FINGERPRINTS, LEMNISCATES AND QUADRATIC DIFFERENTIALS

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Abstract. We discuss some aspects of the theory of recognition of two-dimensional shapes by means of fingerprints of Jordan curves. An interesting approach to problems on shape recognition suggested by P. Ebenfelt, D. Khavinson, and H. Shapiro and extended further by M. Younsi reveals the fact that the fingerprints of polynomial lemniscates and, more generally, fingerprints of rational lemniscates can be obtained as solutions to certain functional equations involving Blaschke products.

Our main goal here is to develop an approach which relates fingerprints of Jordan curves composed of arcs of trajectories and orthogonal trajectories of certain quadratic differentials with solutions of functional equations involving pullbacks of these quadratic differentials under appropriate Riemann mapping functions. In particular, we show that the previous results of P. Ebenfelt, D. Khavinson, and H. Shapiro and the recent results of M. Younsi follow from our more general theorems as special cases.

1. Shapes and fingerprints

The study of two-dimensional “shapes” and their “fingerprints”, initiated by A. Kirillov [7] and developed by E. Sharon and D. Mumford [14], became a rather hot topic in recent publications on applications of complex analysis to problems in pattern recognition. Several authors have explored their own ways to work in this area; see, for instance, a paper [10] of D. Marshall, who used his zipper algorithm and a paper [19] of B. Williams, who applied a circle packing technique. An interesting approach to the fingerprint problem was suggested by P. Ebenfelt, D. Khavinson, and H. Shapiro in [2]. In particular, they showed that fingerprints of polynomial lemniscates (which, by the well-known theorem of D. Hilbert (cf. Chapter 4 in [18]), are dense in the space of all two dimensional shapes) are generated by solutions of functional equations which involve Blaschke products. A simpler proof of the main result in [2] and its generalization to the case of rational lemniscates were presented in a nice short paper by M. Younsi [20]. One more approach, which allows certain reinterpretation of the results of P. Ebenfelt, D. Khavinson, and H. Shapiro and results of M. Younsi, was recently suggested by T. Richards; see [12] and references therein.

My intention here is to emphasize the role of quadratic differentials in this developing theory. In the context of image recognition, quadratic differentials were already used by S. Huckemann, T. Hotz, and A. Munk in their very interesting paper [6]. I want to stress here that our approach is different from the approach used in [6]. First, we introduce, following the presentations in [14] and [2], two-dimensional shapes and their fingerprints. Let Γ be a Jordan curve in the complex plane C and let Ω− and Ω+ denote the bounded and unbounded components of C \ Γ, where C is the complex sphere. Then Ω− and Ω+ are simply connected.

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domains and therefore, by the Riemann mapping theorem, there exist conformal and one-to-one maps $\varphi_- : D \to \Omega_-$ and $\varphi_+ : D_+ \to \Omega_+$, where $D = \{z : |z| < 1\}$ is the unit disk and $D_+ = \overline{C \setminus D}$. Figure 1 illustrates basic notations related to the curve $\Gamma$ and mapping functions $\varphi_- \text{ and } \varphi_+$. 

We suppose that $\varphi_+$ is normalized by the conditions $\varphi_+(\infty) = \infty, \varphi_+'(\infty) > 0$, where $\varphi_+'(\infty) = \lim_{z \to \infty} \varphi_+(z)/z$. The latter normalization defines $\varphi_+$ uniquely. By the Carathéodory theorem on boundary correspondence (cf. Theorem 4′ in [5, Chapter 2, §3]), each of the maps $\varphi_- \text{ and } \varphi_+$ extends as a continuous one-to-one function to the unit circle $T = \partial D$. Therefore, the composition $k = \varphi_+^{-1} \circ \varphi_-$ defines an orientation preserving automorphism of $T$. Since $\varphi_-$ is uniquely determined up to a precomposition with a Möbius automorphism of $D$, the automorphism $k$ is also uniquely determined up to a Möbius automorphism of $D$, i.e. up to a precomposition with maps

\[ \phi(z) = \lambda \frac{z - a}{1 - \overline{a} z}, \quad |\lambda| = 1, \quad a \in \mathbb{D}. \]  

The equivalence class of the automorphism $k$ under the action of the Möbius group of automorphisms (1.1) is called the fingerprint set of $\Gamma$ (many authors prefer a shorter name the fingerprint) and elements of this set are called fingerprints. Furthermore, the fingerprint set of the curve $\Gamma$ is invariant under translations and scalings of $\Gamma$, i.e. under affine maps $L(z) = az + b$ with $a > 0, b \in \mathbb{C}$. The equivalence class of a Jordan curve $\Gamma$ under the action of affine maps of this form is called the shape and $\Gamma$ is a representative of this shape. In what follows, we mainly work with smooth and piecewise smooth Jordan curves $\Gamma$; i.e. we mostly work with Jordan curves $\Gamma$, which have a continuous or piecewise continuous unit tangent vector. More precisely, a Jordan curve $\Gamma$ is said to be piecewise smooth if it consists of a finite number of simple arcs, say $\gamma_1, \ldots, \gamma_n$, such that every arc $\gamma_k$ has a unit tangent vector $\vec{v}(a)$ at every point $a \in \gamma_k$, including one-sided unit tangent vectors at the endpoints of $\gamma_k$ and such that $\vec{v}(a)$ is continuous on $\gamma_k$, including one-sided continuity at the endpoints of $\gamma_k$. In these cases, the corresponding shapes will be called smooth shapes and piecewise smooth shapes, accordingly. Thus, we have a map $F$ from the set of all shapes into the set of all equivalence classes of orientation preserving homeomorphisms of $T$ onto itself. Let $S^1$ denote the class of all smooth shapes in $\mathbb{C}$ and let $\text{Diff}_M(T)$ denote the set of equivalence classes (under the action of the Möbius group of automorphisms (1.1)) of orientation preserving diffeomorphisms of $T$. The following pioneering result was proved by A. Kirillov.

**Theorem 1** ([7]). The map $F$ is a bijection between $S^1$ and $\text{Diff}_M(T)$.

In fact, Theorem 1 is a consequence of the "fundamental theorem of conformal welding" proved by A. Pfluger in 1961, [11].

In other words, Theorem 1 says that $\text{Diff}_M(T)$ parameterizes the set $S^1$ of all smooth shapes. The set of all diffeomorphisms of $T$ is rather large. So, P. Ebenfelt,
D. Khavinson, and H. Shapiro [2] studied possible parameterizations of fingerprints of polynomial lemniscates. A lemniscate of a polynomial \( P(z) \) at level \( c \) is defined as \( L_P(c) = \{ z : |P(z)| = c \} \). Although in this paper we deal mostly with analytic and connected lemniscates, we emphasize here that the polynomial lemniscates are not necessarily connected or Jordan, in general. Later on, we will consider also rational lemniscates and lemniscates associated with nonconstant meromorphic functions. The set of analytic and connected polynomial lemniscates is much smaller than the set of all smooth Jordan curves, nevertheless, by the Hilbert theorem mentioned above, it still can be used to approximate any Jordan shape. The following theorem of P. Ebenfelt, D. Khavinson, and H. Shapiro characterizes exactly which elements of the set of diffeomorphisms of the unit circle \( T \) appear to be the fingerprints of polynomial lemniscates.

**Theorem 2** ([2]). Let \( P(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_0 \) be a polynomial of degree \( n \) with \( c_n > 0 \) such that \( L_P(1) \) is analytic and connected and let \( k : T \to T \) be a fingerprint of \( L_P(1) \). Then \( k(z) \) is given by the equation

\[
k(z) = B(z)^{1/n},
\]

where \( B(z) \) is a Blaschke product of degree \( n \),

\[
B(z) = e^{i\alpha} \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z},
\]

with some real \( \alpha \), where \( a_k = \varphi^{-1}(\zeta_k) \) and \( \zeta_1, \ldots, \zeta_n \) are the zeroes of \( P(z) \) counting multiplicities.

Conversely, given any Blaschke product of degree \( n \), there is a polynomial \( P(z) \) of the same degree whose lemniscate \( L_P(1) \) is analytic and connected and has \( k(z) = B(z)^{1/n} \) as its fingerprint. Moreover, \( P(z) \) is unique up to precomposition with an affine map of the form \( L(z) = az + b \) with \( a > 0 \) and \( b \in \mathbb{C} \).

Throughout the paper, the functional notations like \( k = k(z) \), \( k = k(e^{i\theta}) \), etc. will be reserved to denote fingerprints. However, the letter \( k \) will be also used as an index in several formulas involving products and sums. We believe that this usage will not confuse the reader since the meaning of symbol \( k \) will be clear from the context.

The proof of Theorem 2 given in [2] is rather involved. A shorter proof was given by M. Younsi [20] who also proved a counterpart of this theorem for the case of rational lemniscates, which was conjectured in [2].

**Theorem 3** ([20]). Let \( R(z) \) be a rational function of degree \( n \) with \( R(\infty) = \infty \) such that its lemniscate \( L_R(1) = \{ z : |R(z)| = 1 \} \) is analytic and connected and let \( k : T \to T \) be a fingerprint of \( L_R(1) \). Then \( k(z) \) is given by a solution to the functional equation

\[
A \circ k = B,
\]

where \( A(z) \) and \( B(z) \) are Blaschke products of degree \( n \) and \( A(\infty) = \infty \).

Conversely, given any solution \( k(z) \) to a functional equation \( A \circ k = B \), where \( A(z) \) and \( B(z) \) are Blaschke products of degree \( n \) and \( A(\infty) = \infty \), there exist a rational function \( R(z) \) of degree \( n \) with \( R(\infty) = \infty \) whose lemniscate \( L_R(1) \) is analytic and connected and has \( k(z) \) as its fingerprint.

We will see later that equations (1.2) and (1.3) can be interpreted in a rather natural way in terms of quadratic differentials. Necessary properties of quadratic differentials and their relation with fingerprints are discussed in Section 2. In particular, in Lemma 1 and Theorem 4 we show how to construct a fingerprint of a
piecewise smooth Jordan curve consisting of arcs of trajectories and/or arcs of orthogonal trajectories of certain quadratic differentials. Then in Theorems 5 and 6 we describe the so-called welding procedure used to construct a curve by its fingerprint related to a pair of quadratic differentials, one of which is defined in the unit disk and the other one is defined in the exterior of the unit disk.

In fact, the welding procedure mentioned above can be used to construct a rather broad variety of quadratic differentials with desired topological properties. This approach will be explained in Section 3. Our presentation of the results in Section 3 follows closely to the exposition of results in our paper [15] where this approach was initiated. Our main goal in this section is to set up terminology and present results in the form which might be useful in future work on image recognition.

In Section 4, we work with lemniscates of meromorphic functions. In particular, we show how earlier results of P. Ebenfelt, D. Khavinson, and H. Shapiro in [2] and more recent results of M. Younsi in [20] follow from our theorems presented in Section 2.

Finally in Section 5, we will demonstrate how our method can be applied to study fingerprints of polygonal Jordan curves when the corresponding Riemann mapping functions can be expressed in terms of the well-known Schwarz-Christoffel integrals.

Some interesting problems related to our results presented in this paper are discussed in the manuscript [4] by A. Frolova, D. Khavinson, and A. Vasil’ev, which was communicated to us by one of the referees.

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2. Fingerprints and quadratic differentials

For the definitions and results on quadratic differentials needed for the purposes of this paper the reader may consult the classical monographs of J.A. Jenkins [9] and K. Strebel [17] and papers [8], [15], and [16].

A quadratic differential on a domain \( D \subset \mathbb{C} \) is a differential form \( Q(z) \, dz^2 \) with meromorphic \( Q(z) \) and with the conformal transformation rule

\[
Q_1(\zeta) \, d\zeta^2 = Q(\varphi(z)) \left( \varphi'(z) \right)^2 \, dz^2, \tag{2.1}
\]

where \( \zeta = \varphi(z) \) is a conformal map from \( D \) onto a domain \( G \) in the extended plane of the parameter \( \zeta \). Then the zeros and poles of \( Q(z) \) are critical points of \( Q(z) \, dz^2 \), in particular, zeros and simple poles are finite critical points and poles of order greater than 1 are infinite critical points of \( Q(z) \, dz^2 \). A trajectory (respectively, orthogonal trajectory) of \( Q(z) \, dz^2 \) is a closed analytic Jordan curve or maximal open analytic arc \( \gamma \subset D \) such that

\[
Q(z) \, dz^2 > 0 \text{ along } \gamma \quad \text{(respectively, } Q(z) \, dz^2 < 0 \text{ along } \gamma).\]

A trajectory \( \gamma \) is called critical if at least one of its endpoints is a finite critical point of \( Q(z) \, dz^2 \). If \( \gamma \) is a rectifiable arc in \( D \) then its \( Q \)-length is defined by

\[
|\gamma|_Q = \int_{\gamma} |Q(z)|^{1/2} |dz|. \tag{2.2}
\]

An important property of quadratic differentials is that the transformation rule \( Q_1(\zeta) \, d\zeta^2 = Q(\varphi(z)) \left( \varphi'(z) \right)^2 \, dz^2 \) respects trajectories, orthogonal trajectories and their \( Q \)-lengths, as well as it respects critical points together with their multiplicities and trajectory structure nearby.
Returning to fingerprints, suppose that $\Gamma$ is a piecewise smooth Jordan curve in the plane $\mathbb{C}$ with the parameter $\zeta$ and that $Q(\zeta)\,d\zeta^2$ is a quadratic differential on some neighborhood $G$ of $\Gamma$. The best case scenario is when $G = \mathbb{T}$. More generally, we may assume, without loss of generality, that $G$ is a doubly connected domain bounded by Jordan analytic curves and that $\Gamma$ separates boundary components of $G$. Let $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ denote pullbacks of $Q(\zeta)\,d\zeta^2$ under the conformal maps $\zeta = \varphi_-(z)$ and $\zeta = \varphi_+(z)$ defined in Section 1 and let $\tau_-(\zeta) = \varphi_-^{-1}(\zeta)$ and $\tau_+(\zeta) = \varphi_+^{-1}(\zeta)$. Then
\[ Q(\zeta)\,d\zeta^2 = Q_-(\tau_-(\zeta))(\tau'_-(\zeta))^2\,d\zeta^2 \quad \text{for} \quad \zeta \in (\Omega_- \cap G) \cup \Gamma \tag{2.2} \]
and
\[ Q(\zeta)\,d\zeta^2 = Q_+(\tau_+(\zeta))(\tau'_+(\zeta))^2\,d\zeta^2 \quad \text{for} \quad \zeta \in (\Omega_+ \cap G) \cup \Gamma. \tag{2.3} \]

Since $\Gamma$ is piecewise smooth it follows that each of the maps $\tau_-(\zeta)$ and $\tau_+(\zeta)$ and their derivatives $\tau'_-(\zeta)$ and $\tau'_+(\zeta)$ can be extended by continuity to any smooth arc of $\Gamma$. If $\Gamma$ is smooth at $\zeta$ then equations (2.2) and (2.3) imply that
\[ Q_-(\tau_-(\zeta))(\tau'_-(\zeta))^2\,d\zeta^2 = Q_+(\tau_+(\zeta))(\tau'_+(\zeta))^2\,d\zeta^2. \tag{2.4} \]
Changing the variable in (2.4) via $\zeta = \varphi_-(z)$, we obtain the equivalent equation
\[ Q_-(z)\,dz^2 = Q_+(k(z))(k'(z))^2\,dz^2, \tag{2.5} \]
which holds for all points $z \in \mathbb{T}$ such that $\varphi_-(z)$ belongs to a smooth arc of $\Gamma$. Here $k = \varphi_-^{-1} \circ \varphi_-$ is an homeomorphism from $\mathbb{T}$ onto itself and therefore it is a fingerprint of $\Gamma$ as we defined in Section 1.

