Global gauge conditions in the Batalin–Vilkovisky formalism

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Abstract. In the Batalin–Vilkovisky formalism, gauge conditions are expressed as Lagrangian submanifolds in the space of fields and antifields. We discuss a way of patching together gauge conditions over different parts of the space of fields, and apply this method to extend the light-cone gauge for the superparticle to a conic neighbourhood of the forward light-cone in momentum space.

The Gribov ambiguity concerns the impossibility, for topological reasons, of making a global choice of gauge in non-Abelian Yang-Mills theories. An analogous phenomenon occurs in string theory, where the light-cone gauge only yields a gauge condition on a dense open subset of the forward light-cone in momentum space. In this article, we introduce a technique for patching together gauge conditions over different parts of the space of fields. In the final section, we apply this method to give an extension of the light-cone gauge to all of the forward light-cone; the usual light-cone gauge only makes sense over an open dense subset of the forward light-cone.

We work in the Batalin–Vilkovisky formalism, in which gauge conditions are expressed by means of Lagrangian submanifolds of the superspace of fields and antifields. Our main result shows how the gauge conditions associated to Lagrangian submanifolds may be glued together. In mathematics, this is called descent, and is typically performed using the language of simplicial manifolds and of cosimplicial algebras. This is the approach we adopt here. We call the resulting collection of Lagrangian submanifolds together with families of isotopies between them flexible Lagrangian submanifolds.

The main ingredient in our construction is a homotopy formula for the gauge conditions associated to a family of Lagrangian submanifolds, due to Mikhailov and Schwarz. Imitating Weil’s proof of the de Rham theorem (see [2]), we obtain our formula for integration of half-forms over flexible Lagrangian submanifolds.

We apply this technique to the problem of gauge-fixing the superparticle in the Batalin–Vilkovisky formalism. We construct a flexible Lagrangian submanifold giving a gauge condition in an open conic neighbourhood of the forward light-cone in momentum space, based on

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a pair of open sets \( U_+ \) and \( U_- \) contained in the region \( \{ p_0 > 0 \} \), and gauge conditions determined by the light-like vectors \( (1, 0, \ldots, 0, \pm 1) \in \mathbb{R}^{1,9} \). There are Lagrangian submanifolds \( L_\pm \) lying over \( U_\pm \), and a Lagrangian isotopy between them over the intersection

\[
U_{++} = U_+ \cap U_-.
\]

This flexible Lagrangian submanifold is neither Lorentz invariant nor invariant under supersymmetry. In order to show that our functional integral respects these symmetries, we extend our formula to the case where the theory carries an action of a (finite-dimensional) Lie superalgebra. As in the BRST formalism, this formula takes values in the differential graded commutative algebra of Lie superalgebra cochains, and exhibits the equivariance of our construction with respect to this Lie superalgebra. Applied to the superparticle, this shows that our gauge-fixed functional integral is Lorentz invariant and supersymmetric.

**Convention**

Throughout this paper, we use the terms submanifold and subspace when referring to sub(super)manifolds and sub(super)spaces. Unless otherwise mentioned, we make use of the Einstein summation convention.

1. Odd symplectic structures

In this section, we review several results on the supergroup of symmetries of an odd symplectic superspace. Our references are Manin [10, Section 3.5] and Khudaverdian and Voronov [8].

Let \( V \) be a finite-dimensional superspace, with homogeneous basis \( \{ e_a \}_{a \in I} \): each vector \( e_a \) has a well-defined parity \( p_a = p(e_a) \). The dual superspace \( V^* \) has the dual basis \( \{ e^*_a \}_{a \in I} \), where \( e^*_a \) has the same parity as \( e_a \), and

\[
e^*_a(e_b) = \delta_{ab}.
\]

The sign rule implies that the basis \( \{ e^{**}_a \}_{a \in I} \) of the double dual \( V^{**} \) differs from the original basis \( \{ e_a \}_{a \in I} \) by a sign:

\[
e^{**}_a = (-1)^{p_a} e_a.
\]

Let \( W \) be a second finite-dimensional superspace, with homogeneous basis \( \{ f_b \}_{b \in J} \), with parity \( q_b = p(f_b) \). Given a morphism \( A : V \to W \), we define the matrix elements of \( A \) by

\[
A^b_a = f^*_b(Ae_a).
\]

For a general vector \( v = v^a e_a \), the morphism \( A \) acts by the formula

\[
Av = (A^b_a v^a) f_b.
\]

The supertranspose \( A^* : W^* \to V^* \) of a morphism \( A : V \to W \) is defined by

\[
(A^* f^*_b)(e_a) = (-1)^{p(A) q_b} f^*_b(Ae_a).
\]

The supertranspose has the following properties:

1) \((A + B)^* = A^* + B^*\);
2) \((AB)^* = (-1)^{p(A) p(B)} B^* A^*\);
3) \((A^*)^* = (-1)^{p(A)} A\).

Although \(A \mapsto A^*\) is not in general an involution, its square is.

If \(V\) is a superspace, \(\Pi V\) is the superspace obtained by exchanging the even and odd subspaces of \(V\). Denote \(\Pi V^*\) by \(V^*\). If \(A : V \to W\) is a morphism of superspaces, let \(A^\Pi\) be the induced morphism from \(\Pi V\) to \(\Pi W\). Let \(A^\circ = A^{\Pi^*} : W^* \to V^*\). This operation has the following properties:

1) \((A + B)^\circ = A^\circ + B^\circ;\)
2) \((AB)^\circ = (-1)^{p(A)p(B)} B^\circ A^\circ;\)
3) \((A^\circ)^\circ = A.\)

If \(V\) is a finite-dimensional superspace, then so is its space of endomorphisms \(\text{End}(V)\). The Berezinian is a rational function on \(\text{End}(V)\) that specializes to the determinant when \(V\) is concentrated in even degree, and to the inverse of the determinant when \(V\) is concentrated in odd degree. To give a formula for the Berezinian, we break \(A\) up into blocks mapping between the parity homogeneous components of \(V\): \(A_{pq}, p, q \in \{0, 1\}\), maps \(V_q\) to \(V_p\). We have
\[
\text{Ber}(A) = \frac{\det(A_{00})}{\det(A_{11} - A_{10}A_{01}^{-1}A_{00})} \cdot \frac{\det(A_{00} - A_{01}A_{11}^{-1}A_{10})}{\det(A_{11})}.
\]

The Berezinian has the following properties:

1) \(\text{Ber}(\text{Id}) = 1;\)
2) \(\text{Ber}(AB) = \text{Ber}(A) \text{Ber}(B);\)
3) \(\text{Ber}(A^*) = \text{Ber}(A)\) and \(\text{Ber}(A^\circ) = \text{Ber}(A)^{-1}.\)

The general linear supergroup \(\text{GL}(V)\) is the Zariski open subset of \(\text{End}(V)\) where \(\text{Ber}(A) \neq \{0, \infty\}\).

Let \(V\) be a superspace with an odd symplectic form, that is, a bilinear pairing \(\omega(-,-) : V \times V \to \mathbb{C}\) satisfying

a) \(\omega(v, w) = 0\) unless \(p(v) + p(w) = 1;\)

b) \(\omega(v, w) = -\omega(w, v).\)

A polarization of \(V\) is a decomposition of \(V\) into a direct sum
\[V = L \oplus M,\]
where \(L\) and \(M\) are Lagrangian subspaces for the form \(\omega\), that is, maximal isotropic subspaces. The symplectic form induces a natural isomorphism \(M \cong L^\circ\). We will work in a Darboux basis of \(V\), consisting of a homogeneous basis \(\{e_a\}_{a \in I}\) of \(L\) and the dual basis \(\{f_a\}_{a \in I}\) of \(L^\circ\), where \(p(f_a) = p_a + 1.\)

The odd symplectic supergroup \(\Pi \text{ISP}(V, \omega) \subset \text{GL}(V)\) consists of transformations preserving the symplectic form \(\omega\). Decomposing an endomorphism \(A : V \to V\) into blocks
\[
A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}
\]
where \( P : L \to L, \ Q : L^\circ \to L, \ R : L \to L^\circ, \ \text{and} \ S : L^\circ \to L^\circ \), we may express the conditions to preserve the symplectic form as the following matrix equations:

\[
\begin{align*}
S^\circ P &= Q^\circ R + \text{Id}_L, \\
Q^\circ S &= S^\circ Q, \\
P^\circ R &= R^\circ P, \\
P^\circ S &= R^\circ Q + \text{Id}_{L^\circ}.
\end{align*}
\]

These equations define a superquadric \( Q(V, \omega) \) in \( \text{End}(V) \), closed under composition. The supergroup \( \Pi SP(V, \omega) \) is the open subset of \( Q(V, \omega) \) where \( \text{Ber}(A) \) is nonzero.

**Proposition 1.1** (Khudaverdian and Voronov [8, Theorem 4]). The restriction of the Berezinian to \( Q(V, \omega) \) is the rational function \( \text{Ber}(P)^2 \).

**Proof.** From the equation

\[
\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} \text{Id}_L & 0 \\ RP^{-1} & \text{Id}_{L^\circ} \end{pmatrix} \begin{pmatrix} P & Q \\ 0 & S - RP^{-1}Q \end{pmatrix}
\]

we see that

\[\text{Ber}(A) = \text{Ber}(P) \text{Ber}(S - RP^{-1}Q).\]

By the equation

\[
P^\circ (S - RP^{-1}Q) = P^\circ S - (P^\circ R)P^{-1}Q = (R^\circ Q + \text{Id}_{L^\circ}) - (R^\circ P)P^{-1}Q = \text{Id}_{L^\circ},
\]

we see that \( \text{Ber}(S - RP^{-1}Q) = \text{Ber}(P) \), and the result follows. \( \Box \)

Unlike for a superspace with an even orthosymplectic form, the Lagrangian Grassmannian of a superspace with an odd symplectic form is not connected. The above result holds on each component of this Grassmannian. On the component containing the even parity subspace \( V_0 \) of \( V \), the proposition shows that \( \text{Ber}(A) = \det(P)^2 \), and we may conclude that the restriction of the Berezinian to the quadric \( Q(V, \omega) \) is a polynomial in the entries of \( A \).

