Optimal probabilistic estimation of quantum states

Jaromír Fiurášek
Department of Optics, Palacký University, 17. listopadu 50, 772 00 Olomouc, Czech Republic
E-mail: fiurasek@optics.upol.cz

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Abstract. We extend the concept of probabilistic unambiguous discrimination of quantum states to quantum state estimation. We consider a scenario where the measurement device can output either an estimate of the unknown input state or an inconclusive result. We present a general method how to evaluate the maximum fidelity achievable by the probabilistic estimation strategy. We illustrate our method on two explicit examples: estimation of a qudit from a pair of conjugate qudits and phase covariant estimation of a qubit from $N$ copies. We show that by allowing for inconclusive results it is possible to reach estimation fidelity higher than that achievable by the best deterministic estimation strategy.

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1. Introduction

The laws of quantum mechanics impose fundamental bounds on the amount of information that can be extracted from the measurements on quantum states. In particular, it is not possible to exactly determine an unknown quantum state from a single copy. More generally, non-orthogonal quantum states cannot be perfectly deterministically discriminated if only a single copy of the state is available. The optimal estimation and discrimination of quantum states has attracted a lot of attention during recent years. This interest was largely stimulated by the rapid development of quantum information theory. Quantum measurement forms an essential ingredient of practically every protocol for quantum information transmission and processing since it converts the quantum information carried by a quantum system on to classical information.

Two different strategies were proposed in the literature to optimally discriminate among non-orthogonal quantum states. The first option is to minimize the discrimination error, i.e. the probability of a wrong guess [1]–[5]. In this case the measurement device always provides a guess of the state, which sometimes may be wrong. An alternative approach pioneered by Ivanovic [6], Dieks [7] and Peres [8] is the unambiguous probabilistic discrimination which allows for perfect error-free identification of non-orthogonal states at the expense of a fraction of inconclusive results. It was shown that $n$ pure quantum states from a set \( \{ |\psi_j\rangle \}_{j=1}^n \) can be unambiguously discriminated if and only if they are linearly independent [9]–[13]. Recently, the concept of unambiguous discrimination was extended to mixed quantum states [14]–[18]. More generally, it was shown that by allowing for a fraction of inconclusive results it is possible to reduce the probability of a wrong guess even if it is not possible to achieve the perfect error-free discrimination [19].

The optimal quantum state estimation can be thought of as a limiting case of quantum state discrimination among infinitely many states forming a continuous set. The canonical example of the state estimation task is a determination of an unknown state of a qubit from a single copy if a priori the state could lie anywhere on the surface of the Bloch sphere [20]. The similarity of the (mixed) estimated state $\rho_{est}$ with the true pure state $|\psi\rangle$ is quantified by the fidelity $F = \langle \psi | \rho_{est} | \psi \rangle$ and the optimal state estimation strategy is defined as the one which maximizes the average estimation fidelity. During recent years, optimal estimation strategies were established for a wide class of input sets of states including optimal universal estimation of qubits [20, 21] and qudits [22], optimal phase-covariant estimation of qubits [21] and qudits [23] and optimal estimation of coherent states [24]. Also optimal estimation of mixed states [25] has been studied.

In this paper, we generalize the concept of unambiguous state discrimination to state estimation. We consider a scenario where the measuring apparatus can either output an estimate of the state or an inconclusive result. We shall show on explicit examples that with such probabilistic estimation strategy, it is possible to increase the estimation fidelity above that achievable by the optimal deterministic estimation.

