GLOBAL SMALL SOLUTIONS TO THREE-DIMENSIONAL INCOMPRESSIBLE MHD SYSTEM

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Abstract. In this paper, we consider the global wellposedness of 3-D incompressible magneto-hydrodynamical system with small and smooth initial data. The main difficulty of the proof lies in establishing the global in time $L^1$ estimate for gradient of the velocity field due to the strong degeneracy and anisotropic spectral properties of the linearized system. To achieve this and to avoid the difficulty of propagating anisotropic regularity for the transport equation, we first write our system (1.1) in the Lagrangian formulation (2.20). Then we employ anisotropic Littlewood-Paley analysis to establish the key $L^1$ in time estimates to the velocity and the gradient of the pressure in the Lagrangian coordinate. With those estimates, we prove the global wellposedness of (2.20) with smooth and small initial data by using the energy method. Toward this, we will have to use the algebraic structure of (2.20) in a rather crucial way. The global wellposedness of the original system (1.1) then follows by a suitable change of variables together with a continuous argument. We should point out that compared with the linearized systems of 2-D MHD equations in [22] and that of the 3-D modified MHD equations in [21], our linearized system (3.1) here is much more degenerate, moreover, the formulation of the initial data for (2.20) is more subtle than that in [22].

Keywords: Inviscid MHD system, Anisotropic Littlewood-Paley theory, Dissipative estimates, Lagrangian coordinates

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1. Introduction

In this paper, we investigate the global wellposedness of the following three-dimensional incompressible magnetic hydrodynamical system (or MHD in short) with initial data being sufficiently close to the equilibrium state:

$$\begin{align*}
\partial_t b + u \cdot \nabla b &= b \cdot \nabla u, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -\frac{1}{2} \nabla |b|^2 + b \cdot \nabla b, \\
\text{div } u &= \text{div } b = 0, \\
b|_{t=0} &= b_0, \quad u|_{t=0} = u_0,
\end{align*}$$

(1.1)

where $b = (b^1, b^2, b^3)^T$ denotes the magnetic field, and $u = (u^1, u^2, u^3)^T, p$ the velocity and scalar pressure of the fluid respectively. This MHD system (1.1) with zero diffusivity in the equation for the magnetic field can be applied to model plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions are extremely small. One may check the references [13, 17, 4] for more detailed explanations to this system.

It has been a long-standing open problem that whether or not classical solutions of (1.1) can develop finite time singularities even in the two-dimensional case. In the case when there is full magnetic diffusion in (1.1), Duvaut and Lions [14] established the local existence and uniqueness of solution in the classical Sobolev space $H^s(\mathbb{R}^d), s \geq d$, they also proved the

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global existence of solutions to this system with small initial data; Sermange and Temam \[26\] proved the global unique solution in the two space dimensions; Abidi and Paicu \[1\] proved similar result as in \[14\] for the so-called inhomogeneous MHD system with initial data in the critical spaces. With mixed partial dissipation and additional magnetic diffusion in the two-dimensional MHD system, Cao and Wu \[5\] (see also \[6\]) proved that such a system is globally well posed for any data in $H^2(\mathbb{R}^2)$. Lin and the second author \[21\] proved the global well posedness to a modified three-dimensional MHD system (3-D version of (1.2) below) with initial data sufficiently close to the equilibrium state. Lin and the authors \[22\] established the global existence of small solutions to the two-dimensional MHD equations (1.1).

For the incompressible MHD equations (1.1), whether there is a dissipation or not for the magnetic field is a very important problem also from physics of plasmas. The heating of high temperature plasmas by MHD waves is one of the most interesting and challenging problems of plasma physics especially when the energy is injected into the system at the length scales much larger than the dissipative ones. It has been conjectured that in the three-dimensional MHD system, energy is dissipated at a rate that is independent of the ohmic resistivity \[11\]. In other words, the viscosity (diffusivity) for the magnetic field equation can be zero yet the whole system may still be dissipative. We shall justify this conjecture for (1.1) with initial data close enough to the equilibrium state.

Notice that in two space dimensions, $\text{div} \ b = 0$ implies the existence of a scalar function $\phi$ so that $b = (\partial_2 \phi, -\partial_1 \phi)^T$, and the system (1.1) becomes

$$
\begin{align*}
\partial_t \phi + u \cdot \nabla \phi &= 0, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -\frac{1}{2} \nabla |\nabla \phi|^2 - \text{div} [\nabla \phi \otimes \nabla \phi], \\
\text{div} \ u &= 0, \\
\phi|_{t=0} &= \phi_0(x) = x_1 + \psi_0(x), \\
u|_{t=0} &= u_0,
\end{align*}
$$

The main idea in \[22\] is first to seek another scalar function $\tilde{\phi}(x) = -x_1 + \tilde{\psi}_0$ so that

$$
\det U_0 = 1 \quad \text{for} \quad U_0 = \begin{pmatrix} 1 + \frac{\partial_x \psi_0}{\partial_y \psi_0} & \frac{\partial_x \tilde{\psi}_0}{\partial_y \psi_0} \\ -\frac{\partial_x \psi_0}{\partial_y \psi_0} & 1 - \frac{\partial_x \tilde{\psi}_0}{\partial_y \psi_0} \end{pmatrix},
$$

provided that $\psi_0$ is sufficiently small in some sense. Then the authors of \[22\] looked for a volume preserving diffeomorphism in $\mathbb{R}^2$, $X_0(y) = y + Y_0(y)$, so that

$$
U_0 \circ X_0(y) = \nabla_y X_0(y) = I + \nabla_y Y_0(y).
$$

Let $(Y(t, y), q(t, y))$ be determined by

$$
X(t, y) = X_0(y) + \int_0^t u(s, X(s, y)) \, ds \overset{\text{def}}{=} y + Y(t, y),
$$

$$
q(t, y) \overset{\text{def}}{=} (p + |\nabla \phi|^2) \circ X(t, y).
$$

(1.2) can be equivalently reformulated as

$$
\begin{align*}
Y_{tt} - \nabla_y \cdot \nabla Y_t - \partial_{y_1}^2 Y + \nabla_Y g &= 0, \\
\nabla_Y \cdot Y_t &= 0, \\
Y|_{t=0} &= Y_0, \\
Y_t|_{t=0} &= u_0 \circ X_0(y) \overset{\text{def}}{=} Y_1,
\end{align*}
$$
where $\nabla_Y \text{def} = A^T_Y \nabla_y$ and

$$
(1.7) \quad A_Y \text{def} = \begin{pmatrix}
1 + \partial_{y_2}Y^2 & -\partial_{y_2}Y^1 \\
-\partial_{y_1}Y^2 & 1 + \partial_{y_1}Y^1
\end{pmatrix}.
$$

In particular, the linearized system of (1.6) reads

$$
(1.8) \quad \begin{cases}
Y_{tt} - \Delta_y Y_t - \partial_{y_1}^2 Y = f(Y, q), 
\nabla_y \cdot Y = \rho(Y), 
Y|_{t=0} = Y_0, \quad Y|_{t=0} = Y_1.
\end{cases}
$$

By using anisotropic Littlewood-Paley theory, the authors [22] first established the global wellposedness of (1.6) with small and smooth initial data, then they proved the global wellposedness of (1.2) with sufficiently small data $(\psi_0, u_0)$ through a suitable changes of variables.

However, in the three-dimensional case, we can not find such an equivalent formulation of (1.1) as (1.2). Instead, for $b_0 = e_3$ being sufficient small, we can find a $\Psi = (\psi_1, \psi_2, \psi_3)^T$ so that there holds (2.3). Compared with (1.3), (2.3) is a nonlinear system. With this $\Psi$, we can define $\vec{b}_0$ and $\vec{b}_0$ via (2.4) so that the $3 \times 3$ matrix $U_0 \text{def} = (\vec{b}_0, \vec{b}_0, b_0)$ satisfies

$$
(1.9) \quad \text{div} \hat{b} = \text{div} \hat{b} = 0, \quad \text{and} \quad \det U = 1.
$$

With thus obtained $U_0$, we can find a 3-D volume preserving diffeomorphism $X_0(y) = y + Y_0(y)$, and reformulate (1.1) in the Lagrangian coordinate (2.20) with its linearized system (2.21). We point out that one crucial idea in [22] is to use $\partial_{y_1}Y^1 + \partial_{y_2}Y^2 = \rho(Y)$ to propagate the time dissipative estimate of $\partial_{y_1}Y^1$ to that of $\partial_{y_2}Y^2$. Notice that in the linearized system (2.21), one only has time dissipative estimate for $\partial_{y_3}Y$, and we can not use $\nabla_y \cdot Y = \rho(Y)$ to propagate the time dissipative estimate from $\partial_{y_3}Y^3$ to that of $\partial_{y_1}Y^1, \partial_{y_2}Y^2$, which gives rise to another difficulty in the analysis of three-dimensional MHD system. And we will have to use the nonlinear structure of (2.20) in a rather crucial way so that the source term in (2.21) is still globally integrable in time. As in [22], we shall first establish the global wellposedness of (2.20) with small and smooth initial data, we then prove the global existence of small solution to (1.1) by a suitable changes of variables along with a continuous argument.

We should remark that the system (1.2) is of interest not only because it models the incompressible MHD equations, but also because it arises in many other important applications. Moreover, its nonlinear coupling structure is universal, see the recent survey article [19]. Indeed, the system (1.2) resembles the 2-D viscoelastic fluid system:

$$
(1.10) \quad \begin{cases}
U_t + \mathbf{u} \cdot \nabla U = \nabla \mathbf{u} U, 
\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \Delta \mathbf{u} + \nabla \cdot (UU^T), 
\text{div} \mathbf{u} = 0, 
U|_{t=0} = U_0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0,
\end{cases}
$$

where $U$ denotes the deformation tensor, $\mathbf{u}$ is the fluid velocity and $p$ represents the hydrodynamic pressure (we refer to [20] and the references therein for more details).

In two space dimensions, when $\nabla \cdot U = 0$, it follows from (1.10) that $\nabla \cdot U(t, x) = 0$ for all $t > 0$. Therefore, one can find a vector $\phi = (\phi_1, \phi_2)^T$ such that

$$
U = \begin{pmatrix}
-\partial_2 \phi_1 & -\partial_2 \phi_2 \\
\partial_1 \phi_1 & \partial_1 \phi_2
\end{pmatrix}.
$$
Then (1.10) can be equivalently reformulated as

\begin{align}
\left\{ \begin{array}{l}
\phi_t + u \cdot \nabla \phi = 0, \\
u_t + u \cdot \nabla u + \nabla p = \Delta u - \sum_{i=1}^{2} \text{div} \left[ \nabla \phi_i \otimes \nabla \phi_i \right], \\
\text{div} \ u = 0, \\
\phi|_{t=0} = \phi_0, \quad u|_{t=0} = u_0.
\end{array} \right.
\end{align}

The authors ([20]) established the global existence of smooth solutions to the Cauchy problem in the entire space or on a periodic domain for (1.11) in general space dimensions provided that the initial data is sufficiently close to the equilibrium state (one may check [10, 18] for the 3-D result). One sees the only difference between (1.2) and (1.11) lying in the fact that \( \phi \) is a scalar function in (1.2), while \( \phi = (\phi_1, \phi_2)^T \) is a vector-valued function with the unit Jacobian in (1.11). However, it gives rise to an essential difficulty in the analysis. In fact, there is a damping mechanism of the system (1.11) that can be seen from the linearization of the system \( \partial_t \) (1.11):

\begin{align}
\left\{ \begin{array}{l}
\phi_{tt} - \Delta \phi - \Delta \phi_t + \nabla q = f, \\
u_{tt} - \Delta u - \Delta u_t + \nabla p = F, \\
\text{div} \ u = 0.
\end{array} \right.
\end{align}

We also remark that the linearized system of (1.2) in 3-D reads

\begin{align}
\partial_t^2 \psi - (\partial_{x_1}^2 + \partial_{x_2}^2) \psi - \Delta \partial_t \psi = f.
\end{align}

One may check Remark 1.4 of [21] for details. It is easy to observe that our linearized system in (2.21) is much more degenerate than (1.12) and (1.13).

As in [22], to describe the initial data \( b_0 \) in (1.1), we need the following definition:

**Definition 1.1.** Let \( b_0 = (b_0^1, b_0^2, b_0^3)^T \) be a smooth enough vector field. We define its trajectory \( X(t, x) \) by

\begin{align}
\left\{ \begin{array}{l}
dX(t, x) \over dt = b_0(X(t, x)), \\
X(t, x)|_{t=0} = x.
\end{array} \right.
\end{align}

We call that \( f \) and \( b_0 \) are admissible on a domain \( D \) of \( \mathbb{R}^3 \) if there holds

\[ \int_{\mathbb{R}} f(X(t, x)) \over dt = 0 \quad \text{for all} \quad x \in D. \]

**Remark 1.1.** As in [22], the condition that \( f \) and \( b \) are admissible on some set of \( \mathbb{R}^3 \) is to guarantee that

\begin{align}
b_0^1 \partial_{x_1} \psi + b_0^2 \partial_{x_2} \psi + b_0^3 \partial_{x_3} \psi = f
\end{align}

has a solution \( \psi \) so that \( \lim_{|x| \to \infty} \psi(x) = 0 \). Let us take \( b = (0, 0, 1)^T \) for example. In this case, (1.15) becomes \( \partial_{x_3} \psi = f \), which together with the condition \( \lim_{|x_3| \to \infty} \psi(x) = 0 \) ensures that

\[ \psi(x_h, x_3) = - \int_{x_3}^{\infty} f(x_h, t) \over dt = \int_{-\infty}^{x_3} f(x_h, t) \over dt. \]

We thus obtain that \( \int_{\mathbb{R}} f(x_h, t) \over dt = 0 \), that is, \( f \) and \( (0, 0, 1)^T \) are admissible on \( \mathbb{R}^2 \times \{0\} \).

**Notations:** Let \( X_1, X_2 \) be Banach spaces, the norms \( \| \cdot \|_{X_1 \cap X_2} \overset{\text{def}}{=} \| \cdot \|_{X_1} + \| \cdot \|_{X_2} \) and \( \| \cdot \|_{L^p(\mathbb{R}^+; X_1 \cap X_2)} \overset{\text{def}}{=} \| \cdot \|_{L^p(\mathbb{R}^+; X_1)} + \| \cdot \|_{L^p(\mathbb{R}^+; X_2)} \) for \( p \in [1, \infty] \).
We now state the main result of this paper:

**Theorem 1.1.** Let \( s_1 > \frac{5}{4} \), \( s_2 \in (\frac{1}{2}, \frac{1}{4}) \), and \( p \in (\frac{3}{2}, 2) \). Let \( s \geq s_1 + 2 \), let \((b_0, u_0)\) satisfy \( b_0 - e_3 \in B^{s_1 + \frac{3}{p} + 1}_{p,1} \cap H^s(\mathbb{R}^3) \) for \( e_3 = (0, 0, 1)^T \), and \( u_0 \in \dot{H}^{s_2} \cap \dot{B}^{\frac{3}{p} - 1}_{p,1}(\mathbb{R}^3) \) with
\[
\nabla u_0 \in B^{s_1 + \frac{3}{p} + \frac{1}{2}}_{p,1} \cap H^{s_2 - 1}(\mathbb{R}^3) \text{ and }
\]
(1.16) \[
\|b_0 - e_3\|_{B^{s_1 + \frac{3}{p} + \frac{1}{2}}_{p,1}} + \|u_0\|_{\dot{H}^{s_2} \cap \dot{B}^{\frac{3}{p} - 1}_{p,1}} \leq c_0
\]
for some \( c_0 \) sufficiently small. We assume moreover that \( b_0 - e_3 \) and \( b_0 \) are admissible on \( \mathbb{R}^2 \times \{0\} \) in the sense of Definition 1.1 and \( \text{Supp} (b_0 - e_3)(x_1, x_2, \cdot) \subset [-K, K] \) for some positive constant \( K \). Then (1.1) has a unique global solution \((b, u, p)\) (up to a constant for \( p \)) so that
\[
b - e_3 \in C([0, \infty); \dot{H}^s(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1 + 1} \cap \dot{H}^{s_2 + 1}(\mathbb{R}^3)),
\]
(1.17) \[
\nabla p \in C([0, \infty); \dot{H}^{s - 1}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3)),
\]
\[
u \in C([0, \infty); \dot{H}^{s}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{H}^{s_1 + 2} \cap \dot{B}^{\frac{3}{5}}_{2,1}(\mathbb{R}^3)) \cap L^2_{\text{loc}}(\mathbb{R}^+; \dot{H}^{s+1}(\mathbb{R}^3)).
\]
Furthermore, there holds
\[
\|b - e_3\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2})} + \|u\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2})} + \|b - e_3\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})} + \|u\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+2})} + \|\nabla p\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2})} \\
\leq C(\|b_0 - e_3\|_{B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}} + \|u_0\|_{\dot{H}^{s_2} \cap \dot{B}^{\frac{3}{p}-1}_{p,1}} + \|\nabla u_0\|_{B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}}).
\]

**Remark 1.2.** (1) One may find the definitions of Besov spaces in Subsection 3.2. We remark that those technical assumptions on \( b_0 \) and \( u_0 \) will be used to deal with the low frequency part of \( b \) and \( u \). For simplicity, we do not provide result on the propagation of regularities for \( b_0 - e_3 \in B^{s_1+\frac{3}{p}+1}_{p,1}(\mathbb{R}^3) \) and \( \nabla u_0 \in B^{s_1+\frac{3}{p}-\frac{3}{2}}_{p,1}(\mathbb{R}^3) \).

(2) Here we point out that the estimate of \( \|b - e_3\|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})} \) in (1.18) is not standard for the solutions of the transport equation in (1.1). It is purely due to the coupling structure in (1.1). And this estimate in some sense explains that the magnetic field is indeed time dissipative even without resistivity for the magnetic field. We shall go back to this point in our future work.

(3) We can improve the condition that: \( \text{Supp} (b_0 - e_3)(x_1, x_2, \cdot) \subset [-K, K] \) for some positive number \( K \), in Theorem 1.1 by assuming appropriate decay of \( b_0 - e_3 \) with respect to \( x_3 \) variable. For a clear presentation, we prefer not to present this technical part here.

Let us complete this section by the notation we shall use in this context.

**Notation.** For any \( s \in \mathbb{R} \), we denote by \( H^s(\mathbb{R}^3) \) the classical \( L^2 \) based Sobolev spaces with the norm \( \| \cdot \|_{H^s} \), while \( \dot{H}^s(\mathbb{R}^3) \) the classical homogenous Sobolev spaces with the norm \( \| \cdot \|_{\dot{H}^s} \). Let \( A, B \) be two operators, we denote \([A; B] = AB - BA\), the commutator between \( A \) and \( B \). For \( a \lesssim b \), we mean that we have a uniform constant \( C \), which may be different on different lines, such that \( a \leq Cb \), and \( a \sim b \) means that both \( a \lesssim b \) and \( b \lesssim a \). We shall denote by \((a|b)\) the \( L^2(\mathbb{R}^3) \) inner product of \( a \) and \( b \). \((d_{j,k})_{j,k \in \mathbb{Z}}\) (resp. \((c_j)_{j \in \mathbb{Z}}\)) will be a generic element of \( \ell^1(\mathbb{Z}^2) \) (resp. \( \ell^2(\mathbb{Z}) \)) so that \( \sum_{j,k \in \mathbb{Z}} d_{j,k} = 1 \) (resp. \( \sum_{j \in \mathbb{Z}} c_j^2 = 1 \)). Finally, we denote by \( L^p_T(L^q_h(L^r_{x_h})) \) the space \( L^p([0, T]; L^q(\mathbb{R}^3; L^r(\mathbb{R}^3))) \) with \( x_h = (x_1, x_2) \).
2. Lagrangian formulation of (1.1)

Motivated by [22], we are going to construct two vector fields \( \tilde{b}_0 = (\tilde{b}_0^1, \tilde{b}_0^2, \tilde{b}_0^3)^T \) and \( \tilde{b}_0 = (\tilde{b}_0^1, \tilde{b}_0^2, \tilde{b}_0^3)^T \) so that the 3 \times 3 matrix \( U_0 \overset{\text{def}}{=} (\tilde{b}_0, \tilde{b}_0, 0) \) satisfies (1.9).

**Proposition 2.1.** Let \( s > 2 + \frac{n}{p} \) and \( p \in (\frac{2}{n}, 2) \). Let \( b_0 - e_3 = (b_0^1, b_0^2, b_0^3 - 1)^T \in B^s_{p,1}(\mathbb{R}^3) \) with
\[
\begin{align*}
\text{div} b_0 &= 0 \quad \text{and} \quad \|(b_0^1, b_0^2, b_0^3 - 1)\|_{B^s_{p,1}} \leq \varepsilon_0.
\end{align*}
\]
We assume moreover that \( b_0 - e_3 \) and \( b_0 \) are admissible on \( \mathbb{R}^2 \times \{0\} \) in the sense of Definition 1.1 and \( \text{Supp} (b_0 - e_3)(x_1, x_2, \cdot) \subset [-K, K] \) for some positive constant \( K \). Then for \( \varepsilon_0 \) sufficiently small, there exists a \( \Psi = (\psi_1, \psi_2, \psi_3)^T \) which satisfies
\[
\|(\psi_1, \psi_2, \psi_3)\|_{B^s_{p,1}} \leq C(K, \varepsilon_0)\|(b_0^1, b_0^2, b_0^3 - 1)\|_{B^s_{p,1}},
\]
and
\[
\begin{align*}
b_0^1 &= \partial_{x^2} \psi_1 \partial_{x_3} \psi_2 + \partial_{x^3} \psi_1 (1 - \partial_{x_2} \psi_2), \quad b_0^2 = \partial_{x_3} \psi_1 \partial_{x_1} \psi_2 + \partial_{x^3} \psi_2 (1 - \partial_{x_1} \psi_1), \\
b_0^3 &= (1 - \partial_{x_1} \psi_1)(1 - \partial_{x_2} \psi_2) - \partial_{x_2} \psi_1 \partial_{x_1} \psi_2, \quad \text{and} \quad \text{det}(I - \nabla_2 \Psi) = 1.
\end{align*}
\]
Moreover, we define
\[
\begin{align*}
\tilde{b}_0 &\overset{\text{def}}{=} ((1 - \partial_{x_2} \psi_2)(1 - \partial_{x_3} \psi_3) - \partial_{x_3} \psi_2 \partial_{x_2} \psi_3, \partial_{x_3} \psi_2 \partial_{x_1} \psi_3 + \partial_{x_1} \psi_2 (1 - \partial_{x_2} \psi_3), \\
& \quad \partial_{x_1} \psi_2 \partial_{x_2} \psi_3 + \partial_{x_1} \psi_3 (1 - \partial_{x_2} \psi_2))^T \quad \text{and} \\
\tilde{b}_0 &\overset{\text{def}}{=} ((1 - \partial_{x_3} \psi_3 \partial_{x_2} \psi_2, \partial_{x_3} \psi_2 \partial_{x_1} \psi_3 + \partial_{x_1} \psi_2 (1 - \partial_{x_3} \psi_3), \partial_{x_1} \psi_1 \partial_{x_2} \psi_3 (1 - \partial_{x_2} \psi_2))^T,
\end{align*}
\]
then \( U_0 \overset{\text{def}}{=} (\tilde{b}_0, \tilde{b}_0, b_0) \) satisfies (1.9), and for \( e_1 = (1, 0, 0)^T, e_2 = (0, 1, 0)^T \),
\[
\|(b_0 - e_1)\|_{B^s_{p,1}} + \|(b_0 - e_2)\|_{B^s_{p,1}} \leq C(K, \varepsilon_0)\|(b_0^1, b_0^2, b_0^3 - 1)\|_{B^s_{p,1}}.
\]

The proof of this proposition is postponed in Appendix B.

