Transversal fluctuations for increasing subsequences on the plane

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Abstract. Consider a realization of a Poisson process in $\mathbb{R}^2$ with intensity 1 and take a maximal up/right path from the origin to $(N, N)$ consisting of line segments between the points, where maximal means that it contains as many points as possible. The number of points in such a path has fluctuations of order $N^\chi$, where $\chi = 1/3$, [BDJ]. Here we show that typical deviations of a maximal path from the diagonal $x = y$ is of order $N^\xi$ with $\xi = 2/3$. This is consistent with the scaling identity $\chi = 2\xi - 1$ which is believed to hold in many random growth models.

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1. Introduction and results

The fluctuations in many random growth models, for example in first-passage percolation, are described by two exponents, $\chi$ and $\xi$, see e.g. [KS] and [LNP]. The exponent $\chi$ describes the longitudinal whereas $\xi$ describes the transversal fluctuations. In first-passage percolation the length of a minimizing path from the origin to $(N, N)$ has fluctuations of order $N^\chi$, and the minimizing path has typical deviations from the diagonal $x = y$ of order $N^\xi$. General heuristic arguments (see [KS]) suggest that the scaling identity $\chi = 2\xi - 1$ is valid in any dimension, compare the heuristic argument below. In two dimensions it is predicted that $\chi = 1/3$ and hence we should have $\xi = 2/3$. Since $\xi > 1/2$ one says that the minimizing path is superdiffusive.

We will consider a related model where it is known that $\chi = 1/3$ and prove that in this model we actually have $\xi = 2/3$. The model is a Poissonized version of the problem of the longest increasing subsequence in a random permutation introduced in [Ha], see also [AD]. In this model one considers a Poisson process with intensity 1 in $\mathbb{R}^2_+$ and looks at a maximal up/right path from the origin to $(N, N)$ consisting of line segments between the Poisson points, where maximal means that it contains as many points as possible. The length of a path is the number of Poisson points in the path, and the length of a maximal path has fluctuations of order $N^{1/3}$, see [BDJ]. In this paper we will prove that the typical deviations of the maximal paths from $x = y$ are of order $N^{2/3}$.

The proof uses the line of argument, for first-passage percolation models, initiated in [NP], to prove $\chi' \geq 2\xi - 1$ (where $\chi'$ is closely related to $\chi$), and [LNP] to prove lower (superdiffusive) bounds on a suitably defined $\xi$. A related argument was used to analyze the corresponding problem for crossing Brownian motion in a Poissonian potential in [Wü], and the present paper follows the arguments in [Wü]. A heuristic argument goes as follows. The length of a typical maximal path from the origin to $(x, y)$ is $\sim 2\sqrt{xy}$, see [AD]. Hence, a maximal path from the origin to $(N, N)$ that passes through $(N(t - \delta), N(t + \delta))$, $0 < t < 1, \delta$ small, is shorter by the amount

$$2\sqrt{N(t - \delta)N(t + \delta)} + 2\sqrt{N(1 - t + \delta)N(1 - t - \delta)} - 2\sqrt{N^2}.$$ 

This should be of the same order as the length fluctuations, i.e. $O(N^\chi)$, which gives $\delta^2 = O(N^{\chi-1})$. Thus, $N^\chi \sim N^\xi \sim N^{\chi/2+1/2}$, that is $2\xi - 1 = \chi$ and hence $\xi = 2/3$ since $\chi = 1/3$. The argument used below essentially makes this rigorous.
We will now give the precise definitions. Let $\mathbb{P}$ denote the Poissonian law with fixed intensity 1 on the space $\Omega$ of locally finite, simple, pure point measures on $\mathbb{R}^2$; $\omega = \sum_i \delta_{\zeta_i} \in \Omega$, $\zeta_i = (x_i, y_i)$ are the points in $\omega$. Write $(x, y) \prec (x', y')$ if $x < x'$ and $y < y'$. Given $\omega$ and two points $w \prec w'$ in $\mathbb{R}^2$ an up/right path $\pi$ from $w$ to $w'$ is a subsequence $\{\zeta_{i_k}\}_{k=1}^M$ of points in $\omega$ such that

$$w \prec \zeta_{i_1} \prec \ldots \prec \zeta_{i_M} \prec w'.$$

The length, $|\pi|$, of $\pi$ is $M$, the number of Poisson points in the path. Let $\Pi(w, w'; \omega)$ denote the set of all up/right paths from $w$ to $w'$ in $\omega$. If $K$ is a convex subset of $\mathbb{R}^2$ we let $\Pi^K(w, w'; \omega)$ denote all up/right paths $\pi$ from $w$ to $w'$ inside $K$, i.e. $\pi \subseteq K$ and $w, w' \in K$. Let

