Quantum integrable multi-well tunneling models

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Abstract

In this work we present a general construction of integrable models describing boson tunneling in multi-well systems. We show how the models may be derived through the quantum inverse scattering method and solved by algebraic Bethe ansatz means. From the transfer matrix we find only two conserved operators. However, we construct additional conserved operators through an alternative approach. As a consequence the models admit multiple pseudovacua, each associated with a set of Bethe ansatz equations (BAEs). We show that all sets of BAEs are needed to obtain a complete set of eigenstates.

Keywords: integrability, bethe ansatz, tunneling

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the challenging experimental realization of Bose–Einstein condensates, our understanding about this state of matter has improved in both theoretical and experimental aspects. Nowadays this subject continues to be a focus of intense investigations, with the main aim to understand phenomena that occur at the mesoscopic scale. It is recognized that exactly solvable models allow studies taking into account all the quantum fluctuations that play an important role at this scale, and at ultra low temperature [1]. In this direction the algebraic Bethe ansatz method has been an important tool to build new integrable models. Moreover, this technique was already used to construct a two-mode integrable model (the two-site...
Bose–Hubbard model [2–4] which has been used with success to describe experimental results [5, 6].

Motivated by this, and the following recent developments:

• experimental efforts to investigate two-well systems with two levels in each well to study EPR entanglement [17];
• the theoretical paper A bosonic multi-state two-well model [9] where two solvable models in the sense of algebraic Bethe ansatz method are presented;
• recent discussions about the definition of quantum integrability [7, 8];

in this paper, we revisit [9] to provide a full solution for a class of multi-well tunneling models using the algebraic Bethe ansatz method. These models are defined on complete bipartite graphs $K_{n,m}$. The models are naturally associated with $(n + m)$ modes, and integrability requires $(n + m)$ conserved operators. However the standard algebraic Bethe ansatz method, via the transfer matrix, provides only two of these conserved operators. On the other hand, we show how to obtain the other $(n + m) - 2$ additional independent conserved operators. Another important aspect is that the method allows us to find a set of pseudovacua. All pseudovacua allow us to build a set of Bethe states leading to a complete set of eigenvalues and eigenvectors for the models.

In the next section we present the models and the generalization of the algebraic Bethe ansatz technique. The approach follows the methods of [9], although with different notational conventions.

2. Integrable Hamiltonians

We begin by introducing the Hamiltonian for $(n + m)$ wells in terms of a set of canonical boson operators $a_i, a_i^{\dagger}, N_{a,i} = a_i^{\dagger}a_i, i = 1,...,n$ and another set $b_j, b_j^{\dagger}, N_{b,j} = b_j^{\dagger}b_j, j = 1,...,m$. The Hamiltonian reads

$$H_{n,m} = U(N_A - N_B)^2 + \mu(N_A - N_B) + \sum_{i=1}^{n} \sum_{j=1}^{m} t_{ij}(a_i b_j^{\dagger} + a_i^{\dagger} b_j),$$

where we have defined $A^{\dagger} = \sum_{i=1}^{n} \alpha_i a_i^{\dagger}, B^{\dagger} = \sum_{j=1}^{m} \beta_j b_j^{\dagger}, N_A = \sum_{i=1}^{n} a_i^{\dagger} a_i, N_B = \sum_{j=1}^{m} b_j^{\dagger} b_j$ and $N = N_A + N_B$. Above, the coupling $U$ is the intra-well and inter-well interaction between bosons, $\mu$ is the external potential and $t_{ij} = t \alpha_i \beta_j$ are the constant couplings for the tunneling amplitude. The parameters $\alpha_i, \beta_j$ are real numbers satisfying

$$\sum_{i=1}^{n} \alpha_i^2 = \sum_{j=1}^{m} \beta_j^2 = 1.$$ 

We will show that the above models are integrable in the sense of algebraic Bethe ansatz, but a generalization is needed. As we will show later, we can obtain just two conserved operators from the standard algebraic method through the transfer matrix, but the generalization of the method allows us to identify $(n + m) - 2$ additional constant operators. In this sense we will show that the method presented here is a generalization for the algebraic Bethe ansatz method, and integrability is a consequence of this generalization.

