PARABOLIC LITTLEWOOD-PALEY INEQUALITY FOR A CLASS OF TIME-DEPENDENT OPERATORS OF ARBITRARY ORDER, AND APPLICATIONS TO HIGHER ORDER STOCHASTIC PDE

ILDOO KIM, KYEONG-HUN KIM, AND SUNGBIN LIM

Abstract. In this paper we prove a parabolic version of the Littlewood-Paley inequality for a class of time-dependent local and non-local operators of arbitrary order, and as an application we show this inequality gives a fundamental estimate for the $L^p$-theory of the stochastic partial differential equations.

1. Introduction

The classical Littlewood-Paley inequality says (see [12]) that for any $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^d)$,
\[
\int_{\mathbb{R}^d} \left( \int_0^\infty |\sqrt{-\Delta} e^{t\Delta} f|^2 dt \right)^{p/2} dx \leq N(p) \| f \|_{L^p}^p,
\]
(1.1)
where $e^{t\Delta} f(x) := S_t f = p(t, \cdot) * f(\cdot) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} f(x-y) e^{-|y|^2/4t} dy$. In [5, 7] Krylov proved the following parabolic version, in which $H$ is a Hilbert space: for any $p \in [2, \infty), -\infty \leq a < b \leq \infty, f \in L^p((a,b) \times \mathbb{R}^d, H)$,
\[
\left\| \left( \int_a^b |(\sqrt{-\Delta} e^{(t-s)\Delta} f)(s,x)|^2_H ds \right)^{1/2} \right\|^p_{L^p((a,b) \times \mathbb{R}^d)} \leq N(p) \| f \|_{L^p((a,b) \times \mathbb{R}^d)}^p.
\]
(1.2)
Some related works and the significance of the parabolic Littlewood-Paley inequality in the $L^p$-theory of stochastic PDEs will be discussed later.

If $f = f(x)$ and $H = \mathbb{R}$ then by (1.2) with $a = 0$ and $b = 2$,
\[
\int_{\mathbb{R}^d} \int_0^1 |\sqrt{-\Delta} e^{s\Delta} f|^2 ds |^{p/2} dx \\
\leq \int_{\mathbb{R}^d} \int_0^2 \int_0^t |\sqrt{-\Delta} e^{(t-s)\Delta} f|^2 ds |^{p/2} dt dx \leq 2N(p) \| f \|_{L^p(\mathbb{R}^d)}^p.
\]
This and the scaling $(\sqrt{-\Delta} S_t f(c\cdot))(x) = \sqrt{-\Delta}(cS_{ct} f)(cx)$ yield (1.1). Hence (1.2) is a generalization of (1.1). Note that by putting $K_0(t,x) = \sqrt{-\Delta} p(t,x)$, we get

2010 Mathematics Subject Classification. 42B25, 26D10, 60H15, 35G05, 47G30.

Key words and phrases. Parabolic Littlewood-Paley inequality, Stochastic partial differential equations, Time-dependent high order operators, Non-local operators of arbitrary order.

The research of the second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (20110015961).
where \( \Delta \) is the Laplacian operator. We provide a classification of operators \( K \) and \( a \) so that the solution of (1.3) holds for any \( \Re \). Here \( \Re \) is given by

\[
\Re := \left( \int_{\mathbb{R}^d} |K_0(t-s, \cdot) * f(s, \cdot)|^2 \, ds \right)^{1/2} \leq N ||f||_{L^p((a,b) \times \mathbb{R}^d)} \leq N \left\| f \right\|_{L^p((a,b) \times \mathbb{R}^d)},
\]

In this article we extend (1.3) to a class of time-dependent operators. For a wide class of differential operators \( A(t) \) with symbol \( \psi(t, \xi) \), one can define the kernel

\[
p(t, s, x) = p_A(t, s, x) = F^{-1}(\exp(\int_{t}^{s} \psi(r, \xi) \, dr))(x)
\]

so that the solution of

\[
u_t = A(t)u + f, \quad u(0) = 0
\]

is given by

\[
u = \int_{0}^{t} p(t, s, \cdot) * f(s, \cdot) \, ds.
\]

We provide a classification of operators \( A(t) \) for which (1.3) holds with formally

\[
K_A(t, s, x) = \sqrt{-A(t)}p(t, s, x).
\]

More generally, we provide sufficient conditions on measurable functions \( K(t, s, x) \) on \( \mathbb{R}^{d+2} \) so that

\[
\left\| \int_{s}^{t} |K(t, s, \cdot) * f(s, \cdot)(x)|^2 \, ds \right\|_{L^p((a,b) \times \mathbb{R}^d)} \leq N \left\| f \right\|_{L^p((a,b) \times \mathbb{R}^d)} \leq N \left\| f \right\|_{L^p((a,b) \times \mathbb{R}^d)}
\]

holds for any \( f \in C_0^\infty(\mathbb{R}^{d+1}, H) \) with constant \( N \) independent of \( f, a \) and \( b \). The functions \( K(t, s, x) \) are assumed to satisfy the conditions described in Assumptions 2.1 and 2.2.

For concrete examples we introduce the operators \( A_1(t) \) of \( 2m \)-order \((m = 1, 2, 3, \cdots) \) and \( A_2(t) \) of order \( \gamma \in (0, \infty) \)

\[
A_1(t)u := (-1)^{m-1} \sum_{|\alpha| = |\beta| = m} a^{\alpha\beta}(t) D^{\alpha+\beta} u, \quad A_2(t)u := -a(t)(-\Delta)^{\gamma/2}
\]

where \(-a(t)(-\Delta)^{\gamma/2}\) is the operator with symbol \(-a(t)|\xi|^\gamma\) and the coefficients \( a(t) \) and \( a^{\alpha\beta}(t) \) are bounded measurable in \( t \) and satisfy the ellipticity conditions

\[
0 < \nu < \Re[a(t)] < \nu^{-1},
\]

and

\[
\nu|\xi|^{2m} \leq \sum_{|\alpha| = |\beta| = m} \xi^n \xi^\beta \Re \left[ a^{\alpha\beta}(t) \right] \leq \nu^{-1}|\xi|^{2m}.
\]

Here \( \Re[z] \) is the real part of \( z \). Let \( p_1(t, s, x) \) and \( p_2(t, s, x) \) be the kernels related to \( A_1(t) \) and \( A_2(t) \) respectively. We prove that (1.4) holds with

\[
K_1(t, s, x) := D^m p_1(t, s, x), \quad K_2(t, s, x) := (-\Delta)^{\gamma/4} p_2(t, s, x).
\]

Letting the function \( f \) depend only on \( x \), one can obtain elliptic versions of these results. For instance, we have for any \( \gamma \in (0, \infty) \) and \( f \in L_p(\mathbb{R}^d) \),

\[
\int_{\mathbb{R}^d} \left( \int_{0}^{\infty} \left| (-\Delta)^{\gamma/2} e^{-t(-\Delta)^{\gamma/4}} f \right|^2 \, dt \right)^{p/2} \, dx \leq N(p, \gamma) ||f||_{L^p(\mathbb{R}^d)}, \quad \forall \gamma \in (0, \infty),
\]

which is an extension of (1.1), the classical (elliptic) Littlewood-Paley inequality.
Among many other examples of \([1, 2]\) are the product \(A_1(t)A_2(t)\) and \((-\Delta)^k L_0(t)\) \((k = 0, 1, 2, \ldots)\), where

\[
L_0(t)u = \int_{\mathbb{R}^d} \left( u(x + y) - u(x) - \chi(y)(\nabla u(x), y) \right) m(t, y) \frac{dy}{|y|^{d+\gamma}}, \tag{1.6}
\]

for \(\gamma \in (0, 2)\), \(\chi(y) = I_{\gamma>1} + I_{\gamma=1} I_{|y|\leq1}\), and \(m(t, y) \geq 0\) satisfies a certain condition described in Corollary 2.3. Note that if \(m(t, y) \equiv 1\) then \(L_0 = -(-\Delta)^{\gamma/2}\).

One of important applications of the parabolic Littlewood-Paley inequality is the theory of stochastic partial differential equations of the type

\[
du = A_i(\omega, t)u \, dt + \sum_{k=1}^{\infty} f^k \, dw^k_t, \quad u(0, x) = 0. \tag{1.7}
\]

Here \(f = (f^1, f^2, \ldots)\) is an \(\ell_2\)-valued random function depending on \((t, x)\), and \(w^k_t\) are independent one-dimensional Wiener processes defined on a probability space \((\Omega, P)\). The operators \(A_i = A_i(\omega, t)\) are defined in \([1, 2]\), but this time we allow the coefficients \(a(\omega, t)\) and \(a^{\alpha\beta}(\omega, t)\) to depend also on \(\omega \in \Omega\). It turns out that if \(f = (f^1, f^2, \ldots) \in L_p(\Omega \times (0, \infty) \times \mathbb{R}^d, \ell_2)\) satisfies a certain measurability condition, the solutions of these problems are given by

\[
u_i(t, x) = \sum_{k=1}^{\infty} \int_0^t p_i(t, s, \cdot) * f^k(s, \cdot)(x) \, dw^k_s, \quad i = 1, 2 \tag{1.8}
\]

where \(p_i(t, s, x)\) are introduced above, but they are random due to the randomness of the coefficients. The derivation of formula \([1.8]\) can be found in \([3]\) when \(A_i = \Delta\), and by repeating the arguments in \([3]\) one can derive \([1.8]\) for such \(A_i\). By Burkholder-Davis-Gundy inequality (see \([6]\)), we have

\[
\|D^m u_1(t, \cdot)\|_{L_p(\Omega \times (0, \infty) \times \mathbb{R}^d)}^p \leq N(p) \left\| \int_0^t |D^m p_1(t, s, \cdot) * f(s, \cdot)(x)|_{\ell_2}^2 \, ds \right\|_{L_p(\Omega \times (0, \infty) \times \mathbb{R}^d)}^{1/2}. \tag{1.9}
\]

The corresponding inequality for \(u_2\) also holds with \(p_2\) and \((-\Delta)^{\gamma/4}\) in place of \(p_1\) and \(D^m\) respectively. Actually if \(f\) is not random, then \(u_1\) and \(u_2\) become Gaussian processes and the reverse inequalities also hold. Thus to prove

\[
D^m u_1, \quad (-\Delta)^{\gamma/4} u_2 \in L_p(\Omega \times (0, \infty) \times \mathbb{R}^d)
\]

and to get a legitimate start of the \(L_p\)-theory of stochastic PDEs of type \([1.7]\), one has to estimate the right-hand side of \([1.9]\). Obviously \([1.4]\) with \(K_1\) and \([1.5]\) imply

\[
\|D^m u_1(t, \cdot)\|_{L_p(\Omega \times (0, \infty) \times \mathbb{R}^d)}^p \leq N(p, m) \|f\|_{L_p(\Omega \times (0, \infty) \times \mathbb{R}^d)}^P. \tag{1.10}
\]

