Option Pricing in a Regime Switching Jump Diffusion Model

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Abstract

This paper presents the solution to a European option pricing problem by considering a regime-switching jump diffusion model for the underlying financial asset price dynamics. The regimes are assumed to be the results of an observed pure jump process, driving the values of interest rate and volatility coefficient. The pure jump process is assumed to be a semi-Markov process on finite state space. This consideration helps to incorporate a specific type of memory influence in the asset price. Under this model assumption, the locally risk minimizing prices of the European type path-independent options are found. The Föllmer-Schweizer decomposition is adopted to show that the option price satisfies an evolution problem, as a function of time, stock price, market regime, and the stagnancy period. To be more precise, the evolution problem involves a linear, parabolic, degenerate and non-local system of integro-partial differential equations. We have established existence and uniqueness of classical solution to the evolution problem in an appropriate class.

Keywords: jump-diffusion model, semi-Markov processes, locally risk minimizing pricing, optimal hedging, generalized solution, integro partial differential equation.

Classification No: 60K15, 91B30, 91G20, 91G60.

1 Introduction

There are extensive literature available in the theory and practice of option valuation following the pioneering work by Black and Scholes [3] in 1973. Contrary to the subsequent empirical evidence from the dynamics of financial asset prices, the Black-Scholes-Merton (BSM) model assumed a constant growth rate and a constant or deterministic volatility coefficient. The classical option pricing theory of BSM also relies on complete markets in which the payoff of every contingent claim can be replicated by a self-financing portfolio consisting of investments in the underlying stock and in a money market account paying a risk-less rate of interest. Hence an investor can be completely hedged against the risk of writing an option.

In subsequent studies, to overcome the limitations of BSM model, various option valuation models have been proposed and implemented in tune with increasingly more realistic price dynamics. These include stochastic volatility models, jump-diffusion models, regime-switching models and combinations of these. The markets in these models are incomplete where a perfect hedge is impossible. Various approaches have been adopted in solving option pricing in incomplete markets. The local risk minimization approach is one such and was initiated by [16], [25], [27], [26] and [28]. To hedge a claim in this approach, a unique dynamic strategy is sought for that replicates the claim at the maturity by allowing additional cash flow with continuous trading. This unique strategy minimizes the quadratic residual risk (QRR), a measure of the accumulated additional cash flow, under a certain set of constraints. This minimizing strategy is termed as the optimal hedging. Thus in a complete market, a self financing hedging strategy becomes the optimal hedging and results in zero QRR. The existence of an optimal hedging is shown in [17] to be equivalent to

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that of Föllmer-Schweizer (F-S) decomposition of the relevant discounted claim for a particular class of asset price dynamics.

In recent years, there have been phenomenal advancements in application of regime-switching dynamics in finance. The model parameters here are driven by a finite-state continuous time pure jump process whose states represent the stages of business cycles. One can refer to for example DiMasi et al. [13], Guo [21], Elliott et al. [14], and Siu et al. [30] for development of the theory of option pricing in Markov regime switching model. With the locally risk minimizing approach, the European call option price in a regime switching market is shown to satisfy a generalized B-S-M PDE in the work of [13] and many others subsequently including [12] and [4]. An extension of the Markov modulated regime switching model in [13] appears in [18] and [19], where the holding time of each regime is not restricted to be exponential variable and the regime dynamics follows a semi-Markov process. A semi-Markov process may, in general, exhibit duration dependent transition which a time homogeneous or a time in-homogeneous Markov chain do not capture. An empirical evidence of duration dependent transition of business cycles is presented in [7]. In [6] and [11] the calibrations of a semi-Markov modulated discrete and continuous time models are presented. The sensitivity of the call option price to the calibration error in the transition rate of a semi-Markov process is studied in [20]. The price of a basket option in a semi-Markov modulated time in-homogeneous market is also found in [10] to satisfy an extension of the multi-dimensional version of the BSM PDE. In another recent work [5], the option pricing problem in a local volatility model with semi-Markov switching is solved. All the studies mentioned above assume absence of discontinuities in the asset price dynamics. Option price problem with discontinuous asset price setting was initially addressed in [1][2][9]. A Reader may consult [8] for various aspects of discontinuous asset price dynamics. The study by Elliott et al. [15] and Su [32] involved jump diffusion models with Markov type regime switching. In Siu [31] the jump intensity was modulated by a continuous-time, finite-state, hidden Markov chain. As per our knowledge, no existing model capture both the features, the discontinuity in the asset price path, and the duration dependent regime switching.

In this paper we have first verified that the asset price model involving a semi-Markov regime-switching and jump discontinuities in paths does not allow arbitrage opportunity. It is well known that the parameters in a model with jump discontinuities should satisfy an additional constraint for ensuring no arbitrage (NA). We have obtained such appropriate condition. Furthermore, we have constructed a nontrivial realistic example which satisfies that condition. Having established the NA, we have considered a general European type path independent contract in which the terminal payoff is a Lipschitz continuous function of the final stock value. This class of payoffs is broad enough to include options like call, put and butterfly. We have taken the approach of F-S decomposition for finding the locally risk minimizing pricing. Although this is a standard approach in the literature, the issues become more subtle when asset prices are allowed to jump. The conditions under which the pseudo optimal strategy, obtained from the F-S decomposition, is locally risk minimizing needs a careful checking. Furthermore, the F-S decomposition under the minimal martingale measure may not produce an F-S decomposition under the original measure. We refer to [33] which nicely highlights this aspect. We have appropriately obtained the F-S decomposition under the market measure to construct the pseudo optimal strategy. Subsequently, the pseudo optimal strategy is shown to be optimal.

From the F-S decomposition we have established that the locally risk minimizing price of a Lipschitz terminal payoff can be expressed as a solution to an appropriate evolution problem. The corresponding equation turns out to be a linear, non-local, degenerate parabolic system of integro-PDEs which, when restricted, gives the BSM PDE. However, this evolution problem does not possess a closed form solution and is very different from its well-studied special cases. The methodology used in the literature for settling well-posedness of some of the special cases is also inadequate to deal with the present price equation. The treatment adopted in this paper does also not follow immediately from the existing results in the literature of integro-PDE. In this paper, we establish the existence and uniqueness of the classical solution to the problem using semigroup theory. This is accomplished in two steps. First, the price equation is shown to have a continuous mild solution satisfying an integral equation. Then by studying each term of the integral equation, it is proved that its solution is sufficiently smooth and belongs to the domain of the differential operator in the evolution equation. Thus the mild solution is shown to solve the equation classically.

Rest of the paper is organized as follows. Section 2 has four subsections. The asset price dynamics is demonstrated in the first three whereas the fourth subsection establishes the no arbitrage condition. The
option price equation is introduced at the end of Subsection 2.3. There are two subsections in the third section. The locally risk minimizing pricing using F-S decomposition in an abstract setting is briefly recalled in the first subsection. In Subsection 3.2 we derive the F-S decomposition of a Lipschitz payoff on final stock value under our specific market model. In this part we have temporarily assumed existence of a unique classical solution to the option price equation. That assumption is proved to be true in Section 4. Proof of some technical results are added in the appendix. We end this paper with some concluding remarks in Section 5.

2 Market model

2.1 Probability space

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a complete filtered probability space where the filtration satisfies the usual hypothesis and \(\mathcal{X} = \{1, 2, 3, \ldots, k\}\) a finite subset of \(\mathbb{R}\). Let \(X_0\), a \(\mathcal{X}\) valued random variable, and \(Y_0\), a non-negative random variable, be \(\mathcal{F}_0\) measurable. We further assume that \(\varphi\), a Poisson random measure on \([0, \infty) \times [0, \infty)\) with uniform intensity, \(W := \{W_t\}_{t \geq 0}\), a standard Wiener process, and \(N(\,dz, dt\,)\), a Poisson random measure on \(\mathbb{R} \times [0, \infty)\) with intensity \(\nu(\,dz, dt\,)\) where \(\nu\) is a finite Borel measure, are adapted to the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). We take \(X_0, Y_0, \varphi, W, N(\,dz, dt\,)\) as independent.

The above mentioned random variables, processes and measures would be used to model the inherent randomness in the asset price dynamics in Subsection 2.3. The asset price dynamics in Subsection 2.3. The asset price dynamics in Subsection 2.3.

2.2 Semi-Markov process

We begin this subsection by recollecting the definition of a time-homogeneous semi-Markov process or in short semi-Markov process as we would not consider the time-inhomogeneous case in this paper.

**Definition 2.1.** A \(\mathcal{X}\)-valued process \(X = \{X_t\}_{t \geq 0}\) defined on \((\Omega, \mathcal{F}, P)\) is called a semi-Markov process if

i. for almost every \(\omega \in \Omega\), \(X\) has a piece-wise constant rcll path with discontinuities at an increasing unbounded positive sequence \(\{T_n(\omega)\}_{n \geq 1}\), and

ii. \(P(X_{T_{n+1}} = j, T_{n+1} - T_n \leq y \mid (X_{T_0}, T_0), (X_{T_1}, T_1), \ldots, (X_{T_n}, T_n)) = P(X_{T_{n+1}} = j, T_{n+1} - T_n \leq y \mid X_{T_n})\) for all \(j \in \mathcal{X}\), \(y > 0\), \(n \geq 0\) and for some \(T_0 \leq 0\), where \(X_{T_0} := X_0\).

Given a semi-Markov process \(X = \{X_t\}_{t \geq 0}\), on \(\mathcal{X}\), with transition time sequence \(T_0 \leq 0 < T_1 < T_2, \ldots\); the instantaneous transition rate function of \(X\), if exists, is a collection \(\lambda_{ij} : [0, \infty) \rightarrow [0, \infty), \forall i \neq j \in \mathcal{X}\), given by

\[
\lambda_{ij}(y) := \lim_{\delta \to 0} \frac{1}{\delta} P [X_{T_{n+1}} = j, T_{n+1} - T_n \in (y, y + \delta), X_{T_n} = i, T_{n+1} - T_n > y]
\]

and does not depend on \(n\). We consider the class of semi-Markov processes which admits the above limit satisfying the following additional conditions.

**A1.** (i) For each \(i \neq j \in \mathcal{X}\), \(\lambda_{ij} : [0, \infty) \rightarrow [0, \infty)\) is a bounded, positive and continuously differentiable function. (ii) If \(\lambda_i(y) := \sum_{j \neq i} \lambda_{ij}(y)\) and \(\Lambda_i(y) := \int_0^y \lambda_i(y)dy\), then \(\lim_{y \to \infty} \Lambda_i(y) = \infty\).

Note that the instantaneous transition rates of an irreducible finite-state continuous time Markov chain are only positive constants. Therefore the assumption (A1) includes the Markov counterpart as a special case. Here we recall the following results from [13].

**Proposition 2.1.** Given a collection \(\lambda = \{\lambda_{ij} : [0, \infty) \rightarrow (0, \infty) \mid i \neq j \in \mathcal{X}\}\) of bounded measurable maps, satisfying (A1(ii)), there exist a finite interval \(I_{\lambda}\) and a piecewise linear maps \(h_{\lambda}\) and \(g_{\lambda}\) on \(\mathcal{X} \times [0, \infty) \times I_{\lambda}\) such that the following hold.
(i) The system of coupled stochastic integral equations

\[
X_t = X_0 + \int_{(0,t]} \int_{\mathcal{Y}} h(X_{u-}, Y_{u-}, z) \varphi(dz, du),
\]

\[
Y_t = Y_0 + t - \int_{(0,t]} \int_{\mathcal{Y}} g(X_{u-}, Y_{u-}, z) \varphi(dz, du),
\]

for all \( t > 0 \) has a strong rcll solution \((X,Y)\) for any given \((X_0, Y_0) \in \mathcal{X} \times [0, \infty)\) such that the solution \(X\) is a semi-Markov process on \(\mathcal{X}\) with \(\lambda\) as instantaneous transition rate function and \(Y\) gives the age at the current state. More precisely, \(Y_t = t - T_n(t)\), where \(n(t)\) is the number of transitions during \((0,t]\).

