Close-to-optimal continuity bound for the von Neumann entropy and other quasi-classical applications of the Alicki-Fannes-Winter technique

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Abstract

We consider a quasi-classical version of the Alicki-Fannes-Winter technique widely used for quantitative continuity analysis of characteristics of quantum systems and channels. This version allows us to obtain continuity bounds under constraints of different types for quantum states belonging to subsets of a special form that can be called "quasi-classical".

Several applications of the proposed method are described. Among others, we obtain the universal continuity bound for the von Neumann entropy under the energy-type constraint which in the case of one-mode quantum oscillator is close to the specialized optimal continuity bound presented recently by Becker, Datta and Jabbour.

We obtain semi-continuity bounds for the quantum conditional entropy of quantum-classical states and for the entanglement of formation in bipartite quantum systems with the rank/energy constraint imposed only on one state. Semi-continuity bounds for entropic characteristics of classical random variables and classical states of a multi-mode quantum oscillator are also obtained.

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1 Introduction

Quantitative continuity analysis of characteristics of quantum systems and channels is an important technical task which is necessary for conducting research in various directions of quantum information theory [17, 41]. It suffices to mention the role of the Fannes continuity bound for the quantum entropy and the Alicki-Fannes continuity bound for the quantum conditional entropy in study of different questions that arose when exploring the information abilities of quantum systems and channels.

One of the universal methods of quantitative continuity analysis is the Alicki-Fannes-Winter technique. The first version of this technique was used by Alicki and Fannes to obtain a continuity bound for the quantum conditional entropy in finite-dimensional quantum systems [2]. Then this technique was analysed and improved by different authors [37, 28], its optimal form was proposed by Winter, who applied it to get tight continuity bounds for the quantum conditional entropy and for the bipartite relative entropy of entanglement [43]. In a full generality this technique is described in Section 3 in [35], its new development is proposed in the recent article [11].

In this article we consider a quasi-classical version of the Alicki-Fannes-Winter technique. It allows us to obtain continuity bounds under constraints of different types for quantum states belonging to subsets of a special form that can be called "quasi-classical".

Two basic examples of "quasi-classical" sets of quantum states used in this article are the following:

- the set of all quantum states diagonalizable in some orthonormal basis;
- the set of classical states of a multi-mode quantum oscillator.

We describe several applications of the proposed method. In particular, it allows us to obtain universal continuity bound and semi-continuity bound for the von Neumann entropy under the energy-type constraint which improve the continuity bounds proposed by Winter in [43]. In the case of one-mode quantum oscillator the obtained continuity bound for the von Neumann entropy is close to the specialized optimal continuity bound presented by Becker, Datta and Jabbour in [6].

We also obtain continuity bounds and semi-continuity bounds for the quantum conditional entropy of quantum-classical states under different constraints, which allow...
us to derive (via the standard Nielsen-Winter technique) continuity bounds and semi-continuity bounds for the entanglement of formation in bipartite quantum systems. In particular, we obtain a continuity bound for the entanglement of formation under the energy-type constraint which is essentially sharper than the previously proposed continuity bounds.

The proposed method allows us to significantly refine continuity bounds for characteristics of composite infinite-dimensional quantum systems obtained in [33, 35] by restricting attention to commuting states of these systems. It also allows us to refine continuity bounds for characteristics of a multi-mode quantum oscillator by restricting attention to its classical states.

Other applications considered in the article are continuity bounds and semi-continuity bounds for entropic characteristics of discrete random variables under constrains of different forms.

It is essential that the proposed method applied to nonnegative functions gives semi-continuity bounds for quasi-classical states with rank/energy constraint imposed only on one state.

2 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space, $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$ with the operator norm $\| \cdot \|$ and $\mathfrak{T}(\mathcal{H})$ the Banach space of all trace-class operators on $\mathcal{H}$ with the trace norm $\| \cdot \|_1$. Let $\mathfrak{S}(\mathcal{H})$ be the set of quantum states (positive operators in $\mathfrak{T}(\mathcal{H})$ with unit trace) [17, 23, 41].

Denote by $I_H$ the unit operator on a Hilbert space $\mathcal{H}$ and by $\text{Id}_H$ the identity transformation of the Banach space $\mathfrak{T}(\mathcal{H})$.

We will use the Mirsky inequality

$$\sum_{i=1}^{+\infty} |\lambda^\rho_i - \lambda^\sigma_i| \leq \| \rho - \sigma \|_1$$

valid for any positive operators $\rho$ and $\sigma$ in $\mathfrak{T}(\mathcal{H})$, where $\{\lambda^\rho_i\}_{i=1}^{+\infty}$ and $\{\lambda^\sigma_i\}_{i=1}^{+\infty}$ are sequence of eigenvalues of $\rho$ and $\sigma$ arranged in the non-increasing order (taking the multiplicity into account) [22, 15].

A finite or countable collection $\{\rho_k\}$ of quantum states with a probability distribution $\{p_k\}$ is called (discrete) ensemble and denoted by $\{p_k, \rho_k\}$. The state $\bar{\rho} = \sum_k p_k \rho_k$ is called the average state of $\{p_k, \rho_k\}$.

The von Neumann entropy of a quantum state $\rho \in \mathfrak{S}(\mathcal{H})$ is defined by the formula $S(\rho) = \text{Tr} \eta(\rho)$, where $\eta(x) = -x \ln x$ if $x > 0$ and $\eta(0) = 0$. It is a concave lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H})$ taking values in $[0, +\infty]$ [17, 25, 40]. The von Neumann entropy satisfies the inequality

$$S(p\rho + (1-p)\sigma) \leq pS(\rho) + (1-p)S(\sigma) + h_2(p)$$
valid for any states $\rho$ and $\sigma$ in $\mathfrak{S}(\mathcal{H})$ and $p \in (0, 1)$, where $h_2(p) = \eta(p) + \eta(1-p)$ is the binary entropy [29, 23, 41].

The quantum relative entropy for two states $\rho$ and $\sigma$ in $\mathfrak{S}(\mathcal{H})$ is defined as

$$D(\rho \| \sigma) = \sum_i \langle \varphi_i | \rho \ln \rho - \rho \ln \sigma | \varphi_i \rangle,$$

where $\{\varphi_i\}$ is the orthonormal basis of eigenvectors of the state $\rho$ and it is assumed that $D(\rho \| \sigma) = +\infty$ if $\text{supp} \rho$ is not contained in $\text{supp} \sigma$ [17, 25, 40].

The quantum conditional entropy (QCE) of a state $\rho$ of a finite-dimensional bipartite system $AB$ is defined as

$$S(A|B)_{\rho} = S(\rho) - S(\rho_B).$$

The function $\rho \mapsto S(A|B)_{\rho}$ is concave and

$$|S(A|B)_{\rho}| \leq S(\rho_A)$$

for any state $\rho$ in $\mathfrak{S}(\mathcal{H}_{AB})$ [17, 41]. By using concavity of the von Neumann entropy and inequality (2) it is easy to show that

$$S(A|B)_{\rho p^{1-p} \sigma} \leq p S(A|B)_{\rho} + (1-p) S(A|B)_{\sigma} + h_2(p)$$

for any states $\rho$ and $\sigma$ in $\mathfrak{S}(\mathcal{H}_{AB})$ and $p \in [0, 1]$, where $h_2$ is the binary entropy.

Definition (3) remains valid for a state $\rho$ of an infinite-dimensional bipartite system $AB$ with finite marginal entropies $S(\rho_A)$ and $S(\rho_B)$ (since the finiteness of $S(\rho_A)$ and $S(\rho_B)$ are equivalent to the finiteness of $S(\rho)$ and $S(\rho_B)$). For a state $\rho$ with finite $S(\rho_A)$ and arbitrary $S(\rho_B)$ one can define the QCE by the formula

$$S(A|B)_{\rho} = S(\rho_A) - D(\rho \| \rho_A \otimes \rho_B)$$

proposed and analysed by Kuznetsova in [21] (the finiteness of $S(\rho_A)$ implies the finiteness of $D(\rho \| \rho_A \otimes \rho_B)$). The QCE extended by the above formula to the convex set $\{\rho \in \mathfrak{S}(\mathcal{H}_{AB}) | S(\rho_A) < +\infty\}$ possesses all basic properties of the QCE valid in finite dimensions [21]. In particular, it is concave and satisfies inequalities (4) and (5).

The quantum mutual information of a state $\rho$ in $\mathfrak{S}(\mathcal{H}_{AB})$ is defined as

$$I(A:B)_{\rho} = D(\rho \| \rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho),$$

where the second formula is valid if $S(\rho)$ is finite [26]. Basic properties of the relative entropy show that $\rho \mapsto I(A:B)_{\rho}$ is a lower semicontinuous function on the set $\mathfrak{S}(\mathcal{H}_{AB})$ taking values in $[0, +\infty]$.

It is well known that

$$I(A:B)_{\rho} \leq 2 \min\{S(\rho_A), S(\rho_B)\}$$

The support $\text{supp} \rho$ of a state $\rho$ is the closed subspace spanned by the eigenvectors of $\rho$ corresponding to its positive eigenvalues.
for any state $\rho \in \mathcal{S}(\mathcal{H}_{AB})$ and that factor 2 in (8) can be omitted if $\rho$ is a separable state [26, 41].

The quantum mutual information is not convex or concave, but it satisfies the inequalities

$$I(A:B)_{p\rho+(1-p)\sigma} \geq pI(A:B)_{\rho} + (1-p)I(A:B)_{\sigma} - h_2(p)$$

and

$$pI(A:B)_{\rho} + (1-p)I(A:B)_{\sigma} \geq I(A:B)_{p\rho+(1-p)\sigma} - h_2(p)$$

valid for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H}_{AB})$ and any $p \in (0, 1)$ with possible values $+\infty$ in both sides [33, 35].

Let $H$ be a positive (semi-definite) operator on a Hilbert space $\mathcal{H}$ (we will always assume that positive operators are self-adjoint). Denote by $\mathcal{D}(H)$ the domain of $H$. For any positive operator $\rho \in \mathcal{T}(\mathcal{H})$ we will define the quantity $\text{Tr}H\rho$ by the rule

$$\text{Tr}H\rho = \begin{cases} \sup_n \text{Tr}P_n H\rho & \text{if supp}\rho \subseteq \text{cl}(\mathcal{D}(H)) \\ +\infty & \text{otherwise} \end{cases}$$

where $P_n$ is the spectral projector of $H$ corresponding to the interval $[0, n]$ and $\text{cl}(\mathcal{D}(H))$ is the closure of $\mathcal{D}(H)$. If $H$ is the Hamiltonian (energy observable) of a quantum system described by the space $\mathcal{H}$ then $\text{Tr}H\rho$ is the mean energy of a state $\rho$.

For any positive operator $H$ the set

$$\mathcal{C}_{H,E} = \{ \rho \in \mathcal{S}(\mathcal{H}) \mid \text{Tr}H\rho \leq E \}$$

is convex and closed (since the function $\rho \mapsto \text{Tr}H\rho$ is affine and lower semicontinuous). It is nonempty if $E > E_0$, where $E_0$ is the infimum of the spectrum of $H$.

The von Neumann entropy is continuous on the set $\mathcal{C}_{H,E}$ for any $E > E_0$ if and only if the operator $H$ satisfies the Gibbs condition

$$\text{Tr} e^{-\beta H} < +\infty \quad \text{for all } \beta > 0$$

and the supremum of the entropy on this set is attained at the Gibbs state

$$\gamma_H(E) = e^{-\beta(E)H} / \text{Tr} e^{-\beta(E)H},$$

where the parameter $\beta(E)$ is determined by the equation $\text{Tr}H e^{-\beta H} = E \text{Tr} e^{-\beta H}$ [40]. Condition (11) can be valid only if $H$ is an unbounded operator having discrete spectrum of finite multiplicity. It means, in Dirac’s notation, that

$$H = \sum_{k=0}^{+\infty} E_k |\tau_k\rangle \langle \tau_k|,$$

where $\mathcal{T} = \{ \tau_k \}_{k=0}^{+\infty}$ is the orthonormal system of eigenvectors of $H$ corresponding to the nondecreasing unbounded sequence $\{ E_k \}_{k=0}^{+\infty}$ of its eigenvalues and it is assumed
that the domain \( D(H) \) of \( H \) lies within the closure \( \mathcal{H}_T \) of the linear span of \( T \). In this case

\[
\text{Tr} H \rho = \sum_i \lambda_i \| \sqrt{H} \varphi_i \|^2
\]

for any positive operator \( \rho \) in \( \Sigma(\mathcal{H}) \) with the spectral decomposition \( \rho = \sum_i \lambda_i | \varphi_i \rangle \langle \varphi_i | \) provided that all the vectors \( \varphi_i \) lie in \( D(\sqrt{H}) = \{ \varphi \in \mathcal{H}_T | \sum_{k=0}^{+\infty} E_k | \langle \tau_k | \varphi \rangle |^2 < +\infty \} \). If at least one eigenvector of \( \rho \) corresponding to a nonzero eigenvalue does not belong to the set \( D(\sqrt{H}) \) then \( \text{Tr} H \rho = +\infty \).

We will use the function

\[
F_H(E) = \sup_{\rho \in \mathcal{E}_{H,E}} S(\rho) = S(\gamma_H(E)).
\]

This is a strictly increasing concave function on \([E_0, +\infty)\) \([32, 43]\). It is easy to see that \( F_H(E_0) = \ln m(E_0) \), where \( m(E_0) \) is the multiplicity of \( E_0 \). By Proposition 1 in \([32]\) the Gibbs condition (11) is equivalent to the following asymptotic property

\[
F_H(E) = o(E) \quad \text{as} \quad E \to +\infty.
\]

For example, if \( H = \hat{N} \doteq a^\dagger a \) is the number operator of a quantum oscillator then \( F_H(E) = g(E) \), where

\[
g(x) = (x + 1) h_2 \left( \frac{x}{x + 1} \right) = (x + 1) \ln(x + 1) - x \ln x, \quad x > 0, \quad g(0) = 0.
\]

We will often assume that

\[
E_0 \doteq \inf_{\| \varphi \| = 1} \langle \varphi | H | \varphi \rangle = 0.
\]

In this case the concavity and nonnegativity of \( F_H \) imply that (cf.\([43, \text{Corollary 12}]\))

\[
x F_H(E/x) \leq y F_H(E/y) \quad \forall y > x > 0.
\]

### 3 The Alicki-Fannes-Winter method in the quasi-classical settings

#### 3.1 Basic lemma

In this subsection we describe one general result concerning properties of a real-valued function \( f \) on a convex subset \( \mathcal{G}_0 \) of \( \mathcal{G}(\mathcal{H}) \) satisfying the inequalities

\[
f(p \rho + (1 - p) \sigma) \geq pf(\rho) + (1 - p)f(\sigma) - a_f(p)
\]

and

\[
f(p \rho + (1 - p) \sigma) \leq pf(\rho) + (1 - p)f(\sigma) + b_f(p),
\]

where...
for all states $\rho$ and $\sigma$ in $\mathcal{S}_0$ and any $p \in [0, 1]$, where $a_f(p)$ and $b_f(p)$ are continuous functions on $[0, 1]$ vanishing as $p \to +0$. These inequalities can be treated, respectively, as weakened forms of concavity and convexity. Following [35] we will call functions satisfying both inequalities (18) and (19) locally almost affine (briefly, LAA functions), since for any such function $f$ the quantity $|f(p\rho + (1-p)\sigma) - p f(\rho) - (1-p) f(\sigma)|$ tends to zero as $p \to 0^+$ uniformly on $\mathcal{S}_0 \times \mathcal{S}_0$.

Below we consider a modification of the Alicki-Fannes-Winter method widely used for quantitative continuity analysis of characteristics of quantum systems and channels (the brief history of appearance of this method and its most general description can be found in [35, Section 3]).

