AUTODUALITY HOLDS FOR A DEGENERATING ABELIAN VARIETY

JESSE LEO KASS

ABSTRACT. We prove that certain degenerate abelian varieties, the compactified Jacobian of a nodal curve and a stable quasialbelian variety, satisfy autoduality. We establish this result by proving a comparison theorem that relates the associated family of Picard schemes to the Néron model, a result of independent interest. In our proof, a key fact is that the total space of a suitable family of compactified Jacobians has rational singularities.

In this paper we prove that certain degenerate abelian varieties satisfy autoduality. The classical statement of autoduality is a statement about an abelian variety $J_0$ with an ample divisor $\Theta_0$ that defines a principal polarization. If $\tau_{x_0}: J_0 \to J_0$ denotes translation by a point $x_0$, then the morphism

\begin{align*}
J_0 & \to \text{Pic}^0(J_0/k), \\
x_0 & \mapsto \mathcal{O}(\tau_{x_0}^* \Theta_0 - \Theta_0)
\end{align*}

from $J_0$ to the (identity component of the) Picard scheme is an isomorphism. The Picard scheme $\text{Pic}^0(J_0/k)$ is the dual abelian variety, so the fact that (1) is an isomorphism implies that $J_0$ is self-dual — that the autoduality theorem holds.

Here we prove analogues of this result for certain degenerate abelian varieties: compactified Jacobians of nodal curves and stable quasialbelian varieties. A stable quasialbelian variety $\overline{J}_0$ is a certain degenerate abelian variety with an action of a semialbelian variety $G$ that is studied in moduli of abelian varieties. The compactified Jacobian $\overline{J}_0$ of a nodal curve $X_0$ is the moduli space of degree 0 rank 1, torsion-free sheaves, which are required to satisfy a semistability condition when $X_0$ is reducible. (When $X_0$ is reducible, there are many choices of semistability conditions and thus many compactified Jacobians, a topic discussed at the end of Section [1])

Both stable quasialbelian varieties and compactified Jacobians are (possibly reducible) projective varieties, so we can form the Picard scheme $\text{Pic}^0(\overline{J}_0/k)$ and then (2) again defines a morphism, out of the semialbelian variety $G$ acting on $\overline{J}_0$ when $\overline{J}_0$ is a quasialbelic variety and out of the generalized Jacobian when $\overline{J}_0$ is a compactified Jacobian. We prove that this morphism is an isomorphism:

Main Theorem (Autoduality). The autoduality theorem holds for compactified Jacobians and stable quasialbelian varieties.

This is Corollaries [6] and [7] below.

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One consequence of the Main Theorem is that, when $J_0$ is the compactified Jacobian of a nodal curve $X_0$, $\text{Pic}^0(J_0/k)$ depends only on the curve $X_0$, rather than on the compactified Jacobian $\overline{J}_0$. Recall $\overline{J}_0$ depends on a choice of semistability condition, and different choices produce different schemes. For example, when $X_0$ equals two curves meeting in 3 nodes, one choice produces a $\overline{J}_0$ with two irreducible components, while another produces a $\overline{J}_0$ with three irreducible components. (See [OS79, Example 13.1(3)].)

This autoduality theorem is new when $J_0$ is stable quasiabelian variety or a coarse compactified Jacobian (a type of compactified Jacobian discussed in Section I). Certain compactified Jacobians are known to be stable quasiabelian varieties, but we treat these two types of varieties separately in this paper. The theorem was known when $J_0$ is a fine compactified Jacobian by work we now review. When $X_0$ is irreducible, the theorem was proven by Esteves–Gagné–Kleiman [EGK02, Theorem (Autoduality), pages 5-6]. This result was extended by Esteves–Rocha [ER13, pages 414-415] to tree-like curves and by Melo–Rapagnetta–Viviani [MRV12a, Theorem C] to arbitrary nodal curves. These authors prove results for curves with worse than nodal singularities, and their work has been generalized in various way, e.g., to curves with planar singularities [Ari11] and to results about the compactified Picard scheme of $J_0$ [EK05, Ari13, MRV12b].

For fine compactified Jacobians, the proof of autoduality we give here, which was inspired by [BLR90, Theorem 1, Section 9.7], is different from previous proofs. Given $X_0$ and $J_0$, we realize $J_0$ as the special fiber of a family $\overline{J}/S$ over $S = \text{Spec}(k[[t]])$ that is associated to a family of curves $X/S$ such that the total space $X$ is a regular scheme.

