BERNOULLI OPERATOR AND RIEHMANN’S ZETA FUNCTION

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Abstract. We introduce a Bernoulli operator, let ”B” denote the operator symbol, for n=0,1,2,3,... let $B^n := B_n$ (where $B_n$ are Bernoulli numbers, $B_0 = 1, B_1 = 1/2, B_2 = 1/6, B_3 = 0...$). We obtain some formulas for Riemann’s Zeta function, Euler constant and a number-theoretic function related to Bernoulli operator. For example, we show that

$$B^{1-s} = \zeta(s)(s - 1),$$

$$\gamma = -\log B,$$

where $\gamma$ is Euler constant. Moreover, we obtain an analogue of the Riemann Hypothesis (All zeros of the function $\xi(B + s)$ lie on the imaginary axis). This hypothesis can be generalized to Dirichlet L-functions, Dedekind Zeta function, etc. In fact, we obtain an analogue of Hardy’s theorem (The function $\xi(B + s)$ has infinitely many zeros on the imaginary axis).

In addition, we obtain a functional equation of $\log \Pi(Bs)$ and a functional equation of $\log \zeta(B + s)$ by using Bernoulli operator.

1. Introduction

We introduce a Bernoulli operator, let ”B” denote the operator symbol, for n=0,1,2,3,... let $B^n := B_n$ (where $B_n$ are Bernoulli numbers, $B_0 = 1, B_1 = 1/2, B_2 = 1/6, B_3 = 0...$). Despite the fact that Bernoulli defined $B_1 = 1/2$, some authors set $B_1 = -1/2$. In this paper, it will be convenient to set $B_1 = 1/2$. Using the operator, we are easy to obtain the equation $(1 - B)^n = B^n$ (n is non-negative integer), and the following equation

$$e^{-Bz} = \frac{z}{e^z - 1}.$$

First we introduce some definitions.

If a function’s (equation’s) expression contains the operator symbol ”B”, then we say the function(equation) is a Bernoulli operator function (equation) or simply a function (equation). For example, the function $f(z) = e^{-Bz}$ is a Bernoulli operator function.

If a Bernoulli operator function (equation) equal to another Bernoulli operator function (equation) without taking Bernoulli operator, we say the function (equation) is primitive equivalent. For example, the following equation is primitive equivalent:

$$\sin(\pi B/2) = \sin(\pi B/2 + 2\pi).$$

If a Bernoulli operator function (equation) equal to another Bernoulli operator function (equation) by taking Bernoulli operator, we say the function (equation) is Bernoulli operator equivalent. For example, the following equation is Bernoulli operator equivalent:

$$\sin(\pi B/2) = \sin(\pi B/2 + 2\pi).$$

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Let $Mf$ plane, we use (1.1) that is if the right of the expansion does not converge in a certain areas of the complex Euler-Maclaurin formula (see [2] p.104):

The zeros (poles, singularity) of a Bernoulli operator function without taking Bernoulli operator are called primitive zeros (poles, singularity). For example, the primitive zeros of function $\sin(\pi(B + z)/2)$ are $-B + 4k$ (k is an integer).

The zeros (poles, singularity) of a Bernoulli operator function by taking Bernoulli operator are called Bernoulli operator zeros (poles, singularity). For example, the Bernoulli operator zero of function $ze^{-Bz} = z^2/(e^z - 1)$ is the number 0. When there is no confusion, we say a Bernoulli operator zero (pole, singularity) is a zero (pole, singularity).

In the following paper, we must to distinguish between primitive equivalent and Bernoulli operator equivalent. Otherwise, we will obtain many wrong equations. For example, we taking the logarithms on both sides of a primitive equivalent equation which remains equivalent; But we taking the logarithms on both sides of a Bernoulli operator equivalent. Otherwise, we will obtain many wrong equations. For example, $B^2 - B = B^3 = 0 \neq 1/6 \cdot 1/2$.

From the above discussion, we know the value of $e^{-B}$ is $\frac{1}{e^B}$, we naturally want to know what is the value of a general Bernoulli operator function? Using the Euler-Maclaurin formula (see [2] p.104):

\[
\sum_{n=M}^{N} f'(n) = \int f'(x)dx + \frac{1}{2} [f'(M) + f'(N)] + \frac{B_2}{2} f''(x)|^N_M + \frac{B_4}{4} f'''(x)|^N_M + \ldots + \frac{B_{2v}}{(2v)!} f^{2v}(x)|^N_M + R_{2v},
\]

where

\[
R_{2v} = -\frac{1}{(2v)!} \int_{M}^{N} B_{2v}(x)f^{(2v+1)}(x)dx.
\]

Let $M = 1$ and $N \to \infty$, if $f(N) \to 0, f'(N) \to 0, f^{(2v)}(N) \to 0$ and $R_{2v} \to 0$, then

\[
\sum_{n=1}^{\infty} f'(n) = -f(1) + \frac{1}{2} f(1) - \frac{B_2}{2} f''(1) - \frac{B_4}{4} f'''(1) + \ldots - \frac{B_{2v}}{(2v)!} f^{(2v)}(1) + \ldots
\]

\[
= -f(1 - B).
\]

Since $(1 - B)^n = B^n$, we obtain

\[
(1.1) \quad f(B) = -\sum_{n=1}^{\infty} f'(n).
\]

A Taylor expansion of Bernoulli operator function is $f(B + a) = \sum_{n=0}^{\infty} f^{(n)}(a)(B-a)^n$, if the right of the expansion does not converge in a certain areas of the complex plane, we use (1.1) that is $f(B + a) = -\sum_{n=1}^{\infty} f'(n + a)$ to expand the domain of the function $f(B + a)$; Conversely, if the right of the expansion $f(B + a) = -\sum_{n=1}^{\infty} f'(n + a)$ does not converge in a certain areas of the complex plane, we use the expansion $f(B + a) = \sum_{n=0}^{\infty} f^{(n)}(a)(B-a)^n$ to expand the domain of the function $f(B + a)$. This is a principle of analytic continuation.
For example, the following Bernoulli operator function is analytic at all points of the complex $z$-plane except for some poles at $z = 2\pi ki$,

\[ e^{-Bz} = \sum_{n=0}^{\infty} \frac{(-1)^n B^n z^n}{n!} = \sum_{n=1}^{\infty} ze^{-nz} = \frac{z}{e^z - 1}. \]

From the above ideas and (1.1), we can obtain the formula,

\[ B^{1-s} = \zeta(s)(s-1). \]

Therefore

\[ B^{-1} = \zeta(2) = \pi^2/6. \]

Similarly, we can obtain the following formulas

\[ (B + n)^{1-s} = [\zeta(s) - 1^{-s} - 2^{-s} - ... - n^{-s}](s-1), \]

\[ (B + \alpha)^{1-s} = \zeta(s, \alpha)(s-1), \]

where $\alpha > 0$, $\zeta(s, \alpha)$ is Hurwitz Zeta function. Moreover, using a formula of [3] p.249 and above (1.4), we can obtain the following formula

\[ L(s, \chi) = k^{-s} \sum_{r=1}^{k} \chi(r) \zeta(s, \frac{r}{k}) = k^{-s} \sum_{r=1}^{k} \chi(r) \frac{(B + \frac{r}{k})^{1-s}}{s-1}. \]

We taking the derivative of an equation of Bernoulli operator equivalent carries a new equation of Bernoulli operator equivalent (the Bernoulli operator $B$ can be thought of as a constant). For example,

\[ (e^{-Bz})' = \left( \frac{z}{e^z - 1} \right)', \]

we obtain the equation

\[ -Be^{-Bz} = \frac{e^z - 1 - ze^z}{(e^z - 1)^2}. \]

Similarly, we taking the derivatives on both sides of (1.2), we obtain

\[ -B^{1-s} \log B = \zeta(s) + (s-1)\zeta'(s). \]

And now, we have the following equation

\[ \lim_{s \to 1} [\zeta(s) + (s-1)\zeta'(s)] = \gamma, \]

where $\gamma$ is Euler constant, therefore

\[ \gamma = -\log B. \]

Using (1.7), we will give a proof of formula $\gamma = \sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n}$(The formula can be found in [4]).

