The Elliptic Drinfeld Center of a Premodular Category

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Abstract

Given a tensor category $C$, one constructs its Drinfeld center $Z(C)$ which is a braided tensor category, having as objects pairs $(X, \lambda)$, where $X \in \text{Obj}(C)$ and $\lambda$ is a half-braiding. For a premodular category $C$, we construct a new category $Z_{\text{el}}(C)$ which we call the Elliptic Drinfeld Center, which has objects $(X, \lambda^1, \lambda^2)$, where the $\lambda^i$’s are half-braidings that satisfy some compatibility conditions. We discuss an $\text{SL}_2(\mathbb{Z})$-action on $Z_{\text{el}}(C)$ that is related to the anomaly appearing in Reshetikhin-Turaev theory. This construction is motivated from the study of the extended Crane-Yetter TQFT, in particular the category associated to the once punctured torus.

1 Introduction and Preliminaries

In [CY1993], Crane and Yetter define a 4d TQFT using a state-sum involving 15j symbols, based on a sketch by Ooguri [Oog1992]. The state-sum begins with a coloring of the 2- and 3-simplices of a triangulation of the four manifold by integers from $0, 1, \ldots, r$. These 15j symbols then arise as the evaluation of a ribbon graph living on the boundary of a 4-simplex. The labels $0, 1, \ldots, r$ correspond to simple objects of the Verlinde modular category, the semi-simple subquotient of the category of finite dimensional representations of the quantum group $U_q\mathfrak{sl}_2$ at $q = e^{\pi i / (r+2)}$ as defined in [AP1995]. Later Crane, Kauffman, and Yetter [CKY1997] extend this definition to colorings with objects from a premodular category (i.e. artinian semisimple tortile/ribbon).

The invariant for closed 4-manifolds that one obtains from the Crane-Yetter (CY) state-sum essentially boils down to the signature and Euler characteristic of the manifold, though it is still interesting because it expresses the signature of a 4-manifold in terms of local combinatorial data [CKY1993]. It is believed that the CY TQFT, with a modular category as input, is a boundary theory, in that $Z_{\text{CY}}(M^4)$ for a 4-manifold with boundary is determined by its boundary $\partial M^4$ and classical invariants of $M^4$ like the signature and Euler characteristic.

In [CKY1997], the authors speculate that their theory, when extended to include insertions at surfaces and points, could be related to Donaldson-Floer (DF) theory.
Attempts have been made (e.g. [Yet1993], [Rob1997]) to modify the state-sum in [CKY1997] in the presence of insertions on surfaces. In principle, these insertions are labellings of a codimension-2 submanifold by objects in a category associated to the abstract homeomorphism class of that submanifold. These categories should be related to each other via some gluing axioms.

The construction presented in this paper arose out of studying such categories. Namely, starting with a fixed premodular category \( \mathcal{C} \) (which would be used to produce the CY TQFT), we have an abstract schema of producing a category \( Z_{\text{CY}}(\Sigma) = Z_{\text{CY,C}}(\Sigma) \) for each surface (possibly with punctures and boundaries). The basic objects in \( Z_{\text{CY}}(\Sigma) \) are configurations of finitely many points, each labelled with an object in \( \mathcal{C} \). Morphisms are skein modules with appropriate boundary conditions. One then completes the category by considering the Karoubian closure.

In [BZBJ2015], [BZBJ2016], Ben-Zvi, Brochier, and Jordan use factorization homology to construct such categories, integrating certain algebras over surfaces. We expect that our constructions agree.

Although we have abstractly defined these categories \( Z_{\text{CY}}(\Sigma) \) from skeins, the goal of our studies is to relate them to the input premodular category \( \mathcal{C} \). For example, we have that
\[
\begin{align*}
Z_{\text{CY}}(D^2) &= \mathcal{C} \quad \text{(by default)} \\
Z_{\text{CY}}(S^2) &= Z_{\text{Mü}}(\mathcal{C}), \quad \text{the Müger center} \\
Z_{\text{CY}}(S^1 \times [0,1]) &= Z(\mathcal{C}), \quad \text{the Drinfeld center} \\
Z_{\text{CY}}(T^2_1) &= Z^\text{el}(\mathcal{C}) \\
Z_{\text{CY}}(T^2_1) &= \text{Vec} \quad \text{for } \mathcal{C} \text{ modular}
\end{align*}
\]
where \( Z^\text{el}(\mathcal{C}) \) is the category we construct in this paper, which we call the Elliptic Drinfeld center. We will establish these results in future work, as our main goal in this paper is to define and study \( Z^\text{el}(\mathcal{C}) \).

Briefly, \( Z^\text{el}(\mathcal{C}) \) has as objects \((X, \lambda^1, \lambda^2)\), where \( \lambda^1, \lambda^2 \) are half-braidings on \( X \). \( \lambda^1 \) and \( \lambda^2 \) are required to satisfy certain commutativity relations involving the braiding on \( \mathcal{C} \). Thus we stress that while the Drinfeld center can be defined for any monoidal category, our elliptic Drinfeld center requires that \( \mathcal{C} \) be braided. The other conditions (fusion, ribbon) are not essential for the definition but are needed to define the (extended) TQFT. They also lead \( Z^\text{el}(\mathcal{C}) \) to have nice properties.

Choose a pair of oriented simple closed curves on \( T^2_1 \) so that \( T^2_1 \) deformation retracts onto their union (see Remark 3.21 for a picture). Then there is a functor
\[
Z^\text{el}(\mathcal{C}) \overset{\sim}{\rightarrow} Z_{\text{CY}}(T^2_1)
\]
that sends \((X, \lambda^1, \lambda^2)\) to the image of a projection on the configuration with one
marked point labelled by $X$. The projection is built out of $\lambda^1$ and $\lambda^2$, somehow assigning them to the two chosen curves. In hand-wavy terms, $Z^\text{el}(C)$ is a “coordinate representation” of $Z_{\text{CY}}(\mathbb{T}^2_1)$, in the sense that we have picked a marked point and a pair of such curves in order to express our objects with “coefficients” in $C$, and this “coordinate representation” changes when we change these choices. Further discussions of this can be seen in e.g. Remark 3.12, Section 3.3, Remark 5.5).

Let us give a brief outline of the paper. In Section 2, we first recall some properties of the usual Drinfeld center $Z(C)$. In Section 3, we then discuss various properties of $Z^\text{el}(C)$ in parallel with those of $Z(C)$ laid out in Section 2. We show that $Z^\text{el}(C)$ is monoidal (see Definition-Proposition 3.18). Being the category associated to $\mathbb{T}^2_1$, it naturally carries an action of $\text{SL}_2(\mathbb{Z})$ (see Theorem 3.22). However, some of the arguments are of a topological nature, and is more naturally understood in the context of the extended Crane-Yetter TQFT, hence to limit the scope of the paper, we postpone full proofs to future work.

As mentioned above, when $C$ is modular, it is expected that $Z_{\text{CY}}$ is a boundary theory. Since $Z_{\text{CY}}(\mathbb{D}^2) = C$, one expects $Z_{\text{CY}}(\mathbb{T}^2_1) \cong C$ as well. To this end, we prove in Section 4 that:

**Theorem 4.3** If $C$ is modular, then the composition $$i = \text{I}_1 \circ \iota : C \to Z^\text{el}(C)$$ is an equivalence of abelian categories, where $\iota : C \to Z(C)$ is the functor $X \mapsto (X, c_{-X})$, and $\text{I}_1$ is the intermediate induction functor defined in Proposition 3.9.

In Section 4.1, we discuss the connection of the $\text{SL}_2(\mathbb{Z})$-action on $Z^\text{el}(C)$ with the anomaly in Chern-Simons/Reshetikhin-Turaev theory via Theorem 4.3. However, in part due to the reliance of this action on the $\text{SL}_2(\mathbb{Z})$-action on $Z^\text{el}(C)$, we’ve decided to omit some details and proofs and once again relegate them to future work.

In Section 5, we consider $C = H - \text{mod}$, where $H$ is a Hopf algebra. In this case, the Drinfeld center is equivalent to the category of modules over Drinfeld’s quantum double, $\mathcal{D}(H)$. In the same spirit, when $H$ is braided, we construct an algebra $\mathcal{D}^\text{el}(H)$, which we call the Elliptic Drinfeld double, such that

**Theorem 5.3** For $C = H - \text{mod}$, $Z^\text{el}(C) \cong \mathcal{D}^\text{el}(H) - \text{mod}$ as abelian categories.

Brochier and Jordan [BJ2014] defined an algebra which they also call the elliptic double, and also arising from studying the category associated to the once-punctured torus. These algebras are not isomorphic, but we expect them to be Morita equivalent. In [BJ2014], they also obtain an $\text{SL}_2(\mathbb{Z})$-action on their elliptic double; we touch on this briefly in Remark 5.5.
When $\mathcal{C}$ is symmetric, there is a tensor product on $\mathcal{Z}^\text{el}(\mathcal{C})$ different from the one defined in Section 3.2. When $H$ is cocommutative, $\mathcal{D}^\text{el}(H)$ has a ribbon Hopf structure, thus $\mathcal{D}^\text{el}(H) - \text{mod}$ is a tensor category. Then with respect to these monoidal structures, the equivalence in Theorem 5.3 is one of tensor categories.

In Section 6, we discuss a generalization of our construction of $\mathcal{Z}^\text{el}(\mathcal{C})$, corresponding to considering surfaces other than $\mathbb{T}^2_1$. Finally, the last section is an Appendix, with some useful lemma that are frequently used in computing with string diagrams, and a discussion of group actions on categories given by generators and relations.

To conclude this section, let us compare the structures on the elliptic Drinfeld center with the usual Drinfeld center. Beginning with a monoidal category $\mathcal{C}$, the Drinfeld center $\mathcal{Z}(\mathcal{C})$ is a braided monoidal category. On the other hand, beginning with a braided monoidal category $\mathcal{C}$, the elliptic Drinfeld center $\mathcal{Z}^\text{el}(\mathcal{C})$ is a monoidal category but not braided (this is discussed in Section 3.2, but full proofs will be given in future work). In addition, $\mathcal{Z}^\text{el}(\mathcal{C})$ carries an action of $\text{SL}_2(\mathbb{Z})$.

This difference is a feature of the topology of surfaces: as mentioned above, the Drinfeld center is associated to the annulus, while the elliptic Drinfeld center is associated to the once-punctured torus. In both cases, the monoidal structure arises from a generalized pair of pants $\text{I}$ (see Remark 3.21). For the annulus, this generalized pair of pants is just a thicked pair of pants, so has a homeomorphism swapping the two inputs, making the monoidal structure a braided one, while for the once-punctured torus, this generalized pair of pants does not admit such a swapping operation.

Note that in both cases, the (braided) monoidal structure differs from that of $\mathcal{C}$: thinking of $\mathcal{C}$ as an $E_2$ algebra, the monoidal structure manifests as inclusion of little disks in the little disks operad, so in a very loose sense governs “local behaviour”. However, the monoidal structures on $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}^\text{el}(\mathcal{C})$ are an artefact of global topology of the relevant surfaces. For example, the braided structure on $\mathcal{Z}(\mathcal{C})$ is quite different from that of $\mathcal{C}$ - it is constructed using only the monoidal structure of $\mathcal{C}$. The takeaway is that we should not think of “gaining” or “losing” structures from $\mathcal{C}$ when we construct $\mathcal{Z}(\mathcal{C})$ and $\mathcal{Z}^\text{el}(\mathcal{C})$, but rather observe that they merely reflect the topology of surfaces.

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\footnote{This is not the generalized pair of pants in the sense of Floer theory in symplectic geometry.}
1.1 Notation and Conventions

Throughout, let us fix an algebraically closed field $k$ of characteristic 0.

Let $\mathcal{C}$ be a $k$-linear premodular category, that is, a ribbon fusion category. Some assumptions and notations:

- For simplicity of exposition and minimalil of parentheses, we suppress applications of the associativity constraint unless it leads to confusion.
- We implicitly identify $V^{**}$ with $V$ via the pivotal structure $\delta_V : V \rightarrow V^{**}$.
- We denote the braiding by $c_{A,B} : A \otimes B \rightarrow B \otimes A$.
- $\mathcal{C}^{\text{bop}}$ is the same underlying category as $\mathcal{C}$ but with opposite braiding.
- The set of isomorphism classes of simples is denoted by $J$, and we fix a representative $X_j$ for each $j \in J$. $0 \in J$ will index the unit object, $X_0 = 1$.
- We fix isomorphisms $\varphi_i : X_j^\ast \rightarrow X_j^\ast$, compatible with the pivotal structure, i.e. $\delta_{X_j} = \varphi_i^\ast \circ \varphi_i^{-1} : X_i \rightarrow X_i^\ast$.
- Evaluation, coevaluation maps are
  \[
  ev_X : X^\ast \otimes X \rightarrow 1 \\
  coev_X : 1 \rightarrow X \otimes X^\ast \\
  \overline{ev}_X = ev_{X^\ast} : X \otimes X^\ast \rightarrow 1 \\
  \overline{coev}_X = \circ coev_{X^\ast} : 1 \rightarrow X^\ast \otimes X
  \]
  (where in the third and fourth line we suppressed the pivotal map $\delta_X : X \rightarrow X^{**}$)
- The categorical dimension of $X_j$ is denoted $d_j = \dim X_j \in \text{End}(1)$. For each $j$, we fix a square root $\sqrt{d_j}$. The dimension of $\mathcal{C}$ is denoted $D = \sum_j d_j^2$, and we will assume that $D \neq 0$. We also fix a square root $\sqrt{D}$.
- Quite often we will omit the symbol $\otimes$, so that concatenation of objects denote tensor products, e.g. $c_{A,B} : AB \rightarrow BA$.
- We use an “Einstein convention”: when latin lowercase alphabet appear in dual pairs, they will be summed over the set of simple objects $J$. For example, $X_j X_j^\ast$ is short for $\bigoplus X_j X_j^\ast$.

We will describe morphisms using graphical calculus (see for example [BK2001], [KJ2011}). Here are some conventions:

- All diagrams represent morphisms in $\mathcal{C}$; morphisms in the other categories that show up, $Z(\mathcal{C})$ and $Z^\text{el}(\mathcal{C})$, are subspaces of morphisms in $\mathcal{C}$.
- Our convention will be that morphisms go from the bottom object to the top.
- If a string is shown without orientation, it is going up by default.
- In string diagrams, some strings with be labelled with a lowercase latin alphabet. This means we are meant to sum the diagram over $J$. There is a similar notion for greek letters (see Appendix).
• Dashed lines will stand for the sum over all colorings of an edge/loop by simple objects \( j \), each taken with coefficient \( d_j \):

\[
\begin{array}{c}
\begin{array}{c}
| \\
| \\
| \\
| \\
\end{array}
\end{array}
= d_j \begin{array}{c}
\begin{array}{c}
| \\
| \\
| \\
\end{array}
\end{array}
\]

(note this is to be summed over \( j \in J \), as mentioned above)

• A pair of morphisms labelled with the same greek letter (sometimes with an overline) will denote a sum over a pair of dual bases with respect to a certain pairing - see appendix for details.

We refer the reader to the appendix for examples, useful identities, and further clarification.

Remark 1.1. All of our constructions are purely algebraic, but we try to explain their topological underpinnings. Thus, topological discussion will be a little sloppy; in particular, we will confuse boundaries and punctures on a surface unless they lead to confusion.

2 The Drinfeld Center

Let us recall the construction and properties of the Drinfeld center. There is nothing new here, so the expert may skip to Section 3, we include this so as to make the constructions and proofs for the elliptic Drinfeld center more transparent and to set some notation.

The following construction is due to Drinfeld (unpublished), and appears in [Maj, JS1991]:

Definition 2.1. The Drinfeld center \( \mathcal{Z}(\mathcal{C}) \) of a monoidal category \( \mathcal{C} \) is a category consisting of the following:

An object of \( \mathcal{Z}(\mathcal{C}) \) is a pair \((X, \lambda)\), where \( X \) is an object of \( \mathcal{C} \) and \( \lambda \) is a half-braiding on \( X \), i.e. a natural transformation \( \lambda : - \otimes X \rightarrow X \otimes - \) that respects tensor products, i.e. satisfies the equation on the left below:
The morphisms $\text{Hom}_{Z(C)}((X, \lambda), (Y, \mu))$ are the subspace of those morphisms in $\text{Hom}_C(X, Y)$ that intertwine the half-braidings $\lambda, \mu$ (equation on the right above). \(\triangle\)

A more concise way to simultaneously state the naturality of $\lambda$ and the above condition on $\lambda$ is the following, which will be used frequently to manipulate diagrams and prove equations:

![Diagram]

When $C$ is spherical fusion, a useful alternative description of $\text{Hom}_{Z(C)}((X, \lambda), (Y, \mu))$ is as the image of a projection, which will be very useful for checking that a certain morphism is actually in $Z(C)$:

**Lemma 2.2.** Let $(X, \lambda), (Y, \mu) \in \text{Obj } Z(C)$. Define the operator

$$P_{\lambda, \mu} \subseteq \text{Hom}_C(X, Y)$$

as follows:

$$P_{\lambda, \mu} : f \mapsto \frac{1}{D} \bigcirc f$$

Then $P_{\lambda, \mu}$ is a projector onto the subspace $\text{Hom}_{Z(C)}((X, \lambda), (Y, \mu)) \subseteq \text{Hom}_C(X, Y)$.

**Proof.** See e.g. [BKJ2010, Lemma 2.2] \(\square\)

### 2.1 Properties of $Z(C)$

Let us recall some well-known facts about $Z(C)$. In this section we will work with spherical fusion categories $C$.

**Proposition 2.3.** [Müg2003] Let $C$ be a spherical fusion category. Then $Z(C)$ is modular.

We give a sketch of a proof and relevant constructions since we will be using similar techniques for the new category. Our proof differs slightly from [Müg2003], particularly proof of semisimplicity and finiteness. The expert may wish to skip to the next section and refer back later.
Proof Sketch. We first show it is abelian. It is clearly additive. The kernel, cokernel, and image of a morphism \( f : (X, \lambda) \to (Y, \mu) \) is obtained from the kernel, cokernel, and image of \( f \) thought of as a morphism in \( \mathcal{C} \), and the object inherits a half-braiding from \( X \) or \( Y \). We illustrate this in more detail for the kernel, since we will repeat the construction for the elliptic Drinfeld center. The cokernel and image follow a similar pattern.

For an exact sequence \( 0 \to K \to X \to Y \), we have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & A \otimes K \\
& \downarrow & \downarrow \lambda_A \\
0 & \to & K \otimes A \\
\end{array}
\quad
\begin{array}{ccc}
A \otimes K & \xrightarrow{\text{id}_A \otimes f} & A \otimes Y \\
& \downarrow & \downarrow \mu_A \\
A \otimes X & \xrightarrow{\text{id}_A} & A \otimes Y \\
\end{array}
\]

The top and bottom rows are exact by the exactness of \( A \otimes - \) and \(- \otimes A\). The leftmost vertical arrow exists and is unique by universal property of kernels. The half-braiding condition is automatically satisfied by the uniqueness of this arrow.

