A natural first step in the classification of all ‘physical’ modular invariant partition functions $\sum N_{L,R} \chi_L \chi_R^*$ lies in understanding the commutant of the modular matrices $S$ and $T$. We begin this paper extending the work of Bauer and Itzykson on the commutant from the $SU(N)$ case they consider to the case where the underlying algebra is any semi-simple Lie algebra (and the levels are arbitrary). We then use this analysis to show that the partition functions associated with even self-dual lattices span the commutant. This proves that the lattice method due to Roberts and Terao, and Warner, will succeed in generating all partition functions. We then make some general remarks concerning certain properties of the coefficient matrices $N_{L,R}$, and use those to explicitly find all level 1 partition functions corresponding to the algebras $B_n$, $C_n$, $D_n$, and the 5 exceptionals. Previously, only those associated to $A_n$ seemed to be generally known.
1. Introduction

The relevance of two-dimensional conformal field theories [1] to the study of string theories and second order phase transitions in statistical systems is well-known, hence the considerable attention devoted in the literature to their classification. An important subclass of these are the Wess-Zumino-Witten models [2]. The partition function of one associated with (untwisted affine) Kac-Moody algebra \( \hat{g} = g^{(1)} \) and level \( k \) [3,4] can be written in the following way:

\[
Z(z_L z_R | \tau) = \sum N_{\lambda L \lambda R} \chi_{\lambda L}(z_L, \tau) \chi_{\lambda R}(z_R, \tau^*). \tag{1.1}
\]

\( \chi_{\lambda} \) is the normalized character (see [3]) of the representation of \( \hat{g} \) with (horizontal) highest weight \( \lambda \). The (finite) sum in eq.(1.1) is over all level \( k \) highest weights \( \lambda_L, \lambda_R \). The coefficients \( N_{\lambda L \lambda R} \) are numbers (multiplicities).

Most writers consider the restricted partition functions \( Z(\tau) \equiv Z(0,0|\tau) \) (however in [5] it is argued that for \( c \geq 1 \) the complex vectors \( z \) should be retained because the restricted partition functions cannot carry enough information to specify the theory). In this paper we will retain the vectors \( z \). Of course the restricted partition functions can be recovered at the end by substituting \( z = 0 \).

We will restrict our attention to the case where the right- and left-moving sectors correspond to the same algebra \( \hat{g} \) and level \( k \). For a discussion of the heterotic case, where \( \hat{g}_L, k_L \neq \hat{g}_R, k_R \), see [6].

There are three properties the sum in eq.(1.1) must satisfy in order to be the partition function of a sensible conformal field theory:

- (P1) it must be modular invariant. This is equivalent to the two conditions:
  \[
  Z(z_L z_R | \tau + 1) = Z(z_L z_R | \tau), \tag{1.2a}
  \]
  \[
  \exp[-k\pi i(z_L^2/\tau - z_R^2/\tau^*)] Z(z_L/\tau, z_R/\tau - 1/\tau) = Z(z_L z_R | \tau); \tag{1.2b}
  \]

- (P2) the coefficients \( N_{\lambda L \lambda R} \) in (1.1) must be non-negative integers; and
- (P3) \( N_{00} = 1 \) (the zero vector here is the highest weight of the singlet representation of level \( k \)).

If a function \( Z \) in (1.1) satisfies (P1), we will call it an invariant; if in addition each \( N_{\lambda L \lambda R} \geq 0 \), we shall call it a positive invariant; and if it satisfies (P1), (P2), and (P3) we will call it a physical invariant. This paper is concerned with the problem of finding all physical invariants corresponding to a given choice of algebra and level. It will not address the question of which of these physical invariants are actually realized by a well-defined theory.

The problem of classifying all physical invariants is a difficult one. Several examples of these invariants are known. The usual techniques for finding these include conformal embeddings [7], simple currents [8] and outer automorphisms of the Kac-Moody algebra [9]. The pattern seems to be that for a fixed algebra \( \hat{g} \), there are a finite number of infinite series \( A_k, D_k, \) etc. of physical invariants, each defined for all levels \( k \) lying on certain arithmetic sequences, together with some exceptional invariants \( E_k \), defined only
for finitely many $k$. The most famous example is the A-D-E classification for $\hat{g} = A_1^{(1)}$ [10]. (In Sec.5 we make a small step towards establishing this general pattern, by proving that for each choice of algebras and levels, there are only finitely many physical invariants.)

In spite of the large numbers of known physical invariants, there are very few completeness proofs which determine all physical invariants belonging to a certain class. The most significant example is the $A_1^{(1)}$ completeness proof [10]. Another one concerns the level 1 $A_n^{(1)}$ physical invariants [11]. In ref.[8] it was remarked that for level 1 simply-laced $\hat{g}$, the method of simple currents reduces to the bosonic lattice construction, and for this reason conjectured their list of physical invariants for $k = 1$, $\hat{g} = D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ and $E_8^{(1)}$ was complete. We will prove this conjecture in Sec.5, and complete the search for all level 1 physical invariants by finding all corresponding to the non-simply-laced $\hat{g}$ as well. These $k = 1$ physical invariants are explicitly listed in Thm.5.

A frustrating feature of these completeness proofs is that they tend to be primarily number-theoretic, unlike the more algebraic techniques for finding these invariants. Related to this is that, whereas these algebraic techniques provide a very elegant derivation for several of these invariants, some of the exceptional invariants are difficult to derive in these ways. For these reasons, a method recently given by Warner [12] and, independently, by Roberts and Terao [13], holds much promise. They propose to generate these invariants by using the Weyl-Kac formula and theta functions associated with even self-dual lattices. Their method will be briefly described below. This method is more number-theoretic than algebraic. Moreover, all invariants, exceptional as well as those lying in infinite series, are treated on an equal footing. At least for small levels and algebras, it is quite practical. We will prove that this lattice method succeeds in generating all invariants — in fact, a small subclass $L_*$ of the even self-dual lattices suffice to span all invariants (see Cor.2).

In Sec.2 we will introduce the notation and terminology used in the later sections. In Sec.3 we generalize the arguments of [14] and find a basis of the commutant corresponding to any (semi-)simple algebra and levels. In Sec.4 we will briefly describe the Roberts-Terao-Warner lattice method, and then prove its completeness. At the beginning of Sec.5 we include a few general comments and useful tools concerning physical invariants and their “shape”. We apply these later in the section to find a complete list of all level 1 physical invariants corresponding to $\hat{g} = B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$.

The essence of the proof in Sec.4 that lattice partition functions span the commutant lies in the observation that the integral basis found in [14] for $A_n^{(1)}$ consists essentially of lattice partition functions. All that remains in Secs.3 and 4 then is the reasonably straightforward generalization of this argument to any semi-simple algebra. Sec.5 however is independent of this analysis. The commutant is not only a vector space (this is what [14] is exclusively concerned with), but it also has a much richer algebraic structure. It is difficult to imagine that this additional structure will not also be very valuable to exploit. Sec.5 is a preliminary attempt in that direction.

2. Notation and terminology

Before we begin the main body of this paper, it is necessary to establish some notation
and terminology, some of which is non-standard. For a much more complete description of Kac-Moody algebras, see e.g. [3,4].

Let $g$ be any simple finite-dimensional Lie algebra. Let $M = M_g$ be its coroot lattice. Then the dual lattice $M^*$ is defined to be spanned by the fundamental weights $\beta_1, \ldots, \beta_n$, where $n$ is the rank of $g$ and the dimension of $M$ and $M^*$. Let $\rho = \sum \beta_i$. Let $\hat{g} = g^{(1)}$ denote the untwisted affine Kac-Moody algebra corresponding to $g$. An integrable irreducible representation is given by a positive integer $k$ (called the level) and a highest weight vector $\lambda \in M^*$. The set of all possible highest weights corresponding to level $k$ representations will be called $P_+(g,k)$, and is defined by

$$P_+(g,k) = \left\{ \sum_{i=1}^n \ell_i \beta_i \mid 0 \leq \ell_i, \ell_i \in \mathbb{Z}, \sum \ell_i a_i^\vee \leq k \right\},$$

where the numbers $a_i^\vee$ are called the colabels of $g$. The number $1 + \sum a_i^\vee$ is denoted $h^\vee$ and is called the dual Coxeter number.

The relevant facts about lattices that we require can be found e.g. in [15]. By $M^{(t)}$ we mean the scaled lattice $\sqrt{t}M$. The theta series of a translate $v + \Lambda$ of any Euclidean lattice $\Lambda$ is defined to be

$$\Theta(v + \Lambda)(z|\tau) \overset{\text{def}}{=} \sum_{x \in v + \Lambda} \exp[\pi i \tau x^2 + 2\pi i z \cdot x],$$

where $\tau \in \mathbb{C}$ satisfies $\text{Im} \ \tau > 0$, and where the complex vector $z$ lies in the complexification $\mathbb{C} \otimes \Lambda = \left\{ \sum c_i \cdot x_i \mid c_i \in \mathbb{C}, \ x_i \in \Lambda \right\}$.

The Weyl-Kac character formula gives us a convenient expression for the (normalized) character $\chi^{g,k}_\lambda$ of the representation of $\hat{g}$ with level $k$ and highest weight $\lambda \in P_+(g,k)$:

$$\chi^{g,k}_\lambda(z,\tau) = \frac{\sum_{w \in W(g)} \epsilon(w) \Theta(\frac{\lambda + \rho}{\sqrt{k + h^\vee}} + M^{(k+h^\vee)}(\sqrt{k + h^\vee}w(z)|\tau))}{D_g(z|\tau)},$$

$$D_g(z|\tau) \overset{\text{def}}{=} \sum_{w \in W(g)} \epsilon(w) \Theta(\frac{\rho}{\sqrt{h^\vee}} + M^{(h^\vee)}(\sqrt{h^\vee}w(z)|\tau)).$$

Here, $W(g)$ is the (finite) Weyl group of $g$ and $\epsilon(w) = \det w \in \{\pm 1\}$, and $z \in \mathbb{C} \otimes M$.

