Loosely-Stabilizing Phase Clocks and The Adaptive Majority Problem

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Abstract

We present a loosely-stabilizing phase clock for population protocols. In the population model we are given a system of \( n \) identical agents which interact in a sequence of randomly chosen pairs. Our phase clock is leaderless and it requires \( O(\log n) \) states. It runs forever and is, at any point of time, in a synchronous state w.h.p. When started in an arbitrary configuration, it recovers rapidly and enters a synchronous configuration within \( O(n \log n) \) interactions w.h.p. Once the clock is synchronized, it stays in a synchronous configuration for at least \( \text{poly}(n) \) parallel time w.h.p.

We use our clock to design a loosely-stabilizing protocol that solves the adaptive variant of the majority problem. We assume that the agents have either opinion \( A \) or \( B \) or they are undecided and agents can change their opinion at a rate of \( \frac{1}{n} \). The goal is to keep track which of the two opinions is (momentarily) the majority. We show that if the majority has a support of at least \( \Omega(\log n) \) agents and a sufficiently large bias is present, then the protocol converges to a correct output within \( O(n \log n) \) interactions and stays in a correct configuration for \( \text{poly}(n) \) interactions, w.h.p.

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1 Introduction

In this paper we introduce a loosely-stabilizing leaderless phase clock for the population model and demonstrate its usability by applying the clock to the comparison problem introduced in [3]. Population protocols have been introduced by Angluin et al. [5]. A population consists of \( n \) anonymous agents. A random scheduler selects in discrete time steps pairs of agents to interact. The interacting agents execute a state transition, as specified by the algorithm of the population protocol. Angluin et al. [5] gave a variety of motivating examples for the population model, including averaging in sensor networks, or modeling a disease monitoring system for a flock of birds. In [24] the authors introduce the notion of loose-stabilization. A population protocol is loosely-stabilizing if, from an arbitrary state, it reaches a state with correct output fast and remains in such a state for a polynomial number of interactions. In contrast, self-stabilizing protocols are required to converge to the correct output state from any possible initial configuration and stay in a correct configuration indefinitely. Many population protocols heavily rely on so-called phase clocks which divide the interactions into
blocks of $O(n \log n)$ interactions each. The phase clocks are used to synchronize population protocols. For example, in [22, 16] they are used to efficiently solve leader election and in [15] they are used to solve the majority problem.

In the first part of this paper we present a loosely-stabilizing and leaderless phase clock with $O(\log n)$ many states per agent. We show that this clock can run forever and that, at any point of time, it is synchronized w.h.p.\(^1\) In contrast to related work [1, 7, 15, 22], our clock protocol recovers rapidly in case of an error: from an arbitrary configuration it always enters a synchronous configuration within $O(n \log n)$ interactions w.h.p. Once synchronized it stays in a synchronous configuration for at least $\text{poly}(n)$ interactions, w.h.p. Our phase clock can be used to synchronize population protocols into phases of $O(n \log n)$ interactions, guaranteeing that there is a big overlap between the phases of any pair of agents. Our clock protocol is simple, robust and easy to use.

In the second part of this paper we demonstrate how to apply our phase clock by solving an adaptive majority problem motivated by the work of [3, 4]. Our problem is defined as follows. Each agent has either opinion $A$, $B$, or $U$ for being undecided. We say that agents change their input with rate $r$ if in every time step an arbitrary agent can change its opinion with probability $r$. The goal is to output, at any time, the actual majority opinion. The idea of our approach is as follows. Our protocol simply starts, at the beginning of each phase, a static majority protocol as a black box. This protocol takes as an input the set of opinions at that time and calculates the majority opinion over these inputs. The outcome of the protocol is then used during the whole next phase as majority opinion. In order to highlight the simplicity of our phase clock, we first use the very natural protocol based solely on canceling opposing opinions introduced in [7]. Then we present a variant based on the undecided state dynamics from [8] which works as follows. The agents have one of two opinions $A$ or $B$, or they are undecided. Whenever two agents with the same opinion interact, nothing happens. When two agents with an opposite opinion interact they will become undecided. Undecided agents interacting with an agent with either opinion $A$ or opinion $B$ adopt that opinion. Without loss of generality we assume that $A$ is the majority opinion in the following. When at least $\Omega(\log n)$ agents have opinion $A$, there is a constant factor bias between $A$ and $B$, and the opinions change at most at rate $1/n$ per interaction, the system outputs $A$ w.h.p.

Our protocol requires only $O(\log n)$ many states. For the setting where all agents have either opinion $A$ or opinion $B$ (none of the agent is in the undecided state $U$) and we have an additive bias of $n^{3/4+\varepsilon}$ for some constant $\varepsilon > 0$ is present, the system again converges to $A$ w.h.p. In the latter setting we can tolerate a rate of order $r = \Omega(n^{-1/4+\varepsilon})$.

**Related Work.** Population protocols have been introduced by Angluin et al. [5]. Many of the early results focus on characterizing the class of problems which are solvable in the population model. For example, population protocols with a constant number of states can exactly compute predicates which are definable in Presburger arithmetic [5, 6, 9]. There are many results for majority and leader election, see [20] and [16] for the latest results. In [24] the authors introduce the notion of loose-stabilization to mitigate the fact that self-stabilizing protocols usually require some global knowledge on the population size (or a large amount of states). See [17] for an overview of self-stabilizing population protocols.

In [7] the authors present and analyze a phase clock which divides time into phases of $O(n \log n)$ interactions assuming that a unique leader exists. They also present a generalization using a junta of size $n^\varepsilon$ (for constant $\varepsilon$) instead of a unique leader and analyze the

\[^1\] The expression *with high probability* (w.h.p.) refers to a probability of $1 - n^{-\Omega(1)}$.\]
12. The currently best protocol [20] solves exact majority with \(O(\log n)\) states and use the resulting clock to solve leader election. The protocol can easily be modified such that it requires only a constant number of states after the junta election [15]. In [1] the authors present a leaderless phase clock with \(O(\log n)\) states. In contrast to our leaderless phase clock, the clock from [1] was not proven to be self-stabilizing. The analysis is based on the potential function analysis introduced in [25] for the greedy balls-into-bins strategy where each ball has to be allocated into one out of two randomly chosen bins. This analysis assumes an initially balanced configuration and it cannot be adopted to an arbitrary unbalanced state, which would be required to deal with unsynchronized clock configurations. In [10] the authors consider a variant of the population model, so-called clocked population protocols, where agents have an additional flag for clock ticks. The clock signal indicates when the agents have waited sufficiently long for a protocol to have converged. They show that a clocked population protocol running in less than \(\omega^k\) time for fixed \(k \geq 2\) is equivalent in power to nondeterministic Turing machines with logarithmic space.

Another line of related work considers the problem of *exact majority*, where one seeks to achieve (guaranteed) majority consensus, even if the additive bias is as small as one [21, 1, 14, 12]. The currently best protocol [20] solves exact majority with \(O(\log n)\) states and \(O(\log n)\) stabilization time, both in expectation and w.h.p. The authors of [8] solve the approximate majority problem. They introduce the undecided state dynamics in the population model for two opinions. They show that their 3-state protocol reaches consensus w.h.p. in \(O(n \log n)\) interactions. If the bias is of order \(\omega(\sqrt{n} \cdot \log n)\) the undecided state dynamics converges towards the initial majority w.h.p. In [18] the required bias is reduced to \(\Omega(\sqrt{n \log n})\). For completeness [11] provides a survey about further protocols in the gossip model.

