Analysis of the Ensemble Kalman–Bucy Filter for correlated observation noise

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Abstract. Ensemble Kalman–Bucy filters (EnKBFs) are an important tool in Data Assimilation that aim to approximate the posterior distribution for continuous time filtering problems using an ensemble of interacting particles. In this work we extend a previously derived unifying framework for consistent representations of the posterior distribution to correlated observation noise and use these representations to derive an EnKBF suitable for this setting as a constant gain approximation of these optimal filters. Existence and uniqueness results for both the EnKBF and its mean field limit are provided. The existence and uniqueness of solutions to its limiting McKean-Vlasov equation does not seem to be covered by the existing literature. In the correlated noise case the evolution of the ensemble depends also on the pseudoinverse of its empirical covariance matrix, which has to be controlled for global well posedness. These bounds may also be of independent interest. Finally the convergence to the mean field limit is proven. The results can also be extended to other versions of EnKBFs.

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1 Introduction

The aim of filtering algorithms is to approximate the state of a signal from noisy and potentially incomplete or indirect observations. Ensemble Kalman–Bucy filters

1 The time continuous version of classical Ensemble Kalman filters.
1 INTRODUCTION

dynamics enriched by an added interaction term that nudges them in the direction of the observations.

Ensemble Kalman(–Bucy) Filters were first introduced in [18] and are nowadays widely used for data assimilation tasks in many scientific fields such as meteorology and the geosciences. For uncorrelated observation noise there exists a rich literature treating the mathematical theory of EnKBFs. See for example the recent overview papers [3],[8], and the references found therein. In this work we derive an EnKBF for correlated observation noise and analyse its mean field limit.

More precisely for an arbitrary, but fixed timeframe \( T > 0 \) we consider a \( \mathbb{R}^{d_x} \)-valued signal process \((X_t)_{t \in [0,T]}\) determined by the stochastic differential equation (SDE)

\[
\diff X_t = B_t(X_t)\diff t + C_t(X_t)\diff W_t + \tilde{C}_t(X_t)\diff V_t, \tag{1}
\]

Both \( W \) and \( V \) shall be two independent \( \mathbb{R}^{d_x} \)- and \( \mathbb{R}^{d_v} \)-dimensional Brownian motions. Furthermore we make the following standard assumption regarding the coefficients of the equation.

**Assumption 1.** The drift \( B : [0,T] \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x} \) and the (square root of the) diffusion \( C : [0,T] \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x \times d_w} \) are Borel-measurable and satisfy the usual linear growth and global Lipschitz conditions found for example in [25]. The corresponding Lipschitz constants shall be denoted by \( \text{Lip}(B) \) and \( \text{Lip}(C). \) \( \tilde{C} : [0,T] \times \mathbb{R}^{d_x} \to \mathbb{R}^{d_x \times d_v} \) is assumed to be a continuous matrix valued function.

The \( \mathbb{R}^{d_v} \)-valued observation process \((Y_t)_{t \in [0,T]}\) is then given by

\[
dY_t = H_t X_t \diff t + \Gamma_t \diff V_t,
\]

where both \( H : [0,T] \to \mathbb{R}^{d_y \times d_x} \) and \( \Gamma : [0,T] \to \mathbb{R}^{d_v \times d_w} \) are continuous. As is usual, we assume that \( Y_0 = 0. \)

The goal of stochastic filtering is to compute, or at least approximate, the conditional distribution of \( X_t \) given all past observations \((Y_s)_{s \in [0,t]}\), which we denote by

\[
\eta_t := \mathbb{P} \left( X_t \in \cdot \mid Y_s, s \leq t \right), \tag{2}
\]

for all times \( t \in [0,T] \). \( \eta_t \) is referred to as the posterior distribution or the optimal filter. We shall denote the integral of a testfunction \( \phi \) with respect to \( \eta_t \) by \( \eta_t(\phi) := \int_{\mathbb{R}^{d_x}} \phi(x) \eta_t(dx). \)

For any sufficiently regular and integrable testfunction \( \phi \), the weak form of the Kushner–Stratonovich equation [2], [33]

\[
\diff \eta_t(\phi) = \eta_t(L_t \phi) \diff t + \eta_t(H_t \text{id}_{\mathbb{R}^{d_y}} \phi)^T - \eta_t(H_t \text{id}_{\mathbb{R}^{d_y}} \phi)^T \eta_t(\phi) + \eta_t \left( (\nabla \phi)^T \tilde{C}_t \Gamma_t^T \right) R_t^{-1} \diff t \tag{3}
\]

\[
dI_t = dY_t - \eta_t(H_t \text{id}_{\mathbb{R}^{d_y}} \phi) \diff t
\]

is satisfied. Hereby \( L_t \) is the generator associated to (1) and \( R_t := \Gamma_t \Gamma_t^T \). In many applications the dimension of the signal \( d_x \) is too large, so that an approximation of (3) using standard numerical PDE solvers becomes infeasible and even in one dimension they are not efficient for unstable/transient signals. Sequential Monte-Carlo type methods called particle filters provide an alternative approximation of the optimal filter that is often considered, however these methods can suffer from weight degeneracy which makes them scale poorly with the state dimension. Instead practitioners often rely on EnKBFs, which, even though they only provide an approximation to the optimal filter in the linear Gaussian case, have proven to be a successful tool for Data Assimilation tasks even for nonlinear signals. A particular version of EnKBF that will be the main focus
of this paper is
\[ dX^i_t = B_t \left( X^i_t \right) dt + C_t \left( X^i_t \right) dW^i_t + \tilde{C}_t dV^i_t + \left( P^M_t H^T_t + \tilde{C}_t \Gamma^T_t \right) R_t^{-1} \left( dY_t - \frac{H_t \left( X^i_t + x^M_t \right)}{2} dt \right) \]
\[ - \left( P^M_t H^T_t + \tilde{C}_t \Gamma^T_t \right) R_t^{-1} \Gamma_t \tilde{C}_t \left( P^M_t \right)^+ X^i_t - x^M_t \frac{dt}{2} \]
for \( i = 1, \cdots, M \), with
\[ x^M_t := \frac{1}{M} \sum_{i=1}^{M} X^i_t \quad \text{and} \quad P^M_t := \frac{1}{M-1} \sum_{i=1}^{M} \left( X^i_t - x^M_t \right) \left( X^i_t - x^M_t \right)^T, \]
and where \( \left( P^M_t \right)^+ \) denotes the Moore–Penrose pseudoinverse of \( P^M_t \). For \( \tilde{C} = 0 \) equation (4) is the continuous-time counterpart to the deterministic Ensemble Kalman filter introduced in [37]. We will prove well posedness of both (4) and its mean field limit and also show propagation of chaos. Our analysis can easily be extended to other types of EnKBFs.

Let us now briefly summarize the structure and the contributions of this paper:

- In section 2 we formally derive a McKean–Vlasov equation that is consistent with (3), meaning that if a solution exists, its law will evolve according to (3). For the uncorrelated case \( \tilde{C} = 0 \) this has already been done in [35] in a unifying framework covering existing mean field filters like [11] and the well known feedback particle filter (FPF) [44], [45]. In the correlated noise case FPFs have already been derived in [30] and [33]. In this work we extend the framework in [35] to arbitrary \( \tilde{C} \).

- In section 3 we derive a mean field EnKBF as a special case of the consistent mean field representation of section 2 in a linear Gaussian filtering setting. In the uncorrelated noise framework continuous time McKean–Vlasov interpretations of the filtering equation with Ensemble Kalman filtering mean field interpretations for linear Gaussian models have been discussed for the first time in [15]. We use arguments first found in [15] to prove the well posedness of the mean field EnKBF in the correlated noise framework.

- In section 4 we first interpret the mean field EnKBF in the nonlinear, non-Gaussian setting as an approximation of the optimal filter derived in section 2. In the uncorrelated setting this connection is often referred to as the constant gain approximation of the feedback particle filter [40]. In subsection 4.4 we then show the well posedness of the mean field EnKBF for nonlinear signals and correlated observation noise. Previous results on this matter [12] require the observation function \( H \) to be bounded. Linear observations, which are highly relevant in practice, do not seem to be covered by existing literature, even in the simpler case of uncorrelated observation noise. We prove the well posedness for this case by using a mixture of a fixed point and a deterministic localization argument. Finally, subsection 4.5 discusses other mean field EnKBFs. Our well posedness result can be extended to these McKean–Vlasov equations in a straightforward manner.

- In section 5 we show the well posedness of of the particle system approximating the mean field EnKBF, which we just refer to as the EnKBF. For uncorrelated observation noise this was already proven in [28], however the correlated noise framework requires an additional term depending on the (pseudo) inverse of the ensemble covariance matrix, that may become
singular. We prove existence and uniqueness of strong solutions under suitable assumptions by showing that these singularities will never be hit by the filter. Due to the stochasticity of the ensemble covariance matrix we cannot apply techniques for bounding the spectrum from below found for example in [14] and instead dominate its inverse in terms of the ensemble covariance itself. This technique may be of independent interest.

• In section 6 we derive a time dependent propagation of chaos result, extending previous results in the uncorrelated case found for example in [28]. In the uncorrelated, linear Gaussian setting propagation of chaos has first been proven in [15]. In this setting the mean field limit is the optimal filter and several uniform in time estimates for the convergence exist (e.g.[4],[5],[15],[16]), providing error/inconsistency bounds for the EnKBF. In the nonlinear setting our propagation of chaos result no longer gives these error bounds as the mean field limit does not coincide with the posterior. Nevertheless the mean field limit provides a connection to the optimal filter via the constant gain approximation discussed in subsection 4.1 and as such may be an important tool for quantifying the inconsistency of the EnKBF.

2 Mean field representation of the posterior

Assumption 2. To achieve greater generality in our results throughout this section, we allow both $\tilde{C}$ and $H$ to be state dependent functions. I.e. for every $t \geq 0$ we consider the (nonlinear) maps $x \mapsto C_t(x)$ and $x \mapsto H_t(x)$. This assumption is restricted to section 2.

In [35] a representation of the posterior distribution in the uncorrelated setting by a diffusion process was derived, by matching the Kolmogorov equation of said diffusion to the Kushner–Stratonovich equation. This gives a class of mean field equations that contains many well known optimal filters like the Feedback Particle Filter [44]. In this section we follow the approach of [35] in order to derive consistent mean–field representations in the correlated noise framework. Thus we aim to represent the posterior through a McKean–Vlasov equation, i.e. we want to find a diffusion process $(\tilde{X}_t)_{t \geq 0}$ such that for all times $t > 0$, its (conditional) marginal law $\tilde{\eta}_t$ is given by the posterior $\eta_t$. To this end we assume throughout this section that all functions appearing are sufficiently regular and integrable, so that we can always differentiate and integrate whenever necessary.

Let $\tilde{W}$ be an independent copy of the Brownian motion $W$ and $\tilde{V}$ be an independent copy of $V$. To determine $\tilde{X}$ we make the Ansatz

$$d\tilde{X}_t = B_t(\tilde{X}_t)dt + C_t(\tilde{X}_t)d\tilde{W}_t + \tilde{C}_t(\tilde{X}_t)d\tilde{V}_t + K_t(\tilde{X}_t, Y_{0:t}, \tilde{\eta}_t) dY_t + a_t(\tilde{X}_t, Y_{0:t}, \tilde{\eta}_t) dt,$$

for two functions $K$ and $a$ depending on both the state and the (marginal) law of the process $\tilde{X}$. To make the class of potential processes $\tilde{X}$ even larger, we also allow for a dependence on all past observations $Y_{0:t}$, which as we will later on see, can be dropped. Furthermore let us make the following notational convention, which helps to shorten formulas.

Notation 3. For any random variable $Z$ we denote the conditional expectation with respect to all observations $Y$ on the total time interval $[0,T]$ by

$$\mathbb{E}_Y[Z] := \mathbb{E}[Z | Y_t, t \leq T].$$

Remark 4. Notation 3 is in accordance to [35]. Due to the independence of $Y$ from $\tilde{W}$ and $\tilde{V}$, another suitable way to define $\mathbb{E}_Y$ would be to let it denote the integral with respect to the joint law by $\tilde{W}$ and $\tilde{V}$. In both cases we have

$$\mathbb{E}[Z] = \int \mathbb{E}_Y[Z] P(Y \in dy).$$
Remark 5. Another way to view the McKean–Vlasov equations derived in this and the subsequent sections is to interpret the observations \( Y \) as a deterministic rough path. This is a research direction that has recently received attention in the EnKBF literature (e.g. [12]) and is a natural modelling choice, as in practice one will often only have the data of a single realization/path of \( Y \). Large parts of our calculations, in particular all results in the sections 2, 3 and 4, also hold in this setting. In particular the well posedness proof in subsection 4.4 can easily be adapted to the case when \( Y \) is a deterministic rough path, using the recent well posedness result for rough-stochastic differential equations found in [20].

Let \( \phi : \mathbb{R}^d \to \mathbb{R} \) be some sufficiently regular and bounded testfunction. Using Itô’s formula and taking the (conditional) expectation \( \mathbb{E}_Y \), we derive for \( \bar{\eta}_t(\phi) := \mathbb{E}_Y \left[ \phi \left( \bar{X}_t \right) \right] \) the equation

\[
\begin{align*}
\mathrm{d} \bar{\eta}_t(\phi) &= \bar{\eta}_t(L_t \phi) \mathrm{d} t + \bar{\eta}_t \left( \nabla \phi \cdot K_t (\cdot, Y_{0:t}, \bar{\eta}_t) \right) \mathrm{d} Y_t \\
&\quad + \frac{1}{2} \bar{\eta}_t \left( \text{tr} \left[ \phi'' K_t (\cdot, Y_{0:t}, \bar{\eta}_t) R_t K_t (\cdot, Y_{0:t}, \bar{\eta}_t)^T \right] \right) \mathrm{d} t.
\end{align*}
\]

(6)

Hereby \( \nabla \phi \) denotes the gradient of \( \phi \) and \( \phi'' \) its Hessian.

Since \( \bar{\eta}_t \) shall coincide with the posterior \( \eta_t \), it must also adhere to the Kushner–Stratonovich equation (3). By comparing the terms on the right-hand side of both equations (3) and (6), we get the two consistency conditions

\[
\bar{\eta}_t \left( \nabla \phi \cdot K_t (\cdot, Y_{0:t}, \bar{\eta}_t) \right) = \left( \bar{\eta}_t \left( H_t^T \phi \right) - \bar{\eta}_t \left( H_t^T \bar{\eta}_t (\phi) + \bar{\eta}_t \left( (\nabla \phi)^T \bar{\Gamma}_t \right) \right) \right) R_t^{-1}
\]

(7)

and

\[
\begin{align*}
\bar{\eta}_t \left( \nabla \phi \cdot a_t (\cdot, Y_{0:t}, \bar{\eta}_t) \right) + \frac{1}{2} \bar{\eta}_t \left( \text{tr} \left[ \phi'' K_t (\cdot, Y_{0:t}, \bar{\eta}_t) R_t K_t (\cdot, Y_{0:t}, \bar{\eta}_t)^T \right] \right) \\
&= - \left( \bar{\eta}_t \left( H_t^T \phi \right) - \bar{\eta}_t \left( H_t^T \bar{\eta}_t (\phi) + \bar{\eta}_t \left( (\nabla \phi)^T \bar{\Gamma}_t \right) \right) \right) R_t^{-1} \bar{\eta}_t (H_t).
\end{align*}
\]

(8)

Since the right hand sides of both equations do not depend on past observations \( Y_{0:t} \), we can drop the \( Y \)-dependence in both \( K \) and \( a \). Note that both \( K_t (\cdot, \cdot) \) and \( a_t (\cdot, \cdot) \) are thus purely statistical quantities of the distribution \( \bar{\eta}_t \).

Assuming that \( \bar{\eta}_t \) admits a sufficiently regular density, which shall also be denoted by \( \bar{\eta}_t \), we derive more direct characterizations of the two terms \( K \) and \( a \) in the following.

2.1 Consistency of the Kalman-gain

First we investigate the consistency condition (7) for the Kalman gain term \( K \). To this end we make the following notational conventions that will be used throughout this paper.

Notation 6. The divergence of a matrix-valued function \( A \) shall be interpreted columnwise, i.e. if \( A(x) \in \mathbb{R}^{m \times k} \) we have

\[
\text{div}(A) := (\text{div}(A_{1,:}), \ldots, \text{div}(A_{k,:})) := \left( \sum_{j=1}^m \partial_{x_j} A_{j,1}, \ldots, \sum_{j=1}^m \partial_{x_j} A_{j,k} \right).
\]

Furthermore we also interpret the scalar product between a matrix and a vector columnwise. Thus the scalar product between \( A \) and the gradient \( \nabla \) gives the following row vector-valued differential operator

\[
A \cdot \nabla := (A_{1,:} \cdot \nabla, \ldots, A_{k,:} \cdot \nabla) := \left( \sum_{j=1}^m A_{j,1} \partial_{x_j}, \ldots, \sum_{j=1}^m A_{j,k} \partial_{x_j} \right).
\]
Employing integration by parts and the Fundamental Theorem of Calculus of Variations, we see that (7) can be interpreted as the weak form of the partial differential equation
\[-\text{div} \left( \bar{\eta}_t K_t (\cdot, \bar{\eta}_t) \right) + \text{div} \left( \bar{\eta}_t C_t \Gamma_t^T \right) R_t^{-1} \bar{\eta}_t = (H_t^T - \bar{\eta}_t(H_t^T)) R_t^{-1} \bar{\eta}_t. \tag{9}\]
If we denote the dependence of $K$ on $\tilde{C}$ by $K^{\tilde{C}}$, then we clearly have
\[K^{\tilde{C}}_t = K^0_t + \tilde{C}_t \Gamma_t^T R_t^{-1}, \tag{10}\]
with
\[-\text{div} \left( \bar{\eta}_t K^0_t (\cdot, \bar{\eta}_t) \right) = (H_t^T - \bar{\eta}_t(H_t^T)) R_t^{-1} \bar{\eta}_t. \tag{11}\]
Thus $K^{\tilde{C}}$ is just a translation of $K^0$, the Kalman gain for the uncorrelated case, which in turn is defined uniquely up to $\bar{\eta}_t$-harmonic vector fields.

