V = N$.

Mainly, we deal with braidings $R$ obeying the condition

$$(qI - R)(q^{-1}I + R) = 0, \quad q \in \mathbb{K}. \quad (1.9)$$

If $q^2 = 1$, the corresponding braiding is called an involutive symmetry. Unless, it is called a Hecke symmetry. Below, if $q \neq 1$, $q$ is assumed to be generic, that is $q^n \neq 1$ for any integer $n$.

The best known examples of Hecke symmetries are standard one\footnote{Hereafter, we use the term standard for all objects related to the QG $U_q(sl(N))$.} However, there exist many other involutive and Hecke symmetries, constructed by one of the authors in the 80’s (see [G]).
Also, we assume a basis \( \{ x_i \} \in V \) to be fixed in the space \( V \). Then \( R = R^V \) is represented by the corresponding matrix in the basis \( \{ x_i \otimes x_j \} \in V \otimes 2 \) as follows

\[
R^V(x_i \otimes x_j) = R^{ks}_{ij} x_k \otimes x_s.
\]

The summation over repeated indices is always understood.

The RTT and RE algebras are particular cases of the so-called Quantum Matrix (QM) algebras as defined in [IOP], but their properties differ drastically. The reader is referred to [GPS] for a comparative description of these algebras.

The Yangian-like algebras, we are dealing with in the present paper, are current (i.e. depending on a parameter) counterparts of the mentioned QM algebras. In order to define them we have first to construct a current \( R \)-matrix \( R(u, v) \). By using the so-called Baxterization procedure, we associate \( R(u, v) \) with a given involutive or Hecke symmetry \( R \)

\[
R(u, v) = R + g(u, v) I.
\]

Emphasize that the concrete form of the function \( g(u, v) \) depends on \( R \) (see Proposition [12]). Now, we are able to give the following definition.

**Definition 3** Consider a current \( N \times N \) matrix \( L(u) \) and impose the following relations

\[
R(u, v)L_1(u)RL_1(v) - L_1(v)RL_1(u)R(u, v) = 0. \tag{1.10}
\]

We suppose \( L(u) \) to be a series in \( u \) analogous to (1.4). Then, the unital algebra generated by matrix elements \( l^i_{ij}[k] \) of the Laurent coefficients \( L[k], k \geq 1 \), is called a braided Yangian and is denoted \( \mathbf{Y}(R) \).

Besides, we introduce an RTT-type Yangian, defined by the system

\[
R(u, v)T_1(u)T_2(v) - T_1(v)T_2(u)R(u, v) = 0.
\]

As usual, we assume that \( T(u) \) expands in a Laurent series but we do not impose the condition \( T[0] = I \). Such an RTT-type Yangian is denoted \( \mathbf{Y}_{RRT}(R) \). The RTT-type Yangians corresponding to the standard current \( R \)-matrices will be called \( q \)-Yangians.

In this paper we study some basic properties of the braided Yangians and compare them with those of the RTT-type Yangians. First, we define the evaluation morphisms for the Yangians of both types. It is interesting to note that the concrete form of these morphisms in the braided Yangians depends on the initial symmetry: it differs for the braided Yangians, arising from involutive symmetries and those arising from Hecke ones. Also, we define the bi-algebra structures in the Yangians of both types. In analogy with the RTT algebra, the bi-algebra structure of \( \mathbf{Y}_{RRT}(R) \) is usual, while in the braided Yangian \( \mathbf{Y}(R) \) it is braided in the sense discussed below.

In all our Yangians we introduce quantum analogs of some symmetric polynomials. Mainly, we are interested in the elementary symmetric polynomials and, in particular, in the determinant. The quantum determinant, which is the "highest" elementary symmetric polynomial (it is well-defined if \( R \) is even, see the next section), can be defined for a subclass of the Yangians of both types. We show that it is always central in the braided Yangians, whereas in the Yangians of RTT type it can be central or not depending on the centrality of the determinant in the corresponding RTT algebra. Also, we exhibit a version of the Cayley-Hamilton-Newton identity and the subsequent Cayley-Hamilton and Newton identities for the braided Yangians.

Finally, we consider a bosonization of the braided Yangians. In this connection we discuss analogs of Zamolodchikov-Faddeev algebras, associated with our current \( R \)-matrices.
Note that all properties of the braided Yangians are more similar to those of the usual Yangians than the corresponding properties of the RTT-type Yangians. Also, note that the Drinfeld Yangian $Y(gl(N))$ belongs to both classes since in this case $R = P$.

The paper is organized as follows. In section 2 we recall some properties of braidings and symmetries, and then we reproduce the main properties of the QM algebras in question (section 3). In section 4 we exhibit the so-called Baxterization procedure, giving rise to a family of current $R$-matrices. The evaluation morphisms and bi-algebraic structures of all Yangians in question are discussed in section 5.

In section 6 we define quantum analogs of the matrix powers, skew-matrix powers, some symmetric polynomials, and prove the Cayley-Hamilton-Newton identities for the generating matrices of the braided Yangians. Also, as corollaries we get the Newton and Cayley-Hamilton identities.

In section 7 we perform a bosonization of the braided Yangians and discuss the Zamolodchikov-Faddeev algebras.

In conclusion of the Introduction we would like to mention some possible applications of the braided Yangians. First, the braided $q$-Yangian gives rise to interesting Poisson structure useful for constructing a new version of the Gaudin model [GS1, GS2]. Note that the Hamiltonians for this model can be constructed in the spirit of [T]. Second, by using the Cayley-Hamilton identity for the generating matrices of the braided Yangians it is possible to perform a quantum analog of the Drinfeld-Sokolov reduction (see [GST]).

### 2 Braiding and symmetries

In this section we exhibit some properties of symmetries which are used in the sequel. For a more complete list of their properties we refer the reader to the paper [GPS]. Note that we mainly deal with Hecke symmetries, the corresponding results for involutive ones can be obtained by passing to the limit $q \to 1$.

Below we use a concise notation $R_k := R_{k,k+1}$ and the standard notation for $q$-numbers:

$$k_q = \frac{q^k - q^{-k}}{q - q^{-1}}.$$  

Emphasize, that for generic values of $q$ the $q$-integers are all non-zero: $k_q \neq 0 \forall k \in \mathbb{Z}$. Now, note that the following recursive formula

$$P^{(1)} = I, \quad P^{(k+1)}_{12\ldots k+1} = \frac{k_q}{(k+1)_q} P^{(k)}_{12\ldots k} \left( \frac{q^k}{k_q} I - R_k \right) P^{(k)}_{12\ldots k}, \quad k \geq 1 \quad (2.1)$$

defines (for a generic $q$) some projectors, called *skew-symmetrizers*

$$P^{(k)} : V^\otimes k \to \bigwedge_R^{(k)}(V), \quad k \geq 1. \quad (2.2)$$

Here $\bigwedge_R^{(k)}(V)$ is the $k$-th homogenous component of the $R$-skew-symmetric algebra $\bigwedge_R(V)$ of the space $V$, where

$$\bigwedge_R^{(k)}(V) = T(V)/\langle \text{Im}(q^{-1}I + R) \rangle.$$  

**Definition 4** We say that a symmetry $R$ is even if there exists an integer $m \geq 2$ such that the image of the skew-symmetrizer $P^{(m)}$ is non-trivial while the images of the skew-symmetrizers $P^{(k)}$, $k > m$ are trivial. The couple $(m|0)$ is called the *bi-rank* of this symmetry.

\footnote{For a definition of the bi-rank $(m|n)$ the reader is referred to [GPS].}
**Definition 5** A braiding $R$ is called *skew-invertible*, if there exists an operator $\Psi : V^\otimes 2 \to V^\otimes 2$ such that

$$
\text{Tr}_2 R_{12} \Psi_{23} = P_{13} \Leftrightarrow R_{ij}^{kl} \Psi_{lp}^{jq} = \delta_{i}^{q} \delta_{l}^{j}.
$$

(2.3)

Given a skew-invertible braiding $R : V^\otimes 2 \to V^\otimes 2$, the corresponding operator $\Psi$ enables us to extend $R$ up to a braiding

$$
R^{V\otimes V^*} : (V \oplus V^*)^\otimes 2 \to (V \oplus V^*)^\otimes 2,
$$

(2.4)

where $V^*$ is the space dual to $V$, in such a way that the corresponding pairing $V \otimes V^* \to \mathbb{K}$ is $R^{V\otimes V^*}$-invariant (see [GPS]). Emphasize that this extension is unique. All braidings below are assumed to be skew-invertible.

Also, by means of the operator $\Psi$ we define two linear operators $B$ and $C$ as follows

$$
B = \text{Tr}_1 \Psi \Leftrightarrow B_{i}^{j} = \Psi_{ki}^{ij}, \text{ and } C = \text{Tr}_2 \Psi \Leftrightarrow C_{i}^{j} = \Psi_{ik}^{ji}.
$$

(2.5)

These operators play an important role in all algebras considered below. In particular, the operator $C$ allows one to introduce the corresponding $R$-trace of matrices. Let $X = \|X_i^j\|$ be an $N \times N$ matrix with entries belonging to any algebra. We define its $R$-trace as follows

$$
\text{Tr}_R(X) := \text{Tr}(X \cdot C) = C_{i}^{j} X_j^i.
$$

(2.6)

Let us list some properties of the operators $B$ and $C$ and of the $R$-trace (see [O, GPS]). First, for any skew-invertible braiding $R$ one gets $\text{Tr}_{R(2)} R_{12} = I_1$. Second, for any $N \times N$ matrix $X$ the following holds

$$
\text{Tr}_{R(2)}(R_{1}^{-1} X_1 R_1) = \text{Tr}_{R(2)}(R_{1} X_1 R_{1}^{-1}) = \text{Tr}_R(X) \cdot I_1.
$$

(2.7)

Third, if a skew-invertible symmetry $R$ is even\(^4\) of bi-rank $(m|0)$, then

$$
\text{Tr}_R I = q^{-m} m q.
$$

(2.8)

and

$$
B \cdot C = C \cdot B = q^{-2m} I.
$$

(2.9)

For a skew-invertible braiding $R$ the corresponding $R$-trace possesses the following cyclic property: for any matrix $X$ of an appropriate size and any polynomial $p(R_1, \ldots, R_{k-1})$ we have\(^5\)

$$
\text{Tr}_{R(1 \ldots k)}(X \cdot p(R_1, \ldots, R_{k-1})) = \text{Tr}_{R(1 \ldots k)}(p(R_1, \ldots, R_{k-1}) \cdot X).
$$

(2.10)

Hereafter, we use the following concise notation $\text{Tr}_{R(1 \ldots k)} := \text{Tr}_{R(1)} \ldots \text{Tr}_{R(k)}$.

Indeed, the operators $B$ and $C$ obey the relations (see [O])

$$
R_1 B_1 B_2 = B_1 B_2 R_1, \quad R_1 C_1 C_2 = C_1 C_2 R_1.
$$

(2.11)

It is evident that for any positive integer $k$ we have the same type relation

$$
R_k C_k C_{k+1} = C_k C_{k+1} R_k,
$$

and, therefore, the “string” $C_1 \ldots C_n$ commutes with any polynomial in $R_1, \ldots, R_{n-1}$ and, in particular, with the projectors $\mathcal{P}^{(k)}$, $k \leq n - 1$.