With \( U_0 \) obtained in Proposition 2.1, we shall first investigate the global wellposedness to the following system with sufficiently small \( u_0 \) :
\[
\begin{align*}
\partial_t U + u \cdot \nabla U &= \nabla u U, \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla p &= -\frac{1}{2} \nabla |b|^2 + b \cdot \nabla b, \\
\text{div} u &= 0 \quad \text{and} \quad \text{div} U = 0., \\
U|_{t=0} &= U_0, \quad u|_{t=0} = u_0,
\end{align*}
\]
where the 3 \times 3 matrix \( U = (\tilde{b}, \tilde{b}, b) \), and \( \tilde{b} = (\tilde{b}_0^1, \tilde{b}_0^2, \tilde{b}_0^3)^T, b = (\tilde{b}_0^1, \tilde{b}_0^2, \tilde{b}_0^3)^T \). In particular, for any smooth enough solution \((U, u)\) of (2.6), \((b, u)\) must be a smooth enough solution of (1.1).

The main result concerning the wellposedness of the system (2.6) can be stated as follows:

**Theorem 2.1.** Let \( s_1 > \frac{5}{4}, s_2 \in (-\frac{1}{2}, -\frac{1}{2}) \) and \( p \in (1, 2) \). Let \( u_0 \in \dot{H}^{s_2} \setminus \dot{B}^{\frac{5}{2}}_{p,1}(\mathbb{R}^3) \) with \( \nabla u_0 \in B^{s_1 + \frac{\alpha - 1}{p} - \frac{3}{2}}_{p,1}(\mathbb{R}^3) \) and \( U_0 = (I - \nabla_2 \Psi)^{-1} \) with \( \Psi = (\psi_1, \psi_2, \psi_3)^T \) satisfying \( \nabla \Psi \in \dot{B}^{s_2 + \frac{\alpha - 1}{p} - \frac{3}{2}}_{p,1}(\mathbb{R}^3) \) and \( \text{det}(I - \nabla_2 \Psi) = 1 \). We assume that
\[
\|(\nabla \Psi)\|_{\dot{B}^{s_2 + \frac{\alpha - 1}{p} - \frac{3}{2}}_{p,1}(\mathbb{R}^3)} + \|u_0\|_{\dot{H}^{s_2} \setminus \dot{B}^{\frac{5}{2}}_{p,1}} + \|\nabla u_0\|_{\dot{B}^{s_1 + \frac{\alpha - 1}{p} - \frac{3}{2}}_{p,1}} \leq \varepsilon_0
\]
for some $\varepsilon_0$ sufficiently small. Then (2.6) has a unique global solution $(U, u, p)$ (up to a constant for $p$), with $U = (b, \bar{b}, b)$, so that

$$b - e_1, \bar{b} - e_2 \in C((0, \infty); \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1}(\mathbb{R}^3)),$$

$$b - e_3 \in C((0, \infty); \dot{H}^{s_1+1} \cap \dot{H}^{s_2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1}(\mathbb{R}^3)),$$

$$\mathbf{u} \in C((0, \infty); \dot{H}^{s_1+1} \cap \dot{H}^{s_2}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+1}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{B}_2^{3,1}(\mathbb{R}^3)),$$

$$|\nabla p| \in L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3)).$$

Furthermore, there holds

$$\begin{aligned}
&\| (\bar{b} - e_1, \bar{b} - e_2) \|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})} + \| b - e_3 \|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})} \\
&+ \| u \|_{L^\infty(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})} + \| b - e_3 \|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1})} \\
&+ \| u \|_{L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+1})} + \| u \|_{L^1(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{B}_2^{3,1})} + \| \nabla p \|_{L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2})} \\
&\leq C(\| \nabla \Psi \|_{B_{p,2}^{s_1+1} \cap \dot{B}_2^{3,1}} + \| \Psi \|_{B_{p,1}^{s_1+1} \cap \dot{B}_2^{3,1}} + \| u \|_{B_{p,1}^{s_1+1} \cap \dot{B}_2^{3,1}} + \| \nabla u \|_{B_{p,1}^{s_1+1} \cap \dot{B}_2^{3,1}}).
\end{aligned}$$

In order to avoid the difficulty of propagating anisotropic regularity for the transport equation in the system (2.6), we shall reformulate (2.6) in the Lagrangian coordinates. Toward this, we need first to find a volume preserving diffeomorphism $X_0(y)$ on $\mathbb{R}^3$ so that there holds (1.4).

**Lemma 2.1.** Let $p \in (1, 2)$, $s > 1 + \frac{2}{p}$. Let $\Psi = (\psi_1, \psi_2, \psi_3)^T$ satisfy $\nabla \Psi \in B_{p,1}^{s-1}(\mathbb{R}^3)$, $\det (I - \nabla \Psi) = 1$ and $\| \nabla \Psi \|_{B_{p,1}^{s-1}} \leq \varepsilon_0$ for some $\varepsilon_0$ sufficiently small. Then for $U_0 = (I - \nabla \Psi)^{-1}$, there exists $Y_0(y) = (Y_0^1(y), Y_0^2(y), Y_0^3(y))^T$ so that $X_0(y) = y + Y_0(y)$ satisfies

$$U_0 \circ X_0(y) = \nabla y X_0(y) = I + \nabla y Y_0(y) \quad \text{and} \quad \| \nabla y Y_0 \|_{B_{p,1}^{s-1}} \leq C\| \nabla x \Psi \|_{B_{p,1}^{s-1}}.$$

**Proof.** Let $Y = (Y^1, Y^2, Y^3)^T$, we denote

$$\mathbf{F}(y, Y) \overset{\text{def}}{=} Y - \Psi(y + Y),$$

with $\mathbf{F}(y, Y) = (F^1(y, Y), F^2(y, Y), F^3(y, Y))^T$.

It is easy to observe from the assumption: $\det (I - \nabla \Psi) = \det U_0 = 1$, that

$$\det \frac{\partial (F^1, F^2, F^3)}{\partial (Y^1, Y^2, Y^3)} = \det (I - \nabla \Psi)|_{x = y + Y} = 1,$$

from which, $\| \nabla \Psi \|_{L^\infty} \leq C\varepsilon_0$ for some $\varepsilon_0$ sufficiently small, and the classical implicit function theorem, we deduce that around every point $y$, the function $\mathbf{F}(y, Y) = \mathbf{0}$ determines a unique function $Y_0(y) = (Y_0^1(y), Y_0^2(y), Y_0^3(y))^T$ so that

$$\mathbf{F}(y, Y_0(y)) = \mathbf{0},$$

or equivalently

$$Y_0(y) = \Psi(y + Y_0(y)).$$

Then denoting by $X_0(y) = y + Y_0(y)$, we have

$$\partial_{y_1} Y_0^j(y) = \partial_{x_1} \psi_j \circ X_0(y)(1 + \partial_{y_1} Y_0^1(y)) + \partial_{x_2} \psi_j \circ X_0(y) \partial_{y_1} Y_0^2(y)$$

$$+ \partial_{x_3} \psi_j \circ X_0(y) \partial_{y_1} Y_0^3(y), \quad \text{for} \quad j = 1, 2, 3.$$
Due to the fact that \( \det (I - \nabla_x \Psi) = \det U_0 = 1 \), we conclude that \( I - \nabla_x \Psi \) equals the adjoint matrix of \( U_0 \overset{\text{def}}{=} (b_{ij})_{i,j=1,2,3} \), which along with (2.13) ensures that

\[
\partial_{y_i} Y^1_0(y) = b_{11} \circ X_0(y) - 1, \quad \partial_{y_1} Y^2_0(y) = b_{21} \circ X_0(y), \quad \partial_{y_1} Y^3_0(y) = b_{31} \circ X_0(y).
\]

Along the same line, one has

\[
\partial_{y_i} Y^j_0(y) = b_{ij} \circ X_0(y) - \delta_{ij},
\]

which implies the first part of (2.10). This in particular leads to

\[
\nabla_x (X_0^{-1}(x)) = (\nabla_y X_0) \circ X_0^{-1}(x)^{-1} = U_0^{-1}(x) = I - \nabla \Psi(x),
\]

from which, (2.12), \( \|\nabla \Psi\|_{B^{-1}} \leq \varepsilon_0 \) for \( s > 1 + \frac{3}{p} \) and Lemma A.1, we achieve the second part of (2.10).

With \( X_0(y) = y + Y_0(y) \) obtained in Lemma 2.1, we now define the flow map \( X(t, y) \) by

\[
\begin{aligned}
\frac{dX(t, y)}{dt} &= u(t, X(t, y)), \\
X(t, y)|_{t=0} &= X_0(y),
\end{aligned}
\]

and \( Y(t, y) \) through

\[
X(t, y) = X_0(y) + \int_0^t u(s, X(s, y)) ds \overset{\text{def}}{=} y + Y(t, y).
\]

Then by virtue of Proposition 1.8 of [23] and (2.10), we deduce from (2.6) that

\[
U(t, X(t, y)) = \nabla_y X(t, y) = I + \nabla_y Y(t, y) \quad \text{and} \quad \det (I + \nabla_y Y(t, y)) = 1.
\]

Denoting \( U(t, X(t, y)) \overset{\text{def}}{=} (a_{ij})_{i,j=1,2,3} \) and \( A_Y \overset{\text{def}}{=} (b_{ij})_{i,j=1,2,3} \) with

\[
\begin{aligned}
b_{11} &= (1 + \partial_2 Y^2)(1 + \partial_3 Y^3) - \partial_3 Y^2 \partial_2 Y^3, \quad b_{12} = \partial_2 Y^1 \partial_2 Y^3 - \partial_2 Y^1 (1 + \partial_3 Y^3), \\
b_{13} &= \partial_3 Y^1 \partial_3 Y^2 - \partial_3 Y^1 (1 + \partial_2 Y^2), \quad b_{21} = \partial_3 Y^2 \partial_1 Y^3 - \partial_1 Y^2 (1 + \partial_3 Y^3), \\
b_{22} &= (1 + \partial_1 Y^1)(1 + \partial_3 Y^3) - \partial_3 Y^1 \partial_1 Y^3, \quad b_{23} = \partial_3 Y^1 \partial_1 Y^2 - (1 + \partial_1 Y^1) \partial_3 Y^2, \\
b_{31} &= \partial_1 Y^2 \partial_3 Y^3 - (1 + \partial_2 Y^2) \partial_1 Y^3, \quad b_{32} = \partial_2 Y^1 \partial_1 Y^3 - (1 + \partial_1 Y^1) \partial_2 Y^3, \\
b_{33} &= (1 + \partial_1 Y^1)(1 + \partial_2 Y^2) - \partial_2 Y^1 \partial_1 Y^2.
\end{aligned}
\]

It is easy to observe that \( \sum_{i=1}^3 \frac{\partial b_{ij}}{\partial y_m} = 0 \) (see also Lemma 2.1 of [27]). Moreover, as \( \det U = 1 \), \( A_Y = (I + \nabla_y Y)^{-1} \). Then it follows from (2.16) that

\[
\begin{aligned}
\mathbf{b} \circ X(t, y) &= (\partial_{y_1} Y^1, \partial_{y_2} Y^2, 1 + \partial_{y_3} Y^3)^T \quad \text{and} \quad A_Y (\mathbf{b} \circ X) = (0, 0, 1)^T,
\end{aligned}
\]

from which, we infer

\[
\begin{aligned}
(b \cdot \nabla X) \circ X(t, y) &= [\text{div}_x (b \otimes b)] \circ X(t, y) \\
&= \nabla_y \cdot [A_Y (b \circ X) \otimes (b \circ X)] = \partial_{y_3} (b \circ X) = \partial_{y_3}^2 Y(t, y).
\end{aligned}
\]

Thanks to (2.15) and (2.19), we can equivalently reformulate (2.6) as

\[
\begin{aligned}
&\begin{cases}
Y_{tt} - \nabla Y \cdot \nabla Y_t - \partial_{y_3}^2 Y + \nabla Y q = 0, \\
\nabla Y \cdot Y_t = 0,
\end{cases} \\
Y|_{t=0} = Y_0, \quad Y_t|_{t=0} = u_0 \circ X_0(y) \overset{\text{def}}{=} Y_1,
\end{aligned}
\]
where \( q(t,y) \) is defined as \((p + \frac{1}{2}(b)^2) \circ X(t,y) \) and \( \nabla_Y \) is defined as \( A_Y^T \nabla_y \) with \( A_Y = (a_{ij})_{i,j=1,2,3} \) being determined by (2.17). Here and in what follows, we always assume that \( \|\nabla_y Y\|_{L^\infty} \leq \frac{1}{2} \). Under this assumption, we rewrite (2.20) as

\[
\begin{aligned}
&Y_{tt} - \Delta_y Y_t - \partial_{y_3}^2 Y = f(Y,q), \\
&\nabla_y \cdot Y = \rho(Y), \\
&Y|_{t=0} = Y_0, \quad Y_{tt}|_{t=0} = Y_1.
\end{aligned}
\]

(2.21)

where

\[
\begin{aligned}
f(Y,q) &= (\nabla_Y - \Delta_y)Y_t - \nabla_Y q, \\
\rho(Y) &= \nabla_y \cdot Y_0 - \int_0^t (\nabla_Y - \nabla_y) \cdot Y_s \, ds \\
&= - \sum_{i<j} \partial_i Y^i \partial_j Y^j + \sum_{i<j} \partial_i Y^j \partial_j Y^i - \partial_1 Y^1 \partial_2 Y^2 \partial_3 Y^3 - \partial_3 Y^1 \partial_1 Y^2 \partial_2 Y^3 \\
&\quad - \partial_2 Y^1 \partial_3 Y^2 \partial_1 Y^3 + \partial_1 Y^1 \partial_2 Y^2 \partial_2 Y^3 + \partial_1 Y^1 \partial_2 Y^2 \partial_1 Y^3 + \partial_2 Y^1 \partial_1 Y^2 \partial_3 Y^3.
\end{aligned}
\]

(2.22)

Here we used (2.17) and \( \det (I + \nabla Y_0) = 1 \) to derive the second equality of (2.22). Indeed thanks to (2.17), one has

\[
\begin{aligned}
(\nabla_Y - \nabla_y) \cdot Y_t &= \frac{d}{dt} \left( \sum_{i<j} \partial_i Y^i \partial_j Y^j - \sum_{i<j} \partial_i Y^j \partial_j Y^i + \partial_1 Y^1 \partial_2 Y^2 \partial_3 Y^3 \\
&\quad + \partial_3 Y^1 \partial_2 Y^2 \partial_1 Y^3 + \partial_2 Y^1 \partial_3 Y^2 \partial_1 Y^3 - \partial_1 Y^1 \partial_3 Y^2 \partial_2 Y^3 \\
&\quad - \partial_3 Y^1 \partial_2 Y^2 \partial_1 Y^3 - \partial_2 Y^1 \partial_1 Y^2 \partial_3 Y^3 \right) \\
&= \frac{d}{dt} (\det (I + \nabla_y Y) - 1 - \nabla_y \cdot Y),
\end{aligned}
\]

(2.23)

which together with \( \det (I + \nabla Y_0) = 1 \) ensures the second equality of (2.22). Moreover, the equation \( \nabla_y \cdot Y = \rho(Y) \) implies that \( \det (I + \nabla Y_0) = 1 \) and \( \nabla_Y \cdot Y_1 = 0 \).

For notational convenience, we shall neglect the subscripts \( x \) or \( y \) in \( \partial, \nabla \) and \( \Delta \) in the sequel. We make the convention that whenever \( \nabla \) acts on \( (U, u, p) \), we understand \( (\nabla U, \nabla u, \nabla p) \) as \( (\nabla_x U, \nabla_x u, \nabla_x p) \). While \( \nabla \) acts on \( (Y, q) \), we understand \( (\nabla_Y, \nabla q) \) as \( (\nabla_y Y, \nabla_y q) \). Similar conventions for \( \partial \) and \( \Delta \).

For (2.21)-(2.22), we have the following global wellposedness result:

**Theorem 2.2.** Let \( s_1 > \frac{4}{3}, \ s_2 \in (-\frac{5}{6}, -\frac{1}{6}) \). Let \( (Y_0, Y_1) \) satisfy \( (\partial_3 Y_0, \Delta Y_0) \in \dot{H}^{s_1} \cap \dot{H}^{s_2} \cap B^{-\frac{1}{2},0}_{2,0} \cap B^{s_1,0}(\mathbb{R}^3), \ Y_1 \in \dot{H}^{s_1+1} \cap \dot{H}^{s_2} \cap B^{-\frac{1}{2},0}_{2,0} \cap B^{s_1,0}(\mathbb{R}^3) \) and

\[
\det (I + \nabla Y_0) = 1, \quad \nabla Y_0 \cdot Y_1 = 0, \quad \text{and}
\]

(2.24)

\[
\begin{aligned}
&\|Y_0\|_{\dot{H}^{s_1+2\gamma} \cap \dot{H}^{s_2+2}} + \|\partial_3 Y_0\|_{\dot{H}^{s_2}} + \|Y_1\|_{\dot{H}^{s_1+1} \cap \dot{H}^{s_2}} \\
&\quad + \|Y_0\|_{B^\frac{3}{2},0 \cap B^{s_1+2,0}} + \|\partial_3 Y_0\|_{B^\frac{1}{2},0 \cap B^{s_1,0}} + \|Y_1\|_{B^\frac{5}{2},0 \cap B^{s_1,0}} \leq \varepsilon_0
\end{aligned}
\]

(2.25)
for some $\varepsilon_0$ sufficiently small. Then (2.21)-(2.22) has a unique global solution $(Y, q)$ (up to a constant for $q$) so that

$$Y \in C([0, \infty); \dot{H}^{s_1+2} \cap \dot{H}^{s_2+2} \cap B^{\frac{2}{s_1} \cdot 0} \cap B^{s_1+2,0}(\mathbb{R}^3))$$

and

$$\partial_t Y \in C([0, \infty); \dot{H}^{s_2} \cap B^{\frac{2}{s_1} \cdot 0} \cap B^{s_1,0}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+1} \cap \dot{H}^{s_2+1}(\mathbb{R}^3)),
$$

(2.26)\quad Y_t \in C([0, \infty); \dot{H}^{s_1+1} \cap \dot{H}^{s_2} \cap B^{\frac{4}{s_1} \cdot 0} \cap B^{s_1,0}(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+; \dot{H}^{s_1+2} \cap \dot{H}^{s_2+1}(\mathbb{R}^3))$$

$$\cap L^1(\mathbb{R}^+; B^{\frac{2}{s_1} \cdot 0} \cap B^{s_1+2,0}(\mathbb{R}^3)),
$$

$$\nabla q \in L^2(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3)) \cap L^1(\mathbb{R}^+; \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3)).$$

Moreover, there hold det $(I + \nabla Y) = 1,$ $\nabla Y \cdot Y_t = 0,$ and

$$\|Y\|_{L_t^\infty(H^{s_1+2} \cap \dot{H}^{s_2+2})}^2 + \|\partial_t Y\|_{L_t^\infty(\dot{H}^{s_2})}^2 + \|Y_t\|_{L_t^\infty(H^{s_1+1} \cap \dot{H}^{s_2})}^2 + \|\partial_t Y\|_{L_t^\infty(H^{s_1+2} \cap \dot{H}^{s_2+1})}^2 + \|\partial_t Y\|_{L_t^\infty(B^{\frac{2}{s_1} \cdot 0} \cap B^{s_1,0})}^2 + \|\nabla q\|_{L_t^\infty(H^{s_1} \cap \dot{H}^{s_2})}^2$$

$$\leq C \left(\|\partial_t Y_{s_0}\|_{H^{s_2}}^2 + \|Y_{s_0}\|_{H^{s_1+2} \cap \dot{H}^{s_2+2}}^2 + \|Y_t\|_{H^{s_1+1} \cap \dot{H}^{s_2}}^2 + \|\partial_t Y\|_{H^{s_1+2}}^2 + \|\partial_t Y\|_{B^{\frac{2}{s_1} \cdot 0} \cap B^{s_1,0}}^2 + \|\nabla q\|_{B^{\frac{2}{s_1} \cdot 0} \cap B^{s_1+2,0}}^2 \right).$$

**Remark 2.1.** The norm of $\|\cdot\|_{B^{s,0}}$ is given by Definition 3.2. We should mention once again that the equation $\nabla \cdot Y = \rho(Y)$ in (2.21) plays a key role in the proof of Theorem 2.2. In particular, we need to use this equation to derive the globally $L^1$ in time estimates of $\nabla q$ and $\nabla Y_t$, which will be crucial for us to close the energy estimates for (2.21)-(2.22).

