Characterization of some causality conditions through the continuity of the Lorentzian distance

Ettore Minguzzi\textsuperscript{a,1}

\textsuperscript{a}Dipartimento di Matematica Applicata, Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze

Abstract

A classical result in Lorentzian geometry states that a strongly causal spacetime is globally hyperbolic if and only if the Lorentzian distance is finite valued for every metric choice in the conformal class. It is proved here that a non-total imprisoning spacetime is globally hyperbolic if and only if for every metric choice in the conformal class the Lorentzian distance is continuous. Moreover, it is proved that a non-total imprisoning spacetime is causally simple if and only if for every metric choice in the conformal class the Lorentzian distance is continuous wherever it vanishes. Finally, a strongly causal spacetime is causally continuous if and only if there is at least one metric in the conformal class such that the Lorentzian distance is continuous wherever it vanishes. 2000 MSC: 53C50; 53C80; 83C75

Keywords: Lorentzian distance, time separation

1 Introduction

Let $(M, g)$ be a spacetime, the Lorentzian distance function $d : M \times M \to [0, +\infty]$ is defined by,

$$d(p, q) = \sup_{\gamma} l(\gamma)$$

where $\gamma : [a, b] \to M$ is the generic $C^1$ causal curve connecting $p$ to $q$ and $l$ is the Lorentzian length functional $l(\gamma) = \int_{a}^{b} \sqrt{-g(\dot{\gamma}, \dot{\gamma})} \, dt$. If there is no causal curve connecting $p$ to $q$ then it is understood that $d(p, q) = 0$.

\textsuperscript{1} E-mail address: ettore.minguzzi@unifi.it
The Lorentzian distance is clearly a non-conformally invariant concept as the Lorentzian length changes under multiplication of the metric by a positive function. However, given the function \( d \) the chronological relation is determined through the equivalence 
\[(x, z) \in I^+ \iff d(x, z) > 0.\]
It must therefore be possible to relate more or less directly some properties of the Lorentzian distance with the causality properties of spacetime.

As a matter of fact there can be more than one characterization of a given causality property. For instance, future distinction can be characterized as \([1, \text{Lemma 4.23}]\)

\[
\text{for each pair of distinct } p, q \in M, \text{ there is some } x \in M, \\
\text{such that exactly one of } d(p, x) \text{ and } d(q, x) \text{ is zero,}
\]
which is merely a restatement of the original definition, \( x \neq z \Rightarrow I^+(x) \neq I^+(z) \), taking into account the equivalence 
\[(x, z) \in I^+ \iff d(x, z) > 0.\]
Alternatively, there is the possibility of characterizing a spacetime as future distinguishing through the following theorem which is analogous to \([1, \text{Theorem 4.27}]\) for strong causality.

**Theorem 1.1** A spacetime is future distinguishing iff every point \( x \in M \) admits arbitrary small neighborhoods \( V \) such that the restricted distance \( d(x, \cdot)|_V : V \to [0, +\infty] \) coincides with \( d_V(x, \cdot) \) where \( d_V \) is the Lorentzian distance of spacetime \((V, g|_V)\). (Moreover, in this case \( V \) can be chosen globally hyperbolic, so that \( d(x, \cdot)|_V \) is actually finite and continuous.) An analogous past version holds.

(the proof is postponed to the end of the introduction)

Characterizations of the last type should be preferred because, although sometimes more difficult to obtain, they provide new information on some non-trivial features of the Lorentzian distance, namely its finiteness, continuity and locality properties.

The Lorentzian distance has been successfully used to characterize in this way some causality properties among which the most important are strong causality and global hyperbolicity \([1]\). In the literature no characterization can be found of causal continuity and causal simplicity, a gap which will be filled by this work.

I refer the reader to \([2]\) for most of the conventions used in this work. In particular, I denote with \((M, g)\) a \(C^r\) spacetime (connected, time-oriented Lorentzian manifold), \( r \in \{3, \ldots, \infty\} \) of arbitrary dimension \( n \geq 2 \) and signature \((- , +, \ldots, +)\). On \( M \times M \) the usual product topology is defined. For convenience and generality I often use the causal relations on \( M \times M \) in place of the more widespread point based relations \( I^+(x) \), \( J^+(x) \), \( E^+(x) \) (and past
versions). The subset symbol $\subset$ is reflexive, $X \subset X$.

With $(M, g)$ it is denoted the conformal structure namely the class of spacetimes which share the same time orientation of a representative $(M, g)$ but for which the metric may differ from $g$ by a positive conformal factor $g' = \Omega(x)g$, $\Omega > 0$. With the boldface notation $\mathbf{g}$ it is also denoted the set of metrics conformal to $g$. I shall write “spacetime $(M, g)$” by meaning with this the conformal structure.