Denote by \( \lambda|_K \) the half-braiding on \( K \) constructed above (i.e. the vertical arrow in the commutative diagram above), so that the candidate kernel constructed above is \( (K, \lambda|_i) \), or simply \( (K, \lambda|_K) \) when there is no confusion. We still need to show that this object satisfies the universal property of kernels. Consider the diagram \((W, \zeta) \xrightarrow{g} (X, \lambda) \to (Y, \mu)\) which composes to \( gf = 0 \). As morphisms in \( Z(\mathcal{C}) \) are subsets of morphisms in \( \mathcal{C} \), \( gf = 0 \) in \( Z(\mathcal{C}) \) implies \( gf = 0 \) as morphisms in \( \mathcal{C} \), so \( g \) must factor uniquely in \( \mathcal{C} \) through \( \iota \), so that there exists a unique \( \mathcal{C} \)-morphism \( h : W \to K \) such that \( g = \iota h \).

To see that \( h \) is a morphism in \( Z(\mathcal{C}) \), i.e. intertwines half-braidings, consider the following diagram:

\[
\begin{array}{c}
A \otimes W \\
\downarrow \zeta_A \\
W \otimes A \\
\downarrow h \otimes \text{id}_A \\
K \otimes A \\
\downarrow \iota \otimes \text{id}_A \\
X \otimes A \\
\end{array}
\quad
\begin{array}{c}
A \otimes K \\
\downarrow \lambda_A \\
A \otimes X \\
\downarrow \text{id}_A \otimes g \\
A \otimes Y \\
\end{array}
\quad
\begin{array}{c}
A \otimes A \\
\end{array}
\]

We need to show that the front left parallelogram commutes. This follows from the facts that: (1) all other faces commute, (2) so composing the parallelogram in question with the bottommost arrow \( K \otimes A \xrightarrow{\iota \otimes \text{id}_A} A \otimes K \) commutes, and (3) this arrow \( (\iota \otimes \text{id}_A) \) is monic.

For the cokernel of \( f \), there is a similar construction of half-braiding on \( C = \text{coker}_\mathcal{C}(f) \) using the universal property of cokernels, and we will denote the half-braiding inherited from \( (Y, \mu) \) by \( \overline{\mu}^C \). For the image, one simply notes that the two ways of
constructing a half-braiding on \( I = \text{im} \, f \) (from \( \text{ker}(\text{coker}(f)) \) and \( \text{coker}(\text{ker}(f)) \)) agree, so that \( \lambda^I = \mu|_I \).

**Semisimplicity** of \( \mathcal{Z}(\mathcal{C}) \): Let \( (K, \lambda|_K) \rightarrow (X, \lambda) \) be a subobject (any other subobject \( (K, \mu) \rightarrow (X, \lambda) \) is isomorphic to \( (K, \lambda|_K) \) since by monicity, \( \iota' = \text{ker}(\text{coker}(\iota')) \) in \( \mathcal{Z}(\mathcal{C}) \).

In particular, \( K \rightarrow X \) is a subobject in \( \mathcal{C} \), so by semisimplicity of \( \mathcal{C} \), we have \( X \cong K \oplus K' \), or more precisely, 
\[
K \xrightarrow{\iota} X \xrightarrow{p'} K'.
\]

It remains to check that \( p \) and \( \iota' \) are morphisms in \( \mathcal{Z}(\mathcal{C}) \). For \( p \), this amounts to showing that the half-braiding \( \lambda^K \) inherited from \( X \) along \( p \) is equal to \( \lambda|_K \). But we have that 
\[
(K, \lambda|_K) \rightarrow (X, \lambda) \xrightarrow{p} (X, \lambda^K)
\]
are morphisms in \( \mathcal{Z}(\mathcal{C}) \), and \( p \mu = \text{id}_K \), so \( \text{id}_K \)
intertwines \( \lambda|_K \) and \( \lambda^K \), hence the two half-braidings are equal. Similarly for \( p' \) and \( \iota' \), so that we have 
\[
(K, \lambda|_K) \xrightarrow{\iota} (X, \lambda) \xrightarrow{p} (K', \lambda^{K'})
\]
giving \( (K, \lambda|_K) \) as a direct summand of \( (X, \lambda) \).

\( \mathcal{Z}(\mathcal{C}) \) has **finitely many simple objects**: this is a consequence of Proposition 2.6, which asserts that \( (Y, \lambda) \in \mathcal{Z}(\mathcal{C}) \) is a direct summand of \( \mathcal{I}Y \). So if \( (X, \lambda) \) is a simple object, with \( X = \bigoplus_j X_j \oplus n_j \), then \( (X, \lambda) \subseteq \bigoplus (\mathcal{I}X_j) \oplus n_j \), hence it must be a subobject of some \( \mathcal{I}X_j \). Since \( \text{End}_{\mathcal{Z}(\mathcal{C})}(\mathcal{I}X_j) \) is finite dimensional, there can only be finitely many simple subobjects of \( \mathcal{I}X_j \). Finally, there are only finitely many simples \( X_j \) in \( \mathcal{C} \).

**Tensor structure:** The tensor product of two objects \( (X, \lambda) \) and \( (Y, \mu) \) is given by 
\[
(X, \lambda) \otimes (Y, \mu) := (X \otimes Y, \lambda \otimes \mu)
\]
where the tensor product \( X \otimes Y \) is from \( \mathcal{C} \), and 
\[
(\lambda \otimes \mu)_A := (\text{id}_X \otimes \mu_A) \otimes (\lambda_A \otimes \text{id}_Y) = \lambda \otimes \mu
\]
The associativity constraint is given by the one from \( \mathcal{C} \), and easily seen to respect the half-braiding.

The **unit** object is \((1, \text{id}_-)\) (with the left/right unit constraints from \( \mathcal{C} \)). It has endomorphism ring \( K \), so is simple.

**Rigidity:** \((X, \lambda)\) has left dual \((X^*, \lambda^*)\), where \((\lambda^*)_A = (\lambda_A)^*\). Similarly, the right dual is \((^*X, ^*\lambda)\), where \(^*\lambda_A = (^*(\lambda_A)^*)\). A simple computation shows that the (co)evaluation maps on \( X \) are morphisms \( 1 \rightarrow (X, \lambda) \otimes (X^*, \lambda^*) \) etc. The pivotal
structure on \( \mathcal{C} \) naturally induces one on \( \mathcal{Z}(\mathcal{C}) \), because \( \delta_{X} : X \to X^{**} \) is always a morphism \( \delta_{X} : (X, \lambda) \to (X^{**}, \lambda^{**}) \) (see e.g. [EGNO2015] Exercise 7.13.6)). It is clearly still spherical on \( \mathcal{Z}(\mathcal{C}) \).

The **braiding** is given by the half-braiding of the second factor:

\[
\tilde{c}(X, \lambda, (Y, \mu)) = \mu_{X} : (X, \lambda) \otimes (Y, \mu) \to (Y, \mu) \otimes (X, \lambda)
\]

We do not prove modularity here as it will not be needed later, referring the reader to [Müg2003, EGNO2015 Corollary 8.20.14].

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**Proposition 2.4.** The forgetful functor \( \mathcal{F} : \mathcal{Z}(\mathcal{C}) \to \mathcal{C} \) has a two-sided “induction” adjoint functor \( \mathcal{I} : \mathcal{C} \to \mathcal{Z}(\mathcal{C}) \), where on objects, \( \mathcal{I} \) sends

\[
X \mapsto (X, \lambda, (X, \lambda)^{\ast}, \Gamma),
\]

where

\[
\Gamma = \sum_{j \neq j'} \sqrt{d_{j}} \sqrt{d_{j'}}
\]

(where \( \alpha, \bar{\alpha} \) are defined in the appendix) and on morphisms, \( f \in \text{Hom}_{\mathcal{C}}(X, Y) \),

\[
f \mapsto \sum_{i} \text{id}_{X_{i}} \otimes f \otimes \text{id}_{X_{i}^{\ast}}
\]

The adjunction is given by the functorial isomorphisms

\[
\text{Hom}_{\mathcal{C}}(X, Y) \cong \text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathcal{I}X, (Y, \mu)) \quad \text{Hom}_{\mathcal{C}}(X, Y) \cong \text{Hom}_{\mathcal{Z}(\mathcal{C})}((X, \lambda), \mathcal{I}Y)
\]

---

**Proof.** By Lemma 7.3 \( \Gamma \) is a half-braiding. It is also easy to check that \( \mathcal{I} \) is a well-defined functor, and that the maps between Hom spaces are indeed isomorphisms, natural in each each variable. We refer the reader to [KJ2011 Theorem 8.2] and [BKJ2010 Theorem 2.3] for more details. \( \square \)
Note that $\mathcal{I}$ is not a monoidal functor, while $\mathcal{F}$ is naturally a tensor functor, but is not braided tensor.

A useful consequence of this adjunction is a description of morphisms $\text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathcal{I}X,\mathcal{I}Y)$:

**Corollary 2.5.** $\text{Hom}_{\mathcal{Z}(\mathcal{C})}(\mathcal{I}X,\mathcal{I}Y)$ consists of morphisms of the form

![Diagram]

where $g_k \in \text{Hom}_\mathcal{C}(X, X_k Y X_k^*)$.

The point here is that the “struts” in the middle are labelled by the same simple object, and is independent of $j,j'$ (up to the $\sqrt{d_j} \sqrt{d_{j'}}$ factor hidden in $\alpha$); these are not true for a general morphism $f \in \text{Hom}_\mathcal{C}(X_j X_j^*, X_{j'} Y X_{j'}^*)$.

**Proposition 2.6.** $(Y,\mu) \in \mathcal{Z}(\mathcal{C})$ is a direct summand of $\mathcal{I}Y$. In particular, $\mathcal{Z}(\mathcal{C})$ is the Karoubian completion of the full image of $\mathcal{I}$.

**Proof.** It is easy to see that the morphism defined below (on the left) projects $\mathcal{I}Y$ onto a direct summand that is isomorphic to $(Y,\mu)$. For example, the first half of this projection is a morphism from $\mathcal{I}$ to $(Y,\mu)$ in $\mathcal{Z}(\mathcal{C})$:

![Diagram]

(note the change from $\alpha$ to $\bar{\alpha}$ in the first equality; in the second equality, we use Lemma 7.1).

2.2 Special $\mathcal{C}$’s

In this section we study what happens when we take special $\mathcal{C}$’s, in particular when $\mathcal{C}$ is modular and when $\mathcal{C}$ is given as the category of finite-dimensional representations of a Hopf algebra $H$. We will be considering analogs of these results for the elliptic Drinfeld center in Sections 4 and 5.

2.2.1 $\mathcal{C}$ Modular

Since $\mathcal{Z}(\mathcal{C})$ is modular, one may expect interesting things to happen when $\mathcal{C}$ itself is modular. Indeed, one has the following:
**Proposition 2.7.** [ENO2004] \textit{If }C\textit{ is modular, then }\mathcal{Z}(C) \cong_{br} C \otimes C^{\text{bop}}, \textit{where }C^{\text{bop}}\textit{ is the same underlying fusion category with the opposite braiding, and }\boxtimes \textit{ is Deligne’s tensor product }[\text{DelI990}].

\textit{Proof Sketch.} There are braided tensor functors }C \rightarrow \mathcal{Z}(C)\textit{ and }C^{\text{bop}} \rightarrow \mathcal{Z}(C)\textit{ given by }X \mapsto (X, c_{-,-})\textit{ and }X \mapsto (X, c_{X,-}^{-1})\textit{ respectively, where recall that }c_{-,-}\textit{ is the braiding on }C\textit{. These fit together into a braided tensor functor }C \boxtimes C^{\text{bop}} \rightarrow_{br} \mathcal{Z}(C)\textit{, and the modularity of }C\textit{ ensures that this is fully faithful. To show essential surjectiveness, one checks that this functor hits all the simple objects of }\mathcal{Z}(C)\textit{ by counting dimensions of endomorphism algebras }\text{End}_{\mathcal{Z}(C)}(X_i X_k X^*), \textit{ or one checks that their Frobenius-Perron dimensions are the same, as in }[\text{EGNO2015}, \text{Prop 8.20.12}].

We will use this in studying the elliptic Drinfeld center }\mathcal{Z}^{el}(C)\textit{ when }C\textit{ is modular in Section 4, in particular it will be the key fact in proving Lemma 4.2.

### 2.2.2 }C = H – \text{mod}

Next we consider when }C = H – \text{mod}, \textit{the category of finite-dimensional modules over a finite-dimensional spherical Hopf algebra }H. \textit{We outline the construction of }\mathcal{D}(H), \textit{Drinfeld’s quantum double of }H, \textit{defined in }[\text{Dri1986}], \textit{which is a ribbon Hopf algebra (in the sense of }[\text{RT1990}])\textit{, and show that }\mathcal{Z}(C) \cong \mathcal{D}(H) – \text{mod}. \textit{Since we work with semisimple }C, \textit{we implicitly assume that }H\textit{ is semisimple, even though the construction of }\mathcal{D}(H)\textit{ does not use semisimplicity. Most of this is follows }[\text{EGNO2015}, \text{Section 7.14}]; \textit{see also }[\text{Kas}, \text{Section XIII.5}].

**Definition-Proposition 2.8.** Let \((H, m, 1, \Delta, \varepsilon, S, v)\) be a finite-dimensional spherical Hopf algebra. The \textit{Drinfeld double of }H, \textit{denoted }\mathcal{D}(H)\textit{, is a ribbon Hopf algebra defined as follows:}

- As a coalgebra, it is }H \otimes H^{*,\text{cop}}, \textit{where }H^{*,\text{cop}} = (H, \Delta^*, \varepsilon, (m^*)^{\text{cop}}, 1, - \circ S^{-1})\textit{ is the dual Hopf algebra with opposite comultiplication.}

- As an algebra, the obvious inclusions }H \cong H \otimes 1 \hookrightarrow H \otimes H^{*,\text{cop}}\textit{ and }H^{*,\text{cop}} \cong 1 \otimes H^{*,\text{cop}} \hookrightarrow H \otimes H^{*,\text{cop}}\textit{ are algebra maps, and the commutation relation is given by

\[ fh = (f_3, S^{-1}(h_1))(f_1, h_2)h_2f_2 \]

where we use Sweedler’s notation }\Delta^2(h) = h_1 \otimes h_2 \otimes h_3\textit{ and }\Delta^2(f) = ((m^*)^{\text{cop}})^2(f) = f_3 \otimes f_2 \otimes f_1\textit{ (note the opposite numbering is used in }[\text{EGNO2015}]).

- The antipode is given componentwise, i.e. }S(hf) = f(S^{-1}(\cdot))S(h)\textit{.

- }v \in H \rightarrow \mathcal{D}(H)\textit{ is the pivotal element.

- The }R\text{-matrix is }\sum h_i \otimes h_i^* \in \mathcal{D}(H) \otimes \mathcal{D}(H)\text{, where }\{h_i\}\text{ is a basis of }H, \text{ and }\{h_i^*\}\text{ the dual basis of }H^{*,\text{cop}}.\]
Proof. Straightforward elementary computations, e.g. see [EGNO2015 Section 7.14].

Example 2.9 (Group Algebra). For $H = k[G]$, where $G$ is a finite group, $H^{*, \text{cop}} \cong F(G)^{\text{cop}}$, the Hopf algebra of functions on $G$ with the opposite comultiplication $m^{*, \text{cop}}(\delta_g) = \sum g_1 g_2 = g \delta_{g_1} \otimes \delta_{g_2}$. By definition, $\{g\}_{g \in G}$ serves as a basis for $k[G]$; let $\{\delta_g\}_{g \in G}$ be the corresponding dual basis of $F(G)^{\text{cop}}$. Then in these bases, for $h \in G$ and $\delta_g \in F(G)^{\text{cop}}$, the commutation relations between $k[G]$ and $F(G)^{\text{cop}}$ is simply

$$\delta_g h = h \delta_{h^{-1} g h}$$

Denote $D(G) := D(k[G])$. Using this explicit description of $D(G)$, we can interpret representations of $D(G)$ as $G$-equivariant bundles over $G$, where $G$ acts on itself by conjugation. Briefly, the $\delta_g$ are projections, giving us a (vector space) decomposition of a representation $V$ of $D(G)$ into $\bigoplus_{g \in G} V_{\bar{g}}$, where $V_{\bar{g}} = \delta_{\bar{g}} V$. Then for $v_{\bar{g}} = \delta_{\bar{g}} v \in V_{\bar{g}}$,

$$h \cdot v_{\bar{g}} = h \delta_{\bar{g}} \cdot v = \delta_{h g h^{-1}} h \cdot v \in V_{h g h^{-1}}$$

thus the bundle with $V_{\bar{g}}$ sitting over $g \in G$ is $G$-equivariant.

For each conjugacy class $\bar{g} \in \bar{G}$, the sum $\delta_{\bar{g}} = \sum_{\bar{g} \in \bar{G}} \delta_{\bar{g}}$ is a central idempotent, and the collection of such $\delta_{\bar{g}}$ is pairwise orthogonal and sum to 1. So the category of finite-dimensional representations $D(G)$-mod is semisimple with simple objects $V_{\bar{g}}$, and the collection of such $V_{\bar{g}}$ provides a (vector space) decomposition of a representation of $D(G)$ into $\bigoplus_{\bar{g} \in \bar{G}} V_{\bar{g}}$. We refer the reader to [BK2001 Section 3.2] for further details. (Note that in terms of our set up in Definition 2.8, they are working with $D(F(G)) \cong D(k[G^{op}]^{cop})$.)

Let us also note that $D(k[G]) \cong D(F(G))$, but this will not quite hold true for the elliptic double $D^{el}(H)$ defined later in Definition 5.2 - there the input Hopf algebra $H$ must at least be ribbon, so $D^{el}(F(G))$ is not even defined, unless $G$ is abelian. We will discuss the elliptic analog of this example in Example 5.9.

Proposition 2.10. For a finite-dimensional Hopf algebra $H$, let $C = H$-mod, the category of finite-dimensional left $H$-modules. Then $Z(C) \equiv \text{op}, \text{br} D(H)$-mod.

Proof Sketch. We essentially follow [EGNO2015 Section 7.14], referring the reader to it for more details.

The functor $D(H)$-mod $\to Z(C)$ is constructed as follows. Let $X$ be a left $D(H)$-module. It is in particular an $H$-module, i.e. an object in $C$. As an object of $C$, it has a natural half-braiding, given by the $R$-matrix of $D(H)$: for $A \in C$ some $H$-module, define

$$\lambda_A : A \otimes X \to X \otimes A$$

to be the linear map given by first acting by $R = \sum h_i \otimes h_i^*$ (the action is defined on $A \otimes X$ because the first factors appearing in $R$ are in $H$), and then swapping the factors.
This is a half-braiding on $X$ because $(\Delta \otimes \text{id})(R) = R^{13}R^{23} \in H \otimes H \otimes H^*$. Thus, we have defined a functor $\mathcal{D}(H) - \text{mod} \to \mathcal{Z}(\mathcal{C})$.

