WORST-CASE ANALYSIS OF GINI MEAN DIFFERENCE SAFETY MEASURE

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ABSTRACT. The paper introduces the worst-case portfolio optimization models within the robust optimization framework for maximizing return through either the mean or median metrics. The risk in the portfolio is quantified by Gini mean difference. We put forward the worst-case models under the mixed and interval+polyhedral uncertainty sets. The proposed models turn out to be linear and mixed integer linear programs under the mixed uncertainty set, and semidefinite program under interval+polyhedral uncertainty set. The performance comparison of the proposed models on the listed stocks of Euro Stoxx 50, Dow Jones Global Titans 50, S&P Asia 50, consistently exhibit advantage over their conventional non-robust counterpart models on various risk parameters including the standard deviation, worst return, value at risk, conditional value at risk and maximum drawdown of the portfolio.

1. Introduction. Since the landmark research of Markowitz [30] who put forward the mean-risk portfolio theory taking the variance of returns for risk, several endeavors have been made by researchers to enrich the domain of portfolio optimization by including other forms of risk measures in their studies.

Question on how adequately we can quantify the systematic risk remains complex and relevant at all times. Over the last few decades, there is an increased emphasis on modeling risk in investment. A large body of literature has emerged aiming to study risk measures of varying intricacy. While significant research contributions poured in to develop the theoretical side providing a mathematical framework characterizing risk measures, equally vital research has grown parallelly which concentrated on developing empirically tractable optimization models for computing risk. Among many such studies, a few prominent ones include [45, 23, 46, 1, 36].

Sharpe [44] proposed mean absolute deviation (MAD) in portfolio selection which results in a linear programming model. Konno and Yamazaki [23] showed the superior performance of the MAD portfolio optimization model over the variance model in Tokyo stock market. Bower and Wentz [8] observed that difference between returns from MAD and variance models is not statistically significant, and thus prefer MAD over variance for a computational reason.

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Yitzhaki [45] defined Gini mean difference (GMD) risk measure as the expected value of absolute dispersion between every pair of realizations of return of a portfolio. GMD is an appealing risk measure due to its competency to utilize information from the entire return distribution while still preserving symmetric demeanor. The main feature of GMD which makes it favorable over variance is that it provides a necessary condition for second order stochastic dominance (SSD) which in turn yields it to be consistent with the utility maximization theory [45].

Ogryczak [33] provided a theoretic comparison between MAD and GMD. Gerstenberger and Vogel [14] studied the efficiency of GMD under various distributions and concluded it to be superior to MAD and variance over a wide range of distributions. Shalit and Yitzhaki [40] and Ogryczak [33] investigated various properties of GMD. Shalit and Yitzhaki [41] defined the extended Gini of a portfolio and analyzed the numerical properties of mean-variance, mean-Gini, and extended mean-Gini models. Ji et al. [19] extended the idea of the mean-Gini model to the risk-adjusted mean-Gini ratio model. Furman et al. [13] proposed the Gini shortfall risk measure and explored its advantages over the other existing risk measures. Recently, Berkhouch et al. [4] introduced extended Gini shortfall, which encompassed in it the properties of Gini shortfall as well as variability in the risk-taking capacity of investors.

However, all these risk measures are not designed to model large losses in a portfolio. The tail-based risk measures like value-at-risk (VaR) and conditional value-at-risk (CVaR) have been widely utilized by the risk management community to control the downside risk. VaR$_{\alpha}$ represents the worst loss at tolerance level $\alpha$, and CVaR$_{\alpha}$ [36] is the average of losses beyond VaR$_{\alpha}$. Weighted CVaR (WCVaR$_{\{\alpha_1, \ldots, \alpha_r\}}$) for tolerance set $\{\alpha_1, \ldots, \alpha_r\}$, proposed by Mansini et al. [28], is defined as the weighted sum of various CVaR$_{\alpha_j}$, $j = 1, \ldots, r$. Mansini et al. [28] proved that the mean safety measure corresponding to the tail Gini measure [29] is an approximation to the WCVaR of return distribution of a portfolio.

Apart from risk, another factor that prominently influences portfolio selection is the return. Research in portfolio optimization primarily suggests on maximizing mean of the portfolio returns. Benati [3] showed that the median of return is less sensitive to the outliers, and consequently, a robust statistic in comparison to the mean for maximizing returns, and hence proposed median-risk models for portfolio selection.

1.1. Robust portfolio optimization. The traditional portfolio optimization models customarily surmise that the input data is accessible with certainty. It totally ignores situations where the problem structure along with the input parameters are non-deterministic. More concretely, the input return data and the choice of the probability measure form the two primary sources of uncertainty in portfolio selection. For example, all empirical studies ordinarily work with the closing prices of assets, overlooking the intra-day developments, which causes wary return information. On the other side, the calculation of mean return and risk ineluctably depends on how accurately the moments of the distribution of returns can be predicted [49]. Any incorrect prediction on the probability distribution of returns can bring about wrong investment decisions leading to significant financial losses. Robust optimization (RO) techniques have been widely applied to construct optimal portfolios immune to such uncertainties.

Best and Grauer [6] and Black and Litterman [7] studied the effect of small changes in mean returns of assets on the portfolio from the mean-variance model.
Chen and Tan [10] proposed a robust mean-variance portfolio optimization model by incorporating interval uncertainty in the mean return vector.

Moon and Yao [31] formulated a robust linear MAD model under the framework of Bertsimas and Sim [5]. They considered the uncertainty in returns by varying the mean return of the portfolio in a symmetric interval. Li et al. [25] extended their study by including asymmetric distributions of returns and showed that the model can be casted as a second order cone program (SOCP). Liu [27] proposed a robust MAD model where the asset returns were varied in the intervals obtained by solving a pair of mathematical programming models.

El Ghaoui et al. [11] performed the worst-case analysis of VaR by formulating a semidefinite program (SDP) model. Goldfarb and Iyenger [16] proposed SOCP by developing a robust factor model for asset returns in the mean-variance and mean-VaR frameworks under well-defined uncertainty sets. Natarajan et al. [32] derived the coherent risk measures, like VaR and CVaR, using the well-defined uncertainty sets of the probability distribution. Zhu and Fukushima [48] proposed the worst-case analysis of CVaR under mixed, box, and ellipsoid uncertainty sets for the probability distribution. Gotoh et al. [17] analyzed the worst-case CVaR minimization by considering worst-case in return vectors. Kara et al. [21] proposed a robust portfolio optimization model by constructing parallelepiped uncertainty set and applied it on CVaR of the portfolio. Postek et al. [35] developed uncertainty sets for probability distributions from the goodness of fit test statistics for several risk measures including VaR and CVaR. Using Fenchel duality and results of Ben-Tal et al. [2], they derived different tractable RO models with different risk measures in constraints.

Several authors have also proposed robust models for reward-risk ratios. Keating and Shadwick [22] formulated the Omega ratio as the ratio of expected returns above a certain threshold to the expected returns below the same threshold. The worst-case Omega ratio maximization is studied by Kapsos et al. [20] under the mixed, box, and ellipsoidal uncertainty sets. Sharma et al. [43] presented the Omega ratio maximization model for a loss-averse investor using CVaR of the benchmark index as threshold. The worst-case analysis of the same is performed under the uncertainty sets considered by Zhu and Fukushima [48] and Kapsos et al. [20]. Sehgal and Mehra (2019) [38] put forward robust models for Omega, semi-MAD, and weighted STARR ratios, by varying the uncertain input returns in bounded and symmetric intervals.

Recent times witnessed some attempts to extend RO techniques in index tracking problems. Kwon and Wu [24] proposed a robust portfolio optimization model for enhanced index-tracking based on the three-factor model of Fama and French to define uncertainty sets for expected return and covariance matrix. They concluded that the robust enhanced index tracking model has a better tracking performance as opposed to its non-robust counterpart model. Chen and Kwon [9] proposed an index tracking model by adopting the technique of Bertsimas and Sim [5]. Gharakhhani et al. [15] formulated the robust version of the index tracking model in [37].

1.2. Contributions of the paper. In this paper, we aim to contribute to the literature on robust optimization for portfolio selection by considering reward-risk safety measures. The salient features of the paper are summarized as follows:

(i) We propose to study the safety measures with GMD risk measure and analyze their worst-case execution when only partial information on the underlying
probability distribution is known. We utilize mean as well as the median statistics for maximizing returns and GMD to minimize risk in the optimal portfolios. We aim to maximize the worst-case of the proposed safety measures with GMD risk measure under two well-defined uncertainty sets, namely, the mixed and the interval+polyhedral sets. In each case, the formulated optimization models turned out to be computationally tractable.