The study of the above integrable Hamiltonians is important, in part, because it generalizes models that have been studied already in the literature. To be more precise: the case $n = m = 1$
is the well-known canonical Josephson Hamiltonian [2–4] which has been a useful model in understanding tunneling phenomena and has been studied in many aspects [3, 10, 11]. The case \( n = m = 2 \), in reference [9], was interpreted as an integrable Hamiltonian with two wells, and two levels in each well. However, it was later shown that the solution presented was not complete [12]. On the other hand, non-integrable variants of these types of models were studied in [13–16]. In reference [13], a case with two greatly different tunneling rates was studied as a model for a mesoscopic quantum system in thermal contact. The quantum dynamics for a range of different initial conditions, in terms of the number of distribution among the wells and the quantum statistics, is presented in [14]. The tunneling dynamics at zero temperature was studied in [15] to investigate possible ways in which to achieve mass transport around a loop and persistent current. In [16] it was pointed out that an appropriate control of short-range and dipolar interaction may lead to novel scenarios for the dynamics of bosons in lattices, including the dynamical creation of mesoscopic quantum superposition, which may be employed in the design of Heisenberg-limited atom interferometers.

Physically one can say that the Hamiltonians (1) describe Josephson tunneling for bosonic systems in multiple \((n + m)\) wells. Besides the apparent simplicity, the models show a rich and beautiful mathematical structure as can be seen in the next sections.

2.1. Some particular Hamiltonians

From the above general Hamiltonian, for particular choices of \(n, m\), we obtain the following integrable models:

2.1.1. Two wells.
\[
H_{1,1} = U(N_{a,1} - N_{b,1})^2 + \mu(N_{a,1} - N_{b,1}) + t_{1,1}(a_1^\dagger b_1 + a_1 b_1^\dagger)
\]  

(2)

2.1.2. Three wells.
\[
H_{2,1} = U(N_{a,1} + N_{a,2} - N_{b,1})^2 + \mu(N_{a,1} + N_{a,2} - N_{b,1})
\]
\[
+ t_{1,1}(a_1^\dagger b_1 + a_1 b_1^\dagger) + t_{2,1}(a_2^\dagger b_1 + a_2 b_1^\dagger)
\]  

(3)
2.1.3. Four wells.

\[ H_{3,2} = U(N_{a,1} + N_{a,2} - N_{b,1} - N_{b,2})^2 + \mu(N_{a,1} + N_{a,2} - N_{b,1} - N_{b,2}) + t_{1,1}(a_{1}b_{1}^{\dagger} + a_{2}^{\dagger}b_{1}) + t_{1,2}(a_{1}b_{2}^{\dagger} + a_{2}^{\dagger}b_{2}) + t_{2,1}(a_{2}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{2,2}(a_{2}b_{2}^{\dagger} + a_{1}^{\dagger}b_{2}) \]

\[ H_{3,1} = U(N_{a,1} + N_{a,2} + N_{a,3} - N_{b,1})^2 + \mu(N_{a,1} + N_{a,2} + N_{a,3} - N_{b,1}) + t_{1,1}(a_{1}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{2,1}(a_{2}b_{1}^{\dagger} + a_{2}^{\dagger}b_{1}) + t_{3,1}(a_{3}b_{1}^{\dagger} + a_{3}^{\dagger}b_{1}) \]  

(4)

2.1.4. Five wells.

\[ H_{3,2} = U(N_{a,1} + N_{a,2} + N_{a,3} - N_{b,1} - N_{b,2})^2 + \mu(N_{a,1} + N_{a,2} + N_{a,3} - N_{b,1} - N_{b,2}) + t_{1,1}(a_{1}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{1,2}(a_{1}b_{2}^{\dagger} + a_{2}^{\dagger}b_{1}) + t_{2,1}(a_{2}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{2,2}(a_{2}b_{2}^{\dagger} + a_{2}^{\dagger}b_{2}) + t_{3,1}(a_{3}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{3,2}(a_{3}b_{2}^{\dagger} + a_{2}^{\dagger}b_{2}) \]

\[ H_{4,1} = U(N_{a,1} + N_{a,2} + N_{a,3} + N_{a,4} - N_{b,1})^2 + \mu(N_{a,1} + N_{a,2} + N_{a,3} + N_{a,4} - N_{b,1}) + t_{1,1}(a_{1}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{2,1}(a_{2}b_{1}^{\dagger} + a_{2}^{\dagger}b_{1}) + t_{3,1}(a_{3}b_{1}^{\dagger} + a_{3}^{\dagger}b_{1}) + t_{4,1}(a_{4}b_{1}^{\dagger} + a_{4}^{\dagger}b_{1}) \]

(7)

