Statistical mechanics of self-gravitating systems in general relativity: 
I. The quantum Fermi gas

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We develop a general formalism to determine the statistical equilibrium states of self-gravitating systems in general relativity and complete previous works on the subject. Our results are valid for an arbitrary form of entropy but, for illustration, we explicitly consider the Fermi-Dirac entropy for fermions. The maximization of entropy at fixed mass-energy and particle number determines the distribution function of the system and its equation of state. It also implies the Tolman-Oppenheimer-Volkoff equations of hydrostatic equilibrium and the Tolman-Klein relations. Our paper provides all the necessary equations that are needed to construct the caloric curves of self-gravitating fermions in general relativity as done in recent works. We consider the nonrelativistic limit $c \rightarrow +\infty$ and recover the equations obtained within the framework of Newtonian gravity. We also discuss the inequivalence of statistical ensembles as well as the relation between the dynamical and thermodynamical stability of self-gravitating systems in Newtonian gravity and general relativity.

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I. INTRODUCTION

Self-gravitating fermions play an important role in different areas of astrophysics. They appeared in the context of white dwarfs, neutron stars and dark matter halos where the fermions are electrons, neutrons and massive neutrinos respectively. We start by a brief history of the subject giving an exhaustive list of references.¹

Soon after the discovery of the quantum statistics by Fermi [3, 4] and Dirac [5] in 1926, Fowler [6] used this “new thermodynamics” to solve the puzzle of the extreme high density of white dwarfs, which could not be explained by classical physics [7]. He understood that white dwarfs owe their stability to the quantum pressure of the degenerate electron gas resulting from the Pauli exclusion principle [8]. He considered a completely degenerate electron gas at $T = 0$ based on the fact that the temperature in white dwarfs is much smaller than the Fermi temperature ($T \ll T_F$). He also used Newtonian gravity which is a very good approximation to describe white dwarfs in general. The first models [9–11] of white dwarfs were based on the nonrelativistic equation of state of a Fermi gas and provided the corresponding mass-radius relation. Stoner [9] developed an analytical approach based on a uniform density approximation for the star. Chandrasekhar [11] derived the exact mass-radius relation of nonrelativistic white dwarfs by applying the theory of polytropes of index $n = 3/2$ [12]. It was then realized that special relativity must be taken into account at high densities. When the relativistic equation of state is employed it was found that white dwarfs can exist only below a maximum mass $M_{\text{max}} = 1.42 M_\odot$ [13, 24], now known as the Chandrasekhar limiting mass. Frenkel [13] was the first to mention that relativistic effects become important when the mass of white dwarfs becomes larger than the solar mass but he did not envision the existence of an upper mass limit. The maximum mass of white dwarfs was successively derived by Anderson [14], Stoner [15], Chandrasekhar [16] and Landau [19] using different methods. Anderson [14] first obtained an estimate of the maximum mass but his relativistic treatment of the problem was erroneous. Stoner [15] corrected the mistakes of Anderson and derived the complete mass-radius relation of white dwarfs and their maximum mass. Anderson [14] and Stoner [15] both used an analytical approach based on the uniform density approximation previously introduced by Stoner [9] in the nonrelativistic limit. Chandrasekhar [16] and Landau [19] obtained the exact value of the maximum mass by considering an ultrarelativistic electron gas and applying the theory of polytropes of index $n = 3$ [12]. Finally, Chandrasekhar [25] obtained the complete mass-radius relation of white dwarfs and the maximum mass by numerically solving the equation of hydrostatic equilibrium with the relativistic equation of state. At the maximum mass, the radius of the star vanishes. Close to the maximum mass, general relativity must be taken into account as first considered by Kaplan [26] and Chandrasekhar and Tooper [27]. In that case, the radius of the star at $M_{\text{max}}$ is finite, being equal to $R_* = 1.03 \times 10^3$ km (246 times the corresponding Schwarzschild radius), instead of vanishing as in the Newtonian treatment.

Contrary to the case of white dwarfs, neutron stars were predicted theoretically before being observed. The neutron

¹ A detailed historic of the statistical mechanics of self-gravitating systems (classical and quantum) in Newtonian gravity and general relativity is given in Refs. [1, 2].
was predicted by Rutherford \textsuperscript{28} as early as 1920 and finally discovered by Chadwick in 1932 \textsuperscript{29}. The first explicit prediction of neutron stars with extremely high density and very small radius was made by Baade and Zwicky in December 1933 \textsuperscript{30,32}. Remarkably, they anticipated that neutron stars could result from supernova explosion. The first calculation of neutron star model was performed by Oppenheimer and Volkoff \textsuperscript{33} who assumed matter to be composed of an ideal gas of free neutrons at high density. They worked at zero temperature at which the neutrons are completely degenerate and employed the relativistic equation of state of an ideal fermion gas previously derived in the case of white dwarfs. They used general relativity because of the high mass and density of neutron stars. They found that equilibrium configurations exist only below a maximum mass \( M_{\text{max}} = 0.710 M_\odot \), now known as the Oppenheimer-Volkoff limiting mass.\textsuperscript{2} Their radius at \( M_{\text{max}} \) is \( R_\star = 9.16 \text{ km} \) (4.37 times the corresponding Schwarzschild radius). Detailed studies of neutron stars with more realistic equations of state taking into account the repulsive effect of nuclear forces were made between 1958 and 1967 \textsuperscript{39,60} (see a review in Ref. \textsuperscript{67}), but, apart from these works, neutron stars were not much studied because it was estimated that their residual thermal radiation would be too faint to be observed. The situation changed when pulsars were discovered by Hewish et al. \textsuperscript{68} in 1968. The same year, Gold \textsuperscript{69} proposed that pulsars were rotating neutron stars, and this is generally accepted today. Too massive stars cannot reach a quiescent equilibrium state and rather collapse into a black hole which is a Schwarzschild \textsuperscript{70,71} singularity of spacetime in the nonrotating case or a Kerr \textsuperscript{72} singularity of spacetime in the rotating case. This continued gravitational contraction was originally described by Oppenheimer and Snyder \textsuperscript{73} in 1939 but the concept of black holes as real physical objects took a long time to be accepted. It is now believed that supermassive black holes, surrounded by an accretion disk, lie in the core of galaxies and power active galactic nuclei and quasars.\textsuperscript{3}

The fermionic models of white dwarfs and neutron stars were exported to the case of dark matter halos, assuming that dark matter is made of massive neutrinos as originally proposed by Markov \textsuperscript{81} and Cowsik and McClelland \textsuperscript{81,82}. A lower bound on the fermion mass was obtained by Tremaine and Gunn \textsuperscript{83} using constraints arising from the Vlasov equation. The first models described dark matter halos at \( T = 0 \) using the equation of state of a completely degenerate fermion gas either in the nonrelativistic limit \textsuperscript{82,84,91} or in general relativity \textsuperscript{80,92,100}. Subsequent models considered dark matter halos at finite temperature showing that they have a “core-halo” structure consisting in a dense core (fermion ball) surrounded by a dilute isothermal atmosphere leading to flat rotation curves. Most models were based on the ordinary Fermi-Dirac distribution in Newtonian gravity \textsuperscript{101–118} or general relativity \textsuperscript{2,119–130}. Other models were based on the more realistic fermionic King model (describing tidally truncated fermionic dark matter halos) in Newtonian gravity \textsuperscript{141,143} or general relativity \textsuperscript{146}. Some authors \textsuperscript{102,123,125,126,129,146} have proposed that a fermion ball could mimic a supermassive black hole that is purported to exist at the center of a galaxy but some difficulties with this scenario were pointed out in \textsuperscript{147,149}. The status of the fermion ball scenario is still not clearly settled. Recently, it has been proposed that the fermion ball may represent a large bulge instead of mimicking a black hole \textsuperscript{150}. The self-gravitating Fermi gas was also studied in relation to the violent relaxation of collisionless self-gravitating systems described by the Lynden-Bell \textsuperscript{151} distribution which is formally similar to the Fermi-Dirac distribution \textsuperscript{102–104,143,152–155}.

The study of phase transitions in the self-gravitating Fermi gas is an important problem in itself. To make the mathematical study well-posed, we need to confine the system in a finite region of space in order to have a finite mass. Indeed, the density of an unbounded isothermal self-gravitating system decreases as \( r^{-2} \) at large distances leading to an infinite mass. Phase transitions in the self-gravitating Fermi gas were first analyzed for box-confined models both in the nonrelativistic limit \textsuperscript{102,110,135} (see a review in \textsuperscript{110}) and in general relativity \textsuperscript{2,123,130}. The study of phase transitions in the more realistic case of tidally truncated self-gravitating fermions described by the fermionic King model \textsuperscript{141,143} has been performed in Ref. \textsuperscript{147} in the nonrelativistic limit. These studies describe the transition between a gaseous phase and a condensed phase corresponding to a compact object (white dwarfs and neutron stars).
dwarf, neutron star, fermion ball), or the collapse of the system towards a black hole when its mass is too large. These phase transitions may be related to the onset of the red-giant structure (leading to white dwarfs) in a late phase of stellar evolution and to the supernova phenomenon (leading to neutron stars) \[156\,158\]. General phase diagrams have been obtained in Refs. \[2\,110\,142\].

The basic equations describing a Newtonian self-gravitating gas of fermions at arbitrary temperature consist in the condition of hydrostatic equilibrium combined with the ideal equation of state of the Fermi gas \[132\]. These equations can be obtained all at once from the maximum entropy principle of statistical mechanics. It determines the most probable state of the system at statistical equilibrium. This variational principle was introduced in astrophysics by Ogorodnikov \[174\,175\], Antonov \[176\] and Lynden-Bell and Wood \[177\] in the context of stellar systems and was further studied by several authors \[178\,209\] (see the reviews \[110\,207\,208\]). The statistical equilibrium state of a classical self-gravitating system is obtained by maximizing the Boltzmann entropy at fixed energy and particle number in the microcanonical ensemble or by minimizing the Boltzmann free energy at fixed particle number in the canonical ensemble. Statistical ensembles are inequivalent for self-gravitating systems \[110\,204\,208\] and for other systems with long-range interactions \[209\,210\]. The maximum entropy principle was then extended to the case of self-gravitating fermions by replacing the Boltzmann entropy by the Fermi-Dirac entropy \[102\,110\,134\,145\,147\,151\,153\,155\]. The maximum entropy principle has also been applied to generalized forms of entropy in Refs. \[178\,187\,211\–221\,223\,224\]. For example, the Tsallis entropy \[225\] (which is related to one of the functionals considered by Ipser \[178\] – see \[217\]) leads to power-law distribution functions \[178\,203\,214\,217\,218\,220\,230\]. They correspond to the stellar polytropes introduced by Eddington \[231\] in 1916 as particular steady states of the Vlasov-Poisson equations.

The basic equations describing a spherically symmetric self-gravitating gas of fermions at arbitrary temperature in general relativity consist in the Tolman-Oppenheimer-Volkov \[32\,232\] equations of hydrostatic equilibrium combined with the ideal equation of state of the relativistic Fermi gas \[132\,136\] and the Tolman-Klein \[232\,240\] relations stating that in general relativity the temperature and the chemical potential are not uniform at statistical equilibrium. These equations can be obtained all at once from the maximum entropy principle of statistical mechanics. This variational principle was introduced in relativistic astrophysics by Tolman \[232\] (see Appendix \[11\]) and further developed by several authors \[211\,263\]. The maximum entropy principle is valid for a general form of entropy. It was specifically applied to the self-gravitating black-body radiation in \[232\,234\,251\] to self-gravitating fermions described by the Fermi-Dirac entropy in \[252\] and to classical particles described by the Boltzmann entropy in \[243\,244\,246\,248\,251\]. The statistical equilibrium state of a general relativistic self-gravitating system is obtained by maximizing the entropy at fixed mass-energy and particle number in the microcanonical ensemble or by minimizing the free energy at fixed particle number in the canonical ensemble. Again, the statistical ensembles are inequivalent in general relativity.

In this paper, we synthesize previous works on the subject and develop a general formalism to determine the

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5 These equations are equivalent to the Fermi-Dirac-Poisson equations. They constitute the so-called finite temperature Thomas-Fermi (TF) model. At \( T = 0 \), the Fermi-Dirac distribution reduces to a step function and we recover the TF model. We recall that the original TF model \[159\,162\] describes the distribution of electrons in an atom which results from the balance between the quantum pressure (Pauli’s exclusion principle), the electrostatic repulsion of the electrons, and the electrostatic attraction of the nucleus. For self-gravitating systems, the TF model describes the distribution of fermions in a star or in a dark matter halo which results from the balance between the quantum pressure (Pauli’s exclusion principle) and the gravitational attraction of the fermions. The rigorous mathematical justification of the thermodynamic limit for self-gravitating fermions in a box leading to the finite temperature TF model is given in Refs. \[163\,174\].

6 A functional of the form \( S = - \int C(f) \, dr \, dv \), where \( C(f) \) is any convex function of the coarse-grained distribution function \( f(r,v,t) \), was introduced by Antonov \[211\] for collisionless stellar systems, and called “quasi-entropy” (see Ref. \[211\]). The same functional was reintroduced independently by Tremaine et al. \[214\] in relation to the theory of violent relaxation \[159\] and called “H-function”. A functional of the form \( S = - \int f \, C(f) \, dv \), without the bar on \( f \), was introduced by Ipser \[178\,187\] in his study on the dynamical stability of collisionless stellar systems with respect to the Vlasov-Poisson equations. An effective thermodynamical formalism involving generalized entropic functionals of the form \( S = - \int f \, C(f) \, dv \) was developed by Chavanis \[214\,221\,223\,224\] who gave various interpretations of these functionals.

7 There is debate \[223\,224\] on the name that should be given to these equations: OV or TOV? As far as we can judge, Eq. \[102\] was first written by Tolman \[232\] while Eq. \[103\] was first written by Oppenheimer and Volkoff \[33\]. Therefore, Eq. \[102\] should be called the Tolman equation and Eq. \[103\] should be called the OV equation. If we consider that Eq. \[103\] is a rather direct consequence of Eq. \[102\] then it may be called the TOV equation as well \[224\]. However, Ref. \[225\] stresses some fundamental differences between Eqs. \[102\,103\]. To complement this discussion, we note that Chandrasekhar \[51\,236\] rederived Eq. \[103\] without referring to Oppenheimer and Volkoff \[33\], maybe considering that this equation results almost immediately from the Einstein field equations \[101\,110\] with the metric \[97\]. In his review on the first thirty years of general relativity \[235\], he writes: “Equations (60) and (61) are often referred to as the Oppenheimer-Volkoff equations though they are contained in Schwarzschild’s paper in very much these forms.”

8 The equation of state of a completely degenerate \(( T = 0 )\) Fermi gas at arbitrary densities (i.e. for any degree of relativistic motion) was first derived by Frenkel \[13\] in a not well-known paper. It was rederived independently by Stoner \[13\,20\] and Chandrasekhar \[22\]. The equation of state of a gas of fermions at arbitrary temperature was first derived by Juttner \[13\] extending his earlier work on the relativistic theory of an ideal classical gas \[238\,239\]. These different results are exposed in the classical monograph of Chandrasekhar \[132\] on stellar structure.
statistical equilibrium states of self-gravitating systems in general relativity. Our results are valid for an arbitrary form of entropy but, for illustration, we explicitly consider the case of fermions described by the Fermi-Dirac entropy. Our paper provides all the necessary equations that are needed to construct the caloric curves of self-gravitating fermions in general relativity as done in recent works [2, 123, 130]. The present paper is organized as follows. In Sec. II we develop the statistical mechanics of nonrelativistic self-gravitating fermions. In Sec. III we develop the statistical mechanics of relativistic self-gravitating fermions within the framework of general relativity. In Sec. IV we consider the nonrelativistic limit $c \to +\infty$ and recover from the general relativistic formalism the equations obtained within the framework of Newtonian gravity. Throughout the paper, we discuss the relation between dynamical and thermodynamical stability of self-gravitating systems in Newtonian gravity and general relativity.

II. STATISTICAL MECHANICS OF NONRELATIVISTIC SELF-GRAVITATING FERMIONS

In this section, we consider the statistical mechanics of nonrelativistic self-gravitating fermions. We use a presentation that can be extended in general relativity (see Sec. III). In particular, we assume since the start that the system is spherically symmetric. It can be shown that the maximum entropy state of a nonrotating self-gravitating system is necessarily spherically symmetric so this assumption is not restrictive.

A. Hydrostatic equilibrium of gaseous spheres in Newtonian gravity

1. Newton’s law

The gravitational potential $\Phi(r)$ is determined from the mass density $\rho(r)$ by the Poisson equation

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho. \tag{1}$$

The mass contained within a sphere of radius $r$ is

$$M(r) = \int_0^r \rho(r') 4\pi r'^2 dr' \implies \frac{dM}{dr} = 4\pi \rho r^2. \tag{2}$$

Multiplying the Poisson equation by $r^2$ and integrating between 0 and $r$, we obtain Newton’s law

$$\frac{d\Phi}{dr} = \frac{GM(r)}{r^2}. \tag{3}$$

We note that Newton’s law is valid for any spherically symmetric distribution of matter, steady or unsteady.

Let us assume that the system occupies a region of radius $R$ and contains a mass $M(R) = M$. In the empty space outside the system, Newton’s law becomes

$$\frac{d\Phi}{dr} = \frac{GM}{r^2} \implies \Phi(r) = -\frac{GM}{r} \quad (r \geq R). \tag{4}$$

This leads to the boundary condition

$$\Phi(R) = -\frac{GM}{R}. \tag{5}$$

2. Condition of hydrostatic equilibrium

We now consider a self-gravitating gas at equilibrium. The condition of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho \frac{d\Phi}{dr} \tag{6}$$

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9 The case of classical particles described by the Boltzmann entropy is specifically considered in our companion paper [264] (Paper II).
expresses the balance between the pressure gradient and the gravitational force. Using Newton’s law, we can rewrite the condition of hydrostatic equilibrium as
\[ \frac{dP}{dr} = -\frac{GM(r)}{r^2}. \] (7)

Multiplying Eq. (7) by \( r^2/\rho \), taking the derivative of this relation with respect to \( r \), and using Eq. (2), we obtain the fundamental differential equation of hydrostatic equilibrium:
\[ \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho. \] (8)

This equation can also be obtained by dividing Eq. (6) by \( \rho \), forming the Laplacian, and using the Poisson equation.

3. Barotropic equation of state

If the gas is described by a barotropic equation of state, \( P = P(\rho) \), Eq. (8) determines a differential equation for \( \rho(r) \) of the form
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \rho P'(\rho) \frac{d\rho}{dr} \right) = -4\pi G \rho. \] (9)

On the other hand, integrating Eq. (6), we find that
\[ d\Phi = -\frac{dP}{\rho} \Rightarrow \Phi(r) = -\int_{\rho(0)}^{\rho(r)} \frac{P'(\rho)}{\rho} d\rho, \] (10)

implying that the gravitational potential is a function \( \Phi = \Phi(\rho) \) of the density or, inversely, that the density is a function \( \rho = \rho(\Phi) \) of the gravitational potential. Substituting this relation into the Poisson equation we obtain a differential equation for \( \Phi(r) \) of the form
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \rho(\Phi). \] (11)

We can also introduce the enthalpy \( h \) through the relation
\[ dh = \frac{dP}{\rho} \Rightarrow h(r) = \int_{\rho(0)}^{\rho(r)} \frac{P'(\rho)}{\rho} d\rho. \] (12)

For a barotropic gas, the enthalpy is a function \( h = h(\rho) \) of the density or, inversely, the density is a function \( \rho = \rho(h) \) of the enthalpy. From Eq. (10), we obtain
\[ h(r) + \Phi(r) = \text{cst}. \] (13)

Taking the Laplacian of this relation and using the Poisson equation we obtain a differential equation for \( h(r) \) of the form
\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dh}{dr} \right) = -4\pi G \rho(h). \] (14)

Eqs. (9), (11) and (14) are clearly equivalent. They are also equivalent to the two first order differential equations for \( M(r) \) and \( \Phi(r) \):
\[ \frac{dM}{dr} = 4\pi \rho r^2, \quad \frac{d\Phi}{dr} = \frac{GM(r)}{r^2}, \] (15)
with \( \rho = \rho(\Phi) \). These equations can be combined into the single equation. Using Eq. (13) they can also be expressed in terms of \( h \). Finally, using Eqs. (2) and (7), we can easily obtain the following differential equation
\[ P'(r) \left( \frac{M'}{4\pi r^2} \right) (r^2 M'' - 2r M') + GM(r) M' = 0 \] (16)
for the mass profile \( M(r) \). It has to be supplemented by the boundary conditions \( M(0) = 0 \) and \( M(R) = M \).
B. Local variables

We consider a system of self-gravitating fermions described by the distribution function \( f(r, p) \) such that \( f(r, p) \, dr \, dp \) gives the number density of fermions at position \( r \) with impulse \( p \). We introduce the particle number density

\[
n = \int f \, dp
\]

and the kinetic energy density

\[
\epsilon_{\text{kin}} = \int f E_{\text{kin}}(p) \, dp,
\]

where

\[
E_{\text{kin}}(p) = \frac{p^2}{2m}
\]

is the kinetic energy of a particle. The local pressure is given by

\[
P = \frac{1}{3} \int f p \frac{dE_{\text{kin}}}{dp} \, dp = \frac{1}{3} \int f \frac{p^2}{m} \, dp.
\]

We have the relation

\[
P = \frac{2}{3} \epsilon_{\text{kin}}
\]

between the pressure and the kinetic energy density. It is valid for an arbitrary distribution function (see Appendix A). Finally, we introduce the Fermi-Dirac entropy density

\[
s = -k_B \frac{g}{h^3} \int \left\{ \frac{f}{f_{\text{max}}} \ln \frac{f}{f_{\text{max}}} + \left( 1 - \frac{f}{f_{\text{max}}} \right) \ln \left( 1 - \frac{f}{f_{\text{max}}} \right) \right\} \, dp,
\]

where \( f_{\text{max}} = g/h^3 \) is the maximum possible value of the distribution function fixed by the Pauli exclusion principle and \( g \) is the spin multiplicity of quantum states (\( g = 2 \) for particles of spin 1/2). The Fermi-Dirac entropy can be obtained from a combinatorial analysis. It is equal to the logarithm of the number of microstates (complexions) – characterized by the specification of the position and the impulse of all the fermions \( \{r_i, p_i\} \) – corresponding to a given macrostate – characterized by the (smooth) distribution function \( f(r, p) \) giving the density of fermions in a macrocell \( (r, r + dr; p, p + dp) \), irrespectively of their precise position in the cell. The microstates must respect the Pauli exclusion principle, i.e., there cannot be more than \( g \) particles in the same microcell of volume \( h^3 \). A counting analysis taking into account the Pauli exclusion principle leads to the expression (22) of the entropy (see, e.g., Refs. \[109, 110\] for details).

Remark: In this paper, to be specific, we consider a system of fermions associated with the Fermi-Dirac entropy (22). However, as shown in Appendices A-F, our approach is more general. It is actually valid for any kind of particles described by a (generalized) entropy of the form (C1).

C. Global variables

The particle number is

\[
N = \int n \, 4\pi r^2 \, dr.
\]

The mass is

\[
M = Nm = \int \rho \, 4\pi r^2 \, dr,
\]
where $\rho = nm$ is the mass density. The energy is $E = E_{\text{kin}} + W$, where\(^{10}\)

$$E_{\text{kin}} = \int \epsilon_{\text{kin}} 4\pi r^2 \, dr = \frac{3}{2} \int P 4\pi r^2 \, dr$$

(25)

is the kinetic energy and

$$W = \frac{1}{2} \int \rho \Phi 4\pi r^2 \, dr = - \int \rho \frac{GM(r)}{r} 4\pi r^2 \, dr$$

(26)

is the gravitational potential energy (the second expression – only valid for spherical systems – is derived in Appendix B). The entropy is

$$S = \int s 4\pi r^2 \, dr.$$ 

(27)

A statistical equilibrium state exists only if the system is confined within a box of radius $R$ otherwise it would evaporate\(^{109, 110}\). In the microcanonical ensemble, the particle number $N$ and the energy $E$ are conserved. The statistical equilibrium state of the system is obtained by maximizing the Fermi-Dirac entropy at fixed energy $E$ and particle number $N$:

$$\max \{ S | E, N \text{ fixed} \}.$$ 

(28)

This determines the “most probable” state of an isolated system. To solve this maximization problem, we proceed in two steps.\(^{11}\) We first maximize the entropy density $s(r)$ at fixed kinetic energy density $\epsilon_{\text{kin}}(r)$ and particle number density $n(r)$ with respect to variations on $f(r, p)$. This gives us the Fermi-Dirac distribution\(^{22}\) which corresponds to the condition of local thermodynamic equilibrium. Then, we substitute this distribution in the entropy density\(^{22}\) to express it as a function of $\epsilon_{\text{kin}}(r)$ and $n(r)$. Finally, we maximize the entropy $S$ at fixed energy $E$ and particle number $N$ with respect to variations on $\epsilon_{\text{kin}}(r)$ and $n(r)$. This gives us the mean field Fermi-Dirac distribution\(^{65}\) which is the statistical equilibrium state of the system. In Appendix C2\(^{22}\) we maximize the entropy $S$ at fixed energy $E$ and particle number $N$ with respect to variations on $f(r, p)$ (one-step process) and directly obtain the mean field Fermi-Dirac distribution\(^{65}\).

D. Maximization of the entropy density at fixed kinetic energy density and particle number density

1. Local thermodynamic equilibrium

We first maximize the entropy density\(^{22}\) at fixed kinetic energy density\(^{18}\) and particle number density\(^{17}\). We write the variational problem for the first variations (extremization) under the form

$$\frac{\delta s}{k_B} - \beta(r) \delta \epsilon_{\text{kin}} + \alpha(r) \delta n = 0,$$

(29)

where $\beta(r)$ and $\alpha(r)$ are local (position dependent) Lagrange multipliers. This leads to the Fermi-Dirac distribution function

$$f(r, p) = \frac{\theta}{h^3} \frac{1}{1 + e^{\beta(r)p^2/2m - \alpha(r)}},$$

(30)

where $\alpha(r)$ and $\beta(r)$ are determined in terms of $n(r)$ and $\epsilon_{\text{kin}}(r)$ by substituting Eq.\(^{30}\) into Eqs.\(^{17}\) and\(^{18}\) (see Eqs.\(^{34}\) and\(^{35}\) below). By computing the second variations of $s$, we can easily show (see Appendix C1\(^{22}\)) that Eq.\(^{30}\) is the global maximum of $s(r)$ at fixed $\epsilon_{\text{kin}}(r)$ and $n(r)$. Therefore, Eq.\(^{30}\) corresponds to the condition of

\(^{10}\) We use the same symbol for the kinetic energy of one particle [see Eq.\(^{19}\)] and for the total kinetic energy [see Eq.\(^{25}\)]. In general, there is no ambiguity.

