Resummation at Large $Q$ and at Small $x$

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Abstract

We propose a unified and simple viewpoint to various evolution equations appropriate in different kinematic regions. We show that the resummation technique reduces to the Altarelli-Parisi equation, if the transverse degrees of freedom of a parton are ignored, to the Balitskii-Fadin-Kuraev-Lipatov equation, if the momentum fraction of a parton vanishes, and to the Gribov-Levin-Ryskin equation, if the recombination of gluons is included.
Recently, we applied the resummation technique to hard QCD processes, such as deeply inelastic scattering and Drell-Yan production, and grouped large radiative corrections into Sudakov factors \[1\]. In particular, we have demonstrated how to resum the double logarithms produced in radiative corrections to a quark distribution function. In this letter we shall show that the resummation technique can also deal with single-logarithm cases appropriate in different kinematic regions. It reduces to the conventional evolution equations, such as the Altarelli-Parisi (AP) equation \[2\], which organizes large logarithms $\ln Q$, $Q$ being a momentum transfer, the Balitskii-Fadin-Kuraev-Lipatov (BFKL) equation \[3\], which organizes large $\ln(1/x)$, $x$ being the Bjorken variable, and the Gribov-Levin-Ryskin equation \[4\], which organizes large $\ln Q \ln(1/x)$. It has been shown that the resummation technique is equivalent to the Wilson-loop formalism for the summation of soft logarithms in \[1, 5\]. Hence, we provide a unified viewpoint to all these methods in terms of the concept of the resummation. It will be found that the derivation of the above equations by means of the resummation is simpler.

1. The Altarelli-Parisi equation

We consider a flavor nonsinglet structure function associated with deeply inelastic scattering of a hadron with momentum $p^\mu = p^+ \delta^\mu_\perp$. A quark distribution function $\phi$ is constructed by factorizing collinear radiative gluons with momenta parallel to $p^\perp$ from the hard scattering amplitude onto an eikonal line in an arbitrary direction $n$. In axial gauge $n \cdot A = 0$ these collinear gluons are decoupled from the eikonal line, and the $n$ dependence goes into the gluon propagator $(-i/l^2)N^{\mu\nu}(l)$ with $N^{\mu\nu} = g^{\mu\nu} - (n^{\mu}l^{\nu} + n^{\nu}l^{\mu})/(n \cdot l) + n^2l^{\mu}l^{\nu}/(n \cdot l)^2$.

The key step in the resummation is to obtain the derivative $p^+ \frac{d}{dp^+} \phi(x, p^+, \mu)$. It has been argued that $\phi$ depends only on the ratio $(p \cdot n)^2/n^2 \[1\]. A chain rule relating $p^+ d/\!d p^+$ to $d/\!d n$ leads to the formula \[1\]

$$p^+ \frac{d}{dp^+} \phi(x, p^+, \mu) = 2 \tilde{\phi}(x, p^+, \mu),$$

shown in Fig. 1(a), where $x = -q^2/(2p \cdot q)$ is the Bjorken variable with $q$ the momentum transfer. The argument $p^+$ denotes the large logarithms $\ln(p^+/\mu)$ appearing in $\phi$. The function $\phi$ contains a new vertex at the outer end of the quark line, which is represented by a square and expressed as $\tilde{\v}_\alpha = n^2 v_\alpha/(v \cdot n n \cdot l) \[1\]$ with $v_\alpha = \delta_\alpha^+ + a$ vector along $p$. This vertex is obtained by applying $d/\!d n_\alpha$ to the gluon propagator $N^{\mu\nu}$. The coefficient 2 counts the two valence quark lines.
We show that Eq. (1) reduces to the AP equation, if the transverse degrees of freedom of a parton are ignored. For simplicity, we concentrate on the end-point region with \( x \to 1 \). As stated in \([1]\), the leading regions of the loop momentum flowing through the new vertex are soft and hard, in which \( \tilde{\phi} \) can be factorized into the convolution of the subdiagram containing the new vertex with the original distribution function \( \phi \). \( \tilde{\phi} \) is then written as

