Orthonormal Polynomials on the Unit Circle
and Spatially Discrete Painlevé II Equation

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Abstract

We consider the polynomials \( \phi_n(z) = \kappa_n (z^n + b_{n-1}z^{n-1} + \cdots) \)
ortonormal with respect to the weight \( \exp(\sqrt{\lambda}(z + 1/z))dz/2\pi iz \)
on the unit circle in the complex plane. The leading coefficient \( \kappa_n \) is found to satisfy a difference-differential (spatially discrete) equation which is further proved to approach a third order differential equation by double scaling. The third order differential equation is equivalent to the Painlevé II equation. The leading coefficient and second leading coefficient of \( \phi_n(z) \) can be expressed asymptotically in terms of the Painlevé II function.

1 Introduction

In this paper, we discuss the orthonormal polynomials with respect to the weight \( \exp(\sqrt{\lambda}(z + 1/z))dz/2\pi iz \) on the unit circle in the complex plane [1].

\[
\int \phi_n(z)\phi_m(z)e^{\sqrt{\lambda}(z+\frac{1}{z})} \frac{dz}{2\pi iz} = \delta_{nm}, \quad m, n \geq 0, \tag{1.1}
\]

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where the integral is over the unit circle, and \( \overline{\phi_m(z)} \) means the complex conjugate of \( \phi_m(z) \), \( \lambda \) is a positive parameter. The polynomials \( \phi_n(z) = \kappa_n z^n + \cdots \) have explicit representation in terms of the Toeplitz determinants \( D_n(\lambda) = D_n\left(\exp(2\sqrt{\lambda}\cos \theta)\right) \). Here we are interested in analyzing the properties of the leading coefficient \( \kappa_n(\lambda) \). As discussed in [1], \( \kappa_n^2 \) can be expressed in terms of the Toeplitz determinants, \[ \kappa_n^2 = \frac{D_{n-1}(\lambda)}{D_n(\lambda)}. \] (1.2)

And by the Szegő strong limit theorem, \( \lim_{n \to \infty} D_n(\lambda) = e^{\lambda} \). So we have \( \kappa_n^2 \to 1 \), as \( n \to \infty \).

From the orthonormal property and the recursion formula of the \( \phi_n \)'s, we show that \( \kappa_n \) satisfies the following difference-differential equation

\[
\frac{n + 1}{2s} \kappa_n^2 - \frac{\kappa_n}{\kappa_n} + \frac{1}{4} \left( \frac{\kappa_n}{\kappa_n} \right)^3 - \frac{1}{4} \left( \frac{\kappa_n}{\kappa_n} \right)^3 \right)^2 = 0, \tag{1.3}
\]

where \( s = \sqrt{\lambda} \), and \( (\kappa_n^2)_s = \frac{d}{ds}(\kappa_n^2) \). If we make the scaling

\[
\kappa_n^2 = 1 + \frac{c_1}{(n + 1)^\alpha} R(T(n, s)), \tag{1.4}
\]

with \( T(n, s) = t(n, s) + \epsilon(n, s) \), and

\[
t(n, s) = \frac{(n + 1)^\beta}{c_2} \left( \frac{c_3 s}{n + 1} - 1 \right), \tag{1.5}
\]

then we obtain, as \( n \to \infty \), that the parameters satisfy \( \alpha = 1/3, \beta = 2/3, c_1 = -2^{1/3}, c_2 = -1/2^{1/3}, c_3 = 2, \epsilon(n, s) = O(1/(n + 1)^{1/3}) \), and the function \( R \) satisfies

\[
(R')^2 - 8(R')^3 + 4t(R')^2 - 2R'R'' + o(1) = 0. \tag{1.6}
\]

If we drop the \( o(1) \) term, the equation above is another form of the Painlevé II equation which is discussed by Tracy and Widom in [2]. So (1.3) is called spatially discrete Painlevé II equation in this sense.

Therefore we have another proof of the asymptotic formula

\[
\kappa_n^2 = 1 - \frac{2^{1/3}}{(n + 1)^{2/3}} R(t) + \cdots, \tag{1.7}
\]

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where $R'(t) = -q^2(t)$, and $q(t)$ satisfies the Painlevé II equation $q'' = tq + 2q^3$. This asymptotic formula was first proved by Baik, Deift and Johansson using Riemann-Hilbert problems. They discussed much more asymptotic properties for $\kappa_n$. See [3] for the details. The asymptotic formula (1.7) is used for investigating the distribution of the length $l_n$ of the longest increasing subsequence of a random permutation [3], by discussing the asymptotics of the $D_{n-1}(\lambda) = e^\lambda \prod_{k=n}^{\infty} \kappa_k^2$. The distribution of $l_n$ (as $n \to \infty$) is same as the distribution of the largest eigenvalue of the Gaussian Unitary Ensemble (GUE) in random matrix theory [3] [4] [5]. In [5], Tracy and Widom used a different method to study the distribution of the $l_n$ by directly investigating the asymptotics of $D_n(\lambda)$. They also obtained the distribution of the length of the longest increasing subsequence of an odd permutation [3].