(ii) To verify the performance of the proposed models, we consider various datasets from equity markets across the globe. We apply the rolling window strategy to analyze the out-of-sample performance of the robust portfolio from mixed uncertainty set. We perform a comparative analysis with the corresponding non-robust model on several performance metrics. We also test the performance of the proposed models under different market conditions. Furthermore, using a single-window approach, we analyze the performance of the robust models under interval+polyhedral uncertainty sets and draw a comparison between the mixed and interval+polyhedral uncertainty sets. The different risk models are evaluated on a statistical basis. The empirical analysis indicates lower risk, in terms of standard deviation, VaR, CVaR, and maximum drawdown in the robust portfolios compared to those from the non-robust models.

To the best of our knowledge, the paper is the first endeavor to apply GMD as a safety measure in investigating the worst-case portfolio selection within the RO framework. The examination enables us to address uncertainty in the input parameters. The key take away of our present research is that the worst-case optimization of the safety measures confers a better risk hedging capability to the portfolios.

The uncertainty set holds the key in robust portfolio optimization problems. In this paper, we have worked with the mixed uncertainty set and interval+polyhedral uncertainty set to model the worst-case safety measure corresponding to GMD. The two sets had been applied [48, 20, 43, 34] in the context of RO.

Financial markets customarily experience different phases engendered by bull and bear effects. Mixed uncertainty set naturally embeds the market information inundating scenarios and presents a robust framework to build a model thereupon. On the other hand, an interval+polyhedral uncertainty set takes the probability distribution of returns to belong to the intersection of an asymmetric box and a polyhedral. The area of this intersecting region decides the caliber of robustness that an investor is inclined to consider.

One can indeed use other forms of uncertainty sets, like an ellipsoidal set, which are less conservative than the interval+polyhedral uncertainty set [20]. However, unlike the mixed and interval+polyhedral uncertainty sets, an ellipsoidal set leads to an intractable robust portfolio optimization problem with GMD. The structures of the two considered sets empower us to develop a linear program (LP) and a mixed integer linear program (MILP) models for the mixed uncertainty set, and an SDP model for the interval+polyhedral uncertainty set.

The remainder of this article proceeds as follows. Section 2 revisits the preliminaries of GMD and median optimization. Section 3 introduces four portfolio optimization models with safety measure. Sections 4 and 5 present their worst-case robust counterpart models under the mixed and interval+polyhedral uncertainty sets, respectively. Section 6 includes the empirical analysis and performance comparison of the robust models with their non-robust counterparts. Section 7 concludes the paper and lists a few potential directions for future research.
2. Preliminaries. We shall be using the following notations in the sequel:

- $T$: the total number of scenarios
- $n$: the total number of stocks available for investment
- $x$: the $n \times 1$ vector $(x_1, \ldots, x_n)'$ representing a portfolio; $x_i$ is weight of $i$-th asset, $i = 1, \ldots, n$
- $q_t$: the probability of scenario $t$, $t = 1, \ldots, T$
- $R_t$: the random variable representing returns from $i$-th asset, $i = 1, \ldots, n$
- $r_{it}$: the realization of random variable $R_t$ under scenario $t$, $t = 1, \ldots, T$
- $R$: the random vector $(R_1, \ldots, R_n)'$ of returns from $n$ assets with $t$-th realization $(r_{1t}, \ldots, r_{nt})'$, $t = 1, \ldots, T$
- $R_x$: the random variable representing return from portfolio $x$; $R_x = \sum_{i=1}^{n} x_i R_i$
- $y_t$: the $t$-th realization of random variable $R_x$: $y_t = \sum_{i=1}^{n} r_{it} x_i$, $t = 1, \ldots, T$
- $\mu(x)$: the expected return from portfolio $x$; $\mu(x) = E(R_x) = \sum_{i=1}^{n} x_i E(R_i)$
- $z^+$: equals $\max\{0, z\}$

Throughout the paper, the expected return $\mu(x)$ from the portfolio $x$ is approximated by the sample mean return $\sum_{t=1}^{T} q_t y_t = \sum_{t=1}^{T} \sum_{i=1}^{n} q_t r_{it} x_i$, based on the realizations $y_t$ of $R_x$.

We compute $r_{it} = \frac{P_{it} - P_{it-1}}{P_{it-1}}$, where $P_{it}$ is the closing price of the $i$-th asset in scenario $t$, $t = 1, \ldots, T$.

The set of admissible portfolios is denoted by

$$X = \{ x = (x_1, \ldots, x_n)' : x_i \geq 0, i = 1, \ldots, n, \sum_{i=1}^{n} x_i = 1 \}$$

where it is assumed that short selling is prohibited and the entire capital is allocated to the portfolio.

2.1. Gini mean difference model. For a discrete random variable $R_x$, GMD is defined by $\rho(x) = \frac{1}{2} \sum_{t=1}^{T} \sum_{t=1}^{T} |y_t - y_{t_1}| q_t q_{t_1}$. It is one-half of the average of absolute dispersion between every pair of realizations $y_t$ and $y_{t_1}$.

We introduce the auxiliary variables, $\{g_{it}, t, t_1 = 1, \ldots, T\}$, representing the difference between a pair of realizations. Thus, minimizing GMD can be described by the following LP:

$$\min \frac{1}{2} \sum_{t=1}^{T} \sum_{t=1}^{T} g_{it} q_t q_{t_1}$$

subject to  

$$g_{it} \geq (y_t - y_{t_1}), \quad t, t_1 = 1, \ldots, T, \quad (1)$$

$$g_{it} \geq -(y_t - y_{t_1}), \quad t, t_1 = 1, \ldots, T, \quad (2)$$

$$x \in X.$$

For a continuous distribution, the GMD is expressed as

$$\rho(x) = 2 \int_{0}^{1} (\mu(x) \alpha - F_{R_x}^{(-2)}(\alpha)) d\alpha,$$

where $F_{R_x}^{(-2)}(\alpha) = \max_{\eta \in \mathbb{R}} (\alpha \eta - E((\eta - R_x)^+) )$, is the second quantile of $R_x$. 


Instead of GMD, the tail Gini measure (TGM) is used in [29] to model the
downside risk measure. For a tolerance level $\alpha \in (0,1]$, the TGM is defined by
$$\rho_\alpha(x) = \frac{2}{\alpha^2} \int_0^\alpha (\mu(x)\beta - F_{R_x}^{(-\frac{2}{\alpha})}(\beta)) \, d\beta.$$  
Mansini et al. [28] proved that for a partition $\{0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = \alpha\}$ of $[0,\alpha]$, $\alpha \in (0,1]$, the safety measure corresponding to TGM is given by
$$\mu(x) - \rho_\alpha(x) \approx \sum_{j=1}^{r-1} w_j \text{CVaR}_{\alpha_j}(x),$$
where $w_j = \frac{(\alpha_{j+1} - \alpha_j)}{\alpha_j^2}$, $j = 1,\ldots,r-1$, and CVaR$_{\alpha_j}(x)$ is the CVaR of return distribution of portfolio $x$ at $\alpha$ tolerance level. Note that $\sum_{j=1}^{r} w_j = 1$.

2.2. The median portfolio optimization model. For a random variable $R_x$, the median $Z(x)$ of a portfolio $x$ is the central value of the portfolio returns i. e., $Z(x) = \inf \{ z \mid P(R_x \leq z) \geq 0.5 \}$. Note that $Z(x) = \text{VaR}_{0.5}(x)$. Benati [3] adopted the median statistics for portfolio return maximization with a constraint on risk. The median is observed to be less sensitive to skewness and fat tails in the distribution of return and thus more robust than the mean in portfolio formulation. Benati [3] proposed the following mixed integer linear program (MILP) for median maximization:

$$(MED) \quad \max z$$
subject to $z \leq y_t + Ms_t$, $t = 1,\ldots,T$,
$$\sum_{t=1}^{T} q_t s_t \leq 0.5,$$
$$s_t \in \{0,1\}, \quad t = 1,\ldots,T,$$
$$x \in X,$$
where $M$ is an arbitrary large positive constant. Feng et al. [12] proposed algorithms to assign an appropriate value to $M$ in the above optimization problem.

