Integral inequalities of Hilbert’s type involving Fenchel-Legendre transform with applications

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ABSTRACT
New integral inequalities of Hilbert’s type are introduced through Fenchel-Legendre transform. Some of these inequalities are considered as generalizations as well as improvements of some previously proved results. Applications of the extracted inequalities are presented.

1. Introduction
The classical Hilbert’s integral inequality takes the form [1]:

\[
\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{x+y} \, dx \, dy \leq \frac{\pi}{\sin(\pi/\eta)} \times \left( \int_0^\infty f^n(x) \, dx \right)^{1/n^*} \times \left( \int_0^\infty g^n(x) \, dx \right)^{1/n^*},
\]

(1)

unless \( f \) or \( g \) is equivalent to zero, where \( n^* = \eta/(\eta - 1) \). The constant \( \pi/\sin(\pi/\eta) \) in (1) is optimal, see [1].

Since Hilbert published inequality (1), many mathematicians derived improvements and generalizations of it, see for instance [2–4] and the references therein, these refinements together with the original inequality led to important developments and improvements of many more advanced mathematical branches, see for example [5–7].

The following three theorems state some results that were proved in [8]. These results are considered as extensions to inequality (1).

Theorem 1.1: Let \( x, y \in (0, \infty) \), \( q \geq 1 \), \( p \geq 1 \), \( f(\xi) \) and \( g(\eta) \) be two positive functions on \( (0, x) \) and \( (0, y) \) respectively. Assume \( \kappa(s) = \int_0^s f(\xi) \, d\xi \) and \( \chi(t) = \int_0^t g(\eta) \, d\eta \), for \( 0 < s < x \) and \( 0 < t < y \). Then, the inequality

\[
\int_0^x \int_0^y \frac{\kappa(s) \chi(t) \kappa(t)\chi(s)}{s+t} \, ds \, dt \leq C(x, y, p, q) \left[ \int_0^x \mu(x, s) (\kappa(s) \kappa(t)\chi(s))^2 \, ds \right]^{1/2} \times \left[ \int_0^y \nu(y, t) (\kappa(t)\chi(s))^2 \, dt \right]^{1/2},
\]

holds, unless \( f \) or \( g \) is null, where \( C(x, y, p, q) = \frac{\pi \sqrt{xy}}{2pq} \), \( \mu(x, s) = x-s \), and \( \nu(y, t) = y-t \).

More advanced versions of Theorem 1.1 are the following two theorems.

Theorem 1.2: Let \( f, \kappa, g, \) and \( \chi \) be as assumed in Theorem 1.1, and \( \rho(s) \), \( \psi(t) \) be positive functions. Assume \( P(s) = \int_0^s \rho(\xi) \, d\xi \) and \( Q(t) = \int_0^t \psi(\eta) \, d\eta \). Then, for \( 0 < s < x \), and \( 0 < t < y \) the inequality

\[
\int_0^x \int_0^y \frac{\psi(t) \psi(s) \kappa(s) \chi(t) \kappa(t)\chi(s)}{s+t} \, ds \, dt \leq M(x, y) \left[ \int_0^x \rho(s) \Phi(\frac{\kappa(s) \kappa(t)\chi(s)}{P(s)})^{2} \, ds \right]^{1/2} \times \left[ \int_0^y \psi(t) \Psi(\frac{\kappa(t)\chi(s)}{Q(t)})^{2} \, dt \right]^{1/2},
\]

holds, unless \( f \) or \( g \) is null, where \( \mu(x, s) = x-s \), \( \nu(y, t) = y-t \), and

\[
M(x, y) = \left[ \int_0^x \Phi(\frac{\kappa(s) \kappa(t)\chi(s)}{P(s)})^{2} \, ds \right]^{1/2} \times \left[ \int_0^y \Psi(\frac{\kappa(t)\chi(s)}{Q(t)})^{2} \, dt \right]^{1/2},
\]
provided that \( \Phi \) and \( \Psi \) are non-negative, real-valued, sub-multiplicative, and convex functions defined on \([0, \infty)\).

It is worth mentioning that the inequalities (2) and (3) are consequences of the general relations established in [9] in which an unified treatment to Hilbert-Pachpatte-multiplicative, and convex functions defined on non-homogeneous kernels is presented.

