Fun with Higgsless Theories

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Abstract
Motivated by recent works on “Higgsless theories,” I discuss an $SU(2)_0 \times SU(2)^N \times U(1)$ gauge theory with arbitrary bifundamental (or custodial $SU(2)$ preserving) symmetry breaking between the gauge subgroups and with ordinary matter transforming only under the $U(1)$ and $SU(2)_0$. When the couplings, $g_j$, of the other $SU(2)$s are very large, this reproduces the standard model at the tree level. I calculate the $W$ and $Z$ masses and other electroweak parameters in a perturbative expansion in $1/g_j^2$, and give physical interpretations of the results in a mechanical analog built out of masses and springs. In the mechanical analog, it is clear that even for arbitrary patterns of symmetry breaking, it is not possible (in the perturbative regime) to raise the Higgs mass by a large factor while keeping the $S$ parameter small.

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1 Higgsless theories, deconstruction, masses, springs

So-called “Higgsless” theories \[1, 2\] make use of boundary conditions on an extra dimension to break the electroweak symmetry of the standard model. In a phenomenologically successful model of this kind (if such could be constructed), there would be no light scalars, but instead one would find additional massive vector bosons at the electroweak symmetry breaking scale, the Kaluza-Klein partners of the $W$ and $Z$ from the extra dimension.

A number of groups (see for example \[3\] and \[4\]) have studied Higgsless theories using the technique of deconstruction \[5, 6\] to actualize the extra-dimensional metaphor in conventional four dimensional quantum field theory. These works are the primary motivation for this note. The approach here differs from that of previous works in several ways. I consider a more general pattern of symmetry breaking, preserving a custodial $SU(2)$ symmetry \[7\], but otherwise completely arbitrary.\[1\] I analyze these models in a power series expansion around a standard model limit. This is a strong-coupling expansion in the couplings of the “extra” $SU(2)$ gauge groups. I also make use of what I think is an interesting trick to relate the $W$ and $Z$ properties in this general class of theories. Finally, I discuss a mechanical analog of the field theories in systems of masses and springs. I believe that this is extremely useful in developing intuition about the properties of these theories. In particular, I find physical interpretations of the two most critical issues facing theories of this kind: raising the scale of symmetry breaking and keeping the $S$ parameter small. Sadly, I conclude, in agreement with previous analyses, that the promise of Higgsless theories is unlikely to be realizable, even in this more general class of theories. But I hope that the reader will find that this analysis is sufficiently unusual to justify the term “fun” in the title.

In section 2 I introduce the class of models I discuss in this paper and briefly discuss the scalar sector. In section 3 I introduce the mechanical analog - two systems of masses and springs - one related to the $Z$ mass matrix and the other to the $W$. In sections 4 and section 5 I study the $W$ and $Z$ mass matrices, respectively. The analysis of the light $W$ mass is straightforward in a strong-coupling expansion of the inverse mass matrix around the standard model limit. A similar analysis of the $Z$ is possible after a transformation of the inverse mass matrix. In section 6 I discuss the phenomenology of the class of models by calculating the electroweak parameters, $S$, $T$ and $U$, of which $S$ is the potential problem. Finally in section 7 I give a physical interpretation of $S$ in the mechanical analog that makes it obvious that $S$ is a very strong constraint for all models in the class.

2 Where is the Higgs?

The class of theories that we consider in this paper are $SU(2)_0 \times SU(2)^N \times U(1)_{N+1}$ gauge theories with arbitrary bifundamental (or custodial $SU(2)$ preserving) symmetry breaking.

\[1\] Reference \[2\] generalizes the simple deconstruction in a different direction, including additional $U(1)$ gauge groups, but still retaining the local structure of symmetry breaking associated with deconstruction.
between the gauge subgroups and with ordinary matter transforming only under the $U(1)_{N+1}$ and $SU(2)_0$. This includes the deconstructed version of Higgsless theories, \[3\ \text{and}\ ]4\] the Moose diagram for which is shown in figure 1. We will refer to this special case as the “linear model” for reasons that are probably obvious. More general patterns of symmetry breaking can involve additional links between nodes of the Moose. For example, for $N = 1$, there are three other distinct possibilities. They are shown in figure 2. The number of possible symmetry breaking patterns grows very rapidly with $N$. For $N = 2$, there are fifteen, one of which is shown in figure 3. In this paper, I am not interested in doing away with the Higgs entirely.

Figure 1: The Moose diagram associated with the linear model.

Figure 2: The three “non-linear” (meaning not equivalent to the linear model) symmetry breaking patterns for $N = 1$.

Figure 3: A Moose diagram for general symmetry breaking for $N = 2$. There are fourteen other distinct possibilities (counting the linear model) in which some of the links are missing.

I am happy to think about the symmetry breaking being done by the vacuum expectation values (VEVs) of scalar fields. The question I address is whether we can raise the lightest scalar mass above the TeV scale while retaining the phenomenology of the standard model.
Since this would give rise to a Higgsless effective low energy theory, we will continue to use the term Higgsless.

These theories involve several independent symmetry breaking sectors. This raises the question, given a set of symmetry breaking sectors (SBSs), where do we expect the lightest scalar “Higgs”? Let’s briefly consider this in the simple realization in which each symmetry breaking sector is just a linear $\sigma$-model. Here there is a single neutral custodial $SU(2)$ singlet scalar for each $\sigma$-model, and depending on the structure of the theory, there may be custodial $SU(2)$ triplet scalar pseudo-Goldstones below the cut-off scale. The obvious thing to say, I think, is that if we have a set of SBSs with scales $v_j$, the lightest scalar would be expected at or below the lowest symmetry breaking scale

$$\approx 4\pi \min_{j} v_j$$

(2.1)

If this were just an ordinary field theory with several $\sigma$-models, there would be arbitrary couplings without respect to locality. Then all the VEVs would be of the same order of magnitude unless some fine tuning was going on. But in an extra dimension stretched “between” the $SU(2)_0$ and the $U(1)_{N+1}$, locality is a strong constraint. First of all, with locality, we don’t have to worry about pseudo-Goldstones. They are all eaten. Also because of locality, we can imagine some dependence of the VEV on “position” in the extra dimension. However, in this case, it might be argued that the Higgs would show up at the smallest scale, so it probably makes sense to keep all the scales the same if we are trying to push up the Higgs mass as much as possible.

