BOUNDED MEROMORPHIC FUNCTIONS
ON THE COMPLEX 2-DISC

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Abstract. We describe bounded, holomorphic functions on the complex 2-disc, that admit meromorphic extension to a larger 2-disc. This solves a conjecture of Bickel, Knese, Pascoe and Sola. The key technical ingredient is an old theorem of Zariski about integrally closed ideals in 2-dimensional regular rings.

The aim of this note is to study bounded, holomorphic functions \( \psi \) on the 2-disc \( D^2 := \{ (x, y) : |x|, |y| < 1 \} \subset \mathbb{C}^2 \), that admit a meromorphic extension \( \Psi \) to a larger 2-disc \( \{ (x, y) : |x|, |y| < 1 + \epsilon \} \).

It is easy to see that the polar set of \( \Psi \) intersects the closed 2-disc \( \overline{D^2} \) in finitely many points only, all of which satisfy \( |x| = |y| = 1 \). In Theorem 3 we give a complete local description of \( \Psi \) at these special points, answering [BKPS21, Conj.1.3].

We refer to [BPS21, BKPS21] for a history of this question and to many related results. The higher dimensional cases are of considerable interest, but appear more complicated, see Remark 15.

Up to conformal equivalence, this problem is the same as studying functions \( \Phi(x, y) \) that are meromorphic in a neighborhood of the origin \((0, 0) \in \mathbb{C}^2\) and bounded on \( \mathbb{H}^2 := \{ (x, y) : \Im(x), \Im(y) > 0 \} \subset \mathbb{C}^2 \). From now on we write \( \mathbb{H}^2_{\text{loc}} \) to denote the intersection of \( \mathbb{H}^2 \) with a suitable neighborhood of the origin \((0, 0) \in \mathbb{C}^2\), and similarly for \( \mathbb{R}^2_{\text{loc}}, \mathbb{C}^2_{\text{loc}} \). We can write \( \Phi = f/g \) where \( f, g \in \mathbb{C}\{x, y\} \) are analytic on \( \mathbb{C}^2_{\text{loc}} \) and have no common factors.

The first necessary condition is that \( g(x, y) \) vanishes on \( \overline{\mathbb{H}^2_{\text{loc}}} \) (the closure of \( \mathbb{H}^2_{\text{loc}} \)) only at the origin. A complete description of such analytic functions is given in [BKPS21], in terms of their Puiseux factors as follows.

By Newton, we can write any \( g \in \mathbb{C}\{x, y\} \) (that is not divisible by \( x \)) uniquely as a product of Puiseux series

\[
g = u(x, y) \prod_i (y + \phi_i(x^{1/r_i})),
\]

where \( u(0, 0) \neq 0 \), the \( \phi_i \) are holomorphic, \( \phi_i(0) = 0 \) and \( r_i \geq 1 \).

**Theorem 1.** [BKPS21] p.4] The holomorphic function \( g(x, y) \in \mathbb{C}\{x, y\} \) vanishes on \( \overline{\mathbb{H}^2_{\text{loc}}} \) only at the origin if and only if each of its Puiseux factors is of the form

\[
y + q_i(x) + x^{m_i} \psi_i(x^{1/r_i}),
\]

where \( q_i \in \mathbb{R}\{x\} \), \( q_i(0) = 0 \), \( q_i'(0) > 0 \), the \( m_i \) are even and \( \Im(\psi_i(0)) > 0 \).

This gives a complete description of the possible denominators; see Proposition 16 on how to check this condition in concrete situations.

In order to understand the numerators, [BKPS21] p.7] defines ideals as follows.
Theorem 3. Let \((6.2)\) are clear.

\[ f, g \in H \]

show that \((6.4)\) is equivalent to \((3.2)\).

Using the notation of Theorems 1 and 3, the following are equivalent.

\[ f/g \in \mathcal{I}(g) := \prod_i (y + q_i(x), x^{m_i}) \subset \mathbb{C}\{x, y\}. \]

The following theorem gives a positive answer to [BKPS21 Conj.1.3].

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Then define \(\mathcal{I}(g)\) as their product

\[ \mathcal{I}(g) := \prod_i (y + q_i(x), x^{m_i}) \subset \mathbb{C}\{x, y\}. \]

Remark 4. The answer of Theorem 3 seems asymmetrical in \(x, y\), but this is an accident of our choices.