The rest of the paper is structured as follows. In section 2, we establish a general formalism for the determination of the optimal probabilistic covariant estimation strategy. We shall show that the maximum attainable fidelity can be evaluated as a maximum eigenvalue of a certain operator. In section 3, we apply the method to the determination of optimal probabilistic estimation of a qudit state $|\psi\rangle$ from a pair of complex conjugate qudits $|\psi\rangle |\psi^*\rangle$. This represents an extension of the well-studied problem of optimal estimation of a qubit from a pair of orthogonal qubits [26, 27] to $d$-dimensional quantum system. Recently, the optimal deterministic estimation of $|\psi\rangle$...
from a single copy of \( |\psi\rangle|\psi^*\rangle \) was addressed by Zhou et al [28] who numerically calculated the maximum achievable fidelity. We shall show that this numerically obtained fidelity is actually the fidelity of optimal probabilistic estimation and we will find a simple analytical formula for it. We will also derive from the first principles the maximum deterministic estimation fidelity \( F_{\text{det}} \) for this case. Remarkably, \( F_{\text{det}} \) turns out to coincide with the fidelity corresponding to the analytically found local extremum point in [28]. As a second example we shall consider in section 4, the optimal phase covariant estimation of a qubit from \( N \) copies. We shall show that for \( N > 2 \), the probabilistic estimation strategy achieves strictly larger fidelity than the deterministic one and we shall compare the asymptotic behaviour of the fidelities for large \( N \). Finally, section 5, contains conclusions and a brief summary of the main results.

2. Optimal probabilistic estimation

Consider a set of pure quantum states \( |\Psi(\psi)\rangle \) which are parameterized by a state \( |\psi\rangle \) and let \( S \) denote the set of all admissible \( |\psi\rangle \). The \textit{a priori} probability distribution of \( |\psi\rangle \) labelled by \( d\psi \) satisfies \( \int_S d\psi = 1 \). Using this notation we can treat in a unified way more complex situations such as the estimation of \( |\psi\rangle \) from \( N \) copies of the state, when \( |\Psi\rangle = |\psi\rangle^\otimes N \). The goal of the quantum state estimation is to determine the state \( |\psi\rangle \) as precisely as possible by performing a generalized quantum measurement described by a positive operator valued measure (POVM) \( \Pi(\phi)d\phi \) on \( |\Psi\rangle \). Here, \( |\phi\rangle \in S \) and the detection of \( \Pi(\phi) \) implies that the state \( |\phi\rangle \) is given as the estimate. The optimal measurement strategy generally depends on the set \( S \) and on the \textit{a priori} probability distribution \( d\psi \).

In the probabilistic estimation, one allows for inconclusive results where the machine does not produce any estimate of the state. This null outcome is associated with a POVM element \( \Pi_0 \) and the whole POVM should satisfy the completeness condition:

\[
\int_S \Pi(\phi) d\phi + \Pi_0 = \mathbb{1},
\]

where \( \mathbb{1} \) is the identity operator on the Hilbert space spanned by the states \( |\Psi(\psi)\rangle \). The success of the estimation procedure can be conveniently quantified by the average fidelity. Consider first a particular input state \( |\psi\rangle \). The normalized fidelity of the estimation of \( |\psi\rangle \) can be expressed as

\[
F(\psi) = \frac{1}{P(\psi)} \int_S \langle \Psi(\psi)|\Pi(\phi)|\Psi(\psi)\rangle |\langle \psi|\phi\rangle|^2 d\phi,
\]

where

\[
P(\psi) = \int_S \langle \Psi(\psi)|\Pi(\phi)|\Psi(\psi)\rangle d\phi,
\]

is the probability of producing an estimate of the state and, consequently, \( 1 - P(\psi) \) is the probability of inconclusive outcome. We choose as a figure of merit the normalized average fidelity,

\[
\bar{F} = \frac{\int_S F(\psi) P(\psi) d\psi}{\int_S P(\psi) d\psi}.
\]
The maximization of $\tilde{F}$ is a complicated task since in general it requires the optimization of infinitely many POVM elements $\Pi(\phi)$. The problem simplifies considerably if the states $|\psi\rangle$ and $|\Psi(\psi)\rangle$ form orbits of some group $G$, $S \equiv G$, and if $d\psi$ is an invariant measure induced by the Haar measure on $G$. In the rest of the paper, we will assume that this is the case. We then have $|\psi\rangle = U(\psi)|0\rangle$ and $|\Psi(\psi)\rangle = V(\psi)|\Psi(0)\rangle$, where $U(\psi)$ and $V(\psi)$ denote unitary representations of the group $G$. With slight abuse of notation we use $\psi$ to label the elements of $G$. It can be shown that due to the underlying group structure the optimal POVM which maximizes $\tilde{F}$ can always be chosen to be covariant and all the POVM elements are generated from a single element,

$$\Pi_C(\phi) = V(\phi)\Pi_C V^\dagger(\phi),$$  

(5)

and $\Pi_C \equiv \Pi_C(0)$.