Using \([1.10]\) and following the ideas in \([3]\), one can construct an \(L_p\)-theory of the general \(2m\)-order stochastic PDEs. Similarly one can construct an \(L_p\)-theory of stochastic PDEs with the operator \(A_2(\omega, t)\). We acknowledge that if the coefficients \(a^{\alpha\beta}\) are independent of \(t\) then inequality \([1.10]\) for high order stochastic PDEs is also introduced in \([11]\) on the basis of \(H^\infty\)-functional calculus which is far different from our approach. One of advantages of our approach is that no regularity condition of the coefficients with respect to time variable is required.
Below is a short description on the related works. As mentioned above parabolic Littlewood-Paley inequality related to the Laplacian $\Delta$ was first proved by Krylov in \cite{7}. This result is considered as a foundation of the $L_p$-theory of the second-order stochastic partial differential equations. Recently the parabolic Littlewood-Paley inequality was proved for the fractional Laplacian $(-\Delta)^{\gamma/2}$, $\gamma \in (0, 2)$, in \cite{1,2}, and a slight extension of the result of \cite{1,2} was made to the operator $L_0(t)$ in \cite{10} and to the operator with symbol $-\phi(|\xi|^2)$ in \cite{4}, where $L_0(t)$ is from \cite{10} and $\phi$ is a Bernstein function satisfying
\[
c^{-1}\lambda^a \phi(t) \leq \phi(\lambda t) \leq c\lambda^a \phi(t), \quad \forall \lambda, t \geq 1,
\]
with some constants $c > 1$, $0 < \delta_1 \leq \delta_2 < 1$ and $\delta_3 \in (0, 1]$. The operators considered in \cite{1,2,4,10} are of order less than 2, they (except $L_0(t)$) do not depend on $t$. The novelty of this article is that it extends existing results which have been proved for lower order operators independent of $t$ to the time-dependent local and non-local operators of arbitrary order.

Next we briefly describe our approach to prove \cite{4}. We estimate the sharp function of $(\int_0^t |K(t, s, \cdot)| f(s, \cdot) x|ds)^{1/2}$ in terms of the maximal function of $|f|_H$, then apply Fefferman-Stein theorem and Hardy-Littlewood maximal theorem. The operators considered in \cite{1,2,4} are the infinitesimal generators of certain Lévy processes, and the related kernels $p(t, x)$ are transition densities of these processes. Thus to estimate the sharp function of $(\int_0^t |K(t, s, \cdot)| f(s, \cdot) x|ds)^{1/2}$, appropriate bounds of the transition densities can be used as in \cite{1,2,4}. But for high order operators there is no such related Lévy process and this method cannot be applied. Instead, we modify the idea in \cite{10} and make a good use of Parseval’s identity which enables us to avoid using estimates of the kernels related to the operators.

Finally we introduce some notation used in the article. As usual $\mathbb{R}^d$ stands for the Euclidean space of points $x = (x^1, \ldots, x^d)$, $B_r(x) := \{y \in \mathbb{R}^d : |x - y| < r\}$ and $B_r := B_r(0)$. For multi-indices $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \in \{0, 1, 2, \ldots\}$, $x \in \mathbb{R}^d$, and functions $u(x)$ we set
\[
 u_{x_i} = \frac{\partial u}{\partial x^i} = D_i u, \quad D^\alpha u = D_1^{\alpha_1} \cdots D_d^{\alpha_d} u,
\]
\[
x^\alpha = (x^1)^{\alpha_1} (x^2)^{\alpha_2} \cdots (x^d)^{\alpha_d}, \quad |\alpha| = \alpha_1 + \cdots + \alpha_d.
\]
We also use $D_x^n$ to denote a partial derivative of order $m$ with respect to $x$. For an open set $U \subset \mathbb{R}^d$ and a nonnegative integer $n$, we write $u \in C^n(U)$ if $u$ is $n$-times continuously differentiable in $U$. By $C^n(U)$ (resp. $C_0^n(U)$) we denote the set of all functions in $C^n(U)$ (resp. $C_0^n(U)$) with compact supports. The standard $L_p$-space on $U$ with Lebesgue measure is denoted by $L_p(U)$. Similarly, by $C_0^\infty(\mathbb{R}^d, H)$ we denote the set of $H$-valued infinitely differentiable functions with compact support. We use “ := ” to denote a definition. $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$ and $\lfloor a \rfloor$ is the biggest integer which is less than or equal to $a$. By $\mathcal{F}$ and $\mathcal{F}^{-1}$ we denote the Fourier transform and the inverse Fourier transform, respectively. That is, $\mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} e^{-ix \xi} f(x) dx$ and $\mathcal{F}^{-1}(f)(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \xi} f(\xi) d\xi$. For a Borel set $X \subset \mathbb{R}^d$, we use $|X|$ to denote its Lebesgue measure and by $I_X(x)$ we denote the indicator of $A$. For a sequence $a = (a_1, a_2, a_3, \ldots)$, we define $|a|_{L^2} = (\sum_{k=1}^{\infty} a_k^2)^{1/2}$. 
If we write $N = N(a, \ldots, z)$, this means that the constant $N$ depends only on $a, \ldots, z$.

2. Main results

In this section we prove (1.4), a generalized version of the parabolic Littlewood-Paley inequality, under the following conditions on the kernel $K(t, s, x)$ and provide a classification of operators $A(t)$ for which (1.4) holds with $K = K_A$ (see (1)). Three interesting examples related to the operators $A_1(t), A_2(t)$ and $(-\Delta)^k \mathcal{L}_0(t)$ are also presented.

Assumption 2.1 below is needed to prove (1.4) for $p = 2$.

**Assumption 2.1.** The kernel $K(t, s, x)$ is a measurable function defined on $\mathbb{R}^{d+2}$ satisfying

$$\int_s^\infty |\mathcal{F}\{K(t, s, \cdot)\}(\xi)|^2 dt \leq C_0$$

with constant $C_0$ independent of $(s, \xi)$.

Take a constant $c_2 > \frac{1}{2}$ and denote

$$c_3 := \frac{2(d+1)(c_2 + 1) + 3}{2(d+2)}.$$  

**Assumption 2.2.** (i) For almost all $t$ and each $s < t$, $K(t, s, \cdot)$, $\frac{\partial}{\partial t} K(t, s, x)$ and $\frac{\partial}{\partial x} K(t, s, x)$ are locally integrable functions of $x$.

(ii) There exist functions $F_i(t, s, x)$ and positive constants $\sigma_i, \kappa_i$ ($i = 1, 2, 3$) and $C$ such that for almost all $t$ and each $s < t$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$|D_x K(t, s, x)| \leq C \left| (t-s)^{-\sigma_1} F_1(t, s, (t-s)^{-\kappa_1} x) \right|,$$

$$|D_x^2 K(t, s, x)| \leq C \left( (t-s)^{-\sigma_2} \left| F_2(t, s, (t-s)^{-\kappa_2} x) \right| \wedge (t-s)^{-c_2} \right),$$

$$\left| \frac{\partial}{\partial t} D_x K(t, s, x) \right| \leq C \left( (t-s)^{-\sigma_3} \left| F_3(t, s, (t-s)^{-\kappa_3} x) \right| \wedge (t-s)^{-c_3} \right).$$

(iii) For these $F_i$ ($i = 1, 2, 3$), we have

$$\sup_{s < t} \int_{\mathbb{R}^d} |x|^{\mu_1} |F_1(t, s, x)|^2 dx < \infty,$$

$$\sup_{s < t} \int_{|x| \geq (t-s)^{c_2 + 1 - \kappa_2}} |x|^{\mu_2} |F_2(t, s, x)|^2 dx < \infty,$$

$$\sup_{s < t} \int_{|x| \geq (t-s)^{c_3 + 1 - \kappa_3}} |x|^{\mu_3} |F_3(t, s, x)|^2 dx < \infty,$$

where $\mu_i > d + 2$ ($i = 1, 2, 3$) satisfy the following system.
\[
\begin{align*}
(c_2 - c_3 + 1 - \kappa_1)\mu_1 &= d(\kappa_1 + c_2 - c_3 + 1) + 2(c_2 - \sigma_1) + 3 \\
(c_2 - c_3 + 1 - \kappa_2)\mu_2 &= d(\kappa_2 - c_2 + c_3 - 1) + 2(c_2 - \sigma_2) \\
(c_2 - c_3 + 1 - \kappa_3)\mu_3 &= d(\kappa_3 - c_2 + c_3 - 1) + 2(c_2 - \sigma_3)
\end{align*}
\] (2.9)

Remark 2.3. (i) Suppose
\[
\sup_{s \leq t} \int_{\mathbb{R}^4} |F_i(t, s, x)|^2 \, dx < \infty, \quad (i = 1, 2, 3)
\]
and
\[
\sup_{s \leq t} \int_{\mathbb{R}^4} |x|^\hat{\mu}_i |F_i(t, s, x)|^2 \, dx < \infty \quad (i = 1, 2, 3)
\]
with some \( (\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3) \in \mathbb{R}^3 \). Then obviously (2.6)-(2.8) hold for any \( \mu_i \leq \hat{\mu}_i \) \( (i = 1, 2, 3) \).

(ii) Suppose, for example, \( c_2 - c_3 + 1 - \kappa_2 = 0 \). Then in (2.9) we are assuming
\[
d(\kappa_2 - c_2 + c_3 - 1) + 2(c_2 - \sigma_2) = 0.
\]
In this case, we have a freedom of choosing \( \mu_2 \), that is we can choose arbitrary \( \mu_2 > d + 2 \) satisfying (2.10).

(iii) Put
\[
\delta_0 := c_2 - c_3 + 1, \quad \Theta(\theta, \vartheta) := \theta d - 2\vartheta.
\]
(2.10)

One can easily check
\[
\Theta(\theta_1 + \theta_2, \vartheta_1 + \vartheta_2) = \Theta(\theta_1, \vartheta_1) + \Theta(\theta_2, \vartheta_2),
\]
and (2.10) is equivalent to
\[
\begin{pmatrix}
\delta_0 - \kappa_1 \\
\delta_0 - \kappa_2 \\
\delta_0 - \kappa_3
\end{pmatrix}
\begin{pmatrix}
\mu_1 \\
\mu_2 \\
\mu_3
\end{pmatrix}
= \begin{pmatrix}
\Theta(\kappa_1 + \delta_0, \sigma_1 - \delta_0) + 1 \\
\Theta(\kappa_2 - \delta_0, \sigma_2 - \delta_0) \\
\Theta(\kappa_3 - \delta_0, \sigma_3 - \delta_0)
\end{pmatrix}.
\] (2.11)

Note that to prove (1.4) we may assume \( a = -\infty \) and \( b = \infty \). Recall \( H \) denote a Hilbert space. Here are the main results of this article. The proofs of Theorems 2.4 and 2.5 are given in Sections 4 and 5 respectively.