\[
\tilde{\phi}(x, p^+, \mu) = \int_x^1 d\xi [K(x, \xi, \mu) + G(x, \xi, \mu)] \phi(\xi, p^+, \mu), \tag{2}
\]

where the function \( K, \) absorbing soft divergences, corresponds to Fig. 1(b), and \( G, \) absorbing ultraviolet divergences, corresponds to Fig. 1(c). Their expressions are given by

\[
K = \frac{ig^2 C_F \mu^\varepsilon}{(2\pi)^{4-\varepsilon}} \frac{\delta(\xi - x)}{l^2} + \frac{2\pi i \delta(l^2)}{l^2} \delta(\xi - x - l^+ / p^+) N^\mu\nu, \tag{3}
\]

\[
G = \frac{ig^2 C_F \mu^\varepsilon}{(2\pi)^{4-\varepsilon}} \frac{\delta(\xi - x)}{l^2} - \frac{\nu p^+}{\mu} N^\mu\nu \log \left( \frac{\nu p^+}{\mu} \right), \tag{4}
\]

where \( \delta K \) and \( \delta G \) are additive counterterms. A straightforward calculation gives

\[
K = \frac{\alpha_s}{\pi \xi} C_F \left[ \frac{1}{1 - x/\xi} + \ln \frac{\nu p^+}{\mu} \right], \quad G = -\frac{\alpha_s}{\pi \xi} C_F \ln \frac{\nu p^+}{\mu}, \tag{5}
\]

with \( \nu = \sqrt{(v \cdot n)^2 / |n|^2} \).

We then treat \( K \) and \( G \) by renormalization group (RG) methods \([4]\). The RG solution of \( K + G \) is written as

\[
K(x, \xi, \mu) + G(x, \xi, \mu) = K(x, \xi, p^+) + G(x, \xi, p^+) - \int_{p^+} d\mu \frac{d\hat{\mu}}{\hat{\mu}} \lambda_K(\alpha_s(\hat{\mu})), \tag{6}
\]

where the anomalous dimension of \( K \) is defined by \( \lambda_K = \mu d\delta K/d\mu \). Obviously, the source of double logarithms, \( i.e., \) the integral cantaining \( \lambda_K \), vanishes. If the transverse degrees of freedom are included, the function
\(\delta(\xi - x - l^+/p^+)\) in Eq. (3) will be replaced by \(\delta(\xi - x) \exp(\mathbf{A} \cdot \mathbf{b})\), \(b\) being the transverse distance travelled by the parton \(p\). In this case the scale \(1/b\) is substituted for the lower bound of \(\mu\). Double logarithms then exist, implying that the soft logarithms in \(\phi\) do not cancel exactly. Therefore, the resummation technique can deal with both the double-logarithm and single-logarithm problems.

Inserting Eq. (6) into (2) and solving (1), we derive

\[
\phi(x, \mu) = \phi(x, \Lambda, \mu) + \int_\Lambda^\mu d\mu' \frac{\alpha_s(\mu')}{\pi} C_F \int_x^1 \frac{d\xi}{\xi} \frac{2}{(1-x/\xi)_+} \phi(\xi, \mu, \mu),
\]

(7)

with \(\Lambda\) an infrared cutoff. We have defined \(\phi(x, \mu) \equiv \phi(x, \mu, \mu)\), which does not contain large logarithms. Differentiating Eq. (7) with respect to \(\mu\), and substituting the RG equation \(\mu d\phi/\mu = -2\lambda_q \phi\), \(\lambda_q = -\alpha_s/\pi\) being the quark anomalous dimension in axial gauge, we have

\[
\mu \frac{d}{d\mu} \phi(x, \mu) = \frac{\alpha_s(\mu)}{\pi} C_F \int_x^1 \frac{d\xi}{\xi} \frac{2}{(1-x/\xi)_+} \phi(\xi, \mu) - \lambda_q(\mu) \phi(x, \mu),
\]

(8)

where Eq. (7) has been inserted to obtain the second term on the right-hand side of the above equation. Eq. (8) can be recast into

\[
\mu \frac{d}{d\mu} \phi(x, \mu) = \frac{\alpha_s(\mu)}{\pi} \int_x^1 \frac{d\xi}{\xi} P(x/\xi) \phi(\xi, \mu),
\]

(9)

with the function

\[
P(x) = C_F \left[ \frac{2}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right].
\]

(10)

It is easy to identify \(P\) as the splitting function \(P_{qq}(x) = C_F[(1+x^2)/(1-x)_+ + (3/2)\delta(1-x)]\) in the limit \(x \to 1\). The AP equations for intermediate \(x\) and for other kinds of partons will be discussed elsewhere.