3. Safety measure portfolio optimization. In this section, we introduce the safety measures and formulate two models which seek to maximize these safety measures for portfolio selection.

We consider the following safety measure based portfolio optimization problem:

$$\max_{x \in X} (\psi(x) - \rho(x)),$$
where $\psi(x)$ and $\rho(x)$ represent return and risk in the portfolio $x$, respectively.

In the present paper, we apply $\mu(x)$ and $Z(x)$ for return $\psi(x)$, while $GMD$ is used to quantify the risk measure $\rho(x)$.

The two portfolio optimization models considered in the paper are as follows:

$$(\mu\text{GMD}) \quad \max \sum_{t=1}^{T} y_t q_t - \frac{1}{2} \sum_{t_1=1}^{T} \sum_{t_2=1}^{T} g_{t_1} q_{t_1} q_{t_2},$$
subject to $(1)-(2)$,
$$x \in X,$$
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\[ \text{(MGMD)} \quad \max_z \quad z = \frac{1}{2} \sum_{t_1=1}^{T} \sum_{t=1}^{T} g_{t_1} q_t q_{t_1} \]

subject to \((1) - (5))
\[ x \in X. \]

The \((\mu GMD)\) and \((MGMD)\) models seek to maximize mean and median of returns in the safety measure, respectively. Note that \((MGMD)\) model is an MILP.

We aim to perform the worst-case analysis of the two proposed models, varied from their standard ex-ante models, with the underlying uncertainty sets on the probability distribution being constructed through prior partial information.

**Definition 3.1.** For a general set of probability distribution \(Q\) and a portfolio \(x\), the worst-case mean/median safety measure is given by
\[ \min_{q \in Q} (\psi(x) - \rho(x)), \]

And the corresponding worst-case portfolio optimization model is given by
\[ \max_{x \in X} \min_{q \in Q} (\psi(x) - \rho(x)). \]

In the sections to follow, we analyze the above models under two different structures on the uncertainty set \(Q\), namely, the mixed uncertainty set and the interval+polyhedral uncertainty (also known as budget uncertainty set) [5].

4. Robust optimization with mixed uncertainty set. The probability distribution of return is taken to a convex combination of different probability density functions resulting in a mixture distribution. Hall et al. [18] observed that the thick-tailed behavior of commodity futures is due to a mixture of normal distributions. The geometry and applications of the mixture model is studied by Lindsay [26].

The mixed uncertainty set is defined by:
\[ Q_M = \{ q : q = \sum_{l=1}^{L} \lambda_l q^l, \quad \lambda \in A \}, \]

where
\[ A = \{ \lambda = (\lambda_1, \ldots, \lambda_L)^t : \sum_{l=1}^{L} \lambda_l = 1, \quad \lambda_l \geq 0, \quad l = 1, \ldots, L \}. \]

For each \(l = 1, \ldots, L\), we assume that the number of samples is \(T^l\) and the probability of the \(t\)-th sample for \(l\)-th likelihood distribution \(q^l\) is \(q^l_t\). Thus, the \(l\)-th likelihood probability vector is given by \(q^l = (q^l_1, \ldots, q^l_{T^l})^t : \sum_{t=1}^{T^l} q^l_t = 1, \quad q^l_t \geq 0, \quad l = 1, \ldots, L, \quad t = 1, \ldots, T^l \}. \) Let \(y^l_t\) denote the return from the \(t\)-th sample in \(l\)-th likelihood distribution \(q^l\).

We also set the following notations:
\[ \mu^l(x) = \sum_{t=1}^{T^l} y^l_t q^l_t, \]
\[ GMD^{l,l_1}(x) = \frac{1}{2} \sum_{t_1=1}^{T^l} \sum_{t=1}^{T^l} |y^l_t - y^l_{t_1}| q^l_t q^l_{t_1}. \]
4.1. **Mean-GMD safety model.** Taking note of the following equality

\[
\begin{align*}
\max_{x \in X} & \min_{\lambda \in A} \sum_{t=1}^{T} \sum_{l=1}^{L} y_l^t (\lambda_l q_l^t) - \frac{1}{2} \sum_{t_1=1}^{T} \sum_{l_1=1}^{L} \sum_{t_1=1}^{T} \sum_{l_1=1}^{L} \left| y_l^t - y_{l_1}^t \right| (\lambda_l q_l^t)(\lambda_{l_1} q_{l_1}^t) \\
= & \max_{x \in X} \min_{\lambda \in A} \sum_{l=1}^{L} \lambda_l \lambda_t (\mu_l^I(x) - GMD^{l,t_1}(x)) \\
= & \max_{x \in X} \min_{\lambda \in A, l_1 \in \{1,...,L\}} (\mu_l^I(x) - GMD^{l,t_1}(x)),
\end{align*}
\]

the worst-case robust model of \((\mu GMD)\), under the uncertainty set \(Q_M\), is described as follows:

\[
\max_{x \in X} \{ \theta : \mu_l^I(x) - GMD^{l,t_1}(x) \geq \theta, \ l, \ l_1 = 1,\ldots, L \}. \tag{6}
\]

By introducing the auxiliary variables \(\{g_{q_{l,t_1}}^{1,l_1} \mid l, l_1 = 1,\ldots, L, \ t = 1,\ldots, T^t, \ t_1 = 1,\ldots, T^{t_1}\}\), problem (6) can be equivalently expressed as the following LP model:

\[
\begin{align*}
& (R\mu GMD) \quad \max \ \theta \\
& \text{subject to} \quad \sum_{t=1}^{T^t} y_l^t q_l^t - \frac{1}{2} \sum_{t_1=1}^{T} \sum_{l_1=1}^{L} \sum_{l_1=1}^{L} g_{q_{l,t_1}}^{1,l_1} q_{q_{l_1}}^{t_1} \geq \theta, \ l, l_1 = 1,\ldots, L, \\
& \quad g_{q_{l,t_1}}^{1,l_1} \geq (y_l^t - y_{l_1}^t), \ l, l_1 = 1,\ldots, L, \ t = 1,\ldots, T^t, \ t_1 = 1,\ldots, T^{t_1}, \\
& \quad g_{q_{l,t_1}}^{1,l_1} \geq -(y_l^t - y_{l_1}^t), \ l, l_1 = 1,\ldots, L, \ t = 1,\ldots, T^t, \ t_1 = 1,\ldots, T^{t_1}, \\
& \quad x \in X.
\end{align*}
\]

4.2. **Median-GMD safety model.** Under the mixed uncertainty set \(Q_M\), the worst-case robust median model can be obtained by observing the following equivalent conditions:

\[
\sum_{i=1}^{L} \sum_{l=1}^{T^t} \lambda_l q_l^t s_l^t \leq 0.5 \ \forall \ \lambda \in A \iff \sum_{i=1}^{T^t} q_l^t s_l^t \leq 0.5, \ l = 1,\ldots, L.
\]

Let \(z_l^t\) be the auxiliary variable representing the objective function of \((MED)\) model, and \(s_l^t\) be the integer variable for \(t\)-th sample with respect to the \(l\)-th likelihood distribution. Then the worst-case robust median model is an MILP given as follows:

\[
\begin{align*}
& (RMED) \quad \max \ \theta \\
& \text{subject to} \quad z_l^t \geq \theta, \ l = 1,\ldots, L, \\
& \quad z_l^t \leq y_l^t + Ms_l^t, \ l = 1,\ldots, L, \ t = 1,\ldots, T^t, \\
& \quad \sum_{l=1}^{L} q_l^t s_l^t \leq 0.5, \ l = 1,\ldots, L, \\
& \quad s_l^t \in \{0,1\}, \ x \in X, \ l = 1,\ldots, L, \ t = 1,\ldots, T^t.
\end{align*}
\]