Theorem 2.4. Let \( p \geq 2 \). Suppose that Assumptions 2.4 and 2.5 hold. Then for any \( f \in C_0^\infty(\mathbb{R}^{d+1}, H) \),
\[
\left\| \left( \int_{-\infty}^t |K(t, s, \cdot) * f(s, \cdot)(x)|_H^2 \, ds \right)^{1/2} \right\|_{L_p(\mathbb{R}^{d+1})} \leq N \| f \|_H \| f \|_{L_p(\mathbb{R}^{d+1})},
\]
where \( N \) is independent of \( f \).

Let \( A(t) \) be a non-positive operator with the symbol \( \psi(t, \xi) \), that is
\[
\mathcal{F}(A(t)u)(\xi) = \psi(t, \xi)\mathcal{F}(u)(\xi), \quad \forall u \in C_0^\infty(\mathbb{R}^d).
\]
Define the kernel $p(t, s, x)$ by the formula
\[ p(t, s, x) = I_{0 \leq s < t} F^{-1} \left( \exp \left( - \int_s^t \psi(r, \xi) dr \right) \right)(x). \] (2.12)

**Theorem 2.5.** Fix $p \geq 2$ and $\gamma > 0$. Assume there exist constants $\nu > 0$ such that for any multi-index $|\alpha| \leq \frac{d}{2} + 2$,
\[ \Re[\psi(t, \xi)] \leq -\nu|\xi|^{\gamma}; \] (2.13)
\[ |D^\alpha \psi(t, \xi)| \leq \nu^{-1}|\xi|^{-|\alpha|}; \] (2.14)
hold for almost every $t > 0$ and $\xi \neq 0$. Then for any $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$
\[ \left\| \left( \int_0^t |\Delta^{\gamma/4} p(t, s, \cdot) * f(s, \cdot)(x)|_H^2 ds \right)^{1/2} \right\|_{L_p(\mathbb{R}^{d+1})} \leq N \|f\|_H \|f\|_{L_p(\mathbb{R}^{d+1})}, \] where $N$ depends only on $p, \nu, \gamma$ and $d$.

For applications of Theorem 2.5 we recall the operators $A_i(t)$ from [1,5], that is,
\[ A_1(t)u = (-1)^{m-1} \sum_{|\alpha| = |\beta| = m} a^{\alpha\beta}(t) D^\alpha \beta u, \quad A_2(t) = -a(t)(-\Delta)^{\gamma/2} \] where the coefficients $a^{\alpha\beta}$ and $a(t)$ are bounded complex-valued measurable functions satisfying $\nu < \Re[a(t)] < \nu^{-1}$ and
\[ \nu|\xi|^{2m} \leq \sum_{|\alpha| = |\beta| = m} \xi^\alpha \xi^\beta \Re[a^{\alpha\beta}(t, \cdot)] \leq \nu^{-1}|\xi|^{2m}, \quad \forall \xi \in \mathbb{R}^d. \]

Denote
\[ p_1(t, s, z) = p_{1, m}(t, s, x) = I_{0 \leq s < t} F^{-1} \left( \exp \left\{ - \int_s^t a^{\alpha\beta}(r) \xi^\alpha \xi^\beta dr \right\} \right)(x), \]
\[ p_2(t, s, x) = p_{2, \gamma}(t, s, x) = I_{0 \leq s < t} F^{-1} \left( \exp \left\{ - |\xi|^\gamma \int_s^t a(r) dr \right\} \right)(x). \]

**Corollary 2.6.** Let $p \geq 2$. Then for any $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$,
\[ \left\| \left( \int_0^t |\Delta^{m/2} p_1(t, s, \cdot) * f(s, \cdot)(x)|_H^2 ds \right)^{1/2} \right\|_{L_p(\mathbb{R}^{d+1})} \leq N \|f\|_H \|f\|_{L_p(\mathbb{R}^{d+1})}, \] where $N$ depends only on $p, \nu, m$ and $d$.

**Proof.** It is obvious that the symbol $\psi(t, \xi) = -a^{\alpha\beta}(t) \xi^\alpha \xi^\beta$ satisfies (2.13) and (2.14) with $\gamma = 2m$ and any multi-index $\alpha$. Thus the corollary follows from Theorem 2.5. \qed

**Corollary 2.7.** Let $p \geq 2$. Then for any $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$,
\[ \left\| \left( \int_0^t |\Delta^{\gamma/4} p_2(t, s, \cdot) * f(s, \cdot)(x)|_H^2 ds \right)^{1/2} \right\|_{L_p(\mathbb{R}^{d+1})} \leq N \|f\|_H \|f\|_{L_p(\mathbb{R}^{d+1})}, \] where $N$ depends only on $p, \nu, \gamma$ and $d$.

**Proof.** The symbol related to the operator $A_2(t)$ is $-a(t)|\xi|^\gamma$, and therefore the corollary follows from Theorem 2.5. \qed
Recall we defined $(-\Delta)^{\gamma/2}$ as the operator with symbol $|\xi|^\gamma$ for any $\gamma \in (0, \infty)$. For further applications of Theorem 2.5, we consider a product of $(-\Delta)^{k}$ and an integro-differential operator $L_0 = L_{0, \gamma}$. We remark that in place of $(-\Delta)^{k}$ one can consider many other pseudo-differential or high order differential operators.

Fix $\gamma \in (0, 2)$, and for $k = 0, 1, 2, \cdots$ denote

$L_k(t)u = (-\Delta)^{k}L_{0, \gamma}u :=$

\[
\int_{\mathbb{R}^{d}\setminus\{0\}} \left((-\Delta)^k u(t, x + y) - (-\Delta)^k u(t, x) - \chi(y)\nabla (-\Delta)^k u(t, x), y\right)m(t, y) \frac{dy}{|y|^{d+\gamma}}
\]

where $\chi(y) = I_{\gamma>1} + I_{|y|\leq 1}I_{\gamma=1}$ and $m(t, y) \geq 0$ is measurable function satisfying the following conditions:

(i) If $\gamma = 1$ then

\[
\int_{\partial B_1} \psi m(t, w) S_1(dw) = 0, \quad \forall t > 0,
\]

where $\partial B_1$ is the unit sphere in $\mathbb{R}^d$ and $S_1(dw)$ is the surface measure on it.

(ii) The function $m = m(t, y)$ is zero-order homogeneous and differentiable in $y$ up to $d_0 = [\frac{d}{2}] + 2$.

(iii) There is a constant $K$ such that for each $t \in \mathbb{R}$

\[
sup_{|\alpha| \leq d_0, |y|=1} |D^\alpha y m^{(\alpha)}(t, y)| \leq K.
\]

It turns out that the operator $L_k$ is a pseudo differential operator with symbol

\[
\psi(t, \xi) = -c_1 |\xi|^{2k} \int_{\partial B_1} |(w, \xi)|^\gamma \left[1 - i\varphi^{(\gamma)}(w, \xi)\right] m(t, w) S_1(dw),
\]

\[
\varphi^{(\gamma)}(w, \xi) = c_2 \frac{(w, \xi)}{|(w, \xi)|} I_{\gamma \neq 1} - \frac{2}{\pi} \frac{(w, \xi)}{|(w, \xi)|} \ln |(w, \xi)| I_{\gamma = 1},
\]

and $c_1(\gamma, d), c_2(\gamma, d)$ are certain positive constants.

(iv) There is a constant $N_0 > 0$ such that the symbol $\psi(t, \xi)$ of $L_k$ satisfies

\[
\sup_{t, |\xi|=1} \Re[\psi(t, \xi)] \leq -N_0.
\]

One can check that (2.16) holds if there exists a constant $c > 0$ so that $m(t, y) > c$ on a set $E \subset \partial B_1$ of positive $S_1(dw)$-measure.

**Corollary 2.8.** Let $p \geq 2$ and $p(t, s, x)$ be the kernel related to $L_k(t)$. Then under above conditions (i)-(iv) on $m(t, y)$ it holds that for any $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$

\[
\left\| \left( \int_0^t |\Delta^{k/2+\gamma/4} p(t, s, \cdot) * f(s, \cdot)(x)|_H^2 ds \right)^{1/2} \right\|_{L_p(\mathbb{R}^{d+1})} \leq N\|f\|_H \|f\|_{L_p(\mathbb{R}^{d+1})},
\]

where $N$ depends only on $p, \gamma, k, d, N_0$ and $K$.

**Proof.** Note that for $\xi \neq 0$

\[
\psi(t, \xi) = |\xi|^{2k+\gamma} \psi\left(t, \frac{\xi}{|\xi|}\right) = |\xi|^{2k+\gamma} \tilde{\psi}(t, \xi).
\]
The above equality is obvious if \( \gamma \neq 1 \), and if \( \gamma = 1 \) then by (2.15)
\[
\psi(t, \xi) = |\xi|^{2k+1} \psi\left(t, \frac{\xi}{|\xi|}\right) + |\xi|^{2k} \ln |\xi| \int_{\partial B_1} (w, \xi) m(t, w) S_1(dw)
\]
\[
= |\xi|^{2k+1} \psi\left(t, \frac{\xi}{|\xi|}\right).
\]
By using condition (iii) one can check (see e.g. [9, Remark 2.6]) that for any multi-index \( \alpha \), \( |\alpha| \leq d_0 \), there exists a constant \( N = N(\alpha) \) such that
\[
|D^\alpha \tilde{\psi}(t, \xi)| \leq N |\xi|^{-|\alpha|}.
\]
Thus it is obvious that the given symbol \( \psi \) satisfies (2.13) and (2.14). The corollary is proved.

3. Some preliminary estimates

For \( f \in C_0^\infty(\mathbb{R}^{d+1}, H) \), we define
\[
\mathcal{G}f(t, x) := \left( \int_{-\infty}^{t} \left| K(t, s, \cdot) * f(s, \cdot)(x) \right|^2_H ds \right)^{1/2}.
\]

Lemma 3.1. Let Assumption 2.1 hold and \( f \in C_0^\infty(\mathbb{R}^{d+1}, H) \). Then for any \(-\infty \leq a \leq b \leq \infty\),
\[
\|\mathcal{G}f\|_{L^2((a,b) \times \mathbb{R}^d)}^2 \leq N \|f\|^2_H \|f\|_{L^2((a,b) \times \mathbb{R}^d)}^2,
\]
where \( N = N(d, C_0) \).

