AN ABSOLUTELY STABLE \(hp\)-HDG METHOD FOR THE TIME-HARMONIC MAXWELL EQUATIONS WITH HIGH WAVE NUMBER

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ABSTRACT. We present and analyze an hybridizable discontinuous Galerkin (HDG) method for the time-harmonic Maxwell equations. The divergence-free condition is enforced on the electric field, then a Lagrange multiplier is introduced, and the problem becomes in solving a mixed curl-curl formulation of the Maxwell’s problem. The method is shown to be an absolutely stable HDG method for the indefinite time-harmonic Maxwell equations with high wave number. By exploiting the duality argument, the dependence of convergence of the HDG method on wave number \(\kappa\), mesh size \(h\) and polynomial order \(p\) is obtained.

1. INTRODUCTION

The time-harmonic Maxwell equations read as follows:

\[
\begin{align*}
\text{curl} \, \text{curl} \, \mathbf{u} - \kappa^2 \mathbf{u} &= \tilde{\mathbf{f}} \quad \text{in } \Omega, \\
\text{curl} \, \mathbf{u} \times \mathbf{n} + i\kappa \mathbf{u}^t &= \tilde{\mathbf{g}} \quad \text{on } \partial \Omega,
\end{align*}
\]

where \(\Omega \subset \mathbb{R}^3\) is a bounded Lipschitz domain, the wave number \(\kappa\) is real and positive, \(i\) denotes the imaginary unit, \(\mathbf{n}\) denotes the unit outward normal to \(\partial \Omega\), and \(\mathbf{u}^t = (\mathbf{n} \times \mathbf{u}) \times \mathbf{n}\) denotes the tangential component of the electric field \(\mathbf{u}\). Equation (1.1b) is the standard impedance boundary condition which indicates \(\tilde{\mathbf{g}} \cdot \mathbf{n} = 0\), thus, \(\tilde{\mathbf{g}}^t = \tilde{\mathbf{g}}\). The above Maxwell equations are of considerable importance in the engineering and scientific computation. In this paper we assume the current density is divergence-free (namely \(\text{div} \, \tilde{\mathbf{f}} = 0\)), hence the electric field \(\mathbf{u}\) is also divergence-free. In fact, the numerical method introduced in this paper can also be used for the case when the current density is not divergence-free.

The Maxwell’s operator is strongly indefinite for large wave number \(\kappa\), which brings difficulties both in theoretical analysis and numerical simulation. Various numerical methods which include finite element methods (FEM) \([19, 20, 11, 12, 7, 25]\), discontinuous Galerkin (DG) methods \([22, 23, 3, 14, 15, 21, 13, 10]\) and weak Galerkin FEM method \([18]\) have been developed to solve the Maxwell’s problem. In particular, Feng and Wu \([10]\) recently proposed and analyzed an interior penalty discontinuous Galerkin (IPDG) method for the problem (1.1) with high wave number, which is uniquely solvable without any mesh constraint. DG methods have several attractive features which include the capabilities to handle complex
Two HDG methods were presented in [21] for the numerical solution of the Maxwell problem. The first HDG method enforces the divergence-free condition on the electric field and introduces a Lagrange multiplier. It produces a linear system for the degrees of freedom (DOF) of the approximate traces of both the tangential component of the vector field and the Lagrange multiplier. The second HDG method does not enforce the divergence-free condition and results in a linear system only for the DOF of the approximate trace of the tangential component of the vector field. Compared to the IPDG method for the time-harmonic Maxwell equations in [15, 10], the two HDG methods have less globally coupled unknowns. However, no convergence analysis is given in [21]. In this paper we are interested in the convergence analysis for the HDG method which solves a mixed curl-curl formulation of the time-harmonic Maxwell equations

\begin{align}
\text{curl} \ \text{curl} \ u - \kappa^2 u + \nabla \tilde{\sigma} &= \tilde{f} \quad \text{in } \Omega, \\
\text{div} \ u &= 0 \quad \text{in } \Omega, \\
\text{curl} \ u \times n + i\kappa u_t &= \tilde{g} \quad \text{on } \partial\Omega, \\
\tilde{\sigma} &= 0 \quad \text{on } \partial\Omega,
\end{align}

where $\tilde{\sigma}$ is a scalar Lagrange multiplier used to enforce the divergence-free condition. Taking the divergence of the equation (1.2a) yields $\Delta \tilde{\sigma} = 0$, which together with the boundary condition (1.2d) implies that $\tilde{\sigma} = 0$ throughout the domain. Hence, under the divergence-free condition of the current density, the equations (1.1) and (1.2) are equivalent.

We aim to develop an HDG method which is absolutely stable without any mesh constraint for the above mixed curl-curl formulation (1.2) and reveal the dependence of convergence for the HDG method on wave number $\kappa$, mesh size $h$ and polynomial order $p$. We mention that only simple $L^2$-projections are used in our analysis which is different from the projection-based error analysis in [6], and the $p$-dependence of the stability estimate and the convergence can be derived. We also mention that the stabilization parameters in our HDG method are different from that in [21]. The choice of the stabilization parameters in our HDG method is derived from the stability analysis. The focus of our analysis is to apply the duality argument to establish the rigorous stability estimate and error analysis for the HDG method proposed for the mixed curl-curl formulation (1.2). Intrinsically, the regularity estimate of the solution of the dual problem used in this paper can be obtained due to the introduction of a Lagrange multiplier in the mixed curl-curl formulation. This is also the reason why the Helmholtz decomposition technique can be avoided in the analysis and the $p$-estimate can be derived. Up to our best knowledge, we give the first $p$-estimate of numerical methods using piecewise polynomial solution spaces for solving the time-harmonic Maxwell equations with high wave number.

The remainder of this paper is the following. We give some notations, introduce the HDG method for the mixed curl-curl formulation of the time-harmonic Maxwell equations (1.2) and present the main results of stability estimates and error estimates in the next section. Section 4 and section 5 are devoted to providing detailed proofs of the stability estimates and error estimates respectively.
2. Notation, HDG method and main results

Let \( f := -i\vec{f}, \sigma := -i\vec{\sigma} \) and \( g := -i\vec{g} \). The HDG scheme for the equation (1.2) is based on a first-order system of this equation, which can be rewritten in a mixed formulation as finding \((w, u, \sigma)\) such that

\[
\begin{align*}
\mathbf{i}w - \text{curl } u &= 0 \quad \text{in } \Omega, \quad (2.1a) \\
\text{curl } w + \kappa^2 u + \nabla \sigma &= f \quad \text{in } \Omega, \quad (2.1b) \\
\text{div } u &= 0 \quad \text{in } \Omega, \quad (2.1c) \\
w \times n + \kappa u^t &= g \quad \text{on } \partial \Omega, \quad (2.1d) \\
\sigma &= 0 \quad \text{on } \partial \Omega. \quad (2.1e)
\end{align*}
\]

Throughout the paper we use the standard notations and definitions for Sobolev spaces (see, e.g., Adams [1]). We denote by \( T_h \) a conforming triangulation of \( \Omega \) made of shape-regular simplicial elements. We denote by \( h_T \) the diameter of \( T \in T_h \) and \( h = \max_{T \in T_h} h_T \), the collection of faces is denoted by \( \mathcal{E}_h \), with the collection of interior faces by \( \mathcal{E}_h^0 \) and the collection of boundary faces by \( \mathcal{E}_h^\partial \), the collection of element boundaries by \( \partial T_h := \{ \partial T | T \in T_h \} \). We let \( C \) denote a positive number independent of the mesh size, polynomial order and wave number, but the value of which can take on different values in different occurrences. The corresponding finite element spaces for the HDG method for the first-order system (2.1) are defined to be

\[
\begin{align*}
V_h := \{ r \in L^2(\Omega) : r|_T \in P_p(T), \forall T \in T_h \}, \\
U_h := \{ v \in L^2(\Omega) : v|_T \in P_p(T), \forall T \in T_h \}, \\
M^i_h := \{ \eta \in L^2(\mathcal{E}_h) : \eta|_F \in P_p(F), (\eta \cdot n)|_F = 0, \forall F \in \mathcal{E}_h \}, \\
Q_h := \{ q \in L^2(\Omega) : q|_T \in P_p(T), \forall T \in T_h \}, \\
M_h := \{ \xi \in L^2(\mathcal{E}_h) : \xi|_T \in P_p(\mathcal{E}_h), \forall F \in \mathcal{E}_h \},
\end{align*}
\]

where the polynomial order \( p \geq 1 \), \( L^2(\Omega) = [L^2(\Omega)]^3, L^2(\mathcal{E}_h) = [L^2(\mathcal{E}_h)]^3, P_p(T) = [P_p(T)]^3 \) and \( P_p(F) = [P_p(F)]^3 \). In addition, we set \( M_h(g) := \{ \xi \in M_h : \xi = P_M g \text{ on } \partial \Omega \} \). Here, \( P_M \) denotes the standard \( L^2 \)-projection operator from \( L^2(\mathcal{E}_h) \) onto \( P_p(\mathcal{E}_h) \). Similarly, \( P_M \) denotes the standard \( L^2 \)-projection operator from \( L^2(\mathcal{E}_h) \) onto \( P_p(\mathcal{E}_h) \). We use \( \Pi_V, \Pi_U, \Pi_Q \) to denote the standard \( L^2 \)-projection onto \( V_h, U_h \) and \( Q_h \) respectively. In the analysis, we shall use the following approximation results of \( L^2 \)-projections:

\[
\begin{align*}
\|w - \Pi_V w\|_{\mathcal{E}_h} &\leq Ch^t/p^t \|w\|_{t, \Omega} \quad 0 \leq t \leq p + 1, \quad (2.2a) \\
\|u - \Pi_U u\|_{\mathcal{E}_h} &\leq Ch^s/p^s \|u\|_{s, \Omega} \quad 0 \leq s \leq p + 1, \quad (2.2b) \\
\|\sigma - \Pi_Q \sigma\|_{\mathcal{E}_h} &\leq Ch^\beta/p^\beta \|\sigma\|_{\beta, \Omega} \quad 0 \leq \beta \leq p + 1, \quad (2.2c) \\
\|w - \Pi_V w\|_{0, \partial T} &\leq Ch^{t-1/2}/p^{t-1/2} \|w\|_{t, T} \quad \forall T \in T_h, \ 0 \leq t \leq p + 1, \quad (2.2d) \\
\|u - \Pi_U u\|_{0, \partial T} &\leq Ch^{s-1/2}/p^{s-1/2} \|u\|_{s, T} \quad \forall T \in T_h, \ 0 \leq s \leq p + 1, \quad (2.2e) \\
\|\sigma - \Pi_Q \sigma\|_{0, \partial T} &\leq Ch^{\beta-1/2}/p^{\beta-1/2} \|\sigma\|_{\beta, T} \quad \forall T \in T_h, \ 0 \leq \beta \leq p + 1, \quad (2.2f) \\
\|w - P_M w\|_{\partial \Omega} &\leq Ch^{t-1/2}/p^{t-1/2} \|w\|_{t, \Omega} \quad 0 \leq s \leq p + 1, \quad (2.2g)
\end{align*}
\]
\[ \| \mathbf{u} - P_M \mathbf{u} \|_{\partial T_h} \leq C h^{s - \frac{1}{2}} / \rho^{s - \frac{1}{2}} \| \mathbf{u} \|_{s, \Omega} \quad 0 \leq s \leq p + 1, \quad (2.2h) \]

\[ \| \sigma - P_M \sigma \|_{\partial T_h} \leq C h^{\beta - \frac{1}{2}} / \rho^{\beta - \frac{1}{2}} \| \sigma \|_{\beta, \Omega} \quad 0 \leq \beta \leq p + 1. \quad (2.2i) \]

The above results hold due to the \( hp \) approximation theory of polynomials and trace inequality when \( T_h \) consists of shape-regular simplicial elements. Thus when we only consider \( \kappa - h \)-dependence in our analysis, \( T_h \) can be a conforming mesh consisting of shape-regular polyhedral elements. This is due to the fact that only the approximation results \( (2.2a)-\(2.2d) \) can be deduced recently in the literature (cf. [2]) when the mesh consists of general polyhedral elements. The \( p \)-dependence of convergence for the trace estimate of the polynomial \( L^2 \)-projection was first studied in [4] on simplicial element. It would be an interesting topic of extending the result to the general polyhedral element in the future work.

We define the bilinear forms

\[ (\mathbf{\eta}, \mathbf{\zeta})_{T_h} := \sum_{T \in T_h} (\mathbf{\eta} \cdot \mathbf{n}) T, \quad \langle \mathbf{\eta}, \mathbf{\zeta} \rangle_{\partial T_h} := \sum_{T \in T_h} \langle \mathbf{\eta} \cdot \mathbf{n} \rangle_{\partial T}, \]

where \( \langle \mathbf{\eta}, \mathbf{\zeta} \rangle_D \) (respectively, \( \langle \mathbf{\eta}, \mathbf{\zeta} \rangle_D \)) denotes the integral of \( \mathbf{\eta} \cdot \mathbf{\zeta} \) (respectively, \( \mathbf{\eta} \cdot \mathbf{\zeta} \)) over \( D \subset \mathbb{R}^3 \) and \( \langle \mathbf{\eta}, \mathbf{\zeta} \rangle_D \) (respectively, \( \langle \mathbf{\eta}, \mathbf{\zeta} \rangle_D \)) denotes the integral of \( \mathbf{\eta} \cdot \mathbf{\zeta} \) (respectively, \( \mathbf{\eta} \cdot \mathbf{\zeta} \)) over \( D \subset \mathbb{R}^2 \).

The HDG method for the first-order system \( (2.1) \) yields a solution \((\mathbf{w}_h, \mathbf{u}_h, \hat{\mathbf{u}}_h^{i}, \sigma_h, \tilde{\sigma}_h) \) \( \in \mathbf{V}_h \times \mathbf{U}_h \times \mathbf{M}_h^{i} \times Q_h \times M_h(0) \) such that

\[ (i\mathbf{w}_h, \mathbf{r}_h)_{T_h} - (\mathbf{u}_h, \text{curl} \mathbf{r}_h)_{T_h} + \langle \hat{\mathbf{u}}_h^{i} \times \mathbf{n}, \mathbf{r}_h \rangle_{\partial T_h} = 0 \quad (2.3a) \]

\[ (\mathbf{w}_h, \text{curl} \mathbf{v}_h)_{T_h} - (\hat{\mathbf{w}}_h \times \mathbf{n}, \mathbf{v}_h)_{\partial T_h} + (i\kappa^2 \mathbf{u}_h, \mathbf{v}_h)_{T_h} \]

\[ - (\sigma_h, \text{div} \mathbf{v}_h)_{T_h} + \langle \tilde{\sigma}_h, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial T_h} = (\mathbf{f}, \mathbf{v}_h)_{T_h}, \quad (2.3b) \]

\[ - (\mathbf{u}_h, \nabla q_h)_{T_h} + \langle \hat{\mathbf{u}}_h^{n} \cdot \mathbf{n}, q_h \rangle_{\partial T_h} = 0, \quad (2.3c) \]

\[ \langle \hat{\mathbf{w}}_h \times \mathbf{n}, \mathbf{n} \rangle_{T_h} = 0, \quad (2.3d) \]

\[ \langle \hat{\mathbf{u}}_h^{i} \times \mathbf{n}, \mathbf{\eta}_h \rangle_{\partial T_h} + \langle \kappa \hat{\mathbf{u}}_h^{i}, \mathbf{\eta}_h \rangle_{\partial T_h} = \langle \mathbf{g}, \mathbf{\eta}_h \rangle_{\partial T_h}, \quad (2.3e) \]

\[ \langle \hat{\mathbf{u}}_h^{n} \cdot \mathbf{n}, \mathbf{\xi}_h \rangle_{\partial T_h} = 0, \quad (2.3f) \]

for all \((\mathbf{r}_h, \mathbf{v}_h, q_h, \mathbf{\xi}_h) \) \( \in \mathbf{V}_h \times \mathbf{U}_h \times \mathbf{M}_h^{i} \times Q_h \times M_h(0) \), where

\[ \hat{\mathbf{w}}_h = \mathbf{w}_h + \tau_t (\mathbf{u}_h^{i} - \hat{\mathbf{u}}_h^{i}) \times \mathbf{n}, \quad \hat{\mathbf{u}}_h^{n} = \mathbf{u}_h^{n} + \tau_n (\sigma_h - \tilde{\sigma}_h) \mathbf{n}. \quad (2.4) \]

Here, for any vector \( \mathbf{r} \in \mathbb{R}^3 \), \( \mathbf{r}^{n} = (\mathbf{r} \cdot \mathbf{n}) \mathbf{n} \) denotes the normal component of the vector \( \mathbf{r} \). The parameter \( \tau_t \) and \( \tau_n \) are the so-called local stabilization parameter which have an important effect on both the stability of the solution and the accuracy of the HDG scheme. We always choose \( \tau_t = p/h \) and \( \tau_n = \kappa h/p \) in this paper.

**Remark 2.1.** The mixed curl-curl formulation \((1.2)\) can also be applied to the Maxwell equations \((1.1)\) with \( \text{div} \, \mathbf{f} \neq 0 \). In this case \( \text{div} \, \mathbf{u} = \mathbf{\theta} \neq 0 \) with \( \mathbf{\theta} \) a given variable. Indeed, taking the divergence of the equation \((1.1a)\) implies that \( \mathbf{\theta} \) satisfies that \( -\kappa^2 \mathbf{\theta} = \text{div} \, \mathbf{f} \).

Then taking the divergence of the equation \((1.2a)\) again yields \( \Delta \tilde{\mathbf{\sigma}} = \text{div} \, \mathbf{f} + \kappa^2 \mathbf{\theta} = 0, \)
which together with the boundary condition (1.2d) also implies that $\tilde{\sigma} = 0$. Hence, the HDG scheme in this paper can also be used for the Maxwell equations (1.1) with $\text{div} \, \tilde{f} \neq 0$.