We compare the family $\text{Pic}^0(\overline{J}/S)/S$ with the Néron model of its generic fiber. (The Néron model is an extension of the generic fiber to a $S$-scheme that satisfies a universal mapping property.) Thus pick a resolution of singularities $\beta: \overline{J} \to \overline{J}$ and consider the pullback homomorphism $\beta^*: \text{Pic}^0(\overline{J}/S) \to \text{Pic}^0(\overline{J}/S)$. In Proposition [1] we prove $\overline{J}$ has rational singularities, and this implies the differential of $\beta^*$ — and hence $\beta^*$ itself — is an isomorphism. A theorem of Pépin states that $\text{Pic}^0(\overline{J}/S)$ is the identity component of the Néron model, so we conclude that:

**Theorem (Néron Comparison).** $\text{Pic}^0(\overline{J}/S)/S$ is the identity component of the Néron model of its generic fiber.

This theorem is Theorem 4 below, and it immediately implies the Main Theorem because the universal mapping property of the Néron model implies that the classical autoduality isomorphism of the generic fiber extends over all of $S$.

The proof just sketched deduces autoduality from the fact that $\overline{J}_0$ deforms in a family $\overline{J}/S$ such that $\overline{J}$ has rational singularities. By contrast, in [EGK02] the result is deduced from a description of $\overline{J}_0$ coming from the presentation scheme, in [ER13] from autoduality for irreducible curves, and in [MRV12a] from the computation of the cohomology of a universal family of sheaves, a computation done by putting $\overline{J}_0$ into a suitable miniversal family. In particular, previous work established autoduality for the compactified Jacobian using descriptions of the compactified Jacobian as a moduli space, while the present work establishes autoduality using facts about the singularities of the total space $\overline{J}$.
A word about the characteristic. In this paper we assume:

**Assumption.** $R$ is a discrete valuation ring with characteristic $0$ residue field $k$.

We need to make this assumption because we make use of properties of rational singularities. There is a well-developed theory of rational singularities in characteristic zero, but not in positive characteristic (except for the case of surface singularities). To extend the proof of the main results of this paper to allow $k$ to have positive characteristic, it would be enough to prove that the total space $\hat{J}$ admits a rational resolution and that Corollary 3 remains valid. Based on conversations with singularity theorists, the author believes that these properties are expected to hold in positive characteristic but such results have not been published.

### I. A REVIEW OF COMPACTIFIED JACOBIANS

Here we recall the definition of the compactified Jacobian and related objects. We begin by fixing notation. Let $k$ be a field of characteristic zero. A curve $X_0/\text{Spec}(k)$ is a $k$-scheme that is geometrically connected, geometrically reduced, 1-dimensional, and proper over $k$. We set $g := 1 - \chi(X_0, \mathcal{O}_{X_0})$ equal to the arithmetic genus. When $k = \overline{k}$ is algebraically closed, we say that $X_0/\text{Spec}(k)$ is nodal if the completed local ring $\hat{\mathcal{O}}_{X_0,x_0}$ of $X_0$ at a point not lying in the $k$-smooth locus is isomorphic to $k[[x,y]]/(xy)$. In general, we say that $X_0/\text{Spec}(k)$ is nodal if $X_0 \otimes_k \overline{k}$ is nodal. A family of curves over a scheme $T$ is a $T$-scheme $X/T$ that is proper and flat over $T$ and such that the fibers of $X \to T$ are curves. If the fibers are nodal curves, then we say $X/T$ is a family of nodal curves. We say that a family of curves $X/S$ is regular if $X$ is a regular scheme (i.e. at every closed point the Zariski tangent space has dimension equal to the local Krull dimension of $X$).

If $X_0/\text{Spec}(k)$ is a curve, then we say that a line bundle $L_0$ on $X_0$ has multidegree zero if for all irreducible components $Y_0$ of $X_0 \otimes_k \overline{k}$ the restriction of $L_0 \otimes_k \overline{k}$ to $Y_0$ has degree 0 (i.e. the Euler characteristic of the restriction equals $\chi(Y_0, \mathcal{O}_{Y_0})$). A $O_X$-module $L$ on a family of curves $X/T$ is called a family of multidegree 0 line bundles if $L$ is a line bundle such that the restriction to any fiber of $X \to T$ has multidegree 0. The étale sheafification of the functor that assigns to a $k$-scheme $T$ the set of isomorphism classes of families of multidegree 0 line bundles on $X_0 \times_k T$ is represented by a $k$-scheme $J_0^0$ called the generalized Jacobian of $X_0/\text{Spec}(k)$. The generalized Jacobian is quasi-projective and smooth over $k$.