On inserting the expression (5) into equation (4) we obtain

$$\tilde{F} = \frac{\text{Tr}[R\Pi_C]}{\text{Tr}[A\Pi_C]},$$  

(6)

where

$$R = \int_G |\Psi(\psi)\rangle \langle \Psi(\psi) | |\psi\rangle |\langle \psi|0\rangle|^2 \, d\psi,$$  

(7)

and

$$A = \int_G |\Psi(\psi)\rangle \langle \Psi(\psi) | \, d\psi.$$  

(8)

Note that the expression (6) is formally similar to the formula for the fidelity of the optimal probabilistic completely positive map that approximates some unphysical operation which was derived in [29]. Upon introducing

$$\tilde{\Pi}_C = A^{1/2} \Pi_C A^{1/2},$$  

(9)

we can rewrite equation (6) as

$$\tilde{F} = \frac{\text{Tr}[A^{-1/2}RA^{-1/2}\tilde{\Pi}_C]}{\text{Tr}[\tilde{\Pi}_C]},$$  

(10)

It follows that the fidelity is bounded from above by the maximum eigenvalue $\mu_{\text{max}}$ of the operator $M = A^{-1/2}RA^{-1/2}$. If the Hilbert space spanned by $|\Psi(\psi)\rangle$ is finite dimensional then there exists a POVM which attains the maximum fidelity $F_{\text{max}} = \mu_{\text{max}}$ and produces the estimate of $|\psi\rangle$ with nonzero probability $P > 0$. Let $|\mu_j^{\text{max}}\rangle$, $j = 1, \ldots, J$, be the eigenvectors corresponding to the maximum eigenvalue $\mu_{\text{max}}$. Then the POVM element $\Pi_{C,\text{opt}}$ which generates the optimal covariant POVM can be expressed as

$$\Pi_{C,\text{opt}} = A^{-1/2} \sum_{j,k=1}^J \pi_{jk} |\mu_j^{\text{max}}\rangle \langle \mu_k^{\text{max}} | A^{-1/2}.$$  

(11)
The coefficients $\pi_{jk}$ must be chosen such that $\Pi_{C, \text{opt}} \geq 0$, $\Pi_{C, \text{opt}}^\dagger = \Pi_{C, \text{opt}}$ and
\[
\int_S V(\psi) \Pi_{C, \text{opt}} V^\dagger(\psi) \, d\psi \leq \mathbb{1}. \tag{12}
\]
The coefficients $\pi_{jk}$ may be optimized such as to maximize the average probability of success $P$ under the constraints $\Pi_{C, \text{opt}} \geq 0$ and (12). This is an instance of a semidefinite program which is a convex optimization problem that can be very efficiently solved numerically [29, 30].

If the maximum eigenvalue is non-degenerate, $J = 1$, then the optimal POVM element $\Pi_{C, \text{opt}}$ is proportional to a rank-one projector $\Pi_{C, \text{opt}} = \frac{1}{N} A^{-1/2} |\mu_{\text{max}}\rangle\langle \mu_{\text{max}}| A^{-1/2}$, \tag{13}
and the normalization constant $N$ has to be chosen such that (12) holds.

