On the Construction of Covariant Differential Calculi on Quantum Homogeneous Spaces

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Abstract

Let $A$ be a coquasitriangular Hopf algebra and $X$ the subalgebra of $A$ generated by a row of a matrix corepresentation $u$ or by a row of $u$ and a row of the contragredient representation $u^c$. In the paper left-covariant first order differential calculi on the quantum group $A$ are constructed and the corresponding induced calculi on the left quantum space $X$ are described. The main tool for these constructions are the L-functionals associated with $u$. The results are applied to the quantum homogeneous space $GL_q(N)/GL_q(N-1)$.

0 Introduction

Based on the pioneering work of S.L. Woronowicz [W2], a beautiful theory of bicovariant differential theory on quantum groups has been developed till now. A thorough treatment of this theory can be found in Chapter 14 of the monograph [KS]. The theory of covariant differential calculi on quantum spaces, in contrast, is still at the very beginning and neither general methods for the construction of such calculi nor remarkable general results are known. Covariant differential calculi have been constructed and studied so far only on a few simple quantum spaces ([PW], [WZ], [P1], [P2], [SS1], [AS], [CHZ], [We]).

In this paper we are concerned with the construction of first order differential calculi (abbreviated, FODC) on subalgebras of a coquasitriangular Hopf algebra $A$ which are generated by a row of a fixed corepresentation $u$ or by a row of $u$ and a row of the contragredient corepresentation $u^c$ of $A$. Such a subalgebra is a left quantum space of $A$ with left coaction given by the restriction of the comultiplication. Our method of construction is easy to explain: The FODC on the quantum spaces are induced from appropriate left-covariant differential calculi on the quantum group $A$. The main technical tool for the construction of the left-covariant calculi on $A$ are the L-functionals associated with the corepresentation $u$. We always try to be as simple and close to the classical situation as possible. Our approach has two important advantages: First, because of the close relationship between the calculi on the quantum space and on the quantum group the theory of L-functionals and other Hopf algebra techniques can be applied to the study of the calculi on the quantum space. Secondly, the simplicity of the constructed
left-covariant calculi, in contrast to the usual bicovariant calculi, might be useful for doing explicit computations. Our guiding example are the quantum spheres associated with the quantum group $GL_q(N)$ (see [SV], [NYM] or [KS], 11.6). For these quantum spheres a classification of covariant differential calculi has been recently given by M. Welk [We]. As an application of our method we describe some of the main calculi occurring there as induced from left-covariant calculi on $GL_q(N)$. Strictly speaking, we derive the left-covariant counter-parts of these calculi, because in [We] right quantum spheres and right-covariant calculi are investigated.

This paper is organized as follows. Section 1 contains some preliminaries and collects some notation. In Sections 2 and 3 first order calculi on the left quantum spaces generated a a single row of $u$ and $u^c$, respectively, are investigated. Section 4 deals with the quantum space generated by a row of $u$ and a row of $u^c$. Four families of covariant FODC are constructed and the commutation rules between generators and their differentials are explicitly described. The application of the results to the fundamental corepresentations of the quantum groups $GL_q(N)$ and $SL_q(N)$ are discussed in Section 5. In Section 6 another interesting FODC on the quantum sphere is obtained from a particular bicovariant (!) calculus on $GL_q(N)$. The left-covariant differential calculi on the quantum groups have so far only auxilary tools for the study of the induced FODC on the quantum spaces.

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2 Preliminaries

Throughout this paper $\mathcal{A}$ is a coquasitriangular complex Hopf algebra and $r$ denotes a fixed universal $r$-form of $\mathcal{A}$ (see, for instance, [LT] or [KS], Section 10.1, for these notions). The comultiplication, the counit and the antipode of $\mathcal{A}$ are denoted by $\Delta$, $\varepsilon$ and $S$, respectively. We shall use the Sweedler notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for the comultiplication of $\mathcal{A}$. Let us recall that a Hopf algebra $\mathcal{A}$ is called coquasitriangular if it is equipped with a linear functional $r$ on $\mathcal{A} \otimes \mathcal{A}$ which is invertible with respect to the convolution multiplication and satisfies the following conditions for arbitrary elements $a, b, c \in \mathcal{A}$:

$$r(ab \otimes c) = r(a \otimes c_{(1)})r(b \otimes c_{(2)}), \quad r(a \otimes bc) = r(a_{(1)} \otimes c)r(a_{(2)} \otimes b),$$  \hspace{1cm} (1)

$$r(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)} = r(a_{(2)} \otimes b_{(2)})b_{(1)}a_{(1)}. \hspace{1cm} (2)$$

Such a linear form $r$ is called a universal $r$-form of the Hopf algebra $\mathcal{A}$. The convolution inverse of $r$ is denoted by $\bar{r}$. We shall write $r(a, b) := r(a \otimes b), a, b \in \mathcal{A}$.

Further, $u = (u^j_i)_{i,j=1,...,n}$ denotes a fixed $n$-dimensional matrix corepresentation of $\mathcal{A}$, that is, $u$ is an $n \times n$-matrix of elements $u^j_i$ of $\mathcal{A}$ such that

$$\Delta(u^j_i) = \sum_{k=1}^n u^k_i \otimes u^k_j \quad \text{and} \quad \varepsilon(u^j_i) = \delta_{ij} \quad \text{for } i, j = 1, \cdots, n.$$
We define the L-functionals \( l^\pm_{ij} \) and the R-matrix \( \hat{R} \) associated with the corepresentation \( u \) by

\[
l^+_j(\cdot) = r(\cdot \otimes u_j), \quad l^-_j(\cdot) = \bar{r}(u_j \otimes \cdot), \quad \hat{R}_{iim} := r(u_i, u_m).
\]

The Hopf dual of the Hopf algebra \( A \) is denoted by \( A^\circ \). The L-functionals \( l^\pm_{ij} \) belong to \( A^\circ \). From (I) it follows that

\[
\Delta(l^\pm_{ij}) = \sum_{k=1}^n l^\pm_{ik} \otimes l^\pm_{kj}, \quad i, j = 1, \ldots, n.
\]

These and the following relations will be often used in this paper:

\[
(l^+_{ij}, u^k_l) = \hat{R}^i_{lj} k, \quad (l^-_{ij}, u^k_l) = (\hat{R}^{-1})^k_{lj} = \bar{r}(u^i, u^k_j),
\]

\[
(S(l^+_{ij}, u^k_l) = (\hat{R}^{-1})^k_{lj}, \quad (S(l^-_{ij}, u^k_l) = \hat{R}^k_{lj}).
\]

Formula (I) implies that the matrix \( \hat{R} \) and hence also \( \hat{R}^{-1} \) intertwine the tensor product corepresentation \( u \otimes u \).

Suppose that \( X \) is a subalgebra \( X \) of \( A \) such that \( \Delta(X) \subseteq A \otimes X \). Then \( X \) is a left \( A \)-comodule algebra or equivalently a left quantum space of \( A \) with left coaction \( \varphi \) given by the restriction \( \Delta[X] \) of the comultiplication of \( A \). As in [KS], such a subalgebra \( X \) will be called a left quantum homogenous space of the Hopf algebra \( A \).

A first order differential calculus (abbreviated, a FODC) over \( X \) is an \( X \)-bimodule \( \Gamma \) equipped with a linear mapping \( d : X \to \Gamma \), called the differentiation, such that:

(i) \( d \) satisfies the Leibniz rule \( d(xy) = x \cdot dy + dx \cdot y \) for any \( x, y \in X \),

(ii) \( \Gamma \) is the linear span of elements \( x \cdot dy \cdot z \) with \( x, y, z \in X \).

A FODC \( \Gamma \) over \( X \) is called left-covariant if there exists a linear mapping \( \Phi : \Gamma \to X \otimes \Gamma \) such that \( \Phi(xdy) = \Delta(x)(\text{id} \otimes d)(\Delta(y)) \) for all \( x, y \in X \). For a left-covariant FODC \( \Gamma \) of \( X \) the elements of the vector space \( \text{inv} \, \Gamma = \{ \eta \in \Gamma \mid \Phi(\eta) = 1 \otimes \eta \} \) are called left-invariant one-forms. A left-covariant FODC \( \Gamma \) of \( X \) is called inner if there exists a left-invariant one-form \( \theta \in \text{inv} \, \Gamma \) such that

\[
dx = \theta x - x \theta, \quad x \in X.
\]

Let \( \Gamma \) be a left-covariant FODC on the Hopf algebra \( A \) itself such that \( \text{dim} \, \Gamma := \text{dim}_\text{inv} \, \Gamma \) is finite-dimensional. We briefly recall a few facts from the general theory of these calculi (see [W2], [AS] or [KS], Section 14.1, for more details) that will be used in what follows. Such a FODC \( \Gamma \) is characterized by a finite-dimensional subspace \( T \) of \( A \circ \), called the quantum tangent space of \( \Gamma \), and there is a canonical non-generate bilinear form \( (\cdot, \cdot) \) on \( T \times \text{inv} \, \Gamma \). If \( \{ X_i; i \in I \} \) and \( \{ \theta_i; i \in I \} \) are dual bases of \( T \) and \( \text{inv} \, \Gamma \) with respect to this bilinear form, then the differentiation \( d \) of the FODC \( \Gamma \) can be expressed by

\[
da = \sum_ia_1^{(i)}X_ia_2^{(i)}\theta_i, \quad a \in A.
\]
The commutation relations between the elements of \( \mathcal{A} \) and left-invariant one-forms of \( \Gamma \) are given by

\[
\theta_i a = \sum_k a_{(1)} f^i_k(a_{(2)}) \theta_k, \quad a \in \mathcal{A},
\]

where \( f^i_k \) are the functionals on \( \mathcal{A} \) are determined by the equation

\[
\Delta(X_k) - \varepsilon \otimes X_k = \sum_i X_i \otimes f^i_k.
\]

Let \( \omega : \mathcal{A} \rightarrow inv \Gamma \) be the canonical projection defined by \( \omega(a) = S(a_{(1)}) da_{(2)} \) for \( a \in \mathcal{A} \). Then one has

\[
(X, \omega(a)) = (X, a) \quad \text{for} \quad X \in \mathcal{T} \quad \text{and} \quad a \in \mathcal{A}.
\]

If \( \Gamma \) is a FODC of \( \mathcal{A} \) with differentiation \( d \), then \( \bar{\Gamma} := \mathcal{X} \cdot d\mathcal{X} \cdot \mathcal{X} \) is obviously a FODC of the subalgebra \( \mathcal{X} \) with differentiation \( d[\mathcal{X}] \). We call \( \bar{\Gamma} \) the induced FODC of the FODC \( \Gamma \) of \( \mathcal{A} \). Clearly, if \( \Gamma \) is left-covariant on the quantum group \( \mathcal{A} \), then so is \( \bar{\Gamma} \) on the left quantum space \( \mathcal{X} \).

Our constructions of left-covariant FODC on \( \mathcal{A} \) are based on the following lemma.

**Lemma 1.** A finite-dimensional vector space \( \mathcal{T} \) of \( \mathcal{A}^o \) is the quantum tangent space of a left-covariant FODC of \( \mathcal{A} \) if and only if \( X(1) = 0 \) and \( \Delta(X) - \varepsilon \otimes X \in \mathcal{X} \otimes \mathcal{A}^o \) for all \( X \in \mathcal{X} \).

