Simulation of ice melting in a single-phase statement of the Stefan problem

S D Sleptsov¹, N A Rubtsov¹ and N A Savvinova²

¹Kutateladze Institute of Thermophysics, SB RAS, Novosibirsk, Acad. Lavrentiev ave. 1, 630090, Russia
²Institute of Physics and Technologies, Ammosov North-East Federal University, Yakutsk, Belinsky str. 20, 677000, Russia

E-mail: sleptsov@itp.nsc.ru

Abstract. The problem of radiation-conductive heating and subsequent melting of ice in a climatic chamber in a single-phase approximation of the Stefan problem with allowance for the emerging thin water film on the irradiated surface was solved by mathematical modeling methods. Fields of temperature and flux density of the resulting radiation are obtained, as well as the rate of melting and heating of the non-irradiated ice surface. Comparison with experiment showed satisfactory agreement.

1. Introduction

Modeling of ice melting is necessary both for understanding the processes taking place in nature and for ensuring the safety of building structures, equipment and population in the northern latitudes. The problem of ice melting is a classical problem of phase transition. The history of Stefan problem statements is presented in [1]. Theoretically and experimentally, the two- and three-phase Stefan problems with radiation are well studied [2–7]. Mathematical modeling of the single-phase Stefan problem with radiation was considered in a series of works (see, for instance, [8, 9]), however, there are few experimental works, which can verify the statement and method of solution. The work of [10] is computationally experimental. There, ice was placed on a vertical opaque substrate in the climatic chamber at the constant temperature of 0°C illuminated by two kinds of lamps (halogen with filament temperature \( T_b = 3200 \) K and lamps with a nichrome filament with \( T_b = 800 \) K). Ice melting was considered under the conditions of short-wave and long-wave radiation. In the mathematical model of the process, the authors neglected the presence of a thawed water film on the surface and calculation was carried out in a single-phase statement of the Stefan problem. They compared the rates of melting and heating of the non-irradiated ice side and obtained a satisfactory agreement, using in calculations the fitting parameters and direct integration of radiation transfer according to the Bouguer law. The effect of short-wave radiation on formation of highly rough surfaces in ice was shown.

In this paper, we present a mathematical model of experiment [10] for verification of problem statement and method for solving the radiation part used in [9], with experimental data also obtained in [10].
2. Statement of the problem and methods of solution

The geometric scheme of the problem, assuming an opaque bakelite substrate of thickness $L_1$ with an adhered layer of pure ice, absorbing radiation, with thickness $L_2$, considered in approximation of the gray medium, is shown in figure 1. Radiation from a lamp with filament temperature $T_b = 800 \text{ K}$ falls on the right side of ice. This models approximately solar radiation on a cloudy day; the data obtained are used for subsequent comparison with the results of [10]. The ice boundaries diffusely absorb, reflect and pass radiation in such a way that $A_i + R_i + D_i = 1$, where $A_i$, $R_i$, $D_i$ are absorptive, reflective and transmissive hemispherical abilities of the ice surface, $i = 1, 2$. The left boundary of substrate $T_i(0, t)$ is maintained at a constant temperature $T_{\text{sub}} = 256.15 \text{ K}$, ambient air is kept at $T_\infty = 273.15 \text{ K}$.

The solution of the problem includes two stages. At the first stage, radiative-conductive heat transfer is considered, it continues until the right ice boundary (with initial temperature $T_2(L_2, t)$) reaches the phase transition temperature $T_f$. At the second stage, we consider the Stefan problem with a fixed value of $T_f$, at this stage, a thin film of water flows under the influence of gravitational forces, causing an additional heat load in the form of convection and radiation. The position of interface $L_2(t)$ is determined from the solution to the boundary problem.