### 3. Pair of conjugate qudits

#### 3.1. Optimal probabilistic estimation

In this section, we will investigate the optimal probabilistic estimation of a state of a single qudit $|\psi\rangle$ from a pair of conjugate qudits, $|\Psi(\psi)\rangle = |\psi\rangle |\psi\rangle^\ast$. The a priori distribution $d\psi$ is assumed to be induced by the Haar measure on the group $\text{SU}(d)$. This scenario is a generalization of the estimation of a qubit from a pair of orthogonal qubits [26, 27] to dimensions $d > 2$. The corresponding operators $A$ and $R$ can be easily evaluated with the help of the Schur Lemma. The unitary representation $U^{\otimes N}$ of the group $\text{SU}(d)$ acts irreducibly on the totally symmetric subspace of $N$ qudits. Taking into account that $|\psi\rangle\langle \psi| = (|\psi\rangle\langle \psi|)^T$ and exchanging the order of integration and (partial) transposition we obtain
\[
A = \frac{1}{d(d+1)} (\mathbb{1} + d\Phi^+), \tag{14}
\]
where $\Phi^+ = |\Phi^+\rangle\langle \Phi^+|$ and
\[
|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle |j\rangle \tag{15}
\]
is a maximally entangled state of two qudits. We also find that
\[
R = \frac{1}{D_1(d)} \text{Tr}_3[ |\Pi_{123}^\dagger \rangle\langle 0_1| \otimes |0_2\rangle\langle 0_3| ] \tag{16}
\]
Here, $\Pi_{123}^\dagger$ is the projector on to the symmetric subspace of three qudits, $D_1(d) = \frac{1}{d} d(d+1)$ $(d+2)$ is the dimension of this subspace, $T_2$ denotes partial transposition with respect to the second qudit, and $\text{Tr}_3$ stands for the partial trace over the third qudit. After some algebra, we find that $R$ can be expressed as
\[
R = \frac{1}{d(d+1)(d+2)} [ \mathbb{1}_{12} + d\Phi^+ + \sqrt{d}|\Phi^+\rangle\langle 0_1| + \sqrt{d}|0_1\rangle\langle \Phi^+| + \mathbb{1}_1 \otimes |0_2\rangle\langle 0_1| + |0_1\rangle\langle 0_2| \otimes \mathbb{1}_2 ] \tag{17}
\]
Instead of calculating the eigenvalues of $A^{-1/2}RA^{-1/2}$, we can equivalently look for the eigenvalues of $RA^{-1}$ because the eigenvalues of these two operators coincide and the latter is easier to deal with. The operator $A$ can be easily inverted,

$$A^{-1} = d(d + 1) \left[ \mathbb{1} - \frac{d}{d+1} \Phi^* \right], \quad \text{(18)}$$

and we arrive at

$$RA^{-1} = \frac{1}{d+2} \left[ \mathbb{1}_{12} - \frac{d}{d+1} \Phi^* + \sqrt{d}\Phi^* \langle 00 \rangle - \frac{\sqrt{d}}{d+1} \langle 00 \rangle |00\rangle + \mathbb{1}_1 \otimes |0\rangle_2 + |0\rangle_1 \otimes \mathbb{1}_2 \right]. \quad \text{(19)}$$

This operator possesses only four different eigenvalues which can be expressed analytically for arbitrary $d$. The eigenvalue $\mu_1 = 1/(d+2)$ is $d(d-2)$-fold degenerate and the eigenstates read $|j\rangle_1 |k\rangle_2$, where $j \neq 0, k \neq 0$. The second eigenvalue $\mu_2 = 2/(d+2)$ is $(2d-2)$-fold degenerate with eigenstates $|0\rangle_1 |j\rangle_2$ and $|j\rangle_1 |0\rangle_2$, $j \neq 0$. Finally, the last two eigenvalues are non-degenerate and can be expressed as

$$\mu_{3,4} = 2 \frac{d}{d+2} \left[ 1 \pm \sqrt{\frac{d}{2(d+1)}} \right]. \quad \text{(20)}$$

The maximum eigenvalue is $\mu_3$ for all $d \geq 2$ and the fidelity of the optimal probabilistic estimation of $|\psi\rangle$ from a pair of conjugate qudits is equal to this eigenvalue,

$$F_{\text{max,prob}} = \frac{2}{d+2} \left[ 1 + \sqrt{\frac{d}{2(d+1)}} \right]. \quad \text{(21)}$$

The numerical values for the optimal estimation fidelity $F_{\perp}$ obtained by Zhou et al [28] fully agree with the above analytical formula so their global optimization actually yielded the optimal probabilistic estimation strategy.