\(^4\)If $R$ is not even, analogs of formulae (2.8) and (2.9) are presented in [GPS].

\(^5\) Observe that formula (2.10) is valid even if the matrix $X$ has non-commutative entries. It is so since the matrix $p(R_1, \ldots, R_{k-1})$ is numerical. Emphasize that the classical formula $\text{Tr} AB = \text{Tr} BA$ is valid if the entries of $A$ commute with these of $B$.  

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Now, turn to the role of the operator $B$. Let us endow the space $V^*$ dual to $V$ with the right dual basis $\{x^i\}$. Thus, we have a paring

$$\langle \cdot, \cdot \rangle : V \otimes V^* \to \mathbb{K}, \quad \langle x_i, x^j \rangle = \delta^j_i.$$  

Note that the pairing of these spaces in the reverse order $\langle \cdot, \cdot \rangle : V^* \otimes V \to \mathbb{K}$ has to be defined by $\langle x^j, x_i \rangle = B^j_i$.

We treat elements of the space $V \otimes V^*$ as endomorphisms of the space $V$. To this end we introduce a linear operator by defining the following action of a basis element $b^j_i = x_i \otimes x^j$ onto a vector $x_k \in V$ according to the latter paring:

$$l^j_i(x_k) := x_i \langle x^j, x_k \rangle = x_i B^j_k. \quad (2.12)$$

The fact that the resulting map $V \otimes V^* \to \text{End}(V)$ is an isomorphism of vector spaces follows from the invertibility of the operator $B$.

Moreover, the usual product (i.e. the composition of endomorphisms)

$$\circ : \text{End}(V)^{\otimes 2} \to \text{End}(V)$$

is defined in the basis $\{l^j_i\}$ also via the operator $B$:

$$l^j_i \circ l^j_k = l^j_i B^j_k.$$

If $R$ is a skew-invertible symmetry of bi-rank $(m|0)$, then $\dim \text{Im} \mathcal{P}^{(m)} = 1$ and consequently this skew-symmetrizer can be presented as follows

$$\mathcal{P}^{(m)}(x_1 \otimes \ldots \otimes x_m) = u_{i_1\ldots i_m} v, \quad (2.13)$$

where

$$v = v^{j_1\ldots j_m} x_{j_1} \otimes \ldots \otimes x_{j_m} \quad \text{and} \quad u_{i_1\ldots i_m} v^{i_1\ldots i_m} = 1. \quad (2.14)$$

The tensors $u$ and $v$ are defined up to a non-trivial factor: $u \to u \lambda, v \to v/\lambda, \lambda \neq 0$. The element

$$v = v^{i_1\ldots i_m} x_{i_1} \otimes \ldots \otimes x_{i_m} \in \bigwedge^m_R(V)$$

is a generator of the 1-dimensional component $\bigwedge^m_R(V)$.

It can be proved that for any $x, y \in V$ there exist $z, u \in V$ such that

$$R_m \ldots R_2 R_1 (x \otimes v) = v \otimes z, \quad R_1 R_2 \ldots R_m (v \otimes y) = u \otimes v.$$

Therefore, we can define two linear maps $\mathcal{M}, \mathcal{N} : V \to V$ by setting $\mathcal{M}(x) = z$, and $\mathcal{N}(y) = u$. As was shown in [G] (in a slightly different normalization) the matrices $\|\mathcal{M}^j_i\|$ and $\|\mathcal{N}^j_i\|$ of these operators are given by the following formulae

$$\mathcal{M}^j_i = (-1)^{m-1} q m_q u_{i_1 a_2 \ldots a_{m-1} j} v^{i_1 a_2 \ldots a_{m-1}}, \quad (2.15)$$

$$\mathcal{N}^j_i = (-1)^{m-1} q m_q u_{i_1 a_2 \ldots a_{m-1} j} v^{i_1 a_2 \ldots a_{m-1} j}. \quad (2.16)$$

In general, the operators $\mathcal{M}$ and $\mathcal{N}$ are not scalar, but their product is always a scalar operator. Therefore, the operators $\mathcal{M}$ and $\mathcal{N}$ are scalar simultaneously. The determinant in the RTT algebra corresponding to a skew-invertible even symmetry $R$ is central iff these operators are scalar (see [G] for detail). Below, we prove that it is also true in the corresponding Yangian of RTT type.
Now, we exhibit an example of an involutive symmetry $R \in \text{End}(V^\otimes 2)$, $\dim V = 2$, for which the operator $\mathcal{N}$ (and consequently $\mathcal{M}$) is not scalar:

$$R = \begin{pmatrix}
1 & a & -a & ab \\
0 & 0 & 1 & -b \\
0 & 1 & 0 & b \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad a, b \in \mathbb{K}. \quad (2.17)$$

Checking the fact that $R$ is an involutive symmetry is straightforward. Also, it is even and its bi-rank is $(2|0)$. Moreover, we have

$$u_{11} = a, \quad u_{12} = -1, \quad u_{21} = 1, \quad u_{22} = 0, \quad \nu^{11} = 0, \quad \nu^{12} = -\frac{1}{2}, \quad \nu^{21} = \frac{1}{2}, \quad \nu^{22} = -\frac{b}{2}.$$ 

The matrix of the operator $\mathcal{N}$ reads

$$\|\mathcal{N}^j\| = \begin{pmatrix} 1 & a - b \\ 0 & 1 \end{pmatrix}.$$ 

Thus, the operator $\mathcal{N}$ is scalar iff $a = b$.

This example is very instructive, it shows that the tensor $u$ does not determine the tensor $v$ in a unique way (and vice versa). Indeed, if $b = 0$ the symmetric algebra $\text{Sym}_R(V) = T(V)/\langle \nu \rangle$ is classical (it is the polynomial algebra in two commutative generators). However, the tensor $u$ depends on $a$ and, consequently, we still have a family of the corresponding involutive symmetries, parameterised by $a$.

Also, note that the operator $\mathcal{N}$ constructed for a standard Hecke symmetry is scalar. Consequently, the quantum determinants in the corresponding RTT algebra and the $q$-Yangians of RTT type are central.

The following proposition is important for defining the determinants in the quantum algebras considered below.

**Proposition 6** Let $R$ be a skew-invertible even Hecke symmetry of bi-rank $(m|0)$. Then the tensors $u$ and $v$ (see (2.13), (2.14)) satisfy the following relations:

$$C_{i_1}^{a_1} \ldots C_{i_m}^{a_m} u_{j_1 \ldots j_m} = q^{-m} u_{i_1 \ldots i_m}, \quad C_{i_1}^{a_1} \ldots C_{i_m}^{a_m} \nu_{i_1 \ldots i_m} = q^{-m} \nu_{j_1 \ldots j_m},$$

and consequently

$$C_1 \ldots C_m \mathcal{P}^{(m)} = \mathcal{P}^{(m)} C_1 \ldots C_m = q^{-m} \mathcal{P}^{(m)}.$$

**Proof.** Since the string $C_1 \ldots C_m$ commutes with the projector $\mathcal{P}^{(m)}$, then using (2.13) and (2.14), we get the matrix equality:

$$C_{i_1}^{a_1} \ldots C_{i_m}^{a_m} u_{a_1 \ldots a_m} \nu_{j_1 \ldots j_m} = u_{i_1 \ldots i_m} v_{a_1 \ldots a_m} C_{i_1}^{j_1} \ldots C_{i_m}^{j_m}.$$ 

Contracting the both sides with $u_{j_1 \ldots j_m}$ and taking into account (2.14), we come to

$$C_{i_1}^{a_1} \ldots C_{i_m}^{a_m} u_{a_1 \ldots a_m} = \text{Tr}_{R(1 \ldots m)}(\mathcal{P}^{(m)}) u_{i_1 \ldots i_m}.$$ 

The $R$-trace of $\mathcal{P}^{(m)}$ can be easily calculated on the base of (2.11):

$$\text{Tr}_{R(m)}(\mathcal{P}^{(m)}) = \frac{(m - 1)q}{mq} \mathcal{P}^{(m-1)} \text{Tr}_{R(m)}\left(\frac{q^{m-1}}{(m - 1)q} I - R_{m-1}\right) \mathcal{P}^{(m-1)} = \frac{q^{-m}}{mq} \mathcal{P}^{(m-1)},$$

where we used (2.8) and $\text{Tr}_{R(m)} R_{m-1} = I_{1 \ldots m-1}$. 

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Then, by recurrence we arrive to the following formula

\[
\text{Tr}_{R(k+1...m)}(P^{(m)}_{1...m}) = q^{-m(m-k)} \frac{k_q(m-k)_q!}{m_q!} P^{(k)}_{1...k}, \quad 0 \leq k \leq m - 1.
\]

(2.18)

By setting \( k = 0 \) we get the final result

\[
\text{Tr}_{R(1...m)}(P^{(m)}) = q^{-m^2}.
\]

This entails the first claim of the proposition. The second claim can be proved in a similar way.

**Corollary 7** For any matrix \( X \) of an appropriate size we have

\[
\text{Tr}_{R(1...m)}(P^{(m)} X) = q^{-m^2} \text{Tr}_{R(1...m)}(P^{(m)} X).
\]

(2.19)

Thus, the usual trace and \( R \)-trace of the product \( P^{(m)} X \) differ by a non-trivial factor only.

### 3 Quantum matrix algebras related to symmetries

Before studying the braided Yangians and their RTT counterparts we consider their constant analogs, namely, the RE and RTT algebras. Below, we describe some properties of these algebras and define quantum analogs of symmetric polynomials.

Let \( R \) be any braiding. Together with the RTT and RE algebras consider the unital algebra, generated by elements \( l^j_i \), which are subject to the system of relations

\[
RL_1RL_1 - L_1RL_1R - h(RL_1 - L_1R) = 0, \quad L = \|l^j_i\|.
\]

(3.1)

This filtered algebra is called the **modified RE algebra** and is denoted \( \mathcal{L}(R, h) \) provided \( h \neq 0 \). If \( h = 0 \), then (3.1) turns into defining relations (1.7), the corresponding RE algebra will be denoted \( \mathcal{L}(R) \). In what follows, we mainly deal with the algebra \( \mathcal{L}(R, 1) \). The numeric factor \( h \) in the right hand side of (3.1) enables us to treat the algebra \( \mathcal{L}(R, 1) \) as a deformation of \( \mathcal{L}(R) \).

Note that the map

\[
\mathcal{L}(R) \rightarrow \mathcal{L}(R, h) : \quad L \mapsto L - \frac{h}{q - q^{-1}} I
\]

(3.2)

establishes an isomorphism of the algebras \( \mathcal{L}(R) \) and \( \mathcal{L}(R, h) \), provided \( q \neq \pm 1 \).

