Spectrum of commuting graphs of some classes of finite groups

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Abstract: In this paper, we initiate the study of spectrum of the commuting graphs of finite non-abelian groups. We first compute the spectrum of this graph for several classes of finite groups, in particular AC-groups. We show that the commuting graphs of finite non-abelian AC-groups are integral. We also show that the commuting graph of a finite non-abelian group $G$ is integral if $G$ is not isomorphic to the symmetric group of degree 4 and the commuting graph of $G$ is planar. Further it is shown that the commuting graph of $G$ is integral if the commuting graph of $G$ is toroidal.

Key words: commuting graph, spectrum, integral graph, finite group.

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1 Introduction

Let $G$ be a finite group with centre $Z(G)$. The commuting graph of a non-abelian group $G$, denoted by $\Gamma_G$, is a simple undirected graph whose vertex set is $G \setminus Z(G)$, and two vertices $x$ and $y$ are adjacent if and only if $xy = yx$. Various aspects of commuting graphs of different finite groups can be found in \[3\] \[6\] \[10\] \[11\] \[12\] \[13\].

In this paper, we initiate the study of spectrum of commuting graphs of finite non-abelian groups. Recall that the spectrum of a graph $\mathcal{G}$ denoted by $\text{Spec}(\mathcal{G})$ is the set $\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_n^{k_n}\}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of $\mathcal{G}$ with multiplicities $k_1, k_2, \ldots, k_n$ respectively. A graph $\mathcal{G}$ is called integral if $\text{Spec}(\mathcal{G})$ contains only integers. It is well known that the complete graph

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The notion of integral graph was introduced by Harary and Schwenk [9] in the year 1974. A very impressive survey on integral graphs can be found in [5].

We observe that the commuting graph of a non-abelian finite AC-group is disjoint union of some complete graphs. Therefore, commuting graphs of such groups are integral. In general it is difficult to classify all finite non-abelian groups whose commuting graphs are integral. As applications of our results together with some other known results, in Section 3, we show that the commuting graph of a finite non-abelian group $G$ is integral if $G$ is not isomorphic to $S_4$, the symmetric group of degree 4, and the commuting graph of $G$ is planar. We also show that the commuting graph of a finite non-abelian group $G$ is integral if the commuting graph of $G$ is toroidal. Recall that the genus of a graph is the smallest non-negative integer $n$ such that the graph can be embedded on the surface obtained by attaching $n$ handles to a sphere. A graph is said to be planar or toroidal if the genus of the graph is zero or one respectively. It is worth mentioning that Afkhami et al. [2] and Das et al. [7] have classified all finite non-abelian groups whose commuting graphs are planar or toroidal recently.

2 Computing spectrum

It is well known that the complete graph $K_n$ on $n$ vertices is integral and $\text{Spec}(K_n)$ is given by $\{(−1)^{n−1}, (n−1)^1\}$. Further, if $G = K_{m_1} \sqcup K_{m_2} \sqcup \cdots \sqcup K_{m_l}$, where $K_{m_i}$ are complete graphs on $m_i$ vertices for $1 \leq i \leq l$, then

$$\text{Spec}(G) = \{(−1)^{\sum_{i=1}^{l} m_i−l}, (m_1−1)^1, (m_2−1)^1, \ldots, (m_l−1)^1\}. \quad (2.1)$$

If $m_1 = m_2 = \cdots = m_l = m$ then we write $G = lK_m$ and in that case $\text{Spec}(G) = \{(−1)^{l(m−1)}, (m−1)^l\}$.

In this section, we compute the spectrum of the commuting graphs of different families of finite non-abelian AC-groups. A group $G$ is called an AC-group if $C_G(x)$ is abelian for all $x \in G \setminus Z(G)$. Various aspects of AC-groups can be found in [1] [2] [14]. The following lemma plays an important role in computing spectrum of commuting graphs of AC-groups.

**Lemma 2.1.** Let $G$ be a finite non-abelian AC-group. Then the commuting graph of $G$ is given by

$$\Gamma_G = \bigsqcup_{i=1}^{n} K_{|X_i|−|Z(G)|}$$

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of $G$. 

