Canonical Scalar Field Inflation with String and $R^2$-Corrections

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Assuming that a scalar field controls the inflationary era, we examine the combined effects of string and $f(R)$ gravity corrections on the inflationary dynamics of canonical scalar field inflation, imposing the constraint that the speed of the primordial gravitational waves is equal to that of light's. Particularly, we study the inflationary dynamics of an Einstein-Gauss-Bonnet gravity in the presence of $\alpha R^2$ corrections, where $\alpha$ is a free coupling parameter. As it was the case in the pure Einstein-Gauss-Bonnet gravity, the realization that the gravitational waves propagate through spacetime with the velocity of light, imposes the constraint that the Gauss-Bonnet coupling function $\xi(\phi)$ obeys the differential equation $\ddot{\xi} = H \dot{\xi}$, where $H$ is the Hubble rate. Subsequently, a relation for the time derivative of the scalar field is extracted which implies that the scalar functions of the model, which are the Gauss-Bonnet coupling and the scalar potential, are interconnected and simply designating one of them specifies the other immediately. In this framework, it is useful to freely designate $\xi(\phi)$ and extract the corresponding scalar potential from the equations of motion but the opposite is still feasible. We demonstrate that the model can produce a viable inflationary phenomenology and for a wide range of the free parameters. Also, a mentionable issue is that when the coupling parameter $\alpha$ of the $R^2$ correction term is $\alpha < 10^{-3}$ in Planck Units, the $R^2$ term is practically negligible and one obtains the same equations of motion as in the pure Einstein-Gauss-Bonnet theory, however the dynamics still change, since now the time derivative of $\frac{df}{dR}$ is nonzero. We study in detail the dynamics of inflation assuming that the slow-roll conditions hold true and also we briefly address the constant-roll case dynamics, and by using several illustrative examples, we compare the dynamics of the pure and $R^2$-corrected Einstein-Gauss-Bonnet gravity. Finally the ghosts issue is briefly addressed.

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I. INTRODUCTION

One good question in theoretical physics has always been if the constants of nature are actually constant, or have been dynamically evolving to their value we actually estimate at present time. Tough to answer actually, but it is our opinion that during and possible after the inflationary era, the Universe had settled to a classical state, so all the particles received their mass, and all constants of nature that do not depend inherently on the comoving radius of the Universe, received the value that we measured at present time.

In this line of research, the striking GW170817 event [1] involved a Kilonova which indicated that the gravitational wave speed $c_T$ is equal to that of light, so $c_T^2 = 1$ in natural units. Now the question is whether it is compelling to assume that the speed of the primordial gravitational waves is also equal to that of light’s. To our opinion it is since the constants of nature during the inflationary and the post-inflationary era, must actually be constants. This way of thinking is also adopted in the literature, so as was shown in Ref. [2], many theoretical frameworks were actually put into serious question after the striking GW170817 event.

As we already mentioned previously, the inflationary era is basically a classical era or at least semi-classical, in the sense that the Universe is already four dimensional and expands and simultaneously cools significantly. Since the inflationary era emerged from a quantum era, it is natural to assume that some imprints of the quantum era may be present in the effective Lagrangian that describes the inflationary era. Since string theory in its various aspects is to date the most prominent candidate theory able to describe the quantum era, it is natural to assume that string corrections may actually be present in the effective Lagrangian that describes the inflationary era. A well studied
class of string-corrected gravitational theories, is known under the name Einstein-Gauss-Bonnet theory, and it is quite popular in inflationary and astrophysical studies [3–11]. The Einstein-Gauss-Bonnet theory was crucially affected by the GW170817 results, since the primordial gravitational wave speed for this theory is not equal to unity in natural units, but it is equal to \( c_s^2 = 1 - \frac{Q_f}{2Q_t} \), where \( Q_f = 8c_1(\dot{\xi} - H\dot{\xi}) \). Recently, we pointed out in several articles that if we manage to make the term \( Q_f \) equal to zero, the speed of the primordial gravitational waves can be equal to unity in natural units, for the Einstein-Gauss-Bonnet [31–33, 35, 50] and related theories [51], and this constraint actually relates directly the scalar field potential \( V(\phi) \) and the Gauss-Bonnet scalar coupling function \( \xi(\phi) \). In the same spirit in this work we consider the effects of \( R^2 \)-corrections of the inflationary phenomenology of the Einstein-Gauss-Bonnet theory. The motivation for this is mainly the fact that modified gravity theories [42–47], and particularly, \( f(R) \) gravity, is very successful in describing several evolutionary eras of our Universe, such as the late-time and early-time era, and sometimes can also provide a unified description of the acceleration eras of the Universe see for example Ref. [48] and also Ref. [49] for a recent article in the context of Palatini gravity. Thus it is interesting to see how such \( f(R) \) gravity terms may affect the inflationary phenomenology of the string-corrected scalar field Lagrangian. Basically, it is of the same spirit as the Gauss-Bonnet correction is in the scalar field Lagrangian. The central premise of this article is thus, that the scalar field basically controls the inflationary era, and the scalar field Lagrangian is corrected by string terms and \( f(R) \) gravity terms. We study in detail the inflationary phenomenology of the \( R^2 \)-corrected theory in both the slow-roll and constant-roll cases, and we compare the theory with the latest Planck (2018) observational data [50].

II. THEORETICAL FRAMEWORK OF \( R^2 \) GRAVITY WITH STRING CORRECTIONS

The general expression of the string and \( f(R) \) gravity corrected canonical scalar field gravitational action is the following,

\[
S = \int d^4x\sqrt{-g}\left(\frac{f(R)}{2\kappa^2} - \frac{1}{2}\omega g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) - \xi(\phi)\mathcal{G}\right),
\]

where \( R \) denotes the Ricci scalar, \( \kappa = \frac{1}{M_P} \) with \( M_P \) being the reduced Planck mass, \( V(\phi) \) the scalar potential and \( \xi(\phi) \) the Gauss-Bonnet coupling scalar function. Also for the purposes of this paper we shall assume that \( f(R) = R + \alpha R^2 \), where \( \alpha \) is a constant with mass dimensions [\( m \)]^{-2}, but we keep the \( f(R) \) function notation in order to provide general expressions for the equations of motion, so for the sake of generality. Additionally, \( \mathcal{G} \) denotes the Gauss-Bonnet topological invariant defined as \( \mathcal{G} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \) with \( R_{\mu\nu\rho\sigma} \) and \( R_{\mu\nu} \) being the Riemann curvature tensor and the Ricci tensor respectively. Furthermore, concerning the metric, we shall assume it is a flat Friedman-Robertson-Walker hence the line element read,

\[
ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3}(dx^i)^2,
\]

where \( a(t) \) denotes as usually the scale factor. Consequently, since the cosmological background is flat, the Ricci scalar and the Gauss-Bonnet topological invariant are written in terms of Hubble’s parameter \( H = \frac{\dot{a}}{a} \) as \( R = 6(2H^2 + \dot{H}) \) and \( \mathcal{G} = 24H^2(\dot{H} + H^2) \), where the “dot” as usual implies differentiation with respect to cosmic time \( t \). Also, \( \omega \) shall be equated to unity in order to derive the description of the canonical case, but it shall not be replaced with unity in order to keep general expressions for the subsequent equations containing \( \omega \), and thus having the phantom case corresponding to \( \omega = -1 \) available as well for the interested reader.

