Abstract

This article proposes a new method to generate synthetic data sets based on copula models. Our goal is to produce surrogate data resembling real data in terms of marginal and joint distributions. We present a complete and reliable algorithm for generating a synthetic data set comprising numeric or categorical variables. Applying our methodology to two datasets shows better performance compared to other methods such as SMOTE and autoencoders.

1. Introduction

Governments, organizations, businesses, academia, members of the public and other decision-making bodies all would like easier and unrestricted access to good quality data to make informed and accurate decisions. However, accessing a good, real-world dataset can be difficult, expensive, time-consuming, or simply impossible, and real data is almost always vulnerable to data privacy breach incidents. These data-limited situations can limit the efforts of the data science community to build robust models for organizations to make informed policy decisions and address major challenges. (Kaloskampis, 2019).

In this work, we propose to use synthetic data generation methods to produce surrogate data closely resembling real data in data limited situations, thus minimizing the need to access real data. Synthetic data is a powerful tool when the data required is limited or when it is desired to be shared securely with the parties concerned. The idea of synthetic data, that is, data artificially fabricated rather than obtained by direct measurement, was introduced by (Rubin, 1993), who used multiple imputation to generate a synthetic version of the Decennial Census. As a result, he was able to release samples without disclosing microdata.

(Rubin, 1993) first attempted to generate synthetic data in 1993 using multiple imputation, for which parametric modeling was used. (Drechsler & Reiter, 2011) points out, however, that it is generally difficult and tedious to produce precise models based on standard parametric approaches. Therefore, nonparametric machine learning approaches have been used, such as classification and regression trees, support vector machines, bagging, and random forests, see (Kaloskampis, 2019). With recent developments in deep learning, deep neural networks have started to become an attractive option for generating synthetic data, given their ability to use big data to create powerful models. The main deep learning algorithms, widely used for data synthesis, especially in the fields of image and video, are antagonistic generator networks (GAN), (Goodfellow et al., 2014) And autoencoders (Ballard, 1987).

In this paper, we propose to use a robust statistical model such as copulas (Nelson, 2007) to generate synthetic data sets that closely resemble real data. Copulas are a powerful tool for isolating the structure of dependencies in a multivariate distribution. It is a probability model that represents a multivariate uniform distribution, which examines the association or dependence between many variables. Using copulas and the inverse transformation method, we can produce a synthetic data set that mimics the real data in terms of association or dependence and marginal distribution. In addition, our copula-based method can be used for imputation (replacing missing data with substituted values) and augmentation of real data with synthetic data, to build robust statistical, machine-learning and deep-learning models more rapidly and at lower cost. Another challenge in the present work is to generate a synthetic dataset from an actual dataset that includes categorical features. In this context, we propose an algorithm that converts categorical features to numeric ones, then generates the synthetic copula-based dataset and finally converts the generated features to categorical ones. Thus, our proposed method is a complete and robust algorithm to generate a synthetic dataset including numeric or categorical features, which makes possible to share data more easily and more quickly between government, universities and the private sector.

The rest of the paper is organized as follows. Section II gives an overview of copulas and the algorithm to simulate them. We also present the rank correlation and the relationship with copulas. In Section III, our approach of generating
What makes a copula model above so useful is Sklar’s theorem which is the CDF function of the given marginals. With the following properties

1. \( C(u_1, u_2, \ldots, u_n) \) in non-decreasing in each component, \( u_i \)
2. The \( i \)-th marginal distribution is obtained by setting \( u_j = 1 \) for \( j \neq i \) and since it is uniformly distributed
\[
C(1, \ldots, 1, u_i, 1, \ldots 1) = u_i
\]
3. For \( a_i \leq b_i \), \( P(U_1 \in [a_1, b_1], \ldots, U_n \in [a_n, b_n]) \) must be non-negative.

The reverse is also true in that any function that satisfies properties 1 to 3 is a copula.

What makes a copula model above so useful is Sklar’s famous theorem (Sklar, 1959). It states that for any joint cumulative distribution function (CDF) with a set of continuous random variables \( \{X_i\}_1^n \) and marginal CDF \( F_i(x_i) = P(X_i \leq x_i) \), there exists a unique copula function such that the joint CDF is
\[
F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \quad (2)
\]
By probability integral transform, each marginal CDF is a uniform random variable on \([0, 1]\). Therefore, the above copula is valid. In the opposite direction, consider a copula \( C \) and marginal CDF, \( F_i(x_i) \). Then, \( F \) as defined in (2) is a multivariate CDF with marginals \( F_i(x_i) \). And, for a given joint distribution, we can also find the corresponding copula which is the CDF function of the given marginals.

