Natural extension of hidden $Z_2 \times Z_2$ symmetry toward arbitrary integer spin chains

Isao Maruyama*

Graduate School of Engineering Science, Osaka University, Toyonaka, Osaka 560-8531, Japan

We show how entangled valence-bond singlet pairs are disentangled partially and totally by the Kennedy-Tasaki transformation which reveals the hidden $Z_2 \times Z_2$ symmetry in valence-bond-solid chains as a higher-spin generalization of the previous studies toward the intermediate-$D$ state. The totally disentangled states correspond to four Ising-like states with $Z_2$ variables on the boundary. We present a simple expression of results by using the spin decomposition and the boundary matrix.

KEYWORDS: hidden $Z_2 \times Z_2$ symmetry, valence-bond solid, matrix product state, disentangler

Haldane\(^1\) predicted that the ground state in integer spin quantum antiferromagnetic chains is unique, massive, and disordered. This conjecture has been examined by many numerical, experimental, and rigorous studies. Here, “massive” means a gapped ground state with the Haldane gap, and “disordered” indicates exponentially decaying correlation functions. In spite of disorder, there is the hidden antiferromagnetic order detected by the nonlocal string order parameter.\(^2\) It is regarded as consequence of the hidden $Z_2 \times Z_2$ symmetry breaking.\(^3\)

This hidden order of the disordered ground state is understood by the restricted solid-on-solid(RSOS) model.\(^2\) To illustrate it, let us consider a spin-$1/2$ Ising (or $Z_2$) variable $\sigma$, where $\sigma = \uparrow$, or $\downarrow$. When we randomly generate $\sigma_i$ on $i$th site in a chain, the Ising state written as $|\sigma_1, \ldots, \sigma_{L+1}\rangle$ is disordered. However, defining spin-$1$ Ising (or $Z_3$) variables as $m_i = \sigma_i - \sigma_{i+1}$, one finds that $|m_1, \ldots, m_L\rangle$ has the hidden antiferromagnetic order, i.e., $|m_1, \ldots, m_L\rangle$ is identical to the Néel ordered state $|\ldots, +1, -1, +1, -1, \ldots\rangle$ if we skip all $m_i = 0$. While two Néel states, $|+1, -1, \ldots\rangle$ and $|-1, +1, \ldots\rangle$, are identified by the boundary spin on the first site, the hidden antiferromagnetic ordered states are identified by the two boundary $Z_2$ variables, $\sigma_1$ and $\sigma_{L+1}$. The emergence of the boundary degrees of freedom is an important topological property. In fact, the exact solution in the Affleck-Kennedy-Lieb-Tasaki(AKLT) model,\(^4\) which is one of rigorous studies of Haldane systems, shows four-fold degeneracy of the ground states with the Haldane gap in the open boundary condition(OBC), while there is a unique ground state in the periodic boundary condition(PBC) as Haldane conjectured. This four($4 = 2 \times 2$)-fold degeneracy is due to the hidden $Z_2 \times Z_2$ symmetry. This kind of topological property, emergence

*E-mail address: maru@mp.es.osaka-u.ac.jp
of the edge/surface modes, is not only theoretical concept but also realized experimentally in Haldane systems,\textsuperscript{5} quantum Hall systems,\textsuperscript{6} quantum spin Hall systems,\textsuperscript{7} and topological insulators.\textsuperscript{8} Especially, edge modes in the Haldane system have been applied to the quantum computation\textsuperscript{9} based on the topological entanglement. As a theoretical study on the topological entanglement, AdS/CFT correspondence\textsuperscript{10} is also a hot topic.

Many theoretical models generalized from the AKLT model by means of valence bond solid(VBS) construction\textsuperscript{11} have been studied in higher dimension and/or in various spin symmetries except for $SU(2)$. However, even in the one-dimensional Heisenberg model with $SU(2)$ symmetry, what kind of hidden order exists in higher-spin chains is still unclear. For example, as shown in a rigorous study about the VBS state,\textsuperscript{12} the hidden $Z_2 \times Z_2$ symmetry breaks down only for odd integer $S$ but remains unbroken for even integer. As discussed in the recent numerical study on the $S = 2$ anisotropic chain,\textsuperscript{13} the determination of the phase diagram is still worthwhile especially about existence of the intermediate-$D(ID)$ phase. In this sense, higher-$S$ generalization is not straightforward.