\(^{11}\) We use this “two-steps” procedure because (i) it can be easily extended to general relativity and (ii) it is useful for studying the sign of the second variations of entropy determining the thermodynamical stability of the system (see Appendix C).
local thermodynamic equilibrium. Introducing the local temperature \(T(r)\) and the local chemical potential \(\mu(r)\) by the relations

\[
\beta(r) = \frac{1}{k_B T(r)} \quad \text{and} \quad \alpha(r) = \frac{\mu(r)}{k_B T(r)},
\]

(31)

the Fermi-Dirac distribution \((30)\) can be rewritten as

\[
f(r, p) = \frac{g}{h^3} \frac{1}{1 + e^{p^2/2m - \mu(r)/k_B T(r)}}.
\]

(32)

On the other hand, the variational principle \((29)\) reduces to

\[
ds = \frac{d\epsilon_{\text{kin}}}{T} - \frac{\mu}{T} d n,
\]

(33)

which corresponds to the first law of thermodynamics. This law is valid for an arbitrary form of entropy (see Appendix C).

2. Local variables

Substituting the Fermi-Dirac distribution \((32)\) into Eqs. \((17)\), \((18)\) and \((20)\), we get

\[
n(r) = g \frac{1}{h^3} \int \frac{1}{1 + e^{p^2/2m - \mu(r)/k_B T(r)}} dp,
\]

(34)

\[
\epsilon_{\text{kin}}(r) = g \frac{1}{h^3} \int \frac{p^2/2m}{1 + e^{p^2/2m - \mu(r)/k_B T(r)}} dp,
\]

(35)

\[
P(r) = g \frac{p^2/m}{3h^3} \int \frac{1 + e^{p^2/2m - \mu(r)/k_B T(r)}}{1 + e^{p^2/2m - \mu(r)/k_B T(r)}} dp = g \frac{3}{h^3} k_B T(r) \int \ln \left(1 + e^{-p^2/2m - \mu(r)/k_B T(r)}\right) dp,
\]

(36)

where the second equality in Eq. \((36)\) has been obtained from an integration by parts. Eqs. \((34)\) and \((35)\) determine \(T(r)\) and \(\mu(r)\) as a function of \(n(r)\) and \(\epsilon_{\text{kin}}(r)\). They also determine the equation of state \(P = P[n(r), T(r)]\) in implicit form. On the other hand, substituting the Fermi-Dirac distribution function \((32)\) into Eq. \((22)\), and using Eqs. \((34)-(36)\), we obtain after some calculations the relation

\[
s(r) = \frac{\epsilon_{\text{kin}}(r) + P(r) - \mu(r)n(r)}{T(r)}.
\]

(37)

Actually, this relation, which is called the integrated Gibbs-Duhem relation, is a general relation valid for an arbitrary form of entropy (see Appendix E). Using Eq. \((21)\), it reduces to the form

\[
s(r) = \frac{\frac{5}{3} \epsilon_{\text{kin}}(r) - \mu(r)n(r)}{T(r)}.
\]

(38)

Finally, combining the first law of thermodynamics \((33)\) with the integrated Gibbs-Duhem relation \((37)\) we obtain the identity

\[
d \left(\frac{P}{T}\right) = n d \left(\frac{\mu}{T}\right) - \epsilon_{\text{kin}} d \left(\frac{1}{T}\right).
\]

(39)

We also have the identities

\[
d \left(\frac{\epsilon_{\text{kin}}}{n}\right) = -P d \left(\frac{1}{n}\right) + T d \left(\frac{s}{n}\right) \quad \text{and} \quad sdT - dP + nd\mu = 0
\]

(40)

which correspond to the standard form of the first law of thermodynamics and the local Gibbs-Duhem relation. These results are valid for an arbitrary form of entropy (see Appendix E).
E. Maximization of the entropy at fixed energy and particle number

1. Variational principle

Using the integrated Gibbs-Duhem relation (37), the entropy can be written as

$$S = \int_{0}^{R} \frac{\epsilon_{\text{kin}}(r) + P(r) - \mu(r)n(r)}{T(r)} 4\pi r^2 \, dr.$$  \hfill (41)

The functionals $S$, $E$ and $N$ depend on $\epsilon_{\text{kin}}(r)$ and $n(r)$. We now maximize $S$ at fixed $E$ and $N$. We write the variational problem for the first variations (extremization) under the form

$$\frac{\delta S}{k_B} - \beta_0 \delta E + \alpha_0 \delta N = 0,$$ \hfill (42)

where $\beta_0$ and $\alpha_0$ are global (uniform) Lagrange multipliers. Taking the first variations of $E$ and $N$ from Eqs. (23)-(26), and taking the first variations of $S$ from Eq. (27) by using the first law of thermodynamics (33), we obtain

$$\int \frac{\delta \epsilon_{\text{kin}}}{k_B T} dV - \int \frac{\mu}{k_B T} \delta n dV - \beta_0 \int \delta \epsilon_{\text{kin}} dV - \beta_0 \int m \Phi \delta n dV + \alpha_0 \int \delta n dV = 0,$$ \hfill (43)

where we have introduced the abbreviation $dV = 4\pi r^2 \, dr$ for the volume element. In Eq. (43) the variations on $\delta \epsilon_{\text{kin}}$ and $\delta n$ must vanish individually.

2. Variations on $\delta \epsilon_{\text{kin}}$

The vanishing of Eq. (43) with respect to variations on $\delta \epsilon_{\text{kin}}$ gives

$$\frac{1}{k_B T(r)} = \beta_0,$$ \hfill (44)

implying that the temperature is uniform at statistical equilibrium. In the following, we will denote the temperature by $T$ and we will replace $\beta_0$ by $\beta$. Therefore, the first relation of Eq. 31 becomes

$$\beta = \frac{1}{k_B T}.$$ \hfill (45)

Since the temperature is uniform, the Fermi-Dirac distribution (32) and the local variables (34)-(36) can be rewritten as

$$f(r, p) = \frac{g}{\hbar^3} \frac{1}{1 + e^{(p^2/2m - \mu(r))/k_B T}},$$ \hfill (46)

$$n(r) = \frac{g}{\hbar^3} \int \frac{1}{1 + e^{(p^2/2m - \mu(r))/k_B T}} \, dp,$$ \hfill (47)

$$\epsilon_{\text{kin}}(r) = \frac{g}{\hbar^3} \int \frac{p^2/2m}{1 + e^{(p^2/2m - \mu(r))/k_B T}} \, dp,$$ \hfill (48)

$$P(r) = \frac{g}{3\hbar^3} \int \frac{p^2/m}{1 + e^{(p^2/2m - \mu(r))/k_B T}} \, dp = \frac{g}{\hbar^3} k_B T \int \ln \left(1 + e^{-(p^2/2m - \mu(r))/k_B T}\right) \, dp.$$ \hfill (49)

On the other hand, Eq. (39) reduces to

$$dP = n \, d\mu.$$ \hfill (50)
On the other hand, eliminating formally $\mu(r)$ between Eqs. (47) and (49), we see that the equation of state of the Fermi gas at statistical equilibrium is barotropic: $P(r) = P[n(r), T]$ (see Appendix F 3 for a more explicit expression). Therefore, according to Eq. (50) we have

$$\mu'(n, T) = \frac{P'(n', T)}{n'} \Rightarrow \mu(n, T) = \int^n P'(n', T) \frac{dn'}{n'},$$  \hspace{1cm} (51)

and

$$\frac{dP}{dr} = n \frac{d\mu}{dr}. \hspace{1cm} (52)$$

In Eq. (51), the derivative is with respect to the variable $n$.

### 3. Variations on $\delta n$

The vanishing of Eq. (43) with respect to variations on $\delta n$ gives

$$\frac{\mu(r)}{k_B T} = \alpha_0 - \beta m \Phi(r). \hspace{1cm} (53)$$

Using the second relation of Eq. (51), becoming

$$\alpha(r) = \frac{\mu(r)}{k_B T}, \hspace{1cm} (54)$$

we can rewrite Eq. (53) as

$$\alpha(r) = \alpha_0 - \beta m \Phi(r), \hspace{1cm} \text{or} \hspace{1cm} \mu(r) = \mu_0 - m \Phi(r) \hspace{1cm} (55)$$

with $\mu_0 = \alpha_0 k_B T$. The chemical potential $\mu(r)$ is not uniform at statistical equilibrium when a gravitational potential is present. However, the total chemical potential $\mu_{\text{tot}}(r) \equiv \mu(r) + m \Phi(r)$ is uniform at statistical equilibrium ($\mu_{\text{tot}}(r) = \mu_0$). This is the Gibbs law. Taking the derivative of Eq. (53) with respect to $r$ and using Eq. (3) we get

$$\frac{d\mu}{dr} = -m \frac{d\Phi}{dr} = -\frac{GM(r)m}{r^2}. \hspace{1cm} (56)$$

On the other hand, from Eqs. (5) and (55), we obtain

$$\mu(R) = \mu_0 + \frac{GMm}{R}. \hspace{1cm} (57)$$

### 4. Condition of hydrostatic equilibrium

Combining Eqs. (52) and (56), we obtain the condition of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho \frac{d\Phi}{dr} = -\frac{\rho(r)GM(r)}{r^2}. \hspace{1cm} (58)$$

Therefore, the condition of statistical equilibrium, obtained by extremizing the entropy at fixed energy and particle number, implies the condition of hydrostatic equilibrium. This condition was not assumed in the preceding calculations. It results from the thermodynamical variational principle (maximization of entropy at fixed energy and particle number). The intrinsic reason of this result will be given in Sec. II 1.

### 5. Entropy

Using Eqs. (45) and (55), the entropy density (38) can be rewritten as

$$s(r) = \frac{\frac{5}{2} \epsilon_{\text{kin}}(r) - \mu_0 n(r) + \rho(r)\Phi(r)}{T}. \hspace{1cm} (59)$$

Integrating Eq. (59) over the whole configuration, we find that the entropy is given at statistical equilibrium by

$$S = -\frac{\mu_0}{T} N + \frac{5E_{\text{kin}}}{3T} + \frac{2W}{T}. \hspace{1cm} (60)$$

We emphasize that the results derived in this section are valid for an arbitrary form of entropy.
F. Canonical ensemble: Minimization of the free energy at fixed particle number

In the previous sections, we worked in the microcanonical ensemble in which the particle number and the energy are fixed. We now consider the canonical ensemble where the system is in contact with a heat bath fixing the temperature $T$. In that case, the relevant thermodynamical potential is the free energy

$$ F = E - TS. \quad (61) $$

In the canonical ensemble, the statistical equilibrium state of the system is obtained by minimizing the Fermi-Dirac free energy $F$ at fixed particle number $N$:

$$ \min \{ F | N \text{ fixed} \}. \quad (62) $$

This determines the “most probable” state of a system in contact with a thermal bath.

Minimizing the free energy $F = E - TS$ at fixed $N$ is equivalent to maximizing the Massieu function $J = S/k_B - \beta E$ at fixed $N$ (the Massieu function is the Legendre transform of the entropy with respect to the energy). To solve this maximization problem we proceed in two steps. We first maximize the Massieu function $J$ at fixed particle number $N$, and since the particle number density $n(r)$ determines the kinetic energy $E_{\text{kin}}$, this is equivalent to maximizing the entropy $S$ at fixed $E_{\text{kin}}(r)$ and $n(r)$. This returns the results of Sec. II D. Using these results, we can express the Massieu function $J = S/k_B - \beta E$ in terms of $E_{\text{kin}}(r)$ and $n(r)$. We now maximize $J$ at fixed particle number $N$ under variations of $E_{\text{kin}}(r)$ and $n(r)$. The first variations (extremization) can be written as

$$ \delta \left( \frac{S}{k_B} - \beta E \right) + \alpha_0 \delta N = 0. \quad (63) $$

Since $\beta$ is a constant, this variational principle is equivalent to Eq. (62) so we get the same results as in Sec. II E (for the first variations). Finally, using Eqs. (60) and (61), we find that the free energy is given at statistical equilibrium by

$$ F = \mu_0 N - W - \frac{2}{3} E_{\text{kin}}. \quad (64) $$

G. Equations determining the statistical equilibrium state

In this section, we provide the equations that determine the statistical equilibrium state of a gas of self-gravitating fermions. These equations can be easily extended to a distribution function arising from a generalized form of entropy (see Appendix C).

1. Local variables in terms of $\Phi(r)$

Substituting Eqs. (44) and (53) into Eqs. (32) and (34)-(36), we obtain

$$ f(r, p) = \frac{g}{h^3} \frac{1}{1 + e^{-\alpha_0 \beta (p^2/2m + m\Phi(r))}}, \quad (65) $$

$$ n(r) = \frac{g}{h^3} \int \frac{1}{1 + e^{-\alpha_0 \beta (p^2/2m + m\Phi(r))}} \, dp, \quad (66) $$

$$ \epsilon_{\text{kin}}(r) = \frac{g}{h^3} \int \frac{p^2/2m}{1 + e^{-\alpha_0 \beta (p^2/2m + m\Phi(r))}} \, dp, \quad (67) $$

$$ P(r) = \frac{g}{3h^3} \int \frac{p^2/m}{1 + e^{-\alpha_0 \beta (p^2/2m + m\Phi(r))}} \, dp = \frac{g}{h^3} k_B T \int \ln \left[ 1 + e^{\alpha_0 \beta (p^2/2m + m\Phi(r))} \right] \, dp. \quad (68) $$

These equations determine the statistical equilibrium state of the self-gravitating Fermi gas. Taking the derivative of the pressure from Eq. (68) with respect to $r$, and using Eq. (66), we recover the condition of hydrostatic equilibrium (see Appendix D). We show in Appendix D that this result remains valid for an arbitrary form of entropy.
2. Equation for $\rho(r)$

The density of particles $\rho_\text{f}$ and the pressure $P$ are related to the gravitational potential by

$$
\rho(r) = \frac{4\pi g \sqrt{2} m^{5/2}}{h^3 \beta^{3/2}} I_{1/2} \left[ e^{-\alpha_0 + \beta m \Phi(r)} \right],
$$

(69)

$$
P(r) = \frac{8\pi g \sqrt{2} m^{3/2}}{3 h^3 \beta^{5/2}} I_{3/2} \left[ e^{-\alpha_0 + \beta m \Phi(r)} \right],
$$

(70)

where $I_n$ denotes the Fermi integral

$$
I_n(t) = \int_0^{+\infty} \frac{x^n}{1 + te^x} \, dx.
$$

(71)

We recall the identity

$$
I'_n(t) = -\frac{n}{t} I_{n-1}(t), \quad (n > 0),
$$

(72)

which can be established from Eq. (71) by an integration by parts. Eqs. (69) and (70) determine the equation of state $P(r) = P[\rho(r), T]$ of the nonrelativistic Fermi gas at finite temperature in parametric form with parameter $\alpha(r) = \alpha_0 - \beta m \Phi(r)$. Substituting this equation of state into the fundamental differential equation of hydrostatic equilibrium (8), we obtain a differential equation for the density profile of the form

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\rho}{dr} \right) = -4\pi G \rho,
$$

(73)

where $P(\rho)$ is the Fermi-Dirac equation of state.

3. Equation for $\Phi(r)$

The distribution function of a system of self-gravitating fermions at statistical equilibrium is given by Eq. (65). Substituting this distribution function into the Poisson equation (1), using Eq. (17), we obtain a differential equation for the gravitational potential of the form

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = 4\pi G \frac{2m}{h^3} \int \frac{1}{1 + e^{-\alpha_0 + \beta m \Phi(r)}} \, dp.
$$

(74)

It is called the Fermi-Dirac-Poisson equation or the Thomas-Fermi equation at finite temperature. Using Eq. (69), it can be rewritten as

$$
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = \frac{16\pi^2 g \sqrt{2} G m^{5/2}}{h^3 \beta^{3/2}} I_{1/2} \left[ e^{-\alpha_0 + \beta m \Phi(r)} \right].
$$

(75)

Once the gravitational potential $\Phi(r)$ has been determined by solving Eq. (75), the density $\rho(r)$ is given by Eq. (69). The general procedure to obtain the density profiles and the caloric curves of the self-gravitating Fermi gas at finite temperature from the differential equation (73) is explained in detail in [109]. The study of phase transitions in the nonrelativistic self-gravitating Fermi gas has been performed in [102, 110, 135]. The complete canonical and microcanonical phase diagrams are given in [110]. Phase transitions in the fermionic King model have been studied in [145].

12 Refs. [102, 135] work in the canonical ensemble while Refs. [103, 104] work in the microcanonical ensemble. Refs. [102, 110] consider both canonical and microcanonical ensembles.
4. Equation for fugacity \( z(r) \)

Introducing the fugacity

\[
\begin{align*}
    z(r) &= e^{\alpha(r)} = e^{\beta(r)/k_B T} \\
    \text{(76)}
\end{align*}
\]

and using Eq. (53) giving

\[
\begin{align*}
    z(r) &= e^{\alpha_0} e^{-\beta m \Phi(r)}, \\
    \text{(77)}
\end{align*}
\]

we can rewrite the Fermi-Dirac distribution function (65) as

\[
\begin{align*}
    f(r, p) &= \frac{g}{\hbar^3 \left( 1 + \frac{1}{z(r)} e^{\beta p^2/2m} \right)^3/2}, \\
    \text{(78)}
\end{align*}
\]

On the other hand, the density and the pressure from Eqs. (69) and (70) can be expressed in terms of \( z \) as

\[
\begin{align*}
    \rho(r) &= \frac{4\pi g \sqrt{2m^{5/2}}}{\hbar^3 \beta^{3/2}} I_{1/2} \left[ \frac{1}{z(r)} \right], \\
    \text{(79)}
\end{align*}
\]

\[
\begin{align*}
    P(r) &= \frac{8\pi g \sqrt{2m^{3/2}}}{3 \hbar^3 \beta^{5/2}} I_{3/2} \left[ \frac{1}{z(r)} \right]. \\
    \text{(80)}
\end{align*}
\]

Taking the derivative of Eq. (80) with respect to \( r \) and using Eq. (72), we obtain

\[
\begin{align*}
    P'(r) &= \rho(r) \frac{k_B T}{m} z'(r) \frac{z(r)}{z'(r)}. \\
    \text{(81)}
\end{align*}
\]

Starting from the condition of hydrostatic equilibrium (7), using Eq. (81), multiplying the resulting expression by \( r^2 \), taking the derivative with respect to \( r \) and using Eqs. (2) and (79), we finally obtain the following differential equation for \( z(r) \):

\[
\begin{align*}
    \frac{z''}{z} + \frac{2}{r^2} \frac{z'}{z} - \left( \frac{z'}{z} \right)^2 + \frac{16\pi^2 g \sqrt{2Gm r^{7/2}}}{\hbar^3 \beta^{3/2}} I_{1/2} \left( \frac{1}{z} \right) &= 0. \\
    \text{(82)}
\end{align*}
\]

We can obtain the density profiles and the caloric curves of the self-gravitating Fermi gas at finite temperature from the differential equation (82) as done in [222].

H. Thermodynamical stability and ensembles inequivalence

We have seen that the statistical equilibrium states in the microcanonical and canonical ensembles are the same. Indeed, the extrema of entropy at fixed energy and particle number coincide with the extrema of free energy at fixed particle number. This is a general result of statistical mechanics which is due to the fact that the first order variational problems (42) and (63) coincide [223]. However, the thermodynamical stability of these equilibrium states, which is related to the sign of the second order variations of the appropriate thermodynamical potential (entropy or free energy), may be different in the microcanonical and canonical ensembles. In this sense, the statistical ensembles are inequivalent for self-gravitating systems [110, 207, 208]. This is a specificity of systems with long-range interactions whose energy is nonadditive [209, 210]. It can be shown that canonical stability implies microcanonical stability while the reciprocal is wrong [223]. Therefore, there are more stable equilibrium states in the microcanonical ensemble than in the canonical ensemble.\(^\text{13}\) Basically, this is because the microcanonical ensemble is more constrained (because of the conservation of energy) than the canonical ensemble.

---

\(^\text{13}\) For example, equilibrium states with a negative specific heat may be stable in the microcanonical ensemble while they are always unstable in the canonical ensemble [177, 265] (see Appendix B of [1]).
The thermodynamical stability of an equilibrium state can be investigated by studying the sign of the second order variations of the appropriate thermodynamical potential (entropy or free energy) and reducing this study to an eigenvalue problem. This method allows us to determine the form of the perturbation that triggers the thermodynamical instability. We refer to [176, 195, 200, 202, 203, 207] for a detailed discussion of this stability problem in the case of classical self-gravitating systems.

On the other hand, the thermodynamical stability of the system can be directly settled from the topology of the series of equilibria $\beta(−E)$ by using the Poincaré criterion [266] (see [110, 177, 185] for some applications of this method in relation to the statistical mechanics of self-gravitating systems):

(i) In the microcanonical ensemble, if we plot $\beta(−E)$ at fixed $N$, a change of stability can occur only at a turning point of energy. A mode of stability is lost if the curve $\beta(−E)$ rotates clockwise and gained if it rotates anticlockwise. Since $S$ and $E$ reach their extrema at the same points (in view of the fact that $δS/k_B = βδE$), the curve $S(E)$ displays spikes at its extremal points.

(ii) In the canonical ensemble, if we plot $\beta(−E)$ at fixed $N$, a change of stability can occur only at a turning point of temperature. A mode of stability is lost if the curve $\beta(−E)$ rotates clockwise and gained if it rotates anticlockwise. Since $J$ and $β$ reach their extrema at the same points (in view of the fact that $δJ = −Eδβ$), the curve $J(β)$ displays spikes at its extremal points. We can also consider the curve $α_0(N)$ at fixed $T$. If we plot $α_0(N)$, a change of stability can occur only at a turning point of particle number $N$. A mode of stability is lost if the curve $α_0(N)$ rotates clockwise and gained if it rotates anticlockwise. Since $J$ and $N$ reach their extrema at the same points (in view of the fact that $δJ = −α_0δN$), the curve $J(N)$ displays spikes at its extremal points.

I. Dynamical stability

1. Vlasov-Poisson equations

The distribution function $f(r,v)$ of a system of self-gravitating fermions at statistical equilibrium is given by Eq. [105]. It is of the form $f = f(ε)$ with $f'(ε) < 0$, where $ε = v^2/2 + Φ(r)$ is the energy of a particle by unit of mass. Therefore, at statistical equilibrium, the distribution function depends only on the individual energy $ε$ of the particles and is monotonically decreasing. It is shown in Appendices [4] and [12] that these properties remain valid for a general form of entropy. Since $f(r,v)$ is a function of the energy $ε$, which is a constant of the motion, it is a particular steady state of the Vlasov-Poisson equations. This is a special case of the Jeans theorem [267]. Therefore, a statistical equilibrium state (extremum of entropy at fixed energy and particle number) is a steady state of the Vlasov-Poisson equations. Furthermore, it can be shown that thermodynamical stability implies dynamical stability with respect to the Vlasov-Poisson equations [187]. Therefore, a stable thermodynamical equilibrium state (maximum of entropy at fixed energy and particle number) is always dynamically stable. In general, the reciprocal is wrong. This is the case in Newtonian gravity. Indeed, it can be shown [192, 205, 273] that all the distribution functions of the form $f = f(ε)$ with $f'(ε) < 0$ are dynamically stable with respect to the Vlasov-Poisson equations. As a result, in Newtonian gravity, all the statistical equilibrium states (i.e. all the extrema of entropy at fixed energy and particle number) are dynamically stable, even those that are thermodynamically unstable.

2. Euler-Poisson equations

We have seen that the statistical equilibrium state of a system of self-gravitating fermions is described by a barotropic equation of state of the form $P(r) = P(ρ(r))$ and that it satisfies the condition of hydrostatic equilibrium [10]. It is shown in Appendices [2] and [13] that these properties remain valid for an arbitrary form of entropy. As a result, the system is in a steady state of the Euler-Poisson equations. Furthermore, it can be shown (see [215] and Appendices [5, 11]) that the thermodynamical stability of a self-gravitating system in the canonical ensemble is equivalent to its dynamical stability with respect to the Euler-Poisson equations.

14 According to the Jeans theorem [267], a spherical stellar system in collisionless equilibrium has a distribution function of the form $f = f(ε,L)$ where $ε$ is the energy and $L$ is the angular momentum. We note that an extremum of entropy $S$ at fixed energy $E$ and particle number $N$ leads to a distribution function that depends only on $ε$, not on $L$. Therefore, an extremum of entropy at fixed energy and particle number is necessarily isotropic.

15 This result is very general (being valid for an arbitrary entropic functional and for any long-range potential of interaction) and stems from the fact that the entropy (which is a particular Casimir), the energy and the particle number are conserved by the Vlasov-Poisson equations (see [224] and Appendix [10]).
3. Kinetic equations

It can also be shown that the thermodynamical stability of a self-gravitating system in the microcanonical ensemble is equivalent to its dynamical stability with respect to the Landau-Poisson equations and that the thermodynamical stability of a self-gravitating system in the canonical ensemble is equivalent to its dynamical stability with respect to the Kramers-Poisson, damped Euler-Poisson, and Smoluchowski-Poisson equations. These results are natural since these kinetic equations describe the thermodynamical (secular) evolution of the system in the microcanonical and canonical ensembles respectively. In particular, they satisfy an $H$-theorem for the entropy in the microcanonical ensemble and for the free energy in the canonical ensemble. Using Lyapunov’s direct method it can be shown that they relax towards a stable thermodynamical equilibrium state.

J. Particular limits

The Fermi-Dirac distribution can be written as

$$f(\mathbf{r}, \mathbf{p}) = \frac{g}{h^3} \frac{1}{1 + e^{[E_{\text{kin}}(\mathbf{p}) - \mu(r)]/k_B T}}, \quad (83)$$

where $E_{\text{kin}} = p^2/2m$ and $\mu(r) = \mu_0 - m\Phi(r)$. Let us consider particular limits of this distribution function.

