THE REACTION $\pi N \rightarrow \pi \pi N$ AT THRESHOLD

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ABSTRACT:

We consider the chiral expansion for the reaction $\pi N \rightarrow \pi \pi N$ in heavy baryon chiral perturbation theory. To order $M_\pi$ we derive novel low-energy theorems that compare favorably with recent determinations of the total cross sections for $\pi^+ p \rightarrow \pi^+ \pi^+ n$ and $\pi^- p \rightarrow \pi^0 \pi^0 n$. 

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1. Over the last few years, new data for the reaction $\pi N \to \pi\pi N$ in the threshold region and above have become available, see e.g. [1] [2] [3] [4] [5] and the compilation in [6]. The interest in this reaction stems mostly from the fact that it apparently offers a possibility of determining the low–energy $\pi\pi$ elastic scattering amplitude whose precise knowledge allows to test our understanding of the chiral symmetry breaking of QCD. However, at present no calculation based on chiral perturbation theory is available which links the pion production data to the $\pi\pi \to \pi\pi$ amplitude in a model–independent fashion. Consequently, all presently available determinations of the S–wave $\pi\pi$ scattering lengths from the abovementioned data should be taken cum grano salis. In the framework of relativistic baryon chiral perturbation theory, Beringer [6] has considered tree diagrams. In that approach, however, there is no strict one–to–one correspondence between the loop and the small momentum expansion due to the nonvanishing nucleon mass in the chiral limit [7]. This problem can be circumvented if one considers the nucleons in the non–relativistic limit. This was first used by Gasser and Leutwyler [8] (and others) and later formulated in terms of heavy quark effective field theory methods in ref.[9]. We will make use here of the two–flavor formulation detailed in ref.[10]. The aim of this letter is to show that the first two terms in the chiral expansion of the threshold amplitudes for $\pi N \to \pi\pi N$ lead to a set of low–energy theorems which indeed can be tested against the available data for $\pi^+p \to \pi^+\pi^+n$ and $\pi^-p \to \pi^0\pi^0n$ close to threshold. Naturally, at the next stage one has to consider the following terms in the chiral expansion to make contact with the $\pi\pi$ interaction.

2. To be specific, consider the process $\pi^aN \to \pi^b\pi^cN$, with $N$ denoting the nucleon (proton or neutron) and ‘$a, b, c’$ are isospin indices. At threshold, the transition matrix–element in the $\pi^aN$ centre–of–mass frame takes the form

$$T = i \vec{\sigma} \cdot \vec{k} \left[D_1(\tau^b\delta^{ac} + \tau^c\delta^{ab}) + D_2\tau^a\delta^{bc}\right]$$

(1)

where $\vec{k}$ denotes the three–momentum of the incoming pion and the amplitudes $D_1$ and $D_2$ will be subject to the chiral expansion as discussed below. They are related to the more commonly used amplitudes $A_{2I, I_{\pi\pi}}$, with $I$ the total isospin of the initial $\pi N$ system and $I_{\pi\pi}$ the isospin of the two–pion system in the final state, via

$$A_{32} = 2\sqrt{2}D_1, \quad A_{10} = -2D_1 - 3D_2$$

(2)

which have recently been determined [4]. In what follows, we will also consider the total cross section for the reactions $\pi^+p \to \pi^+\pi^+n$ and $\pi^-p \to \pi^0\pi^0n$. At present, only in these two channels there exist accurate data in the 20...30 MeV region above threshold. The data of ref.[4] for $\pi^+p \to \pi^+\pi^0p$ are still too sparse and inaccurate in the threshold region.

1This result was indirectly contained in ref.[6] but not made explicit and is much more transparent in the formulation used here.
which we are investigating. Assuming that the amplitude in the threshold region can be approximated by the exact threshold amplitude, the total cross section can be written in a compact form,

\[
\sigma_{\text{tot}}(s) = \frac{m^2}{2s} \sqrt{\lambda(s, m^2, M_\pi^2)} \Gamma_3(s) |\eta_1 D_1 + \eta_2 D_2|^2 S
\]

with \( m \) the nucleon and \( M_\pi \) the pion mass, respectively and \( s \) the total centre–of–mass energy squared. \( \Gamma_3(s) \) denotes the conventional three–body phase space and \( \lambda(x, y, z) \) the Källén–function. The \( \eta_{1,2} \) are channel-dependent isospin factors and \( S \) is a Bose symmetry factor. For \( \pi^+ p \to \pi^+ \pi^+ n \) and \( \pi^- p \to \pi^0 \pi^0 n \) we have \( \eta_1 = 2\sqrt{2}, \eta_2 = 0, S = 1/2 \) and \( \eta_1 = 0, \eta_2 = \sqrt{2}, S = 1/2 \), in order. In the threshold region, one can approximate to a high degree of accuracy the three–body phase space and flux factor \[11\] by analytic expressions so that

\[
\sigma_{\text{tot}}(T_\pi) = \frac{M_\pi^2 \sqrt{3(2 + \mu)(2 + 3\mu)}}{128\pi^2(1 + 2\mu)^{11/2}} |\eta_1 D_1 + \eta_2 D_2|^2 S (T_\pi - T_\pi^{\text{thr}})^2
\]

Here, \( T_\pi \) is the pion kinetic energy in the laboratory frame, \( T_\pi = (s - m_1^2 - M_\pi^2)/(2m_1) - M_\pi \) where the subscript \('1'\) denotes the particles in the initial state. We furthermore have introduced the small parameter \( \mu = M_\pi/m \simeq 1/7 \). This completes the necessary formalism.