Taking square roots of both sides of equation (2.5) and then integrating along the unit circle, we obtain
\[ \int_{z_0}^{z} \sqrt{Q_+(k(\tau))k'(\tau)} \,d\tau = \int_{z_0}^{z} \sqrt{Q_-(\tau)} \,d\tau, \tag{2.6} \]
where $z_0 = e^{i\theta_0}$, $z = e^{i\theta}$ with $0 \leq \theta_0 < 2\pi$, $\theta_0 \leq \theta \leq \theta_0 + 2\pi$ and integration in (2.6) is taken along a circular arc $l(\theta_0, \theta) = \{ \tau = e^{it} : \theta_0 \leq t \leq \theta \}$.

We note that equation (2.6) contains integrals of square roots of meromorphic functions and therefore it requires appropriate interpretation. In the case when $\Gamma$ does not contain infinite critical points of $Q(\zeta)\,d\zeta^2$, the integrals in (2.6) are finite and therefore this equation holds for all $z_0$ and $z$ as in (2.7) if the branches of square roots in (2.6) are chosen appropriately.

If $\Gamma$ contains infinite critical points of $Q(\zeta)\,d\zeta^2$ then (2.6) holds if the arc of integration $l(\theta_0, \theta)$ does not contain infinite critical points of $Q_-(z)\,dz^2$. In the presence of infinite critical points of $Q_-(z)\,dz^2$ on $\mathbb{T}$, the integrals in (2.6) blow up and we actually have to consider separate equations for each maximal arc of $\mathbb{T}$ which is free of infinite critical points.

The following lemma summarizes the simple observations made above for the case when infinite critical points are absent.

**Lemma 1.** Let $\Gamma$ be a piecewise smooth Jordan curve and let $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ be pullbacks of the quadratic differential $Q(\zeta)\,d\zeta^2$ introduced above. If $\Gamma$ does not contain infinite critical points of $Q(\zeta)\,d\zeta^2$ then there is a fingerprint $k : \mathbb{T} \to \mathbb{T}$ of $\Gamma$ given by a solution to the functional equation
\[ A \circ k = B, \tag{2.8} \]
where
\[ A(z) = \int_{k(z_0)}^{z} \sqrt{Q_+(\tau)} \,d\tau, \quad B(z) = \int_{z_0}^{z} \sqrt{Q_-(\tau)} \,d\tau. \tag{2.9} \]
with $z_0$ as in (2.7) and with appropriately chosen branches of the radicals.

Since solutions to equations (2.5), (2.6), and (2.8) provide a kind of welding of $\Gamma$ by quadratic differentials, we will call them equations of QD-welding of $\Gamma$. These equations do not provide much information on the fingerprint of $\Gamma$ unless we know how to construct quadratic differentials $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ with the required properties. Fortunately enough, there is a rather general procedure, suggested in our paper [16], which in many important cases allows us to construct the quadratic differentials with the desired properties. This construction, which presents some independent interest, will be discussed in Section 3.

Equations (2.8) and (1.3) have the same form and this, of course, is not a coincidence. In Section 4, we will show that equations (1.2) and (1.3) are special forms of equation (2.5) for the cases of polynomial and rational lemniscates, respectively.

In Lemma 1 we assumed that $\Gamma$ does not contain infinite critical points of $Q(\zeta)\,d\zeta^2$. To address the fingerprint problem in the general case, we need an approximation lemma, which easily follows from the classical convergence theorem of T. Radó. We need the following definition, which is included in the statement of Theorem 2 in [5, Chapter 2, §5].

**Definition 1.** We say that a sequence of Jordan curves $\Gamma_n$, $n = 1, 2, \ldots$, converges to a Jordan curve $\Gamma$ in Radó’s sense if for every $\varepsilon > 0$ there is positive integer $N$ such that for every $n \geq N$ there exists a continuous one-to-one correspondence between the points of $\Gamma_n$ and the points of $\Gamma$ such that the distance between corresponding points of $\Gamma_n$ and $\Gamma$ is less than $\varepsilon$.

**Lemma 2.** Let $\Gamma_n$, $n = 1, 2, \ldots$, be a sequence of Jordan curves which converges in Radó’s sense to a Jordan curve $\Gamma$. Let $\Omega^+_n$ and $\Omega^-_n$ denote the bounded and unbounded components of $\overline{\mathbb{C}} \setminus \Gamma_n$ and let $z_0 \in \Omega_-$ and $z_0 \in \Omega_+^-$ for all $n = 1, 2, \ldots$. Let $k_n = (\varphi^+_n)^{-1} \circ \varphi^-_n$ and $k = \varphi^+_n \circ \varphi^-_n$ denote the fingerprints of $\Gamma_n$ and $\Gamma$ unequally determined by conditions $\varphi^+_n(0) = z_0$, $(\varphi^+_n)'(0) > 0$, and $\varphi^-_-(0) = z_0$, $(\varphi^-_-(0))' > 0$, respectively. Then, $k_n(e^{i\theta})$ converges to $k(e^{i\theta})$ uniformly on $\mathbb{T}$.

**Proof.** Radó’s convergence theorem (see Theorem 2 in [5, Chapter 2, §5]) implies that $\varphi^+_n(z) \to \varphi^+_\infty(z)$ uniformly on $\mathbb{T}$ and that $\varphi^-_n(z) \to \varphi^-\infty(z)$ uniformly on the closed annulus $\{z : 1 \leq |z| \leq 2\}$. Since $k_n = (\varphi^+_n)^{-1} \circ \varphi^-_n$ and $k = \varphi^+_\infty \circ \varphi^-\infty$, the lemma follows.

Now, we can extend Lemma 1 for the case when a piecewise smooth Jordan curve $\Gamma$ contains infinite critical points of $Q(\zeta)\,d\zeta^2$ as follows. For any $a \in \Gamma$ and $\varepsilon > 0$ small enough, let $\gamma_\varepsilon(a)$ denote an arc of $\Gamma$ lying inside the circle $C_\varepsilon(a) = \{\zeta : |\zeta - a| = \varepsilon\}$. Since $\Gamma$ is piecewise smooth the arc $\gamma_\varepsilon(a)$ is defined uniquely for all $\varepsilon > 0$ small enough. Furthermore, let $l_\varepsilon(a)$ denote an arc of $C_\varepsilon(a)$, which lies in the closure of the domain $\Omega_-$ and joins the endpoints of $\gamma_\varepsilon(a)$. Since $\Gamma$ is piecewise smooth the arc $l_\varepsilon(a)$ is also defined uniquely for all $\varepsilon > 0$ small enough.

Let $\Gamma_\varepsilon$ denote the curve obtained from $\Gamma$ by replacing $\gamma_\varepsilon(a)$ with $l_\varepsilon(a)$ for all infinite critical points $a \in \Gamma$. If $\Gamma$ is piecewise smooth and $\varepsilon > 0$ is small enough, then $\Gamma_\varepsilon$ is a piecewise smooth Jordan curve which does not contain infinite critical points of $Q(\zeta)\,d\zeta^2$. Take a sequence $\varepsilon_n \to 0$. By Lemma 1, the fingerprint $k_n$ of $\Gamma_{\varepsilon_n}$ can be obtained from equation (2.8). Then, by Lemma 2, the fingerprint $k$ of $\Gamma$ can be found as the limit

$$k(e^{i\theta}) = \lim_{n \to \infty} k_n(e^{i\theta}).$$

(2.10)

We note here that Jordan curves $\Gamma_\varepsilon$ described above can be constructed in a more general case when $\Gamma$ is smooth except at most finite number of points $a \in \Gamma$ such that for each of these points and every $\varepsilon > 0$ sufficiently small each of the
intersections $\Omega_- \cap C_\epsilon(a)$ and $\Omega_+ \cap C_\epsilon(a)$ consists of a single arc. In particular, this happens in the case when $\Gamma$ consists of a finite number of arcs of trajectories and/or orthogonal trajectories of a quadratic differential. We emphasize that, in the latter case, $\Gamma$ may have points $a \in \Gamma$ such that in a sufficiently small neighborhood of $a$ the arcs of $\Gamma \setminus \{a\}$ behave like logarithmic spirals. Therefore, the limit relation (2.10) remains valid if $\Gamma$ is not necessarily piecewise smooth but consists of a finite number of arcs of trajectories and/or orthogonal trajectories of some quadratic differential (as in our Theorems 4 and 5 presented below).

We admit here that equation (2.8) is of little practical use unless we know how to find functions $A(z)$ and $B(z)$ defined by (2.9) or, equivalently, how to construct quadratic differentials $Q_-(z)dz^2$ and $Q_+(z)dz^2$. In general, such construction looks problematic. Rare cases when this is possible include cases of polynomial and rational lemniscates presented in Theorems 2 and 3. Below in this section, we explain how to find a general form of the quadratic differentials $Q_-(z)dz^2$ and $Q_+(z)dz^2$ making some additional assumptions.

Namely, we suppose now that a quadratic differential $Q(\zeta)d\zeta^2$ is defined on the whole complex sphere $\mathbb{C}$. Then, of course, $Q(\zeta)$ is a rational function. Furthermore, we suppose that $\Gamma$ is a Jordan curve consisting of a finite number of arcs, $\gamma_1, \ldots, \gamma_m$, of trajectories and/or orthogonal trajectories of $Q(\zeta)d\zeta^2$ and their endpoints. In this context, the whole trajectory or orthogonal trajectory is also considered as an arc. In particular, we allow cases when $\Gamma$ consists of a single trajectory or orthogonal trajectory, closed or not. Infinite critical points on $\Gamma$ are also allowed.

First, we discuss briefly the possible structure of $\Gamma$ in a neighborhood of a point $\zeta_0 \in \Gamma$ where two arcs, say $\gamma_1$ and $\gamma_2$, meet. All properties stated in items (1)–(5) below follow from the well-known results on the local structure of trajectories of quadratic differentials; see, for instance, Ch. 3 in [9].

1. If $\zeta_0$ is a regular point of $Q(\zeta)d\zeta^2$, then one of the arcs $\gamma_1$ or $\gamma_2$ is an arc of a trajectory and the other one is an arc of an orthogonal trajectory. In this case, $\gamma_1$ and $\gamma_2$ form a corner of opening $\pi/2$ with respect to one of the domains $\Omega_-$ and $\Omega_+$ and a corner of opening $3\pi/2$ with respect to the other one.

2. If $\zeta_0$ is a zero of $Q(\zeta)d\zeta^2$ of order $n$ then the arcs $\gamma_1$ and $\gamma_2$ form an angle of opening $\pi k/(n+2)$ with some integer $k$, $0 < k < 2(n+2)$. Moreover, if $k$ is odd then one of the arcs $\gamma_1$ or $\gamma_2$ is an arc of a trajectory and the other one is an arc of an orthogonal trajectory. If $k$ is even then both $\gamma_1$ and $\gamma_2$ are either arcs of trajectories or arcs of orthogonal trajectories.

3. If $\zeta_0$ is a simple pole, then one of the arcs $\gamma_1$ or $\gamma_2$ is an arc of a trajectory and the other one is an arc of an orthogonal trajectory and $\Gamma$ is analytic at $\zeta_0$.

4. If $\zeta_0$ is a pole of order two then both $\gamma_1$ and $\gamma_2$ are either arcs of trajectories or arcs of orthogonal trajectories. If they are arcs of trajectories then $Q(\zeta)d\zeta^2$ has a radial or spiral structure of trajectories near $\zeta_0$ and if they are arcs of orthogonal trajectories $Q(\zeta)d\zeta^2$ has circular or spiral structure of trajectories. In case of the radial or circular structure of trajectories, $\gamma_1$ and $\gamma_2$ can form any angle $\alpha$, $0 < \alpha < 2\pi$. In case of the spiral structure of trajectories, $\gamma_1$ and $\gamma_2$ in a neighborhood of $\zeta_0$ look like logarithmic spirals and do not form any angle at $\zeta_0$.

5. Finally, if $\zeta_0$ is a pole of order $n \geq 3$, then $\gamma_1$ and $\gamma_2$ can form any angle of opening $\pi k/(n-2)$ with some integer $k$, $0 \leq k \leq 2(n-2)$. Moreover, if $k$ is odd then one of the arcs $\gamma_1$, or $\gamma_2$ is an arc of a trajectory and the other one is an arc of an orthogonal trajectory. If $k$ is even then both $\gamma_1$ and $\gamma_2$ are either arcs of trajectories or arcs of orthogonal trajectories.
Summarizing, we conclude that $\Gamma$ may contain arcs of logarithmic spirals, and therefore it is not necessarily rectifiable, and it may have interior and exterior cusps and corners of any angle.

Now, we discuss pullbacks $Q_{-}(z)\, dz^2$ and $Q_{+}(z)\, dz^2$ of the quadratic differential $Q(\zeta)\, d\zeta^2$. It is enough to consider the quadratic differential $Q_{-}(z)\, dz^2$, the trajectory structure of $Q_{+}(z)\, dz^2$ is similar. Under our assumptions, $Q_{-}(z)\, dz^2$ is defined on the whole unit disk $\mathbb{D}$. Furthermore, the unit circle $\Gamma$ consists of a finite number of arcs of trajectories and/or orthogonal trajectories of $Q_{-}(z)\, dz^2$ and their endpoints. Since $Q_{-}(z)\, dz^2$ is real on $\mathbb{T}$ (possibly except a finite number of points), it follows from the reflection principle that $Q_{-}(z)\, dz^2$ can be continued to a quadratic differential on $\overline{\mathbb{D}}$, for which we will keep the same notation $Q_{-}(z)\, dz^2$, and its trajectory structure is symmetric with respect to $\mathbb{T}$. In particular, the set of zeros and the set of poles of $Q_{-}(z)\, dz^2$ are both symmetric with respect to $\mathbb{T}$.