The optimal probabilistic covariant POVM is generated by the POVM element $\Pi_{C,\text{opt}} = |\pi_{C,\text{opt}}\rangle \langle \pi_{C,\text{opt}}|$, where

$$|\pi_{C,\text{opt}}\rangle \propto |00\rangle - \sqrt{\frac{2d}{d+1}} \frac{\sqrt{2(d+1)} - \sqrt{d}}{d+2} |\Phi^*\rangle. \quad \text{(22)}$$

3.2. Optimal deterministic estimation strategy

For $d > 2$, the optimal estimation strategy obtained above cannot be made deterministic and there is a nonzero probability of inconclusive results. Thus a question arises what is the optimal deterministic strategy of estimation of $|\psi\rangle$ from a single copy of the state $|\psi\rangle|\psi^*\rangle$. When seeking an answer to this question, we can restrict ourselves to the covariant POVMs. Since the probability
of inconclusive results should vanish, we have $\Pi_0 = 0$ and the completeness condition for the POVM becomes

$$\int_S U(\psi) \otimes U^*(\psi) \Pi_C U^T(\psi) \, d\psi = \mathbb{I}.$$  \hspace{1cm} (23)

Recall that $U(\psi)$ is a unitary operation acting on the Hilbert space of a single qudit and $U(\psi)|0\rangle = |\psi\rangle$. We should maximize the estimation fidelity $F = \text{Tr}[\Pi_C R]$ under the above completeness condition. In the present case, this is equivalent to maximizing $F$ under simpler constraints that can be obtained from (23). In particular, by calculating the trace of equation (23) and by taking into account the invariance $U \otimes U^* |\Phi_+\rangle = |\Phi_+\rangle$ we find that

$$\text{Tr}[\Pi_C] = d^2, \quad \text{Tr}[\Pi_C |\Phi_+\rangle\langle\Phi_+|] = 1,$$  \hspace{1cm} (24)

must hold.

The constraints (24) can be accounted for by introducing two Lagrange multipliers $\lambda_1$ and $\lambda_2$ and our task is to maximize

$$\mathcal{F}[\Pi_C] = \text{Tr}[R\Pi_C] - \lambda_1 \text{Tr}[\Pi_C] - \lambda_2 \text{Tr}[|\Phi_+\rangle\langle\Phi_+| \Pi_C],$$  \hspace{1cm} (25)

under the constraints (24) and $\Pi_C \succeq 0$. This is an instance of a semidefinite program [30]. For this class of convex optimization problems, one can straightforwardly derive the extremal equation for the optimal $\Pi_C$ and we obtain

$$(R - \lambda_1 \mathbb{I} - \lambda_2 |\Phi_+\rangle\langle\Phi_+|) \Pi_C = 0.$$  \hspace{1cm} (26)

Moreover, we also find the optimality condition,

$$\lambda_1 \mathbb{I} + \lambda_2 |\Phi_+\rangle\langle\Phi_+| - R \succeq 0.$$  \hspace{1cm} (27)

If (26) and (27) hold simultaneously, then $\Pi_C$ is the optimal one which maximizes the fidelity. To prove this statement, we take the trace of a product of equation (27) with an arbitrary $\tilde{\Pi}_C$ which satisfies all the constraints imposed on it. We get $\text{Tr}[\tilde{\Pi}_C R] \leq \lambda_1 d^2 + \lambda_2$ hence the Lagrange multipliers provide an upper bound on the achievable fidelity which is saturated if the POVM satisfies equation (26).

