Building trust for continuous variable quantum states

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We first introduce heterodyne quantum state tomography, a reliable method for continuous variable quantum state certification which directly yields the elements of the density matrix of the state considered and analytical confidence intervals, using heterodyne detection. This method neither needs mathematical reconstruction of the data, nor discrete binning of the sample space, and uses a single Gaussian measurement setting.

Beyond quantum state tomography and without its identical copies assumption, we also derive a general protocol for verifying continuous variable pure quantum states with Gaussian measurements against fully malicious adversaries. In particular, we make use of a de Finetti reduction for infinite-dimensional systems \cite{1}. As an application, we consider verified universal continuous variable quantum computing, with a computational power restricted to Gaussian operations and an untrusted non-Gaussian states source.

These results are obtained using a new analytical estimator for the expected value of any operator with samples from heterodyne detection of the state. Many subensembles must be performed and combined to reconstruct the density matrix of the state. The data do not yield the state directly, but rather indirectly through data analysis. Quantum state tomography assumes an independent and identically distributed (i.i.d.) behaviour for the device, i.e. that the density matrix of the output state considered is the same at each round. This assumption may be relaxed with a tradeoff in the efficiency of the protocol \cite{7}. In the following, the task of checking the correct functioning of a quantum device is denoted certification when i.i.d. behaviour is assumed, and verification without the i.i.d. assumption.

When the quantum device is untrusted, possibly controlled by a fully malicious adversary, e.g. in the context of delegated quantum computing, the task of quantum verification is to ensure that either the device behaved properly, or the computation aborts with high probability. While delegated computing is a natural platform for utilising the emerging NISQ devices, one can provide a physical interpretation to this adversarial setting by emphasising the common feature for all of these approaches is to utilise some basic obfuscation scheme that allows to reduce the problem of dealing with a fully general noise model, or a fully general adversarial deviation of the device, to a simple error detection scheme \cite{9}.

In this work, we consider the setting of quantum information with continuous variables \cite{10}, in which quantum states live in an infinite-dimensional Hilbert space. Continuous variable quantum computing is a powerful alter-
native to discrete variable quantum computing. Firstly, it is compatible with standard network optics technology, where more efficient measurements are available. Secondly, it allows for unprecedented scaling in entanglement, with entangled states of up to tens of thousands of subsystems reported [11], generated deterministically. A continuous variable quantum process or state can be described by a quasi-probability distribution in phase space, often the Wigner function [12], but also the Husimi Q function [13] or the Glauber–Sudarshan P function. This allows for a simple and experimentally relevant classification of quantum states: those with a Gaussian quasiprobability distributions are called Gaussian states, and the others non-Gaussian states. By extension, operations mapping Gaussian states to Gaussian states are also called Gaussian. These Gaussian operations and states are the ones implementable with linear optics and quadratic nonlinearities [14], and are hence relatively easy to construct experimentally.

Certification of quantum states, i.e. checking that the output state of a quantum device is close to a target state, may be done with linear optics for continuous variable states using optical homodyne tomography [15]. This method allows to reconstruct the Wigner function of a generic state using only Gaussian measurements, namely homodyne detection. Because of the continuous character of its outcomes, one must proceed to a discrete binning of the sample space, in order to build probability histograms. Then, the state representation in phase space is determined by a mathematical reconstruction. For multimode Gaussian states, more efficient certification methods have been derived [16] with Gaussian measurements. These methods involve the computation of a fidelity witness, i.e. a lower bound on the fidelity, from the measured samples. The cubic phase state certification protocol of [17] also introduces a fidelity witness, and is an example of certification of non-Gaussian state with Gaussian measurements, which assumes an i.i.d. state preparation. The verification protocol for Gaussian continuous variable weighted hypergraph states of [18] removes this assumption.

In this work we address two main issues. Firstly, existing continuous variable homodyne tomography is not reliable in the sense of [17], because errors coming from the reconstruction procedure are indistinguishable from errors coming from the data. Secondly, to the best of our knowledge there is no Gaussian verification protocol for non-Gaussian states without i.i.d. assumption. We thus introduce a general receive-and-measure protocol for building trust for continuous variable quantum states, using solely Gaussian measurements, namely heterodyne detection (Fig. 1). This protocol allows to perform reliable continuous variable quantum state tomography based on heterodyne detection, which we call heterodyne tomography. In particular, this tomography only needs a single fixed measurement setting, compared to homodyne tomography. This protocol also provides a mean for certifying continuous variable quantum states, under the i.i.d. assumption. Finally, the same protocol also allows to verify continuous variable states, without the i.i.d. assumption. For these three applications, the measurements performed are the same. It is only the set of subsystems to be measured and the classical post-processing performed that differ from one application to another.

We detail the structure of the protocol in Section II, and we give in Section III our main technical result: an estimator for the expected value of an operator acting on a state with bounded support over the Fock basis. The estimate is expressed as an expected value under a Gaussian measurement, heterodyne detection [19]. Similar estimates have been obtained in the context of imperfect heterodyne detection [20, 21]. We go beyond these works in two different respects: using this result, we introduce heterodyne tomography in Section IV and compute analytical bounds on its efficiency. We then derive in Section V a continuous variable quantum state receive-and-measure certification protocol with Gaussian measurements against i.i.d. adversary, and we generalise it to a verification protocol against fully malicious adversary in Section VI using a de Finetti reduction for infinite-dimensional systems [1]. As an application, we consider universal continuous variable quantum computing with Gaussian resources and verified non-Gaussian states.

II. DESCRIPTION OF THE PROTOCOL

As mentioned above, continuous variable quantum states live in an infinite-dimensional Hilbert space H, spanned by the Fock basis \{n\}_{n \in \mathbb{N}} and are equivalently represented in phase space by their Husimi Q function [13], a smoother relative of the Wigner function. Given a single-
mode state $\sigma$, its $Q$ function is defined as:

$$Q_\sigma(\alpha) = \frac{1}{\pi} \langle \alpha | \sigma | \alpha \rangle,$$

for all $\alpha \in \mathbb{C}$, where $|\alpha\rangle = e^{-|\alpha|^2} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ is a coherent state. In particular,

$$Q_\sigma(\alpha) = \frac{1}{\pi} \text{Tr} (|\alpha\rangle \langle \alpha| \sigma) = \text{Tr} (\Pi_\alpha \sigma),$$

where $\{\Pi_\alpha\}_{\alpha \in \mathbb{C}} = \{\frac{1}{\pi} |\alpha\rangle \langle \alpha|\}_{\alpha \in \mathbb{C}}$ is the Positive Operator Valued Measure for heterodyne detection [19].

This detection, also called double homodyne or eight-port homodyne, consists in splitting the measured state with a beamsplitter and measuring both ends with homodyne detection (Fig. 1). This corresponds to a joint noisy measurement of quadratures $q$ and $p$. This is a Gaussian measurement, which yields two real outcomes, corresponding to the real and imaginary parts of $\alpha$. The $Q$ function of a single-mode state thus is a probability density function over $\mathbb{C}$, and measuring such a state with heterodyne detection amounts to sampling from its $Q$ function.

Using this detection, one may acquire knowledge about an unknown continuous variable quantum state. More precisely, we define the following receive-and-measure protocol, depicted in Fig. 2: given a quantum state $\sigma^N$ over $N$ subsystems, measure with heterodyne detection some of the subsystems, thus obtaining the samples $\alpha$. Then, post-process these samples to retrieve information about the remaining subsystems. The way the subsystems to be measured are chosen and the post-processing performed depend on the application considered.

We show in the following sections how this protocol may be used to perform reliable continuous variable quantum state tomography, certification, and verification, and we detail the corresponding choice of subsystems and the classical post-processing for each task. In the next section, we first introduce notations and technical results which will be used in the rest of the paper.
bounded support. Then, also where the function \( f \) is defined in Eq. (4) and the constant \( K \) in Eq. (3).

The proof of Corollary 1 comes directly from Theorem 1 as detailed in Appendix C. This result provides an estimator for the fidelity between any continuous variable pure state \( \tau \) and any continuous variable state \( \sigma \) with bounded support over the Fock basis. This estimator is the expected value of a bounded function \( f \) over samples drawn from the probability density corresponding to a Gaussian measurement of \( \sigma \), namely heterodyne detection. The right hand side of Eq. (7) is an energy bound, which may be refined depending on the expression of \( \tau \). In particular the second bound is independent of the target state \( \tau \).

Given these results, one may choose a target pure state \( \tau \), and measure with heterodyne detection various copies of the output state \( \sigma \) of a quantum device with bounded support over the Fock basis. Then, using the samples obtained, one may estimate the expected value of \( f \), thus obtaining an estimate of the fidelity between the states \( \tau \) and \( \sigma \). In the next section, we develop this idea by introducing a reliable method for performing continuous variable quantum state tomography using heterodyne detection.

IV. HETERODYNE QUANTUM STATE TOMOGRAPHY

Quantum state tomography aims at characterising the output state of a quantum device by measuring successive output states and making use of the samples obtained to reconstruct the corresponding density matrix. Current methods for quantum state tomography in continuous variable use outcomes from many different homodyne detection settings, corresponding to measurements of quadratures rotated in phase space, in order to reconstruct the phase information on the coefficients of the density matrix of the measured state. This is done first by introducing a binning of the sample space, then by building probability histograms from the data, and finally by using data analysis to recover an approximation of the phase space representation of the state considered. These methods make two assumptions: firstly the measured states are all equal (i.i.d.), and secondly they have a bounded support over the Fock basis.

We present a complementary method for continuous variable quantum state tomography with Gaussian measurements which has the advantage of providing analytical confidence intervals, without the need for a reconstruction of the phase-space distribution. Our method directly provides approximates of the elements of the state density matrix, phase included. As such, neither mathematical reconstruction of the phase, nor binning of the sample space is needed, since the samples are used only to compute expected values of bounded functions. Moreover, only a single Gaussian measurement setting is needed, namely heterodyne detection. This detection, also called double homodyne detection, consists in splitting the measured state with a beamsplitter and measuring both ends with homodyne detection. Heterodyne detection amounts to measuring both quadratures at the same time, and is effectively sampling from the Q function of the measured state (see [19] for further details on this detection).

Let \( \alpha = \alpha_1, \ldots, \alpha_N \in \mathbb{C} \), for \( N \geq 1 \). For \( \epsilon > 0 \) and \( k, l \in \mathbb{N} \), we define

\[
\hat{F}_{kl}(\alpha) = \frac{1}{N} \sum_{i=1}^{N} f_{i[k]}(\alpha_i, \epsilon, \epsilon) K_{i[k]},
\]

where the function \( f \) is defined in Eq. (4) and the constant \( K \) in Eq. (5). The next result shows that this estimator, when computed using samples from heterodyne detection of \( N \) copies of a quantum state, approximates the matrix elements of this state with high probability. We use the notations of Theorem 1.

Theorem 2. Let \( \epsilon, \epsilon' > 0 \), let \( k, l \geq 0 \), let \( N \geq 1 \), and let \( \alpha = \alpha_1, \ldots, \alpha_N \) be samples obtained by measuring with heterodyne detection \( N \) copies of a state \( \sigma = \sum_{m,n=0}^{\infty} \sigma_{mn} |m\rangle \langle n| \) with bounded support, for \( E \in \mathbb{N} \). Then

\[
|\sigma_{kl} - \hat{F}_{kl}(\alpha)| \leq \epsilon + \epsilon',
\]

with probability greater than

\[
1 - 4e^{-\frac{\epsilon^2}{2C_{kl}^2}},
\]

where the estimate \( \hat{F}_{kl}(\alpha) \) is defined in Eq. (4), and where \( C_{kl} \) is a constant depending on \( k, l \).

The full expression of the constant \( C_{kl} \) is given in Eq. (C22), in Appendix C along with the proof of the result. This proof combines Theorem 1 with Hoeffding inequality, which quantifies the speed of convergence of the sample mean towards the expected value of a bounded random variable. Theorem 2 quantifies the quality of the estimate \( \hat{F}_{kl}(\alpha) \) of a single density matrix element \( \sigma_{kl} \).