1. The completely degenerate Fermi gas (ground state)

The completely degenerate limit corresponds to $T \to 0$, $\mu(r) > 0$ finite, and $\alpha(r) = \mu(r)/k_B T \to +\infty$. In that case, the chemical potential is positive and large compared to the temperature. This yields the Fermi distribution (or Heaviside function):

$$f(\mathbf{r}, \mathbf{p}) = \frac{g}{h^3} \text{ if } E_{\text{kin}}(\mathbf{p}) < E_F(r) \quad (p < p_F(r)), \quad (84)$$

$$f(\mathbf{r}, \mathbf{p}) = 0 \text{ if } E_{\text{kin}}(\mathbf{p}) > E_F(r) \quad (p > p_F(r)), \quad (85)$$

where

$$E_F(r) = \mu(r) = \mu_0 - m\Phi(r), \quad (86)$$

and

$$p_F(r) = \sqrt{2m\mu(r)} = \sqrt{2m(\mu_0 - m\Phi(r))} \quad (87)$$

are the Fermi energy and the Fermi impulse. The density and the pressure are given by

$$\rho = \int f m \, d\mathbf{p} = \int_0^{p_F} \frac{g}{h^3} m 4\pi p^2 \, dp = \frac{4\pi gm}{3h^3} p_F^3(r), \quad (88)$$

$$P = \frac{1}{3} \int f \frac{p^2}{m} \, d\mathbf{p} = \frac{1}{3} \int_0^{p_F} \frac{g}{h^3} \frac{1}{m} p^2 4\pi p^2 \, dp = \frac{4\pi g}{15mh^3} p_F^5(r). \quad (89)$$

Eliminating the Fermi impulse between these two expressions, we find that the equation of state of the nonrelativistic Fermi gas at $T = 0$ is

$$P = K \rho^{5/3}, \quad K = \frac{1}{5} \left( \frac{3h^3}{4\pi gm^4} \right)^{2/3}. \quad (90)$$

\[16\] This is at variance with the Vlasov and Euler equations which do not satisfy an $H$-theorem.
This is the equation of state of a polytrope of index $\gamma = 5/3$ (i.e. $n = 3/2$). Substituting Eq. (90) into Eq. (76), the fundamental equation of hydrostatic equilibrium can be written as

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\rho}{dr} \right) = -\frac{8\pi G}{5} \rho. \quad (91)$$

Alternatively, substituting Eq. (88) with the relation from Eq. (87) into the Poisson equation (1), we obtain the Thomas-Fermi equation

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi}{dr} \right) = \frac{16\pi^2 gGm}{3h^3} \left[ 2m(\mu - m\Phi(r)) \right]^{3/2}. \quad (92)$$

These equations can both be reduced to the Lane-Emden equation [132]. For a given value of $\rho_0 = \rho(0)$ or $\Phi_0 = \Phi(0)$, we can solve these equations until the point where the density vanishes: $\rho(R) = 0$. This determines the radius $R$ of the configuration (when $T = 0$ we do not need a box to confine the system). We can then compute the corresponding mass $M$. By varying $\rho_0$ or $\Phi_0$, we get the mass-radius relation $M(R)$. The mass-radius relation and the corresponding density profiles of the self-gravitating Fermi gas at $T = 0$ are given in [132].

Remark: For the self-gravitating Fermi gas at $T = 0$, the free energy $F$ reduces to the energy $E = E_{\text{kin}} + W$. Therefore, a stable equilibrium state at $T = 0$ is a minimum of $E$ at fixed $N$ (ground state). We have seen that the nonrelativistic Fermi gas at $T = 0$ is described by the polytropic equation of state $E = E_{\text{kin}} + W$. Using Eqs. (21) and (90) the internal energy can be written as $E = E_{\text{kin}} + W$ where $E_{\text{kin}} = \frac{3}{2} \int P \, dr = \frac{3}{2} K \int \rho^{5/3} \, dr$ is the kinetic energy. This is the same as the energy $W = U + W$ of a polytropic gas of index $n = 3/2$, where $U = \int \rho \int^{\rho} \frac{P(\rho')}{\rho'^2} \, d\rho' \, dr = \frac{3}{2} K \int \rho^{5/3} \, dr$ is the internal energy (see Appendix C1). Therefore, for the self-gravitating Fermi gas at $T = 0$, we explicitly recover the fact that the condition of thermodynamical stability coincides with the condition of dynamical stability with respect to the Euler-Poisson equations (see Sec. II 1 2).

2. The nondegenerate Fermi gas (classical limit)

The nondegenerate (classical) limit corresponds to

$$\frac{E_{\text{kin}}(\rho) - \mu(r)}{k_B T} \gg 1 \quad \text{i.e.} \quad \beta E_{\text{kin}}(\rho) - \alpha(r) \gg 1. \quad (93)$$

This yields the Maxwell-Boltzmann distribution

$$f(\mathbf{r}, \mathbf{p}) = \frac{g}{h^3} e^{-(E_{\text{kin}}(\rho) - \mu(r))/k_B T}. \quad (94)$$

The condition from Eq. (93) is always fulfilled when $\alpha(r) \to -\infty$, whatever the value of $\beta E_{\text{kin}}(\rho)$. Therefore, the condition $\mu(r) \to -\infty$, $T$ finite and $\alpha(r) = \mu(r)/k_B T \to -\infty$ implies the nondegenerate (classical) limit. In that case, the chemical potential is negative and large (in absolute value) as compared to the temperature. However, this is not the only case where the Maxwell-Boltzmann distribution is valid. We can be in the classical limit for arbitrary values of $\alpha(r)$ (positive or negative) provided that $\beta E_{\text{kin}}(\rho) - \alpha(r) \gg 1$. The classical limit is specifically studied in Paper II.

III. STATISTICAL MECHANICS OF GENERAL RELATIVISTIC FERMIONS

In this section, we consider the statistical mechanics of self-gravitating fermions within the framework of general relativity. We use a presentation similar to the one developed in Sec. II to treat the statistical mechanics of self-gravitating fermions within the framework of Newtonian gravity.

17 Since $\rho(R) = 0$ we find that $\mu_0 = m\Phi(R) = -GMm/R$. Therefore, $\mu_0 < 0$.
18 See Appendix C of [286] for a more general discussion.
A. Hydrostatic equilibrium of gaseous spheres in general relativity

1. Einstein equations

The Einstein field equations of general relativity are expressed as

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \frac{8\pi G}{c^4} T_{\mu\nu}, \quad (95) \]

where \( R_{\mu\nu} \) is the Ricci tensor, \( T_{\mu\nu} \) is the energy-momentum tensor and \( g_{\mu\nu} \) is the metric tensor defined by

\[ ds^2 = -g_{\mu\nu} dx^\mu dx^\nu, \quad (96) \]

where \( ds \) is the invariant interval between two neighbouring space-time events.

In the following, we shall restrict ourselves to spherically symmetric systems with motions, if any, only in the radial directions. Under these assumptions, the metric can be written in the form

\[ ds^2 = e^\nu c^2 dt^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - e^\lambda dr^2, \quad (97) \]

where \( \nu \) and \( \lambda \) are functions of \( r \) and \( t \) only. The energy-momentum tensor is assumed to be that for a perfect fluid

\[ T_{\mu\nu} = P g_{\mu\nu} + (P + \epsilon) u^\mu u^\nu, \quad (98) \]

where \( u^\mu = dx^\mu / ds \) is the fluid four-velocity, \( P \) is the isotropic pressure and \( \epsilon \) is the energy density including the rest mass. The mass contained within a sphere of radius \( r \) is

\[ M(r) = \frac{1}{c^2} \int_0^r \epsilon(r) 4\pi r^2 dr \quad \Rightarrow \quad \frac{dM}{dr} = \frac{\epsilon}{c^2} 4\pi r^2. \quad (99) \]

The total mass is

\[ M = \frac{1}{c^2} \int_0^R \epsilon(r) 4\pi r^2 dr, \quad (100) \]

where \( R \) is the size of the system. The mass-energy is \( E = Mc^2 \). In the nonrelativistic limit \( c \to +\infty \), using \( \epsilon \sim \rho c^2 \) where \( \rho c^2 \) is the rest-mass energy (see below), Eqs. (99) and (100) return Eqs. (2) and (24).

2. TOV equations

The equations of general relativity governing the hydrostatic equilibrium of a spherical distribution of matter are well-known. They are given by (see, e.g., [275]):

\[ \frac{d}{dr} (re^{-\lambda}) = 1 - \frac{8\pi G}{c^4} r^2 \epsilon, \quad (101) \]

\[ \frac{dP}{dr} = -\frac{1}{2} (\epsilon + P) \frac{d\nu}{dr}, \quad (102) \]

\[ \frac{e^{-\lambda} \nu}{r} = \frac{1}{r^2} (1 - e^{-\lambda}) + \frac{8\pi G}{c^4} P. \quad (103) \]

These equations can be deduced from the Einstein equations (95). However, Eq. (102) can be obtained more directly from the local law of energy-momentum conservation, \( D_\mu T^{\mu\nu} = 0 \), which is also contained in the Einstein equations. It can be interpreted as the condition of hydrostatic equilibrium in general relativity. This equation was first derived and emphasized by Tolman [232]. In the nonrelativistic limit \( c \to +\infty \) it reduces to Eq. (6) (see Sec. IV A).

Equations (101)-(103) can be combined to give

\[ \frac{dP}{dr} = -\frac{\epsilon + P}{c^2} \frac{G M(r)}{r} + \frac{8\pi G P r}{1 - \frac{2GM(r)}{rc^2}}, \quad (104) \]

where \( M(r) \) is given by Eq. (99). This equation was first derived by Oppenheimer and Volkoff [33]. It extends the classical condition of hydrostatic equilibrium for a star to the context of general relativity. In the nonrelativistic limit \( c \to +\infty \), using \( \epsilon \sim \rho c^2 \gg P \), Eq. (104) reduces to Eq. (7).
3. Metric functions

Integrating Eq. (101) with the condition $\lambda(r) \to 0$ at infinity, we obtain

$$e^{-\lambda(r)} = 1 - \frac{2GM(r)}{rc^2}. \quad (105)$$

Then, Eq. (103) can be rewritten as

$$\frac{d\nu}{dr} = \frac{1 + 4\pi Pr^3/M(r)c^2}{r(rc^2/2GM(r) - 1)}. \quad (106)$$

These equations determine the metric functions $\lambda(r)$ and $\nu(r)$. Eq. (105) can be interpreted as a generalization of Newton’s law. In the nonrelativistic limit $c \to +\infty$ it reduces to Eq. (3) (see Sec. IV A).

In the empty space outside the star, $P = \epsilon = 0$. Therefore, if $M = M(R)$ denotes the mass-energy of the star, Eqs. (105) and (106) become for $r > R$:

$$e^{-\lambda(r)} = 1 - \frac{2GM}{rc^2} \quad \text{and} \quad \frac{d\nu}{dr} = \frac{1}{r(rc^2/2GM - 1)}. \quad (107)$$

The second equation is readily integrated into

$$\nu(r) = \ln \left(1 - \frac{2GM}{rc^2}\right), \quad (108)$$

where we have taken the constant of integration to be zero by convention. In this manner $\nu(r) \to 0$ when $r \to +\infty$. We note that $\nu(r) = -\lambda(r)$. From these equations, we get

$$e^{-\lambda(R)} = 1 - \frac{2GM}{Rc^2} \quad \text{and} \quad \nu(R) = \ln \left(1 - \frac{2GM}{Rc^2}\right). \quad (109)$$

Substituting the foregoing expressions for $\lambda$ and $\nu$ into Eq. (97), we obtain the well-known Schwarzschild’s form of the metric in the empty space outside a star [71]:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{dr^2}{1 - 2GM/rc^2}. \quad (110)$$

This form of the metric remains valid even if the star is unsteady as long as it remains spherically symmetric (Jebsen-Birkhoff theorem [276–278]). In this respect, Newton’s theorem according to which the gravitational field external to a spherical distribution of matter depends only on its total mass [see Eq. (3)] is equally true in general relativity. The metric (110) is singular at

$$r = \frac{2GM}{c^2} \equiv R_S, \quad (111)$$

where $R_S$ is the Schwarzschild radius appropriate to the mass $M$. This does not mean that spacetime is singular at that radius but only that this particular metric is. Indeed, the singularity can be removed by a judicious change of coordinate system [278–281]. When $R_S > R$, the star is a black hole and no particle or even light can leave the region $R < r < R_S$. However, for a gaseous sphere in hydrostatic equilibrium the discussion does not arise because $R_S < R$. Indeed, it can be shown that the radius of the configuration is necessarily restricted by the Buchdahl inequality

$$R \geq \frac{9}{8} \frac{2GM}{c^2} = \frac{9}{8} R_S. \quad (112)$$

Therefore, the points exterior to the star always satisfy $r > R_S$.

---

19 This inequality was previously derived by Schwarzschild [71] in the case of equilibrium configurations with uniform energy density.
B. Local variables

We consider a gas of relativistic fermions described by the distribution function \( f(r,p) \) such that \( f(r,p) \, dr \, dp \) gives the number density of fermions at position \( r \) with impulse \( p \). The local particle number density is

\[
\begin{align*}
n &= \int f \, dp. 
\end{align*}
\] (113)

The energy density is

\[
\begin{align*}
\epsilon &= \int f E(p) \, dp, 
\end{align*}
\] (114)

where \( E \) is the total (kinetic + rest mass) energy of a particle given by

\[
\begin{align*}
E(p) &= \sqrt{p^2 c^2 + m^2 c^4}. 
\end{align*}
\] (115)

In can be written as

\[
\begin{align*}
E(p) &= mc^2 + E_{\text{kin}}(p), 
\end{align*}
\] (116)

where

\[
\begin{align*}
E_{\text{kin}}(p) &= mc^2 \left\{ \sqrt{\frac{p^2}{m^2 c^2} + 1} - 1 \right\}, 
\end{align*}
\] (117)

is the kinetic energy of the particle. In the nonrelativistic limit \( c \to +\infty \), the kinetic energy of a particle reduces to Eq. (19). The energy density \( \epsilon \) can be written as

\[
\begin{align*}
\epsilon &= \rho c^2 + \epsilon_{\text{kin}}, 
\end{align*}
\] (118)

where \( \rho = nm \) is the rest-mass density and

\[
\begin{align*}
\epsilon_{\text{kin}} &= \int f E_{\text{kin}}(p) \, dp 
\end{align*}
\] (119)

is the kinetic energy density. In the nonrelativistic limit \( c \to +\infty \), we have \( \epsilon \sim \rho c^2 \). The local pressure is given by

\[
\begin{align*}
P &= \frac{1}{3} \int f p \frac{dE}{dp} \, dp = \frac{1}{3} \int f p^2 c^2 E(p) \, dp. 
\end{align*}
\] (120)

In the nonrelativistic limit \( c \to +\infty \), using \( E(p) \simeq mc^2 + p^2/2m \), we recover Eq. (20). Finally, the Fermi-Dirac entropy density is given by

\[
\begin{align*}
s &= -k_B g \frac{g}{h^3} \int \left\{ \frac{f}{f_{\text{max}}} \ln \frac{f}{f_{\text{max}}} + \left(1 - \frac{f}{f_{\text{max}}} \right) \ln \left(1 - \frac{f}{f_{\text{max}}} \right) \right\} \, dp, 
\end{align*}
\] (121)

as in Sec. II.B.

C. Global variables

The entropy of the fermion gas is given by

\[
\begin{align*}
S &= \int_0^R s(r) \left[ 1 - \frac{2GM(r)}{rc^2} \right]^{-1/2} 4\pi r^2 \, dr, 
\end{align*}
\] (122)
where the entropy density \( s(r) \) has been multiplied by the proper volume element \( e^{\lambda/2}4\pi r^2 \, dr \) obtained by using the expression (105) of the metric coefficient \( \lambda(r) \). Similarly, the particle number is given by

\[
N = \int_0^R n(r) \left[ 1 - \frac{2GM(r)}{rc^2} \right]^{-1/2} 4\pi r^2 \, dr. \tag{123}
\]

We introduce the (binding) energy \( \text{E} \)

\[
E = (M - Nm)c^2, \tag{124}
\]

where \( Mc^2 \) is the mass-energy given by Eq. (100). We note that \( M \), unlike \( S \) and \( N \), involves the coordinate volume, not the proper volume. In the nonrelativistic limit \( c \to +\infty \), the binding energy \( E \) reduces to the Newtonian energy \( E = E_{\text{kin}} + W \) including the kinetic energy and the potential energy (see Sec. IV F).

As in the Newtonian case, a statistical equilibrium state exists only if the system is confined within a box of radius \( R \) otherwise it would evaporate. In the microcanonical ensemble, the mass-energy \( E = Mc^2 \) and the particle number \( N \) are conserved. The statistical equilibrium state of the system is obtained by maximizing the Fermi-Dirac entropy \( S \) at fixed mass-energy \( E = Mc^2 \) and particle number \( N \):

\[
\max \{ S \mid E, N \text{ fixed} \}. \tag{125}
\]

This determines the “most probable” state of an isolated system. To solve this maximization problem, we proceed in two steps as we did previously in the nonrelativistic case (the one-step process is discussed in Appendix C).

D. Maximization of the entropy density at fixed energy density and particle number density

1. Local thermodynamic equilibrium

We first maximize the Fermi-Dirac entropy density (121) at fixed energy density (114) and particle number density (113). We write the variational problem for the first variations (extremization) under the form

\[
\frac{\delta s}{k_B} - \beta(r) \delta \epsilon + \alpha(r) \delta n = 0, \tag{126}
\]

where \( \beta(r) \) and \( \alpha(r) \) are local Lagrange multipliers. This leads to the Fermi-Dirac distribution function

\[
f(r, p) = \frac{g}{h^3} \frac{1}{1 + e^{\beta(r)E(p)-\alpha(r)}}, \tag{127}
\]

which corresponds to the condition of local thermodynamic equilibrium. Introducing the local temperature \( T(r) \) and the local chemical potential \( \mu(r) \) by the relations

\[
\beta(r) = \frac{1}{k_B T(r)} \quad \text{and} \quad \alpha(r) = \frac{\mu(r)}{k_B T(r)}, \tag{128}
\]

the Fermi-Dirac distribution (124) can be rewritten as

\[
f(r, p) = \frac{g}{h^3} \frac{1}{1 + e^{[E(p)-\mu(r)]/k_B T(r)}}. \tag{129}
\]

On the other hand, the variational principle (126) reduces to the first law of thermodynamics

\[
ds = \frac{d\epsilon}{T} - \frac{\mu}{T} dn. \tag{130}
\]

This law is valid for an arbitrary form of entropy (see Appendix C).

---

20 Equation (105) is valid at equilibrium (for a steady state). Therefore, the expressions (122) and (123) of \( S \) and \( N \) are justified only at equilibrium. However, it is shown in [51] that Eq. (105) remains valid for small perturbations about equilibrium up to second order in the motion. This justifies using Eqs. (122) and (123) when we make perturbations about the equilibrium state as in the variational problem considered below.

21 The binding energy is usually defined by \( E_b = (Nm - M)c^2 \) so that \( E = -E_b \). It is the difference between the rest mass energy \( Nmc^2 \) (the energy that the matter of the star would have if dispersed to infinity) and the total mass-energy \( Mc^2 \). In order to simplify the discussion, we shall define the binding energy as \( E_b = (M - Nm)c^2 \) so that \( E = E_b \).
2. Local variables

Substituting the Fermi-Dirac distribution \( \text{Eq. (129)} \) into Eqs. \( \text{113} - \text{120} \) we get

\[
n(r) = \frac{g}{h^3} \int \frac{1}{1 + e^{(E(p) - \mu(r))/k_B T(r)}} \, dp,
\]

\[
\epsilon(r) = \frac{g}{h^3} \int \frac{E(p)}{1 + e^{(E(p) - \mu(r))/k_B T(r)}} \, dp,
\]

\[
\epsilon_{\text{kin}}(r) = \frac{g}{h^3} \int \frac{E_{\text{kin}}(p)}{1 + e^{(E(p) - \mu(r))/k_B T(r)}} \, dp,
\]

\[
P(r) = \frac{g}{3h^3} \int \frac{1}{1 + e^{(E(p) - \mu(r))/k_B T(r)}} \, dp \int \ln \left( 1 + e^{(E(p) - \mu(r))/k_B T(r)} \right) \, dp,
\]

where we used an integration by parts to obtain the last equality in Eq. \( \text{134} \). These equations determine the Lagrange multipliers \( T(r) \) and \( \mu(r) \) in terms of \( \epsilon(r) \) and \( n(r) \). They also determine the equation of state \( P = P[n(r), T(r)] \) in implicit form. On the other hand, substituting the Fermi-Dirac distribution function \( \text{129} \) into Eq. \( \text{121} \), and using Eq. \( \text{131} - \text{134} \), we obtain after some calculations the integrated Gibbs-Duhem relation

\[
s(r) = \frac{\epsilon(r) + P(r) - \mu(r)n(r)}{T(r)}.
\]

This relation is valid for an arbitrary form of entropy (see Appendix E). Finally, combining the first law of thermodynamics \( \text{130} \) and the integrated Gibbs-Duhem relation \( \text{135} \), we obtain the identity

\[
d \left( \frac{P}{T} \right) = n \, d \left( \frac{\mu}{T} \right) - \epsilon \, d \left( \frac{1}{T} \right).
\]

We also have the identities

\[
d \left( \frac{s}{n} \right) = -Pd \left( \frac{1}{n} \right) + Td \left( \frac{s}{n} \right)
\quad \text{and} \quad sdT - dP + nd\mu = 0,
\]

which correspond to the standard form of the first law of thermodynamics and the local Gibbs-Duhem relation (see Appendix E). These results are valid for an arbitrary form of entropy.

E. Maximization of the entropy at fixed mass-energy and particle number

1. Variational principle

If we introduce the Gibbs-Duhem relation \( \text{135} \) into the entropy \( \text{122} \), we obtain

\[
S = \int_0^R \frac{\epsilon(r) + P(r) - \mu(r)n(r)}{T(r)} \left[ 1 - \frac{2GM(r)}{rc^2} \right]^{-1/2} 4\pi r^2 \, dr.
\]

The functionals \( S, M \) and \( N \) depend on \( \epsilon(r) \) and \( n(r) \). We now maximize the entropy \( S \) at fixed mass-energy \( \mathcal{E} = Mc^2 \) and particle number \( N \). From that point, we follow Bilic and Viollier \cite{Bilic96} (see also \cite{Bilic96, Viollier98, Viollier99} for alternative derivations and generalizations).\(^{22}\) We write the variational problem for the first variations (extremization) under the form

\[
\frac{\delta S}{k_B} - \beta_0 c^2 \delta M + \alpha_0 \delta N = 0,
\]

\(^{22}\) We note that Bilic and Viollier \cite{Bilic96} work in the canonical ensemble while we work in the microcanonical ensemble. However, as we have already indicated (see Sec. \( \text{111} \)), the statistical ensembles are equivalent at the level of the first order variations (extremization problem) so they determine the same equilibrium states. For systems with long-range interactions, like self-gravitating systems, ensembles inequivalence may occur at the level of the second order variations of the thermodynamical potential, i.e., regarding the stability of the equilibrium states.
where \( \beta_0 \) and \( \alpha_0 \) are global (uniform) Lagrange multipliers. Taking the first variations of \( M, N \) and \( S \) from Eqs. (100), (122), and (123), and using the first law of thermodynamics (130) and the integrated Gibbs-Duhem relation (135), we obtain

\[
\int \frac{\delta \epsilon}{k_B T} \chi \, dV - \int \frac{\mu}{k_B T} \delta n \, dV + \int \left( \frac{\epsilon + P - \mu n}{k_B T} \right) \delta \chi \, dV - \beta_0 \int \delta \epsilon \, dV + \alpha_0 \int \delta n \, dV + \alpha_0 \int n \, \delta \chi \, dV = 0, \quad (140)
\]

where we have introduced the short-hand notations

\[
\chi(r) = \left[ 1 - \frac{2GM(r)}{rc^2} \right]^{-1/2} \quad \text{and} \quad dV = 4\pi r^2 \, dr. \quad (141)
\]

We note for future reference that

\[
\delta \chi = \frac{\partial \chi}{\partial M} \delta M(r) = \left[ 1 - \frac{2GM(r)}{rc^2} \right]^{-3/2} \frac{G\delta M(r)}{rc^2}, \quad (142)
\]

\[
\delta M(r) = \frac{1}{c^2} \int_0^r \delta \epsilon \, 4\pi r^2 \, dr, \quad \frac{d\delta M(r)}{dr} = \frac{1}{c^2} \delta \epsilon \, 4\pi r^2. \quad (143)
\]

In Eq. (140) the variations on \( \delta n \) and \( \delta \epsilon \) must vanish independently.

2. Variations on \( \delta n \)

The vanishing of Eq. (140) with respect to variations on \( \delta n \) gives

\[
\alpha(r) = \frac{\mu(r)}{k_B T(r)} = \alpha_0. \quad (144)
\]

This relation shows that the ratio between the local chemical potential and the local temperature is a constant. In the sequel, we will denote this constant by \( \alpha \) instead of \( \alpha_0 \). Therefore, we write

\[
\alpha = \frac{\mu(r)}{k_B T(r)}. \quad (145)
\]

Since \( \mu/T \) is constant, Eq. (136) reduces to

\[
d \left( \frac{P}{T} \right) = -\epsilon \, d \left( \frac{1}{T} \right), \quad (146)
\]

implying

\[
\frac{dP}{dr} = \frac{\epsilon + P}{T} \frac{dT}{dr}. \quad (147)
\]

This equation was first obtained by Tolman [232]. Using Eq. (145), we also have

\[
\frac{dP}{dr} = \frac{\epsilon + P}{\mu} \frac{d\mu}{dr}. \quad (148)
\]

---

23 A similar equation \( dP/dT = (\epsilon + P)/T \) is used in cosmology in order to relate the temperature \( T \) of a cosmic fluid described by an equation of state \( P = P(\epsilon) \) to its energy density \( \epsilon \) (see, e.g., [284]). In that context, it is derived from thermodynamical arguments by assuming that \( \mu = 0 \) like in the case of the black-body radiation. By contrast, in the present calculation, we have simply used the fact that \( \mu/T \) is constant, not that \( \mu \) is necessarily equal to zero.
3. Variations on $\delta e$

Using Eq. (144) and focusing now on the variations on $\epsilon$, Eq. (140) reduces to

$$
\int \frac{\delta \epsilon}{k_B T} \chi dV + \int \frac{\epsilon + P}{k_B T} \delta \chi dV - \beta_0 \int \delta \epsilon dV = 0.
$$

(149)

Using the identities (142) and (143), the foregoing equation can be rewritten as

$$
c^2 \int \left( \frac{\chi}{k_B T} - \beta_0 \right) \frac{dM(r)}{dr} dr + \int \frac{\epsilon + P}{k_B T} \frac{\partial \chi}{\partial M} \delta M(r) 4\pi r^2 dr = 0.
$$

(150)

Using an integration by parts, we obtain

$$
c^2 \left[ \frac{\chi(R)}{k_B T(R)} - \beta_0 \right] \delta M(R) - \int \left[ c^2 \frac{d}{dr} \left( \frac{\chi}{k_B T} \right) - \frac{\epsilon + P}{k_B T} \frac{\partial \chi}{\partial M} \right] 4\pi r^2 \delta M(r) dr = 0.
$$

(151)

The two terms in brackets must vanish individually. The vanishing of the first bracket yields

$$
\beta_0 = \frac{\chi(R)}{k_B T(R)} = \frac{1}{k_B T(R) \sqrt{1 - \frac{2GM}{rc^2}}}.
$$

(152)

The vanishing of the second bracket yields

$$
c^2 \frac{d}{dr} \left( \frac{\chi}{T} \right) = \frac{\epsilon + P}{T} \frac{\partial \chi}{\partial M} 4\pi r^2,
$$

(153)

leading to

$$
\frac{1}{T} \frac{dT}{dr} = \frac{1}{c^2} \frac{\frac{GM(r)}{r^2} + \frac{4\pi G}{c^2} Pr}{1 - \frac{2GM(r)}{rc^2}}.
$$

(154)

4. Condition of hydrostatic equilibrium

Combining Eqs. (147) and (154), we obtain the OV equation

$$
\frac{dP}{dr} = -\frac{\epsilon + P}{c^2} \frac{\frac{GM(r)}{r^2} + \frac{4\pi G}{c^2} Pr}{1 - \frac{2GM(r)}{rc^2}},
$$

(155)

expressing the condition of hydrostatic equilibrium. From Eqs. (106) and (155) we obtain the Tolman equation

$$
\frac{dP}{dr} = -\frac{1}{2} (\epsilon + P) \frac{d\nu}{dr},
$$

(156)

which also expresses the condition of hydrostatic equilibrium (see Sec. III A 2). Therefore, the condition of statistical equilibrium, obtained by extremizing the entropy at fixed mass-energy and particle number, implies the condition of hydrostatic equilibrium. This condition was not assumed in the preceding calculations. It results from the thermodynamical variational problem (maximization of entropy at fixed mass-energy and particle number). In this sense, the maximum entropy principle leads to the TOV equations. As explained in Sec. III E 7 below, this should not cause surprise. This was already the case in Newtonian gravity (see Sec. III E 4). The intrinsic reason of this result will be given in Sec. III J 1.

