OPERATOR FORMALISM FOR BOSONIC BETA-GAMMA FIELDS ON GENERAL ALGEBRAIC CURVES

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ABSTRACT

An operator formalism for bosonic $\beta - \gamma$ system on arbitrary algebraic curves is introduced. The classical degrees of freedom are identified and their commutation relations are postulated. The explicit realization of the algebra formed by the fields is given in the Hilbert space equipped with a bilinear form. The construction is based on the "gaussian" representation for $\beta - \gamma$ system on the complex sphere [Alvarez-Gaumé et al, Nucl. Phys. \textbf{B 311} (1988) 333]. Detailed computations are provided for 2 and 4 point correlation functions.

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1. INTRODUCTION

Since the middle of 1980s a big progress has been done in understanding the structure of $D = 2$ conformal field theories (CFT’s) [1]. Most of that work was motivated by string theory, but CFT’s have also been actively investigated because of their role in describing phase transition phenomena. In string theory it is necessary to consider CFT’s defined on topologically nontrivial $D = 2$ manifolds, in particular on closed higher genus Riemann surfaces (RS) of genus $g$ [2]. The properties of CFT’s on the torus are now relatively well known and, at least in the context of string theory, it turns out that the class of acceptable models are those which are modular invariant [3].

There have been many efforts to give a satisfactory description of CFT’s also on higher genus RS, for instance by means of an operator formalism [4]. Usually, an operator formalism of such kind makes it possible to introduce the notions of vacuum, creation and annihilation operators, normal ordering etc. In this way, theories defined on curved spacetimes resemble ordinary quantum field theories in flat space. A valid operator formalism for string theory application should be able to reproduce the relevant correlation functions and their analytical properties. Usually these properties (the structure of zeros and poles) are strong enough to determine these correlators up to an overall constant [5], [6].

One version of operator formalism has been developed for the $b – c$ systems in [7], [8]. In this paper we would like to extend it in such a way that also the bosonic $\beta – \gamma$ systems can effectively be treated.

These systems are in fact interesting examples of CFT’s. They have been extensively studied in the second half of 1980s after they turned out to be a necessary ingredient (as Faddeev-Popov superghosts) in the perturbative approach to strings [9] – [12]. A treatment on hyperelliptic curves, mainly in connection with the computation of amplitudes in superstring theory at two loops, can for instance be found in refs. [13]. $\beta – \gamma$ systems have also been studied in the context of the Wess-Zumino-Witten model [14].

The analysis of the $\beta – \gamma$ systems is in many respects analogous to that of the $b – c$ systems. In fact, the only difference lies in the statistics, since the $\beta – \gamma$ fields commute while the $b – c$ fields anticommute. Following [9], many formulas characterizing both systems can be written in common. If by $b$ (and $c$) one denotes either the $b$ or $\beta$ ($c$ or $\gamma$) fields with conformal dimension $\lambda$ (and $1 – \lambda$), then both theories are defined by the action:
\[ S = \frac{1}{\pi} \int d^2z (b\bar{\partial}c). \quad (1.1) \]

where \( \bar{\partial} \equiv \frac{\partial}{\partial z} \). It follows from (1.1) that the OPE’s for the fields are

\[ c(z)b(w) \sim \frac{1}{z-w}, \quad b(z)c(w) \sim \frac{\epsilon}{z-w} \quad (1.2) \]

where \( \epsilon = 1 \) for \( b-c \) systems and \( \epsilon = -1 \) for \( \beta-\gamma \) systems. The energy-momentum tensor, determined by the dependence of (1.1) on the metric (in order to derive it one has to write the action in the covariant form) carries no dependence on \( \epsilon \):

\[ T = -\lambda b \partial c + (1-\lambda)(\partial b)c. \quad (1.3) \]

The OPE’s of the elementary fields with \( T \) can be described with the same formula both in the fermionic and bosonic case, as they are determined only by conformal dimensions:

\[ T(z)b(w) \sim \frac{1}{(z-w)^2}b(w) + \frac{1}{z-w}\partial b(w) + \ldots \quad (1.4) \]

\[ T(z)c(w) \sim \frac{1-\lambda}{(z-w)^2}c(w) + \frac{1}{z-w}\partial c(w) + \ldots \quad (1.5) \]

The OPE with two energy-momentum tensors depends on \( \epsilon \) since in calculating it one has to (anti)-commute elementary fields:

\[ T(z)T(w) \sim -\epsilon(6\lambda^2 - 6\lambda + 1) \frac{T(w)}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots \quad (1.6) \]

The correlation functions of the \( \beta-\gamma \) systems were intensively studied at the end of the last decade [11], [12].

In string theory computations, one would like to compute the correlators also within the bosonized version of the theory. It turns out that in addition to the scalar fields with quadratic action one has to introduce also auxiliary fermionic fields \( \eta - \xi \) of conformal dimensions 1 and 0. One gets an infinite number of inequivalent representations \( (q \text{ vacua in } [9]) \) of the basic algebra in a Hilbert space.

For \( \lambda \geq 1 \) and \( g \geq 2 \) only the zero modes in the \( b \) fields are present. In both bosonic and fermionic cases one has to insert delta functions for any zero mode in order to obtain a reasonable correlator. Without these insertions one gets vanishing or diverging amplitudes.
in the case of fermionic or bosonic systems respectively. In the path integral approach this follows from the properties of the integrals:

\[
\int_\mathbb{R} dx \to \infty, \quad \int d\zeta = 0 \quad (1.7)
\]

where \( \zeta \) is an element of a Grassmann algebra satisfying \( \zeta^2 = 0 \). In the fermionic case \( \delta(\zeta) = \zeta \), so that any insertion in the amplitudes of a delta function \( \delta(b) \) is equivalent to an insertion of a \( b \) field. In the bosonic case one has instead to make sense of true delta functions (or step functions) containing field operators in their argument.

The manipulations with delta function operators become straightforward in the ”gaussian” representation for \( \beta \) and \( \gamma \) fields [15]. In that representation one sees that inequivalent vacua can be chosen and that they cannot be interchanged under the action of a finite number of elementary excitations. Each vacuum state is determined by a choice of signs in a infinite product of gaussian (or, strictly speaking ”fresnelian”) functions \( \exp(\pm \frac{i}{2}x^2) \). The general property of delta-like operators is that they can map one vacuum state into another [15]. Some operations in the gaussian representation may seem formal, e.g. formally infinite normalization factors have to be included. However, there is always a unique procedure to handle with these factors in order to obtain finite results.

The gaussian representation makes it possible to define an operator formalism for the \( \beta - \gamma \) systems following the ideas of [8], where the case of the \( b - c \) systems has been discussed. In [8], the classical degrees of freedom are identified and the simplest possible anticommutation relations are postulated. Some of the elementary oscillators are defined as annihilation or creation operators. The remaining excitations correspond to the zero modes of the model. It has been demonstrated that these definitions enable the calculation of the correlation functions of the theory. The construction is not restricted to any particular class of RS - it can be done on arbitrary algebraic curve.

The aim of this paper is to present in detail the construction of an analogous operator formalism on RS for \( \beta - \gamma \) systems. The outcome of our computations are correlation functions with all the necessary analytic properties.

The operator formalism for the \( \beta - \gamma \) systems exhibits some similarities and also some interesting differences with respect to the operator formalism found in the case of the \( b - c \) fields in [8]. The starting point is the same, one chooses the same classical degrees of freedom and postulates the simplest possible commutation (instead of anticommutation) relations. One decomposes the elementary excitations into annihilation, creation and zero
mode sectors. A fundamental property of the normal ordering is that the difference between an ordered and an unordered product of two fields is equal to the Weierstrass kernel \( K_\lambda(z, w) \), which is one of the basic building blocks in the construction of the correlation functions. \( K_\lambda(z, w) \) is a \( \lambda \) differential in \( z \), a \( 1 - \lambda \) differential in \( w \) and its only finite singularity in \( z \) is a single pole at \( z = w \) (provided \( z \) and \( w \) parametrize points on the same branch of the RS). In our investigations we need to introduce multivalued fields on the complex sphere. In this sense one can think that new classes of CFT’s containing multivalued models are defined on the sphere [7]. The computation of the correlation functions for \( \beta - \gamma \) systems is quite different from the case of the \( b-c \) systems. In the bosonic case many nontrivial manipulations with Fresnel integrals are required before reaching the final result. This will be shown in detail in Chapters 2-4.