Building on this result, we obtain the efficiency of tomography with heterodyne detection, which we call
heterodyne tomography, which corresponds to the reconstruction of the full density matrix:

**Theorem 3** (Efficiency of heterodyne tomography). Let \( \epsilon, \epsilon' > 0 \), let \( N \geq 1 \), and let \( \alpha = \alpha_1, \ldots, \alpha_N \) be samples obtained by measuring with heterodyne detection \( N \) copies of a state \( \sigma = \sum_{m,n=0}^E \sigma_{mn} |m\rangle \langle n| \) with bounded support, for \( E \in \mathbb{N} \). Then

\[
\left| \sigma_{kl} - \tilde{F}^\epsilon_{kl}(\alpha) \right| \leq \epsilon + \epsilon',
\]

for all \( 0 \leq k, l \leq E \), with probability greater than

\[
1 - \sum_{0 \leq k \leq l \leq E} 4e^{-N \sum_{k,l=0}^{E} \epsilon_{kl}^2} + \epsilon,
\]

where the estimate \( \tilde{F}^\epsilon_{kl}(\alpha) \) is defined in Eq. [3], and where \( C_{kl} \) is a constant depending on \( k, l \).

The proof of this theorem directly comes from applying Theorem 2 for all values of \( k, l \) between 0 and \( E \), together with the union bound. In light of this result, the principle for heterodyne tomography is straightforward and as follows: \( N \) identical copies \( \sigma \otimes^N \) of the output quantum state of some physical experiment are measured with heterodyne detection, yielding the values \( \alpha = \alpha_1, \ldots, \alpha_N \). These values are used to compute the estimates \( \tilde{F}^\epsilon_{kl}(\alpha) \), defined in Eq. [3], for all \( k, l \) in the range of energy of the experiment. Then, Theorem 3 directly provides confidence intervals for all these estimates of \( \sigma_{kl} \), the matrix elements of the density operator \( \sigma \), without the need for a binning of the sample space or any additional data reconstruction. Furthermore, only one measurement setting is needed. This contrasts with homodyne tomography where many measurements settings are necessary. For a desired precision \( \epsilon \) and a failure probability \( \delta \), the number of samples needed scales as \( N = \text{poly}(1/\epsilon, \log(1/\delta)) \).

Both homodyne quantum state tomography and heterodyne quantum state tomography assume a bounded support over the Fock basis for the density matrix of the output state considered, i.e. that all matrix elements are equal to zero beyond a certain value, and that the output quantum states are i.i.d., i.e. all measured output states are independent and identical. While these assumptions are natural when looking at the output of a physical experiment, corresponding to a noisy partially trusted quantum device with bounded energy, they may be questionable in the context of untrusted devices. We show in the next sections how to remove these assumptions: in Section V we drop the bounded support assumption by deriving a certification protocol for continuous variable quantum states of an i.i.d. device with heterodyne detection, and in Section VI we drop both assumptions, by deriving a general verification protocol for continuous variable quantum states against an adversary who can potentially be fully malicious.

V. QUANTUM STATE CERTIFICATION WITH GAUSSIAN MEASUREMENTS

In this section, we consider the certification of the output of an i.i.d. quantum device, i.e. which output state is the same at each round. However, we do not assume that the output states of the device have bounded support over the Fock basis. This is instead ensured probabilistically using the samples from heterodyne detection.

Our continuous variable quantum state certification protocol then is the following: let \( \tau \) be a target pure state, of which one wants to certify \( m \) copies. The value \( E \) is a free parameter of the protocol. One instructs the i.i.d. device to prepare \( N + m \) copies of \( \tau \). One keeps \( m \) copies \( \sigma \otimes^m \), and measures the \( N \) others with heterodyne detection, obtaining the samples \( \alpha = \alpha_1, \ldots, \alpha_N \). One records the number \( R \) of samples such that \( |\alpha_i|^2 > E \). We refer to this step as support estimation. For \( \epsilon > 0 \), one also computes with the same samples the estimate

\[
\tilde{F}^\epsilon_{\tau, m}(\alpha) = \left[ \frac{1}{N} \sum_{i=1}^{N} f_i \left( \frac{\epsilon}{(m+1)R} \right) \right]^m,
\]

where the function \( f \) is defined in Eq. [1] and the constant \( K \) in Eq. [3]. The next result quantifies how close this estimate is from the fidelity between the remaining \( m \) copies of the output state \( \sigma \otimes^m \) of the tested device and \( m \) copies of the target state \( \sigma \otimes^m \).

**Theorem 4** (Gaussian certification of continuous variable quantum states). Let \( \epsilon, \epsilon' > 0 \), let \( N \geq 1 \), and let \( \alpha = \alpha_1, \ldots, \alpha_N \) be samples obtained by measuring with heterodyne detection \( N \) copies of a state \( \sigma \). Let \( E \in \mathbb{N} \), and let \( R \) be the number of samples such that \( |\alpha_i|^2 > E \). Let also \( \tau \) be a pure state. Then for all \( m \in \mathbb{N}^+ \),

\[
\left| F(\tau \otimes^m, \sigma \otimes^m) - \tilde{F}^\epsilon_{\tau, m}(\alpha) \right| \leq \epsilon + \epsilon',
\]

with probability greater than

\[
1 - \left( P_{\text{Support}}^{\text{iid}} + P_{\text{Hoeffding}}^{\text{iid}} \right),
\]

where

\[
P_{\text{Support}}^{\text{iid}} = \frac{1}{N} \sqrt{R + \frac{(n+1)^2}{N}},
\]

\[
P_{\text{Hoeffding}}^{\text{iid}} = 2e^{-N \sum_{k,l=0}^{E} \epsilon_{kl}^2},
\]

where the estimate \( \tilde{F}^\epsilon_{\tau, m}(\alpha) \) is defined in Eq. [13], and where \( C_{\text{Hoeffding}} \) is a constant depending on \( \tau \).

The proof goes along the same lines as the one of Theorem 2 with the addition of the support estimation step, which is detailed in Appendix D 1. The full expression for the constant \( C_{\text{Hoeffding}} \) is given in Eq. (D12) in Appendix D 2 along with the proof of the theorem. This theorem being valid for all continuous variable target pure states \( \tau \), the
VI. QUANTUM STATE VERIFICATION WITH GAUSSIAN MEASUREMENTS

We now consider a cryptographic setting, where a verifier delegates the preparation of a continuous variable quantum state to a potentially malicious party, called the prover. One could consider the verifier as the experimentalist in the lab and the prover the noisy device, where we aim not to make any assumptions about its correct functionality or noise model. Given the absence of any direct error correction mechanism that permits a fault tolerant run of the device, the aim of verification is to ensure that a wrong outcome is not being accepted. In the context of state verification, this amounts to making sure that the output state of the tested device is close enough to an ideal target state.

The prover is not supposed to have i.i.d. behaviour. In particular, when asked for various copies of the same state, he may actually send a large state entangled over all subsystems, possibly also entangled with a quantum memory on his side. In that case, the certification protocol derived in the previous section is not reliable. With usual tomography measurements, the number of samples needed for a given precision of the fidelity estimate scales exponentially in the number of copies to verify. This is an essential limitation of quantum tomography techniques, because they check all possible correlations between the different subsystems.

However we prove that, because of the symmetry of the protocol, the verifier can assume that the prover is sending permutation invariant states, i.e. states that are invariant under any permutation of their subsystems. After a specific support estimation step, reduced states which allows restricting probabilistically to an almost-i.i.d. verifier wants to verify states obtained by measuring K subsystems at random with heterodyne detection and let \( \sigma^N \) be the remaining state after the measurement. Let \( E \) in \( \mathbb{N} \), and let \( R \) be the number of samples such that \( |\beta|^2 > E \). Let also \( S \geq m \), and let \( \sigma^m \) be the state remaining after discarding \( S \) subsystems of \( \sigma^N \) at random, and measuring \( N - S - m \) other subsystems at random with heterodyne detection, yielding the samples \( \alpha = \alpha_1, \ldots, \alpha_{N-S-m} \). Let \( \epsilon, \epsilon' > 0 \) and let \( \epsilon'' = \frac{\pi}{\sqrt{2}} \frac{m(S + m - 1)}{N - S} \). Let \( \tau \) be a target pure state. Then,

\[
| F(\tau^m, \sigma^m) - \hat{F}_r(\alpha) | \leq \epsilon + \epsilon' + \epsilon'' + \frac{\pi}{2} P_{\text{deFinetti}},
\]

with probability greater than

\[
1 - (P_{\text{Support}} + P_{\text{deFinetti}} + P_{\text{Hoeffding}}),
\]

where

\[
P_{\text{Support}} = 8K^{3/2}e^{-\frac{4}{9}(\frac{N}{2S} - \frac{2E}{3})^2},
\]

\[
P_{\text{deFinetti}} = S E^2 / 2 e^{-\frac{S(S + 1)}{N}},
\]

\[
P_{\text{Hoeffding}} = 2 \left( \frac{N - S}{S} \right) \times e^{-\frac{2(N - 2S)}{(m + 1) + 2E}} \left( \frac{1 + E'}{1 + \epsilon''} - \frac{2S(m + 2) + E}{N - S + m} \right)^2,
\]

where the estimate \( \hat{F}_r(\alpha) \) is defined in Eq. (18), and where \( C_\tau \) is a constant depending on \( \tau \).

The full expression of \( C_\tau \) is given in Appendix E along with the proof of the theorem. This proof combines three main ingredients: a support estimation step for permutation-invariant states using samples from heterodyne detection, the de Finetti reduction from [1], and a refined version of Hoeffding inequality for superpositions of almost-i.i.d. states under a product measurement. The three terms appearing in the above expression of the probability correspond to these three ingredients, respectively.

For specific choices of the free parameters of the protocol, detailed in Appendix E, the number of samples needed for a precision of \( \epsilon \) and a failure probability of
δ scale as $N = \text{poly}(m, 1/\epsilon, 1/\delta)$. In particular, the efficiency of the protocol may be greatly refined by taking into account the expression of $\tau$ in the Fock basis.

This verification protocol let the verifier gain confidence about the precision of the estimate of the fidelity in Eq. (18). If the value of the the estimate is close enough to 1, the verifier may then decide to use the state to run a computation. Indeed, statements on the fidelity of a state allow inferring the correctness of any trusted computation done afterwards using this state. Let $\eta > 0$, and let $\mathcal{O}$ be the observable corresponding to the result of the trusted computation performed on $\sigma^m$, the reduced state over $m$ subsystems instead of $\tau^{\otimes m}$, $m$ copies of the target state $\tau$—in other words, $\mathcal{O}$ encodes the resources which the verifier can perform perfectly (ancillary states, evolution and measurements), the imperfections being encoded in $\sigma$—; then $F(\tau^{\otimes m}, \sigma^m) \geq 1 - \eta$ implies the following bound on the total variation distance between the probability densities of the computation output of the actual and the target computations (see Appendix A):

$$\|P_{\tau^{\otimes m}} - P_{\sigma^m}\|_{\text{tvd}} \leq D(\tau^{\otimes m}, \sigma^m) \leq \sqrt{\eta}. \quad (24)$$

What this means is the distribution of outcomes for the state $\sigma^m$ sent by the prover is almost indistinguishable from the distribution of outcomes for $m$ copies of the ideal state $\tau$, when the fidelity is close enough to one.

As an application, we consider verified universal continuous variable quantum computing, with a computational power restricted to Gaussian operations and an untrusted non-Gaussian states source: universal continuous variable quantum computing can be realised with Gaussian computational power, with the addition of non-Gaussian states at the beginning of the computation [10, 23]. A polynomial number of non-Gaussian state injections may be used to engineer a polynomial number of non-Gaussian gates within a Gaussian computation. A verifier with Gaussian computational power may thus achieve verified universal continuous variable quantum computing, by verifying the preparation of various copies of a non-Gaussian state delegated to a prover with the previous protocol, and injecting these copies into its Gaussian computation.

This approach for verifying continuous variable quantum computing is experimentally relevant, as Gaussian quantum processes are the easiest to implement with current technology.

vii. SUMMARY OF RESULTS AND CONCLUSION

Determining an unknown quantum state is difficult especially in continuous variable, where it is described by possibly infinitely many complex parameters. Existing methods like homodyne quantum state tomography require many different measurement settings, and heavy classical post-processing. For that purpose, we have introduced heterodyne quantum state tomography, which uses a single Gaussian measurement setting and allows to retrieve the density matrix of an unknown state without the need for data reconstruction nor binning of the sample space. We expect heterodyne tomography to be less efficient than homodyne tomography, however heterodyne tomography is more reliable, in the sense that it provides a better management of errors and analytical confidence intervals. For data reconstruction methods such as Maximum Likelihood, errors from the reconstruction procedure are usually indistinguishable from errors coming from the quantum device measured. For that reason, such methods do not extend well to the task of verification, unlike our method.

