On Geometrical Methods that Provide a Short Proof of Four Color Theorem

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Abstract In this article we introduce a short and comprehensive proof of four color theorem based on geometrical methods. At the end of the article we will provide a short proof of the De Bruijn Erdos theorem for locally finite infinite graphs.

Keywords: four color theorem, geometrical methods, De Bruijn Erdos theorem

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1. Introduction and Preliminaries

The Four Color Theorem is well known in the mathematical community. It states that any number of connected locally connected regions located in the plane (i.e. they are all subsets of \( R^2 \)), intersecting only on their common boundaries, can be separated by four colors. In order to present a coherent mathematical view of the above theorem we need to state the following definitions and use the basic concepts and facts in topology. For more details on the basic concepts in topology we recommend the book, Topology (1970) by James Dugundji. Throughout this article the paths and closed curves are subsets of finite a graphs hence we use a type of digital geometry to prove some of the arguments. But many of the lemmas and theorems are stated and proved in the most general cases. The Jordan closed curve theorem as stated at [2] is the basis for many of the lemmas and theorems. When dealing with closed curves and paths we define the interior path of the given closed curve to be paths that except one or two points on the boundary the rest of the path lives in the interior of the closed curve. For a connected Graph \( G \), its spectrum, \( \text{spect}(G) \) is the operator norm of the adjacency matrix. We say that two connected graphs \( G_1 \) and \( G_2 \) are equivalent, \( G_1 \sim G_2 \), if their corresponding adjacency matrix are unitary equivalent. this concept is used to define some useful definitions that will help proving the main theorem, theorem (2.21), the paths are all directed in same way. For example the paths located on closed curves have the same direction as the direction on the closed curve which is clockwise direction. Many of the sets we are going to deal with are connected, closed and bounded. These kinds of sets constitute a very important class of sets.

Definition 1.1 A compact connected subspace of a Hausdorff metric space is called continuum.

In particular here by region we mean locally connected continuum sub-set of \( R^2 \). To state equivalent version of four color theorem represent each of the regions by a vertex in \( R^2 \) and connect two points if and only if the corresponding regions intersect. This will provide us with a graph \( G \). Now the four color theorem can be expressed equivalently to this situation as to associate a color to each of the vertices in such a way that no two vertices connected by an arc can have the same color. Note that without loss of generality we can assume that the above graphs are connected. In fact throughout this article all the graphs are connected. For a set \( Q \subseteq R^2 \), let \( Q' \) be the complement of \( Q \) in \( R^2 \), i.e. \( Q' = (x \in R^2, x \; \text{not in} \; Q) \). Also we denote the interior of \( Q \) by \( \text{int}(Q) \) In final part of this article we are going to deal with locally finite infinite graphs. These are infinite graphs with the property that each vertex is connected to finitely many other vertices. For a set \( Q \), we set \( \text{Card}(Q) \) to be equal to the cardinality of the set \( Q \). So by our definition any two region \( A_2 \) and \( A_2 \) from the collection are either disjoint or \( (A_1) \cap (A_2) = \emptyset \). In the later case \( A_1 \) and \( A_2 \) have to take different colors. Finally by Jordan closed curve theorem \( S \) divides the space in two regions, where one of the regions \( SC \subseteq S \) can be be mapped homeomorphically onto the closed disc corresponding to a circle. For simplifying the notation we write \( \text{int}(S) \) instead of writing \( \text{int}(SC) \).

The following definitions are well known

Definition 1.2 A subset of \( S \subseteq R^2 \) which is homeomorphic to a circle is called closed curve. The above homeomorphism can be chosen so that can be extended to the homeomorphism taking \( SC = S \cap \text{int}(S) \) onto the closed disc that corresponds to the above circle.

Similarly the the edge or the arc connecting two vertices of a graph is homeomorphic to a closed interval.

A path is a sub continuum of \( R^2 \) which is the union of number of arcs. It is clear that the path is also homeomorphic to a closed interval. Suppose \( S \) is a closed
curve which is the subset of a connected graph G. Let us adopt
the following notations.

Definition 1.3 Given a connected graph G, we denote
the set of its vertices by V(G), the number of its vertices
by n(G), and the set of its edges by E(G). Suppose S ⊆ G
be a closed curve. We set SCG = SC ∩ G and SCintG, to
be equal to the set SCG minus E(G) i.e. SCintG = SCG − (E(G))

Note that the set SCintG is not necessarily a connected
set. Let G be a connected graph and suppose
\((a_i)_{i=1}^{n(G)} ⊆ G\) be the set of all vertices of G.
Furthermore for the vertices located on a closed curve S ⊆ G
the sequence \((a_i) ⊆ S\) of vertices have clockwise
orientation, ie' a1 is located to the right of a0 and ai+1
are called neighbouring vertices if they are
connected by an edge in V(G).

Definition 1.4 For a given connected graph G, The
function f from V(G) to the set \{1,2,3,4\} is called
colorization if for any two neighboring vertices \(a_i\) and
\(a_{i+1}\), \(f(a_i) ≠ f(a_{i+1})\).

Definition 1.5 Following the same notations as in the
above we define Cl(G), to be the set of all colorizations
for G.

For any two points \(x,y ∈ G\), let d(x,y) be the usual
Euclidean distance of x from y in \(R^2\). Let us set the
following notations, \(d(x,G) = \min(d(x,y), y ∈ G)\) and \(d(G) = \max(d(x,y), x, y ∈ G)\). For each \(x ∈ G\) let \(D(x,d(x,G))\) be
equal to the closed Disk with the center in x and radius
\(d(x,G)\). Let us define \(D(G) = \bigcap_{x ∈ G} D(x,d(x,G))\).