To state our main results, we need a regularity assumption of the dual problem. Let $\Psi$ and $\varphi$ be the solution of the following dual problem:

$$\begin{align*}
\text{curl curl } \Psi - \kappa^2 \Psi + i\nabla \varphi &= iJ \quad \text{in } \Omega, \\
\text{div } \Psi &= 0 \quad \text{in } \Omega, \\
\text{curl } \Psi \times n - i\kappa \Psi^t &= 0 \quad \text{on } \partial \Omega, \\
\varphi &= 0 \quad \text{on } \partial \Omega,
\end{align*}$$

(2.5a) – (2.5d)

where $J \in L^2(\Omega)$. We assume that the solution $(\Psi, \varphi)$ has the following stability estimate:

$$\kappa \|\Psi\|_{1,\Omega} + \|\text{curl } \Psi\|_{1,\Omega} + \kappa^2 \|\Psi\|_{0,\Omega} + \kappa \|\nabla \varphi\|_{0,\Omega} \leq C \kappa \|J\|_{0,\Omega}. \quad (2.6)$$

In the following, we show that (2.6) holds when $\Omega$ is a $C^2$-domain. It is easy to see that $\Psi$ satisfies

$$\begin{align*}
\text{curl curl } \Psi - \kappa^2 \Psi &= i(\text{curl } \varphi) \quad \text{in } \Omega, \\
\text{div } \Psi &= 0 \quad \text{in } \Omega, \\
\text{curl } \Psi \times n - i\kappa \Psi^t &= 0 \quad \text{on } \partial \Omega.
\end{align*}$$

(2.7) – (2.9)

By (2.7), we have

$$(\text{curl curl } \Psi - \kappa^2 \Psi, \nabla q)_\Omega = (i(\text{curl } \varphi), \nabla q)_\Omega \quad \forall q \in C^\infty_0(\Omega).$$

By doing integration by parts and (2.8), we have

$$(\text{curl curl } \Psi - \kappa^2 \Psi, \nabla q)_\Omega = 0 \quad \forall q \in C^\infty_0(\Omega).$$

By density argument, we have

$$(\text{curl curl } \Psi - \kappa^2 \Psi, \nabla q)_\Omega = 0 \quad \forall q \in H^1_0(\Omega).$$

We easily obtain $\text{div} \, (J - \nabla \varphi) = 0$ and $\|\nabla \varphi\|_{0,\Omega} \leq \|J\|_{0,\Omega}$. So, we can conclude that the assumption (2.6) holds when $\Omega$ is a $C^2$-domain (cf. [12]). In general, $\Psi$ may not belong to $H^2(\Omega)$. However, if $\Psi \in H^2(\Omega)$, it has been mentioned in [10] that the following estimate holds

$$\|\Psi\|_{2,\Omega} \leq C \kappa \|J - \nabla \varphi\|_{0,\Omega} \leq C \kappa \|J\|_{0,\Omega}. \quad (2.10)$$

We always assume $\Psi \in H^2(\Omega)$ in this paper.

Now we are ready to outline the main results in the following by showing the stability of the discrete solutions from the HDG method (2.3) and the associated error estimates.
Theorem 2.1. We assume that (2.6) and (2.10) hold. Let \((w_h, u_h, \tilde{u}_h^i, \sigma_h, \tilde{\sigma}_h)\) be the solution of the problem (2.3). We have

\[
\|u_h\|_{\mathcal{T}_h} \leq C \left( 1 + \frac{\kappa^3 h^2}{p^2} + \frac{\kappa h^2}{p^2} \right) \|f\|_{0,\Omega} \left( 1 + \frac{\kappa^2 h}{p} + \frac{\kappa h}{p} \right) \|g\|_{0,\partial\Omega},
\]  

(2.11)

\[
\|w_h\|_{\mathcal{T}_h} \leq C \left( 1 + \kappa^{-1} + \frac{\kappa^3 h^2}{p^2} + \frac{\kappa^2 h^2}{p^2} \right) \|f\|_{0,\Omega} \left( 1 + \frac{\kappa^{-\frac{1}{2}}}{p} + \frac{\kappa^\frac{1}{2} h}{p} \right) \|g\|_{0,\partial\Omega},
\]  

(2.12)

\[
\|\tilde{u}_h^i\|_{\sigma_{\mathcal{T}_h}} \leq C \left( \frac{\kappa h}{p} \right)^\frac{1}{2} \left( 1 + \frac{\kappa^3 h^2}{p^2} + \frac{\kappa^2 h^2}{p^2} \right) \|f\|_{0,\Omega} \left( 1 + \frac{\kappa^2 h}{p} + \frac{\kappa h}{p} \right) \|g\|_{0,\partial\Omega},
\]  

(2.13)

\[
\|\text{curl } u_h\|_{\mathcal{T}_h} \leq C \left( 1 + p^\frac{1}{2} \right) \left( 1 + \kappa^{-1} \right) \left( \kappa + p^\frac{1}{2} \right) \left( \frac{\kappa^2 h^2}{p^2} + \frac{\kappa h^2}{p^2} \right) \|f\|_{0,\Omega}
\]

\[ + C \left( 1 + p^\frac{1}{2} \kappa^{-\frac{1}{2}} \right) \left( 1 + \kappa^{-\frac{1}{2}} \right) \left( \kappa + p^\frac{1}{2} \right) \frac{\kappa^\frac{1}{2} h}{p} \|g\|_{0,\partial\Omega}.
\]  

(2.14)

\[
\|\text{div } u_h\|_{\mathcal{T}_h} \leq C p^\frac{1}{2} \left( 1 + \frac{\kappa^3 h^2}{p^2} + \frac{\kappa^2 h^2}{p^2} \right) \|f\|_{0,\Omega} \left( 1 + \frac{\kappa^2 h}{p} + \frac{\kappa h}{p} \right) \|g\|_{0,\partial\Omega}.
\]  

(2.15)

Theorem 2.2. We assume that (2.6) and (2.10) hold. Let \((w, u, \sigma)\) and \((w_h, u_h, \tilde{u}_h^i, \sigma_h, \tilde{\sigma}_h)\) solve the equations (2.1) and (2.3). We have

\[
\|u - u_h\|_{\mathcal{T}_h} \leq C \left( \frac{\kappa^\frac{1}{2} h^t + 1}{p^t + 1} + \frac{\kappa^2 h + 1}{p^t + 1} \right) \|w\|_{t,\Omega} + C \left( \frac{h^s}{p^s} + \frac{\kappa^\frac{1}{2} h^s}{p^s} + \frac{\kappa^2 h^s}{p^s} \right) \|u\|_{s,\Omega},
\]  

(2.16)

\[
\|w - w_h\|_{\mathcal{T}_h} \leq C \left( \frac{h^t}{p^t + 1} + \frac{\kappa^\frac{1}{2} h^t + 1}{p^t + 1} + \frac{\kappa^2 h^t + 1}{p^t + 1} \right) \|w\|_{t,\Omega}
\]

\[ + C \left( 1 + \kappa^{-\frac{1}{2}} \right) \frac{h^s}{p^s} + \frac{\kappa^\frac{1}{2} h^s}{p^s} + \frac{\kappa^2 h^s}{p^s} \|u\|_{s,\Omega}.
\]  

(2.17)

\[
\|\text{curl } (u - u_h)\|_{\mathcal{T}_h} \leq C \left( \frac{h^t}{p^t + 1} + \frac{\kappa^\frac{1}{2} h^t + 1}{p^t + 1} + \frac{\kappa^2 h^t + 1}{p^t + 1} \right) \|w\|_{t,\Omega}
\]

\[ + C \left( 1 + \kappa^{-\frac{1}{2}} \right) \frac{h^s}{p^s} + \frac{\kappa^\frac{1}{2} h^s}{p^s} + \frac{\kappa^2 h^s}{p^s} \|u\|_{s,\Omega},
\]  

(2.18)

\[
\|\text{div } (u - u_h)\|_{\mathcal{T}_h} \leq C \left( \frac{h^t}{p^t + 1} \|w\|_{t,\Omega} + C \left( \frac{h^s}{p^s + \frac{1}{2}} + \frac{\kappa^\frac{1}{2} h^s}{p^s + \frac{1}{2}} \right) \|u\|_{s,\Omega}.
\]  

(2.19)

Remark 2.2. When we consider only \(\kappa\)- and \(h\)-dependence, the above results hold when \(\mathcal{T}_h\) consists of general polyhedral elements.

Remark 2.3. We assume the solution of the first-order system (2.1) satisfy \(u \in H^2(\Omega)\) and \(w \in H^1(\Omega)\). When \(f\) is divergence-free, it can be shown that (cf. [12] [10])

\[
\|u\|_{2,\Omega} + \|w\|_{1,\Omega} \leq C \kappa M(f, g),
\]
The above estimates indicate that the error when the wave number \( \kappa > 1 \), it further holds that

\[
\| u - u_h \|_{\mathcal{T}_h} \leq C \left( \frac{\kappa h^2}{p^2} + \frac{\kappa^2 h^2}{p^2} \right) M(f, g),
\]

\[
\| w - w_h \|_{\mathcal{T}_h} \leq C \left( \frac{\kappa h}{p} + \frac{\kappa^2 h^2}{p^2} \right) M(f, g).
\]

3. Stability estimate

In this section we shall show that the HDG method (2.3) is absolutely stable. We first present a lemma which shall be used to estimate the stability estimate of \( u_h \).

Lemma 3.1. Let \( (w_h, u_h, \hat{u}_h, \sigma_h, \tilde{\sigma}_h) \) be the solution of the problem (2.3). It holds that

\[
\| \tau_t (u_h - \hat{u}_h) \|_{\mathcal{O} \mathcal{T}_h}^2 + \| \tau_n (\sigma_h - \tilde{\sigma}_h) \|_{\mathcal{O} \mathcal{T}_h}^2 + \kappa \| \hat{u}_h \|_{0, \partial \Omega}^2 \leq \| f \|_{0, \Omega} \| u_h \|_{\mathcal{T}_h} + \frac{1}{2 \kappa} \| g \|_{0, \partial \Omega}^2. \tag{3.1}
\]

\[
\| w_h \|_{\mathcal{T}_h}^2 \leq \kappa^2 \| u_h \|_{\mathcal{T}_h}^2 + 2 \| f \|_{\Omega} \| u_h \|_{\mathcal{T}_h} + \frac{1}{\kappa} \| g \|_{0, \partial \Omega}^2. \tag{3.2}
\]

