STABILITY AND CASCADES FOR THE KOLMOGOROV-ZAKHAROV SPECTRUM OF WAVE TURBULENCE

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Abstract. We consider the kinetic wave equation arising in wave turbulence to describe the Fourier spectrum of solutions to the cubic Schrödinger equation. This equation has two Kolmogorov-Zakharov steady states corresponding to out-of-equilibrium cascades transferring, for the first solution mass from \( \infty \) to 0 (small spatial scales to large scales), and for the second solution energy from 0 to \( \infty \). After conjecturing the generic development of the two cascades, we verify it partially in the isotropic case by proving the nonlinear stability of the mass cascade in the stationary setting. This constructs non-trivial out-of-equilibrium steady states with a direct energy cascade as well as an indirect mass cascade.

1. The Kinetic wave equation and turbulence

1.1. The equation and its explicit solutions. The kinetic wave equation for 4-wave interactions is in the isotropic case

\[ \partial_t f = C(f), \]

where the collision operator is given by

\[ C(f)(\omega_1) = \int \int \int_{\omega_2,\omega_3,\omega_4 \geq 0} W[(f_1 + f_2)f_3f_4 - (f_3 + f_4)f_1f_2] \, d\omega_3 \, d\omega_4. \]

Here, the distribution \( f \) is expressed in terms of the dispersion relation \( \omega(k) = |k|^2 \), where \( k \) is the momentum. Moreover, we use the shorthand \( f_i = f(t, \omega_i) \) and the notations

\[ \omega_2 = \omega_3 + \omega_4 - \omega_1, \quad \text{and} \quad W = \frac{\min(\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4})}{\sqrt{\omega_1}}. \]

The structure is similar to the Boltzmann equation in kinetic theory, where the collision operator is quadratic compared to the cubic interaction in (1.2). Like the Boltzmann collision operator, the evolution preserves formally the mass and energy given by

\[ M(f) = \int_0^\infty f(t, \omega) \omega^{1/2} \, d\omega \quad \text{and} \quad E(f) = \int_0^\infty f(t, \omega) \omega^{3/2} \, d\omega. \]

As an analogue of the Boltzmann H-theorem, the entropy

\[ S(f) = \int_0^\infty \log(f) \omega^{1/2} \, d\omega \]

is formally increasing as

\[ \frac{d}{dt} S(f) = \frac{1}{4} \int \int \min(\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4}) f_1f_2f_3f_4 \left( \frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_3} - \frac{1}{f_4} \right)^2 \, d\omega_1 \, d\omega_3 \, d\omega_4. \]
Turning to formal stationary solutions, the first class is the Rayleigh-Jeans (RJ) solutions $f(k) = \frac{1}{\alpha + \beta \omega}$ with $\alpha, \beta > 0$ where the collision integral vanishes as $(f_1 + f_2)f_3f_4 - (f_3 + f_4)f_1f_2 = 0$. They correspond to statistical equilibria giving equipartition of a linear combination of mass and energy.

More interesting are the Kolmogorov-Zakharov (KZ) solutions which correspond to out-of-equilibrium dynamics. They are given by $\omega^{-7/6}$ and $\omega^{-3/2}$, see Subsection 7.4, and correspond respectively to a constant flux of mass (particles) from infinite to zero frequency, and a constant flux of energy from zero to infinite frequency.

Among all these stationary solutions, the KZ spectrum $\omega^{-7/6}$ is a well-defined solution, while $\omega^{-3/2}$ and the RJ solutions are only formal solutions as the collision integral is not absolutely converging. Details regarding the well-definition of the collision integral are given in Section 3.

1.2. Wave turbulence and dual cascade. The classical theory of two-dimensional hydrodynamic turbulence [25, 20] features a dual cascade: kinetic energy is transported to smaller frequencies, and enstrophy to higher frequencies. This behavior is different from three-dimensional hydrodynamic turbulence, and is explained by the two conservation laws associated to the two-dimensional Euler equation: energy and enstrophy.

Wave turbulence follows a similar pattern: when the microscopic model has two conserved quantities, it is expected that the associated wave kinetic equation will exhibit a dual cascade [41, 31]. This is typically the case for problems whose Hamiltonian only contain even terms, for instance the MMT toy model [38], elasticity [10], gravitational waves [21], or the nonlinear Schrödinger equation [39, 11], which is the focus of the present article.

Still in the class of problems exhibiting dual cascades, the (gravity) water-wave equation has played a very important role in the development of kinetic wave theory, see [32] for a review. It is speculated in [24, 40] that steady solutions of the associated forced kinetic wave equation should display tails at low and high frequencies corresponding to the inverse and direct cascades, respectively.

1.3. Proposed dual cascade scenario. Based on the preceding discussion, we expect that steady solutions of (1.1) will connect the solution $\omega^{-7/6}$ corresponding to an inverse cascade of mass, with the solution $\omega^{-3/2}$ corresponding to a (direct) energy cascade. It is tempting to propose the following scenario for the forced kinetic wave equation; we choose for simplicity in this discussion the force $\phi$ to be smooth and compactly supported on $(0, \infty)$.

Static ideal scenario: The equation $-\mathcal{C}(f) + \phi$ admits stationary solutions $f$ with tails $\omega^{-7/6}$ and $\omega^{-3/2}$, as $\omega \to 0$ and $\infty$ respectively. These tails correspond to outgoing fluxes of mass and energy which equilibrate the input of these quantities through $\phi$.

Dynamic ideal scenario: Solutions of the dynamical problem $\partial_t f = \mathcal{C}(f) + \phi$ with arbitrary data converge to one such stationary solution.

In the present article, we partially validate the static scenario by proving the existence of stationary solutions in a neighborhood of $\omega^{-7/6}$ (mass cascade).

Establishing rigorously the validity of the full ideal scenario seems very challenging. Compared to common studies in kinetic theory, we work in out-of-equilibrium dynamics so that variational formulations seem not to apply. Furthermore, the equation is nonlocal, and the structure of the collision operator is, as we shall see, more intricate than may seem.

As a side remark, let us mention the Smoluchowski coagulation equation [37, 14], which describes the evolution of dust particles that can stick together. There the appearance of a flux towards infinity is well-understood and known as gelation. However, the corresponding collision integral is just quadratic and the evolution only supports one cascade.
1.4. Main result. For any mass flux $j_M$, we define the scaled Kolmogorov-Zakharov spectrum

$$f_{j_M} = \left( \frac{j_M}{j_M^*} \right)^{1/3} \frac{1}{\omega^{7/6}}$$

where $j_M^* > 0$ is a positive constant given by (4.11). It is a steady state solution of (1.1) as $C(f_{j_M})(\omega) = 0$ for all $\omega > 0$; a short proof being given in Lemma 7.6. The time variations of the mass and energy densities present an inverse cascade of mass with a source at $\infty$ and a sink at 0 of $j_M$ mass per unit of time, and no cascade of energy:

$$-\omega^{1/2}C(f_{j_M}) = j_M\delta_\infty - j_M\delta_0$$
$$-\omega^{3/2}C(f_{j_M}) = 0.$$  \(1.4\)

see Definition 4.3 and (4.16) for the precise meaning of these equalities.

Our main result shows that this solution is stable if one perturbs the right-hand side of (1.4) by adding a source term of finite mass, and that this also triggers a direct cascade of energy towards $\infty$.

We formulate the result in terms of the weighted $L^\infty$ norm

$$\|f\|_{\alpha,\beta} = \sup_{0<\omega<1} \omega^\alpha |f(\omega)| + \sup_{\omega>1} \omega^\beta |f(\omega)|$$  \(1.5\)

and denote $E_{\alpha,\beta}(\mathbb{R}_+)$ the corresponding Banach space.

**Theorem 1.1.** For all $0 < \delta < 1/12$ there exists $\epsilon > 0$ such that the following holds true. For all $j_M^\infty > 0$ and $\phi \in L^\infty_{loc}(\mathbb{R}_+)$ satisfying

$$\|\phi\|_{3/2-\delta,3/2+\delta} \leq \epsilon j_M^\infty,$$  \(1.6\)

there exists a solution $f = f_{j_M^\infty} + g$ to

$$-C(f)(\omega) = \phi(\omega), \quad \forall \omega > 0$$

with

$$f \geq 0 \quad \text{and} \quad \|g\|_{7/6,7/6+\delta} \lesssim \epsilon j_M^\infty.$$  

Furthermore, it satisfies:

(i) Stability estimates. For all $\omega > 0$,

$$f(\omega) = \begin{cases} \left( \frac{j_M}{j_M^*} \right)^{1/3} \omega^{-7/6} (1 + O(\epsilon\omega^\delta)) & \text{for } \omega \leq 1, \\ \left( \frac{j_M^\infty}{j_M^*} \right)^{1/3} \omega^{-7/6} (1 + O(\epsilon\omega^{-\delta})) & \text{for } \omega > 1, \end{cases}$$  \(1.7\)

with the mass balance

$$j_M^0 = j_M^\infty + \int_0^\infty \omega^{1/2} \phi(\omega) d\omega.$$  \(1.8\)

(ii) Mass and energy cascades. In the sense of Definition 4.3,

$$-\omega^{1/2}C(f) = \omega^{1/2} \phi + j_M^\infty \delta_\infty - j_M^0 \delta_0$$
$$-\omega^{3/2}C(f) = \omega^{3/2} \phi - \left( \int_0^\infty \tilde{\omega}^{3/2} \phi(\tilde{\omega}) d\tilde{\omega} \right) \delta_\infty,$$  \(1.9\)

where the second identity holds provided $\int \tilde{\omega}^{3/2} \phi = \lim_{R \to \infty} \int_0^R \tilde{\omega}^{3/2} \phi$ exists. Its mass and energy fluxes, defined by (4.3) and (4.9), satisfy:

$$J_M(f)(\omega) = -j_M^0 + \int_0^\omega \tilde{\omega}^{1/2} \phi(\tilde{\omega}) d\tilde{\omega} \quad \text{and} \quad J_E(f)(\omega) = \int_0^\omega \tilde{\omega}^{3/2} \phi(\tilde{\omega}) d\tilde{\omega}.$$  \(1.10\)
Remark 1.2 (Positivity). The solutions we construct are non-negative. As for the forcing $\phi$, it is required to be small, but not to have a sign, even though the physically relevant case is $\phi \geq 0$.

Remark 1.3 (Cascades). The identity (1.9) shows that there is a source of $j_M^\infty$ mass of particles at $\infty$ per unit of time, which, together with the additional mass added by the forcing $\phi$, are dissipated in a sink of $j_M^0 = j_M^\infty + \int \sqrt{\omega} \phi \, d\omega$ mass at 0. The identity (1.10) shows that there is no source of energy at 0 and a sink of energy of $\int \omega^{3/2} \phi \, d\omega$ at $\infty$. Therefore, this solution presents an indirect cascade of mass as well as a direct cascade of energy.

