ON THE EXISTENCE AND BLOW-UP OF SOLUTIONS FOR A MEAN FIELD EQUATION WITH VARIABLE INTENSITIES

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Abstract. We study an elliptic problem with exponential nonlinearities describing the statistical mechanics equilibrium of point vortices with variable intensities. For suitable values of the physical parameters we exclude the existence of blow-up points on the boundary, we prove a mass quantization property and we apply our analysis to the construction of minimax solutions.

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1. Introduction and main result

Motivated by the theory of hydrodynamic turbulence as developed by Onsager [8, 17], we consider the problem:

\[
\begin{aligned}
-\Delta u &= \lambda \left( \frac{e^u}{\int_{\Omega} e^u} + \sigma \gamma \frac{e^{\gamma u}}{\int_{\Omega} e^{\gamma u}} \right) \quad \text{in } \Omega \\
        u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(1.1)

where \( \lambda, \sigma > 0, \gamma \in [-1, 1) \) and \( \Omega \subset \mathbb{R}^2 \) is a smooth bounded domain. Problem (1.1) is derived by statistical mechanics arguments under a “deterministic” assumption on the point vortex intensities [3, 26]. More precisely, the equation derived in [26] is given by

\[
\begin{aligned}
-\Delta u &= \tilde{\lambda} \int_{[-1,1]} \frac{\alpha e^{\alpha u}}{\int_{\Omega} e^{\alpha u} \, dx} \mathcal{P}(d\alpha) \quad \text{in } \Omega \\
        u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(1.2)

where \( u \) is the stream function of the two-dimensional flow, \( \mathcal{P} \) is a Borel probability measure defined on the interval \([-1, 1]\) describing the point vortex intensity distribution and \( \tilde{\lambda} > 0 \) is a constant related to the inverse temperature. In the special case \( \mathcal{P}(d\alpha) = \mathcal{P}_\gamma(d\alpha) \), where

\[
\mathcal{P}_\gamma(d\alpha) = \tau \delta_1(d\alpha) + (1 - \tau) \delta_\gamma(d\alpha),
\]  

(1.3)

and \( \delta_1(d\alpha), \delta_\gamma(d\alpha) \) denote the Dirac measures concentrated at the points 1, \( \gamma \in [-1, 1] \), respectively, and \( \tau \in (0, 1) \), problem (1.2) takes the form

\[
\begin{aligned}
-\Delta u &= \tilde{\lambda} \left( \tau \frac{e^u}{\int_{\Omega} e^u \, dx} + (1 - \tau) \gamma \frac{e^{\gamma u}}{\int_{\Omega} e^{\gamma u} \, dx} \right) \quad \text{in } \Omega \\
        u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(1.4)

Setting

\[
\lambda = \tilde{\lambda} \tau, \quad \sigma = \frac{1 - \tau}{\tau},
\]  

(1.5)

problem (1.4) reduces to (1.1).

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We observe that taking $\gamma = -1$ in problem (1.1) we obtain the sinh-Poisson type problem derived in [19]:
\[
\begin{aligned}
-\Delta u &= \lambda \left( \frac{e^u}{\int_{\Omega} e^u} - \sigma \frac{e^{-u}}{\int_{\Omega} e^{-u}} \right) \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] (1.6)
which has received a considerable interest in recent years, see [1, 10, 12, 13, 16, 20] and the references therein. In particular, the blow-up analysis for (1.6) has been clarified by geometrical arguments involving constant mean curvature surfaces in [13]. However, such an approach seems difficult to extend to our case.

For $\sigma = 0$ problem (1.1) reduces to the standard mean field problem
\[
\begin{aligned}
-\Delta u &= \lambda \int_{\Omega} e^u \, dx \quad \text{in } \Omega \\
\quad u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\] (1.7)
which has been extensively analyzed in view of its connections to differential geometry, physics and biology, see, e.g., [11]. However, even in the “positive case” $\gamma \in (0, 1)$, problem (1.1) does not necessarily exhibit the properties of a perturbation of (1.7). This fact may be seen, by example, by considering the optimal constant for the Moser-Trudinger inequality associated to (1.1), see [21, 28] or the proof of Lemma 3.2 below. In this respect, problem (1.1) significantly differs from its “stochastic” version derived in [14] and recently analyzed in [18, 22, 23, 24, 25]. In fact, our aim in this article is to determine suitable smallness conditions for $|\gamma|$ and $\sigma$ (see (1.8)–(1.9) below) which ensure that the nonlinearity $e^{\gamma u}$ may indeed be treated as “lower-order” with respect to the “principal” term $e^{u}$.

Our main result is the following.

**Theorem 1.1.** Assume that $\mathbb{R}^2 \setminus \Omega$ has a bounded component containing at least one interior point. Fix $0 < |\gamma| < 1/2$ and $0 < \sigma < \sigma_{\gamma}$. Then, there exists a solution to Problem (1.1) for every $\lambda \in (8\pi, \lambda_{\gamma, \sigma})$.

We note that $8\pi < \lambda_{\gamma, \sigma} < 16\pi$ whenever $0 < |\gamma| < 1/2$ and $\sigma \in (0, \sigma_{\gamma})$, see Lemma 3.3 below.