5. Tolman and Klein relations

Combining Eqs. (147) and (156), we get

$$
\frac{1}{T} \frac{dT}{dr} = -\frac{1}{2} \frac{d\nu}{dr}.
$$

(157)
Integrating this equation with respect to $r$, we obtain the Tolman relation between the local temperature and the metric coefficient

$$T(r) = T_\infty e^{-\nu(r)/2},$$

(158)

where $T_\infty$ is a constant of integration. Since $\nu(r) \to 0$ when $r \to +\infty$ according to Eq. (108), we see that $T_\infty$ represents the temperature measured by an observer at infinity. For brevity we will call it the Tolman temperature (or the global temperature). Using Eq. (158), we get the Klein relation

$$\mu(r) = \mu_\infty e^{-\nu(r)/2},$$

(159)

where $\mu_\infty \equiv \alpha k_B T_\infty$ is a constant representing the chemical potential measured by an observer at infinity. For brevity we will call it the Klein chemical potential (or the global chemical potential). For future reference, we note that

$$\alpha = \frac{\mu(r)}{k_B T(r)} = \frac{\mu_\infty}{k_B T_\infty},$$

(160)

where $T(r)$ and $\mu(r)$ are the local temperature and the local chemical potential while $T_\infty$ and $\mu_\infty$ are the global temperature and the global chemical potential. Applying the Tolman relation (158) at the edge of the system, we get

$$T_\infty = T(R)e^{\nu(R)/2}.$$  

(161)

Using the value of $\nu(R)$ from Eq. (109), we obtain

$$T_\infty = T(R)\sqrt{1 - \frac{2GM}{Rc^2}}.$$  

(162)

Comparing this relation with Eq. (152), we conclude that

$$\beta_0 = \frac{1}{k_B T_\infty} \equiv \beta_\infty.$$  

(163)

Therefore, the Lagrange multiplier $\beta_0$ is equal to the inverse of the Tolman temperature. From now on, we shall write $\beta_\infty$ instead of $\beta_0$.

### 6. Entropy

Using Eqs. (160), the entropy density (135) can be rewritten as

$$s(r) = \frac{\epsilon(r) + P(r)}{T(r)} - \frac{\mu_\infty}{T_\infty} n(r).$$  

(164)

Integrating Eq. (164) over the whole configuration, we find that the entropy is given at statistical equilibrium by

$$S = \int_0^R \frac{P(r) + \epsilon(r)}{T(r)} \left[1 - \frac{2GM(r)}{rc^2}\right]^{-1/2} 4\pi r^2 dr - \frac{\mu_\infty}{T_\infty} N.$$  

(165)

We emphasize that the results derived in this section are valid for an arbitrary form of entropy.

### 7. About the derivation of the TOV equations from thermodynamics

It seems that we have obtained the OV equation (155) from a thermodynamical variational principle without using the Einstein field equations (101)-(103). On this account, it has sometimes been suggested in the literature that the Einstein equations could be derived from thermodynamics (see the conclusion). This is, however, not quite true in the present context for the following reasons:

(i) In order to write the total entropy (122) and the total particle number (123), we need the expression of the metric coefficient $\lambda(r)$ that appears in the proper volume element. This is how gravitational effects ($G$) arise in
the thermodynamical variational principle. We have assumed that $\lambda(r)$ is given by Eq. (105). Therefore, we have implicitly used the Einstein equation (101).

(ii) Under the above assumption, the maximum entropy principle yields the OV equation (155). This equation is actually all that we need to determine the equilibrium state of a spherical system (see Secs. III E and III F). In particular, we do not need the metric function $\nu(r)$ – the equivalent of the gravitational potential $\Phi(r)$ in Newtonian gravity. However, the Einstein equations contain more information than just the OV equation. For example, in order to derive the Tolman equation (156) from the OV equation (155), we need to use Eq. (106) which arises from the Einstein equations (101) and (103). This equation relates the metric function $\nu(r)$ to the cumulated mass $M(r)$. This is the generalization of Newton’s law [3]. This important equation cannot be derived from the present thermodynamical approach. Therefore, we have not derived the whole set of Einstein equations (101)-(105). Our point of view is that the maximum entropy principle just yields the condition of hydrostatic equilibrium, not the whole set of Einstein equations.

Similarly, in Newtonian gravity, if we use the expression (B2) of the gravitational energy (which is deduced from the Newton equations for a spherically symmetric system), the maximum entropy principle yields the condition of hydrostatic equilibrium under the form of Eq. (7) – the Newtonian analogue of the OV equation (155). This equation is actually all that we need to determine the equilibrium state of a spherical system. In particular, we do not need the gravitational potential $\Phi(r)$. However, if we use the Newton law (3), which is equivalent to the Poisson equation (1), we get the condition of hydrostatic equilibrium under the form of Eq. (1) – the Newtonian analogue of the Tolman equation (156). The Poisson equation – the Newtonian analogue of the Einstein equations (15) – cannot be derived from the present thermodynamical approach. Again, our point of view is that the maximum entropy principle just yields the condition of hydrostatic equilibrium, not the whole set of Newton equations.

Actually, the fact that the maximum entropy principle yields the condition of hydrostatic equilibrium is a very general result. Indeed, it can be shown that thermodynamical stability implies dynamical stability (see Appendix II). More precisely, an extremum of entropy at fixed energy and particle number is a steady state of the Vlasov equation (furthermore an entropy maximum is dynamically stable). Therefore, it satisfies the condition of hydrostatic equilibrium. For spherically symmetric systems in general relativity, the condition of hydrostatic equilibrium takes the form of the TOV equations. In this sense, it is not surprising that the maximum entropy principle yields the TOV equations. This is not, in our opinion, the manifestation of a fundamental relationship between general relativity and thermodynamics, although such a relationship could arise in some theories of emergent gravity as discussed in the conclusion.

F. Canonical ensemble: Minimization of the free energy at fixed particle number

In the previous sections, we worked in the microcanonical ensemble in which the particle number and the energy are fixed. We now consider the canonical ensemble where the system is in contact with a heat bath fixing the temperature $T$. In that case, the relevant thermodynamical potential is the free energy

$$ F = Mc^2 - T_\infty S, \quad (166) $$

where $T_\infty = 1/k_B\beta_\infty$ is the (constant) temperature of the thermal bath.24 In the canonical ensemble, the statistical equilibrium state of the system is obtained by minimizing the Fermi-Dirac free energy $F$ at fixed particle number $N$:

$$ \min \{ F \mid N \text{ fixed} \}. \quad (167) $$

This determines the “most probable” state of a system in contact with a thermal bath.

Minimizing the free energy $F = Mc^2 - T_\infty S$ at fixed $N$ is equivalent to maximizing the Massieu function $J = S/k_B - \beta_\infty Mc^2$ at fixed $N$ (the Massieu function is the Legendre transform of the entropy with respect to the mass-energy). To solve this maximization problem, we proceed in two steps. We first maximize the Massieu function $J$ at fixed energy density $\epsilon(r)$ and particle number density $n(r)$ under variations of $f(r, v)$. Since the energy density $\epsilon(r)$ determines the mass-energy $M$, this is equivalent to maximizing the entropy $S$ at fixed $\epsilon(r)$ and $n(r)$. This returns the results of Sec. III D. Using these results, we can express the Massieu function $J = S/k_B - \beta_\infty Mc^2$ in terms of $\epsilon(r)$ and $n(r)$. We now maximize $J$ at fixed particle number $N$ under variations of $\epsilon(r)$ and $n(r)$. The first variations

24 We have seen in Sec. III E that the inverse Tolman temperature $\beta_\infty$ is the conjugate variable to the energy. Therefore, this is the quantity to keep constant in the canonical ensemble.
(extremization) can be written as

$$\delta \left( \frac{S}{k_B} - \beta_\infty M c^2 \right) + \alpha_0 \delta N = 0.$$  

(168)

Since $\beta_\infty$ is a constant, this variational problem is equivalent to Eq. (139) so we get the same results as in Sec. III E (for the first variations).

In order to correctly recover the nonrelativistic results in the limit $c \to +\infty$ (see Sec. IV G), it is better to define the free energy by

$$F = E - T_\infty S,$$  

(169)

where $E$ is the binding energy defined by Eq. (124). This is possible since $N$ is fixed. Using Eqs. (124) and (165), we find that the free energy (169) is given at statistical equilibrium by

$$F = (M - N m) c^2 + \mu_\infty N - \int_0^R \frac{T_\infty}{T(r)} (P(r) + \epsilon(r)) \left[ 1 - \frac{2GM(r)}{rc^2} \right]^{-1/2} 4\pi r^2 dr.$$  

(170)

G. Equations determining the statistical equilibrium state in terms of $T(r)$

In this section, we present the full set of equations determining the statistical equilibrium state of self-gravitating fermions within the context of general relativity. We express the results in terms of the local temperature $T(r)$.

1. Local variables in terms of $T(r)$ and $\alpha$

Using Eq. (145), we can rewrite the distribution function (129) and the local variables (131)-(134) as

$$f(r, p) = \frac{g}{h^3} \frac{1}{1 + e^{-\alpha E(p)/k_B T(r)}},$$  

(171)

$$n(r) = \frac{g}{h^3} \int \frac{1}{1 + e^{-\alpha E(p)/k_B T(r)}} dp,$$  

(172)

$$\epsilon(r) = \frac{g}{h^3} \int \frac{E(p)}{1 + e^{-\alpha E(p)/k_B T(r)}} dp,$$  

(173)

$$\epsilon_{\text{kin}}(r) = \frac{g}{h^3} \int \frac{E_{\text{kin}}(p)}{1 + e^{-\alpha E(p)/k_B T(r)}} dp,$$  

(174)

$$P(r) = \frac{g}{3h^3} \int \frac{1}{1 + e^{-\alpha E(p)/k_B T(r)}} dp = \frac{g}{3h^3} k_B T(r) \int \ln \left[ 1 + e^{\alpha E(p)/k_B T(r)} \right] dp,$$  

(175)

where we recall that $\alpha$ is a constant while the temperature $T(r)$ depends on the position (Tolman’s effect). From these equations, we have $n(r) = n[\alpha, T(r)]$, $\epsilon(r) = \epsilon[\alpha, T(r)]$, $\epsilon_{\text{kin}}(r) = \epsilon_{\text{kin}}[\alpha, T(r)]$ and $P(r) = P[\alpha, T(r)]$ leading to a barotropic equation of state of the form $P(r) = P[\alpha, T(r)]$. Taking the derivative of $P(r)$ with respect to $r$, we recover Eq. (147). We show in Appendix D that this result is valid for an arbitrary form of entropy.

2. The TOV equations in terms of $T(r)$

The TOV equations can be written in terms of $T(r)$ as

$$\frac{dM}{dr} = \frac{\epsilon}{c^2} 4\pi r^2,$$  

(176)
with the boundary conditions

\[ M(0) = 0 \quad \text{and} \quad T(0) = T_0. \]

For a given value of \( \alpha \) and \( T_0 \) we can solve Eqs. (176) and (177) between \( r = 0 \) and \( r = R \) with the local variables \( \frac{GM(r)}{rc^2} \). The particle number constraint

\[ N = \int_0^R n(r) \left[ 1 - \frac{2GM(r)}{rc^2} \right]^{-1/2} 4\pi r^2 dr \]

can be used to determine \( T_0 \) as a function of \( \alpha \) (there may be several solutions for the same value of \( \alpha \)). The mass \( M \) and the temperature measured by an observer at infinity \( T_\infty \) are then obtained from the relations

\[ M = M(R) \quad \text{and} \quad T_\infty = T(R) \sqrt{1 - \frac{2GM}{Rc^2}}. \]

In this manner, we get the binding energy \( E = (M - Nm)c^2 \) and the Tolman temperature \( T_\infty \) as a function of \( \alpha \). By varying \( \alpha \) between \(-\infty \) and \(+\infty \), we can obtain the full caloric curve \( T_\infty(E) \) for a given value of \( N \) and \( R \). Finally, the entropy and the free energy are given by Eqs. (165) and (170). Phase transitions and instabilities in the general relativistic Fermi gas have been studied in \( [2, 123, 130] \). The complete canonical and microcanonical phase diagrams are given in \( [1] \).

Remark: We can interpret Eq. (177) in different manners: (i) We can consider that the temperature is related to the pressure by Eq. (147); then Eqs. (154)-(157) describe an equilibrium between the pressure and the temperature measured by an observer at infinity \( T_\infty \). (ii) We can consider that the temperature is a measure of the gravitational potential or the gravitational “force”. This is the correct interpretation in the case of the black-body radiation (see Appendix I). (iii) We can consider that the temperature is related to the pressure by Eq. (147); then Eqs. (154)-(157) describe the balance between the pressure – or temperature – gradient and the gravitational force. This interpretation is suggested by the post-Newtonian approximation of Sec. IV B. More generally, we can consider that, at statistical equilibrium, a temperature gradient forms to balance the weight of matter and heat.

**H. Equations determining the statistical equilibrium state in terms of \( \varphi(r) \)**

In this section we reformulate the previous results in terms of the gravitational potential \( \varphi(r) \). This formulation will allow us, in particular, to correctly recover the completely degenerate limit \( (T = 0) \) in Sec. III K 1.

**1. Gravitational potential \( \varphi(r) \)**

As noted by Tolman \( [232] \), Eq. (156) may be interpreted as the condition of hydrostatic equilibrium in general relativity (see Sec. III A 2). The left hand side is the pressure gradient and the metric coefficient on the right hand side plays the role of the gravitational potential in Newtonian gravity. Instead of working with the metric coefficient \( \nu(r) \) it is convenient to introduce the general relativistic gravitational potential \( \varphi(r) \) defined by

\[ e^{\nu(r)} = \left( \frac{\mu_\infty}{mc^2} \right)^2 \frac{1}{1 + \frac{\varphi(r)}{c^2}} \quad \Rightarrow \quad \nu(r) = -\ln \left( 1 + \frac{\varphi(r)}{c^2} \right) + 2 \ln \left( \frac{\mu_\infty}{mc^2} \right). \]

It satisfies \( \varphi(r) > -c^2 \). Using Eq. (160), we have equivalently

\[ e^{\nu(r)} = \left( \frac{\alpha k_BT_\infty}{mc^2} \right)^2 \frac{1}{1 + \frac{\varphi(r)}{c^2}} \quad \Rightarrow \quad \nu(r) = -\ln \left( 1 + \frac{\varphi(r)}{c^2} \right) + 2 \ln \left( \frac{\alpha k_BT_\infty}{mc^2} \right). \]

We can also relate the gravitational potential \( \varphi(r) \) to the temperature \( T(r) \). Combining Eq. (181) with the Tolman-Klein relations \( [158] \) and \( [159] \) we obtain

\[ k_BT(r) = \frac{mc^2}{|\alpha|} \sqrt{\frac{\varphi(r)}{c^2} + 1} \quad \text{and} \quad |\mu(r)| = mc^2 \sqrt{\frac{\varphi(r)}{c^2} + 1}. \]
Finally, using Eq. [109] or Eq. [162], we find that $\varphi(R)$ is determined by the relation

$$\frac{k_B T_\infty}{mc^2} = \frac{1}{|\alpha|} \sqrt{\frac{\varphi(R)}{c^2}} + 1 \left(1 - \frac{2GM}{Rc^2}\right)^{1/2}.$$  

(184)

2. Local variables in terms of $\varphi(r)$

Using Eq. [183], we can rewrite the distribution function (171) and the local variables (172)-(175) in terms of $\alpha$ and $\varphi(r)$ as

$$f(r,p) = \frac{g}{\hbar^3} \frac{1}{1 + e^{-\alpha} e^{m c^2 \sqrt{1 + \frac{\varphi(r)}{c^2}}}},$$

(185)

$$n(r) = \frac{g}{\hbar^3} \int \frac{1}{1 + e^{-\alpha} e^{m c^2 \sqrt{1 + \frac{\varphi(r)}{c^2}}}} dp,$$

(186)

$$\epsilon(r) = \frac{g}{\hbar^3} \int \frac{E(p)}{1 + e^{-\alpha} e^{m c^2 \sqrt{1 + \frac{\varphi(r)}{c^2}}}} dp,$$

(187)

$$\epsilon_{kin}(r) = \frac{g}{\hbar^3} \int \frac{E_{kin}(p)}{1 + e^{-\alpha} e^{m c^2 \sqrt{1 + \frac{\varphi(r)}{c^2}}}} dp,$$

(188)

$$P(r) = \frac{g}{3\hbar^3} \int \frac{1}{1 + e^{-\alpha} e^{m c^2 \sqrt{1 + \frac{\varphi(r)}{c^2}}}} \frac{dE}{dp} dp = \frac{g}{\hbar^3} \frac{mc^2}{|\alpha|} \sqrt{1 + \frac{\varphi(r)}{c^2}} \int \ln \left(1 + e^{\alpha} e^{-\frac{|\alpha| E(p)}{mc^2 \sqrt{1 + \frac{\varphi(r)}{c^2}}}} \right) dp.$$  

(189)

From these equations, we have $n(r) = n[\alpha, \varphi(r)], \epsilon(r) = \epsilon[\alpha, \varphi(r)], \epsilon_{kin}(r) = \epsilon_{kin}[\alpha, \varphi(r)]$ and $P(r) = P[\alpha, \varphi(r)]$ leading to an equation of state of the form $P(r) = P[\alpha, \epsilon(r)]$.

3. The TOV equations in terms of $\varphi(r)$

Taking the derivative of Eq. [182] with respect to $r$ we obtain

$$\frac{dv}{dr} = -\frac{1}{1 + \frac{\varphi(r)}{c^2}} \frac{1}{c^2} \frac{d\varphi}{dr}.$$  

(190)

Substituting this relation into Eq. [156] we get

$$\frac{dP}{dr} = \frac{1}{2c^2} (\epsilon + P) \frac{1}{1 + \frac{\varphi(r)}{c^2}} \frac{d\varphi}{dr}.$$  

(191)

Alternatively, substituting Eq. [190] into Eq. [157] we obtain

$$\frac{1}{T} \frac{dT}{dr} = \frac{1}{2c^2} \frac{1}{1 + \frac{\varphi(r)}{c^2}} \frac{d\varphi}{dr}.$$  

(192)

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25 This equation can be directly obtained by taking the gradient of Eq. [189].
The TOV equations can be written in terms of $\varphi(r)$ as
\[
\frac{dM}{dr} = \frac{\epsilon}{c^2} 4 \pi r^2 ,
\]
(193)
\[
\frac{d\varphi}{dr} = -2 \left[ 1 + \frac{\varphi(r)}{c^2} \right] \left( \frac{GM(r)}{r^2} + \frac{4 \pi G P r}{2GM(r)} \right) ,
\]
(194)
with the boundary conditions
\[
M(0) = 0 \quad \text{and} \quad \varphi(0) = \varphi_0 > -c^2 .
\]
(195)
For a given value of $\alpha$ and $\varphi_0$ we can solve Eqs. (193) and (194) between $r = 0$ and $r = R$ with the local variables \[186]-[189]. The particle number constraint
\[
N = \int_0^R n(r) \left[ 1 - \frac{2GM(r)}{rc^2} \right]^{-1/2} 4 \pi r^2 dr
\]
(196)
can be used to determine $\varphi_0$ as a function of $\alpha$ (there may be several solutions for the same value of $\alpha$). The mass $M$ and the temperature measured by an observer at infinity $T_\infty$ are then obtained from the relations
\[
M = M(R) \quad \text{and} \quad \frac{k_B T_\infty}{mc^2} = \frac{1}{|\alpha|} \sqrt{\frac{\varphi(R)}{c^2}} + 1 \left( 1 - \frac{2GM}{Rc^2} \right)^{1/2} .
\]
(197)
In this manner, we get the binding energy $E = (M - Nm)c^2$ and the Tolman temperature $T_\infty$ as a function of $\alpha$. By varying $\alpha$ between $-\infty$ and $+\infty$, we can obtain the full caloric curve $T_\infty(E)$ for a given value of $N$ and $R$. Finally, the entropy and the free energy are given by Eqs. [165] and [170] where $T(r)$ is related to $\varphi(r)$ by Eq. [183].

I. Thermodynamical stability and ensembles inequivalence

As discussed in Sec. [H], the statistical equilibrium states in the microcanonical and canonical ensembles (extrema of entropy or free energy) are the same. However, their thermodynamical stability may be different in the microcanonical and canonical ensembles. This corresponds to the concept of ensembles inequivalence for systems with long-range interactions [209, 210].

The thermodynamical stability of an equilibrium state can be investigated by studying the sign of the second order variations of the appropriate thermodynamic potential (entropy or free energy) and reducing this study to an eigenvalue problem. We refer to [241, 249, 254, 257, 258, 260, 263] for a detailed discussion of this stability problem in general relativity.

The thermodynamical stability of the system can also be directly settled from the topology of the series of equilibria by using the Poincaré criterion. The discussion is the same as in Sec. [H] provided that $E$ is replaced by $\mathcal{E} = Mc^2$ or $E = (M - Nm)c^2$ and $\beta$ is replaced by $\beta_\infty$. In short, a change of microcanonical stability can take place at a turning point of energy and a change of canonical stability can take place at a turning point of temperature. We refer to [2, 123, 130, 247, 248, 254, 285] for some applications of the Poincaré criterion in general relativity.

J. Dynamical stability

1. Vlasov-Einstein equations

The distribution function $f(r, p)$ of a system of self-gravitating fermions at statistical equilibrium is given by Eq. [171]. Using the Tolman relation [158], it is of the form $f = f(E e^{\nu(r)/2})$ with $f'(E e^{\nu(r)/2}) < 0$. Therefore, at statistical equilibrium, the distribution function depends only on the energy at infinity $E e^{\nu(r)/2}$ and is monotonically decreasing. It is shown in Appendices [C] and [D] that these properties remain valid for a general form of entropy. Since $f(r, p)$ is a function of the energy at infinity $E e^{\nu(r)/2}$, which is a constant of the motion, it is a particular steady
state of the Vlasov-Einstein equations. This is a special case of the relativistic Jeans theorem.\textsuperscript{26} Therefore, a statistical equilibrium state (extremum of entropy at fixed energy and particle number) is a steady state of the Vlasov-Einstein equations. Furthermore, it can be shown that thermodynamical stability implies dynamical stability with respect to the Vlasov-Einstein equations.\textsuperscript{248} Therefore, a stable thermodynamical equilibrium state (maximum of entropy at fixed energy and particle number) is always dynamically stable.\textsuperscript{27} In general, the reciprocal is wrong. However, in general relativity, Ipser\textsuperscript{248} has shown that dynamical and thermodynamical stability (in the microcanonical ensemble) coincide (see Appendix II). As a result, using the Poincaré criterion (see Sec. III), we generically conclude that the series of equilibria before the first turning point of energy is dynamically stable while it becomes dynamically unstable afterwards. This is in sharp contrast with the Newtonian gravity case where all the statistical equilibrium states are dynamically stable, even those that are thermodynamically unstable (see Sec. III). To solve this apparent paradox, one expects that the growth rate $\lambda$ of the dynamical instability decreases as relativity effects decrease and that it tends to zero in the nonrelativistic limit $c \rightarrow +\infty$.

2. Euler-Einstein equations

We have seen that the statistical equilibrium state of a system of self-gravitating fermions in general relativity is described by a barotropic equation of state of the form $P(r) = P[e(\nu)]$ and that it satisfies the TOV equation\textsuperscript{155} expressing the condition of hydrostatic equilibrium. It is shown in Appendices C and D that these properties remain valid for an arbitrary form of entropy. As a result, the system is in a steady state of the Euler-Einstein equations. Furthermore, it can be shown that the thermodynamical stability of a self-gravitating system in the canonical ensemble is equivalent to its dynamical stability with respect to the Euler-Einstein equations. This generalizes the result obtained by\textsuperscript{219} in Newtonian gravity (see Sec. III).

K. Particular limits

The Fermi-Dirac distribution\textsuperscript{185} can be written as

$$f(\mathbf{r}, \mathbf{p}) = \frac{g}{\hbar^3} \frac{1}{1 + e^{(E(\mathbf{p}) - \mu(\mathbf{r}))/k_B T(\mathbf{r})}},$$

(198)

where $E(p) = \sqrt{p^2 c^2 + m^2 c^4}$ and $|\mu(r)| = mc^2 \sqrt{\varphi(r)/c^2 + 1}$. Let us consider particular limits of this distribution function.

1. The completely degenerate Fermi gas (ground state)

The completely degenerate limit corresponds to $T(\mathbf{r}) \rightarrow 0$, $\mu(\mathbf{r}) > 0$ finite, and $\alpha(\mathbf{r}) = \mu(\mathbf{r})/k_B T(\mathbf{r}) \rightarrow +\infty$. In that case, the chemical potential is positive and large compared to the temperature. This yields the Fermi distribution (or Heaviside function):

$$f(\mathbf{r}, \mathbf{p}) = \frac{g}{\hbar^3} \quad \text{if} \quad E(p) < E_F(\mathbf{r}) \quad (p < p_F(\mathbf{r})), \quad (199)$$

and

$$f(\mathbf{r}, \mathbf{p}) = 0 \quad \text{if} \quad E(p) > E_F(\mathbf{r}) \quad (p > p_F(\mathbf{r})), \quad (200)$$

\textsuperscript{26} According to the relativistic Jeans theorem, a spherical stellar system in collisionless equilibrium has a distribution function of the form $f = f(E e^{\nu(r)/2}, L)$ where $E e^{\nu(r)/2}$ is the energy at infinity and $L$ is the angular momentum. We note that an extremum of entropy $S$ at fixed mass-energy $Mc^2$ and particle number $N$ leads to a distribution function that depends only on $E e^{\nu(r)/2}$, not on $L$. Therefore, an extremum of entropy at fixed mass-energy and particle number is necessarily isotropic. We also note that the relativistic Jeans theorem implies the Tolman relation\textsuperscript{155} when $f$ is the Maxwell-Boltzmann distribution.