Also, note that if \( R = R_q \) is a standard Hecke symmetry, then at the limit \( q \rightarrow 1 \) the defining system of the algebra \( \mathcal{L}(R, 1) \) tends to that

\[
PL_1PL_1 - L_1PL_1P - (PL_1 - L_1P) = 0.
\]

which is nothing but a matrix writing of the relations between the usual generators of the algebra \( U(gl(N)) \).

Moreover, the algebra \( \mathcal{L}(R, h) \) is covariant with respect to the adjoint action of the QG \( U_q(sl(N)) \) (see [GPS]). By contrast, the RTT algebra is not covariant with respect to the adjoint action, it is only covariant with respect to two one-sided actions of \( U_q(sl(N)) \).

A very important property of the action (2.12) consists in the following: it defines a representation (called **covariant**) of the algebra \( \mathcal{L}(R, 1) \) in the space \( V \). Whereas the **contravariant** representation of this algebra in the space \( V^* \) is defined by the rule

\[
l^j_i(x^l) = -R^l_{ki} x^k.
\]

(3.3)
Note that the algebra $\mathcal{L}(R)$ has a braided bi-algebra structure, discovered by Sh.Madjid (see [Maj]). On the generators $l_i^j$ of the algebra $\mathcal{L}(R)$ the coproduct $\Delta : \mathcal{L}(R) \rightarrow \mathcal{L}(R) \otimes \mathcal{L}(R)$ can be presented in the following form
\begin{equation}
\Delta(L) = L \otimes L \quad \Leftrightarrow \quad \Delta(l_i^j) = \sum_k l_i^k \otimes l_k^j.
\end{equation}
(3.4)

However, its extension on the higher monomials in generators should be performed via the braiding $R^{End(V)}$ (see below (3.7)).

By using the isomorphism (3.2) at $h = 1$, it is easy to find the coproduct on the generators of the algebra $\mathcal{L}(R, 1)$:
\begin{equation}
\Delta(l_i^j) = l_i^j \otimes 1 + 1 \otimes l_i^j - (q - q^{-1}) l_i^k \otimes l_k^j.
\end{equation}

It also can be extended on the higher monomials in generators by the same braiding $R^{End(V)}$. Consequently, the tensor products $V^\otimes k \otimes V^\ast \otimes l, \forall k, l \in \mathbb{N}$, can be also endowed with a $\mathcal{L}(R, 1)$-module structures. According to the representation theory of the Hecke algebra for a generic $q$ it is possible to decompose the spaces $V^\otimes k \otimes V^* \otimes l$ into direct sums of some $\mathcal{L}(R, 1)$-invariant subspaces in such a way that the final quasi-tensor rigid category (called Schur-Weyl one and denoted $SW(V)$) looks like the category of $U(gl(m|n))$-modules. Besides, with the map (3.2) we can convert objects of the category $SW(V)$ into $\mathcal{L}(R)$-modules. In the sequel the category $SW(V)$ will also give rise to the corresponding representation categories of the braided Yangians. Emphasize that this construction is valid for any skew-invertible Hecke symmetry $R$ (see [GPS]).

Now, we introduce a useful notation, which allows us to treat the RE and RTT algebras in a similar manner:
\begin{equation}
L_\tau = L_1, \quad L_{\tau + 1} = R_k L_\tau R_k^{-1}, \quad k = 1, 2, ..., p - 1.
\end{equation}
(3.5)

Note that $L_\tau$ are $N^p \times N^p$ matrices with entries belonging to $\mathcal{L}(R)$ or $\mathcal{L}(R, h)$.

Using this notation we can represent the defining relations of the algebra $\mathcal{L}(R, h)$ as
\begin{equation}
RL_\tau L_\tau - L_\tau L_\tau R - h(L_\tau - L_\tau) = 0.
\end{equation}
(3.6)

For $h = 0$ we get the relations in the algebra $\mathcal{L}(R)$ looking like those in the RTT algebra

\begin{equation}
RL_\tau L_\tau - L_\tau L_\tau R = 0.
\end{equation}

Let us exhibit the aforementioned braiding $R^{End(V)}$ by using the same notation
\begin{equation}
R^{End(V)}(L_\tau \otimes L_\tau) = L_\tau \otimes L_\tau.
\end{equation}
(3.7)

As we noticed above, this braiding enters the extension of the coproducts in the algebras $\mathcal{L}(R)$ and $\mathcal{L}(R, h)$ onto higher monomials. Thus, for instance, in the algebra $\mathcal{L}(R)$ we have
\begin{equation}
\Delta(L_\tau L_\tau) = (L_\tau \otimes L_\tau)(L_\tau \otimes L_\tau) = (L_\tau L_\tau) \otimes (L_\tau L_\tau).
\end{equation}
(3.8)

Our next aim is to introduce some distinguished elements of the algebra $\mathcal{L}(R)$, which are quantum analogs of the symmetric polynomials.

**Definition 8** The elements of the algebra $\mathcal{L}(R)$ defined by
\begin{equation}
e_k(L) = \text{Tr}_{R^{(1...k)}}(\mathcal{P}(k) L_\tau L_\tau \ldots L_\tau)
\end{equation}
(3.9)
are called the (quantum) elementary symmetric polynomials. If a Hecke symmetry $R$ is even and its bi-rank is $(m|0)$, the highest order polynomial $e_m(L)$ is called the quantum determinant of $L$:
\begin{equation}
e_m(L) = \det_{RE}(L) = \text{Tr}_{R^{(1...m)}}(\mathcal{P}(m) L_\tau L_\tau \ldots L_\tau).
\end{equation}
An important property of $e_k(L)$ is that they belong to the center of the RE algebra $L(R)$.

Consider the quantum determinant in more details. In virtue of Corollary [4] with $X = L_1 L_2 \ldots L_{m}$ we can conclude that

$$\det_{RE}(L) = q^{-m^2} \text{Tr}_{(1\ldots m)}(P^{(m)} L_1 L_2 \ldots L_{m}).$$

Thus, the quantum determinant can be also defined via the usual trace. Moreover, it can be defined without applying any trace. Namely, we have the following proposition.

**Proposition 9** The following matrix identity holds true:

$$P^{(m)} L_1 L_2 \ldots L_{m} = L_1 L_2 \ldots L_{m} P^{(m)} = q^{m^2} \det_{RE}(L) P^{(m)}.$$  \hspace{1cm} (3.10)

**Proof.** First, we note that the defining relations of the RE algebra $L(R)$ can be “shifted to a higher position”:

$$R_k L_{m} L_{k+1} = L_{m} L_{k+1} R_k, \quad \forall k \geq 1.$$  

Besides, $R_k$ and $L_m$ commute with each other if $k \neq m, k \neq m + 1$. This entails that $P^{(m)}$ and $L_1 L_2 \ldots L_{m}$ commute:

$$P^{(m)} L_1 L_2 \ldots L_{m} = L_1 L_2 \ldots L_{m} P^{(m)}.$$  

On taking into account $P^{(m)} = P^{(m)} P^{(m)}$, we get the following

$$P^{(m)} L_1 \ldots L_{m} = P^{(m)} L_1 \ldots L_{m} P^{(m)} = P^{(m)} \text{Tr}_{(1\ldots m)}(P^{(m)} L_1 \ldots L_{m}).$$

It remains to employ the definition of the quantum determinant and formula (2.19).  \hspace{1cm} \blacksquare

Denote $H_k(q)$ the Hecke algebra with $k - 1$ generators $\sigma_1, \ldots, \sigma_{k-1}$. This algebra is represented in the spaces $V^\otimes k$ in a natural way $\rho_R(\sigma_i) = R_i$.

**Proposition 10** [IP] Let $R$ be a skew-invertible Hecke symmetry. For an element $z \in H_k(q)$ denote

$$\text{ch}(z) := \text{Tr}_{R(1\ldots k)}(Z L_1 \ldots L_{k}) = \text{Tr}_{R(1\ldots k)}(L_1 \ldots L_{k} Z), \quad Z = \rho_R(z).$$

Consider a linear subspace $\text{Ch}_R[L] \subset L(R)$ spanned by the unit and the elements $\text{ch}(z)$ for all $z \in H_k(q), k \geq 1$. The space $\text{Ch}_R[L]$ is a subalgebra of the center of the algebra $L(R)$.

Observe that the last equality in this formula is valid since the matrix $Z$ is a polynomial in $R_1, \ldots, R_{k-1}$.

The subalgebra $\text{Ch}_R[L]$ is called characteristic. Note that this claim is also valid for a skew-invertible involutive symmetry. In this case the role of the algebra $H_k(q)$ is played by the group algebra of the corresponding symmetric group.

Thus, the elementary symmetric polynomials (3.9) belong to the characteristic subalgebra of $L(R)$. Other distinguished elements of the characteristic subalgebra are (quantum) full symmetric polynomials, and more generally, Schur symmetric polynomials.

One more family of symmetric polynomials is the so-called (quantum) power sums defined as follows

$$p_k(L) = \text{Tr}_{R(1\ldots k)}(R_{k-1} \ldots R_{1} L_1 \ldots L_{k}) = \text{Tr}_{R(1\ldots k)}(L_1 \ldots L_{k} R_{k-1} \ldots R_{1}).$$  \hspace{1cm} (3.11)

Emphasize that these power sums can be reduced to the ”classical” form $p_k(L) = \text{Tr}_R(L^k)$. Let us demonstrate this property, for instance, for $k = 3$. Indeed, we have

$$p_3(L) = \text{Tr}_{R(123)} L_1 R_1 L_{T} R_{1}^{-1} R_{2} R_{1} L_{T} R_{1}^{-1} R_{2} R_{1} = \text{Tr}_{R(12)} \text{Tr}_{R(3)} L_1 R_1 L_{T} R_{1}^{-1} R_{2} R_{1} L_{T} =$$
Now, pass to the RTT algebras. In these algebras quantum analogs of the above symmetric polynomials can be defined by similar formulae but the factors $L_1$ have to be replaced by $T_k$.

Thus, the power sums in the RTT algebras can be defined as follows

$$ p_k(T) = \text{Tr}_{R(1...k)}(R_{k-1} \cdots R_1 T_1 \cdots T_k) = \text{Tr}_{R(1...k)}(T_1 \cdots T_k R_{k-1} \cdots R_1). \quad (3.12) $$

Also, the corresponding characteristic algebra can be defined in the same way, mutatis mutandis. The main differences with the RE algebras are the following ones. First, power sums (3.12) cannot be reduced to the form $\text{Tr}_R T_k$ (for $k \geq 2$). Second, the characteristic subalgebra generated by $p_k(T)$ is still commutative but not central. It is called a Bethe subalgebra.

Note that this way of constructing a Bethe subalgebra in an RTT algebra was suggested by J.-M. Maillet [Ma]. However, in the cited paper the usual trace was used instead of its $R$-analog.

Note, however, that if in the defining relations (1.6) of an RTT algebra one replaces the matrix $T$ by $C \cdot T$, these relations remain valid since the product $C_1 C_2$ commutes with $R$. So, the map $T \mapsto C \cdot T$ gives rise to an automorphism of the RTT algebra. This entails that though the corresponding Bethe subalgebras differ, the proof of their commutativity is the same. Since the characteristic subalgebra in the RTT algebra is not central, a natural problem arises to describe the center of this algebra. If $R$ is of bi-rank $(m|0)$, the only non-trivial candidate to the role of a central element is the determinant $\det_{RTT}(T)$. However, as we said above its centrality depends on the symmetry $R$.