Let \( \text{Ber}^{1/2}(A) \) be the function \( \text{Ber}(P) \) on \( Q(V, \omega) \).

**Corollary 1.2.** The function \( \text{Ber}^{1/2}(A) \) satisfies

\[\text{Ber}^{1/2}(AB) = \text{Ber}^{1/2}(A) \text{Ber}^{1/2}(B).\]

**Proof.** This formula holds up to a rational function in the entries of \( A \) and \( B \) whose square equals 1, and is thus constant. Taking \( A \) and \( B \) equal to \( \text{Id} \), the result follows. \( \Box \)

In this article, all superspaces are \( \mathbb{Z} \)-graded: in physics, this grading is referred to as the ghost number. All of the odd symplectic forms considered will have ghost number \(-1\): that is, \( \omega(v, w) \) vanishes on a pair of vectors \( v \) and \( w \) homogeneous with respect to the ghost number unless \( \text{gh}(v) + \text{gh}(w) = 1 \).
2. Odd symplectic supermanifolds

We now recall the definition of the line bundle of half-forms $\Omega^{1/2}$ on a supermanifold $M$ with odd symplectic form $\omega$, and the differential operator $\Delta$ acting on sections of $\Omega^{1/2}$. This material is taken from Khudaverdian [7].

We start with a supermanifold $M$, modelled on a superspace $V$. The line bundle $\Omega$ over $M$ associated to the character $\text{Ber}(A)^{-1}$ of $\text{GL}(V)$ is called the bundle of integral forms. An odd non-degenerate 2-form $\omega$ on $M$ induces a $\Pi\text{SP}(V,\omega)$-structure on the tangent superbundle $TM$. The line bundle $\Omega^{1/2}$ over $M$ associated to the character $\text{Ber}^{1/2}(A)^{-1}$ of $\Pi\text{SP}(V,\omega)$ is called the bundle of half-forms. (Khudaverdian [7] actually studies a similar bundle, associated to the character $|\text{Ber}^{1/2}(A)|^{-1}$, called the bundle of half-densities, but his results hold if this bundle is replaced by the bundle of half-forms.)

A Darboux coordinate chart on $M$ is a coordinate chart $\{x^a, \xi_a\}_{a \in I}$ such that the tangent vectors $\{\partial/\partial x^a\}_{a \in I}$ and $\{\partial/\partial \xi_a\}_{a \in I}$ span Lagrangian subspaces of the tangent superbundle $TM$, and

$$\omega(\partial/\partial \xi_a, \partial/\partial x^b) = \delta^b_a,$$

or equivalently,

$$\omega = \sum_{a \in I} (-1)^{p_a} d\xi_a \wedge dx^a.$$

Such coordinate charts exist locally if the 2-form $\omega$ is closed, in which case, $M$ is called an odd symplectic supermanifold. Let $\{x^a, \xi_a\}_{a \in I}$ be a Darboux coordinate chart: a total order of the indices $I$ (or equivalently, a bijection between $I$ and the set of natural numbers $\{1, \ldots, n\}$, where $n$ is the cardinality of $I$) yields a nonvanishing section

$$dx = dx^1 \ldots dx^n$$

of the line bundle $\Omega^{1/2}$ of half-forms. Let $\Delta_0$ be the second-order differential operator

$$\Delta_0 = \sum_{a \in I} (-1)^{p_a} \frac{\partial^2 f}{\partial x^a \partial \xi_a}.$$ 

Define a second-order differential operator $\Delta$ on sections of $\Omega^{1/2}$ as follows: a section $\sigma$ of $\Omega^{1/2}$ may be written in the Darboux chart as $f dx$, and we define

$$\Delta \sigma = \Delta_0 f dx.$$ 

It is clear that this does not depend on the choice of total ordering of $I$, since a change of ordering only changes $dx$, and hence $f$, by a sign.

Given a function $f$ on $M$, denote by $m(f)$ the operation of multiplication by $f$ on the sheaf of sections of $\Omega^{1/2}$. The following properties of the operator $\Delta$ are easily derived from the explicit formula in a Darboux coordinate chart:

1) $\Delta^2 = \frac{1}{2}[\Delta, \Delta] = 0$;

2) $\Delta$ is a second-order differential operator, that is, if $f$, $g$ and $h$ are functions on $M$, then

$$[[[\Delta, m(f)], m(g)], m(h)] = 0.$$
The Batalin–Vilkovisky antibracket is the Poisson bracket associated to the odd symplectic form $\omega$, given by the formula
\[
m((f, g)) = (-1)^{p(f)}[[\Delta, m(f)], m(g)].
\]

**PROPOSITION 2.1.** The antibracket is antisymmetric
\[
(g, f) = (-1)^{(p(f)+1)(p(g)+1)} (f, g)
\]
and satisfies the Jacobi relation
\[
(f, (g, h)) = ((f, g), h) + (-1)^{(p(f)+1)p(g)} (g, (f, h)).
\]

**PROOF.** To prove antisymmetry, we use the formula $[f, g] = 0$:
\[
m((f, g) + (-1)^{(p(f)+1)(p(g)+1)} (f, g))
\]
\[
= (-1)^{p(f)}[[\Delta, m(f)], m(g)] - (-1)^{p(f)p(g)+p(f)}[[\Delta, m(g)], m(f)]
\]
\[
= (-1)^{p(f)}[[\Delta, m(f)], m(g)] + [m(f), [\Delta, m(g)]]
\]
\[
= (-1)^{p(f)}[[\Delta, m(f)], m(g)].
\]
To prove the Jacobi relation, we use the formulas $[\Delta, [\Delta, m(f)]] = \frac{1}{2}[[\Delta, \Delta], m(f)] = 0$ and $[[[\Delta, m(f)], m(g)], m(h)] = 0$:
\[
0 = -[[\Delta, [[\Delta, m(f)], m(g)], m(h)]
\]
\[
= (-1)^{p(f)}[[[\Delta, m(f)], [\Delta, m(g)]], m(h)]
+ (-1)^{p(f)+p(g)}[[[\Delta, m(f)], m(g)], [\Delta, m(h)]]
\]
\[
= (-1)^{p(f)}[[\Delta, m(f)], [[\Delta, m(g)], m(h)]
+ (-1)^{(p(f)+1)p(g)}[[\Delta, m(g)], [[\Delta, m(f)], m(h)]
+ (-1)^{p(h)(p(f)+p(g)+1)}[[\Delta, m(h)], [[\Delta, m(f)], m(g)]]
\]
\[
= (-1)^{p(g)}((f, (g, h)) + (-1)^{p(f)p(g)+p(f)}(g, (f, h))
+ (-1)^{p(h)(p(f)+p(g)+p(f)}(h, (f, g))
\]
\[
= (-1)^{p(g)}((f, (g, h)) - (-1)^{(p(f)+1)(p(g)+1)}(g, (f, h)) - ((f, g), h)). \quad \square
\]

Let $H_f$ be the first-order differential operator given by the formula
\[
H_f = (-1)^{p(f)}[\Delta, m(f)].
\]

**PROPOSITION 2.2.**

1) $[H_f, m(g)] = m((f, g))$
2) $H_{fg} = m(f)H_g + (-1)^{p(f)p(g)}m(g)H_f + (-1)^{p(g)}m((f, g))$
3) $H_{(f,g)} = [H_f, H_g]$
Proof. The first formula follows immediately from the definition of $H_f$. The second formula is proved as follows:

$$H_{fg} = (-1)^{p(f)+p(g)}[\Delta, m(fg)]$$

$$= (-1)^{p(g)}m(f)[\Delta, m(g)] + (-1)^{p(f)+p(g)}[\Delta, m(f)]m(g)$$

$$= m(f)H_g + (-1)^{p(g)}H_fm(g)$$

$$= m(f)H_g + (-1)^{p(g)+(p(f)+1)p(g)}m(g)H_f + (-1)^{p(g)}m((f, g)).$$

To prove the third formula, we argue as follows:

$$[H_f, H_g] = (-1)^{p(f)+p(g)}[[\Delta, m(f)], [\Delta, m(g)]]$$

$$= (-1)^{p(g)+1}[\Delta, [[\Delta, m(f)], m(g)]] - (-1)^{p(g)+1}[\Delta, [\Delta, m(f)], m(g)].$$

The first term equals $H_{(f,g)}$, and the second term vanishes. □

If $f$ is a function of odd parity on $M$, the operator $g \mapsto (f, g)$ is an even vector field $H_f$ on $M$, called the Hamiltonian vector field associated to $f$. Under the infinitesimal flow $\exp(\epsilon H_f)$, the Darboux coordinate chart transforms to

$$x^a + \epsilon(f, x^a) = x^a - \epsilon \frac{\partial f}{\partial x^a}$$

$$\xi_a + \epsilon(f, \xi_a) = \xi_a + \epsilon \frac{\partial f}{\partial x^a}.$$

The section $dx$ of the bundle of half-forms transforms to

$$\text{Ber}(\text{Id} - \epsilon \frac{\partial^2 f}{\partial x^a \partial \xi_a}) dx = dx - \epsilon \Delta(f dx).$$