Proof. By the continuity of \( f \), the range of \( f \) belongs to a separable subspace of \( H \). Thus by using a countable orthonormal basis of this subspace and the Fourier transform one easily finds
\[
\|\mathcal{G}f\|^2_{L^2((a,b) \times \mathbb{R}^d)}
\]
\[
= (2\pi)^d \int_{\mathbb{R}^d} \int_a^b \int_{-\infty}^t |\mathcal{F}\{K(t, s, \cdot)\}\{(\xi)\}|^2 \mathcal{F}(f)(s, \xi)_{H}^2 dtds d\xi
\]
\[
\leq (2\pi)^d \int_{\mathbb{R}^d} \int_{-\infty}^t \int_{a}^{b} \mathcal{I}_{0 \leq t-s} |\mathcal{F}\{K(t, s, \cdot)\}\{(\xi)\}|^2 dt |\mathcal{F}(f)(s, \xi)_{H}^2 dtds d\xi
\]
\[
\leq (2\pi)^d \int_{\mathbb{R}^d} \int_{-\infty}^{t} \left( \int_{s}^{\infty} |\mathcal{F}\{K(t, s, \cdot)\}\{(\xi)\}|^2 dt \right) |\mathcal{F}(f)(s, \xi)_{H}^2 dtds d\xi.
\]
From (2.1), we have
\[
\|\mathcal{G}f\|^2_{L^2((a,b) \times \mathbb{R}^d)} \leq N \int_{-\infty}^{b} |\mathcal{F}(f)(s, \xi)_{H}^2 ds d\xi.
\]
The last expression is equal to the right-hand side of (3.1), and therefore the lemma is proved. \( \square \)
Corollary 3.2. Let $r_1, r_2 > 0$. Suppose that Assumption 2.4 holds, $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$, and $f(t, x) = 0$ for $x \notin B_{3r_1}$. Then

$$\int_{-2r_2}^0 \int_{B_{r_1}} |\mathcal{G}f(s, y)|^2 dy ds \leq N(d, C_0) \int_{-\infty}^0 \int_{B_{3r_1}} |f(s, y)|^2 dy ds.$$ 

Proof. Applying Lemma 3.1 with $a = -2r_2$ and $b = 0$ and using the condition on $f$, we get

$$\int_{-2r_2}^0 \int_{B_{r_1}} |\mathcal{G}f(s, y)|^2 dy ds \leq \int_{-\infty}^0 \int_{\mathbb{R}^d} |\mathcal{G}f(s, y)|^2 dy ds$$

$$\leq N \int_{-\infty}^0 \int_{\mathbb{R}^d} |f(s, y)|^2 dy ds$$

$$= N \int_{-\infty}^0 \int_{B_{3r_1}} |f(s, y)|^2 dy ds.$$ 

Hence the corollary is proved. \qed

For $R \geq 0$ and real-valued locally integrable functions $h(x)$ on $\mathbb{R}^d$, define the maximal functions

$$\mathbb{M}_x^Rh(x) := \sup_{r > R} \frac{1}{|B_r(x)|} \int_{B_r(x)} |h(y)| dy, \quad \mathbb{M}_x^0 h(x) := \mathbb{M}_x^0 h(x).$$

Similarly, for real-valued locally integrable functions $h = h(t)$ on $\mathbb{R}$ we introduce

$$\mathbb{M}_t^R h(t) := \sup_{r > R} \frac{1}{2r} \int_r^{-r} |h(t + s)| ds, \quad \mathbb{M}_t^0 h(t) := \mathbb{M}_t^0 h(t).$$

For functions $h = h(t, x)$, set

$$\mathbb{M}_x^R h(t, x) := \mathbb{M}_x^R (h(t, \cdot))(x), \quad \mathbb{M}_t^R h(t, x) = \mathbb{M}_t^R (h(\cdot, x))(t).$$

Obviously if $R_1 \geq R_2$, then

$$\mathbb{M}_x^{R_1} h(x) \leq \mathbb{M}_x^{R_2} h(x)$$

and if $R_0 \downarrow R$, then

$$\mathbb{M}_x^{R_0} h(x) \uparrow \mathbb{M}_x^R h(x).$$

The same properties hold for $\mathbb{M}_t^R$.

Let $S_1(dw)$ denote the counting measure on $\{-1, 1\}$ if $d = 1$ and the surface measure on the unit sphere if $d \geq 2$. The following lemma is a slight modification of [10] Lemma 8.

Lemma 3.3. Let $f \in C_0(\mathbb{R}^d)$, and $v(x)$ be a locally integrable and continuously differentiable function on $\mathbb{R}^d$. Let $x, y \in \mathbb{R}^d$, $|x - y| \leq R_1$ and $f(y - z) = 0$ if $|z| \leq R_2$ with some constants $R_1, R_2 \geq 0$ Then it holds that

$$|(f * v)(y)| \leq N(M_x^{R_1} + R_2 f^2(x))^{1/2} \left( \int_{R_2}^{\infty} (R_1 + \rho)^d \left( \int_{\partial B_1} (\nabla v(\rho w), w)^2 S_1(dw) \right)^2 d\rho \right)^{1/2}$$

where $N = N(d)$. 

Proof. Since the case \( d = 1 \) is easier, we assume \( d \geq 2 \). Using the polar coordinates and Fubini’s theorem we get

\[
\int_{|z| > R_2} f(y - z)v(z) \, dz = \int_{R_2} \int_{\partial B_1} f(y - \rho w)v(\rho w)\rho^{d-1} S_1(\rho w)\, d\rho \\
= \int_{\partial B_1} \left[ \int_{R_2} \int_v(\rho w) \left( \frac{d}{d\rho} \int_{R_2}^\rho f(y - \gamma w)\gamma^{d-1} \, d\gamma \right) S_1(\rho w) \right] \, d\rho.
\]

By integration by parts and the assumption on \( v \), we get

\[
\int_{R_2} v(\rho w) \left( \frac{d}{d\rho} \int_{R_2}^\rho f(y - \gamma w)\gamma^{d-1} \, d\gamma \right) \, d\rho
= - \int_{R_2} (\nabla v(\rho w), w) \int_{R_2}^\rho f(y - \gamma w)\gamma^{d-1} \, d\gamma \, d\rho.
\]

In the above we use the fact that there exists a sequence \( \rho_n \to \infty \), which might be dependent on \( w \), so that \( v(\rho_n w) \to 0 \) as \( n \to \infty \) and that \( \int_{R_2}^\rho f(y - \gamma w)\gamma^{d-1} \, d\gamma \) is a bounded function of \( \rho \). Also note that the limits of two improper integrals exist since the first one is actually an integral over finite interval.

By the assumption \( |x - y| \leq R_1 \), for any \( \rho > R_2 \)

\[
\int_{B_\rho} f^2(y - z) \, dz = \int_{B_{\rho} \setminus B_1} f^2(z) \, dz \leq \int_{B_{R_1 + \rho} \setminus B_1} f^2(z) \, dz \\
\leq N(d)(R_1 + \rho)^{d} M_{x}^{R_1 + R_2} f^2(x).
\]

Finally using Fubini’s theorem, Hölder’s inequality, and the assumption that \( f(y - z) = 0 \) if \( |z| \leq R_2 \), we get

\[
| (f * v)(y) | \\
\leq \left| \int_{R_2}^\rho \int_{\partial B_1} (\nabla v(\rho w), w) \int_{R_2}^\rho f(y - \gamma w)\gamma^{d-1} \, d\gamma S_1(\rho w) \, d\rho \right| \\
\leq \int_{R_2}^\rho \left( \int_{\partial B_1} \int_{R_2}^\rho \left| (\nabla v(\rho w), w) \right|^2 \gamma^{d-1} \, d\gamma S_1(\rho w) \right)^{1/2} \\
\times \left( \int_{\partial B_1} \int_{R_2}^\rho f^2(y - \gamma w)\gamma^{d-1} \, d\gamma S_1(\rho w) \right)^{1/2} \, d\rho \\
\leq \int_{R_2}^\rho \rho^{d/2} \left( \int_{\partial B_1} \left| (\nabla v(\rho w), w) \right|^2 S_1(\rho w) \right)^{1/2} \left( \int_{|z| \leq \rho} f^2(y - z) \, dz \right)^{1/2} \, d\rho \\
\leq N(M_{x}^{R_1 + R_2} f^2(x))^{1/2} \int_{R_2}^\rho (R_1 + \rho)^{d} \left( \int_{\partial B_1} \left| (\nabla v(\rho w), w) \right|^2 S_1(\rho w) \right)^{1/2} \, d\rho.
\]

The lemma is proved. \( \square \)

For \( r_1, r_2 > 0 \) denote

\[
Q_{r_2, r_1} := (-2r_2, 0) \times B_{r_1}.
\]

**Lemma 3.4.** Suppose there exist constants \( \sigma, \kappa > 0 \) and \( \mu > d + 2 \) so that

\[
|D_x K(t, s, x)| \leq C \left| (t - s)^{-\sigma} F_1(t, s, (t - s)^{-\kappa} x) \right|,
\]

\[
-2\sigma + \kappa(\mu + d) > -1,
\]

\[
(3.2)
\]

\[
(3.3)
\]
and

\[ H_{1,K}(\mu) := \sup_{s < 1} \int_{|x| > r_1(s-r)^{-\kappa}} |x|^\mu |F_1(t,s,x)|^2 \, dx < \infty. \] (3.4)

Let \( f \in C_0^\infty(\mathbb{R}_d^d, H) \) with support in \((-10r_2, 10r_2) \times \mathbb{R}_d \setminus B_{2r_1}\). Then for any \( x \in B_{r_1} \) we have

\[
\int_{Q_{r_2,r_1}} |Gf(s,y)|^2 \, dsdy \leq NH_{1,K}(\mu)^{-1} r_1^{-2\sigma+\kappa(\mu+d)+1} \int_{-10r_2}^0 M_{3r_1}^2 |f_H|^2(s,x) \, ds,
\]

where \( N = N(d, \mu, \sigma, \kappa, C_0, C) \).