In this letter, we study what is a natural extension toward higher integer spin based on the Kennedy-Tasaki(KT) transformation $\hat{U}$,\textsuperscript{3} which is nonlocal unitary transformation revealing the hidden $Z_2 \times Z_2$ symmetry directly. In short, the transformed Hamiltonian $\tilde{H} = \hat{U} \hat{H} \hat{U}^{-1}$ has clear $Z_2 \times Z_2$ symmetry and the non-local string order in $\hat{H}$ corresponds to conventional ferromagnetic order in $\tilde{H}$. In addition, the four-fold degeneracy which is hidden in $\hat{H}$ with the PBC corresponds to the four ferromagnetic ground states in $\tilde{H}$. Our motivation is to obtain four ferromagnetic ground states generalized to higher-$S$. It is surprising that what this strategy showed us in higher-$S$ models is not the Haldane state but the ID VBS state.\textsuperscript{12} (See Fig. 1.) It is quite natural if we adopt the concept that the $Z_2$ symmetry is originated from the $Z_2$ variables on the boundaries and, in this sense, the Haldane state must have the (hidden) $Z_{S+1} \times Z_{S+1}$ symmetry.

To show the beautiful mathematical structure of our result, this letter is organized inversely as follows. First, we start with the known spin-$S$ VBS Hamiltonian for ID phase\textsuperscript{12} as a generalization of the spin-1 AKLT Hamiltonian. Then, we write down its ground states as a matrix product state(MPS) with the two boundary variables. Using a spin decomposition, we summarize our results for the KT transformation and discuss the role of the KT transformation as the topological disentangler, generalizing the spin-1 topological disentangler studied by Okunishi recently.\textsuperscript{14} In this meaning, the KT transformation for arbitrary integer spin deeply relates not only the hidden $Z_2 \times Z_2$ symmetry and but also the topological entanglement.

Let us illustrate the ID VBS state for arbitrary spin $S$ in the Schwinger boson picture, where one spin-$S$ is decomposed into $2S$ spin-$1/2$’s at each site. To consider the ID VBS states which correspond to the four-fold degenerated ground states, we limit ourselves to the case that the number of the valence bonds between the two nearest neighbor sites is only one. As
shown in Fig. 1(a), at each site \( i \), one spin-1/2 at \( i_ℓ \) couples to the left site, one spin-1/2 at \( i_r \) couples to the right site, and there are \( (S - 1) \)-pairs of up and down spin-1/2’s for all the rest in \( i_z \). The Hamiltonian which has the ID VBS states as the exact ground states is given only with the nearest neighbor interactions as \( \hat{H} = \sum_{i} \hat{H}_{i,i+1} \). Using the spin-S operator at \( i \)th site, \( \hat{S}_i \), and the local \( S_z \) basis given by \( \hat{S}_z |m⟩_i = m|m⟩_i \) with \( ℏ = 1 \), \( \hat{H}_{i,i+1} \) is given as follows:\(^{12}\)

\[
\hat{H}_{ij} = \hat{P}^{(2S)}_{ij} + \hat{H}_{i,i+1}^{loc},
\]

\[
\hat{P}^{(k)}_{ij} = \prod_{m=0}^{k-1} \frac{\left(\hat{S}_i + \hat{S}_j\right)^2 - m(m+1)}{k(k+1) - m(m+1)},
\]

\[
\hat{H}_{i,i+1}^{loc} = 1 - \sum_{m=-1}^{1} |m_i⟩⟨m_i|.
\]