Since string corrections are implemented, gravitational waves propagate through spacetime with a velocity which does not necessarily coincide with the speed of light. Specifically, the speed of the primordial tensor perturbations is in our case equal to [3, 32, 56],

\[
\hat{c}_T = 1 - \frac{Q_f}{2Q_t},
\]

in natural units and in addition, \( Q_f = 16(\dot{\xi} - H\dot{\xi}) \) and \( Q_t = \frac{F}{\kappa^2} - 8\dot{\xi}H \) with \( F = \frac{\partial f}{\partial R} \). As a result, compatibility with the GW170817 event can be restored by demanding that the Gauss-Bonnet coupling satisfies the differential equation,

\[
\dot{\xi} = H\xi,
\]

that \( Q_f = 0 \). As was shown in Refs. [32, 56], rewriting the differential equation [41] in terms of the scalar field and assuming that the slow-roll conditions hold true, meaning that \( \dot{\phi} \ll H\dot{\phi} \) for the scalar field, then we get,

\[
\dot{\phi} = H\xi',
\]
where “prime” denotes differentiation with respect to the scalar field $\phi$. By varying the action $\Pi$ with respect to the metric tensor $g^{\mu \nu}$ and the scalar field $\phi$, the equations of motion are obtained which in this case read,

\[ \frac{3FH^2}{\kappa^2} = \frac{1}{2} \omega \dot{\phi}^2 + V + \frac{FR-f}{2\kappa^2} - \frac{3HF}{\kappa^2} + 24\xi H^3, \]

\[ -\frac{2FH}{\kappa^2} = \omega \dot{\phi}^2 - 16\xi H\dot{H} + \frac{\ddot{F} - H\dot{F}}{\kappa^2}, \]

\[ V' + \omega(\dot{\phi} + 3H\dot{\phi}) + \xi'G = 0. \]

In this case, we shall implement only the slow-roll conditions,

\[ \dot{H} \ll H^2, \quad \frac{1}{2} \omega \dot{\phi}^2 \ll V, \quad \ddot{\phi} \ll H \dot{\phi}, \]

thus, the equations of motion can be simplified greatly. Recalling that $f(R) = R + \alpha R^2$, one obtains elegant simplifications and functional expressions. For instance, since $f(R)$ has this specific form, we have $3FH^2 = 3H^2 + \frac{FR-f}{2\kappa^2}$ and in addition, $2FH = 2\dot{H} + \dot{H}$. Furthermore, $\ddot{F} \sim \dot{H}$ hence it can be neglected from the first equation of motion due to the slow-roll conditions. Hence, the first two equations of motion are written as,

\[ \frac{3H^2}{\kappa^2} = V - \frac{144\alpha H^2 \dot{H}}{\kappa^2} + 24\xi H^3, \]

\[ -\frac{2H}{\kappa^2} = \omega \dot{\phi}^2 + \frac{\ddot{F}}{\kappa^2} - 16\xi H\dot{H}. \]

Here, we need to note that for $\dot{H} \ll H^2$, the Ricci scalar becomes approximately equal to $R \simeq 12H^2$ and therefore $F = 1 + 24\alpha H^2$, $\ddot{F} = 48\alpha H\dot{H}$ and $\ddot{F} = 48\alpha(H^2 + H\dot{H})$. Naturally, from the slow-roll approximations, it stands to reason that $\dot{F}$ and $\ddot{F}$ are relative small. In the following, we shall neglect their contribution from the equations of motion and afterwards ascertain whether such assumption is in fact reasonable. Therefore, assuming that $\dot{F} \ll \omega \dot{\phi}^2$ and discarding the term $144\alpha H^2 \dot{H}$, one obtains the same equations of motion as in the simple Einstein-Gauss-Bonnet case, as shown below,

\[ H^2 \simeq \frac{\kappa^2 V}{3}, \]

\[ \dot{H} \simeq -H^2 \frac{\kappa^2 \omega}{2} \left( \frac{\xi'}{\xi^0} \right)^2, \]

\[ V' + 3\omega H \dot{\phi} + 24\xi' H^4 \simeq 0, \]

Here, we mention again that two assumptions were made. Firstly, the slow-roll conditions, from which the derivatives of the Ricci scalar have been neglected and secondly, the string corrections are negligible in the equations of motion. As a result, Hubble’s parameter is given by a simple expression which as a matter of fact coincides with the case of $\alpha \to 0$ along with Hubble’s derivative and the continuity equation for the scalar field, which in this framework serves as a differential equation satisfied by the scalar potential. Thus, the squared term $\alpha R^2$ participates only indirectly in the inflationary phenomenology, via the expressions of the slow-roll indices and the corresponding observational indices. In order to see this at first hand, we shall examine the same models as in Ref. [36], since these scalar coupling functions lead to simple expressions for the ratio $\xi'/\xi''$. It is expected however that a change will occur since now $\ddot{F} \neq 0$, hence the third slow-roll index shall be present. For $f(R) = R + \alpha R^\gamma$, one obtains,

\[ \frac{3FH^2}{\kappa^2} + \frac{f - FR}{2\kappa^2} = \frac{3H^2}{\kappa^2} + \frac{\alpha(12H^2)^\gamma}{\kappa^2} \left( \frac{\gamma}{4} - \frac{\gamma - 1}{2} \right), \]
The second term is simplified greatly for $\gamma = 2$, hence the reason $f(R) = R + \alpha R^2$ was selected. Since the slow-roll conditions are implemented, or at least are assumed to hold true, the slow-roll indices are defined as follows \[3, 32-36\],

\[
\begin{align*}
\epsilon_1 &= -\frac{\ddot{H}}{H^2}, \\
\epsilon_2 &= \frac{\dot{\phi}}{H\dot{\phi}}, \\
\epsilon_3 &= \frac{\dot{F}}{2HF}, \\
\epsilon_4 &= \frac{\dot{E}}{2HE}, \\
\epsilon_5 &= \frac{\dot{F} + \kappa^2 Q_a}{2H \kappa^2 Q_t}, \\
\epsilon_6 &= \frac{\dot{Q}_t}{2HQ_t} \\
\end{align*}
\]  

where the auxiliary parameters are given by the following expressions,

\[
\begin{align*}
F &= 1 + 2\alpha H = 1 + 24\alpha H^2, \\
Q_a &= -8\dot{\xi}H^2, \\
Q_t &= \frac{F}{\kappa^2} - 8\dot{\xi}H, \\
E &= \frac{F}{\kappa^2}\left(\omega + \frac{3(\dot{F} + \kappa^2 Q_a)^2}{2\kappa^4 \dot{\phi}^2 Q_t}\right), \\
Q_e &= -32\dot{\xi}\dot{H}, \\
\end{align*}
\]