Without locality, the situation is more complicated, but it does not seem to be any better, at least not if the goal is to push up the minimum mass of things in the scalar sector. We will ignore this and make the simple assumption that all the symmetry breaking scales are of the same order of magnitude.

### 3 Springs and masses

There is a mechanical analog to each of the theories we consider. We can think of each gauge group as a degree of freedom with mass $1/g^2$ and each VEV between groups as a massless spring with spring constant $v^2$. Then the masses of the gauge bosons are proportional to the frequencies of the normal modes.

To see how this works more precisely, let’s look at the example of the linear model illustrated in figure 1 where the groups associated with nodes 0-N are $SU(2)_s$ and the group associated with $N + 1$ is a $U(1)_{N+1}$.\(^2\) The mechanical analog of the neutral gauge boson sector is then illustrated in figure 4.

\(^2\)As much as possible, I use the notation of reference 4, though I will discuss only the case of a single $U(1)$.  

4
In the linear model, we can write the neutral gauge boson mass-squared matrix as the $N+2 \times N+2$ matrix
\[ M_n^2 = \frac{1}{4} G V G \] (3.1)
where $G$ is the diagonal matrix of gauge couplings
\[
G = \begin{pmatrix}
g_0 & 0 & \cdots & 0 & 0 \\
0 & g_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & g_N & 0 \\
0 & 0 & \cdots & 0 & g_{N+1}
\end{pmatrix}
\] (3.2)
and the matrix $V$ is
\[
V = \begin{pmatrix}
v_{01}^2 & -v_{01}^2 & \cdots & 0 & 0 \\
-v_{01}^2 & v_{01}^2 + v_{12}^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & v_{N-1,N}^2 + v_{N,N+1}^2 & -v_{N,N+1}^2 \\
0 & 0 & \cdots & -v_{N,N+1}^2 & v_{N,N+1}^2
\end{pmatrix}
\] (3.3)
These VEV’s break the gauge symmetry down to a single diagonal symmetry if all the gauge groups are the same. The squared gauge boson masses, $m_\alpha^2$ and the corresponding mass eigenstates, $\kappa^\alpha$ are eigenvalues and eigenvectors of the gauge boson mass squared matrix,
\[
\frac{1}{4} G V G \kappa^\alpha = m_\alpha^2 \kappa^\alpha 
\] (3.4)
For more general symmetry breaking, the formulas are the same (so long as each symmetry breaking sector preserves a custodial $SU(2)$ and “plaquette” terms are introduced to align the vacuum properly), except that more entries in the VEV matrix $V$ are populated. I will analyze the general case, but will continue also to illustrate the analysis in the simple example of a linear theory space.

For the mechanical analog, the squared normal angular frequencies, $\omega_\alpha^2$ and the corresponding normal modes, $\lambda^\alpha$ are eigenvalues and eigenvectors of the $M^{-1}K$ matrix,
\[
M^{-1}K \lambda^\alpha = \omega_\alpha^2 \lambda^\alpha 
\] (3.5)
where

\[
M = \begin{pmatrix}
  m_0 & 0 & \cdots & 0 & 0 \\
  0 & m_1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & m_N & 0 \\
  0 & 0 & \cdots & 0 & m_{N+1}
\end{pmatrix}
\]  (3.6)

and the matrix \( K \) is

\[
K = \begin{pmatrix}
  K_{01} & -K_{01} & \cdots & 0 & 0 \\
  -K_{01} & K_{01} + K_{12} & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & \cdots & K_{N-1,N} + K_{N,N+1} & -K_{N,N+1} \\
  0 & 0 & \cdots & -K_{N,N+1} & K_{N,N+1}
\end{pmatrix}
\]  (3.7)

We can rewrite (3.4) as

\[
G^2 V (G \kappa_\alpha) = m_\alpha^2 (G \kappa_\alpha)
\]  (3.8)

and in this form it is clear that there is an exact correspondence,

\[
\frac{1}{4} G^2 \leftrightarrow M^{-1} \quad V \leftrightarrow K \quad G \kappa_\alpha \leftrightarrow \lambda_\alpha \quad m_\alpha \leftrightarrow \omega_\alpha
\]  (3.9)

Notice that if the mass of a degree of freedom is very small and if there are only two springs attached, it is as if there is a single, continuous spring with no mass on it between the degrees of freedom on either side. The spring constants then add reciprocally, like capacitances. For example, suppose \( g_2 \) goes to infinity. The effective spring constant between the two nodes 1 and 3 is then

\[
\frac{1}{1/v_{12}^2 + 1/v_{23}^2}
\]  (3.10)

At the level of the Goldstone bosons, the relevant Goldstone kinetic energy term are

\[
\frac{v_{12}^2}{4} \text{Tr} ([\partial^\mu U_{12} - igU_{12}W^\mu][\partial_\mu U_{12} - igU_{12}W_\mu]^\dagger)
\]

\[
+ \frac{v_{23}^2}{4} \text{Tr} ([\partial^\mu U_{23} + igW^\mu U_{23}][\partial_\mu U_{23} + igW_\mu U_{23}]^\dagger)
\]  (3.11)

The eaten Goldstone boson is

\[
\frac{-i}{2} (v_{12}^2 [\partial^\mu U_{12}]^\dagger U_{12} - v_{23}^2 U_{23}[\partial^\mu U_{23}]^\dagger)
\]  (3.12)
And the Goldstone boson kinetic energy can be written as

\[
\frac{1}{4} \frac{1}{v_{12}^2 + 1/v_{23}^2} \text{Tr} \left( [\partial^\mu (U_{12}U_{23})][\partial^\mu (U_{12}U_{23})]^\dagger \right) + \frac{1}{4} \frac{1}{v_{12}^2 + v_{23}^2} \\
\text{Tr} \left( (v_{12}^2 [\partial^\mu U_{12}]^\dagger U_{12} - v_{23}^2 U_{23}[\partial^\mu U_{23}]^\dagger) (v_{12}^2 [\partial^\mu U_{12}]^\dagger U_{12} - v_{23}^2 U_{23}[\partial^\mu U_{23}]^\dagger) \right)
\]  

(3.13)

and you can see that the spring constant of the uneaten Goldstone boson is given by (3.10).