Since conformal mappings preserve critical points of quadratic differentials as well as the local structure of their trajectories, it follows that the set of critical points of $Q_{-}(z)\, dz^2$ inside $\mathbb{D}$ is in a one-to-one correspondence with the set of critical points of $Q(\zeta)\, d\zeta^2$ inside $\Omega_-$ and that $Q_{-}(z)\, dz^2$ and $Q(\zeta)\, d\zeta^2$ have the same local structure near corresponding critical points.

Now, we identify possible critical points of $Q_{-}(z)\, dz^2$ on $\mathbb{T}$. All our conclusions follow from our knowledge of the local structure of trajectories of $Q(\zeta)\, d\zeta^2$ near endpoints of arcs composing $\Gamma$ (and therefore from our knowledge of local structure of trajectories of $Q_{-}(z)\, dz^2$ near corresponding points of $\mathbb{T}$). When working with critical points of $Q_{-}(z)\, dz^2$ situated on $\mathbb{T}$, we consider $\Gamma$ as the boundary of $\Omega_-$ oriented counterclockwise. To emphasize this fact, we use notation $\Gamma_-$ instead of $\Gamma$ in items (a)–(g) below. Also, all angles mentioned in items (a)–(g) are considered with respect to the domain $\Omega_-$.\[\]

(a) If $\zeta_0$ is a regular point of $Q(\zeta)\, d\zeta^2$ where $\Gamma_-$ forms an angle $\pi/2$, then $z_0 = \tau_{-}(\zeta_0)$ is a simple pole of $Q_{-}(z)\, dz^2$.\[\]

(b) If $\zeta_0$ is a regular point of $Q(\zeta)\, d\zeta^2$ where $\Gamma_-$ forms an angle $3\pi/2$, then $z_0 = \tau_{-}(\zeta_0)$ is a simple pole of $Q_{-}(z)\, dz^2$.\[\]

(c) If $\zeta_0$ is a zero of $Q(\zeta)\, d\zeta^2$ of order $n$ where $\Gamma_-$ forms an angle $\pi/2$, then $z_0 = \tau_{-}(\zeta_0)$ is a simple pole of $Q_{-}(z)\, dz^2$ if $k = 1$, a regular point if $k = 2$, and a zero of order $k - 2$ if $3 \leq k \leq 2n + 3$.\[\]

(d) If $\zeta_0$ is a simple pole of $Q(\zeta)\, d\zeta^2$, then $z_0 = \varphi_{-}(\zeta_0)$ is a simple pole of $Q_{-}(z)\, dz^2$.\[\]

(e) If $\zeta_0$ is a pole of order 2 of $Q(\zeta)\, d\zeta^2$ and $\gamma_1$ and $\gamma_2$ are arcs of orthogonal trajectories, then $z_0 = \varphi_{-}(\zeta_0)$ is a pole of $Q_{-}(z)\, dz^2$ of order 2 with circular trajectory structure.\[\]

(f) If $\zeta_0$ is a pole of order 2 of $Q(\zeta)\, d\zeta^2$ and $\gamma_1$ and $\gamma_2$ are arcs of trajectories, then $z_0 = \varphi_{-}(\zeta_0)$ is a pole of $Q_{-}(z)\, dz^2$ of order 2 with a radial trajectory structure.\[\]

(g) If $\zeta_0$ is a pole of order $n \geq 3$ of $Q(\zeta)\, d\zeta^2$ where $\Gamma_-$ forms an angle $\frac{\pi}{2n-2}$, then $z_0 = \varphi_{-}(\zeta_0)$ is a pole of $Q_{-}(z)\, dz^2$ of order $k + 2$, $0 \leq k \leq 2(n - 2)$. In particular, if $k = 0$ and $\gamma_1$ and $\gamma_2$ are arcs of trajectories, then $z_0$ is a pole of $Q_{-}(z)\, dz^2$ of order 2 with radial trajectory structure and if $k = 0$ and $\gamma_1$, $\gamma_2$ are arcs of orthogonal trajectories, then $z_0$ is a pole of $Q_{-}(z)\, dz^2$ of order 2 with the circular trajectory structure.\[\]

All statements (a)–(g) follow from the well-known results on the local structure of trajectories of quadratic differentials. To demonstrate details, we consider, for instance, item (b). In all other cases the argument is similar and therefore is left to the interested reader. In case (b), there is a neighborhood $G \subset \Omega_-$ of $\zeta_0$ (considered
as a boundary point of $\Omega_-$, which is bounded by three arcs, say $l_1$, $l_2$, and $l_3$, of trajectories and three arcs, say $s_1$, $s_2$, and $s_3$, of orthogonal trajectories of $Q(\zeta)\,d\zeta^2$ as it is shown in Figure 2. This neighborhood is filled up with arcs of trajectories of $Q(\zeta)\,d\zeta^2$, one of which, let $l_1'$, is a continuation of a boundary arc $l_1$ through the point $z_0$. Let $G_z^-$ be the image of $G$ under the mapping $\tau_- = \varphi_0^{-1}$ and let $G_z$ denote the extension of the domain $G_z^-$ obtained by reflection with respect to the arc $L_1 \cup S_1 \cup \{z_0\}$ of the unit circle. Here $L_1 = \tau_-(l_1)$, $S_1 = \tau_-(s_1)$, $z_0 = \tau_-(z_0)$. Then $z_0$ is a point of $G_z$, which serves as an end point for three critical trajectories, $L_1$, $L_1' = \tau_-(l_1')$, and $L_1'' = \{z : 1/z \in L_1'\}$, of the quadratic differential $Q_-(z)\,dz^2$ as it is shown in Figure 2. Since no other trajectory of $Q_-(z)\,dz^2$ terminates at $\tau_-(z_0)$ it follows from the known results on the local structure of trajectories (see, for instance, Theorem 3.2 in [9]) that $z_0 = \tau_-(z_0)$ is a simple zero of $Q_-(z)\,dz^2$.

As we have mentioned above the nature of critical points of $Q_+(z)\,dz^2$ on $\mathbb{T}$ is similar.

Since the quadratic differentials $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ were extended to $\overline{\mathbb{C}}$ via reflection principle they are symmetric with respect to the unit circle $\mathbb{T}$ and can be expressed in terms of rational functions as follows. Let $B_0^\mathbb{C}(z) = \prod_{k=1}^{m_0}(z-c_k)(1-c_kz)$ and $B_\infty^\mathbb{C}(z) = \prod_{k=1}^{m_\infty}(z-p_k)(1-p_kz)$, where the products are taken over all zeros (counting multiplicity) of $Q_-(z)\,dz^2$ in the unit disk $\mathbb{D}$ and over all poles (counting multiplicity) of $Q_-(z)\,dz^2$ in the unit disk $\mathbb{D}$, respectively. Also, let $P_0^\mathbb{C}(z) = \prod_{k=1}^{m_0}e^{-i(\tau-\beta_k)/2}(z-e^{i\beta_k})$ and $P_\infty^\mathbb{C}(z) = \prod_{k=1}^{m_\infty}e^{-i(\tau+\beta_k)/2}(z-e^{-i\beta_k})$, where the products are taken over all zeros (counting multiplicity) of $Q_-(z)\,dz^2$ on the unit circle $\mathbb{T}$ and over all poles (counting multiplicity) of $Q_-(z)\,dz^2$ on the unit circle $\mathbb{T}$, respectively. The coefficients $e^{-i(\tau+\beta_k)/2}$ and $e^{-i(\tau-\beta_k)/2}$ in the products representing $P_0^\mathbb{C}(z)$ and $P_\infty^\mathbb{C}(z)$ are chosen in this form to have a real coefficient $C_-$ in the expression for $Q_-(z)\,dz^2$ in the formula (2.11) below. Let $B_0^\mathbb{T}(z)$, $B_\infty^\mathbb{T}(z)$, $P_0^\mathbb{T}(z)$, and $P_\infty^\mathbb{T}(z)$ denote similar products for the quadratic differential $Q_+(z)\,dz^2$ and let $n_0^+, n_\infty^+, m_0^+, m_\infty^+$ denote the number of terms in the corresponding products.

In the notations introduced above, the quadratic differentials $Q_-(z)\,dz^2$ and $Q_+(z)\,dz^2$ can be written as

$$Q_-(z)\,dz^2 = C_- \frac{P_0^\mathbb{C}(z)B_0^\mathbb{C}(z)}{P_\infty^\mathbb{C}(z)B_\infty^\mathbb{C}(z)}\,dz^2, \quad Q_+(z)\,dz^2 = C_+ \frac{P_0^\mathbb{C}(z)B_0^\mathbb{C}(z)}{P_\infty^\mathbb{C}(z)B_\infty^\mathbb{C}(z)}\,dz^2$$

(2.11)

with some nonzero constants $C_-$ and $C_+$. In fact, the constants $C_-$ and $C_+$ are real. Indeed, it follows from a well known formula linking the number of zeros and the number of poles of a quadratic differential on $\overline{\mathbb{C}}$ (see Lemma 3.2 in [9]), that

$$2n_\infty + m_\infty - 2n_0 - m_0 = 4 \quad \text{and} \quad 2n_\infty^+ + m_\infty^+ - 2n_0^+ - m_0^+ = 4.$$

(2.12)
Next, to show that $C_-$ is real, we remind that the unit circle $\mathbb{T}$ consists of arcs of trajectories and/or arcs of orthogonal trajectories of the quadratic differential $Q_-(z)dz^2$ and therefore \( \arg(Q_-(e^{i\theta})dz) = 0 \pmod{\pi} \) if $z = e^{i\theta}$ is not a critical point of $Q_-(z)dz^2$. The latter equation (after routine calculation left to the interested reader) shows that $\arg C_- = 0 \pmod{\pi}$. To perform such calculation one may use the first equation in (2.11) and take into account the first equation in (2.12). and the fact that the tangent vector $dz$ to the unit circle $\mathbb{T}$ at $z = e^{i\theta}$ has the form $dz = i e^{i\theta}$. Same argument shows that $C_+$ is also real.

Combining equation (2.11) with Lemma 1, we obtain the following.

**Theorem 4.** Let $Q(\zeta)d\zeta^2$ be a quadratic differential on $\mathbb{C}$ and let $\Gamma$ be a Jordan curve free of infinite critical points of $Q(\zeta)d\zeta^2$ which consists of a finite number of arcs of trajectories and/or arcs of orthogonal trajectories of $Q(\zeta)d\zeta^2$ and their endpoints. Then the fingerprint $k : \mathbb{T} \to \mathbb{T}$ of $\Gamma$ is given by a solution to the functional equation (2.9) with $A(z)$ and $B(z)$ defined by (2.9) and $Q_-(z)dz^2$ and $Q_+(z)dz^2$ defined by (2.11).

Theorem 4 generalizes the first part of Theorems 2 and 3. The converse statement for this theorem, similar to the converse statements of Theorems 2 and 3, will require additional restrictions which are discussed below.

Suppose that $Q_-(z)dz^2$ and $Q_+(z)dz^2$ are quadratic differentials satisfying conditions (2.11) and (2.12). Then $\mathbb{T}$ consists of a finite number of maximal open arcs, $\alpha_1, \ldots, \alpha_k$, enumerated in the counterclockwise direction on $\mathbb{T}$, such that for every $j$ either $Q_-(z)dz^2 \geq 0$ for all $z \in \alpha_j^-$ or $Q_-(z)dz^2 \leq 0$ for all $z \in \alpha_j^-$. The maximality here means that if one or another inequality holds on $\alpha_j^-$ then it does not hold on any larger arc containing $\alpha_j^-$. As before, we allow here the case when $k_- = 1$ and $\alpha_1^- = \mathbb{T}$.

Let $z_j^- = e^{i\theta_j}$ denote the initial point of $\alpha_j^-$, $1 \leq j \leq k_-$, and let $\alpha_{k_- + 1}^- = \alpha_1^-$ and $z_{k_- + 1}^- = z_1^-$. The points $z_j^-$ can be of two types:

1) If $Q_-(z)dz^2$ has the same sign on arcs $\alpha_{j-1}^-$ and $\alpha_j^-$ then $z_j^-$ is a pole of $Q_-(z)dz^2$ of even order.

2) If $Q_-(z)dz^2$ has different signs on arcs $\alpha_{j-1}^-$ and $\alpha_j^-$ then $z_j^-$ is a critical point of $Q_-(z)dz^2$ of odd order.

Let $\alpha_1^+, \ldots, \alpha_k^+$ and $z_1^+, \ldots, z_k^+$ denote similar systems of arcs and points for the quadratic differential $Q_+(z)dz^2$.

**Definition 2.** We will say that quadratic differentials $Q_-(z)dz^2$ and $Q_+(z)dz^2$ satisfying conditions (2.11) and (2.12) are coordinated on $\mathbb{T}$ if their systems of arcs $\alpha_1^-, \ldots, \alpha_k^-$ and $\alpha_1^+, \ldots, \alpha_k^+$ have the same number of arcs (i.e. if $k_- = k_+ = k \geq 1$) and if they can be enumerated in the counterclockwise direction on $\mathbb{T}$ in such a way that the following conditions are satisfied:

(a) $Q_-(z)dz^2 \geq 0$ on $\alpha_j^-$ if and only if $Q_+(z)dz^2 \geq 0$ on $\alpha_j^+$.

(b) $z_j^-$ is an infinite critical point of $Q_-(z)dz^2$ if and only if $z_j^+$ is an infinite critical point of $Q_+(z)dz^2$.

(c) If the $Q_-$-length $|\alpha_j^+|_{Q_-} = \int_{\alpha_j^-} \sqrt{Q_-(z)}dz$ of $\alpha_j^-$ is finite then the $Q_+$-length $|\alpha_j^+|_{Q_+} = \int_{\alpha_j^+} \sqrt{Q_+(z)}dz$ of $\alpha_j^+$ is also finite and $|\alpha_j^-|_{Q_-} = |\alpha_j^+|_{Q_+}$.

We emphasize here that the arcs $\alpha_j^-$ and $\alpha_j^+$ may contain zeroes of even order of the quadratic differentials $Q_-(z)dz^2$ and $Q_+(z)dz^2$, respectively, and that the numbers of such zeroes contained in $\alpha_j^-$ and $\alpha_j^+$ may be different.
Theorem 5. Suppose that the quadratic differentials $Q_-(z)dz^2$ and $Q_+(z)dz^2$ are coordinated on $\mathbb{T}$. Then there is a quadratic differential $Q(\zeta)d\zeta^2$ defined on $\overline{U}$ and a closed Jordan curve $\Gamma$ consisting of arcs of trajectories and/or orthogonal trajectories of $Q(\zeta)d\zeta^2$ such that the quadratic differentials $Q_-(z)dz^2$ and $Q_+(z)dz^2$ are pullbacks of the quadratic differential $Q(\zeta)d\zeta^2$ under the appropriate mappings $\varphi_-$ and $\varphi_+$ associated with $\Gamma$.

