RELATIONSHIP BETWEEN STOCHASTIC FLOWS AND CONNECTION FORMS

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ABSTRACT. In this article I will prove new representation for the Levi-Civita connection in terms of the stochastic flow corresponding to Brownian motion on manifold.

The idea of reconstructing of geometry of riemannian manifold $M$ from the Brownian motion on $M$ has been productively explored for a long time (see, for instance, expository article of Pinsky [7] and references therein). In [7] different asymptotics of Brownian motion (mean exit time, distribution of exit time from small ball,...) has been produced and it was shown that it is possible to retrieve geometry of the manifold through the asymptotics in low dimensions (generally less than six). Another possibility to deduce geometry of the manifold is through small time asymptotics of logarithm of transition function of Brownian motion on manifold (see [8]). Indeed, Bismut-type formula (see, for example, Corollary 3.2, p. 264 of [3]) allow us to deduce logarithm of the transition function $p : [0, T] \times M \times M \to \mathbb{R}$ of the Brownian motion on the manifold. We have

$$d \log p_t(x, y)(v_0) = \frac{1}{t} \left\{ \int_0^t < T_{\xi_s}(v_0), X(x_s)dB_s > |_{\xi_t(x) = y} \right\}$$

where $\xi : [0, T] \times \Omega \to Diff(M)$ is a stochastic flow of diffeomorphisms generated by the Brownian motion on the manifold

$$dx_s = X(x_s) \circ dB_s + A(x_s)ds,$$

$T_{\xi}$ is a derivative of flow $\xi$, w.r.t. initial condition. Now we can notice that $X(x)dB_s$ is a martingale part of the flow $\xi$ and, consequently, can be easily calculated. It is enough to subtract the drift of the flow $\xi$ which, in its turn, can be calculated using Nelson derivative of the flow.

Another way to deduce connection is arising from the theory of Stochastic flows. It is well known fact that every nondegenerate stochastic flow induces certain connection called Le Jan-Watanabe connection (see example B, section 1.2 in the book [4]). Indeed, connection is defined by formulas 1.2.4 and 1.2.2 in [4]. This connection is a metric connection with respect to the riemannian metric induced by the SDE. It coincides with Levi-Civita connection in the case of gradient Brownian systems.

My contribution is a new representation for the Levi-Civita connection in terms of the stochastic flow, corresponding to the Brownian motion on the manifold. Let $\int x_i dx^j, i, j = 1, \ldots, n$ are areas of projections of smooth curves $\gamma \subset M \subset \mathbb{R}^n$ on planes spanned by two vectors of orthogonal basis of $\mathbb{R}^n$. We consider small
time asymptotic behaviour of area \( \int_{Y_t(\gamma)} x^i dx^j, i, j = 1, \ldots, n \) (where \( Y_t : M \to M \)) is a stochastic flow of diffeomorphisms (a.s.) generated by Brownian motion on \( M \) and find connection form \( \Gamma \) of compact manifold \( M \) through this asymptotic. We would like to mention that, contrary to the Pinsky paper, our method does not depend on dimension of manifold and it is local (i.e. knowledge of stochastic flow in the infinitesimal neighborhood of the point \( x \) immediately allow us to retrieve the connection form in \( x \)).

1. Definitions and Presentation of the Main Result

Let \((M, g)\) be compact riemannian manifold of dimension \( k \) and assume that it is embedded in \( \mathbb{R}^n \), \( T_x M, x \in M \) be tangent space in the point \( x \in M, \Gamma : \mathbb{R}^n \supset M \to GL(n) \), \( \Gamma = \{ \Gamma^i_j(x) \}_{i,j=1}^n = \{ \Gamma^i_j(dx^l) \}_{i,j=1}^n \) be Levi-Civita connection form of manifold \( M \). \( P(x) : \mathbb{R}^n \to T_x M, x \in M \) be an orthogonal projection to the tangent space. We denote \( P^i_j(x) = (P(x)\mathbf{\bar{e}}_i, \mathbf{\bar{e}}_j), i, j = 1, \ldots, n \), where \( \{ e_i \}_{i=1}^n \) is a standard orthonormal basis in \( \mathbb{R}^n \);

\[
S^i_{jl}(x) = \sum_{m=1}^n P^i_m \frac{\partial P^j_m}{\partial x^l},
\]

\[
r^i(x) = \frac{1}{2} \sum_{l=1}^n S^i_{ll}, i, j, l = 1, \ldots, n, x \in M.
\]

