Hard thermal loops effective action for $\pi \to \gamma \gamma$

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To Klaus.

I consider the low temperature correction to the anomalous coupling of a neutral pion to two photons from an effective Lagrangian point of view.

I. INTRODUCTION

In vacuum, the coupling of a neutral pion to two (on mass-shell) photons is directly related to (the coefficient of) the anomalous divergence of the axial current $[1]$. It is a truly wonderful thing that the axial anomaly (as computed to one-loop in vacuum) is not renormalized by higher order quantum corrections $[2]$. As the anomaly emerges from the ultraviolet behaviour of quantum fields, this is also the case for thermal corrections $[3]$.

On the other hand, the relation between the $\pi^0$ decay amplitude and the anomaly can be—and actually is—subject to thermal corrections $[4]$. A rather straightforward way to verify this is to look at the anomalous Ward identities. In these proceedings, however, I prefer to follow an effective Lagrangian approach, partly because it is more suitable for generalisations. (See C. Manuel $[5]$, these proceedings.)

I use chiral perturbation theory ($\chi$PT) $[6]$, and thus consider temperatures (well) below the chiral symmetry breaking scale, $T \ll f_\pi$, with $f_\pi \approx 97\,\text{MeV}$ the pion decay constant. Except when explicitly stated, I work in the chiral limit, $m_\pi = 0$. I consider first the pion one-loop quantum corrections, then discuss the low temperature corrections and the emergence of hard thermal loops. This is only a brief account of the results of $[4]$.

II. VACUUM

In the vacuum, the coupling of a neutral pion to two photons is described by the $\mathcal{O}(P^4)$ effective Lagrangian,

$$\mathcal{L}^{(4)}_{\pi\gamma\gamma} = \left(\frac{e^2 N_c}{48\pi^2}\right) \frac{1}{f_\pi} \pi^0 F_{\alpha\beta} \tilde{F}^{\alpha\beta}$$  \hspace{1cm} (2.1)

From (2.1), the pion decay amplitude is given by

$$M = g_{\pi\gamma\gamma} \epsilon_{\alpha\beta\gamma\delta} \epsilon^\alpha P_1^\beta P_2^\delta$$  \hspace{1cm} (2.2)

with

$$f_\pi g_{\pi\gamma\gamma} = \frac{e^2 N_c}{12\pi^2}$$  \hspace{1cm} (2.3)

and where $\epsilon_{1,2}$ and $P_{1,2}$ are, respectively, the polarisation vectors and momenta of the outgoing photons. The anomalous coupling $g_{\pi\gamma\gamma}$ is independent of the pion mass, and expressed in terms of the decay constant $f_\pi$ and the RHS of (2.3), that arises solely from the anomaly. Without the anomaly and in the chiral limit, $g_{\pi\gamma\gamma}$ would vanish to $\mathcal{O}(P^4)$, which is consistent with the Sutherland-Veltman theorem $[10]$.

Remarkably, in the chiral limit, $m_\pi = 0$, and for photons on mass-shell, $P_1^2 = P_2^2 = 0$, the relation (2.3) is exact $[1]$. This result rests on the non-renormalisation of the axial anomaly $[2]$, and can be derived using the anomalous Ward identities only $[3]$. It also has a translation in the effective Lagrangian language $[12]$. Note first

$^1$ From the effective Lagrangian point of view, (2.1) is only one of the possible anomalous couplings that arise in the gauging of the Wess-Zumino-Witten effective action $[8]$. Most of what I say applies straightforwardly to the other anomalous couplings.
that the effective coupling \( \boxed{2.1} \) saturates the anomaly, \( i.e. \) correctly reproduced the anomaly of the underlying quark degrees of freedom. Under \( \pi/f_\pi \to \pi/f_\pi + \alpha \),

\[
\delta L_{\pi^0\gamma\gamma} = \alpha \left( \frac{e^2 N_c}{48\pi^2} \right) F_{\alpha\beta} \bar{F}_{\alpha\beta} \tag{2.4}
\]

so that

\[
\partial^\mu J_{5,\alpha} = -\frac{e^2 N_c}{48\pi^2} F_{\alpha\beta} \bar{F}_{\alpha\beta} \tag{2.5}
\]

