Efficient Bitruss Decomposition for Large-scale Bipartite Graphs

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Abstract—Cohesive subgraph mining in bipartite graphs becomes a popular research topic recently. An important structure k-bitruss is the maximal cohesive subgraph where each edge is contained in at least k butterflies (i.e., (2, k)-bicliques). In this paper, we study the bitruss decomposition problem which aims to find all the k-bitrusses for k ≥ 0. The existing bottom-up techniques need to iteratively peel the edges with the lowest butterfly support. In this peeling process, these techniques are time-consuming to enumerate all the supporting butterflies for each edge. To relax this issue, we first propose a novel online index — the BE-Index which compresses butterflies into k-blooms (i.e., (2, k)-bicliques). Based on the BE-Index, the new bitruss decomposition algorithm BiT-BU is proposed, along with two batch-based optimizations, to accomplish the butterfly enumeration of the peeling process in an efficient way. Furthermore, the BiT-PC algorithm is devised which is more efficient against handling the edges with high butterfly supports. We theoretically show that our new algorithms significantly reduce the time complexities of the existing algorithms. Also, we conduct extensive experiments on real datasets and the results demonstrate that our new techniques can speed up the state-of-the-art techniques by up to two orders of magnitude.

I. INTRODUCTION

Bipartite networks are widely used in many real-world applications where we need to model relationships between two different types of entities. For example, author-paper relationships (e.g., authors form the upper layer and papers form the lower layer in the network in Figure 1), user-product relationships, etc. Consequently, cohesive subgraph mining in bipartite networks (graphs) becomes a popular research topic recently. In unipartite graphs, there are extensive studies on k-truss decomposition [1]–[4] which constructs the hierarchy of k-trusses (each edge in k-truss is contained in at least k triangles). However, k-truss decomposition cannot be used in bipartite graphs since there is no triangle structure existing in bipartite graphs. Also, since the degree distributions of most real-world bipartite graphs are skewed, it will cause the explosion in the number of edges/triangles if we project bipartite graphs to unipartite graphs [5].

In bipartite graphs, butterfly (i.e., a complete 2 × 2 biclique) [6]–[8] is the smallest non-trivial cohesive structure and is recognised as an analogue of triangle in unipartite graphs. Based on butterfly, k-bitruss is defined as the cohesive subgraph where each edge is contained in at least k butterflies [5], [9]. Consequently, the bitruss number of an edge e, denoted by φe, is defined as the largest k such that a k-bitruss contains e. In this paper, we study the bitruss decomposition problem, which computes the bitruss number for each edge in a bipartite graph. For instance, in Figure 1 the bitruss numbers of the edges in blue color (i.e., (u0, v0), (u0, v1), (u1, v0), (u1, v1), (u2, v0), (u2, v1), (u3, v0), (u3, v1), (u3, v2)) and gray color (i.e., (u2, v3), (u3, v3)) are 2, 1 and 0, respectively. In the literature, the study of bitruss decomposition can be easily adopted in many applications. We list some examples below.

• Fraud detection. In social media such as Facebook, there exist fraudulent users who give fake “like”s. Also, with the improvement of the fraud detection techniques, the cost of opening fake accounts is increased, thus frauds cannot rely on too many fake accounts [10]. Therefore, these malicious users tend to form a closely connected group. Although the size of the cluster of frauds is unknown, the output of bitruss decomposition applied on the bipartite network (e.g., user-page network) can reveal the close communities at different level of granularity for further investigation.

• Identifying nested research groups. Bipartite graphs are natural fits for modelling the relationship between authors and publications. The bitruss decomposition algorithm can reveal the hierarchical relations of researchers by finding a loose connected research group first and further decomposing it into smaller, more cohesive groups [5]. For instance, in Figure 1 all the researchers belong to a loosely connected research group, while {v0, v1, v2} constructs a more cohesive one, and {v0, v1} constructs the most cohesive research group.

• Recommendation system. When applied to bipartite graphs with user-item structure, bitruss decomposition algorithm can effectively identify dense subgraphs in hierarchical manner. The denser the subgraph is, the more similar the users/items are in this subgraph. Finding users/items at different similarity levels is especially helpful to support the construction of recommendation systems [11].

In real-world applications, the graphs can be very large and the state-of-the-art algorithms cannot handle large-scale bipar-
tite graphs efficiently. For example, on the graph Wiki-it with $10^7$ edges, the decomposition algorithm in [5] needs more than 30 hours to solve the bitruss decomposition problem as evaluated. Therefore, the study of more efficient bitruss decomposition algorithms is essential to support large-scale graph analysis.

**Existing techniques.** [5], [9] both propose a bottom-up approach by iteratively peeling the edges with the lowest butterfly support. It has two key steps: (1) in the counting process, for each edge $e$, it counts the number of butterflies containing $e$ (i.e., the butterfly support of $e$ — $X_e$); (2) in the peeling process, it iteratively removes the edge $e$ with minimum $X_e$ and assigns the bitruss number to $e$ as $X_e$. To complete the counting process, a novel algorithm recently proposed in [8] takes $O(\sum_{(u,v)\in E(G)}\min\{d(u),d(v)\})$ time; on the other hand, the peeling process still requires $O(|E(G)|^2)$ time in [9] or $O(\sum_{(u,v)\in E(G)}\max\{d(u),d(w)\})$ time in [5] and, consequently, becomes the performance bottleneck of bitruss decomposition. Here, $E(G)$ denotes the edge set of a graph $G$, $d(v)$ and $N_G(v)$ denote the degree and the neighbor set of a vertex $v$, respectively.

**Motivation and challenges.** In the peeling process, when an edge $e$ is removed, the butterfly supports of the edges which share butterflies with $e$ need to be updated correspondingly. In [5], [9], this edge removal operation needs to enumerate all the butterflies containing $e$. The butterfly enumeration methods used by [5], [9] are inherently the same — enumerate the combinations of four vertices with three edges first, then check whether there exists the forth edge to form a butterfly. The main drawback of the existing combination-based methods is that if the forth edge does not exist (e.g., the butterfly $\{u_1,v_1,u_2,v_2\}$ does not exist in Figure 2(a)), the time of combining and checking is wasted. For instance, considering the graph in Figure 2(a) with 4002 vertices, $u_0$ is connected with $v_0$, $v_1$, and $u_1$ ($v_1$) is connected with $v_0$ to $v_{1000}$ ($u_0$ to $u_{1000}$), and $u_2$ ($v_2$) is connected with $v_{1000}$ to $v_{2000}$ ($u_{1001}$ to $u_{2000}$), respectively. When edge $(u_1,v_1)$ is removed, the existing algorithms enumerate butterflies containing $(u_1,v_1)$ by (1) checking whether there is an edge between $u_1$’s neighbors and $v_1$’s neighbors which needs $d(u_1) \times d(v_1) = 1001 \times 1001$ checks [9]; or (2) checking whether there is an edge between $v_1$’s two-hop neighbors (e.g., $v_{2000}$) and $u_1$ which needs $\sum_{w\in N_G(v_1)}\max\{d(u_1),d(w)\} = 1001 \times 1001$ checks [5]. However, there only exists one butterfly containing $(u_1,v_1)$: $[u_0,v_0,u_1,v_1]$. 

In addition, we observe that the degree distributions of most real-world graphs are skewed (e.g., Wiki-it and Delicious). In these graphs, some edges can have very high butterfly supports (i.e., hub edges), though their bitruss numbers are comparatively much smaller. For example, the maximum bitruss number for an edge is only 6,638 on the Delicious dataset, while its butterfly support reaches 1,219,319. For those hub edges, it requires a large number of butterfly support updates to obtain their bitruss numbers in the peeling process.