**Proof.** Let \( x \in B_{r_1}, (s, y) \in Q_{r_2,r_1} \) and \( r \leq s \). Then \(|x - y| \leq 2r_1\), and \(|z| \leq r_1\) implies \(|y - z| \leq 2r_1\) and \( f(r, y - z) = 0 \) due to the assumption on \( f \). Therefore,

\[
|K(s, r, \cdot) * f(r, \cdot)(y)|_H \leq \int_{|z| \geq r_1} |K(s, r, z)||f|_H(r, y - z) \, dz.
\]

Applying Lemma 3.3 with \( R_1 = 2r_1 \) and \( R_2 = r_1 \), we get

\[
|K(s, r, \cdot) * f(r, \cdot)(y)|_H^2 \leq \frac{NM_{3r_1}^2 |f_H|^2(r, x)}{r_1} \left( \int_{r_1}^{\infty} (2r_1 + \rho)^d \left( \int_{\partial B_1} \left| \nabla K(s, r, \rho \omega) \right|^2 S_1(dw) \right)^{1/2} \, d\rho \right)^2 \leq \frac{NM_{3r_1}^2 |f_H|^2(r, x)}{r_1} \left( \int_{r_1}^{\infty} \rho^d \left( \int_{\partial B_1} \left| \nabla K(s, r, \rho \omega) \right|^2 S_1(dw) \right)^{1/2} \, d\rho \right)^2. \] (3.5)

By (3.2) and the change of variable \((s-r)^{-\kappa } \rho \to \rho \), the last term is less than or equal to constant times of

\[
(s-r)^{-2\sigma + 2\kappa (d+1)} M_{3r_1}^2 |f_H|^2(r, x) \left( \int_{r_1(s-r)^{-\kappa}}^{\infty} \rho^d \left( \int_{\partial B_1} \left| F_1(s, r, \rho \omega) \right|^2 S_1(dw) \right)^{1/2} \, d\rho \right)^2.
\]

By Hölder inequality and the definition of \( H_{1,K}(\mu) \),

\[
\left( \int_{r_1(s-r)^{-\kappa}}^{\infty} \rho^d \left( \int_{\partial B_1} \left| F_1(s, r, \rho \omega) \right|^2 S_1(dw) \right)^{1/2} \, d\rho \right)^2 \leq \left( \int_{r_1(s-r)^{-\kappa}}^{\infty} \rho^{d+1-\mu} \, d\rho \right) \cdot \left( \int_{r_1(s-r)^{-\kappa}}^{\infty} \int_{\partial B_1} \rho^{\mu+d-1} \left| F_1(s, r, \rho \omega) \right|^2 S_1(dw) \, dp \right) \leq Nr_1^{d+2-\mu} (s-r)^{\kappa(\mu-d-2)} H_{1,K}(\mu).
\]
Coming back to (3.5) and remembering the definition of $G_f$, we get

$$
\int_{Q_{r_2,r_1}} |G_f(s, y)|^2 \, dsdy \\
\leq NH_1.K(\mu)r_1^{2d+2-\mu} \int_{-\infty}^0 \int_{-10r_2}^{r_2} (s-r)^{-2\sigma+\kappa(\mu+d)} M_x^{3r_1} |f|_H^2(r, x) drds \\
\leq NH_1.K(\mu)r_1^{2d+2-\mu} \int_{-\infty}^0 \int_{-10r_2}^{r_2} (s-r)^{-2\sigma+\kappa(\mu+d)} ds M_x^{3r_1} |f|_H^2(r, x) dr \\
\leq NH_1.K(\mu)r_1^{2d+2-\mu} \int_{-\infty}^0 \int_{-10r_2}^{r_2} (-r)^{-2\sigma+\kappa(\mu+d)+1} M_x^{3r_1} |f|_H^2(r, x) dr \\
\leq NH_1.K(\mu)r_1^{2d+2-\mu}(-2\sigma+\kappa(\mu+d)+1) \int_{-\infty}^0 M_x^{3r_1} |f|_H^2(r, x) dr.
$$

The lemma is proved. \qed

Recall that $\Theta(\theta, \vartheta) := \theta d - 2\vartheta$.

**Lemma 3.5. Suppose that**

$$
|D^2K(t, s, x)| \leq C \left( (t-s)^{-\sigma}|F_2(t, s, (t-s)^{-\kappa} x) \right) \wedge (t-s)^{-\epsilon} \tag{3.6}
$$

**holds with some constants $\sigma, \kappa, c > 0$ and there exists $\delta > 0$ such that**

$$
r_2^0 = r_1, \quad \Theta(2\delta, c - \delta) < -1.
$$

**Moreover assume that there exists $\mu > d + 2$ so that**

$$
\Theta(\kappa + \delta, \sigma - \delta) - (\delta - \kappa) \mu < -1, \tag{3.7}
$$

**and**

$$
H_{2,K}(\mu) := \sup_{r \leq s} \int_{|x| \geq (s-r)^{\delta-\kappa}} |x|^{\mu}|F_2(s, r, x)|^2 dx < \infty.
$$

Let $f \in C^2_0(\mathbb{R}^{d+1}, H)$, and $f(t, x) = 0$ for $t \geq -8r_2$. Then for any $(t, x) \in Q_{r_2, r_1}$ we have

$$
\sup_{Q_{r_2, r_1}} |\nabla Gf|^2 \\
\leq N \left( H_{2,K}(\mu)r_2^{\Theta(\kappa + \delta, \sigma - \delta) - (\delta - \kappa) \mu + 1} \wedge r_2^{\Theta(2\delta, c - \delta) + 1} \right) M_x^{6r_2} M_x^{2r_1} |f|_H^2(t, x),
$$

**where** $N = N(d, \mu, \delta, c, \sigma, \kappa, C_0, C)$.

**Proof.** Let $(t, x), (s, y) \in Q_{r_2, r_1}$ and $r \leq s$. By Minkowski’s inequality

$$
\left| \frac{\|f(s+h, \cdot)\| - \|f(s, \cdot)\|}{h} \right| \leq \frac{\|f(s+h, \cdot) - f(s, \cdot)\|}{|h|},
$$

the derivative of a norm is less than or equal to the norm of the derivative if both exist. Thus,

$$
\left| \frac{\partial}{\partial x_i} Gf(s, y) \right| = \left| \frac{\partial}{\partial x_i} \left( \int_{-\infty}^{s} |K(s, r, \cdot) * f(r, \cdot)(y)|^2_H dr \right)^{1/2} \right| \\
\leq \left( \int_{-\infty}^{s} \left| \frac{\partial}{\partial x_i} K(s, r, \cdot) * f(r, \cdot)(y) \right|^2_H dr \right)^{1/2}.
$$
Applying Lemma 3.3 with $R_1 = 2r_1$ and $R_2 = 0$ we get
\[
\left| \frac{\partial}{\partial x_1} K(s, r, \cdot) * f(r, \cdot)(y) \right|_H^2 \leq N M_x^{2r_1} |f|_H^2(r, x)(I_1^2 + I_2^2)
\]

where
\[
I_1 = \int_{(s-r)^{\delta}}^{\infty} (2r_1 + \rho)^d \left( \int_{\partial B_1} \left| D_z^2 K(s, r, \rho w) \right|^2 S_1(dw) \right)^{1/2} d\rho,
\]
\[
I_2 = \int_{0}^{(s-r)^{\delta}} (2r_1 + \rho)^d \left( \int_{\partial B_1} \left| D_z^2 K(s, r, \rho w) \right|^2 S_1(dw) \right)^{1/2} d\rho.
\]

Thus,
\[
\left| \frac{\partial}{\partial x_1} G(s, y) \right|^2 \leq N \int_{-\infty}^{s} M_x^{2r_1} |f|_H^2(r, x)(I_1^2 + I_2^2) \, dr.
\]

Since $f(r, x) = 0$ if $r \geq -8r_2$, we may assume $r < -8r_2$. So
\[
|s - r|^{\delta} \geq 6r_2^2 = 6r_1. \tag{3.8}
\]

First, we estimate $I_1$. Due to (3.8) and (3.6),
\[
I_1 = \int_{(s-r)^{\delta}}^{\infty} (2r_1 + \rho)^d \left( \int_{\partial B_1} \left| D_z^2 K(s, r, \rho w) \right|^2 S_1(dw) \right)^{1/2} d\rho
\]
\[
\leq N \int_{(s-r)^{\delta}}^{\infty} \rho^d \left( \int_{\partial B_1} \left| D_z^2 K(s, r, \rho w) \right|^2 S_1(dw) \right)^{1/2} d\rho
\]
\[
\leq N (s-r)^{-\sigma} \int_{(s-r)^{\delta}}^{\infty} \rho^d \left( \int_{\partial B_1} |F_2(s, r, (s-r)^{-\kappa} \rho w)|^2 S_1(dw) \right)^{1/2} d\rho.
\]

By the change of variable $(s-r)^{-\kappa} \rho \rightarrow \rho$, the last therm is less than or equal to
\[
N (s-r)^{-\sigma+\kappa(d+1)} \int_{(s-r)^{\delta-\kappa}}^{\infty} \rho^d \left( \int_{\partial B_1} |F_2(s, r, \rho w)|^2 S_1(dw) \right)^{1/2} d\rho
\]
\[
\leq N (s-r)^{-\sigma+\kappa(d+1)} \left[ \int_{(s-r)^{\delta-\kappa}} \rho^{d-\mu+1} d\rho \right]^{1/2} \left[ \int_{|z| \geq (s-r)^{\delta-\kappa}} |z|^{\mu} |F_2(s, r, z)|^2 dz \right]^{1/2}
\]
\[
\leq N (s-r)^{\theta_2(d, \delta-\kappa-\mu)} \left[ \int_{|z| \geq (s-r)^{\delta-\kappa}} |z|^{\mu} |F_2(s, r, z)|^2 dz \right]^{1/2}
\]
\[
\leq NH_2^{1/2}(\mu)(s-r)^{\theta_2(d, \delta-\kappa-\mu)}.
\]
Note $|s - r| \geq r/2$ for $r \leq -8r_2$. Thus, by the integration by parts and the assumption on $f$,

$$\int_{-\infty}^{s} M_{x}^{2r_1} |f|_{H}^{2}(r, x) \mathcal{I}_1^2 \, dr$$

$$\leq NH_{2, K}(\mu) \int_{-\infty}^{-8r_2} (s - r)^{\Theta(\kappa, \delta, \rho, 1 - (\delta - \kappa))} M_{x}^{2r_1} |f|_{H}^{2}(r, x) \, dr$$

$$\leq NH_{2, K}(\mu) \int_{-\infty}^{-8r_2} |r|^{\Theta(\kappa, \delta, \rho, 1 - (\delta - \kappa))} \left[ \int_{r}^{0} M_{x}^{2r_1} |f|_{H}^{2}(\bar{s}, x) \, d\bar{s} \right] \, dr$$

$$\leq NH_{2, K}(\mu) r_2^{\Theta(\kappa, \delta, \rho, 1 - (\delta - \kappa))} M_{x}^{2r_1} |f|_{H}^{2}(t, x).$$