3. In QCD, the chiral expansion of the amplitude functions \( D_1 \) and \( D_2 \) takes the form

\[
D = f_0 + f_1 \mu + f_2 \mu^2 + \ldots
\]

modulo logarithms. We are interested here in the first two coefficients of this expansion. To calculate them, we make use of heavy baryon chiral perturbation theory as detailed in ref.\[10\]. The pertinent effective Lagrangian has the form

\[
\mathcal{L}_{\text{eff}} = \mathcal{L}_{\pi N}^{(1)} + \mathcal{L}_{\pi N}^{(2)} + \mathcal{L}_{\pi \pi}^{(2)}
\]

with \( \mathcal{L}_{\pi N}^{(1,2)} \) given in ref.\[11\] and the standard meson Lagrangian e.g. in ref.\[12\]. The diagrams with insertions from \( \mathcal{L}_{\pi N}^{(1,2)} \) which are non–vanishing at threshold are shown in fig.1. Notice that the much debated next–to–leading order \( \pi \pi \) interaction does not appear at this order in the chiral expansion. It is important to note that from \( \mathcal{L}_{\pi N}^{(2)} \) only terms which are kinematical \( 1/m \) corrections contribute. None of the low–energy constants \( c_{1,2,3} \) related to elastic \( \pi N \) scattering (in particular to the isospin–even S–wave scattering length \( a^+ \)) \[13\] appear at order \( q^2 \) (here, \( q \) denotes a small momentum or a meson mass). One

\[2\]Here, \( D \) stands as a generic symbol for \( D_{1,2} \).
can therefore write down low–energy theorems for $D_{1,2}$ which only involve well–known physical (lowest order) parameters,

\[
D_1 = \frac{g_A}{8F_\pi^2} \left( 1 + \frac{7M_\pi}{2m} \right) + \mathcal{O}(M_\pi^2) 
\]  

(7)

\[
D_2 = -\frac{g_A}{8F_\pi^2} \left( 3 + \frac{17M_\pi}{2m} \right) + \mathcal{O}(M_\pi^2) 
\]  

(8)

with $g_A$ the axial–vector coupling constant. In what follows, we will always use the Goldberger–Treiman relation $g_A = g_{\pi N} F_\pi / m$ to calculate the numerical values of $D_{1,2}$ (with $g_{\pi N}$ the strong pion–nucleon coupling constant).

![Diagram](image)

Fig.1: Diagrams which give the contributions to $D_{1,2}$ up–to–and–including order $\mathcal{O}(M_\pi)$. The circle–cross denotes an insertion from $\mathcal{L}_{\pi N}^{(2)}$.

There are potentially large contributions from diagrams with intermediate $\Delta(1232)$ states of the type $M_\pi^2 / (m_\Delta - m - 2M_\pi)$, which numerically would be of the order $10 \cdot M_\pi^3$. We have checked that no such terms appear from diagrams involving one or two intermediate $\Delta$ resonances. Consequently, the chiral expansion is well behaved but not too rapidly converging. The order $M_\pi$ corrections give approximatively $50\%$ of the leading term. As we will discuss below, the calculations of Beringer [6] in relativistic baryon chiral perturbation theory indicate that further $1/m$ suppressed kinematical corrections are small. To get an idea about the corrections to eqs.(7,8) we have also calculated the imaginary parts of the threshold amplitudes from the one–loop diagrams shown in fig.2. These start to contribute at order $M_\pi^2$ with the result

\[
\text{Im } D_1 = -\frac{\sqrt{3} g_A^3 M_\pi^2}{128 \pi F_\pi^5} + \mathcal{O}(M_\pi^3) 
\]  

(9)

\[
\text{Im } D_2 = \frac{5 \sqrt{3} g_A^3 M_\pi^2}{64 \pi F_\pi^5} + \mathcal{O}(M_\pi^3) 
\]  

(10)

\footnote{Possible large $\Delta$–contributions starting at order $M_\pi^2$ have yet to be investigated in a systematic fashion together with loop effects and alike.}
Of course, at this order there are other contributions to the real parts of $D_{1,2}$, so one should consider the $M_\pi^2$ corrections given in eqs. (9,10) as indicative.

Fig. 2: One–loop diagrams which give a nonzero $\text{Im } D_{1,2}$ at order $\mathcal{O}(M_\pi^2)$.

