THE HYPOCOERCIVITY INDEX FOR THE SHORT AND LARGE TIME BEHAVIOR OF ODEs

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Abstract

We consider the class of conservative–dissipative ODE systems, which is a subclass of Lyapunov stable, linear time-invariant ODE systems. We characterize asymptotically stable, conservative–dissipative ODE systems via the hypocoercivity (theory) of their system matrices. Our main result is a concise characterization of the hypocoercivity index (an algebraic structural property of matrices with positive semi-definite Hermitian part introduced in [2]) in terms of the short time behavior of the propagator norm for the associated conservative–dissipative ODE system.

1 Introduction

In this paper we shall use hypocoercivity techniques to characterize the short and long time behavior of linear ODE-systems of the form

\[ x'(t) = -Bx(t), \]

with matrices \( B \in \mathbb{C}^{n \times n} \) whose Hermitian part \( B_H := (B + B^*)/2 \) is positive semi-definite, such that \(-B\) is a conservative–dissipative (or semi-dissipative) matrix, see Definition[1.1] An extension to stable systems of the form (1.1), i.e. matrices \( B \) having their spectrum in the closed right-half plane where, additionally, purely imaginary eigenvalues are non-defective[1] will be discussed in a forthcoming article [4].

Concerning the short time behavior we shall be interested in estimates on the propagator of (1.1), \( P(t) := e^{-Bt} \) of the form

\[ \| P(t) \| = 1 - c t^a + O(t^{a+1}) \quad \text{for } t \to 0+, \]

An eigenvalue is non-defective if its algebraic and geometric multiplicities coincide.
where \( c > 0 \) and \( a \in \mathbb{N} \). Concerning the large time behavior we shall be concerned with exponential decay estimates of the form
\[
\|P(t)\| \leq c e^{-\mu t}, \quad t \geq 0,
\]
with some \( c \geq 1 \), and \( \mu > 0 \). In fact, following [19], the matrix \( B \) is called \textit{hypocoercive} if (1.3) holds, and \( B \) is \textit{coercive} iff \( \|P(t)\| \leq e^{-\mu t} \) for some \( \mu > 0 \) and all \( t \geq 0 \).

Our interest in systems (1.1) is motivated by non-equilibrium statistical physics and, in particular, kinetic theory. There, a frequently encountered class of linear equations has the form
\[
\frac{\partial}{\partial t} f(t, x, v) = -v \cdot \nabla_x f(t, x, v) + Q f(t, x, v),
\]
where \( Q \) is a linear operator, the so-called \textit{linearized collision operator}, describing changes in velocity resulting from binary collisions. Take the domain of the \( x \) variable to be a torus \( \mathbb{T} \) of side-length \( L \). The operator \( v \cdot \nabla_x \), called the streaming operator is anti-Hermitian on a weighted \( L^2 \) space on \( \mathbb{T} \times \mathbb{R}^d \), whereas \( Q \) is negative semi-definite on the same weighted \( L^2 \) space. Concerning examples we refer to [12, §1.4], [1, 2] where modal decomposition (in \( x \)) of kinetic BGK equations\(^2\) led to ODE-systems like (1.1); in the case of continuous velocities it actually led to “infinite matrices” \( B \). In [8, §5] the modal decomposition of a Fokker-Planck equation was considered.

The main result of this paper is to give a concise interpretation of the \textit{hypocoercivity index} (an algebraic structural property of the matrix \( B \), introduced in [2]) in terms of the exponent \( a \) in short time estimate (1.2).

A first application of our main result appears in the recent study [9] of Fokker-Planck equations with linear drift, i.e. \( \frac{\partial}{\partial t} f = \text{div}_x \left( D \nabla_x f + C x f \right) \), where \( f = f(t, x), \ t > 0, \ x \in \mathbb{R}^n \), positive-semidefinite, symmetric diffusion matrix \( D \in \mathbb{R}^{n \times n} \) and drift matrix \( C \in \mathbb{R}^{n \times n} \). In [9], it is shown that: Under a suitable coordinate transformation these equations can be normalized such that the diffusion and drift matrices are linked as \( D = C S \), the symmetric part of \( C \). Then, for such a normalized Fokker-Planck equation its \( L^2 \)-propagator norm coincides with the propagator norm of its drift ODE \( \dot{x} = -C x \). Two main consequences are pointed out: First, the sharp (exponential) decay of the PDE is reduced to the same, but much easier question on the ODE level. The second consequence is that the hypocoercivity index (see [7, 2]) of the drift matrix determines the short-time behavior (in the sense of a Taylor series expansion) both of the drift ODE and the FP equation. As a further consequence for solutions of the FP equation the short-time regularization from the weighted \( L^2 \)-space to a weighted \( H^1 \)-space is determined. This result can be seen as an illustration of the fact that for the FP equation hypocoercivity is equivalent to hypoellipticity.

This paper is organized as follows: In § 1.1 we define the class of linear conservative–dissipative ODE systems, which are a subclass of Lyapunov stable ODE systems. To characterize asymptotically stable, linear conservative–dissipative ODE systems, we recall some notions of hypocoercivity theory for ODEs. In Section 2 we present our first main result in Theorem 2.6, i.e. a concise characterization of the hypocoercivity index in terms of the short time behavior of conservative–dissipative ODE systems (1.1).
In §2.1, we recall the original definition of the hypocoercivity index (HC-index) for positive conservative–dissipative matrices and present two (equivalent) variants. In §2.2, we state and prove our main result. For the proof we need an upper bound for the propagator norm, which is formulated in Lemma A.1 and proven in Appendix A.

We use the following notation: The conjugate transpose (transpose) of a matrix $B$ is denoted by $B^*$ ($B^T$). The set of Hermitian matrices in $\mathbb{C}^{n \times n}$ is denoted by $\mathbb{H}_n$. Positive definiteness (semi-definiteness) of $B \in \mathbb{H}_n$ is denoted by $B > 0$ ($B \geq 0$). The set of all positive definite (semi-definite) Hermitian matrices in $\mathbb{C}^{n \times n}$ is denoted by $\mathbb{H}^>_n$ ($\mathbb{H}^\geq_n$).

1.1 Conservative-dissipative systems of ODEs

It is well known that an ODE (1.1) is (Lyapunov) stable if all eigenvalues of $-B$ have non-positive real part and the eigenvalues on the imaginary axis are non-defective, and (1.1) is asymptotically stable if all eigenvalues of $-B$ have negative real part.

Consider a system of ODEs (1.1) with matrix $B \in \mathbb{C}^{n \times n}$. Then the derivative of the squared Euclidean norm of a solution $x(t)$ satisfies

$$\frac{d}{dt} \|x(t)\|^2 = \langle -Bx(t), x(t) \rangle + \langle x(t), -Bx(t) \rangle = -2\langle x(t), B_Hx(t) \rangle.$$  \hspace{1cm} (1.4)

Therefore a sufficient condition for $B$ to generate a stable system (1.1) is that its Hermitian part is positive semi-definite. This fact and the importance of this subclass of stable systems in kinetic theory inspires the following definition:

**DEFINITION 1.1.** A matrix $-B \in \mathbb{C}^{n \times n}$ is called dissipative (resp. conservative–dissipative or semi-dissipative) if the Hermitian part of $-B$ is negative definite (resp. negative semi-definite).

For a (conservative-)dissipative matrix $-B \in \mathbb{C}^{n \times n}$, the associated system of ODEs (1.1) is called a (conservative-)dissipative ODE system.

For practical reasons, a matrix $B \in \mathbb{C}^{n \times n}$ is called positive conservative–dissipative (or positive semi-dissipative) if the Hermitian part of $B$ is positive semi-definite.

This conservative–dissipative property of a matrix $B$ is invariant under unitary transformations, but it is not invariant under similarity transformations (see [4] for details).

The following spectral inclusion between the matrices $B$ and $B_H$ holds:

**LEMMA 1.2.** \cite[Th. 2.1]{[13], Prop. III.5.3, [11], Fact 7.12.24} Let $B \in \mathbb{C}^{n \times n}$. Then

$$\lambda_{min}^B \leq \min \{ \Re \lambda : \lambda \in \sigma(B) \} \leq \max \{ \Re \lambda : \lambda \in \sigma(B) \} \leq \lambda_{max}^B,$$

where $\lambda_{min}^B$ and $\lambda_{max}^B$ denote, respectively, the smallest and largest eigenvalue of $B_H$.

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3 However, the Hermitian part of a matrix $B$ pertaining to a stable system does not have to be positive semi-definite.
Proof. Using the Rayleigh quotient, this follows easily since
\[
\lambda_{\min}^{\scriptscriptstyle BH} = \min_{x \neq 0} \frac{x^* B x}{\|x\|^2} = \min_{x \neq 0} \frac{\Re(x^* B x)}{\|x\|^2} \leq \min_i \frac{\Re(\lambda_i x_i^* x_i)}{\|x_i\|^2} = \min \{ \Re \lambda_i \},
\]
where \(\lambda_i\) are the eigenvalues of \(B\) (not counting multiplicity), and \(x_i \in \mathbb{C}^n\) are the corresponding eigenvectors.