Remark 1.4 (Optimality). The condition (1.6) for some $\delta > 0$ is optimal since for $\delta = 0$ the forcing $\phi$ would have infinite mass. The upper bound $\delta < 1/12$ allows for the shortest proof, avoiding to track logarithmic losses appearing in it; we did not try to optimise the range of $\delta$.

Remark 1.5 (Uniqueness). Finally, our proof shows that the constructed solution is unique in a neighbourhood of $f_{j_M^\infty}$.

1.5. Mathematical literature. The kinetic wave equation (1.1) is closely related to the Uhling-Uhlenbeck equation (also called Boltzmann-Bose-Einstein equation for hard spheres, or bosonic Nordheim equation). Both equations share the same cubic terms, but differ in lower order (quadratic) terms; their dynamics have many common features.

The basic theory of the Uhling-Uhlenbeck equation was established in [26, 27, 28, 18]: existence of weak solutions, development of singularities (Bose-Einstein condensation), and large-time behavior. Escobedo-Mischler-Velazquez [15, 16, 13] further studied singular solutions corresponding to the KZ solution $f_M$. They investigated linear stability (following [2]), and local well-posedness in their neighborhood; see [17] for RJ solutions. These articles will provide the foundation of the present work.

Finally, [18] focused specifically on (1.1), proving the existence of weak solutions, Bose-Einstein condensation, and characterizing large-time dynamics. While all of the above-mentioned works dealt with the spherically symmetric case, [22] gives optimal local well-posedness result for the kinetic wave equation without symmetry assumptions.

The kinetic equation for three-wave interactions involve a quadratic integral kernel. Related phenomena have been studied, see [34, 33] and references therein.

The derivation of the kinetic wave equation from weakly turbulent solutions to the cubic nonlinear Schrödinger equation in suitable regimes has received a large attention recently, see [3, 4, 8, 5, 1] for successive attempts and [29] for an earlier result at equilibrium. Its full validity for certain regimes of parameters has then been obtained in [7, 9]. Additional related recent results on the kinetic description of wave turbulence may be found in [19, 12, 6, 36, 30].

2. Notations

We will simply write $f(\omega) = f_{j_M^\infty}(\omega) = \frac{1}{\omega^{3/2}}$.

For the collision operator we introduce the trilinear operator

\begin{equation}
\hat{C}(f, g, h) = 2 \int_{0 < \omega_3 < \omega_4, 0 < \omega_5} W[(f_1 + f_2)g_3h_4 - (g_3 + g_4)h_1f_2] \, d\omega_3 \, d\omega_4
\end{equation}

and the symmetric form

\begin{equation}
C(f, g, h) = \frac{1}{6} (\hat{C}(f, g, h) + \hat{C}(f, h, g) + \hat{C}(g, f, h) + \hat{C}(g, h, f) + \hat{C}(h, f, g) + \hat{C}(h, g, f))
\end{equation}

By construction, $C(f) = C(f, f, f)$. 
Note that all forthcoming integrals will be implicitly performed on the domain of integration \( \{ \omega_1, \omega_2, \omega_3, \omega_4 > 0, \omega_1 + \omega_2 = \omega_3 + \omega_4, \omega_3 < \omega_4 \} \), which we stop mentioning explicitly. It is possible to assume that \( \omega_3 < \omega_4 \) due to the symmetry between these variables.

For the estimates we decompose the collision integral according to the minimal \((\omega_i)_{i=1,\ldots,4}\) as
\[
\mathcal{C}(f)(\omega_1) = \int \int_{\omega_2,\omega_3,\omega_4 \geq 0, \omega_3 < \omega_4} [\mathbb{I}_{\omega_{1\min}} + \mathbb{I}_{\omega_{2\min}} + \mathbb{I}_{\omega_{3\min}}] \ldots d\omega_3 d\omega_4 = C^1 + C^2 + C^3
\]
where
\[
\mathbb{I}_{\omega_{i\min}} = \mathbb{I}\{\omega_i < \omega_j \text{ for } j \neq i\}
\]
and we extend naturally this notation to other operators, e.g. \( \bar{C}^i \).

According to \( E_{\alpha,\beta} \) defined in (1.5), we define the weight
\[
\rho(\omega) = \rho^{\alpha,\beta}(\omega) = \mathbb{I}_{\omega < 1} \omega^{-\alpha} + \mathbb{I}_{\omega > 1} \omega^{-\beta}.
\]
We obtain the space \( E_{\alpha,\beta}(\mathbb{R}) \) through the change of coordinates
\[
(2.3) \quad \omega = e^x, \quad f(\omega) = F(x);
\]
in other words the norm of \( E_{\alpha,\beta}(\mathbb{R}) \) is
\[
\|F(x)\|_{\alpha,\beta} = \sup_{x < 0} e^{\alpha x}|F(x)| + \sup_{x > 0} e^{\beta x}|F(x)| = \|f(\omega)\|_{\alpha,\beta}.
\]
The spaces \( E_{\alpha,\beta}(\mathbb{R}_+) \) and \( E_{\alpha,\beta}(\mathbb{R}) \) can obviously be identified, and we will systematically abuse notations by simply denoting \( E_{\alpha,\beta} \).

3. Critical powers

3.1. List of critical powers. The equation \( \partial_t f = \mathcal{C}(f) \) has a more complicated structure than the simplicity of the formula for the collision operator might suggest. This becomes clear once one looks at the various powers which act as thresholds; we first give a list of these threshold powers from the least to the most singular at 0

Power \(-1\) As we will see below, the range for local well-posedness is \( \alpha < 1 < \beta \); equivalently, this is the range for which \( \mathcal{C} \) is bounded on \( E_{\alpha,\beta} \). The exponent \( \alpha = \beta = 1 \) corresponds to the scaling of the equation, and it is not clear whether local well-posedness holds for \( E_{1,1} \). However, the Rayleigh-Jeans solutions are stationary solutions with decay \( \omega^{-1} \) as \( \omega \to \infty \). That the RJ solutions actually solve the equation is related to cancellations in the integrand of \( \mathcal{C} \). For this reason, it seems unlikely that the equation is locally well-posed in \( E_{1,1} \), but it is reasonable to expect nonlinear stability (or co-dimensional stability) of the RJ solutions in a stronger topology.

Power \(-7/6\) The range \( \alpha < 7/6 < \beta \) corresponds to solutions which conserve mass. Furthermore, the first KZ solution is \( \omega^{-7/6} \).

Power \(-5/4\) The integral defining the collision operator converges absolutely for \( \alpha < 5/4 \) and \( \beta > 1 \), see Proposition 3.2 below. This is termed “locality” in the work of Zakharov and collaborators.

Power \(-3/2\) The power \( 3/2 \) gives first of all the threshold for the finiteness of mass: solutions for which \( \alpha < 3/2 < \beta \) have finite mass. Second, though \( \mathcal{C} \) may not consist of absolutely convergent integrals, it remains well-defined in a weak sense for \( \alpha < 3/2 \) and \( \beta > 1 \), see Lemma 3.3. Finally, the power \( 3/2 \) corresponds to the second KZ solution, namely \( \omega^{-3/2} \). Since \( 3/2 > 5/4 \), the locality exponent, this KZ solution is deemed physically irrelevant by Balk and Zakharov [2].
3.2. Scaling. The scaling invariance of the collision operator
\[ C[f(\lambda \cdot)] = \lambda^{-2} C[f(\cdot)] \]
implies that the transformation
\[ f(t, \omega) \mapsto \mu^{1/2} \lambda f(\mu t, \lambda \omega) \]
leaves the set of solutions invariant.

The scaling (in \( \lambda \) only) obtained for \( \mu = 1 \), namely \( f(\omega) \mapsto \lambda f(\lambda \omega) \) leaves (amongst others) the norm \( \| \omega f(\omega) \|_\infty \) invariant. One can expect local well-posedness to hold in spaces with this scaling (or slightly more regular). This was indeed confirmed in [22] in the non-radial case.

For the record, we include here the simple proof in the radial case.

**Proposition 3.2.** Assume \( 0 \leq \alpha < 1 < \beta \), then the operator \( C \) is bounded from \( E_{\alpha, \beta} \) to itself. As a consequence, the equation \( \partial_t f = C(f) \) is locally well-posed in \( E_{\alpha, \beta} \).

**Proof.** The estimate for \( C \) follows directly from Corollary 7.3. The local well-posedness in \( E_{\alpha, \beta} \) then follows by standard Cauchy-Lipschitz theory. \( \square \)

3.3. Local property. As defined in Balk and Zakharov [2], a spectrum is called local if each of the summands defining \( C \) is absolutely convergent. These authors suggest that physically relevant spectra must be local.

**Proposition 3.2.** A function \( f \in E_{\alpha, \beta} \) is local if \( \beta > 1 \) and \( 0 \leq \alpha < \frac{5}{4} \).

This is a direct corollary of Lemma 7.1. When either \( \alpha \geq 5/4 \) or \( \beta \leq 1 \), explicit examples can be found following the proof of Lemma 7.1, for which the collision integrals may no longer be absolutely convergent, hence the optimality of the range of parameters in Proposition 3.2.

3.4. Weak formulation. For smooth and localized \( f \) and \( \phi \), symmetrizing leads to the identity
\[ \int \sqrt{\omega_1} C(f) \varphi_1 \, d\omega_1 = \frac{1}{2} \int \int \int \sqrt{\omega_1} W f_1 f_2 (f_3 + f_4) (\varphi_3 + \varphi_4 - \varphi_1 - \varphi_2) \, d\omega_1 \, d\omega_3 \, d\omega_4. \]

This weak formulation features a cancellation in the term \( (\varphi_3 + \varphi_4 - \varphi_1 - \varphi_2) \), which allows to make sense of this identity for a larger set of functions than allowed by Proposition 3.2; this is the content of the following lemma.

**Lemma 3.3.** Assume \( f \in E_{\alpha, \beta} \) with \( \alpha < 3/2 \) and \( \beta > 1 \) and \( \varphi \in C_0^\infty((0, \infty)) \). Then the integral in the right-hand side of (3.1) converges absolutely.

One can check that this result is optimal; for either \( \alpha \geq 3/2 \) or \( \beta \geq 1 \) there exists \( f \in E_{\alpha, \beta} \) for which the integral is not absolutely convergent if \( \varphi \in C_0^\infty((0, \infty)) \) is nonzero.