**Proof.** [SS2], Lemma 1, or [KS], Proposition 14.5. \( \square \)

3 Quantum spaces generated by a row of \( u \)

Let \( \mathcal{X} \) denote the unital subalgebra of \( \mathcal{A} \) generated by the entries of the last row of the matrix \( u \), that is, by the elements \( x_i := u^i_n, i = 1, \ldots, n \). Clearly, \( \mathcal{X} \) is a left quantum homogeneous space of \( \mathcal{A} \) with left coaction \( \varphi = \Delta[\mathcal{X}] \) determined by

\[
\varphi(x_i) \equiv \Delta(u^i_n) = \sum_{j=1}^n u^i_j \otimes x_j, \quad i = 1, \ldots, n.
\]

In this section we shall construct an \( n \)-dimensional left-covariant FODC \( \Gamma \) on the Hopf algebra \( \mathcal{A} \) which induces a FODC \( \Gamma^\mathcal{X} \) on \( \mathcal{X} \) such that the differentials \( dx_1, \ldots, dx_n \) form a free left \( \mathcal{X} \)-module basis of \( \Gamma^\mathcal{X} \).

First we define a FODC \( \Gamma \) of \( \mathcal{A} \). Let \( \mathcal{T}^\mathcal{X} \) be the linear span of functionals

\[
X_i := \alpha^{-1} l^{-n} l^{-m} i, \quad i = 1, \ldots, n - 1, \quad \text{and} \quad X_n := \alpha^{-1} ((l^{-n})^2 - \varepsilon)
\]

on \( \mathcal{A} \), where \( \alpha \) is a non-zero complex number that will be specified by formula (11) below. We assume that

\[
l^{-m} = 0 \quad \text{if} \quad m < n.
\]
Since \( \Delta(l^{-n}) = \sum_{i} l_i^{-n} \otimes l_i^{-n} \), this assumption implies in particular that \( \Delta(l^{-n}) = l^{-n} \otimes l^{-n} \), so that \( l^{-n} \) is a character of the algebra \( \mathcal{A} \) (that is, \( l^{-n}(ab) = l^{-n}(a)l^{-n}(b) \) for \( a, b \in \mathcal{A} \) and \( l^{-n}(1) = 1 \)). Using the relation \( \Delta(l^{-n}) = l^{-n} \otimes l^{-n} \) we get

\[
\Delta(X_i) - \varepsilon \otimes X_i = \sum_{j=1}^{n} X_j \otimes l^{-1}_j l^{-n}, \quad i = 1, \ldots, n - 1,
\]

\[
\Delta(X_n) - \varepsilon \otimes X_n = X_n \otimes (l^{-n})^2.
\]

Because of (8), the latter equations can be written in the compact form

\[
\Delta(X_i) - \varepsilon \otimes X_i = \sum_{j=1}^{n} X_i \otimes l^{-1}_j l^{-n}, \quad i = 1, \ldots, n.
\]

Since obviously \( X(1) = 0 \) and \( \Delta(X) - \varepsilon \otimes X \in \mathcal{T}^X \otimes \mathcal{A}^0 \) for all \( X \in \mathcal{T}^X \) by (4), it follows from Lemma 1 there exists a left-covariant FODC \( \Gamma \) on \( \mathcal{A} \) such that \( \mathcal{T}^X \) is the quantum tangent space of \( \Gamma \).

Let us suppose in addition that

\[
(l^{-n}, u^i_n) = 0 \quad \text{if} \quad i \neq j, \quad i, j = 1, \ldots, n,
\]

\[
\alpha := (l^{-n} l^{-n}, u^i_n) = ((l^{-n})^2, u^i_n) - 1 \neq 0 \quad \text{for} \quad i = 1, \ldots, n - 1.
\]

We abbreviate \( c_- := (l^{-n}_n, u^i_n) \). Then we have \( \alpha = c_-^2 - 1 \).

For \( i = 1, \ldots, n \), let \( \theta_i \) denote the left-invariant 1-form \( \omega(u^i_n) \equiv \sum_k S(u^i_n)du^k_n \) of \( \Gamma \). The assumptions (10) and (11) imply that \( (X_j, u^i_n) = \delta_{ij} \) and so by formula (5) that

\[
(X_j, \theta_i) = (X_j, \omega(u^k_n)) = (X_j, u^i_n) = \delta_{ij}
\]

for \( i, j = 1, \ldots, n \). In particular we conclude that the functionals \( X_1, \ldots, X_n \) are linearly independent, so that the FODC \( \Gamma \) is \( n \)-dimensional. Further, (12) shows that \( \{\theta_1, \ldots, \theta_n\} \) and \( \{X_1, \ldots, X_n\} \) are dual bases of \( \text{inv}(\Gamma) \) and \( \mathcal{T}^X \), respectively. Therefore, comparing (5) and (6) and using (1), (7) and (8), we obtain for \( r, j = 1, \ldots, n \),

\[
\theta_r x_j = \sum_{k,s} u^j_k(l^{-r}_s l^{-n}_n, u^k_n) \theta_s = \sum_{k,m,s} u^j_k(l^{-r}_s, u^k_m)(l^{-n}_n, u^m_s) \theta_s
\]

\[
= \sum_{k,s} c^{-1}(R^{-1})_{ns}^{rk} u^j_k \theta_s.
\]

These relations lead to the following commutation rules between the one-forms \( \theta_r \) and elements of the algebra \( \mathcal{X} \):

\[
\theta_r x = \sum_{s=1}^{n} x_{(1)}(\tilde{r}, x_{(2)})(l^{-n}_n, x_{(3)}) \theta_s, \quad x \in \mathcal{X}.
\]

Indeed, if \( x \) is the generator \( x_j \) of \( \mathcal{X} \), then the third expression of (12) can be rewritten as the right-hand side of (14). Using the facts that \( l^{-n}_n \) is a character and that \( \tilde{r}_{21} \) is also a
universal $r$-form of $\mathcal{A}$ (see [KS], Proposition 10.2(iv)), one easily verifies that (14) holds for a product $x'x''$ provided that it holds for both factors $x'$ and $x''$. Thus, (14) is valid for arbitrary elements $x$ of $\mathcal{X}$.

Next we turn to the FODC $\Gamma^X$ of $\mathcal{X}$.

**Proposition 2.** (i) The FODC $\Gamma$ of $\mathcal{A}$ induces a left-covariant FODC $\Gamma^X$ of $\mathcal{X}$ such that the set \{dx$_1$, ..., dx$_n$\} is a free left $\mathcal{X}$-module basis of $\Gamma^X$. The $\mathcal{X}$-bimodule structure of $\Gamma^X$ is determined by the commutation relations

\[
d x_i \cdot x_j = (l^{-n_i}_n, u^n_n) \sum_{k,m=1}^n (\hat{R}^{-1})^{ij}_{km} x_k \cdot dx_m, \quad i, j = 1, \ldots, n \tag{15}
\]

or equivalently by

\[
d x_i \cdot x = \sum_{m=1}^n r(u^i_m, x(1)) x(2) (l^{-n}_n, x(3)) dx_m, \quad x \in \mathcal{X}. \tag{16}
\]

(ii) For the differentiation $d$ of the FODC $\Gamma^X$ of $\mathcal{X}$ we have

\[
dx = \alpha^{-1}(\theta_n x - x \theta_n), \quad x \in \mathcal{X}. \tag{17}
\]

**Proof.** (i): First we prove formula (15). Since $(X_r, u^n_n) = \delta_{kr}$, it follows from (3) and we have

\[
dx_i \equiv d u^i_n = \sum_{k,r} u^i_k X_r(u^n_n) \theta_r = \sum_r u^i_r \theta_r. \tag{18}
\]

Using (13), (18) and the fact that $\hat{R}^{-1}$ intertwines the tensor product corepresentation $u \otimes u$, we obtain

\[
d x_i \cdot x_j = \sum_k u^i_k \theta_k u^n_n = \sum_{k,m,s} c_- u^i_k u^j_m (\hat{R}^{-1})^{km}_{ns} \theta_s
\]

\[
= \sum_{k,m,s} c_- (\hat{R}^{-1})^{ij}_{km} u^i_k u^m_s \theta_s = \sum_{k,m} c_- (\hat{R}^{-1})^{ij}_{km} x_k \cdot dx_m;
\]

which proves (15). Formula (14) can be derived from (13) similarly as (14) was from (13).

From (15) combined with the Leibniz rule it follows that $\Gamma^X \equiv \mathcal{A} \cdot \mathcal{X} \cdot \mathcal{X}$ is equal to $\text{Lin}\{x \cdot dx_i; \ x \in \mathcal{X}, i = 1, \ldots, n\}$. Suppose that $\sum_i a_i dx_i = 0$ for certain elements $a_i \in \mathcal{X}$. Then we have $\sum_i a_i u^i_k \theta_k = 0$. Since $\{\theta_1, \ldots, \theta_n\}$ is a free left $\mathcal{A}$-module basis of $\Gamma^X$, the latter yields $\sum_i a_i u^i_k = 0$ for $k = 1, \ldots, n$ and hence $\sum_{i,k,j} a_i u^i_k S(u^j_s) = a_j = 0$ for all $j = 1, \ldots, n$. Thus, $\{dx_1, \ldots, dx_n\}$ is a free left $\mathcal{X}$-module basis of $\Gamma^X$.

(ii): By (10) and (11) we have $c_- (\hat{R}^{-1})_{nk}^{n_1} = (l^{-n_s}_s, u^n_s)(l^{-n}_n, u^n_n) = \delta_{ks}(l^{-n}_s l^{-n}_n, u^n_k) = \delta_{ks}\alpha$ and $c_- (\hat{R}^{-1})_{nk}^{n_1} = c_- (l^{-n_s}_s, u^n_k) = \delta_{kn}\epsilon_{s}^{n_1, n_1}$ for $s = 1, \ldots, n - 1$ and $k = 1, \ldots, n$. Inserting this into (13) and using (18) we obtain

\[
\theta_n x_j = \sum_{k=1}^{n-1} \alpha u^j_k \theta_k + c^2_- u^j_n \theta_n
\]

\[
= \sum_{k=1}^{n} \alpha u^j_k \theta_k + (c^2_- - \alpha) u^j_n \theta_n = \alpha dx_j + x_j \theta_n,
\]
which proves (17) in the case $x = x_j$. Since both sides of (17), considered as mappings of $\mathcal X$ to $\Gamma^X$, satisfy the Leibniz rule, (17) holds for all $x \in \mathcal X$.

Remarks: 1.) Since the left-invariant form $\theta_n \in \Gamma$ does not belong to the $\mathcal X$-bimodule $\Gamma^X$, formula (17) does not mean that the FODC $\Gamma^X$ is inner. It expresses rather the differentiation $d$ of $\Gamma^X$ by means of an extended bimodule in the sense of Woronowicz (see [W1]). But for the FODC $\Gamma^Z_1$ of the larger algebra $\mathcal Z$ considered in Section 4 the form $\theta_n$ is in $\Gamma^Z_1$ and makes $\Gamma^Z_1$ into an inner FODC (see Proposition 4(iii) below).

2.) If $\mathcal A$ is one of the coordinate Hopf algebras $\mathcal O(G_q), G_q = GL_q(N), SL_q(N), O_q(N), Sp_q(N)$, then the conditions (8) and (19) below can be assumed without loss of generality. This follows from the particular form of the universal R-matrix for the corresponding Drinfeld-Jimbo algebras (see, for instance, [KS], Theorem 8.17).

3 Quantum spaces generated by a row of $u^c$

Let $\mathcal Y$ be the subalgebra of $\mathcal A$ generated by the elements $y_i := (u^c)^i_n = S(u^n_i), i = 1, \ldots, n$, of the last row of the contragredient corepresentation $u^c$. Then $\mathcal Y$ is a left quantum space of $\mathcal A$ with left coaction $\varphi = \Delta |\mathcal Y$ given on the generators $y_i$ by

$$\varphi(y_i) \equiv \Delta(S(u^n_i)) = \sum_{j=1}^{n} S(u^j_i) \otimes y_j, \ i = 1, \ldots, n.$$

We shall proceed in a similar manner as in the preceding section. But the considerations are technically slightly more complicated, because we have to deal with square and inverse of the antipode of $\mathcal A$.