By the Weyl-folded commutant $\Omega_W(g,k)$ we mean the (complex) space of all functions

$$Z(z_L z_R|\tau) = \sum_{\lambda,\lambda' \in P_+(g,k)} N_{\lambda \lambda'} \chi^{g,k}_\lambda(z_L,\tau) \chi^{g,k}_{\lambda'}(z_R,\tau)^*$$

invariant under the modular group, i.e. those $Z$ in (2.4a) satisfying eqs.(1.2). It is not hard to show that two functions $Z$ and $Z'$ are equal iff their coefficient matrices $N$ and $N'$ are equal; we will use the invariant $Z$ interchangeably with its matrix $N$. No confusion should result.
Our task in the next two sections of this paper is to understand that commutant. A convenient way to get at it is through a closely related space, which we will call the theta-commutant $\Omega_{\text{th}}(g, k)$. It consists of all modular invariant functions

$$Z(z_L z_R | \tau) = \sum_{\lambda, \lambda' \in \Lambda^*/\Lambda} N_{\lambda \lambda'} t^{g,k}_{\lambda}(z_L, \tau) t^{g,k}_{\lambda'}(z_R, \tau)^*, \quad (2.4b)$$

$$t^{g,k}_{\lambda}(z, \tau) = \frac{\Theta(\lambda + \Lambda)(\sqrt{k + h^v z|\tau})}{D_g(z|\tau)}, \quad (2.4c)$$

where $\Lambda = M^{(h^v + k)}$. Here also we have $Z = Z'$ iff $N = N'$. The Weyl-Kac formula (2.3) tells us that by expanding its numerator, any $Z \in \Omega_W(g, k)$ can also be thought of as lying in $\Omega_{\text{th}}(g, k)$. Moreover, we can use (2.3a) to define $\chi_{\lambda}^{g,k}$ for all $\lambda \in M^*$. Then we learn from [3] that either

$$\chi_{\lambda}^{g,k}(z, \tau) = 0 \quad (2.5a)$$

for all $z, \tau$, or there exists a unique $\epsilon \in \{\pm 1\}$ and $\mu \in P_+(g, k)$ such that

$$\chi_{\lambda}^{g,k}(z, \tau) = \epsilon \chi_{\mu}^{g,k}(z, \tau) \quad (2.5b)$$

for all $z, \tau$. Thus (2.3) also defines the process (which we shall call Weyl-folding) by which an element of $\Omega_{\text{th}}$ can be associated with an element of $\Omega_W$ (see e.g. (4.1d)). Note that the theta-commutant $\Omega_{\text{th}}$ is isomorphic to the Hilbert space $E$ considered in [14].

The functions $\chi_{\lambda}$ and $t_{\lambda}$ behave quite nicely under the modular transformations $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$:

$$\chi_{\lambda}^{g,k}(z, \tau + 1) = \sum_{\lambda' \in P_+(g, k)} (T^W(g, k))_{\lambda \lambda'} \chi_{\lambda'}^{g,k}(z, \tau), \quad \text{where} \quad (2.6a)$$

$$(T^W(g, k))_{\lambda \lambda'} = \exp[\pi i (\lambda + \rho)^2/h^v + k - \pi i \rho^2/h^v] \delta_{\lambda \lambda'}; \quad (2.6b)$$

$$\exp[-k\pi iz^2/\tau] \chi_{\lambda}^{g,k}(z, \tau, -1/\tau) = \sum_{\lambda' \in P_+(g, k)} (S^W(g, k))_{\lambda \lambda'} \chi_{\lambda'}^{g,k}(z, \tau), \quad \text{where} \quad (2.6c)$$

$$(S^W(g, k))_{\lambda \lambda'} = \frac{i^{\|\Delta_+\|}}{(h^v + k)^{n/2} \sqrt{|M|}} \sum_{w \in W(g)} \epsilon(w) \exp[-2\pi i \frac{w(\lambda' + \rho) \cdot (\lambda + \rho)}{h^v + k}]; \quad (2.6d)$$

$$t^{g,k}_{\lambda}(z, \tau + 1) = \sum_{\lambda' \in \Lambda^*/\Lambda} (T^{\text{th}}(g, k))_{\lambda \lambda'} t^{g,k}_{\lambda'}(z, \tau), \quad \text{where} \quad (2.6e)$$

$$(T^{\text{th}}(g, k))_{\lambda \lambda'} = \exp[\pi i \lambda^2 - \pi i \rho^2/h^v] \delta_{\lambda \lambda'}; \quad (2.6f)$$

$$\exp[-k\pi iz^2/\tau] t^{g,k}_{\lambda}(z, \tau, -1/\tau) = \sum_{\lambda' \in \Lambda^*/\Lambda} (S^{\text{th}}(g, k))_{\lambda \lambda'} t^{g,k}_{\lambda'}(z, \tau), \quad \text{where} \quad (2.6g)$$

$$(S^{\text{th}}(g, k))_{\lambda \lambda'} = \frac{i^{\|\Delta_+\|}}{(h^v + k)^{n/2} \sqrt{|M|}} \exp[-2\pi i \lambda' \cdot \lambda]. \quad (2.6h)$$
In these equations, $\|\Delta_+\|$ denotes the number of positive roots of $g$, and $|M|$ denotes the determinant of the lattice $M$. As before, $\Lambda = M^{(h^\vee + k)}$ and $n$ is the rank of $g$.

The matrices $T^W(g, k), S^W(g, k), T^{\text{th}}(g, k)$ and $S^{\text{th}}(g, k)$ are unitary and symmetric. One of the main reasons $\Omega_W$ will be studied indirectly through $\Omega_{\text{th}}$ is that the matrix $S^{\text{th}}$ is simpler than $S^W$.

Note that $Z^W = \sum_{\lambda \lambda'} N^W_{\lambda \lambda'} \chi_\lambda \chi^*_{\lambda'}$ lies in $\Omega_W(g, k)$ iff both

\begin{align}
(T^W(g, k))^\dagger N^W(T^W(g, k)) &= N^W, \\
(S^W(g, k))^\dagger N^W(S^W(g, k)) &= N^W;
\end{align}

$Z^{\text{th}} = \sum_{\lambda \lambda'} N^{\text{th}}_{\lambda \lambda'} t_\lambda t^*_{\lambda'}$ lies in $\Omega_{\text{th}}(g, k)$ iff both

\begin{align}
(T^{\text{th}}(g, k))^\dagger N^{\text{th}}(T^{\text{th}}(g, k)) &= N^{\text{th}}, \\
(S^{\text{th}}(g, k))^\dagger N^{\text{th}}(S^{\text{th}}(g, k)) &= N^{\text{th}}.
\end{align}

The extension of these remarks to the semi-simple case is trivial. By a type $\mathcal{T}$ we mean the collection of ordered pairs

$$\mathcal{T} = \{\{g_1, k_1\}, \{g_2, k_2\}, \ldots, \{g_m, k_m\}\},$$

where each $g_i$ is a simple finite-dimensional Lie algebra, and $k_i$ is a positive integer. Define

\begin{align}
P_+(\mathcal{T}) &= P_+(g_1, k_1) \times \cdots \times P_+(g_m, k_m), \\
\Lambda(\mathcal{T}) &= M_{g_1}^{(h^\vee + k_1)} \oplus \cdots \oplus M_{g_m}^{(h^\vee + k_m)}, \\
\chi_\lambda(\mathcal{T})(z, \tau) &= \chi_{\lambda_1}^{g_1, k_1}(z_1, \tau) \cdots \chi_{\lambda_m}^{g_m, k_m}(z_m, \tau), \\
t_\lambda(\mathcal{T})(z, \tau) &= t_{\lambda_1}^{g_1, k_1}(z_1, \tau) \cdots t_{\lambda_m}^{g_m, k_m}(z_m, \tau), \\
W(\mathcal{T}) &= W(g_1) \times \cdots \times W(g_m),
\end{align}

where ‘$\times$’ in (2.8b, f) denotes the cartesian product of sets and ‘$\oplus$’ in (2.8c) denotes the orthogonal direct sum of lattices, and where in (2.8d) $\lambda = (\lambda_1, \ldots, \lambda_m) \in P_+(\mathcal{T})$, in (2.8e) $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda(\mathcal{T})^*/\Lambda(\mathcal{T})$, and in (2.8d, e) $z = (z_1, \ldots, z_m) \in \mathbb{C} \setminus \Lambda(\mathcal{T})$.

The modular matrices $S$ and $T$ become the matrix tensor products

\begin{align}
T^W(\mathcal{T}) &= T^W(g_1, k_1) \otimes \cdots \otimes T^W(g_m, k_m), \\
S^W(\mathcal{T}) &= S^W(g_1, k_1) \otimes \cdots \otimes S^W(g_m, k_m), \\
T^{\text{th}}(\mathcal{T}) &= T^{\text{th}}(g_1, k_1) \otimes \cdots \otimes T^{\text{th}}(g_m, k_m), \\
S^{\text{th}}(\mathcal{T}) &= S^{\text{th}}(g_1, k_1) \otimes \cdots \otimes S^{\text{th}}(g_m, k_m).
\end{align}

The commutants of type $\mathcal{T}$ are written $\Omega_W(\mathcal{T})$ and $\Omega_{\text{th}}(\mathcal{T})$; of course the analogues of eqs.(2.7) remain valid.
Clearly, if \( Z_i \in \Omega_W(g_i, k_i) \) has coefficient matrix \( N_i \), then the function corresponding to \( N_1 \otimes \cdots \otimes N_m \) lies in \( \Omega_W(T) \). The converse is not true, nor do such tensor products even \( \text{span} \) \( \Omega_W(T) \), in general. A trivial example is \( g_1 = g_2 = A_1, k_1 = k_2 = 1 \): each \( \Omega_W(g_i, k_i) \) here is only 1-dimensional, while \( \Omega_W(T) \) is 2-dimensional.

