Summation of certain trigonometric series with logarithmic coefficients

Rufus Boyack

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Abstract The sums of three trigonometric series with logarithmic coefficients are derived by extending an approach first utilized by Lerch. By applying Frullani’s theorem to two of these series, two non-trivial integrals involving hyperbolic functions are evaluated in terms of the digamma function.

Keywords Trigonometric series · Logarithmic coefficients · Digamma function · Frullani’s theorem.

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1 Introduction

Trigonometric series [1,2] are of vital importance in mathematics and physics, especially when such series correspond to a Fourier series. The physical applications of Fourier series in mathematical physics abound, and often it is desirable to obtain a closed-form expression for their sum. One familiar example is Kummer’s expression [3] for the logarithm of the Gamma function, which involves a series whose coefficients include a logarithmic term. In his elegant paper [4] “Über eine Formel aus der Theorie der Gammafunktion”, Lerch derives the sum of another particular trigonometric series with a logarithmic coefficient. The result is expressed as follows: if \(0 < \nu < 1\),

\[
\sum_{k=2}^{\infty} \log \left(1 - \frac{1}{k^2}\right) \cos (2k\nu \pi) = 2 \sin^2 (\nu \pi) (\psi (1) - \psi (\nu) - \log (\pi))
\]

\[
-\frac{\pi}{2} \sin (2\nu \pi) - \log (2).
\] (1)
By implementing similar methodology, the sums of three other trigonometric series with different logarithmic coefficients than that of Eq. (1) can be derived. Thus, it is the purpose of this paper to present a derivation of the following results:

**Theorem 1.** If \(0 < \nu < 1\),

\[
\sum_{k=2}^{\infty} \log \left(\frac{k-1}{k+1}\right) \sin(2k\nu \pi) = \sin(2\nu \pi) (\log(4) - \psi(1) + \psi(\nu)) + \pi \cos^2(\nu \pi). \tag{2}
\]

\[
\sum_{k=2}^{\infty} \log \left(\frac{2k-1}{2k+1}\right) \sin(2k\nu \pi) = \frac{1}{2} \sin(\nu \pi) \left(\psi\left(\frac{\nu}{2}\right) - \psi\left(\frac{\nu+1}{2}\right)\right) + \pi + \log(3) \sin(2\nu \pi). \tag{3}
\]

\[
\sum_{k=2}^{\infty} \log \left(\frac{4(k^2-1)}{4k^2-1}\right) \cos(2k\nu \pi) = \frac{1}{2} \cos(\nu \pi) \left(\psi\left(\frac{\nu+1}{2}\right) - \psi\left(\frac{\nu}{2}\right) - 4 \log(2) \cos(\nu \pi)\right) + \frac{\pi}{2} \sin(2\nu \pi)
\]

\[
+ \cos(2\nu \pi) \left(\log\left(\frac{3\pi}{2}\right) - \psi(1) + \psi(\nu)\right). \tag{4}
\]

Further identities can be obtained upon combining Eq. (1) with Eq. (4) or Eq. (2) with Eq. (3). In Ref. [4], Lerch applies Frullani’s theorem to Eq. (1) to obtain a closed-form expression for the integral of a particular theta function. Applying Frullani’s theorem to the series appearing in Eq. (2) and Eq. (3) leads to two series that can be explicitly summed, and, as will be shown, this produces two non-trivial integration identities. The series given here also provide novel representations of the digamma function, and they can be used to obtain certain integrals of the digamma function [5]. The derivation of these results is now presented.

2 Derivation of results

2.1 An intermediate lemma

The starting point is Eq. (1) of Ref. [4], which states:

**Lemma 1.** If \(0 < x < 1\), \(0 < \nu < 1\),

\[
\sum_{n=0}^{\infty} e^{-2\pi i (\nu+n)} = -\log(2\pi x) + \frac{\Gamma'(\nu)}{\Gamma(\nu)} - \frac{i\pi}{2} \sum_{k=-\infty}^{\infty} e^{2k\pi i} \log\left(\frac{x+k}{k}\right). \tag{5}
\]
Here, the prime notation appearing in the series means that the term \( k = 0 \) is excluded. For convenience, logarithmic derivatives of the Gamma function shall be written in terms of the digamma function: \( \psi(z) \equiv \Gamma'(z) / \Gamma(z) \). In Ref. [6], Lerch proved the result in Eq. (5) by taking the derivative of the series on the left-hand side (with \( ix \) replaced by \( x \)) and then summing the resulting series. Upon integrating the result, Lerch then verified the Lemma. For completeness, an alternative derivation of Eq. (5) is provided here.

