THE FARRELL-JONES CONJECTURE FOR HYPERBOLIC-BY-CYCLIC GROUPS

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Abstract. We prove the Farrell-Jones Conjecture for mapping tori of automorphisms of virtually torsion-free hyperbolic groups. The proof uses recently developed geometric methods for establishing the Farrell-Jones Conjecture by Bartels-Lück-Reich, as well as the structure theory of mapping tori by Dahmani-Krishna.

1. Introduction

The Farrell-Jones Conjecture (which we will frequently abbreviate by FJC) was formulated in [FJ93]. For a group $G$, the Farrell-Jones Conjecture (relative to the family $\mathcal{VCyc}$ of virtually cyclic subgroups) predicts an isomorphism between the $K$-groups (resp. $L$-groups) of the group ring $RG$ and the evaluation of a homology theory on a certain type of classifying space for $G$. Such a computation, at least in principle, gives a way of classifying all closed topological manifolds homotopy equivalent to a given closed manifold of dimension $\geq 5$, as long as the FJC is known for the fundamental group. Particularly significant consequences include the Borel conjecture (a homotopy equivalence between aspherical manifolds can be deformed to a homeomorphism) and the vanishing of the Whitehead group $Wh(G)$ for torsion-free $G$.

There are different versions of the Farrell-Jones conjecture; we will always mean “with coefficients in an additive category with a $G$-action” and relative to $\mathcal{VCyc}$, see e.g. [BB19, Section 4.2]. This version has convenient inheritance properties, for example passing to subgroups and finite index supergroups.

The Farrell-Jones Conjecture has generated much interest in the last decade, due in no small part to the recent development of an axiomatic formulation that both satisfies useful inheritance properties, and provides a method for proving the conjecture. For example, FJC is currently known for hyperbolic groups [BLR08a], relatively hyperbolic groups [Bar17], CAT(0)
groups [Weg12], virtually solvable groups [Weg15], \( GL_n(\mathbb{Z}) \) [BLRR14], lattices in connected Lie groups [KLR16], and mapping class groups [BB19]. The reader is invited to consult the papers [BLR08c, Bar17, Bar16, BB19, BFL14] for more information on applications of FJC and methods of proof.

The primary goal of this paper is the following theorem.

**Theorem 1.1.** Let \( G \) be a virtually torsion-free hyperbolic group and \( \Phi : G \to G \) be an automorphism of \( G \). Then the hyperbolic-by-cyclic group \( G_\Phi = G \rtimes_\Phi \mathbb{Z} \) satisfies the \( K \)- and \( L \)-theoretic Farrell-Jones conjectures.

Conjecturally, all hyperbolic groups are virtually torsion-free. In our proof we quickly reduce to the case of torsion-free hyperbolic groups, which simplifies the arguments. It is possible that a similar proof can be given for general hyperbolic groups, but we decided against attempting such a proof in view of the new technical difficulties and the above conjecture.

Theorem 1.1 generalizes the case when \( G \) is a finite rank free group, which we proved in [BFW], and the present paper supersedes it, so [BFW] will not be published independently. Brück, Kielak and Wu generalized this result to infinitely generated free groups and further to normally poly-free groups in [BKW21].

Perhaps the most important consequence is the following:

**Theorem 1.2.** Let \( G \) be a virtually torsion-free hyperbolic group and let

\[
1 \to G \to H \to Q \to 1
\]

be a short exact sequence of groups. If \( Q \) satisfies the \( K \)-theoretic (resp. \( L \)-theoretic) Farrell-Jones Conjecture, then so does \( H \).

The group \( G_\Phi \) appearing in the statement of Theorem 1.1 depends only on the outer class of \( \Phi \), which we denote by \( \phi \), and sometimes we use \( G_\phi \) for \( G_\Phi \). Outer automorphisms of torsion free hyperbolic groups are well understood (see [Sel97a, Lev05, GL07] for a small sample). Broadly speaking, the tools used to study such automorphisms differ based on the number of ends of \( G \): when \( G \) is one-ended, techniques from JSJ decompositions are used; when \( G \) has infinitely many ends, one uses the Grushko decomposition and the associated relative Outer space. Our proof of Theorem 1.1 is no different; though our techniques in the latter case are further divided according to whether the automorphism \( \phi \) is polynomially growing or exponentially growing.

The structure of this paper is as follows: Section 2 contains the background material on the Farrell-Jones Conjecture necessary for our purposes. In particular, we review the geometric group theoretic means for proving FJC developed in [FJ93] and subsequently refined and generalized in [BLR08a, BLR08b]. In Section 3 we treat the case that \( G \) is a one-ended hyperbolic group. Then in Section 4, we treat polynomially growing automorphisms of groups with infinitely many ends. The case of exponentially growing automorphisms is easy in comparison, thanks to the recent structure.
theorem of Dahmani-Krishna [DK20] which asserts that $G_\Phi$ is relatively hyperbolic; it is handled in Section 5. In Section 6 we prove Theorem 1.2 using standard techniques. The last section contains a proof of a technical result needed for the polynomially growing case.

2. Background

For the remainder of this paper, $G$ will denote a finitely generated group. While the Farrell-Jones Conjecture originated in [FJ93], its present form with coefficients in an additive category is due to Bartels-Reich [BR07] (in the $K$-theory case) and Bartels-Lück [BL10] (in the $L$-theory case). We restrict ourselves to the axiomatic formulation.

A family $F$ of subgroups of $G$ is a non-empty collection of subgroups which is closed under conjugation, taking subgroups and finite index supergroups. For example, the collection $\text{Fin}$ of finite subgroups of $G$ is a family, as is the collection $\text{VCyc}$ of virtually cyclic subgroups of $G$.

2.1. Geometric Axiomatization of FJC. We first recall a regularity condition that has been useful in recent results on the Farrell-Jones conjecture [BLR08b, Bar17, Kno17]. Let $X$ be a space on which $G$ acts by homeomorphisms and $F$ be a family of subgroups of $G$. An open subset $U \subseteq X$ is said to be an $F$-subset if there is $F \in F$ such that $gU = U$ for $g \in F$ and $gU \cap U = \emptyset$ if $g \notin F$. An open cover $U$ of $X$ is $G$-invariant if $gU \in U$ for all $g \in G$ and all $U \in U$. A $G$-invariant cover $U$ of $X$ is said to be an $F$-cover if the members of $U$ are all $F$-subsets. The order (or multiplicity) of a cover $U$ of $X$ is less than or equal to $N$ if each $x \in X$ is contained in at most $N + 1$ members of $U$.