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Proof. Let $G$ be a finite non-abelian AC-group and $X_1, \ldots, X_n$ be the distinct centralizers of non-central elements of $G$. Let $X_i = C_G(x_i)$ where $x_i \in G \setminus Z(G)$ and $1 \leq i \leq n$. Let $x, y \in X_i \setminus Z(G)$ for some $i$ and $x \neq y$ then, since $G$ is an AC-group, there is an edge between $x$ and $y$ in the commuting graph of $G$. Suppose that $x \in (X_i \cap X_j) \setminus Z(G)$ for some $1 \leq i \neq j \leq n$. Then $[x, x_i] = 1$ and $[x, x_j] = 1$. Hence, by Lemma 3.6 of [1] we have $C_G(x) = C_G(x_i) = C_G(x_j)$, a contradiction. Therefore, $X_i \cap X_j = Z(G)$ for any $1 \leq i \neq j \leq n$. This shows that

$$\Gamma_G = \bigsqcup_{i=1}^n K|X_i| - |Z(G)|.$$ 

**Theorem 2.2.** Let $G$ be a finite non-abelian AC-group. Then the spectrum of the commuting graph of $G$ is given by

$$\{(-1)^i, |X_i| - |Z(G)| - 1\}^n_{i=1},$$

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of $G$.

Proof. The proof follows from Lemma 2.1 and (2.1). \qed

**Corollary 2.3.** Let $G$ be a finite non-abelian AC-group and $A$ be any finite abelian group. Then the spectrum of the commuting graph of $G \times A$ is given by

$$\{(-1)^i, |A||X_i| - |Z(G)| - 1\}^n_{i=1},$$

where $X_1, \ldots, X_n$ are the distinct centralizers of non-central elements of $G$.

Proof. It is easy to see that $Z(G \times A) = Z(G) \times A$ and $X_1 \times A, X_2 \times A, \ldots, X_n \times A$ are the distinct centralizers of non-central elements of $G \times A$. Therefore, if $G$ is an AC-group then $G \times A$ is also an AC-group. Hence, the result follows from Theorem 2.2 \qed

Now we compute the spectrum of the commuting graphs of some particular families of AC-groups. We begin with the well-known family of quasidihedral groups.

**Proposition 2.4.** The spectrum of the commuting graph of the quasidihedral group $QD_{2^n} = \langle a, b : a^{2n-1} = b^2 = 1, bab^{-1} = a^{2n-2} - 1 \rangle$, where $n \geq 4$, is given by

$$\text{Spec}(\Gamma_{QD_{2^n}}) = \{(-1)^{2n-2n-2-3}, 1^{2n-2}, (2^n-1)^3\}.$$
Proof. It is well-known that $Z(QD_{2n}) = \{1, a^{2^{n-2}}\}$. Also
\[
C_{QD_{2n}}(a) = C_{QD_{2n}}(a^i) = \langle a \rangle \text{ for } 1 \leq i \leq 2^{n-1} - 1, i \neq 2^{n-2}
\]
and
\[
C_{QD_{2n}}(a^j b) = \{1, a^{2^{n-2}}, a^j b, a^{j+2^{n-2}} b\} \text{ for } 1 \leq j \leq 2^{n-2}
\]
are the only centralizers of non-central elements of $QD_{2n}$. Note that these centralizers are abelian subgroups of $QD_{2n}$. Therefore, by Lemma 2.1
\[
\Gamma_{QD_{2n}} = K_{[C_{QD_{2n}}(a) \setminus Z(QD_{2n})]} \sqcup \bigcup_{j=1}^{2^{n-2}} K_{[C_{QD_{2n}}(a^j b) \setminus Z(QD_{2n})]}.
\]
That is, $\Gamma_{QD_{2n}} = K_{2^n-1, 2} \sqcup 2^{n-2} K_2$, since $|C_{QD_{2n}}(a)| = 2^{n-1}, |C_{QD_{2n}}(a^j b)| = 4$ for $1 \leq j \leq 2^{n-2}$ and $|Z(QD_{2n})| = 2$. Hence, the result follows from (2.1).