Proof. The proof of this theorem is rather standard. We weld (some authors prefer the term “glue”) the unit disk $D$ (with the quadratic differential $Q_-(z)dz^2$ defined on it) and its exterior $D_+$ (with the quadratic differential $Q_+(z)dz^2$ defined on it) along the unit circle $\mathbb{T}$ in such a way that $Q_-(z)dz^2$ and $Q_+(z)dz^2$ will be meromorphic extensions of each other across $\mathbb{T}$.

First, to each of the arcs $\alpha_j^-$ and $\alpha_j^+$ we assign a “representative” $w_j^- = e^{i\beta_j^+}$ and a “representative” $w_j^+ = e^{i\beta_j^-}$, respectively. Precisely, we put $w_j^- = z_j^-$, $w_j^+ = z_j^+$ if $z_j^-$ is a finite critical point and we put $w_j^- = z_{j+1}^-$, $w_j^+ = z_{j+1}^+$ if $z_j^-$ is an infinite critical point but $z_{j+1}^-$ is a finite critical point. If both $z_j^-$ and $z_{j+1}^-$ are infinite critical points of $Q_-(z)dz^2$ then representatives $w_j^- = e^{i\beta_j^-}$ and $w_j^+ = e^{i\beta_j^+}$ can be any points in the arcs $\alpha_j^-$ and $\alpha_j^+$, respectively. Also, if $\alpha_j^- = \mathbb{T}$ then representatives $w_j^- = e^{i\beta_j^-}$ and $w_j^+ = e^{i\beta_j^+}$ can be any points of $\mathbb{T}$. Now the welding procedure can be performed as follows.

1) The initial points $z_j^- \in l(\theta_j^-, \beta_j^-)$ and $z_{j+1}^+ \in l(\theta_j^+, \beta_{j+1}^+)$ are considered as identical as well as representatives $w_j^- \in w_{j+1}^+$ of these arcs (if different).

2) The points $z_- = e^{i\theta_-} \in \alpha_j^-$ and $z_+ = e^{i\theta_+} \in \alpha_j^+$ are considered to be identical if either $z_- \in l(\theta_j^-, \beta_j^-)$, $z_+ \in l(\theta_j^+, \beta_{j+1}^+)$ are such that

$$\int_{l(\theta_j^-, \beta_j^-)} \sqrt{Q_-(z)}dz = \int_{l(\theta_j^+, \beta_{j+1}^+)} \sqrt{Q_+(z)}dz$$

or $z_- \in l(\beta_j^-, \theta_{j+1}^-)$, $z_+ \in l(\beta_j^+, \theta_{j+1}^+)$ are such that

$$\int_{l(\beta_j^-, \theta_j^-)} \sqrt{Q_-(z)}dz = \int_{l(\beta_j^+, \theta_{j+1}^+)} \sqrt{Q_+(z)}dz.$$

After this identification of boundary points we obtain a topological sphere $S$. To turn it into a Riemann sphere we introduce complex structure on $S$ as follows. We assume that the disk $D$ and its exterior $D_+$ serve as charts for their points and thus the complex parameter for these points is given by the identity mapping $z = \tau$.

Suppose now that $z_0 \in S$ is defined by identifying the points $z_0^\pm = e^{i\theta_0} \in \alpha_j^-$ and $z_0^\pm = e^{i\theta_0} \in \alpha_j^+$, which are not critical points of the quadratic differentials $Q_-(z)dz^2$ and $Q_+(z)dz^2$, respectively. Then the complex parameter $\tau$ can be assigned as follows. Let $\varepsilon > 0$ be small enough and let $U_\varepsilon = \{ \tau : |\tau| < \varepsilon \}$. For $\tau \in U_\varepsilon$, we define a function $F : U_\varepsilon \to S$ to be the inverse of the function

$$\tau = \Phi(z) = \begin{cases} \int_{z_0^-}^z \sqrt{Q_-(\xi)}d\xi & \text{if } z \in D \text{ is close enough to } z_0^- \\ \int_{z_0^+}^z \sqrt{Q_+(\xi)}d\xi & \text{if } z \in D_+ \text{ is close enough to } z_0^+ \end{cases}$$

We assume here that the branches of the radicals in (2.13) are chosen such that $\Im \Phi(z) < 0$ if $z \in D$ and $\Im \Phi(z) > 0$ if $z \in D_+$. Furthermore, we stress that the first integral in (2.13) is taken along paths in the unit disk while the second integral is taken along paths in the exterior of the unit disk. Under these conditions $z = F(\tau)$ defines a conformal parameter in $U_\varepsilon$. Indeed, (2.13) implies that $\Phi(z)$ is analytic in a neighborhood of the point $z_0$ under consideration, possibly except points of $S$. 
corresponding to the points of the unit circle. Furthermore, the identification rule 2) stated above in this proof implies that
\[ \int_{z_0}^{z_-} \sqrt{Q_-(t)} \, dt = \int_{z_0}^{z_+} \sqrt{Q_+(t)} \, dt \]
if \( z_- = e^{i\theta_0} \) and \( z_+ = e^{i\theta_1} \) define the same point of \( S \). The latter implies that \( \Phi(z) \)
is continuous in a neighborhood of \( z_0 \). Since \( \Phi(z) \) is analytic in a slit neighborhood of \( z_0 \) and continuous in the whole neighborhood it follows that \( \Phi(z) \) is analytic on this neighborhood. Also it follows from (2.13) that \( \Phi'(z_0) \neq 0 \) if \( z^0_0 \) is not a critical point of \( Q_-(z) \, dz^2 \) and \( z^0_+ \) is not a critical point of \( Q_+(z) \, dz^2 \). Hence, \( \Phi(z) \) is one-to-one in a neighborhood of \( z_0 \) in this case.

Next, differentiating both parts of equation (2.13) and squaring the resulting equation, we obtain the following equation for the quadratic differentials:
\[ dr^2 = \begin{cases} Q_-(z) \, dz^2 & \text{if } z \in \mathbb{D} \\ Q_+(z) \, dz^2 & \text{if } z \in \mathbb{D}_+. \end{cases} \]
The latter equation shows that, in terms of the parameter \( \tau = \Phi(z) \), the quadratic differentials \( Q_-(z) \, dz^2 \) and \( Q_+(z) \, dz^2 \) define the same quadratic differential \( dr^2 \) and therefore they are extensions of each other.

Let \( S' \) denote the surface \( S \) punctured at the points which correspond to critical points situated on the unit circle \( \mathbb{T} \) of the quadratic differentials \( Q_-(z) \, dz^2 \) and/or \( Q_+(z) \, dz^2 \). Since \( S' \) is a topological sphere with punctures, which is supplied with a complex structure, it can be mapped conformally onto a punctured complex sphere (we denote it by \( C' \)) by a one-to-one function \( g : S' \to C' \). Let \( \psi = g^{-1} \) be the inverse of \( g \). Since \( \psi \) is conformal and one-to-one on \( C' \), each puncture is a removable singular point of \( \psi \). Thus, \( \psi \) can be extended to a conformal one-to-one mapping from \( C \) to \( S \). Let \( Q(\zeta) \, d\zeta^2 \) be a quadratic differential on \( C' \) defined by
\[ Q(\zeta) \, d\zeta^2 = \begin{cases} Q_-(\psi'(\zeta))(\psi'(\zeta))^2 \, d\zeta^2 & \text{if } z = \psi(\zeta) \in \mathbb{D} \cup \mathbb{T}' \\ Q_+(\psi'(\zeta))(\psi'(\zeta))^2 \, d\zeta^2 & \text{if } z = \psi(\zeta) \in \mathbb{D}_+ \cup \mathbb{T}' \end{cases}, \tag{2.14} \]
where \( \mathbb{T}' \) denotes the set of points \( e^{i\theta} \in \mathbb{T} \), which are regular for both \( Q_-(z) \, dz^2 \) and \( Q_+(z) \, dz^2 \).

The quadratic differential (2.14) can be extended to the whole complex sphere \( \mathbb{C} \). Indeed, suppose that \( \zeta_0 \notin \mathbb{C} \). We may assume that \( \zeta_0 \) is finite. Then \( \zeta_0 \) corresponds via the mapping \( \psi(\zeta) \) to a critical point \( z^0_- \in \mathbb{T} \) of the quadratic differential \( Q_-(z) \, dz^2 \) or/and to a critical point \( z^0_+ \in \mathbb{T} \) of the quadratic differential \( Q_+(z) \, dz^2 \). Since the trajectory structure of \( Q(\zeta) \, d\zeta^2 \) near \( \zeta = \zeta_0 \) is inherited from the quadratic differentials \( Q_-(z) \, dz^2 \) and \( Q_+(z) \, dz^2 \), it follows that \( \zeta_0 \) is an infinite critical point of \( Q(\zeta) \, d\zeta^2 \) if and only if the points \( z^0_- \in \mathbb{T} \) and \( z^0_+ \in \mathbb{T} \), which correspond to \( \zeta_0 \) under the mapping \( \psi(\zeta) \), are infinite critical points of the quadratic differentials \( Q_-(z) \, dz^2 \) and \( Q_+(z) \, dz^2 \), respectively.

Furthermore, \( \zeta_0 \) is a finite critical point of \( Q(\zeta) \, d\zeta^2 \) if and only if at least one of the corresponding points \( z^0_- \in \mathbb{T} \) and \( z^0_+ \in \mathbb{T} \) is a finite critical point of the quadratic differential \( Q_-(z) \, dz^2 \) or quadratic differential \( Q_+(z) \, dz^2 \), respectively.

Summarizing our previous arguments, we conclude that the image of the unit circle \( \Gamma = \psi^{-1}(\mathbb{T}) \) is a closed Jordan curve consisting of a finite number of arcs of trajectories and/or arcs of orthogonal trajectories of the quadratic differential \( Q(\zeta) \, d\zeta^2 \) and the endpoints of these arcs. The latter completes our proof of Theorem 5.

The critical points of \( Q(\zeta) \, d\zeta^2 \) situated off \( \Gamma \) and the trajectory structure nearby are inherited from the corresponding critical points of the quadratic differentials \( Q_-(z) \, dz^2 \) and \( Q_+(z) \, dz^2 \). To understand the trajectory structure of \( Q(\zeta) \, d\zeta^2 \) near
its critical points on \( \Gamma \) we have to reverse our procedure described in items (a)-(g) above. For instance, if \( Q_- (z) \, dz^2 \) has a simple pole at \( z_0^+ \in \mathbb{T} \) and \( Q_+ (z) \, dz^2 \) has a zero of order \( 2n - 1 \), \( n \geq 1 \), at \( z_0^- \in \mathbb{T} \) then \( \zeta_0 \) is a zero of order \( n - 1 \) of \( Q(\zeta) \, d\zeta^2 \) (cf. items (c) and (d)). In all other possible cases for the points \( z_0^- \) and \( z_0^+ \), the order of the corresponding critical point \( \zeta_0 \) of the quadratic differential \( Q(\zeta) \, d\zeta^2 \) can be found in a similar way by using items (a)-(g).

Because we have a choice of representatives, a quadratic differential \( Q(\zeta) \, d\zeta^2 \) of Theorem 5 is not unique in general. Thus any uniqueness result related to Theorem 5 will need some additional assumptions as in our theorem below.

**Theorem 6.** Suppose that the assumptions of Theorem 5 are satisfied. Then, for any given sets of representatives \( w^- \) and \( w^+ \), a quadratic differential \( Q(\zeta) \, d\zeta^2 \) and a Jordan curve \( \Gamma \) as in Theorem 5 are defined uniquely up to a Möbius transformation.

**Proof.** Suppose that we have two quadratic differentials \( Q_1(\zeta) \, d\zeta^2 \) and \( Q_2(\zeta) \, d\zeta^2 \) and suppose that \( \Gamma_1 \) and \( \Gamma_2 \) denote the corresponding Jordan curves. Suppose further that \( \varphi^- : D \to \Omega^1_1, \varphi^+_1 : D_+ \to \Omega^1_1, \varphi^+_2 : D \to \Omega^2_1 \), and \( \varphi^+ : D_+ \to \Omega^2_1 \) denote canonical mappings onto domains \( \Omega^1_1, \Omega^1_1, \Omega^2_1, \) and \( \Omega^2_1 \), which correspond to curves \( \Gamma_1 \) and \( \Gamma_2 \), respectively. We assume here that \( Q_-(z) \, dz^2 \) is a pullback of \( Q_1(\zeta) \, d\zeta^2 \), \( j = 1, 2 \), under the mapping \( \varphi^+_j (z) \) and that \( Q_+(z) \, dz^2 \) is a pullback of \( Q_j(\zeta) \, d\zeta^2 \), \( j = 1, 2 \), under the mapping \( \varphi^-_j (z) \). Then the composed mapping

\[
\varphi(\zeta) = \begin{cases} 
\varphi^-_2 ( (\varphi^+_1)^{-1} (\zeta) ) & \text{if } \zeta \in \Omega^1_1 \cup \Gamma_1 \\
\varphi^+_2 ( (\varphi^-_1)^{-1} (\zeta) ) & \text{if } \zeta \in \Omega^1_1
\end{cases}
\]

(2.15)
is conformal on \( \Omega^1_1 \) and on \( \Omega^1_1 \). Furthermore, the quadratic differentials \( Q_-(z) \, dz^2 \) and \( Q_+(z) \, dz^2 \) are pullbacks (under appropriate mappings) of each of the quadratic differentials \( Q_1(\zeta) \, d\zeta^2 \) and \( Q_2(\zeta) \, d\zeta^2 \). The latter implies that \( \varphi(\zeta) \) is continuous on \( \Gamma_1 \). Since \( \Gamma_1 \) is piecewise analytic and \( \varphi(\zeta) \) is continuous on \( \Gamma_1 \) it follows that \( \varphi(\zeta) \) is conformal on \( \Gamma_1 \). Thus, \( \varphi(\zeta) \) is conformal and one-to-one on \( \mathbb{C} \) and therefore \( \varphi(\zeta) \) is a Möbius mapping, as required.

**Corollary 1.** Suppose that the assumptions of Theorem 5 are satisfied and suppose additionally that the number of arcs \( \alpha^-_j \) is at least two and at least one of the endpoints of each of the arcs \( \alpha^-_j \) is a finite critical point. Then a quadratic differential \( Q(\zeta) \, d\zeta^2 \) and a Jordan curve \( \Gamma \) as in Theorem 5 are defined uniquely up to a Möbius transformation.

