GRAVITY AS A HIGGS FIELD.
II. FERMION-GRAVITATION COMPLEX

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Abstract

If gravity is a metric field by Einstein, it is a Higgs field. Gravitation theory meets spontaneous symmetry breaking when the structure group of the principal linear frame bundle \( LX \) over a world manifold \( X^4 \) is reducible to the connected Lorentz group \( SO(3, 1) \). The physical underlying reason of this reduction is Dirac fermion matter possessing only exact Lorentz symmetries. The associated Higgs field is a tetrad gravitational field \( h \) represented by a global section of the quotient \( \Sigma \) of \( LX \) by \( SO(3, 1) \). The feature of gravity as a Higgs field issues from the fact that, in the presence of different tetrad fields, there are nonequivalent representations of cotangent vectors to \( X^4 \) by Dirac’s matrices. It follows that, in gravitation theory, fermion fields must be regarded only in a pair with a certain tetrad field. These pairs constitute the so-called fermion-gravitation complex and are represented by sections of the composite spinor bundle \( S \rightarrow \Sigma \rightarrow X^4 \) where values of tetrad gravitational fields play the role of coordinate parameters, besides the familiar world coordinates. In Part 1 [14] of the work, geometry of this composite spinor bundle has been investigated. This Part is devoted to dynamics of fermion-gravitation complex. It is a constraint system to describe which we use the covariant multisymplectic generalization of the Hamiltonian formalism when canonical momenta correspond to derivatives of fields with respect to all world coordinates, not only the time. On the constraint space, the canonical momenta of a tetrad gravitational field as a Higgs field are equal to zero, otherwise in the presence of fermion fields. Fermion fields deform the constraint space in the gravitation sector that leads to modification of the Einstein equations.

1 Introduction

Gravitation theory is theory with spontaneous symmetry breaking established by the equivalence principle reformulated in the terms of Klein-Chern geometries of invariants \([3, 8, 9]\). It postulates that there exist reference frames with respect to which Lorentz invariants can be defined everywhere on a world manifold \( X^4 \). This principle has the adequate mathematical formulation in terms of fibre bundles.

Let \( LX \) be the principal bundle of linear frames in tangent spaces to \( X^4 \). The geometric equivalence principle requires that its structure group

\[ GL_4 = GL^+(4, \mathbb{R}) \]
is reduced to the connected Lorentz group
\[ L = SO(3, 1). \]

It means that there is given a reduced subbundle \( L^h X \) of \( LX \) whose structure group is \( L \). They are atlases of \( L^h X \) with respect to which Lorentz invariants can be defined. In accordance with the well-known theorem, there is the 1:1 correspondence between the reduced \( L \) subbundles \( L^h X \) of \( LX \) and the tetrad gravitational fields \( h \) represented by global sections of the Higgs bundle
\[ \Sigma = LX/L \rightarrow X^4 \]  
with standard fibre \( GL_4/L \).

The underlying physical reason of the geometric equivalence principle is Dirac fermion matter possessing only exact Lorentz symmetries.

Let us consider a bundle of complex Clifford algebras \( C_{3,1} \) over \( X^4 \). Its subbundles are both a spinor bundle \( S_M \rightarrow X^4 \) and the bundle \( Y_M \rightarrow X^4 \) of Minkowski spaces of generating elements of \( C_{3,1} \). There is the bundle morphism
\[ \gamma : Y_M \otimes S_M \rightarrow S_M \]
which defines representation of elements of \( Y_M \) by Dirac’s \( \gamma \)-matrices on elements of the spinor bundle \( S_M \). To describe Dirac fermion fields on a world manifold, one must require that the bundle \( Y_M \) is isomorphic to the cotangent bundle \( T^*X \) of \( X^4 \). It takes place if \( Y_M \) is associated with some reduced \( L \) subbundle \( L^h X \) of the linear frame bundle \( LX \). Then, there exists the representation
\[ \gamma_h : T^*X \otimes S_h \rightarrow S_h \]
of cotangent vectors to a world manifold \( X^4 \) by Dirac’s \( \gamma \)-matrices on elements of the spinor bundle \( S_h \) associated with the lift of \( L^h X \) to a \( SL(2, \mathbb{C}) \) principal bundle. Sections of \( S_h \) describe Dirac fermion fields in the presence of a tetrad gravitational field \( h \).

The crucial point consists in the fact that, for different tetrad fields \( h \) and \( h' \), the representations \( \gamma_h \) and \( \gamma_{h'} \) fail to be equivalent. It follows that every Dirac fermion field must be regarded only in a pair with a certain tetrad gravitational field \( h \). There is the 1:1 correspondence between these pairs and the sections of the composite bundle
\[ S \rightarrow \Sigma \rightarrow X^4 \]  
where \( S \rightarrow \Sigma \) is a spinor bundle associated with the \( L \) principal bundle \( LX \rightarrow \Sigma \).

This Part of the work covers dynamics of the fermion-gravitation complex.

Dynamics of fields represented by sections of a fibred manifold \( Y \rightarrow X \) is phrased in terms of jet manifolds \([2, 3, 6, 11]\).

Recall that the \( k \)-order jet manifold \( J^k Y \) of a fibred manifold \( Y \rightarrow X \) comprises the equivalence classes \( j^k_x s, \ x \in X \), of sections \( s \) of \( Y \) identified by the \((k + 1)\) terms of their
Taylor series at \( x \). It is a finite-dimensional manifold. Jet manifolds have been widely used in the theory of differential operators. Their application to differential geometry is based on the 1:1 correspondence between the connections on a fibred manifold \( Y \to X \) and the global sections of the jet bundle \( J^1Y \to Y \).

In the first order Lagrangian formalism, the jet manifold \( J^1Y \) plays the role of a finite-dimensional configuration space of fields. Given fibred coordinates \((x^\lambda, y^i)\) of \( Y \to X \), it is endowed with the adapted coordinates \((x^\lambda, y^i, y^i_\lambda)\) where coordinates \( y^i_\lambda \) make the sense of values of partial derivatives \( \partial_\lambda y^i(x) \) of field functions \( y^i(x) \). A Lagrangian density on \( J^1Y \) is defined by a form

\[
L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda)\omega, \quad \omega = dx^1 \wedge ... \wedge dx^n, \quad n = \dim X.
\]

If a Lagrangian density is degenerate, the corresponding Euler-Lagrange equations are underdetermined and need supplementary gauge-type conditions. In gauge theory, they are the familiar gauge conditions. In general case, the above-mentioned supplementary conditions remain elusive.

To describe constraint field systems, one can use the covariant multimomentum Hamiltonian formalism where canonical momenta correspond to derivatives of fields with respect to all world coordinates, not only the time \([1, 4, 10, 11, 12]\). Given a fibred manifold \( Y \to X \), the corresponding multimomentum phase space is the Legendre bundle \( \Pi = \wedge^n T^*X \otimes TX \otimes V^*Y \) over \( Y \) into which the Legendre morphisms \( \tilde{L} \) associated with Lagrangian densities \( L \) on \( J^1Y \) take their values. This bundle is provided with the fibred coordinates \((x^\mu, y^i, p^i_\mu)\) such that

\[
(x^\mu, y^i, p^i_\mu) \circ \tilde{L} = (x^\mu, y^i, \pi^\mu_i), \quad \pi^\mu_i = \partial_\mu^\lambda \mathcal{L}.
\]

We shall call them the canonical coordinates. The Legendre bundle \([3]\) carries the generalized Liouville form

\[
\theta = -p^\mu_i dy^i \wedge \omega \otimes \partial_\lambda
\]

and the multisymplectic form

\[
\Omega = dp^\mu_i \wedge dy^i \wedge \omega \otimes \partial_\lambda.
\]

If \( X = \mathbb{R} \), they recover respectively the Liouville form and the symplectic form in mechanics.

The multimomentum Hamiltonian formalism is phrased intrinsically in terms of Hamiltonian connections which play the role similar Hamiltonian vector fields in the symplectic geometry. We say that a connection \( \gamma \) on the fibred Legendre manifold \( \Pi \to X \) is a Hamiltonian connection if the form \( \gamma \rfloor \Omega \) is closed. Then, a Hamiltonian form \( H \) on \( \Pi \) is defined to be an exterior form such that

\[
dH = \gamma \rfloor \Omega
\]
for some Hamiltonian connection $\gamma$. The key point consists in the fact that every Hamiltonian form admits splitting

$$H = p_i^\lambda dy^i \wedge \omega_\lambda - p_i^\lambda \Gamma^i_\lambda \omega - \tilde{H}_\Gamma \omega = p_i^\lambda dy^i \wedge \omega_\lambda - \mathcal{H} \omega$$

(7)

where $\Gamma$ is a connection on the fibred manifold $Y$ and

$$\omega_\lambda = \partial_\lambda | \omega.$$

Given the Hamiltonian form $H$ (7), the equality (6) comes to the Hamilton equations

$$\partial_\lambda y^i(x) = \partial^\lambda \mathcal{H},$$

$$\partial_\lambda p_\lambda^i(x) = -\partial_i \mathcal{H}$$

(8a)

(8b)

for sections $r$ of the fibred Legendre manifold $\Pi \to X$.