Building on these new tomography techniques, and with the addition of cryptographic techniques such as the de Finetti theorem, we derived a protocol for verifying various copies of a continuous variable quantum state, without i.i.d. assumption, with Gaussian measurements. This protocol is robust, as it directly gives a confidence interval on an estimate of the fidelity between the tested state and the target state. For verifying $m$ copies of a target state, with a desired precision $\epsilon$, and a failure probability $\delta$, the number of samples needed scales as $\text{poly}(m, 1/\epsilon, 1/\delta)$. As an application, we considered verified universal continuous variable quantum computing, with a verifier restricted to Gaussian computational power. This verification is complementary to the approach of [18], where a measurement-only verifier performs universal continuous variable quantum computing by delegating the preparation of Gaussian cluster states to a prover, and has to perform non-Gaussian measurements.

Our protocol may be tailored to different uses and assumptions, from tomography to verification, simply by changing the classical post-processing. We expect this protocol to be useful for certifying continuous variable quantum devices in the NISQ [5] era and onward. In particular, an interesting perspective would be fine-tuning the various parameters of the protocol for specific target states in order to optimise its efficiency, thus reducing the number of samples needed for a given confidence interval.

Another interesting prospect would be to extend our main technical result, Theorem 1, which applies to operators, to quantum maps. Also, in the case where the operator is the density matrix of a pure state, our result provide an estimate for the fidelity, and it would be interesting to extend this to target mixed states.

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The fidelity between two states $\tau, \sigma$ is defined as [24]

$$F(\tau, \sigma) = \text{Tr}\left(\sqrt{\sqrt{\tau} \sigma \sqrt{\tau}}\right). \quad (A1)$$

When at least one of the two states is a pure state, this expression reduces to

$$F(\tau, \sigma) = \text{Tr}(\tau \sigma). \quad (A2)$$

We write the Schatten 1-norm of a bounded operator $T$ as

$$\|T\|_1 = \text{Tr}\left(\sqrt{T^*T}\right) = \text{Tr}(|T|). \quad (A3)$$

The fidelity is related to the trace distance $D(\tau, \sigma) = \frac{1}{2}\|\tau - \sigma\|_1 = \frac{1}{2} \text{Tr}(|\tau - \sigma|)$ by [25]

$$1 - \sqrt{F(\tau, \sigma)} \leq D(\tau, \sigma) \leq \sqrt{1 - F(\tau, \sigma)}. \quad (A4)$$

The trace distance verifies

$$D(\tau, \sigma) = \max_O \|P^O_\tau - P^O_\sigma\|_{tvd}, \quad (A5)$$

**Appendix A: Fidelity and trace distance**

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$$D(\tau, \sigma) = \max_O \|P^O_\tau - P^O_\sigma\|_{tvd}. \quad (A5)$$
where $P^\tau$ (resp. $P^\sigma$) is the probability density associated to measuring the observable $O$ for the state $\tau$ (resp. $\sigma$), and where the maximum of the total variation distance is taken over all observables.

Appendix B: Proof of the technical results

1. Proof of Theorem 1

We use the notations of the Theorem.

The function $f_A$ defined in Eq. (4) is a bounded approximation of the Glauber-Sudarshan function $P_A$ of the operator $A$. This approximation is parametrised by a precision $\eta$, and a cutoff value $E$. We have [13]

$$\text{Tr}(A \sigma) = \int Q_\sigma(\alpha) P_A(\alpha) d^2\alpha. \quad (B1)$$

Given that

$$\mathbb{E}_{\alpha \sim Q_\sigma} [f_A(\alpha, \eta)] = \int Q_\sigma(\alpha) f_A(\alpha, \eta) d^2\alpha, \quad (B2)$$

we would expect that $\mathbb{E}_{\alpha \sim Q_\sigma} [f_A(\alpha, \eta)]$ is an approximation of $\text{Tr}(A \sigma)$ parametrised by $\eta$ and $E$. Theorem 1 makes this statement more precise, and we prove it in the following.

With Eq. (4) we obtain

$$\left| \text{Tr} (A \sigma) - \mathbb{E}_{\alpha \sim Q_\sigma} [f_A(\alpha, \eta)] \right| = \left| \sum_{k,l=0}^{E} A_{lk} \text{Tr} (|l\rangle \langle k| \sigma) - \mathbb{E}_{\alpha \sim Q_\sigma} [f_{|l\rangle \langle k|}(\alpha, \eta)] \right| \quad (B3)$$

where we used in the second line the fact that $\sigma$ has a bounded support over the Fock basis. This shows that it is sufficient to prove the Theorem for $A = |l\rangle \langle k|$, for all $k, l$ from 0 to $E$, which we do hereafter.

Let us fix $k, l$ in $0, \ldots, E$. By Eqs. (3, 4) we have, for all $z \in \mathbb{C}$,

$$f_{|l\rangle \langle k|}(z, \eta) = \frac{1}{\eta} e^{\left(1-\frac{1}{\eta}\right)z^*z} \mathcal{L}_{l,k} \left( \frac{z}{\sqrt{\eta}} \right) \quad (B4)$$

Moreover, for all $\alpha \in \mathbb{C}$,

$$Q_\sigma(\alpha) = \frac{1}{\pi} \langle \alpha | \sigma | \alpha \rangle \quad (B5)$$
Combining these expressions we obtain
\[
\mathbb{E}_{\alpha \sim Q_{\sigma}}[f_{[j]}|k_{l}] (\alpha, \eta)] = \int Q_{\sigma}(\alpha) f_{[j]}|k_{l}(\alpha, \eta) d^2 \alpha
\]
\[
= \frac{1}{\pi \eta} \sum_{m,n=0}^{E} \sigma_{mn} \frac{\sqrt{k! \sqrt{l!}}}{\sqrt{m! \sqrt{n!}}} \sum_{p=0}^{\min(k,l)} (-1)^p \left( \frac{1}{\eta} \right)^{k+l-p} \int_{0}^{+\infty} \alpha^{k+n-p} \alpha^{*(l+m-p)} e^{-\alpha^2/2} d^2 \alpha.
\]
Setting \(\alpha = r e^{i \theta}\), the integral on the right hand side of this last equation may be computed as
\[
\int_{0}^{+\infty} \alpha^{k+n-p} \alpha^{*(l+m-p)} e^{-\alpha^2/2} d^2 \alpha = \int_{0}^{2\pi} r^{k+l+m+n-2p+1} e^{-r^2/2} dr \int_{0}^{+\infty} e^{i(k+n-l-m)\theta} d\theta
\]
\[
= \begin{cases} 
\pi \left( k+l+m+n \right) ! \left( \frac{k+l+m+n}{\pi} \right) ! \eta^{k+l+m+n} & \text{for } k - l = m - n, \\
0 & \text{for } k - l \neq m - n.
\end{cases}
\]
Hence
\[
\mathbb{E}_{\alpha \sim Q_{\sigma}}[f_{[j]}|k_{l}] (\alpha, \eta)] = \sum_{m,n=0}^{E} \sigma_{mn} \frac{\sqrt{k! \sqrt{l!}}}{\sqrt{m! \sqrt{n!}}} \sum_{p=0}^{\min(k,l)} (-1)^p \left( \frac{1}{\eta} \right)^{k+l-p} \frac{(k+l+m+n)!}{(k+l+m+n-2p+1)!} \frac{1}{\eta^{m+n-k-l}}
\]
\[
= \sum_{m,n=0}^{E} \sigma_{mn} \eta^{m+n-k-l} \frac{(k+l+m+n)!}{(k+l+m+n-2p+1)!} \frac{1}{\eta^{m+n-k-l}} \sum_{p=0}^{\min(k,l)} (-1)^p \left( \frac{k}{p} \right) \left( \frac{l}{q} \right)!
\]
\[\text{for } q \geq k + l, \]
\[0 \text{ for } q < k + l.
\]
Now for \(k \leq l\) we have, for all \(q \in \mathbb{N}\) (see, e.g., result 7.1 of [26]),
\[
\sum_{p=0}^{k} (-1)^p \left( \frac{k}{p} \right) \left( \frac{l}{q} \right)! = \begin{cases} 
\frac{(k+1)!}{(q+1)!} & \text{for } q \geq k + l, \\
0 & \text{for } q < k + l.
\end{cases}
\]
When \(k \leq l\), Eq. (B8) thus yields
\[
\mathbb{E}_{\alpha \sim Q_{\sigma}}[f_{[j]}|k_{l}] (\alpha, \eta)] = \sum_{m,n=0}^{E} \sigma_{mn} \eta^{m+n-k-l} \frac{(k+l+m+n)!}{(k+l+m+n-2p+1)!} \frac{1}{\eta^{m+n-k-l}} \sum_{p=0}^{\min(k,l)} (-1)^p \left( \frac{k}{p} \right) \left( \frac{l}{q} \right)!
\]
\[
= \sum_{m,n=0}^{E} \sigma_{mn} \eta^{m+n-k-l} \frac{(k+l+m+n)!}{(k+l+m+n-2p+1)!} \frac{1}{\eta^{m+n-k-l}} \sum_{p=0}^{\min(k,l)} (-1)^p \left( \frac{k}{p} \right) \left( \frac{l}{q} \right)!
\]
\[\text{for } q \geq k + l, \]
\[0 \text{ for } q < k + l.
\]
where we used that within the summation \(m - n = k - l\). This formula is also valid for \(l \leq k\), with the same reasoning.
We finally obtain, for any \(k, l \in 0, \ldots, E\)
\[
\mathbb{E}_{\alpha \sim Q_{\sigma}}[f_{[j]}|k_{l}] (\alpha, \eta)] = \sum_{m \geq k, n \geq l}^{E} \sigma_{mn} \eta^{m+n-k-l} \frac{\sqrt{m \choose k} \sqrt{n \choose l}}{\sqrt{(m-k)!(n-l)!}}
\]
\[\text{for } m - n = k - l.
\]
\[
= \sigma_{kl} + \sum_{m \geq k, n \geq l}^{E} \sigma_{mn} \eta^{m+n-k-l} \frac{\sqrt{m \choose k} \sqrt{n \choose l}}{\sqrt{(m-k)!(n-l)!}}
\]
\[\text{for } m - n = k - l.
\]
Hence,
\[
\left| \text{Tr}(|l\rangle \langle k| \sigma) - \mathbb{E}_{\alpha \sim Q_{\alpha}}[f_{|l\rangle \langle k|}(\alpha, \eta)] \right| = \left| \sigma_{kl} - \mathbb{E}_{\alpha \sim Q_{\alpha}}[f_{|l\rangle \langle k|}(\alpha, \eta)] \right|
\]
\[
\leq \sum_{m>k,n>l}^{E} \sum_{m-n=k-l}^{E} |\sigma_{mn}\eta|^{m+n-k-l} \sqrt{\binom{m}{k}\binom{n}{l}}
\]
\[
\leq \sum_{s=1}^{E-\max(k,l)} \sum_{s=1}^{E-\max(k,l)} |\sigma_{s+k,s+l}| \eta^{s} \sqrt{\binom{s+k}{k}\binom{s+l}{l}}
\]
\[
\leq \sum_{s=1}^{E-\max(k,l)} \eta^{s} \sqrt{\binom{s+k}{k}\binom{s+l}{l}} \sqrt{\sigma_{s+k,s+k} \sigma_{s+l,s+l}}.
\]

where we set \( s = m - k = n - l = \frac{m+n-k-l}{2} \) in the third line, and where we used \( |\sigma_{s+k,s+l}| \leq \sqrt{\sigma_{s+k,s+k} \sigma_{s+l,s+l}} \) in the last line, since \( \sigma \) is a hermitian matrix.