**Proof.** Since $B^{2n+1} = 0$ (n is positive integer), applying the Taylor expansion, we readily find that

\[ B \log(1 + B) = -B \log(1 - B), \]

and we have

\[ B \log(1 + \frac{1}{B}) = B \log(1 + B) - B \log B, \]

therefore

\[ B \log(1 + \frac{1}{B}) = -B \log(1 - B) - B \log B. \]
Since \((1 - B)^n = B^n\), we have
\[(1.9) \quad -B \log(1 - B) = -(1 - B) \log B.\]
Using (1.8) and (1.9), we obtain
\[(1.10) \quad B \log(1 + \frac{1}{B}) = -\log B.\]
Using (1.7) and (1.10), we obtain
\[(1.11) \quad \gamma = B \log(1 + \frac{1}{B}).\]
Now we have
\[(1.12) \quad \log(1 + B^{-1}) = \log(1 + B) - \log B,\]
and applying the Taylor expansion, we readily find that
\[\log(1 + B) = 1 + \log(1 - B) = 1 + \log B,\]
therefore
\[(1.13) \quad \log(1 + B^{-1}) = 1 + \log B - \log B = 1.\]
Combining (1.11) and (1.13), we obtain
\[(1.14) \quad \gamma = B \log(1 + \frac{1}{B}) + \log(1 + B^{-1}) - 1 = \sum_{n=2}^{\infty} \frac{(-1)^n B^{1-n}}{n(n - 1)}.\]
By (1.2), we show that
\[(1.15) \quad B^{1-n} = \zeta(n)(n - 1),\]
Combining (1.14) and (1.15), we deduce that
\[(1.16) \quad \gamma = \sum_{n=2}^{\infty} \frac{(-1)^n \zeta(n)}{n}.\]

Similar to complex plane, we define a formal Bernoulli operator plane or simply Bernoulli plane, which is \(B := \{a + bBi | a, b \in \mathbb{R}\}\). If we define an "analytic function" on the Bernoulli plane, and define a contour integral for the analytic function, then we can deduce the classic Cauchy integral theorem, Cauchy integral formula, and Residue theorem. Here, we do not describe in detail.

We can obtain another proof of (1.2) by Euler’s integral for \(\Pi(s - 1)\), where \(\Pi(s - 1) := \int_{0}^{\infty} e^{-x} x^{s-1} \, dx\). Substitution of \(nx\) for \(x\) gives
\[
\int_{0}^{\infty} e^{-nx} x^{s-1} \, dx = \frac{\Pi(s - 1)}{n^s}.\]
We sum this over \(n\) to obtain
\[
\int_{0}^{\infty} \frac{x^{s-1}}{e^x - 1} \, dx = \Pi(s - 1)\zeta(s).\]
Therefore
\[ \Pi(s - 1)\zeta(s) = \int_0^\infty \frac{x^{s-2} \cdot x}{e^x - 1} \, dx = \int_0^\infty x^{s-2} e^{-Bx} \, dx. \]

Let \( Bx \to t \), note that \( B\infty \) can be thought of as \( \infty \), we deduce that
\[ \Pi(s - 1)\zeta(s) = B^{1-s} \int_0^\infty t^{-2} e^{-t} \, dt = B^{1-s} \Pi(s-2) = B^{1-s} \frac{\Pi(s-1)}{s-1}. \]

Therefore we obtain (1.2) again.

**Remark 1.1.** Recently, I find that (1.1) is similar to Ramanujan summation, and he has researched divergent series by his summation, but he did not introduce Bernoulli operator (see [6]). On the other hand, Christophe Vignat told me that the Bernoulli operator somewhat similar to Bernoulli umbra (see [7]).

**Remark 1.2.** If a point of the complex \( z \)-plane is Bernoulli operator pole (or singularity) of a function, then using the transformation \( B \to 1 - B \) will cause errors. For example,
\[ e^{-2\pi i B} = \cos 2\pi B - i \sin 2\pi B, \]
if let \( B \to 1 - B \), then
\[ e^{-2\pi i(1-B)} = e^{-2\pi i + 2\pi i B} = e^{2\pi i B} = \cos 2\pi B + i \sin 2\pi B. \]

Therefore we obtain an error equation
\[ -i \sin 2\pi B = i \sin 2\pi B. \]

### 2. Distribution of Bernoulli operator zeros of the function \( \xi(B + s) \) and \( \sin \pi B \cdot \xi(B + s) \)

**Theorem 2.1.** The function \( \xi(B + s) \) and \( \sin \pi B \cdot \xi(B + s) \) satisfy the following functional equations respectively
\[
(2.1) \quad \xi(B + s) = \xi(B - s),
\]
\[
(2.2) \quad \sin \pi B \cdot \xi(B + s) = \sin \pi B \cdot \xi(B - s).
\]

The values of these two functions are positive in the real axis; and these two functions have infinitely many Bernoulli operator zeros on the imaginary axis.

**Proof.** By \( \xi(s) = \xi(1 - s) \), and let \( s \to B + s \), we have
\[ \xi(B + s) = \xi(1 - B - s). \]
Since \( (1 - B + s)^n = B^n \), we have
\[ \xi(1 - B - s) = \xi(B - s). \]
Using (2.3) and (2.4), we obtain (2.1). Similarly, we have
\[ \sin \pi B \cdot \xi(B + s) = \sin \pi(1 - B) \cdot \xi(1 - B + s) = \sin \pi(1 - B) \cdot \xi(B - s). \]
Since \( \sin \pi(1 - B) \) is primitive equal to \( \sin \pi B \), we have
\[ \sin \pi(1 - B) \cdot \xi(B + s) = \sin \pi B \cdot \xi(B + s). \]
Therefore
\[
\sin \pi B \cdot \xi(B + s) = \sin \pi B \cdot \xi(B - s).
\]

We now prove that the function \(\xi(B + s)\) is positive in the real axis. We have the following equation (see [2] p.17),
\[
\xi(s) = \int_{1}^{\infty} \frac{d[x^{3/2} \psi'(x)]}{dx}(2x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}})dx,
\]
where \(\psi'(x) = \sum_{n=1}^{\infty} e^{-n^2 \pi x} \), therefore
\[
\xi(B + s) = \int_{1}^{\infty} \frac{d[x^{3/2} \psi'(x)]}{dx}(2x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}})dx.
\]

Since \(e^{-Bx} = \frac{x^B}{x^{\frac{1}{2}} - 1}\), we have \(x^{-\frac{1}{2}} = \frac{\log x}{2(x^{\frac{1}{2}} - 1)}\). Let \(\phi(x) = \frac{d[x^{3/2} \psi'(x)]}{dx} \frac{\log x}{x^{1/2} - 1}\), then
\[
\xi(B + s) = \int_{1}^{\infty} \phi(x)(x^B + x^{-B})dx.
\]

Because \(\phi(x) > 0(x \in (1, \infty))\), we obtain \(\xi(B + s) > 0, s \in (-\infty, \infty)\). Using \([11]\), we conclude that \(\sin \pi B \cdot \xi(B + s) > 0, s \in (-\infty, \infty)\).

We now prove that the function \(\xi(B + s)\) has infinitely many Bernoulli operator zeros on the imaginary axis.

Let \(G(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x^2}, H(x) = \frac{d[xG(x) - x - 1]}{dx} \), then we have the following equation (see [2] p. 228),
\[
H(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \xi(a + it)x^{-1}x^{-it} dt,
\]
multiply both sides of above equation by \(x^{1-a}\)
\[
x^{1-a}H(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \xi(a + it)x^{it} dt.
\]
Let \(a \to B\) (B is Bernoulli operator), then
\[
x^{1-B}H(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \xi(B + it)x^{it} dt,
\]
\[
= \frac{1}{\pi} \int_{-\infty}^{\infty} \xi(B + it) \sum_{n=0}^{\infty} \frac{(it \log x)^n}{n!} dt.
\]
Denote \(C_n = \frac{1}{\pi n} \int_{-\infty}^{\infty} \xi(B + it)t^n dt\), then
\[
x^{1-B}H(x) = \frac{x \log x}{x - 1}H(x) = \sum_{n=0}^{\infty} C_n(i \log x)^n.
\]
Since $\xi(B+it) = \xi(B-it)$, we have $C_n = 0$ (n is odd). The differential operator $ix(d/dx)$ doing $\frac{x \log x}{2} H(x)$ any number of times carries $\frac{x \log x}{2} H(x)$ to a function which approaches zero as $x \to i\frac{1}{2}$. Therefore, similar to [2] p.228-229, we can prove that function $\xi(B+s)$ has infinitely many Bernoulli operator zeros on the imaginary axis.