If both endpoints of \( \alpha^-_j \) are infinite critical points of \( Q_-(z) \, dz^2 \) and if both \( Q_-(z) \, dz^2 \) and \( Q_+(z) \, dz^2 \) have zeroes on the arcs \( \alpha^-_j \) and \( \alpha^+_j \) then by choosing the representatives defined in the proof of Theorem 5 appropriately we can obtain quadratic differentials \( Q_1(\zeta) \, d\zeta^2 \) and \( Q_2(\zeta) \, d\zeta^2 \), which have different numbers of distinct zeroes. Thus, \( Q_1(\zeta) \, d\zeta^2 \) and \( Q_2(\zeta) \, d\zeta^2 \) are essentially different and therefore the generated Jordan curves \( \Gamma_1 \) and \( \Gamma_2 \) are also essentially different. Even in the case when \( \alpha^+_1 = \alpha^-_1 = \mathbb{T} \) and \( \alpha^-_1 \) and \( \alpha^+_1 \) do not contain critical points we may have essentially different quadratic differentials for different choices of representatives. Here are some examples.

**Theorem 1.** In the simplest case when

\[
Q_-(z) \, dz^2 = Q_+(z) \, dz^2 = -\frac{dz^2}{(z + 1)^2},
\]

for any choice of representatives \( w^- \) and \( w^+ \), the resulting quadratic differential is \( Q(\zeta) \, d\zeta^2 = -\frac{d\zeta^2}{(\zeta + 1)^2} \) and \( \Gamma \) is the unit circle (up to a Möbius transformation).

Indeed, changing variables via \( z = \frac{1}{\zeta + 1} \), we transform \( Q_-(z) \, dz^2 \) into the quadratic
differential $\frac{1}{4} d\tau^2$ whose trajectories are horizontal lines. Then, the procedure of choosing representatives $w^{-}_1$ and $w^{+}_1$ is equivalent to the geometric procedure of sliding the upper-half plane of a variable $\tau$ with respect to its lower-half plane. Since these slidings don’t affect horizontal lines and Euclidean lengths they don’t change the trajectory structure and $Q$-lengths, and therefore the quadratic differential $\frac{1}{4} d\tau^2$ remains unchanged.

The same geometric argument can be applied to identify the form of the quadratic differential $Q(\zeta) d\zeta^2$ in our examples (2), (3), and (4) below. Necessary technical details in these examples are left to the interested reader.

(2) In the case when

$$Q_-(z) dz^2 = Q_+(z) dz^2 = \frac{dz^2}{(z - 1)^2(z + 1)^2},$$

different choices of pairs of representatives $w^{-}_1$, $w^{-}_2$ and $w^{+}_1$, $w^{+}_2$ generate quadratic differentials $Q(\zeta) d\zeta^2 = e^{i\alpha} \frac{d\zeta^2}{(\zeta - 1)^2(\zeta + 1)^2}$ with $-\pi < \alpha < \pi$ depending on the choice of representatives. In case $\alpha = 0$, the resulting curve $\Gamma$ coincides with the unit circle and in the case $\alpha \neq 0$ it consists of a pair of logarithmic spirals with foci at $\pm 1$; see Figure 3.

(3) Let

$$Q_-(z) dz^2 = Q_+(z) dz^2 = \frac{(z - 1)^2}{(z + 1)^6} dz^2.$$

For this pair of quadratic differentials we have only two possibilities. The first one occurs when $w^{-}_1 = w^{-}_2 = 1$. In this case $Q(\zeta) d\zeta^2 = \frac{(\zeta - 1)^2}{(\zeta + 1)^2} d\zeta^2$ and $\Gamma$ is the unit circle. For any other choice of representatives $w^{-}_1$ and $w^{+}_1$, we have quadratic differentials of the form $Q(\zeta) d\zeta^2 = C \frac{\zeta^2 + 1}{(\zeta + 1)^2} d\zeta^2$ with some nonzero $C \in \mathbb{C}$ depending on the choice of representatives. In this case $\Gamma$ consists of three critical trajectories as it is shown in Figure 4.

(4) For a given $p$, $0 < p < 1$, consider quadratic differentials

$$Q_-(z) dz^2 = Q_+(z) dz^2 = \frac{(z - p)(z - 1/p)}{z^3} dz^2.$$

In this case, for every $p$, we have a family of quadratic differentials of the form

$$Q(\zeta) d\zeta^2 = e^{i\alpha} \frac{(\zeta - p)(\zeta - e^{i\beta}/p)}{\zeta^3} d\zeta^2.$$
with $0 < \rho < 1$ and $0 \leq \beta < \pi$ depending on $p$ and on the choice of representatives and with
\[ \alpha = -\arg \left( i \int_0^{2\pi} \sqrt{(e^{i\theta} - \rho)(e^{i\theta} - e^{i\beta}/\rho)} e^{i\theta/2} d\theta \right). \]

For $\rho = 1/3$ and for three different choices of $\beta$, the critical trajectories of corresponding quadratic differentials and the resulting curves $\Gamma$ are shown in Figure 5.

We note here that even when quadratic differentials have a small number of critical points they still can generate a rich variety of topological structures depending on the positions of these critical points; see, for example, Section 11 in [13], which presents all possible structures of critical trajectories for a quadratic differential $Q(z)dz^2 = e^{i\alpha} \frac{(z-p_1)(z-p_2)}{(z-1)^2} dz^2$.

Two more examples will be given in Section 5, where we discuss polygonal curves. In the latter case, the corresponding quadratic differentials $Q_-(z)dz^2$ and $Q_+(z)dz^2$ are related to the well-known Schwarz-Christoffel integrals.

3. Modulated graphs and quadratic differentials

In this section we discuss a procedure designed for construction of quadratic differentials with required properties. We note that the full generality of results presented here is not required for the rest of this paper but they can be useful for the future work on image recognition. For the purposes of this section, it is convenient to use the terminology of graph theory, although we do not need any deep results from it. Our exposition here follows the lines of the paper [16] with the necessary changes to include doubly-connected domains. Let $G^0 = (V, E, F)$ be a graph embedded in $\mathbb{C}$ with the set of vertices $V = \{v_k\}$, the set of edges $E = \{e_{kj}\}$, and the set of faces $F = \{f_k\}$. We will assume that edges are enumerated in such a way that
way that $e_{kj}$ separates faces $f_k$ and $f_j$. Then, of course, $e_{kj} = e_{jk}$. In case $k = j$, an edge $e_{kk}$ separates $f_k$ from itself. In this case $e_{kk}$ is called a cut edge.

We suppose further that $G^0$ is at most doubly connected. The latter means that each face is either simply connected or a doubly connected domain. For convenience, we assume additionally that at most doubly connected graphs do not have connected components which are cyclic graphs.

If $f_k$ is simply connected then its boundary $\partial f_k$ represents a closed walk in $G^0$ in which each vertex may be visited more than once and each cut edge is traversed twice. To be definite, we assume that all boundary walks considered in this paper are oriented in a counterclockwise direction with respect to corresponding faces $f_k$. If $f_k$ is doubly connected then its boundary consists of two boundary walks. To fix notation for future use, one of these boundary walks will be denoted by $\partial_1 f_k$ and the other one by $\partial_2 f_k$. We note here that, by our definition of at most doubly connected graphs, all boundary walks are distinct. Then every walk along each boundary component of $f_k$ contains at least one vertex which order is not two.

Next, we will load $G^0$ with three sets, the set of weights $W = \{w_{kj}\}$, the set of heights $H = \{h_k\}$, and the set of twists $T = \{t_k\}$. The element $w_{kj}$ of $W$ assigns positive weight to the edge $e_{kj}$. Weighted graphs are standard in the graph theory. Using weights, we can define lengths (or total weights) $\alpha_k$ of boundary walks around simply connected faces $f_k$ as follows:

$$\alpha_k = \sum_{i,j} w_{kj},$$

(3.1)

where the weight $w_{kk}$ of each cut edge $e_{kk} \subset \partial f_k$ is counted twice. If $f_k$ is doubly connected then lengths $\alpha^1_k$ of $\partial_1 f_k$ and $\alpha^2_k$ of $\partial_2 f_k$ are defined in a similar way:

$$\alpha^m_k = \sum_{i,j} w_{kj}, \quad m = 1, 2,$$

(3.2)

where the sum is taken over all weights $w_{kj}$ corresponding to the edges $e_{kj} \subset \partial_m f_k$. As before the weight $w_{kk}$ of each cut edge $e_{kk} \subset \partial_m f_k$ is counted twice. An at most doubly connected graph is called coordinated if $\alpha^1_k = \alpha^2_k$ for each doubly connected face $f_k$. In the case of coordinated graphs we will write $\alpha_k$ instead of $\alpha^1_k$ and $\alpha^2_k$.

Two other assigned sets, $H = \{h_k\}$ with $h_k > 0$ and $T = \{t_k\}$, are not standard in the graph theory. These sets are assigned to doubly connected faces $f_k$ only with the purpose to introduce a sort of a moduli space structure on graphs. Specifically, the height $h_k > 0$ can be thought as a (combinatorial) distance between boundary components $\partial_1 f_k$ and $\partial_2 f_k$. Then the quotient

$$m(f_k) = \frac{h_k}{\alpha_k}$$

(3.3)

can be considered as a (combinatorial) module of a doubly connected face $f_k$ (compare this with the formulas (3.8) and (3.9) below).

To explain what the term "twist of $f_k$" means, we represent the boundary walks $\partial_1 f_k$ and $\partial_2 f_k$ by cyclic graphs $\gamma_{k,1}$ and $\gamma_{k,2}$ with vertices and edges on the unit circle. Any such representation has the form

$$\gamma_{k,m} = c_1(k, m)l_1(k, m)\ldots c_N(k, m)l_N(k, m)c_{N+1}(k, m), \quad m = 1, 2,$$

(3.4)

where

$$c_s(k, m) = e^{i\theta_s(k, m)}, \quad l_s(k, m) = \{e^{i\theta} : \theta_s(k, m) < \theta < \theta_{s+1}(k, m)\}$$

and $c_{N+1}(k, m) = e^{i\theta_{N+1}} = e^{i(\theta_{1}+2\pi)}$. In these formulas, $N = N(k, m)$ denotes the number of edges in the corresponding walk, where each cut edge is counted twice.
In a walk along a boundary component of $f_k$ some vertices can be visited several times. Hence, the same vertex of the graph $G^0$ can be represented by more than one of the vertices $c_k(k,m)$ of the string (3.4). Furthermore, we will assume that the lengths of the arcs $l_s(k,m)$ are proportional to the weights $w_{kj}^i$ of the edges $e_{kj}^i$, which are represented by these arcs. Thus, we assume that

$$\frac{1}{2\pi} (\theta_{s+1}(k,m) - \theta_s(k,m)) = \frac{w_{kj}^i}{\alpha_k}$$

(3.6)

if the arc $l_s(k,m)$ represents the edge $e_{kj}^i \in \partial_m f_k$.

A pair $(\gamma_{k,1}, \gamma_{k,2})$ of cyclic graphs defined by (3.4) – (3.6) will be called a representation of the pair of boundary walks $(\partial_1 f_k, \partial_2 f_k)$. Two such representations $(\gamma_{k,1}', \gamma_{k,2}')$ and $(\gamma_{k,1}'', \gamma_{k,2}'')$ are equivalent if one of them is a rotation of the other around the origin; i.e. if there is real $\alpha$ such that $\gamma_{k,1}' = e^{i\alpha} \gamma_{k,1}$ and $\gamma_{k,2}'' = e^{i\alpha} \gamma_{k,2}$.

**Definition 3.** By a twist $t_k$ of $f_k$ we mean the class of equivalent representations $(\gamma_{k,1}, \gamma_{k,2})$ of the pair of boundary walks $(\partial_1 f_k, \partial_2 f_k)$.

Thus, twist $t_k$ characterizes the relative positions of cyclic graphs $\gamma_{k,1}$ and $\gamma_{k,2}$ and is not a pure numerical characteristic. To assign a numerical value to $t_k$, we may fix a position of one vertex for each of the cyclic graphs $\gamma_{k,1}$ and $\gamma_{k,2}$. For instance, we may choose

$$c_1(k,m) = 1 \quad c_1(k,2) = e^{i\tau_k} \quad \text{with} \quad 0 \leq \tau_k < 2\pi.$$  

(3.7)

Then, the parameter $\tau_k$ is the angle which characterizes rotation of the cyclic graph $\gamma_{k,2}$ with respect to the cyclic graph $\gamma_{k,1}$ but the value of this angle depends on the choice of a pair of “initial” vertices on the graphs $\gamma_{k,1}$ and $\gamma_{k,2}$.

**Definition 4.** A coordinated at most doubly connected graph $G^0 = (V, E, F)$ loaded with the set of weights $W = \{w_{kj}^i\}$ assigned to its edges $e_{kj}^i$ and sets of heights $H = \{h_k\}$ and twists $T = \{t_k\}$ assigned to its doubly connected faces $f_k$ will be called a modulated graph and will be denoted by $G = (V, E, F, W, H, T)$.

Next, we will discuss graphs associated with quadratic differentials. Let $Q(z) \, dz^2$ be a quadratic differential on $\mathbb{C}$. Consider sets $V_Q$, $E_Q$, and $F_Q$, where $V_Q$ is the set of all critical points of $Q(z) \, dz^2$, $E_Q$ is the set of all critical trajectories of $Q(z) \, dz^2$, and $F_Q$ is the set consisting of all open components of $\mathbb{C} \setminus E_Q$. It follows from Jenkins’ Basic Structure Theorem (see Theorem 3.5 in [3]) that $F_Q$ consists of a finite number of circle domains, ring domains, strip domains, and end domains of the quadratic differential $Q(z) \, dz^2$. A ring domain in this context means a doubly connected domain. Furthermore, the interior of $E_Q$ may have a finite number of connected component, which are called density domains.

For the purposes of this paper, we need quadratic differentials with circle domains and ring domains only; i.e. without strip domains, end domains, and density domains. Quadratic differentials with these properties are commonly known as Jenkins-Strebel quadratic differentials.

Suppose now that $Q(z) \, dz^2$ is a Jenkins-Strebel quadratic differential. Then the triple $(V_Q, E_Q, F_Q)$ can be considered as a graph, which is at most doubly connected in our terminology. Furthermore, the $Q$-length of every critical trajectory of a Jenkins-Strebel quadratic differential is finite. Hence, we may suppose further that every critical trajectory $e_{kj}^i \in E_Q$ carries a weight $w_{kj}^i$ equal to the $Q$-length of this critical trajectory. The set of all these weights will be denoted by $W_Q$. The string of four set $V_Q$, $E_Q$, $F_Q$, and $W_Q$ defines the weighted critical graph $G_Q = (V_Q, E_Q, F_Q, W_Q)$ of the quadratic differential $Q(z) \, dz^2$. 