2.1.5. Six wells.

\[ H_{3,3} = U(N_{a,1} + N_{a,2} + N_{a,3} - N_{b,1} - N_{b,2} - N_{b,3})^2 + \mu(N_{a,1} + N_{a,2} + N_{a,3} - N_{b,1} - N_{b,2} - N_{b,3}) + t_{1,1}(a_{1}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{1,2}(a_{1}b_{2}^{\dagger} + a_{2}^{\dagger}b_{1}) + t_{1,3}(a_{1}b_{3}^{\dagger} + a_{3}^{\dagger}b_{1}) + t_{2,1}(a_{2}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{2,2}(a_{2}b_{2}^{\dagger} + a_{2}^{\dagger}b_{2}) + t_{2,3}(a_{2}b_{3}^{\dagger} + a_{3}^{\dagger}b_{2}) + t_{3,1}(a_{3}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{3,2}(a_{3}b_{2}^{\dagger} + a_{2}^{\dagger}b_{2}) + t_{3,3}(a_{3}b_{3}^{\dagger} + a_{3}^{\dagger}b_{3}) \]

\[ H_{4,2} = U(N_{a,1} + N_{a,2} + N_{a,3} + N_{a,4} - N_{b,1} - N_{b,2})^2 + \mu(N_{a,1} + N_{a,2} + N_{a,3} + N_{a,4} - N_{b,1} - N_{b,2}) + t_{1,1}(a_{1}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{1,2}(a_{1}b_{2}^{\dagger} + a_{2}^{\dagger}b_{2}) + t_{2,1}(a_{2}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{2,2}(a_{2}b_{2}^{\dagger} + a_{2}^{\dagger}b_{2}) + t_{3,1}(a_{3}b_{1}^{\dagger} + a_{3}^{\dagger}b_{1}) + t_{3,2}(a_{3}b_{2}^{\dagger} + a_{3}^{\dagger}b_{2}) + t_{4,1}(a_{4}b_{1}^{\dagger} + a_{4}^{\dagger}b_{1}) + t_{4,2}(a_{4}b_{2}^{\dagger} + a_{4}^{\dagger}b_{2}). \]

(9)

\[ H_{5,1} = U(N_{a,1} + N_{a,2} + N_{a,3} + N_{a,4} + N_{a,5} - N_{b,1})^2 + \mu(N_{a,1} + N_{a,2} + N_{a,3} + N_{a,4} + N_{a,5} - N_{b,1}) + t_{1,1}(a_{1}b_{1}^{\dagger} + a_{1}^{\dagger}b_{1}) + t_{2,1}(a_{2}b_{1}^{\dagger} + a_{2}^{\dagger}b_{1}) + t_{3,1}(a_{3}b_{1}^{\dagger} + a_{3}^{\dagger}b_{1}) + t_{4,1}(a_{4}b_{1}^{\dagger} + a_{4}^{\dagger}b_{1}) + t_{5,1}(a_{5}b_{1}^{\dagger} + a_{5}^{\dagger}b_{1}). \]

(10)

Schematic representations of \( H_{5,1} \) and \( H_{4,2} \) are provided in figure 1.
3. Exact Bethe ansatz solution

We start this section applying the quantum inverse scattering method [19–22] to discuss the exact Bethe ansatz solution for the model (1).

We begin with the standard $su(2)$-invariant $R$-matrix, depending on the spectral parameter $u$:

$$R(u) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b(u) & c(u) & 0 \\
0 & c(u) & b(u) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (11)$$

with $b(u) = u/(u + \eta)$ and $c(u) = \eta/(u + \eta)$. Above, $\eta$ is a free real parameter. It is easy to check that $R(u)$ satisfies the Yang–Baxter equation

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v). \quad (12)$$

Here $R_{jk}(u)$ denotes the matrix acting non-trivially on the $j$th and $k$th spaces and as the identity on the remaining space.

3.1. General realization of Yang–Baxter algebra

We start with the general Lax operator

$$L^X(u) = \begin{pmatrix}
u + \eta N_X & X \\
X^\dagger & \eta^{-1}
\end{pmatrix}, \quad X = A, B$$

satisfying

$$R_{12}(u - v)L^X_1(u)L^X_2(v) = L^X_2(v)L^X_1(u)R_{12}(u - v), \quad (13)$$

as a result of the following algebra being satisfied

$$[X, X^\dagger] = I, \quad [N_X, X] = -X, \quad [N_X, X^\dagger] = X^\dagger, \quad X = A, B.$$ 