If a gauge coupling goes to zero, which is equivalent to having no gauge symmetry at all, this corresponds to an infinite mass, which is like a fixed wall. So for example, in the linear model, the mechanical analog for the neutral gauge bosons looks like figure 4, but for the charged gauge bosons, the mechanical is shown in figure 5. The more general case would

\[
\begin{array}{ccccccc}
0 & \frac{\cdots}{\frac{\cdots}{\cdots}} & 1 & \frac{\cdots}{\frac{\cdots}{\cdots}} & \cdots & \frac{\cdots}{\frac{\cdots}{\cdots}} & N \\
m_0 & K_{01} & m_1 & & & & m_N \\
\end{array}
\]

Figure 5: The mechanical analog for the charged gauge boson mass matrix.

have every spring connected to the $N+1$st mass in the $Z$ analog connected to the fixed wall in the $W$ analog. The corresponding gauge boson mass squared matrix for the charged gauge bosons is the $N+1 \times N+1$ matrix, obtained from (3.11) by eliminating the $N+2$nd row and column,

\[
M_c^2 = \frac{1}{4} \tilde{G} \tilde{V} \tilde{G}
\]  

(3.14)

where $\tilde{G}$ is the diagonal matrix of gauge couplings without $g_{N+1}$

\[
\tilde{G} = \begin{pmatrix}
g_0 & 0 & \cdots & 0 & 0 \\
0 & g_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & g_{N-1} & 0 \\
0 & 0 & \cdots & 0 & g_N
\end{pmatrix}
\]  

(3.15)

and the matrix $\tilde{V}$ in the linear model is

\[
\tilde{V} = \begin{pmatrix}
v_{01}^2 & -v_{01}^2 & \cdots & 0 & 0 \\
-v_{01}^2 & v_{01}^2 + v_{12}^2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & v_{N-2,N-1}^2 + v_{N-1,N}^2 & -v_{N-1,N}^2 \\
0 & 0 & \cdots & -v_{N-1,N}^2 & v_{N-1,N}^2 + v_{N,N+1}^2
\end{pmatrix}
\]  

(3.16)
with corresponding $\tilde{M}$ and $\tilde{K}$ for the mechanical analog of figure 5.

Again, the general formula is analogous. $\tilde{V}$ is obtained from $V$ in the same way, by removing the $N+2$nd row and column.

The low energy charged-current weak interactions are determined by the inverse of $\tilde{V}$, \(^3\)

$$\sqrt{2} G_F = \frac{1}{v^2} = [\tilde{V}^{-1}]_{00} \sim \sum_{j=0}^{N} \frac{1}{v^2_{j,j+1}}$$  \quad (3.17)

where I have indicated the value of $[\tilde{V}^{-1}]_{00}$ in the linear example by the symbol $\sim$. I will continue to use this notation below.

The low energy neutral-current weak interactions are then given in terms of $\tilde{V}^{-1}$ by the Georgi-Weinberg construction \([8]\) (assuming that matter couples only to 0 and $N+1$)

$$\sum_{j,k=0}^{N} [\tilde{V}^{-1}]_{jk} \left[ T_3 \delta_{j0} - \frac{e^2}{g_j^2} Q \right] \left[ T_3 \delta_{k0} - \frac{e^2}{g_k^2} Q \right]$$  \quad (3.18)

For convenience in the following, I will sometimes abbreviate the matrix elements $[\tilde{V}^{-1}]_{jk}$ as follows:

$$[\tilde{V}^{-1}]_{00} = \frac{1}{v^2} \equiv \chi_0, \quad [\tilde{V}^{-1}]_{j0} = [\tilde{V}^{-1}]_{0j} \equiv \chi_j, \quad [\tilde{V}^{-1}]_{jk} \equiv \chi_{jk} \quad \text{for } j, k = 1 \text{ to } N$$  \quad (3.19)

In the linear theory,

$$\chi_j \sim \sum_{\ell=j}^{N} \frac{1}{v^2_{\ell,\ell+1}} \quad \text{and} \quad \chi_{jk} \sim \chi_{\text{max}(j,k)}$$  \quad (3.20)

The premise of Higgsless models (in their deconstructed form) is that by extending the gauge group to include additional copies of $SU(2)$ we can raise the scale of all the symmetry-breaking breaking physics above a TeV, thus pushing the Higgs boson out of the low energy theory, while leaving the $W$ and $Z$ mass and the low energy weak interactions unchanged. In such a model, the job of unitarizing $W-W$ scattering at a TeV would be done by the extra massive vector bosons, some of which would necessarily appear below the TeV scale.

The mechanical analog of the raising of the symmetry-breaking scale is the following. You have only very stiff springs (corresponding to a high symmetry breaking scale), and you want to build a system that has low frequency normal modes (the $W$ and $Z$) with the same properties as those in a system with a single more flexible spring! It easy to see how we can do this, at least classically. The linear model works very well for this purpose. If we string stiff springs together in series with light or massless connections, the result behaves for low frequencies like a single flexible spring. Thus if we could make the gauge couplings $g_1-g_N$ very large, we could break up the spring into segments, each of which has a larger spring constant and therefore larger Higgs mass. In the limit

$$g_j \to \infty \quad \text{for } j = 1 \text{ to } N$$  \quad (3.21)

\(^3\)Because of our numbering of the gauge groups, to agree with reference \[4\], it is convenient to label the rows and columns of our vectors and matrices beginning with 0 rather than 1, so that is what we will do.
only $[\tilde{V}^{-1}]_{00}$ is relevant to the low energy weak interactions. This is a deconstructed version the strong coupling limit of a Higgsless model.\footnote{See for example \cite{3}.}

\section{The light W mass}

We can find the light $W$ gauge boson mass by diagonalizing the inverse mass squared matrix,

\[ 4 \tilde{G}^{-1} \tilde{V}^{-1} \tilde{G}^{-1} \approx \begin{pmatrix} 4 \chi_0 / g_0^2 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 4 / g_0^2 v^2 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \]  

(4.1)

which in the linear model depends only on the sum of the reciprocal VEVs (see (3.17)). The advantage of working with the inverse mass squared matrix rather than the mass squared matrix itself is that in the limit we are considering, the inverse light $W$ mass squared dominates the inverse matrix which as you see in (4.1), is automatically diagonal in the limit.

Somewhat less naively, we should not allow the other couplings to be infinitely large. Presumably the picture ceases to make sense if the $g_j$ are larger than or of order $4 \pi$. This means that we cannot take our connectors in the mechanical analog to be massless. They have some minimum possible mass (not such an unreasonable toy model).