In [6], Hisakado discussed the same polynomials as in this paper, and he also get Painlevé II equation from a discrete equation which is called discrete string equation in [6]. The difference is as follows. In [6], Hisakado discussed that the constant term of polynomial $\phi_n(z)/\kappa_n$ satisfies a discrete equation (discrete string equation). In this paper, we discuss $\kappa_n$, the coefficient of the leading term of $\phi_n(z)$, which satisfies a spatially discrete equation (difference-differential equation). And the discrete string equation in [6] is convergent to the original Painlevé II ($q'' = xq + 2q^3$). In this paper, the spatially discrete equation is convergent to a third order differential equation. As discussed by Tracy and Widom in [3], this third order equation is equivalent to the Painlevé II equation.

It is known that the Painlevé equations have discrete analogs. See [7] [8] [9] [10] and references therein for the discrete or q-difference Painlevé equations. In this paper, we show that the Painlevé II equation has spatially discrete version. So far, to my knowledge, the spatially discrete versions for Painlevé equations are not known very well. In [1], Fokas, Its and Kitaev discussed another spatially discrete integrable equation, which is obtained from orthonormal polynomials on real line satisfying a recursion formula in the form (2.11). It is discussed in [7] that the simultaneous solution of the spatially discrete equation in [7] and of discrete Painlevé I equation solves a special case of Painlevé IV equation.

This report is organized as follows. In the next section, we state the recursion formula for the orthonormal polynomials on the unit circle, which is proved in [1]. In Sect. 3, we use the recursion formula and the orthonormal property of the polynomials to investigate the leading coefficient $\kappa_n(s)$, which is found to satisfy a difference-differential equation, or spatially dis-
crete equation. And we also get a formula for the second leading coefficient of \( \phi_n(z) \) in terms of \( \kappa_n \). In Sect. 4, the difference-differential equation is proved to tend to the second Painlevé equation in a third order equation form. Then we get the asymptotics of the coefficients for the leading term and for the second leading term of \( \phi_n(z) \).

2 Recursion Formula for the Polynomials

Let us consider the orthonormal polynomials \( \phi_n(n = 0, 1, 2, \cdots) \) on the unit circle in the complex plane defined by (1.1). If we let \( f(e^{i\theta}) = \exp(2\sqrt{\lambda} \cos \theta) \) \( = \exp(\sqrt{\lambda}(z + 1/z)) \), \( z = e^{i\theta} \), then (1.1) becomes

\[
\int_{-\pi}^{\pi} \phi_n(e^{i\theta})\overline{\phi_m(e^{i\theta})}f(e^{i\theta})\frac{d\theta}{2\pi} = \delta_{nm}, \quad m, n \geq 0. \tag{2.1}
\]

Since \( 2\sqrt{\lambda} \cos \theta \) is even in \( \theta \), the coefficients of \( \phi_n \) are real [1]. It is also proved in [1] that \( \phi_n \) satisfy the recursion formulas

\[
\kappa_{n-1} z \phi_{n-1}(z) = \kappa_n \phi_n(z) - \phi_n(0) \phi_n^*(z), \tag{2.2}
\]

\[
\kappa_n \phi_{n+1}(z) = \kappa_{n+1} z \phi_n(z) + \phi_{n+1}(0) \phi_{n+1}^*(z). \tag{2.3}
\]

Here the "*' is defined as

\[
\phi^*_n(z) = \overline{a_n} + \overline{a_{n-1}} z + \cdots + \overline{a_0} z^n, \tag{2.4}
\]

if \( \phi(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \). Notice that for \( \phi_n(z) \), the coefficients are real as mentioned above.

Since we are interested in the leading coefficient \( \kappa_n \), we set \( \phi_n(z) = \kappa_n p_n(z) \), and for simplicity we denote \( s = \sqrt{\lambda} \). Then (2.1) becomes

\[
\int_{-\pi}^{\pi} p_n(e^{i\theta})\overline{p_m(e^{i\theta})}f(e^{i\theta})\frac{d\theta}{2\pi} = \delta_{nm} \frac{1}{\kappa_n^2}, \quad m, n \geq 0. \tag{2.5}
\]

If we eliminate \( \phi^*_n(z) \) in (2.2) and (2.3), we get

\[
z \left( \frac{\kappa_{n+1}}{\kappa_n} \phi_n(z) - \frac{\phi_{n+1}(0)}{\phi_n(0)} \frac{\kappa_{n-1}}{\kappa_n} \phi_{n-1}(z) \right) = \phi_{n+1}(z) - \frac{\phi_{n+1}(0)}{\phi_n(0)} \phi_n(z). \tag{2.6}
\]

By \( \phi_n(z) = \kappa_n p_n(z) \), (2.6) becomes

\[
z \left( p_n(z) - \frac{p_{n+1}(0)}{p_n(0)} \kappa^2_n p_{n-1}(z) \right) = p_{n+1}(z) - \frac{p_{n+1}(0)}{p_n(0)} p_n(z). \tag{2.7}
\]
For the usage in later discussions, let us record this as a lemma, which is due to Szegö [1].