**Proof.** By symmetry, we can restrict to the case \( \omega_1 < \omega_2 \) and \( \omega_3 < \omega_4 \) and consider
\[ I = \int \int \min (\sqrt{\omega_1}, \sqrt{\omega_2}) f_1 f_2 (f_3 + f_4) (\varphi_3 + \varphi_4 - \varphi_1 - \varphi_2) \, d\omega_1 \, d\omega_3 \, d\omega_4. \]

Since \( \varphi \) is a compactly supported test function, it can be assumed to have supp \( \varphi \subset [2\delta, \delta^{-1}] \) for some \( \delta > 0 \). Then \( \omega_2 \geq \delta \) if \( \varphi_3 + \varphi_4 - \varphi_1 - \varphi_2 \neq 0 \). We decompose accordingly
\[ I = \int_{\omega_2=\delta}^{\infty} \int_{\omega_3=0}^{\delta} \min (\sqrt{\omega_1}, \sqrt{\omega_3}) f_1 f_2 (f_3 + f_4) (\varphi_4 - \varphi_2) \, d\omega_3 \, d\omega_1 \, d\omega_2. \]
\[ + \int_{\omega_2=\delta}^{\infty} \int_{\max(\omega_1, \omega_3) \geq \delta} \min (\sqrt{\omega_1}, \sqrt{\omega_3}) f_1 f_2 (f_3 + f_4) (\varphi_4 - \varphi_2 + \varphi_3 - \varphi_1) \, d\omega_3 \, d\omega_1 \, d\omega_2. \]
The integral $I_1$ contains the contribution of the mass for small $\omega$. As $\varphi$ is Lipschitz we find that $|\varphi_4 - \varphi_2| \leq |L| |\omega_4 - \omega_2| = L |\omega_3 - \omega_1|$. Using also $|f_3 + f_4| \lesssim \rho_3$ and $\int_\delta^\infty |f_2| d\omega < \infty$ as $\beta > 1$, the integral $I_1$ is absolutely converging as soon as

$$
\int_\delta^\infty \int_\delta^\infty \min (\sqrt{\omega_1}, \sqrt{\omega_3}) \rho_1 \rho_3 |\omega_1 - \omega_3| d\omega_3 d\omega_1
$$

is integrable. It can be bounded by

$$
\int_\delta^\infty \int_\delta^\infty \min (\sqrt{\omega_1}, \sqrt{\omega_3}) \rho_1 \rho_3 |\omega_1 - \omega_3| d\omega_3 d\omega_1 \leq 4 \int_\delta^\infty \int_\delta^\infty \sqrt{\omega_1 \omega_3}^\alpha \omega_3^{-\alpha} d\omega_3 d\omega_1,
$$

which is finite as soon as $\alpha < 3/2$. Alternatively, the integral is bounded if $M(\|\omega < 1\|) < \infty$.

As for the second integral $I_2$, we bound by brute force $\min (\sqrt{\omega_1}, \sqrt{\omega_3}) \lesssim \min (1, \sqrt{\omega_1}, \sqrt{\omega_3})$ on the support of $(\varphi_4 - \varphi_2 + \varphi_3 - \varphi_1)$ and $|\varphi_4 - \varphi_1 + \varphi_3 - \varphi_1| \lesssim 1$ so that

$$
|I_2| \lesssim \int_\omega_2=\delta \int \min (1, \sqrt{\omega_1}, \sqrt{\omega_3}) \rho_1 \rho_2 \rho_3 \omega_1 d\omega_3 d\omega_3 \omega_2 d\omega_3 < \infty. \quad \square
$$

3.5. How to interpret the equation. Consider a function

$$
f \in C^1([0, T], E_{\alpha, \beta}).
$$

When is it a solution of the kinetic wave equation (1.1)? In light of the discussion above, we see that the equation can be given different meanings, depending on the range of $\alpha$ and $\beta$. We discuss the evolution problem, but the extension to stationary solutions is obvious.

If $\alpha < 1 < \beta$, by Proposition 3.1, the equation (1.1) can be interpreted as an equality in $C^1([0, T], E_{\alpha, \beta})$.

If $\alpha < \frac{3}{4}$, $\beta > 1$, by Proposition 3.2, the equation (1.1) can be understood pointwise in $[0, T] \times (0, \infty)$.

If $\alpha < \frac{3}{2}$ or finite mass and $\beta > 1$, by Lemma 3.3, the equation (1.1) can be understood weakly (tested against compactly supported Lipschitz functions).

Remark 3.4. This has not been mentioned yet, but the equation (1.1) is always considered in the present paper in the non-interacting condensate regime, following the terminology of [18]. This means that the mass which is transfered to zero momentum is not allowed to interact with the rest of the solution, or equivalently that it is instantaneously absorbed by an infinite sink.

4. Cascades, fluxes of mass and energy

The fluxes of the equation are a common tool for studying the equation [11, 35, 18] and they provide a natural notion of solution.

4.1. Definition of the fluxes. By Proposition 3.2 and the computation of Section 7.4, we have that $C(f)(\omega)$ is well-defined (i.e. by a convergent integral), and equal to 0 for every $\omega$. However, “$C(f)(\omega) = 0$” is not an appealing identity, since one expects $f$ to be a steady state with constant mass flux from $\infty$ to 0. Rather, we propose here in Definition 4.3 a weak definition of $\omega^{1/2}C(f)$ and $\omega^{3/2}C(f)$ (which would be the time variation of the mass and energy densities for a solution of (1.1)) for general $f \in E_{7/6, 7/6}$. The KZ solution $f$ will then present a source of mass at $\infty$ and a sink at 0. These definitions of $\omega^{1/2}C(f)$ and $\omega^{3/2}C(f)$ will moreover be proved to be natural, since corresponding to the approximation by finite size systems, i.e. to the limit of $\omega^{1/2}C(f_n)$ and $\omega^{3/2}C(f_n)$ for any sequence $f_n \to f$ of smooth and localised functions $f_n$. 
We first derive the mass flux. After symmetrizing, for \( f \) smooth and localised we obtain that

\[
\int \sqrt{\omega_1} C(f) \varphi_1 \, d\omega_1 = \int_0^{\omega_3, \omega_4 \geq 0} \sqrt{\omega_1} W [(f_1 + f_2) f_3 f_4 - (f_3 + f_4) f_1 f_2] \varphi_1 \, d\omega_1 \, d\omega_3 \, d\omega_4
\]

\[
= \int \sqrt{\omega_1} W f_1 f_2 f_3 (\varphi_3 + \varphi_4 - \varphi_1 - \varphi_2) \, d\omega_1 \, d\omega_3 \, d\omega_4.
\]

(4.1)

We now write

\[
\varphi_3 - \varphi_2 = \mathbb{I}_{\omega_3 > \omega_2} \int_{\omega_3}^{\omega_2} \varphi' - \mathbb{I}_{\omega_2 > \omega_3} \int_{\omega_2}^{\omega_3} \varphi'
\]

\[
\varphi_4 - \varphi_1 = \mathbb{I}_{\omega_4 > \omega_1} \int_{\omega_4}^{\omega_1} \varphi' - \mathbb{I}_{\omega_1 > \omega_4} \int_{\omega_1}^{\omega_4} \varphi'.
\]

Using the formulas above to express \((\varphi_3 + \varphi_4 - \varphi_1 - \varphi_2)\), inserting the result and rearranging, we find that

\[
\int \sqrt{\omega} C(f) \varphi \, d\omega = \int J_M(f) \varphi' \, d\omega.
\]

(4.2)

Above, \( J_M \) is a flux whose orientation is from 0 to \( \infty \) with

\[
J_M(f) = J_1(f) + J_2(f) - J_3(f) - J_4(f)
\]

(4.3)

\[
J_1(f)(\omega) = \int_{\omega > 0}^{\omega_1} \sqrt{\omega_1} W f_1 f_2 f_3 \, d\omega_1 \, d\omega_2 \, d\omega_3
\]

(4.4)

\[
J_2(f)(\omega) = \int_{\omega > 0}^{\omega_2} \sqrt{\omega_1} W f_1 f_2 f_3 \, d\omega_1 \, d\omega_2 \, d\omega_3
\]

(4.5)

\[
J_3(f)(\omega) = \int_{\omega > 0}^{\omega_3} \sqrt{\omega_1} W f_1 f_2 f_3 \, d\omega_1 \, d\omega_2 \, d\omega_3
\]

(4.6)

\[
J_4(f)(\omega) = \int_{\omega > 0}^{\omega_4} \sqrt{\omega_1} W f_1 f_2 f_3 \, d\omega_1 \, d\omega_2 \, d\omega_3
\]

(4.7)

Next, integrating by parts in (4.2), and using \( \int \sqrt{\omega} C(f) = 0 \) we get

\[
\int \omega^{3/2} C(f) \varphi \, d\omega = \int J_E(f) \varphi' \, d\omega,
\]

(4.8)

where

\[
J_E(f)(\omega) = \omega J_M(\omega) - \int_{0}^{\omega} J_M(\tilde{\omega}) \, d\tilde{\omega}.
\]

(4.9)

4.2. Locality of the fluxes.

**Lemma 4.1.** If \( \alpha < \frac{5}{4} \) and \( \beta > 1 \) then the fluxes \((J_i)_{1 \leq i \leq 4}\) are local (i.e. the integrals converge absolutely).

**Proof.** By a rescaling of variables, it suffices to take \( \omega = 1 \) and prove \( J_n(\rho)(1) < \infty \) for \( 1 \leq n \leq 4 \). We introduce \( \bar{\omega}_i = \sqrt{\omega_i} \) and \( \bar{W}_{i,j} = \min(\sqrt{\omega_i}, \sqrt{\omega_j}) \).