**Remark 7.** By using partial integration to rewrite (11) into flux form
\[\int_D \bar{\eta}_t K^0_t (\cdot, \bar{\eta}_t) \cdot (-\nu_D) ds = \int_D (H_t^T - \bar{\eta}_t(H_t^T)) R_t^{-1} \bar{\eta}_t dx,\]
where $D$ is an arbitrary domain and $\nu_D$ is its outer normal vector, we see that $K^0_t (\cdot, \bar{\eta}_t)$ can be interpreted as the velocity (speed and direction) by which particles of density $\bar{\eta}_t$ must travel, such that for every domain $D$ the flux into $D$ is equal to the difference between the expected observation in that domain and the global expected observation. Therefore particles are pushed into areas where the observations are expected to deviate from average observation in the whole ensemble. Thus $K^0_t (\cdot, \bar{\eta}_t)$ can be seen as a quantity of the distribution $\bar{\eta}_t$ that corresponds to the exploration of the state space with respect to the observation function $H$.

### 2.2 Consistency of the correctional transport term

Next we investigate the consistency equation (8) for the correctional transport term $a$. Note that while $K$ also shows up in this equation, it is already fully determined by (7).

By again using integration by parts and the Fundamental Theorem of Calculus of Variations, we derive the strong form of consistency condition (7)
\[-\text{div} \left( \bar{\eta}_t a_t (\cdot, \bar{\eta}_t) \right) + \frac{1}{2} \sum_{i,j=1}^{d_x} \partial^2_{x_i x_j} \left( \bar{\eta}_t K_t (\cdot, \bar{\eta}_t) R_t K_t (\cdot, \bar{\eta}_t)^T \right)_{ji} \]
\[-\text{div} \left( \bar{\eta}_t C_t \Gamma_t^T \right) R_t^{-1} \bar{\eta}_t (H_t) = - (H_t^T - \bar{\eta}_t(H_t^T)) R_t^{-1} \bar{\eta}_t(H_t) \bar{\eta}_t. \tag{12}\]
Using
\[\sum_{i,j=1}^{d_x} \partial^2_{x_i x_j} \left( \bar{\eta}_t K_t (\cdot, \bar{\eta}_t) R_t K_t (\cdot, \bar{\eta}_t)^T \right)_{ji} = \text{div} \left( K_t (\cdot, \bar{\eta}_t) R_t \text{div} (\bar{\eta}_t K_t (\cdot, \bar{\eta}_t)^T) \right) + \text{div} \left( \bar{\eta}_t ((K_t (\cdot, \bar{\eta}_t) \cdot \nabla) R_t K_t (\cdot, \bar{\eta}_t)^T) \right) \]
and the consistency equation of the Kalman gain term (9) gives us the identity
\[\sum_{i,j=1}^{d_x} \partial^2_{x_i x_j} \left( \bar{\eta}_t K_t (\cdot, \bar{\eta}_t) R_t K_t (\cdot, \bar{\eta}_t)^T \right)_{ji} = -\text{div} \left( \bar{\eta}_t K_t (\cdot, \bar{\eta}_t) (H_t - \bar{\eta}_t(H_t)) \right) + \text{div} \left( K_t (\cdot, \bar{\eta}_t) \text{div} \left( \bar{\eta}_t C_t \Gamma_t^T \right) \right) \]
\[+ \text{div} \left( \bar{\eta}_t ((K_t (\cdot, \bar{\eta}_t) \cdot \nabla) R_t K_t (\cdot, \bar{\eta}_t)^T) \right).\]
Thus we can rewrite (12) into

\[- \text{div} \left( \tilde{\eta} a_t (\cdot, \tilde{\eta}_t) \right) - \frac{\text{div} \left( \tilde{\eta}_t K_t (\cdot, \tilde{\eta}_t) (H_t - \tilde{\eta}_t(H_t)) \right)}{2} - \text{div} \left( \tilde{\eta}_t \tilde{C}_t \Gamma_t^T \right) R_t^{-1} \tilde{\eta}_t(H_t) + \frac{\text{div} \left( K_t (\cdot, \tilde{\eta}_t) \text{div} \left( \tilde{\eta}_t \tilde{C}_t \Gamma_t^T \right) \right)^T}{2} + \frac{\text{div} \left( \tilde{\eta}_t \left( (K_t (\cdot, \tilde{\eta}_t) \cdot \nabla) R_t K_t^T (\cdot, \tilde{\eta}_t) \right)^T \right)}{2} \]

\[= - \left( H_t^T - \tilde{\eta}_t (H_t^T) \right) R_t^{-1} \tilde{\eta}_t(H_t) \tilde{\eta}_t.\]

Just as in [35] one sees immediately that

\[a_t (\cdot, \tilde{\eta}_t) = - \frac{K_t (\cdot, \tilde{\eta}_t) (H_t + \tilde{\eta}_t(H_t))}{2} + \frac{\left( (K_t (\cdot, \tilde{\eta}_t) \cdot \nabla) R_t K_t^T (\cdot, \tilde{\eta}_t) \right)^T}{2} + \Omega^0_t,\]

where \(\Omega^0_t\) is an arbitrary \(\tilde{\eta}_t\)-harmonic field. Thus the full equation governing the evolution of \(\tilde{X}\) is given by

\[d\tilde{X}_t = B_t(\tilde{X}_t)dt + C_t(\tilde{X}_t) dW_t + \tilde{C}_t(\tilde{X}_t) dV_t + \frac{H_t(\tilde{X}_t) + \int H_t(x) \tilde{\eta}_t(x) dx}{2} dt + \frac{\left( (K_t (\tilde{X}_t, \tilde{\eta}_t) \cdot \nabla) R_t K_t (\tilde{X}_t, \tilde{\eta}_t) \right)^T}{2} dt + \frac{\text{div} \left( \tilde{\eta}_t \tilde{C}_t \Gamma_t^T \right)^T (\tilde{X}_t)}{2} \tilde{\eta}_t dt + \Omega^0_t(\tilde{X}_t) dt.\]

**Remark 8.** We note that

\[\frac{\text{div} \left( \tilde{\eta}_t \tilde{C}_t \Gamma_t^T \right)}{\tilde{\eta}_t} = (\nabla \log \tilde{\eta}_t)^T \tilde{C}_t \Gamma_t^T + \text{div} \left( \tilde{C}_t \Gamma_t^T \right)\]

and thus the correction in (14) for the correlated observation noise can be rewritten using the gradient of the logarithmic density. This also shows that, just as stated in [35], if \(\tilde{C}_t(x) = \tilde{C}_t\) for all \(t \geq 0\), the correlated observation case can be interpreted in terms of the uncorrelated case with a modified observation map \(\tilde{H}_t := H_t - \Gamma_t \tilde{C}_t^T \nabla \log \tilde{\eta}_t\).

**Remark 9.** Note that in the case of one-dimensional observations \(d_y = 1\), the term \(R_t\) is scalar and \(K_t (\cdot, \tilde{\eta}_t)\) is a \(\mathbb{R}^d\)-valued function. One then sees immediately that

\[\left( (K_t (\cdot, \tilde{\eta}_t) \cdot \nabla) R_t K_t^T (\cdot, \tilde{\eta}_t) \right)^T = R_t \left( (K_t (\cdot, \tilde{\eta}_t) \cdot \nabla) K_t (\cdot, \tilde{\eta}_t) \right).\]

This is the convective change of the field \(K\) under a flow with velocity \(K\).

**Remark 10.** We note that for the McKean–Vlasov process \(\tilde{X}\) in (14) (and more general any diffusion process driven by \(Y\)) the tower property of the conditional expectation gives

\[\tilde{\eta}_t(x) := \mathbb{P} \left( \tilde{X}_t \in dx \mid Y_{0:T} \right) = \mathbb{P} \left( \tilde{X}_t \in dx \mid Y_{0:t} \right) \text{ for all } t \geq 0.\]

Thus one does not need to fix a timeframe \([0, T]\) a priori and can actually compute \(\tilde{X}\) online.
3 The mean field EnKBF in the linear Gaussian setting

Notation 11. For any matrix $A$ its symmetric part is denoted by $\text{Sym}(A) := \frac{A + A^T}{2}$.

When $B_t$ and $H_t$ are linear and $C_t, \bar{C}_t$ are constant matrices for all $t \geq 0$, it is well known that if $\eta_0$ is Gaussian, then the posterior $\eta_t$ will also be Gaussian for all times $t \geq 0$. We denote its (conditional) mean by $\bar{m}_t := \mathbb{E}_Y \left[ X_t \right] \in \mathbb{R}^{d_x}$ and its (conditional) covariance by $\bar{P}_t := \mathbb{Cov}_Y \left[ X_t, X_t \right] \in \mathbb{R}^{d_x \times d_x}$, i.e. $\bar{\eta}_t = \eta_t = \mathcal{N}(\bar{m}_t, \bar{P}_t)$.

Note that in this case one can easily verify that for any matrix $A \in \mathbb{R}^{d_x \times k}$ with $k \in \mathbb{N}$ arbitrary, it holds that

$$\text{div} ( A \bar{\eta}_t ) = - (x - \bar{m}_t)^T \bar{P}_t^{-1} A \bar{\eta}_t. \quad (15)$$

This motivates the Ansatz $K_t^0 = \bar{P}_t H_t^T R_t^{-1}$, which in turn results in the Kalman gain

$$K_t = \left( \bar{P}_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1}.$$

Since $K_t$ only depends on the distribution $\bar{\eta}_t$, but not on the state variable $X_t$, it follows that

$$\left( (K_t \cdot, \bar{\eta}_t) \cdot \nabla \right) R_t K_t^0 (\cdot, \bar{\eta}_t)^T = 0$$

and therefore we obtain by using (15) and by setting $\Omega^0 = 0$

$$a_t (x, \bar{\eta}_t) = - \left( \bar{P}_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \left( \frac{H_t (x + \bar{m}_t)}{2} + \Gamma_t \bar{C}_t \bar{P}_t^{-1} \left( \frac{x - \bar{m}_t}{2} \right) \right).$$

Thus in the linear Gaussian case the following corollary holds.

Corollary 12. Assume that $X_0$ is Gaussian, and that for all times $t \geq 0$ both $B_t, H_t$ are linear and both $C_t, \bar{C}_t$ are constant. Assuming that $\bar{X}$ is the unique solution of the McKean-Vlasov equation

$$d\bar{X}_t = B_t \bar{X}_t dt + C_t dW_t + \bar{C}_t d\bar{V}_t$$

$$+ \left( \bar{P}_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \left( dY_t - \frac{H_t (\bar{X}_t + \bar{m}_t)}{2} dt \right) \quad (16)$$

$$- \left( \bar{P}_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \Gamma_t \bar{C}_t \bar{P}_t^{-1} \left( \frac{\bar{X}_t - \bar{m}_t}{2} \right) dt,$$

where again $\bar{P}_t := \mathbb{Cov}_Y \left[ X_t \right]$ and $\bar{m}_t := \mathbb{E}_Y \left[ X_t \right]$, satisfying the initial condition Law $\left( \bar{X}_0 \right) = \eta_0 = \text{Law} \left( X_0 \right)$.

Then it holds that Law $\left( \bar{X}_t \right) = \eta_t$ for all times $t \geq 0$. Thus $\bar{X}$ is a consistent mean field representation of the posterior $\eta$.

Note that in Corollary 16 we have assumed that equation (16) is well posed. Indeed since (16) is a singular McKean–Vlasov equation, that even in the uncorrelated case only satisfies local Lipschitz conditions, this equation falls outside of the standard theory for the well posedness of McKean–Vlasov equations (for example found in [7]). To prove existence and uniqueness of solutions, one can use the relation of (16) to the famous Kalman–Bucy filter, which is another representation of the posterior in the linear Gaussian setting.
It is straightforward to derive that the (conditional) mean $\bar{m}$ and the (conditional) covariance $\bar{P}$ of the process $\bar{X}$ in Corollary 12 satisfy

\[ \frac{d\bar{m}_t}{dt} = B_t \bar{m}_t dt + \left( \bar{P}_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \left( dY_t - H_t \bar{m}_t dt \right) \quad (17a) \]

\[ \frac{d\bar{P}_t}{dt} = B_t \bar{P}_t + \bar{P}_t B_t^T + C_t \bar{C}_t^T + \bar{C}_t \Gamma_t^{2T} - \left( \bar{P}_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \left( H_t \bar{P}_t + \Gamma_t \bar{C}_t^T \right). \quad (17b) \]

These are the famous Kalman–Bucy equations [24, chapter 7, page 228] for the correlated noise setting. This again shows the consistency of $\bar{X}$ in the linear Gaussian case. Note that (17b) is completely decoupled from the observation process and is thus a deterministic quantity.

Of course one can consider system (17) detached from (16) and easily show the global existence and uniqueness of solutions, which can then be used to prove global existence and uniqueness of solutions to (16) via a fixed point argument. For the uncorrelated case a well posedness result can also be found in [10, Remark 2.1], using an argument that was first derived in [15, Lemma 5.2] and can easily be applied to the correlated framework as well. It proves the following Lemma. For the sake of completeness and the convenience of the reader we also state the proof, arguing just as in [15].

**Lemma 13.** Assume that $\bar{P}_0 := \text{Cov} \ [X_0]$ is invertible. Then there exists a unique solution $\bar{X}$ to (16) satisfying the initial condition $X_0 = \bar{X}_0$.

**Proof.** We make a fixed point argument. Let $(P_t)_{t \geq 0}$ be the solution to (17b) with initial condition $P_0$. Since the solutions to the matrix Riccati equation (17b) stay positive definite, $P_t$ is invertible for every time $t \geq 0$. Now we define the process $\bar{X}^P$ to be the solution to the linear McKean–Vlasov equation

\[ d\bar{X}^P_t = B_t \bar{X}^P_t dt + C_t d\bar{W}_t + \bar{C}_t d\bar{V}_t \]

\[ + \left( P_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \left( dY_t - \frac{H_t \left( \bar{X}^P_t + \text{E}_Y [\bar{X}^P_t] \right)}{2} dt \right) \]

\[ - \left( P_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \Gamma_t \bar{C}_t^T P_t^{-1} \frac{\bar{X}^P_t - \text{E}_Y [\bar{X}^P_t]}{2} dt \]

with initial condition $\bar{X}^P_0 = X_0$.

We set $P_t := \text{Cov} \ [\bar{X}^P_t, \bar{X}^P_t]$. It is easy to see that $\bar{P}$ satisfies the linear equation

\[ \frac{d\bar{P}_t}{dt} = B_t \bar{P}_t + \bar{P}_t B_t^T + C_t \bar{C}_t^T + \bar{C}_t \Gamma_t^T - 2 \text{Sym} \left( \left( P_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \left( H_t + \Gamma_t \bar{C}_t^T P_t^{-1} \right) \right). \]

But this equation is also satisfied by $P$, the solution to the Riccati equation (17b). Thus by uniqueness of linear equations we derive $P = \bar{P}$ and therefore $\bar{X}^P$ is a solution to (16). Given two solutions $\bar{X}^i$, $i = 1, 2$ of (16), their (conditional) covariance matrices $\text{Cov} \ [\bar{X}^i_t, \bar{X}^i_t]$, $i = 1, 2$ satisfy the Riccati equation (17b). Thus by the uniqueness of Riccati equations $\text{Cov} \ [\bar{X}^1_t, \bar{X}^1_t] = \text{Cov} \ [\bar{X}^2_t, \bar{X}^2_t]$. Due to the uniqueness of the linear equation (18), we can thus conclude $\bar{X}^1 = \bar{X}^2$.