Let $\{X, \mathcal{F}\}$ be a measurable space and $\hat{\omega}(x)$ a $\mathcal{F}$-measurable $\mathcal{S}(\mathcal{H})$-valued function on $X$. Denote by $\mathcal{P}(X)$ the set of all probability measures on $X$ (more precisely, on $\{X, \mathcal{F}\}$). We will assume that the function $\hat{\omega}(x)$ is integrable (in the Pettis sense [38]) w.r.t. any measure in $\mathcal{P}(X)$. Consider the set of states

$$
\mathcal{Q}_{X,\mathcal{F},\omega} = \left\{ \rho \in \mathcal{S}(\mathcal{H}) \mid \exists \mu_\rho \in \mathcal{P}(X) : \rho = \int_X \hat{\omega}(x) \mu_\rho(dx) \right\}.
$$

(20)

We will call any measure $\mu_\rho$ in $\mathcal{P}(X)$ such that $\rho = \int_X \hat{\omega}(x) \mu_\rho(dx)$ a representing measure for a state $\rho$ in $\mathcal{Q}_{X,\mathcal{F},\omega}$.

We will use the total variation distance between probability measures $\mu$ and $\nu$ in $\mathcal{P}(X)$ defined as

$$
TV(\mu, \nu) = \sup_{A \in \mathcal{F}} |\mu(A) - \nu(A)|.
$$

(21)

Consider two examples used below:

- if $X = \mathbb{N}$, $\mathcal{F}$ is the $\sigma$-algebra of all subsets of $\mathbb{N}$ and $\hat{\omega}(n) = |n\rangle\langle n|$, where $\{|n\rangle\}_{n \in \mathbb{N}}$ is an orthonormal basic in $\mathcal{H}$ then $\mathcal{Q}_{X,\mathcal{F},\omega}$ is the set of all states in $\mathcal{S}(\mathcal{H})$ diagonizable in the basic $\{|n\rangle\}$. In this case a representing measure $\mu_\rho$ is determined by the spectrum of $\rho$.

- if $X = \mathbb{C}$, $\mathcal{F}$ is the Borel $\sigma$-algebra on $\mathbb{C}$ and $\hat{\omega}(z) = |z\rangle\langle z|$ – the coherent state of a quantum oscillator corresponding to a complex number $z$ then $\mathcal{Q}_{X,\mathcal{F},\omega}$ is the set of all classical states of a quantum oscillator. In this case $\mu_\rho(dz) = P_\rho(z)dz$, where $P_\rho$ is the $P$-function of $\rho$ [16, 36, 14].

The following lemma gives semi-continuity bound for LAA functions on a set of quantum states having form (20). It is proved by obvious modification of the Alicki-Fannes-Winter technique.

**Lemma 1.** Let $\mathcal{Q}_{X,\mathcal{F},\omega}$ be the set defined in (20) and $\mathcal{S}_0$ a convex subset of $\mathcal{S}(\mathcal{H})$ with the property

$$
\rho \in \mathcal{S}_0 \cap \mathcal{Q}_{X,\mathcal{F},\omega} \Rightarrow \{ \sigma \in \mathcal{Q}_{X,\mathcal{F},\omega} \mid \exists \epsilon > 0 : \epsilon \sigma \leq \rho \} \subseteq \mathcal{S}_0.
$$

(22)

Let $f$ be a function on the set $\mathcal{S}_0$ taking values in $(-\infty, +\infty]$ that satisfies inequalities (18) and (19) with possible value $+\infty$ in both sides. Let $\rho$ and $\sigma$ be states in
We may assume that then the set \( Q \) and the left hand side of (23) may be equal to \(-\infty\).

If the function \( f \) is nonnegative then inequality (23) holds with \( C_f(\rho, \sigma|\varepsilon) \) replaced by

\[
C_f^+(\rho|\varepsilon) = \sup \{ f(\varrho) \mid \varrho \in \Omega_{X,\tilde{\sigma},\tilde{\omega}}, \varepsilon \varrho \leq \rho \}.
\]

Proof. We may assume that \( f(\sigma) < +\infty \), since otherwise (23) holds trivially. By the condition we have

\[
2TV(\mu_\rho, \mu_\sigma) = [\mu_\rho - \mu_\sigma]_+(X) + [\mu_\rho - \mu_\sigma]_-(X) = 2\varepsilon,
\]

where \([\mu_\rho - \mu_\sigma]_+\) and \([\mu_\rho - \mu_\sigma]_-\) are the positive and negative parts of the measure \( \mu_\rho - \mu_\sigma \) (in the sense of Jordan decomposition theorem [10]). Since \( \mu_\rho(X) = \mu_\sigma(X) = 1 \), it follows from the above equality that \( [\mu_\rho - \mu_\sigma]_+(X) = [\mu_\rho - \mu_\sigma]_-(X) = \varepsilon \). Hence, \( \nu_\pm = \varepsilon^{-1}[\mu_\rho - \mu_\sigma]_\pm \in \mathcal{P}(X) \). Moreover, it is easy to show, by using the definition of \([\mu_\rho - \mu_\sigma]_\pm\) via the Hahn decomposition of \( X \), that

\[
\varepsilon \nu_+ = [\mu_\rho - \mu_\sigma]_+ \leq \mu_\rho \quad \text{and} \quad \varepsilon \nu_- = [\mu_\rho - \mu_\sigma]_- \leq \mu_\sigma.
\]

Consider the states

\[
\tau_+ = \int_X \tilde{\omega}(x)\nu_+(dx) \quad \text{and} \quad \tau_- = \int_X \tilde{\omega}(x)\nu_-(dx).
\]

Since the inequalities in (26) imply that \( \varepsilon \tau_+ \leq \rho \) and \( \varepsilon \tau_- \leq \sigma \), these states belong to the set \( \Omega_{X,\tilde{\sigma},\tilde{\omega}} \cap \mathcal{G}_0 \) due to condition (22). Then we have

\[
\frac{1}{1+\varepsilon} \rho + \frac{\varepsilon}{1+\varepsilon} \tau_- = \omega_* = \frac{1}{1+\varepsilon} \sigma + \frac{\varepsilon}{1+\varepsilon} \tau_+,
\]

where \( \omega_* \) is a state in \( \mathcal{G}_0 \). By the finiteness of \( f(\rho) \) and \( f(\sigma) \) the operator inequalities \( \varepsilon \tau_+ \leq \rho \) and \( \varepsilon \tau_- \leq \sigma \) along with inequalities (18) and (19) imply the finiteness of \( f(\tau_+), f(\tau_-) \) and \( f(\omega_*) \).

By applying inequalities (18) and (19) to the decompositions of \( \omega_* \) in (28) we obtain

\[
(1-p)(f(\rho) - f(\sigma)) \leq p(f(\tau_+) - f(\tau_-)) + a_f(p) + b_f(p),
\]

where \( p = \varepsilon/(1+\varepsilon) \). If follows that

\[
f(\rho) - f(\sigma) \leq \varepsilon(f(\tau_+) - f(\tau_-)) + D_f(\varepsilon).
\]

Since \( \varepsilon \tau_+ \leq \rho \) and \( \varepsilon \tau_- \leq \sigma \), this implies inequality (23).

The last claim of the lemma is obvious. \(\square\)
Remark 1. In general, the condition $TV(\mu_\rho, \mu_\sigma) = \varepsilon$ in Lemma 1 can not be replaced by the condition $TV(\mu_\rho, \mu_\sigma) \leq \varepsilon$, since the functions $\varepsilon \mapsto C_f(\rho, \sigma | \varepsilon)$ and $\varepsilon \mapsto C_f^+(\rho | \varepsilon)$ may be decreasing and, hence, special arguments are required to show that the r.h.s. of (23) is a nondecreasing function of $\varepsilon$.

Remark 2. Condition (22) in Lemma 1 can be replaced by the condition

$$\rho, \sigma \in \mathcal{G}_0 \cap \mathcal{Q}_{X, \tilde{X}, \tilde{\omega}} \Rightarrow \tau_+, \tau_- \in \mathcal{G}_0,$$

where $\tau_+$ and $\tau_-$ are the states defined in (27) via the representing measures $\mu_\rho$ and $\mu_\sigma$.

In this case one should correct the definitions of $C_f(\rho, \sigma | \varepsilon)$ and $C_f^+(\rho | \varepsilon)$ by replacing $\mathcal{Q}_{X, \tilde{X}, \tilde{\omega}}$ in (24) and (25) with $\mathcal{Q}_{X, \tilde{X}, \tilde{\omega}} \cap \mathcal{G}_0$.

3.2 General results

Among characteristics of a $n$-partite finite-dimensional quantum system $A_1...A_n$ there are many that satisfy inequalities (18) and (19) with the functions $a_f$ and $b_f$ proportional to the binary entropy (defined after (2)) and the inequality

$$-c_f^- C_m(\rho) \leq f(\rho) \leq c_f^+ C_m(\rho), \quad C_m(\rho) = \sum_{k=1}^{m} S(\rho_{A_k}), \quad m \leq n,$$

for any state $\rho$ in $\mathcal{G}(\mathcal{H}_{A_1...A_n})$, where $c_f^-$ and $c_f^+$ are nonnegative numbers.

Following [33, 35] introduce the class $L^m_n(C, D|\mathcal{G}_0)$, $m \leq n$, of functions on a convex subset $\mathcal{G}_0$ of $\mathcal{G}(\mathcal{H}_{A_1...A_n})$ satisfying inequalities (18) and (19) with $a_f(p) = d_f^- h_2(p)$ and $b_f(p) = d_f^+ h_2(p)$ and inequality (30) for any states in $\mathcal{G}_0$ with the parameters $c_f^-$ and $d_f^+$ such that $c_f^- + c_f^+ = C$ and $d_f^- + d_f^+ = D$.

If $A_1,...,A_n$ are arbitrary infinite-dimensional quantum systems then we define the classes $L^m_n(C, D|\mathcal{G}_0)$, $m \leq n$, by the same rule assuming that all functions in $L^m_n(C, D|\mathcal{G}_0)$ take values in $(-\infty, +\infty)$ on $\mathcal{G}_0$ and that all the inequalities in (18), (19) and (30) hold with possible infinite values in both sides.

We also introduce the class $\mathcal{L}^m_n(C, D|\mathcal{G}_0)$ obtained by adding to the class $L^m_n(C, D|\mathcal{G}_0)$ all functions of the form

$$f(\rho) = \inf_{\lambda} f_\lambda(\rho) \quad \text{and} \quad f(\rho) = \sup_{\lambda} f_\lambda(\rho),$$

where $\{f_\lambda\}$ is some family of functions in $L^m_n(C, D|\mathcal{G}_0)$.

If $\mathcal{G}_0 = \mathcal{G}_m(\mathcal{H}_{A_1...A_n})$, where

$$\mathcal{G}_m(\mathcal{H}_{A_1...A_n}) \doteq \{\rho \in \mathcal{G}(\mathcal{H}_{A_1...A_n}) \mid S(\rho_{A_1}), ..., S(\rho_{A_m}) < +\infty\},$$

then we will denote the classes $L^m_n(C, D|\mathcal{G}_0)$ and $\mathcal{L}^m_n(C, D|\mathcal{G}_0)$ by $L^m_n(C, D)$ and $\mathcal{L}^m_n(C, D)$ for brevity.\(^2\)

\(^2\)The necessity to consider the case $\mathcal{G}_0 \neq \mathcal{G}_m(\mathcal{H}_{A_1...A_n})$ will be shown in Section 4.
For example, the von Neumann entropy belongs to the class $L^1(1, 1|S(H))$, while the (extended) quantum conditional entropy $S(A_1|A_2)$ (defined in (6)) lies in $L^1(2, 1) = L^2(2, 1|S(1(H), A_2))$. This follows from the concavity of these characteristics and inequalities (2), (4) and (5). The nonnegativity of $S(A_1|A_2)$ on the convex set $S_{sep}$ of separable states in $S_1(H_{A_1A_2})$ implies that this function also belongs to the class $L^1(1, 1|S_{sep})$. More complete list of characteristics of quantum systems belonging the classes $L^m_n(C, D)$ and $\tilde{L}^m_n(C, D)$ can be found in [33, 35].

Now we apply Lemma 1 in Section 3.1 to obtain continuity bounds for functions from the classes $L^m_n(C, D|S_0)$ under the rank constraint on the marginal states corresponding to the subsystems $A_1, ..., A_m$.

**Theorem 1.** Let $\mathfrak{Q}_{X, \tilde{\omega}}$ be the set of states in $S(H_{A_1...A_n})$ defined in (20) via some $S(H_{A_1...A_n})$-valued function $\tilde{\omega}(x)$ and $S_0$ a convex subset of $S(H_{A_1...A_n})$ possessing property (22).

A) If $f$ is a function in $\tilde{L}^m_n(C, D|S_0)$ then

$$|f(\rho) - f(\sigma)| \leq C \varepsilon \ln d_m(\rho, \sigma) + D g(\varepsilon)$$

(31)

for any states $\rho$ and $\sigma$ in $\mathfrak{Q}_{X, \tilde{\omega}} \cap S_0$ such that $\text{TV}(\mu_\rho, \mu_\sigma) \leq \varepsilon$ and $d_m(\rho, \sigma) \leq \max \{\prod_{k=1}^m \text{rank} \rho_{A_k}, \prod_{k=1}^m \text{rank} \sigma_{A_k}\}$ is finite, where $\mu_\rho$ and $\mu_\sigma$ are measures in $\mathcal{P}(X)$ representing the states $\rho$ and $\sigma$ and $g$ is the function defined in (15).

B) If $f$ is a nonnegative function in $\tilde{L}^m_n(C, D|S_0)$ and $\rho$ is a state in $\mathfrak{Q}_{X, \tilde{\omega}} \cap S_0$ such that $d_m(\rho) \equiv \prod_{k=1}^m \text{rank} \rho_{A_k}$ is finite then

$$f(\rho) - f(\sigma) \leq C \varepsilon \ln d_m(\rho) + D g(\varepsilon)$$

(32)

for any state $\sigma$ in $\mathfrak{Q}_{X, \tilde{\omega}} \cap S_0$ such that $\text{TV}(\mu_\rho, \mu_\sigma) \leq \varepsilon$ (the l.h.s. of (32) may be equal to $-\infty$).

**Remark 3.** It is essential that in part B of Theorem 1 we impose no restrictions on the state $\sigma$ other than the requirement $\text{TV}(\mu_\rho, \mu_\sigma) \leq \varepsilon$.

**Remark 4.** Both claims of Theorem 1 remain valid if $S_0$ is any convex subset of $S(H_{A_1...A_n})$ provided that condition (29) holds for the states $\rho$ and $\sigma$. This follows from the proof of this theorem and Remark 2.

**Proof.** In the proofs of both parts of the theorem we may assume that $f$ is a function in $L^m_n(C, D|S_0)$ and that $\text{TV}(\mu_\rho, \mu_\sigma) = \varepsilon$. Indeed, the expressions in r.h.s. of (31) and (32) are non-decreasing functions of $\varepsilon$ and depend only on the parameters $C$ and $D$ and the characteristics of the states $\rho$ and $\sigma$.

A) Note first that the condition $d_m(\rho, \sigma) < +\infty$ implies that $f(\rho), f(\sigma) < +\infty$ by inequality (30). Since $\text{rank} \rho_{A_k} \leq \text{rank} \rho_{A_k}$ and $\text{rank} \sigma_{A_k} \leq \text{rank} \sigma_{A_k}, k = 1, m$, for any states $\rho$ and $\sigma$ in $\mathfrak{Q}_{X, \tilde{\omega}}$ such that $\varepsilon \rho \leq \rho$ and $\varepsilon \sigma \leq \sigma$, inequality (30) implies that

$$|f(\rho) - f(\sigma)| \leq C \ln d_m(\rho, \sigma)$$

(33)
for any such states $\rho$ and $\varsigma$. So, the quantities $C_f(\rho, \sigma | \varepsilon)$ and $C_f(\sigma, \rho | \varepsilon)$ defined in (24) do not exceed the r.h.s. of (33). Thus, by applying Lemma 1 twice we obtain (31).