Similar to the formulation of \((R\mu GMD)\) model, the worst-case robust model of \((MGMD)\) under the mixed uncertainty set \(Q_M\) is the following MILP:

\[
\begin{align*}
& (RMGMD) \quad \max \ \theta \\
& \text{subject to} \quad z_l^t - \frac{1}{2} \sum_{t_1=1}^{T^t} \sum_{l_1=1}^{L} \sum_{t_1=1}^{T^t} g_{q_{l,t_1}}^{1,l_1} q_{q_{l_1}}^{t_1} \geq \theta, \ l, l_1 = 1,\ldots, L.
\end{align*}
\]
\[ z^l \leq y^l_i + Ms^l, \quad l = 1, \ldots, L, \quad t = 1, \ldots, T^l, \]
\[ \sum_{t=1}^{T} q^l_t s^l_t \leq 0.5, \quad l = 1, \ldots, L, \]
\[ g^{l_1}_{l_1 t} \geq (y^l_i - y^l_{i_1}), \quad l, l_1 = 1, \ldots, L, \quad t = 1, \ldots, T^l, \quad t_1 = 1, \ldots, T^{l_1}, \]
\[ g^{l_1}_{l_1 t} \geq -(y^l_i - y^l_{i_1}), \quad l, l_1 = 1, \ldots, L, \quad t = 1, \ldots, T^l, \quad t_1 = 1, \ldots, T^{l_1}, \]
\[ s^l_t \in \{0,1\}, \quad l = 1, \ldots, L, \quad t = 1, \ldots, T^l, \]
\[ x \in X. \]

5. **Robust optimization under interval+polyhedral uncertainty set.** In this section, we formulate the RO models when the uncertainty set is described by an interval+polyhedral set \( Q_B \), defined as follows:

\[ Q_B = \{ q : q_t = q^0_t + v_t, \quad 1 \leq t \leq T, \quad \sum_{t=1}^{T} v_t = 0, \quad \nu \leq v_t \leq \nu, \quad \sum_{t=1}^{T} |v_t| \leq a \}, \]

where \( q^0_t, \quad 1 \leq t \leq T, \) is a nominal distribution at time \( t, \) \( \sum_{t=1}^{T} v_t = 0 \) ensures that \( \sum_{t=1}^{T} q_t = 1, \) the interval parameters \( \nu \) and \( \nu \) can be suitably chosen so that \( 0 \leq q_t \leq 1, \quad 1 \leq t \leq T, \) and \( a \in (0, 1]. \)

5.1. **Mean-GMD safety model.** Under the uncertainty set \( Q_B \), the robust optimization of \((\mu GMD)\) model is described as follows:

\[ (P1) \quad \max \theta \]
\[ \text{subject to} \quad \min_{q \in Q_B} \left( \sum_{t=1}^{T} y_t q_t - \frac{1}{2} \sum_{t=1}^{T} \sum_{t_1=1}^{T} g_{t t_1} q_t q_{t_1} \right) \geq \theta, \]
\[ \quad (1)-(2), \]
\[ \quad g_{t t_1} = g_{t_1 t}, \quad t, t_1 = 1, \ldots, T, \]
\[ \quad x \in X. \]

The inner minimization problem in (7) can be simplified to the following quadratic programming problem (QPP) by taking \( q_t = q^0_t + v_t \), and using the properties of \( v_t \) from the set \( Q_B \):

\[ (P2) \quad \min_{\nu, \gamma} \sum_{t=1}^{T} y_t v_t - \sum_{t=1}^{T} \sum_{t_1=1}^{T} g_{t t_1} q^0_t v_{t_1} - \frac{1}{2} \sum_{t=1}^{T} \sum_{t_1=1}^{T} g_{t t_1} v_t v_{t_1} \]
\[ \text{subject to} \quad v_t^2 - (\nu + v) v_t + \nu v \leq 0, \quad t = 1, \ldots, T, \]
\[ \quad \sum_{t=1}^{T} v_t = 0, \]
\[ \quad \sum_{t=1}^{T} \gamma_t \leq a, \]
\[ \quad v_t - \gamma_t \leq 0, \quad t = 1, \ldots, T, \]
\[ \quad -v_t - \gamma_t \leq 0, \quad t = 1, \ldots, T. \]
Note that the variable $\gamma_1$ help in removing nonlinearity in the constraint $\sum_{t=1}^{T} |v_t| \leq a$.

However, solving problem $(P2)$ is of concern as it is a non-convex QPP. We follow the approach of Zheng et al. [47], who proposed the dual of a general QPP and derived the necessary and sufficient conditions for the zero duality gap. Using the fact that $g_{tt_1} = g_{t_1t}$, $t, t_1 = 1, \ldots, T$, we compare problem $(P2)$ with problem $(A1)$ in the Appendix to formulate the dual of $(P2)$. Under appropriate conditions, the duality gap between $(P2)$ and its dual is zero.

We retrace our steps back and apply the dual of problem $(P2)$ to resolve constraint $(7)$ in problem $(P1)$. We obtain the following SDP model as the robust optimization variant of $(\mu GMD)$ model under the uncertainty set $Q_B$: 

$$(R_B\mu GMD) \quad \max \theta$$

subject to 

$$\sum_{t=1}^{T} y_tq_t^\circ - \frac{1}{2} \sum_{t_1=1}^{T} \sum_{t_1=1}^{T} g_{tt_1}q_t^\circ q_{t_1}^\circ + \gamma \geq \theta,$$

$$M_k - \tau \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{k=1}^{3T+3} \mu_k M_k \succeq 0,$$

$$(1) - (2),$$

$$g_{tt} = g_{t_1t}, \quad t, t_1 = 1, \ldots, T,$$

$$\mu_k \geq 0, \quad k = 1, \ldots, 3T + 3,$$

$$\gamma \in \mathbb{R}, \quad x \in X.$$

Here, the matrices $M_0$ and $M_k$, $k = 1, \ldots, 3T + 3$, and the variables $\gamma$ and $\mu_k$, $k = 1, \ldots, 3T + 3$, are obtained by comparing problem $(A1)$ in the Appendix with problem $(P2)$. Thus,

$$M_k = \begin{pmatrix} c_k & b_k' \\ b_k & A_k \end{pmatrix}, \quad k = 0, \ldots, 3T + 3,$$  \hspace{1cm} (8)

and $'$ denotes the transpose of the matrix,

$$A_0 = \begin{bmatrix} -\frac{1}{2}g_{tt_1} & 0 \\ 0 & 0 \end{bmatrix}_{2T \times 2T},$$

where $[-g_{tt_1}]_{T \times T}$ is a symmetric matrix;

$A_k$ is a $2T \times 2T$ matrix with 1 on the $(k, k)$th entry, for $k = 1, \ldots, T$;

$A_k$ is a $2T \times 2T$ zero matrix, for $k = T + 1, \ldots, 3T + 3$;

$$b_0 = 0.5(y_1 - \sum_{t_1=1}^{T} g_{t_1}q_{t_1}^\circ, \ldots, y_T - \sum_{t_1=1}^{T} g_{t_1}q_{t_1}^\circ, 0, \ldots, 0)_{2T \times 1};$$

$$b_k$$ is a $2T \times 1$ vector with $-0.5(\nu + \tau)$ at the $k^{th}$ position, $k = 1, \ldots, T$;

$$b_{T+1} = 0.5(1, 1, 0, \ldots, 0)_{2T \times 1};$$

$$b_{T+2} = -0.5(1, \ldots, 1, 0, \ldots, 0)_{2T \times 1};$$

$$b_{T+3} = 0.5(0, \ldots, 0, 1, \ldots, 1)_{2T \times 1};$$

$b_{T+3+k}$ is a $2T \times 1$ vector with 0.5 at the $k^{th}$ position and $-0.5$ at the $(k + T)^{th}$ position, $k = 1, \ldots, T$;

$$b_{2T+3+k} = -0.5(1, \ldots, 1)_{2T \times 1}, \quad k = 1, \ldots, T;$$

$c_0 = 0$
\( c_k = v, \quad k = 1, \ldots, T; \)
\( c_{T+3} = -a; \)
\( c_k = 0, \quad k = T + 1, T + 2, T + 4, \ldots, 3T + 3. \)

5.2. Median-GMD safety model. The robust optimization of \((MGMD)\) model, under the uncertainty set \(Q_B\), is described as follows:

\[ (P3) \quad \max \theta \]

subject to \( \min_{q \in Q_B} (z - \frac{1}{2} \sum_{t_1=1}^{T} \sum_{t=1}^{T} g_{tt_1} q_t q_{t_1}) \geq \theta, \]
\[ (1) - (5), \]
\( g_{tt_1} = g_{t_1 t}, \quad t, t_1 = 1, \ldots, T, \]
\( x \in X. \)

