On Relative Entropy and Global Index

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Abstract

Certain duality of relative entropy can fail for chiral conformal net with nontrivial representations. In this paper we quantify such statement by defining a quantity which measures the failure of such duality, and identify this quantity with relative entropy and global index associated with multi-interval subfactors for a large class of conformal nets. In particular we show that the duality holds for a large class of conformal nets if and only if they are holomorphic. The same argument also applies to CFT in two dimensions. In particular we show that the duality holds for a large class of CFT in two dimensions if and only if they are modular invariant. We also obtain various limiting properties of relative entropies which naturally follow from our formula.

*Supported in part by NSF grant DMS-1764157.
1 Introduction

In the last few years there has been an enormous amount of work by physicists concerning entanglement entropies in QFT, motivated by the connections with condensed matter physics, black holes, etc.; see the references in [6] for a partial list of references. See [5], [12], [11], [13], [14], [19], [22] and [23] for a partial list of recent mathematical work.

This paper is motivated by a very simple fact about von Neumann entropy. In finite dimensional case the von Neumann entropy of a pure state for a matrix algebra $M$ and its commutant $M'$ are equal, a simple exercise in linear algebra. In the case of conformal net the algebra $M$ is replaced by the algebra of observables localized on disjoint union of intervals $I$ denoted by $\mathcal{A}(I)$. The vacuum state is a pure state. Hence one may expect that the von Neumann entropy of vacuum state for $\mathcal{A}(I)$ and its commutant are equal. But for type $\text{III}$ factors von Neumann entropy is always infinity so this is not very interesting. By the work of [1] and [14] one can define a regularized von Neumann entropy (cf. Def. 2.9) for $\mathcal{A}(I)$, denoted by $G(I)$, which is finite but not positive, yet verifies equations similar to von Neumann entropy in the finite dimensional case. When the global dimension of $\mathcal{A}$ is one, $\mathcal{A}(I)' = \mathcal{A}(I')$, one can therefore ask if the regularized von Neumann entropy for $\mathcal{A}(I)$ and $\mathcal{A}(I)' = \mathcal{A}(I')$ is the same. This is what we called a duality relation.

It was observed in §3 of [14] that the regularized von Neumann entropy for $\mathcal{A}(I)$ and $\mathcal{A}(I')$ are different when the global dimension of $\mathcal{A}$ is greater than one, and it is natural to conjecture that duality relation above holds if and only if the conformal net has global index equal to 1. The only currently known example that verify such a relation is the free fermion net for which we have explicit formulas for mutual information in general as in [14]. One of the goals of this paper is to prove that this conjecture is true for a large class chiral CFT (Cor. 2.16) and also CFT in two dimensions which are modular invariant (Cor. 3.7). For an example, it follows from Cor. 2.16 that such duality relation is true for conformal nets associated with any even positive unimodular lattices. The number of such lattices grow very fast as their rank increase.

To prove such results we are led to consider a quantity called deficit, which is simply the difference $D_\mathcal{A}(I) = G(I) - G(I')$, and conjecture (cf. 2.12) that $D_\mathcal{A}(I)$ is equal to another quantity $\hat{D}_\mathcal{A}$ which is defined by using the data associated with the inclusion $\mathcal{A}(I) \subset \mathcal{A}(I')$ (cf. [9]). Our key observation is Th. 2.13 that $D_\mathcal{A}(I) - \hat{D}_\mathcal{A}(I)$ remain the same for a pair of conformal nets $\mathcal{A} \subset \mathcal{B}$ with finite index. Recall that $D_\mathcal{A}(I) - \hat{D}_\mathcal{A}(I)$ for free fermion nets can be verified by explicit formulas of [14]. It follows that any conformal net $\mathcal{A}$ that is chain related to free fermion net $\mathcal{A}_r$, i.e., there exists a sequence of conformal nets $\mathcal{B}_1, \ldots, \mathcal{B}_n$ such that $\mathcal{B}_1 = \mathcal{A}, \mathcal{B}_n = \mathcal{A}_r$ and either $\mathcal{B}_i \subset \mathcal{B}_{i+1}$ or $\mathcal{B}_{i+1} \subset \mathcal{B}_i, 1 \leq i \leq n - 1$, and all inclusions are of finite index must verify our conjecture (cf. Cor. 2.14 and Cor. 3.6).

To give the reader an idea what kind of equalities are proved in this paper let us consider a special case of Cor. 2.14 for a conformal net $\mathcal{A}$ that is chain related to free
fermion net $\mathcal{A}_r$. Then for $I = I_1 \cup I_2, I' = J_1 \cup J_2$ we have

$$S(\omega, \omega_{J_1} \otimes \omega_{J_2}) - S(\omega, \omega_{I_1} \otimes \omega_{I_2}) - \frac{c}{6} \ln \eta = S(\omega, \omega F_I) - \frac{1}{2} \ln \mu_\mathcal{A}$$

where $S$ is the relative entropy, $\omega$ is the vacuum state, $c$ is the central charge, $\mu_\mathcal{A}$ is the global index of $\mathcal{A}$, $\eta = \frac{r_{J_1} r_{J_2}}{r_{I_1} r_{I_2}}$ is a cross ratio, and $F_I : \mathcal{A}(J_1 \cup J_2)' \to \mathcal{A}(I_1 \cup I_2)$ is the conditional expectation. Previously relations among relative entropies, central charge and global index are given in asymptotic form in Th. 4.2 of [14]. The above relation is an identity. The duality condition as described above holds when the righthand side is 0.

The rest of this paper is as follows: In §1 after introducing relative entropy, spatial derivatives, index for general von Neumann algebras, we prove a property of relative entropy 2.4 which is motivated by our conjecture above. In §2 we consider chiral conformal net. We first define a quantity $\mathcal{D}$ which is Deficit to measure the failure of duality and we prove our main theorem Th. 2.13. We deduce Cor. 2.14, Cor. 2.15 as consequences of Th. 2.13. In sections 2.4 and 2.5 we apply Th. 2.13 to study a number of natural problems on relative entropy.