When \( m = 1 \) in (2.8) we call \( T \) a simple type; otherwise \( T \) is called semi-simple. The classification of physical invariants of semi-simple type unfortunately does not seem to reduce in any convenient way to the classification of physical invariants of simple type.

3. The theta-commutant as a vector space

Our goal in this section is to extend the work of [14]. In particular, they generated the \( A_n \) commutant using certain orbits of \( \text{SL}(2, \mathbb{Z}) \). We will extend their analysis, which was done only for \( A_n \), to any semi-simple algebra. This will set the stage for the following section where we prove the lattice partition functions of the Roberts-Terao-Warner method span the commutant.

It should be noted that [14] is only concerned with the structure of the commutant as a vector space. The commutant has a much richer structure than that (e.g. it is an algebra), and some of this additional structure will be exploited in Sec.5.

We will be concerned here solely with the theta-commutant. Because of that, some of the labels ‘th’ will be dropped.

Because we will be referring to these so frequently, call

\[
G = \Lambda(T)^*/\Lambda(T), \quad G_2 = (\Lambda(T))^{(2)*}/(\Lambda(T))^{(2)},
\]

where as usual the superscript ‘(2)’ refers to scaling the lattice by \( \sqrt{\Sigma} \). For any \( \mu \in G_2 \), by ‘\( \sqrt{2\mu} \)’ we will always mean the coset \( \sqrt{2\mu} + \Lambda(T) \in G \).

To find a convenient description of \( \Omega_{th} \), it will be necessary to simplify the action of the modular matrices \( S \) and \( T \). To do this, for each pair \( \mu, \mu' \in G_2 \) define a matrix \( \{\mu, \mu'\} \) by

\[
\{\mu, \mu'\}_{\lambda, \lambda'} = \delta_{\lambda, \sqrt{2\mu} + \lambda'} \exp[2\pi i (\mu \cdot \mu' + \sqrt{2\mu'} \cdot \lambda')],
\]

for all \( \lambda, \lambda' \in G \). This \( |\Lambda(T)||\Lambda(T)| \) matrix is the coefficient matrix of

\[
\sum_{\lambda, \lambda'} \{\mu, \mu'\}_{\lambda, \lambda'} t_\lambda(T) t_{\lambda'}(T)^* = e^{-2\pi i \mu \cdot \mu'} \frac{A(\sqrt{2\mu}; 0). \sqrt{2\mu'}; 0)}{D(T) D(T)^*} \Lambda_D,
\]

where the function \( A^{-\cdot-}(\Lambda) \), introduced in [6], is defined by

\[
A^{u, v}(\Lambda)(z_L z_R | \tau) = \sum_{(x_L, x_R) \in \Lambda} \exp[\pi i \tau (x_L + u_L)^2 - \pi i \tau^* (x_R + u_R)^2] \cdot \exp[2\pi i ((z_L + v_L) \cdot (x_L + u_L) - (z_R + v_R) \cdot (x_R + u_R))],
\]

and where \( \Lambda_D \) is the diagonal gluing for \( \Lambda(T) \), i.e. the even self-dual lattice

\[
\Lambda_D = \Lambda_D(T) \overset{\text{def}}{=} \bigcup_{\lambda \in G} (\lambda; \lambda).
\]
These are easily shown to span all $|\Lambda(\mathcal{T})| \times |\Lambda(\mathcal{T})|$ complex matrices $M_{\Lambda,\mathcal{N}}$ (see eqs. (3.7b, c) below), but they are not linearly independent (see (3.7) below). Their value as linear generators of the matrices rests with these relations:

\[
T^\dagger \{\mu, \mu'\} T = \{\mu, \mu' + \mu\} \overset{\text{def}}{=} \{\mu, \mu'\} 
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix},
\]

(3.3a)

\[
S^\dagger \{\mu, \mu'\} S = \{\mu', -\mu\} \overset{\text{def}}{=} \{\mu, \mu'\} 
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

(3.3b)

The derivation of (3.3a) is straightforward, while (3.3b) follows most easily from the transformation properties of $A^{-\tau}(\Lambda)$ under $\tau \to -1/\tau$ (see eq.(3.8b) in [6]). As we know [16], the matrices $\left(\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array}\right)$ and $\left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right)$ generate $\Gamma=\text{SL}(2,\mathbb{Z})$. This immediately suggests a description of $\Omega_{th}$. In particular, let $N = |\Lambda(\mathcal{T})^{(2)}| = \|G_2\|$. Then for all $\mu, \mu' \in G_2$,

\[
\{\mu, \mu'\} \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \overset{\text{def}}{=} \{a\mu + c\mu', b\mu + d\mu'\}
\]

(3.4a)

depends only on the values of $a, b, c, d \pmod{N}$, so the matrix

\[
\sum_{K \in \Gamma_{2N}} \{\mu, \mu'\} K,
\]

(3.4b)

where $\Gamma_{2N} = \text{SL}(2,\mathbb{Z}_N)$, commutes with $S$ and $T$ and lies in $\Omega_{th}(\mathcal{T})$. In fact, because the matrices in (3.1a) span all complex matrices, we immediately get that the matrices in (3.4b) span $\Omega_{th}(\mathcal{T})$. Note that if we let $\vartheta$ be the $\Gamma$-orbit in $G_2 \times G_2$ of $(\mu, \mu')$ — i.e. the set $\vartheta = \{(\mu, \mu')K \mid K \in \Gamma\}$ — then (3.4b) is an integral multiple of

\[
N_{\vartheta} \overset{\text{def}}{=} \sum_{(\mu_1, \mu_2) \in \vartheta} \{\mu_1, \mu_2\},
\]

(3.4c)

and so these $N_{\vartheta}$ span $\Omega_{th}(\mathcal{T})$.

That observation is all that will be required for the following section, and the proof that lattice partition functions span the commutant. However, we will continue to follow [14] for now and explicitly find a basis for $\Omega_{th}$. Our goal for the remainder of this section will be to generalize their result for $\mathcal{T} = \{\{A_n, k\}\}$ (Thm.2 in [14]) to any type $\mathcal{T}$.

Let $\vartheta_1, \vartheta_2$ be two $\Gamma$-orbits in $G_2 \times G_2$, and suppose some pairs $(\mu_i, \mu'_i) \in \vartheta_i$ satisfy

\[
\sqrt{2}(\mu_1, \mu'_1) = \sqrt{2}(\mu_2, \mu'_2), \quad \text{modulo } \Lambda(\mathcal{T}) \times \Lambda(\mathcal{T}) \text{ of course}. \quad \text{Choose any } K = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.
\]

Because $\Lambda(\mathcal{T})$ is an even lattice, norms of $\lambda \in G$ are well-defined (mod 2). Hence, $\mu_1^2 \equiv \mu_2^2, \mu_1'^2 \equiv \mu_2'^2, 2\mu_1 \cdot \mu'_1 \equiv 2\mu_2 \cdot \mu'_2 \pmod{1}$. That means that (using $ad - bc = 1$)

\[
(a\mu_1 + c\mu'_1) \cdot (b\mu_2 + d\mu'_2) - \mu_1 \cdot \mu'_1 \equiv ab\mu^2_1 + cd\mu'^2_1 + 2b\mu_2 \cdot \mu'_1 \equiv (a\mu_2 + c\mu'_2) \cdot (b\mu_2 + d\mu'_2) - \mu_2 \cdot \mu'_2 \pmod{1},
\]

(3.5a)
which, together with (3.7d) below, gives us the matrix equality
\[ \exp[2\pi i \mu_1 \cdot \mu'_1] N_{\vartheta_1} = r \cdot \exp[2\pi i \mu_2 \cdot \mu'_2] N_{\vartheta_2}, \]  
(3.5b)

where \( r \) is a positive rational number calculated from the number of times \( \sqrt{2} \vartheta_i \) covers the orbits \( \Gamma(\sqrt{2} \mu_i, \sqrt{2} \mu'_i), i = 1, 2 \). Thus the invariants corresponding to \( \vartheta_1 \) and \( \vartheta_2 \) differ only by a constant factor.

Thus to any \( \Gamma \)-orbit \( \tilde{\vartheta} \) in \( G \times G \) (as opposed to \( G_2 \times G_2 \)), we can assign an invariant \( M_{\tilde{\vartheta}} \) in \( \Omega_{th}(T) \), well-defined up to a constant phase, defined in the following manner: let \( \vartheta \) be any \( \Gamma \)-orbit in \( G_2 \times G_2 \) for which \( \sqrt{2} \vartheta = \tilde{\vartheta} \), and define
\[ M_{\tilde{\vartheta}} \overset{\text{def}}{=} N_{\vartheta}. \]  
(3.6)

Obviously, many such orbits \( \vartheta \subset G_2 \times G_2 \) exist; (3.5b) shows that which orbit \( \vartheta \) is chosen will only affect the answer \( M_{\tilde{\vartheta}} \) by a constant (hence irrelevant) phase factor.