We note here that the weighted critical graph of a Jenkins-Strebel quadratic differential is at most doubly connected and coordinated.

Information provided by the weighted graph $G^0_Q$ is usually sufficient when we work with Jenkins-Strebel quadratic differentials which faces are simply connected domains. To have graphs carrying more information about Jenkins-Strebel quadratic differentials with doubly connected faces we will load their weighted graphs with two additional sets, the set of heights $H_Q$ and the set of twists $T_Q$. As we mentioned above, heights and twists are defined for ring domains of $Q(z)\,dz^2$ only.

Let $f_k \in F_Q$ be a ring domain of $Q(z)\,dz^2$. By the height $h_k$ of $f_k$ we mean the $Q$-distance between boundary components of $f_k$. The latter coincides with the $Q$-length of an arc of any orthogonal trajectory, which joins the boundary components of $f_k$. Thus, $0 < h_k < \infty$. Now we can define the set of heights of $Q(z)\,dz^2$ as $H_Q = \{h_k\}$.

Let $m(f_k)$ denote the module of a ring face $f_k \in F_Q$ with respect to the family of curves separating boundary components of $f_k$. Then

$$m(f_k) = \frac{h_k}{\alpha_k}, \quad (3.8)$$

where $\alpha_k$ denotes the length of the boundary walk around a boundary component of the face $f_k$. Formula (3.8) relates the set of weights and the set of heights with the set of moduli of doubly connected faces $f_k \in F_Q$. The latter explains our choice of the term “modulated graphs”.

To introduce the twist $t_k \in T_Q$ of a ring face $f_k \in F_Q$, we will consider a canonical conformal mapping $\Phi_k(z)$ from $f_k$ onto a circular annulus $A(1, R_k)$, where $A(1, R) = \{z : 1 < |z| < R\}$. The latter mapping is given by

$$\Phi_k(z) = \exp\left(-\frac{2\pi i}{\alpha_k} \int_{z_0}^z \sqrt{Q(\tau)} \,d\tau\right), \quad z \in f_k.$$ 

We assume here that the initial point of integration, $z_0$, belongs to $\partial f_k$ and that the branch of the radical is chosen such that $|\Phi_k(z_0)| > 1$. Since the module of a doubly connected domain is conformally invariant and $m(A(1, R)) = \frac{1}{2\pi} \log R$, we conclude from (3.8) that

$$\frac{h_k}{\alpha_k} = \frac{1}{2\pi} \log R_k. \quad (3.9)$$

The function $\Phi_k(z)$ maps the boundary components $\partial_1 f_k$ and $\partial_2 f_k$ onto circles $C_1$ and $C_{R_k}$. Moreover, this mapping is continuous and one-to-one in the sense of boundary correspondence. Therefore, the normalized function $\varphi_k(z) = \frac{\Phi_k(z)}{||\Phi_k(z)||}$ maps boundary walks $\partial_1 f_k$ and $\partial_2 f_k$ onto cyclic graphs, call them $\gamma^Q_{k,1}$ and $\gamma^Q_{k,2}$, placed on the unit circle. Now, the pair $(\gamma^Q_{k,1}, \gamma^Q_{k,2})$ of these cyclic graphs defines a twist $t^Q_k$ which will be called the twist of the ring domain $f_k$ (of the quadratic differential $Q(z)\,dz^2$). Let $T_Q = \{t_k\}$ denote the set of twists of the quadratic differential $Q(z)\,dz^2$.

**Definition 5.** The weighted graph $G^0_Q = (V_Q, E_Q, F_Q, W_Q)$ of the quadratic differential $Q(z)\,dz^2$ loaded with two additional sets, the set of heights $H_Q = \{h_k\}$ and the set of twists $T_Q = \{t_k\}$ assigned as above to ring domains of $Q(z)\,dz^2$, will be called the critical modulated graph of $Q(z)\,dz^2$ and will be denoted by $G_Q = (V_Q, E_Q, F_Q, W_Q, H_Q, T_Q)$.

Before we state the main result of this section we recall the following. A function $\tau : \mathbb{C} \times [0, 1] \to \mathbb{C}$ is called a deformation of $\mathbb{C}$ if the following conditions are fulfilled:

1) $\tau$ is continuous on $\mathbb{C} \times [0, 1]$,

2) $\tau(\cdot, 0)$ is the identity mapping.
3) for every fixed \( t \in [0, 1] \), \( \tau(\cdot, t) \) is a homeomorphism from \( \mathbb{C} \) onto itself.

Every deformation \( \tau \) transforms a given modulated graph \( G \) embedded in \( \mathbb{C} \) into a homeomorphic modulated graph \( G_\tau \) embedded in \( \mathbb{C} \). We assume additionally that deformation respect the weights of edges of \( G \) and the heights and twists of doubly connected faces of \( G \).

**Theorem 7.** For every at most doubly connected modulated graph

\[ G = \{ V, E, F, W, H, T \} \]

embedded into \( \mathbb{C} \) there exists a Jenkins-Strebel quadratic differential \( Q(z) \, dz^2 \) on \( \mathbb{C} \), the critical modulated graph of which is homeomorphic to \( G \) on \( \mathbb{C} \).

Furthermore, \( Q(z) \, dz^2 \) is defined uniquely up to a Möbius automorphism of \( \mathbb{C} \).

This theorem is also valid for quadratic differentials and modulated graphs embedded into an arbitrary compact Riemann surface; see [16], [3]. The study of weighted graphs and quadratic differentials on compact Riemann surfaces with relation to extremal problems was initiated by this author and for graphs with simply connected faces the result stated above was proved in Theorem 1 in [16]. Then an extension of this theorem to include quadratic differentials with doubly connected domains with essentially the same proof was given by E. Emel’yanov who used a slightly different terminology; see [3].

### 4. Lemniscates and Quadratic Differentials

Let \( f(z) \) be a nonconstant meromorphic function on a domain \( D \subset \mathbb{C} \). For \( 0 \leq c \leq \infty \), the lemniscate of \( f(z) \) at level \( c \) is defined by equation

\[ L_f(c) = \{ z \in D : |f(z)| = c \}. \]

Every connected component of \( L_f(c) \) will be called a lemniscate component. Thus, \( L_f(c) \) is just a level set of \( f(z) \) with value \( c \). Since \( L_f(0) \) is the set of zeroes of \( f(z) \) and \( L_f(\infty) \) is the set of its poles, each lemniscate component of \( L_f(0) \) and \( L_f(\infty) \) is a point.

Suppose now that \( 0 < c < \infty \) and that \( L_f(c) \) is not empty. Then each lemniscate component of \( L_f(c) \) is either a closed Jordan analytic curve or it consists of Jordan analytic arcs and their endpoints in \( D \).

Let \( L_f'(c) \) be a lemniscate component having a tangent vector \( dz \) at its point \( z \). The gradient of the real valued function \( \log |f(z)| \) at \( z \) can be calculated as follows:

\[ \text{grad}(\log |f(z)|) = 2 \frac{\partial}{\partial z} \log |f(z)| = \left( \frac{f'(z)}{f(z)} \right). \]

Hence, the tangent vector \( dz \) to \( L_f'(c) \) at \( z \) is given by

\[ dz = i \left( \frac{f'(z)}{f(z)} \right). \]

Multiplying both sides of this equation by \( \frac{i}{\pi} \frac{f'(z)}{f(z)} \) and then squaring, we obtain

\[ -\frac{1}{4\pi^2} \left( \frac{f'(z)}{f(z)} \right)^2 \, dz^2 = \frac{1}{4\pi^2} \left| \frac{f'(z)}{f(z)} \right|^2. \]  \hspace{1cm} (4.1)

The left hand-side of (4.1) defines a quadratic differential, which will be denoted by \( Q_f(z) \, dz^2 \); i.e.,

\[ Q_f(z) \, dz^2 = -\frac{1}{4\pi^2} \left( \frac{f'(z)}{f(z)} \right)^2 \, dz^2 \]  \hspace{1cm} (4.2)
Now equation (4.1) shows that

\[ Q_f(z) \, dz^2 > 0, \]

if \( dz \) is a tangent vector to the lemniscate of \( f(z) \) passing through \( z \).

From here we conclude that, for \( 0 < c < \infty \), each lemniscate component is a trajectory of the quadratic differential (4.2). Furthermore, each critical point of \( Q_f(z) \, dz^2 \) has even order and that endpoints of the analytic arcs constituting lemniscate components are zeroes of \( f'(z) \), which are not zeroes of \( f(z) \).

Since the quadratic differential \( Q_f(z) \, dz^2 \) is generated by the logarithmic derivative its trajectory structure is not of a general form. Let \( z_0 \neq \infty \) be a zero or pole of \( f(z) \) of order \( s \geq 1 \). Then the logarithmic derivative \( f'(z)/f(z) \) can be represented as

\[
\frac{f'(z)}{f(z)} = \pm \frac{s}{z-z_0} + \text{higher powers of } (z-z_0) \tag{4.3}
\]

with “+” sign for zeros and “−” sign for poles. Similarly, if \( z_0 = \infty \) then

\[
\frac{f'(z)}{f(z)} = \pm \frac{s}{z} + \text{higher powers of } z \tag{4.4}
\]

with “−” sign for zeros and “+” sign for poles. In what follows we will assume that \( z_0 \neq \infty \), the case \( z_0 = \infty \) requires only minor notational changes.

It follows from (4.3) that the quadratic differential \( Q_f(z) \, dz^2 \) has the following representation near \( z_0 \):

\[
Q_f(z) \, dz^2 = -\frac{s^2}{4\pi^2} \frac{1}{(z-z_0)^2} + \text{higher powers of } (z-z_0).
\]

The latter implies that every zero or pole of \( f(z) \) represents a second order pole of the quadratic differential \( Q_f(z) \, dz^2 \) with circular structure of trajectories near \( z_0 \). Moreover, the \( Q_f \)-length of each trajectory \( \gamma \) separating \( z_0 \) from the boundary of \( D \) and other critical points is

\[
|\gamma|_{Q_f} = \int_{\gamma} |Q_f(z)|^{1/2} |dz| = s.
\]

Our discussion above implies, in particular, that the trajectory structure of \( Q_f(z) \, dz^2 \) does not include end domains and strip domains having poles on their boundaries in \( D \). Also, the trajectory structure of \( Q_f(z) \, dz^2 \) does not include density domains since otherwise some level set of \( f(z) \) would be dense in such a domain. Then, \( |f(z)| \) must be constant on a density domain and therefore \( f(z) \) must be constant on \( D \) contradicting our assumption.

Suppose now that \( \gamma \) is a closed trajectory of \( Q_f(z) \, dz^2 \) and that \( f(z) \) is meromorphic on a connected component \( D_0 \) of \( \mathbb{C} \setminus \gamma \). Then the \( Q_f \)-length of \( \gamma \) can be calculated as follows:

\[
|\gamma|_{Q_f} = \int_{\gamma} \sqrt{|Q_f(z)|} \, dz = \frac{1}{2\pi} \left| \int_{\gamma} \frac{f'(z)}{f(z)} \, dz \right| = |N_P - N_Z|, \tag{4.5}
\]

where \( N_P \) and \( N_Z \) denote the number of poles and number of zeroes of \( f(z) \) in \( D_0 \) (counting multiplicity), respectively.

Formula (4.5) shows, in particular, that a regular trajectory of \( Q_f(z) \, dz^2 \) cannot enclose an equal number of poles and zeroes of \( f(z) \) (if \( f(z) \) is meromorphic on \( D_0 \)). In terms of lemniscates, this fact can be stated as follows. Suppose that a Jordan curve \( \gamma \) is a subset of some lemniscate \( L_f(c) \). Suppose further that \( f(z) \) is meromorphic on a component \( D_0 \) of \( \mathbb{C} \setminus \gamma \) and \( N_P = N_Z \). Then \( \gamma \) contains a critical point of \( f(z) \) or, in other words, \( Q_f(z) \, dz^2 \) has a zero on \( \gamma \).
Our previous discussions in this section deal with meromorphic functions on an arbitrary domain on the complex sphere. All these results and formulas can be extended to the case of meromorphic functions defined on Riemann surfaces.

In the rest of this section, we will focus specifically on the case when \( D = \mathbb{C} \).

In this case, every meromorphic function is a rational function and therefore we will denote such a function by \( R(z) \) rather than \( f(z) \). Let \( \Gamma \subset \mathbb{C} \) be a connected component of \( L_R(c) \) which does not contain critical points of \( R(z) \). Let \( N_p^- \) and \( N_Z^- \) denote the number of poles and zeros of \( R(z) \) in the domain \( \Omega^- \) and let \( N_p^+ \) and \( N_Z^+ \) denote the number of poles and zeros of \( R(z) \) in the domain \( \Omega^+ \). Then (4.5) implies that

\[
|N_p^- - N_Z^-| = |N_p^+ - N_Z^+| \neq 0.
\]

Below we include three results on Blaschke products which follow from our previous considerations. These results are familiar to experts teaching Complex Analysis and could be good exercises for their students. The parts (1) and (2) will be used in the proof of the converse statement of Theorem 3. The part (3) provides additional information on critical points of related quadratic differentials.

Let \( A(z) = \prod_{k=1}^n \frac{1}{1-\sigma_k} \) and \( B(z) = \prod_{k=1}^n \frac{1}{1-\delta_k} \) be Blaschke products of order \( n \geq 1 \) each. Then the following holds.

1. \( A(z) \) (and \( B(z) \)) does not have critical points on the unit circle \( \mathbb{T} \); i.e. \( A'(z) \neq 0 \) (and \( B'(z) \neq 0 \)) if \( |z| = 1 \). Therefore, \( \mathbb{T} \) is a regular trajectory of \( Q_A(z) \) \( dz^2 \) (and of \( Q_B(z) \) \( dz^2 \)).

2. \( |T_{Q_A}| = |T_{Q_B}| = n \). Therefore \( Q_A(z) \) \( dz^2 \) and \( Q_B(z) \) \( dz^2 \) are coordinated in a sense of Definition 2.

3. The quotient function \( R(z) = \frac{A(z)}{B(z)} \) has a critical point on \( \mathbb{T} \); i.e. \( R'(e^{i\theta}) = 0 \) for some \( \theta, 0 \leq \theta < 2\pi \).