**Definition 2.1.** Let $F$ be a family of subgroups of $G$. An action $G \acts X$ is said to be $N$-$F$-amenable if for any finite subset $S$ of $G$ there exists an open $F$-cover $U$ of $G \times X$ (equipped with the diagonal $G$-action) with the following properties:

- the multiplicity of $U$ is at most $N$;
- for all $x \in X$ there is $U \in U$ with $S \times \{x\} \subseteq U$.

An action that is $N$-$F$-amenable for some $N$ is said to be **finitely F-amenable**. We remark that such covers have been called wide in some of the literature.

2.2. The class $AC(\text{VNil})$. Following [BB19] we now define the class of groups $AC(\text{VNil})$ that satisfy suitable inheritance properties and all satisfy FJC. Let $\text{VNil}$ denote the class of finitely generated virtually nilpotent groups. Set $\text{ac}^0(\text{VNil}) = \text{VNil}$ and inductively $\text{ac}^{n+1}(\text{VNil})$ consists of groups $G$ that admit a finitely $F$-amenable action on a compact Euclidean retract (ER) with all groups in $F$ belonging to $\text{ac}^n(\text{VNil})$. The action on a point shows that $\text{ac}^n(\text{VNil}) \subseteq \text{ac}^{n+1}(\text{VNil})$ and we set $AC(\text{VNil}) = \bigcup_{n=0}^{\infty} \text{ac}^n(\text{VNil})$.

**Proposition 2.2** ([BB19]). (i) $AC(\text{VNil})$ is closed under taking subgroups, taking finite index supergroups, and finite products.
(ii) All groups in $\text{AC(VNil)}$ satisfy the Farrell-Jones Conjecture.

Our main result can now be stated as follows.

**Theorem 2.3.** Let $G$ be a torsion-free hyperbolic group and let $\Phi : G \to G$ be an automorphism. Then $G_\Phi = G \rtimes_\Phi \mathbb{Z}$ belongs to $\text{AC(VNil)}$.

If $G$ is only virtually torsion-free, there is a finite index characteristic subgroup $H < G$ which is torsion-free, and then $H_\Phi$ has finite index in $G_\Phi$, so Theorem 1.1 follows from Theorem 2.3 and Proposition 2.2(i).

The following two theorems of Knopf and of Bartels will be crucial. Recall that an isometric action of a group $G$ on a tree $T$ is acylindrical if there exists $D > 0$ such that whenever $x, y \in T$ are at distance $\geq D$ then the stabilizer of $[x, y]$ has cardinality $\leq D$.

**Theorem 2.4 ([Kno17, Corollary 4.2]).** Let $G$ act acylindrically on a simplicial tree $T$ with finitely many orbits of edges. If all vertex stabilizers belong to $\text{AC(VNil)}$ then so does $G$.

**Theorem 2.5 ([Bar17]).** Suppose $G$ is hyperbolic relative to a finite collection of subgroups, each of which is in $\text{AC(VNil)}$. Then $G$ also belongs to $\text{AC(VNil)}$.

## 3. One-ended groups $G$

This section is dedicated to proving our main result for one-ended groups using JSJ decompositions and an analysis of the action of $G_\Phi$ on the associated Bass-Serre tree. Throughout this section, $G$ will denote a torsion-free one-ended Gromov hyperbolic group.

**Proposition 3.1.** Let $G$ be a one-ended torsion-free hyperbolic group and $\Phi \in \text{Aut}(G)$. Then the hyperbolic-by-cyclic group $G_\Phi$ is in $\text{AC(VNil)}$.

### 3.1. JSJ decompositions

The group $G$ (after Rips and Sela [RS94, Sel97b], see also [Bow98, DS99, FP06, GL17]) has a JSJ decomposition, $\Lambda$, which is a finite graph of groups such that $\pi_1(\Lambda) = G$, each edge group is cyclic, and each vertex group is of one of the following types:

- cyclic,
- quadratically hanging (or QH), or
- rigid.

QH vertex groups are represented by compact surfaces that carry a pseudo-Anosov homeomorphism (pA) and whose boundary components exactly represent the incident edge groups. Rigid vertex groups are non-cyclic quasi-convex subgroups. A degenerate case occurs when $G$ is a closed surface group (that carries a pA); in that case the JSJ decomposition is a single QH vertex. The key property of JSJ decompositions is that they are “maximal” such decompositions, and we will need the following manifestation of it. Let $\text{Out}_A(G)$ be the group of “visible” automorphisms, i.e. the subgroup of $\text{Out}(G)$ generated by Dehn twists in edge groups and in 1-sided simple closed curves in surfaces representing the QH vertices.
Theorem 3.2 ([Sel97a]). \( \text{Out}_\Lambda(G) \) has finite index in \( \text{Out}(G) \).

An important special case, when \( G \) has no splittings over \( \mathbb{Z} \), was proved earlier by Paulin [Pau91].

If we let \( T \) be the Bass-Serre tree associated with \( \Lambda \), then for every Dehn twist above, and hence for every element \( \phi \in \text{Out}_\Lambda(G) \), and for every lift \( \Phi \in \text{Aut}(G) \), there is an isometry \( t : T \to T \) which is \( \Phi \)-equivariant. It follows that the group \( G_\Phi \) also acts by isometries on the same tree \( T \). This action may not be acylindrical, and our proof consists of modifying the JSJ decomposition in order to construct an acylindrical action and apply Theorem 2.4. All our modifications will have the property that \( t \) and \( G_\Phi \) continue to act by isometries, and we retain the name \( t \).

Since \( G_{\phi^k} \) has index \( k \) in \( G_\phi \) when \( k > 0 \), inheritance to finite index supergroups implies that we are free to replace \( \phi \) by a power when proving Theorem 2.3. To begin with, we are assuming, by passing to a power, that

- \( \phi \in \text{Out}_\Lambda(G) \), and
- the restriction of \( \phi \) to any QH-vertex group is represented by a mapping class of the punctured surface that fixes a (possibly empty) multicurve and in each complementary component it is either identity or pseudo-Anosov.

**Modification 1.** We refine \( \Lambda \) by replacing each QH-vertex group with the graph of groups dual to the reducing multicurve in the corresponding surface. The resulting graph of groups representing \( G \) will be called \( \Lambda_1 \). Thus the number of edges of \( \Lambda_1 \) is equal to the number of edges of \( \Lambda \) plus the sum of the numbers of curves in reducing multicurves in the QH vertex groups, and each QH vertex group is replaced by a collection of vertex groups, one for each complementary component of the reducing multicurve. It will now be convenient not to distinguish between rigid vertex groups and noncyclic vertex groups coming from complementary components where \( \phi \) is identity. We will use the following classification of vertex groups in \( \Lambda_1 \).