Finally, we remark that problem (1.1) shares some similarity in structure with Liouville systems and Toda-type systems. Indeed, setting $v_1 = G \ast e^{u}$, $v_2 = G \ast e^{\gamma u}$, $a^{11} = \lambda / \int_{\Omega} e^u$, $a^{12} = \lambda \sigma / \int_{\Omega} e^{\gamma u}$, $a^{21} = \lambda \sigma / \int_{\Omega} e^u$, $a^{22} = \gamma^2 \sigma \lambda / \int_{\Omega} e^{\gamma u}$, we obtain $u = \lambda v_1 / \int_{\Omega} e^u \, dx + \lambda \sigma v_2 / \int_{\Omega} e^{\gamma u}$ and problem (1.1) takes the form $-\Delta v_i = \exp\{\sum_{j=1,2} a^{ij} v_j\}$, $i = 1, 2$, which is a system of Liouville type, as analyzed in [4, 9]. On the other hand, setting $w_1 = u$, $w_2 = \gamma u$, $b^{11} = \lambda / \int_{\Omega} e^{w_1}$, $b^{12} = \lambda \sigma / \int_{\Omega} e^{w_2}$, $b^{21} = \lambda / \int_{\Omega} e^{w_1}$, $b^{22} = \lambda \sigma \gamma / \int_{\Omega} e^{w_2}$, we obtain the system $-\Delta w_i = \sum_{j=1,2} b^{ij} e^{w_j}$, $i = 1, 2$, which has a “Toda-like” structure when $\gamma < 0$, see [1] and the references therein. However, Theorem 1.1 does not follow directly from the results for systems of Liouville and Toda type mentioned above, due to the substantially different assumptions for the coefficients $(a^{ij})$ and $(b^{ij})$, $i, j = 1, 2$.

This note is organized as follows. In Section 2 we use Brezis-Merle estimates [2] to exclude the existence of blow-up points on the boundary and to derive a mass quantization property, for suitably small values of $\gamma$ and $\sigma$. We note that the exclusion of boundary blow-up points could also be derived by extending the argument in [22]. Here, we provide a simple ad hoc proof which exploits the smallness assumptions on $|\gamma|$ and $\sigma$. In Section 3 we derive an
improved Moser-Trudinger type inequality. We prove Theorem 1.1 by suitably adapting an argument in [7] and by applying the blow-up results derived in Section 2.

**Notation.** Henceforth, all integrals are taken with respect to the Lebesgue measure. We may omit the integration variables if they are clear from the context. We denote by $C$ a general constant whose actual value may vary from line to line.

## 2. Blow-up results

In this section we show that for suitably small values of $|\gamma|$ and $\sigma$ the blow-up analysis for problem (1.1) is similar to the blow-up analysis for the standard mean field equation (1.7). We first exclude the existence of boundary blow-up points. Then, we prove a mass quantization property.

More precisely, let $(u_n, \lambda_n)$ be a solution sequence for (1.1) with $\lambda_n \to \lambda_0 \geq 0$. We define

$$S_\pm = \{ x_0 \in \overline{\Omega} : \exists x_n \to x_0 \text{ such that } u_n(x_n) \to \pm \infty \}$$

and we set $S = S_+ \cup S_-.$

### 2.1. Boundary blow-up exclusion

In the case where $\gamma > 0$ the boundary blow-up is readily excluded in view of the moving plane argument in [9], p. 223. Therefore, throughout this subsection, we consider the “asymmetric sinh-case” of (1.1), namely

$$\begin{cases}
-\Delta u = \lambda \left( \frac{e^u}{\int_{\Omega} e^u} - \sigma |\gamma| \frac{e^{-|\gamma|u}}{\int_{\Omega} e^{-|\gamma|u}} \right) & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega.
\end{cases} \quad (2.1)$$

We make the following assumption:

$$\lambda (1 + \sigma |\gamma|) < \frac{4\pi}{|\gamma|}. \quad (2.2)$$

In this subsection we show the following.

**Proposition 2.1.** Let $(u_n, \lambda_n)$ be a solution sequence for problem (2.1) with $\lambda_n \to \lambda_0 \geq 0$ and assume that $\lambda_0$ satisfies (2.2). Then, $S \cap \partial \Omega = \emptyset$.

We first reduce problem (2.1) to a mean field type problem with smooth weight function. Let $G = G(x, y)$ be the Green’s function defined for $x, y \in \Omega$ by

$$\begin{cases}
-\Delta G(\cdot, y) = \delta_y & \text{in } \Omega \\
 G(\cdot, y) = 0 & \text{on } \partial \Omega.
\end{cases}$$

Let $u := u_+ - u_-$, where

$$u_+ = G \ast \lambda \frac{e^u}{\int_{\Omega} e^u}$$

$$u_- = G \ast \lambda \sigma |\gamma| \frac{e^{-|\gamma|u}}{\int_{\Omega} e^{-|\gamma|u}}.$$