For all \( k, l \in 0, \ldots, E \) and for all \( s \in 2, \ldots, E - \max(k,l) \), we have
\[
\frac{\sqrt{s+k}}{s} \leq \frac{E}{2}.
\]

This in turn implies that for all \( s \in 2, \ldots, E - \max(k,l) \)
\[
\eta^{s} \sqrt{\binom{s+k}{k}\binom{s+l}{l}} = \eta^{s} \sqrt{\binom{s+k}{k}\binom{s+l}{l}} \eta^{s-1} \sqrt{\binom{s-1+k}{k}\binom{s-1+l}{l}}
\]
\[
\leq \frac{\eta E}{2} \eta^{s-1} \sqrt{\binom{s-1+k}{k}\binom{s-1+l}{l}}
\]
\[
\leq \eta^{s-1} \sqrt{\binom{s-1+k}{k}\binom{s-1+l}{l}},
\]

since we assumed \( \eta \leq \frac{2}{E} \). Hence, for all \( s \in 2, \ldots, E - \max(k,l) \),
\[
\eta^{s} \sqrt{\binom{s+k}{k}\binom{s+l}{l}} \leq \eta \sqrt{(k+1)(l+1)}.
\]

Combining this with Eq. \([B12]\) yields
\[
\left| \text{Tr}(|l\rangle \langle k| \sigma) - \mathbb{E}_{\alpha \sim Q_{\alpha}}[f_{|l\rangle \langle k|}(\alpha, \eta)] \right| \leq \eta \sqrt{(k+1)(l+1)} \sum_{s=1}^{E-\max(k,l)} \sqrt{\sigma_{s+k,s+k} \sigma_{s+l,s+l}}
\]
\[
\leq \eta \sqrt{(k+1)(l+1)} \sum_{s=1}^{E-\max(k,l)} \sum_{s=1}^{E-\max(k,l)} \sqrt{\sigma_{s+k,s+k} \sigma_{s+l,s+l}}
\]
\[
\leq \eta \sqrt{(k+1)(l+1)},
\]

for all \( k, l \in 0, \ldots, E \), where we used Cauchy-Schwarz inequality and the fact that \( \text{Tr}(\sigma) = 1 \). Together with Eq. \([B3]\) and triangular inequality we obtain
\[
\left| \text{Tr}(A\sigma) - \mathbb{E}_{\alpha \sim Q_{\alpha}}[f_{A}(\alpha, \eta)] \right| \leq \eta \sum_{k,l=0}^{E} |A_{kl}| \sqrt{(k+1)(l+1)}.
\]

With Eq. \([5]\), this proves Theorem \([\ref{thm1}]\).

2. Proof of Corollary 1

In order to prove Corollary 1 we apply Theorem 1 for \( A = \tau \) a pure state. We obtain

\[
\left| \text{Tr}(\tau \sigma) - \mathbb{E}_{\alpha \sim Q, \eta} [f_\tau(\alpha, \eta)] \right| \leq \eta K \tau
\]

\[
= \eta \sum_{k,l=0}^{E} |\tau_k \tau_l| \sqrt{(k+1)(l+1)}
\]

\[
= \eta \left( \sum_{n=0}^{E} |\tau_n| \sqrt{n+1} \right)^2 \tag{B18}
\]

\[
\leq \eta \sum_{n=0}^{E} |\tau_n|^2 \sum_{n=0}^{E} (n+1)
\]

\[
\leq \frac{\eta}{2} (E+1)(E+2),
\]

where we used Cauchy-Schwarz inequality, and \( \text{Tr}(\tau) = 1 \). Since \( \tau \) is a pure state, we have \( F(\tau, \sigma) = \text{Tr}(\tau \sigma) \) \[24\], which concludes the proof.

\[\square\]

Appendix C: Proof of Theorem 2 (determining a matrix element)

The law of large numbers ensures that the sample average from independently and identically distributed (i.i.d.) random variables converges to the expected value of these random variables, when the number of samples goes to infinity. The following lemma refines this statement and quantifies the speed of convergence:

Lemma 1. (Hoeffding) Let \( \epsilon' > 0 \), let \( N \geq 1 \), let \( Z_1, \ldots, Z_N \) be i.i.d. complex random variables from a probability density \( D \) over \( \mathbb{R} \), and let \( f : \mathbb{C} \mapsto \mathbb{R} \) such that \( |f(z)| \leq M \), for \( M > 0 \) and all \( z \in \mathbb{C} \). Then

\[
\Pr \left[ \left| \frac{1}{N} \sum_{i=1}^{N} f(Z_i) - \mathbb{E}_{Z \sim D} [f(Z)] \right| \geq \epsilon' \right] \leq 2 e^{-\frac{N \epsilon'^2}{2 M^2}}. \tag{C1}
\]

This comes directly from Hoeffding inequality \[27\] applied to the real bounded i.i.d. random variables \( f(Z_1), \ldots, f(Z_N) \).

When dealing with complex random variables, we will use the following result instead:

Lemma 2. (Hoeffding for complex random variables) Let \( \epsilon' > 0 \), let \( N \geq 1 \), let \( Z_1, \ldots, Z_N \) be i.i.d. complex random variables from a probability density \( D \) over \( \mathbb{C} \), and let \( f : \mathbb{C} \mapsto \mathbb{C} \) such that \( |f(z)| \leq M \), for \( M > 0 \) and all \( z \in \mathbb{C} \). Then

\[
\Pr \left[ \left| \frac{1}{N} \sum_{i=1}^{N} f(Z_i) - \mathbb{E}_{Z \sim D} [f(Z)] \right| \geq \epsilon' \right] \leq 4 e^{-\frac{N \epsilon'^2}{4 M^2}}. \tag{C2}
\]

Proof. For all \( a > 0 \) and all \( z \in \mathbb{C} \), \( |z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2} \geq a \) implies \( |\text{Re}(z)| > a/\sqrt{2} \) or \( |\text{Im}(z)| > a/\sqrt{2} \). Hence,

\[
\Pr [|z| > a] \leq \Pr \left[ |\text{Re}(z)| > \frac{a}{\sqrt{2}} \right] + \Pr \left[ |\text{Im}(z)| > \frac{a}{\sqrt{2}} \right], \tag{C3}
\]

so applying twice Lemma 1 for the real random variables \( \text{Re}(f(Z)) \) and \( \text{Im}(f(Z)) \), respectively, yields Lemma 2.

\[\square\]

In order to apply this Lemma to the functions \( z \mapsto f_{|k\rangle\langle l|}(z, \eta) \) defined in Eq. \[4\], we first prove the following bound:
Lemma 3. For all $k, l \geq 0$, define
\[
M_{kl} := \sqrt{2|l-k|} \left( \max\left(\frac{k}{l}, \frac{l}{k}\right) \right). \tag{C4}
\]
Then for all $k, l$ and all $z \in \mathbb{C}$,
\[
|f_{[k,l]}(z, \eta)| \leq \frac{M_{kl}}{\eta^{1+\frac{1}{2}\alpha}}. \tag{C5}
\]

Proof. For $k, l > E$ the lemma is trivial. For all $k, l \leq E$ and all $z \in \mathbb{C}$,
\[
|f_{[k,l]}(z, \eta)| = \frac{1}{\eta} e^{(1-\frac{1}{n})|z|^2} \left| L_{k,l} \left( \frac{z}{\sqrt{\eta^2}} \right) \right|^{1/2}
\]
\[
= \frac{1}{\eta} e^{(1-\frac{1}{n})|z|^2} \left( \sum_{p=0}^{\min(k,l)} \frac{(-1)^p k!}{p! (k-p)! (l-p)!} \frac{1}{\eta^{k+l-p}|z|^{k-l} e^{\eta z^2}} \right), \tag{C6}
\]
where we used Eq. [2]. Now for all $z \in \mathbb{C}^*$ and all $a > 0$ we have [22]
\[
\sum_{p=0}^{\min(k,l)} \frac{(-1)^p k!}{p! (k-p)! (l-p)!} a^{k+l-p} z^{k-l} e^{\eta z^2} = a^{k!} z^{k-l} L_{k}^{(k-l)}(a|z|^2) \tag{C7}
\]
where
\[
L_{n}^{(\alpha)}(x) = \sum_{q=0}^{n} \frac{(-1)^q}{q!} \left( \frac{n+\alpha}{n-q} \right) x^q \tag{C8}
\]
are the generalised Laguerre polynomials [28], defined for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$. Plugging this relation into Eq. (C6) we obtain
\[
|f_{[k,l]}(z, \eta)| = e^{(1-\frac{1}{n})|z|^2} \left| \frac{z}{\eta^{1+k}} \frac{\sqrt{k!}}{\sqrt{l!}} \left( \frac{|z|^2}{\eta} \right) \right|^{1/2} L_{k}^{(k-l)}(a |z|^2) \tag{C9}
\]
for all $z \in \mathbb{C}$. The generalised Laguerre polynomials are bounded as [29]
\[
|L_{n}^{(\alpha)}(x)| \leq \frac{\Gamma(n+\alpha+1)}{n! \Gamma(\alpha+1)} e^{\frac{x}{2}}, \tag{C10}
\]
for all $x \geq 0$, all $\alpha \geq 0$ and all $n \in \mathbb{N}$, and as
\[
|L_{n}^{(\alpha)}(x)| \leq 2^{-\alpha} e^{\frac{x}{2}}, \tag{C11}
\]
for all $x \geq 0$, all $\alpha \leq -\frac{1}{2}$ and all $n \in \mathbb{N}$.

Let $a > 0$. Assuming $k < l$, we have $|z|^{l-k} \leq a^{l-k}$ for $|z| \leq a$, and $|z|^{k-l} \leq a^{k-l}$ for $|z| \geq a$. Thus, the first line of Eq. (C9), together with Eq. (C10), give
\[
|f_{[k,l]}(z, \eta)| \leq e^{(1-\frac{1}{n})|z|^2} \frac{a^{l-k}}{\eta^{1+l}} \frac{1}{\sqrt{l!}} \frac{1}{\sqrt{(l-k)!}} \frac{|z|^2}{\eta^{1+k}} \tag{C12}
\]
\[
\leq \frac{a^{l-k}}{\eta^{1+l}} \frac{1}{\sqrt{l!}} \frac{1}{\sqrt{(l-k)!}} \frac{|z|^2}{\eta^{1+k}},
\]
for all $x \geq 0$, all $\alpha \leq -\frac{1}{2}$ and all $n \in \mathbb{N}$.
for $|z| \leq a$ and $k < l$. Similarly, the second line of Eq. (C9), together with Eq. (C11), give
\[
|f_{[k]}(z, \eta)| \leq e^{(1 - \frac{1}{\eta})|z|^2} \frac{a^{k-l}}{\eta^{k+l+1}} \frac{e^{\frac{|z|^2}{2\eta^2}}}{\sqrt{k!} \sqrt{l!}} \\
\leq \frac{a^{k-l}}{\eta^{k+l+1}} \frac{e^{\frac{|z|^2}{2\eta^2}}}{\sqrt{k!} \sqrt{l!}},
\]
for $|z| \geq a$ and $k < l$. These two last bounds (C12, C13) are equal for $a^{l-k} = (2\eta)^{l-k} \sqrt{(l-k)!}$, yielding the bound
\[
|f_{[k]}(z, \eta)| \leq \frac{a^{l-k}}{\eta^{k+l+1}} \frac{\binom{l}{k}}{\sqrt{k!} \sqrt{l!}} ,
\]
for all $z \in \mathbb{C}$ and $k < l$. For $l < k$ the same reasoning gives
\[
|f_{[k]}(z, \eta)| \leq \frac{a^{k-l}}{\eta^{k+l+1}} \frac{\binom{k}{l}}{\sqrt{k!} \sqrt{l!}} .
\]
Finally, for $k = l$ the previous bounds also hold, by combining Eqs. (C9, C10), and this proves Lemma 3.