$$d(w, w'; \omega) = \max\{|\pi|; \pi \in \Pi(w, w'; \omega)|,$$

and

$$d^K(w, w'; \omega) = \max\{|\pi|; \pi \in \Pi^K(w, w'; \omega)|.$$

Let $\ell_N(\sigma)$ denote the length of a longest increasing subsequence in a random permutation $\sigma \in S_N$ (uniform distribution). If $i_1 < \ldots < i_n$ and $\sigma(i_1) < \ldots < \sigma(i_n)$ we have an increasing subsequence of length $n$ and $\ell_N(\sigma)$ is the length of the longest such sequence. We define the Poissonized distribution function by

$$\phi_n(\lambda) = e^{-\lambda} \sum_{N=0}^{\infty} \frac{\lambda^N}{N!} P[\ell_N(\sigma) \leq n],$$

$[\ell_0(\sigma) \equiv 0]$. Let $a(w, w')$ denote the area of the rectangle $[w, w']$ with corners at $w$ and $w'$. Now,

$$\mathbb{P}[d(w, w') \leq n] = \sum_{N=0}^{\infty} \mathbb{P}[d(w, w') \leq n \mid \omega([w, w']) = N] \mathbb{P}[\omega([w, w']) = N],$$

and, see [Ha] or [AD], $\mathbb{P}[d(w, w') \leq n \mid \omega([w, w']) = N] = P[\ell_N(\sigma) \leq n]$. Hence

$$\mathbb{P}[d(w, w') \leq n] = \phi_n(a(w, w')). \quad (1.1)$$

By Lemma 7.1 in [BDJ] we have a very good control of the function $\phi_n(\lambda)$. Let

$$t = 2^{1/3}(n + 1)^{-1/3} (n + 1 - 2\sqrt{\lambda}). \quad (1.2)$$
Then for any fixed $t$ in $\mathbb{R}$,
\[
\lim_{\lambda \to \infty} \phi_n(\lambda) = F(t),
\]
(1.3)
where $F(t)$ is the Tracy-Widom largest eigenvalue distribution for GUE, see [TW] and [BDJ]. The distribution function $F(t)$ is given by
\[
F(t) = \exp\left(-\int_t^\infty (x-t)u(x)^2 dx\right),
\]
where $u(x)$ is the solution of the Painlevé II equation
\[
\frac{d^2 u}{dx^2} = 2u(x)^3 + xu(x), \quad \text{and } u(x) \sim \text{Ai}(x) \text{ as } x \to \infty,
\]
where $\text{Ai}(x)$ is the Airy function. From this formula and the asymptotics of $u(x)$, see [BDJ], it follows that $0 < F(0) < 1$, which will be used below. Furthermore we have the following estimates. There are positive constants $\delta, T_0, c_1, c_2$ so that if $T_0 \leq t \leq 2 - 2/3(n+1)^{2/3}$, then
\[
|\log \phi_n(\lambda)| \leq c_1 \exp(-c_2t^{3/2}),
\]
(1.4)
and if $-\delta(n+1)^{2/3} \leq t \leq -T_0$, then
\[
\phi_n(\lambda) \leq c_1 \exp(c_2t^3),
\]
(1.5)
for all sufficiently large $n$. The estimate (1.4) also follows from the results in [Se]. These estimates will be important in the proof of our theorem.

Let $C(\gamma, N)$ be the cylinder of width $N^\gamma$ from $0$ to $w_N = (N, N)$:
\[
C(\gamma, N) = \{(x, y); 0 \leq x + y \leq 2N, -\sqrt{2}N^\gamma \leq -x + y \leq \sqrt{2}N^\gamma\}.
\]
Denote by
\[
\Pi_{\text{max}}(w, w'; \omega) = \{\pi \in \Pi(w, w'; \omega); |\pi| = d(w, w'; \omega)\},
\]
the set of maximal paths from $w$ to $w'$. We are interested in the size of the fluctuations of maximal paths around the diagonal $x = y$, the transversal fluctuations. Let $A_N^{\gamma}$ be the event that all maximal paths from $0$ to $w_N$ are contained in the cylinder $C(\gamma, N)$,
\[
A_N^{\gamma} = \{\omega \in \Omega; \text{for all } \pi \in \Pi_{\text{max}}(0, w_N; \omega) \text{ we have } \pi \subseteq C(\gamma, N)\}.
\]
The exponent of transversal fluctuations, $\xi$, is then defined by
\[
\xi = \inf\{\gamma > 0; \liminf_{N \to \infty} \mathbb{P}[A_N^{\gamma}] = 1\}.
\]
(1.6)
We can now state the main result of the paper.
Theorem 1.1. For the model defined above the exponent of transversal fluctuations $\xi = 2/3$.