**Theorem 1.3:** Let \( P, g, f, Q, \rho, \) and \( \psi \) be as assumed in Theorem 1.2. Assume \( \kappa(s) = (1/P(s)) \int_0^s \rho(\xi) \, d\xi, \) and \( \chi(t) = (1/Q(t)) \int_0^t \psi(r) \, dr. \) Then, for \( 0 < s < x, \) and \( 0 < t < y \) the inequality

\[
\int_0^x \int_0^y \frac{\psi(\chi(t)) \Phi(\kappa(s)) Q(t) P(s)}{t + s} \, dt \, ds \leq C(x, y, 1, 1) \left[ \int_0^x \left[ \rho(s) \Phi(\kappa(s)) \right]^2 \mu(x, s) \, ds \right]^{1/2} \times \left[ \int_0^y \left[ \psi(t) \Psi(g(t)) \right]^2 \nu(y, t) \, dt \right]^{1/2},
\]

holds, unless \( f \) or \( g \) is null, where \( C(x, y, 1, 1) = \frac{1}{\sqrt{xy}}, \mu(x, s) = x - s, \) and \( \nu(y, t) = y - t, \) provided that \( \Phi \) and \( \Psi \) are non-negative, real-valued, and convex functions defined on \([0, \infty)\).

The theorems that have been stated and proved in this paper, i.e. Theorems 3.1–3.4, give, to some extent, advanced and more improved versions of the results given in Theorems 1.1–1.3. The inequalities derived in this paper are attained by utilizing Jensen’s inequality and Schwarz inequality as well as by taking advantage of Fenchel-Legendre transform. In addition, some new Hilbert-type inequalities, that are different from those in Theorems 1.1–1.3, will be derived. The paper is entailed in some interesting applications.

Before stating and proving the main results of this paper we, in the following section, shed some lights on some important tools that will be used in proofs.

**2. Preliminaries**

In this section the Fenchel-Legendre transform along with Jensen’s inequality are introduced. We would like to draw the reader’s attention here that the Fenchel-Legendre transform has an influential role in later sections. For more details on this transform, the reader is referred to [10–12].

**Definition 2.1:** Suppose \( h : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\} \) is a function such that \( h \neq +\infty, \) i.e. the domain of \( h \) is \( \text{dom}(h) = \{x \in \mathbb{R} | h(x) < +\infty\} \neq \emptyset. \) Then the transform:

\[
h^* : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}
\]

\[
y \rightarrow h^*(y) = \sup \{xy - h(x), x \in \text{dom}(h)\},
\]

is called Fenchel-Legendre transform. In addition, the mapping \( h \rightarrow h^* \) will often be called the conjugate operation.

Furthermore, \( h^* \) has a domain \( \text{dom}(h^*), \) that is the set that contains slopes of all the affine functions minorizing the function \( h \) over \( \mathbb{R}. \)

On the other hand, a transform called Legendre transform, that is equivalent formula to (5), can be given with additional hypotheses on \( h \) as follows:

**Corollary 2.2:** Let \( h : \mathbb{R} \rightarrow \mathbb{R} \) be differentiable, strictly convex, and \( 1 \)-coercive function. Then

\[
h^*(y) = yh^{-1}(y) - h[h^{-1}(y)] \quad \forall y \in \text{dom}(h^*),
\]

where \( h^{-1} \) is the inverse function of \( h'. \)

**Corollary 2.3 (Fenchel-Young inequality [12]):** Suppose \( h \) is a function and \( h^* \) is its Fenchel-Legendre transform then

\[
xy \leq h(x) + h^*(y),
\]

for all \( x \in \text{dom}(h), \) and \( y \in \text{dom}(h^*). \)

**Corollary 2.4 (Jensen’s inequality [13]):** Let \( \Phi : [a, b] \rightarrow \mathbb{R} \) be a convex function, \( h : [a, b] \rightarrow (0, \infty), \) and \( u : [a, b] \rightarrow (0, \infty) \) be integrable functions. Then, the following inequality known as Jensen’s inequality holds

\[
\Phi \left( \frac{1}{b-a} \int_a^b h(x) \, dx \int_a^b u(x) \, dx \right) \leq \frac{1}{b-a} \int_a^b h(x) \Phi(u(x)) \, dx.
\]

**Definition 2.5:** The function \( \Phi \) is a submultiplicative function on the interval \((0, \infty)\) if it satisfies the condition:

\[
\Phi(xy) \leq \Phi(x) \Phi(y), \quad \text{for all } x, y \geq 0.
\]

**3. Main results**

In this section we state and prove our results in the following series of theorems.

