Asymptotic behavior of solutions toward the constant state to the Cauchy problem for the non-viscous diffusive dispersive conservation law

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Abstract
In this paper, we investigate the asymptotic behavior of solutions to the Cauchy problem for the scalar non-viscous diffusive dispersive conservation laws where the far field states are prescribed. We proved that the solution of the Cauchy problem tends toward the constant state as time goes to infinity.

Keywords: diffusive dispersive conservation laws, non-viscous diffusive flux, convex flux, asymptotic behavior, constant state
AMS subject classifications: 35K55, 35B40, 35L65

1. Introduction and main theorems

In this paper, we consider the asymptotic behavior of solutions to the Cauchy problem for a one-dimensional scalar diffusive dispersive conservation laws without viscous flux

\[
\begin{aligned}
\partial_t u + \partial_x \left( f(u) + \delta \partial_x^2 u + \nu \partial_x^3 u \right) &= 0 \quad (t > 0, x \in \mathbb{R}), \\
u(0, x) &= u_0(x) \to \bar{u} \quad (x \to \pm \infty),
\end{aligned}
\]

where, \( u = u(t, x) \) is the unknown function of \( t > 0 \) and \( x \in \mathbb{R} \), the so-called conserved quantity, \( \bar{u} \in \mathbb{R} \) is the constant state,

\[
f(u) + \delta \partial_x^2 u + \nu \partial_x^3 u \quad (\delta \in \mathbb{R}, \nu \geq 0)
\]
is the total flux (that is, the functions \( f(u), \delta \partial_x^2 u \) and \( \nu \partial_x^3 u \) stand for the convective flux, dispersive one and diffusive one, respectively), \( u_0 \) is the initial data, and \( u_{\pm} \in \mathbb{R} \) are the prescribed far field states. We suppose that \( f \) is
a smooth function. It is noted that, the equation in the problem (1.1) is the non-viscous case ($\mu = 0$) for the following equation:

$$\partial_t u + \partial_x \left( f(u) - \mu \partial_x u + \delta \partial_x^2 u + \nu \partial_x^3 u \right) = 0,$$

(1.2)

where $\mu \partial_x u$ is viscous/diffusive flux ($\mu$ is the so-called viscous coefficient or anti-diffusion coefficient). It should be noted that, in the case $\mu > 0$, $\delta = 0$, $\nu = 0$, (1.2) becomes the viscous conservation law/generalized viscous Burgers equation, in the case $\mu = 0$, $\delta \in \mathbb{R}$, $\nu = 0$, the one does the Korteweg-de Vries equation as one of the dispersive conservation laws, in the case $\mu > 0$, $\delta \in \mathbb{R}$, $\nu > 0$, the one does the generalized Korteweg-de Vries-Burgers equation, in the case $\mu > 0$, $\delta \in \mathbb{R}$, $\nu > 0$, the one does the generalized Korteweg-de Vries-Burgers-Kuramoto equation or the derivative form of the Kuramoto-Sivashinsky equation. We also note that the Korteweg-de Vries equation can be categorized as dispersive conservation laws, and the Korteweg-de Vries-Burgers equation and the Korteweg-de Vries-Burgers-Kuramoto equation or the derivative form of the Kuramoto-Sivashinsky equation the diffusive dispersive conservation laws.

There have been known the various of the stability results concerning with the conservation laws (see [1], [3], [7], [9], [10], [11], [13], [14], [15], [17], [19], [21], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [41], [42], and so on, cf. [4], [5], [6], [12], [16], [18], [20], [22], [23], [32], [33]). In particular, for the Cauchy problem of (1.2), Duan-Fan-Kim-Xie [8], Ruan-Gao-Chen [31], showed some stabilities of the rarefaction waves and Yoshida [40] showed the global stabilities of the constant state and the rarefaction wave. However, the any stabilities have not yet been known for more difficult non-viscous case, (1.1).

Our main theorem is stated as follows.

**Theorem 1.1 (Main Theorem ).** Assume the convective flux $f \in C^2(\mathbb{R})$ satisfy

$$|f''(u)| \leq O(1) \left( 1 + |u|^q \right) \quad (0 \leq q \leq 5)$$

(1.3)

and the initial data satisfy $u_0 - \tilde{u} \in L^2$ and $\partial_x u_0 \in H^1$. Then the Cauchy problem (1.1) has a unique global in time solution $u$ satisfying

$$\left\{ \begin{array}{l}
u - \tilde{u} \in C^0([0, \infty); H^2), \\
\partial_x^2 u \in L^2(0, \infty; H^2),
\end{array} \right.$$ 

and the asymptotic behavior

$$\lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}} |u(t, x) - \tilde{u}| + \sup_{x \in \mathbb{R}} |\partial_x u(t, x)| \right) = 0.$$ 

The proofs of Theorem 1.1 is given by a technical energy method.