**Lemma 1** The orthogonal polynomials \( p_n(z) \) defined by (2.4) on the unit circle in the complex plane satisfy the following recursion formula

\[
z \left( p_n(z) + C_n p_{n-1}(z) \right) = p_{n+1}(z) + B_n p_n(z),
\]

where

\[
C_n = \frac{p_{n+1}(0)}{p_n(0)} \frac{\kappa_{n-1}^2}{\kappa_n^2},
\]

\[
B_n = -\frac{p_{n+1}(0)}{p_n(0)}.
\]

As a remark, the recursion formula for the orthogonal polynomials on the real line takes the form [1] [11]

\[
z p_n = a_n p_{n+1} + b_n p_n + c_n p_{n-1}.
\]

But we have seen that on the unit circle with the weight \( d\mu \), the coefficient of \( p_{n-1} \) in the recursion formula contains \( z \).

### 3 Spatially Discrete Equation for \( \kappa_n \)

In this section, we use the orthogonal relation (2.5) and the recursion relation (2.8) to show that \( \kappa_n(s) \) satisfies a spatially discrete equation (difference-differential equation). In the next section we show that the continuous limit of this spatially discrete equation gives a third order ordinary differential equation, which is equivalent to Painlevé II equation.

Let us write (2.5) in the form

\[
\langle p_n, p_m \rangle \equiv \oint p_n(z) \overline{p_m(z)} f(z) \frac{dz}{z} = \delta_{nm} h_n, \quad m, n \geq 0,
\]

where \( h_n = 2\pi i / \kappa_n^2 \), and the integral \( \oint \) is taken over the unit circle in the complex plane. We will use the notation \( d\mu = f(z)dz/z \) for simplicity. First it is easy to show the following identities

\[
\oint z^k \frac{\partial \overline{p_m(z)}}{\partial \overline{z}} p_n(z) d\mu = \oint z^{-k} \frac{\partial p_m(z)}{\partial z} \overline{p_n(z)} d\mu,
\]

\[
\oint z^k \overline{p_m(z)} p_n(z) d\mu = \oint z^{-k} p_m(z) \overline{p_n(z)} d\mu,
\]

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by using \( z = e^{i\theta} \), and \( \theta \) replaced by \( -\theta \). Recall \( p_n(z) = p_n(\bar{z}) \).

Now, consider (3.1) with \( m = n \),

\[
h_n = \oint p_n(e^{i\theta}) p_n(\bar{e}^{i\theta}) e^{s(z + \frac{1}{z})} \frac{dz}{z}.
\]

Then by integration by parts and using (3.2) and (3.3), we have

\[
h_n = \frac{1}{s} \oint z^2 \frac{\partial p_n(z)}{\partial z} p_n(z) d\mu + \oint z^2 p_n(z) p_n(\bar{z}) d\mu + \frac{1}{s} \oint z p_n(z) p_n(\bar{z}) d\mu. \tag{3.4}
\]

Since \( p_n(z) = p(\bar{z}) = p(1/z) \) on the unit circle, by the Cauchy integral formula, the right hand side of the above equation can be calculated, and the result involves all the coefficients of \( p_n(z) \). That does not help too much to investigate the \( \kappa_n \). Here, by using the recursion formula and the orthonormal property, the right side of (3.4) can be expressed in terms of \( h_{n-1}, h_n, h_{n+1} \) and their first derivatives.

We need to consider two type integrals,

\[
\oint z^k \frac{\partial p_n(z)}{\partial z} p_n(\bar{z}) d\mu, \quad \oint z^k p_n(z) p_n(\bar{z}) d\mu
\]

for (3.4), where \( k \) is integer. In this paper, we do not discuss the general formula. We just consider the integrals in (3.4), and the formulas are given in the following lemmas.

**Lemma 2**

\[
\oint z p_n(z) p_n(\bar{z}) d\mu = (B_n - C_n) h_n. \tag{3.5}
\]

*Proof.* Multiply \( p_n(\bar{z}) \) on both sides of (2.8). Then integrating on both sides yields the result. \( \blacksquare \)

**Lemma 3**

\[
\oint z^2 p_n(z) p_n(\bar{z}) d\mu = -(C_{n+1} (B_n - C_n) - (B_n - C_n)^2 + C_n (B_{n-1} - C_{n-1})) h_n. \tag{3.6}
\]
Proof. Let $I = \oint z^2 p_n(z) \overline{p_n(z)} d\mu$. By Lemma 1, we have

$$z^2 p_n = p_{n+2} + B_{n+1} p_{n+1} - C_{n+1} z p_n + B_n z p_n - C_n z^2 p_{n-1},$$

which implies

$$I = \oint \left( (-C_{n+1} + B_n) z p_n - C_n z^2 p_{n-1} \right) \overline{p_n(z)} d\mu,$$

where we have used $< p_{n+2}, p_n > = < p_{n+1}, p_n > = 0$. Again by Lemma 1,

$$z^2 p_{n-1} = z p_n + B_{n-1} p_n + B^2_{n-1} p_{n-1} - z B_{n-1} C_{n-1} p_{n-2} - C_{n-1} z^2 p_{n-2}.$$