**Proof.** To begin, the following result (due to Kronecker) can be obtained from page 123 of Ref. [3]. If \( 0 < \nu < 1 \),

\[
\lim_{N \to \infty} \sum_{k=-N}^{N} \frac{e^{2k\nu \pi i}}{k+y} = \frac{2\pi i}{1-e^{-2\pi i y}} - \frac{1}{y}. \tag{6}
\]

Now integrate both sides of Eq. (6) with respect to \( y \), with the domain of integration between \( y = \frac{\epsilon}{2\pi} \) and \( y = x \), where \( 0 < x < 1 \), and at the end of the calculation take the limit \( \epsilon \to 0 \). Using term-by-term integration, the left-hand side becomes

\[
\lim_{\epsilon \to 0} \int_{\frac{x}{2\pi}}^{x} dy \sum_{k=-\infty}^{\infty} \frac{e^{2k\nu \pi i}}{k+y} = \sum_{k=-\infty}^{\infty} e^{2k\nu \pi i} \log \left( \frac{x+k}{k} \right).
\]

Perform the same procedure on the right-hand side of Eq. (6), and then substitute \( t = 2\pi iy \) to obtain

\[
\lim_{\epsilon \to 0} \int_{\frac{x}{2\pi}}^{x} dt \left( 2\pi i \frac{e^{-2\pi i \nu y}}{1-e^{-2\pi i y}} - \frac{1}{y} \right) = \lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2\pi}}^{2\pi i x} dt \left( \frac{e^{-\nu t}}{1-e^{-t}} - \frac{1}{t} \right)
\]

\[
= \lim_{\epsilon \to 0} \left[ \int_{\frac{\epsilon}{2\pi}}^{2\pi i x} dt \left( \frac{e^{-\nu t}}{1-e^{-t}} - \log \left( \frac{2\pi x}{\epsilon} \right) \right) \right].
\]

The remaining integral is now evaluated by converting it to a contour integral in the complex plane. Indeed, consider

\[
\tilde{I} = \oint_{C} dz \frac{e^{-\nu z}}{1-e^{-z}}.
\]

Here, the closed contour \( C \) is the concatenation of the contours \( C_1, C_2, C_3, \) and \( C_{\epsilon} \), where \( C_1 \) is the contour along \( (0,i\epsilon) \) to \( (0,2\pi ix) \), \( C_2 \) is the contour along \( (0,2\pi ix) \) to \( (\infty,0) \), \( C_3 \) is the contour along \( (\infty,0) \) to \( (\epsilon,0) \), and \( C_{\epsilon} \) is the quarter-circle contour skirting round from \( (\epsilon,0) \) to \( (0,i\epsilon) \). By Cauchy’s theorem \( \tilde{I} = 0 \). The integral \( I_{C_{\epsilon}} \) can be computed using the quarter-circle contour, and in the limit \( \epsilon \to 0 \) the result is \( I_{C_{\epsilon}} = \frac{-i\pi}{2} \). Combining these results together then implies

\[
\lim_{\epsilon \to 0} \int_{\frac{\epsilon}{2\pi}}^{2\pi i x} dt \left( \frac{e^{-\nu t}}{1-e^{-t}} \right) = \lim_{\epsilon \to 0} \left[ \int_{\epsilon}^{\infty} dt \left( \frac{e^{-\nu t}}{1-e^{-t}} \right) - \int_{2\pi i x}^{\infty} dt \left( \frac{e^{-\nu t}}{1-e^{-t}} \right) - \frac{i\pi}{2} \right]. \tag{7}
\]
The first integral on the right-hand side of Eq. (7) is
\[
\lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} dt \frac{e^{-\nu t}}{1 - e^{-t}} = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} dt \frac{e^{-\nu t} - e^{-t} + e^{-t}}{1 - e^{-t}}
\]
\[
= \int_{0}^{\infty} dt \frac{e^{-\nu t} - e^{-t}}{1 - e^{-t}} + \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} dt \frac{e^{-t}}{1 - e^{-t}}
\]
\[
= \psi(1) - \psi(\nu) - \lim_{\epsilon \to 0} \log(\epsilon).
\]

In the third line, the integral representation of the digamma function (page 247 of Ref. [3]) has been used:
\[
\psi(z) = \int_{0}^{\infty} dt \left( \frac{e^{-t}}{t} - \frac{e^{-zt}}{1 - e^{-t}} \right), \quad \text{Re}(z) > 0.
\]