The symmetrizing projection \( \hat{P}^{(k)}_{ij} \) is usually written in \( \sum_m α_m (\hat{S}_i \cdot \hat{S}_j)^m \), where coefficients \( α_m \) are easily obtained due to \( (\hat{S}_i)^2 = S(S+1) \). Here the notation of local states is defined as \( |m⟩_i = |m⟩_i \) at \( i \)th site, including notations like \( |0⟩_i \) or \( |↑⟩_i \). Local operators such as \( |m⟩_i ⟨m_i| \) can be extended to operators for the total Hilbert space, \( 1 ⊗ 1 ⊗ \ldots ⊗ |m⟩_i ⟨m_i| ⊗ \ldots ⊗ 1 \), automatically depending on the context. The Hamiltonian has a flexibility: one can multiply positive coefficient to \( \hat{H}_{i,i+1}^{loc} \) and replace it with \( \hat{H}_{i,i+1}^{loc} = (S_z^i)^2 \{(S_z^i)^2 - 1\} \) including the negative large-D term, \( D \sum_i (S_z^i)^2 \).

The ID-VBS states are given in the Schwinger boson representation.\(^{12}\) Explicitly, the states can be written in the MPS form as \( \Psi = \prod_{i=1}^{L} A_i \) with the local matrix

\[
A_i = \left( \begin{array}{cc} -\sqrt{S} |0⟩_i & S+1 |1⟩_i \\ -\sqrt{S+1} |0⟩_i & S |1⟩_i \end{array} \right) /\sqrt{4S-2}.
\]

\( \Psi \) is a \( 2 × 2 \) matrix corresponding to the 4-fold degenerated ground states, which are unnormalized and non-orthogonal. It is easy to check \( \hat{H}_{ij} A_i A_j \) becomes the zero matrix, which means
that the ground states are zero-energy states. Since the Hamiltonian is written as sum of the projections, the other excited states have non-zero eigen-values. In the $S = 1$ case, due to $H_{i}^{\text{loc}} = 0$, Eq.(1) becomes $\hat{H}_{ij} = \hat{P}_{ij}^{(2)}$, which is nothing but the original AKLT Hamiltonian.\textsuperscript{4}

In fact, $A_i$ for $S = 1$ corresponds to that of the spin-1 Haldane state.\textsuperscript{12,15}

Under the OBC, the four-fold degenerated ground states are written as

$$|\Psi_{\alpha, \sigma_{L+1}}\rangle = (\Psi)_{\sigma_1, \sigma_{L+1}},$$

with $(\sum_{i=1}^{L-1} \hat{H}_{i,i+1})|\Psi_{\sigma_1, \sigma_{L+1}}\rangle = 0$. However, under the PBC, $\hat{H} = \hat{H}^{\text{L},1} + \sum_{i=1}^{L-1} \hat{H}_{i,i+1}$ requires $\hat{H}^{\text{L},1} \mathbf{A}_L \mathbf{A}_1 = 0$. Then, the unique ground state is written as $|\Psi_{\text{PBC}}\rangle = \text{Tr} \hat{\Psi} = |\Psi_{\uparrow\uparrow}\rangle + |\Psi_{\downarrow\downarrow}\rangle$. In general, as Östlund and Rommer introduced,\textsuperscript{16} the ground state is written with the boundary matrix $B$ as, $|\Psi(B)\rangle = \text{Tr} \hat{\Psi} B = \sum_{\sigma_1, \sigma_{L+1}} (B)_{\sigma_{L+1}, \sigma_1} |\Psi_{\sigma_1, \sigma_{L+1}}\rangle$, and $|\Psi(1)\rangle = |\Psi_{\text{PBC}}\rangle$ for the identity matrix. Even in the PBC, importance of this boundary matrix has been noticed again recently in the numerical\textsuperscript{17} and rigorous studies.\textsuperscript{18,19}

Higher-spin generalization of the KT transformation is given as\textsuperscript{12}

$$\hat{U} = \prod_{i<j} e^{i\pi \hat{S}_i^\alpha \hat{S}_j^\alpha},$$

with the spin-$S$ operator $\hat{S}_i^\alpha$. The hidden $Z_2 \times Z_2$ symmetry is revealed by $\hat{U}$ because the ferromagnetic correlation function $\hat{S}_i^\alpha \hat{S}_j^\alpha$ is transformed into the string operator for $\alpha = x, z$.\textsuperscript{12} For example, $\hat{U} \hat{S}_i^x \hat{S}_j^z (\hat{U})^{-1} = \hat{S}_i^x e^{i\pi \sum_{k=1}^{L-1} \hat{S}_k^x \hat{S}_j^z}$ in arbitrary integer $S$.