In §3 we consider the two dimensional CFT cases while essentially all results of §2 hold with small modifications.

# 2 Preliminaries

## 2.1 Spatial derivatives, relative entropy and index theory for general subfactors

Let $\psi$ be a normal state on a von Neumann algebra $M$ acting on a Hilbert space $H$ and $\phi'$ be a normal faithful state on the von Neumann algebra $M'$. The Connes spatial derivative, usually denoted by $\frac{d\psi}{d\phi'}$, is a positive operator (cf. [3]). We will use the simplified notation of [15] and write $\frac{d\psi}{d\phi'} = \Delta(\psi \phi')$. If $\psi$ is faithful, we have

$$\Delta(\psi \phi')^it m \Delta(\psi \phi')^{-it} = \sigma_t^\psi (m), \forall m \in M, \Delta(\psi \phi')^it m \Delta(\psi \phi')^{-it} = \sigma_{-t}^{\phi'} (m), \forall m \in M'$$

where $\sigma_t^\psi, \sigma_{-t}^{\phi'}$ are modular automorphisms.

$$[D \psi_1 : \psi_2]_t := \Delta(\psi_1 \phi')^it \Delta(\psi_2 \phi')^{-it}$$

is independent of the choice of $\phi'$ and is called Connes cocycle.

Also if $\psi_1 \geq \psi_2$ then

$$\Delta(\psi_1 \phi') \geq \Delta(\psi_2 \phi')$$

By Page 476 of [21] this is equivalent to

$$\frac{1}{1 + \Delta(\psi_1 \phi')} \leq \frac{1}{1 + \Delta(\psi_2 \phi')}$$
as bounded operators.

Suppose $M$ acts on a Hilbert space $H$ and $\omega$ is a vector state given by $\Omega \in H$. The relative entropy (cf. 5.1 of [18]) in this case is $S(\omega, \phi) = -\langle \Delta(\phi/\omega')\Omega, \Omega \rangle$ for any pair of normal faithful weights $\phi$ and $\omega'$ where $\omega'$ is the vector state on $M'$ defined by vector $\Omega$ and $\Delta(\phi/\omega') := \frac{d\phi}{d\omega'}$ is Connes spatial derivative. When $\Omega$ is not in the support of $\phi$ we set $S(\omega, \phi) = \infty$.

A list of properties of relative entropies that will be used later can be found in [18] (cf. Th. 5.3, Th. 5.15 and Cor. 5.12 [18]):

**Theorem 2.1.** (1) Let $M$ be a von Neumann algebra and $M_1$ a von Neumann subalgebra of $M$. Assume that there exists a faithful normal conditional expectation $E$ of $M$ onto $M_1$. If $\psi$ and $\omega$ are states of $M_1$ and $M$, respectively, then $S(\omega, \psi \cdot E) = S(\omega \upharpoonright M_1, \psi) + S(\omega, \omega \cdot E)$;

(2) Let be $M_i$ an increasing net of von Neumann subalgebras of $M$ with the property $(\bigcup_i M_i)^\prime = M$. Then $S(\omega_1 \upharpoonright M_i, \omega_2 \upharpoonright M_i)$ converges to $S(\omega_1, \omega_2)$ where $\omega_1, \omega_2$ are two normal states on $M$;

(3) Let $\omega$ and $\omega_1$ be two normal states on a von Neumann algebra $M$. If $\omega_1 \geq \mu \omega$, then $S(\omega, \omega_1) \leq \ln \mu^{-1}$;

(4) Let $\omega$ and $\phi$ be two normal states on a von Neumann algebra $M$, and denote by $\omega_1$ and $\phi_1$ the restrictions of $\omega$ and $\phi$ to a von Neumann subalgebra $M_1 \subset M$ respectively. Then $S(\omega_1, \phi_1) \leq S(\omega, \phi)$;

(5) Let $\phi$ be a normal faithful state on $M_1 \otimes M_2$. Denote by $\phi_i$ the restriction of $\phi$ to $M_i, i = 1, 2$. Let $\psi_i$ be normal faithful states on $M_i, i = 1, 2$. Then

$$S(\phi, \psi_1 \otimes \psi_2) = S(\phi_1, \psi_1) + S(\phi_2, \psi_2) + S(\phi, \phi_1 \otimes \phi_2)$$

Let $E : M \to N$ be a normal faithful conditional expectation onto a subalgebra $N$. $E^{-1} : N' \to$ is in general an operator valued weight which verifies the following equation: for any pair of normal faithful weights $\psi$ on $N$ and $\phi'$ on $M'$ we have

$$\Delta(\psi E) = \Delta(\psi \phi' E^{-1})$$

Kosaki (cf. [8]) defined index of $E$, denoted by $\text{Ind} E$ to be $E^{-1}(1)$. When 1 is in the domain of $E^{-1}$, we say that $E$ has finite index. When both $N, M$ are factors and $E$ has finite index, we have the (cf. [20]) Pimsner-Popa inequality

$$E(m) \geq \lambda m, \forall m \in M_+,$$

where $\lambda = (\text{Ind} E)^{-1}$. The action of the modular group $\sigma^E_\psi$ on $N' \cap M$ is independent of the choice of $\psi$. When $E$ is the minimal conditional expectation such action is trivial on $N' \cap M$. Also the compositions of minimal conditional expectations are minimal (cf. [10]).

### 2.2 A result on relative entropy

**Lemma 2.2.** Let $A, B$ be positive unbounded operators on a Hilbert space such that $A \geq B$, and $\Omega$ is a unit vector such that $B\Omega = c\Omega$ where $c > 0$ is a constant, $\langle A\Omega, \Omega \rangle = \langle B\Omega, \Omega \rangle$. Then

$$\langle A\Omega, \Omega \rangle - \langle B\Omega, \Omega \rangle \geq \langle A\Omega, \Omega \rangle - \langle B\Omega, \Omega \rangle.$$
1. Let $m_A$ be the spectral measure of $A$ associated with $\Omega$. Then $\int_0^\infty (\ln \lambda)^2 dm_A(\lambda) < \infty$.