Proof. We first choose \( r_h = w_h, v_h = u_h, \eta_h = \hat{u}_h, q_h = \sigma_h, \xi_h = \tilde{\sigma}_h \) in (2.5a)-(2.5l) to get the following equalities:

\[
(iw_h, w_h)_{\mathcal{T}_h} - (u_h, \operatorname{curl} w_h)_{\mathcal{T}_h} + (\hat{u}_h \times n, w_h)_{\mathcal{T}_h} = 0, \tag{3.3a}
\]

\[
(\operatorname{curl} w_h, u_h)_{\mathcal{T}_h} + (\hat{u}_h \times n, u_h)_{\mathcal{T}_h} + (\hat{u}_h \times n, u_h)_{\mathcal{T}_h} = 0, \tag{3.3b}
\]

\[
(\operatorname{div} u_h, \sigma_h)_{\mathcal{T}_h} - (\hat{u}_h \cdot n, \sigma_h)_{\mathcal{T}_h} + (\hat{u}_h \cdot n, \sigma_h)_{\mathcal{T}_h} = 0, \tag{3.3c}
\]

\[
(\hat{u}_h \times n, \hat{u}_h)_{\mathcal{T}_h} = 0, \tag{3.3d}
\]

\[
(\hat{u}_h \times n, \hat{u}_h)_{\mathcal{T}_h} + (\kappa \hat{u}_h, \hat{u}_h)_{\mathcal{T}_h} = g, \hat{u}_h)_{\partial \Omega}, \tag{3.3e}
\]

\[
(\hat{u}_h \cdot n, \sigma_h)_{\mathcal{T}_h} = 0. \tag{3.3f}
\]

where (3.3b) and (3.3c) are obtained by integration by parts. Furthermore, noting the definitions of \( \hat{u}_h \) and \( \hat{u}_h \) in (2.4) and applying the complex conjugation of (3.3a), (3.3c) and (3.3f), we get the following equalities after simple manipulations:

\[
- (iw_h, w_h)_{\mathcal{T}_h} - (\operatorname{curl} w_h, u_h)_{\mathcal{T}_h} - (\tau_t (u_h - \hat{u}_h) \hat{u}_h)_{\mathcal{T}_h} + (\kappa \hat{u}_h, \hat{u}_h)_{\mathcal{T}_h} = g, \hat{u}_h)_{\partial \Omega}, \tag{3.3b}
\]

\[
(\sigma_h, \operatorname{div} u_h)_{\mathcal{T}_h} - (\sigma_h, \hat{u}_h)_{\mathcal{T}_h} + (\sigma_h, \hat{u}_h)_{\mathcal{T}_h} = 0, \tag{3.3c}
\]

\[
- (\sigma_h, \hat{u}_h)_{\mathcal{T}_h} = 0. \tag{3.3f}
\]
Adding the above three equalities and (3.3b) together and noting that \((\mathbf{u}_h^t - \mathbf{\tilde{u}}_h)^t = \mathbf{u}_h^t - \mathbf{\tilde{u}}_h^t\), we have
\[
- (i\mathbf{w}_h, \mathbf{w}_h)_{\mathcal{T}_h} + \langle \tau_t (\mathbf{u}_h^t - \mathbf{\tilde{u}}_h^t), \mathbf{u}_h^t - \mathbf{\tilde{u}}_h^t \rangle_{\mathcal{O}_h} + \langle \kappa \mathbf{\tilde{u}}_h^t, \mathbf{\tilde{u}}_h^t \rangle_{\partial \Omega} + \langle \tau_n (\sigma_h - \mathbf{\tilde{\sigma}}_h), \sigma_h - \mathbf{\tilde{\sigma}}_h \rangle_{\partial \Omega} + (i\kappa^2 \mathbf{u}_h, \mathbf{u}_h)_{\mathcal{T}_h} = (\mathbf{f}, \mathbf{u}_h)_{\mathcal{T}_h} + \langle \mathbf{g}, \mathbf{\tilde{u}}_h^t \rangle_{\partial \Omega},
\]
which implies the lemma by the Cauchy-Schwarz inequality. □

Next we shall utilize the dual argument to give the \(L^2\)-norm estimate of \(\mathbf{u}_h\). Given \(\mathbf{u}_h \in L^2(\Omega)\), we introduce the first-order system of the dual problem (2.5) with \(J = \mathbf{u}_h\):
\[
\begin{align*}
\mathbf{i}\Phi - \text{curl} \Psi &= 0 \quad \text{in } \Omega, \quad (3.4a) \\
\text{curl} \Phi + i\kappa^2 \Psi + \nabla \varphi &= \mathbf{u}_h \quad \text{in } \Omega, \quad (3.4b) \\
\text{div} \Psi &= 0 \quad \text{in } \Omega, \quad (3.4c) \\
\Phi \times \mathbf{n} - \kappa \mathbf{\Psi}^t &= 0 \quad \text{on } \partial \Omega, \quad (3.4d) \\
\varphi &= 0 \quad \text{on } \partial \Omega. \quad (3.4e)
\end{align*}
\]
Due to \(\varphi \in H^1_0(\Omega)\), we easily obtain
\[
\|\varphi\|_{1, \Omega} \leq C \|\mathbf{u}_h\|_{\mathcal{T}_h}. \quad (3.5)
\]
Taking \(J = \mathbf{u}_h\) in (2.6) and (2.10) we have
\[
\|\Psi\|_{2, \Omega} + \kappa \|\Psi\|_{1, \Omega} + \|\text{curl } \Psi\|_{1, \Omega} + \kappa^2 \|\Psi\|_{0, \Omega} \leq C \|\mathbf{u}_h\|_{\mathcal{T}_h}, \quad (3.6)
\]
which implies
\[
\|\Phi\|_{1, \Omega} \leq C \|\mathbf{u}_h\|_{\mathcal{T}_h}. \quad (3.7)
\]
By the equation (2.5a) with \(J = \mathbf{u}_h\), we directly have
\[
(\text{curl } \text{curl} \Psi, \Psi)_{\Omega} - \kappa^2 (\Psi, \Psi)_{\Omega} + (i\nabla \varphi, \Psi)_{\Omega} = (i\mathbf{u}_h, \Psi)_{\Omega},
\]
which together with the fact \((\nabla \varphi, \Psi)_{\Omega} = 0\) and the boundary condition (2.5c) yields
\[
(\text{curl } \Psi, \text{curl } \Psi)_{\Omega} - \kappa^2 (\Psi, \Psi)_{\Omega} - (i\kappa \Psi^t, \Psi^t)_{\partial \Omega} = (i\mathbf{u}_h, \Psi)_{\Omega}.
\]
Thus, taking the imaginary part of the left-hand side of the above equation and using (3.6) we have
\[
\kappa \|\Psi^t\|_{0, \partial \Omega}^2 \leq \|\mathbf{u}_h\|_{\mathcal{T}_h} \|\Psi\|_{0, \Omega} \leq C \kappa^{-1} \|\mathbf{u}_h\|_{\mathcal{T}_h}^2. \quad (3.8)
\]

Next we present a key equality.

**Lemma 3.2.** Let \((\Phi, \Psi, \varphi) \in H^1(\Omega) \times H^2(\Omega) \times H^1_0(\Omega)\) be the solution of the dual problem (3.4). We have
\[
\|\mathbf{u}_h\|_{\mathcal{T}_h}^2 = \sum_{k=1}^{6} T_k.
\]
where

\[
T_1 = \langle u_h^t \times n - \hat{u}_h^t \times n, \Phi - \Pi V \Phi \rangle_{\partial \Gamma_h},
\]

\[
T_2 = \langle u_h^t \cdot n - \hat{u}_h^t \cdot n, \varphi - \Pi Q \varphi \rangle_{\partial \Gamma_h},
\]

\[
T_3 = -\langle \tau(u_h^t - \hat{u}_h^t), \Psi - \Pi U \Psi \rangle_{\partial \Gamma_h},
\]

\[
T_4 = -\langle \kappa \hat{u}_h^t + \hat{w}_h^t \times n, \Psi^t \rangle_{\partial \Omega},
\]

\[
T_5 = -(f, \Pi U \Psi)_{\partial \Gamma_h},
\]

\[
T_6 = (\sigma_h, \text{div} (\Psi - \Pi U \Psi))_{\partial \Gamma_h} - \langle \hat{\sigma}_h, (\Psi - \Pi U \Psi) \cdot n \rangle_{\partial \Gamma_h}.
\]

**Proof.** Using the dual first-order system (3.4), we obtain

\[
\| u_h \|^2_{\Gamma_h} = (u_h, \text{curl} \Phi + i \kappa \Psi + \nabla \varphi)_{\Gamma_h} + (w_h, i \Phi - \text{curl} \Psi)_{\Gamma_h}
\]

\[
= (u_h, \text{curl} \Pi V \Phi + i \kappa \Pi U \Phi - \nabla \Pi_Q \varphi)_{\Gamma_h} + (w_h, i \Pi V \Phi - \text{curl} \Pi U \Psi)_{\Gamma_h}
\]

\[
+ (u_h, \text{curl} (\Phi - \Pi V \Phi))_{\Gamma_h} + (u_h, i \kappa (\Psi - \Pi U \Psi))_{\Gamma_h} + (\psi_h, \nabla (\varphi - \Pi Q \varphi))_{\Gamma_h}
\]

\[
+ (w_h, i (\Phi - \Pi V \Phi))_{\Gamma_h} - (w_h, \text{curl} (\Psi - \Pi U \Psi))_{\Gamma_h}.
\]  

By the definitions of \( \Pi_U \) and \( \Pi_V \), we have \((u_h, i \kappa (\Psi - \Pi U \Psi))_{\Gamma_h} = 0 \) and \((w_h, i (\Phi - \Pi V \Phi))_{\Gamma_h} = 0 \). Integrating by parts and applying the property of the \( L^2 \)-projections yields

\[
(u_h, \text{curl} (\Phi - \Pi V \Phi))_{\Gamma_h} = (\text{curl} u_h, \Phi - \Pi V \Phi)_{\Gamma_h} + (u_h \times n, \Phi - \Pi V \Phi)_{\partial \Gamma_h}
\]

\[
= (u_h \times n, \Phi - \Pi V \Phi)_{\partial \Gamma_h}
\]

\[
= (u_h^t \times n, \Phi - \Pi V \Phi)_{\partial \Gamma_h},
\]  