**Theorem 3.1:** Let \( x, y \in (0, \infty), q \geq 1, p \geq 1. \) Let \( f(\xi) \) and \( g(r) \) be two positive functions on \([0, x)\) and \([0, y)\) respectively. Assume \( \kappa(s) = \int_0^s f(\xi) \, d\xi, \) and \( \chi(t) = \int_0^t g(r) \, dr, \) for \( 0 < s < x, \) and \( 0 < t < y. \) Then, the inequalities

\[
\int_0^x \int_0^y \frac{\kappa^p(s) \chi^q(t)}{h(\sqrt{s}) + h^*(\sqrt{t})} \, dt \, ds \leq C_1(x, y, p, q) \left[ \int_0^x \mu(x, s) (\kappa^{p-1}(s) f(s))^2 \, ds \right]^{1/2} \times \left[ \int_0^y \nu(y, t) (\chi^{q-1}(t) g(t))^2 \, dt \right]^{1/2},
\]

where \( C_1 \) is a constant.
and,
\[
\int_0^x \int_0^y \frac{k^p(s)^2q_2^2(t)}{h(s) + h^*(t)} \, dt \, ds \\
\leq C_2(p, q) \left[ \int_0^x \mu(x,s)(k^{p-1}(s)f(s))^2 \, ds \right] \\
\times \left[ \int_0^y \nu(y,t)(\chi^{q-1}(t)g(t))^2 \, dt \right], \quad (11)
\]

hold, unless \(f\) or \(g\) is null, where \(C_1(x,y,p,q) = pq \sqrt{xy}\), \(C_2(p, q) = p^2q^2, \mu(x,s) = x - s, \) and \(\nu(y,t) = y - t\).

**Proof:** By assumption and the fact that \((d/ds)k^p(s) = pk^{p-1}(s)f(s)\) we observe that
\[
k^p(s) = p \int_0^s k^{p-1}(\zeta)f(\zeta) \, d\zeta, \quad (12)
\]
similarly, we have
\[
\chi^q(t) = q \int_0^t \chi^{q-1}(r)g(r) \, dr. \quad (13)
\]

Using (12) and (13) along with Schwarz inequality yields
\[
k^p(s)\chi^q(t) = pq \left( \int_0^s k^{p-1}(\zeta)f(\zeta) \, d\zeta \right) \\
\times \left( \int_0^t \chi^{q-1}(r)g(r) \, dr \right) \\
\leq pq \left( \int_0^s \left( k^{p-1}(\zeta)f(\zeta) \right)^2 \, d\zeta \right)^{1/2} \\
\times \left( \int_0^t \left( \chi^{q-1}(r)g(r) \right)^2 \, dr \right)^{1/2}. \quad (14)
\]

Squaring both sides of inequality (14) gives
\[
k^{2p}(s)\chi^{2q}(t) \leq p^2q^2st \left[ \int_0^s \left( k^{p-1}(\zeta)f(\zeta) \right)^2 \, d\zeta \right] \\
\times \left[ \int_0^t \left( \chi^{q-1}(r)g(r) \right)^2 \, dr \right]. \quad (15)
\]

Applying Fenchel-Young inequality (7) on the right hand sides of inequalities (14) and (15) produces the following two inequalities respectively
\[
k^p(s)\chi^q(t) \leq pq \left( h(\sqrt{s}) + h^*(\sqrt{t}) \right) \\
\times \left[ \int_0^s \left( k^{p-1}(\zeta)f(\zeta) \right)^2 \, d\zeta \right]^{1/2} \\
\times \left[ \int_0^t \left( \chi^{q-1}(r)g(r) \right)^2 \, dr \right]^{1/2}, \quad (16)
\]
and,
\[
k^{2p}(s)\chi^{2q}(t) \leq p^2q^2 \left( h(s) + h^*(t) \right) \\
\times \left[ \int_0^s \left( k^{p-1}(\zeta)f(\zeta) \right)^2 \, d\zeta \right] \\
\times \left[ \int_0^t \left( \chi^{q-1}(r)g(r) \right)^2 \, dr \right]. \quad (17)
\]