Then it is easy to check that D(G) is a closed
set containing G.

Definition 1.6 Keeping the same notations as in the
above, the vertex \(x ∈ G\) is called to belong to the exterior
boundary of G, denoted by EBound(G), if \(x\) can be
connected via a path in \(R^2 \cap (G)^t\) to any point in \(D(G)^c = R^2 \cap D(G) = \{y ∈ R^2, y \notin D(G)\}\).

In the following arguments in order to shorten the
notations for a given closed curve S we write \(int(S)\)
instead of \(int(SC)\).

Definition 1.7 Keeping the same notations as in the
above, The closed curve S ⊆ G is called maximal closed
curve if and only if \(S ⊆ EBound(G)\). It is called max
closed curve if given some other closed curve \(S_1\), with
\(int(S_1) ⊋ int(S)\), then \(S_1 = S\).

Suppose S is a closed curve, \(S ⊆ G\). Now let
\(x_1, x_2, ..., x_m\) be number of vertices on \(S ∩ G\), where xi can
be reached from \(x_{i-1}\) moving clockwise on encounter
once we move from \(x_i\) to \(x_{i+1}\) including the two end
points. By \(G(x_i,x_{i+1})\) we mean the set of all the vertices
of \(G ∩ S\) that we encounter once we move from \(x_i\) to
\(x_{i+1}\) excluding the two end points. Note that since we are
moving clockwise on the closed curve we have the
following identifications, \(x_{m+1} = x_1, x_{m+2} = x_2, ..., \) and
so on. Note that moving clock wise along S, the intervals
\([x_i, x_{i+1}]\), (respectively \((x_i, x_{i+1})\)) are well defined only if
\(|i−j| ≤ m−1\). Suppose \(a_1, a_2, ..., a_m\) be the set of all the
vertices on S moving clockwise on S. Then for each
integer \(1 ≤ i\), the vertices \(a_i\) and \(a_{i+1}\) are called
neighbouring vertices. Note that S is a metric space with
the metric inherited from \(R^2\) . Given two points \(x_1, x_2 ⊆ S\) where \(x_2\) is to the right side of \(x_1\) (ie' moving
clockwise from \(x_1\) to \(x_2\), we set an order \(x_1 ≤ x_2\),
the closed and open intervals on the closed curve S have
the same definition as the ones on real line, just with
the points belong to S, ie, \([x,y] = (z ∈ S), x ≤ z ≤ y, \) and
\((x,y) = (z ∈ S), x < z < y\). In order to identify the
intervals on S clearly, we let the set \(\{a_i\} ⊆ S\) to take their
indices from the well known ring \(Z_m\). The following
theorem is going to be used throughout the rest of
the article.

Theorem 1.8 Keeping the same notations as in the
above, two max closed curves that are subsets of G can
not intersect in more than one point,

Proof Suppose \(S_1\) and \(S_2\) are are max closed curves
that are subset of G. Let us give clockwise orientation to
the above closed curves. Since \(S_1\) and \(S_2\) are not equal
there exits a point x in \(S_1\) not belonging to \(S_2\). Moving
from point \(x\) along \(S_1\) clockwise suppose \(S_1\) intersect \(S_2\)
at vertices \(y_1\) and \(y_2\). With \(y_1\) the first vertex on \(S_2\) that
\(S_1\) will encounter and \(y_2\) the last one. Now consider the
paths, \([y_1, y_2] ⊆ S_1\) and \([y_2, x] ⊆ S_1\). It is
clear that \(S_3\) the union of these three, paths is a closed
curve. In order to prove the theorem it is enough to show
the following condition, \(int(S_1) ⊋ (int(S_1) ∪ int(S_2))\)
holds. We call the above condition the U-condition. Let
\(v ∈ D(G)^c\) and w be a point in \(int(S_1) ∪ int(S_2)\). For
U-condition to hold it is enough to show that any path s
connecting v to w will intersect \(S_1\). To complete the proof
we need to prove the following lemmas.

Lemma 1.8.1 Keeping the above notations suppose
\(S_1 ⊋ \{y_1, y_2\} = S_1 \cap \{y_1, y_2\} \subseteq \{S_1\} \cap \{S_2\}\) then
the U-condition holds. Furthermore \(S_1 \cap G\) contains \(x_1\).

Proof Let \(x ∈ int(S_1)\) and \(v ∈ D(G)^c\). Suppose s is a
path connecting v to x. Then s has to intersect the
boundary of \(S_1\). Hence s either hits \([y_2, y_1] ⊆ S_1\) or hits
\([y_1, y_2] ⊆ S_1\). Suppose in its way to v it hits
\(y ∈ [y_1, y_2] ⊆ S_1\) for the last time at point
\(y ∈ [y_1, y_2] ⊆ S_1\). Now using the fact that
\(int(S_1) ∩ int(S_2) = \phi\), at point y, s either enters the
interior of \(S_1\) or enters the interior of \(S_2\). In the first case
it will hit \([y_2, y_1] ⊆ S_1\) before reaching v. In the second
case it will hit \([y_1, y_2] ⊆ S_2\) before reaching v. This
implies that \(int(S_1) ∪ int(S_2) ⊋ int(S_1)\) thus the U-
condition hold. Now we claim that
\(int(S_3) ⊋ \{y_1, y_2\} ⊆ S_1\). To show that suppose
\( x \in \{y_1, y_2\} \subseteq S_1 \) Then for \( y \in \text{int}(S_1) \) there exists a path \( s_2 = [x, y] \subseteq \text{int}(S_1) \). But if \( x \) is in \( (S_2 \cap C)^c \), then \( s_2 \) must intersect \( S_3 \). At this point the fact that \( S_3 \) does not intersects \( \{y_1, y_2\} \subseteq S_1 \) implies that \( \text{int}(S_3) \supseteq \{y_1, y_2\} \subseteq S_1 \).