If a Lagrangian density $L$ is regular, there exists the unique Hamiltonian form $H$ such that the first order Euler-Lagrange equations and the Hamilton equations are equivalent, otherwise in case of degenerate Lagrangian densities. Given a degenerate Lagrangian system when the constraint space is $Q = \hat{L}(J^1 Y)$, one must consider a family of different Hamiltonian forms $H$ associated with the same Lagrangian density $L$ in order to exhaust solutions of the Euler-Lagrange equations.

Lagrangian densities of field models are almost always quadratic and affine in derivative coordinates $y^\mu_i$. In this case, we have the comprehensive relation between solutions of the Euler-Lagrange equations and the Hamilton equations. Given an associated Hamiltonian form $H$, every solution of the corresponding Hamilton equations which lives on the constraint space $Q$ yields a solution of the Euler-Lagrange equations. Conversely, for any solution of the Euler-Lagrange equations, there exists the corresponding solution of the Hamilton equations for some associated Hamiltonian form. In particular, Hamilton equations are separated in the dynamic equations and the above-mentioned gauge-type conditions independent of momenta $p_\lambda^i$.

It is the multimomentum Hamiltonian formalism which enables one to analyse dynamics of field systems on composite manifolds, in particular, dynamics of fermion-gravitation complex.

In the gauge gravitation theory, classical gravity is described by pairs $(h, A_h)$ of tetrad gravitational fields $h$ and gauge gravitational potentials $A_h$ identified with principal connections on the reduced $L$ subbundles $L^h X$ of the linear frame bundle $LX$. Every connection on $L^h X$ is extended to a Lorentz connection on $LX$ which however fails to be reducible to a principal connection on another reduced subbundle $L^{h'} X$ if $h \neq h'$. It follows that gauge gravitational potentials also must be regarded in pairs with a certain tetrad gravitational field $h$. Following the general procedure [11, 14], one can describe these pairs $(h, A_h)$ by sections of the composite bundle

$$C_L = J^1 LX/L \to J^1 \Sigma \to \Sigma \to X^4.$$  

(9)
It is endowed with the local fibred coordinates

$$(x^\mu, \sigma^\mu_a, k^{ab}_\lambda = -k^{ba}_\lambda, \sigma^\mu_{a\lambda})$$

where $(x^\mu, \sigma^\mu_a, \sigma^\mu_{a\lambda})$ are coordinates of the jet bundle $J^1\Sigma$. Given a section $s$ of $C_L$, we recover familiar tetrad functions and Lorentz gauge potentials

$$(\sigma^\mu_a \circ s)(x) = h^\mu_a(x), \quad (k^{ab}_\lambda \circ s)(x) = A^{ab}_\lambda(x)$$

respectively. The corresponding configuration space is the jet manifold $J^1C_L$.

The total configuration space of the fermion-gravitation complex is the product

$$J^1C_L \times J^1S$$

(10)

where $S$ is the composite spinor bundle endowed with the coordinates $(x^\mu, \sigma^\mu_a, y^A)$.

As a test case, we shall restrict our consideration to the Hilbert-Einstein Lagrangian density. Then, on the configuration space $(10)$, the feature of the fermion-gravitation complex lies only in the generalization

$$\overline{D}_\lambda = y^A_\lambda - \frac{1}{2}(k^{ab}_\lambda + N^{ab}_\lambda)I_{ab}^A B^B$$

of the familiar covariant differential of Dirac fermion fields by means of the composite term

$$N^{ab}_\lambda = A^{abc}_\mu(\sigma^\mu_{c\lambda} - \Gamma^\mu_{c\lambda}).$$

Here, $\Gamma^\mu_{c\lambda}$ is a connection on the Higgs bundle $\Sigma$ and

$$A^{abc}_\mu = \frac{1}{2}(\eta^{ca}_\mu b^b - \eta^{cb}_\mu a^a)$$

(11)

where $\eta$ denotes the Minkowski metric is the canonical connection on the bundle $GL_4 \rightarrow GL_4/L$.

(12)

As an immediate consequence, the standard gravitational constraints

$$p^{c\lambda}_\mu = 0$$

(13)

where $p^{c\lambda}_\mu$ are canonical momenta of tetrad fields are replaced by the relations

$$p^{c\lambda}_\mu + \frac{1}{2}A^{abc}_\mu I^{ab}_B B^B p^{c\lambda}_A = 0$$

(14)

where $p^{c\lambda}_A$ are canonical momenta of fermion fields. When restricted to the constraint space $(13)$, the Hamilton equations

$$\partial_\lambda p^{c\lambda}_\mu = \partial_\mu H$$

(5)
come to the familiar Einstein equations, otherwise on the constraint space (14).

Thus, the goal is modification of the Einstein equations for the total system of fermion fields and gravity because of deformation of gravitational constraints. This deformation makes also contribution into the energy-momentum conservation law. In the framework of the multimomentum Hamiltonian formalism, we have the fundamental identity whose restriction to a constraint space can be treated as the energy-momentum conservation law (11). In Part III of the work, percularity of this conservation law in gravitation theory will be considered.

2 Technical preliminary

Given a fibred manifold $Y \rightarrow X$, the first order jet manifold $J^1Y$ of $Y$ is both the fibred manifold $J^1Y \rightarrow X$ and the affine bundle $J^1Y \rightarrow Y$ modelled on the vector bundle $T^*X \otimes_Y VY$. The adapted coordinates $(x^\lambda, y^i, y^i_\lambda)$ of $J^1Y$ are compatible with these fibrations:

$$x^\lambda \rightarrow x'^\lambda(x^\mu), \quad y^i \rightarrow y'^i(x^\mu, y^j), \quad y^i_\lambda = \left(\frac{\partial y'^i}{\partial y^j} y^j_\lambda + \frac{\partial y'^i}{\partial x^\mu} \frac{\partial x^\mu}{\partial x'^\lambda}\right).$$

There is the canonical bundle monomorphism (the contact map)

$$\lambda : J^1Y \rightarrow T^*Y \otimes_Y TY, \quad \lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i).$$

Let $\Phi$ be a fibred morphism of $Y \rightarrow X$ to $Y' \rightarrow X$ over a diffeomorphism of $X$. Its jet prolongation reads

$$J^1\Phi : J^1Y \rightarrow J^1Y', \quad y^i_\mu \circ J^1\Phi = (\partial_\lambda \Phi^i + \partial_j \Phi^j y^j_\lambda) \frac{\partial x^\lambda}{\partial x'^\mu}.$$

The jet prolongation of a section $s$ of $Y \rightarrow X$ is the section $y^i_\lambda \circ J^1s = \partial_\lambda s^i$ of $J^1Y \rightarrow X$.

The repeated jet manifold $J^1J^1Y$, by definition, is the first order jet manifold of $J^1Y \rightarrow X$. It is provided with the adapted coordinates $(x^\lambda, y^i, y^i_\lambda, y^i_\mu, y^i_{\lambda\mu})$. Its subbundle $\tilde{J}^2Y$ with $y^i_{(\lambda)} = y^i_\lambda$ is called the sesquiholonomic jet manifold. The second order jet manifold $J^2Y$ of $Y$ is the subbundle of $\tilde{J}^2Y$ with $y^i_{\lambda\mu} = y^i_{\mu\lambda}$.

Given a fibred manifold $Y \rightarrow X$, a jet field $\Gamma$ on $Y$ is defined to be a section of the jet bundle $J^1Y \rightarrow Y$. A global jet field is a connection on $Y$. By means of the contact map $\lambda$, every connection $\Gamma$ on $Y$ can be represented by the tangent-valued form $\lambda \circ \Gamma$ on $Y$. For the sake of simplicity, we denote this form by the same symbol

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i).$$

The Legendre manifold $\Pi$ (3) of a fibred manifold $Y$ is the composite manifold

$$\pi_{\Pi X} = \pi \circ \pi_{\Pi Y} : \Pi \rightarrow Y \rightarrow X$$
endowed with the fibred coordinates \((x^\lambda, y^i, p^\lambda_i)\):

\[ p^\lambda_i = J \frac{\partial y^i}{\partial y^\mu} \frac{\partial x^\mu}{\partial x^\lambda} p_j^\mu, \quad J^{-1} = \det \left( \frac{\partial x^\lambda}{\partial x^\mu} \right). \]

By \(J^1 \Pi\) is meant the first order jet manifold of \(\Pi \to X\). It is provided with the adapted fibred coordinates \((x^\lambda, y^i, p^\lambda_i, y^i_{(\mu)}, p^\lambda_{i\mu})\).