Observe first the following integrals are finite since \( \alpha < 5/4 \) and \( \beta > 1 \):

\[
\int_{\omega_1 > 1} \rho_1 \, d\omega_1 < \infty, \quad \int_{\omega_1 < 1} \bar{W}_{i} \rho_1 \, d\omega_1 < \infty, \quad \int_{\omega_i, \omega_j < 1} \bar{W}_{i,j} \rho_i \rho_j \, d\omega_i \, d\omega_j < \infty.
\]

(4.10)

\(^1\) We suspect that there is a typo in [18] for \( J_1 \).
For the first flux:
\[
J_1(\rho)(1) = \int_{\omega_1, \omega_2 < 1} \sqrt{\omega_1} W_1 \rho_1 \rho_2 \rho_3 \, d\omega_1 
+ \int_{\omega_1 > 1} \sqrt{\omega_1} W_1 \rho_1 \rho_2 \rho_3 
\]
\[
\leq \int_{\omega_1, \omega_2 < 1} \tilde{W}_1,2 \rho_1 \rho_2 \rho_3 
+ \int_{\omega_1 > 1} \tilde{W}_1,2 \rho_1 \rho_2 \rho_3 
< \infty
\]
using a combination of the identities in (4.10). For the second, similarly:
\[
J_2(\rho)(1) = \int_{\omega_1, \omega_2 < 1} \tilde{W}_1,3 \rho_1 \rho_2 \rho_3 
+ \int_{\omega_1 > 1} \tilde{W}_2,3 \rho_1 \rho_2 \rho_3 
< \infty.
\]
For the third and fourth, analogously:
\[
J_3(\rho)(1) = \int_{\omega_1, \omega_3 < 1} \tilde{W}_1,3 \rho_1 \rho_2 \rho_3 
+ \int_{\omega_1 > 1} \tilde{W}_3 \rho_1 \rho_2 \rho_3 
< \infty,
\]
\[
J_4(\rho)(1) = \int_{\omega_1, \omega_3 < 1} \tilde{W}_2,3 \rho_1 \rho_2 \rho_3 
\]
\[
\leq \int_{\omega_1 > 1} \rho_1 \rho_2 
+ \int_{\omega_1 > 1} \tilde{W}_2,3 \rho_1 \rho_2 \rho_3 
< \infty
\]
where for the first sum in \( J_4 \) we used that \( \rho_3 \sqrt{\omega_1} W \lesssim 1 \) in the domain of integration. This concludes the proof. \( \square \)

By a rescaling of the variables, one easily checks that the flux of mass of the KZ solution \( f = \omega^{-7/6} \) is constant:
\[
\forall \omega > 0, \quad J_M(f)(\omega) = -j_M^* < 0.
\]
Its negativity, see [35, 18], indicates mass flowing from infinity to 0. Combining (4.3) and (4.11) shows there is no flux of energy:
\[
\forall \omega > 0, \quad J_E(f)(\omega) = 0.
\]

**Corollary 4.2.** Let \( f \in E_{7/6,7/6} \). Then
(i) There holds \( J_M(f) \in L^\infty(0, \infty) \) and \( \omega^{-1} J_E(f) \in L^\infty(0, \infty) \) with
\[
\|J_M(f)\|_{L^\infty(0, \infty)} + \|\omega^{-1} J_E(f)\|_{L^\infty(0, \infty)} \lesssim \|f\|_{7/6,7/6}^2.
\]
(ii) If \( f_n \) is a bounded sequence in \( E_{7/6,7/6} \) such that \( f_n \to f \) almost everywhere, then \( J_M(f_n) \to J_M(f) \) and \( J_E(f_n) \to J_E(f) \) uniformly on every compact sets of \( (0, \infty) \).

**Proof.** We mentioned right before the corollary that the integrals defining \( J_k(f)(\omega) \) for \( k = 1, 2, 3, 4 \) are convergent, with a value that does not depend on \( \omega \). This directly implies part (i) of the corollary.
For the second one, fix $0 < \omega_+ < \omega_-$ and $k \in \{1, 2, 3, 4\}$. Decompose any function $g$ as $g = g^1_1 + g^1_2$ where $g^1_1 = 1(\epsilon < \omega < \epsilon^{-1})g$. Then

$$J_k(f) - J_k(f_n) = \sum_{1 \leq i_1, i_2, i_3 \leq 2} J_k(f^+_{i_1}, f^+_{i_2}, f^+_{i_3}) - J_k(f^-_{i_1}, f^-_{i_2}, f^-_{i_3})$$

where $J_k(a, b, c)$ is given by the corresponding expression in (4.4)-(4.7), with $f_1, f_2, f_3$ replaced by $a, b, c$ respectively. On the one hand, by Lemma 4.1, if $(i_1, i_2, i_3) \neq (1, 1, 1)$ then $J_k(f^+_{i_1}, f^+_{i_2}, f^+_{i_3}) \to 0$ and $J_k(f^-_{i_1}, f^-_{i_2}, f^-_{i_3}) \to 0$ uniformly for $\omega \in [\omega_-, \omega_+]$ and $n \in \mathbb{N}$. On the other hand, for each $\epsilon > 0$, still by Lemma 4.1, $J_k(f^+_{i_1}, f^+_{i_2}, f^+_{i_3}) \to J_k(f^+_{i_1}, f^+_{i_2})$ uniformly on $[\omega_-, \omega_+]$ as $n \to \infty$. Combining, we obtain that $J_M(f_n) \to J_M(f)$ uniformly on $[\omega_-, \omega_+]$. Using this convergence for any $0 < \omega_- < \omega_+$, the identity (4.3) and the first bound in (4.13) shows that $J_E(f_n) \to J_E(f)$ as well uniformly on any compact.

4.3. Cascades. We are able to make sense of mass and energy cascades by considering their fluxes in the framework of distributions. We define the space of test functions $\mathcal{D}([0, \infty))$ as the set of all functions $\varphi \in C^\infty([0, \infty))$ such that $\varphi'$ has compact support. This space can be equipped with a similar topology as the space of usual test functions for usual distributions. Hence, we abuse notations and still call distributions linear forms on $\mathcal{D}$.

**Definition 4.3.** For $f \in E_{7/6,7/6}$, we define $\sqrt{\omega}C(f)$ and $\omega^{3/2}C(f)$ in the distributional sense by:

$$\forall \varphi \in \mathcal{D}, \quad \langle \sqrt{\omega}C(f), \varphi \rangle = \int_0^\infty J_M(f)(\omega) \varphi'(\omega) \, d\omega,$$

$$\forall \varphi \in \mathcal{D}, \quad \langle \omega^{3/2}C(f), \varphi \rangle = \int_0^\infty J_E(f)(\omega) \varphi'(\omega) \, d\omega.$$

**Remark 4.4.** These weak definitions $\sqrt{\omega}C(f)$ and $\omega^{3/2}C(f)$ for any $f \in E_{7/6,7/6}$ are natural because of Corollary 4.2. Indeed, let us approximate $f$ by a sequence $f_n$ of smooth and localised functions. Then for all $\varphi \in \mathcal{D}$ we have, on the one hand by item (ii) that $\langle \sqrt{\omega}C(f_n), \varphi \rangle \to \langle \sqrt{\omega}C(f), \varphi \rangle$, and on the other hand that $\langle \sqrt{\omega}C(f_n), \varphi \rangle = \int C(f_n) \varphi$ because of (4.2). Thus the formula (4.14) is such that $\langle \sqrt{\omega}C(f), \varphi \rangle = \lim_{n \to \infty} \int C(f_n) \varphi$ is the limit obtained from any approximation of $f$ by non-singular functions $f_n$. The same approximation holds true for $\omega^{3/2}f$ as well.

Thanks to (4.11)-(4.12) we have for the KZ steady state $\tilde{f}$ that, in the sense of (4.14)-(4.15):

$$-\sqrt{\omega}C(f) = j_M^+ \delta_0 - j_M^- \delta_\infty, \quad -\omega^{3/2}C(f) = 0,$$

where we introduced the Dirac deltas:

$$\forall \varphi \in \mathcal{D}, \quad \langle \delta_0, \varphi \rangle = \varphi(0) \quad \text{and} \quad \langle \delta_\infty, \varphi \rangle = \lim_{\omega \to \infty} \varphi(\omega).$$

This shows an indirect cascade of mass from $\infty$ to 0 and no cascade of energy.

The next proposition relates the fluxes of a solution to $-C(f) = \phi$ to its asymptotic behaviours near 0 and $\infty$.

**Proposition 4.5.** Assume $f \in E_{7/6,7/6}$ and $\phi \in L^1(\sqrt{\omega}d\omega)$ solve

$$\forall \omega > 0, \quad -C(f)(\omega) = \phi(\omega),$$

with

$$f(\omega) = c_0 \omega^{-7/6} + o(\omega^{-7/6}) \text{ as } \omega \downarrow 0 \quad \text{and} \quad f(\omega) = c_\infty \omega^{-7/6} + o(\omega^{-7/6}) \text{ as } \omega \to \infty$$

for some $c_0, c_\infty \in \mathbb{R}$. Then, in the sense of distributions given by Definition 4.3:

$$\sqrt{\omega}C(f) = c_\infty^3 j_M^+ \delta_\infty - c_0^3 j_M^- \delta_0 + \omega^{1/2}\phi,$$

$$\omega^{3/2}C(f) = \omega^{3/2}\phi - E(\phi)\delta_\infty.$$
where the second identity holds provided $E(\phi) = \lim_{R \to \infty} \int_0^R \omega^{3/2} \phi \, d\omega$ exists, and there holds:

\begin{equation}
\epsilon_0^3 = \epsilon_\infty^3 + \frac{M(\phi)}{j_M^*}.
\end{equation}

Moreover,

\begin{equation}
J_M(f)(\omega) = -c_0^3j_M^* + \int_0^\omega \omega^{1/2} \phi(\omega) \, d\omega; \quad J_E(f)(\omega) = \int_0^\omega \omega^{3/2} \phi(\omega) \, d\omega.
\end{equation}

**Proof.** Let $f_\lambda(\omega) = \lambda^{-7/6} f(\omega/\lambda)$ for $\lambda > 0$. Then $f_\lambda$ is bounded uniformly in $E_{7/6, 7/6}$ for all $\lambda > 0$, with $f \to c_0 \omega^{-7/6}$ as $\lambda \to \infty$ and $f \to c_\infty \omega^{-7/6}$ as $\lambda \to 0$, in $L^\infty_{loc}((0, \infty))$. By Lemma 4.1 and (4.11), this implies $J(f_\lambda)(1) \to c_0^3j_M^*$ and $J(f(\lambda))(1) \to c_\infty^3j_M^*$ respectively. Since by rescaling, $J(f(\lambda))(\lambda) = J(f(\lambda^{-1}))(1)$ we get

\begin{equation}
\lim_{\omega \to 0} J(f)(\omega) = -c_0^3j_M^* \quad \text{and} \quad \lim_{\omega \to \infty} J(f)(\omega) = -c_\infty^3j_M^*.
\end{equation}

Fix now $\varphi \in D$. Let $\chi$ be a smooth cut-off function with $\chi(\omega) = 1$ for $\omega \leq 1$ and $\chi(\omega) = 0$ for $\omega \geq 2$. For $R > 0$ we let $\chi_R(\omega) = \chi(\omega/R)$. For $\epsilon > 0$ we decompose $\varphi = \varphi_1^\epsilon + \varphi_2^\epsilon + \varphi_3^\epsilon$ where $\varphi_1^\epsilon = \epsilon^4 \varphi$ and $\varphi_3^\epsilon = (1 - \chi^{1/\epsilon}) \varphi$ so that

\begin{equation}
\langle \sqrt{\omega} C(f), \varphi \rangle = \sum_{k=1}^3 \langle \sqrt{\omega} C(f), \varphi_k^\epsilon \rangle.
\end{equation}

On the one hand, by (4.14) and (4.21) we have:

\begin{equation}
\langle \sqrt{\omega} C(f), \varphi_1^\epsilon \rangle \to c_0^3j_M^* \varphi(0) \quad \text{and} \quad \langle \sqrt{\omega} C(f), \varphi_3^\epsilon \rangle \to -c_\infty^3j_M^* \lim_{\omega \to \infty} \varphi(\omega)
\end{equation}

as $\epsilon \to 0$. On the other hand, for each fixed $\epsilon > 0$, the derivation of the formula (4.2) is valid for $\varphi_2^\epsilon$ as it has compact support in $(0, \infty)$ and using Lemma 4.1, which combined with the identity $-C(f)(\omega) = \phi(\omega)$ for all $\omega > 0$ gives:

\[ \langle \sqrt{\omega} C(f), \varphi_2^\epsilon \rangle = -\int_0^\infty \sqrt{\omega} \phi \varphi_3^\epsilon \, d\omega. \]

Hence \( \langle \sqrt{\omega} C(f), \varphi_2^\epsilon \rangle \to -\int_0^\infty \sqrt{\omega} \phi \varphi_3 \, d\omega \) as $\epsilon \to 0$, which, combined with (4.22) and (4.23) implies (4.17).