The experience from the \( \beta - \gamma \) systems can hopefully be used in dealing with more complicated and important systems, such for instance the bosonic theories with quadratic action. These theories provide the basis of the Coulomb gas formulation of minimal models. Therefore, the construction of an operator formalism for the \( \beta - \gamma \) systems can shed some light in the treatment of the minimal models on arbitrary RS represented as algebraic curve. Until now, only the case of hyperelliptic and \( Z_N \) symmetric curves have been studied in this context [16], [17].

2. \( \lambda = -1 \) COMPUTATIONS ON THE SPHERE

To better explain the ideas of the calculations, it is useful to start with the simpler situation of the complex sphere. We choose in this Chapter \( g = 0 \) and \( \lambda = -1 \) in order to have \( \beta \) rather then \( \gamma \) field zero modes.

The classical degrees of freedom are identified as

\[
\beta(z) = \sum_{j=-\infty}^{+\infty} \beta_j z^{-j+1} d z^\lambda
\]  \hspace{1cm} (2.1)

\[
\gamma(z) = \sum_{k=-\infty}^{+\infty} \gamma_k z^{-k-2} d z^{1-\lambda}
\]  \hspace{1cm} (2.2)

In quantum theory one postulates the following commutation relations:
\[ [\gamma_k, \beta_j] = \delta_{j+k,0}. \]  

(2.3)

The normal ordering is defined by assigning to the elementary excitations \( \beta_j \) and \( \gamma_k \) the property that they are creation or annihilation operators. Annihilation operators correspond to \( j \geq 2 \) and \( k \geq -1 \). Creation operators are those with \( j \leq -2 \) and \( k \leq -2 \). The remaining \( \beta_j \) operators \((j = 0, \pm 1)\) correspond to the three zero modes (holomorphic vector fields).

The above assignment implies:

\[
\gamma(z)\beta(w) := \gamma(z)\beta(w) + \frac{1}{z - w}. \quad (2.4)
\]

which should be understood as an identity in the radial ordered correlation functions.

Another important identity is (in \( \exp(i\beta(z)) \) the introduction of normal ordering makes no modification):

\[
\gamma(z)e^{ip\beta(w)} := \gamma(z)e^{ip\beta(w)} + \frac{ip}{z - w}e^{ip\beta(w)}. \quad (2.5)
\]

Let us start calculating propagator of the theory. It is known from the general discussion and it will also be very clearly seen in the course of the computations that the propagator requires 3 (the number of \( \beta \) field zero modes) insertions of delta functions of \( \beta \) fields.

Using the Wick theorem one calculates:

\[
<0|\gamma(z)\beta(w)\exp\left(\sum_{r=1}^{3} p_r \beta(w_r)\right)|0> = \frac{1}{z - w} <0|\exp\left(\sum_{r=1}^{3} p_r \beta_j w_r^{-j+1}\right)|0> + \sum_{r=1}^{3} \frac{ip_r}{z - w_r} <0|\beta(w)\exp\left(\sum_{t=1}^{3} \sum_{j=-\infty}^{+\infty} p_t \beta_j w_t^{-j+1}\right)|0>. \quad (2.6)
\]

Finally, the integration over \( \prod_{r=1}^{3} \frac{dp_r}{2\pi} \) has to be performed.

The computations will be done in the "gaussian" representation for the \( \beta - \gamma \) systems \[15\]. The basic definitions of that construction are summarized in Appendix A.

In order to evaluate (2.6) one calculates
\[ \exp \left( i \sum_{n=-\infty}^{+\infty} A_n \beta_n \right) |0> \]  \hspace{1cm} (2.7)

with the state \(|0>\) taken to be the state \(\Phi^{-1}\) defined in (A.8)

\[ |0> = \Phi^{-1} = \exp \frac{i}{2} \left( \sum_{n \geq -1} x_n^2 - \sum_{n < -1} x_n^2 \right). \]  \hspace{1cm} (2.8)

A concrete realization of the operators \(\beta_n\) is given by (see (A.2)):

\[ \beta_n = -\frac{i}{\sqrt{2}} (x_n - i \frac{\partial}{\partial x_n}) \]  \hspace{1cm} (2.9)

The operators \(\beta_n\) involve only \(x_{-n}\) and acting on \(\exp \left( -\frac{i}{2} x_{-n}^2 \right)\) give 0.

It follows that

\[ \exp \left( i \sum_{n=-\infty}^{+\infty} A_n \beta_n \right) |0> = \exp \left( -\frac{i}{2} \sum_{m<1} x_m^2 \right) \exp \left( i \sum_{n \leq 1} A_n \beta_n \right) \exp \left( \frac{i}{2} \sum_{m \geq -1} x_m^2 \right). \]  \hspace{1cm} (2.10)

Repeating steps similar to those presented in the end of Appendix A one finds

\[ \exp \left( -\frac{i}{2} \sum_{m<1} x_m^2 \right) \exp \left( i \sum_{n \leq 1} A_n \beta_n \right) \exp \left( \frac{i}{2} \sum_{m \geq -1} x_m^2 \right) = \]  \hspace{1cm} (2.11)

\[ \exp \left( -\frac{i}{2} \sum_{m<1} x_m^2 \right) \exp \left( \frac{i}{2} \sum_{n \leq 1} \left( A_n + i\sqrt{2} x_{-n} \right)^2 + x_{-n}^2 \right) \]

To define a bilinear form we introduce the right vacuum:

\[ <0| = \exp \frac{i}{2} \left( \sum_{m>1} x_m^2 - \sum_{m \leq 1} x_m^2 \right). \]  \hspace{1cm} (2.12)

A proper normalization of scalar product is included in a (formally infinite) factor \(V\):

\[ <0| \prod_{s=1}^{3} \delta(\beta(z_s)) |0> = \frac{1}{V} \int \prod_{s=1}^{3} dp_s \int \prod_{k=-\infty}^{+\infty} dx_k \exp \left( -i \sum_{m \leq 1} x_m^2 \right) \times \]
\[ \times \exp \left( -\frac{i}{2} \sum_{m>1} \left( A_{-m} + i\sqrt{2}x_m \right)^2 \right) \exp \left( -\frac{i}{2} \sum_{m=0,\pm1} \left( (A_m + i\sqrt{2}x_{-m})^2 + 2x_m^2 \right) \right) \]

(2.13)

for \( A_m = \sum_{s=1}^3 p_s z_{s-m} \). It will soon become clear that \( V \) should be defined as

\[ V = \left( \prod_{m<1} \int \exp \left( -ix_m^2 \right) dx_m \right) \left( \prod_{n>1} \int \exp \left( ix_n^2 \right) dx_n \right) \times \]

\[ \times \left( \prod_{k=0,\pm1} \int \exp \left( -\frac{i}{2}p_k^2 \right) dp_k \right) \left( \prod_{l=0,\pm1} \int \exp \left( -ix_l^2 \right) dx_l \right) . \]  

(2.14)

It is most convenient to integrate first over \( \prod_m dx_m \) for \( m < 1 \) and then for \( m > 1 \). Both integrations are trivial and contribute only to overall normalization. Next one should integrate over \( \prod_{s=1}^3 dp_s \) after a convenient change of variables that will produce an appropriate Jacobian. The trivial integration over \( \prod_{m=0,\pm1} dx_m \) will be performed later.

The above mentioned change of variables is:

\[ p_j \rightarrow A_m, \quad j = 1, 2, 3, \quad m = 0, \pm1 . \]  

(2.15)

\[ \int \prod_{j=1}^3 dp_j = \int \prod_{m=-1}^1 dA_m \det \left( \frac{\partial A_n}{\partial p_k} \right)^{-1} \]  

(2.16)

where

\[ \det \left( \frac{\partial A_n}{\partial p_k} \right) = (w_1 - w_2)(w_1 - w_3)(w_2 - w_3) . \]  

(2.17)

The integration over \( dp_s \) and \( dx_n \) cancel precisely the factor \( V \) and the final result is

\[ < 0 | \prod_{s=1}^3 \delta(\beta(w_s)) | 0 > = \frac{1}{(w_1 - w_2)(w_1 - w_3)(w_2 - w_3)} . \]  

(2.18)

In the computation of the second term in (2.6) it is sufficient to concentrate ourselves on the most important point without repeating many of the above arguments.

\[ \beta(w) \exp \left( i \sum_{n=-\infty}^{+\infty} A_n \beta_n \right) | 0 > = \]
\[ -\frac{i}{\sqrt{2}} \sum_{j=-\infty}^{+\infty} w^{-j+1} (x_{-j} - i \frac{\partial}{\partial x_{-j}}) \exp \left( -\frac{i}{2} \sum_{m<-1} x_m^2 \right) \times \]

\[ \times \exp \left( -\frac{i}{2} \sum_{n\leq 1} \left( (A_n + i\sqrt{2}x_{-n})^2 + x_{-n}^2 \right) \right) = \]

\[ = -\exp \left( -\frac{i}{2} \sum_{m<1} x_m^2 \right) \sum_{j\leq 1} w^{-j+1} (A_j + i\sqrt{2}x_{-j}) \times \]

\[ \times \exp \left( -\frac{i}{2} \sum_{n\leq 1} \left( (A_n + i\sqrt{2}x_{-n})^2 + x_{-n}^2 \right) \right) \quad (2.19) \]