Also associated to $X_0/\text{Spec}(k)$ is its compactified Jacobian. Suppose that $k = \overline{k}$ is algebraically closed and $A_0$ is an ample line bundle on $X_0$. A coherent sheaf $I_0$ on $X_0$ is rank 1, torsion-free if the restriction of $I_0$ to a dense open subset is locally free of rank 1 and $I_0$ does not contain a $\mathcal{O}_{X_0}$-submodule supported on a 0-dimensional subset. The slope of a coherent sheaf $F_0$ is defined to be $\mu(I_0) := \chi(X_0, I_0)/\tau(I_0)$ where $\tau(I_0)$ is the leading term of the Hilbert polynomial of $I_0$ with respect to $A_0$. When $k = \overline{k}$ is algebraically closed, we say that a rank 1, torsion-free sheaf $I_0$ on $X_0$ is semistable (resp. stable) with respect to $A_0$ if the slope satisfies $\mu(J_0) \leq \mu(I_0)$ (resp. $\mu(J_0) < \mu(I_0)$) for all nonzero subsheaves $J_0 \subset I_0$. If $X/T$ is a family of curves with a family of ample line bundles $A$ (i.e. $A$ is a line bundle
on $X$ such that the restriction to every geometric fiber of $X \to T$ is ample), then a **family of rank 1, torsion-free sheaves** on $X/T$ that is semistable with respect to $A$ is a $O_T$-flat, finitely presented $O_X$-module $I$ such that the restriction to every geometric fiber of $X \to T$ is a rank 1, torsion-free sheaf semistable with respect to the restriction of $A$.

The degree $d$ **compactified Jacobian** $\overline{J}_0/\text{Spec}(k)$ associated to a curve $X_0/\text{Spec}(k)$ and an ample line bundle $A_0$ is the $k$-scheme that universally corepresents the functor that assigns to a $k$-scheme $T$ the set of isomorphism classes of families of degree $d$ rank 1, torsion-free sheaves on $X_0 \times_k T$ that are semistable with respect to $A_0 \otimes_k O_T$. The compactified Jacobian exists and is projective over $k$ by [Sim94, Theorem 1.21] (Note: the moduli space described in loc. cit. includes pure sheaves that fail to have rank 1, and $\overline{J}_0$ is a connected component of this larger moduli space; when stability coincides with semistability, this is shown in [Kas13 Section 4.2], and the semistable case can be treated by applying the argument in loc. cit. to a suitable Quot scheme). We call $\overline{J}_0$ the **compactified Jacobian**. We say that $\overline{J}_0$ is a **fine** compactified Jacobian if every degree 0 semistable rank 1, torsion-free sheaf is stable. Otherwise we say that $\overline{J}_0$ is **coarse**.

Suppose now that $k$ is the residue field of a discrete valuation ring $R$ with field of fractions $K$ and $S := \text{Spec}(R)$. Given a family of curves $X/S$ we define the associated **family of generalized Jacobians** $\overline{J}_0/S$ to be the $S$-scheme that represents the étale sheafification of the functor assigning to a $S$-scheme $T$ the set of isomorphism classes of families of multidegree zero line bundles on $X \times_S T$. The family of generalized Jacobians $\overline{J}_0/S$ exists as a $S$-scheme that is smooth and quasi-projective over $S$. The fibers of $\overline{J}_0 \to S$ are the generalized Jacobians of the fibers of $X \to S$ (because the formation of the functor $\overline{J}_0$ represents commutes with fiber products). Given a family of ample line bundles $A$ on $X/S$, the functor that assigns to a $S$-scheme $T$ the set of isomorphism classes of families of rank 1, torsion-free sheaves on $X \times_S T$ that are semistable with respect to $A$ is universally corepresented by a $S$-scheme $\overline{J}_0/S$ that is projective over $S$. We call $\overline{J}_0/S$ the **family of compactified Jacobians** associated to $X/S$ and $A$. The fibers of $\overline{J}_0 \to S$ are the compactified Jacobians of the fibers of $X \to S$ (because the formation of the functor $\overline{J}_0$ universally corepresents commutes with fiber products).

A word about the compactified Jacobians we study in this paper. They are more properly called Simpson compactified Jacobians or slope semistable compactified Jacobians. Other compactified Jacobians have been constructed (see e.g. [Kas13] for a brief survey). We restrict our attention to slope semistable compactified Jacobians only to keep our review of compactified Jacobians short. The author expects that the results of this paper remain valid for the other compactified Jacobians that have been constructed. Indeed, the key results we use about $\overline{J}$ are Propositions 1 and 2 and the proofs of these propositions remain valid for any family of compactified Jacobians that is a moduli space of rank 1, torsion-free sheaves that is either fine or is constructed using Geometric Invariant Theory, and to the author’s knowledge, this includes all families of compactified Jacobians in the literature.
II. THE SINGULARITIES OF $\mathcal{J}$

Here we prove some results about the singularities of a family of compactified Jacobians associated to a family of nodal curves. These results show that compactified Jacobians satisfy Hypothesis 1 of Section III, i.e. that the Néron comparison theorem applies.

In this section, $S$ is the spectrum of a fixed discrete valuation ring $R$ with residue field $k$, $X/S$ is a regular family of nodal curves, and $\mathcal{J}/S$ is a family of compactified Jacobians associated to $X/S$. We assume $k$ has characteristic zero.