(3.10)

and

\[
(u_h, \nabla (\varphi - \Pi Q \varphi))_{\Gamma_h} = -(\text{div} u_h, \varphi - \Pi Q \varphi)_{\Gamma_h} + (u_h^n \cdot n, \varphi - \Pi Q \varphi)_{\partial \Gamma_h}
\]

\[
= (u_h^n \cdot n, \varphi - \Pi Q \varphi)_{\partial \Gamma_h},
\]  

(3.11)

and

\[
-(w_h, \text{curl} (\Psi - \Pi U \Psi))_{\Gamma_h} = -(\text{curl} w_h, \Psi - \Pi U \Psi)_{\Gamma_h} - (w_h \times n, \Psi - \Pi U \Psi)_{\partial \Gamma_h}
\]

\[
= -(w_h^t \times n, \Psi - \Pi U \Psi)_{\partial \Gamma_h}.
\]  

(3.12)

Taking \( r_h = \Pi V \Phi \) in the equation (2.3a), noting that \( \hat{u}_h^t \times n \) is continuous across each interior face and using the boundary condition (3.4d), we obtain

\[
(u_h, \text{curl} \Pi V \Phi)_{\Gamma_h} = (i w_h, \Pi V \Phi)_{\Gamma_h} + (\hat{u}_h^t \times n, \Pi V \Phi)_{\partial \Gamma_h}
\]

\[
= (i w_h, \Pi V \Phi)_{\Gamma_h} + (\hat{u}_h^t \times n, \Pi V \Phi - \Phi)_{\partial \Gamma_h} - (\hat{u}_h^t, \Phi \times n)_{\partial \Omega}
\]

\[
= (i w_h, \Pi V \Phi)_{\Gamma_h} + (\hat{u}_h^t \times n, \Pi V \Phi - \Phi)_{\partial \Gamma_h} - (\hat{u}_h^t, \kappa \Psi^t)_{\partial \Omega}.
\]  

(3.13)

Taking \( q_h = \Pi Q \varphi \) in (2.3c) and noting that \( \hat{u}_h^n \cdot n \) is continuous across each interior face and \( \varphi \in H_0^1(\Omega) \) we have

\[
(u_h, \nabla \Pi Q \varphi)_{\Gamma_h} = (\hat{u}_h^n \cdot n, \Pi Q \varphi)_{\partial \Gamma_h} = (\hat{u}_h^n \cdot n, \Pi Q \varphi - \varphi)_{\partial \Gamma_h}.
\]  

(3.14)
We further take \( v_h = \Pi_U \Psi \) in (2.3b) to get
\[
-w_h, \text{curl} \Pi_U \Psi \rangle_{\mathcal{T}_h} - \langle w_h \times n, \Pi_U \Psi \rangle_{\partial \mathcal{T}_h} + (i\kappa^2 u_h, \Pi_U \Psi)_{\mathcal{T}_h} - (\sigma_h, \text{div} \Pi_U \Psi)_{\mathcal{T}_h}
\]
\[
+ \langle \sigma_h, \Pi_U \Psi \cdot n \rangle_{\partial \mathcal{T}_h} - (f, \Pi_U \Psi)_{\mathcal{T}_h}
\]
\[
= -\langle w_h \times n - \tau_t(u_h^t - \tilde{u}_h^t), \Pi_U \Psi \rangle_{\partial \mathcal{T}_h} + (i\kappa^2 u_h, \Pi_U \Psi)_{\mathcal{T}_h} - (\sigma_h, \text{div} \Pi_U \Psi - \text{div} \Psi)_{\mathcal{T}_h}
\]
\[
+ \langle \sigma_h, \Pi_U \Psi \cdot n - \Psi \cdot n \rangle_{\partial \mathcal{T}_h} - (f, \Pi_U \Psi)_{\mathcal{T}_h},
\]
(3.15)
where the above second equality holds due to the fact that \( \text{div} \Psi = 0 \), \( \tilde{\sigma}_h \) is continuous across each interior face and \( \tilde{\sigma}_h = 0 \) on \( \mathcal{E}_h^\partial \). Inserting the above equalities (3.10)-(3.15) into the right-hand side of (3.9), we obtain the result. This completes the proof. \( \square \)

We can now give the proof of Theorem 2.1

Proof. (Proof of Theorem 2.1) We derive the upper bounds for \( T_1, \ldots, T_5 \) in Lemma 3.2. By the Cauchy-Schwarz inequality, the approximation properties of standard \( L^2 \)-projections and the inequalities (3.3)-(3.5), we have
\[
T_1 \leq C\|\tau_t^\frac{1}{2}(u_h^t - \tilde{u}_h^t)\|_{\mathcal{T}_h} \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \| \Psi \|_{1,\Omega} \leq C\|\tau_t^\frac{1}{2}(u_h^t - \tilde{u}_h^t)\|_{\mathcal{T}_h} \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \| \Psi \|_{1,\Omega},
\]
\[
T_2 = -\langle \tau_t(\sigma_h - \tilde{\sigma}_h), \varphi - \Pi_Q \varphi \rangle_{\partial \mathcal{T}_h} \leq C\|\tau_t^\frac{1}{2}(\sigma_h - \tilde{\sigma}_h)\|_{\partial \mathcal{T}_h} \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \| \varphi \|_{1,\Omega}
\]
\[
\leq C\|\tau_t^\frac{1}{2}(\sigma_h - \tilde{\sigma}_h)\|_{\partial \mathcal{T}_h} \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \| u_h \|_{\mathcal{T}_h},
\]
\[
T_3 \leq C\|\tau_t^\frac{1}{2}(u_h^t - \tilde{u}_h^t)\|_{\partial \mathcal{T}_h} \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \| \Psi \|_{2,\Omega} \leq C\|\tau_t^\frac{1}{2}(u_h^t - \tilde{u}_h^t)\|_{\partial \mathcal{T}_h} \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \| u_h \|_{\mathcal{T}_h}.
\]
Taking \( \eta_h = P_M \Psi^t \) in (2.3a) and using the property of the \( L^2 \)-projection operator \( P_M \) on \( \mathcal{E}_h^\partial \) and the inequality (3.8) yields
\[
T_4 = -\langle \kappa \tilde{u}_h^t + \tilde{w}_h^t \times n, P_M \Psi^t \rangle_{\partial \mathcal{O}} = -\langle g, P_M \Psi^t \rangle_{\partial \mathcal{O}} \leq \| g \|_{0,\partial \Omega} \| \Psi^t \|_{0,\partial \Omega} \leq C\| g \|_{0,\partial \Omega} \| u_h \|_{\mathcal{T}_h}.
\]
For the estimate of \( T_5 \), we easily deduce
\[
T_5 \leq \| f \|_{0,\Omega} \| \Psi \|_{0,\Omega} \leq C\| f \|_{0,\Omega} \| u_h \|_{\mathcal{T}_h}.
\]
Applying integration by parts on parts \( T_6 \), we have
\[
T_6 = -\langle \nabla \sigma_h, \Psi - \Pi_U \Psi \rangle_{\mathcal{T}_h} + \langle \sigma_h - \tilde{\sigma}_h, (\Psi - \Pi_U \Psi) \cdot n \rangle_{\partial \mathcal{T}_h} = \langle \sigma_h - \tilde{\sigma}_h, (\Psi - \Pi_U \Psi) \cdot n \rangle_{\partial \mathcal{T}_h}
\]
\[
\leq C\|\tau_t^\frac{1}{2}(\sigma_h - \tilde{\sigma}_h)\|_{\partial \mathcal{T}_h} \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \| \Psi \|_{2,\Omega} \leq C\|\tau_t^\frac{1}{2}(\sigma_h - \tilde{\sigma}_h)\|_{\partial \mathcal{T}_h} \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \| u_h \|_{\mathcal{T}_h}.
\]
Combining the above estimates for \( T_1, \ldots, T_6 \), we obtain
\[
\| u_h \|_{\mathcal{T}_h}^2 \leq C\| f \|_{0,\Omega} + \| g \|_{0,\partial \Omega} \| u_h \|_{\mathcal{T}_h}
\]
\[
+ C\left( \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \kappa + \tau_t^\frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \kappa \right) \| \tau_t^\frac{1}{2}(u_h^t - \tilde{u}_h^t)\|_{\partial \mathcal{T}_h} \| u_h \|_{\mathcal{T}_h}
\]
\[
+ C\left( \frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} + \frac{1}{2} \left( \frac{h}{p} \right)^\frac{3}{2} \kappa \right) \| \tau_t^\frac{1}{2}(\sigma_h - \tilde{\sigma}_h)\|_{\partial \mathcal{T}_h} \| u_h \|_{\mathcal{T}_h}.
\]
We choose $\tau_t = 2h$ and $\tau_n = \frac{\kappa h}{p}$ to get the minimum of $\frac{\kappa h}{p} \frac{1}{2} \kappa + \frac{1}{2} \frac{\kappa}{p} + \frac{1}{2} \frac{\kappa}{p} + \frac{1}{2} \frac{\kappa}{p}$ respectively. By the Young’s inequality, we have

$$\|u_h\|_{\mathcal{T}_h}^2 \leq C \left( \kappa^{-2} \left\| f \right\|_{0,\Omega}^2 + \kappa^{-2} \left\| g \right\|_{0,\partial\Omega}^2 + \left( \frac{\kappa h}{p} \right)^2 \left\| u_h^t - \hat{u}_h^t \right\|_{\partial\mathcal{T}_h}^2 + \kappa \left( \frac{h}{p} \right)^2 \right\| \sigma_h - \hat{\sigma}_h \|^2_{\mathcal{T}_h}. \tag{3.1}$$

Combining the above estimate and (3.1), we obtain (2.11) by the Young’s inequality.

Using integration by parts on the above equation, we have

$$\|w_h\|_{\mathcal{T}_h}^2 \leq \|\mathbf{w}_h\|_{\mathcal{T}_h}^2 + \frac{1}{\kappa} \left\| g \right\|_{0,\partial\Omega}^2 + \frac{1}{\kappa} \left\| g \right\|_{0,\partial\Omega}^2. \tag{2.12}$$

Then (2.12) is derived by (2.11). Furthermore, combining the fact (cf. [24]) $\tau$ and the triangular inequality yields (2.13).