The function $Q_e$ does not participate in the slow-roll indices but is introduced since it participates in subsequent equations. Concerning $F$, we replaced the Ricci scalar with only Hubble’s parameter squared due to the slow-roll conditions. As a result, the slow-roll parameters are rewritten as,

\[
\begin{align*}
\epsilon_1 &= \frac{\kappa^2 \omega}{2} \left(\frac{\xi'}{\xi''}\right)^2, \\
\epsilon_2 &= 1 - \epsilon_1 - \frac{\xi'\xi'''}{\xi''^2}, \\
\epsilon_3 &= \frac{24\alpha \dot{H}}{1 + 24\alpha H^2}, \\
\epsilon_4 &= \frac{1}{2} \frac{\xi' E'}{\xi'' E}, \\
\epsilon_5 &= \frac{24\alpha \xi'' H^2 - 4\kappa^2 \xi'^2 H^2}{\xi''(1 + 24\alpha H^2) - 8\kappa^2 \xi'^2 H^2}, \\
\epsilon_6 &= \frac{24\alpha \xi'' \dot{H} - 4\kappa^2 \xi'^2 H^2(1 - \epsilon_1)}{\xi''(1 + 24\alpha H^2) - 8\kappa^2 \xi'^2 H^2}.
\end{align*}
\]

Moreover, the auxiliary parameters introduced previously are rewritten, according to the approximated equations of motion, as follows,

\[
\begin{align*}
F &= 1 + 8\alpha \kappa^2 V, \\
Q_a &= -8\frac{\xi'^2}{\xi'' \kappa^2} \left(\frac{V}{3}\right)^{\frac{2}{3}},
\end{align*}
\]
issue formally. This issue was addressed in Ref. [51] in detail, where we demonstrated that the condition \( \dot{\epsilon} = 0 \) is superfluous and misleading, and we devote one article for exactly explaining this delicate problem.

But these conditions are superfluous and misleading, and we devote one article for exactly explaining this delicate problem.

Using the slow-roll indices, we can evaluate the observed indices, and specifically the spectral index of the primordial scalar curvature perturbations \( n_s \), the tensor-to-scalar ratio \( r \) and the tensor spectral index \( n_T \) as,

\[
  n_S = 1 - 2 \frac{2\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4}{1 - \epsilon_1} \quad \quad n_T = -2 \frac{\epsilon_1 + \epsilon_6}{1 - \epsilon_1} \quad \quad r = 16 \left| \frac{\kappa^2 Q_T}{4HF - \epsilon_1 - \epsilon_3} \frac{F c_A^3}{\kappa^2 Q_t} \right| ,
\]

where \( c_A \) denotes the sound wave velocity given by the expression,

\[
  c_A^2 = 1 + \frac{\kappa^2 Q_e (\dot{F} + \kappa^2 Q_e)}{2\omega^4 Q_e \phi^2 + 3(\dot{F} + \kappa^2 Q_e)^2} .
\]

These are similar equations with the ones used in Ref. [30], but the main difference is that \( \epsilon_3 \neq 0 \) in the case at hand. This is also the reason why the exact same models shall be studied, in order to ascertain the effects of the \( R^2 \) corrections, and examine how this correction affects the observational quantities. Since each function is now written as a function of the scalar field \( \phi \), it shall be used in order to extract information about the inflationary era.

Specifically, by letting \( \epsilon_1 \sim O(1) \), the final value of the scalar field is derived. Subsequently, by using the e-foldings number defined as \( N = \int_{t_i}^{t_f} H dt \) where \( t_f - t_i \) signifies the duration of inflation, and expressing it in terms of the scalar field \( \phi \) as,

\[
  N = \int_{\phi_i}^{\phi_f} \frac{\xi''}{\xi'} d\phi ,
\]

the expression for the scalar field during the first horizon crossing is produced. Hence, by using it as an input in Eq. (33), we shall examine whether there exist values for the free parameters which produce compatible results with the latest Planck 2018 collaboration [54], which specifically constrain \( n_s \) and \( r \) as follows,

\[
  n_s = 0.9649 \pm 0.0042 \quad \quad r < 0.064 ,
\]

with 68\%C.L and 95\%C.L respectively. Referring to the tensor spectral index \( n_T \), no specific value is expected since B-Modes have yet to be observed. In the following, we present the results for certain model coupling functions which are capable of simplifying the ratio \( \xi' / \xi'' \).

An important comment is in order, related to the final form of the observational indices [33]. In our case, a crucial assumption for the derivation of the observational indices [33] was the satisfaction of the slow-roll condition \( \epsilon_i \ll 1 \), \( i = 1, 3, 4, 5, 6 \), that is, for all the observational indices. A confusion occurs in the literature, since the authors of Ref. [3] derived the expressions for the observational indices and specifically for the spectral indices appearing in [33], by using the additional condition \( \dot{\epsilon}_1 = 0 \), or in other texts, this is modified to the more general \( \dot{\epsilon}_1 = \text{const} \). But these conditions are superfluous and misleading, and we devote one article for exactly explaining this delicate issue formally. This issue was addressed in Ref. [51] in detail, where we demonstrated that the condition \( \dot{\epsilon}_1 = 0 \) is redundant and even can be misleading, and certainly causes confusion in the literature. As we showed in detail in [51], it is possible to obtain exactly the same expressions for the spectral indices (tensor and scalar) as those appearing in Eq. [33], without assuming \( \dot{\epsilon}_1 = 0 \) or even \( \dot{\epsilon}_1 = \text{const} \), by simply making use of Karamata’s theorem for regularly varying functions, and the slow-roll conditions for the slow-roll indices \( \epsilon_i \ll 1 \), \( i = 1, 3, 4, 5, 6 \). Thus the inconsistency caused by the condition \( \epsilon_1 = \text{const} \), is artificial and can be formally alleviated by using the slow-roll assumptions and Karamata’s theorem.