A useful representation that we can get by Sklar’s theorem for a continuous copula is,
\[
C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)) \quad (3)
\]
One of the most important result in the theory of copula is the theorem of Invariance Under Monotonic Transformations. It basically states that if we apply a monotonic transformation to each component of \( X = (X_1, \ldots, X_n) \) then the copula of the resulting multivariate distribution remains the same.

Consider a copula \( C(u) = C(u_1, \ldots, u_n) \); by the theorem of Fréchet-Hoeffding bounds, we have the following result
\[
\max \left\{ 1 - n - \sum_{i=1}^{n} u_i, 0 \right\} \leq C(u) \leq \min \{u_1, \ldots, u_n\} \quad (4)
\]
The upper Fréchet-Hoeffding bound is tight for all \( n \) while the lower Fréchet-Hoeffding Bound is tight only when \( n = 2 \). These bounds correspond to cases of extreme of dependency, i.e. comonotonicity and countermonotonicity. The comonotonic copula is given by
\[
M(u) = \min \{u_1, \ldots, u_n\} \quad (5)
\]
which is the Fréchet-Hoeffding upper bound. This corresponds to the case of extreme positive dependence. The countermonotonic copula is the 2-dimensional copula that is the Fréchet-Hoeffding lower Bound. It is given by
\[
W(u) = \max \{u_1 + u_2 - 1, 0\} \quad (6)
\]
and corresponds to the case of perfect negative dependence.

The independence copula satisfies
\[
\Pi(u) = \prod_{i=1}^{n} u_i \quad (7)
\]
and this corresponds to the case of independence random variables.

There are other families of copulas such as elliptical copulas (Gaussian copula, t copula), Archimedean copulas (Gumbel copula, Clayton copula), Extreme Value copulas, etc., see (McNeil et al., 2005), (Joe, 1997) and (Nelsen, 1999). In this study, we will focus on the elliptical copulas such as Gaussian and t copulas that are used in this paper to generate synthetic dataset.

2. Overview of copulas

Copula (Nelson, 2007) is defined as a probability distribution over a high-dimensional unit cube \([0, 1]^n\) whose univariate marginal distributions are uniform on \([0, 1]\). Copula can model the dependencies of large dimensional random variables and is very efficient in risk management or other tasks that require dependency modelling, see (McNeil et al., 2005), (Kole et al., 2007), etc... Formally, given a set of uniformly distributed random variables, \( U_1, U_2, \ldots, U_n \), a copula is a joint cumulative distribution defined as
\[
C(u_1, u_2, \ldots, u_n) = P(U_1 \leq u_1, \ldots, U_n \leq u_n) \quad (1)
\]
with the following properties

1. \( C(u_1, u_2, \ldots, u_n) \) in non-decreasing in each component, \( u_i \)
2. The \( i \)-th marginal distribution is obtained by setting \( u_j = 1 \) for \( j \neq i \) and since it is uniformly distributed
\[
C(1, \ldots, 1, u_i, 1, \ldots 1) = u_i
\]
3. For \( a_i \leq b_i \), \( P(U_1 \in [a_1, b_1], \ldots, U_n \in [a_n, b_n]) \) must be non-negative.

The conclusion is drawn in Section V.

2.1. Elliptical copulas: Gaussian and t copulas

2.1.1. Definition and simulation of Gaussian copulas

Recall that when the marginal CDFs are continuous, we have from (3) that
\[
C(u_1, \ldots, u_n) = F(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n))
\]
Suppose that \( X \sim \text{MN}(0, P) \) with \( P \) is the correlation matrix of \( X \). then the corresponding Gaussian copula is
The general multivariate $t$ matrix can be defined by the stochastic representation of freedom (d.o.f), corresponding correlation matrix $\Sigma$. For an arbitrary covariance matrix $\Sigma$ we denote $X \sim t_{\nu,\mu,\Sigma}$; for more details, see (Hofert, 2013), (Shaw, 2006), etc.