The second integral on the right-hand side of Eq. (7) is computed by expanding the denominator as a geometric series. Thus,
\[
\int_{2\pi i x}^{\infty} dt \frac{e^{-\nu t}}{1 - e^{-t}} = \sum_{n=0}^{\infty} e^{-2\pi i x (\nu + n)} \frac{1}{\nu + n}.
\]

Combining the previous results then produces
\[
\sum_{k=-\infty}^{\infty} e^{2k\nu i \pi} \log \left( \frac{x + k}{k} \right) = -\log(2\pi x) + \psi(1) - \psi(\nu)
\]
\[
- \frac{i\pi}{2} - \sum_{n=0}^{\infty} e^{-2\pi i x (\nu + n)} \frac{1}{\nu + n}.
\]

Rearranging this expression then gives the desired result in Eq. (5).

2.2 Proof of theorem 1

We now proceed to prove the results in Eqs. (2)-(4).

Proof. The first step is to set \( x = \frac{1}{2} \) in Eq. (5) and multiply both sides by \( e^{i\nu \pi} \). After simplifying, the expression obtained is
\[
\sum_{k=-\infty}^{\infty} e^{(2k+1)\nu i \pi} \log \left( \frac{2k+1}{2k} \right) = e^{i\nu \pi} \left( \log(\pi) - \psi(1) + \psi(\nu) + \frac{i\pi}{2} \right) + \sum_{n=0}^{\infty} \frac{(-1)^n}{\nu + n}. \]
The sum on the right-hand side of Eq. (8) can be explicitly computed using the digamma function \[3,7\], which leads to

\[
\sum_{k=-\infty}^{\infty} e^{(2k+1)i\nu\pi} \log \left( \frac{2k + 1}{2k} \right) = e^{i\nu\pi} \left( \log (\pi) - \psi(1) + \psi(\nu) + \frac{i\pi}{2} \right) + \frac{1}{2} \left( \psi \left( \frac{\nu + 1}{2} \right) - \psi \left( \frac{\nu}{2} \right) \right).
\]

By equating the real and imaginary parts of each side of this equation, two results are obtained. In the case of the imaginary parts, after rewriting the \( k \)-summation over a single sum from \( k = 1 \) to \( k = \infty \), the following expression is produced

\[
\sum_{k=1}^{\infty} \log \left( \frac{k}{k+1} \right) \sin \left( (2k + 1)\nu\pi \right) = \sin (\nu\pi) (\log (2\pi) - \psi(1) + \psi(\nu)) + \frac{\pi}{2} \cos (\nu\pi).
\]

Now multiply this result by \( 2\sin (\nu\pi) \), then use the identity \( 2\sin (a)\sin (b) = \cos (a - b) - \cos (a + b) \), and finally relabel indices to obtain

\[
2\sin^2 (\nu\pi) (\log (2\pi) - \psi(1) + \psi(\nu)) + \frac{\pi}{2} \sin (2\nu\pi) = \sum_{k=1}^{\infty} \log \left( \frac{k}{k+1} \right) \left[ \cos (2k\nu\pi) - \cos ((2k + 2)\nu\pi) \right]
\]

\[
= \sum_{k=2}^{\infty} \log \left( \frac{k}{k+1} \right) \cos (2k\nu\pi) + \log \left( \frac{1}{2} \right) \cos (2\nu\pi)
\]

\[
= -\sum_{k=2}^{\infty} \log \left( \frac{1}{k^2} \right) \cos (2k\nu\pi) - \log (2) \cos (2\nu\pi).
\]

Simplifying this equation then gives the result in Eq. (11).

It is possible to obtain three more results from Eq. (9), either by multiplying the imaginary part by \( 2\cos (\nu\pi) \), or by taking the real part and then multiplying by \( 2\sin (\nu\pi) \) or \( 2\cos (\nu\pi) \). Indeed, upon multiplying Eq. (10) by \( 2\cos (\nu\pi) \), then using the identity \( 2\sin (a)\cos (b) = \sin (a + b) + \sin (a - b) \), and after finally relabelling indices, the result obtained is

\[
\sin (2\nu\pi) (\log (2\pi) - \psi(1) + \psi(\nu)) + \pi \cos^2 (\nu\pi) = \sum_{k=1}^{\infty} \log \left( \frac{k}{k+1} \right) [\sin ((2k + 2)\nu\pi) + \sin (2k\nu\pi)]
\]

\[
= \sum_{k=2}^{\infty} \log \left( \frac{k-1}{k+1} \right) \sin (2k\nu\pi) + \log \left( \frac{1}{2} \right) \sin (2\nu\pi).
\]