Using the Lax operator presented above and the co-multiplication property [18], we can obtain a new realization for the monodromy matrix that satisfies the Yang–Baxter equation through

$$T(u) = L^A(u + \omega)L^B(u - \omega) = \begin{pmatrix} A(u) & B(u) \\
C(u) & D(u) \end{pmatrix},$$

where

$$A(u) = (u + \omega + \eta N_A)(u - \omega + \eta N_B) + AB^\dagger$$
$$B(u) = (u + \omega + \eta N_A)B + \eta^{-1}A$$
$$C(u) = (u - \omega + \eta N_B)A^\dagger + \eta^{-1}B^\dagger$$
$$D(u) = A^\dagger B + \eta^{-2}.$$ 

It can be directly shown that the monodromy matrix satisfies the Yang–Baxter equation

$$R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v). \quad (14)$$

From this identity, there are many commutation relations between the operators $A(u), B(u), C(u)$ and $D(u)$. We present those that are important to our discussion:
\[ A(u)C(v) = \frac{u - v + \eta}{u - v} C(v)A(u) - \frac{\eta}{u - v} C(u)A(v) \]
\[ D(u)C(v) = \frac{u - v - \eta}{u - v} C(v)D(u) + \frac{\eta}{u - v} C(u)D(v). \]

Finally, defining the transfer matrix
\[ \tau(u) = \text{trace}(T(u)) = A(u) + D(u) = c_0 + c_1 u + c_2 u^2. \]

It follows from (14) that the transfer matrix commutes for different values of the spectral parameter
\[ [\tau(u), \tau(v)] = 0. \]

Above \( c_i, \ i = 0, 1, 2, \) are conserved operators given by:
\begin{align*}
  c_0 &= \tau(0) = \frac{H_{n,m}}{t} + (\eta^2 N^2 - \omega^2 + \eta^{-2})I \\
  c_1 &= \frac{d}{du} \tau(u)|_{u=0} = 2\eta N \\
  c_2 &= \frac{1}{2} \frac{d^2}{du^2} \tau(u)|_{u=0} = I,
\end{align*}
where \( I \) is the identity operator, the following identification has been made for the coupling constants
\[ U = -\frac{\eta^2}{4}, \quad \mu = -t\omega\eta, \]
and the commutation relations \([c_i, c_j] = 0, \ i, j = 0, 1, 2\) are satisfied.

We observe that independent of \( n \) and \( m \) the transfer matrix gives us just two independent conserved operators \((H_{n,m}, N)\). The next step is to derive the eigenvalues of the transfer matrix (16). First, to apply the algebraic Bethe ansatz method we have to find a pseudovacuum. In the next section, we show that to have a complete solution of the model the algebraic method demands a set of pseudovacua.

### 3.2. Pseudovacua

The Bethe states of the system are obtained using a set of \( n \) dimensional orthonormal vectors, including \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \beta_2, \ldots, \beta_m) \). Consider \( \mu_j = (\mu_{j,1}, \mu_{j,2}, \ldots, \mu_{j,m}) \) and \( \nu_j = (\nu_{j,1}, \nu_{j,2}, \ldots, \nu_{j,m}) \) satisfying
\begin{align*}
  \langle \mu_j, \mu_k \rangle &= \delta_{j,k}, \quad \langle \mu_j, \alpha \rangle = 0, \quad j, k = 1, 2, \ldots, n - 1, \\
  \langle \nu_j, \nu_k \rangle &= \delta_{j,k}, \quad \langle \nu_j, \beta \rangle = 0, \quad j, k = 1, 2, \ldots, m - 1,
\end{align*}
where \( \langle x, y \rangle = \sum_{i=1}^n x_i y_i \). Now we define the operators
\[ \Gamma_i = \langle \mu_i, a \rangle, \quad \Gamma_j = \langle \nu_j, b \rangle, \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m \]
where \( a = (a_1, a_2, \ldots, a_n) \) and \( b = (b_1, b_2, \ldots, b_m) \) and the following commutation relations are satisfied
\[
\begin{align*}
[\Gamma^j_+, \Gamma^j_-] &= 0, \\
[\Gamma^j_+, A^j_-] &= 0, \\
[\Gamma^j_-, A^j_+] &= 0, \\
[\Gamma^j_-, B^j] &= 0, \\
[\Gamma^j, C(u)] &= 0, \\
[\Gamma^j, C^l(u)] &= 0, \\
[\Gamma^j, C(\eta)] &= \eta \Gamma^j A^j, \\
[\Gamma^j, C^l(\eta)] &= \eta \eta \Gamma^j, \\
[N_A, (\Gamma^j_+)^k] &= k(\Gamma^j_+)^k, \\
[N_A, (\Gamma^j_-)^k] &= k(\Gamma^j_-)^k.
\end{align*}
\]