Suppose that $X = (X_1, \ldots, X_n) \sim t_{\nu,0,\rho}$ where $\nu > 2$ and $\rho$ its correlation matrix. The corresponding $t$-copula is defined as

$$C^t_{\nu,\rho} = t_{\nu,0,\rho}(t^{-1}_{\nu}(u_1), \ldots, t^{-1}_{\nu}(u_n))$$

where $t_{\nu}$ is the univariate CDF of $t$-distribution with mean zero and dispersion parameter equal 1 and d.o.f, $\nu > 2$.

A higher value for $\nu$ decreases the probability of tail events. When the $t$-copula converges to the Gaussian copula for $\nu \rightarrow \infty$, the $t$ copula assigns more probability to tail events than the Gaussian copula. In addition, the $t$-copula exhibits tail dependence even if the correlation coefficients are equal to zero.

### Simulation of $t$ Copulas

Let $\Sigma$ be the covariance matrix of the input data $X = (X_1, \ldots, X_n)$ and $P = \text{cor}(X)$ its corresponding correlation matrix. The steps to generate the $t$ copulas are following:

1. Generate $W \sim \text{MN}(0, P)$
2. For $\nu > 2$, generate $\epsilon \sim \chi^2_\nu$ independent of $W$
3. Return $U = (t_{\nu} \left( \frac{W_1}{\sqrt{\nu}} \right), \ldots, t_{\nu} \left( \frac{W_n}{\sqrt{\nu}} \right))$, where $t_{\nu}$ is the CDF of a standard univariate $t$ distribution with d.o.f, $\nu$. The distribution of $U$ is in fact the $t$-copula $C^t_{\nu,\rho}$.

Figures 1 to 3 show the simulations of the bivariate Gaussian and $t$-copulas with the associated random samples using the above algorithms. We consider different correlation coefficients such as $\rho = -0.2$, $\rho = 0.9$ and $\rho = -0.9$.

#### 2.1.2. Definition and Simulation of $t$ Copulas

The general multivariate $t$ distribution $X$, with $\nu$ the degree of freedom (d.o.f), $\mu$ the location vector and $\Sigma$ the scale matrix can be defined by the stochastic representation

$$X = \mu + \sqrt{V} B Z$$

where

$$V = \frac{\nu}{\chi^2_\nu}$$

$\chi^2_\nu$ follows a chi-squared distribution with $\nu > 0$ and is independent of $Z \sim \text{MN}(0, I)$; $B$ is such that $BB^T = \Sigma$. The steps to generate the $t$ copula are following:

1. Compute the Cholesky decomposition, $A$ of $P$ so that $P = AA^T$ with $A^T$ the transpose of $A$.
2. Generate $Z \sim \text{MN}(0, I)$ with $I$ being the identity matrix.
3. Set $W = AZ$. In fact, $W \sim \text{MN}(0, P)$
4. Return $U = (\phi(W_1), \ldots, \phi(W_n))$. The distribution of $U$ is the Gaussian copula $C^\text{Gau}_{P}$.
Spearman’s rho
Let’s $X_1$ and $X_2$ two random variables and $\rho$ the linear (Pearson) correlation coefficient. The Spearman’s rho is defined as

$$\rho_s(X_1, X_2) = \rho(F_1(X_1), F_2(X_2))$$ (11)

Spearman’s rho is the linear correlation of the probability-transformed random variables. If $X_1$ and $X_2$ have continuous marginals, it can be shown the following relationship between the Spearman’s rho and the unique copula $C$

$$\rho_s(X_1, X_2) = 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2) du_1 du_2$$ (12)

In the case of bivariate Gaussian copula, (12) is reduced to

$$\rho_s(X_1, X_2) = \frac{6}{\pi} \arcsin \left( \frac{\rho}{2} \right) \approx \rho$$ (13)

Kendall’s tau
Let’s $X_1$ and $X_2$ two random variables. The Kendall’s tau is defined as

$$\tau(X_1, X_2) = E \left[ \text{sign} \left( (X_1 - X_1^*)(X_2 - X_2^*) \right) \right]$$ (14)

where $(X_1^*, X_2^*)$ is independent of $(X_1, X_2)$ but has the same joint distribution as $(X_1, X_2)$. Kendall’s tau can also be written as

$$\tau(X_1, X_2) = P \left( (X_1 - X_1^*)(X_2 - X_2^*) > 0 \right) - P \left( (X_1 - X_1^*)(X_2 - X_2^*) < 0 \right)$$