with \( J_{5,\alpha} = f_\pi \partial_\pi \pi^0 + \ldots \), in term of the pion field. This implies in particular that higher order terms in the momentum expansion of \( \chi PT \), and thus quantum corrections to \( \boxed{2.1} \), are non-anomalous. According to the power counting of \( \chi PT \), the anomalous Lagrangian \( \boxed{2.1} \) is \( O(\chi^4) \), so that one-loop corrections with insertions of one anomalous and one \( O(\chi^2) \) vertex induce terms that are \( O(\chi^6) \). One such term is

\[
L^{(6)}_{\pi^0\gamma\gamma} \propto \frac{N_c \alpha}{f_\pi} \epsilon_{\mu\nu\alpha\beta} \partial_\lambda F^{\mu\nu} F^{\alpha\lambda} \pi_0 + \ldots \tag{2.6}
\]

If the photons are on mass-shell, as is clear for dimensional reason, all such operators can only give a contribution that is proportional to the pion mass. Thus, to \( O(\chi^6) \)

\[
f_\pi \propto g_{\pi\gamma\gamma} = \frac{e^2 N_c}{12\pi^2} + O(m_\pi^2) \tag{2.7}
\]

on mass-shell, which implies that \( \boxed{2.3} \) is exact in the chiral limit, \( m_\pi^2 = 0 \).

III. THERMAL BATH

The discussion of the last section suggests that non-trivial corrections to the pion coupling can arise at finite temperature – despite the fact that the anomaly itself is not affected – for the presence of a thermal bath introduces, beside the pion and photon momenta, a new scale set by the temperature, \( T \). Temperature can modify the relation \( \boxed{2.3} \), pretty much like the pion mass does, Eq. \( \boxed{2.7} \). Of course, unlike a pion mass, finite temperature preserves chiral symmetry. What will eventually matter here is that manifest Lorentz invariance is lost in a thermal bath.

First, the power counting of \( \chi PT \) has to be modified to also take into account the powers of \( T \). Consider for instance the correction to the pion decay constant \( f_\pi \), that enters in \( \boxed{2.1} \). In the vacuum, the one-loop correction (corresponding to a tadpole diagram) is a homogeneous function of the pion mass, \( O(m_\pi^2) \). Thus, in the chiral limit, \( f_\pi^{1-loop} = f_\pi^{bare} \). (Actually, this is true to all orders.) At finite temperature however, the correction is \( O(T^2) \), for \( m_\pi \ll T \), and

\[
f_\pi(T) = (1 - T^2/12f_\pi^2) f_\pi, \tag{3.1}
\]

for two light quark flavours \( [12] \). The new scale set by the temperature induces non-trivial corrections, even in the chiral limit. If one na"ively substitutes \( f_\pi(T) \) in \( \boxed{2.3} \), one gets from \( \boxed{2.3} \) a coupling \( g_{\pi\gamma\gamma}(T) \) that increases with temperature, which is a strange – but not inconceivable– result. However, this low temperature behaviour can be contrasted to the one near \( T_c \), obtained using a linear sigma model with constituent quarks \( [13] \): in the chiral limit, the anomalous coupling vanishes above \( T_c \).

At finite, but low temperature, the expansion of the effective Lagrangian is thus in powers of \( m_\pi^2 \), \( T \) and the external momenta. The thermal correction to the pion decay constant is \( O(T^2) \) and arises from pions of the thermal bath with momentum \( P \sim T \). This is similar to the thermal corrections –known as hard thermal loops (HTL)– that arise in hot gauge theories \( [14] \). This suggests to consider the following hierarchy of scale:

\[
P \ll T \ll f_\pi, \tag{3.2}
\]

where \( P \) stands for external momenta. Then, as the leading contribution to the anomalous coupling is \( O(\chi^4) \), the low temperature correction to the pion decay constant in \( \boxed{2.1} \) gives a contribution that is \( O(\chi^4 T^2) \), and a priori dominates over the \( O(\chi^6) \) quantum corrections (and a fortiori the \( O(\chi^4 m_\pi^2) \) ones that vanish in the chiral limit). This leads us to consider the \( O(T^2) \) effective action of soft pions, \( P \ll T, f_\pi \), in a cool thermal bath, \( T \ll f_\pi \). (As the number density of these soft pions in the thermal bath is \( n \sim T/p > 1 \), this seems like a reasonable thing to consider.) To \( O(T^2) \) there are a priori many more terms that one can write that do not vanish.