\textsuperscript{27} This result is very general and stems from the fact that the entropy (which is a particular Casimir), the mass-energy and the particle number are conserved by the Vlasov-Einstein equations (see\textsuperscript{223} and Appendix III).
where
\[ E_F(r) \equiv \mu(r) = mc^2 \sqrt{\frac{\varphi(r)}{c^2}} + 1, \tag{201} \]
and
\[ p_F(r) = m \sqrt{\varphi(r)} \tag{202} \]
are the Fermi energy and the Fermi impulse. We note that Eq. (202) imposes the condition \( \varphi \geq 0 \). Setting
\[ x = \frac{p_F}{mc} = \frac{\sqrt{\varphi(r)}}{c}, \tag{203} \]
and using the results of \[132\] we find that the equation of state of the relativistic Fermi gas at \( T = 0 \) is given by
\[ n = \frac{4 \pi g m^3 c^3}{3 \hbar^3} x^3, \tag{204} \]
\[ \epsilon = \frac{\pi g m^4 c^5}{2 \hbar^3} \left[ x(2x^2 + 1)(1 + x^2)^{1/2} - \sinh^{-1}(x) \right], \tag{205} \]
\[ P = \frac{\pi g m^4 c^5}{6 \hbar^3} \left[ x(2x^2 - 3)(1 + x^2)^{1/2} + 3 \sinh^{-1}(x) \right]. \tag{206} \]
For a given value of \( n_0 = n(0) \), we can solve the TOV equations \([19]\) and \([14]\) with the equation of state \([201, 200]\) until the point where the density vanishes: \( n(R) = 0 \). This determines the radius \( R \) of the configuration (when \( T = 0 \) we do not need a box to confine the system). We can then compute the corresponding mass \( M \) and particle number \( N \). By varying \( n_0 \), we get the mass-radius relation \( M(R) \). The mass-radius relation and the corresponding density profiles of the general relativistic Fermi gas at \( T = 0 \) have been obtained by Oppenheimer and Volkoff \([33]\) and Chandrasekhar and Tooper \([27]\). These results can also be obtained by solving Eqs. \([193, 194]\) where \( \epsilon \) and \( P \) are expressed in terms of \( \varphi \) by using Eqs. \([205, 206]\) with Eq. \([203]\) \([2, 123]\).

**Remark:** For the general relativistic Fermi gas at \( T = 0 \), the free energy \( F \) reduces to the binding energy \( E = (M - Nm)c^2 \). Therefore, a stable equilibrium state at \( T = 0 \) is a minimum of \( E \) at fixed \( N \) or, equivalently, a minimum of \( M \) at fixed \( N \). We have seen that the general relativistic Fermi gas at \( T = 0 \) is described by a barotropic equation of state of the form \( P = P(\epsilon) \) determined in parametric form by Eqs. \([203, 206]\). Therefore, for the self-gravitating Fermi gas at \( T = 0 \), the condition of thermodynamical stability coincides with the condition of dynamical stability with respect to the Euler-Einstein equations (see Appendix C.2). Actually, this equivalence is valid at arbitrary temperature in the canonical ensemble (see Sec. III.J.2).

2. The nondegenerate Fermi gas (classical limit)

The nondegenerate (classical) limit corresponds to
\[ \frac{E(p) - \mu(r)}{k_BT(r)} \gg 1 \quad i.e. \quad \beta(r)E(p) - \alpha \gg 1. \tag{207} \]
This yields the Maxwell-Juttner distribution
\[ f(r, p) = \frac{g}{\hbar^3} e^{-[E(p) - \mu(r)]/k_BT(r)}. \tag{208} \]
The condition \( \alpha \rightarrow -\infty \), whatever the value of \( \beta(r)E(p) \). Therefore, the condition \( \mu(r) \rightarrow -\infty, \ T(r) \) finite and \( \alpha = \mu(r)/k_BT(r) \rightarrow -\infty \) implies the nondegenerate (classical) limit. In that case, the chemical potential is negative and large (in absolute value) compared to the temperature. However, this is not the only case where the Boltzmann distribution is valid. We can be in the classical limit for arbitrary values of \( \alpha \) (positive or negative) provided that \( \beta(r)E(p) - \alpha \gg 1 \). The classical limit is specifically studied in Paper II.
IV. THE NONRELATIVISTIC LIMIT

A. Relation between $\nu$ and $\Phi$

The condition of hydrostatic equilibrium in general relativity is given by the Tolman equation (102) where the metric coefficient $\nu(r)$ plays the role of the gravitational potential $\Phi(r)$ in Newtonian gravity. In the nonrelativistic limit $c \to +\infty$, using $\epsilon \simeq \rho c^2 \gg P$, it reduces to

$$\frac{dP}{dr} \simeq -\frac{1}{2} \rho c^2 \frac{d\nu}{dr}. \quad (209)$$

Comparing this equation with the condition of hydrostatic equilibrium in Newtonian gravity given by Eq. (5), we find that

$$\nu(r) \simeq \frac{2\Phi(r)}{c^2} + C, \quad (210)$$

where $C$ is a constant. On the other hand, according to Eq. (109) we have

$$\nu(R) \simeq -\frac{2GM}{Rc^2}. \quad (211)$$

Comparing Eqs. (210) and (211) with Eq. (5), we see that the constant $C$ in Eq. (210) is equal to zero. Therefore, we get

$$\nu(r) \simeq \frac{2\Phi(r)}{c^2} \quad \text{or} \quad e^{\nu(r)} \simeq 1 + \frac{2\Phi(r)}{c^2}. \quad (212)$$

On the other hand, in the nonrelativistic limit $c \to +\infty$, using $\epsilon \simeq \rho c^2 \gg P$, the OV equations (100) and (104) reduce to the Newtonian equations (2) and (7). Then, Eqs. (6) and (7) can be combined to recover the Newton law (3). This equation is also directly obtained from Eq. (106) in the limit $c \to +\infty$. Finally, Eqs. (2) and (3) return the Poisson equation (1).

Remark: The relation (212) and the Poisson equation $\Delta \Phi = 4\pi G\rho$ can be obtained directly from the Einstein equations in the nonrelativistic limit $c \to +\infty$. The metric takes the form

$$ds^2 = (c^2 + 2\Phi)dt^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - dr^2, \quad (213)$$

which is a consequence of the correspondence principle [237, 290].

B. Tolman-Klein relations in the post-Newtonian approximation

From Eqs. (158), (159) and (212) we find that the local temperature and the local chemical potential are given in the post-Newtonian approximation by

$$\frac{T(r)}{T_\infty} \simeq \frac{\mu(r)}{\mu_\infty} \simeq 1 - \frac{\Phi(r)}{c^2} + O(1/c^4). \quad (214)$$

If $g = -\nabla \Phi$ denotes the gravitational force by unit of mass (acceleration) one has

$$\frac{\nabla T}{T} = \frac{g}{c^2}. \quad (215)$$

This relation was first given by Tolman [232] as a preamble of his general result (158). In the nonrelativistic limit $c \to +\infty$, using Eq. (214) and $\epsilon \simeq \rho c^2 \gg P$, we find that Eq. (147) reduces to Eq. (6). On the other hand, the TOV equations (176) and (177) reduce to Eq. (15). The Tolman relation (214) clearly shows that, in the post-Newtonian approximation, the gravitational potential $\Phi(r)$ is “hidden” in the inhomogeneous temperature $T(r)$ (see also the Remark at the end of Sec. III G 2).
C. Distribution function and $\alpha$

The distribution function of a gas of self-gravitating fermions at statistical equilibrium is given by Eq. (171), where the energy of a particle is given by Eq. (115). In the nonrelativistic limit $c \to +\infty$, we have

$$E(p) \simeq mc^2 + \frac{p^2}{2m} + O(1/c^2).$$  \hspace{1cm} (216)

Using Eq. (214), we get

$$\frac{E(p)}{k_B T(r)} \simeq \frac{1}{k_B T_\infty} \left[ mc^2 + \frac{p^2}{2m} + m\Phi(r) \right].$$  \hspace{1cm} (217)

Therefore, the distribution function (171) becomes

$$f(r, p) \simeq \frac{g}{h^3} \frac{1}{1 + e^{-\alpha mc^2/k_B T_\infty} e^{p^2/2m k_B T_\infty} e^{m\Phi(r)/k_B T_\infty}}.$$  \hspace{1cm} (218)

In order to obtain an expression independent of $c$ and consistent with the expression (65) obtained in Newtonian gravity, i.e.,

$$f(r, p) = \frac{g}{h^3} \frac{1}{1 + e^{-\alpha_{NR}^0} e^{(p^2/2m + m\Phi(r))/k_B T_\infty}},$$  \hspace{1cm} (219)

we have to write\(^{28}\)

$$\alpha = \frac{mc^2}{k_B T_\infty} + \alpha_{NR}^0 + O(1/c^2).$$  \hspace{1cm} (220)

We can then define

$$\alpha_{NR}(r) = \alpha_{NR}^0 - \beta_\infty m\Phi(r)$$  \hspace{1cm} (221)

in agreement with Eq. (55).

D. Chemical potential

From Eqs. (160) and (220) we get

$$\mu_\infty = mc^2 + \mu_{NR}^0 + O(1/c^2)$$  \hspace{1cm} (222)

with

$$\mu_{NR}^0 = \alpha_{NR}^0 k_B T_\infty.$$  \hspace{1cm} (223)

Using Eqs. (214) and (222), we obtain

$$\mu(r) = mc^2 + \mu_{NR}^0 - m\Phi(r) + O(1/c^2).$$  \hspace{1cm} (224)

Therefore, we can write\(^{29}\)

$$\mu(r) = mc^2 + \mu_{NR}(r) + O(1/c^2)$$  \hspace{1cm} (225)

with

$$\mu_{NR}(r) = \mu_{NR}^0 - m\Phi(r)$$  \hspace{1cm} (226)

in agreement with Eq. (55).

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\(^{28}\) We note that $\alpha \sim mc^2/k_B T \to +\infty$ in the nonrelativistic limit $c \to +\infty$ (i.e. $k_B T \ll mc^2$), and is therefore positive, while $\alpha_{NR}^0$ may be of any sign.

\(^{29}\) We note that $\mu(r) \sim mc^2 \to +\infty$ in the nonrelativistic limit $c \to +\infty$, and is therefore positive, while $\mu_{NR}(r)$ may be of any sign.
E. Relation between $\varphi$ and $\Phi$

Combining Eqs. (182) and (220), we get

$$\nu(r) \simeq -\frac{\varphi(r)}{c^2} + \frac{2\alpha_{0}^{NR}k_{B}T_{\infty}}{mc^2}. \quad (227)$$

Comparing this equation with Eq. (212), we find that

$$\varphi(r) \simeq -2\Phi(r) + \frac{2\alpha_{0}^{NR}k_{B}T_{\infty}}{m}. \quad (228)$$

Using Eq. (228), we note that Eq. (191) reduces to the Newtonian condition of hydrostatic equilibrium (6) when $c \to +\infty$. On the other hand, the TOV equations (193) and (194) reduce to Eq. (15).

F. Mass, particle number and energy

Using Eq. (118), the mass-energy (100) can be written as

$$Mc^2 = \int \rho c^2 dV + \int \epsilon_{\text{kin}} dV. \quad (229)$$

Here, no approximation has been made. On the other hand, for $c \to +\infty$, using the approximation

$$\left[1 - \frac{2GM(r)}{rc^2}\right]^{-1/2} \simeq 1 + \frac{GM(r)}{rc^2} + O(1/c^4) \quad (230)$$

the rest mass (123) can be written as

$$Nmc^2 = \int \rho c^2 dV + \int \rho \frac{GM(r)}{r} dV + O(1/c^2). \quad (231)$$

Therefore, in the nonrelativistic limit $c \to +\infty$, the binding energy (124) takes the form

$$E = Mc^2 - Nmc^2 = \int \epsilon_{\text{kin}} dV - \int \rho \frac{GM(r)}{r} dV = E_{\text{kin}} + W \quad (c \to +\infty), \quad (232)$$

where we have used the expression (132) of the Newtonian gravitational energy valid for a spherically symmetric distribution of matter. As a result, in the nonrelativistic limit $c \to +\infty$, the binding energy (124) reduces to the Newtonian energy which is equal to the sum of the kinetic and potential energies.

G. Entropy and free energy

In the nonrelativistic limit $c \to +\infty$, using Eqs. (118), (214), (222), (230) and (B2), we find that the entropy (165) takes the form

$$T_{\infty}S = -N\mu_{0}^{NR} + 2W + \int P dV + E_{\text{kin}} \quad (c \to +\infty). \quad (233)$$

Using Eqs. (232) and (233), the free energy reduces to

$$F = E - T_{\infty}S = N\mu_{0}^{NR} - W - \int P dV \quad (c \to +\infty). \quad (234)$$

Recalling Eq. (A3), we recover the expressions (60) and (64) obtained in the Newtonian approach.
V. CONCLUSION

In this paper, elaborating upon previous works on the subject, we have developed a general formalism to determine the statistical equilibrium state of a system of particles in general relativity. Although we have considered a gas of fermions described by the Fermi-Dirac entropy for illustration, our formalism is valid for an arbitrary form of entropy. This shows that the notion of “generalized thermodynamics” developed in recent years in statistical mechanics can be extended to the context of general relativity. For spherically symmetric systems, the extremization of the entropy $S$ at fixed mass-energy $Mc^2$ and particle number $N$ yields the TOV equations expressing the condition of hydrostatic equilibrium and the Tolman-Klein relations (155) and (156). In Newtonian gravity, the maximum entropy principle implies the uniformity of the temperature [see Eq. (44)] and of the total chemical potential (Gibbs law) [see Eq. (55)], and the condition of hydrostatic equilibrium (58).

Research in general relativity has shown that there is a deep connection between gravitation and thermodynamics. It is reflected, e.g., in the thermodynamical interpretation given by Bekenstein and Hawking of the four laws of black hole mechanics which were derived from the Einstein equations. This led to the concept of Hawking radiation. There have been also some attempts by Jacobson, Padmanabhan and Verlinde to derive the Einstein equations from thermodynamics. All these attempts are based on some deep underlying principles, like the holographic principle or the concept of emergent gravity, which could lay the foundation for a theory of quantum gravity. In this connection, the fact that the TOV equations can be derived from the maximum entropy principle (as discussed in Sec. III E 7) has sometimes been regarded as a strong evidence for the fundamental relationship between general relativity and thermodynamics. However, for systems of particles described by classical general gravity such as the ones that we have considered in this paper, this result is not really surprising and does not reflect, we believe, a special connection between gravity and thermodynamics.

Indeed, it is well-known that a statistical equilibrium state is always a steady state of the Vlasov equations and that thermodynamical stability implies dynamical stability (see Appendix A). This is due to the fact that the entropy $S$ (a particular Casimir functional), the energy $E$ and the particle number $N$ which appear in the maximum entropy principle are conserved by the Vlasov equation. As a result, an extremum of $S$ at fixed $E$ and $N$ is a steady state of the Vlasov equation and a maximum of $S$ at fixed $E$ and $N$ is dynamically stable (in addition of being thermodynamically stable). Therefore, the maximum entropy principle implies the condition of hydrostatic equilibrium. For spherically symmetric systems in general relativity, this leads to the TOV equations. This result – the fact that the maximum entropy principle implies the condition of hydrostatic equilibrium – is very general and is valid for all systems with long-range interactions. In the context of general relativity, it was first noted by Tolman and rediscovered by many other authors in the sequel. It does not bear a deeper significance. In particular, we emphasized in Sec. III E 7 that the maximum entropy principle implies the condition of hydrostatic equilibrium (yielding the TOV equations for spherically symmetric systems) but does not provide the whole set of Einstein equations. Similarly, in Newtonian gravity, the maximum entropy principle implies the condition of hydrostatic equilibrium, but not the whole set of Newton equations.

The equations derived in this paper have been used to construct the caloric curves of self-gravitating fermions and study phase transitions between gaseous states and condensed states in Newtonian gravity and general relativity. A rather complete understanding of these phase transitions has now been reached and general phase diagrams have been obtained in two-dimensional phase diagrams. In the case of classical self-gravitating systems there is nothing to halt the collapse so that an equilibrium state with a high density core (condensed phase) is never reached. The caloric curves of classical self-gravitating systems have been obtained both in Newtonian gravity and in general relativity. In Paper II, we adapt the present formalism to the case of classical particles described by the Boltzmann entropy and give all the necessary equations to understand these studies. We also investigate precisely the nonrelativistic and ultrarelativistic limits of the classical self-gravitating gas.

Appendix A: General relations between the pressure and the energy density

In this Appendix, we provide general relations between the pressure and the energy density in the nonrelativistic and ultrarelativistic limits. They are valid for an arbitrary distribution function.

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30 This also suggests that what we call “generalized thermodynamics” is just “standard thermodynamics” with a generalized form of entropy taking into account microscopic constraints.
1. Nonrelativistic limit

In the nonrelativistic limit, the kinetic energy of a particle is
\[ E_{\text{kin}} = \frac{p^2}{2m}. \]  
(A1)

In that case, the kinetic energy density and the pressure are given by
\[ \epsilon_{\text{kin}} = \int f \frac{p^2}{2m} \, dp \quad \text{and} \quad P = \frac{1}{3} \int f \frac{p^2}{m} \, dp. \]  
(A2)

We have the general relations
\[ P = \frac{2}{3} \epsilon_{\text{kin}}, \quad E_{\text{kin}} = \frac{3}{2} \int P \, dr. \]  
(A3)

2. Ultrarelativistic limit

In the ultrarelativistic limit, the energy of a particle is
\[ E = E_{\text{kin}} = pc. \]  
(A4)

In that case, the energy density and the pressure are given by
\[ \epsilon = \epsilon_{\text{kin}} = \int fpc \, dp \quad \text{and} \quad P = \frac{1}{3} \int fpc \, dp. \]  
(A5)

We have the general relations
\[ P = \frac{1}{3} \epsilon = \frac{1}{3} \epsilon_{\text{kin}}, \quad \mathcal{E} = E_{\text{kin}} = \frac{3}{2} \int P \, dr. \]  
(A6)

Appendix B: Virial theorem for Newtonian systems

In this Appendix, we establish the general expression of the equilibrium scalar virial theorem for Newtonian systems. For the sake of generality, we allow the particles to be relativistic in the sense of special relativity. It can be shown that the virial of the gravitational force is equal to the gravitational energy (see, e.g., Appendix G of [298]):
\[ -\int \rho r \cdot \nabla \Phi \, dr = W. \]  
(B1)

This equation is general, being valid for steady and unsteady configurations. It does not depend whether the system is spherically symmetric or not. If we now consider a spherically symmetric system (still allowed to be unsteady), using the Newton law \([300]\), we find from Eq. (B1) that
\[ W = -\int \rho \frac{GM(r)}{r} \, dV. \]  
(B2)

This formula is useful to calculate the gravitational potential energy of a spherically symmetric distribution of matter. It can be directly obtained by approaching from infinity a succession of spherical shells of mass \(dM(r) = \rho(r)4\pi r^2 \, dr\) with potential energy \(-GM(r)dM(r)/r\) in the field of the mass \(M(r)\) already present, and integrating over \(r\) (see, e.g., Ref. [299]).

We now consider a self-gravitating system at equilibrium. Substituting the condition of hydrostatic equilibrium
\[ \nabla P + \rho \nabla \Phi = 0 \]  
(B3)

into Eq. (B1), and integrating by parts, we get
\[ W = -3 \int P \, dr + \oint P r \cdot dS. \]  
(B4)
If the system is not submitted to an external pressure, the second term on the right hand side vanishes. On the other hand, if the external pressure is uniform on the boundary of the system, i.e. \( P(r) = P_b = \text{cst} \), we have

\[
\oint Pr \cdot dS = P_b \oint r \cdot dS = P_b \int \nabla \cdot r \, dr = 3P_b V.
\] (B5)

More generally, this relation can be taken as a definition of \( P_b \). Combining the foregoing relations, we obtain the general form of the equilibrium scalar virial theorem

\[
3 \int P \, dr + W = 3P_b V.
\] (B6)

For nonrelativistic particles, using Eq. (A3), the equilibrium scalar virial theorem becomes

\[
2E_{\text{kin}} + W = 3P_b V.
\] (B7)

Using \( E = E_{\text{kin}} + W \), it can be rewritten as

\[
E = -E_{\text{kin}} + 3P_b V.
\] (B8)

For ultrarelativistic particles, using Eq. (A6), the equilibrium scalar virial theorem becomes

\[
E = E_{\text{kin}} + W = 3P_b V.
\] (B9)

**Appendix C: Derivation of the statistical equilibrium state for a general form of entropy**

In this Appendix, we derive the statistical equilibrium state of a self-gravitating system for a general form of entropy

\[
s = -k_B \int C(f) \, dp.
\] (C1)

where \( C(f) \) is any convex function (i.e. \( C''(f) > 0 \)). This is what we call a generalized entropy \[214, 221, 224\]. These functionals appeared in relation to the notion of “generalized thermodynamics” pioneered by Tsallis \[225\] who introduced a particular form of non-Boltzmannian entropy (of a power-law type) called the Tsallis entropy. These functionals (which are particular Casimir integrals) are also useful to obtain sufficient conditions of nonlinear dynamical stability \[178, 187, 219, 223, 248\] with respect to the Vlasov equation describing a collisionless evolution of the system (see Appendix H). Below we show that the maximum entropy principle can be applied to an arbitrary form of entropy. We consider both Newtonian and general relativistic systems. We first present a two-steps derivation as in the main text, then a one-step derivation.

1. **Two-steps derivation**

To maximize the entropy \( S \) at fixed energy \( E \) and particle number \( N \), we proceed in two steps as in Secs. II and III. We first maximize the entropy density \( s(r) \) at fixed energy density \( \epsilon(r) \) and particle number density \( n(r) \) with respect to variations on \( f(r, p) \) following the steps of Secs. II.D and III.D. The variational principle \[126\] for the extremization problem yields

\[
C''(f) = -\beta(r)E(p) + \alpha(r).
\] (C2)

Since \( C \) is convex, this relation can be inverted. It determines a distribution function of the form

\[
f(r, p) = F[\beta(r)E(p) - \alpha(r)],
\] (C3)

where

\[
F(x) = (C')^{-1}(-x).
\] (C4)

Since

\[
\delta^2 s = -k_B \int C''(f) \frac{\delta f}{2} \, dp < 0,
\] (C5)

this distribution function is the global maximum of the entropy density at fixed energy density and particle number density. This corresponds to the condition of local thermodynamical equilibrium. Using the integrated Gibbs-Duhem relation \[E10\] which is valid for a general form of entropy (see Appendix E), we can express the entropy \( S \) as a functional of \( n(r) \) and \( \epsilon(r) \). We now maximize the entropy \( S \) at fixed energy and particle number with respect to variations on \( n(r) \) and \( \epsilon(r) \).
a. Newtonian gravity

We first consider the Newtonian gravity regime but, for the sake of generality, we allow the particles to be relativistic in the sense of special relativity. In Newtonian gravity, Eq. (C3) is replaced by

$$f(r, p) = F \left[ \beta(r) E_{\text{kin}}(p) - \alpha(r) \right],$$  

(C6)

where $E_{\text{kin}}(p)$ is given by Eq. (117). Repeating the steps of Sec. II E, we obtain Eqs. (44) and (55) expressing the uniformity of the temperature and of the total chemical potential (Gibbs law). We also obtain the condition of hydrostatic equilibrium (58). As a result, the equilibrium distribution function at statistical equilibrium is given by

$$f(r, p) = F \left[ \beta E_{\text{kin}}(p) + m \Phi(r) - \alpha_0 \right].$$  

(C7)

In the nonrelativistic regime where $E_{\text{kin}} = p^2/2m$ this is a function of the form

$$f(r, v) = f \left[ \epsilon(r, v) \right] \text{ with } f'(\epsilon) < 0,$$  

(C8)

where $\epsilon(r, v) = v^2/2 + \Phi(r)$ is the energy of a particle by unit of mass and we have introduced the velocity $v = p/m$ instead of the impulse $p$. We note that an extremum of entropy at fixed energy and particle number is necessarily isotropic. Repeating the arguments of Sec. II E, we can show that the gas corresponding to the distribution function (C7) is described by a barotropic equation of state $P(r) = P(\rho(r), T)$, where the function $P(\rho, T)$ is determined by the function $C(f)$ characterizing the entropy.

b. General relativity

We now consider the general relativity case. Repeating the steps of Sec. III E, we obtain Eq. (144). We also obtain the TOV equations (155) and (156) expressing the condition of hydrostatic equilibrium and the Tolman-Klein relations (158) and (159). As a result, the equilibrium distribution function at statistical equilibrium is given by

$$f(r, p) = F \left[ \beta \epsilon(r, p)/2 E(p) - \alpha \right].$$  

(C9)

Using Eq. (158), it can be written as

$$f(r, p) = F \left[ \beta E(r)/2 E(p) - \alpha \right].$$  

(C10)

This is a function of the form

$$f(r, p) = f \left[ \epsilon(r)/2 E(p) \right] \text{ with } f' \left[ \epsilon(r)/2 E(p) \right] < 0,$$  

(C11)

where $E(p)$ is the energy of a particle. We note that an extremum of entropy at fixed mass-energy and particle number is necessarily isotropic. Repeating the arguments of Sec. III G 1, we can show that the gas corresponding to the distribution function (C9) is described by a barotropic equation of state $P(r) = P(\alpha, \epsilon(r))$, where the function $P(\epsilon, \alpha)$ is determined by the function $C(f)$ characterizing the entropy.

2. One-step derivation

We now present a one-step derivation of the preceding results. We first consider the Newtonian gravity case. The generalized entropy is

$$S = -k_B \int C(f) \, dr \, dp.$$  

(C12)

The particle number and the mass are given by

$$M = Nm = m \int f \, dr \, dp = \int mn \, dr = \int \rho \, dr.$$  

(C13)
The energy is given by

\[ E = E_{\text{kin}} + W = \int fE_{\text{kin}}(p) \, dr \, dp + \frac{1}{2} \int \rho \Phi \, dr, \]  

(C14)

where \( E_{\text{kin}} \) is the kinetic energy and \( W \) is the potential (gravitational) energy.