Let \((\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \geq 0}, \mathbb{P})\) be complete probability space with right continuous filtration, \( \{ W_t \}_{t \geq 0} \) be standard Wiener process in \( \mathbb{R}^n \). Then a Brownian motion on the manifold \( M \) is a stochastic process \( Y_t \) which satisfies following equation:

\[
dY_t(x) = P(Y_t(x)) \circ dW_t, Y_0(x) = x, x \in M, t \in \mathbb{R}^+,
\]

where equation is understood in Stratonovich sense, see e.g. [6], [2]. We will use the same letter i.e. \( Y_t \) for the stochastic flow of diffeomorphisms corresponding to the Brownian motion. In the notation introduced above \( Y_t(A), A \subset M \) denotes image of the set \( A \subset M \) by diffeomorphism \( Y_t \).

**Theorem 1.1.** Let us denote

\[
q^i(x) = [\frac{d}{dt} \mathbb{E} Y_t^i(x)]|_{t=0}, x \in M, i = 1, \ldots, n,
\]

\[
\Psi^{ij}(t, \gamma) = \mathbb{E} \int_{Y_t(\gamma)} x^i dx^j, i, j = 1, \ldots, n, t \geq 0, \gamma \in C^1([0, 1], M)
\]

Then \( \Psi^{ij}(\cdot, \gamma), \gamma \in C^1([0, 1], M) \) is differentiable and we have following formula:

\[
\int_{\gamma} \Gamma_{jk}^i(x) dx_k = \frac{\partial \Psi^{ij}}{\partial t}(0, \gamma) - \frac{\partial \Psi^{ij}}{\partial t}(0, \gamma) - 2 \int_{\gamma} (q^j dx^j - q^i dx^i)
\]

\[-(\gamma^j(1)q^i(\gamma(1)) - \gamma^i(0)q^j(\gamma(0))) + (\gamma^j(1)q^i(\gamma(1)) - \gamma^j(0)q^i(\gamma(0)))
\]

\[
i, j = 1, \ldots, k; \gamma \in C^1([0, 1], M).
\]

\( ^1 \Gamma_{jk}^i, i, j, l = 1, \ldots, n \) are Christoffel symbols of our connection.
Remark 1.2. It will be shown below that function $\bar{q} = (q^1, \ldots, q^n)$ can also be written as follows:

\begin{equation}
q^i(x) = \frac{1}{4} \sum_{l=1}^{n} \Gamma^i_{il}(x) + \sum_{l=1}^{n} \frac{\partial F^i_{il}}{\partial x_l}, x \in M, i = 1, \ldots, n.
\end{equation}

Remark 1.3. The formula (1.5) allow us to find the value of Christoffel symbols $\Gamma^i_{jl}, i, j, l = 1, \ldots, n$. Indeed, if $\gamma$ is a closed loop we can apply Stokes Theorem to the left part of equality (1.5), divide the result on the area of the surface and tend the size of the surface to 0. As the result we get the Levi-Civita connection form up to the exact form. The remaining exact form can be calculated by considering of curves with fixed initial point and varying end point.