Let \( \Phi \in \text{Aut}(G) \) be a representative of \( \phi \in \text{Out}(G) \).

(i) \( R \)-vertices: associated groups are noncyclic, and the restriction of \( \Phi \) is a conjugation by an element of \( G \),

(ii) \( pA \)-vertices: these are represented by a surface with edge groups corresponding exactly to the boundary components, and the restriction of \( \Phi \) is represented by a pseudo-Anosov mapping class, composed with conjugation,

(iii) \( Z \)-vertices: these are cyclic, and the restriction of \( \Phi \) acts by conjugation.

Let \( T_1 \) be the Bass-Serre tree corresponding to \( \Lambda_1 \).

**Modification 2.** Here we arrange that distinct edges incident to the same \( R \)-vertex in \( T_1 \) have non-commensurable stabilizers. Note that this is already
true for $pA$-vertices. Given an $R$-vertex, consider the set of incident edges whose stabilizers are in a fixed commensurability class and fold together thirds of these edges incident to the vertex (only a third is to make sure that if folding from both ends there is no intersection). Do this for all commensurability classes. The new vertices created by folding are declared to be $Z$-vertices. Call the resulting tree $T_2$. The edge stabilizers are still cyclic, and (i)-(iii) still hold. In addition we have

(iv) Distinct edges incident to a vertex of type $R$ or $pA$ have non-commensurable stabilizers.

Before the next modification, we take a closer look at the $Z$-vertices.

**Lemma 3.3.** Let $E$ be a maximal cyclic subgroup of $G$. The set of edges of $T_2$ whose stabilizer is a subgroup of $E$ is finite and the union of these edges is a subtree $Z(E)$ of $T_2$. All vertices of $Z(E)$ of valence $> 1$ are of $Z$-type and the tree $Z(E)$ intersects the closure of $T_2 \setminus Z(E)$ in the set of vertices of $R$- and $pA$-type, all of which are of valence 1 in $Z(E)$.

**Proof.** If a nontrivial subgroup of $E$ fixes two edges in an edge-path, then it fixes all edges between them, so the claim that $Z(E)$ is a tree follows. Any vertex of $Z(E)$ of valence $> 1$ must be of $Z$-type by (iv) and therefore of finite valence. It remains to show that $Z(E)$ has finite diameter. We will show that it does not exceed $2N$, where $N$ is the number of edges in the quotient graph $\Lambda_2 = T_2/G$. Indeed, assume there is a reduced edge path of $2N+1$ edges in $Z(E)$. Then there are two oriented edges $a, b$ that map to the same edge in $\Lambda_2$ with the same orientation, i.e. there is $g \in G$ with $g(a) = b$ and so that $a$ points to $b$ while $b$ points away from $a$. This implies that $g$ is a hyperbolic isometry of $T_2$, and it conjugates $E$ to $E$. Since maximal cyclic subgroups of $G$ are malnormal and $g \notin E$ this is a contradiction. \[\square\]

We say that two $R$-vertices are **equivalent** if $\Phi : G \to G$ conjugates each of them by the same $g \in G$ (which is unique since the $R$-vertex groups are not cyclic). Note that the action of $G$ preserves the equivalence classes, as does the $\Phi$-equivariant isometry $t : T_2 \to T_2$ induced by $t : T \to T$.

**Modification 3.** Here we arrange that in each $Z(E)$ the convex hulls of equivalent $R$-vertices are pairwise disjoint. Say $V$ is the set of $R$-type vertices in $Z(E)$ and $W$ is the set of $pA$-type vertices in $Z(E)$. All of them have valence 1. Write $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ by grouping equivalent vertices into one $V_i$. Remove the interior $Z(E) \setminus (V \cup W)$ of $Z(E)$. Then for each $V_i$ introduce a vertex $v_i$ and an edge joining $v_i$ to each vertex in $V_i$. Finally introduce a new vertex $v$ and an edge joining $v$ to each $v_i$ and to each $pA$-type vertex. Perform the same modification on all trees $Z(E)$, so $G$ continues to act. The stabilizer of the vertex $v$ above is $E$ and the edge stabilizers are subgroups of $E$. All of the newly introduced vertices are of type $Z$. Call the resulting tree $T_3$. We retain the previous notation, so now in addition to (i)-(iv) we also have
(v) $Z(E)$ is a tree of diameter $\leq 4$ and the convex hulls of equivalent $R$-vertices in $Z(E)$ are pairwise disjoint.

Before the final modification, we make some observations. First, $\Phi$ takes the stabilizers of vertex and edge groups of $T_3$ to other such stabilizers and induces an isometry $t : T_3 \to T_3$, which is type preserving. Thus

$$G_\Phi = \langle G, t \mid tgt^{-1} = \Phi(g) \rangle$$

acts on $T_3$. This action may not be acylindrical; for example the edges whose stabilizer is fixed by $\Phi$ will be fixed by $t$, so we have to ensure that such edges form a bounded set.

For $g \in G$ let $R(g)$ be the convex hull of the set of type $R$ vertices whose stabilizers are conjugated by $g$.

**Lemma 3.4.** There are no $pA$-vertices in $R(g)$ and all $R$-vertices in $R(g)$ are equivalent, so $\Phi$ conjugates all vertex and edge groups in $R(g)$ by $g$. For $g \neq g'$ the trees $R(g)$ and $R(g')$ are disjoint.

**Proof.** We may assume $g = 1$ by rechoosing $\Phi \in Aut(G)$. Let $V, W$ be two $R$-vertex groups fixed by $\Phi$ elementwise and consider the edge-path $[v, w]$ joining the two vertices $v, w$. Thus $t$ fixes $[v, w]$ and $\Phi$ leaves all vertex groups along $[v, w]$ invariant. Let $U$ be such a vertex group corresponding to a vertex $u \in [v, w]$, $u \neq v, w$. If $U$ is cyclic, then a finite index subgroup appears as an edge group, and so it is fixed elementwise by $\Phi$. If $U$ is of $R$-type, $\Phi|U$ is conjugation by some element $h \in G$, but since the two incident edge groups along $[v, w]$ are non-commensurable and are both fixed by $\Phi$, we must have $h = 1$, i.e. $U$ is fixed elementwise. By a similar argument, $U$ cannot be of $pA$-type (if $\Phi$ fixes the cyclic group corresponding to one boundary component, it must nontrivially conjugate all the others).