A half-form $\sigma = g dx$ transforms to

$$(g + \epsilon(f, g)) dx - \epsilon g \Delta(f dx) = \sigma - \epsilon [[\Delta, f], g] dx - \epsilon g \Delta(f dx)$$

$$= \sigma + \epsilon H_f \sigma.$$

We may interpret the differential operator $H_f$ as the lift of the vector field $H_f$ to the bundle of half-forms. The operator $\Delta$ is invariant under this action, by the formula

$$[H_f, \Delta] = -[\Delta, [\Delta, m(f)]] = 0.$$

A diffeomorphism preserving the antibracket is called a (Batalin–Vilkovisky) canonical transformation. These transformations form a pseudogroup, which is generated by nonautonomous Hamiltonian flows (associated to time-dependent Hamiltonians), in the sense that for any canonical transformation $f : U \to V$, each point $x \in U$ has an open neighborhood on which the restriction of $f$ may be written as the composition of flows associated to nonautonomous Hamiltonian vector fields. It follows that the operator $\Delta$ is independent of the choice of Darboux coordinate chart. This is the strategy adopted by Khudaverdian in his proof of this theorem [7 Section 2]; Ševera [14] has given another proof, which identifies Khudaverdian’s operator $\Delta$ with the differential on the $E_2$ page of the spectral sequence associated to the Hodge filtration (filtration by degree of differential forms) in the de Rham complex of $M$ with deformed differential $d + \omega$. 7
An orientation of an odd symplectic manifold $M$ is a nowhere-vanishing section $\sigma$ of the bundle of half-forms $\Omega^{1/2}$ such that $\Delta \sigma = 0$ (Behrend and Fantechi [1]). In particular, $\sigma$ defines a global trivialization of $\Omega^{1/2}$. If we choose a Darboux coordinate chart and express $\sigma$ as $e^S dx$, we may write the equation $\Delta \sigma = 0$ as

$$\Delta_0 S + \frac{1}{2} (S, S) = 0.$$  

This equation is known as the quantum master equation. In applications of this equation to quantum field theory, there is an additional parameter $\hbar$. In this case, $\sigma$ equals $e^{S/\hbar} dx$, where $S$ is itself a power series in $\hbar$:

$$S = \sum_{n=0}^{\infty} \hbar^n S_n.$$  

In this setting, the quantum master equation becomes

$$\hbar \Delta_0 S + \frac{1}{2} (S, S) = 0.$$  

Expanding in powers of $\hbar$, this equation is seen to be equivalent to the series of equations

$$\begin{cases}  
(S_0, S_0) = 0,  \\
\Delta_0 S_{n-1} + (S_0, S_n) + \frac{1}{2} \sum_{i=1}^{n-1} (S_i, S_{n-i}) = 0, \quad n > 0. 
\end{cases}$$  

3. Simplicial supermanifolds

A simplicial supermanifold $M_\bullet$ consists of the following data: for each $k \geq 0$, $M_k$ is a supermanifold, and there are face maps $d_i : M_k \to M_{k-1}$ and degeneracy maps $s_i : M_k \to M_{k+1}$ satisfying the usual simplicial relations. Let $[k] = \{0, \ldots, k\}$ be the set of vertices of the $k$-simplex. If $\mu : [k] \to [\ell]$ is a function preserving the ordering of the vertices, then there is a differentiable map $\mu^* : M_\ell \to M_k$ satisfying $(\mu \nu)^* = \nu^* \mu^*$. The face map $d_i$ is associated to the function

$$\mu(j) = \begin{cases}  
j, & j < i,  
\; j + 1, & j \geq i, 
\end{cases}$$  

while the degeneracy map $s_i$ is associated to the function

$$\mu(j) = \begin{cases}  
j, & j \leq i,  
\; j - 1, & j > i. 
\end{cases}$$  

For example, suppose that $M$ is a supermanifold and $\mathcal{U} = \{U_\alpha\}$ is a locally finite open cover of $M$. If $(\alpha_0 \ldots \alpha_k)$ is a sequence of indices of the open sets in the cover $\mathcal{U}$ of $M$, we denote by

$$U_{\alpha_0 \ldots \alpha_k}$$  

their intersection. We obtain a simplicial supermanifold $\mathcal{U}_\bullet$ by setting

$$\mathcal{U}_k = \prod_{\alpha_0 \ldots \alpha_k} U_{\alpha_0 \ldots \alpha_k}.$$
where \( \mu^* \) is the inclusion of \( U_{\alpha_0\ldots\alpha_k} \subset \mathcal{U}_k \) into \( U_{\alpha_{\mu(0)}\ldots\alpha_{\mu(k)}} \subset \mathcal{U}_k \). In particular, the face map \( d_i : \mathcal{U}_k \to \mathcal{U}_{k-1} \) is the open embedding of \( U_{\alpha_0\ldots\alpha_k} \) into

\[
U_{\alpha_i\ldots\alpha_k} \subset \mathcal{U}_{k-1},
\]

while the degeneracy map \( s_i : \mathcal{U}_k \to \mathcal{U}_{k+1} \) is the identification of \( U_{\alpha_0\ldots\alpha_k} \) with \( U_{\alpha_0\ldots\alpha_i\alpha_i\ldots\alpha_k} \subset \mathcal{U}_{k+1} \).

Every map \( \mu^* \) may be factored into a finite sequence of face maps followed by a sequence of degeneracy maps, so it actually suffices to consider just these two sets of maps.

The \( k \)-simplex \( \Delta^k \) is the convex hull of the unit coordinate vectors in \( \mathbb{R}^{|k|} = \mathbb{R}^{\{0,\ldots,k\}} \). We denote the coordinates on \( \mathbb{R}^{|k|} \) by \( (t_0, \ldots, t_k) \); on \( \Delta^k \), they satisfy \( t_0 + \cdots + t_k = 1 \) and \( t_i \geq 0 \).

A function \( \mu : [k] \to [\ell] \) preserving the order of the vertices induces a map \( \mu^* : \Delta^k \to \Delta^\ell \): the vertices of \( \Delta^k \) are mapped to the vertices of \( \Delta^\ell \) following the function \( \mu \), and the map is the affine extension to the convex hull of these points. In particular, the coface map \( d_i : \Delta^{k-1} \to \Delta^k \), \( 0 \leq i \leq k \), is given by formula

\[
d_i(t_0, \ldots, t_{k-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}),
\]

and the codegeneracy map \( s_i : \Delta^{k+1} \to \Delta^k \), \( 0 \leq i \leq k \), is given by formula

\[
s_i(t_0, \ldots, t_{k+1}) = (t_0, \ldots, \tilde{t}_i, \ldots, t_{k+1}).
\]

These maps go in the opposite direction to the maps in a simplicial manifold: \( \Delta^\bullet \) is an example of a cosimplicial manifold (with corners). Let \( \Omega_k \) be the de Rham complex of \( \Delta^k \), with differential \( \delta \): this is a simplicial complex of superspaces (and even a simplicial differential graded algebra).

If \( V \) is a vector bundle on a supermanifold \( M \), there is an inclusion with dense image

\[
\Gamma(M, V) \otimes \Omega_k \to \Gamma(M \times \Delta^k, V \boxtimes \Lambda^* T^* \Delta^k).
\]

The target of this morphism may be thought of as a completed tensor product

\[
\Gamma(M \times \Delta^k, V \boxtimes \Lambda^* T^* \Delta^k) = \Gamma(M, V) \hat{\otimes} \Omega_k.
\]

**Definition 3.1.** A morphism \( f : M \to N \) of supermanifolds is \( \text{étale} \) if it is locally an open embedding, or equivalently, the tangent maps \( d_x \mu^* \) are isomorphisms at all points \( x \in M \), and a \( \text{cover} \) if it is \( \text{étale} \) and surjective.

A simplicial odd symplectic supermanifold is a simplicial manifold \( M^\bullet \) such that

a) each supermanifold \( M_k \) is odd symplectic, and

b) the morphisms \( \mu^* : M_\ell \to M_k \) are \( \text{étale} \) and preserve the odd symplectic structures.

An example of a simplicial odd symplectic supermanifold is the Čech nerve \( \mathcal{U}_\bullet \) associated to an open cover \( \mathcal{U} \) of an odd symplectic supermanifold.

Associated to a simplicial odd symplectic supermanifold are the cosimplicial commutative superalgebra \( \mathcal{O}(M_\bullet) \) of functions on \( M_\bullet \) and the cosimplicial \( \mathcal{O}(M_\bullet) \)-supermodule
The Thom-Whitney normalization of $\mathcal{O}(M)$ is the differential graded commutative superalgebra

$$\text{Tot} \mathcal{O}(M) = \left\{ (f_k) \in \prod_{k=0}^{\infty} \mathcal{O}(M_k) \otimes \Omega_k \right\}$$

for all $\mu : [k] \to [\ell]$, we have $(\mu^* \otimes 1) f_\ell = (1 \otimes \mu_*) f_k \in \mathcal{O}(M_k) \otimes \Omega_\ell$}

with differential $\delta$ induced by the de Rham differential $\delta$ on $\Omega_k$ for different $k$. In [3], we showed that topological terms may be incorporated into the Batalin–Vilkovisky formalism by replacing the classical master equation

$$\frac{1}{2}(S,S) = 0$$

for an element of $\mathcal{O}(M)$ by the classical master equation

$$\delta S + \frac{1}{2}(S,S) = 0$$

in $\text{Tot} \mathcal{O}(M)$. (In [3], we denote this totalization by $\text{Tot}_{\text{TW}}$; since it is the only totalization functor employed in this article, we will write $\text{Tot}$ instead.)