**Proposition 2.5.** The spectrum of the commuting graph of the projective special linear group $PSL(2,2^k)$, where $k \geq 2$, is given by
\[
\{(-1)^{2k-2} - 2^{k-1} - 2, (2k - 1)^{2k-1} - 1, (2k - 2)^{2k-1} - 1, (2k - 3)^{2k-1} - 1\}.
\]

**Proof.** We know that $PSL(2,2^k)$ is a non-abelian group of order $2^k (2^{2k} - 1)$ with trivial center. By Proposition 3.21 of [1], the set of centralizers of non-trivial elements of $PSL(2,2^k)$ is given by
\[
\{xP\bar{x}^{-1}, xA\bar{x}^{-1}, xB\bar{x}^{-1} : x \in PSL(2,2^k)\}
\]
where $P$ is an elementary abelian $2$-subgroup and $A, B$ are cyclic subgroups of $PSL(2,2^k)$ having order $2^k, 2^k - 1$ and $2^k + 1$ respectively. Also the number of conjugates of $P, A$ and $B$ in $PSL(2,2^k)$ are $2^k + 1, 2^{k-1}(2^k + 1)$ and $2^{k-1}(2^k - 1)$ respectively. Note that $PSL(2,2^k)$ is a AC-group and so, by Lemma 2.1, the commuting graph of $PSL(2,2^k)$ is given by
\[
(2^k + 1)K_{[xP\bar{x}^{-1}]^{-1}} \sqcup 2^{k-1}(2^k + 1)K_{[xA\bar{x}^{-1}]^{-1}} \sqcup 2^{k-1}(2^k - 1)K_{[xB\bar{x}^{-1}]^{-1}}.
\]
That is, $\Gamma_{PSL(2,2^k)} = (2^k + 1)K_{2^k - 1} \sqcup 2^{k-1}(2^k + 1)K_{2^k - 2} \sqcup 2^{k-1}(2^k - 1)K_{2^k}$. Hence, the result follows from (2.1).

**Proposition 2.6.** The spectrum of the commuting graph of the general linear group $GL(2, q)$, where $q = p^h > 2$ and $p$ is a prime integer, is given by
\[
\{(-1)^q q^3 + 2q^2 - q, (q^2 - 2q + 1)\frac{q(q+1)}{2}, (q^2 - q - 1)\frac{q(q-1)}{2}, (q^2 - 2q)q + 1\}.
\]
Proof. We have $|GL(2,q)| = (q^2 - 1)(q^2 - q)$ and $|Z(GL(2,q))| = q - 1$. By Proposition 3.26 of [1], the set of centralizers of non-central elements of $GL(2,q)$ is given by

$$\{x Dx^{-1}, xIx^{-1}, x P Z(GL(2,q)) x^{-1} : x \in GL(2,q)\}$$

where $D$ is the subgroup of $GL(2,q)$ consisting of all diagonal matrices, $I$ is a cyclic subgroup of $GL(2,q)$ having order $q^2 - 1$ and $P$ is the Sylow $p$-subgroup of $GL(2,q)$ consisting of all upper triangular matrices with 1 in the diagonal. The orders of $D$ and $P Z(GL(2,q))$ are $(q - 1)^2$ and $q(q - 1)$ respectively. Also the number of conjugates of $D$, $I$ and $P Z(GL(2,q))$ in $GL(2,q)$ are $\frac{q(q+1)}{2}$, $\frac{q(q-1)}{2}$ and $q+1$ respectively. Since $GL(2,q)$ is an AC-group (see Lemma 3.5 of [1]), by Lemma 2.1 we have $\Gamma_{GL(2,q)} = \frac{q(q + 1)}{2} K_{x Dx^{-1} [-q+1]} \sqcup \frac{q(q - 1)}{2} K_{xIx^{-1} [-q+1]} \sqcup (q + 1) K_{x P Z(GL(2,q)) x^{-1} [-q+1]}$.

That is, $\Gamma_{GL(2,q)} = \frac{q(q+1)}{2} K_{q^2 - 3q + 2} \sqcup \frac{q(q-1)}{2} K_{q^2 - q} \sqcup (q + 1) K_{q^2 - 2q + 1}$. Hence, the result follows from (2.1).

**Theorem 2.7.** Let $G$ be a finite group and $\frac{G}{Z(G)} \cong Sz(2)$, where $Sz(2)$ is the Suzuki group presented by $(a, b : a^5 = b^4 = 1, b^{-1} ab = a^2)$. Then

$$\text{Spec}(\Gamma_G) = \{(-1)^{19}|Z(G)|^{-6}, (4|Z(G)| - 1)^1, (3|Z(G)| - 1)^5\}.$$

**Proof.** We have

$$\frac{G}{Z(G)} = \langle a Z(G), b Z(G) : a^5 Z(G) = b^4 Z(G) = Z(G), b^{-1} ab Z(G) = a^2 Z(G) \rangle.$$