(ii) The infinitesimal generator \(A\) of \((X,Y)\) is given by \(A\varphi(i, y) = \frac{\partial \varphi}{\partial y}(i, y) + \sum_{j \neq i} \lambda_{ij}(y) (\varphi(j, 0) - \varphi(i, y))\) for every function \(\varphi : \mathcal{X} \times [0, \infty) \to \mathbb{R}\) with bounded continuous derivatives.

(iii) Consider \(F : [0, \infty) \times \mathcal{X} \to [0, 1]\), defined as \(F(y|i) := 1 - e^{-\Lambda_i(y)}\), where \(\Lambda_i\) is as in (A1)(ii). Then under A1(i), \(F(\cdot | i)\) is a twice differentiable function and is the conditional cumulative distribution function of the holding time of \(X\) given that the present state is \(i\).

(iv) We define \(p_{ij}(y) := \frac{\lambda_{ij}(y)}{\Lambda_i(y)}\) for \(j \neq i\) with \(p_{ii}(y) = 0\) for all \(i\) and \(y\). Then for each \(i, j\) and \(y\), \(p_{ij}(y)\) denotes the conditional probability of transition to \(j\) given that the process transits from \(i\) at the age \(y\).

(v) We define \(f(y|i) := \frac{\partial}{\partial y}F(y|i)\), then \(\lambda_{ij}(y) = p_{ij}(y) \cdot \frac{f(y|i)}{F(y|i)}\) holds for all \(i \neq j\).

From (i) in the above proposition, \(Y_0 = -T_0\). However, for the sake of simplicity, we assume that the initial age, i.e., \(Y_0\) is zero. This causes no loss of generality in our treatment. Indeed, since the regimes are assumed to be observed, for valuation of options with maturity \(\tau\) one may, in the case of nonzero initial age, set the value of initial time as the initial age \(y^*\) (say) instead of zero, so as to make \(Y_{y^*} = y^*\), or in other words, \(Y_0 = 0\). Hence, the new maturity time \(T\) is \(y^* + \tau\). Of course as a consequence \(T_0\) (as in Definition 2.2) is zero and \(Y_t \in [0, T]\) for all \(t \in [0, T]\). We also assume the following irreducibility condition.

(A1). (iii) Set \(\hat{p}_{ij} := \int_0^\infty p_{ij}(y) dF(y|i)\). The matrix \((\hat{p}_{ij})_{k \times k}\) is irreducible.

Note that \((\hat{p}_{ij})_{k \times k}\) denotes the transition probability matrix of the embedded discrete time Markov chain.

### 2.3 Asset price dynamics

In this subsection the mathematical model of a financial market dynamics is presented. We consider a market having two types of securities, one is a risk-less asset, also called money market instrument, and another is a risky asset, called stock. Let \(X = \{X_t\}_{t \geq 0}\) be a semi-Markov process on the state space \(\mathcal{X}\) satisfying A1 and (2.1)-(2.2), \(r_t := r(X_t)\) be the spot interest rate of risk-less asset whose price is \(B_t\) at time \(t\) with \(B_0 = 1\). Then we have \(B_t = e^{\int_0^t r_u du}\). Now let \(S_t := \{S_t\}_{t \geq 0}\) be the price process of the risky asset which is governed by a semi-Markov modulated jump diffusion model as given below

\[
dS_t = S_{t-} (\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}} \eta(z) N(dz, dt)),
\]

where \(t > 0\), \(S_0 > 0\), \(\mu_t := \mu(X_t)\) denotes the drift, \(\sigma_t := \sigma(t, X_t)\) the volatility coefficient, and a bounded continuous function \(\eta : \mathbb{R} \to (-1, \infty)\) the jump size coefficient. Furthermore, the time-inhomogeneous volatility function \(\sigma : [0, \infty) \times \mathcal{X} \to (0, \infty)\) is assumed to be continuous.

**Theorem 2.2.** The SDE (2.3) has an almost sure unique strong solution which is given by

\[
S_t = S_0 \exp \left( \int_0^t (\mu(X_{u-}) - \frac{1}{2} \sigma(u, X_{u-})^2) du + \int_0^t \sigma(u, X_{u-}) dW_u + \int_0^t \int_{\mathbb{R}} \ln(1 + \eta(z)) N(dz, du) \right).
\]

Furthermore, the solution is positive valued and square integrable.
Proof of the above theorem is deferred to the Appendix.

**Corollary 2.3.** (i) The discounted asset price process \( S^* = \{S^*_t\}_{t \geq 0} \) given by \( S^*_t = \varepsilon t = \exp \left( -\int_0^t r_u du \right) S_t \), where \( S \) is as in (2.3), which is also the solution to (2.2) can be rewritten as \( S^*_t = S_0 + C_t + G_t \) where \( G = \{G_t\}_{t \geq 0} \) is a square integrable martingale with \( G_0 = 0 \) and \( C = \{C_t\}_{t \geq 0} \) a predictable continuous process of finite variation. (ii) The conditional quadratic variation process \( t \mapsto \langle G \rangle_t \) is strictly increasing on \([0,T]\) almost surely. (iii) \( C \) is absolutely continuous w.r.t. \( \langle G \rangle \) with a density \( \delta_C \) such that the mean-variance tradeoff process \( \bar{K} \) given by \( \bar{K}_t = \int_0^t \delta_C^2(t) d\langle G \rangle_t \) has finite expectation i.e., \( E\bar{K}_T < \infty \).

**Proof.** From (2.3), we directly get that

\[
dS^*_t = \exp \left( -\int_0^t r_u du \right) dS_t - S_t \exp \left( -\int_0^t r_u du \right) r_t dt
\]

\[
= (1/B_t) (dS_t - S_t - r_t dt)
\]

\[
= (1/B_t) \left( S_t - \left( \mu_t - \sigma_t dW_t + \int_{\mathbb{R}} \eta(z)\mathcal{N}(dz, dt) \right) - S_t - r_t dt \right)
\]

\[
= (\mu_t - r_t) S_t^* dt + \sigma_t S_t^* dW_t + \int_{\mathbb{R}} \eta(z)\mathcal{N}(dz, dt).
\]

Thus \( S^*_t = S_0 + C_t + G_t \) where

\[
C_t := \int_0^t (\mu_u - r_u + \int_{\mathbb{R}} \eta(z)\nu(dz)) S_u^* du, \quad \text{and} \quad G_t := \int_0^t \sigma_u S_u^* dW_u + \int_0^t S_u^* \int_{\mathbb{R}} \eta(z)\mathcal{N}(dz, du),
\]

where \( S \) is as in (2.3) and \( \tilde{N}(dz, du) := N(dz, du) - \nu(dz) du \) is the compensated Poisson random measure. The square integrability of \( G \) follows from the square integrability of \( S \). Thus the conditional quadratic variation of \( G \) is given by

\[
\langle G \rangle_t = \int_0^t \sigma_u^2 S_u^* - 2 du + \int_0^t S_u^* \int_{\mathbb{R}} \eta(z)^2 \nu(dz) du.
\]

Since \( \sigma \) is assumed to be strictly positive, \( \langle G \rangle_t \) is strictly increasing. The density \( \delta_C \) is defined by

\[
\delta_C(t) = \frac{dC_t}{d\langle G \rangle_t} = \frac{\mu_u - r_u + \int_{\mathbb{R}} \eta(z)\nu(dz)}{S_t^* \left( \sigma_t - \int_{\mathbb{R}} \eta(z)^2 \nu(dz) \right)}.
\]

Thus

\[
E\bar{K}_T = E \int_0^T \left( \frac{\mu_u - r_u + \int_{\mathbb{R}} \eta(z)\nu(dz)}{\sigma_u^2 + \int_{\mathbb{R}} \eta(z)^2 \nu(dz)} \right)^2 dt < \infty
\]

since \( \eta \) is bounded, \( \nu \) is finite and \( \inf_{[0,T] \times \mathbb{R}} \sigma(t, i) > 0 \) for finite \( T \).

The results obtained in the above corollary is stronger than the so called Structure Conditions (SC). We would use these results in the next section.

We would end this subsection with the following observations. The Dynkin’s formula states that if \( \{A_t\}_{t \geq 0} \) is the infinitesimal generator of \( \{(S_t, X_t, Y_t)\}_{t \geq 0} \), then \( t \mapsto \psi(S_t, X_t, Y_t) - \psi(S_0, X_0, Y_0) - \int_0^t A_u \psi(S_{u-}, X_{u-}, Y_{u-}) du \) is an \( \{\mathcal{F}_t\}_{t \geq 0} \)-martingale for any \( \psi \in \mathcal{C}^\infty \). By denoting the above martingale by \( \{M_t\}_{t \geq 0} \), we get

\[
\psi(S_t, X_t, Y_t) = \psi(S_0, X_0, Y_0) + \int_0^t A_u \psi(S_{u-}, X_{u-}, Y_{u-}) du + M_t.
\]

From (2.1), (2.2) and (2.3) and using Proposition (2.1) ii) we get

\[
A_t \psi(s, i, y) = \left( \mu(s) \frac{\partial}{\partial s} + \frac{1}{2} \sigma^2(t, i) s \frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial y} \right) \psi(s, i, y) + \sum_{j \neq i} \lambda_{ij}(y) \left( \psi(s, j, 0) - \psi(s, i, y) \right) + \int_{\mathbb{R}} \left( \psi(s(1 + \eta(z)), i, y) - \psi(s, i, y) \right) \nu(dz).
\]
In Section 3 we consider, with an appropriate terminal condition at time $T$, the following integro-partial differential equations (IPDE) for $(t, s, i, y) \in D := \bigcup_{0 \leq t < T} \{ \{t\} \times (0, \infty) \times X \times (0, t) \}$

$$
\left( \frac{\partial}{\partial t} + A_t + (r(i) - \mu(i) + \beta_1(t, i))s \frac{\partial}{\partial s} \right) \varphi(t, s, i, y)
+ \int_{R} (\beta_2(t, z, i) - 1) (\varphi(t, s(1 + \eta(z)), i, y) - \varphi(t, s, i, y)) \nu(dz) = r(i) \varphi(t, s, i, y)
$$

(2.6)

where the continuous functions $\beta_1(t, i)$ and $\beta_2(t, z, i)$ do not depend on $\varphi$ and are yet to be chosen.

### 2.4 No arbitrage

An arbitrage is an indication of the instability in the market. A market is said to have an arbitrage when it enables an investor to get positive profit without an initial capital and possibility of loss. So we need to check whether this model has no arbitrage (NA) under a reasonably large class of admissible strategies. From Theorem VII.2c.2 of [29], one obtains that the existence of an equivalent martingale measure (EMM) implies NA in the sense of NFLVR (no free lunch with vanishing risk) under a class of admissible strategies. Thus to ensure an arbitrage free model, exhibition of an EMM is called for. An EMM is commonly constructed using a Radon-Nikodym process involving a Doléans-Dade exponential. However, in general, a Doléans-Dade exponential of a martingale is a super martingale giving rise to a sub-probability measure. A Novikov’s type condition, if satisfied, resolves this deficit by asserting that the stochastic exponential is a martingale. We appeal to the results in [23] for a similar condition in the jump diffusion setting in order to show that a sufficiently large class of Doléans-Dade exponential are martingales in the following lemma.

**Lemma 2.4.** Let $Z = \{Z_t\}_{t \in [0, T]}$ be an adapted process, defined as

$$
Z_t := \exp \left( \int_0^t \phi_u dW_u - \frac{1}{2} \int_0^t \phi_u^2 du + \int_0^t \int_R \ln \Gamma_u(z) N(dz, du) - \int_0^t \int_R (\Gamma_u(z) - 1) \nu(dz) du \right),
$$

(2.7)

where $\phi = \{\phi_t\}_{t \in [0, T]}$ and $\Gamma = \{\Gamma_t(z)\}_{t \in [0, T]}$ are predictable and bounded processes with $\Gamma > 0$. Then $Z$ is a positive martingale under $P$ with $Z_0 = 1$.