**Scheme of the proof and organization of the paper.**

To avoid the difficulty caused by propagating anisotropic regularity for the transport equation in (2.6), we shall first prove the global wellposedness of the Lagrangian formulation (2.21)-(2.22) with small initial data.

Let $(Y, q)$ be a smooth enough solution of (2.21), applying standard energy estimate to (2.21) leads to

$$\frac{d}{dt} \left( \frac{1}{2} \|Y_t\|_{\dot{H}^s}^2 + \|Y_t\|_{H^{s+1}}^2 + \|\partial_t Y\|_{\dot{H}^s}^2 + \|\partial_t Y\|_{H^{s+1}}^2 + \frac{1}{4} \|Y\|_{H^{s+2}}^2 \right)
$$

$$- \frac{1}{4} \langle Y_t, \Delta Y_t \rangle_{\dot{H}^{s}} + \frac{3}{4} \|Y_t\|_{H^{s+1}}^2 + \|Y_t\|_{H^{s+2}}^2 + \frac{1}{4} \|\partial_t Y\|_{H^{s+1}}^2 = (f, Y_t - \frac{1}{4} \Delta Y_t - \Delta Y_t)_{\dot{H}^{s}}$$

where $(a \mid b)_{\dot{H}^{s}}$ denotes the standard $\dot{H}^s$ inner product of $a$ and $b$. (2.28) shows that $\partial_t Y$ belongs to $L^2(\mathbb{R}^+; \dot{H}^{s+1}(\mathbb{R}^3))$, however, there is no time dissipative estimate of $\Delta Y$. Therefore, in order to close the energy estimate in (2.28), we would require the source term $f$ in (2.21) belonging to $L^1(\mathbb{R}^+; \dot{H}^{s}(\mathbb{R}^3))$. To achieve this, we need also the $L^1(\mathbb{R}^+; B^{\frac{2}{s_1} \cdot 0} \cap B^{s_1+2,0}(\mathbb{R}^3))$ estimate of $Y_t$. Toward this, we shall use the dissipative estimates for $\partial_t Y$ as well as the fact that $\nabla \cdot Y = \rho(Y)$ in a rather crucial way.

In the first part of Section 3, we shall present a heuristic analysis to the linearized system of (2.21)-(2.22), which motivates us to use anisotropic Littlewood-Paley theory below, then we shall collect some basic facts on functional framework and Littlewood-Paley analysis in Subsection 3.2.
In Section 4, we apply anisotropic Littlewood-Paley theory to explore the dissipative mechanism for a linearized model of (2.21)-(2.22).

In Section 5, we present the proof of Theorem 2.2, and we present the proof of Theorems 2.1 and 1.1 in Section 6.

Finally, we present the proofs of some technical lemmas in the Appendices.

3. Preliminary

3.1. Spectral analysis to the linearized system of (2.21)-(2.22).

We first investigate heuristically the spectrum properties to the following linearized system of (2.21)-(2.22):

\[
\begin{align*}
Y_{tt} - \Delta Y_t - \partial_3^2 Y &= f, \\
Y|_{t=0} &= Y_0, \quad Y_t|_{t=0} = Y_1.
\end{align*}
\]

Note that the symbolic equation corresponds to (3.1) reads

\[
\lambda^2 + |\xi|^2 \lambda + \xi_3^2 = 0 \quad \text{for} \quad \xi = (\xi_h, \xi_3) \quad \text{and} \quad \xi_h = (\xi_1, \xi_2).
\]

It is easy to calculate that this equation has two different eigenvalues

\[
\lambda_{\pm} = -\frac{|\xi|^2 \pm \sqrt{|\xi|^4 - 4\xi_3^2}}{2}.
\]

The Fourier modes correspond to \(\lambda_+\) decays like \(e^{-t|\xi|^2}\). Whereas the decay property of the Fourier modes corresponding to \(\lambda_-\) varies with directions of \(\xi\) as

\[
\lambda_- (\xi) = -\frac{2\xi_3^2}{|\xi|^2(1 + \sqrt{1 - 4\xi_3^2/|\xi|^4})} \rightarrow -1 \quad \text{as} \quad |\xi| \rightarrow \infty
\]

only in the \(\xi_3\) direction. This shows that smooth solution of (3.1) decays in a very subtle way. In order to capture this delicate decay property for the linear equation (3.1), we shall decompose our frequency space into two parts: \(\{\xi = (\xi_h, \xi_3) : |\xi|^2 \leq 2|\xi_3|\}\) and \(\{\xi = (\xi_h, \xi_3) : |\xi|^2 > 2|\xi_3|\}\).

This heuristic analysis shows that the dissipative properties of the solutions to (3.1) may be more complicated than that for the linearized system of isentropic compressible Navier-Stokes system in [12], and this brief analysis also suggests us to employ the tool of anisotropic Littlewood-Paley theory as in [22] for 2-D incompressible MHD system and [21] for a modified 3-D MHD system, which has also been used in the study of the global wellposedness to 3-D anisotropic incompressible Navier-Stokes equations [7, 8, 9, 15, 16, 24, 25, 28]. One may check Section 4 below for the detailed rigorous analysis corresponding to this scenario.

3.2. Littlewood-Paley theory.

The proof of Theorem 2.2 requires a dyadic decomposition of the Fourier variables, or the Littlewood-Paley decomposition. Let \(\varphi\) and \(\chi\) be smooth functions supported in \(\mathcal{C} \overset{\text{def}}{=} \{\tau \in \mathbb{R}^+ : \frac{3}{4} \leq \tau \leq \frac{8}{3}\}\) and \(\mathcal{B} \overset{\text{def}}{=} \{\tau \in \mathbb{R}^+, \tau \leq \frac{4}{3}\}\) such that

\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1 \quad \text{for} \quad \tau > 0 \quad \text{and} \quad \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1 \quad \text{for} \quad \tau \geq 0.
\]
For $a \in \mathcal{S}'(\mathbb{R}^2)$, we set
\begin{equation}
\Delta_k^h a \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-k}|\xi|)\hat{a}), \quad S_k^h a \overset{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-k}|\xi|)\hat{a})
\end{equation}
(3.4)
\begin{equation}
\Delta_j^\ell a \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi|)\hat{a}), \quad S_j^\ell a \overset{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi|)\hat{a}), \quad \text{and}
\end{equation}
\begin{equation}
\Delta_j a \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\hat{a}), \quad S_j a \overset{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\hat{a}),
\end{equation}
where $Fa$ and $\hat{a}$ denote the Fourier transform of the distribution $a$. The dyadic operators satisfy the property of almost orthogonality:
\begin{equation}
\Delta_k \Delta_j a = 0 \quad \text{if} \quad |k - j| \geq 2 \quad \text{and} \quad \Delta_k(S_{j-1} a \Delta_j b) = 0 \quad \text{if} \quad |k - j| \geq 5.
\end{equation}
Similar properties hold for $\Delta_k^h$ and $\Delta_j^\ell$.

**Definition 3.1.** (Definition 2.15 of [2]) Let $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}_r'(\mathbb{R}^3)$, (see Definition 1.26 of [2]), which means $u \in \mathcal{S}'(\mathbb{R}^d)$ and $\lim_{j \to -\infty} \|\chi(2^{-j}D)u\|_{L^\infty} = 0$, we set
\begin{equation}
\|u\|_{\dot{B}_{p,r}^s} \overset{\text{def}}{=} \left(2^{js} \|\Delta_j u\|_{L^p}ight)^{1/j}.
\end{equation}

- For $s < \frac{3}{p}$ (or $s = \frac{3}{p}$ if $r = 1$), we define $\dot{B}_{p,r}^s(\mathbb{R}^3) \overset{\text{def}}{=} \left\{ u \in \mathcal{S}_r'(\mathbb{R}^3) \mid \|u\|_{\dot{B}_{p,r}^s} < \infty \right\}$.
- If $k \in \mathbb{N}$ and $\frac{3}{p} + k - 1 \leq s < \frac{3}{p} + k$ (or $s = \frac{3}{p} + k$ if $r = 1$), then $\dot{B}_{p,r}^s(\mathbb{R}^3)$ is defined as the subset of distributions $u \in \mathcal{S}_r'(\mathbb{R}^3)$ such that $\partial^\beta u \in \dot{B}_{p,r}^{s-k}(\mathbb{R}^3)$ whenever $|\beta| = k$.

Inhomogenous Besov spaces $\mathcal{B}_{p,r}^s(\mathbb{R}^3)$ can be defined similarly (see Definition 2.68 of [2]). For simplicity, we shall abbreviate $\dot{B}_{2,1}^s(\mathbb{R}^3)$ (resp. $\dot{B}_{2,1}^s(\mathbb{R}^3)$) as $\dot{B}^s(\mathbb{R}^3)$ (resp. $\dot{B}^s(\mathbb{R}^3)$) in all that follows.

**Remark 3.1.** (1) It is easy to observe that $\dot{B}_{2,2}^s(\mathbb{R}^3) = \dot{H}^s(\mathbb{R}^3)$.

(2) Let $(p, r) \in [1, +\infty]^2$, $s \in \mathbb{R}$ and $u \in \mathcal{S}'(\mathbb{R}^3)$. Then $u \in \dot{B}_{p,r}^s(\mathbb{R}^3)$ if and only if there exists $\{c_{j,r}\}_{j \in \mathbb{Z}}$ such that $\|c_{j,r}\|_{L^r} = 1$ and
\begin{equation}
\|\Delta_j u\|_{L^p} \leq C_{c_{j,r}} 2^{-js} \|u\|_{\dot{B}_{p,r}^s} \quad \text{for all} \quad j \in \mathbb{Z}.
\end{equation}

(3) Let $s, s_1, s_2 \in \mathbb{R}$ with $s_1 < s < s_2$ and $u \in \dot{H}^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3)$. Then $u \in \dot{B}^s(\mathbb{R}^3)$, and there holds
\begin{equation}
\|u\|_{\dot{B}^s} \lesssim \|u\|_{\dot{H}^{s_1}}^{s-s_1} \|u\|_{\dot{H}^{s_2}}^{s-s_2} \lesssim \|u\|_{\dot{H}^{s_1}} + \|u\|_{\dot{H}^{s_2}}.
\end{equation}

For the convenience of the readers, we recall the following Bernstein type lemma from [2, 9, 24]:

**Lemma 3.1.** Let $\mathcal{B}_h$ (resp. $\mathcal{B}_v$) be a ball of $\mathbb{R}^2$ (resp. $\mathbb{R}$), and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) a ring of $\mathbb{R}^2$ (resp. $\mathbb{R}$); let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then there holds:
If the support of $\hat{a}$ is included in $2^k \mathcal{B}_h$, then
\begin{equation}
\|\partial_h^\alpha a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{k(|\alpha| + 2\left(\frac{1}{p_2} - \frac{1}{p_1}\right))} \|a\|_{L^{p_2}_h(L^{q_2}_v)} \quad \text{for} \quad \partial_h = (\partial_1, \partial_2).
\end{equation}
If the support of $\hat{a}$ is included in $2^l \mathcal{B}_v$, then
\begin{equation}
\|\partial_v^\ell a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{l\left(\beta + \left(\frac{1}{p_2} - \frac{1}{q_2}\right)\right)} \|a\|_{L^{p_1}_h(L^{q_1}_v)} \quad \text{for} \quad \partial_v = (\partial_1, \partial_2).
\end{equation}
If the support of $\hat{a}$ is included in $2^k \mathcal{C}_h$, then
\begin{equation}
\|a\|_{L^{p_1}_h(L^{q_1}_v)} \lesssim 2^{-kN} \|\partial_h^N a\|_{L^{p_1}_h(L^{q_1}_v)}.
\end{equation}
If the support of \( \hat{a} \) is included in \( 2^k \mathcal{C}_v \), then
\[
\|a\|_{L^p_k(L^q_k)} \lesssim 2^{-kN} \|\partial^N_b a\|_{L^p_k(L^q_k)}.
\]

In order to obtain the \( L^1(\mathbb{R}^+; \text{Lip}(\mathbb{R}^3)) \) estimate of \( Y_t \) for the linearized equation (3.1), we recall the following anisotropic Besov type space from [21, 22]:

**Definition 3.2.** Let \( s_1, s_2 \in \mathbb{R} \) and \( u \in \mathcal{S}'(\mathbb{R}^3) \), we define the norm
\[
\|u\|_{B^{s_1,s_2}} \overset{\text{def}}{=} \sum_{j,k \in \mathbb{Z}^2} 2^{js_1} 2^{ks_2} \|\Delta_j \Delta_k^u\|_{L^2}.
\]

Then we have the following three dimensional version of Lemma 3.2 in [22]:

**Lemma 3.2.** Let \( s_1, s_2, \tau_1, \tau_2 \in \mathbb{R} \), which satisfy \( s_1 < \tau_1 + \tau_2 < s_2 \) and \( \tau_2 > 0 \). Let \( a \in H^{s_1} \cap \dot{H}^{s_2}(\mathbb{R}^3) \). Then \( a \in B^{s_1-s_2}_2(\mathbb{R}^3) \), and there holds
\[
\|a\|_{B^{s_1-s_2}_2} \lesssim \|a\|_{H^s_1} + \|a\|_{\dot{H}^s_2}.
\]

**Proof.** By virtue of Definition 3.2 and the fact: \( j \geq k - N_0 \) for some fixed positive integer \( N_0 \) in dyadic operator \( \Delta_j \Delta_k^u \), we infer
\[
\|a\|_{B^{s_1-s_2}_2} = \sum_{j,k \in \mathbb{Z}^2} 2^{js_1} 2^{ks_2} \|\Delta_j \Delta_k^u\|_{L^2} \lesssim \sum_{j \in \mathbb{Z}} 2^{js_1} \|\Delta_j a\|_{L^2} \sum_{k \leq j + N_0} 2^{ks_2} \lesssim \sum_{j \in \mathbb{Z}} 2^{(j_1 + \tau_2)} \|\Delta_j a\|_{L^2} \lesssim \|a\|_{\dot{H}^{s_1}_2},
\]
which together with (3.6) completes the proof of the lemma.

In order to obtain a better description of the regularizing effect for the transport-diffusion equation, we will use Chemin-Lerner type spaces \( \dot{L}^q_T(\mathcal{B}^s_{p,r}(\mathbb{R}^3)) \) (see [2] for instance).

**Definition 3.3.** Let \( (r, q, p) \in [1, +\infty]^3 \) and \( T \in (0, +\infty) \). We define the \( \dot{L}^q_T(\mathcal{B}^s_{p,r}(\mathbb{R}^3)) \) by
\[
\|u\|_{\dot{L}^q_T(\mathcal{B}^s_{p,r})} \overset{\text{def}}{=} \left( \sum_{j \in \mathbb{Z}} 2^{jr} \|\Delta_j u\|_{L^q_T(L^p)}^q \right)^{1/q}, \quad \|u\|_{\dot{L}^q_T(\mathcal{B}^{s_1,s_2})} \overset{\text{def}}{=} \sum_{j,k \in \mathbb{Z}^2} 2^{js_1} 2^{ks_2} \|\Delta_j \Delta_k^u\|_{L^q_T(L^2)},
\]
with the usual change if \( r = \infty \).

**Remark 3.2.** The proof of Lemma 3.2 ensures that
\[
(3.7) \quad \|a\|_{\dot{L}^q_T(\mathcal{B}^{s_1,s_2})} \lesssim \|u\|_{\dot{L}^q_T(\mathcal{B}^{s_1,s_2})} \lesssim \|u\|_{\dot{L}^q_T(\dot{H}^{s_1})} + \|u\|_{L^2(\dot{H}^{s_2})},
\]
for \( \tau_1, \tau_2 \) and \( s_1, s_2 \) given by Lemma 3.2.

We also recall the isotropic para-differential decomposition of Bony from [3]: let \( a, b \in \mathcal{S}'(\mathbb{R}^3) \),
\[
ab = T(a, b) + R(a, b), \quad \text{or} \quad ab = T(a, b) + \bar{T}(a, b) + R(a, b), \quad \text{where}
\]
\[
T(a, b) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} S_j a \Delta_j b, \quad \bar{T}(a, b) \overset{\text{def}}{=} T(b, a), \quad R(a, b) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \Delta_j a S_{j+2} b, \quad \text{and}
\]
\[
(3.8) \quad R(a, b) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \Delta_j a \tilde{\Delta}_j b, \quad \text{with} \quad \tilde{\Delta}_j b \overset{\text{def}}{=} \sum_{\ell = j-1}^{j+1} \Delta_\ell b.
\]
Considering the special structure of the functions in $\mathcal{B}_{s_1,s_2}(\mathbb{R}^3)$, we sometime use both isentropic Bony’s decomposition (3.8) and (3.8) for the vertical variable $x_3$ simultaneously.

As an application of the above basic facts on Littlewood-Paley theory, we present the following result in space $\mathcal{B}_{s_1,s_2}(\mathbb{R}^3)$.

**Lemma 3.3.** Let $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$, which satisfy $s_1, s_2 \leq 1$, $\tau_1, \tau_2 \leq \frac{1}{2}$ and $s_1 + s_2 > 0$, $\tau_1 + \tau_2 > 0$. Then for $a \in \mathcal{B}_{s_1,\tau_1}(\mathbb{R}^3)$ and $b \in \mathcal{B}_{s_2,\tau_2}(\mathbb{R}^3)$, $ab \in \mathcal{B}_{s_1+s_2-1,\tau_1+\tau_2-\frac{1}{2}}(\mathbb{R}^3)$ and there holds

\[(3.9)\quad \|ab\|_{\mathcal{B}_{s_1+s_2-1,\tau_1+\tau_2-\frac{1}{2}}} \lesssim \|a\|_{\mathcal{B}_{s_1,\tau_1}} \|b\|_{\mathcal{B}_{s_2,\tau_2}}.\]

**Proof.** The proof of this lemma is identical to that of Lemma 3.3 in [22], we omit the details here. \qed

**Lemma 3.4.** Let $\delta \in [0, \frac{1}{2})$, $s_1 \leq \frac{3}{2} - \delta$, $s_2 \leq 1 + \delta$ and $s_1 + s_2 > \frac{1}{2}$. Then one has

\[(3.10)\quad \|ab\|_{\mathcal{B}^{s_2,0}} \lesssim \|a\|_{\mathcal{B}^{s_1-\frac{1}{2}+\delta,0}} \|b\|_{\mathcal{B}^{s_2-\delta}} \lesssim \|a\|_{\mathcal{B}^{s_1}} \|b\|_{\mathcal{B}^{s_2-\delta}}.\]

**Proof.** By virtue of Lemma 3.2 and Lemma 3.3, we have

\[\|ab\|_{\mathcal{B}^{s_2,0}} \lesssim \|a\|_{\mathcal{B}^{s_1} \ast b} \|b\|_{\mathcal{B}^{s_2,0}}, \quad \text{and} \quad \|ab\|_{\mathcal{B}^{s_2,0}} \lesssim \|a\|_{\mathcal{B}^{s_2}} \|b\|_{\mathcal{B}^{s_1+\frac{1}{2}}} \text{ for } -1 < s \leq 1.\]

This completes the proof of the lemma. \qed

**Remark 3.3.** It follows from Lemma 3.3 and Lemma 3.4 that

\[(3.11)\quad \|ab\|_{\mathcal{B}^{s_2,0}} \lesssim \|a\|_{\mathcal{B}^{s_1}} \|b\|_{\mathcal{B}^{s_2}}, \quad \text{and} \quad \|ab\|_{\mathcal{B}^{s_2,0}} \lesssim \|a\|_{\mathcal{B}^{s_1}} \|b\|_{\mathcal{B}^{s_2+\frac{1}{2}}} \|a\|_{\mathcal{B}^{s_2}},\]

for $\delta_1, \delta_2 \in (0, \frac{1}{2})$.

**Proof.** We first get, by using Bony’s decomposition (3.8) and (3.8) for the vertical variable, that

\[(3.12)\quad ab = (TT^v + T\bar{T}^v + TR^v + T\bar{T}^v + \bar{T}T^v + \bar{T}R^v + TR^v + RT^v + RR^v) (a, b).\]

We shall present the detailed estimates to typical terms above. Indeed applying Lemma 3.1 gives

\[\|\Delta_j \Delta_k (TR^v(a, b))\|_{L^2} \lesssim 2^{j \frac{1}{2}} \sum_{|j' - j| \leq 4, k' \geq k-N_0} \|S_{j' - 1} \Delta_k a\|_{L^\infty(L^{\frac{2}{3}})} \|\Delta_j \bar{\Delta}_k b\|_{L^2}\]

\[\lesssim 2^{j \frac{1}{2}} \sum_{|j' - j| \leq 4, k' \geq k-N_0} d_{j',k'} 2^{-j's} 2^{-k' \frac{1}{2}} \|a\|_{\mathcal{B}^{1,\frac{1}{2}}} \|b\|_{\mathcal{B}^{s,0}} \lesssim d_{j,k} 2^{-j's} \|a\|_{\mathcal{B}^{1,\frac{1}{2}}} \|b\|_{\mathcal{B}^{s,0}},\]

as $\|S_{j' - 1} \Delta_k a\|_{L^\infty(L^{\frac{2}{3}})} \lesssim 2^{-j' \frac{1}{2}} \|a\|_{\mathcal{B}^{1,\frac{1}{2}}}$. Similar estimate holds for $\Delta_j \Delta_k (\bar{T}R^v(a, b))$.\]
Along the same line, we have
\[
\| \Delta_j \Delta_k^v(\mathcal{R}R^v(a, b)) \|_{L^2} \lesssim 2^j 2^k \sum_{j' \geq j - N_0 \atop k' \geq k - N_0} \| \Delta_j' \Delta_{k'}^v a \|_{L^2} \| \check{\Delta}_j' \check{\Delta}_{k'}^v b \|_{L^2}
\]
\[
\lesssim 2^j 2^k \sum_{j' \geq j - N_0 \atop k' \geq k - N_0} d_{j', k'} 2^{-j'(s+1)2^{-s-2}} \| a \|_{B_t^{s+\frac{3}{2}}} \| b \|_{B_t^{s, 0}}
\]
\[
\lesssim d_{j, k} 2^{-j s} \| a \|_{B_t^{s+\frac{3}{2}}} \| b \|_{B_t^{s, 0}}
\]
due to the fact: \( s + 1 > 0 \). The estimate to the remaining terms in (3.13) is identical, and we omit the details here.

