EVOLUTION OF INTRA-CAVITY FIELDS AT NON-STEADY STATE
IN DUAL RECYCLED INTERFEROMETER

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Abstract

We describe how exactly the intra-cavity fields in a dual recycling cavity build up their power before achieving a steady state value. We restricted our analysis here to interferometers with lossless mirrors and beam-splitter. The complete series representation of the intra-cavity lights at any stage of evolution in non-steady state have been presented.

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Laser interferometric gravitational wave observatories[1] are currently being designed or constructed in several countries (LIGO, VIRGO, GEO, AIGO). The interferometer, in essence, is a rather elaborate transducer from optical path difference to output power. So, monitoring the change in output power, it would be possible to detect the changing curvature of spacetime induced by the passage of a gravitational wave. In order to reduce the shot noise level, an integral feature of these detectors will be the use of high power laser, in conjunction with variants of optical technique known as light recycling. The first one of these techniques is called power recycling[2], in which, at a dark fringe operation of the interferometer, the outgoing laser light is recycled back into the interferometer by putting a mirror in front of the source, thus enhancing the laser power. When a gravitational wave passes through an interferometer, it modulates the phase of the laser light, thus producing sidebands which travel towards the photodetector [3]. So, another variant of optical technique called dual recycling incorporates a second recycling mirror placed in front of the photodetector as shown in Fig.1. This arrangement can store signal sidebands for sufficiently long time to allow optimum photon noise sensitivity within a restricted bandwidth [4].

In this note, we describe, how with the help of Mathematica[5], one can do calculation for the evolution of intra-cavity fields in the two recycling cavities of a dual recycled interferometer and finally achieve the expressions for the intra-cavity fields at the steady state condition. This may find its application in investigating various kinds of transient states in such interferometers.

We first introduce the notations used here. In all the following figures, rays of light labelled by $a$, $b$, $x$, $y$ represent the complex amplitude of the beam’s electromagnetic field. **Mirrors:** In this note, we make the following assumptions on the mirrors and beam-splitter: (i) these are lossless, (ii) the substrate and the dielectric layers in mirrors are linear media, (iii) mirrors are time-reversal invariant, (iv) any mirror has a reflection symmetry with respect to exchange of inputs and outputs on the same side of the mirror, (v) there is no differential time delay between the transmitted and reflected lights and also the overall time delay across the mirror is negligible. One can then arrive at the following simple input-output mirror relation (refer to Fig.2):

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix} = \begin{pmatrix} t & -r \\ r & t \end{pmatrix} \begin{pmatrix} b_0 \\ a_0 \end{pmatrix},$$

(1)

where $t$ and $r$ represent the transmission and reflection coefficients of the mirror. For a
lossless mirror, \(|t|^2 + |r|^2 = 1\). We consider all the beam-splitters to be of 50:50 type.

**Fabry-Perot Cavity:** The flowchart of the light paths in such a cavity has been shown in Fig.3. Light beams \(a_0\) and \(b_0\) enter the cavity from corner mirror (CM) and end mirror (EM) sides respectively. The beam \(x\) represents the intra-cavity field that grows up in every bounce it gets from CM or EM, provided it adds up in phase with the incoming light there. The index \(j\) represents the number of round-trips of the \(x\)-field inside the cavity. The circle with the quantity \(L\) inside it represents the change in phase of the intra-cavity light due to the traversal of cavity of length \(L\). After a sufficient number of bounces, the intra-cavity field, \(x\), achieves a steady state when it stops growing any more (i.e., the change after any bounce would be negligibly small).