Next we estimate $\mathcal{I}_2$. Using $\text{(3.6)}$ and $\text{(3.8)}$,

$$\mathcal{I}_2 \leq \int_{0}^{(s - r)^{\delta}} (2r_1 + \rho)^{d} \left[ \int_{\partial B_1} \left| D_{x}^{2} K(s, r, \rho w) \right|^{2} S_{1}(dw) \right]^{1/2} \, dp$$

$$\leq N(s - r)^{-c} \int_{0}^{(s - r)^{\delta}} (2r_1 + \rho)^{d} \, dp \leq N(s - r)^{\Theta(\delta, c - \delta)}.$$ 

Applying the integration by parts again, we obtain

$$\int_{-\infty}^{s} M_{x}^{2r_1} |f|_{H}^{2}(r, x) \mathcal{I}_2^2 \, dr$$

$$\leq N \int_{-\infty}^{-8r_2} (s - r)^{\Theta(\delta, c - \delta)} M_{x}^{2r_1} |f|_{H}^{2}(r, x) \, dr$$

$$\leq N \int_{-\infty}^{-8r_2} |r|^{\Theta(\delta, c - \delta) - 1} \left[ \int_{r}^{0} M_{x}^{2r_1} |f|_{H}^{2}(\bar{s}, x) \, d\bar{s} \right] \, dr$$

$$\leq N M_{x}^{6r_2} M_{x}^{2r_1} |f|_{H}^{2}(t, x) \int_{-\infty}^{-8r_2} |r|^{\Theta(\delta, c - \delta)} \, dr$$

$$\leq N r_2^{\Theta(\delta, c - \delta) + 1} M_{x}^{6r_2} M_{x}^{2r_1} |f|_{H}^{2}(t, x).$$

Finally, we get

$$\left| \frac{\partial}{\partial x_i} G(f(s, y)) \right|^2$$

$$\leq N \left( H_{2, K}(\mu)r_2^{\Theta(\kappa, \delta, \rho, 1 - (\delta - \kappa))} \wedge r_2^{\Theta(\delta, c - \delta) + 1} \right) M_{x}^{6r_2} M_{x}^{2r_1} |f|_{H}^{2}(t, x).$$

The lemma is proved. \hfill \Box

**Lemma 3.6.** Suppose that

$$\left| \frac{\partial^2}{\partial x \partial t} K(t, s, x) \right| \leq C(t - s)^{-\sigma} \left| F_3(t, s, (t - s)^{-\kappa} x) \right| \wedge (t - s)^{-c}$$

holds with some constants $\sigma, \kappa, c > 0$ and there exists a constant $\delta > 0$ such that $r_2^\delta = r_1$, $\Theta(2\delta, c - \delta) < -1$. 


Moreover assume that there exists $\mu > d + 2$ so that

$$\Theta(\kappa + \delta, \sigma - \delta) - (\delta - \kappa)\mu < -1,$$

and

$$H_{3,K}(\mu) := \sup_{r \leq s} \int_{J\left| x \right| \geq (s-r)^{d-\kappa}} |x|^\mu |F_3(s,r,x)|^2 dx < \infty.$$  

Let $f \in C^\infty_0(\mathbb{R}^{d+1}, H)$ and $f(t,x) = 0$ for $t \geq -8r_2$. Then for any $(t,x) \in Q_{r_2,r_1}$ we have

$$\sup_{Q_{r_2,r_1}} |D_t G f|^2 \leq N \left( H_{3,K}(\mu) r_2^{\Theta(\kappa + \delta, \sigma - \delta) - (\delta - \kappa)\mu + 1} \wedge r_2^{\Theta(2\delta, c - \delta) + 1} \right) M_2^6 M_2^{r_2} |f|_H^2(t,x),$$

where $N = N(d, \mu, \delta, c, \sigma, \kappa, C_0, C)$. 

Proof. The proof of this lemma is quite similar to the previous one. Note that by Minkowski’s inequality

$$|D_s G f(s,y)| = |D_s \left[ \int_{-\infty}^{-8r_2} |K(s,r,\cdot) * f(r,\cdot)(y)|^2_H dr \right]|^{1/2} \leq \left[ \int_{-\infty}^{-8r_2} |D_s K(s,r,\cdot) * f(r,\cdot)(y)|^2_H dr \right]^{1/2}.$$ 

The other parts are easily obtained by following the proof of the previous lemma. \[\square\]

4. PROOF OF THEOREM 2.4

First, observe that from (2.2) and (2.9) we have

$$-2\sigma_1 + \kappa_1(\mu_1 + d) > -1. \quad (4.1)$$

Indeed,

$$-2\sigma_1 + \kappa_1(\mu_1 + d) = \mu_1(c_2 - c_3 + 1) - d(c_2 - c_3 + 1) - 2(c_2 - c_3) - 3 = (c_2 - c_3 + 1)(\mu_1 - d - 2) - 1 > -1,$$

since $c_2 - c_3 + 1 = \frac{2c_2 - 1}{2(d+2)} > 0$ and $\mu_1 > d + 2$.

Also, we can derive the following relation from (2.2) (note that $c_2 > \frac{1}{2}$)

$$\Theta(2\delta_0, c_2 - \delta_0) = -2\delta_0 - 1 = \frac{1 - 2c_2}{d + 2} - 1 < -1 \quad (4.2)$$

and

$$\Theta(2\delta_0, c_3 - \delta_0) = \Theta(2\delta_0, c_2 - \delta_0) + 2(c_2 - c_3) = -3. \quad (4.3)$$

Take $\delta_0$ from (2.11). If Assumption 2.2 holds, then $\delta_0 > 0$ due to (4.2). For $R > 0$ set

$$Q_R = (-2R, 0) \times B_{R^2}.$$ 

By $\int_{Q_R} f \, ds dy$ we denote the mean average of $f$ on $Q_R$, i.e.

$$\int_{Q_R} f \, ds dy := \frac{1}{|Q_R|} \int_{Q_R} f(s,y) \, ds dy.$$
Recall
\[ \mathcal{G} f(t, x) := \left( \int_{-\infty}^{t} |K(t, s, \cdot) * f(s, \cdot)(x)|^2_H ds \right)^{1/2}. \]

To continue the proof we need the following lemma.

**Lemma 4.1.** Suppose that Assumption 2.1 and 2.2 hold. Then for any \((t, x) \in Q_R\)
\[ \frac{1}{|Q_R|^2} \int_{Q_R} \int_{Q_R} |\mathcal{G} f(s, y) - \mathcal{G} f(r, z)|^2 \, dsdydrdz \leq N \mathcal{M}_t \mathcal{M}_x |f|^2_H(t, x), \]
where the constant \(N\) is independent of \(f, R,\) and \(t, x). \]

**Proof.** Let \((t, x) \in Q_R\). We take a function \(\zeta \in C_0^\infty(\mathbb{R})\) such that \(0 \leq \zeta \leq 1, \zeta = 1\) on \([-8R, 8R]\), and \(\zeta = 0\) outside of \([-10R, 10R]\). Define
\[ A(s, y) := f(s, y)\zeta(s), \quad B(s, y) := f(s, y) - A(s, y) = f(s, y)(1 - \zeta(s)). \]

Then
\[ K(t, s, \cdot) * A(s, \cdot) = \zeta(s)K(t, s, \cdot) * f(s, \cdot), \quad \mathcal{G} f \leq \mathcal{G} A + \mathcal{G} B \quad \text{and} \quad \mathcal{G} B \leq \mathcal{G} f. \]

The first inequality comes from Minkowski’s inequality. The second inequality comes from the fact \(|K(t, s, \cdot) * B(s, \cdot)(y)| = (1 - \zeta(s))|K(t, s, \cdot) * f(s, \cdot)(y)| \) and \(|1 - \zeta(s)| \leq 1\). So for any constant \(c\),
\[ |\mathcal{G} f - c| \leq |\mathcal{G} A| + |\mathcal{G} B - c|. \]

This is because if \(\mathcal{G} f \geq c\), then
\[ |\mathcal{G} f - c| = \mathcal{G} f - c \leq \mathcal{G} A + \mathcal{G} B - c \leq |\mathcal{G} A| + |\mathcal{G} B - c| \]
and if \(\mathcal{G} f < c\), then
\[ |\mathcal{G} f - c| = c - \mathcal{G} f \leq c - \mathcal{G} B \leq |\mathcal{G} A| + |\mathcal{G} B - c|. \]

First we prove
\[ \int_{Q_R} |\mathcal{G} A(s, y)|^2 \, dsdy \leq N |Q_R| \mathcal{M}_t \mathcal{M}_x |f|^2_H(t, x). \]

Take \(\eta \in C_0^\infty(\mathbb{R}^d)\) such that \(0 \leq \eta \leq 1, \eta = 1\) in \(B_{2R_{\delta_0}}\), and \(\eta = 0\) outside of \(B_{3R_{\delta_0}}\). Set \(A_1 = \eta A\) and \(A_2 = (1 - \eta)A\). By Minkowski’s inequality, \(\mathcal{G} A \leq \mathcal{G} A_1 + \mathcal{G} A_2\). \(\mathcal{G} A_1\) can be estimated by Corollary 3.2. Indeed,
\[ \int_{-2R}^{0} \int_{B_{R_{\delta_0}}} |\mathcal{G} A_1(s, y)|^2 \, dsdy \leq N \int_{-\infty}^{0} \int_{B_{R_{\delta_0}}} |A_1(s, y)|^2_H \, dsdy \]
\[ \leq N \int_{-10R}^{0} \int_{B_{4R_{\delta_0}}} |A_1(s, y)|^2_H \, dsdy \]
\[ \leq N R^{6d} \int_{-10R}^{0} \mathcal{M}_x |A_1(t, x)|^2_H \, dt \]
\[ \leq N R^{1 + 6d} \mathcal{M}_t \mathcal{M}_x |f(t, x)|^2_H. \]

Hence it only remains to show [4.5] for \(\mathcal{G} A_2\) instead of \(\mathcal{G} A\).
Due to (4.4), (3.3) holds for $\mu = \mu_1$ and $(\sigma, \kappa) = (\sigma_1, \kappa_1)$. Thus from Lemma 3.3 with $(r_2, r_1) = (R, R^\delta)$ we have

$$\int_{Q_R} |G A_2(s, y)|^2 \, dsdy \leq N R^{\delta_0(2d+2-\mu_1)-2\sigma_1+i(\mu_1+d)+2 M_i M_x} |f|^2_H(t, x) \leq N R^{\Theta(2\delta_0+\kappa_1, \sigma_1-\delta_0)+2-(\delta_0-\kappa_1)\mu_1} M_i M_x |f|^2_H(t, x).$$