4. Let us now turn to the numerical results. We use $F_\pi = 93$ MeV, $g_{\pi N} = 13.4$, $m = 938.27$ MeV and $M_\pi = 139.57$ MeV. This amounts to $D_1 = 2.4$ fm$^3$ and $D_2 = -6.8$ fm$^3$ or using eq. (2)

$$A_{32} = 2.4 M_\pi^{-3}, \quad A_{10} = 5.5 M_\pi^{-3}$$ (11)

which compare favourably with the recent determinations of ref. [2], $A_{32} = 2.07 \pm 0.10 M_\pi^{-3}$ and $A_{10} = 6.55 \pm 0.16 M_\pi^{-3}$. If one assumes that the imaginary parts eqs. (9,10) set the magnitude for the order $M_\pi^2$ corrections of $\text{Re } D_{1,2}$, then one expects $D_1$ to change very little and $D_2$ by approximatively 30%. We have also calculated the cross sections for $\pi^+ p \rightarrow \pi^+ \pi^+ n$ and $\pi^- p \rightarrow \pi^0 \pi^0 n$ using eq. (3). These are shown in fig. 3 in comparison to the existing data. Notice that we have calculated the matrix–elements in the isospin limit, in fig. 3 we have shifted the resulting cross sections to account for the corresponding thresholds as proposed by Beringer [6]. To a high degree of accuracy, one can parametrize the cross sections calculated from the first two terms of the chiral expansion of $D_{1,2}$ by the simple forms using eq. (4)

$$\sigma_{\text{tot}}^{\pi^+ p \rightarrow \pi^+ \pi^+ n}(T_\pi) = 0.225 \mu b \left( \frac{T_\pi - T_{\text{thr}}}{10 \text{ MeV}} \right)^2 \quad (T_{\text{thr}} = 172.4 \text{ MeV})$$ (12)

$$\sigma_{\text{tot}}^{\pi^- p \rightarrow \pi^0 \pi^0 n}(T_\pi) = 0.442 \mu b \left( \frac{T_\pi - T_{\text{thr}}}{10 \text{ MeV}} \right)^2 \quad (T_{\text{thr}} = 160.5 \text{ MeV})$$ (13)

as shown by the dashed lines in fig. 3. The solid lines differ very little from the ones in ref. [3] indicating that higher order $1/m$ corrections (which are summed up in the relativistic approach) are fairly small. This also means that the approximation of using the exact threshold amplitude in the threshold region is a very good one for the first 30 MeV. The advantage of the heavy mass approach used here is the strict one–to–one correspondence
between the loop and small momentum expansion. The abovementioned expectations of higher order corrections from $\text{Im } D_{1,2}$ are indeed such that they can improve the description of the data since the first/second channel allows to test $D_1/D_2$, respectively.

Fig. 3: Total cross sections for $\pi^+ p \rightarrow \pi^+ \pi^+ n$ and $\pi^- p \rightarrow \pi^0 \pi^0 n$ in comparison to the data. Squares: ref.[3], diamonds: ref.[4] and octagon: ref.[5].

5. We have considered the first two coefficients of the chiral expansion for the threshold $\pi N \rightarrow \pi \pi N$ amplitudes and derived a set of low–energy theorems, eqs.(7,8,9,10), which only involve well–known physical parameters. We have also shown that the corresponding cross sections agree with the empirical ones close to threshold. Of course, to go further, one has to consider loop diagrams as well as contributions from resonance exchange (like e.g. the $\Delta(1232)$). Ultimately, this will tell how accurately one can in fact get to the
elastic $\pi\pi$ amplitude from data on single pion production. Work along these lines is in progress.

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References

[1] D. Počanić et al., Phys.Rev.Lett.\textbf{72}, 1156 (1994).

[2] H. Burkhardt and J. Lowe, Phys.Rev.Lett.\textbf{67}, 2622 (1991).

[3] M.E. Sevior et al., Phys.Rev.Lett.\textbf{66}, 2569 (1991).

[4] G. Kernel et al., Z.Phys. \textbf{C48}, 201 (1990).

[5] J. Lowe et al., Phys.Rev.\textbf{C44}, 956 (1991).

[6] J. Beringer, $\pi N$ Newsletter \textbf{7}, 33 (1993).

[7] J. Gasser, M.E. Sainio and A. Švarc, Nucl.Phys.\textbf{B307}, 779 (1988).

[8] J. Gasser and H. Leutwyler, Phys.Rep.\textbf{87}, 77 (1982).

[9] E. Jenkins and A.V. Manohar, Phys.Lett.\textbf{B255}, 558 (1991).

[10] V. Bernard, N. Kaiser, J. Kambor and Ulf-G. Meißner, Nucl.Phys.\textbf{B388}, 315 (1992).

[11] V. Bernard, N. Kaiser, Ulf-G. Meißner and A. Schmidt, ”Threshold Two–Pion Photo– and Electroproduction: More neutrals than expected”, preprint CRN-94/14, 1994.

[12] J. Gasser and H. Leutwyler, Ann.Phys.(NY)\textbf{158}, 124 (1984).

[13] V. Bernard, N. Kaiser and Ulf-G. Meißner, Phys.Lett.\textbf{B309}, 421 (1993).
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