Similarly, we can prove that function $\sin \pi B \cdot \xi(B+s)$ have infinitely many Bernoulli operator zeros on the imaginary axis.

**Conjecture 2.2.** All zeros of the function $\xi(B+s)$ ($\sin \pi B \cdot \xi(B+s)$) lie on the imaginary axis.

We know that Riemann Hypothesis generalized to Dirichlet L-functions, Dedekind Zeta function and zeta functions of varieties over finite fields, etc. Similarly, the conjecture 2.2 can be generalized to other Zeta functions (L-functions). Here, we do not describe in detail.

### 3. An application of Bernoulli operator to number-theoretic function

**Theorem 3.1.** We define a number-theoretic function:

$$
\psi(x) = \frac{1}{2} \left[ \sum_{p^n < x} \frac{\log p}{p^n - 1} + \sum_{p^n < x} \frac{\log p}{p^n - 1} \right],
$$

we have

$$\psi(x) = \log(x-1) - \lim_{\varepsilon \to 0} \sum_{\rho \neq \rho > 0} \left( \int_{0}^{1} \frac{t^{\rho-1} + t^{-\rho}}{t^{1-\varepsilon} - t^{1+\varepsilon}} dt + \int_{1}^{x} \frac{t^{\rho-1} + t^{-\rho}}{t^{1-\varepsilon} - t^{1+\varepsilon}} dt \right)$$

$$+ \int_{x}^{\infty} \frac{dt}{t(t-1)(t+1)} + \log[-\zeta(B)].$$

where $\rho$ are zeros of $\xi(s)$, and $x > 1$.

**Proof.** By Euler product formula

$$\log \zeta(s) = \sum_{p} \left[ \sum_{n=1}^{\infty} \frac{1}{n} p^{-ns} \right], \text{ Re } s > 1,$$

let $s \to s + B$, therefore

$$\log \zeta(s + B) = \sum_{p} \left[ \sum_{n=1}^{\infty} \frac{1}{n} p^{-ns-nB} \right] = \sum_{p} \left[ \sum_{n=1}^{\infty} \frac{1}{n} p^{-ns} \log p \right] p^{n - 1}$$

$$< \sum_{p} \left[ \sum_{n=1}^{\infty} \frac{1}{n} p^{n(s+1)} \right].$$

The above series on the right is absolutely convergent for Re $s > 0$. Therefore write this sum as a Stieltjes integral

$$\log \zeta(s + B) = \int_{0}^{\infty} x^{-s} d\psi(x) = s \int_{0}^{\infty} \psi(x)x^{-s-1} dx. \text{ (Re } s > 0)$$
We apply Fourier inversion to the above formula, to conclude
\[
\psi(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^s \log \zeta(s + B) \frac{ds}{s} \ 	ext{)(a > 0)}
\]

We integrate by parts to obtain
\[
(3.1) \quad \psi(x) = -\frac{1}{2\pi i \log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left( \frac{\log \zeta(s + B)}{s} \right) x^s ds.
\]

On the other hand, we have
\[
\xi(B + s) = \Pi \left( \frac{B + s}{2} \right) \pi^{-\frac{s}{2}} (s + B - 1) \zeta(s + B),
\]
where \( \Pi(s) := \int_{0}^{\infty} e^{-x} x^s dx \),

and
\[
\xi(B + s) = \xi(B) \prod_{\rho} \left( 1 - \frac{s}{\rho - B} \right). \ 	ext{(\( \rho \) are zeros of \( \xi(s) \), and \( \rho - B \) are primitive zeros of \( \xi(s + B) \)).}
\]

Therefore
\[
(3.2) \quad \log \zeta(s + B) = -\log \Pi \left( \frac{B + s}{2} \right) + \frac{s + B}{2} \log \pi - \log(s + B - 1) + \log \xi(B) + \sum_{\rho} \log \left( 1 - \frac{s}{\rho - B} \right).
\]

Using (3.1) and (3.2), we obtain
\[
(3.3) \quad \psi(x) = -\frac{1}{2\pi i \log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \log \left( \frac{s}{s - B} \right) + \log \left( \frac{s}{B - 1} \right) \right] x^s ds.
\]

Similar to [2] p.26-31, we show that
\[
\frac{1}{2\pi i \log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \log \left( \frac{s}{s - B} \right) + \log \left( \frac{s}{B - 1} \right) \right] x^s ds = \frac{1}{2\pi i \log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \log \left( 1 - \frac{s}{s - B} \right) + \log \left( \frac{s}{B - 1} \right) \right] x^s ds
\]

\[= \frac{1}{2\pi i \log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \log \left( 1 - \frac{s}{s - B} \right) + \log \left( 1 - \frac{s}{B - 1} \right) \right] x^s ds
\]

\[= \lim_{\varepsilon \to 0} \left( \int_{0}^{1-\varepsilon} \frac{t^{B-1}}{\log t} dt + \int_{1+\varepsilon}^{x} \frac{t^{B-1}}{\log t} dt \right) - \log(1 - B)
\]

\[= \lim_{\varepsilon \to 0} \left( \int_{0}^{1-\varepsilon} \frac{t^{1-B}}{\log t} dt + \int_{1+\varepsilon}^{x} \frac{t^{1-B}}{\log t} dt \right) - \log(1 - B)
\]
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(3.4) \[ = \lim_{\varepsilon \to 0} \left( \frac{1}{0} \int_{t=0}^{x} \frac{1}{t-1} \, dt + \int_{1+\varepsilon}^{x} \frac{1}{t-1} \, dt \right) - \log(1 - B). \]

If \( x > 1 \), then

\[ \lim_{\varepsilon \to 0} \left( \frac{1}{0} \int_{t=0}^{x} \frac{1}{t-1} \, dt + \int_{1+\varepsilon}^{x} \frac{1}{t-1} \, dt \right) = \log(x - 1). \]

Therefore

(3.5) \[ \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{\log(s + B - 1)}{s} x^s \, ds = \log(x - 1) - \log(1 - B). \]

Similar to [2] p.26-31, we show that

\[ \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{\sum \log(1 - \frac{s + B}{s})}{s} x^s \, ds = \sum_{\Im \rho > 0} \left[ \text{Li}(x^{\rho - B}) + \text{Li}(x^{1-\rho - B}) \right] \]

\[ = \sum_{\Im \rho > 0} \left[ \text{Li}(x^{\rho - B}) + \text{Li}(x^{B - \rho}) \right]. \]

Using a formula from [2] p.8, we show that

\[ \log \Pi\left( \frac{s + B}{2} \right) = \sum_{n=1}^{\infty} \left[ \log(1 + \frac{s + B}{2n}) + \frac{s + B}{2n} \log(1 + \frac{1}{n}) \right]. \]

Since

\[ \log(1 + \frac{s + B}{2n}) = \log(1 + \frac{B}{2n} + \frac{s}{2n}) = \log(1 + \frac{B}{2n}) + \log(1 + \frac{s}{B + 2n}), \]

we obtain

\[ \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{\log \Pi\left( \frac{s + B}{2} \right)}{s} x^s \, ds \]

\[ = \frac{1}{2\pi i} \log x \sum_{n=1}^{\infty} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ -\log(1 + \frac{B}{2n}) - \log(1 + \frac{s}{B + 2n}) + \frac{s + B}{2n} \log(1 + \frac{1}{n}) \right] x^s \, ds \]

\[ = \frac{1}{2\pi i} \log x \sum_{n=1}^{\infty} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ -\log(1 + \frac{B}{2n}) - \log(1 + \frac{s}{B + 2n}) + \frac{s + B}{2} \log(1 + \frac{1}{n}) \right] x^s \, ds \]

\[ = \sum_{n=1}^{\infty} \left[ \log(1 + \frac{B}{2n}) - \frac{B}{2} \log(1 + \frac{1}{n}) \right] - \frac{1}{2\pi i} \log x \sum_{n=1}^{\infty} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \log(1 + \frac{s}{B + 2n}) \right] x^s \, ds \]

\[ = -\log \Pi\left( \frac{B}{2} \right) - \frac{1}{2\pi i} \log x \sum_{n=1}^{\infty} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \log(1 + \frac{s}{B + 2n}) \right] x^s \, ds \]
\[
(3.7) \quad \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left( \log \Pi \left( \frac{s+B}{2} \right) \right) x^s ds = -\log \Pi \left( \frac{B}{2} \right) - \int_{x}^{\infty} \frac{dt}{t(t-1)(t^2-1)}.
\]