Now suppose that $g' \neq 1$ and consider $R(1) \cap R(g')$. This intersection is a tree, and every vertex in it must be of $Z$-type. It follows from (v) that the intersection is empty. \hfill $\square$

**Modification 4.** Collapse all $R(g)$’s to get a tree $T_4$. Since $t : T_3 \to T_3$ sends $R(g)$ to $R(\Phi(g))$, it induces an isometry, still called $t : T_4 \to T_4$. Thus $G_\Phi$ acts on $T_4$, and the new vertices corresponding to $R(g)$ are of $R$-type.

**Proposition 3.5.** All vertex and edge stabilizers of the action of $G_\Phi$ on $T_4$ are in $AC(VNil)$, and the action is acylindrical.

**Proof.** The edge stabilizers are $Z \times Z$, hence in $AC(VNil)$. The $G$-tree $T_4$ is minimal with cyclic edge stabilizers. It follows that vertex groups are finitely generated and quasi-convex in $G$, hence hyperbolic. There are 3 types of vertices in the $G_\Phi$-tree $T_4$, corresponding to the vertex types of the $G$-tree. If $v$ is a $Z$-type vertex in the $G$-tree, the corresponding vertex stabilizer in $G_\Phi$ is $Z \times Z$. If $v$ is an $R$-type vertex with stabilizer $V$, the corresponding stabilizer in $G_\Phi$ is $V \times Z$, which belongs to $AC(VNil)$ by Proposition 2.2(i). If $v$ is of $pA$-type, the stabilizer in $G_\Phi$ is the semi-direct product $V \rtimes Z$, which
which belongs to $\mathcal{AC}(\text{VN}i\text{l})$ (it is the fundamental group of a finite volume hyperbolic 3-manifold, so the statement follows from Theorem 2.5, see also Roushon [Rou08]).

It remains to argue that the action of $G_\Phi$ is acylindrical, that is, we need a uniform bound on the diameter of $\text{Fix}(\gamma)$ for every nontrivial $\gamma \in G_\Phi$. We consider different cases.

**Case 1:** $\gamma = g \in G$. Then $\text{Fix}(\gamma)$ is empty, a point, or contained in $Z(E)$, where $E$ is the maximal cyclic subgroup of $G$ that contains $\gamma$, so it has diameter $\leq 4$ by (v).

**Case 2:** $\gamma = t$. Then $\text{Fix}(\gamma)$, if not a single vertex, is the union of edges whose stabilizer is fixed by $\Phi$. If $R(1) = \emptyset$, i.e. if $T_3$ has no $R$-vertices whose stabilizer is fixed by $\Phi$, then $\text{Fix}(t)$ is contained in a single $Z(E)$ so has diameter $\leq 4$ (otherwise the vertex in the intersection of two adjacent $Z(E)$’s containing edges of $\text{Fix}(t)$ would be an $R$-vertex whose stabilizer is elementwise fixed by $\Phi$). Otherwise, by the same argument, every such edge is contained in some $Z(E)$ that contains an $R$-vertex in $R(1)$, and so in $T_4$ the set $\text{Fix}(t)$ is contained in the 4-neighborhood of the vertex $R(1)$, so it has diameter $\leq 8$.

**Case 3:** $\gamma = ht$ for $h \in G$. Then $\gamma$ fixes an edge $e$ if and only if $\Phi(g) = h^{-1}gh$ for $g \in \text{Stab}_G(e)$. In other words, if we replace $\Phi$ by the conjugate $h\Phi h^{-1}$ this case reduces to Case 2.

**Case 4:** $\gamma = t^k$ for $k \neq 0$. If we replace $\Phi$ with $\Phi^k$ and repeat the modifications 1-4 to the JSJ-tree, the resulting tree is exactly the same. For example, the equivalence relation on the set of $R$-vertices doesn’t change, since $g \neq h$ implies $g^k \neq h^k$ in torsion-free hyperbolic groups. (This is one place where torsion-free hypothesis is used in a substantial way.) Thus $\text{Fix}(t^k)$ has diameter $\leq 8$.

**Case 5:** $\gamma = ht^k$ for $k \neq 0$, $h \in G$. This is the general case, and it follows by combining Cases 3 and 4, that is, applying the argument to a different representative of $\phi^k$. □

**Proof of Proposition 3.1.** Follows from Theorem 2.4 applied to $G_\phi$ acting on $T_4$, using Proposition 3.5. □

We also record the following consequence, which may be of independent interest.

**Corollary 3.6.** Let $G$ be a 1-ended torsion-free hyperbolic group and $\phi \in \text{Out}(G)$. Then there is a power $\phi^N$, $N > 0$, so that for every lift $\Phi \in \text{Aut}(G)$ of $\phi^N$ every $\Phi$-periodic element of $G$ is fixed (i.e. $\Phi^k(g) = g$, $k > 0$ imply $\Phi(g) = g$).

**Proof.** The power $\phi^N$ is the one we take in order to make the Modifications 1-4 above. We may assume $g \neq 1$, and consider $T_4$, where we set $t$ to be the isometry induced by $\Phi$. If $g$ is hyperbolic then $t^k$ leaves the axis of $g$ invariant. If $t^k$ is elliptic, it fixes the whole axis contradicting acylindricity. If $t$ and $t^k$ are loxodromic, then for suitable $m, s \neq 0$ the isometries $(t^k)^m$...
and $g^s$ have the same action on this axis. This implies that $t^{km}g^{-s}$ fixes the axis, again contradicting acylindricity. Thus $g$ is elliptic. If $g$ fixes an edge or a vertex of the form $R(h)$, then $\Phi$ acts on $g$ by conjugation, so either $g$ is fixed or it is not periodic (this uses that $G$ is torsion-free). If $g$ fixes a $pA$ vertex but no edges, then $g$ is represented by a nonperiodic curve in a surface and $\phi^N$ acts as a pseudo-Anosov. Thus even the conjugacy class of $g$ is non-periodic. □

4. The PG case ($\infty$-ly many ends)

We continue with the standing assumption that $G$ is a torsion-free Gromov hyperbolic group. The goal of the present section is the following proposition, whose proof closely mirrors the analogous result in the free group case, treated in [BFW].

**Proposition 4.1.** Let $G$ be a torsion-free hyperbolic group with infinitely many ends and assume $\phi \in \text{Out}(G)$ has polynomial growth. Then the group $G_{\phi}$ belongs to $\text{AC}(\text{VNil})$.

4.1. Polynomial growth. We start by reviewing the notion of polynomial growth in this setting; for more details see [DK20].