In fact, the same expression for the spectral index as in Eq. [33] can be derived without assuming \( \dot{\epsilon}_1 = 0 \), as we showed in Ref. [51].
III. TESTING THE THEORETICAL FRAMEWORK WITH THE LATEST OBSERVATIONAL DATA

In this section, we shall ascertain the validity of certain model functions. Firstly, the Gauss-Bonnet scalar coupling function shall be defined properly, aiming for a simple ratio $\xi'/\xi''$ as mentioned previously. Afterwards, the scalar potential shall be derived from Eq. (14). Ideally, the string corrections have been neglected from the equations of motion without altering the results simply because relation (5) holds true. Thus, in the following examples, we shall use a different, more simplified differential equation for the scalar field as shown below,

$$V' + 3 \omega H^2 \frac{\xi'}{\xi''} \simeq 0 ,$$

(37)

which is valid only if $24 \xi H^4$ is indeed negligible compared to the rest terms. This new form is inspired from neglecting of $24 \xi H^3$ and $16 \xi H \dot{H}$ from the first and second equation of motion in (10) and (11) respectively, and it leads to a much more elegant functional form of the scalar potential as we shall demonstrate in the following examples. Furthermore, the opposite case is also interesting, where some of the string corrections in the differential equation are kept while the kinetic term is omitted, meaning that,

$$V' + 24 \xi' H^4 \simeq 0 ,$$

(38)

due to the fact that $H$ is proportional to the scalar potential $V$ and also $\kappa \xi'/\xi'' \ll 1$, hence once obtains an ordinary differential equation, however this case will not be studied in the present paper. In the following we demonstrate certain viable models.

A. Error Function Coupling Under The Slow-Roll Assumption

We begin our examples by designating the Gauss-Bonnet coupling as follows,

$$\xi(\phi) = \frac{2 \lambda_1}{\sqrt{x}} \int_0^{\gamma_1 \kappa \phi} e^{-x^2} \, dx ,$$

(39)

where $\lambda_1$ and $\gamma_1$ are dimensionless constants to be specified later, whereas $x$ is an auxiliary integration variable. In Ref. [36], the complete differential equation (14) was used and the produced scalar potential was at the least lengthy. Since Hubble’s parameter and the differential equation for the potential have not been altered in the $R^2$ case, as it was demonstrated in the previous section, the same potential is sure to be produced in this case as well. Hence, we shall use Eq. (37) in order to examine whether the string corrections are as negligible as we stated they were. Using Eq. (37), the scalar potential reads,

$$V(\phi) = V_1 (\kappa \phi)^{\frac{\omega}{2}},$$

(40)

where $V_1$ is an arbitrary integration constant with mass dimensions $[m]^4$ and signifies the amplitude of the potential. Hence, by omitting the string corrections from Eq. (14), one obtains a simple power-law form for the scalar potential with the exponent not necessarily being an integer. Furthermore, $F$ might not be directly present in the equations of motion, however, it alters the dynamics of the model as it participates in the observational indices, hence a difference between the pure Einstein-Gauss-Bonnet case and the $R^2$ approach shall be demonstrated. Concerning the slow-roll indices, these are,

$$\epsilon_1 = \frac{\omega}{2(2 \gamma_1^2 \kappa \phi)^2} ,$$

(41)

$$\epsilon_2 = \frac{(2 \gamma_1)^2 - \omega}{2(2 \gamma_1^2 \kappa \phi)^2} ,$$

(42)

$$\epsilon_3 = -\frac{\alpha \omega \kappa^2 V(\phi)}{(\gamma_1^2 \kappa \phi)^2 (8 \alpha \kappa^2 V(\phi) + 1)} ,$$

(43)

$$\epsilon_5 = \frac{\kappa^2 V(\phi) \left( 4 \gamma_1^2 \kappa^2 \lambda_1 \kappa \phi - 3 \sqrt{\pi} \alpha \omega e^{(\gamma_1 \kappa \phi)} \right)}{(3 \sqrt{\pi} \gamma_1 \kappa \phi e^{(\gamma_1 \kappa \phi)} + 8 \kappa^2 V(\phi) \left( 3 \sqrt{\pi} \alpha \gamma_1 \kappa \phi e^{(\gamma_1 \kappa \phi)} + \kappa^2 \lambda_1 \right)} ,$$

(44)
\[ \epsilon_6 = \frac{2\lambda_1 k^4 V(\phi) (2(\gamma_1 \kappa \phi)^2 + 1) - 2\kappa \phi e V'(\phi) \left(3\sqrt{\pi} \alpha \gamma_1 \kappa \phi e (\gamma_1 \kappa \phi)^2 + \kappa^2 \lambda_1 \right)}{(\gamma_1 \kappa \phi)^2 \left(3\sqrt{\pi} \gamma_1 \kappa \phi e (\gamma_1 \kappa \phi)^2 + 8\kappa^2 V(\phi) \left(3\sqrt{\pi} \alpha \gamma_1 \kappa \phi e (\gamma_1 \kappa \phi)^2 + \kappa^2 \lambda_1 \right) \right)}, \] \hspace{1cm} (45)

It is clear that the first three slow-roll indices have simple functional forms, due to the simplified ratio \( \xi'/\xi'' \), however this is not the case for \( \epsilon_4 \), since the produced form is quite lengthy, it was deemed suitable to neglect such index but it still participates in the scalar spectral index. Utilizing the first slow-roll index and the \( \epsilon \)-foldings number, meaning equations (41) and (35), the initial and final value of the scalar field read,

\[ \phi_i = \pm \sqrt{\frac{N + (\gamma_1 \kappa \phi_0)^2}{\gamma_1 \kappa}}, \] \hspace{1cm} (46)

\[ \phi_f = \pm \sqrt{\frac{1}{2} \frac{\omega}{N}}, \] \hspace{1cm} (47)

In the following, we shall limit our work only in the positive values of the scalar field. Assigning the following values for the free parameters of the model in Planck Units, meaning \( \kappa^2 = 1, (\omega, \lambda_1, N, \gamma_1, V_1, \alpha) = (1, 1, 60, 0.5, 1, 10^{-3}) \), then the resulting observational indices are equal to \( n_S = 0.966841, r = 0.045034 \) and \( n_T = -0.005677 \) which are obviously acceptable values. In addition, since \( \epsilon_4 = 1 \), the model is free of ghost instabilities. Lastly, we mention that the numerical values of the slow-roll indices, which are indicative of the validity of the slow-roll conditions, are \( \epsilon_1 = 0.0082644, \epsilon_2 = 3.3 \cdot 10^{-16}, \epsilon_3 = -0.00545, \epsilon_4 = -0.005534 \) and \( \epsilon_5, \epsilon_6 \) are both equal to \( \epsilon_3 \). This in turn implies that the slow-roll conditions hold true. Also in Fig. 1 we plot the spectral index of primordial curvature perturbations \( n_S \) (left) and the tensor-to-scalar ratio \( r \) (right) depending on parameters \( \gamma_1 \) and \( V_1 \) ranging from \([0.5, 2]\) and \([0.1, 2]\) respectively. It is interesting to make comparisons between the \( R^2 \)-corrected Einstein-Gauss-Bonnet gravity studied in this paper with the pure Einstein studied in Ref. [36]. Firstly, the scalar potential in this case is extremely simple, as it turns out that \( V(\phi) \sim \phi^2 \) for the previous designation. Hence, the exponent is indeed integer. The major difference is that the previous designation is not fine tuned, which was not the case in Ref. [36]. In that case, the integration constant \( c \) of Ref. [36], which in this case can be thought of as the inverse of the potential amplitude, was of order \( \mathcal{O}(10^{-20}) \). In the present paper, the only parameter with a small value happens to be \( \alpha \). This particular designation is somewhat necessary in order to not only extract compatible results, but also satisfy the approximations which were imposed previously. The spectral index between the case studied in [37] and the \( R^2 \)-corrected case remains the same and as a matter of fact, the main difference is the tensor-to-scalar ratio, which in the case at hand is greater than the case studied in Ref. [36].