The proof of the theorem occupies the next section.

Remark 1.2. We can consider the analogous problem for the growth model introduced in [Jo]. Let $w(i, j), (i, j) \in \mathbb{Z}_+^2$, be independent geometrically (or exponentially) distributed random variables and consider

$$G(N) = \max\{ \sum_{(i,j) \in \pi} w(i, j) ; \pi \text{ an up/right path from (1,1) to (N,N)} \}.$$  

In [Jo] it is proved that there are positive constants $a$ and $b$ so that $(G(N) - aN)/bN^{1/3}$ converges in distribution to a random variable with distribution function $F(t)$. In analogy with above we can consider the transversal deviations of a maximal path and define the exponent $\xi$. If we had large deviation estimates for $\mathbb{P}[G(N) \leq n]$ analogous to (1.4) and (1.5) we could copy the proof given in the next section and show that $\xi = 2/3$ in this case also. In [Jo] an estimate like (1.4) is proved, but (1.5) is open. It follows from [BR] that $\mathbb{P}[G(N) \leq n]$ is given by a certain $n \times n$ Toeplitz determinant just as $\phi_n(\lambda)$, and it might be possible to prove the analogue of (1.5) using Riemann-Hilbert techniques as in [BDJ].

2. Proof of $\xi \geq 2/3$

We will first prove that $\xi \geq 2/3$. Pick $\gamma \in (\xi, 1)$ and $\epsilon > 0$ (small). That $\xi < 1$ follows from the proof in sect. 3 that $\xi \leq 2/3$, which is independent of the present section. By the definition of $\xi$ there is an $N_0$ such that

$$\mathbb{P}[A_N^\gamma] \geq 1 - \epsilon \quad (2.1)$$

for all $N \geq N_0$. If $\omega \in A_N^\gamma$, then every maximal path from 0 to $w_N$ is contained in the cylinder $C(\gamma, N)$, so writing $C_1 = C(\gamma, N)$, we see that $d^{C_1}(0, w_N; \omega) = d(0, w_N; \omega)$. Hence, by (2.1),

$$\mathbb{P}[d^{C_1}(0, w_N) = d(0, w_N)] \geq 1 - \epsilon, \quad (2.2)$$

if $N \geq N_0$.

Set $v_1 = (1/\sqrt{2},1/\sqrt{2})$ and $v_2 = (-1/\sqrt{2},1/\sqrt{2})$. Let $m_N = 3N\gamma v_2$ and let $C_2$ be the cylinder $C_2 = C_1 + m_N$. Pick a $b$ such that $\gamma < b < 1$, and assume that
$N$ is so large that $N^b - 4N^\gamma > 0$. Define the points $A, B, C$ on the sides of $C_2$ by

$$
\overline{OA} = (N^b + 2N^\gamma)v_1 + 2N^\gamma v_2, \\
\overline{OB} = (N^b + 4N^\gamma)v_1 + 4N^\gamma v_2, \\
\overline{OC} = N^b v_1 + 4N^\gamma v_2.
$$

$ABC$ is a right angle triangle with the right angle at $A$, the side $AB$ is vertical with $A$ on the lower side of $C_2$ and $B$ on the upper side. Divide the vertical side $AB$ into $K = K(N)$ segments $z_{i-1}z_i, i = 1, \ldots, K$, where $z_0 = A$ and $z_K = B$. Let $L_i$ be the part of the straight line through $z_i$, parallel to the $x$-axis, lying in $C_2$. The parallelogram between $L_{i-1}$ and $L_i$ in $C_2$ is denoted by $F_i, i = 1, \ldots, K$. We also define the analogous geometrical objects at the other end of the cylinder, close to $m_N + w_N$, by translating the whole picture by $t_N = \sqrt{2N} - 6N^\gamma - 2N^b$,

$$
z_i' = z_i + t_Nv_1, \; F_i' = F_i + t_Nv_1, \; \overline{OA'} = \overline{OA} + t_Nv_1 \text{ and } \overline{OB'} = \overline{OB} + t_Nv_1.
$$