B) The condition $d_m(\rho) < +\infty$ implies that $f(\rho) < +\infty$ by inequality (30). So, this assertion follows from the last claim of Lemma 1, since $\text{rank}_A \leq \text{rank}_B \leq +\infty$, $k = 1, m$, for any state $\varrho$ in $\Omega_{X, \tilde{\omega}}$ such that $\varepsilon \varrho \leq \rho$. Thus, inequality (30) implies that

$$f(\varrho) \leq C \ln d_m(\rho)$$

(34)

for any such state $\varrho$. So, the quantity $C_f^+(\rho | \varepsilon)$ defined in (25) does not exceed the r.h.s. of (34).

Now we apply Lemma 1 in Section 3.1 to obtain continuity bounds for functions from the classes $\tilde{L}_n^m(C, D | \mathcal{S}_0)$ under the constraint corresponding to the positive operator

$$H_m = H_{A_1} \otimes I_{A_2} \otimes \cdots \otimes I_{A_m} + \cdots + I_{A_1} \otimes \cdots \otimes I_{A_m-1} \otimes H_{A_m}$$

(35)

is a positive operator on the space $\mathcal{H}_{A_1 \cdots A_m}$ determined by positive operators $H_{A_1}, \ldots, H_{A_m}$ on the spaces $\mathcal{H}_{A_1}, \ldots, \mathcal{H}_{A_m}$ (it is assumed that $H_1 = H_{A_1}$). It is essential that the operator $H_m$ satisfies condition (11) if all the operators $H_{A_1}, \ldots, H_{A_m}$ satisfy this condition (this follows from the equivalence of (11) and (14), see the proof of [33, Lemma 4]). Note that $\text{Tr} H_m \rho = \sum_{k=1}^m \text{Tr} H_{A_k} \rho_{A_k}$ for any $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1 \cdots A_m})$. We will use the function

$$F_{H_m}(E) = \sup \{ S(\rho) | \rho \in \mathcal{S}(\mathcal{H}_{A_1 \cdots A_m}), \text{Tr} H_m \rho \leq E \}.$$ 

(36)

If all the operators $H_{A_1}, \ldots, H_{A_m}$ are isometrically equivalent to each other then $F_{H_m}(E) = m F_{H_{A_1}}(E/m)$, where $F_{H_{A_1}}$ is the function defined in (13) [33, Lemma 4].

**Theorem 2.** Let $H_{A_1}, \ldots, H_{A_m}$ be positive operators on the Hilbert spaces $\mathcal{H}_{A_1}, \ldots, \mathcal{H}_{A_m}$ satisfying condition (11) and (16) and $F_{H_m}$ the function defined in (36). Let $\Omega_{X, \tilde{\omega}}$ be the set of states in $\mathcal{S}(\mathcal{H}_{A_1 \cdots A_n})$ defined in (20) via $\mathcal{S}(\mathcal{H}_{A_1 \cdots A_n})$-valued function $\tilde{\omega}(x)$ and $\mathcal{S}_0$ a convex subset of $\mathcal{S}(\mathcal{H}_{A_1 \cdots A_n})$ with property (22).

A) If $f$ is a function in $\tilde{L}_n^m(C, D | \mathcal{S}_0)$ then

$$|f(\rho) - f(\sigma)| \leq C \varepsilon F_{H_m}(mE/\varepsilon) + Dg(\varepsilon)$$

(37)

for any states $\rho$ and $\sigma$ in $\Omega_{X, \tilde{\omega}} \cap \mathcal{S}_0$ such that $\sum_{k=1}^m \text{Tr} H_{A_k} \rho_{A_k} \leq mE$ and $\text{TV}(\mu_{\rho}, \mu_{\sigma}) \leq \varepsilon$, where $\mu_{\rho}$ and $\mu_{\sigma}$ are measures in $\mathcal{P}(X)$ representing the states $\rho$ and $\sigma$ and $g$ is the function defined in (15).

B) If $f$ is a nonnegative function in $\tilde{L}_n^m(C, D | \mathcal{S}_0)$ and $\rho$ is a state in $\Omega_{X, \tilde{\omega}} \cap \mathcal{S}_0$ such that $\sum_{k=1}^m \text{Tr} H_{A_k} \rho_{A_k} \leq mE$ then

$$f(\rho) - f(\sigma) \leq C \varepsilon F_{H_m}(mE/\varepsilon) + Dg(\varepsilon)$$

(38)

for any state $\sigma$ in $\Omega_{X, \tilde{\omega}} \cap \mathcal{S}_0$ such that $\text{TV}(\mu_{\rho}, \mu_{\sigma}) \leq \varepsilon$ (the l.h.s. of (38) may be equal to $-\infty$). If the set $\Omega_{X, \tilde{\omega}}$ consists of commuting states then (38) holds with $mE$
replaced by \( mE - \sum_{k=1}^{m} \text{Tr} H_{A_k} \rho_{A_k}^\varepsilon \), where \( \rho^\varepsilon = [\rho - \varepsilon I_{A_1 \ldots A_n}]_+ \) is the positive part of the Hermitian operator \( \rho - \varepsilon I_{A_1 \ldots A_n} \).

**Remark 5.** Since the operator \( H_m \) satisfies conditions (11) and (16), the equivalence of (11) and (14) and inequality (17) show that the r.h.s. of (37) and (38) are non-decreasing functions of \( \varepsilon \) tending to zero as \( \varepsilon \to 0 \).

**Remark 6.** It is essential that in part B of Theorem 2 we impose no restrictions on the state \( \sigma \) other than the requirement \( \text{TV}(\mu_\rho, \mu_\sigma) \leq \varepsilon \).

**Remark 7.** Both claims of Theorem 2 remain valid if \( S_0 \) is any convex subset of \( \mathcal{S}(H_{A_1 \ldots A_n}) \) provided that condition (29) holds for the states \( \rho \) and \( \sigma \). This follows from the proof of this theorem and Remark 2.

**Proof.** In the proofs of both parts of the theorem we may assume that \( f \) is a function in \( L_m^n(C, D|\mathcal{S}_0) \) and that \( \text{TV}(\mu_\rho, \mu_\sigma) = \varepsilon \). Indeed, by Remark 5 the expressions in r.h.s. of (37) and (38) are non-decreasing functions of \( \varepsilon \) and depend only on the parameters \( C \) and \( D \) and the characteristics of the states \( \rho \) and \( \sigma \).

A) Since \( \text{Tr} H_m[\vartheta \otimes \ldots \otimes \vartheta] = \sum_{k=1}^{m} \text{Tr} H_{A_k} \vartheta_{A_k} \), we have

\[
\sum_{k=1}^{m} S(\vartheta_{A_k}) = S(\vartheta \otimes \ldots \otimes \vartheta) \leq F_{H_m}(mE)
\]

for any state \( \vartheta \in \mathcal{S}_0 \) such that \( \text{Tr} H_m \vartheta_{A_1 \ldots A_n} = \sum_{k=1}^{m} \text{Tr} H_{A_k} \vartheta_{A_k} \leq mE \). Hence, for any such state \( \vartheta \) inequality (30) implies that

\[
-c_f^+ F_{H_m}(mE) \leq f(\vartheta) \leq c_f^+ F_{H_m}(mE). \tag{39}
\]

It follows, in particular, that \( f(\rho) \) and \( f(\sigma) \) are finite.

Assume that \( \varrho \) and \( \varsigma \) are states in \( \mathcal{S}_X, \mathcal{S}_\bar{\omega} \) such that \( \varepsilon \varrho \leq \rho \) and \( \varepsilon \varsigma \leq \sigma \). Then

\[
\varepsilon \sum_{k=1}^{m} \text{Tr} H_{A_k} \varrho_{A_k} \leq \sum_{k=1}^{m} \text{Tr} H_{A_k} \rho_{A_k} \leq mE \tag{40}
\]

and

\[
\varepsilon \sum_{k=1}^{m} \text{Tr} H_{A_k} \varsigma_{A_k} \leq \sum_{k=1}^{m} \text{Tr} H_{A_k} \sigma_{A_k} \leq mE.
\]

Thus, it follows from (39) that

\[
|f(\varrho) - f(\varsigma)| \leq CF_{H_m}(mE/\varepsilon).
\]

So, the quantities \( C_f(\rho, \sigma|\varepsilon) \) and \( C_f(\sigma, \rho|\varepsilon) \) defined in (24) do not exceed \( CF_{H_m}(mE/\varepsilon) \). Thus, by applying Lemma 1 twice we obtain (37).

B) The main assertion of part B follows from the last claim of Lemma 1, since the arguments from the proof of part A show that \( f(\rho) \leq +\infty \) and that

\[
f(\varrho) \leq CF_{H_m}(mE/\varepsilon) \tag{41}
\]
for any state \( \varrho \) in \( \mathcal{Q}_{X,\mathfrak{g},\mathfrak{w}} \) such that \( \varepsilon \varrho \leq \rho \). So, the quantity \( C_f^+(\rho|\varepsilon) \) defined in (25) does not exceed the r.h.s. of (41).

If a state \( \varrho \) commutes with the state \( \rho \) then the condition \( \varepsilon \varrho \leq \rho \) implies that
\[
\varepsilon \varrho \leq \rho - [\rho - \varepsilon I_{A_1 \ldots A_n}]_+,
\]
since \( \varrho \leq I_{A_1 \ldots A_n} \). By incorporating this to the estimates in (40) it is easy to prove the last claim of part B. \( \square \)

Theorems 1 and 2 allows us to refine the continuity bounds for functions from the classes \( \hat{L}_n^m(C, D) \) presented in [33, 35] in the case of commuting states, i.e. such states \( \rho \) and \( \sigma \) in \( \mathcal{G}(H_{A_1 \ldots A_n}) \) that
\[
[\rho, \sigma] = \rho \sigma - \sigma \rho = 0.
\]
Indeed, since for any commuting states \( \rho \) and \( \sigma \) in \( \mathcal{G}(H_{A_1 \ldots A_n}) \) there exists an orthonormal basis \( \{|k\rangle\}_{k \in \mathbb{N}} \) in \( H_{A_1 \ldots A_n} \) in which these states are diagonalizable, they belong to the set \( \mathcal{Q}_{X,\mathfrak{g},\mathfrak{w}} \), where \( X = \mathbb{N}, \mathfrak{g} \) is the \( \sigma \)-algebra of all subsets of \( \mathbb{N} \) and \( \mathfrak{w}(k) = |k\rangle\langle k| \).

In this case the set \( P(X) \) can be identified with the set of probability distributions on \( \mathbb{N} \), the representing measures \( \mu_\rho \) and \( \mu_\sigma \) correspond to the probability distributions formed by the eigenvalues of \( \rho \) and \( \sigma \) (taken the multiplicity into account) and
\[
\text{TV}(\mu_\rho, \mu_\sigma) = \frac{1}{2}\|\rho - \sigma\|_1.
\]

Thus, Theorem 1 implies the following

**Corollary 1.** Let \( \mathcal{G}_0 \) be a convex subset of \( \mathcal{G}(H_{A_1 \ldots A_n}) \) with the property\(^3\)
\[
\frac{[\rho - \sigma]_\pm}{\text{Tr}[\rho - \sigma]_\pm} \in \mathcal{G}_0 \quad \text{for any } \rho \text{ and } \sigma \text{ in } \mathcal{G}_0 \text{ such that } [\rho, \sigma] = 0 \text{ and } \rho \neq \sigma,
\]
where \([\rho - \sigma]_\pm\) are the positive and negative parts of the Hermitian operator \( \rho - \sigma \).

A) If \( f \) is a function in \( \hat{L}_n^m(C, D|\mathcal{G}_0) \) then
\[
|f(\rho) - f(\sigma)| \leq C\varepsilon \ln d_m(\rho, \sigma) + Dg(\varepsilon)
\]
for any states \( \rho \) and \( \sigma \) in \( \mathcal{G}_0 \) s.t. \( d_{m}(\rho, \sigma) \doteq \max \{\prod_{k=1}^{m} \text{rank } \rho_{A_k}, \prod_{k=1}^{m} \text{rank } \sigma_{A_k}\} \) is finite, \([\rho, \sigma] = 0\) and \( \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \).

B) If \( f \) is a nonnegative function in \( \hat{L}_n^m(C, D|\mathcal{G}_0) \) and \( \rho \) is a state in \( \mathcal{G}(H_{A_1 \ldots A_n}) \) such that \( d_{m}(\rho) \doteq \prod_{k=1}^{m} \text{rank } \rho_{A_k} \) is finite then
\[
f(\rho) - f(\sigma) \leq C\varepsilon \ln d_{m}(\rho) + Dg(\varepsilon)
\]
for any state \( \sigma \) in \( \mathcal{G}_0 \) such that \([\rho, \sigma] = 0\) and \( \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \) (the l.h.s. of (43) may be equal to \(-\infty\)).

**Proof.** By the observation before the corollary all its claims follow from Theorem 1 and Remark 4, since property (42) guarantees the validity of condition (29) for any commuting states in \( \mathcal{G}_0 \) as in this case \( \tau_\pm = [\rho - \sigma]_\pm/\text{Tr}[\rho - \sigma]_\pm \).

\( \square \)

\(^3\)This is a weakened form of the \( \Delta \)-invariance property [35, Section 3].
Theorem 2 implies the following

**Corollary 2.** Let $\mathcal{S}_0$ be a convex subset of $\mathcal{S}(\mathcal{H}_{A_1\ldots A_n})$ with property (42). Let $H_{A_1}, \ldots, H_{A_m}$ be positive operators on the Hilbert spaces $\mathcal{H}_{A_1}, \ldots, \mathcal{H}_{A_m}$ satisfying conditions (11) and (16) and $F_{H_m}$ the function defined in (36).

A) If $f$ is a function in $\tilde{L}_n^m(C, D|\mathcal{S}_0)$ then

$$|f(\rho) - f(\sigma)| \leq C\varepsilon F_{H_m}(mE/\varepsilon) + Dg(\varepsilon)$$

for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H}_{A_1\ldots A_n})$ s.t. $\sum_{k=1}^m \text{Tr}H_{A_k}\rho_{A_k}$, $\sum_{k=1}^m \text{Tr}H_{A_k}\sigma_{A_k} \leq mE$, $[\rho, \sigma] = 0$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$.

B) If $f$ is a nonnegative function in $\tilde{L}_n^m(C, D|\mathcal{S}_0)$ and $\rho$ is a state in $\mathcal{S}(\mathcal{H}_{A_1\ldots A_n})$ such that $\sum_{k=1}^m \text{Tr}H_{A_k}\rho_{A_k} \leq mE$ then

$$f(\rho) - f(\sigma) \leq C\varepsilon F_{H_m}((mE - E_c(\rho))/\varepsilon) + Dg(\varepsilon) \leq C\varepsilon F_{H_m}(mE/\varepsilon) + Dg(\varepsilon) \quad (44)$$

for any state $\sigma$ in $\mathcal{S}_0$ such that $[\rho, \sigma] = 0$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, where\(^4\)

$$E_c(\rho) = \sum_{k=1}^m \text{Tr}H_{A_k} \langle [\rho - \varepsilon I_{A_1\ldots A_n}]_+ \rangle_{A_k}$$

and the left hand sides of (44) may be equal to $-\infty$.

**Proof.** By the observation before Corollary 1 all the claims of this corollary follow from Theorem 2 and Remark 7. It suffices to note that property (42) guarantees the validity of condition (29) for any commuting states $\rho$ and $\sigma$ in $\mathcal{S}_0$, since in this case $\tau_\pm = [\rho - \sigma]_+/\text{Tr}[\rho - \sigma]_\pm$.

**Note:** Since $E_c(\rho)$ tends to $E(\rho) = \sum_{k=1}^m \text{Tr}H_{A_k}\rho_{A_k}$ as $\varepsilon \to 0$, the first estimate in (44) is essentially sharper than the second one provided that $mE = E(\rho)$ and $\varepsilon \ll 1$.

**Example 1.** Let $f = I(A_1 : A_2)$ be the quantum mutual information defined in (7). This is a function on the whole space $\mathcal{S}(\mathcal{H}_{A_1A_2})$ taking values in $[0, +\infty]$ and satisfying the inequalities (9) and (10) with possible value $+\infty$ in both sides. This and upper bound (8) show that $I(A_1 : A_2)$ belongs to the class $L_2^1(2, 2|\mathcal{S}(\mathcal{H}_{A_1A_2}))$.