Since $< p_{n-1}, p_n > = < z p_{n-2}, p_n > = 0$, we then obtain from (3.7)

$$I = \oint \left( (-C_{n+1} + B_n - C_n) (z p_n + B_n p_n - z C_n p_{n-1}) - C_n B_{n-1} p_n + C_n C_{n-1} z^2 p_{n-2} \right) \overline{p_n(z)} d\mu$$

$$= (-C_{n+1} + B_n - C_n) (B_n - C_n) h_n - C_n B_{n-1} h_n + C_n C_{n-1} h_n,$$

where $< z p_{n-1}, p_n > = < z^2 p_{n-2}, p_n > = h_n$. □

In order to evaluate the integral $< z^2 \frac{\partial p_n}{\partial z}, p_n >$ in (3.4), we need to consider the second leading coefficient of $p_n(z)$. Suppose

$$p_n(z) = z^n + b_{n-1} z^{n-1} + \cdots. \tag{3.8}$$

Because $z^2 \frac{\partial p_n(z)}{\partial z} = n z^{n+1} + (n-1) b_{n-1} z^n + \cdots$, we set

$$z^2 \frac{\partial p_n(z)}{\partial z} = n (p_{n+1} + \mu_n p_n + \cdots).$$

Then by

$$z^{n+1} + b_n z^n + \cdots + \mu_n (z^n + b_{n-1} z^{n-1} + \cdots)$$

$$= z^{n+1} + (1 - \frac{1}{n}) b_{n-1} z^n + \cdots,$$

it follows that

$$\mu_n = \left(1 - \frac{1}{n}\right) b_{n-1} - b_n. \tag{3.9}$$

Since $< z^2 \frac{\partial p_n}{\partial z}, p_n > = n \mu_n h_n$, we have proved the following lemma.
Lemma 4

\[ \oint z^2 \frac{\partial p_n(z)}{\partial z} p_n(z) d\mu = (n^2 (b_{n-1} - b_n) - b_n) h_n. \]  

(3.10)

Because we want an equation for the leading coefficient \( \kappa_n \), we need to express the second leading coefficient \( b_n \) in terms of \( \kappa_n \). First, the \( b_n \) can be expressed in terms of \( B_n \) and \( C_n \).

Lemma 5

For the second leading coefficient \( b_n \) of \( p_n \) defined by (3.8), there are the following properties

(i) \( b_{n-1} - b_n = B_n - C_n \),

(ii) \[ \frac{1}{s} b_{n-1} = -C_n (B_{n-1} - C_{n-1}) - \frac{h_n}{h_{n-1}}. \]

(3.11)

(3.12)

Proof. By Lemma 1, \( z p_n(z) = p_{n+1}(z) + B_n p_n(z) - C_n z p_{n-1}(z) \). By (3.8),
\( z p_n = p_{n+1} + (b_{n-1} - b_n) p_n + \cdots \). Then we have \((b_{n-1} - b_n) h_n = B_n h_n - C_n h_n\) by considering \(< z p_n, p_n >\) in the two ways.

To prove the second formula (3.12), consider

\[ I = \oint p_{n+1}(z) \frac{1}{s} \bar{p}_n(z) e^{s(z+\frac{1}{s})} dz. \]

(3.13)

Using integration by parts,

\[ I = \frac{1}{s} \oint p_{n+1}(z) \frac{1}{s} \bar{p}_n(z) e^{s(z+\frac{1}{s})} dz - \frac{1}{s} \oint (z \frac{\partial p_{n+1}}{\partial z}) \bar{p}_n d\mu + \frac{1}{s} \oint p_{n+1} \left( \bar{z} \frac{\partial p_n}{\partial z} \right) d\mu + \oint p_{n+1} \left( \bar{z} \frac{\partial p_n}{\partial z} \right) d\mu, \]

where \(< p_{n+1}, z \frac{\partial p_n}{\partial z} >= 0\), and \(< p_{n+1}, z^2 p_n >= h_{n+1}\). And \( z \frac{\partial p_{n+1}}{\partial z} = (n + 1) (z^{n+1} + \frac{n}{n+1} b_n z^n + \cdots) \). Set \( z \frac{\partial p_{n+1}}{\partial z} = (n + 1) (p_{n+1} + \gamma_n p_n + \cdots) \), which implies \( z \frac{\partial p_{n+1}}{\partial z} = (n + 1) (p_{n+1} + \gamma_n p_n + \cdots) \). Then by

\[ p_{n+1} + \gamma_n p_n + \cdots = z^{n+1} + \frac{n}{n+1} b_n z^n + \cdots, \]

we get \( \gamma_n + b_n = \frac{n}{n+1} b_n \), so \( \gamma_n = -\frac{1}{n+1} b_n \). Thus

\[ I = \frac{1}{s} b_n h_n + h_{n+1}. \]

(3.14)
On the other hand, the integral $I$ can be evaluated by using the recursion formula (see (2.8))

$$z p_{n+1} = p_{n+2} + B_{n+1} p_{n+1} - C_{n+1} p_{n+1} - C_{n+1} B_n p_n + C_{n+1} C_n z p_{n-1},$$

which gives

$$I = < z p_{n+1}, p_n > = -C_{n+1} (B_n - C_n) h_n. \quad (3.15)$$

By (3.14) and (3.15), we get (3.12). $\blacksquare$

We have seen by the last four lemmas that the right hand side of (3.4) can be expressed in terms of $B_n$ and $C_n$. The following lemma gives the formulas of $B_n, C_n$ in terms of $\kappa_n$.