**Proof.** From [27], it is obvious that $Z > 0$ with $Z_0 = 1$. We derive the following

$$
\Delta Z_t = Z_t - Z_{t-}
$$

$$
= \exp \left( \int_0^t \phi_u dW_u - \frac{1}{2} \int_0^t \phi_u^2 du + \int_0^t \int_R (\Gamma_u(z) - 1) \nu(dz) du \right)
\exp \left( \int_{[0,t]} \int_R \ln \Gamma_u(z) N(dz, du) \right)
$$

$$
- \exp \left( \int_0^t \phi_u dW_u - \frac{1}{2} \int_0^t \phi_u^2 du + \int_0^t \int_R (\Gamma_u(z) - 1) \nu(dz) du \right)
\exp \left( \int_{[0,t]} \int_R \ln \Gamma_u(z) N(dz, du) \right)
$$

$= Z_{t-} \left( \int_R (\Gamma_t(z) - 1) N(dz, \{t\}) \right).
$

We define $y_t := \int_0^t \phi_u dW_u - \frac{1}{2} \int_0^t \phi_u^2 du + \int_0^t \int_R \ln \Gamma_u(z) N(dz, du) - \int_0^t \int_R (\Gamma_u(z) - 1) \nu(dz) du$. Hence, $\Delta y_t = \int_R \ln \Gamma_t(z) N(dz, \{t\})$ and an application of Itô formula on $Z_t = \exp(y_t)$ gives,

$$
Z_t - Z_0 = \int_0^t Z_u - d[y_u]_t + \frac{1}{2} \int_0^t Z_u - d[y_u]^2_t + \sum_{0 < u \leq t} \left( Z_u - Z_{u-} - Z_{u-} - \Delta y_u \right)
$$

$$
Z_{t-} = \int_0^t Z_u - d[y_u]_t + \frac{1}{2} \int_0^t Z_u - d[y_u]^2_t + \int_0^t \int_R \ln \Gamma_u(z) N(dz, du) - \int_0^t \int_R (\Gamma_u(z) - 1) \nu(dz) du
$$

(2.8)
From above, we can see that $Z$ satisfies
\begin{equation}
    dZ_t = Z_{t-} d\tilde{M}_t, \quad Z_0 = 1
\end{equation}
where $\tilde{M} := \{\tilde{M}_t\}_{t \geq 0}$ is a $P-$local martingale given by
\[ \tilde{M}_t := \int_0^t \phi_u dW_u + \int_0^t \int_{\mathbb{R}} \left( \Gamma_u(z) - 1 \right) \tilde{N}(dz, du). \]

However, our assumptions on $\phi$ and $\Gamma$ imply that $\tilde{M}$ is a square integrable martingale with jump size $(\Delta \tilde{M})$ greater than $-1$. Thus in view of (2.9), $Z$ is the Doléans-Dade exponential of $\tilde{M}$ and satisfies all the conditions given in Theorem 9 of [23]. Hence, applying that theorem we conclude that $Z$ is a positive martingale.

The following lemma, which is essentially borrowed from the Theorem 3.2 of [9], presents the change in law of underlying processes under a new measure $Q$, constructed via the Radon-Nikodym process $Z$ as above. The result can be viewed as a consequence of a version of the Girsanov theorem. This lemma is useful to pin down an EMM for our specific model.

**Lemma 2.5.** Let $Q$ be defined on $\mathcal{F}_T$ by $\frac{dQ}{dP} = Z_T$, where $Z$ is as in (2.7). Then the process $\tilde{W} := W - \int_0^t \phi_u du$ is a Wiener process under $Q$ and
\[ \int_0^t \int_{\mathbb{R}} \left( \Gamma_u(z) - 1 \right) \left( \tilde{N}(dz, du) - \Gamma_u(z) \nu(dz) du \right) \]
is a $Q-$martingale with respect to its natural filtration. The compensator measure of $N(dz, dt)$ under $Q$ is given by $\tilde{\nu}(dz, dt) := \Gamma_t(z) \nu(dz) dt$.

Lemma 2.5 implies that $\tilde{M}(dz, dt) := N(dz, dt) - \tilde{\nu}(dz, dt)$ is a compensated Poisson random measure with respect to the measure $Q$. We rewrite the SDE (2.6) using $\tilde{W}$, $\tilde{M}$ and the unspecified processes $\phi$ and $\Gamma$ to get
\begin{align*}
    dS^*_t &= \left( \mu_{t-} - r_{t-} \right) S^*_t dt + \sigma_{t-} S^*_t (d\tilde{W}_t + \phi_t dt) + S^*_t \int_{\mathbb{R}} \eta(z) \left( \tilde{M}(dz, dt) + \tilde{\nu}(dz, dt) \right) \\
    &= \left( \mu_{t-} - r_{t-} + \sigma_{t-} \phi_t + \int_{\mathbb{R}} \eta(z) \Gamma_t(z) \nu(dz) \right) S^*_t dt + \sigma_{t-} S^*_t d\tilde{W}_t + S^*_t \int_{\mathbb{R}} \eta(z) \tilde{M}(dz, dt). \quad (2.10)
\end{align*}

Now we wish to specify $\Gamma$ and $\phi$ so as to make $S^*$ a martingale under $Q$. In view of Lemma 2.5, it is possible only when the drift term in (2.10) is zero. Thus we have
\begin{equation}
    \mu_{t-} - r_{t-} + \sigma_{t-} \phi_t + \int_{\mathbb{R}} \eta(z) \Gamma_t(z) \nu(dz) = 0. \quad (2.11)
\end{equation}

This is a single equation with two unknowns. Hence (2.11) leads to many different possibilities corresponding to different pairs of $(\phi, \Gamma)$ satisfying (2.11). We would like to select one which satisfies an additional relation such that (2.8) can be represented as
\begin{align*}
    dZ_t &= \Psi_{t-} Z_{t-} (\sigma_{t-} dW_t + \int_{\mathbb{R}} \eta(z) \tilde{N}(dz, dt)), \quad Z_0 = 1. \quad (2.12)
\end{align*}

Now by comparing (2.8) and (2.12), we get
\[ \phi_t = \Psi_{t-} \sigma_{t-}, \quad \text{and} \quad \Gamma_t(z) - 1 = \Psi_{t-} \eta(z). \]

Hence by substituting above in (2.11), we obtain
\begin{align*}
    \Psi_t \sigma^2_t &= r_t - \mu_t - \int_{\mathbb{R}} \eta(z)(1 + \Psi_t \eta(z)) \nu(dz) \\
    &= r_t - \mu_t - \int_{\mathbb{R}} \eta(z) \nu(dz) - \Psi_t \int_{\mathbb{R}} \eta^2(z) \nu(dz)
\end{align*}
for all \( t \). Therefore \( \Psi_t \) can be written as

\[
\Psi_t = \frac{r_t - \mu_t - \int_{\mathbb{R}} \eta(z) \nu(dz)}{\sigma_t^2 + \int_{\mathbb{R}} \eta^2(z) \nu(dz)},
\]

which results in

\[
\begin{align*}
\Gamma_t(z) &= \frac{r_t - \mu_t - \int_{\mathbb{R}} \eta(z) \nu(dz)}{\sigma_t^2 + \int_{\mathbb{R}} \eta^2(z) \nu(dz)} \eta(z) + 1 \\
\phi_t &= \frac{r_t - \mu_t - \int_{\mathbb{R}} \eta(z) \nu(dz)}{\sigma_t^2 + \int_{\mathbb{R}} \eta^2(z) \nu(dz)} \sigma_t^2.
\end{align*}
\]

(2.13)

It is easy to see that under the assumption

\textbf{A2.} \( \frac{r(i) - \mu(i) - \int_{\mathbb{R}} \eta(z) \nu(dz)}{\sigma^2(t,i) + \int_{\mathbb{R}} \eta^2(z) \nu(dz)} \eta(z) > -1 \) for all \( t \in [0, T] \), \( i \in \mathcal{X}, z \in \mathbb{R} \),

the conditions on \( \phi \) and \( \Gamma \) in Lemma 2.4 hold true. We assume (A2) throughout the paper and illustrate some of its implications in Remark 2.1. Thus under (A2), using (2.13) and Lemma 2.4, the measure \( Q \) as in Lemma 2.3 is a probability measure equivalent to \( P \). Furthermore, the substitution of (2.13) in (2.10), gives

\[
dS_t^* = S_t^* \left( \sigma(t, X_t) d\bar{W}_t + \int_{\mathbb{R}} \eta(z) \bar{M}(dz, dt) \right),
\]

a Doléans-Dade exponential, which is, by the virtue of Lemma 2.6 and Theorem 9 of [23], a positive martingale under \( Q \). Thus we have proved the following theorem.

**Theorem 2.6.** Under (A2), the substitution of the values of \( \phi_t \) and \( \Gamma_t(z) \) from (2.13) in (2.10) leads to an equivalent martingale measure \( Q \) which is given in Lemma 2.3. Hence the market is arbitrage free.

**Remark 2.1.** An assumption similar to (A2) is standard in the literature. We refer to [50] for an instance. But in the BSM model, or in its regime switching generalizations no such assumption is required. It is evident that the BSM model is a special case of the present model in which the asset price is a continuous function of time, or in other words, \( \eta \) is identically zero. It is important to note that if \( \eta \equiv 0 \), (A2) holds for any choice of \( r \), \( \sigma \), and \( \nu \) and thus imposes no further constraint on any model parameter. However, for a nontrivial \( \eta \), (A2) puts an upper bound on the drift coefficient \( \mu \). That is, \( \mu(i) < \left( r(i) - \int_{\mathbb{R}} \eta(z) \nu(dz) \right) \eta(z) + \sigma^2(t,i) + \int_{\mathbb{R}} \eta^2(z) \nu(dz) \) for all \( i, t, \) and \( z \). In view of the fact that any real risky asset has greater drift value than the ideal bank rate, it is important to cross check if the above mentioned upper bound leads to an unrealistic model assumption. In order to illustrate the implication of this upper bound, we consider a nontrivial example where \( \eta(z) = \max(\min(z,1), -\frac{1}{2}) \), \( \nu \) is the Lebesgue measure on \( [-\frac{1}{2}, 1] \), and \( r(i) < 3/8 \) for all \( i \). Then \( \int_{\mathbb{R}} \eta(z) \nu(dz) = \int_{\mathbb{R}} \eta^2(z) \nu(dz) = 3/8 \). Therefore, (A2) implies that \( \mu(i) < \left( r(i) - 3/8 \right) \max(\min(z,1), -\frac{1}{2}) + \sigma^2(t,i) + 3/8 \) for all \( i, t, \) and \( z \). This is equivalent to the condition \( \mu(i) < r(i) + \min_{[0,T]} \sigma^2(t,i) \). This bound is clearly not unrealistic.

### 3 Pricing and Optimal Hedging

#### 3.1 Locally Risk Minimizing Approach

Let \( \xi_t \) and \( \varepsilon_t \) be the number of units invested in assets with prices \( S_t \) and \( B_t \) respectively at time \( t \). The value of the resulting portfolio at time \( t \) is given by

\[
V_t := \xi_t S_t + \varepsilon_t B_t.
\]

An admissible strategy is defined to be a predictable process \( \pi = \{ \pi_t = (\xi_t, \varepsilon_t) \}_{t \in [0, T]} \) which satisfies the following conditions.
(i) $\int_0^T \xi_t^2 \, d\langle G \rangle_t + E(\int_0^T |\xi_t| \, dC_t)^2 < \infty$, where $G$ and $C$ are as in Corollary \[28]\.

(ii) $E(\varepsilon_t^2) < \infty$.