Motivated by the above observations, in this paper, we aim to significantly improve the efficiency of bitruss decomposition by addressing the following two major challenges:

**Our approaches.** To address Challenge 1, we observe that the bloom structure (i.e., a biclique with exactly 2 vertices in one layer) is the combination of butterflies which may have the ability to be used in compacting the butterflies. For example, in Figure 3(a), the graph is a 1001-bloom (also a $(2,1001)$-biclique) which contains $2^{1001-1}$ butterflies. Thus, given a bipartite graph $G$, we can compact all the butterflies in $G$ into blooms. Besides, to guarantee that each butterfly is contained in exactly one bloom, we only identify the maximal priority-obeyed blooms — the maximal bloom where the vertex with the largest priority belongs to the layer with only two vertices. Here, the higher the degree, the higher the priority; and the ties are broken by vertex ID. Then, the index is constructed by linking the maximal priority-obeyed blooms with the edges they contain; that is the Bloom-Edge-Index (BE-Index). For example, for the graph in Figure 3(a), we can construct the corresponding BE-Index as shown in Figure 3(b). The BE-Index can be efficiently constructed after the counting process which needs only $O(\sum_{(u,v)\in E(G)}\min\{d(u),d(v)\})$ time. Then, when we perform an edge removal operation for $e$, we can directly find all the affected edges through the blooms in BE-Index rather than enumerating the butterflies containing $e$ using combination-based methods as what existing techniques do. For example, to remove $(u_1,v_1)$ in Figure 3(a), we can directly find the 4 edges to be updated in BE-Index as shown in Figure 3(b) instead of using $1001 \times 1001$ butterfly checks in existing solutions. Also as shown in Figure 3 we can also directly find all the affected edges if one of those edges is removed. Based on BE-Index, the total peeling process needs only $O(|X_G|)$ time where $X_G$ is the number of butterflies in the graph $G$. 

![Figure 2. Observations](image)

![Figure 3. (a) a bipartite graph (also a 1001-bloom), (b) the corresponding BE-Index.](image)
To address Challenge 2, we propose the progressive compression approach BiT-PC based on the observation that $X_e$ is a lower bound of $\phi_e$ for an edge $e$. Unlike the bottom-up algorithms which process the edges with minimum butterfly supports first, BiT-PC handles a bunch of edges with high butterfly supports (i.e., hub edges) first within cohesive subgraphs and compresses those edges after assigning bitruss numbers to them. In this manner, BiT-PC can significantly reduce the number of butterfly support updates, especially for those hub edges. This is because after assigning the bitruss number for a hub edge, we only need to preserve its support in the BE-Index and do not need to update its butterfly supports when edges with lower bitruss numbers are removed.

**Contribution.** Our principal contributions are summarized as follows.

- We propose a novel online index — the BE-Index. Based on the BE-Index, our new bitruss decomposition algorithm BiT-BU significantly reduces the time complexities of the existing algorithms as shown in Section V-A. We also propose two batch-based optimizations to further enhance the performance of BiT-BU.
- To deal with the hub edge issue, we propose the BiT-PC algorithm which processes the hub edges within cohesive subgraphs and compresses the processed edges progressively. In this manner, BiT-PC greatly reduces the number of butterfly support updates for those hub edges.
- We conduct extensive experiments on real bipartite graphs. The result shows that the proposed algorithm BiT-PC outperforms the state-of-the-art algorithm [5] by up to two orders of magnitude. For instance, the BE-Index algorithm can solve the bitruss decomposition problem within 20 minutes on Wiki-it dataset with 10^7 edges, while the state-of-the-art algorithm [5] runs more than 30 hours.

**Organization.** The rest of the paper is organized as follows. The related work directly follows. Section II presents the problem definition. Section III introduces the existing algorithms BiT-BS. The BE-Index is presented in Section IV. Section V introduces the BE-Index-based algorithms including BiT-BU, BiT-BU++, and BiT-PC. Section VI reports the experimental results. Section VII concludes the paper.

**Related Work.** In the literature, there are many cohesive subgraph models and recent works on graph decomposition are based on these models [12].

**Unipartite graphs.** In unipartite networks, many models are defined to capture the cohesiveness of subgraphs such as $k$-core [13], $k$-truss [1] and clique [16]. Furthermore, researchers also study the core decomposition [17]–[19] and truss decomposition [1]–[4] algorithms. Among those works, truss decomposition is the most similar topic. The reason is that the cohesive structure used in the truss decomposition (i.e., triangle) is the smallest non-trivial clique in unipartite networks, while the cohesive structure used in the bitruss decomposition (i.e., butterfly) is the smallest non-trivial biclique in bipartite networks. However, the structures are different (4-hops’ circle vs 3-hops’ circle) and the applied networks are different (bipartite network vs unipartite network). Thus, the truss decomposition techniques are not applicable.

**Bipartite graphs.** In bipartite networks, some studies are conducted towards core-like (e.g., ($a$, $b$)-core [20], ($p$, $q$)-core [21], fractional $k$-core [22], truss-like (e.g., bitruss [5], [9]), and clique-like (e.g., ($p$, $q$)-biclique [23], quasi-biclique [24]) cohesive structures. Among those works, the core-like and clique-like structures are inherently different from bitruss. For instance, ($a$, $b$)-core is the maximal subgraph where the degree of each vertex in the upper/lower layer is at least $a/b$; biclique [24] is the maximal complete subgraph. Thus, the techniques in these works cannot be used to solve our problem. In [25], the authors project the bipartite graph into a unipartite graph and apply the $k$-truss decomposition algorithm. As we mentioned before, this will cause the explosion of edges/triangles. Thus, the study in this paper aims to improve the recent works in [5], [9] which directly solve the bitruss decomposition problem.

II. PROBLEM DEFINITION

In this section, we formally introduce the notations and definitions. Mathematical notations used throughout this paper are summarized in Table I.

| Notation | Definition |
|----------|------------|
| $G$      | a bipartite graph |
| $V(G)/E(G)$ | the vertex/edge set of $G$ |
| $U(G),L(G)$ | a vertex layer of $G$ |
| $u,v,w,x$ | a vertex in a bipartite graph |
| $(u,v,e)$ | an edge in a bipartite graph |
| $B/k-B$ | a bloom/k-bloom in a bipartite graph |
| $B^*$ | a maximal priority-obeying bloom |
| $(u,v,w)$ | a wedge formed by $u$, $v$, $w$ |
| $[u,v,w]$ | a butterfly formed by $u$, $v$, $w$ |
| $d(u)/p(u)$ | the degree/priority of $u$ |
| $N_G(u)$ | the set of neighbors of $u$ |
| $X_e$ | the number of butterflies containing $e$ |
| $X_B/\overline{X}_G$ | the number of butterflies in $B/\overline{G}$ |
| $G_{\ge k}$ | $G_{\ge k} \subseteq G$ where $X_e \ge k$ for each $e \in G_{\ge k}$ |
| $n,m$ | the number of vertices and edges in $G$ ($m > n$) |

Our problem is defined over an undirected bipartite graph $G(V = (U,L), E)$, where $U(G)$ denotes the set of vertices in the upper layer, $L(G)$ denotes the set of vertices in the lower layer, $U(G) \cap L(G) = \emptyset$, $V(G) = U(G) \cup L(G)$ denotes the vertex set, and $E(G) \subseteq U(G) \times L(G)$ denotes the edge set. An edge between two vertices $u$ and $v$ in $G$ is denoted as $(u,v)$ or $(v,u)$. The set of neighbors of a vertex $u$ in $G$ is denoted as $N_G(u) = \{v \in V(G) | (u,v) \in E(G)\}$, and the degree of $u$ is denoted as $d(u) = |N_G(u)|$. Each vertex $u$ has a unique id and we assume for every pair of vertices $u \in U(G)$ and $v \in L(G)$, $u.\text{id} > v.\text{id}$.