Unlike for the consistent filter in the general nonlinear setting (14), it is clear how (16) can be approximated using an interacting particle system. For $M \in \mathbb{N}$ let $W^i$, $i = 1, \ldots, M$ and
4 The mean field EnKBF for nonlinear signals

For nonlinear signals the posterior cannot be expected to be Gaussian and thus the gain term will in general not allow for a explicit description, that admits a simple statistical approximation by interacting particles. Thus deriving suitable numerical approximations to \( (14) \) is challenging. Even though some progress on this subject has been made in recent years (see for example \[6\] for a Galerkin based approach, \[42\] for a diffusion maps approach and \[34\] for a neural networks based approach), in practice the EnKBF (19), and its discrete time analogue, is still widely used in the uncorrelated case, even for nonlinear signals, where it will not approximate the optimal filter. A common justification for this is that if the posterior \( \eta_t \) and by that also the density of the optimal filter \( \bar{\eta}_t \) are close to a Gaussian, then the mean field EnKBF

\[
\frac{dX_t}{dt} = B_t (X_t) dt + C_t (X_t) dW_t + \hat{C}_t dV_t \\
+ \left( \hat{P}_t H_t^T + \hat{C}_t \Gamma_t^T \right) R_t^{-1} \left( d\hat{Y}_t - H_t (X_t + \hat{m}_t) dt \right) \\
- \left( \hat{P}_t H_t^T + \hat{C}_t \Gamma_t^T \right) R_t^{-1} \Gamma_t \hat{C}_t^T \left( \hat{P}_t \right)^+ \frac{X_t - \hat{m}_t}{2} dt,
\]

\( \text{where} \)

\[
x_t^M := \frac{1}{M} \sum_{i=1}^{M} X_t^i \text{ and } P_t^M := \frac{1}{M-1} \sum_{i=1}^{M} (X_t^i - x_t^M)(X_t^i - x_t^M)^T
\]

denote the ensemble average and the ensemble covariance matrix. \( (P_t^M)^+ \) denotes the Moore–Penrose pseudoinverse of \( P_t^M \). The pseudoinverse has to be used as for small ensemble sizes \( M \leq d_x \), which are commonly used in practice, the ensemble covariance \( P_t^M \) can not be invertible.

\[4\]

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\[4\]
Therefore we want to discuss another connection between the EnKBF (20) and the consistent filter (14) that in literature is often referred to as the constant gain approximation [40].

4.1 Constant gain approximation of the optimal filter

By using partial integration and (11) one derives immediately the identity

\[
\mathbb{E}_Y[K_t^0(\bar{X}_t, \bar{\eta}_t)] = \int_{\mathbb{R}^d_x} K_t^0(x, \bar{\eta}_t) \bar{\eta}_t(x) \, dx = \int_{\mathbb{R}^d_x} (x - \bar{m}_t) \left(-\div(\bar{\eta}_t K_t^0(\cdot, \cdot))\right) \, dx
\]

\[
= \int_{\mathbb{R}^d_x} (x - \bar{m}_t) (H_t x - H_t \bar{m}_t)^T R_t^{-1} \bar{\eta}_t(x) \, dx = \tilde{P}_t H_t^T R_t^{-1}.
\]

Thus the average of \( K \) is given by

\[
\mathbb{E}_Y[K_t(\bar{X}_t, \bar{\eta}_t)] = \left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1}.
\] (22)

**Remark 14.** For nonlinear observations \( H_t \) and state dependent \( \tilde{C}_t \) the constant gain relation (22) also holds and becomes

\[
\mathbb{E}_Y[K_t(\bar{X}_t, \bar{\eta}_t)] = \left(\mathbb{E}_{\mathbb{W}_Y}[\bar{X}_t, H_t(\bar{X}_t)] + \mathbb{E}_Y[\tilde{C}_t(\bar{X}_t)] \Gamma_t^T\right) R_t^{-1}.
\]

While (22) explains the constant gain term in the EnKBF (20), it does not motivate the approximation of \( \nabla \log \bar{\eta}_t(\bar{X}_t) \) by \( \tilde{P}_t^{-1}(\bar{X}_t - \bar{m}_t) \). To this end we note that for any affine function \( p(x) := \alpha x + \beta \) with \( \alpha \in \mathbb{R}^{d_x \times d_x} \) and \( \beta \in \mathbb{R}^{d_x} \) one derives by partial integration that

\[
\int_{\mathbb{R}^d_x} (\nabla \log \bar{\eta}_t(x) \cdot p(x)) \bar{\eta}_t(x) \, dx = \int_{\mathbb{R}^d_x} (\nabla \bar{\eta}_t(x) \cdot p(x)) \, dx = \int_{\mathbb{R}^d_x} \div(p) \bar{\eta}_t(x) \, dx
\]

\[
= -\text{tr} \alpha = -\text{tr} \alpha^T = -\text{tr} \left[\tilde{P}_t^{-1} P_t \alpha^T\right] = -\mathbb{E}_Y \left[\text{tr} \left[\tilde{P}_t^{-1}(\bar{X}_t - \bar{m}_t)(\bar{X}_t - \bar{m}_t)^T \alpha^T\right]\right]
\]

\[
= -\mathbb{E}_Y \left[\tilde{P}_t^{-1}(\bar{X}_t - \bar{m}_t) \cdot (\alpha \bar{X}_t - \alpha \bar{m}_t)\right] = \int_{\mathbb{R}^d_x} (-\tilde{P}_t^{-1}(x - \bar{m}_t)) \cdot p(x) \bar{\eta}_t(x) \, dx.
\]

Therefore \( -\tilde{P}_t^{-1}(x - \bar{m}_t) \) is the \( \bar{\eta}_t \)-orthogonal projection of \( \nabla \log \bar{\eta}_t \) onto the space of affine functions.

To summarize, if we denote the \( \bar{\eta}_t \)-orthogonal projection onto the space of (vector-valued) polynomials of order \( k \) by \( \pi^k[\bar{\eta}_t] \), then

\[
\pi^0[\bar{\eta}_t] K_t(\cdot, \bar{\eta}_t) = \left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1}
\]

\[
\pi^1[\bar{\eta}_t] \nabla \log \bar{\eta}_t = -\tilde{P}_t^{-1}(\cdot - \bar{m}_t).
\] (23)

Thus, the mean field EnKBF can be seen as an approximation of the optimal filter by projecting its coefficients onto polynomials of certain order. These projections of course cause an inconsistency in the nonlinear setting, however they still result in a similar evolution of the moments. While the projection \( \pi^0 \) will cause a similar evolution of the mean, projections onto the affine functions \( \pi^1 \) will even result in a similar evolution of the covariance. Note that due to the nonlinearity of the signal, this course of does not mean that the mean of the mean field EnKBF is the equal to the one of the optimal filter. However it does explain why the evolution equation of its mean

\[
d\bar{m}_t = \mathbb{E}_Y[B_t(\bar{X}_t)] \, dt + \left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1} \left(dY_t - H_t \bar{m}_t \, dt\right),
\]

is of the same form as the one for the mean of the consistent filter.
Remark 15. One of course could use relation (23) to improve the approximation of the optimal filter by deriving higher order polynomial projections. This question is out of the scope of this paper. In the case of the gain term $K$ this in particular seems to be a non trivial task as it seems to require to look at what happens in (7) if $\nabla \phi$ is replaced by non-gradient type vector fields.

Viewing the mean field EnKBF (20) as a constant gain approximation to the optimal filter (14) makes it attractive to quantify its inconsistency by an appropriate coupling of the two differential equation (20) and (14). This task is not within the scope of this paper and so we will not investigate this further. However we want to briefly note here that another more classical way of quantifying the inconsistency is to compare the associated Kolmogorov equation of (20), which we derive in the next section, with the KSE.

4.2 Associated stochastic partial differential equation

Let $\phi$ be an arbitrary smooth test function. Using Itô’s rule we derive for the law $\bar{\eta}_t$ of $\bar{X}_t$ defined by equation (20)

$$
\begin{align*}
\mathrm{d}\bar{\eta}_t(\phi) &= \bar{\eta}_t(L_t \phi) \mathrm{d}t + \bar{\eta}_t \left( \nabla \phi \cdot \left( \bar{P}_t \bar{H}_t^T + \bar{C}_t \Gamma_t^T \right) \right) R_t^{-1} \mathrm{d}Y_t \\
&\quad - \bar{\eta}_t \left( \nabla \phi \cdot \left( \bar{P}_t \bar{H}_t^T + \bar{C}_t \Gamma_t^T \right) \right) R_t^{-1} \frac{H' \left( \cdot + \bar{m}_t \right)}{2} \mathrm{d}t \\
&\quad - \bar{\eta}_t \left( \nabla \phi \cdot \left( \bar{P}_t \bar{H}_t^T + \bar{C}_t \Gamma_t^T \right) \right) R_t^{-1} \frac{\mathrm{tr} \left( \bar{P}_t \bar{H}_t^T + \bar{C}_t \Gamma_t^T \right)}{2} \mathrm{d}t \\
&\quad + \frac{1}{2} \bar{\eta}_t \left( \nabla \phi \cdot \left( \bar{P}_t \bar{H}_t^T + \bar{C}_t \Gamma_t^T \right) \right) R_t^{-1} \left( H_t \bar{P}_t + \Gamma_t \bar{C}_t^T \right) \mathrm{d}t,
\end{align*}
$$

(24)

which in general will not coincide with the KSE. In the Gaussian case however we already know that this indeed coincides with the KSE. This can also be shown by using integration by parts on the last four terms in the equation above, as for Gaussian distributions this allows to translate the first and second order differential terms into terms of order zero.

Before we investigate the well posedness of (20), we derive some a priori bounds of the (conditional) covariance matrix $\bar{P}$ that will be key for the analysis.

4.3 Covariance bounds for the mean field EnKBF

To make formulas in this section a bit more concise, we use the following notation.

Notation 16. For any random vector $Z$ in $\mathbb{R}^d$, we denote

$$
[B_t | Z] := \mathbb{E}_Y \left( (B_t(Z) - \mathbb{E}_Y[B_t(Z)]) \left( \bar{X}_t - \mathbb{E}_Y[\bar{X}_t] \right) \right)^	op + \mathbb{E}_Y \left( (\bar{X}_t - \mathbb{E}_Y[\bar{X}_t]) (B_t(Z) - \mathbb{E}_Y[B_t(Z)]) \right).
$$

(25)

Notation 17. Throughout this paper $| \cdot |$ shall denote the standard Euclidian norm. If the input is a matrix $A \in \mathbb{R}^{m \times k}$ the result is the Frobeniusnorm $|A| = \sqrt{\sum_{i=1}^m \sum_{j=1}^k A_{ij}^2} = \sqrt{\text{tr}[AA^\top]}$. It is easy to see that for any square matrix $A \in \mathbb{R}^{k \times k}$, the trace can be bounded by the Frobeniusnorm in the following way $\text{tr} A \leq \sqrt{k} |A|$. If, on the other hand, $A \in \mathbb{R}^{k \times k}$ is symmetric positive semidefinite, then, by using the singular value decomposition, one can verify easily that $|A| \leq \text{tr} A$.

It is clear that the conditional mean $\bar{m}_t$ evolves according to

$$
\mathrm{d}\bar{m}_t = \mathbb{E}_Y \left[ B_t \left( \bar{X}_t \right) \right] \mathrm{d}t + \left( \bar{P}_t \bar{H}_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \left( \mathrm{d}Y_t - H_t \bar{m}_t \mathrm{d}t \right).
$$

(26)
Thus Itô’s product formula gives us for the centralized process \( \bar{X} - \bar{m} \) that
\[
\begin{align*}
    d (\bar{X}_t - \bar{m}_t) (\bar{X}_t - \bar{m}_t)^T & = 2 \text{Sym} \left[ (\bar{X}_t - \bar{m}_t) (B(\bar{X}_t) - \mathbb{E} [ B(\bar{X}_t)]) \right] dt + \left( (C(\bar{X}_t)C(\bar{X}_t)^T) + \bar{C}_t \bar{C}_t^T \right) dt \\
    & = \text{Sym} \left[ \left( \bar{P}_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \left( H_t + \Gamma_t \bar{C}_t^T \bar{P}_t^{-1} \right) (\bar{X}_t - \bar{m}_t) (\bar{X}_t - \bar{m}_t)^T \right] dt \\
    & = 2 \text{Sym} \left[ (\bar{X}_t - \bar{m}_t) C_t(\bar{X}_t) dW_t \right] + 2 \text{Sym} \left[ (\bar{X}_t - \bar{m}_t) \bar{C}_t dW_t \right].
\end{align*}
\]

Now by taking the (conditional) expectation \( \mathbb{E}_Y \), we see that
\[
\frac{d \bar{P}_t}{dt} = \| B_t | X_t \| + \mathbb{E}_Y \left[ C_t (\bar{X}_t) C_t (\bar{X}_t)^T \right] + \bar{C}_t \bar{C}_t^T
\]
and therefore by using that \( \text{tr} \left[ \left( \bar{P}_t H_t^T + \bar{C}_t \Gamma_t^T \right) R_t^{-1} \left( H_t \bar{P}_t + \Gamma_t \bar{C}_t^T \right) \right] \geq 0 \) one derives
\[
\frac{d \text{tr} \bar{P}_t}{dt} \leq 2 \text{Lip} (B) \text{tr} \bar{P}_t + \| C_t \|_2^2 + \| \bar{C}_t \|_2^2.
\]

Using the deterministic Grönwall Lemma this differential inequality lets us derive
\[
\sup_{t \leq T} \text{tr} \bar{P}_t \leq \Psi (T) := \exp \left( 2 T \text{Lip} (B) \left( \text{tr} \bar{P}_0 + \int_0^T \| C_t \|_2^2 + \| \bar{C}_t \|_2^2 \right) dt \right).
\]

**Remark 18.** We have thus found an upper bound for the spectrum of \( \bar{P} \). Note that this is the same as the upper bound for the variance of the signal \( X \), i.e.
\[
\sup_{t \leq T} \text{tr} \text{Cov} [X_t, X_t] \leq \Psi (T).
\]

When \( X \) is a consistent representation of the posterior, i.e. in the linear Gaussian case, this is a stronger version of the inequality implied by the law of total variance. In the inconsistent setting this shows that even though (20) is not the optimal filter, it still satisfies the same conditional variance bound as the true posterior.

The following positive scalar will play a role in deriving the invertibility of \( \bar{P}_t \), \( t \geq 0 \).

**Definition 19.** For any \( t \in [0, T] \) let
\[
\gamma_t := \lambda_{\min} \left( \bar{C}_t (I - \Gamma_t^T R_t^{-1} \Gamma_t) \bar{C}_t^T \right) + \inf_x \lambda_{\min} (C_t(x)C_t(x)^T).
\]

Note that \( \lambda_{\min} \left( \bar{C}_t (I - \Gamma_t^T R_t^{-1} \Gamma_t) \bar{C}_t^T \right) \geq 0 \).

The next Lemma shows that the (conditional) covariance matrix \( \bar{P} \) of solutions to (20) stays invertible for all times \( t \geq 0 \) if \( \bar{P}_0 \) is already invertible. This hints that the singularity on the right hand side of (20) does not cause a problem, as it will never be seen by the solution \( \bar{X} \).

**Lemma 20.** Let \( \bar{X} \) be a solution of (20) with conditional covariance matrix \( \bar{P} := \text{Cov} [\bar{X}, \bar{X}] \). Assume that \( \bar{P}_0 \) is invertible. If the standard Lipschitz conditions in Assumption 1 hold and if
\[
\inf_{t \leq T} \gamma_t > 0
\]
then \( \bar{P}_t \) is invertible for all times \( t \in [0, T] \). The norm of its inverse can be uniformly bounded by a constant \( \Psi (T) \), i.e. \( \sup_{t \leq T} \left( \bar{P}_t \right)^{-1} \leq \Psi (T) \).
\textbf{Proof.} To not worry too much about the inverse $\hat{P}^{-1}$ in the following computations, we first replace it in equation (20) by the Moore–Penrose pseudoinverse $\hat{P}_t^+$, which is always well defined but may develop singularities. As $\hat{X}$ is assumed to be a solution to (20), it must also satisfy
\begin{align}
\frac{d\hat{X}_t}{dt} &= B_t \left( \hat{X}_t \right) dt + C_t \left( \hat{X}_t \right) d\hat{W}_t + \hat{C}_t d\hat{V}_t \\
&\quad + \left( \hat{P}_t H_t^T + \hat{C}_t \Gamma_t^T \right) R_t^{-1} \left( \frac{dY_t}{dt} - \frac{H_t \left( \hat{X}_t + \bar{m}_t \right)}{2} \right) \\
&\quad - \left( \hat{P}_t H_t^T + \hat{C}_t \Gamma_t^T \right) R_t^{-1} \Gamma_t \hat{C}_t^T \hat{P}_t^+ \frac{\hat{X}_t - \bar{m}_t}{2} dt.
\end{align}
\hspace{1cm} (31)

Since for every $t \geq 0$ the term $\hat{P}_t$ is a symmetric positive semidefinite matrix, there exists an orthogonal matrix $Q_t$ and a diagonal matrix $\Lambda_t := \text{diag} \left( \lambda_1^t, \ldots, \lambda_d^t \right)$ such that
\begin{equation}
\hat{P}_t = Q_t^T \Lambda_t Q_t \quad \text{and} \quad \lambda_i^t \geq 0 \quad \text{for all} \quad t \geq 0, \quad i = 1, \ldots, d_x.
\end{equation}
\hspace{1cm} (32)

It can be shown (see [17]) that $Q$ is continuous and $\Lambda_t$ is differentiable and satisfies
\begin{equation}
\frac{d\Lambda_t}{dt} = \text{diag} \left( Q_t \frac{d\hat{P}_t}{dt} Q_t^T \right).
\end{equation}
\hspace{1cm} (33)

Denote by $e_i$ the $i$-th unit vector in $\mathbb{R}^{d_x}$, then this and (27) implies
\begin{align}
\frac{d\lambda_i}{dt} &= e_i^T Q_t \frac{d\hat{P}_t}{dt} Q_t^T e_i \\
&= e_i^T Q_t \left[ B_t | \hat{X}_t \right] Q_t^T e_i + \mathbb{E}_Y \left[ e_i^T Q_t C_t \left( \hat{X}_t \right) C_t^T \left( \hat{X}_t \right)^T Q_t^T e_i \right] \\
&\quad + e_i^T Q_t \hat{C}_t \hat{C}_t^T Q_t^T e_i - e_i^T Q_t \left( \hat{P}_t H_t^T + \hat{C}_t \Gamma_t^T \right) R_t^{-1} \left( H_t + \Gamma_t \hat{C}_t^T \hat{P}_t^+ \right) \frac{\hat{X}_t - \bar{m}_t}{2} e_i.
\end{align}

We want to estimate $\lambda_i$ from below. To this end we bound all three terms on the right-hand side from below to derive a differential inequality.