If \(G(x, y) = ax + by + \cdots \) with \(ab \neq 0\) then, by the Weierstrass preparation theorem, for every \(m > 1\) there are unique polynomials \(q_1(x), q_2(y)\) of degree < \(m\) such that \(q_i(0) = 0\), and

\[ (G, (x, y)^m) = (y + q_1(x), x^m) = (x + q_2(y), y^m). \]

Example 5. [BKPS21 2.18] The simplest non-trivial examples come from

\[ (y + x + i x^2)^2 - x^{4m+1} = (y + x + i x^2 + x^{2m+1/2})(y + x + i x^2 - x^{2m+1/2}). \]

The corresponding ideal is \(y + x, x^{2m})^2 = ((y + x)^2, (y + x)x^{2m}, x^{4m}). \) Thus we get that meromorphic functions with denominator \((y + x + i x^2)^2 - x^{4m+1}\), that are bounded on \(H^2\) are of the form

\[ \frac{v_1(x, y)(y + x)^2 + v_2(x, y)(y + x)x^{2m} + v_3(x, y)x^{4m}}{(y + x + i x^2)^2 - x^{4m+1}}, \]

where the \(v_i(x, y)\) are holomorphic.

In the first step of the proof, we transform the boundedness of \(f/g\) on \(H^2\) into an inequality over \(C^2\).

Theorem 6. Using the notation of Theorems 1 and 3, the following are equivalent.

1. \( f/g \) is bounded on \(H^2\).
2. \( |f| \leq C \cdot \prod_i (|y + q_i(x)| + |x^{m_i}|) \) for \((x, y) \in H^2\).
3. \( |f| \leq C \cdot \prod_i (|y + q_i(x)| + |x^{m_i}|) \) for \((x, y) \in R^2\).
4. \( |f| \leq C \cdot \prod_i (|y + q_i(x)| + |x^{m_i}|) \) for \((x, y) \in C^2\).

Then we use the theory of integral closure of ideals and a theorem of [Zar38] to show that (3.4) is equivalent to (3.2).

7. (Proof of Theorem 3 beginning). The implications (3.2) \(\Rightarrow\) (3.3) and (3.4) \(\Rightarrow\) (3.2) are clear.
are mutually bounded by constant multiples of each other. Thus the conditions

\[ \sum_{i}(|y + q_i(x)| + |x^{m_i}|) \leq C \cdot |y + \phi_i(x)| \quad \text{for } (x, y) \in \mathbb{H}^2_{\text{loc}}. \]

Since the reverse bound is clear, we get that \( |g(x, y)| \) and \( \prod_i(|y + q_i(x)| + |x^{m_i}|) \) are mutually bounded by constant multiples of each other. Thus the conditions

\[ |f(x, y)| \leq C \cdot |g(x, y)| \quad \text{and} \quad |f(x, y)| \leq C \cdot \prod_i(|y + q_i(x)| + |x^{m_i}|) \]

are equivalent on \( \mathbb{H}^2_{\text{loc}}. \)

The advantage of the form (6.4) is that it ties in with the notion of integral dependence of holomorphic functions.

**Definition 8** (Integral dependence). Let \( X \) be a complex space, \( x \in X \) a point and \( \mathcal{O}_{X,x} \) the ring of germs of holomorphic functions at \( x \). Let \( J = (g_1, \ldots, g_m) \subset \mathcal{O}_{X,x} \) be an ideal and \( h \in \mathcal{O}_{X,x} \). Then \( h \) is integral over \( J \) if \( |h| \leq C \cdot \sum_i |g_i| \) for some \( C > 0 \) in some neighborhood of \( x \in X \). This notion is independent of the generators chosen.

\( J \) is called integrally closed if it contains every holomorphic function that is integral over \( J \).

Most of the literature on integral dependence is algebraic, with a very different definition. Among the standard books, the equivalence is stated in [Laz04, Rem.9.6.10] and [SH06, Thm.7.1.7], with further references for proofs.

We need the valuative criterion of integral dependence. However, we need not only its statement as in [Laz04,SH06], but a more precise version treated in [LJT08].

**9 (Valuative criterion of integral dependence).** [LJT08] Let \( J = (g_1, \ldots, g_r) \subset \mathcal{O}_{X,x} \) be an ideal sheaf and \( f \) a holomorphic function. How do we check that \( f \) is integral over \( J \)?

First, if \( X = \mathbb{C}^n \) and \( J = (\prod_i x_i^{m_i}) \) is generated by a monomial, then clearly \( f \) is integral over \( J \) iff it vanishes along \( (x_i = 0) \) with multiplicity \( \geq m_i \) for every \( i \). This can be checked along \( n \) arcs as follows.