Moreover due to (2.11) and (2.10),

$$\Theta(2\delta_0+\kappa_1, \sigma_1-\delta_0) + 2 - (\delta_0-\kappa_1)\mu_1 = \Theta(\delta_0, 0) + 1 = \delta_0 + 1$$

and so (1.5) is obtained. To go further, recall (2.11) and (4.2),

$$\Theta(\kappa_2+\delta_0, \sigma_2-\delta_0) - (\delta_0-\kappa_2)\mu_2 = \Theta(2\delta_0, c_2 - \delta_0) = -2\delta_0 - 1 < -1$$

so (3.7) holds with $\mu = \mu_2$ and $(\sigma, \kappa, c) = (\sigma_2, \kappa_2, c_2)$. Hence applying Lemma 3.5 with $(r_2, r_1) = (R, R^\delta)$,

$$\sup_{Q_R} |\nabla GB|^2 \leq N \left( R^{\Theta(\kappa_2+\delta_0, \sigma_2-\delta_0) - (\delta_0-\kappa_2)\mu_2 + 1} \right) M_i M_x |B|^2_H(t, x) \leq N R^{-\delta_0} M_i M_x |B|^2_H(t, x).$$

Hence

$$\sup_{Q_R} |R^{\delta_0} \nabla GB|^2 \leq N M_i M_x |B|^2_H(t, x). \quad (4.6)$$

Similarly Lemma 3.6 with $(r_2, r_1) = (R, R^\delta)$, $(\mu, \delta, \sigma, \kappa, c) = (\mu_3, \delta_0, \sigma_3, \kappa_3, c_3)$ gives

$$\sup_{Q_R} \left| R \frac{\partial}{\partial t}(GB) \right|^2 \leq N M_i M_x |B|^2_H(t, x). \quad (4.7)$$

To apply Lemma 3.3 and Lemma 3.6 above we used the fact that $GB(s, y) = G(I_{(-\infty, 0)}B)(s, y)$ on $Q_R$. Next by (4.4),

$$\frac{1}{|Q_R|^2} \int_{Q_R} \int_{Q_R} |Gf(s, y) - Gf(r, z)|^2 \, dsdydrdz \leq 2 \int_Q |Gf - c|^2 \, dsdy \leq 4 \int_Q |GA|^2 \, dsdy + 4 \int_Q |GB - c|^2 \, dsdy.$$

Taking $c = GB(t, x)$, from (4.5), (4.6), and (4.7) we get

$$\frac{1}{|Q_R|^2} \int_{Q_R} \int_{Q_R} |Gf(s, y) - Gf(r, z)|^2 \, dsdydrdz \leq 4 \int_{Q_R} |GA|^2 \, dsdy + 4 \int_{Q_R} |GB - GB(t, x)|^2 \, dsdy \leq NM_i M_x |f|^2_H(t, x) + 4 \int_{Q_R} |GB - GB(t, x)|^2 \, dsdy \leq NM_i M_x |f|^2_H(t, x) + N \sup_{Q_R} \left( |RD GB|^2 + |R^{\delta_0} \nabla GB|^2 \right) \leq NM_i M_x |f|^2_H(t, x).$$

The lemma is proved. $\Box$
We continue the proof of the theorem. For measurable functions \( h(t, x) \) on \( \mathbb{R}^{d+1} \), we define the sharp function \( h^\#(t, x) \)

\[
  h^\#(t, x) = \sup_Q \frac{1}{|Q|} \int_Q |f(r, z) - f_Q| \, drdz,
\]

where \( f_Q := \frac{1}{|Q|} \int_Q f(r, z) \, drdz \), and the sup is taken all \( Q \) containing \((t, x)\) of the type

\[
  Q = (s - R, s + R) \times B_{R^\delta_0}(y), \quad R > 0.
\]

By Fefferman-Stein Theorem \[12\] Theorem 4.2.2, for any \( h \in L_p(\mathbb{R}^{d+1}) \),

\[
  \|h\|_{L_p(\mathbb{R}^{d+1})} \leq N \|h^\#\|_{L_p(\mathbb{R}^{d+1})}.
\]

Now we claim

\[
  (\mathcal{G} f)^\#(t, x) \leq N(M_t M_x|f|_H^2)^{1/2}(t, x). \tag{4.8}
\]

By Jensen’s inequality, to prove (4.8) it suffices to prove that for each \( Q \in \mathcal{Q} \) and \((t, x) \in Q\),

\[
  \int_Q |\mathcal{G} f - (\mathcal{G} f)_Q|^2 \, dyds \leq N(M_t M_x|f|_H^2)(t, x).
\]

Note that for any \( h_1 \in \mathbb{R} \) and \( h_2 \in \mathbb{R}^d \),

\[
  \mathcal{G} f(t - h_1, x - h_2) = \mathcal{G} \tilde{f}(t, x) = \left( \int_{-\infty}^t |\tilde{K}(t, s, \cdot) * \tilde{f}(s, \cdot)(x)|^2 H ds \right)^{1/2},
\]

where \( \tilde{f}(t, x) = f(t - h_1, x - h_2) \) and \( \tilde{K}(t, s, y) = K(t - h_1, s - h_1, y) \). Since \( \tilde{K} \) also satisfies Assumptions 2.1 and 2.2 with the same constants, we may assume \( Q = [-2R, 0] \times B_{R^\delta_0} \). Thus Lemma 4.1 proves (4.8) because

\[
  \int_Q |\mathcal{G} f - (\mathcal{G} f)_Q|^2 \, dyds \leq \frac{1}{|Q R^2|} \int_{Q_R} \int_{Q_R} |\mathcal{G} f(s, y) - \mathcal{G} f(r, z)|^2 \, dsdydrdz.
\]

Finally, combining the Fefferman-Stein theorem and Hardy-Littlewood maximal theorem \[12\] Theorem 1.3.1, we conclude (recall \( p/2 > 1 \))

\[
  \|u\|_{L_p(\mathbb{R}^{d+1})}^p \leq N\|(|M_t M_x|f|_H^2)^{1/2}\|_{L_p(\mathbb{R}^{d+1})}^p = N \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^d} (M_t M_x|f|_H^2)^{p/2} \, dt \, dx
  \leq N \int_{\mathbb{R}^{d+1}} \int_{\mathbb{R}^d} (M_x|f|_H^2)^{p/2} \, dx \, dt
  \leq N \|f\|_{L_p(\mathbb{R}^{d+1}, H)}^p.
\]

Therefore, the theorem is proved. \( \square \)
5. **Proof of Theorem 2.3**

Denote

\[ K(t, s, x) = (-\Delta)^{\gamma/4} p(t, s, x) = \int_0^{t_0} \psi(r, \xi) dr \]

We prove that Assumptions 2.1 and 2.2 hold with \( K \) of \( \kappa \).

The first assertion comes from (2.13). Indeed, since there exists a constant \( \kappa_1 \),

\[ F_1(t, s, x) = \int_0^{t_0} \left| F^{-1} \left( \frac{\psi(t, \xi)}{\gamma \gamma/2} \exp \left\{ \int_s^t \psi(r, \xi) dr \right\} \right)(x) \right|, \]

\[ F_2(t, s, x) = \int_0^{t_0} \left| F^{-1} \left( \frac{\psi(t, \xi)}{\gamma \gamma/2} \exp \left\{ \int_s^t \psi(r, \xi) dr \right\} \right)(x) \right|, \]

and

\[ F_3(t, s, x) = \int_0^{t_0} \left| F^{-1} \left( \frac{\psi(t, \xi)}{\gamma \gamma/2} \exp \left\{ \int_s^t \psi(r, \xi) dr \right\} \right)(x) \right|, \]

where \( M(t, s, \xi) := \int_s^t \psi(r, \xi) dr \).

In the following lemma we first prove (2.1)-(2.5) with \( \kappa = \kappa_2 = \kappa_3 = \gamma^{-1}, \quad \sigma_1 = \frac{d + 1}{\gamma} + \frac{1}{2}, \quad \sigma_2 = c_2 = \frac{d + 2}{\gamma} + \frac{1}{2}, \quad \sigma_3 = c_3 = \frac{d + 1}{\gamma} + \frac{3}{2} \).

**Lemma 5.1.** There exists a constant \( N = N(d, \gamma, \nu) > 0 \) such that

\[ \int_0^\infty \left| F \left( K(t, s, \cdot) \right)(\xi) \right|^2 dt \leq N, \]

\[ |D_1K(t, s, x)| \leq N(t-s)^{-\frac{d}{2} - \frac{1}{2} - \frac{1}{4}} \left( |F_1(t, s, x)| \wedge 1 \right), \]

\[ |D_2^2K(t, s, x)| \leq N(t-s)^{-\frac{d}{2} - \frac{1}{4} - \frac{3}{4}} \left( |F_2(t, s, x)| \wedge 1 \right), \]

and

\[ \left| \frac{\partial}{\partial t} D_1 K(t, s, x) \right| \leq N(t-s)^{-\frac{d}{2} - \frac{1}{4} - \frac{1}{4}} \left( |F_3(t, s, x)| \wedge 1 \right). \]

**Proof.** The first assertion comes from (2.13). Indeed, since \( \Re \psi(t, \xi) \leq -\nu |\xi|^\gamma \),

\[ \int_0^\infty \left| F \left( K(t, s, \cdot) \right)(\xi) \right|^2 dt = \int_0^{\infty} \left| \int_s^t \psi(r, \xi) dr \right|^2 dt \]

\[ \leq N \int_0^{\infty} |\xi|^\gamma e^{-2\nu t|\xi|^\gamma} dt \leq N. \]

Next because of the similarity, we only prove the last assertion. From the definition of \( K(t, s, x) \) and \( M(t, s, \xi) \),

\[ \left| \frac{\partial}{\partial x^i} \frac{\partial}{\partial t} K(t, s, x) \right| = \int_0^{t_0} \left| F^{-1} \left( \psi(t, \xi) \xi^i |\xi|^{\gamma/2} \exp \left( \int_s^t \psi(r, \xi) dr \right) \right)(x) \right| \]

\[ = \int_0^{t_0} (t-s)^{-\frac{d}{2} - \frac{1}{4} - \frac{1}{4}} \left| F^{-1} \left( \psi(t, \xi) \xi^i |\xi|^{\gamma/2} \exp \left( \int_s^t \psi(r, \xi) dr \right) \right)(x) \right| \]

\[ \leq (t-s)^{-\frac{d}{2} - \frac{1}{4} - \frac{1}{4}} \left| F_3(t, s, x) \right|, \]
Furthermore, by (2.13) and (2.14),
\[|\mathcal{F}^{-1}\left((t-s)\sigma\left(t, \frac{t}{(t-s)^{1/\gamma}}\right)\xi^4|\xi|^{\gamma/2} \exp\{M(t, s, \xi)\}\right)(x)| \]
\[\leq N \int_{\mathbb{R}^d} \left|\left(t-s)\sigma\left(t, \frac{t}{(t-s)^{1/\gamma}}\right)\xi^4|\xi|^{\gamma/2} \exp\left(\int_s^t \sigma(r, \frac{t}{(t-s)^{1/\gamma}})dr\right)\right| d\xi \]
\[\leq N \int_{\mathbb{R}^d} |\sigma|^{\gamma/2} \exp\left(-\nu|\sigma|^{\gamma}\right) d\xi \leq N.\]