Kendall’s tau is simply the probability of concordance minus the probability of discordance. So if both probabilities are equal then $\tau(X_1, X_2) = 0$. If $X_1$ and $X_2$ have continuous marginals, it can be shown that

$$\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1$$ (15)

It may also be shown that for a bivariate Gaussian copula

$$\tau(X_1, X_2) = \frac{2}{\pi} \arcsin(\rho)$$ (16)

where $\rho$ is the linear (Pearson) correlation coefficient. (16) can be very useful to estimate $\rho$ for fat-tailed elliptical distribution such as the $t$-distribution. This has been experienced by (Haugh, 2016) who showed that (16) provides much more robust estimates of $\rho$ than the usual Pearson estimator. The following result from (Ruppert & Matteson, 2015) can be useful for fitting the Gaussian and $t$ Copula. Let $Y = (Y_1, ..., Y_n)^T$ have a meta-Gaussian distribution$^1$ with

$$C_{\rho}^{Ga}\text{u}$$ and let $P_{i,j} = [P]_{i,j}$; Then,

$$\rho_s(Y_i, Y_j) = \frac{2}{\pi} \arcsin(P_{i,j})$$ (17)

and

$$\rho_s(Y_i, Y_j) = \frac{6}{\pi} \arcsin \left( \frac{P_{i,j}}{2} \right) \approx P_{i,j}$$ (18)

If instead $Y$ has a meta-$t$ distribution$^2$ with continuous univariate marginal distributions and copula $C_{\nu, P}^{t}$ then only (17). For more details about Kendall’s tau and Spearman’s rho and their estimation, we refer to (Haugh, 2016), (Kendall & Stuart, 1979), (Kruskal, 1958), (Lehmann, 1975).

3. Synthetic data using Copulas
3.1. Generating Synthetic data with numerical features
Our aim in this section is to propose a copula-based methodology to generate data containing numerical (continuous) features. We will first isolate the dependency structure of the dataset by copulas. As defined above, we will focus on the elliptic copulas and in this context the dependency between the features can be explained by the linear correlation matrix. Second, we will estimate the marginal distributions of the different features and then we can use the inverse transform method to generate them. Recall the Inverse Transform method.

$^1$A meta-Gaussian distribution is any distribution that has the Gaussian copula

$^2$A meta-$t$ distribution is any distribution that has the $t$-copula
Inverse Transform Method

Let \( F(x), x \in \mathbb{R} \) denote any cumulative distribution function (cdf) (continuous or discrete). Let \( F^{-1}(y), y \in [0, 1] \) denote the inverse function defined by

\[
F^{-1}(y) = \min \{ x : F(x) \geq y \}, y \in [0, 1]
\]

(19)

Define \( X = F^{-1}(U) \) where \( U \) has the uniform continuous distribution over the interval \((0, 1)\). Then \( X \) is distributed as \( F \), that is, \( P(X \leq x) = F(x), x \in \mathbb{R} \).

Now, let \( X = (X_1, ..., X_n) \) be the real dataset and one would like to produce substitute data closely resembling to the real data in terms of marginal and joint distributions. Let \( P = \text{cor}(X) \) be its corresponding Pearson correlation matrix. A robust way to estimate the Pearson correlation \( P_{i,j} = [P]_{i,j} \) is to estimate the Kendall’s tau in the first step and to use (17). Using the Inverse Transform Method and the Sklar’s theorem, we can generate the synthetic dataset by the following steps:

1. Generate the elliptic copulas \( C \) such as the Gaussian or t copulas using the correlation matrix \( P \), as explained in section 2.1. Let \( U = (U_1, ..., U_n) \) be the generated set of Uniform random variables that has the joint cdf, the copula \( C \).

2. Find the marginal cumulative distributions, \( G_i \) for each feature. In the case \( G_i \) is not known, one can estimate them by the empirical cumulative distribution function (ecdf), \( \hat{G}_i \).

3. Generate multivariate \( \eta = (\eta_1, ..., \eta_n) \) such that \( \eta_i = \hat{G}_i^{-1}(U_i) \). \( \eta \) is in fact the synthetic data that shares the same distribution (marginal or joint) with the real dataset \( X \). \( \eta \) has a Meta-elliptical distribution since it is generated by the elliptical copulas.