**Lemma 6** The coefficients $B_n, C_n$ in the recursion formula (see (2.9), (2.10)) can be expressed as follows.

(i) $B_n - C_n = -\frac{(\kappa^2_n)_{s}}{2 \kappa^2_n}.$ \quad (3.16)

(ii) $B_n = -\frac{(\kappa^2_n)_{s}}{2(\kappa^2_n - \kappa^2_{n-1})}.$ \quad (3.17)

(iii) $C_n = -\frac{\kappa^2_{n-1}}{2 \kappa^2_n} \frac{(\kappa^2_n)_{s}}{\kappa^2_n - \kappa^2_{n-1}}.$ \quad (3.18)

where $\kappa_n$ is the leading coefficient of $\phi_n$ defined by (1.1), and $(\kappa^2_n)_{s} = \frac{d\kappa^2_n}{ds}$.

*Proof.* Since $h_n = < p_n, p_n >$, we have

$$\frac{dh_n(s)}{ds} = \oint p_n(z) \overline{p_n(z)} \left( z + \frac{1}{z} \right) d\mu = 2 \oint z p_n \overline{p_n} d\mu, \quad (3.19)$$

where we have used $< \frac{\partial p_n}{\partial s}, p_n > = < p_n, \frac{\partial p_n}{\partial s} > = 0$, since $\deg(\frac{\partial p_n}{\partial s}) \leq n - 1$, $\deg(\overline{\frac{\partial p_n}{\partial s}}) \leq n - 1$. Then by Lemma 1 and the notation $h_n = 2\pi i/\kappa_n^2$, we have the first formula (3.16). By (2.9) and (2.10), there is $C_n = B_n \kappa_{n-1}/\kappa_n^2$. We then get (3.17) and (3.18). $\blacksquare$

By Lemma 2, 3, 4, (3.4) is changed to

$$1 = \frac{n}{s}(b_{n-1} - b_n) - \frac{1}{s} b_{n-1}$$

$$- C_{n+1}(B_n - C_n) + (B_n - C_n)^2 - C_n(B_{n-1} - C_{n-1}) + \frac{1}{s}(B_n - C_n). \quad (3.20)$$
By Lemma 5, this equation further becomes
\[
\frac{n+1}{s}(B_n - C_n) + \frac{h_n}{h_{n-1}} - 1 - C_{n+1}(B_n - C_n) + (B_n - C_n)^2 = 0. \tag{3.21}
\]

Therefore by Lemma 6, we have proved the following theorem.

**Theorem 1** The leading coefficient \(\kappa_n(s)\) of the orthonormal polynomials \(\phi_n(z)\) defined by \((1.1)\) satisfies the following spatially discrete equation (difference-differential equation)
\[
\frac{n+1}{2s} (\kappa_n^2)_s - \frac{\kappa_{n-1}^2 - \kappa_n^2}{\kappa_n^2} + \frac{1}{4} \frac{(\kappa_{n+1}^2)_s}{\kappa_{n+1}^2} - \frac{1}{4} \left( \frac{(\kappa_n^2)_s}{\kappa_n^2} \right)^2 = 0, \tag{3.22}
\]

where \(s = \sqrt{\lambda}\), and \((\kappa_n^2)_s = \frac{d}{ds}(\kappa_n^2)\).

This is a new result for \(\kappa_n(s)\), the leading coefficients of the orthonormal polynomials \(\phi_n(z, s)\), with the weight \(\exp\left(s(z + \frac{1}{2})\right)\) on the unit circle.

In the next section, we show that as \(n \to \infty\), this equation is reduced to a third order ordinary differential equation which is equivalent to the Painlevé II equation.

### 4 Painlevé II Equation

We have shown that \(\kappa_n^2\) satisfies the equation \((3.22)\). As mentioned in Sect. 1, \(\kappa_n^2\) satisfies the boundary condition \(\kappa_n^2 = 1 + o(1)\), as \(n \to \infty\). In this section, we compute the asymptotics of \(\kappa_n^2 - 1\), as \(n \to \infty\). We will see that the asymptotics involves the second Painlevé function.

Equation \((3.22)\) has two independent variables \(n\) and \(s\). To study the asymptotics, we use “similarity” reduction, or the so called double scaling method. By comparing the first two terms in \((3.22)\), we consider the case when \(n+1\) and \(s\) are in the same order as they are large, and let
\[
\frac{c_3 s}{n+1} = 1 + \frac{c_2 t}{(n+1)^3}, \tag{4.1}
\]
\[
\kappa_n^2 = 1 + \frac{c_1}{(n+1)^\alpha} R(T(n, s)), \tag{4.2}
\]
where \(T(n, s) = t(n, s) + \epsilon(n, s)\), the leading term \(t(n, s)\) is defined by \((1.1)\), and it will be shown that \(\epsilon(n, s)\) is a smaller term as \(n \to \infty\). We want to
determine the constants $\alpha, \beta, c_1, c_2, c_3$, such that as $n \to \infty$, (3.22) is reduced to a differential equation.