By a momentum morphism, we call a fibred morphism \(\Phi : \Pi \to J^1 Y\); \((x^\lambda, y^i, y^i_{(\lambda)}) \circ \Phi = (x^\lambda, y^i, \Phi^i_{(\lambda)}(p))\).

Given a momentum morphism \(\Phi\), its composition with the contact map \(\lambda\) is represented by the horizontal pullback-valued 1-form

\[ \Phi = dx^\lambda \otimes (\partial_{\lambda} + \Phi^i_{(\lambda)}(p) \partial_i) \] (15)

on \(\Pi \to X\). For instance, let \(\Gamma\) be a connection on \(Y \to X\). Then, \(\hat{\Gamma} = \Gamma \circ \pi_{Y^1}\) is a momentum morphism. Conversely, every momentum morphism \(\Phi\) of the Legendre manifold \(\Pi\) of \(Y\) defines the associated connection \(\Gamma^\Phi = \Phi \circ \hat{0}\) on \(Y \to X\) where \(\hat{0}\) is the global zero section of the Legendre bundle \(\Pi \to Y\).

### 3 Lagrangian formalism

Given a Lagrangian density \(L\), one can construct the exterior form

\[ \Lambda_L = (y^i_{(\lambda)} - y^i_{(\lambda)}) d\pi^\lambda_i \wedge \omega + (\partial_i - \hat{\partial}_{\lambda} \partial_i^\lambda) \mathcal{L} dy^i \wedge \omega, \] (16)

\[ \lambda = dx^\lambda \otimes \partial_{\lambda}, \quad \hat{\partial}_{\lambda} = \partial_{\lambda} + y^i_{(\lambda)} \partial_i + y^i_{(\mu)} \partial^\mu_i, \]
on the repeated jet manifold \(J^1 J^1 Y\). Its restriction to the second order jet manifold \(J^2 Y\) reproduces the familiar variational Euler-Lagrange operator

\[ \mathcal{E}_L = (\partial_i - \hat{\partial}_{\lambda} \partial_i^\lambda) \mathcal{L} dy^i \wedge \omega, \quad \hat{\partial}_{\lambda} = \partial_{\lambda} + y^i_{(\lambda)} \partial_i + y^i_{(\mu)} \partial^\mu_i. \] (17)

The restriction of the form (16) to the sesquiholonomic jet manifold \(J^2 Y\) of \(Y\) defines the sesquiholonomic extension

\[ \mathcal{E}'_L : J^2 Y \to T^* Y \] (18)
of the Euler-Lagrange operator (17). It has the form (17) with nonsymmetric coordinates \(y^i_{(\mu)}\).

Let \(\Sigma\) be a section of the fibred jet manifold \(J^1 Y \to X\) such that its first order jet prolongation \(J^1 \Sigma\) takes its values into \(\text{Ker} \mathcal{E}'_L\). Then, it satisfies the system of first order Euler-Lagrange equations

\[ \partial_{\lambda} \Sigma^i = \Sigma^i_{(\lambda)}, \quad \partial_i \mathcal{L} - (\partial_{\lambda} + \Sigma^j_{(\lambda)} \partial_j + \partial_{\mu} \Sigma^j_{(\mu)} \partial^\mu_j) \partial^\lambda_i \mathcal{L} = 0. \] (19)

They are equivalent to the familiar second order Euler-Lagrange equations

\[ \partial_i \mathcal{L} - (\partial_{\lambda} + \partial_{\lambda} s^j \partial_j + \partial_{\mu} s^j \partial^\mu_j) \partial^\lambda_i \mathcal{L} = 0 \] (20)

for sections \(s\) of \(Y \to X\). We have \(\Sigma = J^1 s\).
4 Multimomentum Hamiltonian formalism

Let \( \Pi \) be the Legendre manifold (3) provided with the generalized Liouville form \( \theta \) (4) and the multisymplectic form \( \Omega \) (5).

We say that a connection
\[
\gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^i_\lambda \partial_i + \gamma^{\mu}_i \partial^i_\mu)
\]
on the Legendre manifold \( \Pi \to X \) is a Hamiltonian connection if the exterior form
\[
\gamma \lrcorner \Omega = dp^\lambda_i \wedge dy^i + \gamma^\lambda_i dy^i \wedge \omega - \gamma^i_\lambda dp^\lambda_i \wedge \omega
\]
is closed.

Hamiltonian connections constitute an affine subspace of connections on \( \Pi \to X \). The following construction shows that this subspace is not empty.

Every connection \( \Gamma \) on \( Y \to X \) is lifted to the connection
\[
\tilde{\Gamma} = dx^\lambda \otimes [\partial_\lambda + \Gamma^\lambda_i(y) \partial_i + (-\partial_j \Gamma^\lambda_i(y)p^\mu_i - K^\mu_{\nu \lambda}(x)p^\nu_j + K^\alpha_{\alpha \lambda}(x)p^\mu_i) \partial^i_\mu]
\]
on \( \Pi \to X \) where \( K \) is a linear symmetric connection on the bundles \( TX \) and \( T^*X \). We have the equality
\[
\tilde{\Gamma} \lrcorner \Omega = d(\hat{\Gamma} \lrcorner \theta)
\]
which shows that \( \tilde{\Gamma} \) is a Hamiltonian connection.

An exterior \( n \)-form \( H \) on the Legendre manifold \( \Pi \) is called a Hamiltonian form if there exists a Hamiltonian connection for \( H \) satisfying the equation \( \gamma \lrcorner \Omega = dH \).

Note that Hamiltonian forms throughout are considered modulo closed forms since closed forms do not make any contribution into the Hamilton equations.

It follows that Hamiltonian forms constitute an affine space modelled on a linear space of the exterior horizontal densities \( \tilde{H} = \tilde{\mathcal{H}} \omega \) on \( \Pi \to X \). A glance at the equality (21) shows that this affine space is not empty. Given a connection \( \Gamma \) on a \( Y \to X \), its lift \( \tilde{\Gamma} \) on \( \Pi \to X \) is a Hamiltonian connection for the Hamiltonian form
\[
H_\Gamma = \tilde{\Gamma} \lrcorner \theta = p^\lambda_i dy^i \wedge \omega_\lambda - p^\lambda_i \Gamma^i_\lambda(y) \omega.
\]
It follows that every Hamiltonian form on the Legendre manifold \( \Pi \) can be given by the expression (7).

Moreover, a Hamiltonian form has the canonical splitting (7) as follows. Every Hamiltonian form \( H \) defines the associated momentum morphism
\[
\tilde{H} : \Pi \to J^1Y, \quad y^i_\lambda \circ \tilde{H} = \partial^i_\lambda \mathcal{H},
\]
and the associated connection \( \Gamma_H = \tilde{H} \circ \hat{0} \) on \( Y \to X \). As a consequence, we have the canonical splitting
\[
H = H_{\Gamma_H} - \tilde{H}.
\]
The Hamilton operator \( \mathcal{E}_H \) for a Hamiltonian form \( H \) is defined to be the first order differential operator

\[
\mathcal{E}_H = dH - \hat{\Omega} = [(y^i_{(\lambda)} - \partial_\lambda^i H)dp^i - (p^i_{(\lambda)} + \partial_\lambda^i H)dy^i] \wedge \omega
\]  

(24)

where \( \hat{\Omega} \) is the pullback of the multisymplectic form \( \Omega \) onto \( J^1\Pi \).

For any connection \( \gamma \) on the Legendre manifold \( \Pi \), we have

\[
\mathcal{E}_H \circ \gamma = dH - \gamma \| \Omega.
\]

It follows that \( \gamma \) is a Hamiltonian jet field for a Hamiltonian form \( H \) if and only if it takes its values into \( \text{Ker} \mathcal{E}_H \), that is, satisfies the algebraic Hamilton equations

\[
\gamma^\lambda_i = \partial^i_\lambda H, \quad \gamma^\lambda_{i\lambda} = -\partial_i H.
\]

(25)

Let \( r \) be a section of the fibred Legendre manifold \( \Pi \to X \) such that its jet prolongation \( J^1r \) takes its values into \( \text{Ker} \mathcal{E}_H \). Then, the Hamilton equations (23) are brought to the first order differential Hamilton equations (8a) and (8b).