followed by integrating over \(s\) from 0 to \(x\) to obtain
\[
\int_0^x \int_0^y \frac{k^p(s)\chi^q(t)}{h(\sqrt{s}) + h^*(\sqrt{t})} \, dt \, ds \\
\leq pq \left[ \int_0^x \left[ \int_0^s (k^{p-1}(s)f(s))^2 \, ds \right]^{1/2} \, ds \right] \\
\times \left[ \int_0^y \left[ \int_0^t (\chi^{q-1}(r)g(r))^2 \, dr \right]^{1/2} \, dt \right]. \quad (18)
\]

Apply Schwarz inequality on inequality (18) then interchange the order of integration to have
\[
\int_0^x \int_0^y \frac{k^p(s)\chi^q(t)}{h(\sqrt{s}) + h^*(\sqrt{t})} \, dt \, ds \\
\leq pq \sqrt{xy} \left[ \int_0^x \left[ \int_0^s (k^{p-1}(s)f(s))^2 \, ds \right]^{1/2} \, ds \right] \\
\times \left[ \int_0^y \left[ \int_0^t (\chi^{q-1}(r)g(r))^2 \, dr \right]^{1/2} \, dt \right] \\
\leq pq \sqrt{xy} \left[ \int_0^x \left( k^{p-1}(s)f(s) \right)^2 \mu(x,s) \, ds \right]^{1/2} \\
\times \left[ \int_0^y \left( \chi^{q-1}(t)g(t) \right)^2 \nu(y,t) \, dt \right]^{1/2}, \quad (19)
\]

which is inequality (10). In order to prove inequality (11), divide both sides of (17) by \(h(s) + h^*(t)\), then integrate the result over \(s\) from 0 to \(x\) preceded by the integration over \(t\) from 0 to \(y\) to get
\[
\int_0^x \int_0^y \frac{k^p(s)\chi^q(t)}{h(s) + h^*(t)} \, dt \, ds \\
\leq pq \sqrt{xy} \left[ \int_0^x \left[ \int_0^s (k^{p-1}(s)f(s))^2 \, ds \right] \, ds \right]^{1/2} \\
\times \left[ \int_0^y \left[ \int_0^t (\chi^{q-1}(r)g(r))^2 \, dr \right] \, dt \right]^{1/2}. \quad (20)
\]

Now interchange the order of integration to obtain
\[
\int_0^x \int_0^y \frac{k^p(s)\chi^q(t)}{h(s) + h^*(t)} \, dt \, ds \\
\leq pq \sqrt{xy} \left[ \int_0^x \left( k^{p-1}(s)f(s) \right)^2 \mu(x,s) \, ds \right] \\
\times \left[ \int_0^y \left( \chi^{q-1}(t)g(t) \right)^2 \nu(y,t) \, dt \right], \quad (21)
\]

which is (11), this makes the proof completed. \[\square\]

**Theorem 3.2:** Under the hypotheses of Theorem 3.1 the following inequalities hold
\[
\int_0^x \int_0^y \frac{k^p(s)\chi^q(t)}{h(\sqrt{s}) + h^*(\sqrt{t})} \, dt \, ds \\
\leq C_1(x,y,p,q) \left[ h \left( \int_0^x (k^{p-1}(s)f(s))^2 \mu(x,s) \, ds \right) \\
+ h^* \left( \int_0^y (\chi^{q-1}(t)g(t))^2 \nu(y,t) \, dt \right) \right]^{1/2}, \quad (22)
\]
and,
\[ \int_0^x \int_0^y \frac{k^{2p}(s) x^{2q}(t)}{h(s) + h^*(t)} \, dt \, ds \leq C_2(p, q) \left( h \left( \int_0^x (k^{p-1}(s) f(s))^2 \mu(x, s) \, ds \right) + h^* \left( \int_0^y (\chi^{p-1}(t) g(t))^2 \nu(y, t) \, dt \right) \right), \] (23)

unless \( f \) or \( g \) is null, where \( C_1(x, y, p, q) = pq \sqrt{xy}, C_2(p, q) = p^2 q^2, \mu(x, s) = x - s, \) and \( \nu(y, t) = y - t. \)

**Proof:** To obtain (22) and (23), apply Fenchel-Young inequality (7) on the inequalities (10) and (11). This completes the proof.