As the main result of this letter, we will write down the four states after the KT transformation. It is interesting that we can obtain a simple form if we use the boundary matrix $B = \Omega$ defined as

$$\Omega = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$  

Then, the MPS after the KT transformation can be written as

$$\hat{U} \hat{\Psi} \Omega = \left( \begin{array}{cc} (-1)^{L} \hat{S}^e & (-1)^{L} e^{i\pi \hat{S}^e} \\ -e^{i\pi \hat{S}^e} & e^{i\pi \hat{S}^e} \end{array} \right) |\phi\rangle,$$

with $\hat{S}^\alpha = \sum_{i=1}^{L} \hat{S}_i^\alpha$ and

$$|\phi\rangle = \prod_{i} \left( \frac{\sqrt{S} |0\rangle_i + (-1)^{(S+1)(i+L)} \sqrt{S} + 1 |1\rangle_i}{\sqrt{4S - 2}} \right).$$

These four Ising states are generalization of ferromagnetic states in the $S = 1$ case.\textsuperscript{3} Using this formula, one can easily transform an MPS with any boundary. For example, in the PBC for even $L$, one can obtain $\hat{U} |\Psi_{\text{PBC}}\rangle = \text{Tr}[(\hat{U} \hat{\Psi}) \Omega^{-1}] = (1 + e^{i\pi \hat{S}^z})(1 + e^{i\pi \hat{S}^x})|\phi\rangle/2$.

The $2 \times 2$ matrix in Eq.(8) reveals the hidden $Z_2 \times Z_2$ symmetric operations about 180° rotation via $x$ and $z$-axis for four (anti)ferromagnetic states thanks to the boundary matrix $\Omega$. Note that these four states are written in the classical (or Ising) states, i.e., the direct-
product states of the local states. In this meaning, $\hat{U}$ is the total disentangler. That is, the KT transformation can disentangle each valence bond singlet having the log 2 entanglement entropy corresponding to the $Z_2$ variable, $\sigma_i$. In the following, we will illustrate it generalizing Okunishi’s result\textsuperscript{14} to higher-$S$ in a unified way using a spin decomposition.

Let us illustrate the spin decomposition. The spin-$S$ operator at $i$th site can be decomposed into spin-($S-1$) operator $\hat{S}_{is}$ and spin-1 operator $\hat{S}_{iv}$, when we introduce the relation between the local basis as

$$|m\rangle_i = \sum_{m'm''} |m'\rangle_{iv} |m''\rangle_{is} \langle 1,m';S-1,m''|S,m\rangle, \quad (10)$$