First we note that
\begin{align}
e_i^T Q_t \left[ B_t | \hat{X}_t \right] Q_t^T e_i \\
&= 2 \mathbb{E}_Y \left[ (e_i^T Q_t \left( B_t(\hat{X}_t) - B_t(\bar{m}_t) \right)) (e_i^T Q_t \left( \hat{X}_t - \bar{m}_t \right)) \right] \\
&\geq -2 \sqrt{\mathbb{E}_Y \left[ |e_i^T Q_t^T \left( B_t(\hat{X}_t) - B_t(\bar{m}_t) \right)|^2 \right]} \sqrt{\mathbb{E}_Y \left[ |e_i^T Q_t^T \left( \hat{X}_t - \bar{m}_t \right)|^2 \right]} \\
&\geq -2 \text{Lip}(B) \sqrt{\text{tr} \hat{P}_t} \sqrt{\lambda_i}.
\end{align}

By the variational definition of the minimal eigenvalue we also see that
\begin{align}
\mathbb{E}_Y \left[ e_i^T Q_t C_t \left( \hat{X}_t \right) C_t^T \left( \hat{X}_t \right)^T Q_t^T e_i \right] \geq \inf_x \lambda_{\min} \left( C(x)C(x)^T \right).
\end{align}

Finally it remains to bound
\begin{align}
e_i^T Q_t \hat{C}_t \hat{C}_t^T Q_t^T e_i - e_i^T Q_t \left( \hat{P}_t H_t^T + \hat{C}_t \Gamma_t^T \right) R_t^{-1} \left( H_t + \Gamma_t \hat{C}_t^T \hat{P}_t^+ \right) \frac{\hat{X}_t - \bar{m}_t}{2} e_i \\
&= e_i^T Q_t \left( \hat{C}_t \Gamma_t^T R_t^{-1} \Gamma_t \hat{C}_t^T \hat{P}_t^+ \right) Q_t^T e_i - e_i^T Q_t \hat{P}_t H_t^T R_t^{-1} H_t \frac{\hat{X}_t - \bar{m}_t}{2} e_i \\
&- e_i^T Q_t \left( \hat{C}_t \Gamma_t^T R_t^{-1} H_t + \hat{P}_t H_t^T R_t^{-1} \Gamma_t \hat{C}_t^T \hat{P}_t^+ \right) \frac{\hat{X}_t - \bar{m}_t}{2} e_i.
\end{align}
from below. The decomposition (32) of \( \dot{P} \), lets us also easily derive
\[
e_i^T Q_t \dot{P} t H_t^T R_t^{-1} H_t \dot{P} t Q_t^T e_i \leq \lambda_{\max} (H_t^T R_t^{-1} H_t) (\lambda_t')^2
\]
\[
e_i^T Q_t (\dot{C}_t \Gamma_t^T R_t^{-1} H_t \dot{P} t + \dot{P} t H_t^T R_t^{-1} \Gamma_t \dot{C}_t^T P_t^+ P_t) Q_t^T e_i \leq 2 |\dot{C}_t \Gamma_t^T R_t^{-1} H_t| \lambda_t'
\]
\[
e_i^T Q_t (\dot{C}_t \dot{C}_t^T - \dot{C}_t \Gamma_t^T R_t^{-1} \Gamma_t \dot{C}_t^T P_t^+ P_t) Q_t^T e_i \geq \lambda_{\min} (\dot{C}_t (I - \Gamma_t^T R_t^{-1} \Gamma_t) \dot{C}_t^T)
\].

Note that for the last inequality we also used that \( \dot{C}_t \Gamma_t^T R_t^{-1} \Gamma_t \dot{C}_t^T \) is positive semidefinite. And thus we derive
\[
\frac{d \lambda_t'}{dt} \geq -2 \text{Lip}(B) \sqrt{tr P_t \sqrt{\lambda_t'}}
\]
\[
+ \inf_{x} \lambda_{\min} (C_t(x) C_t(x)^T) + \lambda_{\min} (\dot{C}_t (I - \Gamma_t^T R_t^{-1} \Gamma_t) \dot{C}_t^T) \geq \inf_{t \leq T} \gamma_s
\]
\[
- \lambda_{\max} (H_t^T R_t^{-1} H_t) (\lambda_t')^2 - 2 |\dot{C}_t \Gamma_t^T R_t^{-1} H_t| \lambda_t'.
\]

Let \( \lambda \) denote a solution of
\[
\frac{d \lambda}{dt} = -2 \text{Lip}(B) \sqrt{tr P \sqrt{\lambda}} + \inf_{s \leq t} \gamma_s / 2
\]
\[
- \lambda_{\max} (H_t^T R_t^{-1} H_t) (\lambda')^2 - 2 |\dot{C}_t \Gamma_t^T R_t^{-1} H_t| \lambda'.
\]

with initial condition \( \lambda_0 = \min_{t \leq T} \lambda_t \). The existence of solutions (which might not be unique) is guaranteed by the Peano theorem. Then by [43, II. Lemma, page 64], \( \lambda \) is a lower bound for all \( \lambda_t \).

By assumption (30), the right-hand side of (35) is strictly bigger than zero, when \( \lambda_0 = 0 \). Thus for \( t_0 > 0 \) with \( \lambda_0 = 0 \), we have \( \lambda'(t_0) > 0 \). According to [43, II. Lemma, page 64], this indeed implies \( \lambda_t > 0 \) for all \( t \geq 0 \), if \( \lambda_0 > 0 \), which is the case if \( \dot{P} \) is regular. Since \( |\dot{P} t^{-1}| = \sum_{i=1}^{d_x} (1/\lambda_i)^2 \) this indeed gives us a bound \( \Psi(T) := \sqrt{\sum_{i=1}^{d_x} 1/\lambda_i} \) for the inverse of the covariance.

**Remark 21.** Note that if \( \dot{P} \) is not invertible, then one can still show that \( \dot{P} \) is invertible for all \( t > 0 \). To accomplish this one uses the proof above and argues by contradiction. As [43, II. Lemma, page 64] also implies that, if there existed a \( t_0 > 0 \) with \( \lambda_0 = 0 \), then there would be an \( s \) such that \( \lambda_s \leq 0 \) for all \( s \geq t \). We know that \( \lambda_t \) is non-negative, thus it would hold that \( \lambda_t = 0 \) for all \( t \leq t \).

However by (35) and assumption (30) this would also mean that \( \frac{d \lambda_s}{ds} > 0 \) on \( (0, t) \), which is a contradiction to \( \lambda_t \) being constant on \( (0, t) \). Therefore we indeed can conclude that \( \lambda_t > 0 \) and thus \( \lambda_i > 0 \) for all \( t > 0 \), \( i = 1, \ldots, d_x \). Therefore \( \dot{P} \) is regular.

Next we investigate the well posedness of (20).

### 4.4 Well posedness of the mean field Ensemble Kalman–Bucy filter

One difficulty in the analysis of (20) that is apparent at first sight is, that this equation is (potentially) singular, i.e. the inverse of the (conditional) covariance on the right hand side is not well defined for all probability distributions and blows up when \( \dot{P}_t \) becomes non invertible for some point in time \( t_0 \).

However even in the uncorrelated setting \( \dot{C} = 0 \) we note that the coefficients in (20) are only locally Lipschitz and thus this equation falls outside the standard theory of McKean–Vlasov equations that is e.g. found in [7]. For ordinary SDEs with local Lipschitz conditions the main tool for
proving well posedness is the introduction of stopping times that localize the equation and reduce its analysis to the case of global Lipschitz conditions. However looking at a stopped stochastic process of course changes its time marginal laws and due to the law dependence of McKean–Vlasov equations such stopping times would thus change the dynamics of the equation. So far there does not seem to exist a localization argument that is suited to prove the well posedness of a general class of McKean–Vlasov equations under local Lipschitz assumptions. Indeed [39] shows that such an argument can not hold in full generality as it constructs locally Lipschitz McKean–Vlasov differential equations (without the influence of noise) where uniqueness does not hold. Nevertheless in recent years some progress has been made regarding the well posedness of McKean–Vlasov equations under local Lipschitz conditions. For example in [23] existence of weak solutions has been shown by the usage of Lyapunov techniques. [22] show the existence of solutions for a fairly general class of equations and determine some conditions for their uniqueness, but even in the uncorrelated case these results do not apply to (20), in parts due to missing growth conditions.

Before we show our main well posedness result in the correlated noise framework, we first look at the uncorrelated case and make some restricting assumptions regarding the observations, as it will motivate the partial stopping argument used later on in the proof for the general case.

**Lemma 22.** Beside the standard Lipschitz conditions (see assumption 1), we assume that $C$ is bounded and that all coefficients are continuously differentiable with respect to $t$. Furthermore we assume that $\tilde{C} = 0$ and that there exists some scalar valued function $\alpha$ such that $H_t^T R_t^{-1} H_t = \alpha_t I$ for all $t \geq 0$, where $I$ again denotes the identity matrix. Then there exists a unique solution $\tilde{X}$ of (20) on the time interval $[0, T]$.

To prove Lemma 22 we will make a fixed point argument just as we did in the linear case. However, since the equation of the (conditional) covariance matrix $\tilde{P}$ does not decouple from $\tilde{X}$ for nonlinear $B$, we will not be able to simply guess the right fixed point. Instead we have to work with a priori bounds for the first two moments which we derive now, before proving Lemma 22.

First we define the set of suitable fixed points.

**Definition 23.** We denote by $\text{SPSD}_1^X([0, T])$ the set of all differentiable processes $P$ such that
- $P$ is adapted to the natural filtration of $Y$,
- $P_t$ is a symmetric positive semidefinite matrix for every $t \geq 0$,
- $P_0 = \tilde{P}_0$.

For any $P \in \text{SPSD}_1^X([0, T])$ we define the stochastic process $\tilde{X}^P$ to be the solution of

\[
\mathrm{d}\tilde{X}_t^P = B_t(\tilde{X}_t^P)\mathrm{d}t + C_t(\tilde{X}_t^P)\mathrm{d}W_t + P_t H_t^T R_t^{-1} \left( \mathrm{d}Y_t - \frac{H_t (\tilde{X}_t^P + \tilde{m}_t^P)}{2} \right) dt.
\]

Hereby we denote $\tilde{m}_t^P := E_Y [\tilde{X}_t^P]$. Since this is a McKean–Vlasov equation satisfying the usual Lipschitz conditions, there indeed exists a unique strong solution [7].

A solution of (20) is given by a fixed point, i.e. a suitable process $P$ such that

$\text{Cov}_Y [\tilde{X}_t^P] = P_t$ for all $t \in [0, T]$.

We will prove its existence and uniqueness on small time intervals with the Banach fixed point theorem. The extension to arbitrary time intervals then follows from a standard glueing argument. In order to prove that the map

$P \mapsto \text{Cov}_Y [\tilde{X}^P]$
is a strict contraction on a suitable subset of $\text{SPSD}_t^+([0,T])$, a priori estimates on the (conditional) covariance matrix $\text{Cov}_{\mathcal{Y}}[\bar{X}_t^P]$, independent of $P$ and $t$, will be key.

We note that the (conditional) mean and covariance matrix of $\bar{X}_t^P$ satisfy the following two equations

$$dm_t^P = \mathbb{E}_Y \left[B_t \left(\bar{X}_t^P\right)\right] dt + P_t H_t^T R_t^{-1} \left(dy_t - H_t m_t^P dt\right), \tag{36}$$

and

$$\frac{d\text{Cov}_{\mathcal{Y}}[\bar{X}_t^P]}{dt} = \left[B_t |\bar{X}_t^P\right] + \mathbb{E}_Y \left[C_t \left(\bar{X}_t^P\right) C_t \left(\bar{X}_t^P\right)^T\right] - \frac{\alpha_t}{2} \left(P_t \text{Cov}_{\mathcal{Y}}[\bar{X}_t^P] + \text{Cov}_{\mathcal{Y}}[\bar{X}_t^P]P_t\right). \tag{37}$$

**Remark 24.** We note that if $H_t^T R_t^{-1} H_t$ is not a scalar matrix, that is there exists no scalar-valued function $\alpha$ such that $H_t^T R_t^{-1} H_t = \alpha_t I$, then we cannot guarantee that $P_t H_t^T R_t^{-1} H_t$ is positive definite, let alone symmetric. Thus we can not simply take the trace in (37), use the Grönwall lemma and expect to derive bounds independent of $P$. In other words, the covariance matrix is a Lyapunov function of (20), that, for general coefficients, is not robust to perturbations in the covariance matrix.

When the assumptions of Lemma 22 hold, equation (37) reduces to

$$\frac{d\text{Cov}_{\mathcal{Y}}[\bar{X}_t^P]}{dt} = \left[B_t |\bar{X}_t^P\right] + \mathbb{E}_Y \left[C_t \left(\bar{X}_t^P\right) C_t \left(\bar{X}_t^P\right)^T\right] - \frac{\alpha_t}{2} \left(P_t \text{Cov}_{\mathcal{Y}}[\bar{X}_t^P] + \text{Cov}_{\mathcal{Y}}[\bar{X}_t^P]P_t\right). \tag{38}$$

We note that by the cyclical invariance of the trace we have

$$\text{tr} \left(P_t \text{Cov}_{\mathcal{Y}}[\bar{X}_t^P]\right) = \mathbb{E}_Y \left[\text{tr} \left(P_t \left(\bar{X}_t^P - m_t^P\right) \left(\bar{X}_t^P - m_t^P\right)^T\right)\right]$$

$$= \mathbb{E}_Y \left[\left(\bar{X}_t^P - m_t^P\right)^T P_t \left(\bar{X}_t^P - m_t^P\right)\right] \geq 0,$$

which gives us the differential inequality

$$\frac{d\text{tr}\text{Cov}_{\mathcal{Y}}[\bar{X}_t^P]}{dt} \leq 2 \text{Lip}(B_t) \text{tr}\text{Cov}_{\mathcal{Y}}[\bar{X}_t^P] + \|C_t\|^2_{\infty},$$

and thus we derive by the deterministic Grönwall inequality

$$\sup_{t \leq T} \text{tr} \text{Cov}_{\mathcal{Y}}[\bar{X}_t^P] \leq \exp \left(2T \text{Lip}(B)\right) \left(\text{tr} \hat{P}_0 + \int_0^T \|C_t\|^2_{\infty} dt\right) =: \kappa_0(T). \tag{39}$$

This means that we can assume that $\sup_{t \leq T} \text{tr} P_t \leq \kappa_0(T)$. Using (38) we thus obtain

$$\sup_{t \leq T} \frac{d\text{tr} \text{Cov}_{\mathcal{Y}}[\bar{X}_t^P]}{dt} \leq \sup_{t \leq T} \left(2\text{Lip}(B) + \alpha_t\right) \kappa_0(T)^2 =: \kappa_1(T). \tag{40}$$

With the two bounds (39) and (40), we define the set

$$\mathcal{X}_T := \{P \in \text{SPSD}_t^+([0,T]) : \|P\|_{\infty} \leq \kappa_0(T), \|\partial_t P\|_{\infty} \leq \kappa_1(T)\}. \tag{41}$$

The following Lemma will be useful for deriving a contraction property for the fixed point map $P \mapsto \text{Cov}_{\mathcal{Y}}[\bar{X}_t^P, \bar{X}_t^P]$. 

Lemma 25. For every fixed $T > 0$ there exists a constant $\kappa_Y(T)$, such that
\[
\sup_{t \leq T} \left| \int_0^t (P_s^1 - P_s^2) H_s^T R_s^{-1} dY_s \right| \leq \kappa_Y(T) \| P^1 - P^2 \|_{C^1}
\]
holds for all $P^1, P^2 \in \text{SPSD}_1^1([0, T])$. Furthermore these constants satisfy $\kappa_Y(T) \xrightarrow{T \to 0} 0$.

Proof. We note that all $P^1, P^2 \in \text{SPSD}_1^1([0, T])$ are processes of bounded variation and thus one gets by integration by parts
\[
\int_0^t (P_s^1 - P_s^2) H_s^T R_s^{-1} dY_s = (P_1^1 - P_1^2) H_1^T R_1^{-1} Y_1 \int_0^t \frac{d (P_s^1 - P_s^2) H_s^T R_s^{-1}}{dt} Y_s ds,
\]
where the boundary term at $t = 0$ disappears as $Y_0 = 0$.