Now, denoting \(\phi_{\{l,k\}} \equiv \phi_{l_1,l_2,\ldots,l_{n-1};k_1, k_2,\ldots,k_m}\), the whole set of pseudovacua can be defined as

\[|\phi_{\{l,k\}}\rangle = \prod_{i=1}^{n-1} (\Gamma_i^+)^{l_i} \prod_{j=1}^{m-1} (\Gamma^+_j)^{k_j} |0\rangle, \quad r = \sum_{i=1}^{n-1} l_i + \sum_{j=1}^{m-1} k_j \leq N,\]

satisfying the conditions needed for the algebraic Bethe ansatz method to work, that is,

\[
A(u)|\phi_{\{l,k\}}\rangle = (u + \omega + \eta \sum_{i=1}^{n-1} l_i)(u - \omega + \eta \sum_{i=1}^{m-1} k_i)|\phi_{\{l,k\}}\rangle \\
B(u)|\phi_{\{l,k\}}\rangle = 0 \\
C(u)|\phi_{\{l,k\}}\rangle \neq 0 \\
D(u)|\phi_{\{l,k\}}\rangle = \eta^{-2}|\phi_{\{l,k\}}\rangle.
\]

Denoting \(\psi_{\{l,k\}} \equiv \psi_{l_1,l_2,\ldots,l_{n-1};k_1, k_2,\ldots,k_m}\), the Bethe states are given by

\[|\psi_{\{l,k\}}\rangle = \begin{cases} 
\prod_{i=1}^{n-1} C(v_i) \prod_{j=1}^{m-1} (\Gamma^+_j)^{k_j} \prod_{j=1}^{m-1} (\Gamma^+_j)^{k_j} |0\rangle, & \text{if } r < N \\
\prod_{i=1}^{n-1} (\Gamma_i^+)^{l_i} \prod_{j=1}^{m-1} (\Gamma^+_j)^{k_j} |0\rangle, & \text{if } r = N
\end{cases}\]

where \(|0\rangle = |0, 0, \ldots, 0\rangle\) is the tensor product of the \(n + m\) vacua for each mode.

The transfer matrix eigenvalue problem is

\[
\tau(u)|\psi_{\{l,k\}}\rangle = \lambda_{\{l,k\}}(u)|\psi_{\{l,k\}}\rangle
\]

where for \(r = N\) the eigenvalues are given by

\[
\lambda_{\{l,k\}}(u) = (u + \omega + \eta \sum_{i=1}^{n-1} l_i)(u - \omega + \eta \sum_{i=1}^{m-1} k_i) + \eta^{-2}
\]

while for \(r < N\) the eigenvalues are

\[
\lambda_{\{l,k\}}(u) = \left(u + \omega + \eta \sum_{i=1}^{n-1} l_i\right)\left(u - \omega + \eta \sum_{i=1}^{m-1} k_i\right) \prod_{j=1}^{N-r} \frac{u - v_j + \eta}{u - v_j} + \eta^{-2} \prod_{j=1}^{N-r} \frac{u - v_j - \eta}{u - v_j}
\]

subject to the Bethe ansatz equations (BAEs) given by
\[ \eta^2 \left( v_i + \omega + \eta \sum_{i=1}^{n-1} l_i \right) \left( v_i - \omega + \eta \sum_{i=1}^{m-1} k_i \right) = \prod_{j \neq i}^{N-r} \frac{v_i - v_j - \eta}{v_i - v_j + \eta}, \quad r < N. \] (18)

We remark that in the case \( r = N \) there are no associated BAEs and the energy expression (20) takes the simple form

\[ E_{n,m} = t \left[ \left( \sum_{i=1}^{n-1} l_i \right) \left( \sum_{i=1}^{m-1} k_i \right) \eta^2 + \omega \eta \left( \sum_{i=1}^{m-1} k_i - \sum_{i=1}^{n-1} l_i \right) - \frac{\eta^2 N^2}{4} \right] \]

\[ = U \left( \sum_{i=1}^{n-1} l_i - \sum_{i=1}^{m-1} k_i \right)^2 + \mu \left( \sum_{i=1}^{n-1} l_i - \sum_{i=1}^{m-1} k_i \right). \] (19)

Now it is straightforward to check that the Hamiltonian (1) is related to the transfer matrix \( \tau(u) \) (16) through

\[ H_{n,m} = t \left( \tau(u) + \omega^2 - u^2 - \eta^{-2} - u \tau'(0) - \frac{\tau'(0)^2}{4} \right), \]

where \( \tau'(0) \) is the derivative in function of the spectral parameter. The energies of the Hamiltonian (1) are given by

\[ E_{n,m} = t \left( \lambda_{\{l,k\}}(u) + \omega^2 - u^2 - \eta^{-2} - \omega N - \frac{\eta^2 N^2}{4} \right), \] (20)

where \( \lambda_{\{l,k\}}(u) \) is the eigenvalue of the transfer matrix.

