ON SINGULARITIES OF THE GAUSS MAP COMPONENTS OF SURFACES IN $\mathbb{R}^4$

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Abstract. The Gauss map of a generic immersion of a smooth, oriented surface into $\mathbb{R}^4$ is an immersion. But this map takes values on the Grassmannian of oriented 2-planes in $\mathbb{R}^4$. Since this manifold has a structure of a product of two spheres, the Gauss map has two components that take values on the sphere. We study the singularities of the components of the Gauss map and relate them to the geometric properties of the generic immersion. Moreover, we prove that the singularities are generically stable, and we connect them to the contact type of the surface and $J$-holomorphic curves with respect to an orthogonal complex structure $J$ on $\mathbb{R}^4$. Finally, we get some formulas of Gauss-Bonnet type involving the geometry of the singularities of the components with the geometry and topology of the surface.

1. Introduction

The Gauss map singularities of an immersion of a smooth, oriented surface $M$ into $\mathbb{R}^3$ have been a subject of research because they represent a strong relationship between the singularity theory and the geometry of the immersions of a surface. Since this map can be locally described as a map from the plane to the plane, the Whitney theory ([21]) on singularities of differentiable maps is applied. These singularities arise from the contact of the surface with its tangent plane. The analysis of this contact goes back to classic differential geometry of surfaces and has been developed in several works; for instance, [2], [5], and [12] for an extensive list of references. The parabolic set, where the Gaussian curvature vanishes, is the singular set of the map. Most interesting properties from the smooth point of view occur near this set. If $M$ is closed, the parabolic set of the image of a generic immersion consists of a finite number of embedded circles constituted by stable singularities. That is, fold points and isolated cusp points [5]. This structure of the parabolic set allows us to find geometric properties of generic immersions of $M$ into $\mathbb{R}^3$.

If $M$ is immersed into $\mathbb{R}^4$ the Gauss map $g$ takes values into the Grassmannian of oriented 2-planes in $\mathbb{R}^4$. This manifold has a structure of the product $S_1 \times S_2$ of 2-dimensional spheres $S_1 = S_2 = S^2\left(\frac{1}{\sqrt{2}}\right)$ of radius $\frac{1}{\sqrt{2}}$ in $\mathbb{R}^3$. Then, $g$ has two component-maps from the surface into the sphere. Namely, $g = (g_1, g_2) = (\pi_1 \circ G, \pi_2 \circ G)$, where $\pi_i : S_1 \times S_2 \to S_i$ is the projection for $i = 1, 2$. Despite several authors studied the Gauss map in this setting, see for instance [4], [9], [11], and [20], an analysis of the singularities of $g_i$, $i = 1, 2$ has not been developed yet.

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This article aims to settle down the first steps on the study of the singularities of the components of the Gauss map and relate them to geometric properties of the immersed surface. We begin by characterizing the kernel of the derivative of $g_i$, $i = 1, 2$, at each point, as the kernel of a Pfaffian system defined by the connection forms independently of the coordinate chart (Theorem 1). In [11] (see also [20]), the authors proved that the singular set of the component $g_i = \pi_i \circ G$ for $i = 1$ (or $i = 2$) is the zero locus of the function defined as the sum (or the difference, respectively) of the Gaussian and normal curvatures. In the present article, following the ideas from [4], we provide an alternate proof of this result (Corollary 1) which is rather simple and more convenient to the approach of this research.

The Gauss map of a smooth, oriented surface immersed into $\mathbb{R}^4$ might not be regular, i.e., it might not be an immersion at some points. The lack of regularity imposes strong restrictions on the surface. Specifically, these non-regular points are characterized as the points where the Gaussian curvature and the discriminant of the second fundamental form vanish simultaneously, see [13] for other equivalent conditions. We prove that for a generic immersion of $M$ into $\mathbb{R}^4$, the Gauss map is regular (Proposition 3). Further on, we define three properties $(G_1)$, $(G_2)$ and $(G_3)$ for immersions of $M$ into $\mathbb{R}^4$ that determine the structure of the singular set of the components of the Gauss map. Namely, for immersions having these properties the singular set of $g_i$, $i = 1, 2$ consists of a finite number of smooth embedded circles constituted by stable singularities. That is, fold points and isolated cusp points. Moreover, the corresponding singular set’s image under $g_i$, $i = 1, 2$ has stable intersections. Then, we prove the genericity theorem. Namely, immersions having these properties are generic (Theorem 3). Contrasting with the case of the Gauss map of surfaces in $\mathbb{R}^3$, the singularities of the Gauss map components of surfaces in $\mathbb{R}^4$ arise from the contact of the surface with a $J$-holomorphic curve with respect to an orthogonal complex structure $J$ in $\mathbb{R}^4$ that better approaches the surface locally. We introduce the study of this type of contact in the present article. So, we provide a classification of regular and generic singular points of these components employing the normal forms of jets of contact maps (Theorem 5). Further on, we apply certain theorems proved in [17], [18] and [19] to get formulas of Gauss-Bonnet type relating the geometry and topology of the surface to singularities of the components of $g$ (Theorem 6 and Corollary 3). The article is organized as follows. In section 2, we present some preliminary results including the description of the second order invariants of an immersion of a surface into $\mathbb{R}^4$ in terms of the connection forms. Moreover, we describe the model of the oriented Grassmannian of 2-planes in $\mathbb{R}^4$ that will be employed in the research. Section 3 is devoted to studying the singular sets of the components of the Gauss map; we prove Theorem 4 and Corollary 4. In section 4, we define the genericity conditions $(G_1)$, $(G_2)$ and $(G_3)$ and we discuss their meaning concerning the structure of the singular sets of $g_i$, $i = 1, 2$. In section 5, we show the local description of the components of the Gauss map. We use this fundamental tool in the proof of the genericity theorem. In section 6, we deal with the generic properties of the Gauss map and its components. Specifically, we prove Proposition 6 and Theorem 6. Section 7 aims to prove the classification theorem of regular and generic singular point. In section 8, we demonstrate Theorem 7 and Corollary 7 that provide the Gauss-Bonnet type formulas. We conclude by analyzing the example of the double
torus mentioned above. Finally, we point out that the research presented in this article has a relevant extension by using the approach presented in [9] for the class of oriented surfaces with boundary.

2. Preliminaries

2.1. The second-order invariants. Let $M$ be a surface immersed into $\mathbb{R}^4$. We denote by $\mathcal{X}(M)$ and $\mathcal{X}(M)^\perp$ the spaces of tangent and normal smooth vector fields on $M$, respectively. The second fundamental form of $M$ is the tensor field defined as

$$ II : \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M)^\perp, \quad II(X, Y) = (dY(X))^\perp, $$

where $Z^\perp$ denotes the orthogonal projection on $\mathcal{X}(M)^\perp$ of a smooth vector field $Z : M \rightarrow \mathbb{R}^4$. This tensor field determines at each point $p \in M$ a quadratic map:

$$ II_p : T_pM \rightarrow T_pM^\perp, \quad II_p(X) = II(X, X)_p $$

whose invariants under the isometries of $T_pM$ and $T_pM^\perp$, respectively, are the second-order invariants of $M$. They are known as the norm of the mean curvature vector, the Gaussian curvature, the normal curvature and the discriminant of the second fundamental form. We denote them by

$$ (1) \quad |H|^2, \ K, \ K^N, \ \Delta, $$

respectively; see [13] and [3].

Let $U \subset \mathbb{R}^2$ be an open set parameterized by $(u, v)$. We suppose that $x : U \rightarrow M$ is a local parameterization of $M$ endowed with a Darboux frame $(e_1, e_2, e_3, e_4)$. Namely, a positive orthonormal frame of $\mathbb{R}^4$ on $x(U)$ such that, for every $(u, v) \in U$, $(e_1(u, v), e_2(u, v))$ is a positive orthonormal basis of $T_qM$, and $(e_3(u, v), e_4(u, v))$ is an orthonormal basis of $T_qM^\perp$, where $q = x(u, v)$. We denote by $\theta_i = \langle dx, e_i \rangle$ for $i = 1, 2, 3, 4$ the dual forms of the frame. Notice that $\theta_3 = \theta_4 = 0$. Moreover, the connection 1-forms $\omega^i_j$ for $i, j = 1, \ldots, 4$ are defined in the following way

$$ (2) \quad d\theta_i(X) = \sum_{j=1}^{4} \omega^i_j(X) e_j, $$

where $X$ is a tangent vector field on $M$. Thus, the curvatures $K$ and $K^N$ satisfy

$$ (3) \quad d\omega^2_i = -K \theta_1 \wedge \theta_2, \quad d\omega^3_i = -K^N \theta_1 \wedge \theta_2, $$

where $\theta_1 \wedge \theta_2$ is the area form defined by the tangent frame.

The second fundamental form is expressed as

$$ (4) \quad II(X) = (\omega^3_1(e_1)(\theta_1)^2 + (\omega^3_2(e_1) + \omega^3_1(e_2))\theta_1\theta_2 + \omega^3_2(e_2)(\theta_2)^2)e_3 + (\omega^4_1(e_1)(\theta_1)^2 + (\omega^4_2(e_1) + \omega^4_1(e_2))\theta_1\theta_2 + \omega^4_2(e_2)(\theta_2)^2)e_4. $$

2.2. The oriented Grassmannian $Gr^+(2, 4)$. We denote by $Gr^+(2, 4)$ the space of all oriented 2-dimensional subspaces (planes) of $\mathbb{R}^4$. Let $(e_1, e_2, e_3, e_4)$ be a positive orthonormal basis for $\mathbb{R}^4$, i.e., $(e_i, e_j) = \delta_{ij}$ for $i, j = 1, 2, 3, 4$ and $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ is the orientation of $\mathbb{R}^4$. We use Plücker coordinates in $Gr^+(2, 4)$ (see [6]). Let $\Lambda^2\mathbb{R}^4$ be the space of bivectors of $\mathbb{R}^4$. Thus, if $V$ is an oriented plane of $\mathbb{R}^4$ and $(v_1, v_2)$ is a positive orthonormal basis of $V$, the bivector

$$ (5) \quad v_1 \wedge v_2 = \alpha_{12}e_1 \wedge e_2 + \alpha_{23}e_2 \wedge e_3 + \alpha_{34}e_3 \wedge e_4 + \alpha_{13}e_1 \wedge e_3 + \alpha_{14}e_1 \wedge e_4 + \alpha_{24}e_2 \wedge e_4, $$

respectively; see [13] and [3].

Let $U \subset \mathbb{R}^2$ be an open set parameterized by $(u, v)$. We suppose that $x : U \rightarrow M$ is a local parameterization of $M$ endowed with a Darboux frame $(e_1, e_2, e_3, e_4)$.
in the basis \( e_i \wedge e_j \) used in this expression, represents \( V \) and the coordinates \( \alpha_{ij} \) are called Plücker coordinates. They are independent of the choice of a positive orthonormal basis of \( V \) and satisfy the following equations

\[
(6) \quad \alpha_{12}\alpha_{34} + \alpha_{23}\alpha_{14} + \alpha_{31}\alpha_{24} = 0, \\
\alpha_{12}^2 + \alpha_{23}^2 + \alpha_{14}^2 + \alpha_{31}^2 + \alpha_{24}^2 = 1.
\]

The first equation of system (6) follows from \((v_1 \wedge v_2) \wedge (v_1 \wedge v_2) = 0\). Moreover, we extend the dot product on \( \mathbb{R}^4 \) to \( \Lambda^2 \mathbb{R}^4 \) by the standard formula

\[
(7) \quad \langle u_1 \wedge u_2, w_1 \wedge w_2 \rangle = \det \begin{pmatrix} \langle u_1, w_1 \rangle & \langle u_1, w_2 \rangle \\ \langle u_2, w_1 \rangle & \langle u_2, w_2 \rangle \end{pmatrix},
\]

for \( u_1, u_2, w_1, w_2 \in \mathbb{R}^4 \). Then, the second equation of (6) follows from \( \langle v_1 \wedge v_2, v_1 \wedge v_2 \rangle = 1 \).

Straightforward computations imply that if the coordinates \( \alpha_{ij} \) of \( w \in \Lambda^2 \mathbb{R}^4 \) satisfy system (6), then \( w = v_1 \wedge v_2 \) for \( v_1, v_2 \in \mathbb{R}^4 \), such that \( \langle v_i, w_j \rangle = \delta_{ij} \) for \( i, j = 1, 2 \).