**Definition 1 (Wedge).** Given a bipartite graph $G(V,E)$ and vertices $u,v,w \in V(G)$, a path starting from $u$, going through $v$ and ending at $w$ is called a wedge which is denoted as $(u,v,w)$. For a wedge $(u,v,w)$, we call $u$ the start-vertex, $v$ the middle-vertex and $w$ the end-vertex.
Definition 2 (Butterfly). Given a bipartite graph $G$ and four vertices $u, v, w, x \in V(G)$ where $u, w \in U(G)$ and $v, x \in L(G)$, a butterfly induced by the vertices $u, v, w, x$ is a $(2, 2)$-biclique of $G$; that is, $u$ and $w$ are both connected to $v$ and $x$, respectively, by edges $(u, v), (u, x), (w, v), (w, x) \in E(G)$.

Definition 3 (Bloom/-k-Bloom). Given a bipartite graph $G(V, E)$, a bloom denoted as $B$ is a biclique in $G$ where there are exactly two vertices in $U(B)$ (or $L(B)$). Given a positive integer $k$, a $k$-bloom denoted as $B_k$ is a $(2, k)$-biclique in $G$; that is, there are two vertices in $U(k-B)$ (or $L(k-B)$) connected with $k$ vertices in $L(k-B)$ (or $U(k-B)$). For a $k$-bloom, we call $k$ the bloom number. Given a set of vertices $S \subseteq V(G)$ such that the induced subgraph of $S$ is a bloom, we denote this bloom as $B(S)$.

A butterfly induced by vertices $u, v, w, x$ is denoted as $[u, v, w, x]$. We denote the number of butterflies containing an edge $e$ as $\mathbb{X}_e$, the number of butterflies in a bloom $B$ as $\mathbb{X}_B$, and the number of butterflies in $G$ as $\mathbb{X}_G$. Also $\mathbb{X}_e$ is called the butterfly support of $e$.

Definition 4 (k-bitruss). Given a bipartite graph $G$ and a positive integer $k$, a $k$-bitruss denoted as $H_k$ is a maximal subgraph of $G$ where $\mathbb{X}_e \geq k$ for each edge $e \in H_k$.

Definition 5 (Bitruss number). Given a bipartite graph $G$, the bitruss number of an edge $e$ denoted as $\phi(e)$ is the largest $k$ such that a $k$-bitruss in $G$ contains $e$.

Problem Statement. Given a bipartite graph $G(V, E)$, our bitruss decomposition problem is to compute $\phi(e)$ for each edge $e \in E(G)$.

![Figure 4](image)

(a) the bipartite graph $G$, (b) the 1-bitruss of $G$, $H_1$; (c) the 2-bitruss of $G$, $H_2$

Example 1. Considering the bipartite graph $G$ in Figure 4, the bitruss numbers of the edges in $H_2$ are 2, the bitruss numbers of the edges in $E(H_1) \setminus E(H_2)$ are 1 and the bitruss numbers of the other edges are 0.

III. EXISTING SOLUTIONS

In this section, we briefly discuss the existing algorithms to solve the bitruss decomposition problem. [3], [5] both propose bitruss decomposition algorithms. Since these two algorithms follow the same paradigm with inherently the same peeling idea (as illustrated in the introduction), here we only outline the state-of-the-art algorithm Bit-T-BS of [5] in Algorithm 1.

As shown in [5], the time complexity of Bit-T-BS is $O(\sum_{u \in L(G)} \sum_{v_1, v_2 \in N_G(u)} \max\{d(v_1), d(v_2)\} + \sum_{(u, v) \in E(G)} \sum_{w \in N_G(v)} \max\{d(u), d(w)\})$ where the first term is for the counting process and the second term is for the peeling process. The time complexity of the counting process can be reduced to $O(\sum_{(u, v) \in E(G)} \min\{d(u), d(v)\})$ using the algorithm in [3].

The performance bottleneck of Bit-T-BS. Here we analyse the dominant cost of Bit-T-BS. We first define the edge removal operation as follows.

Definition 6 (Edge removal operation). Given a bipartite graph $G(V, E)$ and an edge $e \in E(G)$, an edge removal operation for $e$ denoted as $r(e)$ has two steps. Firstly, find all the edges which share at least one butterfly with $e$ in $G$ and compute their butterfly supports in $G \setminus e$. Secondly, remove $e$ from $G$.

![Figure 5](image)

Figure 5. Time cost of Bit-T-BS on different datasets

As shown in Figure 5, the dominant cost of Bit-T-BS is to accomplish the peeling phase on the testing datasets. Moreover, the dominant cost in the peeling phase is incurred when performing the edge removal operations as shown in Algorithm 1.

IV. A NOVEL BE-INDEX

In this section, we try to explore a compact online index to speed up the edge removal operation.

A. Index Overview

Since a butterfly is a $(2, 2)$-biclique and a $k$-bloom is a $(2, k)$-biclique, we have the following lemma.

Lemma 1. A $k$-bloom contains exactly $\frac{k(k-1)}{2}$ butterflies.

Proof. According to Definition 2 and 3, this lemma holds. □

For example, as shown in Figure 4(c), the 3-bloom $H_2$ contains 3 butterflies $[u_0, v_0, u_1, v_1]$, $[u_0, v_0, u_2, v_1]$, and...
In addition, from the above lemma, we can immediately get the following lemma:

**Lemma 2.** For each edge $e$ contained in a $k$-bloom, there exist $k - 1$ butterflies containing $e$.

For example, as shown in Figure 4(b), the edge $(u_2, v_1)$ is contained in a 3-bloom $(H_2)$ and a 1-bloom $([v_1, u_2, v_2, u_3])$, thus there are 2+1 butterflies containing it. Consequently, using blooms instead of butterflies to construct an index should be an effective way to speed up the edge removal operations.

**The structure of BE-Index.** Before introducing a more compact index, we first give the following definitions.

**Definition 7 (Priority).** Given a bipartite graph $G(V, E)$, for a vertex $u \in V(G)$, the priority $p(u)$ is an integer where $p(u) \in [1, |V(G)|]$. For two vertices $u, v \in V(G)$, $p(u) > p(v)$ if

- $d(u) > d(v)$, or
- $d(u) = d(v)$, $u.id > v.id$.

**Definition 8 (Maximal priority-obeyed bloom).** Given a bipartite graph $G$, a bloom is a maximal priority-obeyed bloom $B^*$ $(V(U, L), E)$ if it satisfies the following constrains:

1. if $v$ has the largest priority in $V(B^*)$, $v \in U(B^*)$ where $|U(B^*)|$ or $|L(B^*)| = 2$; the layer containing $v$ is called the dominant layer of $B^*$.
2. there exists no another bloom $B'$ covering $B^*$ satisfying 1.