Let $k, l \geq 0$. Applying Lemma 2 to the function $f_{[l]}(z)$, with the bound from Lemma 3 we obtain
\[
\Pr \left[ \frac{1}{N} \sum_{i=1}^{N} f_{[l]}(\alpha_i, \eta) - E_{\alpha \sim \sigma} f_{[l]}(\alpha, \eta) \geq \epsilon \right] \leq 4e^{-\frac{\eta^{2+k+l+1}}{2M_{kl}^2} - \frac{1}{4M_{kl}^2}} .
\]
Applying Theorem 1 for $A = [l] \langle k \rangle$ we also obtain
\[
\left| \sigma_{kl} - E_{\alpha \sim \sigma} f_{[l]}(\alpha, \eta) \right| \leq \eta \sqrt{k + 1} \sqrt{l + 1} .
\]
Let $N \in \mathbb{N}$ and let $\alpha_1, \ldots, \alpha_N$ be samples from the Q function of $\sigma$. Combining Eqs. (C16, C17), we obtain
\[
\left| \sigma_{kl} - \frac{1}{N} \sum_{i=1}^{N} f_{[l]}(\alpha_i, \eta) \right| \leq \eta \sqrt{k + 1} \sqrt{l + 1} + \epsilon',
\]
with probability greater than
\[
1 - 4e^{-\frac{\eta^{2+k+l+1}}{2M_{kl}^2} - \frac{1}{4M_{kl}^2}} .
\]
We have $K_{[l]}(\eta) = \sqrt{(k + 1)(l + 1)}$ by Eq. (5). Taking $\eta = \frac{\epsilon}{K_{[l]}(\eta)}$ yields
\[
\left| \sigma_{kl} - \frac{1}{N} \sum_{i=1}^{N} f_{[l]}(\alpha_i, \eta) \left( \alpha_i, \frac{\epsilon}{K_{[l]}(\eta)} \right) \right| \leq \epsilon + \epsilon',
\]
with probability greater than
\[
1 - 4e^{-\frac{\eta^{2+k+l+1}}{2M_{kl}^2} - \frac{1}{4M_{kl}^2}} ,
\]
where we defined
\[
C_{kl} := 4 \left[ (k + 1)(l + 1) \right]^{1 + \frac{k+l}{2}} M_{kl}^2
\]
\[
= 4 \left[ (k + 1)(l + 1) \right]^{1 + \frac{k+l}{2}} 2^{l-k} \left( \max(k, l) \right)^{\min(k, l)} .
\]
Appendix D: State certification with Gaussian measurements

1. Support estimation for i.i.d. states

Let us define the following operators for $E \geq 0$:

$$U = \sum_{n=E+1}^{+\infty} |n\rangle \langle n| = 1 - \Pi_{\leq E}, \quad (D1)$$

where $\Pi_{\leq E} = \sum_{n=0}^{E} |n\rangle \langle n|$ is the projector onto the Hilbert space of states with less than $E$ photons, and

$$T = \frac{1}{\pi} \int_{|\alpha|^2 \geq E} |\alpha\rangle \langle \alpha| d^2\alpha, \quad (D2)$$

where $|\alpha\rangle$ is a coherent state. We have the following result, proven in [30] by expanding $T$ in the Fock basis:

$$U \leq 2T. \quad (D3)$$

Hence, the probability $P_R$ that exactly $R$ among $N$ values of $|\alpha|^2$ are bigger than $E$ and $N - R$ values are lower, and that the projection of the state $\sigma$ onto the Hilbert space of states with less than $E$ photons fails is bounded as

$$P_R = \binom{N}{R} \text{Tr} \left[ (1 - \Pi_{\leq E}) T^R (1 - T)^{N - R} \sigma \otimes (N+1) \right]$$

$$\leq 2 \binom{N}{R} \text{Tr} \left[ UT^R (1 - T)^{N - R} \sigma \otimes (N+1) \right] \text{Tr} (1 - T)^{N - R}$$

$$\leq 2 \binom{N}{R} \max_p |p^{R+1} (1 - p)^{N - R}|$$

$$= 2 \binom{N}{R} \left( \frac{R + 1}{N + 1} \right)^{R+1} \left( 1 - \frac{R + 1}{N + 1} \right)^{N - R} \quad (D4)$$

$$\leq 2 \frac{N^R}{R!} \left( \frac{R + 1}{N + 1} \right)^{R+1} \left( 1 - \frac{R + 1}{N + 1} \right)^{N - R}$$

$$\leq 2 \frac{N^R}{R!} \left( \frac{R + 1}{N + 1} \right)^{R+1} e^{-\frac{(N-R)(R+1)}{N+1}}$$

$$\leq 2 \frac{R + 1}{N} \sqrt{2 \pi (R + 1)} e^{-\frac{(R+1)^2}{2(R+1)}}$$

$$\leq \frac{1}{N} \sqrt{R + 1} e^{-\frac{(R+1)^2}{2(R+1)}},$$

where we used $(R + 1)! \geq \sqrt{2\pi(R + 1)}(R + 1)^{R+1}e^{-(R+1)}$. The fifth line gives a less compact but tighter bound, also independent of $\sigma$.

2. Proof of Theorem 4

The proof goes along the same lines as the one of Theorem 2 detailed in Appendix C, with the addition of the support estimation from the previous section. We first prove Theorem 4 for $m = 1$, from which we deduce the general case.

The function $f_\tau$ is real-valued, since $\tau$ is hermitian. It is bounded as

$$|f_\tau(\alpha, \eta)| = \left| \sum_{k,l=0}^{E} \tau_k \tau_l^* f_{|k\rangle\langle l|}(\alpha, \eta) \right|$$
\[
\sum_{k,l=0}^{E} \left| \tau_k^* \left( f_{|k|} \right) \right| (\alpha, \eta) \leq \sum_{k,l=0}^{E} |\tau_k \tau_l| \frac{M_{kl}}{\eta^{1+\frac{E}{2}}} \\
= \frac{1}{\eta^{1+E}} \sum_{k,l=0}^{E} |\tau_k \tau_l| \eta^{E-(k+l)/2} M_{kl} \\
= \frac{M_\tau(\eta)}{\eta^{1+E}},
\]

where we used Lemma 3 and where we defined
\[
M_\tau(\eta) := \sum_{k,l=0}^{E} |\tau_k \tau_l| \eta^{E-(k+l)/2} M_{kl}.
\]

Applying Lemma 1 to the real-valued function \( f_{\tau,E} \) thus yields
\[
\Pr \left[ \left| \tilde{F} - \mathbb{E}_{\alpha \sim Q_\sigma} [f_{\tau}(\alpha, \eta)] \right| \geq \epsilon' \right] \leq 2e^{-\frac{N\epsilon^2+2E\epsilon'^2}{2M_\tau^2(\eta)}},
\]

for \( \epsilon' > 0 \), where \( \tilde{F} = \frac{1}{N} \sum_{i=1}^{N} f_{\tau}(\alpha_i, \eta) \).

Let \( R \) be the number of samples \( \alpha_i \) such that \( |\alpha_i|^2 \geq E \). With the results from Appendix D 1, the state \( \sigma \) has a support over the Fock basis bounded by \( E \) with probability greater than \( 1 - \frac{1}{N} \sqrt{R + 1}e^{-\frac{(R+1)^2}{N+2}} \). Let \( \eta > 0 \), by Corollary 1 we thus have
\[
\left| F(\tau, \sigma) - \mathbb{E}_{\alpha \sim Q_\sigma} [f_{\tau}(\alpha, \eta)] \right| \leq \eta K_\tau,
\]

with probability greater than \( 1 - \frac{1}{N} \sqrt{R + 1}e^{-\frac{(R+1)^2}{N+2}} \).

Combining Eqs. (D6,D7) together with the union bound yields
\[
\left| F(\tau, \sigma) - \tilde{F} \right| \leq \eta K_\tau + \epsilon',
\]

with probability greater than
\[
1 - \frac{1}{N} \sqrt{R + 1}e^{-\frac{(R+1)^2}{N+2}} - 2e^{-\frac{N\epsilon^2+2E\epsilon'^2}{2M_\tau^2(\eta)}},
\]

Setting \( \eta K_\tau = \epsilon \) yields
\[
\left| F(\tau, \sigma) - \tilde{F}_\epsilon(\alpha) \right| \leq \epsilon + \epsilon',
\]

with probability greater than
\[
1 - \left( \frac{1}{N} \sqrt{R + 1}e^{-\frac{(R+1)^2}{N+2}} + 2e^{-\frac{N\epsilon^2+2E\epsilon'^2}{C_{\text{iid}}^2}} \right),
\]

where \( \tilde{F}_\epsilon(\alpha) = \frac{1}{N} \sum_{i=1}^{N} f_{\tau} \left( \alpha_i, \epsilon \right) \), and where we defined
\[
C_{\text{iid}} = 2K_\tau^{2+2E}M_\tau^{2} \left( \frac{\epsilon}{K_\tau} \right)
\leq 2 \left( \sum_{k,l=0}^{E} |\tau_k \tau_l| \epsilon^{E-\frac{k+l}{2}} K_\tau \right)^{1+\frac{k+l}{2}} \sqrt{2[1-k]} \left( \frac{\max(k,l)}{\min(k,l)} \right)^2.
\]

This concludes the proof of Theorem 4 when \( m = 1 \). We note that the failure probability may be greatly reduced using tighter bounds and taking into account the expression of \( \tau \).

We introduce the following simple result to obtain the theorem for all \( m \geq 1 \):
Lemma 4. Let \( \eta > 0 \) and \( a, b \in [-1, 1] \) such that \( |a - b| \leq \eta \). Then
\[
|a^m - b^m| \leq (m + 1)\eta. \tag{D13}
\]

Proof. With the notations of the lemma,
\[
|a^m - b^m| = |a - b| \left| \sum_{j=0}^{m} a^j b^{m-j} \right|
\leq \eta \sum_{j=0}^{m} |a^j b^{m-j}|
\leq (m + 1)\eta. \tag{D14}
\]

Combining Lemma 4 and Eq. (D10) for \( a = F(\tau, \sigma) \) and \( b = \tilde{F} \), we obtain
\[
\left| F(\tau, \sigma)^m - \left[ \tilde{F}_\tau(\alpha) \right]^m \right| \leq (m + 1)(\epsilon + \epsilon'), \tag{D15}
\]
with probability greater than
\[
1 - \left( \frac{1}{N} \sqrt{R + 1} e^{\frac{(R+1)^2}{2}} + 2e^{-\frac{N^2+2E\epsilon^2}{C_{\tau}}} \right). \tag{D16}
\]

Note that we excluded the pathological case \( \tilde{F} \geq 1 \): when that is the case we instead set \( \tilde{F} = 1 \). The target state \( \tau \) is pure so \( F(\tau \otimes m, \sigma \otimes m) = F(\tau, \sigma)^m \). Hence, setting \( \epsilon = \epsilon/(m+1) \) and \( \epsilon' = \epsilon'/(m+1) \) concludes the proof, with
\[
\tilde{F}^{\tau, m}(\alpha) = \left[ \tilde{F}_\tau^{\tau \otimes m}(\alpha) \right]^m = \left[ \frac{1}{N} \sum_{i=1}^{N} f_{\tau} \left( \alpha_i, \frac{\epsilon}{(m + 1)K_{\tau}} \right) \right]^m. \tag{D17}
\]

Appendix E: State verification with Gaussian measurements

1. Support estimation for permutation-invariant states

We first derive a support estimation for permutation-invariant states and combine it with the de Finetti theorem from [1]. We will use in this section the following operators, already introduced in Appendix D 1: for \( E \geq 0 \):
\[
U = \sum_{n=E+1}^{+\infty} |n\rangle \langle n| = 1 - \Pi_{\leq E}, \tag{E1}
\]
where \( \Pi_{\leq E} = \sum_{n=0}^{E} |n\rangle \langle n| \) is the projector onto the Hilbert space \( \tilde{H} \) of states with at most \( E \) photons, and
\[
T = \frac{1}{\pi} \int_{|\alpha|^2 \geq E} |\alpha\rangle \langle \alpha| d^2 \alpha, \tag{E2}
\]
where \( |\alpha\rangle \) is a coherent state. We also recall the following result, from Eq. (D3) and [30]:
\[
U \leq 2T. \tag{E3}
\]

We recall a few notations and results from [1, 31]: let \( \mathcal{A} = \{A_0, A_1\}, \mathcal{B} = \{B_0, B_1\} \) be two binary POVMs over \( \mathcal{H} \). Define for \( \delta > 0 \),
\[
\gamma_{A \rightarrow B}(\delta) = \sup \{ \text{Tr}(B\tau) \mid \text{Tr}(A\tau) \leq \delta \}. \tag{E4}
\]
In particular,
\[
\gamma_{\mathcal{T} \to \mathcal{U}}(\delta) \leq 2\delta, \tag{E5}
\]
by Eq. (E5). We have the following result (Lemma III.1 of [31]):

**Lemma 5.** Let \( N \geq 2K \), let \( \mathcal{A} = \{A_0, A_1\} \) and \( \mathcal{B} = \{B_0, B_1\} \) be two binary POVMs over \( \mathcal{H} \), and let \( X_1, \ldots, X_{N+K} \) the \((N + K)\)-partite classical outcome of the measurement \( \mathcal{A}^{\otimes N} \otimes \mathcal{B}^{\otimes K} \) applied to any permutation-invariant state \( \sigma^{N+K} \). Then

\[
\Pr\left[ \frac{X_1, \ldots, X_N}{N} > \gamma_{B_1 \to A_1} \left( \frac{X_{N+1}, \ldots, X_{N+K}}{K} + \delta \right) + \delta \right] \leq 8K^{3/2} e^{-K\delta^2}. \tag{E6}
\]

This result is a refined version of Serfling’s bound [32]. It relates the outcomes of a measurement on some subsystems of a symmetric state with the outcomes of a related measurement on the rest of the subsystems.