**Remark 11** It is possible to define the RTT and RE algebras via the braidings $R$ of the Birman-Wenzl-Murakami type. However, by contrast with the algebras considered above, if $R$ comes from a QG $U_q(A_N)$ of a series different from $A_N$, the RTT and RE algebras are not deformations of the algebra $\text{Sym}(\gg)$. This claim can be proved by checking that the corresponding semiclassical counterparts are not Poisson brackets.

### 4 Baxterization of symmetries

Now, let us describe the *Baxterization* procedure, which associates a current $R$-matrix with a given involutive or Hecke symmetry $R$. In this section we assume that $\mathbb{K} = \mathbb{C}$.

**Proposition 12** Consider the sum

$$ R(u, v) = R + g(u, v)I, \quad (4.1) $$

where $g(u, v)$ is a function of two arguments. Assume that the function $g(u, v)$ depends only on the difference of the arguments: $g(u, v) = f(u - v)$ and that $f(z)$ is a non-constant meromorphic function. If $R$ is involutive, then $R(u, v)$ is an $R$-matrix iff

$$ g(u, v) = \frac{a}{u - v}. \quad (4.2) $$

If $R$ is a Hecke symmetry, then $R(u, v)$ is an $R$-matrix iff

$$ g(u, v) = \frac{q - q^{-1}}{bu - v - 1}. \quad (4.3) $$

Here $a$ and $b \neq 1$ are arbitrary nonzero complex numbers.
proof. assume $R$ to be a Hecke symmetry. Consider the sum (4.1) and set $g(u, v) = f(u - v)$. Then, imposing the quantum Yang-Baxter equation\footnote{We use two notations $R(u, v)$ and $R(x)$ for the $R$-matrices under consideration. Here $x = u - v$ if the function $g(u, v)$ is defined as in (4.2) or (4.5) and $x = u/v$ if $g(u, v)$ is of the form (4.3). We hope that this notation does not lead to a misunderstanding.} 

$$R_{12}(x)R_{23}(x + y)R_{12}(x) = R_{23}(y)R_{12}(x + y)R_{23}(x),$$

on the sum $R(x) = R + f(x)I$ we get the following functional equation on $f(x)$:

$$(q - q^{-1})f(x + y) + f(x + y)(f(x) + f(y)) = f(x)f(y).$$

Introducing a new function $h(x) = \frac{g(q^{-1})}{f(x)}$ we get the equation:

$$h(x)h(y) + h(x) = h(x + y). \tag{4.4}$$

Consequently, the function $\varphi(x) = h(x) + 1$ meets the equation $\varphi(x + y) = \varphi(x)\varphi(y)$. This entails $\varphi(x) = b^x$, $b \neq 1$. The case of an involutive symmetry $R$ is left to the reader.

The current $R$-matrix corresponding to (4.2) is called rational, whereas that corresponding to (4.3) is called trigonometric. Note that the Yang $R$-matrices and their super-analogs are particular cases of rational braidings. Also, note that the first examples of the Baxterization procedure for general involutive and Hecke symmetries were presented in [G].

In general, the parameter $b$ in (4.3) is independent of $q$, but if we want a trigonometric $R$-matrix to turn into the corresponding rational $R$-matrix as $q \to 1$ (provided a Hecke symmetry $R_q$ tends to an involutive symmetry when $q \to 1$) we put $b = q^{-2/a}$. Then we have

$$R(x) = R_q - \frac{q^x}{(\frac{a}{q})_q}I, \tag{4.5}$$

where $x = u - v$. As $q \to 1$ this $R$-matrix tends to

$$R_1 = \frac{a}{x}I. \tag{4.6}$$

Below, we deal with the current $R$-matrices (4.5) and (4.6), where for the sake of simplicity we take $a = 1$.

Changing the variables $b^{-u} \to u$, $b^{-v} \to v$ in (4.3), we get the following form of the trigonometric current $R$-matrix:

$$R(u, v) = R - \frac{u(q - q^{-1})}{u - v}I. \tag{4.7}$$

Evidently, it depends only on the quotient $x = u/v$.

Consider an example when $R$ is the standard Hecke symmetry with $N = 2$. The corresponding current $R$-matrix is (here $x = u/v$):

$$R(x) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} - \frac{(q - q^{-1})x}{x - 1}I = \begin{pmatrix} \frac{q^{-x}x}{x - 1} & 0 & 0 & 0 \\ 0 & \frac{(q^{-1} - q)}{1 - x} & 1 & 0 \\ 0 & 1 & \frac{(q^{-1} - q)x}{1 - x} & 0 \\ 0 & 0 & 0 & \frac{q^{-x}x}{x - 1} \end{pmatrix}. \tag{4.8}$$
It is easy to see that the trigonometric current $R$-matrix \((4.7)\) obeys the following relation

$$R(u, v)R(v, u) = \left(1 - \frac{uv(q - q^{-1})^2}{(u - v)^2}\right)I,$$  \hspace{1cm} (4.9)

whereas the rational current $R$-matrix \((4.6)\) with $x = u - v$ is subject to

$$R(u, v)R(v, u) = \left(1 - \frac{1}{(u - v)^2}\right)I.$$  \hspace{1cm} (4.10)

Therefore, the following current $R$-matrices

$$\mathbf{R}(u, v) = \frac{R(u, v)}{q - \frac{a(q - q^{-1})}{(u - v)}} \quad \text{and} \quad \mathbf{R}(u, v) = \frac{R(u, v)}{1 - \frac{1}{(u - v)}},$$  \hspace{1cm} (4.11)

corresponding to Hecke symmetries and involutive ones respectively turn out to be involutive in the following sense: $\mathbf{R}(u, v)\mathbf{R}(v, u) = 1$.

We use the braidings $\mathbf{R}(u, v)$ in the last section in order to define a version of bosonic algebras.

5 Evaluation morphisms, bi-algebra structures

Let $R : V^\otimes 2 \to V^\otimes 2$, $\dim V = N$, be a skew-invertible involutive or Hecke symmetry. Introduce the following notation

$$[A, B]_R = RA_{\mathfrak{T}}B_{\mathfrak{T}} - B_{\mathfrak{T}}A_{\mathfrak{T}} = RA_1RB_1R^{-1} - B_1RA_1,$$

where $A$ and $B$ are $N \times N$ matrices. Thus, the expression $[A, B]_R$ is an $N^2 \times N^2$ matrix.

For further considerations it is useful to keep in mind the following simple properties valid for an involutive $R$:

$$[I, I]_R = [A, I]_R = [I, A]_R = 0,$$

where $I$ is the unit matrix and $A$ is an arbitrary $N \times N$ matrix. However, for a Hecke symmetry the last identity is changed:

$$[I, I]_R = [A, I]_R = 0, \quad [I, A]_R = (q - q^{-1})(A_{\mathfrak{T}} - A_{\mathfrak{T}}).$$

Now, consider the braided Yangian $\mathbf{Y}(R)$, corresponding to an involutive symmetry $R$. Taking into account the explicit form of the rational current $R$-matrix $R(u, v)$, one can write the defining relations of the braided Yangian \((1.10)\) as follows:

$$[L(u), L(v)]_R = \frac{1}{u - v} \left(L_{\mathfrak{T}}(u)L_{\mathfrak{T}}(v) - L_{\mathfrak{T}}(v)L_{\mathfrak{T}}(u)\right).$$  \hspace{1cm} (5.1)

Let us expand the matrix $L(u)$ in a series \((1.4)\). On substituting this series in \((5.1)\) and equating the coefficients of $u^{-r}v^{-s}$, we get

$$\left[L[r + 1], L[s]\right]_R - \left[L[r], L[s + 1]\right]_R = L_{\mathfrak{T}}[r]L_{\mathfrak{T}}[s] - L_{\mathfrak{T}}[s]L_{\mathfrak{T}}[r], \quad \forall r, s \geq 0.$$

These relations are similar to those from \[Mo\]. Also, in the same way as in \[Mo\] we can prove the following claim.

**Proposition 13** Let $R$ be a skew-invertible involutive symmetry. Then the defining relations of the braided Yangian $\mathbf{Y}(R)$ are equivalent to the system

$$\left[L[r], L[s]\right]_R = \sum_{a=1}^{\min(r, s)} (L_{\mathfrak{T}}[a - 1]L_{\mathfrak{T}}[r + s - a] - L_{\mathfrak{T}}[r + s - a]L_{\mathfrak{T}}[a - 1]), \quad r, s \geq 1.$$  \hspace{1cm} (5.2)
The next proposition is important for the representation theory of the braided Yangians.

**Proposition 14** Let $M$ be the generating matrix of the modified RE algebra $\mathcal{L}(R,1)$ (see (3.1)) corresponding to a given skew-invertible involutive symmetry $R$. Then the map

$$L(u) \mapsto I + \frac{M}{u}$$

(5.3)

defines a surjective morphism $Y(R) \to \mathcal{L}(R,1)$.

Besides, the map $M \mapsto L[1]$ defines an injective morphism $\mathcal{L}(R,1) \to Y(R)$.

**Proof.** The defining relations of the algebra $\mathcal{L}(R,1)$ are given by (3.6) at $h = 1$:

$$[M, M]_R = M^2 - M_1.$$

(5.4)

We have to check that the matrix (5.3) does satisfy the relation (5.1), provided that (5.4) is fulfilled. This, in turn, is equivalent to checking the relation

$$(u - v) \left[ \frac{M}{u}, \frac{M}{v} \right]_R = \left( I + \frac{M}{u} \right) \left( I + \frac{M}{v} \right) - \left( I + \frac{M}{v} \right) \left( I + \frac{M}{u} \right),$$

which is straightforward.

The second claim of the Proposition is a direct consequence of (5.2) written for $r = s = 1$ (recall that $L[0] = I$):

$$[L[1], L[1]]_R = L[2][1] - L[1],$$

which is equivalent to the defining relations of the algebra $\mathcal{L}(R,1)$.

**Definition 15** Similarly to the classical case, the map (5.3) is called the evaluation morphism.

Now, let us assume $R$ to be a skew-invertible Hecke symmetry. Taking into account the explicit form (4.7) of the current $R$-matrix $R(u,v)$, we have the following defining relations for the braided Yangian $Y(R)$:

$$(u - v) [L(u), L(v)]_R = u(q - q^{-1}) (L[1][u] L[2](v) - L[1](v) L[2][u]).$$

(5.5)

By expanding the matrix $L(u)$ in a series (1.4), we arrive to the following system

$$[L[r + 1], L[s]]_R - [L[r], L[s + 1]]_R = (q - q^{-1}) (L[r + 1][s] L[2][s] - L[r + 1][s] L[2][s]), \quad r, s \geq 0.$$

Upon dividing the relation (5.5) by $u - v$ and using the expansion $\frac{u}{u - v} = \sum_{p=0}^{\infty} \frac{v^p}{u^p}$ we get the following proposition.