2. The BFKL equation

We show that the master equation in the resummation reduces to the BFKL equation in the small \(x\) region. We start with the derivative of the gluon distribution function \(F(p_L^+, p_T)\) shown in Fig. 1(a), where the valence partons are regarded as gluons with longitudinal momentum \(p_L^+\) and transverse momentum \(p_T\). It is convenient to work in covariant gauge \(\partial A = 0\),
under which the square vertex is replaced by an eikonal line in the direction $n$ along with a new vertex $\hat{n}_\alpha = (n^2 / v \cdot n) [v \cdot l n_\alpha / n \cdot l - v_\alpha]$ on it. This vertex is represented by the symbol $\times$ in Fig. 2. It comes from the derivative of the Feynman rules for the eikonal line with respect to $n_\alpha$. We have shown that the resummation results obtained in axial and covariant gauges are the same.

We express the master equation as

$$p_L^+ \frac{d}{dp_L^+} F(p_L^+, p_T) \equiv x \frac{d}{dx} F(x, p_T) = 4 \tilde{F}(x, p_T), \quad (11)$$

where the variable change $p_L^+ = xp^+$ has been employed. Note that the coefficient 4, which is twice of the corresponding coefficient in the AP case, is due to the one more attachment of the radiative gluon to the quark lines in the box diagram associated with the hard scattering. Working on Eq. (11), we do not need to go into the detailed analysis of angular ordering of radiative gluons as in the conventional derivation of the BFKL equation. All the possible orderings have resided in Eq. (11). Therefore, it is simpler to understand the evolution equations by means of the concept of the resummation.

Similarly, the soft divergences of the subdiagram containing the new vertex are collected by Fig. 1(b), and the ultraviolet divergences by Fig. 1(c). Using the relation $f_{abc} t_b t_c = (i/2) N t_a$ for the color structure, $N = 3$ being the number of colors, Fig. 1(b) gives

$$\tilde{F}_{\text{soft}}(x, p_T) = \frac{i}{2} Ng^2 \int \frac{d^4 l}{(2\pi)^4} \frac{\Gamma^{\mu\nu\lambda} \hat{n}_\nu}{(-2p_L^+ v \cdot l) n \cdot l} \left[ \frac{\theta(p_T^2 - l_T^2)}{l^2} F(x, p_T) \right. \right.$$

$$
\left. + 2\pi i \delta(l^2) F(x, |p_T + l_T|) \right], \quad (12)$$

where the triple-gluon vertex for the vanishing loop momentum $l$ is given by $\Gamma^{\mu\nu\lambda} = p_L^+ (g^{\mu\nu} v^\lambda + g^{\nu\lambda} v^\mu - 2g^{\lambda\mu} v^\nu)$. The first term in Eq. (12) corresponds to the virtual gluon emission, where the $\theta$ function guarantees a small $l_T$. Because of this $\theta$ function, the integral is ultraviolet finite, and it is not necessary to introduce a renormalization scale $\mu$ here. The second term corresponds to the real gluon emission, where $F(x, |p_T + l_T|)$ implies that the gluon entering the hard scattering carries a transverse momentum $p_T$, after emitting a real radiative gluon of momentum $l_T$. 