Definition 1.8.2 The encounter of the closed curves \( S_i \), \( i = 1, 2 \), as stated in the above lemma is called of type-I.

Lemma 1.8.3 Keeping the above notations suppose that \( S_1 \) and \( S_2 \) intersect only on two points \( y_1 \) and \( y_2 \). Furthermore assume that moving clockwise from \( x \) on \( S_1 \) we enter the interior of \( S_2 \) at \( y_1 \) and leave it at \( y_2 \). Then the U-condition holds.

Proof Note, it is easy to see that moving clockwise on \( S_2 \) it will enter the interior of \( S_1 \) at \( y_2 \) and leave it at \( y_1 \). Let us consider the following paths, \( p_1 = \{[y_1, y_2] \subseteq S_1\} \) and \( p_2 = \{[y_2, y_1] \subseteq S_2\} \). Next define the closed curve \( S_4 = p_1 \cup p_2 \). Considering \( S_1 \) and \( S_4 \), note that \( S_1 \cap S_4 = p_1 \). This using lemma 1.8.1 implies that \( \text{int}(S_1) \supseteq \text{int}(S_4) \). Similarly we can show that \( \text{int}(S_1) \supseteq \text{int}(S_2) \), and this complete the proof.

Definition 1.8.4 The encounter of the closed curves \( S_i \), \( i = 1, 2 \), as stated in the above lemma is called of type-II.

Lemma 1.8.5 Keeping the above notations suppose that \( S_1 \) and \( S_2 \) intersect only on two points \( y_1 \) and \( y_2 \). Furthermore suppose \( S_1 \cap \text{int}(S_2) = S_1 \cap \text{int}(S_2) = \emptyset \). Then the U-condition holds.

Proof Let us define the closed curves \( S_4 \) and \( S_5 \), \( S_4 \) being equal to the union of \([y_2, y_1] \subseteq S_2 \) and \([y_1, y_2] \subseteq S_1 \). \( S_5 \) being equal to the union of \([y_1, y_2] \subseteq S_2 \) and \([y_1, y_2] \subseteq S_1 \). It is easy to see that \( \text{int}(S_5) \cap \text{int}(S_4) = \emptyset \), hence by the above lemma \( \text{int}(S_5) \supseteq \text{int}(S_1) \cup \text{int}(S_2) \cup ([y_2, y_1] \subseteq S_2) \). Where the directions of paths on \( S_i \), \( i = 1, 2 \) are the clockwise directions on \( S_i \). Since \( S_1 \cap S_5 = s_1 = \{[y_1, y_2] \subseteq S_1\} \) by lemma 1.8.1, \( \text{int}(S_3) \supseteq \text{int}(S_1) \cup \text{int}(S_2) \) and this complete the proof.

Definition 1.8.6 The encounter of the closed curves \( S_i \), \( i = 1, 2 \), as stated in the above lemma is called of type-III.

In general following the same procedure as in the above moving clockwise on \( S_1 \) from \( x \), suppose \( S_1 \) intersect \( S_2 \) first time at \( y_1 \) and last time at \( y_2 \). Then let us define the closed curve \( S_3 \), to be equal to the union of \([y_2, y_1] \subseteq S_2 \) and \([y_1, y_2] \subseteq S_1 \). \( S_5 \) to be equal to the union of \([y_1, y_2] \subseteq S_2 \) and \([y_1, y_2] \subseteq S_1 \). But \( S_1 \cap S_2 = \{[y_2, y_1] \subseteq S_1\} \). Thus since \( \text{int}(S_1) \cap \text{int}(S_2) = \emptyset \), the encounter of \( S_1 \) and \( S_2 \), is of type-I. Hence by lemma 1.8.1 \( \text{int}(S_3) \supseteq \text{int}(S_1) \), similarly we can show that \( \text{int}(S_3) \supseteq \text{int}(S_2) \) and this complete the proof of theorem 1.8.

Lemma 1.9 Suppose \( S_1 \) is a closed curve which is the subset of a connected graph \( G \). Then there exist a max closed curve \( S \subseteq G \), such that \( \text{int}(S) \) contains \( \text{int}(S_1) \).

Proof For a given closed curve \( S_2 \subseteq G \) if \( \text{int}(S_2) \supseteq \text{int}(S_1) \) then we replace \( S_1 \) by \( S_2 \). Next Using lemma 1.8 if \( S_1 \) intersect another closed curve \( S_2 \subseteq G \) in more than two points, then by the above arguments there exits another closed curve \( S_3 \subseteq G \) such that \( \text{int}(S_3) \supseteq \text{int}(S_1) \cup \text{int}(S_2) \). Now proceed inductively, after finite number of steps we will end up having a closed curve \( S \subseteq G \), such that if \( S \) intersects any other closed curve \( S_k \subseteq G \) at more than one point then \( \text{int}(S) \supseteq \text{int}(S_k) \). Further more if \( S_i \subseteq G \) is any other closed curves with \( \text{int}(S) \subseteq \text{int}(S_i) \), then \( S = S_i \).