Using (3.3), (3.5), (3.6) and (3.8), we obtain

\[
(3.8) \quad \frac{1}{2\pi i} \log x \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left( \log \Pi \left( \frac{s+B}{2} \right) \right) x^s ds = -\log \Pi \left( \frac{B}{2} \right) - \int_{x}^{\infty} \frac{dt}{t(t-1)(t^2-1)}.
\]

On the other hand, we have

\[
\log \xi(B) = \log \Pi \left( \frac{B}{2} \right) + \log(1-B) - \frac{B}{2} \log \pi + \log \left( -\zeta'(B) \right).
\]

By the above equation and (3.9), we have proved the Theorem 3.1. \(\square\)

**Remark 3.2.**

\[
\sin \pi B \cdot \log \zeta(B + s) = \sin \pi B \cdot \sum_{p} \left( \sum_{n=1}^{\infty} \frac{1}{p^{-n(s+B)}} \right),
\]

Using (1.1), we conclude that \(\sin \pi B \cdot p^{-nB} = \frac{\pi}{p^{n+1}}\), therefore

\[
\sin \pi B \cdot \log \zeta(B + s) = \pi \sum_{p} \left( \sum_{n=1}^{\infty} \frac{1}{p^{-n}} \frac{1}{p^n + 1} \right).
\]

The above series on the right is absolutely convergent for \(\text{Re} s > 0\). Similar to Theorem 3.1, we can obtain another number-theoretic function relate to Bernoulli operator, here, we do not calculate in detail.

4. The values of \(\frac{B}{2} \log B, \log \sin \frac{\pi B}{2}, \log \Pi(B), \frac{\zeta'(B)}{\zeta(B)}, \text{ and } \sin \pi B \cdot \frac{\zeta'(B)}{\zeta(B)}\)

**Proposition 4.1.** The values of \(\frac{B}{2} \log B, \log \sin \frac{\pi B}{2}, \log \Pi(B), \frac{\zeta'(B)}{\zeta(B)}\) and \(\sin \pi B \cdot \frac{\zeta'(B)}{\zeta(B)}\) are

\[
\frac{1 - \log 2\pi}{2}, \frac{1 - \log 2}{2}, \frac{\log 2\pi - 1}{2} - \gamma, \frac{1 + \gamma + \log 2\pi}{2} + \frac{\pi^2}{16}
\]

and \(\frac{\pi}{4}(1 + \gamma + \log 4\pi)\) respectively.
Lemma 4.2. If the function \( f(z) \) has the following Taylor expansion
\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (z - a)^n,
\]
then
\[
(4.1) \quad f(a + Bz) = f(a - Bz) + f'(a)z,
\]
\[
(4.2) \quad Bf'(a + Bz) = -Bf'(a - Bz) + f'(a),
\]
\[
(4.3) \quad B^2 f''(a + Bz) = B^2 f''(a - Bz).
\]

Proof. It is clear from the properties of Bernoulli numbers.

Remark 4.3. We can obtain (1.1) by (4.1).

\[
f(1 - B) = f(1 + B) - f'(1),
\]

Since \( f(B) = f(1 - B) \), we have
\[
f(B) = f(1 + B) - f'(1).
\]

Therefore
\[
f(B) = f(1 + 1 - B) - f'(1) = f(2 - B) - f'(1).
\]

using (4.1) again,
\[
f(B) = f(2 + B) - f'(1) - f'(2).
\]

Therefore
\[
f(B) = f(n + B) - \sum_{k=1}^{n} f'(k).
\]

Let \( n \to \infty \), if \( f(n + B) \to 0 \), then we obtain (1.1) again.

Now we prove that Proposition 4.1. Using (1.5), let \( s = 0 \), we obtain
\[
(4.4) \quad -B \log B = \zeta(0) - \zeta'(0) = -\frac{1}{2} - \zeta'(0).
\]

And we have the following equation (see [2] p.135),
\[
(4.5) \quad \zeta'(0) = -\frac{1}{2} \log 2\pi.
\]

Therefore
\[
(4.6) \quad B \log B = \frac{1 - \log 2\pi}{2}.
\]

Here we will use another way to prove (4.6). We take the logarithms of both sides of the Riemann’ Zeta functional equation, let \( s \to B \), we obtain
\[
\log[-\zeta(B)] + \log(1 - B) = \log \Pi(1 - B) + (B - 1) \log 2 + \log \sin \frac{\pi B}{2} + \log[-\zeta(1 - B)]
\]

Therefore
\[
(4.7) \quad -\gamma = \log \Pi(B) - \frac{\log 2\pi}{2} + \log 2 + \log \sin \frac{\pi B}{2}.
\]

Using the Stirling formula (see [2] p.109),
\[
(4.8) \quad \log \Pi(B) = (B + \frac{1}{2}) \log B - B + \frac{\log 2\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{2n}^2 (2\pi)^{2n}}{2n \cdot 2 \cdot (2n)!}.
\]
Note that we have used the following formula
\[ B^{1-2k} = \zeta(2k)(2k - 1) = \frac{(-1)^{k+1}(2\pi)^{2n}B_{2k}}{2 \cdot (2k)!}, \]
where k is positive integer. Since
\[ \frac{\cos z}{\sin z} = \frac{1}{z} - \sum_{n=1}^{\infty} \frac{((-1)^{n+1}2^{2n}B_{2n})}{2n \cdot (2n)!} z^{-2n}, \]
taking the integral on both sides to get
\[ (4.9) \log \sin z = \log z - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2^{2n}B_{2n}}{2n \cdot (2n)!} z^{-2n}. \]
Let \( z \to B\pi \), we have
\[ (4.10) \log \sin B\pi = \log B + \log \pi - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}B_{2n}^2(2\pi)^{2n}}{2n \cdot (2n)!}. \]
Using (4.8) and (4.10), we obtain
\[ (4.11) \log \Pi(B) = \left( B + \frac{1}{2} \right) \log B - B + \frac{\log 2\pi}{2} + \log B + \log \pi - \log \sin B\pi. \]
Using (4.7) and (4.11), we obtain
\[ -\gamma = \left( B + \frac{1}{2} \right) \log B - B + \frac{\log 2\pi}{2} + \log B + \log \pi - \log \sin B\pi - \frac{\log 2\pi}{2} + \log 2 + \log \sin \frac{\pi B}{2}. \]
Therefore
\[ B \log B = \frac{1}{2} - \log 2 - \frac{\log \pi}{2} + \frac{\log \sin B\pi - 2 \log \sin \frac{B\pi}{2}}{2} \]
\[ = \frac{1}{2} - \log 2 - \frac{\log \pi}{2} + \frac{\log \left( 2 \sin \frac{B\pi}{2} \cos \frac{B\pi}{2} \right) - 2 \log \sin \frac{B\pi}{2}}{2}. \]
(4.12)
And we have
\[ \log \cos \frac{B\pi}{2} = \log \sin \frac{(1 - B)\pi}{2} = \log \sin \frac{B\pi}{2}. \]
Using (4.12) and (4.13), we obtain (4.6).
We began to calculate the value of \( \log \sin \frac{\pi B}{2} \), we have
\[ B \log \sin \frac{\pi B}{2} = (1 - B) \log \sin \left( \frac{\pi}{2} (1 - B) \right) = (1 - B) \log \cos \frac{\pi B}{2} \]
\[ = \log \cos \frac{\pi B}{2} - B \log \cos \frac{\pi B}{2} \]
(4.14)
Using the Taylor expansion, we have
\[ \log \cos \frac{\pi B}{2} = 0. \]
(4.15)
Using (4.9), we have
\[ B \log \sin \frac{\pi B}{2} = B \log \frac{\pi B}{2} = B \log \pi B - B \log 2 = B \log \pi + B \log B - \frac{\log 2}{2} \]
(4.16)
\[ = \frac{\log \pi}{2} + 1 - \frac{\log 2}{2} - \frac{\log 2}{2} = \frac{1}{2} - \log 2. \]

Using (4.14), (4.15) and (4.16), we obtain
\[ \log \sin \frac{\pi B}{2} = \frac{1}{2} - \log 2. \]
(4.17)

Using (4.17) and (4.7), we obtain
\[ -\gamma = \log \Pi(B) - \frac{\log 2}{2} + \log 2 + \frac{1}{2} - \log 2. \]