**Theorem 1:**  
(a) For any semi-simple \( T \), let \( \mathcal{C}(T) \) be the set of all \( \Gamma \)-orbits \( \tilde{\vartheta} \) of \( G \times G \) for which \( M_{\tilde{\vartheta}} \) is not the zero matrix. Then the set
\[ \{M_{\tilde{\vartheta}} | \tilde{\vartheta} \in \mathcal{C}(T)\} \]

is a basis for \( \Omega_{th} \).

(b) For \( g = A_{2\ell}, B_{4\ell}, D_{4\ell}, E_6, E_8, F_4 \) and \( G_2 \), \( \mathcal{C} \) is the set of all \( \Gamma \)-orbits of \( G \times G \). For \( g = C_n, E_7 \), and the remaining \( B_n \) and \( D_n \), \( \mathcal{C} \) consists of those orbits \( \tilde{\vartheta} \) satisfying:
\[ (\lambda, \lambda') \in \tilde{\vartheta} \Rightarrow (k + h^\vee)\lambda^2 \equiv (k + h^\vee)\lambda'^2 \equiv 0 \pmod{1}. \]

For \( g = A_{2\ell-1} \), the condition on \( \tilde{\vartheta} \) is:
\[ (\lambda, \lambda') \in \tilde{\vartheta} \Rightarrow (k + h^\vee)\lambda^2 \equiv (k + h^\vee)\lambda'^2 \equiv 0 \pmod{\frac{2}{\ell}}. \]

[14] proved this theorem for \( g = A_n \), and used it to calculate the dimension of \( \Omega_{th}(A_2, k) \) (and hence \( \Omega_{th}(G_2, k - 1) \)). [17] later used the theorem to compute the dimension for all \( A_n \). The calculation was sufficiently general that, together with the above theorem, it should now be possible to extend this dimension calculation to any simple type. However, the formulae obtained in [17] for \( n > 2 \) was sufficiently complicated that the value of extending this work to all other simple types is questionable.

**Proof** First define, for each \( \lambda, \lambda' \in G \), the matrix \( \{\lambda, \lambda'\} \) by
\[ \{\lambda, \lambda'\}_{\lambda L, \lambda R} \overset{\text{def}}{=} \delta_{\lambda L, \lambda + \lambda R} \exp[2\pi i (\lambda \cdot \lambda' + \lambda' \cdot \lambda_R)], \]  
(3.7a)
where \( \lambda_L, \lambda_R \in G \). It is easy to show that these span all \(| \Lambda(T) | \times | \Lambda(T) |\) matrices: for any \( \lambda_1, \lambda_2 \in G \) let \( E(\lambda_1, \lambda_2) \) denote the matrix

\[
E(\lambda_1, \lambda_2) = \delta_{\lambda_1 \lambda_L} \cdot \delta_{\lambda_2 \lambda_R}
\]

consisting of zeros everywhere except for one ‘1’ at \((\lambda_1, \lambda_2)\). These \( E(\lambda_1, \lambda_2) \) span all matrices. The usual projection argument gives

\[
E(\lambda_1, \lambda_2) = \frac{1}{\|G\|} \sum_{\lambda \in G} e^{-2 \pi i \lambda \cdot \lambda} \{ \lambda_1 - \lambda_2, \lambda \}.
\]

Therefore the \( \{ \lambda, \lambda' \} \) must also span all matrices. Furthermore, a dimension check tells us they constitute a basis.

Incidently, note that for any \( \mu, \mu' \in G_2 \),

\[
\{ \mu, \mu' \} = e^{-2 \pi i \mu \cdot \mu'} \{ \sqrt{2} \mu, \sqrt{2} \mu' \},
\]

so if \( \sqrt{2} \mu_1 \equiv \sqrt{2} \mu_2, \sqrt{2} \mu'_1 \equiv \sqrt{2} \mu'_2 \) (mod \( \Lambda(T) \)), then

\[
\{ \mu_2, \mu'_2 \} = e^{2 \pi i (\mu_1 \cdot \mu'_1 - \mu_2 \cdot \mu'_2)} \{ \mu_1, \mu'_1 \}.
\]

Because the \( \{ \lambda, \lambda' \} \) are linearly independent, eq.(3.7d) generates all linear relations among the \( \{ \cdot, \cdot \} \).

(a) now follows: different orbits \( \vartheta \) of \( G \times G \) correspond to \( M_{\vartheta} \) being linear combinations of disjoint sets of terms \( \{ \lambda, \lambda' \} \), and since these terms are all linearly independent, so must the nonzero \( M_{\vartheta} \). We already know they span \( \Omega_{\ell h} \), and hence they form a basis.

To prove (b), we will first make some general calculations, and then conclude the proof by looking explicitly at \( g = C_n \). The other algebras can be done similarly (in most cases, somewhat more easily, too).

Choose any \( \Gamma \)-orbit \( \vartheta \) in \( G_2 \times G_2 \). We want to find what conditions \( \sqrt{2} \vartheta \) must satisfy for \( N_{\vartheta} \) to be non-zero. For any \( \lambda, \lambda' \in G \), define \([\lambda, \lambda']\) to be the (possibly empty) set of all pairs \((\mu, \mu') \in \vartheta \) for which \((\sqrt{2} \mu, \sqrt{2} \mu') = (\lambda, \lambda')\). Then

\[
N_{\vartheta} = \sum_{\lambda, \lambda' \in G} \{ \lambda, \lambda' \} \sum_{(\mu, \mu') \in [\lambda, \lambda']} \exp[-2 \pi i \mu \cdot \mu'] \overset{\text{def}}{=} \sum_{\lambda, \lambda' \in G} s_{\lambda, \lambda'} \{ \lambda, \lambda' \}.
\]

\( N_{\vartheta} \neq 0 \) iff some \( s_{\lambda, \lambda'} \neq 0 \), which turns out to happen iff for some \((\lambda, \lambda') \in \sqrt{2} \vartheta \), \( \mu \cdot \mu' \) is constant (mod 1) for all \((\mu, \mu') \in [\lambda, \lambda']\).

Those comments hold for any (simple or semi-simple) \( \mathcal{T} \). Now, turn to \( g = C_n \). Let \( \ell = k + h' \). \( M^{(\ell)} \) here is the orthogonal lattice \( A_1^{(\ell)} \oplus \cdots \oplus A_1^{(\ell)} \). Let \( e_i \) be an orthonormal basis, then \( G_2 \) is generated by the cosets

\[
g_i + M^{(\ell)}, \quad \text{where} \quad g_i = \frac{e_i}{\sqrt{4 \ell}}.
\]
Each of these cosets has order $4\ell$.

Now let $(\mu, \mu'), (\mu, \mu')K$ both lie in $[\lambda, \lambda']$, where $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We may write $\mu = \sum x_i g_i, \mu' = \sum x_i' g_i$ for $x_i, x_i' \in \mathbb{Z}_{4\ell}$. Then we have

$$ax_i + cx_i' \equiv x_i, \quad bx_i + dx_i' \equiv x_i' \pmod{2\ell}. \quad (3.9a)$$

We are interested in computing

$$(a\mu + c\mu') \cdot (b\mu + d\mu') - \mu \cdot \mu' = \frac{1}{4\ell} \sum_{i=1}^{n} [(ax_i + cx_i') \cdot (bx_i + dx_i') - x_i x_i'] , \quad (3.9b)$$

because we know that $s_{\lambda\lambda'}$ will be nonzero iff all $K \in \Gamma$ satisfying (3.9a) necessarily have (3.9b) congruent to 0 (mod 1).

Suppose an odd number of $x_i$ are odd. Then consider $K = \begin{pmatrix} 1 & 2\ell \\ 0 & 1 \end{pmatrix}$. Clearly $K$ satisfies (3.9a). Eq.(3.9b) becomes

$$\frac{1}{4\ell} \sum_{i=1}^{n} x_i' \cdot 2\ell x_i = \frac{1}{2} \sum_{i=1}^{n} x_i^2 \equiv \frac{1}{2} \pmod{1},$$

and so $s_{\lambda\lambda'}$ would vanish. A similar calculation holds when an odd number of $x_i'$ are odd.