Theorem 1.10 Let \( G \) be a connected graph. Then there exist a set of max closed curves \( (S_i)_{i=1}^{m} \), and paths \( (s_i)_{i=1}^{j=m} \), such that \( G \) is equal to the union of the above paths and the set \( Q = \bigcup_{i=1}^{j=m} (S_i \cap C \cap G) \). Furthermore \( \text{EBound}(G) = \bigcup_{i=1}^{j=m} \bigcup_{j=1}^{i=m} (s_i) \) \( (s_j) \)

Proof In order to prove the above theorem we need the following definition and lemma.

Let us consider a max closed curve \( S_i \subseteq G \), with \( G \) a connected graph. Suppose we have a sequence of max closed curves \( S_i \subseteq G \), \( i = 1, 2, \ldots, n \). Suppose for each \( i < n \), \( S_i \) is connected to \( S_{i+1} \), via a path in \( G \).

Definition 1.11 Keeping the same notation as in the above. The sequence of closed curves \( (S_i) \), \( i = 2, 3, \ldots, n \) are called descenders of \( S_i \).

Lemma 1.12 Keeping the same notation as in the above, \( S_i \subseteq G \), \( i = 1, 2, 3, \ldots, n \) be a sequence of max closed curves. Then the above max curves can be connected to each other only by unique path \( s \subseteq G \). Furthermore if \( S_2 \) and \( S_3 \) are both connected to \( S_1 \) via paths in \( G \), then \( S_2 \) and \( S_3 \) or any of their descenders cannot be connected to each other via paths in \( G \).

Proof Follows from the definition of max closed curves and the arguments of lemma 1.8.

Note that the above statements holds if we consider the vertices of the path connecting the max closed curves as max closed curves. In particular if we retract each one of the max closed curves to a point the resulting will be a tree.

Finally theorem 3.22 implies that any one of max closed curve or corresponding connecting paths between them are subset of \( \text{EBound}(G) \). But by theorem 1.10 any vertex of \( z \in G \) is either a member of \( S_i \cap C \cap G \) for some max closed curve \( S_i \subseteq G \) or is a vertex on one of the paths \( s_{i,j} \) connecting two max closed curves \( S_i \subseteq G \) and \( S_j \subseteq G \). This complete the proof of theorem 1.10.

The following lemmas can be verified immediately therefore we skip the proofs.

Lemma 1.13 Keeping the same notations as before. Given a path \( s \subseteq G \), then the vertices on \( s \) can be separated by using two colors.
Lemma 1.14 Given a closed curve \( S \subseteq G \), with \( G \) a connected graph. Then if the number of vertices on \( S \), is even then the vertices on \( S \) can be separated by two colors. Otherwise three color would be sufficient to separate all the vertices on \( S \).

At this point suppose That \( G \) is a connected graph and \( S \subseteq G \) be a max closed curve. Now let \( S_1 \subseteq G \) be another closed curve with \( S_1 \cap S \subseteq SC \), and such that \( S_1 \cap \text{int} G = \phi \). Then \( G \) is equal to the union of \( (S_1 \cap G)^c \) and \( S_1 \). Assuming that \( G \) is located on the surface of an sphere implies that there exists another connected graph \( G_1 \sim G \) such that \( S_1 \) is max closed curve in \( G_1 \). We denote \( S_1 \) a semi max close curve.

Definition 1.15 We say a connected graph \( G \) has property \( \Gamma \) if there exist a colorization \( cl \in Cl(G) \) such that on every semi max close curve \( S \subseteq G \),

\[
\text{Card}(cl(S)) \leq 3.
\]

Now it is clear to see that if \( G_1 \) is a connected graph with property \( \Gamma \) and \( G_2 \) another connected graph such that \( G_1 \sim G_2 \), then \( G_2 \) has property \( \Gamma \) too. Next let \( G \) be a connected graph and \( S \subseteq G \) a closed curve. Let \( (a_i) \) \( i = 1, 2, \ldots, m \) be a sequence of all the vertices on \( S \).

Definition 1.16 Let \( a_i, a_j \in V(S) \). Given a path \( s \subseteq SC \cap G \), connecting \( a_i \) to \( a_j \). Note consider the closed curve \( S_1 \subseteq G \) that is the union of \( \{ [a_i, a_j] \subseteq S \} \) and \( S \). We call the path \( S \) reducible if there exists another internal path \( S_1 \subseteq SC \cap G \), between two different points \( c \leq d \in G[a, b] \subseteq S \), such that for the closed curve \( S_2 = \bigcup [c, d] \subseteq S \cup S_1 \), \( S_2 \cap C \) is a proper subset of \( S \cap C \). Otherwise \( S \) is called irreducible. Suppose there exist two vertices \( a_i, a_j \in V(S) \) and irreducible path in \( \text{int}(S) \), connecting \( a_i \) to \( a_j \). If \( j - i > 1 \) we say \( S \) is of type-I. Otherwise if \( j = i + 1 \), we say \( S \) is of type-II. If \( S \) is not of type-I or type-II, we say \( S \) is of type-III.

