Nonexistence of the non–Gaussian fixed point predicted by the RG field theory in $4 - \epsilon$ dimensions

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Abstract. The Ginzburg-Landau phase transition model is considered in $d = 4 - \epsilon$ dimensions within the renormalization group (RG) approach. The problem of existence of the non-Gaussian fixed point is discussed. An equation is derived from the first principles of the RG theory (under the assumption that the fixed point exists) for calculation of the correction-to-scaling term in the asymptotic expansion of the two-point correlation (Green’s) function. It is demonstrated clearly that, within the framework of the standard methods (well justified in the vicinity of the fixed point) used in the perturbative RG theory, this equation leads to an unremovable contradiction with the known RG results. Thus, in its very basics, the RG field theory in $4 - \epsilon$ dimensions is contradictory. To avoid the contradiction, we conclude that such a non-Gaussian fixed point, as predicted by the RG field theory, does not exist. Our consideration does not exclude existence of a different fixed point.

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1. Introduction

Since the famous work of K.G. Wilson and M.E. Fisher [1] the renormalization group (RG) field theory has been widely used in calculations of critical exponents [4, 8]. The basic hypothesis of this theory is the existence of a certain fixed point for the RG transformation. However, the existence of such a stable fixed point for the Ginzburg–Landau model (which lies in the basis of the field theory) has not been proven mathematically in the case of the spatial dimensionality $d < 4$. The fact that such a fixed point can be found within a scheme of self–consistent approximations, assuming its existence at the very beginning, cannot be regarded as a mathematical proof. An attempt to prove the existence of the non–Gaussian fixed point in $4 - \epsilon$ dimensions has been made in Ref. [4]. The authors have considered some rather artificial $\varphi^4$–type model, supposing that it simulates the Ginzburg–Landau model in $4 - \epsilon$ dimensions. The method of proof is to solve the problem for a finite system (of linear size $L$), considering the thermodynamic limit afterwards. However, we cannot, in principle, find the non–Gaussian fixed point at a finite $L$ for a system with real interaction, and then consider $L \to \infty$. The problem is that, due to absence of phase transition at $u > 0$, a stable
fixed point with nonzero coupling constant $u$ of $\varphi^4$ interaction cannot exist in a case of finite system. All such efforts to prove the existence of the non-Gaussian fixed point, predicted in the Ginzburg–Landau model by RG field theory, are futile, since there exists an obviously simple proof of nonexistence presented hereafter.

2. Fundamental equations

We consider the Ginzburg–Landau phase transition model. The Hamiltonian of this model in the Fourier representation reads

$$\frac{H}{T} = \sum_k \left( r_0 + c k^2 \right) |\varphi_k|^2 + uV^{-1} \sum_{k_1,k_2,k_3} \varphi_{k_1}\varphi_{k_2}\varphi_{k_3}\varphi_{-k_1-k_2-k_3},$$

where $\varphi_k = V^{-1/2} \int \varphi(x) \exp(-ikx) dx$ are Fourier components of the scalar order parameter field $\varphi(x)$, $T$ is the temperature, and $V$ is the volume of the system. In the RG field theory $[2,3]$ Hamiltonian (1) is renormalized by integration of $\exp(-H/T)$ over $\varphi_k$ with $\Lambda/s < k < \Lambda$, followed by a certain rescaling procedure providing a Hamiltonian corresponding to the initial values of $V$ and $\Lambda$, where $\Lambda$ is the upper cutoff of the $\varphi^4$ interaction. Due to this procedure, additional terms appear in the Hamiltonian (1), so that in general the renormalized Hamiltonian contains a continuum of parameters. The basic hypothesis of the RG theory in $d < 4$ dimensions is the existence of a non-Gaussian fixed point $\mu = \mu^*$ for the RG transformation $R_s$ defined in the space of Hamiltonian parameters, i.e.,

$$R_s \mu^* = \mu^*.$$

The fixed-point values of the Hamiltonian parameters are marked by an asterisk ($r_0^*$, $c^*$, and $u^*$, in particular). Note that $\mu^*$ is unambiguously defined by fixing the values of $c^*$ and $\Lambda$. According to the RG theory, the main terms in the renormalized Hamiltonian in $d = 4 - \epsilon$ dimensions are those contained in (1) with $r_0^*$ and $u^*$ of the order $\epsilon$, whereas the additional terms are small corrections of order $\epsilon^2$.