Since $H_s^T R_s^{-1}$ is a $C^1([0, T]; \mathbb{R}^{d_s \times d_s})$ function we immediately derive the desired inequality with
\[
\kappa_Y(T) := \sup_{t \leq T} \left| H_1^T R_1^{-1} Y_1 \right| + \int_0^T \left| H_s^T R_s^{-1} Y_s \right| + \left| \frac{dH_s^T R_s^{-1}}{dt} Y_s \right| dt.
\]
By the continuity of $Y$ and $Y_0 = 0$ we also immediately see that $\kappa_Y(T) \xrightarrow{T \to 0} 0$.

As a corollary one also immediately derives for any $P \in \mathcal{X}_T$ that
\[
\sup_{t \leq T} \left| \int_0^t P_s H_s^T R_s^{-1} dY_s \right| \leq \kappa_Y(T) \sqrt{\kappa_0(T)^2 + \kappa_1(T)^2}. \tag{42}
\]

Using the Lipschitz continuity of $B$ we see that for any $P \in \mathcal{X}_T$
\[
|m_t^P| \leq |m_0^P| + \int_0^t |B_s(0)| ds + \int_0^t \text{Lip}(B) \mathbb{E}_Y \left[ |X_s^P - m_s^P| \right] ds
\]
\[
+ \int_0^t P_s H_s^T R_s^{-1} dY_s \bigg|_{s \leq \kappa_0(T)} + \left( \int_0^t \text{Lip}(B) \left[ H_s^T R_s^{-1} H_s \right] + \text{Lip}(B) \right) |m_s^P| ds.
\]

Applying the deterministic Grönwall inequality together with
\[
\mathbb{E}_Y \left[ |X_s^P - m_s^P| \right] \leq \sqrt{\mathbb{E}_Y \left[ X_s^P - m_s^P \right]^2} = \sqrt{\text{tr Cov}_Y [X_s^P] \leq \sqrt{\kappa_0(T)}}
\]
and (42), we obtain for $P \in \mathcal{X}_T$
\[
\sup_{t \leq T} |m_t^P| \leq \kappa_m(T) := \exp \left( \int_0^t \left( \kappa_0(T) \left| H_s^T R_s^{-1} H_s \right| + \text{Lip}(B) \right) ds \right)
\]
\[
\times \left( |m_0^P| + \int_0^t |B_s(0)| + \text{Lip}(B) \sqrt{\kappa_0(T)} ds + \kappa_Y(T) \sqrt{\kappa_0(T)^2 + \kappa_1(T)^2} \right).
\]

Now we are in the position to prove Lemma 22.

Proof of Lemma 22. As already noted we look for a fixed point $\bar{P}$ such that $\bar{P} = \text{Cov}_Y \left[ \bar{X}^P \right]$. Due to the previously derived bounds (39) and (40), we can restrict our search to the set $\mathcal{X}_T$. As a closed subset of a Banach space, $\mathcal{X}_T$ is itself a complete metric space.

Thus we can employ the Banach fixed point theorem and it only remains to show the contraction property of the fixed point map $P \mapsto \text{Cov}_Y [X^P, \bar{X}^P]$ on $\mathcal{X}_T$ for sufficiently small $T$. To this end
Let \( P^1, P^2 \in \mathcal{X}_T \) be given and denote by \( X^1, X^2 \) the corresponding solutions and by \( m^1, m^2 \) their means. Then it is clear that

\[
X^1_t - X^2_t = \int_0^t (B_s(X^1_s) - B_s(X^2_s)) \, ds + \int_0^t (P^1_s - P^2_s) \, H_s^T R_s^{-1} \, dY_s
\]

Thus we derive from the Gronwall lemma

\[
\left| \sum_{i=1}^k a_i \right|^2 \leq k \sum_{i=1}^k |a_i|^2 \quad \text{and} \quad \left| \int_0^t f(t) \, dt \right|^2 \leq t \int_0^t |f(t)|^2 \, dt,
\]

we get

\[
\left| X^1_t - X^2_t \right|^2 \leq 5t \left( \int_0^t |B_s(X^1_s) - B_s(X^2_s)|^2 \, ds + 5 \int_0^t |P^1_s - P^2_s| \, H_s^T R_s^{-1} \, dY_s \right)^2
\]

Using the estimates \( \left| \sum_{i=1}^k a_i \right|^2 \leq k \sum_{i=1}^k |a_i|^2 \) and \( \left| \int_0^t f(t) \, dt \right|^2 \leq t \int_0^t |f(t)|^2 \, dt \), we get

\[
\left| X^1_t - X^2_t \right|^2 \leq 5t \left( \int_0^t |B_s| \, ds + 5 \int_0^t |P^1_s - P^2_s| \, H_s^T R_s^{-1} \, dY_s \right)^2
\]

Note that

\[
|m^1_t - m^2_t|^2 \leq \mathbb{E}_Y \left( |X^1_t - X^2_t|^2 \right) \quad \text{and} \quad \mathbb{E}_Y \left( |X^1_t|^2 \right) + |m^1_t|^2 \leq \kappa_0(T) + 2\kappa_m(T)^2.
\]

Using the Lipschitz continuity of \( B \) and \( C \), as well as Itô isometry, we thus derive

\[
\mathbb{E}_Y \left[ |X^1_t - X^2_t|^2 \right]
\]

Thus we derive from the Gronwall lemma

\[
\sup_{t \leq T} \mathbb{E}_Y \left[ |X^1_t - X^2_t|^2 \right]
\]

Where clearly \( \kappa_{\text{contr}}(T) \to 0 \). By employing

\[
|\text{Cov}_Y \left[ X^1_t \right] - \text{Cov}_Y \left[ X^2_t \right]| \leq 2\sqrt{\text{tr Cov}_Y \left[ X^1_t \right] + \text{tr Cov}_Y \left[ X^2_t \right]} \sqrt{\mathbb{E}_Y \left[ |X^1_t - X^2_t|^2 \right]}
\]
we therefore obtain
\[
\sup_{t \leq T} \left| \text{Cov}_Y \left[ X_t^1 \right] - \text{Cov}_Y \left[ X_t^2 \right] \right| \leq 2 \sqrt{2 \kappa_0(T) \kappa_{\text{contr}}(T)} \| P^1 - P^2 \|_{C^1},
\]

Using the differential equation for the evolution of the covariance matrix (38) and the previously derived bounds, one also derives the existence of \( q(T) > 0 \) with \( q(T) \xrightarrow{T\to0} 0 \) such that
\[
\sup_{t \leq T} \left| \frac{d\text{Cov}_Y \left[ X_t^1 \right]}{dt} - \frac{d\text{Cov}_Y \left[ X_t^2 \right]}{dt} \right| \leq q(T) \| P^1 - P^2 \|_{C^1}.
\]

Thus we have proven the contraction property for sufficiently small \( T > 0 \) and therefore Lemma 22 holds for small time domains. To prove this fact for arbitrary timeframes, one now simply uses a standard gluing argument.

Next we generalize Lemma 22 to the correlated noise framework without assuming that \( H_t^T R_t^{-1} H_t \) is scalar. To this end we have to introduce a partial stopping argument that makes the trace of the covariance matrix robust as a Lyapunov function (see Remark 24), since this was the main argument that allowed us to use the Banach fixed point theorem. We now are in the position to show our most general well posedness result for (20).

**Theorem 26.** Beside the standard Lipschitz conditions (see assumption 1), let us again assume that \( \bar{C} \) is bounded. Furthermore we assume that all coefficients are continuously differentiable with respect to \( t \). If \( \bar{C} \neq 0 \) we also assume that \( P_0 \) is invertible and that (30) holds. Then there exists a unique solution \( \bar{X} \) of (20) on the time interval \([0, T]\).

**Proof.** As we have already pointed out before, for general \( H \) and \( R \), we can not expect the trace of the covariance matrix \( \text{tr} \text{Cov}_Y \left[ X^P \right] \) to be bounded independent of \( P \).

To make up for this, we partially localize the dynamics in a manner that only depends on the observations \( Y \). To this end define for arbitrary \( k \in \mathbb{N} \)
\[
\tilde{1}_k : \mathbb{R} \to [0, 1] : x \mapsto \left( 1_{[-1/2, k-1/2]} \ast \rho \right)(x),
\]
where \( \rho \) is a standard mollifier.

With this we define the process \( (\tilde{X}_t^k)_{t \in [0, T]} \) to be the solution of
\[
\begin{align*}
d\tilde{X}_t^k &= B_t(\tilde{X}_t^k)dt + C_t(\tilde{X}_t^k)dW_t + \tilde{C}_tdV_t \\
&\quad + \tilde{1}_k \left( P_t^k H_t^T + \tilde{C}_t \Gamma_t^T \right) R_t^{-1} \left( dY_t - \frac{H_t(\tilde{X}_t^k + \tilde{\mu}_t^k)}{2} dt \right) \\
&\quad - \tilde{1}_k \frac{P_t^k H_t^T + \tilde{C}_t \Gamma_t^T}{2} R_t^{-1} \Gamma_t \tilde{C}_t^T \tilde{1}_{-k} \left( P_t^k \right)^{-1} \left( \tilde{X}_t^k - \tilde{\mu}_t^k \right) dt,
\end{align*}
\]
where
\[
\tilde{\mu}_t^k := \mathbb{E}_Y \left[ X_t^k \right], \quad P_t^k := \text{Cov}_Y \left[ X_t^k \right], \quad \tilde{1}_k := \tilde{1}_k \left( |P_t^k|^2 \right) \quad \text{and} \quad \tilde{1}_{-k} := \tilde{1}_k \left( |(P_t^k)|^2 \right).
\]

Note that (44) still falls outside of the standard framework for the analysis of McKean–Vlasov equations (found e.g. in [7]), as the product of a bounded and an unbounded Lipschitz function may still not be Lipschitz, and thus the coefficients in (44) therefore are only locally Lipschitz.
However the existence and uniqueness of such a solution $\tilde{X}^k$ for every $k \in \mathbb{N}$ can be proven just as in the proof of Lemma 22 by making a fixed point argument with respect to the covariance matrix $P^k$, as the involvement of $\tilde{1}_k$ bounds the covariance matrix independent of the argument in the fixed point map. Let $X_{t:k}$ be the unique solution of equation

$$dX_t^{P,k} = B_t\left(X_t^{P,k}\right) dt + C_t\left(X_t^{P,k}\right) dW_t + \tilde{C}_tdW_t$$

$$+ \tilde{1}_k(|P_t|^2) \left(P_tH_t^T + \tilde{C}_t\Gamma_t^T\right) R_t^{-1} \left(\begin{array}{c} dY_t - \frac{H_t\left(X_t^{P,k} + \tilde{m}_t^{P,k}\right)}{2} dt \\
\end{array}\right)$$

$$- \tilde{1}_k(|P_t|^2) \left(P_tH_t^T + \tilde{C}_t\Gamma_t^T\right) R_t^{-1}\Gamma_t\tilde{C}_t^T \tilde{1}_k\left(|P_t|^{-2}\right) P_t^{-1} \frac{\tilde{m}_t^{P,k}}{2} dt,$$

for any given $P \in \text{SPSD}_Y([0, T])$.

In this case we have

$$d\text{Cov}_{Y}\left[X_t^{P,k}\right] = \left[B_t\left(X_t^{P,k}\right) - \tilde{1}_k(|P_t|^2) \left(P_tH_t^T + \tilde{C}_t\right) R_t^{-1} H_t \text{Cov}_{Y}\left[X_t^{P,k}\right]\right]$$

$$\frac{2}{2} - \text{Cov}_{Y} \left[X_t^{P,k}\right] H_t^T R_t^{-1} \left(R_t P_t + \Gamma_t \tilde{C}_t^T\right) \tilde{1}_k(|P_t|^2)$$

$$- \tilde{1}_k(|P_t|^2) \frac{P_tH_t^T + \tilde{C}_t\Gamma_t^T}{2} R_t^{-1}\Gamma_t\tilde{C}_t^T \tilde{1}_k\left(|P_t|^{-2}\right) P_t^{-1} \text{Cov}_{Y}\left[X_t^{P,k}\right]$$

$$- \text{Cov}_{Y} \left[X_t^{P,k}\right] P_t^{-1} \tilde{1}_k\left(|P_t|^{-2}\right) \tilde{C}_t\Gamma_t^T R_t^{-1} H_t P_t + \Gamma_t \tilde{C}_t^T \tilde{1}_k\left(|P_t|^2\right)$$

$$+ \mathbb{E}_Y \left[C\left(X_t^{P,k}\right) C\left(X_t^{P,k}\right)^T\right] + \tilde{C}\tilde{C}^T$$

(45)

Since $\tilde{1}_{[0, k-1]} \leq \tilde{1}_k \leq \tilde{1}_{[1, k]}$, it holds that

$$\tilde{1}_k\left(|P_t|^2\right) \left(P_tH_t^T + \tilde{C}_t\Gamma_t^T\right) \leq \sqrt{k} |H_t^T| + |\tilde{C}_t\Gamma_t^T|$$

$$\tilde{1}_k\left(|P_t|^{-2}\right) P_t^{-1} \leq \sqrt{k}.$$

and one can derive for every fixed $k$ the boundedness of $\text{tr} \text{Cov}_{Y}\left[X_t^{P,k}\right]$ independent of $P$ by using the Grönwall inequality. As both

$$P \mapsto \tilde{1}_k\left(|P|^2\right) \left(PH^T + \tilde{C}\Gamma_T^T\right)$$

and $P \mapsto \tilde{1}_k\left(|P|^2\right) \frac{PH^T + \tilde{C}\Gamma_T^T}{2} R_t^{-1}\Gamma_t\tilde{C}_t^T \tilde{1}_k\left(|P|^{-2}\right) P_t^{-1}$

are smooth functions with compact support (and therefore Lipschitz) one can now derive the existence and uniqueness of $X^k$ for every fixed $k$ just as in the proof of Lemma 22.

For the fixed point $\tilde{X}^k$ equation (45) gives us

$$\frac{d\text{tr}P_t^{P,k}}{dt} = \left[B_t\left(X_t^k\right) - \tilde{1}_k P_t^{P,k} H_t^T R_t^{-1} H P_t^{P,k} - \tilde{1}_k \left(1 + \tilde{1}_{-k}\right) \text{Sym} \left(\tilde{C}_t\Gamma_t^T R_t^{-1} H P_t^{P,k}\right)\right]$$

$$+ \mathbb{E}_Y \left[C\left(X_t^{P,k}\right) C\left(X_t^{P,k}\right)^T\right] + \tilde{C}_t \left(1 - \tilde{1}_k \tilde{1}_{-k} \Gamma_t^T R_t^{-1}\Gamma_t\right) \tilde{C}_t^T$$

(46)
and thus one derives just as in subsection 4.3
\[
\frac{dtr \bar{P}_t^k}{dt} \leq \left( 2 \text{Lip}(B) + d_x \mathbb{1}_k (1 + \mathbb{1}_{-k}) \left| \tilde{C}_t \Gamma_t^{-1} R_t \right| \right) tr \bar{P}_t^k \\
+ \| C_t \|_\infty^2 + \left| \tilde{C}_t \right|^2 + d_x \mathbb{1}_{-k} \left| \tilde{C}_t \right| \Gamma_t \bar{C}_t^T \\
\leq 2 \left( \text{Lip}(B) + d_x \left| \tilde{C}_t \Gamma_t^{-1} R_t \right| \right) tr \bar{P}_t^k + \| C_t \|_\infty^2 + \left| \tilde{C}_t \right|^2 + d_x \left| \tilde{C}_t \Gamma_t^{-1} \Gamma_t \bar{C}_t^T \right|,
\]
which is similar to the differential inequality (28). Therefore we derive by the Grönnwall lemma that
\[
\sup_{t \leq T} tr \bar{P}_t^k \leq \exp(2 \int_0^T d_x \text{Lip}(B) + \left| \tilde{C}_t \Gamma_t^{-1} R_t \right| dt) \\
\times \left( \bar{P}_0 + \int_0^T \left| C_t \right|^2 + \left| \tilde{C}_t \right|^2 + d_x \left| \tilde{C}_t \Gamma_t^{-1} \Gamma_t \bar{C}_t^T \right| dt \right) = \hat{\Psi}(T)
\]
(the constant is defined in (29)) and we see that if
\[
k \geq \hat{\Psi}(T)^2 + 1,
\]
then \( \mathbb{1}_k (|\bar{P}_t^k|^2) = 1 \) for all \( t \in [0, T] \).