Lemma 1.17 Given a closed curve \( S \subseteq G \) with \( G \) a connected graph. Let \( (a_i) \), \( i = 1, 2, \ldots, m \) be the set of vertices of \( S \) moving clockwise on \( S \). Suppose there exists a vertex \( a_j \subseteq S \), such that the connected component of \( SC \) that contains \( a_j \) and does not contains any edges in \( S \), contains another vertex \( S \equiv a_k \neq a_j \), then there exits an irreducible path in \( SC \cap G \) connecting \( a_j \) to a vertex \( a_j \neq a_k \subseteq S \).

Proof If a connected component \( G_1 \subseteq G \) as in the above contains \( a_j \) and another vertex \( a_k \neq a_j \), then by the fact that \( G_1 \) is connected graph too, we have a path \( s \subseteq G_1 \) that connect \( a_j \) to \( a_k \). Then it is clear that there exits a vertex \( a_i \in S \) and irreducible path \( s \subseteq G \), connecting \( a_j \) to a vertex \( a_j \neq a_i \subseteq S \).

At this point we are going to introduce an special subset \( \Gamma \subseteq R^2 \), \( \Gamma = (n(G), \text{Spect}(G)) \), \( G \) a connected graph.

We impose ordering \( \leq \) on \( \Gamma \) by \( \Gamma \equiv (x_1, y_1) \leq (x_2, y_2) \in \Gamma \) if \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \). It is easy to see that \( \leq \) is partial ordering on \( \Gamma \). Now using the fact that \( k \) the set \( \text{spec}(k) = (\text{spec}(k), G) \) a connected graph and \( n(G) = k \) is a finite set, we can order the above set by the magnitude of its members. Hence set \( \text{spec}(k) = (r_i)_{i=1}^{\phi} \).

Where for \( i > j \), \( r_i > r_j \), \( r_1 = \text{lower bound} \text{spec}(k) \) \( r_n = \text{upper bound} \text{spec}(k) \). Also to facilitate our notations for a closed curve \( S \subseteq G \) we replace \( C(SCG) \) by \( Cl(S) \).

Lemma 1.18 Let \( S_1 \) and \( S_2 \) be subsets of connected graph \( G \) both being max closed curves. Suppose the above two closed curves are connected by a path \( S \subseteq G \) or a common vertex \( q \) and that they have colorizations \( cl_i \in Cl(S_i) \) \( i = 1, 2 \). Let us define connected graph \( G_i = S_1 \cup S_2 \cup S \). Then there exists a colorization \( cl_i \in Cl(G_i) \) extending \( cl_1 \) or \( cl_2 \).

Proof By lemma 1.12 Points on \( S \) can be separated by two colors, \( \text{Cl}_1 \) can be extended to \( S_1 \cup S \) in an obvious way. Suppose the point \( b \in S_2 \cup S \) is connected by an edge in \( s \) to the point \( c \neq b \) in \( s \). If \( cl_1(b) = i = cl_2(b) = j \) then we are done. Otherwise if \( i \neq j \) we define \( cl_1 \in Cl(S_2) \) by \( cl_1(a) = i \), if \( cl_2(a) = j \), \( cl_1(a) = j \), if \( cl_2(a) = i \) or else \( cl_1(a) = cl_2(a) \). Now \( cl_1 \) can be extended to \( G_1 \) and we are done in this case. Finally If \( S_1 \) and \( S_2 \) are connected by a common vertex \( q \) and \( cl_1(q) = i = cl_2(q) = j \) then we are done. Otherwise repeating the above arguments we can define a colorization \( cl_1 \) on \( S_2 \cup S_1 = G_1 \) and we are done.

At this point note that every connected subset the connected graph \( G \) is also a connected graph. As before we define an interior graph to be a subgraph of \( G \) that does not contain any element from the set \( E(G) \).

Theorem 1.19 Given a closed curve \( S \subseteq G \) Suppose every max closed curve, \( S \subseteq G \) has property \( R \). Then \( G \) has a colorization.

Proof Suppose \( G \) is a connected graph and let \( (S_j)_{j=1}^{\phi} \) be the set of all max closed curves in \( G \). Let \( (s_j)_{j=1}^{\phi} \) be the set of all paths in \( G \) connecting max closed curves. Where some of these max closed curves consists of single vertex. Therefore by theorem 1.10 and lemma 1.12 we can assume that \( G \) is equal to the union of the max closed curves, the part of \( G \) they contain in their interior and the paths in \( G \) that connecting them. By lemma 1.12 there are unique paths between the maximal closed curves and for any two maximal closed curve that are connected to a third one their descendents are not connected to each others via paths or common vertices. Now our assumption states that for any one of max closed curve \( S \subseteq G \), \( SC \) has property \( R \). Then each of the maximal closed curves \( S \subseteq G \) has a colorization \( cl_1 \) acting on \( SC \). Next let us begin from the max closed curve \( S_1 \) with a colorization \( cl_1 \in SC \subseteq G \) and all max closed curves \( S_{1j} \subseteq G \) that are connected to \( S_1 \) by the paths \( S_{1j} \subseteq G \). For each of the max closed curves \( S_{1j} \), set \( G_1 = S_1CG \cup S_{1j} \). Now by lemma 1.12, we can extend \( cl_1 \) to become a colorization on \( G_1 = S_1CG \cup G_{1j} \). Now repeating the above process from...
each of the max closed curves $S_{1r}$, using lemma 1.18 we are going to extend $c_{1r}$ to become a colorization on each of the descenders of $S_{1r}$ and ultimately after repeating the above process finitely many times we will get a colorization acting on $G$.

2. The Main Result

In this section we are going to state the main theorem and its proof.