The main tool for the proof of Theorem 1.1 will be the following proposition proved in [5], theorem 4, p. 115:

**Proposition 1.4.** Let $\sigma(t, \cdot) \in C^{2,\alpha}_b([0, T])$, $u(t, \cdot) \in C^{1,\alpha}_b([0, T] \times \mathbb{R}^n)$, $\gamma(t, \cdot) \in C^{1,\alpha}_b([0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n)$. Assume that $\gamma_0$ is a curve of $C^1$ class in $\mathbb{R}^n$ which connects points $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$. Let $F \in C^{1,2}([0, T] \times \mathbb{R}^n \times \mathbb{R}^n)$, $X = X_0(x, \omega) : [0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^n$ be defined by:

\[dX_t(x) = u(t, X_t(x))dt + \sigma(t, X_t(x))dW_t,
X_0(x) = x.\]

Then

\begin{equation}
\int_{X_t(\gamma_{a,b})} \sum_{k=1}^{n} F^k(t, x) dx_k = \int_{X_0(\gamma_{a,b})} \sum_{k=1}^{n} F^k(0, x) dx_k + \int_0^t \int_{X_s(\gamma_{a,b})} \sum_{k=1}^{n} \left( \frac{\partial F^k}{\partial t} + \sum_{j=1}^{n} u^j \left( \frac{\partial F^k}{\partial x_j} - \frac{\partial F^j}{\partial x_k} \right) \right) dx_k ds
+ \int_0^t \int_{X_s(\gamma_{a,b})} \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \sum_{m=1}^{n} \sigma^{im} \sigma^{jm} \frac{\partial F^k}{\partial x_l} \right) dx_k ds
+ \frac{1}{2} \int_0^t \int_{X_s(\gamma_{a,b})} \sum_{k=1}^{n} \left( \sum_{j=1}^{n} \sum_{m=1}^{n} \sigma^{im} \sigma^{jm} \frac{\partial F^k}{\partial x_l} \right) dx_k ds \int_{X_s(\gamma_{a,b})} \sum_{k=1}^{n} \left( \sum_{l=1}^{n} \frac{\partial F^k}{\partial x_l} \sigma^{il} \right) dx_k dW^l_s.
\end{equation}

We also need

**Lemma 1.5.**

\begin{equation}
\Gamma^i_{jk}(x) = S^i_{jk}(x) - S^i_{kj}(x), i, j, k = 1, \ldots, n, x \in M.
\end{equation}

\footnote{We would like to note that in [5] only the special case of closed loops (i.e. $a = b$) has been considered. The general case considered here is proved similarly.}
Proof of lemma 1.5. Define $S : \mathbb{R}^n \supset M \rightarrow GL(n)$ as follows

\[(1.9) \quad S(x) = \{S^i_j(x)\}_{i,j=1}^n = \{\sum_{k=1}^n S^i_jk(x)dx_k\}_{i,j=1}^n, x \in M.\]

We have by (1.1) that

\[S(x) = P(x) dP^*(x).\]

In the same time, we have ([1], formula 3.65) that

\[(1.10) \quad \Gamma = dQP + dPQ = -dPP + dP(Id - P) = dP - dPP - dPP = d(P^2) - 2dPP = PdP - dPP = S - S^*,\]

where $Q = Id - P$ and we have used that $P$ is orthogonal projection (i.e. $P^* = P$, $P^2 = P$).

\[\square\]

2. Proof of the Theorem 1.1

We apply Proposition 1.4 with $F^k(t, x) = x^i \delta_{jk}$ where $i, j = 1, \ldots, n$ and get

\[(2.1) \quad \mathbb{E} \int_{X_{t}(\gamma_{a,b})} x^i dx_j = \int_{\gamma_{a,b}} x^i dx_j + \mathbb{E} \int_0^t \int_{X_s(\gamma_{a,b})} (u^i dx_j - u^j dx_i) ds \]

\[+ \int_0^t \int_{X_s(\gamma_{a,b})} \sum_{k,l=1}^n x^l \frac{\partial \sigma_{im}}{\partial x_k} dx_k dW^l_s + \int_0^t \int_{X_s(\gamma_{a,b})} \sigma_{im} \frac{\partial \sigma_{im}}{\partial x_k} dx_k ds.\]