The equations for the conservation of energy of a substrate with temperature $T_1(z, t)$ and ice with temperature $T_2(x, t)$ are written as follows:

$$
\rho_i c_i \frac{\partial T_1(z, t)}{\partial t} = \lambda_1 \frac{\partial^2 T_1(z, t)}{\partial z^2}, \quad 0 < z < L_1 \quad (1a)
$$

$$
\rho_2 c_2 \frac{\partial T_2(x, t)}{\partial t} = \frac{\partial}{\partial x} \left( \lambda_2 \frac{\partial T_2(x, t)}{\partial x} - E(x, t) \right), \quad 0 < x < L_2 \quad (1b)
$$

Here $c_{pi}$ is the specific heat at constant pressure, $\rho_i$ is density, $\lambda_i$ is thermal conductivity coefficient ($i=1, 2$), $E(x, t) = 2\pi \int_{-1}^{1} I(\tau, \mu) u d\mu$ is the flux density of the resulting radiation, written in terms of radiation intensities $I(\tau, \mu)$, where $\tau = \alpha \cdot x$ is the optic thickness, $\alpha$ is the volumetric absorption coefficient, $\mu$ is the cosine of the angle between the direction of propagation of radiation and the coordinate axis $x$.

The boundary conditions for the first stage are written as relationships:

$$
T_1 = T_{\text{sub}}, \quad z = 0 \quad (2a)
$$

$$
T_2 = T_\infty, \quad t > 0, \quad x = L_2 \quad (2b)
$$

$$
T_2 = 0, \quad t > 0, \quad z = 0 \quad (2c)
$$

$$
T_2(x, t) = 0, \quad t > 0, \quad x = L_2 \quad (2d)
$$

$$
\frac{\partial T_1}{\partial z} = 0, \quad t > 0, \quad z = L_1 \quad (2e)
$$

$$
\frac{\partial T_1}{\partial z} = 0, \quad t > 0, \quad z = 0 \quad (2f)
$$

$$
E(x, t) = E_0, \quad t > 0, \quad x = 0 \quad (2g)
$$

$$
E(x, t) = 0, \quad t > 0, \quad x = L_2 \quad (2h)
$$
\[-\lambda_1 \frac{\partial T}{\partial z} + D \frac{\partial T}{\partial x} (x,t) = \lambda_2 \frac{\partial T}{\partial x} + |E_{res,1}|, \quad z = L_1 \text{ and } x = 0 \quad (2b)\]

\[\lambda_2 \frac{\partial T}{\partial x} - h(T^* - T_\infty) - |E_{res,2}| = 0, \quad x = L_2 \quad (2c)\]

The radiation component $|E_{res,i}|$ in (2b) and (2c) is determined as:

\[|E_{res,1}| = A_i \sigma_0 T_i^4(z,t) - \varepsilon_i \sigma_0 T_i^4(x,t), \quad |E_{res,2}| = A_i E^* - \varepsilon_i \sigma_0 T_i^4(x,t),\]

where $h$ is the heat transfer coefficient, $\sigma_0$ is the Stefan-Boltzmann constant, $\varepsilon_i$ is the emissivity, $E^*$ is the constant incident flux, $i = 1, 2$. In (2b), the second term in the left is responsible for radiation penetration through ice; the terms included into $|E_{res,1}|$ are the flux densities determining absorption and radiation from the substrate and ice surfaces, respectively; at that, the validity of Kirchhoff law is assumed: $A_i = \varepsilon_i$. The system of equations (1) and (2) is supplemented with the initial condition:

\[T_i(z,0) = T_j(x,0) = T_{sub}.\]

At the stage of melting ice, the temperature of the right boundary is fixed:

\[T(x,t) = T_j, \quad x = L_\Delta(t). \quad (3)\]

The boundary condition (2c) is transformed into the Stefan condition taking into account a thin layer of ice water that appears on the surface. We assume that the film temperature is isothermal, there is no temperature gradient, and thickness of the film itself is much smaller than the ice thickness:

\[\lambda_2 \frac{\partial T}{\partial x} - h(T_{fil} - T_\infty) - |E_{res,fil}| = \rho_2 \gamma \frac{\partial L_2}{\partial t}. \quad (4)\]

where $|E_{res,fil}|$ takes the following form:

\[|E_{res,fil}| = A_i E^* + \varepsilon_i \sigma_0 \left(T_{fil}^4 - T_2^4(x,t)\right), x = L_2(t). \quad (5)\]

Here, $T_j = 273.15 \text{ K}$ is melting temperature for ice, $T_{fil} = 277.15 \text{ K}$ is water film temperature. Condition (4) takes into account the convection from the outer surface of the water film, and in (5), the intrinsic radiation of the film and the right surface are taken into account.