**Lemma 3.** A butterfly must be contained in one and exactly one maximal priority-obeyed bloom.

**Proof.** According to Definition 3, since a butterfly itself is also a bloom, we only need to prove that a butterfly cannot be contained in more than one maximal priority-obeyed bloom. We prove it by contradiction. Suppose we have a butterfly $[u, v, w]$ where $u$ has the largest priority in it, $w$ is in the same layer with $u$, and there are two different maximal priority-obeyed blooms $B^1_1$ and $B^2_1$ both containing $[u, v, w]$. By Definition 8, $u$ and $w$ must belong to the dominant layers of $B^1$ and $B^2$ as $u$ has the largest priority. Since $\{u, w\} \times (V(B^1_1) \setminus \{u, w\}) \in E(B^1_1)$ and $\{u, w\} \times (V(B^2_1) \setminus \{u, w\}) \in E(B^2_1)$, we have $\{u, w\} \times (V(B^1_1) \cup V(B^2_1) \setminus \{u, w\}) \in E(B^1_1) \cup E(B^2_1)$, i.e., $B' = B^1_1 \cup B^2_1$ must be a bloom which also satisfies constraint 1 of Definition 8. $B' \geq B^1_1$ and $B' \geq B^2_2$; a contradiction to the constraint 2 of Definition 8. Thus, this lemma holds.

Now, we propose the BE-Index (Bloom-Edge-Index) to speed up the edge removal operation. Given a bipartite graph $G$, a BE-Index denoted as $I(V(U, L), E)$ links all the maximal priority-obeyed blooms with all the edges in $G$. The structure of the BE-Index $I$ is summarized as follows:

Each vertex in $U(I)$ corresponds to a maximal priority-obeyed bloom $B^*$ in $G$ and contains the following information:

- the id of $B^*$;
- $\mathcal{X}_{B^*}$

Each vertex in $L(I)$ corresponds to an edge $e$ in $G$ and contains the following information:

- $\mathcal{X}_e$.
- the id of $e$;
- $\mathcal{X}_{e}$.

Two vertices in $V(I)$ are linked together if a maximal priority-obeyed bloom $B^*$ contains an edge $e$ in $G$. We use $N_I(e)$ to denote the set of maximal priority-obeyed blooms linked to $e$ in $I$, and we use $N_I(B^*)$ to denote the set of edges linked to $B^*$ in $I$. For each $(B^*, e)$ pair in $E(I)$, we also record the twin edge of $e$ in $B^*$ which is defined as follows.

**Definition 9 (Twin edge).** Given a maximal priority-obeyed bloom $B^*$ and an edge $e \in B^*$, the twin edge of $e$ in $B^*$ denoted as $\text{twin}(B^*, e)$ is the edge sharing a vertex $v$ with $e$, where $v$ is in the non-dominant layer of $B^*$.

**Lemma 4.** For each edge $e$ in a maximal priority-obeyed bloom $B^*$, it has exactly one twin edge in $B^*$.

**Proof.** This lemma immediately follows from Definition 3 and Definition 9.

Now, we give an example of the BE-Index. From the graph $G$ in Figure 4(a), we can construct the BE-Index $I$ as shown in Figure 5. We denote $(u_0, v_0), (u_0, v_1), (u_1, v_1), (u_2, v_0), (u_2, v_1), (u_2, v_2), (u_3, v_1), (u_3, v_2)$ as $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ and $e_8$, respectively. $B^1_0$ is equal to $H_2$ and $B^1_1$ is equal to $[u_2, v_1, u_3, v_2]$ in Figure 4. In $I$, $\mathcal{X}_{B^1_0}$ is recorded and in $L(I)$, $\mathcal{X}_e$ is recorded (e.g., $\mathcal{X}_{B^1_0} = 3$ and $\mathcal{X}_{e_0} = 2$). In $E(I)$, the twin edges are recorded (e.g., the twin edge of $e_0$ in $B^1_0$ is $e_1$).

**Perform edge removal operations using BE-Index.** The key advantage of BE-Index is that it compresses butterflies into maximal priority-obeyed blooms without losing any butterfly support information. Thus, we can efficiently perform an edge removal operation using BE-Index as shown in Algorithm 2.

**Algorithm 2: RemoveEdge(e)**

1. foreach $B^* \in N_I(e)$ do
2. compute $k$ from $(k) = \mathcal{X}_{B^*}$
3. foreach $e' \in N_I(B^*)$ do
4. if $\mathcal{X}_{e'} > \mathcal{X}_e$ then
5. if $e' = \text{twin}(B^*, e)$ then
6. $\mathcal{X}_{e'} \leftarrow \mathcal{X}_{e'} - (k - 1)$
7. $E(I) \leftarrow E(I) \setminus (B', e')$
8. else
9. $\mathcal{X}_{e'} \leftarrow \mathcal{X}_{e'} - (k - 1)$
10. $E(G) \leftarrow E(G) \setminus (k - 1)$
11. $E(I) \leftarrow E(I) \setminus e$
12. $L(I) \leftarrow L(I) \setminus e$
Given a bipartite graph $G$, the corresponding BE-Index $I$ for $G$, and an edge $e \in G$, we first find all the maximal priority-obeyed blooms linked to $e$ in $I$ (i.e., $N_I(e)$). For each $B^* \in N_I(e)$, since it contains $X_{B^*} = \frac{k}{k(k-1)} e^*$ butterflies if it is a $k$-bloom according to Lemma 1, we can compute the bloom number $k$ using the equation in line 2. Then, we find the set of edges $N_I(B^*)$ for each $B^* \in N_I(e)$ (line 3). For each edge $e' \in N_I(B^*) \setminus e$, we update the butterfly support $X_{e'}$ if $X_e' > X_e$ (line 4). If $e' = \text{twin}(B^*, e)$, $e'$ will be contained in no butterfly in $B^*$ after removing $e$, we decrease $X_{e'}$ by $(k-1)$ according to Lemma 2. If $e'$ is removed from $B^*$ and $B_e^*$ becomes a $(k-1)$-bloom, we decrease $X_{B_e^*}$ by $(k-1)$.

Here is an example of removing an edge with the BE-Index.

**Example 2.** Consider the bipartite graph $G$ in Figure 2(a) and the BE-Index of $G$ in Figure 6. Suppose we remove the edge $e_5$ as shown in Figure 6. There are 3 affected edges (i.e., $e_5, e_7$, and $e_8$) that can be found through $B_1^*$ in $I$. Since $e_5$ is the twin edge of $e_6$ in $B_1^*$ and $B_6^*$ is a 2-bloom, we need to update $X_{e_5}$ to $3-(2-1) = 2$. Then, because the butterfly supports of the edges $e_7$ and $e_8$ are equal to $X_{e_6} = 1$, we do not need to update their butterfly supports.

**Analysis of the BE-Index.** Below, we give some theoretical analysis of the BE-Index.

**Theorem 1.** Given a bipartite graph $G$, the corresponding BE-Index $I$ for $G$, and an edge $e \in G$, Algorithm 2 correctly performs an edge removal operation for $e$ using $I$.

**Proof.** According to Definition 6, here we only need to prove that the butterfly supports of all the affected edges (i.e., the edges which share butterflies with $e$) are correctly updated. Firstly, we retrieve a set of maximal priority-obeyed blooms containing $e$ from the BE-Index; it is obvious from Lemma 3 that all the affected edges are contained by these blooms. Secondly, according to Lemma 1 and Lemma 4, the butterfly supports of the affected edges are correctly updated as in Algorithm 2 lines 5 - 9. Thus, this theorem holds.

**Lemma 5.** Given a bipartite graph $G$ and $e \in G$, it needs $O(X_e)$ time to perform Algorithm 2 for $e$.

**Proof.** Since there are $O(X_e)$ butterflies associated with $e$, the number of affected edges is $O(X_e)$. By using index $I$, it takes constant time to assess and update an affected edge. Consequently, the overall time for Algorithm 2 is $O(X_e)$.

Before introducing Lemma 5, we give the below definition.

**Definition 10** (Priority-obeyed wedge). A priority-obeyed wedge is a wedge where the priority of start-vertex is larger than the priorities of middle-vertex and end-vertex.