A classical result by Beem and Ehrlich [3, Theorem 3.5], [1, Theorem 4.30], states that a strongly causal spacetime $(M, g)$ is globally hyperbolic if and only if for every $g \in \mathbf{g}$, $d_g : M \times M \to [0, +\infty]$ is finite valued where $d_g$ is the Lorentzian distance of $(M, g)$. In this case the Lorentzian distance is also continuous [1, Lemma 4.5], and moreover, whenever $(p, q) \in J^+$, it is maximized by a suitable connecting geodesic $\eta$, $d(p, q) = l(\eta)$. These good properties are lost even by considering the property of causal simplicity which stays just below globally hyperbolicity in the causal ladder of spacetimes [113]. Nevertheless, I shall prove in this work that both causal simplicity and causal continuity admit a characterization through the continuity properties of the Lorentzian distance function. Moreover, the continuity of the Lorentzian distance in all the conformal class can also be used, in the same way as the finiteness property, to characterize global hyperbolicity (see theorem 3.6).

Recall that a spacetime is causally simple if [615] it is causal and $\bar{J}^+ = J^+$, while it is causally continuous if [7] it is weakly distinguishing (i.e. $I^+(x) = I^+(y)$ and $I^-(x) = I^-(y) \Rightarrow x = y$) and reflective (i.e. $I^+(y) \subset I^+(x) \Leftrightarrow I^-(x) \subset I^-(y)$). A spacetime is future distinguishing if $I^+(x) = I^+(y) \Rightarrow x = y$. Analogously, it is past distinguishing if $I^-(x) = I^-(y) \Rightarrow x = y$. A spacetime is distinguishing if it is both future and past distinguishing. A spacetime is non-total imprisoning if no future inextendible causal curve is contained in a compact (replacing future with past gives the same property [89]). A spacetime is non-partial imprisoning if there is no inextendible causal curve which returns, in the past or future direction, indefinitely into a compact.

It will be useful to keep in mind the chain of implications: global hyperbolicity $\Rightarrow$ causal simplicity $\Rightarrow$ causal continuity $\Rightarrow$ stable causality $\Rightarrow$ strong causality $\Rightarrow$ non-partial imprisoning $\Rightarrow$ distinction $\Rightarrow$ past or future distinction $\Rightarrow$ weak distinction $\Rightarrow$ non-total imprisoning $\Rightarrow$ causality (see [10,2,9]). Finally, recall that a set is an arbitrarily small neighborhood of $x \in M$, if it can be chosen to be contained in any given neighborhood of $x$. The reader will be assumed to be familiar with the limit curve theorem given in [5] (which generalizes and strengthens the limit curve theorem given in [1]). This is the only limit curve theorem to which we shall make reference throughout the paper.
We end the section by giving the proof to theorem 1.1.

PROOF. [of theorem 1.1]
⇒. In this direction the proof has been given by the author in [11, Theorem 2.12]. It is included here for completeness.

Since $M$ is future distinguishing [2, Lemma 3.10] for every open set $U \ni x$ there is a neighborhood $V \subset U$, $V \ni x$ such that every causal curve starting from $x$ and ending at $y \in V$, is necessarily contained in $V$ (this is stated also in [10, Sect. 6.4]). As a consequence $d(x, \cdot)_{|V} : V \to [0, +\infty]$ coincides with $d_V(x, \cdot)$, where $d_V$ is the Lorentzian distance on the spacetime $(V, g|_V)$.

Moreover, the same proof [2, Lemma 3.10] shows that in this case $V$ can be chosen globally hyperbolic when regarded as a spacetime with the induced metric. Since $V$ is globally hyperbolic $d_V(x, \cdot)$ is continuous and finite and so is $d(x, \cdot)_{|V} : V \to [0, +\infty]$.

⇐. It is obvious that for every point $x$ and open neighborhood $V \ni x$, $I^+(x, V) \subset I^+(x) \cap V$. The converse is true provided we choose $V$ so that $d_V(x, \cdot) = d(x, \cdot)_{|V}$ because if $y \in I^+(x) \cap V$ then $d(x, y) > 0$ which implies $d_V(x, y) > 0$ and hence $y \in I^+(x, V)$. Thus for every $x$ there is an arbitrary small open set $V$ such that $I^+(x, V) = I^+(x) \cap V$. This property characterizes future distinction, see [2, Lemma 3.10 and Remark 3.12].

2 Continuity on the vanishing distance set

The Lorentzian distance vanishes on the vanishing distance set $(M \times M) \setminus I^+$. This set is clearly conformally invariant, despite the fact that the Lorentzian distance is not. The idea is to use the continuity properties of the Lorentzian distance on $(M \times M) \setminus I^+$ to characterize causal simplicity and causal continuity. Note that the Lorentzian distance vanishes on the open set $(M \times M) \setminus \tilde{I}^+$ thus it is there continuous. Hence the Lorentzian distance is continuous on the vanishing distance set if and only if it is continuous on $\tilde{I}^+ = J^+$ (for a proof of this equality and $I^+ = J^+$ see [2]).