We prove the main results of this section using a local description of $\mathcal{J}/S$ obtained from deformation theory. When $k = \overline{k}$ is algebraically closed and $\mathcal{J}/S$ is a family of fine compactified Jacobians, the completed local ring of $\mathcal{J}$ at a closed point $x_0 \in \mathcal{J}$ can be described as:

$$\hat{\mathcal{O}}_{\mathcal{J},x_0} \cong \hat{R}[[u_1, v_1, \ldots, u_n, v_n, w_1, \ldots, w_m]]/(u_1v_1 - \pi, \ldots, u_nv_n - \pi)$$

for some uniformizer $\pi \in R$ and some integers $n, m \in \mathbb{N}$. This is [Kas09, Lemma 6.2], a result proven using the techniques used in [CMKV12].

When $\mathcal{J}/S$ is a family of coarse compactified Jacobians, the Luna Slice argument used in the loc. cit. shows that there is a multiplicative torus $G_k^m$ acting on the ring appearing on the right-hand side of Equation (3) such that the torus invariant subring is isomorphic to $\hat{\mathcal{O}}_{\mathcal{J},x}$. Using this result, we prove:

**Proposition 1.** $\mathcal{J}$ has rational singularities, and $\mathcal{J} \to S$ is flat.

**Proof.** We can assume $k = \overline{k}$ because it is enough to prove the result after passing from $R$ to its strict henselization $R^\text{sh}$. With this assumption, suppose first that $\mathcal{J}/S$ is a family of fine compactified Jacobians. The morphism $\mathcal{J} \to S$ is flat because the ring appearing in Equation (3) is the quotient of a power series ring by elements whose images in $\hat{R}[[u_1, v_1, \ldots, w_m]]/(\pi)$ form a regular sequence [Mat89, Corollary to Theorem 22.5]. To see that $\mathcal{J}$ has rational singularities, observe that $\hat{\mathcal{O}}_{\mathcal{J},x_0}$ is isomorphic to the completion of $k[[u_1, v_1, \ldots, u_n, v_n, w_1, \ldots, w_m]]/(u_1v_1 - u_2v_2, \ldots, u_1v_1 - u_nv_n)$, which is the coordinate ring of an affine toric variety. (An isomorphism is determined by a choice of coefficient field $k \subset R$.) Since toric varieties have rational singularities, so does $\mathcal{J}$, proving the proposition when $\mathcal{J}$ is fine.

When $\mathcal{J}/S$ is coarse, the argument just given shows that, if $x_0 \in \mathcal{J}$ is a closed point, then $\hat{\mathcal{O}}_{\mathcal{J},x_0}$ is the torus invariant subring of a ring that is $R$-flat and has rational singularities. In particular, $\hat{\mathcal{O}}_{\mathcal{J},x_0}$ has rational singularities by [Bou87, Corollary, page 66] and is flat over $R$ as it is a direct summand of a flat module. □

From Equation (3), we deduce that when $k$ is algebraically closed and $\mathcal{J}_0$ is a fine compactified Jacobian, the completed local ring of $\mathcal{J}_0$ at a closed point $x_0 \in \mathcal{J}_0$ is:

$$\hat{\mathcal{O}}_{\mathcal{J}_0,x_0} \cong k[[u_1, v_1, \ldots, u_n, v_n, w_1, \ldots, w_m]]/(u_1v_1, \ldots, u_nv_n).$$
When \( \tilde{J}_0 \) is coarse, \( \hat{O}_{\tilde{J}_0, x_0} \) is isomorphic to the invariant subring of the ring appearing on the right-hand side of Equation (4) for some action of a multiplicative torus \( G_m^k \). We use these descriptions to prove:

**Proposition 2.** \( \tilde{J}_0 \) has Du Bois singularities.

**Proof.** We can assume \( k = \overline{k} \). When \( \tilde{J}_0 \) is fine, Equation (4) shows that the completed local ring of \( \tilde{J}_0 \) at a closed point \( x_0 \) is a completed product of double normal crossing singularity rings and power series rings, and such a completed product is Du Bois by [Doh08, Example 3.3, Theorem 3.9]. When \( \tilde{J}_0 \) is coarse, the completed local ring of \( \tilde{J}_0 \) at a closed point is a torus invariant subring of a Du Bois local ring and hence is itself Du Bois by [Kov99, Corollary 2.4] (the left inverse hypothesis is satisfied because a torus invariant subring is a direct summand).

From Proposition 2 we deduce:

**Corollary 3.** The higher direct image \( R^i p_* O_{\tilde{J}} \) of \( O_{\tilde{J}} \) under the projection \( p : \tilde{J} \to S \) is a locally free \( O_S \)-module of rank \( (g_i) \), and its formation commutes with arbitrary base change.

**Proof.** The fibers of \( \tilde{J} \to S \) have Du Bois singularities, so the result is [DB81, Théorème 4.6] together with [Mum08, Corollary 2, page 121].