Given a Borel set $F$, $\omega(F)$ is the number of Poisson points in $F$. Let $\pi = \{\zeta_1, \ldots, \zeta_M\}, \zeta_1 \prec \cdots \prec \zeta_M$, be a maximal path in $\Pi^{C^2}(m_N, m_N + w_N; \omega)$ and let $\pi^*$ be the curve obtained by joining $\zeta_i$ and $\zeta_{i+1}, i = 0, \ldots, K$, by straight line segments, $\zeta_0 = m_N$ and $\zeta_{K+1} = m_N + w_N$. The curve $\pi^*$ intersects $AB$ at some point $P$ and $A'B'$ at some point $Q$. The point $P$ belongs to $\bar{F}_i$ and $Q$ to $\bar{F}'_j$ for some $i, j$. We will write $z(\omega) = z_i$ and $z'(\omega) = z'_j$. (If $P = z_i$ for some $i$ we let $z(\omega) = z_i$ and analogously for $Q$.) If we set $D_N(\omega) = \max_i \omega(\bar{F}_i) + \max_j \omega(\bar{F}'_j)$, then

$$
d^{C^2}(m_N, m_N + w_N) \leq d^{C^2}(m_N, z(\omega)) + d^{C^2}(z(\omega), z'(\omega)) + d^{C^2}(z'(\omega), m_N + w_N) + D_N(\omega). \quad (2.3)
$$

Note that $z(\omega) \in A = \{z_0, \ldots, z_K\}$ and $z'(\omega) \in A' = \{z'_0, \ldots, z'_K\}$.

**Lemma 2.1.** Let $K = [8N^{2\gamma}] + 1$. Then

$$
P[D_N(\omega) \geq d] \leq C(8N^{2\gamma} + 1)e^{-d/2}, \quad (2.4)
$$

for all $d \geq 1$, where $C$ is a numerical constant.

**Proof:** Since

$$
\{D_N(\omega) \geq d\} \subseteq \{\max_i \omega(\bar{F}_i) \geq \frac{d}{2}\} \cup \{\max_j \omega(\bar{F}'_j) \geq \frac{d}{2}\}
$$

we have

$$
P[D_N(\omega) \geq d] \leq 2KP[\omega(\bar{F}_i) \geq d/2]. \quad (2.5)
$$
Here we have used the fact that all the random variables \( \omega(\bar{F}_i), \omega(\bar{F}_j') \) are identically distributed. The area of \( \bar{F}_1 \) is \( 8N^{2\gamma}/K = \lambda \), and thus

\[
\mathbb{P}[\omega(\bar{F}_1) \geq d/2] \leq \sum_{j=[d/2]}^{\infty} e^{-\lambda j/j^1} \leq C \sum_{j=[d/2]}^{\infty} e^{-\lambda f(j/\lambda)}, \tag{2.6}
\]

where \( C \) is a numerical constant and \( f(x) = x \log x + 1 - x \). Here we have used Stirling’s formula. Note that \( f(x) \geq x \) if \( x \geq 9 \) say. Choose \( K = [8N^{2\gamma}] + 1 \), so that \( \lambda \leq 1 \), and assume that \( d \geq 18 \). Then, by (2.6),

\[
\mathbb{P}[\omega(\bar{F}_1) \geq d/2] \leq C \sum_{j=[d/2]}^{\infty} e^{-j} \leq Ce^{-d/2}
\]

and introducing this estimate into (2.5) yields

\[
\mathbb{P}[\omega(\bar{F}_1) \geq d] \leq C(1 + 8N^{2\gamma})e^{-d/2}
\]

for all \( N \geq 1, d \geq 1 \).

Q.E.D

It follows from the estimate (2.4) that

\[
\mathbb{P}[D_N(\omega) \leq 5 \log N] \geq 1 - \epsilon, \tag{2.7}
\]

for all sufficiently large \( N \).

Next, choose \( \kappa_1 \) and \( \kappa_2 \) so that \( 0 < \kappa_1 < 1/3 < \kappa_2 < 1 \).

**Lemma 2.2.** Assume that (2.1) holds. There is a numerical constant \( \eta \in (0, 1) \), such that if \( \epsilon \leq \eta \) and \( N \) is sufficiently large, then

\[
\mathbb{P}[d^{C_1}(0, w_N) - d^{C_2}(m_N, m_N + w_N) \leq -N^{\kappa_1}] \geq \eta. \tag{2.8}
\]

Furthermore, for \( N \) sufficiently large,

\[
\mathbb{P}[|d(0, z(\omega)) - 2\sqrt{a(0, z(\omega))}| \leq N^{b\kappa_2}] \geq 1 - \epsilon, \tag{2.9}
\]

\[
\mathbb{P}[|d(m_N, z(\omega)) - 2\sqrt{a(m_N, z(\omega))}| \leq N^{b\kappa_2}] \geq 1 - \epsilon, \tag{2.10}
\]

\[
\mathbb{P}[|d(z'(\omega), w_N) - 2\sqrt{a(z'(\omega), w_N)}| \leq N^{b\kappa_2}] \geq 1 - \epsilon, \tag{2.11}
\]