We observe that

$$\begin{cases}
-\Delta u_+ = \lambda \frac{h(x)e^{u_+}}{\int_{\Omega} h(x)e^{u_+}} & \text{in } \Omega \\
u_+ = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $h(x) = e^{-u_-}$ satisfies $\|h\|_{C^{1,\alpha}(\Omega)} \leq C$, $h \equiv 1$ on $\partial \Omega$. In fact, we have

$$\begin{cases}
-\Delta u_- = \lambda \sigma |\gamma| \frac{e^{-|\gamma|u}}{\int_{\Omega} e^{-|\gamma|u}} & \text{in } \Omega \\
u_- = 0 & \text{on } \partial \Omega,
\end{cases}$$

where $\lambda \sigma |\gamma| \frac{e^{-|\gamma|u}}{\int_{\Omega} e^{-|\gamma|u}}$ is $L^q$-bounded for some $q > 1$. To see this fact, recall from [2] that if $u$ satisfies:

$$\begin{cases}
-\Delta u = f & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{cases}$$
for some \( f \in L^1(\Omega) \), then for any small \( \delta > 0 \) we have
\[
\int_{\Omega} \exp \left( \frac{4\pi - \delta}{\| f \|_{L^1(\Omega)}} |u| \right) \leq \frac{4\pi^2}{\delta} (\text{diam} \Omega)^2. \tag{2.3}
\]

Hence, by elliptic estimates, \( \| u_- \|_{L^\infty(\Omega)} \leq C \). Now we write the equation for \( u_- \) in the form
\[
\begin{aligned}
-\Delta u_- &= \lambda \sigma |y| \frac{e^{\gamma |u_-| - |u_+|}}{\int_{\Omega} e^{-\gamma |u|}} \quad & \text{in } \Omega \\
u_- &= 0 & \text{on } \partial \Omega
\end{aligned}
\tag{2.4}
\]
and we observe that since \( u_+ \geq 0 \) we have \( e^{-\gamma |u_+|} \leq 1 \). Hence, the right hand side in (2.4) is \( L^\infty(\Omega) \)-bounded. It follows that \( \| u_- \|_{W^2, p(\Omega)} \leq C \) for every \( p \in (1, +\infty) \). In particular, \( \| u_- \|_{C^{1,\alpha}(\overline{\Omega})} \leq C \).

**Proof of Proposition 2.7** We adopt an argument of [1] p. 223 to our case. Let \( x_0 \in \partial \Omega \) and let \( D_r \) be a closed disc touching \( \overline{\Omega} \) only at \( x_0 \). For convenience we assume \( D_r = D(0, r) \) and \( x_0 = (r, 0) \). Then, the inversion mapping \( x \to y = r^2 x / |x|^2 \) fixes the boundary of \( D_r \) and maps \( \Omega \) to a region \( y(\Omega) \) contained inside \( D_r \). Setting \( v(y) = u^+(x) \), recalling that
\[
D_x = \frac{r^2}{|y|^4} \begin{pmatrix} -y_1^2 + y_2^2 & -2 y_1 y_2 \\ -2 y_1 y_2 & y_1^2 - y_2^2 \end{pmatrix}, \quad D_x^T D_x = \frac{r^4}{|y|^4} I,
\tag{2.5}
\]
where \( I \) denotes the identity mapping, we obtain the following equation for \( v \):
\[
\Delta v + \frac{r^4}{|y|^4} h(x(y)) e^v = 0 \quad \text{in } y(\overline{\Omega}).
\]

**Claim.** For every \( x_0 \in \partial \Omega \) there exist \( r(x_0) > 0 \), \( \delta(x_0) > 0 \) and \( \delta'(x_0) > 0 \) such that the function
\[
H(y) = \frac{r^4}{|y|^4} h(x(y))
\]
is decreasing in \( y_1 \)-direction in the set
\[
\left\{ \frac{r(x_0)}{1 + \delta(x_0)} \leq |y| \leq r(x_0), \quad y_1 > 0, \quad |y_2| \leq \delta'(x_0) \right\} \subset y(\overline{\Omega}),
\]
provided that \( r \leq r(x_0) \).

**Proof of Claim.** In view of (2.4), we compute:
\[
\partial_{y_1} \frac{1}{|y|^4} = -\frac{4y_1}{|y|^6},
\]
\[
\partial_{y_1} h(x(y)) = \frac{r^2}{|y|^4} \left\{ \left( \partial_{x_1} h(x(y)) \right)(-y_1^2 + y_2^2) + (\partial_{x_2} h(x(y)))(-2y_1 y_2) \right\}
\]
and therefore
\[
\frac{1}{r^4} \partial_{y_1} H(y) = \frac{1}{|y|^6} \left\{ -4y_1 h(x(y)) + \frac{r^2}{|y|^2} [ \left( \partial_{x_1} h(x(y)) \right)(-y_1^2 + y_2^2) + (\partial_{x_2} h(x(y)))(-2y_1 y_2) ] \right\}
\]
\[
= \frac{1}{|y|^6} \left\{ -4y_1 h(x(y)) + \frac{r^2}{|y|^2} (\partial_{x_1} h)(y_1) + \frac{r^2}{|y|^2} y_2^2 (\partial_{x_2} h)(-2y_1 y_2) \right\}.
\]

We estimate, for \( |y| \geq r/(1 + \delta), \ |y_2| < \delta' \):
\[
\frac{r^2}{|y|^2} (\partial_{x_1} h)(y_1) \leq (1 + \delta)^2 \| h \|_{C^1(\overline{\Omega})} r
\]
\[
\frac{r^2}{|y|^2} y_2^2 (\partial_{x_2} h)(-2y_1 y_2) \leq (1 + \delta)^2 \| h \|_{C^1(\overline{\Omega})} r \delta'.
\]