Whence thanks to (3.13), we arrive at
\[
\| \Delta_j \Delta_k^v (ab) \|_{L^2} \lesssim d_{j, k} 2^{-j s} \left( \| a \|_{B_t^{s+\frac{3}{2}}} \| b \|_{B_t^{s, 0}} + \| b \|_{B_t^{s+\frac{3}{2}}} \| a \|_{B_t^{s, 0}} \right),
\]
which implies the first inequality of (3.11). Exactly along the same line, we can prove the second inequality of (3.11). Finally notice from Lemma 3.2 that \( \check{B}_t^{s+\delta}(\mathbb{R}^3) \rightarrow B_t^{1, \frac{3}{2}}(\mathbb{R}^3) \) and \( \check{B}_t^{s+\delta}(\mathbb{R}^3) \rightarrow B_t^{s, \delta}(\mathbb{R}^3) \) for \( \delta \in (0, \frac{1}{2}) \), the proof of (3.12) is identical to that of (3.11), we omit the details here. This concludes the proof of Lemma 3.5.

4. \( L_t^1(B_t^{s+2, 0}) \) estimate of \( Y_t \) for \( s = \frac{1}{2} \) and \( s > 1 \)

4.1. The estimate of \( \| Y_t \|_{L_t^1(B_t^{s+2, 0})} \) for the linearized system (3.1).

Proposition 4.1. Let \( Y \) be a smooth enough solution of (3.1) on \( [0, T] \). Then for any \( s \in \mathbb{R} \), there holds
\[
\left\| Y_t \right\|_{L_t^\infty(B_t^{s, 0})} + \left\| \partial_3 Y \right\|_{L_t^\infty(B_t^{s, 0})} + \left\| Y \right\|_{L_t^\infty(B_t^{s+2, 0})} + \left\| Y_t \right\|_{L_t^1(B_t^{s+2, 0})} + \left\| \partial_3 Y_t \right\|_{L_t^\infty(B_t^{s+1, 0})} \lesssim \left\| Y \right\|_{B_t^{s, 0}} + \left\| \partial_3 Y_0 \right\|_{B_t^{s, 0}} + \left\| Y_0 \right\|_{B_t^{s+2, 0}} + \| f \|_{L_t^1(B_t^{s, 0})}.
\]

Proof. We first get, by applying \( \Delta_j \Delta_k^v \) to (3.1), that
\[
\Delta_j \Delta_k^v Y_t - \Delta \Delta_j \Delta_k^v Y_t - \partial_{3j}^2 \Delta_j \Delta_k^v Y = \Delta_j \Delta_k^v f.
\]
Taking the \( L^2 \) inner product of (4.2) with \( \Delta_j \Delta_k^v Y_t \) gives
\[
\frac{1}{2} \frac{d}{dt} \left( \left\| \Delta_j \Delta_k^v Y_t \right\|_{L^2}^2 + \left\| \partial_3 \Delta_j \Delta_k^v Y_t \right\|_{L^2}^2 \right) + \| \nabla \Delta_j \Delta_k^v Y_t \|_{L^2}^2 = \left( \Delta_j \Delta_k^v f \mid \Delta_j \Delta_k^v Y_t \right).
\]
While taking the \( L^2 \) inner product of (4.2) with \( \Delta \Delta_j \Delta_k^v Y \) leads to
\[
\left( \Delta_j \Delta_k^v Y_t \mid \Delta \Delta_j \Delta_k^v Y \right) - \frac{1}{2} \frac{d}{dt} \left( \left\| \Delta_j \Delta_k^v Y_t \right\|_{L^2}^2 - \left\| \partial_3 \nabla \Delta_j \Delta_k^v Y_t \right\|_{L^2}^2 \right) = \left( \Delta_j \Delta_k^v f \mid \Delta \Delta_j \Delta_k^v Y \right).
\]
Notice that
\[
\left( \Delta_j \Delta_k^v Y_t \mid \Delta \Delta_j \Delta_k^v Y \right) = \frac{d}{dt} \left( \Delta_j \Delta_k^v Y_t \mid \Delta \Delta_j \Delta_k^v Y \right) - \left( \Delta_j \Delta_k^v Y_t \mid \Delta \Delta_j \Delta_k^v Y \right),
\]
so that there holds
\[
\frac{d}{dt} \left( \frac{1}{2} \left\| \Delta \Delta_j \Delta_k^v Y \right\|_{L^2}^2 - \left( \Delta_j \Delta_k^v Y_t \mid \Delta \Delta_j \Delta_k^v Y \right) \right) - \| \nabla \Delta_j \Delta_k^v Y_t \|_{L^2}^2 = \left( \Delta_j \Delta_k^v f \mid \Delta \Delta_j \Delta_k^v Y \right).
\]
(4.3) + \frac{1}{4}(4.4) gives rise to

\[ \frac{d}{dt} g_{j,k}^2(t) + \frac{3}{4} \| \nabla \Delta_j \Delta_k^u Y_t \|_{L^2}^2 + \frac{1}{4} \| \partial_t \nabla \Delta_j \Delta_k^u Y \|_{L^2}^2 \]

\[ = (\Delta_j \Delta_k^u f \mid \Delta_j \Delta_k^u Y_t - \frac{1}{4} \Delta \Delta_j \Delta_k^u Y), \]

where

\[ g_{j,k}^2(t) \overset{\text{def}}{=} \frac{1}{2} \left( \| \Delta_j \Delta_k^u Y_t(t) \|_{L^2}^2 + \| \partial_j \Delta_j \Delta_k^u Y(t) \|_{L^2}^2 + \frac{1}{4} \| \Delta \Delta_j \Delta_k^u Y(t) \|_{L^2}^2 - \frac{1}{4}(\Delta_j \Delta_k^u Y_t(t) | \Delta_j \Delta_k^u Y(t)). \right) \]

It is easy to observe that

\[ g_{j,k}^2(t) \sim \| \Delta_j \Delta_k^u Y_t(t) \|_{L^2}^2 + \| \partial_j \Delta_j \Delta_k^u Y(t) \|_{L^2}^2 + \| \Delta \Delta_j \Delta_k^u Y(t) \|_{L^2}^2. \]

With (4.5), (4.6), according to the heuristic discussions in Subsection 3.1 and similar to that in [21, 22], we shall separate the analysis of (4.5) into two cases: one is when \( j \leq \frac{k + 1}{2} \), and the other one is when \( j > \frac{k + 1}{2} \).

**Case (1):** \( j \leq \frac{k + 1}{2} \). In this case, we infer from Lemma 3.1 and (4.6) that

\[ g_{j,k}^2(t) \sim \| \Delta_j \Delta_k^u Y_t(t) \|_{L^2}^2 + \| \partial_j \Delta_j \Delta_k^u Y(t) \|_{L^2}^2, \]

and

\[ \| \nabla \Delta_j \Delta_k^u Y_t(t) \|_{L^2}^2 + \| \partial \nabla \Delta_j \Delta_k^u Y(t) \|_{L^2}^2 \]

\[ \geq c 2^j \left( \| \Delta_j \Delta_k^u Y_t(t) \|_{L^2}^2 + \| \partial_j \Delta_j \Delta_k^u Y(t) \|_{L^2}^2 \right) \geq c 2^j g_{j,k}^2(t), \]

from which, for any \( \epsilon > 0 \), dividing (4.5) by \( g_{j,k}(t) + \epsilon \), then taking \( \epsilon \rightarrow 0 \) and integrating the resulting equation over \([0, T]\), we obtain

\[ \| \Delta_j \Delta_k^u Y_t \|_{L^\infty_t(L^2)} + \| \partial_j \Delta_j \Delta_k^u Y \|_{L^\infty_t(L^2)} + \| \Delta \Delta_j \Delta_k^u Y \|_{L^\infty_t(L^2)} \]

\[ + c 2^j \left( \| \Delta_j \Delta_k^u Y_t \|_{L^1_t(L^2)} + \| \partial_j \Delta_j \Delta_k^u Y \|_{L^1_t(L^2)} \right) \leq \| \Delta_j \Delta_k^u Y_1 \|_{L^2} + \| \partial_j \Delta_j \Delta_k^u Y_0 \|_{L^2} + \| \Delta_j \Delta_k^u f \|_{L^1_t(L^2)}. \]

**Case (2):** \( j > \frac{k + 1}{2} \). Notice from Lemma 3.1 that in this case, one has

\[ g_{j,k}^2(t) \sim \| \Delta_j \Delta_k^u Y_t(t) \|_{L^2}^2 + \| \Delta \Delta_j \Delta_k^u Y(t) \|_{L^2}^2, \]

and

\[ \| \nabla \Delta_j \Delta_k^u Y_t(t) \|_{L^2}^2 + \| \partial \nabla \Delta_j \Delta_k^u Y(t) \|_{L^2}^2 \]

\[ \geq c 2^k \left( \| \Delta_j \Delta_k^u Y_t(t) \|_{L^2}^2 + \| \Delta \Delta_j \Delta_k^u Y(t) \|_{L^2}^2 \right) \geq c 2^k g_{j,k}^2(t), \]

from which and (4.5), we deduce by a similar derivation of (4.7) that

\[ \| \Delta_j \Delta_k^u Y_t \|_{L^\infty_t(L^2)} + \| \partial_j \Delta_j \Delta_k^u Y \|_{L^\infty_t(L^2)} + \| \Delta \Delta_j \Delta_k^u Y \|_{L^\infty_t(L^2)} \]

\[ + c 2^k \left( \| \Delta_j \Delta_k^u Y_t \|_{L^1_t(L^2)} + \| \Delta \Delta_j \Delta_k^u Y \|_{L^1_t(L^2)} \right) \leq \| \Delta_j \Delta_k^u Y_1 \|_{L^2} + \| \Delta \Delta_j \Delta_k^u Y_0 \|_{L^2} + \| \Delta_j \Delta_k^u f \|_{L^1_t(L^2)}. \]

On the other hand, standard energy estimate applied to (4.2) yields that

\[ \frac{1}{2} \frac{d}{dt} \| \Delta_j \Delta_k^u Y_t(t) \|_{L^2}^2 + \| \nabla \Delta_j \Delta_k^u Y(t) \|_{L^2}^2 = \left( \partial_j \Delta_j \Delta_k^u Y + \Delta_j \Delta_k^u f \mid \Delta_j \Delta_k^u Y_t \right), \]
from which, Lemma 3.1 and (4.8), we infer
\[ \| \Delta_j \Delta_k^w Y \|_{L^2_t(L^2)} + c 2^j \| \Delta_j \Delta_k^w Y \|_{L^2_t(L^2)} \]
\[ \leq \| \Delta_j \Delta_k^w Y \|_{L^2_t(L^2)} + C(2^{2k} \| \Delta_j \Delta_k^w Y \|_{L^2_t(L^2)} + \| \Delta_j \Delta_k^w f \|_{L^2_t(L^2)}) \]
\[ \lesssim \| \Delta_j \Delta_k^w Y \|_{L^2_t(L^2)} + \| \Delta_j \Delta_k^w Y \|_{L^2_t(L^2)} + \| \Delta_j \Delta_k^w f \|_{L^2_t(L^2)} \text{ for } j > \frac{k + 1}{2}. \]

Therefore according to Definitions 3.2 and 3.3, we get, by summing up (4.7), (4.8) and (4.9), that
\[ \| Y_t \|_{L^\infty_t(B^{s,0})} + \| \partial_3 Y \|_{L^\infty_t(B^{s,0})} + \| Y \|_{L^\infty_t(B^{s+2,0})} + \| Y_t \|_{L^2_t(B^{s+2,0})} \]
\[ \lesssim \| Y \|_{B^{s,0}} + \| \partial_3 Y \|_{B^{s,0}} + \| Y \|_{B^{s+2,0}} + \| f \|_{L^1_t(B^{s,0})}. \]

On the other hand, it follows from (4.5) that
\[ \| \partial_3 \Delta_j \Delta_k^w Y \|_{L^2_t(L^2)} \lesssim \| \Delta_j \Delta_k^w f \|_{L^2_t(L^2)} + \| \Delta_j \Delta_k^w Y \|_{L^2_t(L^2)} + \| \Delta \Delta_j \Delta_k^w Y \|_{L^\infty_t(L^2)}, \]
so that
\[ \| \partial_3 Y \|_{L^1_t(B^{s+1,0})} \lesssim \| f \|_{L^1_t(B^{s,0})} + \| Y \|_{L^\infty_t(B^{s,0})} + \| Y \|_{L^\infty_t(B^{s+2,0})}, \]
which together with (4.10) concludes the proof of (4.1). □

4.2. $L^1_t(B^{s,0})$ estimate of $f(Y, q)$ given by (2.22).

**Proposition 4.2.** Let $(Y, q)$ be a smooth enough solution of (2.20) (or equivalently of (2.21)-(2.22)) on $[0, T]$ with the initial data $(Y_0, Y_1)$ satisfying $\det(I + \nabla Y_0) = 1$ and $\nabla Y_0 \cdot Y_1 = 0$. If we assume moreover that
\[ \| \nabla Y \|_{L^\infty_t(B^{\frac{d}{2}})} \leq c_0 \text{ and } \langle \| Y \|_{L^\infty_t(B^{s+\frac{d}{2}})} \rangle \leq 1 \]
for some $c_0$ sufficiently small. Then there holds
\[ \| \nabla Y \|_{L^1_t(B^{s,0})} \lesssim \| Y_t \|_{L^2_t(B^{\frac{d}{2}})} \| Y_t \|_{L^2_t(B^{s+\frac{d}{2}})} + \| \partial_3 Y \|_{L^2_t(B^{\frac{d}{2}})} \| \partial_3 Y \|_{L^2_t(B^{s+\frac{d}{2}})} \]
if $0 < s \leq 1$, and
\[ \| \nabla Y \|_{L^1_t(B^{s,0})} \lesssim \| Y_t \|_{L^2_t(B^{\frac{d}{2}})} + \| Y_t \|_{L^2_t(B^{s+\frac{d}{2}})} + \| \partial_3 Y \|_{L^2_t(B^{\frac{d}{2}})} \| \partial_3 Y \|_{L^2_t(B^{s+\frac{d}{2}})} \]
\[ + \| \partial_3 Y \|_{L^2_t(B^{s+\frac{d}{2}})} \| \partial_3 Y \|_{L^2_t(B^{s+\frac{d}{2}})} \| \partial_3 Y \|_{L^2_t(B^{s+\frac{d}{2}})} \text{ if } s > 1. \]
Here and in all that follows, $A_s = A_s + (C_s) s > s_0$ means $A_s = A_s$ if $s \leq s_0$ and $A_s = A_s + C_s$ if $s > s_0$.

**Proof.** By virtue of (2.20), we get, by taking $\partial_t$ to $\nabla Y \cdot Y_t = 0$, that
\[ \nabla Y \cdot Y_t = -\partial_t A_{\frac{d}{2}} \nabla Y_t. \]
Note that for $c_0$ in (4.11) being so small that
\[ \| \nabla Y \|_{L^\infty_t(L^\infty)} \leq C \| \nabla Y \|_{L^\infty_t(B^{\frac{d}{2}})} \leq C c_0 \leq \frac{1}{2}, \]

$X(t, y)$ determined by (2.15) has a smooth inverse map $X^{-1}(t, x)$ with $X(t, X^{-1}(t, x)) = x$ and $X^{-1}(t, X(t, y)) = y$. Moreover, as $\det(I + \nabla Y_0) = 1$, we deduce from (2.23) that
\[ \det(I + \nabla Y) = 1, \]
which together with $\nabla Y \cdot Y_t = 0$ ensures that
\[ \nabla Y \cdot (\nabla Y \cdot \nabla Y_t) = [\nabla_x \cdot \Delta_x (Y_t \circ X^{-1}(t, x))] \circ X(t, y) = \nabla Y \cdot \nabla Y (\nabla Y \cdot Y_t) = 0, \]
from which and (4.14), we get, by taking $\nabla_Y \cdot$ to the first equation of (2.20), that
$$\nabla_Y \cdot \nabla_Y q = \partial_t A_Y^T \nabla \cdot Y_t + \nabla_Y \cdot \partial_3^2 Y,$$
or equivalently
$$\begin{aligned}
\Delta q &= -((\nabla_Y - \nabla) \cdot \nabla_Y q - \nabla \cdot (\nabla_Y - \nabla) q + \partial_t A_Y^T \nabla \cdot Y_t + \nabla_Y \cdot \partial_3^2 Y) \\
&= -\nabla \cdot ((A_Y - I) A_Y^T \nabla q) - \nabla \cdot ((A_Y^T - I) \nabla q) + \nabla \cdot (\partial_t A_Y Y_t) + \nabla_Y \cdot \partial_3^2 Y,
\end{aligned}$$
from which, Lemma 3.1 and Definition 3.2, we infer
$$\begin{aligned}
\|\nabla q\|_{L^1_T(B^{s,0})} &\leq \|(A_Y - I) A_Y^T \nabla q\|_{L^1_T(B^{s,0})} + \|(A_Y^T - I) \nabla q\|_{L^1_T(B^{s,0})} \\
&\quad + \|\partial_t A_Y Y_t\|_{L^1_T(B^{s,0})} + \|\nabla_Y \cdot \partial_3^2 Y\|_{L^1_T(B^{s,0})}.
\end{aligned}$$
(4.16)

Applying (3.10) and (2.17) gives for $0 < s \leq 1$
$$\begin{aligned}
\|(A_Y^T - I) \nabla q\|_{L^1_T(B^{s,0})} + \|(A_Y - I) A_Y^T \nabla q\|_{L^1_T(B^{s,0})} \\
&\lesssim \left( \|A_Y^T - I\|_{L^\infty_T(B^{\frac{s}{2}})} + \|(A_Y - I) A_Y^T\|_{L^\infty_T(B^{\frac{s}{2}})} \right) \|\nabla q\|_{L^1_T(B^{s,0})} \\
&\lesssim \left( 1 + \|\nabla Y\|_{L^\infty_T(B^{\frac{s}{2}})} \right)^3 \|\nabla q\|_{L^1_T(B^{s,0})} \\
&\quad + \|\partial_t A_Y Y_t\|_{L^2_T(B^{s,0})} + \|\nabla_Y \cdot \partial_3^2 Y\|_{L^1_T(B^{s,0})},
\end{aligned}$$
and
$$\begin{aligned}
\|\partial_t A_Y Y_t\|_{L^2_T(B^{s,0})} &\lesssim \|\partial_t A_Y\|_{L^2_T(B^{s,0})} \|Y_t\|_{L^2_T(B^{s,0})} \\
&\lesssim \left( 1 + \|\nabla Y\|_{L^\infty_T(B^{\frac{s}{2}})} \right) \|\partial_t A_Y\|_{L^2_T(B^{s,0})} \|Y_t\|_{L^2_T(B^{s,0})}.
\end{aligned}$$

While thanks to (2.17), (2.21) and (2.22), a tedious yet interesting calculation shows that
$$\begin{aligned}
\nabla_Y \cdot \partial_3^2 Y &= \nabla \cdot ((A_Y - I) \partial_3^2 Y) + \partial_3^2 \rho(Y) = Q(\nabla \partial_3 Y, \nabla \partial_3 Y, \nabla Y),
\end{aligned}$$
where $Q(\nabla \partial_3 Y, \nabla \partial_3 Y, \nabla Y)$ is a linear combination of quadratic terms like $\partial_3 \partial_t Y^i \partial_3 \partial_t Y^l$ and cubic terms like $\partial_3 Y^q \partial_3 \partial_t Y^l \partial_3 \partial_t Y^l$. Then applying (3.10) to (4.17) ensures that for $0 < s \leq 2$
$$\begin{aligned}
\|\nabla Y \cdot \partial_3^2 Y\|_{L^1_T(B^{s-1,0})} &\lesssim \|(I + \nabla Y) \partial_3 \nabla Y\|_{L^2_T(B^{\frac{s}{2}})} \|\partial_3 \nabla Y\|_{L^2_T(B^{s,0})} \\
&\lesssim \left( 1 + \|\nabla Y\|_{L^\infty_T(B^{\frac{s}{2}})} \right) \|\partial_3 \nabla Y\|_{L^2_T(B^{s,0})} \|\partial_3 Y\|_{L^2_T(B^{s,0})},
\end{aligned}$$
(4.18)

Thus for $0 < s \leq 1$, resuming the above estimates into (4.16) gives rise to
$$\begin{aligned}
\|\nabla q\|_{L^1_T(B^{s,0})} &\lesssim \left( 1 + \|\nabla Y\|_{L^\infty_T(B^{\frac{s}{2}})} \right)^3 \|\nabla q\|_{L^1_T(B^{s,0})} \\
&\quad + \|Y_t\|_{L^2_T(B^{\frac{s}{2}})} \|Y_t\|_{L^2_T(B^{s,0})} + \|\partial_3 Y\|_{L^2_T(B^{s,0})} \|\partial_3 Y\|_{L^2_T(B^{s,0})},
\end{aligned}$$
which along with (4.11) implies (4.12).