In [2] the authors define the *catalytic input model* (CI model). In this model the agents are divided into the two groups catalytic agents and non-catalytic agents. Non-catalytic agents perform pairwise interactions and change their state. Catalytic agents never change their state. Additionally to the normal state changes non-catalytic agents can perform spurious state changes; the so-called leak rate specifies the frequency of the spurious reactions. The goal of the non-catalytic agents is to compute a function over the states of the catalytic agents. The authors develop an algorithm for their model to detect whether there is a catalytic agent in a given state \(D\) or not. Note that, due to the leaky transactions non-catalysts can compute false-positives. In [4] the authors use the catalytic input model with \(n\) catalysts and \(m\) non-catalysts which they call worker agents (\(N = n + m\)). They solve the approximate majority problem for two opinions w.h.p. in \(O(N \log N)\) interactions when the initial bias among the catalysts is \(\Omega(\sqrt{N \log N})\) and \(m = \Theta(n)\). They show that the size of the initial bias is tight up to a \(O(\sqrt{\log N})\) factor. Additionally, they consider the approximate majority problem in the CI model and in the population model with leaks. Their protocols tolerate a leak rate of at most \(\beta = O(\sqrt{N \log N}/N)\) in the CI model and a leak rate of at most \(\beta = O(\sqrt{n \log n}/n)\) in the population model. They also show a separation between the computational power of the CI model and the population model.

In [3] the authors consider the CI model and introduce the *robust comparison problem*. The catalytic agents are either in state \(A\) or \(B\) and the goal of the worker agents is to decide which of the two states \(A\) and \(B\) has the larger support. In their dynamic version the number of agents in state \(A\) or \(B\) can change during the execution as long as the counts for \(A\) and \(B\) remain stable for a sufficiently long period allowing the algorithm to stabilize on an output. If at time \(t\) at least \(\Omega(\log n)\) catalytic agents are in either \(A\) or \(B\) and the ratio between the numbers of agents supporting agents \(A\) and \(B\) is at least a constant, then most non-catalytic agents (up to \(O(n/\log n)\) agents) outputs w.h.p. the correct majority. The protocol needs with \(O(\log n \cdot \log \log n)\) states per agent, assuming that the number of catalytic agents in \(A\) and \(B\) does not change in the meantime. They also mention that with standard population
splitting $O(\log n + \log \log n)$ states are sufficient under the constraint that only a constant fraction of the agents store the output. Additionally the authors show that their protocol is robust to leaky transitions at a rate of $O(1/n)$. If the initial support of $A$ and $B$ states is $\Omega(\log^2 n)$ the authors can strengthen their results such that a ratio between the two base states of $1 + o(1)$ is sufficient.

## 2 Population Model and Problem Definitions

In the population model we are given a set $V$ of $n$ anonymous agents. At each time step two agents are chosen independently and uniformly at random randomly to interact. We assume that interactions between two agents $(u, v)$ are ordered and call $u$ the initiator and $v$ the responder. The interacting agents update their states according to a common transition function of their previous states. Formally, a population protocol is defined as a tuple of a finite set of states $Q$, a transition function $\delta : Q \times Q \to Q \times Q$, a finite set of output symbols $\Sigma$, and an output function $\omega : Q \to \Sigma$ which maps every state to an output. A configuration is a mapping $C : V \to Q$ which specifies the state of each agent. An execution of a protocol is an infinite sequence $C_0, C_1, \ldots$ such that for all $C_i$ there exist two agents $v_1, v_2$ and a transition $(q_1, q_2) \to (q'_1, q'_2)$ such that $C_i(v_1) = q_1, C_i(v_2) = q_2, C_{i+1}(v_1) = q'_1, C_{i+1}(v_2) = q'_2$ and $C_i(w) = C_{i+1}(w)$ for all $w \neq v_1, v_2$. The main quality criteria of a population protocol are the required number of states and the running time measured in interactions.

The goal of this paper is to develop protocols that are *loosely-stabilizing* according to the definitions of [24]. Let $C$ denote an arbitrary subset of all possible configurations. Consider an infinite sequence of configurations $C_0, C_1, \ldots$. For an arbitrary configuration $C_i \notin C$ the *convergence time* is defined as the smallest $t$ such that $C_{i+t} \in C$. Intuitively, the convergence time bounds the time it takes to reach a configuration in $C$ when starting from a configuration not in $C$. For an arbitrary configuration $C_i \in C$ the *holding time* $t_2$ is defined as the largest $t$ such that $C_{i+t} \notin C$. Intuitively, the holding time bounds the time during which the protocol remains in a configuration in $C$ when starting from a configuration in $C$.

> **Definition 1.** A protocol is loosely-stabilizing wrt. to a subset of configurations $C$ if the maximum convergence time over all possible configurations is w.h.p. less than $t_1$ and the minimum holding time over all configurations in $C$ is w.h.p. at least $t_2$.

### Phase Clocks

Phase clocks are used to synchronize population protocols. We assume a phase clock is implemented by simple counters clock[$u_1$], \ldots, clock[$u_n$] modulo $|Q|$ (see, e.g., [1, 7, 15, 19, 22]). Whenever clock[$u$] crosses zero, agent $u$ receives a so-called signal. These signals will divide the time into phases of $\Theta(n \log n)$ interactions each. We say a $(\tau, w)$-phase clock is synchronous in the time interval $[t_1, t_2]$ if every agent gets a signal every $\Theta(n \log n)$ interactions. More formally:

- Every agent receives a signal in the first $2 \cdot (w + 1) \cdot \tau \cdot n$ steps of the interval.
- Assume an agent $u$ receives a signal at time $t \in [t_1, t_2]$.
- For all $v \in V$, agent $v$ receives a signal at time $t_v$ with $|t - t_v| \leq \tau \cdot n$.
- Agent $u$ receives the next signal at time $t'$ with $(w + 1) \cdot \tau \cdot n \leq |t - t'| \leq 2 \cdot (w + 1) \cdot \tau \cdot n$.

The above definition divides the time interval $[t_1, t_2]$ into a sequence of subintervals that alternates between so-called burst-intervals and overlap-intervals.

- A burst-interval has length at most $\tau \cdot n$ and every agent gets exactly one signal.
- An overlap-interval consists of those time steps between two burst-intervals where none of the agents gets a signal. It has length at least $w \cdot \tau \cdot n$.

A burst-interval together with the subsequent overlap-interval forms a phase.
To define loosely-stabilizing phase clocks, we need to define the set of synchronous configurations \( \mathcal{C} \). Intuitively, we call a state \( C_t \) of a \((\tau, w)\)-phase clock at time \( t \) synchronous if the counters of all pairs of agents do not deviate much. More precisely, \( \text{clock}[u](t) - \text{clock}[v](t) \leq |Q| \cdot f(w, \tau) \) for all pairs of agents \((u, v)\) (Here, “\( \leq |Q| \)” denotes smaller w.r.t. the circular order modulo \(|Q|\).) We define \( f \) and give the formal definition of a synchronous configuration in the next section.

3 Clock Algorithm

![Figure 1] Schematic representation of the clock states.

In this section we introduce our phase clock protocol. For ease of notation, we assume in this section that the state space of an agent is \( Q \) Our \((\tau, w)\)-phase clock has a state space \( Q = \{0, \ldots, (21 + w + 6 \cdot \sqrt{10} + w) \cdot \tau - 1\} \). The clock states are divided into \((21 + w + 6 \cdot \sqrt{10} + w)\) hours, and each hour consists of \( \tau = \tau(c) = 36 \cdot (c + 4) \cdot \ln n \) minutes. The parameter \( c \geq 6 \) determines the error probability in each phase and thus the holding time (see Theorem 2). The parameter \( w \geq 0 \) can be chosen as needed by the application. As we will see, \( \tau \) is a multiple of the running time of the one-way epidemic (see Lemma 4) and \( w \cdot \tau \cdot v \) is the number of interactions in which our agents are synchronized. We divide the hours into three consecutive intervals (see Figure 1): the launching interval \( I_{\text{launch}} \) (first hour), the working interval \( I_{\text{work}} \) (last \( 6 + 2 \cdot \sqrt{10} + w \) hours) and the gathering interval \( I_{\text{gather}} \) (last \( 14 + w + 4 \cdot \sqrt{10} + w \) hours). We say that agent \( u \) is in one of the intervals whenever its clock counter \( \text{clock}[u] \) is in that interval. If the agents are either all in \( I_{\text{gather}} \), all in \( I_{\text{work}} \), or all in \( I_{\text{launch}} \), we say the configuration is homogeneous. For two agents \( u \) and \( v \) we define a distance \( d(u, v) = \min(\lvert\text{clock}[u] - \text{clock}[v]\rvert, \lvert Q \rvert - \lvert\text{clock}[u] - \text{clock}[v]\rvert) \) that takes the cyclic nature of the clock into account. This allows us to define synchronous configurations as follows.