Given a fibred manifold \( Y \to X \), let \( L \) be a first order Lagrangian density. We shall say that a Hamiltonian form \( H \) (7) is associated with a Lagrangian density \( L \) if \( H \) satisfies the relations

\[
\begin{align*}
p^\lambda_j &= \partial_j^\lambda \mathcal{L}(x^\mu, y^i, \partial^i_\mu H), \\
H - p^\lambda_{i\lambda} &= \mathcal{L}(x^\mu, y^i, \partial^i_\mu H).
\end{align*}
\]

(26a)  (26b)

In the terminology of constraint theory, we call \( Q = \hat{L}(J^1Y) \) the constraint space.

If a Lagrangian density \( L \) is regular, there always exists the unique Hamiltonian form associated with \( L \), otherwise in general case. In particular, all Hamiltonian forms \( H_\Gamma \) are associated with the Lagrangian density \( L = 0 \).

Contemporary field theories are almost never regular. We shall restrict our consideration to semiregular Lagrangian densities \( L \) when the preimage \( \hat{L}^{-1}(p) \) of any point \( p \in Q \) is a connected submanifold of \( J^1Y \). This notion of degeneracy seems most appropriate. Lagrangian densities of fields are almost always semiregular. In this case, one can get the workable relations between Lagrangian and multimomentum Hamiltonian formalisms.

(i) A Hamiltonian form associated with a semiregular Lagrangian density \( L \) meets the condition

\[
\pi^\lambda_i y^i - L = \mathcal{H}(x^\mu, y^i, \partial^i_\mu H, \pi^\lambda_i).
\]

(ii) Let \( H \) be a Hamiltonian form associated with a semiregular Lagrangian density \( L \). The Hamilton operator \( \mathcal{E}_H \) for \( H \) satisfies the relation

\[
\Lambda_L = \mathcal{E}_H \circ J^1\hat{L}.
\]

(iii) Let a section \( r \) of \( \Pi \to X \) be a solution of the Hamilton equations (8a) and (8b) for a Hamiltonian form \( H \) associated with a semiregular Lagrangian density \( L \). If \( r \) lives
on the constraint space $Q$, the section $\mathfrak{s} = \hat{H} \circ r$ of $J^1Y \to X$ satisfies the first order Euler-Lagrange equations (19). Conversely, given a semiregular Lagrangian density $L$, let $\mathfrak{s}$ be a solution of the first order Euler-Lagrange equations (19). Let $H$ be a Hamiltonian form associated with $L$ so that

$$\hat{H} \circ \hat{L} \circ \mathfrak{s} = \mathfrak{s}. \quad (27)$$

Then, the section $r = \hat{L} \circ \mathfrak{s}$ of $\Pi \to X$ is a solution of the Hamilton equations (8a) and (8b) for $H$. It lives on the constraint space $Q$. Moreover, for every sections $\mathfrak{s}$ and $r$ satisfying the above-mentioned conditions, we have the relations

$$\mathfrak{s} = J^1s, \quad s = \pi_{\Pi Y} \circ r$$

where $s$ is a solution of the second order Euler-Lagrange equations (20).

We shall say that a family of Hamiltonian forms $H$ associated with a semiregular Lagrangian density $L$ is complete if, for each solution $\mathfrak{s}$ of the first order Euler-Lagrange equations (19), there exists a solution $r$ of the Hamilton equations (8a) and (8b) for some Hamiltonian form $H$ from this family so that

$$r = \hat{L} \circ \mathfrak{s}, \quad \mathfrak{s} = \hat{H} \circ r, \quad \mathfrak{s} = J^1(\pi_{\Pi Y} \circ r).$$

In virtue of assertion (iii), such a complete family exists if and only if, for each solution $\mathfrak{s}$ of the Euler-Lagrange equations for $L$, there exists a Hamiltonian form $H$ from this family so that the condition (27) holds.

In field models where Lagrangian densities are quadratic or affine in velocities, there always exist complete families of associated Hamiltonian forms.

5 Hamiltonian gauge theory

In the rest of the article, the manifold $X$ is assumed to be oriented. It is provided with a nondegenerate fibre metric $g_{\mu\nu}$ in the tangent bundle of $X$. We denote $g = \det(g_{\mu\nu})$.

Let $P \to X$ be a principal bundle with a structure Lie group $G$ which acts on $P$ on the right by the law

$$r_g : P \to Pg, \quad g \in G.$$  

A principal connection is defined to be a $G$-equivariant global jet field $A$ on $P$:

$$A \circ r_g = J^1r_g \circ A, \quad g \in G.$$  

There is the 1:1 correspondence between the principal connections $A$ on $P$ and the global sections of the bundle $C = J^1P/G$. It is the affine bundle modelled on the vector bundle

$$\mathcal{V} = T^*X \otimes V^G P, \quad V^G P = VP/G.$$  

Given a bundle atlas $\Psi^P$ of $P$, the bundle $C$ is provided with the fibred coordinates $(x^\mu, k^m_\mu)$ so that

$$(k^m_\mu \circ A)(x) = A^m_\mu(x).$$
are coefficients of the local connection 1-form of a principal connection $A$ with respect to the atlas $\Psi^P$.

The first order jet manifold $J^1C$ of the bundle $C$ is provided with the adapted coordinates $(x^\mu, k^m_\mu, k^m_{\mu\lambda})$. There exists the canonical splitting
\begin{equation}
J^1C = C_+ \oplus C_- = (J^2P/G) \oplus \bigwedge^2 T^*X \otimes V^G P),
\end{equation}
over $C$. There are the corresponding canonical surjections:
(i) $S : J^1C \to C_+$.
(ii) $F : J^1C \to C_-$ where
\begin{equation}
F = \frac{1}{2} F^m_\lambda dx^\lambda \wedge dx^\mu \otimes e_m, \quad F^m_\lambda = k^m_\mu - k^m_{\mu\lambda} - c^m_{nlk_{\lambda\mu}}.
\end{equation}

The Legendre manifold of the bundle $C$ of principal connections reads
\[ \Pi = \bigwedge^n T^*X \otimes TX \otimes [C \times \bar{C}]^* \]
It is provided with the fibred coordinates $(x^\mu, k^m_\mu, p^{\mu\lambda}_m)$ and has the canonical splitting
\[ \Pi = \Pi_+ \oplus \Pi_- , \]
\begin{equation}
(p^{\mu\lambda}_m) = (p^{(\mu\lambda)}_m) = \frac{1}{2} [p^{\mu\lambda}_m + p^{\lambda\mu}_m] + (p^{[\mu\lambda]}_m = \frac{1}{2} [p^{\mu\lambda}_m - p^{\lambda\mu}_m]).
\end{equation}

On the configuration space (28), the conventional Yang-Mills Lagrangian density $L_{YM}$ is given by the expression
\begin{equation}
L_{YM} = \frac{1}{4\varepsilon^2} a^G_{mn} g^{\lambda\mu} g^{\beta\nu} F^m_\lambda F^m_\nu \sqrt{|g|} \omega \tag{29}
\end{equation}
where $a^G$ is a nondegenerate $G$-invariant metric in the Lie algebra of $G$. It is almost regular and semiregular. The Legendre morphism associated with the Lagrangian density (29) takes the form
\begin{equation}
P_m^{(\mu\lambda)} \circ \hat{L}_{YM} = 0, \tag{30a}
P_m^{[\mu\lambda]} \circ \hat{L}_{YM} = \varepsilon^{-2} a^G_{mn} g^{\lambda\alpha} g^{\beta\beta} F^m_\alpha F^m_\beta \sqrt{|g|} . \tag{30b}
\end{equation}

Let us consider connections on the bundle $C$ of principal connections which take their values into Ker $\hat{L}_{YM}$:
\begin{equation}
S : C \to C_+, \quad S^{m}_{\mu\lambda} - S^{m}_{\lambda\mu} - c^m_{nlk_{\lambda\mu}k_{\mu}} = 0. \tag{31}
\end{equation}
For all these connections, the Hamiltonian forms
\[ H = p^\mu_\lambda dk^m_\mu \wedge \omega_\lambda - p^\mu_\lambda S^m_\mu_\lambda \omega - \tilde{H}_{YM} \omega, \] (32)

\[ \tilde{H}_{YM} = \frac{\varepsilon^2}{4} a^{mn}_G g_{\mu\nu} g_{\lambda\beta} [p^\mu_\lambda [p^{\nu\beta}]_m | g |^{-1/2}, \]

are associated with the Lagrangian density \( L_{YM} \) and constitute the complete family. Moreover, we can minimize this complete family if we restrict our consideration to connections (31) of the following type. Given a symmetric linear connection \( K \) on the cotangent bundle \( T^*X \) of \( X \), every principal connection \( B \) on \( P \) is lifted to the connection \( S_B \) (31) such that
\[ S_B \circ B = \mathcal{S} \circ J^1 B, \]
\[ S^m_{\mu_\lambda} = \frac{1}{2} \left[ c^m_{l_\mu \lambda} [l^l_{\mu_\lambda} + \partial_\mu B^m_{\lambda} + \partial_\lambda B^m_{\mu} - c^m_{nl} (k^m_{\mu_\lambda} + k^m_{\lambda_\mu}) - K^\beta_{\mu_\lambda} (B^m_{\beta} - k^m_{\beta}) \right]. \] (33)

We denote the Hamiltonian form (32) for the connections \( S_B \) (33) by \( H_B \).