3.2. Generating Synthetic data with categorical features

Most of the research on the copulas has been based upon the assumption that the univariate margins are continuous. Without this assumption, many results are no longer valid and the interaction between the copula and the margins is much less well understood. In fact, when discontinuities are allowed, there exists a whole class of copulas which comply with the famous Sklar’s theorem, see (Schweizer & Sklar, 1983). This is generally not a problem in applied contexts, as researchers use copulas because the joint distribution is either not known or is difficult to work with. A much more serious problem with discontinuous margins is that estimates of the dependence parameters are biased and therefore our generated dataset is far to keep the dependence structure of the original dataset. In fact, when the margins are discontinuous, they present jumps that cause their inverse to have plateaus and this potentially leads to biased estimates, see (Genest & Neslehova, 2007).

In this context, to generate categorical features using copulas, we propose to encode the categorical features into multivariate continuous features first and then we can apply the method developed in (3.1). The process is the following:

1. Consider the dataset with \( n \) rows and \( m \) columns; we suppose that the \( m \) columns are exclusively categorical features denoted by \( C_k \) with \( 1 \leq k \leq m \). Let \( C_{jk} \) the \( j \) value of the categorical feature \( C_k \) with \( 1 \leq j \leq n \). In other words, \( n \) is the size of the dataset.

2. Let \( m_k \) the number of levels (groups) in the feature \( C_k \). We estimate the proportions for each level \( i \) in the categorical feature \( C_k \). Let \( \hat{P}_k^i \) the estimated proportion; we have \( \sum_{i=1}^{m_k} \hat{P}_k^i = 1 \).

3. The confidence interval for the proportion \( \hat{P}_k^i \) is computed with the following formula

\[
\text{CI}_k^i = \hat{P}_k^i \pm z \sqrt{\frac{\hat{P}_k^i (1 - \hat{P}_k^i)}{n}}
\]

where \( z \) is the \( Z \)-value for the desired confidence level (e.g. \( z = 1.96 \) for 95% confidence) and \( n \) is assumed sufficiently large.

4. To convert each level \( i \) to continuous value, we replace the estimated proportion, \( \hat{P}_k^i \), with a value \( \hat{V}_k^i \) sampled from a normal distribution with mean \( \hat{P}_k^i \) and variance \( \frac{\hat{P}_k^i (1 - \hat{P}_k^i)}{n} \). So, for each feature \( C_k \) one obtains \( n \) numerical values \( \hat{V}_k^j \).
Table 1. Table with two categorical features: vehicle type and vehicle color

| VEHICLE | COLOR |
|---------|-------|
| 1       | CAR   |
| 2       | BUS   |
| 3       | BICYCLE |
| 4       | BUS   |
| 5       | CAR   |
| 6       | BUS   |
| 7       | CAR   |
| 8       | BICYCLE |
| 9       | BICYCLE |
| 10      | BUS   |

5. Getting the numerical features, one can generate the full dataset by applying the method based on the copulas that is described in section (3.1). We denote $\hat{G}^i_k$ the simulated value of $V^i_k$ by taking account the dependency between the numerical features.

6. After transforming the categorical variables into numerical variables and generating them, the main question is how to convert the generated numerical features into categorical features in order to find the structure of the original dataset?

We propose the following steps:

7. For each $\hat{G}^j_k$, we calculate

$$D^{i,j}_k = |\hat{G}^i_k - \hat{P}^i_k|$$

for all $\hat{P}^i_k$ with $1 \leq i \leq m_k$

8. Let $a^j_k = \min_i (D^{i,j}_k)$; find the level $i$ or set of levels $S_k$ for which $D^{i,j}_k = a^j_k$. In other words, we are looking for the levels $i$ for which the proportions are close to the simulated value $\hat{G}^i_k$.

9. If two or more levels are found and $C^i_k \in S_k$ then we select $C^i_k$. If $C^i_k \notin S_k$, then we select randomly one level in $S^i_k$.

Let illustrate the above algorithm with the following example in Table 1. In this table, we have 2 categorical features of which the first feature relates to the vehicle type and the second feature relates to the color of the vehicle.