**Dual Recycling Cavity:** The corresponding flowchart for a single delay-line dual-recycled cavity has been shown in Fig.4. To understand this figure one may also refer to Fig.1. The circle with the quantity \(l_1\) or \(l_2\) inside it represents the change in phase of the intra-cavity light due to traversal of power recycling cavity (PRC) of length \(l_1\) between the beam-splitter (BS) and the power recycling mirror (PRM) or Signal Recycling Cavity (SRC) of length \(l_2\) between BS and signal recycling mirror (SRM) respectively. The electromagnetic field \(a_0\) represents the coherent beam of laser light entering through PRM. The field \(b_0\) is in vacuum state in ordinary interferometers, but as suggested by Caves[6], in advanced interferometers, this may be made to be in squeezed vacuum state to reduce the shot noise. The fields \(x_4\) and \(y_4\) represent intra-cavity light beams in the Power Recycling Cavity (PRC) and Signal Recycling Cavity (SRC) respectively. We represent the amplitude transmission and reflection coefficients of the PRM (SRM) by \(t_1(t_2)\) and \(r_1(r_2)\) respectively. The box with the label ‘Propagation in Arms’ represents the matrix relation:

\[
\begin{pmatrix}
  x_3[j] \\
  y_3[j]
\end{pmatrix}
= \exp \left[ \frac{i2\omega_0 L_a}{c} \right] \begin{pmatrix}
  e^{-i\theta} & 0 \\
  0 & e^{+i\theta}
\end{pmatrix}
\begin{pmatrix}
  x_2[j] \\
  y_2[j]
\end{pmatrix},
\]

where \(k_0 = \omega_0/c\) and \(\omega_0, L_a, c\) are the circular frequency of light beam, the arm length of the interferometer and the velocity of light respectively. The quantity \(\theta\) represents the differential phase shift of light in the two arms of the interferometer. The end mirrors of dual recycling cavity are assumed to be perfectly reflecting ones.

A small programme in *Mathematica* can be easily written to implement this recursive procedure. After a sufficient number of bounces, \(j\) (say, 10), one can look at the expressions for \(x_4[j]\) and \(y_4[j]\). We have set \(x_0[0] = y_0[0] = 0\) in the final expressions of \(x_4[j]\) and \(y_4[j]\) after finishing the recursive calculation because the initial intra-cavity fields were zero.
Now, grouping the related terms together, we can guess the formation of a series in each case.

The series obtained by us are as follows:

\[ x_4[j] = b_0 \left[ \frac{t_2 Q_0 \sin \theta}{K+1} \sum_{k=0}^{K+1} (-r_1 r_2 Q_0^2 \sin^2 \theta)^k \right. \]
\[ \times \sum_{n=1}^{N} \{n+k-1 C_k\} (r_1 Q_1 \cos \theta)^{n-1} \sum_{m=1}^{M} \{m+k-1 C_k\} (r_2 Q_2 \cos \theta)^{m-1} \]
\[ + a_0 \left. \left[ t_1 Q_1 \cos \theta \sum_{p=1}^{j} (r_1 Q_1 \cos \theta)^{p-1} - t_1 r_2 Q_0^2 \sin^2 \theta \sum_{k=0}^{K} (-r_1 r_2 Q_0^2 \sin^2 \theta)^k \right. \right. \]
\[ \times \sum_{n=1}^{N} \{n+k C_{k+1}\} (r_1 Q_1 \cos \theta)^{n-1} \sum_{m=1}^{M} \{m+k C_{k+1}\} (r_2 Q_2 \cos \theta)^{m-1} \right. \]
\[ + a_0 \left. \left. \left[ \frac{t_2 Q_2 \cos \theta}{K+1} \sum_{k=0}^{K} (-r_1 r_2 Q_0^2 \sin^2 \theta)^k \right. \right. \]
\[ \times \sum_{n=1}^{N} \{n+k-1 C_k\} (r_1 Q_1 \cos \theta)^{n-1} \sum_{m=1}^{M} \{m+k-1 C_k\} (r_2 Q_2 \cos \theta)^{m-1} \right. \]
\[ \left. \right] \left. \right] \] (3a)