**Proof.** By Page 476 of [21] we have that $\frac{1}{1/n+A} \leq \frac{1}{1/n+B}$, $\forall n > 0$ and it follows

$$\int_0^\infty \frac{1}{1/n+\lambda} dm_A(\lambda) \leq \frac{1}{1/n+c}, \forall n > 0$$

Let $n$ goes to infinity and by Monotone convergence theorem we have

$$\int_0^\infty \frac{1}{\lambda} dm_A(\lambda) \leq \frac{1}{c}, \forall n > 0$$

We note that $(\ln \lambda)^2$ is bounded by a constant times $1/\lambda$ on $(0,1)$, and a constant times $\lambda$ on $[1,\infty)$. Since by assumption $\int_0^\infty \lambda dm_A(\lambda) = 1$, we have shown that

- $\int_0^\infty (\ln \lambda)^2 dm_A(\lambda) < \infty$,
- $\int_1^\infty (\ln \lambda)^2 dm_A(\lambda) < \infty$, and the proof is complete. 

**Lemma 2.3.** Let $A$ be a self adjoint operator on a Hilbert space, and $\Omega$ be a vector in the domain of $A$. Let $f(t)$ be a strong operator continuous function in a neighborhood of 0 with value in the space of bounded operators such that $f(0)$ is identity. Then

$$\lim_{t \to 0} -\frac{i}{t} \langle (e^{itA} - 1)f(t)\Omega, \Omega \rangle = \langle A\Omega, \Omega \rangle$$

**Proof.** By assumption it is enough to check that

$$\lim_{t \to 0} -\frac{i}{t} \langle (e^{itA} - 1)(f(t) - 1)\Omega, \Omega \rangle = 0$$

We note that

$$|| -\frac{i}{t} (e^{itA} - 1)\Omega ||^2 = \int |\frac{1}{t} (e^{it\lambda} - 1)|^2 dm_A(\lambda) \leq \int |\lambda|^2 dm_A(\lambda) < \infty$$

|| (f(t) - f(0))\omega ||

goes to 0 as $t$ goes to 0, and the lemma is proved. 

**Proposition 2.4.** Let $M$ be a factor and $\omega$ a normal faithful state on $M$ acting on the standard representation space $H$, and $\Omega$ the corresponding vector such that $\langle m\Omega, \Omega \rangle = \omega(m), \forall m \in M$. We shall use the same notation $\omega$ to denote the vector state on $B(H)$ and its restriction to subalgebras of $B(H)$.

Let $E_1 : M \to M_1, E_2 : M' \to M_2$ be normal conditional expectation with finite index, where $M_1, M_2$ are also factors. Then

$$S(\omega, \omega E_1) - S(\omega, \omega E_2) = S(\omega, \omega E_1 E_2^{-1})$$

and this equation can also be written as

$$S(\omega, \omega E_1) + S(\omega, \omega E_2^{-1}) = S(\omega, \omega E_1 E_2^{-1})$$
Proof. Ad (1): By definition we have

\[ S(\omega, \omega E_1) - S(\omega, \omega E_2) = \lim_{t \to 0} \frac{-i}{t} (\Delta(\frac{\omega E_2}{\omega})^it - (\Delta(\frac{\omega E_1}{\omega'})^it)\Omega, \Omega) \]

We note that

\[ \Delta(\frac{\omega E_1}{\omega'})^it \Delta(\frac{\omega}{\omega'})^it = \Delta(\frac{\omega E_1}{\omega'})^it \Delta(\frac{\omega}{\omega'})^it \]

It follows that

\[ S(\omega, \omega E_1) - S(\omega, \omega E_2) = \lim_{t \to 0} \frac{-i}{t} (\Delta(\frac{\omega E_1}{\omega E_2})^it - 1) \Delta(\frac{\omega E_1}{\omega E_2})^it\Omega, \Omega) \]

Note that \( \Delta(\frac{\omega E_1}{\omega E_2})^it \geq \mu \Delta(\frac{\omega}{\omega'})^it \), for some \( \mu > 0 \). Here the spatial derivative \( \Delta(\frac{\omega}{\omega'})^it \) is determined by state \( \omega \) on \( M_0' \) and \( M_2 \) respectively.

By Lemma 2.2 and Lemma 2.3 we have proved the first equation. Apply this equation with \( E_1 \) equal to identity we get

\[ S(\omega, \omega) - S(\omega, \omega E_2) = S(\omega, \omega E_2^{-1}) \]

and the second equation follows.

It is convenient to formulate the second equation of the above Prop. in the following form:

**Corollary 2.5.** Let \( N_3 \subset N_2 \subset N_1 \) be factors on a Hilbert space \( H \) and \( \omega \) is a vector state on \( B(H) \) given by a vector \( \Omega \in H \). Let \( F_i, N_i \to N_{i+1}, i = 1, 2 \) be conditional expectation with finite index. Assume that \( \Omega \) is cyclic and separating for \( N_2 \). Then

\[ S(\omega, \omega F_2 F_1) = S(\omega, \omega F_2) + S(\omega, \omega F_1) \]

Proof. This is just a reformulation of the second equation of Prop. 2.4 by noting that we can rename \( N_1 = M_2', N_2 = M, N_3 = M_1, F_1 = (\text{Ind } E_2)^{-1} E_2^{-1}, F_2 = E_1 \).

**Remark 2.6.** Under the conditions of the above Cor. \( S(\omega, \omega F) \) is additive under compositions of conditional expectations, just like \( \ln \text{Ind } E \). But of course \( S(\omega, \omega F) \) also depends on the state \( \omega \). This fact plays important role in the proof of Th. 2.13 and Th. 2.20 in the following.
2.3 Chiral CFT case

Let \( \mathcal{A} \) be a conformal net (cf. \cite{9} and \cite{14}). It is always split (cf. \cite{16}). Let \( \mathcal{P} \mathcal{I} \) be the set whose elements are disjoint union of intervals. If \( I \) is an interval on the circle with two end points \( a, b \), \( r_I := |b - a| \) is called the length of \( I \).