On the other hand, we get, by applying (3.11) and (3.12), that for $s > 1$
$$\begin{aligned}
\|(A_Y^T - I) \nabla q\|_{L^1_T(B^{s,0})} + \|(A_Y - I) A_Y^T \nabla q\|_{L^1_T(B^{s,0})} \\
&\lesssim \left( \|A_Y^T - I\|_{L^\infty_T(B^{\frac{s}{2}})} + \|(A_Y - I) A_Y^T\|_{L^\infty_T(B^{\frac{s}{2}})} \right) \|\nabla q\|_{L^1_T(B^{s,0})} \\
&\quad + \left( \|A_Y^T - I\|_{L^\infty_T(B^{s+\frac{1}{2}})} + \|(A_Y - I) A_Y^T\|_{L^\infty_T(B^{s+\frac{1}{2}})} \right) \|\nabla q\|_{L^1_T(B^{1,0})} \\
&\lesssim \left( 1 + \|\nabla Y\|_{L^\infty_T(B^{\frac{s}{2}})} \right)^3 \|\nabla q\|_{L^1_T(B^{s,0})} + \|\nabla Y\|_{L^\infty_T(B^{s+\frac{1}{2}})} \|\nabla q\|_{L^1_T(B^{1,0})},
\end{aligned}$$
(4.19)
and

$$
\| \partial_t A Y_t \|_{L^2_t(B^{s,0})} \lesssim \| \partial_t A Y \|_{L^2_t(B^{\frac{s}{4}})} \| Y_t \|_{L^2_t(B^{s + \frac{1}{2}})} + \| \partial_t A Y \|_{L^2_t(B^{s + \frac{1}{2}})} \| Y_t \|_{L^2_t(B^{s + \frac{1}{2}})}
$$

(4.20)

Moreover, applying (3.12) to (4.17) leads to

$$
\| \nabla Y \cdot \partial_3^2 Y \|_{L^1_t(B^{s-1,0})} \lesssim \left( 1 + \| \nabla Y \|_{L^\infty_t(B^{\frac{5}{2}})} + \| Y \|_{L^\infty_t(B^{s + \frac{1}{2}})} \right) \left( \| \nabla Y_t \|_{L^2_t(B^{s + \frac{1}{2}})} \| Y_t \|_{L^2_t(B^{s + \frac{1}{2}})} + \| \nabla Y_t \|_{L^2_t(B^{s + \frac{1}{2}})} \| Y_t \|_{L^2_t(B^{s + \frac{1}{2}})} \right).
$$

(4.21)

hence, by applying Bony's decomposition (3.8), one has

$$
\| (I + \nabla Y) \partial_3^2 Y \|_{L^1_t(B^{s-1,0})} \lesssim \left( 1 + \| \nabla Y \|_{L^\infty_t(B^{\frac{5}{2}})} + \| Y \|_{L^\infty_t(B^{s + \frac{1}{2}})} \right) \left( \| \partial_3 Y \|_{L^2_t(B^{s + \frac{1}{2}})} + \| \partial_3 Y \|_{L^2_t(B^{s + \frac{1}{2}})} + \| \partial_3 Y \|_{L^2_t(B^{s + \frac{1}{2}})} \right),
$$

(4.22)

Whence plugging (4.19), (4.20), (4.18) and (4.21) into (4.16), we obtain

$$
\| \nabla q \|_{L^1_t(B^{s,0})} \lesssim \left( 1 + \| \nabla Y \|_{L^\infty_t(B^{\frac{5}{2}})} + \| Y \|_{L^\infty_t(B^{s + \frac{1}{2}})} \right) \left( \| \nabla Y_t \|_{L^2_t(B^{s + \frac{1}{2}})} + \| Y_t \|_{L^2_t(B^{s + \frac{1}{2}})} + \| \partial_3 Y \|_{L^2_t(B^{s + \frac{1}{2}})} \right) \left( \| \partial_3 Y \|_{L^2_t(B^{s + \frac{1}{2}})} + \| \partial_3 Y \|_{L^2_t(B^{s + \frac{1}{2}})} + \| \partial_3 Y \|_{L^2_t(B^{s + \frac{1}{2}})} \right)
$$

for $s > 1$,

which together with (4.11) and (4.12) implies (4.13). This completes the proof of Proposition 4.2.

**Corollary 4.1.** Under the assumption of Proposition 4.2, one has

$$
\| \nabla q \|_{L^1_t(B^{s,0})} \lesssim \| Y_t \|_{L^2_t(B^{s + \frac{1}{2}})}^2 + \| \partial_3 Y \|_{L^2_t(B^{s + \frac{1}{2}})}.
$$

(4.23)

**Proof.** We first deduce from (4.15) that

$$
\| \nabla q \|_{L^1_t(B^{s,0})} \lesssim \| (A_Y - I) A_Y^T \nabla q \|_{L^1_t(B^{s,0})} + \| (A_Y^T - I) \nabla q \|_{L^1_t(B^{s,0})}
$$

(4.24)

It follows from product laws in Besov space (see [2] for instance) that

$$
\| (A_Y - I) A_Y^T \nabla q \|_{L^1_t(B^{s,0})} + \| (A_Y^T - I) \nabla q \|_{L^1_t(B^{s,0})}
$$

$$
\lesssim \left( 1 + \| \nabla Y \|_{L^\infty_t(B^{s,0})} \right)^3 \| \nabla Y \|_{L^\infty_t(B^{s,0})} \| \nabla q \|_{L^1_t(B^{s,0})}.
$$

(4.25)
Thanks to (2.22), we split

\[ \text{Proof.} \]

Resuming the above estimates into (4.23) leads to (4.22).

**Proposition 4.3.** Let \( s > 1 \), and \( f(Y, q) \) be given by (2.22). Then under the assumptions of Proposition 4.2 and

\[ ||Y||_{L_t^\infty(B^{s+2,0})} \leq 1, \]

one has

\[ \|f(Y, q)\|_{L_t^1(B^{s,0})} + \|f(Y, q)\|_{L_t^1(B^{s,0})} \lesssim \|\nabla q\|_{L_t^1(B^{s,0})} + \|\nabla Y\|_{L_t^\infty(B^{s,0})} \|Y\|_{L_t^1(B^{s,0})} \]

\[ + \|\nabla q\|_{L_t^1(B^{s,0})} + ||Y||_{L_t^1(B^{s,0})} \left( \|Y\|_{L_t^\infty(B^{s+2,0})} + \|\nabla Y\|_{L_t^\infty(B^{s,0})} \right). \]

**Proof.** Thanks to (2.22), we split \( f(Y, q) \) as follows:

\[ f(Y, q) = \tilde{f}(Y) + \hat{f}(Y, q), \quad \text{with} \]

\[ \tilde{f}(Y) \stackrel{\text{def}}{=} (\nabla_Y - \nabla_Y - \Delta)Y_t \quad \text{and} \quad \hat{f}(Y, q) \stackrel{\text{def}}{=} -\nabla_Y q. \]

As \( \tilde{f}(Y, q) = -(AT - I)\nabla q - \nabla q \), by virtue of (2.17), we deduce from (3.10) and (4.11) that

\[ \|\tilde{f}(Y, q)\|_{L_t^1(B^{s,0})} \lesssim (1 + \|\nabla Y\|_{L_t^\infty(B^{s+2,0})}) \|\nabla q\|_{L_t^1(B^{s,0})} \]

\[ \lesssim (1 + \|\nabla Y\|_{L_t^\infty(B^{s+2,0})}) \left( ||Y||_{L_t^\infty(B^{s+2,0})} + \|\nabla Y\|_{L_t^\infty(B^{s,0})} \right) + \|\nabla Y\|_{L_t^\infty(B^{1,0})} + \|\nabla q\|_{L_t^1(B^{s,0})} \]

\[ \lesssim \|\nabla q\|_{L_t^1(B^{s,0})} + \|\nabla Y\|_{L_t^1(B^{s,0})}, \]

and for \( s > 1 \), we infer from (3.11) and (4.11) that

\[ \|\hat{f}(Y, q)\|_{L_t^1(B^{s,0})} \lesssim \|AT - I\|_{L_t^\infty(B^{1,0})} \|\nabla q\|_{L_t^1(B^{s,0})} \]

\[ + \|AT - I\|_{L_t^\infty(B^{s+1,0})} \|\nabla Y\|_{L_t^1(B^{s,0})} + \|\nabla Y\|_{L_t^\infty(B^{s+1,0})} \|\nabla q\|_{L_t^1(B^{s,0})} \]

\[ \lesssim \|\nabla q\|_{L_t^1(B^{s,0})} + \|\nabla Y\|_{L_t^1(B^{s,0})}, \]

where in the last step, we used the trivial fact that

\[ \|a\|_{B^{s,0}} \lesssim \|a\|_{B^{s+1,0}} + \|a\|_{B^{s+1,0}} \quad \text{for any} \quad s \in [s_1, s_2]. \]

On the other hand, notice that

\[ \tilde{f}(Y) = (\nabla_Y - \nabla_Y) \cdot \nabla_Y Y_t + \nabla \cdot (\nabla_Y - \nabla_Y) Y_t \]

\[ = \nabla \cdot [(AT - I) A^T \nabla Y_t + (A^T - I) \nabla Y_t], \]

which leads to

\[ \|\hat{f}(Y)\|_{L_t^1(B^{s,0})} \lesssim \|(AT - I) A^T \nabla Y_t\|_{L_t^1(B^{s+1,0})} + \|(A^T - I) \nabla Y_t\|_{L_t^1(B^{s+1,0})}. \]
Whereas according to (2.17), we get, by applying (3.11), that
\[
\| (A_Y^T - I) \nabla Y_t \|_{L^1_t(B_s^{+1,0})} \\
\lesssim \| A_Y^T - I \|_{L^1_t(B_s^{+1,0})} \| \nabla Y_t \|_{L^1_t(B_s^{+1,0})} + \| A_Y^T - I \|_{L^1_t(B_s^{+1,0})} \| \nabla Y_t \|_{L^1_t(B_s^{+1,0})}
\lesssim (1 + \| \nabla Y \|_{L^1_t(B_s^{+1,0})}) \left( \| \nabla Y \|_{L^1_t(B_s^{+1,0})} \| Y_t \|_{L^1_t(B_s^{+1,2,0})} + \| Y_t \|_{L^1_t(B_s^{+1,0})} \| \nabla Y \|_{L^1_t(B_s^{+1,0})} \right).
\]
Along the same line, and thanks to (4.11) and (4.24), we have
\[
\| (A_Y - I) A_Y^T \nabla Y_t \|_{L^1_t(B_s^{+1,0})} \lesssim \| \nabla Y \|_{L^1_t(B_s^{+1,0})} \| Y_t \|_{L^1_t(B_s^{+1,2,0})} + \| Y_t \|_{L^1_t(B_s^{+1,0})} \| \nabla Y \|_{L^1_t(B_s^{+1,0})}.
\]
Resuming the above two estimates into (4.31) gives rise to
\[
\| \tilde{f}(Y) \|_{L^1_t(B_s^{+1,0})} \lesssim \| \nabla Y \|_{L^1_t(B_s^{+1,0})} \| Y_t \|_{L^1_t(B_s^{+1,2,0})} + \| Y \|_{L^1_t(B_s^{+1,0})} \| \nabla Y \|_{L^1_t(B_s^{+1,2,0})},
\]
which together with (4.27) and (4.28) ensures (4.25). This concludes the proof of Proposition 4.3. \( \square \)

4.3. \( L^1_t(B_s^{+2,0}(\mathbb{R}^3)) \) estimate of \( Y_t \) for \( s = \frac{1}{2} \) and \( s > 1 \).

**Proposition 4.4.** Let \( s > 1 \), then under assumptions of Proposition 4.2 and
\[
(4.32) \quad \| Y \|_{L^\infty_t(B_s^{+2,0})} \leq c_0,
\]
for some \( c_0 \) sufficiently small, we have
\[
\| Y_t \|_{L^\infty_t(B_s^{+2,0})} + \| \partial_3 Y \|_{L^\infty_t(B_s^{+2,0})} + \| Y \|_{L^\infty_t(B_s^{+2,0})}
+ \| Y_t \|_{L^1_t(B_s^{+2,0})} + \| \partial_3 Y \|_{L^1_t(B_s^{+2,0})} + \| \partial_3 Y \|_{L^2_t(B_s^{+2,0})}
\leq \| Y_t \|_{L^1_t(B_s^{+2,0})}^2 + \| Y_t \|_{L^1_t(B_s^{+2,0})}^2 + \| \partial_3 Y \|_{L^1_t(B_s^{+2,0})}^2 + \| \partial_3 Y \|_{L^2_t(B_s^{+2,0})}^2
+ \| \partial_3 Y \|_{L^2_t(B_s^{+2,0})}^2 + \| Y \|_{L^2_t(B_s^{+2,0})}.
\]

**Proof.** Thanks to Propositions 4.1 and 4.3, we conclude that for \( s > 1 \),
\[
\| Y_t \|_{L^\infty_t(B_s^{+2,0})} + \| \partial_3 Y \|_{L^\infty_t(B_s^{+2,0})} + \| Y \|_{L^\infty_t(B_s^{+2,0})}
+ \| Y_t \|_{L^1_t(B_s^{+2,0})} + \| \partial_3 Y \|_{L^1_t(B_s^{+2,0})} + \| \partial_3 Y \|_{L^2_t(B_s^{+2,0})}
\leq \| Y_t \|_{L^1_t(B_s^{+2,0})} + \| \partial_3 Y \|_{L^1_t(B_s^{+2,0})} + \| Y_t \|_{L^1_t(B_s^{+2,0})} + \| \nabla q \|_{L^1_t(B_s^{+2,0})}
+ \| \nabla q \|_{L^1_t(B_s^{+2,0})} + \| \nabla Y \|_{L^1_t(B_s^{+2,0})} + \| Y_t \|_{L^1_t(B_s^{+2,0})} + \| Y \|_{L^2_t(B_s^{+2,0})}.
\]
which together with (4.11), (4.32) and the fact that \( \|Y_t\|_{L^2_t(B^{1/2}_2 \cap B^{a+2}_0)} \lesssim \|Y_t\|_{L^2_t(B^{1/2}_2 \cap B^{a}_0)} \) ensures that

\[
\|Y_t\|_{L^2_t(B^{1/2}_2 \cap B^{a+2}_0)} + \|\partial_3 Y\|_{L^2_t(B^{1/2}_2 \cap B^{a}_0)} + \|Y\|_{L^2_t(B^{1/2}_2 \cap B^{a+2}_0)} + \|\partial_3 Y\|_{L^2_t(B^{1/2}_2 \cap B^{a+2}_0)} + \|\nabla q\|_{L^2_t(B^{1/2}_2 \cap B^{a}_0)} \lesssim \|Y_1\|_{B^{1/2}_2 \cap B^{a}_0} + \|\partial_3 Y_0\|_{B^{1/2}_2 \cap B^{a}_0} + \|Y_0\|_{B^{1/2}_2 \cap B^{a+2}_0} + \|\nabla q\|_{L^2_t(B^{1/2}_2 \cap B^{a}_0)}.
\]

Hence applying Proposition 4.2 gives rise to (4.33). This complete the proof of Proposition 4.4. \( \square \)

5. The proof of Theorem 2.2

5.1. A priori estimate of (2.20). The goal of this subsection is to present the a priori energy estimate to smooth enough solutions of (2.20).

Lemma 5.1. Let \( Y \) be a smooth enough solution of (2.21)-(2.22) (or equivalently (2.20)) on \([0, T]\). Then there holds

\[
\left\| \Delta_j Y_t \right\|_{L^2_t(B^{1/2}_2)}^2 + \left\| \nabla \Delta_j Y_t \right\|_{L^2_t(L^2)}^2 + \left\| \partial_3 \Delta_j Y \right\|_{L^2_t(L^2)}^2 + \left\| \Delta \Delta_j Y \right\|_{L^2_t(L^2)}^2
+ \left\| \nabla \Delta_j Y_t \right\|_{L^2_t(L^2)}^2 + \left\| \Delta \Delta_j Y_t \right\|_{L^2_t(L^2)}^2 + \left\| \partial_3 \nabla \Delta_j Y \right\|_{L^2_t(L^2)}^2
\lesssim \left\| \Delta_j Y_1 \right\|_{L^2}^2 + \left\| \nabla \Delta_j Y_1 \right\|_{L^2}^2 + \left\| \partial_3 \Delta_j Y_0 \right\|_{L^2}^2 + \left\| \Delta \Delta_j Y_0 \right\|_{L^2}^2
+ \left| \int_0^T (\Delta_j f | \Delta_j Y_t - \frac{1}{4} \Delta \Delta_j Y - \Delta \Delta_j Y_t) \, dt \right|.
\]

Proof. Applying \( \Delta_j \) to (2.21) gives

\[
\Delta_j Y_{tt} - \Delta \Delta_j Y_t - \frac{1}{4} \Delta \Delta_j Y = \Delta_j f.
\]

Taking the \( L^2 \) inner product of (5.2) with \( \Delta_j Y_t - \frac{1}{4} \Delta \Delta_j Y - \Delta \Delta_j Y_t \), we get, by a similar derivation of (4.5), that

\[
\frac{d}{dt} \left( \frac{1}{2} \left\| \Delta_j Y_t \right\|_{L^2}^2 + \left\| \nabla \Delta_j Y_t \right\|_{L^2}^2 + \left\| \partial_3 \Delta_j Y \right\|_{L^2}^2 + \left\| \partial_3 \nabla \Delta_j Y \right\|_{L^2}^2 + \left\| \Delta \Delta_j Y \right\|_{L^2}^2 \right)
- \frac{1}{4} (\Delta_j Y_t | \Delta \Delta j Y)
= \left( \Delta_j f | \Delta_j Y_t - \frac{1}{4} \Delta \Delta j Y - \Delta \Delta j Y_t \right).
\]

However, as

\[
\frac{1}{2} \left( \left\| \Delta_j Y_t \right\|_{L^2}^2 + \left\| \nabla \Delta_j Y_t \right\|_{L^2}^2 + \left\| \partial_3 \Delta j Y \right\|_{L^2}^2 + \left\| \partial_3 \nabla \Delta j Y \right\|_{L^2}^2 + \frac{1}{4} \left\| \Delta \Delta j Y \right\|_{L^2}^2 \right)
- \frac{1}{4} (\Delta_j Y_t | \Delta \Delta j Y)
\sim \left\| \Delta_j Y_t(t) \right\|_{L^2}^2 + \left\| \nabla \Delta_j Y_t \right\|_{L^2}^2 + \left\| \partial_3 \Delta j Y(t) \right\|_{L^2}^2 + \left\| \Delta \Delta j Y(t) \right\|_{L^2}^2,
\]

by integrating (5.3) over \([0, T]\), we obtain (5.1). \( \square \)

To deal with the last line of (5.1), we need to estimate the \( f(Y, q) \) given by (2.22). Toward this, we first deal with the the pressure term in (2.21).
Lemma 5.2. Under the assumptions of Proposition 4.2, for any $s > -\frac{1}{2}$, one has
\[
\|\nabla q(t)\|_{\dot{B}^{s}_{2,\infty}} \lesssim \|Y(t)\|_{\dot{H}^{s+1}} \|\nabla q(t)\|_{\dot{B}^{s}_{2,\infty}} + \|Y(t)\|_{\dot{B}^{s}_{2,\infty}} + \|\partial_{3}Y(t)\|_{\dot{B}^{s}_{2,\infty}} + \|\partial_{3}Y(t)\|_{\dot{B}^{s}_{2,\infty}} \|\partial_{3}Y(t)\|_{\dot{H}^{s+1}}
\]
for all $t \in [0, T]$. 

Proof. For any $t \in [0, T]$, we deduce from (4.15) that
\[
\|\nabla q(t)\|_{\dot{B}^{s}_{2,\infty}} \lesssim \|(\mathcal{A}_{Y} - I)A_{Y}^{T}\nabla q(t)\|_{\dot{H}^{s}} + \|(\mathcal{A}_{Y}^{T} - I)\nabla q(t)\|_{\dot{H}^{s}}
\]
\[
\quad + \|\partial_{3}A_{Y}Y(t)\|_{\dot{H}^{s}} + \|(\nabla \cdot \partial_{3}^{2}Y)(t)\|_{\dot{H}^{s-1}}.
\]
By virtue of (2.17), we get, by using Bony’s decomposition (3.8), that for any $s > -\frac{1}{2}$,
\[
\|(\mathcal{A}_{Y} - I)A_{Y}^{T}\nabla q(t)\|_{\dot{H}^{s}} + \|(\mathcal{A}_{Y}^{T} - I)\nabla q(t)\|_{\dot{H}^{s}}
\]
\[
\lesssim (1 + \|\nabla Y(t)\|_{\dot{B}^{s}_{2,\infty}})^{3}\left(\|\nabla Y(t)\|_{\dot{B}^{s}_{2,\infty}}\|\nabla q(t)\|_{\dot{H}^{s}} + \|\nabla Y(t)\|_{\dot{H}^{s+1}}\|\nabla q(t)\|_{\dot{B}^{s}_{2,\infty}}\right),
\]
and
\[
\|\partial_{3}A_{Y}Y(t)\|_{\dot{H}^{s}} \lesssim (1 + \|\nabla Y(t)\|_{\dot{B}^{s}_{2,\infty}})\|Y(t)\|_{\dot{H}^{s+1}} + \|\nabla Y(t)\|_{\dot{H}^{s+1}}\|\partial_{Y}(t)\|_{\dot{B}^{s}_{2,\infty}}^{2}.
\]
Along the same line, due to (4.17), we obtain for any $s > -\frac{1}{2}$,
\[
\|(\nabla \cdot \partial_{3}^{2}Y)(t)\|_{\dot{H}^{s-1}} \lesssim Q(\nabla \partial_{3}Y, \nabla \partial_{3}Y, \nabla Y)(t)\|_{\dot{H}^{s-1}}
\]
\[
\lesssim \|(I + \nabla Y)\partial_{3}\nabla Y(t)\|_{\dot{B}^{s}_{2,\infty}} + \|\partial_{3}\nabla Y(t)\|_{\dot{H}^{s+1}} + \|\partial_{3}\nabla Y(t)\|_{\dot{B}^{s}_{2,\infty}}
\]
\[
\lesssim (1 + \|\nabla Y(t)\|_{\dot{B}^{s}_{2,\infty}})\|\partial_{3}Y(t)\|_{\dot{B}^{s}_{2,\infty}} + \|\partial_{3}Y(t)\|_{\dot{H}^{s+1}}\|\partial_{3}Y(t)\|_{\dot{B}^{s}_{2,\infty}}^{2}.
\]
Resuming the above estimates into (5.5) and using (4.11) ensures that for any $s > -\frac{1}{2}$,
\[
\|\nabla q(t)\|_{\dot{H}^{s}} \lesssim \|\nabla Y(t)\|_{\dot{B}^{s}_{2,\infty}}\|\nabla q(t)\|_{\dot{H}^{s}} + \|\nabla Y(t)\|_{\dot{H}^{s+1}}\|\nabla q(t)\|_{\dot{B}^{s}_{2,\infty}} + \|\partial_{3}Y(t)\|_{\dot{B}^{s}_{2,\infty}}\|\partial_{3}Y(t)\|_{\dot{H}^{s+1}}
\]
\[
\quad + \|\partial_{3}Y(t)\|_{\dot{B}^{s}_{2,\infty}}\|\partial_{3}Y(t)\|_{\dot{H}^{s+1}} + \|\nabla Y(t)\|_{\dot{B}^{s}_{2,\infty}}\|\partial_{3}Y(t)\|_{\dot{H}^{s+1}} + \|\partial_{3}Y(t)\|_{\dot{B}^{s}_{2,\infty}}\|\partial_{3}Y(t)\|_{\dot{H}^{s+1}},
\]
for any $t \in [0, T]$, which together (4.11) leads to (5.4).