5
It can be shown that as $v^\lambda$ and $v^\mu$ are contracted with a vertex in the hard scattering amplitude and a vertex in the gluon distribution function, respectively, the resulting contributions are suppressed by a power $1/s$, $s = (p + q)^2$, compared to the contribution from the last term $v^\nu$. Absorbing the metric tensor $g^\lambda\mu$ into the gluon distribution function, and evaluating the integral straightforwardly, Eq. (12) becomes

$$
\tilde{F}_{\text{soft}}(x, p_T) = N\alpha_s \frac{\pi}{4} \int \frac{d^2l_T}{\pi l_T^2} \left[ \theta(p_T^2 - l_T^2)F(x, p_T) - F(x, |p_T + l_T|) \right]. \quad (13)
$$

It has been argued that when the fractional momentum $p_L^\perp$ of a parton is small, the logarithms from a loop momentum parallel to $p$ are suppressed [1]. Hence, if we neglect the less important contribution from Fig. 1(c), and adopt the approximation $\tilde{F} = \tilde{F}_{\text{soft}}$, Eq. (13) is written as

$$
\frac{dF(x, p_T)}{d\ln(1/x)} = \frac{N\alpha_s}{\pi} \int \frac{d^2l_T}{\pi l_T^2} \left[ F(x, |p_T + l_T|) - \theta(p_T^2 - l_T^2)F(x, p_T) \right], \quad (14)
$$

which is exactly the BFKL equation. It is then understood that the subdiagrams shown in Fig. 1(b) and 1(c) play the role of the kernel of the BFKL equation.

If including Fig. 1(c), we shall obtain a more accurate equation. In the region with intermediate $x$, it can be shown that the master equation (14) reduces to the Ciafaloni-Catani-Fiorani-Marchesini equation [6]. We leave these subjects to a separate work.

3. The GLR equation

In the region with both large $Q$ and small $x$, many gluons are radiated by partons with small spatial separation among them. A new effect from the annihilation of two gluons into one gluon becomes important. Taking into account this effect, the BFKL equation is modified by a nonlinear term, and leads to the GLR equation [4]

$$
\frac{\partial^2 xG(x, Q^2)}{\partial \ln(1/x) \partial \ln Q^2} = \frac{N\alpha_s}{\pi} xG(x, Q^2) - \frac{\gamma\alpha_s}{Q^2 R_N^2} [xG(x, Q^2)]^2 . \quad (15)
$$

where the gluon density $xG$ is defined in terms of the gluon distribution function by

$$
xG(x, Q^2) = \int \frac{d^2l_T}{\pi} \theta(Q - l_T)F(x, l_T) . \quad (16)
$$
The constant $\gamma = 81/16$ can be regarded as the effective coupling of the annihilation process, and the radius $R_N$ characterizes the correlation length of gluons [7]. It is obvious that the second term is of higher twist, which, however, becomes important as $R_N$ is small. The minus sign in front of it indicates that the annihilation decreases the number of gluons, and that the rise of the gluon density at small $x$ due to the first term might saturate.

We show that the resummation technique can include the annihilation effect, and generates the GLR equation in a natural way. The starting point is still the master equation described by Fig. 1(a),

$$
\frac{\partial^2 xG(x, Q^2)}{\partial x \partial \ln Q^2} = Q^2 \frac{\partial F(x, Q)}{\partial x} = 4Q^2 \tilde{F}(x, Q) .
$$

(17)

However, when factorizing the subdiagram containing the new vertex, we add one more diagram in which four gluons coming out of the hadron attach the subdiagram as shown in Fig. 2(a). This new diagram describes the annihilation process correctly.

The first diagram in Fig. 2(a) leads to Eq. (13), whose first term is in fact plays the role of a soft subtraction, such that the integral is free of infrared divergences. Hence, it is a reasonable approximation to drop the first term, and to replace the denominator $l_T^2$ by $Q^2$. We have

$$
\tilde{F}_{\text{soft}}(x, Q) \approx -\frac{N\alpha_s}{4\pi Q^2} \int_0^Q \frac{dl_T}{\pi} F(x, l_T) = -\frac{N\alpha_s}{4\pi Q^2} xG(x, Q^2) ,
$$

(18)

which gives the linear part of the GLR equation.