Now we take $\mathbf{r}_h = \text{curl } \mathbf{u}_h$ in (2.3a) to get

$$\langle i w_h, \text{curl } \mathbf{u}_h \rangle_{\mathcal{T}_h} - (\mathbf{u}_h, \text{curl } \nabla \mathbf{u}_h)_{\mathcal{T}_h} + \langle \hat{u}_h^t \times \mathbf{n}, \text{curl } \mathbf{u}_h \rangle_{\mathcal{T}_h} = 0. \tag{2.14}$$

Using integration by parts on the above equation, we have

$$\langle i w_h, \text{curl } \mathbf{u}_h \rangle_{\mathcal{T}_h} - (\mathbf{u}_h, \text{curl } \nabla \mathbf{u}_h)_{\mathcal{T}_h} - \langle \hat{u}_h^t \times \mathbf{n}, \text{curl } \mathbf{u}_h \rangle_{\mathcal{T}_h} = 0,$$

which directly yields

$$\|\text{curl } \mathbf{u}_h\|_{\mathcal{T}_h}^2 \leq \|\mathbf{w}_h\|_{\mathcal{T}_h} \|\text{curl } \mathbf{u}_h\|_{\mathcal{T}_h} + C \left\| \tau_t^2 \left( u_h^t - \hat{u}_h^t \right) \right\|_{\partial\mathcal{T}_h} \frac{1}{\kappa} \|\mathbf{w}_h\|_{\mathcal{T}_h} \|\text{curl } \mathbf{u}_h\|_{\mathcal{T}_h}. \tag{3.2}$$

Combining the above inequality, (3.1) and (2.11), we get

$$\|\text{curl } \mathbf{u}_h\|_{\mathcal{T}_h}^2 \leq \|\mathbf{w}_h\|_{\mathcal{T}_h} \|\text{curl } \mathbf{u}_h\|_{\mathcal{T}_h} + C \left\| \tau_t^2 \left( u_h^t - \hat{u}_h^t \right) \right\|_{\partial\mathcal{T}_h} \frac{1}{\kappa} \|\mathbf{w}_h\|_{\mathcal{T}_h} \|\text{curl } \mathbf{u}_h\|_{\mathcal{T}_h}. \tag{3.3}$$

Combining the above inequality, (3.1) and (2.11), we can get the upper bound for $\|\tau_t^2 (\sigma_h - \hat{\sigma}_h)\|_{\partial\mathcal{T}_h}$. Moreover, taking $\mathbf{v}_h = \nabla \sigma_h$ in (2.3b) and applying integration by parts, the Cauchy-Schwarz inequality, trace inequality and the estimates in Lemma 3.1 and Theorem 2.1, we can also get the stability estimate for $\|\nabla \sigma_h\|_{\mathcal{T}_h}$. \hfill \Box

**Remark 3.1.** By the estimates (3.1) and (2.11), we can get the upper bound for $\|\tau_t^2 (\sigma_h - \hat{\sigma}_h)\|_{\partial\mathcal{T}_h}$. Moreover, taking $\mathbf{v}_h = \nabla \sigma_h$ in (2.3b) and applying integration by parts, the Cauchy-Schwarz inequality, trace inequality and the estimates in Lemma 3.1 and Theorem 2.1, we can also get the stability estimate for $\|\nabla \sigma_h\|_{\mathcal{T}_h}$.

**Remark 3.2.** When $\mathbf{f} = 0$ and $\mathbf{g} = 0$ in the first-order system (2.11), Theorem 2.1 and Lemma 3.1 imply $\mathbf{w}_h = 0, \mathbf{u}_h = 0$ on $\mathcal{T}_h$ and $\hat{u}_h^t = 0, \sigma_h = \hat{\sigma}_h$ on $\partial \mathcal{T}_h$. It then follows from (2.3b) that for any $\mathbf{v}_h \in \tilde{U}_h$,

$$-(\sigma_h, \text{div } \mathbf{v}_h)_{\mathcal{T}_h} + \langle \hat{\sigma}_h, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} = (\nabla \sigma_h, \mathbf{v}_h)_{\mathcal{T}_h} + \langle \hat{\sigma}_h, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial\mathcal{T}_h} = 0,$$

which implies $\sigma_h$ is piecewise constant on $\mathcal{T}_h$. Due to the fact that $\sigma_h = \hat{\sigma}_h = 0$ on $\partial \Omega$, we have $\sigma_h = 0$ on $\mathcal{T}_h$ and $\hat{\sigma}_h = 0$ on $\partial \mathcal{T}_h$. Hence, the well-posedness of the HDG method (2.3) always holds without imposing any mesh constraint.
4. Error analysis

In this section we provide detailed proofs of the a priori error estimates in Theorem 2.2.

We denote

\[ e_w = \Pi_Y w - w_h, \quad e_\omega^t = P_M w^t - \hat{w}_h^t, \quad e_u = \Pi_U u - u_h, \quad e_\sigma^t = P_M u^t - \hat{u}_h^t, \]

\[ e_\sigma^n = P_M u^n - \hat{u}_h^n, \quad e_\sigma = \Pi\sigma - \sigma_h, \quad e_\sigma = P_M\sigma - \hat{\sigma}_h. \]

In the following we first present the error equation for the analysis.

**Lemma 4.1.** Let \((w, u, \sigma)\) and \((w_h, u_h, \hat{u}_h, \sigma_h, \hat{\sigma}_h)\) solve the equations (2.1) and (2.3). We have

\[
\begin{align*}
(i e_w, r_h)_{\Sigma_h} - (e_u, \text{curl} r_h)_{\Sigma_h} + (e_\omega^t \times n, r_h)_{\partial \Sigma_h} &= 0, \\
(e_w, \text{curl} v_h)_{\Sigma_h} - (e_\omega^t \times n, v_h)_{\partial \Sigma_h} + (\kappa e_\omega^t, v_h)_{\Sigma_h} \\
&- (e_\sigma, \text{div} v_h)_{\Sigma_h} + (e_\sigma, v_h \cdot n)_{\partial \Sigma_h} = 0, \\
(e_\omega^t \times n, \hat{\eta}_h)_{\partial \Sigma_h} &= 0 \\
(e_\omega^t \times n, \hat{\eta}_h)_{\partial \Sigma_h} &= 0 \\
(e_\omega^t \times n, \hat{\xi}_h)_{\partial \Sigma_h} &= 0,
\end{align*}
\]

for all \((r_h, v_h, \eta_h, q_h, \xi_h) \in V_h \times U_h \times M_h \times Q_h \times M_h(0)\).

**Proof.** We notice that the exact solution \((w, u, u'|_{\Sigma_h}, \sigma, \sigma'|_{\Sigma_h})\) also satisfies the equation (2.3). Hence, due to the property of standard \(L^2\)-projection, the solutions \(w_h, \hat{w}_h'|_{\Sigma_h}, u_h, \hat{u}_h'|_{\Sigma_h},\)

\(\hat{u}_h'|_{\Sigma_h}, \sigma_h, \hat{\sigma}_h'|_{\Sigma_h}\) in the equation (2.3) can be replaced by \(\Pi_Y w, P_M w'|_{\Sigma_h}, \Pi_u u, P_M u'|_{\Sigma_h},\)

\(P_M u''|_{\Sigma_h}, \Pi\sigma, P_M\sigma|_{\Sigma_h}\) respectively to derive a new equation, which subtracts the equation (2.3) to yield the result. \(\square\)

Next we are going to present our first error estimate.

**Lemma 4.2.** If we choose \(\tau_t = \frac{h}{h^2}\) and \(\tau_h = \frac{h^p}{p}\), we have

\[
\|\kappa^2 e_\omega^t\|_{(\cdot, 0, \Omega)} + \|\tau_t^2 (e_u^t - e_u^{t-1})\|_{\Sigma_h} + \|\tau_h (e_\sigma - e_\sigma')\|_{\partial \Sigma_h}
\leq C \left( \frac{h^t}{p^t} \|w\|_{t, \Omega} + (1 + \kappa^{-\frac{1}{2}}) \frac{h^{s-1}}{p^{s-1}} \|u\|_{s, \Omega} \right),
\]

\[
\|e_u\|_{\Sigma_h} \leq \kappa \|e_u\|_{\Sigma_h} + C \left( \frac{h^t}{p^t} \|w\|_{t, \Omega} + (1 + \kappa^{-\frac{1}{2}}) \frac{h^{s-1}}{p^{s-1}} \|u\|_{s, \Omega} \right).
\]

**Proof.** Let \(r_h = e_w, v_h = e_u, \eta_h = e_\omega^t, q_h = e_\sigma, \xi_h = e_\sigma'\) in the error equation (4.1). Then we get the following equalities after some simple manipulations which includes applying
integration by parts:

\[ - (i e_w, e_w)_{\mathcal{T}_h} - (e_w, \text{curl } e_u)_{\mathcal{T}_h} + (e^t_w \times n, e^t_u)_{\partial \mathcal{T}_h} - \langle e^t_w \times n, e^t_u \rangle_{\partial \mathcal{T}_h} = 0, \quad (4.4a) \]

\[ \langle e_w, \text{curl } e_u \rangle_{\mathcal{T}_h} - \langle e^t_w \times n, e^t_u \rangle_{\partial \mathcal{T}_h} + (i \kappa^2 e_u, e_u)_{\mathcal{T}_h} - \langle e^t_u, \text{div } e_u \rangle_{\mathcal{T}_h} + \langle e^t_u, e^t_u \cdot n \rangle_{\partial \mathcal{T}_h} = 0, \quad (4.4b) \]

\[ \langle e_{e_t}, \text{div } e_u \rangle_{\mathcal{T}_h} - \langle e^t_{e_t}, e^n_u \rangle_{\partial \mathcal{T}_h} + \langle e^t_{e_t}, e^n_u \cdot n \rangle_{\partial \mathcal{T}_h} = 0, \quad (4.4c) \]

\[ \langle e_{e_t} \times n, e_{e_t} \rangle_{\partial \mathcal{T}_h} = 0, \quad (4.4d) \]

\[ \langle e_{e_t} \times n, e_{e_t} \rangle_{\partial \Omega} + \langle \kappa e_{e_t}, e_{e_t} \rangle_{\partial \Omega} = 0, \quad (4.4e) \]

\[ - \langle e_{e_t}, e^n_{e_t} \cdot n \rangle_{\partial \mathcal{T}_h} = 0. \quad (4.4f) \]