Lemma 5.3. Under the assumptions of Proposition 4.2, for any $s > -\frac{1}{2}$, we have
\[
\|f(Y, q)\|_{L_{t}^{3}(H^{s})} \lesssim \|Y\|_{L_{t}^{\infty}(H^{s+2})}\left(\|\nabla q\|_{L_{t}^{3}(\dot{B}^{s}_{2,\infty})} + \|Y\|_{L_{t}^{3}(\dot{B}^{s}_{2,\infty})} + \|\partial_{3}Y\|_{L_{t}^{3}(\dot{B}^{s}_{2,\infty})}\|\partial_{3}Y\|_{L_{t}^{3}(\dot{B}^{s}_{2,\infty})} \right)
\]
\[
\quad + \|\partial_{3}Y\|_{L_{t}^{2}(\dot{B}^{s}_{2,\infty})}\|\partial_{3}Y\|_{L_{t}^{3}(\dot{B}^{s}_{2,\infty})} + \|\partial_{3}Y\|_{L_{t}^{2}(\dot{B}^{s}_{2,\infty})}\|\partial_{3}Y\|_{L_{t}^{3}(\dot{H}^{s+1})} + \|\nabla Y\|_{L_{t}^{2}(\dot{B}^{s}_{2,\infty})}\|\partial_{3}Y\|_{L_{t}^{3}(\dot{H}^{s+1})} + \|\partial_{3}Y\|_{L_{t}^{2}(\dot{B}^{s}_{2,\infty})}\|\partial_{3}Y\|_{L_{t}^{3}(\dot{H}^{s+1})},
\]
and for $-\frac{1}{2} < s \leq \frac{1}{2}$,
\[
\|f(Y, q)\|_{L_{t}^{3}(H^{s})} \lesssim \|Y\|_{L_{t}^{\infty}(H^{s+2})}\left(\|\nabla q\|_{L_{t}^{3}(\dot{B}^{s}_{2,\infty})} + \|Y\|_{L_{t}^{3}(\dot{B}^{s}_{2,\infty})} + \|\partial_{3}Y\|_{L_{t}^{3}(\dot{B}^{s}_{2,\infty})}\right)
\]
\[
\quad + \|\partial_{3}Y\|_{L_{t}^{2}(\dot{B}^{s}_{2,\infty})}\|\partial_{3}Y\|_{L_{t}^{3}(\dot{H}^{s+1})} + \|\partial_{3}Y\|_{L_{t}^{2}(\dot{B}^{s}_{2,\infty})}\|\partial_{3}Y\|_{L_{t}^{3}(\dot{H}^{s+1})} + \|\nabla Y\|_{L_{t}^{2}(\dot{B}^{s}_{2,\infty})}\|\partial_{3}Y\|_{L_{t}^{3}(\dot{H}^{s+1})}.
\]

Proof. According to (4.26), we split the estimate of $f(Y, q)$ into that of $\tilde{f}(Y, q)$ and $\tilde{f}(Y)$.

- **Estimates on $\tilde{f}(Y, q) = -\nabla Y q$.**
Thanks to (2.17), we get, by using product laws in Besov spaces ([2]), that $s > -\frac{1}{2}$,

\[
\|\tilde{f}(Y, q)(t)\|_{H^s} \lesssim (1 + \|\nabla Y(t)\|_{\dot{B}_{\infty}^{\frac{3}{2}}}^3) \left( \|\nabla Y(t)\|_{\dot{B}_{\infty}^{\frac{3}{2}}} \|\nabla q(t)\|_{\dot{B}_{\infty}^{\frac{3}{2}}} + \|\nabla Y(t)\|_{\dot{H}^{s+1}} \|\nabla q(t)\|_{\dot{B}_{\infty}^{\frac{3}{2}}} + \|\nabla q(t)\|_{\dot{H}^s} \right),
\]

which along with (4.11) implies that for any $s > -\frac{1}{2}$,

(5.8) \[ \|\tilde{f}(Y, q)(t)\|_{H^s} \lesssim \|\nabla q(t)\|_{H^s} + \|Y(t)\|_{\dot{H}^{s+2}} \|\nabla q(t)\|_{\dot{B}_{\infty}^{\frac{3}{2}}}. \]

- **Estimates on** $f(Y) = (\nabla_Y \cdot \nabla_Y - \Delta)Y_t$.

It follows from (4.30) that

\[
\|\tilde{f}(Y)(t)\|_{H^s} \lesssim \left(1 + \|\nabla Y(t)\|_{\dot{B}_{\infty}^{\frac{3}{2}}}ight)^3 \left( \|\nabla Y(t)\|_{\dot{B}_{\infty}^{\frac{3}{2}}} \|\nabla Y(t)\|_{\dot{H}^{s+1}} + \|\nabla Y(t)\|_{\dot{H}^{s+1}} \|\nabla Y(t)\|_{\dot{B}_{\infty}^{\frac{3}{2}}} \right),
\]

which along with (4.11) implies that for any $s > -\frac{5}{2}$,

(5.9) \[ \|\tilde{f}(Y)(t)\|_{L^2_t(\dot{H}^s)} \lesssim \|\nabla Y(t)\|_{L^\infty_t(\dot{B}_{\infty}^{\frac{3}{2}})} \|\nabla Y(t)\|_{L^\infty_t(\dot{H}^{s+1})} + \|\nabla Y(t)\|_{L^\infty_t(\dot{H}^{s+1})} \|\nabla Y(t)\|_{L^2_t(\dot{B}_{\infty}^{\frac{3}{2}})}. \]

On the other hand, we get by using Bony’s decomposition (3.8) that for $-\frac{5}{2} < s \leq \frac{1}{2}$,

\[
\|(A_Y^T - I)\nabla Y(t)\|_{L^1_t(\dot{H}^{s+1})} \lesssim \|A_Y^T - I\|_{L^\infty_t(\dot{H}^{s+1})} \|\nabla Y(t)\|_{L^1_t(\dot{B}_{\infty}^{\frac{3}{2}})} \lesssim (1 + \|\nabla Y(t)\|_{L^\infty_t(\dot{B}_{\infty}^{\frac{3}{2}})^3} \|\nabla Y(t)\|_{L^\infty_t(\dot{H}^{s+1})} \|\nabla Y(t)\|_{L^1_t(\dot{B}_{\infty}^{\frac{3}{2}})}.
\]

Together with (4.11), this gives

\[
\|(A_Y^T - I)\nabla Y(t)\|_{L^1_t(\dot{H}^{s+1})} \lesssim \|\nabla Y(t)\|_{L^\infty_t(\dot{H}^{s+1})} \|\nabla Y(t)\|_{L^1_t(\dot{B}_{\infty}^{\frac{3}{2}})}.
\]

The same estimate holds for term $\|(A_Y - I)A_Y^T \nabla Y(t)\|_{\dot{H}^{s+1}}$. We thus obtain for $-\frac{5}{2} < s \leq \frac{1}{2}$,

(5.10) \[ \|\tilde{f}(Y)(t)\|_{L^2_t(\dot{H}^s)} \lesssim \|\nabla Y(t)\|_{L^\infty_t(\dot{H}^{s+1})} \|\nabla Y(t)\|_{L^2_t(\dot{B}_{\infty}^{\frac{3}{2}})}.
\]

By summing up (5.8) and (5.9), we obtain for $s > -\frac{1}{2}$,

\[
\|f(Y, q)\|_{L^2_t(\dot{H}^s)} \lesssim \|\nabla q\|_{L^2_t(\dot{H}^s)} + \|Y\|_{L^\infty_t(\dot{H}^{s+2})} \left( \|\nabla q\|_{L^2_t(\dot{B}_{\infty}^{\frac{3}{2}})} + \|\nabla Y(t)\|_{L^\infty_t(\dot{H}^{s+1})} \|\nabla Y(t)\|_{L^2_t(\dot{B}_{\infty}^{\frac{3}{2}})} \right),
\]

which together with (5.4) yields (5.6).

On the other hand, by summing up (5.8) and (5.10), we get for $-\frac{1}{2} < s \leq \frac{1}{2}$

\[
\|f(Y, q)\|_{L^2_t(\dot{H}^s)} \lesssim \|\nabla q\|_{L^2_t(\dot{H}^s)} + \|Y\|_{L^\infty_t(\dot{H}^{s+2})} \left( \|\nabla q\|_{L^2_t(\dot{B}_{\infty}^{\frac{3}{2}})} + \|\nabla Y(t)\|_{L^\infty_t(\dot{H}^{s+1})} \right),
\]

from which and (5.4), we achieve (5.7). This concludes the proof of Lemma 5.3.  

\[\square\]
5.2. The proof of Theorem 2.2. The proof of Theorem 2.2 is based on the following proposition:

**Proposition 5.1.** Let \( s_1 > \frac{5}{4} \) and \( s_2 \in (-\frac{1}{2}, -\frac{1}{4}) \). Let \((Y, q)\) be a smooth enough solution of (2.21)-(2.22) on \([0, T]\). We denote
\[
E_{s_1, s_2}^{Y, q}(t) \overset{\text{def}}{=} E_0^{s_1} + E_0^{s_2} + \|Y_t\|^2_{L_T^2(B_{s_1}^s)} + \|\partial_3 Y\|^2_{L_T^2(B_{s_1}^s)}
\]
and applying Proposition 4.2 and Proposition 4.3 leads to
\[
E_0^{s_1, s_2}(t) \overset{\text{def}}{=} E_0^{s_1} + E_0^{s_2} + \|\partial_3 Y_0\|^2_{L_T^2(B_{s_1}^s)} + \|\partial_3 Y\|^2_{L_T^2(B_{s_1}^s)}
\]
where
\[
E_0^{s_1} \overset{\text{def}}{=} \|Y_t\|^2_{L_T^2(B_{s_1}^s)} + \|\partial_3 Y\|^2_{L_T^2(B_{s_1}^s)} + \|\partial_3 Y_0\|^2_{L_T^2(B_{s_1}^s)} + \|\partial_3 Y\|^2_{L_T^2(B_{s_1}^s)}
\]
and
\[
E_0^{s_2} \overset{\text{def}}{=} \|Y_t\|^2_{L_T^2(B_{s_1}^s)} + \|\partial_3 Y_0\|^2_{L_T^2(B_{s_1}^s)} + \|\partial_3 Y\|^2_{L_T^2(B_{s_1}^s)}
\]

We assume that
\[
\|\nabla Y\|_{L_T^\infty(B_{s_1}^s)} + \|Y\|_{L_T^\infty(B_{s_1}^s)} \leq c_0 \quad \text{and} \quad \|Y\|_{L_T^\infty(B_{s_1}^s)} \leq 1,
\]
for some \( c_0 \) sufficiently small, then there holds
\[
E_{s_1, s_2}^{Y, q}(t) \leq C_1 E_0^{s_1, s_2} + \left( C_1 E_0^{s_1, s_2}(t) + E_0^{s_1, s_2}(t) \right)^2 E_{s_1, s_2}^{Y, q}(t),
\]
for some uniform constant \( C_1 \).

**Proof.** Under the assumptions (5.12), for \( s = s_1 \) and \( s = s_2 \), we deduce from Lemmas 5.1 and 5.2 that
\[
E_0^{s_1, s_2}(t) \leq E_0^s + \|Y_t\|^2_{L_T^2(B_{s_1}^s)} (\|\partial_3 Y\|^2_{L_T^2(B_{s_1}^s)} + \|\nabla q\|^2_{L_T^2(B_{s_1}^s)})
\]
and
\[
\|\nabla Y\|^2_{L_T^2(B_{s_1}^s)} \leq (\|\partial_3 Y\|^2_{L_T^2(B_{s_1}^s)} + \|\partial_3 Y_t\|^2_{L_T^2(B_{s_1}^s)}) (1 + \|Y_t\|^2_{L_T^2(B_{s_1}^s)})
\]
and applying Proposition 4.2 and Proposition 4.3 leads to
\[
\|\nabla Y\|_{L_T^2(B_{s_1}^s)} \leq \|\nabla Y\|_{L_T^2(B_{s_1}^s)} \leq \|\nabla Y\|_{L_T^2(B_{s_1}^s)} \leq \|\nabla Y\|_{L_T^2(B_{s_1}^s)}
\]
where we used the fact that $\|Y_t\|_{L^r_T(\mathcal{B}^{s}_2)} \lesssim \|Y_t\|_{L^r_T(\mathcal{B}^{s}_2 \mathcal{H}^s)}$. While as $s_1 > \frac{5}{4}$ and $s_2 \in (-\frac{1}{2}, -\frac{1}{4})$, one has

\[
\|Y_t\|_{L^r_T(\mathcal{B}^{s}_2)} + \|\partial_3 Y_t\|_{L^r_T(\mathcal{B}^{s}_2)} + \|\partial_3 Y_t\|_{L^r_T(\mathcal{B}^{s+1}_2)} \lesssim \|\partial_3 Y_t\|_{L^r_T(\mathcal{B}^{s+1}_2)} + \|\partial_3 Y_t\|_{L^r_T(\mathcal{B}^{s}_2)}.
\]

Along the same line, we deduce from Corollary 4.1 and its proof that

\[
\|\nabla q\|_{L^r_T(\mathcal{B}^{s}_2)} + \|\nabla q\|_{L^r_T(\mathcal{B}^{s+1}_2)} \lesssim \|\nabla Y_t\|_{L^r_T(\mathcal{B}^{s+1}_2)} + \|\nabla Y_t\|_{L^r_T(\mathcal{B}^{s}_2)}.
\]

As a consequence, we obtain

\[
\|\mathbf{f}\|_{L^r_T(\mathcal{H}^{s+1})} + \|\mathbf{f}\|_{L^r_T(\mathcal{B}^{s}_2)} \lesssim \left( E^{s_1,s_2}_T(Y,q) + E^{s_1,s_2}_T(Y,q) \right)^{\frac{1}{2}} + E^{s_1,s_2}_T(Y,q).
\]

Resuming the above estimates into (5.14) yields

\[
E^{s_1}_T(Y,q) \lesssim E^0 + \left( E^{s_1,s_2}_T(Y,q) \right)^{\frac{1}{2}} + E^{s_1,s_2}_T(Y,q) + E^{s_1,s_2}_T(Y,q) \right)^{\frac{1}{2}}.
\]

On the other hand, it follows from Lemma 5.3 and the fact that $\|Y_t\|_{L^r_T(\mathcal{B}^{s}_2)} \lesssim \|Y_t\|_{L^r_T(\mathcal{B}^{s}_2 \mathcal{H}^s)}$ that

\[
\|\mathbf{f}\|_{L^r_T(\mathcal{H}^{s+2})} + \|\mathbf{f}\|_{L^r_T(\mathcal{H}^{s+2})} \lesssim \left( E^{s_1,s_2}_T(Y,q) \right)^{\frac{1}{2}} + E^{s_1,s_2}_T(Y,q) \right)^{\frac{1}{2}}.
\]

from which, we infer, by a similar derivation of (5.15), that

\[
E^{s_2}_T(Y,q) \lesssim E^0 + \left( E^{s_1,s_2}_T(Y,q) \right)^{\frac{1}{2}} + E^{s_1,s_2}_T(Y,q) + E^{s_1,s_2}_T(Y,q) \right)^{\frac{1}{2}}.
\]

Therefore, by summing up (5.15), (5.16) and (4.33) with $s = s_1$, we achieve (5.13). This completes the proof of Proposition 5.1.

Now we are in a position to complete the proof of Theorem 2.2.

Proof of Theorem 2.2. Given initial data $(Y_0, Y_1)$ satisfying the assumptions listed in Theorem 2.2, we deduce by Proposition 5.1 and a standard argument that (2.21)-(2.22) has a unique solution $(Y, q)$ on $[0, T]$, which satisfies (2.26) on $[0, T]$. Let $T^*$ be the largest possible time so that (2.26) holds. Then to complete the proof of Theorem 2.2, we only need to show that $T^* = \infty$ and there holds (2.27) under the assumptions of (2.24) and (2.25). Otherwise, if $T^* < \infty$, we denote

\[
T^* \stackrel{\text{def}}{=} \max\{ T < T^*: E^{s_1,s_2}_T(Y,q) \leq \eta_0^2 \},
\]

for $\eta_0$ so small that $C(\eta_0 + \eta_0^2 + \eta_0^3) \leq \frac{1}{2}$, and

\[
\|\nabla Y\|_{L^r_T(\mathcal{B}^{s+2}_2)} + \|\nabla Y\|_{L^r_T(\mathcal{B}^{s+2}_2)} \leq C_2 E^{s_1,s_2}_T(Y,q)^{\frac{1}{2}} \leq C_2 \eta_0 \leq 1,
\]

for some $C_2$ as in (5.13).

We shall prove that $T = \infty$ provided that $\varepsilon_0$ is sufficiently small in (2.25). In fact, thanks to (5.18), we get by applying Proposition 5.1 that

\[
E^{s_1,s_2}_T(Y,q) \leq 2C_1 E^0^{s_1,s_2}.
\]
In particular, if we take \( \varepsilon_0 \) so small that \( 2C_1\varepsilon_0^2 \leq \frac{1}{2} \eta_0^2 \), (5.19) contradicts with (5.17) if \( \bar{T} < \infty \). This in turn shows that \( \bar{T} = T^* = \infty \), and there holds (2.27). This completes the proof of Theorem 2.2.

6. THE PROOF OF THEOREM 2.1 AND THEOREM 1.1

With Theorem 2.2 and Lemma A.1 in the Appendix A in hand, we can now present the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Under the assumptions of Theorem 2.1, we get, by applying Lemma 2.1, that there exists a vector-valued function \( Y_0(y) = (Y_0^1(y), Y_0^2(y), Y_0^3(y))^T \) so that

\[
X_0(y) = I + Y_0(y) \quad \text{and} \quad U_0 \circ X_0(y) = \nabla_y X_0(y) = I + \nabla_y Y_0(y),
\]

which in particular implies

\[
\frac{\partial X_0^{-1}(x)}{\partial x} = (I + \nabla_y Y_0)^{-1} \circ X_0^{-1}(x) = U_0^{-1}(x) = I - \nabla_x \Psi.
\]

Let \( Y_1(y) \equiv u_0(X_0(y)) \). Applying Lemma A.1 with \( \Phi(x) = X_0^{-1}(x) \) gives

\[
\|Y_1\|_{H^{s+1}} + \|Y_1\|_{H^{s+2}} \leq C\left(\|\nabla \Psi\|_{B^{s+1}}\right)\left(\|u_0\|_{H^{s+2}} + \|u_0\|_{H^{s+1}} + \|\nabla \Psi\|_{H^{s+1}} + \frac{1}{2}\|u_0\|_{H^2}
\]

\[
+ (1 + \|\Delta \Psi\|_{H^{s+1}})\|\nabla u_0\|_{H^{s+1}}\right)
\]

\[
\leq C\left(\|\nabla \Psi\|_{B^{s+1}}\right)\left(\|u_0\|_{H^{s+2}} + (1 + \|\nabla \Psi\|_{B^{s+1}})\|\nabla u_0\|_{B^{s+1}}\right),
\]

where in the last step, we used Lemma 3.1 so that

\[
\|\nabla u_0\|_{H^{s+1}} \lesssim \|\nabla u_0\|_{B^{s+1} + \frac{3}{2} - \frac{1}{2}} \quad \text{for} \quad p \in (1, 2).
\]

Similarly as \( \|\partial_3 Y_0\|_{B^{s,0}} \lesssim \|Y_0\|_{B^{s,1}} \lesssim \|Y_0\|_{B^{s+2}}, \) and by virtue of (2.12), \( Y_0 = \Psi(X_0(y)) \), one has

\[
\|Y_0\|_{H^{s+2}} + \|\partial_2 Y_0\|_{H^{s+2}} + \|\partial_3 Y_0\|_{B^{s,0}}
\]

\[
\leq C\left(\|\nabla \Psi\|_{B^{s+1}}\right)\left(\|\nabla \Psi\|_{B^{s+2}} + (1 + \|\nabla \Psi\|_{B^{s+1}})\|\nabla u_0\|_{B^{s+1}}\right),
\]

Whereas as \( p < 2 \), it follows from Definition 3.2 and Lemma 3.1 that

\[
\|a\|_{B^{s,0}} \lesssim \sum_{j,k \in \mathbb{Z}^2} 2^{js/2} \|\Delta_j \Delta_k^a\|_{L^2} \lesssim \sum_{j \in \mathbb{Z}} 2^{j(s+2\frac{3}{p} - 1)} \|\Delta_j a\|_{L^p} \left( \sum_{k \in \mathbb{Z}^N} 2^{k(s-2\frac{3}{p} - 1)} \right)
\]

\[
\lesssim \sum_{j \in \mathbb{Z}} 2^{j(s+2\frac{3}{p} - 2)} \|\Delta_j a\|_{L^p} \lesssim \|a\|_{B^{s+\frac{3}{p} - \frac{1}{2}}}
\]

(6.2)

Thanks to (6.2), we get, by applying Lemma A.1, that

\[
\|Y_0\|_{B^{s,0} \cap B^{s+2,0}} \lesssim \|Y_0\|_{B^{s+1} + \frac{3}{2} \cap B^{s+1} + \frac{3}{2}}
\]

\[
\leq C\left(\|\nabla \Psi\|_{B^{s+1}}\right)\left(\|\nabla \Psi\|_{B^{s+1} + \frac{3}{2}} + (1 + \|\nabla \Psi\|_{B^{s+1} + \frac{3}{2}})\|\nabla u_0\|_{B^{s+1} + \frac{3}{2} - \frac{1}{2}}\right),
\]
Then we define
\[ A_{\psi} (6.4) \]
with respect to the proof of Theorem 2.1, it amounts to prove (2.9). For this, we first notice from (6.5) that
\[ \|u_0\|_{B_{p,1}^{s+1+\frac{2}{p} - \frac{1}{2}}} \leq C \left( \|\nabla \Psi\|_{B_{p,1}^{s+1+\frac{2}{p} - \frac{1}{2}}} \right) \left( \|u_0\|_{B_{p,1}^{s+1+\frac{2}{p} - \frac{1}{2}}} + (1 + \|\nabla \Psi\|_{B_{p,1}^{s+1+\frac{2}{p} - \frac{1}{2}}}) \right). \]

Therefore, thanks to (2.7), we conclude that
\[ \|Y_0\|_{H^{s+2} \cap H^{s+2} + \|Y_0\|_{H^{s+2}} + \|\partial_3 Y_0\|_{H^{s+2}} + \|\partial_3 Y_0\|_{B_{p,1}^{s+1+\frac{2}{p} - \frac{1}{2}}} \}
\]
\[ + \|Y_1\|_{H^{s+1}} + \|Y_1\|_{H^{s+1}} + \|Y_1\|_{B_{p,1}^{s+1+\frac{2}{p} - \frac{1}{2}}} \}
\]
\[ \leq C \left( \|\nabla \Psi\|_{B_{p,1}^{s+1+\frac{2}{p} - \frac{1}{2}}} + \|u_0\|_{H^{s+2} \cap B_{p,1}^{s+1+\frac{2}{p} - \frac{1}{2}}} + \|\nabla u_0\|_{B_{p,1}^{s+1+\frac{2}{p} - \frac{1}{2}}} \right) \leq C \varepsilon_0, \]
from which, and Theorem 2.2, we deduce that the system (2.20) (equivalently (2.21)-(2.22)) has a unique global solution \((Y, q)\) which satisfies (2.26) and (2.27) provided that \(\varepsilon_0\) in (2.7) is sufficiently small.