The corresponding Hamilton equations for sections \( r \) of \( \Pi \to X \) read
\[ \partial_\lambda p^\mu_\lambda = -c^m_{l_\mu \lambda} [l^l_{\mu_\lambda} + \partial_\mu B^m_{\lambda} + \partial_\lambda B^m_{\mu} - c^m_{nl} (k^m_{\mu_\lambda} + k^m_{\lambda_\mu}) - K^\beta_{\mu_\lambda} (B^m_{\beta} - k^m_{\beta})], \] (34)
\[ \partial_\lambda k^m_\mu + \partial_\mu k^m_\lambda = 2 S^m_{\mu_\lambda}, \] (35)

plus Eqs. (30b). The equations (30b) and (34) restricted to the constraint space (30a) are the familiar Yang-Mills equations for \( A = \pi_{IC} \circ r \). Different Hamiltonian forms \( H_B \) lead to different Eqs. (35) which play the role of the gauge-type condition.

## 6 Hamiltonian Dirac equations

As a test case let us consider Dirac fermion fields in the presence of a background tetrad field \( h \). Recall that they are represented by global sections of the spinor bundle \( S_h \) associated with the \( L_s \)-lift of the reduced Lorentz subbundle \( L^h X \) of the linear frame bundle \( LX \). Their Lagrangian density is defined on the configuration space \( J^1 (S_h \oplus S_h^*) \) provided with the adapted coordinates
\[ (x^\mu, y^A, y^+_A, y^+_{A\mu}, y^-_{A\mu}). \]

It reads
\[ L_D = \left\{ \frac{i}{2} y^+_A (\gamma^0 \gamma^\mu)^A B (y^B_B - A^B \gamma^C_{\mu \lambda}) - (y^+_A - A^+_{A\mu} y^+_C) (\gamma^0 \gamma^\mu)^A B y^B \right\} h^{-1} \omega, \]
\[ -m y^+_A (\gamma^0)^A B y^B \} h^{-1} \omega, \] (36)
\[ \gamma^\mu = h^\mu_a (x) \gamma^a, \quad h = \det(h^\mu_a), \]
where
\[ A^A_{B\mu} = \frac{1}{2} A^{ab}_{\mu \lambda} (x) I^A_{ab} B \]
is a principal connection on the principal spinor bundle $P_h$.

The Legendre bundle $\Pi_h$ over the spinor bundle $S_h \oplus S^*_h$ is provided with the canonical coordinates

$$(x^\mu, y^A, y^+_A, p^\mu_A, p^+_{\mu A}).$$

Relative to these coordinates, the Legendre morphism associated with the Lagrangian density (36) is written

$$p^\mu_A = \pi^\mu_A = \frac{i}{2} y^+_B (\gamma^0 \gamma^\mu)^B_A h^{-1},$$

$$p^+_{\mu A} = \pi^+_{\mu A} = -\frac{i}{2} (\gamma^0 \gamma^\mu)^A_B y_B h^{-1}. \tag{37}$$

It defines the constraint subspace of the Legendre bundle $\Pi_h$. Given a soldering form $S = S^A_{B \mu}(x) y_B^B dx^\mu \otimes \partial_A$
on the bundle $S_h$, let us consider the connection $A + S$ on $S_h$. The corresponding Hamiltonian forms associated with the Lagrangian density (36) read

$$H_S = (p^\mu_A dy^A + p^+_{\mu A} dy^+_A) \wedge \omega_\mu - \mathcal{H}_S \omega,$$

$$\mathcal{H}_S = p^\mu_A A^A_{B \mu} y^B + y^+_B A^+ B_{A \mu} p^+_{\mu A} + m y^+_B (\gamma^0)^A_B y^B h^{-1} +$$

$$\left(p^\mu_A - \pi^\mu_A \right) S^A_{B \mu} y^B + y^+_B S^+ B_{A \mu} (p^+_{\mu A} - \pi^+_{\mu A}). \tag{38}$$

The corresponding Hamilton equations for a section $r$ of the fibred Legendre manifold $\Pi \rightarrow X$ take the form

$$\partial_\mu y^+_A = y^+_B (A^+ B_{A \mu} + S^+ B_{A \mu}), \tag{39a}$$

$$\partial_\mu p^\mu_A = -p^\mu_B A^B_{A \mu} - (p^+_B - \pi^+_B) S^B_{A \mu} -$$

$$(my^+_B (\gamma^0)^B_A + \frac{i}{2} y^+_B S^+ B_{A \mu} (\gamma^0 \gamma^\mu)^C_A) h^{-1}. \tag{39b}$$

plus the equations for the components $y^A$ and $p^\mu_A$. The equation (39a) and the similar equation for $y^A$ imply that $y$ is an integral section for the connection $A + S$ on the spinor bundle $S_h$. It follows that the Hamiltonian forms (38) constitute the complete family.

On the constraint space (37), Eq. (39b) reads

$$\partial_\mu \pi^\mu_A = -\pi^\mu_B A^B_{A \mu} - (my^+_B (\gamma^0)^B_A + \frac{i}{2} y^+_B S^+ B_{A \mu} (\gamma^0 \gamma^\mu)^C_A) h^{-1}. \tag{40}$$

Substituting Eq. (39a) into Eq. (40), we obtain the familiar Dirac equation in the presence of a tetrad gravitational field $h$. 

13
7 Hamiltonian systems on composite manifolds

In this Section, by \( Y \) throughout is meant a composite manifold

\[
\pi := \pi_{\Sigma X} \circ \pi_{Y \Sigma} : Y \to \Sigma \to X
\]

provided with the fibred coordinates \((x^\lambda, \sigma^m, y^i)\) where \((x^\mu, \sigma^m)\) are fibred coordinates of the fibred manifold \( \Sigma \to X \). We further suppose that the fibred manifold

\[
Y_\Sigma := Y \to \Sigma
\]

is a bundle.

Given the composite manifold (41), let \( J^1 \Sigma \), \( J^1 Y_\Sigma \) and \( J^1 Y \) be the first order jet manifolds of \( \Sigma \to X \), \( Y \to \Sigma \) and \( Y \to X \) respectively which are endowed with the corresponding adapted coordinates:

\[
(x^\lambda, \sigma^m, \sigma^m_\lambda),
(x^\lambda, \sigma^m, y^i, \tilde{y}_i^\lambda, y^i_m),
(x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda).
\]

Recall the following assertions [11, 14].

(i) Given a global section \( h \) of the fibred manifold \( \Sigma \to X \), the restriction

\[
Y_h = h^* Y_\Sigma
\]

of the bundle \( Y_\Sigma \) to \( h(X) \) is a fibred imbedded submanifold of the fibred manifold \( Y \to X \).

(ii) There is the 1:1 correspondence between the global sections \( s_h \) of \( Y_h \) and the global sections of the composite manifold \( Y \) which cover the section \( h \).

(iii) There exists the canonical surjection

\[
\rho : J^1 \Sigma \times \sum J^1 Y_\Sigma \to J^1 Y,
\]

\[
y^i_\lambda \circ \rho = y^{i_m} \sigma^m_\lambda + \tilde{y}_i^\lambda.
\]

Let

\[
A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \tilde{A}^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i)
\]

be a connection on the bundle \( Y \to \Sigma \) and

\[
\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^m_\lambda \partial_m)
\]

a connection on the fibred manifold \( \Sigma \to X \). Building on the morphism (13), one can construct the composite connection

\[
A = dx^\lambda \otimes \left[ \partial_\lambda + \Gamma^m_\lambda \partial_m + (A^i_m \Gamma^m_\lambda + \tilde{A}^i_\lambda)\partial_i \right]
\]
on the composite manifold $Y$. It possesses the following property.