**Proposition 16** Let $R$ be a skew-invertible Hecke symmetry. Then the defining relations of the braided Yangian $Y(R)$ are equivalent to the system

$$[L[r], L[s]]_R = (q - q^{-1}) \sum_{a=0}^{\min(r, s-1)} (L[a][L[r + s - a] - L[r + s - a][L[a]]), \quad r, s - 1 \geq 0.$$

In this case there also exists an evaluation morphism but its surjectivity is not clear, since now the map $M \mapsto L[1]$ does not define a morphism $\mathcal{L}(R) \to Y(R)$.  

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Proposition 17 Let \( M \) be the generating matrix of the RE algebra \( \mathcal{L}(R) \) corresponding to a given skew-invertible Hecke symmetry \( R \). Then the map

\[
L(u) \mapsto I + \frac{M}{u}
\]

(5.6)
defines a morphism \( Y(R) \to \mathcal{L}(R) \).

Proof. We have to check that if the matrix \( M \) is subject to the relation \([M_1, M_2]_R = 0\), then

\[
(u - v) \left( R \left( I + \frac{M_1}{u} \right) R \left( I + \frac{M_1}{v} \right) - \left( I + \frac{M_1}{v} \right) R \left( I + \frac{M_1}{u} \right) \right) =
\]

(5.7)

\[
(q - q^{-1}) u \left( \left( I + \frac{M_1}{u} \right) R \left( I + \frac{M_1}{v} \right) - \left( I + \frac{M_1}{v} \right) R \left( I + \frac{M_1}{u} \right) \right).
\]

This equality can be easily verified by a straightforward calculation with the use of the relation \( R^2 = (q - q^{-1}) R + I \). \( \blacksquare \)

Let us emphasize the difference between the braided Yangians related to involutive and Hecke symmetries. In the former case we have the evaluation morphism, which maps the braided Yangian \( Y(R) \) into the modified RE algebra \( \mathcal{L}(R, 1) \), whereas in the latter case it maps the braided Yangian \( Y(R) \) into the RE algebra \( \mathcal{L}(R) \).

By using these evaluation morphisms and Schur-Weyl categories of finite dimensional \( \mathcal{L}(R, 1) \)-modules discussed in section 3, we can construct some representations of the braided Yangians. Given an object \( U \) of this category, we can covert it into a module over the braided Yangian by putting

\[
\tilde{l}_i^j(u) \triangleright x = x \delta_i^j + \frac{1}{u} \rho_U(\tilde{l}_i^j) \triangleright x, \quad x \in U
\]

(5.7)

where by \( \triangleright \) we denote an action of a linear operator. The symbol \( \rho_U \) stands for the corresponding representation of the algebra \( \mathcal{L}(R, 1) \) if \( R \) is involutive, and for that of \( \mathcal{L}(R) \) if \( R \) is a Hecke symmetry. Thus, the entries of the matrix \( L[1] \) are represented in \( U \) according to \( \rho_U \), whereas \( \rho_U(L[k]) = 0 \) for \( k \geq 2 \).

The representation \( \tilde{l}_i^j(u) \triangleright x \) of the braided Yangian is called the evaluation representation.

Remark 18 If \( R \) is involutive, the map \( L(u) \to L(u - a), \ a \in \mathbb{K} \) is an automorphism of the braided Yangian \( Y(R) \). Consequently, the map \( \tilde{l}_i^j(u) \triangleright x \) remains a representation of the braided Yangian \( Y(R) \), if we replace the denominator \( u \) in \( \tilde{l}_i^j(u) \) by \( u - a \) with any number \( a \in \mathbb{K} \).

If \( R \) is a Hecke symmetry and the corresponding braided Yangian \( Y(R) \) is presented in the form \( \mathcal{L}(R, 1) \), then the map \( L(u) \to L(au), \ a \in \mathbb{K}, \ a \neq 0 \) is an automorphism of the braided Yangian \( Y(R) \). Consequently, a similar claim is valid for the braided Yangian \( Y(R) \), if we replace the denominator \( u \) in \( \tilde{l}_i^j(u) \) by \( au \).

Also, remark that since in general the braided Yangians do not admit triangular decompositions, analogs of the Drinfeld modules cannot be defined.

The representations of the braided Yangians described in this section can be multiplied with the use of the following braided bi-algebra structure of \( Y(R) \). On the generators of \( Y(R) \) the coproduct is defined by:

\[
\Delta(L(u)) = L(u) \otimes L(u).
\]

To extend the above definition on monomials in generators, we again use the operator \( R^{\text{End}(V)} \) discussed in section 3. Thus, similarly to \( (3.8) \) we have

\[
\Delta(L_1(u)L_2(v)) = (L_1(u) \otimes L_1(u))(L_2(v) \otimes L_2(v)) = (L_1(u)L_2(v)) \otimes (L_1(u)L_2(v))
\]
and so on. Thus, the coproduct is similar to that in the RE algebra \( \mathcal{L}(R) \).

As for the RTT-type Yangians, the corresponding coproduct is similar to that in the RTT algebras and the bi-algebra structure is usual.

However, in order to define the evaluation morphisms for these Yangians we do not impose the condition \( T[0] = I \) in the expansion of the function \( T(u) \). Then we introduce the evaluation map by

\[
T(u) \to T + \frac{T}{u},
\]

It is a morphism of algebras if the target algebra is defined by the relations

\[
RT_1 T_2 = T_1 T_2 R, \quad R T \bar{T}_1 \bar{T}_2 = \bar{T}_1 \bar{T}_2 R, \quad R \bar{T}_1 T_2 = T_1 \bar{T}_2 R,
\]

provided \( R \) is a Hecke symmetry, and

\[
RT_1 T_2 = T_1 T_2 R, \quad RT \bar{T}_1 \bar{T}_2 - T_1 T_2 R = T_1 \bar{T}_2 - \bar{T}_1 T_2, \quad R \bar{T}_1 T_2 = T_1 \bar{T}_2 R,
\]

provided \( R \) is involutive. In the standard case the former algebra is similar to the QG \( U_q(gl(m)) \), as presented in [Mo]. We do not know any treatment of the latter algebra.

## 6 Determinant, symmetric polynomials, CHN identities

In this section we define quantum analogs of some symmetric polynomials in the braided Yangians \( Y(R) \) and show that the quantum determinant of the generating matrix \( L(u) \) is always central in \( Y(R) \). Recall that the quantum determinant is the highest order elementary symmetric polynomial, which is well-defined provided the initial Hecke symmetry \( R \) is even. Also, we establish the so-called Cayley-Hamilton-Newton identities for the matrix \( L(u) \) and their corollaries — the Newton relations and the Cayley-Hamilton identity.

Assume \( R \) to be current \( R \)-matrix associated with a Hecke symmetry. Then according to footnote 6 we present the defining relations of the corresponding Yangian \( Y(R) \) as follows

\[
R(u/v)L_T(u)L_{\bar{T}}(v) = L_T(v)L_{\bar{T}}(u)R(u/v), \quad R(x) = R - \frac{(q - q^{-1})x}{x - 1}. \tag{6.1}
\]

Note, that setting \( x = q^{2k} \), we get \( R(q^{2k}) = R - \frac{q^k}{q^{-k}}I \).

Now, we exhibit some technical formulæ which will be used below. They can be easily proved by induction.

\[
P_{1\ldots k+1}^{(k+1)} = \frac{(-1)^k}{(k + 1)_q} R_1(q^2)R_2(q^4)\ldots R_k(q^{2k}) P_{1\ldots k}^{(k)}
\]

\[
= \frac{(-1)^k}{(k + 1)_q} R_k(q^2)R_{k-1}(q^4)\ldots R_1(q^{2k}) P_{2\ldots k+1}^{(k)} \tag{6.2}
\]

\[
= \frac{(-1)^k}{(k + 1)_q} P_{1\ldots k}^{(k)} R_k(q^{2k})R_{k-1}(q^{2(k-1)})\ldots R_1(q^{2})
\]

\[
= \frac{(-1)^k}{(k + 1)_q} P_{2\ldots k+1}^{(k)} R_1(q^{2k})R_2(q^{2(k-1)})\ldots R_k(q^{2}).
\]

To simplify the writing we introduce the following compact notation

\[
R_{i}^{(\pm)}(u) = \begin{cases} R_i(u)R_{i+1}(q^{\pm 2}u)\ldots R_j(q^{\pm 2(j-i)}u), & \text{if } j \geq i \\ R_i(u)R_{i-1}(q^{\pm 2}u)\ldots R_j(q^{\pm 2(i-j)}u), & \text{if } i > j. \end{cases} \tag{6.3}
\]
Proof. The braided Yangian defining relations (6.1) allow one to write the following identity
\[ u \begin{pmatrix} \mathcal{P}_{1\ldots k}^{(k+1)} = \frac{(-1)^k}{(k+1)^q} R_{1\to k}^{(+)}(q^2) \mathcal{P}_{1\ldots k}^{(k)} \end{pmatrix}, \]

With the use of (6.2) we can prove the following lemma.

**Lemma 19** For any value of the parameter \( u \) and for any integer \( k \geq 1 \) the following holds
\[ R_{1\to k}^{(+)}(q^{-2(k-1)}u) \mathcal{P}_{1\ldots k}^{(k)} = \mathcal{P}_{2\ldots k+1}^{(k)} R_{1\to k}^{(-)}(u). \]

If the rank of a Hecke symmetry is \((m|0)\), then for the highest order skew-symmetrizer \( \mathcal{P}^{(m)} \) the following identities hold true:
\[ R_{1\to m}^{(+)}(q^{-2(m-1)}u) \mathcal{P}_{1\ldots m}^{(m)} = (-1)^{m-1} q m q \left( \frac{u - q^2m}{q^2u - q^2m} \right) \mathcal{P}_{2\ldots m+1}^{(m)} \mathcal{P}_{1\ldots m}^{(m)}, \]
\[ \mathcal{P}_{2\ldots m+1}^{(m)} R_{1\to m}^{(-)}(u) = (-1)^{m-1} q m q \left( \frac{u - q^2m}{q^2u - q^2m} \right) \mathcal{P}_{2\ldots m+1}^{(m)} \mathcal{P}_{1\ldots m}^{(m)}. \]

Now, we define analogs of the elementary symmetric polynomials for the braided Yangian \( \mathbf{Y}(R) \) associated with a skew-invertible Hecke symmetry of bi-rank \((m|n)\).

**Definition 20** Let \( L(u) \) be the generating matrix of a braided Yangian, associated with a Hecke symmetry \( R \). The elements
\[ e_0(u) = 1, \quad e_k(u) = \text{Tr}_{R(1\ldots k)} \left( \mathcal{P}_{1\ldots k}^{(k)} L_\mathcal{Y}(u) L_\mathcal{Y}(q^{-2}u) \ldots L_\mathcal{Y}(q^{-2(k-1)}u) \right), \quad k \geq 1 \]
will be called the elementary symmetric polynomials.

If the rank of a Hecke symmetry is \((m|0)\), then the highest order nonzero elementary symmetric polynomial \( e_m(u) \) is called the determinant of \( L(u) \) and denoted \( \det_{\mathbf{Y}(R)}(u) \).