The Thom-Whitney normalization of the cosimplicial superspace $\Omega^{1/2}(M)$ is the differential graded $\text{Tot} \mathcal{O}(M)$-module

$$\text{Tot} \Omega^{1/2}(M) = \left\{ (\sigma_k) \in \prod_{k=0}^{\infty} \Omega^{1/2}(M_k) \otimes \Omega_k \right\}$$

for all $\mu : [k] \to [\ell]$, we have $(\mu^* \otimes 1) \sigma_\ell = (1 \otimes \mu_*) \sigma_k \in \Omega^{1/2}(M_k) \otimes \Omega_\ell$}

The operator $\delta + \Delta$ descends to $\text{Tot} \Omega^{1/2}(M)$, turning it into a complex. In this setting, the quantum master equation becomes

$$(\delta + \Delta) \sigma = 0.$$ 

In the presence of the parameter $\hbar$, this is modified to $$(\delta + \hbar \Delta) \sigma = 0.$$ 

4. Families of Lagrangian submanifolds

Suppose that $M$ is an odd symplectic supermanifold, and that $\iota : L \subset M$ is a Lagrangian submanifold. In other words, the restriction of the odd symplectic form $\omega$ to $L$ induces an isomorphism between the tangent superbundle $TL$ of $L$ and the conormal superbundle $N^*L$. In particular, a Lagrangian submanifold is coisotropic: the ideal of functions vanishing on $L$ is closed under the antibracket.

It is a basic result of odd symplectic geometry that given a Lagrangian submanifold $L$ and a point $x \in L$, there is a Darboux coordinate chart $U$ around $x$ such that $U \cap L$ is the submanifold $\{ \xi_a = 0 \}$. In other words, a neighborhood of $x$ is identified with a neighbourhood of $x$ in the odd cotangent bundle $\Pi T^*L$. The proof is identical to the proof in the even case (Weinstein [18]).

The restriction $\iota^* \Omega^{1/2}$ of the bundle of half-forms $\Omega^{1/2}$ on $M$ to a Lagrangian submanifold $L$ is isomorphic to the bundle of integral forms on $L$, that is, the Berezinian bundle $\text{Ber}(T^*L)$ of the cotangent bundle of $L$. (If $L$ is a manifold, this is the same as the bundle of differential forms of top degree on $L$.) The integral is an invariantly defined linear form on the space
of integral forms of compact support on $L$: thus, the restriction map induces a linear form $\int_L: \Omega^{1/2}_c(M) \to \mathbb{C}$ on the bundle of compactly supported half-forms on $M$.

We now consider the generalization of this construction when $\iota: L \times \Delta^k \to M$ is a family of Lagrangian submanifolds parametrized by the $k$-simplex $\Delta^k$. Taking the derivative of the map $\iota$ in the simplicial direction, we obtain a family of vector fields $X \in \Gamma(L, \iota^*TM) \otimes \Omega_k$ over $L$, parametrized by one-forms on $\Delta^k$. Take the contraction with the odd symplectic form $\omega$ to convert this vector field into a differential in the ambient manifold $M$.

The condition that $L_t$ is Lagrangian for all $t \in \Delta^k$ is equivalent to the condition that this family of one-forms is closed: $d\iota^*(X \lrcorner \omega) = 0 \in \Omega^2(L) \otimes \Omega_k$.

Here, we denote by $d$ the differential in the first factor $L$ of a product $L \times \Delta^k$ of a supermanifold with a simplex, and by $\delta$ the differential in the second factor $\Delta^k$. Since the de Rham cohomology of $L$ vanishes in nonzero ghost number, there is a uniquely determined family of one-forms $\eta \in \mathcal{O}_L \otimes \Omega_k$ such that $d\eta = \iota^*(X \lrcorner \omega)$ and $\delta \eta = 0$.

Let us rewrite this equation in a Darboux coordinate system on $M$. Thus, suppose that $L$ has coordinates $x^a$ and $\iota$ is given in a Darboux coordinate system on $M$ by the equations

$$\begin{align*}
x^a &= x^a(x, t), \\
\xi_a &= \xi_a(x, t),
\end{align*}$$

where $x^a(x, 0) = x^a$ and $\xi_a(x, 0) = 0$. The one-form $\eta = \eta(x, t) dt^i$ satisfies the differential equation

$$\frac{\partial \xi_a(x, t)}{\partial t^i} = \frac{\partial \eta(x, t)}{\partial x^a}.$$ 

This implies that $\delta \eta$ is independent of $x$: thus, if $\delta \eta$ vanishes at any point in $L$, it vanishes everywhere. This may always be arranged, by replacing $\eta$ by

$$\eta + \sum_{i=0}^k t^i d\eta_i(x_0, t).$$

Since we are only concerned with families of Lagrangian submanifolds up to reparametrization, we may assume that, at least in a neighborhood of $(x_0, 0) \in L \times \Delta^k$, we have $x^a = x^a$. In this case, $L$ is (locally) a family of sections of the odd cotangent bundle $\Pi T^*L$:

$$\begin{align*}
x^a &= x^a, \\
\xi_a &= \xi_a(x, t).
\end{align*}$$
The following theorem is a mild generalization of a result of Mikhalkov and Schwarz [12 (3.6)].

**Theorem 4.1.** Let \( \iota : L \times \Delta^k \to M \) be a proper family of Lagrangian submanifolds of \( M \), and let \( \sigma \in \Omega^{1/2}_{c}(M) \otimes \Omega_k \) be a family of compactly supported half-forms on \( M \). Then

\[
\delta \int_{L} e^{-\eta/h} \iota^* \sigma = \int_{L} e^{-\eta/h} \iota^* (\delta + h\Delta) \sigma.
\]

**Proof.** Let \( x_0 \) be a point in \( L \), and consider a Darboux coordinate chart \((x^a, \xi_a)\) around \( \iota(x_0, 0) \in M \). If \( \sigma = f \, dx \), we have

\[
(i^* \sigma)(x, t) = f(x^a(x, t), \xi_a(x, t), t, dt) \text{ Ber} \left( \frac{\partial x^a(x, t)}{\partial x^b} \right) \, dx.
\]

We may assume that \( x^a(x, t) = x^a \), in which case we have

\[
(i^* \sigma)(x, t) = f(x^a, \xi_a(x, t), t, dt) \, dx.
\]

Applying the differential \( \delta \) and multiplying by the inhomogeneous differential form \( e^{-\eta/h} \), we obtain

\[
\delta(e^{-\eta/h} i^* \sigma)(x, t) = e^{-\eta(x, t)/h} \left( i^* \delta \sigma + \sum_{a \in I} \delta \xi_a(x, t) \frac{\partial f(x^a, \xi_a(x, t), t, dt)}{\partial \xi_a} \, dx \right)
\]

\[
= e^{-\eta(x, t)/h} \left( i^* \delta \sigma + \sum_{a \in I} \frac{\partial \eta(x, t)}{\partial x^a} \frac{\partial f(x^a, \xi_a(x, t))}{\partial \xi_a} \, dx \right)
\]

\[
= e^{-\eta(x, t)/h} i^*(-\delta + h\Delta) \sigma - h \sum_{a \in I} (-1)^{p_a} \frac{\partial}{\partial x^a} \left( e^{-\eta(x, t)/h} \frac{\partial f(x^a, \xi_a(x, t))}{\partial \xi_a} \right) \, dx.
\]

Integrating over \( L \), we obtain the result. \( \square \)

5. Lagrangian submanifolds of simplicial odd symplectic supermanifolds

In this section, we explain what we mean by a Lagrangian submanifold of a simplicial odd symplectic supermanifold \( M_\bullet \): this is our formulation in the Batalin–Vilkovisky setting of a global gauge condition pieced together from local gauge conditions.

In order to define a Lagrangian submanifold in this generalized sense, we start with a family of graded supermanifolds \( \{L_k\}_{k \geq 0} \) indexed by the natural numbers. For each function \( \mu : [k] \to [\ell] \) preserving the order of the vertices, we are given a morphism of graded supermanifolds

\[
\begin{array}{ccc}
L_\ell \times \Delta^k & \xrightarrow{\mu^*} & L_k \times \Delta^k \\
\Delta^k & \searrow & \Delta^k \\
\end{array}
\]
such that if $\mu : [k] \rightarrow [\ell]$ and $\nu : [j] \rightarrow [k]$ are a pair of functions preserving the order of the vertices, the following diagram commutes:

$$
\begin{array}{ccc}
L_\ell \times \Delta^j & \xrightarrow{\mu^* \times \Delta^j} & L_k \times \Delta^j \\
\downarrow & \searrow & \downarrow \\
L_j \times \Delta^j & \xrightarrow{\nu^*} & L_k \times \Delta^j
\end{array}
$$

The fibred product $\mu^* \times \Delta^j \Delta^j$ is taken with respect to the morphism of simplices $\nu_* : \Delta^j \rightarrow \Delta^k$.

The other data needed to specify a Lagrangian submanifold of $M_*$ are morphisms $\iota_k : L_k \times \Delta^k \rightarrow M_k$ satisfying the following conditions:

a) for every point $t \in \Delta^k$, the restriction of $\iota_k$ to $L_k \times t$ is a proper Lagrangian embedding;

b) for each morphism $f : [k] \rightarrow [\ell]$ in the simplicial category, the following diagram commutes:

$$
\begin{array}{ccc}
L_\ell \times \Delta^k & \xrightarrow{\mu^*} & L_\ell \times \Delta^\ell \\
\downarrow & \searrow & \downarrow \\
L_k \times \Delta^k & \xrightarrow{\iota_k} & M_k
\end{array}
$$

We only consider the case in which $M_* = U_*$ is the Čech nerve associated to an open cover $U = \{U_\alpha\}$ of an odd symplectic supermanifold $M$. In this case, $L_k$ decomposes into a disjoint union

$$L_k = \coprod_{\alpha_0 \ldots \alpha_k} L_{\alpha_0 \ldots \alpha_k},$$

and $\iota_k$ decomposes into families of proper Lagrangian embeddings

$$\iota_{\alpha_0 \ldots \alpha_k} : L_{\alpha_0 \ldots \alpha_k} \times \Delta^k \rightarrow U_{\alpha_0 \ldots \alpha_k}.$$

Denote by $\eta_{\alpha_0 \ldots \alpha_k} \in \mathcal{O}(L_{\alpha_0 \ldots \alpha_k}) \otimes \Omega_k \subset \Omega^1(L_{\alpha_0 \ldots \alpha_k} \times \Delta^k)$ the one-form associated to the family of Lagrangian submanifolds determined by $\iota_k$.