Note that equation (17) is obtained by a standard application of the algebraic Bethe ansatz [19–22], through which the BAEs (18) are obtained as the requirement for unwanted terms to cancel. It is important to mention that in such a calculation it is assumed that the roots \( \{v_j\} \) are pairwise distinct. If a pair of roots are assumed to coincide, there are additional constraints that need to be satisfied [23, 24]. This means that there may be solutions of (18) which are invalid because the additional constraints are not met. Such solutions are referred to as spurious. For solutions where the roots are pairwise distinct, it is still in principle possible to obtain spurious solutions due to the Bethe state becoming a null vector. This is a well-known feature of the Heisenberg model, e.g. see [25]; but this is not a consideration for the models under investigation here. It is not possible for the Bethe state \( |\psi_{\{l,k\}}\rangle \) to become a null vector due to the bosonic nature of the operator \( C(u) \). The coefficient of \( B^\dagger \) in \( C(u) \) is not dependent on \( u \), and the kernel of \( (B^\dagger)^k \) is trivial for all \( k = 1, \ldots, \infty \). Consequently, there are no spurious solutions of (18) where roots are pairwise distinct, because the unwanted terms will necessarily cancel, and the Bethe vector is not a null vector. In our numerical investigations we have found spurious solutions where at least two Bethe roots coincide, but these are easily identified and discarded.

It remains to show that the above method can generate a complete set of eigenvalues and eigenvectors for the model.

4. Completeness and degeneracy

We directly diagonalize the Hamiltonian (1) for two particular cases, and compare the results with those obtained from the algebraic Bethe ansatz. See the appendix for details of a three-well and a six-well case, and also [12] for a four-well case. By numerical inspection we
observe that for each BAE we have \( N - r + 1 \) valid solutions while the other solutions are spurious. For fixed \( r \) there are
\[
\frac{(r + n + m - 3)!}{(n + m - 3)!r!}
\]
BAEs, taking into account the degenerate equations with \( l = \sum_{i=1}^{n-1} l_i \) fixed. Considering the above comment that each BAE provides \( N - r + 1 \) eigenstates of the Hamiltonian, the total number of eigenstates obtained is
\[
\sum_{r=0}^{N}(N - r + 1)\frac{(r + n + m - 3)!}{(n + m - 3)!r!} = \frac{(N + n + m - 1)!}{(n + m - 1)!N!}
\]
which is the dimension of the Hilbert space for \( N \) particles.

When \( n \geq 2 \) and \( m \geq 2 \), note that for \( l = \sum_{i=1}^{n-1} l_i \) fixed it implies that \( k = \sum_{i=1}^{m-1} k_i \) is also fixed. Then there are
\[
\frac{(n - 2 + l)!}{l!(n - 2)!} \frac{(m - 2 + k)!}{k!(m - 2)!}
\]
pseudovacua corresponding to the same BAE. Therefore, the Bethe states obtained from these pseudovacua are degenerate. This observation agrees with numerical diagonalization results given in the appendix for the six-well case. When \( m = 1 \) the number of pseudovacua with the same BAE is
\[
\frac{(n - 2 + l)!}{l!(n - 2)!},
\]
with an analogous formula for \( n = 1 \).

5. Additional conserved operators

Each Hamiltonian \( H_{n,m} \) is associated with \( n + m \) modes, so integrability requires the existence of \( n + m \) independent conserved operators. The method applied above yields only two independent conserved operators, \( H_{n,m} \) and \( N \), from the transfer matrix. To obtain the other \( n + m - 2 \) independent conserved operators, we define the operators
\[
Q_i = \Gamma^\dagger_i \Gamma_i, \quad \overline{Q}_j = \Gamma^\dagger_j \Gamma_j, \quad i = 1, 2, \ldots, n - 1 \quad j = 1, 2, \ldots, m - 1.
\]
These operators satisfy the commutation relations
\[
[H_{n,m}, Q_i] = [H_{n,m}, \overline{Q}_j] = [N, Q_i] = [N, \overline{Q}_j] = 0 \quad [Q_i, Q_k] = [Q_j, \overline{Q}_k] = [\overline{Q}_j, \overline{Q}_k] = 0,
\]
so the above \( n + m - 2 \) operators together with the Hamiltonian \( H_{n,m} \) and the number operator \( N \) are the \( n + m \) independent conserved operators for the model.