In the inverse mass squared matrix, we can easily include the effects of the other couplings to second order using ordinary perturbation theory for the inverse mass squared matrix. Because the largest eigenvalue is non-degenerate, we can immediately write down the corrections for this state. To second order in $g_0 / g_j$ (for $j = 1$ to $N$), the $W$ eigenvector $\tilde{\kappa}^0$ is approximately given by

\[ [\tilde{\kappa}^0]_j \propto \left[ \tilde{G}^{-1} \tilde{\lambda}^0 \right]_j \propto \left[ \tilde{K}^{-1} \right]_{j0} / g_j = \frac{\chi_j}{g_j} \approx \frac{1}{g_j} \sum_{k=j}^{N} \frac{1}{v_{k,k+1}^2} \]  

(4.2)

and the eigenvalue is

\[ \frac{1}{M_W^2} = \frac{4 \chi_0}{g_0^2} + \frac{4}{\chi_0} \sum_{j=1}^{N} \frac{\chi_j^2}{g_j^2} = \frac{4}{\chi_0} \sum_{j=0}^{N} \frac{\chi_j^2}{g_j^2} = 4 v^2 \sum_{j=0}^{N} \frac{\chi_j^2}{g_j^2} \]  

(4.3)

It is a little curious that in this expression, the large $1 / g_0^2$ term just seems to be one of a series of terms with the same structure.

There is a simple physical argument for (4.2) based on the mechanical analog (you can refer to figure 5 to see how this work in the linear model, but remember that the discussion works for the general case). It is clear that the low frequency mode in the limit in which $m_0$ is much bigger than all the other masses is approximately just a static stretching of the springs, with no force on any of the masses except 0. Thus this mode satisfies

\[ F_j \propto [\tilde{K} \tilde{\lambda}]_j \propto \delta_{j0} \]  

(4.4)
and thus
\[ [\hat{\lambda}]_j^0 \propto [\hat{K}^{-1}]_{j,0} = \chi_j \] (4.5)

The dictionary (3.9) then immediately implies (4.2). The mass (4.3) is the expectation value of the inverse mass-squared matrix in the state (4.2).

The heavy states are initially degenerate, and to second order the $N \times N$ inverse mass squared matrix has matrix elements
\[
4 \frac{1}{g_j} \left( \chi_{jk} - \chi_j \chi_k \chi_0 \right) \frac{1}{g_k} \quad \text{for } j, k = 1 \text{ to } N. \tag{4.6}
\]

An interesting quantity that I will discuss later is the sum of the inverse mass squares, given by the trace. This simplifies in the linear model:
\[
\sum_{\text{heavy}} \frac{1}{4M^2} = \sum_{j=0}^{N} \frac{\chi_{jj}}{g_j^2} - \frac{1}{\chi_0} \sum_{j=0}^{N} \frac{\chi_j^2}{g_j^2} \sim \sum_{j=0}^{N} \frac{\chi_j}{g_j} - \frac{1}{\chi_0} \sum_{j=0}^{N} \frac{\chi_j^2}{g_j^2} \tag{4.7}
\]

5 The light $Z$ mass

Now we need to find the light $Z$ mass. The neutral mass squared matrix given by (3.1) has, of course, a zero eigenvalue associated with the photon. The photon eigenstate, as usual, is
\[
\kappa^{N+1} = \begin{pmatrix}
e/g_0 \\
\vdots \\
e/g_j \\
\vdots \\
e/g_{N+1}
\end{pmatrix} \sim \begin{pmatrix}1 \\
\vdots \\
1\end{pmatrix} \lambda^{N+1} \tag{5.1}
\]

Again, it is easiest to work with the inverse mass squares. The neutral gauge boson mass-squared matrix is not invertible because of the photon, but we can invert it on the subspace orthogonal to the photon eigenvector, $\kappa^{N+1}$, and this can be written in terms of $\tilde{V}^{-1}$. This is the basis of the Georgi-Weinberg construction. [8]

Define $\tilde{V}$ and $\tilde{V}^{-1}$ as\(^5\)
\[
\begin{pmatrix}
\tilde{V} \\
\tilde{V}^{-1}
\end{pmatrix}_{jk} = \begin{pmatrix}
1 \\
\vdots \\
1
\end{pmatrix} \quad \text{for } j, k = 0 \text{ to } N \quad \text{and} \\
\tilde{V}^{N+1, k} = 0
\tag{5.2}
\]

\[
\begin{pmatrix}
\tilde{V}^{-1} \\
\tilde{V}^{N+1, k}
\end{pmatrix}_{jk} = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} \quad \text{for } j, k = 0 \text{ to } N \quad \text{and} \\
\tilde{V}^{N+1, k} = 0 \tag{5.3}
\]

\(^5\)I have put the superscript $-1$ in quotes to indicate that is only the inverse on the subspace orthogonal to the $N + 1$ direction.
Then the inverse of the neutral gauge boson mass-squared matrix on the subspace orthogonal to the photon eigenvector is

\[ M_{nn}^{2-1} = 4 \left( I - \kappa^{N+1} \kappa^{N+1T} \right) G^{-1} \tilde{V}^{-1} G^{-1} \left( I - \kappa^{N+1} \kappa^{N+1T} \right) \] (5.4)

This can also be written as

\[ 4 G^{-1} \left( I - \lambda^{N+1} \lambda^{N+1T} e^2 G^{-2} \right) \tilde{V}^{-1} G^{-1} \left( I - e^2 G^{-2} \lambda^{N+1} \lambda^{N+1T} \right) G^{-1} \] (5.5)

In this basis, (5.5) is not diagonal as \( g_j \to \infty \). We need to diagonalize before we apply perturbation theory. But there is a slightly peculiar trick that allows us to do this automatically, changing to a more convenient basis without making a mess of the matrix. I will first describe the trick in general, and then apply it to (5.5).