For any \( I \in \mathcal{P} \mathcal{I} \), \( \omega_I \) denotes the restriction of \( \omega \) to \( \mathcal{A}(I) \). It follows that \( \omega_I \otimes \ldots \otimes \omega_{I_n} \) is a normal state on \( \mathcal{A}(I) \).

Since we will be concerned with relative entropy of various states, we introduce some definitions to simplify notations. For \( I = I_1 \cup I_2 \cup \ldots \cup I_n \in \mathcal{P} \mathcal{I} \) where \( I_i \) are disjoint intervals,

\[ \omega^\otimes := \omega_{I_1} \otimes \omega_{I_2} \otimes \ldots \otimes \omega_{I_n}. \]

A state \( \psi \) on \( \mathcal{A}(I) \) is said to be related to vacuum state \( \omega \) if we can partition \( I \) into disjoint union \( I = J_1 \cup J_2 \cup \ldots \cup J_m, J_i \in \mathcal{P} \mathcal{I}, 1 \leq i \leq m \), such that \( \psi = \omega_{J_1} \otimes \omega_{J_2} \otimes \ldots \otimes \omega_{J_m} \).

We shall consider conformal net whose mutual information for vacuum state are always finite.

**Definition 2.7.** A conformal net \( \mathcal{A} \) is said to have finite mutual information if

\[ S(\omega, \omega_I^\otimes) < \infty, \forall I \in \mathcal{P} \mathcal{I}. \]

Suppose \( \mathcal{A} \subset \mathcal{B} \) is an inclusion of conformal nets with finite index. We shall denote by \( E_I : \mathcal{B}(I) \rightarrow \mathcal{A}(I) \) the unique conditional expectation which preserves the vacuum state when \( I \) is an interval. When \( I = I_1 \cup I_2 \cup \ldots \cup I_n \) is a disjoint union of \( n \) intervals, we shall use \( E_I \) to denote \( E_{I_1} \otimes \ldots \otimes E_{I_n} \) which is the unique conditional expectation from \( \mathcal{B}(I) \) to \( \mathcal{A}(I) \) which preserves \( \omega_{I_1} \otimes \ldots \otimes \omega_{I_n} \).

**Lemma 2.8.** (1) If \( \mathcal{A} \) has finite mutual information, then \( S(\omega, \psi) < \infty \) for all \( \psi \) on \( \mathcal{A}(I) \) that is related to vacuum state \( \omega \).

(2) If \( \mathcal{A} \subset \mathcal{B} \) and \( \mathcal{B} \) has finite mutual information, then \( \mathcal{A} \) also has finite mutual information;

(3) If \( \mathcal{A} \subset \mathcal{B} \) has finite index and \( \mathcal{A} \) has finite mutual information, then \( \mathcal{A} \) also has finite mutual information.

**Proof.** By (5) of Th. 2.1 we have

\[ S(\omega, \omega^\otimes_{J \cup I}) = S(\omega, \omega^\otimes_J) + S(\omega, \omega^\otimes_I) + S(\omega, \omega_I \otimes \omega) \]

and

\[ S(\omega, \psi_I \otimes \phi_I) = S(\omega, \psi_I) + S(\omega, \phi_I) + S(\omega, \omega_I \otimes \omega_I) \]

It follows that any \( S(\omega, \psi) \) can be expressed as linear combination of \( S(\omega, \omega^\otimes_J) \) for suitable intervals \( J \subset I \) and (1) is proved.

(2) follows from definition and monotonicity of relative entropy in Th. 2.1.

By Th. 2.1 \( S_B(\omega, \omega^\otimes_I) - S_A(\omega, \omega^\otimes_I) = S(\omega, \omega E_I) \). Since \( S(\omega, \omega E_I) \leq \ln(\text{Ind} E_I) \), (3) is proved.

It is proved on Page 13 of \cite{23} that essentially all known conformal net (and probably all) has finite mutual information.
A conformal net is called rational if for some $I = I_1 \cup I_2, I_1 \cap I_2 = \emptyset$ where the $A(I) \subset A(I')$ has finite index which is called Global index and is denoted by $\mu_A$.

Two conformal nets $A$ and $B$ are said to be chain related if there exists a sequence of conformal nets $B_1, \ldots, B_n$ such that $B_1 = A, B_n = B$ and either $B_i \subset B_{i+1}$ or $B_{i+1} \subset B_i$, $1 \leq i \leq n - 1$, and all inclusions are of finite index. See §4 of [14] for a large class of conformal nets that are chain related to free fermion nets.

For a conformal net $A$ with central charge $c$ and finite mutual information, we define:

**Definition 2.9.** The regularized von Neumann entropy of vacuum state for $A(I), I \in \mathcal{P}I$ is defined as follows: For an interval $I$ we let $G(I) := c/6 \ln r_I$, $r_I$ is the length of interval $I$, and

$$G(I_1 \cup I_2 \cup \ldots \cup I_n) = G(I_1) + \ldots + G(I_n) - S(\omega, \omega_{I_1} \otimes \omega_{I_2} \otimes \ldots \otimes \omega_{I_n})$$

Note that von Neumann entropy for type III factors are always infinity, and regularized von Neumann entropy as defined are motivated by the results of [1] and §3 of [14]. Note unlike relative entropy, the regularized von Neumann entropy is not always non negative and not invariant under the conformal transformations on $I$.

When $\mu_A = 1$, $A(I) = A(I')', \forall I \in \mathcal{P}I$, and the vacuum state $\omega$ is a pure vector state, we expect that the von Neumann entropy of $\omega$ for $A(I), I \in \mathcal{P}I$ and $A(I'), I \in \mathcal{P}I$ should be the same. Of course both are infinity, but what is more interesting is to conjecture that

$$G(I) = G(I'), \forall I \in \mathcal{P}I$$

if $\mu_A = 1$. In §3 of [14] we have shown that in general

$$G(I) \neq G(I')$$

if $\mu_A > 1$. Hence we expect that

$$G(I) = G(I'), \forall I \in \mathcal{P}I$$

if and only if $\mu_A = 1$. At present the only known example which verifies $\mu_A = 1$ and

$$G(I) = G(I'), \forall I \in \mathcal{P}I$$

is the free fermion net (cf. §2 of [14]) for which $G(I), \forall I \in \mathcal{P}I$ is known. To investigate the general cases we define the following

**Definition 2.10.** We define the deficit for $A(I), I \in \mathcal{P}I$ to be $D_A(I) := G_A(I) - G_{A(I')}$.

Let $F_I : A(I')' \to A(I)$ be the condition expectation of index $\mu_A^{n-1}$ (cf. [9]). When there are a pair of nets involved we shall use the notation $F_{I,A}$ to avoid confusions.