The following bivectors

\[
(8) \quad \begin{cases} 
  x_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), \\
  y_1 = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4), \\
  x_2 = \frac{1}{\sqrt{2}}(e_2 \wedge e_3 + e_1 \wedge e_4), \\
  y_2 = \frac{1}{\sqrt{2}}(e_2 \wedge e_3 - e_1 \wedge e_4), \\
  x_3 = \frac{1}{\sqrt{2}}(e_3 \wedge e_1 + e_2 \wedge e_4), \\
  y_3 = \frac{1}{\sqrt{2}}(e_3 \wedge e_1 - e_2 \wedge e_4)
\end{cases}
\]

form an orthonormal basis of \( \Lambda^2 \mathbb{R}^4 \). System (6) implies that coordinates of \( v_1 \wedge v_2 = \sum_{i=1}^3 \beta_i x_i + \gamma_i y_i \) satisfy the following equations

\[
(9) \quad \sum_{i=1}^3 \beta_i^2 = \frac{1}{2}, \quad \sum_{i=1}^3 \gamma_i^2 = \frac{1}{2}.
\]

So, \( Gr^+(2, 4) \) is the product of two 2-dimensional spheres \( S_1 \times S_2 \) of radius \( \frac{1}{\sqrt{2}} \) in \( \mathbb{R}^3 \).

### 3. The singular sets of Gauss map components.

Let \( (e_1, e_2, e_3, e_4) \) be a local Darboux frame on \( M \). Then Cartan’s first structural equations, \( d\theta = \omega \wedge \theta \) imply for \( j = 3, 4 \)

\[
0 = d\theta_j = \omega_j^1 \wedge \theta_1 + \omega_j^2 \wedge \theta_2,
\]

which means that we have

\[
(10) \quad \omega_j^1(e_2) = \omega_j^2(e_1) \quad \text{for} \quad j = 3, 4.
\]

By Cartan’s second structural equations \( d\omega = \omega \wedge \omega \), we have

\[
\begin{aligned}
d\omega_1^2 &= \omega_1^3 \wedge \omega_2^3 + \omega_1^4 \wedge \omega_2^4 = -\omega_1^3 \wedge \omega_2^3 - \omega_1^4 \wedge \omega_2^4, \\
d\omega_2^3 &= \omega_3^3 \wedge \omega_2^4 + \omega_3^4 \wedge \omega_2^4 = -\omega_3^3 \wedge \omega_2^3 - \omega_3^4 \wedge \omega_2^3.
\end{aligned}
\]

Therefore,

\[
(11) \quad d\omega_1^2 + d\omega_2^3 = -\omega_1^3 \wedge \omega_2^3 + \omega_1^4 \wedge \omega_2^4, \quad d\omega_2^3 - d\omega_1^2 = (\omega_1^4 - \omega_2^4) \wedge (\omega_1^3 + \omega_2^3).
\]
The kernels of Pfaffian systems

Lemma 1. The kernels of Pfaffian systems \( \{\omega_1^3 - \omega_2^3 = 0, \omega_1^4 + \omega_2^3 = 0\} \) and \( \{\omega_1^3 - \omega_2^3 = 0, \omega_3^3 + \omega_4^3 = 0\} \) on an open subset \( U \) of \( M \) do not depend on the choice of a local Darboux frame \( (e_1, e_2, e_3, e_4) \) on \( U \).

Proof. Any local Darboux frame on \( U \) has the following form \( (Ae_1, Ae_2, Ae_3, Ae_4) \), where

\[
A = \begin{pmatrix}
\cos \alpha & -\sin \alpha & 0 & 0 \\
\sin \alpha & \cos \alpha & 0 & 0 \\
0 & 0 & \cos \beta & -\sin \beta \\
0 & 0 & \sin \beta & \cos \beta
\end{pmatrix}.
\]

If \( \omega \) is a connection form for \( (e_1, e_2, e_3, e_4) \) then \( \tilde{\omega} = dAA^{-1} + A\omega A^{-1} \) is a connection form for \( (Ae_1, Ae_2, Ae_3, Ae_4) \).

Therefore, we get

\[
\begin{align*}
\tilde{\omega}_1^3 - \tilde{\omega}_2^3 &= \cos(\alpha + \beta)(\omega_1^3 - \omega_2^3) - \sin(\alpha + \beta)(\omega_1^4 + \omega_2^3), \\
\tilde{\omega}_1^4 + \tilde{\omega}_2^3 &= \sin(\alpha + \beta)(\omega_1^3 - \omega_2^3) + \cos(\alpha + \beta)(\omega_1^4 + \omega_2^3), \\
\tilde{\omega}_3^3 + \tilde{\omega}_4^3 &= \cos(\beta - \alpha)(\omega_1^3 + \omega_2^3) - \sin(\beta - \alpha)(\omega_1^4 - \omega_2^3), \\
\tilde{\omega}_1^4 - \tilde{\omega}_3^3 &= \sin(\beta - \alpha)(\omega_1^3 + \omega_2^3) + \cos(\beta - \alpha)(\omega_1^4 - \omega_2^3),
\end{align*}
\]

which finish the proof. \( \Box \)

Let \( (e_1, e_2, e_3, e_4) \) be a local Darboux frame on \( U \). The Gauss map (in the Plücker coordinates) has the following form

\[
g : M \to Gr^+(2, 4), \quad p \mapsto e_1(p) \wedge e_2(p)
\]

So, we express the differential of the Gauss map as

\[
dg = d(e_1 \wedge e_2) = de_1 \wedge e_2 - de_2 \wedge e_1 = \omega_1^3 e_3 \wedge e_2 + \omega_1^4 e_4 \wedge e_2 + \omega_3^3 e_1 \wedge e_3 + \omega_4^3 e_1 \wedge e_4.
\]

Therefore, we conclude:

Proposition 2. The Gauss map is an immersion at \( q \in M \) if and only if the dimension of the space span\{\omega_1^3|_q, \omega_1^4|_q, \omega_3^3|_q, \omega_4^3|_q\} \) is greater than 1.

Proof. By (16), a tangent vector \( v \in T_qM \) is in the kernel of the Gauss map if and only if \( \omega_1^3|_q(v) = \omega_1^4|_q(v) = \omega_3^3|_q(v) = \omega_4^3|_q(v) = 0 \). If \( v \neq 0 \) the dimension of the space span\{\omega_1^3|_q, \omega_1^4|_q, \omega_3^3|_q, \omega_4^3|_q\} \) is not greater than 1. \( \Box \)
Lemma 1 implies that the dimension of the space span\{ω^1_i|_q, ω^2_i|_q, ω^3_i|_q, ω^4_i|_q\} does not depend on the choice of a local Darboux frame. Thus, the dimension of span\{ω^1_i|_q, ω^2_i|_q, ω^3_i|_q, ω^4_i|_q\} is 0 if and only if \(ω^1_i|_q = 0\) for \(i = 1, 2\) and \(j = 3, 4\). Let us assume that \(ω^1_i|_q \neq 0\). Then \(\dim \text{span}\{ω^1_i|_q, ω^2_i|_q, ω^3_i|_q, ω^4_i|_q\} = 1\) if and only if

\[
ω^1_i|_q ∧ ω^2_i|_q = ω^3_i|_q ∧ ω^4_i|_q = 0.
\]

We use (17) to prove that the Gauss map is an immersion for a generic surface immersed into \(\mathbb{R}^4\) (see Proposition 3).

We describe the Gauss map (15) as

\[
g(p) = e_1(p) ∧ e_2(p) = \frac{1}{\sqrt{2}} x_1(p) + \frac{1}{\sqrt{2}} y_1(p),
\]

where \((x_1, x_2, x_3, y_1, y_2, y_3)\) is an orthonormal basis on \(Λ^2\mathbb{R}^4\) given by \(\sigma\). Equation (9) implies that

\[
\frac{1}{\sqrt{2}} x_1 ∈ S_1 ⊂ \text{span}(x_1, x_2, x_3), \quad \frac{1}{\sqrt{2}} y_1 ∈ S_2 ⊂ \text{span}(y_1, y_2, y_3).
\]

Therefore, the components of the Gauss map \(g = (g_1, g_2) : M → S_1 × S_2\) have the following form

\[
g_1(p) = \frac{1}{\sqrt{2}} x_1(p) = \frac{1}{2} (e_1(p) ∧ e_2(p) + e_3(p) ∧ e_4(p)),
g_2(p) = \frac{1}{\sqrt{2}} y_1(p) = \frac{1}{2} (e_1(p) ∧ e_2(p) − e_3(p) ∧ e_4(p)).
\]

It is straightforward to prove that \(g_1\) and \(g_2\) do not depend on the choice of a local Darboux frame. By deriving them, we get

\[
dg_1 = \frac{1}{\sqrt{2}} (ω^2_i − ω^3_i)x_2 + \frac{1}{\sqrt{2}} (−ω^1_i − ω^2_i)x_3,
dg_2 = \frac{1}{\sqrt{2}} (−ω^1_i − ω^2_i)y_2 + \frac{1}{\sqrt{2}} (ω^1_i − ω^2_i)y_3.
\]

Therefore, we state the following.

**Theorem 1.** The kernel of \(dg_1\) coincides with the kernel of the Pfaffian system \(\{ω^1_i = 0, ω^1_i + ω^2_i = 0\}\). Moreover, the kernel of \(dg_2\) coincides with the kernel of the Pfaffian system \(\{ω^2_i + ω^3_i = 0, ω^1_i − ω^2_i = 0\}\).

Let \(σ\) be the area form on \(S_i, i = 1, 2\) and \(dA\) be the area form on \(M\). Given a smooth map \(f : M → S_i\), we denote by \(J(f)\) the Jacobian of \(f\). Namely, the smooth function defined by the following equation \(f^∗σ = J(f)dA\). Formulas (20) imply the following classical result.

**Theorem 2** \((\text{[4]}, \text{Proposition 4.5 in [11], [20]})\). Let \(J(g_i)\) be the Jacobian of \(g_i\) for \(i = 1, 2\) and \(K, K^N\) the Gaussian and normal curvature, respectively. Then

\[
J(g_1) = \frac{1}{2}(K + K^N), \quad J(g_2) = \frac{1}{2}(K − K^N).
\]

**Proof.** Equation (9) implies that \(x_2(p), x_3(p)\) span the tangent space to \(S_1\) at the point \(g_1(p) = \frac{x_1(p)}{\sqrt{2}}\), and \(y_2(p), y_3(p)\) span the tangent space to \(S_2\) at \(g_2(p) = \frac{y_1(p)}{\sqrt{2}}\). Hence, we apply system (20) to get

\[
g_1σ = \frac{1}{\sqrt{2}}(ω^2_i − ω^3_i) ∧ \frac{1}{\sqrt{2}} (−ω^1_i − ω^2_i) = \frac{1}{2}(ω^3_i − ω^4_i) ∧ (ω^2_i + ω^3_i),
g_2σ = \frac{1}{\sqrt{2}}(−ω^1_i − ω^2_i) ∧ \frac{1}{\sqrt{2}}(ω^1_i − ω^2_i) = \frac{1}{2}(ω^1_i − ω^2_i) ∧ (ω^3_i + ω^4_i).
\]
Thus, system (11) implies the result. \qed

Consequently, we state the following.

**Corollary 1.** The singular set of the component $g_i$, $i = 1, 2$ of the Gauss map has the following form

$$\Sigma_{g_i} = \{ p \in M | (K + (-1)^{i+1}K^N)(p) = 0 \}.$$ 

If the Gauss map $g = (g_1, g_2)$ of a surface $M \subset \mathbb{R}^4$ is not an immersion at $p$, the kernel of $dg_p$ has dimension greater than zero, so $g_1$ and $g_2$ are singular at $p$ since they are the projections of $g$ onto the spheres.

The converse of this assertion is not valid. Corollary 1 implies that the set of points where $g_1$ and $g_2$ are singular simultaneously is the intersection of the set of flat points, where $K = 0$, with the set of semiumbilic points, where $K^N = 0$.

4. Generic properties

Let $\xi(t)$ be a regular local parameterization of a smooth curve $C$ of $M$ such that $\xi(0) = p$. Let $V$ be a smooth unit vector field along $\xi$, i.e., $V(t) \in T_{\xi(t)}M$ and $V(t)p, V(t)p) = 1$. Then $V$ is 1-tangent to $C$ at $p = \xi(0)$ if $\xi'(0)$ and $V(0)$ are linearly dependent and $\frac{d}{dt}|_{t=0}(\xi(t) \wedge V(t)) \neq 0$. A Pfaffian system of codimension 1 along $\xi$ is 1-tangent to $C$ at $p = \xi(0)$ if the unit generator of its kernel is 1-tangent to $C$ at $p = \xi(0)$.

Let $C^\infty(M, N)$ be the set of all $C^\infty$ maps from a compact manifold $M$ to an arbitrary manifold $N$ with the Whitney $C^\infty$ topology i.e., the topology of uniform convergence of each $k$-jet $(k = 0, 1, 2, 3, \cdots)$. Let $I$ be an open subset of $C^\infty(M, N)$. We call a property of maps in $I$ generic if the subset of maps in $I$ with this property is open and dense in $I$.