Let \( \sigma^{N+K} \) be a state over \( N + K \) subsystems. Applying a random permutation to this state and measuring its last \( K \) subsystems with heterodyne detection is equivalent to measuring \( K \) subsystems at random. We thus assume in the following that the state \( \sigma^{N+K} \) is a permutation-invariant state, without loss of generality, and that the verifier measures its last \( K \) subsystems with heterodyne detection.

Let \( \mathcal{T} = \{1-T, T\} \) and \( \mathcal{U} = \{1-U, U\} \). Let \( X_1, \ldots, X_{N+K} \) the \((N + K)\)-partite classical outcome of the measurement \( \mathcal{U}^{\otimes N} \otimes \mathcal{T}^{\otimes K} \) applied to the permutation-invariant state \( \sigma^{N+K} \) sent by the prover. A value \( X_i = 1 \) for \( i = 1, \ldots, N \) means that the projection of the \( i \)th subsystem onto \( \mathcal{H} \) failed, while a value \( X_j = 1 \) for \( j \in N + 1, \ldots, N + K \) means that the value \( |\beta_i|^2 \) obtained when measuring the \( j \)th subsystem with heterodyne detection is bigger than \( E \). In particular, the number of values \( \beta_i \) satisfying \(|\beta_i|^2 > E\), is expressed as \( X_{N+1} + \cdots + X_{N+K} \).

Let \( F_Q^N \) be the event that the projection onto \( \mathcal{H} \) fails for more than \( Q \) subsystems of the remaining state \( \sigma^N \), and let \( T_R^K \) be the event that exactly \( R \) values \( \beta_i \) satisfy \(|\beta_i|^2 > E\). Then,

\[
\Pr\left[ F_Q^N \cap T_R^K \right] = \Pr\left[ (X_1 + \cdots + X_N > Q) \land (X_{N+1} + \cdots + X_{N+K} = R) \right]. \tag{E7}
\]

With Eq. (E5), we have for all \( \delta > 0 \)
\[
\gamma_{\mathcal{T} \to \mathcal{U}} \left( \frac{X_{N+1} + \cdots + X_{N+K}}{K} \right) + \delta \leq 2 \frac{X_{N+1} + \cdots + X_{N+K}}{K} + 3\delta. \tag{E8}
\]

Hence taking \( \delta = \frac{1}{3} \left( \frac{Q}{N} - \frac{2R}{K} \right) \) we obtain

\[
\Pr\left[ F_Q^N \cap T_R^K \right] \leq \Pr\left[ \left( \frac{X_1 + \cdots + X_N}{N} > \frac{Q}{N} \right) \land \left( \frac{X_{N+1} + \cdots + X_{N+K}}{K} = \frac{R}{K} \right) \right]
\leq \Pr\left[ \left( \frac{X_1 + \cdots + X_N}{N} > \gamma_{\mathcal{T} \to \mathcal{U}} \left( \frac{X_{N+1} + \cdots + X_{N+K}}{K} + \delta \right) + \delta \right) \land \left( \frac{X_{N+1} + \cdots + X_{N+K}}{K} = \frac{R}{K} \right) \right]
\leq 8K^{3/2} e^{-\frac{1}{2} \left( \frac{Q}{N} - \frac{2R}{K} \right)^2}, \tag{E9}
\]

where we used Lemma 5 for \( \mathcal{A} = \mathcal{U} = \mathcal{T} = \mathcal{H} \).

What this means is that as long as the value \( R \) is small, then with high probability all but \( Q \) of the subsystems of the reduced state over \( N \) modes lie in a lower dimensional subspace.

We now combine this support estimation with the de Finetti reduction from [1]. We first introduce the following notation: for all \( n, r \geq 0 \) and all \( |v\rangle \in \mathcal{H} \otimes \mathcal{H} \) we write \( S_{n-r}^v \) the set of almost-i.i.d. states along \( |v\rangle \), i.e. the span of all states over \( n \) subsystems that are, up to reorderings, of the form \( |v\rangle^{\otimes (n-r)} \otimes |\Psi\rangle \), for an arbitrary \( \Psi \in (\mathcal{H} \otimes \mathcal{H})^{\otimes r} \).

**Theorem 6** (de Finetti reduction with heterodyne measurements). Let \( \sigma^{N+K} \) be a permutation-invariant state over \( N + K \) subsystems. Let \( \beta_1, \ldots, \beta_K \) be samples obtained by measuring \( K \) subsystems of \( \sigma^{N+K} \) with heterodyne detection. Let \( E \in \mathbb{N} \), and let \( P \) be the number of values \(|\beta_i|^2 > E\). Then for all \( Q \) there exist a purification \( \tilde{\sigma}^{N-4Q} \) of the reduced state \( \sigma^{N-4Q} \), a finite set \( \mathcal{V} \) of unit vectors \(|v\rangle \in \mathcal{H} \otimes \mathcal{H} \), a probability distribution \( \{p_v\}_{v \in \mathcal{V}} \) over \( \mathcal{V} \), and almost-i.i.d. states \( \sigma_v^{N-4Q} \in S_{v^{N-4Q},4Q} \) such that

\[
F \left( \tilde{\sigma}^{N-4Q}, \sum_{v \in \mathcal{V}} p_v \sigma_v^{N-4Q} \right) > 1 - (4Q)^E^2 e^{-\frac{8Q(4Q+1)}{N}}, \tag{E10}
\]
with probability greater than \(1 - 8K^{3/2}e^{-K\left(\frac{2}{\alpha} - \frac{2}{\lambda}\right)^2}\).

Tracing out the purifying subsystems, Theorem 4 implies that the reduced state \(\sigma^{N-4Q}\) is close in fidelity to a mixture of states that are i.i.d. on \(N-8Q\) subsystems, with high probability, provided the support estimation \(R\) is small enough. Taking, e.g., \(K = Q = M^2\) and \(N = M^4\) in Eq. (E10) yields an exponential bound in \(M\) for the fidelity and the probability, whenever \(R < M^2/2\).

2. Proof of Theorem 5

We first introduce a few technical results and combine them to prove the Theorem.

We recall here Lemma 1 in the context of a product measurement applied to an i.i.d. state \(|v\rangle\langle v|\otimes^n:\n
**Lemma 6. (Hoeffding)** Let \(A < B \in \mathbb{R}\) and let \(f: C \mapsto \mathbb{R}\) be a function bounded as \(|f(\alpha)| \leq M\) for all \(\alpha \in C\). Let \(\lambda > 0\), let \(p \in \mathbb{N}^*\), and let \(|v\rangle \in \mathcal{H}\). Let \(M = \{M_{\alpha}\}_{\alpha \in C}\) be a POVM on \(\mathcal{H}\) and let \(D_{(v)}\) be the probability density function of the outcomes of the measurement \(M\) applied to \(|v\rangle\langle v|\). Then

\[
\Pr_{\alpha} \left[\frac{1}{p} \sum_{i=1}^{p} f(\alpha_i) - \mathbb{E}_{\beta \leftarrow D_{(v)}} [f(\beta)] \geq \lambda \right] \leq 2e^{-\frac{2p\lambda^2}{M^2}}, \tag{E11}
\]

where the probability is taken over the outcomes \(\alpha = (\alpha_1, \ldots, \alpha_p)\) of the product measurement \(M^{\otimes p}\) applied to \(|v\rangle\langle v|\otimes^p\).

The next result gives an equivalent statement for almost-i.i.d. states along a state \(|v\rangle\), measured with a product measurement. It generalises Theorem 4.5.2 of [33], where the probability distributions over finite sets, corresponding to product measurements with finite number of outcomes, are replaced by continuous variable probability densities, corresponding to product measurements with continuous variable outcomes. Frequencies estimators are also replaced with estimators of expected values of bounded functions. We will use this result for the POVM corresponding to a product heterodyne detection.

**Lemma 7.** Let \(A < B \in \mathbb{R}\) and let \(f: C \mapsto \mathbb{R}\) be a function bounded as \(|f(\alpha)| \leq M\) for all \(\alpha \in C\). Let \(\epsilon' > 0\), let \(1 \leq r, m < n\), such that

\[
(n - \min(r, m))\epsilon' > 2M \max(r, m). \tag{E12}
\]

Let also \(|v\rangle \in \mathcal{H}\) and \(|\Psi\rangle \in \mathcal{S}_v^{n-r}\). Let \(M = \{M_{\alpha}\}_{\alpha \in C}\) be a POVM on \(\mathcal{H}\) and let \(D_{(v)}\) be the probability density function of the outcomes of the measurement \(M\) applied to \(|v\rangle\langle v|\). Then

\[
\Pr_{\alpha} \left[\frac{1}{n-m} \sum_{i=1}^{n-m} f(\alpha_i) - \mathbb{E}_{\beta \leftarrow D_{(v)}} [f(\beta)] \geq \epsilon' \right] \leq 2\binom{n}{r} e^{-2(n-r)\left(\frac{\epsilon'}{n} - \frac{2\max(r, m)}{n-\min(r, m)}\right)^2}, \tag{E13}
\]

where the probability is taken over the outcomes \(\alpha = (\alpha_1, \ldots, \alpha_{n-m})\) of the product measurement \(M^{\otimes (n-m)}\) applied to \(|\Psi\rangle\langle \Psi|\).

This lemma says that measurement on all but \(m\) subsystems of an almost-i.i.d. state along a state \(|v\rangle\) will yield statistics that are similar to the ones that would be obtained by measuring the i.i.d. state \(|v\rangle\otimes (n-m)\).

**Proof.** By Lemma 4.1.6 of [33], there exist a finite set \(S\) of size at most \(^n\binom{n}{m}\), a family of “junk” states \(|\tilde{\Psi}^s\rangle \in \mathcal{H}^{\otimes r}\) for \(s \in S\), complex amplitudes \(\{\gamma_s\}_{s \in S}\) and permutations \(\{\pi_s\}_{s \in S}\) over \([1, \ldots, n]\) such that

\[
|\Psi\rangle := \sum_{s \in S} \gamma_s |\tilde{\Psi}^s\rangle \tag{E14}
\]

With the notations of the theorem, let us define for \(\mu > 0:\n
\Omega_{\mu} = \left\{ \alpha \in \mathbb{C}^{n-m}: \left| \frac{1}{n-m} \sum_{i=1}^{n-m} f(\alpha_i) - \mathbb{E}_{\beta \leftarrow D_{(v)}} [f(\beta)] \right| > \mu \right\}. \tag{E15}\n
We recall here Lemma 4.5.1 of [33].
Lemma 8. Let $|\psi\rangle = \sum_{x \in X} |\psi^x\rangle$ and let $A$ be a non-negative operator. Then

$$
\langle \psi | A | \psi \rangle \leq |X| \sum_{x \in X} (\psi^x | A | \psi^x).
$$

(E16)

In particular, using Eq. (E14) and this lemma when $A$ is a POVM element of the product measurement $M\alpha \equiv M_{\alpha_0} \otimes \cdots \otimes M_{\alpha_{n-m}}$, we obtain:

$$
\Pr_{\alpha \sim \bar{|\psi\rangle}} [\alpha \in \Omega_\mu] = \int_{\Omega_\mu} \langle \psi | M\alpha | \psi \rangle d^{2(n-m)}\alpha
\leq \int_{\Omega_\mu} |S| \sum_{s \in S} |\gamma_s|^2 \langle \Psi^s | M\alpha | \Psi^s \rangle d^{2(n-m)}\alpha
\leq |S| \sum_{s \in S} |\gamma_s|^2 \Pr_{\alpha \sim \bar{|\psi\rangle}} [\alpha \in \Omega_\mu],
$$

(E17)

where we write $\alpha \sim \bar{|\psi\rangle}$ to indicate that $\alpha = (\alpha_1, \ldots, \alpha_{n-p})$ is distributed according to the outcomes of the product measurement $M^{\otimes (n-m)}$ applied to $|\psi\rangle$.