For parameter \( \alpha \), we need to note that in order for Eq. (18) to be valid, \( \alpha \) must be approximately of order \( \mathcal{O}(10^{-3}) \) and smaller, otherwise \( \tilde{F} \) becomes equal to, or greater than, the kinetic term \( \omega \phi^2 \) in Eq. (7).
It is also worth mentioning that changing parameter $\alpha$ to $\alpha = 0.019$ produces observational indices which both respect the approximate expressions of the $R^2$ gravity, however the approximations in the equation of motion are violated, specifically $144\alpha H^2H$ and $V$ are of the same order while $\dot{F}$ and $\omega \dot{\phi}^2$ are differing only by one order of magnitude, hence we avoided to give these values to the parameter $\alpha$.

Furthermore, it is worthy to discuss the correspondence of the system under a change in a free parameter. For $\gamma_1$, an increase leads to a subsequent increase in $n_S$ while simultaneously $r$ decreases. In addition, the amplitude of the scalar potential manages to increase $n_S$ and decrease $r$, faster than $\gamma_1$, once $V_1$ increases. The opposite obviously happens when either $\gamma_1$ or $V_1$ decreases. Keeping the current values for the free parameters and changing only $\alpha$, we stated that a decrease manages to produce incompatible results while an increase makes the approximations in the equations of motion invalid. Although this is true, we mention that changing both $\alpha$ and $V_1$, for instance $\alpha = 10^{-4}$ and $V_1 = 10$ produces the exact same observational indices, however, the order of magnitude of each term in the equations of motion experiences the same change, hence their ratio’s remain the same.

Finally, we mention that all the approximations imposed in the equations of motion, meaning the slow-roll conditions and the string corrections are justifiable, for the values of the free parameters we used in order to obtain viability of the model with the observational data. Firstly, as it was hinted from the numerical values of the slow-roll indices, we mention that the slow-roll approximations hold since $\dot{H} \sim \mathcal{O}(10^{-1})$ while $H^2 \sim \mathcal{O}(10)$, $\frac{1}{2} \omega \dot{\phi}^2 \sim \mathcal{O}(10^{-1})$ while $V \sim \mathcal{O}(100)$ and finally, $\dot{\phi} \sim \mathcal{O}(10^{-17})$ whereas $H \dot{\phi} \sim \mathcal{O}(10)$. Furthermore, the following terms are indeed negligible since $24\xi H^3 \sim \mathcal{O}(10^{-21})$, $16\xi H \dot{H} \sim \mathcal{O}(10^{-25})$ and $24\xi' H^4 \sim \mathcal{O}(10^{-22})$ while $V' \sim \mathcal{O}(10)$. Thus, it was reasonable to assume that $\xi' \sim \mathcal{O}(10)$ is the right choice for producing the scalar potential. Finally, since $\alpha = 10^{-3}$, as stated before, we mention that $144\alpha H^2H \sim \mathcal{O}(1)$, $\dot{F} \sim \mathcal{O}(10^{-2})$ thus neglecting such terms relative to $V$ and $\omega \dot{\phi}^2$ is justifiable.

**B. Advanced Exponential As a Coupling Function Under The Slow-Roll Assumption**

Let us now consider another model, so in this case we assume that,

$$
\xi(\phi) = \kappa \lambda_2 \int_0^{\kappa \phi} e^{-\gamma_2 x^2} \, dx,
$$

where now $\lambda_2$ and $\gamma_2$ are now the dimensionless parameters of the model. This choice also simplifies the ratio $\xi'/\xi''$ hence the selection. Using Eq. (57), the resulting scalar potential reads,

$$
V(\phi) = V_2 e^{-\frac{(\kappa \phi)^2 - m}{\gamma_2 x^2}}
$$

where as before, $V_2$ is the integration constant with mass dimensions $[m]^4$. In this case, the scalar potential has an exponential form and not a power-law form as before, however in this case as well the produced potential is very simple functionally. As a result, the slow-roll indices take the following expressions,

$$
\epsilon_1 = \frac{\omega (\kappa \phi)^{2-2m}}{2(\gamma_2)^2},
$$

$$
\epsilon_2 = -\frac{\omega (\kappa \phi)^2 + 2m\gamma_2 (m - 1)(\kappa \phi)^m}{2m^2 \gamma_2^2 (\kappa \phi)^{2m}},
$$

$$
\epsilon_3 = \frac{4\alpha \omega (\kappa \phi)^{2-2m} \kappa^2 V}{\gamma_2 (m \gamma_2)^2 (1 + 8\alpha \kappa^2 V)},
$$

$$
\epsilon_5 = \frac{4\kappa^4 V(\phi)(\kappa \phi)^{-m} \left(3\omega \phi^2 + \gamma_2 (\kappa \phi)^m \phi (\kappa \phi)^m e^{\gamma_2 (\kappa \phi)^m} \right)}{\gamma_2 m \left(3\gamma_2 m (\kappa \phi)^m - 8\kappa^2 V(\phi) (\kappa^3 \lambda_2 \phi \delta^{\gamma_2 (\kappa \phi)^m} - 3\alpha \gamma_2 m (\kappa \phi)^m)\right)}.
$$

Once again, the first three slow-roll indices have simple and elegant forms whereas the rest are intricate. From the first slow-roll index however, one obtains the following values for the scalar field,

$$
\phi_i = \frac{1}{\kappa} \left( (\kappa \phi_i)^m - \frac{N}{\gamma_2} \right),
$$
\[ \dot{\phi}_f = \frac{1}{\kappa} \left( \frac{2m^2\gamma_2}{\omega} \right)^{\frac{1}{2m-3}}. \]  

(55)

By giving the following values for the free parameters in reduced Planck units \((\omega, \lambda_2, N, \gamma_2, V_2, m, \alpha) = (1, -1, 60, -0.5, 1, -6, 10^{-3})\), then the observational indices obtain the values \(n_S = 0.961203, n_T = -1.5 \cdot 10^{-6}\) and \(r = 1.23 \cdot 10^{-5}\) so once again the results are compatible. Additionally, \(c_A = 1\) thus the model is free of ghosts and finally, \(\epsilon_1 = 7 \cdot 10^{-7}, \epsilon_2 = 0.0165\) and indices \(\epsilon_3\) through \(\epsilon_6\) are equal to \(-8 \cdot 10^{-11}\). Hence, the slow-roll conditions indeed apply, in the case of the second index however they are marginally applicable. In Fig. 2 we plot the spectral index of primordial curvature perturbations \(n_S\) (left) and the tensor-to-scalar ratio \(r\) (right) depending on parameters \(\gamma_2\) and \(m\) ranging from \([-2, -0.5]\) and \([-10, -4]\) respectively, and we can see that the viability of the model can be ensured for a wide range of the parameters used. We need to note that the approximations we made are valid, but parameter \(\alpha\) must be of order \(O(10^{-3})\) or smaller in order for the approximations to be valid. When it comes to the rest parameters, the situation is similar to the previous model. This was also the reason this coupling function was chosen, as it shares some common characteristics with the error function.