where $\langle J,m,J',m';J'',m''\rangle$ are Clebsh-Gordan coefficients. With using the projection operator defined in Eq.(2), the equation for spin operators is written as $\hat{S}_i = \hat{P}^{(S)}(\hat{S}_{is} + \hat{S}_{iv})$. Here we note our ambiguity in this equation: the r.h.s. must be $2S + 1$ dimensional matrices with the Hilbert space spanned by $| - S \rangle_i, \ldots, |S\rangle_i$, but, the l.h.s. equals to the product of $3(2S - 1)$ dimensional matrices with direct-product states $|m\rangle_{iv} |m'\rangle_{is}$ for $|m| \leq 1$ and $|m'| \leq S - 1$. In this case, $|m\rangle_i$ must be written in $|m\rangle_{iv} |m'\rangle_{is}$ via Eq.(10). In the following, the Hilbert space of each equation is not explicitly defined and automatically changed depending on the context. The same ambiguity in the identical equation $\hat{S}_i = \hat{P}^{(S)}(\hat{S}_{is} + \hat{S}_{iv})(\hat{P}^{(S)})^\dagger$ might be removed by rewriting the projection operator defined in Eq.(2) as $\hat{P}^{(S)} = \sum_{m'm''}|m_i\rangle\langle S,m_i|1,m'_i;S-1,m''_i\rangle|m'_i,m''_i\rangle\langle 1,m';S-1,m''\rangle$. Another useful formula is $\hat{S}_i\hat{P}^{(S)} = \hat{P}^{(S)}(\hat{S}_{is} + \hat{S}_{iv})$. This is the spin decomposition in the opposite of the spin composition $\hat{S}_{ij} = \hat{S}_i + \hat{S}_j$. For readability, we summarize notations of operators. All operators have the hat notation, $\hat{\cdot}$, and subscripts denoting the region where the operator is defined: single site operator $\hat{S}_i^z$ and two site operator $\hat{H}_{ij}$, except for total operator $\hat{S}^z$. This rule for subscript is also applied to the MPS, such as $A_i$, and $\Psi$.

Using the spin decomposition, the local MPS $A_i$ for $S > 1$ can be constructed from spin-1 MPS $A_{iv}$ as

$$A_i = \hat{P}_{iiv}^{(S)}|0\rangle_{iv} A_{iv}. \quad (11)$$

In addition, we can decompose a spin-$S$ into spin-1/2’s. This is nothing but the schematic picture of 1D-VBS state\textsuperscript{11} as shown in Fig. 1(a). For simplicity, we decompose a spin-1 only at $i_v$-site into two spin-1/2’s as

$$A_{iv} = \hat{P}_{iiv}^{(1)} \begin{pmatrix} |\uparrow\rangle_{iv} \\ |\downarrow\rangle_{iv} \end{pmatrix} \begin{pmatrix} |\uparrow\rangle_{is}, |\downarrow\rangle_{is} \end{pmatrix} e^{i\pi S \theta}. \quad (12)$$

Here we use the $2 \times 2$ matrix $s^y = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, which is the matrix representation of the
spin-1/2 operator. A product of the matrices, $A_{i_r} A_{j_v}$ includes the valence bond singlet

$$|s\rangle_{i_r,j_v} = (|\uparrow\rangle_{i_r} |\downarrow\rangle_{j_v}) e^{i\pi s_y} \left( |\uparrow\rangle_{j_v} |\downarrow\rangle_{i_r} \right) / \sqrt{2}.$$ 

Before we mention the disentangler, we summarize the $Z_2$ property of the spin decomposed MPS, recalling the introduction of this letter. The hidden antiferromagnetic order and $Z_2 \times Z_2$ symmetry come from the random variables, $\sigma_i$. These spin-1/2 Ising (or $Z_2$) variables correspond to the artificial degrees of freedom of the matrix space as $|\Psi_{\sigma_1,\sigma_{L+1}}\rangle = \sum_{s} (A_1)_{\sigma_1} (A_2)_{\sigma_2} (A_L)_{\sigma_{L}} |\psi\rangle_{\sigma_{L+1}}$, which is defined in Eq. (5). After the spin decomposition in Eq. (11), a matrix element $(A_i)_{\sigma_i \sigma_{i+1}}$ is proportional to $|\sigma_i|_{\iota_r} - \sigma_{i+1}|_{\iota_r}$ in the spin decomposed basis. In the original spin basis, $(A_i)_{\sigma_i \sigma_{i+1}}$ corresponds to $|m_i|$, with $m_i = \sigma_i - \sigma_{i+1}$. The correspondence can be checked from Eq. (4). This is the explicit correspondence to the RSOS model as already mentioned in this letter. We emphasize that $\sigma_i$ is the source of the log 2 entanglement entropy of the valence bond singlet $|s\rangle_{i_r,j_v}$, which is the target of the disentangler.