An important property of the elements \( e_k(u) \) is that they commute with each other and consequently their Laurent coefficients in the corresponding series in \( u^{-1} \) generate a commutative subalgebra in the braided Yangian \( \mathbf{Y}(R) \). The proof of this fact, as well as the analysis of the structure of the commutative subalgebra, will be given in our subsequent publication.

Now, we consider the highest order elementary symmetric polynomial \( e_m(u) = \det_{\mathbf{Y}(R)}(u) \) and prove its main property.

**Proposition 21** If \( R \) is a skew-invertible even Hecke symmetry of bi-rank \((m|0)\) then in the corresponding braided Yangian \( \mathbf{Y}(R) \) the determinant \( e_m(u) = \det_{\mathbf{Y}(R)}(u) \) defined by (6.5) with \( k = m \) is central, i.e.
\[ \det_{\mathbf{Y}(R)}(u)L(v) = L(v)\det_{\mathbf{Y}(R)}(u), \quad \forall u, v. \]

**Proof.** The braided Yangian defining relations (6.1) allow one to write the following identity which is valid for arbitrary values of the parameters \( u \) and \( v \):
\[ R_{1\to m}^{(-)}(u/v)L_\mathcal{Y}(u)L_\mathcal{Y}(q^{-2}u) \ldots L_\mathcal{Y}(q^{-2(m-1)}u)L_{m+1}(v) \]
\[ = L_\mathcal{Y}(v)L_\mathcal{Y}(u)L_\mathcal{Y}(q^{-2}u) \ldots L_{m+1}(q^{-2(m-1)}u)R_{1\to m}^{(-)}(u/v). \]

We multiply the both sides of this identity by \( \mathcal{P}_{2\ldots m+1}^{(m)} \) on the left hand side and then use the second relation of (6.4):
\[ \phi(u/v)\mathcal{P}_{2\ldots m+1}^{(m)} \]
\[ = L_\mathcal{Y}(v)\mathcal{P}_{2\ldots m+1}^{(m)}L_\mathcal{Y}(u) \ldots L_{m+1}(q^{-2(m-1)}u)L_{m+1}(v) \]
\[ = L_\mathcal{Y}(v)\mathcal{P}_{2\ldots m+1}^{(m)}L_\mathcal{Y}(u) \ldots L_{m+1}(q^{-2(m-1)}u)L_{m+1}(v), \]

Now, we consider the highest order elementary symmetric polynomial \( e_m(u) = \det_{\mathbf{Y}(R)}(u) \) and prove its main property.
where
\[ \phi(u) = (-1)^{m-1} q m_q \frac{(u - q^{2m})}{(q^{2u} - q^{2m})} \]
stands for the scalar factor in the right hand side of (6.4).

Taking into account the defining relations (6.1) and the recurrent formulae (6.2) we get
\[ \mathcal{P}_{1...m}^{(m)} L_T(u) L(q^{2u}) \ldots L_{m-1}(q^{2(m-1)u}) = L_T(q^{-2(m-1)u}) \ldots L_{m-1}(q^{2u}) L(u) \mathcal{P}_{1...m}^{(m)}. \quad (6.8) \]
Then in full analogy with Proposition 9 we conclude that
\[ \mathcal{P}_{1...m}^{(m)} L_T(u) \ldots L_{m-1}(q^{-2(m-1)u}) = q^{m^2} e_m(u) \mathcal{P}_{1...m}^{(m)}. \quad (6.9) \]
So, the left hand side of (6.7) takes the form:
\[ \phi(u/v)q^{m^2} \mathcal{P}_{2...m+1}^{(m)} e_m(u) L_{m+1}(v). \]
Transforming the right hand side of (6.7) in the same way we obtain:
\[ L_T(v) \mathcal{P}_{2...m+1}^{(m)} L_2(u) \ldots L_{m+1}(q^{-2(m-1)u}) R_1^{-1}(u/v) = \phi(u/v)q^{m^2} L_T(v) e_m(u) \mathcal{P}_{2...m+1}^{(m)} \mathcal{P}_{1...m}^{(m)}. \]
Therefore, we come to the following intermediate result:
\[ \mathcal{P}_{2...m+1}^{(m)} \mathcal{P}_{1...m}^{(m)} e_m(u) L_{m+1}(v) = L_T(v) e_m(u) \mathcal{P}_{2...m+1}^{(m)} \mathcal{P}_{1...m}^{(m)}. \quad (6.10) \]
Now, on taking into account that \( L_{m+1}(v) = R_m \ldots R_1 L_1(v) R_1^{-1} \ldots R_m^{-1} \), we multiply the identity (6.10) by \( R_m \ldots R_1 \) from the right and use the following properties of the highest skew-symmetrizer (see [II-I])
\[ \mathcal{P}_{1...m}^{(m)} R_m \ldots R_1 = (-1)^{m-1} q m_q \mathcal{P}_{1...m}^{(m)} \mathcal{P}_{2...m+1}^{(m)}; \]
\[ \mathcal{P}_{2...m+1}^{(m)} \mathcal{P}_{1...m}^{(m)} \mathcal{P}_{2...m+1}^{(m)} = m_q^2 \mathcal{P}_{2...m+1}^{(m)}. \]
Finally, we get
\[ \mathcal{P}_{2...m+1}^{(m)} e_m(u) L_1(v) = L_1(v) e_m(u) \mathcal{P}_{2...m+1}^{(m)}. \]
Since \( \text{Tr} R(2...m+1) \mathcal{P}_{2...m+1}^{(m)} = q^{-m^2} \) is nonzero, then on applying the corresponding \( R \)-trace to the both sides of the above equality we come to the matrix identity
\[ e_m(u) L_1(v) = L_1(v) e_m(u). \]
This completes the proof.

**Remark 22** Note, that the same considerations can be carried out for any RTT-type Yangian with the defining relations
\[ R(u/v) T_1(u) T_2(v) = T_1(v) T_2(u) R(u/v). \]
In the same way we can come to the intermediate formula analogous to (6.10):
\[ \mathcal{P}_{2...m+1}^{(m)} \mathcal{P}_{1...m}^{(m)} e_m(u) T_{m+1}(v) = T_1(v) e_m(u) \mathcal{P}_{2...m+1}^{(m)} \mathcal{P}_{1...m}^{(m)}; \]
where the elementary symmetric polynomials \( e_k(u) \) in this case are defined by the relation
\[ e_k(u) = \text{Tr} R(1...k) \left( \mathcal{P}_{1...k}^{(k)} T_1(u) T_2(q^{-2}u) \ldots T_k(q^{-2(k-1)}u) \right). \quad (6.11) \]
Writing the skew-symmetrizer \( \mathcal{P}^{(m)} \) in terms of structural tensors \( u \) and \( v \) (2.13) and using the definition (2.16) of the matrix \( \mathcal{N} \) we can transform the above identity to the following form:
\[ \mathcal{N} \cdot T(v) e_m(u) = e_m(u) T(v) \cdot \mathcal{N}. \]
So, in the RTT-type Yangian the determinant \( e_m(u) \) is central iff the matrix \( \mathcal{N} \) (and, therefore, \( \mathcal{M} \)) is scalar.
Now, we go back to the Yangians \( Y(R) \). Let us introduce the skew-symmetric powers of the generating matrix \( L(u) \) by the rule:

\[
L^{\wedge 1}(u) = L(u), \quad L^{\wedge k}(u) = \text{Tr}_{R(2\ldots k)} \left( \mathcal{P}^{(k)}_{12\ldots k} L(u) L(q^{-2}(k-1) u) \right) \quad k \geq 2. \tag{6.12}
\]

If the bi-rank of the Hecke symmetry \( R \) is \( (m|0) \), then \( L^{\wedge k}(u) \) is equal to zero for \( k > m \). Also, it is evident that \( e_k(u) = \text{Tr}_R(L^{\wedge k}(u)) \) for any \( k \geq 1 \).

Besides, we define quantum analogs of the matrix powers of the generating matrix:

\[
L^{[k]}(u) = \text{Tr}_{R(2\ldots k)} \left( L(q^{-2(k-1)} u) L(q^{-2(k-2)} u) \ldots L(u) R_{k-1} \ldots R_1 R_1 \right), \quad k \geq 1. \tag{6.13}
\]

Taking into account the definition of \( L \) and the property \( \text{Tr}_R(s+1) R_s = I \), the above expression for the matrix power can be reduced to the product

\[
L^{[k]}(u) = L(q^{-2(k-1)} u) L(q^{-2(k-2)} u) \ldots L(u). \tag{6.14}
\]

This reducing is realised in the same way as it was done in Section 3 while computing \( p_3(L) \). However, in the braided Yangians the quantum matrix powers \( L^{[k]}(u) \) differ from the classical ones \( L^k(u) \) by multiplicative shifts of the parameters in the matrices \( L(u) \).

By analogy with the elementary symmetric polynomials we define the power sums:

\[
p_k(u) = \text{Tr}_R(L^{[k]}(u)). \tag{6.15}
\]

**Proposition 23** The skew-symmetric and matrix powers of \( L(u) \) satisfies the series of Cayley-Hamilton-Newton identities

\[
(-1)^{k+1} q_k L^{\wedge k}(u) = \sum_{p=0}^{k-1} (-q)^p L^{[k-p]}(u) q^{-2p} u e_p(u), \quad k \geq 1. \tag{6.16}
\]

**Proof.** First of all, taking into account (6.13), we can identically rewrite the definition (6.12) in the form:

\[
L^{\wedge k}(u) = \text{Tr}_{R(2\ldots k)} \left( L(q^{-2(k-1)} u) L(q^{-2(k-2)} u) \ldots L(u) \mathcal{P}^{(k)}_{12\ldots k} \right). \tag{6.17}
\]

Then we use the recurrent formula for the skew-symmetrizers \( \mathcal{P}^{(k)} \) equivalent to (2.1):

\[
k_q \mathcal{P}^{(k)}_{1\ldots k} = \mathcal{P}^{(k-1)}_{2\ldots k} \left( q^{k-1} - (k-1)_q R_1 \right) \mathcal{P}^{(k-1)}_{2\ldots k}.
\]

So, on applying the above relation to (6.17) we obtain:

\[
k_q L^{\wedge k}(u) = q^{k-1} L(q^{-2(k-1)} u) e_{k-1}(u) - (k-1)_q \text{Tr}_{R(2\ldots k)} \left( L(q^{-2(k-1)} u) L(q^{-2(k-2)} u) \ldots L(u) \mathcal{P}^{(k-1)}_{2\ldots k} R_1 \mathcal{P}^{(k-1)}_{2\ldots k} \right). \tag{6.18}
\]

Here, to get the first term in the right hand side we applied the identity

\[
\text{Tr}_{R(r\ldots r+k)} \left( \mathcal{P}^{(k-1)}_{r\ldots r+k} L(u) \ldots L_{r+k}(x+k) \right) = e_{k+1}(u) I_{1\ldots r+1}, \quad \forall r \geq 1.
\]