\[
\mathbb{P}[|d(z'(\omega), w_N + m_N) - 2\sqrt{a(z'(\omega), w_N + m_N)}| \leq N^{b\kappa_2}] \geq 1 - \epsilon, \tag{2.12}
\]
Proof: The random variables $d^{C_1}(0, w_N)$ and $d^{C_2}(m_N, m_N + w_N)$ are independent. Thus

$$\mathbb{P}[d^{C_1}(0, w_N) - d^{C_2}(m_N, m_N + w_N) \leq -N^{\kappa_1}]$$

$$\geq \mathbb{P}[d^{C_1}(0, w_N) - 2N \leq 0 \text{ and } d^{C_2}(m_N, m_N + w_N) - 2N \geq N^{\kappa_1}]$$

$$= \mathbb{P}[d^{C_1}(0, w_N) - 2N \leq 0] \cdot \mathbb{P}[d^{C_1}(0, w_N) - 2N \geq N^{\kappa_1}]. \quad (2.13)$$

If $\omega \in A_N^\gamma$, then $d^{C_1}(0, w_N) = d(0, w_N)$, and consequently the last expression in (2.13) is greater than or equal to

$$\mathbb{P}[\{d(0, w_N) - 2N \leq 0\} \cap A_N^\gamma] \cdot \mathbb{P}[\{d(0, w_N) - 2N \geq N^{\kappa_1}\} \cap A_N^\gamma]$$

$$\geq (\mathbb{P}[d(0, w_N) - 2N \leq 0] + \mathbb{P}[A_N^\gamma] - 1) \times (\mathbb{P}[d(0, w_N) - 2N \geq N^{\kappa_1}] + \mathbb{P}[A_N^\gamma] - 1). \quad (2.14)$$

By (1.1),

$$\mathbb{P}[d(0, w_N) - 2N \leq 0] = \phi_{2N}(N^2).$$

It follows from (1.3) that $\phi_{2N}(N^2) \to F(0)$ as $N \to \infty$. Furthermore, since $\kappa_1 < 1/3$,

$$\mathbb{P}[d(0, w_N) - 2N \geq N^{\kappa_1}] = 1 - \phi_{[2N+N^{\kappa_1}]}(N^2) \to 1 - F(0),$$

as $N \to \infty$, again by (1.3) and the fact that $\phi_n(\lambda)$ is increasing in $n$. Let $\eta = \frac{1}{3} F(0)(1 - F(0)) > 0$. If $N$ is sufficiently large then $\mathbb{P}[d(0, w_N) - 2N \leq 0] \geq F(0) - \eta$ and $\mathbb{P}[d(0, w_N) - 2N \geq N^{\kappa_1}] \geq 1 - F(0) - \eta$. Since $\mathbb{P}[A_N^\gamma] \geq 1 - \epsilon$ by (2.1) we see that the right hand side of (2.14) is $\geq (F(0) - 2\eta)(1 - F(0) - 2\eta) \geq \eta$. This proves (2.8).

Next, we will prove (2.9). The proofs of (2.10), (2.11) and (2.12) are completely analogous. Note that

$$\mathbb{P}[\{d(0, z(\omega)) - 2\sqrt{a(0, z(\omega))} \leq N^{b\kappa_2}\}] \geq \mathbb{P}[\bigcap_{j=0}^K \{d(0, z_j) - 2\sqrt{a(0, z_j)} \leq N^{b\kappa_2}\}]$$

so it suffices to show that

$$\sum_{j=1}^K \mathbb{P}[d(0, z_j) - 2\sqrt{a(0, z_j)} \geq N^{b\kappa_2}] \leq \epsilon \quad (2.15)$$
for all sufficiently large \( N \). We have \( z_j = \frac{1}{\sqrt{2}}(N^b, N^b + r_j) \), where \( 4N^\gamma \leq r_j \leq 8N^\gamma \), so \( a(0, z_j) = \frac{1}{2}(N^{2b} + N^b r_j) = a_j \). Now,

\[
P[d(0, z_j) - 2\sqrt{a_j} \leq -N^{b\kappa_2}] = \phi[2\sqrt{a_j} - N^{b\kappa_2}](a_j).
\]

In this case \( t \) defined by (1.2) is \( \sim -2^{1/6}N^{b\kappa_2 - b/3} \) and since \( 1/3 < \kappa_2 < 1 \), the condition for (1.5) is fulfilled if \( N \) is sufficiently large and we get

\[
P[d(0, z_j) - 2\sqrt{a_j} \leq -N^{b\kappa_2}] \leq c_3 \exp(-c_4 N^{3b(\kappa_2 - 1/3)}) \tag{2.16}
\]

for some positive constants \( c_3, c_4 \) and all \( j \). Similarly we can use (1.4) to prove that

\[
P[d(0, z_j) - 2\sqrt{a_j} \geq N^{b\kappa_2}] \leq c_5 \exp(-c_6 N^{3b(\kappa_2 - 1/3)}) \tag{2.17}
\]

for some positive constants \( c_5, c_6 \) if \( N \) is sufficiently large. Using (2.16) and (2.17) we see that (2.15) holds if \( N \) is sufficiently large since \( K = [8N^{2\gamma}] + 1 \). This completes the proof of the lemma.