Let $\beta$ be a non-zero complex number and let $\mathcal T^\mathcal Y$ be the linear span of functionals

$$Y_i := \beta^{-1} S(l^{+i}_n)l^{-n}_n, i = 1, \ldots, n - 1, \text{ and } Y_n := \beta^{-1}((l^{-n}_n)^2 - \varepsilon).$$

We assume that

$$l^{+n}_m l^{-m}_n = 0 \text{ if } m < n \text{ and } S(l^{\pm n}_n) = l^{\mp n}_n. \quad (19)$$

Similarly as in the preceding section, we then get

$$\Delta(Y_i) - \varepsilon \otimes Y_i = \sum_{j=1}^{n} Y_j \otimes S(l^{+i}_j)l^{-n}_n, \ i = 1, \ldots, n \quad (20)$$

and $\mathcal T^\mathcal Y$ is the quantum tangent space of a left-covariant FODC $\Gamma$ on $\mathcal A$.

Let us suppose in addition that there are numbers $\gamma_i \neq 0, i = 1, \ldots, n$, such that

$$S^2(u^i_j) = \gamma_i u^i_j \gamma_j^{-1}, i, j = 1, \ldots, n, \quad (21)$$

and that

$$(l^{+i}_n, u^j_n) = 0 \text{ for } i \neq j, i, j = 1, \ldots, n, \quad (22)$$
\[ \beta := (l_{+i}^n l_{+i}^n, u_{+i}^n) = ((l_{+i}^n)^2, u_{+i}^n) - 1 \neq 0 \text{ for } i = 1, \ldots, n - 1. \]  

We set \( c := (l_{+i}^n, u_{+i}^n) \) and \( \eta_i := \omega(S^{-1}(u_{+i}^n)) = \sum_j u_j^i dS^{-1}(u_{+i}^n) \) for \( i = 1, \ldots, n \). Since \( S(l_{+i}^n) = l_{+i}^n \) by (19), we have \( c_- = (l_{-i}^n, u_{+i}^n) = c^{-1} \) and \( \beta = c^2 - 1 \). It is straightforward to check that (19), (22) and (23) imply that

\[ (Y_j, \eta_i) = (Y_j, S^{-1}(u_{+i}^n)) = \delta_{ij} \text{ for } i, j = 1, \ldots, n. \]

Therefore, the FODC \( \Gamma^Y \) is \( n \)-dimensional. From (20), (19) and (21) we get

\[
\eta_r y_j = \sum_{k, m, s} S(u_j^k)(S(l_{+r}^s), S(u_k^m))S(l_{-r}^n, S(u_m^n))Y_{rs} = \sum_{k, s} c\gamma_n\gamma_k^{-1} \hat{R}_{kr}^n S(u_j^k) \eta_s.
\]

for \( j, r = 1, \ldots, n \). The first equality combined with the formulas \( (S(l_{+r}^s), \cdot) = r(S(\cdot), u_{+i}^s) = \bar{r}(\cdot, u_{+i}^s) \) leads to the following form of the commutation relations

\[ \eta_r y = \sum_{s=1}^n y(1) \bar{r}(y(2), u_{+i}^s)(l_{-r}^n, y(3)) \eta_s, \quad y \in Y. \]

Let \( \Gamma^Y := Y \cdot dY \cdot Y \) be the FODC on \( Y \) induced by the FODC \( \Gamma \) on \( A \).

**Proposition 3.** (i) \( \Gamma^Y \) is a left-covariant FODC on \( Y \) with the free left \( Y \)-module basis \( \{dy_1, \ldots, dy_n\} \) and with \( Y \)-bimodule structure given by the relations

\[ dy_i \cdot y_j = (l_{+i}^n, u_{+i}^n) \sum_{k, m=1}^n \hat{R}_{ji}^{mk} y_k dy_m, \quad i, j = 1, \ldots, n, \]

or equivalently by

\[ dy_i \cdot y = \sum_{m=1}^n \bar{r}(y(1), u_{+i}^m) y(2)(l_{-r}^n, y(3))dy_m, \quad y \in Y. \]

(ii) For any \( y \in Y \) we have \( dy = \beta^{-1}(\eta_0 y - y \eta_0) \).

**Proof.** (i): It suffices to prove formula (26). First we note that (24) and (21) imply that

\[
dy_i = \sum_{k, r} S(u_i^k)(Y_r, S(u_k^r)) \eta_r = \sum_{k, r} \gamma_n \gamma_r^{-1} S(u_i^k)(Y_r, S^{-1}(u_k^r)) \eta_r
\]

\[ = \sum_r \gamma_n \gamma_r^{-1} S(u_i^r) \eta_r. \]

If \( r \) is a universal \( r \)-form of \( A \), then so is \( \bar{r}_{21} \) and we have \( \bar{r}(a, S(b)) = r(a, b) \) and \( r(S(a), b) = \bar{r}(a, b) \), where \( \bar{r}_{21}(a, b) := \bar{r}(b, a) \) and \( a, b \in A \) (see, for instance, [KS]).
Using these facts and the formulas (2) applied to $\bar{r}_{21}$, (23), (22) and (21), we compute

\[
d_{y_j} y_j = \sum_{r} \gamma_n \gamma_r^{-1} S(u^r_i) \eta_r y_j \\
= \sum_{k,m,r,s} \gamma_n \gamma_r^{-1} S(u^r_j) S(u^k_j) (l^{+1+}_n, u^n_n)(l^{+s}_r, S^2(u^n_n)) u^m_s d S^{-1}(u^n_m) \\
= \sum_{k,m} c \gamma_n \gamma_r^{-1} S(u^r_j) \left( \sum_{k,s} S(u^k_j) u^m_s \bar{r}_{21}(u^s_r, S(u^n_k)) \right) d S^{-1}(u^n_m) \\
= \sum_{k,m} c \gamma_n \gamma_i^{-1} S^{-1}(u^i_j) \left( \sum_{k,s} u^i_s S(u^n_k) \bar{r}_{21}(u^m_s, S(u^i_k)) \right) d S^{-1}(u^n_m) \\
= \sum_{k,m} c S(u^n_k) \bar{r}(S(u^i_j), \gamma_m u^m_i \gamma_i^{-1}) d S^{-1}(\gamma_n u^n_m \gamma_i^{-1}) \\
= \sum_{k,m} c S(u^n_k) \bar{r}(S(u^i_j), S^2(u^i_i)) d S(u^n_m) \\
= \sum_{k,m} c \hat{R}_{ji}^{mk} y_k \cdot d y_m.
\]

(ii): Since $c \hat{R}_{kn} = (l^{+1+}_n, u^n_n)(l^{+s}_n, u^n_n) = \delta_{kn} \delta_{l^{+s}_n} \delta_{l^{+1+}_n}$ and $c \hat{R}_{kn} = \delta_{kn} c^2$ by (22) and (23) for $s = 1, \ldots, n - 1$ and $k = 1, \ldots, n$, it follows from (23) and (28) that

\[
\eta_n y_j = \sum_{k=1}^{n-1} \gamma_n \gamma_k^{-1} \beta S(u^k_j) \eta_k + c^2 S(u^n_j) \eta_n \\
= \sum_{k=1}^{n} \gamma_n \gamma_k^{-1} \beta S(u^k_j) \eta_k + (c^2 - \beta) S(u^n_j) \eta_n = \beta d y_j + y_j \eta_n
\]

which implies the assertion. \hfill \square

4 Quantum spaces generated by a row of $u$ and of $u^c$

Let $Z$ denote the subalgebra of $A$ generated by the elements $x_i = u_i^n$ and $y_i = S(u^n_i)$, $i = 1, \ldots, n$. That is, $Z$ is the subalgebra of $A$ generated by the algebras $X$ and $Y$. Our aim in this section is to construct four classes $\Gamma^Z_j$, $j = 1, 2, 3, 4$, of left-covariant FODC of $Z$.

First let us fix some notations and assumptions which will be kept in force throughout the whole section. Let $Z_n$ be a fixed group-like element of $A^o$, that is, $Z_n(1) = 1$ and $\Delta(Z_n) = Z_n \otimes Z_n$. Then $Z_n$ is invertible in $A^o$ with inverse $Z_n^{-1} = S(Z_n)$. We retain the assumptions (10), (19), (21) and (22). In addition we suppose that

\[
(S(l^{+1+}_n), u^n_i) = (l^{-1+}_i, S^{-1}(u^n_j)) = 0 \text{ for } (i, j) \neq (n, n), i, j = 1, \ldots, n, \quad (29)
\]

\[
\gamma := (l^{-n}_n, l^{+n}_n, u^n_i) \neq 0 \text{ and } \zeta := (l^{-n}_n, l^{+n}_i, u^n_i) \neq 0 \text{ are independent of } i = 1, \ldots, n - 1, \quad (30)
\]
\( (l^+_n, u^j_n) = (l^-_n, u^j_n) = (Z_n, u^j_n) = (Z^{-1}_n, u^j_n) = 0 \) if \( i \neq n \),
\[
\delta := (Z_n, u^j_n) \neq 1.
\] (32)

Clearly, we then have \( \delta^{-1} = (Z^{-1}_n, v^n_n) \).

We now begin with the construction of the FODC \( \Gamma^Z_1 \). Let \( \mathcal{T}^Z_1 \) denote the linear span of functionals
\[
X_i := \gamma^{-1}\delta^{-1} l^{-n}_i l^{+n}_i Z_n \text{ and } Y_i := \zeta^{-1}\delta S(l^{+i}_n) l^{-n}_i Z_n, \ i = 1, \ldots, n - 1,
\]
\[
X_n := (\delta - 1)^{-1} (Z_n - \varepsilon) \text{ and } Y_n := -\delta X_n = (\delta^{-1} - 1)^{-1} (Z_n - \varepsilon).
\]

For \( i = 1, \ldots, n - 1 \), we have
\[
\Delta(X_i) - \varepsilon \otimes X_i = \sum_{j=1}^{n-1} X_j \otimes l^{-j}_n l^{+n}_i Z_n + X_n \otimes (\delta - 1)X_i,
\] (33)
\[
\Delta(Y_i) - \varepsilon \otimes Y_i = \sum_{j=1}^{n-1} Y_j \otimes S(l^{+j}_n) l^{-n}_i Z_n + Y_n \otimes (\delta^{-1} - 1)Y_i,
\] (34)
\[
\Delta(X_n) - \varepsilon \otimes X_n = X_n \otimes Z_n, \quad \Delta(Y_n) - \varepsilon \otimes Y_n = Y_n \otimes Z_n.
\] (35)