(iii) $\exists a > 0$ s.t. $P(V_t \geq -a, \forall t \in [0, T]) = 1$. It is shown in \[17]\ that if the market is arbitrage free, under some conditions the existence of an optimal hedging for replicating an $\mathcal{F}_T$ measurable finite-variance payoff $H$, is equivalent to the existence of F-S decomposition of the discounted payoff $H^* := B_T^{-1} H$ in the form

$$H^* = H_0 + \int_0^T \xi_t^H \, dS_t^* + L_t^H,$$

where $H_0 \in L^2(\Omega, \mathcal{F}_0, P)$, $L_t^H := \{L_t^H\}_{t \in [0, T]}$ is a square integrable martingale starting with zero and orthogonal to the martingale part of $S$, and $\xi_t^H = \{\xi_t^H\}_{t \in [0, T]}$ satisfies (i). A set of sufficient conditions (See Theorem 3.3 of \[28]\ for more details) indicated in the above sentence are given below.

(i) The quadratic variation of the martingale part of $S$, i.e., $\langle G \rangle$ is strictly increasing,

(ii) $t \mapsto C_t$ is continuous, and

(iii) $E_K < \infty$.

These conditions do hold in our setting. Indeed each of these conditions is established in Corollary \[28]\ In \[17]\, it is further asserted that the optimal hedging $\pi = (\xi, \varepsilon)$ is given by

$$\xi_t := \xi_t^H,$$

$$V_t^* := H_0 + \int_0^t \xi_u \, dS_u^* + L_u^H,$$

$$\varepsilon_t := V_t^* - \xi_t S_t^*,$$

and $V_t := B_t V_t^*$ represents the locally risk minimizing price at time $t$ of the contract having a terminal payoff $H$. Hence for the proposed market model, F-S decomposition is the key thing to settle the pricing and hedging problems under the locally risk minimizing approach. In this connection we recall that in the earlier section we have constructed an equivalent martingale measure (EMM) in order to prove that the proposed model does not admit arbitrage in the sense of NFLVR. Once such a risk neutral measure is obtained, one can of course obtain a no arbitrage price of a contract by taking conditional expectation of the discounted contingent claim w.r.t. the EMM. It is also not difficult to show that $Q$, the EMM obtained in the earlier section is indeed the minimal martingale measure (MMM). Thus the price obtained thereby is the locally risk minimizing price. Often it is also tempting to apply Itô’s Lemma on the price function thus obtained to derive a differential equation for the price function. However, apparently it is not obvious that this function has sufficient regularity required for an application of Itô’s Lemma. Besides, those calculations do not give an answer to the hedging problem also. For these reasons we have avoided that path in this paper. Instead we have first considered an ad-hoc equation and have established the existence and uniqueness of classical solution in Section 4. The analysis of the equation is deferred only to retain the focus on option pricing in the present section. In the next subsection we use that classical solution to derive the desired F-S decomposition. Thereby we conclude that the equation under consideration is indeed the price equation. In this way we settle both the pricing and hedging problem.

### 3.2 Derivation of F-S decomposition

In order to price an option with a terminal payoff $H = K(S_T)$ where $K : [0, \infty) \to \mathbb{R}$ is a Lipschitz continuous function, we derive the F-S decomposition directly under $P$. To this end we consider the evolution problem given by \[2.6] with the terminal condition

$$\varphi(T, s, i, y) = K(s).$$

(3.1)

It is important to note that the functional parameters $\beta_1(t, i)$ and $\beta_2(t, z, i)$ are yet to be chosen. For the time being, we assume that there is a nonempty collection of pairs of $(\beta_1, \beta_2)$ so that the above evolution problem has a unique classical solution with at most linear growth for each pair in that collection. We denote
Hence there may not exist any such $\beta_t$ minimizing price at time $t$ respectively. In view of the F-S decomposition mentioned in the previous subsection, we are looking for a solution by we would also pin down an appropriate choice of of Subsection 3.1, that constitutes the optimal hedging for see that $\bar{\phi}$ equal to $\hat{\phi}$ denotes the conditional quadratic covariation. We note that the quadratic covariation

$$ dL_t = d\left(\frac{\partial \phi_t}{B_t}\right) - \xi_t dS^*_t $$

becomes square integrable $P$–martingale and orthogonal to $\tilde{M}$, the martingale part of $S^*$. It is easy to see that $\tilde{M}_t = \int_0^t \sigma_u - S^*_u \, dW_u + \int_0^t S^*_u \, \int_\mathbb{R} \eta(z) N(dz, du)$ for all $t \geq 0$. If such process $\xi$ is found, in view of Subsection 3.1, that constitutes the optimal hedging for $\varphi_T$ i.e., $K(S_T)$ and $\varphi_t$ gives the locally risk minimizing price at time $t$ of a European style contract with the final payoff $K(S_T)$ at time $T$. However, there may not exist any such $\xi$ corresponding to an arbitrary choice of $\beta_1$ and $\beta_2$. Hence, in the due course we would also pin down an appropriate choice of $\beta_1$ and $\beta_2$ which allow existence of such $\xi$. By applying Itô formula on $\varphi(t, S_t, X_t, Y_t)$, and using eqs. (2.5) and (2.6) we get

$$ dL_t = d\left(\frac{\partial \phi_t}{B_t}\right) - \xi_t dS^*_t $$

$$ = \left(\frac{\partial \phi_t}{B_t}\right) \varphi_t \, dt + \frac{1}{B_t} \left(\frac{\partial \varphi_t}{\partial t} + A_{-1} \varphi_t\right) dt + \frac{1}{B_t} \left(\frac{\partial \varphi_t}{\partial s} \sigma_t \varphi_t \, dW_t + 1\right) \int\left(\varphi(t, S_t(1 + \eta(z)), X_t, Y_t) \right) \left(\frac{\partial \varphi_t}{\partial t} \sigma_t - \sigma_t \varphi_t \, dW_t + S_t \int \eta(z) N(dz, dt) \right)$$

$$ = \left(\mu_{-1} - \beta_1(t, X_{-1}) S^*_t \frac{\partial \phi_t}{\partial s} - \int \left(\frac{\beta_2(t, z, X_{-1}) - 1}{B_t}\right) \left(\varphi(t, S_t(1 + \eta(z)), X_t, Y_t) \right) \left(\eta(z) \nu(dz)\right)$$

$$ - \xi_t \left(\mu_{-1} - \beta_1(t, X_{-1}) S^*_t \right) \int \eta(z) \nu(dz)\right) dt + \left(\sigma_t S^*_t \frac{\partial \varphi_t}{\partial s} - \xi_t \nu(dz)\right) dW_t$$

$$ + \int\left(\varphi(t, S_t(1 + \eta(z)), X_t, Y_t) \right) \left(\varphi_t - \xi_t S^*_t \eta(z)\right) \eta(dz, dt) + \frac{1}{B_t} d\tilde{M}_t$$

where $\tilde{M}_t := \int_0^t \int \mathbb{R} \left(\varphi(u, S_u, X_u + h(X_u, Y_u, z), Y_u - g(X_u, Y_u, z)) - \varphi_u\right) \phi(dz, du)$ and $\tilde{\phi}(dz, dt) := \phi(dz, dt) - dtdz$, the compensated Poisson random measure of $\varphi$. Hence the local martingale part of $L$ is equal to

$$ \int_0^t \left[\sigma_u - S^*_u \left(\frac{\partial \varphi_u}{\partial s} - \xi_u\right) dW_u \right.$$

$$ \left. + \int \mathbb{R} \varphi(u, S_u(1 + \eta(z)), X_u, Y_u - \varphi_u - \xi_u S^*_u \eta(z)) \eta(dz, du) + \frac{1}{B_t} d\tilde{M}_t\right].$$

Now we find a suitable $\xi$ such that $L$ becomes orthogonal to $\tilde{M}$, i.e., $\langle L, \tilde{M} \rangle_t = 0$ for all positive $t$, where $\langle \cdot, \cdot \rangle$ denotes the conditional quadratic covariation. We note that the quadratic covariation

$$ d[L, \tilde{M}]_t = \sigma^2_t S^*_t \left(\frac{\partial \varphi_t}{\partial s} - \xi_t\right) dt$$

$$ + S^*_t \int \mathbb{R} \left(\varphi(t, S_t(1 + \eta(z)), X_t, Y_t) - \varphi_t - \xi_t S^*_t \eta^2(z)\right) \eta(dz, dt).$$

Hence

$$ d[L, \tilde{N}]_t = \sigma^2_t S^*_t \left(\frac{\partial \varphi_t}{\partial s} - \xi_t\right) dt$$

$$ + S^*_t \int \mathbb{R} \left(\varphi(t, S_t(1 + \eta(z)), X_t, Y_t) - \varphi_t - \xi_t S^*_t \eta^2(z)\right) \nu(dz).$$
Thus \( \langle L, \bar{M} \rangle = 0 \) if for all positive \( t \)

\[
\sigma^2_t S^*_t \frac{\partial \varphi_{t-}}{\partial s} + \frac{S^*_t}{B_t} \int \left( \varphi(t, S_{t-} (1 + \eta(z)), X_{t-}, Y_{t-}) - \varphi_{t-} \right) \eta(z) \nu(dz)
\]

\[= \sigma^2_t S^*_t \xi_t + \frac{S^*_t}{B_t} \int \eta^2(z) \nu(dz) \]

holds. Therefore, for all \( t > 0 \), \( \langle L, \bar{M} \rangle_t \) is zero if \( \xi_t \) is chosen as

\[
\xi_t = \frac{\sigma^2_t S^*_t \frac{\partial \varphi_{t-}}{\partial s} + \frac{1}{B_t} \int \left( \varphi(t, S_{t-} (1 + \eta(z)), X_{t-}, Y_{t-}) - \varphi_{t-} \right) \eta(z) \nu(dz)}{\left( \sigma^2_t + \frac{1}{B_t} \int \eta^2(z) \nu(dz) \right) S^*_t}. \tag{3.4}
\]

Thus we have essentially proved that the above choice of \( \xi \) makes the local martingale part of \( L \) orthogonal to \( \bar{M} \), irrespective of the choice of \( \beta_1(t, i) \) and \( \beta_2(t, z, i) \). However, we have not yet established existence of a particular pair \( (\beta_1, \beta_2) \) for which \( L \) is a square integrable martingale. It is evident that for ensuring \( L \) to be a local martingale, the coefficient of \( dt \) term in \( (3.3) \) should be zero, that is,

\[
\left( \mu_{t-} - r_{t-} - \beta_1(t, X_{t-}) \right) S^*_t \frac{\partial \varphi_{t-}}{\partial s} - \left( \mu_{t-} - r_{t-} \right) \xi_t S^*_t
\]

\[= \frac{1}{B_t} \int \left( \varphi(t, S_{t-} (1 + \eta(z)), X_{t-}, Y_{t-}) - \varphi_{t-} \right) \eta(z) \nu(dz) - \frac{\beta_2(t, z, X_{t-}) - 1}{B_t} \int \xi_t S^*_t \eta(z) \nu(dz).
\]

This follows if we have

\[
\left( \mu_{t-} - r_{t-} + \int \eta(z) \nu(dz) \right) \xi_t S^*_t = \left( \mu_{t-} - r_{t-} - \beta_1(t, X_{t-}) \right) S^*_t \frac{\partial \varphi_{t-}}{\partial s} - \frac{1}{B_t} \int \beta_2(t, z, X_{t-}) - 1 \left( \varphi(t, S_{t-} (1 + \eta(z)), X_{t-}, Y_{t-}) - \varphi_{t-} \right) \eta(z) \nu(dz).
\]

This follows if we have

\[
\left( \mu_{t-} - r_{t-} + \int \eta(z) \nu(dz) \right) \xi_t S^*_t = \left( \mu_{t-} - r_{t-} - \beta_1(t, X_{t-}) \right) S^*_t \frac{\partial \varphi_{t-}}{\partial s} - \frac{1}{B_t} \int \beta_2(t, z, X_{t-}) - 1 \left( \varphi(t, S_{t-} (1 + \eta(z)), X_{t-}, Y_{t-}) - \varphi_{t-} \right) \eta(z) \nu(dz).
\]