In the microcanonical ensemble, the statistical equilibrium state is obtained by maximizing the entropy \( S \) at fixed energy \( E \) and particle number \( N \) with respect to variations on \( f(r, p) \). We write the variational problem for the first variations (extremization) as

\[ \frac{\delta S}{k_B} - \beta \delta E + \alpha_0 \delta N = 0, \]  

(C15)

where \( \beta \) and \( \alpha_0 \) are global Lagrange multipliers. Taking the variations with respect to \( f(r, p) \), we obtain

\[ C'(f) = -\beta(E_{\text{kin}}(p) + m\Phi(r)) + \alpha_0, \]  

(C16)

leading to

\[ f(r, p) = F \left[ \frac{1}{k_B T} (E_{\text{kin}}(p) + m\Phi(r) - \mu_0) \right], \]  

(C17)

where \( F(x) \) is defined by Eq. (C4). This returns the result from Eq. (C7). The temperature \( T \) and the chemical potential \( \mu_0 \) are related to \( \beta \) and \( \alpha_0 \) by

\[ \beta = \frac{1}{k_B T}, \quad \alpha_0 = \frac{\mu_0}{k_B T}. \]  

(C18)

We can then rewrite Eq. (C17) as

\[ f(r, p) = \frac{g}{\hbar^3} \int \left\{ \frac{f}{f_{\text{max}}} \ln \frac{f}{f_{\text{max}}} + \left(1 - \frac{f}{f_{\text{max}}} \right) \ln \left(1 - \frac{f}{f_{\text{max}}} \right) \right\} \, dr \, dp, \]  

(C20)

we obtain the mean field Fermi-Dirac distribution

\[ f(r, p) = \frac{1}{1 + e^{-(E_{\text{kin}}(p) + m\Phi(r)) / k_B T}} \]  

(C21)

or, equivalently,

\[ f(r, p) = \frac{g}{\hbar^3} \frac{1}{1 + e^{(E_{\text{kin}}(p) + m\Phi(r) - \mu_0) / k_B T}}. \]  

(C22)

For the Boltzmann entropy

\[ S = -k_B \int f \left[ \ln \left( \frac{f}{f_{\text{max}}} \right) - 1 \right] \, dr \, dp, \]  

(C23)

we obtain the mean field Maxwell-Boltzmann distribution

\[ f(r, p) = \frac{g}{\hbar^3} e^{\alpha_0} e^{-\beta(E_{\text{kin}}(p) + m\Phi(r))} \]  

(C24)

or, equivalently,

\[ f(r, p) = \frac{g}{\hbar^3} e^{-(E_{\text{kin}}(p) + m\Phi(r) - \mu_0) / k_B T}. \]  

(C25)

This one-step derivation of the statistical equilibrium state, valid for a generalized entropy of the form (C1), was given in Newtonian gravity by Ipser [178, 187], Tremaine et al. [212] and Chavanis [214], directly leading to Eq. (C7). It was extended in general relativity by Ipser [248], directly leading to Eq. (C10).
Appendix D: Condition of hydrostatic equilibrium for a general form of entropy

In this Appendix, we show by a direct calculation that the condition of statistical equilibrium, obtained by extremizing the entropy at fixed energy and particle number, implies the condition of hydrostatic equilibrium. We consider a general form of entropy given by Eq. (C1).

1. Newtonian gravity

We first consider the Newtonian gravity case but, for the sake of generality, we allow the particles to be relativistic in the sense of special relativity. The extremization of the entropy $S$ at fixed particle number $N$ and energy $E$ leads to a distribution function of the form (see Appendix C)

$$f(r, p) = F\left[\beta (E_{\text{kin}}(p) + m\Phi(r)) - \alpha_0\right], \quad (D1)$$

where $F$ is defined by Eq. (C4) and where $\beta$ and $\alpha_0$ are constant. According to Eqs. (120) and (D1) the pressure is given by

$$P = \frac{1}{3} \int f p \frac{dE_{\text{kin}}}{dp} \, dp = \frac{1}{3} \int F \left[\beta (E_{\text{kin}}(p) + m\Phi(r)) - \alpha_0\right] p \frac{dE_{\text{kin}}}{dp} \, dp. \quad (D2)$$

Taking its gradient with respect to $r$, we get

$$\nabla P = \frac{1}{3} \beta m \nabla \Phi \int F' \left[\beta (E_{\text{kin}}(p) + m\Phi(r)) - \alpha_0\right] p \frac{dE_{\text{kin}}}{dp} \, dp. \quad (D3)$$

This can also be written as

$$\nabla P = \frac{1}{3} m \nabla \Phi \int \mathbf{p} \cdot \frac{\partial F}{\partial \mathbf{p}} \left[\beta (E_{\text{kin}}(p) + m\Phi(r)) - \alpha_0\right] \, dp. \quad (D4)$$

Integrating by parts, we can rewrite the foregoing equation as

$$\nabla P = -m \nabla \Phi \int F' \left[\beta (E_{\text{kin}}(p) + m\Phi(r)) - \alpha_0\right] \, dp. \quad (D5)$$

Since the density is given by

$$\rho = m \int f \, dp = m \int F \left[\beta (E_{\text{kin}}(p) + m\Phi(r)) - \alpha_0\right] \, dp, \quad (D6)$$

we finally obtain the condition of hydrostatic equilibrium

$$\nabla P = -\rho \nabla \Phi. \quad (D7)$$

2. General relativity

We now consider the general relativity case. The extremization of the entropy $S$ at fixed mass-energy $E$ and particle number $N$ leads to a distribution function of the form (see Appendix C)

$$f(r, p) = F \left[\frac{E(p)}{k_B T(r)} - \alpha\right], \quad (D8)$$

where $F$ is defined by Eq. (C4) and where $\alpha$ is constant. According to Eqs. (120) and (D8) the pressure is given by

$$P = \frac{1}{3} \int f p \frac{dE}{dp} \, dp = \frac{1}{3} \int F \left[\frac{E(p)}{k_B T(r)} - \alpha\right] p \frac{dE}{dp} \, dp. \quad (D9)$$

Taking its derivative with respect to $r$, we obtain

$$\frac{dP}{dr} = -\frac{1}{3} \frac{1}{k_B T(r)^2} \frac{dT}{dr} \int F' \left[\frac{E(p)}{k_B T(r)} - \alpha\right] p \frac{dE}{dp} E(p) \, dp. \quad (D10)$$
This can also be written as
\[
\frac{dP}{dr} = -\frac{1}{3 \frac{dT}{dr}} \int_0^{+\infty} \frac{\partial F}{\partial p} \left[ \frac{E(p)}{k_B T(r)} - \alpha \right] p E(p) 4\pi p^2 dp. \tag{D11}
\]

Integrating by parts, we can rewrite the foregoing equation as
\[
\frac{dP}{dr} = \frac{1}{3} \frac{dT}{dr} \int_0^{+\infty} F \left[ \frac{E(p)}{k_B T(r)} - \alpha \right] \left( 3E(p)p^2 + p^3 \frac{dE}{dp} \right) 4\pi dp. \tag{D12}
\]

Since the pressure is given by Eq. (D9) and the energy density by
\[
\epsilon = \int f E(p) dp = \int F \left[ \frac{E(p)}{k_B T(r)} - \alpha \right] E(p) dp, \tag{D13}
\]
we finally obtain the equation
\[
\frac{dP}{dr} = \frac{\epsilon(r) + P(r) \frac{dT}{dr}}{T(r)}. \tag{D14}
\]

Combined with Eq. (154) it leads to the OV equation (155). From Eqs. (101), (103) and (155) we then obtain the Tolman equation (156) which expresses the condition of hydrostatic equilibrium.

### Appendix E: Gibbs-Duhem relation

In this Appendix, we derive the Gibbs-Duhem and integrated Gibbs-Duhem relations. We first recall the usual derivation of these relations which explicitly uses the extensivity of the entropy. Then, we provide a direct derivation of the integrated Gibbs-Duhem relation for an arbitrary form of entropy without explicitly using the extensivity assumption. This shows that our thermodynamical formalism is valid for an arbitrary form of entropy.

#### 1. Standard derivation

The first law of thermodynamics can be written as
\[
dE = -P dV + T dS + \mu dN. \tag{E1}
\]

An extensive variable (energy, entropy,...) is proportional to the absolute size of the system. In other words, if one doubles all extensive variables, all other extensive quantities also become twice as large. For example,
\[
E(\alpha S, \alpha V, \alpha N) = \alpha E(S, V, N), \tag{E2}
\]
where \(\alpha\) is the enlargement factor. One calls functions which have this property homogeneous functions of first order. All extensive variables are homogeneous functions of first order of the other extensive variables. On the other hand, the intensive variables (temperature, pressure...) are homogeneous functions of zeroth order of the extensive variables, i.e., they do not change if we divide or duplicate the system. For example,
\[
T(\alpha S, \alpha V, \alpha N) = T(S, V, N). \tag{E3}
\]

According to the Euler theorem, we have
\[
E = -PV + TS + \mu N. \tag{E4}
\]

Differentiating this expression and using the first law of thermodynamics (E1), we get the Gibbs-Duhem relation
\[
SdT - VdP + Nd\mu = 0. \tag{E5}
\]

We note that the energy does not appear in this expression. The Euler equation (E4) for thermodynamic variables is also called the integrated Gibbs-Duhem relation.
Defining \( s = S/V \), \( n = N/V \) and \( \epsilon = E/V \), the local Gibbs-Duhem and the integrated Gibbs-Duhem relations can be written as

\[
s dT - dP + n d\mu = 0 \tag{E6}
\]

and

\[
\epsilon = -P + Ts + \mu n \tag{E7}
\]

Introducing \( S = sV \), \( N = nV \) and \( E = \epsilon V \) in the first law of thermodynamics \( \text{(E1)} \), developing the expression, and using the integrated Gibbs-Duhem relation \( \text{(E7)} \), we obtain

\[
d\epsilon = T ds + \mu dn \tag{E8}
\]

We note that the pressure does not explicitly appear in this expression. This is the local form of the first law of thermodynamics \[\text{[see Eq. (130)].}\]

2. Direct derivation of the integrated Gibbs-Duhem relation for a general form of entropy

The local condition of thermodynamical equilibrium, obtained by maximizing the local entropy at fixed energy density and particle number density, is given by Eq. \( \text{(C3)} \) with Eq. \( \text{(C4)} \). Substituting Eq. \( \text{(C3)} \) into Eqs. \( \text{(113)}, \text{(114)} \) and \( \text{(120)} \) we find that the particle number density, the energy density and the pressure are given by

\[
n(r) = \int_0^{+\infty} F [\beta(r)E(p) - \alpha(r)] 4\pi p^2 dp, \tag{E9}
\]

\[
\epsilon(r) = \int_0^{+\infty} F [\beta(r)E(p) - \alpha(r)] E(p) 4\pi p^2 dp, \tag{E10}
\]

\[
P(r) = \frac{1}{3} \int_0^{+\infty} F [\beta(r)E(p) - \alpha(r)] pE'(p) 4\pi p^2 dp. \tag{E11}
\]

On the other hand, the entropy density \[\text{[see Eq. (C1)]}\] is given by

\[
s(r) = -k_B \int_0^{+\infty} C \{ F [\beta(r)E(p) - \alpha(r)] \} 4\pi p^2 dp. \tag{E12}
\]

Integrating this equation by parts and using \( C'[F(x)] = -x \), we get

\[
s(r) = -k_B \int_0^{+\infty} [\beta(r)E(p) - \alpha(r)] F' [\beta(r)E(p) - \alpha(r)] \beta(r)E'(p) 4\pi \frac{p^3}{3} dp. \tag{E13}
\]

Integrating by parts on more time, we obtain

\[
s(r) = k_B \int_0^{+\infty} \beta(r)E'(p)F [\beta(r)E(p) - \alpha(r)] \frac{4\pi}{3} p^3 dp + k_B \int_0^{+\infty} [\beta(r)E(p) - \alpha(r)] F [\beta(r)E(p) - \alpha(r)] 4\pi p^2 dp. \tag{E14}
\]

Comparing Eq. \( \text{(E14)} \) with Eqs. \( \text{(E9)}, \text{(E10)}, \text{(E11)} \), we find that

\[
\frac{s(r)}{k_B} = \beta(r)P(r) + \beta(r)\epsilon(r) - \alpha(r)n(r). \tag{E15}
\]

Using Eq. \( \text{(128)} \), we finally obtain the integrated Gibbs-Duhem relation

\[
s(r) = \frac{\epsilon(r) + P(r) - \mu(r)n(r)}{T(r)}. \tag{E16}
\]
This calculation emphasizes the fact that the integrated Gibbs-Duhem relation is valid for an arbitrary form of entropy and for an arbitrary level of relativity.\textsuperscript{31}

Remark: The calculations presented in this Appendix are equivalent to those performed in Appendix B of \textsuperscript{219}, in Appendix C of \textsuperscript{286} and in Appendix D of \textsuperscript{221} although the connection with the integrated Gibbs-Duhem relation was not realized at that time.

Appendix F: Entropy and free energy as functionals of the density for Newtonian self-gravitating systems

We consider a Newtonian self-gravitating system but, for the sake of generality, we allow the particles to be relativistic in the sense of special relativity. We also consider a general form of entropy given by Eq. (C12). The statistical equilibrium state is obtained by maximizing the entropy at fixed energy and particle number in the microcanonical ensemble, or by minimizing the free energy at fixed particle number in the canonical ensemble. In Appendix C2, we have introduced entropy and free energy functionals of the distribution function \( f(r, v) \). In Sec. II E and in Appendix C1 a, we have introduced entropy and free energy functionals of the local density \( n(r) \) and local kinetic energy \( \epsilon_{\text{kin}}(r) \). In this Appendix, we introduce entropy and free energy functionals of the local density \( n(r) \).

1. Microcanonical ensemble

In the microcanonical ensemble, the statistical equilibrium state is obtained by maximizing the entropy \( S[f] \) at fixed energy \( E \) and particle number \( N \). To solve this maximization problem, we proceed in two steps. We first maximize \( S[f] \) at fixed \( E, N \) and particle density \( n(r) \). Since \( n(r) \) determines the particle number \( N[n] \) and the gravitational energy \( W[n] \), this is equivalent to maximizing \( S[f] \) at fixed kinetic energy \( E_{\text{kin}} \) and particle density \( n(r) \). The variational problem for the first variations (extremization) can be written as

\[
\frac{\delta S}{k_B} - \beta \delta E_{\text{kin}} + \int \alpha(r) \delta n \, dr = 0, \quad (F1)
\]

where \( \beta \) is a global (uniform) Lagrange multiplier and \( \alpha(r) \) is a local (position dependent) Lagrange multiplier. This variational problem, which is equivalent to

\[
\frac{\delta s}{k_B} - \beta \delta \epsilon_{\text{kin}} + \alpha(r) \delta n = 0, \quad (F2)
\]

returns the results of Appendix C1 except that \( \beta(r) \) is replaced by \( \beta \). Therefore, it yields

\[
f(r, p) = F[\beta E_{\text{kin}}(p) - \alpha(r)]. \quad (F3)
\]

In this manner, we immediately find that \( T \) is uniform at statistical equilibrium. This results from the conservation of energy. As in Appendix C1 we can show that the distribution \( F3 \) is the global maximum of \( S[f] \) at fixed \( E_{\text{kin}} \) and \( n(r) \). Substituting Eq. (F3) into Eqs. (17), (18) and (20), we get

\[
n(r) = \int F[\beta E_{\text{kin}}(p) - \alpha(r)] \, dp, \quad (F4)
\]

\[
\epsilon_{\text{kin}}(r) = \int F[\beta E_{\text{kin}}(p) - \alpha(r)] E_{\text{kin}}(p) \, dp, \quad (F5)
\]

\[
P(r) = \frac{1}{3} \int F[\beta E_{\text{kin}}(p) - \alpha(r)] p E_{\text{kin}}'(p) \, dp. \quad (F6)
\]

\textsuperscript{31} This is an interesting result because there is a lot of polemic related to the notion of “generalized thermodynamics” introduced by Tsallis \textsuperscript{225}. The present calculation shows that standard thermodynamics is actually valid for an arbitrary form of entropy \textsuperscript{214, 215}. 

The Lagrange multiplier $\alpha(r)$ is determined by the density $n(r)$ according to Eq. (F14). On the other hand, the temperature $T$ is determined by the kinetic energy $E_{\text{kin}}[n(r),T] = E - W[n(r)]$ using Eq. (F4) integrated over the volume. In other words, the temperature is determined by the energy constraint

$$E = E_{\text{kin}}[n(r),T] + W[n(r)].$$  \hfill (F7)

We note that $T$ is a functional of the density $n(r)$ but, for brevity, we shall not write this dependence explicitly.

Repeating the steps of Appendix E2 we can derive the integrated Gibbs-Duhem relation (E10), except that $T(r)$ is replaced by $T$. Therefore, we get

$$s(r) = \frac{\epsilon_{\text{kin}}(r) + P(r) - \mu(n)r}{T} \quad \text{with} \quad \mu(r) = \alpha(r)k_B T.$$  \hfill (F8)

Since $T$ is uniform, Eq. (39) reduces to

$$d\mu = \frac{dP}{n}. \quad \text{(F9)}$$

On the other hand, eliminating formally $\alpha(r)$ between Eqs. (F4) and (F6), we see that the equation of state is barotropic: $P(r) = P[n(r),T]$ (we have explicitly written the temperature $T$ because it is uniform but not constant when we consider variations of $n(r)$ as explained above). Therefore, according to Eq. (F9) we have $\mu(r) = \mu[n(r),T]$ with

$$\mu'(n,T) = \frac{P'(n,T)}{n}, \quad \text{i.e.} \quad \mu(n,T) = \int^n P'(n',T) \frac{dn'}{n'}, \quad \text{(F10)}$$

where the derivative is with respect to $n$. \hfill (32)

We can now simplify the expression of the entropy. Using the integrated Gibbs-Duhem relation (F8), we have

$$S = \frac{1}{T} \left( E_{\text{kin}} + \int P(r) \, dr - \int \mu(r)n(r) \, dr \right). \quad \text{(F11)}$$

The entropy can be written as a functional of the density as

$$S[n(r),T] = \frac{1}{T} \left( E_{\text{kin}}[n(r),T] - U[n(r),T] \right), \quad \text{(F12)}$$

or, using Eq. (F7), as

$$S[n(r),T] = \frac{1}{T} \left( E - W[n(r)] - U[n(r),T] \right), \quad \text{(F13)}$$

where $U[n(r),T]$ is the internal energy given by

$$U[n(r),T] = \int V(n(r),T) \, dr \quad \text{with} \quad V(n,T) = n\mu(n,T) - P(n,T). \quad \text{(F14)}$$

Combining Eqs. (F10) and (F14), we get

$$V'(n,T) = \mu(n,T). \quad \text{(F15)}$$

Therefore, the pressure $P(n,T)$ is related to the density of internal energy $V(n,T)$ by

$$P(n,T) = n\mu(n,T) - V(n,T) = nV'(n,T) - V(n,T) = n^2 \left[ \frac{V(n,T)}{n'} \right]'. \quad \text{(F16)}$$

Inversely, the density of internal energy is determined by the equation of state $P[n(r),T]$ according to the relation

$$V(n,T) = n \int^n P(n',T) \frac{dn'}{n'^2}. \quad \text{(F17)}$$

\hfill (32) This relation determines the chemical potential $\mu$ up to an additive constant that may depend on the temperature $T$. The complete expression of the chemical potential can be obtained from Eq. (F3).
We note the identities
\[ V'(n, T) = \int^n P'(n', T) \frac{dn'}{n'} \quad \text{and} \quad V''(n, T) = \frac{P'(n, T)}{n}. \] (F18)

The internal energy can be written explicitly as
\[ U[n(r), T] = \int n \int^n P(n', T) \frac{dn'}{n'} dn'r. \] (F19)

Finally, the statistical equilibrium state in the microcanonical ensemble is obtained by maximizing the entropy \( S[n] \) at fixed particle number \( N \), the energy constraint being taken into account in the determination of the temperature \( T[n] \) through the relation (F7). The variational problem for the first variations (extremization) can be written as
\[ \frac{\delta S}{k_B} + \alpha_0 \delta N = 0. \] (F20)

The conservation of energy implies [see Eq. (F7)]:
\[ 0 = \delta E_{\text{kin}} + \int m \Phi \delta n \, dr. \] (F21)

Using Eqs. (F11) and (F21), we get
\[ \frac{\delta S}{k_B} = -\beta \int m \Phi \delta n \, dr - \int \alpha(r) \delta n \, dr. \] (F22)

As a result, the variational problem (F20) yields
\[ \alpha(r) = \alpha_0 - \beta m \Phi(r). \] (F23)

We then recover all the results of Sec. II. The interest of this formulation it that it allows us to solve more easily the stability problem related to the sign of the second variations of entropy. This problem has been studied in detail in [195, 202, 207] for the Boltzmann entropy and in [216, 228, 229] for the Tsallis entropy. It has also been studied in [253] for the Boltzmann entropy within the framework of special relativity.

2. Canonical ensemble

In the canonical ensemble, the statistical equilibrium state is obtained by minimizing the free energy \( F[f] = E[f] - TS[f] \) at fixed particle number \( N \), or equivalently, by maximizing the Massieu function \( J[f] = S[f]/k_B - \beta E[f] \) at fixed particle number \( N \). To solve this maximization problem, we proceed in two steps. We first maximize \( J[f] = S[f]/k_B - \beta E[f] \) at fixed \( N \) and particle density \( n(r) \). Since \( n(r) \) determines the particle number \( N[n] \) and the gravitational energy \( W[n] \), this is equivalent to maximizing \( S[f]/k_B - \beta E_{\text{kin}}[f] \) at fixed particle density \( n(r) \). The variational problem for the first variations (extremization) can be written as
\[ \delta \left( \frac{S}{k_B} - \beta E_{\text{kin}} \right) + \int \alpha(r) \delta n \, dr = 0, \] (F24)

where \( \alpha(r) \) is a local (position dependent) Lagrange multiplier. Since \( \beta \) is constant in the canonical ensemble, this is equivalent to the conditions (F1) and (F2) yielding the distribution function (F3). This distribution is the global maximum of \( S[f]/k_B - \beta E_{\text{kin}}[f] \) at fixed \( n(r) \). We then obtain the same results as in Appendix F1 except that \( T \) is fixed while it was previously determined by the conservation of energy (F7).

We can now simplify the expression of the free energy. The entropy is given by Eq. (F12) and the energy by Eq. (F7). Since \( F = E - TS \), we obtain
\[ F[n(r), T] = U[n(r), T] + W[n(r)], \] (F25)

where \( U[n] \) is the internal energy given by Eq. (F14). The statistical equilibrium state in the canonical ensemble is obtained by minimizing the free energy \( F[n] \) at fixed particle number \( N \). The variational problem for the first variations (extremization) can be written as
\[ \delta J + \alpha_0 \delta N = 0. \] (F26)
Decomposing the Massieu function as $J[f] = S[f]/k_B - \beta E_{\text{kin}}[f] - \beta W[n]$ and using Eq. (F24), we get

$$\delta J = - \int \alpha(r) \delta n \, dr - \beta \int m \Phi \delta n \, dr.$$  \hspace{1cm} (F27)

As a result, the variational problem (F26) yields

$$\alpha(r) = \alpha_0 - \beta m \Phi(r).$$  \hspace{1cm} (F28)

We then recover all the results of Sec. II. The interest of this formulation is that it allows us to solve more easily the stability problem related to the sign of the second variations of free energy. This problem has been studied in detail in [200, 202] for the Boltzmann free energy and in [216, 227, 229] for the Tsallis free energy. It has also been studied in [253] for the Boltzmann free energy within the framework of special relativity.

Remark: Using Eq. (F19), we see that the free energy (F25) can be written as

$$F[n] = \int \rho P(\rho) \, d\rho + \frac{1}{2} \int \rho \Phi \, d\rho.$$  \hspace{1cm} (F29)

We have not explicitly written the temperature $T$ since it is a constant in the canonical ensemble. Up to the kinetic term, Eq. (F29) coincides with the energy functional (G1) associated with the Euler-Poisson equations describing a gas with a barotropic equation of state $P = P(\rho)$ (see Appendix G1). As a result, the thermodynamical stability of a self-gravitating system in the canonical ensemble is equivalent to the dynamical stability of the corresponding barotropic gas described by the Euler-Poisson equations. This returns the general result established in [219]. It is valid for an arbitrary form of entropy. According to the Poincaré turning point criterion, the series of equilibria becomes both thermodynamically unstable (in the canonical ensemble) and dynamically unstable with respect to the Euler-Poisson equations at the first turning point of temperature (or, equivalently, at the first turning point of mass).

### 3. Scaling of the equation of state in the nonrelativistic and ultrarelativistic limits

We have seen that the equation of state implied by the distribution function (F3) is of the form $P(r) = P[n(r), T]$. A simple scaling of this equation of state can be obtained in the nonrelativistic and ultrarelativistic limits.

In the nonrelativistic limit, using $E_{\text{kin}} = \frac{p^2}{2m}$, Eqs. (F4)-(F6) reduce to

$$n(r) = \int F \left[ \frac{\beta p^2}{2m} - \alpha(r) \right] \, dp,$$  \hspace{1cm} (F30)

$$\epsilon_{\text{kin}}(r) = \int F \left[ \frac{\beta p^2}{2m} - \alpha(r) \right] \frac{p^2}{2m} \, dp,$$  \hspace{1cm} (F31)

$$P(r) = \frac{1}{3} \int F \left[ \frac{\beta p^2}{2m} - \alpha(r) \right] \frac{p^2}{m} \, dp.$$  \hspace{1cm} (F32)

Making the change of variables $x = (\beta/m)^{1/2} p$, we obtain the scaling

$$P(n, T) = T^{5/2} \Pi_{\text{NR}} \left( \frac{n}{T^{3/2}} \right).$$  \hspace{1cm} (F33)

Therefore, the internal energy (F19) takes the form

$$U[n(r), T] = T \int n \int n^{n/T^{3/2}} \frac{\Pi(x)}{x^2} \, dx \, dr.$$  \hspace{1cm} (F34)

For the Boltzmann entropy in phase space $S_B[f]$, leading to the isothermal equation of state $P = nk_BT$, the free energy is of the form $F[n] = W[n] - TS_B[n]$ where $S_B[n]$ is the Boltzmann entropy in configuration space (see [202]).

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33 A more direct derivation of this result is given in Appendix K3.
for details). For the Tsallis entropy in phase space \( S_q[f] \), leading to the polytropic equation of state \( P = K(T)n^\gamma \), the free energy is of the form \( F[n] = W[n] - K(T)S_q[n] \) where \( S_q[n] \) is the Tsallis entropy in configuration space (see [216] for details). In general, we do not have \( F[n] = W[n] - \Theta(T)S_q[n] \) (except for Boltzmann) nor \( F[n] = W[n] - \Theta(T)S_q[n] \) (except for Tsallis).

In the ultrarelativistic limit, using \( E_{\text{kin}} = pc \), Eqs. (F4)-(F6) reduce to

\[
n(r) = \int F[\beta pc - \alpha(r)] \, dp,
\]

\[
\epsilon_{\text{kin}}(r) = \int F[\beta pc - \alpha(r)] \, pc \, dp,
\]

\[
P(r) = \frac{1}{3} \int F[\beta pc - \alpha(r)] \, pc \, dp.
\]

Making the change of variables \( x = \beta pc \), we obtain the scaling

\[
P(n, T) = T^4 \Pi_{\text{UR}} \left( \frac{n}{T^4} \right).
\]

Therefore, the internal energy (F19) takes the form

\[
U[n(r), T] = T \int n \int n/T^3 \Pi(x) \frac{\rho'}{x^2} d\rho' dr.
\]

4. General relativity

Let us briefly consider the general relativity case. In the microcanonical ensemble, the statistical equilibrium state is obtained by maximizing the entropy \( S[f] \) at fixed mass-energy \( M c^2 \) and particle number \( N \). To solve this maximization problem, we proceed in two steps. We first maximize \( S[f] \) at fixed \( M c^2, N \) and energy density \( \epsilon(r) \). Since \( \epsilon(r) \) determines \( M c^2 \), this is equivalent to maximizing \( S[f] \) at fixed \( N \) and \( \epsilon(r) \). The variational problem for the first variations (extremization) can be written as

\[
\frac{\delta S}{k_B} = - \tilde{\beta}(r) \delta \epsilon \, dr + \alpha \delta N = 0,
\]

where \( \tilde{\beta}(r) \) is a local (position dependent) Lagrange multiplier and \( \alpha \) is a global (uniform) Lagrange multiplier. Noting that \( M(r) \) – which appears in the expressions of \( S \) and \( N \) – is fixed since it is determined by \( \epsilon(r) \), this variational problem yields

\[
C'(f) = - \beta(r) E(p) + \alpha
\]

with \( \beta(r) \equiv \tilde{\beta}(r)[1 - 2GM(r)/rc^2]^{1/2} \), leading to Eq. (F39). In this manner, we immediately find that \( \alpha \) is uniform at statistical equilibrium. This results from the conservation of \( N \). Substituting Eq. (F41) into the expressions of \( S, M \) and \( N \) may help solving the stability problem.