This paper is organized as follows. In Section 2, We reformulate the problem in terms of the deviation from the asymptotic state. In order to show the
asymptotics, we establish the \textit{a priori} estimates by using the technical energy method in Section 3.

**Some Notation.** We denote by $C$ generic positive constants unless they need to be distinguished. In particular, use $C_{\alpha, \beta, \cdot \cdot \cdot}$ when we emphasize the dependency on $\alpha, \beta, \cdot \cdot \cdot$.

For function spaces, $L^p = L^p(\mathbb{R})$ and $H^k = H^k(\mathbb{R})$ denote the usual Lebesgue space and $k$-th order Sobolev space on the whole space $\mathbb{R}$ with norms $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^k}$, respectively.

## 2. Reformulation of the problem

In this section, we reformulate our problem (1.1) in terms of the deviation from the asymptotic state. Now letting

$$u(t, x) = \tilde{u} + \psi(t, x),$$

we reformulate the problem (1.1) in terms of the deviation $\psi$ from $\tilde{u}$ as

\[
\begin{align*}
\partial_t \psi + \partial_x \left( f(\psi + \tilde{u}) \right) &= -\delta \partial_x^3 \psi - \nu \partial_x^4 \psi \quad (t > 0, x \in \mathbb{R}), \\
\psi(0, x) &= \psi_0(x) := u_0(x) - \tilde{u} \to 0 \quad (x \to \pm \infty).
\end{align*}
\]

Then we look for the unique global in time solution $\psi$ which has the asymptotic behavior

\[
\sup_{x \in \mathbb{R}} |\psi(t, x)| \to 0, \quad \sup_{x \in \mathbb{R}} |\partial_x \psi(t, x)| \to 0 \quad (t \to \infty).
\]

Here we note that $\psi_0 \in H^2$. Then the corresponding theorems for $\psi$ to Theorems 1.1 we should prove is stated as follows.

**Theorem 2.1.** (Global Existence). Assume the convective flux $f \in C^2(\mathbb{R})$ satisfy (1.3) and the initial data satisfy $\psi_0 \in H^2$. Then the Cauchy problem (2.2) has a unique global in time solution $\psi$ satisfying

\[
\begin{align*}
\psi &\in C^0 \left( [0, \infty); H^2 \right), \\
\partial_t^2 \psi &\in L^2(0, \infty; H^2),
\end{align*}
\]

and the asymptotic behavior

\[
\lim_{t \to \infty} \left( \sup_{x \in \mathbb{R}} |\psi(t, x)| + \sup_{x \in \mathbb{R}} |\partial_x \psi(t, x)| \right) = 0.
\]

To accomplish the proofs of Theorem 2.1, we prepare the local existence precisely, we formulate the problem (2.2) at general initial time $\tau \geq 0$:

\[
\begin{align*}
\partial_t \psi + \partial_x \left( f(\psi + \tilde{u}) \right) &= -\delta \partial_x^3 \psi - \nu \partial_x^4 \psi \quad (t > \tau, x \in \mathbb{R}), \\
\psi(\tau, x) &= \psi_\tau(x) := u_\tau(x) - \tilde{u} \to 0 \quad (x \to \pm \infty).
\end{align*}
\]
Then the local existence theorem is stated as follows (since the proof is standard, we state only here (cf. [8], [9], [31], [34]).)

**Theorem 2.2** (Local Existence). For any $M > 0$, there exists a positive constant $t_0 = t_0(M)$ not depending on $\tau$ such that if $\psi_\tau \in H^2$ and

$$\| \psi_\tau \|_{L^2} + \| \partial_x^2 \psi_\tau \|_{L^2} \leq M,$$

then the Cauchy problem (2.4) has a unique solution $\psi$ on the time interval $[\tau, \tau + t_0(M)]$ satisfying

$$\left\{ \begin{array}{l}
\psi \in C^0([\tau, \tau + t_0]; H^2), \\
\partial_x^2 \psi \in L^2(\tau, \tau + t_0; H^2), \\
\sup_{t \in [\tau, \tau + t_0]} \left( \| \psi(t) \|_{L^2} + \| \partial_x^2 \psi(t) \|_{L^2} \right) \leq 2M.
\end{array} \right.$$ 

Next, we state the *a priori* estimates as follows.