Therefore
\[ \log \Pi(B) = \frac{\log 2 \pi - 1}{2} - \gamma. \]
(4.18)

We calculate the values of \( \frac{\zeta'(B)}{\zeta(B)} \) and \( \sin \pi B \cdot \frac{\zeta'(B)}{\zeta(B)} \) by the following equation,
\[ \frac{\zeta'(s)}{\zeta(s)} = -\frac{\Pi'(-s)}{\Pi(-s)} + \log 2 \pi + \pi \frac{\cos(\pi s/2)}{2 \sin(\pi s/2)} \frac{\zeta'(1 - s)}{\zeta(1 - s)}. \]
(4.19)

Let \( s \to B \), both sides multiplied by \( B \), we obtain
\[ B \frac{\zeta'(B)}{\zeta(B)} = -B \frac{\Pi'(-B)}{\Pi(-B)} + B \log 2 \pi + \pi \frac{B \cos(\pi B/2)}{\sin(\pi B/2)} - B \frac{\zeta'(1 - B)}{\zeta(1 - B)} \]
\[ = -B \frac{\Pi'(-B)}{\Pi(-B)} + B \log 2 \pi + \frac{\pi}{2} \frac{B \cos(\pi B/2)}{\sin(\pi B/2)} - (1 - B) \frac{\zeta'(B)}{\zeta(B)}. \]

Hence
\[ \frac{\zeta'(B)}{\zeta(B)} = -B \frac{\Pi'(-B)}{\Pi(-B)} + B \log 2 \pi + \frac{\pi}{2} \frac{B \cos(\pi B/2)}{\sin(\pi B/2)}. \]
(4.20)

Firstly, we calculate the value of \( \frac{\pi B \cos(\pi B/2)}{\sin(\pi B/2)} \). We have
\[ B^2 \frac{\sin(\pi B/2)}{\cos(\pi B/2)} = (1 - B)^2 \frac{\sin(\pi (1 - B)/2)}{\cos(\pi (1 - B)/2)} = (1 - 2B + B^2) \frac{\cos(\pi B/2)}{\sin(\pi B/2)}, \]
(4.21)

Using the Taylor expansion, we can prove the following equations,
\[ \frac{\cos(\pi B/2)}{\sin(\pi B/2)} = \frac{2}{\pi B} - \frac{4}{6} - \frac{1}{2} \]
\[ = \frac{2}{\pi B} - \frac{\pi B}{6} = \frac{\pi^2}{6} - \pi = \frac{\pi}{4}, \]
(4.22)

\[ B^2 \frac{\sin(\pi B/2)}{\cos(\pi B/2)} = 0, \]
(4.23)

\[ B^2 \frac{\cos(\pi B/2)}{\sin(\pi B/2)} = \frac{2B^2}{\pi B} = \frac{2B}{\pi} = \frac{1}{\pi}. \]
(4.24)

Using (4.21), (4.22), (4.23) and (4.24), yields
\[ \frac{\pi}{2} B \frac{\cos(\pi B/2)}{\sin(\pi B/2)} = \frac{\pi^2}{16} + \frac{1}{4}. \]
(4.25)
Let us now calculate the value of \( -\frac{\Pi'(B)}{\Pi(-B)} \), taking the logarithmic derivative on both sides of the equation \( \frac{\pi s}{\Pi(s)\Pi(-s)} = \sin \pi s \). Let \( s \to B \), and both sides multiplied by \( B \), yields

\[
(4.26) \quad B \frac{\Pi'(B)}{\Pi(B)} - B \frac{\Pi'(-B)}{\Pi(-B)} = B \frac{\pi B \cos \pi B}{\sin \pi B} = 1 - \frac{\pi B \cos \pi B}{\sin \pi B}.
\]

And now calculate the value of \( \frac{\pi B \cos \pi B}{\sin \pi B} \). Since \( \frac{\cos \pi B}{\sin \pi B} = \frac{\cos(1-B)}{\sin(1-B)} = -\cos B \), we have

\[
\frac{\cos \pi B}{\sin \pi B} = 0
\]

and

\[
\pi B^2 \cos \pi B \sin^{-1} B = \pi(1-B)^2 \cos \pi(1-B) = -\pi(1-2B+B^2) \cos \pi \frac{B}{\sin \pi B}.
\]

Therefore

\[
\pi B^2 \cos \pi B \sin^{-1} B = \pi B \cos \pi B \sin \pi B.
\]

Since \( \pi B^2 \cos \pi B \sin^{-1} B = \frac{B^2}{\sin \pi B} = B = \frac{1}{2}, \) we obtain

\[
(4.27) \quad \pi B \cos \pi B \sin^{-1} B = \frac{1}{2}.
\]

Using (4.26) and (4.27), we obtain

\[
(4.28) \quad B \frac{\Pi'(B)}{\Pi(B)} - B \frac{\Pi'(-B)}{\Pi(-B)} = \frac{1}{2}.
\]

Using (4.2) of Lemma 4.2, we obtain

\[
(4.29) \quad \frac{\Pi'(B)}{\Pi(B)} = -B \frac{\Pi'(-B)}{\Pi(-B)} + \frac{\Pi'(0)}{\Pi(0)} = \frac{\Pi'(0)}{\Pi(-B)} = \gamma,
\]

where \( \gamma \) is Euler constant. Using (4.28) and (4.29), we obtain

\[
(4.30) \quad B \frac{\Pi'(B)}{\Pi(B)} = \frac{1}{4} - \frac{\gamma}{2} - B \frac{\Pi'(-B)}{\Pi(-B)} = \frac{1}{4} + \frac{\gamma}{2}.
\]

Using (4.20), (4.25) and (4.30), we obtain

\[
\frac{\zeta'(B)}{\zeta(B)} = \frac{1}{4} + \frac{\gamma}{2} + B \log 2 \pi + \frac{\pi^2}{16} + \frac{1}{4} = \frac{1 + \gamma + \log 2 \pi}{2} + \frac{\pi^2}{16}.
\]

And now calculate the value of \( \sin \pi B \cdot \frac{\zeta'(B)}{\zeta(B)} \). Both sides of (4.19) multiplied by \( \sin \pi B \), and let \( s \to B \), we obtain

\[
\sin \pi B \cdot \frac{\zeta'(B)}{\zeta(B)} = -\sin \pi B \cdot \frac{\Pi'(-B)}{\Pi(-B)} + \log 2 \pi \cdot \sin \pi B + \frac{\pi \cos(\pi B/2)}{2 \sin(\pi B/2)} \sin \pi B - \sin \pi B \cdot \frac{\zeta'(1-B)}{\zeta(1-B)}
\]

\[
= -\sin \pi B \cdot \frac{\Pi'(-B)}{\Pi(-B)} + \log 2 \pi \cdot \pi B + \pi \cos^2(\pi B/2) - \sin(1-B) \cdot \frac{\zeta'(B)}{\zeta(B)}
\]

\[
= -\sin \pi B \cdot \frac{\Pi'(-B)}{\Pi(-B)} + \frac{\pi \log 2 \pi}{2} + \frac{1 + \cos \pi B}{2} - \sin \pi B \cdot \frac{\zeta'(B)}{\zeta(B)}.
\]

Therefore

\[
(4.31) \quad \sin \pi B \cdot \frac{\zeta'(B)}{\zeta(B)} = -\frac{\sin \pi B}{2} \frac{\Pi'(-B)}{\Pi(-B)} + \frac{\pi \log 2 \pi}{4} + \frac{\pi}{4}.
\]
Firstly, we calculate the value of \( \sin \pi B \cdot \frac{\Pi'(B)}{\Pi(B)} \). Using (4.1) of Lemma 4.2, we have

\[
(4.32) \quad \sin \pi B \cdot \frac{\Pi'(B)}{\Pi(B)} = \sin(-\pi B) \cdot \frac{\Pi'(B)}{\Pi(-B)} - \sin \pi B \cdot \frac{\Pi'(B)}{\Pi(-B)} - \pi \gamma.
\]

Taking the logarithmic derivative on both sides of the equation \( \frac{\pi s}{\Pi(s)\Pi(-s)} = \sin \pi s \).