The main result of this article is the definition of a linear form $Z : \text{Tot} \Omega^{1/2}(U_*) \rightarrow \mathbb{C}$ of degree 0, which is closed in the sense that

$$Z((\delta + h\Delta)\sigma) = 0.$$

The formula for $Z$ specializes, in the case of a Lagrangian submanifold $L \subset M$, to the formula of Batalin and Vilkovisky,

$$Z(\sigma) = \int_L \iota^* \sigma.$$

They interpret $Z(\sigma)$ as the partition function for a quantum field theory, and $L$ as a gauge condition. Our extension of their formula allows the use of more general gauge conditions,
and opens the door to the use of the Batalin–Vilkovisky formalism when the action contains topological terms.

The definition of \( Z \) depends on the auxiliary data of a partition of unity \( \{ \varphi_\alpha \} \) for the cover \( \mathcal{U} \). In other words, \( \varphi_\alpha \in \mathcal{O}_c(U_\alpha) \), and

\[
\sum_\alpha \varphi_\alpha = 1.
\]

Denote the commutator \( [\Delta, m(\varphi_\alpha)] = H_{\varphi_\alpha} \) by \( H_\alpha \). Since \( [\Delta, m(1)] = 0 \), we see that

\[
\sum_\alpha H_\alpha = 0.
\]

From the partition of unity \( \{ \varphi_\alpha \} \), we define a series of differential operators acting on \( \Omega^{1/2}(U_{\alpha_0...\alpha_k}) \):

\[
\Phi_{\alpha_0...\alpha_k} = \frac{\hbar^k}{k + 1} \sum_{i=0}^k (-1)^i H_{\alpha_0} \cdots H_{\alpha_{i-1}} m(\varphi_{\alpha_i}) H_{\alpha_{i+1}} \cdots H_{\alpha_k}.
\]

**Lemma 5.1.**

\[
[\delta + \hbar \Delta, \Phi_{\alpha_0...\alpha_k}] = \sum_{i=0}^{k+1} (-1)^i \sum_\alpha \Phi_{\alpha_0...\alpha_{i-1} \alpha_i ... \alpha_k}
\]

**Proof.** We have

\[
[\delta + \hbar \Delta, \Phi_{\alpha_0...\alpha_k}] = \hbar^{k+1} H_{\alpha_0} \cdots H_{\alpha_k}.
\]

On the other hand, we have

\[
(-1)^i \sum_\alpha \Phi_{\alpha_0...\alpha_{i-1} \alpha_i ... \alpha_k}
\]

\[
= \frac{\hbar^{k+1}}{k + 2} \sum_{j=0}^{i-1} (-1)^{j+i} \sum_\alpha H_{\alpha_0} \cdots m(\varphi_{\alpha_j}) \cdots H_{\alpha_{i-1}} H_\alpha H_{\alpha_i} \cdots H_{\alpha_k}
\]

\[
+ \frac{\hbar^{k+1}}{k + 2} \sum_\alpha H_{\alpha_0} \cdots H_{\alpha_{i-1}} m(\varphi_{\alpha_i}) H_{\alpha_i} \cdots H_{\alpha_k}
\]

\[
+ \frac{\hbar^{k+1}}{k + 2} \sum_{j=i}^k (-1)^{j+i+1} \sum_\alpha H_{\alpha_0} \cdots H_{\alpha_{i-1}} H_\alpha H_{\alpha_i} \cdots m(\varphi_{\alpha_j}) \cdots H_{\alpha_k}.
\]

and by (5.1) and (5.2), this equals

\[
\frac{\hbar^{k+1}}{k + 2} H_{\alpha_0} \cdots H_{\alpha_k}.
\]

Summing over \( i \), the lemma follows. \( \square \)

We now come to the main result of this paper.

**Theorem 5.2.** Define a linear form \( Z \) on \( \text{Tot} \Omega^{1/2}(M_\bullet) \) by the formula

\[
Z(\sigma_\bullet) = \sum_{k=0}^\infty (-1)^k \sum_{\alpha_0...\alpha_k} \int_{\Delta_k} \int_{L_{\alpha_0...\alpha_k}} e^{-\eta_{\alpha_0...\alpha_k}/\hbar} l^{*}_{\alpha_0...\alpha_k} (\Phi_{\alpha_0...\alpha_k} \sigma_{\alpha_0...\alpha_k}).
\]
Then $Z$ is closed: $Z((\delta + h\Delta)\sigma_\bullet) = 0$.

**Proof.** By Lemma [5.1](#) we have

$$
Z((\delta + h\Delta)\sigma_\bullet) = \sum_{k=0}^{\infty} (-1)^k \sum_{\alpha_0...\alpha_k} \int_{\Delta^k} \int_{L_{\alpha_0...\alpha_k}} e^{-y_{\alpha_0...\alpha_k}/h} l_{\alpha_0...\alpha_k}^*(\varphi_{\alpha_0...\alpha_k}(\delta + h\Delta)\sigma_{\alpha_0...\alpha_k})
$$

$$
= \sum_{k=0}^{\infty} \sum_{\alpha_0...\alpha_k} \left( \int_{\Delta^k} \int_{L_{\alpha_0...\alpha_k}} e^{-y_{\alpha_0...\alpha_k}/h} l_{\alpha_0...\alpha_k}^*(\varphi_{\alpha_0...\alpha_k}\sigma_{\alpha_0...\alpha_k}) \right)
$$

By Theorem [4.1](#) and Stokes's Theorem, the first sum equals

$$
\sum_{k=0}^{\infty} \sum_{\alpha_0...\alpha_k} \int_{\Delta^k} \delta \int_{L_{\alpha_0...\alpha_k}} e^{-y_{\alpha_0...\alpha_k}/h} l_{\alpha_0...\alpha_k}^*(\varphi_{\alpha_0...\alpha_k}\sigma_{\alpha_0...\alpha_k})
$$

$$
= \sum_{k=0}^{\infty} \sum_{i=0}^{k} (-1)^i \sum_{\alpha_0...\alpha_k} \int_{\Delta^k} \int_{L_{\alpha_0...\alpha_i...\alpha_k}} e^{-y_{\alpha_0...\alpha_i...\alpha_k}/h} l_{\alpha_0...\alpha_i...\alpha_k}^*(\varphi_{\alpha_0...\alpha_k}\sigma_{\alpha_0...\alpha_i...\alpha_k}).
$$

The result follows. 

We may generalize the above construction when the partition of unity $\varphi_\alpha$ is parametrized by points in an auxiliary $\ell$-simplex $\Delta_\ell$. Let $d$ be the de Rham differential on $\Delta_\ell$, and generalize the differential operator $\Phi_{\alpha_0...\alpha_k}$ to have coefficients in $\Omega_\ell$, the differential graded algebra of differential forms on the auxiliary simplex:

$$
(5.3) \quad \Phi_{\alpha_0...\alpha_k} = \frac{1}{k+1} \sum_{i=0}^{k} (-1)^i (m(d\varphi_{\alpha_0}) + hH_{\alpha_0}) \ldots (m(d\varphi_{\alpha_{i-1}}) + hH_{\alpha_{i-1}})
$$

$$
m(\varphi_{\alpha_i})(m(d\varphi_{\alpha_{i+1}}) + hH_{\alpha_{i+1}}) \ldots (m(d\varphi_{\alpha_k}) + hH_{\alpha_k}).
$$

With no change in its proof, Lemma [5.1](#) now takes the more general form

$$
[d + \delta + h\Delta, \Phi_{\alpha_0...\alpha_k}] = \sum_{i=0}^{k+1} (-1)^i \sum_{\alpha} \Phi_{\alpha_0...\alpha_i-1\alpha_i...\alpha_k}.
$$

We may now define a trace $Z$ with values in $\Omega_\ell$, by the same formula as before. Again with no change in the proof, Theorem [5.2](#) becomes the following:

$$
Z((d + \delta + h\Delta)\sigma_\bullet) + dZ(\sigma_\bullet) = 0
$$

An observable in the Batalin–Vilkovisky formalism is a bosonic half-form $\sigma_\bullet$ of ghost number 0 that is a cocycle in the Thom-Whitney complex:

$$
(d + \delta + h\Delta)\sigma_\bullet = 0.
$$

The following is an immediate consequence of this parametrized generalization of Theorem [5.2](#)
Corollary 5.3. If $\sigma_\bullet$ is an observable in the Batalin–Vilkovisky formalism, then $Z(\sigma_\bullet)$ is independent of the partition of unity used in the definition of $Z$.

6. Application to the superparticle

In this section, we construct a flexible Lagrangian submanifold that imposes the light-cone gauge for the superparticle. The superparticle is a supersymmetric analogue of the relativistic particle in ten-dimensional spacetime $\mathbb{R}^{1,9}$. Recall the action of the relativistic particle. (See [4] for further details.)