Definition (Synchronous Configuration). A configuration \( C \) is called synchronous if and only if for all pairs of agents \((u, v)\) we have \( d(u, v) < \lvert I_{\text{launch}} \rvert + \lvert I_{\text{gather}} \rvert = (7 + 2 \cdot \sqrt{10} + w) \cdot \tau \).

Our clock works as follows. Assume agents \((u, v)\) interact. With two exceptions, agent \( u \) increments its counter \( \text{clock}[u] \) by one minute modulo \( |Q| \) (Rules 1 and 2). If, however, \( u \) is in \( I_{\text{gather}} \) and \( v \) is in \( I_{\text{launch}} \) then agent \( u \) adopts \( \text{clock}[v] \) (Rule 3): we say the agent hops. If \( u \) is in \( I_{\text{gather}} \) and \( v \) is in \( I_{\text{work}} \) then agent \( u \) returns to the beginning of \( I_{\text{gather}} \) (Rule 4): we say that the agent resets. We define that agent \( u \) receives a signal whenever its clock crosses the wrap-around from \( I_{\text{gather}} \) to \( I_{\text{launch}} \). Formally, our clock uses the following transitions.
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\[(q_1, q_2) \in (Q \setminus \{I_{\text{gather}}\}) \times Q: (q_1, q_2) \rightarrow (q_1 + 1, q_2) \text{ (step forward)} \]
\[(q_1, q_2) \in I_{\text{gather}} \times I_{\text{gather}}: (q_1, q_2) \rightarrow (q_1 + 1 \mod |Q|, q_2) \text{ (step forward)} \]
\[(q_1, q_2) \in I_{\text{gather}} \times I_{\text{launch}}: (q_1, q_2) \rightarrow (q_2, q_2) \text{ (hopping)} \]
\[(q_1, q_2) \in I_{\text{gather}} \times I_{\text{work}}: (q_1, q_2) \rightarrow (|I_{\text{launch}}| + |I_{\text{work}}|, q_2) \text{ (reset)} \]

Note that \(|I_{\text{work}}| = 2(|I_{\text{launch}}| + |I_{\text{gather}}|) + w \cdot \tau \) to have \(w \cdot \tau \cdot n\) homogeneous working configurations between two signals. On the other hand, \(|I_{\text{gather}}|/\tau = \Theta(\sqrt{|I_{\text{work}}|/\tau})\) which is necessary to apply Chernoff bounds. We chose \(|I_{\text{launch}}| = \tau\) for simplicity. On an intuitive level, the clock works as follows. Assume the clock is synchronized and all agents are in \(I_{\text{launch}}\). Now consider the next \(k = \Theta(n \cdot |Q|)\) interactions. All agents step forward according to Rule 1 until they reach \(I_{\text{gather}}\). The maximum distance between any agents grows during the \(k\) interactions but can still bounded by \(O(\sqrt{k/n}) = O(\sqrt{|w| \cdot \log n})\), w.h.p. via Chernoff bounds. Hence, due to the choice of \(w\) there is no agent left behind in \(I_{\text{launch}}\) when the first agent reaches \(I_{\text{gather}}\). Additionally, due to the size of \(I_{\text{gather}}\) when the last agent enters \(I_{\text{gather}}\) all of the other agents are still in \(I_{\text{gather}}\). As soon as the first agent reaches \(I_{\text{launch}}\), Rule 3 (agents hop onto agents in \(I_{\text{launch}}\)) ensures that all agents start the next phase without a large gap. Hence, there is an interaction after which all agents are in \(I_{\text{launch}}\) which brings us back to our initial configuration (all agents in \(I_{\text{launch}}\)).

Now we consider an asynchronous configuration \(C_t\) where the agents can be arbitrarily distributed over the \(|Q|\) states of the clock. The main idea for the recovery of our clock is as follows. We show that after \(O(n \log n)\) interactions there is a time \(t\) where \(I_{\text{launch}}\) is empty. After \(O(n \log n)\) additional steps most of the agents are in \(I_{\text{gather}}\): agents cannot hop since \(I_{\text{launch}}\) is empty, and they reset as soon as they interact with an agent in \(I_{\text{work}}\). They enter \(I_{\text{launch}}\) as soon as the first agent crosses 0 by increasing its clock counter.

We will show that the following two properties hold for our clock.

\[\text{Theorem 2.}\] Let \(\tau = 36 \cdot (c + 4) \cdot \ln n\) and let \(w\) be a sufficiently large constant. Let \(t_1, t_2\) with \(t_1 \leq t_2\) be two points in time and assume that the configuration \(C_{t_1}\) at time \(t_1\) is a homogeneous launching configuration and \(t_2 - t_1 \leq n^c\). Then the clock counters of the agents implement a synchronous \((\tau, w)\)-phase clock in the time interval \([t_1, t_2]\) w.h.p.

\[\text{Theorem 3.}\] The clock counters of the agents implement a \((O(n \cdot \log n), \Omega(\poly n))\)-loosely-stabilizing \((\Theta(\log n), w)\)-phase clock.

Note that our simulations suggest that the algorithm also works if \(\tau\) is smaller by a constant fraction. We prove Theorem 2 in Section 4 and Theorem 3 in Section 5.

**Auxiliary Results.** The one-way-epidemic is a population protocol with state space \(\{0, 1\}\) and transitions \((q_1, q_2) \rightarrow (\max(q_1, q_2), q_2)\). An agent in state 0 is called susceptible and an agent in state 1 is called infected. We say agent \(v\) infects agent \(u\) if \(v\) is infected and \(u\) initiates an interaction with \(v\). The following result is folklore, see, e.g., [7]. Additional details can be found in the full version of this paper.

\[\text{Lemma 4 (One-way-epidemic).}\] Assume an agent starts the one-way epidemic in step 1. All agents are infected after \(t = \tau/4 \cdot n\) many steps with probability at least \(1 - n^{-7+2c}\).

The following lemma bounds the number of interactions initiated by some fixed agent \(u\) among a sequence of \(t\) interactions. It is used throughout Sections 4 and 5 and follows immediately from Chernoff bounds (see [23]).
Lemma 5. Consider an arbitrary sequence of \( t \) interactions and let \( X_u \) be the number of interactions initiated by agent \( u \) within this sequence. Then
\[
\Pr[X_u < (1 + \delta) \cdot t/n] \geq 1 - n^{-\frac{12 \cdot (c+4) \cdot c^2}{n^2}} \quad \text{and} \quad \Pr[X_u > (1 - \delta) \cdot t/n] \geq 1 - n^{-\frac{18 \cdot (c+4) \cdot c^2}{n\cdot w \cdot \tau}}.
\]

4 Maintenance: Proof of Theorem 2

In this section we first show the following main result. At the end of the section we show how Theorem 2 follows from this proposition.

Proposition 6 (Maintenance). Consider our \((\tau, w)\)-phase clock for \( n \) agents with \( \tau = 36 \cdot (c+4) \cdot \ln n \) for any \( c \geq 6 \) and sufficiently large \( w \). Let configuration \( C_{t_1} \) be a homogeneous launching configuration. Then, with probability at least \( 1 - n^{-(c+1)} \), there exists a \( t_2 = \Theta(n \cdot w \cdot \log n) \) such that the following holds:

1. \( C_{t_1 + t_2} \) is a homogeneous launching configuration,
2. \( \forall t \in [t_1, t_1 + t_2] : \text{\( C_t \) is synchronous}, \)
3. in the time interval \( [t_1, t_1 + t_2] \) there exists a contiguous sequence of homogeneous working configurations of length \( w \cdot \tau \cdot n \).