The identity (4.19) then directly follows upon using (4.17) with the test function $\varphi = 1$.

The identity (4.17) implies the first identity in (4.20). Injecting (4.19) and the first identity in (4.20) in the expression (4.3) for $J_E$ shows the second identity in (4.20). This, in turns, injected in (4.15), implies the identity (4.18).

\[ \square \]

5. **Proof of the main result**

This section is devoted to the proof of Theorem 1.1. By scaling invariance, it suffices to prove the result in the case $j_M^* = j_M^*$, and the general result follows. We thus aim at solving

\begin{equation}
\forall \omega > 0, \quad -C(f)(\omega) = \phi(\omega),
\end{equation}

in a neighborhood of $f(\omega) = \omega^{-7/6}$, and with a forcing satisfying

\begin{equation}
\|\phi\|_{3/2 - \delta, 3/2 + \delta} \leq \epsilon.
\end{equation}

Expressed in terms of the perturbation $g$ around $f$, i.e. $f \mapsto f + g$, the equation (5.1) becomes

\begin{equation}
Lg - Q(g) - C(g) = \phi,
\end{equation}

where $Q(g)$ is the operator defined in (3.10), and $C(g)$ is a nonlinear term whose action is to compensate the diffusion term. The existence of a solution to this equation is guaranteed by the following result.

\[ \square \]
where we denote $\mathcal{L}$, $\mathcal{Q}$ and $\mathcal{C}$ for the linear, quadratic and cubic terms. Using the power-law behaviour of $f$ the linear operator can be expressed as
\begin{equation}
\mathcal{L}g(\omega) = \frac{a}{\omega^{4/3}}g - \frac{1}{\omega^{4/3}} \int_0^\infty k \left( \frac{r}{\omega} \right) f(r) \, dr
\end{equation}
where $a > 0$ and $k$ can be expressed by explicit but lengthy integrals as in [15, Appendix A]. Using the notation of (2.2), the quadratic term can be expressed as
\[ \mathcal{Q}(g) = 3\mathcal{C}(f, g, g). \]

This equation is more naturally expressed in the variables of (2.3) with $\omega = e^x$, $g(k) = G(x)$, $\phi(\omega) = \Phi(x)$, $\mathcal{K}(x) = e^{-x}k(e^{-x})$ so that (5.3) becomes
\begin{equation}
\mathcal{L}G - e^{x/3}\mathcal{Q}(G) - e^{x/3}\mathcal{C}(G) = e^{x/3}\Phi,
\end{equation}
where
\[ \mathcal{L} = a - \mathcal{K}_*, \]
\[ \mathcal{Q}(G)(x) = \mathcal{Q}(g)(\omega), \]
\[ \mathcal{C}(G)(x) = \mathcal{C}(g)(\omega). \]

**Notation 5.1.** The reader will have noticed that we abuse notations by denoting $\mathcal{Q}$ and $\mathcal{C}$ for the quadratic and cubic operators, regardless of the coordinate choice being considered ($\omega$ or $x$). But this will be clear from the context, and upper case unknowns are always functions of $x$, while lower case indicates functions of $\omega$.

Inverting $\mathcal{L}$, we arrive at the equivalent fixed-point problem
\begin{equation}
G = \mathcal{L}^{-1}e^{x/3} [\Phi + \mathcal{Q}(G) + \mathcal{C}(G)].
\end{equation}

**Remark 5.2.** The naive approach would be to apply the Banach fixed-point theorem in a space containing the constant term $\mathcal{L}^{-1}e^{x/3}\Phi$. Due to the constant term in $\mathcal{L}$, one expects that $\mathcal{L}^{-1}$ is a bounded operator from some $E_{\alpha,\beta}$ to itself. For controlling the cubic term we thus would arrive to the following series of inequalities
\[ \|\mathcal{L}^{-1}e^{x/3}\mathcal{C}(G)\|_{\alpha,\beta} \lesssim \|e^{x/3}\mathcal{C}(G)\|_{\alpha,\beta} = \|\mathcal{C}(G)\|_{\alpha+\frac{1}{3},\beta+\frac{1}{3}} \lesssim \|G\|_{\alpha,\beta}^3, \]
where the last step only holds if $\beta \geq 7/6$ by Corollary 7.3. The same series of inequalities is needed for the quadratic term, where the last step only holds if $\alpha \leq 7/6$. But the function $e^{\frac{x}{7}}$ is in the kernel of the linear operator $\mathcal{L}$, which therefore cannot be inverted.

Of course, this obstacle should have been expected: it is related to the scaling invariance of the problem, or equivalently the need to modulate along the family of solutions $\omega \omega^{-7/6}$.

The argument can be salvaged by improving it in two respects: first, by peeling off the leading order behavior, we find an inverse $\mathcal{L}^{-1}$ from $E_{\alpha,\beta}$ to itself for $\alpha = 7/6 - \delta_1$ and $\beta = 7/6 + \delta_2$ and second, by using the cancellations present in the equation, control the leading order contribution.

We introduce $\chi$ a smooth cut-off function with $\chi(x) = 1$ for $x \geq 1$ and $\chi(x) = 0$ for $x \leq 0$ and $G_0(x) = \chi(x)e^{-\frac{x}{7}}$.

In Section 6, we will show that $\mathcal{L}^{-1}$ can be decomposed in the leading order.

**Proposition 5.3.** Let $0 < \delta_1, \delta_2 < 1/6$. Then for any $\Phi \in E_{3/2-\delta_1,3/2+\delta_2}$, it is possible to decompose $\mathcal{L}^{-1}[e^{x/3}\Phi]$ into leading order (as $x \to \infty$) plus remainder as
\[ \mathcal{L}^{-1}[e^{x/3}\Phi] = c_1 M(\Phi)G_0(x) + \ell_1(\Phi), \]
where \( c_1 = -\frac{1}{3^2 M} \). \( M(\Phi) \) is the mass integral (1.3) and \( \ell_1 \) is a bounded linear operator from \( E_{3/2 - \delta_1, 3/2 + \delta_2} \) to \( \dot{E}_{7/6 - \delta_1, 7/6 + \delta_2} \).

The idea is now to extract the leading order term from the left and right-hand sides of (5.6): to this effect, we split the sought perturbation \( G \) into

\[
G = cG_0 + H.
\]

Using Proposition 5.3, we express the right-hand side of (5.6) for \( F \in E_{\alpha, \beta} \) with \( \alpha = 3/2 - \delta_1 \) and \( \beta = 3/2 + \delta_2 \) as

\[
\mathcal{L}^{-1} e^{x/3} F = \ell_0(F)G_0 + \ell_1(F).
\]

where \( \ell_0(F) = c_1 M(F) \). Introducing

(5.7)

\[
\ell = \begin{pmatrix} \ell_0 \\ \ell_1 \end{pmatrix}
\]

we can now write the fixed point problem as

\[
\begin{pmatrix} c \\ H \end{pmatrix} = \begin{pmatrix} \ell_0 \\ \ell_1 \end{pmatrix} (\Phi + Q(G) + C(G)) = \ell(\Phi) + \ell(N(c, H)),
\]

where

\[
N(c, H) = Q(G) + C(G)
\]

\[
= c^2 Q(G_0) + Q(H) + 2cQ(G_0, H) + c^3 C(G_0) + C(H) + 3c^2 C(G_0, G_0, H) + 3c C(G_0, H, H).
\]

Adopting the natural norm on \( \mathbb{R} \times E_{\alpha, \beta} \)

\[
\|(c, H)\|_{\alpha, \beta} = |c| + \|H\|_{\alpha, \beta},
\]

the nonlinear term acts as a contraction:

**Proposition 5.4.** Assume \( c \in \mathbb{R} \) and \( H \in E_{\alpha, \beta} \) with \( \alpha = 7/6 - \delta_1 \) and \( \beta = 7/6 + \delta_2 \) where \( 0 < \delta_1, \delta_2 < 1/12 \). Then \( \ell(N(c, H)) \in E_{\alpha + \delta_1, \beta + \delta_2} \) with:

(5.8)

\[
\|\ell(N(c, H))\|_{\alpha + \delta_1, \beta} \lesssim \|(c, H)\|^2_{\alpha, \beta} + \|(c, H)\|_{\alpha, \beta}^3.
\]

**Proof.** We decompose \( N(c, H) = N_1(c) + N_2(c, H) + C(H) \) where

\[
N_1(c) = c^2 Q(G_0) + c^3 C(G_0),
\]

\[
N_2(c, H) = Q(H) + 2c Q(G_0, H) + 3c^2 C(G_0, G_0, H) + 3c C(G_0, H, H).
\]

Using the nonlinear estimates from Section 7, we can estimate the different terms.