**Lemma 6.** For a bipartite graph $G$, storing the BE-Index $I$ of $G$ needs $O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\})$ space.

**Proof.** The BE-Index compresses butterfly into maximal priority-obeyed blooms; the space usage is dominated by the summed number of edges in these maximal priority-obeyed blooms. Within one maximal priority-obeyed bloom, each edge is contained by exactly one priority-obeyed wedge according to Definition 8 and Definition 10. Also, it can be proved in a similar way as Lemma 3, one priority-obeyed wedge exists in at most one maximal priority-obeyed blooms; we equivalently prove that the total number of priority-obeyed wedges is bounded by $O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\})$.

Considering an edge $(u, v) \in E(G)$ with $p(u) > p(v)$ (i.e., $d(u) \geq d(v)$), $u$ should be the start-vertex for all priority-obeyed wedges containing $(u, v)$, and the number of such wedges is $O(d(v))$ since there are at most $d(v)$ end-vertices linking with the middle-vertex $v$. Consequently, the total number of priority-obeyed wedges in $G$ is $O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\}) = O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\})$, this theorem holds.

**B. Index Construction**

**Algorithm 3:** IndexConstruction

Input: $G(V = (U, L), E)$: the input bipartite graph

Output: $I$: the BE-Index

1 // $X_e$ for each $e \in E(G)$ is pre-computed
2 Compute $p(u)$ for each $u \in V(G)$ // Definition 7
3 foreach $u \in V(G)$ do
4      initialize hashmap count_wedge with zero
5      foreach $v \in N_C(u)$ : $p(v) < p(u)$ do
6          count_wedge($w$) = count_wedge($w$) + 1
7      foreach $v \in N_C(u)$ : $p(v) < p(u)$ do
8          count_wedge($w$) = count_wedge($w$) + 1
9      if count_wedge($w$) > 1 then
10          $B^*$ = the bloom anchored by $u$ and $w$
11          if $B^* \notin U(I)$ then
12              $X_{B^*} \leftarrow \langle \text{count_wedge}(w) \rangle$
13              add $B^*$ and $X_{B^*}$ into $U(I)$
14          if $e = (u,v) \notin L(I)$ then
15              add $e$ and $X_e$ into $L(I)$
16          if $e = (v,u) \notin L(I)$ then
17              add $e$ and $X_e$ into $L(I)$
18          link $B^*$ with $(u,v)$ in $E(I)$
19          link $B^*$ with $(v,u)$ in $E(I)$
20          twin($B^*$, $(u,v)$) $\leftarrow (v, w)$
21          twin($B^*$, $(v,u)$) $\leftarrow (w, v)$
22 return $I$

To build the BE-Index, the key step is to get all the maximal priority-obeyed blooms. We have the following lemma.

**Lemma 7.** A maximal priority-obeyed bloom with the bloom number equal to $k$ must be the combination of $k$ priority-obeyed wedges.

**Proof.** This lemma immediately follows from Definition 1, Definition 3 and Definition 8.

For example, the bloom in Figure 4(c) is combined by the priority-obeyed wedges $(v_1, u_0, r_b), (v_1, u_1, b_1)$ and $(v_1, u_2, b_0)$. Based on the above observations, we propose the Index Construction algorithm as shown in Algorithm 3. Given a bipartite graph $G$, we first assign a priority to each vertex $u \in V(G)$. After that, we process the wedges from each vertex $u \in V(G)$ and initialize the hashmap count_wedge with zero.
For each \( v \in N_G(u) \), we process \( v \) if \( p(v) < p(u) \) according to Definition 3. Then, to avoid duplicate visiting, we only process \( w \in N_G(u) \) with \( p(w) < p(u) \). After running lines 4 - 7, we get \( |N_G(u) \cap N_G(w)| \) (i.e., count_wedge(w)) for \( u \) and \( w \). According to Definition 3 if count_wedge(w) > 1, it means that there is a maximal priority-obeyed bloom \( B^* \) contains the vertices \( u \) and \( w \) in the dominant layer of \( B^* \). Then, if \( B^* \notin U(I) \), we compute \( X_{B^*} \), put \( B^* \) id and \( X_{B^*} \) into \( U(I) \) (lines 8 - 14). After that, if an edge \( e \in B^* \notin L(I) \), we put \( e \) id and \( X_e \) into \( L(I) \). Also, in \( E(I) \), we link \( e \) with \( B^* \) and record \( \text{twin}(B^*, e) \).

### Time complexity of constructing the BE-Index

The time complexity of constructing the BE-Index (i.e., Algorithm 3) is bounded by the time complexity to find all the maximal priority-obeyed blooms and the edges in them. According to Lemma 7, this can be done by finding all the priority-obeyed wedges. Since finding the priority-obeyed wedges in \( G \) needs \( O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\}) \) time which can be proved similarly as the proof of Lemma 6, this theorem holds.

### V. Decomposition Algorithms

In this section, we present three BE-Index-based bitruss decomposition algorithms. We first present a bottom-up approach BiT-BU which starts the peeling process from the smallest \( k \). Then, we present the algorithm BiT-BU++ which uses two batch-based optimizations to speed up BiT-BU. After that, a progressive compression approach BiT-PC is proposed.

#### A. A bottom-up approach

We firstly introduce the BiT-BU algorithm using the BE-Index which are shown in Algorithm 4.

**Algorithm 4: BiT-BU**

**Input:** \( G(V = (U, L), E) \): the input bipartite graph

**Output:** \( \phi_e \) for each \( e \in E(G) \)

1. compute \( X_e \) for each \( e \in E(G) \)
2. call IndexConstruction // Algorithm 3
3. \( k \leftarrow 0 \)
4. while exist unassigned edges in \( G \) do
   5. while exist unassigned \( e = (u, v) \) with \( X_e \leq k \) do
      6. \( \phi_e \leftarrow k \)
      7. call RemoveEdge(e) // Algorithm 3
      8. mark \( e \) as assigned
   9. \( k \leftarrow k + 1 \)
10. return \( \phi_e \) for each \( e \in E(G) \)

Given a bipartite graph \( G \), BiT-BU first computes \( X_e \) for each edge \( e \in E(G) \) (i.e., the counting phase) using the algorithm in [8]. Then, it calls Algorithm 3 to construct the BE-Index \( I \). After that, in the peeling phase, BiT-BU iteratively removes an unassigned edge \( e \) with \( X_e \) less than the current \( k \) value and \( \phi_e \) of \( e \) is assigned as \( k \).

**Analysis of the BiT-BU algorithm.** Below we show the correctness and time/space complexities of BiT-BU.

**Theorem 2.** The BiT-BU algorithm correctly solves the bitruss decomposition problem.

**Proof.** This theorem directly follows from Theorem 1.

**Time complexity.** BiT-BU has three parts. The counting part needs \( O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\}) \) [8], the index construction part also needs \( O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\}) \) time as illustrated before. For the peeling part, since the removing of an edge \( e \) needs \( O(\phi_e) \) time as proved in Lemma 5 and we need to remove all the edges in \( G \). Thus, the time complexity of the peeling process is \( O(\sum_{(u,v) \in E(G)} \phi_e) = O(\sum_{(u,v) \in E(G)} \max\{d(u), d(v)\}) + X_G) \).