Asymptotically stable systems. A stable system (1.1) with matrix \(B\) is asymptotically stable if all solutions approach the origin in the large time limit. This is equivalent to the condition that all eigenvalues of \(-B\) have negative real part. Starting with the classical notion of a coercive operator/matrix, we introduce/recall the definition of a hypocoercive matrix (see e.g. [2]):

**DEFINITION 1.3.** A matrix \(B \in \mathbb{C}^{n \times n}\) is called coercive if its Hermitian part \(B_H\) is positive definite, and it is called hypocoercive if the spectrum of \(B\) lies in the open right half plane.

For practical reasons, a matrix \(-B \in \mathbb{C}^{n \times n}\) is called negative hypocoercive if the spectrum of \(-B\) lies in the open left half plane.

Hypocoercive matrices are often called positive stable, whereas negative hypocoercive matrices are often called stable (or Hurwitz). We use the notion of hypocoercivity to emphasize the analogous situation in partial differential equations, see [2, 7, 19].

Consider an ODE (1.1) with matrix \(B \in \mathbb{C}^{n \times n}\). Then \(B\) is hypocoercive iff the solutions \(x(t)\) of (1.1) satisfy for all \(x(0) \in \mathbb{C}^n\):
\[
\|x(t)\| \leq c e^{-\mu t} \|x(0)\|, \quad t \geq 0, \quad (1.5)
\]
for some \(c < \infty\) and \(\mu > 0\); whereas \(B\) is coercive iff the solutions \(x(t)\) of (1.1) satisfy \(\|x(t)\| \leq e^{-\mu t} \|x(0)\|\) for some \(\mu > 0\) and all \(t > 0\). In [2, Proposition 1(B3)], hypocoercivity of positive conservative–dissipative matrices \(B\) was proven to be equivalent to the following characterization:

“No non-trivial subspace of \(\ker(B_H)\) is invariant under \(B_A\),” \(\quad (1.6)\)
where \(B_A := (B - B^*)/2\) denotes the anti-Hermitian part of \(B\). This characterization of hypocoercivity also follows from [17, Lemma 3.1]. For further, equivalent hypocoercivity conditions we refer to [2, Proposition 1].

It is well-known, see Lemma 1.2, that for positive conservative–dissipative matrices \(B \in \mathbb{C}^{n \times n}\), the spectrum of \(B\) lies in the closed right half plane, but there may be purely imaginary eigenvalues. In this paper we give simple explicit, necessary and sufficient conditions that exclude such eigenvalues, and thus characterize hypocoercivity (on top of those from [2, Proposition 1]). The characterization (1.6) implies the following result.

**LEMMA 1.4.** Let \(B \in \mathbb{C}^{n \times n}\) be positive conservative–dissipative. Then \(B = B_A + B_H\) is hypocoercive iff \(\epsilon B_A + B_H\) is hypocoercive for all \(\epsilon \neq 0\).
But even if hypocoercivity of $B$ is known, it is not trivial to obtain an exponential decay estimate (1.5) with a quantitative (or even optimal) decay rate $\mu$. Indeed, a simple energy estimate (i.e. pre-multiplying (1.1) by $x^*$) or using Trotter’s product formula only yields conservative–dissipativity of the system, but no decay: So, its propagator $P(t) = e^{-Bt}$ satisfies at least $\|P(t)\| \leq 1$ for all $t \geq 0$.

Let us also comment on the short time behavior. If a decay estimate (1.2) holds with some exponent $a > 1$, then an estimate of the form $\|P(t)\| \leq e^{-\mu t}$ (as it is typical for coercive matrices $B$) is impossible (consider the Taylor expansion of $\|P(t)\|$ around $t = 0$). In such cases the system can only be hypocoercive, along with an estimate (1.3) with $c > 1$.

## 2 Hypocoercivity index and the short-time decay of conservative–dissipative ODE systems

In this section we shall present our first main result, i.e. a concise characterization of the hypocoercivity index in terms of the short time behavior of conservative–dissipative ODE systems (1.1).

### 2.1 Hypocoercivity index

First we recall from [2, §2.2] the definition of the hypocoercivity index:

**DEFINITION 2.1.** Let $B \in \mathbb{C}^{n \times n}$ be positive conservative–dissipative. Its hypocoercivity index (HC-index) $m_{HC}$ is defined as the smallest integer $m \in \mathbb{N}_0$ (if it exists) such that the matrix

$$T_m := \sum_{j=0}^{m} B_A^j B_H (B_A^*)^j$$

(2.1)

is positive definite. Matrix $B$ is coercive iff $m_{HC} = 0$, hypocoercive iff $m_{HC} \in \mathbb{N}_0$, and for non-hypocoercive matrices $B$ we set $m_{HC} = \infty$.

For practical reasons, we define the HC-index $m_{HC}$ also for conservative–dissipative matrices $-B \in \mathbb{C}^{n \times n}$ as the HC-index of $B$.

The definition (2.1) of the Hermitian matrix $T_m$ readily shows that the hypocoercivity index of a (positive) conservative–dissipative matrix $B$ is invariant under unitary congruence transformations but, in general, not under similarity transformations (see [3] for details).

In [6, Lemma 2.3] it was proven that this index equals the smallest integer $m \in \mathbb{N}_0$ such that

$$\text{rank} \left\{ \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right\} = n,$$

(2.2)

which is often called Kalman rank condition (see also [2] Proposition 1(B1)).

**Remark 2.2.** When considering rather $\epsilon B_A + B_H$, its hypocoercivity property and its index of hypocoercivity are independent of $\epsilon \neq 0$, which follows trivially from (2.2). Due to the above mentioned equality of the indices, the positive definiteness of $\sum_{j=0}^{m_{HC}} (\epsilon B_A)^j B_H (\epsilon B_A^*)^j$ is hence also independent of $\epsilon \neq 0$. 
Next we shall present two (equivalent) variants of the Kalman rank condition:

**Lemma 2.3.** Let $B \in \mathbb{C}^{n \times n}$ be positive conservative–dissipative. Consider the following three Kalman rank conditions:

\[\exists m \in \mathbb{N}_0: \quad \text{rank} \left( \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right) = n, \quad (2.3a)\]

\[\exists m \in \mathbb{N}_0: \quad \text{rank} \left( \sqrt{B_H}, B \sqrt{B_H}, \ldots, B^m \sqrt{B_H} \right) = n, \quad (2.3b)\]

\[\exists m \in \mathbb{N}_0: \quad \text{rank} \left\{ C_0, C_1, \ldots, C_m \right\} = n \quad \text{with} \quad C_0 := \sqrt{B_H}; \quad C_{j+1} := [C_j, B_A], \quad j \in \mathbb{N}_0. \quad (2.3c)\]

The conditions (2.3a)–(2.3c) are equivalent in the sense that, if there exists $m \in \mathbb{N}_0$ such that one condition holds, then the other two conditions hold as well for the same $m$.

**Proof.** In fact, we prove that for all $m \in \mathbb{N}_0$ the ranges of all three matrices in (2.3) are equal,

\[
\text{range} \left( \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right) = \text{range} \left( \sqrt{B_H}, B \sqrt{B_H}, \ldots, B^m \sqrt{B_H} \right) = \text{range} \left\{ C_0, C_1, \ldots, C_m \right\},
\]

(2.4a)

which can be done inductively:

The base case $m = 0$ is trivial, since all three matrices in (2.4) are equal to $\sqrt{B_H}$. We start with the induction step for the equivalence of the first identity in (2.4): Assume (2.4a) holds for some $m \in \mathbb{N}_0$. We compute the following representations $(B^{m+1} - B_A^{m+1}) \sqrt{B_H} = \sum_{j=0}^{m} B_A^j \sqrt{B_H} X_j$, with appropriate matrices $X_j \in \mathbb{C}^{n \times n}$ and, e.g., $X_m = B_H$. Therefore

\[
\text{range} \left( (B^{m+1} - B_A^{m+1}) \sqrt{B_H} \right) \subset \text{range} \left( \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right)
\]

and, hence,

\[
\text{range} \left( \sqrt{B_H}, B \sqrt{B_H}, \ldots, B^m \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right) = \text{range} \left( \sqrt{B_H}, B \sqrt{B_H}, \ldots, B^m \sqrt{B_H} \right)
\]

\[
= \text{range} \left( \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right)
\]

\[
= \text{range} \left( \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right)
\]

So $\text{range} \left( \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right) = \text{range} \left( \sqrt{B_H}, B \sqrt{B_H}, \ldots, B^m \sqrt{B_H} \right)$ follows.