Q.E.D

Denote by \( B_N^\gamma \) the set of \( \omega \) that satisfy all the inequalities inside \( \mathbb{P}[ \ ] \) in (2.7) - (2.12). Then, by (2.7) and Lemma 2.2,

\[
P[B_N^\gamma] \geq \eta - 5\epsilon. \tag{2.18}
\]

Note that for any \( \omega \),

\[
d(0, w_N) \geq d(0, z(\omega)) + d(z(\omega), z'(\omega)) + d(z'(\omega), w_N). \tag{2.19}
\]

The inequalities (2.3) and (2.19) give

\[
d^{C_2}(m_N, m_N + w_N) - d(0, w_N) \leq d^{C_2}(m_N, z(\omega)) + d^{C_2}(z'(\omega), m_N + w_N) - d(0, z(\omega)) - d(z'(\omega), w_N) + D_N(\omega). \tag{2.20}
\]

Now, using (2.20), we see that for \( \omega \in B_N^\gamma \),

\[
d^{C_1}(0, w_N) - d(0, w_N)
\]

\[
= d^{C_1}(0, w_N) - d^{C_2}(m_N, m_N + w_N) + d^{C_2}(m_N, m_N + w_N) - d(0, w_N)
\]

\[
\leq -N^{\kappa_1} + 4N^{b\kappa_2} + 2\sqrt{a(m_N, z(\omega))} + 2\sqrt{a(z'(\omega), m_N + w_N)} - 2\sqrt{a(0, z(\omega))} - 2\sqrt{a(z'(\omega), w_N)} + 5\log N. \tag{2.21}
\]

To proceed we need the following purely geometric lemma.
Lemma 2.3. For all sufficiently large \( N \),

\[
\sqrt{a(m_N, z)} - \sqrt{a(0, z)} \leq 10N^{2\gamma - b} \tag{2.22}
\]

for any \( z \in A \) and

\[
\sqrt{a(z', w_N + m_N)} - \sqrt{a(z', w_N)} \leq 10N^{2\gamma - b} \tag{2.23}
\]

for any \( z' \in A' \).

Proof: We will prove (2.22). The inequality (2.23) then follows by symmetry.

Now, \( a(m_N, z_j) = (N^b + 3N^\gamma)(N^b - 3N^\gamma + r_j)/2, a(0, z_j) = (N^{2b} + r_jN^b)/2 \) and hence

\[
\sqrt{a(m_N, z)} - \sqrt{a(0, z)} = \frac{a(m_N, z) - a(0, z)}{\sqrt{a(m_N, z)} + \sqrt{a(0, z)}} \leq \frac{3r_jN^\gamma}{2\sqrt{2}N^b} \leq 10N^{2\gamma - b},
\]

since \( r_j \leq 8N^\gamma \).

Q.E.D.

Introducing the estimates (2.22) and (2.23) into (2.21) we obtain

\[
d^{C_1}(0, w_N) - d(0, w_N) \leq -N^{\kappa_1} + 5N^{b\kappa_2} + 40N^{2\gamma - b}
\]

for all \( \omega \in B_N^\gamma \) if \( N \) is sufficiently large. Thus, by (2.18),

\[
P[d^{C_1}(0, w_N) - d(0, w_N) \leq -N^{\kappa_1} + 5N^{b\kappa_2} + 40N^{2\gamma - b}] \geq \eta - 5\epsilon \geq \frac{\eta}{2}, \tag{2.24}
\]

if \( \epsilon < \eta/10 \) and \( N \) is sufficiently large. But we also have the estimate (2.2). These estimates are consistent for large \( N \) only if

\[
\kappa_1 \leq \max\{b\kappa_2, 2\gamma - b\}. \tag{2.25}
\]

In this inequality we can let \( \kappa_1 /\nearrow 1/3 \) and \( \kappa_2 \searrow 1/3 \) to get \( 1/3 \leq \max\{b/3, 2\gamma - b\} \) and since \( b < 1 \), we must have \( 1/3 \leq 2\gamma - b \). Here we can let \( \gamma \searrow \xi \) and \( b /\nearrow 1 \) to get \( 1/3 \leq 2\xi - 1 \), i.e. \( \xi \geq 2/3 \).