Let us consider the approximate expressions of $(\kappa^2_n)_s$, $\kappa^2_{n+1} - \kappa^2_n$ in terms of $R'$ and maybe also higher order derivatives if we need. Here $R'$ means the limit of $(R(T + \Delta T) - R(T))/\Delta T$. By (4.1), for fixed $s$, as $n$ increases by 1, $t$ is increased by a order $1/(n + 1)^{1 - \beta}$. For the asymptotics of $\kappa^2_{n+1} - \kappa^2_n$ and $(\kappa^2_{n+1})_s - (\kappa^2_n)_s$, the higher order terms also need to be concerned in order to have all the terms in the first three leading orders in the expansion of left side of (3.22). Therefore we have the following

\[
t(n + 1, s) - t(n, s) = \frac{c_4}{(n + 1)^{1 - \beta}} + \cdots,
\]

\[
\kappa^2_{n+1} - \kappa^2_n = R' \frac{c_5}{(n + 1)^{1 + \alpha - \beta}} + R'' \frac{c_6}{(n + 1)^{2 + \alpha - 2\beta}} + R''' \frac{c_7}{(n + 1)^{3 + \alpha - 3\beta}} + R \frac{c_8}{(n + 1)^{1 + \alpha}} + \cdots,
\]

\[
\kappa^2_{n-1} - \kappa^2_n = -R' \frac{c_5}{(n + 1)^{1 + \alpha - \beta}} + R'' \frac{c_6}{(n + 1)^{2 + \alpha - 2\beta}} - R''' \frac{c_7}{(n + 1)^{3 + \alpha - 3\beta}} - R \frac{c_8}{(n + 1)^{1 + \alpha}} + \cdots,
\]

\[
(\kappa^2_n)_s = \frac{c_1 c_3}{c_2} R' \frac{1}{(n + 1)^{1 + \alpha - \beta}} + \cdots,
\]

\[
(\kappa^2_{n+1})_s - (\kappa^2_n)_s = \frac{c_1 c_3}{c_2 (n + 1)^{1 + \alpha - \beta}} \left( R'' \frac{c_4}{(n + 1)^{1 - \beta}} + R''' \frac{c_5}{2 (n + 1)^{2 - 2\beta}} \right) + \cdots,
\]

where $c_4$ is a constant, and

\[
c_5 = c_1 c_4, \quad c_6 = \frac{1}{2} c_1 c_4^2, \quad c_7 = \frac{1}{6} c_1 c_4, \quad c_8 = -c_1 \alpha.
\]

Now write (3.22) in the following form

\[
\frac{n + 1}{2s} (\kappa^2_n)_s (\kappa^2_{n+1} - \kappa^2_n) - (\kappa^2_{n-1} - \kappa^2_n)(\kappa^2_{n+1} - \kappa^2_n) + \frac{1}{4} (\kappa^2_n)_s (\kappa^2_{n+1})_s - \frac{1}{4} \frac{\kappa^2_{n+1} - \kappa^2_n}{\kappa^2_{n+1}} (\kappa^2_n)_s (\kappa^2_{n+1})_s - \frac{1}{4} \frac{((\kappa^2_n)_s)^2}{\kappa^2_{n+1}} (\kappa^2_{n+1} - \kappa^2_n) = 0.
\]

By substituting the asymptotic formulas above into this equation, we get

\[
S_1 + S_2 + S_3 + S_4 + o(1) = 0,
\]

(4.5)
where

\[
S_1 = \frac{c_1c_3^2}{2c_2} \left(1 - \frac{c_2t}{(n+1)^\beta}\right) \frac{R'}{(n+1)^{1+\alpha-\beta}} \times \\
\left\{ \frac{c_5R'}{(n+1)^{1+\alpha-\beta}} + \frac{c_6R''}{(n+1)^{2+\alpha-2\beta}} + \frac{c_7R'''}{(n+1)^{3+\alpha-3\beta}} + \frac{c_8R}{(n+1)^{1+\alpha}} \right\},
\]

\[
S_2 = \frac{c_5^2(R')^2}{(n+1)^{2+2\alpha-2\beta}} + \frac{2c_5c_7R'R''}{(n+1)^{4+2\alpha-4\beta}} + \frac{2c_5c_8R'R}{(n+1)^{2+2\alpha-\beta}} - \frac{c_5^2(R'')^2}{(n+1)^{4+2\alpha-4\beta}},
\]

\[
S_3 = \frac{c_1c_3}{4c_2} \frac{R'}{(n+1)^{1+\alpha-\beta}} \times \\
\left\{ \frac{c_1c_3}{c_2} \frac{R'}{(n+1)^{1+\alpha-\beta}} + \frac{c_1c_3}{c_2(n+1)^{1+\alpha-\beta}} \left( \frac{c_4R''}{(n+1)^{1+\beta}} + \frac{c_2R'''}{2(n+1)^{2-2\beta}} \right) \right\},
\]

\[
S_4 = -\frac{c_5}{4\kappa^2_{n+1}} \left( \frac{c_1c_3}{c_2} \right)^2 \frac{(R')^3}{(n+1)^{3+3\alpha-3\beta}} \times \\
-\frac{c_5}{4\kappa^2_{n}} \left( \frac{c_1c_3}{c_2} \right)^2 \frac{(R')^3}{(n+1)^{3+3\alpha-3\beta}}.
\]