Therefore, by Lemma 1, there exists a left-covariant FODC \( \Gamma^Z_1 \) on \( \mathcal{A} \) with quantum tangent space \( \mathcal{T}^Z_1 \). As in Sections 2 and 3, we set
\[
\theta_j := \omega(u^j_n) \text{ and } \eta_j := \omega(S^{-1}(u^j_n)), \ j = 1, \ldots, n,
\]
for the FODC \( \Gamma^Z \). From the assumptions \((\text{i}), (\text{ii}), (\text{iii}), (\text{iv})\) and the definition of the functionals \( X_i, Y_i \) we immediately derive
\[
(X_r, u^s_n) = (Y_r, S^{-1}(u^s_n)) = \delta_{rs} \quad \text{and} \quad (X_i, S^{-1}(u^s_n)) = (Y_i, u^s_n) = 0
\] (36)
and so
\[
(X_r, \theta_s) = (Y_r, \eta_s) = \delta_{rs} \quad \text{and} \quad (X_i, \eta_j) = (Y_i, \theta_j) = 0
\]
for all \( i, j, r, s = 1, \ldots, n \) such that \( (i, j) \neq (n, n) \). That is, \( \{\theta_1, \ldots, \theta_n, \eta_1, \ldots, \eta_{n-1}\} \) and \( \{X_1, \ldots, X_n, Y_1, \ldots, Y_{n-1}\} \) and likewise \( \{\theta_1, \ldots, \theta_{n-1}, \eta_1, \ldots, \eta_n\} \) and \( \{X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_n\} \) are dual bases of \( \text{inv} \Gamma_1 \) and \( \mathcal{T}^Z_1 \), respectively. In particular, we see that the FODC \( \Gamma_1 \) has the dimension \( \dim \mathcal{T}^Z_1 = 2n-1 \). Moreover, the latter facts imply that formula \((\text{III})\) and \((\text{IV})\) hold for the differentiation \( d \) of the FODC \( \Gamma_1 \) as well. Further, from the formulas \((\text{I})\) and \((\text{III}) - (\text{IV})\) we obtain the following commutation relations between the basis elements of \( \text{inv} \Gamma_1 \) and elements \( a \in \mathcal{A} \):
\[
\theta_i a = \sum_{s=1}^{n-1} a_{(1)}(l^{-s}_n l^{+n}_i Z_n, a_{(2)}) \theta_s, \quad \eta_i a = \sum_{s=1}^{n-1} a_{(1)}(S(l^{+s}_n) l^{-n}_i Z_n, a_{(2)}) \eta_s,
\] (37)
\[
\theta_n a = a_{(1)}(Z_n, a_{(2)}) \theta_n + (\delta - 1) \sum_{s=1}^{u-1} a_{(1)}((X_s, a_{(2)}) \theta_s + (Y_s, a_{(2)}) \eta_s),
\] (38)
\[
\eta_n a = a_{(1)}(Z_n, a_{(2)}) \eta_n + (\delta^{-1} - 1) \sum_{s=1}^{n-1} a_{(1)}((X_s, a_{(2)}) \theta_s + (Y_s, a_{(2)}) \eta_s).
\] (39)
for \( r = 1, \ldots, n - 1 \).

Let \( \Gamma^Z_1 \) denote the FODC of \( Z \) which induced by the FODC \( \Gamma_1 \) of \( A \).

**Proposition 4.** (i) For the \( Z \)-bimodule \( \Gamma^Z_1 \) we have the commutation relations

\[
\begin{align*}
dx_i \cdot x_j &= c\delta \sum_{k,m=1}^{n} (\hat{R}^{-1})^{ij}_{km} x_k \cdot dx_m + (\delta - \gamma \delta - 1)(x_i \cdot dx_j - x_i x_j \theta_n), \\
y_i \cdot y_j &= c^{-1}\delta^{-1} \sum_{k,m=1}^{n} \hat{R}^{nk}_{mj} y_k \cdot dy_m + (\delta - \zeta \delta - 1)(y_i \cdot dy_j - y_i y_j \eta_n), \\
x_i \cdot y_j &= c^{-1}\delta^{-1} \sum_{k,m=1}^{n} \hat{R}^{ki}_{mj} y_k \cdot dx_m + (\delta - 1)(x_i \cdot dy_j - x_i y_j \eta_n), \\
y_i \cdot x_j &= c\delta \sum_{k,m=1}^{n} (\hat{R}^{-1})^{ij}_{km} x_k \cdot dy_m + (\delta - 1)(y_i \cdot dx_j - y_i x_j \theta_n),
\end{align*}
\]

where \((\hat{R}^{-1})^{ij}_{km} := \mathfrak{r}(u^i_k, S^2(u^m_j))\), \( i, j, k, m = 1, \ldots, n \).

(ii) The set \( \{dx_1, \ldots, dx_n, dy_1, \ldots, dy_n\} \) generates \( \Gamma^Z_1 \) as a left \( Z \)-module. For arbitrary elements \( a_1, \ldots, a_n, b_1, \ldots, b_n \in Z \), the relation

\[
\sum_{i=1}^{n} (a_i dx_i + b_i dy_i) = 0
\]

is equivalent to the following set of equations:

\[
a_j = \left( \sum_{i=1}^{n} a_i x_i \right) y_j, \quad b_j = \left( \sum_{i=1}^{n} b_i x_i \right) x_j \gamma_j \gamma_n^{-1} \text{ for } j = 1, \ldots, n, \\
\sum_{i=1}^{n} a_i x_i = (l^{+n}_n, u^n_n) \sum_{i=1}^{n} b_i y_i.
\]

(iii) \( \Gamma^Z_1 \) is an inner FODC of \( Z \) with respect to the left-invariant one-form \( \theta_n = -\delta \eta_n \), that is, we have

\[
dz = (\delta - 1)^{-1} (\theta_n z - z \theta_n) \text{ for } z \in Z.
\]

**Proof.** (i): We carry out the proofs of the second and the fourth relations and work with the dual bases \( \{\theta_1, \ldots, \theta_{n-1}, \eta_1, \ldots, \eta_n\} \) and \( \{X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_n\} \). The two other relations follow by a similar slightly simpler reasoning. Using the formulas (33), (37), (39) and the above assumptions we compute

\[
\begin{align*}
dy_i \cdot y_j &= \sum_{r} \gamma_n \gamma_r^{-1} S(u^r_i) \eta_r S(u^j_n) \\
&= \sum_{r=1}^{n-1} \sum_{k,s=1}^{n} \gamma_n \gamma_r^{-1} S(u^r_i) S(u^j_k) S(l^{+s}_r) l^{+s}_n Z_n, S(u^j_k)) \eta_s
\end{align*}
\]
The first sum is treated as in the proof of Proposition 3. In this manner it becomes equal to 

\[ c^{-1}\delta^{-1} \sum_{k,m} R_{ji}^{mk} y_k dy_n. \]

Put \( \tilde{\zeta} := \zeta \delta^{-1} + 1 - \delta^{-1} \). Since \( Y_s, S(u^n_k) = \gamma_n \gamma_k^{-1}(Y_s, S^{-1}(u^n_k)) = \gamma_n \gamma_k^{-1}\delta_{ks} \) by (34) and \( \gamma_n \gamma_k^{-1}\eta_k = \gamma_n \gamma_k^{-1}\omega(S^{-1}(u^n_k)) = \omega(S(u^n_k)) = \sum_r S^2(u^n_k) dS(u^n_r) \), the second expression yields

\[ \sum_{k=1}^{n-1} -\tilde{\zeta} S(u^n_i) S(u^n_j) \gamma_n \gamma_k^{-1}\eta_k \]

\[ = \tilde{\zeta} S(u^n_i) S(u^n_n) \eta_n - \sum_{k,r=1}^{n-1} \tilde{\zeta} S(u^n_i) S(u^n_j) S^2(u^n_k) dS(u^n_r) \]

\[ = \tilde{\zeta} y_i y_j \eta_n - \tilde{\zeta} y_i \cdot dy_i. \]

Putting both terms together we obtain the second relation. In order to prove the fourth relation we proceed in a similar manner. Using the facts that \( (S(l^{-1}_n)l^{+n}_n Z_n, u^n_k) = \zeta \delta^{-1}(Y_s, u^n_k) = 0 \) and \( (X_s, u^n_k) = \delta_{ks} \) for \( s = 1, \ldots, n - 1 \), we obtain

\[ dy_i \cdot x_j = \sum_{r=1}^{n-1} \sum_{k,s=1}^{n} \gamma_n \gamma_r^{-1} S(u^n_r) u^n_k (S(l^{+s}_n) l^{+n}_n Z_n, u^n_k) \eta_s \]

\[ + \sum_{s=1}^{n-1} \sum_{k=1}^{n} S(u^n_i) u^n_k (\delta^{-1} - 1)((X_s, u^n_k) \theta_s + (Y_s, u^n_k) \eta_s) \]

\[ + \sum_{k=1}^{n} S(u^n_i) u^n_k (Z_n, u^n_n) \eta_n \]

\[ = \sum_{k,m,r,s=1}^{n-1} c\delta \gamma_n \gamma_r^{-1} S(u^n_r) u^n_k u^n_m (\hat{R}^{-1})_{rm}^k dS^{-1}(u^n_m) \]

\[ + \sum_{k=1}^{n-1} (\delta^{-1} - 1) S(u^n_i) u^n_k \theta_k \]

\[ = \sum_{k,m,r,s=1}^{n-1} c\delta \gamma_n \gamma_r^{-1} S(u^n_r) u^n_k u^n_m (\hat{R}^{-1})_{rm}^k dS^{-1}(u^n_m) \]
(ii): Since \( \theta_n = \sum_i y_i dx_i \) and \( \eta_n = \sum_i \gamma_n \gamma_i^{-1} x_i dy_i \), the four relations in (i) imply that the set \( \{ dx_1, \ldots, dx_n, dy_1, \ldots, dy_n \} \) generates \( \Gamma_1^Z \) as a left \( \mathcal{Z} \)-module. It remains to verify that (10) is equivalent to (11) and (12). Since \( X_n = -\delta Y_n \), the element \( S^{-1}(u_n^* + \delta u_n^*) \) annihilated by the whole quantum tangent space \( T_1^Z \) and hence 0 = \( \omega(S^{-1}(u_n^*) + \delta u_n^*) \) = \( \eta_n + \delta \theta_n \). Inserting the relations \( \eta_n = -\delta \theta_n \), (18) and (28) into (40) we see that (40) reads as

\[
\sum_{i=1}^{n} \left( \sum_{r=1}^{n-1} (a_i u_r^* \theta_r + b_i \gamma_n \gamma_r^{-1} S(u_r^*) \eta_r) + (a_i u_r^* - c^2 b_i S(u_r^*)) \theta_r \right) = 0. 
\]

Since the set \( \{ \theta_1, \ldots, \theta_n, \eta_1, \ldots, \eta_{n-1} \} \) is a free left \( \mathcal{A} \)-module basis of \( \Gamma_1^Z \), the latter is equivalent to the relations

\[
\sum_i a_i u_i^* = \sum_i b_i S(u_i^*) = 0 \quad \text{for} \quad r = 1, \ldots, n-1, \\
\sum_i (a_i u_i^* - \delta b_i S(u_i^*)) = 0. 
\]

Multiplying \( \sum_i a_i u_i^* \) by \( S(u_r^*) \) and \( \sum_i b_i S(u_r^*) \) by \( S^2(u_r^*) = \gamma_k \gamma_i^{-1} u_r^* \) and summing over \( r \), (14) implies (11). Formula (13) is nothing but (12). Using the relations \( \sum_i y_i x_i = \sum_i x_i y_i \gamma_i \gamma_i^{-1} = 1 \), equation (11) in turn implies (14).