With the above definition of the right vacuum \( <0| \) one gets

\[ <0|\beta(w) \exp \left( i \sum_{n=-\infty}^{+\infty} A_n \beta_n \right) |0> = -\frac{1}{V} \int \prod_{s=1}^{3} dp_s \int \prod_{k=-\infty}^{+\infty} dx_k \sum_{j\leq 1} w^{-j+1} (A_j + i\sqrt{2}x_{-j}) \]

\[ \times \exp \left( -i \sum_{m<1} x_m^2 \right) \exp \left( -\frac{i}{2} \sum_{m>1} (A_{-m} + i\sqrt{2}x_m)^2 \right) \times \]

\[ \times \exp \left( -\frac{i}{2} \sum_{m=0,\pm1} \left( (A_m + i\sqrt{2}x_m)^2 + 2x_{-m}^2 \right) \right) \quad (2.20) \]

As before, it is most convenient to perform first the integration over \( \prod_{m<-1} dx_m \) and then over \( \prod_{m>1} dx_m \). The sum over \( j \leq 1 \) reduces to three terms with \( j = 0, \pm 1 \). All the other terms vanish due to integration over antisymmetric function in \( x_n \) (\( n > 1 \)).

Having in mind future generalizations, the following notation is useful:

\[ A_j = \sum_{k=1}^{3} R_{jk} p_k. \quad (2.21) \]

Thus, in computing the second term in (2.6) one is left with the evaluation of:

\[ -i \frac{1}{V} \int \prod_{j=0,\pm1} dA_j \int \prod_{n=0,\pm1} dx_n \det \left( \frac{\partial A_m}{\partial p_r} \right)^{-1} \times \]
\[
\times \sum_{s=1}^{3} \sum_{j,k=-1}^{1} \frac{R_{sk}^{-1} A_k}{z - w_s} w^{1-j} (A_j + i\sqrt{2}x_{-j}) \exp \left( -\frac{i}{2} \sum_{l=-1}^{1} \left( (A_l + i\sqrt{2}x_{-l})^2 + 2x_l^2 \right) \right) . \tag{2.22}
\]

with

\[
V' = ( \prod_{k=0,\pm 1} \int \exp \left( -\frac{i}{2} p_k^2 \right) dp_k ) ( \prod_{l=0,\pm 1} \int \exp \left( -ix_l^2 \right) dx_l ) . \tag{2.23}
\]

It is useful to perform still another change of variables:

\[
\tilde{A}_j = A_j + i\sqrt{2}x_{-j} \tag{2.24}
\]

Thus one can rewrite (2.22) as

\[
-i \frac{1}{V'} \int \prod_{j=0,\pm 1} d\tilde{A}_j \int \prod_{n=0,\pm 1} dx_n \left( \det \left( \frac{\partial \tilde{A}_m}{\partial p_r} \right) \right)^{-1} \times
\]

\[
\times \sum_{s=1}^{3} \sum_{j,k=-1}^{1} \frac{R_{sk}^{-1}(\tilde{A}_k - i\sqrt{2}x_{-k})}{z - w_s} w^{1-j} \tilde{A}_j \exp \left( -\frac{i}{2} \sum_{l=-1}^{1} \left( \tilde{A}_l^2 + 2x_l^2 \right) \right) \tag{2.25}
\]

The integration over \( \prod_{j=-1}^{1} d\tilde{A}_j \) gives nonzero result only for \( k = j \). Terms with \( i\sqrt{2}x_{-k} \) drop out (they give rise to integral linear in \( \tilde{A}_j \)). The integration over \( \prod_{l=-1}^{1} dx_l \) is then trivial and one obtains:

\[
-\sum_{s=1}^{3} \sum_{j=-1}^{1} \frac{R_{sj}^{-1}}{z - w_s} w^{1-j} . \tag{2.26}
\]

In order to get the numerical factor in (2.26) one has to use

\[
\int_{\mathbb{R}} A^2 e^{-\frac{i}{2} A^2} dA = -i \int_{\mathbb{R}} e^{-\frac{i}{2} A^2} dA \tag{2.27}
\]

The matrix \( R^{-1} \) is of the form:

\[
R^{-1} = \frac{1}{(w_1 - w_2)(w_1 - w_3)(w_2 - w_3)} \left( \begin{array}{cccc}
w_2 - w_3 & w_3^2 - w_2^2 & w_2w_3(w_2 - w_3) \\
w_3 - w_1 & w_2^2 - w_3^2 & w_1w_3(w_3 - w_1) \\
w_1 - w_2 & w_3^2 - w_1^2 & w_1w_2(w_1 - w_2) \\
\end{array} \right) \tag{2.28}
\]
and the final result is (one has to include $\det \frac{\partial \tilde{A}}{\partial p}$):

$$< 0|\gamma(z)\beta(w)\prod_{s=1}^{3}\delta(\beta(w_s))|0 >=$$

$$= \frac{1}{(w_1-w_2)(w_1-w_3)(w_2-w_3)} \left( \frac{1}{z-w} - \frac{(w-w_2)(w-w_3)}{(z-w_1)(w_1-w_2)(w_1-w_3)} + \frac{(w-w_1)(w-w_3)}{(z-w_2)(w_1-w_2)(w_2-w_3)} - \frac{(w-w_1)(w-w_2)}{(z-w_3)(w_1-w_3)(w_2-w_3)} \right) =$$

$$= \frac{1}{z-w} \prod_{s=1}^{3} \frac{w-w_s}{z-w_s} \prod_{m<n} (w_m-w_n)^{-1} \quad (2.29)$$

It is clear that the above expression has all the analytical properties that the propagator of the bosonic $\beta - \gamma$ systems should possess.

3. THE PROPAGATOR ON ARBITRARY ALGEBRAIC CURVES

In this Chapter the $\beta - \gamma$ system with $\lambda \geq 1$ given by (1.1) is defined on arbitrary RS represented by means of algebraic equation

$$F(z, y) = y^N P_N(z) + y^{N-1} P_{N-1}(z) + ... + P_0(z) = 0 \quad (3.1)$$

where $P_j(z)$ are polynomials in $z$. $y$ can be viewed either as a singlevalued functions on the RS or as multivalued function on the complex sphere. The monodromy properties of $y(z)$ then define the RS.

The classical degrees of freedom are identified as

$$\beta(z) = \sum_{s=0}^{N-1} \sum_{j=-\infty}^{+\infty} \beta_{s,j} z^{-j-\lambda} f_s(z) \quad (3.2)$$

$$\gamma(z) = \sum_{p=0}^{N-1} \sum_{k=-\infty}^{+\infty} \gamma_{p,k} z^{-k+\lambda-1} \phi_p(z) \quad (3.3)$$
where
\[
f_k(z) = \frac{y^{N-1-k}(z)dz}{(F_y(z, y(z)))^{\lambda}}
\] (3.4)

and
\[
\phi_k(z) = \frac{dz^{1-\lambda}}{(F_y(z, y(z)))^{1-\lambda}} \left[ y^k(z) + y^{k-1}(z)P_{N-1}(z) + \ldots + P_{N-k}(z) \right].
\] (3.5)

The basic commutation relations are:
\[
[\gamma_{p,k}, \beta_{s,j}] = \delta_{s,p} \delta_{j+k,0}.
\] (3.6)

The price paid for the simple commutation relations (3.6) is the complicated definition of basis (3.5).

The normal ordering is introduced by assigning to the elementary excitations $\beta_{s,j}$ and $\gamma_{s,k}$ the property that they are creation or annihilation operators. Annihilation operators are those with $j \geq 1 - \lambda$ and $k \geq \lambda$ and creation operators with $j \leq -\lambda - M_s$ and $k \leq \lambda - 1$ ($M_s$ is a number of zero modes in the $s$ sector of the theory). The remaining operators $\beta_{s,j}$ with $-\lambda - M_s < j < 1 - \lambda$ correspond to zero modes.