For the other way, let $(X, \lambda) \in \mathcal{Z}(\mathcal{C})$. By functoriality of $\lambda$ and finite-dimensionality, $\lambda$ is completely determined by $\lambda_H : H \otimes X \to X \otimes H$. Then we define for $f \in H^*$,

$$f \cdot x = (\text{id}_X \otimes f, \lambda_H(1 \otimes x))$$

Said otherwise, it is the action of $H^*$ on $X$ such that $\lambda = P \cdot R$. There are commutation relations between these actions, which can be derived from looking at the Yang-Baxter equation on $H \otimes X \otimes H^*$, and one sees that they are precisely those as imposed on $\mathcal{D}(H)$.

Since morphisms of $\mathcal{Z}(\mathcal{C})$ are precisely those that intertwine half-braidings, it is easy to see that they are also precisely those that intertwine the $H^*$-actions, so that we have a fully faithful functor back $\mathcal{Z}(\mathcal{C}) \to \mathcal{D}(H) - \text{mod}$, and it’s not hard to see that it is an equivalence.

So far we haven’t used the coalgebra structure of $\mathcal{D}(H)$. The monoidal structure on $\mathcal{Z}(\mathcal{C})$ should carry over to $\mathcal{D}(H) - \text{mod}$, and the claim is that it agrees with that which arise from the coalgebra structure on $\mathcal{D}(H)$. We can see this as follows: The tensor product of $(X, \lambda)$ and $(Y, \mu)$ is $(X \otimes Y, \lambda \otimes \mu)$, where recall $\lambda \otimes \mu$ is just braiding by $\lambda$ and then $\mu$, so

$$(\lambda \otimes \mu)_H : 1 \otimes x \otimes y \mapsto h_i^* \cdot x \otimes h_j^* \cdot y \otimes h_i = (m^*)^\text{cop}(h_k^*) \cdot (x \otimes y) \otimes h_k$$

so the action of $f$ is

$$f \cdot (x \otimes y) = (1 \otimes f, (\lambda \otimes \mu)_H(1 \otimes (x \otimes y))) = \langle f, h_k \rangle (m^*)^\text{cop}(h_k^*) \cdot (x \otimes y) = (m^*)^\text{cop}(f) \cdot (x \otimes y)$$

So the functors we have defined here are tensor functors, and in fact braided. The pivotal structure on $\mathcal{C}$ is naturally a pivotal structure on $\mathcal{Z}(\mathcal{C})$, and these functors clearly respect the pivotal structures. Once again we refer the reader to [EGNO2015, Section 7.14] for more details.

**Corollary 2.11.** If $H$ is semisimple, then so is $\mathcal{D}(H)$.

### 3 The Elliptic Drinfeld Center

In this section we define the *Elliptic Drinfeld Center* $\mathcal{Z}^{\text{el}}(\mathcal{C})$ of a *braided* monoidal category $\mathcal{C}$. It is analogous to the Drinfeld center $\mathcal{Z}(\mathcal{C})$, in that objects consist of an object of $\mathcal{C}$ and, not one, but two half-braidings which are related by some equation involving the braiding on $\mathcal{C}$. As mentioned in the introduction, the motivation for constructing $\mathcal{Z}^{\text{el}}(\mathcal{C})$ comes from studying the value of the extended Crane-Yetter TQFT on a once-punctured torus. We discuss some of the properties of the elliptic Drinfeld center parallel to those of the Drinfeld center as discussed in Section 2. We put a
monoidal structure on $Z^\text{el}(\mathcal{C})$ in Section 3.2 and discuss an $\text{SL}_2(\mathbb{Z})$-action on $Z^\text{el}(\mathcal{C})$ in Section 3.3. Just as with $Z(\mathcal{C})$, $Z^\text{el}(\mathcal{C})$ has particularly nice descriptions when $\mathcal{C}$ is modular and when $\mathcal{C} = H - \text{mod}$ for a quasi-triangular Hopf algebra $H$, which we discuss in Sections 4 and 5 respectively.

3.1 Definition and Properties of $Z^\text{el}(\mathcal{C})$

**Definition 3.1.** Given a braided monoidal category $\mathcal{C}$, the Elliptic Drinfeld Center of $\mathcal{C}$, denoted $Z^\text{el}(\mathcal{C})$, is the category with objects of the form $(X, \lambda_1, \lambda_2)$, where $X$ is an object in $\mathcal{C}$, and $\lambda_1, \lambda_2 : - \otimes X \to X \otimes -$ are half-braidings on $X$ satisfying the following compatibility condition which we refer to as COMM:

$$\text{COMM:}$$

![Diagram of COMM](image.png)

We point out that on the l.h.s one has the universal braiding $c_{-, -}$, while on the r.h.s. it is the reverse $c_{-, -}^{-1}$. We also note that COMM is a condition on an ordered pair of half-braidings, in that if $(\lambda_1, \lambda_2)$ satisfies COMM, it does not imply that $(\lambda_2, \lambda_1)$ satisfies COMM; however in some sense $\lambda_1, \lambda_2$ should be treated on equal footing - see Remark 3.13.

The morphisms from $(X, \lambda_1, \lambda_2)$ to $(Y, \mu_1, \mu_2)$ are given by those in $\text{Hom}_{\mathcal{C}}(X, Y)$ that intertwine both half-braidings, i.e.

$$\text{Hom}_{Z^\text{el}(\mathcal{C})}((X, \lambda_1, \lambda_2), (Y, \mu_1, \mu_2)) := \text{Hom}_{Z(\mathcal{C})}((X, \lambda_1), (Y, \mu_1)) \cap \text{Hom}_{Z(\mathcal{C})}((X, \lambda_2), (Y, \mu_2))$$

**Remark 3.2.** As a visual aid, it is helpful to think of COMM as allowing the strands labelled $A$ and $B$ to pass through each other - at the moment they meet, they should be transverse to each other, so that if the $A$ strand was above, then after meeting, the $A$ strand would be below. Morally, the $A$ strand goes around a meridian in the once-punctured torus, while the $B$ strand goes around a longitude, and a pair of meridian and longitude can be isotoped to intersect once and intersect transversally. Later in Section 6 we discuss a variant of COMM, where instead of being transverse, the strands become tangent at the moment of meeting. This reflects the fact that two embedded closed curves in a thrice-punctured sphere can be isotoped to not intersect each other.

For the rest of the section, we will work with premodular $\mathcal{C}$. A simple consequence of COMM is the following analog of Lemma 2.2.
Lemma 3.3. Let \((X, \lambda^1, \lambda^2), (Y, \mu^1, \mu^2)\) be objects in \(Z^{el}(C)\). The projections \(P_{\lambda^1, \mu^1}, P_{\lambda^2, \mu^2} \circ \text{Hom}_C(X, Y)\) defined in Lemma 2.2 commute, and hence we have
\[
\text{Hom}_{Z^{el}(C)}((X, \lambda^1, \lambda^2), (Y, \mu^1, \mu^2)) = \text{im}(P_{\lambda^1, \mu^1} \circ P_{\lambda^2, \mu^2}) \subseteq \text{Hom}_C(X, Y).
\]

Proof.

\[
P_{\lambda^2, \mu^2}(P_{\lambda^1, \mu^1}(f)) = \frac{1}{D^2} = \frac{1}{D^2} = \frac{1}{D^2} = P_{\lambda^1, \mu^1}(P_{\lambda^2, \mu^2}(f))
\]

The second equality uses COMM.

\[\square\]

Proposition 3.4. Let \(C\) be a premodular category. Then \(Z^{el}(C)\) is abelian and semisimple, and has finitely many simple objects.

Proof. The proof is similar to that of Proposition 2.3, and we will use some notation established there.

\(Z^{el}(C)\) is clearly additive. The proof that \(Z^{el}(C)\) is abelian is pretty much the same as for \(Z(C)\). For example, to get the kernel of a morphism \(f: (X, \lambda^1, \lambda^2) \to (Y, \mu^1, \mu^2)\), first find the kernel of \(f\) as a morphism in \(C\), say \(\iota: K \to X\). \(K\) inherits two half-braidings \(\lambda^1|_K, \lambda^2|_K\) from \(X\) by restricting along \(\iota\) (see proof sketch of Proposition 2.3). These two half-braidings satisfy COMM naturally from the universal property of kernels:

\[
\begin{align*}
KBA & \hookrightarrow XBA & XBA & \hookrightarrow KBA \\
& \uparrow \text{id}_X \otimes c_{A,B} & \uparrow \lambda_B^1 \otimes \text{id}_A & \uparrow \\
KAB & \hookrightarrow XAB & BXA & \hookrightarrow BKA \\
& \uparrow \lambda_A^1 \otimes \text{id}_B & \uparrow \text{id}_B \otimes \lambda_A^1 & \uparrow \\
AKB & \hookrightarrow AXB & BAX & \hookrightarrow BAK \\
& \uparrow \text{id}_A \otimes \lambda_B^2 & \uparrow c_{B,A}^1 \otimes \text{id}_X & \uparrow \\
ABK & \hookrightarrow ABX & ABX & \hookrightarrow ABK
\end{align*}
\]

(The inner octagon commutes because \(\lambda^1, \lambda^2\) satisfy COMM by definition, so the outer octagon commutes, hence \(\lambda^1|_K, \lambda^2|_K\) satisfy COMM. The equal signs should really be some associativity constraints, but we suppress them.)

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So \( \ker_{Z^{el}(C)}(f) = (K, \lambda^1|_K, \lambda^2|_K) \). Likewise, \( \text{coker}_{Z^{el}(C)}(f) = (C, \mu^1, \mu^2) \), and \( \text{im}(f) = (I, \lambda^1_f = \mu^1|_I, \lambda^2_f = \mu^2|_I) \), where \( C, I \) are respectively the cokernel, image in \( \mathcal{C} \).

**Semisimplicity** of \( Z^{el}(\mathcal{C}) \) follows from the same argument for \( Z(\mathcal{C}) \) almost verbatim - we again start with \((K, \lambda^1|_K, \lambda^2|_K) \). By the arguments in the proof of Proposition 2.3, we have that \( \iota \) and \( p' \) are morphisms in the following diagram in \( Z^{el}(\mathcal{C}) \):

\[
(K, \lambda^1|_K, \lambda^2|_K) \xrightarrow{\iota} (X, \lambda^1, \lambda^2) \xrightarrow{p'} (K', \lambda^{1'}|_K', \lambda^{2'}|_K')
\]

hence \((K, \lambda^1|_K, \lambda^2|_K) \) is a direct summand of \((X, \lambda^1, \lambda^2) \).

\( Z^{el}(\mathcal{C}) \) has **finitely many simple objects**: similarly to the proof of Proposition 2.3, this is a consequence of Proposition 3.8, which asserts that \((Y, \mu^1, \mu^2) \) is a direct summand of \( I^{el}Y \). Then if \((X, \lambda^1, \lambda^2) \) is simple with \( X = \bigoplus_j X_j^{\oplus n_j} \), then \((X, \lambda^1, \lambda^2) \subseteq \bigoplus_j (I^{el}X_j)^{\oplus n_j} \), hence it must be a subobject of some \( I^{el}X_j \). Since \( \text{End}_{Z^{el}(\mathcal{C})}(I^{el}X_j) \) is finite dimensional, there can only be finitely many simple subobjects of \( I^{el}X_j \). Finally, there are only finitely many simples \( X_j \) in \( \mathcal{C} \).

The following is the analog of Proposition 2.4

**Proposition 3.5.** The forgetful functor \( F^{el} : Z^{el}(\mathcal{C}) \to \mathcal{C} \) has a two-sided adjoint \( I^{el} : \mathcal{C} \to Z^{el}(\mathcal{C}) \), where on objects, \( I^{el} \) sends

\[
X \mapsto (\bigoplus_{i,j} X_iX_jX^*_jX^*_i, \Gamma^1, \Gamma^2)
\]

where

\[
\Gamma^1 = \begin{array}{c}
\alpha \\
\alpha \\
\alpha \\
\end{array}, \quad \Gamma^2 = \begin{array}{c}
\beta \\
\beta \\
\beta \\
\end{array}
\]

(where \( \alpha \) is defined in the appendix) and on morphisms, \( f \in \text{Hom}_{\mathcal{C}}(X, Y) \),

\[
I^{el}(f) = \bigoplus_{i,j} \text{id}_{X_iX_j} \otimes f \otimes \text{id}_{X^*_jX^*_i}
\]
The adjunction is given by the functorial isomorphisms

\[ \text{Hom}_{\mathcal{C}}(X,Y) \cong \text{Hom}_{Z_{\text{el}}(\mathcal{C})}(I_{\text{el}}X, (Y, \mu^1, \mu^2)) \]

\[ \text{Hom}_{\mathcal{C}}(X,Y) \cong \text{Hom}_{Z_{\text{el}}(\mathcal{C})}((X, \lambda^1, \lambda^2), I_{\text{el}}Y) \]

Remark 3.6. Note that \( \Gamma^1 \) can also be described as the half-braiding given by induction \( I : \mathcal{C} \to Z(\mathcal{C}) \), that is, \( (\oplus_{i,j} X_i X_j X_i^* X_j^*, \Gamma^1) \cong I(\oplus_{j} X_j X_j^*) \) in \( Z(\mathcal{C}) \). There is a similar description for \( \Gamma^2 \) that we will elaborate on later. See Proposition 3.9, Remark 3.10.

Proof. By Lemma 7.3, \( \Gamma^1, \Gamma^2 \) are half-braidings. To see that \( I_{\text{el}} \) is a well-defined functor, one easily checks that the two half-braidings \( \Gamma^1, \Gamma^2 \) satisfy COMM (but see Remark 3.7), and that on morphisms, \( I_{\text{el}}(f) \) intertwines both \( \Gamma^1 \) and \( \Gamma^2 \), and \( I_{\text{el}} \) respects identity and composition. The maps between the Hom spaces are also easily checked to be inverses.

Remark 3.7. It is trivial to check that \( \Gamma^1, \Gamma^2 \) satisfy COMM, but more illuminating is the “reason” that they do. \( I_{\text{el}}X = (\oplus_{i,j} X_i X_j X_i^* X_j^*, \Gamma^1, \Gamma^2) \) is the prototype of an object in \( Z_{\text{el}}(\mathcal{C}) \). Indeed, in the skein-theoretic approach to the extended Crane-Yetter TQFT that we are studying, such objects \( I_{\text{el}}X \) arise naturally in the category \( Z_{\text{CY}}(T^2_1) \) as the configuration consisting of one marked point in \( T^2_1 \) labelled with \( X \in \mathcal{C} \). The other objects are obtained by taking all images of idempotents, i.e. we consider the Karoubian closure. In the current context, this takes the form of Proposition 3.8. In other words, the definition of compatible half-braidings was cooked up to capture the essential features of this prototypical object.

The following is the analog of Proposition 2.6:

Proposition 3.8. \((Y, \mu^1, \mu^2) \in Z_{\text{el}}(\mathcal{C})\) is a direct summand of \( I_{\text{el}}Y \). In particular, \( Z_{\text{el}}(\mathcal{C}) \) is the Karoubian completion of the full image of \( I_{\text{el}} \).

Proof. The morphism below projects \( I_{\text{el}}Y \) onto \((Y, \mu^1, \mu^2)\) (see proof of Proposition 3.8).
So far we have only discussed the relation between $Z^\text{el}(C)$ and $C$. As Remark 3.6 suggests, $\mathcal{I}^\text{el}$ actually factors through $Z(C)$. Indeed, observe that $\mathcal{F}^\text{el}$ factors through an intermediate forgetful functor $\mathcal{F}_1 : Z^\text{el}(C) \to Z(C)$, which forgets only the first braiding, $(X, \lambda^1, \lambda^2) \mapsto (X, \lambda^2)$, so that $\mathcal{F}^\text{el} = \mathcal{F} \circ \mathcal{F}_1$.

**Proposition 3.9.** The intermediate forgetful functor $\mathcal{F}_1 : Z^\text{el}(C) \to Z(C)$ which forgets the first braiding, $\mathcal{F}_1 : (X, \lambda^1, \lambda^2) \mapsto (X, \lambda^2)$, has a two-sided adjoint $\mathcal{I}_1 : Z(C) \to Z^\text{el}(C)$, such that $\mathcal{I}^\text{el}$ factors through it, i.e. $\mathcal{I}^\text{el} = \mathcal{I}_1 \circ \mathcal{I}$.

On objects, $\mathcal{I}_1$ sends

$$(X, \lambda) \mapsto (X, XX^*_i, \Gamma, \bar{\lambda})$$

where $\Gamma$ was defined in Proposition 2.4 and

$$\bar{\lambda} = \lambda$$

and on morphisms, $f \in \text{Hom}_{Z(C)}((X, \lambda), (Y, \mu))$,

$$\mathcal{I}_1(f) = \text{id}_{X^i} \otimes f \otimes \text{id}_{X^*_i}$$

which clearly intertwines $\bar{\lambda}$ and $\bar{\mu}$.

The adjunction

$$\text{Hom}_{Z^2}(\mathcal{I}_1(X, \lambda), (Y, \mu)) \cong \text{Hom}_{Z^\text{el}}(\mathcal{I}_1(X, \lambda), (Z, \mu^1, \mu^2))$$

$$\text{Hom}_{Z(C)}((X, \lambda^2), (Y, \mu)) \cong \text{Hom}_{Z^\text{el}}((X, \lambda^1, \lambda^2), \mathcal{I}_1(Y, \mu))$$

is given by the same maps as in Proposition 2.4, just with $\lambda$’s and $\mu$’s replaced by $\lambda^1$ and $\mu^1$’s, respectively.

**Remark 3.10.** In Remark 3.6 we noted that $\Gamma^1$ can be described as the half-braiding given by $\mathcal{I}$. Now, we can describe $\Gamma^2$, the second braiding of $\mathcal{I}^\text{el}X$, as $\bar{\Gamma}$, where recall $\Gamma$ is the braiding $\mathcal{I} \circ \mathcal{I} X$. Indeed, this had better be the case since $\mathcal{I}^\text{el}X = (X, XX^*_i, \Gamma^1, \Gamma^2)$ while $\mathcal{I}(\mathcal{I}X) = \mathcal{I}_1(X, XX^*_i, \Gamma) = (X, XX^*_j X^*_i, \Gamma^1, \bar{\Gamma})$, and the proposition claims they are the same.
Proof. Similar to Proposition 3.5.

The forgetful functor $F_1: \mathcal{Z}^{\text{el}}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C})$ was defined to be forgetting the first half-braiding. There is nothing special about the first braiding compared to the second (see Remark 3.13). We define $F_2: \mathcal{Z}^{\text{el}}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C})$ to be the forgetful functor forgetting the second half-braiding. One then also has a two-sided adjoint functor $I_2$:

**Proposition 3.11.** The intermediate forgetful functor $F_2: \mathcal{Z}^{\text{el}}(\mathcal{C}) \to \mathcal{Z}(\mathcal{C})$ which forgets the second braiding, $F_2: (X, \lambda^1, \lambda^2) \mapsto (X, \lambda^1)$, has a two-sided adjoint $I_2: \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}^{\text{el}}(\mathcal{C})$, such that $I^{\text{el}} = I_2 \circ I$.