Now suppose an even number of both $x_i$ and $x_i'$ are odd. We wish to show $s_{\lambda\lambda'} \neq 0$ in this case. Then $\mu, \mu' \in 1/\sqrt{4\ell}D_n$, so we can express these in terms of the simple root basis $\alpha_i$ of $D_n$ (enumerated as in Table Fin of [3]):

$$\mu = \sum_{i=1}^{n-1} y_i \frac{\alpha_i}{\sqrt{4\ell}} + y_n \frac{2\epsilon_n}{\sqrt{4\ell}}, \quad \mu' = \sum_{i=1}^{n-1} y_i' \frac{\alpha_i}{\sqrt{4\ell}} + y_n' \frac{2\epsilon_n}{\sqrt{4\ell}} \quad (3.10a)$$

(we substituted $2\epsilon_n$ for $\alpha_n$ to simplify eqs.(3.10b, c) below). Eq.(3.9a) becomes

$$ay_i + cy_i' \equiv y_i, \quad by_i + dy_i' \equiv y_i' \pmod{2\ell}, \quad i = 1, \ldots, n -1, \quad (3.10b)$$

$$ay_n + cy_n' \equiv y_n, \quad by_n + dy_n' \equiv y_n' \pmod{\ell}. \quad (3.10c)$$

Eq.(3.9b) now becomes

$$\frac{2}{4\ell} \sum_{i=1}^{n-1} [(ay_i + cy_i')(by_i + dy_i') - y_i y_i'] + \frac{4}{4\ell} [(ay_n + cy_n')(by_n + dy_n') - y_n y_n'] \quad (3.10d)$$

$$- \frac{1}{4\ell} \sum_{i=1}^{n-2} [(ay_i + cy_i')(by_{i+1} + dy_{i+1}') - y_i y_{i+1}'] - \frac{1}{4\ell} \sum_{i=1}^{n-2} [(ay_{i+1} + cy_{i+1}') (by_i + dy_i') - y_{i+1} y_i']$$

$$- \frac{2}{4\ell} [(ay_n - 1 + cy_n')(by_n + dy_n') - y_{n-1} y_n'] + \frac{4}{4\ell} [(ay_n + cy_n')(by_n - 1 + dy_n' - 1) - y_n y_{n-1}'].$$
The following argument shows this must necessarily be \( \equiv 0 \) (mod 1). Each \((ay_i + cy'_i) \cdot (by_i + dy'_i) - y_iy'_i \equiv 0 \) (mod \(2\ell\)) by (3.10b), so (mod 1) the first term vanishes. Similarly, (3.10c) implies the second term vanishes (mod 1). (3.10b) tells us each \((ay_{i+1} + cy'_{i+1}) \cdot (by_i + dy'_i) - y_iy'_i \equiv 0 \) (mod \(2\ell\)), so (mod 1) we may replace the \(-\frac{1}{4\ell}\) coefficient of the fourth term with \(+\frac{1}{4\ell}\), with the result (using \(ad - bc = 1\)) that the third term exactly cancels with this modified fourth term. A similar calculation shows that the final two terms cancel (mod 1).

The condition that there be an even number of odd \(x_i\) and \(x'_i\) is clearly equivalent to the condition that \(\lambda^2 \equiv \lambda'^2 \equiv 0 \) (mod \(1/\ell\)), and if it holds for one pair \((\lambda, \lambda') \in \sqrt{2\vartheta}\), it holds for all pairs. QED

[14] went on to prove that \(\Omega_{th}\) had an integral basis; we will do this in the following section by showing that lattice partition functions span the commutant.

4. Lattice partition functions and the commutant

For some purposes (e.g. calculating dimensions), knowing an explicit basis can be helpful. However, the matrix elements of the \(N_\vartheta\) or \(M_\vartheta\) defined in the previous section will in general be complex, and in any event those matrices are far from conducive to practical calculations, so are of limited value in the search for physical invariants.

A considerably more practical means of generating invariants in \(\Omega_{th}\) or \(\Omega_W\) is the lattice method of Roberts-Terao-Warner. We will very briefly review it below. A more thorough presentation is provided in [6,12,13].

An integral lattice is one in which all dot products are integers; it is called even if in addition all its norms are even. A self-dual lattice \(\Lambda\) is one which equals its dual \(\Lambda^*\). \(\Lambda\) is self-dual iff it is integral and also has determinant \(|\Lambda| = 1\). Finally, by a gluing \(\Lambda\) of \(\Lambda_0\), we mean that \(\Lambda_0\) is a sublattice in \(\Lambda\), and \(\Lambda/\Lambda_0\) is a finite group. By \((\Lambda_0; \Lambda_0)\) we mean the indefinite lattice with elements \(x = (x_L; x_R), x_L, x_R \in \Lambda_0\), whose dot products are defined by \(x \cdot x' = x_L \cdot x'_L - x_R \cdot x'_R\).

Consider any lattice \(\Lambda\). Define its type \(T\) partition function

\[
Z_\Lambda(T)(z_L, z_R | \tau) \overset{\text{def}}{=} \sum_{(x_L; x_R) \in \Lambda} \exp\left[\pi i \tau x^2_L - \pi i \tau^* x^2_R + 2\pi i (z'_L \cdot x_L - z'_R \cdot x_R)\right] \frac{D(T)(z_L | \tau) \cdot D(T)(z_R | \tau)^*}{D(T)(z_L | \tau) \cdot D(T)(z_R | \tau)^*},
\]

(4.1a)

where we use \(z' = (\sqrt{k_1 + h_1^2} z_1, \ldots, \sqrt{k_m + h_m^2} z_m)\). If \(\Lambda\) is a gluing of \((\Lambda(T); \Lambda(T))\), we may write this as (again using \(G = \Lambda(T)^* / \Lambda(T)\))

\[
Z_\Lambda(T)(z_L, z_R | \tau) \overset{\text{def}}{=} \sum_{(\lambda_L, \lambda_R) \in G} (N_\Lambda)_{\lambda_L, \lambda_R} t_{\lambda_L}(T)(z_L | \tau) \cdot t_{\lambda_R}(T)(z_R | \tau)^*
\]

(4.1b)

where \((N_\Lambda)_{\lambda_L, \lambda_R} = \begin{cases} 1 & \text{if } (\lambda_L; \lambda_R) \subset \Lambda \\ 0 & \text{otherwise} \end{cases}\)

(4.1c)

We will call \(N_\Lambda\) the coefficient matrix of \(Z_\Lambda(T)\) or, more briefly, the coefficient matrix of \(\Lambda\).
If $\Lambda$ is in addition both even and self-dual, $Z_{\Lambda}(T)$ will be modular invariant. Thus, finding all even self-dual gluings $\Lambda$ of $(\Lambda(T); \Lambda(T))$, and computing each of their partition functions $Z_{\Lambda}(T)$, constitutes a method of generating elements in $\Omega_{th}$. We will go about this by defining a set of lattices $\Lambda(\Lambda' \mu')$ proving (4.3) involves invariants $\mathcal{W} \mathcal{Z}$ can be written as a sum of functions of translates of the lattice $\Lambda$. We know the $\Lambda(\Lambda' \mu')$ and then inverting those relations so as to express the partition functions $\chi_{\Lambda}(\mu \mu')$ tells us that $\Omega_{th}$ are far from linearly independent, and following this theorem we will describe a small class of lattices for which the $\Lambda(\Lambda')$ still span the commutant.

Let $\Omega_{th}(T)$ be the (complex) space spanned by all $Z_{\Lambda}(T)$, and let $\Omega_{th}(T)$ be the (complex) space spanned by all $WZ_{\Lambda}(T)$, for $\Lambda \in \mathcal{L}(T)$ — they will be called the lattice theta-commutant and lattice Weyl-folded commutant, respectively. The previous discussion tells us that $\Omega_{th}(T) \subseteq \Omega_{th}(T)$ and $\Omega_{th}(T) \subseteq \Omega_{th}(T)$.

**Theorem 2:** (a) The lattice theta-commutant equals the theta-commutant for any type $T$:

$$\Omega_{th}(T) = \Omega_{th}(T);$$

(b) the lattice Weyl-folded commutant equals the Weyl-folded commutant, for any $T$:

$$\Omega_{th}(T) = \Omega_{th}(T).$$

Hence the Roberts-Terao-Warner lattice method is complete. The $\Lambda$ are far from linearly independent, and following this theorem we will describe a small class of lattices for which the $\Lambda$ still span the commutant.

The Weyl-Kac formula tells us that any $Z \in \Omega_{th}$ can be written as the sum of partition functions of translates of the lattice $(\Lambda(T); \Lambda(T))$. However, it does not follow immediately from modular invariance that those translates can be grouped together in such a way that $Z$ can be written as a sum of $WZ_{\Lambda}$ for $\Lambda \in \mathcal{L}(T)$. In other words, the direct approach to proving (4.3b) does not appear promising.

This theorem proves that $\Omega_{th}(T)$ and $\Omega_{th}(T)$ always have integral bases. We are more interested in (4.3b), but it is a trivial corollary of (4.3a), the equality we will prove. We will go about this by defining a set of lattices $\Lambda(\mu \mu', a) \in \mathcal{L}(T)$ in (4.4), expressing in (4.7) their coefficient matrices $N_{\Lambda(\mu \mu', a)}$ (defined as in (4.1c)) in terms of the $N_{\theta}$ of (3.4c), and then inverting those relations so as to express the $N_{\theta}$ in terms of the $N_{\Lambda(\mu \mu', a)}$. Since we know the $N_{\theta}$ span $\Omega_{th}$, this would imply that the $N_{\Lambda(\mu \mu', a)}$ span $\Omega_{th}$, and hence that $\Omega_{th} = \Omega_{th}$. 

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Proof. Consider any $\Gamma$-orbit $\vartheta$ in $G_2 \times G_2$. We want to show $N_\vartheta \in \Omega_{\ell h}^L$. For any $(\mu, \mu') \in \vartheta$ we can define the lattice $L(\vartheta)$ spanned (over the integers) by $\mu, \mu'$ and $\Lambda(\mathcal{T})^{(2)}$; this lattice is independent of which pair in $\vartheta$ is chosen. $L(\vartheta)$ will in general be neither self-dual nor integral.

Call the size of $\vartheta$ the number $s(\vartheta) \overset{\text{def}}{=} |\Lambda(\mathcal{T})^{(2)}|/|L(\vartheta)|$; it will always be an integer, and in fact a perfect square.

Our proof will be by induction on the size of $\vartheta$.

For $\vartheta$ with size $s(\vartheta) = 1$, $L(\vartheta) = \Lambda(\mathcal{T})^{(2)}$ and $\vartheta = \{(0,0)\}$. In this case, $N_\vartheta = \{0,0\} = I$, the identity matrix, which we know equals $N_{\Lambda,D} \in \Omega_{\ell h}^L$.

Now consider any $\Gamma$-orbit $\vartheta$ in $G_2 \times G_2$, and assume that $N_{\vartheta'} \in \Omega_{\ell h}^L$ for any orbit $\vartheta'$ with $s(\vartheta') < s(\vartheta)$.