**Definition 2.11.** Let $I \in \mathcal{P}I$ be a disjoint union of $n$ intervals, define

$$\dot{D}_A(I) := S(\omega, \omega F_I) - \frac{n-1}{2} \ln \mu_A.$$
The main conjecture of this paper is

**Conjecture 2.12.** For a rational conformal net

\[ D_A(I) = \hat{D}_A(I) \]

Note that when \( \mu_A = 1 \), the above conjecture implies that

\[ G(I) = G(I'), \forall I \in \mathcal{P}I \]

Suppose \( A \subset B \) is an inclusion of conformal nets with finite index. Recall that \( E_I : B(I) \to A(I) \) is the unique conditional expectation which preserves the vacuum state when \( I \) is an interval. When \( I = I_1 \cup I_2 \cup \ldots \cup I_n \) is a disjoint union of \( n \) intervals, \( E_I \) denotes \( E_{I_1} \otimes \ldots \otimes E_{I_n} \) which is the unique conditional expectation from \( B(I) \) to \( A(I) \) which preserves \( \omega_{I_1} \otimes \ldots \otimes \omega_{I_n} \).

We will prove Conj. 2.12 for a large class of conformal nets. The idea is the following: Since we have an important example of free fermion net \( A_r \) for which we already know \( D_{A_r}(I) = \hat{D}_{A_r}(I) \), and there are many conformal nets that are chain related to \( A_r \), if we can show that for a pair of conformal nets \( A \subset B \) with finite index that

\[ D_A(I) - \hat{D}_A(I) = D_B(I) - \hat{D}_B(I) \]

then it follows that Conj. 2.12 is true for conformal nets that are chain related to \( A_r \). To state the theorem in more general terms, we note that assuming that all the quantities involved on the left hand side are finite, then

\[ D_A(I) - \hat{D}_A(I) = D_B(I) - \hat{D}_B(I) \]

is equivalent to

\[ S(\omega, \omega E_I) - S(\omega, \omega E_{I'}) = \hat{D}_A(I) - \hat{D}_B(I) \]

Then the following Th. does exactly this:

**Theorem 2.13.** (1) Let \( A \subset B \) be rational conformal nets with finite index, then

\[ S(\omega, \omega E_I) - S(\omega, \omega E_{I'}) = \hat{D}_A(I) - \hat{D}_B(I) \]

(2) (1) also holds when \( B \) is free fermion net \( A_r \).

**Proof.** Fix \( I \in \mathcal{P}I \) which is a disjoint union of \( n \) intervals.

Ad (1): Let \( E := (\text{Ind} E_{I'}\text{Ind} F_{I', B})^{-1} E_{I'} F_{I', B}^{-1} E_{I'}^{-1} \) be the condition expectation from \( A(I')' \to A(I) \). Set \( E_1 := E_{I'} F_{I', B}^{-1} E_{I'}^{-1} \).

Let us compute \( S(\omega, \omega E_I) - S(\omega, \omega F_{I', B}) - S(\omega, \omega E_{I'}) \). Note that \( \Omega \) is separating and cyclic for \( B(I')' \). By Prop. 2.4 we have

\[ S(\omega, \omega E_I) - S(\omega, \omega F_{I', B}) - S(\omega, \omega E_{I'}) = S(\omega, \omega E_1) \]
By §4 of \cite{8} and \cite{10} \(E\) restricts to trace on \(\mathcal{A}(I')\cap\mathcal{A}(I')'\). Let \(P_A\) be the projection in \(\mathcal{A}(I')\cap\mathcal{A}(I')'\) which projects onto the closure of \(\mathcal{A}(I)\Omega\). Then we have

\[
\Delta \left( \frac{\omega F}{\omega'} \right)^{it} P_A \Delta \left( \frac{\omega F}{\omega'} \right)^{-it} = P_A, \forall t
\]

where \(\omega'\) is the state on \(\mathcal{A}(I')\) given by \(\Omega\). It follows that \(\Delta \left( \frac{\omega F}{\omega'} \right)\) commutes with \(P_A\).

We note that when restricted to \(P_A \mathcal{A}(I')' P_A\), \(\omega E\) is given by \(E(P_A)\omega E P_A\) where

\[
E_P: P_A \mathcal{A}(I')' P_A \rightarrow P_A \mathcal{A}(I)
\]

is the unique conditional expectation and can be identified with \(F_{I,A}: \mathcal{A}(I')' \rightarrow \mathcal{A}(I)\)

where the algebras are on \(P_A H_B = H_A\). Note that \(E(P_A) = [B : A]^{-1} = \frac{\mu_A}{\mu_B}\). Hence

\[
\langle \ln \Delta \left( \frac{\omega F}{\omega'} \right) \Omega, \Omega \rangle = \ln E(P_A) + \langle \ln \Delta \left( \frac{\omega F_{P_A}}{\omega} \right) \Omega, \Omega \rangle = \ln E(P_A) + \langle \ln \Delta \left( \frac{\omega F_{I,A}}{\omega'} \right) \Omega, \Omega \rangle
\]

Note that

\[
\text{Ind} E_I = \left( \frac{\mu_A}{\mu_B} \right)^{n/2}, \text{Ind} F_{I',B} = \mu_B^{-1}
\]

Putting the above pieces together we have shown that

\[
S(\omega, \omega E_I) - S(\omega, \omega F_{I',B}) - S(\omega, \omega E_{I'}) = S(\omega, \omega F_{I,A}) - \frac{n-1}{2} (\ln \mu_A + \ln \mu_B)
\]

Finally by Prop. \ref{2.4} we have

\[-S(\omega, \omega F_{I,B}) = S(\omega, \omega) - S(\omega, \omega F_{I,B}) = S(\omega, \omega F_{I,B}^{-1}) = S(\omega, \omega F_{I,B}) - (n - 1) \ln \mu_B\]

and the proof of the theorem is complete.