On objects, $I_2$ sends

$$(X, \lambda) \mapsto (X, XX^*, \lambda, \Gamma)$$

where

$$(\lambda) =$$

(\text{note the different braidings used compared to } \lambda \text{ in Proposition 3.9})

and on morphisms, $f \in \text{Hom}_{\mathcal{Z}(\mathcal{C})}((X, \lambda), (Y, \mu))$,

$$I_2(f) = \text{id}_{XX^*} \otimes f \otimes \text{id}_{X^*}$$

The adjunction

$$\text{Hom}_{\mathcal{Z}(\mathcal{C})}((X, \lambda), (Y, \mu)) \cong \text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{C})}(I_2(X, \lambda), (Y, \mu))$$

is given by the same maps as in Proposition 2.4, just with $\lambda$’s and $\mu$’s replaced by $\lambda^2$ and $\mu^2$’s, respectively.

Proof. Similar to Proposition 3.5. □

**Remark 3.12.** In fact, there are infinitely many such pairs of induct-forget pairs, indexed by elements of $\text{SL}_2(\mathbb{Z})$. Indeed, in Section 3.3 we exhibit an $\text{SL}_2(\mathbb{Z})$-action on $\mathcal{Z}^{\text{el}}(\mathcal{C})$. Then precomposing $F_1$ and postcomposing $I_1$ with the action of a group element gives another forget-induction adjoint pair. For example, $F_2 = F_1 \circ T_s$, and $I_2 \cong T_s^{-1} \circ I_1$. This has a simple explanation in terms of the Crane-Yetter TQFT. Namely, each forgetful functor corresponds to one way to cut the once-punctured torus along an embedded arc into an annulus (the arc necessarily connects puncture to puncture), while the corresponding induction functor would be an inclusion of the annulus into the once-punctured torus that avoids the cut. This description is inspired by the description given in [KJ2011] for the forget-induction adjunction between $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$ in terms of the string-nets interpretation of Turaev-Viro theory. We will say more in forthcoming work. △
Remark 3.13. As pointed out in Definition 3.1, the COMM relation is not commutative, so it may appear that the first and second half-braidings somehow have distinct characteristics. However, this is merely a manifestation of the dependence of the equivalence $Z^{el}(\mathcal{C}) \to Z_{CV}(\mathbb{T}_2^1)$ on certain “coordinate” choices (see Introduction). The action of $s$ in the $\text{SL}_2(\mathbb{Z})$-action discussed in Section 3.3 lends further credence to this, swapping the first and second half-braidings on an object, but to satisfy COMM, one of them is half-twisted. △

In summary, we have the following (2-)commutative diagram:

$$
\begin{array}{c}
\mathcal{C} & \overset{\mathcal{I}}{\longrightarrow} & Z(\mathcal{C}) \\
\downarrow{\mathcal{F}} & \Downarrow{\mathcal{F}^{el}} & \downarrow{\mathcal{F}_1} \\
Z(\mathcal{C}) & \overset{\mathcal{I}_2}{\longleftarrow} & Z^{el}(\mathcal{C})
\end{array}
$$

3.2 Tensor Product on $Z^{el}(\mathcal{C})$

Recall that for a monoidal category $\mathcal{C}$, the Drinfeld center $Z(\mathcal{C})$ is a braided monoidal category, with tensor product given by

$$(X, \lambda) \otimes (Y, \mu) := (X \otimes Y, \lambda \otimes \mu)$$

(see Proposition 2.3). One might naively try to define a tensor product on $Z^{el}(\mathcal{C})$ by

$$(X, \lambda^1, \lambda^2) \otimes (Y, \mu^1, \mu^2) = (X \otimes Y, \lambda^1 \otimes \mu^1, \lambda^2 \otimes \mu^2)$$

Unfortunately, this doesn’t work, because the two half-braidings $\lambda^1 \otimes \mu^1$ and $\lambda^2 \otimes \mu^2$ do not satisfy COMM (unless $\mathcal{C}$ is symmetric, which we discuss in Section 5.1).

All is not lost, however. A closer look at the tensor product on the Drinfeld center reveals that it arises from considering the pair of pants in Turaev-Viro. There is a modified version of the pair of pants for the once-punctured torus (see Remark 3.21) that leads to the following definition:

**Definition 3.14.** Let $\lambda, \mu$ be half-braidings on $X, Y$, respectively. Define $X \lambda \otimes_{\mu} Y$ to be the image of the projection $Q_{\lambda,\mu} \circ X \otimes Y$, where

$$Q_{\lambda,\mu} := \frac{1}{D}$$

We call $X \lambda \otimes_{\mu} Y$ the *reduced tensor product* of $X$ and $Y$ (with respect to $\lambda$ and $\mu$). △
Remark 3.15. Strictly speaking, in defining $X_\lambda \otimes_\mu Y$ as the image of some projection, we have to make a choice of some object along with certain maps relating it to $X \otimes Y$. We make such a choice arbitrarily for each $X,Y$, but identify morphisms out of $X_\lambda \otimes_\mu Y$ with morphisms out of $X \otimes Y$ that are invariant under precomposing with $Q_{\lambda,\mu}$, and similarly for morphisms into $X_\lambda \otimes_\mu Y$:

$$\text{Hom}_C(X_\lambda \otimes_\mu Y, N) \cong \{ g \in \text{Hom}_C(X \otimes Y, N) | g = g \circ Q_{\lambda,\mu} \} = \text{Hom}_C(X \otimes Y, N) \circ Q_{\lambda,\mu}$$

For brevity, when describing a morphism $g : X_\lambda \otimes_\mu Y \to N$, we will sometimes omit $Q_{\lambda,\mu}$, or more accurately we implicitly precompose $g$ with $Q_{\lambda,\mu}$. Likewise for morphisms $f : M \to X_\lambda \otimes_\mu Y$.

Lemma 3.16. Let $\lambda, \mu$ be half-braidings on $X,Y$, respectively. For $A \in C$, we have the following equality of morphisms $A \otimes (X_\lambda \otimes_\mu Y) \to (X_\lambda \otimes_\mu Y) \otimes A$:

So defined, $\lambda \otimes_\mu$ is a half-braiding on $X_\lambda \otimes_\mu Y$.

Proof. The first and third equality uses the fact that $\lambda$ is a half-braiding. The second equality uses Lemma 7.1. This also easily implies the fact that $\lambda \otimes_\mu$ is a half-braiding. The last equality follows from similar sequence of equalities.

The third equality also easily implies that $\lambda \otimes_\mu$ is a half-braiding. △

Corollary 3.17. Let $\lambda, \mu, \zeta$ be half-braidings on $X,Y,Z \in C$, respectively. Then

$$(X_\lambda \otimes_\mu Y) \otimes_\mu \otimes_\zeta Z = \text{im}(Q_{\lambda,\mu,\zeta} \circ (X \otimes Y) \otimes Z)$$

where $Q_{\lambda,\mu,\zeta} = (\text{id}_X \otimes Q_{\mu,\zeta}) \circ (Q_{\lambda,\mu} \otimes \text{id}_Z)$.

More generally, if we have half-braidings $\lambda_1, \ldots, \lambda_k$ on $X_1, \ldots, X_k$, respectively, then

$$(\ldots((X_1_\lambda \otimes_\lambda \lambda_2 X_2) \ldots)_\lambda \lambda_k X_k)$$

is the image of the projection

and similarly for any order of taking reduced tensor products.
Definition-Proposition 3.18. \( \mathcal{Z}^\text{el}(\mathcal{C}) \) admits the following monoidal structure:

For objects \((X, \lambda^1, \lambda^2), (Y, \mu^1, \mu^2) \in \mathcal{Z}^\text{el}(\mathcal{C})\), their tensor product is defined by

\[
(X, \lambda^1, \lambda^2) \otimes (Y, \mu^1, \mu^2) = (X_{\lambda^1 \overline{\mu}^1} Y, \lambda^1 \overline{\mu}^1, \lambda^2 \otimes \mu^2)
\]

where \(X_{\lambda^1 \overline{\mu}^1} Y\) was defined in Definition \[3.14\] and \(\lambda^1 \overline{\mu}^1\) was defined in Lemma \[3.16\].

The associativity constraint \(a\) is given by the associativity constraint of \(\mathcal{C}\) (understood as in Remark \[3.15\]).

The unit object \(1^\text{el}\) is \(\mathcal{Z}_1(1, \text{id}_-\) = \((X_J X^*_J, \Gamma, \Omega)\), where \(\Omega = \overline{\text{id}}_-\), with left and right unit constraint given by

\[
\lambda^1 \otimes \mu^1 = \sqrt{d_i} \lambda^1 \mu^1 ; \quad \lambda^2 \otimes \mu^2 = \sqrt{d_i} \lambda^2 \mu^2
\]

Proof. Let \(\mathcal{X} = (X, \lambda^1, \lambda^2), \mathcal{Y} = (Y, \mu^1, \mu^2), \mathcal{Z} = (Z, \zeta^1, \zeta^2)\). It is easy to see that \(\lambda^1 \overline{\mu}^1\) and \(\lambda^2 \otimes \mu^2\) satisfy \(\text{COMM}\), so the definition of \(\mathcal{X} \otimes \mathcal{Y}\) makes sense.

By Corollary \[3.17\], the associativity constraint \(a_{\mathcal{X}, \mathcal{Y}, \mathcal{Z}}\) is just \(Q_{\lambda^1, \mu^1, \zeta^1} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\), so it is an isomorphism \((X_{\lambda^1 \overline{\mu}^1}, Y_{\lambda^2 \overline{\mu}^1, \zeta^1}, Z) \to X_{\lambda^1 \overline{\mu}^1, \zeta^1} (Y_{\mu^1, \zeta^1}, Z)\). The pentagon axiom also follows from Corollary \[3.17\].

To see that \(l_{\mathcal{X}}, r_{\mathcal{X}}\) are isomorphisms, we provide morphisms

\[
\overline{l}_{\mathcal{X}} = \sqrt{d_i} \lambda^1 ; \quad \overline{r}_{\mathcal{X}} = \sqrt{d_i} \lambda^2
\]

and check that they are inverses: \(l_{\mathcal{X}} \circ \overline{l}_{\mathcal{X}} = \text{id}_X, \overline{l}_{\mathcal{X}} \circ l_{\mathcal{X}} = Q_{\Gamma, \lambda^1}\), and likewise for \(r_{\mathcal{X}}\). For example,

\[
\overline{l}_{\mathcal{X}} \circ l_{\mathcal{X}} = \sqrt{d_i} \lambda^1 \lambda^1 = \zeta^1 \lambda^1 = Q_{\Gamma, \lambda^1}
\]

The triangle axiom for unit constraints is a simple consequence of Lemma \[3.16\] (in particular the last equality). Let us note that the \(\Omega\) in \(1^\text{el}\) is the same \(\Omega\) as in the proof of Lemma \[4.1\] but specifically for \(X = 1\), so that \(1^\text{el} = i1\).\(\square\)
Proposition 3.19. For \((X,\lambda),(Y,\mu) \in \mathcal{Z}(\mathcal{C})\), let \(J_{(X,\lambda),(Y,\mu)} = J_{X,Y} : \mathcal{I}_1(X,\lambda) \otimes \mathcal{I}_1(Y,\mu) \to \mathcal{I}_1((X,\lambda) \otimes (Y,\mu))\) be defined by

\[
J_{X,Y} := \sqrt{d_i}
\]

Then \((\mathcal{I}_1, J) : \mathcal{Z}(\mathcal{C}) \to \mathcal{Z}^{\text{el}}(\mathcal{C})\) is a monoidal functor.

Proof. By construction, \(\mathcal{I}_1\) sends unit to unit. We check that \(J_{X,Y}\) has an inverse, \(\overline{J_{X,Y}}\), as follows:

\[
\begin{align*}
J_{X,Y} &\cdot \sqrt{d_j} = \sqrt{d_k} \cdot \sqrt{d_j} = \sqrt{d_k} \cdot \sqrt{d_j} = d_k \cdot d_j \\
\overline{J_{X,Y}} &\cdot \sqrt{d_i} = \sqrt{d_i} \cdot \sqrt{d_j} = \sqrt{d_i} \cdot \sqrt{d_j} = d_i \cdot d_j \\
&= Q_{k,g}
\end{align*}
\]

(using Lemma 7.4 repeatedly), and similarly check that \(J_{X,Y} \circ \overline{J_{X,Y}} = \text{id}\). That \(J\) satisfies the hexagon axiom follows from

\[
J_{X \otimes Y, Z} \circ (J_{X,Y} \otimes \text{id}_Z) = \sqrt{d_i} \sqrt{d_j} \sqrt{d_k} \sqrt{d_l} \sqrt{d_m} = J_{X,Y \otimes Z} \circ (\text{id}_X \otimes J_{Y,Z})
\]

We work out in Example 5.10 what this tensor product looks like when \(\mathcal{C} = \text{Rep}(G)\), the category of finite-dimensional representations of a finite group \(G\).

Remark 3.20. The presence of the projection in Definition 3.14 leads this tensor product to be highly non-commutative, and also makes \(\mathcal{Z}^{\text{el}}(\mathcal{C})\) multi-tensor but generally not tensor. \(\triangle\)
Remark 3.21. Definition \[3.14\] of the tensor product looks rather ad hoc, but it actually arises from understanding the functor that \(Z_{CY}\) associates to the following cobordism \(Y\) from \(T^2_1 \cup T^2_1\) to \(T^2_1\):

In the diagram, \(T^2_1\) is drawn as the “plumbing” of two annuli, that is, it is the union of two annuli, the gray 1-annulus (annulus labelled with a 1) and the black 2-annulus, glued along the common gray square in a transverse manner (the 1-annulus and 2-annulus do not meet away from the gray square in the middle). We use the names 1- and 2-annuli suggestively, to indicate that they are related to the first and second half-braidings respectively. While the picture has a boundary, we should imagine it to be at infinity, so that we are actually plumbing two open annuli to get a once-punctured torus (and not a torus with one boundary component).

In the picture for \(Y\), we omit most of the 1-annulus, just drawing the bit that meets the 2-annulus. The first (left) half of the cobordism goes from \(T^2_1 \cup T^2_1\) to \(T^2_2\), the twice punctured torus, using a thickened pair of pants on the 2-annuli, leaving the 1-annuli untouched. The second (right) half of the cobordism goes from \(T^2_2\) to \(T^2_1\), leaves the 2-annulus untouched, and “stacks” the two first annuli (see next picture and explanation of \(Y_1\)).

We can also picture \(Y\) as the union of the two cobordisms \(Y_2\) and \(Y_1\), shown below, each of which is a cobordism from two annuli to one annulus. \(Y_2\) and \(Y_1\) are identified along the dark gray portions \(Y\), which is a three-dimensional thickened ‘\(Y\)’, thought of as a cobordism from two disks to one disk.

The cobordism on the right, \(Y_1\), governs the behaviour of the 1-annuli. It is obtained by taking a two-dimensional thickened ‘\(Y\)’ (like the light gray part) and crossing with \(S^1\). (In the first picture of \(Y\), only a small neighbourhood of \(Y\) in \(Y_1\) appears.) We call this “stacking” because the cobordism takes in two annuli, each thought of as a
cylinder going from one circle boundary to the other, and glues the outgoing boundary of one cylinder to the incoming boundary of the other. The cobordism on the left, $Y_2$, is just a thickened pair of pants, and governs the behaviour of the 2-annuli.

Recall that the tensor product of Definition 3.18 is the usual $\lambda^2 \otimes \mu^2$ for the second half-braiding, and the “reduced tensor product” $\lambda^1 \otimes \mu^1$ for the first half-braiding. This is reflected in (or more accurately, a consequence of) the cobordism $Y$: the 2-annuli are governed by the usual pair of pants $Y_2$, while the 1-annuli are governed by this “stacking” cobordism $Y_1$.

The cobordism $Y : T^2_1 \cup T^2_1 \to T^2_1$ is not “commutative” in the way that the usual pair of pants is - that is, there is no homeomorphism from the cobordism to itself that flips the two input boundary components. However, it is still associative, so leads to a tensor product structure.

In the Introduction, we mention that the category we associate to the annulus is the Drinfeld center. The cobordism $Y_2$ leads to the usual tensor product, while the cobordism $Y_1$ leads to a different (multitensor) monoidal structure on $Z(C)$. This has been discussed in [BZBJ2016], though not in terms of half-braidings. This will be the subject of an upcoming paper.

3.3 $\text{SL}_2(\mathbb{Z})$-action on $Z^{\text{el}}(C)$

As briefly discussed in the introduction, $Z^{\text{el}}(C)$ is the category we associate to a once-punctured torus in an extended Crane-Yetter TQFT, so we expect properties of the once-punctured torus to manifest in $Z^{\text{el}}(C)$. For example, in Remark 3.12 we reveal that the intermediate induction functors $I_1, I_2$ has a topological interpretation.

Here, we will discuss an $\text{SL}_2(\mathbb{Z})$-action on $Z^{\text{el}}(C)$, since $\text{SL}_2(\mathbb{Z})$ is the mapping class group of $T^2_1$. We define the action by generators and relations, but do not prove that they are coherent in the appropriate sense. This is because our proof uses the topology of surfaces in a crucial way, so we postpone full proofs to future work in the context of the extended Crane-Yetter TQFT.

Recall that $\text{SL}_2(\mathbb{Z})$ is generated by two matrices

\[ s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]

subject to the relations

\[ r_1 : s^4 = 1 \]
\[ r_2 : st = t^{-1}st^{-1} \]
so that $\text{SL}_2(\mathbb{Z}) = \langle s, t | r_1, r_2 \rangle$.

Since $\text{SL}_2(\mathbb{Z})$ is finitely presented, we would like to write an action of $\text{SL}_2(\mathbb{Z})$ on $\mathcal{Z}^{el}(\mathcal{C})$ with a finite amount of data, i.e. describe the action by “generators and relations”. The general framework for this is spelled out in the Appendix Section 7.2 so here we just provide the relevant data, and discuss concretely what needs to be proved.

To begin, we need to say how $s, t$ act on $\mathcal{Z}^{el}(\mathcal{C})$. We define auto-equivalences $U_s, U_t \in \text{Aut}(\mathcal{Z}^{el}(\mathcal{C}))$ on objects as follows:

$$U_s : (X, \lambda^1, \lambda^2) \mapsto (X, (\lambda^2)^\dagger, \lambda^1)$$
$$U_t : (X, \lambda^1, \lambda^2) \mapsto (X, \lambda^1, \lambda^2 \times \lambda^1)$$

where

$$\lambda^1 = \begin{array}{c} \lambda \\ \lambda \end{array} \quad ; \lambda \times \mu = \begin{array}{c} \lambda \\ \mu \end{array}$$

(Intuitively, $\lambda^1$ is obtained from $\lambda$ by twisting the $X$ ribbon strand by a half-twist, and dragging the cross-strand along.) On morphisms, they will act “trivially” in the following sense: it is easy to check that

$$\text{Hom}_{\mathcal{Z}^{el}(\mathcal{C})}((X, \lambda^1, \lambda^2), (Y, \mu^1, \mu^2)) = \text{Hom}_{\mathcal{Z}^{el}(\mathcal{C})}(U_s(X, \lambda^1, \lambda^2), U_s(Y, \mu^1, \mu^2)) \subseteq \text{Hom}_{\mathcal{C}}(X, Y)$$

so we specify that $U_s$ acts as identity on this vector space (likewise for $U_t$). $U_s, U_t$ are in fact isomorphisms, so they have canonical inverses $U_s^{-1}, U_t^{-1}$.