On the other hand, the conserved operators satisfy the following commutation relations
\[
[Q_i, C(u)] = 0, \quad [\overline{Q}_j, C(u)] = 0.
\]

It is seen that the Bethe states \( |\psi_{(l,k)}\rangle \) as defined above are eigenstates of the conserved operators \( Q_i \) and \( \overline{Q}_j \), that is
\[ Q_j |\psi_{\{l,k\}} \rangle = l_j |\psi_{\{l,k\}} \rangle \]
\[ \bar{Q}_j |\psi_{\{l,k\}} \rangle = k_j |\psi_{\{l,k\}} \rangle. \]

(23)

We note that the operators $\Gamma_j^\dagger$ ($\Gamma_j$) behave like creation (annihilation) operators, that is, they satisfy the commutation relation $[\Gamma_j, \Gamma_j^\dagger] = 1$, while the conserved operators $Q_j = \Gamma_j^\dagger \Gamma_j$ have the action of a number operator.

6. Conclusion

In this work we presented a formulation for quantum integrable multi-well tunneling models through the quantum inverse scattering method and algebraic Bethe ansatz techniques. Integrability of the Hamiltonian $H_{n,m}$ requires the existence of $n + m$ conserved operators, however the transfer matrix gives just two of them. We show how to compute the other $n + m - 2$ conserved operators and associated with these additional conserved operators is a set of pseudovacuum states. Each pseudovacuum generates a set of BAEs. It has been argued that all pseudovacua are required to obtain a complete set of eigenvalues and eigenvectors for each model.

As we were completing this work the preprint [26] appeared, which discusses the same class of models.

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Appendix

Here we compare the results of numerical diagonalization of the Hamiltonian and numerical solution of the BAEs for some illustrative examples.

A.1. $H_{2,1}$ for $N = 3$

In matrix form the Hamiltonian is expressible as

\[
H_{2,1} = \begin{pmatrix}
9U + 2\mu & t_{2,1}\sqrt{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_{2,1}\sqrt{3} & U + 2\mu & 0 & t_{2,1} & 0 & t_{1,1} & 0 & 0 & 0 & 0 \\
0 & 2t_{2,1} & U - \mu & t_{2,1}\sqrt{3} & 0 & t_{1,1}\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & t_{2,1}\sqrt{3} & 9U - 3\mu & 0 & 0 & t_{1,1}\sqrt{3} & 0 & 0 & 0 \\
0 & t_{1,1} & 0 & 0 & 9U + 3\mu & t_{2,1}\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & t_{1,1}\sqrt{3} & 0 & 0 & t_{2,1}\sqrt{3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & t_{1,1}\sqrt{3} & 0 & 0 & t_{2,1}\sqrt{3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2t_{2,1} & t_{2,1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{1,1}\sqrt{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t_{1,1}\sqrt{3} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9U + 3\mu
\end{pmatrix}
\]
Choosing the coupling parameter values

\[ U = 1, \quad \mu = 0.5, \quad t = -0.5, \quad \alpha_1 = \alpha_2 = \frac{1}{\sqrt{2}}, \quad \beta_1 = 1 \]

we obtain the ordered eigenspectrum below through numerical diagonalization:

\[
\begin{align*}
E_1 &= -0.207868448700000014 \\
E_2 &= 0.123564632300000005 \\
E_3 &= 1.47230743099999994 \\
E_4 &= 1.82091556600000004 \\
E_5 &= 2.01646645300000005 \\
E_6 &= 7.607900530000000022 \\
E_7 &= 10.50000000000000000 \\
E_8 &= 10.52769256999999993 \\
E_9 &= 10.5555198000000008 \\
E_{10} &= 10.58350146999999999.
\end{align*}
\]