We then consider the second diagram in Fig. 2(a), whose lowest-order diagram along with its soft subtraction, are shown in Fig. 2(b). The color structure of Fig. 2(b) is given by $f_{dhgf}f_{be}l_hl_f$. Assuming that the two gluons, labeled by the color indices $c$ and $d$, form a color-singlet pair, we extract the color factor $C = (-i/2)C_A^2/(N^2 - 1)$. With the same analysis of the contributions from each term of the triple-gluon vertices as employed in the BFKL case, the first diagram in Fig. 2(b) leads to

$$
\tilde{F}_{\text{soft}}^{(2)} = \frac{Cg^4}{(2\pi)^4} \int d^4l \Gamma^\gamma_{\nu_\delta} \Gamma^\lambda_{\alpha} \Gamma^{\delta\xi} \hat{n}_\xi \frac{2\pi i \delta((p_L^+ \nu + l)^2)}{n \cdot l(l^2)^2} \int d^2l'_T F^{(2)}(x, l_T, l'_T) = \frac{C_A^2}{N^2 - 1} \frac{g^4}{(2\pi)^3} \int d^4l \frac{n^2 \delta((p_L^+ \nu + l)^2)}{(n \cdot l)^2(l^2)^2} \int d^2l'_T F^{(2)}(x, l_T, l'_T) .
$$

(19)
To arrive at the second expression, we have absorbed the relevant metric tensors into the gluon distribution function $F^{(2)}(x, l_T^2, l_{T'}^2)$, which describes the probability of finding two gluons with equal momentum fraction $x$ but different transverse momenta $l_T$ and $l_{T'}$.

Integrating over $l^-$ and then $l^+$, Eq. (19) becomes

$$\tilde{F}^{(2)} = \frac{C_A^2}{N^2 - 1} \int \frac{d^2l_T}{l_T^2} \int d^2l_{T'} F^{(2)}(x, l_T, l_{T'}) ,$$

in which the vector $n$ has been chosen to be $(1, -1, 0)$ [1]. The contribution from the second diagram can be included simply by substituting $Q^4$ for the denominator $l_T^4$. Eq. (20) is then cast into

$$\tilde{F}^{(2)}_{\text{soft}} = \frac{\pi^3}{N^2 - 1} \left( \frac{\alpha_s C_A}{\pi} \right)^2 \frac{1}{Q^4} x^2 G^{(2)}(x, Q^2) ,$$

with the definition

$$x^2 G^{(2)}(x, Q^2) \equiv \int_0^Q \frac{d^2l_T}{\pi} \frac{d^2l_{T'}}{\pi} F^{(2)}(x, l_T, l_{T'}) .$$

Substituting $\tilde{F} = \tilde{F}_{\text{soft}} + \tilde{F}^{(2)}_{\text{soft}}$ into Eq. (17), we obtain

$$\frac{\partial^2 xG(x, Q^2)}{\partial \ln(1/x) \partial \ln Q^2} = \frac{N\alpha_s}{\pi} xG(x, Q^2) - \frac{4\pi^3}{N^2 - 1} \left( \frac{\alpha_s C_A}{\pi} \right)^2 \frac{1}{Q^2} x^2 G^{(2)}(x, Q^2) ,$$

which is exactly the same as the corresponding formula in [8]. Naively employing the relation $G^{(2)} = (3/2)R_N n G^2$ [8] for a very loosely bound nucleus, where $n = (4\pi R_N^3/3)^{-1}$ is the nuclear number density, we derive the GLR equation [3]. Obviously, the number of diagrams involved in the resummation technique is much fewer and the calculation performed here is simpler than in [8].

In conclusion, the master equation of the resummation technique relates the derivative of the distribution function to a new function involving a new vertex. The summation of various large logarithms is embedded in this new function without resort to the complicated diagrammatic analysis. When expressing the new function as a factorization formula, the subdiagram containing the new vertex is exactly the kernel associated with the evolution
equation. With this work, we are sure that the simpler resummation technique is applicable to a large class of QCD processes. These applications will be published elsewhere.

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Figure Captions

Fig. 1. (a) The derivative $p^+ d\phi / dp^+$ in axial gauge. (b) The $O(\alpha_s)$ function $K$. (c) The $O(\alpha_s)$ function $G$.

Fig. 2. (a) The contributions to $\tilde{F}$ including the annihilation effect in covariant gauge. (b) The soft structure of the $O(\alpha_s)$ subdiagram for the first diagram of (a).