The assumption that a thin water film exists on the ice surface does not contradict the one-phase approximation of the Stefan problem, since there is no energy transfer in the film and it acts only as an additional boundary condition on the interface with the constant values. The thermal problem is solved only in the thickness of ice on a vertical substrate.

The modified method of mean flows [3, 7] represents a wide range of possibilities for the simplicity of solution and effectiveness of result obtaining. In the framework of this method, the radiation transfer equation is reduced to the system of two nonlinear differential equations for a plane layer of a semitransparent absorbing medium. The differential dimensionless analog of the radiation transfer equation for hemispherical flows $\Phi^* = E^* \left(4 \sigma_0 T_j^4\right)$, included as the source term $E(x, t)$ into equation (1), can be represented in the form [3]:

\[\frac{d}{d\tau} (\Phi^*(\tau, \eta) - \Phi^*(\tau, \eta)) + \left( m'(\tau) \Phi^*(\tau, \eta) - m^*(\tau) \Phi^*(\tau, \eta) \right) = n^2 \Phi_0, \quad (6)\]

\[\frac{d}{d\tau} (m'(\tau) \Phi^*(\tau, \eta) - m^*(\tau) \Phi^*(\tau, \eta)) + \left( \Phi^*(\tau, \eta) - \Phi^*(\tau, \eta) \right) = 0.\]
The boundary conditions for the system of equations (6) in dimensionless variables are written as:

\[
\Phi'(0,\eta) = A n^2 \frac{\theta^2(0,\eta)}{4} + D \frac{\theta^2(1,\eta)}{4} + \left(1 - \frac{R_1}{n^2}\right)\Phi'(0,\eta),
\]

\[
\Phi'(1,\eta) = A_2 n^2 \frac{\theta^2(1,\eta)}{4} + D_2 F + \left[1 - \frac{R_2}{n^2} - A_2 \left(1 + n^2\right)\right]\Phi'(1,\eta).
\] (7)

Here \( \Phi'(\tau,\eta) = \frac{2\pi}{4\sigma_0 T_r^4} \int_{0}^{1(0)} I(\tau,\mu)\mu d\mu \), \( m^4(\tau) = \int_{0}^{1(0)} I(\tau,\mu)\mu^4 d\mu \), \( l^4(\tau) = \int_{0}^{1(0)} I(\tau,\mu)\mu^2 d\mu \), \( \Phi_0 = B_v/(4\sigma_4 T_r^4) \) is dimensionless density of equilibrium radiation, \( B_v \) is the Planck function of blackbody radiation, \( n \) is refractive index; the values of coefficients \( m^4, l^4 \) are determined from the recurrent relationship obtained by means of a formal solution to the equation of radiation transfer \([3, 7]\). \( \theta_1 = \frac{T_1}{T_r} \) and \( \theta_2 = \frac{T_2}{T_r} \) are the dimensionless temperatures of substrate and ice, \( i = 1, 2 \). \( \Phi^* = E^*/(4\sigma_4 T_r^4) \) is dimensionless density of radiation flux falling on a plate from the right.

The solution of the boundary value problem is reduced to determination of the temperatures \( T_1 \) and \( T_2 \) and the densities resultant radiation fluxes in a region that is a plane layer of pure, absorbing, radiating and non-scattering ice. The position of the phase transition front varies from 1 to 0. The boundary value problem \((1) - (5)\) is solved by the finite-difference method, the non-linear system of implicit difference equations is solved by the sweep and iteration method. When solving the radiation problem, iterations are used, when the boundary value problems \((6) \) and \((7)\) are solved by the matrix factorization method at each step. Fast convergence of such a solution method allows obtaining results with a high degree of accuracy.