Next, we need to relate the actions of two words which are equal in $\text{SL}_2(\mathbb{Z})$. To this end, for each of the relations $r_1, r_2$, we specify a natural isomorphism:

$$\gamma_1 : U_s^4 \sim \text{id}_{\mathcal{Z}^{el}(\mathcal{C})}$$
$$\gamma_2 : U_s U_t U_s \sim U_t^{-1} U_s U_t^{-1}$$

These $\gamma$’s are given by

$$(\gamma_1)_{(X, \lambda^1, \lambda^2)} = \theta_X : U_s^4(X, \lambda^1, \lambda^2) = (X, (\lambda^1)^{\dagger}, (\lambda^2)^{\dagger}) \rightarrow (X, \lambda^1, \lambda^2)$$

$$\gamma_2 : \frac{1}{D} \begin{array}{c} \lambda^1 \\ \lambda^2 \end{array} = P_{\mu^1, \mu^2} \gamma_2$$

where $\mu^2 = (\lambda^2)^{\dagger}$ is the second half-braiding of $U_s U_t U_s(X, \lambda^1, \lambda^2)$, and $\theta$ is the twist or balancing structure on $\mathcal{C}$. (Intuitively, $\theta$ is a full twist that turns $(\lambda^1)^{\dagger}$ back to $\lambda^1$.)
Theorem 3.22. \( U_s, U_t, \gamma_1, \gamma_2 \) generate an \( SL_2(\mathbb{Z}) \)-action on \( Z^{el}(C) \).

We will prove this in future work, but let us briefly discuss how one generates the action, and what is entailed in the proof of the theorem.

To a word \( w \) in \( s,t \) (and inverses \( s^{-1}, t^{-1} \), we associate the appropriate composition of \( U \)'s, which we denote \( U_w \). Each group element \( g \in SL_2(\mathbb{Z}) \) has many word representatives, so the associated \( U_w \)'s of various word representatives should be in some sense equal. Being functors, they should not be expected to be equal on the nose, but related by a natural isomorphism - these are provided by applying \( \gamma_1 \) and \( \gamma_2 \).

The data in Theorem 3.22 gives us two natural isomorphisms \( U_4^s \rightarrow id_{Z^{el}(C)} \):

\[
\begin{array}{ccc}
U_4^s & \xrightarrow{U_2^s \gamma_1} & \gamma_1 \\
\downarrow \gamma_1 & & \downarrow \gamma_1 \\
U_4^s & \rightarrow & id
\end{array}
\]

and in order for the group action to be well-defined, this diagram must commute.

In general, for any pair of words \( w_1, w_2 \), any path of applying relations from \( w_1 \) to \( w_2 \) gives rise to a natural isomorphism, and all paths must result in the same natural isomorphism. In the Appendix Section 7.2, we show that when this is satisfied, one obtains an \( SL_2(\mathbb{Z}) \)-action on \( Z^{el}(C) \) (see Proposition 7.9, Corollary 7.11).

Once again, our proof of Theorem 3.22, that is, checking the above condition, uses the topology of surfaces in an essential way, hence we find it best to present the proof in the context of the extended Crane-Yetter TQFT, which will appear in future work.

Remark 3.23. The action described above does not respect the monoidal structure defined in Section 3.2. Instead, it intertwines the many tensor product structures one can put on \( Z^{el}(C) \), each coming from choosing how to identify \( \mathbb{T}_1^2 \) as the plumbing of two annuli (see Remark 3.21).

Remark 3.24. Later in Theorem 4.3, we prove that when \( C \) is modular, \( Z^{el}(C) \approx C \), so we should have an \( SL_2(\mathbb{Z}) \)-action on \( C \) as well. It is not yet clear to us how to write this action on \( C \) explicitly without reference to \( Z^{el}(C) \).
4 Modular Case

In this section, we prove Theorem 4.3 which says that when $\mathcal{C}$ is modular, $Z^{el}(C) \cong C$. As discussed in the introduction, this is a natural result to expect from the supposition that the Crane-Yetter TQFT is a boundary theory.

We will need two lemmas:

Lemma 4.1. Let $i$ be the composition

$$i = \mathcal{I}_1 \circ \iota : \mathcal{C} \to Z^{el}(\mathcal{C})$$

where $\iota : \mathcal{C} \to Z(\mathcal{C})$ is the functor $X \mapsto (X, c_{-X})$, and $\mathcal{I}_1$ is the intermediate induction functor defined in Proposition 3.9. Then $i$ is fully faithful.

Proof. The functor $i$ sends $X \mapsto (X_j X_j^*, \Gamma, \Omega)$, where $\Omega = c_{-X}$ (the $\Omega$ in Definition-Proposition 3.18 is for $X = 1$), and on morphisms, $i$ sends $f : X \to Y$ to $i(f) = \sum_j \text{id}_{X_j} \otimes f \otimes \text{id}_{X_j^*}$.

Faithfulness follows without modularity of $\mathcal{C}$: indeed, the way $i$ is defined means the following diagram commutes:

$$\begin{array}{ccc}
\text{Hom}_{Z^{el}(\mathcal{C})}(iX, iY) & \xleftarrow{i} & \text{Hom}_{\mathcal{C}}(X_j X_j^*, X_j Y_j^*) \\
\text{Hom}_{\mathcal{C}}(X, Y) & \xrightarrow{i} & \\
\end{array}$$

so $i$ must be an injection.

To show that $i$ is full, let us study $\text{Hom}_{Z^{el}(\mathcal{C})}(iX, iY)$ as the intersection $\text{Hom}_{Z(\mathcal{C})}((X_j X_j^*, \Gamma), (X_j Y_j^*, \Gamma)) \cap \text{Hom}_{Z(\mathcal{C})}((X_j X_j^*, \Omega), (X_j Y_j^*, \Omega))$. The first space is just $\text{Hom}_{Z(\mathcal{C})}(\mathcal{I}X, \mathcal{I}Y)$, so it is of the form seen in Corollary 2.5.

Further intersecting with the second space, or equivalently finding the image of the first space under the projection $P_{\Omega, \Omega}$,

$$P_{\Omega, \Omega} = \frac{1}{D} \begin{array}{cccc}
\bar{a} & k & g & \bar{a} \\
\, & k & g & \bar{a} \\
\end{array} = \frac{1}{D} \begin{array}{cccc}
\bar{a} & k & g & \bar{a} \\
\, & k & g & \bar{a} \\
\end{array}$$

so all morphisms in $\text{Hom}_{Z^{el}(\mathcal{C})}(iX, iY)$ are indeed of the form $i(f), f \in \text{Hom}_{\mathcal{C}}(X, Y)$, hence $i$ is full. This computation depends on the modularity of $\mathcal{C}$, specifically in the third equality, where we use Lemma 7.5. Note that it was crucial that in the first step
we deduced from Corollary 2.5 that the “struts” between the middle strand and the outer strands are labelled by the same thing, so that killing one also kills the other.

\[ \square \]

**Lemma 4.2.** If \( \mathcal{C} \) is modular, then for any object \( (Y, \mu) \in \mathcal{Z}(\mathcal{C}) \),

\[ \mathcal{I}_1(Y, \mu) \cong iY \]

Observe that the lemma implies \( \mathcal{I}_1(Y, \mu) \cong iY = \mathcal{I}_1(Y, c_{-Y}) \), so this lemma is saying that when \( \mathcal{C} \) is modular, \( \mathcal{I}_1 \) kills the difference in braiding in \( \mathcal{Z}(\mathcal{C}) \).

**Proof.** By Proposition 2.7, we have \( \mathcal{Z}(\mathcal{C}) \cong \mathcal{C} \otimes \mathcal{C}^{\text{op}} \), so we may write

\[ (Y, \mu) \cong \bigoplus_k \bigl( (A_k, c_{-A_k}) \otimes (B_k, c_{B_k}^{-1}) \bigr) = \bigoplus_k (A_k B_k, \sigma) \]

where

\[ \sigma = \sum_k \frac{1}{A_k B_k} \]

Let \( \varphi = \sum_k \varphi_k : Y \to \bigoplus_k B_k A_k \) be an isomorphism in \( \mathcal{C} \) (say \( \varphi = c \circ \varphi' \)), and let \( \psi = \sum_k \psi_k : \bigoplus_k B_k A_k \to Y \) be its inverse. Consider the morphisms \( \tilde{\varphi} : X_j Y X_j^* \to X_j \bigl( \bigoplus_k A_k B_k \bigr) X_j^* \) and \( \tilde{\psi} : X_j \bigl( \bigoplus_k A_k B_k \bigr) X_j^* \to X_j Y X_j^* \) described below:

\[ \tilde{\varphi} = \sum_k \frac{1}{A_k B_k} \bigoplus_k \bigl( A_k, c_{-A_k} \bigr) \bigoplus_k \bigl( B_k, c_{B_k}^{-1} \bigr) \]

\[ \tilde{\psi} = \sum_k \frac{1}{A_k B_k} \bigoplus_k \bigl( B_k, c_{B_k}^{-1} \bigr) \bigoplus_k \bigl( A_k, c_{-A_k} \bigr) \]

These are in fact morphisms in \( \mathcal{Z}^{\text{el}}(\mathcal{C}) \), that is,

\[ \tilde{\varphi} \in \text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{C})}(iY, \mathcal{I}_1 \bigl( \bigoplus_k A_k B_k, \sigma \bigr)) \]

\[ \tilde{\psi} \in \text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{C})}(\mathcal{I}_1 \bigl( \bigoplus_k A_k B_k, \sigma \bigr), iY) \]

For the first braiding, it follows from properties of \( \tilde{\alpha} \), (see proof of Lemma 7.3), while for the second braiding, it is apparent from the diagrams.
Finally, we see that $\varphi$ and $\psi$ are inverses:

$$\varphi \circ \psi = \sum_k \sum_{k} \text{ Ald } b_k = \sum_k \text{ Ald } b_k = 1$$

(the first equality follows from the fact that $\Gamma$ is a half-braiding - we just swapped the relative positions of the "walls"), and the other way is similar.

Thus $I_1(Y, \mu) = I_1(\bigoplus_k A_k B_k, \sigma) \cong iY$.

Now we can prove:

**Theorem 4.3.** If $\mathcal{C}$ is modular, then the composition

$$i = I_1 \circ i : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$$

is an equivalence of abelian categories, where $i : \mathcal{C} \to \mathcal{Z}(\mathcal{C})$ is the functor $X \mapsto (X, c_{-X})$, and $I_1$ is the intermediate induction functor defined in Proposition 3.9.

**Proof.** Recall that $i$ sends $X \mapsto (X_j X_j^*, \Gamma, \Omega)$, where $\Omega = \overline{c_{-X}}$, and on morphisms, $i$ sends $f : X \to Y$ to $i(f) = \sum_j \text{id}_{X_j} \otimes f \otimes \text{id}_{X_j^*}$.

By Lemma 4.1, $i$ is fully faithful. It remains to show that $i$ is essentially surjective. Observe that Lemma 4.2 implies that any object of the form $T^{el} X$ is isomorphic to some object $iX'$, $X' \in \mathcal{C}$ - just take $(Y, \mu)$ in Lemma 4.2 to be $(X_j X_j^*, \Gamma) = IX$ (see Proposition 2.4), so that

$$T^{el} X = I_1(I X) = I_1(Y, \mu) \cong iY$$

By Proposition 3.8, any object $A$ of $\mathcal{Z}^{el}(\mathcal{C})$ is a direct summand of some $T^{el} X$, hence by the above observation, $A$ is a direct summand of some $iX' \cong T^{el} X$. But by Lemma 4.1, $i$ is fully faithful, so if $Q_A$ is a projection in $\text{End}_{\mathcal{Z}^{el}(\mathcal{C})}(iX)$ such that $\text{im}(Q_A) \cong A$, then $Q_A = i(q_A)$ for some projection $q_A \in \text{End}_{\mathcal{C}}(X)$, so $i(\text{im}(q_A)) \cong \text{im}(Q_A) \cong A$. Hence, $i$ is essentially surjective, and we are done.

The last paragraph in the proof above can be phrased in more abstract but conceptually clearer terms, encapsulated in the following diagram:
where $E$ is the full image of $T^\text{el} : \mathcal{C} \to Z^\text{el}(\mathcal{C})$, $D$ is the essential image of $i$, and f.f. stands for fully faithful. Let us explain this diagram and its relation to the proof above. The observation that any object $T^\text{el}X$ is isomorphic to some $iX'$ can be restated as the fact that the full image $E$ of $T^\text{el}$ is contained in the essential image $\mathcal{D}$ of $i$. It follows that the Karoubian completion of $\mathcal{D}$ (which can be thought of as a subcategory of $Z^\text{el}(\mathcal{C})$ since $Z^\text{el}(\mathcal{C})$ is abelian) contains the Karoubian completion of $E$. By Lemma 4.1, $\mathcal{D}$ is a full subcategory of $Z^\text{el}(\mathcal{C})$, so the Karoubian completion of $\mathcal{D}$ is $\mathcal{D}$ itself. By Proposition 3.3, any object in $Z^\text{el}(\mathcal{C})$ is a direct summand of some $T^\text{el}X$, hence the Karoubian completion of $E$ is the entire $Z^\text{el}(\mathcal{C})$. So $Z^\text{el}(\mathcal{C}) = E^\oplus \subseteq D^\oplus \subseteq Z^\text{el}(\mathcal{C})$, hence $D = D^\oplus = Z^\text{el}(\mathcal{C})$. In other words, $i$ is essentially surjective.

By Proposition 3.19, $I_1$ is a monoidal functor, and $i$ is also monoidal in a natural way. Thus,

**Corollary 4.4.** If $\mathcal{C}$ is modular, then $(i,j) = (I_1,J) \circ \iota : \mathcal{C} \to Z^\text{el}(\mathcal{C})$ is a monoidal equivalence.

### 4.1 Connection to Reshetikhin-Turaev Central Charge Anomaly

One manifestation of the central charge anomaly in Reshetikhin-Turaev theory at $\mathcal{C}$, $Z^\text{RT,\mathcal{C}}$, already discussed in Witten’s work [Wit1989], is that one has a projective action of the mapping class group of a closed surface $\Sigma$ on $Z^\text{RT,\mathcal{C}}(\Sigma)$, and the deviation from being an honest (i.e. not projective) action is known as the central charge anomaly. In particular, for the torus $\Sigma = \mathbb{T}^2$, we have $Z^\text{RT,\mathcal{C}}(\Sigma) = \text{Hom}_\mathcal{C}(1,X_j X_j^*)$, and there are morphisms $\xi_8, \xi_\ell$ in $\text{End}_\mathcal{C}(X_j X_j^*)$ so that post-composing with them gives an action of $\text{SL}_2(\mathbb{Z})$ on $\text{Hom}_\mathcal{C}(1,X_j X_j^*)$, but only factors through a projective action of $\text{SL}_2(\mathbb{Z})$ on $\text{Hom}_\mathcal{C}(1,X_j X_j^*)$. One also gets projective actions on $\text{Hom}_\mathcal{C}(U,X_j X_j^*)$ for simple $U$ (see e.g. [BK2001] Section 3.1).

We recover this projective representation in $Z^\text{el}(\mathcal{C})$, in part from the $\text{SL}_2(\mathbb{Z})$-action on $Z^\text{el}(\mathcal{C})$ as defined in Section 3.3. Consider the Hom space

$$\text{Hom}_{Z^\text{el}(\mathcal{C})}(i1,T^\text{el}1)$$

By Lemma 4.2, $T^\text{el}1 = I_1(I1) \cong I_1(\iota X_j X_j^*) = i(X_j X_j^*)$, so

$$\text{Hom}_{Z^\text{el}(\mathcal{C})}(i1,T^\text{el}1) \cong \text{Hom}_{Z^\text{el}(\mathcal{C})}(i1,i(X_j X_j^*)) \cong \text{Hom}_\mathcal{C}(1,X_j X_j^*)$$

where the second equality follows from Theorem 4.3 (or just Lemma 4.1).

So we want to describe a projective $\text{SL}_2(\mathbb{Z})$-action on $\text{Hom}_{Z^\text{el}(\mathcal{C})}(i1,T^\text{el}1)$. To this end, we put a projectively-$\text{SL}_2(\mathbb{Z})$-equivariant structure on $i1$, and an $\text{SL}_2(\mathbb{Z})$-equivariant structure on $T^\text{el}1$.  

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For $\mathcal{I}^{\text{el}} \mathbf{1}$, let
\[ \nu_s : U_s(\mathcal{I}^{\text{el}} \mathbf{1}) \to \mathcal{I}^{\text{el}} \mathbf{1} \]
\[ \nu_t : U_t(\mathcal{I}^{\text{el}} \mathbf{1}) \to \mathcal{I}^{\text{el}} \mathbf{1} \]
be the isomorphisms

For $\mathbf{i} \mathbf{1}$, let
\[ \mu_s : U_s(\mathbf{i} \mathbf{1}) \to \mathbf{i} \mathbf{1} \]
\[ \mu_t : U_t(\mathbf{i} \mathbf{1}) \to \mathbf{i} \mathbf{1} \]
be the isomorphisms

Proposition 4.5. With respect to the $SL_2(\mathbb{Z})$-action on $\mathcal{Z}^{\text{el}}(\mathcal{C})$ defined in Section 3.3, $\nu_s, \nu_t$ define an $SL_2(\mathbb{Z})$-equivariant structure on $\mathcal{I}^{\text{el}} \mathbf{1}$, and $\mu_s, \mu_t$ define a projectively-$SL_2(\mathbb{Z})$-equivariant structure on $\mathbf{i} \mathbf{1}$.

We do not prove this here, since we have not shown the validity of the $SL_2(\mathbb{Z})$-action on $\mathcal{Z}^{\text{el}}(\mathcal{C})$ in Section 3.3. Even if we grant that the action is well-defined, some of the computations are lengthy and belie their topological origins.