Finally, we discuss the validity of the approximations imposed. When it comes to the slow-roll approximations, we mention that \(\dot{H} \sim O(10^{-7})\) while \(H^2 \sim O(10^{-1})\), \(\frac{3}{2} \omega \dot{\phi}^2 \sim O(10^{-7})\) while \(V \sim O(1)\) and lastly \(\dot{\phi} \sim O(10^{-6})\) whereas \(H \dot{\phi} \sim O(10^{-4})\) so indeed the slow-roll approximations hold when the free parameters of the theory obtain the previous values. Concerning the additional approximations, \(24\xi H^3 \sim O(10^{-29})\), \(16\xi H \dot{H} \sim O(10^{-35})\) and \(24\xi H^4 \sim O(10^{-26})\) compared to \(V' \sim O(10^{-3})\). Moreover, concerning the derivatives of \(F\), we have \(144\alpha H^2 \dot{H} \sim O(10^{-8})\) and \(\ddot{F} \sim O(10^{-10})\) thus neglecting such terms compared to the kinetic term is justifiable.

C. Power-Law Coupling Under the Constant-Roll Assumption

In this final example, we shall assume that the Gauss-Bonnet coupling function is given by the following expression, \[ \xi (\phi) = \lambda_3 (\kappa \phi)^n, \]  

(56)

Here, we shall also assume that the scalar field obeys the constant-roll condition \(\ddot{\phi} = \beta H \dot{\phi}\) where \(\beta\) is the constant-roll parameter to be specified later. As a result, the condition for the time derivative of the scalar field, derived from Eq. 3 reads,  

\[ \dot{\phi} = H(1 - \beta) \frac{\xi'}{\xi n}, \]  

(57)

Substituting \(\beta = 0\) restores the previous equations. Also, it is worth stating that exponent \(n\) cannot obtain the value \(n = 1\), implying that the Gauss-Bonnet coupling cannot be a linear function of \(\phi\). This is because the constant-roll
parameter becomes equal to $\beta = 1$ hence the slow-roll conditions cannot be implemented. Consequently, the equations of motion (12) through (14) are written as,

$$H^2 = \frac{\kappa^2 V}{3},$$

$$\dot{H} = -H^2 \frac{(1 - \beta)^2 \kappa^2 \omega}{2} \left(\frac{\xi'}{\xi''}\right)^2,$$

$$V' + (3 + \beta)(1 - \beta)\omega H^2 \frac{\xi'}{\xi''} = 0,$$

Thus, only the last two equations were affected by the constant-roll condition directly, however since the scalar potential is derivable from Eq. (60), then Eq. (58) shall also depend on $\beta$. Furthermore, the slow-roll indices in this framework, for an unspecified Gauss-Bonnet coupling function, are given by the following expressions,

$$\epsilon_1 = \frac{(1 - \beta)^2 \kappa^2 \omega}{2} \left(\frac{\xi'}{\xi''}\right)^2,$$

$$\epsilon_2 = \beta,$$

$$\epsilon_3 = \frac{24\alpha \dot{H}}{1 + 24\alpha H^2},$$

$$\epsilon_4 = \frac{1 - \beta \xi' E'}{\xi'' E'},$$

$$\epsilon_5 = \frac{24\alpha \xi'' H^2 - 4(1 - \beta)\kappa^2 \xi'^2 H^2}{\xi''(1 + 24\alpha H^2) - 8(1 - \beta)\kappa^2 \xi'^2 H^2},$$

$$\epsilon_6 = \frac{24\alpha \xi'' H^2 - 4(1 - \beta)\kappa^2 \xi'^2 H^2(1 - \epsilon_1)}{\xi''(1 + 24\alpha H^2) - 8(1 - \beta)\kappa^2 \xi'^2 H^2},$$

We shall refrain from rewriting the auxiliary parameters $Q_i$ and $E$ since they differ only by a factor of $1 - \beta$, but we mention that the $e$-foldings number is given by the expression,

$$N = \frac{1}{1 - \beta} \int_{\phi_i}^{\phi_f} \xi'' d\phi.$$

Let us proceed with the model at hand. From equations (66) and (67), one obtains the following scalar potential,

$$V(\phi) = V_3 e^{-\frac{(\alpha^2 + 2\beta - 3)(\xi')^2}{2(\kappa^2 H^2)}},$$

where $V_3$ is the integration constant with mass dimensions $[m]^4$ for consistency. The resulting potential is a exponential and has a simple functional form. Consequently, the slow-roll indices for this specific pair of scalar functions are written as follows,

$$\epsilon_1 = \frac{\omega}{2} \left(\frac{(1 - \beta)\kappa\phi}{n - 1}\right)^2,$$

$$\epsilon_2 = \beta.$$
\[
\epsilon_3 = -\frac{4(1 - \beta)^2 \alpha \phi^2 \kappa^4 V}{(n - 1)^2 (1 + 8 \alpha \kappa^2 V)},
\]

\[
\epsilon_5 = -\frac{4(1 - \beta) \kappa^4 V(\phi) (3 \alpha (1 - \beta) \phi^2 \omega + n(n - 1) \lambda_3 (\kappa \phi)^n)}{(n - 1) (8 \kappa^2 V(\phi) (3 \alpha (n - 1) - (1 - \beta) \kappa^2 \lambda_3 n(\kappa \phi)^n) + 3(n - 1))}.
\]

From Eq. (69) and Eq. (35), the expressions for the initial and final value of the scalar field are derived, which in this case read,

\[
\phi_i = \phi_f e^{-\frac{N(1-\beta)}{1-\beta}}.
\]

\[
\phi_f = \pm \frac{1}{\sqrt{2}} \kappa^{-1} \sqrt{\frac{n - 1}{\omega (1 - \beta)}},
\]