Let us simplify the second term in the right hand side of (6.18). For this purpose we use the cyclic property of the \( R \)-trace:

\[
\text{Tr}_{R(r\ldots r+k)} \left( \mathcal{P}^{(k-1)}_{r\ldots r+k} X \right) = \text{Tr}_{R(r\ldots r+k)} \left( X \mathcal{P}^{(k-1)}_{r\ldots r+k} \right),
\]

then

\[
\text{Tr}_{R(r\ldots r+k)} \left( \mathcal{P}^{(k-1)}_{r\ldots r+k} L(u) \ldots L_{r+k}(x+k) \right) = \epsilon_{k+1}(u) I_{1\ldots r+1}, \quad \forall r \geq 1.
\]
where $X$ stands for any matrix of an appropriate size. Consequently, we have

\[
\text{Tr}_{R(2\ldots k)} \left( P_{2\ldots k}^{(k-1)} L_{T}^{q^{-2(k-1)u}} L_{T}^{q^{-2(k-2)u}} \ldots L_{T}^{q^{-2(u)}} \right) = (\text{applying } (6.18))
\]

\[
= \text{Tr}_{R(2\ldots k)} \left( P_{2\ldots k}^{(k-1)} L_{T}^{q^{-2(k-1)u}} L_{T}^{q^{-2(k-2)u}} \ldots L_{T}^{q^{-2(u)}} \right) = (\text{applying } (6.18) \text{ again})
\]

\[
= \text{Tr}_{R(2\ldots k)} \left( L_{T}^{q^{-2(k-1)u}} L_{T}^{q^{-2(k-2)u}} \ldots L_{T}^{q^{-2(u)}} \right) = (\text{applying } (6.18) \text{ again})
\]

Besides, in the second line we have taken into account that $P^{(s)} P^{(s)} = P^{(s)}$ for any $s$. So, we come to the following intermediate result:

\[
k_{q} L^{k}(u) = q^{k-1} L_{T}^{q^{-2(k-1)u}} e_{k-1}(u)
\]

\[
- (k-1)_{q} \text{Tr}_{R(2\ldots k)} \left( L_{T}^{q^{-2(k-1)u}} L_{T}^{q^{-2(k-2)u}} \ldots L_{T}^{q^{-2(u)}} \right) = (\text{applying } (6.8))
\]

Now, we apply the same transformations to $(k-1)_{q} P_{2\ldots k}^{(k-1)}$ and so on. The required result (6.16) is proved by induction with the use of definitions of $e_{k}(u)$ and $L^{[k]}(u)$.

**Remark 24** Note, that formula (6.16) is valid for any positive integer $k$ and for any skew-invertible Hecke symmetry $R$. However, if the bi-rank of a Hecke symmetry is $(m|0)$, then we have $L^{k} \equiv 0$ for $k > m$, and all matrix identities given by (6.16) at $k > m$ are consequences of the Cayley-Hamilton identity corresponding to $k = m$ (see Proposition 26 below).

As a simple corollary of the Proposition 23 we get analogs of the Newton identities, connecting the elementary symmetric polynomials and power sums.

**Corollary 25** The elementary symmetric polynomials (6.5) and power sums (6.15) are connected by the set of identities ($k \geq 1$):

\[
k_{q} e_{k}(u) - q^{k-1} p_{1}(q^{-2(k-1)u}) e_{k-1}(u) + q^{k-2} p_{2}(q^{-2(k-2)u}) e_{k-2}(u) + \ldots + (-1)^{k} p_{k}(u) = 0. (6.19)
\]

These relations can be easily proved by taking the $R$-trace of the identities (6.16).

Now, consider the Cayley-Hamilton-Newton identity (6.16) at $k = m$. Actually, it only contains the matrix powers $L^{[k]}$ and is called the Cayley-Hamilton identity for the matrix $L(u)$.

**Proposition 26** The generating matrix of the braided Yangian $Y(R)$ associated with the Hecke symmetry $R$ of bi-rank $(m|0)$ meets the matrix Cayley-Hamilton identity:

\[
\sum_{p=0}^{m} (-q)^{p} L^{[m-p]}(q^{-2p} u) e_{p}(u) = 0,
\]

where by definition we set $L^{[0]}(u) := I$. **Proof.** Let us set $k = m$ in (6.16):

\[
(-1)^{m} m_{q} L^{m}(u) + \sum_{p=0}^{m-1} (-q)^{p} L^{[m-p]}(q^{-2p} u) e_{p}(u) = 0. (6.21)
\]

Transform now the first term of this identity, using the relation (6.9) valid for the highest order skew-symmetrizer:

\[
m_{q} L^{m}(u) = m_{q} \text{Tr}_{R(2\ldots m)} \left( P_{12\ldots m}^{(m)} L_{T}^{q^{-2u}} \ldots L_{T}^{q^{-2(m-1)u}} \right)
\]

\[
= q^{m} m_{q} e_{m}(u) \text{Tr}_{R(2\ldots m)} \left( P_{12\ldots m}^{(m)} \right) = q^{m} e_{m}(u) I_{1}.
\]
Here, at the last step we use the value of a multiple trace of the skew-symmetrizer

$$m_q \text{Tr}_{R(2 \ldots m)} \left( P_{12 \ldots m}^{(m)} \right) = q^{-m(m-1)} I_1,$$

which is a particular case of (2.13).

Substituting the relation $m_q L^{\wedge m}(u) = q^m e_m(u) I$ into (6.21) we get the identity (6.20).

**Remark 27** For the RTT-type Yangian the Cayley-Hamilton identity slightly differs in the last term, corresponding to the zero power of $T(u)$. Indeed, for this type Yangian we have

$$m_q T^{\wedge m}(u) = q^m e_m(u) \mathcal{N},$$

and the CH identity in this case reads

$$q^m e_m(u) \mathcal{N} + \sum_{p=0}^{m-1} (-q)^p T^{[m-p]}(q^{-2p} u) e_p(u) = 0,$$

where the elements $e_k(u)$ are defined by (6.11) and $T[k]$ by

$$T[k](u) = \text{Tr}_{R(2 \ldots k)} \left( T_1(q^{-2(k-1)} u) T_2(q^{-2(k-2)} u) \ldots T_k(u) R_{k-1} R_{k-2} \ldots R_1 \right).$$

The case when $R$ is involutive and consequently the corresponding current $R$-matrix is rational can be treated in a similar manner. Let us define the matrix skew-powers $L^{\wedge k}(u)$ and matrix powers $L[k](u)$ by the formulae analogous to (6.12) and (6.13):

$$L^{\wedge 1}(u) = L(u), \quad L^{\wedge k}(u) = \text{Tr}_{R(2 \ldots k)} \left( P_{12 \ldots k}^{(k)} L_{\wedge 1}(u) L_{\wedge 2}(u-1) \ldots L_{\wedge k}(u-k+1) \right) \quad k \geq 2, \quad (6.22)$$

$$L[k](u) = \text{Tr}_{R(2 \ldots k)} \left( L_{\wedge 1}(u-k+1) L_{\wedge 2}(u-k+2) \ldots L_{\wedge k}(u) R_{k-1} R_{k-2} \ldots R_1 \right) \quad k \geq 1. \quad (6.23)$$

Here the projectors $P^{(k)}$ are obtained as the limit $q \to 1$ of (2.1). The corresponding elementary symmetric polynomials and power sums are respectively defined by $e_k(u) = \text{Tr}_R L^{\wedge k}(u)$ and $p_k(u) = \text{Tr}_R L[k](u)$.

Note that in this case the quantum matrix powers $L[k](u)$ can be reduced to a form similar to (6.14) but with additive shifts of the parameter.

**Proposition 28** In the braided Yangian $Y(R)$ associated with an involutive skew-invertible symmetry $R$ of bi-rank $(m|0)$ the following set of Cayley-Hamilton-Newton relations exists:

$$(-1)^{k+1} k L^{\wedge k}(u) = \sum_{p=0}^{k-1} (-1)^p L^{[k-p]}(u-p) e_p(u), \quad k \geq 1. \quad (6.24)$$

At $k = m$ we get the Cayley-Hamilton identity

$$\sum_{p=0}^{m} (-1)^p L^{[m-p]}(u-p) e_p(u) = 0. \quad (6.25)$$

The elementary symmetric polynomials and power sums are connected by the Newton relations

$$k e_k(u) - p_1(u-k+1) e_{k-1}(u) + p_2(u-k+2) e_{k-2}(u) + \ldots + (-1)^k p_k(u) = 0, \quad k \geq 1. \quad (6.26)$$

The highest order elementary symmetric polynomial (called determinant) commutes with the generating matrix $L(u)$

$$e_m(u) L(v) = L(v) e_m(u), \quad \forall u, v,$$

therefore, the Laurent coefficients of $e_m(u)$ are central elements of $Y(R)$.

**Remark 29** If the involutive symmetry $R$ coincides with the flip $P$, then the braided Yangian $Y(R)$ is nothing but the Drinfeld Yangian $Y(gl(N))$ and Proposition 28 reproduces the well-known results for it (see [Mo]).
7 Bosonization of braided Yangians

In this section we realize a bosonization of the braided Yangians, i.e. represent these Yangians in a braided analog of the Fock space. This representation is also based on the evaluation map.

First, introduce the Fock space and the corresponding bosonic algebra, associated with a skew-invertible Hecke symmetry $R$. The role of the Fock space $\mathcal{F}(R)$ is played by the $R$-symmetric algebra $\text{Sym}_R(V)$ of the basic space $V$. The elements of the basis $\{x_i\}$ considered in section 2, generate this algebra. Also, the unit of this algebra plays the role of the vacuum vector.

In the space $V^*$ we use the both bases: the right dual basis $\{x^i\}$ and the left dual one $\{\tilde{x}^j\}$, i.e. such that $\langle x^i, x_j \rangle = \delta^i_j$. The symmetric algebra of the space $V^*$ is defined by

$$\text{Sym}_R(V^*) = T(V^*) / \langle q x^i x^j - R_{ij}^{kl} x^k x^l \rangle. \quad (7.1)$$

Emphasize that in the left dual basis the algebra $\text{Sym}_R(V^*)$ is defined by the same formula (up to replacing $x^i$ by $\tilde{x}^j$). To prove this claim we use the following relation between two dual bases (see [GPS])

$$x^i = \tilde{x}^k B^i_k \quad (7.2)$$

and use the fact that the product $B_1 B_2$ commutes with the braiding $R$.

Besides, we introduce the permutation relations between the generators of the algebras $\text{Sym}_R(V^*)$ and $\text{Sym}_R(V)$ by putting

$$x^i x_j = B^i_j + q^{-1} x_k x^l \Psi_{ij}^k l, \quad (7.3)$$

where $\Psi$ is defined in (2.3).

**Proposition 30** The ideals coming in the definitions of the algebras $\text{Sym}_R(V)$ and $\text{Sym}_R(V^*)$ are preserved by the permutation relations (7.3). Consequently, the map

$$\text{Sym}_R(V^*) \otimes \text{Sym}_R(V) \rightarrow \text{Sym}_R(V) \otimes \text{Sym}_R(V^*), \quad (7.4)$$

arising from successive permutations of all factors of a monomial $p \in \text{Sym}_R(V^*)$ and those of a monomial $q \in \text{Sym}_R(V)$, is well-defined.