Lemma 2.1 Let $(M, g)$ be a non-total imprisoning spacetime and let $(x, z) \in \tilde{J}^+ \setminus J^+$. There is a representative of the conformal class such that the Lorentzian distance is not finite in any neighborhood of $(x, z)$, in particular it has an infinite discontinuity at $(x, z)$ where the Lorentzian distance vanishes.

PROOF. Let $(M, g)$ be any representative in the conformal class. Let $\gamma_n$ be a sequence of timelike curves with endpoints $(x_n, z_n) \to (x, z)$. There is no
compact set which contains an infinite subsequence of $\gamma_n$ (otherwise by non-total imprisonment and according to the limit curve theorem [5] there would be a limit continuous causal curve connecting $x$ to $z$ which is impossible because $(x, z) \notin J^+$) thus for every compact set all but a finite number of $\gamma_n$ are not contained in the compact set. Let $h$ be an auxiliary complete Riemannian metric on $M$, let $\rho$ be the associated distance, and let $B_n(x)$ be the open balls centered at $x$ and of radius $n$ with respect to $h$. It is possible to pass to a subsequence, denoted again $\gamma_n$, such that there is $p_n \in \gamma_n \cap [M \setminus B_n(x)]$. Let $\Omega_n \geq 1$ be a smooth function equal to 1 outside $A_n = \{r \in M : n - 1 < \rho(x, r) < n\}$ and sufficiently large in $A_n$ so that the Lorentzian length with respect to the metric $\Omega_n g$ of the timelike segment of $\gamma_n$ connecting $x_n$ to $p_n$ is larger than $n$.

Let $y \in I^-(x)$, $w \in I^+(z)$, so that for sufficiently large $n$, $y \ll x_n \ll p_n$, $d_{\Omega_n g}(y, p_n) \geq d_{\Omega_n g}(x_n, p_n) > n$. Defined $\Omega = \Pi_n \Omega_n$, it holds $d_{\Omega g}(y, p_n) > n$ for sufficiently large $n$. Since for sufficiently large $n$, $y \ll p_n \ll w$,

$$d_{\Omega g}(y, w) \geq d_{\Omega g}(y, p_n) + d_{\Omega g}(p_n, w) \geq d_{\Omega g}(y, p_n) > n$$

thus $d_{\Omega g}(y, w) = +\infty$ and since $y$ and $w$ can be chosen arbitrarily close to $x$ and $z$, where $d_{\Omega g}(x, z) = 0$ as $(x, z) \notin J^+$, there follows the infinite discontinuity of $d_{\Omega g}$ at $(x, z)$.

Theorem 2.2 The non-total imprisoning spacetime $(M, g)$ is causally simple if and only if for every metric $g$ in the conformal class the Lorentzian distance is continuous on the vanishing distance set.

**Proof.** $\Rightarrow$. In this direction the proof has been given by the author in [11, Theorem 3.10]. It is included here for completeness.

Assume that $(M, g)$ is causally simple and let $(x, z) \in \hat{I}^+$ so that $d(x, z) = 0$. If $(x, z)$ is a discontinuity point for $d$, then there is a $\epsilon > 0$ and a sequence $(x_n, z_n) \to (x, z)$, such that $d(x_n, z_n) > \epsilon > 0$. In particular $(x_n, z_n) \in \hat{I}^+$ and $(x, z) \in \hat{I}^+ \setminus \hat{I}^+ = \hat{I}^+ = E^+$, by causal simplicity [2, Lemma 3.67]. Let $\gamma_n$ be causal curves connecting $x_n$ to $z_n$ and such that $\limsup_{n \to +\infty} l(\gamma_n) \geq \epsilon$ (for instance let $l(\gamma_n) > d(x_n, z_n) - \frac{1}{n}$ if $d(x_n, z_n) < +\infty$ and $l(\gamma_n) > n$ if $d(x_n, z_n) = +\infty$). By the limit curve theorem there is a continuous causal curve $\gamma$ passing through $x$ and a distinguishing subsequence $\gamma_j$ which converges to it. But by construction any event $y \neq x, z$, of $\gamma$ is the limit of events $y_j \in \gamma_j$, $(x_j, y_j) \in J^+$, hence $(x, y) \in J^+ = J^+$ and analogously $(y, z) \in J^+$, thus $\gamma$ must be a lightlike geodesic connecting $x$ to $z$, otherwise $(x, z) \in I^+$. Finally, by using the upper semi-continuity of the length functional

$$d(x, z) \geq l(\gamma) \geq \limsup_{j \to +\infty} l(\gamma_j) \geq \epsilon > 0.$$
The contradiction proves that \((x, z)\) cannot be a discontinuity point for the Lorentzian distance.

\[\leftarrow.\] Since \((M, g)\) is non-total imprisoning it is causal. Assume that \((M, g)\) is not causally simple, then since a causally simple spacetime is a causal spacetime for which \(\bar{J}^+ = J^+\), it must be \(\bar{J}^+ \neq J^+\), that is there is a pair \((x, z) \in \bar{J}^+ \setminus J^+\). The thesis is now a consequence of lemma 2.1. □

Recall that the timelike diameter of a spacetime \((M, g)\) is defined by

\[
\text{diam}(M, g) = \sup \{ d(p, q) : p, q \in M \}
\]

that is, it is the least upper bound of the Lorentzian lengths of the \(C^1\) causal curves on spacetime.