Ad (2): Note that in this case \(F_{I,B}\) is identity, so we only need to evaluate

\[
S(\omega, \omega E_I) - S(\omega, \omega E_{I'})
\]

Note that \(E_{I'}^{-1}: \mathcal{A}(I')' \rightarrow \mathcal{A}_r(I')' = k \mathcal{A}_r(I) k^{-1}\) where \(k\) is the Klein transform. Let us define

\[
\hat{E}_I(ka^{-1}) = E_I(a), \forall a \in \mathcal{A}_r(I)
\]

Since \(k \Omega = \Omega\), it follows that

\[
\omega(\hat{E}_I(ka^{-1})) = \omega(E_I(a)), \omega(kak^{-1}) = \omega(a)
\]

and \(S(\omega, \omega E_I) = S(\omega, \omega \hat{E}_I)\). Hence by (2) of Prop. \ref{2.4}

\[
S(\omega, \omega E_I) - S(\omega, \omega E_{I'}) = S(\omega, \omega \hat{E}_I) - S(\omega, \omega E_{I'}) = S(\omega, \omega \hat{E}_I E_{I'}^{-1})
\]

The rest of the proof is the same as in (1) above.

By Th. \ref{2.13} we immediately have
Corollary 2.14. If $\mathcal{A}$ is chain related to $\mathcal{A}_r$, then Conj. 2.12 is true for $\mathcal{A}$.

We also have

Corollary 2.15. If $\mathcal{A}$ is chain related to $\mathcal{A}_r$, then $D\mathcal{A} = 0$ if and only if $\mu_\mathcal{A} = 1$.

Proof. If $\mu_\mathcal{A} = 1$, then $D\mathcal{A} = 0$ by Cor. 2.14. Now suppose that $D\mathcal{A}(I_1 \cup I_2) = 0$. By (2) of Th. 4.2 in [14], it follows that $\mu_\mathcal{A} = 1$. $\blacksquare$

Corollary 2.16. Conj. 2.12 is true for conformal nets associated with even positive definite lattices.

Proof. First we prove this for rank one lattices. Let $\mathcal{A}_{U(1)a}$ be the conformal net associated with rank one lattice with $a$ a positive even integer. Denote by $D_1(a) := D_{\mathcal{A}_{U(1)a}}(I) - \hat{D}_{\mathcal{A}_{U(1)a}}(I)$. We prove by induction on $k$ that

$$D_1(ka) = D_1(a), \forall k \geq 1$$

When $k = 1$ this is trivial. Assume the above equation is true for $k$. Consider the following finite index inclusions:

$$U(1)_{(k+1)a} \times U(1)_{(k+1)ka} \subset U(1)_{ka} \times U(1)_{a}$$

where $U(1)_{(k+1)a}$ is diagonally embedded in $U(1)_{ka} \times U(1)_{a}$ and its commutant in $U(1)_{ka} \times U(1)_{a}$ is $U(1)_{(k+1)ka}$.

By Th. 2.13 and induction hypothesis we have

$$2D_1((k + 1)a) = 2D_1(a)$$

and it follows by induction we have proved

$$D_1(ka) = D_1(a), \forall k \geq 1.$$

Now from the inclusion

$$U(1)_2 \times U(1)_2 \subset U(1)_{1} \times U(1)_1$$

and Th. 2.13 we conclude that $D_1(2) = 0$. It follows that $D_1(a) = 0$ for all even $a$.

Now assume that the Corollary is proved for all rank $k$ lattices. If $L$ is an even positive definite lattice, choose a nonzero element $e \in L$ and consider sublattices $L_1 = Ze$ of $L$ and $L_2$ of $L$ which is orthogonal to $L_1$ with rank equal to $k$. Apply Th. 2.13 to the finite index inclusions

$$\mathcal{A}_{L_1} \otimes \mathcal{A}_{L_2} \subset \mathcal{A}_L$$

and induction hypothesis, we have proved the Corollary. $\blacksquare$
2.4 Some continuous properties

Let us first fix a rational conformal net $\mathcal{A}$ with finite mutual information.

By (2) of Th. 2.1 relative entropies are continuous from “inside”. As an application of Th. 2.13 we will prove that relative entropies in Th. 2.13 are also continuous from “outside”. First we have:

**Lemma 2.17.** If $I \subset J, I, J \in \mathcal{PI}$, then $F_J$ restrict to $F_I$ on $\mathcal{A}(I)$ and hence $S(\omega, \omega F_I)$ increase with $I$;

**Proof.** This is proved in §2 of [9] for $n = 2$, but the same argument works for any $n$. ■

**Corollary 2.18.** Let $\mathcal{A} \subset \mathcal{B}$ be as in Th. 2.13. Then $S(\omega, \omega E_I)$ is continuous from “outside”, i.e., if $I_n$ is a decreasing sequence of intervals such that $\cap I_n = I$, and $E_{I'}$ restrict to $E_{I'_n}$, then

$$\lim_{n \to \infty} S(\omega, \omega E_{I_n}) = S(\omega, \omega E_I)$$

**Proof.** This follows from Th. 2.13 and Lemma 2.17. ■

2.5 Singular limits

It is usually an interesting problem to study the limiting properties of relative entropies when intervals get close together. One can find such studies in §3 and §4 of [14]. In the same spirit we will consider such singular limits for related entropy $S(\omega, \omega F_I)$ for a conformal net $\mathcal{A}$.