Using the expression of \( \xi_t \) as in \( (3.4) \), the above can be rewritten as

\[
\left( \mu_{t-} - r_{t-} + \int \eta(z) \nu(dz) \right) \sigma^2_t S^*_t \frac{\partial \varphi_{t-}}{\partial s}
\]

\[+ \left( \mu_{t-} - r_{t-} + \int \eta(z) \nu(dz) \right) \frac{1}{B_t} \int \left( \varphi(t, S_{t-} (1 + \eta(z)), X_{t-}, Y_{t-}) - \varphi_{t-} \right) \eta(z) \nu(dz)
\]

\[= \left( \sigma^2_t + \int \eta^2(z) \nu(dz) \right) \left( \mu_{t-} - r_{t-} - \beta_1(t, X_{t-}) \right) S^*_t \frac{\partial \varphi_{t-}}{\partial s}
\]

\[- \left( \sigma^2_t + \int \eta^2(z) \nu(dz) \right) \frac{1}{B_t} \int \left( \beta_2(t, z, X_{t-}) - 1 \left( \varphi(t, S_{t-} (1 + \eta(z)), X_{t-}, Y_{t-}) - \varphi_{t-} \right) \eta(z) \nu(dz). \right.
\]

We rearrange the terms to get

\[
\left( \mu_{t-} - r_{t-} + \int \eta(z) \nu(dz) \right) \sigma^2_t S^*_t \frac{\partial \varphi_{t-}}{\partial s} - \left( \mu_{t-} - r_{t-} \right) \int \eta^2(z) \nu(dz)
\]

\[+ \beta_1(t, X_{t-}) \left( \sigma^2_t + \int \eta^2(z) \nu(dz) \right) - \left( \mu_{t-} - r_{t-} \right) \sigma^2_t S^*_t \frac{\partial \varphi_{t-}}{\partial s}
\]

\[= - \frac{1}{B_t} \int \left( \mu_{t-} - r_{t-} + \int \eta(z) \nu(dz) \right) \eta(z) + \left( \sigma^2_t + \int \eta^2(z) \nu(dz) \right) \left( \beta_2(t, z, X_{t-}) - 1 \right) \times
\]

\[\left( \varphi(t, S_{t-} (1 + \eta(z)), X_{t-}, Y_{t-}) - \varphi_{t-} \right) \eta(z) \nu(dz). \]

The above identity holds true irrespective of the solution function \( \varphi \) if \( \beta_1 \) and \( \beta_2 \) are such that the coefficients
on both sides vanish for all time t. A direct calculation shows that such $\beta_1$ and $\beta_2$ exist and are given by

$$
\begin{align*}
\beta_1(t, i) &= \frac{(\mu(i) - r(i)) \int_{\mathbb{R}} \eta^2 d\nu - \sigma(t, i)^2 \int_{\mathbb{R}} \eta d\nu}{\sigma^2(t, i) + \int_{\mathbb{R}} \eta^2 d\nu} \\
\beta_2(t, z, i) &= 1 - \frac{(\mu(i) - r(i) + \int_{\mathbb{R}} \eta d\nu) \eta(z)}{\sigma^2(t, i) + \int_{\mathbb{R}} \eta^2 d\nu}.
\end{align*}
$$

(3.6)

Thus we have proved the parts (i) and (ii) of the following theorem.

**Theorem 3.1.** Assuming existence of a classical solution $\varphi$ to (2.4), (2.6) and (3.1) with at most linear growth, the adapted process $\xi$ as in (3.1) is such that $L$ as in (3.2) is

i) a $P$-local martingale,

ii) orthogonal to $\mathcal{M}$, the martingale part of $S^*$,

iii) a square integrable martingale.

Thus $\varphi(t, S_t, X_t, Y_t)$ is the locally risk minimizing price of a contract with terminal payoff $K(S_T)$ and $\xi$ constitutes the optimal hedging.

**Proof.** We have already established the parts (i) and (ii) and obtained

$$
\frac{K(S_T)}{B_T} = \varphi_T \left. \varphi_T \right|_{B_T} = \varphi_0 + \int_0^T \xi_u dS_u^* + L_T
$$

(3.7)

using (3.2). Due to the square integrability of $S$ and the Lipschitz property of $K$, the left side expression of the above equation is in $L^2(P)$, hence the right side is also in $L^2(P)$. Among three additive terms, the first one $\varphi_0 = \varphi(0, S_0, X_0, Y_0)$ is observed hence deterministic and remaining other two terms are orthogonal to each other. Therefore using Itô’s isometry, $\int_0^T (\xi_u)^2 d(S^*)_u + L_T^2$ has finite expectation. Thus both the facts have been established, namely $\xi$ is an admissible strategy and $L$ is indeed a square integrable martingale. Therefore, (3.7) is indeed the desired F-S decomposition. Hence the result follows from the discussion in the subsection 3.1. $\Box$

4 Pricing equation

We begin with a lemma concerning a parameter appearing in the equation (2.6). This lemma would be used in the subsequent analysis.

**Lemma 4.1.** The map $\beta_1 : [0, T] \times \mathcal{X} \rightarrow \mathbb{R}$ given by (3.6) is bounded and continuous. The map $\beta_2 : [0, T] \times \mathbb{R} \times \mathcal{X} \rightarrow \mathbb{R}$ given by (3.6) is (i) bounded and (ii) continuous in $t$ uniformly in $z$.

**Proof.** As $\sigma$ is positive valued continuous function on compact set $[0, T]$, $\inf_{[0,T] \times \mathcal{X}} \sigma(t, i) > 0$ for finite $T$. Hence the denominator in the expressions of $\beta_1$ and $\beta_2$ is away from zero. Thus using finiteness of $\nu$ and boundedness of $\eta$, $\beta_1$ and $\beta_2$ are bounded and also continuous in $t$. Again since $\eta$ is bounded, $\sup_{z, i} |\beta_2(t, z, i) - \beta_2(t', z, i)| \rightarrow 0$ as $t' \rightarrow t$. Hence the proof. $\Box$

Consider the evolution problem (2.6) on $\mathcal{D}$ with the terminal condition (3.1) and $\beta_1, \beta_2$ are according to (3.6). A typical expression of $K$, as in the case of call option takes the form $K(s, i, y) = (s - K_1)^+$, where $K_1$ is the strike price. Here $K$ is not differentiable in $s$, thus not in the domain of the operator present in the equation. So a classical solution to eqs. (2.6), (3.1) and (3.6) is not assured. However $K$ belongs to the following space

$$
V := \left\{ \varphi : (0, \infty) \times \mathcal{X} \times (0, T) \rightarrow \mathbb{R} \text{ continuous} \mid \|\varphi\|_V := \sup_{s, i, y} \frac{\varphi(s, i, y)}{1 + s} < \infty \right\}.
$$

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Clearly \((V, \|\cdot\|_V)\) is a Banach space consisting of continuous functions with at most linear growth.

In this section, via an investigation of the mild solution to eqs. \((2.0), \ (3.1)\) and \((3.6)\) in the class \(C([0,T];V)\), we aim to establish the existence and uniqueness of a classical solution to the evolution problem eqs. \((2.0), \ (3.1)\) and \((3.6)\). To this end, we would first rewrite the evolution problem in a manner, suitable for applying general theory of abstract evolution problems. Then we establish the existence and uniqueness of the continuous mild solution in Corollary \((4.1)\). For this, we now introduce another SDE on the same probability space

\[
d\hat{S}_t = \hat{S}_t \left( (r(X_t) + \beta_1(t, X_t))dt + \sigma(t, X_t)dW_t \right),
\]

where \(\{W_t\}_{t \geq 0}\) is the Brownian motion and \(\{X_t\}_{t \geq 0}\) is as in \((2.1)-(2.2)\). The above SDE has a strong positive continuous solution whose proof is much simpler than that of Theorem \((2.2)\). Indeed the solution is given by

\[
\hat{S}_t = \hat{S}_0 \exp \left( \int_0^t (r(X_u) + \beta_1(u, X_u) - \frac{1}{2} \sigma(u, X_u_-)^2)du + \int_0^t \sigma(u, X_u_-)dW_u \right).
\]

The solution \(\hat{S} = \{\hat{S}_t\}_{t \geq 0}\) along with \(X\) and \(Y\) jointly is strongly Markov. We denote the infinitesimal generator of \((\hat{S}_t, X_t, Y_t)\) by \(\{\hat{A}_t\}_{t \geq 0}\) which is given by

\[
\hat{A}_t \psi(s, i, y) := \left( \frac{\partial}{\partial y} + (r(i) + \beta_1(t, i))s + \frac{1}{2} \sigma^2(t, i)s^2 \frac{\partial^2}{\partial s^2} \right) \psi(s, i, y) + \sum_{j \neq i} \lambda_{ij}(y)(\psi(s, j, 0) - \psi(s, i, y)).
\]

The Feller property of \((\hat{S}, X, Y)\) implies that there exists a continuous evolution system (ES) \(\{U(u, t)\}_{0 \leq t \leq u \leq T}\) associated to \(\{\hat{A}_t\}_{t \geq 0}\) (see Definition \(5.1.3 [22]\)). Indeed \(U(u, t)\psi(s, i, y)\) is given by \(E(\psi(\hat{S}_u, X_u, Y_u) | \hat{S}_t = s, X_t = i, Y_t = y)\) for every \(\psi \in C^\infty\). For the interest of our problem, we wish to confirm if \(\{U(u, t)\}_{0 \leq t \leq u \leq T}\) is an ES on \(V\) associated to \(\{\hat{A}_t\}_{t \geq 0}\). This, along with some differentiability properties of the above ES is presented below whose proofs are given at the end of this section.

**Lemma 4.2.** Assume A1(i), (ii) and (iii). For any \(\psi \in V\) and \(0 \leq t < u \leq T\), \(\Psi(t, s, i, y) := U(u, t)\psi(s, i, y)\) has the following properties.

i. \(\Psi\) has at most linear growth in \(s\) and is bounded in other variables. More precisely,

\[
\|\Psi(t, \cdot, \cdot, \cdot)\|_V \leq \|\psi\|_V \left( 1 + e^{\int_0^t \sup_{\psi \in [0,T]} (r(i) + \beta_1(t, i))} \right) \quad \text{for all } 0 \leq t < u \leq T.
\]

ii. \(\Psi\) satisfies the following integral equation

\[
\Psi(t, s, i, y) = \frac{1 - F(u-t+y|i)}{1 - F(y|i)} \int_0^\infty \psi(x, i, y + u - t)\alpha(x; s, i, t, u-t)dx
\]

\[
+ \int_0^{u-t} \frac{f(y+v|i)}{1 - F(y|i)} \sum_{j \neq i} p_{ij}(y+v) \int_0^{\infty} \Psi(t+v, x, j, 0)\alpha(x; s, i, t, v)dx dv
\]

for all \(t \in [0,u], s > 0, i \in \mathcal{X}, y \in [0,t]\) where \(x \mapsto \alpha(x; s, i, t, v)\) is the probability density function of the log-normal distribution

\[
\text{Lognormal}\left( \ln s + r(i)v + \int_t^{t+v} \left( \beta_1(t', i) - \frac{1}{2} \sigma^2(t', i) \right)dt', \int_t^{t+v} \sigma^2(t', i)dt' \right).
\]

iii. \(\Psi\) is the unique solution to \((4.2)\) with values \(\Psi(t, \cdot, \cdot, \cdot)\) in \(V\) for \(t \in [0,u]\). Thus \(\{U(u, t)\}_{0 \leq t \leq u \leq T}\) is an ES on \(V\).

iv. \(\Psi : \bigcup_{0 \leq t < u} \{t\} \times (0, \infty) \times \mathcal{X} \times \{0,t\} \to \mathbb{R}\) has continuous second order partial derivative w.r.t. \(s\) and \(\Psi\) is in the domain of \(D_{t,y}\). Furthermore, \(D_{t,y}\Psi(t, s, i, y)\) is continuous where \(D_{t,y}\theta\) is defined as

\[
\lim_{h \to 0} \frac{1}{h} \left( \theta(t+h, s, i, y+h) - \theta(t, s, i, y) \right)
\]

provided the limit exists.
We can rewrite (2.3) by substituting the expression of $A_t$ as

$$
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial y} + (r(i) + \beta_1(t, i)) s \frac{\partial}{\partial s} + \frac{1}{2} \sigma^2(t, i) s^2 \frac{\partial^2}{\partial s^2} \right) \varphi(t, s, i, y) + \sum_{j \neq i} \lambda_{ij}(y) (\varphi(t, s, j, 0) - \varphi(t, s, i, y)) + \int_{\mathbb{R}} \beta_2(t, z, i) \left( \varphi(t, s(1 + \eta(z)), i, y) - \varphi(t, s, i, y) \right) d\nu = r(i) \varphi(t, s, i, y).
$$

(4.3)

Hence, using the expression of $\hat{A}_t$ in the above equation, we have the following evolution problem

$$
\begin{align*}
\left( \frac{\partial}{\partial t} + \hat{A}_t \right) \varphi(t) + B(t) \varphi(t) &= R \varphi(t) \\
\varphi(T) &= K
\end{align*}
$$

(4.4)

where for each $t \in [0, T]$, $\varphi(t) \in V$, and $\varphi(t)(s, i, y)$ is also written as $\varphi(t, s, i, y)$; further more, for any $\psi \in V$, $B(t) \psi(s, i, y) := \int_{\mathbb{R}} \beta_2(t, z, i) \left( \psi(s(1 + \eta(z)), i, y) - \psi(s, i, y) \right) \nu(dz)$ and $R \psi(s, i, y) := r(i) \psi(s, i, y)$. Using Lemma 4.1, $B(t)$ is a bounded linear map for each $t \in [0, T]$ and $t \mapsto B(t)$ is continuous. The proof of this fact appears in the Appendix, Lemma 4.2.