Lemma 2.20 Suppose G is a connected graph and S a subset of $G$ which is the max closed curve with $SCG = G$. Then $S$ has property R.

Proof In order to prove the theorem we need to state and proof the following lemmas

Lemma 2.20.1 Keeping the same notations as in the above suppose $S$ is of type-II. Then there are two vertices $a_{1} \in G(S)$, $i = 1, 2$ and an irreducible path $s_{1} \subseteq int(S)$ connecting the above two vertices. Furthermore there exist two connected graphs $G^{1}$ and $G^{2}$ each containing a max closed curve, $S_{1} \subseteq G_{1}$, $S_{1}^{1} \subseteq G^{1}$ such that $S_{1}CG^{1} = G^{1}$ and $S_{1}CG^{1} = G^{1}$. Finally we have $int(S) \supseteq int(int(S)) \cup int(s_{1}) \cup int(S_{1}) \supseteq int(int(S))$ and $G - G_{1}$.

Proof Set $S_{1} = s_{1} \cup \{a_{1}, a_{2}\} \subseteq S$, then $S_{1} \subseteq G$ is a closed curve. Next we can construct a path $s_{1} \subseteq G^{c}$ connecting $a_{1}$ to $a_{2}$. Let $G^{1} = S_{1}CG^{1}$, $G_{1} = G^{1} \cup S_{1}^{2}$ and $S_{1} = s_{1} \cup S_{2}$. It is clear that $S_{1} \subseteq G_{1}$ is a max closed curve in $G_{1}$ and $S_{1}CG_{1} = G_{1}$. Next following their constructions, $int(S_{1}) \supseteq int(int(S)) \cup int(s_{1})$ and $int(S) \supseteq int(int(S))$.

Before proceeding to the next lemma we need to bring the following definition

Definition 2.20.2 Let $G_{1}$ and $G_{2}$ be a connected graph and $S_{1} \subseteq G_{1}$, $i = 1, 2$ be a max closed curves with $S_{1}CG = G_{1}$, $i=1,2$, then we call $S_{1}$ a full closed curve. In particular if $G_{1} \sim G_{2}$ then we write $S_{1} \sim S_{2}$.

Lemma 2.20.3 Let $S \subseteq G$ be as in the above. Then $S$ is either equivalent to type-I or to type-III closed curve.

Proof If $S$ is type-I or type-III we are done. Otherwise following notations and arguments of lemma 2.20.1, consider the full closed curve $S_{1} \subseteq G_{1}$ and $S_{1}^{1} \subseteq G_{1}$. $G_{1}$ connected graph. We had $S - S_{1}$, $int(S_{1}) \supseteq int(S_{1})$, $V(S_{1}) = V(S) = V(G) \cup S_{1}$, $int(S) \supseteq int(S)$. Furthermore any path in the interior of $S_{1}$ connecting to vertices of $S_{1}$ lives in the interior of $S_{1}$. Now if $S_{1}$ is of type-I or type-III we are done, otherwise proceeding as in the above we have two closed curve $S_{2}$ and $S_{2}^{1} \subseteq G$ with the following properties. $S_{2}$ is a full max closed curve subset of a connected graph $G_{2} \sim G$, $int(S_{2}) \supseteq int(S_{2})$, $int(S_{2}) \supseteq int(S_{2})$, $int(S_{2}) \supseteq int(S_{2})$ and the interior paths connecting vertices of $S_{2}$ live in the interior of $S_{2}$. Proceeding by induction and because $G$ is a finite graph after finitely many steps we get the sequence of closed curves, $S$, and $S_{1} \leq S \leq m_{0}$ with the following properties. $S_{1} - S$, $int(S_{1}) \supseteq int(S_{1})$, $int(S_{1}) \supseteq int(S_{1})$ and such that any interior paths connecting the vertices of $S_{1}$ lives in the interior of $S_{1}$. Hence by the fact that $G$ is finite at some stage $m_{0}$, either $G_{m_{0}}$ is of type-I or of type-III.

The proof of the lemma is complete.

To complete the proof of the lemma we use induction on $n(G)$. So suppose given an integer $k$ such that the statement of lemma hold for all connected graph having only one max closed curve $S$ with $n(G) \leq k$. Let $a_{1}, a_{2}, \ldots, a_{m}$ be a set of all vertices on $S \subseteq G$, where $ai$ can be reached from $a_{i-1}$ moving clockwise on $S$. If all connected component of $G$ intersect $S$ at most at one point, ie, $S$ is of type-III. In this case by lemma 1.14 in we can construct a colorization $cl \in Cl(S)$, $cl(S) \leq 3$. Now we want to extend $c_{1}$ to become a colorization on all $G$. Since for each of the connected components $G_{i} \subseteq G$, $n(G_{i}) \leq k$, then by induction and theorem 1.19, $G_{i}$ has property R. Now using lemma 1.18 we can extend $c_{1}$ to become a colorization for $G$ thus $G$ has property R. Otherwise by the above lemma 2.20.3 we can assume that $S$ is of type-I. Hence for some vertex $aj \in S$ there exists an irreducible interior path $s \subseteq SCintG$ from $aj$ to $al$ where $aj$ and al vertices of $S$, moving clockwise on $S$ with $j + 1 < l$. Therefore we will get a closed curve $S \subseteq G$ which is the union of $s$ and the path $[aj, al] \subseteq S$. Now note that $S_{1}CG$ is a connected graph with $n(S_{1}) = S_{1}(G \cap G) = k$. Hence by induction assumption $G_{1} = S_{1}CG$ has property R therefore there exists a colorization $cl \in Cl(G_{1})$ such that $Card(cl(S_{1})) \leq 3$. Now note that any connected component of $G$ that contains a vertex in $G[a_{j}, a_{l}] \subseteq S$, does not intersect $S$ at any other point. Hence using lemma 1.14 and lemma 1.18 we can extend $c_{1}$ to become a colorization on $G$ with $Card(cl(S)) \leq 3$. This complete the proof of lemma 2.20.