From the above assignment it follows that:
\[
\gamma(z)\beta(w) =: \gamma(z)\beta(w) : + \frac{1}{z-w} \sum_{n=0}^{N-1} f_n(z)\phi_n(w) =: \gamma(z)\beta(w) : + K_\lambda(z, w).
\] (3.7)

It is an identity in radial ordered correlation functions. $K_\lambda(z, w)$ is the Weierstrass kernel (see Introduction).

A simple implication of (3.7) is
\[
\gamma(z)e^{ip\beta(w)} =: \gamma(z)e^{ip\beta(w)} : + ipe^{ip\beta(w)}K_\lambda(z, w)e^{ip\beta(w)}.
\] (3.8)

We wish to calculate the propagator for $\beta-\gamma$ system. In order to have finite correlators one has to insert a number $M_{tot}$ of delta functions in the $\beta$ fields, where $M_{tot}$ is the total number of $\beta$ zero modes:
\[
M_{tot} = \sum_{s=0}^{N-1} M_s = (g-1)(2\lambda - 1)
\] (3.9)
The Wick theorem implies

\[ <0|\gamma(z)|\beta(w)\exp\left(i\sum_{r=1}^{M_{\text{tot}}} p_r \beta(w_r)\right)|0> \]

\[ = K_\lambda(z, w) <0|\exp\left(iM_{\text{tot}} \sum_{r=1}^{N-1} \sum_{j=-\infty}^{+\infty} p_r \beta_{s,j} w_r^{-j-\lambda} f_s(w_r)\right)|0> + \sum_{r=1}^{M_{\text{tot}}} ip_r K_\lambda(z, w_r) <0|\beta(w)\exp\left(i\sum_{t=1}^{M_{\text{tot}}} \sum_{s=0}^{N-1} \sum_{j=-\infty}^{+\infty} p_t \beta_{s,j} w_t^{-j-\lambda} f_s(w_t)\right)|0>. \] (3.10)

The propagator is obtained after integration over \( \prod_{r=1}^{M_{\text{tot}}} \frac{dp_r}{2\pi} \).

As in Chapter 2, the computations will be performed in the "gaussian" representation for the \( \beta - \gamma \) system [15]. However, complications can be expected on a Riemann surface due to appearance of \( M_s \) zero modes in the sector \( s \) of the theory.

A first attempt to accommodate these zero modes is to modify the way in which the \( \beta_{s,j} \) and \( \gamma_{p,n} \) excitations are realized:

\[ \beta_{s,j} = -\frac{i}{\sqrt{2}}(x_{s,-j-\nu_s} - i \frac{\partial}{\partial x_{s,-j-\nu_s}}) \] (3.11)

\[ \gamma_{p,n} = \frac{1}{\sqrt{2}}(x_{p,n+\nu_p} + i \frac{\partial}{\partial x_{p,n+\nu_p}}) \] (3.12)

where \( \nu_s \) is to be related to the number \( M_s \) of zero modes in the \( s \) sector.

Let us suppose that the vacuum state is taken as:

\[ \Phi^\kappa(x) = \exp\left(i \sum_{s=0}^{N-1} \left( \sum_{n \geq \kappa_s} x_{s,n}^2 - \sum_{n < \kappa_s} x_{s,n}^2 \right) \right) \] (3.13)

where \( \kappa_s \) is again chosen in an appropriate way. Unfortunately, it turns out that nothing can be achieved in this way. The conditions that \( \beta_{s,j} \) and \( \gamma_{s,k} \) are annihilation operators only for \( j \geq 1 - \lambda, k \geq \lambda \) imply in fact that

\[ \kappa_s = \lambda, \quad \nu_s = 0. \] (3.14)

We conclude that the only way to accommodate the generic structure of zero modes is to redefine the scalar product in an appropriate way. We shall come back to this point shortly.
In calculating (3.10) one follows most of steps described in Chapter 2. The vacuum state is that introduced in (3.13) with $\kappa_s = \lambda$. One finds

$$\exp \left( i \sum_{s=0}^{N-1} \sum_{j=\infty} A_{s,j} \beta_{s,j} \right) |0> = \exp \left( i \sum_{s=0}^{N-1} \sum_{j \leq \lambda} A_{s,j} \beta_{s,j} \right) |0> =$$

$$= \exp \left( -\frac{i}{2} \sum_{s=0}^{N-1} \left( \sum_{n<\lambda} x_{s,n}^2 + \sum_{j \leq -\lambda} \left( (A_{s,j} + i\sqrt{2}x_{s,-j})^2 + x_{s,-j}^2 \right) \right) \right)$$

(3.15)

with

$$A_{s,j} = \sum_{r=1}^{M_{tot}} p_r w_r^{-j-\lambda} f_s(w_r)$$

(3.16)

The correct definition of $<0|$ is

$$<0| = \exp \left( \frac{i}{2} \sum_{s=0}^{N-1} \left( \sum_{n \geq \lambda+M_s} x_{s,n}^2 - \sum_{n<\lambda+M_s} x_{s,n}^2 \right) \right)$$

(3.17)

A possible way of thinking about $<0|$ is that under the conjugation

$$x_{s,n}^\dagger = x_{s,-n+2\lambda+M_s-1}.$$ 

(3.18)

The conjugation on elementary excitations can be deduced from (3.11) and (3.12) to be

$$\beta_{s,n}^\dagger = -\beta_{s,-n-2\lambda-M_s+1}, \quad \gamma_{s,n}^\dagger = \gamma_{s,-n+2\lambda+M_s-1}.$$ 

(3.19)

This definition of conjugation is analogous to that obtained for the $b-c$ system on arbitrary RS [18]. For $g = 0$ and $\lambda = -1$ (the situation discussed in Chapter 2) $M_{tot} = 3$ and $\beta_n^\dagger = -\beta_n, \gamma_n^\dagger = \gamma_n$.

The first term in (3.10) is

$$\frac{1}{V_\lambda} \int \prod_{s=0}^{N-1} \prod_{j=\infty} dx_{s,j} \int \prod_{r=1}^{M_{tot}} dp_r \exp \left( \frac{i}{2} \sum_{s=0}^{N-1} \left( \sum_{n \geq \lambda+M_s} x_{s,n}^2 - \sum_{n<\lambda+M_s} x_{s,n}^2 \right) \right) \times$$

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\begin{align*}
\times \exp \left( -\frac{i}{2} \sum_{s=0}^{N-1} \left( \sum_{n<\lambda} x_{s,n}^2 + \sum_{j\geq\lambda} \left( (A_{s,-j} + i\sqrt{2}x_{s,j})^2 + x_{s,j}^2 \right) \right) \right). \tag{3.20}
\end{align*}

The normalization factor (formally infinite) $V_{\lambda}$ is

\begin{align*}
V_{\lambda} &= \left( \int \prod_{s=0}^{N-1} \prod_{n<\lambda} \exp \left( -ix_{s,n}^2 \right) dx_{s,n} \right) \left( \int \prod_{s=0}^{N-1} \prod_{n>\lambda} \exp \left( ix_{s,n}^2 \right) dx_{s,n} \right) \\
\times \left( \int \prod_{r=1}^{M_{\text{tot}}} \exp \left( -\frac{i}{2} p_r^2 \right) dp_j \right) \left( \int \prod_{s=0}^{N-1} \prod_{n=\lambda}^{\lambda+M_s-1} \exp \left( -ix_{s,n}^2 \right) dx_{s,n} \right) \tag{3.21}
\end{align*}

One integrates first over $\prod_{s=0}^{N-1} \prod_{n<\lambda} dx_{s,n}$ and then over $\prod_{s=0}^{N-1} \prod_{n>\lambda+M_s} dx_{s,n}$.