Appendix G: Dynamical stability of a self-gravitating barotropic gas with respect to the Euler equation

1. Newtonian gravity: Euler-Poisson equations

We consider a Newtonian gaseous star with a barotropic equation of state \( P = P(\rho) \) described by the Euler-Poisson equations. These equations conserve the mass \( M \) [see Eq. (21)] and the energy

\[
\mathcal{W}[\rho, u] = \frac{1}{2} \int \rho u^2 \, dr + \int \rho \int P(\rho') \rho'^2 \, d\rho' \, dr + \frac{1}{2} \int \rho \Phi \, dr,
\]

\[(G1)\]
which is the sum of the kinetic energy $\Theta_c$, the internal energy $U$, and the gravitational energy $W$ (see Appendix K.1). It can be shown that the minimization problem

$$\min \{ W \mid M \text{ fixed} \} \quad \text{(G2)}$$

determines an equilibrium state of the Euler-Poisson equations that is dynamically stable $[219, 300, 301]$. This is a criterion of nonlinear dynamical stability resulting from the fact that $W$ and $M$ are conserved by the Euler-Poisson equations $[268]$. It provides a necessary and sufficient condition of dynamical stability since it takes into account all the invariants of the Euler-Poisson equations.

The variational principle for the first variations (extremization) can be written as

$$\delta W - \frac{\mu_0}{m} \delta M = 0, \quad \text{(G3)}$$

where $\mu_0$ is a Lagrange multiplier. This yields $u = 0$ and

$$\int P'(\rho') \frac{d\rho'}{\rho'} + \Phi(r) - \frac{\mu_0}{m} = 0. \quad \text{(G4)}$$

Taking the gradient of this relation, we obtain the condition of hydrostatic equilibrium

$$\nabla P + \rho \nabla \Phi = 0. \quad \text{(G5)}$$

Therefore, an extremum of $W$ at fixed $M$ is a steady state of the Euler-Poisson equations. Then, considering the second variations of $W$, it can be shown that the star is linearly stable with respect to the Euler-Poisson equations if, and only if, it is a local minimum of $W$ at fixed mass $M$. This is also equivalent to its spectral stability. Indeed, the complex pulsations $\omega$ of the normal modes of the linearized Euler-Poisson equations $[219, 301–305]$ satisfy $\omega^2 > 0$ for all modes if, and only if, $\delta^2 W > 0$ for all perturbations that conserve $M$. Using the Poincaré criterion $[266]$, we can generically conclude $[219]$ that the series of equilibria is dynamically stable before the turning points of mass $M$ or energy $W$ or energy $W$ (they coincide) and that it becomes dynamically unstable afterwards. Furthermore, the curve $W(M)$ displays spikes at its extremal points (since $\delta W = 0 \Leftrightarrow \delta M = 0$). We refer to $[219, 300, 301]$ for the derivation of these results.

**Remark:** In the case of isothermal and polytropic equations of states, the marginal mode of instability has been explicitly determined in $[200, 203, 227]$.

2. **General relativity: Euler-Einstein equations**

We consider a relativistic gaseous star with a barotropic equation of state $P = P(\epsilon)$ described by the Euler-Einstein equations. We restrict ourselves to spherically symmetric systems. The Euler-Einstein equations conserve the mass-energy $Mc^2$ [see Eq. (100)] and the particle number $N$ [see Eq. (123)]. Here, the energy density is equal to $\epsilon = \rho c^2 + u$ where $u$ is the density of internal energy (see Appendix K.2). It can be shown that the minimization problem

$$\min \{ M \mid N \text{ fixed} \} \quad \text{(G6)}$$

or, equivalently, the maximization problem

$$\max \{ N \mid M \text{ fixed} \} \quad \text{(G7)}$$

determine an equilibrium state of the Euler-Einstein equations that is dynamically stable $[47, 67, 241, 275]$. These are criteria of nonlinear dynamical stability resulting from the fact that $M$ and $N$ are conserved by the Euler-Einstein equations $[268]$. They provide a necessary and sufficient condition of dynamical stability since they take into account all the invariants of the Euler-Einstein equations.

The variational principle for the first variations (extremization) can be written as

$$\delta M - \sigma \delta N = 0 \quad \text{or} \quad \delta N - \frac{1}{\sigma} \delta M = 0, \quad \text{(G8)}$$

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34 The optimization problems (G6) and (G7) are equivalent to $\min \{ E \mid N \text{ fixed} \}$ and $\min \{ E \mid M \text{ fixed} \}$, where $E = (M - Nm)c^2$ is the binding energy. In the nonrelativistic limit, they reduce to the optimization problem (G2). Indeed, when $c \to +\infty$, repeating the steps of Sec. IV.F with $u$ in place of $\epsilon_{\text{kin}}$, we get $E \to U + W$. 


The Poincaré turning point criterion [266] is equivalent to the mass-radius theorem of Wheeler [67] introduced in the physics of compact objects like white dwarfs and neutron stars. Here, the analogy with thermodynamics is effective.

where \( \sigma \) is a Lagrange multiplier. It leads to the TOV equations [187] and [300] which express the condition of hydrostatic equilibrium. Then, considering the second variations of \( N \), it can be shown that the star is linearly stable with respect to the Euler-Einstein equations if, and only if, it is a local maximum of \( M \). This is also equivalent to its spectral stability. Indeed, the complex pulsations \( \omega \) of the normal modes of the linearized Euler-Einstein equations [51, 300] satisfy \( \omega^2 > 0 \) for all modes if, and only if, \( \delta^2 N < 0 \) for all perturbations that conserve \( M \). Using the Poincaré criterion [266], we can generically conclude that the series of equilibria is dynamically stable before the turning points of mass-energy \( M \), particle number \( N \), or binding energy \( E \) (they all coincide) and that it becomes dynamically unstable afterwards.\(^{35}\) Furthermore, the curve \( M(N) \) displays spikes at its extremal points (since \( \delta M = 0 \Rightarrow \delta N = 0 \)). We refer to [67, 241, 253, 254] for the derivation of these results.

**Remark:** In the case of a linear equation of state \( P = q \epsilon \), the marginal mode of instability has been explicitly determined in [253, 254].

Appendix H: Dynamical stability of collisionless self-gravitating systems with respect to the Vlasov equation

1. Newtonian gravity: Vlasov-Poisson equations

We consider a Newtonian collisionless stellar system described by the Vlasov-Poisson equations. These equations conserve the energy \( E \) [see Eq. (C14)] and an infinite number of Casimir integrals \( I_h = \int h(f) \, d\mathbf{r} \, dv \), where \( h \) is an arbitrary function, including the particle number \( N \) [see Eq. (C13)]. The minimization problem

\[
\min \{ E \mid I_h \text{ fixed for all } h \}
\]

(H1)

determines a steady state of the Vlasov-Poisson equations that is dynamically stable [187, 223, 273, 300, 307, 309]. This is a criterion of nonlinear dynamical stability resulting from the fact that \( E \) and \( I_h \) are conserved by the Vlasov-Poisson equations [268]. It provides a necessary and sufficient condition of dynamical stability since it takes into account all the invariants of the Vlasov-Poisson equations.

It can be shown that any steady state of the Vlasov-Poisson equations extremizes the energy \( \delta E = 0 \) under phase-preserving, or symplectic, perturbations (those that conserve all the Casimirs). Restricting ourselves to steady states of the form \( f = f(\epsilon) \) with \( f'(\epsilon) < 0 \) and considering the second variations of \( E \), it can be shown that a stellar system is linearly stable with respect to the Vlasov-Poisson equations if, and only if, it is a local minimum of \( E \) under symplectic (phase-preserving) perturbations. This is also equivalent to its spectral stability. Indeed, the complex pulsations \( \omega \) of the normal modes of the linearized Vlasov-Poisson equations [310] satisfy \( \omega^2 > 0 \) (one can show that \( \omega^2 \) is real) for all modes if, and only if, \( \delta^2 E > 0 \) for all perturbations that conserve the Casimirs at first order. We refer to [187, 223, 273, 300, 307, 309] for the derivation of these results.

It can be shown furthermore that the maximization problem (“microcanonical” criterion)

\[
\max \{ S \mid E, N \text{ fixed} \}
\]

(H2)

and the minimization problem (“canonical” criterion)

\[
\min \{ F = E - TS \mid N \text{ fixed} \}
\]

(H3)

for a generalized “entropy” of the form [C1] provide sufficient conditions of dynamical stability with respect to the Vlasov-Poisson equations.\(^{36}\) These are criteria of nonlinear dynamical stability resulting from the fact that \( S, E, F \) and \( N \) are conserved by the Vlasov-Poisson equations [268]. They provide just sufficient conditions of dynamical stability because they take into account the conservation of only certain invariants of the Vlasov-Poisson equations, not all of them. It can be shown that “canonical stability” implies “microcanonical stability” which implies “dynamical stability”. We have

\[
\text{(H3)} \Rightarrow \text{(H2)} \Rightarrow \text{(H1)}. \tag{H4}
\]

This is similar to a situation of ensembles inequivalence in thermodynamics. These results were established in [187, 213, 223]. Since “microcanonical” stability implies dynamical stability, using the Poincaré criterion [266], we

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\(^{35}\) The Poincaré turning point criterion [266] is equivalent to the mass-radius theorem of Wheeler [67] introduced in the physics of compact objects like white dwarfs and neutron stars.

\(^{36}\) Here, the analogy with thermodynamics is effective.
can generically conclude that the series of equilibria is dynamically stable at least until the turning point of energy [187, 219]. This is a general result valid for all systems with long-range interactions [223]. Now, in the case of Newtonian self-gravitating systems, it can be shown [192, 269, 273] that all the distribution functions of the form \( f = f(\epsilon) \) with \( f'(\epsilon) < 0 \) are dynamically stable with respect to the Vlasov-Poisson equations. Therefore, the whole series of equilibria is dynamically stable, even the equilibrium states that lie after the first turning point of energy.

**Antonov first law**: Let us consider an isotropic stellar system with a distribution function of the form \( f = f(\epsilon) \) with \( f'(\epsilon) < 0 \). Introducing the density \( \rho(r) = \int f(\nu^2/2 + \Phi(r)) d\nu \) and the pressure \( P(r) = \frac{1}{\nu} \int f(\nu^2/2 + \Phi(r)) \nu^2 d\nu \), and eliminating formally \( \Phi(r) \) between these two expressions, we find that the corresponding gas is barotopic: \( P(r) = P[\rho(r)] \).

Then, proceeding as in Appendix D2 we can show that it satisfies the condition of hydrostatic equilibrium \( \nabla P + \rho \nabla \Phi = 0 \). Therefore, to any stellar system described by a distribution function of the form \( f = f(\epsilon) \) with \( f'(\epsilon) < 0 \) we can associate a corresponding barotropic star with an equation of state \( P = P(\rho) \) that satisfies the condition of hydrostatic equilibrium. Using the Schwarz inequality, Antonov [311] and Lynden-Bell and Sanitt [301] have shown that a stellar system with \( f = f(\epsilon) \) and \( f'(\epsilon) < 0 \) is stable with respect to the Vlasov-Poisson equations whenever the corresponding barotropic star is stable with respect to the Euler-Poisson equations. This is what Binney and Tremaine [301] have called the Antonov first law. We can recover this result with a different method related to the concept of ensembles inequivalence. It can be shown (see [219]) and Appendices D2 and D1 that the “canonical” criterion of dynamical stability [113] for a stellar system described by the Vlasov-Poisson equations is equivalent to the criterion of dynamical stability [H2] for the corresponding barotropic star described by the Euler-Poisson equations. Since “canonical stability” implies dynamical stability for collisionless stellar systems [see Eq. (H1)], we conclude that

\[
\boxed{\text{(H5)}}
\]

Therefore, the dynamical stability with respect to the Euler-Poisson equations implies the dynamical stability with respect to the Vlasov-Poisson equations. However, the converse is wrong [38]. This provides a new derivation of the Antonov first law [219] in terms of ensembles inequivalence. In particular, this derivation is valid for nonlinear dynamical stability while the original proof [301, 301, 311] was restricted to linear (spectral) dynamical stability.

2. **General relativity: Vlasov-Einstein equations**

The preceding results [H1]-[H4] can be extended to the context of general relativity [248, 312, 314, 316]. In particular, since “microcanonical” stability implies dynamical stability, using the Poincaré criterion [260], we can generically conclude that the series of equilibria is dynamically stable at least until the turning point of binding energy [248]. Now, there is a conjecture by Ipser [248] that, in general relativity, the “microcanonical” criterion [H2] is equivalent to the criterion of dynamical stability [H1], contrary to the Newtonian case. We have

\[
\boxed{\text{(H6)}}
\]

Accordingly, the series of equilibria is dynamically stable before the turning point of binding energy and becomes unstable afterwards. This result has been established numerically for heavily truncated isothermal distributions and stellar polytropes [313, 317]. The conjecture consists in extending its validity to all distribution functions.

**Relativistic Antonov first law**: Let us consider a star cluster with an isotropic distribution function of the form \( f = f(Ee^{\nu(r)/2}) \) with \( f'(Ee^{\nu(r)/2}) < 0 \). Introducing the energy density \( \epsilon(r) = \int f(E) e^{\nu(r)/2} E(p) dp \) and the pressure \( P(r) = (1/3) \int f(E) e^{\nu(r)/2} p E'(p) dp \), and eliminating formally \( \nu(r) \) between these two expressions, we find that the corresponding gas is barotopic: \( P(r) = P[\epsilon(r)] \). Then, proceeding as in Appendix D2 we can show that it satisfies the TOV equations expressing the condition of hydrostatic equilibrium [102]. Using the Schwarz inequality, Ipser [314] has obtained a relativistic generalization of the linear Antonov first law. On the other hand, it can be shown [257, 258, 260, 263] that the “canonical” criterion of dynamical stability [H3] for a star cluster described by the Vlasov-Einstein equations is equivalent to the criterion of dynamical stability [G6] for the corresponding barotropic

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37 We can use this method to show graphically (without calculation) that all the stellar polytropes are stable [219]. This result was originally proven by Antonov [311] with rather complicated calculations.

38 There is an exception. In the case of an infinite and homogeneous medium, collisionless stellar systems (Vlasov) and self-gravitating fluids (Euler) behave in the same way with respect to the Jeans instability in the sense that they lead to the same criterion for instability [313].

39 To make the correspondance with Appendix D2 we just need to replace \( k_B T(r) \) by \( e^{-\nu(r)/2} \). In that case, Eq. (H4) reduces to Eq. (102).
star described by the Euler-Einstein equations. Since “canonical stability” implies dynamical stability for collisionless star clusters [see Eq. (H6)], we conclude that

\[ \Theta \Leftrightarrow H3 \Rightarrow H1. \]  

(H7)

Therefore, the dynamical stability with respect to the Euler-Einstein equations implies the dynamical stability with respect to the Vlasov-Einstein equations. This provides a generalization of the nonlinear Antonov first law obtained in Newtonian gravity [219].

Appendix I: Black-body radiation in general relativity

In this Appendix, we consider a gas of photons (black-body radiation) that is so intense that general relativity must be taken into account. This leads to the concept of “photon stars” or self-gravitating black-body radiation. This problem has been studied in detail in [249, 254]. Below, we recall the basic equations determining the statistical equilibrium state of a gas of photons in general relativity and compare these results with those obtained in Sec. III for material particles such as self-gravitating fermions.

1. Thermodynamics of the black-body radiation

The distribution function of a gas of photons is

\[ f(p) = \frac{1}{\hbar^3} \frac{1}{e^{\beta pc} - 1}. \]  

(I1)

This corresponds to the Bose-Einstein statistics in the ultrarelativistic limit \((E = pc)\) and with a vanishing chemical potential \((\mu = 0)\). These simplifications arise because the photons have no rest mass. Using Eq. (114), we find that the energy density is related to the temperature by the Stefan-Boltzmann law

\[ \epsilon = \frac{24\pi}{h^3c^3}(k_B T)^4 \frac{\pi^4}{90}. \]  

(I2)

The factor in front of \(T^4\) is the Stefan-Boltzmann constant. Using Eq. (113), we find that the pressure is given by

\[ P = \frac{8\pi}{h^3c^3}(k_B T)^3 \frac{\pi^4}{90}. \]  

(I3)

It is related to the energy density by the linear equation of state

\[ P = \frac{1}{3} \epsilon. \]  

(I4)

This linear relationship, with a coefficient 1/3, is valid for an arbitrary ultrarelativistic gas (see Appendix A2). Using Eq. (113), we find that the particle density is given by

\[ n = \frac{8\pi}{h^3c^3}(k_B T)^3 \zeta(3), \]  

where \(\zeta(3) = 1.202056\ldots\) is the Apéry constant (Riemann zeta function in \(x = 3\)). The pressure is related to the particle density through the polytropic equation of state

\[ P = Kn^{4/3} \quad \text{with} \quad K = \frac{\pi^4}{90} \left[ \frac{hc}{8\pi\zeta(3)^4}\right]^{1/3}. \]  

(I6)

Finally, using the integrated Gibbs-Duhem relation (135) with \(\mu = 0\), we find that the entropy density is given by

\[ s = k_B \frac{32\pi^5}{90h^3c^3}(k_B T)^3. \]  

(I7)

It can also be obtained by substituting Eq. (11) into the the Bose-Einstein entropy

\[ s = -k_B \int f \left( f \ln f - (1 + f) \ln(1 + f) \right) dp \]  

where \(f = 1/h^3\).
We see that the entropy density is proportional to the particle density:

\[ s = \lambda n k_B \quad \text{with} \quad \lambda = \frac{4\pi^4}{90\zeta(3)}. \]  

(I8)

More details about these relations and their consequences can be found in Ref. [254].

2. Mechanical derivation of the Tolman relation

Substituting the relation \( \epsilon = 3P \) from Eq. (I4) into Tolman’s equation of hydrostatic equilibrium (102), we get

\[ \frac{d\ln P}{dr} = -2\frac{d\nu}{dr}. \]  

(I9)

On the other hand, according to Eq. (I3), we have

\[ \frac{d\ln P}{dr} = 4\frac{d\ln T}{dr}. \]  

(I10)

These two equations directly imply the Tolman relation

\[ \frac{d\ln T}{dr} = -\frac{1}{2}\frac{d\nu}{dr} \quad \Rightarrow \quad T(r)e^{\nu(r)/2} = \text{cst}. \]  

(I11)

This derivation is valid only for the black-body radiation. It was given by Tolman [232] as a particular example of his relation before considering the general case of an arbitrary perfect fluid.

**Remark:** This derivation presupposes the condition of hydrostatic equilibrium (102). In the following section, we show that this equation can be obtained from the maximization of the entropy \( S \) at fixed mass-energy \( Mc^2 \).

3. Equivalence between dynamical and thermodynamical stability for the self-gravitating black-body radiation

According to Eq. (I8), the entropy of the black-body radiation is proportional to the particle number:

\[ S = \lambda N k_B \quad \text{with} \quad \lambda = \frac{4\pi^4}{90\zeta(3)}. \]  

(I12)

The condition of thermodynamical stability, corresponding to the maximization of the entropy at fixed mass-energy:

\[ \max \{ S | \mathcal{E} = Mc^2 \text{ fixed} \}, \]  

(I13)

turns out to be equivalent to the maximization of the particle number at fixed mass-energy:

\[ \max \{ N | M \text{ fixed} \}, \]  

(I14)

which is itself equivalent to the minimization of the mass-energy at fixed particle number:

\[ \min \{ M | N \text{ fixed} \}, \]  

(I15)

**Remark:**

**Remark:**

**Remark:**

This is a particular case where Ipser’s conjecture [248] (see Appendix [142]) can be easily demonstrated.41

41 However, it is important to realize that, for the self-gravitating black-body radiation, dynamical stability refers to the Euler-Einstein equations while, for a collisionless star cluster, it refers to the Vlasov-Einstein equations. This is a difference of fundamental importance.
The maximization problem determining the thermodynamical stability of the self-gravitating black-body radiation in general relativity was first studied by Tolman [232], and later by Cocke [241], Sorkin et al. [249] and Chavanis [254]. The variational principle for the first variations (extremization) can be written as

\[
\delta S - \frac{1}{T_\infty} \delta \mathcal{E} = 0 \quad \Rightarrow \quad \delta N - \frac{1}{\lambda} \beta_\infty c^2 \delta \mathcal{M} = 0,
\]

where \(1/T_\infty\) is a Lagrange multiplier. It leads to the TOV equations (equivalent to the condition of hydrostatic equilibrium) and to the Tolman relation (the Lagrange multiplier \(T_\infty\) corresponds to the Tolman temperature). Then, considering the second variations of \(S\), it can be shown that the self-gravitating black-body radiation is linearly stable with respect to the Euler-Einstein equations if, and only if, it is a local maximum of \(S\) at fixed \(M\). This is also equivalent to its spectral stability. Indeed, the complex pulsations \(\omega\) of the normal modes of the linearized Euler-Einstein equations [51, 306] satisfy \(\omega^2 > 0\) for all modes if, and only if, \(\delta^2 S < 0\) for all perturbations that conserve \(M\). Using the Poincaré criterion [266], we can show [254] that the series of equilibria is thermodynamically and dynamically stable before the turning points of mass-energy \(M\), particle number \(N\), binding energy \(E\), or entropy \(S\) (they all coincide) and that it becomes thermodynamically and dynamically unstable afterwards. Furthermore, the curve \(S(\mathcal{E})\) displays spikes at its extremal points (since \(\delta S = 0 \Leftrightarrow \delta \mathcal{E} = 0\)). We refer to [241, 249, 254] for the derivation of these results.

Remark: In the case of material particles, the statistical equilibrium state is obtained by maximizing the entropy at fixed mass-energy and particle number. In the case of the self-gravitating black-body radiation, the statistical equilibrium state is obtained by maximizing the entropy, which is proportional to the particle number, at fixed mass-energy. How can we understand this difference? First, we have to realize that, in the case of the black-body radiation, the particle number is not fixed. What is fixed instead is the ratio between the chemical potential and the temperature. Therefore, the correct manner to treat the thermodynamics of the self-gravitating black-body radiation is to work in the grand microcanonical ensemble [190, 203] where \(\alpha = \mu/k_B T\) and \(E = M c^2\) are fixed. In that ensemble, the thermodynamic potential is \(\mathcal{K} = S + \alpha k_B N\). The statistical equilibrium state is then obtained by maximizing \(\mathcal{K}\) at fixed mass-energy:

\[
\max \{ \mathcal{K} \mid \mathcal{E} = M c^2 \text{ fixed} \}.
\]

The extremization problem (first variations) yields

\[
\delta \mathcal{K} - \frac{1}{T_\infty} \delta \mathcal{E} = 0.
\]

Now, for (massless) photons, the chemical potential vanishes: \(\mu = 0\). This implies \(\alpha = 0\) and \(\mathcal{K} = S\). In that case, the maximization problem (117) reduces to (113).

Appendix J: The Tolman-Klein relations

In this Appendix, we review the main results given in the seminal papers of Tolman [232] and Klein [240].

1. Tolman’s (1930) paper

In a paper published in 1930, Tolman [232] investigated “the weight of heat and thermal equilibrium in general relativity”. His main finding is that, even at thermodynamic equilibrium, the temperature is inhomogeneous in the presence of gravitation. He discovered a definite relation connecting the distribution of temperature \(T(r)\) throughout the system to the gravitational potential (or metric coefficient) \(\nu(r)\). Tolman’s relation [see Eq. (158)] between equilibrium temperature and gravitational potential was something essentially new in thermodynamics since, until his work, uniform temperature throughout any system which has come to thermal equilibrium had hitherto been taken as an inescapable part of thermodynamic theory.

Tolman first considered the case of a weak gravitational field described by Newtonian gravitation. By maximizing the entropy for an isolated system he obtained an approximate relation between the temperature distribution and the Newtonian gravitational potential [see Eq. (214)]. This can be viewed as a post-Newtonian relation since the temperature gradient is inversely proportional to the square of the velocity of light.

He then considered the case of the black-body radiation. By performing a purely mechanical treatment of temperature distribution based on the Einstein equations (using Eq. (102) representing the general relativistic extension
of the condition of hydrostatic equilibrium) he obtained an exact general relativistic relation between the proper temperature and the metric coefficient \( \nu \) [see Eq. (114)].

He then recovered this result by maximizing the entropy of the self-gravitating black-body radiation by using the formalism of relativistic thermodynamics that he had developed a few years earlier. Therefore, in the simple case of the black-body radiation where a mechanical treatment can be given, the thermodynamical and mechanical treatments of temperature distribution under the action of gravity lead to the same result.

Finally, he generalized his thermodynamical approach (maximum entropy principle) to the case of any perfect fluid and obtained the Tolman relation (158) in a general setting.

He noted at the end of his paper that the maximum entropy principle implies the general relativistic condition of hydrostatic equilibrium [see Eq. (102)] contained in the Einstein equations. He noted: “It may seem strange that this purely mechanical equation holding within the interior of the system should be derivable from the application of thermodynamics to the system as a whole. The result, however, is the relativity analogue to the equation for change in pressure with height obtained by Gibbs (“Scientific Papers,” Longmans, Green 1906, equation 230, p. 145) in his thermodynamic treatment of the conditions of equilibrium under the influence of gravity. Indeed the whole treatment of this article may be regarded as the relativistic extension of this part of Gibbs’ work.”

2. Klein’s (1949) paper

In a paper entitled “On the thermodynamical equilibrium of fluids in gravitational fields” published in 1949, Klein managed to derive the Tolman relation (158), together with a similar relation between the chemical potential \( \mu(r) \) and the metric coefficient \( \nu(r) \) [see Eq. (159)] with almost no calculation, by using essentially the Gibbs-Duhem relation and the first principle of thermodynamics. We give below a summary of Klein’s calculations.