By Proposition 7.5, we conclude that \( N_1(c) \in E_{7/6, 5/3} \) with

(5.9)

\[
\|N_1(c)\|_{7/6, 5/3} \lesssim |c|^2 + |c|^3.
\]

We now estimate all terms in \( N_2 \). We apply Corollary 7.4 and obtain using 0 < \( \delta_1, \delta_2 < 1/12 \) that \( Q(H) \in E_{\alpha_1, \beta_1}', Q(G_0, H) \in E_{\alpha_2, \beta_2}', C(G_0, G_0, H) \in E_{\alpha_3, \beta_3}' \) and \( C(G_0, H, H) \in E_{\alpha_4, \beta_4}' \) with:

\[
\begin{align*}
\alpha_1 &= \frac{3}{2} - 2\delta_1, & \beta_1' &= \frac{3}{2} + 2\delta_2, & \|Q(H)\|_{\alpha_1', \beta_1'} & \lesssim \|H\|_{\alpha, \beta}^2, \\
\alpha_2 &= \frac{4}{3} - \delta_1, & \beta_2' &= \frac{3}{2} + \delta_2, & \|Q(G_0, H)\|_{\alpha_2, \beta_2} & \lesssim \|H\|_{\alpha, \beta}, \\
\alpha_3 &= \frac{7}{6} - \delta_1, & \beta_3' &= \frac{3}{2} + \delta_2, & \|C(G_0, G_0, H)\|_{\alpha_3', \beta_3'} & \lesssim \|H\|_{\alpha, \beta}, \\
\alpha_4 &= \frac{4}{3} - \delta_1, & \beta_4' &= \frac{3}{2} + 2\delta_2, & \|C(G_0, H, H)\|_{\alpha_4', \beta_4'} & \lesssim \|H\|_{\alpha, \beta}^2.
\end{align*}
\]
Combining, we obtain \( N_2(c, H) \in E^{3/2-2\delta, 3/2+\delta}_1 \) with
\[
\| N_2(c, H) \|_{E^{3/2-2\delta, 3/2+\delta}_1} \lesssim \| H \|_{(\alpha, \beta)} (|c| + c^2 + \| H \|_{(\alpha, \beta)} + |c| \| H \|_{(\alpha, \beta)})
\]
We finally apply Corollary 7.3 using \( 0 < \delta_1, \delta_2 < 1/12 \) that \( C(H) \in E^{\alpha_5, \beta_5}_0 \) with
\[
\alpha_5' = \frac{3}{2} - 3\delta_1, \quad \beta_5' = \frac{3}{2} + 3\delta_2, \quad \| C(H) \|_{(\alpha_5', \beta_5')} \lesssim \| H \|_{(\alpha, \beta)}^{3}.
\]
The three estimates (5.9), (5.10) and (5.11) imply
\[
\| N(c, H) \|_{E^{3/2-2\delta, 3/2+\delta}_1} \lesssim \| (c, H) \|^2_{(\alpha, \beta)} + \| (c, H) \|^3_{(\alpha, \beta)}.
\]
We finally apply Proposition 5.3 and get \( \ell (N(c, H)) \in E^{7/6-2\delta, 7/6+\delta}_1 \) with the desired estimate (5.8).

We can now end the proof of our main theorem.

Proof of Theorem 1.1. Consider the map
\[\Theta : \mathbb{R} \times E^{7/6-\delta, 7/6+\delta}_1 \to \mathbb{R} \times E^{7/6-\delta, 7/6+\delta}_1 \]
\[\ell (\Phi) + \ell (N(c, H)) \]
Combining Proposition 5.3, Proposition 5.4 and the estimate (5.2) shows that for all \( 0 < \delta < 1/12 \), there exist \( \epsilon^* > 0 \) and \( K > 0 \) such that, for all \( 0 < \epsilon \leq \epsilon^* \), the map \( \Theta \) is a contraction on the ball of \( \mathbb{R} \times E^{7/6-\delta, 7/6+\delta} \) of radius \( K \epsilon \). By the Banach fixed-point theorem, there exists a fixed point \( (c^*, H^*) \). We define accordingly
\[
f = f + g, \quad g = c^* G_0 + H^*,
\]
so that
\[
f(\omega) = \begin{cases} 
\omega^{-7/6} + O(\epsilon \omega^{-7/6+\delta}) & \text{for } \omega \leq 1, \\
(1 + c^*) \omega^{-7/6} (1 + O(\epsilon \omega^{-\delta})) & \text{for } \omega > 1.
\end{cases}
\]
We have by definition of \( \Theta \) that \( f \) solves the equation on \((0, \infty)\):
\[
\forall \omega > 0, \quad C(f)(\omega) = \phi(\omega).
\]
The estimate (5.13) is the desired estimate (1.7) of the theorem. The remaining identities (1.8), (1.9), (1.10) and (1.11) are all consequences of Proposition 4.5. This ends the proof of Theorem 1.1 in the case \( j_0^M = j_M^\infty \) i.e. \( j_M^\infty = j_M^* - \int \sqrt{\omega} \phi \, d\omega \), and the general case \( j_M^\infty \neq j_M^* - \int \sqrt{\omega} \phi \, d\omega \) follows by scaling invariance.

6. The linearized problem around \( \omega^{-7/6} \)

In this section we prove Proposition 5.3. The kernel \( k \) of the linearised operator in (5.4) is an analytic function on \((0, 1) \cup (1, \infty)\) with bounds
- \( |k(\lambda)| \lesssim \lambda^{1/3} \) for \( \lambda \ll 1 \)
- \( |k(\lambda)| \lesssim |\lambda - 1|^{-5/6} \) for \( \lambda \to 1 \)
- \( |k(\lambda)| \lesssim 1 \) for \( \lambda \to \infty \)
This follows directly from the definition of \( k \) as written in [15, Appendix A]. Note that [15] obtain further decay, but these more direct estimates are sufficient for us.

In terms of the new variable \( x \), this translates into the fact that
\[
|K(x)| \lesssim e^{-x} \quad \text{as } x \to -\infty \quad \text{and} \quad |K(x)| \lesssim e^{-4x/3} \quad \text{as } x \to \infty.
\]
Hence for any \( b \in (1, 3/2) \) we find that
\[
\int_{x = -\infty}^{\infty} |K(x)| e^{bx} \, dx < \infty.
\]
For the Fourier transform with the convention
\[ \hat{K}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} K(x) \, dx \]
we therefore find that it is bounded and analytic in the strip
\[ S_{\delta_1,\delta_2} = \{ z \in \mathbb{C} : \Re z \in [7/6 - \delta_1, 7/6 + \delta_2] \} \]
for \( 0 < \delta_1, \delta_2 < 1/6 \). Moreover, by a variant of the Riemann-Lebesgue Lemma, see e.g. [23, Theorem 2.8, Chapter 2], \( \hat{K}(\xi) \to 0 \) as \( |\Re \xi| \to \infty \) over \( S_{\delta_1,\delta_2} \).

To solve the linear problem
\[ (a - \hat{K})(\xi) G(\xi) = \Psi(\xi) \]
yielding the spectral condition that \( a - \hat{K} \neq 0 \). Due to the analytic structure of \( a - \hat{K} \), it is straightforward to numerically verify the spectral condition by the argument principle on the strip \( S_{\delta_1,\delta_2} \) and as shown in [15] the only zero in \( S_{\delta_1,\delta_2} \) for \( 0 < \delta_1, \delta_2 < 1/6 \) is at \( \xi = 7i/6 \), is a simple root, and corresponds to the stationary solution \( \hat{S}_L \).

For \( b \in (1, 3/2) \) with \( (a - \hat{K})(x + ib) \neq 0 \) for all \( x \in \mathbb{R} \), the bound (6.1) implies that the Paley-Wiener theorem [23, Chapter 2] (as used often for Volterra equations) yields a resolvent \( R_b \) with
\[ \int_{-\infty}^{\infty} |R_b(x)| e^{bx} \, dx < \infty \]
such that, if \( 1 \leq p \leq \infty \) and \( \Psi \in L^p(\mathbb{R}, e^{bx} \, dx) \), then the problem (6.2) has a solution \( G \in L^p(\mathbb{R}, e^{bx} \, dx) \) given by
\[ G(x) = \frac{1}{a}(\Psi(x) + R_b \ast \Psi(x)). \]

Given \( 0 < \delta_1, \delta_2 < 1/6 \), we can therefore find resolvents \( R_- \) and \( R_+ \) for the linear problem (6.2) with the bounds
\[ \int_{-\infty}^{\infty} |R_-(x)| e^{(7/6-\delta_1)x} \, dx < \infty \quad \text{and} \quad \int_{-\infty}^{\infty} |R_+(x)| e^{(7/6+\delta_2)x} \, dx < \infty. \]
By the integrability and as \( R_- \) solves the linear problem, it can be expressed through Fourier as
\[ R_-(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iz(x+ib_-)} \frac{\hat{K}}{a - \hat{K}}(z + ib_-) \, dz \quad \text{for } b_- = 7/6 - \delta_1. \]
Likewise for \( R_+ \) we find
\[ R_+(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iz(x+ib_+)} \frac{\hat{K}}{a - \hat{K}}(z + ib_+) \, dz \quad \text{for } b_+ = 7/6 + \delta_2. \]

As \( \hat{K} \) is analytic and bounded in the strip \( S_{\delta_1,\delta_2} \) with decay as \( |\Re z| \to \infty \) and \( a - \hat{K} \) has only a simple root in the strip, we find, similar to [23, Theorem 2.1, Chapter 7], for a constant \( c_1 \) (from the residue) that
\[ R_-(x) = R_+(x) + ac_1 e^{-7x/6}. \]
Hence we can write the solution to the linear problem (6.2) as
\[ G(x) = \frac{\Psi(x)}{a} + \frac{1}{a} \int_{y<0} R_-(y)\Psi(x-y) \, dy + \frac{1}{a} \int_{y>0} R_-(y)\Psi(x-y) \, dy \]
\[ = \frac{\Psi(x)}{a} + \frac{1}{a} \int_{y<0} R_-(y)\Psi(x-y) \, dy + \frac{1}{a} \int_{y>0} R_+(y)\Psi(x-y) \, dy + \int_{y>0} c_1 e^{-7y/6}\Psi(x-y) \, dy. \]
The first three terms in the final expression satisfy the desired decay for the remainder, i.e. each define a bounded linear operator from $E_{7/6-\delta,7/6+\delta}$ onto itself. The leading order comes from the last integral which we express as

$$\int_{y>0} c_1 e^{-7y/6} \Psi(x-y) \, dy = c_1 \chi(x) e^{-7x/6} \int_{y \in \mathbb{R}} \Psi(x-y) e^{7(x-y)/6} \, dy$$

$$- c_1 \chi(x) e^{-7x/6} \int_{y<0} \Psi(x-y) e^{7(x-y)/6} \, dy$$

$$+ (1-\chi(x)) c_1 e^{-7x/6} \int_{y>0} \Psi(x-y) e^{7(x-y)/6} \, dy.$$

The first term is the sought leading order while the last two terms can be included in $\ell_1$ of Proposition 5.3.

This shows the claimed splitting of Proposition 5.3.

Remark 6.1 (Value of $c_1$). We are going to use Theorem 1.1 and its proof, which are actually valid without knowing the exact value of $c_1$, to determine it. Indeed, pick any $\Phi \in C_0^\infty((0, \infty))$ with $M(\Phi) = 1$ and for $\epsilon > 0$ let $\Phi_\epsilon = \epsilon \Phi$ and

$$(6.3) \quad j_{M,\epsilon}^\infty = 1 - \epsilon.$$

For $\epsilon$ small, applying Theorem 1.1, using the fixed-point relation (5.12), we get a solution of the form $f = \tilde{f} + c_\epsilon^* G_0 + H_\epsilon^*$ with

$$(6.4) \quad |c_\epsilon^*| + \|H_\epsilon^*\|_{7/6-\delta,7/6+\delta} \lesssim \epsilon$$

for some $\delta^* > 0$ independent of $\epsilon$, and with $j_M^0 = 1$ by mass balance (1.8). Still by the fixed-point relation (5.12), using (5.7), we obtain $c_\epsilon^* = \ell_0(\Phi_\epsilon + N(c_\epsilon^*, H_\epsilon^*))$. Using Proposition 5.3, Proposition 5.4 and (6.4) this gives $c_\epsilon^* = c_1 \epsilon + O(\epsilon^2)$. Injecting this identity in the flux identity (4.21) shows

$$(6.5) \quad j_{M,\epsilon}^\infty = j_M^* (1 + c_\epsilon^*)^3 = j_M^* + 3j_M^* c_1 \epsilon + O(\epsilon^2).$$

The identities (6.3) and (6.5) impose that $c_1 = -\frac{1}{3j_M^*}$ as desired.