Thus, Corollary 1B implies that

$$I(A_1 : A_2)_\rho - I(A_1 : A_2)_\sigma \leq 2\varepsilon \ln \text{rank}\rho_{A_1} + 2g(\varepsilon) \quad (45)$$

for any commuting states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H}_{A_1A_2})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$.

The semi-continuity bound (45) and the symmetry arguments show that

$$-(2\varepsilon \ln \text{rank}\sigma_{A_k} + 2g(\varepsilon)) \leq I(A_1 : A_2)_\rho - I(A_1 : A_2)_\sigma \leq 2\varepsilon \ln \text{rank}\rho_{A_1} + 2g(\varepsilon) \quad (46)$$

\(^4[\rho - \varepsilon I_{A_1\ldots A_n}]_+\) is the positive part of the Hermitian operator $\rho - \varepsilon I_{A_1\ldots A_n}$.  

\(14\)
for any commuting states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H}_{A_1A_2})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, where $x$ is either 1 or 2. The continuity bound (46) gives more accurate estimates than the universal continuity bound for $I(A_1 : A_2)$ presented in [35, Proposition 6]. Indeed, by using the last continuity bound one can only show that

$$ |I(A_1 : A_2)_{\rho} - I(A_1 : A_2)_{\sigma}| \leq 2\varepsilon \ln(\text{rank} \rho_{A_1} + \text{rank} \sigma_{A_1}) + 2g(\varepsilon) $$

for any states $\rho$ and $\sigma$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$.

The possibility to use in (46) the ranks of marginal states corresponding to different subsystems (the case $x = 2$) gives additional flexibility in getting more accurate estimate for the quantity $I(A_1 : A_2)_{\rho} - I(A_1 : A_2)_{\sigma}$.

Let $H$ be any positive operator on $\mathcal{H}_{A_1}$ satisfying conditions (11) and (16). Corollary 2B implies that

$$ I(A_1 : A_2)_{\rho} - I(A_1 : A_2)_{\sigma} \leq 2\varepsilon F_H(E/\varepsilon) + 2g(\varepsilon) $$

for any state $\rho$ in $\mathcal{S}(\mathcal{H}_{A_1A_2})$ such that $\text{Tr}_H \rho_{A_1} \leq E$ and arbitrary state $\sigma$ in $\mathcal{S}(\mathcal{H}_{A_1A_2})$ such that $[\rho, \sigma] = 0$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$. The semi-continuity bound (47) can be refined by replacing $E$ with $E - \text{Tr}_H (\rho - \varepsilon I_{A_1A_2})_{+} A_1$.

It follows from (47) that

$$ |I(A_1 : A_2)_{\rho} - I(A_1 : A_2)_{\sigma}| \leq 2\varepsilon F_H(E/\varepsilon) + 2g(\varepsilon) $$

for any commuting states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H}_{A_1A_2})$ such that $\text{Tr}_H \rho_{A_1}, \text{Tr}_H \sigma_{A_1} \leq E$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$. Comparing the continuity bound (48) with the universal continuity bounds for $I(A_1 : A_2)$ under the energy constraint presented in [35, Section 4.2.2] we see an even greater gain in accuracy than in the first part of this example. Of course, this gain is due to the fact that we restrict attention to commuting states.

4 Applications to characteristics of quantum and classical systems

4.1 Von Neumann entropy

The first continuity bound for the von Neumann entropy in finite-dimensional quantum systems was obtained by Fannes in [13]. This continuity bound and its optimized version obtained by Audenaert in [4] are widely used in quantum information theory. Audenaert’s optimal continuity bound claims that

$$ |S(\rho) - S(\sigma)| \leq \varepsilon \ln(d - 1) + h_2(\varepsilon) $$

for any states $\rho$ and $\sigma$ of a $d$-dimensional quantum system such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1 - 1/d$. If $\frac{1}{2}\|\rho - \sigma\|_1 > 1 - 1/d$ then the r.h.s. of (49) must be replaced by $\ln d$. 

...
In infinite dimensions two different continuity bounds for the von Neumann entropy under the energy constraints were obtained by Winter in [43]. The first of them claims that

$$|S(\rho) - S(\sigma)| \leq 2\varepsilon F_H(E/\varepsilon) + h_2(\varepsilon)$$  \hspace{1cm} (50)$$

for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H})$ such that $\text{Tr} H \rho, \text{Tr} H \sigma \leq E$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, where $H$ is a positive operator on $\mathcal{H}$ satisfying conditions (11) and (16) and $F_H$ is the function defined in (13).\(^5\) The r.h.s. of (50) tends to zero as $\varepsilon \to 0$ due to the equivalence of (11) and (14).

Continuity bound (50) is simple but not tight (because of factor 2 in the r.h.s.). On the contrary, Winter’s second continuity bound presented in Meta-Lemma 16 in [43] is not simple, but asymptotically tight for large energy $E$ provided that the function $F_H(E)$ has a logarithmic growth rate as $E \to +\infty$ (this property holds for Hamiltonians of many quantum systems, in particular, for multi-mode quantum oscillator [5]).

Recently, Becker, Datta and Jabbour obtained in [6] optimal continuity bound for the von Neumann entropy under the constraint induced by the number operator $\hat{N} = a^\dagger a$ of a one mode quantum oscillator. Namely, these authors proved that

$$|S(\rho) - S(\sigma)| \leq E h_2(\varepsilon/E) + h_2(\varepsilon)$$  \hspace{1cm} (51)$$

for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H})$ such that $\text{Tr} \hat{N} \rho, \text{Tr} \hat{N} \sigma \leq E$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq E/(E + 1)$. They also showed that for any given $0 < E < +\infty$ and $\varepsilon$ in $[0, E/(E + 1)]$ there exist states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H})$ such that $\text{Tr} \hat{N} \rho, \text{Tr} \hat{N} \sigma \leq E$ and $\frac{1}{2}\|\rho - \sigma\|_1 = \varepsilon$ for which an equality holds in (51).

The results of numerical calculations presented in [6] show that continuity bound (51) is much more accurate than both of Winter’s continuity bounds in the case $H = \hat{N}$.

By using the results of Section 3 one can obtain the following continuity bound for the von Neumann entropy under the constraint induced by any positive operator $H$ satisfying conditions (11) and (16):

$$|S(\rho) - S(\sigma)| \leq \varepsilon F_H(E/\varepsilon) + g(\varepsilon)$$  \hspace{1cm} (52)$$

for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H})$ such that $\text{Tr} H \rho, \text{Tr} H \sigma \leq E$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, where $F_H$ and $g$ are the functions defined, respectively, in (13) and (15).

Comparing continuity bound (52) with Winter’s continuity bound (50), we see the disappearance of factor 2 in r.h.s. what is paid for by replacing the term $h_2(\varepsilon)$ with the term $g(\varepsilon) \geq h_2(\varepsilon)$. But keeping in mind that $\varepsilon F_H(E/\varepsilon) \gg g(\varepsilon)$ for real values of $E$ and $\varepsilon$ and noting that $g(\varepsilon) \sim h_2(\varepsilon)$ for small $\varepsilon$, we see that the continuity bound (52) is more accurate in general. Moreover, continuity bound (52) is asymptotically tight for large energy $E$ for arbitrary positive operator $H$ satisfying conditions (11) and (16). This follows from the last claim of Proposition 1 below.

If $H = \hat{N} = a^\dagger a$ then $F_H(E) = g(E)$ and continuity bound (52) becomes

$$|S(\rho) - S(\sigma)| \leq \varepsilon g(E/\varepsilon) + g(\varepsilon) = (E + \varepsilon)h_2\left(\frac{\varepsilon}{E + \varepsilon}\right) + (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right).$$  \hspace{1cm} (53)$$

\(^5\)In [43] the function $F_H(E)$ is denoted by $S(\gamma(E))$.\(\)
We see that (53) is less accurate but very close to the specialized optimal continuity bound (51) for $E \gg \varepsilon$ and small $\varepsilon$. The difference between the right hand sides of (53) and (51) is equal to $2(g(\varepsilon) - h_2(\varepsilon))$ for $E = 1$, and quickly decreases to $g(\varepsilon) - h_2(\varepsilon)$ with increasing $E$. This and the last claim of Proposition 1 below give a reason to believe that the universal continuity bound (52) is "close-to-optimal".

Continuity bound (52) is a corollary of the following more general result.

**Proposition 1.** Let $H$ be a positive operator on $\mathcal{H}$ satisfying conditions (11) and (16). If $\rho$ is a state in $\mathcal{S}(\mathcal{H})$ such that $\text{Tr} H \rho \leq E$ then

$$S(\rho) - S(\sigma) \leq \varepsilon F_H((E - E_{H,\varepsilon}(\rho))/\varepsilon) + g(\varepsilon) \leq \varepsilon F_H(E/\varepsilon) + g(\varepsilon)$$

for any state $\sigma$ in $\mathcal{S}(\mathcal{H})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, where $E_{H,\varepsilon}(\rho) \doteq \text{Tr} H[\rho - \varepsilon I_\mathcal{H}]_+$ and the l.h.s. of (54) may be equal to $-\infty$.

The quantity $E_{H,\varepsilon}(\rho)$ in (54) can be replaced by its easily-computable lower bound

$$E_{H,\varepsilon}^*(\rho) \doteq \sum_{k=0}^{+\infty} E_k^H[\lambda_k^\rho - \varepsilon]_+ \quad ([x]_+ \doteq \max\{x, 0\})$$

where $\{\lambda_k^\rho\}_{k=0}^{+\infty}$ and $\{E_k^H\}_{k=0}^{+\infty}$ are the sequences of eigenvalues of the state $\rho$ and the operator $H$ arranged, respectively, in the non-increasing and the non-decreasing orders.

For each $E > 0$ and $\varepsilon \in (0, 1]$ there exist states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H})$ such that

$$S(\rho) - S(\sigma) > \varepsilon F_H(E/\varepsilon), \quad \text{Tr} H \rho, \text{Tr} H \sigma \leq E \quad \text{and} \quad \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon.$$

**Note A:** In Proposition 1 we impose no restrictions on the state $\sigma$ other than the requirement $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$.

**Note B:** The quantity $E_{H,\varepsilon}(\rho)$ monotonically tends to $\text{Tr} H \rho$ as $\varepsilon \to 0$. So, the first estimate in (54) with $E = \text{Tr} H \rho$ may be essentially sharper than the second one for small $\varepsilon$.

**Proof.** Let $\rho$ and $\sigma$ be states in $\mathcal{S}(\mathcal{H})$ such that $\text{Tr} H \rho \leq E$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$.

Let $\{\lambda_k^\rho\}_{k=0}^{+\infty}$ and $\{\lambda_k^\sigma\}_{k=0}^{+\infty}$ be the sequences of eigenvalues of the states $\rho$ and $\sigma$ arranged in the non-increasing order and $\{\varphi_k\}_{k=0}^{+\infty}$ the basis in $\mathcal{H}$ such that

$$\rho = \sum_{k=0}^{+\infty} \lambda_k^\rho |\varphi_k\rangle \langle \varphi_k|.$$  

Consider the state

$$\hat{\sigma} = \sum_{k=0}^{+\infty} \lambda_k^\sigma |\varphi_k\rangle \langle \varphi_k|.$$  

$[\rho - \varepsilon I_\mathcal{H}]_+$ is the positive part of the Hermitian operator $\rho - \varepsilon I_\mathcal{H}$.  

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6 $[\rho - \varepsilon I_\mathcal{H}]_+$ is the positive part of the Hermitian operator $\rho - \varepsilon I_\mathcal{H}$.  

---
The Mirsky inequality (1) implies that

\[ \frac{1}{2}\| \rho - \sigma \|_1 \leq \frac{1}{2}\| \rho - \hat{\sigma} \|_1 \leq \varepsilon. \]

So, since the von Neumann entropy belongs to the class \( L^1_1(1, 1|\mathcal{S}(\mathcal{H})) \) and the states \( \rho \) and \( \hat{\sigma} \) commute, Corollary 2 with \( \mathcal{S}_0 = \mathcal{S}(\mathcal{H}) \) shows that

\[ S(\rho) - S(\sigma) = S(\rho) - S(\hat{\sigma}) \leq \varepsilon F_H((E - \operatorname{Tr}\rho\varepsilon I_\mathcal{H} + \varepsilon I_\mathcal{H})/\varepsilon) + g(\varepsilon). \]

The inequality \( E_{H,\varepsilon}^{\star}(\rho) \leq E_{H,\varepsilon}(\rho) \) follows from the Courant-Fisher theorem (see Proposition 2.3. in [6]).

To prove the last claim one can take the states \( \rho = \varepsilon \gamma_H(E/\varepsilon) + (1 - \varepsilon)|\tau_0\rangle\langle \tau_0| \), where \( \gamma_H(E/\varepsilon) \) is the Gibbs state (12) corresponding to the “energy” \( E/\varepsilon \) and \( \tau_0 \) is the eigenvector of \( H \) corresponding to the minimal eigenvalue \( E_0 = 0 \).

Proposition 1 implies the following refinement of continuity bound (52) in the case of states with different energies.

Corollary 3. If \( H \) is a positive operator on \( \mathcal{H} \) satisfying conditions (11) and (16) then

\[ -(\varepsilon F_H(E_\sigma/\varepsilon) + g(\varepsilon)) \leq S(\rho) - S(\sigma) \leq \varepsilon F_H(E_\rho/\varepsilon) + g(\varepsilon) \]

for any states \( \rho \) and \( \sigma \) in \( \mathcal{S}(\mathcal{H}) \) such that \( E_\rho \doteq \operatorname{Tr}H\rho < +\infty \), \( E_\sigma \doteq \operatorname{Tr}H\sigma < +\infty \) and \( \frac{1}{2}\| \rho - \sigma \|_1 \leq \varepsilon \).

By using Corollary 1 one can prove the version of Proposition 1 in which the condition \( \operatorname{Tr}H\rho < +\infty \) is replaced by the condition \( \operatorname{rank}\rho < +\infty \). It states that

\[ S(\rho) - S(\sigma) \leq \varepsilon \ln(\operatorname{rank}\rho) + g(\varepsilon) \]

for any state \( \sigma \) in \( \mathcal{S}(\mathcal{H}) \) such that \( \frac{1}{2}\| \rho - \sigma \|_1 \leq \varepsilon \). But by using Audenaert’s optimal continuity bound (49) one can prove more sharp inequality.

Proposition 2. Let \( \rho \) be a finite rank state in \( \mathcal{S}(\mathcal{H}) \).\(^7\) Then

\[ S(\rho) - S(\sigma) \leq \varepsilon \ln(\operatorname{rank}\rho - 1) + h_2(\varepsilon) \]

for any state \( \sigma \) in \( \mathcal{S}(\mathcal{H}) \) such that \( \frac{1}{2}\| \rho - \sigma \|_1 \leq \varepsilon \leq 1 - 1/\operatorname{rank}\rho \) (the l.h.s. of (57) may be equal to \(-\infty\)).

Proof. By the arguments from the proof of Proposition 1 we may assume that the states \( \rho \) and \( \sigma \) have representations (55) and (56) correspondingly.

Let \( n = \operatorname{rank}\rho < +\infty \). Consider the quantum channel

\[ \Phi(\varrho) = \sum_{k=0}^{+\infty} W_k \varrho W_k^*, \quad \varrho \in \mathcal{S}(\mathcal{H}), \]

\(^7\)We assume that \( \dim \mathcal{H} = +\infty \).
where $W_k = \sum_{i=0}^{n-1} |\varphi_i\rangle\langle\varphi_{i+nk}|$ is a partial isometry such that $W_kW_k^*$ and $W_k^*W_k$ are the projectors on the linear spans of the sets $\{\varphi_0, ..., \varphi_{n-1}\}$ and $\{\varphi_{nk}, ..., \varphi_{nk+n-1}\}$ correspondingly.