Observe that

$$C_G(a) = Z(G) \sqcup a Z(G) \sqcup a^2 Z(G) \sqcup a^3 Z(G) \sqcup a^4 Z(G),$$
$$C_G(ab) = Z(G) \sqcup ab Z(G) \sqcup a^4 b^2 Z(G) \sqcup a^3 b^3 Z(G),$$
$$C_G(a^2 b) = Z(G) \sqcup a^2 b Z(G) \sqcup a^3 b^2 Z(G) \sqcup a^4 b Z(G),$$
$$C_G(a^2 b^2) = Z(G) \sqcup a^2 b^2 Z(G) \sqcup a^4 b^2 Z(G) \sqcup a^2 b^3 Z(G),$$
$$C_G(a^2 b^3) = Z(G) \sqcup a^2 b^3 Z(G) \sqcup a^4 b^3 Z(G) \sqcup a^2 b^4 Z(G),$$
$$C_G(b) = Z(G) \sqcup b Z(G) \sqcup b^2 Z(G) \sqcup b^3 Z(G) \text{ and}$$
$$C_G(a^3 b) = Z(G) \sqcup a^3 b Z(G) \sqcup a^2 b^2 Z(G) \sqcup a^4 b^3 Z(G)$$

are the only centralizers of non-central elements of $G$. Also note that these centralizers are abelian subgroups of $G$. Thus $G$ is an AC-group. By Lemma 2.1 we have

$$\Gamma_G = K_{4|Z(G)|} \sqcup 5 K_{3|Z(G)|}$$

since $|C_G(a)| = 5|Z(G)|$ and

$$|C_G(ab)| = |C_G(a^2 b)| = |C_G(a^2 b^2)| = |C_G(b)| = |C_G(a^3 b)| = 4|Z(G)|.$$

Therefore, by (2.1), the result follows. \qed
Proposition 2.8. Let $F = GF(2^n), n \geq 2$ and $\vartheta$ be the Frobenius automorphism of $F$, i.e., $\vartheta(x) = x^2$ for all $x \in F$. Then the spectrum of the commuting graph of the group

$$A(n, \vartheta) = \left\{ U(a, b) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & \vartheta(a) & 1 \end{bmatrix} : a, b \in F \right\}.$$ 

under matrix multiplication given by $U(a, b)U(a', b') = U(a + a', b + b' + a'\vartheta(a))$ is

$$\Gamma_{A(n, \vartheta)} = \{(-1)^{(2^n-1)^2}, (2^n - 1)^{2^n-1}\}.$$

Proof. Note that $Z(A(n, \vartheta)) = \{U(0, b) : b \in F\}$ and so $|Z(A(n, \vartheta))| = 2^n - 1$. Let $U(a, b)$ be a non-central element of $A(n, \vartheta)$. It can be seen that the centralizer of $U(a, b)$ in $A(n, \vartheta)$ is $Z(A(n, \vartheta)) \sqcup U(a, 0)Z(A(n, \vartheta))$. Clearly $A(n, \vartheta)$ is an AC-group and so by Lemma 2.1 we have $\Gamma_{A(n, \vartheta)} = (2^n - 1)K_{2^n}$. Hence the result follows from (2.1). \qed

Proposition 2.9. Let $F = GF(p^n), p$ be a prime. Then the spectrum of the commuting graph of the group

$$A(n, p) = \left\{ V(a, b, c) = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix} : a, b, c \in F \right\}.$$ 

under matrix multiplication $V(a, b, c)V(a', b', c') = V(a + a', b + b' + ca', c + c')$ is

$$\Gamma_{A(n, p)} = \{(-1)^{p^{3n-2p^n-1}}, (p^{2n} - p^n - 1)^{p^n+1}\}.$$

Proof. We have $Z(A(n, p)) = \{V(0, b, 0) : b \in F\}$ and so $|Z(A(n, p))| = p^n$. The centralizers of non-central elements of $A(n, p)$ are given by

(i) If $b, c \in F$ and $c \neq 0$ then the centralizer of $V(0, b, c)$ in $A(n, p)$ is $\{V(0, b', c') : b', c' \in F\}$ having order $|p^{2n}|$.

(ii) If $a, b \in F$ and $a \neq 0$ then the centralizer of $V(a, b, 0)$ in $A(n, p)$ is $\{V(a', b', 0) : a', b' \in F\}$ having order $|p^{2n}|$.