Now Jensen’s inequality (8) will be exploited in the following theorem in order to obtain a useful generalization to inequality (10) obtained in Theorem 3.1. In what follows, suppose that \( \Phi \) and \( \Psi \) are non-negative, convex and submultiplicative functions on \( [0, \infty). \)

**Theorem 3.3:** Let \( f, k, g, \) and \( \chi \) be as assumed in Theorem 3.1, and \( \rho(s), \psi(r) \) be positive functions. Assume \( P(s) = \int_0^s \rho(t) \, dt, \) and \( Q(t) = \int_0^t \psi(r) \, dr. \) Then, the inequalities

\[ \int_0^x \int_0^y \frac{\Phi(k(s)) \Psi(\chi(t))}{h(\sqrt{s}) + h^*(\sqrt{t})} \, dt \, ds \leq M(x, y) \left( \int_0^x \left( \rho(s) \Phi \left( \frac{f(s)}{\rho(s)} \right) \right)^2 \mu(x, s) \, ds \right)^{1/2} \]

\[ \times \left( \int_0^y \left( \psi(t) \Psi \left( \frac{g(t)}{\psi(t)} \right) \right)^2 \nu(y, t) \, dt \right)^{1/2}, \] (24)

and,

\[ \frac{1}{M(x, y)} \int_0^x \int_0^y \frac{\Phi(k(s)) \Psi(\chi(t))}{h(\sqrt{s}) + h^*(\sqrt{t})} \, dt \, ds \leq h \left( \int_0^x \left( \rho(s) \Phi \left( \frac{f(s)}{\rho(s)} \right) \right)^2 \mu(x, s) \, ds \right)^{1/2} \]

\[ + \frac{1}{M(x, y)} \left( \int_0^y \left( \psi(t) \Psi \left( \frac{g(t)}{\psi(t)} \right) \right)^2 \nu(y, t) \, dt \right)^{1/2}, \] (25)

hold for \( 0 < s < x, \) and \( 0 < t < y \) unless \( f, g, \rho \) or \( \psi \) is null, where \( \mu(x, s) = x - s, \nu(y, t) = y - t, \) and

\[ M(x, y) = \left( \int_0^x \left( \Phi \left( \frac{f(s)}{P(s)} \right) \right)^2 \, ds \right)^{1/2} \]

\[ \times \left( \int_0^y \left( \Psi \left( \frac{g(t)}{Q(t)} \right) \right)^2 \, dr \right)^{1/2}. \]

**Proof:** Using the assumption that \( \Phi \) is a submultiplicative function leads to

\[ \Phi(k(s)) = \Phi \left( \frac{P(s)}{Q(t)} \right) \frac{\rho(t) f(t)}{\rho(t)} \, ds \]

\[ \leq \Phi(P(s)) \Phi \left( \frac{f(t)}{\rho(t)} \right) \, ds \] (26)

Now apply Jensen’s inequality (8) on (26) then apply Schwarz inequality to have

\[ \Phi(k(s)) \leq \frac{\Phi(P(s))}{P(s)} \int_0^s \rho(t) \Phi \left( \frac{f(t)}{\rho(t)} \right) \, dt \]

\[ \leq \frac{\Phi(P(s))}{P(s)} s^{1/2} \int_0^s \left[ \rho(\xi) \Phi \left( \frac{f(\xi)}{\rho(\xi)} \right) \right]^2 \, d\xi \] (27)

Similarly, we can get

\[ \Psi(\chi(t)) \leq \frac{\Psi(Q(t))}{Q(t)} t^{1/2} \int_0^t \left[ \psi(r) \Psi \left( \frac{g(r)}{\psi(r)} \right) \right]^2 \, dr \] (28)