Rearranging this equation then produces the result in Eq. (2).
Returning now to Eq. (9), by taking the real part of this equation, and rewriting the $k$-summation over a single sum from $k = 1$ to $k = \infty$, the result obtained is

\[-\sum_{k=1}^{\infty} \log \left( \frac{(2k+1)^2}{2^{2k}(k+1)} \right) \cos ((2k+1)\nu\pi) = \cos (\nu\pi) \left( \log \left( \frac{\pi}{2} \right) - \psi (1) + \psi (\nu) \right) - \frac{\pi}{2} \sin (\nu\pi) + \frac{1}{2} \left( \psi \left( \frac{\nu+1}{2} \right) - \psi \left( \frac{\nu}{2} \right) \right) \, . \tag{11} \]

Multiply this result by $2 \sin (\nu\pi)$, then again use the identity $2 \sin (a) \cos (b) = \sin (a+b) + \sin (a-b)$, and finally relabel indices to obtain

\[\sin (2\nu\pi) \left( \log \left( \frac{\pi}{2} \right) - \psi (1) + \psi (\nu) \right) - \pi \sin^2 (\nu\pi) + \sin (\nu\pi) \left( \psi \left( \frac{\nu+1}{2} \right) - \psi \left( \frac{\nu}{2} \right) \right) = -\sum_{k=2}^{\infty} \log \left( \frac{(2k-1)^2}{2^{2k}(k+1)} \right) \sin (2k\nu\pi) + \log \left( \frac{9}{8} \right) \sin (2\nu\pi) \, . \tag{12} \]

After simplifying this expression and using the result in Eq. (2), the result in Eq. (3) is obtained.

Now multiply Eq. (11) by $2 \cos (\nu\pi)$, then use the identity $2 \cos (a) \cos (b) = \cos (a+b) + \cos (a-b)$, and finally relabel indices to obtain

\[2 \cos^2 (\nu\pi) \left( \log \left( \frac{\pi}{2} \right) - \psi (1) + \psi (\nu) \right) - \frac{\pi}{2} \sin (2\nu\pi) + \cos (\nu\pi) \left( \psi \left( \frac{\nu+1}{2} \right) - \psi \left( \frac{\nu}{2} \right) \right) = -\sum_{k=2}^{\infty} \log \left( \frac{(4k^2-1)^2}{16k^2(k^2-1)} \right) \cos (2k\nu\pi) - \log \left( \frac{9}{8} \right) \cos (2\nu\pi) \, . \tag{12} \]

After simplifying this expression and using the result in Eq. (1), the result in Eq. (4) is obtained.

This completes the derivation of the results in Eqs. (1)-(4); the first result was derived in Ref. [4], whereas the other three results have been derived in this paper.
3 Applications

3.1 Applying Frullani’s theorem

Frullani’s theorem (see page 479 of Ref. [7] or page 656 of Ref. [8]; for further discussion of a more general class of Frullanian integrals see page 195 of Ref. [9]) leads to the following result (see pages 116-117 of Ref. [3]); if \( a, b > 0 \),

\[
\log \left( \frac{b}{a} \right) = \int_0^\infty \frac{dx}{x} \left( e^{-ax} - e^{-bx} \right).
\]

Using this result, two integration identities shall be proved.

**Theorem 2.** If \( 0 < \nu < 1 \),

\[
\int_0^\infty \frac{dx}{x} \sinh(x) e^{-x} \left( \frac{e^{-x} - 2 \cos(2\nu \pi)}{\cosh(x) - \cos(2\nu \pi)} \right) = \log(4\pi) - \psi(1) + \psi(\nu) + \frac{\pi}{2} \cot(\nu \pi). \tag{13}
\]

**Proof.** Consider the result in Eq. (2). Applying Frullani’s theorem to the logarithm in the summand leads to

\[
\sin (2 \nu \pi) (\log(4\pi) - \psi(1) + \psi(\nu)) + \pi \cos^2(\nu \pi)
\]

\[
= \sum_{k=2}^{\infty} \int_0^\infty \frac{dx}{x} \left( e^{-(k+1)x} - e^{-(k-1)x} \right) \sin (2k\nu \pi)
\]

\[
= - \int_0^\infty \frac{dx}{x} 2 \sinh(x) \sum_{k=2}^{\infty} e^{-kx} \sin (2k\nu \pi). \tag{14}
\]