\[ y_4[j] = b_0 \left[ \frac{t_2 Q_2 \cos \theta}{K+1} \sum_{k=0}^{K+1} (-r_1 r_2 Q_0^2 \sin^2 \theta)^k \right. \]
\[ \times \sum_{n=1}^{N} \{n+k-1 C_k\} (r_1 Q_1 \cos \theta)^{n-1} \sum_{m=1}^{M} \{m+k-1 C_k\} (r_2 Q_2 \cos \theta)^{m-1} \]
\[ + a_0 \left[ \frac{i t_1 Q_1 \sin \theta}{K+1} \sum_{k=0}^{K+1} (-r_1 r_2 Q_0^2 \sin^2 \theta)^k \right. \]
\[ \times \sum_{n=1}^{N} \{n+k-1 C_k\} (r_1 Q_1 \cos \theta)^{n-1} \sum_{m=1}^{M} \{m+k-1 C_k\} (r_2 Q_2 \cos \theta)^{m-1} \right. \]
\[ \left. \right] \left. \right] \] (3b)

where

\[ Q_0 = \exp[i k_0 (2 L_a + l_1 + l_2)] \]
\[ Q_1 = \exp[i k_0 (2 L_a + 2 l_1)] \]
\[ Q_2 = \exp[i k_0 (2 L_a + 2 l_2)] \]

are the phase factors which satisfy \( Q_0^2 = Q_1 Q_2 \). The upper limits of the integers \( k, n \) and \( m \) are represented by \( K, N, \) and \( M \) respectively. For any value of \( j \), we obtain all the terms in the series that are characterised by sets of values of these limits satisfying the following relationship:

\[ j = 2K + M + N. \] (3d)

So, in the complete expression for \( x_4 \), say, there would be several terms each involving the product of three summations over \( k, m \) and \( n \), one term for each combination of \( K, M \)
and \( N \) satisfying Eq.(3d). So, as \( j \) increases, the number of triple products of summations in Eqs.(3a) and (3b) also increase. The explanation of how these upper limits arise is provided below.

In this connection, interested readers may also like to know the physical interpretation of all the terms appearing in series (3a) and (3b). One may note that any light gets multiplied by either \((i \sin \theta)\) or \((\cos \theta)\) depending on whether it changes its cavity (from PRC to SRC and vice versa) or not after passing through the BS and one arm.

For example, we may consider the series in the second term of the coefficient of \(a_0\) in the intra-cavity field \(x_4[j]\). Any term in this series represent the contribution of \(a_0\) light in \(x_4\) after it completed \((k + n)\) number of round trips in PRC and \((k + m)\) number of round trips in SRC. Since \(j\) represents the sum of the maximum number of round trips in two cavities, this explains why we obtain the relation (3d) for the series described in Eqs.(3a) and (3b). The number of possible ways of completing such a trip would be \(\{n + k\}^{m + k - 1}C_k\}. As an easy example, let us take the contribution corresponding to \(k = 0, n = 2, m = 1\), i.e., \(-2r_1r_2t_1\cos \theta \sin^2 \theta\), which may be expanded as

\[
(r_1 \cos \theta)(i \sin \theta)r_2(i \sin \theta)t_1 + (i \sin \theta)r_2(i \sin \theta)(r_1 \cos \theta)t_1.
\]

This means that this term has been contributed by two parts of \(a_0\) light which made two round trips of PRC and one of SRC in the following two ways:

(i) \(\rightarrow\)PRM; transmitted inside \(\rightarrow\)BS, arm \(\rightarrow\)SRC \(\rightarrow\)SRM; reflected \(\rightarrow\)BS, arms \(\rightarrow\)PRC \(\rightarrow\)PRM; reflected \(\rightarrow\)BS, arm \(\rightarrow\)PRC.

(ii) \(\rightarrow\)PRM; transmitted inside \(\rightarrow\)BS, arm \(\rightarrow\)PRC \(\rightarrow\)PRM; reflected \(\rightarrow\)BS, arms \(\rightarrow\)SRC \(\rightarrow\)SRM; reflected \(\rightarrow\)BS, arm \(\rightarrow\)PRC.