By choosing \( r = r(x_0) \) sufficiently small, we achieve
\[
4h(x(y)) + \frac{r^2}{|y|^2} (\partial_{x_1} h)(y_1) \geq 2.
\]
Then, for \( \delta' \) sufficiently small, we have
\[
- y_1 \left[ 4h(x(y)) + \frac{r^2}{|y|^2}(\partial_{x_1})_y + \frac{r^2}{|y|^2}y_2^2(\partial_{x_2})_y + \frac{r^2}{|y|^2}(\partial_{x_2})_y(-2y_1y_2) \right]
\leq -2y_1 + (1 + \delta)^2\|h\|_{C^1(C)}(\delta')^2 + 2(1 + \delta)^2\|h\|_{C^1(C)} r \delta' \\
\leq -\frac{r}{2} < 0.
\]
Now the argument in [11] concludes the proof. \( \square \)

### 2.2. Mass quantization.
In view of Proposition 2.1.2 we have \( \mathcal{S} \cap \partial \Omega = \emptyset \). Therefore, by local blow-up results from [15, 23] we know that setting
\[
\mu_1(dx) := \lambda e^{u} \int_{\Omega} e^{u} dx, \quad \mu_\gamma(dx) := \lambda e^{\gamma u} \int_{\Omega} e^{\gamma u} dx
\]
we have
\[
\mu_1(dx) \triangleright \sum_{p \in \mathcal{S}} m_1(p)\delta_p(dx) + r_1(x) dx \\
\mu_\gamma(dx) \triangleright \sum_{p \in \mathcal{S}} m_\gamma(p)\delta_p(dx) + r_\gamma(x) dx.
\]

**Lemma 2.1.** At every fixed \( p \in \mathcal{S} \) we have the quadratic identity:
\[
8\pi (m_1(p) + \sigma m_\gamma(p)) = (m_1(p) + \sigma \gamma m_\gamma(p))^2.
\]

**Proof.** We recall from [15] that if \((u_k, \lambda_k)\) is a solution sequence for (1.4) with
\[
\tilde{\lambda} \int_{\Omega} e^{ur_k} \triangleright \sum_{p \in \mathcal{S}} \tilde{m}_1(p)\delta_p(dx) + \tilde{r}_1(x), \quad \tilde{\lambda} \int_{\Omega} e^{\gamma ur_k} \triangleright \sum_{p \in \mathcal{S}} \tilde{m}_\gamma(p)\delta_p(dx) + \tilde{r}_\gamma(x),
\]
where \( \delta_p(dx) \) denotes the Dirac mass centered at \( p \in \Omega \), then the following relation holds:
\[
8\pi (\tau \tilde{m}_1(p) + (1 - \tau)\tilde{m}_\gamma(p)) = (\tau \tilde{m}_1(p) + (1 - \tau)\gamma \tilde{m}_\gamma(p))^2,
\]
for every \( p \in \mathcal{S} \). In view of (1.5) we have \( \tau \tilde{m}_1(p) = m_1(p) \) and \( (1 - \tau)\tilde{m}_\gamma(p) = \sigma m_\gamma(p) \). Hence, we derive the asserted identity. Alternatively, we may derive identity (2.7) by applying the Pohozaev identity in a standard way. \( \square \)

**Lemma 2.2.** Let \( u_n \) be a solution sequence for (1.1). For any \( \gamma \in [-1, 1] \) we have
\[
\int_{\Omega} e^{\gamma u_n} \geq c_0 > 0.
\]

**Proof.** If \( \gamma > 0 \), we have \( u_n \geq 0 \) in \( \Omega \) by the maximum principle and therefore
\[
\int_{\Omega} e^{\gamma u_n} \geq |\Omega| > 0.
\]
Therefore, we assume \( \gamma < 0 \). We note that since \( \|u_n\|_{W^{1,q}_0(\Omega)} \leq C \) for any \( q \in [1, 2) \), there exists \( u_0 \in W^{1,q}_0(\Omega) \) such that \( u_n \to u_0 \) weakly in \( W^{1,q}_0(\Omega) \), strongly in \( L^p(\Omega) \) for any \( p \geq 1 \) and a.e. in \( \Omega \). In view of Fatou’s lemma, we derive
\[
\liminf_{n \to \infty} \int_{\Omega} e^{\gamma u_n} \geq \int_{\Omega} e^{\gamma u_0} > 0.
\]
\( \square \)

**Proposition 2.2** (Mass quantization). Let \((u_n, \lambda_n)\) be a solution sequence for (1.1) with \( \lambda_n \to \lambda_0 \). Assume that \( |\gamma| < 1/2 \) and \( \sigma \in (0, \sigma_\gamma) \), where \( \sigma_\gamma \) is defined in (1.8). Moreover, assume that
\[
8\pi < \lambda_0 < \frac{4\pi}{|\gamma|(1 + |\gamma|\sigma)}.
\]
Then, we have \( m_\gamma(p) = 0 \) and consequently \( m_1(p) = 8\pi, r_1 \equiv 0 \) and \( \lambda_0 \in 8\pi \mathbb{N} \).
implies e known improved Moser-Trudinger inequality [5] readily implies an improved inequality for An improved Moser-Trudinger inequality. Lemma 3.1 (Improved Moser-Trudinger Inequality) Throughout this proof we omit the subscript n. Similarly as above, in view of with f = λ(e_u + σγ e_u), \|f\|_{L^1(\Omega)} \leq \Lambda(1 + \sigma |\gamma|), we have that \|eγu\|_{L^q(\Omega)} is bounded if 1 < q < 4π/[|\gamma|(1 + |\sigma|)]. The existence of such a q > 1 follows from (2.8). Moreover, since by assumption we have λ > 8π, we derive that necessarily

\[ 8\pi < \frac{4\pi}{|\gamma|(1 + |\sigma|)} \]