$L(\vartheta)/\Lambda(\mathcal{T})^{(2)}$ will be an abelian group isomorphic to $\mathbb{Z}_d \times \mathbb{Z}_{dd'}$, for some $d, d'$. (Incidently, $s(\vartheta) = d^4d'^2$.)

Choose any $(\mu, \mu') \in \vartheta$, and for $a = 0, 1, \ldots, d - 1$ define the lattice $\Lambda(\mu, \mu'; a)$ to be the shifting

$$\Lambda(\mu, \mu'; a) \overset{\text{def}}{=} \Lambda_D \{((\sqrt{2} \mu; 0), (\sqrt{2} \mu'; 0)), \zeta\},$$

where $\zeta = (\begin{pmatrix} -\mu^2 & -\mu \cdot \mu' - \frac{a}{d} \\ -\mu \cdot \mu' + \frac{a}{d} & -\mu'^2 \end{pmatrix})$;

i.e. $\Lambda(\mu, \mu'; a)$ is the lattice given by

$$\Lambda(\mu, \mu'; a) = \{(\lambda + \ell \sqrt{2} \mu + \ell' \sqrt{2} \mu'; \lambda) + (\Lambda(\mathcal{T}); \Lambda(\mathcal{T})) \mid \ell, \ell' \in \mathbb{Z}, \lambda \in \Lambda(\mathcal{T})^*,$$

and $\sqrt{2} \lambda \cdot \mu \equiv \ell \zeta_{11} + \ell' \zeta_{12}, \sqrt{2} \lambda \cdot \mu' \equiv \ell \zeta_{21} + \ell' \zeta_{22} \pmod{1}\}$. (4.4c)

Shifting is discussed in reasonable detail and generality in [18]. For now it suffices to remark that $\Lambda(\mu, \mu'; a)$ will be self-dual (this can also be read off from (4.7) below, as was done in the claim in Thm.E of [6]). It is clear from its definition that it is also even, and is a gluing of $(\Lambda(\mathcal{T}); \Lambda(\mathcal{T}))$.

Claim: $\Lambda(\mu, \mu'; a)$ has the coefficient matrix (see (4.1c))

$$N_{\Lambda(\mu, \mu'; a)}(\mathcal{T}) = c \sum \exp[2\pi i \frac{a}{d}(-\ell_1 \ell'_2 + \ell_2 \ell'_1)] \{\gamma_1, \gamma_2\},$$

where the sum is over all $\gamma_1 = \ell_1 \mu + \ell'_1 \mu'$, $\gamma_2 = \ell_2 \mu + \ell'_2 \mu' \in L(\vartheta)$, and where $c = 1/\sqrt{s(\vartheta)}$.

The proof involves using (3.1a) to rewrite the RHS of (4.5), showing that it equals the matrix $N_{\Lambda(\mu, \mu'; a)}$ corresponding to eq.(4.4c). The key point in this straightforward derivation is the observation that for any $\lambda \in G$,

$$\sum_{\gamma_2} \exp[2\pi i \frac{a}{d}(-\ell_1 \ell'_2 + \ell_2 \ell'_1) + \sqrt{2} \gamma_2 \cdot \lambda + \gamma_1 \cdot \gamma_2]$$

(4.6a)
vanishes iff there exists a $\gamma_3 = \ell_3 \mu + \ell_3' \mu' \in L(\vartheta)$ such that
\[
\frac{a}{d}(-\ell_1' \ell_3' + \ell_3' \ell_1') + \sqrt{2} \gamma_3 : \lambda + \gamma_1 : \gamma_3 \neq 0 \pmod{1}.
\]
(4.6b)

The expression in (4.6b) will be $\equiv 0 \pmod{1}$ for all $\gamma_3$, iff it is $\equiv 0 \pmod{1}$ for $\gamma_3 = \mu$ and $\gamma_3 = \mu'$ — these are precisely the two congruences in (4.4c).

We can rewrite (4.5) in a more convenient form, using the easily verified fact (Lemma 2(i) of [14]) that, for any $\Gamma$-orbit $\vartheta'$ with $L(\vartheta') \subseteq L(\vartheta)$, $B(\vartheta') \overset{\text{def}}{=} \ell_1' \ell_2' - \ell_2 \ell_1'$ is independent (mod $d$) of which pair $(\ell_1 \mu + \ell_1' \mu', \ell_2 \mu + \ell_2' \mu')$ is chosen from $\vartheta'$. The result is:
\[
N_{\Lambda(\mu, \mu'; a)} = c \sum_{\vartheta'} \exp[-2\pi i B(\vartheta') \frac{a}{d}] N_{\vartheta'}
\]
(4.7)

where the sum is over all $\Gamma$-orbits $\vartheta'$ whose lattices $L(\vartheta')$ are sublattices of $L(\vartheta)$.

It is important to realize that there is precisely one orbit $\vartheta'$ with $L(\vartheta') = L(\vartheta)$, for each $0 < m \leq d$ relatively prime to $d$ (the assignment is given by $m = B(\vartheta')$) (see p.622 of [14]). Of course $\vartheta' = \vartheta$ is the unique one corresponding to $m = B(\vartheta') = 1$.

By the induction hypothesis we know $N_{\vartheta''} \in \Omega_{th}^L$ for any orbit $\vartheta''$ in (4.7) with size $s(\vartheta'')$ less than $s(\vartheta)$. $N_{\vartheta}$ can be solved for in terms of these $N_{\vartheta''}$ and the $N_{\Lambda(\mu, \mu'; a)}$, by multiplying (4.7) by $\exp[2\pi ia/d]$ and summing over all $a$.

Hence $N_{\vartheta} \in \Omega_{th}^L$. QED

The class of lattices in (4.4) shown to span $\Omega_{th}$ is small compared to the whole class $\mathcal{L}$ of even self-dual gluings of $(\Lambda(T); \Lambda(T))$. We learn from the proof of Thm.2 that $\Omega_{th}$ is spanned by the partition functions $Z_{\Lambda(\mu, \mu'; a)}(T)$, where $\mu, \mu' \in G_2$. Eq.(4.7) shows that this is a function of $\vartheta$ and $a$ — i.e. if $(\mu_1, \mu'_1)$ and $(\mu_2, \mu'_2)$ lie on the same $\Gamma$-orbit $\vartheta$, then $\Lambda(\mu_1, \mu'_1; a) = \Lambda(\mu_2, \mu'_2; a)$. A similar argument shows that if $L(\vartheta_1) = L(\vartheta_2)$, then $\Lambda(\vartheta_1; a) = \Lambda(\vartheta_2; B_{\vartheta_1}(\vartheta') a)$. Moreover, if $(\sqrt{2} \mu_1, \sqrt{2} \mu'_1) = (\sqrt{2} \mu_2, \sqrt{2} \mu'_2)$, then (3.5b) and (4.7) tell us that the $N_{\vartheta_2}$ in (4.7) can be recovered from the $\Lambda(\mu_1, \mu'_1; a)$.

Finally, note that there was an element of overkill in the proof of Thm.2. In particular, there we had ‘$a$’ range from 0 to $d - 1$. Let $k_1, \ldots, k_n$ be the $n = \phi(d)$ integers between 0 and $d$ relatively prime to $d$. All that was relevant for the ‘inverting’ step in the final paragraph of the proof was that the $d \times n$ matrix with entries
\[
B_{aj} = \exp[-2\pi i a \cdot k_j],
\]
(4.8)
for $a = 0, 1, \ldots, d - 1$, $j = 1, \ldots, n$, be of rank $n$. It was that fact which allowed us to solve (4.7) for the $N_{\vartheta'}$ in terms of the $N_{\Lambda(\mu \mu'; a)}$. However, the $n \times n$ submatrix obtained by restricting (4.8) to $a = 0, \ldots, n - 1$, can also be shown to be of rank $n$ (i.e. invertible). For otherwise it would have a zero eigenvalue, which would mean there would exist $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, not all zero, for which
\[
0 = \sum_{j=1}^{n} \alpha_j x^{j-1}
\]
(4.9)
for \( x = \exp[-2\pi ik_1], \ldots, \exp[-2\pi ik_n] \). In other words, the \( n - 1 \)-th degree polynomial in (4.9) would have \( n \) distinct roots. This is impossible. Thus our \( n \times n \) submatrix is invertible.

We can use these remarks to significantly reduce the number of lattices we need to consider. The result is as follows.

Consider each (not necessarily integral) lattice \( L \) of the form

\[
L = \bigcup_{i,j \in \mathbb{Z}} i\lambda + j\lambda' + \Lambda(T) \overset{\text{def}}{=} \Lambda(T)[\lambda, \lambda']
\]

for some \( \lambda, \lambda' \in \Lambda(T)^* \). Put \( \mu = \lambda/\sqrt{2}, \mu' = \lambda'/\sqrt{2} \), and define \( d_L \) by

\[
L'/\Lambda(T)^{(2)} \cong \mathbb{Z}_{d_L} \times \mathbb{Z}_{d_L' d_L'}, \text{ where } L' = \bigcup_{i,j \in \mathbb{Z}} i\mu + j\mu' + \Lambda(T)^{(2)}.
\]

Now construct the lattices \( \Lambda(L, a) \overset{\text{def}}{=} \Lambda(\mu, \mu'; a) \) defined in eqs.(4.4), but now for \( a = 0, 1, \ldots, \phi(d_L) - 1 \).

Finally, let \( \mathcal{L}_*(T) \) denote the resulting collection of all these \( \sum_L \phi(d_L) \) even self-dual lattices \( \Lambda(L, a) \).