Let $G = G_1 \ast G_2 \ast \cdots \ast G_n \ast F_r$ be the Grushko free product decomposition of $G$, where the $G_i$ are 1-ended hyperbolic groups and $F_r$ is the free group of rank $r$ (we allow $n = 0$ or $r = 0$). The Grushko rank of $G$ is $n + r$. By a Grushko tree we mean a minimal simplicial $G$-tree where the edge stabilizers are trivial and the vertex stabilizers are precisely the conjugates of the $G_i$ (and possibly the trivial group). Fix a Grushko tree $T$. When $g \in G$ define

$$||g|| = \min d_T(v, g(v))$$

This is a conjugacy invariant and it is a measure of the length of a conjugacy class relative to the vertex stabilizers. Let $\phi \in \text{Out}(G)$. Then $\phi$ permutes the vertex stabilizers of any Grushko tree. We say that $g \in G$ grows polynomially under $\phi$ if there exists a polynomial $P_g$ such that

$$||\phi^n(g)|| \leq P_g(n)$$

We say that $\phi$ has polynomial growth if every $g \in G$ grows polynomially. These definitions do not depend on the choice of the Grushko tree.

In [CT94], generalizing the case of free groups in [BH92], Collins and Turner proved the following train track theorem. For refinements and generalizations see [FH18, FM15, Lym].

**Theorem 4.2 ([CT94]).** Let $G$ be as above and let $\phi \in \text{Out}(G)$ be a polynomially growing automorphism and $\Phi$ a lift of $\phi$ to $\text{Aut}(G)$. Then there is a Grushko tree $T$ and a $\Phi$-equivariant map $f : T \to T$ that sends vertices to vertices and such that

1. there is a $G$-invariant filtration

$$T_0 \subset T_1 \subset \cdots \subset T_{k-1} \subset T_k = T$$
of $T$ into subgraphs (these are forests and need not be connected), with $T_0$ consisting of all vertices with nontrivial stabilizer, and

(2) for every edge $e \in T_j$, $j = 1, 2, \ldots, k$, $f(e)$ is an edge-path that crosses exactly one edge in $T_j \setminus T_{j-1}$ while all the other edges are in $T_{j-1}$.

It is useful to visualize what this is saying about the quotient graph of groups $T/G$. There is a filtration by subgraphs starting with the vertices representing the $G_i$’s, all these subgraphs are invariant under the induced map $\overline{f} : T/G \to T/G$, and the image of each edge in the stratum $T_j/G \setminus T_{j-1}/G$ crosses one edge of that stratum and the rest of the image is in $T_{j-1}/G$. In other words, the transition matrix of $\overline{f}$ is an upper triangular block matrix with all diagonal blocks permutation matrices.

The Collins-Turner theorem is more general and applies to all automorphisms, in general there are also exponentially growing and zero strata, but they don’t arise in the polynomially growing case.

Since we are allowed to take powers of $\phi$, the statement becomes a bit simpler. The permutation matrices can be taken to be $1 \times 1$.

**Proposition 4.3.** Let $G$ be as above and $\phi \in \text{Out}(G)$ polynomially growing. Then there is a power $\phi^N$ such that for every lift $\Phi$ of $\phi^N$ the following holds.

(i) Up to conjugacy, $\Phi$ preserves each $G_i$.

(ii) There is a Grushko tree $T$ and $f : T \to T$ satisfying the conclusions of Theorem 4.2 for $\Phi$ such that, in addition, each stratum $T_j \setminus T_{j-1}$ consists of exactly one orbit of (oriented) edges, and $f$ sends each such edge to an edge-path that crosses an edge of $T_j \setminus T_{j-1}$ with the same orientation.

We say that an automorphism $\Phi : G \to G$ is neat if $\Phi^k(g) = g$ for $k \neq 0, g \in G$ implies $\Phi(g) = g$. An outer automorphism $\phi \in \text{Out}(G)$ is neat if every lift of $\phi$ to $\text{Aut}(G)$ is neat.

We shall need the following theorem, whose proof is deferred to Section 7.

**Theorem 4.4.** Let $G$ be a torsion free hyperbolic group and let $\phi \in \text{Out}(G)$ be a polynomially growing outer automorphism (with respect to the Grushko tree). Then there is $N > 0$ such that $\phi^N$ is neat.

The proof will show that $N$ depends only on $G$, not on $\phi$.

**Proof of Proposition 4.1.** We will induct on the Grushko rank. The base of the induction is Proposition 3.1. Replace $\phi$ by a power $\phi^N$ so that a lift $\Phi$ of $\phi^N$ is realized as in Proposition 4.3, and so that $\Phi$ is neat. The highest edge in the filtration defines a splitting of $G$ as $G_1 * G_2$ or as $G *_1$, with $G_1, G_2, G'$ of strictly smaller Grushko rank and $\phi$ inducing polynomially growing outer automorphisms on these factors, so by induction their mapping tori are in $\text{AC}(\text{VNil})$. 
We will treat the case where $G = G_1 \ast G_2$; the argument requires only notational changes when $G$ is an HNN extension. We can geometrically realize $\phi$ as follows. For $i \in \{1, 2\}$, let $X_i$ be a presentation complex for $G_i$: that is a 2-dimensional CW complex with $\pi_1(X_i) = G_i$. The automorphism $\phi|_{G_i}$, then tells us how to build a homotopy equivalence $f_i: X_i \to X_i$ such that $(f_i)_* = \phi|_{G_i}$. Attaching $X_1$ and $X_2$ with an edge $e$ yields a graph of spaces $X$ with $\pi_1(X) = G$. We can realize $\phi$ as a homotopy equivalence $f: X \to Y$ restricting to $f_i$ on $X_i$ and such that $f(e) = \alpha_1 e_2 \alpha_2$ for $\alpha_i$ a path in $X_i$. In particular, $f^{-1}(e) \subset e$, so after perturbing $f$ by a homotopy supported on $e$, we may assume there is an interval $J \subset e$ fixed by $f$ and such that $f^{-1}(J) = J$.

We now form the topological mapping torus $X_F := X \times [0,1]/(x,0) \sim (f(x),1)$, whose fundamental group is $G_\phi$. In $X_F$, the interval $J$ gives rise to an embedded annulus, $J \times S^1$, and hence (by Van Kampen’s Theorem) to a splitting of $\pi_1(X_F) = G_\phi$ over an infinite cyclic group: $G_\phi = H_1 \ast_\mathbb{Z} H_2$. Moreover, $H_i$ is precisely the mapping torus of $\phi|_{G_i}: G_i \to G_i$, and is therefore in AC($\text{VNil}$), by induction.

Let $T$ be the Bass-Serre tree associated to this splitting. It is not necessarily the case that $G_\phi \curvearrowright T$ is acylindrical.