The optimal POVM has qualitatively similar structure as the optimal probabilistic POVM (22), namely, $\Pi_{C,\det} = |\pi_{C,\det}\rangle\langle\pi_{C,\det}|$ where

$$|\pi_{C,\det}\rangle = \sqrt{d} d(1 - \sqrt{\frac{1}{d} + 1})|00\rangle - (\sqrt{d} + 1 - 1) |\Phi_+\rangle.$$  \hspace{1cm} (28)

By construction, this POVM satisfies the completeness condition (23). On inserting $\Pi_{C,\det}$ into equation (26), we can solve for the Lagrange multipliers and we get

$$\lambda_1 = \frac{4 - \left(1 - \sqrt{\frac{1}{d} + 1}\right) \left(1 + \frac{2}{d}\right)}{d(d+1)(d+2)}, \quad \lambda_2 = \frac{1}{d(d+1)(d+2)} \left[\frac{d^3 + 2d^2 - 2d - 4}{d\sqrt{d+1}} + \frac{4}{d} \right].$$  \hspace{1cm} (29)
This choice guarantees that equation (26) holds for any $d$. To prove the optimality of the POVM (28), it remains to check the inequality (27). Since $\lambda_1 > 2/[d(d + 1)(d + 2)]$ the only nontrivial part is the verification of the positive semidefiniteness of the operator in the two-dimensional subspace spanned by $|00\rangle$, $|\Phi^+\rangle$. Let $\Pi_2$ denote the projector on to this subspace and consider the $2 \times 2$ matrix $K = \Pi_2(\lambda_1 \mathbb{I} + \lambda_2 \Phi^* - R) \Pi_2$. One eigenvalue of $K$ is zero due to the optimality condition (26). To prove that $K \geq 0$, it thus suffices to show that $\text{Tr} K \geq 0$ and after some algebra we find

$$\text{Tr} K = 2\lambda_1 + \lambda_2 - \frac{d + 6}{d(d + 1)(d + 2)}. \quad (30)$$

It can be shown that $\text{Tr} K$ is a growing function of $d$ and that it is positive for all integer $d \geq 2$. This concludes the optimality proof.

The fidelity of the optimal deterministic estimation corresponding to the optimal covariant POVM reads

$$F_{\text{max, det}} = \frac{1}{d^2(d + 2)} \left[ 3d^2 - 4d + 4 + \frac{2d^2 + 2d - 4}{\sqrt{d + 1}} \right]. \quad (31)$$

This expression agrees with the formula given by Zhou et al [28]. In that paper, the authors claimed that this is only a local maximum of the fidelity and they calculated the global maximum of the fidelity numerically. Our findings provide a precise interpretation of their results. The local maximum is in fact the maximum achievable fidelity of deterministic estimation from a pair of conjugate qudits while the global maximum given in [28] corresponds to the optimal probabilistic estimation strategy which allows for inconclusive results.

It is instructive to explicitly evaluate the probability that the machine outputs an estimate $|0\rangle$ for an input state $|\psi\psi^*\rangle$. We have

$$P_{\perp}(0|\psi) = \frac{1}{d} |d\sqrt{d + 1}|\langle\psi|0\rangle|^2 - \sqrt{d + 1} + 1|^2. \quad (32)$$

Note that this probability is zero if the overlap of the true state $|\psi\rangle$ with the estimated state $|0\rangle$ is equal to $|\langle\psi|0\rangle|^2 = (\sqrt{d + 1} - 1)/(d\sqrt{d + 1})$. It is interesting to compare this with the optimal estimation from a pair of identical qudits $|\psi\psi\rangle$, where the optimal covariant POVM is generated by $|\pi_{C,\perp}\rangle = \sqrt{d(d + 1)}/2|00\rangle$ and the corresponding probability of guessing $|0\rangle$ for the input state $|\psi\rangle$ reads $P_{\perp}(0|\psi) = \frac{1}{d}(d + 1)|\langle\psi|0\rangle|^2$ which vanishes only if the state $|\psi\rangle$ is orthogonal to $|0\rangle$. In particular, for $d = 2$ the probability $P_{\perp} = 0$ only if $|\psi\rangle = |1\rangle$, while $P_{\perp}$ is zero for all states on a certain circle of the Bloch sphere. This observation gives some more insight into why the state $|\psi\rangle$ can be estimated with higher precision from $|\psi\psi^*\rangle$ than from $|\psi\psi\rangle$.