Following the approach analogous to the previous subsection, the inner minimization problem in (9) can be resolved to the following QPP:

\[ (P4) \quad \min_{v_t, \gamma_t} -\sum_{t_1=1}^{T} \sum_{t=1}^{T} g_{tt_1} q_t q_{t_1}^2 v_t - \frac{1}{2} \sum_{t_1=1}^{T} \sum_{t=1}^{T} g_{tt_1} v_t v_{t_1} \]

subject to \( v_t^2 - (\tau + \mu) v_t + \tau v \leq 0, \quad t = 1, \ldots, T, \)
\( \sum_{t=1}^{T} v_t = 0, \)
\( \sum_{t=1}^{T} \gamma_t \leq a, \)
\( v_t - \gamma_t \leq 0, \quad t = 1, \ldots, T, \)
\( -v_t - \gamma_t \leq 0, \quad t = 1, \ldots, T, \)
\( \sum_{t=1}^{T} (q_t^2 s_t + v_t s_t) \leq 0.5. \)

In the same spirit, as for \((R_B \mu GMD)\) model, we utilize the dual of problem \((P4)\) in problem \((P3)\), under conditions ensuring zero duality gap, to obtain the following robust median model under \(Q_B\):

\[ (R_B MGMD) \quad \max \theta \]

subject to \( z - \frac{1}{2} \sum_{t_1=1}^{T} \sum_{t=1}^{T} g_{tt_1} q_t^2 q_{t_1}^2 + \tau \geq \theta, \)
\( M_0 - \tau \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{k=1}^{3T+4} \mu_k M_k \geq 0, \]
\[ (1) - (3) \& (5), \]
\( g_{tt_1} = g_{t_1 t}, \quad t, t_1 = 1, \ldots, T, \)
\( \mu_k \geq 0, \quad k = 1, \ldots, 3T + 4, \)
\( \tau \in \mathbb{R}. \)
The matrices $M_i$, $i = 1, \cdots, 3T + 4$, can be obtained in a similar way applying model (11) in Appendix. For the sake of brevity, we do not mention the matrices’ expressions. However, the noticeable point is that the matrix $M_{3T+4}$ contains the variable $s_t$ giving rise to a nonlinear constraint in (10). Thus, under the uncertainty set $Q_B$, the robust variant of ($MGMD$) model turns out to be a nonlinear problem which is challenging to solve.

6. Empirical analysis. All experiments are conducted on a 64-bit Windows 10 with 8 GB RAM and Intel(R) Core 3.41 GHz processor. We use GAMS software with MOSEK solver to solve LP and MILP problems and LINGO 16.0 software for solving the SDP problems.

Data

The weekly closing prices of the constituents of the four markets are obtained from Thomson Reuters EIKON data-stream:

- Dataset 1 (DI): Euro Stoxx 50 (Eurozone);
- Dataset 2 (DII): Dow Jones Global Titans 50 (Global);
- Dataset 3 (DIII): S &P Asia 50 (Asia);
- Dataset 4 (DIV): Nifty 50 (India).

Performance Metrics

The following metrics are used to test performance of the portfolios on the out-of-sample returns:

- average: mean of returns;
- sd: standard deviation of returns;
- median: median of returns;
- min: worst return;
- max: maximum return;
- neg returns: count of $y_t$ such that $y_t < 0$;
- VaR$_\alpha$: VaR at tolerance levels $\alpha = 0.03, 0.05$;
- CVaR$_\alpha$: CVaR at tolerance levels $\alpha = 0.03, 0.05$;
- MD: Maximum drawdown of returns;

The maximum drawdown measures the downside risk of a portfolio and is referred as the maximum loss incurred by an investor from a peak to a trough of returns graph of a portfolio. We calculate the maximum drawdown using the ‘Performance Analytics’ package of R software.

6.1. Mixture distribution.

6.1.1. Window analysis. Weekly closing prices of the four datasets are considered from 24/04/2015–12/10/2018. The number of assets for (DI), (DII), (DIII), and (DIV) in this period are 48, 54, 50 and 50, respectively. We adopt the rolling window strategy maintaining the in-sample period of 27 weeks, and the following week is taken to be the out-of-sample period. We re-balance the portfolio by shifting the in-sample period ahead by one week to obtain a total of 156 windows. The out-of-sample returns are concatenated to obtain 156 out-of-sample weekly returns.

The performance comparison of optimal portfolios from the robust optimization models is carried out with their non-robust counterparts. Consequently, we analyze the out-of-sample statistics for four models namely ($\mu GMD$), ($MGMD$), ($R\mu GMD$) and ($RMGMD$).

The mixed uncertainty set applies when returns follow different probability distributions in different periods. Following Zhu and Fukushima [48], we try to find a
pattern in return series. For example, we took 27 weeks returns of the S&P Asia 50 market index corresponding to the dataset (DIII) from 13/10/2017 to 13/04/2018. This period also forms one of the in-sample period in our rolling window analysis. From Figure 1, we notice that the time period can be divided into 3 parts, each comprising of approximately 9 weeks, such that returns vary largely across different periods. This observation is also confirmed from statistics in Table 1. We noted that the standard deviation of period 1 and period 2 vary largely while the average return in period 3 showed a sharp decline. Taking above observation into account and the works in ([48], [20], [43]), we decided to divide each in-sample window of 27 weeks into 3 equal parts by setting $L = 3$ and $q^t_l = \frac{1}{3}$, $t = 1, \ldots, 9, l = 1, 2, 3$, in the two robust models.

For the non-robust conventional models ($\mu GMD$) and ($MGMD$), we take the in-sample window with $T = 27$ and $q^t_l = \frac{1}{27}$, $t = 1, \ldots, 27$. 

\footnote{We have solved four optimization problems in 156 different windows on four datasets. It is practically not possible to get the same return pattern and graph illustration for each case.}
Table 2. Out-of-sample statistics ($\times 10^{-3}$) of (DI), (DII), (DIII) and (DIV) on rolling window analysis under mixed uncertainty set $Q_M$

|                  | $\mu_{GMD}$ | $\mu_{MGMD}$ | $R_{\mu_{GMD}}$ | $R_{\mu_{MGMD}}$ |
|------------------|-------------|--------------|-----------------|-----------------|
| **(DI)** average | 0.458       | 1.541        | -0.143          | 0.127           |
| sd               | 21.973      | 22.432       | 21.183          | 22.164          |
| median           | -0.404      | 3.035        | 1.032           | -0.717          |
| min              | -59.257     | -67.184      | -57.742         | -51.915         |
| max              | 55.669      | 63.717       | 57.2492         | 65.108          |
| neg returns      | 80          | 72           | 77              | 82              |
| $CVaR_{0.03}$    | 53.369      | 55.862       | 50.934          | 49.827          |
| $CVaR_{0.05}$    | 50.236      | 48.982       | 46.816          | 47.84           |
| $VaR_{0.03}$     | 49.945      | 44.695       | 43.091          | 46.16           |
| $VaR_{0.05}$     | 40.909      | 35.241       | 38.74           | 43.921          |
| MD               | 213.407     | 156.96       | 173.27          | 224.3           |
| **(DII)** average | 5.548       | 5.315        | 3.652           | 3.934           |
| sd               | 24.384      | 23.277       | 19.905          | 21.320          |
| median           | 7.812       | 5.212        | 5.081           | 5.634           |
| min              | -69.647     | -82.617      | -71.400         | -52.323         |
| max              | 145.054     | 104.310      | 63.899          | 81.426          |
| neg returns      | 56          | 57           | 55              | 61              |
| $CVaR_{0.03}$    | 57.914      | 53.802       | 57.633          | 47.689          |
| $CVaR_{0.05}$    | 49.836      | 47.188       | 48.391          | 43.867          |
| $VaR_{0.03}$     | 43.523      | 38.888       | 37.368          | 41.209          |
| $VaR_{0.05}$     | 33.153      | 34.584       | 31.053          | 37.39           |
| MD               | 118.11      | 146.64       | 111.057         | 149.49          |
| **(DIII)** average | 4.188       | 5.561        | 3.968           | 2.365           |
| sd               | 23.907      | 30.314       | 19.597          | 24.278          |
| median           | 2.817       | 3.380        | 4.128           | 2.592           |
| min              | -93.670     | -108.797     | -82.422         | -88.448         |
| max              | 87.815      | 79.020       | 62.667          | 74.162          |
| neg returns      | 68          | 72           | 62              | 70              |
| $CVaR_{0.03}$    | 61.937      | 73.741       | 52.024          | 63.487          |
| $CVaR_{0.05}$    | 51.816      | 61.994       | 40.559          | 52.514          |
| $VaR_{0.03}$     | 46.458      | 47.761       | 28.247          | 51.358          |
| $VaR_{0.05}$     | 28.933      | 41.034       | 22.291          | 32.79           |
| MD               | 157.024     | 106.34       | 190.67          | 188.84          |
| **(DIV)** average | 1.926       | 1.852        | 1.120           | 0.232           |
| sd               | 23.337      | 25.528       | 21.178          | 23.817          |
| median           | 3.853       | 2.276        | 1.655           | 1.307           |
| min              | -65.330     | -80.298      | -67.686         | -68.472         |
| max              | 54.355      | 72.800       | 50.498          | 60.039          |
| neg returns      | 71          | 70           | 73              | 76              |
| $CVaR_{0.03}$    | 57.727      | 61.945       | 58.193          | 58.013          |
| $CVaR_{0.05}$    | 50.202      | 56.738       | 50.081          | 52.9            |
| $VaR_{0.03}$     | 52.377      | 52.549       | 48.15           | 48.715          |
| $VaR_{0.05}$     | 36.036      | 48.283       | 31.815          | 43.657          |
| MD               | 180.98      | 233.87       | 170.43          | 191.327         |
Table 2 presents the out-of-sample statistics of the four data sets. We present the out-of-sample performance analysis comparing robust models vis a vis their conventional counterparts on various performance metrics.