(iii) If $a, b, c \in F$ and $a \neq 0, c \neq 0$ then the centralizer of $V(a, b, c)$ in $A(n, p)$ is $\{V(a', b', ca'a^{-1}) : a', b' \in F\}$ having order $|p^{2n}|$.

It can be seen that all the centralizers of non-central elements of $A(n, p)$ are abelian. Hence $A(n, p)$ is an AC-group and so

$$\Gamma_{A(n, p)} = K_{p^{2n} - p^n} \sqcup K_{p^{2n} - p^n} \sqcup (p^n - 1)K_{p^{2n} - p^n} = (p^n + 1)K_{p^{2n} - p^n}.$$ 

Hence the result follows from (2.1). \qed
We would like to mention here that the groups considered in Proposition 2.8-2.9 are constructed by Hanaki (see [8]). These groups are also considered in [4], in order to compute their numbers of distinct centralizers.

3 Some applications

In this section, we show that the commuting graph of a finite non-abelian group $G$ is integral if $G$ is not isomorphic to $S_4$ and the commuting graph of $G$ is planar. We also show that the commuting graph of a finite non-abelian group $G$ is integral if the commuting graph of $G$ is toroidal. We shall use the following results.

**Theorem 3.1.** Let $G$ be a finite group such that $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where $p$ is a prime integer. Then

$$\text{Spec}(\Gamma_G) = \{(-1)^{p^2-1}|Z(G)|-p-1, (p-1)|Z(G)|-1)^{p+1}\}.$$

**Proof.** The result follows from Theorem 2.2 noting that $G$ is an AC-group with $p+1$ distinct centralizers of non-central elements and all of them have order $p|Z(G)|$. 

**Proposition 3.2.** Let $D_{2m} = \langle a, b : a^m = b^2 = 1, bab^{-1} = a^{-1} \rangle$ be the dihedral group of order $2m$, where $m > 2$. Then

$$\text{Spec}(\Gamma_{D_{2m}}) = \begin{cases} \{(-1)^{m-2}, 0^m, (m-2)^1\} & \text{if } m \text{ is odd} \\ \{(-1)^{3m-3}, 1^{m}, (m-3)^1\} & \text{if } m \text{ is even} \end{cases}.$$

**Proof.** Note that $D_{2m}$ is a non-abelian AC-group. If $m$ is even then $|Z(D_{2m})| = 2$ and $D_{2m}$ has $\frac{m}{2} + 1$ distinct centralizers of non-central elements. Out of these centralizers one has order $m$ and the rests have order $4$. Therefore $\Gamma_{D_{2m}} = K_{m-2} \sqcup \frac{m}{2}K_2$. If $m$ is odd then $|Z(D_{2m})| = 1$ and $D_{2m}$ has $m + 1$ distinct centralizers of non-central elements. In this case, one centralizer has order $m$ and the rests have order $2$. Therefore $\Gamma_{D_{2m}} = K_{m-1} \sqcup mK_1$. Hence the result follows from (2.1).

**Proposition 3.3.** The spectrum of the commuting graph of the generalized quaternion group $Q_{4n} = \langle x, y : y^{2n} = 1, x^2 = y^n, yxy^{-1} = y^{-1} \rangle$, where $n \geq 2$, is given by

$$\text{Spec}(\Gamma_{Q_{4n}}) = \{(-1)^{3n-3}, 1^n, (2n-3)^1\}.$$

**Proof.** Note that $Q_{4n}$ is a non-abelian AC-group with $n + 1$ distinct centralizers of non-central elements. Out of these centralizers one has order $2n$ and the rests have order $4$. Also $|Z(Q_{4n})| = 2$. Therefore $\Gamma_{Q_{4n}} = K_{2n-2} \sqcup nK_2$. Hence the result follows from (2.1).
As an application of Theorem 3.1 we have the following lemma.

**Lemma 3.4.** Let $G$ be a group isomorphic to any of the following groups

1. $\mathbb{Z}_2 \times D_8$
2. $\mathbb{Z}_2 \times Q_8$
3. $M_{16} = \langle a, b : a^8 = b^2 = 1, bab = a^5 \rangle$
4. $\mathbb{Z}_4 \rtimes \mathbb{Z}_4 = \langle a, b : a^4 = b^4 = 1, bab^{-1} = a^{-1} \rangle$
5. $D_8 \ast \mathbb{Z}_4 = \langle a, b, c : a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = a^2 cb \rangle$
6. $SG(16, 3) = \langle a, b : a^4 = b^4 = 1, ab = b^{-1}a^{-1}, ab^{-1} = ba^{-1} \rangle$.