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2In other words, the \( O(\chi^4) \) effective Lagrangian \( \boxed{2.1} \) is the only possible term that is compatible with the violation of the Sutherland-Veltman theorem induced by the anomaly.
in the chiral limit. Not only is manifest Lorentz invariance explicitly broken – which gives more local terms – but the non-analyticities, characteristic of finite $T$ amplitudes, transcribe into non-local terms in the chiral limit. Not only is manifest Lorentz invariance explicitly broken – which gives more local terms – but the non-analyticities, characteristic of finite $T$ amplitudes, transcribe into non-local terms.

The complete, $O(T^2)$, effective action is

$$\mathcal{L}_{\pi^0\gamma\gamma}(T) = \left(\frac{e^2 N_c}{48\pi^2}\right) \frac{1}{f_\pi(T)} \pi^0 F_{\alpha\beta} \tilde{F}^{\alpha\beta}$$

$$- \frac{T^2}{12 f_\pi^2} \left(\frac{e^2 N_c}{48\pi^2}\right) \int \frac{d\Omega}{4\pi} H_{\gamma\alpha} \hat{K}^\alpha \hat{K}_\beta F_{\gamma\beta},$$

where $\tilde{F}^{\alpha\beta} = \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}/2$, $H_{\alpha\beta} = \partial_\alpha H_\beta - \partial_\beta H_\alpha$, and

$$H_\alpha = \frac{1}{f_\pi} \epsilon_{\alpha\beta\gamma\delta} F_{\beta\gamma} \partial_\delta \pi^0.$$  \hspace{1cm} (3.4)

The vector $\hat{K} = (i, \hat{k})$; one integrates over all angles $\hat{k}$. This integration effectively represents the hard, massless pions in the one loop integral. One can verify that the second term of (3.3) is indeed invariant under an axial transformation, $\pi/f_\pi(T) \rightarrow \pi/f_\pi(T) + \alpha$, and thus, as required, is non-anomalous.

It nevertheless contributes to the effective anomalous coupling. Together with the correction to the pion decay constant in the first term of (3.3), it finally gives

$$g_{\pi\gamma\gamma}(T) = (1 - T^2/12 f_\pi^2) g_{\pi\gamma\gamma}$$  \hspace{1cm} (3.5)

The coupling thus decreases with temperature, a result that – if nothing else – is consistent with the behaviour found near $T_c$ \cite{13}. It is quite interesting that the coupling decreases with $T$ precisely like $f_\pi(T)$. I don’t know whether this is a coincidence.

**IV. CONCLUSIONS**

Further understanding of the results sketched in the previous sections can be gained from an analysis of the anomalous Ward identities at finite temperature \cite{1}. For instance, explicit one-loop calculations allow to see that, unlike in vacuum, in a thermal bath the anomalous divergence of the axial current is not saturated by the one-pion pole. (This is what allows the temperature dependence of the effective coupling to be non-trivial.) Also, the decay of a neutral pion into two photons is only one particular anomalous amplitude. The complete HTL contribution to the Wess-Zumino-Witten (gauged) effective action have been computed by C. Manuel \cite{17}. (For further application of HTL in chiral dynamics, see also \cite{19}.)

I would like to emphasise that the rather academic problem discussed here – the effective anomalous coupling of a neutral pion at finite temperature – is only one simple aspect of the broader class of problems dealing with the relation between axial anomalies and their phenomenological manifestation at finite temperature. For example, shifts in anomalous couplings of hadronic states, like the $\omega$ and $\phi$, could be relevant in heavy ion collisions. Another instance is offered by the ‘t Hooft anomaly matching conditions \cite{19}, for which manifest Lorentz covariance is a crucial ingredient \cite{1}; finite temperature could allow for more exotic solutions, maybe parity doublets \cite{21}. A very interesting problem concerns the relation, at finite temperature, between the breaking of the $U(1)_A$ symmetry and the meson spectrum, in particular the $\eta'$. For three or more flavours, effective Lagrangian \cite{22} and instantons \cite{23} arguments suggests and effective restoration of the $U(1)_A$ symmetry at the critical temperature of chiral symmetry breaking. Two flavours is marginal but a large $N_c$ argument seems to lead to similar conclusions \cite{24}.

\footnote{That the thermal corrections to the WZW effective action are non-anomalous has been proven in \cite{13}.}

\footnote{Effective means here that violation of $U(1)_A$ is relegated to irrelevant operators.}

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