Let $s \in S$. Let $\alpha \sim \bar{|\psi^s\rangle}$. We have $|\Psi^s\rangle = \pi_s \left( |\nu\rangle^{\otimes (n-r)} \otimes |\bar{\psi}\rangle \right)$, and in particular $(\alpha_{s(1)}, \ldots, \alpha_{s(n-r)})$ is distributed according to the outcomes of the product measurement $M^{\otimes (n-r)}$ applied to $|\nu\rangle^{\otimes (n-r)}$, so by Lemma 8 with $p = n - r$,

$$
\Pr \left[ \left| \frac{1}{n-r} \sum_{i=1}^{n-r} f(\alpha, \nu(i)) - \mathbb{E}_{\beta \sim \bar{D}_\nu} [f(\beta)] \right| \geq \lambda \right] \leq 2e^{-\frac{2(n-r)^2}{(\lambda-\beta)^2}},
$$

(E18)

for all $\lambda > 0$. We also have, for $|f| \leq M$,

$$
\left| \frac{1}{n-m} \sum_{i=1}^{n-m} f(\alpha_i) - \frac{1}{n-r} \sum_{i=1}^{n-r} f(\alpha_{s(i)}) \right| = \left| \frac{1}{n-m} \sum_{i=1}^{n-m} f(\alpha_i) - \frac{1}{n-r} \left( \sum_{i=1}^{n} f(\alpha_i) - \sum_{i=n-r+1}^{n} f(\alpha_{s(i)}) \right) \right|
\leq \left| \frac{1}{n-r} - \frac{1}{n-m} \right| \sum_{i=1}^{n-m} f(\alpha_i) + \frac{1}{n-r} \left( \sum_{i=n-r+1}^{n} f(\alpha_i) - \sum_{i=n-r+1}^{n} f(\alpha_{s(i)}) \right)
\leq \left| \frac{r-m}{n-r} \right| M + \frac{(m+r)}{n-r} M
\leq \frac{2 \max(r, m)}{n-r} M.
$$

(E19)

The same reasoning works when swapping $r$ and $m$, so we obtain

$$
\left| \frac{1}{n-m} \sum_{i=1}^{n-m} f(\alpha_i) - \frac{1}{n-r} \sum_{i=1}^{n-r} f(\alpha_{s(i)}) \right| \leq \frac{2 \max(r, m)}{n-min(r, m)} M.
$$

(E20)
Now for all \( s \in \mathcal{S} \),

\[
\Pr_{\alpha \leftarrow \ket{\Psi}} \left[ \alpha \in \Omega_\mu \right] = \Pr_{\alpha \leftarrow \ket{\Psi}} \left[ \left| \frac{1}{n-m} \sum_{i=1}^{n-m} f(\alpha_i) - \mathbb{E}_{\beta \leftarrow D_\psi} [f(\beta)] \right| > \mu \right] \\
\leq \Pr_{\alpha \leftarrow \ket{\Psi}} \left[ \left| \frac{1}{n-r} \sum_{i=1}^{n-r} f(\alpha_{\pi_i}) - \mathbb{E}_{\beta \leftarrow D_\psi} [f(\beta)] \right| + \left| \frac{1}{n-m} \sum_{i=1}^{n-m} f(\alpha_i) - \frac{1}{n-r} \sum_{i=1}^{n-r} f(\alpha_{\pi_i}) \right| > \mu \right] \\
\leq \Pr_{\alpha \leftarrow \ket{\Psi}} \left[ \left| \frac{1}{n-r} \sum_{i=1}^{n-r} f(\alpha_{\pi_i}) - \mathbb{E}_{\beta \leftarrow D_\psi} [f(\beta)] \right| > \mu - \frac{2\max(r,m)}{n-\min(r,m)} M \right] \\
\leq 2e^{-2(n-r)\left( \frac{\mu^2}{\pi^2} \frac{2\max(r,m)}{n-\min(r,m)} \right)^2},
\]  

where we used triangular inequality in the second line, Eq. (E20) in the third line and Eq. (E18) in the fourth line with \( \lambda = \mu - \frac{2\max(r,m)}{n-\min(r,m)} M > 0 \). Combining this last equation with Eq. (E17), and using \( |\mathcal{S}| \leq \binom{n}{r} \) we finally obtain, with \( \mu = \epsilon' \),

\[
\Pr_{\alpha \leftarrow \ket{\Psi}} \left[ \left| \frac{1}{n} \sum_{i=1}^{n} f(\alpha_i) - \mathbb{E}_{\beta \leftarrow D_\psi} [f(\beta)] \right| \geq \epsilon' \right] \leq 2 \binom{n}{r} e^{-2(n-r)\left( \frac{\mu^2}{\pi^2} \frac{2\max(r,m)}{n-\min(r,m)} \right)^2}.
\]  

(22)

We recall the bound on \( f_r \) obtained in Appendix D detailed in Eq. (D5):

\[
|f_r(\alpha, \eta)| \leq \frac{M_f(\eta)}{\eta^{1+E}}.
\]  

(23)

Let \( \eta > 0 \), \( E > 0 \), and let \( \alpha = \alpha_1, \ldots, \alpha_{N-S-m} \in \mathbb{C} \) be samples obtained by measuring \( N-S-m \) subsystems of a state \( \rho \) with heterodyne detection. We define the estimate

\[
\tilde{F}_r^\rho(\alpha) = \frac{1}{N-S-m} \sum_{i=1}^{N-S-m} f_r(\alpha_i, \eta).
\]  

(24)

Let \( \ket{\psi} \in \mathcal{H} \otimes \mathcal{H} \), and let \( \ket{\Psi}_v \in \mathcal{S}_v^{N-r} \). Applying Lemma 7 for the real-valued function \( f_r \), for \( r = S \), for \( n = N-S \), for \( D_\psi = Q_{\psi}(\psi) \), and with the bound from Eq. (23), we obtain

\[
\Pr_{\alpha} \left[ \left| \tilde{F}_r^{\ket{\Psi}_v}(\alpha) - \mathbb{E}_{\alpha \leftarrow Q_{\psi}(\psi)} [f_r(\alpha, \eta)] \right| \geq \epsilon_H \right] \leq \delta_H,
\]  

(25)

where

\[
\delta_H = 2 \binom{N-S}{S} e^{-2(N-2S)} \left( \frac{N-S}{\min(r,m)} \right)^2 e^{-2(n-r)\left( \frac{\mu^2}{\pi^2} \frac{2\max(r,m)}{n-\min(r,m)} \right)^2},
\]  

(26)

and where the probability is over the outcomes of a product heterodyne measurement on \( N-S-m \) subsystems of \( \ket{\Psi}_v \).

We will also use the following result:

**Lemma 9.** Let \( 0 < \eta < 1 \). Let \( \sigma_1, \sigma_2 \) be two states such that \( F(\sigma_1, \sigma_2) > 1 - \eta \). Let \( \tau \) be a third state, then

\[
|F(\tau, \sigma_1) - F(\tau, \sigma_2)| \leq \cos^{-1} \sqrt{1-\eta} \leq \frac{\pi}{2} \sqrt{\eta}.
\]  

(27)

**Proof.** Writing \( F \) the fidelity, \( \theta = \cos^{-1} \sqrt{F} \) is a metric [21]. We thus have \( \theta_{\tau \sigma_1} \leq \theta_{\tau \sigma_2} + \theta_{\sigma_1 \sigma_2} \) and \( \theta_{\tau \sigma_2} \leq \theta_{\tau \sigma_1} + \theta_{\sigma_1 \sigma_2} \), so

\[
\cos^{-1} \sqrt{F(\tau, \sigma_1)} - \cos^{-1} \sqrt{F(\tau, \sigma_2)} = |\theta_{\tau \sigma_1} - \theta_{\tau \sigma_2}| \leq \theta_{\sigma_1 \sigma_2} = \cos^{-1} \sqrt{F(\sigma_1, \sigma_2)} \leq \cos^{-1} \sqrt{1-\eta}.
\]  

(28)
Let \( g(x) = \cos^{-1} \sqrt{1 - x} \) for \( x \in [0, 1] \). We have \( g'(x) = \frac{1}{2\sqrt{x(1-x)}} \) for all \( x \in (0, 1) \). In particular, \( g'(x) \geq 1 \), so for all \( x, y \in [0, 1] \),

\[
|x - y| \leq |g(x) - g(y)|.
\] (E29)

Moreover, setting \( \sqrt{x} = \sin \phi \), one easily deduces \( g(x) \leq \frac{2}{\pi} \sqrt{x} \) for all \( x \in [0, 1] \). Combining with Eq. (E28) leads to

\[
|F(\tau, \sigma_1) - F(\tau, \sigma_2)| \leq \cos^{-1} \sqrt{F(\tau, \sigma_1) - \cos^{-1} \sqrt{F(\tau, \sigma_2)}} \leq \cos^{-1} \sqrt{1 - \eta} \leq \frac{\pi}{2} \sqrt{\eta}.
\] (E30)

Let \( \tau \) be a target pure state, and let \( \sigma_{N+K} \) be the state sent by the prover. Let \( \beta_1, \ldots, \beta_K \) be samples obtained by measuring \( K \) random subsystems of \( \sigma_{N+K} \) with heterodyne detection. Let \( \sigma_{N-S} \) be the permutation-invariant state obtained by applying a random permutation to the subsystems of remaining state and tracing out the first \( S \) subsystems. Let \( E \in \mathbb{N} \), and let \( R \) be the number of values \( |\beta_i|^2 > E \). Then by Theorem 6 with \( S = 4Q \), there exist a purification \( \tilde{\sigma}_{N-S} \) of \( \sigma_{N-S} \), a finite set \( V \) of unit vectors \( |v\rangle \in H \otimes \mathcal{H} \), a probability distribution \( \{p_v\}_{v \in V} \) over \( V \), and almost-i.i.d. states \( \sigma_{N-S}^{\nu} \in \mathcal{S}_{N-S, S}^{\nu} \) such that

\[
F\left(\tilde{\sigma}_{N-S}, \sigma_{N-S}\right) > 1 - \epsilon_F,
\] (E31)

with probability greater than \( 1 - \delta_F \), where \( \tilde{\sigma}_{N-S} = \sum_{v \in V} p_v \sigma_{N-S}^{\nu} \), where

\[
\epsilon_F = SE^2 e^{-\frac{2S(S+1)}{N}},
\]

and where

\[
\delta_F = 8K^{3/2} e^{-\frac{4}{N}(\pi^2 - 2\delta)^2}.
\]

Let \( \alpha_1, \ldots, \alpha_{N-S-m} \) be the samples obtained by measuring the first \( N-S-m \) subsystems of \( \sigma_{N-S} \) with heterodyne detection. Tracing out over the purifying subsystems and these \( N-S-m \) additional subsystems, Eq. (E31) implies with Lemma 9 that

\[
|F(\tau^{\otimes m}, \sigma^m) - F(\tau^{\otimes m}, \tilde{\sigma}^m)| \leq \frac{\pi}{2} \sqrt{\epsilon_F},
\]

where \( \sigma^m \) and \( \tilde{\sigma}^m \) are the remaining states over \( m \) subsystems.

Let us define the completely positive trace-preserving map \( \mathcal{E}_{N-S} \) associated to the classical post-processing of the protocol. It maps any density operator \( \rho_{N-S} \) over \( N-S \) subsystems to the classical state encoding the estimate \( \left( \hat{F}_{\tau}^\sigma \right)^m \), defined at Eq. (E24):

\[
\mathcal{E}_{N-S}(\rho) = \sum_{e} \text{Pr}\left[\left( \hat{F}_{\tau}^\sigma(\alpha) \right)^m = e \right] |e\rangle \langle e|.
\] (E35)

The sum ranges over the values \( e \) that the estimate may take. In terms of trace distance, Eq. (E31) may be expressed as

\[
\frac{1}{2} \left\| \tilde{\sigma}_{N-S} - \tilde{\sigma}_{N-S} \right\|_1 \leq \sqrt{\epsilon_F}.
\]

(E36)

The trace distance is non-increasing under quantum operations, so Eq. (E31) implies

\[
\frac{1}{2} \left\| \left( \mathcal{E}_{N-S} \otimes \mathbb{1}_{N-S+m} \right)(\tilde{\sigma}_{N-S}) - \left( \mathcal{E}_{N-S} \otimes \mathbb{1}_{N-S+m} \right)(\sigma_{N-S}) \right\|_1 \leq \sqrt{\epsilon_F}.
\]

(E37)

Using the definition of the map \( \mathcal{E}_{N-S} \), we obtain a bound in total variation distance:

\[
\left\| P\left[\left( \hat{F}_{\tau}^\sigma(x) \right)^m \right] - P\left[\left( \hat{F}_{\tau}^{\tilde{\sigma}_{N-S}}(x) \right)^m \right] \right\|_{tvd} \leq \sqrt{\epsilon_F}.
\]