Since the equivalence of $\text{range} \left( \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right)$ and $\text{range} \left\{ C_0, C_1, \ldots, C_m \right\}$ follows very similarly, we only give the key steps: Note that the term of $C_{j+1}$ with the leading maximum exponent of $B_A$ is of the form $(-B_A)^{j+1} \sqrt{B_H}$. This allows to compute $C_{m+1} - (-B_A)^{m+1} \sqrt{B_H} = \sum_{j=0}^{m} B_A^j \sqrt{B_H} X_j$, with appropriate matrices $X_j \in \mathbb{C}^{n \times n}$ and, e.g., $X_m = (-1)^m(m+1)B_A$. Therefore

\[
\text{range} \left( C_{m+1} - (-B_A)^{m+1} \sqrt{B_H} \right) \subset \text{range} \left( \sqrt{B_H}, B_A \sqrt{B_H}, \ldots, B_A^m \sqrt{B_H} \right)
\]

and the rest of the proof is as before. \[\square\]

Next we shall present two (equivalent) variants of Definition 2.1 for the hypocoercivity index that are related to similar concepts in the literature: In [7] §2, the definition of the hypocoercivity index of a
degenerate Fokker-Planck equation with linear drift involves a positive stable matrix \( B \) and its Hermitian part \( B_H \geq 0 \) (rather than the Hermitian part and anti-Hermitian part of \( B \) as in Definition 2.1 here). In order to connect these two situations we shall establish the equivalence of these two definitions in the subsequent lemma.

The condition \( T_m > 0 \) from Definition 2.1 can also be related to Hörmander’s “rank \( r \)” bracket condition for hypoellipticity, cf. [15]. In particular, in [19] iterated commutators were used to establish hypocoercivity of kinetic PDEs by constructing an appropriate Lyapunov functional. In Lemma 2.4 Equation (2.5c), below we shall mimic condition (3.5) of [19] for the ODE-system (1.1).

**Lemma 2.4.** Let \( B \in \mathbb{C}^{n \times n} \) be positive conservative–dissipative. Consider the following three hypocoercivity conditions:

\begin{align*}
\exists m \in \mathbb{N}_0 : \quad T_m &:= \sum_{j=0}^{m} B_A^j B_H (B_A^*)^j > 0, \quad (2.5a) \\
\exists m \in \mathbb{N}_0 : \quad \widetilde{T}_m &:= \sum_{j=0}^{m} B^j B_H (B^*)^j > 0, \quad (2.5b) \\
\exists m \in \mathbb{N}_0 : \quad \hat{T}_m &:= \sum_{j=0}^{m} C_j^* C_j > 0 \text{ with } C_0 := \sqrt{B_H}; \quad C_{j+1} := [C_j, B_A], \ j \in \mathbb{N}_0. \quad (2.5c)
\end{align*}

The conditions (2.5a)–(2.5c) are equivalent in the sense that, if there exists \( m \in \mathbb{N}_0 \) such that one condition holds, then the other two conditions hold as well for the same \( m \).

**Proof.** According to [6, Lemma 2.3], each of these three matrix inequalities (2.5a)–(2.5c) is equivalent to the corresponding Kalman rank conditions (2.3a)–(2.3c) where we used in the last case that \( C_j^* = C_j \) (verifiable by a simple induction). Moreover, the Kalman rank conditions (2.3a)–(2.3c) are equivalent due to Lemma 2.3.

This lemma shows that the hypocoercivity index can equally be defined as the smallest integer \( m \in \mathbb{N}_0 \) such that \( \widetilde{T}_m > 0 \) or \( \hat{T}_m > 0 \). Hence, it also gives the maximum number of iterated commutators of \( B_H \geq 0 \) with the matrix \( B_A \) such that their ranges span all of \( \mathbb{C}^n \) – in the spirit of Hörmander’s hypoellipticity theorem.

As an example we consider two matrices with the same Hermitian part \( B_H = \text{diag}(0, 0, 1, 1) \) such that \( \text{rank}(\ker(B_H)) = 2 \) and two different anti-Hermitian parts such that one and, respectively, two iterated commutators are needed:

**Example 2.5.** Consider \( B^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \) such that \( B_A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \). \( \sqrt{B_H^{(1)}}, B_A^{(1)} \) is given by \( \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \).
and

\[ B^{(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ such that } B^{(2)}_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad [\sqrt{B^{(2)}_H}, B^{(2)}_A] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

Hence \( \text{rank } \{ \sqrt{B^{(1)}_H}, [\sqrt{B^{(1)}_H}, B^{(1)}_A] \} = 4, \) but \( \text{rank } \{ \sqrt{B^{(2)}_H}, [\sqrt{B^{(2)}_H}, B^{(2)}_A] \} = 3. \) Thus, by Lemma 2.3, \( m_{HC}(B^{(1)}) = 1 \) and \( m_{HC}(B^{(2)}) = 2. \)

2.2 Short-time decay of conservative–dissipative ODE systems

Here we shall prove that the hypocoercivity index of a conservative–dissipative ODE system characterizes the decay of its solutions for short time. We denote the solution semigroup pertaining to (1.1) by \( P(t) := e^{-Bt} \in \mathbb{C}^{n \times n}, \) and its spectral norm by \( \| P(t) \|_2. \) Its short-time decay is related to the hypocoercivity index as follows:

**THEOREM 2.6.** Let the ODE system (1.1) be conservative–dissipative with coercive or hypocoercive matrix \( B. \) Its (finite) hypocoercivity index is \( m_{HC} \in \mathbb{N}_0 \) iff

\[ \| P(t) \|_2 = 1 - ct^a + O(t^{a+1}) \quad \text{ for } t \to 0+, \]  

(2.6)

with \( a = 2m_{HC} + 1 \) and some \( c > 0. \)

**Proof.** For the “\( \Rightarrow \) direction”, let the hypocoercivity index of \( B \) be \( m_{HC}. \) To prove (2.6) we need to estimate \( \| P(t) \|_2^2 = \lambda_{\max}(Q(t)) \) for \( t \geq 0, \) i.e. the largest eigenvalue of the matrix

\[ Q(t) := P^*(t) P(t) = e^{B^*t} e^{-Bt} \]

\[ = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{t^j}{j! (j - k)!} (-B^*)^k (-B)^{j-k} \]

\[ = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \frac{t^j}{j!} \left( \begin{array}{c} j \\ k \end{array} \right) (-B^*)^k (-B)^{j-k}. \]

(2.7)

The Taylor series for the matrix family \( Q(t) \) converges uniformly on bounded \( t \)-intervals, hence we have

\[ \| Q(t) \|_2 = \| Q_j(t) \|_2 + O(t^{j+1}), \]  

(2.8)

where \( Q_j(t) \) denotes the above Taylor expansion, but truncated after the \( t^j \)-term.

**Step 1:** Let the matrices \( U_j, j \in \mathbb{N} \) denote the coefficients of \( \frac{t^j}{j!} \) in the Taylor expansion (2.7),

\[ U_j := (-1)^j \sum_{k=0}^{j} \left( \begin{array}{c} j \\ k \end{array} \right) (B^*)^k B^{j-k}. \]

(2.9)
$U_j$ consists of a sum of $j$-fold matrix products. Due to Lemma A.3, each $U_j$ is Hermitian and satisfies for $j \in \mathbb{N}_0$

$$U_{j+1} := (-1)^{j+1}2 \sum_{k=0}^{j} \binom{j}{k} (B^*)^k B_H B^{j-k}. \quad (2.10)$$

**Step 2:** We start with the coercive case $m_{HC} = 0$: Using $Q_1(t) = U_0 + U_1 t = I - 2B_H t$, we have for $t > 0$

$$\|Q(t)\|_2 = \|I - 2B_H t\|_2 + O(t^2) = 1 - 2\lambda_1 t + O(t^2),$$

where $\lambda_1 > 0$ is the smallest eigenvalue of $B_H > 0$. Hence, the result (2.6) follows with $\|P(t)\|_2 = 1 - \lambda_1 t + O(t^2)$.

**Step 3:** For the hypocoercive case $m_{HC} > 0$, we recall that $m = m_{HC}$ is the smallest integer such that the Kalman rank condition (2.3b) holds; due to Definition 2.1, Lemma 2.4 and [6, Lemma 2.3]. Hence, there exists a normalized vector

$$x_0 \in \ker(B_H) = \ker(\sqrt{B_H})$$

with $\sqrt{B_H} B x_0 = \ldots = \sqrt{B_H} B^{m_{HC}-1} x_0 = 0$ and $\sqrt{B_H} B^{m_{HC}} x_0 \neq 0$, \quad (2.11)

but none that would satisfy instead also $\sqrt{B_H} B^{m_{HC}} x_0 = 0$. For $1 \leq j \leq 2m_{HC}$, we write (2.10) as

$$U_j = (-1)^j/2 \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{j-1}{k} (B^*)^k B_H B^{j-1-k} + \sum_{k=\lfloor j/2 \rfloor+1}^{j-1} \binom{j-1}{k} (B^*)^k B_H B^{j-1-k},$$

where $\lfloor j/2 \rfloor \leq \lfloor \frac{2m_{HC}-1}{2} \rfloor = m_{HC} - 1$. Hence, all summands of $x_0^* U_j x_0$ get annihilated due to (2.11) (or its transpose). By contrast, for $j = 2m_{HC} + 1$ we have $\lfloor j/2 \rfloor = m_{HC}$ such that

$$x_0^* U_{2m_{HC}+1} x_0 = (-1)^{2m_{HC}+1} 2 \left( \frac{2m_{HC}}{m_{HC}} \right) x_0^* (B^*)^{m_{HC}} B_H B^{m_{HC}} x_0 < 0, \quad (2.12)$$

holds, due to $B_H \geq 0$ and (2.11).