Consider the Fourier transform $G(k,\mu)$ of the two-point correlation (Green’s) function, corresponding to a point $\mu$. Under the RG transformation $R_s$ this function transforms as follows $[2]$

$$G(k,\mu) = s^{2-\eta} G(sk, R_s \mu).$$

Let $G(k,\mu) \equiv G(k,\mu)$ (at $k \neq 0$ and $V \to \infty$) be defined within $k \leq \Lambda$. Since Eq. (3) holds for any $s > 1$, we can set $s = \Lambda/k$, which at $\mu = \mu^*$ yields

$$G(k,\mu^*) = a k^{-2+\eta} \text{ for } k < \Lambda,$$

where $a = \Lambda^{2-\eta} G(\Lambda,\mu^*)$ is the amplitude and $\eta$ is the universal critical exponent. According to the universality hypothesis, the infrared behavior of the Green’s function is described by the same universal value of $\eta$ at any $\mu$ on the critical surface (with the only requirement that all parameters of Hamiltonian (1) are present), i.e.,

$$G(k,\mu) = b(\mu) k^{-2+\eta} \text{ at } k \to 0,$$
where
\[ b(\mu) = \lim_{k \to 0} k^{2-\eta} G(k, \mu) . \] (6)
According to Eq. (3), which holds for any \( s = s(k) > 1 \), Eq. (6) reduces to
\[ b(\mu) = \lim_{k \to 0} k^{2-\eta} s(k)^{2-\eta} G(sk, R_{s}\mu) . \] (7)
By setting \( s(k) = \Lambda/k \), we obtain
\[ b(\mu) = \Lambda^{2-\eta} \lim_{k \to 0} G(\Lambda, R_{\Lambda/k}\mu) = \Lambda^{2-\eta} G(\Lambda, \mu^*) = a , \] (8)
if the fixed point \( \mu^* = \lim_{s \to \infty} R_{s}\mu \) exists. Let us define the function \( X(k, \mu) \) and the self–energy \( \Sigma(k, \mu) \) as follows
\[ X(k, \mu) = k^{-2} G^{-1}(k, \mu) , \] (9)
\[ k^2 X(k, \mu) = 2(r_0 + c k^2) + \Sigma(k, \mu) . \] (10)
Equation (10) is usually used in the perturbation theory, since the self–energy has a suitable representation by Feynman diagrams. According to Eqs. (4), (5), and (8), we have (for \( k < \Lambda \))
\[ X(k, \mu^*) = \frac{1}{a} k^{-\eta} \] (11)
and
\[ X(k, \mu) = \frac{1}{a} k^{-\eta} + \delta X(k, \mu) , \] (12)
where \( \mu \) belongs to the critical surface, \( \mu^* = \lim_{s \to \infty} R_{s}\mu \), and \( \delta X(k, \mu) \) denotes the correction–to–scaling term. From (11) and (12) we obtain the equation
\[ \delta X(k, \mu^* + \delta \mu) = X(k, \mu^* + \delta \mu) - X(k, \mu^*) , \] (13)
where \( \delta \mu = \mu - \mu^* \). This equation, of course, makes sense only if the fixed point \( \mu^* \) exists and \( \mu \) includes all the relevant Hamiltonian parameters to ensure the universal infrared critical behavior (5) of the correlation function.

3. Proof of the nonexistence

On the basis of fundamental equations obtained in the previous section, we prove here the nonexistence of the fixed point predicted by RG field theory, i.e., we assume the existence and derive a contradiction. Since Eq. (3) is true for any small deviation \( \delta \mu \) satisfying the relation
\[ \mu^* = \lim_{s \to \infty} R_{s}(\mu^* + \delta \mu) , \] (14)
we choose \( \delta \mu \) such that \( \mu^* \Rightarrow \mu^* + \delta \mu \) corresponds to the variation of the Hamiltonian parameters \( r_0^* \Rightarrow r_0^* + \delta r_0, c^* \Rightarrow c^* + \delta c, \) and \( u^* \Rightarrow u^* + \epsilon \times \Delta \), where \( \Delta \) is a small constant. The values of \( \delta r_0 \) and \( \delta c \) are choosen to fit the critical surface and to meet the condition (14) at fixed \( c^* = 1 \) and \( \Lambda = 1 \). In particular, quantity \( \delta c \) is found
\[ \delta c = B \epsilon^2 + o(\epsilon^3) , \] (15)
with some (small) coefficient \( B = B(\Delta) \), to compensate the shift in \( c \) of the order \( \epsilon^2 \) due to the renormalization (cf. [2]). The formal \( \epsilon \)-expansion of \( \delta X(k, \mu) \) can be obtained in the usual way from the perturbation theory. In this case Eq. (13) reduces to

\[
\delta X(k, \mu) = 2 \delta c + k^{-2} [\delta \Sigma(k, \mu) - \delta \Sigma(0, \mu)] ,
\]

where \( \delta \Sigma(k, \mu) \) is the variation of self–energy due to the substitution \( \mu^* \Rightarrow \mu^* + \delta \mu \). A simple calculation yields

\[
\delta X(k, \mu) = \epsilon^2 \left[ 2B - 12(2A \Delta + \Delta^2) k^{-2}(I(k) - I(0)) \right] + o(\epsilon^3) ,
\]

where

\[
I(k) = (2\pi)^{-8} \int_{k_1 < 1} d^4k_1 \int_{k_2 < 1} d^4k_2 \times k^{-2} k_1^{-2} k_2^{-2} \left| k - k_1 - k_2 \right|^{-2} \times \theta(1 - \left| k - k_1 - k_2 \right|) \]