**Corollary 2:** The \( Z_{\Lambda}(T) \) for \( \Lambda \in \mathcal{L}_*(T) \), span \( \Omega_{th} \); the corresponding \( \mathbb{W}Z_{\Lambda}(T) \) span \( \Omega_{\mathbb{W}}(T) \).

Particularly for algebras of large rank, the set \( \mathcal{L}_*(T) \) is considerably smaller than the set \( \mathcal{L}(T) \) of all even self-dual gluings of \( (\Lambda(T); \Lambda(T)) \). Moreover, for a fixed choice of algebra, it should be possible to use Thm.1 to reduce \( \mathcal{L}_* \) further so that the corresponding \( Z_{\Lambda} \) will constitute a basis for \( \Omega_{th} \).

One final remark will be made here. Q. Ho-Kim [19] is currently running a program designed to compute all lattice partition functions \( \mathbb{W}Z_{\Lambda} \) for \( \Lambda \in \mathcal{L}(g, k) \), for \( g = A_2, C_2, G_2, A_1 + A_1 \) and with levels \( k \) up to around 30. He will in this way determine \( \Omega_{\mathbb{W}}^{(1)} \) and hence \( \Omega_{\mathbb{W}} \) for these types, and thus compute all physical invariants. The analysis is not yet complete, but when it is it will apparently constitute the only completeness proof for these \( g \), for small \( k \), apart from \( k = 1 \) (see [11], as well as Thm.5 below), and \( k + 3 \) prime with \( g = A_2 \) [20]. (Various other computer searches have been undertaken before, e.g. in [13], but they do not exhaust the commutant and so leave open the possibility that an unknown physical invariant may have escaped detection.) One thing apparent from the work in [19] is the practicality of this lattice method, at least for small \( g \) and \( k \).

5. General observations, and applications to level 1

In this section we make a few general observations relevant to the classification problem, and use some of these to complete the classification of all level 1 physical invariants of simple type.
Using the finite-dimensional Weyl denominator formula, a simpler expression can be found (see eq. (13.8.10) in [3]) for the $\lambda = 0$ row and column of the $S = S^W(T)$ matrix in (2.6): if $T = (\{g, k\})$, then for any $\lambda \in P_+(g, k)$ we have

$$S_{\lambda, 0} = S_{0, \lambda} = \frac{1}{\sqrt{|M| \cdot (k + h^\vee)^{n/2}}} \prod_{\alpha \in \Delta_+} 2 \sin \frac{\pi (\lambda + \rho) \cdot \alpha}{k + h^\vee}, \quad (5.1)$$

where $\Delta_+$ is the set of positive roots of $g$. The corresponding formula for semi-simple types is obtained from (5.1) by multiplication (see (2.9)). Because each $0 \leq \lambda \cdot \alpha \leq k$ and $0 < \rho \cdot \alpha < h^\vee$, we immediately get that each $S_{\lambda, 0}$ is strictly positive. Also, $S_{\lambda, 0} \geq S_{0, 0}$ and $\sum S_{\lambda, 0}^2 = 1$.

This positivity has a number of easy consequences. For one thing, the number of physical invariants of a given type $T$ must be finite. To see this let $s = S^W(T)_{00} = \min \{S^W(T)_{\lambda, 0}\}$ and let $N_{\lambda\lambda'}$ be the coefficient matrix of any physical invariant of type $T$. Then

$$1 = N_{00} = \sum_{\lambda, \lambda'} S_{\lambda, 0} S_{\lambda', 0} N_{\lambda\lambda'} \geq s^2 \sum_{\lambda, \lambda'} N_{\lambda\lambda'}. \quad (5.2)$$

Since each $N_{\lambda\lambda'}$ must be a non-negative integer, which by (5.2) is bounded above by $1/s^2$, the desired finiteness follows. Another consequence of this positivity is that if $N$ is any positive invariant (not necessarily physical), then a similar calculation to that given in (5.2) shows $N_{00} > 0$.

A second observation connects more directly with the lattice formalism of the previous section. Consider any type $T$, and let $\Lambda_0$ denote the indefinite lattice $(\Lambda(T); \Lambda(T))$ defined in Sec. 4. Choose any $x = (x_L; x_R) \in \Lambda_0^*$ and let its order be $m$ (i.e. $\ell x \in \Lambda_0$ iff $m$ divides $\ell$). Let $\Lambda_1$ be any gluing of $\Lambda_0$ (i.e. the quotient group $\Lambda_1/\Lambda_0$ exists and is finite). Then for any $\ell$ relatively prime to $m$, $\ell$ has an inverse (mod $m$), so $x \in \Lambda_1$ iff $\ell x \in \Lambda_1$.

Now for any $y = (y_L; y_R) \in \Lambda_1^*$ consider the function

$$c(y)(z_L z_R | \tau) \overset{\text{def}}{=} \sum \epsilon(w) \epsilon(w') t_w(y_L)(T)(z_L, \tau) t_{w'}(y_R)(T)(z_R, \tau)^*, \quad (5.3a)$$

where the sum is over all $w, w'$ in the Weyl group $W(T)$. Using eqs. (2.5) we get that either

$$c(y)(z_L z_R | \tau) = 0 \quad (5.3b)$$

for all $z_L, z_R, \tau$, or there exist unique $\epsilon_y \in \{\pm 1\}, \lambda_y, \lambda_y' \in P_+(T)$ such that

$$c(y)(z_L z_R | \tau) = \epsilon_y \chi_{\lambda_y}(z_L, \tau) \chi_{\lambda_y'}(z_R, \tau)^* \quad (5.3c)$$

for all $z_L, z_R, \tau$. It can be shown that for $\ell$ relatively prime to $m$, $c(x) = 0$ iff $c(\ell x) = 0$.

So what does all this tell us? Note that we may write

$$WZ_{\Lambda_1}(T) = \sum_{y \in \Lambda_1/\Lambda_0} c(y) = \sum_{\lambda, \lambda' \in P_+(T)} N_{\lambda\lambda'} \chi_\lambda(T) \chi_{\lambda'}(T)^* \quad (5.3d)$$

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Then this argument implies for any $x \in \Lambda_1/\Lambda_0$ for which $c(x) \neq 0$,  
\[ \epsilon_x N_{\lambda_x x} = \epsilon_{\ell x} N_{\lambda_{\ell x} x}. \]  
(5.3e)

By Thm.2, eq.(5.3e) also holds for the coefficient matrix $N$ of any $Z \in \Omega_W(T)$, and for any $x \in \Lambda_0*/\Lambda_0$. 

For example, for $g = A_2$, $k = 5$, (5.3e) gives us relations such as 
\[ N_{00,00} = N_{22,22} \text{ and } N_{10,10} = N_{01,01} = N_{40,40} = N_{04,04}, \]  
(5.4a)

where we write the subscript ‘$ij$’ for the vector $i\beta_1 + j\beta_2$ with Dynkin weights $i$ and $j$. For $g = A_2$, $k = 9$, we get relations such as 
\[ N_{01,05} = -N_{16,04} = -N_{12,54} = N_{08,45} = N_{18,40} = -N_{62,50} \]
\[ = N_{10,50} = -N_{61,40} = -N_{21,45} = N_{80,54} = N_{81,04} = -N_{26,05}. \]  
(5.4b)

For example, (5.4b) comes from $x = (\pm \frac{\beta_1 + 2\beta_2}{\sqrt{12}}, \pm \frac{\beta_1 + 6\beta_2}{\sqrt{12}})$, which has order $m = 36$, and $\ell = 1, 5, 7, 11, 13, 17, -1, -5, -7, -11, -13, -17$, respectively. Note that because of the sign changes in (5.4b), any positive (hence any physical) invariant of type $\{A_2, 9\}$ must have $0 = N_{01,05} = \cdots = N_{26,05}$. 

Eq.(5.3e), particularly when $\epsilon_x \cdot \epsilon_{\ell x} = -1$, has significance both for simplifying computer searches as in [19], and for theoretical considerations (we will give one at the end of this section).

We will complete this general discussion with two theorems. In the following, we will repeatedly make use of the fact that, if $N_1, N_2$ are the coefficient matrices of invariants of type $T$, then so will be 
\[ aN_1 + bN_2, \ N_1^T, \ N_1^\dagger, \ N_1 N_2 \]  
for any complex numbers $a, b$.

Call a (not necessarily physical) invariant 0-decoupled if $N_{\lambda 0} = N_{0\lambda} = 0$ for all $\lambda \neq 0$ in $P_+$. A 0-decoupled invariant $Z$ can be written in the form
\[ Z = a|\chi_0|^2 + Z', \]
where $Z'$ is independent of $\chi_0$ and $\chi_0^*$, and $a$ is any constant. For example, the only 0-decoupled physical invariants for $g = A_1$ are $A_k \forall k$, and those lying in the $D_k$ series with level $k \equiv 2 \pmod{4}$ (the $A_1$ physical invariants are given in [10]). For $g = A_2$, $D_k$ is 0-decoupled $\forall k \equiv 0 \pmod{3}$ (the known $A_2$ physical invariants are given e.g. in [14]).