Our goal is to give an idea of how to simulate this table using the above method. Of course, we have only 10 observations and it would be difficult to get a good estimate and therefore a good simulation. Here, $m = 2$ and $n = 10$. The numbers of levels in the feature $C_1$ and in the feature $C_2$ are $m_1 = 3$ and $m_2 = 2$ respectively. In the feature $C_1$, we denote $1 = \text{CAR}$, $2 = \text{BUS}$, $3 = \text{BICYCLE}$. In the feature $C_2$, $1 = \text{BLUE}$, $2 = \text{GREEN}$. We have the following estimations of the proportions:

$$P^1_1 = \frac{3}{10}, P^2_1 = \frac{4}{10}, P^3_1 = \frac{3}{10}, P^1_2 = \frac{5}{10}, P^2_2 = \frac{5}{10}$$

Based on the estimated proportions, one can generate numerical features by applying the steps 4 to 5 of the above algorithm. Suppose that we obtain the following numerical table

Now, the last step is to convert this table into categorical one. Taking two values in the table, say $\hat{G}^1_1 = \frac{7}{10}$ and $\hat{G}^3_2 = \frac{1}{3}$, let’s apply the conversion process.

- For $\hat{G}^1_1 = \frac{7}{10}$

$$P^1_1 = \frac{3}{10}, P^2_1 = \frac{4}{10}, P^3_1 = \frac{3}{10}$$

and therefore $\hat{G}^1_1$ will be replaced by the level 2 of the feature $C_1$ which is BUS.

- For $\hat{G}^3_2 = \frac{1}{3}$

$$P^1_2 = \frac{5}{10}, P^2_2 = \frac{5}{10}$$

and therefore $\hat{G}^3_2$ will be replaced by GREEN.

Applying the same process to the rest of the table gives the final categorical Table 2.

4. Application to Datasets

To access the ability of our model to generate synthetic data, we focused on a public and private datasets. For publicly available datasets, we used a Kaggle dataset; this datasets contains transactions made by credit cards in September 2013 by European cardholders and presents transactions that occurred in two days. It contains 30 numerical features and among them 28 are the results of PCA transformations. The only features that have not been transformed by the PCA are the "Time" and the "amounts". Feature "Time" contains
Table 2. Generating Synthetic tables based on Table 1

| VEHICLE | COLOUR |
|---------|--------|
| 1       | BUS    |
| 2       | BUS    |
| 3       | BICYCLE|
| 4       | CAR    |
| 5       | BUS    |
| 6       | CAR    |
| 7       | CAR    |
| 8       | BUS    |
| 9       | BUS    |
| 10      | CAR    |

the seconds elapsed between each transaction and the first transaction in the datasets. This feature being deterministic, we remove it from the original dataset because it can bias our simulation by copulas. The second dataset is private and consists of transactions for customers of a financial institution. The private datasets include 15 features, two of which are categorical. Our goal is to generate both datasets by the proposed methodology.

4.1. Generating the Kaggle datasets

The method described in section (3.1) is applied to Kaggle datasets because these datasets include only numerical features. For simplicity, we focus on Gaussian copulas. As noted above, we removed the feature “Time” from the original dataset because this feature is not random. Our results are compared to those generated by autoencoders and SMOTE.

Autoencoders

An autoencoder (AE) is an artificial neural network (or neural network) used to learn a representation of a set of data, (Liou et al., 2008). It has an input layer, an output layer and one or more hidden layers with the purpose of reconstructing its original input. There are two parts of an autoencoder; the encoder and the decoder. The encoder is the subnetwork from the input layer to the middle layer and reduces the dimension of the input data to learn simplified representation of it. The decoder is the subnetwork from the middle layer to the output layer and reconstructs the input data, (Zheng & Peng, 2018).

To generate synthetic datasets, we consider an autoencoder with three hidden layers of 16, 8 and 16 hidden neurons respectively. For the parameters, we fix a batch size of 128, 100 epochs and learning rate of 1e-7.

Synthetic Minority Over Sampling (SMOTE)

Synthetic Minority Over Sampling (SMOTE) synthesizes new data instances between existing (real) instances and was developed to address the problems in applying classification algorithms to unbalanced datasets. Proposed in 2002 by (Chawla et al., 2002) it is now an established method, with over 85 extensions of the basic method reported in specialised literature. A way to visualise how the basic concept works is to imagine drawing a line between two existing instances. SMOTE then creates new synthetic instances somewhere on these lines.