In the following, we shall limit our work only to the positive value of the scalar field. Assigning the following values to the free parameters, always in reduced Planck units, \((\omega, \lambda_3, N, V_3, \alpha, n, \beta) = (1, 1, 60, 1, 0.001, 15, 0.0165)\) then the observational indices take the values \(n_S = 0.96612, r = 0.00346392\) and \(n_T = -0.000433029\) which are obviously compatible with the latest Planck data \([50]\). Concerning the scalar field, we mention that \(\phi_3 = 0.297384\) and \(\phi_f = 20.1312\) hence an increasing with time field is present. Also, as expected \(c_A = 1\) and lastly, \(c_1 = 0.000218, c_3 = -1.7 \cdot 10^{-6}\) and the rest indices are approximately equal to \(c_3\) hence the slow-roll conditions are valid assumptions. In Fig. 3 we plot the spectral index of primordial curvature perturbations \(n_S\) (left) and tensor-to-scalar ratio \(r\) (right) depending on exponent \(n\) and constant-roll parameter \(\beta\), ranging from [10,17] and [0.015, 0.02] respectively. As it can be inferred, there exist multiple pairs which lead to viable results. In this simple model once again \(\alpha\) must be equal to or smaller that \(10^{-3}\) in order for the approximated equations of motion to be valid. Also, the constant-roll parameter \(\beta\) affects the spectral index of primordial curvature perturbations greatly while the tensor-to-scalar ratio mildly. Specifically, a decrease in \(\beta\) leads to an increase in both \(n_S\) and \(r\). In contrast, the exponent \(n\) affects the tensor-to-scalar ratio more than the scalar spectral index. Decreasing \(n\) leads to a decrease in \(r\) while it increases \(n_S\). Also, \(\alpha\) affects the tensor-to-scalar ratio, but it is not a significant change. For instance, letting \(\alpha = 10^{-5}\), a decrease in order to be consistent with the approximations in the equations of motion, leads to a numerical change in the fifth decimal of \(r\). Moreover, \(V_3\) affects once again only the tensor-to-scalar ratio. Assigning \(V_3 = 10\) leads to \(r = 0.003234\) while \(V_3 = 0.1\) results in \(r = 0.003489\). The parameter \(\lambda_3\) does not affect the observational indices, but is used only to decrease the order of magnitude of the string corrections.

Finally, we discuss the validity of the approximations which were made in the equations of motion. Referring to the slow-roll conditions, we mention that \(\dot{H} \sim O(10^{-5})\) whereas \(H^2 \sim O(10^{-1})\) and \(\frac{1}{2} \omega \phi^2 \sim O(10^{-5})\) while
$V \sim \mathcal{O}(1)$. In addition, the following terms are indeed negligible since $24\dot{c}^2H^3 \sim \mathcal{O}(10^{-8}),$ $16\dot{c}^2H \dot{H} \sim \mathcal{O}(10^{-12})$ and $24\dot{c}^2H^4 \sim \mathcal{O}(10^{-6})$ in comparison to $V' \sim \mathcal{O}(10^{-2})$. Their order of magnitude can also further be decreased by altering $\lambda_3$, and specifically by decreasing $\lambda_3$. Lastly, we mention that $144\alpha H^2 \dot{H} \sim \mathcal{O}(10^{-6})$ while $\ddot{F} \sim \mathcal{O}(10^{-7})$ thus it is reasonable to neglect such terms. Moreover, a decrease in $\alpha$ leads to a subsequent decrease in these terms as well.

\section{IV. THE GHOST INSTABILITIES ISSUE: ARE THERE ANY GHOST MODES}

Let us now discuss an important issue for the $R^2$- and string-corrected canonical scalar theory we studied in this paper, and specifically the ghost issue. So the question is, are there any ghost propagating modes? At the cosmological level, this issue is easily answered by examining the values that the sound wave velocity $c_A$ appearing in Eq. (34) can take. Specifically, tachyonic propagating modes can occur if the sound speed $c_A$ takes negative values or values $c_A > 1$. Let us recall how the sound speed enters the cosmological perturbations evolution equations, and how it determines whether ghost modes occur or not. The cosmological perturbations are obtained by actually deforming the flat FRW background $g_{3\alpha\beta}^{(3)}$ as follows [3],

$$d^2s = -a^2(1 + \alpha)d\eta^2 - 2a^2\beta_{,\alpha}d\eta dx^\alpha + a^2 (g_{3\alpha\beta}^{(3)} + 2\varphi g_{3\alpha\beta}^{(3)} + 2\gamma_{,\alpha,\beta} + 2C_{\alpha\beta}) \ dx^\alpha dx^\beta,$$  \hspace{1cm} (75)

where the FRW background $g_{3\alpha\beta}^{(3)}$ is,

$$g_{3\alpha\beta}^{(3)} dx^\alpha dx^\beta = db^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (76)

and also $\eta$ and $a$ in Eq. (34) stand for the conformal time and the scale factor respectively. Also $\alpha$, $\beta$, $\gamma$ and $\varphi$ in Eq. (34) are scalar type order variables of the perturbations, while $C_{\alpha\beta}$ is the tensor perturbation is trace-free and transverse. The evolution of the scalar perturbation variable $\Phi = \varphi_{,\delta\phi}$, which is related to the scalar gauge invariant variable $\delta\phi = -\frac{1}{2}\varphi_{,\delta\phi}$, for the case of the $R^2$- and string-corrected canonical scalar theory, is governed by the following differential equation,

$$\frac{(H + \ddot{F} + Q_c + 2\dot{F})^2}{a^3(\omega\dot{\phi}^2 + 3\frac{(\dot{F} + Q_c)^2}{2F + Q_c} + Q_c)\Phi} \frac{d}{dt} \left( a^3 (\omega\dot{\phi}^2 + 3\frac{(\dot{F} + Q_c)^2}{2F + Q_c} + Q_c) \Phi \right) = c_A^2 \frac{\Delta}{a^2 \Phi},$$  \hspace{1cm} (77)

where $\Delta$ is the Laplacian operator corresponding to the spatial section of the FRW metric, and $c_A$ is the sound wave speed defined in Eq. (34). As it is obvious from the above differential equation, ghost propagating modes can occur if the sound wave speed $c_A$ can develop negative values or values $c_A > 1$. We performed a full numerical analysis for the values of $c_A$ for all the models we studied in this paper, and for the values of the free parameters that yield compatibility of the theory with the observational constraints for inflation coming from the Planck data. Our analysis showed that the sound wave speed is remarkably equal to unity for all the models, as we already mentioned in the text too. Let us here examine one case, just for illustrative purposes. We shall consider the advanced exponential model of the next to previous subsection, and let us see how the sound wave speed $c_A$ behaves as we vary for example the parameter $\gamma_2$, so in Fig. 4 we plot the behavior of the sound wave speed $c_A$ as a function of the parameter $\gamma_2$, for $(\omega, \lambda_2, N, \gamma_2, V_2, m, \alpha) = (1, -1, 60, 1, -6, 10^{-3})$ and , and recall that these values of the free parameters the inflationary phenomenology of this specific model is rendered viable. As shown in Fig. 4, the sound wave speed is equal to unity, thus no ghost instabilities occur for this model. The same results hold true for the other models we studied in this paper, but we do not present these here for brevity.