The topological disentangler,\textsuperscript{14} defined as

$$\hat{D}_n = \prod_{i=1}^{n-1} e^{i\pi \hat{S}^z_{i} \hat{S}^z_{i+1}} \prod_{i=n+1}^{L} e^{i\pi \hat{S}^z_{i} \hat{S}^z_{i-1}}, \quad (12)$$

has the same form in the higher-spin generalization. Then, using the boundary $Z_2$ variables $\mu = \sigma_1, \nu = \sigma'_{L+1}$ and the total projection $\hat{P} = \prod_{i=1}^{L} \hat{P}_{i_{\iota_r}i_{\iota_v}}^{(s)} \hat{P}_{i_{\iota_v}i_{\iota_r}}^{(1)}$, we have

$$(\Psi \Omega)_{\mu\nu} = -\sqrt{2L} \hat{P}_{i_{\iota_r}i_{\iota_v}}^{(s)} \prod_{i=1}^{L-1} |s\rangle_{i_{\iota_r}(i+1)_{\iota_v}} \langle \nu|_{\iota_v} \prod_{i} |0_{\iota_r}\rangle,$$

where $|\sigma\rangle_k = e^{-i\hat{S}^z_k |\sigma\rangle_k}$ for $k = i_r$ or $k = i_v$ are local spin-1/2 states in spin-$x$ basis. Since we can write $(\Psi \Omega)_{\sigma_{L+1}} = \sum_{\sigma_{L+1}} (\Psi)_{\sigma_1 \sigma_{L+1}} \times (\Omega)_{\sigma_{L+1} \sigma'_{L+1}} = \sum_{\sigma_{L+1}} |\Psi_{\sigma_1 \sigma_{L+1}}\rangle \langle \Omega|_{\sigma_{L+1} \sigma'_{L+1}}$, $\Omega$ is a transformation (or mapping) between $\sigma_{L+1}$ and $\sigma'_{L+1}$. In fact, due to $\Omega = \sqrt{2} e^{i\frac{\pi}{2} s^x}$,
\(\sigma'_{L+1}\) is a variable in spin-\(x\) axis. This is the meaning of \(\Omega\) and the local basis at \(L_e\) is written in \(|\nu\rangle_L^n\). The local states on both boundaries are determined by \(\mu = \sigma_1, \nu = \sigma'_L\), but each singlet is entangled and has a free (or random) variable \(\sigma_i\) as shown in Fig. 2(a). For odd \(S\), this MPS is disentangled as

\[
\hat{D}_n(\Psi \Omega)_{\mu \nu} = \sqrt{2^L(\Omega)_{\mu \nu}} \hat{P}|\mu\rangle_{1\ell} |\mu\rangle_{n\ell} |\nu\rangle_{n1}^x |\nu\rangle_{L},
\]

and

\[
\hat{U}(\Psi \Omega)_{\mu \nu} = (-\sqrt{2})^L(\Omega)_{\mu \nu}^{L+1} \hat{P} \prod_i |\mu\rangle_{i1} |\nu\rangle_{i2}^x |0\rangle_{i2},
\]

as shown in Fig. 2. In short, the target local states are determined by the boundary variables \(\mu, \nu\).

Equations (13) and (14) are valid only for odd \(S\). For general \(S\), we can obtain the same result if we replace \(e^{i\pi \hat{S}^x_1 \hat{S}^x_{\ell}}\) with \(e^{i\pi \hat{S}^x_1 (\hat{S}^x_{\ell} + S - 1)}\) in the definition of \(\hat{U}\) and \(\hat{D}_n\). This is the reason why \(|\phi\rangle\) in Eq.(9) has antiferromagnetic-like alternating behavior for even \(S\). These results can be proved easily by using the spin decomposition.\(^{20}\)