The map $f \in C^\infty(M, N)$ is stable if there exists an open neighborhood $U$ of $f$ in $C^\infty(M, N)$, such that for any $h$ in $U$, $h$ is conjugated to $f$. That is, there exist diffeomorphisms $\Phi : M \rightarrow M$ and $\Psi : N \rightarrow N$, that satisfy $h = \Psi \circ f \circ \Phi$.

In this research $M$ is a smooth, closed, oriented surface and $N = \mathbb{R}^4$. We mainly study the set $I(M, \mathbb{R}^4)$ of all immersions of $M$ into $\mathbb{R}^4$, which is an open subset of $C^\infty(M, \mathbb{R}^4)$.

Let $\{e_1, e_2, e_3, e_4\}$ be a local Darboux frame on a closed, oriented surface $M$ immersed into $\mathbb{R}^4$ and $\omega_i^j = \langle de_i, e_j \rangle$.

We define the following properties of immersions in $I(M, \mathbb{R}^4)$.

(G1) The 1-form $d(K \pm K^N)$ never vanishes on the set $\{ K \pm K^N = 0 \}$.

(G2) Property (G1) holds, and the kernel of the Pfaffian system $\{ \omega_1^2 \mp \omega_2^3 = 0, \omega_1^3 \pm \omega_2^4 = 0 \}$ is transverse to the curve $\{ K \pm K^N = 0 \}$ except at a finite number of points where it is 1-tangent.

(G3) Properties (G1) and (G2) hold and the bivector $\frac{1}{2}(e_1(p) \wedge e_2(p) \pm e_3(p) \wedge e_4(p))$ at any point $p$ such the kernel of the Pfaffian system $\{ \omega_1^2 \mp \omega_2^3 = 0, \omega_1^3 \pm \omega_2^4 = 0 \}$ is 1-tangent to the curve $\{ K \pm K^N = 0 \}$ does not coincide with $e_1(q) \wedge e_2(q) \pm e_3(q) \wedge e_4(q)$ at any other point $q$ of the curve $\{ K \pm K^N = 0 \}$. There are, at most, finitely many pairs (and no triplets, etc.) of points of the curve $\{ K \pm K^N = 0 \}$ at which values of $\frac{1}{2}(e_1 \wedge e_2 \pm e_3 \wedge e_4)$ coincide and the image of $\{ K \pm K^N = 0 \}$ by the map $\frac{1}{2}(e_1 \wedge e_2 \pm e_3 \wedge e_4)$ are transverse at these pairs.
Property \((G_1)\) implies that the set \(\{ K - (-1)^i K^N = 0 \}\) is a smooth curve embedded in \(M\) for \(i = 1, 2\). Since \(M\) is closed, it is a finite disjoint collection of smoothly embedded circles. By Theorem \([2]\) the set \(\{ K - (-1)^i K^N = 0 \}\) is the set singular points of \(g_i\) for \(i = 1, 2\).

The generic local rank one singular points are characterized by having normal forms given in some pair of local coordinates at \(p\) and \(f(p)\) by

\[
(u, v) \mapsto (u^2, v) \text{ or } (u, v) \mapsto (uv \pm v^3, u).
\]

In the first case, \(p\) is called a fold point; in the second, \(p\) is called a cusp point.

By Theorem \([1]\) the kernel of the Pfaffian system \(\{ \omega_1^2 + (-1)^i \omega_2^2 = 0, \omega_1^4 - (-1)^i \omega_2^3 = 0 \}\) coincides with the kernel of \(dg_i\) for \(i = 1, 2\). The kernel of \(dg_i\) is transverse to the regular curve \(\Sigma_{g_i} = \{ K - (-1)^i K^N = 0 \}\) of singular points of \(g_i\) at \(p\) if and only if \(p\) is a fold point of \(g_i\). The kernel of \(dg_i\) is 1-tangent to \(\Sigma_{g_i}\) at \(p\) if and only if \(p\) is a cusp point of \(g_i\). Thus, property \((G_2)\) implies that the singular points of \(g_i\) are fold points except by a finite number of cusp points.

By observing that \(g_i = \frac{1}{2}(e_1 \wedge e_2 - (-1)^i e_3 \wedge e_4)\) (see \([13]\)), we conclude that property \((G_3)\) means that the set of singular values of \(g_i\) does not intersect itself in the image of the cusp point; there are, at most, finitely many pairs (and no triple, etc.) of images of fold points at which the set of singular values intersects transversally.

By the very definition (see \([21]\)), the map \(f\), from \(M\) into another surface, is good if and only if the Jacobian and the gradient of the Jacobian do not vanish simultaneously. Moreover, \(f\) is excellent if and only if it is good and the singular set consists of fold points or isolated cusp points. Thus, an immersion \(x \in \mathcal{I}(M, \mathbb{R}^4)\) satisfies \((G_1)\) if and only if the components of the Gauss map are good. Moreover, \(x\) satisfies \((G_2)\) if and only if it is excellent. We will prove that properties \((G_1)\), \((G_2)\) and \((G_3)\) are generic in Section \(6\).

These properties are generalizations to our setting of those provided in \([5]\) for immersions of \(M\) into \(\mathbb{R}^3\).

5. LOCAL DESCRIPTION OF THE GAUSS MAP COMPONENTS.

We provide a local description of the Gauss map of a surface \(M\) immersed into \(\mathbb{R}^4\) near a point \(p\). To do so, we translate \(p\) to the origin and parameterize \(M\) as the graph of a function in an open neighborhood \(U \subset \mathbb{R}^2\) of the origin, with parameters \((u, v)\), in the following way

\[
x(u, v) = (u, v, a(u, v), b(u, v)),
\]

where \(a\) and \(b\) are local smooth functions. For simplicity, we introduce the following notation

\[
a_{ij} = \frac{\partial^{i+j}a}{\partial u^i \partial v^j}(u, v) \quad \text{and} \quad b_{ij} = \frac{\partial^{i+j}b}{\partial u^i \partial v^j}(u, v),
\]

where \(i, j \in \mathbb{N} \cup \{0\}\). If we assume that \(a_{ij}\) and \(b_{ij}\) vanish at the origin for \(i + j = 1\), the tangent plane at this point is the \(uv\)-plane, and we say that \(M\) is parameterized locally in Monge form at the origin. The tangent frame on \(M\) induced by the parameterization is

\[
x_u = (1, 0, a_{10}, b_{10}) \quad \text{and} \quad x_v = (0, 1, a_{01}, b_{01}).
\]

Then, we define the following orthonormal tangent frame with positive orientation...
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\[ e_1 = \frac{1}{\sqrt{1 + a_{10}^2 + b_{10}^2}} \begin{pmatrix} 1 \\ 0 \\ a_{10} \\ b_{10} \end{pmatrix}, \]

\[ e_2 = \frac{x_v - (x_v \cdot e_1) e_1}{| (x_v - x_v \cdot e_1) e_1 |} = \frac{1}{h \sqrt{1 + a_{10}^2 + b_{10}^2}} \begin{pmatrix} -(a_{10} a_{01} + b_{10} b_{01}) \\ 1 + a_{10}^2 + b_{10}^2 \\ a_{01} - a_{10} b_{10} b_{01} + a_{01} b_{10}^2 \\ b_{01} - a_{10} a_{01} b_{10} + b_{01} a_{10}^2 \end{pmatrix}, \]

where

\[ h = \sqrt{1 + a_{10}^2 + a_{01}^2 + b_{10}^2 + b_{01}^2 + (a_{01} b_{10} - a_{10} b_{01})^2}. \]

Now, we complete the tangent frame \( \{e_1, e_2\} \) with the normal frame defined as

\[ e_3 = \frac{1}{\sqrt{1 + a_{10}^2 + a_{01}^2}} \begin{pmatrix} -a_{10} \\ -a_{01} \\ 1 \\ 0 \end{pmatrix}, \]

\[ e_4 = \frac{1}{h \sqrt{1 + a_{10}^2 + a_{01}^2}} \begin{pmatrix} a_{01} a_{10} b_{01} - b_{10} (1 + a_{01}^2) \\ a_{01} a_{10} b_{10} - b_{01} (1 + a_{10}^2) \\ -a_{01} b_{01} - a_{10} b_{10} \\ 1 + a_{10}^2 + a_{01}^2 \end{pmatrix}, \]

where \( h \) is given by \( (22) \). We observe that \( e_1 \wedge e_2 \wedge e_3 \wedge e_4 \) is the positive orientation of \( \mathbb{R}^4 \). So, the frame

\[ (e_1, e_2, e_3, e_4) \]

is a Darboux frame on \( M \). Moreover, for the standard positive orthonormal basis \( E_1 = (1, 0, 0, 0), E_2 = (0, 1, 0, 0), E_3 = (0, 0, 1, 0), E_4 = (0, 0, 0, 1) \) of \( \mathbb{R}^4 \), we define the following orthonormal basis of \( \Lambda^2 \mathbb{R}^4 \)

\[ \{ X_1 = \frac{1}{\sqrt{2}} (E_1 \wedge E_2 + E_3 \wedge E_4), Y_1 = \frac{1}{\sqrt{2}} (E_1 \wedge E_2 - E_3 \wedge E_4), \]

\[ X_2 = \frac{1}{\sqrt{2}} (E_2 \wedge E_3 + E_1 \wedge E_4), Y_2 = \frac{1}{\sqrt{2}} (E_2 \wedge E_3 - E_1 \wedge E_4), \]

\[ X_3 = \frac{1}{\sqrt{2}} (E_3 \wedge E_1 + E_2 \wedge E_4), Y_3 = \frac{1}{\sqrt{2}} (E_3 \wedge E_1 - E_2 \wedge E_4). \]

Then, \( g_1 = \frac{x_v}{\sqrt{2}} = \frac{1}{\sqrt{2}} (e_1 \wedge e_2 + e_3 \wedge e_4) \) and \( g_2 = \frac{y_v}{\sqrt{2}} = \frac{1}{\sqrt{2}} (e_1 \wedge e_2 - e_3 \wedge e_4) \) have the following form in the basis \( (24) \)

\[ g_1 = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3, \quad g_2 = \gamma_1 Y_1 + \gamma_2 Y_2 + \gamma_3 Y_3, \]

where

\[ \beta_1 = \frac{1 + a_{10} b_{01} - a_{01} b_{10}}{\sqrt{2} h}, \quad \beta_2 = \frac{b_{01} - a_{10}}{\sqrt{2} h}, \quad \beta_3 = \frac{-a_{01} + b_{10}}{\sqrt{2} h}, \]

\[ \gamma_1 = \frac{1 - a_{10} b_{01} + a_{01} b_{10}}{\sqrt{2} h}, \quad \gamma_2 = \frac{-a_{10} + b_{01}}{\sqrt{2} h}, \quad \gamma_3 = \frac{b_{10} - a_{01}}{\sqrt{2} h}. \]

It is easy to see that these functions satisfy \( \Omega \). By applying the stereographic projection to \( (\beta_1, \beta_2, \beta_3) \in S_1 \),

\[ (\beta_1, \beta_2, \beta_3) \mapsto \left( \frac{\beta_2}{\sqrt{2} + \beta_1}, \frac{\beta_3}{\sqrt{2} + \beta_1} \right) \in \mathbb{R}^2 \]
and analogously to $(\gamma_1, \gamma_2, \gamma_3) \in S_2$, we get that the components of the Gauss map have the following local form

\[(25)\]
\[g_1(u, v) = r(-a_{10} + b_{01}, -a_{01} - b_{10}), \quad g_2(u, v) = p(-a_{10} - b_{01}, b_{10} - a_{01}),\]

where

\[(26)\]
\[r = \frac{1}{1 - a_{01}b_{10} + a_{10}b_{01} + h}, \quad p = \frac{1}{1 + a_{01}b_{10} - a_{10}b_{01} + h},\]

and $h$ is as (22). We notice that for any $(u, v)$

\[1 - a_{01}b_{10} + a_{10}b_{01} + h > 1, \quad 1 + a_{01}b_{10} - a_{10}b_{01} + h > 1,\]

since

\[h > |a_{01}b_{10} - a_{10}b_{01}|.\]

These inequalities imply that the components of the Gauss map in (25) are smooth map-germs $\mathbb{R}^2 \to \mathbb{R}^2$. To simplify notation, we denote the components of these map-germs by $g_1$ and $g_2$.

6. **Generic singularities of the components of the Gauss map**

In [13] Little proved that the Gauss map of an immersion $M$ into $\mathbb{R}^4$ is not an immersion at $p$ if and only if $\Delta = 0$ and $K = 0$ at $p$. We begin by showing that the Gauss map is generically an immersion in the setting. Namely,

**Proposition 3.** If $M$ is a generic, closed, oriented surface immersed into $\mathbb{R}^4$ then the Gauss map $g : M \to Gr^+(2, 4)$ is an immersion.