Similarly we can bound \( \lambda_{\min} (\bar{P}_t^k) \) from below. Using (46), we see, by employing the same inequalities as in Lemma 20, that for the \( i \)-th eigenvalue \( \lambda_{k,i} \) of \( \bar{P}_t^k \), the differential inequality
\[
\frac{d \lambda_{k,i}}{dt} \geq -2 \text{Lip}(B) \sqrt{\hat{\Psi}(T)} \sqrt{\lambda_{k,i}^2 - \mathbb{1}_k \lambda_{\max} (\tilde{H}_t^T R_t^{-1} H_t) (\lambda_i^2)} \\
- 2 \mathbb{1}_k (1 + \mathbb{1}_{-k}) \lambda_{\max} \left( \text{Sym} \left( \tilde{C}_t \Gamma_t^{-1} R_t \right) \right) \lambda_i^2 \\
+ \inf_x \lambda_{\min} \left( C_t(x) C_t(x)^T \right) + \lambda_{\min} \left( C_t (I - \mathbb{1}_k \mathbb{1}_{-k} \Gamma_t^T R_t^{-1} \Gamma_t) \bar{C}_t^T \right)
\]
holds. Since \( 0 \leq \mathbb{1}_k, \mathbb{1}_{-k} \leq 1 \), we have, due to the variational characterization of the smallest eigenvalue,
\[
\lambda_{\min} \left( \tilde{C}_t (I - \mathbb{1}_k \mathbb{1}_{-k} \Gamma_t^T R_t^{-1} \Gamma_t) \bar{C}_t^T \right) = \min_{|v| = 1} v^T \tilde{C}_t \bar{C}_t^T v - \mathbb{1}_k \mathbb{1}_{-k} v^T \tilde{C}_t \Gamma_t^T R_t^{-1} \Gamma_t \bar{C}_t^T v \\
\geq \min_{|v| = 1} v^T \tilde{C}_t \bar{C}_t^T v - v^T \tilde{C}_t \Gamma_t^T R_t^{-1} \Gamma_t \bar{C}_t^T v \geq \lambda_{\min} \left( \tilde{C}_t (I - \Gamma_t^T R_t^{-1} \Gamma_t) \bar{C}_t^T \right) .
\]

This lets us derive the same inequality as in the proof of Lemma 20
\[
\frac{d \lambda_{k,i}}{dt} \geq -2 \text{Lip}(B) \sqrt{\hat{\Psi}(T)} \sqrt{\lambda_{k,i}^2} \\
- \lambda_{\max} (\tilde{H}_t^T R_t^{-1} H_t) (\lambda_i^2) - 4 \lambda_{\max} \left( \text{Sym} \left( \tilde{C}_t \Gamma_t^{-1} R_t \right) \right) \lambda_i^2 + \gamma_t .
\]
This inequality allows us to bound \( \lambda_{\min} (\bar{P}_t^k) \) from below, independently of \( k \). To see this, let \( \lambda_{i} \) be a solution of the initial value problem
\[
\begin{cases}
\frac{d \lambda_{i}}{dt} = -2 \text{Lip}(B) \sqrt{\hat{\Psi}(T)} \sqrt{\lambda_{i}^2} \\
- \lambda_{\max} (\tilde{H}_t^T R_t^{-1} H_t) (\lambda_{i}^2) - 4 \lambda_{\max} \left( \text{Sym} \left( \tilde{C}_t \Gamma_t^{-1} R_t \right) \right) \lambda_{i}^2 + \frac{\gamma_{i}}{2} \\
\lambda_{0} = \frac{\lambda_{i}}{2} .
\end{cases}
\]
where $\lambda_0^i$ denotes the $i$-th eigenvector of $\bar{P}_0$. Note that the Peano theorem guarantees the existence of such a solution, while its uniqueness is not guaranteed. Thus we have $(\lambda^k, x^k)'(t_0) > (A_0)'(t_0)$, if $\lambda_k^i = \bar{\lambda}_0^i$. Now [43, II. Lemma, page 64] guarantees that $\lambda_k^i \geq \bar{\lambda}_0^i$ and furthermore it also guarantees that $\bar{\lambda}_0^i > 0$ for all times $t$. Therefore we have found a uniform (both in $k$ and in $i$) lower bound.

This lets us conclude that for $k \in \mathbb{N}$ large enough such that $k \geq \frac{1}{d_k \left( \min_{i \leq T} \bar{\lambda}_i \right)}$, we have

$$1_k \left( \left| \left( \bar{P}_k^i \right)^{-1} \right|^2 \right) = 1$$

and $\bar{X}^k$ is the unique solution of (20) on $[0, T]$.

\[ \square \]

4.5 Generalization to other EnKBFs

Note that for the uncorrelated case $\bar{C} = 0$ the solution of (20) is not the only McKean–Vlasov equation that describes the mean field limit of an EnKBF. Indeed in the literature it is sometimes referred to as the deterministic (mean field) EnKBF [4][10]. This is the continuous time counterpart of the filter defined by Sakov and Oke in [37] and was studied in [4], [5] as well as in [13]. Given a fixed realization/path of the observations $Y$, it is a deterministic perturbation of the signal. However, even for a fixed path of $Y$, it does not define a deterministic equation due to the involvement of the Brownian motions $\bar{V}, \bar{W}$. In filtering one often replaces Brownian motions with deterministic inflation terms [13][14] to derive deterministic equations. This is based on the following observation that was derived in [36].

**Lemma 27.** Let $Z$ be the solution of the SDE

$$dZ_t = \alpha_t(Z_t)dt + \beta_t dW_t.$$  \hspace{1cm} (47)

For every time $t$ denote by $\eta^Z_t$ the density of the law of $Z_t$. Furthermore we assume that there exists a solution $\tilde{Z}$ to

$$d\tilde{Z}_t = \alpha_t(\tilde{Z}_t)dt - \frac{\beta_t \beta^T_t}{2} \nabla \log \eta^Z_t(\tilde{Z})dt,$$

where $\eta^Z_t$ denotes the density of the law of $\tilde{Z}_t$. We assume that both $\eta^Z, \eta^\tilde{Z}$ are well defined densities and that both are strictly positive and smooth. Then if the Kolmogorov backward equation

$$\partial_t \eta^Z_t = \text{div} \left( \frac{\beta_t \beta^T_t}{2} \nabla \eta^Z_t \right) - \text{div} \left( \alpha_t \eta^Z_t \right)$$  \hspace{1cm} (48)

associated to (47) has a unique (strong) solution, it holds that $\eta^Z_t = \eta^\tilde{Z}_t$.

**Proof.** Assuming that $\eta^\tilde{Z}_t$ is sufficiently smooth, it satisfies

$$\partial_t \eta^\tilde{Z}_t = -\text{div} \left( -\frac{\beta_t \beta^T_t}{2} \nabla \log \eta^\tilde{Z}_t + \alpha_t \right) \eta^\tilde{Z}_t = \text{div} \left( -\frac{\beta_t \beta^T_t}{2} \nabla \eta^\tilde{Z}_t \right) - \text{div} \left( \alpha_t \eta^\tilde{Z}_t \right),$$

and thus by the uniqueness of solutions to (48), we derive $\eta^Z_t \eta^\tilde{Z}_t$ for all $t \geq 0$. \[ \square \]

If we now assume for every point in time $t$ the matrix $C_t$ does not depend on state $\bar{X}_t$, we could thus replace $C_t dW_t + \tilde{C}_t dV_t$ in (20) by $-\frac{C_t C_t^T + \tilde{C}_t C_t^T}{2} \nabla \log \bar{\eta}_t$. If one uses (23) to approximate $\nabla \log \bar{\eta}_t$ by its affine projection $\pi^T(\bar{\eta}_t)\nabla \bar{\eta}_t = -\bar{P}_t^{-1}(\cdot - \bar{m}_t)$, one thus derives the equation

$$d\bar{X}_t = B_t(\bar{X}_t)dt + \left( C_t C_t^T + \tilde{C}_t C_t^T \right) \bar{P}_t^{-1} (\bar{X}_t - \bar{m}_t) \frac{dt}{2}$$

$$+ \left( \bar{P}_t H_t^T + \tilde{C}_t \Gamma_t^T \right) R_t^{-1} \left( dY_t - \frac{H_t (\bar{X}_t + \bar{m}_t)}{2} dt \right)$$

$$- \left( \bar{P}_t H_t^T + \tilde{C}_t \Gamma_t^T \right) R_t^{-1} \Gamma_t \bar{C}_t \bar{P}_t^{-1} (\bar{X}_t - \bar{m}_t) \frac{dt}{2}.$$  \hspace{1cm} (49)
This correspondence of Brownian motion and the inflation term $\tilde{P}_t^{-1}(\cdot - \tilde{m}_t)$ can also be used in the other direction to derive a generalization of the classical EnKBF with randomized innovation term, that also eliminates the singular inflation term which stems from the correlated observation noise. To this end we note that (49) can be rewritten as

$$
d\tilde{X}_t = B_t(\tilde{X}_t) dt + \left(C_t C_t^T + \tilde{C}_t \tilde{C}_t^T\right) \tilde{P}_t^{-1} \frac{\tilde{X}_t - \tilde{m}_t}{2} dt \\
+ \left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1} (dY_t - H_t \tilde{X}_t dt) \\
+ \left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1} \left(\tilde{H}_t \tilde{P}_t - \Gamma_t \tilde{C}_t^T\right) \tilde{P}_t^{-1} \frac{\tilde{X}_t - \tilde{m}_t}{2} dt.
$$

Again we use (23) to replace $-\tilde{P}_t^{-1}(\tilde{X}_t - \tilde{m}_t)$ by $\nabla \log \tilde{\eta}(\tilde{X}_t)$ and thus turn (49) into

$$
d\tilde{X}_t = B_t(\tilde{X}_t) dt - \frac{C_t C_t^T + \tilde{C}_t \tilde{C}_t^T}{2} \nabla \log \tilde{\eta}(\tilde{X}_t) dt \\
+ \left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1} (dY_t - H_t \tilde{X}_t dt) \\
- \frac{\left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1} \left(\tilde{H}_t \tilde{P}_t - \Gamma_t \tilde{C}_t^T\right)}{2} \nabla \log \tilde{\eta}(\tilde{X}_t) dt.
$$

(50)

Next we note that since the matrix $\mathcal{M} := \tilde{C}_t \Gamma_t^T R_t^{-1} H_t \tilde{P}_t - \tilde{P}_t H_t^T R_t^{-1} \Gamma_t \tilde{C}_t^T$ is skew symmetric, we have

$$
\text{div}(\tilde{\eta}_t \mathcal{M} \nabla \log \tilde{\eta}_t) = \text{div}(\mathcal{M} \nabla \log \tilde{\eta}_t) = \mathcal{M} : \tilde{\eta}_t'' = \text{tr} \left[\mathcal{M}^T \tilde{\eta}_t''\right] = \text{tr} \left[\mathcal{M} \tilde{\eta}_t''\right] = -\text{tr} \left[\mathcal{M}^T \tilde{\eta}_t''\right] = -\text{div}(\tilde{\eta}_t \mathcal{M} \nabla \log \tilde{\eta}_t) = 0.
$$

Thus we can erase $\mathcal{M}$ (49) for (50) without changing the associated Kolmogorov equation. Therefore solutions of

$$
d\tilde{X}_t = B_t(\tilde{X}_t) dt - \frac{C_t C_t^T}{2} \nabla \log \tilde{\eta}(\tilde{X}_t) dt \\
+ \left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1} (dY_t - H_t \tilde{X}_t dt) \\
- \frac{\left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1} \left(\tilde{H}_t \tilde{P}_t - \Gamma_t \tilde{C}_t^T\right)}{2} \nabla \log \tilde{\eta}(\tilde{X}_t) dt.
$$

(51)

have the same conditional laws $(\tilde{\eta}_t)_{t \geq 0}$ as solutions to (50). Since $\Gamma_t^T R_t^{-1} \Gamma_t$, $\Gamma_t^T$ is a projection onto the range of $\Gamma_t^T$, it is easy to see that we can rewrite (51) into

$$
d\tilde{X}_t = B_t(\tilde{X}_t) dt - \frac{C_t C_t^T}{2} \nabla \log \tilde{\eta}(\tilde{X}_t) dt + \left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1} (dY_t - H_t \tilde{X}_t dt) \\
- \frac{\left(\tilde{C}_t \left(I - \Gamma_t^T R_t^{-1} \Gamma_t\right) - \tilde{P}_t H_t^T R_t^{-1} \Gamma_t\right) \tilde{C}_t^T}{2} \nabla \log \tilde{\eta}(\tilde{X}_t) dt.
$$

By now applying Lemma 27 we derive the equation

$$
d\tilde{X}_t = B_t(\tilde{X}_t) dt + C_t dW_t + \tilde{C}_t d\tilde{W}_t \\
+ \left(\tilde{P}_t H_t^T + \tilde{C}_t \Gamma_t^T\right) R_t^{-1} (dY_t - H_t \tilde{X}_t dt - \Gamma_t d\tilde{W}_t).
$$

(52)

This is a generalization of the classical EnKBF with randomized innovation term to the correlated noise case. This mean field equation was also derived by different arguments in [33] for state and parameter estimation problems. Note that (52) does not contain any singular terms. This simplifies the analysis and in [12] a well posedness proof for (52) can be found under the restriction that the observation function $H$ must be bounded. Our well posedness result of (20) in Theorem 26 can easily be adapted to all McKean–Vlasov equations in this section, in particular to the mean field filters (52) and (49).
Remark 28. We note that since for Gaussian \( \bar{\eta} \), it holds that \( \pi^1[\bar{\eta}][\nabla \log \bar{\eta}] = \nabla \log \bar{\eta} \), and by Lemma 27, all mean field filters derived in this subsection are optimal in the linear Gaussian setting. In particular (52) and (49) are consistent representations of the true posterior in this case.

5 The well posedness of the EnKBF

Just as for the linear Gaussian case, one can approximate (20) in a straightforward manner by the interacting particle system

\[
\begin{align*}
dX_i^t &= B_t \left( X_i^t \right) dt + C_t \left( X_i^t \right) dW_i^t + \bar{C}_t dV_i^t \\
&+ \left( P_t^M H_t^T + \bar{C}_t G_t^T \right) R_t^{-1} \left( dY_t - \frac{H_t \left( X_i^t + x_t^M \right)}{2} dt \right) \\
&- \left( P_t^M H_t^T + \bar{C}_t G_t^T \right) R_t^{-1} \Gamma_t \bar{C}_t^T \left( P_t^M \right)^+ \frac{X_i^t - x_t^M}{2} dt
\end{align*}
\]

for \( i = 1, \ldots, M \). Where again

\[
x_t^M := \frac{1}{M} \sum_{i=1}^{M} X_i^t \quad \text{and} \quad P_t^M := \frac{1}{M-1} \sum_{i=1}^{M} \left( X_i^t - x_t^M \right) \left( X_i^t - x_t^M \right)^T
\]

denote the ensemble average and the ensemble covariance matrix. \( (P_t^M)^+ \) is the Moore–Penrose pseudoinverse of \( P_t^M \).

In the uncorrelated case \( \bar{C} = 0 \) system (53) is referred to as the (deterministic) EnKBF and we will also refer to it as such in the correlated noise framework. Only in the linear Gaussian setting it will be an approximation of the optimal filter, otherwise its mean field limit (20) will not be a representation of the posterior.

Equation (53) is a system of nonlinear SDEs, in which the coefficients do not satisfy linear growth properties and which therefore falls outside the standard existence theorem for SDEs found for example in [25]. Nevertheless in [28] the existence and uniqueness of solutions was proven for uncorrelated observation noise. More precisely it was assumed that \( C \) is constant and \( \bar{C} = 0 \).

However, generalizing this result to the correlated case (53) is not straightforward, as the pseudoinverse \( (P_t^M)^+ \) does not depend continuously on the ensemble members \( X^t, i = 1, \ldots, M \) and actually develops singularities where \( P_t^M \) changes its rank. Indeed for \( M = 2 \) it is easy to see that

\[
(P_t^M)^+ \left( X_i^t - x_t^M \right) = \mathbb{1}_{|X_i^t - x_t^M| > \epsilon} \frac{X_i^t - x_t^M}{2 |X_i^t - x_t^M|^2}, \quad i = 1, 2,
\]

which becomes singular when the two particles collide. This makes the analysis of (53) particularly challenging as some of the most general existing well posedness results require at least local integrability. Indeed there exist counter examples of SDEs containing drift terms that are structurally similar to (54) that do not even admit weak solutions [9, Example 1.17]. The following theorem shows that despite these difficulties (53) is well posed under suitable assumptions.

**Theorem 29.** Beside the standard Lipschitz assumptions 1, let us assume that \( C \). If \( \bar{C} \neq 0 \) we furthermore assume that \( P_0^M \) is invertible and that \( M \) is large enough, so that

\[
Z^M := \inf_{t \leq T} \left( \gamma_t - \frac{2}{M-1} \left( \frac{2}{M-1} \right) \left( \|C_t\|^2 + |\bar{C}_t|^2 \right) \right) > 0.
\]

holds. Then there exists a unique strong solution to (53).
Remark 30. Before we prove Theorem 29 we want to remark that the assumption that $P_0^M$ is invertible implies $M \geq d_x + 1$. Thus this result is restricted to large ensemble sizes.

Remark 31. Note that in the limit $M \to \infty$ assumption (55) is consistent with condition (30), which was required for the well posedness of the mean field EnKBF (20) if $\mathcal{C} \neq 0$.

Proof of Theorem 29. The proof consists of three major steps. First we derive equations for the empirical moments of the ensemble. Next we show that $P^M$ stays regular as long as it does not blow up. Finally we prove the boundedness of $P^M$ and by that also the existence and uniqueness of solutions.

Denote by $\xi$ the explosion time of the ensemble and by $\underline{\xi}$ the time $P^M$ becomes non invertible.