(iii): It suffices to prove (14) for the generators \( z = x_i, y_i \). Because \( (Y_s, u_n^*) = 0 \) and \( (X_s, u_n^*) = \delta_{js} \) for \( s = 1, \ldots, n-1 \), it follows from (38) and (18) that

\[
\theta_n x_i = u_i^n(Z_n, u_n^n) \theta_n + (\delta - 1) \sum_{s=1}^{n-1} u_s^i \theta_s \\
= u_i^n(\delta - (\delta - 1)) \theta_n + (\delta - 1) \sum_{s=1}^{n} u_s^i \theta_s \\
= x_i \theta_n + (\delta - 1) dx_i, 
\]

which gives (13) in the case \( z = x_i \). Similarly, using the formulas (12) and (28) we get \( dy_i = (\delta - 1)^{-1}(\eta_i y_i - Y_i \eta_n) \), so that \( dy_i = (\delta - 1)^{-1}(\theta_n y_i - y_i \theta_n) \).

\[ \square \]

Next we turn to the FODC \( \Gamma_4^Z \) on \( \mathcal{Z} \). We take the linear span \( T_4^Z \) of functionals

\[
X_i := \gamma^{-1} l^{-n+1} l_{-i}^{n+1} \text{ and } Y_i := \zeta^{-1} l(l^{i+1}) l_{-i}^{n+1}, \quad i = 1, \ldots, n-1,
\]

\[
X_n = (\delta - 1)^{-1}(Z_n - \varepsilon) \text{ and } Y_n = (\delta - 1)^{-1}(Z_n - \varepsilon).
\]
For $i = 1, \ldots, n - 1$, we then have

$$\Delta(X_i) - \varepsilon \otimes X_i = \sum_{j=1}^{n-1} X_j \otimes l^{-j} l^{+n}_n, \quad \Delta(X_n) - \varepsilon \otimes X_n = X_n \otimes Z_n,$$

$$\Delta(Y_i) - \varepsilon \otimes Y_i = \sum_{j=1}^{n-1} Y_j \otimes S(l^{-j}) l^{+n}_n, \quad \Delta(Y_n) - \varepsilon \otimes Y_n = Y_n \otimes Z_n.$$  \hfill (46)

These formulas and the relations $(X_i, u^a_j) = (Y_i, S^{-1}(u^a_j)) = \delta_{ij}$ and $(X_i, S(u^a_k)) = (Y_i, u^k_i) = 0$ for $i, j = 1, \ldots, n$ and $k = 1, \ldots, n - 1$ imply that $T^Z_4$ is the quantum tangent space of a $(2n-1)$-dimensional left-covariant FODC $\Gamma_4$ of $A$. The commutation relations of this FODC between the one-forms $\theta_r, \eta_s$ and elements of $A$ are

$$\theta_r a = \sum_{s=1}^{n-1} a_{(1)}(l^{-r} l^{+n}_n, a_{(2)} \theta_s), \quad \eta_r a = \sum_{s=1}^{n-1} a_{(1)}((S(l^{+s}) l^{+n}_n, a_{(2)} \eta_s),$$

$$\theta_r a = a_{(1)}(Z_n, a_{(2)}) \theta_n, \quad \eta_r a = a_{(1)}(Z_n, a_{(2)}) \eta_n$$

for $r, s = 1, \ldots, n - 1$. Let $\Gamma^Z_4$ denote the FODC of $Z$ which is induced by the FODC $\Gamma_4$ of $A$. By similar computations as carried out above one prove the following cross commutation relations of the $Z$-bimodule $\Gamma^Z_4$:

$$dx_i \cdot x_j = c \sum_{k, m=1}^{\infty} (\hat{R}^{-1})^{ij}_{km} x_k \cdot dx_m - \gamma x_i \cdot dx_j + \gamma x_i x_j \theta_n,$$

$$dy_i \cdot y_j = c^{-1} \sum_{k, m=1}^{\infty} \hat{R}^{nk}_{ji} y_k \cdot dy_m - \zeta y_i \cdot dy_j + \zeta y_i y_j \eta_n,$$

$$dx_i \cdot y_j = c^{-1} \sum_{k, m=1}^{\infty} \hat{R}^{ki}_{mj} y_k \cdot dx_m,$$

$$dy_i \cdot x_j = c \sum_{k, m=1}^{\infty} (\hat{R}^{-1})^{ij}_{km} x_k \cdot dy_m.$$

These are precisely the relations which are obtained by setting formally $\delta = 1$ in the commutation relations for the FODC $\Gamma^Z_1$ (see Proposition 4(i)). That is, the FODC $\Gamma^Z_4$ can be viewed as the limit of the FODC $\Gamma^Z_1$ as $\delta \to 1$. Note that the FODC $\Gamma^Z_1$ has no direct meaning in the case $\delta = 1$.

By "mixing" the elements of the quantum tangent spaces of the FODC $\Gamma^Z_1$ and $\Gamma^Z_4$ one obtains two other FODC $\Gamma^Z_1$ and $\Gamma^Z_4$ on $Z$. We briefly describe the quantum Lie algebras of the corresponding FODC $\Gamma_2$ and $\Gamma_3$ of $A$ and the commutation rules of these calculi. Let $T^Z_2$ be the linear span of functionals

$$X_i := \gamma^{-1} \delta^{-1} l^{-i} l^{+n}_n Z_n \text{ and } Y_i := \zeta^{-1} S(l^{+i}) l^{+n}_n, \quad i = 1, \ldots, n - 1,$$

$$X_n := (\delta - 1)^{-1} (Z_n - \varepsilon) \text{ and } Y_n := -\delta X_n = (\delta^{-1} - 1)^{-1} (Z_n - \varepsilon).$$
From the formulas (33), (34), (46) and (47) we see that these formulas we also read off the following commutation relations between the left-invariant one-forms $\theta_i, \eta_k$ and elements $a \in A$:

$$
\begin{align*}
\Gamma_2 : \quad & \theta_r a = \sum_{s=1}^{n-1} a_{(1)}(l^{-r}_s l^n Z_n, a_{(2)}) \theta_s, \quad \eta_r a = \sum_{s=1}^{n-1} a_{(1)}(S l^+_r l^n Z_n, a_{(2)}) \eta_s, \\
\theta_s a &= a_{(1)}(Z_n, a_{(2)}) \theta_s + (\delta - 1) \sum_{s=1}^{n-1} a_{(1)}(X_s, a_{(2)}) \theta_s, \\
\eta_s a &= a_{(1)}(Z_n, a_{(2)}) \eta_s + (\delta^{-1} - 1) \sum_{s=1}^{n-1} a_{(1)}(X_s, a_{(2)}) \eta_s, \\
\Gamma_3 : \quad & \theta_r a = \sum_{s=1}^{n-1} a_{(1)}(l^{-r}_s l^n Z_n, a_{(2)}) \theta_s, \quad \eta_r a = \sum_{s=1}^{n-1} a_{(1)}(S l^+_r l^n Z_n, a_{(2)}) \eta_s, \\
\theta_s a &= a_{(1)}(Z_n, a_{(2)}) \theta_s + (\delta - 1) \sum_{s=1}^{n-1} a_{(1)}(Y_s, a_{(2)}) \eta_s, \\
\eta_s a &= a_{(1)}(Z_n, a_{(2)}) \eta_s + (\delta^{-1} - 1) \sum_{s=1}^{n-1} a_{(1)}(Y_s, a_{(2)}) \eta_s,
\end{align*}
$$

where $r = 1, \ldots, n - 1$. As earlier, the FODC on $\mathcal{Z}$ induced by the FODC $\Gamma_j$ on $A$ is denoted by $\Gamma^Z_j$, $j = 2, 3$. From the preceding set of formulas one gets the following commutation rules for the $\mathcal{Z}$-bimodule $\Gamma^Z_j$:

$$
\begin{align*}
\Gamma^Z_2 : \quad & dx_i \cdot x_j = c \delta \sum_{k,m=1}^{n} (\hat{R}^{-1}_{k}^m)^i_j x_k \cdot dx_m + (\delta - \gamma \delta - 1)(x_i \cdot dx_j - x_i x_j \theta_n), \\
& \delta y_i \cdot y_j = c^{-1} \sum_{k,m=1}^{n} \hat{R}^m_{ji} y_k \cdot dy_m - \zeta y_i \cdot dy_j + \zeta y_i y_j \eta_n, \\
& \delta x_i \cdot y_j = c^{-1} \delta^{-1} \sum_{k,m=1}^{n} \hat{R}^k_{mj} y_k \cdot dx_m, \\
& \delta y_i \cdot x_j = c \sum_{k,m=1}^{n} (\hat{R}^{-})^{ij}_{km} x_k \cdot dy_m + (\delta^{-1} - 1)y_i \cdot dx_j \\
\Gamma^Z_3 : \quad & dx_i \cdot x_j = c \sum_{k,m=1}^{n} (\hat{R}^{-1}_{k}^m)^i_j x_k \cdot dx_m - \gamma x_i \cdot dx_j + \gamma x_i x_j \theta_n,
\end{align*}
$$

and $\mathcal{T}^Z_3$ the span of functionals

$$
X_i := \gamma^{-1} l^{-n}_i l^n Z_n \quad \text{and} \quad Y_i := \zeta^{-1} \delta S l^{+n} Z_n, \quad i = 1, \ldots, n - 1, \\
X_n := (\delta - 1)^{-1} (Z_n - \varepsilon) \quad \text{and} \quad Y_n := -\delta X_n = (\delta^{-1} - 1)^{-1} (Z_n - \varepsilon).
$$
\[
\begin{aligned}
\mathbf{dy}_i \wedge \mathbf{y}_j &= c^{-1} \delta^{-1} \sum_{k,m=1}^n \hat{R}_{k;i}^{mk} \mathbf{y}_k \wedge \mathbf{d}y_m + (\delta^{-1} - \zeta \delta^{-1} - 1)(\mathbf{y}_i \wedge \mathbf{d}y_j - \mathbf{y}_j \wedge \mathbf{d}y_i) + \mathbf{y}_j \mathbf{d}y_i - \mathbf{y}_i \mathbf{d}y_j, \\
\mathbf{dx}_i \wedge \mathbf{y}_j &= c^{-1} \sum_{k,m=1}^n \hat{R}_{m;j}^{ki} \mathbf{y}_k \wedge \mathbf{d}x_m + (\delta - 1) \mathbf{x}_i \wedge \mathbf{d}y_j, \\
\mathbf{dy}_i \wedge \mathbf{x}_j &= c\delta \sum_{k,m=1}^n (\hat{R}^{-1})_{km}^{-ij} \mathbf{y}_k \wedge \mathbf{d}y_m.
\end{aligned}
\]

Recall that by Proposition 4(iii) the FODC \( \Gamma^Z_1 \) of \( Z \) is inner. It turns out that none of the three other FODC \( \Gamma^Z_2, \Gamma^Z_3, \Gamma^Z_4 \) is inner. Indeed, from the above commutation rules one easily derives that

\[\Gamma^Z_2: \quad \theta_n y_i = \delta^{-1} y_i \theta_n, \quad \Gamma^Z_3: \quad \theta_n x_i = x_i \theta_n, \quad \Gamma^Z_4: \quad \theta_n y_i = y_i \theta_n \quad (48)\]

for all \( i = 1, \ldots, n \). Further, for all four FODC \( \Gamma^Z_j \) we have \( \theta_n = -\delta \eta_n \) and this is up to complex multiples the only left-invariant one-form of \( \Gamma^Z_j \). Therefore, we conclude at once from (48) that none of the FODC \( \Gamma^Z_j, j = 2, 3, 4 \), of \( Z \) is inner.

All four left-covariant FODC \( \Gamma^Z_j \) of \( Z \) depend on the group-like element \( Z_n \in A^0 \). It can be freely chosen such that it satisfies the conditions (31) and (32). This dependence is reflected by the appearance of the parameter \( \delta = (Z_n, u_n) \) in the above formulas. For the FODC \( \Gamma^Z_1 \) a distinguished choice of \( Z_n \) is \( Z_n = (l^{-1} n)^2 \). In this case \( \mathcal{T}^Z_1 \) is just the sum of the quantum tangent spaces \( \mathcal{T}^X \) and \( \mathcal{T}^Y \) considered in Sections 2 and 3 and the FODC \( \Gamma^Z_1 \) might be thought as gluing together the FODC \( \Gamma^X \) and \( \Gamma^Y \). Further, if we assume in addition the conditions (11) and (23), then we have \( \alpha = \gamma c = \delta - 1, \beta = \gamma c^2 = \delta^{-1} - 1, \) and \( \delta = c^{-2} \), so that \( \gamma \delta + 1 - \delta = \zeta \delta^{-1} + 1 - \delta^{-1} = 0 \). Thus, in this case the first two relations for the FODC \( \Gamma^Z_1 \) in Proposition 4(i) become even linear.