**Theorem 2.3** (*A Priori* Estimates). Under the same assumptions as in Theorem 2.1, for any initial data $\psi_0 \in H^2$, there exists a positive constant $C_{\psi_0}$ such that if the Cauchy problem (2.1) has a solution $\phi$ on the time interval $[0, T]$ satisfying

$$\left\{ \begin{array}{l}
\phi \in C^0([0, T]; H^2), \\
\partial_x^2 \psi \in L^2(0, T; H^2),
\end{array} \right.$$ 

for some positive constant $T$, then it holds that

$$\| \psi(t) \|^2_{H^2} + \int_0^t \| \partial_x^2 \psi(\tau) \|^2_{H^2} \, d\tau + \int_0^t \left( \sup_{x \in \mathbb{R}} |\psi(\tau, x)| \right)^4 \, d\tau$$

$$+ \int_0^t \left( \sup_{x \in \mathbb{R}} |\partial_x \psi(\tau, x)| \right)^{\frac{8}{3}} \, d\tau \leq C_{\psi_0}, \quad (t \in [0, T]).$$

Combining the local existence Theorem 2.2 together with the each *a priori* estimates, Theorem 2.3, we can obtain global existence Theorem 2.1. In fact, we can obtain the unique global in time solutions $\psi$ to (2.2) in Theorem 2.2 satisfying

$$\left\{ \begin{array}{l}
\psi \in C^0([0, \infty); H^2), \\
\partial_x^2 \psi \in L^2(0, \infty; H^2),
\end{array} \right.$$ 

and

$$\sup_{t \geq 0} \| \psi(t) \|^2_{H^2} + \int_0^\infty \| \partial_x^2 \psi(t) \|^2_{H^2} \, dt < \infty$$ (2.6)
which yields
\[ \int_0^\infty \left| \frac{d}{dt} \| \partial_x^2 \psi(t) \|^2_{L^2} \right| dt < \infty. \tag{2.7} \]
We immediately have from (2.6) and (2.7) that
\[ \| \partial_x \psi(t) \|_{L^2} \to 0 \quad (t \to \infty). \tag{2.8} \]
Further from (2.8) with \( T \to \infty \) and
\[ \sup_{x \in \mathbb{R}} \| \partial_x \psi(t, x) \|_{L^2} \leq \sqrt{2} \| \partial_x^2 \psi(t) \|_{L^2} \quad (t \geq 0), \tag{2.9} \]
we obtain the asymptotic behavior (2.3).
Thus Theorem 2.1 is proved.

3. *A priori estimates*

In this section, under the assumption
\[ |f''(u)| \leq O(1) \left( 1 + |u|^q \right) \quad (q \geq 0), \tag{3.1} \]
we show the following *a priori* estimate for \( \psi \) in Theorem 2.3. To do that, we prepare the following basic estimate.

**Proposition 3.1.** For \( q \geq 0 \), it follows that
\[ \| \psi(t) \|_{L^2}^2 + 2 \nu \int_0^t \| \partial_x^2 \psi(\tau) \|_{L^2}^2 d\tau = \| \psi_0 \|_{L^2}^2 \quad (t \in [0, T]). \]

**Proof of Proposition 3.1.** Multiplying the equation in (2.2) by \( \phi \) and integrating it with respect to \( x \), we have, after integration by parts,
\[ \frac{1}{2} \frac{d}{dt} \| \psi(t) \|_{L^2}^2 + \nu \| \partial_x^2 \psi(t) \|_{L^2}^2 = 0. \tag{3.2} \]
Next, integrating (3.2) with respect to \( t \), we immediately get the desired estimate.
Thus, we complete the proof of Proposition 3.1.

From Proposition 3.1, we have the next lemma.

**Lemma 3.2.** There exists a positive constant \( C_{\psi_0} \) such that
\[ \int_0^t \left( \sup_{x \in \mathbb{R}} |\psi(\tau, x)| \right)^8 d\tau + \int_0^t \left( \sup_{x \in \mathbb{R}} |\partial_x \psi(\tau, x)| \right)^{\frac{8}{3}} d\tau \leq C_{\psi_0} \quad (t \in [0, T]). \]
Proof of Lemma 3.2. By using the Sobolev inequality and the integration by parts, we get

\[
\sup_{x \in \mathbb{R}} |\psi(t, x)| \leq \sqrt{2} \|\psi(t)\|_{L^2}^{\frac{3}{4}} \|\partial_x \psi(t)\|_{L^2}^{\frac{3}{4}} \quad (t \in [0, T]),
\]

(3.3)

\[
\sup_{x \in \mathbb{R}} |\psi(t, x)| \leq \sqrt{2} \|\partial_x \psi(t)\|_{L^2}^{\frac{3}{2}} \|\partial_x^2 \psi(t)\|_{L^2}^{\frac{3}{2}} \quad (t \in [0, T]).
\]

(3.4)

From (3.3) and (3.4), noting Proposition 3.1, we immediately have the desired estimate.