In the proof of Theorem 3 below, we will use rational functions depending on a variable \( \zeta \). Accordingly, we will use the following notations: \( R(\zeta) = C \frac{P_1(\zeta)}{P_2(\zeta)} \), where \( C \neq 0 \) and

\[
P_1(\zeta) = \prod_{k=1}^{m_1} (\zeta - a_k)^{p_k}, \quad P_2(\zeta) = \prod_{k=1}^{m_2} (\zeta - b_k)^{q_k}.
\]

We assume in (4.6) that \( a_1, \ldots, a_{m_1}, b_1, \ldots, b_{m_2} \) are distinct points on \( \mathbb{C} \) and \( p_k, q_k \) are positive integers. Let \( n_1 = \sum_{k=1}^{m_1} p_k, n_2 = \sum_{k=1}^{m_2} q_k \). Calculating initial terms of the power series expansion of \( \frac{P_1(\zeta)}{P_2(\zeta)} \) as in (4.4), we conclude that \( Q_R(\zeta) \) \( d\zeta^2 \) has a circular domain centered at \( \zeta = \infty \) if and only if \( n_1 \neq n_2 \). Therefore, the domain configuration of \( Q_R(\zeta) \) \( d\zeta^2 \) includes \( N = m_1 + m_2 + 1 \) circle domains if \( n_1 \neq n_2 \) and it includes \( N = m_1 + m_2 \) circle domains if \( n_1 = n_2 \geq 2 \). In addition, the domain configuration of \( Q_R(\zeta) \) \( d\zeta^2 \) may include a finite number of ring domains.

Next, we show how to use quadratic differentials to prove Theorem 3 for the rational function \( R(\zeta) \) defined above. Then Theorem 2 will follow as a special case.

**Proof of Theorem 3.** Let \( R(\zeta) = C \frac{P_1(\zeta)}{P_2(\zeta)} \) be a rational function of degree \( n \geq 2 \). Suppose further that \( L_R(1) \) is analytic and connected and that \( n_1 > n_2 \). Then \( n = n_1 \) and \( R(\infty) = \infty \). Let \( \Omega^- \) and \( \Omega^+ \) be inner and exterior domains for \( L_R(1) \). Then \( \Omega^- \) contains all zeros \( a_k \) of \( R(\zeta) \) and \( \Omega^+ \) contains all poles \( b_k \) of \( R(\zeta) \) including the pole at \( \zeta = \infty \).

The lemniscate \( L_R(1) \) is a regular closed trajectory of the quadratic differential \( Q_R(\zeta) \) \( d\zeta^2 \) defined by (4.2). Transplanting \( Q_R(\zeta) \) \( d\zeta^2 \) from \( \Omega^+ \) and \( \Omega^- \) to \( \mathbb{D}^+ \) and
\[ Q_{A_1}(z)\,dz^2 = Q_R(\varphi_+(z))(\varphi'_+(z))^2\,dz^2 = -\frac{1}{4\pi^2} \left( \frac{R(\varphi_+(z))}{A'_1(z)} \right)^2 \left( \varphi'_+(z) \right)^2 \,dz^2 = -\frac{1}{4\pi^2} A'_1(z)^2 \,dz^2, \quad (4.7) \]

where \( A_1(z) = R(\varphi_+(z)) \) for \( z \in \mathbb{D} \), and the quadratic differential

\[ Q_{B_1}(z)\,dz^2 = Q_R(\varphi_-(z))(\varphi'_-(z))^2\,dz^2 = -\frac{1}{4\pi^2} \left( \frac{R(\varphi_-(z))}{B'_1(z)} \right)^2 \left( \varphi'_-(z) \right)^2 \,dz^2 = -\frac{1}{4\pi^2} B'_1(z)^2 \,dz^2, \quad (4.8) \]

where \( B_1(z) = R(\varphi_-(z)) \) for \( z \in \mathbb{D} \).

Since \(|A_1(z)| = |R(\varphi_+(z))| = 1\) for \( |z| = 1\) and since \( A_1(z) \) does not have zeroes in \( \mathbb{D} \), it follows that \( A_1(z) \) is a Blaschke product of order \( n \) with poles at the points \( \varphi_+^{-1}(b_k) \in \mathbb{D} \) and \( z = \infty \). Similarly, since \(|B_1(z)| = |R(\varphi_-(z))| = 1\) for \(|z| = 1\) and since \( B_1(z) \) does not have poles in \( \mathbb{D} \) it follows that \( B_1(z) \) is a Blaschke product of order \( n \) with zeroes at the points \( \varphi^-^{-1}(a_k) \in \mathbb{D} \).

Applying equation \((2.8)\) of Lemma 1, we conclude that the fingerprint \( k(z) \), \( z = e^{i\theta} \), can be found as a solution to the equation

\[ \log(A_1(k(z))/A_1(k(z_0))) = \log(B_1(z)/B_1(z_0)). \quad (4.9) \]

Since \(|A_1(k(z_0))| = 1 = |B_1(z_0)| = 1\), the latter equation implies that \( k(z) \) satisfies equation \((\mathbb{L}_3)\) with Blaschke products \( A(z) = A_1(z)/A_1(k(z_0)) \) and \( B(z) = B_1(z)/B_1(z_0) \) of order \( n \) each. This proves the first part of Theorem 3.

To prove the converse statement of Theorem 3, we consider two Blaschke products \( A(z) = \prod_{k=1}^n \frac{z-a_k}{z-b_k} \) and \( B(z) = \prod_{k=1}^n \frac{z-a_k}{z-b_k} \) of order \( n \geq 2 \) each such that \( A(\infty) = \infty \). Let \( Q_A(z)\,dz^2 \) and \( Q_B(z)\,dz^2 \) be quadratic differentials defined by \((4.2)\). As we had mentioned above in the remark (1), Blaschke products do not have critical points on the unit circle. Thus, \( T \) is a regular trajectory of each of the quadratic differentials \( Q_A(z)\,dz^2 \) and \( Q_B(z)\,dz^2 \). Furthermore it follows from the remark (2) above that the quadratic differentials \( Q_A(z)\,dz^2 \) and \( Q_B(z)\,dz^2 \) are coordinated on \( T \).

Equation \((\mathbb{L}_3)\) is equivalent to equation \((2.8)\) with \( \mathcal{A} \) and \( \mathcal{B} \) defined by \((2.9)\) with

\[ Q_+(z)\,dz^2 = Q_A(z)\,dz^2 \quad \text{and} \quad Q_-(z)\,dz^2 = Q_B(z)\,dz^2. \]

Thus, if \( k(z) \) is a solution to \((\mathbb{L}_3)\) then \( k(z) \) is a solution to \((2.8)\). Now, since \( Q_A(z)\,dz^2 \) and \( Q_B(z)\,dz^2 \) are coordinated on \( T \), it follows from Theorem \(3\) that there is a quadratic differential \( Q(\zeta)\,d\zeta^2 \) defined on \( \mathbb{C} \) and its regular trajectory \( \Gamma \) such that

\[ Q(\zeta)\,d\zeta^2 = \left\{ \begin{array}{ll} Q_A(\tau_+(\zeta)) \left( \tau'_+(\zeta) \right)^2 \,d\zeta^2 & \text{for } \zeta \in \Omega_+ \cup \Gamma, \\ Q_B(\tau_-(\zeta)) \left( \tau'_-(\zeta) \right)^2 \,d\zeta^2 & \text{for } \zeta \in \Omega_- \cup \Gamma, \end{array} \right. \quad (4.10) \]

where \( \tau_+ : \Omega_+ \rightarrow \mathbb{D}_+ \) and \( \tau_- : \Omega_- \rightarrow \mathbb{D}_- \) are inverses of the conformal mappings \( \varphi_+ \) and \( \varphi_- \), respectively. Thus, the quadratic differentials \( Q_A(z)\,dz^2 \) and \( Q_B(z)\,dz^2 \) are pullbacks of the quadratic differential \( Q(\zeta)\,d\zeta^2 \) and therefore, by Theorem \(3\), the fingerprint \( k = \varphi_+^{-1} \odot \varphi_- \) of \( \Gamma \) given by equation \((2.8)\) or, equivalently, by equation \((\mathbb{L}_3)\).

It still remains to show that the trajectory \( \Gamma \) defined above is a lemniscate of a rational function with required properties. Equations \((4.10)\) and \((4.7)\) and \((4.8)\) imply that

\[ Q(\zeta)\,d\zeta^2 = \left\{ \begin{array}{ll} -\frac{1}{4\pi^2} \left( \frac{A'_1(\zeta)}{A_1(\zeta)} \right)^2 \,d\zeta^2 & \text{for } \zeta \in \Omega_+ \cup \Gamma, \\ -\frac{1}{4\pi^2} \left( \frac{B'_1(\zeta)}{B_1(\zeta)} \right)^2 \,d\zeta^2 & \text{for } \zeta \in \Omega_- \cup \Gamma, \end{array} \right. \quad (4.11) \]

where \( A_1(\zeta) = A(\tau_+(\zeta)) \) and \( B_1(\zeta) = B(\tau_-(\zeta)) \).
Since \( Q(\zeta) \, d\zeta^2 \) is a quadratic differential on \( \mathbb{C} \), \( Q(\zeta) \) is a rational function. It follows from (4.11) that all zeros and poles of \( Q(\zeta) \) are of even orders and therefore there is a rational function \( R_1(\zeta) \) such that

\[
Q(\zeta) \, d\zeta^2 = -\frac{1}{4\pi^2} R_1^2(\zeta) \, d\zeta^2.
\]

Furthermore, since \( R_1^2(\zeta) = (A_1(\zeta)/A_1'(\zeta))^2 \) for \( \zeta \in \Omega_+ \), and \( R_1^2(\zeta) = (B_1(\zeta)/B_1'(\zeta))^2 \) for \( \zeta \in \Omega_- \), it follows that all finite poles of \( R_1(\zeta) \) are simple with real integer residues. Also, \( A_1(\zeta) = A(\tau_+)(\zeta) \) has a pole at \( \zeta = \infty \) and therefore \( \zeta = \infty \) is a simple zero of \( R_1(\zeta) \) and the limit \( \lim_{\zeta \to \infty} \zeta R_1(\zeta) \) is a positive integer. The latter information implies that the function \( R(\zeta) \) defined as

\[
R(\zeta) = \exp \left( \int_{\zeta_0}^\zeta R_1(t) \, dt \right)
\]

is a rational function such that \( R'(\zeta)/R(\zeta) = R_1(\zeta) \) and \( R(\infty) = \infty \). This shows that

\[
Q(\zeta) \, d\zeta^2 = Q_R(\zeta) \, d\zeta^2 = -\frac{1}{4\pi^2} \left( \frac{R'(\zeta)}{R(\zeta)} \right)^2 \, d\zeta^2 \quad \text{(4.13)}
\]

and therefore \( \Gamma \) is a lemniscate of the rational function \( R(\zeta) \). In particular, if in (4.12) we choose \( \zeta_0 \in \Gamma \) then \( R(\zeta_0) = 1 \) and therefore \( \Gamma = L_R(1) \). This completes the proof of Theorem 3.

**Proof of Theorem 2.** Taking \( P_2(\zeta) \equiv 1 \) in the first part of the proof of Theorem 3 given above, we conclude that (4.9) holds with \( A_1(z) = cz^n \) where \( |c| = 1 \). Thus, (4.9) is equivalent to (4.2) in this case.

Similarly, taking \( A(z) = z^n \) in the second part of the proof of Theorem 3 given above, we conclude that (4.13) holds with the rational function \( R(\zeta) \) whose the only pole is at \( \infty \). Hence, \( R(\zeta) \) is a polynomial of order \( n \) in this case.

The uniqueness statement of the converse part of Theorem 2 easily follows from the maximum modulus principle. Indeed, if there are two polynomials \( P_1(\zeta) \) and \( P_2(\zeta) \) of degree \( n \geq 2 \) with positive leading coefficients, which share the lemniscate \( \Gamma \), then their quotient \( P_1(\zeta)/P_2(\zeta) \) is analytic and non-vanishing on \( \Omega_+ \) (including the point \( \zeta = \infty \)) and satisfies the equation \(|P_1(\zeta)/P_2(\zeta)| = 1\) on \( \Gamma = \partial\Omega_+ \). Hence, \( P_1(\zeta) = P_2(\zeta) \) as required.

We note here that equation (4.12) defines a unique rational function \( R(\zeta) \) having \( \Gamma \) as its lemniscate \( L_R(1) \). However, (unlike in Theorem 2) a rational function \( R(\zeta) \), which existence is quarantined by the converse part of Theorem 3, is not unique in general. Indeed, any two Blaschke products \( B_1(\zeta) \) and \( B_2(\zeta) \) of degree \( n \) such that \( B_1(\infty) = \infty, B_2(\infty) = \infty \) share the unit circle \( \mathbb{T} \) as their lemniscates \( L_{B_1}(1) \) and \( L_{B_2}(1) \). Whether or not there are non-circular rational lemniscates shared by two essentially distinct rational functions of the same degree remains an open question to this author.

5. **Polygons, Lemniscates and Related Quadratic Differentials**

In this section, we discuss two particular cases of quadratic differentials \( Q_-(z) \, dz^2 \) and \( Q_+(z) \, dz^2 \) and corresponding curves \( \Gamma \) generated by fingerprints \( k(z) \) defined by formulas (2.8) and (2.9).

(a) **Cartesian polygon curves.** By a Cartesian polygon curve we understand a Jordan curve consisting of a finite number of horizontal and vertical segments. Any such curve \( \Gamma \) is a boundary of a standard polygon \( \Omega_- \) having an even number of sides and even number of vertices, \( v_1, \ldots, v_{2n} \). We suppose here that vertices are always oriented in the counterclockwise direction and that \( v_{2n+1} = v_1 \).
$v_0 = v_{2n}$. We assume additionally that the boundary segment $[v_1, v_2]$ represents a horizontal side of $\Omega_-$. Let $\alpha_k \pi$ be the angle of $\Omega_-$ at its vertex $v_k$. Then either $\alpha_k = \frac{\pi}{4}$ or $\alpha_k = \frac{3\pi}{2}$. Since the sum of angles of any polygon with $2n$ vertices is $2\pi(n - 1)$, one can easily see that $\Omega_-$ must have $n + 2$ vertices with angles $\frac{\pi}{2}$ and $n - 2$ vertices with angles $\frac{3\pi}{2}$.