We split the proof of Proposition 6 into two parts, Lemmas 7 and 8. The formal proof follows.

Proof. Assume the configuration \( C_{t_1} \) at time \( t_1 \) is a homogeneous launching configuration. Statements 1 and 2 of Proposition 6 follow immediately from Lemmas 7 and 8:

- It follows from Lemma 7 that the agents transition via a sequence of synchronous configurations into a homogeneous gathering configuration within \( \Theta(n \cdot w \cdot \log n) \) time w.h.p.
- It follow from Lemma 8 that the agents transition via a sequence of synchronous configurations back into a homogeneous launching configuration within \( \Theta(n \cdot w \cdot \log n) \) further time w.h.p.

It remains to show Statement 3. Recall that in a synchronous configuration all pairs of agents have distance (w.r.t. the circular order modulo \( |Q| \)) at most \( \Delta = (7 + 2 \cdot \sqrt{10 + w}) \cdot \tau \). Since \( |I_{work}| = w \cdot \tau + 2\Delta \) it immediately follows that there must be \( w \cdot \tau \cdot n \) interactions where all agents are in \( I_{work} \). This concludes the proof.

The following lemma establishes that w.h.p. all agents transition from a homogeneous launching configuration into a homogeneous gathering configuration via a sequence of synchronous configurations.

Lemma 7. Let \( C_t \) be a homogeneous launching configuration. Let \( t' = n \cdot \frac{|I_{launch}| + |I_{work}|}{1 - \frac{1}{2}} - \frac{1}{|I_{work}|/\tau} \). Then the following holds with probability at least \( 1 - n^{-(c+1)/2} \):

1. \( C_{t+t'} \) is a homogeneous gathering configuration and
2. \( \forall t'' \in [t, t+t'] : \text{\( C_{t''} \) is synchronous}. \)

Proof. In the following we assume w.l.o.g. \( t = 0 \). We prove the two statements separately.

Statement 1. Our goal is to show that after \( t' \) interactions all agents are in \( I_{gather} \) when we start from a homogeneous launching configuration \( C_0 \) at time \( t = 0 \). We first show that there is no agent left in \( I_{launch} \) when the first agent enters \( I_{gather} \). Let \( t_a \) be the first interaction in which an agent enters \( I_{gather} \). Note that before \( t_a \) all agents are either in \( I_{launch} \) or in \( I_{work} \) and thus the agents increase their counter by one whenever they initiate an interaction.

First we show that w.h.p. \( t_a \geq 2 \cdot \tau \cdot n \). Let \( X_u(2 \cdot \tau \cdot n) \) denote the number of interactions agent \( u \) initiates before time \( 2 \cdot \tau \cdot n \). From Lemma 5 it follows with \( \delta = 1 \) that \( X_u(2 \cdot \tau \cdot n) < 4 \cdot \tau \) with probability at least \( 1 - n^{-24 \cdot (c+4)} \). Since \( 4 \cdot \tau < |I_{work}| \), it holds that
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clock[u](2 \cdot \tau \cdot n) < |I_{\text{launch}}| + |I_{\text{work}}| in this case. Hence, agent u has not yet reached \( I'_{\text{gather}} \) with probability at least \( 1 - n^{-24(c+4)} \) at time \( t_a \). It follows from a union bound over all agents that no agent has reached \( I_{\text{gather}} \) with probability at least \( 1 - n^{-24(c+4)+1} \) at time \( t_a \).

Next we show that w.h.p. at time \( 2 \cdot \tau \cdot n \) all agents have left \( I_{\text{launch}} \). As before, let \( X_u(2 \cdot \tau \cdot n) \) denote the number of interactions agent \( u \) initiates before time \( 2 \cdot \tau \cdot n \). From Lemma 5 it follows with \( \delta = 1 \) that \( X_u(2 \cdot \tau \cdot n) > \tau \) with probability at least \( 1 - n^{-36(c+4)} \). Since \( \tau = |I_{\text{launch}}| \), it holds that clock[u](t + 2 \cdot \tau \cdot n) \geq |I_{\text{launch}}| in this case. Hence, agent \( u \) has left \( I_{\text{launch}} \) with probability at least \( 1 - n^{-36(c+4)} \) at time \( t_a \). Again, it follows from a union bound over all agents that all agents have left \( I_{\text{launch}} \) with probability at least \( 1 - n^{-36(c+4)+1} \) at time \( t_a \).

Let now \( t_b \) be the first interaction in which an agent enters the last minute of \( I_{\text{gather}} \) and observe that \( t_b > t_a \). Then, w.h.p. no agent is in \( I_{\text{launch}} \) during the time interval \([t + t_a, t + t_b]\). Therefore, agents cannot hop. Thus, by definition of \( t_b \), no agent can leave \( I_{\text{gather}} \) before time \( t_b \). All initiators must therefore either increase their counter by one or reset.

First we show that w.h.p. \( t_b > t' \). From Lemma 5 it follows with \( \delta = \left( \frac{2 \cdot \sqrt{|I_{\text{work}}|/\tau}}{3} \right)^{-1} \) that \( X_u(t') < |I_{\text{work}}| + |I_{\text{gather}}| \) with probability at least \( 1 - n^{-3(c+4)} \). (Note that we use \( (1 + \delta)/(1 - \delta) < 1/(1 - 2 \cdot \delta) \) for \( \delta < 0.5 \) and \( \{(I_{\text{launch}} + |I_{\text{work}}|)/(1 - 2 \cdot \delta) \geq |I_{\text{work}}| + (w + 5 \cdot \sqrt{10 + w + 16})/(\sqrt{10 + w + 16}) \} \). \( \tau < |I_{\text{work}}| + |I_{\text{gather}}| \). Thus, \( \text{clock}[u](t') \leq \text{clock}[u](0) + X_u(t') < |I_{\text{launch}}| - 1 + |I_{\text{work}}| + |I_{\text{gather}}| \) (which is the last state of \( I_{\text{gather}} \)) with probability at least \( 1 - n^{-3(c+4)} \). By a union bound, this holds for all agents with probability at least \( 1 - n^{-3(c+4)+1} \). No agent has reached \( I_{\text{gather}} \) before time \( t' \). All agents have left \( I_{\text{launch}} \) and thus, w.h.p. at time \( t' \) all agents have reached \( I_{\text{gather}} \). From Lemma 5 it follows for our choice of \( \delta \) that \( X_u(t') > |I_{\text{launch}}| + |I_{\text{work}}| \) with probability at least \( 1 - n^{-9(c+4)/2} \). Thus, \( \text{clock}[u](t') \geq \text{clock}[u](0) + X_u(t') > 0 + |I_{\text{launch}}| + |I_{\text{work}}| \), with probability at least \( 1 - n^{-9(c+4)/2} \). By a union bound, this holds for all agents with probability at least \( 1 - n^{-17(c+4)/2} \).

Together it follows that at time \( t' \) no agent has left \( I_{\text{gather}} \) but all agents have entered it with probability at least \( 1 - n^{-5(c+3)} \). Therefore, \( C_T \) is a homogeneous gathering configuration.

**Statement 2.** Recall that a synchronous configuration \( C \) is defined as a configuration where max\( \{d(u,v)\} < |I_{\text{launch}}| + |I_{\text{gather}}| \). As before, let \( X_u(i) \) denote the number of interactions agent \( u \) initiates before time \( i \). Now fix a time \( t \leq t' \) and a pair of agents \( (u,v) \) with \( X_u(t) < X_v(t) \). Use Lemma 5 to bound the deviation of \( X_u(t) \) and \( X_v(t) \) at time \( t \) as follows: \( \Pr[X_u(t) > t/n - |I_{\text{gather}}|/2] \geq 1 - n^{-6(c+4)} \) and \( \Pr[X_v(t) < t/n + |I_{\text{gather}}|/2] \geq 1 - n^{-4(c+4)} \). Therefore, \( |X_u(t) - X_v(t)| < |I_{\text{gather}}| \) with probability at least \( 1 - n^{-4(c+4) - 6(c+4)} \).