This inequality holds in view of the assumption σ ∈ (0, σ_γ). Therefore we have that 1 < 4π/[|\gamma|(1 + |\sigma|)] and \|eγu\|_{L^q(\Omega)} is bounded for some q > 1. Hence, m_σ(p) = 0. Now (2.8) implies m_1(p) = 8π. We decompose u = w_1 + w_2, with

\[ -\Delta w_1 = \lambda e_u \quad \text{in } \Omega \]

\[ w_1 = 0 \quad \text{on } \partial \Omega, \]

and

\[ -\Delta w_2 = \sigma \gamma \lambda e_{\gamma u} \quad \text{in } \Omega \]

\[ w_2 = 0 \quad \text{on } \partial \Omega. \]

Then, setting

\[ \phi = \sigma \gamma \lambda e_{\gamma u} \]

we have \|\phi\|_{L^q(\Omega)} ≤ C for some q > 1 and therefore \|w_2\|_{L^\infty(\Omega)} ≤ C. It follows that e_u = he_{w_1} with h = e_{w_2} ≥ e_{\inf w_2} ≥ e^{-C(\lambda)} > 0. Moreover,

\[ w_1 \rightarrow G \ast (m_1(p)\delta_p + r_1) = 4\log \frac{1}{|x-p|} + \omega + G \ast r_1 \]

with \omega smooth in the closure of a neighbourhood U of p. Therefore, by Fatou’s lemma:

\[ \liminf \int U e^\omega \geq \int U \liminf e^u \geq e^{-C(\lambda)} \int U e^\omega \frac{dx}{|x-p|^4} = +\infty. \]

Hence r_1 ≡ 0 since u is locally uniformly bounded in \Omega \setminus S. Now, the first equation in (2.6) implies the mass quantization λ ∈ 8πN.

\[ \square \]

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 by suitably adapting a variational argument due to [7]. The variational functional for Problem 1.1 is given by:

\[ J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \lambda \ln \int_\Omega e^u \, dx - \lambda \sigma \ln \int_\Omega e_{\gamma u} \, dx. \]

3.1. An improved Moser-Trudinger inequality. We observe that the standard well-known improved Moser-Trudinger inequality [5] readily implies an improved inequality for J_\lambda. For any fixed a_0, d_0 > 0 we consider the set

\[ A_{a_0,d_0} = \left\{ u \in H_0^1(\Omega) \mid \exists \Omega_1, \Omega_2 \subset \Omega \text{ s.t. dist}(\Omega_1, \Omega_2) \geq d_0 \text{ and } \frac{\int_{\Omega_1} e^u \, dx}{\int_{\Omega_2} e^u \, dx} \geq a_0, \quad i = 1, 2 \right\}. \]

Lemma 3.1 (Improved Moser-Trudinger Inequality). The functional J_\lambda is bounded from below on A_{a_0,d_0} if

\[ \lambda < \frac{16\pi}{1 + 2\sigma \gamma^2}. \]

(3.1)

Proof. For ε ∈ (0, 1) to be fixed later we decompose:

\[ J_\lambda(u) = (1 - \epsilon)\left( \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \lambda \ln \int_\Omega e^u \right) \]

\[ + \frac{\epsilon \gamma^2}{2} \int_\Omega |\nabla \gamma u|^2 \, dx - \frac{\lambda \sigma \gamma^2}{\epsilon} \ln \int_\Omega e_{\gamma u} \]

\[ := K^1(u) + K^\gamma(u) \]

(3.2)
In view of the Improved Moser-Trudinger inequality, the functional $K^1$ is bounded below on $A_{\delta_0, d_0}$ if
\[ \frac{\lambda}{1 - \epsilon} < 16\pi. \]  
(3.3)
On the other hand, the functional $K^\gamma$ is bounded below on $H^1_0(\Omega)$ if
\[ \frac{\lambda \sigma \gamma^2}{\epsilon} \leq 8\pi. \]  
(3.4)
Considering (3.3) and (3.4) we can take a suitable $\epsilon \in (0, 1)$ satisfying
\[ \epsilon < 1 - \frac{\lambda}{16\pi} \quad \text{and} \quad \epsilon \geq \frac{\lambda \sigma \gamma^2}{8\pi} \]
if
\[ \frac{\lambda \sigma \gamma^2}{8\pi} < 1 - \frac{\lambda}{16\pi} \]
which is equivalent to (3.4).

□

**Remark 3.1.** Actually, we expect boundedness below of $J_\lambda$ for all $\lambda \in (8\pi, 16\pi)$.

**Lemma 3.2.** For every $0 < |\gamma| < 1/2$ and for every $0 < \sigma < (1 - 2|\gamma|)/(2\gamma^2) = \sigma_\gamma$, the functional $J_\lambda$ is bounded below on $H^1_0(\Omega)$ if and only if $\lambda \leq 8\pi$.