Theorem 2.21 Every connected graph $G$, has property R thus has a colorization.

Proof By lemma 2.20 each one of max closed curves has property R. Finally Theorem 1.19 completes the proof.

Finally we are going to bring a short proof of De Bruijn Erdos theorem for locally finite infinite planar graphs

Theorem(De Bruijn Erdos) Suppose $G$ is a locally finite infinite graph. Then $G$ has a colorization.

Proof Note that without loss of generality we can assume that $G$ is connected. Let $v \in G$ be a vertex in $G$. Consider all the paths $s \in E(G)$ of length $m \in N$ from $v$ to other vertices in $G$. The union of these paths is a finite connected subgraph $G_{m} \subseteq G$. By theorem(2.21), there exists a colorization $cl_{m} \in Cl(G_{m})$. Next $Cl(G_{m})$ can be considered as a subset of $R^{4}(G_{m})$ with integer entries. Let us pick randomly one of these graphs say $G_{1} = G_{m0}$. Thus there exist an infinite subsequence $(m_{i})_{i=0}^{\infty}$ of $(N)$, and a colorization $cl_{\infty} \in Cl(G^{i})$ such that for each Gini the restriction of $cl_{m_{i}}$ to $G_{1}$ will agree with $cl_{\infty}$. Proceeding inductively we can construct an increasing sequence of finite subgraphs of $G$, $G^{1} \subseteq G^{2} \subseteq \ldots \subseteq G^{\infty} \subseteq \ldots$, where for each $\leq k$, $G_{k}$ has a colorization $cl_{k}\infty$, with the property that the restriction of $cl_{k\infty}$ to $G^{k}$ will be equal to $cl_{k\infty}$. But $G = S(G^{1})$, and this complete the proof.

3. Appendix

In this section we are going to proof some technical lemmas needed in the proof of number of the lemmas and theorems in the above

Theorem 3.22 Keeping the same notations as in the above. let $S \subseteq G$ be a max closed curve. Then $S \subseteq EBound(G)$.