**Proof.** For simplicity, we assume that an immersed surface $M$ into $\mathbb{R}^4$ is parameterized locally in Monge form at the origin. Namely, it is locally described as a graph of a smooth map-germ

\[(27) \quad F : \mathbb{R}^2 \to \mathbb{R}^2, \quad (u, v) \mapsto (a(u, v), b(u, v)),\]

such that

\[a_{10}(0, 0) = a_{01}(0, 0) = b_{10}(0, 0) = b_{01}(0, 0) = 0.\]

Thus, we use the Gauss map’s local expression (25) of

\[g = (g_1, g_2) : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2\]

to develop the analysis of the space of jets.

Let $J^k(2, 2)$ be the space of $k$-jets of smooth maps $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$. The Gauss map is not an immersion at $(0, 0)$ if its differential

\[(28) \quad dg_{(0,0)} = \begin{pmatrix}
S^1_{10}(0, 0) & S^1_{01}(0, 0) \\
S^2_{10}(0, 0) & S^2_{01}(0, 0) \\
T^1_{10}(0, 0) & T^1_{01}(0, 0) \\
T^2_{10}(0, 0) & T^2_{01}(0, 0)
\end{pmatrix}\]

has a rank 0 or 1, where

\[S^1_{10} = r_{00}(a_{20} - b_{11}) + r_{10}(a_{10} - b_{01}), \quad S^1_{01} = r_{00}(a_{11} - b_{02}) + r_{01}(a_{10} - b_{01}),\]
\[S^2_{10} = p_{00}(a_{20} + b_{11}) + p_{10}(a_{10} + b_{01}), \quad S^2_{01} = p_{00}(a_{11} + b_{02}) + p_{01}(a_{10} + b_{01}),\]
\[T^1_{10} = r_{00}(a_{11} + b_{20}) + r_{10}(a_{01} + b_{10}), \quad T^1_{01} = r_{00}(a_{02} + b_{11}) + r_{01}(a_{01} + b_{10}),\]
\[T^2_{10} = p_{00}(a_{11} - b_{20}) + p_{10}(a_{01} - b_{10}), \quad T^2_{01} = p_{00}(a_{02} - b_{11}) + p_{01}(a_{01} - b_{10}),\]
vanish. For simplicity, we assume that \((S_3)\) vanish at \((0,0)\), intersect this set.

Thus, \(\Gamma\) is a submanifold of codimension 3. So, for a generic \(F\), \(j^3F\) does not intersect this set.

If the rank of \(dg_{(0,0)}\) is 1, then at least one of the rows of matrix \(25\) does not vanish. For simplicity, we assume that \((S_{10}, S_{01}) \neq (0,0)\). Then, the rank of \(dg_{(0,0)}\) is 1 if and only if

\[
W_1 = S_{10}^1 T_{10}^1 - S_{01}^1 T_{01}^1, \quad W_2 = S_{10}^1 S_{01} - S_{01}^1 S_{10}, \quad W_3 = S_{10}^2 T_{01}^2 - S_{01}^2 T_{10}^2
\]

vanish at \((0,0)\). We consider the following subset

\[
\Gamma^1 = \{(S_{10}^1)^2 + (S_{01}^1)^2 \neq 0, W_1 = W_2 = W_3 = 0\}
\]

of \(J^3(2,2)\). A direct calculation implies that

\[
dW_1 \wedge dW_2 \wedge dW_3 \neq 0 \text{ for } (S_{10}^1)^2 + (S_{01}^1)^2 \neq 0 \text{ and } a_{10} = a_{01} = b_{10} = b_{01} = 0.
\]

Thus, \(\Gamma^1\) is a submanifold of codimension 3. So, for a generic \(F\), \(j^3F\) does not intersect this set.

If the rank of \(dg_{(0,0)}\) is 0, then

\[
S_{10}^i = T_{10}^i = S_{01}^i = T_{01}^i = 0, \text{ for } i=1,2 \text{ at } (0,0).
\]

It is easy to check that a subset of \(J^3(2,2)\) described by \(31\) is contained in the following set

\[
C = \{S_{10}^1 = T_{10}^1 = S_{01}^1 = T_{01}^1 = 0\}.
\]

Moreover, \(C\) is a submanifold of codimension 6 in \(J^2(2,2)\), because

\[
dS_{10}^1 \wedge dT_{10}^1 \wedge dS_{01}^1 \wedge dT_{01}^1 \neq 0
\]

for \(a_{10} = a_{01} = b_{10} = b_{01} = 0\). Hence, for a generic \(F\), \(j^2F\) does not intersect this set. Finally, we point out that this local argument extends in the standard way for immersions of closed surfaces into \(\mathbb{R}^4\). \(\square\)

Now we state the main theorem.

**Theorem 3.** Properties \((G_1), (G_2)\) and \((G_3)\) are generic.

**Proof.** We only present a proof for the first component \(g_1\) of the Gauss map since a proof for the second component \(g_2\) is similar. We use the local descriptions \(25\) and \(27\) and employ Wolfram Mathematica \(22\) for the more complicated computations.
Let $J^k(2, 2)$ be the space of $k$-jets of smooth maps $C^\infty(\mathbb{R}^2, \mathbb{R}^2)$. The 4-jet of $F$ at $(u, v)$ has the following form

$$J^4_{(u,v)} F = \left( \sum_{i+j=0}^4 \frac{\partial u^i \partial v^j}{\partial u^i \partial v^j} \right),$$

where $\Delta u = (\bar{u} - u)$ and $\Delta v = (\bar{v} - v)$. By (25), the 3-jet of $g_1$ at $(u, v)$ has the following form

$$J^3_{(u,v)} g_1 = \left( \sum_{i+j=0}^3 \frac{\partial u^i \partial v^j}{\partial u^i \partial v^j} \right),$$

where

$$s_{kl} = -\frac{\partial^{k+i}}{\partial u^i \partial v^j} (r(u,v)(a_{10}(u,v) - b_{01}(u,v)),$$

$$t_{kl} = -\frac{\partial^{k+i}}{\partial u^i \partial v^j} (r(u,v)(a_{01}(u,v) + b_{10}(u,v))).$$

We apply formulas (32)-(33) to define the map

$$\mu : J^4(2, 2) \rightarrow J^3(2, 2), \; J^4_{(u,v)} F \mapsto J^3_{(u,v)} g_1.$$

Then, we will show that $\mu$ is a submersion. In the natural coordinate system

$$x = (u, v, a_{ij}, b_{ij})_{i+j=0, \ldots , 4}, \; y = (u, v, s_{ij}, t_{ij})_{i+j=0, \ldots , 3}$$

on $J^4(2, 2)$ and $J^3(2, 2)$, respectively, the map $\mu$ has the following form

$$\mu(x) = (u, v, S_{00}(x), T_{00}(x), S_{10}(x), T_{10}(x), \ldots , S_{03}(x), T_{03}(x)),$$

where by (25), we have

$$S_{00}(x) = -r_{00}(a_{10} - b_{01}), \; T_{00}(x) = -r_{00}(a_{01} + b_{10}),$$

$r_{00}$ is given by (29), and $S_{ij}(x), T_{ij}(x)$ are obtained by applying to formulas (33) the following rules

$$\frac{\partial^{k+l}}{\partial u^i \partial v^j} a_{ij} = a_{(i+k)(j+l)}, \; \frac{\partial^{k+l}}{\partial u^i \partial v^j} b_{ij} = b_{(i+k)(j+l)}.$$

For example, the functions $S_{10}, T_{10}, S_{01}, T_{01}$ have the following form

$$S_{10}(x) = -r_{00}(a_{20} - b_{11}) - r_{10}(a_{10} - b_{01}), \; S_{01}(x) = -r_{00}(a_{11} - b_{02}) - r_{01}(a_{10} - b_{01}),$$

$$T_{10}(x) = -r_{00}(a_{11} + b_{20}) - r_{10}(a_{01} + b_{10}), \; T_{01}(x) = -r_{00}(a_{02} + b_{11}) - r_{01}(a_{01} + b_{10}),$$

where $r_{10}$ and $r_{01}$ are given by (29). Observe that these formulas coincide with those in (28), after omitting superindex 1. The explicit formulas for $S_{ij}, T_{ij}$ are relatively easy to calculate, but they should be shorter to present all of them. For a smooth map $(w_1, \ldots , w_n) \mapsto (P_1(w_1, \ldots , w_n), \ldots , P_m(w_1, \ldots , w_n))$, we denote by $\frac{\partial (P_1, \ldots , P_m)}{\partial (w_{i_1}, \ldots , w_{i_k})}$ the matrix $(\frac{\partial P_{i_q}}{\partial w_{i_{q+1}}})_{1 \leq i_q \leq k}$. Let us notice that $T_{ij}$ are $S_{ij}$ do not depend on $a_{kl}$ and $b_{kl}$ for $k + l > i + j + 1$. Hence, the square matrix of first order partial derivatives of $\mu$ concerning variables $u, v, a_{ij}, b_{ij}$ for $i \neq 0$ has the following
By direct computations, we get

\[ \mu_i \]

where

\[ i \]

Hence,

\[ \mu_i \]

By direct computations, we get

\[ \det \left( \frac{\partial (S_{i0}, T_{i0})}{\partial (a_{i0}, b_{i0})} \right) = \epsilon_k \left( \frac{1 + a_{i0}^2 + b_{i0}^2}{h_{00}} \right)^{k+1}, \]

where \( \epsilon_k = -1 \) for \( k = 1, 2 \) and \( \epsilon_k = 1 \) for \( k = 0, 3 \). Thus by (35), we obtain that

\[ \det \left( \frac{\partial \mu_i}{\partial (u, v, a_{ij}, b_{ij})} \right)_{i \neq 0} > 0. \]

Hence, \( \mu : J^4(2, 2) \to J^3(2, 2) \) is a submersion. It implies that for any smooth submanifold \( \Sigma \) of \( J^3(2, 2) \), the set \( \mu^{-1}(\Sigma) \) is a smooth submanifold of \( J^4(2, 2) \) and codim \( \mu^{-1}(\Sigma) = \text{codim} \Sigma \). The map \( j^3 g_1 : (u, v) \mapsto j^3_{(u,v)} g_1 \in J^3(2, 2) \) is transverse to \( \Sigma \) if and only if the map \( j^4 F : (u, v) \mapsto j^4_{(u,v)} F \in J^4(2, 2) \) is transverse to \( \mu^{-1}(\Sigma) \) of \( J^4(2, 2) \) since \( j^3 g_1 = \mu \circ j^4 F \) and \( \mu \) is a submersion.

Let \( d g_1 = \left( \begin{array}{c} s_{10}, s_{01} \\ t_{10}, t_{01} \end{array} \right) \) and

\[ Q_{g_1}(\Delta u, \Delta v) = \left( \begin{array}{c} s_{20}\Delta u^2 + 2s_{11}\Delta u \Delta v + s_{02}\Delta v^2 \\ t_{20}\Delta u^2 + 2t_{11}\Delta u \Delta v + t_{02}\Delta v^2 \end{array} \right). \]

In this notation, the 3-jet of \( g_1 \) at \( (u, v) \) has the following form

\[ j^3_{(u,v)} g_1 = \left( \begin{array}{c} s_{00} \\ t_{00} \end{array} \right) + d g_1 \left( \begin{array}{c} \Delta u \\ \Delta v \end{array} \right) + \frac{1}{2} Q_{g_1}(\Delta u, \Delta v) + \frac{1}{6} \left( \begin{array}{c} s_{30}\Delta u^3 + 3s_{21}\Delta u^2 \Delta v + 3s_{12}\Delta u \Delta v^2 + s_{03}\Delta v^3 \\ t_{30}\Delta u^3 + 3t_{21}\Delta u^2 \Delta v + 3t_{12}\Delta u \Delta v^2 + t_{03}\Delta v^3 \end{array} \right). \]

We consider \( \Sigma^i = \{ \text{dim ker } d g_1 = i \} \subset J^4(2, 2) \). Then, \( J^4(2, 2) = \bigcup_{i=0}^2 \Sigma^i \) and \( \Sigma^i \) is a smooth submanifold of codimension \( i^2 \). So, the point \( (u, v) \) is a regular point of \( g_1 \) if and only if \( j^3_{(u,v)} g_1 \) belongs to \( \Sigma^0 = \{ s_{10} t_{01} - s_{01} t_{10} \neq 0 \} \).