Across all datasets, the robust models possess low standard deviation indicating a higher concentration of returns from portfolios near their mean. The worst returns from the robust models are generally higher than those from the corresponding non-robust models indicating the reduced downside risk in optimal portfolios from robust models. This feature is especially observed in the optimal portfolios from \((MGMD)\) and \((RMGMD)\) models on all data sets. However, we also observe a decrease in the maximum returns in the robust models.

In a nutshell, under the mixed uncertainty set, the range of returns of portfolios from robust models show low spread as opposed to its non-robust counterparts. We also observe lower maximum drawdown for \((R\mu GMD)\) model as opposed to portfolios obtained by \((\mu GMD)\) model. The lower values of standard deviation and a low width returns range are ideal for an investor seeking low volatility in returns and hence better performance in portfolio risk.

Robust portfolios also yield lower values for tail-based risk \(CVaR_\alpha\) and \(VaR_\alpha\) on almost all datasets compared to non-robust portfolios. A significant reduction can specifically be seen in \(CVaR_\alpha\) values for datasets \((DI)\) and \((DIII)\).

Figure 2 displays the out-of-sample downside risk \([42]\) from \((\mu GMD)\) and \((R\mu MGMD)\) models and \((MGMD)\) and \((RMGMD)\) models on dataset \((DIII)\)\(^2\). We obtained these graphs by arranging the out-of-sample returns in ascending order and plotting the sorted cumulative return. The downside risk curves of robust models are above the corresponding curves from the non-robust models for almost first 110 out-of-sample periods in Figure 2(b). Though eventually, due to higher average returns by the non-robust models, the two curves swap their positions. Our analysis indicates a lower downside risk in robust models for a longer duration in the period of investment.

6.1.2. Performance analysis under different market scenarios. In this subsection, we propose to examine the performance of the proposed models under different market conditions. We consider a period between 06/08/1999 and 16/04/2004, contemplating different phases of markets in all four datasets. We refer to the increasing (decreasing) trend of the market by UP (DOWN) phase. The entire data in each dataset is divided by noting the following:

- **UP-DOWN**: Increasing trend in the in-sample phase and decreasing trend in the out-sample phase
- **DOWN-DOWN**: Decreasing trend in the in-sample phase and decreasing trend in the out-sample phase
- **DOWN-UP**: Decreasing trend in the in-sample phase and increasing trend in the out-sample phase
- **UP-UP**: Increasing trend in the in-sample phase and increasing trend in the out-sample phase

The number of assets during the time periods mentioned in Figure 3 against datasets \((DI)\), \((DII)\), \((DIII)\) and \((DIV)\) are 44, 47, 30 and 36, respectively. In each phase, the in-sample period is 39 weeks, and the out-of-sample period is 20 weeks for all four datasets.

\(^2\)We have four models and four datasets, so four downside risk graphs. For illustrative purpose, we present the graphs for dataset \((DIII)\) only.
The non-robust models are solved using $T = 39$ and $q_t = \frac{1}{39}$. Again, to maintain the consistency throughout the paper, the robust models are solved by dividing each in-sample period into 3 equal parts and setting $q^l_t = \frac{1}{13}, t = 1, \ldots, 13, l = 1, 2, 3$.

Since there are only 20 out-of-sample data points, the VaR$_\alpha$ and CVaR$_\alpha$ for $\alpha = 0.03, 0.05$, are approximately the same as the worst return, and thus only the worst out-of-sample returns are reported.

Tables 3 and 4 present the out-of-sample statistics for datasets (DIII), (DIV), respectively.

We highlight some observations from our analysis in different phases of markets.

**UP-DOWN**
We observe higher values for average, median, worst out-of-sample return and maximum drawdown metrics, and lower standard deviation from robust portfolios compared to their non-robust counterparts. Thus, the robust models perform well in terms of maximizing returns and reducing the downside risk in portfolios.

**DOWN-DOWN**
The robust models possess lower standard deviation as opposed to non-robust models except for dataset (DII).

**DOWN-UP**
Worst returns from robust models are higher for almost all datasets indicating a reduced downside risk in portfolios. However, the maximum returns also decrease. A significant reduction in standard deviation from non-robust to robust portfolios can especially be seen for datasets (DIII) in (MGMD) and (RMGMD) models. We also observe a lower maximum drawdown for robust models in most cases. This phenomenon is especially observed in (DII) Also, the robust models yield better
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Figure 3. Cumulative returns of market indices corresponding to (DI), (DII), D(III) and (DIV) depicting different phases of markets.

out-of-sample average returns for (DI) and (DII) whereas no clear conclusion is available for (DIII) and (DIV).

Except for the dataset (DIV), we observe higher average returns and lower standard deviation in robust portfolios as opposed to the non-robust counterparts in most of the cases. The downside risk also diminishes in robust portfolios, which in turn generates higher worst returns and lower maximum drawdown values.

Figures 4 and 5 illustrate the out-of-sample cumulative return and downside risk respectively generated by portfolios from robust and corresponding non-robust models in the UP-DOWN phase on dataset (DIII). Figure 4 depicts that the out-of-sample cumulative returns of robust model dominate the non-robust counterpart model for (DIII) at each time point. In Figure 1(b), we note a major difference in the downside risk of robust and non-robust models.

We conclude that in all market phases, robust models yield lower values of standard deviation as opposed to the non-robust counterpart model. Also, except for the DOWN-DOWN phase, robust models outperform the conventional models in terms of higher average returns, median, and lower downside risk.

To understand the reasons for the poor performance of robust models in the DOWN-DOWN phase, we take a look at the allocations to assets in portfolios. We observed instances of stark differences in allocation patterns in robust and non-robust models. The robust model allocates an optimal weight of more than 50% to assets performing well in the in-sample periods of the DOWN phase. However,

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3There are four models, four datasets and four market scenarios. It is practically not possible to provide the graphs for each phase and each dataset. Thus, to maintain uniformity, we illustrate the out-of-sample cumulative returns and downside risk for dataset (DIII) in one market phase.
these assets fail to live up to their good show in the DOWN phase out-of-sample period leading to a sharp decline in average returns and maximum drawdown. In other words, a higher concentration in fewer assets by robust models seems a plausible cause for their poor performance in the DOWN-DOWN phase. We can avoid this situation by building more realistic constraints in robust models for better diversification, especially in the market meltdown phase.

6.2. Interval+polyhedral uncertainty set and numerical comparison of uncertainty sets. We perform a single window analysis to test the performance of $(RB_{\mu}GMD)$, because of it being computationally far-reaching to tackle. In addition, the $(RB_{B}MGMD)$ model is nonlinear and hard to solve, we could not perform any empirical investigation of the same. Furthermore, we solve the $(R_{\mu}GMD)$ model to compare the performance of both uncertainty sets on the same datasets in the same time frame.