So let us grant Proposition 4.5 for now. $SL_2(\mathbb{Z})$ acts on $\text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{C})} (\mathbf{i} \mathbf{1}, \mathcal{I}^{\text{el}} \mathbf{1})$ as follows: for $g \in SL_2(\mathbb{Z})$ and $\psi \in \text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{C})} (\mathbf{i} \mathbf{1}, \mathcal{I}^{\text{el}} \mathbf{1})$,
\[ \rho_g(\psi) = \nu_g \circ \psi \circ \mu_g^{-1} : \mathbf{i} \mathbf{1} \xrightarrow{\mu_g^{-1}} U_g(\mathbf{i} \mathbf{1}) \xrightarrow{U_g(\psi)=\psi} U_g(\mathcal{I}^{\text{el}} \mathbf{1}) \xrightarrow{\nu_g} \mathcal{I}^{\text{el}} \mathbf{1} \]

We use the explicit isomorphism $\Phi : \mathcal{I}^{\text{el}} \mathbf{1} \to \mathbf{i}(X_j X_j^*)$,
\[ \Phi = \frac{\sqrt{d_i} \sqrt{d_j}}{\sqrt{D}} \]
to identify $\text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{C})} (\mathbf{i} \mathbf{1}, \mathcal{I}^{\text{el}} \mathbf{1}) \to \text{Hom}_{\mathcal{Z}^{\text{el}}(\mathcal{C})} (\mathbf{i} \mathbf{1}, \mathbf{i}(X_j X_j^*)).$
The actions of \( s, t \in \text{SL}_2(\mathbb{Z}) \) on \( \text{Hom}_\mathcal{C}(\mathbf{1}, X_j X_j^* ) \) are given by post-composing with

\[
\xi_s = \mu_s^{-1} \\
\xi_t = (\theta^{-1} \otimes \text{id}) \circ \mu_s \circ (\theta^{-1} \otimes \text{id})
\]

respectively, where we just think of \( \mu_s \) as a morphism in \( \mathcal{C} \), and \( \theta \) is the balancing structure of \( \mathcal{C} \). (Note that in [BK2001], the actions of \( s, t \) are given by \( \xi_s, \theta \otimes \text{id} \), respectively. These actions are related by twisting by the automorphism of \( \text{SL}_2(\mathbb{Z}) \) sending \( s \mapsto s, t \mapsto t^{-1}s^{-1}t^{-1} \).

Finally, we may state the connection of our work to the central charge anomaly in Reshetikhin-Turaev theory as follows:

**Proposition 4.6.**

\[
\text{Hom}_\mathcal{C}(\mathbf{1}, X_j X_j^*) \rightarrow \text{Hom}_{\text{Zel}}(\mathcal{C})(i_1, I\text{el}^1) \\
\varphi \mapsto \Phi^{-1} \circ i(\varphi)
\]

intertwines the projective \( \text{SL}_2(\mathbb{Z}) \)-action in Reshetikhin-Turaev theory and the one described above.

Once again we do not prove this here for the same reasons as before, but we verify it for the action of \( s \in \text{SL}_2(\mathbb{Z}) \). Since \( \{\text{coev}_{X_j}\}_{j \in J} \) forms a basis of \( \text{Hom}_\mathcal{C}(\mathbf{1}, X_j X_j^*) \), it suffices to consider \( \varphi_j := \text{coev}_{X_j} \), for which we check that \( \Phi \circ \rho_s(\Phi^{-1} \circ i(\varphi_j)) = i(\xi_s \circ \varphi_j) \) below (we leave out coefficients for readability, and diagrams read left to right):
The computation for \( t \) is similar.

To conclude this section, let us say a few words in relation to the extended Crane-Yetter TQFT. \( \mathcal{I}^\text{el} \) is the “empty configuration” in \( Z_{\text{CY}}(\mathbb{T}^2) \), with no marked points or projections, so it is natural to expect that it has an \( \text{SL}_2(\mathbb{Z}) \)-equivariant structure. \( i^1 \) is the object \( Z_{\text{CY}}(M) \), where \( M \) is the solid torus with a point removed from its interior. \( M \) depends on the choice of an isotopy class of a simple closed curve on \( \mathbb{T}^2 \), the one that is made contractible in \( M \). Clearly the action of \( \text{SL}_2(\mathbb{Z}) \) on \( \partial M \cong \mathbb{T}^2 \) doesn’t extend to \( M \), since it doesn’t fix isotopy classes of curves on \( \mathbb{T}^2 \), so while \( U_s i^1 \cong T_i^1 \cong i^1 \), these isomorphisms are not expected to be coherent with the action of \( \text{SL}_2(\mathbb{Z}) \) on \( Z_{\text{CY}}(\mathbb{T}^2) \cong \mathcal{I}^\text{el}(C) \).

5 \( \mathcal{C} = H - \text{mod} \) and Elliptic Drinfeld Double

In this section, we treat the case when \( \mathcal{C} = H - \text{mod} \) for a finite-dimensional quasi-triangular Hopf algebra \( H \) (as in Section 2.2.2, once again we implicitly assume semisimplicity of \( H \) to keep \( \mathcal{C} \) semisimple, but we never really use it).

In Section 2.2.2, we saw that for a Hopf algebra \( H \), \( Z(H-\text{mod}) \cong \mathcal{D}(H)-\text{mod} \), where \( \mathcal{D}(H) \) is Drinfeld’s quantum double (see Proposition 2.10). Here we will construct a similar algebra \( \mathcal{D}^\text{el}(H) \) associated to a quasi-triangular Hopf algebra \( H \), which we call the **Elliptic Drinfeld double**. We then show that \( Z^2(H - \text{mod}) \cong \mathcal{D}^\text{el}(H) - \text{mod} \) (see Theorem 5.3 below). While \( \mathcal{D}(H) \) is a ribbon algebra, in general, \( \mathcal{D}^\text{el}(H) \) has no obvious coalgebra structure, but it does when \( H \) is cocommutative.

**Remark 5.1.** As mentioned in the introduction, [BJ2014] defines a similar algebra which the authors also call the elliptic double, and we expect these elliptic doubles to be Morita equivalent.

Let us first define \( \mathcal{D}^\text{el}(H) \) as an algebra:

**Definition-Proposition 5.2.** Let \((H, m, \Delta, \varepsilon, S, R, v, u)\) be a finite-dimensional ribbon Hopf algebra, where \( v, u \) are the pivotal and ribbon elements respectively. The **Elliptic Drinfeld double** of \( H \), denoted \( \mathcal{D}^\text{el}(H) \), is defined as the vector space \( H \otimes H_1^{*,\text{cop}} \otimes H_2^{*,\text{cop}} \) (where \( H_1 = H_2 = H \), indexing for clarity) with the following algebra structure:

- the three obvious inclusions of \( H \), \( H_1^{*,\text{cop}} \), and \( H_2^{*,\text{cop}} \) into \( H \otimes H_1^{*,\text{cop}} \otimes H_2^{*,\text{cop}} \) are algebra maps.
- Each copy of \( H_1^{*,\text{cop}} \) commutes past \( H \) in the same manner as in \( \mathcal{D}(H) \): e.g. for \( h \in H \) and \( f \in H_1^{*,\text{cop}} \),

\[
fh = \langle f_3, S^{-1}(h_1) \rangle \langle f_1, h_3 \rangle h_2 f_2
\]
where we use Sweedler’s notation $\Delta^2(h) = h_1 \otimes h_2 \otimes h_3$ and $\Delta^2(f) = ((m^*)^{\text{cop}})^2(f) = f_3 \otimes f_2 \otimes f_1$. In other words, the inclusions

\[ \mathcal{D}(H) \cong H \otimes H_1^{*,\text{cop}} \subseteq \mathcal{D}^{\text{el}}(H) \]
\[ \mathcal{D}(H) \cong H \otimes H_2^{*,\text{cop}} \subseteq \mathcal{D}^{\text{el}}(H) \]

are algebra maps.

- The two copies of $H_1^{*,\text{cop}}$ commute by the following relation: writing $\mathcal{R} = s_a \otimes t_a$ (suppressing the sum), for $f^1 \in H_1^{*,\text{cop}}, f^2 \in H_2^{*,\text{cop}}$,

\[ f^2 f^1 = (f^1_1, t_a)(f^1_3, s_a)(f^2_1, s_a)(f^2_3, t_a) f^1_2 f^2_2 \]

Note that the coproduct on $H^{*,\text{cop}}$ is not used here, but will be important later when considering the (symmetric) tensor product structures on $\mathcal{Z}^{\text{el}}(\mathcal{C})$ and $\mathcal{D}^{\text{el}}(H) - \text{mod}$.

**Theorem 5.3.** For $\mathcal{C} = H - \text{mod}$, $\mathcal{Z}^{\text{el}}(\mathcal{C}) \cong \mathcal{D}^{\text{el}}(H) - \text{mod}$ as abelian categories.

**Proof.** The proof will be very similar to $\mathcal{Z}(H - \text{mod}) \cong \mathcal{D}(H) - \text{mod}$ (see Proposition 2.10).

The functor $\mathcal{D}^{\text{el}}(H) - \text{mod} \to \mathcal{Z}^{\text{el}}(\mathcal{C})$ is constructed as follows. Let $X$ be a left $\mathcal{D}^{\text{el}}(H)$-module. It is in particular an $H$-module, i.e. an object in $\mathcal{C}$. The action of $H_1^{*,\text{cop}}$ on $X$ gives us one half-braiding: for $A$ another $H$-module,

\[ \lambda^1_A = P \circ R_1 : A \otimes X \to X \otimes A \]

where $P$ is the swapping of factors, and $R_1$ stands for acting by $R_1 = (\iota_1 \otimes \iota_1)(R) = \sum h_i \otimes \iota_1(h^*_i)$, where recall $\iota_1$ is the first inclusion of algebras $\mathcal{D}(H) \cong H \otimes H_1^{*,\text{cop}} \subseteq \mathcal{D}^{\text{el}}(H)$ (we suppress $\iota_1$ on $H$ because $H \to \mathcal{D}^{\text{el}}(H)$ is unambiguous). Likewise, we can define a second half-braiding by

\[ \lambda^2_A = P \circ R_2 : A \otimes X \to X \otimes A \]

where $R_2 = (\iota_2 \otimes \iota_2)(R)$.

We need to show that $\lambda^1, \lambda^2$ satisfy COMM, and it suffices to check it for $A = B = H$ by the naturality of half-braidings. This boils down to checking that

\[ \mathcal{R}^{12} R_1^{13} R_2^{23} = R_2^{23} R_1^{13} (\mathcal{R}^{-1})^{21} \in H \otimes H \otimes \text{End}_k(X) \]

or equivalently,

\[ \mathcal{R}^{-1} R_1^{13} R_2^{23} \mathcal{R} = R_2^{23} R_1^{13} \]

Here $\mathcal{R}^{21} = (\mathcal{R}^{\text{op}})^{12}$. Once again we point out that in COMM, one side has $c_{-,-}$ while the other has $c_{-,-}^{-1}$, hence the appearance of $(\mathcal{R}^{-1})^{\text{op}}$. So

\[ s_{a} h_i t_{a'} \otimes t_a h_j s_{a'} \otimes \iota_1(h^*_1) \iota_2(h^*_2) = h_i \otimes h_k \otimes \iota_2(h^*_2) \iota_1(h^*_1) \]
where $\mathcal{R} = s_a \otimes t_a$. For $f^1, f^2 \in H^*$, applying $f^2 \otimes f^1 \otimes \text{id}$, we get
\[
\iota_2(f^2)\iota_1(f^1) = (f^2, s_a h_{i} t_a')(f^1, t_a h_{j} s_a')\iota_1(h^*_i)\iota_2(h^*_j) \\
= (f^2, s_a)(t_a')(f^1, t_a)(f^2, s_a')\iota_1(f^1)\iota_2(f^2)
\]
which is implied by the commutation relation between $H_1^{*,\text{cop}}$ and $H_2^{*,\text{cop}}$ in $\mathcal{D}^\text{el}(H)$.

For the other way, let $(X, \lambda^1, \lambda^2)$. Using the same methods in the proof of Proposition 2.10, for each half-braiding $\lambda^1, \lambda^2$, we cook up two $H^*$-actions on $X$, so that we have an action of $H \ast H_1^* \ast H_2^*$ on $X$, where the $\ast$ denotes free product of algebras. To see that this action factors through $\mathcal{D}^\text{el}(H)$, we check that the commutation relations between factors $H, H_1^{*,\text{cop}}, H_2^{*,\text{cop}}$ of $\mathcal{D}^\text{el}(H)$ are satisfied in their actions on $X$. For commutation relations between $H_1^{*,\text{cop}}$ and $H_2^{*,\text{cop}}$, it basically follows from the same computations above but in reverse, while for the other two pairs, it follows from the proof of Proposition 2.10.

**Corollary 5.4.** If $H$ is semisimple, then so is $\mathcal{D}^\text{el}(H)$.

**Remark 5.5.** The elliptic double in [BJ2014] carries an action of $\overline{\text{SL}_2(\mathbb{Z})}$, but ours do not. Via the equivalence in Theorem 5.3, the $\text{SL}_2(\mathbb{Z})$-action on $\mathcal{Z}^\text{el}(C)$ laid out in Section 3.3 defines an $\text{SL}_2(\mathbb{Z})$-action on $\mathcal{D}^\text{el}(H) \mod$, hence some sort of action on $\mathcal{D}^\text{el}(H)$ as well. However, due to the flexibility of group actions on categories (manifested in the extra data of natural isomorphisms $\gamma_1, \gamma_2$), we don’t get an honest action on $\mathcal{D}^\text{el}(H)$.

More precisely, from reconstruction theory, we have equivalences
\[
\begin{array}{ccc}
\text{End}(F) - \text{mod} & \longrightarrow & \text{End}(F \circ U_s) - \text{mod} \\
\uparrow & & \uparrow \\
\mathcal{Z}^\text{el}(C) & \overset{U_s}{\longrightarrow} & \mathcal{Z}^\text{el}(C)
\end{array}
\]
where $F$ is the forgetful functor to $\text{Vec}$ (forgetting the half-braidings and $H \mod$ structure. Since $U_s$ only changes the half-braidings, $F = F \circ U_s$). Thus $s \in \text{SL}_2(\mathbb{Z})$ acts on the right by
\[
\psi_s : \mathcal{D}^\text{el}(H) \xrightarrow{\sim} \text{End}(F) \xrightarrow{-\circ U_s} \text{End}(F \circ U_s) = \text{End}(F) \xrightarrow{\sim} \mathcal{D}^\text{el}(H)
\]
and similarly for $t$. Concretely, since $U_s$ changes the half-braidings $(X, \lambda^1, \lambda^2) \mapsto (X, (\lambda^2)^\dagger, \lambda^1)$, it changes the resulting action of $H_1^{*,\text{cop}}, H_2^{*,\text{cop}}$ on $X$, and $\psi_s$ encodes this (and likewise for $U_t$).

However, for the relation $r_1 : s^4 = 1$, we don’t get $\psi_s^4 = \text{id}_{\mathcal{D}^\text{el}(H)}$, but we have
\[
\begin{array}{ccc}
\text{End}(F \circ U_s) - \text{mod} & \xrightarrow{\gamma_1 = \theta} & \text{End}(F) - \text{mod} \\
\text{id} & \circlearrowright & \text{id}
\end{array}
\]

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So instead we have
\[ \theta : U^4_s(X, \lambda^1, \lambda^2) \to (X, \lambda^1, \lambda^2) \]
hence \( u \), the ribbon element of \( H \), intertwines:
\[ u \cdot \psi^4_s(x) = x \cdot u \]
for every \( x \in \mathcal{D}^{el}(H) \). This suggests an \( \text{SL}_2(\mathbb{Z}) \)-action, but the other relation \( r_2 : sts = t^{-1}st^{-1} \) also requires a functorial isomorphism \( \gamma_2 \). So instead we have the action of the mapping class group of a genus 1 surface with one puncture and one boundary component, which is some extension of \( \text{SL}_2(\mathbb{Z}) \) by \( \pi_1(T^1_0) \cong \mathbb{Z} \ast \mathbb{Z} \); this is not too surprising since our definition of \( \mathcal{Z}^{el}(\mathcal{C}) \) (and hence \( \mathcal{D}^{el}(H) \)) comes from fixing a point and a pair of meridian and longitude (see Introduction), giving rise to a unnaturalness of the action. We will investigate this and connections to the elliptic double in [BJ2014] further in upcoming work.

5.1 \( H \) cocommutative, \( \mathcal{C} \) symmetric

In the usual Drinfeld center case, we have a braided tensor equivalence \( \mathcal{Z}(H - \text{mod}) \cong \mathcal{D}(H) - \text{mod} \) (see Proposition 2.10). The proof presented there implicitly uses the perspective of reconstruction theory of finite dimensional Hopf algebras, in that we start with the forgetful functor \( F : \mathcal{Z}(H - \text{mod}) \to \text{Vec} \), and \( \mathcal{D}(H) \) appears as the Hopf algebra of endomorphisms of \( F \) (see also Remark 5.3). We can try to do the same thing with the tensor product structure on \( \mathcal{Z}^{el}(\mathcal{C}) \) sketched in Section 3.2. However, since the tensor product on \( \mathcal{Z}^{el}(\mathcal{C}) \) defined in Section 3.2 is generally multitensor but not tensor, the obvious forgetful functor to \( \text{Vec} \) cannot be tensor, so there’s no clear way to use reconstruction theory to recover a coalgebra structure on \( \mathcal{D}^{el}(H) \).

If we restrict ourselves to \( H \) cocommutative (so that it is quasi-triangular with \( \mathcal{R} = 1 \otimes 1 \) and \( \mathcal{C} \) is symmetric), we can consider the naive tensor product
\[ (X, \lambda^1, \lambda^2) \otimes (Y, \mu^1, \mu^2) = (X \otimes Y, \lambda^1 \otimes \mu^1, \lambda^2 \otimes \mu^2) \]
on \( \mathcal{Z}^{el}(\mathcal{C}) \), and here \( \lambda^1 \otimes \mu^1, \lambda^2 \otimes \mu^2 \) indeed satisfy COMM:
\[ \begin{align*}
\lambda^1 \otimes \mu^1 \otimes \lambda^2 \otimes \mu^2 &= \lambda^1 \otimes \mu^1 \otimes \lambda^2 \otimes \mu^2 \\
\end{align*} \]

The associativity constraint will just be the one from \( \mathcal{C} \). The obvious forgetful functor to \( \text{Vec} \) has an obvious tensor structure, so we can apply reconstruction theory again. In particular, we may upgrade \( \mathcal{D}^{el}(H) \) to a ribbon Hopf algebra:
Definition-Proposition 5.6. Let \((H, m, 1, \Delta, \varepsilon, S, v)\) be a finite-dimensional cocommutative ribbon Hopf algebra (with \(\mathcal{R} = 1 \otimes 1\) and \(u = 1\)). The elliptic Drinfeld double \(D^\text{el}(H)\) as defined in Definition 5.2 admits the following additional structure, making it a ribbon Hopf algebra:

- As a coalgebra, it is simply \(H \otimes H^{\ast,\cop} \otimes H^{\ast,\cop}\), i.e. \(\Delta(h \otimes f^1 \otimes f^2) = (h_1 \otimes f_1^2 \otimes f_2^2) \otimes (h_2 \otimes f_1^1 \otimes f_2^2)\),
- The antipode is also given componentwise, i.e.
  \[
  S(hf_{f_1}f_{f_2}) = S^{-1}(f_2)S^{-1}(f_1)S(h)
  \]
  where \(f_{f_1} \in H^{\ast,\cop}, f_{f_2} \in H^{\ast,\cop}\).
- \(v \in H \hookrightarrow D^\text{el}(H)\) is the pivotal element.

**Proof.** Checking compatibility between the various structures boils down to familiar computations. \(\square\)

Observe that in this case of \(H\) cocommutative, the actions of \(H^{\ast,\cop}\) commute, since \(\mathcal{R} = 1 \otimes 1\), so

\[
  f^2f^1 = (f_1^1, 1)(f_3^1, 1)(f_1^2, 1)(f_3^2, 1)f_1^2f_2^2 = \varepsilon(f_1^1)\varepsilon(f_1^2)\varepsilon(f_2^1)\varepsilon(f_2^2)f_1^2f_2^2 = f^1f^2
\]

**Remark 5.7.** By Deligne’s theorem on tensor categories \([\text{Del}1990], [\text{Del}2002]\), any symmetric fusion category is tensor equivalent to \(\text{Rep}(G)\) for some finite group \(G\), and is braided tensor equivalent to it up to a twist by some central element \(z\) of order 2. Thus, \(H \mod\) covers basically all symmetric fusion categories.