The submanifold \( \Sigma^1 \) can be described in the following way

\[ \Sigma^1 = \{ s_{10} t_{01} - s_{01} t_{10} = 0, s_{10}^2 + t_{10}^2 + s_{01}^2 + t_{01}^2 \neq 0 \}. \]

We observe that the immersion \( (u, v) \mapsto (u, v, F(u, v)) \) has property \( (G_1) \) if and only if \( j^3 g_1 \) is transverse to \( \Sigma^1 \). Then the set \( \{(u, v)|j^3_{(u,v)} g_1 \in \Sigma^1 \} \) is a smooth submanifold of codimension 1. The point \( (u, v) \) is a singular point of \( g_1 \) of corank 1 if and only if \( j^3_{(u,v)} g_1 \) belongs to \( \Sigma^1 \). The singular point \( (u, v) \) of corank 1 is a fold point if the kernel of \( d g_1(u, v) \) is transverse to the submanifold \( \{(u, v)|j^3_{(u,v)} g_1 \in \Sigma^1 \} \). It means that the restriction of \( g_1 \) to the smooth curve of singular points is a regular map at \( (u, v) \); or equivalently (see Section 7.1 of [14])

\[ Q_{g_1}(\ker d g_1(u, v)) \not\subset d g_1(u, v)(\mathbb{R}^2). \]
Since $s_{10}^2 + t_{10}^2 + s_{01}^2 + t_{01}^2 \neq 0$, we may assume that $s_{10} \neq 0$. In other cases the proofs are similar. Since $s_{10} \neq 0$ and $s_{10}t_{01} - s_{01}t_{10} = 0$, $\ker dg_1$ is spanned by the vector $(-s_{01}, s_{10})$, and the image of $dg_1$ is spanned by $(s_{10}, t_{10})$. Then, condition (30) has the following form
\[
\det \begin{pmatrix}
20s_{21}^2 - 2s_{11}s_{01}s_{10} + s_{02}s_{10}^2, & s_{10} \\
t_{20}s_{21}^2 - 2t_{11}s_{01}s_{10} + t_{02}s_{10}^2, & t_{10}
\end{pmatrix} \neq 0.
\]
We denote by $\Sigma^{11} \subset \Sigma^1$ the following subset of $J^3(2, 2)$
\[
\Sigma^{11} = \{ \dim \ker dg_1 = 1, \ Q_g, (\ker dg_1) \subset \ker \Delta \}.
\]
Since $s_{10} \neq 0$,
\[
\Sigma^{11} = \left\{ s_{10}t_{01} - s_{01}t_{10} = 0, \ \det \begin{pmatrix}
20s_{21}^2 - 2s_{11}s_{01}s_{10} + s_{02}s_{10}^2, & s_{10} \\
t_{20}s_{21}^2 - 2t_{11}s_{01}s_{10} + t_{02}s_{10}^2, & t_{10}
\end{pmatrix} = 0 \right\}
\]
is a smooth submanifold of codimension 2 of $J^3(2, 2)$. Therefore, the immersion $(u, v) \mapsto (u, v, F(u, v))$ has property $(G_2)$ if and only if $j^3g_1$ is transverse to $\Sigma^1$ and $\Sigma^{11}$.

Let us denote $\Sigma^{10} = \Sigma^1 \setminus \Sigma^{11}$. The point $(u, v)$ is a fold point of $g_1$ if and only if $j^3_{(u, v)}g_1$ belongs to $\Sigma^{10}$ (see Section 7.3 of [14]). The point $(u, v)$ is a cusp point of $g_1$ if and only if $j^3_{(u, v)}g_1$ belongs to $\Sigma^{11}$ and the map $j^3g_1$ is transverse to $\Sigma^1$ at $(u, v)$. By Thom’s Transversality Theorem, the map $g_1$ is generic if $j^3g_1$ is transverse to submanifolds $\Sigma^0$, $\Sigma^{10}$, $\Sigma^{11}$, $\Sigma^2$. Since the codimension of $\Sigma^2$ is 4, then $j^3g_1$ is transverse to $\Sigma^2$ if its image does not intersect $\Sigma^2$. Hence, generically singularities of $g_1$ are fold points and cusp points (see Theorem 7.4 in [14]).

Since $\mu$ is a submersion, $j^3g_1$ is transverse to the submanifold $\Sigma^0$ $(\Sigma^{10}$, $\Sigma^{11}$, $\Sigma^2$ respectively) if and only if $j^4F$ is transverse to the submanifold $\mu^{-1}(\Sigma^0)$ $(\mu^{-1}(\Sigma^{10})$, $\mu^{-1}(\Sigma^{11})$, $\mu^{-1}(\Sigma^2)$ respectively). The set of points of $M$ at which $j^4F$ transversally intersects $\mu^{-1}(\Sigma^1)$ is a 1-dimensional smooth submanifold of the surface $M$. Since $M$ is closed, it is a finite disjoint collection of smoothly embedded circles. Since $\operatorname{codim} \mu^{-1}(\Sigma^{11}) = 2$, the set of points of $M$ at which $j^4F$ intersects transversally $\mu^{-1}(\Sigma^{11})$ is a 0-dimensional smooth submanifold of the surface $M$. So, it consists of a finite set of points (cusps points) on the singular set of $g_1$. All the other singular points of $g_1$ are fold points because $\operatorname{codim} \mu^{-1}(\Sigma^2) = 4$. Thus, by Thom’s Transversality Theorem, we finish the proof of genericity of $(G_1)$ and $(G_2)$.

To prove the genericity of $(G_3)$, we first define the sets
\[
M^{(r)} = \{(p_1, \ldots, p_r) \in M^r | p_i \neq p_j \text{ for } i \neq j\},
\]
\[
\Delta_r = \{(q, \ldots, q) \in N^r | q \in N\},
\]
where $M$ and $N$ are smooth manifolds. Let $\alpha : J^k(M, N) \rightarrow M$ be the canonical projection. Then, we define
\[
\alpha' = \alpha \times \cdots \times \alpha : (J^k(M, N))^r \rightarrow M^r, \ \ (j_{p_1}^k, \ldots, j_{p_r}^k) \mapsto (p_1, \ldots, p_r),
\]
and determine a $r$-fold $k$-jet bundle as $rJ^k(M, N) = (\alpha')^{-1}(M^{(r)})$. Thus, for a map $f \in C^\infty(M, N)$, we define
\[
_{r}J^k f : M^{(r)} \rightarrow rJ^k(M, N), \ (p_1, \ldots, p_r) \mapsto (j_{p_1}^k f, \ldots, j_{p_r}^k f).
\]
The $P = (\mathbb{R}^2)^2 \times \Delta_2 \times (\Sigma^1)^2$ is a submanifold of $2J^3(2, 2)$ of codimension 4, $C = (\mathbb{R}^2)^2 \times \Delta_2 \times (\Sigma^1)^2$ is a submanifold of $2J^3(2, 2)$ of codimension 5, and $T = (\mathbb{R}^2)^3 \times \Delta_3 \times (\Sigma^1)^3$ is a submanifold of $3J^3(2, 2)$ of codimension 7. Thus,
Theorem 4.13 in Chapter II \[10\]), property \((G)\) if the intersection of the following result.

By evaluating the 4-form \((37)\) on \((X, 1, X, 1, X, 1)\) to the tangent map vanishes at every point. Analogously, we study the singularities of \(J\)-holomorphic curves. To do so, we first characterize a \(J\)-holomorphic curve, with respect to an orthogonal complex structure \(J\) on \(R^4\), as a surface with a constant Gauss map component (see Theorem 4.13 in Chapter II \[10\]). Property \((G)\) is generic.

By Theorem 6.3 in Chapter VII \[10\] (see also Proposition 2.4 in \[5\]), we get the following result.

**Corollary 2.** Let \(M\) be an oriented, closed surface. Let \(M \rightarrow R^4\) be an embedding that has property \((G)\). If \(g : M \rightarrow S_1 \times S_2\) is the Gauss map then, the component \(g_i, i = 1, 2\) is stable.

7. **Singularities of \(g_i\) and Contact with Holomorphic Curves.**

Singularities of the Gauss map of a surface in \(R^3\) are studied by contact of the surface with its tangent planes. The Gauss map of an affine plane is constant, so its tangent map vanishes at every point. Analogously, we study the singularities of the Gauss map components of a surface in \(R^4\) by the contact of this surface with \(J\)-holomorphic curves. To do so, we first characterize a \(J\)-holomorphic curve, with respect to an orthogonal complex structure \(J\) on \(R^4\), as a surface with a constant Gauss map component (see Theorem 5.3 (b) in \[11\]).

We recall that an orthogonal complex structure on the euclidean space \(R^4\) is an integrable almost complex structure \(J\) on \(R^4\) such that \(\langle Jv, Jw \rangle = \langle v, w \rangle\) for any \(x \in R^4\) and any \(v, w \in T_xR^4\). An example of an orthogonal complex structure
on \( \mathbb{R}^4 \) is the standard complex structure on \( \mathbb{R}^4 = \mathbb{C}^2 \) denoted by \( \mathcal{J}_0 \). It is worth noticing that every orthogonal complex structure on \( \mathbb{R}^4 \) is constant i.e. \( d\mathcal{J} = 0 \) (see proof of Proposition 6.6 in [23] and Corollary 3.5 in [17]).

**Proposition 4.** A connected surface \( M \) is a \( \mathcal{J} \)-holomorphic curve with respect to an orthogonal complex structure \( \mathcal{J} \) on an euclidean space \( \mathbb{R}^4 \) if and only if \( g_i, \ i = 1 \) or 2 is constant. If \( i=1 \), \( \mathcal{J} \) preserves the orientation of \( \mathbb{R}^4 \); otherwise, it reverses it. Moreover, in any case, the mean curvature \( H \) vector vanishes at any point.

**Proof.** Let us assume that the orthogonal complex structure \( \mathcal{J} \) preserves the orientation of \( \mathbb{R}^4 \). Since \( M \) is a \( \mathcal{J} \)-holomorphic curve, we can choose a Darboux frame \((e_1, e_2, e_3, e_4)\) with \( e_2 = \mathcal{J}e_1 \) and \( e_4 = \mathcal{J}e_3 \). Since \( \mathcal{J} \) is constant, we have

\[
\omega_2^3 = < de_2, e_3 > = < d(\mathcal{J}e_1), e_3 > = < \mathcal{J}de_1, e_3 > = < \mathcal{J}de_1, -\mathcal{J}^2 e_3 > = - < de_1, \mathcal{J}e_3 > = - < de_1, e_4 > = - \omega_1^3,
\]

and

\[
\omega_2^4 = < de_2, e_4 > = < d(\mathcal{J}e_1), \mathcal{J}e_3 > = < \mathcal{J}de_1, \mathcal{J}e_3 > = < de_1, e_3 > = \omega_1^4.
\]

Thus, we obtain that

\[
(39) \quad \omega_2^3 + \omega_1^4 = \omega_2^3 - \omega_1^3 = 0.
\]

It implies that \( dg_1 = 0 \), by (20). Conversely, if \( g_1 \) is constant, we use the local expression of \( g_1 \) in (25) to conclude, after evaluating \((u, v) = (0, 0)\), that the map \((a(u, v), b(u, v))\) satisfies the Cauchy-Riemann conditions. Now, we observe that the mean curvature vector \( H \) is given by

\[
(40) \quad H = \frac{1}{2} \left( (\omega_1^3(e_1) + \omega_2^3(e_2))e_3 + (\omega_1^4(e_1) + \omega_2^4(e_2))e_4 \right).
\]

By (39) and (10), we get that

\[
\omega_2^3(e_1) = \omega_2^3(e_2) = -\omega_2^3(e_2)
\]

and

\[
\omega_1^4(e_1) = -\omega_1^3(e_1) = -\omega_1^3(e_1) = -\omega_1^3(e_2).
\]

Thus, we get that \( \omega_2^3(e_1) + \omega_2^3(e_2) = \omega_1^4(e_1) + \omega_2^4(e_2) = 0 \), which implies that \( H = 0 \) by (40). Suppose the orthogonal complex structure reverses the orientation of \( \mathbb{R}^4 \) and \( M \) is a holomorphic curve. In that case, we can choose a Darboux frame \((e_1, e_2, e_3, e_4)\) such that \( e_2 = \mathcal{J}e_1 \) and \( e_4 = -\mathcal{J}e_3 \) and proceed in the same way described above.

We recall some basic definitions and results on contact equivalence (for details see [1], [2]).

**Definition 1.** The map-germs \( f, \tilde{f} : (\mathbb{R}^s, 0) \rightarrow (\mathbb{R}^t, 0) \) are \( K \)-equivalent if there exists a diffeomorphism-germ \( \phi : (\mathbb{R}^s, 0) \rightarrow (\mathbb{R}^s, 0) \) and a map-germ \( A : (\mathbb{R}^s, 0) \rightarrow GL(\mathbb{R}^t) \) such that \( \tilde{f} = A \cdot (f \circ \phi) \).