**Step 4:** For $m_{HC} \geq 1$, $a := 2m_{HC} + 1 \geq 3$. For $t > 0$ small enough, we have $\|Q_a(t)\|_2 = \lambda_{\max}(Q_a(t)) = \max_{\|x\|=1} x^* Q_a(t) x$, where

$$Q_a(t) = I - 2B_H t + 2[B_H B + B^* B_H] t^2 - 2[B_H B^2 + 2B^* B_H B + (B^*)^2 B_H] t^3 + \ldots + U_a t^a, \quad (2.13)$$

which shows the explicit form of $U_0$, $U_1$, $U_2$, $U_3$. For each fixed $x \neq 0$, the first non-vanishing coefficient of $t_j^a$, for some $j \geq 1$ in (2.13), i.e. $x^* U_j x$, has to be negative, since (1.1) is assumed to be conservative–dissipative. Also, for each fixed $x \neq 0$, the index of this first non-vanishing coefficient (after $x^* I x$) satisfies $j \leq a$, which can be seen inductively from Step 3: If $x^* U_j x = 0$ we have $x \in \ker(B_H)$, and hence $x^* U_2 x = 0$ due to (2.10), $x^* U_3 x = -4x^* B^* B_H B x$. Then, either $x^* U_3 x \neq 0$ or $x \in \ker(\sqrt{B_H} B)$. 


This argument can be continued up to, at maximum, \( j = a \). There, \( x \) would satisfy \((2.11)\) and hence \((2.12)\) would hold.

It is now easy to see that vectors satisfying \((2.11)\) give a lower bound of \( \|Q_a(t)\|_2 \); they yield the largest number of vanishing coefficients right after the term \( I \). This gives

\[
\|Q_a(t)\|_2 = \max_{x \in \mathbb{C}^n : \|x\|=1} x^*Q_a(t)x \geq 1 + \max_{\|x\|=1, \text{satisfying } (2.11)} \left( -2 \left( \frac{2m_{HC}}{m_{HC}} \right)^a \right) B_B^*B_B^m_{HC}x_0^t a!
\]

Hence, \( \|Q(t)\|_2 \geq 1 - \hat{c}t^a + \mathcal{O}(t^{a+1}) \) for some \( \hat{c} > 0 \) and sufficiently small \( t \geq 0 \) due to \((2.8)\). Due toLemma A.1 we also have \( \|Q(t)\|_2 \leq 1 - \hat{ct}^a + \mathcal{O}(t^{a+1}) \) for some \( \hat{c} > 0 \) and sufficiently small \( t \geq 0 \). Moreover, there exists a time \( t_0 > 0 \) and a real analytic function \( \Phi : [0, t_0] \to \mathbb{R} \) such that \( \|Q(t)\|_2 = \Phi(t) \) for all \( t \in [0, t_0] \), e.g. see [16, Lemma 1]. This proves \((2.6)\).

For the reverse direction let \( P(t) \) decay like \((2.6)\) for some \( a \in 2\mathbb{N}_0 + 1 \). Assume now that the hypocoercivity index \( m_{HC} \) of \( B \) would be different from \((a − 1)/2 \). By the first part of this proof, this would imply a different decay behavior than assumed, hence contradicting \((2.6)\). This finishes the second direction of the proof.

\[\square\]

**Remark 2.7.** Special cases of the above theorem were pointed out to us by Laurent Miclo: In §1 of [18] the short-time decay behavior of the Goldstein-Taylor model (a linear transport equation with relaxation term) was determined as \( 1 − \frac{t^3}{3} + o(t^3) \). Actually, this model is a PDE. But since it is considered on a torus in \( x \), each of its spatial Fourier modes (except of the 0-mode) satisfies a conservative–dissipative ODE system with hypocoercivity index 1 (see [1] for details of this modal decomposition). Hence, mode by mode, the result from [18] is an example for Theorem 2.6. For closely related BGK-models with hypocoercivity index 2 and 3 we refer to [2].

In [14] the short-time decay behavior of a kinetic Fokker-Planck equation on the torus in \( x \) was computed as \( 1 − \frac{t^3}{12} + o(t^3) \). Again, in Fourier space and by using a Hermite function basis in velocity, this model can be written as an (infinite dimensional) conservative–dissipative system with hypocoercivity index 1 (see §2.1 of [14]). In that paper it was also mentioned that the decay exponent in \((2.6)\) can be seen as some “order of hypocoercivity” of the generator.

For degenerate Fokker-Planck equations, the hypocoercivity index can also be related to the regularization rate for short times: In [19, Theorem A.12] the regularization of initial data from a weighted \( L^2 \) space into a weighted \( H^1 \) space is derived, and in [19, Theorem A.15], [7, Theorem 4.8] it is generalized to entropy functionals and their corresponding Fisher informations. In all these cases the regularization rate is \( t^{-a} \) with \( a = 2m_{HC} + 1 \) (somewhat related to Theorem 2.6 above).

### A Auxiliary results to prove short-time decay of propagator norm

To finish the proof of Theorem 2.6 we prove the following upper bound for the propagator norm:

**Lemma A.1.** Let the ODE system \((1.1)\) be conservative–dissipative, and let the system matrix \( B \) be (hypo)coercive with hypocoercivity index \( m_{HC} \in \mathbb{N}_0 \). Then, there exists \( c > 0 \) such that

\[
\|e^{-Bt}\|_2 \leq 1 - ct^a + \mathcal{O}(t^{a+1}) \quad \text{for } t \to 0^+,
\]

\[\text{(A.1)}\]
where \( a = 2m_{HC} + 1 \).

The proof of this lemma is deferred to the end of Appendix [A].

The propagator norm of an ODE (1.1) is usually not given by the norm of a specific solution, rather it is the envelope of the norm of a family of solutions, see e.g. [5] and Figure 1.

**EXAMPLE A.2.** We consider ODE (1.1) with matrix

\[
B = \begin{pmatrix}
1 & -3/10 \\
3/10 & 0
\end{pmatrix}.
\tag{A.2}
\]

The eigenvalues of \( B \) are \( \lambda_1 = 1/10 \) and \( \lambda_2 = 9/10 \), and the eigenvalues of \( B_H \) are 0 and 1. Thus, matrix \( B \) is hypocoercive with hypocoercivity index \( m_{HC} = 1 \). Following (the first part of) the proof of Theorem 2.6 solutions starting in \( x_0 \) satisfying (2.11) are used to establish the desired lower bound of the propagator norm. The kernel of \( B_H \) is one-dimensional and it is spanned by the normalized vector \( x_0 = (0, 1) \). The solution of (1.1) with initial condition \( x(0) = x_0 \) is given by

\[
x(t) = \frac{1}{8} \begin{pmatrix} 3 & -3 \\ 9 & -1 \end{pmatrix} \begin{pmatrix} e^{-t/10} \\ e^{-9t/10} \end{pmatrix}
\]

and its squared norm satisfies

\[
\|x(t)\|^2_2 = \frac{45}{32} e^{-t/5} - \frac{9}{16} e^{-t} + \frac{5}{32} e^{-9t/5} \sim 1 - 0.06 t^3 + \mathcal{O}(t^4) \quad \text{for } t \to 0^+.
\]

By contrast, due to [5, Proposition 4.2], the squared propagator norm satisfies

\[
\|e^{-Bt}\|^2_2 \leq e^{-t} \left( \sqrt{(25 \cosh(8t/10) - 9)^2 - 16^2} + 25 \cosh(8t/10) - 9 \right)
\]

\[
\sim 1 - 0.015 t^3 + \mathcal{O}(t^4) \quad \text{for } t \to 0^+.
\]

Thus the propagator norm decays slower than the solution starting at the vector \( x_0 \) which satisfies (2.11) with \( m_{HC} = 1 \), see also Figure 1.

To understand, why the asymptotic expansion of the propagator norm is of the form (2.6), we take a closer look at its Taylor expansion: The norm of the propagator \( e^{-Bt} \) for (1.1) satisfies \( \|e^{-Bt}\|^2_2 = \lambda_{\max}(Q(t)) \) for \( t \geq 0 \), where

\[
Q(t) = e^{-B^*t}e^{-Bt} = \sum_{j=0}^{\infty} \frac{t^j}{j!} U_j \quad \text{with } U_j = (-1)^j \sum_{k=0}^{j} \binom{j}{k} (B^*)^k B^{l-k}, \quad j \in \mathbb{N}_0.
\]

Thus, we observe the following properties of the coefficients \( U_j \):

**LEMMA A.3.** Let \( B \in \mathbb{C}^{n \times n} \) be positive conservative–dissipative. For \( j \in \mathbb{N}_0 \), the matrices

\[
U_j : = \sum_{k=0}^{j} \binom{j}{k} (-B^*)^k (-B)^{l-k} \tag{A.3}
\]

are Hermitian matrices in \( \mathbb{C}^{n \times n} \).
Figure 1: Comparison between the squared propagator norm (red line), the squared norm of the solution of (1.1) starting at $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (blue line) and another solution (green line) of (1.1) with matrix $B$ in (A.2).
If statement follows from a direct computation:

\[ U_{j+1} = -B^*U_j - U_j B. \]  \hfill (A.4)

Consequently, if \( x \in \ker U_j \) for some \( j \in \mathbb{N} \) then \( x^*U_{j+1}x = 0 \) and \( x^*U_{j+2}x = 2(Bx)^*U_j Bx \).