The following Theorem is a reformulation of Proposition 3.25 of [14]:

**Theorem 2.19.** Assume that $M_n$ is an increasing sequence of factors act on a fixed Hilbert space, $N_n \subset M_n$ are subfactors and $\omega$ is a vector state associated with a vector $\Omega$. Suppose that $E_n : M_n \to N_n, n \geq 1$ is a sequence of conditional expectations such that when restricting to $M_n$, $E_{n+1} = E_n, n \geq 1$, and $\text{Ind} E_n = \lambda$ is a positive real number independent of $n$. If strong operator closure of $\cup_n N_n$ contains $M_1$, then

$$\lim_{n \to \infty} S(\omega, \omega E_n) = \ln \lambda$$

**Proof.** Set $\phi_n := \omega E_n$.

It is sufficient to prove the following as in Proposition 3.25 of [14]: Given any $\epsilon > 0$, we need to find $e \in M_n$ for sufficiently large $n$, such that

$$|\omega(e) - 1| < \epsilon, |\omega(e^*) - 1| < \epsilon, |\omega(e^* e) - 1| < \epsilon, |\phi_n(ee^*) - \lambda| < \epsilon$$

Let $e_1 \in M_1$ be the Jones projection for $E_1 : M_1 \to N_1$, and $v \in N_1$ be the isometry such that $\lambda v^* e_1 v = 1$. By assumptions we can find a sequence of elements $e_n \in N_n, n \geq 2$ which converges in strong star topology to $e_1$. Now choose $x_n =$
\[ \lambda^{-1} v^* e_1 e_n v. \] Then \( x_n \to 1 \) in strong star topology, and so \( \omega(x_n), \omega(x_n x_n^*) \) converges to 1. On the other hand by definition
\[
F_n(x_n^* x_n) = v^* e_n^* e_n v
\]
converges to \( v^* v = \lambda^{-1} \) strongly. Hence given any \( \epsilon > 0 \), we can choose \( n \) sufficiently large such that if we set \( e = x_n^* \), then \( e \in M_n \), and
\[
|\omega(e) - 1| < \epsilon, |\omega(e^*) - 1| < \epsilon, |\omega(e^* e) - 1| < \epsilon, |\phi_n(e e^*) - \lambda| < \epsilon.
\]

Let \( I = I_1 \cup I_2 \cup \cdots \cup I_n \in \mathcal{PL} \) and \( I' = \hat{I}_1 \cup \hat{I}_2 \cup \cdots \cup \hat{I}_n \). Let us arrange indices such that \( \hat{I}_i \) share end points with \( I_i, I_{i+1}, 1 \leq i \leq n - 1 \). We are interested in shrinking \( I' \). Let us first introduce some terminology. By a contraction of \( I \) along \( \hat{I}_1 \) we mean keep \( I_1 \cup \overline{I}_1 \cup I_2 := I_{12} \) fixed and let the length of \( \hat{I}_1 \) go to 0. We will use a sequence \( I_1(k), \hat{I}_1(k), I_2(k) \) such that \( \hat{I}_1(k) \) is decreasing to describe such a process. Such a sequence is called a contraction sequence along \( \hat{I}_1 \). Let \( C_1(I) = I_{12} \cup I_3 \cup \cdots \cup I_n \in \mathcal{PL} \).

**Theorem 2.20.** Choosing a contracting \( I_1(k), \hat{I}_1(k), I_2(k) \) sequence along \( \hat{I}_1 \). Then
\[
\lim_{k \to \infty} S(\omega, \omega F_{I_1}) = S(\omega, \omega F_{C_1(I)}) + \ln \mu
\]

**Proof.** Observe that when restricting \( F_{C_1(I)} \) to \( \mathcal{A}(I')' \), we get a conditional expectation simply denoted only in the proof by \( F_k : \mathcal{A}(I')' \to \mathcal{A}(C_1(I)) \cap \mathcal{A}(\hat{I}_1)' \). Let \( E_k : \mathcal{A}(C_1(I)) \cap \mathcal{A}(\hat{I}_1)' \to \mathcal{A}(I) \) be the conditional expectation such that \( E_k \) restricts to identity on \( \mathcal{A}(I_3 \cup \cdots \cup I_n) \), and on \( \mathcal{A}(I_{12}) \cap \mathcal{A}(\hat{I}_1)' \) the unique conditional expectation onto \( \mathcal{A}(I_1 \cup I_2) \). Note that the index of \( E_k \) is \( \mu \). Notice that \( \Omega \) is cyclic and separating for \( \mathcal{A}(C_1(I)) \cap \mathcal{A}(\hat{I}_1)' \). By Cor. 2.5 we have
\[
S(\omega, \omega F_{I_1}) = S(\omega, \omega E_k F_k) = S(\omega, \omega E_k) + S(\omega, \omega F_k)
\]

By (2) of Th. 2.1 we have \( \lim_k S(\omega, \omega F_k) = S(\omega, \omega F_{C_1(I)}) \). To finish the proof it is sufficient to show that
\[
\lim_k S(\omega, \omega E_k) = \ln \mu.
\]

This follows from Th. 2.19 since \( \cup_k I_1(k) \cup I_2(k) \) is equal to \( I_{12} \) minus a point. ■

We note that we can apply Th. 2.20 a few times to shrink intervals \( \hat{I}_2, \ldots, \hat{I}_{n-1} \) successively. This way we see that
\[
\lim_k S(\omega, \omega F_{I_k}) = \frac{n - 1}{2} \ln \mu_A
\]
where one take an increasing of disjoint intervals \( I_k \), each one is a disjoint union \( n \) intervals such that \( \cup_k I_k \) is equal to \( S^1 \) minus finitely many points. This can of course be proved directly using Th. 2.19

Now consider the case of \( \mathcal{A} \subset \mathcal{B} \) with finite index.
Lemma 2.21. Choosing a contracting $I_1(k)$, $\hat{I}_1(k)$, $I_2(k)$ sequence along $\hat{I}_1$. Then
\[
\lim_{k \to \infty} S(\omega, \omega E_I) = 1/2(\ln \mu_A - \ln \mu_B) + S(\omega, \omega E_{C_1(I)})
\]