## III. Comparison with the Néron model

Here we prove a comparison theorem relating a family of Picard schemes to the Néron model of its generic fiber. We fix the spectrum \( S \) of a discrete valuation ring \( R \) with residue field \( k \) and field of fractions \( K \). We assume \( k \) has characteristic zero.

The comparison theorem we prove is stated so that it applies to both the compactified Jacobians studied in Section IV and the degenerate abelian varieties studied in Section V. The theorem applies to a \( S \)-flat and \( S \)-projective \( S \)-scheme \( J/S \) such that the generic fiber \( \tilde{J}_K \) is an abelian variety of dimension \( g \) and the following hypothesis is satisfied:

**Hypothesis 1.** The scheme \( \tilde{J} \) has rational singularities, and the higher direct images \( R^i p_* O_{\tilde{J}} \) under the projection \( p : \tilde{J} \to S \) is a locally free \( O_S \)-module of rank \( (g_i) \) whose formation commutes with base change.

Observe that in the previous section we proved this hypothesis holds for a family of compactified Jacobians.

Given \( \tilde{J}/S \) satisfying Hypothesis 1, we can form the associated family of Picard schemes \( \text{Pic}(\tilde{J}/S)/S \). This is the \( S \)-scheme that represents the fppf sheafification of the functor that assigns to a \( S \)-scheme \( T \) the set \( \text{Pic}(\tilde{J}_T) \) of isomorphism classes of line bundles on \( \tilde{J}_T \). The family of Picard schemes exists as a (possibly nonseparated) \( S \)-group space that is locally of finite presentation over \( S \). Indeed, because the formation of the pushforward \( p_* O_{\tilde{J}} \) by \( p : \tilde{J} \to S \) commutes with base change, this representability result is [Ray70, (1.5)]. Because
$R^1p_*\mathcal{O}_J$ is locally free and its formation commutes with base change, $\text{Pic}(\tilde{J}/S)/S$ contains the **identity component** $\text{Pic}(\tilde{J}/S)/S$, an open $S$-subgroup scheme that is of finite type and smooth over $S$ and has the property that the fibers of $\text{Pic}(\tilde{J}/S) \to S$ are the identity components of the fibers of $\text{Pic}(\tilde{J}/S) \to S$ by [Kle05 Corollary 5.14, Proposition 5.20]. We denote the identity component by $P^0/S = P^0(\tilde{J}/S)$ and define $P/S = P(\tilde{J}/S)/S$ to be the closure of the generic fiber $P^0_k$ in $\text{Pic}(\tilde{J}/S)$. Because $P^0$ is smooth over $S$, it is contained in $P$.

We compare $P^0$ to the **Néron model** of its generic fiber $P_k$. The Néron model $N/S$ of $P_k$ is a $S$-scheme that is smooth over $S$, contains $P_k$ as the generic fiber, and satisfies the Néron mapping property; that is, for every smooth morphism $T \to S$ the natural map

$$\text{Hom}_S(T, N) \to \text{Hom}_K(T_k, P_k)$$

is bijective. By a theorem of Néron $N/S$ exists and is separated and of finite type over $S$ [BLR90 Corollary 2, Section 9.7]. The **identity component** $N^0/S$ is defined to be the complement of the connected components of the special fiber $N_k$ that do not contain the group identity element $e \in N_0(k)$. By construction $N^0$ is an open $S$-group subscheme of $N$ such that the fibers of $N^0 \to S$ are connected.

The identity morphism $\text{id}_K: P_k \to P_k$ extends uniquely to a $S$-morphism

$$P^0 \to N^0,$$

and we prove:

**Theorem 4** (Néron Comparison). The morphism (6) is an isomorphism.

**Proof.** We prove this theorem by choosing a regular $S$-model $\tilde{J}/S$ of $\bar{J}/S$, using a theorem of Pépin to relate the family of Picard schemes of $\tilde{J}/S$ to the Néron model, and then using the rational singularities hypothesis to show that $\tilde{J}/S$ and $\bar{J}/S$ have isomorphic families of Picard schemes.

Let $p: \tilde{J} \to S$ be the structure morphism and $\beta: \bar{J} \to \tilde{J}$ a resolution of singularities. Because $\tilde{J}$ has rational singularities, the higher direct images $R^j\beta_*\mathcal{O}_{\bar{J}}$ vanish for $j > 0$ and the direct image satisfies $\beta_*\mathcal{O}_{\bar{J}} = \mathcal{O}_\bar{J}$. The Leray spectral sequence $R^ip_* \circ R^j\beta_*\mathcal{O}_{\bar{J}} \Rightarrow R^{i+j}(p \circ \beta)_*\mathcal{O}_J$ thus degenerates at the $E_2$ page, so the natural homomorphisms

$$R^ip_*\mathcal{O}_{\bar{J}} = R^i\beta_*\mathcal{O}_{\bar{J}} \to R^i(p \circ \beta)_*\mathcal{O}_J$$

are isomorphisms. In particular, the direct image $R^i(p \circ \beta)_*\mathcal{O}_J$ is locally free of rank $g$ and its formation commutes with base change.