Thus, the proof is complete.

Next, we state the a priori estimate for \(\partial_x^2 \psi\) as follows.

Proposition 3.3. For \(0 \leq q \leq 5\), there exists a positive constant \(C_{\psi_0}\) such that

\[
\|\partial_x^2 \psi(t)\|_{L^2}^2 + \int_0^t \|\partial_x^4 \psi(\tau)\|_{L^2}^2 \, d\tau \leq C_{\psi_0} \quad (t \in [0, T]).
\]

Once Proposition 3.3 holds true, by using Proposition 3.1, we can estimate as follows.

\[
\|\partial_x \psi\|_{L^2}^2 \leq \|\psi\|_{L^2} \|\partial_x^2 \psi\|_{L^2} \leq C_{\phi_0},
\]

(3.5)

\[
\int_0^t \|\partial_x^3 \psi\|_{L^2}^2 \, d\tau \leq \left( \int_0^t \|\partial_x^3 \psi\|_{L^2}^2 \, d\tau \right)^{\frac{3}{2}} \left( \int_0^t \|\partial_x^4 \psi\|_{L^2}^2 \, d\tau \right)^{\frac{1}{2}} \leq C_{\phi_0},
\]

(3.6)

for \(t \in [0, T]\). From the uniform estimates (3.5) and (3.6), we immediately get the a priori estimate for \(\partial_x \psi\) as follows.

Proposition 3.4. For \(0 \leq q \leq 5\), there exists a positive constant \(C_{\psi_0}\) such that

\[
\|\partial_x \psi(t)\|_{L^2}^2 + \int_0^t \|\partial_x^3 \psi(\tau)\|_{L^2}^2 \, d\tau \, dx \, d\tau \leq C_{\psi_0} \quad (t \in [0, T]).
\]

From Propositions 3.1, 3.3-3.4, by using the Sobolev inequality, we have the following uniform boundedness of \(\psi\) and \(\partial_x \psi\) as follows.

Lemma 3.5. There exists a positive constant \(C_{\psi_0}\) such that

\[
\sup_{x \in \mathbb{R}} |\psi(t, x)| \leq C_{\psi_0}, \quad \sup_{x \in \mathbb{R}} |\partial_x \psi(t, x)| \leq C_{\psi_0} \quad (t \in [0, T]).
\]

By combining Propositions 3.1, 3.3-3.4 and Lemma 3.2, we can obtain Theorem 2.3. Therefore, in order to complete the proof of Theorem 2.3, we finally prove Proposition 3.3.
Proof of Proposition 3.3. Multiplying the equation in (2.2) by \( \partial_x^4 \psi \), and integrating the resultant formula with respect to \( x \), we have

\[
\frac{1}{2} \frac{d}{dt} \| \partial_x \psi(t) \|_{L^2}^2 + \nu \| \partial_x^3 \psi(t) \|_{L^2}^2 = - \int_{-\infty}^{\infty} \partial_x^4 \psi \partial_x \left( f(\psi + \bar{u}) \right) dx. \tag{3.7}
\]

The right-hand side of (3.7) becomes

\[
- \int_{-\infty}^{\infty} \partial_x^4 \psi \partial_x \left( f(\psi + \bar{u}) \right) dx
= \int_{-\infty}^{\infty} \partial_x^4 \psi | \partial_x \psi |^2 \partial_x^2 \psi dx + \int_{-\infty}^{\infty} f'(\psi + \bar{u}) \partial_x^2 \psi \partial_x^3 \psi dx \tag{3.8}
= \int_{-\infty}^{\infty} \partial_x^4 \psi | \partial_x \psi |^2 \partial_x^2 \psi dx - \frac{1}{2} \int_{-\infty}^{\infty} \partial_x^4 \psi \partial_x^2 \psi | \partial_x^2 \psi |^2 dx.
\]

From (3.1), by making use of the Cauchy-Schwarz, Sobolev and Young inequalities, and integration by parts, we can estimate the first term on the right-hand side of (3.8) as follows.