**Lemma 4.5.** If $E, E'$ are two edges in $T$ then either $\text{Stab}(E) = \text{Stab}(E')$ or $\text{Stab}(E) \cap \text{Stab}(E') = 1$.

**Proof.** Consider the standard presentation,

$$G_\Phi = \langle x_1, \ldots, x_n, t \mid tx_it^{-1} = \Phi(x_i) \rangle,$$

where $x_1, \ldots, x_n$ is a generating set for $G$. We recall that every element $h \in G_\phi$ can be written uniquely as $h = gt^k$ for some integer $k$ and some element $g \in G$. Now the fundamental group of the embedded annulus, $\pi_1(J \times S^1)$, is identified in $G_\phi$ with the stable letter, $t$. Evidently the edge stabilizers in $T$ are precisely the conjugates of $(t)$, $g(tg^{-1}) = g\Phi(g^{-1})t$. Thus if $\text{Stab}(E) \cap \text{Stab}(E') \neq 1$, then for some $n > 0$ we have

$$(g\Phi(g^{-1})t)^n = t^n$$

(the exponents must be equal by looking at the homomorphism to $\mathbb{Z}$ that kills the $x_i$’s). By a straightforward induction the left-hand side can be written as $g\Phi^n(g^{-1})t^n$ so the equation says $\Phi^n(g) = g$. By the neatness of $\Phi$ we have $\Phi(g) = g$, so the equation holds for $n = 1$, i.e. $\text{Stab}(E) = \text{Stab}(E')$. \hfill $\square$

Consider an action $G \curvearrowright T$ on a simplicial tree by isometries. Recall that a **transverse covering** [Gui04, Definition 4.6] of $T$ is a $G$-invariant family $\mathcal{Y}$ of non-degenerate closed subtrees of $T$ such that any two distinct subtrees in $\mathcal{Y}$ intersect in at most one point. A transverse covering $\mathcal{Y}$ gives rise to a new tree, $S$, called the **skeleton of $\mathcal{Y}$** as follows: the vertex set $V(S) = V_0 \cup V_1$ where the elements of $V_0$ are in one-to-one correspondence with elements of $\mathcal{Y}$, and the elements of $V_1$ are in correspondence with nonempty intersections...
(necessarily consisting of a single point) of distinct elements of \( \mathcal{Y} \). Edges are determined by set containment: there is an edge from \( x \in V_1 \) to \( Y \in V_2 \) if \( x \in Y \). The action of \( G \) on \( \mathcal{Y} \) determines an action \( G \curvearrowright S \).

We use Lemma 4.5 to define a transverse covering of \( T \) whose skeleton will be acylindrical. For an edge \( E \) of \( T \), we let \( T_E \) be the forest in \( T \) consisting of edges whose stabilizer is equal to \( \text{Stab}(E) \). That \( T_E \) is connected follows from Lemma 4.5. We are interested in understanding the stabilizer \( \text{Stab}(T_E) \), so suppose that \( E, E' \in T_E \). Without loss of generality, assume that \( \text{Stab}(E) = \langle t \rangle \) and that \( w \in \text{Stab}(T_E) \) is such that \( w^{-1} \cdot E = E' \in T_E \). As above, we can write \( w \) uniquely as \( w = g t^k \). The definition of \( T_E \) provides that the stabilizer of \( E' \) is equal to \( \langle t \rangle \). On the other hand,

\[
\text{Stab}(E') = \langle wt w^{-1} \rangle = \langle (gt^k) t (t^{-k} g^{-1}) \rangle = \langle gtg^{-1} \rangle = \langle g \Phi(g)^{-1} t \rangle.
\]

In particular, \( g = \Phi(g) \), and we conclude that \( \text{Stab}(T_E) \simeq \langle t \rangle \times \text{Fix}(\Phi) \).

Neumann proved [Neu92] that for a hyperbolic group \( G \), \( \text{Fix}(\Phi) \) is quasi-convex in \( G \) and therefore is itself a hyperbolic group. Since \( \text{AC}(\text{VNil}) \) is closed under taking products, and hyperbolic groups belong to \( \text{AC}(\text{VNil}) \), we conclude that \( \text{Stab}(T_E) \) belongs to \( \text{AC}(\text{VNil}) \).

The subtrees \( \{T_E\}_{E \in T} \) form a transverse covering \( \mathcal{Y} \) of \( T \). Let \( S \) denote the skeleton of this transverse cover (this is the tree of cylinders of [GL11]).

We observe the following:

**Lemma 4.6.** The action \( G_{\Phi} \curvearrowright S \) is acylindrical.

**Proof.** Let \( v, v' \in V(S) \) be vertices with \( d(v, v') \geq 6 \) and suppose \( g \in \text{Stab}(v) \cap \text{Stab}(v') \). By moving to adjacent vertices if necessary, we may assume that \( v, v' \in V_1 \) so that they are labeled by intersections of subtrees in \( \mathcal{Y} \) (i.e., points in \( T \)) rather than by trees themselves. We will again denote by \( v \) and \( v' \) the corresponding points in \( T \). Now \( g \) must fix the segment in \( T \) connecting \( v \) to \( v' \), and even after moving to adjacent vertices we still have \( d_S(v, v') \geq 4 \). In particular, there are two vertices in \( V_0 \) on the segment in \( S \) between \( v \) and \( v' \); hence the segment connecting \( v \) and \( v' \) in \( T \) contains edges in two distinct subtrees \( T_E \) and \( T_{E'} \). So \( g \in \text{Stab}(T_E) \cap \text{Stab}(T_{E'}) \) must stabilize two edges with different stabilizers and so \( g = 1 \) by Lemma 4.5. \( \square \)

To conclude our proof of Proposition 4.1 we recall that the vertex stabilizers in \( S \) come in two flavors: stabilizers of vertices in \( V_1 \) are subgroups of vertex stabilizers in \( T \), which belong to \( \text{AC}(\text{VNil}) \) by induction; and stabilizers of vertices in \( V_0 \), which are isomorphic to \( \text{Fix}(\Phi) \times \mathbb{Z} \) and also belong to \( \text{AC}(\text{VNil}) \). We have thus produced an acylindrical action of \( G_{\Phi} \) on a tree \( S \) in which all stabilizers belong to \( \text{AC}(\text{VNil}) \). Theorem 2.4 implies that \( G_{\Phi} \) belongs to \( \text{AC}(\text{VNil}) \) as well. \( \square \)

5. **The EG case (\( \infty \)-ly many ends)**

The following proposition is all that remains of Theorem 1.1 and is the goal of the present section.
Proposition 5.1. Let $G$ be a torsion-free hyperbolic group with infinitely many ends and let $\Phi \in \text{Out}(G)$ be an automorphism that does not grow polynomially. Then $G_\Phi \in \text{AC} (\text{VNil})$.