We collected the data of the same four markets as earlier with an in-sample window of 27 weeks from 16/02/2018–17/08/2018 and an out-of-sample window of 4 weeks from 24/08/2018–14/09/2018.

The parameters in the uncertainty set $Q_B$ are set to $q_t^c = \frac{1}{27}$, $t = 1, \ldots, 27$, $\nu = \frac{1}{27}$, $\nu = -\frac{1}{27}$ and $a = 0.75$. Because of short out-of-sample period, we do not report VaR, CVaR and maximum drawdown in this section. For uncertainty set $Q_M$, we divide each in-sample window of 27 weeks into 3 equal parts by taking $L = 3$ and $q_t^l = \frac{1}{3}$, $t = 1, \ldots, 9$, $l = 1, 2, 3$. As the zero duality gap condition is practically hard to accomplish for SDP model in any of the datasets, we allow a maximum acceptable $3 \times 10^{-3}$ duality gap in SDP for empirical purpose.
Table 3. Out-of-sample statistics ($\times 10^{-3}$) of $(DI)$ and $(DII)$ under mixed uncertainty set $Q_M$ for market directions based four phases data

|        | $\mu GMD$ | $MGMD$ | $R\mu GMD$ | $RMGMD$ |
|--------|-----------|--------|------------|----------|
| UP-DOWN |           |        |            |          |
| average| -6.039    | -6.859 | 1.118      | -4.655   |
| sd     | 43.196    | 52.886 | 24.373     | 40.885   |
| median | -19.16    | -18.64 | -2.775     | -10.72   |
| min    | -80.7     | -87.88 | -41.22     | -64.31   |
| max    | 112.33    | 139.21 | 67.997     | 109.74   |
| neg returns | 12  | 13 | 11  | 12 |
| MD     | 151.538   | 151.107| 52.69      | 95.87    |
|        |           |        |            |          |
| DOWN-DOWN |        |        |            |          |
| average| -3.382    | -1.58  | -3.02      | -2.335   |
| sd     | 39.36     | 44.553 | 38.817     | 39.861   |
| median | -8.466    | -3.124 | -7.898     | -7.301   |
| min    | -76.31    | -122.3 | -77.25     | -93.32   |
| max    | 116.87    | 73.054 | 113.99     | 80.892   |
| neg returns | 12  | 12 | 12  | 11 |
| MD     | 209.357   | 241.872| 206.61     | 240.81   |
|        |           |        |            |          |
| DOWN-UP |        |        |            |          |
| average| 4.5948    | 6.5533 | 6.3614     | 9.0281   |
| sd     | 17.897    | 29.42  | 18.545     | 21.693   |
| median | 6.7725    | 4.9728 | 6.5631     | 11.843   |
| min    | -26.7     | -32.28 | -17.9      | -29.67   |
| max    | 42.982    | 60.549 | 50.299     | 37.346   |
| neg returns | 9   | 7 | 9   | 7 |
| MD     | 36.5      | 41.155 | 31.584     | 29.673   |
|        |           |        |            |          |
| UP-UP   |        |        |            |          |
| average| 4.5768    | 4.0462 | 5.1412     | 5.5771   |
| sd     | 19.647    | 28.427 | 18.428     | 16.52    |
| median | 2.6147    | 5.4864 | 6.6005     | 6.9434   |
| min    | -28.86    | -39.84 | -25.75     | -35.76   |
| max    | 36.136    | 50.381 | 37.729     | 29.958   |
| neg returns | 10  | 8  | 10  | 6 |
| MD     | 48.181    | 87.913 | 35.078     | 40.38    |
|        |           |        |            |          |
| UP-DOWN |        |        |            |          |
| average| -5.029    | -3.828 | -5.007     | 1.4072   |
| sd     | 66.172    | 72.696 | 61.369     | 58.82    |
| median | -5.895    | -0.19  | 8.9474     | 3.7326   |
| min    | -179.4    | -193.8 | -154.4     | -159.6   |
| max    | 129.46    | 146.11 | 126.09     | 140.28   |
| neg returns | 10  | 10 | 9   | 9 |
| MD     | 235.221   | 236.219| 222.13     | 150.582  |
|        |           |        |            |          |
| DOWN-DOWN |        |        |            |          |
| average| -1.545    | 0.006  | -2.109     | -2.249   |
| sd     | 34.09     | 30.876 | 35.189     | 33.007   |
| median | -4.819    | -3.358 | -4.806     | -4.546   |
| min    | -67.06    | -48.79 | -61.39     | -90.7    |
| max    | 115.75    | 72.059 | 122.57     | 76.609   |
| neg returns | 11  | 12 | 11  | 13 |
| MD     | 162.256   | 113.982| 173.88     | 159.283  |
|        |           |        |            |          |
| DOWN-UP |        |        |            |          |
| average| -6.301    | -6.031 | -2.023     | -4.701   |
| sd     | 17.066    | 24.228 | 15.888     | 19.831   |
| median | -4.875    | -9.129 | -3.986     | -6.138   |
| min    | -37.04    | -47    | -33.08     | -39.68   |
| max    | 28.022    | 53.495 | 38.54      | 58.312   |
| neg returns | 15  | 12 | 14  | 14 |
| MD     | 132.87    | 142.629| 92.60      | 119.743  |
|        |           |        |            |          |
| UP-UP   |        |        |            |          |
| average| 2.7559    | 2.8771 | 5.6724     | 7.6174   |
| sd     | 21.022    | 28.97  | 12.812     | 18.045   |
| median | 5.1358    | 2.4316 | 4.687      | 4.2454   |
| min    | -38.74    | -46.22 | -18.15     | -31.4    |
| max    | 51.007    | 65.855 | 36.726     | 48.432   |
| neg returns | 9   | 8 | 6   | 6 |
| MD     | 47.811    | 109.512| 18.517     | 36.649   |
Table 4. Out-of-sample statistics ($\times 10^{-3}$) of (DIII) and (DIV) under mixed uncertainty set $Q_M$ for market directions based four phases data