The horizontal and vertical sides of $\Omega_-$ are arcs of trajectories and, respectively, arcs of orthogonal trajectories of the quadratic differential $Q(\zeta) \frac{d\zeta^2}{1 \cdot d\zeta^2}$. Transplanting this quadratic differential via the mapping $\varphi_- : \mathbb{D} \rightarrow \Omega_-$, we obtain the following quadratic differential:

$$Q_-(z) \, dz^2 = C_- e^{i\gamma_-} \prod_{k=1}^{2n} (z - e^{i\beta_k^-})^{2(\alpha_k - 1)} \, dz^2, \quad z \in \mathbb{D}, \quad (5.1)$$

with some $C_- > 0$, $\gamma_- \in \mathbb{R}$, and with $e^{i\beta_k^\pm} = \tau_- (v_k)$, where $0 \leq \beta^-_1 < \beta^-_2 < \cdots < \beta^-_{2n} < \beta^+_1 + 2\pi$. Figure 6 presents an example of a Cartesian polygonal curve $\Gamma$ and shows critical trajectories of the corresponding quadratic differential $Q_-(z) \, dz^2$.

Since the unit circle $T$ consists of arcs of trajectories and/or arcs of orthogonal trajectories of $Q_-(z) \, dz^2$, the quadratic differential $Q_-(z) \, dz^2$ is real in all its regular points of $T$. This after simple calculation gives the following value for $\gamma_-:

$$\gamma_- = \sum_{k=1}^{2n} (1 - \alpha_k) \beta_k^-. \quad (5.2)$$

Now, the value of a positive constant $C_-$, which does not affect the trajectory structure of $Q_-(z) \, dz^2$, can be obtained from the equation for the $Q_-$-length of the arc $\{e^{i\theta} : \beta^-_1 \leq \theta < \beta^-_2\}$, which gives the following:

$$C_- = -e^{-i\gamma_-} |v_2 - v_1|^2 \left( \int_{\beta^-_1}^{\beta^-_2} 2n \prod_{k=1}^{2n} (e^{i\theta} - e^{i\beta_k^-})^{\alpha_k - 1} e^{i\theta} \, d\theta \right)^{-2}.$$ 

Similarly, transplanting $Q(\zeta) \, d\zeta^2 = 1 \cdot d\zeta^2$ via the mapping $\varphi_+ : \mathbb{D}_+ \rightarrow \Omega_+$, we obtain the second required quadratic differential:

$$Q_+(z) \, dz^2 = C_+ e^{i\gamma_+} z^{-2} \prod_{k=1}^{2n} (z - e^{i\beta_k^+})^{2(1 - \alpha_k)} \, dz^2, \quad z \in \mathbb{D}_+, \quad (5.3)$$

where $e^{i\beta_k^\pm} = \tau_+ (v_k)$ with $0 \leq \beta^+_1 < \beta^+_2 < \cdots < \beta^+_{2n} < \beta^+_1 + 2\pi$. The constants $\gamma_+ \in \mathbb{R}$ and $C_+ > 0$ in $(5.3)$ are given by equations:

$$\gamma_+ = \sum_{k=1}^{2n} (\alpha_k - 1) \beta_k^+$$

and

$$C_+ = -e^{-i\gamma_+} |v_2 - v_1|^2 \left( \int_{\beta^+_1}^{\beta^+_2} 2n \prod_{k=1}^{2n} (e^{i\theta} - e^{i\beta_k^+})^{1 - \alpha_k} e^{-i\theta} \, d\theta \right)^{-2}.$$ 

We note that quadratic differentials $Q_-(z) \, dz^2$ and $Q_+(z) \, dz^2$ given by formulas $(5.1)$, $(5.2)$ are coordinated in the sense of Definition 2 if and only if the following equations are satisfied for all $k = 1, \ldots, 2n$:

$$\int_{\beta^-_1}^{\beta^-_j} \prod_{i=1}^{2n} \left( e^{i\theta} - e^{i\beta_k^-} \right)^{1 - \alpha_j} e^{-i\theta} \, d\theta = C e^{i\gamma}, \quad (5.3)$$

where $C = \sqrt{C_-/C_+}$ and $\gamma = (\gamma_- - \gamma_+)/2$.

Finally, if $Q_-(z) \, dz^2$ and $Q_+(z) \, dz^2$ given by formulas $(5.1)$, $(5.2)$ are coordinated, i.e. if they satisfy equation $(5.3)$, then by Theorem 6 they define a polygonal
curve $\Gamma$ whose fingerprint $k : T \to \mathbb{T}$ can be found from equations (2.8) and (2.9) for the quadratic differentials $Q_-(z) \, dz^2$ and $Q_+(z) \, dz^2$ given by (5.1) and (5.2), respectively.

(b) Polar polygonal curves. We start with the quadratic differential

$$Q(\zeta) \, d\zeta^2 = -d\zeta^2 / \zeta^2. \quad (5.4)$$

Then the radial segments of the form $\{\zeta = r e^{i\alpha} : r_1 \leq r \leq r_2\}$ with some $\alpha \in \mathbb{R}$ and $0 < r_1 < r_2 < \infty$ are closed arcs on the orthogonal trajectories of $Q(\zeta) \, d\zeta^2$ and the closed arcs of circles centered at $\zeta = 0$ are closed arcs on the trajectories of $Q(\zeta) \, d\zeta^2$.

By a polar polygonal curve $\Gamma$ we mean a closed Jordan curve bounded by a finite number of radial segments and circular arcs as above. The components $\Omega_-$ and $\Omega_+$ of $\mathbb{C} \setminus \Gamma$ will be called a polar polygon and an outer polar polygon, respectively. We will assume additionally that $0 \in \Omega_-$. We want to mention here that polar polygons, also known as gearlike domains, were used by R.W. Barnard and his collaborators in their study of Goodman’s omitted area problem; see, for instance, [1].

Each of the polygons $\Omega_-$ and $\Omega_+$ has an even number of sides, say $2n$, and accordingly they have an even number of vertices and same even number of corresponding angles. For all these objects we will use notations introduced in part (a) of this section. Since $0 \in \Omega_-$ one can easily find that $\Omega_-$ has $n$ vertices $v_k$ with angles $\alpha_k \pi = \frac{1}{2} \pi$ and $n$ vertices $v_k$ with angles $\alpha_k \pi = \frac{3}{2} \pi$.

Transplanting $Q(\zeta) \, d\zeta^2$ via the mapping $\varphi_- : \mathbb{D} \to \Omega_-$ and assuming that $\varphi(0) = 0$, we obtain the following quadratic differential:

$$Q_-(z) \, dz^2 = -C_- e^{i\gamma_-} z^{-2} \prod_{k=1}^{2n} (z - e^{i\beta^-_k})^{2(\alpha_k - 1)} \, dz^2, \quad z \in \mathbb{D}, \quad (5.5)$$

where $e^{i\beta^-_k} = \tau_-(v_k)$ with $0 \leq \beta^-_1 < \beta^-_2 < \cdots < \beta^-_{2n} < \beta^-_1 + 2\pi$. Figure 7 displays an example of a polar polygonal curve $\Gamma$ and shows critical trajectories of the corresponding quadratic differential $Q_-(z) \, dz^2$.

Comparing the coefficients for term $\zeta^{-2}$ in (5.4) and for the term $z^{-2}$ in (5.5), we obtain the following equation:

$$C_- e^{i(\gamma^- + 2\sum_{k=1}^{n} (\alpha_k - 1)\beta^-_k)} = 1.$$ 

The latter implies that $C_- = 1$ and

$$\gamma^- = 2 \sum_{k=1}^{2n} (1 - \alpha_k)\beta^-_k.$$
Next, the unit circle $T$ consists of arcs of trajectories and orthogonal trajectories of $Q_-(z) \, dz^2$. Hence $Q_-(z) \, dz^2$ is real at all its regular points on $T$. This, after some algebra, leads to the following condition:

$$
e^{-i\gamma_0} = -e^{2i \sum_{k=1}^{2n} (1-\alpha_k) \beta_k^-}.
$$

Furthermore, we will assume that the boundary arc of $\partial \Omega_-$ with endpoints $v_1$ and $v_2$ is a circular arc. Then the quadratic differential $Q_-(z) \, dz^2$ must be positive on the arc $\{e^{i\theta} : \beta_1^- < \theta < \beta_2^- \}$. The latter condition, after routine calculation, implies the following:

$$\sum_{k=1}^{2n} (1-\alpha_k) \beta_k^- = \pi \quad (\mod 2\pi). \tag{5.6}$$

This equation is equivalent to the following

$$\sum_1^k \beta_k^- - \sum_2^k \beta_k^- = 2\pi \quad (\mod 4\pi), \tag{5.7}$$

where the first sum is taken over all $k$ such that $e^{i\beta_k^-}$ is a zero of $Q_-(z) \, dz^2$ and the second sum is taken over all $k$ such that $e^{i\beta_k^-}$ is a pole of $Q_-(z) \, dz^2$.

Combining our observations, we conclude that $Q_-(z) \, dz^2$ can be represented in the form

$$Q_-(z) \, dz^2 = -z^{-2} \prod_{k=1}^{2n} (z - e^{i\beta_k^-})^{2(\alpha_k-1)} \, dz^2$$

with $\beta_1^- \ldots , \beta_{2n}^-$ satisfying equation $(5.6)$ or, equivalently, equation $(5.7)$.

The same argument as above shows that the quadratic differential $Q_+(z) \, dz^2$ has the form

$$Q_+(z) \, dz^2 = -z^{-2} \prod_{k=1}^{2n} (z - e^{i\beta_k^+})^{2(1-\alpha_k)} \, dz^2, \quad z \in \mathbb{D}_+ \tag{5.8}$$

where $e^{i\beta_k^+} = \tau_+(v_k)$ with $0 \leq \beta_1^+ < \beta_2^+ < \ldots < \beta_{2n}^+ < \beta_1^+ + 2\pi$. Furthermore, the parameters $\beta_1^+, \ldots , \beta_{2n}^+$ satisfy the following equation:

$$\sum_{k=1}^{2n} (\alpha_k - 1) \beta_k^+ = \pi \quad (\mod 2\pi).$$

As in the previous case, quadratic differentials $(5.5)$ and $(5.8)$ are coordinated in the sense of Definition 2 if and only if

$$\frac{f_{\beta_k^+}^{\beta_{k-1}^+} \prod_{j=1}^{2n} \left( e^{i\theta} - e^{i\beta_j^+} \right)^{1-\alpha_j} \, d\theta}{f_{\beta_k^-}^{\beta_{k-1}^-} \prod_{j=1}^{2n} \left( e^{i\theta} - e^{i\beta_j^-} \right)^{\alpha_j-1} \, d\theta} = 1 \quad \text{for } k = 1, 2, \ldots , 2n. \tag{5.9}$$

As in part (a), we conclude that if the quadratic differentials $Q_-(z) \, dz^2$ and $Q_+(z) \, dz^2$ given by formulas $(5.5)$ and $(5.8)$ are coordinated, i.e. if they satisfy equations $(5.9)$, then they define a polar polygonal curve $\Gamma$ whose fingerprint $k : T \to T$ can be found from equations $(2.8)$ and $(2.9)$.

(c) A few remarks are in order now.

(1) The mapping functions $\phi_-$ and $\phi_+$ used in parts (a) and (b) of this section can be represented by the well-known Schwarz-Christoffel integrals. In fact, our formulas $(5.3)$ and $(5.9)$ can be obtained as Schwarz-Christoffel integrals taken over the corresponding arcs of the unit circle.
(2) A simple fact of geometry that, in the case (a), the polygon $\Omega_-$ has $n + 2$ vertices with angle $\frac{\pi}{2}$ and $n - 2$ vertices with angles $\frac{3\pi}{2}$ can be interpreted as an elementary corollary of the following formula:

$$p - q = 4 - 4g - 2s,$$

see, for example, Lemma 3.2 in [9]). This formula relates the number of poles $p$ and number of zeros $q$ (both counted with multiplicities) of the quadratic differential $Q(z)\,dz^2$ defined on a domain $D$ lying on a compact Riemann surface of genus $g$ when $\partial D$ consists of $s$ non-degenerate curves and $Q(z)\,dz^2 > 0$ on $\partial D$. Of course, in the case of the complex sphere this formula reduces to $p - q = 4$.

Now, since the vertices $v_k$ of $\Omega_-$ with angles $\frac{\pi}{2}$ correspond to simple poles of $Q_-(z)\,dz^2$ and the vertices $v_k$ with angles $\frac{3\pi}{2}$ correspond to simple zeroes of $Q_-(z)\,dz^2$, we conclude from (5.1) that the difference between the number of vertices of $\Omega_-$ with angles $\frac{\pi}{2}$ and the number of its vertices with angles $\frac{3\pi}{2}$ equals 4. The same conclusion can be made by comparing the number of poles and zeroes of the quadratic differential (5.2).

The same argument, being applied to the quadratic differentials (5.5) and (5.8), shows that, in the case of polar polygons, the number of vertices of $\Omega_-$ with angle $\frac{\pi}{2}$ should be equal to the number of its vertices with angle $\frac{3\pi}{2}$.

(3) Equations (5.3) and (5.9) give necessary and sufficient conditions which guarantee that the Schwarz-Christoffel integrals representing functions $\varphi_-$ and $\varphi_+$ define one-to-one mappings from $D$ and $D_+$ onto polygons $\Omega_-$ and $\Omega_+$, respectively. Of course, experts know that a similar fact holds true for the Schwarz-Christoffel mappings from $D$ and $D_+$ onto any two complementary polygons with common Jordan boundary.

Surprisingly to this author, the latter fact is not mentioned in standard textbooks on Complex Analysis. Thus, we state it here.

**Proposition 1.** For $n \geq 3$, let $0 \leq \beta_1^- < \beta_2^- < \cdots < \beta_n^- < \beta_1^+ + 2\pi$ and let $0 < \alpha_k < 2$, $k = 1, 2, \ldots, n$, be such that $\sum_{k=1}^n \alpha_k = n - 2$.

Then the Schwarz-Christoffel integral

$$F(z) = \int_0^z \prod_{k=1}^n \left( \tau - e^{i\beta_k^-} \right)^{\alpha_k} d\tau$$

maps $D$ conformally and one-to-one onto some polygon if and only if there are points $z_k^\pm = e^{i\beta_k^\pm}$ with $0 \leq \beta_1^+ < \beta_2^+ < \cdots < \beta_n^+ < \beta_1^+ + 2\pi$ such that the equations (5.3) with some $C > 0$ and $\gamma \in \mathbb{R}$ are satisfied for all $k = 1, 2, \ldots, n$. 

**Fig 7.** Polar polygonal curve and critical trajectories of $Q_-(z)\,dz^2$. 

(2) A simple fact of geometry that, in the case (a), the polygon $\Omega_-$ has $n + 2$ vertices with angle $\frac{\pi}{2}$ and $n - 2$ vertices with angles $\frac{3\pi}{2}$ can be interpreted as an elementary corollary of the following formula:
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