Step 1: First we note that by the linearity of stochastic differentials the following equation for the mean holds up to time $\xi \land \underline{\xi}$

$$
\mathrm{d}x^M_t = \frac{1}{M} \sum_{i=1}^{M} B_t(X^i_t) \mathrm{d}t + \frac{1}{M} \sum_{i=1}^{M} C_t(X^i_t) \mathrm{d}W^i_t + \tilde{C}_t \mathrm{d}V^i_t \\
\quad + \left( P^M_t H^T_t + \tilde{C}_t \Gamma^T_t \right) R_t^{-1} \left( \mathrm{d}Y_t - H_t x^M_t \mathrm{d}t \right),
$$

which gives us

$$
\mathrm{d} \left( x^i_t - x^M_t \right) = \left( B_t(X^i_t) - b^M_t \right) \mathrm{d}t \\
\quad + \frac{1}{M} \sum_{j=1}^{M} \left( C_t(X^j_t) \mathrm{d}W^j_t - C_t(X^i_t) \mathrm{d}W^i_t \right) + \frac{1}{M} \sum_{j=1}^{M} \left( \tilde{C} \mathrm{d}V^j_t - \tilde{C} \mathrm{d}V^i_t \right) \\
\quad - \left( P^M_t H^T_t + \tilde{C}_t \Gamma^T_t \right) R_t^{-1} \left( H_t + \Gamma_t \tilde{C}_t^T (P^M_t)^+ \right) \frac{X^i_t - x^M_t}{2} \mathrm{d}t.
$$

By employing Itô’s product rule we derive the following evolution equation for the empirical covariance matrix up to time $\xi \land \underline{\xi}$

$$
\mathrm{d}P^M_t = [B_t|X^i_t]_M \mathrm{d}t + \left( \frac{1}{M} \sum_{i=1}^{M} C_t(X^i_t)C_t(X^i_t)^T + \tilde{C}_t \tilde{C}_t^T \right) \mathrm{d}t + \mathrm{d}\text{im}_t, \quad (56)
$$

where

$$
[B_t|X^i_t]_M := \frac{2}{M-1} \sum_{i=1}^{M} \text{Sym} \left( (B_t(X^i_t) - b^M_t) (X^i_t - x^M_t)^T \right) \mathrm{d}t.
$$

and $\text{im}$ denotes the local martingale starting at zero given by

$$
\mathrm{d}\text{im}_t := \frac{2}{M-1} \sum_{j=1}^{M} \text{Sym} \left( X^i_t - x^M_t \right) \left( C(X^j_t) \mathrm{d}W^j_t \right)^T \\
\quad + \frac{2}{M-1} \sum_{j=1}^{M} \text{Sym} \left( X^i_t - x^M_t \right) \left( \tilde{C} \mathrm{d}V^j \right)^T, \quad (57)
$$

Step 2: Using this equation we want to prove that $P^M$ is always invertible, as long as it stays bounded. To this end we define for every symmetric positive semidefinite matrix $P$ its regularized
inverse $P^{+\epsilon} := (P + \epsilon I)^{-1}$ for any $\epsilon > 0$ and where $I$ denotes the identity matrix. Note that if $P$ is not invertible, then $\text{tr} P^{+\epsilon}$ will blow up for $\epsilon \to 0$. Therefore we now try to dominate $(P^M)^{+\epsilon}$ uniformly in $\epsilon$.

In the following we will need the first and second order Fréchet derivatives of $T_\epsilon(P) := P^{+\epsilon}$ which are given by

$$T'_\epsilon(P)[A] = -P^{+\epsilon}AP^{+\epsilon}$$
$$T''_{\epsilon\epsilon}(P)[A, B] = 2 \text{Sym} (P^{+\epsilon}AP^{+\epsilon}BP^{+\epsilon})$$

for any two matrices $A$ and $B$. To simplify notations in the following we also define for any vector $\hat{x}$

$$S(A, \hat{x}, B) := \sum_k \text{tr} \left[ A \text{Sym} (\hat{x}B_k^T) \text{Sym} (\hat{x}B_k^T) A \right],$$

where $B_k$ denotes the $k$-th column of $B$.

Now we note that if we define the local martingale

$$dM'_\epsilon := -\text{tr} \left[ (P^M)^{+\epsilon} \ dm_t \ (P^M)^{+\epsilon} \right],$$

then by Itô's rule one can see

$$\text{tr} (P^M)^{+\epsilon} = -\text{tr} \left[ (P^M)^{+\epsilon} \ [B_t X_t]_M \ (P^M)^{+\epsilon} \right] dt$$
$$+ \text{tr} \left[ (P^M)^{+\epsilon} \left( P^M H_t^T + C_t G_t^T \right) R_t^{-1} \left( H_t + \Gamma_t \hat{C}_t \ (P^M)^{+\epsilon} \right) P^M \ (P^M)^{+\epsilon} \right] dt$$
$$- \text{tr} \left[ (P^M)^{+\epsilon} C_t \hat{C}_t^T \ (P^M)^{+\epsilon} \right] dt$$
$$- \frac{1}{M} \sum_{i=1}^M \text{tr} \left[ (P^M)^{+\epsilon} C_t (X^i) C_t (X^i)^T \ (P^M)^{+\epsilon} \right] dt$$
$$+ \frac{4}{(M-1)^2} \sum_{j=1}^M S \left( (P^M)^{+\epsilon}, X_j^i - x_j^M, C_t (X^j) \right) dt$$
$$+ \frac{4}{(M-1)^2} \sum_{j=1}^M S \left( (P^M)^{+\epsilon}, X_j^i - x_j^M, \hat{C}_t \right) dt + dM'_\epsilon.$$  \hspace{1cm} (59)

We aim to estimate this equation linearly and then use a Grönwall argument to bound $\text{tr} (P^M)^{+\epsilon}$.

To this end let us denote by $e_i$ the $i$-th canonical basis vector of $\mathbb{R}^{d_x}$ and by $P^M = Q^T \text{diag} \left( \lambda_1, \ldots, \lambda_d^M \right) Q$ the singular value decomposition of $P^M$. Then we first note that for arbitrary $k$

$$\sum_{j=1}^M \text{tr} \left[ (P^M)^{+\epsilon} (X^j_k - x^M_k) C(X^j)^T_k (P^M)^{+\epsilon} (X^j_k - x^M_k) C(X^j)^T_k (P^M)^{+\epsilon} \right]$$
$$= \sum_{i=1}^{d_x} \frac{1}{\lambda_i^M + \epsilon} \sum_{j=1}^M \left| e_i^T Q \left( X^j_k - x^M_k \right) \right| \left| (P^M)^{+\epsilon} (X^j_k - x^M_k) \right|$$
$$\leq \|C_k\| \frac{1}{\lambda_i^M + \epsilon} \sum_{j=1}^M \left| e_i^T Q \left( X^j_k - x^M_k \right) \right| \left| \left( P^M \right)^{+\epsilon} (X^j_k - x^M_k) \right|$$
$$\leq (M-1) \|C_k\| \sum_{i=1}^{d_x} \frac{\sqrt{\lambda_i^M}}{\lambda_i^M + \epsilon} \sqrt{\sum_{i=1}^{d_x} \lambda_i^M}. $$
For the last inequality we used that
\[
\frac{1}{M - 1} \sum_{j=1}^{M} \left| e_i^T Q \left( X_j^i - x_j^M \right) \right|^2 = \lambda_i^2 \quad \text{and that}
\]
\[
\frac{1}{M - 1} \sum_{j=1}^{M} \left| (P_t^M)^{+2\epsilon} \left( X_j^i - x_j^M \right) \right|^2 = \sum_{i=1}^{d_x} \frac{\lambda_i^2}{(\lambda_i^2 + \epsilon)^2}.
\]

Using the Cauchy–Schwarz inequality two times one then can verify immediately that
\[
\sum_{i=1}^{d_x} \frac{\lambda_i^2}{(\lambda_i^2 + \epsilon)^2} \leq \sqrt{\sum_{i=1}^{d_x} \lambda_i^2} \sqrt{\sum_{i=1}^{d_x} \frac{1}{(\lambda_i^2 + \epsilon)^2}}.
\]

With this and the estimate
\[
\sum_{j=1}^{M} \left( (P_t^M)^{+2\epsilon} \left( X_j^i - x_j^M \right) C(X_j^i)^T \right) \left( P_t^M \right)^{+2\epsilon} C(X_j^i)^T \left( X_j^i - x_j^M \right)^T \left( P_t^M \right)^{+2\epsilon}
\]
\[
= \sum_{i=1}^{d_x} \sum_{j=1}^{M} \left( e_i^T Q \left( X_j^i - x_j^M \right)^2 \left( C(X_j^i)^T \right) \left( P_t^M \right)^{+2\epsilon} C(X_j^i)^T \right)
\]
\[
\leq \|C\|_\infty^2 \left( \sum_{i=1}^{d_x} \frac{\lambda_i^2}{(\lambda_i^2 + \epsilon)^2} \right) \leq \|C\|_\infty^2 \sum_{i=1}^{d_x} \frac{1}{(\lambda_i^2 + \epsilon)^2}
\]

we can bound the Itô correction term
\[
\frac{4}{(M - 1)^2} \sum_{j=1}^{M} \left( S \left( \left( P_t^M \right)^{+2\epsilon}, X_j^i - x_j^M, C(X_j^i) \right) + S \left( \left( P_t^M \right)^{+2\epsilon}, X_j^i - x_j^M, \tilde{C}_t \right) \right)
\]
\[
\leq \frac{2}{M - 1} \left( 1 + \sqrt{d_x} \right) \left( \|C\|_\infty^2 + \|\hat{C}\|^2 \right) \sum_{i=1}^{d_x} \frac{1}{(\lambda_i^2 + \epsilon)^2}.
\] (60)

Similarly one shows that
\[
\text{tr} \left[ \left( P_t^M \right)^{+2\epsilon} \left( P_t^M H_t^T + \tilde{C}_t \Gamma_t^T \right) R_t^{-1} \left( H_t + \Gamma_t \tilde{C}_t \right) \left( P_t^M \right)^{+2\epsilon} \right]
\]
\[
\leq d_x \lambda_{\max} \left( H_t^T R_t^{-1} H_t \right) + 2 \left( H_t^T R_t^{-1} \Gamma_t \tilde{C}_t \right) \text{tr} \left[ \left( P_t^M \right)^{+2\epsilon} \right]
\]
\[
+ \sum_{i=1}^{d_x} \frac{1}{(\lambda_i^2 + \epsilon)^2} e_i^T Q \tilde{C}_t R_t^{-1} \tilde{C}_t^T Q^T e_i
\]
\[
\text{and}
\text{tr} \left[ \left( P_t^M \right)^{+2\epsilon} \tilde{C} \tilde{C}_t^T \left( P_t^M \right)^{+2\epsilon} \right]
\]
\[
+ \frac{M - 1}{M} \sum_{i=1}^{M} \text{tr} \left[ \left( P_t^M \right)^{+2\epsilon} C(X_j^i) \left( P_t^M \right)^{+2\epsilon} \right]
\]
\[
+ \frac{1}{M^2(M - 1)} \sum_{i=1}^{M} \sum_{j \neq i} \text{tr} \left[ \left( P_t^M \right)^{+2\epsilon} C(X_j^i) \left( P_t^M \right)^{+2\epsilon} \right]
\]
\[
\geq \sum_{i=1}^{d_x} \frac{e_i^T Q \tilde{C}_t \tilde{C}_t^T Q^T e_i}{(\lambda_i^2 + \epsilon)^2} + \inf_{x} \lambda_{\min} \left( C(x) C(x)^T \right) \sum_{i=1}^{d_x} \frac{1}{(\lambda_i^2 + \epsilon)^2}.
\] (62)
Combining these inequalities we derive the stochastic differential inequality
\[
dtr (P_t^M)^{+\epsilon} \leq -\tr \left[ (P_t^M)^{+\epsilon} \left[ B_t | X_t \right] M (P_t^M)^{+\epsilon} \right] dt - \frac{1}{2} \sum_{j=1}^{d_x} \frac{1}{(\lambda_j^t + \epsilon)^2} dt \\
+ d_x \lambda_{\max} \left( H_t^T R_t^{-1} H_t \right) dt + 2 \left| H_t^T R_t^{-1} \Gamma_t \tilde{C}_t^T \right| \tr \left[ (P_t^M)^{+\epsilon} \right] dt + dM_t^*.
\]

Using Cauchy–Schwarz and the Taylor inequality we derive for any arbitrary $\delta > 0$
\[
-\tr \left[ (P_t^M)^{+\epsilon} \left[ B_t | X_t \right] M (P_t^M)^{+\epsilon} \right] \\
\leq \sum_{j=1}^{d_x} \sqrt{\frac{2}{M-1}} \sum_{i=1}^{M} (e_j^T Q (B_t (X_t^i) - b_i^M))^2 \sqrt{\frac{2}{M-1}} \sum_{i=1}^{M} (e_j^T Q (X_t^i - x_t^M))^2 \\
\leq \sum_{j=1}^{d_x} \frac{2 \sqrt{\lambda_j^t \text{Lip}(B)} \sqrt{\tr P_t^M}}{(\lambda_j^t + \epsilon)^2} \leq \sum_{j=1}^{d_x} \frac{\text{Lip}(B)^2 \tr P_t^M}{\delta (\lambda_j^t + \epsilon)^2} + \delta \tr (P_t^M)^{+\epsilon}.
\]

By setting $\delta := \frac{\text{Lip}(B)^2 \tr P_t^M}{2M}$ we can thus bound the evolution of $\tr (P_t^M)^{+\epsilon}$ even further by
\[
dtr (P_t^M)^{+\epsilon} \leq \left( \delta + 2 \left| H_t^T R_t^{-1} \Gamma_t \tilde{C}_t^T \right| \right) \tr (P_t^M)^{+\epsilon} dt \\
+ d_x \lambda_{\max} \left( H_t^T R_t^{-1} H_t \right) dt + dM_t^*.
\]

(63)

We set $\xi^\kappa := \inf \left\{ t \geq 0 : \tr P_t^M > \kappa \right\}$. Then using the stochastic Grönwall lemma as found in [38, Theorem 4] on the differential inequality (63) implies
\[
E \left[ \sup_{t \leq T \wedge \xi^\epsilon} \left( \tr (P_t^M)^{+\epsilon} \right)^{1/2} \right] \leq (\pi + 1) \exp \left( \int_0^T \left| H_s^T R_s^{-1} \Gamma_s \tilde{C}_s^T \right| + \frac{\text{Lip}(B)^2 \xi}{2\gamma} ds \right) \\
\times E \left[ \left( \tr (P_0^M)^{+\epsilon} + d_x \int_0^T \lambda_{\max} \left( H_t^T R_t^{-1} H_s \right) ds \right)^{1/2} \right].
\]

We note that in particular this bound is independent of the local martingale $M^*$ and its quadratic variation, which is the key advantage of the stochastic Grönwall Lemma compared to standard bounds based on the Burkholder–Davis–Gundy inequality.

Letting $\epsilon$ go to zero thus shows that $\sup_{t \leq T \wedge \xi^\epsilon} \tr (P_t^M)^{+}$ can not blow up for arbitrary $\kappa > 0$. Thus on the interval $[0, \xi]$ the empirical covariance matrix $P^M_t$ will always be regular, which also implies $\xi \leq \xi^\epsilon$. Therefore it is left to show that blow ups can not occur in order to prove the well posedness.

**Step 3:** The rest of the proof is similar to the one found in [28] for the uncorrelated case.

We note that since $P_t^M H_t^T R_t^{-1} H_t^T P_t^M$ is symmetric positive semidefinite and since $\left| (P_t^M)^{+} P_t^M \right| \leq$
\[ \sqrt{d_x}, \text{ we have} \]
\[ -\text{tr} \left[ \left( P_t^M H_t^T + \hat{C}_t \Gamma_t^T \right) R_t^{-1} \left( H_t + \Gamma_t \hat{C}_t^T \left( P_t^M \right)^+ \right) P_t^M \right] \]
\[ = -\text{tr} \left[ P_t^M H_t^T R_t^{-1} H_t P_t^M \right] - \text{tr} \left[ P_t^M H_t^T R_t^{-1} \Gamma_t \hat{C}_t^T \left( P_t^M \right)^+ P_t^M \right] \]
\[ - \text{tr} \left[ \hat{C}_t \Gamma_t^T R_t^{-1} H_t P_t^M \right] - \text{tr} \left[ \hat{C}_t \Gamma_t^T R_t^{-1} \Gamma_t \hat{C}_t^T \left( P_t^M \right)^+ P_t^M \right] \]
\[ \leq 2d_x \left| \hat{C}_t \Gamma_t^T R_t^{-1} H_t \right| \text{tr} \left[ P_t^M \right] + d_x \left| \hat{C}_t \Gamma_t^T R_t^{-1} \Gamma_t \hat{C}_t^T \right|. \]

This gives us the stochastic differential inequality
\[ \text{d}tr P_t^M \leq 2 \left( \text{Lip} (B) + d_x \left| \hat{C}_t \Gamma_t^T R_t^{-1} H_t \right| \right) \text{tr} P_t^M \text{dt} + 2d_x \left| \hat{C}_t \Gamma_t^T R_t^{-1} \Gamma_t \hat{C}_t^T \right| \text{dt} \]
\[ + \sup_{s \leq t} \text{tr} \hat{C}_t \Gamma_t^T \text{dt} + \sup_{s \leq t} \| C_s \|^2 \text{dt} + \text{d}m_t \]

Using the stochastic Grönwall Lemma [38, Theorem 4] on the differential inequality (64) we can thus bound the covariance matrix by
\[ \mathbb{E} \left[ \sup_{t \leq T \wedge \xi} \sqrt{\text{tr} P_t^M} \right] \leq (\pi + 1) \exp \left( \int_0^T \text{Lip} (B) + d_x \left| \hat{C}_s \Gamma_s^T R_s^{-1} H_s \right| \text{ds} \right) \]
\[ \sqrt{d_x T} \sup_{t \leq T} \left| \hat{C}_t \Gamma_t^T R_t^{-1} \Gamma_t \hat{C}_t^T \right| + \frac{TM}{M - 1} \sup_{t \leq T} \left( \left| \hat{C}_t \right|^2 + \| C_s \|^2 \right). \]

Due to the assumptions, the right-hand side of this inequality is finite. It is easy to see that on the set of paths where \( P_t^M \) is bounded, the empirical mean evolves at most linearly and thus one can easily verify the bound
\[ \mathbb{E} \left[ \sup_{t \leq T \wedge \xi} \sqrt{|x_t^M|} \right] < +\infty, \]

which shows that \( \mathbb{P} (\xi < T) = 0 \). Thus explosion can not occur in the time interval \([0, T]\), which let’s us use standard localization arguments (see for example [32, Theorem 3.4 and its proof]) to conclude the existence of a unique global solution to the EnKBF.