By the definition of \( t \) and asymptotics of \( \Delta t \), we have \( 0 < \beta < 1 \). Since \( \kappa_n^2 \to 1 \), as \( n \to \infty \), we have \( \alpha > 0 \). Consider the orders of the terms on the left side of (4.3). The coefficients of \((R')^2, R'R', R'R'', (R'')^2\) have orders \(2(1+\alpha-\beta), 2(1+\alpha-\beta)+(1-\beta), 2(1+\alpha-\beta)+2(1-\beta), 2(1+\alpha-\beta)+2(1-\beta)\) respectively. The coefficients of \(t(R')^2, RR', (R')^3\) have orders \(2(1+\alpha-\beta)+\beta, 2(1+\alpha-\beta)+\beta, 2(1+\alpha-\beta)+(1+\alpha-\beta)\) respectively. And the \( o(1) \) in (4.3) contains higher order terms which we do not concern. To determine the values of \( \alpha \) and \( \beta \), the only choice is to set the coefficients of \( R'R'', (R'')^2, t(R')^2, RR' \) and \( (R')^3 \) be in the same order. So we have \( 2(1-\beta) = \beta = 1+\alpha-\beta \). The solution is unique \( \alpha = 1/3, \beta = 2/3 \). So (4.5) becomes

\[
A_1 \frac{(R')^2}{(n+1)^{4/3}} + A_2 \frac{R'R''}{(n+1)^{5/3}} + \left( A_3 R'R + A_4 R'R'' + A_5 (R'')^2 + A_6 t(R')^2 + A_7 (R')^3 \right) \frac{1}{(n+1)^2}
\]

\[
+ O \left( \frac{1}{(n+1)^{7/3}} \right) = 0,
\]

where

\[
A_1 = \frac{c_3}{2} \frac{c_1c_3}{c_2} c_5 + c_5^2 + \frac{1}{4} \left( \frac{c_1c_3}{c_2} \right)^2,
\]

\[A_2 = \frac{c_5^2}{(n+1)^{2}}\]
\begin{align*}
A_2 &= \frac{c_3}{2} \frac{c_1 c_3}{c_2} c_6 + \frac{1}{4} \left(\frac{c_1 c_3}{c_2}\right)^2 c_4, \\
A_3 &= \frac{c_3}{2} \frac{c_1 c_3}{c_2} c_8 + 2 c_5 c_8, \\
A_4 &= \frac{c_3}{2} \frac{c_1 c_3}{c_2} c_7 + 2 c_5 c_7 + \frac{1}{8} \left(\frac{c_1 c_3}{c_2}\right)^2 c_4^2, \\
A_5 &= -c_6^2, \\
A_6 &= -\frac{1}{2} c_1 c_3 c_5, \\
A_7 &= -\frac{1}{2} \left(\frac{c_1 c_3}{c_2}\right)^2 c_5.
\end{align*}

Look at the equation (4.6). If \( A_1 \) or \( A_2 \) is not zero, then we get asymptotics of \( \kappa_n \) contradicting to the results proved in [1]. In [1], the asymptotics of \( \kappa_n \) contains exponential function of \( s \). So we must have \( A_1 = A_2 = 0 \). The coefficient of the \( 1/(n+1)^2 \) in (4.6) has the pattern of Painlevé II equation in [2]. So we want to choose proper constant numbers \( c_1, c_2, c_3, c_4 \), such that as \( n \to \infty \), (4.6) reduces to the Painlevé II equation. In [3], Tracy and Widom discussed two forms of Painlevé II equation

\begin{align*}
\frac{1}{2} \frac{R''}{R'} - \frac{1}{2} \frac{(R')^2}{(R')^2} - \frac{R}{R} + R' &= 0, \\
(R'')^2 + 4 R' ((R')^2 - t R' + R) &= 0,
\end{align*}

where \( R'(t) = -q(t)^2 \), and \( q(t) \) satisfies the original Painlevé II \( q'' = t q + 2 q^3 \). Equation (4.8) is called Jimbo-Miwa-Okamoto \( \sigma \) form for Painlevé II. Eliminating the \( R \) in (4.7) and (4.8) gives another form

\begin{equation}
-2 R' R'' + (R'')^2 + 4 t (R')^2 - 8(R')^3 = 0.
\end{equation}

In (4.6), we set \( A_1 = A_2 = A_3 = 0 \), \( A_4 = -2 A_5 \), \( A_6 = 4 A_5 \), and \( A_7 = -8 A_5 \). By using (4.3) and (4.4), the solution of these nonlinear equations is found to be

\begin{align*}
c_1 &= -2^{1/3}, \\
c_2 &= -\frac{1}{2^{1/3}}, \\
c_3^2 &= 4, \\
c_4 &= 2^{1/3}.
\end{align*}
Since we consider positive \( n \) and \( s \), \( c_3 \) is positive, i.e., \( c_3 = 2 \). And it is seen that \( \epsilon(n, s) = O(1/(n + 1)^{1/3}) \) because the last term in (4.6) is 1/3 order higher than the proceeding term.