The intra-cavity fields, \(x_4\) and \(y_4\) achieve their steady state value as \(j\) tends to infinity. Now, identifying

\[
\sum_{n=1}^{\infty} \{n + k\}^{m + k - 1}C_k\}(r_1Q_1 \cos \theta)^{n-1} = \frac{1}{(1 - r_1Q_1 \cos \theta)^{k+1}}, \quad (4)
\]

etc. and after doing some algebraic manipulations, we finally arrive at the steady state expressions:

\[
x_4[\text{steady}] = x_4[j \rightarrow \infty] = \frac{1}{\chi}[a_0t_1Q_1(\cos \theta - r_2Q_2) + ib_0t_2Q_0 \sin \theta], \quad (5a)
\]

\[
y_4[\text{steady}] = y_4[j \rightarrow \infty] = \frac{1}{\chi}[ia_0t_1Q_0 \sin \theta + b_0t_2Q_2(\cos \theta - r_1Q_1)], \quad (5b)
\]
where
\[ \chi = 1 - (r_1 Q_1 + r_2 Q_2) \cos \theta + r_1 r_2 Q_1 Q_2. \] (5c)

The steady-state expressions (5) can, of course, be calculated very easily by assuming that, under steady-state condition, the change in the intra-cavity fields after every round-trip of the cavities is negligibly small [7]. One can then write the following two equations which can be solved to obtain the above-mentioned expressions.

\[ x_4[\text{steady}] = a_0 t_1 Q_1 \cos \theta + x_4[\text{steady}] r_1 Q_1 \cos \theta \]
\[ + i b_0 t_2 Q_0 \sin \theta + i y_4[\text{steady}] r_2 Q_0 \sin \theta, \] (6a)

\[ y_4[\text{steady}] = b_0 t_2 Q_2 \cos \theta + y_4[\text{steady}] r_2 Q_2 \cos \theta \]
\[ + i a_0 t_1 Q_0 \sin \theta + i x_4[\text{steady}] r_1 Q_0 \sin \theta. \] (6b)

The intra-cavity fields can achieve their maximum values at the resonance condition of the dual recycling cavity, i.e., when \( l_1 = l_2 \) and \( Q_0 = Q_1 = Q_2 = 1 \). This implies that the intra-cavity fields \( x_4 \) and \( y_4 \) add up in phase with the incoming light. For a sufficiently low value of \( r_2 \), this condition also represents the broadband mode of operation of a dual recycled cavity [4]. The expressions for the case of power recycling (when no SRM is used) can be easily obtained by setting \( r_2 = 0 \). Similar exercise can of course be carried out for interferometers incorporating Fabry-Perot cavities in the arms.

For exactly equal armlengths of the interferometer (\( \theta = 0 \)), the dual recycling cavity, at any frequency, gets decoupled into two equivalent three mirror cavities. If \( Q_1 = 1 \), the laser light, \( a_0 \) becomes resonant with the three mirror power recycling cavity (PRM and two EMs), whereas the sidebands of laser light (generated by the gravitational wave[3]) travel down the SRC. One can make one of these sidebands (say, \( \omega_0 + \omega_g \)) resonant with the three mirror signal recycling cavity (SRM and two EMs) by adjusting \( l_2 \), such that \( Q_2 \exp[+i \omega_g (2L_a + 2l_2)] = 1 \). This is called the narrowband mode of operation. Obviously, under this mode of operation, neither the \( b_0 \) light nor its sidebands (which, after getting created, travel down the PRC) can become resonant with any of the cavities.