Note that Lemma 5 allows us to bound the deviation in the numbers of interactions initiated by agents \( u \) and \( v \). However, this does not immediately give a bound on the difference of the clock counters \( |\text{clock}[v](t) - \text{clock}[u](t)| \). To bound the deviation of clock counters (by \( |I_{\text{launch}}| + |I_{\text{gather}}| \)), we therefore distinguish three cases.

First, assume that neither \( u \) nor \( v \) has reached \( I_{\text{gather}} \) at time \( t \). Then \( \text{clock}[u](t) = \text{clock}[u](0) + X_u(t) \) and \( \text{clock}[v](t) = \text{clock}[v](t) + X_v(t) \). Observe that by the assumption of the lemma, both \( u \) and \( v \) are in \( I_{\text{launch}} \) at time \( t = 0 \) and thus \( |\text{clock}[v](0) - \text{clock}[u](0)| < |I_{\text{launch}}| \). Together with the above bound on \( |X_u(t) - X_v(t)| \) we get \( |\text{clock}[v](t) - \text{clock}[u](t)| < |I_{\text{launch}}| + |I_{\text{gather}}| \).

Secondly, assume that \( u \) has not reached \( I_{\text{gather}} \) but \( v \) has reached \( I_{\text{gather}} \) at time \( t \). Then \( \text{clock}[u](t) = \text{clock}[u](0) + X_u(t) \). For \( \text{clock}[v](t), \) however, it might have occurred that \( v \) has reset in some interactions before time \( t \). Nevertheless, the clock counter of \( v \) is bounded by
the number of initiated interactions such that \( \text{clock}[v](t) \leq \text{clock}[u](t) + X_v(t) \). (Note that \( v \) can only increment its clock counter or reset its value; hopping is not possible since we have shown in the proof of the first statement that \( I_{\text{launch}} \) is empty when the first agent enters \( I_{\text{gather}} \).) Therefore, we get again \( |\text{clock}[v](t) - \text{clock}[u](t)| < |I_{\text{launch}}| + |I_{\text{gather}}| \).

Finally, assume that both \( u \) and \( v \) are in \( I_{\text{gather}} \) at time \( t \). Then \( |\text{clock}[v](t) - \text{clock}[u](t)| \leq |I_{\text{gather}}| < |I_{\text{launch}}| + |I_{\text{gather}}| \) is trivially true.

There are no further cases: in the proof of the first statement we have shown that all agents transition from a homogeneous launching configuration to a homogeneous gathering configuration during the time interval \([0, \tilde{t}']\). The result now follows from a union bound over all \( o(n^2) \) points in time \( t \leq \tilde{t}' \) and all \( n \cdot (n - 1) \) pairs of agents.

The following lemma is the main technical contribution of this section. It establishes that w.h.p. all agents transition from a homogeneous gathering configuration into a homogeneous launching configuration via a sequence of synchronous configurations. Consider a homogeneous gathering configuration and recall that whenever an agent hops from \( I_{\text{gather}} \) into \( I_{\text{launch}} \) it adopts the state of the responder. The main difficulty is to show that all agents hop into \( I_{\text{launch}} \) before the first agent leaves \( I_{\text{launch}} \).

**Lemma 8.** Let \( C_t \) be a homogeneous gathering configuration. Then with probability at least \( 1 - n \cdot (c + 1)/2 \) the following holds:

1. There exists a \( t_0 = O(n \cdot \sqrt{w} \cdot \log n) \) such that the first agent enters \( I_{\text{launch}} \) at time \( t + t_0 \).
2. There exists a \( t' \leq \tau/4 \cdot n \) such that \( C_{t+t_0+t'} \) is a homogeneous launching configuration.
3. \( \forall t' \in [t, t + t'] : C_{t'} \) is synchronous.

**Proof.** We prove first show Statement 1, Statement 2 and Statement 3 are shown together.

**Statement 1.** Let \( t_0 \) be defined such that the first agent \( u \) leaves \( I_{\text{gather}} \) at time \( t + t_0 \). Since \( C_t \) is a homogeneous gathering configuration, \( I_{\text{launch}} \) is empty at time \( t \) and hence agent \( u \) can only leave \( I_{\text{gather}} \) by increasing its counter. In every interaction before time \( t + t_0 \) some agent has to increase its state by one. Thus \( t_0 \leq n \cdot |I_{\text{gather}}| = O(n \cdot \sqrt{w} \cdot \log n) \).

**Statement 2+3.** We continue our analysis at time \( t_0 \) and again assume w.l.o.g. for the sake of brevity of notation that \( t_0 = 0 \). Note that at that time exactly one agent is in state 0 and all remaining agents are still in \( I_{\text{gather}} \). We show the following: there exists a time \( \tilde{t} = \tau/4 \cdot n \) such that at time \( \tilde{t} \) all agents are in \( I_{\text{launch}} \) (Recall that \( |I_{\text{launch}}| = \tau \cdot n \)). To do so we first define a simplified process with the same state space \( Q \), however, we refer to the last state of \( I_{\text{launch}} \) as \( \text{stop} \). Agents in \( \text{stop} \) never change their state (which renders the states of \( I_{\text{work}} \) unreachable). The formal definition of the simplified process is as follows. Rule 2 and 3 are identical to the original process and Rule 1 and 4 are modified as follows.

\[
\begin{align*}
(q_1, q_2) \in (I_{\text{launch}} \setminus \{ \text{stop} \}) \times Q : & \quad (q_1, q_2) \to (q_1 + 1, q_2) & \quad \text{(step forward)} & \quad (1) \\
(q_1, q_2) \in \{ \text{stop} \} \times Q : & \quad (q_1, q_2) \to (q_1, q_2) & \quad \text{(stopping)} & \quad (4)
\end{align*}
\]

For this simplified process we show a lower bound: after \( \tilde{t} = \Theta(n \cdot \log n) \) interactions all agents are in \( I_{\text{launch}} \). Then we show (for the simplified process) an upper bound: in \( C_t \) none of the agents are in state \( \text{stop} \). A simple coupling of the simplified process and the original process shows that under these circumstances none of the agents entered \( I_{\text{work}} \) for our original process. This finishes the proof with \( \tilde{t}' = \tilde{t} \).

**Lower Bound.** In the simplified process agents can enter \( I_{\text{launch}} \) either via hopping or by making enough steps forward on their own. From Lemma 4 it follows that all agents enter \( I_{\text{launch}} \) after at most \( \tilde{t}' = \tau/4 \cdot n \) interactions with probability at least \( 1 - n \cdot (5 + c) \). (For
the upper bound, one can simply discard setting the clock counter to zero when an agent enters $I_{\text{launch}}$ by increasing its counter.) Showing that none of the agents are in state $\text{stop}$ is much harder. Due to the hopping the clock counters of agents in $I_{\text{launch}}$ are highly correlated. Nevertheless, we can show that the clock counters of each agent can be majorized by independent binomially distributed random variables as follows.

**Upper Bound.** Let $u_i$ be the $i$'th agent that enters $I_{\text{launch}}$ and let $t_i$ be the time when $u_i$ enters $I_{\text{launch}}$. Let furthermore $X_i(t)$ be a random variable for the clock counter of agent $u_i$ in $I_{\text{launch}}$ in the time interval $[0, t]$. Formally, we define for a time step $t$ that $X_i(t) = 0$ if $u_i$ is in $I_{\text{gather}}$ and $X_i(t) = \text{clock}[u_i](t)$ if $u_i$ is in $I_{\text{launch}}$. We show by induction on $i$ that $X_i(t)$ is majorized by a random variable $Z_i(t)$ with binomial distribution $Z_i(t) \sim \text{Bin}(t, 1/n \cdot (1 + 1/(n - 1))^{i-1})$, i.e., $\Pr[X_i(t) > x] \leq \Pr[Z_i(t) > x]$ for all $x \geq 0$. Ultimately, our goal is to apply Chernoff bounds to $Z_i(t)$ which shows that agent $u_i$ does not reach $\text{stop}$ w.h.p. The statement for the simplified process then follows from a union bound over all agents w.h.p.