**Proof.** We rewrite
\[ J_\lambda(u) = \tilde{J}_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \tilde{\lambda} \tau \ln \int_{\Omega} e^u \, dx - \tilde{\lambda}(1 - \tau) \ln \int_{\Omega} e^{\gamma u} \, dx. \]  
(3.5)
where:
\[ \sigma = \frac{1 - \tau}{\tau} \quad \text{and} \quad \tilde{\lambda} = \frac{\lambda}{\tau} \quad \tau \in (0, 1]. \]
We use a result from [21] for the functionals of the form
\[ J^P_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \tilde{\lambda} \int_{\Omega} \left( \log \int_{\Omega} e^{\alpha u} \, dx \right) P(d\alpha), \]
u \in H^1_0(\Omega). Note that $J^P_\lambda$ is the Euler-Lagrange functional for problem (1.2). In view of Theorem 4 in [21] (see also [27]) we have that $J^P_\lambda$ is bounded below if and only if $\lambda \leq \lambda^P$ where
\[ \lambda^P = 8\pi \inf \left\{ \frac{\mathcal{P}(K_\pm)}{(\int_{K_\pm} \alpha \mathcal{P}(d\alpha))^2}; K_\pm \subset I_\pm \cap \text{supp} \mathcal{P} \right\}, \]
$I_+ = [0, 1], I_- = [-1, 0]$ and $\mathcal{P} = \mathcal{P}_\gamma$ is defined by (1.3), i.e., $\mathcal{P}_\gamma(d\alpha) = \tau \delta_1(d\alpha) + (1 - \tau)\delta_1(d\alpha)$.

Assume $\gamma \geq 0$. In this case, we have
\[ \frac{\mathcal{P}_\gamma(K)}{(\int_{K} \alpha \mathcal{P}_\gamma(d\alpha))^2} = \begin{cases} 
1, & \text{if } K = \{1\} \\
\frac{\tau}{\gamma(1 - \tau)} = \frac{1}{\sigma_\gamma}, & \text{if } K = \{\gamma\} \\
\frac{\tau}{(\tau + \gamma(1 - \tau))^2} = \frac{1 + \sigma}{(1 + \gamma)^2}, & \text{if } K = \{\gamma, 1\}.
\end{cases} \]
Hence, if $\gamma > 0$ we have $\tau \lambda^P = 8\pi$ whenever $0 < \sigma \leq \frac{1 - 2\gamma}{2\gamma^2} = \sigma_\gamma$.

Analogously, for $\gamma < 0$ we have
\[ \frac{\mathcal{P}_\gamma(K)}{(\int_{K} \alpha \mathcal{P}_\gamma(d\alpha))^2} = \begin{cases} 
1, & \text{if } K = \{1\} \\
\frac{\tau}{\gamma(1 - \tau)} = \frac{1}{\sigma_\gamma}, & \text{if } K = \{\gamma\}.
\end{cases} \]
Hence, if $\gamma < 0$ we have that $\tau \lambda^P = 8\pi$ if $0 < \sigma < \sigma_\gamma$.

□

**Lemma 3.3.** Let $0 < |\gamma| < 1/2$ and let $0 < \sigma < \sigma_\gamma$, where $\sigma_\gamma$ is defined in (1.8). Then, we have $8\pi < \lambda_{\gamma, \sigma} \leq 16\pi$, where $\lambda_{\gamma, \sigma}$ is defined in (1.9).
Proof. The upper bound is clear. Therefore, we only prove the lower bound \( \lambda_{\gamma, \sigma} > 8\pi \). We readily check that
\[
\frac{16\pi}{1 + 2\sigma^2} > 8\pi \quad \text{if and only if} \quad \sigma < \frac{1}{2\gamma^2}
\]
and
\[
\frac{4\pi}{|\gamma|(1 + \sigma|\gamma|)} > 8\pi \quad \text{if and only if} \quad \sigma < \sigma_\gamma = \frac{1 - 2|\gamma|}{2\gamma^3} < \frac{1}{2\gamma^2}.
\]
The claim follows. \(\square\)

For every \( u \in H^1_0(\Omega) \) we consider the measure:
\[
\mu_u = \frac{e^u}{\int_{\Omega} e^u dx} dx \in \mathcal{M}(\Omega)
\]
and the corresponding “center of mass”:
\[
\overline{x}_\mu(u) = \int_{\Omega} x d\mu_u \in \mathbb{R}^2.
\]

**Lemma 3.4.** Let \( \lambda \in (8\pi, \frac{16\pi}{1 + 2\gamma^2}) \) and let \( \{u_n\} \subset H^1_0(\Omega) \) be a sequence such that \( J_\lambda(u_n) \to -\infty \) and \( \overline{x}_\mu(u_n) \to x_0 \in \mathbb{R}^2 \). Then, \( x_0 \in \overline{\Omega} \) and
\[
\mu_{u_n} \rightharpoonup^* \delta_{x_0} \quad \text{weakly}^* \quad \text{in} \quad C(\overline{\Omega})'.
\]

**Proof.** For every fixed \( r > 0 \) we denote by \( Q_n(r) \) the concentration function of \( \mu_n \), i.e.
\[
Q_n(r) = \sup_{x \in \Omega} \int_{B(x, r) \cap \Omega} \mu_n.
\]
For every \( n \), there exists \( \tilde{x}_n \in \overline{\Omega} \) such that
\[
Q_n(r/2) = \int_{B(\tilde{x}_n, r/2) \cap \Omega} \mu_n.
\]
Upon taking a subsequence, we have that \( \tilde{x}_n \to \tilde{x}_0 \in \overline{\Omega} \).