(b) Moreover, the matrices \( U_{j+1}, j \in \mathbb{N}_0 \) satisfy

\[ U_{j+1} = \sum_{k=0}^{j} \binom{j}{k} (-B^*)^k U_1 (-B)^{j-k}. \]  \hfill (A.5)

Consequently, if \( x \in \bigcap_{k=1}^{j} \ker U_{2k-1} \) for some \( j \in \mathbb{N} \) then \( B^{j-1}x \in \ker U_1 = \ker B_H \) and \( x^*U_{2j+1}x = (\binom{2j}{j}) (B^j x)^*U_1 B^j x \leq 0 \).

**Proof.** The matrices \( U_j, j \in \mathbb{N}_0 \), are Hermitian matrices in \( \mathbb{C}^{n \times n} \) since

\[ U_j^* = \left( \sum_{k=0}^{j} \binom{j}{k} (-B^*)^k (-B)^{j-k} \right)^* = \sum_{k=0}^{j} \binom{j}{k} (-B^*)^{j-k} (-B)^k = U_j, \]

due to \( \binom{j}{k} = \binom{j}{j-k} \). We prove identity \((A.4)\) by induction: At the induction start, for \( j = 0 \), we have \( U_0 = I \) and \( U_1 = (-B^*) + (-B) = (-B^*)U_0 + U_0(-B) \). We assume statement \((A.4)\) for \( j - 1 \in \mathbb{N} \), and prove the identity \((A.4)\) for \( j \):

\[ (-B^*)U_j + U_j (-B) \]

\[ = \sum_{k=0}^{j} \binom{j}{k} (-B^*)^{k+1} (-B)^{j-k} + \sum_{k=0}^{j} \binom{j}{k} (-B^*)^k (-B)^{j+1-k} \]

\[ = \sum_{k=1}^{j+1} \binom{j}{k-1} (-B^*)^k (-B)^{j+1-k} + \sum_{k=0}^{j} \binom{j}{k} (-B^*)^k (-B)^{j+1-k} \]

\[ = \sum_{k=1}^{j} \left( \left( \binom{j}{k-1} + \binom{j}{k} \right) (-B^*)^k (-B)^{j+1-k} + (-B^*)^{j+1} + (-B)^{j+1} \right) \]  \hfill (A.6)

\[ = \sum_{k=1}^{j} \binom{j+1}{k} (-B^*)^k (-B)^{j+1-k} + (-B^*)^{j+1} + (-B)^{j+1} \]

\[ = \sum_{k=0}^{j+1} \binom{j+1}{k} (-B^*)^k (-B)^{j+1-k} \]

\[ = U_{j+1}. \]

If \( x \in \ker U_j \) for some \( j \in \mathbb{N} \) then \( x^*U_{j+1}x = 0 \) follows now from identity \((A.4)\). The second part of the statement follows from a direct computation:

\[ x^*U_{j+2}x = x^* \left( (-B^*)U_{j+1} + U_{j+1} (-B) \right) x \]

\[ = x^* \left( (-B^*)^2 U_j + 2(-B^*)U_j (-B) + U_j (-B)^2 \right) x \]

\[ = 2(Bx)^*U_j (-Bx), \]
where we used identity (A.4) to express $U_{j+2}$ and $U_{j+1}$, as well as $x \in \ker U_j$.

Next, we prove the representation (A.5). For all $j \in \mathbb{N}_0$, we observe that

$$U_{j+1} = (-B^*)U_j + U_j(-B) = \sum_{k=0}^{j} \binom{j}{k} (-B^*)^{k+1}(-B)^{j-k} + \sum_{k=0}^{j} \binom{j}{k} (-B^*)^k(-B)^{j+1-k}$$

$$= \sum_{k=0}^{j} \binom{j}{k} (-B^*)^{k} [(-B^*) + (-B)](-B)^{j-k} = \sum_{k=0}^{j} \binom{j}{k} (-B^*)^k U_1(-B)^{j-k}.$$ 

We verify the first part of the final statement via mathematical induction: At the induction start, for $j = 1$, we have $x \in \ker U_1$ and $(-B)^0 x = x \in \ker U_1$. We assume the statement for $j \in \mathbb{N}$, and prove it for $j + 1$: Due to the assumption $x \in \bigcap_{k=1}^{j+1} \ker U_{2k-1}$ and the induction hypothesis we deduce that

$$B^{k-1}x \in \ker U_1 = \ker B_H \quad \forall k \in \{1, \ldots, j\}. \quad (A.7)$$

Then, $x \in \bigcap_{k=1}^{j+1} \ker U_{2k-1}$, identity (A.5), and (A.7) imply that

$$0 = x^* U_{2j+1} x = \sum_{k=0}^{2j} \binom{2j}{k} x^* (-B^*)^{k} U_1(-B)^{2j-k} x = \left( \binom{2j}{j} \right) (B^j x)^* U_1 B^j x. \quad (A.8)$$

Thus, $B^j x \in \ker U_1$. The second part of the final statement follows directly: Under the assumption $x \in \bigcap_{k=1}^{j} \ker U_{2k-1}$, we conclude from (A.7) and (A.8) that

$$x^* U_{2j+1} x = \left( \binom{2j}{j} \right) (B^j x)^* U_1 B^j x \leq 0,$$

since $U_1 = -2B_H$ is a negative semi-definite matrix. \hfill $\square$

Lemma (A.3) shows for initial conditions $x_0$ satisfying (2.11) that for the corresponding solutions $x(t)$ of (1.1) only a bound like $\|x(t)\|_2^2 \leq 1 - cr^{2m_{HC}+1} + O(r^{2m_{HC}+2})$ for some $c > 0$ is to be expected. Since the propagator norm is the envelope of all solution norms, the coefficient $c$ may, theoretically, vanish. Next, we show that this can not happen and we prove the upper bound (A.1):

Proof of Lemma (A.1) First we note that the hypocoercivity of $B$ implies $B_H \not= 0$ due to (1.6). Following the proof of Theorem (2.6) we consider $\|e^{-B^j t}\|_2^2 = \lambda_{\max}(Q(t))$ for small $t > 0$, where

$$Q(t) = e^{-B^j t} e^{-B^j t} = \sum_{j=0}^{\infty} \frac{t^j}{j!} U_j$$

with

$$U_j = (-1)^j \sum_{k=0}^{j} \binom{j}{k} (B^*)^k B^{j-k}, \quad \text{satisfying } \|U_j\| \leq (2\|B\|)^j, \quad j \in \mathbb{N}_0. \quad (A.10)$$

To compute $\lambda_{\max}(Q(t)) = \max_{\|x\|_2 = 1} x^* Q(t) x$, we consider the $t$-dependent function $g(x; t) := x^* Q(t) x - 1$ with $x$ in the sphere $S := \{ x \in \mathbb{C}^n \mid \|x\|_2 = 1 \}$. For $a = 2m_{HC} + 1$, we denote the Taylor series for $Q(t)$
and \( g(t) \) truncated after the \( t'/a! \) term with \( Q_a(t) \) and \( g_a(x; t), \) respectively. We recall that \( U_0 = I \) and \( U_1 = -2B_H. \)

First we outline the strategy of the proof, say for the case \( m_{HC} = 1, \) i.e. \( a = 3: \) If \( x \in \ker U_1 = \ker B_H \) with \( ||x|| = 1, \) then Lemma A.3 and (2.11) imply \( x^*U_3x = 0, \) \( x^*U_3x < 0 \) for \( t > 0. \) By contrast, if \( x \notin \ker U_1, \) we have \( x^*U_1x = -2x^*B_Hx < 0. \) Hence, for \( x \notin \ker U_1, \)

\[
g_a(x; t) = -\hat{c}t + \mathcal{O}(t^2) \leq -\hat{c}t^3 \quad \text{for} \quad t \to 0^+
\]

follows for some \( \hat{c}, \hat{c} > 0 \) that depend on \( x. \) Since \( g_a(x; t) \) depends continuously on \( x, \) it is possible to combine these two estimates with a constant \( c \) that is independent of \( x \in S. \) Since \( (\ker U_1)^c \cap S \) is not compact, we do not obtain a uniform estimate “automatically”. So, the key aspect is here to obtain a uniform decay estimate for \( x \) “close to \( \ker U_1 \)”, in the sense that \( -\epsilon \leq x^*U_1x \leq 0. \)

**Step 1.** Matrices with hypo coercivity index \( m_{HC} = 1. \) We suppose that matrix \( B \) has hypo coercivity index \( m_{HC} = 1, \) i.e. \( a = 3. \) Our goal is to estimate \( g(x; t) \) on \( S \) where we decompose \( S = C_0 \cup C_1 \) into the sets

\[
C_0 := \{x \in S \mid x^*U_1x \leq -\epsilon\}, \quad C_1 := \{x \in S \mid -\epsilon \leq x^*U_1x\}, \quad (A.11)
\]

for some positive parameter \( \epsilon \) to be determined later.