Let $K$ be in $V$ as before and $f : [0, T] \times V \to V$ be a continuous function in $t$ and uniformly Lipschitz continuous on $V$. Theorem 6.1.2 of [22] implies that the initial value problem

$$
\frac{\partial \varphi(t)}{\partial t} = \hat{A}_t \varphi(t) + f(t, \varphi(t)), \quad \varphi(0) = K,
$$

has a unique continuous mild solution which solves another integral equation given by

$$
\varphi(t) = U(t, 0) K + \int_0^t U(t, u) f(u, \varphi(u)) du,
$$

where $\{U(t, u)\}_{0 \leq u \leq T}$ is the ES associated to $\{\hat{A}_t\}_{t \geq 0}$. The obvious counterpart of the above statement for a terminal value problem is stated below.

**Proposition 4.3.** Let there exist an ES $\{U(u, t)\}_{0 \leq t \leq u \leq T}$ on the Banach space $V$ associated to a family of operators $\{\hat{A}_t\}_{0 \leq t \leq T}$. Then for every $K$ in $V$, the evolution problem $\frac{\partial \varphi(t)}{\partial t} + \hat{A}_t \varphi(t) + f(t, \varphi(t)) = 0$, $\varphi(T) = K$ has a unique continuous mild solution which solves

$$
\varphi(t) = U(T, t) K + \int_t^T U(u, t) f(u, \varphi(u)) du,
$$

(4.5)

provided $f$ is continuous in $t$, on $[0, T]$ and uniformly Lipschitz continuous on $V$.

It is easy to see that the evolution problem (4.4) is a special case of the above problem where

$$
f(t, \varphi(t)) = (B(t) - R) \varphi(t),
$$

where $B(t)$ and $R$ are as in (4.4). Indeed $R$ is a bounded linear operator and $B \in C([0, T]; \mathfrak{B}(V))$ (see Lemma 4.2), and hence the above $f$ satisfies the conditions of Proposition 4.3. Thus using (4.5), the following integral equation

$$
\varphi(t) = U(T, t) K + \int_t^T U(u, t) (B(u) - R) \varphi(u) du
$$

(4.6)

has a continuous solution and the solution is the mild solution to (4.4). We state this as a corollary below.

**Corollary 4.4.** The evolution problem (4.4) has a unique continuous mild solution and that solves (4.6).
Having proved this, it remains to establish the regularity of the mild solution to justify that the mild solution is indeed the classical solution. That is immediate for the cases when either \( \{ A_t \} \) are bounded operators or \( K \) is in the domain of \( \hat{A}_t \). However, none of these conditions are true in our setting.

**Theorem 4.5.** (i) Assume (A1). Let \( \varphi \) be a continuous solution to (4.6), then \( \varphi(\cdot)(\cdot, \cdot) \) in \( C(D) \) is twice differentiable in \( s \); and for every \( s, i \), \( \varphi(\cdot)(s, i, \cdot) \) is in the domain of \( D_{t,y} \) (see Lemma 4.2(iv) for the definition of \( D_{t,y} \)).

(ii) The IPDE (2.6), (3.6) and (3.1) is rewritten as (4.4). Thus by using Corollary 4.4 the IPDE has a unique classical solution with at most linear growth.

**Proof.** (i) The proof follows by applying Lemma 4.2 on each additive term on the right side of (4.6).

(ii) The IPDE (2.6), (3.6) and (3.1) is rewritten as (4.4). Thus by using Corollary 4.4 the IPDE has a unique continuous mild solution \( \varphi \) in \( C([0, T]; V) \) which solves (4.6). From (i), this mild solution is in the domain of operators in (2.6). Hence that mild solution solves (2.6) classically. \( \square \)

**proof of Lemma 4.2** (i) The measurability of \( \Psi \) follows from its representation as a conditional expectation. Let \( \{ F^X_t \} \) denote the filtration generated by the process \( X \). Consider \( \tilde{S} \) as in (4.1). Then

\[
E \left[ \frac{S_t}{S_0} \bigg| F^X_t \right] = E \left[ a.e. \lim_{N \to \infty} \prod_{i=1}^{N} \frac{\tilde{S}^{T_{i,\wedge} t}}{S_{T_{i-1,\wedge} t}} \bigg| F^X_t \right] \leq \lim_{N \to \infty} E \left[ \prod_{i=1}^{N} \frac{\tilde{S}^{T_{i,\wedge} t}}{S_{T_{i-1,\wedge} t}} \bigg| F^X_t \right],
\]

by Fatou’s lemma, where \( T_i \) is as in section 2.2 and denotes the \( i \)th transition time of \( X \). Now since for each \( i = 1, \ldots, N \), \( \frac{\tilde{S}^{T_{i,\wedge} t}}{S_{T_{i-1,\wedge} t}} \) are conditionally independent to each other given \( F^X_t \), using (4.11) the above limit can be rewritten as

\[
\lim_{N \to \infty} \prod_{i=1}^{N} e^{\int_{T_{i-1,\wedge} t}^{T_{i,\wedge} t} \left( r(X_s, u, X_u) \right) ds},
\]

which is same as \( \exp \left( \int_0^t \left( r(X_s, u, X_u) \right) ds \right) \), a bounded random variable. Thus using the above observation, we get

\[
|\Psi(t, s, i, y)| \leq E[|\psi(\tilde{S}_s, X_s, Y_s)|] \bigg| \tilde{S}_t = s, X_t = i, Y_t = y
\]

\[
= \|\psi\|_V E(1 + \tilde{S}_t) \bigg| \tilde{S}_t = s, X_t = i, Y_t = y
\]

\[
= \|\psi\|_V + \left( \|\psi\|_V E \left( e^{\int_{t}^{T} (r(X_{s}, u, X_u)) ds} \right) \right) s
\]

\[
\leq \|\psi\|_V \left( 1 + e^{T} \sup_{t', v \in [0, t]} (r(t', u, v)) s \right)
\]

for all \( 0 \leq u < t \leq T; \ i \in X; \ y \in [0, t] \). Now using Lemma A.1 the result follows.

(ii) Using the functions \( F, f, \) and \( n(t) \) as in Proposition 2.1 and tower property of conditional expectation, for all \( u > t \)

\[
U(u, t)\psi(s, i, y) = E[\psi(\tilde{S}_s, X_s, Y_s)|\tilde{S}_t = s, X_t = i, Y_t = y]
\]

\[
= E \left[ E \left( \psi(\tilde{S}_s, X_s, Y_s) \right) \tilde{S}_t, X_t, Y_t, T_{n(t)+1} \right] |\tilde{S}_t = s, X_t = i, Y_t = y
\]

\[
= \int_{v \in (0, \infty)} E \left( \psi(\tilde{S}_s, X_s, Y_s) |\tilde{S}_t = s, X_t = i, Y_t = y, T_{n(t)+1} = t + v \right) \frac{f(y + v|i)}{1 - F(y|i)} dv.
\]
The conditional distribution of $\tilde{S}_u$ given $\{T_{n(t)+1} > u\}$ and $\mathcal{F}_t$, is lognormal. Thus we get for all $v > u - t$

$$
E \left( \psi(\tilde{S}_u, X_u, Y_u) | \tilde{S}_t = s, X_t = i, Y_t = y, T_{n(t)+1} = t + v \right) = \int_0^{\infty} \psi(x, i, y + u - t) \alpha(x; s, i, t, u - t) dx
$$

where $\alpha$ is as in (4.2). Therefore, by decomposing the domain of $v$ into $(0, u - t)$ and $(u - t, \infty)$ in (4.2) and by the tower property of expectation,

$$
U(u, t) \psi(s, i, y) = \int_{u-t}^{\infty} \frac{f(y + v| i)}{1 - F(y| i)} \int_0^{\infty} \psi(x, i, y + u - t) \alpha(x; s, i, t, u - t) dx dv + \int_0^{u-t} \frac{f(y + v| i)}{1 - F(y| i)} \sum_{j \neq i} p_{ij}(y + v) \int_0^{\infty} E \left( \psi(\tilde{S}_u, X_u, Y_u) | \tilde{S}_{t+v} = x, X_{t+v} = j, Y_{t+v} = 0 \right) \alpha(x; s, i, t, v) dx dv.
$$

See Figure 2 for the case $v < u - t$. During $[T_{n(t)}, t + v)$, $X$ does not change and the age $Y$ increases monotonically. Finally, $Y_{t+v} = 0$ certainly since $T_{n(t)+1} = t + v$ implies that a transition takes place at time $t + v$, or in other words $Y_{t+v}$, the age at $t + v$ is zero. Using the conditional distribution of $X_{t+v}, Y_{t+v}$, and $S_{t+v}$, we get

$$
U(u, t) \psi(s, i, y) = \frac{1 - F(y + u - t| i)}{1 - F(y| i)} \int_0^{\infty} \psi(x, i, y + u - t) \alpha(x; s, i, t, u - t) dx + \int_0^{u-t} \frac{f(y + v| i)}{1 - F(y| i)} \sum_{j \neq i} p_{ij}(y + v) \int_0^{\infty} E \left( \psi(\tilde{S}_u, X_u, Y_u) | \tilde{S}_{t+v} = x, X_{t+v} = j, Y_{t+v} = 0 \right) \alpha(x; s, i, t, v) dx dv.
$$

Thus, for $u$ fixed and $y < t$, $U(u, t) \psi(s, i, y)$ satisfies the following integral equation

$$
U(u, t) \psi(s, i, y) = \frac{1 - F(u - t + y| i)}{1 - F(y| i)} \int_0^{\infty} \psi(x, i, y + u - t) \alpha(x; s, i, t, u - t) dx + \int_0^{u-t} \frac{f(y + v| i)}{1 - F(y| i)} \sum_{j \neq i} p_{ij}(y + v) \int_0^{\infty} U(u + v) \psi(x, j, 0) \alpha(x; s, i, t, v) dx dv.
$$

Hence (ii) follows.

(iii) In (i) and (ii) it is shown that $\Psi$ is a measurable solution to (1.2) having at most linear growth in $s$ and bounded in other variables. Now for fixed $t$, and a given $\psi \in V$, due to the property of lognormal density, the right side of (1.2) is a continuous function of $s$, and $y$. Thus the left side, $\Psi$ is also continuous for each $t$ and hence $\Psi(t, \cdot, \cdot, \cdot)$ belongs to $V$. Furthermore, in a standard manner, (1.2) can be expressed as a fixed point problem of a contraction map on $V$. We recommend the reader to see the proof of Lemma 3.1(i) of [19] for the details in a very similar context. Subsequently, a direct application of Banach fixed point theorem completes the proof of (iii).