Let \( \mathcal{E}_s \) denote the local ring of smooth function-germs at \( 0 \) on \( \mathbb{R}^s \).

**Remark 1** ([1]). For the \( K \)-equivalence of two map-germs it is necessary and sufficient that the two ideals generated by the components of these map-germs may be mapped one to the other by an isomorphism of \( \mathcal{E}_s \) induced by a diffeomorphism-germ of the source space \((\mathbb{R}^s, 0)\).
Let \( M_i, N_i \) be germs at \( x_i \) for \( i = 1, 2 \) of smooth surfaces in the space \( \mathbb{R}^4 \). We describe them in the following way.

\[ M_i = \phi_i^{-1}(0), \quad \text{where} \quad \phi_i : (\mathbb{R}^4, x_i) \to (\mathbb{R}^2, 0) \text{ is a submersion-germ for } i = 1, 2, \text{ and} \]

\[ N_i = \psi_i(\mathbb{R}^2), \quad \text{where} \quad \psi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^4, x_i) \text{ is an embedding-germ for } i = 1, 2. \]

**Definition 2.** The contact of \( M_1 \) and \( N_1 \) at \( x_1 \) is of the same contact-type as the contact of \( M_2 \) and \( N_2 \) at \( x_2 \) if there exists a diffeomorphism-germ \( \Phi : (\mathbb{R}^4, x_1) \to (\mathbb{R}^4, x_2) \) such that \( \Phi(M_1) = M_2 \) and \( \Phi(N_1) = N_2 \). We denote the contact-type of \( M_1 \) and \( N_1 \) at \( x_1 \) by \( \mathcal{K}(M_1, N_1, x_1) \).

The contact-type can be studied by a contact-map.

**Definition 3.** A contact map between surface-germs \( M_1, N_1 \) is the map-germ \( \phi_1 \circ \psi_1 : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \).

**Theorem 4** ([16]). \( \mathcal{K}(M_1, N_1, x_1) = \mathcal{K}(M_2, N_2, x_2) \) if and only if the contact maps \( \phi_1 \circ \psi_1 \) and \( \phi_2 \circ \psi_2 \) are \( \mathcal{K} \)-equivalent.

We study the contact of the surface with holomorphic curves with respect to an orthogonal complex structure \( \mathcal{J} \) on \( \mathbb{R}^4 \). We present the results for \( g_1 \). The results for \( g_2 \) are similar. Thus we assume that \( \mathcal{J} \) preserves the orientation of \( \mathbb{R}^4 \) and the surface \( M \) is parameterized locally in Monge form at 0 in \( \mathbb{R}^4 \). It implies that the tangent space to \( M \) at 0 is spanned by \((1,0,0,0)\) and \((0,1,0,0)\) and the normal space at 0 is spanned by \((0,0,1,0)\) and \((0,0,0,1)\). We consider holomorphic curves which are tangent to \( M \) at 0. Thus we have that \( \mathcal{J}(1,0,0,0) = (0,1,0,0) \) and \( \mathcal{J}(0,1,0,0) = (0,0,1,0) \). Namely, \( \mathcal{J}|_{\mathbb{R}^4|_{\mathbb{R}^2}} = J_0|_{\mathbb{R}^2|} \). Since \( \mathcal{J} \) is an orthogonal complex structure on \( \mathbb{R}^4 \), \( \mathcal{J} \) is constant. Thus we have \( \mathcal{J} = J_0 \). Hence we may apply the standard identification of \( \mathbb{R}^4 \) with \( \mathbb{C}^2 \) given by

\[ \mathbb{R}^4 \ni (x_1, x_2, x_3, x_4) \mapsto (z, w) = (x_1 + ix_2, x_3 + ix_4) \in \mathbb{C}^2. \]

We may assume that a holomorphic curve in \( \mathbb{C}^2 \) is the graph of a holomorphic function \( w = f(z) \), since \( M \) is parameterized locally in Monge form at the origin. Thus, we analyze the contact map \( k \) of \( M \) in this parameterization with a holomorphic curve at the origin, and we determine the coefficients of degree less than \( s + 1 \) of such holomorphic curve that has the highest order of contact in \( \mathcal{J}^s(\mathbb{R}^2, \mathbb{R}^2) \) with \( M \) for \( s = 2, 3, 4 \). Thus,

\[ k(x_1, x_2) = \left( \frac{a(x_1, x_2) - \text{Re} f(x_1 + ix_2)}{b(x_1, x_2) - \text{Im} f(x_1 + ix_2)} \right). \]

The 4-jet of this map at 0 has the following form

\[ j_0^4 k(x_1, x_2) = \left( \frac{\sum_{i+j=2} \frac{a_{ij} x_1^i x_2^j}{b_{ij} x_1^i x_2^j} - \text{Re} j_0^4 f(x_1 + ix_2)}{\sum_{i+j=2} \frac{b_{ij} x_1^i x_2^j}{b_{ij} x_1^i x_2^j} - \text{Im} j_0^4 f(x_1 + ix_2)} \right), \]

where

\[ \text{Re} j_0^4 f(x_1 + ix_2) = \alpha_1 x_1^2 + 2\alpha_2 x_1 x_2 - \alpha_2 x_2^2 - \beta_1 x_1^3 - 3\beta_2 x_1 x_2^2 - 3\beta_3 x_1 x_2^3 + \beta_4 x_2^3 + \gamma_1 x_1^4 + 4\gamma_2 x_1^3 x_2 + 4\gamma_3 x_1^2 x_2^2 + \gamma_4 x_2^4, \]

\[ \text{Im} j_0^4 f(x_1 + ix_2) = \alpha_1 x_1^2 + 2\alpha_2 x_1 x_2 - \alpha_2 x_2^2 + \beta_1 x_1^3 + 3\beta_2 x_1 x_2^2 - 3\beta_3 x_1 x_2^3 - \beta_4 x_2^3 + \gamma_1 x_1^4 + 4\gamma_2 x_1^3 x_2 - 4\gamma_3 x_1^2 x_2^2 + \gamma_4 x_2^4. \]
In these expressions, we use the following notation
\[ a_{ij} = \frac{\partial^{i+j} a}{\partial u^i \partial v^j}(0,0) \quad \text{and} \quad b_{ij} = \frac{\partial^{i+j} b}{\partial u^i \partial v^j}(0,0). \]

Let \( \mathcal{E}_{2,2} \) denote a \( \mathcal{E}_2 \)-module of smooth map-germs \( (\mathbb{R}^2,0) \to \mathbb{R}^2 \). Let \( \text{Jac}(k) \) be the Jacobian module of \( k \), i.e., \( \text{Jac}(k) = \mathcal{E}_2 \left( \frac{\partial k}{\partial x_1}, \frac{\partial k}{\partial x_2} \right) \) and \( I_k \) be the ideal \( (k_1, k_2) \) in \( \mathcal{E}_2 \) generated by components of \( k \). Then \( \text{Jac}(k) \) and \( I_k \cdot \mathcal{E}_{2,2} \) are submodules of \( \mathcal{E}_{2,2} \). The \( K \)-tangent space to \( k \) is the \( \mathcal{E}_2 \)-module \( T_k = \text{Jac}(k) + I_k \cdot \mathcal{E}_{2,2} \). Since \( k(0) = \frac{\partial k}{\partial x_2}(0) = \frac{\partial k}{\partial x_1}(0) = 0 \), the space \( j_0^k(T_k) \) is a subspace of the vector space \( J_0^*(T_k) \) of \( s \)-jets of map-germs \( (\mathbb{R}^2,0) \to (\mathbb{R}^2,0) \). It is easy to see that the dimension of \( J_{0,0}^s(2,2) \) equals to \( s^2 + 3s \). We define the \( s \)-jet \( K \)-codimension of \( k \) as a codimension of \( j_0^k(T_k) \) in \( J_{0,0}^s(2,2) \). A holomorphic curve has the highest possible contact with a surface \( M \) in \( s \)-jets at 0 if the \( s \)-jet \( K \)-codimension of its contact map-germ \( k \) is maximal among all holomorphic curves passing through the point 0.

We determine the coefficients \( \alpha_1, \alpha_2 \) to get that the quadratic form
\[ (43) \quad \left( \begin{array}{c} \frac{1}{2}(a_{20} - 2a_1) \\ \frac{1}{2}(b_{02} - 2b_3) \\ \end{array} \right), \]
has rank 1. We assume that the mean curvature vector \( H \) of \( M \) at the origin does not vanish. In this parameterization \( |H|^2 \) has the following form
\[ (44) \quad |H|^2 = (a_{20} + a_{02})^2 + (b_{20} + b_{02})^2. \]

We determine coefficients \( \alpha_1, \alpha_2 \) such that the rank of the matrix \( \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \) is not greater than 1. By direct computation we obtain that
\[ \alpha_1 = \frac{b_{11}(a_{20} + a_{02})^2 + (a_{20}(b_{02} - a_{11}) - a_{02}(a_{11} + b_{20}))(b_{20} + b_{02})}{2((a_{20} + a_{02})^2 + (b_{20} + b_{02})^2)} \]
\[ \alpha_2 = \frac{-a_{11}(b_{20} + b_{02})^2 + (b_{20}(a_{02} + b_{11}) + b_{02}(b_{11} - a_{20}))(a_{20} + a_{02})}{2((a_{20} + a_{02})^2 + (b_{20} + b_{02})^2)} \]
and the second jet of the contact map at the origin has the following form
\[ (45) \quad j_0^k(x_1, x_2) = \frac{Ax_1^2 + 2Bx_1x_2 + Cx_2^2}{2((a_{20} + a_{02})^2 + (b_{20} + b_{02})^2)} \left( \begin{array}{c} a_{20} + a_{02} \\ b_{20} + b_{02} \end{array} \right), \]
where
\[ A = (a_{20} + a_{02})(a_{20} - b_{11}) + (a_{11} + b_{20})(b_{20} + b_{02}), \]
\[ B = a_{20}(a_{11} - b_{02}) + b_{11}(b_{20} + b_{02}) + a_{02}(a_{11} + b_{20}), \]
\[ C = (a_{20} + a_{02})(a_{02} + b_{11}) - (a_{11} - b_{02})(b_{20} + b_{02}). \]

**Proposition 5.** The discriminant \( \delta \) of the quadratic form in \( |H|^2 \) of the contact map \( k \) at the origin has the following form
\[ (46) \quad \delta = -(K + K^N)|H|^2. \]

**Proof.** We describe the curvatures and the discriminant when the surface is parameterized locally in Monge form at the origin as follows:
\[ K = -a_{11}^2 + a_{20}b_{02} - b_{11}^2 + b_{20}b_{02}, \quad K^N = b_{11}(a_{20} - a_{02}) - a_{11}(b_{20} - b_{02}), \]
\[ \delta = -((a_{20} - b_{11})(a_{02} + b_{11}) - (a_{11} - b_{02})(a_{11} + b_{20}))((a_{20} + a_{02})^2 + (b_{20} + b_{02})^2). \]
Thus, using the expression of \( |H|^2 \) given by \( (44) \) a direct substitution completes the proof. \( \square \)
Definition 4. We say that a point \( p \in M \) is \( g_1 \)-elliptic if \((K + K^N)(p) > 0\). We say that a point \( p \in M \) is \( g_1 \)-hyperbolic if \((K + K^N)(p) < 0\).

We provide a description of the first jets of the contact map \( k \) using the normal forms under the action of the contact group of local diffeomorphisms.

Theorem 5 (Classification of regular and generic singular points of \( g_1 \)). Let \( k \) be a contact map of \( M \) and a holomorphic curve with the highest contact in \( s \)-jets at \( p \) for \( s = 2 \) in (1)-(3) and (6), \( s = 3 \) in (4) and \( s = 4 \) in (5).

1. A point \( p \) is \( g_1 \)-elliptic if and only if the contact map-germ \( k \) at \( p \) is \( K \)-equivalent to a map-germ at 0 with the 2-jet of the form
   \( (x_1^2 + x_2^2, 0) \).

2. A point \( p \) is \( g_1 \)-hyperbolic if and only if the contact map-germ \( k \) at \( p \) is \( K \)-equivalent to a map-germ at 0 with the 2-jet of the form
   \( (x_1x_2, 0) \).

3. A point \( p \) is a corank-one singular point of \( g_1 \) if and only if the contact map-germ \( k \) at \( p \) is \( K \)-equivalent to a map-germ at 0 with the 2-jet of the form
   \( (x_1^2, 0) \).

4. A point \( p \) is a fold point of \( g_1 \) if and only if the contact map-germ \( k \) at \( p \) is \( K \)-equivalent to a map-germ at 0 with the 3-jet of the form
   \( (x_1^2 + x_2^3, 0) \).

5. A point \( p \) is a cusp point of \( g_1 \) if and only if the contact map-germ \( k \) at \( p \) is \( K \)-equivalent to a map-germ at 0 with the 4-jet of the form
   \( (x_1^2 \pm x_2^2, 0) \).