This shows that the hypothesis of [Ray70 (1.5)] holds for $\tilde{J}$, so the family of Picard schemes $\text{Pic}(\tilde{J}/S)/S$ exists as a $S$-group space that is locally of finite presentation over $S$. Define $P^0(\tilde{J}/S)/S$ and $P(\tilde{J}/S)/S$ in analogy with $P^0(\bar{J}/S)/S$ and $P(\bar{J}/S)/S$. The identity component of the $S$-group smoothening (in the sense of [BLR90 page 174]) of $P(\tilde{J}/S)$ is isomorphic to $N^0$ by [Pép13 Proposition 10.3]. In fact, it is equal to its $S$-group smoothening. Indeed, $P(\tilde{J}/S)$ is smooth over $S$ because it is flat (as its generic fiber is dense) and the
fibers of $P(\overline{J}/S) \to S$ are smooth (by [Kle05, Corollary 5.15] and the fact that $R^1(p \circ \beta)_*, O_{\overline{J}}$ satisfies the analogue of Corollary 3), so Pépin’s result asserts that the morphism

$$P^0(\overline{J}/S) \to N^0$$

extending the identity map is an isomorphism. Thus to prove the theorem, it is enough to show that

$$\beta^*: P^0(\overline{J}/S) \to P^0(\overline{J}/S),$$

$$M \mapsto \beta^*(M)$$

is an isomorphism.

Consider the map $\beta^*$ induces on Lie algebras. The map on Lie algebras is the natural homomorphism

$$R^1p_*O_{\overline{J}} \to R^1(p \circ \beta)_*, O_{\overline{J}},$$

and we already observed that this is an isomorphism. We conclude that $\beta^*$ is étale. In particular, $\beta^*$ has finite fibers. The morphism is also birational ($\beta^*$ is an isomorphism), so $\beta^*$ must be an open immersion by Zariski’s main theorem. Because the fibers of $P^0(\overline{J}/S) \to S$ are connected, the only open $S$-subgroup scheme of $P^0(\overline{J}/S)$ is $P^0(\overline{J}/S)$, and so $\beta^*$ is an isomorphism. □

Remark 5. The Néron Comparison Theorem is sharp in the following sense. The theorem shows that the identity component of the Néron model is isomorphic to an open $S$-subgroup scheme of $Pic(\overline{J}/S)$, and one can ask if there is a larger open subgroup scheme that is isomorphic to the Néron model. Without additional hypotheses, no such larger subgroup scheme exists. We demonstrate this with the following example, which is both a compactified Jacobian and a stable quasiabelian variety.

Let $S$ equal $\text{Spec}(\mathbb{C}[t]_{(t)})$ (the localization of $\mathbb{C}[t]$ at $(t)$), $X$ the minimal regular model of $\text{Spec}((\mathbb{R}[x, y]/(y^2 - x^3 - x^2 - t^2)))$, and $\overline{J}/S$ the family of degree 0 compactified Jacobians associated to any family of ample line bundles. (A computation shows that in this special case the semistability condition is independent of the ample line bundle.) Observe that $\text{Spec}((\mathbb{R}[x, y]/(y^2 - x^3 - x^2 - t^2)))$ has a singularity at the closed point $(x, y, t)$, so $X$ is a blow up of a compactification of $\text{Spec}((\mathbb{R}[x, y]/(y^2 - x^3 - x^2 - t^2)))$, and a computation shows that the special fiber $X_0$ of $X$ consists of two rational curves meeting in two nodes.

The family $X/S$ is a family of genus 1 nodal curves with reducible special fiber, and $\overline{J}/S$ is a family of genus 1 curves with irreducible special fiber. Since $\overline{J}/S$ is a family of curves, $Pic^0(\overline{J}/S)$ is flat over $S$ and thus $Pic^0(\overline{J}/S)$ is equal to the closure of its generic fiber in $Pic(\overline{J}/S)$. We can conclude that $Pic^0(\overline{J}/S)/S$ is the largest subgroup scheme of $Pic(\overline{J}/S)$ that contains the identity component $Pic^0(\overline{J}/S)$ and is isomorphic to an open subgroup scheme of the Néron model (for any open scheme of the Néron model has dense generic fiber by $S$-smoothness).