Suppose \( \kappa \) and \( \hat{e} \) are unit vectors. Then

\[ P_\pm = \frac{1}{2} \left( 1 \pm \kappa^T \hat{e} \right) \left( \kappa \pm \hat{e} \right)^T \] (5.6)

are projection operators onto the one-dimensional subspaces spanned by \( \kappa \pm \hat{e} \) respectively. Then

\[ U_\pm = I - \frac{1}{\left( 1 \pm \kappa^T \hat{e} \right)} \left( \kappa \pm \hat{e} \right) \left( \kappa \pm \hat{e} \right)^T \] (5.7)

are symmetric orthogonal matrices, with eigenvalue \(-1\) on the one-dimensional subspace and \(1\) elsewhere, so they satisfy

\[ U_\pm = U_\pm^T = U_\pm^{-1} \] (5.8)

Since these give opposite signs on \( \kappa + \hat{e} \) and \( \kappa - \hat{e} \), they just interchange \( \kappa \) and \( \hat{e} \). One finds

\[ U_\pm \kappa = \mp \hat{e} , \quad U_\pm \hat{e} = \mp \kappa \] and transposes. (5.9)

We are interested in applying this transformation to objects which are annihilated by \( \kappa \). We want to make use of the fact that the components of such an object in the \( e \) direction can always be eliminated in terms of the other components. We can write

\[ (I - \kappa \kappa^T) = (I - \kappa \kappa^T) \left[ I - \kappa \hat{e}^T / (\hat{e}^T \kappa) \right] \] (5.10)

because the second term in square brackets vanishes, and

\[ = (I - \kappa \kappa^T) (I - \hat{e} \hat{e}^T) \left[ I - \kappa \hat{e}^T / (\hat{e}^T \kappa) \right] \] (5.11)

because the second term in the middle factor vanishes.

So now we look at

\[ (I - \kappa \kappa^T) U_\pm = (I - \kappa \kappa^T) (I - \hat{e} \hat{e}^T) \left[ I - \kappa \hat{e}^T / (\hat{e}^T \kappa) \right] U_\pm (I - \hat{e} \hat{e}^T) \] (5.12)

\[ = (I - \kappa \kappa^T) (I - \hat{e} \hat{e}^T) \left[ I - \kappa \hat{e}^T / (\hat{e}^T \kappa) \right] \left( I \mp \frac{\hat{e} \kappa^T}{(1 \pm \kappa \hat{e}^T \hat{e})} \right) (I - \hat{e} \hat{e}^T) \] (5.13)
\[
(I - \kappa \kappa^T) \left( I - \hat{e} \hat{e}^T \right) \left( I \pm \frac{\kappa \kappa^T}{(\hat{e}^T \kappa) (1 \pm \kappa^T \hat{e})} \right) \left( I - \hat{e} \hat{e}^T \right) \quad (5.14)
\]

We can’t average these because \( U_+ \) and \( U_- \) are not equal, though both have similar properties. We can write (5.14) equivalently as

\[
(I - \kappa \kappa^T) \ U_\pm = \left( I - \kappa \kappa^T \right) \ H_\pm
\]

where

\[
H_\pm \equiv \left( I - \hat{e} \hat{e}^T \right) \left( I - \kappa \kappa^T \right) \left( \kappa - \hat{e} (\kappa^T \hat{e}) \right) \left( \kappa^T - (\kappa^T \hat{e}) \hat{e}^T \right) \left( \hat{e}^T \kappa \right) \left( 1 \pm \kappa^T \hat{e} \right)
\]

\[\text{where we don’t need the projectors onto the subspace orthogonal to } \hat{e}.\]

We are interested in the case where \( \kappa = \kappa_{N+1} \), the photon eigenvector, and \( \hat{e} = \hat{e}_{N+1} \), the unit vector in the \( N + 1 \) direction. Before proceeding, let’s check the result for \( N = 0 \). This is also the 0th contribution to the \( Z \) mass in the general theory, so we have to do it anyway. In this case,

\[
\kappa = \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \quad \text{and} \quad \hat{e} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

Then (5.16) becomes

\[
H_\pm = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pm \frac{1}{\cos \theta (1 \pm \cos \theta)} \begin{pmatrix} \sin \theta \\ 0 \end{pmatrix} \begin{pmatrix} \sin \theta & 0 \end{pmatrix} \quad (5.18)
\]

\[
= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pm \frac{1}{\cos \theta (1 \pm \cos \theta)} \begin{pmatrix} \sin^2 \theta & 0 \\ 0 & 0 \end{pmatrix} \quad (5.19)
\]

\[
= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pm \frac{1}{\cos \theta (1 \pm \cos \theta)} \begin{pmatrix} 1 - \cos^2 \theta & 0 \\ 0 & 0 \end{pmatrix} \quad (5.20)
\]

\[
= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \pm \frac{1}{\cos \theta} \begin{pmatrix} 1 \mp \cos \theta & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \pm 1/ \cos \theta & 0 \\ 0 & 0 \end{pmatrix} \quad (5.21)
\]

There is one such factor from each side of the mass matrix, so this gives the usual factor of \( 1/ \cos^2 \theta \) in the \( Z \) mass compared to the \( W \) mass.\(^6\)

Another good check is to derive Georgi-Weinberg this way. To do this we note that the inverse of \( H_\pm \) on the subspace orthogonal to \( \hat{e} \) is

\[
H_{\pm}^{-1} = \left( I - \hat{e} \hat{e}^T \right) - \frac{\kappa - \hat{e} (\kappa^T \hat{e})}{(\kappa^T \hat{e}) (1 \pm \kappa^T \hat{e})} \left( \kappa^T - (\kappa^T \hat{e}) \hat{e}^T \right)
\]

\[\text{where in the mechanical analog for } N = 0, \text{ the } \cos^2 \theta \text{ factor in the } Z \text{ mass squared is just the ratio of the reduced mass of the system to the mass of 0. In this case, the trick is just the standard analysis using the reduced mass. It is not clear to me what the correspondence is for } N > 0.\]

\[6\text{Notice that in the mechanical analog for } N = 0, \text{ the } \cos^2 \theta \text{ factor in the } Z \text{ mass squared is just the ratio of the reduced mass of the system to the mass of 0. In this case, the trick is just the standard analysis using the reduced mass. It is not clear to me what the correspondence is for } N > 0.\]
Now applying this to the $Z$ mass matrix, we use these matrices with
\[ \kappa = \kappa_{N+1} \quad \text{and} \quad \hat{e} = \hat{e}_{N+1} \] (5.23)
we can transform the neutral gauge boson mass squared matrix to a basis in which it is orthogonal to $\hat{e}_{N+1}$ as follows:\footnote{\text{Remember from (5.8) that $U^{-1} = U$.}}
\[ M^2_n = \frac{1}{4} G V G \rightarrow U^\pm M^2_n U^\pm = \frac{1}{4} U^\pm G V G U^\pm = \frac{1}{4} H^\pm \bar{\tilde{V}} \tilde{G} H^\pm \] (5.24)
The expression (5.24) is interesting for two reasons. Firstly, we have reduced the $N+2 \times N+2$ problem to an $N+1 \times N+1$ problem. Secondly, the expressions are a bit bizarre, with the $(1 \pm \kappa^T \hat{e})$. Somehow, the arbitrary $\pm$ sign must cancel in all physical results.