Now, let us set
\[
\Omega_1^n = B(\tilde{x}_n, r/2) \cap \Omega \quad \text{and} \quad \Omega_2^n = \Omega \setminus B(\tilde{x}_n, r),
\]
so that
\[
\text{dist}(\Omega_1^n, \Omega_2^n) \geq r/2.
\]
Since \( J_\lambda(u_n) \to -\infty \) and since \( \lambda < \frac{16\pi}{1 + 2\gamma^2} \), in view of (3.2) necessarily we have \( K^1(u_n) \to -\infty \). Therefore, in view of the standard Improved Moser-Trudinger inequality (1), we conclude that
\[
\min\{\mu_n(\Omega_1^n), \mu_n(\Omega_2^n)\} \to 0.
\]
In particular, \( \min\{Q_n(r/2), 1 - Q_n(r)\} \leq \min\{\mu_n(\Omega_1^n), \mu_n(\Omega_2^n)\} \to 0 \).

On the other hand, for every fixed \( r > 0 \) let \( k_r \in \mathbb{N} \) be such that \( \Omega \) is covered by \( k_r \) balls of radius \( r/2 \). Then, \( 1 = \mu_n(\Omega) \leq k_r Q_n(r/2) \), so that \( Q_n(r/2) \geq k_r^{-1} \) for every \( n \). We conclude that necessarily \( Q_n(r) \to 1 \) as \( n \to \infty \). Since \( r > 0 \) is arbitrary, we derive in turn that \( 1 - Q_n(r/2) = \mu_n(\Omega \setminus B(\tilde{x}_n, r/2)) \to 0 \) as \( n \to \infty \). That is, \( \mu_{u_n} \rightharpoonup^* \delta_{\tilde{x}_0} \). It follows that \( \overline{x}_\mu(u_n) = \int_{\Omega} x d\mu_{u_n} \to \tilde{x}_0 = x_0 \in \Omega \), as asserted. \(\square\)

At this point, in order to prove Theorem 1.1, we shall adapt a construction in (4). Let \( \Gamma_1 \subset \Omega \) be a non-contractible curve which exists since \( \Omega \) is non-simply connected. Let \( \mathbb{D} = \{(r, \theta) : 0 \leq r < 1, \ 0 \leq \theta < 2\pi\} \) be the unit disc. Define
\[
\mathcal{D}_\lambda := \left\{ h \in C(\mathbb{D}, H^1_0(\Omega)) \ \text{s.t.}: \right\}
\]
\[
(i) \quad \lim_{r \to 1} \sup_{\theta \in (0, 2\pi)} J_\lambda(h(r, \theta)) = -\infty
\]
\[
(ii) \quad \overline{x}_\mu(h(r, \theta)) \text{ can be extended continuously to } \overline{\mathbb{D}}
\]
\[
(iii) \quad \overline{x}_\mu(h(1, \cdot)) \text{ is one-to-one from } \partial \mathbb{D} \text{ onto } \Gamma_1
\]

**Lemma 3.5.** For every \( \lambda \in (8\pi, 16\pi) \) the set \( \mathcal{D}_\lambda \) is non-empty.
We shall prove that $c \subset \{25\}$, Proposition 4.1.

We define “truncated Green’s function”:

$$V_r(X) = \begin{cases} 4 \log \frac{1}{1-r} & \text{for } X \in B(0,1-r) \\ 4 \log \frac{1}{|X|} & \text{for } X \in B(0,1) \setminus B(0,1-r) \end{cases}$$

and

$$v_{r,\theta}(x) = \begin{cases} 0 & \text{for } x \in \Omega \setminus B(\gamma_1(\theta), \varepsilon_0) \\ V_r(\varphi_{\theta}(x)) & \text{for } x \in B(\gamma_1(\theta), \varepsilon_0). \end{cases}$$

We set

$$h(r, \theta)(x) = v_{r,\theta}(x), \ x \in \Omega.$$ (3.6)

The function $h$ defined in (3.6) satisfies $h \in \mathcal{D}_\Lambda$. To see that $h$ verifies the (i)-condition it is sufficient to note that

$$\int_{\Omega} e^{\gamma h} \, dx \geq |\Omega| - \pi \varepsilon_0 > 0.$$ Then the claim follows by [7].

Define

$$c_\lambda := \inf_{h \in \mathcal{D}_\Lambda} \sup_{(r, \theta) \in \Omega} J_\lambda(h(r, \theta)).$$ (3.7)

We shall prove that $c_\lambda$ defines a critical value for $J_\lambda$ using the Struwe Monotonicity Trick contained in [23], Proposition 4.1.

In view of Lemma 3.4 we have $c_\lambda < +\infty$.