Considering a sample $x_i$, a new sample will be generated considering its $k$ nearest neighbours. The steps are the following:

- Take the difference between the feature vector sample under consideration and one of those $k$ nearest neighbours, $x_{ki}$.
- Multiple the difference by a random number $\gamma$ between 0 and 1 and add this to the sample vector to generate a new sample $x_{\text{new}}$

$$x_{\text{new}} = x_i + \gamma(x_{ki} - x_i)$$

Variations of the technique have been developed by varying the way the samples and their neighbours are selected before generating new ones. For more details, see (Haibo et al., 2008), etc...

To access the quality of the algorithms for generating synthetic datasets, we used two criteria of comparison:

1. Determine whether the relationships between the variables in the original data are preserved in the synthetic data.
2. Determine whether the marginal distributions of the features in the original dataset are preserved in the synthetic dataset.

4.1.1. Testing the Quality of Dependency or Association on Generated Synthetic Datasets

To evaluate the quality of the synthetic datasets to preserve the dependency structure of the Kaggle datasets we proceed as follows:

- We evaluate the pairwise correlation between the $n$ variables of the real datasets. Let $P_{\text{real}}$ be the correlation matrix
- We evaluate the pairwise correlation between the $n$ variables of the synthetic datasets. Let $P_{\text{syn}}$ be the correlation matrix
- We calculate $P_{\text{diff}} = P_{\text{real}} - P_{\text{syn}}$
To evaluate how the generated datasets preserve the marginal distributions of the Kaggle datasets, we estimate some descriptive statistics such as the quartiles and the standard deviation on the feature variables of the Kaggle and synthetic datasets. Next, we calculate the errors between these statistics that are shown in Figure 5. We also proceed to the Kolmogorov-Smirnov (K-S) test that compares the empirical cumulative distribution functions between the feature variables of the Kaggle and synthetic datasets. Figure 6 shows the results of the p-values. Lower values of p-values (< 5%) means that we reject the hypothesis that the distribution is the same. The analysis of the two figures will capture the proximity in terms of marginal distribution between the kaggle and the synthetic datasets.

The analysis of Figure 5 reveals significant errors between the statistics of the autoencoders methods and the Kaggle datasets while the errors are in general close to zeros for the copulas and the SMOTE. This means that both copulas and the SMOTE are able to preserve the statistic measures of the Kaggle datasets during the generation of datasets. Analysis of the p-values of the K-S test in Figure 6 shows that the hypothesis of same distribution between the variables of the Kaggle and the copulas datasets can not be rejected at the significance level of 5%. The small values for the p-values of the autoencoders and the SMOTE generally leads to reject the assumption of same distribution on the feature variables between these datasets and the Kaggle dataset. We can conclude that among the 3 methods, the copula method is the most efficient in generating synthetic datasets while preserving correlations and marginal distributions.

4.2. Generating the private categorical datasets

Private datasets are provided by Netguardians and consist of transactions for customers of a financial institution and includes 15 features of which 2 are categorical. It contains a total of 1,000 transactions made in various currencies to different destinations. For confidentiality reasons the names of the financial institution are not mentioned. Our
goal is to generate these datasets using the copulas method, autoencoders and SMOTE and compare their performance. To assess performance, we will especially focus on how the generated datasets capture the association between the two categorical features.

The datasets generation process here consists of performing some transformations to the categorical features to obtain numerical features before applying the 3 datasets generation methods. Then, we apply the second phase of the transformation methods to obtain the categorical features as described in section (3.2). The 2 categorical features concern the currency and the destination on the transactions for certain clients of the financial institution. Table 3 shows the cross-tabulation of two categorical variables in the datasets. The ”Destination” variable has 8 levels and the variable ”currency” has 6 levels and the columns define the number of transactions that are made in different currencies for a given destination. We transform the two categorical features into numerical ones, then we apply the 3 synthetic generation methods to these datasets following the algorithms in section (3.2). The results of the cross-tabulations between the 2 categorical generated features are shown in Tables 4 to 6.