**Lemma 2.3** Let \(h\) be an auxiliary complete Riemannian metric on \(M\) and let \(\rho\) be the associated distance. Let \(q \in M\) and let \(B_n(q) = \{ r : \rho(q, r) < n \}\) be the open balls of radius \(n\) centered at \(q\). If \((M, g)\) is strongly causal, then there is a representative \(g\), such that \(\text{diam}(M, g)\) is finite and for every \(\epsilon > 0\) there is a \(n \in \mathbb{N}\) such that if \(\gamma : I \to M\) is any \(C^1\) causal curve,

\[
\int_{I \cap \gamma^{-1}(M \setminus \bar{B}_n)} \sqrt{-g(\dot{\gamma}, \dot{\gamma})} \, dt < \epsilon
\]

that is, its many connected pieces contained in the open set \(M \setminus \bar{B}_n\) have a total Lorentzian length less than \(\epsilon\).

**PROOF**. Let \(\tilde{g}\) be a representative in the conformal class. Every point \(p\) admits a causally convex neighborhood \(U(p)\) such that its closure is compact and contained in a globally hyperbolic neighborhood \(V \supset \tilde{U}\). Since \(U\) is causally convex in \(M\) the Lorentzian distance of spacetime \((U, \tilde{g}|_U)\) coincides with \(\tilde{d}|_{U \times U}\) where \(\tilde{d}\) is the Lorentzian distance of \((M, \tilde{g})\). However, since \(V \subset M\), \(U\) is also causally convex in \(V\), thus this same distance coincides with the restriction of \(\tilde{d}_{(V, \tilde{g}|_V)}\) to the set \(U \times U\), but since \(\tilde{U} \times \tilde{U} \subset V \times V\) is compact and \(\tilde{d}_{(V, \tilde{g}|_V)}\) is continuous, \(\tilde{d}|_{U \times U}\) is actually bounded and continuous.

Thus there is a constant that bounds the Lorentzian length of all the causal curves contained in \(U\).

Every compact set \(W_n = \bar{B}_{n+2}(q) \setminus B_{n+1}(q)\) can be covered with a finite number of such causally convex neighborhoods \(U(p_i)\) with compact closure contained in \(B_{n+3}(q) \setminus \bar{B}_n(q)\). Let \(A_n = \cup_i U(p_i), A_n \subset B_{n+3}(q) \setminus \bar{B}_n(q)\). Since every causal curve \(\eta\) can pass through \(U(p_i)\) only once, and the segment there contained is bounded by a constant depending on the causally convex neighborhood, the length of (the many pieces of) \(\eta \cap W_n\) is bounded from above by a constant \(k_n\) independent of \(\eta\). Let \(\epsilon > 0\) be given and let \(0 < \Omega_n \leq 1\) be a conformal
factor such that $\Omega_n = 1$ outside $B_{n+3}(q) \setminus B_n(q)$ and sufficiently small on $A_n$ that $k_n < \varepsilon / 2^n$.

Defining $g = (\Pi_n \Omega_n)\bar{g}$ the Lorentzian distance $d$ of $(M, g)$ is such that if $\gamma$ is a causal curve the length of the many pieces of $\gamma \cap M \setminus B_{n+1}(q)$ is bounded by $\sum_{i=n}^{+\infty} k_i = \varepsilon / 2^{n-1}$. From this fact the thesis follows.  

**Theorem 2.4** The strongly causal spacetime $(M, g)$ is causally continuous if and only if there is a metric $g$ in the conformal class such that the Lorentzian distance is continuous on the vanishing distance set.

**Proof.** $\Leftarrow$. In this direction the proof has been given by the author in [11 Corollary 3.4]. Actually it suffices to assume weak (or even feeble) distinction instead of strong causality. I repeat the proof here for completeness. Assume $(M, g)$ has a continuous Lorentzian distance on the vanishing distance set. Note that we have only to prove that $(M, g)$ is reflective. If $(M, g)$ were not reflective then it would be non past or non future reflective. We can assume the first possibility as the other case can be treated similarly. Thus there is a pair $(x, z)$ and an event $y$ such that $I^+(x) \supset I^+(y)$ but $y \in I^-(x)$ while $y \notin I^-(z)$. In particular, $d(y, z) = 0$. Since $I^+(z) \subset I^+(x)$, $z \in \bar{I}^+(x)$. Let $z_n \to z$, $z_n \in I^+(x)$, then

$$d(y, z_n) \geq d(y, x) + d(x, z_n) > d(y, x) > 0,$$

thus there is a discontinuity at $(y, z)$, where $d(y, z) = 0$, a contradiction.