Hence the assertion is proved. \(\square\)

**Lemma 5.2.** Let \(h \in C^2(\mathbb{R}^d \setminus \{0\})\) satisfy
\[|h(x)| \leq N_0|x|e^{-c|x|^\gamma}, \quad \forall x \in \mathbb{R}^d \setminus \{0\},\]
with some constants \(c, N_0 > 0, c > \frac{d}{2}\) and \(\gamma > 0\). Further assume that either
\[\eta \in [0, 1) \quad \text{and} \quad |Dh(x)| \leq N_0|x|^{\gamma-1}e^{-c|x|^\gamma}, \quad \forall x \in \mathbb{R}^d \setminus \{0\}\]
or
\[\eta \in [1, 2) \quad \text{and} \quad |D^2h(x)| \leq N_0|x|^{\gamma-2}e^{-c|x|^\gamma}, \quad \forall x \in \mathbb{R}^d \setminus \{0\}\]
holds. Then
\[\|(-\Delta)^{\eta/2}h\|_{L^2(\mathbb{R}^d)} < N < \infty,\]
where \(N = N(N_0, \eta, c, \varsigma, \gamma)\).

**Proof.** See [3] Lemma 5.1. \(\square\)

**Corollary 5.3.** Suppose
\[
\left\{ \begin{array}{ll}
2[\frac{d}{2}] + 1 & \leq 2[\frac{d}{2}] + 2 \quad \text{if} \quad \frac{d}{2} - 2[\frac{d}{2}] \in [0, 1) \\
2[\frac{d}{2}] + 2 & \leq 2[\frac{d}{2}] + 2 \quad \text{if} \quad \frac{d}{2} - 2[\frac{d}{2}] \in [1, 2) \\
\end{array} \right.
\]
Then,
\[
\sup_{s < t} \int_{\mathbb{R}^d} |x|^\mu |F_1(t, s, x)|^2 dx < \infty, \quad \text{if} \quad \mu < \gamma + d + 2;
\]
\[
\sup_{s < t} \int_{\mathbb{R}^d} |x|^\mu |F_2(t, s, x)|^2 dx < \infty, \quad \text{if} \quad \mu < (\gamma + d + 4);
\]
\[
\sup_{s < t} \int_{\mathbb{R}^d} |x|^\mu |F_3(t, s, x)|^2 dx < \infty, \quad \text{if} \quad \mu < (3\gamma + d + 2).
\]

**Proof.** Because of the similarity of proofs, we only prove the last assertion. By Parseval’s identity, it suffices to show
\[
\sup_{s < t} \int_{\mathbb{R}^d} \left|(-\Delta)^{\mu/4} \tilde{F}_3(t, s, \xi)\right|^2 d\xi < \infty, \quad \forall i,
\]
where
\[
\tilde{F}_3(t, s, \xi)(\xi) = I_{0 \leq s < t}(t-s)\sigma\left(t, \frac{t}{(t-s)^{1/\gamma}}\right)\xi^4|\xi|^{\gamma/2} \exp\{M(t, s, \xi)\}.
\]
Using (2.13) and (2.14), one can check that there exists a constant \(N = N(\nu, m)\) such for each \(0 < s < t, \xi \neq 0, \text{and} \, \mu < [\frac{d}{2}] + 2 \)
\[|(-\Delta)^{\mu/4} \tilde{F}_3(t, s, \xi)| \leq N|\xi|^{2\gamma/2 + 2}e^{-\nu|\sigma|^{\gamma}}.\]
Moreover
\[
\frac{\partial^2}{\partial \xi^i}(-\Delta)^{\lfloor \mu/4 \rfloor} \tilde{F}^3_i(s, t, \xi) \leq N|\xi|^{\frac{3\gamma}{2} - 2\lfloor \mu/4 \rfloor} e^{-\nu|\xi|^\gamma},
\]
if \( \frac{\mu}{4} - \lfloor \frac{\mu}{4} \rfloor \in [0, 1) \), and
\[
\left| \frac{\partial^2}{\partial \xi^i \partial \xi^j}(-\Delta)^{\lfloor \mu/4 \rfloor} \tilde{F}^3_i(s, t, \xi) \right| \leq N|\xi|^{\frac{3\gamma}{2} - 1 - 2\lfloor \mu/4 \rfloor} e^{-\nu|\xi|^\gamma}.
\]
if \( \frac{\mu}{4} - \lfloor \frac{\mu}{4} \rfloor \in [1, 2) \). Finally we set
\[
\eta = \mu/2 - 2\lfloor \mu/4 \rfloor, \quad \zeta = \frac{3\gamma}{2} + 1 - 2\lfloor \mu/4 \rfloor.
\]
Then, for \( \mu < 3\gamma + d + 2 \), we have
\[
\eta - \frac{d}{2} < \zeta.
\]
Therefore Lemma 5.2 is applicable, and the assertion is proved.

We continue the proof of the theorem. Recall that we defined
\[
\kappa_1 = \kappa_2 = \kappa_3 = \frac{1}{\gamma}, \quad \sigma_1 = \frac{d}{\gamma} + \frac{1}{2} + \frac{1}{\gamma},
\sigma_2 = \frac{d}{\gamma} + \frac{1}{2} + \frac{2}{\gamma}, \quad \sigma_3 = \frac{d}{\gamma} + \frac{3}{2} + \frac{1}{\gamma}.
\]
So obviously
\[
\delta_0 = c_2 - c_3 + 1 = \frac{1}{\gamma}, \quad c_2 > \frac{1}{2},
\]
\[
\Theta(\kappa_1 + \delta_0, \sigma_1 - \delta_0) + 1 = \frac{2d}{\gamma} - 2\left(\frac{d}{\gamma} + \frac{1}{2}\right) + 1 = 0,
\]
\[
\Theta(\kappa_2 - \delta_0, \sigma_2 - c_2) = \Theta(0, 0) = 0,
\]
and
\[
\Theta(\kappa_3 - \delta_0, \sigma_3 - c_3) = \Theta(0, 0) = 0.
\]
Thus (2.9) (or equivalently (2.10)) is satisfied for any \( (\mu_1, \mu_2, \mu_3) \in \mathbb{R}^3 \). Next we choose \( (\mu_1, \mu_2, \mu_3) \) such that
\[
d + 2 < \mu_1 < \gamma + d + 2,
\]
\[
d + 2 < \mu_2 < \gamma + d + 4,
\]
and
\[
d + 2 < \mu_3 < 3\gamma + d + 2
\]
so that for all \( 1 \leq i \leq 3 \)
\[
\begin{align*}
2\frac{\mu_i}{4} + 1 & \leq \left| \frac{\mu_i}{4} \right| + 2 \quad \text{if } \frac{\mu_i}{4} - 2\left| \frac{\mu_i}{4} \right| \in [0, 1) \\
2\frac{\mu_i}{4} + 2 & \leq \left| \frac{d}{4} \right| + 2 \quad \text{if } \frac{\mu_i}{4} - 2\left| \frac{\mu_i}{4} \right| \in [1, 2).
\end{align*}
\]
Then due to Corollary 5.3, we see that (2.6), (2.7), and (2.8) hold for these \( \mu_1, \mu_2, \) and \( \mu_3 \) hence Assumption 2.2 holds. The theorem is proved.
LITTLEWOOD-PALEY INEQUALITY FOR HIGHER ORDER

References

[1] T. Chang and K. Lee, On a stochastic partial differential equation with a fractional Laplacian operator, Stochastic Process. Appl., 122 (2012), 3288-3311.

[2] I. Kim and K. Kim, A generalization of the Littlewood-paley inequality for the fractional Laplacian $\Delta^{\alpha/2}$, J. Math. Anal. Appl., 388 (2012), no.1, 175-190.

[3] I. Kim, K. Kim, and S. Lim, Parabolic BMO estimates for pseudo-differential operators of arbitrary order, to appear in J. Math. Anal. Appl., doi:10.1016/j.jmaa.2015.02.065.

[4] I. Kim, K. Kim, and P. Kim, Parabolic Littlewood-Paley inequality for $\phi(\Delta)$-type operators and applications to Stochastic integro-differential equations, Advances in Math, 249 (2013), 161-203.

[5] N.V. Krylov, A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial differential equations, Ulam Quaterly, 2 (1994), no.4, 16-26.

[6] N.V. Krylov, Introduction to the theory of diffusion processes, Translations in Mathematical Monographs, 142, AMS, Providence, RI, 1995.

[7] N.V. Krylov, On the foundation of the $L_p$-Theory of SPDEs, Stochastic partial differential equations and applications—VIII, 179-191, Lect. Notes Pure Appl. Math., 245, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[8] N.V. Krylov, An analytic approach to SPDEs, pp. 185-242 in Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs, 64 (1999), AMS, Providence, RI.

[9] R. Mikulevicius and H. Pragarauskas, On the cauchy problem for certain integro-differential operators in Sobolev and Hölder spaces, Lithuanina Math. J, 32 (1992), no.2, 238-264.

[10] R. Mikulevicius and H. Pragarauskas, On $L_p$-estimates of some singular integrals related to jump processes, SIAM J. Math. Anal., 44 (2012), no.4, 2305-2328.

[11] J. Neerven, M. Veraar, and L. Weis, Maximal $L^p$-regularity for stochastic evolution equations, SIAM J. Math. Anal., 44 (2012), no.3, 1372-1414.

[12] E. Stein, Harmonic analysis : real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, 1993.

Department of Mathematics, Korea University, 1 Anam-dong, Sungbuk-gu, Seoul, 136-701, Republic of Korea
E-mail address: valdoo@korea.ac.kr

Department of Mathematics, Korea University, 1 Anam-dong, Sungbuk-gu, Seoul, 136-701, Republic of Korea
E-mail address: kyeonghun@korea.ac.kr

Department of Mathematics, Korea University, 1 Anam-dong, Sungbuk-gu, Seoul, 136-701, Republic of Korea
E-mail address: sungbin@korea.ac.kr