7. Nonlinear estimates

7.1. Estimates on the trilinear operator $\tilde{C}$. Recall the trilinear operator $\tilde{C}$ from (2.1). In the definition, we estimate the absolute value of the integral as

$$(7.1) \quad |\tilde{C}(f,g,h)| \leq \tilde{C}(f,g,h) = 2 \int_{0<\omega_3<\omega_4,0<\omega_2} W[(f_1 + f_2)g_3 h_4 + (g_3 + g_4)h_1 f_2] \, d\omega_3 \, d\omega_4.$$

The different estimates are then deduced from the following lemma.

Lemma 7.1. Suppose that $f \in E_{\alpha_f,\beta_f}$, $g \in E_{\alpha_g,\beta_g}$ and $h \in E_{\alpha_h,\beta_h}$ for $0 \leq \alpha_f, \alpha_g, \alpha_h < 5/4$ with $\alpha_f, \alpha_g, \alpha_h \neq 1$ and $\beta_f, \beta_g, \beta_h > 1$ with $\beta_f + \beta_g \neq 5/2$. Then for the absolute value of the collision integral $\tilde{C}$ we have

$$\|\tilde{C}(f,g,h)\|_{\alpha',\beta'} \lesssim \|f\|_{\alpha_f,\beta_f} \|g\|_{\alpha_g,\beta_g} \|h\|_{\alpha_h,\beta_h}$$

for

$$\alpha' = \max(\alpha_f, \alpha_g, \alpha_f + \alpha_g + \alpha_h - 2, \alpha_f + \alpha_g - 1, \alpha_h + \alpha_g - 1)$$

and

$$\beta' = \min\left(\beta_f + \beta_g + \beta_h - 2, \beta_f + 1/2\right).$$

Remark 7.2. The restrictions $\alpha_f, \alpha_g, \alpha_h \neq 1$ and $\beta_f + \beta_g \neq 5/2$ prevent logarithmic losses.
Proof. It suffices to show that \( \| \tilde{C}(f, g, h) \|_{\alpha', \beta'} \lesssim 1 \) when \( f \leq \rho_{\alpha_f, \beta_f}, g \leq \rho_{\alpha_g, \beta_g} \) and \( h \leq \rho_{\alpha_h, \beta_h} \). We then estimate the different parts of the integral depending on whether \( \omega_1, \omega_2 \) or \( \omega_3 \) is minimal.

\( \omega_1 \) minimal: In this case we find
\[
|\tilde{C}_1| \lesssim \int_{\omega_3=\omega_1}^{\infty} \int_{\omega_4=\omega_3}^{\infty} \rho_{f,1} \rho_{g,3} \rho_{h,4} + \int_{\omega_3=\omega_1}^{\infty} \int_{\omega_2=2\omega_1-\omega_3}^{\infty} \rho_{h,1} \rho_{g,3} \rho_{f,2}.
\]

For \( \omega_1 \geq 1 \) we thus find
\[
|\tilde{C}_1| \lesssim \omega_1^{2-\beta_f-\beta_g-\beta_h} \lesssim \omega_1^{-\beta'}.
\]

For \( \omega_1 \leq 1 \) we find
\[
|\tilde{C}_1| \lesssim \omega_1^{-\alpha_f} \left( 1 + \int_{\omega_3=\omega_1}^{1} \left( 1 + \int_{\omega_4=\omega_3}^{1} \rho_{h,4} \, d\omega_4 \right) \rho_{g,3} \, d\omega_3 \right)
\]
\[
+ \omega_1^{-\alpha_h} \left( 1 + \int_{\omega_3=\omega_1}^{1} \left( 1 + \int_{\omega_2=\omega_3}^{1} \rho_{f,2} \, d\omega_4 \right) \rho_{g,3} \, d\omega_3 \right)
\]
\[
\lesssim \omega_1^{-\alpha_f} + \omega_1^{-\alpha_h} + \omega_1^{1-\alpha_f-\alpha_g} + \omega_1^{1-\alpha_g-\alpha_h} + \omega_1^{2-\alpha_f-\alpha_g-\alpha_h} \lesssim \omega_1^{-\alpha'}.
\]

\( \omega_2 \) minimal: Here we find
\[
|\tilde{C}_2| \lesssim \int_{\omega_4=0}^{\omega_1} \int_{\omega_3=\omega_2}^{(\omega_1+\omega_2)/2} \sqrt{\omega_2 \omega_1} \left[ \rho_{f,2} \rho_{g,3} \rho_{h,4} + \rho_{g,3} \rho_{h,1} \rho_{f,2} \right].
\]

In this case \( \omega_4 \geq \omega_1/2 \) so that
\[
|\tilde{C}_2| \lesssim \rho_{h,1} \int_{\omega_2=0}^{\omega_1} \int_{\omega_3=\omega_2}^{(\omega_1+\omega_2)/2} \sqrt{\omega_2 \omega_1} \rho_{g,3} \rho_{f,2}.
\]

For \( \omega_1 \leq 1 \) this shows
\[
|\tilde{C}_2| \lesssim \omega_1^{2-\alpha_f-\alpha_g-\alpha_h} \lesssim \omega_1^{-\alpha'}.
\]

For \( \omega_1 \geq 1 \) we find
\[
|\tilde{C}_2| \lesssim \omega_1^{-\beta_h-\frac{1}{2}} \int_{\omega_2=0}^{\omega_1} \sqrt{\omega_2 \rho_{f,2} (\mathbb{1}_{\omega_2 \leq 1} (1 + \omega_2^{1-\alpha_g}) + \mathbb{1}_{\omega_2 \geq 1} \omega_2^{1-\beta_g})} \, d\omega_2 \lesssim \omega_1^{-\beta_h-\frac{1}{2}} + \omega_1^{2-\beta_f-\beta_g-\beta_h} \lesssim \omega_1^{-\beta'}.
\]

\( \omega_3 \) minimal: Here we find
\[
|\tilde{C}_3| \lesssim \int_{0<\omega_3<\omega_1, \omega_2=\omega_4} \sqrt{\omega_3 \omega_1} \left[ (\rho_{f,1} + \rho_{f,2}) \rho_{g,3} \rho_{h,4} + \rho_{g,3} \rho_{h,1} \rho_{f,2} \right].
\]

In this case \( \omega_4 \geq \omega_1 \) so that
\[
|\tilde{C}_3| \lesssim \int_{\omega_3=0}^{\omega_1} \int_{\omega_4=\omega_3}^{\infty} \sqrt{\omega_3 \omega_1} \rho_{f,1} \rho_{g,3} \rho_{h,4} + \int_{\omega_3=0}^{\omega_1} \int_{\omega_2=\omega_3}^{\infty} \sqrt{\omega_3 \omega_1} \rho_{h,1} \rho_{g,3} \rho_{f,2}.
\]

Hence we find for \( \omega_1 \leq 1 \) that
\[
|\tilde{C}_3| \lesssim \omega_1^{-\alpha_f-\frac{1}{2}} (1 + \omega_1^{1-\alpha_h}) \int_{\omega_3=0}^{\omega_1} \omega_3^{-\alpha_g} \, d\omega_3 + \omega_1^{-\alpha_h-\frac{1}{2}} \int_{\omega_3=0}^{\omega_1} \omega_3^{-\alpha_g} (1 + \omega_3^{1-\alpha_f}) \, d\omega_3 \lesssim \omega_1^{-\alpha'}.
\]

For \( \omega_1 \geq 1 \) we find that
\[
|\tilde{C}_3| \lesssim \omega_1^{-\beta_f-\frac{1}{2}} \omega_1^{1-\beta_h} (1 + \omega_1^{\frac{3}{2}-\beta_g}) + \omega_1^{-\beta_h-\frac{1}{2}} (1 + \omega_1^{\frac{5}{2}-\beta_f-\beta_g}) \lesssim \omega_1^{-\beta'}.
\]

Hence we have found the required bound for all parts of the integral.
7.2. Nonlinear estimates on the collision operator. We can deduce the immediate corollary for the collision operator.

**Corollary 7.3.** Let $0 \leq \alpha < 5/4$ and $\beta > 1$ with $\alpha \neq 1$ and $\beta \neq 5/4$. Then the collision operator $C$ is bounded from $E_{\alpha, \beta}$ to $E_{\alpha', \beta'}$, where

$$\alpha' = \alpha + \max(0, 2\alpha - 2) \quad \text{and} \quad \beta' = \beta + \min(2\beta - 2, \frac{1}{2}).$$

For $\alpha < 1$ there is no loss at small $\omega$ as in this case $\alpha' = \alpha$.

On the remaining quadratic and cubic terms of the perturbation around $f$ we find the following direct corollary.

**Corollary 7.4.** Let $0 \leq \alpha < 5/4$ and $\beta > 1$ with $\alpha \neq 1$ and $\beta \notin \{5/4, 4/3\}$. Then one has the collection of estimates for $H \in E_{\alpha, \beta}$:

\[
(7.2) \quad \|Q(H, H)\|_{\alpha_1', \beta_1'} \lesssim \|H\|^2_{\alpha_2, \beta_2}, \quad \alpha_1' = \max\left(2\alpha - \frac{5}{6}, \frac{7}{6}\right), \quad \beta_1' = \min\left(\frac{5}{3}, \beta + 1, \frac{2\beta - 5}{6}\right),
\]

\[
(7.3) \quad \|Q(G_0, H)\|_{\alpha_2', \beta_2'} \lesssim \|H\|_{\alpha_3, \beta_3}, \quad \alpha_2' = \max\left(\alpha + \frac{1}{6}, \frac{7}{6}\right), \quad \beta_2' = \min\left(\frac{5}{3}, \beta + 1\right),
\]

\[
(7.4) \quad \|C(G_0, H)\|_{\alpha_3', \beta_3'} \lesssim \|H\|_{\alpha_4, \beta_4}, \quad \alpha_3' = \alpha, \quad \beta_3' = \min\left(\frac{5}{3}, \beta + 1\right),
\]

\[
(7.5) \quad \|C(G_0, H)\|_{\alpha_4', \beta_4'} \lesssim \|H\|^2_{\alpha_5, \beta_5}, \quad \alpha_4' = \max(\alpha, 2\alpha - 1), \quad \beta_4' = \min\left(\frac{5}{3}, \beta + 1, 2\beta - \frac{5}{6}\right).
\]

**Proof.** Recall the identity (2.2). The first inequality (7.2) is obtained by applying Lemma 7.1 to $(f, g, h) \in \{(f, H, H), (H, f, H), (H, H, f)\}$, and with $(\alpha_f, \alpha_h, \beta_h)$ and $(\beta_f, \beta_g, \beta_h)$ the corresponding permutations of $(7/6, \alpha, \alpha)$ and $(7/6, \beta, \beta)$. The remaining estimates are obtained similarly, using in addition that $G_0 \in E_{0, 7/6}$. \hfill \qed

7.3. Nonlinear estimates involving $g_0$.

**Proposition 7.5.** The function $C(g_0)$ belongs to $E_{0, 5/3}$ and the function $C(f, g_0, g_0)$ belongs to $E_{7/6, 5/3}$.