**Base Case.** For the base case we consider all agents that enter $I_{\text{launch}}$ on their own by incrementing their counters to 0 (modulo $|Q|$) in $I_{\text{gather}}$. Fix such an agent $u_i$. It holds that $X_i(t)$ for $t \geq t_i$ has binomial distribution $X_i(t) \sim \text{Bin}(t - t_i, 1/n)$. Therefore, $X_i(t) \prec Z_i(t)$ as claimed.\textsuperscript{2} (Intuitively, this means that the clock counter of any other agent $u_i$ with $i > 1$ that enters $I_{\text{launch}}$ at time $t_i > 0$ is majorized by the clock counter of an agent which enters $I_{\text{launch}}$ at time $t_1 = 0$ and increments its counter with probability $1/n$.)

**Induction Step.** For the induction step we now consider all agents that enter $I_{\text{launch}}$ by hopping onto some other agent in $I_{\text{launch}}$. Fix such an agent $u_i$. Let $S_i$ be the event that agent $u_i$ is the $i$'th agent that enters $I_{\text{launch}}$. Let furthermore $t_i$ be the time when $u_i$ enters $I_{\text{launch}}$. We condition on $S_i$ and observe that agent $u_i$ enters $I_{\text{launch}}$ by hopping onto some other agent $u_j \in \{ u_1, \ldots, u_{i-1} \}$. Intuitively, we would now like to exploit the fact that the counter of agent $u_i$ is copied at time $t_i$ from agent $u_j$ such that $X_i(t_i) = X_j(t_i)$. Unfortunately, we must be extremely careful here: conditioning on $S_i$ alters the probability space! (For example, under $S_i$ the agent $u_i$ with $i \geq 2$ cannot initiate an interaction with agent $u_1$ before agent $u_2$ does, since $S_i$ rules out that $u_i$ enters $I_{\text{launch}}$ before agent $u_2$.) We account for the modified probability space as follows.

Let $\Omega_{S_i}(t)$ be the probability space of possible interactions conditioned on $S_i$ at time $t \leq \hat{t}$. Without the conditioning on $S_i$, the probability space $\Omega(t)$ at time $t$ contains all (ordered) pairs of agents with $|\Omega(t)| = n \cdot (n - 1)$. When conditioning on $S_i$, the event $S_i$ rules out that agent $u_i$ interacts with any other agent $u_j \in I_{\text{launch}}$ before time $t_i$. In particular, agent $u_i$ cannot interact with another agent $u_j$ with $j < i$ during the time interval $[t_j, t_i]$. In order to give a lower bound on $|\Omega_{S_i}(t)|$, we exclude all $(n - 1)$ interactions $(u_i, u_j)$ for $j \in [n]$ from $\Omega(t)$. Hence $|\Omega_{S_i}(t)| \geq n \cdot (n - 1) - (n - 1) = (n - 1)^2$ for any time $t \leq t_i$. (The probability space after time $t_i$ is not affected by conditioning on $S_i$, but the majorization holds nonetheless.) We now consider the event $E_{\hat{t}}$ for $\hat{t} \leq t_i$ that the interaction at time $\hat{t}$ increments $X_j(t)$ by 1 (recall that $u_j$ is the agent onto which $u_i$ hopped). It then holds for the reduced probability space $\Omega_{S_i}$ that $\Pr[E_{\hat{t}} \mid S_i] \leq \Pr[E_{\hat{t}}] \cdot |\Omega(t)|/|\Omega_{S_i}(t)|$. (Note that $\Omega_{S_i}$ is still a uniform probability space.) We calculate

$$\frac{|\Omega(t)|}{|\Omega_{S_i}(t)|} = \frac{n \cdot (n - 1)}{(n - 1)^2} = 1 + \frac{1}{n - 1}$$

\textsuperscript{2} The expression $X \prec Y$ means that the random variable $X$ is majorized by the random variable $Y$. 
Conclusions. From the induction it follows that for each agent $u_i$ with probability $\Pr[E_i | S_i] \leq \Pr[E_i] \cdot (1 + 1/(n-1))$ for $\hat{t} \leq t_i$. Therefore, we use the induction hypothesis (that describes $X_j(t_i)$) and get $X_i(t_i) \sim Z_i(t_i)$, where $Z_i(t_i) \sim \text{Bin}\left(t_i, 1/n \cdot (1 + 1/(n-1))^{t_i-1}\right)$. Similarly, we define $E_i$ for $\hat{t} \geq t_i$ to be the event that $u_i$ increments its counter in $I_{\text{launch}}$. Observe that $\Pr[E_i] \leq 1/n$ for $\hat{t} > t_i$. It follows that $X_i(t) \sim Z_i(t)$ with distribution $Z_i(t) \sim \text{Bin}(\hat{t}, 1/n \cdot (1 + 1/(n-1))^{\hat{t}-1})$ for $t \leq \hat{t}$ as claimed. This concludes the induction.

**Proof of Theorem 2.** The proof of Theorem 2 follows readily from the main result of this section, Proposition 6.

Assume the configuration at time $t_1$ is a homogeneous launching configuration. Then from Proposition 6 it follows w.h.p. that after $t_2 = \Theta(n \cdot w \cdot \log n)$ interactions the configuration $C_{t_2}$ is again a homogeneous launching configuration, and all configurations in $[t_1, t_2]$ are synchronous. From Statement 3 it follows that no agent receives a signal in a contiguous subinterval $[t_1', t_2'] \subset [t_1, t_2]$ of length $t_2' - t_1' = w \cdot \tau \cdot n$. This shows that we have w.h.p. the required overlap according to the definition of synchronous $(\tau, w)$-phase clocks.

From Lemma 8 it follows w.h.p. that all agents transition from a homogeneous gathering configuration into a homogeneous launching configuration within $\tau/4 \cdot n$ interactions. Recall that whenever an agent crosses zero, it receives a signal. Therefore, when all agents transition from a homogeneous gathering configuration into a homogeneous launching configuration via a sequence of synchronous configurations, all agents receive exactly one signal, and the time between two signals of two agents $(u, v)$ is w.h.p. at most $\tau/4 \cdot n$. This shows that we have w.h.p. the required bursts according to the definition of synchronous $(\tau, w)$-phase clocks.

Together, the counters of our clock implement a synchronous $(\tau, w)$-phase clock in $[t_1, t_2]$ with probability $n^{-c}$. It follows from an inductive argument that the clock counters implement a synchronous $(\tau, w)$-phase clock during the $n^c$ interactions that follow time $t_1$ w.h.p.

**5 Recovery: Proof of Theorem 3**

In this section we first show the following main result. At the end of the section we show how Theorem 3 follows from this proposition.
Proposition 9 (Recovery). Consider our \((\tau, w)\)-phase clock with \(n\) agents and sufficiently large \(c\) and \(w\). Let \(C_t\) be an arbitrary configuration. Then with probability at least \(1 - 1/n\), there exists a \(t_2 = O(n \cdot w \cdot \log n)\) such that \(C_{t_1 + t_2}\) is a homogeneous launching configuration.

We say a configuration is an almost homogeneous gathering configuration if no agent is in \(I_{\text{launch}}\) and at least \(0.9 \cdot n\) many agents are in \(I_{\text{gather}}\). We start our analysis by showing that within \(t = O(n \cdot w \cdot \log n)\) interactions, we reach an almost homogeneous gathering configuration \(C_{t_1 + t_2}\).

Lemma 10. Let \(C_t\) be an arbitrary configuration. Then with probability at least \(1 - 1/(3n)\), there exists a \(t' = O(n \cdot w \cdot \log n)\) such that \(C_{t + t'}\) is an almost homogeneous gathering configuration.