**Lemma 3.6.** For any $\lambda \in (8\pi, \lambda_{\gamma, \sigma})$, $c_\lambda > -\infty$.

**Proof.** Denote by $B$ a bounded component of $\mathbb{R}^2 \setminus \Omega$ with at least an interior point and such that $\Gamma_1$ encloses $B$. By continuity and by the (iii)-property defining $\mathcal{D}_\lambda$, we have $\varphi_{\theta}(h(\mathbb{D})) \supset B$ for all $h \in \mathcal{D}_\lambda$. By contradiction, assume that $c_\lambda = -\infty$. Then, there exists a sequence $\{h_n\} \subset \mathcal{D}_\lambda$ such that $\sup_{(r, \theta) \in \Omega} J_\lambda(h_n(r, \theta)) \to -\infty$. Let $x_0$ be an interior point of $B$. For every $n$ we take $(r_n, \theta_n) \in \Omega$ such that $\varphi_{\theta}(h_n(r_n, \theta_n)) = x_0$. In view of Lemma 3.4 it should be $x_0 \in \overline{B} \cap \overline{\Omega} = \emptyset$, a contradiction. □

At this point we set

$$\mathcal{G}(u) = \ln \int_{\Omega} e^u \, dx + \sigma \ln \int_{\Omega} e^{\gamma u} \, dx$$ (3.8)

so that our functional $\mathcal{J}_\lambda$ takes the form

$$J_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \lambda \mathcal{G}(u).$$

**Lemma 3.7.** For $8\pi < \lambda_1 \leq \lambda_2 < 16\pi$, we have $\mathcal{D}_{\lambda_1} \subseteq \mathcal{D}_{\lambda_2}$.

**Proof.** It is sufficient to note that whenever $J_\lambda(u) \leq 0$ it is $\mathcal{G}(u) \geq 0$, with $\mathcal{G}$ given by (3.8) and this implies that

$$J_{\lambda_1}(u) \geq J_{\lambda_2}(u) \quad \text{for } 8\pi < \lambda_1 \leq \lambda_2 < 16\pi \quad \text{if } J_{\lambda_1}(u) \leq 0.$$ Hence, $\mathcal{D}_{\lambda_1} \subseteq \mathcal{D}_{\lambda_2}$ for every $8\pi < \lambda_1 \leq \lambda_2 < 16\pi$. □

**Lemma 3.8.** The function $\mathcal{G} : H_0^1(\Omega) \to \mathbb{R}$ defined by (3.8) satisfies:

1) $\mathcal{G} \in \mathcal{C}^2(H_0^1(\Omega); \mathbb{R})$
2) $\mathcal{G}'$ is compact
3) $\langle \mathcal{G}''(u) \varphi, \varphi \rangle \geq 0$ for every $u, \varphi \in H_0^1(\Omega)$, where $\langle \cdot, \cdot \rangle$ is the $L^2$-inner product.
Proof. For every $u, \varphi \in H^1_0(\Omega)$ we have:

$$G'(u)\varphi = \frac{\int_\Omega \varphi e^u dx}{\int_\Omega e^u dx} + \sigma \int_\Omega \gamma \varphi e^{\gamma u} dx$$

and therefore the compactness of $G'$ follows by the compactness of the Moser-Trudinger embedding. Moreover, for every $u, \varphi \in H^1_0(\Omega)$ we have, using the Schwarz inequality,

$$\langle G''(u)\varphi, \varphi \rangle = \frac{1}{\left( \int_\Omega e^u dx \right)^2} \left[ \left( \int_\Omega e^u \varphi^2 dx \right) \left( \int_\Omega e^u \varphi dx \right) - \left( \int_\Omega e^u \varphi dx \right)^2 \right]$$

$$+ \frac{\gamma^2\sigma}{\left( \int_\Omega e^{\gamma u} dx \right)^2} \left[ \left( \int_\Omega e^{\gamma u} \varphi^2 dx \right) \left( \int_\Omega e^{\gamma u} \varphi dx \right) - \left( \int_\Omega e^{\gamma u} \varphi dx \right)^2 \right] \geq 0.$$

$$\Box$$

Now we are able to prove the following.

**Proposition 3.1.** Let $\sigma > 0$ and assume that (1.8) holds. For almost every $\lambda \in (8\pi, \lambda_\gamma, \sigma)$, $c_\lambda > -\infty$ given by (8.7) is a saddle-type critical value for $J_\lambda$.

**Proof of Proposition 3.1.** In view of Lemma 3.8, Lemma 3.5, Lemma 3.6 and Lemma 3.7 we may apply the well known Struwe’s monotonicity trick to derive the existence of the desired critical value. See [7] or [25], Proposition 4.1 with $\mathcal{H} = H^1_0(\Omega)$, $V = \mathbb{D}$, $A = -\infty$ and $F_\lambda = D_\lambda$. \hfill $\Box$

**Proof of Theorem 1.1.** (Completion by blow-up results). We fix $\lambda_0 \in (8\pi, \lambda_\gamma, \sigma)$. In view of Proposition 3.1 there exists $\lambda_n \to \lambda_0$ such that problem (1.1) with $\lambda = \lambda_n$ admits a solution $u_n$. By the blow up analysis as stated in Proposition 2.2 we have the compactness of solution sequences. Therefore, up to subsequences, we obtain that $u_n \to u_0$ with $u_0$ a solution to (1.1) with $\lambda = \lambda_0$. \hfill $\Box$

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