In particular, we can trivially invert (5.24) on the subspace perpendicular to $\hat{e}_{N+1}$ to get
\[ U^\pm M^{n-1^\prime}_n U^\pm = 4 H^{n-1^\prime}_\pm \tilde{G}^{-1} \bar{\tilde{V}} \tilde{G}^{-1} H^{n-1^\prime}_\pm \] (5.25)
Then we can use $U^\pm$ again to transform back to the original basis,
\[ 4 U^\pm H^{n-1^\prime}_\pm \tilde{G}^{-1} \bar{\tilde{V}} \tilde{G}^{-1} H^{n-1^\prime}_\pm U^\pm \] (5.26)
Somewhat miraculously, when one calculates $H^{n-1^\prime}_\pm U^\pm$ explicitly, all the $\pm$ dependence cancels and one finds
\[ H^{n-1^\prime}_\pm U^\pm = (I - \hat{e} \hat{e}^T) (I - \kappa \kappa^T) \] (5.27)
and thus (5.26) reproduces (5.4).

More generally, the expression (5.25) affords a systematic way of evaluating the inverse mass squared matrix perturbatively. The non-zero matrix elements of $H^{n-1^\prime}_\pm$ are, for $j, k = 0$ to $N$,
\[ \left[ H^{n-1^\prime}_\pm \right]_{jk} = \delta_{jk} - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_j g_k} \] (5.28)
\[ \left[ \tilde{G}^{-1} H^{n-1^\prime}_\pm \right]_{jk} = \frac{1}{g_j} \delta_{jk} - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_j^2 g_k} \] (5.29)
Now separating the 0 components and keeping only the terms that contribute to second order, we have
\[ \left[ \tilde{G}^{-1} H^{n-1^\prime}_\pm \right]_{00} = \frac{1}{g_0} - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} = \frac{1}{g_0} \left[ 1 - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right] \] (5.30)
\[ \left[ \tilde{G}^{-1} H^{n-1^\prime}_\pm \right]_{j0} = - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_j^2 g_0} \] (5.31)
\[ \left[ \tilde{G}^{-1} H^{n-1^\prime}_\pm \right]_{0k} = - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2 g_k} \] (5.32)
\[ \left[ G^{-1} H_{\pm}^{-1} \right]_{jk} = \frac{1}{g_j} \delta_{jk} \]  

(5.33)

Expanding (5.25) to second order in \( e/g_j \) for \( j = 1 \) to \( N \) and using (3.20) and (5.2), we can collect the relevant terms of the transformed matrix as follows:

\[
\frac{1}{4} \left[ U_{\pm} M_n^{2,1} U_{\pm} \right]_{00} = \frac{\chi_0}{g_0^2} \left[ 1 - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right]^2
\]  

(5.34)

\[
-2 \sum_{j=1}^{N} \frac{\chi_j}{g_j^2} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \left[ 1 - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right]
\]  

(5.35)

\[
\frac{1}{4} \left[ U_{\pm} M_n^{2,1} U_{\pm} \right]_{0j} = \left[ \frac{\chi_j}{g_0 g_j} - \frac{\chi_0}{g_0 g_j} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right] \left[ 1 - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right]
\]

\[
+ \frac{\chi_0}{g_j g_k} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} + \chi_0 \left( \frac{1}{1 \pm e/g_{N+1}} \right)^2 \frac{e^2}{g_0^2} \]

(5.36)

\[
\frac{1}{4} \left[ U_{\pm} M_n^{2,1} U_{\pm} \right]_{jk} = \frac{\chi_j}{g_j g_k} - \frac{\chi_j + \chi_k}{g_j g_k} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} + \chi_0 \left( \frac{1}{1 \pm e/g_{N+1}} \right)^2 \frac{e^2}{g_0^2}
\]

(5.37)

The light Z mass to second order is then given by

\[
\frac{1}{4 M_Z^2} = \frac{\chi_0}{g_0^2} \left[ 1 - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right]^2
\]  

(5.39)

\[
-2 \sum_{j=1}^{N} \frac{\chi_j}{g_j^2} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \left[ 1 - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right]
\]  

(5.40)

\[
+ \sum_{j=1}^{N} \frac{1}{\chi_0} \left[ \frac{\chi_j}{g_j} - \frac{\chi_0}{g_j} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right]^2
\]

(5.41)

Expanding and rearranging a bit, this is (note that one of the sums now starts at \( j = 0 \))

\[
\frac{1}{4 M_Z^2} = \frac{1}{\chi_0} \sum_{j=0}^{N} \frac{\chi_j^2}{g_j^2}
\]  

(5.42)

\[
-2 \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \left[ 2 - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right] \sum_{j=1}^{N} \frac{\chi_j}{g_j^2}
\]

\[
- \frac{\chi_0}{g_0^2} \left[ \frac{1}{1 \pm e/g_{N+1}} \right] \frac{e^2}{g_0^2} \left[ 2 - \left( \frac{1}{1 \pm e/g_{N+1}} \right) \left( \frac{e^2}{g_0^2} + \sum_{j=1}^{N} \frac{e^2}{g_j^2} \right) \right]
\]

(5.43)

(5.44)

We can now use the fact that

\[
\frac{e^2}{g_0^2} = 1 - \frac{e^2}{g_{N+1}} - \sum_{j=1}^{N} \frac{e^2}{g_j^2}
\]  