Proof Sketch. The main idea of the proof is as follows. If there are not too many agents in \(I_{\text{gather}}\), the reset rule prevents agents from reaching the end of \(I_{\text{gather}}\). Agents may still enter \(I_{\text{launch}}\) by hopping, but if no agent enters state 0, eventually there is no agent left in state 0 to hop on. Then the same argument applies to state 1, and so on. Eventually, there are no agents left in \(I_{\text{launch}}\) to hop onto. This means the agents are trapped in \(I_{\text{gather}}\) until a sufficiently large number of agents enters \(I_{\text{gather}}\) which renders resetting quite unlikely again. The resulting configuration is what we call an almost homogeneous gathering configuration.

Next, we show that from an almost homogeneous gathering configuration we reach a homogeneous gathering configuration in \(O(n \cdot w \cdot \log n)\) interactions. From Lemma 8 in Section 4 it then follows that we reach a homogeneous launching configuration in an additional number of \(O(n \cdot \log n)\) interactions.

Lemma 11. Let \(C_t\) be an almost homogeneous gathering configuration. Then with probability at least \(1 - 1/(3n)\), there exists a \(t' = \Theta(n \cdot w \cdot \ln n)\) such that \(C_{t + t'}\) is a homogeneous gathering configuration.

Proof Sketch. If \(C_t\) is an almost homogeneous gathering configuration, then there are no agents in \(I_{\text{launch}}\) and at least \(0.9 \cdot n\) many agents in \(I_{\text{gather}}\). Thus, agents cannot hop until an agent enters \(I_{\text{launch}}\) on its own. Now there are two cases. If no agent enters \(I_{\text{launch}}\) on its own before the last agent enters \(I_{\text{gather}}\), we are done: this is by definition of a homogeneous gathering configuration. Otherwise, we will show that a large fraction of agents leave \(I_{\text{gather}}\) together. This large fraction behaves similar as in the proof of the maintenance. The remaining agents have a small head start but then they are again trapped in \(I_{\text{gather}}\) until the bulk of agents arrives. Once the bulk of agents enters \(I_{\text{gather}}\) we have reached a homogeneous gathering configuration and all agents start to run through the clock synchronously.

Proof of Theorem 3. The proof of Theorem 3 follows readily from the main result of this section, Proposition 9. Observe that \(\tau = \Theta(\log n)\). According to Proposition 9, our clock recovers to a homogeneous launching configuration in \(O(n \cdot \log n)\) interactions. By Theorem 2, this marks the beginning of a time interval in which the agents implement a synchronous \((\tau, w)\)-phase clock. It follows immediately from Theorem 2 that this interval has length \(n^c\). Together, this implies that our \((\tau, w)\)-phase clock is a \((O(n \cdot \log n), \Omega(\text{poly}(n)))\)-loosely-stabilizing \((\Theta(\log n), w)\)-phase clock.

Adaptive Majority Problem

In this section we consider the adaptive majority problem. At any time, every agent has as input either an opinion \((A\) or \(B)\) or it has no input, in which case we say it is undecided \((U)\). During the execution of the protocol, the opinions of the agents can change. In the
adaptive majority problem, the goal is that all agents output (at all times) the opinion which is dominant among all inputs. In this setting we present a loosely-stabilizing protocol that solves the adaptive majority problem. We define a loosely-stabilizing adaptive majority protocol according to Definition 1 by defining $C$ as all configurations where all agents output the correct majority opinion. Recall that the performance of a loosely-stabilizing protocol is measured in terms of the convergence time and the holding time. Note that the loose-stabilization comes from an application of our phase clock. The phase clocks guarantee synchronized phases for polynomial time. During this time we say a configuration $C$ is correct w.r.t. the adaptive majority problem if the following conditions hold. Suppose there is a sufficiently large bias towards one opinion. Then every agent in a correct configuration outputs the majority opinion. Otherwise, if there is no sufficiently large bias, we consider any output of the agents as correct. In this setting, we show the following result: We show that a $(O(\log n), \text{poly}(n))$-loosely-stabilizing algorithm exists that solves adaptive majority, using $O(\log n)$ states per agent.

6.1 Our Protocol

Our protocol is based on the $(\tau, w)$-phase clock defined in Section 3 with $w = 566$. In addition to the states required by the clock, every agent $v$ has three variables $\text{input}[v]$, $\text{opinion}[v]$, and output $\text{output}[v]$. The variable $\text{input}[v]$ always reflects the current input to the agent, $\text{opinion}[v]$ holds the current opinion of agent $v$, and $\text{output}[v]$ defines the current output value of agent $v$. All three variables take values in $\{A, B, U\}$. $A$ and $B$ stand for the corresponding opinions and $U$ stands for undecided. The state space of the protocol is $Q_c \times \{A, B, U\}^3$ where $Q_c$ is the state space of our clock for $\tau = 36 \cdot (c+4) \ln n$ and $w = 566$.

We use the $(\tau, w)$-phase clock to synchronize the agents. Then it follows from Proposition 6 that all configurations are synchronous w.h.p. Observe that in a synchronous configuration for our choice of parameters the clock counters of agents do not deviate by more than $\Delta = 55 \cdot \tau$. This allows us to define three subphases of $I_{\text{work}}$, where agents execute three different protocols, as follows. We split the working interval $I_{\text{work}}$ into six contiguous subintervals of equal length. The clock counters $\text{clock}[u]$ allows us to define a simple interface to the phase clock for each agent $u$ as follows. The variable $\text{subphase}[u]$ for each agent $u$ is then defined as follows. We set $\text{subphase}[u] = 1$ if $\text{clock}[u]$ is in the first subinterval of $I_{\text{work}}$, $\text{subphase}[u] = 2$ if $\text{clock}[u]$ is in the third subinterval of $I_{\text{work}}$, and $\text{subphase}[u] = 3$ if $\text{clock}[u]$ is in the fifth subinterval of $I_{\text{work}}$. Otherwise, $\text{subphase}[u] = \bot$. The clock now assures a clean separation into these subphases such that no two agents perform a different protocol at any time w.h.p. Additionally, we will show the overlap within each subphase is long enough such that the subprotocols for the corresponding subphases succeed w.h.p.

On an intuitive level, our protocol works as follows. At the beginning of the phase, the input is copied to the opinion variable. In the first protocol, the support of opinions $A$ and $B$ is amplified until no undecided agents are left. We call this the Pólya Subphase. In the second protocol, agents with opposite opinions cancel each other out, becoming undecided. We call this the Cancellation Subphase. Finally, in the third protocol the single remaining opinion is amplified again. We call this the Broadcasting Subphase. The resulting opinion is copied to the output variable after the working interval $I_{\text{work}}$. Formally, our protocol is specified in Algorithm 1.

In the remainder of this section, we let $A_t$ and $B_t$ denote the number of agents $u$ with $\text{opinion}[u] = A$ and $\text{opinion}[u] = B$, respectively, at time $t$. Analogously, we let $A^u_t$ and $B^u_t$ denote the number of agents $u$ with $\text{input}[u] = A$ and $\text{input}[u] = B$, respectively, at time $t$. We now state our main result for this section.
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Algorithm 1 Interaction of agents \((u, v)\) in the adaptive majority protocol.

1. update clock\([u]\) according to Rules 1–4 with \(w = 566\)
2. if Agent \(u\) receives a signal then opinion\([u]\) ← input\([u]\)
3. if subphase\([u] = 1 \land opinion\([u]\) = \(U\) then opinion\([u]\) ← opinion\([v]\)
4. if subphase\([u] = 2 \land opinion\([v]\) ≠ opinion\([u]\) ∧ opinion\([u]\), opinion\([u]\) ≠ \(U\) then
5. opinion\([u]\), opinion\([v]\) ← \(U\)
6. if subphase\([u] = 3 \land opinion\([u]\) = \(U\) then opinion\([u]\) ← opinion\([v]\)
7. if clock\([u]\) ≥ \(|I_{\text{launch}}| + |I_{\text{work}}|\) then output\([u]\) ← opinion\([u]\)

Theorem 12. Algorithm 1 is a \((O(n \log n), \Omega(\text{poly}(n)))\)-loosely stabilizing adaptive majority protocol.