Let $\tilde{\sigma} = \Phi(\sigma)$. Both states $\rho$ and $\tilde{\sigma}$ are supported by the $n$-dimensional subspace generated by the vectors $\varphi_0, ..., \varphi_{n-1}$. Since $\rho = \Phi(\rho)$ by the construction, we have

$$\|\rho - \tilde{\sigma}\|_1 = \|\Phi(\rho) - \Phi(\sigma)\|_1 \leq \|\rho - \sigma\|_1 \leq 2\varepsilon$$

due to monotonicity of the trace norm under action of a channel. Thus, to derive the claim of Proposition 2 from Audenaert’s continuity bound (49) it suffices to show that $S(\tilde{\sigma}) \leq S(\sigma)$. This can be done by noting that

$$\tilde{\sigma} = \sum_{k=0}^{+\infty} \lambda_{kn}^\rho |\varphi_0\rangle\langle\varphi_0| + \sum_{k=0}^{+\infty} \lambda_{kn+i}^\rho |\varphi_i\rangle\langle\varphi_i| + ... + \sum_{k=0}^{+\infty} \lambda_{kn+n-1}^\rho |\varphi_{n-1}\rangle\langle\varphi_{n-1}|,$$

and hence the state $\sigma$ is majorized by the state $\tilde{\sigma}$ in the sense of [8, Section 13.5].

By combining Propositions 1 and 2 one can obtain the ”mixed” continuity bound:

$$-(\varepsilon \ln(d-1) + h_2(\varepsilon)) \leq S(\rho) - S(\sigma) \leq \varepsilon F_H(E/\varepsilon) - g(\varepsilon)$$

for any states $\rho$ and $\sigma$ in $\mathcal{G}(\mathcal{H})$ such that $\operatorname{Tr}H\rho \leq E$, $\operatorname{rank}\sigma \leq d$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$ provided that $\varepsilon \leq 1 - 1/d$ and $H$ is a positive operator on $\mathcal{H}$ satisfying conditions (11) and (16).

### 4.2 Quantum conditional entropy of quantum-classical states

In this subsection we apply the results of Section 3 to obtain continuity bounds and semi-continuity bounds for the (extended) quantum conditional entropy (QCE) defined in (6) restricting attention to quantum-classical (q-c) states of a bipartite system $AB$, i.e. states $\rho$ and $\sigma$ having the form

$$\rho = \sum_k p_k \rho_k \otimes |k\rangle\langle k| \quad \text{and} \quad \sigma = \sum_k q_k \sigma_k \otimes |k\rangle\langle k|, \quad (58)$$

where $\{p_k, \rho_k\}$ and $\{q_k, \sigma_k\}$ are ensembles of states in $\mathcal{G}(\mathcal{H}_A)$ and $\{|k\rangle\}$ a fixed orthonormal basis in $\mathcal{H}_B$. By using definition (6) one can show (see the proof of Corollary 3 in [42]) that

$$S(A|B)_\rho = \sum_k p_k S(\rho_k) \leq +\infty \quad \text{and} \quad S(A|B)_\sigma = \sum_k q_k S(\sigma_k) \leq +\infty. \quad (59)$$

The expressions in (59) allow us to define the QCE on the set $\mathcal{G}_{qc}$ of all q-c states (including the q-c states $\rho$ with $S(\rho_A) = +\infty$ for which definition (6) does not work). The properties of the von Neumann entropy imply that the QCE defined in such a way
on the convex set \( \mathcal{G}_{qc} \) satisfies inequalities (18) and (19) with \( a_f = 0 \) and \( b_f = h_2 \) with possible values \( +\infty \) in both sides.

In finite dimensions the first continuity bound for the QCE was obtained by Alicki and Fannes in [2]. Then Winter optimized this continuity bound and obtained its specification for q-c states [43]. In 2019, by using the Alhejji-Smith optimal continuity bound for the Shannon conditional entropy (presented in [1]) Wilde proved optimal continuity bound for the QCE restricted to q-c states [42].

The first continuity bound for the QCE in infinite-dimensional quantum system \( AB \) under the energy-type constraint imposed on subsystem \( A \) was obtained by Winter [43]. Another continuity bound for the QCE under the same constraint is presented in Proposition 5 in [35, Section 4.1.4]. Both continuity bounds are universal and cannot be improved by restricting attention to q-c states.

To explain our approach take any \( \varepsilon > 0 \) and assume that \( \rho \) and \( \sigma \) are q-c states with representation (58) such that

\[
||\rho - \sigma||_1 = \sum_k ||p_k \rho_k - q_k \sigma_k||_1 \leq 2\varepsilon.
\]

For each \( k \) let \( \{\varphi_k^i\} \) be the orthonormal basis in \( \mathcal{H}_A \) such that \( \rho_k = \sum_{i} \lambda_i^{\rho_k} |\varphi_i^k\rangle \langle \varphi_i^k| \), where \( \{\lambda_i^{\rho_k}\}_i \) is a non-increasing sequence of eigenvalues of \( \rho_k \). Consider the q-c state

\[
\tilde{\sigma} = \sum_k q_k \tilde{\sigma}_k \otimes |k\rangle \langle k|, \quad \tilde{\sigma}_k = \sum_{i} \lambda_i^{\sigma_k} |\varphi_i^k\rangle \langle \varphi_i^k|,
\]

where \( \{\lambda_i^{\sigma_k}\}_i \) is the non-increasing sequence of eigenvalues of \( \sigma_k \). By Mirsky inequality (1) we have

\[
||\rho - \tilde{\sigma}||_1 = \sum_k ||p_k \rho_k - q_k \tilde{\sigma}_k||_1 \leq \sum_k ||p_k \rho_k - q_k \sigma_k||_1 \leq 2\varepsilon.
\]

Note also that the representation (59) implies that

\[
S(A|B)_\sigma = \sum_k q_k S(\sigma_k) = \sum_k q_k S(\tilde{\sigma}_k) = S(A|B)_{\tilde{\sigma}}.
\]
**Proposition 3.** Let $AB$ be an infinite-dimensional bipartite quantum system.

A) If $\rho$ is a q-c state in $\mathcal{G}(\mathcal{H}_{AB})$ such that $\text{rank}\rho_A$ is finite then

$$S(A|B)_\rho - S(A|B)_\sigma \leq \varepsilon \ln(\text{rank}\rho_A) + g(\varepsilon)$$

(62)

for any q-c state $\sigma$ in $\mathcal{G}(\mathcal{H}_{AB})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$ (the l.h.s of (62) may be equal to $-\infty$).

B) Let $H$ be a positive operator on $\mathcal{H}_A$ satisfying conditions (11) and (16). If $\rho$ is a q-c state in $\mathcal{G}(\mathcal{H}_{AB})$ such that $\text{Tr}H\rho_A \leq E$ then

$$S(A|B)_\rho - S(A|B)_\sigma \leq \varepsilon F_H((E - E_{H,\varepsilon}(\rho))/\varepsilon) + g(\varepsilon) \leq \varepsilon F_H(E/\varepsilon) + g(\varepsilon)$$

(63)

for any q-c state $\sigma$ in $\mathcal{G}(\mathcal{H}_{AB})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, where

$$E_{H,\varepsilon}(\rho) \doteq \sum_k \text{Tr}H[p_k\rho_k - \varepsilon I_A]_+,$$

$[p_k\rho_k - \varepsilon I_A]_+$ is the positive part of the Hermitian operator $p_k\rho_k - \varepsilon I_A$ and the l.h.s. of (63) may be equal to $-\infty$.

For each $E > 0$ and $\varepsilon \in (0, 1]$ there exist q-c states $\rho$ and $\sigma$ in $\mathcal{G}(\mathcal{H}_{AB})$ such that

$$S(A|B)_\rho - S(A|B)_\sigma > \varepsilon F_H(E/\varepsilon), \quad \text{Tr}H\rho_A, \text{Tr}H\sigma_A \leq E \quad \text{and} \quad \frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon.$$

The last claim of Proposition 3 can be proved by taking the q-c states $\rho \otimes |0\rangle\langle 0|$ and $\sigma \otimes |0\rangle\langle 0|$, where $\rho$ and $\sigma$ are the states from the last claim of Proposition 1.

**Note:** The quantity $E_{H,\varepsilon}(\rho)$ monotonically tends to $\text{Tr}H\rho_A$ as $\varepsilon \to 0$. So, the first estimate in (63) with $E = \text{Tr}H\rho_A$ may be essentially sharper than the second one for small $\varepsilon$.

It follows from Proposition 3A that

$$-(\varepsilon \ln(\text{rank}\sigma_A) + g(\varepsilon)) \leq S(A|B)_\rho - S(A|B)_\sigma \leq \varepsilon \ln(\text{rank}\rho_A) + g(\varepsilon)$$

(64)

for any q-c states $\rho$ and $\sigma$ in $\mathcal{G}(\mathcal{H}_{AB})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$.

Since the dimension of the subspace generated by the supports of $\rho_A$ and $\sigma_A$ may be close to $\text{rank}\rho_A + \text{rank}\sigma_A$, in some cases the continuity bound (64) may be sharper than Wilde’s optimal continuity bound for QCE of q-c states proposed in [42].

Proposition 3B implies that

$$|S(A|B)_\rho - S(A|B)_\sigma| \leq \varepsilon F_H(E/\varepsilon) + g(\varepsilon)$$

(65)

for any q-c states $\rho$ and $\sigma$ in $\mathcal{G}(\mathcal{H}_{AB})$ such that $\text{Tr}H\rho_A, \text{Tr}H\sigma_A \leq E$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, where $H$ is a positive operator on $\mathcal{H}_A$ satisfying conditions (11) and (16). It is easy to see that continuity bound (65) is *essentially sharper* than the existing continuity bounds for the QCE under the energy-type constraints mentioned before. The last claim of Proposition 3 shows that continuity bound (65) is *close to optimal*. It can be treated as a generalization of continuity bound (52) for the von Neumann entropy: inequality (65) with the q-c states $\rho \otimes |0\rangle\langle 0|$ and $\sigma \otimes |0\rangle\langle 0|$ turns into inequality (52).

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8Throughout this subsection we assume that $\rho$ and $\sigma$ are q-c states having representation (58).
4.3 Entanglement of formation

The entanglement of formation (EoF) is one of the basic entanglement measures in bipartite quantum systems [7, 20, 30]. In a finite-dimensional bipartite system $AB$ the EoF is defined as the convex roof extension to the set $\mathcal{S}(\mathcal{H}_{AB})$ of the function $\omega \mapsto S(\omega_A)$ on the set $\text{ext}\mathcal{S}(\mathcal{H}_{AB})$ of pure states in $\mathcal{S}(\mathcal{H}_{AB})$, i.e.

$$E_F(\omega) = \inf \sum_k p_k S([\omega_k]_A),$$

where the infimum is over all finite ensembles $\{p_k, \omega_k\}$ of pure states in $\mathcal{S}(\mathcal{H}_{AB})$ with the average state $\omega$ [7]. In this case $E_F$ is a uniformly continuous function on $\mathcal{S}(\mathcal{H}_{AB})$ possessing basic properties of an entanglement measure [20, 30, 27].

The first continuity bound for the EoF in a finite-dimensional bipartite system $AB$ was obtained by Nielsen [24]. Then it was improved by Winter who used his continuity bound for the QCE of q-c states and updated Nielsen’s arguments [43]. Finally, Wilde embedded his optimal continuity bound for the QCE of q-c states into Nielsen-Winter construction to obtain the most accurate continuity bound for the EoF to date [42].

If both systems $A$ and $B$ are infinite-dimensional then there are two versions $E^d_F$ and $E^c_F$ of the EoF defined, respectively, by means of discrete and continuous convex roof extensions

$$E^d_F(\omega) = \inf \sum_k p_k S([\omega_k]_A),$$

$$E^c_F(\omega) = \int \omega' \mu(d\omega') = \omega \int S(\omega'_A) \mu(d\omega'),$$

where the infimum in (66) is over all countable ensembles $\{p_k, \omega_k\}$ of pure states in $\mathcal{S}(\mathcal{H}_{AB})$ with the average state $\omega$ and the infimum in (67) is over all Borel probability measures on the set of pure states in $\mathcal{S}(\mathcal{H}_{AB})$ with the barycenter $\omega$ (the last infimum is always attained) [34, Section 5]. It follows from the definitions that $E^d_F(\omega) \geq E^c_F(\omega)$ for any state $\omega \in \mathcal{S}(\mathcal{H}_{AB})$. In [34] it is shown that the function $E^d_F$ has better properties (as an entanglement measure), in particular, it is lower semicontinuous on $\mathcal{S}(\mathcal{H}_{AB})$ and is equal to zero at any separable state in $\mathcal{S}(\mathcal{H}_{AB})$ (as far as I know, the equality $E^d_F(\omega) = 0$ has not yet been proven if $\omega$ is a countably-non-decomposable separable state such that $S(\omega_A) = S(\omega_B) = +\infty$ [34, Remark 6]).

In [34] it is shown that $E^d_F(\omega) = E^c_F(\omega)$ for any state $\omega$ in $\mathcal{S}(\mathcal{H}_{AB})$ such that $\min\{S(\omega_A), S(\omega_B)\} < +\infty$. The conjecture of coincidence of $E^d_F$ and $E^c_F$ on $\mathcal{S}(\mathcal{H}_{AB})$ is an interesting open question.

Different continuity bounds for the functions $E^d_F$ and $E^c_F$ under the energy-type constraints are obtained by using the Nielsen-Winter technique and different continuity bounds for the QCE. They are described in Section 4.4.2 in [35].

By using the semi-continuity bounds for the QCE of q-c states presented in Proposition 3 and the Nielsen-Winter technique one can obtain semi-continuity bounds for the functions $E^d_F$ and $E^c_F$. 
Proposition 4. Let $AB$ be an infinite-dimensional bipartite quantum system.

A) If $\rho$ is a state in $\mathfrak{S}(\mathcal{H}_{AB})$ such that $\text{rank}\rho_A$ is finite then

$$E^*_F(\rho) - E^*_F(\sigma) \leq \delta \ln(\text{rank}\rho_A) + g(\delta), \quad E^*_F = E^d_F, E^c_F,$$

for any state $\sigma$ in $\mathfrak{S}(\mathcal{H}_{AB})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1$, where $\delta = \sqrt{\varepsilon(2 - \varepsilon)}$ and the l.h.s. of (68) may be equal to $-\infty$. 

B) If $\rho$ is a state in $\mathfrak{S}(\mathcal{H}_{AB})$ such that $\text{Tr}H\rho_A \leq E$, where $H$ is a positive operator on $\mathcal{H}_A$ satisfying conditions (11) and (16), then

$$E^*_F(\rho) - E^*_F(\sigma) \leq \delta F_H(E/\delta) + g(\delta), \quad E^*_F = E^d_F, E^c_F,$$

for any state $\sigma$ in $\mathfrak{S}(\mathcal{H}_{AB})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1$, where $\delta = \sqrt{\varepsilon(2 - \varepsilon)}$ and the l.h.s. of (69) may be equal to $-\infty$.

Note A: The conditions imposed in both parts of Proposition 4 on the state $\rho$ imply that $E^c_F(\rho) = E^d_F(\rho) < +\infty$ (since they imply that $S(\rho_A) < +\infty$).

Proof. Both claims of the proposition for $E^*_F = E^d_F$ are obtained from the semi-continuity bounds for the QCE of q-c states presented in Proposition 3 by using the Nielsen-Winter technique (see the proof of Corollary 4 in [43]). The only difference consists in the necessity to use an $\epsilon$-optimal decomposition of the state $\sigma$, because an optimal decomposition may not exist in the infinite-dimensional case.

The claims for $E^*_F = E^c_F$ can be proved by a simple approximation. We may assume that $E^c_F(\sigma) < +\infty$ (since otherwise (68) and (69) obviously hold). We may also assume that $\delta = \sqrt{\varepsilon(2 - \varepsilon)} \neq 0$, where $\varepsilon = \frac{1}{2}\|\rho - \sigma\|_1$.

Let $\{P_n\}$ be any sequence of finite-rank projectors in $\mathfrak{B}(\mathcal{H}_A)$ strongly converging to the unit operator $I_A$. Then the sequence of states $\sigma_n = c_n^{-1}(P_n \otimes I_B)\sigma(P_n \otimes I_B)$, $c_n = \text{Tr}P_n\sigma_A$, (well defined for all $n$ large enough) converges to the state $\sigma$.