**Lemma 8.** Given a bipartite graph \( G(V, E) \), we have the following equations:

\[
X_G \leq m^2 
\]

(1)

\[
X_G \leq \sum_{(u,v) \in E(G)} \sum_{w \in N_G(u)} \max\{d(u), d(w)\}
\]

(2)

**Proof.** Apparently, for each edge \( (u,v) \in E(G) \), there can be at most \( E(G) - 1 \) butterflies that contain \( (u,v) \), as the edge \( (w,x) \) of a butterfly \( [u,v,w,x] \) cannot be shared with any other butterflies containing \( (u,v) \). Thus, we can get that \( X_G \leq \sum_{(u,v) \in E(G)} m - 1 \leq m^2 \); the first equation holds. For the second equation above, we alternatively prove a stricter equation. Since \( X_G = \sum_{(u,v) \in E(G)} X_{(u,v)} / 4 \) (one butterfly is a \((2, 2)\)-biclique containing 4 edges), we will show \( \forall (u,v) \in E(G) \), \( X_{(u,v)} \leq \sum_{w \in N_G(u)} \max\{d(u), d(w)\} \).

For each edge \( (u,v) \in E(G) \), there can be at most \( (d(u) - 1) \times (d(v) - 1) \) butterflies that contain \( (u,v) \). This is because there should exist a vertex \( w \in N_G(u) \) or a vertex \( x \in N_G(v) \) to form a butterfly with \( u \) and \( v \). Thus, we get the equation \( X_{(u,v)} \leq (d(u) - 1) \times (d(v) - 1) \). For each edge \( (u,v) \in E(G) \):

1. if \( d(v) = 1 \), \( X_{(u,v)} = \sum_{w \in N_G(u)} \max\{d(u), d(w)\} = 0 \);
2. if \( d(v) > 1 \), \( X_{(u,v)} \leq (d(u) - 1) \times (d(v) - 1) \leq \sum_{w \in N_G(u)} \max\{d(u), d(w)\} \).

Thus, this lemma holds.

From the above lemma, we can get that BiT-BU reduces the time complexities of the existing algorithms [5], [9].

**Space complexity.** In the BiT-BU algorithm, we need \( O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\}) \) space to store the BE-Index as proved in Lemma 6 and \( O(m) \) space to store the butterfly supports and bitruss numbers for edges. Thus, the space complexity is \( O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\}) \).

#### B. Batch-based optimizations

Here, we introduce two batch-based optimizations to further improve BiT-BU.

**Batch edge processing.** The batch edge processing optimization is based on the following lemma.

**Lemma 9.** In BiT-BU, the removing of an edge \( e \) does not change \( \phi_e \) if \( X_e = X_e' \).

**Proof.** This lemma is immediate.

**Lemma 9** is immediate since we only update \( X_e' \) if \( X_e' > X_e \) in Algorithm 3 (lines 7 - 8). Based on Lemma 9, we can process a set \( C \) of edges which contains all the edges with the same butterfly supports in each iteration of peeling. Then, we can compute the total butterfly supports for each edge affected by the removing of edges in \( C \) and update the
butterfly supports for each affected edge in one step. Thus, the number of butterfly support updates can be reduced. The details of this optimization are shown in Algorithm 5 later.

**Batch bloom processing.** According to Algorithm 2 when removing an edge \( e \), we need to go through \( B^* \) to get the affected edge \( e' \). Using the batch edge processing strategy, it may need to go through the same bloom many times in BE-Index. Thus, we consider also processing the blooms in batch. We use an array to record the number of accesses for each bloom in BE-Index. Then, we process all the accessed \( B^* \) and update butterflies counts for the affected edges.

The details of the algorithm BiT-BU++ which utilizes the above two strategies are shown in Algorithm 5. Given a bipartite graph \( G \), BiT-BU++ first computes \( X_e \) for each edge \( e \in E(G) \) and constructs the BE-Index \( J \). Then, in the peeling phase, BiT-BU++ first puts all the unassigned edges with minimum butterfly supports into a set \( S \) and initializes \( MBS \) to record the minimum butterfly support in this iteration. We also initialize \( C(B^*) \) for each \( B^* \in U(I) \) to record the number of edge-pairs removed (i.e., a removed edge and its twin edge) of \( B^* \) in one iteration (lines 3 - 5). Then, for each \( e \in S \), we assign \( \phi_e \) and for each \( B^* \in N_I(e) \), we increase \( C(B^*) \) by 1 and remove \( e' = \text{twin}(B^*, e) \) from \( I \) if \( e' \) is not assigned (lines 8 - 13). This is because when an edge is removed from \( B^* \), its twin edge also loses all the supports from \( B^* \) and a pair of twin edges should only count once. Next, if \( C(B^*) > 0 \), we also need to update \( X_{B^*} \) and \( X_{e'} \) for each unassigned \( e' \in N_I(B^*) \) according to Lemma 10. Then, we mark all the edges in \( S \) as assigned and remove them (lines 19 - 21).

**Example 3.** Consider the bipartite graph \( G \) in Figure 2(a) and the BE-Index of \( G \) in Figure 6. Using the batch-based optimizations, BiT-BU++ firstly processes all the edges with butterfly support equal to 1 (i.e., \( e_6 \) to \( e_8 \) in Figure 6). Since \( B_1^* \) is a 2-bloom and \( e_5 \) is the twin edge of \( e_6 \) in \( B_1^* \), \( X_{e_6} \) becomes \( 3 - (2 - 1) = 2 \) as shown in Algorithm 5 lines 11 - 13. Then, since no other unassigned edges are affected, we only need to update \( X_{B_1^*} \) to 0 and assign the bitruss numbers of \( e_6, e_7 \) and \( e_8 \) as 1. Similarly, in the next peeling iteration, we can process \( e_6 \) to 5 together. Since they form three pairs of twin edges, we just update \( X_{B_2^*} \) to 0 and assign the bitruss numbers of them as 2.

Note that, the worst case time and space complexities of BiT-BU++ are the same as BiT-BU since the batch-based strategies are used to find potential cost-sharing.

**C. A progressive compression approach**

**Motivation.** As discussed above, the algorithm BiT-BU++ using batch-based optimizations already reduces the number of butterfly support updates comparing with BiT-BU. However, in BiT-BU++, we still cost lots of time to update butterfly supports for those edges with high butterfly supports (i.e., hub edges). For example, as shown in Figure 7 about 80% update operations are performed for hub edges (i.e., edges with original butterfly supports > 20,000) in BiT-BU++. To solve this issue, we have the observation of the following lemma.

![Figure 7](image7.png)

**Lemma 10.** Given a bipartite graph \( G \), the \( k \)-bitruss of \( G \) is contained in a subgraph of \( G \) denoted as \( G_{\geq k} \) where for each edge \( e \in G_{\geq k} \), \( X_e \geq k \).

**Proof.** This lemma directly follows from Definition 4. \( \square \)

Based on Lemma 10 in this section, we introduce a progressive compression approach BiT-PC which aims to reduce the number of butterfly support updates for hub edges.

![Figure 8](image8.png)

**Figure 8.** Illustrating the BiT-PC algorithm

**The algorithmic framework.** We first introduce the algorithmic framework of BiT-PC. Given a bipartite graph \( G \) and the parameter \( \epsilon \) (i.e., the butterfly supports threshold in an iteration), BiT-PC has the follows steps:

1) extract the candidate subgraph \( G_{\geq \epsilon} \):
2) peel $G_{≥ε}$ similarly as BiT-BU++ to obtain $ε$-bitruss; 
3) decrease $ε$, repeat steps 1 - 2 until $ε = 0$.

According to the above framework, in each iteration of BiT-PC, we only handle the edges with butterfly supports $≥ ε$ and get the bitruss numbers of the edges in $ε$-bitruss. For example, as shown in Figure 8 we first consider the candidate graph $G_{≥k_{max}}$, where $k_{max}$ is the largest possible bitruss number. We can get the $k_{max}$-bitruss from $G_{≥k_{max}}$ by only considering the edges in $E(G_{≥k_{max}})$. In this manner, the bitruss number of hub edges can be computed within a cohesive subgraph of $G$, and we can avoid performing unnecessary updates (caused by the edges with low butterfly supports). The details of BiT-PC are shown in Algorithm 7.

