A short proof of the equivalence of any Reidemeister oriented move 3

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May 5, 2014

Abstract

In this note we present a short proof that the 4 oriented Reidemeister moves of type 2 together with any one of the 8 oriented Reidemeister moves of type 3 are sufficient to imply the other 7.

1 Introduction

In a recent paper, Polyak presented a minimal set of Reidemeister moves sufficient to generate all the others of types 1,2,3, [2]. Moves of type 1 are special, in the sense that looking for invariants of 3-manifolds we do not want to impose invariance under move of type 1. This is because the 3-manifold given by surgery in a framed link do change under such a move [1]. Therefore, it is convenient to have a minimal set of regular isotopy moves disregarding completely moves of type 1. The author is aware that the result is well known. However, the proof that follows is so short that it deserves to be mentioned.

2 Proof of equivalence

A Reidemeister move of type 3 has a triangular region over which is put upside down when the move is applied. There are 16 configurations of triangular regions involved in oriented type 3 moves. The boundary of the triangular region is formed by three curves which is ordered in the following way: first the curve whose immediate extension goes up and up, second the curve whose immediate extension goes once up and once down, third the curve whose immediate extension goes down and down. These curves can deformed to become line segments. A triangular configuration present in a link diagram is locally rotated so that the up-up segment becomes horizontal and the boundary forms a letter ∆ (not a ∇). We encode such a triangular region by a sequence of 3 arrows. The first arrow is horizontal and can be from west to east or from east to west; the second arrow has its direction coinciding with the second line segment and the third arrow has its direction coinciding with the third line segment. The 16 triangular configurations are encoded and named as follows:

\[ a: \rightarrow\uparrow\rightarrow\wedge \quad b: \rightarrow\downarrow\rightarrow\wedge \quad c: \rightarrow\downarrow\leftarrow\wedge \quad d: \rightarrow\uparrow\leftarrow\wedge \quad A: \rightarrow\uparrow\rightarrow\wedge \quad B: \rightarrow\uparrow\leftarrow\wedge \quad C: \rightarrow\downarrow\leftarrow\wedge \quad D: \rightarrow\downarrow\rightarrow\wedge \]

A property of the 3-arrow notation is that it defines the eight oriented forms of Reidemeister moves of type 3: a move \( x \) is the passage from \( x^\uparrow \) to \( x^\downarrow \) (or vice-versa), where \( x \in \{ a,b,c,d,A,B,C,D \} \). The three arrows are reversed in the move. Another property is that we get the mirror of a configuration interchanging lower and upper case of the same symbol in \( \{ a,c,A,C \} \). Similarly for the symbols in \( \{ b,d,B,D \} \) but for these in addiction we must also interchange \( x^\uparrow \) and \( x^\downarrow \), see Fig. 1. A consistent digon is one whose 2 crossings have distinct signs. We say that a strand that crosses the digon is consistent if the crossings are up-up or down-down. A consistent digon crossed by a consistent strand is named a \( \Theta \)-configuration.

(2.1) Proposition. There are sixteen \( \Theta \)-configurations.

Proof. By rotating we may suppose that the horizontal strand goes from east to west in each \( \Theta \)-configuration. The horizontal strand can be up or down. The upper strand of the digon can be at the left or at the right. Finally, the 4 directions of the two strands of the digon may occur independently. This yields \( 2 \times 2 \times 4 = 16 \) \( \Theta \)-configurations.

Suppose the two triangular regions of a \( \Theta \)-configuration be labeled by \( x^\uparrow \) (the upper triangle) and \( y^\downarrow \) (the lower triangle). Then we say that \( (x^\uparrow, y^\downarrow) \) are \( \Theta \)-related. Fig. 1 shows that if \( (x^\uparrow, y^\downarrow) \) are \( \Theta \)-related, then \( (y^\downarrow, x^\uparrow) \) are \( \Theta \)-related.
For $x, y \in \{a, b, s, d, A, B, C, D\}$, we say that $y \Rightarrow_3 x$ if $x^\uparrow$ is obtained from $x^\downarrow$ by moves of type 2 and moves $y^\downarrow \rightarrow y^\uparrow$ and their inverses.

(2.2) Lemma. If $(x^\uparrow, y^\downarrow)$ are $\Theta$-related, then $y \Rightarrow_3 x$.

Proof. The $x^\uparrow \rightarrow x^\downarrow$ move can be factored by a complexifying type 2 move, the move $y^\downarrow \rightarrow y^\uparrow$ and a simplifying move of type 2, see Fig. 2. This establishes the result. □

(2.3) Corollary. If $(x^\uparrow, y^\downarrow)$ are $\Theta$-related, then $y \Leftrightarrow_3 x$.

Proof. By Lemma 2.2, $y \Rightarrow_3 x$. If $(x^\uparrow, y^\downarrow)$ are $\Theta$-related, then $(y^\downarrow, x^\uparrow)$ are $\Theta$-related by Fig. 1. As $(y^\downarrow, x^\uparrow)$ are $\Theta$-related, by Lemma 2.2, $x \Rightarrow_3 y$. In this way, $y \Leftrightarrow_3 x$. □

At this point, in face of the above corollary and of Fig. 1 we have $a \Leftrightarrow b \Leftrightarrow c \Leftrightarrow d$ and $A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D$.

(2.4) Theorem. The set of 4 oriented Reidemeister moves of type 2 together with any one of the 8 oriented Reidemeister moves of type 3 implies the other 7.

Proof. In Fig. 3 we prove that $A \Rightarrow c$. Taking the mirror of the moves involved we get that $a \Rightarrow c$. Thus, the 8 oriented Reidemeister moves of type 3 are equivalent given that moves of type 2 are available. □

References

[1] L.H. Kauffman and S. Lins. Temperley-Lieb Recoupling Theory and Invariants of 3-manifolds. *Annals of Mathematical Studies, Princeton University Press*, 134:1–296, 1994.

[2] M. Polyak. Minimal generating sets of Reidemeister moves. *Quantum Topology*, 1:399–411, 2010.