By the monotonicity of $E_F^c$ under selective LOCC operations (which follows from part A-2 of Theorem 2 in [34]) we have $c_nE^c_F(\sigma_n) \leq E^c_F(\sigma)$ for all $n$. So, since $c_n \to 1$ as $n \to +\infty$, the lower semicontinuity of $E^c_F$ (mentioned after (67)) implies that

$$\lim_{n \to +\infty} E^c_F(\sigma_n) = E^c_F(\sigma) < +\infty.$$ 

(70)

Since $E^c_F(\sigma_n) = E^d_F(\sigma_n)$ for all $n$ (because $S([\sigma_n]_A) < +\infty$) and $E^c_F(\rho) = E^d_F(\rho)$ (by Note A), the proved claims of the proposition show that the inequalities (68) and (69) with $E^*_F = E^c_F$ hold provided that $\sigma$ and $\delta$ are replaced, respectively, by $\sigma_n$ and $\delta_n = \sqrt{\varepsilon_n(2 - \varepsilon_n)}$, $\varepsilon_n = \frac{1}{2}\|\rho - \sigma_n\|_1$, for all $n$. Since $\delta_n$ tends to $\delta > 0$ as $n \to +\infty$, by taking the limit as $n \to +\infty$ one can prove, due to (70), the validity of inequalities (68) and (69) with $E^*_F = E^c_F$ (as $F_H$ is a continuous function on $(0, +\infty)$ by the condition (16)).

Proposition 4A implies that

$$-(\delta \ln(\text{rank}\sigma_A) + g(\delta)) \leq E_F(\rho) - E_F(\sigma) \leq \delta \ln(\text{rank}\rho_X) + g(\delta)$$ 

(71)
for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H}_{AB})$ such that $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1$ and $\text{rank}\rho_X, \text{rank}\sigma_A < +\infty$, where $\delta = \sqrt{\varepsilon(2 - \varepsilon)}$, $X$ is either $A$ or $B$ and $E_F(\omega) \doteq E_F^d(\omega) = E_F^e(\omega), \omega = \rho, \sigma$ (the equalities $E_F^d(\omega) = E_F^d(\omega), \omega = \rho, \sigma$, follow from the conditions $\text{rank}\rho_X, \text{rank}\sigma_A < +\infty$).

Continuity bound (71) with $X = A$ may be sharper than the continuity bound for the EoF obtained by Wilde in [42] in the case when the dimension of the subspace generated by the supports of states $\rho_A$ and $\sigma_A$ is close to $\text{rank}\rho_A + \text{rank}\sigma_A$. Continuity bound (71) with $X = B$ has no analogues and gives a way to obtain more accurate estimate for the quantity $E_F(\rho) - E_F(\sigma)$.

Let $H$ be a positive operator on $\mathcal{H}_A$ satisfying conditions (11) and (16). Proposition 4B implies that

$$|E_F(\rho) - E_F(\sigma)| \leq \delta F_H(E/\delta) + g(\delta) \quad (72)$$

for any states $\rho$ and $\sigma$ in $\mathcal{S}(\mathcal{H}_{AB})$ such that $\text{Tr}H\rho_A, \text{Tr}H\sigma_A \leq E < +\infty$ and $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon \leq 1$, where $\delta = \sqrt{\varepsilon(2 - \varepsilon)}$ and $E_F(\omega) \doteq E_F^d(\omega) = E_F^e(\omega), \omega = \rho, \sigma$ (the equalities $E_F^d(\omega) = E_F^d(\omega), \omega = \rho, \sigma$, follow from the conditions $\text{Tr}H\rho_A, \text{Tr}H\sigma_A \leq E$, since they imply that $S(\rho_A), S(\sigma_A) < +\infty [40]$). Continuity bound (72) is essentially sharper than all the existing continuity bounds for the EoF under the energy-type constraint (described in Section 4.4.2 in [35]).

Remark 8. In all the continuity bounds and the semi-continuity bounds for the EoF presented in this subsection the condition $\frac{1}{2}\|\rho - \sigma\|_1 \leq \varepsilon$, where $\varepsilon$ is such that $\delta = \sqrt{\varepsilon(2 - \varepsilon)}$, can be replaced by the condition $F(\rho, \sigma) \geq 1 - \delta^2$, where $F(\rho, \sigma) \doteq \|\sqrt{\rho} - \sqrt{\sigma}\|_1^2$ is the fidelity of $\rho$ and $\sigma$ [17, 41]. This follows from the Nielsen-Winter arguments used in their proofs. The use of fidelity as a measure of closeness of the states $\rho$ and $\sigma$ makes these continuity bounds essentially sharper in some cases.

### 4.4 Characteristics of discrete random variables

Assume that $\{\tau_i^n\}_{i \in \mathbb{N}}, \ldots, \{\tau_i^n\}_{i \in \mathbb{N}}$ are given orthonormal bases in separable Hilbert spaces $\mathcal{H}_{A_1}, \ldots, \mathcal{H}_{A_n}$ correspondingly. Then the set $\mathcal{S}_\tau$ of states in $\mathcal{S}(\mathcal{H}_{A_1 \cdots A_n})$ diagonalizable in the basis

$$\{\tau_{i_1}^1 \otimes \cdots \otimes \tau_{i_n}^n\}_{(i_1, \ldots, i_n) \in \mathbb{N}^n} \quad (73)$$

can be identified with the set $\mathcal{P}_n$ of $n$-variate probability distributions $\{p_{i_1, \ldots, i_n}\}_{(i_1, \ldots, i_n) \in \mathbb{N}^n}$.

Any $n$-variate probability distribution $\bar{p} = \{p_{i_1, \ldots, i_n}\}_{(i_1, \ldots, i_n) \in \mathbb{N}^n}$ can be treated as a joint distribution of some discrete random variables $X_1, \ldots, X_n$. So, a value of some entropic characteristic of $n$-partite quantum system $A_1 \cdots A_n$ at the state $\omega \in \mathcal{S}_\tau$ identified with the distribution $\bar{p}$ coincides with the value of the corresponding classical characteristic of the random variables $X_1, \ldots, X_n$. For example, the quantum mutual information $I(A_1 : A_2)$ of a bipartite state $\omega = \sum_{i,j} p_{i,j} |\tau_i^1 \otimes \tau_j^2\rangle \langle \tau_i^1 \otimes \tau_j^2|$ is equal to the classical mutual information $I(X_1 : X_2)$ of random variables $X_1, X_2$ having joint distribution $\{p_{i,j}\}$.
Thus, since the set $\mathcal{S}_\tau$ consists of commuting states, one can apply Corollaries 1 and 2 in Section 3.2 to obtain continuity bounds for entropic characteristics of classical discrete random variables under constraints of different types.

To formulate the main results of this section we have to introduce "classical" analogues of the classes $L_n^m(C, D|\mathcal{S}_0)$ described in Section 3.2. Denote by $T_n^m(C, D|\mathcal{P}_0)$ the class of all functions on a convex subset $\mathcal{P}_0$ of $\mathcal{P}_n$ taking values in $(-\infty, +\infty]$ that satisfy the inequalities

$$f(\lambda \bar{p} + (1 - \lambda) \bar{q}) \geq \lambda f(\bar{p}) + (1 - \lambda) f(\bar{q}) - d_f^+ h_2(\lambda)$$

and

$$f(\lambda \bar{p} + (1 - \lambda) \bar{q}) \leq \lambda f(\bar{p}) + (1 - \lambda) f(\bar{q}) + d_f^- h_2(\lambda)$$

for all distributions $\bar{p}$ and $\bar{q}$ in $\mathcal{P}_0$ and any $\lambda \in [0, 1]$ with possible infinite values in both sides, where $d_f^+$ and $d_f^-$ are nonnegative numbers such that $d_f^+ + d_f^- = D$, and the double inequality

$$-c_f^- C_m(\bar{p}) \leq f(\bar{p}) \leq c_f^+ C_m(\bar{p}), \quad C_m(\bar{p}) = \sum_{k=1}^m H(\bar{p}_k), \quad (74)$$

for any distribution $\bar{p}$ in $\mathcal{P}_0$ with possible infinite values in all sides, where $\bar{p}_k$ denotes the marginal distribution of $\bar{p}$ corresponding to the $k$-th component, i.e. $[\bar{p}_k]_i = \sum_{(i_1, \ldots, i_n) \backslash \{i_k\}} p_{i_1, \ldots, i_{k-1}, i, i_{k+1}, \ldots, i_n}$, $H(\cdot)$ is the Shannon entropy and $c_f^-$ and $c_f^+$ are nonnegative numbers such that $c_f^- + c_f^+ = C$.

We will denote by $T_n^m(C, D)$ the class $T_n^m(C, D|\mathcal{P}_n^m)$, where

$$\mathcal{P}_n^m = \{ \bar{p} \in \mathcal{P}_n \mid H(\bar{p}_k) < +\infty, k = 1, m \}$$

is the maximal set on which all the functions satisfying (74) are finite.

Within this notation the Shannon entropy belongs to the class $T_1^1(1, 1|\mathcal{P}_1)$ (due to its concavity and the classical version of inequality (2)). The Shannon conditional entropy $H(X_1|X_2)$ (also called equivocation) extended to the set $\mathcal{P}_n^1$ by the classical version of formula (6) belongs to the class $T_2^1(1, 1) = T_2^1(1, 1|\mathcal{P}_2)$. This follows from its nonnegativity, concavity and the classical versions of inequalities (4) and (5).

The mutual information $I(X_1: \ldots : X_n)$ (also called total correlation [39]) defined on the set $\mathcal{P}_n$ by the classical version of formula (7) belongs to the classes $T_n^{n-1}(1, n|\mathcal{P}_n)$ and $T_n^n(1 - 1/n, n|\mathcal{P}_n)$. This can be shown by using its nonnegativity, the inequalities\footnote{The definitions of all the notions from the classical information theory used in this section can be found in [12].}

$$I(X_1: \ldots : X_n) \lambda_{\bar{p} + (1 - \lambda) \bar{q}} \geq \lambda I(X_1: \ldots : X_n)_{\bar{p}} + (1 - \lambda) I(X_1: \ldots : X_n)_{\bar{q}} - h_2(\lambda),$$

$$I(X_1: \ldots : X_n) \lambda_{\bar{p} + (1 - \lambda) \bar{q}} \leq \lambda I(X_1: \ldots : X_n)_{\bar{p}} + (1 - \lambda) I(X_1: \ldots : X_n)_{\bar{q}} + (n - 1)h_2(\lambda)$$

These inequalities are the classical versions of the inequalities in [33, formula (10)].
valid for all distributions $\tilde{p}$ and $\tilde{q}$ in $\mathfrak{P}_n$ and any $\lambda \in [0, 1]$ with possible value $+\infty$ in both sides, and the upper bounds
\[
I(X_1; \ldots; X_n)_\tilde{p} \leq \sum_{k=1}^{n-1} H(\tilde{p}_k), \quad I(X_1; \ldots; X_n)_\tilde{p} \leq \frac{n-1}{n} \sum_{k=1}^{n} H(\tilde{p}_k)
\]
(the first upper bound follows from the classical version of the representation in [33, formula (9)] with trivial system $C$ and upper bound (8) along with the remark after it, the second one is obtained from the first by simple symmetry arguments).

The one-to-one correspondence between the set $\mathfrak{P}_n$ of all $n$-variate probability distributions and the set $\mathfrak{G}_\tau$ of quantum states in $\mathfrak{G}(\mathcal{H}_{A_1\ldots A_n})$ diagonalizable in basis (73) allows us to identify the class $T_n^m(C, D|\mathfrak{P}_0)$ with the class $L_n^m(C, D|\mathfrak{P}_0^\varepsilon)$, where $\mathfrak{P}_0^\varepsilon$ is the subset of $\mathfrak{G}_\tau$ corresponding to the subset $\mathfrak{P}_0$. So, we may directly apply Corollaries 1 and 2 to obtain continuity bounds for functions from the classes $T_n^m(C, D|\mathfrak{P}_0)$.

We will use the total variation distance (21) between $n$-variate probability distributions $\tilde{p} = \{p_{i_1\ldots i_n}\}$ and $\tilde{q} = \{q_{i_1\ldots i_n}\}$ (considered as probability measures on $\mathbb{N}^n$). In this case we have
\[
TV(\tilde{p}, \tilde{q}) = \frac{1}{2} \sum_{i_1, \ldots, i_n} |p_{i_1\ldots i_n} - q_{i_1\ldots i_n}| = \frac{1}{2} \|\rho - \sigma\|_1,
\]
where $\rho$ and $\sigma$ are the states in $\mathfrak{G}_\tau$ corresponding to the distributions $\tilde{p}$ and $\tilde{q}$.

Corollary 1 Section 3.2. with $\mathfrak{G}_0 = \mathfrak{P}_0^\varepsilon$ implies the following

**Proposition 5.** Let $\mathfrak{P}_0$ be a convex subset of $\mathfrak{P}_n$ with the property
\[
\Delta^\pm(\tilde{p}, \tilde{q}) \in \mathfrak{P}_0 \quad \text{for any } \tilde{p} \text{ and } \tilde{q} \in \mathfrak{P}_0 \text{ such that } \tilde{p} \neq \tilde{q},
\]
where $\Delta^+(\tilde{p}, \tilde{q})$ and $\Delta^-(\tilde{p}, \tilde{q})$ are the $n$-variate probability distributions with the entries $c[p_{i_1\ldots i_n} - q_{i_1\ldots i_n}]_+$ and $c[q_{i_1\ldots i_n} - p_{i_1\ldots i_n}]_+$, $c = 1/TV(\tilde{p}, \tilde{q})$ ($[x]_+ = \max\{x, 0\}$).

A) If $f$ is a function in $T_n^m(C, D|\mathfrak{P}_0)$ then
\[
|f(\tilde{p}) - f(\tilde{q})| \leq C\varepsilon \ln d_m(\tilde{p}, \tilde{q}) + Dg(\varepsilon)
\]
for any distributions $\tilde{p}$ and $\tilde{q}$ in $\mathfrak{P}_0$ such that $d_m(\tilde{p}, \tilde{q}) = \max\{\prod_{k=1}^{m} |\tilde{p}_k|, \prod_{k=1}^{m} |\tilde{q}_k|\}$ is finite and $TV(\tilde{p}, \tilde{q}) \leq \varepsilon$, where $|\tilde{p}_k|$ and $|\tilde{q}_k|$ are the numbers of nonzero entries of the 1-variate marginal distributions $\tilde{p}_k$ and $\tilde{q}_k$ corresponding to the $k$-th component.

B) If $f$ is a nonnegative function in $T_n^m(C, D|\mathfrak{P}_0)$ and $\tilde{p}$ is a distribution in $\mathfrak{P}_0$ such that $d_m(\tilde{p}) = \prod_{k=1}^{m} |\tilde{p}_k|$ is finite then
\[
f(\tilde{p}) - f(\tilde{q}) \leq C\varepsilon \ln d_m(\tilde{p}) + Dg(\varepsilon)
\]
for any distribution $\tilde{q}$ in $\mathfrak{P}_0$ such that $TV(\tilde{p}, \tilde{q}) \leq \varepsilon$ (the l.h.s. of (77) may be equal to $-\infty$).

**Remark 9.** It is easy to see that property (75) holds for the convex sets $\mathfrak{P}_n$ and $\mathfrak{P}_m$. 26
Note A: In Proposition 5B we impose no restrictions on the distribution $\bar{q}$ other than the requirement $\text{TV}(\bar{p}, \bar{q}) \leq \varepsilon$.

The "energy-constrained" version of Proposition 5 will be obtained for characteristics belonging to the classes $T_\varepsilon^1(C, D|\mathfrak{P}_0)$ (for simplicity).