Proof. For the ease of notations we set $\omega := \omega_{I_3} \otimes \ldots \otimes \omega_{I_n}$. By (5) of Th. 2.1
\[
S(\omega, \omega I_1 \otimes I_2 \otimes \omega_2) = S(\omega, \omega I_1 \otimes I_2) + S(\omega, \omega_2) + S(\omega, \omega I_1 \cup I_2 \otimes \omega_2)
\]
We note that as $k$ goes to infinity, $I_1 \cup I_2$ increase to $I_{12}$, hence
\[
\lim_{k} S(\omega, \omega I_1 \cup I_2 \otimes \omega_2) = S(\omega, \omega I_{12} \otimes \omega_2)
\]
Hence
\[
\lim_{k} S(\omega, \omega E_I) = \lim_{k} S(\omega, \omega E_{I_1 \cup I_2}) + S(\omega, \omega E_{C_1(I)})
\]
The lemma now follows from Th. 4.4 of [14]. ■

Proposition 2.22. Let $A \subset B$ be as in Th. 2.13. Choosing a contracting $I_1(k), \hat{I}_1(k), I_2(k)$ sequence along $\hat{I}_1$. Then
\[
\lim_{k \to \infty} S(\omega, \omega E_{I}) = S(\omega, \omega E_{C_1(I)})
\]
This follows from Th. 2.13, Th. 2.20 and Lemma 2.21. ■

The above Cor. can be phrased as follows: Let $I_k = I_{1k} \cup I_2 \cup \ldots \cup I_n \in \mathcal{PI}$ be such that $I_{1k}$ is a decreasing sequence such that the length of $I_{1k}$ tends to 0 as $n$ goes to infinity. Then
\[
\lim_{k \to \infty} S(\omega, \omega E_{I_k}) = S(\omega, \omega E_{I_2 \cup \ldots \cup I_n})
\]
It follows that if either $A$ or $B$ has the property that
\[
\lim_{k \to \infty} S(\omega, \omega_{I_k}) = S(\omega, \omega_{I_2 \cup \ldots \cup I_n})
\]
then the other net also has this property. In particular all conformal nets that are chain related to free fermion nets have this property since free fermion nets verify such property. It will be interesting to see if this can be proved under more general conditions.

3 CFT in two dimensions

For a formulation of CFT in two dimensions we refer to §2 of [7] for more details.

A double cone $C$ is defined to be $I \times J$ where $I, J$ are intervals on the circle $S^1$, and we consider $C$ to be a subset of $S^1 \times S^1$. Denote by $\mathcal{PC}$ the set which consists of finite disjoint union of double cones. We shall use $C'$ to denote the casual complement of $C$.

We will consider the case $A \subset B$ where $A(I \times J) = A_L(I) \times A_R(J)$, both $A_L$ and $A_R$ are rational, and $A \subset B$ has finite index. Denote by $c_L, c_R$ the central charges of $A_L$ and $A_R$. 14
Definition 3.1. For a double cone $C = I \times J$ we let $G(C) := c_L/6 \ln r_I + c_R/6 \ln r_J$, and

$$G(C_1 \cup C_2 \cup ... \cup C_n) = G(C_1) + ... + G(C_n) - S(\omega, \omega_{C_1} \otimes \omega_{C_2} \otimes ... \otimes \omega_{C_n})$$

Definition 3.2. We define the deficit for $B(C), C \in \mathcal{P}C$ to be $D_B(C) := G_B(C) - G_B(C')$.

Note that when the two dimensional net is tensor product $A_L \otimes A_R$, and $C = C_1 \cup C_2 \cup ... \cup C_n, C_i = I_i \times J_i, 1 \leq i \leq n$, we have

$$G_{A_L \otimes A_R}(C) = G_{A_L}(I_1 \cup I_2 \cup ... \cup I_n) + G_{A_R}(J_1 \cup J_2 \cup ... \cup J_n)$$

Let $F_C : B(C') \to B(C)$ be the condition expectation of index $\mu_B^{n-1}$.

Definition 3.3. When $C$ is a disjoint union of $n$ double cones, define

$$\hat{D}_B(C) := S(\omega, \omega F_C) - \frac{n-1}{2} \ln \mu_B.$$

The Conj. 2.12 for $B$ is now

Conjecture 3.4. For a rational two dimensional conformal net

$$D_B(C) = \hat{D}_B(C)$$

The proof of Th. 2.13 applies verbatim to the case of two dimensional conformal nets $A \subset B$, and we have the following

Theorem 3.5. (1) Let $A \subset B$ be rational two dimensional conformal nets with finite index, then

$$S(\omega, \omega E_C) - S(\omega, \omega E_C') = \hat{D}_A(C) - \hat{D}_B(C)$$

Corollary 3.6. Suppose $B$ is chain related to $A_L \otimes A_R$, where both $A_L$ and $A_R$ are chain related to $A_r$, then Conj. 3.4 is true for $B$.

We also have

Corollary 3.7. (1) Suppose $B$ is chain related to $A_L \otimes A_R$, where both $A_L$ and $A_R$ are chain related to $A_r$ then $D_B = 0$ if and only if $\mu_B = 1$;

(2) Suppose that $A_L \otimes A_R \subset B$, and both $A_L$ and $A_R$ are chain related to $A_r$, then $D_B = 0$ if and only if $B$ is modular invariant.

Proof. The proof of (1) is the same as the proof of (1) of Cor. 2.15 (2) follows from Th. 4.2 of [15].

A large class of examples with $\mu_B = 1$ can be obtained as follows: take any conformal net $\mathcal{A}$ which is chain related to free fermion net and take the Longo-Rehren two dimensional net (which corresponds to identity modular invariant), it follows by the above corollary that such net verifies $D_B = 0$. 
Remark 3.8. The computation of entropies in physics literature is usually done (cf. [2]) with replica trick using path integrals, and when the underlying CFT can be described by a Lagrangian it is usually assumed that the CFT is modular invariant. In cases where such computations are done, one finds that the deficit vanishes. Hence (2) of the above Cor. is a rigorous formulation of such intuitions.

Finally we note that the results of sections 2.4 and 2.5 apply to two dimensional conformal nets as well, with essentially the same proof.

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