(iv) The proof of this part relies on the results obtained in (i), (ii) and (iii). We use the fact that $\Psi$ satisfies the Volterra integral equation (1.2) of second kind. We establish the desired smoothness of $\Psi$ by establishing that of the right side expression of (1.2). On the right hand of (1.2) lognormal densities $\alpha$ appear in every integral term. We use the smoothness, finite second moment property and estimates of partial derivatives of $\alpha$ for establishing desired smoothness of the integral terms. The details of the proof is presented in the following three steps.
Step 1. In this step we would check the applicability of $D_{t,y}$ on the first additive term on the right of (4.2). From A1(ii), $F$ is twice differentiable, therefore it is enough to verify that the domain of $D_{t,y}$ contains

$$\int_0^\infty \psi(x, i, y + u - t)\alpha(x; s, i, t, u - t)dx.$$ 

It is important to note that $\psi$ in the integrand is merely continuous and thus need not be differentiable in $t$ or $y$. However, the image of $D_{t,y}$ on the above function is the limit of

$$\frac{1}{\varepsilon} \left[ \int_0^\infty \psi(x, i, y + u - t)(\alpha(x; s, i, t + \varepsilon, u - t - \varepsilon) - \alpha(x; s, i, t, u - t)) \right]dx$$

as $\varepsilon$ tends to zero, provided that the limit exists. Due to the continuous differentiability of the p.d.f. $\alpha(x; s, i, t, v)$, w.r.t. $t$ and $v$, the above expression can be rewritten using Mean Value Theorem as

$$\int_0^\infty \psi(x, i, y + u - t) \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial v} \right) \alpha(x; s, i, t + \varepsilon_1, u - t - \varepsilon_1)dx$$

(4.8)

for some $0 < \varepsilon_1(x, s, i, t, u) < \varepsilon$. Again a direct calculation shows that both $\frac{\partial \alpha}{\partial t}$ and $\frac{\partial \alpha}{\partial v}$ are of the form $\alpha(x; s, i, t, v) O(\ln^2 |x|)$. It is important to note that due to the presence of $\varepsilon_1$ at the last two arguments of $\frac{\partial \alpha}{\partial t}$ and $\frac{\partial \alpha}{\partial v}$, those arguments also depend on the $x$ variable. However, $\varepsilon_1 \in (0, \varepsilon)$, and for any given $u > 0$ and $t$ in $(0, u)$, $\varepsilon$ can be chosen so that $0 < \varepsilon < u - t$. Therefore, due to the monotonicity of the tail decay of lognormal density w.r.t. the parameter values, there is an $x'$, large enough and some $t' \in [t, t + \varepsilon]$, $v' \in [u - t - \varepsilon, u - t]$ such that

$$\sup_{\varepsilon_1 \in (0, \varepsilon)} \alpha(x; s, i, t + \varepsilon_1, u - t - \varepsilon_1) = \alpha(x; s, i, t', v')$$

for all $x \geq x'$. On the other hand, $\sup_{\varepsilon_1 \in (0, \varepsilon)} \alpha(x; s, i, t + \varepsilon_1, u - t - \varepsilon_1)$ is bounded on $[0, x']$. Hence we write the integral (4.8) as sum of two integrals by decomposing the domain $(0, \infty)$ as union of $(0, x')$ and $[x', \infty)$. For the first integral with a finite domain and bounded integrand, the convergence is obvious due to the dominated convergence theorem. Now we consider the remaining part. As $\psi$ is of at most linear growth with respect to $x$, there exists a positive constant $c_1$ such that the absolute value of the integrand in (4.8) is dominated by $c_1(1 + x)(1 + \ln^2 |x|)\alpha(x; s, i, t', v')$ on the interval $[x', \infty)$ for some $t' \in [t, t + \varepsilon]$, $v' \in [u - t - \varepsilon, u - t]$. The integral of this with respect to $x$ over $[x', \infty)$ is finite. The finiteness is immediate, since $\alpha$ is the p.d.f. of a random variable with finite variance and since there is a sufficiently large $c_2$ such that $(1 + x)(1 + \ln^2 |x|) \leq c_2(1 + x^2)$, for all $x \geq 0$. Thus, if

$$\lim_{(t', v') \to (t, v)} \int_{x'}^\infty (1 + x^2)\alpha(x; s, i, t', v')dx = \int_{x'}^\infty \lim_{(t', v') \to (t, v)} (1 + x^2)\alpha(x; s, i, t', v')dx < \infty,$$

(4.9)

using the General Lebesgue Convergence Theorem (Theorem 4.17 [24]), and the assertion of convergence of integrals on $(0, x')$, we can conclude that (4.8) converges to

$$\int_0^\infty \psi(x, i, y + u - t) \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial v} \right) \alpha(x; s, i, t, u - t)dx$$

(4.10)

as $\varepsilon$ tends to zero. The equality (4.9) can be justified by applying Vitali’s convergence theorem. To this end, we note that the integrand on the left side of (4.9) is $(1 + x^2)\alpha(x; s, i, t', v')$, a product of a quadratic function and a log normal density. This is a uniformly integrable family of functions in $x$ with family-parameters $(t', v')$ which vary on a bounded set away from $\mathbb{R} \times \{0\}$. This family is also tight as a consequence of tightness of Gaussian measures with bounded means and variances. Therefore Vitali’s convergence theorem is applicable in establishing (4.10). Hence, we conclude that (4.10) is the limit of (4.8). Using a very similar argument as above, one can prove the continuity of (4.10) also. Thus, first additive term on the right of (4.2) is in the domain of $D_{t,y}$ and the image of $D_{t,y}$ is also continuous.

Step 2. The second term on the right of (4.2) to be denoted by $\gamma$ now onward, is more involved than the first term. This term is a double integral with one of the limits depending on $t$ variable. The variable $t$
appears in the argument of continuous function $Ψ$, but not in the form of $t - y$. We check if this term is in the domain of $D_{t,y}$. The analysis is somewhat similar to the treatment of the first term. Here $D_{t,y}$ is the limit of the following expression

$$
\frac{1}{\varepsilon} \left[ \int_0^{u-t} \frac{f(y + v + \varepsilon | i)}{1 - F(y + \varepsilon | i)} \sum_{j \neq i} p_{ij}(y + v + \varepsilon) \int_0^\infty \Psi(t + v + \varepsilon, x, j, 0) \alpha(x; s, i, t + \varepsilon, v) dx dv 

- \int_0^{u-t} \frac{f(y + v | i)}{1 - F(y | i)} \sum_{j \neq i} p_{ij}(y + v) \int_0^\infty \Psi(t + v, x, j, 0) \alpha(x; s, i, t, v) dx dv \right]
$$

provided the limit exists. After a suitable substitution, the above expression can be rewritten as

$$
\int_\varepsilon^{u-t} \sum_{j \neq i} p_{ij}(y + v) \int_0^\infty \Psi(t + v, x, j, 0) \beta_\varepsilon(x, s, i, t, y) dx dv - \frac{1}{\varepsilon} \int_0^{u-t} \frac{f(y + v | i)}{1 - F(y | i)} \sum_{j \neq i} p_{ij}(y + v) \int_0^\infty \Psi(t + v, x, j, 0) \alpha(x; s, i, t, v) dx dv
$$

(4.11)

where $\beta_\varepsilon(x, s, i, t, y) = \frac{1}{\varepsilon} \left[ \frac{f(y + v | i)}{1 - F(y | i)} \alpha(x; s, i, t + \varepsilon, v - \varepsilon) - \frac{f(y + v | i)}{1 - F(y | i)} \alpha(x; s, i, t, v) \right]$. The expression in (4.11) has two additive terms. For showing convergence of first term involving repeated integrals, we observe the following. Since $f$ and $\alpha$ are continuously differentiable, using Mean value theorem we can rewrite

$$
\beta_\varepsilon = f(y + v | i) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial y} - \frac{\partial}{\partial v} \right) \frac{\alpha(x; s, i, t + \varepsilon, v - \varepsilon)}{1 - F(y + \varepsilon | i)}
$$

for some, $0 < \varepsilon < \varepsilon$. Due to A1.(i), $f$ is bounded and $\frac{1}{1 - F}$ is bounded on compact. Thus the inner integral of the first term looks similar to (4.8) with one major difference that there $\psi$ in $V$ but here $Ψ$ is a $V$ valued function of $t$. However, in the part (i), it is established that $Ψ(t, \cdots, \cdots)$ is dominated by a fixed member in $V$ uniformly in $t$. Therefore, one can mimic the arguments presented in step 1, to prove that $\lim_{\varepsilon \to 0} \int_0^\infty \Psi(t + v, x, j, 0) \beta_\varepsilon(x, s, i, t, v) dx$ exists and is equal to $\int_0^\infty \Psi(t + v, x, j, 0) \beta_0(x, s, i, t, v) dx$. Since the outer integral is on a bounded domain, to assure the convergence of this term it would be sufficient if we have $\sup_{t' \in (t, t+\varepsilon), v' \in (0, u-t)} \int_0^\infty (1 + x^2) \alpha(x; s, t, t', v') dx$ finite. This follows as the density functions are corresponding to distributions with variances lying on a bounded set. Indeed, for a fixed $s$ and $i$ this supremum is less than or equal to $1 + s^2 \left( \exp(T \sup_{t' \in [0,T]} \sigma^2(t', i)) - 1 \right) \exp(2T \sup_{t' \in [0,T]} \left( v(i) + \beta_1(t', i) \right))$.

Now we address the convergence issue of the second term in (4.11). Clearly (4.9) implies continuity of the map

$$
v \mapsto \int_0^\infty \Psi(t + v, x, j, 0) \alpha(x; s, i, t, v) dx
$$

and existence of right limit at $v = 0$. This integral is multiplied by a continuous function in $v$. Hence the second term converges to $- \frac{f(y | i)}{1 - F(y | i)} \sum_{j \neq i} p_{ij}(y) \Psi(t, s, j, 0)$ as $\varepsilon$ goes to zero. Thus, $γ$ is in the domain of $D_{t,y}$ and hence from Steps 1 and 2, the left of Equation (4.2) is in the domain of $D_{t,y}$. Or in other words, $Ψ$ is in the domain of $D_{t,y}$. Finally from the above derivations, $D_{t,y}Ψ(t, s, i, y)$ is equal to

$$
\int_0^{u-t} \sum_{j \neq i} p_{ij}(y + v) f(y + v | i) \int_0^\infty \Psi(t + v, x, j, 0) \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial y} - \frac{\partial}{\partial v} \right) \frac{\alpha(x; s, i, t, v)}{1 - F(y | i)} dx dv

- \frac{f(y | i)}{1 - F(y | i)} \sum_{j \neq i} p_{ij}(y) \Psi(t, s, j, 0).
$$

The continuity of this function w.r.t. $t$, $s$ and $y$ can easily be obtained using the estimates of partial derivatives of $α$ as before. However, we omit the details.

**Step 3.** In this step we would check the second order partial differentiability of $Ψ$ w.r.t. $s$ variable as well as the continuity of the derivative. We recall that $α$ is twice differentiable w.r.t. $t$ and $ψ$ and $Ψ(t, \cdots, \cdots)$
are in $V$ (continuous and at most of linear growth in first variable). Further more, $\frac{\partial n}{\partial t}(x; s, i, t, v) = \frac{1}{s}O(\ln |x|)\alpha(x; s, i, t, v)$. Thus by mimicking the earlier steps, for checking partial differentiability of right side of (4.2) w.r.t. $s$, we need to consider the following family of dominating integrands $(v, x) \mapsto \frac{1}{s + \varepsilon} |x|^2 \alpha(x; s + \varepsilon, i, t, v)$. In view of the General Lebesgue Convergence Theorem (Theorem 4.17 [24]) we need to investigate its integral’s convergence as $\varepsilon$ tends to zero. We appeal to Vitali’s convergence theorem in this connection. To this end we notice that $\frac{1}{s + \varepsilon} |x|^2 \alpha(x; s + \varepsilon, i, t, v)$ is uniformly integrable family of functions in $x$ with family-parameter $\varepsilon \ll 1$. This family is also tight, as $\varepsilon$ is taken from a bounded set. Indeed, for our case it is enough to consider $|\varepsilon| < s/2$. Next, we also notice that

$$v \mapsto \int_0^\infty \frac{1}{s + \varepsilon} |x|^2 \alpha(x; s + \varepsilon, i, t, v) dx,$$

is a bounded function and a member of a uniformly bounded family for $\varepsilon \ll 1$. Since the integration w.r.t. $v$ is on a finite range, the dominated convergence theorem can be applied at this stage so as to pass the limit and finally obtain the partial differentiability of right side of (4.2) w.r.t. $s$. In a similar manner existence of partial derivative with respect to $s$ of any higher order can be shown successively.