$\Rightarrow$. Let $(M, g)$ be causally continuous and let $q \in M$. Consider the representative $(M, g)$ in the conformal class with the properties mentioned in the statement of lemma 2.3 and let $h$ be the complete Riemannian metric there mentioned. We have to show that $d$ is continuous at $(x, z) \in I^+$, that is if $(x_k, z_k) \to (x, z)$ then for any given $\epsilon > 0$, for sufficiently large $k$, $d(x_k, z_k) \leq \epsilon$. If it were not then there would be a $\epsilon > 0$ and a subsequence $(x_n, z_n) \to (x, z)$ such that $d(x_n, z_n) > \epsilon$. Since the timelike diameter is finite $d(x_n, z_n) < +\infty$ and we can consider a limit maximizing sequence $[15]$ of timelike curves $\gamma_n$ connecting $x_n$ to $z_n$ with length $l(\gamma_n) > \epsilon$ (limit maximizing means $d(x_n, z_n) - l(\gamma_n) \to 0$). Note that $x \neq z$, indeed if $x = z$ since $l(\gamma_n) > \epsilon$ there is a neighborhood $U$ of $x$ such that none of the curves $\gamma_n$ are contained in $U$ (the bound to the Lorentzian length is a consequence, see [5] proof of (b) theorem 2.4], of the bound on the Riemannian length, which goes to zero with the size of the neighborhood see [1] Sect. 3.3 [5, Lemma 2.5]). However, by strong causality at $x$ this is impossible, thus it must be $x \neq z$.

Parametrize $\gamma_n$ with respect to $h$-length, so that they have domain $[a_n, b_n]$, $x_n = \gamma_n(a_n)$, $z_n = \gamma_n(b_n)$. We are going to apply the limit curve theorem [5 Theorem 3.1] case 2. The limit causal curve cannot connect $x$ to $z$ for otherwise
this causal curve would have length \( \lim_{n \to +\infty} l(\gamma_n) = \lim_{n \to +\infty} d(x_n, z_n) > \epsilon \) (see [13]) and hence \((x, z) \in I^+\) a contradiction (thus subcase \(b < +\infty\) of [5, Theorem 3.1] case 2 does not apply).

We are now going to prove that there is a subsequence (denoted in the same way) such that for sufficiently large \(n\), \(l(\gamma_n) \leq \epsilon\) which again is a contradiction.

![Fig. 1. The sequence of causal curves \(\gamma_n\) has only a finite number of limit curves intersecting the compact set \(\bar{B}_N\). Indeed, since the convergence is uniform on compact subsets to each limit curve, given a causally convex neighborhood intersected by the limit curve \(U \subset B_{N+1}\), the limit sequence has to enter and escape it. This can happen only a finite number of times (at most once for each causally convex set \(U\) of the covering of \(\bar{B}_N\)) thus the limit curves are finite in number. Since the limit causal curves are all lightlike geodesics, by the upper semi-continuity of the length functional the contribution to the length of \(\gamma_n\) coming from \(B_{N+1}\) can be controlled.

Choose \(N\) so that the compact set \(\bar{B}_N(q)\) is such that the Lorentzian length of \(\gamma \cap M \backslash \bar{B}_N(q)\) is smaller than \(\epsilon/2\), where \(\gamma\) is a generic causal curve. Passing to a subsequence it is possible to assume that \(\lim(b_n - a_n)\) exists, in particular since subcase \(b < +\infty\) of [5, Theorem 3.1] case 2 does not apply it must be \(\lim(b_n - a_n) = +\infty\). If all but a finite number of \(\gamma_n\) do not intersect \(\bar{B}_N(q)\), then there is nothing to prove.

More generally there will be some limit point in \(\bar{B}_N\) and hence some limit curve (see figure [1]). Observe that every limit curve must be either a lightlike line or a lightlike ray (starting from \(x\) or ending at \(z\)). Indeed, assume on the contrary that the limit curve has two points \(p \ll q\), then since \((x, p) \in \bar{J}^+\) by future reflectivity \(p \in \bar{J}^+(x)\), analogously, \((q, z) \in \bar{J}^+\) and by past reflectivity \(q \in \bar{J}^-(z)\), finally, since \(I^+\) is open \(x \ll z\) a contradiction.