Adding the above equalities (4.4a)-(4.4f) together yields

\[ - (i e_w, e_w)_{\mathcal{T}_h} + (i \kappa^2 e_u, e_u)_{\mathcal{T}_h} + (e^t_w - e^t_{e_t}) \times n, e^t_u - e_{e_t} \rangle_{\partial \mathcal{T}_h} + \langle \kappa e_{e_t}, e_{e_t} \rangle_{\partial \Omega} - \langle e^t_{e_t}, e_{e_t} \cdot n \rangle_{\partial \mathcal{T}_h} = 0. \quad (4.5) \]

By the definition of \( \hat{\omega}_h \) in (2.4), we have

\[ (e^t_w - e^t_{e_t}) \times n = (\Pi V w - \omega_h) \times n - (P_M w - \hat{\omega}_h) \]

\[ = (\Pi V w) \times n - P_M w \times n - \tau_t(u_h - \hat{u}_h) \]

\[ = (\Pi V w) \times n - P_M w \times n - \tau_t((\Pi U u - e_u) - (P_M u - e_{e_t})) \]

\[ = (\Pi V w) \times n - P_M w \times n + \tau_t(e^t_u - e_{e_t}) - \tau_t((\Pi U u) - P_M u). \quad (4.6) \]

Moreover, by the definition of \( \bar{u}_h^n \) in (2.4), we have

\[ (e^n_u - e^n_{e_t}) \cdot n = (\Pi U u - u_h) \cdot n - (P_M u^n - \bar{u}_h^n) \cdot n \]

\[ = (\Pi U u - P_M u^n) \cdot n + \tau_n(e_h - \bar{e}_h) \]

\[ = (\Pi U u - P_M u^n) \cdot n - \tau_n(e - \bar{e}) + \tau_n(P_Q \sigma - P_M \sigma). \quad (4.7) \]

Inserting (4.6) and (4.7) into (4.5), we obtain

\[ - i ||e_w||^2_{\mathcal{T}_h} + \kappa^2 ||e_u||^2_{\mathcal{T}_h} + \kappa ||e_{e_t}\|^2_{0,0,\Delta h} + \|\tau_t (e^t_u - e^t_{e_t})\|^2_{0,\mathcal{T}_h} + \|\tau_n (e - \bar{e})\|^2_{\partial \mathcal{T}_h} \]

\[ = - \langle (\Pi V w) \times n - P_M w \times n, e^t_u - e_{e_t} \rangle_{\partial \mathcal{T}_h} + \langle \tau_t((\Pi U u) - P_M u), e^t_u - e_{e_t} \rangle_{\partial \mathcal{T}_h} \]

\[ + \langle e - \bar{e}, (\Pi U u - P_M u^n) \cdot n \rangle_{\partial \mathcal{T}_h} + \langle e - \bar{e}, \tau_n(P_Q \sigma - P_M \sigma) \rangle_{\partial \mathcal{T}_h} \]

\[ = - \langle (\Pi V w - \omega) \times n, e^t_u - e_{e_t} \rangle_{\partial \mathcal{T}_h} + \langle \tau_t(\Pi U u - u), e^t_u - e_{e_t} \rangle_{\partial \mathcal{T}_h} \]

\[ + \langle e - \bar{e}, (\Pi U u - u) \cdot n \rangle_{\partial \mathcal{T}_h} + \langle e - \bar{e}, \tau_n(P_Q \sigma - \sigma) \rangle_{\partial \mathcal{T}_h}, \quad (4.8) \]

where the second equality is derived by the properties of \( L^2 \)-projections \( P_M \) and \( P_M \). Based on (4.5), taking the real part and imaginary part of the left-hand side of (4.8) respectively, the estimates (4.2) and (4.3) can be obtained by the approximation properties of standard \( L^2 \)-projections, the Young’s inequality and the fact that \( \sigma = 0 \). This completes the proof. ☐
Now we start to use the duality argument to get an estimate for $e_u$. Given $e_u \in L^2(\Omega)$, we introduce the first-order system of the dual problem (2.5) with $J = e_u$:

\[
\begin{align*}
    i\Phi - \text{curl } \Psi &= 0 \quad \text{in } \Omega, \quad (4.9a) \\
    \text{curl } \Phi + i\kappa^2\Psi + \nabla \varphi &= e_u \quad \text{in } \Omega, \quad (4.9b) \\
    \text{div } \Psi &= 0 \quad \text{in } \Omega, \quad (4.9c) \\
    \Phi \times n - \kappa \Psi^t &= 0 \quad \text{on } \partial \Omega, \quad (4.9d) \\
    \varphi &= 0 \quad \text{on } \partial \Omega. \quad (4.9e)
\end{align*}
\]

Similar to the estimates in (3.5), (3.6) and (3.7), we have

\[
\begin{align*}
\|\varphi\|_{1, \Omega} &\leq C\|e_u\|_{\mathcal{T}_h}, \quad (4.10) \\
\|\Phi\|_{1, \Omega} + \|\Psi\|_{2, \Omega} + \kappa\|\Psi\|_{1, \Omega} + \|\text{curl } \Psi\|_{1, \Omega} + \kappa^2\|\Psi\|_{0, \Omega} &\leq C\kappa\|e_u\|_{\mathcal{T}_h}. \quad (4.11)
\end{align*}
\]

Next we first present an important equality.

**Lemma 4.3.** Let $(\Phi, \Psi, \varphi) \in H^1(\Omega) \times H^2(\Omega) \times H^1_0(\Omega)$ be the solution of the dual problem (4.9). It holds that

\[
\|e_u\|_{\mathcal{T}_h}^2 = \sum_{k=1}^5 E_k, \quad (4.12)
\]

where

\[
\begin{align*}
E_1 &= (e_u^t - e_u^w) \times n, \Phi - \Pi_V \Phi)_{\partial \mathcal{T}_h}, \\
E_2 &= (e_u^n - e_u^n) \cdot n, \varphi - \Pi_Q \varphi)_{\partial \mathcal{T}_h}, \\
E_3 &= (e_w^t - e_w^t) \times n, \Psi - \Pi_U \Psi)_{\partial \mathcal{T}_h}, \\
E_4 &= -(e_w^t - e_w^t) \times n + \kappa e_w^t, \Psi^t)_{\partial \Omega}, \\
E_5 &= (e_u, \text{div } (\Psi - \Pi_U \Psi))_{\mathcal{T}_h} - (e_u, (\Psi - \Pi_U \Psi) \cdot n)_{\partial \mathcal{T}_h}.
\end{align*}
\]

**Proof.** By the dual problem (4.9), we have

\[
\begin{align*}
\|e_u\|_{\mathcal{T}_h}^2 &= (e_u, \text{curl } \Phi + i\kappa^2\Psi + \nabla \varphi)_{\mathcal{T}_h} + (e_w, i\Phi - \text{curl } \Psi)_{\mathcal{T}_h} \\
&= (e_u, \text{curl } \Pi_V \Phi + i\kappa^2 \Pi_U \Psi + \nabla \Pi_Q \varphi)_{\mathcal{T}_h} + (e_w, i\Pi_V \Phi - \text{curl } \Pi_U \Psi)_{\mathcal{T}_h} \\
&\quad + (e_u, \text{curl } (\Phi - \Pi_V \Phi))_{\mathcal{T}_h} + (e_u, i\kappa^2(\Psi - \Pi_U \Psi))_{\mathcal{T}_h} + (e_w, \nabla (\varphi - \Pi_Q \varphi))_{\mathcal{T}_h} \\
&\quad + (e_w, i(\Phi - \Pi_V \Phi))_{\mathcal{T}_h} - (e_w, \text{curl } (\Psi - \Pi_U \Psi))_{\mathcal{T}_h}. \quad (4.13)
\end{align*}
\]

By the definitions of $\Pi_U$ and $\Pi_V$, we have $(e_u, i\kappa^2(\Psi - \Pi_U \Psi))_{\mathcal{T}_h} = 0$ and $(e_w, i(\Phi - \Pi_V \Phi))_{\mathcal{T}_h} = 0$. Similar to the derivations of (3.10)-(3.12), we have

\[
\begin{align*}
(e_u, \text{curl } (\Phi - \Pi_V \Phi))_{\mathcal{T}_h} &= (e_u^t \times n, \Phi - \Pi_V \Phi)_{\partial \mathcal{T}_h}, \quad (4.14) \\
(e_u, \nabla (\varphi - \Pi_Q \varphi))_{\mathcal{T}_h} &= (e_u^n \cdot n, \varphi - \Pi_Q \varphi)_{\partial \mathcal{T}_h}, \quad (4.15) \\
-(e_w, \text{curl } (\Psi - \Pi_U \Psi))_{\mathcal{T}_h} &= -(e_w^t \times n, \Psi - \Pi_U \Psi)_{\partial \mathcal{T}_h}. \quad (4.16)
\end{align*}
\]
Taking $r_h = \Pi_V \Phi$ in the equation (4.1a), noting that $e_{\hat{a}_i} \times n$ is continuous across each interior face and using the boundary condition (4.9d), we obtain

$$(e_u, \text{curl } \Pi_V \Phi)_{\tau_h} = (ie_w, \Pi_V \Phi)_{\tau_h} + (e_{\hat{a}_i} \times n, \Pi_V \Phi)_{\partial \tau_h}$$

$$= (ie_w, \Pi_V \Phi)_{\tau_h} + (e_{\hat{a}_i} \times n, \Pi_V \Phi - \Phi)_{\partial \tau_h} - (e_{\hat{a}_i}, \Phi \times n)_{\partial \Omega}$$

$$= (ie_w, \Pi_V \Phi)_{\tau_h} + (e_{\hat{a}_i} \times n, \Pi_V \Phi - \Phi)_{\partial \tau_h} - (e_{\hat{a}_i}, \kappa \Psi')_{\partial \Omega}. \quad (4.17)$$