5 Conclusion

In this paper we have considered regime switching extension of the geometric Lévy process for asset price modelling. Although the Markov regimes are more common in the literature, we have considered the age dependent semi-Markov process which gives a rather general type of regime switching. Under this model assumption we derive a theoretical fair price of a class of European style contracts using the locally risk minimizing approach. The price function is shown to satisfy an IPDE (4.3). The existence and uniqueness of a classical solution to the equation is also established. We have also found out an expression of the optimal hedging in (3.4). Or in other words, if $\xi(t, s, i, y)$ denotes the number of units invested in the risky asset with immediate (left limit) price $s$ at time $t$ when the immediate market regime is at $i$ with age $y$ then

$$\xi(t, s, i, y) = \frac{\sigma(t, i)^2 \partial \varphi}{\partial s}(t, s, i, y) + \frac{1}{\sigma(t, i)^2 + \int_\mathbb{R} \eta^2(z) \nu(dz)} \frac{\eta(z) \nu(dz)}{\varphi(t, s(1 + \eta(z)), i, y) - \varphi(t, s, i, y)}.$$

We have also obtained an alternative way to write the option price function using integral equations. This equation appears at (4.4). In this the evolution operators appear which can again be calculated by solving another set of integral equations as in (4.2). Thus it is possible to write the option price function as a solution of a system of integral equations. This observation might lead instead of a finite difference method to an alternative numerical scheme involving quadrature method for finding the option price. A systematic critical comparison of these methods might be an interesting research direction. The computation of hedging strategy which involves a partial derivative and an integration of price function can also be discussed under both of the numerical approaches.

There could be many other subsequent studies depending on these results by extending the asset price model. Local volatility extension is of course one possibility. The component-wise semi-Markov regime (CSM) could be another. In the CSM setting, each of the regime dependent market parameters namely, $\mu$, $\sigma$ and $r$ is allowed to be driven by a separate semi-Markov process. These processes could be independent or correlated.

It is interesting to note that the expression of the option price function and that of the hedging function involve the drift parameter $\mu$. However, the estimation of $\mu$ is rather tricky when the dynamics has regime switching and the volatility is not negligibly small. Therefore, a relevant filtering problem should be able to find an application to the derivative pricing problem in the present context.
A Appendix

Proof of Theorem 2.2 If (2.3) has a solution \( \{S_t\} \), the jumps of this process originates from the last term on the RHS of (2.3). So we can write

\[
\Delta S_t = S_t - S_{t-} = S_{t-} \int_\mathbb{R} \eta(z)N(dz, \{t\}).
\]  

(A.1)

Thus for any finite measurable function \( f \),

\[
f \left( \frac{\Delta S_t}{S_{t-}} \right) = \int_\mathbb{R} f(\eta(z))N(dz, \{t\}).
\]

(A.2)

From (2.3), we have

\[
dS_t^c = S_{t-}(\mu_{t-}dt + \sigma_{t-}dW_t)
\]

(A.3)

since only the first two terms on RHS of (2.3) contributes to the continuous part. Hence

\[
d[S_t]^c = S_{t-}^2 \sigma_{t-}^2 dt.
\]

(A.4)

Let \( \tau := \min\{t > 0 : S_t \leq 0\} \) is a stopping time and let \( Z_t = \ln S_t \) for \( t < \tau \). Applying Itô’s formula on \( \ln S_t \) for \( 0 \leq t < \tau \) and using (A.1)-(A.4), we get

\[
dZ_t = \frac{dS_t^c}{S_{t-}^2} - \frac{1}{2} \frac{d[S_t]}{S_{t-}} + \ln S_t - \ln S_{t-} = \mu_{t-}dt + \sigma_{t-}dW_t - \frac{1}{2} \sigma_{t-}^2 dt + \ln(1 + \frac{\Delta S_t}{S_{t-}}).
\]

Using (A.2), we get,

\[
dZ_t = (\mu_{t-} - \frac{1}{2} \sigma_{t-}^2) dt + \sigma_{t-}dW_t + \int_\mathbb{R} \ln(1 + \eta(z))N(dz, \{t\}).
\]

By integrating from 0 to \( t \), where \( 0 \leq t < \tau \),

\[
\frac{S_t}{S_0} = \exp(Z_t - Z_0) = \exp \left( \int_0^t (\mu_{u-} - \frac{1}{2} \sigma_{u-}^2)du + \int_0^t \sigma_{u-}dW_u + \int_0^t \int_\mathbb{R} \ln(1 + \eta(z))N(dz, du) \right).
\]

Hence the solution to SDE (2.3) has the above form for \( 0 \leq t < \tau \). Due to the finiteness of \( \nu \) and the lower bound of \( \eta \), namely \( \eta(z) > -1 \) for all \( z \) in \( \mathbb{R} \), \( \int_0^\nu \int_\mathbb{R} \ln(1 + \eta(z))N(dz, du) \) is finite for all \( t \) and almost every \( \omega \). If possible, choose \( \omega \in \Omega \) such that \( \tau(\omega) \) is finite. Thus by letting \( t \uparrow \tau(\omega) \) in the above expression of \( S, S_{\tau(\omega)-} > 0 \). Hence non-positivity may occur only by a jump. Equation (A.1) makes it clear that non-positivity of \( S_t \) does not happen due to a jump with our assumption \( \eta(Z_1) > -1 \). Hence \( \tau = \infty \) P a.s. and \( S_t > 0 \) P a.s. for all \( t \in (0, \infty) \).

The above analysis implies that if solution exists it is unique and positive. The existence is straight forward since a direct calculation would show that the above mentioned expression of \( S \) satisfies the SDE. For establishing the square integrability, we note the following. Clearly \( S_t \) can be written as a product of a conditionally log-normal random variable and the term \( \exp(\int_0^t \int_\mathbb{R} \ln(1 + \eta(z))N(dz, du)) \) where both are independent. We further note that the log-normal random variable has bounded parameters on \([0, T]\). Therefore it is sufficient to check if \( E \left[ \exp(2 \int_0^t \int_\mathbb{R} \ln(1 + \eta(z))N(dz, du)) \right] \) is bounded on \([0, T]\).


We first note that $|N_t| := N(\mathbb{R} \times [0, t])$ is finite a.s. as $|\nu| < \infty$. Therefore

$$E \left[ \exp \left( 2 \int_0^t \int_\mathbb{R} \ln(1 + \eta(z)) N(dz, du) \right) \right] = E \left[ \prod_{i=1}^{|N_t|} (1 + \eta(z_i))^2 \right]$$

$$= E \left[ E \left[ \prod_{i=1}^{|N_t|} (1 + \eta(z_i))^2 \bigg| |N_t| \right] \right]$$

where $\{(z_i, t_i) | i = 1, \ldots, |N_t|\}$ are the point masses of $N$ on $\mathbb{R} \times [0, t]$. Since $(1 + \eta(z_1)), \ldots, (1 + \eta(z_{|N_t|}))$ are conditionally independent and identically distributed given $|N_t| = n$, the right side is equal to

$$\sum_{n=1}^\infty [E(1 + \eta(z_1))^2]^n P(|N_t| = n) = \sum_{n=1}^\infty (1 + c)^n e^{-t|\nu|} \frac{(t|\nu|)^n}{n!} = e^{-t|\nu|} \exp (t|\nu|(1 + c)) = \exp (ct|\nu|)$$

which is clearly bounded on $[0, T]$, where $E \left[ (1 + \eta(z_1))^2 - 1 \right] = c < \infty$ as $\eta$ is bounded.

**Lemma A.1.** For any nonnegative $c$,

$$\sup_{s \in (0, \infty)} \frac{1 + cs}{1 + s} \leq 1 + c.$$

**Proof.** We write $(0, \infty) = (0, 1] \cup (1, \infty)$. We check for supremum over $(0, 1]$ and $(1, \infty)$ separately. First we note that

$$\sup_{s \in (0, 1]} \frac{1 + cs}{1 + s} \leq 1 + c.$$

Again, since $0 < \frac{1}{s} < 1$ for $s \in (1, \infty)$, we have

$$\sup_{s \in (1, \infty)} \frac{1 + cs}{1 + s} = \sup_{s \in (1, \infty)} \frac{\frac{1 + c}{s} + 1}{0 + 1} \leq 1 + c.$$

Thus

$$\sup_{s \in (0, \infty)} \frac{1 + cs}{1 + s} = \max \left( \sup_{s \in (0, 1]} \frac{1 + cs}{1 + s}, \sup_{s \in (1, \infty)} \frac{1 + cs}{1 + s} \right) \leq 1 + c.$$

**Lemma A.2.** Let $\beta_2 : [0, T] \times \mathbb{R} \times \mathcal{X} \to \mathbb{R}$ be a bounded continuous function, such that $\beta_2(t, z, i)$ is continuous in $t$ uniformly in $z$ then $B : [0, T] \to \mathcal{B}(V)$ as in (4.4) is continuous, where $\mathcal{B}(V)$ is the space of bounded linear maps from $V$ to $V$ with subordinate norm.

**Proof.** We define $\overline{\beta}_2 := \sup_{(t, z, i)} |\beta_2(t, z, i)|$. Using Lemma A.1 with $c = 1 + \eta(z)$ for each $z$,

$$\|B(t)\psi\|_V = \sup_{(s, i, y) \in (0, \infty) \times \mathbb{R} \times (0, \infty)} \left| \int_\mathbb{R} \beta_2(t, z, i) \frac{\psi(s(1 + \eta(z)), i, y) - \psi(s, i, y)}{1 + s} \nu(dz) \right|$$

$$\leq \overline{\beta}_2 \sup_{(s, i, y)} \left[ \int_\mathbb{R} \frac{1 + s(1 + \eta(z))}{1 + s} \frac{\psi(s(1 + \eta(z)), i, y)}{1 + s(1 + \eta(z))} \right. + \left. \frac{\psi(s, i, y)}{1 + s} \nu(dz) \right]$$

$$\leq \overline{\beta}_2 \left[ \int_\mathbb{R} (2 + \eta(z)) \|\psi\|_V + \|\psi\|_V \nu(dz) \right]$$

$$= \overline{\beta}_2 \|\psi\|_V \left( 3\nu(\mathbb{R}) + \int \eta \nu(d\nu) \right) < \infty,$$
since $\nu$ is a finite measure and $\eta$ is a bounded function. Hence $\|B(t)\|_{\mathcal{B}(V)} \leq \beta_2 \left( 3\nu(R) + \int \eta \nu \right)$. Thus $B(t)$ is a bounded linear map for each $t \in [0, T]$.

Again since $\beta_2(t, z, i)$ is continuous in $t$ uniformly in $z$, in the similar manner as above, for nonzero $\psi$

$$\frac{\|B(t)\psi - B(t')\psi\|_V}{\|\psi\|_V} \leq \sup_{z, i} |\beta_2(t, z, i) - \beta_2(t', z, i)| \left( 3\nu(R) + \int \eta \nu \right) \to 0$$

as $t'$ tends to $t$. Thus $\lim_{t' \to t} \|B(t) - B(t')\|_{\mathcal{B}(V)} \to 0$. Hence the result.

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