6. A point \( p \) is a corank-two singular point of \( g_1 \) if and only if the contact map-germ \( k \) at \( p \) is \( K \)-equivalent to a map-germ at 0 with the 2-jet of the form
   \( (0, 0) \).

Proof. We use the following notation \( x = (x_1, x_2) \) and \( k(x) = (k_1(x), k_2(x)) \). Let us assume that \( p = 0 \). First, we consider the case \(|H(0)|^2 \neq 0\). Then, without loss of generality, we suppose that
\[
(47) \quad a_{20} + a_{02} \neq 0.
\]

We apply the transformation
\[
k(x) \mapsto (k_1(x), k_2(x) - \frac{b_{20} + b_{02}}{2((a_{20} + a_{02})^2 + (b_{20} + b_{02})^2)k_1(x)})
\]
to get \( j_0^2k(x) = (j_0^2k_1(x), 0) \). Then, Proposition 5 and classical results on quadratic forms imply (1) and (2). In both cases the 2-jet \( K \)-codimension equals to 4.

Now, we assume that 0 is a corank-one singular point, i.e., the corank of the matrix
\[
(48) \quad dg_{10} = \begin{pmatrix}
  a_{20} - b_{11} & a_{11} - b_{02} \\
  a_{11} + b_{20} & a_{02} + b_{11}
\end{pmatrix}
\]
is 1. It implies that
\[
(49) \quad \det dg_{10} = (K + K^N)(0) = (a_{20} - b_{11})(a_{02} + b_{11}) - (a_{11} - b_{02})(a_{11} + b_{20}) = 0.
\]
By (48), we may assume without loss of generality that
\( a_{20} - b_{11} \neq 0. \)

Then, we solve the equation (49) with respect to \( a_{02} \). Replacing \( a_{02} \) with this solution in \( k \), we have
\[
j_0^2 k(x) = \left( 1 + \left( \frac{a_{11} - b_{02}}{a_{20} - b_{11}} \right)^2 \right) \left( \left( a_{20} - b_{11} \right)x_1 + \left( a_{11} - b_{02} \right)x_2 \right)^2, 0 \right).
\]

Thus, we change coordinates in the following way
\[(x_1, x_2) \mapsto \left( x_1 - \left( a_{11} - b_{20} \right)x_2, \frac{a_{20} - b_{11}}{a_{20} - b_{11}} \right), \]

to get that the 2-jet of \( k \) has the following form
\[
j_0^2 k(x) = \left( \frac{a_{20} - b_{11}}{a_{20} - b_{11}} \right)^2 \left( \frac{a_{20} - b_{11}}{a_{20} - b_{11}} \right)^2, 0 \right).
\]

Hence, we apply the transformation
\[k(x) \mapsto \left( \frac{(a_{20} - b_{11})^2}{(a_{20} - b_{11})^2 + (a_{11} + b_{20})^2} \right), \]

to obtain that \( j_0^2 k(x) = (x_1^2, 0) \). Its 2-jet \( K \)-codimension equals to 6. We point out that this codimension is maximal by the construction of the normal form. Thus, we finish the proof of (3).

The 3-jet of \( k \) at 0 has the following form
\[
j_0^3 k(x) = \left( x_1^2 + \sum_{i=0}^{3} k_{3-i}^0 x_1^3 - i, x_2^2 \sum_{i=0}^{3} k_{3-i}^1 x_1^3 - i x_2^2 \right),
\]

where coefficients \( k_{3-i}^l \) depend linearly on \( \beta_1, \beta_2 \) for \( l = 1, 2 \) and \( i = 0, 1, 2, 3 \), see (12). Using the term \( x_1^3 \) in \( k_1(x_1, x_2) \), we can reduce \( k_{20}^0 x_1^3 + k_{21}^0 x_1^3 x_2 \) but we are not able to do the same with \( k_{22}^2 x_1^2 x_2^2 \). Therefore, first, we find \( \beta_1 \) and \( \beta_2 \) such that
\[
k_{12}^2 = k_{03}^2 = 0.
\]
The determinant of \( 2 \times 2 \) linear system (51) has the following form
\[
\frac{(3)(a_{11} - b_{02})^2 + (a_{20} - b_{11})^2)^3((a_{20} - b_{11})^2 + (a_{11} + b_{20})^2)}{(a_{20} - b_{11})^2(a_{20} + a_{20})^2}.
\]

By (47) and (50), this determinant is well defined and does not vanish. Then, we apply Wolfram Mathematica (22) to find solutions \( \beta_1 \) and \( \beta_2 \). The formulas are too long to present them. Although, we compute, assuming that \( \beta_1 \) and \( \beta_2 \) satisfy (51), to get that
\[
k_{03}^1 = \frac{(a_{11} - b_{02})^2 + (a_{20} - b_{11})^2}{3(a_{20} - b_{11})(a_{20} - b_{11})^2 + (a_{11} + b_{20})^2} d(K + K^N)_0(X(0)),
\]

where \( X \) is the germ of a smooth vector field which spans \( \text{ker} \, d g_1 \) on \( \{ K + K^N = 0 \} \).

By (50), we conclude that 0 is a fold point of \( g_1 \) iff \( k_{03}^1 \) does not vanish.

Now, we apply the transformation
\[k(x) \mapsto (k_1(x), k_2(x) - (k_{30}^2 x_1 + k_{31}^2 x_2)k_1(x)) \]
to obtain $j^3_0k(x) = (x_1^2 + k_{30}^1x_1^3 + k_{21}^1x_1^2x_2 + k_{12}^1x_1x_2^2 + k_{13}^1x_2^3, 0)$. So, we assume that $k_{03}^1 \neq 0$ and we change coordinates in the following way

$$(x_1, x_2) \mapsto \left( x_1, \frac{1}{\sqrt{k_{03}}} x_2 - \frac{k_{12}^1}{3k_{03}^1} x_1 \right)$$

to express $j^3_0k$ in the following form $j^3_0k(x) = (x_1^2 + k_{30}^1x_1^3 + k_{21}^1x_1^2x_2 + x_2^3, 0)$. Then, we apply the transformation

$$k(x) \mapsto \left( \frac{k_1(x)}{1 + k_{30}^1x_1 + k_{21}^1x_2}, k_2(x) \right)$$

to get that $j^3_0k(x_1, x_2) = (x_1^2 + x_2^3, 0)$. Thus, the 3-jet $K$-codimension of $(x_1^2 + x_2^3, 0)$ equals to 8. We point out that this codimension is maximal by the construction of the normal form. Thus, we finish the proof of $\mathbf{41}$.

Now, we assume that $k_{03}^1 = 0$. By $\mathbf{42}$, this assumption is equivalent to

$$d(K + K^N)_0(X(0)) = 0,$$

which has the following form

$$(a_{20} - b_{11})^2(a_{03} + b_{12}) - (a_{20} - b_{11})^2(a_{12} - b_{03})(a_{11} + b_{20}) + 2(a_{20} - b_{11})(a_{11} - b_{02})(a_{21} - b_{12})(a_{11} + b_{20}) - 2(a_{20} - b_{11})^2(a_{11} - b_{02})(a_{12} + b_{21}) + (a_{20} - b_{11})(a_{11} - b_{02})^2(a_{21} + b_{30}) - (a_{11} - b_{02})^2(a_{12} + b_{20})(a_{30} - b_{21}) = 0.$$  

Then, we solve the equation $\mathbf{54}$ with respect to $a_{03}$. Replacing $a_{03}$ with this solution in $k$, we have that $j^3_0k(x) = (x_1^2 + k_{30}^1x_1^3 + k_{21}^1x_1^2x_2 + k_{12}^1x_1x_2^2, 0)$. So, we change coordinates in the following way $(x_1, x_2) \mapsto (x_1 - (1/2)k_{12}^1x_2^2, x_2)$ to get that $j^3_0k(x) = (x_1^2 + k_{30}^1x_1^3 + k_{21}^1x_1^2x_2, 0)$ in the new coordinates. Moreover, by applying transformation $\mathbf{53}$, we reduce $k$ to such form that

$$j^4_0k(x) = \left( x_1^2 + \sum_{i=0}^4 k_{4-i,i}^1 x_1^{4-i} x_2^i, \sum_{i=0}^4 k_{4-i,i}^2 x_1^{4-i} x_2^i \right),$$

where coefficients $k_{4-i,i}^l$ depend linearly on $\beta_1, \beta_2$ for $l = 1, 2$ and $i = 0, \ldots, 4$, see $\mathbf{42}$. Now, we proceed in the same way as before. By using the term $x_1^3$ in $k_1(x_1, x_2)$, we can reduce $k_{22}^l x_1^3 x_2 + k_{22}^l x_1 x_2^2$ but we are not able to do the same with $k_{13}^1 x_1 x_2^3 + k_{13}^2 x_2^4$. Therefore, first, we determine $\gamma_1$ and $\gamma_2$ such that

$$k_{13}^1 = k_{13}^2 = 0.$$

The determinant of $2 \times 2$ linear system $\mathbf{55}$ has the following form

$$4((a_{11} - b_{02})^2 + (a_{20} - b_{11})^2)^4((a_{20} - b_{11})^2 + (a_{11} + b_{20})^2).$$

By $\mathbf{47}$ and $\mathbf{50}$, this determinant is well defined and does not vanish. Then, we apply Wolfram Mathematica $\mathbf{22}$ to find solutions $\gamma_1$ and $\gamma_2$. The formulas are too long to present them. Although, we compute, assuming that $\gamma_1$ and $\gamma_2$ satisfy $\mathbf{55}$, to get that

$$k_{04}^1 = \frac{(a_{11} - b_{02})^2 + (a_{20} - b_{11})^2}{12(a_{20} - b_{11})^2((a_{20} - b_{11})^2 + (a_{11} + b_{20})^2)} d(K + K^N)(X)_0(X(0)).$$

By $\mathbf{54}$, we obtain that $0$ is a cusp point of $g_1$ iff $k_{04}^1$ does not vanish for $\gamma_1, \gamma_2$ satisfying $\mathbf{55}$. We assume that $k_{04}^1 \neq 0$. Let us notice that the sign of $k_{04}^1$ depends
only on the sign of \( d(d(K + K^N)(X))_0(X(0)) \) since 
\[
\frac{(a_{11} - b_{02})^2 + (a_{20} - b_{11})^2}{(a_{20} - b_{11})^2 + (a_{20} - b_{11})^2}
\] 
is positive. We apply the transformation 
\[
k(x) \rightarrow (k_1(x), k_2(x) - (k_{10}^2 + k_{31}^2 x_1 x_2 + k_{22}^2 x_2^2)k_1(x)),
\]
to obtain that 
\[
\tilde{j}_0^k(x) = (x_1^2 + k_{10}^2 + k_{11}^2 x_1 x_2 + k_{22}^2 x_2^2 + k_{13}^2 x_1 x_2^3 + k_{04}^2 x_2^4, 0).
\]
Then, we change coordinates in the following way 
\[
(x_1, x_2) \mapsto \left(x_1, \frac{1}{\sqrt{k_{04}}} x_2 - \frac{k_{13}}{4k_{04}} x_1\right)
\]
to express the 4-jet of \( k \) at 0 in the following form 
\[
\tilde{j}_0^k(x_1, x_2) = (x_1^2 + k_{10}^2 + k_{11}^2 x_1 x_2 + k_{22}^2 x_2^2 + \text{sgn}(k_{04}) x_2^4, 0),
\]
in new coordinates. Thus, we apply the transformation
\[
(57) \quad k(x) \mapsto \left(1 + k_{10}^2 x_1^2 + k_{31}^2 x_1 x_2 + k_{22}^2 x_2^2, k_2(x)\right)
\]
to get that 
\[
\tilde{j}_0^k(x_1, x_2) = (x_1^2 \pm x_2^2, 0).
\]
Its 4-jet \( K \)-codimension equals to 10. Thus, we finish the proof of (5).

Now, we assume that 0 is a corank-two singular point. Then, the following equations hold
\[
(58) \quad a_{20} = b_{11} = -a_{02}, \quad b_{02} = a_{11} = -b_{20}.
\]
These conditions imply that \( |H(0)| = 0 \), and so, the quadratic form of \( g_1 \) has following form
\[
(59) \quad \begin{pmatrix}
\frac{1}{2}(a_{20} - 2a_1) & -(b_{20} - 2a_2) & -\frac{1}{2}(a_{20} - 2a_1) \\
\frac{1}{2}(b_{20} - 2a_2) & a_{20} - 2a_1 & -\frac{1}{2}(b_{20} - 2a_2)
\end{pmatrix}.
\]
Hence, we take \( \alpha_1 = \frac{1}{2}b_{20} \) and \( \alpha_2 = \frac{1}{2}b_{20} \) to obtain (10). Its 2-jet \( K \)-codimension equals to 10.