(18)

and \( A \) is the expansion coefficient in the \( \epsilon \)-expansion of the renormalized coupling constant \( u^* \), i.e.,

\[
u^* = A \epsilon + o(\epsilon^2) .
\]

The theta function appears in Eq. (18) due to the cutting of the integration region at \( k = \Lambda = 1 \). Term (18) is well known [2]. It behaves like \( \text{const} + k^2 \ln k \) at small \( k \). The expansion coefficient at \( \epsilon^2 \) in Eq. (17) is exact, since uncontrolled parameters of order \( \epsilon^2 \) contained in the renormalized Hamiltonian \( H^* \), which are absent in Eq.(1), give a contribution of order \( \epsilon^3 \) to \( \delta \Sigma(k, \mu) - \delta \Sigma(0, \mu) \). In such a way, at small \( k \) the expansion is unambiguous, i.e.,

\[
\delta X(k, \mu) = \epsilon^2 \left[ C_1(\Delta) + C_2(\Delta) \ln k \right] + o(\epsilon^3) \text{ at } k \to 0 ,
\]

where \( C_1(\Delta) \) and \( C_2(\Delta) \) are coefficients independent on \( \epsilon \).

It is commonly accepted in the RG field theory to make an expansion like (17), obtained from the diagrammatic perturbation theory, to fit an asymptotic expansion at \( k \to 0 \), thus determining the critical exponents. In general, such a method is not rigorous since, obviously, there exist such functions which do not contribute to the asymptotic expansion in \( k \) powers at \( k \to 0 \), but give a contribution to the formal \( \epsilon \)-expansion at any fixed \( k \). Besides, the expansion coefficients do not vanish at \( k \to 0 \). A trivial example of such a function is \( \epsilon^m [1 - \tanh(\epsilon k^{-\epsilon})] \) where \( m \) is integer. Nevertheless, according to the general ideas of the RG theory, in the vicinity of the fixed point the asymptotic expansion

\[
X(k, \mu) = \frac{1}{a} k^{-\eta} + b_1 k^{\epsilon + o(\epsilon^2)} + b_2 k^{2 + o(\epsilon)} + ... \]

is valid not only at \( k \to 0 \), but within \( k < \Lambda \). The latter means that terms of the kind \( \epsilon^m [1 - \tanh(\epsilon k^{-\epsilon})] \) are absent or negligible. In such a way, if there exists a fixed point, then we can obtain correct \( \epsilon \)-expansion of \( \delta X(k, \mu) \) at small \( k \) by expanding the term \( b_1 k^{\epsilon + o(\epsilon^2)} \) (with \( b_1 = b_1(\epsilon, \Delta) \)) in Eq. (21) in \( \epsilon \) powers, and the result must agree with
at small $\Delta$, at least. The latter, however, is impossible since Eq. (20) never agree with

$$\delta X(k, \mu) = b_1(\epsilon, \Delta) \left[ 1 + \epsilon \ln k + o(\epsilon^2) \right]$$

obtained from (21) at $k \to 0$. Thus, we have arrived to an obvious contradiction, which means that the initial assumption about existence of a certain fixed point, predicted by the RG field theory in $4 - \epsilon$ dimensions, is not valid. The only reason why this contradiction has not been detected before, seems, is the fact that Eq.(13) never has been considered in literature. Since the results of the RG field theory in $4 - \epsilon$ dimensions completely are based on the formal $\epsilon$-expansion, the predicted ”fixed point”, obviously, is a set of Hamiltonian parameters at which Eq. (2) is satisfied in the limit $\epsilon \to 0$ at a fixed $s$, but is not satisfied in the limit $s \to \infty$ at a fixed $\epsilon$.

4. Conclusions

We have demonstrated clearly that, in its very basics, the RG field theory in $4 - \epsilon$ dimensions is contradictory. Based on a mathematically correct (according to the general arguments of the RG theory) method, we have shown that such a fixed point in $4 - \epsilon$ dimensions, as predicted by the RG field theory, does not exist. It should be noticed, however, that our consideration does not exclude existence of a different fixed point.

References

[1] K.G. Wilson, M.E. Fisher, *Phys.Rev.Lett.* 28, 240 (1972)
[2] Shang–Keng Ma, *Modern Theory of Critical Phenomena* (W.A. Benjamin, Inc., New York, 1976)
[3] J. Zinn–Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1996)
[4] D. Brydges, J. Dimock, T.R. Hurd, *archived as 96-681 of mp_arc in Texas*