It is instructive to show how the KT transformation fails to disentangle the Haldane state for \(S > 1\). For example, let us consider the Haldane state for \(L = 2, S = 2\). We can decompose a \(S = 2\) spin at \(i_\ell\) site into two spin-1's at \(i_\ell\) and \(i_{\ell+1}\) with the valence bond singlet \(|s\rangle_{i_\ell, (i+1)_{\ell}} = |1\rangle_{i_\ell} |1\rangle_{(i+1)_{\ell}} - |0\rangle_{i_\ell} |0\rangle_{(i+1)_{\ell}} + |1\rangle_{i_\ell} |1\rangle_{(i+1)_{\ell}}\), which is the unique ground state of the Hamiltonian \(\mathbf{S}_{i_\ell} \cdot \mathbf{S}_{(i+1)_{\ell}}\) and has log 3 entanglement entropy. Then, one of 3 \(\times 3\) states, \(|\Psi_{00}\rangle = \hat{P}_{tot} |0\rangle_{1\ell} |s\rangle_{1,2\ell} |0\rangle_{2\ell}\), is transformed as \(e^{i\pi \hat{S}^x_1 \hat{S}^x_{\ell}} |\Psi_{00}\rangle = \hat{P}_{tot} |0\rangle_{1\ell} |1\rangle_{1_\ell} |1\rangle_{2\ell} - |0\rangle_{1\ell} |0\rangle_{2\ell} + |1\rangle_{1\ell} |1\rangle_{2\ell} |0\rangle_{2\ell}\). As shown in this example, the KT transformation fails to disentangle the singlet corresponds to \(Z_3\) symmetry.

In summary, we have rigorously found that four-fold degenerated Ising-like states generalized to arbitrary integer spins correspond to four-fold degenerated ID-states via the KT transformation as the total disentangler. In the viewpoint of the one-site disentangler, we have given the higher-spin generalization of Okunishi’s paper\(^{14}\) using the spin decomposition representation as an alternative to the Schwinger boson representation.

The spin decomposition approach reminds us the decomposition of a \(S = 1\) spin into ferromagnetically coupled \(S = 1/2\) spins, discussed in the \(S = 1/2\) quantum spin chain with bond alternation.\(^{21,22}\) The ground state is adiabatically connected to the direct product of the local singlets. Adding \(|0\rangle_{i_\ell}\) to it, we can make a corresponding model with local singlets. However, such local singlets can be disentangled easily by a unitary operator, such as \(e^{i\pi \hat{S}^x_1 \hat{S}^x_{\ell}}\). This means the entanglement of the singlet can easily destroyed by a two-site unitary operator, while we need the nonlocal KT transformation to disentangle the singlet in the spin-\(S\) chain. This is because we have operators like \(\hat{S}^x_{i_\ell} = \hat{P}(\hat{S}^x_{i_\ell} + \hat{S}^x_{i_{\ell+1}} + \hat{S}^x_{i_\ell})\hat{P}_{\ell}\) only. In this sense, a highly
non-trivial task is to obtain the disentangler, and it is surprising that the disentangler for the VBS states in the $S = 1$ ALKT model is the KT transformation.\textsuperscript{14} Moreover, as written in this letter, the higher-spin generalization can be obtained and leads us to the intermediate-$D$ states with $Z_2$ boundary variables. Since the KT transformation fails to disentangle the Haldane state for $S > 1$, we think this way is a natural extension in the sense that the KT transformation can play a role of the disentangler of the log 2 entanglement entropy.

This Hamiltonian breaks the spin $SU(2)$ symmetry but has $U(1)$ rotational symmetry via spin $z$-axis. Then, we adopt the usual $Z_2$ Berry phase via gauge twist on $z$-axis,\textsuperscript{23} which corresponds to the twisted boundary condition used in the level spectroscopy in the $S = 2$ chain.\textsuperscript{13} Since the gauge twist just modifies the boundary matrix, we can prove that the ID state gives non-trivial $\pi$ Berry phase due one singlet on the bond.

As a future problem, one can consider the $q$-deformed model with $U_q(su(2))$ quantum group.\textsuperscript{15} In addition, construction of the disentangler corresponding to the hidden $Z_{S+1} \times Z_{S+1}$ symmetry is still an open question, but this (dis)entanglement view of point will be important in this generalization.

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