Let $\{E_\mu\}_{0 \leq \mu \leq 9}$ be a basis for $\mathbb{R}^{1,9}$, with inner product

$$(E_\mu, E_\nu) = \begin{cases} \delta_{\mu\nu}, & \mu + \nu > 0, \\ -1, & \mu = \nu = 0 \end{cases}$$

with dual basis $\{E^\mu\}_{0 \leq \mu \leq 9}$. Let $\eta^{\mu\nu} = (E_\mu, E_\nu)$.

The world-line is an oriented parametrized one-dimensional manifold, and fields are differential forms on this manifold. Denote differentiation along the world-line by $\partial$.

The physical fields in this model (the fields of ghost number 0) are the position $x^\mu$ and momentum $p_\mu$, which are world-line scalars taking values in $\mathbb{R}^{1,9}$ and its dual $(\mathbb{R}^{1,9})^\vee$, and the einbein (or gravitational field) $e$, which is a nowhere-vanishing world-line one-form. The antifields $x_+^\mu$ and $p^+\mu$ are world-line one-forms taking values in $(\mathbb{R}^{1,9})^\vee$ and $\mathbb{R}^{1,9}$ respectively, and the antifield $e^+$ is a world-line scalar; all have ghost number $-1$ and odd total parity.

There is also a ghost field $c$, associated to parametrization of the world-line. This field is a world-line scalar of ghost number 1 and odd total parity; its antifield $c^+$ is a world-line one-form of ghost number $-2$ and even total parity.

The Batalin–Vilkovisky action of the relativistic particle is

$$S_0 = \int \left( p_\mu \partial x^\mu - \frac{1}{2} e(p, p) + (\partial e^+ - (x^+, p)) c \right) dt. \quad (6.1)$$

The simply-connected cover $\text{Spin}(1,9)$ of the Lorentz group $\text{SO}(1,9)$ has a pair of 16-dimensional real representations, the left- and right-handed Majorana–Weyl spinors $S_\pm$. Denote the Clifford action of the standard basis of $\mathbb{R}^{1,9}$ by $\gamma^\mu$, so that

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu}.$$ 

The action of the Lie algebra $\mathfrak{so}(1,9)$ on $S_\pm$ is realized by the elements of the Clifford algebra

$$\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu).$$

Denote the non-degenerate pairing between $S_\pm$ and $S_\mp$ by $T(\alpha, \beta)$, and let

$$T^\mu(\alpha, \beta) = T(\gamma^\mu \alpha, \beta) = T(\alpha, \gamma^\mu \beta)$$

and

$$T^{\mu\nu}(\alpha, \beta) = T(\gamma^{\mu\nu} \alpha, \beta) = -T(\alpha, \gamma^{\mu\nu} \beta).$$

The superparticle has, in addition to the field content of the relativistic particle, a sequence $\theta_n$ of world-line scalar fields of ghost number $n$, which for even $n$ take values in $S_+$ and have odd total parity, and for odd $n$ take values in $S_-$ and have even total parity. For
each $n$ there is the corresponding antifield $\theta^+_n$, of ghost number $-1-n$, which is a world-line
one-form that for even $n$ takes values in $\mathbb{S}_+$ and has odd total parity, and for odd $n$ takes
values in $\mathbb{S}_-$ and has even total parity. Let $N$ be the graded supermanifold with coordinates
$\{x^{\mu},p_\nu,e,c,\theta_0\}$, and let $M = T^*[1]N$ be its shifted cotangent bundle, whose fibres have
coordinates the corresponding antifields.

The Batalin–Vilkovisky extension of the classical action of the superparticle has the form
$S = S_0 + S_1$, where $S_0$ is the classical action of the particle [6.1], and $S_1$ depends only on
the fields and antifields

$$\{p_\mu,\theta_0\} \cup \{x^{\mu}_+, e^+, c^+, \theta^+_0\}$$

and their derivatives. The salient term $-\frac{1}{2} \int p_\mu T^\mu(\theta^+_0, \theta^+_0) \, dt$ of $S_1$ is the dimensional reduction of the topological term in the Green–Schwarz action for the superstring.

A light-like vector $\mathbf{m}$ is a non-zero vector such that

$$(\mathbf{m}, \mathbf{m}) = 0.$$ 

The space of solutions of this equation has two components, the forward and backward light-cones, depending on the sign of the time-component $\mathbf{m}_0$. Fix such a light-like vector, for example $\mathbf{m} = \frac{1}{2}(E^0 + E^8)$.

Let $\mathbf{n}$ be another light-like vector satisfying the equation $(\mathbf{m}, \mathbf{n}) = 1/2$, for example $\mathbf{n} = \frac{1}{2}(E^0 - E^8)$. Let $c(\mathbf{m})$ be Clifford multiplication by $\mathbf{m}$, given by contraction with the $\gamma$-matrices,

$$c(\mathbf{m}) = \mathbf{m} \cdot \gamma^\mu : \mathbb{S}_+ \to \mathbb{S}_+$$

and similarly for $c(\mathbf{n})$. We have $c(\mathbf{m})^2 = c(\mathbf{n})^2 = 0$, and $c(\mathbf{m})c(\mathbf{n}) + c(\mathbf{n})c(\mathbf{m}) = 1$.

The orthogonal complement to the plane spanned by the vectors $\{\mathbf{m}, \mathbf{n}\}$ is the eight-dimensional Euclidean space spanned by $\{E_1, \ldots, E_8\}$. Associated to this subspace is a pair of spinor representations $\mathbb{S}_\pm$, also eight-dimensional, which we may identify with solutions in $\mathbb{S}_\pm$ of the equation

$$c(\mathbf{m})\theta_\pm = 0.$$ 

The map $\theta \mapsto c(\mathbf{m})c(\mathbf{n})\theta$ projects from $\mathbb{S}_\pm$ to $\mathbb{S}_\pm$.

**Lemma 6.1.** If $\theta_\pm \in \mathbb{S}_\pm$, then $p_\mu T^\mu(\theta^+_+, \theta^-_-) = 2(p, \mathbf{m}) \mathbf{n}_\nu T^\nu(\theta^+_+, \theta^-_-)$.

**Proof.** Let $q = p - 2(p, \mathbf{n})\mathbf{m} - 2(p, \mathbf{m})\mathbf{n}$. We have

$$c(p) = c(p)c(\mathbf{m})c(\mathbf{n}) + c(p)c(\mathbf{n})c(\mathbf{m})$$

$$= (-c(\mathbf{m})c(q)c(\mathbf{n}) + 2(p, \mathbf{m})c(\mathbf{n})c(\mathbf{m})c(\mathbf{n}))$$

$$+ (-c(\mathbf{n})c(q)c(\mathbf{m}) + 2(p, \mathbf{n})c(\mathbf{m})c(\mathbf{n})c(\mathbf{m}))$$

$$= -c(\mathbf{m})c(q)c(\mathbf{n}) - c(\mathbf{n})c(q)c(\mathbf{m}) + 2(p, \mathbf{m})c(\mathbf{n}) + 2(p, \mathbf{n})c(\mathbf{m}).$$

It follows that

$$p_\mu T^\mu(\theta^+_+, \theta^-_-) = -q_\mu(T^\mu(c(\mathbf{n})\theta^+_+, c(\mathbf{m})\theta^-_-) + T^\mu(c(\mathbf{m})\theta^+_+, c(\mathbf{n})\theta^-_-))$$

$$+ 2(p, \mathbf{m})T(c(\mathbf{n})\theta^+_+, \theta^-_-) + 2(p, \mathbf{n})T(c(\mathbf{m})\theta^+_+, \theta^-_-)$$

$$= 2(p, \mathbf{m})T(c(\mathbf{n})\theta^+_+, \theta^-_-).$$

$\square$
Let $U(m)$ be the open subset of $M$ where the inequality $(p, m) > 0$ holds. Let $L(m)$ be the Lagrangian submanifold in $U(m)$ defined by the equations

\[
\begin{align*}
x^+ \mu &= 0, \\
p^+ \mu &= 0, \\
e &= 1, \\
c^+ &= 0, \\
c(m)\theta &= 0, \\
c(m)\theta^+ &= 0,
\end{align*}
\]

and let $\iota$ be the inclusion of $L(m)$ in $U(m)$.

The explicit formula for the Batalin–Vilkovisky extension of the classical action of the superparticle and Lemma 6.1 give the following proposition.

**Lemma 6.2.** The gauge-fixed action $\iota^* S$ for the superparticle equals

\[
\int \left( p_\mu \partial x^\mu - \frac{1}{2} (p, p) + \partial e^+ c + (p, m) \mathbf{n}_\mu \left( - T^\mu (\theta_0, \partial \theta_0) + 2 \sum_{n=0}^{\infty} T^\mu (\theta^+_n, \theta_{n+1}) \right) \right) dt.
\]

In particular, there is no dependence on the function $\Phi(p)$ at the classical level.