What remains is

\begin{align*}
\frac{1}{V'_{\lambda}} &\int \prod_{s=0}^{N-1} \prod_{n=\lambda}^{\lambda+M_s-1} dx_{s,n} \int \prod_{r=1}^{M_{\text{tot}}} \exp \left( -\frac{i}{2} p_r^2 \right) dp_j \\
\times \exp \left( -\frac{i}{2} \sum_{s=0}^{N-1} \sum_{n=\lambda}^{\lambda+M_s-1} \left( (A_{s,-n} + i\sqrt{2}x_{s,n})^2 + 2x_{s,n}^2 \right) \right). \tag{3.22}
\end{align*}

with

\begin{align*}
V'_{\lambda} &= \left( \int \prod_{r=1}^{M_{\text{tot}}} \exp \left( -\frac{i}{2} p_r^2 \right) dp_j \right) \left( \int \prod_{s=0}^{N-1} \prod_{n=\lambda}^{\lambda+M_s-1} \exp \left( -ix_{s,n}^2 \right) dx_{s,n} \right). \tag{3.23}
\end{align*}

The number of integrations to be done over $dx_{s,n}$ is equal $M_{\text{tot}}$. It is very convenient to introduce an index $L$ which runs over all the allowed pairs $(s, n)$ $L = 1, \ldots, M_{\text{tot}}$. There is still minor technical complications due to the fact that in expressions with $x_{s,n}$ are combined with $A_{s,-n}$. The following definition will be adopted: if the index $J$ denotes a pair $(s, n)$ then

\begin{align*}
x_{J} \equiv x_{s,n}, \quad A_{J} \equiv A_{s,-n}. \tag{3.24}
\end{align*}

A change of variables is to be done

\begin{align*}
p_j \rightarrow A_{r,-k} = \sum_{n=1}^{M_{\text{tot}}} p_n w_n^{k-\lambda} f_r(w_n) = \sum_{n=1}^{M_{\text{tot}}} p_n \Omega_L(w_n) \tag{3.25}
\end{align*}
\( \Omega_L(w) \) are \( \beta \) field zero modes, i.e. holomorphic \( \lambda \) differentials. Introducing a similar notation as in Chapter 2 one can write

\[
A_L = \sum_{j=1}^{M_{\text{tot}}} R_{Lj} p_j
\]  

(3.26)

where

\[
R_{Lj} = \begin{pmatrix}
\Omega_1(w_1) & \Omega_1(w_2) & \ldots & \Omega_1(w_{M_{\text{tot}}}) \\
\ldots & \ldots & \ldots & \ldots \\
\Omega_{M_{\text{tot}}}(w_1) & \Omega_{M_{\text{tot}}}(w_2) & \ldots & \Omega_{M_{\text{tot}}}(w_{M_{\text{tot}}})
\end{pmatrix}
\]  

(3.27)

is a matrix of zero modes. The result is that only the Jacobian arising due to the change of variables (3.25) gives nontrivial contribution. The remaining integrations cancel the normalization factor \( V'_{\lambda} \) and the first term in (3.10) is

\[
\frac{K_\lambda(z, w)}{\det \Omega_L(w_j)}.
\]  

(3.28)

In order to calculate the next term in (3.10) one should follow the procedure presented in Chapter 2 with the above explained modifications due to presence of \( N \) sectors and \( M_s \) zero modes in each of them. The result is

\[
- \det (\Omega_L(w_j)) \sum_{s,N=1}^{M_{\text{tot}}} K_\lambda(z, w_s) R_{sN}^{-1} \Omega_N(w).
\]  

(3.29)

The final result can be written in compact form

\[
<0|\gamma(z)\beta(w) \prod_{r=1}^{M_{\text{tot}}} \delta(\beta(w_r))|0> = \det \left( \begin{array}{cccc}
K_\lambda(z, w) & K_\lambda(z, w_1) & \ldots & K_\lambda(z, w_{M_{\text{tot}}}) \\
\Omega_1(w) & \Omega_1(w_1) & \ldots & \Omega_1(w_{M_{\text{tot}}}) \\
\ldots & \ldots & \ldots & \ldots \\
\Omega_{M_{\text{tot}}}(w) & \Omega_{M_{\text{tot}}}(w_1) & \ldots & \Omega_{M_{\text{tot}}}(w_{M_{\text{tot}}})
\end{array} \right) (\det \Omega_L(w_j))^{-2}.
\]  

(3.30)

It has all the necessary analytical properties of the propagator: poles as \( z \to w \), \( z \to w_r, w_{r_1} \to w_{r_2} (r_1 \neq r_2) \) and zeroes as \( w \to w_j \).

One can also define

\[
\frac{<0|\gamma(z)\beta(w) \prod_{r=1}^{M_{\text{tot}}} \delta(\beta(w_r))|0>}{<0|\prod_{r=1}^{M_{\text{tot}}} \delta(\beta(w_r))|0>} = \]

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4. THE 4-POINT CORRELATOR

The operator formalism allows also the calculation of higher order correlation functions. In this Chapter the computation of the four-point correlator

\[
< 0|\gamma(z_1)\gamma(z_2)\beta(w_1)\beta(w_2) \prod_{s=1}^{M_{tot}} \delta(\beta(u_s))|0 > \quad (4.1)
\]

is presented. The Wick theorem implies

\[
< 0|\gamma(z_1)\gamma(z_2)\beta(w_1)\beta(w_2) \prod_{s=1}^{M_{tot}} \exp(ip_s\beta(u_s))|0 > =
\]

\[
= (K_{\lambda}(z_1, w_1)K_{\lambda}(z_2, w_2) + K_{\lambda}(z_1, w_2)K_{\lambda}(z_2, w_1)) < 0| \prod_{s=1}^{M_{tot}} \exp(ip_s\beta(u_s))|0 > +
\]

\[
+ K_{\lambda}(z_1, w_1) \sum_{t=1}^{M_{tot}} ip_t K_{\lambda}(z_2, u_t) < 0|\beta(w_2) \prod_{s=1}^{M_{tot}} \exp(ip_s\beta(u_s))|0 >
\]

\[
+ K_{\lambda}(z_1, w_2) \sum_{t=1}^{M_{tot}} ip_t K_{\lambda}(z_2, u_t) < 0|\beta(w_1) \prod_{s=1}^{M_{tot}} \exp(ip_s\beta(u_s))|0 >
\]

\[
+ K_{\lambda}(z_2, w_1) \sum_{t=1}^{M_{tot}} ip_t K_{\lambda}(z_1, u_t) < 0|\beta(w_2) \prod_{s=1}^{M_{tot}} \exp(ip_s\beta(u_s))|0 >
\]

\[
+ K_{\lambda}(z_2, w_2) \sum_{t=1}^{M_{tot}} ip_t K_{\lambda}(z_1, u_t) < 0|\beta(w_1) \prod_{s=1}^{M_{tot}} \exp(ip_s\beta(u_s))|0 >
\]

\[
- \sum_{t_1, t_2=1}^{M_{tot}} p_{t_1} p_{t_2} K_{\lambda}(z_1, u_{t_1}) K_{\lambda}(z_2, u_{t_2}) < 0|\beta(w_1)\beta(w_2) \prod_{s=1}^{M_{tot}} \exp(ip_s\beta(u_s))|0 > \quad (4.2)
\]
Only the last term in (4.2) requires a new analysis.

It is necessary to compute

\[
\beta(w_1)\beta(w_2) \exp \left( i \sum_{s=0}^{N-1} \sum_{j=-\infty}^{+\infty} A_{s,j} \beta_{s,j} \right) |0> = \\
= -\frac{1}{2} \sum_{r,s=0}^{N-1} \sum_{j,k=-\infty}^{+\infty} \psi_{r,-j} \psi_{s,-k} \left( x_{r,j} - i \frac{\partial}{\partial x_{r,j}} \right) \left( x_{s,k} - i \frac{\partial}{\partial x_{s,k}} \right) \times \\
\times \exp \left( -\frac{i}{2} \sum_{s'=0}^{N-1} \sum_{n'<\lambda} x_{s',n'}^2 \right) \exp \left( -\frac{i}{2} \sum_{q=0}^{N-1} \sum_{m \geq \lambda} \left( A_{q,-m} + i \sqrt{2} x_{q,m} \right)^2 + x_{q,m}^2 \right) = \\
= \sum_{r,s=0}^{N-1} \sum_{j,k \geq \lambda} \psi_{r,-j} \psi_{s,-k} \left( A_{r,-j} + i \sqrt{2} x_{r,j} \right) \left( A_{s,-k} + i \sqrt{2} x_{s,k} \right) + i \delta_{jk} \times \\
\times \exp \left( -\frac{i}{2} \sum_{s'=0}^{N-1} \sum_{n'<\lambda} x_{s',n'}^2 \right) \exp \left( -\frac{i}{2} \sum_{q=0}^{N-1} \sum_{m \geq \lambda} \left( A_{q,m} + i \sqrt{2} x_{q,-m} \right)^2 + x_{q,-m}^2 \right) \tag{4.3}
\]

where

\[
\beta(w) = \sum_{s=0}^{N-1} \sum_{j=-\infty}^{+\infty} \psi_{s,j} \beta_{s,j}, \quad \psi_{s,j}(w) = f_s(w) w^{-j-\lambda} \tag{4.4}
\]

so that

\[
A_{r,j} = \sum_{s=1}^{M_{\text{tot}}} p_s \psi_{r,j}(u_s). \tag{4.5}
\]