**Step 1: candidate subgraph generation.** Given a bipartite graph $G$, to generate the first candidate subgraph $G_{≥ε_1}$, we need to compute $ε_1$ which equals to the largest possible bitruss number $k_{max}$ in $G$. We set $k_{max}$ as the largest integer if there exists at least $k_{max}$ edges in $G$ with their butterfly supports $≥ k_{max}$. It can be easily computed after sorting the edges in non-ascending order of their butterfly supports. In other iteration with $i > 1$, $ε_i$ is computed in Step 3. In iteration $i$, we extract the candidate subgraph $G_{≥ε_i}$, where $X_e ≥ ε_i$ for each edge $e$ in $G_{≥ε_i}$. Then, we recompute $X_e$ for each edge $e$ on $G_{≥ε_i}$, and remove $e$ from $G_{≥ε_i}$ if $X_e < ε_i$. 

**Step 2: compressed index construction and index-based computation.** According to Lemma 10 we can compute the $ε$-bitruss $H_e$ on $G_{≥ε}$ and obtain the bitruss numbers of all the edges in $H_e$. Following the similar idea as BiT-BU++, in iteration $i$, we construct the BE-Index $I_{≥ε_i}$ based on $G_{≥ε_i}$, and run the peeling process. Note that, for the first iteration with $ε_1 = k_{max}$, we just construct the BE-Index $I_{≥ε_1}$ based on $G_{≥ε_1}$ and run the peeling process similar as BiT-BU++. For the other iterations with $i > 1$, since there may exist assigned edges (i.e., edges with their bitruss numbers assigned in previous iterations) in the candidate graph $G_{≥ε_i}$, we do not insert these assigned edges into $I_{≥ε_i}$ but preserve the blooms they supported in $I_{≥ε_i}$. As shown in Algorithm 9 lines 8 - 14, we only insert the unassigned edges into $I_{≥ε_i}$, but the blooms are preserved in $I_{≥ε_i}$. In this manner, (1) we can get the correct butterfly supports of unassigned edges in $G_{≥ε_i}$; (2) when removing an unassigned edge $e$ in $G_{≥ε_i}$, we do not update the supports of the assigned edges which share butterfly with $e$. Then, we run the peeling process similar as BiT-BU++. Note that, for an unassigned edge $e ∈ G_{≥ε_i}$, we assign $φ_e$ to $e$ and mark $e$ as assigned only if $X_e ≥ ε_i$. This is because in each iteration, BiT-PC only computes the bitruss numbers for the edges in the $ε$-bitruss.

**Step 3: preparation for the next iteration.** For an iteration $i$, after running steps 1 - 2, we can obtain the bitruss number of all the edges in $H_e$. Then, we decrease $ε_i$ and run steps 1 - 2 until all the edges are assigned. To reduce the number of iterations, we can decrease $ε_i$ by an integer larger than 1, which means that we compute $ε_{i+1} = ε_i − α$ where $α ≥ 1$ is an integer. Consequently, we can take all the edges with butterfly supports $≥ ε_{i+1}$ into consideration and compute the $k$-bitrusses with $ε_{i+1} ≤ k < ε_i$ in one iteration. In BiT-PC, we set $α$ as $\lceil k_{max} × τ \rceil$ where $k_{max}$ is the largest possible bitruss number and $τ \in (0, 1]$. Thus, the total number of iterations in BiT-PC can be reduced from $k_{max}$ to $\lceil \frac{k_{max} × τ}{k_{max} × τ} \rceil$. We also provide a guideline of choosing $τ$ in Section 6.

**Algorithm 6: CompressedIndexConstruction**

**Input:** $G_{≥ε_i}$: the candidate graph in iteration $i$  
**Output:** $G_{≥ε_i}$: the BE-Index of $G_{≥ε_i}$

1) // $X_e$, for each $e ∈ E(G_{≥ε_i})$ is pre-computed
2) Compute $p(v)$ for each $v ∈ V(G_{≥ε_i})$ // Definition 7
3) foreach $u ∈ V(G_{≥ε_i})$ do
4) run Algorithm 3 lines 4 - 7, replace $G$ with $G_{≥ε_i}$
5) foreach $v ∈ N_{G_{≥ε_i}}(u) : p(v) < p(u)$ do
6) if $count\_edge(u) > 1$ then
7) run Algorithm 3 11-14, replace $I$ with $I_{≥ε_i}$
8) if unassigned $e = (u, v) ∉ L(I_{≥ε_i})$ then
9) add $e.id$ and $X_e$ into $L(I_{≥ε_i})$
10) if unassigned $e = (v, u) ∉ L(I_{≥ε_i})$ then
11) add $e.id$ and $X_e$ into $L(I_{≥ε_i})$
12) if $(u, v) or (v, u)$ is unassigned then
13) run Algorithm 5 lines 19 - 22, replace $I$ with $I_{≥ε_i}$
14) return $I_{≥ε_i}$

**Algorithm 7: BiT-PC**

**Input:** $G(V = (U, L), E)$: the input bipartite graph, $τ \in (0, 1]$

**Output:** $φ_e$ for each $e ∈ E(G)$

1) compute $X_e$ for each $e ∈ E(G)$
2) compute the largest possible bitruss number $k_{max}$ in $G$
3) $i ← 1; \; ε_i ← k_{max}$
4) while exist unassigned edge in $G$ do
5) extract $G_{≥ε_i}$ from $G$ where $X_e ≥ ε_i$ for each $e ∈ E(G_{≥ε_i})$
6) recompute $X_e$ for each $e ∈ E(G_{≥ε_i})$ on $G_{≥ε_i}$, and remove $e$ from $G_{≥ε_i}$ if $X_e < ε_i$
7) call CompressedIndexConstruction // Algorithm 6
8) while exist unassigned edges in $G_{≥ε_i}$ do
9) run Algorithm 5 lines 3 - 21, replace $I, G$ with $I_{≥ε_i}, G_{≥ε_i}$
10) $i ← i + 1; \; ε_{i+1} ← \max\{\epsilon_i − \alpha, 0\}$
11) return $φ_e$ for each $e ∈ E(G)$

**Analysis of the BiT-PC algorithm.** Below we show the correctness and time/space complexities of BiT-PC.

**Theorem 3.** The BiT-PC algorithm correctly solves the bitruss decomposition problem.

**Proof.** This theorem immediately follows from Lemma 10 and Theorem 2.
time since we can avoid updating the edges which were already assigned in previous iterations.

**Space complexity.** For BiT-PC, we need $O(m)$ space to store the bitruss numbers for edges. Also, in iteration $i$, when handling a candidate subgraph $G_{2^i}$, we need to construct a compressed BE-Index for $G_{2^i}$, and release it before the next iteration. Since $G_{2^i} \subseteq G$ and the BE-Index of $G_{2^i}$ uses $O(\sum_{(u,v) \in E(G_{2^i})} \min\{d(u), d(v)\})$ space, the space complexity of BiT-PC is $O(\sum_{(u,v) \in E(G)} \min\{d(u), d(v)\})$.

VI. EXPERIMENTS

In this section, we report the evaluation of bitruss decomposition algorithms on 15 real-world datasets.

A. Experiments setting

**Algorithms.** Our empirical studies have been conducted against the following algorithms: 1) the state-of-the-art BiT-BS in [5] deployed with the new counting algorithm in [8] as the baseline algorithm, 2) the bottom-up algorithm BiT-BU in Section [V] 3) the bottom-up algorithm with batch-based optimizations BiT-BU++ in Section [V] 4) the progressive compression algorithm BiT-PC in Section [V].