Let \( s \to B \), and both sides multiplied by \( \sin \pi B \), we obtain

\[
\sin \pi B \cdot \frac{\Pi'(B)}{\Pi(B)} = \sin \pi B \cdot \frac{\Pi'(B)}{\Pi(B)} - \pi \gamma.
\]

Using (4.32) and (4.33), we obtain

\[
(4.34) \quad \sin \pi B \cdot \frac{\Pi'(B)}{\Pi(B)} = \frac{\pi \log 2}{2}.
\]

Using (4.31) and (4.34), we obtain

\[
(4.35) \quad \frac{\zeta'(B)}{\zeta(B)} = \frac{\pi \log 2}{4} + \frac{\pi \log 2}{2} + \frac{\pi}{4} = \frac{1 + \gamma + \log 4\pi}{4}.
\]

Thus, we have

\[
\lambda_1 = \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{\infty} \frac{1}{(2n+k)^2} - \frac{1}{2n} \right], \quad \lambda_2 = \sum_{n=1}^{\infty} \left[ \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2n+k} - \frac{1}{4n} \right],
\]

\[
\lambda_1 = \frac{1 + \log 2}{2} - \frac{5}{48\pi^2}, \quad \lambda_2 = \frac{1 - 2\log 2}{4}.
\]

\[
\sum_{\rho} \frac{1}{\rho - B} = 0.
\]

Using (1.3), we have

\[
(4.38) \quad (B + 2n)^{-1} = (2-1)(\zeta(2) - 1^{-2} - 2^{-2} - 3^{-2} - \ldots - (2n)^{-2}) = \sum_{k=1}^{\infty} \frac{1}{(2n + k)^2}.\]
Using (4.36), (4.37) and (4.38), we obtain

\[ (4.39) \quad \frac{\zeta'(B)}{\zeta(B)} = \frac{-1}{B-1} + \frac{\gamma + \log \pi}{2} + \lambda_1 = \frac{1}{B} + \frac{\gamma + \log \pi}{2} + \lambda_1 = \frac{\pi^2}{6} + \frac{\gamma + \log \pi}{2} + \lambda_1. \]

By Proposition 4.1 and (4.39), we obtain

\[ \lambda_1 = \frac{1 + \log 2}{2} - \frac{5}{48} \pi^2. \]

Both sides of (4.35) multiplied by \( \sin \pi B \), and let \( s \to B \), we obtain

\[ \sin \pi B \cdot \frac{\zeta'(B)}{\zeta(B)} = \sin \pi B \cdot \frac{1}{B-1} + \frac{\gamma + \log \pi}{2} + \sum \frac{\sin \pi B}{B - \rho} + \sum_{n=1}^{\infty} \sin \pi B \cdot \left( \frac{1}{B+2n} - \frac{1}{2n} \right) \]

\[ = \frac{\sin \pi (1-B)}{B} + \frac{\gamma + \log \pi}{2} \pi B + \sum \frac{\sin \pi B}{B - \rho} + \sum_{n=1}^{\infty} \left( \frac{\sin \pi B}{B+2n} - \frac{\pi B}{2n} \right) \]

\[ = \frac{\sin \pi B}{B} + \frac{\gamma + \log \pi}{4} + \sum \frac{\sin \pi B}{B - \rho} + \sum_{n=1}^{\infty} \left( \frac{\sin \pi B}{B+2n} - \frac{\pi}{4n} \right) \]

\[ (4.40) \quad = \pi \log 2 + \frac{\gamma + \log \pi}{4} + \sum \frac{\sin \pi B}{B - \rho} + \sum_{n=1}^{\infty} \left( \frac{\sin \pi B}{B+2n} - \frac{\pi}{4n} \right). \]

Similarly, we can prove that

\[ (4.41) \quad \sum_{\rho} \frac{\sin \pi B}{B - \rho} = 0. \]

Using (1.1), we have

\[ (4.42) \quad \sin \pi B \frac{\zeta'(B)}{\zeta(B)} = -\sum_{k=1}^{\infty} \frac{\pi (2n + k) \cos k \pi - \sin k \pi}{(2n+k)^2} = \pi \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2n+k}. \]

Using (4.40), (4.41) and (4.42), we obtain

\[ \sin \pi B \frac{\zeta'(B)}{\zeta(B)} = \pi \log 2 + \frac{\gamma + \log \pi}{4} + \pi \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2n+k} - \frac{1}{4n} \]

\[ = \pi \log 2 + \frac{\gamma + \log \pi}{4} + \pi \lambda_2. \]

By Proposition 4.1 and (4.43), we obtain

\[ \lambda_2 = \frac{1 - 2 \log 2}{4}. \]
5. The functional equation of $\log \Pi(Bs)$

Proposition 5.1. The function $\log \Pi(Bs)$ satisfy the following functional equation

\begin{equation}
\log \Pi(Bs) = s \log \Pi\left(\frac{B}{s}\right) + \frac{s+1}{2} \log s + (1-s) \frac{\log 2\pi}{2}.
\end{equation}

Proof. Using the Stirling formula (see [2] p.109), let $s \to B/s$, we obtain

\begin{equation}
\log \Pi\left(\frac{B}{s}\right) = \left(\frac{B}{s} + \frac{1}{2}\right) \log \frac{B}{s} - \frac{B}{s} + \frac{\log 2\pi}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2n \cdot 2 \cdot (2n)!} s^{2n-1}.
\end{equation}

Note that we have used the following formula

\begin{equation}
B^{1-2k} = \zeta(2k)(2k-1) = \frac{(-1)^{k+1} (2\pi)^{2n} B_{2k}}{2 \cdot (2k)!},
\end{equation}

where $k$ is positive integer. Using (4.9), we have

\begin{equation}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2n \cdot 2 \cdot (2n)!} s^{2n-1} = -\log \sin \frac{\pi Bs}{2s} + \frac{\log \pi}{2s} + \frac{\log Bs}{2s}.
\end{equation}

Using (5.2) and (5.3), we have

\begin{equation}
\log \Pi\left(\frac{B}{s}\right) = \left(\frac{B}{s} + \frac{1}{2}\right) \log \frac{B}{s} - \frac{B}{s} + \frac{\log 2\pi}{2} - \frac{\log \sin \frac{\pi Bs}{2s}}{2s} + \frac{\log \pi}{2s} + \frac{\log Bs}{2s}.
\end{equation}

By

\begin{equation}
\frac{\pi s}{\Pi(s)\Pi(-s)} = \sin \pi s,
\end{equation}

we have

\begin{equation}
\log \Pi(Bs) + \log \Pi(-Bs) = \log \pi Bs - \log \sin \pi Bs.
\end{equation}

Using (4.1) of Lemma 4.2, we have

\begin{equation}
\log \Pi(Bs) = \log \Pi(-Bs) + \frac{\Pi'(0)}{\Pi(0)} s.
\end{equation}

Therefore

\begin{equation}
\log \Pi(Bs) = \log \Pi(-Bs) - \gamma s.
\end{equation}

Using (5.5) and (5.6), we have

\begin{equation}
\log \Pi(Bs) = \frac{1}{2} (-\gamma s - \gamma + \log \pi s - \log \sin \pi Bs).
\end{equation}

Using (5.4) and (5.7), we obtain (5.1).

\[ \Box \]

Remark 5.2. Recently, I find that Proposition 5.1 is equivalent to Ramanujan and A.P. Guinand’s result, but the proofs are not the same (see [1], [5]).
6. The functional equation of \( \log \zeta(B+s) \)

**Theorem 6.1.** The function \( \log \zeta(B+s) \) satisfy the following functional equation

\[
\log \zeta(B-s) - \log \zeta(B+s) = \frac{\Pi'(s)}{\Pi(s)} - \frac{1}{2} s - s \log 2\pi + \frac{\pi}{2} \frac{\cos \pi s}{\sin \pi s} + \frac{\pi}{4} \frac{2 \cos \pi s}{\sin \pi s} \cos(\pi s/2),
\]

moreover the function \( \log \zeta(B+s) \) has poles at the non-positive integers (i.e. at \( s = 0, -1, -2, -3, \ldots \)).

**Lemma 6.2.**

\[
\log(B+s) = \frac{\Pi'(s)}{\Pi(s)}.
\]

**Proof.** By \( \Pi(s) = s\Pi(s-1) \), we have

\[
\log \Pi(B+s) = \log(B+s) + \log \Pi(B+s-1)
\]

(6.2) \[ = \log(B+s) + \log \Pi(-B+s). \]

Using (4.1), we have

\[
(6.3) \log \Pi(B+s) = \log \Pi(-B+s) + \frac{\Pi'(s)}{\Pi(s)}. \]

Using (6.2) and (6.3), we have proved Lemma 6.2. \( \square \)

**Remark 6.3.** In particular, let \( s = 0 \), we obtain \((1.7)\) again by Lemma 6.2. The function \( \log(B+s) \) has poles at the negative integers (i.e. at \( s = -1, -2, -3, \ldots \)).