The series above can be exactly summed, as shown forthwith.

\[
\sum_{k=2}^{\infty} e^{-kx} \sin (2k\nu \pi) = \text{Im} \sum_{k=2}^{\infty} e^{-k(x-2i\nu \pi)}
\]

\[
= \text{Im} \frac{\exp(4i\nu \pi - 2x)}{1 - \exp(2i\nu \pi - x)}
\]

\[
= \frac{1}{2} e^{-x} \sin (2\nu \pi) \left( \frac{2 \cos(2\nu \pi) - e^{-x}}{\cosh(x) - \cos(2\nu \pi)} \right).
\]

Inserting this identity into Eq. (14) and then simplifying leads to the result in Eq. (13). \( \square \)

**Theorem 3.** If \( 0 < \nu < 1 \),

\[
\int_0^\infty \frac{dx}{x} \sinh(x) e^{-2x} \left( \frac{e^{-2x} - 2 \cos(2\nu \pi)}{\cosh(2x) - \cos(2\nu \pi)} \right) = \frac{\pi}{2} \cosec(2\nu \pi) + \log(3) + \frac{1}{4} \sec(\nu \pi) \left( \psi \left( \frac{\nu}{2} \right) - \psi \left( \frac{\nu + 1}{2} \right) \right). \tag{15}
\]
Proof. Applying Frullani’s theorem to the series in Eq. (3) gives

\[
\frac{1}{2} \sin (\nu \pi) \left( \psi \left( \frac{\nu}{2} \right) - \psi \left( \frac{\nu + 1}{2} \right) \right) + \frac{\pi}{2} + \log (3) \sin (2\nu \pi)
\]

\[
= \sum_{k=2}^{\infty} \int_{0}^{\infty} \frac{dx}{x} \left( e^{-(2k+1)x} - e^{-(2k-1)x} \right) \sin (2k\nu \pi)
\]

\[
= - \int_{0}^{\infty} \frac{dx}{x} 2 \sinh (x) \sum_{k=2}^{\infty} e^{-2kx} \sin (2k\nu \pi).
\]  \hspace{1cm} (16)

The sum can be computed as shown previously, and the result is

\[
\sum_{k=2}^{\infty} e^{-2kx} \sin (2k\nu \pi) = \frac{1}{2} e^{-2x} \sin (2\nu \pi) \left( \frac{2 \cos (2\nu \pi) - e^{-2x}}{\cosh (2x) - \cos (2\nu \pi)} \right).
\]

Inserting this identity into Eq. (16) and then simplifying leads to the result in Eq. (15).

These two integration results are non-trivial since neither Maple nor Mathematica appear to be able to calculate the closed-form expressions. Nevertheless, numerical evaluation of these integrals for arbitrary values of \( \nu \in (0,1) \) agrees with the analytical results.

3.2 Particular results for the cases \( \nu = 0 \) and \( \nu = 1 \)

As discussed in Ref. \[4\], the validity of the result in Eq. (1) can be extended from \( 0 < \nu < 1 \) to \( 0 \leq \nu \leq 1 \). For \( \nu = 0 \) and \( \nu = 1 \), this leads to the result

\[
\sum_{k=2}^{\infty} \log \left( 1 - \frac{1}{k^2} \right) = - \log (2).
\]  \hspace{1cm} (17)

For the series in Eq. (4), the logarithm tends to \(-3/(4k^2)\), as \( k \to \infty \), and thus the series is absolutely convergent. Hence, the validity of the result in Eq. (4) can be extended from \( 0 < \nu < 1 \) to \( 0 \leq \nu \leq 1 \). Setting \( \nu = 0 \) (or \( \nu = 1 \)) then leads to the summation identity

\[
\sum_{k=2}^{\infty} \log \left( \frac{4(k^2 - 1)}{4k^2 - 1} \right) = \log \left( \frac{3\pi}{16} \right).
\]  \hspace{1cm} (18)

The proof of this results is as follows.
Proof. Consider the $\nu = 0$ limit of Eq. (1):
\[
\sum_{k=2}^{\infty} \log \left( \frac{4 \left( k^2 - 1 \right)}{4k^2 - 1} \right)
= \lim_{\nu \to 0} \left[ \frac{1}{2} \cos (\nu \pi) \left( \psi \left( \frac{\nu + 1}{2} \right) - \psi \left( \frac{\nu}{2} \right) - 4 \log (2) \cos (\nu \pi) \right) - \frac{\pi}{2} \sin (2\nu \pi)
+ \cos (2\nu \pi) \left( \log \left( \frac{3\pi}{2} \right) - \psi (1) + \psi (\nu) \right) \right]
= \frac{1}{2} \left( \psi \left( \frac{1}{2} \right) - \psi (1) - 4 \log (2) \right) + \log \left( \frac{3\pi}{2} \right)
= \log \left( \frac{3\pi}{16} \right).
\]