|          | µGMD | MGMD | RµGMD | RMGMD |
|----------|------|------|-------|-------|
| **DIII** |      |      |       |       |
| UP-DOWN  |      |      |       |       |
| average  | -0.2681 | -3.322 | 0.499 | -0.047 |
| sd       | 19.1129 | 35.824 | 18.919 | 18.732 |
| median   | 2.6173 | 1.5277 | 2.3188 | -0.6 |
| min      | -44.737 | -55.41 | -40.08 | -45.67 |
| max      | 34.6569 | 77.502 | 35.718 | 32.997 |
| neg returns | 9 10 | 9 11 | 68.053 | 64.932 |
| MD       | 74.493 | 159.476 | 117.954 | 122.734 |
| DOWN-DOWN |      |      |       |       |
| average  | -3.162 | 0.4164 | -4.351 | -4.109 |
| sd       | 21.5749 | 28.782 | 19.827 | 21.41 |
| median   | 2.12342 | -0.945 | -2.471 | -2.008 |
| min      | -48.014 | -48.23 | -63.18 | -66.7 |
| max      | 30.412 | 66.157 | 25.402 | 25.383 |
| neg returns | 9 10 | 11 10 | 39.14  | 48.21 |
| MD       | 119.543 | 106.662 | 127.845 | 124.788 |
| **DOWN-UP** |      |      |       |       |
| average  | 11.2417 | 30.789 | 4.1896 | 8.3275 |
| sd       | 20.3046 | 48.878 | 16.079 | 19.256 |
| median   | 11.6296 | 39.14 | 7.2174 | 8.4065 |
| min      | -27.77 | -71 | -33.04 | -40.3 |
| max      | 41.2383 | 98.964 | 25.703 | 46.593 |
| neg returns | 4 6 | 7 5 | 48.851 | 48.21 |
| MD       | 27.77 | 78.851 | 36.129 | 48.21 |
| **UP-UP** |      |      |       |       |
| average  | 1.05369 | -3.017 | 0.2775 | -0.183 |
| sd       | 38.8387 | 42.249 | 34.946 | 31.665 |
| median   | 2.73823 | 3.0881 | 1.0448 | 3.6441 |
| min      | -77.25 | -79.58 | -69.68 | -59.33 |
| max      | 66.1191 | 66.947 | 56.4 | 40.851 |
| neg returns | 9 10 | 10 9 | 40.851 | 48.21 |
| MD       | 131.452 | 160.753 | 127.845 | 124.788 |
| **DIV** |      |      |       |       |
| UP-DOWN  |      |      |       |       |
| average  | -23.233 | -28.74 | -24.08 | -23.28 |
| sd       | 94.011 | 107.97 | 87.816 | 88.499 |
| median   | -27.104 | -37.74 | -26.68 | -29.55 |
| min      | -197.61 | -222.9 | -192.2 | -170 |
| max      | 224.571 | 271.88 | 225.58 | 216.6 |
| neg returns | 13 13 | 13 14 | 489.529 | 489.529 |
| MD       | 517.924 | 576.729 | 507.51 | 489.529 |
| DOWN-DOWN |      |      |       |       |
| average  | 0.58685 | 1.1156 | -0.835 | 0.5133 |
| sd       | 28.4234 | 58.191 | 29.021 | 33.774 |
| median   | -3.4995 | -1.348 | -2.763 | -7.861 |
| min      | -36.563 | -79.36 | -46.89 | -47.1 |
| max      | 65.927 | 152.43 | 70.592 | 96.089 |
| neg returns | 9 10 | 10 11 | 96.089 | 96.089 |
| MD       | 137.989 | 285.251 | 160.875 | 164.657 |
| DOWN-UP  |      |      |       |       |
| average  | 7.63253 | 3.1941 | 6.4851 | 10.284 |
| sd       | 16.2213 | 15.549 | 15.238 | 19.419 |
| median   | 8.10496 | 2.7367 | 9.7434 | 10.025 |
| min      | -27.858 | -24.52 | -24.91 | -30.12 |
| max      | 47.6872 | 28.554 | 34.307 | 39.199 |
| neg returns | 5 7 | 7 7 | 96.089 | 96.089 |
| MD       | 42.35 | 40.646 | 35.51 | 53.741 |
| **UP-UP** |      |      |       |       |
| average  | 31.0544 | 30.368 | 17.863 | 15.659 |
| sd       | 56.3185 | 63.198 | 47.088 | 67.943 |
| median   | 51.5261 | 43.3 | 20.626 | 19.016 |
| min      | -124.52 | -136.7 | -97.17 | -109.3 |
| max      | 100.258 | 152.24 | 111.24 | 174.18 |
| neg returns | 4 7 | 6 7 | 124.517 | 139.654 |
Table 5 exhibits the out-of-sample statistics for \((\mu GMD)\), \((MGMD)\), \((R\mu GMD)\), \((RMGMD)\), and \((RB\mu GMD)\), on all four datasets.

We first compare the performance of \((\mu GMD)\) and \((RB\mu GMD)\) models. Even though the portfolio returns are negative for \((DI)\), \((DII)\), and \((DIV)\), we observe higher average returns from \((RB\mu GMD)\) than the one from non-robust model except for dataset \((DII)\). The same is noticed for the median returns wherein virtually in all cases the robust portfolios outperform the non-robust portfolios. Moreover, the \((RB\mu GMD)\) model observes a fewer number of negative returns for all four datasets.

While no model is consistent in generating the best average and median returns, \((RB\mu GMD)\) model yields the highest return by taking a lower risk in terms of sd as compared to \((R\mu GMD)\) model in datasets \((DIII)\) and \((DIV)\). However, the opposite is noted on the performance of the two models on datasets \((DI)\) and \((DII)\). Also, comparing the three robust models, the \((RMGMD)\) model performed poorly on the risk-return profile on all datasets.

To summarize the analysis under the interval+polyhedral uncertainty set, the robust models yield higher average returns and lower risk compared to the non-robust model. The question on which uncertainty set to use is hard to answer. Our observation-based on experiments suggests that \((RB\mu GMD)\) model fairs reasonably well on all datasets.
Table 5. Out-of-sample statistics ($\times 10^{-3}$) of (DI), (DII), (DIII) and (DIV) on a single window analysis under mixed and interval+polyhedral uncertainty sets

|       | $\mu_{GMD}$ | $MGMD$ | $R_{\mu GMD}$ | $RMGMD$ | $RB_{\mu GMD}$ |
|-------|-------------|--------|---------------|---------|----------------|
| (DI)  |             |        |               |         |                |
| average | -9.9697 | -17.83 | -9.51 | -5.5184 | -8.024 |
| sd     | 7.95041 | 12.595 | 5.629 | 19.863 | 12.937 |
| med    | -8.499 | -13.149 | -7.412 | -6.5741 | -6.643 |
| min    | -20.925 | -36.073 | -17.7 | -27.913 | -25.07 |
| max    | -1.9555 | -8.9473 | -5.514 | 18.987 | 6.2582 |
| (DII)  |             |        |               |         |                |
| average | -1.505 | -0.2718 | -0.303 | -5.5184 | -1.547 |
| sd     | 15.727 | 16.813 | 15.154 | 19.863 | 17.94 |
| med    | 4.52 | 0.3627 | 2.1474 | -6.5741 | 1.4017 |
| min    | -24.706 | -20.442 | -20.77 | -27.913 | -26.07 |
| max    | 9.631 | 18.629 | 15.258 | 18.987 | 17.077 |
| (DIII) |             |        |               |         |                |
| average | 4.60903 | 4.288 | 0.1445 | 0.3184 | 5.2191 |
| sd     | 23.0986 | 28.943 | 17.512 | 25.937 | 17.18 |
| med    | 8.31506 | 9.593 | 3.2971 | 2.3598 | 9.9122 |
| min    | -24.436 | -33.33 | -21.39 | -33.356 | -18.75 |
| max    | 26.2417 | 31.299 | 15.372 | 29.91 | 19.8 |
| (DIV)  |             |        |               |         |                |
| average | -12.383 | -18.126 | -11.54 | -18.126 | -7.655 |
| sd     | 14.327 | 24.613 | 18.997 | 24.613 | 18.297 |
| med    | -13.526 | -6.7408 | -9.971 | -6.7408 | -6.39 |
| min    | -28.661 | -54.996 | -35.76 | -54.996 | -30.83 |
| max    | 6.17971 | -4.0262 | 9.5596 | -4.0262 | 12.992 |

7. Conclusions. This research pursued applications of GMD in constructing safety measure based portfolio optimization problems. Our proposed return-risk safety models use mean and median for quantifying returns and GMD for measuring risk. We introduce the robust counterparts optimization models under the mixed and interval+polyhedral uncertainty sets. The models under mixed uncertainty set are LP and MILP, while those under interval+polyhedral uncertainty set are SDP models.

We tested the performance of robust models on four datasets of Euro Stoxx 50, Dow Jones Global Titans 50, S&P Asia 50, and Nifty 50. The empirical results indicate that portfolios from the robust optimization models are better equipped to hedge the investment against risk. The robust models under the mixed uncertainty set significantly reduce the standard deviation and downside risk of portfolios than those from the conventional non-robust analogs. An interval+polyhedral robust model exhibit higher average returns as well as lower risk in optimal portfolios as opposed to their non-robust counterparts.

The examination can be reached out to incorporate different sorts of uncertainty sets as well as by considering uncertainty in returns of assets similar to the one proposed by Bertsimas and Sim [5]. Besides, we can likewise investigate the viability of GMD based measures on linear chance-constrained optimization problem using RO in portfolio selection on the lines of research by Sengupta and Kumar [39].

Appendix. Consider the following non-convex QPP:

$$(A1) \quad \min \ x^T A_\phi x + 2b^T_\phi x$$
subject to $x^T A_k x + 2 b_k^T x + c_k \leq 0, \quad k = 1, \ldots, m,$

where $A_i, \ i = 0, 1, \ldots, m,$ are real symmetric matrices. Zheng et al. [47] explain the dual of (A1) as the following SDP:

$$(A2) \quad \max \tau \quad \text{subject to} \quad M_0 - \tau \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{k=1}^m \mu_k M_k \succeq 0,$$

$$\mu_k \geq 0, \quad k = 1, \ldots, m,$$

$$\tau \in \mathbb{R},$$

where, $$M_k = \begin{pmatrix} c_k & b_k^T \\ b_k & A_k \end{pmatrix}, \quad k = 0, \ldots, m, \quad c_0 = 0,$$ (11)

and $A \succeq B$ means $A - B$ is a positive semidefinite matrix. The sufficient conditions for zero duality gap are derived in Theorem 1 by Zheng et al. [47].

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