The identity component $Pic^0(\overline{J}/S)$ is not, however, the Néron model of its generic fiber because the Néron model has disconnected special fiber. (The elliptic curve $J_K$ has reduction type $I_2$ in Kodaira’s classification [Sil94, Theorem 8.2].) The theorems [Pép13,
Theorem 9.3] and [Ray70, Theoreme 8.1.4] suggest that one should not ask for an open subgroup of $\text{Pic}(\mathcal{J}/S)$ isomorphic to the Néron model, but rather for an open subgroup scheme whose maximal separated quotient is isomorphic to the Néron model. In the example just discussed, $\text{Pic}(\mathcal{J}/S)$ is separated, so again no such open subgroup scheme exists.

IV. AUTODUALITY FOR COMPACTIFIED JACOBIANS

Here we use Theorem 4, the Néron Comparison Theorem, to prove that the compactified Jacobian of a nodal curve satisfies autoduality. We fix the spectrum $S$ of a discrete valuation ring $R$ with residue field $k$ and field of fractions $K$. We let $X/S$ be a family of curves such that $X$ is regular and $X_K/\text{Spec}(K)$ is nonsingular and $\mathcal{J}/S$ an associated family of compactified Jacobians. We assume $k$ has characteristic zero.

In the introduction we defined the autoduality isomorphism (1) in terms of an ample divisor, but for Jacobians the isomorphism can alternatively be defined in terms of the Abel map. If $L_K$ is a degree $-1$ line bundle on $X_K$, then the rule

$$(7) \quad \alpha_K : x_K \mapsto L_K(x_K)$$

defines a morphism $\alpha_K = \alpha_{L_K} : X_K \rightarrow J_K$ that is the Abel map associated to $L_K$. The pullback morphism

$$(8) \quad \alpha^*_K : \text{Pic}^0(J_K/K) \rightarrow J_K$$

is, up to sign, the inverse of the autoduality isomorphism (1) from the introduction.

If $L$ is a line bundle on $X$ that extends $L_K$, then the rule $x \mapsto L(x)$ defines a morphism $\alpha$ extending $\alpha_K$ provided the fibers of $X/S$ are irreducible [EGK02 2.2], but when $X_0$ is reducible, the expression can fail to define a morphism because $L(x)$ can fail to be semistable. The problem of constructing a $L$ such that $L(x)$ is always semistable (i.e. of constructing an Abel map for a reducible curve) is nontrivial. This and related problems are studied in [Cap07, CE07, CCE08, CP10], and we direct the reader to those papers for results about the existence of an Abel map for a reducible curve.

Regardless of whether the Abel map extends, Propositions 1 and 2 show that Hypothesis 1 holds for $\mathcal{J}/S$, so we can form the family $\text{Pic}^0(\mathcal{J}/S)$ of Picard schemes. We prove that this family is related to the generalized Jacobian by an autoduality isomorphism:

**Corollary 6 (Autoduality).** If $\mathcal{J}/S$ is a family of compactified Jacobians extending $J_K$, then (8) extends to an isomorphism

$$(9) \quad \text{Pic}^0(\mathcal{J}/S) \cong \mathcal{J}^3$$

that is pullback by an Abel map when an Abel map is defined.
We prove the result by using comparisons with the Néron model and the Néron mapping property. There are $S$-isomorphisms
\begin{align}
J^\oplus \cong (N^\vee)^\oplus & \text{ by [BLR90] Theorem 1, page 286} \\
\text{Pic}^0(J/S) \cong N^\oplus & \text{ by Theorem 4}
\end{align}
uniquely determined by the requirement that they restrict to the identity on the generic fiber. Here $N^\vee$ is the Néron model of $J_K$ and $N$ is the Néron model of $P_K$.

The autoduality isomorphism $J_K \cong \text{Pic}^0(J_K/K)$ extends to an isomorphism
\begin{equation}
N^\vee \cong N
\end{equation}
by the Néron mapping property. The isomorphism (12) restricts to an isomorphism between identity components, and the composition of this restriction with the isomorphisms (10) and (11) is the desired isomorphism $\text{Pic}^0(J_0/k) \cong J_0^\oplus$. This isomorphism must equal pullback by an Abel map when an Abel map is defined because both morphisms agree on the generic fiber.

\begin{proof}
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\end{proof}

V. AUTODUALITY FOR DEGENERATE ABELIAN VARIETIES

Here we use the Néron Comparison Theorem, Theorem 4, to prove that stable quasialbelian varieties, certain degenerations of principally polarized abelian variety, satisfies an autoduality theorem. In this section $S$ is the spectrum of a discrete valuation ring $R$ with field of fractions $K$ and residue field $k$ that we assume has characteristic zero.