4. Optimal probabilistic estimation of equatorial qubits

In this section, we will investigate the optimal phase-covariant probabilistic estimation of a qubit. We shall assume that it is a priori known that the qubit state is located on the equator of the Poincare sphere, $|\psi(\varphi)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\varphi}|1\rangle)$. The state is thus characterized by a single parameter—the relative phase $\varphi$. Starting from the seed state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ all states $|\psi(\varphi)\rangle$ can be obtained as the orbit of the Abelian group $\text{U}(1)$ which generates rotations of the Bloch sphere.
about z-axis. To make our treatment general, we will consider optimal estimation of $|\psi(\varphi)\rangle$ from $N$ input copies.

This scenario corresponds to a typical phase-shift measurement, where $N$ particles pass through an interferometer which applies an unknown relative phase shift $\varphi$ to one of the states of the particle and the goal is to determine $\varphi$ as precisely as possible. The optimal phase-estimation strategies which reach the so-called Heisenberg limit $\Delta \varphi \approx 1/N$ require entangled input states of $N$ particles, see e.g. [31]. Here, we show that even for the product state $|\psi(\varphi)\rangle \otimes N$ it is possible to probabilistically improve the precision of $\varphi$ estimation, as witnessed by the improved asymptotic scaling of the optimal fidelity $1 - F_{\text{max,prob}} \propto 1/N^2$. However, it should be noted that this apparent improvement is achieved only for the sub-ensemble of conclusive measurement outcomes, while the inconclusive outcomes are neglected.

The input state $|\psi(\varphi)\rangle \otimes N$ belongs to the $N + 1$ dimensional symmetric (bosonic) subspace of the Hilbert space of $N$ qubits and it can be written as follows,

$$
|\psi(\varphi)\rangle \otimes N = \frac{1}{2^{N/2}} \sum_{k=0}^{N} \sqrt{\binom{N}{k}} e^{ik\varphi} |N; k\rangle.
$$

(33)

Here, $|N; k\rangle$ denotes a normalized fully symmetric state of $N$ qubits with $k$ qubits in state $|1\rangle$ and $N - k$ qubits in state $|0\rangle$.

The operators $A$ and $R$ can be determined from the formulae (8) and (7), where we have to integrate over the phase shift $\varphi$, $\int d\psi = \int_{0}^{2\pi} \frac{1}{2\pi} d\varphi$. After the integration we obtain

$$
A = \frac{1}{2^N} \sum_{k=0}^{N} \binom{N}{k} |N; k\rangle \langle N; k|,
$$

(34)

and

$$
R = \frac{1}{2^{N+1}} \sum_{k=0}^{N} \binom{N}{k} |N; k\rangle \langle N; k| + \frac{1}{2^{N+2}} \sum_{k=1}^{N} \sqrt{\binom{N}{k} \binom{N}{k-1}} (X_k + X_k^\dagger),
$$

(35)

where $X_k = |N; k\rangle \langle N; k - 1|$. Since the operator $A$ is diagonal in the basis $|N; k\rangle$, the operator $M$ whose maximum eigenvalue determines the maximum achievable estimation fidelity can be easily calculated and we have

$$
M = A^{-1/2} R A^{-1/2} = \frac{1}{2} \mathbb{1} + \frac{1}{4} \sum_{k=1}^{N} (X_k + X_k^\dagger).
$$

(36)