**Remark 32.** We could drop the assumptions \( \gamma^M > 0 \) and \( P_0^M \) being invertible in the uncorrelated case \( \hat{C} = 0 \), where well posedness was proven already in [28]. We could also drop these assumptions if we replaced the Moore–Penrose inverse \( (P_t^M)^+ \) in (53) by a regularized inverse of the form
\[ (P_t^M)^{+_{\epsilon, n}} := \left( (P_t^M)^n + \epsilon I \right)^{-1} \left( P_t^M \right)^{n-1}. \]

for arbitrary but fixed \( \epsilon > 0 \) and \( n \in \mathbb{N} \). For \( n = 1 \) this is similar to Ensemble inflation, a well known technique in data assimilation [36], whereas for \( n = 2 \) this is a canonic approximation of the Pseudoinverse. Since this regularization depends Lipschitz-continuously on \( P_t^M \) it would have then sufficed to only carry out steps 1 and 3 in the proof of theorem 29.

If one tried to prove the well posedness for (53) without the regularization for small ensemble sizes \( M \leq d_x \), i.e. without the assumption that \( P_t^M \) is invertible, one would try to show that \( P_t^M \) never changes its rank. This could be done similar to step 2 in the proof of theorem 29. However one would need to show that \( (P_t^M)^{+_{\epsilon, 2}} \), instead of just bounding \( (P_t^M)^{+_{\epsilon, 1}} \) as it was done in the proof, as the latter one will always explode for non-invertible \( P_t^M \) when \( \epsilon \to 0 \), whereas \( (P_t^M)^{+_{\epsilon, 2}} \) only does this when the matrix rank changes.

Now we again shift our focus to the mean field limit of (53).
6 Propagation of Chaos

In this section we aim to prove that the particles defined by the regularized EnKBF (53) indeed converge to the solution of (20) for $M \to \infty$. A proof for the uncorrelated case with time independent signal diffusion $C_t(x) = C$, $\tilde{C}_t = 0$ can be found in [28]. The first results in the linear Gaussian setting were obtained in [15].

We consider $M$ independent copies $\tilde{X}_i^t$, $i = 1, \cdots, M$ of the solution $\tilde{X}$ to (20) satisfying

\[
\begin{aligned}
d\tilde{X}_i^t &= B_t(\tilde{X}_i^t)dt + C_t(\tilde{X}_i^t)dW_i^t + \tilde{C}_tdV_i^t \\
&+ \left( \tilde{P}_tH_t^T + \tilde{C}_t\Gamma_t^T \right) R_t^{-1} \left( dY_t - \frac{H_t(\tilde{X}_i^t + \tilde{m}_t)}{2} dt \right) \\
&- \frac{\tilde{P}_tH_t^T + \tilde{C}_t\Gamma_t^T}{2} R_t^{-1} \Gamma_t \tilde{C}_t^T \tilde{P}_t^+ (\tilde{X}_i^t - \tilde{m}_t) dt.
\end{aligned}
\]  

(65)

**Theorem 33.** We make the same assumptions as in Theorem 26. Define the error term $r_i^t := X_i^t - \tilde{X}_i^t$, then the following result holds

\[
\sup_{t \leq T} \frac{1}{M} \sum_{i=1}^{M} |r_i^t|^2 \xrightarrow{M \to \infty} 0
\]  

(66)

in probability (and almost surely with respect to $Y$).

**Proof.** By Remark 31 the condition $\gamma > 0$ also implies $\gamma^M > \frac{\gamma}{2} > 0$ for sufficiently large $M$. Thus we can assume in the following that $M$ is large enough so that the well posedness of the ensemble filter proven in Theorem 29 holds.

For the sake of brevity in formulas let us define

\[
\psi_s(P) := \frac{PH_t^T + \tilde{C}_t\Gamma_t^T}{2} R_t^{-1} \Gamma_t \tilde{C}_t^T P^+.
\]  

(67)

We note that since both $X_i$ and $\tilde{X}_i$ share the same initial condition. Then we have

\[
\begin{aligned}
r_i^t &= \int_0^t \left( B_s(X_i^s) - B_s(\tilde{X}_i^s) \right) ds + \int_0^t \left( C_s(X_i^s) - C_s(\tilde{X}_i^s) \right) dW_i^s \\
&+ \int_0^t \left( P_s - \tilde{P}_s \right) H_s^T R_s^{-1} \left( dY_s - \frac{H_s(\tilde{X}_i^s + \tilde{m}_s)}{2} ds \right) \\
&- \frac{1}{2} \int_0^t \left( \tilde{P}_s H_s^T + \tilde{C}_s \Gamma_s^T \right) R_s^{-1} H_s \left( (X_i^s - \tilde{X}_i^s) + (x_i^M - \tilde{m}_s) \right) ds \\
&+ \int_0^t \psi_s(\tilde{P}_s) \left( (X_i^s - \tilde{X}_i^s) + (x_i^M - \tilde{m}_s) \right) ds \\
&- \int_0^t \left( \psi_s(P_{i}^M) - \psi_s(\tilde{P}_s) \right) (X_i^s - x_i^M) ds.
\end{aligned}
\]

To take care of the integral involving $Y$ one could later on use a change of measure to turn $Y$ into a Brownian motion. We will not do this and instead use the explicit representation $dY_s =
Using Itô’s rule, the Lipschitz properties of the coefficients, as well as the fact that $2a \cdot b \leq |a|^2 + |b|^2$, we get

$$r^i_s = \int_0^t d \left( X^i_s - \bar{X}^i_s \right)$$

$$= \int_0^t \left( B_s(X^i_s) - B_s(\bar{X}^i_s) - \frac{1}{2} \bar{P}_s H_s^T R_s^{-1} H_s \left( X^i_s - \bar{X}^i_s \right) \right) ds$$

$$+ \int_0^t \left( C_s(X^i_s) - C_s(\bar{X}^i_s) \right) dW^i_s + \int_0^t \left( P^M_s - \bar{P}_s \right) H_s^T \Gamma_s dV_s$$

$$+ \int_0^t \left( P^M_s - \bar{P}_s \right) H_s^T R_s^{-1} H_s \left( X^\text{ref}_s - \frac{\bar{X}^i_s + \bar{m}_s}{2} \right) ds$$

$$- \frac{1}{2} \int_0^t \left( \bar{P}_s H_s^T + \bar{C}_s \Gamma_s^T \right) R_s^{-1} H_s (x^M_s - \bar{m}_s) ds$$

$$+ \int_0^t \psi(\bar{P}_s) \left( (X^i_s - \bar{X}^i_s) - (x^M_s - \bar{m}_s) \right) ds$$

$$- \int_0^t \left( \psi(P^M_s) - \psi(\bar{P}_s) \right) (X^i_s - x^M_s) ds.$$

Using Itô’s rule, the Lipschitz properties of the coefficients, as well as the fact that $2a \cdot b \leq |a|^2 + |b|^2$, we get

$$\frac{1}{M} \sum_{i=1}^M |r^i_s|^2 \leq \int_0^t \mathcal{L}_s \frac{1}{M} \sum_{i=1}^M |r^i_s|^2 ds + \frac{2}{M} \sum_{i=1}^M \int_0^t r^i_s \cdot \left( C_s(X^i_s) - C_s(\bar{X}^i_s) \right) dW^i_s$$

$$+ \frac{2}{M} \sum_{i=1}^M \int_0^t r^i_s \cdot \left( P^M_s - \bar{P}_s \right) H_s^T \Gamma_s dV_s$$

$$+ \int_0^t \left| P^M_s - \bar{P}_s \right|^2 \left| H_s^T R_s^{-1} H_s \right|^2 \left( |X^\text{ref}_s|^2 + \frac{1}{2M} \sum_{i=1}^M |\bar{X}^i_s|^2 + \frac{\bar{m}_s^2}{2} \right) ds$$

$$- \int_0^t \left( \frac{1}{M} \sum_{i=1}^M r^i_s \right) \cdot A_s (x^M_s - \bar{m}_s) ds + \int_0^t \left| H_s^T R_s^{-1} H_s \right| \left| P^M_s - \bar{P}_s \right|^2 ds$$

$$+ \int_0^t \psi_s(P^M_s) - \psi(\bar{P}_s) \left( \frac{1}{M} \sum_{i=1}^M |X^i_s - x^M_s|^2 \right) ds,$$

where we denote

$$\mathcal{L}_s := 2 \text{ Lip}(B) + \text{ Lip}(C) + |\bar{P}_s H_s^T R_s^{-1} H_s| + 2 |\psi_s(\bar{P}_s)| + 2 \bar{A}_s$$

$$A_s := \bar{P}_s H_s^T R_s^{-1} H_s + \bar{C}_s \Gamma_s^T R_s^{-1} H_s + 2 \psi_s(\bar{P}_s)$$

for the sake of brevity.

First we note that

$$- (x^M_s - \bar{x}^M_s) \cdot \left( \bar{P}_s H_s^T R_s^{-1} H_s + \bar{C}_s \Gamma_s^T R_s^{-1} H_s + 2 \psi_s(\bar{P}_s) \right) (x^M_s - \bar{m}_s)$$

$$\leq \frac{1}{M} \sum_{i=1}^M |r^i_s|^2 + \left| \bar{P}_s H_s^T R_s^{-1} H_s + \bar{C}_s \Gamma_s^T R_s^{-1} H_s + 2 \psi_s(\bar{P}_s) \right|^2 |\bar{x}^M_s - \bar{m}_s|^2$$
and therefore if we denote
\[
\mathrm{Im}_t^M := \frac{2}{M} \sum_{i=1}^{M} \int_0^t r_s^i \cdot (C_s(X_s^i) - C_s(\bar{X}_s^i)) \, dW_s^i + \frac{2}{M} \sum_{i=1}^{M} \int_0^t r_s^i \cdot (P_s^M - \bar{P}_s) H_s^T R_s^{-1} \Gamma_s \, dV_s,
\]
we derive
\[
\frac{1}{M} \sum_{i=1}^{M} |r_t^i|^2 \leq \int_0^t (\mathcal{L}_s + 1) \frac{1}{M} \sum_{i=1}^{M} |r_t^i|^2 \, ds + \int_0^t |P_s^M - \bar{P}_s|^2 |H_s^T R_s^{-1} H_s|^2 \left( |X_s^i|^2 + \frac{1}{2M} \sum_{i=1}^{M} |\bar{X}_s^i|^2 + \frac{|\bar{m}_s|^2}{2} \right) \, ds + \int_0^t |\bar{P}_s H_s^T R_s^{-1} H_s + \bar{C}_s \Gamma_s^T R_s^{-1} H_s + \psi_s(\bar{P}_s)|^2 |\bar{x}_s^M - \bar{m}_s|^2 \, ds + \int_0^t |\psi_s(P_s^M) - \psi_s(\bar{P}_s)|^2 \, tr P_s^M \, ds + \mathrm{Im}_t^M.
\]
Clearly \( \psi_s \) is locally Lipschitz if both arguments are invertible. Let us denote its Lipschitz constant on
\[
\mathcal{S}_\kappa = \{ A \in \mathbb{R}^{d_x \times d_x} \mid |A| \leq \kappa \text{ and } |A^{-1}| \leq \kappa \}
\]
by \( \text{Lip}_{loc}(\psi, \kappa) \). Since we have a priori bounds for \( \bar{P} \), both from above and below, we can choose \( \kappa \) large enough so that \( \bar{P}_t \in \mathcal{S}_\kappa \) for all times \( t \leq T \). On the event that both \( P^M \) also stays in the set \( \mathcal{S}_\kappa \) this gives us the bound
\[
|\psi(P_s^M) - \psi(\bar{P}_s)| \leq \text{Lip}_{loc}(\psi_s, \kappa) |P_s^M - \bar{P}_s|.
\]
Now we note that
\[
\frac{1}{M} \sum_{i=1}^{M} |\bar{X}_s^i|^2 \leq \frac{2}{M} \sum_{i=1}^{M} |\bar{X}_s^i - \bar{x}_s^m|^2 + 2 |\bar{x}_s^m|^2 \leq \frac{2(M - 1)}{M} \text{tr} P_s^M + 2 |\bar{x}_s^m|^2,
\]
and that
\[
|P_s^M - \bar{P}_s| \leq \left( \frac{M}{M - 1} + \frac{M}{M - 1} \right) \sqrt{\text{tr} P_s^M + \text{tr} \bar{P}_s} \left( \frac{1}{M} \sum_{i=1}^{M} |r_t^i|^2 \right)^{1/2} + |P_s^M - \bar{P}_s|,
\]
where \( \bar{x}_s^M \) and \( P^M \) denote the empirical mean and covariance matrix of the i.i.d. copies \( \bar{X}_s^i, i = 1, \ldots, M \).

Therefore we derive that on the event that \( P^M \) stays in \( \mathcal{S}_\kappa \) we have
\[
\frac{1}{M} \sum_{i=1}^{M} |r_t^i|^2 \leq \int_0^t \mathcal{L}_s^1 \frac{1}{M} \sum_{i=1}^{M} |r_t^i|^2 \, ds + \int_0^t \mathcal{L}_s^2 |P_s^M - \bar{P}_s|^2 \, ds + \int_0^t \mathcal{L}_s^3 |\bar{x}_s^M - \bar{m}_s|^2 \, ds + \mathrm{Im}_t^M,
\]
where
\[
L_s^1 := L_s + 1 + (1 + \text{Lip}_{\text{loc}}(\psi_s, \kappa))^2 \left( \sqrt{\frac{M}{M - 1} + \frac{M}{M - 1}} \right)^2 \left( \text{tr}P_s^M + \text{tr}\bar{P}_s^M \right) L_s^2,
\]
\[
L_s^2 := 2 \left( \left| H_s^T R_s^{-1} H_s \right|^2 + 1 \right) \left( \left| X_s^\text{ref} \right|^2 + \frac{(M - 1)}{M} \left| \bar{P}_s^M \right|^2 + \left| \bar{m}_s \right|^2 + 1 \right),
\]
\[
L_s^3 := \left| \bar{P}_s \right|^2 \left| H_s^T R_s^{-1} H_s \right|^2.
\]

For arbitrary constant $\kappa > 0$ we define the stopping time
\[
\zeta_\kappa := \inf \{ t \geq 0 : P_t^M \notin \mathcal{S}_\kappa \text{ or } \text{tr}\bar{P}_t^M \leq \kappa \}.
\]

Then on the stochastic interval $[0, \zeta_\kappa]$ we can bound the three constants $L_t^1, L_t^2, L_t^3$ uniformly by some $C_\kappa$. Thus the stochastic Grönwall Lemma implies that there exists a constant $c > 0$ such that
\[
\mathbb{E} \left[ \sup_{t \leq T < \zeta_\kappa} \left| \frac{1}{M} \sum_{i=1}^M r_i^2 \right| \right] \leq c \sqrt{C_\kappa} \exp \left( \frac{TC_\kappa}{2} \right) \mathbb{E} \left[ \left| X_s^\text{ref} - \bar{m}_s \right|^2 + \left| \bar{P}_s^M - \bar{P}_s \right|^2 \right].
\]

For fixed $\kappa$ the right hand side will converge to zero for $M \to +\infty$ due to the law of large numbers. Thus for every fixed $\kappa$ the error term converges in probability against zero. Since $\zeta_\kappa \to +\infty$ for $\kappa \to +\infty$. This concludes the proof.

Finally we want to note that both the well posedness result in Theorem 29 and the propagation of chaos result in Theorem 33 can easily be adapted to the particle systems approximating the transport based mean field EnKBF (49) and the vanilla EnKBF (52).

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