The degenerations we study are degenerations to a stable quasialbelian variety, a type of degeneration defined by Alekeev and Nakamura in [AN99]. In proving that stable quasialbelian varieties satisfy autoduality, we make use of relatively few properties of these varieties. Alexeev and Nakamura construct stable quasialbelian varieties using Mumford’s construction: starting with a family of semialbelian varieties $\tilde{G}/S$, they form an explicit sheaf of graded $O_G$-algebras $\mathcal{A}$, define an action of a discrete group $Y$ on $\text{Proj}(\mathcal{A})$, and then define the quotient of $\text{Proj}(\mathcal{A})$ by $Y$ to be the family of stable quasialbelian varieties $J/S$. In particular, the construction provides an explicit description of the local structure of $J$, and we use that description to prove that $J$ satisfies Hypothesis 4 and then we prove the autoduality theorem using Theorem 4.

While we only make use of the construction in [AN99], Alexeev further developed the properties of quasialbelian varieties in [Ale02]. In that paper (where the term semialbelic is used instead of quasialbelian; see [Ale06, Definition 5.2]), Alexeev defines a stable quasialbelian variety to be a (possibly reducible) seminormal projective variety $J/\text{Spec}(k)$ together with an action of a semialbelian variety $G$ of the same dimension as $J$ such that $J$ has only finitely many $G$-orbits and the stabilizer of any point is connected, reduced, and contained in the maximal multiplicative torus [Ale02, Definition 1.1.5].

Principally polarized abelian varieties satisfy a stable reduction theorem that we now state. A principally polarized abelian variety can be identified with a triple $(G_K, J_K, \Theta_K)$ consisting of an abelian variety $G_K$, a $G_K$-torsor $J_K$, and a Cartier divisor $\Theta_K \subset J_K$ defining
a principal polarization, and after possibly passing to a ramified extension of \( \mathbb{R} \), the pair \((G, \tilde{J})\) can be extended to a pair of \( S \)-flat \( S \)-schemes \((G, \tilde{J})\) such that \( G \) is a semi-abelian variety acting on \( \tilde{J} \) with special fiber a stable quasiabelian variety. This extension is unique if one additionally requires that \( \Theta_k \) extends in a suitable way (that \((G_k, J_k, \Theta_k)\) extends to a family of stable quasiabelian pairs \((G, \tilde{J}, \Theta)\) rather than a family of stable quasiabelian varieties \((G, \tilde{J})\)).

As was explained in the introduction, the ample divisor \( \Theta_k \) determines an autoduality isomorphism

\[
G_k \to \text{Pic}^0(\tilde{J}_k)
\]

\[
g \mapsto \mathcal{O}(\tau_g^*(\Theta_k) - \Theta_k),
\]

where \( \tau_g \) is translation by \( g \). We prove that this isomorphism extends for stable quasiabelic varieties:

**Corollary 7.** If \((G, \tilde{J})\) is a family of stable quasiabelic varieties extending \((G_k, J_k)\), then \((13)\) extends to an isomorphism

\[
G \cong \text{Pic}^0(\tilde{J}/S).
\]

*Proof.* The properties of stable quasiabelic established in [AN99] include Hypothesis 1, and the remainder of the proof is as in Corollary 6. To see this, observe that the scheme \( \tilde{J} \) has rational singularities because its completed local rings are isomorphic to completions of coordinate rings of affine toric varieties, essentially by the construction of \( \tilde{J} \) [AN99, Theorem 3.8(i)] (in that theorem, it is assumed that the special fiber \( G_0 \) is a multiplicative torus, but the local structure for general \( G_0 \) is the same as for \( G_0 \) a multiplicative torus; see e.g. the proof of [AN99, Lemma 4.1]). Similarly, the higher direct image \( R^i p_* \mathcal{O}_{\tilde{J}} \) is a locally free \( \mathcal{O}_S \)-module of rank \( \binom{n}{i} \) whose formation commutes with base change by [AN99, Theorem 4.3]. (The theorem states only that \( h^i((\tilde{J}_0), \mathcal{O}_{\tilde{J}_0}) = \binom{n}{i} \), but the proof shows that \( R^i p_* \mathcal{O}_{\tilde{J}} \) can be identified with the analogous direct image for a family of abelian varieties.)

This shows that Hypothesis 1 is satisfied, so by Theorem 4, \( \text{Pic}^0(\tilde{J}/S)/S \) is the identity component of its Néron model. Similarly \( G/S \) is the identity component of its Néron model by [BLR90, Proposition 3, page 182], and we complete the proof by arguing as in the proof of Corollary IV.

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**Remark 8.** Observe that in Corollary 7 we do not require that the divisor \( \Theta_k \) extends, i.e. we do not require that there is a divisor \( \Theta \) such that \((G, \tilde{J}, \Theta)\) is a family of stable quasiabelian pairs. Consequently, for a given \((G_k, J_k, \Theta_k)\), there can be many extensions of \((G_k, J_k)\) to a family \((G, \tilde{J})\) of stable semiabelic varieties.

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**DEPT. OF MATHEMATICS, UNIVERSITY OF SOUTH CAROLINA, COLUMBIA SC**

*E-mail address: kassj@math.sc.edu*