Let $\mathcal{S} = \{E_i\}_{i=1}^{+\infty}$ be a nondecreasing sequence of nonnegative numbers such that

$$
\sum_{i=1}^{+\infty} e^{-\beta E_i} < +\infty \quad \forall \beta > 0.
$$

Consider the function

$$
F_\mathcal{S}(E) = \sup \left\{ H(\{p_i\}) \mid \{p_i\} \in \mathfrak{P}_1, \sum_{i=1}^{+\infty} E_i p_i \leq E \right\}
$$

$$
= \beta(E)E + \ln \sum_{i=1}^{+\infty} e^{-\beta(E)E_i},
$$

where $\beta(E)$ is defined by the equation $\sum_{i=1}^{+\infty} E_i e^{-\beta E_i} = E \sum_{i=1}^{+\infty} e^{-\beta E_i}$ [40],[32, Prop.1].

It is easy to see that the function $F_\mathcal{S}(E)$ coincides with the function $F_H(E)$ defined in (13) provided that $H$ is a positive operator with the spectrum $\mathcal{S}$. If $\mathcal{S} = \{0, 1, 2, \ldots\}$ then $F_\mathcal{S}(E) = g(E)$ – the function defined in (15). So, Proposition 1 in [32] shows that condition (78) is equivalent to the property

$$
F_\mathcal{S}(E) = o(E) \quad \text{as} \quad E \to +\infty.
$$

Thus, Corollary 2 in Section 3.2 with $\mathcal{G}_0 = \mathfrak{P}_0^\varepsilon$ implies the following

Proposition 6. Let $\mathfrak{P}_0$ be a convex subset of $\mathfrak{P}_n$ with the property (75). Let $\mathcal{S} = \{E_i\}_{i=1}^{+\infty}$ be a nondecreasing sequence of nonnegative numbers such that $E_1 = 0$ satisfying condition (78) and $F_\mathcal{S}$ the function defined in (79).

A) If $f$ is a function in $T_\varepsilon^1(C, D|\mathfrak{P}_0)$ then

$$
|f(\bar{p}) - f(\bar{q})| \leq C\varepsilon F_\mathcal{S}(E/\varepsilon) + Dg(\varepsilon)
$$

for any distributions $\bar{p}$ and $\bar{q}$ in $\mathfrak{P}_0$ such that $\sum_{i=1}^{+\infty} E_i [\bar{p}_1]_i$, $\sum_{i=1}^{+\infty} E_i [\bar{q}_1]_i \leq E$ and $\text{TV}(\bar{p}, \bar{q}) \leq \varepsilon$, where $[\bar{p}_1]_i$ denotes the $i$-th entry of the marginal distribution $\bar{p}_1$.

B) If $f$ is a nonnegative function in $T_\varepsilon^1(C, D|\mathfrak{P}_0)$ and $\bar{p}$ is a distribution in $\mathfrak{P}_0$ such that $\sum_{i=1}^{+\infty} E_i [\bar{p}_1]_i \leq E$ then

$$
f(\bar{p}) - f(\bar{q}) \leq C\varepsilon F_\mathcal{S}((E - E_\varepsilon(\bar{p}))/\varepsilon) + Dg(\varepsilon) \leq C\varepsilon F_\mathcal{S}(E/\varepsilon) + Dg(\varepsilon)
$$

for any distribution $\bar{q}$ in $\mathfrak{P}_0$ such that $\text{TV}(\bar{p}, \bar{q}) \leq \varepsilon$, where

$$
E_\varepsilon(\bar{p}) = \sum_{i=1}^{+\infty} E_i [\bar{p}_1^\varepsilon]_i, \quad [\bar{p}_1]_i = \sum_{i_2, \ldots, i_n} [p_{i_1, i_2, \ldots, i_n} - \varepsilon]_+ \quad ([x]_+ = \max\{x, 0\}),
$$

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and the left hand side of (82) may be equal to $-\infty$.

**Note B:** The equivalence of (78) and (80) implies that the r.h.s. of (81) and (82) tend to zero as $\varepsilon \to 0$.

**Note C:** The quantity $E_\varepsilon(\bar{p})$ monotonically tends to $E(\bar{p}) = \sum_{i=1}^{+\infty} E_i[\bar{p}_i]$, as $\varepsilon \to 0$. So, the first estimate in (82) may be essentially sharper than the second one for small $\varepsilon$ and $E(\bar{p})$ close to $E$.

If $\bar{X} = (X_1, ..., X_n)$ and $\bar{Y} = (Y_1, ..., Y_n)$ are vectors of random variables with probability distributions $\bar{p}$ and $\bar{q}$ in $\mathcal{P}_n$ such that the random variables $X_1$ and $Y_1$ takes values in $\mathcal{S}$ then the constraint $\sum_{i=1}^{+\infty} E_i[\bar{p}_i] \leq E$ (respectively, $\sum_{i=1}^{+\infty} E_i[\bar{q}_i] \leq E$) means that $\mathbb{E}(X_1) \leq E$ (respectively, $\mathbb{E}(Y_1) \leq E$).

**Example 2.** It was mentioned before that the Shannon conditional entropy (equivocation) $H(X_1|X_2)$ defined on the set $\mathcal{P}_2^1$ by the classical version of formula (6) belongs to the class $T_2^1(1, 1)$. Since it is a nonnegative function, Proposition 5B with $\mathcal{P}_0 = \mathcal{P}_2^1$ and Remark 9 imply that

$$H(X_1|X_2) - H(Y_1|Y_2) \leq \varepsilon \ln|\bar{p}_1| + g(\varepsilon)$$

(83)

for any distribution $\bar{p}$ in $\mathcal{P}_2^1$ with finite $|\bar{p}_1|$ and arbitrary distribution $\bar{q}$ in $\mathcal{P}_2^1$ such that $\text{TV}(\bar{p}, \bar{q}) \leq \varepsilon$.

It follows from (83) that

$$-(\varepsilon \ln|\bar{q}_1| + g(\varepsilon)) \leq H(X_1|X_2) - H(Y_1|Y_2) \leq \varepsilon \ln|\bar{p}_1| + g(\varepsilon)$$

(84)

for any distributions $\bar{p}$ and $\bar{q}$ in $\mathcal{P}_2^1$ such that $|\bar{p}_1|, |\bar{q}_1| < +\infty$ and $\text{TV}(\bar{p}, \bar{q}) \leq \varepsilon$.

If $\bar{p}$ and $\bar{q}$ are 2-variate probability distributions such that $[\bar{p}_1]_i = [\bar{q}_1]_i = 0$ for all $i > n$ and $\text{TV}(\bar{p}, \bar{q}) \leq \varepsilon$ then (84) implies that

$$|H(X_1|X_2) - H(Y_1|Y_2)| \leq \varepsilon \ln n + g(\varepsilon)$$

(85)

(the classical version of Winter’s continuity bound for the quantum conditional entropy [43]), while the optimal continuity bound for the Shannon conditional entropy obtained by Alhejji and Smith in [1] claims that

$$|H(X_1|X_2) - H(Y_1|Y_2)| \leq \varepsilon \ln(n - 1) + h_2(\varepsilon)$$

(86)

provided that $\varepsilon \leq 1 - 1/n$.

It is clear that (86) is sharper than (85) but the difference is not too large for $n \gg 1$ and small $\varepsilon$. At the same time, continuity bound (84) may give more accurate estimates for the quantity $H(X_1|X_2) - H(Y_1|Y_2)$ than continuity bound (86).

Assume that the random variables $X_1$ and $Y_1$ takes the values 0, 1, 2, ... Proposition 6B with $\mathcal{P}_0 = \mathcal{P}_2^1$ and Remark 9 imply that

$$H(X_1|X_2) - H(Y_1|Y_2) \leq \varepsilon g(E/\varepsilon) + g(\varepsilon)$$

(87)

for any distributions $\bar{p}$ and $\bar{q}$ in $\mathcal{P}_2^1$ such that $\mathbb{E}(X_1) \doteq \sum_{i=1}^{+\infty} (i - 1)|\bar{p}_1|i \leq E$ and $\text{TV}(\bar{p}, \bar{q}) \leq \varepsilon$.  

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To show the accuracy of the semi-continuity bound (87) take any $E > 0$ and $\varepsilon \in [0, E/(E + 1)]$ and consider the probability distributions $\bar{p} = \{r_i, t_j\}_{i,j \geq 1}$ and $\bar{q} = \{s_i, t_j\}_{i,j \geq 1}$, where $\{t_j\}_{j \geq 1}$ is any probability distribution, while $\{r_i\}_{i \geq 1}$ and $\{s_i\}_{i \geq 1}$ are the probability distributions such that

$$H(\{r_i\}) = Eh_2(\varepsilon/E) + h_2(\varepsilon), \quad \sum_{i=1}^{+\infty} (i - 1)r_i = E, \quad H(\{s_i\}) = 0, \quad \sum_{i=1}^{+\infty} (i - 1)s_i = 0$$

and $\text{TV}(\{r_i\}, \{s_i\}) = \varepsilon$ constructed after Theorem 2 in [6]. Then $E(X_1) = E$, $E(Y_1) = 0$, TV($\bar{p}, \bar{q}$) = $\varepsilon$ and

$$H(X_1|X_2)_{\bar{p}} - H(Y_1|Y_2)_{\bar{q}} = H(\{r_i\}) - H(\{s_i\}) = Eh_2(\varepsilon/E) + h_2(\varepsilon).$$

We see that in this case the l.h.s. of (87) is very close to the r.h.s. of (87) equal to

$$(E + \varepsilon)h_2\left(\frac{\varepsilon}{E + \varepsilon}\right) + (1 + \varepsilon)h_2\left(\frac{\varepsilon}{1 + \varepsilon}\right)$$

provided that $\varepsilon$ is small and $E \gg \varepsilon$.

The semi-continuity bound (87) implies that

$$-(\varepsilon g(E_q/\varepsilon) + g(\varepsilon)) \leq H(X_1|X_2)_{\bar{p}} - H(Y_1|Y_2)_{\bar{q}} \leq \varepsilon g(E_p/\varepsilon) + g(\varepsilon)$$

for any distributions $\bar{p}$ and $\bar{q}$ in $\mathfrak{P}_2$ such that $E(X_1) \doteq \sum_{i=1}^{+\infty} (i - 1)[\bar{p}_i]_i \leq E_p$, $E(Y_1) \doteq \sum_{i=1}^{+\infty} (i - 1)[\bar{q}_i]_i \leq E_q$ and TV($\bar{p}, \bar{q}$) $\leq \varepsilon$.

The main advantage of the proposed technique consists in the possibility to obtain semi-continuity bounds (83) and (87) with no constraint on the distribution $\bar{q}$.

Both statements of Proposition 6 can be generalized to functions from the classes $T^n_m(C, D|\mathfrak{P}_0)$, $m > 1$, by using Corollary 2 in Section 3.2 via the mentioned above identification of $n$-variate probability distributions with quantum states in $\mathcal{S}(\mathcal{H}_{A_1...A_n})$ diagonalizable in the basis (73).

### 4.5 Characteristics of classical states of quantum oscillators

Assume that $A_1...A_n$ is a $n$-mode quantum oscillator and $\mathcal{S}_{cl}(\mathcal{H}_{A_1...A_n})$ is the set of classical states – the convex closure of the family $\{|\tilde{z}\rangle\langle\tilde{z}|\}_{\tilde{z} \in \mathbb{C}^n}$ of coherent states [17, 14, 3]. Each state $\rho$ in $\mathcal{S}_{cl}(\mathcal{H}_{A_1...A_n})$ can be represented as

$$\rho = \int_{\mathbb{C}^n} |\tilde{z}\rangle\langle\tilde{z}|\mu_\rho(dz_1...dz_n), \quad \tilde{z} = (z_1, ..., z_n),$$  \hspace{1cm} (88)$$

where $\mu_\rho$ is a Borel probability measure on $\mathbb{C}^n$. Representation (88) is called Glauber–Sudarshan $P$-representation and generally is written as

$$\rho = \int_{\mathbb{C}^n} |\tilde{z}\rangle\langle\tilde{z}|P_\rho(\tilde{z})dz_1...dz_n,$$
where $P_\rho$ is the $P$-function of a state $\rho$, which in this case is nonnegative and can be treated as a generalized probability density function on $\mathbb{C}^n$ (in contrast to the standard probability density, the function $P_\rho$ may be singular, since the measure $\mu_\rho$ may be not absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{C}^n$) [16, 36].

Representation (88) means that $\mathcal{G}_{cl}(\mathcal{H}_{A_1...A_n}) = \mathcal{O}_{X,\mathfrak{S},\mathfrak{F}}$ - the set defined in (20), where $X = \mathbb{C}^n$, $\mathfrak{S}$ is the Borel $\sigma$-algebra on $\mathbb{C}^n$ and $\mathfrak{F}(z) = |z\rangle\langle z|$.

Let $H_{A_k} = \hat{N}_k = a_k^\dagger a_k$ be the number operator of the $k$-th mode, $k = 1, m$, $m \leq n$. Then the operator $H_m$ defined in (35) is the total number operator for the subsystem $A_1...A_m$ and the function $F_{H_m}$ defined in (36) has the form

$$F_{H_m}(E) = mF_{\hat{N}_1}(E/m) = mg(E/m),$$

where $g$ is the function defined in (15).

Denote by $\rho(\mu)$ the classical state having representation (88) with a given measure $\mu$ in $\mathcal{P}(\mathbb{C}^n)$. It is easy to see that

$$\text{Tr}\hat{N}_k[\rho(\mu)]_{A_k} = \int_{\mathbb{C}} |z|^2 \mu_k(dz), \quad k = 1, 2, ... , n, \quad \text{(89)}$$

where $\mu_k$ is the marginal measure of $\mu$ corresponding to the $k$-th component of $\bar{z}$, i.e. $\mu_k(A) = \mu(\mathbb{C}_1 \times ... \times \mathbb{C}_{k-1} \times A \times \mathbb{C}_{k+1} \times ... \times \mathbb{C}_n)$ for any $A \subseteq \mathbb{C}$.

By applying Theorem 2 in Section 3.2 we obtain the following

**Proposition 7.** Let $\mathcal{G}_0$ be a convex subset of $\mathcal{G}(\mathcal{H}_{A_1...A_n})$ with the property

$$\rho \in \mathcal{G}_0 \cap \mathcal{G}_{cl}(\mathcal{H}_{A_1...A_n}) \Rightarrow \{ \sigma \in \mathcal{G}_{cl}(\mathcal{H}_{A_1...A_n}) | \exists \varepsilon > 0 : \varepsilon \sigma \leq \rho \} \subseteq \mathcal{G}_0 \quad \text{(90)}$$

and $\mathcal{P}_{\mathcal{G}_0}(\mathbb{C}^n)$ the subset of $\mathcal{P}(\mathbb{C}^n)$ consisting of measures $\mu$ such that $\rho(\mu) \in \mathcal{G}_0$.

A) If $f$ is a function in $\hat{L}^m_n(C, D|\mathcal{G}_0)$ then

$$|f(\rho(\mu)) - f(\rho(\nu))| \leq C\varepsilon mg(E/\varepsilon) + Dg(\varepsilon)$$

for any measures $\mu$ and $\nu$ in $\mathcal{P}_{\mathcal{G}_0}(\mathbb{C}^n)$ such that $\sum_{k=1}^m \int_{\mathbb{C}} |z|^2 \mu_k(dz) \leq mE$, $\sum_{k=1}^m \int_{\mathbb{C}} |z|^2 \nu_k(dz) \leq mE$ and $\text{TV}(\mu, \nu) \leq \varepsilon$.

B) If $f$ is a nonnegative function in $\hat{L}^m_n(C, D|\mathcal{G}_0)$ and $\mu$ is a measure in $\mathcal{P}_{\mathcal{G}_0}(\mathbb{C}^n)$ such that $\sum_{k=1}^m \int_{\mathbb{C}} |z|^2 \mu_k(dz) \leq mE$ then

$$f(\rho(\mu)) - f(\rho(\nu)) \leq C\varepsilon mg(E/\varepsilon) + Dg(\varepsilon) \quad \text{(91)}$$

for any measure $\nu$ in $\mathcal{P}_{\mathcal{G}_0}(\mathbb{C}^n)$ such that $\text{TV}(\mu, \nu) \leq \varepsilon$ (the left hand side of (91) may be equal to $-\infty$).