(5.45)
and substitute this inside the square brackets in (5.43) and (5.44) to rewrite (5.42-5.44) as

\[
\frac{1}{4M_Z^2} = \frac{1}{\chi_0} \sum_{j=0}^N \frac{\chi_j^2}{g_j^2} \tag{5.46}
\]

\[- 2 \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \left[ 2 - \left( \frac{1 - e^2/g_{N+1}^2}{1 \pm e/g_{N+1}} \right) \right] \sum_{j=1}^N \frac{\chi_j}{g_j^2} \tag{5.47}
\]

\[- \frac{\chi_0}{g_0^2} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \left[ 2 - \left( \frac{1 - e^2/g_{N+1}^2}{1 \pm e/g_{N+1}} \right) \right] \tag{5.48}
\]

where we have neglected higher order terms in (5.44). Now inevitably, but still apparently miraculously, the ± signs disappear, and we have\(^8\)

\[
\frac{1}{4M_Z^2} = \frac{1}{\chi_0} \sum_{j=0}^N \frac{\chi_j^2}{g_j^2} - 2 \frac{e^2}{g_0^2} \sum_{j=0}^N \frac{\chi_j}{g_j^2} + \frac{\chi_0}{g_0^2} \left( \frac{e^2}{g_0^2} \right)^2 \tag{5.49}
\]

To the order to which we are working, we can replace the factors of \(e^2/g_0^2\) in (5.49) with any expression that has the same zeroth order value. It is convenient to substitute

\[
\frac{e^2}{g_0^2} \rightarrow \frac{e^2}{\chi_0} \sum_{j=0}^N \frac{\chi_j}{g_j^2} \tag{5.50}
\]

which simplifies (5.49) further to

\[
\frac{1}{4M_Z^2} = \frac{1}{\chi_0} \sum_{j=0}^N \frac{\chi_j^2}{g_j^2} - \frac{e^2}{\chi_0} \left( \sum_{j=0}^N \frac{\chi_j}{g_j^2} \right)^2 \tag{5.51}
\]

Note also that the inverse mass matrix for the heavy neutral gauge bosons to leading order is an \(N \times N\) matrix with matrix elements

\[
4 \left( \frac{\chi_{jk}}{g_j g_k} - \frac{\chi_j + \chi_k}{g_j g_k} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} + \frac{\chi_0}{g_j g_k} \left( \frac{1}{1 \pm e/g_{N+1}} \right)^2 \left( \frac{e^2}{g_0^2} \right)^2 \right) \tag{5.52}
\]

\[- \frac{1}{\chi_0} \left[ \frac{\chi_j}{g_j} - \frac{\chi_0}{g_j} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right] \left[ \frac{\chi_k}{g_k} - \frac{\chi_0}{g_k} \left( \frac{1}{1 \pm e/g_{N+1}} \right) \frac{e^2}{g_0^2} \right] \tag{5.53}
\]

\[= 4 \frac{1}{g_j} \left( \frac{\chi_{jk} - \chi_j \chi_k}{\chi_0} \right) \frac{1}{g_k} \tag{5.54}
\]

for \(j, k = 1\) to \(N\). This looks just like (5.54), as it should to this order.

\(^8\)Again we have changed the lower limit of a sum.
6 Alphabet soup

Let’s review where we are. From the low energy charged-current weak interactions, we have (pulling some equations from previous sections with their original numbers)

\[ \sqrt{2} G_F = \frac{1}{v^2} = [\tilde{V}^{-1}]_{00} \sim \sum_{j=0}^{N} \frac{1}{v_{j,j+1}^2} \]  

(3.17)

The low energy neutral-current weak interactions are

\[ \sum_{j,k=0}^{N} [\tilde{V}^{-1}]_{jk} \left[ T_3 \delta_{j0} - \frac{e^2}{g_{j}^2} Q \right] \left[ T_3 \delta_{k0} - \frac{e^2}{g_{k}^2} Q \right] \]  

(3.18)

where the matrix elements \([\tilde{V}^{-1}]_{jk}\) are given by

\[ [\tilde{V}^{-1}]_{00} = \frac{1}{v^2} \equiv \chi_0, \quad [\tilde{V}^{-1}]_{j0} = [\tilde{V}^{-1}]_{0j} \equiv \chi_j, \quad [\tilde{V}^{-1}]_{jk} \equiv \chi_{jk} \quad \text{for } j, k = 1 \text{ to } N \]  

(3.19)

Note that the normalization of the \(T_2^2\) term satisfies custodial \(SU(2)\) symmetry, so the correction to the \(\rho\) parameter is small. The analog of \(\sin^2 \theta\) as determined by the low energy weak interactions is determined by the coefficient of \(T_3 Q\) in (3.18) to be

\[ \sin^2 \theta = e^2 v^2 \sum_{j=0}^{N} \frac{\chi_j}{g_{j}^2} \]  

(6.1)

The \(W\) and \(Z\) masses are determined to next-to-leading order in the small couplings by

\[ \frac{1}{M_W^2} = 4v^2 \sum_{j=0}^{N} \frac{\chi_j^2}{g_{j}^2} \]  

(4.3)

\[ \frac{1}{M_Z^2} = 4v^2 \left( \sum_{j=0}^{N} \frac{\chi_j^2}{g_{j}^2} - \frac{e^2}{\chi_0} \left( \sum_{j=0}^{N} \frac{\chi_j}{g_{j}^2} \right)^2 \right) \]  

(6.2)

One way of describing this is to say that we can write all quantities to this order in terms of four parameters (as usual, shown in general and with their values in the linear model):

\[ e^2 = \left( \sum_{k=0}^{N+1} \frac{1}{g_{k}^2} \right)^{-1} \]  

(6.3)

\[ v^2 = \frac{1}{\chi_0} \sim \left( \sum_{k=0}^{N} \frac{1}{v_{k,k+1}^2} \right)^{-1} \]  

(6.4)

\[ s_1^2 \equiv e^2 v^2 \sum_{j=0}^{N} \frac{\chi_j}{g_{j}^2} \sim \frac{e^2}{g_{0}^2} + \sum_{j=1}^{N} \frac{e^2}{g_{j}^2} \left( \sum_{k=j}^{N} \frac{v_{k,k+1}^2}{v_{j}^2} \right) \]  

(6.5)

\[ s_2^2 \equiv e^2 v^4 \sum_{j=0}^{N} \frac{\chi_j^2}{g_{j}^2} \sim \frac{e^2}{g_{0}^2} + \sum_{j=1}^{N} \frac{e^2}{g_{j}^2} \left( \sum_{k=j}^{N} \frac{v_{k,k+1}^2}{v_{j}^2} \right)^2 \]  

(6.6)
The two parameters, $s_1^2$ and $s_2^2$ both reduce to $\sin^2 \theta$ in the standard model limit in which the extra gauge couplings go to infinity. In terms of these, we can write

$$\sin^2 \theta_{\text{neutral current}} = s_1^2$$  \hspace{1cm} (6.8)

$$M_W^2 = \frac{e^2 v^2}{4 s_2^2}$$  \hspace{1cm} (6.9)

$$M_Z^2 = \frac{e^2 v^2}{4(s_2^2 - s_1^4)} = \frac{s_2^2}{s_2^2 - s_1^4} M_W^2$$  \hspace{1cm} (6.10)

Note that $s_1^2 > s_2^2$.