5. Application to the quantum homogeneous space \( \text{GL}_q(N) / \text{GL}_q(N - 1) \)

In this section let \( A \) denote the Hopf algebra \( \mathcal{O}(\text{GL}_q(N)) \), \( u = (u^i_j)_{i,j=1,\ldots,N} \) the fundamental corepresentation of \( A \) and \( \hat{R} \) the corresponding \( R \)-matrix given by (see [FRT])

\[\hat{R}^{ij}_{kl} \equiv \hat{R}^{ij}_{k; l} \equiv q^{\delta_{ij} \delta_{kl}} + (q - q^{-1}) \theta(j - i) \delta_{ik} \delta_{jl}, \quad i,j,k,l = 1, \ldots, N. \quad (49)\]

The Hopf algebra \( A \) is coquasitriangular with universal \( r \)-form \( \mathbf{r} \) determined by

\[\mathbf{r}(u_j^i, u_l^k) = \hat{R}^{ki}_{jl}, \quad i,j,k,l = 1, \ldots, N. \quad (50)\]

Further, We suppose that \( Z_n \) is a monomial in the main diagonal \( L \)-functionals \( l^{+i}_j \).

Using (11) and (23) one easily verifies that the above assumptions (8), (11), (14), (19), (21), (22), (23), (29), (30), (31) and (32) are then fulfilled with \( n = N, \alpha = \ldots \).
\(-\zeta = q^{-2} - 1, \beta = -\gamma = q^2 - 1, c = q \) and \(\gamma_i = q^{2i} \). Therefore, all results obtained in the preceding three sections are valid in this case. Here we shall add only a few remarks concerning these results rather than restating them in the present situation. The quantum homogeneous space \(X\) is then, of course, isomorphic to the quantum vector space \(O(C^N_q)\) ([KS], Proposition 9.11) and the FODC \(\Gamma^X\) is one of the two well-known covariant calculi on \(O(C^N_q)\) discovered in [PW] and [WZ]. However, the approach given in Section 2 might be still of interest. The FODC \(\Gamma^Z_{\cdot j}, j = 1,2,3,4\), developed in the preceding section are left-covariant FODC on the subalgebra \(Z\) of \(A\) generated by the element \(x_i \equiv u^i_N\) and \(y_i \equiv S(u^i_N), i = 1,\ldots,N\). All four FODC have the property that \(\Gamma^Z_{\cdot j}\) as a left \(Z\)-module is generated by the differentials \(dx_1,\ldots, dx_N, dy_1,\ldots, dy_N\). The FODC \(\Gamma^Z_1\) described by Proposition 4 is inner. In the special case \(Z_n = (l^n-1)^2\) it coincides with the distinguished calculus considered in [We] (more precisely with its left-covariant counter-part).

The importance of the left quantum space \(Z\) stems from the fact that it is (isomorphic to) the quantum homogeneous space \(GL_q(N)/GL_q(N-1)\). Indeed, there is a unique surjective Hopf algebra homomorphism \(\pi : GL_q(N) \to GL_q(N-1)\) such that

\[
\pi(u^j_i) = w^j_i, \quad i, j = 1,\ldots,N-1, \\
\pi(u^N_i) = \pi(u^N_j) = 0, \quad i = 1,\ldots,N-1, \quad \pi(u^N_N) = 1,
\]

where \(w^j_i, i, j = 1,\ldots,N-1\), denote the matrix entries of the fundamental matrix for the quantum group \(GL_q(N-1)\). Then the set

\[
O(GL_q(N)/GL_q(N-1)) := \{ a \in O(GL_q(N)) : (id \otimes \pi) \circ \Delta(a) = a \otimes 1 \}
\]

of all right \(GL_q(N-1)\)-invariant elements of \(O(GL_q(N))\) is a subalgebra and a left quantum space for \(O(GL_q(N))\) with respect to the coaction \(\Delta[O(GL_q(N))].\) Obviously, the elements \(x_i\) and \(y_i\) are in \(O(GL_q(N)/GL_q(N-1))\), so that \(Z \subseteq O(GL_q(N)/GL_q(N-1))\). If \(q\) is not a root of unity, then we have the equality \(Z = O(GL_q(N)/GL_q(N-1))\). (For the corresponding right quantum space \(GL_q(N-1)\backslash GL_q(N)\) this is proved in [NYM], Proposition 4.4, or [KS], Section 14.6. The proof for the left quantum space \(GL_q(N)/GL_q(N-1)\) is completely analogous.)

Suppose now that \(q\) is a real number and \(q \neq 0,\pm 1\). Then it is well-known that the Hopf algebra \(O(GL_q(N))\) is a Hopf *-algebra, denoted by \(O(U_q(N))\), with involution determined by \((u^i_j)^* = S(u^j_i)\), \(i, j = 1,\ldots,N\). Further, the algebra \(O(GL_q(N)/GL_q(N-1))\) is a *-subalgebra such that \(x^i_j \equiv (u^i_j)^* = y_i \equiv S(u^i_N)\) and a left *-quantum space for \(O(U_q(N))\). It is denoted by \(O(U_q(N)/U_q(N-1))\) and called the coordinate *-algebra of the quantum sphere associated with the quantum group \(U_q(N)\). In this case the two left-covariant FODC \(\Gamma_1\) and \(\Gamma_4\) of \(O(U_q(N))\) and hence their induced FODC \(\Gamma^Z_1\) and \(\Gamma^Z_4\) on \(Z = O(U_q(N)/U_q(N-1))\) are *-calculi. We prove these assertions for \(\Gamma_1\) and \(\Gamma^Z_1\). First note that \((l^{\pm 1})^* = S(l^{\pm 1})\) (see [KS], formula (10.47)) for the corresponding involution of the Hopf dual \(O(GL_q(N))^\circ\). Hence we obtain \(X^* = X_N\) and \(X^* = (l^{-N}_N l^{-N}_N Z_N)^* = Z_N l^{-N}_N S(l^{\pm 1})_N\) for \(i = 1,\ldots,N-1\). Since \(Z_N\) is a monomial in the L-functionals \(l^{\pm 1}_N\), \(Z_N l^{-N}_N S(l^{\pm 1}_N)\) is a complex multiple of \(S(l^{\pm 1}_N) l^{-N}_N Z_N = Y_i\). Therefore, we have \(X^* \in T^2_1\) for all \(X \in T^2_1\), so that \(\Gamma^Z_1\) is a *-calculus of \(O(U_q(N))\) by Proposition 14.6 in [KS]. Since \(Z\) is a *-subalgebra of \(O(U_q(N))\), the induced FODC \(\Gamma^Z_1\) is also a *-calculus. Thus,
the FODC $\Gamma_3^Z$ and $\Gamma_4^Z$ are $*$-calculi on the coordinate $*$-algebra $Z = \mathcal{O}(U_q(N)/U_q(N-1))$ of the quantum sphere. Note that because these FODC are $*$-calculi it suffices to prove only one of the commutation relations for $dx_i, x_j$ and $dy_i, y_j$ and one of the relations for $dx_i, y_j$ and $dy_i, x_j$. The two others follow then by applying the involution and inverting the corresponding $R$-matrix. The FODC $\Gamma_2^Z$ and $\Gamma_3^Z$ are not $*$-calculi on $Z$, but one has $(\mathcal{T}_2^Z)^* = \mathcal{T}_3^Z$.

Let us return to the general case where $q$ is a complex number such that $q \neq 0, \pm 1$. From its very construction it is clear that the left-covariant $(2n-1)$-dimensional FODC $\Gamma_1$ of the Hopf algebra $\mathcal{O}(GL_q(N))$ is a useful tool for the study of the induced FODC $\Gamma_3^Z$ on the subalgebra $Z$. However, $\Gamma_1$ is not suitable as a FODC of the Hopf algebra $\mathcal{O}(GL_q(N))$ itself, because the generators $X_i, Y_i$ of the quantum tangent space $T^Z$ are only supported on the last row and column of the fundamental matrix $u = (u_{ij}^l)$. To remedy this defect, one can construct an $N^2$-dimensional left-covariant FODC $\Gamma$ on $A = \mathcal{O}(GL_q(N))$ that induces the FODC $\Gamma_3^Z$ on $Z$ as well. We restrict ourselves to the distinguished calculus $\Gamma_2^Z$ with $Z_n = (l_n^{-1})^2$. Let $T$ be the linear span of linear functionals

$$X_{ij} = (q^2 - 1)^{-1}l_{ij}^{-1}l_{ji}^{-1}, \quad i < j, \quad (51)$$
$$Y_{ji} = (q^2 - 1)^{-1}S(l_{ij}^{-1})l_{ji}^{-1}, \quad i < j, \quad (52)$$
$$X_{ii} = (q^2 - 1)^{-1}((l_{ii}^{-1})^2 - \varepsilon), \quad Y_{ii} = -q^{-2}X_{ii} \quad (53)$$

on $A$. For $i \leq j, i, j = 1, \ldots, N$, we then have

$$\Delta(X_{ij}) - \varepsilon \otimes X_{ij} = \sum_{k \leq j} X_{kj} \otimes l_{kj}^{-1}l_{ji}^{-1}, \quad (54)$$
$$\Delta(Y_{ji}) - \varepsilon \otimes Y_{ji} = \sum_{k \leq j} Y_{jk} \otimes S(l_{ki}^{-1})l_{ji}^{-1} \quad (55)$$

Thus, by Lemma 1, there is a left-covariant FODC $\Gamma$ of $A$ which has the quantum tangent space $T$. From the explicit form $[19]$ of the matrix $R$ and its inverse $R^{-1} = (q - q^{-1})R + I$ we compute

$$(X_{ij}, u_s^r) = \delta_{ir}\delta_{js} \text{ and } (Y_{ji}, S^{-1}(u_s^r)) = \delta_{ir}\delta_{js} \text{ for } i \leq j, \quad (56)$$

Setting $\theta_{ij} := \omega(u_{ij}^l) = \sum_k S(u_{ij}^k)du_{ij}^k$ and $\eta_{ij} := \omega(S^{-1}(u_{ij}^l)) = \sum_k u_{ij}^kS^{-1}(l_{ki}^j)$ for $i \leq j$, the formulas $[3]$ and $[6]$ imply that

$$(X_{ij}, \theta_{rs}) = (Y_{ji}, \eta_{sr}) = \delta_{ir}\delta_{js} \text{ for } i \leq j, i, j, r, s = 1, \ldots, N. \quad (57)$$

In particular, the functionals $X_{ij}, Y_{rs}, i \leq j, s < r$, are linearly independent, so that the FODC $\Gamma$ has dimension $N^2$. Further, it follows from $[3]$ and $[7]$ that the sets $\{\theta_{ij}, \eta_{rs}; i \leq j, s < r\}$ and $\{X_{ij}, Y_{rs}; i \leq j, s < r\}$ and also the sets $\{\theta_{ij}, \eta_{rs}; i < j, s \leq r\}$ and $\{X_{ij}, Y_{rs}; i < j, s \leq r\}$ are dual bases of $\text{in}_n\Gamma$ and $T$, respectively. It is not difficult to verify that the two calculi $\Gamma$ and $\Gamma_1$ with $Z_n = (l_n^{-1})^2$ of $A$ induce the same FODC $\Gamma_3^Z$ on the quantum space $Z$.

For $j = 1, \ldots, N$, let $\mathcal{T}_j$ denote the linear span of functionals $X_{ij}$ and $Y_{ji}, i \leq j$. From $[74]$ and $[55]$ we conclude that there is a $(2j-1)$-dimensional left-covariant FODC $\Gamma_j^Z$ on
$O(\text{GL}_q(N))$ which has the quantum tangent space $\mathcal{T}_j$. The FODC $\Gamma^N$ is nothing but the FODC $\Gamma_1$ developed in Section 4 (as always throughout this discussion, with $Z_n = (l^n - 1)^2$).