We compare these results with those obtained from the BAEs, which are displayed in table A1. Note that in this table we do not present spurious solutions, namely those where roots of the BAE are equal. Column 5 identifies the results from the BAEs with those from exact

| 1 | Pseudovacuum | BAE | BAE solution | Energy |
|---|---|---|---|---|
| 0 | | \( \gamma^2(v_1^2 - \omega^2) = \frac{(v_1 - v_2 - \eta)}{(v_1 - v_2 + \eta)} \frac{(v_2 - v_3 - \eta)}{(v_2 - v_3 + \eta)} \) | \( v_1 = 0.3643816442 \) | \( E_1 \) |
| | | \( \gamma^2(v_2^2 - \omega^2) = \frac{(v_2 - v_3 - \eta)}{(v_2 - v_3 + \eta)} \frac{(v_3 - v_1 - \eta)}{(v_3 - v_1 + \eta)} \) | \( v_1 = 0.3359698043 \) | \( E_2 \) |
| | | \( \gamma^2(v_3^2 - \omega^2) = \frac{(v_3 - v_1 - \eta)}{(v_3 - v_1 + \eta)} \frac{(v_1 - v_2 - \eta)}{(v_1 - v_2 + \eta)} \) | \( v_1 = 0.3535693555 \) | \( E_3 \) |
| | | \( \gamma^2(v_1^2 - \omega^2) = \frac{(v_1 - v_2 - \eta)}{(v_1 - v_2 + \eta)} \frac{(v_2 - v_3 - \eta)}{(v_2 - v_3 + \eta)} \) | \( v_1 = 0.3207690545 \) | \( E_4 \) |
| | | \( \gamma^2(v_2^2 - \omega^2) = \frac{(v_2 - v_3 - \eta)}{(v_2 - v_3 + \eta)} \frac{(v_3 - v_1 - \eta)}{(v_3 - v_1 + \eta)} \) | \( v_1 = 0.356912007 \) | \( E_5 \) |
| | | \( \gamma^2(v_3^2 - \omega^2) = \frac{(v_3 - v_1 - \eta)}{(v_3 - v_1 + \eta)} \frac{(v_1 - v_2 - \eta)}{(v_1 - v_2 + \eta)} \) | \( v_1 = 0.3643816442 \) | \( E_6 \) |
diagonalization (A.1). It can be seen that there is a one-to-one correspondence between the results of the two approaches.

A.2. $H_{3,3}$ for $N = 3$

We set the parameters as

$$N = 3, \quad U = 1.3, \quad \mu = 0.5, \quad t = -3.7,$$

$$\phi_1 = \pi/3, \quad \theta_1 = \pi/6, \quad \phi_2 = \pi/4, \quad \theta_2 = \pi/7,$$

| $r$ | $l_1$ | $l_2$ | $k_1$ | $k_2$ | Energy       |
|-----|------|------|------|------|-------------|
| 0   | 0    | 0    | 0    | 0    | -8.20422, 3.59673, 13.3926, 17.2149 |
| 1   | 0    | 0    | 1    | 0    | -4.70392, 4.99326, 15.5107 |
| 2   | 0    | 0    | 0    | 1    | -5.01626, 4.81132, 13.0049 |
| 2   | 1    | 0    | 0    | 0    | -2.43363, 5.03363 |
| 2   | 1    | 0    | 0    | 0    | 0.704413, 14.2956 |
| 2   | 0    | 0    | 1    | 1    | -0.481639, 11.4816 |
| 3   | 1    | 1    | 1    | 0    | |
| 3   | 1    | 1    | 0    | 1    | |
| 3   | 2    | 0    | 1    | 0    | 1.8 |
| 3   | 2    | 0    | 0    | 1    | |
| 3   | 0    | 2    | 1    | 0    | |
| 3   | 0    | 2    | 0    | 1    | |
| 3   | 3    | 0    | 0    | 0    | 13.2 |
| 3   | 2    | 0    | 0    | 1    | |
| 3   | 2    | 0    | 0    | 0    | |
| 3   | 0    | 3    | 0    | 0    | 10.2 |
| 3   | 0    | 0    | 3    | 0    | |
| 3   | 0    | 0    | 2    | 1    | |

Table A2. Energy spectrum obtained by the BAE for $H_{3,3}$ for $N = 3$. 

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with\[
\begin{align*}
\alpha_1 &= \sin \phi_1 \cos \theta_1, \quad \alpha_2 = \sin \phi_1 \sin \theta_1, \quad \alpha_3 = \cos \phi_1, \\
\beta_1 &= \sin \phi_1 \cos \theta_1, \quad \beta_2 = \sin \phi_2 \sin \theta_2, \quad \beta_3 = \cos \phi_2.
\end{align*}
\]

Ignoring spurious solutions, table A2 lists the spectrum obtained by numerically solving the BAEs. These are grouped for the different sectors determined by the various choices of pseudo-vacua for each fixed \( J \) and \( k \). The results have been compared with those obtained by direct numerical diagonalization, and it was again found that there is a one-to-one correspondence. In particular, the degeneracies are found to be in complete agreement with the formula (21).

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