Let $h$ be an integral section of the connection $\Gamma$ on $\Sigma \to X$, that is,

$$\Gamma \circ h = J^1 h,$$
$$\partial_\mu h^m = \Gamma^m_\mu.$$

In this case, the composite connection (43) on $Y$ is reducible to the connection

$$A_h = dx^\lambda \otimes [\partial_\lambda + (A^i_m \partial_\lambda h^m + (\tilde{A} \circ h)^i_\lambda)\partial_i]$$

(46)
on the fibred submanifold $Y_h$ (42) of $Y \to X$.

Given a composite manifold $Y$ (41), every connection (44) on the bundle $Y\Sigma$ determines:

- the horizontal splitting

$$VY = VY_{\Sigma} \oplus (Y \times V\Sigma),$$

(47)

$$\dot{y}^i \partial_i + \dot{\sigma}^m \partial_m = (\ddot{y}^i - A^i_m \dot{\sigma}^m)\partial_i + \dot{\sigma}^m(\partial_m + A^i_m \partial_i),$$

of the vertical tangent bundle $VY$ of $Y \to X$;

- the dual horizontal splitting

$$V^*Y = V^*Y_{\Sigma} \oplus (Y \times V^*\Sigma),$$

(48)

$$\ddot{y}_i \dddot{y}^i + \dot{\sigma}_m \dddot{\sigma}^m = \dot{y}_i(\dddot{y}^i - A^i_m \dddot{\sigma}^m) + (\dot{\sigma}_m + A^i_m \dot{y}_i)\dddot{\sigma}^m,$$

of the vertical cotangent bundle $V^*Y$ of $Y \to X$.

It is readily observed that the splitting (47) is uniquely characterized by the form

$$\omega \wedge A_{\Sigma} = \omega \wedge d\sigma^m \otimes (\partial_m + A^i_m \partial_i).$$

(49)

Building on the horizontal splitting (47), one can construct the following first order differential operator on the composite manifold $Y$:

$$\tilde{D} = pr_1 \circ D_A : J^1 Y \to T^* X \otimes VY \to T^* X \otimes VY_{\Sigma},$$

$$\tilde{D} = d\lambda^\lambda \otimes [y^i_\lambda - A^i_\lambda - A^i_m (\sigma^m_\lambda - \Gamma^m_\lambda)]\partial_i =$$

$$dx^\lambda \otimes (y^i_\lambda - \tilde{A}^i_\lambda - A^i_m \sigma^m_\lambda)\partial_i,$$

(50)

where $D_A$ is the covariant differential relative to the composite connection $A$ which is composition of $A_{\Sigma}$ and some connection $\Gamma$ on $\Sigma \to X$. We shall call $\tilde{D}$ the vertical covariant differential.

Let $h$ be an integral section of the connection $\Gamma$ and $Y_h$ the portion of $Y_{\Sigma}$ over $h(X)$. It is readily observed that the vertical covariant differential (50) restricted to $J^1 Y_h \subset J^1 Y$
comes to the familiar covariant differential for the connection $A_h$ on the portion $Y_h \to X$.

Thus, it is the vertical covariant differential that we may utilize in order to construct a Lagrangian density

$$L : J^1Y \xrightarrow{\bar{D}} T^*X \otimes VY \to \wedge^n T^*X$$

for fields on a composite manifold. It should be noted that such a Lagrangian density is never regular because of the constraint conditions

$$\pi^\mu A_i^m = \pi^m_i.$$ Therefore, the multimomentum Hamiltonian formalism must be applied.

The major feature of Hamiltonian systems on a composite manifold $Y$ lies in the following. The horizontal splitting yields immediately the corresponding splitting of the Legendre bundle $\Pi$ over the composite manifold $Y$. As a consequence, the Hamilton equations for sections $h$ of the fibred manifold $\Sigma \to X$ reduce to the gauge-type conditions independent of momenta. Thereby, these sections play the role of parameter fields.

Let $Y$ be a composite manifold. The Legendre bundle $\Pi$ over $Y$ is endowed with the canonical coordinates

$$(x^\lambda, \sigma^m, y^i, p^\lambda_i, p^\lambda_m).$$

Let $A_\Sigma$ be a connection on the bundle $Y \to \Sigma$. With a connection $A_\Sigma$, the splitting

$$\Pi = \wedge^n T^*X \otimes TX \otimes [V^*Y \oplus (Y \times V^*)]$$

of the Legendre bundle $\Pi$ is performed as an immediate consequence of the splitting. We call this the horizontal splitting of $\Pi$. Given the horizontal splitting, the Legendre bundle $\Pi$ is provided with the coordinates

$$\bar{p}^\lambda_i = p^\lambda_i, \quad \bar{p}^\lambda_m = p^\lambda_m + A_i^m P^\lambda_i$$

which are compatible with this splitting.

Let $h$ be a global section of the fibred manifold $\Sigma \to X$. Given the horizontal splitting, the submanifold

$$\{\sigma = h(x), \bar{p}^\lambda_m = 0\}$$

of the Legendre bundle $\Pi$ over $Y$ is isomorphic to the Legendre bundle $\Pi_h$ over the portion $Y_h$ of $Y_\Sigma$.

Let the composite manifold $Y$ be provided with the composite connection determined by connections $A_\Sigma$ on $Y \to \Sigma$ and $\Gamma$ on $\Sigma \to X$. Relative to the coordinates...
compatible with the horizontal splitting (52), every Hamiltonian form on the Legendre bundle \( \Pi \) over \( Y \) can be given by the expression

\[
H = (p^\lambda_i dy^i + p^\mu_m d\sigma^m) \wedge \omega - [\overline{p}^\lambda_i \overline{A}^i_\lambda + \overline{p}^\lambda_m \Gamma^m_\lambda + \overline{\mathcal{H}}(x^\mu, \sigma^m, y^i, \overline{p}_m, \overline{p}^\mu)] \omega
\]

where

\[
\overline{p}^\lambda_i \overline{A}^i_\lambda + \overline{p}^\lambda_m \Gamma^m_\lambda = p^\lambda_i A^i_\lambda + p^\lambda_m \Gamma^m_\lambda.
\]

The corresponding Hamilton equations are

\[
\begin{align*}
\partial_\lambda p^\lambda_i &= -p^\lambda_j [\partial_i \overline{A}^j_\lambda + \partial_{\lambda j} \overline{A}^i_\lambda (\Gamma^m_\lambda + \partial^m_\lambda \overline{\mathcal{H}})] - \partial_i \overline{\mathcal{H}}, \\
\partial_\lambda y^i &= \overline{A}^i_\lambda + A^i_m (\Gamma^m_\lambda + \partial^m_\lambda \overline{\mathcal{H}}) + \partial^i_\lambda \overline{\mathcal{H}}, \\
\partial_\lambda p^\lambda_m &= -p^\lambda_i [\partial_m \overline{A}^i_\lambda + \partial_{i m} \overline{A}^i_\lambda (\Gamma^m_\lambda + \partial^m_\lambda \overline{\mathcal{H}})] - \overline{p}^\lambda_m \partial_m \Gamma^m_\lambda - \partial_m \overline{\mathcal{H}}, \\
\partial_\lambda \sigma^m &= \Gamma^m_\lambda + \partial^m_\lambda \overline{\mathcal{H}}
\end{align*}
\]

and plus constraint conditions.

Let the Hamiltonian form (55) be associated with a Lagrangian density (51) which contains the velocities \( \sigma^m_\mu \) only inside the vertical covariant differential (50). Then, the Hamiltonian density \( \overline{\mathcal{H}} \omega \) appears independent of the momenta \( \overline{p}^\mu_m \) and the Lagrangian constraints read

\[
\overline{p}^\mu_m = 0.
\]

In this case, Eq. (56d) comes to the gauge-type condition

\[
\partial_\lambda \sigma^m = \Gamma^m_\lambda
\]

independent of momenta.

Let us consider now a Hamiltonian system in the presence of a background parameter field \( h(x) \). Substituting Eq. (56d) into Eqs. (56a) - (56c) and restricting them to the submanifold (54), we obtain the equations

\[
\begin{align*}
\partial_\lambda p^\lambda_i &= -p^\lambda_j [\partial_i (\overline{A} \circ h)_j^i + \partial_{i m} h^m] - \partial_i \overline{\mathcal{H}}, \\
\partial_\lambda y^i &= (\overline{A} \circ h)_j^i + A^i_m \partial_\lambda h^m + \partial^i_\lambda \overline{\mathcal{H}}
\end{align*}
\]

for sections of the Legendre manifold \( \Pi_h \rightarrow X \) of the bundle \( Y_h \) endowed with the connection (10). Equations (59) are the Hamilton equations corresponding to the Hamiltonian form

\[
H_h = p^\lambda_i dy^i \wedge \omega - [p^\lambda_i A^i_\lambda + \overline{\mathcal{H}}(x^\mu, h^m(x), y^i, \overline{p}^\mu, \overline{p}^\mu_m = 0)] \omega
\]
on \( \Pi_h \) which is induced by the Hamiltonian form (55) on \( \Pi \).
8 Fermion-gravitation complex

At first, we consider gravity without matter.