\[
\left| \int_{-\infty}^{\infty} \partial_x^4 \psi | \partial_x \psi |^2 \partial_x^2 \psi dx \right| \leq C_q \int_{-\infty}^{\infty} | \partial_x \psi |^2 | \partial_x^2 \psi | dx + C_q \int_{-\infty}^{\infty} | \psi |^q | \partial_x \psi |^2 | \partial_x^2 \psi | dx, \tag{3.9}
\]

\[
C_q \int_{-\infty}^{\infty} | \partial_x \psi |^2 | \partial_x^2 \psi | dx
\leq C_q \| \partial_x \psi \|_{L^2}^2 \| \partial_x^2 \psi \|_{L^2}^2 \| \partial_x^4 \psi \|_{L^2}^{\frac{1}{2}} \tag{3.10}
\]

\[
\leq C_q \| \psi \|_{L^q} \| \partial_x^2 \psi \|_{L^\frac{4}{3}} \| \partial_x \psi \|_{L^{\frac{8}{3}}} \| \partial_x^3 \psi \|_{L^2}^{\frac{1}{2}}
\]

\[
\leq C_q \| \psi \|_{L^q} || \partial_x^2 \psi ||_{L^{\frac{4}{3}}} \| \partial_x \psi \|_{L^{\frac{8}{3}}} \| \partial_x^3 \psi \|_{L^2}^{\frac{1}{2}}.
\]

\[
C_q \int_{-\infty}^{\infty} | \psi |^q | \partial_x \psi |^2 | \partial_x^2 \psi | dx
\leq C_q \| \psi \|_{L^q} \| \partial_x \psi \|_{L^2}^2 \| \partial_x^2 \psi \|_{L^\infty} \| \partial_x^4 \psi \|_{L^\infty}^{\frac{1}{2}}
\]

\[
\leq C_q \| \psi \|_{L^q} \| \partial_x \psi \|_{L^2} \| \partial_x^2 \psi \|_{L^{\frac{8}{3}}} \| \partial_x^3 \psi \|_{L^2}^{\frac{1}{2}} \tag{3.11}
\]

\[
\leq \frac{\nu}{8} || \partial_x^2 \psi ||_{L^2}^2 + C_{q,\nu} \| \psi \|_{L^\infty} \| \psi \|_{L^{\frac{8}{3}}} \| \partial_x^2 \psi \|_{L^2}^2
\]

\[
\leq \frac{\nu}{8} || \partial_x \psi ||_{L^2}^2 + C_{q,\nu} \left( 1 + \| \psi \|_{L^\infty}^{\frac{8}{3}} \right) \| \psi \|_{L^2}^2 \| \partial_x^2 \psi \|_{L^2}^2 \quad (0 \leq q \leq 5).
\]

Similarly, the second term on the right-hand side of (3.8) can be estimated as
follows.

\[
\left| \frac{1}{2} \int_{-\infty}^{\infty} f''(\psi + \tilde{u}) \partial_x \psi \left| \partial_x^2 \psi \right|^2 \, dx \right| \\
\leq C_q \int_{-\infty}^{\infty} |\partial_x \psi| \left| \partial_x^2 \psi \right|^2 \, dx + C_q \int_{-\infty}^{\infty} |\psi|^q |\partial_x \psi| \left| \partial_x^2 \psi \right|^2 \, dx,
\]

\[
C_q \int_{-\infty}^{\infty} |\partial_x \psi| \left| \partial_x^2 \psi \right|^2 \, dx \leq C_q \left\| \partial_x \psi \right\|_{L^2}^{\frac{q}{2}} \left\| \partial_x^2 \psi \right\|_{L^2}^{\frac{q}{2}}
\]

\[
\leq C_q \left( \left\| \psi \right\|_{L^\infty}^{\frac{q}{2}} + \left\| \partial_x \psi \right\|_{L^\infty}^{\frac{q}{2}} \right) \left\| \partial_x^2 \psi \right\|_{L^2}^2
\]

\[
\leq C_q \left( 1 + \left\| \psi \right\|_{L^\infty}^{\frac{q}{2}} + \left\| \partial_x \psi \right\|_{L^\infty}^{\frac{q}{2}} \right) \left\| \partial_x^2 \psi \right\|_{L^2}^2 \quad (0 \leq q \leq 5).
\]

Noting Lemma 3.2, substituting (3.8)-(3.14) into (3.7), integrating the resultant formula with respect to \( t \) and further using the Gronwall inequality, we obtain the desired formula, Proposition 3.3.

Thus, we complete the proof of Theorem 2.3 from Propositions 3.1, 3.3-3.4.

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