**Step 1a.** For all \( x \in C_0 \) and \( t \in [0,1], \) we deduce

\[
g(x; t) := x^* \left( \sum_{j=1}^{\infty} \frac{t^j}{j!} U_j \right)x = tx^*U_1x + \sum_{j=2}^{\infty} \frac{t^j}{j!} x^*U_jx \leq -\epsilon t + M_2 t^2, \quad (A.12)
\]

since \( \| \sum_{j=2}^{\infty} \frac{t^j}{j!} x^*U_jx \| \leq t^2 M_2 \) with \( M_2 := \sum_{j=2}^{\infty} \frac{1}{j!} (2\|B\|)^j. \) Then, \( -\epsilon t + M_2 t^2 \leq -\frac{\epsilon}{2M_2} t^2 \) for all \( 0 \leq t \leq \frac{\epsilon}{2M_2}. \) For any given \( c > 0, \) estimate \( -\frac{\epsilon}{2} t \leq -\epsilon t^3 \) holds if \( 0 \leq t \leq \sqrt{\frac{\epsilon}{2c}}. \) Thus, we obtain

\[
g(x; t) \leq -\epsilon t^3 \quad \text{for all} \quad x \in C_0 \quad \text{and} \quad t \in [0,t_0], \quad (A.13)
\]

with \( t_0 := \min \{ \frac{\epsilon}{2M_2}, \sqrt{\frac{\epsilon}{2c}} \}. \)

**Step 1b.** For \( x \in C_1 \) and \( t \in [0,1], \) the key idea (to estimate \( g(x; t) \)) is to collect the terms \( t^j \) of order \( j \) less than \( a = 2m_{HC} + 1 = 3 \) in a quadratic form which is non-positive. Therefore, we use (A.5) and Lemma A.4 with \( U = -B^*, V = U_1, W = -B \) and \( m = 0, \) to derive

\[
g(x; t) = x^* \left( \sum_{j=1}^{\infty} \frac{t^j}{j!} U_j \right)x
\]

\[
= \sum_{j=1}^{\infty} \sum_{k=0}^{j-1} \frac{1}{j!} \binom{j-1}{k} ((-B)^k x)^*U_1(-B)^{j-k-1}x
\]

\[
= t \left( \sum_{k=0}^{\infty} \frac{1}{(k+1)!} t^k ((-B)^k x)^* U_1 \left( \sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)!} t^\ell (-B)^\ell x \right) \right)
\]

\[
+ \frac{t^3}{3!} \frac{1}{4} ((-B)x)^*U_1(-B)x + \sum_{j=4}^{\infty} \frac{t^j}{j!} \sum_{k=1}^{j-2} \binom{j-1}{k} \Delta_{j,k}^{(1)} ((-B)^k x)^*U_1(-B)^{j-k-1}x
\]

\[
\leq \frac{t^3}{3!} ((-B)x)^*U_1(Bx) t^3 + M_4 t^4,
\]
where \( M_4 := \sum_{j=0}^{\infty} \frac{1}{j!} \| B \|^j \). In the final estimate we used \( U_1 \leq 0 \), and hence the first term is non-positive. Moreover, (A.5), (A.10), and \( \Delta_{i,k}^{(l)} < 1 \) are used to estimate the third term.

To establish a uniform negative upper bound for \((Bx)^* U_1 Bx\) for all \( x \in C_1 \), we consider the orthogonal projection \( \Pi_0 \) onto the \( \ker U_1 = \ker B_H \) and decompose the vector \( C_1 \ni x = x_1 + x_2 \) such that \( x_1 \in \ker U_1 \) and \( x_2 \in (\ker U_1)^\perp \), i.e. \( x_1 = \Pi_0 x \). The Hermitian matrix \( U_1 = -2B_H \) is negative semi-definite, hence its largest non-zero eigenvalue \( \lambda_{\max}^{U_1} < 0 \) is negative. Therefore,

\[
(Bx)^* U_1 Bx \leq \lambda_{\max}^{U_1} \| (I - \Pi_0) B(x_1 + x_2) \|^2. \tag{A.15}
\]

Using \( \| y - z \| \geq \| y \| - \| z \| \) for all \( y, z \in \mathbb{C}^n \), we derive

\[
\| (I - \Pi_0) B(x_1 + x_2) \| \geq \| (I - \Pi_0) Bx_1 \| - \| (I - \Pi_0) Bx_2 \|.
\]

On the one hand, for \( x \in C_1 \), we observe \( -\epsilon \leq x^* U_1 x = x_2^* U_1 x_2 \leq \lambda_{\max}^{U_1} \| x_2 \|^2 \leq 0 \), which implies

\[
\| x_2 \|^2 \leq \frac{\epsilon}{\lambda_{\max}^{U_1}}, \quad \text{hence} \quad \| (I - \Pi_0) Bx_2 \| \leq \| (I - \Pi_0) \| \| B \| \| x_2 \| \leq \| B \| \frac{\epsilon}{\lambda_{\max}^{U_1}}.
\]

On the other hand, for \( x \in C_1 \), it follows that \( 1 \geq \| x_1 \|^2 = 1 - \| x_2 \|^2 \geq 1 - \frac{\epsilon}{\lambda_{\max}^{U_1}} \). Moreover,

\[
\| (I - \Pi_0) Bx_1 \| \geq \frac{1}{\| \sqrt{B_H} \|} \| \sqrt{B_H} (I - \Pi_0) Bx_1 \| = \frac{1}{\| \sqrt{B_H} \|} \| \sqrt{B_H} Bx_1 \| \geq \frac{1}{\sqrt{\lambda_{\max}^{B_H}}} \| \sqrt{B_H} Bx_1 \|,
\]

where \( \lambda_{\max}^{B_H} > 0 \) is the largest eigenvalue of \( B_H \). Finally, we estimate \( \| \sqrt{B_H} Bx_1 \| \geq \beta_\epsilon \) where \( \beta_\epsilon := \min_{x \in C_1} \| \sqrt{B_H} Bx \| \) is a continuous, decreasing function with respect to \( \epsilon \) (since the set \( C_1 \) grows with \( \epsilon \)).

Estimate of \( \beta_\epsilon \): First, we consider \( \beta_0 \) and prove that \( \beta_0 > 0 \). For \( \epsilon = 0 \), we have \( C_1 = S \cap \ker B_H \) such that all \( z \in C_1 \) satisfy (2.11). Therefore, \( \| \sqrt{B_H} Bz \| \leq \beta_0 \). Since \( C_1 \) is a compact set, \( \| \sqrt{B_H} Bz \| \) has a uniformly positive lower bound for all \( z \in C_1 \), i.e. \( \beta_0 = \min_{z \in C_1} \| \sqrt{B_H} Bz \| > 0 \). Due to continuity of \( \beta_\epsilon \), we have for all \( \epsilon > 0 \) sufficiently small, \( \beta_\epsilon > 0 \). Therefore, it is possible to choose \( \epsilon_0 > 0 \) sufficiently small such that

\[
c_0 := \frac{\beta_\epsilon}{\sqrt{\lambda_{\max}^{B_H}}}, \quad \frac{\epsilon_0}{\sqrt{\lambda_{\max}^{U_1}}}, \quad \text{hence},
\]

\[
\| (I - \Pi_0) B(x_1 + x_2) \| \geq \| (I - \Pi_0) Bx_1 \| - \| (I - \Pi_0) Bx_2 \| \geq c_0 > 0 \text{.}
\]

Together with (A.14) and (A.15), we obtain

\[
g(x; t) \leq -ct^3 + M_4 t^4 \quad \text{for all } x \in C_1, \tag{A.16}
\]

where \( c := \frac{\lambda_{\max}^{U_1} \epsilon_0^2}{12} > 0 \). For \( m_{HC} = 1 \), estimate (A.1) follows now from (A.13) and (A.16).
Step 2. Matrices with hypocoercivity index \( m_{HC} \geq 2 \). For matrices \( B \) with hypocoercivity index \( m_{HC} \geq 2 \), we generalize this procedure as follows: We decompose \( S \) into the (non-disjoint) closed subsets

\[
C_0 := \{ x \in S \mid x^t U_1 x \leq -\epsilon \}, \\
C_1 := \{ x \in S \mid -\epsilon \leq x^t U_1 x \wedge (B x)^t U_1 B x \leq -\epsilon \}, \\
C_2 := \{ x \in S \mid -\epsilon \leq x^t U_1 x \wedge -(B x)^t U_1 B x \wedge (B^2 x)^t U_1 B^2 x \leq -\epsilon \}, \\
\vdots \\
C_m := \{ x \in S \mid \forall k \in \{0, \ldots, m - 1 \} : -\epsilon \leq (B^k x)^t U_1 B^k x \wedge (B^m x)^t U_1 B^m x \leq -\epsilon \}, \\
\vdots \\
C_{m_{HC} - 1} := \{ x \in S \mid \forall k \in \{0, \ldots, m_{HC} - 2 \} : -\epsilon \leq (B^k x)^t U_1 B^k x \\
\wedge (B^{m_{HC} - 1} x)^t U_1 B^{m_{HC} - 1} x \leq -\epsilon \},
\]

as well as

\[
C_{m_{HC}} := \{ x \in S \mid \forall k \in \{0, \ldots, m_{HC} - 1 \} : -\epsilon \leq (B^k x)^t U_1 B^k x \},
\]

for some positive parameter \( \epsilon \) to be determined later.