According to the particle data group, \[9\]

$$M_Z^2 = \frac{1 - \alpha T}{1 - G_F m_{Z0}^2 S / 2 \sqrt{2} \pi}$$  \hspace{1cm} (6.11)

$$M_W^2 = \frac{1}{1 - G_F m_{W0}^2 (S + U) / 2 \sqrt{2} \pi}$$

“where $M_{Z0}$ and $M_{W0}$ are the SM expressions (as functions of $m_t$ and $M_H$) in the $\overline{MS}$ scheme.” Or more sloppily (but simpler for our purposes)

$$M_Z^2 = \frac{(1 - \alpha T)e^2 v^2 / 4}{\sin^2 \theta \cos^2 \theta - \alpha S / 4}$$  \hspace{1cm} (6.12)

$$M_W^2 = \frac{e^2 v^2 / 4}{\sin^2 \theta - \alpha (S + U) / 4}$$

All this implies the following.

1. The custodial $SU(2)$ relation for the ratio of NC to CC low energy weak interactions, along with the fact (see the discussion of (6.19) below) that the heavy gauge bosons make a negligible contribution implies that to this order,

$$T = 0$$  \hspace{1cm} (6.13)

2. From the expressions for $M_W^2$ and $\sin^2 \theta$, (6.10) and (6.8), we get $S$,$^9$

$$s_2^2 = s_1^2 - \alpha S / 4$$  \hspace{1cm} (6.14)

or

$$S = \frac{4}{\alpha} (s_1^2 - s_2^2)$$  \hspace{1cm} (6.15)

In particular this shows that $S > 0$. This result is well known for the linear model. [3, 4]

We now see that it is true for more general symmetry breaking patterns, at least near the standard model limit.

\[9\]This formula was derived in the special case of the linear model in [10].

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3. Now from the expression from $M_Z^2$ and the rest, we can write

$$s^2_2 - s^4_1 = s^2_1 - s^4_1 - \alpha(S + U)/4 \quad (6.16)$$

or

$$U = 0 \quad (6.17)$$

The right hand side of $U = 0$ is higher order in the small quantities $e^2/g^2_j$ for $j = 1$ to $N$.

Comparing (6.15), (6.5), (6.6) and (4.7), we can write

$$S - 4\pi^2 v^2 \sum_{\text{heavy } W_4} \frac{1}{M^2} = 16\pi^2 v^2 \left( \sum_{j=0}^{N} \frac{\chi_j^2}{g^2_j} - \sum_{j=0}^{N} \frac{\chi_{jj}}{g^2_j} \right) \sim 0 \quad (6.18)$$

Thus the $S$ parameter is related to the sum of the inverse mass squares of the heavy gauge bosons only in the linear model. In general these are independent.

The couplings of the $W$ and $Z$ to fermions are determined by the mass eigenstates. We know the $W$ eigenstate from (4.2). We find for the coupling

$$g^2_W = \frac{\chi_0^2}{\sum_{j=0}^{N} \frac{\chi_j^2}{g^2_j}} = \frac{M^2_W}{v^2} \quad (6.19)$$

which is the same as the tree-level standard model result. This does not get corrections to this order because the heavier states are not only heavier, they are more weakly coupled to ordinary matter. Presumably, the $Z$ couplings behave the same way for the same reason. This is more complicated to work out, so I won’t do it here.

7. The mechanical analog of $S$

How small can we make $S$? To think about this, let us first rewrite (6.15) as

$$S = \frac{1}{\pi} \sum_{j=1}^{N} \left( \frac{4\pi^2}{g^2_j} \right) \left( \frac{\chi_j}{\chi_0} \right) \left( 1 - \frac{\chi_j}{\chi_0} \right) \quad (7.1)$$

Even if we take the coupling factors to be of order 1, the terms in the sum are each of order $1/4$ unless the $\chi_j/\chi_0$ is close to 0 or 1, in which case the contribution is small.\(^{10}\)

In the linear model with approximately equal $v^2_{jj+1}$ along the line, each $v^2_{jj+1}$ is approximately $(N + 1)v^2$, so the Higgs mass can be raised by a factor of $\sqrt{N + 1}$, and

$$\frac{\chi_j}{\chi_0} \approx \frac{N + 1 - j}{N + 1} \quad (7.2)$$

\(^{10}\)A similar formula is written down in [11]. I would like to thank the authors for their comments about this.
so that
\[ S \gtrsim \frac{N^2 - 1}{6\pi N} \]  \hspace{1cm} (7.3)
which suggests that \( N \) cannot be large.

It might not be immediately obvious that one cannot do significantly better than this in a completely general model. But in fact, this becomes quite clear if you think about what this means in the mechanical analog. The \( \chi_j \) are the components of the low-frequency normal mode of the mechanical analog, so \( \chi_j/\chi_0 \) is the displacement of the \( j \)th mass as a fraction of the displacement of the 0th mass when the system is stretched slowly by pulling on the 0th mass. Thus to get small \( S \), we want a system in which all the masses either move very little when the 0th mass is pulled, or else move along with the 0th mass. What we don’t want is a number of masses whose motions interpolate between the motion of mass 0 and the fixed wall, because these give the maximum contribution to (7.1). Now it is perfectly possible to have a system of springs with the properties that give small \( S \) (for example, the systems analogous to figures 2a and 2b, where the spring to mass 1 would not be stretched at all), but unfortunately it is not consistent with the fundamental goal of producing a low frequency mode with only stiff springs (that is - raising the Higgs mass). The low frequency mode arises precisely because the stretching of the system can be spread over many stiff springs, so that each stretches only a small amount. But that means that the displacements vary from zero to the full displacement.

Thus I conclude that raising the Higgs mass is essentially equivalent to increasing \( S \), not just in the linear system, or in any other deconstruction of one or more extra dimensions, but for the completely general structure of \( SU(2)s \) and a single \( U(1) \). And while we have had some fun with Higgsless theories, it seems unlikely that nature has chosen this amusing approach to electroweak symmetry breaking.

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