Note that every limit curve, since the spacetime is non-partial imprisoning, has to escape \(\bar{B}_{N+1}\) to never reenter it. Moreover, let \(\{U_1, \ldots, U_k\}, U_i \subset B_{N+1}\), be a covering of \(\bar{B}_N\) with causally convex subsets with compact closures. The limit curve intersects and escapes at least one of these sets and thus, for large \(n\),
the same is true for the curves $\gamma_n$, which converge to the limit curve uniformly on compact subsets (with respect to a complete Riemannian metric). Thus it is possible to find a subsequence (denoted in the same way) which has at most $k$ limit causal curves passing through $\bar{B}_N$ and such that all the limit points belong to one of these curves. Since the convergence is uniform on compact subsets, the length functional is upper semi-continuous and the limit curves are all lightlike, for sufficiently large $n$ the contribution to the length of $\gamma_n$ coming from the open set $B_{N+1}$ is less than $\epsilon/2$ for sufficiently large $n$. Moreover the length coming from the open set $M\setminus \bar{B}_N$ is again $\epsilon/2$, thus $l(\gamma_n) \leq \epsilon$, for sufficiently large $n$, and the thesis is proved. □

3 Characterizations of global hyperbolicity

In this section a new characterization of global hyperbolicity in terms of the Lorentzian distance is obtained.

**Lemma 3.1** In any spacetime the two properties

(i) $\forall x, z, \overline{J^+(x) \cap J^-(z)}$ is compact,

(ii) $\forall x, z, \overline{I^+(x) \cap I^-(z)}$ is compact,

are equivalent, and implied by

(iii) $\forall x, z, J^+(x) \cap J^-(z)$ is compact.

In a non-total imprisoning spacetime they are all equivalent.

**PROOF.** (i) ⇒ (ii). Since $I^+(x) \subset J^+(x)$, $I^-(z) \subset J^-(z)$, the closed set $\overline{I^+(x) \cap I^-(z)}$ being a closed subset of the compact set $\overline{J^+(x) \cap J^-(z)}$ is compact.

(ii) ⇒ (i). Take $x' \ll x, z' \gg z$, then $\overline{I^+(x') \cap I^-(z')}$ is compact. Since $J^+(x) \subset I^+(x')$, $J^-(z') \subset I^-(z)$, the closed set $\overline{J^+(x) \cap J^-(z)}$ being a closed subset of the compact set $\overline{I^+(x') \cap I^-(z')}$ is compact.

(iii) ⇒ (i). Trivial.

(i) ⇒ (iii) in the non-total imprisoning case. Assume $(M, g)$ is non-total imprisoning. We are going to prove that (i) implies the closure of $J^+(y)$ and $J^-(y)$ for all $y$. From that it follows for every $x, z, \overline{J^+(x) \cap J^-(z)} = J^+(x) \cap J^-(z)$ and thus the compactness of this last set. Assume for instance that $J^+(y)$ is not closed and let $w \in J^+(y) \setminus J^+(y)$. Take $r \gg w$, and a sequence $r_n$, such that $w \ll r_n \ll r$, $r_n \to w$. Let $\sigma_n$ be causal curves connecting $y$ with $r_n$. They are all contained in the compact set $C = \overline{J^+(y) \cap J^-(r)}$. By the limit curve theorem [5] there is a limit causal curve $\sigma$ starting from $y$, which necessarily joins $y$ to $w$, otherwise $\sigma$ would be future inextendible but contained in the
compact $C$, in contradiction with non-total imprisonment. Thus $w \in J^+(y)$ again a contradiction which proves that $\bar{J}^+(y) = J^+(y)$. □

**Remark 3.2** In the previous lemma non-total imprisonment cannot be weakened to causality, see Carter’s example [10, Fig. 39].

As a preliminary step we obtain this new characterization of global hyperbolicity.

**Corollary 3.3** A spacetime is globally hyperbolic iff it is non-total imprisoning and such that for every pair $x,z \in M$, $\bar{I}^+(x) \cap I^-(z)$ is compact.

**Proof.** It suffices to recall that a spacetime is globally hyperbolic iff it is causal and for every $x, z$, $J^+(x) \cap J^-(z)$ is compact [26]. □

**Remark 3.4** Non-total imprisonment cannot be weakened to causality, see again Carter’s example [10, Fig. 39].

**Lemma 3.5** A causally simple spacetime is globally hyperbolic or it is possible to find events $x, z \in M$, $x \ll z$, such that $J^+(x) \cap J^-(z)$ is not compact and there is a sequence of causal curves $\sigma_n : [0, a_n] \to M$ of endpoints $x$ and $z$, such that no subsequence of $\sigma_n$ is contained in a compact set, $\sigma_n$ converges uniformly (with respect to a complete Riemannian metric) on compact subsets to a future lightlike ray $\sigma^x$ starting from $x$, and the reparametrized sequence $\tilde{\sigma}_n(t) = \sigma_n(t-a_n), \tilde{\sigma}_n : [-a_n, 0] \to M$ converges uniformly on compact subsets to a past lightlike ray $\sigma^z$ ending at $z$.