Then $\text{Spec}(\Gamma_G) = \{(-1)^9, 3^3\}$.

**Proof.** If $G$ is isomorphic to any of the above listed groups, then $|G| = 16$ and $|Z(G)| = 4$. Therefore, $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus the result follows from Theorem 3.1.

The next lemma is also useful in this section.

**Lemma 3.5.** Let $G$ be a non-abelian group of order $pq$, where $p$ and $q$ are primes with $p \mid (q - 1)$. Then

$$\text{Spec}(\Gamma_G) = \{(-1)^{pq-q-1}, (p-2)^q, (q-2)^1\}.$$  

**Proof.** It is easy to see that $|Z(G)| = 1$ and $G$ is an AC-group. Also the centralizers of non-central elements of $G$ are precisely the Sylow subgroups of $G$. The number of Sylow $q$-subgroups and Sylow $p$-subgroups of $G$ are one and $q$ respectively. Therefore, by Lemma 2.1 we have $\Gamma_G = K_{q-1} \sqcup qK_{p-1}$. Hence, the result follows from (2.1).

Now we state and proof the main results of this section.

**Theorem 3.6.** Let $\Gamma_G$ be the commuting graph of a finite non-abelian group $G$. If $G$ is not isomorphic to $S_4$ and $\Gamma_G$ is planar then $\Gamma_G$ is integral.

**Proof.** By Theorem 2.2 of [2] we have that $\Gamma_G$ is planar if and only if $G$ is isomorphic to either $D_6, D_8, D_{10}, D_{12}, Q_8, Q_{12}, \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, M_{16}, \mathbb{Z}_4 \rtimes \mathbb{Z}_4, D_8 \ast \mathbb{Z}_4, SG(16, 3), A_4, A_5, S_4, SL(2, 3)$ or $Sz(2) = \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^3 \rangle$.

If $G \cong D_6, D_8, D_{10}$ or $D_{12}$ then by Proposition 3.2, one may conclude that $\Gamma_G$ is integral. If $G \cong Q_8$ or $Q_{12}$ then by Proposition 3.3 $\Gamma_G$ becomes integral. If
\[ G \cong \mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times Q_8, M_{16}, \mathbb{Z}_4 \times \mathbb{Z}_4, D_8 \ast \mathbb{Z}_4 \text{ or } SG(16, 3) \] then by Lemma 3.4 \( \Gamma_G \) becomes integral.

If \( G \cong A_4 = \langle a, b : a^2 = b^3 = (ab)^3 = 1 \rangle \) then the distinct centralizers of non-central elements of \( G \) are \( C_G(a) = \{1, a, bab^2, b^2ab\}, C_G(b) = \{1, b, b^2\}, C_G(ab) = \{1, ab, b^2a\}, C_G(ba) = \{1, ba, ab^2\} \) and \( C_G(aba) = \{1, aba, bab\} \). Note that these centralizers are abelian subgroups of \( G \). Therefore, \( \Gamma_G = K_3 \sqcup 4K_2 \) and

\[
\text{Spec}(\Gamma_G) = \{(-1)^6, 2^1, 1^4\}.
\]

Thus \( \Gamma_G \) is integral.

If \( G \cong Sz(2) \) then by Theorem 2.7 we have

\[
\Gamma_G = \{(-1)^3, (3)^1, (2)^5\}.
\]

Hence, \( \Gamma_G \) is integral.

If \( G \) is isomorphic to \( SL(2, 3) = \langle a, b, c : a^3 = b^4 = 1, b^2 = c^2, c^{-1}bc = b^{-1}, a^{-1}ba = b^{-1}c^{-1}, a^{-1}ca = b^{-1} \rangle \)