We denote \(X(t, y) \overset{\text{def}}{=} y + Y(t, y)\). Then it follows from (2.27) that \(X(t, y)\) is invertible with respect to \(y\) variables and we denote its inverse mapping by \(X^{-1}(t, x)\). Since \(\det(1 + \nabla Y) = 1\), the adjoint matrix \(A_Y\) of \(I + \nabla Y\) satisfies
\[ \nabla \cdot A_Y = 0 \quad \text{and} \quad A_Y = (I + \nabla Y)^{-1} \]
which implies
\[ \nabla_x \cdot [(I + \nabla Y) \circ X^{-1}] = (\nabla \cdot [A_Y (I + \nabla Y)]) \circ X^{-1} = 0. \]

Then we define \(U(t, x) = (\tilde{b}(t, x), \tilde{b}(t, x), \tilde{b}(t, x))\) and \((u(t, x), p(t, x))\) through
\[ U(t, x) = (\tilde{b}, \tilde{b}, \tilde{b})(t, x) \overset{\text{def}}{=} (I + \nabla Y)(t, X^{-1}(t, x)) \quad \text{and} \]
\[ u(t, x) \overset{\text{def}}{=} Y_1(t, X^{-1}(t, x)), \quad p(t, x) = q(t, X^{-1}(t, x)) - \frac{1}{2} |b(t, x)|^2, \]
from which and (6.4), we infer that
\[ \text{div} \tilde{b} = \text{div} \tilde{b} = \text{div} b = 0. \]

Hence according to Section 2, \((U, u, p)\) thus defined globally solves (2.6). Then to complete the proof of Theorem 2.1, it amounts to prove (2.9). For this, we first notice from (6.5) that
\[ (\nabla u) \circ X(t, y) = \nabla_y Y_1(t, y)(I + \nabla_y Y(t, y))^{-1}, \]
which along with the proof of (A.4) in the Appendix A implies
\[ \|u\|_{L^1(R^+, \dot{B}^{s+\frac{2}{p}}_p)} \leq (1 + \|\nabla Y\|_{L^\infty(R^+, \dot{B}^{s+\frac{2}{p}}_p)}) \|\nabla Y_1\|_{L^1(R^+, \dot{B}^{s+\frac{2}{p}}_p)} \]
\[ \leq C \|\nabla Y\|_{L^\infty(R^+, \dot{B}^{s+\frac{2}{p}}_p)} \|\nabla Y_1\|_{L^1(R^+, \dot{B}^{s+\frac{2}{p}}_p)}. \]

Again thanks to (6.5), we get, by applying Lemma A.1 with \(\Phi = X(t, y)\), that
\[ \|b - e_1, \tilde{b} - e_2\|_{L^\infty(R^+, \dot{H}^{s+2} + 1)} + \|b - e_3\|_{L^\infty(R^+, \dot{H}^{s+2})} \]
\[ + \|u\|_{L^\infty(R^+, \dot{H}^{s+2})} + \|b - e_3\|_{L^2(R^+, \dot{H}^{s+2} + 1)} + \|u\|_{L^2(R^+, \dot{H}^{s+2} + 1)} \]
\[ \leq C \|\nabla Y\|_{L^\infty(R^+, \dot{B}^{s+\frac{2}{p}}_p)} \left( \|\partial_1 Y, \partial_2 Y\|_{L^\infty(R^+, \dot{H}^{s+2} + 1)} + \|\partial_3 Y\|_{L^\infty(R^+, \dot{H}^{s+2})} \right) \]
\[ + \|Y_1\|_{L^\infty(R^+, \dot{H}^{s+2} + 1)} + \|\partial_3 Y\|_{L^2(R^+, \dot{H}^{s+2} + 1)} + \|Y_1\|_{L^2(R^+, \dot{H}^{s+2} + 1)}. \]
Along the same line, one has

\[(6.8) \quad \|u\|_{L^1(\mathbb{R}^+; H^{s+1})} \leq C(\|\nabla Y\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s+\frac{1}{2}})}(1 + \|\Delta Y\|_{L^\infty(\mathbb{R}^+; H^{s+1})})\|\nabla Y_t\|_{L^1(\mathbb{R}^+; H^{s+1})},\]

and

\[(6.9) \quad \|(\tilde{b} - e_1, \tilde{b} - e_2)\|_{L^\infty(\mathbb{R}^+; H^{s+1})} + \|\tilde{b} - e_3\|_{L^\infty(\mathbb{R}^+; H^{s+1})}
+ \|u\|_{L^\infty(\mathbb{R}^+; H^{s+1})} + \|\tilde{b} - e_3\|_{L^2(\mathbb{R}^+; H^{s+1})} + \|u\|_{L^2(\mathbb{R}^+; H^{s+1})}
\leq C(\|\nabla Y\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s+\frac{1}{2}})}(\|\nabla Y\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s+1})} + \|Y_t\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s+1})}
+ \|\partial_3 Y\|_{L^2(\mathbb{R}^+; H^{s+1})} + \|Y\|_{L^\infty(\mathbb{R}^+; H^{s+1})}(\|\Delta Y\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s+1})} + \|Y_t\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s+1})}
+ \|\partial_3 Y\|_{L^2(\mathbb{R}^+; H^{s+1})} + (1 + \|\Delta Y\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s+1})})\|\Delta Y\|_{L^\infty(\mathbb{R}^+; \dot{H}^{s+1})}
+ \|\nabla Y_t\|_{L^\infty(\mathbb{R}^+; H^{s+1})} + \|\partial_3 \nabla Y\|_{L^2(\mathbb{R}^+; H^{s+1})} + \|\nabla Y_t\|_{L^2(\mathbb{R}^+; H^{s+1})})),\]

Consequently, we deduce from (2.27), (3.6), (6.3), (6.6) to (6.9) and the fact that \(\|u\|_{H^s} \lesssim \|u\|_{B^{s,0}}\) that

\[(6.10) \quad \|\partial_3 \nabla\|_{L^2(\mathbb{R}^+; H^{s+1+1\cap H^{s+1}})} + \|\nabla \|_{L^2(\mathbb{R}^+; H^{s+1+1\cap H^{s+1}})}
+ \|u\|_{L^2(\mathbb{R}^+; H^{s+1\cap H^{s+1}})} + \|u\|_{L^2(\mathbb{R}^+; H^{s+2\cap H^{s+1}})}
\leq C(\|\nabla Y\|_{B^{s+1\cap H^{s+1}}(\mathbb{R}^+; \dot{H}^{s+1\cap H^{s+1}})} + \|\nabla u_0\|_{H^{s+1\cap H^{s+1}}(\mathbb{R}^+; \dot{H}^{s+1\cap H^{s+1}})} + \|\nabla u_0\|_{B^{s+1\cap H^{s+1}}(\mathbb{R}^+; \dot{H}^{s+1\cap H^{s+1}})})\]

provided that \(\epsilon_0\) is sufficiently small in (2.7).

On the other hand, taking space divergence to the momentum equations of (1.1) gives rise to

\[(6.11) \quad \nabla p = -\frac{1}{2}\nabla(|b|^2) + \nabla(-\Delta)^{-1}\text{div}(u \cdot \nabla u - b \cdot \nabla b),\]

then applying product laws in Sobolev spaces gives rise to

\[\|\nabla p\|_{L^2(\mathbb{R}^+; H^{s+1\cap H^{s+2}})} \leq C(\|u\|_{L^2(\mathbb{R}^+; H^{s+1\cap H^{s+2}})} + \|u\|_{L^2(\mathbb{R}^+; H^{s+1\cap H^{s+2}})}\|\nabla u\|_{L^2(\mathbb{R}^+; \dot{H}^{s+\frac{1}{2}})}
+ (1 + \|b - e_3\|_{L^2(\mathbb{R}^+; \dot{H}^{s+1})})\|\nabla b\|_{L^2(\mathbb{R}^+; \dot{H}^{s+1})} + \|b - e_3\|_{L^2(\mathbb{R}^+; H^{s+1})}\|\nabla b\|_{L^2(\mathbb{R}^+; \dot{H}^{s+\frac{1}{2}})},\]

which together with (6.10) ensures that

\[\|\nabla p\|_{L^2(\mathbb{R}^+; H^{s+1\cap H^{s+2}})} \leq C(\|\nabla Y\|_{B^{s+1\cap H^{s+1}}(\mathbb{R}^+; \dot{H}^{s+1\cap H^{s+1}})} + \|u_0\|_{H^{s+1\cap H^{s+1}}(\mathbb{R}^+; \dot{H}^{s+1\cap H^{s+1}})} + \|u_0\|_{B^{s+1\cap H^{s+1}}(\mathbb{R}^+; \dot{H}^{s+1\cap H^{s+1}})}),\]

provided that \(\epsilon_0\) is sufficiently small in (2.7). This completes the proof of (2.9) and thus Theorem 2.1.

Before we present the proof of Theorem 1.1, we shall first prove the following blow-up criterion for smooth enough solutions of (1.1).

**Proposition 6.1.** Let \(b_0 - e_3 \in H^s(\mathbb{R}^3)\) and \(u_0 \in H^s(\mathbb{R}^3)\) for \(s > \frac{3}{2}\), (1.1) has a unique solution \((b, u)\) on \([0, T]\) for some \(T > 0\) so that \(b - e_3 \in C([0, T]; H^s(\mathbb{R}^3))\), \(u \in C([0, T]; H^s(\mathbb{R}^3))\)
with \( \nabla u \in L^2((0,T); \dot{H}^{s+1}(\mathbb{R}^3)) \) and \( \nabla p \in C([0,T]; H^{s-1}(\mathbb{R}^3)) \). Moreover, if \( T^* \) is the life span to this solution, and \( T^* < \infty \), one has

\[
\int_0^{T^*} \left( \| \nabla u(t) \|_{L^\infty} + \| b(t) \|_{L^\infty}^2 \right) dt = \infty.
\]

**Proof.** It is well-known that the existence of solution to a nonlinear PDE basically follows from the uniform estimates to some smooth enough approximate solutions. For simplicity, we may only present a priori estimates to smooth enough solutions of (1.1) (one may check [23] for the detailed proof to the related system). As a matter of fact, let \( \bar{b} \overset{\text{def}}{=} b - e_3 \), we first get, by using a standard energy estimate for (1.1), that

\[
\frac{1}{2} \frac{d}{dt} (\| \bar{b} \|_{L^2}^2 + \| u \|_{L^2}^2) + \| \nabla u \|_{L^2}^2 = 0.
\]

Along the same line, applying \( \Delta_j \) to the system (1.1) and then taking \( L^2 \) inner product of the resulting equations with \( (\Delta_j \bar{b}, \Delta_j u) \), we obtain

\[
\frac{1}{2} \frac{d}{dt} (\| \Delta_j \bar{b} \|_{L^2}^2 + \| \Delta_j u \|_{L^2}^2) + \| \nabla \Delta_j u \|_{L^2}^2 = - (\Delta_j (u \cdot \nabla \bar{b}) | \Delta_j \bar{b}) + (\Delta_j (\bar{b} \cdot \nabla u | \Delta_j \bar{b})
\]

\[- (\Delta_j (u \cdot \nabla u | \Delta_j u) + (\Delta_j (\bar{b} \cdot \nabla \bar{b}) | \Delta_j u).
\]

By virtue of the commutator estimates (see Section 2.10 of [2]), we write

\[
\| (\Delta_j (u \cdot \nabla b) | \Delta_j b) \| \lesssim c_j(t)^{2-2j} \| \nabla u(t) \|_{L^\infty} \| \bar{b}(t) \|_{H^s} + \| \bar{b}(t) \|_{L^\infty} \| \nabla u(t) \|_{H^s} \| \bar{b}(t) \|_{H^s},
\]

and

\[
\| (\Delta_j (\bar{b} \cdot \nabla b) | \Delta_j b) \| = \| (\Delta_j (\bar{b} \cdot \nabla b) | \Delta_j u) \| \lesssim c_j(t)^{2-2j} \| \bar{b}(t) \|_{L^\infty} \| \bar{b}(t) \|_{H^s} \| \nabla u(t) \|_{H^s}.
\]

Resuming the above estimates into (6.14) and using (6.13), we conclude that for any \( s > 0 \),

\[
\| u(t) \|_{H^s}^2 + \| \bar{b}(t) \|_{H^s}^2 + \| \nabla u(t) \|_{L^2(H^s)}^2 \leq \| u_0 \|_{H^s}^2 + \| \bar{b}_0 \|_{H^s}^2 + C \int_0^t \left( \| \nabla u(t') \|_{H^s} + \| \bar{b}(t') \|_{H^s}^2 \right) dt'.
\]

Notice that \( s > \frac{3}{2} \), one has, \( \| \nabla u(t) \|_{H^s} \lesssim \| \nabla u(t) \|_{H^s} \), we thus achieve

\[
\| u(t) \|_{H^s}^2 + \| \bar{b}(t) \|_{H^s}^2 + \| \nabla u(t) \|_{L^2(H^s)}^2 \leq \| u_0 \|_{H^s}^2 + \| \bar{b}_0 \|_{H^s}^2 + C \int_0^t (\| u(t') \|_{H^s} + \| \bar{b}(t') \|_{H^s}^2) (\| u(t') \|_{H^s}^2 + \| \bar{b}(t') \|_{H^s}^2) dt',
\]

from which, we infer that there exists a positive time \( T^* \), so that there holds

\[
\| u(t) \|_{H^s}^2 + \| \bar{b}(t) \|_{H^s}^2 + \| \nabla u(t) \|_{L^2(H^s)}^2 \leq C_T \| u_0 \|_{H^s}^2 + \| \bar{b}_0 \|_{H^s}^2 \quad \text{for any } T < T^*.
\]

which along with (6.11) ensures that \( \nabla p \in C([0,T]; H^{s-1}(\mathbb{R}^3)) \) for any \( T < T^* \). This concludes the existence part of Proposition 6.1.
Finally, applying Gronwall’s inequality to (6.15) yields
\[
\|u(t)\|^2_{H^s} + \|\tilde{b}(t)\|^2_{H^s} + \|\nabla u\|^2_{L^2_t(H^s)} \\
\leq \left(\|u_0\|^2_{H^s} + \|\tilde{b}_0\|^2_{H^s}\right) \exp\left(C \int_0^t \left(\|\nabla u(t')\|_{L^\infty} + \|\tilde{b}(t')\|_{L^\infty}\right) dt'\right) \quad \text{for} \quad t < T^*,
\]
which together with a classical continuous argument implies (6.12). This completes the proof of Proposition 6.1.

Now we are in a position to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Under the assumption of Theorem 1.1, we deduce from Proposition 2.1 that there exists a \(\Psi = (\psi_1, \psi_2, \psi_3)^T\) so that there holds (2.1)-(2.5). Notice that for \(s_2 \in (-\frac{1}{2}, -\frac{1}{4})\) and \(p \in (1, 2)\), \(s_2 + \frac{3}{p} - \frac{1}{2} > 0\), then it is easy to observe that
\[
\|\nabla \Psi\|_{B^{s_2+\frac{3}{p}-\frac{1}{2}}_{p,1}} \lesssim \|\Psi\|_{B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}} \lesssim \|b_0 - e_3\|_{B^{s_1+\frac{3}{p}+\frac{1}{2}}_{p,1}}.
\]
Therefore, under the assumption of (1.16), for \(U_0 = (I - \nabla \Psi)^{-1}\), we infer from Theorem 2.1 that (2.6) has a unique global solution \((U, u, p)\) so that there holds (2.8) and (2.9). Let \(U = (\tilde{b}, \tilde{b}, \tilde{b})\), then according to the discussions at the beginning of Section 2, \((\tilde{b}, \tilde{u}, \tilde{p})\) thus obtained satisfies (1.1), which is in fact the unique solution of (1.1) with initial data \((b_0, u_0)\), and there holds (1.18).

On the other hand, by virtue of Proposition 6.1, given initial data \((b_0, u_0)\) with \(b_0 - e_3 \in H^s(\mathbb{R}^3)\) and \(u_0 \in H^{s+2}(\mathbb{R}^3)\) (since \(u_0 \in H^{s+2}(\mathbb{R}^3)\) and \(\nabla u_0 \in H^{s-1}(\mathbb{R}^3)\)) for \(s \geq s_1 + 2\), (1.1) has a unique solution \((b, u, p)\) with \(b - e_3 \in C([0, T]; H^s(\mathbb{R}^3))\), \(\nabla p \in C([0, T]; H^{s-1}(\mathbb{R}^3))\), and \(u \in C([0, T]; H^s(\mathbb{R}^3))\) with \(\nabla u \in L^2((0, T); H^{s+1}(\mathbb{R}^3))\) for any fixed \(T < T^*\). Furthermore, if \(T^* < \infty\), there holds (6.12). Due to the uniqueness, this solution must coincide with the one obtained in the last paragraph. Then thanks to (1.18), (6.12) can not be true for any finite \(T^*\), and therefore \(T^* = \infty\) and there holds (1.17). This completes the proof of Theorem 1.1.

\[\square\]

**Appendix A. The Besov estimates to functions composed with a measure preserving diffeomorphism**

**Lemma A.1.** Let \(\Phi(y) = y + \Psi(y)\) be a smooth volume preserving diffeomorphism on \(\mathbb{R}^3\). Then for \(u, v \in \mathcal{S}(\mathbb{R}^3)\), there hold
\[
\|u \circ \Phi\|_{\dot{B}_{p,r}^{s}} \leq C\left(\|\nabla \Psi\|_{B_{p,1}^{s_2}}\right)\|u\|_{\dot{B}_{p,r}^{s}} \quad \text{and} \\
\|u \circ \Phi^{-1}\|_{B_{p,r}^{s}} \leq C\left(\|\nabla \Psi\|_{B_{p,1}^{s_2}}\right)\|u\|_{B_{p,r}^{s}} \quad \text{for} \quad s \in (-1, 2),
\]
\[
\|u \circ \Phi\|_{\dot{B}_{p,r}^{s}} \leq C\left(\|\nabla \Psi\|_{B_{p,1}^{s_2}}\right)(\|u\|_{B_{p,r}^{s}} + \|\Psi\|_{B_{p,r}^{s_1+\frac{3}{p}}})\|u\|_{\dot{B}_{p,r}^{s}} \quad \text{and}
\]
\[
\|u \circ \Phi^{-1}\|_{B_{p,r}^{s}} \leq C\left(\|\nabla \Psi\|_{B_{p,1}^{s_2}}\right)\left(\|v\|_{B_{p,r}^{s}} + \|\Psi\|_{B_{p,r}^{s_1+\frac{3}{p}}})\|v\|_{B_{p,r}^{s}} \quad \text{for} \quad s \in (2, 3),
\]
\[
\|u \circ \Phi\|_{\dot{B}_{p,r}^{s}} \leq C\left(\|\nabla \Psi\|_{B_{p,1}^{s_2}}\right)(1 + \|\Delta \Psi\|_{B_{p,r}^{s-2}})\|\nabla u\|_{B_{p,r}^{s-1}} \quad \text{and}
\]
\[
\|u \circ \Phi^{-1}\|_{B_{p,r}^{s}} \leq C\left(\|\nabla \Psi\|_{B_{p,1}^{s_2}}\right)(1 + \|\Delta \Psi\|_{B_{p,r}^{s-2}})\|\nabla v\|_{B_{p,r}^{s-1}} \quad \text{for} \quad s > 3,
\]
where \(C(\lambda)\) denotes a positive constant non-decreasingly depending on \(\lambda\).
Proof. Let
\[ \mathcal{A} = (a_{ij})_{i,j=1,2,3} \overset{\text{def}}{=} I + \nabla_y \Psi, \quad \mathcal{B} = (b_{ij})_{i,j=1,2,3} \overset{\text{def}}{=} (I + \nabla_y \Psi)^{-1}. \]