In the correlator

\[
<0|\beta(w_1)\beta(w_2) \exp \left( i \sum_{r=1}^{M_{\text{tot}}} p_r \beta(u_r) \right) |0> = \\
= \frac{1}{V}\int \prod_{s=0}^{N-1} \prod_{j=-\infty}^{+\infty} dx_{s,j} \int \prod_{r=1}^{M_{\text{tot}}} dp_r \exp \left( \frac{i}{2} \sum_{t=0}^{N-1} \left( \sum_{n \geq \lambda+M_s} x_{s,n}^2 - \sum_{n<\lambda+M_s} x_{s,n}^2 \right) \right) \times
\]
\[
\times \sum_{r,s=1}^{N-1} \sum_{j,k \geq \lambda} \Psi_{r,-j} \Psi_{s,-k} \left( (A_{r,-j} + i\sqrt{2}x_{r,j})(A_{s,-k} + i\sqrt{2}x_{s,k}) + i\delta_{j,k}\delta_{r,s} \right) \times \\
\times \exp \left( -\frac{i}{2} \sum_{s=0}^{N-1} \left( \sum_{n<\lambda} x_{s,n}^2 + \sum_{j \geq \lambda} (A_{s,-j} + i\sqrt{2}x_{s,j})^2 + x_{s,j}^2 \right) \right) 
\]

the integrations over \( dx_{s,n} \) for \( n < \lambda \) are trivial. In the integrations over \( dx_{s,n} \) for \( n \geq \lambda + M_s \) a crucial observation is that in the sum over \( j, k > \lambda \) the only terms that survive are those for which \( j, k \leq \lambda + M_s - 1 \). The reason is that (2.27) holds. One obtains therefore

\[
< 0|\beta(w_1)\beta(w_2)\exp \left( i \sum_{r=1}^{M_{\text{tot}}} p_r \beta(u_r) \right) = \frac{1}{V'} \int \prod_{s=0}^{N-1} \prod_{n=\lambda}^{\lambda+M_s-1} dx_{s,n} \int \prod_{r=1}^{M_{\text{tot}}} dp_r \times \\
\times \sum_{r,s=1}^{N-1} \sum_{j,k=\lambda}^{\lambda+M_s-1} \Psi_{r,-j} \Psi_{s,-k} \left( (A_{r,-j} + i\sqrt{2}x_{r,j})(A_{s,-k} + i\sqrt{2}x_{s,k}) + i\delta_{j,k}\delta_{r,s} \right) \times \\
\times \exp \left( -\frac{i}{2} \sum_{s=0}^{N-1} \sum_{n=\lambda}^{\lambda+M_s-1} \left( A_{s,-n} + i\sqrt{2}x_{s,n} \right)^2 + 2x_{s,n}^2 \right) \right). 
\]

At this point it is very convenient to rewrite above expression using the indices \( L \) introduced in Chapter 3.

\[
< 0|\beta(w_1)\beta(w_2)\exp \left( i \sum_{r=1}^{M_{\text{tot}}} p_r \beta(u_r) \right) |0 >= \frac{1}{V'} \int \prod_{J=1}^{M_{\text{tot}}} \int \prod_{r=1}^{M_{\text{tot}}} dp_r \times \\
\times \sum_{K,L=1}^{M_{\text{tot}}} \Psi_K \Psi_L \left( (A_K + i\sqrt{2}x_K)(A_L + i\sqrt{2}x_L) + i\delta_{K,L} \right) \times \\
\times \exp \left( -\frac{i}{2} \sum_{J=1}^{M_{\text{tot}}} \left( A_J + i\sqrt{2}x_J^2 + 2x_J^2 \right) \right). 
\]

One should not forget that in (4.2) this expression is multiplied by \((-p_{t_1}p_{t_2})\). Performing the standard changes of variables \( p_r \to A_L \) and \( A_J \to \tilde{A}_J = A_J + i\sqrt{2}x_J \), the remaining integrals are...
\[ \frac{1}{V^\lambda} \int \prod_{K=1}^{M_{\text{tot}}} dx_K \int \prod_{J=1}^{M_{\text{tot}}} d\tilde{A}_J \exp \left( -\frac{i}{2} \sum_{J=1}^{M_{\text{tot}}} \left( \tilde{A}_J^2 + 2x_J^2 \right) \right) \times \]

\[ \times R_{t_1 I}^{-1} R_{t_2 N}^{-1} (\tilde{A}_M - i\sqrt{2}x_M)(\tilde{A}_N - i\sqrt{2}x_N) \sum_{K,L=1}^{M_{\text{tot}}} \Psi_K \Psi_L \left( \tilde{A}_K \tilde{A}_L + i\delta_{K,L} \right) \quad (4.9) \]

One can replace \( \Psi_J(z) \) with \( \Omega_J(z) \) as for the allowed values of \( J \) only the zero modes appear in the above expression, which we denote with the symbol \( \Omega_J \).

Integrals to be evaluated are

\[ \int \prod_{Q=1}^{M_{\text{tot}}} d\tilde{A}_Q (\tilde{A}_M - i\sqrt{2}x_M)(\tilde{A}_N - i\sqrt{2}x_N)(\tilde{A}_K \tilde{A}_L + i\delta_{K,L}) \exp \left( -\frac{i}{2} \sum_{S=1}^{M_{\text{tot}}} \tilde{A}_S^2 \right) \quad (4.10) \]

It is clear that terms with \( x_M \) drop out. Terms linear in \( x_M \) give rise to terms linear or of the third order in \( \tilde{A}_P \) which yield no contribution after integration. Also the quadratic term containing \( x_M x_N \) gives rise to integral with \( \tilde{A}_K \tilde{A}_L + i\delta_{K,L} \) which integrated give 0 because of (2.27).

The tensor structure of (4.10) implies that it must be of the form

\[ \text{const} (\delta_{N,K}\delta_{M,L} + \delta_{N,L}\delta_{M,K}) \int \prod_{S=1}^{M_{\text{tot}}} d\tilde{A}_S \exp \left( -\frac{i}{2} \sum_{S=1}^{M_{\text{tot}}} \tilde{A}_S^2 \right) . \]

Using

\[ \int dp \exp \left( -\frac{i}{2} p^2 \right) = -3 \int dp \exp \left( -\frac{i}{2} p^2 \right) \quad (4.11) \]

and

\[ \int dp \exp \left( -\frac{i}{2} p^2 \right) = -i \int dp \exp \left( -\frac{i}{2} p^2 \right) . \quad (4.12) \]

it is easy to verify that \( \text{const} = -1. \)

After performing the remaining trivial integrations the final expression for the last term in (4.2) is (apart from the Jacobian \( \det \left( \frac{\partial A_I}{p_r} \right) \))
\[
\sum_{t_1, t_2=1}^{M_{\text{tot}}} K_\lambda(z_1, u_{t_1}) K_\lambda(z_2, u_{t_2}) \left( R_{t_1 L}^{-1} \Omega_L(w_2) R_{t_2 K}^{-1} \Omega_K(w_1) + R_{t_1 K}^{-1} \Omega_K(w_1) R_{t_2 L}^{-1} \Omega_L(w_2) \right)
\]

 Altogether (4.12) is equal to

\[
\frac{1}{\det \Omega_I(u_J)} \left( K_\lambda(z_1, w_1) K_\lambda(z_2, w_2) + K_\lambda(z_1, w_2) K_\lambda(z_2, w_1) + \right.
\]

\[
- \sum_{t=1}^{M_{\text{tot}}} \sum_{K=1}^{M_{\text{tot}}} \left( K_\lambda(z_1, w_1) K_\lambda(z_2, u_t) R_{t K}^{-1} \Omega_K(z_2) + K_\lambda(z_1, w_2) K_\lambda(u_t, w_1) R_{t K}^{-1} \Omega_K(z_2) + \right.
\]

\[
+ K_\lambda(z_2, w_1) K_\lambda(u_t, w_2) R_{t K}^{-1} \Omega_K(z_1) + K_\lambda(z_1, w_1) K_\lambda(u_t, w_2) R_{t K}^{-1} \Omega_K(z_1) + \]

\[
+ \sum_{t_1, t_2=1}^{M_{\text{tot}}} \sum_{L, K=1}^{M_{\text{tot}}} \left( K_\lambda(u_{t_1}, w_1) K_\lambda(u_{t_2}, w_2) R_{t_1 L}^{-1} \Omega_L(z_2) R_{t_2 K}^{-1} \Omega_K(z_1) + \right.
\]

\[
+ K_\lambda(u_{t_1}, w_1) K_\lambda(u_{t_2}, w_2) R_{t_1 K}^{-1} \Omega_K(z_1) R_{t_2 L}^{-1} \Omega_L(z_2) \right)
\]