For each \( i \) let \( \gamma_i : \mathbb{D} \to \mathbb{C}^n \) be an analytic arc whose image intersects the hyperplane \( (x_i = 0) \) at a single point \( \gamma_i(0) \), and is disjoint from the \( (x_j = 0) \) for \( j \neq i \). Then \( f \) is integral over \( J = (g : = \prod_i x_i^{m_i}) \) iff the \( (f/g) \circ \gamma_i \) are all bounded near \( 0 \in \mathbb{D} \). Note also that since \( f/g \) is meromorphic, it is enough to check boundedness on a set that accumulates to \( 0 \in \mathbb{D} \).

Most ideals are not monomial, but Hironaka’s resolution theorem says that for any reduced complex space \( X \) and ideal sheaf \( J \subset \mathcal{O}_X \), there is a proper, birational morphism \( \pi : Y \to X \) such that \( Y \) is smooth and the preimage \( J_Y := \pi^* J \subset \mathcal{O}_Y \) is locally monomial. This is called a **monomialization** of \( J \).

Combining with our previous discussions, we get the valuative criterion:

**Theorem 3.1.** \( f \) is integral over \( J = (g_1, \ldots, g_r) \subset \mathcal{O}_X \) iff, for every analytic arc \( \gamma : \mathbb{D} \to X \), there is a \( C_\gamma > 0 \) such that

\[ |f \circ \gamma(t)| \leq C_\gamma \cdot \sum_i |g_i \circ \gamma(t)|. \quad (\text{3.1.1}) \]

Note, however, that our previous discussions give a more precise result. Let \( E_j \subset Y \) be the \( \pi \)-exceptional divisors. Choose arcs \( \gamma_j : \mathbb{D} \to Y \) such that \( \gamma_j(0) \) intersects \( E_j \) at a single point \( \gamma_j(0) \) and is disjoint from the other \( E_k \) for \( k \neq j \).
Refined Theorem 4.2. $f$ is integral over $J = (g_1, \ldots, g_r) \subset \mathcal{O}_{X,x}$ iff, for the above analytic arcs $\pi \circ \gamma_j : \mathbb{D} \to X$, there are $C_j > 0$ and sets $U_j \subset \mathbb{D}$ accumulating to $0 \in \mathbb{D}$, such that
\[ |f \circ \gamma_j(t)| \leq C_j \cdot \sum_i |g_i \circ \gamma_j(t)| \quad \text{for } t \in U_j. \tag{9.2.1} \]

Note that the ideals $\mathcal{I}(g)$ in Definition 2 are generated by real power series. This turns out to be quite important, so we need some general facts about real power series and real structures on complex manifolds.

10 (Reality questions). Let $h \in \mathbb{R}\{x,y\}$ be an irreducible power series. It either stays irreducible in $\mathbb{C}\{x,y\}$, or it decomposes as a product of two irreducible factors. A typical example is $y^2 + x^2$.

Let $C_h := (h = 0) \subset \mathbb{C}^2_{\text{loc}}$ be the corresponding curve. Then $h$ is irreducible in $\mathbb{C}\{x,y\}$ iff the normalization $\bar{C}_h \to C_h$ has a unique point $c \in \bar{C}_h$ lying over the origin. Complex conjugation of $\mathbb{C}^2$ lifts to an antiholomorphic involution in $\bar{C}_h$ and $c$ is a fixed point.

If $p \in \mathbb{C}^2$ is a real point, then complex conjugation lifts to the blow-up $B_p \mathbb{C}^2$, giving a real structure. The exceptional curve of $B_p \mathbb{C}^2 \to \mathbb{C}^2$ is then isomorphic to $\mathbb{CP}^1$ with its standard real structure $(x:y) \mapsto (\bar{x}:\bar{y})$. The same applies after any number of blow-ups of real points $\pi : Y \to \mathbb{C}^2$.

Proposition 11. Let $J = (g_1, \ldots, g_m) \subset \mathbb{R}\{x,y\}$ be a real ideal and $f \in \mathbb{C}\{x,y\}$ an analytic function. Assume that $J$ has a monomialization $\pi : Y \to \mathbb{C}^2$ where we blow up only real points. Then the following are equivalent.
(11.1) $|f(x,y)| \leq C \cdot \sum_i |g_i(x,y)|$ for all $(x,y) \in \mathbb{R}^2_{\text{loc}}$ for some $C > 0$.
(11.2) $|f(x,y)| \leq C \cdot \sum_i |g_i(x,y)|$ for all $(x,y) \in \mathbb{C}^2_{\text{loc}}$ for some $C > 0$.

Proof. The implication (11.2) $\Rightarrow$ (11.1) is clear.