**Step 2a.** For all \( \ell \in \{0, \ldots, m_{HC} - 1\} \), \( x \in C_\ell \) and \( t \in [0,1] \), the key idea (to estimate \( g(x;t) \)) is to collect the terms \( t^j \) of order \( j \) less than \( 2\ell + 1 \) in a quadratic form which is non-positive. Therefore, we use again (A.5) and Lemma A.4 with \( U = -B^\ast \), \( V = U_1 \), \( W = -B \) and \( m = \ell - 1 \), to obtain

\[
g(x; t) = x^t \left( \sum_{j=1}^{\infty} \frac{t^j}{j!} U_j \right) x \\
= \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{k=0}^{j-1} \binom{j-1}{k} ((-B)^k x)^t U_1 (-B)^{j-k-1} x \\
= \sum_{j=0}^{\ell-1} \frac{t^{2j+1}}{(2j+1)!} \left( \sum_{k=0}^{(2j+1)!} \binom{k+j}{j} t^k ((-B)^{k+j} x)^t \right) U_1 \left( \sum_{k=0}^{(2j+1)!} \binom{k+j}{j} t^k ((-B)^{k+j} x)^t \right) \\
+ \frac{t^{2\ell+1}}{(2\ell+1)!} \Delta_{2\ell+1,\ell} (-B)^{\ell} x^t U_1 (-B)^{\ell} x \\
+ \sum_{j=2\ell+2}^{\infty} \frac{t^j}{j!} \sum_{k=0}^{j-1} \binom{j-1}{k} \Delta_{j,k} (-B)^k x^t U_1 (-B)^{j-k-1} x \\
\leq -\epsilon \left( \sum_{j=1}^{2\ell+1} \frac{t^j}{j!} \right) + M_{2\ell+2} t^{2\ell+2},
\]

using that \( U_1 = -2B_H \) is a negative semi-definite matrix, the assumption \( x \in C_\ell \), and \( \Delta_{2\ell+2} := \sum_{j=2\ell+2}^{\infty} \frac{1}{j! (2\|B\|)^j} > 0 \) (to estimate the first, second and third term, respectively). In the last step we used (A.5), (A.10), and \( \Delta_{\ell,0} \leq 1 \) for \( 0 \leq \ell \leq k \leq j - k - 1 \). For given \( \epsilon > 0 \), there exists \( t_\ell > 0 \) such that

\[
g(x; t) \leq -\epsilon t^{2m_{HC}+1} \quad \text{for all } x \in C_\ell \text{ and } t \in [0, t_\ell]. \tag{A.17}
\]

**Step 2b.** In case \( x \in C_{m_{HC}} \) and \( t \in [0,1] \), we derive as in **Step 2a:**

\[
g(x; t) \leq \frac{(B^{m_{HC}} x)^t U_1 B^{m_{HC}} x}{(2m_{HC} + 1)! \left( \frac{m_{HC}}{m_{HC}} \right)} (2m_{HC} + 1)! \left( \frac{m_{HC}}{m_{HC}} \right) t^{2m_{HC}+1} + M_{2m_{HC}+2} t^{2m_{HC}+2}, \tag{A.18}
\]
with $M_{2m_Hc+2} := \sum_{j=2m_Hc+2}^{\infty} \frac{1}{j!} (2\|B\|)^j > 0$. To establish a uniform upper bound for $(B^{m_{HC}})^* U_1 B^{m_{HC}} x$ for all $x \in C_{m_{HC}}$, we recall that there exist normalized vectors $x_0$ satisfying (2.11), but none that would satisfy instead also $\sqrt{B_H} B^{m_{HC}} x_0 = 0$. Due to Lemma A.3(b), for all vectors $x_0$ satisfying (2.11), we deduce

$$x_0^* U_{2j+1} x_0 = \begin{pmatrix} \frac{2j}{j} \end{pmatrix} (B^j x_0)^* U_1 B^j x_0 \begin{cases} 0 & \text{if } 0 \leq j \leq m_{HC} - 1, \\ < 0 & \text{if } j = m_{HC}. \end{cases}$$  

(A.19)

To treat the general case $x \in C_{m_{HC}}$, we consider the orthogonal projection $\Pi_0$ onto the ker $U_1 = \ker B_H$ and decompose the vector $C_{m_{HC}} \ni x = x_1 + x_2$ such that $x_1 \in \ker U_1$ and $x_2 \in (\ker U_1)\perp$, i.e. $x_1 = \Pi_0 x$. The Hermitian matrix $U_1 = -2B_H$ is negative semi-definite, hence its largest non-zero eigenvalue $\lambda_{\max}^{U_1} < 0$ is negative. Therefore,

$$(B^{m_{HC}})^* U_1 B^{m_{HC}} x \leq \lambda_{\max}^{U_1} \| (I - \Pi_0) B^{m_{HC}} (x_1 + x_2) \|^2.$$

(A.20)

Using $\|y - z\| \geq \|y\| - \|z\|$ for all $y, z \in \mathbb{C}^n$, we derive

$$\| (I - \Pi_0) B^{m_{HC}} (x_1 + x_2) \| \geq \| (I - \Pi_0) B^{m_{HC}} x_1 \| - \| (I - \Pi_0) B^{m_{HC}} x_2 \|.$$

On the one hand, for $x \in C_{m_{HC}}$, we observe $-\epsilon \leq x^* U_1 x = x_2^* U_1 x_2 \leq \lambda_{\max}^{U_1} \|x_2\|^2 \leq 0$, which implies that

$$\|x_2\|^2 \leq \frac{\epsilon}{|\lambda_{\max}^{U_1}|}, \quad \text{hence, } \| (I - \Pi_0) B^{m_{HC}} x_2 \| \leq \| (I - \Pi_0) \| B^{m_{HC}} \| x_2 \| \leq \| B^{m_{HC}} \| \sqrt{\frac{\epsilon}{|\lambda_{\max}^{U_1}|}}.$$

On the other hand, for $x \in C_{m_{HC}}$, it follows that $1 \geq \|x_1\|^2 = 1 - \|x_2\|^2 \geq 1 - \frac{\epsilon}{|\lambda_{\max}^{U_1}|}$. Moreover,

$$\| (I - \Pi_0) B^{m_{HC}} x_1 \| \geq \frac{1}{\sqrt{\|B_H\|}} \| \sqrt{B_H} (I - \Pi_0) B^{m_{HC}} x_1 \| = \frac{\| \sqrt{B_H} B^{m_{HC}} x_1 \|}{\sqrt{\|B_H\|}} \geq \frac{\| \sqrt{B_H} B^{m_{HC}} x_1 \|}{\sqrt{\lambda_{\max}^{B_H}}},$$

where $\lambda_{\max}^{B_H} > 0$ is the largest eigenvalue of $B_H$. Finally, we estimate $\| \sqrt{B_H} B^{m_{HC}} z \| \geq \beta_{\epsilon}$ where $\beta_{\epsilon} := \min_{z \in C_{m_{HC}}} \| \sqrt{B_H} B^{m_{HC}} z \|$ is a continuous, decreasing function with respect to $\epsilon$ (since the set $C_{m_{HC}}$ grows with $\epsilon$).

**Estimate of $\beta_{\epsilon}$:** First, we consider $\beta_0$ and prove that $\beta_0 > 0$. For $\epsilon = 0$ we have $C_{m_{HC}} = S \cap (\bigcap_{k=0}^{\infty} \ker B^k)^{-1}$ such that all $z \in C_{m_{HC}}$ satisfy (2.11). Therefore, $\| \sqrt{B_H} B^{m_{HC}} z \|^2 = (B^{m_{HC}})^* B_H B^{m_{HC}} z > 0$. Since $C_{m_{HC}}$ is a compact set, $\| \sqrt{B_H} B^{m_{HC}} z \|$ has a uniformly positive lower bound for all $z \in C_{m_{HC}}$, i.e. $\beta_0 = \min_{z \in C_{m_{HC}}} \| \sqrt{B_H} B^{m_{HC}} z \| > 0$. Due to continuity of $\beta_{\epsilon}$, we have for all $\epsilon > 0$ sufficiently small, $\beta_{\epsilon} > 0$. Therefore, it is possible to choose $\epsilon_0 > 0$ sufficiently small such that

$$c_0 := \frac{\beta_{\epsilon_0}}{\sqrt{\lambda_{\max}^{B_H}}} - \| B^{m_{HC}} \| \sqrt{\frac{\epsilon_0}{|\lambda_{\max}^{U_1}|}} > 0,$$

hence,

$$\| (I - \Pi_0) B^{m_{HC}} (x_1 + x_2) \| \geq \| (I - \Pi_0) B^{m_{HC}} x_1 \| - \| (I - \Pi_0) B^{m_{HC}} x_2 \| \geq c_0 > 0.$$
Together with (A.18) and (A.20), we obtain
\[ g(x; t) \leq -c t^{2m_{HC}+1} + M_{2m_{HC}+2} t^{2m_{HC}+2} \quad \text{for all } x \in C_{m_{HC}} \] 
(A.21)
where \( c := \frac{|\ell_{1}|^{2} + |\ell_{2}|^{2}}{2m_{HC}+1} \). For \( m_{HC} \geq 2 \), estimate (A.1) follows from (A.17) and (A.21).