From (27) and (28) and utilizing Fenchel-Young inequality (7), we have

\[ \Phi(k(s)) \Psi(\chi(t)) \leq \left( h(\sqrt{s}) + h^*(\sqrt{t}) \right) \frac{\Phi(P(s))}{P(s)} \]

\[ \times \left( \int_0^s \left[ \rho(\xi) \Phi \left( \frac{f(\xi)}{\rho(\xi)} \right) \right]^2 \, d\xi \right)^{1/2} \]

\[ \times \frac{\Psi(Q(t))}{Q(t)} \left[ \int_0^t \left[ \psi(r) \Psi \left( \frac{g(r)}{\psi(r)} \right) \right]^2 \, dr \right]^{1/2}. \] (29)

For both sides of inequality (29), divide by \( h(\sqrt{s}) + h^*(\sqrt{t}) \), then integrate over \( s \) from 0 to \( x \) followed by integrating over \( s \) from 0 to \( y \) to reach the following inequality

\[ \int_0^x \int_0^y \frac{\Phi(k(s)) \Psi(\chi(t))}{h(\sqrt{s}) + h^*(\sqrt{t})} \, dt \, ds \]

\[ \leq \int_0^x \frac{\Phi(P(s))}{P(s)} \left[ \int_0^s \left[ \rho(\xi) \Phi \left( \frac{f(\xi)}{\rho(\xi)} \right) \right]^2 \, d\xi \right]^{1/2} \]

\[ \times \frac{\Psi(Q(t))}{Q(t)} \left[ \int_0^t \left[ \psi(r) \Psi \left( \frac{g(r)}{\psi(r)} \right) \right]^2 \, dr \right]^{1/2} \, dt. \] (30)
A direct application of Schwarz inequality yields
\[
\int_0^X \int_0^Y \Phi(k(s))\Psi(x(t)) \Phi(P(s)) \Psi(Q(t)) \, dt \, ds \\
= \left[ \int_0^X \left( \frac{\Phi(P(s))}{P(s)} \right)^2 \right]^{1/2} \\
\leq \left[ \int_0^X \left( \frac{\rho(\zeta)}{\rho(\zeta)} \right) \right]^{1/2} \\
\times \left[ \int_0^Y \left( \frac{\Psi(Q(t))}{Q(t)} \right) \right]^{1/2} \\
\times \left[ \int_0^Y \left( \frac{\varphi(r)}{\varphi(r)} \right) \right]^{1/2}.
\]

Now take \( M(x, y) \) as
\[
M(x, y) = \left[ \int_0^X \left( \frac{\Phi(P(s))}{P(s)} \right)^2 \right]^{1/2} \\
\times \left[ \int_0^Y \left( \frac{\Psi(Q(t))}{Q(t)} \right) \right]^{1/2},
\]
then, interchange the order of integration to obtain
\[
\int_0^X \int_0^Y \Phi(k(s))\Psi(x(t)) \Phi(P(s)) \Psi(Q(t)) \, dt \, ds \\
\leq M(x, y) \left[ \int_0^X \left( \frac{\rho(\zeta)}{\rho(\zeta)} \right) \right]^{1/2} \\
\times \left[ \int_0^Y \left( \frac{\varphi(r)}{\varphi(r)} \right) \right]^{1/2}.
\]
which implies that
\[
\int_0^X \int_0^Y \Phi(k(s))\Psi(x(t)) \Phi(P(s)) \Psi(Q(t)) \, dt \, ds \\
\leq M(x, y) \left[ \int_0^X \left( \frac{\rho(\zeta)}{\rho(\zeta)} \right) \right]^{1/2} \mu(x, s) \, ds \\
\times \left[ \int_0^Y \left( \frac{\varphi(r)}{\varphi(r)} \right) \right]^{1/2} v(y, t) \, dt.
\]
This proves (24). Concerning inequality (25), it follows through applying Fenchel-Young inequality (7) on inequality (24).