**Lemma 6.4.**

\[
(B+s) \log(B+s) - B \log B - s = \log \Pi(s).
\]

**Proof.** By Lemma 6.2, we have

\[
\int \log(B+s)ds = \int \frac{\Pi'(s)}{\Pi(s)} ds.
\]

Therefore

\[
(B+s) \log(B+s) - s = \log \Pi(s) + c.
\]

Let \( s = 0 \), we obtain \( c = B \log B \). \( \square \)

**Remark 6.5.** We can obtain a very short proof of Stirling formula by Lemma 6.4.

**Proof.** Using Lemma 6.4, we have

\[
\log \Pi(s) = (B+s) \log(1 + \frac{B}{s}) + (B+s) \log s - s - B \log B
\]

\[= (B+s) \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B^n}{ns^n} + \left( \frac{1}{2} + s \right) \log s - s - \frac{1 - \log 2\pi}{2}. \]

\( \square \)

**Remark 6.6.** We can obtain another proof of Proposition 5.1 by Lemma 6.4.
Proof. Using Lemma 6.4, let \( s \rightarrow B'/s \), where \( B' \) is another Bernoulli operator, we have

\[
\log \Pi(B'/s) = (B + B'/s) \log(B + B'/s) - B \log B - B'/s
\]

Therefore

\[
s \log \Pi(B'/s) = (Bs + B') \log(Bs + B') - (Bs + B') \log s - B \log B - B'/s.
\]

On the other hand, using Lemma 6.4, let \( s \rightarrow B's \), where \( B' \) is another Bernoulli operator, we have

\[
\log \Pi(B's) = (B + B's) \log(B + B's) - B \log B - B's.
\]

Since

\[
( Bs + B' ) \log(Bs + B') = ( B + B's ) \log(B + B's),
\]

we have

\[
s \log \Pi(B'/s) = (B + B's) \log(B + B's) - \frac{1 + s}{2} \log s - \frac{1 - \log 2\pi}{2} s - \frac{1}{2}
\]

\[
= \log \Pi(B's) + \frac{1 - \log 2\pi}{2} + \frac{s}{2} - \frac{1 + s}{2} \log s - \frac{1 - \log 2\pi}{2} s - \frac{1}{2}
\]

\[
= \log \Pi(B's) + \frac{\log 2\pi}{2} s - \frac{\log 2\pi}{2} - \frac{1 + s}{2} \log s.
\]

\[\square\]

Lemma 6.7.

\[
\log \Pi(s - B) = s \frac{\Pi'(s)}{\Pi(s)} - \frac{1 - \log 2\pi}{2} - s.
\]

Proof. By \( \Pi(s) = s\Pi(s - 1) \), we have

\[
B \log \Pi(B + s) = B \log(B + s) + B \log \Pi(B + s - 1)
\]

(6.4)

\[
= B \log(B + s) + (1 - B) \log \Pi(-B + s).
\]

Using (4.2), we have

(6.5)

\[
B \log \Pi(B + s) = -B \log \Pi(-B + s) + \log \Pi(s).
\]

Combining (6.4) and (6.5), we obtain

\[
\log \Pi(s) = B \log(B + s) + \log \Pi(s - B).
\]

By Lemma 6.2 and Lemma 6.4, we have

\[
\log \Pi(s) = \log \Pi(s) + B \log B + s - s \log(B + s) + \log \Pi(s - B)
\]

\[
= \log \Pi(s) + \frac{1 - \log 2\pi}{2} + s - s \frac{\Pi'(s)}{\Pi(s)} + \log \Pi(s - B).
\]

\[\square\]

Lemma 6.8.

\[
\log \Pi\left( \frac{B + s}{2} \right) = \frac{\Pi'(s/2)}{4\Pi(s/2)} - \frac{s}{2} \log 2 + s \frac{\Pi'(s)}{2\Pi(s)} - \frac{s}{2} + \frac{\Pi'(s)}{2\Pi(s)} - \frac{1}{4} + \frac{\log \pi}{2}.
\]
Proof. By
\[ \Pi(s) = 2^s \Pi\left(\frac{s}{2}\right) \Pi\left(\frac{s-1}{2}\right) \pi^{-1/2}, \]
we have
\[ \log \Pi(B + s) = (B + s) \log 2 + \log \Pi\left(\frac{B + s}{2}\right) + \log \Pi\left(\frac{s-B}{2}\right) - \frac{1}{2} \log \pi \]
\[ = \left(\frac{1}{2} + s\right) \log 2 + \log \Pi\left(\frac{B + s}{2}\right) + \log \Pi\left(\frac{s+B}{2}\right) - \frac{1}{2} \Pi'(s/2) \frac{1}{2} \Pi(s/2) - \frac{1}{2} \log \pi. \]
Therefore
\[ 2 \log \Pi\left(\frac{B + s}{2}\right) = \log \Pi(B + s) - \left(\frac{1}{2} + s\right) \log 2 + \frac{1}{2} \Pi'(s/2) \frac{1}{2} \Pi(s/2) + \frac{1}{2} \log \pi \]
\[ = \log \Pi(-B + s) + \frac{\Pi'(s)}{\Pi(s)} - \left(\frac{1}{2} + s\right) \log 2 + \frac{1}{2} \Pi'(s/2) \frac{1}{2} \Pi(s/2) + \frac{1}{2} \log \pi. \]
By Lemma 6.7, we obtain
\[ 2 \log \Pi\left(\frac{B + s}{2}\right) = s \frac{\Pi'(s)}{\Pi(s)} - \frac{1}{2} \log \pi + \log \frac{\pi}{s} \log - \frac{s}{2} \frac{\cos \pi s}{\sin \pi s} + \frac{\pi}{4} \frac{2 \cos \pi s}{\sin \pi s} - \frac{\cos(\pi s/2)}{\sin(\pi s/2)}. \]
Lemma 6.9.
\[ \log \sin \frac{\pi(B - s)}{2} = -\log 2 + \frac{\pi s \cos \pi s}{2 \sin \pi s} + \frac{\pi}{4} \frac{2 \cos \pi s}{\sin \pi s} - \frac{\cos(\pi s/2)}{\sin(\pi s/2)}. \]
Proof. Since
\[ \frac{\pi s}{\Pi(s) \Pi(-s)} = \sin \pi s, \]
we have
\[ \log \sin \frac{\pi(B - s)}{2} = \log \frac{\pi(B - s)}{2} - \log \Pi\left(\frac{B - s}{2}\right) - \log \Pi\left(-\frac{B + s}{2}\right) \]
\[ = \log \frac{\pi}{2} + \frac{\Pi'(s)}{\Pi(s)} - \log \Pi\left(\frac{B - s}{2}\right) - \log \Pi\left(\frac{s+B}{2}\right) + \frac{1}{2} \Pi'(s/2) \frac{1}{2} \Pi(s/2). \]
By Lemma 6.8, we obtain
\[ \log \sin \frac{\pi(B - s)}{2} = \log \frac{\pi}{2} + \frac{\Pi'(s)}{\Pi(s)} - \frac{\Pi'(s/2)}{4 \Pi(-s/2)} - \frac{s}{2} \log 2 + \frac{s \Pi'(s)}{2 \Pi(s)} - \frac{s}{2} \frac{1}{2} \Pi'(s/2) \frac{1}{2} \Pi(-s/2) \]
\[ + \frac{1}{2} - \frac{1}{2} \log \pi - \frac{1}{4} \frac{\Pi'(s/2)}{\Pi(s/2)} + \frac{s}{2} \log 2 + \frac{s \Pi'(s)}{2 \Pi(s)} + \frac{1}{2} \Pi'(s) + \frac{1}{4} \log \frac{\pi}{2} + \frac{1}{2} \Pi'(s/2) \frac{1}{2} \Pi(s/2) \]
\[ = \log \frac{\pi}{2} + \frac{\Pi'(s)}{\Pi(-s)} - \frac{1}{4} \frac{\Pi'(s/2)}{\Pi(s/2)} + \frac{\Pi'(s/2)}{\Pi(-s/2)} + \frac{s}{2} \frac{\Pi'(s)}{\Pi(-s)} - \frac{\Pi'(s)}{\Pi(s)} \]
\[ \left(\frac{1}{2} \frac{\Pi'(s)}{\Pi(s)} + \frac{\Pi'(s)}{\Pi(-s)} \right) + \frac{1}{2} - \log \pi + \frac{1}{2} \frac{\Pi'(s/2)}{\Pi(s/2)}. \]
Moreover
\[ \frac{\Pi'(s/2)}{2 \Pi(s/2)} - \frac{\Pi'(s/2)}{2 \Pi(-s/2)} = \frac{1}{s} - \frac{\pi \cos \pi s}{2 \sin(\pi s/2)}, \]
By lemma 6.7 and lemma 6.9, we have

\[
\Pi'(s) \Pi(s) - \Pi'(-s) \Pi(-s) = \frac{1}{s} - \pi \frac{\cos \pi s}{\sin \pi s}.
\]

Using (6.7), (6.8) and (6.9), we obtain (6.6). □

We now prove the Theorem 6.1.