Finally, we consider the case where the mean curvature vanishes at \( p \). This condition implies that the 2-jet of the map \( F \) satisfies:
\[
(60) \quad a_{20} + a_{02} = 0, \quad b_{20} + b_{02} = 0.
\]
Thus we get that the quadratic part of \( k \) has the following form (see (13))
\[
(61) \quad \begin{pmatrix}
\frac{1}{2}(a_{20} - 2a_1) & a_{11} + 2a_2 & -\frac{1}{2}(a_{20} - 2a_1) \\
\frac{1}{2}(b_{20} - 2a_2) & b_{11} - 2a_1 & -\frac{1}{2}(b_{20} - 2a_2)
\end{pmatrix}.
\]
The discriminants of the the first and the second components are non-negative
\[
(a_{11} + 2a_2)^2 + (a_{20} - 2a_1)^2 \geq 0, \quad (b_{11} - 2a_1)^2 + (b_{20} - 2a_2)^2 \geq 0.
\]
We assume that at least one of them is positive, since they both vanish if and only if \( a_{11} + b_{20} = a_{20} - b_{11} = 0 \), which together with (60) means that 0 is a corank-two singular point. This case has already been considered. Let us choose \( \alpha_1 = \frac{1}{2}b_{11} \) and \( \alpha_2 = \frac{1}{2}b_{20} \). Then (61) has the following form
\[
(62) \quad \begin{pmatrix}
\frac{1}{2}(a_{20} - b_{11}) & a_{11} + b_{20} & -\frac{1}{2}(a_{20} - b_{11}) \\
0 & 0 & 0
\end{pmatrix}.
\]
Since \( (a_{11} + b_{20})^2 + (a_{20} - b_{11})^2 > 0 \), we get that the point is \( g_1 \)-hyperbolic and \( \tilde{j}_0^k \) is \( K \)-equivalent to \( \# \).
8. The Gauss-Bonnet Type Formulas

The cusp point \( p \) of \( g_i \) is called positive (negative, respectively) if \( dg_{i,q} : T_qM \to T_{g(q)}S_i \) preserves (reverses, respectively) the orientation for every \( q \) in a small neighborhood \( U \) of \( p \) such that \( g_i|_U \) is injective at \( q \). We parameterize the singular set \( \Sigma_{gi} = \{ p \in M | (K - (-1)^iK^N)(p) = 0 \} \) by \( \xi_i \) such that \( g_i(M) \) lies on the left hand side \( g_i \circ \xi_i \) and let \( \tau \) be the arc length parameter of \( g_i \circ \xi_i \) on \( S_i \). Then the geodesic curvature \( \kappa_\xi \) of \( g_i \circ \xi_i \) on \( S_i \) is well defined in fold points since the kernel of \( dg_i \) is transverse to \( \Sigma_{gi} \), at these points. Let \( dA \) be the area form on \( M \) induced by the immersion. For \( N \subset M \), \( \chi(N) \) denotes the Euler characteristic of \( N \).

**Theorem 6** (Gauss-Bonnet formulas). Let \( M \) be an oriented, closed surface. If \( M \to \mathbb{R}^4 \) is an immersion that has property \((G_2)\), then

\[
2 \pi \chi(M) = \int_M |K \pm K^N|dA + 2 \int_{(K \pm K^N = 0)} \kappa_g(\tau)d\tau, \tag{63}
\]

\[
\frac{1}{\pi} \int_M KdA = \sum_{i=1}^2 \chi(M_i^+) - \chi(M_i^-) + S_{gi}^+ - S_{gi}^-, \tag{64}
\]

\[
\frac{1}{\pi} \int_M K^NdA = \sum_{i=1}^2 (-1)^i \chi(M_i^+) - \chi(M_i^-) + S_{gi}^+ - S_{gi}^- \tag{65},
\]

where \( M_i^+ = \{ p \in M | K(p) > (-1)^i K^N(p) \} \), \( M_i^- = \{ p \in M | K(p) < (-1)^i K^N(p) \} \), and \( S_{gi}^+, (S_{gi}^-) \), respectively) is the number of positive (negative, respectively) cusp points of \( g_i \) for \( i = 1, 2 \).

**Proof.** By Theorem 5 the singular sets of the components \( g_i : M \to S_i \) for \( i = 1, 2 \) consist of fold points and cusp points. Therefore, a straightforward application of Quine’s Theorem (see [17]) to \( g_i \) for \( i = 1, 2 \) implies

\[
2 \deg(g_i) = \chi(M_i^+) - \chi(M_i^-) + S_{gi}^+ - S_{gi}^-,
\]

since \( \chi(S_i) = 2 \). Moreover, Theorem 4.6 in [11] (see also Proposition 5 in [20]) states

\[
\int_M KdA = 2 \pi(\deg(g_1) + \deg(g_2)), \quad \int_M K^NdA = 2 \pi(\deg(g_1) - \deg(g_2)).
\]

Thus, we obtain equations (64)-(65).

Since the singular sets of \( g_1 \) and \( g_2 \) consist of fold points and cusp points, we may apply Proposition 3.1 and Proposition 3.7 in [19] to conclude that \( \int_{\Sigma_{gi}} \kappa_g(\tau)d\tau \) is bounded. So, since the Gaussian curvature of \( S_i \) equals to 2 at any point, the following equation holds

\[
2 \pi \chi(M) = \int_M 2|g_i^* \sigma| + 2 \int_{\Sigma_{gi}} \kappa_g(\tau)d\tau, \tag{66}
\]

where \( \sigma \) is the area form on \( S_i \). But by Proposition 4.5 in [11] \( g_i^* \sigma = \frac{1}{2}(K - (-1)^iK^N)dA \), and considering that \( \Sigma_{gi} = \{ K - (-1)^iK^N = 0 \} \), we conclude that (66) implies (63). \( \square \)

If the map \( M \to \mathbb{R}^4 \) is an embedding, then \( \deg(g_1) = \deg(g_2) = \frac{1}{4\pi} \int_M KdA = \frac{1}{2} \chi(M) \) (6, [11], [20]). Thus, as a consequence of Theorem 6 we state.
Corollary 3. Let $M$ be an oriented, closed surface. Let $M \to \mathbb{R}^4$ be an embedding that has property $(G_2)$. If $g : M \to S_1 \times S_2$ is the Gauss map of this embedding, then for $i = 1, 2$

\begin{equation}
\chi(M) = \chi(M_1^i) + \frac{1}{2}(S_{g_i}^+ - S_{g_i}^-), \quad \chi(M_2^-) = \frac{1}{2}(S_{g_2}^+ - S_{g_2}^-),
\end{equation}

where $g_i, i = 1, 2$ are the components of $g$.

Proof. Since the map $M \to \mathbb{R}^4$ is an embedding, we have that

\[ \frac{1}{\pi} \int_M K^N dA = 0 \text{ and } \frac{1}{2\pi} \int_M KdA = \chi(M) \]

(see [20], [11]). Hence by (65), we have that

\[ \chi(M_1^+) - \chi(M_1^-) + S_{g_1}^+ - S_{g_1}^- = \chi(M_2^+) - \chi(M_2^-) + S_{g_2}^+ - S_{g_2}^-.
\]

Thus (65) implies that for $i = 1, 2$

\begin{equation}
\chi(M) = \chi(M_1^i) - \chi(M_1^-) + S_{g_i}^+ - S_{g_i}^-
\end{equation}

On the other hand, we have

\begin{equation}
\chi(M) = \chi(M_1^+) + \chi(M_1^-) + \chi(\Sigma_{g_i}).
\end{equation}

But $\Sigma_{g_i}$ is a disjoint union of simple, regular, closed, smooth curves. This property implies that $\chi(\Sigma_{g_i}) = 0$. Thus from (65)-(68), we get

\[ \chi(M_1^-) = \frac{1}{2}(S_{g_i}^+ - S_{g_i}^-) \quad \text{and} \quad \chi(M) = \chi(M_1^+) + \frac{1}{2}(S_{g_i}^+ - S_{g_i}^-),
\]

which finishes the proof. \hfill \Box

We finish by describing the singular set of a closed surface of genus two embedded in the standard 3-dimensional sphere of unit radius $S^3 \subset \mathbb{R}^4$.

Example 1.

Let $T_2$ be the surface given as the transversal intersection of the hypersurfaces $F^{-1}(0)$ and $G_r^{-1}(0)$, where

\[ F : \mathbb{R}^4 \to \mathbb{R}, \quad F(u, v, x, y) = x^2 - y^2 + u^3 - 3uv^2, \]

\[ G_r : \mathbb{R}^4 \to \mathbb{R}, \quad G_r(u, v, x, y) = x^2 + y^2 + u^2 + v^2 - r^2. \]

The projection

\[ \pi : \mathbb{R}^4 \to \mathbb{R}^2, \quad (u, v, x, y) \mapsto (u, v) \]

maps $T_2$ onto a hexagon $H_{uv}$ defined by the following inequalities on the $uv$-plane, see Figure 1

\[ h(u, v) \geq 0, \quad k(u, v) \geq 0, \]

where,

\[ h(u, v) = \frac{r^2 - u^2 - v^2 + u^3 - 3uv^2}{2}, \quad k(u, v) = \frac{r^2 - u^2 - v^2 + u^3 - 3uv^2}{2}.
\]

Thus, we consider four parameterizations

\[ x_{\pm \pm}(u, v) = \left( u, v, \pm \sqrt{h(u, v)}, \pm \sqrt{k(u, v)} \right), \]

with the same domain. Namely, the interior of $H_{uv} \subset \mathbb{R}^2$. 

Figure 1. \( h = 0 \) the solid curve, \( k = 0 \) the dashed curve, the cups points \( c_1, \ldots, c_6 \) on the singular set \( \hat{\Sigma}_{g_i} = \gamma_1 \cup \gamma_2 \).

We denote the image of \( H_{uv} \) under \( x_{\pm, \pm} \) by \( H_{k, \pm} \). This model of the double torus is obtained as the union of four identical pieces; each one is the closure of a hexagonal region. Namely,

\[
T_2 = H_{++} \cup H_{+-} \cup H_{-+} \cup H_{--},
\]

see Figure 1. A description of the symmetries and the parameterization of this model of the double torus is presented in [15]. Moreover, a direct computation implies that the invariant \( \Delta \), see (1), is negative everywhere except at the four points parameterized by the origin in the four coordinate charts, where the Gaussian curvature is not zero. So, Theorem 1.2 in [13] implies that the Gauss map is a global immersion of \( T_2 \) into the Grassmannian.

Now, we analyze the singularities of the components of the Gauss map. Since the surface is spheric, the normal curvature vanishes. Thus, the singular set of both components \( g_1 \) and \( g_2 \) are coincident; it is the zero locus of the Gaussian curvature. We parameterize its projection on \( H_{uv} \) by using polar coordinates as the intersection of the zero locus of the curve

\[
\hat{\Sigma}_{g_i} = 9r^{10} - 248r^8 + 232r^6 - 464r^4 + 352r^2 - 32 + (9r^4 - 86r^2 + 16)r^2 \cos(6\varphi),
\]

where \( i = 1, 2 \) with the region \( H_{uv} \), see Figure 1.

By applying the symmetries of the surface, specifically, the reflections on the lines through the origin with angles

\[
\phi = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6},
\]

is easy to get the following description.

**Proposition 6.** The curve \( \hat{\Sigma}_{g_i} \) consists of two connected components \( \gamma_i \), \( i = 1, 2 \), where \( \gamma_i \) is diffeomorphic to \( S^1 \). Both have index 1 for any point in the interior of the bounded region of its complement.

We observe that the exterior component \( \gamma_2 \) intersects \( H_{uv} \) in a slight curve near each vertex. So, each intersection is glued, at the boundary of the hexagon, with the other three parameterized in different coordinate charts. In this way, we get six closed curves on \( T_2 \). If we add four components parameterized by \( \gamma_1 \), we conclude
that the singular set of the components of the Gauss map consists of ten closed curves diffeomorphic to a circle. By direct computations we verify that $K < 0$ in the interior of the discs bounded by these closed curves, while $K > 0$ in the complement of the closure of the union of them. Therefore, the following equations hold.

$$\chi(M^+_1) = \chi(M^+_2) = -12, \text{ and } \chi(M^-_1) = \chi(M^-_2) = 10.$$  

Moreover, by the very definition of a cusp point we determine six cusp points as the intersection of the singular set with the rays from the origin with angles $\phi = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{9\pi}{6}$, see Figure 1. We verify that all of them are negative, by computing of $J_\gamma$ near each cusp point. So, there are twenty four negative cusp points on the whole surface. Therefore, equations (67) are satisfied in this example.

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