It is known from train-track theory that if $||\phi^n(g)||$ does not grow polynomially then it grows exponentially, but we will not need this. The result follows immediately from the work above and the following theorem of Dahmani-Krishna.

Theorem 5.2 ([DK20, Theorem 1.1 and 1.2]). If $G$ is a torsion-free hyperbolic group, and $\Phi \in \text{Aut}(G)$, then $G_\Phi$ is hyperbolic relative to a family $\{P_1, \ldots, P_k\}$ of subgroups, each of which has the form $H_i \rtimes \psi_i Z$ for a finitely generated torsion-free hyperbolic group $H_i$ and a polynomially growing automorphism $\psi_i: H_i \to H_i$. Moreover, an element $h \in G$ grows polynomially under $\Phi$ if and only if it is conjugate into some $H_i$.

Proof of Proposition 5.1. Let $G$ and $\Phi$ be as in the statement. Theorem 5.2 shows that $G_\Phi$ is hyperbolic relative to a collection of subgroups, each of which is a mapping torus of a polynomially growing automorphism of a torsion-free hyperbolic group. By Proposition 4.1 (and Proposition 3.1), such groups are in $\text{AC} (\text{VNil})$. An application of Theorem 2.5 then shows that $G_\Phi \in \text{AC} (\text{VNil})$. 

6. Extensions

In this section, we prove Theorem 1.2.

Theorem 1.2. Let $G$ be a virtually torsion-free hyperbolic group and let

$$1 \longrightarrow G \longrightarrow H \xrightarrow{\pi} Q \longrightarrow 1$$

be a short exact sequence of groups. If $Q$ satisfies the Farrell-Jones conjecture, then so does $H$.

Proof. We use the general fact (see [BFL14, Theorem 2.7]) that if $Q$ satisfies FJC and for every virtually cyclic subgroup $Z < Q$ the preimage $\pi^{-1}(Z) < H$ satisfies FJC, then $H$ satisfies FJC. Further, it suffices to show that a finite index subgroup of $\pi^{-1}(Z)$ satisfies FJC. The preimage of a finite index cyclic subgroup of $Z$ will have the form $G \rtimes Z$ and will have finite index in $\pi^{-1}(Z)$. Theorem 1.1 implies that $G \rtimes Z$ satisfies FJC.

7. Proof of Theorem 4.4

The proof will run by induction on the Grushko rank, so we will be considering the splitting $G = G_1 * G_2$ or $G = G' *_1$ given by the topmost edge in Proposition 4.3. The basis of the induction is Corollary 3.6. To summarize the inductive step, we will prove the following.

Proposition 7.1. Suppose a torsion-free group $G$ is acting on a tree $T$ with trivial edge stabilizers, $\Phi \in \text{Aut}(G)$ is an automorphism, and $t : T \to T$ is a $\Phi$-equivariant isometry, meaning that $t(g(x)) = \Phi(g)t(x)$ for any
\( x \in T, g \in G \). Also assume that \( t \) preserves the \( G \)-orbits of oriented edges. Suppose further that whenever \( v \) is a vertex of \( T \) such that \( t(v) = v \) then \( \Phi|V : V \to V \) is neat, where \( V = Stab(v) \). Then \( \Phi \) is neat.

We then apply this to the splitting of \( G \) given by the topmost edge in Proposition 4.3. Proposition 7.1 is proved in a sequence of three lemmas.

**Lemma 7.2.** Suppose a lift \( \Phi \in Aut(G) \) of \( \phi \) is realized as a loxodromic \( \Phi \)-equivariant isometry \( t : T \to T \). Then \( \Phi \) is neat.

**Proof.** Suppose \( \Phi^n(g) = g \) for some \( n > 0 \) and \( g \in G, g \neq 1 \). If \( g \) is elliptic in \( T \), its fixed point set is a single vertex since edge groups are trivial. Thus \( t^n \) fixes this vertex contradicting the assumption that \( t \) is loxodromic. Thus \( g \) is loxodromic, and its axis is preserved by \( t^n \). It follows that \( t \) and \( g \) have the same axis. Thus \( g \) and \( tgt^{-1} = \Phi(g) \) have the same axis and the same (signed) translation length, so \( g^{-1}\Phi(g) \) fixes many edges and must be trivial. \( \square \)

Now we assume that \( t : T \to T \) realizing \( \Phi \) is an elliptic isometry. The next lemma says that in this case periodic edges incident to a fixed vertex are fixed.

**Lemma 7.3.** If \( t \) fixes a vertex \( v \) of \( T \), then the incident edges are either fixed by \( t \), or the \( t \)-orbit is infinite.

**Proof.** Let \( V = Stab(v) \) and let \( c \) be an edge incident to \( v \). Then \( t(c) \) is in the same \( V \)-orbit as \( c \), so we may write \( t(c) = a(c) \) for some \( a \in V \). Suppose now that \( t^n(c) = c \). Then \( t^n \) fixes the whole \( t \)-orbit of \( c \), so \( t^n(a(c)) = a(c) \). This implies \( \Phi^n(a)t^n(c) = a(c) \), i.e. \( \Phi^n(a) = a \). Since \( \Phi|V \) is neat, this implies \( \Phi(a) = a \). Now we see that \( t^2(c) = t(a(c)) = \Phi(a)t(c) = a^2(c) \) and by induction \( t^k(c) = a^k(c) \). In particular \( c = t^n(c) = a^n(c) \) so \( a^n = 1 \). Since \( G \) is torsion-free we have \( a = 1 \) so \( t(c) = c \). \( \square \)

The next lemma finishes the proof of Proposition 7.1 and Theorem 4.4.

**Lemma 7.4.** Suppose \( t : T \to T \) is elliptic, \( g \in G \), \( \Phi^n(g) = g \) for some \( n > 0 \). Then \( \Phi(g) = g \).

**Proof.** First suppose that \( g \) is loxodromic in \( T \). Then \( t^n \) fixes the axis of \( g \). Lemma 7.3 implies that the subtree fixed by \( t^n \) is the same as \( Fix(t) \), so \( t \) also fixes the axis of \( g \). Thus \( g \) and \( tgt^{-1} \) have the same axis and the same (signed) translation length, so they are equal, i.e. \( \Phi(g) = g \).