**Proof.** Assume $(M, g)$ is not globally hyperbolic then since it is non-total imprisoning there are $p \ll g$ such that $\bar{I}^+(p) \cap I^-(q)$ is not compact thus since, $\bar{I}^+(p) \cap I^-(q) \subset \bar{I}^+(p) \cap \bar{I}^-(q) = J^+(p) \cap J^-(q)$, this last set is not compact. Let $\gamma : \mathbb{R} \to M$ be a timelike curve connecting $p$ to $q$, such that $p = \gamma(0), q = \gamma(1)$. Let $B$ be the set of closed intervals $[a, b], a, b \in [0, 1]$, such that $J^+(\gamma(a)) \cap J^-(\gamma(b))$ is not compact (by causality $a < b$). The set $B$ is not empty because $[0, 1] \in B$. Moreover, the set $B$ is ordered by inclusion and we want to prove that it admits a minimal element. By Hausdorff’s maximum principle there is a maximal (totally ordered) chain $C$. The arbitrary intersection of convex intervals is convex, thus the intersection of the elements in the maximal chain, being the intersection of connected non-empty compact intervals is a non-empty (this is a standard result in topology [12]) connected compact interval $[\tilde{a}, \tilde{b}], \tilde{a}, \tilde{b} \in [0, 1], \tilde{a} \leq \tilde{b}$.

Actually, $\tilde{a} < \tilde{b}$ because of the following argument. Assume $\tilde{a} = \tilde{b}$, the point $\gamma(\tilde{a})$ admits a causally convex neighborhood $V$ with compact closure, and there is $\epsilon > 0$, such that the image of $\gamma|[\tilde{a} - \epsilon, \tilde{a} + \epsilon]$ is contained in $V$. The interval
$[\tilde{a}, \tilde{a} + \epsilon]$ cannot belong to all the elements of the maximal chain, nor can the interval $[\tilde{a} - \epsilon, \tilde{a}]$ thus there is an interval $[a', b']$ belonging to the maximal chain such that $[a', b'] \subset (\tilde{a} - \epsilon, \tilde{a} + \epsilon)$, thus $J^+(\gamma(\tilde{a} - \epsilon)) \cap J^-(\gamma(\tilde{a} + \epsilon))$ is not compact which is impossible because it is contained in $V$. The contradiction proves that $\tilde{a} < \tilde{b}$.

Define $x = \gamma(\tilde{a})$ and $z = \gamma(\tilde{b})$. In order to prove the minimality of $[\tilde{a}, \tilde{b}]$ on $B$ we have only to show that $D = J^+(x) \cap J^-(z)$ is not compact, the minimality of $[\tilde{a}, \tilde{b}]$ would follow trivially from the maximality of the chain $C$.

Since $[\tilde{a}, \tilde{b}]$ is the intersection of the elements of $C$, it is possible to construct (by arguing as above) sequences $a_n \to a$, $b_n \to b$, $a > a_{n+1} \geq a_n$, $b < b_{n+1} \leq b_n$, such that $J^+(\gamma(a_n)) \cap J^-(\gamma(b_n))$ is not compact.

Note that $\bigcap_n J^-(\gamma(b_n)) \subset J^-(z)$ indeed, if $w \in J^-(\gamma(b_n))$ for all $n$, then $\gamma(b_n) \in J^+(w)$ and since $\gamma(b_n) \to z$, $z \in J^+(w) = J^+(w)$ which implies $w \in J^-(z)$. Analogously, $\bigcap_n J^+(\gamma(a_n)) \subset J^+(x)$. We conclude

$$\bigcap_n [J^+(\gamma(a_n)) \cap J^-(\gamma(b_n))] \subset J^+(x) \cap J^-(z). \quad (1)$$

Assume that $D = J^+(x) \cap J^-(z)$ is compact, let $h$ be a complete Riemannian metric and let $B_n(p)$ be the open ball of $h$-radius $n$ centered at $p$. Let $N$ such that $D \subset B_N(p)$ and let $E = B_{N+1}(p) \setminus B_N(p)$ be a compact shell which contains $D$ in its interior. Note that $E \cap J^+(\gamma(a_n)) \cap J^-(\gamma(b_n))$ is non-empty because the causal curves issued from $\gamma(a_n)$ and reaching $\gamma(b_n)$ are not all included in a compact set (recall that the sets $J^+(\gamma(a_n)) \cap J^-(\gamma(b_n))$ are non compact) and thus some of them cross $E$. The sets $E \cap J^+(\gamma(a_n)) \cap J^-(\gamma(b_n))$ give a nested family of non-empty compact subsets whose intersection is, by Eq. (1), the empty set which is impossible. The contradiction proves that $D$ is not compact, and thus $[\tilde{a}, \tilde{b}]$ is a minimal element for $B$.