This finishes the proof. \( \square \)

The proof of Lemma A.1 uses the following identity:

**LEMMA A.4.** Let \( U, V, W \in C^{nxn} \). For all \( m \in \mathbb{N}_{0} \), the following identity holds

\[
\sum_{j=1}^{\infty} \frac{t^{j}}{j!} \sum_{k=0}^{j-1} \binom{j-1}{k} U^{k} V W^{j-k-1}
\]

\[
= \sum_{j=0}^{m} \frac{t^{2j+1}}{(2j+1)!} \left( \sum_{k=0}^{(2j+1)!} \binom{2j+1}{j} \binom{k+j}{j} t^{k} U^{k+j} \right) V \left( \sum_{\ell=0}^{\infty} \frac{\ell!}{(\ell+2j+1)!} \binom{\ell + j}{j} t^{\ell} W^{\ell+j} \right)
\]

\[
+ \sum_{j=m+3}^{\infty} \frac{t^{j}}{j!} \sum_{k=m+1}^{j-2} \binom{j-1}{k} \Delta_{j,k}^{(m+1)} U^{k} V W^{j-k-1},
\]

where \( \Delta_{j,k}^{(m)} := \frac{\binom{k}{j-1}}{\binom{j}{k} \binom{j}{k-1}^{m}} \) for all \( j \geq 2m+3 \) and \( m+1 \leq k \leq j - m - 2 \).

**Proof.** We will prove the identity by induction. For \( m = 0 \), we have to prove the identity

\[
\sum_{j=1}^{\infty} \frac{t^{j}}{j!} \sum_{k=0}^{j-1} \binom{j-1}{k} U^{k} V W^{j-k-1}
\]

\[
= t \left( \sum_{k=0}^{\infty} \frac{1}{(k+1)!} t^{k} U^{k} \right) V \left( \sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)!} t^{\ell} W^{\ell} \right) + \sum_{j=3}^{\infty} \frac{t^{j}}{j!} \sum_{k=1}^{j-2} \binom{j-1}{k} \frac{k(j-k-1)}{(k+1)(j-k)} U^{k} V W^{j-k-1}
\]

since \( \Delta_{j,k}^{(1)} = \binom{j-1}{k} \binom{j}{k} = \frac{k(j-k-1)}{(k+1)(j-k)} \). The first term on the right hand side can be written by the Cauchy product formula as

\[
t \left( \sum_{k=0}^{\infty} \frac{1}{(k+1)!} t^{k} U^{k} \right) V \left( \sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)!} t^{\ell} W^{\ell} \right)
\]

\[
= t \sum_{j=0}^{\infty} \frac{1}{(j+1)!} U^{j} V \frac{1}{(j-k+1)!} W^{j-k} = \sum_{j=0}^{\infty} \frac{1}{(j-k+1)!} U^{j} V \frac{1}{(j-k+1)!} W^{j-k}
\]

\[
= \sum_{j=0}^{\infty} \frac{t^{j+1}}{(j-k+1)!} U^{j} V W^{j-k} = \sum_{j=0}^{\infty} t^{j+1} \sum_{k=0}^{j} \frac{1}{(j-k+1)!} U^{j} V W^{j-k}
\]

\[
= \sum_{j=1}^{\infty} \frac{t^{j}}{j!} \sum_{k=0}^{j-1} \frac{1}{(j-k)!} U^{k} V W^{j-k-1} = \sum_{j=1}^{\infty} \frac{t^{j}}{j!} \sum_{k=0}^{j-1} \frac{1}{(j-k)!} U^{k} V W^{j-k-1}
\]

\[
= \sum_{j=1}^{\infty} \frac{2}{j!} \sum_{k=0}^{j-1} \binom{j-1}{k} U^{k} V W^{j-k-1} + \sum_{j=3}^{\infty} \frac{t^{j}}{j!} \sum_{k=0}^{j-1} \binom{j-1}{k} \frac{j}{(j-k)!} U^{k} V W^{j-k-1}
\].
Therefore,

\[
\sum_{j=0}^\infty \left( \sum_{k=0}^\infty \frac{1}{(k+1)!} t^k U^k \right) \left( \sum_{\ell=0}^\infty \frac{1}{(\ell+1)!} t^\ell W^\ell \right) + \sum_{j=3}^\infty \frac{t^j}{j!} \sum_{k=1}^{j-2} \frac{(j-1)}{(j-k-1)(k+1)(j-k)} U^k V W^{j-k-1} = \sum_{j=1}^\infty \sum_{k=0}^{j-1} \frac{(j-1)}{(j-k)k} U^k V W^{j-k-1}.
\]

We assume that the formula holds for \( m \in \mathbb{N}_0 \) and prove it for \( m + 1 \). First, we use again the Cauchy product formula to derive

\[
\frac{t^{2m+3}}{(2m+3)!} \sum_{j=0}^\infty \frac{1}{(2m+2)!} \sum_{k=0}^{j-1} \frac{(2m+3)!}{(k+2m+3)!} \frac{1}{(k+2m+3)!} U^k V W^{j-k-1} = \frac{t^{2m+3}}{(2m+3)!} \sum_{j=0}^\infty \frac{1}{(2m+2)!} \sum_{k=0}^{j-1} \frac{(2m+3)!}{(k+2m+3)!} \frac{1}{(k+2m+3)!} U^k V W^{j-k-1}.
\]

Therefore,

\[
\sum_{j=0}^{m+1} \frac{t^{2j+1}}{(2j+1)!} \frac{1}{(2j)!} \left( \sum_{k=0}^{j+1} \frac{(2j+1)!}{(k+2j+1)!} \left( \begin{array}{c} k+j+1 \end{array} \right) \frac{1}{j} t^k U^{k+j} \right) V \left( \sum_{\ell=0}^\infty \frac{(2j+1)!}{(\ell+2j+1)!} \left( \begin{array}{c} \ell+j \end{array} \right) t^\ell W^{\ell+j} \right)
\]

\[
+ \sum_{j=m+5}^\infty \frac{t^j}{j!} \sum_{k=m+3}^{j-1} \frac{(j-1)}{(j-k)k} U^k V W^{j-k-1} = \sum_{j=0}^{m} \frac{t^{2j+1}}{(2j+1)!} \frac{1}{(2j)!} \left( \sum_{k=0}^{j+1} \frac{(2j+1)!}{(k+2j+1)!} \left( \begin{array}{c} k+j+1 \end{array} \right) \frac{1}{j} t^k U^{k+j} \right) V \left( \sum_{\ell=0}^\infty \frac{(2j+1)!}{(\ell+2j+1)!} \left( \begin{array}{c} \ell+j \end{array} \right) t^\ell W^{\ell+j} \right)
\]

\[
+ \sum_{j=m+5}^\infty \frac{t^j}{j!} \sum_{k=m+3}^{j-1} \frac{(j-1)}{(j-k)k} U^k V W^{j-k-1}.
\]

(A.23)
Using $\Delta_{j,k}^{(m+2)} = \Delta_{j,k}^{(m+1)} \frac{(k-m-1)(j-k-m-2)}{(k+m+2)(j-k+m+1)}$, we deduce that \textbf{(A.23)} equals:

\[
= \sum_{j=0}^{m} \frac{t^{2j}}{(2j+1)!} \frac{1}{(2j)!} \left( \sum_{k=0}^{\infty} \frac{(2j+1)!}{(k+2j+1)!} \left( k + j \right)^{k U^k W^j} \right) V \left( \sum_{\ell=0}^{\infty} \frac{(2j+1)!}{(\ell+2j+1)!} \left( \ell + j \right)^{\ell W^{\ell+j}} \right) \\
+ \sum_{j=2m+3}^{\infty} \frac{t^j}{j!} \sum_{k=m+1}^{j-2} \binom{j-1}{k} \Delta_{j,k}^{(m+1)} \frac{(2m+3)!}{(k+m+2)(j-k+m+1)} U^k V W^{j-k-1} \\
+ \sum_{j=2m+5}^{\infty} \frac{t^j}{j!} \sum_{k=m+2}^{j-3} \binom{j-1}{k} \frac{(j-m-1)(j-k-m-2)}{(k+m+2)(j-k+m+1)} U^k V W^{j-k-1} \\
= \sum_{j=0}^{m} \frac{t^{2j}}{(2j+1)!} \frac{1}{(2j)!} \left( \sum_{k=0}^{\infty} \frac{(2j+1)!}{(k+2j+1)!} \left( k + j \right)^{k U^k W^j} \right) V \left( \sum_{\ell=0}^{\infty} \frac{(2j+1)!}{(\ell+2j+1)!} \left( \ell + j \right)^{\ell W^{\ell+j}} \right) \\
+ \sum_{j=2m+3}^{\infty} \frac{t^j}{j!} \sum_{k=m+1}^{j-1} \binom{j-1}{k} \frac{(j-1)!}{(j-k-1)!} U^k V W^{j-k-1} \\
= \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{k=0}^{j-1} \binom{j-1}{k} U^k V W^{j-k-1} ,
\]

where we used the induction hypothesis, i.e. \textbf{(A.22)}, in the final equality. This finishes the proof. \hfill \Box

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