**Proof.** Here we use the left dual basis. By means of the map (7.2), we can write the relation (7.3) in terms of the left dual basis. Namely, we have

$$\tilde{x}^i x_j = \delta^i_j + q^{-1} x_k x^l \Psi_{ij}^k l. \quad (7.5)$$

The fact that the systems (7.3) and (7.5) are equivalent can be easily deduced from the relation (see [O])

$$B_1 \Psi_{12} = R_{21}^{-1} B_2 \iff B^i_j \Psi_{jm}^m = (R^{-1})_{ji}^p B^m_p. \quad (7.6)$$

With the use of (7.6) and the Yang-Baxter equation for the Hecke symmetry $R$ we get:

$$\tilde{x}^k (q x_i x_j - R_{ij}^{ab} x_a x_b) = q^{-2} (R^{-1})_{ji}^p (R^{-1})_{ir}^{ka} (q x_a x_b - R_{ab}^{cd} x_c x_d) x^s;$$

This means that the permutation relations preserve the ideal generated by the relations

$$q x_i x_j - R_{ij}^{ab} x_a x_b,$$

therefore, the permutation relations are compatible with the algebraic structure of $\text{Sym}_R(V)$. The same is true for the ideal in the quotient (7.1). Details are left to the reader. □
Now, convert the generators of the algebra $\text{Sym}_R(V^*)$ into operators acting on the space $\mathcal{F}(R)$. Thus, in order to apply $\bar{x}^i$ to an element $f \in \mathcal{F}(R)$ we send the product $\bar{x}^i \otimes f$ to the space $\text{Sym}_R(V) \otimes \text{Sym}_R(V^*)$ by the map $(7.3)$. Then we apply the counit $\varepsilon$ to the factors from $\text{Sym}_R(V^*)$. This counit kills all elements of the algebra $\text{Sym}_R(V^*)$ except for those of the ground field $\mathbb{K}$ and acts as the identity operator on $\mathbb{K}$. Thus, we get an element $g \in \text{Sym}_R(V) \otimes \mathbb{K} \cong \text{Sym}_R(V)$. Finally, we define the action as $\bar{x}^i(f) = g$. In particular, we have $\bar{x}^i(1) = 0$, $\bar{x}^i(x_i) = \delta^i_j$.

Now, define the creation and annihilation operators $a_i^+, a_i \in \text{End}(\mathcal{F}(R))$ acting on the space $\mathcal{F}(R)$ as follows

$$a_i^+(f) = x_i f, \quad a_i(f) = \bar{x}^i(f), \quad \forall f \in \mathcal{F}(R).$$

(7.6)

**Proposition 31** The creation and annihilation operators meet the following permutation relations

$$qa_i^+ a_j^+ = R_{ij}^k a_k^+ a_i^+, \quad qa_i a_j = R_{ik}^j a_k a_i^+ \quad a_i^+ a_j^+ - q^{-1}(R^{-1})_{ik}^j a_i^+ a_k^+ = \delta^i_j.$$  

(7.7)

**Proof.** The first relation is clear. The second relation follows from the fact that the permutation of elements of the algebra $\text{Sym}_R(V^*)$ with a fixed element $f \in \text{Sym}_R(V)$ is a morphism of the algebra $\text{Sym}_R(V)$ onto itself. Also, the counit $\varepsilon$ is a morphism. So, the map $\bar{x}^i \mapsto a_i^+$ is a representation of the algebra $\text{Sym}_R(V^*)$. The third relation is valid in virtue of the definition of the annihilation operators.

The algebra generated by the creation and annihilation operators is called the $R$-bosonic algebra and is denoted $\mathcal{B}(R)$.

Now, we pass to the bosonization of the modified RE algebra $\mathcal{L}(R, 1)$.

**Proposition 32** The elements $l_i^j = a_i^+ a_k B_k^j$ meet the defining relations of the algebra $\mathcal{L}(R, 1)$. Consequently, the map

$$\pi : l_i^j \mapsto a_i^+ a_k B_k^j,$$

which is called a bosonization of the algebra $\mathcal{L}(R, 1)$, defines a representation of this algebra in the Fock space $\mathcal{F}(R)$.

**Proof.** Here, we use the notations $x_i$ and $\bar{x}^j$ instead of $a_i^+$ and $a^p B_p^j$ respectively. Let us present the relations (7.3) in an equivalent form

$$x^a R_{ai}^{bj} x_b = \delta^j_i + q^{-1} x_i x^j.$$

Now, transform the matrix $\pi(R_1 L_1 R_1 L_1)$ as follows (underlined are the terms affected by transformations on the next step)

$$\pi(R_{ij}^{ab} c_{kd} R_{cd}^{kl}) = R_{ij}^{ab} x_a x_c R_{cd}^{kl} x_d x_k = R_{ij}^{al} x_a x^k + q^{-1} R_{ij}^{ab} x_a x_b x^l x^k = \pi(R_{ij}^{al} x_a x^l x^k) + x_i x_j x^l x^k.$$

Analogously, we transform the entries of the matrix $L_1 R_1 L_1$:

$$\pi(l_i^a R_{ai}^{bc} d_{jd}^{kl}) = x_i x^a R_{ai}^{bc} x_b x^d R_{de}^{kl} = x_i x^d R_{ij}^{kl} + q^{-1} x_i x_j x^c x^d R_{de}^{kl} = \pi(l_i^a R_{ij}^{kl}) + x_i x_j x^l x^k.$$

Subtracting the latter expression from the former one we come to the result.

Now, we are able to realize a bosonization of the braided Yangians based on the evaluation morphism. Let us represent the generating matrix $L(u)$ of a given braided Yangian $\mathcal{Y}(R)$ via
the map $L(u) \mapsto I + \frac{M}{u}$, where $M$ is either the generating matrix of the RE algebra $L(R)$, if $R$ is a Hecke symmetry, or that of $L(R, 1)$ if $R$ is an involutive symmetry.

First, consider the latter case. Using the bosonization of the algebra $L(R, 1)$ described above, we immediately get a bosonization of the braided Yangian $Y(R)$. More precisely, we have the following proposition.

**Proposition 33** The map

$$l^j_i(u) \mapsto \delta^j_i + \frac{1}{u}a^+_ia^kB^i_j,$$

where $a^+_i$ and $a^j$ are respectively the creation and annihilation operators acting on the Fock space $F(R)$ and subject to (7.7) with $q = 1$, is a representation of the braided Yangian $Y(R)$, associated with an involutive symmetry $R$.

In terms of the Laurent coefficients the map (7.8) can be presented as follows

$$l^j_i[1] \mapsto a^+_ia^kB^i_j, \quad L[k] \mapsto 0, \quad k \geq 2.$$

Note that the denominator in (7.8) can be replaced by $u - u_0$, $u_0 \in \mathbb{K}$. Such representations with different $u_0$ can be multiplied by means of the coproduct, defined in the braided Yangians. Note that the vacuum vector of the product of the corresponding Fock spaces is the product of the vacuum vectors of the factors.

If $R$ is a Hecke symmetry, then by using the map inverse to (3.2) with $h = 1$, we arrive to the following bosonization of the braided Yangian $Y(R)$ (also, presented by means of the Laurent coefficients)

$$l^j_i[1] \mapsto a^+_ia^kB^i_j + \frac{\delta^j_i}{q - q^{-1}}, \quad L[k] \mapsto 0, \quad k \geq 2.$$

**Proposition 34** The map (7.9) is a representation of the braided Yangian $Y(R)$ associated with a Hecke symmetry $R$.

This map does not admit a limit $q \to 1$ since the evaluation maps for the braided Yangians corresponding to involutive symmetries and Hecke ones essentially differ. However, it is possible by renormalizing the map (7.9) to get the following representation

$$l^j_i[1] \mapsto (q - q^{-1})a^+_ia^kB^i_j + \delta^j_i, \quad L[k] \mapsto 0, \quad k \geq 2,$$

which has a limit. However, this limit is out of interest.

In the classical case, i.e. while $R(u, v)$ is the Yang $R$-matrix, the role of the bosonic algebra is often attributed to the Zamolodchikov-Faddeev (ZF) algebra (see [ZZ] [F]). It appeared in the framework of the second quantization of some dynamical models. In particular, it comes to the Nonlinear Schrodinger hierarchy (see [MRSZ]). Nevertheless, a consequent definition of a bosonic algebra in the spirit of the ZF one meets some difficulties, which we comment below. Also, we suggest a modified version of the bosonic algebra valid for all current $R$-matrices in question.

As was pointed out in [LM], a consistent Fock space can be only associated with an involutive current $R$-matrix, i.e. that subject to the condition: $R(u, v)R(v, u) = I$. The authors of [LM] found such a current $R$-matrix in the form $g(u, v)I$, where $g(u, v)$ is an appropriate function. However, this braiding is out of our interest.

So, assuming $R(u, v)$ to be one of the current $R$-matrices constructed in section 4, we first normalize it as suggested there, and get an involutive braiding $R(u, v)$, given by one of formulae (4.11). Then we define the Fock space $F(R)$ as an $R$-symmetric algebra of a space $V(w)$. The
space $\mathcal{F}(R)$ consists of finite linear combinations of products $x_{i_1}(u_1) \otimes x_{i_2}(u_2) \otimes \ldots \otimes x_{i_k}(u_k)$, modulo the ideal generated by quadratic elements

$$x_{i}(u)x_{j}(v) = R_{ij}^{kl}(u,v)x_{k}(v)x_{l}(u). \quad (7.10)$$

Rewrite these relations in a more detailed form

$$q x_{i}(u)x_{j}(v) - R_{ij}^{kl}(u,v)x_{k}(v)x_{l}(u) = g(u,v)(x_{i}(u)x_{j}(v) - x_{i}(v)x_{j}(u)) \quad (7.11)$$

where $g(u,v)$ is one of the functions

$$g(u,v) = \frac{1}{u - v} \quad \text{or} \quad g(u,v) = \frac{u(q - q^{-1})}{u - v}.$$ 

In the former case we also set $q = 1$ in (7.11).

In a similar manner, we define the $R$-symmetric algebra of the dual space $V^*(w)$ by imposing the following system of relations:

$$\tilde{x}^i(u)\tilde{x}^k(v) = R_{ij}^{kl}(u,v)\tilde{x}^j(v)\tilde{x}^l(u). \quad (7.12)$$

Now, according to the above pattern, define the corresponding permutation relations by the formula similar to (7.3):

$$\tilde{x}^j(u)x_i(v) = R_{ik}^{jl}(v,u)x_l(v)\tilde{x}^k(u) + \delta^j_i \delta(u - v). \quad (7.13)$$

We leave to the reader checking that these permutation relations preserve the ideals coming in the definitions of the above symmetric algebras. Finally, we define the corresponding bosonic algebra by the relations (7.10), (7.12), and (7.13).

Observe that even in the case when $R(u,v)$ is the Yang $R$-matrix, our definition of the bosonic algebra differs from that of the ZF algebra.

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