Klein started from the first principle of thermodynamics

\[
dE = -PdV + TdS + \mu dN. \tag{J1}
\]

Since \( E \) is a homogeneous function of the first degree in the three variables \( V, S \) and \( N \), the Euler theorem implies that

\[
E = -PV + TS + \mu N, \tag{J2}
\]

which is the Gibbs-Duhem relation (see Appendix E). From Eqs. (J1) and (J2), we get

\[
d\left(\frac{P}{T}\right) = \frac{N}{V}d\left(\frac{\mu}{T}\right) - \frac{E}{V}d\left(\frac{1}{T}\right). \tag{J3}
\]

Written under a local form, with the variables \( s = S/V, n = N/V \) and \( \epsilon = E/V \), Eqs. (J1), (J2) return Eq. (130) and Eqs. (135)-(137). In turn, Eq. (136) can be written as

\[
\frac{dP}{dr} = \epsilon + \frac{PdT}{T} + nT \frac{d}{dr}\left(\frac{\mu}{T}\right). \tag{J4}
\]

Combined with the condition of hydrostatic equilibrium from Eq. (102), we get

\[
\frac{1}{2} \frac{d\nu}{dr} + \frac{1}{T} \frac{dT}{dr} = -nT \frac{d}{dr}\left(\frac{\mu}{T}\right). \tag{J5}
\]

At that point, Klein considered several independent substances present in the same gravitational field and argued that an equation of the type (J5) holds for each of them separately with the same values of \( \nu \) and \( T \). As one such substance, we always have the radiation for which \( \mu = 0 \). Thus, we get

\[
\frac{1}{2} \frac{d\nu}{dr} + \frac{1}{T} \frac{dT}{dr} = 0 \quad \Rightarrow \quad T(r)e^{\nu(r)/2} = \text{cst}, \tag{J6}
\]

which is Tolman’s relation. Then, for all other substances

\[
\frac{\mu(r)}{k_B T(r)} = \text{cst} \quad \Rightarrow \quad \mu(r)e^{\nu(r)/2} = \text{cst}, \tag{J7}
\]

which is Klein’s relation. As emphasized by Klein, this relation constitutes the relativistic generalization of the well-known Gibbs condition for the equilibrium in a gravitational field.

\[42\text{ The calculations of Tolman based on the maximum entropy principle are comparatively much more complicated.}\]
Appendix K: Thermodynamic identities

In this Appendix, we regroup useful thermodynamic identities valid for Newtonian and general relativistic barotropic gases.

1. Newtonian isentropic or cold barotropic gases

The first principle of thermodynamics writes

\[ d \left( \frac{u}{n} \right) = -P d \left( \frac{1}{n} \right) + T d \left( \frac{s}{n} \right), \]  

(K1)

where \( u \) is the density of internal energy. We assume that \( T d(s/n) = 0 \). This corresponds to cold \((T = 0)\) or isentropic \((s/n = \lambda = \text{cst})\) gases. In that case, Eq. (K1) reduces to

\[ d \left( \frac{u}{n} \right) = -P d \left( \frac{1}{n} \right) = P \frac{n^2}{n^2} dn. \]  

(K2)

For a barotropic equation of state \( P = P(n) \), Eq. (K2) can be integrated into

\[ u(n) = n \int^n P(n') \frac{n'}{n'^2} dn'. \]  

(K3)

The internal energy is

\[ U = \int n \int^n P(n') \frac{n'}{n'^2} dn' dr. \]  

(K4)

We also have

\[ du = \frac{P + u}{n} \frac{n}{n^2} dn = h \frac{n}{n^2} dn \]  

(K5)

and

\[ dh = \frac{dP}{n}, \]  

(K6)

where

\[ h = \frac{P + u}{n} \]  

(K7)

is the enthalpy. We note the identities

\[ u'(n) = h(n), \quad u''(n) = h'(n) \]  

(K8)

and

\[ P(n) = nh(n) - u(n) = nu'(n) - u(n). \]  

(K9)

The energy of a Newtonian isentropic or cold barotropic self-gravitating gas is \( W = U + W \) where \( U \) is the internal energy and \( W \) is the gravitational energy. A stable equilibrium state of the Euler-Poisson equations is a minimum of energy \( W \) at fixed particle number \( N \) (see Appendix G1).43 If the pressure can be written as \( P(n) = T \Pi(n) \), we get

\[ W = W - TS_{\text{eff}} \]  

where \( S_{\text{eff}} = -n \int^n \frac{n}{n'} P(n') \frac{n'}{n'^2} dn' dr \) is a generalized entropy of the density \( n \).221 222

Remark: For an ideal gas at \( T = 0 \), the thermodynamic identities of Sec. II reduce to \( d\epsilon_{\text{kin}} = \mu \frac{n}{n'} dn', \epsilon_{\text{kin}} + P - \mu n = 0 \) and \( dP = n d\mu \). We can check that they coincide with Eqs. (K2)-(K9) with \( u = \epsilon_{\text{kin}} \) and \( h = \mu \).

43 In the isentropic case \( s/n = \lambda \) we have \( S = \lambda N \). Therefore a minimum of \( W \) at fixed \( N \) is also a maximum of \( S \) at fixed \( W \).
2. General relativistic isentropic or cold barotropic gases

In general relativity, the first principle of thermodynamics writes

\[ d\left(\frac{\epsilon}{n}\right) = -Pd\left(\frac{1}{n}\right) + Td\left(\frac{s}{n}\right), \]  

(K10)

where \( \epsilon = \rho c^2 + u \) is the mass-energy density and \( \rho = nm \) is the rest-mass density. The relations of Appendix [K1] remain valid with \( u \) or with \( \epsilon \). When \( Td(s/n) = 0 \), Eq. (K10) reduces to

\[ d\epsilon = P + \epsilon \frac{dn}{n}. \]  

(K11)

For a barotropic equation of state of the form \( P = P(n) \), we obtain

\[ \epsilon(n) = nm c^2 + n \int P(n') \frac{dn'}{n'^2}. \]  

(K12)

Since \( \epsilon \) is a function of \( n \), the pressure is a function \( P = P(\epsilon) \) of the energy density. Therefore, Eq. (K11) can be integrated into

\[ n(\epsilon) = e^{\int P(\epsilon') n' \frac{dn'}{n'^2}}. \]  

(K13)

The binding energy of a general relativistic isentropic or cold barotropic gas is \( E = (M - Nm)c^2 \) where \( M \) is the mass and \( N \) is the particle number. A stable equilibrium state of the Euler-Einstein equations is a minimum of energy \( E \) at fixed particle number \( N \) (see Appendix [G2]). In the Newtonian limit, \( E \rightarrow U + W = \mathcal{W} \) (see Sec. IV F with \( u \) in place of \( \epsilon_{kin} \)) and we recover the results of Appendix [K1].

Remark: For a linear equation of state

\[ P = q\epsilon \quad \text{with} \quad q = \gamma - 1, \]  

(K14)

and for \( Td(s/n) = 0 \) we obtain

\[ P = Kn^\gamma \quad \text{and} \quad \epsilon = \frac{K}{q} n^\gamma, \]  

(K15)

where \( K \) is a constant of integration. When \( \mu = 0 \) the integrated Gibbs-Duhem relation (135) reduces to

\[ s = \frac{\epsilon + P}{T}. \]  

(K16)

When \( T = 0 \) we obtain \( P = -\epsilon \). Therefore, the equation of state of dark energy corresponds to a relativistic gas at \( T = 0 \) with \( \mu = 0 \). When \( s = \lambda n \) we obtain

\[ T = \frac{q + 1}{q} \frac{K}{\lambda} n^{\gamma-1}. \]  

(K17)

The case \( \mu = 0 \) applies to the black-body radiation for which \( q = 1/3 \). In that case, we recover the relations of Appendix [I] but the constant \( K \) is not determined by the present method.

3. Newtonian self-gravitating gases at statistical equilibrium

We consider a Newtonian self-gravitating gas at statistical equilibrium (see Sec. [I]). The first principle of thermodynamics writes

\[ d\left(\frac{\epsilon_{kin}}{n}\right) = -Pd\left(\frac{1}{n}\right) + Td\left(\frac{s}{n}\right). \]  

(K18)

\[ 44 \] In the isentropic case \( s/n = \lambda \) we have \( S = \lambda N \). Therefore a minimum of \( E \) at fixed \( N \) is also a maximum of \( S \) at fixed \( E \) (or \( \epsilon \)).
Since the temperature \( T \) is uniform at statistical equilibrium, we have
\[
d\left( \frac{\epsilon_{\text{kin}} - Ts}{n} \right) = -Pd\left( \frac{1}{n} \right) = \frac{P}{n^2}dn.
\]
(K19)

On the other hand, we have seen in Sec. [II] that the gas has a barotropic equation of state \( P = P(n) \). Therefore, the foregoing equation can be integrated into
\[
\epsilon_{\text{kin}}(n) - Ts(n) = n \int^n P(n') \frac{dn'}{n'}.
\]
(K20)

Introducing the internal energy defined by Eq. (K3) we obtain the important relation
\[
\epsilon_{\text{kin}}(n) - Ts(n) = u(n).
\]
(K21)

Integrating this relation over the whole configuration, we find that the entropy is given by
\[
S = \frac{E_{\text{kin}} - U}{T},
\]
(K22)

which returns Eq. (F12). On the other hand, the total energy is given by \( E = E_{\text{kin}} + W \). In the microcanonical ensemble, a stable equilibrium state is a maximum of entropy \( S \) at fixed energy \( E \) and particle number \( N \). On the other hand, in the canonical ensemble, a stable equilibrium state is a minimum of free energy \( F \) at fixed particle number \( N \). Using Eq. (K22), we find that the free energy is given by
\[
F = E - TS = E_{\text{kin}} + W - TS = U + W,
\]
(K23)

which returns Eq. (F25). We see that
\[
F = \mathcal{W}.
\]
(K24)

Therefore, the criterion of thermodynamical stability in the canonical ensemble (minimum of \( F \) at fixed \( N \)) coincides with the criterion of dynamical stability with respect to the Euler-Poisson equations (minimum of \( \mathcal{W} \) at fixed \( N \)).

Remark: At \( T = 0 \) we see from Eq. (K21) that \( \epsilon_{\text{kin}}(n) = u(n) \). This implies that \( E_{\text{kin}} = U \) so that \( E = E_{\text{kin}} + W = U + W = \mathcal{W} \). This is a particular case of the general relation (K24). At \( T = 0 \), the equilibrium state is obtained either by minimizing \( E = E_{\text{kin}} + W \) at fixed particle number \( N \) or by minimizing \( \mathcal{W} = U + W \) at fixed particle number \( N \).
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[23] S. Chandrasekhar, The Observatory 57, 93 (1934)
[24] S. Chandrasekhar, The Observatory 57, 373 (1934)
[25] S. Chandrasekhar, MNRAS 95, 207 (1935)
[26] S.A. Kaplan, Uch. Zap. L'vov, Univ 15, 109 (1949)
[27] S. Chandrasekhar, R.F. Tooper, Astrophys. J. 139, 1396 (1964)
[28] E. Rutherford, Proc. Roy. Soc. London A 97, 374 (1920)
[29] J. Chadwick, Nature 129, 312 (1932)
[30] W. Baade, F. Zwicky, Proc. Nat. Ac. Sc. 20, 254 (1934)
[31] W. Baade, F. Zwicky, Proc. Nat. Ac. Sc. 20, 259 (1934)
[32] W. Baade, F. Zwicky, Phys. Rev. 46, 76 (1934)
[33] J.R. Oppenheimer, G.M. Volkoff, Phys. Rev. 54, 374 (1939)
[34] F. Zwicky, Astrophys. J. 88, 522 (1938)
[35] F. Zwicky, Phys. Rev. 55, 726 (1939)
[36] M. Ortega-Rodríguez et al., arXiv:1703.04234
[37] H. Kragh, Arch. Hist. Exact Sci. 57, 395 (2003)
[38] P.H. Chavanis, Phys. Dark Univ. 24, 100271 (2019)
[39] B.K. Harrison, M. Wakano, J.A. Wheeler, Onz. Cons. de Physique Solvay, Stoops, Brussels, p. 124 (1958)
[40] A.G.W. Cameron, Astrophys. J. 130, 884 (1959)
[41] Ya. B. Zel'dovich, J. Exptl. Theoret. Phys. (U.S.S.R.) 10, 403 (1960)
[42] T. Hamada, E.E. Salpeter, Astrophys. J. 134, 683 (1961)
[43] Ya. B. Zel'dovich, J. Exptl. Theoret. Phys. (U.S.S.R.) 41, 1609 (1961)
[44] V.A. Ambartsumyan, G.S. Saakyan, Soviet Astron. 5, 601 (1962)
[45] V.A. Ambartsumyan, G.S. Saakyan, Soviet Astron. 5, 779 (1962)
[46] Ya. B. Zel'dovich, J. Exptl. Theoret. Phys. (U.S.S.R.) 42, 641 (1962)
[47] Ya. B. Zel'dovich, J. Exptl. Theoret. Phys. (U.S.S.R.) 42, 1667 (1962)
[48] J.A. Wheeler, in Relativity and Gravitation, H.Y. Chiu and W.F. Hoffmann, eds., Chap. 10 (Benjamin, New-York, 1963)
[49] G.S. Saakyan, Soviet Astron. 7, 60 (1963)
[50] N.A. Dmitriev, S.A. Kholin, Voprosy kosmogoni 9, 254 (1963)
[51] S. Chandrasekhar, Astrophys. J. 140, 417 (1964)
[52] R.F. Tooper, Astrophys. J. 140, 434 (1964)
[53] H.-Y. Chiu, Ann. Phys. 26, 364 (1964)
[54] C.W. Misner, H.S. Zapolsky, Phys. Rev. Lett. 12, 635 (1964)
[55] G.S. Saakyan, Yu. L. Vartanyan, Soviet Astron. 8, 147 (1964)
[56] C.L. Inman, Astrophys. J. 141, 187 (1965)
[57] R.F. Tooper, Astrophys. J. 142, 1541 (1965)
[58] B.K. Harrison, Phys. Rev. 137, 1644 (1965)
[59] E.R. Harrison, Astrophys. J. 142, 1643 (1965)
[60] V.A. Ambartsumyan, G.S. Saakyan, Astrofizika 1, 1 (1965)
[61] K.S. Thorne, Science 150, 1671 (1965)
[62] S. Tsuruta, A.G.W. Cameron, Canad. J. Phys. 44, 1895 (1966)
[63] D. Meltzer, K. Thorne, Astrophys. J. 145, 514 (1966)
[64] J. Bardeen, K. Thorne, D. Meltzer, Astrophys. J. 145, 505 (1966)
[65] J.A. Wheeler, Ann. Rev. Astron. Astrophys. 4, 393 (1966)
[66] P. Cazzola, L. Lucaroni, C. Scarinci, Nuovo Cimento 52, 411 (1967)
[67] B.K. Harrison, K.S. Thorne, M. Wakano, J.A. Wheeler, Gravitation Theory and Gravitational Collapse, (Chicago University Press, Chicago, 1965)
[68] A. Hewish, S.J. Bell, J.D.H. Pilkington, P.F. Scott, R.A. Collins, Nature 217, 709 (1968)
[69] T. Gold, Nature 218, 731 (1968)
[70] K. Schwarzschild, Berliner Sitzungsbesichte, 189 (1916)
[71] K. Schwarzschild, Berliner Sitzungsbesichte, 424 (1916)
[72] R.P. Kerr, Phys. Rev. Lett. 11, 237 (1963)
[73] J.R. Oppenheimer, H. Snyder, Phys. Rev. 56, 455 (1939)
[74] J.A. Wheeler, American Scientist 56, 1 (1968)
[75] J.A. Wheeler, The American Scholar 37, 248 (1968)
[76] A. Ewing, Sci. News Lett. 85, 39 (1964)
[77] A. Rosenfeld, Life Magazine 24, 11 (1964)
[78] C.A.R. Herdeiro, J.P.S. Lemos, arXiv:1811.06587
[79] H.-Y. Chiu, Physics Today 17, 21 (1964)
[80] M.A. Markov, Phys. Lett. 10, 122 (1964)
[81] R. Cowxik and J. McClelland, Phys. Rev. Lett. 29, 669 (1972)
[82] R. Cowxik and J. McClelland, Astrophys. J. 180, 7 (1973)
[83] S. Tremaine and J.E. Gunn, Phys. Rev. Lett. 42, 407 (1979)
[84] R. Ruffini, Lett. Nuovo Cim. 29, 161 (1980)
[85] A. Crollalanza, J.G. Gao, R. Ruffini, Lett. Nuovo Cim. 32, 411 (1981)
[147] R. Schödel et al., Nature 419, 694 (2002)
[148] M.J. Reid, Int. J. Mod. Phys. D 18, 889 (2009)
[149] R. Genzel, F. Eisenhauer, S. Gillessen, Rev. Mod. Phys. 82, 3121 (2010)
[150] P.H. Chavanis, arXiv:1810.08948
[151] D. Lynden-Bell, Mon. Not. Roy. Astr. Soc. 136, 101 (1967)
[152] A. Kull, R.A. Treumann, H. Böhringer, Astrophys. J. 466, L1 (1996)
[153] P.H. Chavanis, J. Sommeria, R. Robert, Astrophys. J. 471, 385 (1996)
[154] A. Kull, R.A. Treumann, H. Böhringer, Astrophys. J. 484, 58 (1997)
[155] P.H. Chavanis, Statistical mechanics of violent relaxation in stellar systems, in Multiscale Problems in Science and Technology, edited by N. Antonić, C.J. van Duijn, W. Jäger, and A. Mikelić (Springer, 2002)
[156] Y. Pomeau, M. Le Berre, P.H. Chavanis and B. Denet, Eur. Phys. J. E 37, 26 (2014)
[157] P.H. Chavanis, J. Sommeria, R. Robert, Astrophys. J. 471, 385 (1996)
[158] A. Kull, R.A. Treumann, H. Böhringer, Astrophys. J. 484, 58 (1997)
[159] P.H. Chavanis, Statistical mechanics of violent relaxation in stellar systems, in Multiscale Problems in Science and Technology, edited by N. Antonić, C.J. van Duijn, W. Jäger, and A. Mikelić (Springer, 2002)
[160] Y. Pomeau, M. Le Berre, P.H. Chavanis and B. Denet, Eur. Phys. J. E 37, 26 (2014)
[161] P.H. Chavanis, G. Alberti, arXiv:1908.10303
[162] P. Hertel and W. Thirring, Commun. Math. Phys. 24, 22 (1971)
[163] P. Hertel, H. Narnhofer and W. Thirring, Commun. Math. Phys. 28, 159 (1972)
[164] J. Messer, Z. Physik 33, 313 (1979)
[165] B. Baumgartner, Commun. Math. Phys. 48, 207 (1976)
[166] H. Narnhofer and G.L. Sewell, Commun. Math. Phys. 71, 1 (1980)
[167] W. Braun and K. Hepp, Commun. Math. Phys. 56, 101 (1977)
[168] J. Messer, J. Math. Phys. 22, 2910 (1981)
[169] J. Katz, Mon. Not. R. Astron. Soc. 190, 497 (1980)
[170] M. Lecar, J. Katz, Astrophys. J. 243, 983 (1981)
[171] J. Messer, H. Spohn, J. Stat. Phys. 29, 561 (1982)
[172] J.F. Luciani, R. Pellat, Astrophys. J. 233, 299 (1978)
[173] J. Katz, Mon. Not. R. Astron. Soc. 138, 495 (1968)
[174] J.R. Ipser, Astrophys. J. 193, 463 (1974)
[175] L. Taff, H. van Horn, Astrophys. J. 197, L23 (1975)
[176] G. Horwitz, J. Katz, Astrophys. J. 211, 226 (1977)
[177] Y. Nakada, Publ. Astron. Soc. Japan 30, 57 (1978)
[178] I. Hachisu, D. Sugimoto, Prog. Theor. Phys. 60, 123 (1977)
[179] G. Horwitz, J. Katz, Astrophys. J. 222, 941 (1977)
[180] J. Katz, G. Horwitz, A. Dekel, Astrophys. J. 223, 299 (1978)
[181] J. Katz, Mon. Not. R. Astron. Soc. 188, 765 (1978)
[182] J. Katz, MNRAS 189, 817 (1979)
[183] J.R. Ipser, G. Horwitz, Astrophys. J. 232, 863 (1979)
[184] S. Inagaki, Publ. Astron. Soc. Japan 32, 213 (1980)
[185] J. Katz, MNRAS 190, 497 (1980)
[186] M. Lecar, J. Katz, Astrophys. J. 243, 983 (1981)
[187] J. Katz, Mon. Not. R. Astron. Soc. 138, 495 (1968)
[188] J.R. Ipser, Astrophys. J. 193, 463 (1974)
[189] L. Taff, H. van Horn, Astrophys. J. 197, L23 (1975)
[190] G. Horwitz, J. Katz, Astrophys. J. 211, 226 (1977)
[191] Y. Nakada, Publ. Astron. Soc. Japan 30, 57 (1978)
[192] I. Hachisu, D. Sugimoto, Prog. Theor. Phys. 60, 123 (1977)
[193] G. Horwitz, J. Katz, Astrophys. J. 222, 941 (1977)
[194] J. Katz, G. Horwitz, A. Dekel, Astrophys. J. 223, 299 (1978)
[195] J. Katz, Mon. Not. R. Astron. Soc. 188, 765 (1978)
[196] J. Katz, MNRAS 189, 817 (1979)
[197] J.R. Ipser, G. Horwitz, Astrophys. J. 232, 863 (1979)
[198] S. Inagaki, Publ. Astron. Soc. Japan 32, 213 (1980)
[199] J. Katz, MNRAS 190, 497 (1980)
[200] M. Lecar, J. Katz, Astrophys. J. 243, 983 (1981)
[201] J. Messer, H. Spohn, J. Stat. Phys. 29, 561 (1982)
[202] J.F. Luciani, R. Pellat, Astrophys. J. 237, 241 (1987)
[203] M. Kiessling, J. Stat. Phys. 55, 203 (1989)
[204] T. Padmanabhan, Astrophys. J. Supp. 71, 651 (1989)
[205] H.J. de Vega, N. Sanchez, F. Combes, Phys. Rev. D 54, 6008 (1996)
[206] J. Katz, I. Okamoto, MNRAS 317, 163 (2000)
[207] H.J. de Vega, N. Sanchez, Nucl. Phys. B 625, 409 (2002)
[208] H.J. de Vega, N. Sanchez, Nucl. Phys. B 625, 460 (2002)
[209] P.H. Chavanis, Astron. Astrophys. 381, 340 (2002)
[210] P.H. Chavanis, C. Rosier, C. Sire, Phys. Rev. E 66, 036105 (2002)
[211] C. Sire, P.H. Chavanis, Phys. Rev. E 66, 046133 (2002)
[212] P.H. Chavanis, Astron. Astrophys. 401, 15 (2003)
[213] P.H. Chavanis, Astron. Astrophys. 432, 117 (2005)
[214] L. Cassetti, C. Nardini, Phys. Rev. E 85, 061105 (2012)
[215] P.H. Chavanis, M. Lemou, F. Méhats, Phys. Rev. D 91, 063531 (2015)
[216] T. Padmanabhan, Phys. Rep. 188, 285 (1990)
[217] J. Katz, Found. Phys. 33, 223 (2003)
[218] A. Campa, T. Dauxois, S. Ruffo, Physics Reports 480, 57 (2009)
[273] H. Kandrup, Astrophys. J. 370, 312 (1991)
[274] P.H. Chavanis, Physica A 332, 89 (2004)
[275] S. Weinberg, Gravitation and Cosmology (John Wiley & Sons, 1972)
[276] J.T. Jelosen, Arkiv für Matematik, Astronomi och Fysik. 15, 1 (1921)
[277] G.D. Birkhoff, Relativity and Modern Physics (Cambridge Massachusetts, Harvard University Press, 1923)
[278] S. Deser, J. Franklin, American Journal of Physics 73, 261 (2005)
[279] A.S. Eddington, Nature 113, 192 (1924)
[280] D. Finkelstein, Phys. Rev. 110, 965 (1958)
[281] C. Fronsdal, Phys. Rev. 116, 778 (1959)
[282] M.D. Kruskal, Phys. Rev. 119, 1743 (1960)
[283] H.A. Buchdahl, Phys. Rev. 116, 1027 (1959)
[284] P.H. Chavanis, Eur. Phys. J. Plus 129, 38 (2014)
[285] Z. Roupas, Class. Quantum Grav. 32, 135023 (2015)
[286] P.H. Chavanis, Phys. Rev. D 76, 023004 (2007)
[287] Y.B. Zeldovich, M.A. Podurets, Soviet Astron. – AJ 9, 742 (1966)
[288] G.S. Bisnovatyi-Kogan, K.S. Thorne, Astrophys. J. 160, 875 (1970)
[289] E. Fackerell, Astrophys. J. 153, 643 (1968)
[290] L.D. Landau, E.M. Lifshitz, The Classical Theory of Field (Pergamon, 1951)
[291] C. Tsallis, Introduction to Nonextensive Statistical Mechanics (Springer, 2009)
[292] J.D. Bekenstein, Phys. Rev. D 7, 2333 (1973)
[293] S. Hawking, Commun. Math. Phys. 43, 199 (1975)
[294] J.M. Bardeen, B. Carter, S. Hawking, Commun. Math. Phys. 31, 161 (1973)
[295] T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995)
[296] T. Padmanabhan, Rep. Prog. Phys. 73, 046901 (2010)
[297] E. Verlinde, J. High Energy Phys. 04, 029 (2011)
[298] P.H. Chavanis, Eur. Phys. J. Plus 132, 248 (2017)
[299] L.D. Landau, E.M. Lifshitz, Statistical Physics (Pergamon, 1959)
[300] D. Lynden-Bell, N. Sanitt, Mon. Not. Roy. Astr. Soc. 143, 167 (1969)
[301] Binney, J., & Tremaine, S. 1987, Galactic Dynamics (Princeton Series in Astrophysics)
[302] A.S. Eddington, MNRAS 79, 2 (1918)
[303] P. Ledoux, C.L. Pekeris, Astrophys. J. 94, 124 (1941)
[304] S. Chandrasekhar, Astrophys. J. 138, 896 (1963)
[305] S. Chandrasekhar, Astrophys. J. 139, 664 (1964)
[306] S. Yabushita, Mon. Not. R. Astron. Soc. 165, 17 (1973)
[307] D. Lynden-Bell, Mon. Not. Roy. Astr. Soc. 144, 189 (1969)
[308] P. Bartholomew, Mon. Not. Roy. Astr. Soc. 151, 333 (1971)
[309] J.P. Doremus, M.R. Feix, G. Baumann, Astrophys. Space Sci. 13, 478 (1971)
[310] V.A. Antonov, Sov. Astron. 4, 859 (1961)
[311] V.A. Antonov, Vest. Leningr. Gos. Univ. 19, 96 (1962)
[312] J.R. Ipser, K.S. Thorne, Astrophys. J. 154, 251 (1968)
[313] P.H. Chavanis, Eur. Phys. J. B 85, 229 (2012)
[314] J.R. Ipser, Astrophys. J. 156, 509 (1969)
[315] J.R. Ipser, Astrophys. J. 158, 17 (1969)
[316] E.D. Fackerell, Astrophys. J. 162, 1053 (1970)
[317] S.L. Shapiro, S.A. Teukolsky, Astrophys. J. 208, 58 (1985)
[318] W. Gibbs, Collected Works (Yale University Press, New Haven, 1948) I, p. 144