Proof As we demonstrated, if $(S_{i})_{i=1,2,\ldots}$ be the set of all max closed curves, then $G$ is the union of the sets
S\_CG together with the paths s\_ij, i\_j = 1, 2, \ldots, m, where s\_ij is the unique path connecting S\_i to S\_j. By the finiteness of the Graph G we can assume that each of the max closed curves is a circle and the connecting paths are straight lines going through the centers of corresponding circles. Next for each of the circles Si there exists a circle S\_j, with the same center but strictly larger than Si. By taking S\_(i,j) close enough to Si we can make sure that S\_(i,j) does not intersect any other max closed curves or the paths connecting them to any other max closed curve other than Si. set b\_ij be the intersection of s\_ij with S\_j. Now moving clockwise on S\_(i,j) consider two points b\_i\_j\_1 < b\_i\_j < b\_i\_j\_2 on S\_(i,j). We can choose them close enough so that if we draw lines s\_i\_j\_1 and s\_i\_j\_2, parallel to s\_i\_j intersecting S\_(i,j) at b\_i\_j\_1 and b\_i\_j\_2 respectively, with b\_i\_j\_2 > b\_i\_j, then s\_i\_j\_k, k = 1, 2 intersect G at S\_i and S\_j only. Next let a\_i\_j\_k, k = 1, 2, be the intersection of s\_i\_j\_k with S\_i,1, 2 with S\_j respectively. let us define the close curved S\_(i,j) to be defined to be the union of [a\_i\_j\_1, a\_i\_j\_2] ⊆ s\_i\_j,1, [a\_i\_j\_2, a\_i\_j\_1] ⊆ s\_i\_j,2 and [a\_i\_j\_2, a\_i\_j\_1] ⊆ s\_i\_j,2. Then it is clear that S\_(i,j) contains s\_ij in its interior. At this point we assume that Si is connected to the sequence of max closed curves (S\_j, i = 2, \ldots, m), where each of the above closed curves is connected to Si only. Next for each integer 2 ≤ j ≤ m we define a loop Ω\_(i,j) ⊆ (G\_j) to be a path which is the union of following paths, [b\_i\_j\_1, b\_i\_j\_2] ⊆ s\_i\_j,1, [b\_i\_j\_2, b\_i\_j\_1] ⊆ s\_i\_j,2 and [b\_i\_j\_1, b\_i\_j\_2] ⊆ s\_i\_j,1. Now suppose S\_j is connected to more than one closed cure S\_k \subseteq G, k = 2, \ldots, m. Ordering the points of intersection of s\_ik with S\_j. Let a\_ik, j = 1, 2, \ldots, m, be the sequence of the above vertices. At this point we assume that The sequence of closed curves that are connected to S\_i, j = 1, 2, \ldots, m, are not connected to any other max closed curve. Next for a fixed 2 ≤ j ≤ m we define a loop Ω\_(i,j) ⊆ (G\_j) to be a path going from b\_i\_j\_1 ∈ S\_j\_i\_j\_1 to S\_j\_i\_j\_2 and back to S\_j\_i\_j\_1. Now we want to extend the definition of loop to more complicated case. we want to define a path Ω\_(i,j) ⊆ (G\_j) going from b\_i\_j\_1 ∈ S\_j\_i\_j\_1 to S\_j\_i\_j\_2 and back to S\_j\_i\_j\_1. Set Ω\_(i,j) to be equal to the union of the above paths, [b\_i\_j\_1, b\_i\_j\_2] ⊆ s\_i\_j,1, [b\_i\_j\_2, b\_i\_j\_1] ⊆ s\_i\_j,2 and \[b\(i,j,1\), b\(i,j,2\)] ⊆ s\_i\_j,1. Using our assumptions each one of the paths Ω\_(i,j,k), is well defined hence the above formula is going to identify the path Ω\_(i,j) ⊆ G\_j. We call the above formula the loop formula. We saw that the loop formula is valid for two special cases. We want to show that the loop formula will identify the path Ω\_(i,j) ⊆ G\_j from S\_i\_j\_i\_j\_1 to S\_j\_i\_j\_2 and back to S\_j\_i\_j\_1 in the most general case. By the loop formula we can identify Ω\_(i,j) ⊆ G\_j, hence we can identify Ω\_(i,j,k) for all max closed curves connected to S\_j\_i\_j\_1 except Si. Similarly to identify Ω\_(i,j, k), it is enough to identify all the loops Ω\_(j, k) where S\_j\_j\_j\_j\_j\_k is one of the max closed curves that are connected to S\_j\_i\_j\_1, moreover for each 2 ≤ k, S\_i\_j\_j\_j\_k is one of the max closed curves that is connected to S\_j\_j\_j\_j\_j\_k. We call this the stage (j\_i\_j\_j\_j\_j\_j\_j, j\_j\_j\_j\_j\_j\_j\_j) of calculation. To complete the proof of the theorem we have to show that for any point x ∈ G, v ∈ D(G\_j), there exists a path s \subseteq G\_j connecting x to v. Next we need to employ some new notation. As we mentioned we say two max closed curves S\_1 \subseteq G and S\_2 \subseteq G are connected, if they are connected by the path S\_1\_2. By chain of max closed curves we mean a sequence (S\_i), i = 1, 2, \ldots, m such that for each i ≤ m, Si and S\_i\_1 are connected. Now without loss of generality we can assume S\_1 \supseteq x = a\_i\_j\_1 for some max closed curve S\_i \subseteq G, where S\_j\_i\_j\_1 is a closed curve connected to S\_i. The other cases can be overcome using some minor technicalities. Now there exists a path s\_i \subseteq (S\_i)\_i\_j\_1 from a\_i\_j\_1 to h\_i\_j\_1, furthermore by the structure of G we can assume that in fact s\_i \subseteq G\_j. Also there is a path s\_2 \subseteq S\_2\_i\_j\_1 from b\_i\_j\_1 to v. Moving from v to b\_i\_j\_1 on s\_2 suppose the first point that we intersect G is y = a\_i\_j\_1 ∈ S\_1. Where S\_1, S\_i\_1 \subseteq G are connected max closed curves. The other cases can be overcome with minor technicalities. At this point without loss of generality we can assume that s\_2 will intersects S\_i\_1 first time at b\_i\_j\_1. But by the structure of G, the exists a chain of closed curves (S\_i\_1)\_i\_j\_1 \subseteq S\_i\_1, \ldots, S\_i\_1\_j\_1 \subseteq S\_i\_1\_j\_2 \subseteq S\_i\_1\_j\_1 \subseteq S\_i\_1\_j\_2 \subseteq S\_i\_1\_j\_1. Next let us define the path s\_3 \subseteq G\_j, by s\_3 = \bigcup_{i\_j\_1=1}^{i\_j\_2} \Omega\_(i\_j\_1-1,i\_j\_2) \bigcup [b\_i\_j\_1,b\_i\_j\_2]. In this paths all the intervals are on S\_i\_1, and it takes point b\_i\_j\_1 to point b\_i\_j\_2. Note since we number the max closed curves connected to S\_1\_1, it takes point b\_i\_j\_1 to point b\_i\_j\_2. Let us denote s\_3 by N\_(h\_i\_j\_1,h\_i\_j\_2). (It is also clear that if none of the points in the interval [b\_i\_j\_1,b\_i\_j\_2] are connected to another max closed curve then N\_(h\_i\_j\_1,h\_i\_j\_2) = [b\_i\_j\_1,b\_i\_j\_2]. Now for an integer i ≤ k, consider the path s(i\_i\_1 + 2) \subseteq G\_j, that is then union of
\[ \{b_{k,j+1,1}, b_{2,3}\} \subseteq S_{ij} \quad \text{and} \quad N\{b_{i+1,j,2}, b_{i+1,j+2,1}\}. \]

Finally consider the paths
\[ s_4 = \left( \bigcup_{i=2}^{k-2} s(i, i + 2) \right) \subseteq G^c, \]
\[ s_5 = \left[ b_{k,k+1,1}, v \right] \subseteq s \cap G^c \quad \text{and} \quad s_6 = \left[ a_{k,k+1}, b_{k,k+1} \right] \subseteq G^c. \]

At this end the path
\[ s_7 = s_5 \cup s_6 \cup s_4 \cup s_3 \cup s_1 \]
is a path in \( G^c \) taking \( x \) to \( v \) and this complete the proof of the theorem.

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