**Proof.** Taking the logarithms on both sides of functional equation of Riemann Zeta, let \( s \to B - s \), we have

\[
\log \zeta(B-s) = \log \Pi(s-B) + \log 2\pi - (s+B-1) + \log 2 + \log \sin \left( \frac{\pi}{2} (B-s) \right) + \log \zeta(1-B+s).
\]

By lemma 6.7 and lemma 6.9, we have

\[
\log \zeta(B-s) = s \frac{\Pi'(s)}{\Pi(s)} - \frac{1}{2} \log 2\pi - s \left( \frac{1}{2} + s \right) \log 2 \pi + \log 2 - \log 2 + \frac{\pi s \cos \pi s}{2 \sin \pi s} + \frac{\pi}{4} \left( \frac{2 \cos \pi s}{\sin \pi s} - \frac{\cos(\pi s/2)}{\sin(\pi s/2)} \right) + \log \zeta(B+s)
\]

\[
= s \frac{\Pi'(s)}{\Pi(s)} - \frac{1}{2} - s \log 2\pi + \frac{\pi s \cos \pi s}{2 \sin \pi s} + \frac{\pi}{4} \left( \frac{2 \cos \pi s}{\sin \pi s} - \frac{\cos(\pi s/2)}{\sin(\pi s/2)} \right) + \log \zeta(B+s).
\]

Because \( \log \zeta(B+s) \) is analytic at \( \text{Re } s > 0 \), using the functional equation (6.1), it is clear that the function \( \log \zeta(B+s) \) has poles at the negative integers (i.e. at \( s = -1, -2, -3, \ldots \)). Because

\[
\log \zeta(B) = \log[\zeta(B)(B-1)] - \log(B-1),
\]

by (1.1), we have

\[
\log \zeta(B) = -\gamma - \sum_{n=2}^{\infty} \frac{\zeta(n)}{\zeta(n)(s-1)} + \sum_{n=2}^{\infty} \frac{\zeta(n)}{\zeta(n)} + \lim_{s \to 1} \frac{1}{s-1} + \sum_{n=2}^{\infty} \frac{1}{n-1}
\]

therefore the function \( \log \zeta(B+s) \) have a pole at \( s = 0 \). By the way, we give

\[
\log[-\zeta(B)] = \log[\zeta(B)(B-1)] - \log(1-B)
\]

\[
= -\gamma - \sum_{n=2}^{\infty} \frac{\zeta(n)}{\zeta(n)} + \log B - \log B
\]

\[
= -\gamma - \sum_{n=2}^{\infty} \frac{\zeta(n)}{\zeta(n)}
\]

therefore

\[
\log[-\zeta(B)] = -\gamma - \sum_{n=2}^{\infty} \frac{\zeta(n)}{\zeta(n)} = -\gamma + \log \zeta(1+B)
\]

\[
= -\gamma + \sum_{n=2}^{\infty} \Lambda(n)(\log n)^{-1} n^{-1} - B = -\gamma + \sum_{n=2}^{\infty} \Lambda(n)(\log n)^{-1} n^{-1} \log n
\]

where \( \Lambda(n) \) is von Mangoldt function. By calculating, we can find the value of \( \log[-\zeta(B)] \) is close to the value of \( \log(-\zeta(1/2)) \). □
Remark 6.10. If Riemann Hypothesis is true, then \( \log \zeta(1/2 + s) \) is analytic at all points of \( \text{Re } s > 0 \) except for \( s = 1/2 \). On the other hand, the Bernoulli operator function \( \log \zeta(B + s) \) is an analogue of \( \log \zeta(1/2 + s) \). Because \( \log \zeta(B + s) \) is analytic at \( \text{Re } s > 0 \), this seems to believe that the Riemann hypothesis is true. In fact, if we can prove function \( \log \zeta(B + s) \) has singularities at \( \rho - \frac{1}{2} \) (where \( \rho \) are zeros of \( \xi(s) \)), then Riemann Hypothesis is proved. Using (3.2), let \( s \to \rho' - \frac{1}{2} \), where \( \rho' \) is a zero of \( \xi(s) \), we have

\[
\log \zeta(B + \rho' - \frac{1}{2}) = -\log \Pi(B + \rho' - \frac{1}{2}) + \frac{B + \rho' - 1/2}{2} \log \pi \\
-\log(\rho' - \frac{1}{2} + B - 1) + \log \zeta(B) + \sum_{\rho} \log(1 - \rho' - 1/2) \\
= -\log \Pi(B + \rho' - \frac{1}{2}) + \frac{B + \rho' - 1/2}{2} \log \pi \\
-\log(\rho' - \frac{1}{2} + B - 1) + \log \zeta(B) + \sum_{\rho} \log(\rho - B' + 1/2) .
\]

when \( \rho = \rho' \), the right of above equation will lead to a singular item \( \log(1/2 - B) \) (But need a proof). Hence \( \text{Re } \rho' = 1/2 \).

Corollary 6.11.

\[ -s \frac{\zeta'(B)}{\zeta(B)} = Bs \frac{\Pi'(Bs)}{\Pi(Bs)} - \frac{1 + s + s \log 2\pi}{2} + \frac{B\pi s \cos \pi Bs}{2 \sin \pi Bs} - \frac{\pi^2}{16}s. \]

Proof. Setting \( s \to B' \) (\( B' \) is also a Bernoulli operator) in (6.1), we have

\[
\log \zeta(B - B') - \log \zeta(B + B') = B' \frac{\Pi'(B')}{\Pi(B')} - \frac{1}{2} - B' - B' \log 2\pi \\
+ \frac{\pi B' \cos \pi B'}{2 \sin \pi B'} + \frac{\pi}{4} \left( \frac{2 \cos \pi B'}{\sin \pi B'} - \frac{\cos(\pi B'/2)}{\sin(\pi B'/2)} \right) \\
\]

(6.10)

\[
= Bs \frac{\Pi'(Bs)}{\Pi(Bs)} - \frac{1}{2} - Bs - Bs \log 2\pi + \frac{\pi Bs \cos \pi Bs}{2 \sin \pi Bs} + \frac{\pi}{4} \left( \frac{2 \cos \pi Bs}{\sin \pi Bs} - \frac{\cos(\pi Bs/2)}{\sin(\pi Bs/2)} \right) .
\]

Using (4.1), we have

(6.11)

\[
\log \zeta(B - B') - \log \zeta(B + B') = \frac{\zeta'(B)}{\zeta(B)}. 
\]

By the Taylor expansion, we give

(6.12)

\[
\frac{\pi}{4} \left( \frac{2 \cos \pi Bs}{\sin \pi Bs} - \frac{\cos(\pi Bs/2)}{\sin(\pi Bs/2)} \right) = \frac{\pi}{4} \left[ \frac{1}{\pi Bs} - \frac{2^2}{2} B_2 \pi Bs - \frac{2}{\pi Bs} + \frac{2^2}{2} B_2 \pi Bs \right] = -\frac{\pi^2}{16}s.
\]

Using (6.10), (6.11) and (6.12), we obtain Corollary 6.11. \( \square \)

In particular, setting \( s = 1 \) in Corollary 6.11, using (4.27) and (4.30), we obtain the following equation again

\[
\frac{\zeta'(B)}{\zeta(B)} = \frac{1 + \gamma + \log 2\pi}{2} + \frac{\pi^2}{16}.
\]
7. **Appendix**

This paper is not too rigorous, the author hopes that other mathematicians to complete.

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