With manipulations analogous to those used in the derivation of (4.29) it is possible to write down (4.14) in a compact form. The details are given in Appendix B. The four-point correlator is

\[
\frac{<0|\beta(z_1)\beta(z_2)\gamma(w_1)\gamma(w_2) \prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>}{<0|\prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>} =
\]

\[
= \frac{<0|\beta(z_1)\gamma(w_1) \prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>}{<0|\prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>} + \frac{<0|\beta(z_2)\gamma(w_2) \prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>}{<0|\prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>}
\]

\[
+ \frac{<0|\beta(z_1)\gamma(w_2) \prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>}{<0|\prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>} + \frac{<0|\beta(z_2)\gamma(w_1) \prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>}{<0|\prod_{s=1}^{M_{\text{tot}}} \delta(\beta(u_s))|0>}
\]

(4.15)

Appendix A. GAUSSIAN REPRESENTATION

The aim of this Appendix is to provide a short but selfconsistent presentation of the construction of "gaussian" representation for $\beta - \gamma$ system proposed in [13]. The notation correspond to that used in Chapter 2 where the RS is a complex sphere.
The elementary excitations for $\beta$ and $\gamma$ satisfy the commutation relations:

$$[\gamma_n, \beta_m] = \delta_{n+m,0}.$$  \hspace{1cm} (A.1)

They are represented as

$$\beta_n = -\frac{i}{\sqrt{2}} (x_{-n} - i \frac{\partial}{\partial x_{-n}})$$  \hspace{1cm} (A.2)

$$\gamma_n = \frac{1}{\sqrt{2}} (x_n + i \frac{\partial}{\partial x_n})$$  \hspace{1cm} (A.3)

The operators $\beta_n$ and $\gamma_m$ act on functions of $x_n$. The vector space they act on is a product of "gaussian" factors $\exp(\pm \frac{i}{2} x_n^2)$ multiplied by polynomials in $x_m$. In this vector "Fock space" the scalar product is defined in the following way:

$$< \Phi_1 | \Phi_2 > = \frac{1}{V_\lambda} \int \prod_n dx_n (\Phi_1(x))^* \Phi_2(x).$$  \hspace{1cm} (A.4)

It is assumed that $x_n^* = x_{-n}$ (but in the case of general RS this rule will have to be modified). $V_\lambda$ is a formally infinite normalization factor equal to

$$V_\lambda = \left( \frac{\int dp \exp\left( -\frac{i}{2} p^2 \right)}{2\pi} \right)^{2\lambda-1} \left( \prod_{m<\lambda} \int \exp(-ix_m^2) dx_m \right) \left( \prod_{n\geq 1-\lambda} \int \exp(ix_n^2) dx_n \right)$$  \hspace{1cm} (A.5)

It is understood that (A.4) is defined in such a way that the two formally infinite terms in numerator and denominator cancel each other.

A choice of the vacuum state is a choice of "Bose sea level". A vacuum state $\Phi^{(\lambda)}$ can be constructed such that

$$< \Phi^{(\lambda)} | \Phi^{(\lambda)} > = \infty$$  \hspace{1cm} (A.6)

(the explanation for that infinity can be found in Introduction) but (assume that $\lambda \geq 2$)

$$< \Phi^{(\lambda)} | \prod_{j=1-\lambda}^{\lambda-1} \delta(\gamma_j) | \Phi^{(\lambda)} > = 1.$$  \hspace{1cm} (A.7)

The correct defintion is
\[ \Phi^\lambda(x) = \exp \frac{i}{2} \left( \sum_{n \geq \lambda} x_n^2 - \sum_{n < \lambda} x_n^2 \right). \]  
(A.8)

It satisfies

\[ \left( \Phi^{(\lambda)}(x) \right)^* = \Phi^{1-\lambda}(x). \]  
(A.9)

so that

\[ \Phi_0^{(\lambda)} \left( \Phi_0^{(\lambda)} \right)^* = \exp i \left( \sum_{n \geq \lambda} x_n^2 - \sum_{n < -\lambda} x_n^2 \right) \]  
(A.10)

The integration over \( dx_{\lambda-1}...dx_{1-\lambda} \) in (A.4) lead to (A.6). On the other hand

\[ \delta(\gamma_n) \exp \left( -\frac{i}{2} x_n^2 \right) = \frac{1}{2\pi} \int dp \exp \left( ip\gamma_n \right) \exp \left( -\frac{i}{2} x_n^2 \right) = \]

\[ = \frac{1}{2\pi} \int dp \exp \left( \frac{i}{\sqrt{2}} p(x_n + i \frac{\partial}{\partial x_n}) \right) \exp \left( -\frac{i}{2} x_n^2 \right) = \frac{1}{2\pi} \int dp \exp \left( \frac{i}{2} x_n^2 - \frac{i}{2} (p - i \sqrt{2} x_n)^2 \right) = \]

\[ = \frac{1}{2\pi} \int dp \exp \left( -\frac{1}{2} p^2 \right) \exp \left( \frac{i}{2} x_n^2 \right) \]  
(A.11)

Therefore, thanks to the appropriate definition of the normalization factor \( V_\lambda \), (A.7) holds.

An important identity used above is

\[ \exp \left( a \frac{\partial}{\partial x_n} \right) \exp \left( \frac{i}{2} x_n^2 \right) = \exp \left( iax_{-n} + \frac{i}{2} a^2 \right) \exp \left( \frac{i}{2} x_{-n}^2 \right). \]  
(A.12)

Appendix B. DERIVATION OF (4.15)

The aim of this Appendix is to provide a proof that the 4-point correlation function can be represented by (4.15). The starting point is (4.14) and the basic trick is that used in the derivation of (3.29):
\[
\sum_{s,N=1}^{M_{\text{tot}}} K_\lambda(z, w_s) R_{sN}^{-1} \Omega_N(w) = K_\lambda(z, w) - \\
\det \begin{pmatrix}
K_\lambda(z, w) & K_\lambda(z, w_1) & \cdots & K_\lambda(z, w_{M_{\text{tot}}}) \\
\Omega_1(w) & \Omega_1(w_1) & \cdots & \Omega_1(w_{M_{\text{tot}}}) \\
\vdots & \vdots & \ddots & \vdots \\
\Omega_{M_{\text{tot}}}(w) & \Omega_{M_{\text{tot}}}(w_1) & \cdots & \Omega_{M_{\text{tot}}}(w_{M_{\text{tot}}})
\end{pmatrix} \left( \det \Omega_L(w_j) \right)^{-1}.
\]

(B.1)

Thanks to (B.1) all the summations in (4.14) can be eliminated. One obtains (everything has to be multiplied by \(\frac{1}{\det \Omega_I(u, j)}\))