Instead of directly working with $M$ let us consider the operator $\tilde{M} = 4M - 2\mathbb{1}$ and the eigenvalues $\mu_j$ of $M$ are then related to the eigenvalues $\tilde{\mu}_j$ of $\tilde{M}$ by $\mu_j = (\tilde{\mu}_j + 2)/4$. The matrix $\tilde{M}$ is tridiagonal and its characteristic polynomial is given by the Tchebychev polynomial of the second kind, $\det(\tilde{M} - \lambda \mathbb{1}) = U_{N+1}(-\frac{1}{2})$. Maximum eigenvalue of $\tilde{M}$ is thus given by the largest root of $U_N$ and with the help of the definition $U_N(\cos \theta) = \sin [(N + 1)\theta]/\sin \theta$, we arrive at [32]

$$
\tilde{\mu}_{\text{max}} = 2 \cos \left( \frac{\pi}{N + 2} \right).
$$

(37)
The maximum fidelity can be determined as $(\tilde{\mu}_{\text{max}} + 2)/4$ and we finally obtain

$$ F_{\text{max, prob}} = \frac{1}{2} \left[ 1 + \cos \left( \frac{\pi}{N + 2} \right) \right]. $$

(38)

It is instructive to compare the fidelity $F_{\text{max, prob}}$ with the fidelity of the optimal deterministic phase covariant estimation of a qubit from $N$ copies [21],

$$ F_{\text{max, det}} = \frac{1}{2} + \frac{1}{2^{N+1}} \sum_{k=1}^{N} \sqrt{\binom{N}{k} \binom{N}{k-1}}. $$

(39)

It follows that for $N = 1$ and $N = 2$ $F_{\text{max, prob}} = F_{\text{max, det}}$, hence it is not possible to improve the fidelity of estimation by allowing for some fraction of inconclusive results. However, if $N \geq 3$ then $F_{\text{max, prob}} > F_{\text{max, det}}$ and the optimal probabilistic estimation strategy attains a strictly larger fidelity than the optimal deterministic strategy. This is illustrated in figure 1 which shows the difference between the fidelities (38) and (39).

Since the maximum eigenvalue of $M$ is non-degenerate, the optimal covariant POVM is generated by the rank-one projector (13). The maximization of the success probability $P = \text{Tr}[\Pi_{C, \text{opt}} A]$ is equivalent to the minimization of $\mathcal{N}$ under the constraint (12). Let $|\mu_{\text{max}}\rangle = \sum_{k=0}^{N} c_k |N; k\rangle$ be the normalized eigenvector of $M$ with the eigenvalue $\mu_{\text{max}}$. Then it is optimal to choose

$$ \mathcal{N} = 2^N \max_k \binom{N}{k}^{-1} |c_k|^2. $$

(40)

Numerical calculation reveals that $P$ decreases exponentially with growing $N$, see figure 2.
To further facilitate the comparison of the fidelities, let us analyse their asymptotic behaviour for large $N$. For the probabilistic estimation strategy we obtain

$$F_{\text{max, prob}} \approx 1 - \frac{\pi^2}{4} \frac{1}{(N + 2)^2},$$

and we can write $1 - F_{\text{max, prob}} = O(N^{-2})$. On the other hand, for the deterministic estimation, we find that $1 - F_{\text{max, det}} \approx O(N^{-1})$. We can see that with growing $N$ $F_{\text{max, prob}}$ converges to unity much faster than $F_{\text{max, det}}$. This superior scaling is achieved at the expense of a decreasing probability of successful estimation $P$.

### 5. Conclusions

In the present paper, we have generalized the concept of unambiguous quantum state discrimination to the realm of quantum state estimation. We have shown that by allowing for inconclusive results, it is possible to increase the fidelity of estimation evaluated for the sub-ensemble of conclusive outcomes of the estimation process. We have established a general formula for the maximum fidelity achievable by the probabilistic state estimation. The method was illustrated on two explicit examples. First, we have studied the optimal estimation of a qudit from a pair of conjugate qudits and we have provided an exact interpretation of the results recently obtained by Zhou et al [28]. As a second example, we have investigated the phase covariant estimation of a qubit from $N$ copies of the state. The present quantum-state estimation scheme could find applications in quantum communication and it may potentially help to probabilistically improve the sensitivity and precision of measurements performed at the quantum limit.
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