In the gauge gravitation theory, dynamic gravitational variables are pairs of tetrad gravitational fields \( h \) and gauge gravitational potentials \( A_h \) identified with principal connections on \( P_h \). Following general procedure, one can describe these pairs \((h, A_h)\) by sections of the bundle \((6)\). The corresponding configuration space is the jet manifold \( J^1C_L \) of \( C_L \). The Legendre bundle

\[
\Pi = \bigwedge^4 T^*X^4 \otimes_{C_L} TX^4 \otimes_{C_L} V^*C_L.
\]

(60)

over \( C_L \) plays the role of a phase space of the gauge gravitation theory.

The bundle \( C_L \) is endowed with the local fibred coordinates

\[
(x^\mu, \sigma^\lambda_a, k^{ab}_\lambda = -k^{ba}_\lambda, \sigma_{a\mu}^\lambda)
\]

where

\[
(x^\mu, \sigma^\lambda_a, \sigma_{a\mu}^\lambda)
\]

are coordinates of the jet bundle \( j^1\Sigma \). The jet manifold \( J^1C_K \) of \( C_K \) is provided with the corresponding adapted coordinates

\[
(x^\mu, \sigma^\lambda_a, k^{ab}_\lambda = -k^{ba}_\lambda, \sigma_{a\mu}^\lambda = \sigma_{a\nu}^\lambda, k^{ab}_\mu, \sigma_{a\lambda}^\nu)
\]

The associated coordinates of the Legendre manifold (60) are

\[
(x^\mu, \sigma^\lambda_a, k^{ab}_\lambda, \sigma_{a\nu}^\lambda, p^{\alpha\mu}_a, p^{\lambda\mu}_a, p^{\alpha\nu\mu}_a)
\]

where \((x^\mu, \sigma^\lambda_a, p^{\alpha\mu}_a)\) are coordinates of the Legendre manifold of the bundle \( \Sigma \).

For the sake of simplicity, we here consider the Hilbert-Einstein Lagrangian density of classical gravity

\[
L_{HE} = -\frac{1}{2\kappa} \mathcal{F}^{ab}_{\mu\lambda} \sigma^\mu_a \sigma^\lambda_b \sigma^{-1}\omega,
\]

\[
\mathcal{F}^{ab}_{\mu\lambda} = k^{ab}_{\mu\lambda} - k^{ab}_{\mu\lambda} + k^{a}_{\epsilon\mu} k^{cb}_{\lambda} - k^{a}_{\epsilon\lambda} k^{cb}_{\mu},
\]

\[
\sigma = \det(\sigma^\alpha_a).
\]

The corresponding Legendre morphism \( \hat{L}_{HE} \) is given by the coordinate expressions

\[
p^{[\lambda\mu]}_a = \pi^{[\lambda\mu]}_a = \frac{-1}{\kappa \sigma} \sigma^{[\lambda\mu]}_a \sigma^\alpha_b.
\]

(62a)

\[
p^{(\lambda\mu)}_a = 0, \quad p^{\alpha\mu}_a = 0, \quad p^{\alpha\nu\mu}_a = 0.
\]

(62b)

We construct the complete family of multimomentum Hamiltonian forms associated with the affine Lagrangian density (61). Let \( K \) be a world connection associated with a
principal connection $B$ on the linear frame bundle $LX$. To minimize the complete family, we consider the following connections on the bundle $C_K$:

$$
\Gamma_{\alpha\mu}^\lambda = B_{\lambda\mu}^\nu \sigma^\nu - K_{\alpha\mu}^\nu \psi^\nu, \\
\Gamma_{\alpha\nu\mu}^\lambda = \partial_\mu B_{\lambda\alpha\nu}^\delta - \partial_\nu K_{\alpha\mu}^\delta \sigma^\delta_a - B_{\alpha\mu\nu}^\lambda \sigma^\lambda_b - K_{\alpha\mu}^\delta \sigma^\delta_b, \\
\Gamma_{\mu\lambda\nu}^{ab} = \frac{1}{2} [k_c^{a\lambda} k_{\mu\nu}^{cb} - k_{c\mu}^{a\lambda} k_{\mu\nu}^{cb}] + \partial_\nu B_{\mu\lambda\nu}^a + \partial_\mu B_{\mu\lambda\nu}^a, \\
\Gamma_{\mu\lambda\nu}^{ab} = \frac{1}{2} R_{\mu\lambda\nu}^{ab}(p_{ab}^{\lambda\mu} - \pi_{ab}^{\lambda\mu}).
$$

where $R$ is the curvature of the connection $B$. The complete family of multimomentum Hamiltonian forms associated with the Lagrangian density (61) consists of the forms given by the coordinate expressions

$$
H_{HE} = \left( p_{ab}^{\lambda\mu} d\sigma_{\mu\nu}^a + p_{\lambda\mu}^{a\nu} d\sigma_{\mu\nu}^a \right) \wedge (\omega_\mu - \mathcal{H}_{HE} \omega), \\
\mathcal{H}_{HE} = \left( p_{ab}^{\lambda\mu} \Gamma_{\mu\lambda}^{ab} + p_{\lambda\mu}^{a\nu} \Gamma_{\mu\lambda}^{\nu a} \Gamma_{\nu\mu}^{ab} \Gamma_{\lambda\mu}^{ab} \right) + \frac{1}{2} R_{\mu\lambda\nu}^{ab}(p_{ab}^{\lambda\mu} - \pi_{ab}^{\lambda\mu}).
$$

The Hamilton equations corresponding to such a multimomentum Hamiltonian form read

$$
\mathcal{F}_{\mu\lambda}^{ab} = R_{\mu\lambda}^{ab}, \\
\partial_\lambda k_{ab}^{\mu} + \partial_\mu k_{ab}^{\lambda} = 2\Gamma_{(\mu\lambda)}^{ab}, \\
\partial_\lambda \sigma_{a}^{\lambda} = \Gamma_{\mu\lambda}^{\lambda}, \\
\partial_\mu \sigma_{a}^{\lambda} = \Gamma_{a\mu}^{\lambda}, \\
\partial_\mu p_{ac}^{\lambda\mu} = -\frac{\partial \mathcal{H}_{HE}}{\partial \sigma_{ac}^{\lambda}}, \\
\partial_\mu p_{\lambda a}^{\mu} = -\frac{\partial \mathcal{H}_{HE}}{\partial \sigma_{\lambda a}^{\mu}},
$$

plus the equations which are reduced to the trivial identities on the constraint space (62a). The equations (63a) - (63d) make the sense of gauge-type conditions. The equation (63c) has the solution

$$
\sigma_{a\mu}^{\lambda} = \partial_\nu \sigma_{a\mu}^{\lambda}.
$$

The gauge-type condition (63b) has the solution

$$
k(x) = B.
$$

19
It follows that the forms $H_{HE}$ really constitute the complete family of multimomentum Hamiltonian forms associated with the Hilbert-Enstein Lagrangian density (61).

On the constraint space, Eqs. (63e) and (63f) are brought to the form

$$\partial_{\mu} \pi^{\lambda}_{ac} = 2k^{b}{}_{c\mu} \pi^{\lambda}_{ab} + \pi^{\beta\gamma}_{ac} \Gamma^{\lambda}{}_{\beta\gamma},$$

(64a)

$$R^{cb}{}_{\beta\mu} \partial_{a} \pi^{cb}{}_{\beta\mu} = 0.$$  

(64b)

The equation (64a) shows that $k(x)$ is the Levi-Civita connection for the tetrad field $h(x)$. Substitution of Eqs. (63a) into Eqs. (64b), leads to the familiar Einstein equations.

Turn now to the fermion matter. Given the $L_{s}$-principal lift $P_{\Sigma}$ of $LX_{\Sigma}$, let us consider the composite spinor bundle

$$S := \pi_{\Sigma X} \circ \pi_{S \Sigma} : (P_{\Sigma} \times V)/L_{s} \to \Sigma \to X^{4}$$

(65)

where

$$S_{\Sigma} := S \to \Sigma$$

is the spinor bundle associated with the $L_{s}$ principal bundle $P_{\Sigma}$. It is readily observed that, given a global section $h$ of the Higgs bundle $\Sigma \to X^{4}$, the restriction $S \to \Sigma$ to $h(X^{4})$ consists with the spinor bundle $S_{h}$ whose sections describe Dirac fermion fields in the presence of the background tetrad field $h$.