In the last step we have used [7]: $\psi (1) = -\gamma$ and $\psi \left( \frac{1}{2} \right) = -\gamma - 2 \log (2)$, where $\gamma$ is the Euler-Mascheroni constant. The same result is obtained if the limit $\nu \to 1$ is taken in Eq. (4). As another example, setting $\nu = 0$ (the same result is obtained if $\nu = 1$) in Eq. (12) gives
\[
\sum_{k=2}^{\infty} \log \left( \frac{(4k^2 - 1)^2}{16k^2(k^2 - 1)} \right) = \log \left( \frac{128}{9\pi^2} \right).
\]

Proof. Consider the $\nu = 0$ limit of Eq. (12):
\[
\sum_{k=2}^{\infty} \log \left( \frac{(4k^2 - 1)^2}{16k^2(k^2 - 1)} \right)
= -\lim_{\nu \to 0} \left[ \cos (\nu \pi) \left( \psi \left( \frac{\nu + 1}{2} \right) - \psi \left( \frac{\nu}{2} \right) \right) + \log \left( \frac{9}{8} \right) \cos (2\nu \pi)
+ 2 \cos^2 (\nu \pi) \left( \log \left( \frac{\pi}{2} \right) - \psi (1) + \psi (\nu) \right) - \frac{\pi}{2} \sin (2\nu \pi) \right]
= \psi (1) - \psi \left( \frac{1}{2} \right) - \log \left( \frac{9}{8} \right) - 2 \log \left( \frac{\pi}{2} \right)
= \log \left( \frac{128}{9\pi^2} \right).
\]

3.3 Infinite products

The infinite series results in Eqs. (17)-(19) are particular cases of a more general result [10] involving infinite products. Indeed, as an example, if we let
\( k = m + 2 \), then the result in Eq. (19) can be expressed as the following infinite product

\[
\prod_{m=0}^{\infty} \left[ \frac{(m + \frac{5}{2})(m + \frac{3}{2})}{(m + 3)(m + 2)(m + 1)} \right] = \frac{128}{9\pi^2}. \tag{20}
\]

According to theorem 1.1 of Ref. [10], the infinite product above evaluates to \( \Gamma(3) \Gamma(2) / \Gamma(5/2)\Gamma(3/2) \Gamma(1) = 2^7/(9\pi^2) \), in agreement with Eq. (20). The results in Eqs. (17)-(20) can also be confirmed by using Maple or Mathematica.

Since the exponential mapping \( x \mapsto \exp(x) \) is a continuous function and a group isomorphism from \((\mathbb{R},+)\) to \((\mathbb{R}_{>0},\times)\) all of the results in Theorem 1 can be equivalently expressed as infinite products, by taking the exponential of both sides of the equations. As one non-trivial example, Wallis’ formula [7] for \( \pi \) can be derived. The proof is as follows. Set \( \nu = \frac{1}{4} \) in Eq. (2) to obtain

\[
\sum_{k=2}^{\infty} \log \left( \frac{k-1}{k+1} \right) \sin \left( \frac{k\pi}{2} \right) = \sum_{i=1}^{\infty} \frac{(-1)^i \log \left( \frac{l}{l+1} \right)}{i+1} = \log (4\pi) - \psi (1) + \psi \left( \frac{1}{4} \right) + \frac{\pi}{2} = \log \left( \frac{\pi}{2} \right). 
\]

In the last step we have used [7]: \( \psi \left( \frac{1}{4} \right) = -\frac{\pi}{2} - 3 \log (2) + \psi (1) \). Taking the exponential of both sides of the equation above then gives Wallis’ formula:

\[
\prod_{l=1}^{\infty} \left[ \frac{\left( \frac{l}{l+1} \right)^{(-1)^l}}{\left( \frac{l}{l+1} \right)^{(-1)^l}} \right] = \prod_{k=1}^{\infty} \left( \frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \right) = \frac{\pi}{2}.
\]

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**Compliance with ethical standards**

**Conflict of interest**: The author declares that he has no conflict of interest.  
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