Since the linear quantum tangent space $\mathcal{T}$ is the direct sum of vector spaces $\mathcal{T}_1, \ldots, \mathcal{T}_N$, the FODC $\Gamma$ is the direct sum of FODC $\Gamma^1, \ldots, \Gamma^N$. These and other properties indicate that the FODC $\Gamma_i$ is a promising tool for the study of the interplay between the quantum group $\text{GL}_q(N)$ and the quantum homogeneous spaces $\text{GL}_q(j)/\text{GL}_q(j-1)$, $j = 2, \ldots, N$.

The FODC $\Gamma$ is only left-covariant, but not bicovariant. However, because of its particular and simple structure the FODC $\Gamma$ might be even more important and useful for applications and computations than the bicovariant calculi of the Hopf algebra $O(\text{GL}_q(N))$. We shall return to this matter in Section 7.

At the end of this section, let us briefly turn to the quantum group $\text{SL}_q(N)$. The Hopf algebra $O(\text{SL}_q(N))$ is also quasitriangular with universal $r$-form $r$ such that

$$r(u_i^j, u_k^l) = z \hat{R}^k_{jl}, i, j, k, l = 1, \ldots, N,$$

where $\hat{R}$ is given by (49) and $z$ is a complex $N$-th root of $q^{-1}$. Then the linear span of functionals $X_{ij}, Y_{ij}, i < j$, and $X_{rr}, r = 2, \ldots, N$, defined by (51)–(53) is also the quantum tangent space of a $(N^2 - 1)$-dimensional FODC on $O(SL_q(N))$. It should be emphasized that because of the appearance of the number $z$ in (58) the equalities (56) are no longer valid for $O(SL_q(N))$. Some $(N^2 - 1)$-dimensional left-covariant FODC on $O(SL_q(N))$ with reasonable properties have been constructed in [SS3]. This FODC is different from those in [SS3], but it is based on a similar idea.

6. A left-covariant FODC on $\text{GL}_q(N)/\text{GL}_q(N-1)$ induced from a bicovariant FODC on $\text{GL}_q(N)$

In this section we retain the notation of the preceding section. Let $\Gamma_{bi}$ be the bicovariant FODC on $A = O(\text{GL}_q(N))$ constructed by the bicovariant bimodule $(u^e \otimes u, L^+ \otimes L^-)$. (Details can be found, for instance, in [KS], Sections 14.5 and 14.6). Here we only need the two facts (see [KS], 14.6.3 and Example 14.8) that the set $\{\omega_{ij} := \omega(u_i^j) = \sum_k S(u_k^j)du_k^i, i, j = 1, \ldots, N\}$ is a basis of the vector space $\text{inv}(\Gamma_{bi})$ of left-invariant one-forms of $\Gamma_{bi}$ and that the commutation rules between the forms $\omega_{ij}$ and an element $a \in A$ are given by

$$\omega_{ij}a = \sum_{r,s} a_{(1)}^i l^+_r(a_{(2)})S(l^-_s)(a_{(3)})\omega_{rs}. \quad (59)$$

**Proposition 5.** The FODC $\Gamma_{bi}$ induces a left-covariant FODC $\Gamma^Z$ on the quantum space $Z$ such that

$$dx_i \cdot x_j = q \sum_{k,m} \hat{R}^{ij}_{km} x_k \cdot dx_m,$$

$$dy_i \cdot y_j = q^{-1} \sum_{k,m} (\hat{R}^{-1})^{ij}_{km} y_k \cdot dy_m,$$
\[
\begin{align*}
\text{dx}_i \cdot \text{y}_j &= q^{-1} \sum_{k,m} (\hat{R}^{-1})_{mj}^{ki} y_k \cdot \text{dx}_m, \\
\text{dy}_i \cdot \text{x}_j &= q \sum_{k,m} \hat{R}^{ij}_{km} x_k \cdot \text{dy}_m,
\end{align*}
\]

where \(\hat{R}^{ij}_{km} := r(S^2(u^j_i), u^i_m)\), \(i, j, k, m = 1, \ldots, N\). Further, we have \(\omega_{NN} x_i = q^2 x_i \omega_{NN}\) and \(\omega_{NN} y_i = q^{-2} y_i \omega_{NN}\) for \(i = 1, \ldots, N\).

**Proof.** We verify, for instance, the third commutation relation. From the explicit form (49) of the matrix \(\hat{R}\) it follows that \((\hat{R}^{-1})_{sN}^{lN} = q^{-1} \delta_{sN} \delta_{lN}\) for \(s, l = 1, \ldots, N\). Using essentially this fact and formula (59) we compute

\[
\text{dx}_i \cdot \text{y}_j = \sum_k u^i_k \omega_{kN} S(u_j^N)
\]

\[
= \sum_{k,m,l,r,s} u^i_k S(u_j^m) (l^{+k}_r, S(u^l_m))(S(l^{-s}_N), S(u^r_i)) \omega_{rs}
\]

\[
= \sum_{r,s,l} \left( \sum_{k,m} u^i_k S(u_j^m) (\hat{R}^{-1})_{rm}^{lk} \right) q^{2N-2l} (\hat{R}^{-1})_{sN}^{lN} \omega_{rs}
\]

\[
= \sum_{r,s,l} \left( \sum_{k,m} S(u^i_k) u^m_r (\hat{R}^{-1})_{mj}^{ki} \right) q^{2N-2l} q^{-1} \delta_{sN} \delta_{lN} \omega_{rs}
\]

\[
= \sum_q^{-1} (\hat{R}^{-1})_{mj}^{ki} S(u^N_k) u^m_r \omega_{rN}
\]

\[
= \sum_q^{-1} (\hat{R}^{-1})_{mj}^{ki} y_k \cdot \text{dx}_m.
\]

The other relations follow by similar reasonings as above or as used earlier. We shall not carry out the details. \(\square\)

The FODC \(\Gamma^Z\) is another good candidate of a reasonable differential calculus on the quantum homogeneous space \(\mathcal{Z}\). It is a \(*\)-calculus if \(q\) is real and the involution of \(\mathcal{Z}\) is given by \(x^*_i = y_i\), \(i = 1, \ldots, N\), because the FODC \(\Gamma_{bi}\) on \(O(GL_q(N))\) is known to be a \(*\)-calculus with respect to the involution \((u^i_j)^* = S(u^j_i), i, j = 1, \ldots, N\). But there is a striking difference between the two distinguished calculi \(\Gamma^Z\) and \(\Gamma^Z_1\): \(\Gamma^Z_1\) is inner, while \(\Gamma^Z\) is not. In order to verify the latter, it suffices to note that \(\omega_{NN} x_i - x_i \omega_{NN} = (q^2-1)x_i \omega_{NN}\) is obviously not a multiple of \(\text{dx}_i\).

### 7. A recipe for the construction of left-covariant FODC

The first order differential calculi on quantum homogeneous spaces developed above are induced from left-covariant calculi on the quantum group. All these left-covariant calculi on the corresponding Hopf algebra are built by the same simple recipe that will be elaborated more explicitly in this section. As always, \(\mathcal{A}\) is a coquasitriangular Hopf algebra
and $l^±_j$ are the $L$-functionals on $\mathcal{A}$ with respect to a fixed corepresentation $u = (u^i_j)_{i=1,...,n}$ of $\mathcal{A}$. Throughout this section we retain assumption (19).

Let $i, j \in \{1, \ldots, n\}$ be two indices such that $i \leq j$ and let $Z$ be a group-like element of $\mathcal{A}^\otimes$. Define

$$X^+_r = t^{i+r}_i l^{-r}_i Z, \quad r = i+1, \ldots, j, \quad \text{and} \quad X^+_i = Z - \varepsilon,$$
$$X^-_r = l^{-r}_i t^{r+i}_j Z, \quad r = i, \ldots, j-1, \quad \text{and} \quad X^-_j = Z - \varepsilon,$$
$$Y^+_r = S(l^{ij}_j) t^{r+j}_j Z, \quad r = i, \ldots, j-1, \quad \text{and} \quad Y^+_j = Z - \varepsilon,$$
$$Y^-_r = S(l^{ji}_i) l^{r+i}_i Z, \quad r = i+1, \ldots, j, \quad \text{and} \quad Y^-_i = Z - \varepsilon,$$
$$\mathcal{T}^\pm_{ij}(Z) = \text{Lin} \{X^\pm_r; i \leq r \leq j\}, \quad \mathcal{T}^\pm_{ji}(Z) = \text{Lin} \{Y^\pm_r; i \leq r \leq j\}.$$

Using (19) one easily verifies that each vector space $\mathcal{T} = \mathcal{T}^\pm_{ij}(Z), \mathcal{T}^\pm_{ji}(Z)$ has the properties that $X(1) = 0$ and $\Delta(X) - \varepsilon \otimes X \in \mathcal{T} \otimes \mathcal{A}^\otimes$ for all $X \in \mathcal{T}$. Hence, by Lemma 1 each space $\mathcal{T}^\pm_{ij}(Z), \mathcal{T}^\pm_{ji}(Z)$ is the quantum tangent space of a left-covariant FODC $\Gamma^\pm_{ij}, \Gamma^\pm_{ji}$ on $\mathcal{A}$. Let us call the first order calculi of the form $\Gamma^\pm_{ij}, \Gamma^\pm_{ji}$ elementary FODC. All left-covariant FODC on $\mathcal{A}$ occurring in this paper are direct sums of elementary FODC (with possible different group-like elements $Z$!). By forming sums of elementary FODC one gets a large supply of left-covariant FODC which have a very simple structure and are easy to handle. FODC of this form have been introduced in [SS3]. Note that the commutation rules of the elements of the quantum tangent spaces obtained in this manner are not necessarily quadratically closed and that the dimensions of the spaces of higher forms may be different from the corresponding classical dimensions (see [SS2] for such examples).

For the group-like elements $Z$ one may take, for instance, a monomial in the main diagonal $L$-functionals $l^\pm_i, i = 1, \ldots, n$. Interesting choices of $Z$ are, of course, $Z = l^\pm_i$ for $\mathcal{T}^\pm_{ij}(Z)$ and $Z = l^\pm_i$ for $\mathcal{T}^\pm_{ji}(Z)$ or $Z = \varepsilon$ for all four FODC. Let us illustrate this by simple examples and set

$$\mathcal{T}^+ = \sum_i \mathcal{T}^+_i(l^+_i) = \text{Lin} \{l^{i+j}_j; i \leq j, i, j = 1, \ldots, n\},$$
$$\mathcal{T}^- = \sum_j \mathcal{T}^-_j(l^-_j) = \text{Lin} \{l^{i+j}_i; i \leq j, i, j = 1, \ldots, n\},$$
$$\mathcal{T}^+ = \sum_j \mathcal{T}^+_j(l^+_j) = \text{Lin} \{S(l^{i+j}_j); i \leq j, i, j = 1, \ldots, n\},$$
$$\mathcal{T}^- = \sum_i \mathcal{T}^-_i(l^-_i) = \text{Lin} \{S(l^{i+j}_i); i \leq j, i, j = 1, \ldots, n\},$$

Then, $\mathcal{T}^+, \mathcal{T}^-, \mathcal{T}^+, \mathcal{T}^+ + \mathcal{T}^-$ and $\mathcal{T}^- + \mathcal{T}^+$ are quantum tangent spaces of left-covariant FODC on $\mathcal{A}$.

Now we want to be more specific and suppose that $\mathcal{A}$ is one of the Hopf algebras $\mathcal{O}(G_q), G_q = GL_q(N), SL_q(N), O_q(N), S_p(N)$, and $u$ is the fundamental corepresentation.