3. Result analysis

Below, we present the results of numerical modeling of ice melting on a substrate with the following physical parameters: substrate thickness \( L_1 = 0.015 \text{ m} \), initial ice thickness \( L_2 = 0.045 \text{ m} \), temperature of the left boundary of substrate and initial temperature of substrate and ice \( T_{\text{sub}} = 256.15 \text{ K} \), temperature of the atmosphere inside the chamber is maintained at constant value \( T_{\infty} = 273.15 \text{ K} \), equal to ice melting temperature \( T_r \), and constant density of the incident radiation flux \( E^* = 1162.22 \text{ W/m}^2 \). Thermophysical properties of substrate and ice have the following values: thermal conductivity of bakelite and ice, respectively, \( \lambda_1 = 0.232 \text{ and } \lambda_2 = 1.9 \text{ W/(m-K)} \);\( a_2 = 9.3 \times 10^{-7} \text{ m}^2/\text{s} \);latent heat of phase transition \( \gamma = 335 \text{ kJ/kg} \). The optical parameters of ice are as follows: refractive index \( n = 1.31 \), coefficient of volumetric absorption taking into account the long-wavelength region \( a = 1000 \text{ m}^3 \) \([12]\), reflection coefficients \( r_{1,2} = 0.063 \), and emissivity at the right boundary \( \varepsilon_2 = 0.97 \).

When solving the problem, two parameters (emissivity \( \varepsilon_1 \) on the left boundary and coefficient of heat transfer from the right boundary of ice) were varied. At the first stage, emissivity of the left boundary, conjugated with the substrate, is assumed to be \( \varepsilon_1 = 0.002 \). At the second stage, \( \varepsilon_1 = 0 \). Such an approach does not allow the temperature of the left boundary to reach the melting temperature ahead of time. The heat transfer coefficient at the first stage is assumed to be \( h = 7 \text{ W/(m}^2\text{-K}) \), which equals heat transfer from the vertical wall. At the second stage, \( h = 80 \text{ W/(m}^2\text{-K}) \) is assumed, as in calculations of \([10]\).
The temperature field in the ice layer at different instants of time is shown in figure 2. The curves between 1 and 2 refer to the heating stage. The high absorption coefficients on the right boundary ($\varepsilon_r$) and coefficient of volumetric absorption ($\alpha$) in the medium lead to deep ice heating. The effect of radiation is also observed on the left boundary. At the melting stage, the temperature curves (lines between 2 and 3) take on the character of the curves, in which the heat transfer by thermal conductivity prevails.

Calculation of the rate of ice melting and its comparison with experimental data of [10] are shown in figure 3 (for the convenience of comparison with the experiment, the measurement units are given in accordance with those presented in [10]). It can be seen that calculation, which takes into account a thin water film on the surface, is in good agreement with experiment. Left boundary heating with time is shown in figure 4. Here, the calculation results diverge quantitatively with the experimental data, but there is a qualitative agreement. This divergence is related to the mathematical model used in this paper. In the model, the left boundary of ice does not absorb, but reflects the radiation. In turn, radiation, reflected from the boundaries, equalizes the temperature over the entire volume of the medium (curves between 2 and 3 in figure 2), including the left boundary.

![Figure 2](image1.png)  
**Figure 2.** Temperature field during heating and melting of ice. 1 - process beginning of heating, 2 - melting beginning, 3 - end of melting.  

![Figure 3](image2.png)  
**Figure 3.** Ice melting rate (line - calculation, ● - data of [10]).

![Figure 4](image3.png)  
**Figure 4.** Rate of heating of the left boundary of ice when it melts (line - calculation, ● - data of [10]).

![Figure 5](image4.png)  
**Figure 5.** The melting (left ordinate) and heating (right ordinate) of the left boundary of the ice without water film.
The importance of taking into account the flowing water film is shown in figure 5, which shows the curves of virtual melting rate and heating of the left boundary of ice without a film. In this case, the melting stage is strongly stretched in time and agreement with experiment is not observed.

4. Conclusion
A mathematical model of the experiment on heating and subsequent melting of ice at its radiation by a long-wave source was constructed. To solve the radiation part of the problem, a gray medium model was used. Consideration of the presence of a thin film of ice water on the irradiated surface is in good agreement with the experimental data on the rate of ice melting.

Matching of calculated and experimental data makes it possible to consider verification of the single-phase Stefan problem for a semitransparent medium implemented. At the same time, the indicated problem with increasing temperature at the substrate-ice boundary requires further calculations and improvements of the proposed mathematical model.

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