Note that $e_{\hat{a}_i} \cdot n$ is continuous across each interior face and $\varphi \in H_0^1(\Omega)$. We let $q_h = \Pi_Q \varphi$ in (4.1c) to obtain

$$(e_u, \nabla \Pi_Q \varphi)_{\tau_h} = (e_{\hat{a}_i} \cdot n, \Pi_Q \varphi)_{\partial \tau_h} = (e_{\hat{a}_i} \cdot n, \Pi_Q \varphi - \varphi)_{\partial \tau_h}. \quad (4.18)$$

We further take $v_h = \Pi_U \Psi$ in (4.1b) to get

$$- (e_w, \text{curl } \Pi_U \Psi)_{\tau_h} = -(e_{\hat{a}_i} \cdot n, \Pi_U \Psi)_{\partial \tau_h} + (i\kappa^2 e_u, \Pi_U \Psi)_{\tau_h} - (e_{\sigma}, \text{div } \Pi_U \Psi)_{\tau_h} + (e_{\hat{a}_i} \cdot n, \Pi_U \Psi)_{\partial \tau_h},$$

$$= -(e_{\hat{a}_i} \cdot n, \Pi_U \Psi - \Psi)_{\partial \tau_h} - (e_{\hat{a}_i} \cdot n, \Psi')_{\partial \Omega} + (i\kappa^2 e_u, \Pi_U \Psi)_{\tau_h}$$

$$- (e_{\sigma}, \text{div } \Pi_U \Psi - \text{div } \Psi)_{\tau_h} + (e_{\hat{a}_i} \cdot n, \Pi_U \Psi - \Psi \cdot n)_{\partial \tau_h}, \quad (4.19)$$

where the above second equality holds due to the fact that $\text{div } \Psi = 0$, $e_{\hat{a}_i} \times n$ and $e_{\hat{a}_i}$ are continuous across each interior face, and $e_{\hat{a}_i} = 0$ on $e_{\hat{a}_i}^b$. Then, inserting (4.14)-(4.19) into (4.13) yields the result.

Based on the above lemma, we can obtain the estimate for $e_u$.

**Lemma 4.4.** If the regularity properties (4.10) and (4.11) hold, we have

$$\|e_u\|_{\tau_h} \leq C \left( \frac{\kappa h^{l+1}}{p^{l+1}} + \frac{\kappa h^{l+1}}{p^{l+1}} \right) \|w\|_{s,\Omega} + C \left( \frac{\kappa h^s}{p^s} + \frac{\kappa h^s}{p^s} \right) \|u\|_{s,\Omega}. \quad (4.20)$$

**Proof.** We need to derive the upper bounds for $E_1, \cdots, E_5$ in Lemma 4.3. By the Cauchy-Schwarz inequality and the approximation property of $\Pi_V$, we obtain

$$E_1 \leq C \|\tau^{\frac{1}{2}}(e_u' - e_{\hat{a}_i})\|_{\partial \tau_h} \|\tau^{\frac{1}{2}}\Phi\|_{1,\Omega} \leq C \|\tau^{\frac{1}{2}}(e_u' - e_{\hat{a}_i})\|_{\partial \tau_h} \frac{kh}{p} \|e_u\|_{\tau_h}.$$ 

By the identity (4.7) for $(e_u^n - e_{\hat{a}_i}) \cdot n$ and the fact that $\sigma = 0$, we can derive that

$$E_2 = \langle (\Pi_U u - u + u^n - P_M u^n) \cdot n - \tau_n(e_{\sigma} - e_{\hat{a}_i}) + \tau_n(\Pi_Q \sigma - P_M \sigma), \varphi - \Pi_Q \varphi \rangle_{\partial \tau_h},$$

$$\leq C \left( \frac{h^s}{p} \right)^{\frac{1}{2}} \|u\|_{s,\Omega} + \tau_n \|e_{\sigma} - e_{\hat{a}_i}\|_{\partial \tau_h} \left( \frac{h}{p} \right)^{\frac{1}{2}} \|\varphi\|_{1,\Omega}$$

$$\leq C \left( \frac{h^s}{p} \right)^{\frac{1}{2}} \|u\|_{s,\Omega} + \frac{\kappa h}{p} \|e_{\sigma} - e_{\hat{a}_i}\|_{\partial \tau_h} \|e_u\|_{\tau_h}.$$ 

Moreover, by the identity (4.6) for $(e_u' - e_{\hat{a}_i}) \times n$ and the triangular inequality, we get

$$E_3 = -\langle \tau_t(e_u' - e_{\hat{a}_i}), \Psi - \Pi_U \Psi \rangle_{\partial \tau_h} + \langle P_M w^t \times n - (\Pi_V w)^t \times n, \Psi - \Pi_U \Psi \rangle_{\partial \tau_h}$$

$$+ \tau_t(\langle \Pi_U u \rangle^t - P_M u^t, \Psi - \Pi_U \Psi \rangle_{\partial \tau_h}$$

$$\leq C \left( \frac{kh}{p} \right)^{\frac{1}{2}} \|e_u' - e_{\hat{a}_i}\|_{\sigma \tau_h} + \frac{\kappa h^{l+1}}{p^{l+1}} \|w\|_{t,\Omega} + \frac{\kappa h^s}{p^s} \|u\|_{s,\Omega} \|e_u\|_{\tau_h}.$$
By the boundary condition (4.1e), we have $E_4 = -\langle e_{\Omega} \times n + \kappa e_{\Omega}, P_M \Psi \rangle_{\partial \Omega} = 0$. Applying integration by parts, we obtain the estimate for $E_5$ as follows:

$$E_5 = (\nabla e_\sigma, \Psi - \Pi_U \Psi) + \langle e_\sigma - e_{\hat{\sigma}}, (\Psi - \Pi_U \Psi) \cdot n \rangle_{\partial \Omega}$$

$$= \langle e_\sigma - e_{\hat{\sigma}}, (\Psi - \Pi_U \Psi) \cdot n \rangle_{\partial \Omega}$$

$$\leq C h^{\frac{3}{2}} \|e_\sigma\|_{\partial \Omega} \|e_u\|_{\partial \Omega}.$$

Finally, combining the above estimates for $E_1, \cdots, E_5$ and the estimate (4.2), we can conclude the result.

We can now give the proof of Theorem 2.2.

**Proof.** (Proof of Theorem 2.2) By the triangular inequality, we have

$$\|u - u_h\|_{\mathcal{T}_h} \leq \|u - \Pi_U u\|_{\mathcal{T}_h} + \|e_u\|_{\partial \Omega}.$$

The error estimate (2.16) can be obtained by the approximation property of $\Pi_U$ and the estimate (2.18) for $\|e_u\|_{\partial \Omega}$. Similarly, (2.17) can be obtained by the triangular inequality, the approximation property of $\Pi_V$ (4.13) and (4.20). Similar to the techniques used in the $L^2$-norm estimates for $\text{curl} u_h$ and $\text{div} u_h$. We take $r_h = \text{curl} e_u$ in (4.1a) to obtain

$$\|\text{curl} e_u\|_{\mathcal{T}_h} \leq \|e_u\|_{\mathcal{T}_h} + C h^{t} \|\tau^{t} \text{div} e_u\|_{\mathcal{T}_h}$$

and take $q_h = \text{div} e_u$ in (4.1b) to get

$$\|\text{div} e_u\|_{\mathcal{T}_h} \leq C h^{s} \|\text{curl} e_u\|_{\mathcal{T}_h}.$$

Moreover, we have the following approximation properties of the $L^2$-projections:

$$|w - \Pi_U w|_{1,T} \leq Ch^{t-1/p} \|w\|_{s,T} \quad \forall T \in \mathcal{T}_h, \ 0 \leq t \leq p + 1,$$

$$|u - \Pi_U u|_{1,T} \leq Ch^{s-1/p} \|u\|_{s,T} \quad \forall T \in \mathcal{T}_h, \ 0 \leq s \leq p + 1,$$

which can be deduced by the technique used in Lemma 6.4 of [8]. Then the error estimate (2.18) can be derived by the triangular inequality, (2.2), (4.13), (4.20) and the approximation property of $L^2$-projection, and (2.19) can also be deduced by the triangular inequality, the identity (4.17), the estimate (2.12) and the approximation property of $L^2$-projection.

**Remark 4.1.** Besides we get the error estimate for $\|\tau^{t}(e_\sigma - e_{\hat{\sigma}})\|_{\partial \Omega}$ in (4.2), we can also obtain the error estimate for $\|\nabla e_\sigma\|_{\mathcal{T}_h}$. Actually, this can be similarly derived as the stability estimate for $\|\nabla \sigma_h\|_{\mathcal{T}_h}$ (cf. Remark 3.1) by taking $v_h = \nabla e_\sigma$ in the error equation (4.1b). Then the error estimate for $\|\nabla (\sigma - \sigma_h)\|_{\mathcal{T}_h}$ can be further deduced by the triangular inequality.

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