then \( Z(G) = \{1, b^2\} \). It can be seen that

\[
\begin{align*}
C_G(b) & = \{1, b, b^2, b^3\} = \langle b \rangle, \\
C_G(c) & = \{1, c, c^2, c^3\} = \langle c \rangle, \\
C_G(bc) & = \{1, b^2, bc, cb\} = \langle bc \rangle, \\
C_G(a^2b^2) & = \{1, b^2, a, a^2b^2, ab^2\} = \langle a^2b^2 \rangle, \\
C_G(ac) & = \{1, b^2, ac, ca^2, a^2bc, ab^2c\} = \langle ac \rangle, \\
C_G(ca) & = \{1, b^2, ca, a^2c, ba^2, ab\} = \langle ca \rangle \quad \text{and} \\
C_G(a^2b) & = \{1, b^2, a^2b, ba, b^3a, (ba)^2\} = \langle a^2b \rangle
\end{align*}
\]

are the only distinct centralizers of non-central elements of \( G \). Note that these centralizers are abelian subgroups of \( G \). Therefore, \( \Gamma_G = 3K_2 \sqcup 4K_4 \) and

\[
\text{Spec}(\Gamma_G) = \{(-1)^{15}, 1^3, 3^4\}.
\]

Thus \( \Gamma_G \) is integral.

If \( G \cong A_5 \) then by Proposition 2.5 we have

\[
\text{Spec}(\Gamma_G) = \{(-1)^{38}, 1^{10}, 2^5, 3^6\}
\]

noting that \( PSL(2, 4) \cong A_5 \). Thus \( \Gamma_G \) is integral.
Finally, if $G \cong S_4$ then it can be seen that the characteristic polynomial of $\Gamma_G$ is $(x - 1)^7(x + 1)^{10}(x^2 - 5)(x^2 - 3x - 2)$ and so

$$\text{Spec}(\Gamma_G) = \left\{ 1^7, (-1)^{10}, (\sqrt{5})^2, (-\sqrt{5})^2, \left(\frac{3 + \sqrt{17}}{2}\right)^1, \left(\frac{3 - \sqrt{17}}{2}\right)^1 \right\}. $$

Hence, $\Gamma_G$ is not integral. This completes the proof. 

In [2, Theorem 2.3], Afkhami et al. have classified all finite non-abelian groups whose commuting graphs are toroidal. Unfortunately, the statement of Theorem 2.3 in [2] is printed incorrectly. We list the correct version of [2, Theorem 2.3] below, since we are going to use it.

**Theorem 3.7.** Let $G$ be a finite non-abelian group. Then $\Gamma_G$ is toroidal if and only if $\Gamma_G$ is projective if and only if $G$ is isomorphic to either $D_{14}, D_{16}, Q_{16}, QD_{16}, D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_7 \times \mathbb{Z}_3$.

**Theorem 3.8.** Let $\Gamma_G$ be the commuting graph of a finite non-abelian group $G$. Then $\Gamma_G$ is integral if $\Gamma_G$ is toroidal.

**Proof.** By Theorem 3.7 we have that $\Gamma_G$ is toroidal if and only if $G$ is isomorphic to either $D_{14}, D_{16}, Q_{16}, QD_{16}, D_6 \times \mathbb{Z}_3, A_4 \times \mathbb{Z}_2$ or $\mathbb{Z}_7 \times \mathbb{Z}_3$.

If $G \cong D_{14}$ or $D_{16}$ then by Proposition 3.2 one may conclude that $\Gamma_G$ is integral. If $G \cong Q_{16}$ then by Proposition 3.3 $\Gamma_G$ becomes integral. If $G \cong QD_{16}$ then by Proposition 2.3 $\Gamma_G$ becomes integral. If $G \cong \mathbb{Z}_7 \times \mathbb{Z}_3$ then $\Gamma_G$ is integral, by Lemma 3.5. If $G$ is isomorphic to $D_6 \times \mathbb{Z}_3$ or $A_4 \times \mathbb{Z}_2$ then $\Gamma_G$ becomes integral by Corollary 2.3 since $D_6$ and $A_4$ are AC-groups. This completes the proof.

We shall conclude the paper with the following result.

**Proposition 3.9.** Let $\Gamma_G$ be the commuting graph of a finite non-abelian group $G$. Then $\Gamma_G$ is integral if the complement of $\Gamma_G$ is planar.

**Proof.** If the complement of $\Gamma_G$ is planar then by Proposition 2.3 of [1] we have that $G$ is isomorphic to either $D_6, D_8$ or $Q_8$. If $G \cong D_6$ or $D_8$ then by Proposition 3.2 $\Gamma_G$ is integral. If $G \cong Q_8$ then by Proposition 3.3 $\Gamma_G$ becomes integral. This completes the proof.

\[\square\]
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