We should note that the analytic study of the wave speed is not feasible for the model under study, due to the complexity of the resulting functional form of the wave speed $c_A$. Let us see why, the wave speed for the present model has the form,

$$c_A = \frac{\sqrt{32V(\phi)^3\xi(\phi)^3V''(\phi)^2(27\alpha^2-2\xi(\phi)^2)+192\alpha V(\phi)^3\xi(\phi)^4V''(\phi)^2+24V(\phi)^3\xi(\phi)^2(1-24\alpha V(\phi)^3)1^2V(\phi)^3V''(\phi)^2+72\alpha V(\phi)^3\xi(\phi)^2\xi^3(\phi)^2-288\alpha^2 V(\phi)^3\xi(\phi)^4V''(\phi)^2+3V(\phi)^3\xi(\phi)^2(1-24\alpha V(\phi)^3)1^2V(\phi)^3V''(\phi)^2+72\alpha V(\phi)^3\xi(\phi)^2\xi^3(\phi)^2}}{\sqrt{3}},$$  \hspace{1cm} (78)

where we took into account relation (37). As it is obvious from the expression (78), the analytic treatment of $c_A$ is impossible, since there is no restriction constraining directly $\xi'$ and the scalar potential and its derivative with respect to the scalar field, namely $V$ and $V'$. Thus only a numerical study is possible for the model under study in order
Ψ defined as, We shall use the notation of [3]. The other scalar perturbation associated with the theoretical framework at hand is Gauss-Bonnet model at hand, and also we need to discuss from which perturbed action these perturbations originate. However we need to further clarify this, since there is also another scalar component which is perturbed in the Einstein-flat FRW case, in their table I, last page of their article.

The sound wave speed

\[ c = \bar{\varepsilon} \text{s} \]

where \( \bar{\varepsilon} \) is defined as follows,

\[ \bar{\varepsilon} = \frac{E}{H + \frac{F + Q_b}{2F + Q_b}} \frac{a}{a^2} = \frac{\bar{\varepsilon}}{\bar{\varepsilon}} \] (81)

As it can be seen from both the wave equations (77) and (80), the quantity \( c^2 \) plays the role of wave speed of both the fluctuating field and of the perturbed metric. Since we assumed that the spacetime has a flat spatial part, the sound wave speed \( c^2 = \frac{2}{3} \) is identical with the sound wave speed \( c_A \). This feature can also be seen in Ref. [3] for the flat FRW case, in their table I, last page of their article.

To further see that the quantity \( c^2 \) acts as a wave speed for the evolution of the perturbed metric and for the field perturbations, we shall bring Eqs. (77) and (80) to the more familiar Mukhanov-Sasaki form. Following [3], we define \( \xi = c_A \zeta \), where \( \zeta \) is defined as follows,

\[ \zeta = \frac{a \phi}{(1 + \epsilon_5)H} \sqrt{\frac{E}{F}} \] (82)

where \( E \) is defined in Eq. (20) for the model at hand. Also upon defining \( \tilde{v} = \zeta \Phi \) and \( u = \frac{a}{\kappa H} \frac{1}{\bar{\varepsilon}} \Psi \), the wave equations (77) and (80) can be cast to the more familiar Mukhanov-Sasaki form, which is,

\[ \tilde{v}'' - \left( c_A^2 \Delta + \frac{z''}{\zeta} \right) \tilde{v} = 0 \] (83)
\[ u'' - \left( c_A^2 \Delta + \frac{(1/\bar{z})''}{1/\bar{z}} \right) u = 0, \quad (84) \]

and clearly \( c_A \) has the role of wave speed for both the fluctuating fluid or field, and the perturbed metric. Finally, let us note that the perturbed action for the fluctuating field takes the form,

\[ \delta^2 S = \frac{1}{2} \int a \bar{z}^2 \left( \Phi'^2 - c_A^2 - \frac{2}{\bar{a}^2} \dot{\Phi} \dot{\Phi} \right) dt d^3x, \quad (85) \]

or in terms of the conformal time, and the variable \( \bar{v} \), the above becomes,

\[ \delta^2 S = \frac{1}{2} \int \left( \bar{v}^2 - c_A^2 \bar{v} \bar{v} - \frac{z''}{z} \bar{v}^2 \right) d\eta d^3x. \quad (86) \]

Thus it is apparent that for the flat spacetime case, the quantity \( c_A \) plays the role of the wave speed, which is identical to the sound speed, only for the flat spacetime case. For more details on this issue, we refer the reader to Ref. \[3\].

V. CONCLUSIONS

In this paper we studied a string and \( R^2 \) corrected canonical minimally coupled scalar field theory and we demonstrated that the \( R^2 \) gravity corrections do also provide a phenomenologically viable theory. In this framework, by restoring the compatibility of the theory with the GW170817 event, we showed that the scalar potential and the Gauss-Bonnet scalar coupling functions are interrelated via a specific differential equation, hence they cannot be arbitrarily chosen, otherwise the continuity equation of the scalar field is not satisfied. Furthermore, it becomes apparent that some of the terms related to the string corrections, are inferior compared to the rest terms in the equations of motion, thus discarding them simplifies the equations greatly. Doing so does not imply that their presence is also discarded as string corrections, since these are also present in the kinetic term due to the time derivative of the scalar field, a relation which is derived from the realization that gravitational waves propagate through spacetime with the speed of light. Moreover, when the constant coupling \( \alpha \) of the \( R^2 \) correction term, satisfies \( \alpha < 10^{-3} \) in Planck units, or in other words, for mass scales greater that \( 10^{20} \) GeV in natural units, the contribution of the \( R^2 \) term becomes inferior compared to the string corrections, and thus, such terms may be omitted from the equations of motion as well. Although negligible, the presence of the \( R^2 \) term alters the dynamics of the theory, since now the time derivative of the term \( \partial_t R^2 \) is nonzero, thus the slow-roll index \( \epsilon_3 \) is non-zero. Also the term \( \partial_t R^2 \), for such choices of the parameter \( \alpha \), is close to, or equal to, unity in Planck Units. As a last comment, it is worth stating that a different choice of the Gauss-Bonnet scalar coupling function \( \xi(\phi) \) may need in principle, different \( \alpha \) values, in order for this framework to produce a viable phenomenology. Also, the same could be said about the case of having a non-minimally coupled scalar theory with \( R^2 \) and string corrections, of the form \( f(R, \phi) = h(\phi) R + \alpha R^2 \) gravity. We hope to address this issue in a future work. Also, let us comment that the newly introduced branch of astronomy, namely the gravitational wave astronomy, can shed light on the issue of gravitational wave speed, both for astrophysical and primordial gravitational waves, and several interesting works have already appeared focusing on this issue \[52\ [54\].

Finally, one issue that we briefly discussed is the possible occurrence of ghost degrees of freedom in the context of the \( R^2 \) corrected Einstein-Gauss-Bonnet gravity. As we showed, at a cosmological level, the perturbations of the Jordan frame theory is free of ghosts, since quantitatively the speed of sound of the scalar perturbations is equal to unity.

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