To see the converse, we follow the approach explained in (9.2). By assumption $Y$ has a real structure compatible with the standard one on $\mathbb{C}^2$, and all the $\pi$-exceptional curves $E_j \subset Y$ are real isomorphic to $\mathbb{CP}^1$. Thus, for every $E_j$ we can choose a real analytic arc $\gamma_j : \mathbb{D} \to Y$ that intersects $E_j$ transversally but is disjoint from the other exceptional curves. Thus $\pi \circ \gamma_j$ maps $\mathbb{D} \cap \mathbb{R}$ to $\mathbb{R}^2$.

If (11.1) holds then (9.2.1) is satisfied with $U_j := \mathbb{D} \cap \mathbb{R}$. Thus $f$ is integral over $J$ by Theorem 9.2, which is (11.2).

12 (Monomialization for ideals in $\mathbb{C}\{x,y\}$). While the general monomialization method of Hironaka is quite complicated, there is a simple algorithm for ideals $J = (g_1, \ldots, g_m) \subset \mathbb{C}\{x,y\}$.

Start with the curve $C = (g_1 \cdots g_m = 0)$ and $Y_0 := \mathbb{C}^2$. If $\pi_i : Y_i \to \mathbb{C}^2$ is already constructed, then we get $Y_{i+1} \to Y_i$ by blowing up all the intersection points of the $\pi_i$-exceptional curve and of the birational transform $C_i \subset Y_i$ of $C$; that is, the closure of $\pi_i^{-1}(C \setminus \{(0,0)\})$. After finitely many steps, the preimage of the ideal $J$ becomes monomial. (This algorithm depends on the choice of the $g_i$, but works well for our purposes. See [Kol07, Chap.1] for a discussion of various other methods.)

Assume now that the $g_i$ are real and write $\prod_i g_i = \prod_j p_j$ as a product of irreducible factors in $\mathbb{R}\{x,y\}$. Then we repeatedly blow up the intersection points of the exceptional curve and the union of the birational transforms of the curves $B_j := (p_j = 0)$. If the $p_j$ are also irreducible in $\mathbb{C}\{x,y\}$, then, as we noted in
Paragraph 10 for each $B_j$ the intersection consists of a single real point. Then at each step we blow up only real points. We have thus proved the following.

Claim 12.1. Let $J = (g_1, \ldots, g_m) \subset \mathbb{R}\{x, y\}$ be an ideal. Assume that every $\mathbb{R}\{x, y\}$-irreducible factor of $\prod_i g_i$ is irreducible in $\mathbb{C}\{x, y\}$. Then $J$ has a monomialization $\pi : Y \to \mathbb{C}^2$ where we blow up only real points. 

13 (Proof of Theorem 6 end). We prove the last implication $(\text{6.3}) \Rightarrow (\text{6.4})$.

Our ideal $\mathcal{I}(g)$ is the product of ideals of the form $(y + q_i(x), x^m)$. Thus it has a generating set $g_j$ where each $g_j$ is a product of a power of $x$ with some $\prod_i (y + q_i(x))$. Thus the irreducible factors of $\prod_j g_j$ are $x$ and the $y + q_i(x)$. These are irreducible both in $\mathbb{R}\{x, y\}$ and $\mathbb{C}\{x, y\}$.

Thus $\mathcal{I}(g)$ has a monomialization $\pi : Y \to \mathbb{C}^2$ where we blow up only real points by Claim 12.1. Therefore Proposition 11 applies to $\mathcal{I}(g)$, and shows that $(\text{6.3})$ implies $(\text{6.4})$.

14 (Proof of Theorem 8). If we expand $\prod_i (|y + q_i(x)| + |x^m|)$, we get the sum of the absolute values of the generators of the ideal $\mathcal{I}(g)$. Thus Theorem 6.4 says that $f$ is integral over $\mathcal{I}(g)$. If $\mathcal{I}(g) = \prod_i (y + q_i(x), x^m)$ is integrally closed, then $f \in \mathcal{I}(g)$, which is what we want.

Note that $(y + q_i(x), x^m)$ is, after a coordinate change, the same as $(y, x^m)$, which is clearly integrally closed. By a theorem of Zariski, in a two-dimensional regular local ring, products of integrally closed ideals are integrally closed. This is proved in [Zar38]: a more accessible reference is [SH03] Thm.14.4.4.

Since $\mathbb{C}\{x, y\}$ is two-dimensional and regular, we obtain that $\prod_i (y + q_i(x), x^m)$ is integrally closed. Therefore $f \in \prod_i (y + q_i(x), x^m) = \mathcal{I}(g)$.