Taking mathematical expectation of both parts of formula (2.1) we get

\[(2.2) \quad \mathbb{E} \int_{X_{t}(\gamma_{a,b})} x^i dx_j = \int_{\gamma_{a,b}} x^i dx_j + \mathbb{E} \int_0^t \int_{X_s(\gamma_{a,b})} (u^i dx_j - u^j dx_i) ds \]

\[+ \mathbb{E} \int_0^t \int_{X_s(\gamma_{a,b})} (X^i_s(b)w^j(s, X_s(b)) - X^i_s(a)w^j(s, X_s(a))) ds \]

\[+ \mathbb{E} \int_0^t \int_{X_s(\gamma_{a,b})} \sum_{k,l=1}^n x^l \frac{\partial \sigma_{im}}{\partial x_k} dx_k dW^l_s.\]
Let us rewrite equation (1.3) for Brownian motion \{Y_t\}_{t\geq 0} on \(M\) in the Ito form. We have

\[
dY_t^i(x) = \sum_{j=1}^{n} P^{ij}(Y_t(x)) \circ dW_t^j
\]

\[
= \sum_{j=1}^{n} P^{ij}(Y_t(x))dW_t^j + \frac{1}{2} \sum_{i,j=1}^{n} P^{ij} \frac{\partial P^{ij}}{\partial x_l}(Y_t(x))dt
\]

\[
= \sum_{j=1}^{n} P^{ij}(Y_t(x))dW_t^j + \frac{1}{2} \sum_{l=1}^{n} S^j_l(Y_t(x))dt
\]

(2.3)

\[
r^i(Y_t(x))dt + \sum_{j=1}^{n} P^{ij}(Y_t(x))dW_t^j, i = 1, \ldots, n, t \geq 0.
\]

Now we can put \(X_t = Y_t\) in the formula (2.2) and we get

\[
\Psi^{ij}(t, \gamma_{a,b}) = \int_{\gamma_{a,b}} x^i dx_j + \mathbb{E} \int_0^t (r^i dx_j - r^j dx_i)ds
\]

\[
+ \mathbb{E} \int_0^t (Y_s^i(b)r^j(Y_s(b)) - Y_s^i(a)r^j(Y_s(a)))ds + \mathbb{E} \int_0^t \int_{Y_s(\gamma_{a,b})} S^i_{jk}(x)dx_kds.
\]

Therefore, we have

\[
\frac{\partial \Psi^{ij}}{\partial t}(0, \gamma_{a,b}) = \int_{\gamma_{a,b}} (r^i dx_j - r^j dx_i) + (b^jr^j(b) - a^jr^j(a))
\]

\[
+ \int_{\gamma_{a,b}} S^i_{jk}(x)dx_k, i, j = 1, \ldots, n.
\]

(2.5)

It remains to show that

\[
r^i = q^i, i = 1, \ldots, n
\]

and formula (1.5) will immediately follow from (2.5) and (1.8). We can notice that \(r\) is a drift of Brownian motion by formula (2.3). Now applying mathematical expectation to formula (2.3) we immediately get the result.

\[\square\]

\textit{Proof of Remark 1.2} We have

\[
S + S^* = PdP + dPP = d(P^2) = dP
\]

i.e.

\[
S^j_{jm}(x) + S^j_{im}(x) = \frac{\partial P^{ij}}{\partial x_m}, ij, m = 1, \ldots, n.
\]

(2.6)

Therefore, from (1.8) and (2.6) it follows that

\[
r^i = \frac{1}{2} \sum_{l=1}^{n} S^l_i = \frac{1}{4} \sum_{l=1}^{n} [(S^l_i - S^i_l) + (S^i_l - S^l_i)]
\]

\[
= \frac{1}{4} \sum_{l=1}^{n} [r^l_i + \frac{\partial P^{il}}{\partial x_l}] = q_i, i = 1, \ldots, n.
\]

(2.7)
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