Now suppose \( g \) is elliptic in \( T \). Again we have that \( t^n \), and hence \( t \), fixes \( Fix(g) \). It follows that there is vertex \( v \) of \( T \) fixed by both \( g \) and \( t \), and so we have \( \Phi(g) = g \) from the fact that \( \Phi|Stab(v) \) is neat. \( \square \)

**Remark 7.5.** Theorem 4.4 holds for all \( \phi \in Out(G) \), not only polynomially growing ones. This follows from Theorem 5.2 and the polynomially growing case since any periodic conjugacy class is peripheral.
References

[Bar16] Arthur Bartels. On proofs of the Farrell-Jones conjecture. In Topology and geometric group theory, volume 184 of Springer Proc. Math. Stat., pages 1–31. Springer, [Cham], 2016.

[Bar17] A. Bartels. Coarse flow spaces for relatively hyperbolic groups. Compos. Math., 153(4):745–779, 2017.

[BB19] Arthur Bartels and Mladen Bestvina. The Farrell-Jones conjecture for mapping class groups. Invent. Math., 215(2):651–712, 2019.

[BFL14] A. Bartels, F. T. Farrell, and W. Lück. The Farrell-Jones conjecture for cocompact lattices in virtually connected Lie groups. J. Amer. Math. Soc., 27(2):339–388, 2014.

[BFW] Mladen Bestvina, Koji Fujiwara, and Derrick Wigglesworth. Farrell-Jones Conjecture for free-by-cyclic groups. arXiv:1906.00069v1.

[BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. Ann. of Math. (2), 135(1):1–51, 1992.

[BKW21] Benjamin Brück, Dawid Kielak, and Xiaolei Wu. The Farrell-Jones Conjecture for normally poly-free groups. Proc. Amer. Math. Soc., 149:2349–2356, 2021.

[BL10] Arthur Bartels and Wolfgang Lück. On crossed product rings with twisted involutions, their module categories and $L$-theory. In Cohomology of groups and algebraic $K$-theory, volume 12 of Adv. Lect. Math. (ALM), pages 1–54. Int. Press, Somerville, MA, 2010.

[BLR08a] Arthur Bartels, Wolfgang Lück, and Holger Reich. Equivariant covers for hyperbolic groups. Geom. Topol., 12(3):1799–1882, 2008.

[BLR08b] Arthur Bartels, Wolfgang Lück, and Holger Reich. The $K$-theoretic Farrell-Jones conjecture for hyperbolic groups. Invent. Math., 172(1):29–70, 2008.

[BLR08c] Arthur Bartels, Wolfgang Lück, and Holger Reich. On the Farrell-Jones conjecture and its applications. J. Topol., 1(1):57–86, 2008.

[BLRR14] Arthur Bartels, Wolfgang Lück, Holger Reich, and Henrik Rüping. $K$- and $L$-theory of group rings over $\text{GL}_n(\mathbb{Z})$. Publ. Math. Inst. Hautes Études Sci., 119:97–125, 2014.

[Bow98] Brian H. Bowditch. Cut points and canonical splittings of hyperbolic groups. Acta Math., 180(2):145–186, 1998.

[BR07] Arthur Bartels and Holger Reich. Coefficients for the Farrell-Jones conjecture. Adv. Math., 209(1):337–362, 2007.

[CT94] D. J. Collins and E. C. Turner. Efficient representatives for automorphisms of free products. Michigan Math. J., 41(3):443–464, 1994.

[DK20] François Dahmani and Suraj M S Krishna. Relative hyperbolicity of hyperbolic-by-cyclic groups. 2020.

[DS99] M. J. Dunwoody and M. E. Sageev. JSJ-splittings for finitely presented groups over slender groups. Invent. Math., 135(1):25–44, 1999.

[FI18] Mark Feighn and Michael Handel. Algorithmic constructions of relative train track maps and CTs. Groups Geom. Dyn., 12(3):1159–1238, 2018.

[FJ93] F. T. Farrell and L. E. Jones. Isomorphism conjectures in algebraic $K$-theory. J. Amer. Math. Soc., 6(2):249–297, 1993.

[FM15] Stefano Francaviglia and Armando Martino. Stretching factors, metrics and train tracks for free products. Illinois J. Math., 59(4):859–899, 2015.

[FP06] K. Fujiwara and P. Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. Geom. Funct. Anal., 16(1):70–125, 2006.

[GL07] Vincent Guirardel and Gilbert Levitt. The outer space of a free product. Proc. Lond. Math. Soc. (3), 94(3):695–714, 2007.

[GL11] Vincent Guirardel and Gilbert Levitt. Trees of cylinders and canonical splittings. Geom. Topol., 15(2):977–1012, 2011.
[GL17] Vincent Guirardel and Gilbert Levitt. JSJ decompositions of groups. Astérisque, (395):vii+165, 2017.

[Gui04] Vincent Guirardel. Limit groups and groups acting freely on $\mathbb{R}^n$-trees. Geom. Topol., 8:1427–1470 (electronic), 2004.

[KLR16] Holger Kammeyer, Wolfgang Lück, and Henrik Rüping. The Farrell-Jones conjecture for arbitrary lattices in virtually connected Lie groups. Geom. Topol., 20(3):1275–1287, 2016.

[Kno17] S. Knopf. Acylindrical Actions on Trees and the Farrell-Jones Conjecture. ArXiv e-prints, April 2017.

[Lev05] Gilbert Levitt. Automorphisms of hyperbolic groups and graphs of groups. Geom. Dedicata, 114:49–70, 2005.

[Lym] Rylee Alanza Lyman. Train track maps and CTs on graphs of groups. arXiv:2102.02848.

[Neu92] Walter D. Neumann. The fixed group of an automorphism of a word hyperbolic group is rational. Invent. Math., 110(1):147–150, 1992.

[Pau91] Frédéric Paulin. Outer automorphisms of hyperbolic groups and small actions on $\mathbb{R}$-trees. In Arboreal group theory (Berkeley, CA, 1988), volume 19 of Math. Sci. Res. Inst. Publ., pages 331–343. Springer, New York, 1991.

[Rou08] S. K. Roushon. The Farrell-Jones isomorphism conjecture for 3-manifold groups. J. K-Theory, 1(1):19–82, 2008.

[RS94] E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. I. Geom. Funct. Anal., 4(3):337–371, 1994.

[Sel97a] Z. Sela. Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II. Geom. Funct. Anal., 7(3):561–593, 1997.

[Sel97b] Z. Sela. Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II. Geom. Funct. Anal., 7(3):561–593, 1997.

[Weg12] Christian Wegner. The $K$-theoretic Farrell-Jones conjecture for CAT(0)-groups. Proc. Amer. Math. Soc., 140(3):779–793, 2012.

[Weg15] Christian Wegner. The Farrell-Jones conjecture for virtually solvable groups. J. Topol., 8(4):975–1016, 2015.

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