Let $\sigma_n$ be a sequence of causal curves not all contained in a compact connecting $x$ to $z$. By the limit curve theorem [5] there is a subsequence $\sigma_n : [0, a_n] \to M$ and limit curves $\sigma^x$ and $\sigma^z$ as in the statement of this theorem, but for their ‘lightlike ray’ nature which we have still to prove. This is indeed a consequence of the minimality of $[\tilde{a}, \tilde{b}]$. Assume for instance that $\sigma^x$ is not a lightlike ray, then there is $w \in \sigma^x \setminus \{x\}$, such that $x \ll w$, and since $I^+$ is open there is $\delta > 0$, and $x' = \gamma(\tilde{a} + \delta)$, such that $(x', w) \in I^+$. Since $w$ is a limit point for $\sigma_n$, it is possible to construct a sequence of causal curves $\sigma'_n$ not entirely contained in a compact, which connects $x'$ to $z$. This fact implies that, $J^+(x') \cap J^-(z)$ is not compact and hence that $[\tilde{a} + \delta, \tilde{b}] \in B$ in contradiction with the minimality of $[\tilde{a}, \tilde{b}]$ in $B$. \hfill \Box

**Theorem 3.6** A non-total imprisoning spacetime $(M, g)$ is globally hyperbolic

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Fig. 2. The argument of lemma 3.5 which allows us to construct lightlike rays $\sigma^x$ and $\sigma^z$ in a causally simple non-globally hyperbolic spacetime (here given by 1+1 Minkowski spacetime with the usual coordinates $(t, x)$ restricted to $x > 0$).

*if and only if for every metric choice in the conformal class the Lorentzian distance is continuous.*

**PROOF.** $\Rightarrow$. This implications is well known, see [1, Corollary 4.7].

$\Leftarrow$. For the converse, assume the spacetime $(M, g)$ is non-total imprisoning and that for every metric choice in the conformal class the Lorentzian distance is continuous. By theorem 2.2 the spacetime $(M, g)$ is causally simple. Let $h$ be a complete Riemannian metric on $M$ and let $q \in M$. Take as representative $g$ the metric selected by the statement of lemma 2.3, and let $B_n(q)$ be the open balls of $h$-radius $n$ centered at $q$. By lemma 2.3 diam$(M, g)$ is finite. Assume that $(M, g)$ is not globally hyperbolic, then there are $x, z \in M$, $x \ll z$, as in the statement of lemma 3.5. Since diam$(M, g)$ is finite, $d_g(x, z) < \infty$. The idea is to conformally change the metric outside the closed set $J^+(x) \cap J^-(z)$. Any such change necessarily leaves unaltered the distance between $x$ and $z$ but the conformal factor can be chosen so that there is a infinite discontinuity for the new Lorentzian distance. Let $z_n \gg z$ be a sequence of points such that $z_n \to z$, and let $w_n \in \sigma^z$ be a sequence of points such that $w_n \to +\infty$ (i.e. it escapes every compact; recall that $(M, g)$ is non-total imprisoning thus the past ray $\sigma^z$ escapes every compact). There are timelike curves $\eta_n$ connecting $w_n$ to $z_n$ and this curve has no intersection with $J^-(z)$ but for the starting point $w_n$, for otherwise $w_n \in I^-(z)$ which is impossible because $\sigma^z$ is a lightlike ray. Without loss of generality we can assume that there is $N > 0$ such that for $n > N$, $\eta_n$ intersects the open set $A_n = B_{n+1}(q) \setminus [B_n(q) \cup [J^+(x) \cap J^-(z)]]$ (just pass to a subsequence and relabel it). Let $\Omega_n : M \to [1, +\infty)$, be a function equal to 1 outside $A_n$ and sufficiently large on $A_n$ that the Lorentzian length of $\eta_n$ with respect to $\Omega_n g$ is greater than $n$. Define $\bar{g} = \Pi_n \Omega_n g$, then $d_{\bar{g}}(w_n, z_n) > n$, and thus since $w_n \in J^+(x)$, $d_{\bar{g}}(x, z_n) > n$ which implies that there is an infinity discontinuity on the Lorentzian distance $d_{\bar{g}}$ as $(x, z_n) \to (x, z)$. The contradiction proves that the spacetime is globally hyperbolic. $\square$
4 Conclusions

It has been proved that causal simplicity and causal continuity can be characterized through the continuity properties of the Lorentzian distance at those pairs of events where it vanishes. The non-total imprisoning spacetime is causally simple if and only if for every metric choice in the conformal class the Lorentzian distance is continuous wherever it vanishes. Similarly, a strongly causal spacetime is causally continuous if and only if there is at least one choice of metric in the conformal class such that the Lorentzian distance is continuous wherever it vanishes. Using some preliminary lemmas it has also been shown that a non-total imprisoning spacetime is globally hyperbolic if and only if the Lorentzian distance is continuous for every choice of metric in the conformal class.

Other results obtained in this work are the characterization of the distinction condition given by theorem 1.1 and that of global hyperbolicity given by corollary 3.3.

Acknowledgments

This work has been partially supported by GNFM of INDAM.

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