**Example 3.** Consider two-mode quantum oscillator $A_1A_2$. Let $f = I(A_1 : A_2)$ be the quantum mutual information defined in (7). This is the function on the whole space $\mathcal{G}(\mathcal{H}_{A_1A_2})$ taking values in $[0, +\infty]$, which belongs to the class $L^1_2(1, 2|\mathcal{G}_{cl})$. This
follows from the inequalities (9) and (10) and the remark after upper bound (8), since all the states in $\mathcal{G}_{cl}$ are separable. Thus, Proposition 7B with $\mathcal{G}_0 = \mathcal{G}_{cl}$ implies that

$$I(A_1 : A_2)_{\rho(\mu)} - I(A_1 : A_2)_{\rho(\nu)} \leq \varepsilon g(E/\varepsilon) + 2g(\varepsilon)$$  \hspace{1cm} (92)

for any measure $\mu$ in $\mathcal{P}(\mathbb{C}^2)$ such that $\int |z|^2 \mu_1(dz) \leq E$ and arbitrary measure $\nu$ in $\mathcal{P}(\mathbb{C}^2)$ such that $\text{TV}(\mu, \nu) \leq \varepsilon$. The condition $\int |z|^2 \mu_1(dz) \leq E$ can be replaced by the symmetrical condition $\int |z|^2 \mu_1(dz) + \int |z|^2 \mu_2(dz) \leq 2E$ by noting that the function $f = I(A_1 : A_2)$ also belongs to the class $L^2(\mathbb{C}^2)$, which can be proved easily by using upper bound (8), the remark after it and the symmetry arguments.

Continuity bound (92) complements semi-continuity bound (47) for commuting states. Note that universal semi-continuity bound for the quantum mutual information has not been constructed yet (as far as I know).

The semi-continuity bound (92) implies that

$$|I(A_1 : A_2)_{\rho(\mu)} - I(A_1 : A_2)_{\rho(\nu)}| \leq \varepsilon g(E/\varepsilon) + 2g(\varepsilon)$$  \hspace{1cm} (93)

for any measures $\mu$ and $\nu$ in $\mathcal{P}(\mathbb{C}^2)$ such that $\int |z|^2 \mu_1(dz), \int |z|^2 \nu_1(dz) \leq E$ and $\text{TV}(\mu, \nu) \leq \varepsilon$.

The advantage of continuity bound (93) is its simplicity and accuracy in contrast to the universal continuity bounds for QMI under the energy constraint (described in [35, Section 4.2.2]). Its obvious drawback is the use of the total variation distance between representing measures as a quantity describing closeness of classical states (instead of the trace norm distance).

If the measures $\mu$ and $\nu$ representing classical states $\rho$ and $\sigma$ are absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{C}^n$ then $\text{TV}(\mu, \nu) = \frac{1}{2} \| P_{\rho} - P_{\sigma} \|_{L_1}$ - the $L_1$-norm distance between the $P$-functions of $\rho$ and $\sigma$.

5 Remarks on other applications

In this article we proposed a modification of the Alicki-Fannes-Winter technique designed for quantitative continuity analysis of locally almost affine functions on convex sets of states called ”quasi-classical” that can be represented as the set $\Omega_{X, F, \tilde{\omega}}$ defined in (20) by means of some measurable space $\{X, F\}$ and a $F$-measurable $\mathcal{G}(\mathcal{H})$-valued function $\tilde{\omega}(x)$ on $X$. We have used ”quasi-classical” sets of two types:

- the set of all states in $\mathcal{G}(\mathcal{H})$ diagonalizable in a particular basic in $\mathcal{H}$;
- the set of classical states of a multi-mode quantum oscillator.
The scope of potential applications of the proposed method can be expanded using the following

**Lemma 2.** Let $\mathcal{S}_0$ be a closed convex subset of $\mathcal{S}(\mathcal{H})$. Then

$$\mathcal{S}_0 = \mathcal{N}_{\text{cl}(\text{ext}\mathcal{S}_0), \mathcal{B}, \text{Id}},$$

where $\text{cl}(\text{ext}\mathcal{S}_0)$ is the closure of the set of extreme points of $\mathcal{S}_0$, $\mathcal{B}$ is the Borel $\sigma$-algebra on $\text{cl}(\text{ext}\mathcal{S}_0)$ and $\text{Id}$ is the identity map on $\text{cl}(\text{ext}\mathcal{S}_0)$.

The set $\text{cl}(\text{ext}\mathcal{S}_0)$ in (94) can be replaced by any closed subset $\mathcal{S}_*$ of $\mathcal{S}_0$ the convex closure of which coincides with $\mathcal{S}_0$.\(^{11}\)

**Proof.** Both claims of the lemma follow from the $\mu$-compactness of the set $\mathcal{S}(\mathcal{H})$ in terms of [31]. This property of $\mathcal{S}(\mathcal{H})$ established in [18, Proposition 2] can be formulated as

$$\{\mathcal{S} \text{ is a compact subset of } \mathcal{S}(\mathcal{H})\} \iff \{\mathcal{P}_\mathcal{S} \text{ is a compact subset of } \mathcal{P}((\mathcal{S}(\mathcal{H}))\},$$

(95)

where $\mathcal{P}((\mathcal{S}(\mathcal{H}))$ is the set of all Borel probability measures on $\mathcal{S}(\mathcal{H})$ equipped with the topology of weak convergence and $\mathcal{P}_\mathcal{S}$ is the subset of $\mathcal{P}((\mathcal{S}(\mathcal{H}))$ consisting of measures with the barycenter in $\mathcal{S}$ [9, 10]. The nontrivial implication in (95) is "$\Rightarrow$", since "$\Leftarrow$" follows from continuity of the barycenter map w.r.t. the weak convergence.

The $\mu$-compactness of $\mathcal{S}(\mathcal{H})$ allows us to prove for this noncompact set some general results valid for compact sets, in particular, several results from the Choquet theory.\(^{12}\) For example, the $\mu$-compactness of $\mathcal{S}(\mathcal{H})$ implies, by Proposition 5 in [31], that

$$\mathcal{S}_0 = \{b(\mu) \mid \mu \in \mathcal{P}(\text{cl}(\text{ext}\mathcal{S}_0))\},$$

(96)

where $b(\mu) = \int \rho \mu(d\rho)$ is the barycenter of $\mu$ and $\mathcal{P}(\text{cl}(\text{ext}\mathcal{S}_0))$ is the set of all Borel probability measures on $\text{cl}(\text{ext}\mathcal{S}_0)$. This proves (94).

To prove the last claim of the lemma we have to show that (96) holds with $\text{cl}(\text{ext}\mathcal{S}_0)$ replaced by $\mathcal{S}_*$. Assume that $\rho_0$ is a state in $\mathcal{S}_0$. Then there is a sequence $\{\rho_n\}$ from the convex hull of $\mathcal{S}_*$ converging to $\rho_0$. It means that $\rho_n = b(\mu_n)$ for all $n$, where $\mu_n$ is a measure in $\mathcal{P}(\mathcal{S}_*)$ with finite support. Since the set $\{\rho_n\} \cup \{\rho_0\}$ is compact, the implication "$\Rightarrow"$ in (95) shows that the sequence $\{\mu_n\}$ is relatively compact in the topology of weak convergence and hence has a partial limit $\mu_*$. The continuity of the barycenter map implies that $\rho_0 = b(\mu_*)$. \(\square\)

Lemma 2 claims that any state $\rho$ in a closed convex subset $\mathcal{S}_0$ of $\mathcal{S}(\mathcal{H})$ can be represented as

$$\rho = b(\mu), \quad \mu \in \mathcal{P}(\text{cl}(\text{ext}\mathcal{S}_0)).$$

(97)

If $\mathcal{S}_0$ is the set of all states in $\mathcal{S}(\mathcal{H})$ diagonizable in a basic $\{|n\rangle\langle n|\}_{n=0}^{+\infty}$ in $\mathcal{H}$ then $\text{cl}(\text{ext}\mathcal{S}_0) = \text{ext}\mathcal{S}_0 = \{|n\rangle\langle n|\}_{n=0}^{+\infty}$ and representation (97) is the spectral decomposition of $\rho$.

\(^{11}\)The convex closure of a set $S$ in a Banach space is the minimal closed convex set containing $S$.

\(^{12}\)Another corollaries of the $\mu$-compactness of $\mathcal{S}(\mathcal{H})$ are presented in the first part of [34].
If $\mathcal{S}_0$ is the set of classical states of a multi-mode quantum oscillator then $\text{cl}(\text{ext}\mathcal{S}_0) = \text{ext}\mathcal{S}_0 = \{|\bar{z}\rangle\langle\bar{z}|\}_{\bar{z} \in \mathbb{C}^n}$ and (97) is the equivalent form of the $P$-representation (88).

As a nontrivial application of Lemma 2 one can consider the case when $\mathcal{S}_0$ is the set of separable (non-entangled) states of a bipartite quantum system $AB$ defined as the convex closure of the set of product states of $AB$. In this case $\text{cl}(\text{ext}\mathcal{S}_0) = \text{ext}\mathcal{S}_0$ is the set of all pure product states and representation (97) means that any separable state can be represented as a "continuous convex mixture" of pure product states despite the existence of separable states that can not be represented as a countable (discrete) convex mixture of pure product states [19].

By Lemma 2 we may apply the technique developed in this article for quantitative continuity analysis of locally almost affine functions on any closed convex subset $\mathcal{S}_0$ of $\mathcal{S}(\mathcal{H})$ using the total variation distance between representing measures as a quantity describing closeness of the states in $\mathcal{S}_0$ (as it was made for the set of classical states of a quantum oscillator in Section 4.5).

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References

[1] Alhejji, M.A., Smith, G.: A Tight Uniform Continuity Bound for Equivocation, IEEE International Symposium on Information Theory (ISIT), Los Angeles, CA, USA, 2270-2274 (2020); arXiv:1909.00787

[2] Alicki, R., Fannes, M.: Continuity of quantum conditional information, J. Phys. A Math. Gen. 37(5), L55-L57 (2004)

[3] Amosov, G.G.: On Various Functional Representations of the Space of Schwarz Operators. J. Math. Sci. (N.Y.) 252(1), 1–7, (2021)

[4] Audenaert, K.M.R.: A sharp continuity estimate for the von Neumann entropy. J. Math. Phys. A: Math. Theor. 40(28), 8127-8136 (2007)

[5] Becker, S., Datta, N.: Convergence rates for quantum evolution and entropic continuity bounds in infinite dimensions. Commun. Math. Phys. 374, 823-871 (2019); arXiv:1810.00863

[6] Becker, S., Datta, N., Jabbour, M.G.: From Classical to Quantum: Uniform Continuity Bounds on Entropies in Infinite Dimensions. arXiv:2104.02019 (2021)

[7] Bennett, C.H., DiVincenzo, D.P., Smolin, J.A., Wootters, W.K.: Mixed-state entanglement and quantum error correction. Phys. Rev. A 54, 3824-3851 (1996)
[8] Bengtsson, I., Zyczkowski, K.: Geometry of Quantum States: An Introduction to Quantum Entanglement. Cambridge University Press (2017)

[9] Billingsley P.: Convergence of Probability Measures. New York (1968).

[10] Billingsley P.: Probability and Measure. New York (1995).

[11] Bluhm, A., Capel, A., Gondolf, P., Perez-Hernandez, A.: Continuity of quantum entropic quantities via almost convexity. arXiv:2208.00922 (2022).

[12] Cover, T.M., Thomas, J.A.: Elements of information theory. New York (2006)

[13] Fannes, M.: A continuity property of the entropy density for spin lattice systems. Commun. Math. Phys. 31 291-294 (1973)

[14] Gerry, Ch., Knight, P.L.: Introductory Quantum Optics. Cambridge University Press (2005)

[15] Ghourchian, H., Gohari, A., Amini, A.: Existence and continuity of differential entropy for a class of distributions. IEEE Commun. Lett. 21(7), 1469–1472 (2017). https://doi.org/10.1109/LCOMM.2017.2689770

[16] Glauber, R.J.: Coherent and Incoherent States of the Radiation Field. Phys. Rev. 131, 2766 (1963)

[17] Holevo, A.S.: Quantum systems, channels, information. A mathematical introduction. Berlin, DeGruyter (2012)

[18] Holevo, A.S., Shirokov, M.E.: Continuous ensembles and the capacity of infinite-dimensional quantum channels. Theory Probab. Appl. 50(1) 86–98 (2005); arXiv:quant-ph/0408176.

[19] Holevo, A.S., Shirokov, M.E., Werner, R.F.: On the notion of entanglement in Hilbert spaces. Russian Math. Surveys. 60:2, 359–360 (2005); arXiv: quant-ph/0504204.

[20] Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev.Mod.Phys. 81, 865-942 (2009)

[21] Kuznetsova, A.A.: Quantum conditional entropy for infinite-dimensional systems. Theory of Probability and its Applications. 55(4), 709-717 (2011)

[22] Mirsky, L.: Symmetric gauge functions and unitarily invariant norms. Quart. J. Math.Oxford 2(11), 50-59 (1960)

[23] Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press (2000)
[24] Nielsen, M.A.: Continuity bounds for entanglement. Physical Review A 61(6), 064301 (2000)

[25] Lindblad, G.: Expectation and Entropy Inequalities for Finite Quantum Systems. Commun. Math. Phys. 39(2), 111-119 (1974)

[26] Lindblad, G.: Entropy, information and quantum measurements. Commun. Math. Phys. 33, 305-322 (1973)

[27] Khatri, S., Wilde, M.M.: Principles of Quantum Communication Theory: A Modern Approach. arXiv:2011.04672 (2020)

[28] Mosonyi, M., Hiai, F.: On the quantum Renyi relative entropies and related capacity formulas. IEEE Trans. Inf. Theory 57(4), 2474-2487 (2011)

[29] Ohya, M., Petz, D.: Quantum Entropy and Its Use, Theoretical and Mathematical Physics. Springer Berlin Heidelberg (2004)

[30] Plenio, M.B., Virmani, S.: An introduction to entanglement measures. Quantum Inf. Comput. 7(1-2), 1-51 (2007)

[31] Protasov, V.Yu., Shirokov, M.E.: Generalized compactness in linear spaces and its applications. Sb. Math. 200(5), 697–722 (2009); arXiv: 1002.3610

[32] Shirokov, M.E.: Entropy characteristics of subsets of states I. Izv. Math. 70(6), 1265-1292 (2006); arXiv: quant-ph/0510073

[33] Shirokov, M.E.: Uniform continuity bounds for characteristics of multipartite quantum systems. J. Math. Phys. 62(9), 92206 , 31 pp. (2021); arXiv:2007.00417

[34] Shirokov, M.E.: On properties of the space of quantum states and their application to the construction of entanglement monotones. Izv. Math. 74(4), 849–882 (2010); arXiv: 0804.1515

[35] Shirokov, M.E.: Quantifying continuity of characteristics of composite quantum systems. Phys. Scr. 98 042002 (2023)

[36] Sudarshan, E.C.G.: Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams. Phys. Rev. Lett. 10, 277 (1963)

[37] Synak-Radtke, B., Horodecki, M.: On asymptotic continuity of functions of quantum states. Journal of Physics A General Physics. 39(26), L423 (2006)

[38] Talagrand, M.: Pettis Integral and Measure Theory. Mem. Amer. Math. Soc. 51(307), ix+224 (1984)

[39] Watanabe, S.: Information theoretical analysis of multivariate correlation. IBM Journal of Research and Development 4, 66–82 (1960)
[40] Wehrl, A.: General properties of entropy. Rev. Mod. Phys. 50, 221-250 (1978)

[41] Wilde, M.M.: Quantum Information Theory. Cambridge, UK: Cambridge Univ. Press, (2013)

[42] Wilde, M.M.: Optimal uniform continuity bound for conditional entropy of classical-quantum states. Quantum Information Processing. 19, Article no. 61 (2020)

[43] Winter, A.: Tight uniform continuity bounds for quantum entropies: conditional entropy, relative entropy distance and energy constraints. Commun. Math. Phys. 347(1), 291-313 (2016); arXiv:1507.07775 (v.6)