**Proof.** We first deal with $C(g_0)$. Notice that $g_0 \in E_{0, 7/6}$, so that by applying Corollary 7.3 we get

$$C(g_0) \in E_{0, 3/2}.$$

This shows that $C(g_0)$ stays uniformly bounded for $\omega$ small. To deal with large frequencies $\omega \gg 1$, we let

$$g_1 = f - g_0 \quad \text{so that} \quad \text{supp} \ g_1 = [0, c], \text{ for some } c > 0$$

and we rely on the identity $C(f) = 0$, whose proof is recalled in Section 7.4 to write

$$C(g_0) = -C(g_1) - 3C(g_0, g_1, g_1) - 3C(g_0, g_0, g_1).$$

Above, we have $g_1 \in E_{7/6, M}$ where $M > 0$ is any arbitrarily large constant, so that applying Corollary 7.3 to $g_1$, and then the estimates (7.4) and (7.5) we obtain

$$C(g_1) \in E_{3/2, M}, \quad C(g_0, g_1, g_1) \in E_{3/3, 5/3}, \quad C(g_0, g_0, g_1) \in E_{7/6, 5/3}.$$

Injecting (7.8) in (7.7) shows $C(g_0) \in E_{3/2, 5/3}$. Combining with (7.6) we obtain $C(g_0) \in E_{0, 3/2} \cap E_{3/2, 5/3} = E_{0, 5/3}$. This is the first estimate of the proposition.

We now deal with $C(f, g_0, g_0)$ in a similar way. As $g_0 \in E_{0, 7/6}$, applying estimate (7.2) gives

$$C(f, g_0, g_0) \in E_{7/6, 3/2}.$$

(7.9)
Again, this is sufficient for \( \omega \) small. For \( \omega \) large we once more use the identity \( C(f) = 0 \) to produce
\[
C(f, g_0, g_0) = -2C(f, g_0, g_1) - C(f, g_1, g_1).
\]
We apply (7.3) and (7.2) and get
\[
(\ref{eq:7.11}) \quad \text{for all} \quad C \in E_{4/3, 5/3}, \quad C(f, g_1, g_1) \in E_{3/2, 5/3}.
\]
Injecting (7.11) in (7.10) shows \( C(f, g_0, g_0) \in E_{3/2, 5/3} \). Combined with (7.9) this shows \( C(f, g_0, g_0) \in E_{7/6, 5/3} \). This is the second estimate and ends the proof. \( \square \)

7.4. The basic cancellation. The KZ spectrum \( f = \omega^{-7/6} \) is a stationary solution of (1.1). We give here a proof relying solely on the scaling invariance of the equation and on the locality of the integrals of collision and mass flux (in the spirit of the dimensional analysis argument of [41] and [18, Prop. 2.41]), before recalling the classical one.

Lemma 7.6. For all \( \omega > 0 \) one has \( C(f)(\omega) = 0 \).

Proof. By Proposition 3.2 and Lemma 4.1, we have for all \( \omega > 0 \) that \( C(f)(\omega) \) as well as \( J_M(f)(\omega) \) are well-defined, i.e. by convergent integrals. By rescaling the variables in the integrals in (4.3), we get that \( J(f)(\omega) = J(f)(1) \) is actually independent of \( \omega > 0 \). Hence by differentiating, \( \partial_\omega J(f) = 0 \). Thus (4.2) shows that \( \sqrt{\omega} C(f)(\omega) = 0 \) almost everywhere. \( \square \)

We recall now the classical conformal change of coordinates due to Zakharov [39], which shows that \( \omega^{-7/6} \) and \( \omega^{-3/2} \) are solutions of \( C(f) = 0 \). In the former case, the spectrum is local (Proposition 3.2), and the computation is rigorously justified; in the latter case, it is merely formal.

We assume for the solution the form \( f(\omega) = \omega^\alpha \) so that the collision operator (1.2) reads
\[
C(f)(\omega) = \int_{\omega_2, \omega_3, \omega_4 \geq 0} W_1 \omega_1^\alpha \omega_2^\alpha \omega_3^\alpha \omega_4^\alpha \left[ \omega_1^{-\alpha} + \omega_2^{-\alpha} - \omega_3^{-\alpha} - \omega_4^{-\alpha} \right] \, d\omega_3 \, d\omega_4.
\]
Then we split the domain of the integral defining the collision operator (1.2) into four subdomains as
\[
C(f) = \int_{\omega_2, \omega_3, \omega_4 \geq 0} \, d\omega_3 \, d\omega_4 = \sum_{i=1}^4 \int_{\Delta_i} \, d\omega_3 \, d\omega_4,
\]
where
- \( \Delta_1 = \{(\omega_3, \omega_4) \text{ such that } \omega_3 + \omega_4 \geq \omega_1, \ 0 \leq \omega_3 \leq \omega_1, \ 0 \leq \omega_4 \leq \omega_1 \} \)
- \( \Delta_2 = \{(\omega_3, \omega_4) \text{ such that } \omega_3 \geq \omega_1, \ 0 \leq \omega_4 \leq \omega_1 \} \)
- \( \Delta_3 = \{(\omega_3, \omega_4) \text{ such that } \omega_3 \geq \omega_1, \ \omega_4 \geq \omega_1 \} \)
- \( \Delta_4 = \{(\omega_3, \omega_4) \text{ such that } 0 \leq \omega_3 \leq \omega_1, \ \omega_4 \geq \omega_1 \} \)

which is illustrated in Fig. 1.

The region \( \Delta_2 \) can be mapped to \( \Delta_1 \) by the the change of variables
\[
(\omega_1, \omega_2, \omega_3, \omega_4) = \frac{\omega_1}{\omega_3} (\omega_3, \omega_4, \omega_1, \omega_2).
\]
The Jacobian is
\[
\det \frac{\partial (\omega_2', \omega_4')}{\partial (\omega_3, \omega_4)} = -\frac{\omega_3^3}{\omega_1^3}
\]
and the new coordinates satisfy the conservation of momentum as \( \omega_1 + \omega_2' = \omega_3 + \omega_4' \). By applying this change of variables and omitting the primes, we thus find
\[
\int_{\Delta_2} W_1 \omega_1^\alpha \omega_2^\alpha \omega_3^\alpha \omega_4^\alpha \left[ \omega_1^{-\alpha} + \omega_2^{-\alpha} - \omega_3^{-\alpha} - \omega_4^{-\alpha} \right] \, d\omega_3 \, d\omega_4
\]
\[
= -\int_{\Delta_1} W_1 \omega_1^\alpha \omega_2^\alpha \omega_3^\alpha \omega_4^\alpha \left[ \omega_1^{-\alpha} + \omega_2^{-\alpha} - \omega_3^{-\alpha} - \omega_4^{-\alpha} \right] \left( \frac{\omega_1}{\omega_3} \right) ^{3\alpha + \frac{2}{3}} \, d\omega_3 \, d\omega_4.
\]
Figure 1. Illustration of the regions $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$ in the collision integral for $(\omega_3, \omega_4)$. Through an appropriate change of variables they will all map onto $\Delta_1$ where they can cancel.

The region $\Delta_3$ can be mapped to $\Delta_1$ by the change of variables

$$(\omega_1, \omega_2, \omega'_3, \omega'_4) = \frac{\omega_1}{\omega_2} (\omega_2, \omega_1, \omega_3, \omega_4).$$

The Jacobian is

$$\det \frac{\partial (\omega'_3, \omega'_4)}{\partial (\omega_3, \omega_4)} = -\frac{\omega_3^2}{\omega_1^3}$$

and the new coordinates satisfy the conservation of momentum as $\omega_1 + \omega'_2 = \omega_3 + \omega'_4$. By applying this change of variables and omitting the primes, we thus find

$$\int_{\Delta_3} W \omega_1^\alpha \omega_2^\alpha \omega_3^\alpha \omega_4^\alpha [\omega_1^{-\alpha} + \omega_2^{-\alpha} - \omega_3^{-\alpha} - \omega_4^{-\alpha}] \, d\omega_3 \, d\omega_4$$

$$= \int_{\Delta_1} W \omega_1^\alpha \omega_2^\alpha \omega_3^\alpha \omega_4^\alpha [\omega_1^{-\alpha} + \omega_2^{-\alpha} - \omega_3^{-\alpha} - \omega_4^{-\alpha}] \left( \frac{\omega_1}{\omega_2} \right)^{3\alpha + \frac{7}{2}} \, d\omega_3 \, d\omega_4.$$

The integral over $\Delta_4$ is symmetric to $\Delta_2$. Hence collecting the mappings we obtain that

$$C(f) = \int_{\Delta_1} W \omega_1^\alpha \omega_2^\alpha \omega_3^\alpha \omega_4^\alpha [\omega_1^{-\alpha} + \omega_2^{-\alpha} - \omega_3^{-\alpha} - \omega_4^{-\alpha}]$$

$$\left[ 1 + \left( \frac{\omega_1}{\omega_2} \right)^{3\alpha + \frac{7}{2}} - \left( \frac{\omega_1}{\omega_3} \right)^{3\alpha + \frac{7}{2}} - \left( \frac{\omega_1}{\omega_4} \right)^{3\alpha + \frac{7}{2}} \right] \, d\omega_3 \, d\omega_4.$$

From this expression, one can immediately read off the values of $\alpha$ for which $k^{-\alpha}$ is a (formal) stationary solution:

- Either $-\alpha = 0$ or $1$, which gives the RJ solutions $\alpha = 0$ and $\alpha = -1$.
- Or $-3\alpha - \frac{7}{2} = 0$ or $-3\alpha - \frac{7}{2} = -1$, which gives the KZ solution $\alpha = -\frac{7}{6}$ and $\alpha = -\frac{3}{2}$.

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