Note that we match the results of [3] for \(r = 1/n\), a multiplicative bias of \(1 + \beta = 1 + 1/\log n\) and \(\Omega(\log^2 n)\) many agents \((\alpha \geq \log n)\). In contrast to their protocol, every agent outputs at any point of time the correct majority opinion, w.h.p. But, again in contrast to their work, we do not consider leaky transitions.

6.2 Analysis

In the following analysis, we consider an arbitrary but fixed phase. We condition on the event that the clock is synchronized according to Proposition 6. We show the following main result, and later in this section we describe how Theorem 12 follows from it this proposition. The proofs for the statements in this section can be found in the full version of this paper.

Proposition 13. Assume that at time \(t_1\) the agents are in a homogeneous launching configuration and we have \(A_t_1 ≥ \alpha \cdot \log n\) and \(A_{t_2} ≥ (1 + \beta) \cdot B_{t_1}\). If \(\alpha\) and \(\beta\) are large enough constants, then there exists a \(t_2 = \Theta(n \cdot w \cdot \log n)\) such that all agents output \(A\) in configuration \(C_{t_1 + t_2}\) with probability \(1 - n^{-c}\).

The analysis is split into three parts, one for the Pólya Subphase, one for the Cancellation Subphase, and one for the Broadcasting Subphase. First, we assume that no changes in the input occur. Then we generalize our results: we adopt the undecided state dynamics introduced in [8], and show how we can tolerate input changes at various rates.

Observe that we get a separation between the subphases from the guarantees of the phase clock in Theorem 2: no two agents are more than \(1/6\) of \(I_{\text{work}}\) apart. We also know that every agent has copied its input at the beginning of the phase before the first agent enters the first subphase. The total time for the three subphases (including the separation time) is sufficiently large such that every agent has finished its work before the next phase starts.

When we refer to a distribution before a subphase, we mean the distribution at the time just before the first agent performs an interaction in that subphase. Analogously, when we refer to a distribution after a subphase, we mean the distribution at the time when the last agent has performed an interaction in that subphase. Recall that in the following analysis, we let \(A_t\) and \(B_t\) denote the number of agents \(u\) with opinion\([u]\) = \(A\) and opinion\([u]\) = \(B\), respectively, at time \(t\). Furthermore, we let \(s_i\) and \(e_i\) (for start and end) be the first and the last time, respectively, when an agent performs an interaction in the \(i\)th subphase.

Subphases. We first consider the Pólya Subphase, where we model the process by means of so-called Pólya urns. Pólya urns are defined as follows. Initially, the urn contains \(a\) red balls and \(b\) blue balls. In each step, a ball is drawn uniformly at random from the urn. The
ball’s color is observed, and it is returned into the urn along with an additional ball of the same color. The Pólya-Eggenberger distribution $PE(a, b, m)$ describes the total number of red balls after $m$ steps of this urn process.

This observation allows us to apply concentration bounds to the opinion distribution after the Pólya Subphase. Recall that $s_1$ and $e_1$ are the first and the last time steps, respectively, when an agent performs an interaction in the Pólya Subphase. We get the following lemma.

\begin{lemma}
Let $a = A_{s_1-1}$ and $b = B_{s_1-1}$. For any constant $\beta > 0$ there exists a constant $\alpha$ such that if $a > \alpha \cdot \log n$ and $a > (1 + \beta) \cdot b$ then $A_{e_1} - B_{e_1} = \Omega(n)$ with probability at least $1 - n^{-(c+2)}$.
\end{lemma}

Next we consider the Cancellation Subphase. The goal is to remove any occurrence of the minority opinion. Whenever an agent with opinion $A$ interacts with another agent with opinion $B$, both agents become undecided. Formally, we show the following lemma.

\begin{lemma}
If $A_{s_2-1} - B_{s_2-1} = \Omega(n)$ then $A_{e_2} = \Omega(n)$ and $B_{e_2} = 0$ with probability at least $1 - n^{-(c+2)}$.
\end{lemma}

Finally we consider the Broadcasting Subphase. The goal is to spread the (unique) remaining opinion to all other agents. Whenever an undecided agent $u$ interacts with another agent $v$ that has an opinion, agent $u$ adopts the opinion of agent $v$. This leads to a configuration where every agent has the majority opinion w.h.p. Formally, we show the following lemma.

\begin{lemma}
If $A_{e_2} = \Omega(n)$ and $B_{e_2} = 0$, then $A_{e_3} = n$ and $B_{e_3} = 0$ with probability at least $1 - n^{-(c+2)}$.
\end{lemma}

We have now everything we need to prove Proposition 13 and in turn Theorem 12.

\textbf{Proof of Proposition 13.} We assume the configuration at time $t_1$ is a homogeneous launching configuration. From Proposition 6 it follows that all configurations in the time interval $[t_1, t_1 + t_2]$ for some $t_2 = \Theta(n \cdot w \cdot \log n)$ are synchronous with probability at least $1 - n^{-(c+1)}$. This means that the three subphases are strictly separated as explained above. It therefore follows (each with probability at least $1 - n^{-(c+2)}$) from Lemma 14 that after the Pólya Subphase no agent is undecided, from Lemma 15 that after the Cancellation Subphase no agent has opinion $B$, and from Lemma 16 that after the Broadcasting Subphase all agents have opinion $A$. Once all agents have opinion $A$, this becomes the output when the agents enter $I_{\text{gather}}$. Together, this shows that all agents output the majority opinion after $\Theta(n \cdot w \cdot \log n)$ interactions with probability at least $1 - n^{-c}$. \hfill $\blacksquare$

\textbf{Proof of Theorem 12.} Here we show the result without input changes. Fix a time $t_1$ and assume the agents are in an arbitrary configuration at time $t_1$. From Theorem 3 it follows the agents enter a synchronous configuration within $O(n \log n)$ interactions and stay in synchronous configurations for $\poly(n)$ time w.h.p.

Now we consider a fixed synchronized phase $i < \poly(n)$ of our phase clock. It follows from Proposition 13 that all agents enter a correct configuration at the end of phase $i$ with probability at least $1 - n^{-c}$. (Recall that in a correct configuration all agents have to output the majority opinion if there is a sufficiently large bias. Without a bias, any output constitutes a correct configuration.) From the guarantees of the phase clock it follows that the first synchronized phase starts within $O(n \log n)$ time after time $t_1$ w.h.p. This shows a convergence time of $O(n \log n)$. From a union bound over at most $n^{c-1}$ phases it follows that the protocol is in a correct configuration for $\poly(n)$ interactions w.h.p. This shows a holding time of $\poly(n)$. 

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The proof with input changes can be found in the full version. The main idea is that we bound the number of input changes in $\Theta(n \log n)$ interactions by a simple application of Chernoff bounds.

**Improving the Bound.** In order to show-case the simplicity of the application of our phase clock, we have presented a simplistic protocol, where we assumed a constant factor bias towards the majority opinion. If we replace the Cancellation Subphase and the Broadcasting Subphase (lines 6 to 9 in Algorithm 1) with the undecided state dynamics introduced in [8] we can show a tighter result.

Formally, we show the following statement, the proof can be found in the full version of this paper.

**Observation 17.** If we use the undecided state dynamics, Proposition 13 also holds for $\alpha = \Omega(\beta^{-2})$ provided that $\beta = \Omega(n^{-1/4+\varepsilon})$.

This means that we can solve the adaptive majority problem with a multiplicative bias of $1 + \beta = 1 + 1/\log n = 1 + o(1)$ and asymptotically at least $\Omega(\log^2 n)$ many agents with opinion $A$ or $B$ (assuming sufficiently large constants). Hence we achieve similar results as in [3] for a model without leaky transitions.

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