(E38)
where $P$ denotes the probability distributions for the values of the estimates $\left(\tilde{F}_\tau(\alpha)\right)^m$ for $\sigma$ and $\tilde{\sigma}$. In particular, this bound implies that for all $\epsilon > 0$

$$\left| \Pr \left[ F(\tau^\otimes m, \sigma^m) - \left(\tilde{F}_\tau^\sigma(\alpha)\right)^m \right] > \epsilon \right| - \Pr \left[ F(\tau^\otimes m, \sigma^m) - \left(\tilde{F}_\tau^{\tilde{\sigma}^{N-S}}(\alpha)\right)^m \right] > \epsilon \right| \leq \sqrt{\epsilon F}.$$  

(E39)

Hence, setting $\epsilon = \nu + \frac{\pi}{2} \sqrt{\epsilon F}$, for $\nu > 0$, Eq. (E31) implies

$$\Pr \left[ F(\tau^\otimes m, \sigma^m) - \left(\tilde{F}_\tau^\sigma(\alpha)\right)^m \right] > \nu + \frac{\pi}{2} \sqrt{\epsilon F} \leq \sqrt{\epsilon F} + \Pr \left[ F(\tau^\otimes m, \sigma^m) - \left(\tilde{F}_\tau^{\tilde{\sigma}^{N-S}}(\alpha)\right)^m \right] > \nu + \frac{\pi}{2} \sqrt{\epsilon F}$$

$$\leq \sqrt{\epsilon F} + \Pr \left[ F(\tau^\otimes m, \sigma^m) - F(\tau^\otimes \tilde{\sigma}^m) \right] > \nu + \frac{\pi}{2} \sqrt{\epsilon F}$$

$$\leq \sqrt{\epsilon F} + \Pr \left[ F(\tau^\otimes m, \sigma^m) - F(\tau^\otimes \tilde{\sigma}^m) \right] > \nu + \frac{\pi}{2} \sqrt{\epsilon F}$$

$$\text{(E40)}$$

where we used the triangular inequality in the second line. With Eq. (E34) we obtain

$$\Pr \left[ F(\tau^\otimes m, \sigma^m) - \left(\tilde{F}_\tau^\sigma(\alpha)\right)^m \right] > \nu + \frac{\pi}{2} \sqrt{\epsilon F} \leq \sqrt{\epsilon F} + \Pr \left[ F(\tau^\otimes m, \tilde{\sigma}^m) - \left(\tilde{F}_\tau^{\tilde{\sigma}^{N-S}}(\alpha)\right)^m \right] > \nu,$$

(E41)

whenever Eq. (E31) holds. Now this happens with probability greater than $1 - \delta_F$, so with the union bound,

$$\Pr \left[ F(\tau^\otimes m, \sigma^m) - \left(\tilde{F}_\tau^\sigma(\alpha)\right)^m \right] > \nu + \frac{\pi}{2} \sqrt{\epsilon F} \leq \sqrt{\epsilon F} + \delta_F + \Pr \left[ F(\tau^\otimes m, \tilde{\sigma}^m) - \left(\tilde{F}_\tau^{\tilde{\sigma}^{N-S}}(\alpha)\right)^m \right] > \nu.$$  

(E42)

We now bound the quantity

$$\Pr \left[ F(\tau^\otimes m, \Psi^m) - \left(\tilde{F}_\tau^{\Psi^\Psi}(\Psi^\Psi)(\alpha)\right)^m \right] > \nu,$$

(E43)

for all $|\Psi\rangle \in S_{N-S}^S$, where $\Psi^m$ is the reduced state obtained from $|\Psi\rangle \langle \Psi|$ by tracing all but $m$ subsystems. We have

$$\left| F(\tau^\otimes m, \Psi^m) - \left(\tilde{F}_\tau^{\Psi^\Psi}(\Psi^\Psi)(\alpha)\right)^m \right| \leq \left| F(\tau^\otimes m, \Psi^m) - F(\tau^\otimes m, |v\rangle \langle v|)^m \right|$$

$$+ \left| F(\tau^\otimes m, |v\rangle \langle v|^m) - \left(\mathbb{E}_{\alpha \sim Q_{(|v\rangle \langle v|)}} [f_\tau(\alpha, \eta)]^m \right) \right|$$

$$+ \left| \left(\mathbb{E}_{\alpha \sim Q_{(|v\rangle \langle v|)}} [f_\tau(\alpha, \eta)]^m \right) - \left(\tilde{F}_\tau^{\Psi^\Psi}(\Psi^\Psi)(\alpha)\right)^m \right|$$

$$\leq \left| F(\tau^\otimes m, \Psi^m) - F(\tau^\otimes m, |v\rangle \langle v|^m) \right|$$

$$+ (m + 1) \left| F(\tau, |v\rangle \langle v|) - \mathbb{E}_{\alpha \sim Q_{(|v\rangle \langle v|)}} [f_\tau(\alpha, \eta)] \right|$$

$$+ (m + 1) \left| \mathbb{E}_{\alpha \sim Q_{(|v\rangle \langle v|)}} [f_\tau(\alpha, \eta)] - \tilde{F}_\tau^{\Psi^\Psi}(\Psi^\Psi)(\alpha) \right|,$$

(E44)

where we used $F(\tau^\otimes m, |v\rangle \langle v|^m) = F(\tau, |v\rangle \langle v|m$ and Lemma 4 (whenever $\tilde{F} \geq 1$ we instead set $\tilde{F} = 1$). We bound these three terms in the following.

When selecting at random $m$ subsystems from an almost-i.i.d. state over $N - S$ subsystems which is i.i.d. on $N - 2S$ subsystems, the probability that all of the selected states are from the $N - 2S$ i.i.d. subsystems is

$$\binom{N-2S}{m} = \frac{(N-2S)(N-2S-1) \ldots (N-2S-m+1)}{(N-S)(N-S-1) \ldots (N-S-m+1)},$$

(E45)
and we have
\[
1 - \frac{(N - 2S)(N - 2S - 1) \ldots (N - 2S - m + 1)}{(N - S)(N - S - 1) \ldots (N - S - m + 1)} \leq 1 - \frac{(N - 2S - m + 1)^m}{(N - S)^m} \\
= 1 - \left(1 - \frac{S + m - 1}{N - S}\right)^m \\
\leq \min\left(1, \frac{m(S + m - 1)}{N - S}\right)
\]
(E46)

where we used \(1 - (1 - x)^a \leq ax\) for all \(a \geq 1\) and \(x \in [0, 1]\). In particular, for \(|\Psi\rangle \in S_{e}^{N - S, S}\) we have
\[
F(|v\rangle \langle v|^m, \Psi^m) = \text{Tr}(|v\rangle \langle v|^m \Psi^m) \\
\geq \frac{(N - 2S)}{(N - m)} \\
\geq 1 - \frac{m(S + m - 1)}{N - S},
\]
(E47)

where we used the definition of \(S_{e}^{N - S, S}\), and Eq. (E46). Using Lemma [9] we obtain the bound for the first term:
\[
|F(\tau^\otimes m, \Psi^m) - F(\tau^\otimes m, |v\rangle \langle v|^m)| \leq \frac{\pi}{2} \sqrt{\frac{m(S + m - 1)}{N - S}}.
\]
(E48)

The bound for the second term is directly given by Corollary [1] applied to the state \(|v\rangle\):
\[
|F(\tau, |v\rangle \langle v|) - \mathbb{E}_{\alpha \leftarrow Q_{\tau}(|v|)} [f_{\tau}(\alpha, \eta)]| \leq \eta K_{\tau}.
\]
(E49)

The bound for the third term is probabilistic, directly given by Eq. (E25):
\[
|\mathbb{E}_{\alpha \leftarrow Q_{\tau}(|v|)} [f_{\tau}(\alpha, \eta)] - \bar{F}_{\tau}^{|\Psi\rangle\langle\Psi|}(\alpha)| \leq \varepsilon_H,
\]
(E50)

with probability greater than \(\delta_H = 2(N - S)e^{-2(N - 2S)(\sqrt{\varepsilon_F + \mu} - \frac{2\max(S, m)}{N - S - \min(S, m)})^2}\).

We now bring together the previous results in order to prove Theorem [5] Combining Eqs. (E44, E48, E49, E50) yields
\[
\text{Pr}\left[\left|F(\tau^\otimes m, \Psi^m) - \bar{F}_{\tau}^{|\Psi\rangle\langle\Psi|}(\alpha)^m\right| > \nu\right] \leq \delta_H,
\]
(E51)

where we have set
\[
\nu := \frac{\pi}{2} \sqrt{\frac{m(S + m - 1)}{N - S}} + \eta(m + 1)K_{\tau} + (m + 1)\varepsilon_H.
\]
(E52)

Writing \(\tilde{\sigma}_{N-S} = \sum_{i,v} q_i |\Psi_v\rangle \langle \Psi_v|\), with \(\sum_i q_i = 1\), this implies by convexity
\[
\text{Pr}\left[\left|F(\tau^\otimes m, \tilde{\sigma}^m) - \left(\bar{F}_{\tau}^{\tilde{\sigma}_{N-S}}(\alpha)\right)^m\right| > \nu\right] \leq \delta_H.
\]
(E53)

With Eq. (E42) we obtain
\[
\text{Pr}\left[\left|F(\tau^\otimes m, \sigma^m) - \left(\bar{F}_{\tau}^{\sigma}(\alpha)\right)^m\right| > \mu\right] \leq \sqrt{\varepsilon_F} + \delta_F + \delta_H,
\]
(E54)

where we have set
\[
\mu := \frac{\pi}{2} \sqrt{\frac{m(S + m - 1)}{N - S}} + \eta(m + 1)K_{\tau} + (m + 1)\varepsilon_H + \frac{\pi}{2} \sqrt{\varepsilon_F}.
\]
(E55)
Thus

\[
F(\tau^m, \sigma^m) - \left( \tilde{F}_\tau^\sigma(\alpha) \right)^m \leq \frac{\pi}{2} \sqrt{\frac{m(S+m-1)}{N-S}} + \eta(m+1)K_\tau + (m+1)\epsilon_H + \frac{\pi}{2} \sqrt{\epsilon_F},
\]  

(E56)

with probability greater than

\[
1 - (\sqrt{\epsilon_F} + \delta_F + \delta_H),
\]  

(E57)

where

\[
K_\tau = \sum_{k,l=0}^E |\tau_k \tau_l| \sqrt{(k+1)(l+1)},
\]  

(5)

\[
\delta_H = 2 \left( \frac{N-S}{S} \right) e^{-2(N-2S)} \left( \frac{e^{1+\epsilon_H} - 1}{m \epsilon_H} - \frac{2 \max(S,m)}{N-S - \min(S,m)} \right)^2,
\]  

(E26)

\[
M_\tau(\eta) = \sum_{k,l=0}^E |\tau_k \tau_l| \eta^{-(k+l)/2} M_{kl},
\]  

(D5)

\[
M_{kl} = \sqrt{2^{l-k} \max(k,l) \min(k,l)},
\]  

(C4)

\[
\epsilon_F = S^E e^{-\frac{E(S+1)}{N}},
\]  

(E32)

\[
\delta_F = 8K^{3/2} e^{-\frac{K}{2} \left( \frac{S-1}{N} - \frac{2}{N} \right)^2},
\]  

(E33)

and where \( E, \eta, \epsilon_H, N, m, S, K \) are free parameters of the protocol, and \( R \) is the result of the support estimation.

Setting \( \epsilon = \eta(m+1)K_\tau, \epsilon' = (m+1)\epsilon_H, \epsilon'' = \frac{\pi}{2} \sqrt{\frac{m(S+m-1)}{N-S}} \), and \( S \geq m \) in Eq. (E56) concludes the proof of Theorem [5] with

\[
\tilde{F}_\tau(\alpha) := \left( \tilde{F}_\tau^\sigma(\alpha) \right)^m = \left[ \frac{1}{N-S-m} \sum_{i=1}^{N-S-m} f_\tau \left( \alpha_i, \frac{\epsilon}{(m+1)K_\tau} \right) \right]^m.
\]  

(E58)

With the notations of the main text, we then have \( P_{\text{Support}} = \delta_F, P_{\text{deFinetti}} = \sqrt{\epsilon_F}, P_{\text{Hoeffding}} = \delta_H \), and \( C_\tau = K_\tau^{1+E} M_\tau(\eta) \).

\[ \square \]

In terms of scaling of the parameters, taking, e.g., \( E = O(1), R = O(1), N = O(m^{8+8E}), K = O(m^{7+8E}), S = O(m^{5+4E}) \), \( \eta = \epsilon_H = O(\frac{1}{m^2}) \) gives an error polynomially small in \( m \) and a probability exponentially close in \( m \) to 1, by plugging the different scalings in Eq. (E56). We note that there is a lot of room for optimisation of the free parameters of the protocol.

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