**Theorem 3.4:** Let \( P, Q, f, g, \rho, \) and \( \psi \) be as assumed in Theorem 3.3. Assume \( k(x) = (1/P(s))^2 \) \( \rho(\zeta) \) \( dx \), and \( x(t) = (1/Q(t))^2 \) \( \varphi(r) \) \( dr \). Then, for \( 0 < s < x, \) and \( 0 < t < y \) the inequalities
\[
\int_0^X \int_0^Y \Phi(k(s))\Psi(x(t)) \Phi(P(s)) \Psi(Q(t)) \, dt \, ds \\
\leq C(x, y, 1, 1) \left[ \int_0^X \left( \frac{\rho(\zeta)}{\rho(\zeta)} \right) \right]^{1/2} \mu(x, s) \, ds \\
\times \left[ \int_0^Y \left( \frac{\varphi(r)}{\varphi(r)} \right) \right]^{1/2} v(y, t) \, dt,
\]
and,
\[
\frac{1}{C(x, y, 1, 1)} \int_0^X \int_0^Y \Phi(k(s))\Psi(x(t)) \Phi(P(s)) \Psi(Q(t)) \, dt \, ds \\
\leq h \left[ \int_0^X \left( \frac{\rho(\zeta)}{\rho(\zeta)} \right) \right]^{1/2} \mu(x, s) \, ds \\
+ h^* \left[ \int_0^Y \left( \frac{\varphi(r)}{\varphi(r)} \right) \right]^{1/2} v(y, t) \, dt,
\]
hold, unless \( f \) or \( g \) is null, where \( C(x, y, 1, 1) = \frac{1}{2} \sqrt{xy} \mu, \mu(x, s) = x - s, \) and \( v(y, t) = y - t. \)

The proof of Theorem 3.4 can be obtained from the proof of Theorem 3.4 with adequate modifications. Here, the details are omitted.

**4. Some applications**

In this section we try to show the beauty behind our results. The inequalities obtained in this paper cover a wide spectrum of known inequalities as well as new inequalities. This objective can be attained through inequality (10) and by utilizing Table 1 that shows some examples of \( h \) and \( h^* \).

In what follows three applications of inequality (10) are presented.

1. Putting \( h(x) = x^2/2 \) (this means that \( h^*(x) = x^2/2 \) in inequality (10)) gives
\[
\int_0^X \int_0^Y \frac{x(t)k(s)}{t + s} \, dt \, ds \\
\leq \frac{1}{2} C_1(x, y, p, q) \left[ \int_0^X \mu(x, s)(t^q - 1) f(s) \, ds \right]^{1/2} \\
\times \left[ \int_0^y v(y, t)(t^q - 1) g(t) \, dt \right]^{1/2},
\]
which is exactly the inequality (2) given in Theorem 1.1. In addition, the same choice of \( h \) in the inequalities (24) and (35) gives us the inequalities in Theorems 1.2 and 1.3.
If we take $h(s) = |s|^{\alpha}/\alpha; \alpha > 1$, then $h^*(t) = |t|^\beta/\beta$ where $1/\alpha + 1/\beta = 1$ and $s, t \in \mathbb{R}_+$, then inequality (10) produces

$$
\int_0^x \int_0^y \frac{k^p(s) \chi^q(t)}{\beta \sqrt{s + \alpha t}} \, ds \, dt
\leq \frac{1}{\alpha \beta} C_1(x, y, p, q) \left[ \int_0^x \mu(x, s) (k^{p-1}(s) f(s))^2 \, ds \right]^{1/2}
\times \left[ \int_0^y v(y, t) (\chi^{q-1}(t) g(t))^2 \, dt \right]^{1/2}.
$$

(38)

When $h(s) = e^s$, then $h^*(t) = t \log(t) - t$, then inequality (10) yields

$$
\int_0^x \int_0^y \frac{k^p(s) \chi^q(t)}{e^s + \sqrt{t} \log(\sqrt{t}) - \sqrt{t}} \, ds \, dt
\leq C_1(x, y, p, q) \left[ \int_0^x \mu(x, s) (k^{p-1}(s) f(s))^2 \, ds \right]^{1/2}
\times \left[ \int_0^y v(y, t) (\chi^{q-1}(t) g(t))^2 \, dt \right]^{1/2}.
$$

(39)

## 5. Conclusion

Fenchel-Young inequality (7) gave a great assistance in deriving more general Hilbert’s type inequalities by selecting the functions $h(x)$ and $h^*(x)$ in an appropriate way. No doubt, using other transforms opens the door to extracting new inequalities not only of Hilbert’s type but also of any other types.

Discrete analogous to all theorems in this paper can be derived following, to some extent, the same strategy of the proofs mentioned here.

## Disclosure statement

No potential conflict of interest was reported by the authors.

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