Then due to \( \det \mathcal{A} = 1 \), the matrix \( \mathcal{B} \) equals the adjoint matrix of \( \mathcal{A} \). This leads to
\[ (A.2) \quad (\partial_x u) \circ \Phi = \sum_{j=1}^{3} b_{ij} \partial_y_j (u \circ \Phi) \quad \text{and} \quad (\partial_y v) \circ \Phi^{-1} = \sum_{j=1}^{3} a_{ij} \circ \Phi^{-1} \partial_x_j (v \circ \Phi^{-1}). \]

In what follows, we shall only present the proof of the related estimates involving \( u \circ \Phi \), and the ones involving \( v \circ \Phi^{-1} \) are identical. We first deduce from Lemma 2.7 of [2] that
\[ \| \Delta_j ((\Delta_k u) \circ \Phi) \|_{L^p} \leq C \min (2^{j-k}, 2^{k-j}) \| \nabla \Psi \|_{L^\infty} \| \Delta_k u \|_{L^p} \quad \text{for all} \quad j, k \in \mathbb{Z}, \]
so that for \( s \in (-1, 1) \) and \( u \in \dot{B}^s_{p,r} (\mathbb{R}^3) \), one has
\[ \| \Delta_j (u \circ \Phi) \|_{L^p} \leq \sum_{k \in \mathbb{Z}} \| \Delta_j ((\Delta_k u) \circ \Phi) \|_{L^p} \]
\[ \leq C \left( \sum_{k \leq j} 2^{k-j} + \sum_{k > j} 2^{j-k} \right) \| \nabla \Psi \|_{L^\infty} \| \Delta_k u \|_{L^p} \]
\[ \leq C \| \nabla \Psi \|_{L^\infty} 2^{-j s} \left( \sum_{k \leq j} c_k r 2^{(k-j)(1-s)} + \sum_{k > j} c_k r 2^{(j-k)(1+s)} \right) \| u \|_{\dot{B}^s_{p,r}} \]
\[ \leq C c_{j,r} 2^{-j s} \| \nabla \Psi \|_{L^\infty} \| u \|_{\dot{B}^s_{p,r}} \quad \text{for} \quad (j_{c_j,r}) \in \mathcal{I}'(\mathbb{Z}). \]

This gives
\[ (A.3) \quad \| u \circ \Phi \|_{\dot{B}^s_{p,r}} \leq C (\| \nabla \Psi \|_{L^\infty}) \| u \|_{\dot{B}^s_{p,r}} \quad \text{for} \quad s \in (-1, 1). \]

Whereas we deduce from (A.2) and (A.3) that
\[ \| u \circ \Phi \|_{\dot{B}^1_{p,r}} \leq \| (\nabla_x u) \circ \Phi \|_{\dot{B}^0_{p,r}} \leq C (\| \nabla \Psi \|_{L^\infty}) \left( 1 + \| \nabla \Psi \|_{\dot{B}^p_{p,1}} \right)^2 \| u \|_{\dot{B}^1_{p,r}}. \]

For \( 1 < s \leq 2 \), we get, by using (A.2) and product laws in Besov spaces, that
\[ \| u \circ \Phi \|_{\dot{B}^s_{p,r}} \leq \| (\nabla_x u) \circ \Phi \|_{\dot{B}^{s-1}_{p,1}} \leq C \left( 1 + \| \mathcal{B} - I \|_{\dot{B}^{s-1}_{p,1}} \right) \| u \|_{\dot{B}^s_{p,r}}. \]

This proves the first line of (A.1).

On the other hand, it is easy to observe from Bony’s decomposition (3.8) that
\[ \| a b \|_{\dot{B}^\tau_{p,r}} \lesssim \| a \|_{L^\infty} \| b \|_{\dot{B}_r^{\tau}} + \| a \|_{\dot{B}^{\tau+\frac{1}{2}}_r} \| b \|_{\dot{B}^{\tau}_{p,\infty}} \quad \text{for} \quad \tau > 0, \]
from which, (A.2), and the first line of (A.1), we infer that for \( s \in (2, 3) \)
\[ \| u \circ \Phi \|_{\dot{B}^s_{p,r}} \lesssim \| (\nabla_x u) \circ \Phi \|_{\dot{B}^{s-1}_{p,1}} \]
\[ \lesssim (1 + \| \nabla \Psi \|_{L^\infty})^2 \| \nabla u \|_{\dot{B}^s_{p,r}} + (1 + \| \nabla \Psi \|_{L^\infty}) \| \nabla \Psi \|_{\dot{B}^{s-1}_{p,r}} \| u \|_{\dot{B}^s_{p,r}} \]
\[ \leq C \left( \| \nabla \Psi \|_{\dot{B}^p_{p,1}} \right) \left( \| u \|_{\dot{B}^s_{p,r}} + \| \nabla \Psi \|_{\dot{B}^{s-1}_{p,r}} \right) \| u \|_{\dot{B}^s_{p,r}}. \]

This proves the third line of (A.1).
Inductively we assume that for $k \in \mathbb{N}$ and $k + 1 < s - 1 \leq k + 2$,

\begin{equation}
(A.5) \quad \|u \circ \Phi\|_{B^s_{p,r}} \leq C\left(\|\nabla \Psi\|_{B^p_{p,1}}\right)\left(\|u\|_{B^s_{p,r}} + \sum_{j=0}^{k-1}\|\Psi\|_{B^{s-\frac{1}{2}}_{p,r}}\|u\|_{B^{s+2}_{p,r}}\right).
\end{equation}

Then by virtue of (A.2) and (A.5), we deduce that

\[
\begin{align*}
\|u \circ \Phi\|_{B^s_{p,r}} &\lesssim \|\nabla_x u \circ \Phi\|_{B^s_{p,r}} \\
&\leq C\left(\|\nabla \Psi\|_{B^p_{p,1}}\right)\left(\|\nabla u \circ \Phi\|_{B^s_{p,r}} + \|\nabla \Psi\|_{B^{s-1}_{p,r}}\|\nabla u \circ \Phi\|_{L^\infty}\right) \\
&\leq C\left(\|\nabla \Psi\|_{B^p_{p,1}}\right)\left(\|u\|_{B^s_{p,r}} + \sum_{j=0}^{k-1}\|\Psi\|_{B^{s-1}_{p,r}}\|\nabla u\|_{B^{s+2}_{p,r}} + \|\Psi\|_{B^{s-1}_{p,r}}\|\nabla u\|_{L^\infty}\right) \\
&\leq C\left(\|\nabla \Psi\|_{B^p_{p,1}}\right)(1 + \|\Delta \Psi\|_{B^{s-2}_{p,r}}\|\nabla u\|_{B^{s-1}_{p,r}}),
\end{align*}
\]

which leads to the fifth inequality of (A.1). This concludes the proof of Lemma A.1. \qed

**APPENDIX B. THE PROOF OF PROPOSITION 2.1**

The proof of Proposition 2.1 will be based on the following lemma:

**Lemma B.1.** Let $s > \frac{3}{2}$, $p \in (\frac{3}{2}, 2)$, and $f \in B^s_{p,1}(\mathbb{R}^3)$ with $\text{Supp } f(x_1, x_2, \cdot) \subset [-K, K]$ for some positive constant $K$. We assume moreover that $f$ and $b_0$ are admissible on $\mathbb{R}^2 \times \{0\}$ in the sense of Definition 1.1 and (2.1) holds. Then (1.15) has a solution $\psi \in B^s_{p,1}(\mathbb{R}^3)$ so that

\begin{equation}
(B.1) \quad \|\psi\|_{B^s_{p,1}} \leq C(K, \|\nabla b_0\|_{B^s_{p,1}})\|f\|_{B^s_{p,1}}.
\end{equation}

**Proof.** Due to (2.1), (1.14) has a unique global solution on $\mathbb{R}$ so that for all $t \in \mathbb{R},$

\begin{equation}
(B.2) \quad \|\nabla X(t, \cdot)\|_{L^\infty} \leq \exp\left(\|\nabla b_0\|_{L^\infty}|t|\right) \quad \text{and} \quad \det\left(\frac{\partial X(t, x)}{\partial x}\right) = 1.
\end{equation}

While it follows from (1.15) and (1.14) that

\[
\frac{d}{dt}\psi(X(t, x)) = f(X(t, x)),
\]

from which, we define

\[
\psi(x) = \begin{cases} 
-\int_{0}^{\infty} f(X(t, x)) dt & \text{if } x_3 \geq 0, \\
\int_{-\infty}^{0} f(X(t, x)) dt & \text{if } x_3 \leq 0.
\end{cases}
\]

Thanks to the assumption that $f$ and $b_0$ are admissible on $\mathbb{R}^2 \times \{0\}$ in the sense of Definition 1.1, the values of $\psi(x)$ at $(x_1, x_2, 0)$ are compatible. We remark that $b_0^3\partial_3 \psi = -b_0^1 \partial_1 \psi - b_0^2 \partial_2 \psi + f$ and $b_0^3 \geq \frac{1}{2}$ implies that the derivatives of $\psi$ in the $x_1, x_2$ variables yields the derivatives of $\psi$ with respect to $x_3$ variable. Therefore, we do not require any admissible condition for the derivatives of $f$ and $b_0$. 
On the other hand, it follows from (2.1) that $b_0^3 \geq \frac{1}{2}$ as long as $\varepsilon_0$ is small enough. So that we deduce from (1.14) that
\[
X_3^2(t, x) \geq x_3 + \frac{t}{2} \geq K \quad \text{if } t \geq 2K, \quad x_3 \geq 0, \\
X_3^2(t, x) \leq x_3 + \frac{t}{2} \leq -K \quad \text{if } t \leq -2K, \quad x_3 \leq 0,
\]
which together with the assumption: Supp $f(x_1, x_2, \cdot) \subset [-K, K]$ for some positive constant $K$, implies that
\[
\psi(x) = \begin{cases} 
-\int^{|t|}_0 f(X(t, x)) \, dt & \text{if } x_3 \geq 0, \\
\int^{|t|}_{-2K} f(X(t, x)) \, dt & \text{if } x_3 \leq 0.
\end{cases}
\]
(B.3)

With this solution formula for (1.15), it amounts to prove (B.1) in order to complete the proof of Lemma B.1. Indeed for any $s > 0$, we deduce from (1.14) and product laws in Besov spaces that for any $t \in [-2K, 2K]$
\[
\|\nabla_x X(t, \cdot) - I\|_{\dot{B}^s_p} \lesssim \int_0^{|t|} \left( \|\nabla b_0\|_{L^\infty} \|\nabla_x X(t', \cdot) - I\|_{\dot{B}^s_p} + \|\nabla b_0\|_{L^\infty} \left( 1 + \|\nabla X(t', \cdot) - I\|_{L^\infty} \right) \right) dt',
\]
from which, (A.3) and (B.2), we get, by using Gronwall’s inequality, that
\[
\max_{t \in [-2K, 2K]} \|\nabla_x X(t, \cdot) - I\|_{\dot{B}^s_p} \leq C(K, \|\nabla b_0\|_{L^\infty}) \|\nabla b_0\|_{\dot{B}^s_p}, \quad \text{for } s \in (0, 1).
\]

Then for $s \in (1, 2)$ and $t \in [-2K, 2K]$, we infer
\[
\|f(X(t, \cdot))\|_{\dot{B}^s_p} = \|\nabla f(X(t, \cdot)) \nabla X(t, \cdot)\|_{\dot{B}^{s-1}_p} \\
\lesssim \|\nabla f\|_{L^\infty} \|\nabla_x X(t, \cdot) - I\|_{\dot{B}^{s-1}_p} + \|\nabla f(X(t, \cdot))\|_{\dot{B}^{s-1}_p} \left( 1 + \|\nabla X(t, \cdot) - I\|_{L^\infty} \right) \\
\leq C(K, \|\nabla b_0\|_{L^\infty}) \left( \|\nabla f\|_{L^\infty} \|b_0\|_{\dot{B}^s_p} + \|f\|_{\dot{B}^s_p} \right),
\]

Notice that for $p \in (\frac{3}{2}, 2), \frac{3}{p} \in (\frac{3}{2}, 2)$, we thus deduce from (B.4) that
\[
\|\nabla_x X(t, \cdot) - I\|_{\dot{B}^{\frac{3}{p}}_p} \lesssim \int_0^{|t|} \|\nabla b_0\|_{L^\infty} \|\nabla_x X(t', \cdot) - I\|_{\dot{B}^{\frac{3}{p}}_p} \, dt' \\
+ C(K, \|\nabla b_0\|_{L^\infty}) \left( \|\nabla^2 b_0\|_{L^\infty} \|b_0\|_{\dot{B}^{\frac{3}{p}}_p} + \|\nabla b_0\|_{\dot{B}^{\frac{3}{p}}_p} \right),
\]
for $t \in [-2K, 2K]$. Applying Gronwall’s inequality gives rise to
\[
\max_{t \in [-2K, 2K]} \|\nabla X(t, \cdot) - I\|_{\dot{B}^{\frac{3}{p}}_p} \leq C(K, \|\nabla b_0\|_{\dot{B}^{1+\frac{3}{p}}_p}) \|\nabla b_0\|_{\dot{B}^{\frac{3}{p}}_p}.
\]
(B.5)

While it is easy to observe from (A.4) that
\[
\|u \circ \Phi\|_{\dot{B}^{\frac{3}{p}}_p} \leq C \left( \|\nabla \Psi\|_{\dot{B}^\frac{3}{p}_p} \right) \left( \|u\|_{\dot{B}^\frac{3}{p}_p} + \|\nabla \Psi\|_{\dot{B}^\frac{3}{p}_p} \|u\|_{\dot{B}^\frac{3}{p}_p} \right)
\]

for $s \in (2, 3]$, so that for $2 < s - 1 < 3$, (B.4) implies
\[
\| \nabla X(t, \cdot) - I \|_{B^{s-1}_{p,1}} \lesssim \int_0^t \left( \| \nabla b_0 \|_{L^\infty} \| \nabla X(t', \cdot) - I \|_{B^{s-1}_{p,1}} \\
+ C(K, \| \nabla b_0 \|_{B^{1+\frac{3}{p}}_{p,1}}) (\| \nabla b_0 \|_{B^{s-1}_{p,1}} + \| \nabla X(t', \cdot) - I \|_{B^{s-1}_{p,1}}) dt'.
\]
for $t \in [-2K, 2K]$, applying Gronwall's inequality gives
(B.6) \[
\max_{t \in [-2K, 2K]} \| \nabla X(t, \cdot) - I \|_{B^{s-1}_{p,1}} \leq C(K, \| \nabla b_0 \|_{B^{1+\frac{3}{p}}_{p,1}}) \| \nabla b_0 \|_{B^{s-1}_{p,1}} \quad \text{for} \quad s \in (3, 4].
\]
For $s > 4$, we deduce from (A.1) and (B.4) that
\[
\| \nabla X(t, \cdot) - I \|_{L^p} \lesssim \int_0^t \left( \| \nabla b_0 \|_{L^\infty} \| \nabla X(t', \cdot) - I \|_{L^p} + \| \nabla b_0 \|_{L^p} \right) dt'.
\]
As a consequence, we obtain
\[
\| \nabla X(t, \cdot) - I \|_{B^{s-1}_{p,1}} \leq C(K, \| \nabla b_0 \|_{B^{s-1}_{p,1}}) \int_0^t \left( \| \nabla X(t', \cdot) - I \|_{B^{s-1}_{p,1}} + \| \nabla b_0 \|_{B^{s-1}_{p,1}} \right) dt'.
\]
Applying Gronwall's inequality leads to
(B.7) \[
\max_{t \in [-2K, 2K]} \| \nabla X(t, \cdot) - I \|_{B^{s-1}_{p,1}} \leq C(K, \| \nabla b_0 \|_{B^{s-1}_{p,1}}) \| \nabla b_0 \|_{B^{s-1}_{p,1}} \quad \text{for} \quad s > 4.
\]
Finally we deduce from (A.1) and (B.3) that
\[
\| \psi \|_{B^{s}_{p,1}} \leq \int_{-2K}^{2K} C(\| \nabla X(t', \cdot) - I \|_{B^{\frac{3}{p}}_{p,1}}) \left( 1 + \| \nabla X(t', \cdot) - I \|_{B^{s-1}_{p,1}} \right) \| \nabla f \|_{B^{s-1}_{p,1}} dt',
\]
which together with
\[
\| \psi \|_{L^p} \leq \| f \|_{L^p}
\]
and (B.6) and (B.7) concludes the proof of (B.1). \qed

Proof of Proposition 2.1. Under the assumption of (2.1), we would first like to find a solution $(\psi_1, \psi_2) \in B^s_{p,1}(\mathbb{R}^3)$ to the following system:
(B.8) \[
\begin{cases}
(1 - \partial_{x_2} \psi_2) \partial_{x_3} \psi_1 + \partial_{x_3} \partial_{x_3} \psi_2 \psi_1 = b^{1}_{0}, & \text{for } x \in \mathbb{R}^3, \\
(1 - \partial_{x_1} \psi_1) \partial_{x_3} \psi_2 + \partial_{x_3} \partial_{x_1} \psi_1 \psi_2 = b^{2}_{0},
\end{cases}
\]
If we use the standard iteration scheme to solve the above problem, the iterated solutions will lose derivative on each step. However, notice that
\[
\partial_{x_1} \left( \partial_{x_2} \psi_1 \partial_{x_3} \psi_2 + \partial_{x_3} \psi_1 (1 - \partial_{x_2} \psi_2) \right) + \partial_{x_2} \left( \partial_{x_3} \partial_{x_1} \psi_1 \psi_2 + \partial_{x_3} \psi_2 (1 - \partial_{x_1} \psi_1) \right) + \partial_{x_3} \left( (1 - \partial_{x_1} \psi_1) (1 - \partial_{x_2} \psi_2) - \partial_{x_2} \partial_{x_1} \psi_1 \psi_2 \right) = 0.
\]
This along with $\text{div } b_0 = 0$ ensures that
\[
\partial_{x_3} (b^{3}_{0} - (1 - \partial_{x_1} \psi_1) (1 - \partial_{x_2} \psi_2) + \partial_{x_2} \psi_1 \partial_{x_1} \psi_2) = 0.
\]
Since $b_0^3 - 1 \in B^s_{p,1}(\mathbb{R}^3)$, we conclude that

$$b_0^3 = (1 - \partial_x \psi_1)(1 - \partial_x \psi_2) - \partial_x \psi_1 \partial_x \psi_2. \tag{B.9}$$

With (B.9), we solve (B.8) for $\partial_x \psi_1$ and $\partial_x \psi_2$ from (B.8)

$$\begin{pmatrix} \partial_x \psi_1 \\ \partial_x \psi_2 \end{pmatrix} = \frac{1}{b_0^3} \begin{pmatrix} 1 - \partial_x \psi_1 & -\partial_x \psi_1 \\ -\partial_x \psi_2 & 1 - \partial_x \psi_2 \end{pmatrix} \begin{pmatrix} b_0^1 \\ b_0^2 \end{pmatrix},$$

or equivalently

$$\begin{align*}
\frac{b_0^1}{b_0^3} \partial_x \psi_1 + \frac{b_0^2}{b_0^3} \partial_x \psi_1 + \frac{b_0^3}{b_0^3} \partial_x \psi_1 &= b_0^1, \\
\frac{b_0^1}{b_0^3} \partial_x \psi_2 + \frac{b_0^2}{b_0^3} \partial_x \psi_2 + \frac{b_0^3}{b_0^3} \partial_x \psi_2 &= b_0^2. \tag{B.10}
\end{align*}$$

Thanks to (2.1) and Lemma B.1, (B.10) has a solution $(\psi_1, \psi_2)$ so that

$$\| (\psi_1, \psi_2) \|_{B^s_{p,1}} \leq C(K, \varepsilon_0) \| (b_0^1, b_0^2) \|_{B^s_{p,1}}. \tag{B.11}$$

Whereas for $\Psi = (\psi_1, \psi_2, \psi_3)^T$, we deduce from $\det(I - \nabla \Psi) = 1$ that

$$\begin{align*}
(1 - \partial_1 \psi_1 - \partial_2 \psi_2 + \partial_1 \psi_1 \partial_2 \psi_2 - \partial_2 \psi_1 \partial_1 \psi_2) & \partial_3 \psi_3 + (\partial_2 \psi_1 \partial_3 \psi_2 + \partial_3 \psi_1 (1 - \partial_2 \psi_2)) \partial_1 \psi_3 \\
+ (\partial_3 \psi_1 \partial_1 \psi_2 + (1 - \partial_1 \psi_1) \partial_3 \psi_2) \partial_2 \psi_3 &= -\partial_1 \psi_1 - \partial_2 \psi_2 + \partial_1 \psi_1 \partial_2 \psi_2 - \partial_2 \psi_1 \partial_1 \psi_2,
\end{align*}$$

which together with (B.8) and (B.9) yields

$$\begin{align*}
\frac{b_0^1}{b_0^3} \partial_x \psi_3 + \frac{b_0^2}{b_0^3} \partial_x \psi_3 + \frac{b_0^3}{b_0^3} \partial_x \psi_3 &= b_0^3 - 1. \tag{B.12}
\end{align*}$$

Along the same line to the proof of (B.11), (B.12) has a solution $\psi_3$ so that

$$\| \psi_3 \|_{B^s_{p,1}} \leq C(K, \varepsilon_0) \| b_0^3 - 1 \|_{B^s_{p,1}}.$$ 

This together with (B.11) leads to (2.2). And thus (2.5) follows from (2.4).

Finally observing that $U_0$ defined in Proposition 2.1 is in fact the adjoint matrix of $I - \nabla \Psi$, $U_0$ automatically satisfies (1.9). This finishes the proof of Proposition 2.1. \hfill \Box

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