Therefore we have a formal proof of the following theorem which was first proved by Baik, Deift and Johansson [3] by using Riemann-Hilbert method.

**Theorem 2** As \( n \to \infty \), \( \kappa_n^2 \) has the following asymptotic formula

\[
\kappa_n^2 = 1 - \frac{2^{1/3}}{(n + 1)^{1/3}} R(t) + O \left( \frac{1}{(n + 1)^{2/3}} \right),
\]

where \( t \) is defined by

\[
\frac{2\sqrt{\lambda}}{n+1} = 1 - \frac{t}{2^{1/3}(n+1)^{2/3}}, \quad R'(t) = -q^2(t), \quad \text{and} \quad q(t) \text{ satisfies Painlevé II } q'' = tq + 2q^3.
\]

As discussed in [3], the Painlevé II function \( q(t) \) in Theorem 2 satisfies the boundary condition \( q(t) \sim -Ai(t) \), as \( t \to \infty \), where \( Ai(t) \) is the Airy function. This boundary condition can also be obtained by the asymptotics of \( \kappa_n \) in terms of exponential function [1] and the Painlevé II equation that \( q(t) \) satisfies. The Painlevé II function \( q(t) \) with this boundary condition is discussed by Hastings and McLeod in [12].

As a remark, there is no resolution if ones want to get (4.7) or (4.8) from (4.6). So these three equations (4.7), (4.8) and (4.9) play different roles. It is interesting that in [13] there is a similar augument for three forms of Painlevé VI equation. It is discussed in [13] that by eliminiting the third order derivative in the equation obtained in [13] and in the equation in [11], ones get the \( \sigma \) form of Painlevé VI.

Finally by Lemma 5, Lemma 6 and Theorem 2, we obtain the asymptotics for the second leading coefficient of \( \phi_n(z) \).

**Theorem 3** For the polynomial \( \phi_n(z) = \kappa_n (z^n + b_{n-1} z^{n-1} + \cdots) \) defined by (1.4), the second leading coefficient \( \kappa_n b_{n-1} \) has the asymptotic formula

\[
\frac{\kappa_n b_{n-1}}{\sqrt{\lambda}} = -1 + \frac{1}{2^{2/3}(n + 1)^{1/3}} R(t) + O \left( \frac{1}{(n + 1)^{2/3}} \right),
\]

as \( n \to \infty \), where \( t \) and \( R(t) \) are same as in Theorem 2.
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References

[1] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications, Vol 23, 4th Ed, New York, 1975.

[2] C. A. Tracy and H. Widom, *Level-spacing distributions and the Airy kernel*, Comm. Math. Phys., 159 (1994), 151 – 174.

[3] J. Baik, P. Deift and K. Johansson, *On the distribution of the length of the longest increasing subsequence of random permutations*, preprint, LANL archives, math. CO/9810105.

[4] C. A. Tracy and H. Widom, *Universality of the distribution functions of random matrix theory*, preprint.

[5] C. A. Tracy and H. Widom, *Random unitary matrices, permutations and Painlevé*, preprint.

[6] M. Hisakado, *Unitary matrix model and Painlevé III*, Mod. Phys. Lett. A11 (1996), 3001 – 3010.

[7] A. S. Fokas, A. R. Its and A. V. Kitaev, *Discrete Painlevé equations and their appearance in quantum gravity*, Comm. Math. Phys. 142 (1991), 313 – 344.

[8] A. Ramani, B. Grammaticos and J. Hietarinta, *Discrete versions of the Painlevé equations*, Phys. Rev. Lett. 67 (1991), 1829 – 1832.

[9] A. Bassom and A. P. Clarkson, *New exact solutions of the discrete fourth Painlevé equation*. Phys. Lett. A, 194 (1994), no.5-6, 358 – 370.

[10] M. Jimbo and H. Sakai, *A q-analog of the sixth Painlevé equation*, Lett. Math. Phys. 38 (1996), 145 – 154.
[11] C. A. Tracy and H. Widom, *Fredholm determinants, differential equations and matrix models*, Comm. Math. Phys., **163** (1994), 33 – 72.

[12] S. P. Hastings and J. B. McLeod, *A boundary value problem associated with the second Painlevé transcendent and the Korteweg de Vries equation*, Arch. Rat. Mech. Anal. **73** (1980), 31 – 51.

[13] L. Haine and J.-P. Semengue, *The Jacobi polynomial ensemble and the Painlevé VI equation*, preprint.