Table 4 shows that the copulas method focused on the levels of the two categorical features that are more associated and neglected the levels that are less associated. In other words, the copula method during the process of generating synthetic datasets reproduces essentially the characteristics of the levels of the categorical features that are more associated. In Table 5, it seems that the SMOTE method tends to keep most of the levels for the two features but there is the presence of noises in the distribution which will probably reduce the association between the two features. In Table 6, the autoencoders method focuses only on one level of the currency and its repartition through the destinations changes significantly compared to the original datasets. The Chi-squared test for independence is applied to the contingency tables and shows that the independence between the categorical features is rejected for all datasets except the one generated by autoencoders. Finally, Figure 7 shows the Kendall’s tau correlation between the continuous features in the synthetic datasets after the transformation of the categorical ones. It reveals that among the 3 methods, the copulas method captures more closely the correlation in the original datasets between the two continuous features. The autoencoders method shows a high negative correlation that differs significantly from the original datasets.

5. Conclusion

We present a method of generating synthetic data based on copulas and in this context it is shown how to deal with categorical variables. Our model is experimented on Kaggle and private datasets and its performance is compared to that of the SMOTE and autoencoders models. The results show that the copulas approach clearly outperforms other approaches because it better preserves the dependency structure and marginal distributions of the original datasets. The autoencoders method produces the least accurate results but its performance could eventually be improved with further hyperparameters optimization. It is important to note that this paper describes the first steps in our data synthesis work. For the categorical variables generation, we conducted our experiments on a private dataset consisting of approximately 1,000 records and 15 variables, of which we focused on 2 categorical variables. We are aware that in many cases it is necessary to synthesise categorical datasets with hundreds of thousands records and several hundreds of categorical variables. We therefore acknowledge that further experimentation is required to draw definitive conclusions with respect to the ability of the tested algorithms to generate synthetic data especially in the context of categorical variables. This is left for future work.
Table 3. Cross-tabulation between the features Currency and Destination in the original datasets

| Currency | Destination | Dst1 | Dst2 | Dst3 | Dst4 | Dst5 | Dst6 | Dst7 | Dst8 |
|----------|-------------|------|------|------|------|------|------|------|------|
| Cur1     |             | 0    | 3    | 0    | 0    | 0    | 0    | 0    | 0    |
| Cur2     |             | 550  | 0    | 5    | 0    | 45   | 0    | 0    | 0    |
| Cur3     |             | 100  | 0    | 1    | 1    | 0    | 0    | 198  | 0    |
| Cur4     |             | 2    | 0    | 0    | 0    | 0    | 91   | 0    | 1    |
| Cur5     |             | 0    | 0    | 0    | 0    | 0    | 0    | 0    | 1    |
| Cur6     |             | 0    | 2    | 0    | 0    | 0    | 0    | 0    | 0    |

Table 4. Cross-tabulation between the features Currency and Destination in the synthetic datasets generated by the copulas

| Currency | Destination | Dst1 | Dst2 | Dst3 | Dst4 | Dst5 | Dst6 | Dst7 |
|----------|-------------|------|------|------|------|------|------|------|
| Cur2     |             | 452  | 1    | 1    | 0    | 17   | 7    | 21   |
| Cur3     |             | 120  | 0    | 0    | 0    | 0    | 0    | 257  |
| Cur4     |             | 1    | 0    | 0    | 1    | 1    | 120  | 1    |

Table 5. Cross-tabulation between the features Currency and Destination in the synthetic datasets generated by SMOTE

| Currency | Destination | Dst1 | Dst2 | Dst3 | Dst4 | Dst5 | Dst6 | Dst7 |
|----------|-------------|------|------|------|------|------|------|------|
| Cur1     |             | 1    | 0    | 0    | 0    | 0    | 0    | 2    |
| Cur2     |             | 450  | 1    | 0    | 0    | 12   | 33   | 102  |
| Cur3     |             | 153  | 3    | 3    | 0    | 20   | 44   | 82   |
| Cur4     |             | 24   | 0    | 0    | 2    | 6    | 19   | 30   |
| Cur5     |             | 0    | 0    | 0    | 0    | 0    | 1    | 0    |
| Cur6     |             | 0    | 0    | 1    | 0    | 0    | 1    | 1    |

Table 6. Cross-tabulation between the features Currency and Destination in the synthetic datasets generated by the autoencoders method

| Currency | Destination | Dst2 | Dst3 | Dst4 | Dst5 | Dst6 | Dst7 | Dst8 |
|----------|-------------|------|------|------|------|------|------|------|
| Cur2     |             | 57   | 127  | 559  | 8    | 24   | 187  | 38   |
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