Gravitational radiation from a coalescing compact binary system has a chirp waveform which continually increases in frequency and amplitude as the two stars spiral in towards each other. To observe this broadband waveform, dynamical dual recycling technique[8] was proposed, which incorporated two complementary techniques to improve the sensitivity of observation: (i) narrow band observation to achieve high signal to noise ratio at a
given frequency, and (ii) tracking the chirp frequency by adjusting the tuning frequency by changing $l_2$ as long as the signal lies within the detector bandwidth. To implement this technique, it is important to understand how exactly the signal (i.e., the sideband that is tracked down) builds up its power in the presence of errors in tuning the SRC. A programme simulating this technique in the way described above, therefore, may come to be handy and useful.

In actual LIGO type interferometers, the Fabry Perot cavities in the arms may be physically different in length by several meters, and the storage times in the arms different by as much as 1.0% due to differing mirror transmissions or deviations from the operating point [9]. It may be noted that this way of doing the calculation exactly tracks down various light beams and thus provides an answer to the question ‘how many times a particular beam traverses through any particular arm or cavity before a specific point of time in a transient state’. The arbitrary phase noise (mainly caused by various nonlinear effects, e.g. scattering, thermoelastic effects on the mirrors etc.) induced on the beams in different arms may lead to leakage of laser light to the SRC. Also, an important issue is the control of the resonance conditions in the various cavities in these type of interferometers made of suspended mirrors. For this purpose it is useful to compute the transient behaviour of the light amplitudes at various locations in the interferometer for given motions of mirrors. This particular algorithm will help in investigating such kinds of transient state of the interferometer and in tracking down the contribution of different cavities in the total phase noise in an effective way. Investigations are currently in progress in this line and results will be communicated in future.

One point of caution may be worth noting here, although that is not important for the present designs of the interferometric gravitational wave detectors. If the Fabry Perot cavities in the arms of the interferometer are significantly different in length, then within a given time period, the number of bounces of light in one cavity will become different from the number of bounces in the other. The analysis in the form presented here will break down in that case because the iteration parameter $j$ will differ for different beams depending on the path history.

We point out again that the series compaction to arrive at Eqs.(3a) and (3b) was performed by manual inspection of the recursive series expansion produced by Mathematica. Employing Mathematica to find these automatically is, in general, impossible since there may be infinite number of functions with identical series expansions up to a certain degree.
Abbott[10] has studied different aspects of this Mathematica problem and showed a method of compacting series using generating functions, if the corresponding recurrence relation is available. Investigation is in order to find a way to combine the complicated recursive process of the present problem with the method described by Abbott[10] for simpler cases, with the purpose of employing Mathematica to arrive at Eqs(3a) and (3b) automatically. It should be noted, however, that this is not a problem for the applicability of the method of calculation described in this paper to realistic situations where one has to do numerical computation.

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**Figure Caption**

FIG.1: Optical arrangement for dual recycling. (BS) - Beam Splitter, (EM) - End Mirror, (PRM) - Power Recycling Mirror, (SRM) - Signal Recycling Mirror, (PD) - Photodetector. The light beams: $a_0$ - input laser beam, $b_0$ - ordinary or squeezed vacuum state (*see text*); $x_1, x_4$ - intracavity fields in power recycling cavity; $y_1, y_4$ - intracavity fields in signal recycling cavity; $x_1, x_2, y_2, y_3$ - light fields inside the arms; $a_1, b_1$ - output beams through PRM and SRM respectively.

FIG.2: Mirror and light beams. $a_0, b_0$ - input beams; $a_1, b_1$ - output beams.

FIG.3: Flowchart of a Fabry-Perot cavity at non-steady state. CM - Corner Mirror, EM - End Mirror. The circle with $L$ inside represents phase change due to traversal of cavity length, $L$.

FIG.4: Flowchart of a single delay-line dual recycled cavity at non-steady state. PRM - Power Recycling Cavity, SRM - Signal Recycling Cavity, BS - Beam Splitter. Circle with $l_1$ or $l_2$ inside represents phase changes due to traversal of length, $l_1$ or $l_2$ between BS and PRM or SRM respectively.
Initialize: $j=0$

Propagation in Arms