3. Proof of \( \xi \leq 2/3 \)

We turn now to the proof of the opposite inequality \( \xi \leq 2/3 \). By the definition (1.6) of \( \xi \) we see that we have to show that if \( \gamma > 2/3 \), then

\[
\lim_{N \to \infty} P[\Omega \setminus A_N^\gamma] = 0. \tag{3.1}
\]
If $\omega \in \Omega \setminus A_N^\gamma$, then there is a path $\pi_0 \in \Pi_{\text{max}}(0, w_N; \omega)$ such that $\pi_0$ is not contained in $C(\gamma, N)$. We take one such path. Fix $\gamma \in (2/3, 1)$. Let $\pi_0^*$ be the curve associated to $\pi_0$. Then $\pi_0^*$ intersects the upper and/or the lower sides of $C(\gamma, N)$. Assume that it intersects the upper side. Define a sequence of points on the upper side of $C(\gamma, N)$, $z_j = \left(\frac{jM}{K}, \frac{jM}{K} + \sqrt{2N^\gamma}\right)$, $0 \leq j \leq K$, where $M = N - \sqrt{2N^\gamma}$ and $K = \left[2\sqrt{2N^{1+\gamma}}\right]+1$. Let $D_j$ be the parallelogram with corners at $z_{j-1}$, $z_j$, $(jM/K, jM/K - \sqrt{2N^\gamma})$ and $((j-1)M/K, (j-1)M/K - \sqrt{2N^\gamma})$, $1 \leq j \leq K$.

The curve $\pi_0^*$ intersects the upper side for the first time, going from 0 to $w_N$, in the line segment $z_{j-1}z_j$ for some $j$. We set $z(\omega) = z_{j-1}$. By the choice of $z(\omega)$ we have that

$$d(0, w_N) \leq d(0, z(\omega)) + d(z(\omega), w_N) + \max_{1 \leq j \leq K} \omega(D_j). \quad (3.2)$$

In the case when $\pi_0^*$ does not intersect the upper side but only the lower side, there is a last time where it intersects the lower side and we can assign a point $z(\omega)$ on the lower side so that (3.2) holds. This case is the image under the map $T_N : (x, y) \to (N - x, N - y)$ of the first case. Let $C = \{z_j\}_{j=0}^K$ and let $C'$ be the image of $C$ under $T_N$.

**Lemma 3.1.** Set

$$\Lambda_N = \{\omega; \max_{1 \leq j \leq K} \omega(D_j) \leq 2\log N\},$$

and for each $z \in C \cup C'$, $\delta \in (1/3, 2\gamma - 1)$,

$$E_z = \{\omega; d(0, z) \leq 2\sqrt{a(0, z) + a(0, z)\delta^2/2 + N^\delta},$$

and $d(z, w_N) \leq 2\sqrt{a(z, w_N) + a(z, w_N)\delta^2/2 + N^\delta}\}.$

For any given $\epsilon > 0$, there is an $N_0$ such that if $N \geq N_0$, then

$$\mathbb{P}\left[ \bigcup_{z \in C \cup C'} (\Omega \setminus E_z) \cup (\Omega \setminus \Lambda_N) \right] \leq \epsilon. \quad (3.3)$$

**Proof:** An argument analogous to the one used in the proof of Lemma 2.1 shows that there is a numerical constant $C$ so that

$$\mathbb{P}[\Omega \setminus \Lambda_N] \leq CN^{\gamma-1}.$$  

We consider $z \in C$, the case $z \in C'$ is analogous by symmetry. Recall that $[z, w]$ denotes the rectangle with corners at $z$ and $w$. If $a(0, z) \leq N^{\delta/2}$, then $\mathbb{P}[\omega([0, z]) \geq$
\[ N^\delta \leq C \exp(-N^{\delta}/2) \] for some numerical constant \( C \), by Chebyshev’s inequality. Since we trivially have \( d(0, z; \omega) \leq \omega([0, z]) \), we obtain

\[
P[d(0, z) > 2\sqrt{a(0, z)} + a(0, z)^{\delta/2} + N^\delta] \leq C \exp(-N^{\delta}/2),\tag{3.4}
\]

provided \( a(0, z) \leq N^{\delta/2} \). Now, with \( a = a(0, z) \),

\[
P[d(0, z) > 2\sqrt{a + a^{\delta/2} + N^\delta} \leq 1 - \phi_{[2\sqrt{a} + a^{\delta/2}]}(a).
\]

This last expression can be estimated using (1.4), which gives

\[1 - \phi_{[2\sqrt{a} + a^{\delta/2}]}(a) \leq c'_1 \exp(-c'_2 a^{(\delta-1)/2}).\]

If \( a \geq N^{\delta/2} \), the right hand side is \( \leq c'_1 \exp(-c'_2 N^{\delta(\delta-1)/4}) \) and thus

\[
P[d(0, z) > 2\sqrt{a + a^{\delta/2} + N^\delta} \leq c'_1 \exp(-c'_2 N^{\delta(\delta-1)/4}).\tag{3.5}
\]

We can prove estimates analogous to (3.4) and (3.5) with \( d(0, z) \) replaced by \( d(z, w_N) \) in the same way. Bringing everything together we see that (3.3) holds if \( N \) is sufficiently large. The lemma is proved.