Case 1. $\mathcal{A} = \mathcal{O}(GL_q(N))$

Then the vector spaces $\mathcal{T}^+ + \mathcal{T}^-$ and $\mathcal{T}^- + \mathcal{T}^+$ defined above are the quantum tangent spaces of two $N^2$-dimensional left-covariant FODC on $\mathcal{O}(GL_q(N))$. It is easily seen that the commutation relations of the elements of both quantum tangent spaces are quadratically
closed. Further, it can be shown that the dimensions of the spaces of $k$-forms for the associated universal higher order differential calculi (see [KS], 14.3, for this notion) are \( \binom{N^2}{k} \) as in the classical case.

**Case 2.** \( \mathcal{A} = \mathcal{O}(SL_q(N)) \)

In this case, \( T^+ + \mathcal{T}_- \) and \( T^- + \mathcal{T}_+ \) are also \( N^2 \)-dimensional FODC on \( \mathcal{O}(SL_q(N)) \), but we are interested in FODC that have the classical group dimension \( N^2-1 \). It is rather easy to construct such FODC: Let \( \mathcal{T}_{od} \) be the sum of \( \mathcal{T}_{i_1, j_1}(\epsilon), \mathcal{T}_{i_j}(\epsilon) \), \( i = 1, \ldots, n \), and let \( \mathcal{T}_{md} \) be the vector space spanned by \( N-1 \) of the \( N \) functionals \( l_i^+ - \epsilon \). Then, \( \mathcal{T} = \mathcal{T}_{od} + \mathcal{T}_{md} \) is the quantum tangent space of an \((N^2-1)\)-dimensional FODC on \( \mathcal{O}(SL_q(N)) \). This first order calculus strongly resembles the ordinary differential calculus on the Lie group \( SL(N) \) in many aspects. But it has the disadvantage that the commutation rules between elements of the quantum tangent space (for instance, \( l_i^+ l_i^{-1} \) and \( l_j^+ l_j^{-1} \)) do not close quadratically. \((N^2-1)\)-dimensional FODC on \( \mathcal{O}(SL_q(N)) \) that do not have this defect have been constructed in [SS3]. However, using the same idea as in [SS3], the quantum tangent space \( \mathcal{T} \) can be modified by multiplying the secondary diagonal elements such that commutation relations close quadratically.

In order to be more precise, let \( f_i \) and \( g_i \), \( i = 1, \ldots, N \), be monomials in the main diagonal \( L \)-functionals \( l_i^{\pm} \). Let \( \mathcal{T}_{od} \) be the linear span of \( X_{ij} := l_i^{+1} l_i^{-1} f_i \) and \( X_{ji} := l_j^{+1} l_j^{-1} g_j \), \( i < j \), and let \( \mathcal{T}_{md} \) be an \((N-1)\)-dimensional vector space generated by functionals of the form \( f_l - \epsilon \), where \( f \) is a monomial in \( l_i^{\pm} \), \( j = 1, \ldots, N \). Suppose that \( f_i, g_i \in \mathbb{C} \epsilon \oplus \mathcal{T}_{md} \) for \( i = 1, \ldots, N \). Then one easily verifies that \( \mathcal{T} := \mathcal{T}_{od} + \mathcal{T}_{md} \) is the quantum tangent space of an \((N^2-1)\)-dimensional FODC on \( \mathcal{O}(SL_q(N)) \). Further, the commutation relations for elements of \( \mathcal{T} \) are quadratically closed if and only if \( f_i^{-1} g_i (l_i^{+1})^2 \) is independent of \( i = 1, \ldots, N \). (This assertion and the explicit form of commutation rules can be derived from the relations \( L_1^+ L_2^+ R = RL_2^+ L_1^+ \) and \( L_1^- L_2^- R = R L_2^- L_1^- \) using (19). We omit the details.) These conditions can be fulfilled as follows: Fix an index \( k \in \{1, \ldots, N\} \) and set \( g_i = (l_i^+)^2 (l_i^-)^2 \) and \( f_i = \epsilon \) for \( i = 1, \ldots, N \). Another possible choice is \( f_i = (l_i^+)^2 (l_i^-)^2 \) and \( g_i = \epsilon \) for \( i = 1, \ldots, N \). These two special cases are in fact the two FODC \( \Gamma_1 \) and \( \Gamma_2 \) constructed in [SS3].

In order to come into contact with the considerations in Sections 4 and 5, we carry out the same consideration based in the generators \( X_r^-, Y_r^+ \) rather than \( X_r^+, X_r^- \). We suppose that the elements \( f_i, g_i \) and the vector space \( \mathcal{T}_{md} \) satisfy the assumptions stated in the first half of the preceding paragraph. Now let \( \mathcal{T}_{od} \) be the vector space generated by the functionals \( X_{ji} = l_j^{+1} l_i^{-1} f_i \) and \( X_{ij} = S(t_i^{+1}) l_j^{+1} g_j \), \( i < j \). Then \( \mathcal{T} := \mathcal{T}_{od} + \mathcal{T}_{md} \) is again the quantum tangent space of an \((N^2-1)\)-dimensional FODC on \( \mathcal{O}(SL_q(N)) \). The commutation relations for \( \mathcal{T} \) close quadratically if and only if \( f_i g_i (l_i^{+1})^2 \) does not depend on \( i = 1, \ldots, N \).

**Case 3: \( \mathcal{A} = \mathcal{O}(O_q(N)) \) and \( \mathcal{A} = \mathcal{O}(Sp_q(N)) \)**

In this case the fundamental matrix \( u \) fulfills the metric condition

\[
 u C u^t C^{-1} = C u C^{-1} u = I \tag{60}
\]

and the R-matrix is given by

\[
 R_{mn}^{ji} = q^{\delta_{ij} - \delta_{jm}} \delta_{im} \delta_{jn} + (q - q^{-1}) \theta(i - m) (\delta_{jm} \delta_{in} - \epsilon C_i^j C_m^i), \tag{61}
\]
where \( i' := n+1-i, \epsilon = 1 \) for \( O_q(N) \), \( \epsilon = -1 \) for \( Sp_q(N) \) and \( C = (C^i_j) \) is the corresponding matrix of the metric (see [FRT] or [KS] for details). We shall essentially use the fact that \( C^i_j = 0 \) if \( i \neq j' \).

Before we turn to the construction of the FODC, let us look for a moment at the "ordinary" first order calculus on the Lie groups \( O(N) \) and \( Sp(N) \). Then the matrix

\[
\theta = S(u)du = (\theta_{ij} \equiv \sum_k S(u^i_k)du^k_j))_{i,j=1,...,N}
\]

satisfies the relation

\[
\theta = -C^{-1} \theta C, \quad \text{that is, } \theta_{ij} = (C^{-1})^i_j \theta_j^{i'} C^{i'}_j \quad \text{for } i, j = 1, \ldots, N.
\]

We briefly sketch the proof of this well-known fact. Indeed, differentiating the condition \( uC^{-1}u = C^{-1} \), we obtain

\[
du^tC^{-1}u + u^tC^{-1}du = 0.
\]

From \( C^t u^t C^{-1} u = I \) we get \( C^{-1} S(u) = u^t C^{-1} u \), and so \( u^t C^{-1} du = C^{-1} \theta \). For the metric \( C \) of the Lie groups \( O(N) \) and \( Sp(N) \) we have \( (C^{-1})^t = \epsilon C^{-1} \). Hence the relation \( C^{-1} S(u) = u^t C^{-1} \) implies that \( C^{-1} u = S(u)u^t C^{-1} \). Because functions and forms commute (!) for the classical differential calculus, we can write \( du^t C^{-1} u = (S(u)du)^t C^{-1} = \theta^t C^{-1} \).

Inserting these expressions into \((64)\) we obtain \((65)\).

For the construction of the left-covariant FODC we shall restrict ourselves to the quantum group \( O_q(N) \). In the case of \( Sp_q(N) \) one has to omit the elements \( X_{ii} \) supporting the secondary diagonal entries \( u^i_j \) in order to be in accordance with the ordinary calculus on the classical group \( Sp(N) \). The remaining parts are verbatim the same.

Let us abbreviate \( I := \{(i,j) : i' \leq j, i, j = 1, \ldots, N\} \). Then the elements \( u^i_j \) with \((i,j) \in I \) are precisely those entries of the matrix \( u \) that are below or on the secondary diagonal. Now we define

\[
X_{ji} := l^{-j} l^{i+j} Z_j \quad \text{and} \quad X_{ij} := l^{i+j} l^{i-j} Z_{j'} \quad \text{for } j' \leq j, i, j = 1, \ldots, n,
\]

where \( Z_j \) and \( Z_{j'} \) are group-like elements of the Hopf dual \( \mathcal{O}(O_q(N))^\circ \). These functionals \( X_{ji}, X_{ij} \) separate the elements \( u^i_j \) such that \((i,j) \in I \) and \( i \neq j \). In order to separate also the main diagonal entries \( u^i_i, i' \leq i \), we choose group-like elements \( Y_i, i' \leq i \), of \( \mathcal{O}(O_q(N))^\circ \) such that

\[
(Y_i - \varepsilon, u^r_s) = \delta_{rs} \delta_{ir} \quad \text{for } (r, s) \in I, i' \leq i,
\]

and put

\[
X_{ii} := Y_i - \varepsilon \quad \text{for } i' \leq i, i = 1, \ldots, N.
\]

Further, we suppose that

\[
Z_j, Z_{j'} \in \text{Lin } \{Y_{ii}; i' \leq i\} \quad \text{for } j' < j.
\]

Then the vector space \( T = \text{Lin } \{X_{rs}; (s, r) \in I\} \) is the quantum tangent space of a left-covariant FODC \( \Gamma \) on \( \mathcal{O}(O_q(N)) \). From the construction and the explicit form of the
matrix $R$ it is straightforward to check that that $(X_{rs}, u^i_j) \neq 0$ if and only if $(r, s) = (j, i)$ for arbitrary indices $(s, r) \in I$ and $(i, j) \in I$. This implies that the FODC $\Gamma$ has the dimension $N(N + 1)/2$ and that the elements $\theta_{ij} = \omega(u^i_j), (i, j) \in I,$ form a basis of the space of left-invariant one-forms $\mathfrak{im}\Gamma$. These facts are in accordance with the ordinary first order calculus on the Lie group $O(N)$. Note that the FODC $\Gamma$ just constructed depends on the group-like elements $Z_j, Z'_j, j' < j,$ and $Y_i, i' \leq i,$ of $O_q(O(N)) \circ$ which can be freely chosen such that they satisfy the assumptions (65) and (66). These conditions can be easily fulfilled by taking monomials in the main diagonal L-functionals $l^{\pm i}$. We make all that more explicit by an example.

Example: $O_q(5)$

Then the 15 generators of the quantum tangent space $\mathcal{T}$ are

$$
X_{15} = t^{+1} l^{+1} Z_1, \quad X_{25} = t^{+1} l^{+1} Z_1, \quad X_{35} = t^{+1} l^{+1} Z_1, \quad X_{45} = t^{+1} l^{+1} Z_1,
$$

$$
X_{24} = t^{+2} l^{+1} Z_2, \quad X_{34} = t^{+2} l^{+1} Z_2,
$$

$$
X_{51} = t^{+5} l^{+1} Z_5, \quad X_{52} = t^{+5} l^{+1} Z_5, \quad X_{53} = t^{+5} l^{+1} Z_5, \quad X_{54} = t^{+5} l^{+1} Z_5,
$$

$$
X_{33} = Y_3 - \varepsilon, \quad X_{44} = Y_4 - \varepsilon, \quad X_{55} = Y_5 - \varepsilon
$$

and assumption (66) means that $Z_1, Z_2, Z_4, Z_5 \in \text{Lin} \{Y_3, Y_4, Y_5\}$. \hfill \Box

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