\[
K_\lambda(z_1, w_1)K_\lambda(z_2, w_2) + K_\lambda(z_1, w_2)K_\lambda(z_2, w_1) + \\
-K_\lambda(z_1, w_1)K_\lambda(z_2, w_2) + \\
+ \frac{1}{\det \Omega_I(u, j)} K_\lambda(z_1, w_1) \det \begin{pmatrix}
K_\lambda(z_2, w_2) & K_\lambda(z_2, w_1) & \cdots & K_\lambda(z_2, w_{M_{\text{tot}}}) \\
\Omega_1(w_2) & \Omega_1(w_1) & \cdots & \Omega_1(w_{M_{\text{tot}}}) \\
\vdots & \vdots & \ddots & \vdots \\
\Omega_{M_{\text{tot}}}(w_2) & \Omega_{M_{\text{tot}}}(w_1) & \cdots & \Omega_{M_{\text{tot}}}(w_{M_{\text{tot}}})
\end{pmatrix} + \\
-K_\lambda(z_1, w_2)K_\lambda(z_2, w_1) + \\
+ \frac{1}{\det \Omega_I(u, j)} K_\lambda(z_1, w_2) \det \begin{pmatrix}
K_\lambda(z_2, w_1) & K_\lambda(z_2, u_1) & \cdots & K_\lambda(z_2, w_{M_{\text{tot}}}) \\
\Omega_1(w_1) & \Omega_1(u_1) & \cdots & \Omega_1(w_{M_{\text{tot}}}) \\
\vdots & \vdots & \ddots & \vdots \\
\Omega_{M_{\text{tot}}}(w_1) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(w_{M_{\text{tot}}})
\end{pmatrix} + \\
-K_\lambda(z_2, w_1)K_\lambda(z_1, w_2) + \\
+ \frac{1}{\det \Omega_I(u, j)} K_\lambda(z_2, w_1) \det \begin{pmatrix}
K_\lambda(z_1, w_2) & K_\lambda(z_1, u_1) & \cdots & K_\lambda(z_1, w_{M_{\text{tot}}}) \\
\Omega_1(w_2) & \Omega_1(u_1) & \cdots & \Omega_1(w_{M_{\text{tot}}}) \\
\vdots & \vdots & \ddots & \vdots \\
\Omega_{M_{\text{tot}}}(w_2) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(w_{M_{\text{tot}}})
\end{pmatrix} + \\
-K_\lambda(z_2, w_2)K_\lambda(z_1, w_2) + \\
+ \frac{1}{\det \Omega_I(u, j)} K_\lambda(z_2, w_2) \det \begin{pmatrix}
K_\lambda(z_1, w_1) & K_\lambda(z_1, u_1) & \cdots & K_\lambda(z_1, w_{M_{\text{tot}}}) \\
\Omega_1(w_1) & \Omega_1(u_1) & \cdots & \Omega_1(w_{M_{\text{tot}}}) \\
\vdots & \vdots & \ddots & \vdots \\
\Omega_{M_{\text{tot}}}(w_1) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(w_{M_{\text{tot}}})
\end{pmatrix} + \\
\left( K_\lambda(z_1, w_2) - \frac{1}{\det \Omega_I(u, j)} \right)^{-1} \det \begin{pmatrix}
K_\lambda(z_1, w_2) & K_\lambda(z_1, u_1) & \cdots & K_\lambda(z_1, w_{M_{\text{tot}}}) \\
\Omega_1(w_2) & \Omega_1(u_1) & \cdots & \Omega_1(w_{M_{\text{tot}}}) \\
\vdots & \vdots & \ddots & \vdots \\
\Omega_{M_{\text{tot}}}(w_2) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(w_{M_{\text{tot}}})
\end{pmatrix} \times
\]
\[ \times \left( K_{\lambda}(z_2, w_1) - \frac{1}{\det \Omega_I(u_j)} \det \begin{pmatrix} K_{\lambda}(z_2, w_1) & K_{\lambda}(z_2, u_1) & \cdots & K_{\lambda}(z_2, u_{M_{\text{tot}}}) \\ \Omega_1(w_1) & \Omega_1(u_1) & \cdots & \Omega_1(u_{M_{\text{tot}}}) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{M_{\text{tot}}}(w_1) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(u_{M_{\text{tot}}}) \end{pmatrix} \right) + \\
+ \left( K_{\lambda}(z_1, w_1) - \frac{1}{\det \Omega_I(u_j)} \det \begin{pmatrix} K_{\lambda}(z_1, w_1) & K_{\lambda}(z_1, u_1) & \cdots & K_{\lambda}(z_1, u_{M_{\text{tot}}}) \\ \Omega_1(w_1) & \Omega_1(u_1) & \cdots & \Omega_1(u_{M_{\text{tot}}}) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{M_{\text{tot}}}(w_1) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(u_{M_{\text{tot}}}) \end{pmatrix} \right) \times \\
\times \left( K_{\lambda}(z_2, w_2) - \frac{1}{\det \Omega_I(u_j)} \det \begin{pmatrix} K_{\lambda}(z_2, w_2) & K_{\lambda}(z_2, u_1) & \cdots & K_{\lambda}(z_2, u_{M_{\text{tot}}}) \\ \Omega_1(w_2) & \Omega_1(u_1) & \cdots & \Omega_1(u_{M_{\text{tot}}}) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{M_{\text{tot}}}(w_2) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(u_{M_{\text{tot}}}) \end{pmatrix} \right) \]

All except of two terms cancel each other and \(1.14\) turns out to be equal

\[ \frac{1}{(\det \Omega_I(u_j))^3} \left( \det \begin{pmatrix} K_{\lambda}(z_1, w_2) & K_{\lambda}(z_1, u_1) & \cdots & K_{\lambda}(z_1, u_{M_{\text{tot}}}) \\ \Omega_1(w_2) & \Omega_1(u_1) & \cdots & \Omega_1(u_{M_{\text{tot}}}) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{M_{\text{tot}}}(w_2) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(u_{M_{\text{tot}}}) \end{pmatrix} \right) \times \]

\[ \times \left( \det \begin{pmatrix} K_{\lambda}(z_2, w_1) & K_{\lambda}(z_2, u_1) & \cdots & K_{\lambda}(z_2, u_{M_{\text{tot}}}) \\ \Omega_1(w_1) & \Omega_1(u_1) & \cdots & \Omega_1(u_{M_{\text{tot}}}) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{M_{\text{tot}}}(w_1) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(u_{M_{\text{tot}}}) \end{pmatrix} \right) + \\
+ \left( \det \begin{pmatrix} K_{\lambda}(z_1, w_1) & K_{\lambda}(z_1, u_1) & \cdots & K_{\lambda}(z_1, u_{M_{\text{tot}}}) \\ \Omega_1(w_1) & \Omega_1(u_1) & \cdots & \Omega_1(u_{M_{\text{tot}}}) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{M_{\text{tot}}}(w_2) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(u_{M_{\text{tot}}}) \end{pmatrix} \right) \times \\
\times \left( \det \begin{pmatrix} K_{\lambda}(z_2, w_2) & K_{\lambda}(z_2, u_1) & \cdots & K_{\lambda}(z_2, u_{M_{\text{tot}}}) \\ \Omega_1(w_2) & \Omega_1(u_1) & \cdots & \Omega_1(u_{M_{\text{tot}}}) \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{M_{\text{tot}}}(w_2) & \Omega_{M_{\text{tot}}}(u_1) & \cdots & \Omega_{M_{\text{tot}}}(u_{M_{\text{tot}}}) \end{pmatrix} \right) \] \hspace{1cm} (B.3)

By looking at (3.31) one immediately gets (1.13).
References

[1] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer Verlag, 1996.

[2] A. M. Polyakov, *Phys. Lett.* 103B (1981), 207, 211.

[3] J. Cardy, *Nucl. Phys.* B270[FS16] (1986), 186.

[4] L. Bonora, A. Lugo, M. Matone and J. Russo, *Comm. Math. Phys.* 123 (1989), 329; C. Vafa, *Phys. Lett.* 190B (1987), 47; L. Alvarez-Gaumé, C. Gomez, G. Moore and C. Vafa, *Nucl. Phys.* B303 (1988), 455; A. M. Semikhatov, *Phys. Lett.* 212B (1988), 357.

[5] E. Verlinde and H. Verlinde, *Nucl. Phys.* B288 (1987), 357

[6] A. K. Raina, *Helv. Phys. Acta* 63 (1990), 694; *Comm. Math. Phys.*, 140 (1991), 373.

[7] F. Ferrari and J. Sobczyk, *Int. Jour. Mod. Phys.* A11 (1996), 2213.

[8] F. Ferrari and J. Sobczyk, *Journal Geom. Phys.* 19 (1996), 287.

[9] D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* B271 (1986), 93.

[10] J. Atick and A. Sen, *Nucl. Phys.* B293 (1987), 317.

[11] E. Verlinde and H. Verlinde, *Phys. Lett.* 192B (1987), 95.

[12] A. Losev, *Phys. Lett.* 226B (1989), 62; O. Lechtenfeld, *Phys. Lett.* 232B (1989), 193; P. di Vecchia, *Phys. Lett.* 248B (1990), 329; U. Carow-Watamura, Z. F. Ezawa, K. Harada, A. Tezuku and S. Watamura, *Phys. Lett.* 227B (1989), 73.

[13] D. Montano, *Nucl. Phys.* B297 (1988), 125; E. Gava, *Conformal Fields on Complex Curves*, Preprint LPTENS-88/01; E. Gava, R. Iengo and G. Sotkov, *Phys. Lett.* 207B (1988), 283; D. Lebedev and A. Morozov, *Nucl. Phys.* B302 (1988), 163; R. Iengo and C-J Zhu, *Phys. Lett.* 212B (1988), 313.

[14] A. Gerasimov, A. Marshakov, A. Morozov et al., *Int. Jour. Mod. Phys.* A5 (1990), 2495.

[15] L. Alvarez-Gaumé, C. Gomez, P. Nelson, G. Sierra and C. Vafa, *Nucl. Phys.* B311 (1988), 333.

[16] C. Crnkovic, G. M. Sotkov and M. Stanishkov, *Phys. Lett.* 220B (1989), 397.

[17] S. A. Apikyan and C. J. Ethimiou, *Minimal Models of CFT’s on $\mathbb{Z}_N$ Surfaces*, [hep-th/9610051].

[18] F. Ferrari, J. T. Sobczyk, *Monodromy Properties of Energy-Momentum Tensor on General Algebraic Curves*, Preprint PAR-LPTHE 97-38, [hep-th/9709162].