Q.E.D.

Set

\[ B_N^\gamma = (\Omega \setminus A_N^\gamma) \cap \bigcap_{z \in C \cup C'} E_z \cap \Lambda_N.\]

By Lemma 3.1, for \( N \geq N_0 \),

\[
P[\Omega \setminus A_N^\gamma] \leq \epsilon + \P[B_N^\gamma].\tag{3.6}
\]

Since \( a(0, z) \leq N^2 \) and \( a(z, w_N) \leq N^2 \) for any \( z \in C \cup C' \), we see from (3.2) that for \( \omega \in B_N^\gamma \),

\[
d(0, w_N) \leq 2 \log N + 4N^\delta + 2\sqrt{a(0, z(\omega)) + a(z(\omega), w_N)}.\tag{3.7}
\]

We need one more geometric lemma.

**Lemma 3.2.** For any \( z \in C \cup C' \),

\[
\sqrt{a(0, z)} + \sqrt{a(z, w_N)} - \sqrt{a(0, w_N)} \leq -N^{2\gamma-1},\tag{3.8}
\]
if $N$ is sufficiently large.

**Proof:** Again, by symmetry, it suffices to consider the case $z \in C$. Now, $a(0, z_j) = j \frac{N}{K}(j \frac{N}{K} + \sqrt{2N}\gamma)$ and $a(z_j, w_N) = (N - j \frac{N}{K})(N - j \frac{N}{K} - \sqrt{2N}\gamma)$. where $1 \leq j \leq K = \lceil 2\sqrt{2N^{1+\gamma}} \rceil + 1$ and $M = N - \sqrt{2N}\gamma$. Write $x = jM/KN$ and $y = \sqrt{2N}\gamma^{-1}$, so that $0 \leq x \leq 1 - y$. Then,

$$\sqrt{a(0, z)} + \sqrt{a(z, w_N)} - \sqrt{a(0, w_N)} = Nf(x, y), \quad (3.9)$$

where

$$f(x, y) = \sqrt{x^2 + xy + (1 - x)^2} - (1 - x)y.$$  

For a fixed $y \in (0, 1)$ this function assumes its maximum in $[0, 1 - y]$ at $x = (1 - y)/2$, which gives $f(x, y) \leq -y^2/2$. Inserting this estimate into (3.9) and taking $y = \sqrt{2N}\gamma^{-1} < 1$, which is true if $N$ is large enough, proves the lemma.

Q.E.D.

Combining the estimates (3.7) and (3.8), we see that

$$\mathbb{P}[B_N] \leq \mathbb{P}[d(0, w_N) - 2\sqrt{a(0, w_N)} \leq 2 \log N + 4N^\delta - 2N^{2\gamma-1}]. \quad (3.10)$$

To finish the proof we need

**Lemma 3.3.** If $\delta \in (1/3, 2\gamma - 1)$, $\gamma > 2/3$, then

$$\lim_{N \to \infty} \mathbb{P}[d(0, w_N) - 2\sqrt{a(0, w_N)} \leq 2 \log N + 4N^\delta - 2N^{2\gamma-1}] = 0. \quad (3.11)$$

**Proof:** Since $\delta < 2\gamma - 1$, we have that $2 \log N + 4N^\delta - 2N^{2\gamma-1} \leq -N^{2\gamma-1}$ if $N$ is sufficiently large. Thus, by (1.1),

$$\mathbb{P}[d(0, w_N) \leq 2N + 2 \log N + 4N^\delta - 2N^{2\gamma-1}] \leq \mathbb{P}[d(0, w_N) \leq 2N - N^{2\gamma-1}] = \phi_{[2N-N^{2\gamma-1}]}(N^2).$$

The identity (1.2) with $n = [2N - N^{2\gamma-1}]$ and $\lambda = N^2$ gives $t \sim N^{2\gamma-4/3}$, and hence (1.5) gives us the estimate

$$\phi_{[2N-N^{2\gamma-1}]}(N^2) \leq c_1 \exp(-c_2'N^{6\gamma-4}),$$

where $c_2' > 0$. This proves the lemma.

Q.E.D.

Combining (3.11) with (3.6) and (3.10) we have proved (3.1). Thus $\xi \leq 2/3$ and we are done.
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