**Theorem 5.8.** When \(H\) is cocommutative, the equivalence in Theorem 5.3 is a tensor equivalence.

**Proof.** Essentially the same as in the proof of Proposition 2.10 \(\square\)

**Example 5.9 (Group Algebra).** Recall the setup of Example 2.9: \(H = \mathbb{k}[G], H^{\ast,\cop} = F(G)^{\cop}\), so we have

\[
  D^\text{el}(H) = \mathbb{k}[G] \otimes F(G_1^{\cop}) \otimes F(G_2^{\cop}) = \mathbb{k}[G] \otimes F(G_1^{\cop} \times G_2^{\cop})
\]

as coalgebras, where of course \(G_1 = G_2 = G\) (the second equality is justified because the actions of \(F(G_1^{\cop})\) and \(F(G_2^{\cop})\) commute). Then the commutation relations read

\[
  \delta_{(g_1 \cdot g_2)h} = h\delta_{(h^{-1}g_1h, h^{-1}g_2h)}
\]

Denote \(D^\text{el}(G) := D^\text{el}(\mathbb{k}[G])\). Similar to Example 2.9, representations of \(D^\text{el}(G)\) can be interpreted as \(G\)-equivariant vector bundles over \(G \times G\), where \(G\) acts on \(G \times G\) by conjugation on each factor.
The diagonal map $G \hookrightarrow G \times G$ is $G$-equivariant, and pulls back $G$-equivariant bundles, giving us a restriction functor $\mathcal{D}\text{el}(G) \otimes \text{mod} \rightarrow \mathcal{D}(G) \otimes \text{mod}$. On the level of algebras, this corresponds to the inclusion

\[ \mathcal{D}(G) \hookrightarrow \mathcal{D}\text{el}(G) \]

\[ g \mapsto g \]

\[ \delta_g \mapsto \delta_{(g,g)} \]

This is not a coalgebra map, or equivalently, the restriction functor is not tensor. For example, bundles supported on the orbits of $(g,1)$ and $(1,g)$, respectively, would each restrict to 0 on the diagonal, but their tensor product would have a non-trivial vector space over $(g,g)$.

However, the diagonal inclusion $G \hookrightarrow G \times G$ induces a push-forward functor

\[ \mathcal{D}(G) \otimes \text{mod} \rightarrow \mathcal{D}\text{el}(G) \otimes \text{mod} \]

and this is tensor; equivalently, it is easy to verify that the projection $\mathcal{D}\text{el}(G) \rightarrow \mathcal{D}(G)$ induced from the central idempotent $\sum_g \delta_{(g,g)} \in \mathcal{D}\text{el}(G)$ is a coalgebra map. In terms of half-braidings, this functor

\[ \mathcal{Z}(G \otimes \text{mod}) \rightarrow \mathcal{Z}^2(G \otimes \text{mod}) \]

is given by

\[ (X,\lambda) \rightarrow (X,\lambda,\lambda) \]

\(\triangle\)

**Example 5.10.** We may consider the group algebra example above, but instead consider the tensor product discussed in Section 3.2. So our objects are still $G$-equivariant bundles over $G \times G$, but the tensor product of two bundles $V = \bigoplus_{g_1,g_2} V_{(g_1,g_2)}$ and $W = \bigoplus_{h_1,h_2} W_{(h_1,h_2)}$ is the image of the usual $V \otimes W$ under the projection $Q_{\lambda^1,\mu^1}$ (see Definition 3.18).

Recall from Theorem 5.3 that to interpret a $\mathcal{D}\text{el}(H)$-module $V$ as an object in $\mathcal{Z}\text{el}(H \otimes \text{mod})$, the first half-braiding is given by $\lambda^1 = P \circ R_1$, where $P$ is the usual swapping of factors, and $R_1 = \sum_j h_j \otimes t_1(h_j^*)$, and similarly for the second half-braiding. Here $H = k[G]$, so $R_1 = \sum_g g \otimes \delta_{(g,*)}$, where $\delta_{(g,*)} := t_1(\delta_g) = \sum_h \delta_{g,h}$.

Concretely, $Q_{\lambda^1,\mu^1}$ works out to the following. We write it as a sum $Q_{\lambda^1,\mu^1} = \frac{1}{|G|} \sum_j \dim X_j \cdot Q^j$ (recall the dashed line represents a sum over simples, weighted by $d_j = \dim X_j$, and $\mathcal{D} = \sum_j d_j^2 = |G|$).
For each $j$, $Q^j$ works out to be

$$v \otimes w \mapsto e_j \otimes e_j^* \otimes v \otimes w$$

$$\mapsto \sum_g e_j \otimes \delta(g,*) \cdot v \otimes g \cdot e_j^* \otimes w$$

$$\mapsto \sum_g \delta(g,*) \cdot v \otimes e_j \otimes w \otimes g \cdot e_j^*$$

$$\mapsto \sum_g \delta(g,*) \cdot v \otimes \delta(h,*) \cdot w \otimes h \cdot e_j \otimes g \cdot e_j^*$$

$$\mapsto \sum_{g,h} (g \cdot e_j^* \cdot h \cdot e_j) \delta(g,*) \cdot v \otimes \delta(h,*) \cdot w$$

$$= \sum_{g,h} \text{tr}_{X_j}(g^{-1}h) \delta(g,*) \cdot v \otimes \delta(h,*) \cdot w$$

Then $Q_{\lambda^1,\mu^1}$ is

$$v \otimes w \mapsto \frac{1}{|G|} \sum_{j,g,h} \dim X_j \text{tr}_{X_j}(h^{-1}g) \delta(g,*) \cdot v \otimes \delta(h,*) \cdot w$$

$$= \frac{1}{|G|} \sum_{g,h} \text{tr}_{k[G]}(h^{-1}g) \delta(g,*) \cdot v \otimes \delta(h,*) \cdot w$$

$$= \sum_{g,h} \delta(g,*) \cdot v \otimes \delta(g,*) \cdot w$$

In short, $Q_{\lambda^1,\mu^1}$ is projection onto those $V_{(g_1,g_2)} \otimes W_{(h_1,h_2)} \subseteq V \otimes W$ such that $g_1 = h_1$, so that

$$(V \otimes W)_{(g,h)} = \sum_{h_1 h_2 = h} V_{(g,h_1)} \otimes W_{(g,h_2)}$$

Thus, under this tensor product, $Z^{el}(\mathcal{C})$ decomposes as a direct sum of $|G|$ copies of $Z(\mathcal{C})$ as monoidal categories:

$$Z^{el}(\mathcal{C}) \cong \bigoplus_g Z(\mathcal{C})$$

$V \mapsto (\delta(g,*) \cdot V)_g$

\[ \triangle \]

6 Concluding remarks, Future directions

Recall that our motivation for constructing $Z^{el}(\mathcal{C})$ is to understand the extended Crane-Yetter TQFT, and $Z^{el}(\mathcal{C}) \cong Z_{CY}(T^2_1)$. We have a similar construction of the category associated to each open surface. For example, the thrice-punctured sphere can also be
thought of as built out of two annuli, except that instead of plumbing to get the once-punctured torus (see figures in Remark 3.21), you identify a segment on the boundary of each annulus. This results in a category with similar looking objects, \((X, \lambda^1, \lambda^2)\), except that the compatibility relation between the half-braidings \(\lambda^1, \lambda^2\) is different: instead of \text{COMM}, they should satisfy the following variant, which was mentioned in Remark 3.2.

\[
\begin{array}{ccc}
X & B & A \\
\hat{\lambda} & & \hat{\lambda} \\
A & B & X
\end{array}
= \begin{array}{ccc}
X & B & A \\
\hat{\lambda} & & \hat{\lambda} \\
A & B & X
\end{array}
\]

Note now the braidings used on both sides are the same, where they were different in \text{COMM}. While both the once-punctured torus and the thrice-punctured sphere can be obtained from a disk by attaching two 1-handles to the boundary, the crucial difference is that the 1-handles “link” in the former but don’t in the latter. In general, for a surface \(\Sigma_{g,n}\) of genus \(g\) with \(n > 0\) punctures, upon presenting it as a disk with \(2g + n - 1\) 1-handles attached to the boundary, the associated category should consist of objects of the form \((X, \lambda^1, \ldots, \lambda^{2g+n-1})\), where \(\lambda^i\) are half-braidings on \(X\), and pairs of \(\lambda^i\)’s satisfy some variant of \text{COMM} or the above relation depending on whether the corresponding 1-handles are “linked” or not. The idea here is reminiscent of the way “gluing patterns” of a surface are used to compute factorization homology in \[BZBJ2015\].

7 Appendix

7.1 Useful Lemmas for Computing with String Diagrams

We record some useful results about string diagrams, adapted mostly from \[KJ2011, BKJ2010\].

Let us denote
\[
\langle V_1, \ldots, V_n \rangle = \text{Hom}_C(1, V_1 \otimes \ldots \otimes V_n)
\]

There is a symmetric non-degenerate pairing
\[
\begin{array}{ccc}
\langle V_1, \ldots, V_n \rangle & \otimes & \langle V^*_1, \ldots, V^*_n \rangle \\
\downarrow P & & \downarrow = \\
\langle V^*_n, \ldots, V^*_1 \rangle & \otimes & \langle V_1, \ldots, V_n \rangle
\end{array} \longrightarrow k
\]

where \(P\) is the usual swapping \(W \otimes W' \to W' \otimes W\) of vector spaces, and the horizontal arrows are given by \((V = V_1 \otimes \ldots \otimes V_n, \varphi \in \langle V \rangle, f \in \langle V^* \rangle)\)

\[
\varphi(f) = (1 \cong 1 \otimes \underbrace{\varphi \circ f}_{\text{ev}} \rightarrow V \otimes V^* \xrightarrow{\text{ev}} 1)
\]

\[
(f, \varphi) = (1 \cong 1 \otimes \underbrace{f \circ \varphi}_{\text{ev}} \rightarrow V^* \otimes V \xrightarrow{\text{ev}} 1)
\]
We will use the following convention: if a figure contains a pair of vertices, one with outgoing edges labelled $V_1, \ldots, V_n$, and the other with outgoing edges labelled $V_1^*, \ldots, V_n^*$, and the vertices are labelled by the same greek letter, say $\alpha$, it will stand for

\[
\begin{array}{c}
\begin{array}{cc}
V_1 & \cdots & V_n \\
\downarrow & & \downarrow \\
\alpha & & \alpha
\end{array} \\
\otimes \\
\begin{array}{cc}
V_1^* & \cdots & V_n^* \\
\downarrow & & \downarrow \\
\varphi_\alpha & & \varphi_\alpha^*
\end{array}
\end{array}
= \sum_{\alpha} \begin{array}{cc}
V_1 & \cdots & V_n \\
\downarrow & & \downarrow \\
\varphi_\alpha & & \varphi_\alpha^*
\end{array} \otimes \begin{array}{cc}
V_1^* & \cdots & V_n^* \\
\downarrow & & \downarrow \\
\alpha & & \alpha^*
\end{array}
\]

where $\{\varphi_\alpha\}, \{\varphi_\alpha^*\}$ are a pair of dual bases of $(V_1, \ldots, V_n), (V_1^*, \ldots, V_n^*)$ respectively, dual respect with respect to the pairing above.

We also establish the following convention: when $\alpha$’s (or any pair of greek letters) appear with a bar $\overline{\alpha}$, and two pairs of edges are labelled with small-case latin alphabets it will stand for the following sum:

\[
\begin{array}{c}
\begin{array}{cc}
V_i & \cdots & V_j \\
\downarrow & & \downarrow \\
\overline{\alpha} & & \overline{\alpha}
\end{array} \\
\otimes \\
\begin{array}{cc}
V_i^* & \cdots & V_j^* \\
\downarrow & & \downarrow \\
\overline{\alpha} & & \overline{\alpha}
\end{array}
\end{array}
= \sum_{i,j} \sqrt{d_i} \sqrt{d_j} \begin{array}{cc}
V_i & \cdots & V_j \\
\downarrow & & \downarrow \\
\overline{\alpha} & & \overline{\alpha}
\end{array} \otimes \begin{array}{cc}
V_i^* & \cdots & V_j^* \\
\downarrow & & \downarrow \\
\overline{\alpha} & & \overline{\alpha}
\end{array}
\]

The $i, j$ will also often be omitted when the context is clear.

Here are some lemmas that are mostly adapted from [BKJ2010] and [KJ2011]. We leave out the proofs, which are standard.

**Lemma 7.1.**

\[
\begin{array}{c}
\begin{array}{cc}
V_1 V_2 \cdots V_n \\
\downarrow & & \downarrow \\
\varnothing & & \varnothing
\end{array}
\end{array}
\]

(Recall the convention of dashed line in the introduction)

**Lemma 7.2.** (Variant of Lemma 3.6 in [KJ2011]) For $\Phi : V \to W$,

\[
\left\langle V \Phi \begin{array}{cc}
W & \\
\alpha & \\
U_n D_1 & U_n D_2
\end{array} \right\rangle \otimes \left\langle U_1 \cdots U_n \begin{array}{cc}
W & \\
\alpha & \\
D_1 & D_2
\end{array} \right\rangle = \left\langle V \Phi \begin{array}{cc}
U_1 \cdots U_n & \\
\beta & \\
D_1 & D_2
\end{array} \right\rangle \otimes \left\langle U_1 \cdots U_n \begin{array}{cc}
V & \\
\beta & \\
\Phi & \end{array} \right\rangle
\]

**Lemma 7.3.** $\Gamma$, as defined in Proposition 2.4, is a half-braiding.

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Proof. Naturality is immediate from Lemma 7.2 above, and respecting tensor product is checked as follows:

\[
\sqrt{d_i}d_j\sqrt{d_k} \langle \begin{array}{c}
B \\
\downarrow
d_i \\
\uparrow
A
\end{array} \rangle \otimes \langle \begin{array}{c}
B \\
\downarrow
d_j \\
\uparrow
A
\end{array} \rangle = \sqrt{d_i}d_j\sqrt{d_k} \langle \begin{array}{c}
B \\
\downarrow
d_i \\
\uparrow
A
\end{array} \rangle \otimes \langle \begin{array}{c}
B \\
\downarrow
d_j \\
\uparrow
A
\end{array} \rangle
\]

(Recall our convention of summing over all latin lower case labels.) The first equality uses Lemma 7.2 with \(\Phi = \beta\), and the second follows from Lemma 7.1.

The following lemma is often used when the half-braiding \(\Gamma\) shows up, allowing us to switch the “main branch” with the “side branch” (see proof of Proposition 3.19).

Lemma 7.4.

Lemma 7.5. (Charge conservation) When \(\mathcal{C}\) is modular,

\[
\frac{1}{D} = \delta_{i,0} \text{ (i not summed)}
\]

Proof. See e.g. [BK2001, Cor 3.1.11].

7.2 Group Actions on Categories by Generators and Relations

Let \(G\) be a group acting on a category \(\mathcal{A}\), in the sense of [EGNO2015, Section 2.7]. This consists of an auto-equivalence \(T_g : \mathcal{A} \to \mathcal{A}\) for each \(g \in G\), and natural isomorphisms \(\gamma_{g,h} : T_g \circ T_h \to T_{gh}\) satisfying the cocycle condition

\[
T_y T_h T_k \xrightarrow{T_y \gamma_{h,k}} T_y T_{hk} \\
\downarrow \gamma_{g,k} T_k \quad \downarrow \gamma_{g,h,k} \\
T_{gh} T_k \xrightarrow{\gamma_{gh,k}} T_{ghk}
\]
For convenience, we will refer to this as the *usual* definition of a group action, and in particular refer to the above as the *usual cocycle condition*.

It is convenient to rephrase this as simply a monoidal functor

\[(F, J) : \text{Cat}(G) \to \otimes \text{Aut}(A)\]

where \(\text{Cat}(G)\) is the monoidal category whose objects are elements of \(G\), with only identity morphisms, and the monoidal structure is the group operation. Then in this interpretation, \(T_g = F(g)\), the \(\gamma\)'s correspond to the natural isomorphism \(J_{g,h} : F(g) \circ F(h) \to F(gh)\), and the cocycle condition is just the hexagon axiom relating the associativity constraints with \(J\) (both \(\text{Cat}(G)\) and \(\text{Aut}(A)\) are strict, so the hexagon is just a square).

When \(G\) is presented by generators and relations, say \(\text{SL}_2(\mathbb{Z}) = \langle s, t | r_1, r_2 \rangle\) as in Section 3.3 we would like to describe a \(G\)-action on \(A\) by generators and relations as well. In a typical group action, say on some set \(X\), it suffices to provide an automorphism of \(X\) for each generator, and check that the relations are satisfied. For an action on a category, one provides an auto-equivalence for each generator, a natural isomorphism for each relation, and check certain equalities between compositions of such natural isomorphisms; these are the analogs of \(T_g, \gamma_{g,h}\), and the cocycle condition in the usual definition of a group action on a category given above. It is the goal of this note to spell this out in more detail.

In this note, we fix a group \(G\) and a presentation of it, \(G = \langle g_i | r_j \rangle\). Since we will be working with unreduced words, we will include among the relations the trivial ones \(g_i g_i^{-1} = 1, g_i^{-1} g_i = 1\). All words (henceforth assumed to be unreduced) will be in the generators \(g_i\) (and their inverses \(g_i^{-1}\)). We will think of a relation \(r_j\) as a *move* to transform one word into another; more precisely, it is given by a pair of words \(v_{j,1}, v_{j,2}\), so that for any words \(x, y\), we may transform \(xv_{j,1}y\) into \(xv_{j,2}y\) when working in \(G\). When we need to be precise, we will denote this move by \(xr_jy\), and the inverse move by \(xr_j^{-1}y\). (Ambiguities can arise, for example, “applying \(r_j\) to \(v_{j,1}v_{j,1}\)” could mean \(r_jv_{j,1} : v_{j,1}v_{j,1} \to v_{j,2}v_{j,1}\) or \(v_{j,1}r_j : v_{j,1}v_{j,1} \to v_{j,1}v_{j,2}\).)

First, let us give a definition:

**Definition 7.6.** Let \(G = \langle g_i | r_j \rangle\) be a group presentation as above, and let \(A\) be a category. A *pre-\(G\)-action on \(A\) given by generators and relations* consists of the following data:

- For each generator \(g_i\), auto-equivalences \(U_{g_i}, U_{g_i}^{-1} : A \otimes\).
- For each relation \(r_j : v_{j,1} = v_{j,2}\), a natural isomorphism \(\gamma_j : U_{v_{j,1}} \to U_{v_{j,2}}\), where we write \(U_w = U_{a_1} \ldots U_{a_k}\) for a word \(w = a_1 \ldots a_k\).

\[\triangle\]
From $\gamma_j$, we also get, for words $x, y$, a natural isomorphism $U_x \gamma_j U_y : U_{xv_j} y \to U_{xv_j} y$. We will sometimes abuse notation and denote this natural isomorphism as $\gamma_j$ too.