The algorithms are implemented in C++ and the experiments are run on a Linux server with Intel Xeon E5-2698 processor and 512GB main memory. We terminate an algorithm if the running time is more than 30 hours.

**Datasets.** We use 15 real datasets in our experiments and all the datasets we used can be found in KONECT [1].

The summary of datasets is shown in Table [II] $U$ and $L$ are vertex layers, $|E|$ is the number of edges. $X_e$ is the number of butterflies. $\mathbf{X}_{max}$ and $\phi_{max}$ are the largest butterfly support and largest bitruss number of an edge in a dataset, respectively.

**Parameters.** The experiments are conducted using different settings on 2 parameters: $n$ (graph size), $\tau$ (the parameter used in BiT-PC). When varying the graph size $n$, we randomly sample 20% to 100% vertices of the original graphs, and construct the induced subgraphs using these vertices. We vary $\tau$ from 0.02 to 1 and set $\tau$ as 0.02 by default.

B. Performance Evaluation

In this section, we evaluate the performance of the proposed algorithms. First, we evaluate the performance of BiT-BS, BiT-BU, BiT-BU++ and BiT-PC on all the datasets. After that, we evaluate the number of butterfly support updates and the size of online indexes of BiT-BU, BiT-BU++ and BiT-PC. Then, we test the scalability of our algorithms. Also, we evaluate the batch-based optimizations. Finally, we evaluate the parameter $\tau$ used in BiT-PC.

**Evaluating the performance on all the datasets.** In Figure [9] we show the performance of the BiT-BS, BiT-BU, BiT-BU++ and BiT-PC algorithms on different datasets. We can observe that BiT-BU, BiT-BU++ and BiT-PC are scalable. The computation costs of these algorithms increase as the percentage of vertices increases. As discussed before, BiT-PC significantly outperforms BiT-BU and BiT-PC on D-style and Wiki-it.

**Evaluating the total number of butterfly support updates.** In Figure [11] we show the number of updates of our algorithms on four representative datasets Github, D-label, D-style and Wiki-it. Here, the number of updates means the total number of butterfly support updates for edges (i.e., the sum of updates of $\mathbf{X}_e$ for each edge $e$). We can observe that, on all these datasets, the number of updates of BiT-BS++ is less than the number of updates of BiT-BU because of the batch-based optimizations. BiT-PC reduces more than 90% updates than BiT-BU and BiT-BU++. This is because BiT-PC processes the hub edges within a more cohesive subgraph and generates compressed BE-Index in later iterations. Thus, it significantly reduces the number of butterfly support updates for those hub edges as discussed in Section [V].

**Evaluating the size of online indexes.** In Figure [6] we show the size of online indexes constructed by our algorithms on four representative datasets Github, D-label, D-style and Wiki-it. We can observe that on each dataset, the size of online index of BiT-PC is less than the size of online indexes of BiT-BU and BiT-BU++. This is because BiT-PC processes a more cohesive subgraph and generates compressed BE-Index in each iteration. In addition, we can see that, on Wiki-it with 601,291 million butterflies, the online indexes of all our algorithms only need less than 4GB space.

**Scalability.** In Figure [12] we study the scalability of BiT-BU, BiT-BU++ and BiT-PC by varying the graph size $n$ on the Github, D-label, D-style and Wiki-it datasets. When varying $n$, we randomly sample 20% to 100% vertices of the original graphs, and construct the induced subgraphs using these vertices. We can observe that, the algorithms BiT-BU, BiT-BU++ and BiT-PC are scalable. The computation costs of these algorithms increase as the percentage of vertices increases. As discussed before, BiT-PC significantly outperforms BiT-BU and BiT-PC on D-style and Wiki-it.

**Evaluate the batch-based optimizations.** In Figure [13] we evaluate the efficiency of our batch-based optimizations (i.e., batch edge processing and batch bloom processing in Section [V-B]) on Github, D-label, D-style and Wiki-it.
Table II
SUMMARY OF DATASETS

| Dataset          | $|E|$   | $|U|$   | $|L|$   | $X_G$ | $X_{e_{max}}$ | $\phi_{e_{max}}$ |
|------------------|-------|-------|-------|-------|-------------|-----------------|
| Condmat          | 58,595| 16,726| 22,015| 70,549| 127         | 63              |
| Marvel           | 96,662| 6,486 | 12,942| 10,709,594| 6,612     | 1761            |
| DBPedia          | 293,697| 172,091| 53,407| 3,761,594| 1,720     | 852             |
| Github           | 440,237| 56,519| 120,867| 50,894,505| 40,675   | 1014            |
| Twitter          | 1,890,661| 175,214| 530,418| 206,508,691| 29,708   | 5864            |
| D-label          | 5,302,276| 1,754,823| 270,771| 5,261,758,502| 625,418  | 15948           |
| D-style          | 5,740,842| 1,617,943| 383   | 77,383,418,076| 1,279,105| 52,015          |
| Amazon           | 5,743,258| 2,146,057| 1,230,915| 35,849,304| 8,827    | 551             |
| DBLP             | 8,649,016| 4,000,150| 1,425,813| 201,991,038,864| 4,500,590| 40,675          |
| Wiki-it          | 12,544,802| 2,225,180| 137,693| 298,492,670,057| 2,994,802| 166,785         |
| Delicious        | 22,900,703| 288,275| 4,022,276| 601,291,038,864| 4,500,590| 231,253         |
| Live-journal     | 101,798,957| 833,081| 33,778,223| 56,892,252,403| 1,219,319| 6,638           |
| Wiki-en          | 112,307,385| 3,201,203| 7,489,073| 3,297,158,439,527| 70,549   | 4,022,276       |
| Tracker          | 140,613,762| 27,665,730| 12,756,244| 20,067,567,209,850| 46,747,317| 2,462,013       |

Figure 9. Performance on different datasets

We can see that, the batch edge processing optimization significantly reduces the computation cost while the batch bloom processing optimization further enhances the performance.

Evaluate the effect of $\tau$. The algorithm BiT-PC needs a parameter $\tau$ to decide the decrease of $k'$ after each iteration of processing. In Figure 14 we evaluate the effect of $\tau$ on datasets Github, D-label, D-style and Wiki-it. Figure 14 (b) shows the number of updates increases when $\tau$ increases. This is because when $\tau$ is small, the original
In this paper, we study the bitruss decomposition problem. To solve this problem efficiently, we propose a novel online BE-Index which compresses the butterflies into blooms. Based on the BE-Index, we first propose a bottom-up algorithm BiT-BU which reduces the time complexities of the existing algorithms. Also, two batch-based optimizations are deployed on BiT-BU to enhance the performance. Then, to efficiently handle edges with high butterfly supports, we propose the BiT-PC algorithm which handles and compresses the graph progressively. We conduct extensive experiments on real datasets and the result shows that our algorithms significantly outperform the state-of-the-art algorithm.

VII. CONCLUSION

In this paper, we study the bitruss decomposition problem. To solve this problem efficiently, we propose a novel online BE-Index which compresses the butterflies into blooms. Based on the BE-Index, we first propose a bottom-up algorithm BiT-BU which reduces the time complexities of the existing algorithms. Also, two batch-based optimizations are deployed on BiT-BU to enhance the performance. Then, to efficiently handle edges with high butterfly supports, we propose the BiT-PC algorithm which handles and compresses the graph progressively. We conduct extensive experiments on real datasets and the result shows that our algorithms significantly outperform the state-of-the-art algorithm.

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