Remark 15. Zariski’s theorem holds only in dimension 2. The following example, based on [BPS21] and e-mails of A. Sola and I. Swanson, shows that higher dimensional versions of Theorem 8 are more complicated.

The ideals $(z + x + y, x^2 + y^2)$ and $(z + x + y, x^2 + 2y^2)$ are integrally closed, but their product

$$( (z + x + y)^2, (z + x + y)x^2, (z + x + y)y^2, (x^2 + y^2)(x^2 + 2y^2))$$

is not since $(z + x + y)xy$ is integral over it. This gives that

$$\frac{(z + x + y)xy}{(z + x + y + ix^2 + iy^2)(z + x + y + ix^2 + 2iy^2)}$$

is bounded on $\mathbb{H}^3_{\text{loc}}$, but it is not a linear combination of products of bounded functions with denominators $(z + x + y + ix^2 + iy^2)$ and $(z + x + y + ix^2 + 2iy^2)$.

The conclusion of Theorem 11 can be checked without writing down the Puiseux series, working with 1 irreducible factor $p(x, y)$ of $g$ at a time. Newton’s method of rotating rulers (see, for example, [New60] pp.126-127] or [JK16] Thm.18) for a detailed treatment) constructs the Puiseux series solution of $p(x, y) = 0$ term by term as

$$y = a_1 x + a_2 x^2 + \cdots + a_m x^m + \cdots .$$

The formulas become more complicated once a fractional power of $x$ appears, but the method is very transparent while we have only integer powers. We obtain the following.
Proposition 16. Let \( p(x, y) \in \mathbb{C}\{x, y\} \). Assume that \( p(0, 0) = 0 \) and the lowest \( y \) power in \( p \) is \( y^r \) (with coefficient 1). Let \( h(x) \in \mathbb{C}[x] \) be a polynomial of degree \( m \) such that \( h(0) = 0 \). Then the following hold.

(16.1) If \( p \) is irreducible and one Puiseux factor of \( p \) has the form
\[
y + h(x) + (\text{higher terms}), \tag{16.1}
\]
then so is every Puiseux factor.

(16.2) Every Puiseux factor of \( p \) has the form (16.1) iff we can write
\[
p(x, y) = (y + h(x))^r + \sum_{i+j/m>r}(y+h(x))^ix^j.
\]

(16.3) The implicit function theorem solves
\[
\frac{1}{r!} \frac{\partial^{r-1}p}{\partial y^{r-1}} = 0 \quad \text{as} \quad y = -h(x) - (\text{higher terms}).
\]

(16.4) If \( p \) is a Weierstrass polynomial in \( y \) (of degree \( r \)) then
\[
\frac{1}{r!} \frac{\partial^{r-1}p}{\partial y^{r-1}} = y + h(x) + (\text{higher terms}).
\]

Note, however, that in (16.1.1) the higher terms form a Puiseux series (thus involve fractional powers of \( x \)), but in (16.3–4) the higher terms form a power series.

When we apply this to Theorem 1, we use \( h(x) = q(x) + \psi(0)x^m \).

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References

[BKPS21] Kelly Bickel, Greg Knese, James Eldred Pascoe, and Alan Sola, Local theory of stable polynomials and bounded rational functions of several variables, arXiv:2109.07507, 2021.

[BPS21] Kelly Bickel, James Eldred Pascoe, and Alan Sola, Singularities of rational inner functions in higher dimensions, Amer. J. Math. (2021) (to appear)

[JK16] Jennifer M. Johnson and János Kollár, Arcology, Amer. Math. Monthly 123 (2016), no. 6, 519–541.

[Kol07] János Kollár, Lectures on resolution of singularities, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007.

[Laz04] Robert Lazarsfeld, Positivity in algebraic geometry. I-II, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge., vol. 48–49, Springer-Verlag, Berlin, 2004. MR 2095471 (2005k:14001a)

[LJT08] Monique Lejeune-Jalabert and Bernard Teissier, Clôture intégrale des idéaux et équisingularité, Ann. Fac. Sci. Toulouse Math. (6) 17 (2008), no. 4, 781–859, With an appendix by Jean-Jacques Risler. MR 2499856

[New60] Isaac Newton, The correspondence of Isaac Newton, Vol. II: 1676–1687, Cambridge Univ. Press, New York, 1960.

[SH06] Irena Swanson and Craig Huneke, Integral closure of ideals, rings, and modules, London Mathematical Society Lecture Nots, vol. 336, Cambridge Univ. Pres, 2006.

[Zar38] Oscar Zariski, Polynomial ideals defined by infinitely near base points, Amer. J. Math. 60 (1938), 151–204.

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