**Definition 7.7.** In the set up of Definition 7.6, a $G$-action on $A$ given by generators and relations is a pre-$G$-action that satisfies the following cocycle condition: a sequence of moves (i.e. application of relation) $w_1 \xrightarrow{r_{j_1}} \ldots \xrightarrow{r_{j_p}} w_2$ gives rise to a sequence of natural isomorphisms $U_{w_1} \xrightarrow{\gamma_{j_1}} \ldots \xrightarrow{\gamma_{j_p}} U_{w_2}$ whose composition is some natural isomorphism $\gamma : U_{w_1} \to U_{w_2}$; the cocycle condition says that any sequence of moves from $w_1$ to $w_2$ results in the same $\gamma$. 

**Remark 7.8.** We do not need to include the trivial relations $g_i g_i^{-1} = 1$ if we can guarantee that the $U_{g_i}$'s are isomorphisms, and $U_{g_i^{-1}} = U_{g_i}$. This was the case in Section 3.3.

Now we justify this definition. Let us first sketch an approach that is more intuitive. Consider the following CW-complex $\Xi$: the 0-skeleton/vertices is the set of unreduced words, and for words $x, y$ and relation $r_j$, there is a 1-cell $xr_jy$ going from $xv_j 1 y$ to $xv_j 2 y$. The set of connected components is clearly in bijective correspondence with $G$. (We can even impose an H-group structure on $\Xi$ so that the obvious quotient map $\Xi \to G$ is a map of H-groups, but since we are just giving intuition, we leave this discussion to the more formal discussions to come.)

Given a pre-$G$-action (as in Definition 7.6), we can assign to a vertex $w$ the auto-equivalence $U_w$, and to a 1-cell $xv_j y \to xv_j 2 y$ we can assign the natural isomorphism $U_x \gamma_j U_y : U_{xv_j} y \to U_{xv_j} y$. Then the cocycle condition in Definition 7.7 means that for any loop beginning and ending at a vertex $w$, the composition of natural isomorphisms encountered along the loop is just the identity $\text{id}_{U_w} : U_w \circ$.

So if we have a $G$-action in the sense of Definition 7.7, we can get a usual action of $G$ by picking a representative word $w_g$ for each $g \in G$, and specifying $T_g = U_{w_g}$, and $\gamma_{g,h} : T_g T_h = U_{w_g w_h} \to U_{w_g h} = T_{gh}$, where $\gamma$ is the appropriate composition of natural isomorphisms - by the cocycle condition in the sense of Definition 7.7, any choice gives the same natural isomorphism. The cocycle condition for $T_g, \gamma_{g,h}$ is automatically satisfied.

Now let us give a more precise discussion. Let $G = \langle g_i | r_j \rangle$ as above. Consider the following monoidal category $\mathcal{G}$: its objects are all unreduced words in $g_i$'s. The morphisms are given by compositions of applications of $r_j$'s; more precisely, we consider the arrows $q_{i,x,y} : xv_{j_i} y \to xv_{j_i} y$ for all words $x, y$, and then a morphism $w_1 \to w_2$ in $\mathcal{G}$ is just a (possibly empty) composable sequence of such arrows or their reverse, reduced in the sense that an arrow and its reverse cancel out. (If we didn’t include the trivial relations among the $r_j$'s, we may end up with no morphisms between $g_i g_i^{-1}$ and the empty word.) Alternatively, the set of morphisms from $w_1$ to $w_2$ is the set of homotopy classes of paths from $w_1$ to $w_2$ in $\Xi$, the CW-complex considered above. The
morphisms consisting of a single \( q_{j,x,y} \) will be called \textit{simple}. The monoidal structure on \( \tilde{G} \) is given by concatenation of words for objects, and for morphism, if we have morphisms \( f : w_1 \to w_2 \) and \( g : w'_1 \to w'_2 \), we set

\[
f \otimes f' = f \circ f' : w_1 w'_1 \to w_2 w'_2 \]

It is easy to see that this forms a well-defined monoidal category.

Next consider \( G \) to be the category with the same objects, but with fewer morphisms: there is a unique morphism \( w_1 \to w_2 \) in \( G \) if there is at least one morphism in \( \tilde{G} \), and there are no morphisms otherwise. In other words,

\[
\text{Hom}_G(w_1,w_2) = \begin{cases} \ast & \text{if } \text{Hom}_{\tilde{G}}(w_1,w_2) \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}
\]

The monoidal structure on \( G \) is also concatenation. There is an obvious monoidal “quotient functor” \( Q : \tilde{G} \to G \) which is the identity on objects, and identifies all morphisms with common source and target.

There is a canonical functor \( \pi : G \to \text{Cat}(G) \) that sends a word to the corresponding element in \( G \), and the morphisms are sent to the identity morphisms. Clearly this functor is fully faithful and essentially surjective, and is monoidal (in a unique way), so is an equivalence of monoidal categories.

Thus, we can describe a \( G \)-action on \( A \) by giving a monoidal functor

\[(F,J) : G \to \otimes \text{Aut}(A)\]

The point of using \( G \) instead of \( \text{Cat}(G) \) is that we know \( G \) is somehow built out of generators and relations. Precomposing such a monoidal functor with an inverse to \( \pi : G \to \text{Cat}(G) \) gives a monoidal functor \( (F',J') : \text{Cat}(G) \to \otimes \text{Aut}(A) \), recovering a group action in the usual sense. Note that any two such inverses are naturally isomorphic by a unique natural isomorphism. We say more at the end of this section.

Let us see how \( (F,J) \) gives us a \( G \)-action in the sense of Definition 7.7. \( F \) in particular gives us, for each generator \( g_i \), auto-equivalences

\[
U_{g_i} := F(g_i) : A \to A
\]

\[
U_{g_i^{-1}} := F(g_i^{-1}) : A \to A
\]

For a word \( w = a_1 \ldots a_k \), where \( a_i = g_i \) or \( g_i^{-1} \), we write \( U_w = U_{a_1} \ldots U_{a_k} \). \( J \) gives us \( U_{a_1 a_2} = F(a_1) F(a_2) \to F(a_1 a_2) \), and similarly, successive applications of \( J \)’s gives us

\[
U_w = F(a_1) \ldots F(a_k) \xrightarrow{F(a_1) \ldots F(a_k-1) J_{a_{k-1} a_k}} F(a_1) \ldots F(a_{k-2}) F(a_{k-1} a_k) \\
\vdots\\n\xrightarrow{J_{a_1 a_2 \ldots a_k}} F(a_1 \ldots a_k) = F(w)
\]
For brevity, we call the composition of these natural isomorphisms $J$ too. (By the hexagon axiom, the order by which we group the $a_i$’s together is immaterial.) Then for each relation $r_j: v_{j,1} = v_{j,2}$, we have

$$
\gamma_j: U_{v_{j,1}} J F(v_{j,1}) U(r_j) F(v_{j,2}) J^{-1} U_{v_{j,2}}
$$

So $U_g, U_g^{-1}, \gamma_j$ defines a pre-$G$-action (as in Definition 7.6), and in fact satisfies the cocycle condition, so it is a $G$-action (in the sense of Definition 7.7). Indeed, suppose we have a path $w_1 \xrightarrow{r_{j_1}} \ldots \xrightarrow{r_{j_p}} w_2$. The resulting sequence of isomorphism compose to simply $U_{w_1} J F(w_1) F(r) F(w_2) J^{-1} U_{w_2}$, where $r$ is the unique morphism in $\text{Hom}_G(w_1, w_2)$. Since this natural isomorphism is independent of the path we started with, we see that we indeed have a $G$-action.

Conversely, suppose we are provided with a pre-$G$-action $U_g, U_g^{-1}, \gamma_j$ (see Definition 7.6). We can easily construct a monoidal functor

$$(\bar{F}, \bar{J}): \bar{G} \to \otimes \text{Aut}(A)$$

as follows: for a word $w = a_1 \ldots a_k$, we define

$$\bar{F}(w) := U_{a_1} \ldots U_{a_k}$$

and make $\bar{J} = \text{id}$. For the simple morphism $q_{j,x,y}: xv_{j,1}y \to xv_{j,2}y$ we define

$$\bar{F}(q_{j,x,y}): \bar{F}(xv_{j,1}y) \xrightarrow{\bar{F}(x)\gamma_j \bar{F}(y)} \bar{F}(xv_{j,2}y)$$

Since a morphism $q$ in $\bar{G}$ is a sequence of simple $q_{j,x,y}$ and their reverses, we take $\bar{F}(q)$ to be the composition of the appropriate $\bar{F}(q_{j,x,y})$’s (these are natural isomorphisms, so if the sequence uses a reversed arrow, we associate the inverse natural isomorphism).

It is easy to see that this gives a well-defined monoidal functor $(\bar{F}, \bar{J}): \bar{G} \to \otimes \text{Aut}(A)$. Then we would get a monoidal functor $(F, J): G \to \otimes \text{Aut}(A)$ if $(\bar{F}, \bar{J})$ factors through $Q$:

$$
\begin{array}{c}
\bar{G} \\
Q \downarrow \quad (F, J) \\
\bar{G} \quad (F, J) \quad \text{Aut}(A)
\end{array}
$$

In concrete terms, factoring through $Q$ means that for each pair of unreduced words $w_1, w_2$, any sequence of applications of relations to get from $w_1$ to $w_2$ (i.e. a morphism in $\bar{G}$ from $w_1$ to $w_2$), will result in the same natural isomorphism $\bar{F}(w_1) \to \bar{F}(w_2)$. In other words, it is equivalent to the statement that $U_g, U_g^{-1}, \gamma_j$ satisfy the cocycle condition of Definition 7.7. In summary,
**Proposition 7.9.** Given a monoidal functor \((F, J) : G \to \otimes \text{Aut}(A)\), the values of \(F\) on generators and relations, interpreted appropriately with \(J\), defines a group action in the sense of Definition 7.7.

Conversely, from a group action in the sense of Definition 7.7, one can construct a monoidal functor \((\bar{F}, \bar{J}) : \bar{G} \to \otimes \text{Aut}(A)\) that factors through \(Q\), and hence defines a monoidal functor \((F, J) : G \to \otimes \text{Aut}(A)\).

Furthermore, beginning with some \((F, J) : G \to \otimes \text{Aut}(A)\), applying the first construction and then the second, we get a new monoidal functor \((F', J') : G \to \otimes \text{Aut}(A)\), and \((F, J), (F', J')\) are naturally isomorphic as monoidal functors.

**Proof.** It remains to prove the last part, that \((F, J), (F', J')\) are naturally isomorphic as monoidal functors. It is easy to check that \(\eta_w = J^{-1}\) works, where recall we abuse notation for \(J\) to also mean successive applications of \(J\)'s: for \(w = a_1 \ldots a_k\), \(F'(w) = U_w = F(a_1) \ldots F(a_k) \to \ldots \to F(a_1 \ldots a_k) = F(w)\).

\[\square\]

**Remark 7.10.** If we take the trivial presentation \(G = \langle \bar{g} \rangle\) for \(g \in G\) with \(r_{g,h} : \bar{g}h = \bar{g}h\), we find that we recover the usual notion of a group action of \(G\) on \(A\), as first discussed at the beginning of this section.

Finally, let us relate this back to the usual notion of group action on a category:

**Corollary 7.11.** Given a \(G\)-action on \(A\) by generators and relations \(U_{g_1}, U_{g_2}, \gamma_i\), as in Definition 7.7, we can obtain a group action in the usual sense by first applying Proposition 7.9 to get a monoidal functor \((F, J) : G \to \otimes \text{Aut}(A)\), then choosing an inverse to the canonical \(\pi : G \to \text{Cat}(G)\).

More concretely, one chooses, for each \(g \in G\), a word \(w_g\), and sets \(T_g = F(w_g) = U_{w_g}\), and for \(g, h \in G\), set \(\gamma_i' = F(r) : T_gT_h = F(w_gF(w_h) \to F(w_{gh})) = T_{gh}\), where \(r\) is the unique morphism in \(\text{Hom}_Q(w_gw_h, w_{gh})\). The uniqueness of \(r\) also guarantees that the usual cocycle condition is satisfied by \(T_g, \gamma_i'_{g,h}\).

Any two inverses to \(\pi\) are naturally isomorphic by unique natural isomorphism, so the resulting group actions are equivalent.

**Remark 7.12.** In general, it is not so clear how to check that a pre-\(G\)-action in the sense of Definition 7.6 satisfies the cocycle conditions of Definition 7.7 to be a \(G\)-action. If we are to rely solely on algebra, we need to understand the presentation \(G = \langle g_i \mid r_j \rangle\) a little better, somehow know the “relations between relations”, not unlike the second syzygies of a module as studied in homological algebra.

Let us clarify. Recall the CW-complex \(\Xi\) defined above as justification for Definition 7.7. Suppose we have checked that two paths \(r, r'\) from \(w_1\) to \(w_2\) in \(\Xi\) lead to the same natural isomorphism \(U_{w_1} \to U_{w_2}\); we call this a second-order relation. We attach
a 2-cell along the loop $r^{-1}r'$. For any words $x,y$, we automatically have that the two paths $xry, x'r'y$ from $xw_1y$ to $xw_2y$ in $\Xi$ also lead to the same natural isomorphism $U_{xw_1y} \to U_{xw_2y}$, so we also attach a 2-cell along the loop $(xry)^{-1}(x'r'y)$.

Now suppose we have found several second-order relations, so that upon attaching the corresponding 2-cell and its “translates” as above for each one, each connected component of the new CW-complex $\Xi'$ is simply connected. Then it is easy to see that this implies that the pre-$G$-action we started with is actually a $G$-action. Indeed, this means that any loop is contractible via a sequence of 2-cells; each time a loop homotopes through a 2-cell, the second-order relation implies that the corresponding natural isomorphism doesn’t change. Since the constant loop is associated the identity natural isomorphism, we find that the original loop is also associated the identity, hence the cocycle condition is satisfied.

7.2.1 Equivariant Objects

Suppose we are given a group $G$ acting on a category $\mathcal{A}$ in the usual sense, as described at the beginning of the previous Section 7.2. Then recall that a $G$-equivariant object of $\mathcal{A}$ is an object $A \in \mathcal{A}$ with isomorphisms $\mu_g : T_g(A) \to A$, and $\mu_-$ is required to be compatible with $T_-$ in the sense that

$$
\begin{align*}
T_g(T_h(A)) & \xrightarrow{T_g(T_h)} T_g(A) \\
\downarrow^{\gamma_{g,h}} & \downarrow^{\mu_g} \\
T_{gh}(A) & \xrightarrow{\mu_{gh}} A
\end{align*}
$$

commutes for every pair of $g, h \in G$ (see for example [EGNO2015, Section 2.7]).

If $G$ and its action are presented by generators and relations as in Definition 7.7, we would like to describe a $G$-equivariant object by making use of the presentation. In particular, if $G$ is finitely presented, one would hope to be able to describe a $G$-equivariant object by a finite amount of data that satisfy a finite number of equations. The goal of this subsection is to spell this out in detail.

We begin with a $G$-action on $\mathcal{A}$ that is given by the following data: for each generator $g_i$, auto-equivalences $U_{g_i}, U_{g_i^{-1}}$, and for each relation $r_j : v_{j,1} = v_{j,2}$, we have $\gamma_j : U_{v_{j,1}} \to U_{v_{j,2}}$.

Suppose we are given an object $A \in \mathcal{A}$, and for each generator $g_i$ of $G$, we are given an isomorphism $\mu_{g_i} : U_{g_i}(A) \to A$

From this, we can define $\mu_{g_i^{-1}} : U_{g_i^{-1}}(A) \xrightarrow{U_{g_i}^{-1}(\mu_{g_i})} U_{g_i^{-1}}(U_{g_i}(A)) \xrightarrow{\gamma_{l_i}} A$ where $\gamma_{l_i}$ is
the natural isomorphism corresponding to the relation \(g_i^{-1}g_i = 1\). We can then construct...

\[ \mu_w : U_w(A) \rightarrow A \]

Now just by construction, we have that, for words \(w_1, w_2\),

\[ U_{w_1}(U_{w_2}(A)) \xrightarrow{U_{w_1}(\mu_{w_2})} U_{w_1}(A) \]

so it seems that there is nothing to prove.

Thinking back to the usual definition of a \(G\)-equivariant object as in the beginning of this section, one should have exactly one \(\mu_g : U_g(A) \rightarrow A\) for each \(g\), so we should expect that if two words \(w_1, w_2\) are equal in \(G\), then \(\mu_{w_1}\) should be equal to \(\mu_{w_2}\). This cannot literally be true, being morphisms from different objects. We may however impose that for each relation \(r_j : v_{j,1} = v_{j,2}\),

\[ U_{v_{j,1}}(A) \xrightarrow{(\gamma_j)_A} U_{v_{j,2}}(A) \]

\[ U_{v_{j,1}}(A) \xrightarrow{\mu_{v_{j,1}}} A \quad \xleftarrow{\mu_{v_{j,2}}} U_{v_{j,2}}(A) \]

This implies a similar commutative diagram for \(xr_jy\) relating \(xv_{j,1}y\) to \(xv_{j,2}y\):

**Definition 7.13.** Let an action of \(G = \langle g_i | r_j \rangle\) on a category \(A\) be given by generators and relations in the sense of Definition 7.7. Then a generators-and-relations presentation of a \(G\)-equivariant object structure on \(A\) consists of an isomorphism \(\mu_g : U_g(A) \rightarrow A\) for each \(g_i\), subject to the equation \(R_j\) above for each relation \(r_j\). △
Note that, in contrast to checking the cocycle condition of Definition 7.7, there is no need to use “second order relations” (as discussed at the end of the previous section).

Finally, let us relate this to the usual notion of group action on categories and equivariant objects. Recall that in Corollary 7.11, we see that to go from a generators-and-relations description of a group action to the usual one, we make a choice of word $w_g$ for each $g \in G$, and set $T_g = U_{w_g}$, and $\gamma'_{g,h} : T_g T_h \to T_{gh}$ is determined from $\gamma_j$’s. Then we can obtain a usual $G$-equivariant structure on $A$ from a generators-and-relations description as in Definition 7.13 as follows: Take $\nu_g = \mu_{w_g} : T_g(A) = U_{w_g}(A) \to A$. The compatibility of $\nu_-$ with the group action $T_-, \gamma'$ (i.e. whether $\nu_-$ defines a $G$-equivariant structure on $A$) is equivalent to the commutativity of the diagram

$$
\begin{array}{ccc}
U_{w_g}(U_{w_h}(A)) & \xrightarrow{U_{w_g}(\mu_{w_h})} & U_{w_g}(A) \\
\downarrow \gamma & & \downarrow \mu_{w_h} \\
U_{w_{gh}}(A) & \xrightarrow{\mu_{w_{gh}}} & A
\end{array}
$$

where $\gamma$ is the composition of some $\gamma_j$’s corresponding to a path from $w_g w_h$ to $w_{gh}$. We see that this diagram is commutative precisely because we have imposed the equations $R_j$ on $\mu_j$.

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