In terms of the redefined spinor fields

\[
\Theta_n = (p, m)^{n+1/2} \theta_n, \quad \Theta^+_n = (p, m)^{-n-1/2} \theta^+_n,
\]

this gauge-fixed theory has the same action as a free massless relativistic superparticle, with classical action

\[
\int \left( p_\mu \partial x^\mu - \frac{1}{2} (p, p) + \partial e^+ c - \mathbf{n}_\mu T^\mu (\Theta_0, \partial \Theta_0) + 2 \sum_{n=0}^{\infty} \mathbf{n}_\mu T^\mu (\Theta^+_n, \Theta_{n+1}) \right) dt.
\]

At least after regularization, the Berezinian of the canonical transformation (6.2) is seen to be proportional to the value of the Dirichlet $L$-function

\[
L(s) = 1 - 3^{-s} + 5^{-s} - 7^{-s} + 9^{-s} - \ldots
\]

at $s = -1$. As Hurwitz showed [6],

\[
L(-1) = 1 - 3 + 5 - 7 + 9 - \cdots = \sum_{n \in \mathbb{Z}} n
\]

vanishes. In terms of the Hurwitz zeta-function

\[
\zeta(s, a) = a^{-s} + (a + 1)^{-s} + (a + 2)^{-s} + (a + 3)^{-s} + \ldots,
\]

we clearly have

\[
L(s) = 4^{-s} (\zeta(s, 1/4) - \zeta(s, 3/4)).
\]

The values of the Hurwitz zeta-function at negative integers is related to the Bernoulli polynomials $B_n(a)$, by the formula

\[
\zeta(1 - n, a) = -B_n(a)/n.
\]

(This is a consequence of the equation $\partial \zeta(s, a)/\partial s = -s \zeta(s + 1, a)$ and Euler’s formulas $\zeta(1 - n) = (-1)^n B_n/n$, $n > 0$, and $\operatorname{Res}_{s=1} \zeta(s) = 1$.) Since $B_n(1 - a) = (-1)^n B_n(a)$, and in particular $B_2(a) = a^2 - a + 1/6$, the vanishing of $L(-1)$ follows. This L-function was first presented by Schlömilch in the guise of a problem for university students [11], and we reproduce here the original text.
Arguably the physics of the superparticle takes place in the vicinity of the forward light-cone \( \{ (p,p) = 0 \} \cap \{ p_0 > 0 \} \). The open set \( U(m) \) does not cover the whole of the forward light-cone: in fact, it omits the ray where \( p_0 = p_9 \) and the transverse momenta vanish. This is a symptom of a Gribov ambiguity in the use of the light-cone gauge, which appears to have received little attention in the literature.\(^1\)

The open set
\[
U = \{ p_1^2 + \cdots + p_8^2 > \frac{1}{2} p_9^2 \} \cap \{ p_0 > 0 \}
\]
is a neighbourhood of the forward light-cone. Consider the light-like vectors
\[
m_\pm = \frac{1}{2} (E^0 \pm E^9),
\]
and the associated open sets \( U(m_\pm) = \{ p_0 > \pm p_9 \} \). We consider the subsets
\[
U_\pm = \{ p_0 > \pm 2 p_9 \} \cap U \subset U(m_\pm).
\]
The open sets \( U_+ \) and \( U_- \) cover \( U \), and have intersection
\[
U_{+-} = U_+ \cap U_- = \{ p_1^2 + \cdots + p_8^2 > \frac{1}{2} p_9^2 > 2 p_9^2 \}.
\]

We adopt the convention that indices in the range \( \{ 1, \ldots, 8 \} \), that is, transverse to the light-cone, are denoted \( p_a, p^{+b} \), etc. The Einstein summation convention will also be applied for these indices. Consider the Hamiltonian flow \( \Phi_\tau \) associated to the Hamiltonian
\[
(6.3)
\]
\[
\psi = -\frac{\pi}{p_9} \sum_{n=0}^{\infty} p_a T^{a9}(\theta_n^+, \theta_n).
\]

\(^1\)An exception is Siegel \cite{15}. He resolves the problem in the absence of supersymmetry by a method akin to stochastic quantization. He adjoins additional coordinates in the target spacetime: a pair of bosonic fields, with metric of signature \( (1,1) \), and a pair of compensating fermionic fields. The effect is to replace the Lorentz group \( \text{SO}(1,9) \) by the orthosymplectic supergroup \( \text{SOSp}(2,9|2) \). Extending this method from the particle to the superparticle would seem to require the introduction of spinors for the superspace \( \mathbb{R}^{1,1|2} \), or equivalently, differential forms on an auxiliary line. We will not pursue this approach further here.
where \( p_* = (\eta^{ab}p_ap_b)^{1/2} = (p_1^2 + \cdots + p_8^2)^{1/2} \). This flow leaves all of the fields invariant except \( p^+a, \theta_n \) and \( \theta_n^+ \). In terms of the one-parameter group in Spin(1, 9),

\[
g(\tau) = \cos(\pi \tau / 2) - \frac{\sin(\pi \tau / 2)}{p_*} p_a \gamma^{a9},
\]

these fields transform as follows:

\[
\theta_n(\tau) = g(\tau)\theta_n, \quad \theta_n^+(\tau) = g(\tau)\theta_n^+ \quad \text{and}
\]

\[
p^{+a}(\tau) = p^{+a} + \sum_{n=0}^{\infty} \mathcal{T} \left( \frac{\partial g(\tau)}{\partial p_a} r_{n}^{+}(\theta_n^+, \theta_n) \right).
\]

The formulas for \( \theta_n(\tau) \) and \( \theta_n^+(\tau) \) are clear, and the formula for \( p^{+a} \) on solving for the antibrackets \( (p_a, p^{+b}(\tau)) = \delta^b_a, \quad (\theta_n(\tau), p^{+a}(\tau)) = 0 \) and \( (\theta_n^+(\tau), p^{+a}(\tau)) = 0 \).

Consider the Lagrangian submanifolds

\[
L_{\pm} = L(\mathbf{m}_\pm) \cap U_{\pm},
\]

with inclusions \( i_{\pm}: L_{\pm} \to U_{\pm} \). Let \( L_{++} = L_{+} \cap U_{+} \), and let \( L(\tau) \) be the image of \( L_{++} \) under the canonical transformation \( \Phi_\tau \). A calculation in the Clifford algebra shows that

\[
g(\tau) c(\mathbf{m}) g(\tau)^{-1} = c(\mathbf{m}(\tau)),
\]

where \( \mathbf{m}(\tau) \) is the light-like vector

\[
\mathbf{m}(\tau) = \frac{1}{2} \left( E^0 + \cos(\pi \tau) E^9 - \frac{\sin(\pi \tau)}{p_*} p_a E^a \right).
\]

On \( L(\tau) \), we have \( c(\mathbf{m}(\tau)) \theta_n(\tau) = 0 \) and \( c(\mathbf{m}(\tau)) \theta_n^+(\tau) = 0 \), and hence \( \theta_n = c(\mathbf{m}(\tau)) c(\mathbf{n}(\tau)) \), where \( \mathbf{n}(\tau) \) is the light-like vector

\[
\mathbf{n}(\tau) = \frac{1}{2} \left( E^0 - \cos(\pi \tau) E^9 + \frac{\sin(\pi \tau)}{p_*} p_a E^a \right).
\]

The Lagrangian \( L(\tau) \) is cut out by the equations \( x_\mu^+ = 0, \quad p^{+\mu}(\tau) = 0, \quad e = 1, \quad c^+ = 0, \quad c(\mathbf{m}(\tau)) \theta_n = 0 \) and \( c(\mathbf{m}(\tau)) \theta_n^+ = 0 \).

**Lemma 6.3.** On \( L(\tau) \), we have

\[
p^{+a}(\tau) = p^{+a} + \frac{\sin(\pi \tau)}{2p_*} \left( p_b \eta^{ab} \frac{1}{2p_*} p_c \mathcal{T}^c_d (\theta_n^+, \theta_n) - \mathcal{T}^{a0}(\theta_n^+, \theta_n) \right).
\]

**Proof.** Since \( \theta_n(\tau) = c(\mathbf{m}(\tau)) c(\mathbf{n}(\tau)) \theta_n(\tau) \), we see that

\[
p^{+a}(\tau) = p^{+a} + \sum_{n=0}^{\infty} \mathcal{T} \left( \frac{\partial g(\tau)}{\partial p_a} r_{n}^{+}(\theta_n^+, \theta_n) \right)
\]

\[
= p^{+a} + \sum_{n=0}^{\infty} \mathcal{T} \left( \frac{\partial g(\tau)}{\partial p_a} g(\tau)^{-1} \theta_n^+(\tau), c(\mathbf{m}(\tau)) c(\mathbf{n}(\tau)) \theta_n(\tau) \right)
\]

\[
= p^{+a} + \sum_{n=0}^{\infty} \mathcal{T} \left( g(\tau) c(\mathbf{n}) c(\mathbf{m}) g(\tau)^{-1} \frac{\partial g(\tau)}{\partial p_a} g(\tau)^{-1} \theta_n^+(\tau), \theta_n(\tau) \right).
\]

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Observe that
\[
c(\mathbf{n}) c(\mathbf{m}) g(\tau)^{-1} = \cos(\pi \tau / 2) c(\mathbf{n}) c(\mathbf{m}) + \frac{\sin(\pi \tau / 2) p_0 \gamma^9}{p_*} c(\mathbf{m}) c(\mathbf{n}).
\]
Since
\[
\frac{\partial g(\tau)}{\partial p_a} = \frac{\sin(\pi \tau / 2)}{p_*} \left( \frac{p_b \eta^{ab}}{(p_*)^2} \rho c \gamma^9 - \gamma^9 \right),
\]
we see that
\[
c(\mathbf{m}) c(\mathbf{n}) \frac{\partial g(\tau)}{\partial p_a} g(\tau)^{-1} \theta^+_{\mu}(\tau) = \frac{\partial g(\tau)}{\partial p_a} c(\mathbf{n}) c(\mathbf{m}) g(\tau)^{-1} \theta^+_{\mu}(\tau)
= \frac{\partial g(\tau)}{\partial p_a} g(\tau)^{-1} c(\mathbf{n}(\tau)) c(\mathbf{m}(\tau)) \theta^+_{\mu}(\tau)
= 0
\]
and
\[
c(\mathbf{n}) c(\mathbf{m}) \frac{\partial g(\tau)}{\partial p_a} g(\tau)^{-1} \theta^+_{\mu}(\tau) = \frac{\partial g(\tau)}{\partial p_a} c(\mathbf{m}) c(\mathbf{n}) g(\tau)^{-1} \theta^+_{\mu}(\tau)
= \frac{\partial g(\tau)}{\partial p_a} g(\tau)^{-1} c(\mathbf{m}(\tau)) c(\mathbf{n}(\tau)) \theta^+_{\mu}(\tau)
= \frac{\partial g(\tau)}{\partial p_a} g(\tau)^{-1} \theta^+_{\mu}(\tau).
\]
The lemma follows. □

Note that \( m(0) = m_+ \) and \( m(1) = m_- \), while \( n(0) = m_- \) and \( n(1) = m_+ \). In conjunction with the lemma, this implies the following corollary.