Let us provide the principal bundle $LX$ with a holonomic atlas $\{U_{\xi}, \psi_{T \xi}\}$ and the principal bundles $P_{\Sigma}$ and $LX_{\Sigma}$ with associated atlases $\{U_{e}, z^{s}_{e}\}$ and

$$\{U_{e}, z_{e} = r \circ z^{s}_{e}\}$$

respectively. Relative to these atlases, the composite spinor bundle (65) is endowed with the fibred coordinates

$$(x^{\lambda}, \sigma^{\mu}_{a}, y^{A})$$

where $(x^{\lambda}, \sigma^{\mu}_{a})$ are fibred coordinates of the Higgs bundle $\Sigma \to X$ which is coordinatized by matrix components $\sigma^{\mu}_{a}$ of the group elements

$$GL_{4} \ni (\psi_{T \xi} \circ z_{e})(\sigma) : R^{4} \to R^{4}, \quad \sigma \in U_{e}, \quad \pi_{\Sigma X}(\sigma) \in U_{\xi}.$$  

Given a section $h$ of $\Sigma \to X^{4}$, we have

$$z_{e}^{h}(x) = (z_{e} \circ h)(x),$$

$$(\sigma_{a}^{\lambda} \circ h)(x) = h_{a}^{\lambda}(x),$$

$$h(x) \in U_{e}, \quad x \in U_{\xi},$$

where $h_{a}^{\lambda}(x)$ are the tetrad functions.

The jet manifolds $J^{1}\Sigma$, $J^{1}S_{\Sigma}$ and $J^{1}S$ of the bundles $\Sigma$, $S_{\Sigma}$ and $S$ respectively are provided with the adapted coordinates

$$(x^{\lambda}, \sigma^{\mu}_{a}, \sigma^{\mu}_{a\lambda}),$$

$$(x^{\lambda}, \sigma^{\mu}_{a}, y^{A}, \bar{y}^{A}_{\lambda}, y^{A}_{\mu}),$$

$$(x^{\lambda}, \sigma^{\mu}_{a}, y^{A}, \sigma^{\mu}_{a\lambda}, y^{A}_{\lambda}).$$

20
Let us consider the bundle of Minkowski spaces

$$(LX \times M)/L \to \Sigma$$

associated with the $L$-principal bundle $LX\Sigma$. It is isomorphic to the pullback $\Sigma \times T^*X$ which we denote by the same symbol $T^*X$. We have the bundle morphism

$$\gamma_\Sigma : T^*X \otimes S_\Sigma = (P_\Sigma \times (M \otimes V))/L_s \to (P_\Sigma \times \gamma(M \otimes V))/L_s = S_\Sigma; \quad (66)$$

$$\hat{dx}^\lambda = \gamma_\Sigma(dx^\lambda) = \sigma^\lambda_a \gamma^a;$$

where $dx^\lambda$ is the basis for the fibre of $T^*X$ over $\sigma \in \Sigma$. Owing to the canonical vertical splitting

$$VS_\Sigma = S_\Sigma \times S_\Sigma,$$

the morphism (66) implies the corresponding morphism

$$\gamma_\Sigma : T^*X \otimes VS_\Sigma \to VS_\Sigma. \quad (67)$$

To construct a connection on the composite spinor bundle (65), let us consider a connection on the composite bundle

$$LX \to \Sigma \to X^4. \quad (68)$$

Given a principal connection

$$A_\Sigma = (\tilde{A}^{ab}_\mu, A^{abc}_\mu)$$

on $LX \to \Sigma$ and a connection $\Gamma^\nu_{c\mu}$ on $\Sigma$, let

$$A = (A^{ab}_\mu, \Gamma^\nu_{c\mu}), \quad A^{ab}_\mu = \tilde{A}^{ab}_\mu + \Gamma^\nu_{c\mu} A^{abc}_\mu,$$

be the composite connection (45) on (68). We require that, given a tetrad gravitational field $h$, its reduction (46)

$$A_h^{ab}_\mu = \tilde{A}^{ab}_\mu + \partial_\mu h^\nu_{ab} A^{abc}_\nu$$

consists with the Levi-Civita connection. Then, it is readily observed that $A^{ab}_\mu$ must be given by the relation (11) and

$$\tilde{A}^{ab}_\mu = \frac{1}{2} K^{\nu}_{\lambda \mu} \sigma^\lambda_c (\eta^e_c \sigma^b_{\mu} - \eta^c_{eb} \sigma^a_{\mu}) \quad (69)$$

where $K$ is some symmetric connection on $TX$. Then, the associated connection on the spinor bundle $S \to \Sigma$ reads

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \frac{1}{2} \tilde{A}^{ab}_\lambda I_{ab} A^A B \partial_B) + d\sigma^\mu_c \otimes (\partial^c_{\mu} + \frac{1}{2} A^{abc}_\mu I_{ab} B A^A \partial_B).$$
It determines the canonical horizontal splitting (47) of the vertical tangent bundle $VS_\Sigma$ given by the form (49)

$$\omega \wedge \otimes [\partial^C_\mu + A^{BC}_\mu \partial_B].$$

The total configuration space of the fermion-gravitation complex is the product

$$J^1S \times J^1C_L.$$ 

On this configuration space, the Lagrangian density $L_{FG}$ of the fermion-gravitation complex is the sum of the Hilbert-Einstein Lagrangian density $L_{HE}$ (61) and the modification $L_{DF'}$ of the Lagrangian density (36) of fermion fields:

$$L_{DF'} = \{ \frac{1}{2} [y^+_A (\gamma^0 \gamma^\mu)^A B (y^B_\mu - \frac{1}{2} (k^{ab}_\mu - A^{abc}_\mu (\sigma^{\nu}_{c\mu} - \Gamma^\nu_{c\mu})) I_{ab}^+ C_\mu y^C) - (y^+_A - \frac{1}{2} (k^{ab}_\mu - A^{abc}_\mu (\sigma^{\nu}_{c\mu} - \Gamma^\nu_{c\mu})) I_{ab}^{+C} A_\mu y^C (\gamma^0 \gamma^\mu)^A B y^B] - my^+_A (\gamma^0 \gamma^\mu)^A B y^B \} \sigma^{-1}\omega$$

where

$$\gamma^\mu = \sigma^\mu_\alpha \gamma^\alpha.$$ 

The total phase space $\Pi$ of the fermion-gravitation complex is coordinatized by

$$(x^\lambda, \sigma^\mu_\epsilon, \sigma^\mu_\epsilon y^A, y^+_A, k^{ab}_\mu, p^{cA}_\lambda, p^{cA}_\lambda, p^{AB}_\epsilon, p^{AB}_\epsilon, p^{ab}_\mu)$$

and admits the corresponding splitting (52). The Legendre morphism associated with the Lagrangian density $L_{FG}$ defines the constraint subspace of $\Pi$ given by the relations (37), (62a) and conditions

$$p^{ab}_\mu (\lambda \mu) = 0,$$

$$p^{c\mu}_\lambda = 0,$$

$$\frac{1}{2} p^A_{\lambda \mu} A^{abc}_\mu I_{ab}^+ C_\mu y^C + \frac{1}{2} p^{A+}_\mu A^{abc}_\mu I_{ab}^{+C} A_\mu y^C + p^{c\mu}_\lambda = 0. \quad (70)$$

Hamiltonian forms associated with the Lagrangian density $L_{FG}$ are the sum of the Hamiltonian forms $H_{HE}$ and $H_S$ (68) where

$$A^A_{B\mu} = \frac{1}{2} k^{ab}_\mu I_{ab}^+ A B y^B. \quad (71)$$

The corresponding Hamilton equations for spinor fields consist with Eqs. (39a) and (39b) where $A$ is given by the expression (71). The Hamilton equations (63a) - (63d) remain true. The Hamilton equations (63c) and (63d) contain additional matter sources. On the constraint space

$$p^{a\mu}_\lambda = 0$$

the modified equations (63) would come to the familiar Einstein equations

$$G^a_\mu + T^a_\mu = 0,$$

22
where $T$ denotes the energy-momentum tensor of fermion fields, otherwise on the constraint space (70). In that latter case, we have

$$D_\lambda p^{\alpha}_\mu = G^{\alpha}_\mu + T^{\alpha}_\mu$$

where $D_\mu$ denotes the covariant derivative with respect to the Levi-Civita connection which acts on the indices $c^{\alpha}_\mu$. Substitution of (70) into (72) leads to the modified Einstein equations for the total system of fermion fields and gravity:

$$-{1\over 2} J^\lambda_{ab} D_\lambda A^{abc}_\mu = G^c_\mu + T^c_\mu$$

where $J$ is the spin current of the fermion fields.

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