**Corollary 6.4.** \( L(1) = L_- \cap U_+ \)

Let \( \iota_{++} : L_{++} \times \Delta^1 \to U_{++} \) be the family of Lagrangians equal to \( L(\tau) \) at \( \tau \in \Delta^1 \). Together with the Lagrangians \( L_{\pm} \), we obtain a flexible Lagrangian for the cover \( \{ U_+, U_- \} \).

By Lemma 6.2, we have
\[
\iota^*_{++} S = \int \left( p_\mu \partial x^\mu - \frac{1}{2} (p, p) + \partial e^+ c + p(\tau) n_\mu(\tau) \left(-T^\mu(\partial \theta_0, \partial \theta_0) + 2 \sum_{n=0}^{\infty} T^\mu(\theta^+_{n+1}, \theta_{n+1}) \right) \right) dt,
\]
where \( p(\tau) = (p, m(\tau)) \). Here, we have used that the action \( S \) of the superparticle does not depend on \( p^+ \).

**Lemma 6.5.** The functions \( p(\tau), 0 \leq \tau \leq 1, \) and \( p_* \) are positive on \( U_{++} \).

**Proof.** On \( U_{++} \), we have
\[
p(\tau) = \frac{1}{2} (p_0 - \cos(\pi \tau) p_9 + \sin(\pi \tau) p_*) \geq \frac{1}{2} (p_0 - |p_0|) > 0.
\]
Likewise,
\[
\frac{1}{2} p_0^2 < p_9^2 + p_*^2 < p_9^2 + \frac{1}{4} p_0^2,
\]
and hence \( p_*^2 > \frac{1}{4} p_0^2 \), showing that \( p_* \) is positive on \( U_{++} \). □
We now perform the change of variables (6.2):
\[ \Theta_n = p(\tau)^{n+1/2}\theta_n, \quad \Theta_n^+ = p(\tau)^{-n-1/2}\theta_n^+. \]

The resulting gauge-fixed action \( \iota^*_+S \) is independent of the parameter \( \tau \). The one-form \( \eta_- = \psi \, d\tau \) in the contribution of \( U_+ \) to the functional integral \( Z(\sigma_\bullet) \) equals
\[ \eta_- = -\frac{\pi}{p_*} \sum_{n=0}^{\infty} p_n T^a(\Theta_n^+; \Theta_n) \, d\tau. \]

To complete the formula for \( Z(\sigma_\bullet) \), we need a partition of unity for the cover \( U_\pm \) of \( U \). Choose a function \( \varphi \in C^\infty(\mathbb{R}) \) that vanishes for \( t \leq \frac{1}{4} \) and such that \( \varphi(t) + \varphi(1-t) = 1 \). A suitable partition of unity is
\[ \varphi_\pm(p) = \varphi((p_9 \mp p_0)/2p_0). \]

7. Global symmetries

The superparticle is Lorentz invariant and supersymmetric. On the other hand, the flexible Lagrangian that we constructed in the last section is not invariant under these symmetries. In this section, we give an equivariant extension of Theorem 5.2. There is now a Hamiltonian action of a (finite-dimensional) Lie superalgebra \( g \) on the Batalin–Vilkovisky supermanifold \( M \). We do not assume any compatibility between this action and either the cover \( \mathcal{U} \) or the simplicial Lagrangian \( L_\bullet \). Instead, we express the covariance of the linear form \( Z \) by adapting the BRST formalism. In practice, this means that we replace the complex numbers by the commutative superalgebra \( C^*(g) \) of cochains of the Lie superalgebra \( g \).

The action of \( g \) on \( M \) is determined by a moment map, that is, a morphism of Lie superalgebras \( \rho : g \to \mathcal{O}(M)[-1] \). In other words, if \( \xi_1, \xi_2 \in g \), we have
\[ (\rho(\xi_1), \rho(\xi_2)) = \rho([\xi_1, \xi_2]). \]

We now introduce the differential graded commutative superalgebra \( C^*(g) \) of Lie superalgebra cochains on \( g \). This is the free graded commutative superalgebra generated by the dual superspace \( g^\vee[-1] \) to \( g \) placed at ghost number 1. If \( \{\xi_a\} \) is a (homogeneous) basis of \( g \), then \( C^*(g) \) is generated by elements \( \{e^a\} \) of ghost number 1, having the opposite total degree to \( \xi_a \): if \( \xi_a \) is even (respectively odd), \( e^a \) is an exterior (resp. polynomial) generator.

The structure coefficients of \( g \) are defined as follows:
\[ [\xi_a, \xi_b] = C^c_{ab} \xi_c. \]

The differential on \( C^*(g) \) is given by the formula
\[ \delta_g e^a = \frac{1}{2} \sum_{b,c} (-1)^{(p(\xi_b)+1)p(\xi_c)} C^a_{bc} e^b e^c. \]

The element \( \mu = \sum_a p(\xi_a) e^a \in \mathcal{O}(M) \otimes C^*(g) \) satisfies the Maurer-Cartan equation:
\[ \delta_g \mu + \frac{1}{2} (\mu, \mu) = 0. \]
This implies the following identity for differential operators on $\Omega^{1/2}(M) \otimes C^*(\mathfrak{g})$: 
\[
e^{\mu/h} \circ (\delta_{\mathfrak{g}} + H_\mu + h\Delta) = (\delta_{\mathfrak{g}} + h\Delta) \circ e^{\mu/h}.
\]
In particular, $\delta_{\mathfrak{g}} + H_\mu + h\Delta$ is a differential on $\Omega^{1/2}(M) \otimes C^*(\mathfrak{g})$.

We have the following equivariant extension of Theorem 5.2.

**Theorem 7.1.** Define a linear form $Z_\mathfrak{g}$ on $\text{Tot} \mathcal{O}(M_\bullet) \otimes \Omega_\ell \otimes C^*(\mathfrak{g})$ with values in $\Omega_\ell \otimes C^*(\mathfrak{g})$ by the formula
\[
Z_\mathfrak{g}(\sigma_\bullet) = \sum_{k=0}^{\infty} (-1)^k \sum_{\alpha_0...\alpha_k} \int_{\Delta^k} \int_{L_{\alpha_0...\alpha_k}} e^{-\eta_{\alpha_0...\alpha_k}/h} \theta_{\alpha_0...\alpha_k} \left( \theta_{\alpha_0...\alpha_k} \left( e^{\mu/h} \sigma_{\alpha_0...\alpha_k} \right) \right).
\]

Then $Z_\mathfrak{g}$ is closed: $Z_\mathfrak{g}((d + \delta_{\mathfrak{g}} + H_\mu + \delta + h\Delta)\sigma_\bullet) + (d + \delta_{\mathfrak{g}})Z_\mathfrak{g}(\sigma_\bullet) = 0$.

We apply this theorem to the superparticle. The Lie superalgebra $\mathfrak{g}$ is the sum of three subspaces: translations, parametrized by a covariant vector in $\mathbb{R}^{1,9}$, supersymmetries, parametrized by a Majorana–Weyl spinor in $\mathbb{S}_-$, and Lorentz transformations, parametrized by $\mathfrak{so}(1,9)$, or equivalently, the second exterior power $\Lambda^2 \mathbb{R}^{1,9}$. The momentum for translation symmetry equals
\[
\int x^+ \mu \, dt.
\]
The restriction of $x^+ \mu$ to the flexible Lagrangian of the previous section vanishes. Thus, translations may be ignored in the calculation of $\mu$.

The momentum for supersymmetry equals
\[
\int \left( \theta^+_0 - \frac{1}{2} x^+ \mu \gamma^0 \theta_0 \right) dt.
\]
On restriction to the flexible Lagrangian by any of the maps $\iota_\pm$ or $\iota_{+-}$, the second term vanishes. Denoting the corresponding BRST ghosts, of ghost number 1 and even total parity, by $\epsilon \in \mathbb{S}_+$, we obtain a contribution of $\int T(\theta^+_0, \epsilon) dt$ to $\mu$ in all three cases.

The momentum for Lorentz symmetries is
\[
\int \left( \eta^{\lambda_\mu} x^+_\lambda - \eta^{\lambda_\mu} p^+_\mu \gamma^0 \theta_0 - \sum_{n=0}^{\infty} T^{\mu \nu}(\theta^+_n, \theta_0) \right) dt.
\]

Its contribution to $\iota^*_+ \mu$ and $\iota^*_+ \mu$ equals
\[
- \sum_{n=0}^{\infty} \int \left( T^{a_0}(\theta^+_n, \theta_0) \epsilon_{a_0} + T^{a_9}(\theta^+_n, \theta_0) \epsilon_{a_9} \right) dt,
\]
since $x^+_\mu$ and $p^+ \mu$ vanish on $L_+$ and $L_-$, and $\gamma^{ab}$ and $\gamma^{09}$ commute with $c(m_+)$ and $c(m_-)$.

Its contribution to $\iota^*_{+-} \mu$ may be derived from the formula of Lemma 6.3 for $p^+ \mu(\tau)$.

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