By a permutation invariant we mean an invariant 
\[ Z = \sum_{\lambda \in P_+} \chi_\lambda \chi_\sigma^*, \ i.e. \ N_{\lambda \lambda'} = \delta_{\lambda', \sigma \lambda} \]
for some permutation $\sigma$ of $P_+$. Any permutation invariant is necessarily physical (see the proof of Thm.3 below). Of course, the permutation $\sigma$ must be a symmetry of both
the $S$ and $T$ matrices, and by Verlinde’s formula \cite{21}, it also is a symmetry of the fusion coefficients:

$$S_{\lambda\lambda'} = S_{\sigma\lambda,\sigma\lambda'}, \quad T_{\lambda\lambda'} = T_{\sigma\lambda,\sigma\lambda'}, \quad N^\lambda_{\lambda\lambda'} = N^\sigma_{\sigma\lambda,\sigma\lambda'}.$$

**Theorem 3:** $N$ is a 0-decoupled physical invariant iff it is a permutation invariant.

**Proof** Since a permutation invariant $N$ is necessarily positive, it must satisfy $N_{00} > 0$ by an observation made earlier in this section. Hence $N_{00} = 1$, $N$ is physical, and $N$ must be 0-decoupled.

Now consider $\tilde{N} = NN^T$, for some 0-decoupled physical invariant $N$. $\tilde{N}$ will also correspond to a 0-decoupled physical invariant, as will any of its powers $(\tilde{N})^\ell$. Also,

$$\tilde{N}_{\lambda\lambda} = \sum_{\lambda'} N^{2}_{\lambda\lambda'} \geq \sum_{\lambda'} N_{\lambda\lambda'} \overset{\text{def}}{=} r_{\lambda} \geq 0.$$

Suppose for contradiction that the row sum $r_{\lambda} > 1$ for some $\lambda$, and look at the powers $(\tilde{N})^\ell$. An easy calculation shows that $(\tilde{N})^\ell_{\lambda\lambda} \to \infty$ as $\ell \to \infty$, contradicting (5.2). Therefore every entry in the $\lambda$-row is 0, except for at most one 1.

A similar argument applies to columns, by considering $N^T N$. Hence $\tilde{N}$ is diagonal, with 1’s and 0’s on the diagonal. But

$$1 = \tilde{N}_{00} = \sum_{\lambda,\lambda'} S_{\lambda 0} S_{\lambda' 0} \tilde{N}_{\lambda\lambda'} = \sum_{\lambda} S_{\lambda 0}^2 \tilde{N}_{\lambda\lambda} \leq \sum_{\lambda} S_{\lambda}^2 = 1$$

and each $S_{\lambda 0} > 0$, so equality can hold only if each $\tilde{N}_{\lambda\lambda} = 1$. QED

By a *block* we mean something of the form $m|\chi_{\lambda_1} + \cdots + \chi_{\lambda_\ell}|^2$. $m$ is called the *scale*, and $\ell$ the *length*. We will demand $m, \ell \neq 0$. Call an invariant $Z$ a *block-diagonal* if it is the sum of pairwise disjoint blocks. For example, the diagonal invariant is always a block-diagonal, with each block having length $\ell_i = 1$ and scale $m_i = 1$. In fact, it is the only physical invariant which is both 0-decoupled and block-diagonal. The block-diagonal physical invariants of $g = A_1$ are the diagonal invariants, along with the invariants in the $D_k$ series with level $k \equiv 0 \pmod{4}$, and the exceptional invariants $E_6$ and $E_8$. For $g = A_2$, the block-diagonal invariants include $D_k$ for $k \equiv 0 \pmod{3}$, as well as exceptional invariants of level $k = 5, 9$ and 21.

Most of the known physical invariants seem to be either permutation invariants, block-diagonal, or products of two such invariants.

**Theorem 4:** Let $Z$ be any invariant (not necessarily physical) which is a block-diagonal, and let the $i$th block have scale $m_i$ and length $\ell_i$. Then:

(i) $\chi_0$ must belong to some block;
(ii) all products $[m_i]_{\ell_i}$ are equal;
(iii) all $m_i$ must be of equal sign;
(iv) if $Z$ is also *physical*, then each $\ell_i$ must divide the length of the block containing $\chi_0$. 

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Proof The proof here is similar to that used in proving the previous theorem; the key idea is that the coefficient matrix $N$ of $Z$, after a re-ordering of the indices, is a direct sum of $\ell_i \times \ell_i$ matrices looking like

$$m_i \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

with $\ell_i$ 1’s in each row and column. Raising $N$ to the $\ell$-th power then gives a direct sum of

$$m_i^{\ell} \ell_i^{\ell-1} \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}.$$ 

To prove (i), simply square $N$ and use the fact that $N_{00}$ must be positive.

To prove (ii), let $L = \max_i |m_i|\ell_i$ and look at the limit $N_\infty$ of $\left(N^2/L^2\right)^\ell$ as $\ell \to \infty$. The only blocks that will have survived would be those with $L^2 = m_i^2 \ell_i^2$. If $N_\infty \neq N^2/L^2$, then either $N_\infty$ or $N^2/L^2 - N_\infty$ will be a block-diagonal positive invariant with no block containing $\chi_0$. Therefore $N_\infty$ must equal $N^2/L^2$.

(iii) follows from a similar argument, by looking at $N^2/L \pm N$.

(iv) follows immediately from (ii). QED

Obviously Thms.3,4 are two among many that can be proved using similar techniques. Thm.3 will play an important role in the complete classification of the level 1 physical invariants given in Thm.5.

The remainder of this section will be devoted to the level 1 physical invariants, and in particular, the proof of the following theorem.

We will use the conventions for numbering Dynkin nodes given in Table Fin in [3]. In eqs.(5.5b-p) we will write $\chi_i$ for $\chi_{\beta_i}(T)$.

**Theorem 5:** (i) For $g = B_n, E_7, E_8, F_4$ and $G_2$, the only level 1 physical invariant is the diagonal invariant:

$$Z_A = \sum_{\lambda \in P_+(g,1)} \chi_{\lambda}^{g,1} \chi_{\lambda}^{g,1*}. \quad (5.5a)$$

(ii) For $g = E_6$, $k = 1$, the physical invariants are the diagonal one (5.5a), along with

$$Z_{\sigma} = \chi_0 \chi_0^* + \chi_1 \chi_5^* + \chi_5 \chi_1^*. \quad (5.5b)$$

(iii) For $g = C_n$, $k = 1$, the physical invariants are: the diagonal one (5.5a) for each $n$; for $n \equiv 0 \pmod{4}$ the invariant

$$Z_n' = 2|\chi_{n/2}|^2 + \sum_{i=0}^{n/4-1} |\chi_{2i} + \chi_{n-2i}|^2; \quad (5.5c)$$
for $n \equiv 2 \pmod{4}$, $n \geq 6$, the invariant
\[
Z'_n = |\chi_{n/2}|^2 + \sum_{i=0}^{n/2} |\chi_{2i}|^2 + \sum_{i=0}^{n-6} (\chi_{2i+1} \chi_{n-2i-1}^* + \chi_{n-2i-1} \chi_{2i+1}^*);
\]
(5.5d)

for $n = 10, 16, 28$, respectively, the additional invariants
\[
\begin{align*}
Z''_{10} &= |\chi_0 + \chi_6|^2 + |\chi_3 + \chi_7|^2 + |\chi_4 + \chi_{10}|^2; \\
Z''_{16} &= |\chi_0 + \chi_{16}|^2 + |\chi_4 + \chi_{12}|^2 + |\chi_6 + \chi_{10}|^2 + |\chi_8|^2 + \\
&\quad \chi_8 (\chi_2^* + \chi_{14}^*) + (\chi_2 + \chi_{14}) \chi_8^*; \\
Z''_{28} &= |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2.
\end{align*}
\]
(5.5f)

(iv) For $g = D_n$, $k = 1$, the physical invariants are: for $n \neq 0 \pmod{4}$, the diagonal invariant (5.5a) along with
\[
Z_\sigma = \chi_0 \chi_0^* + \chi_1 \chi_1^* + \chi_{n-1} \chi_{n-1}^* + \chi_n \chi_n^*;
\]
(5.5h)

for $n \equiv 4 \pmod{8}$ the physical invariants are given by eqs.(5.5a, h) and
\[
\begin{align*}
Z_3 &= \chi_0 \chi_0^* + \chi_1 \chi_{n-1}^* + \chi_{n-1} \chi_1^* + \chi_n \chi_n^*, \\
Z_4 &= \chi_0 \chi_0^* + \chi_1 \chi_n^* + \chi_{n-1} \chi_{n-1}^* + \chi_n \chi_1^*, \\
Z_5 &= \chi_0 \chi_0^* + \chi_1 \chi_{n-1}^* + \chi_{n-1} \chi_n^* + \chi_n \chi_1^*, \\
Z_6 &= \chi_0 \chi_0^* + \chi_1 \chi_n^* + \chi_{n-1} \chi_{n-1}^* + \chi_n \chi_n^*;
\end{align*}
\]
(5.5i)–(5.5k)

and for $n \equiv 0 \pmod{8}$ the physical invariants are given by eqs.(5.5a, h) and
\[
\begin{align*}
Z'_3 &= \chi_0 \chi_0^* + \chi_0 \chi_{n-1}^* + \chi_{n-1} \chi_0^* + \chi_n \chi_{n-1}^*, \\
Z'_4 &= \chi_0 \chi_0^* + \chi_0 \chi_n^* + \chi_n \chi_0^* + \chi_n \chi_n^*, \\
Z'_5 &= \chi_0 \chi_0^* + \chi_0 \chi_{n-1}^* + \chi_{n-1} \chi_0^* + \chi_n \chi_{n-1}^*, \\
Z'_6 &= \chi_0 \chi_0^* + \chi_0 \chi_n^* + \chi_{n-1} \chi_0^* + \chi_{n-1} \chi_n^*.
\end{align*}
\]
(5.5m)–(5.5p)

Thus, together with the $g = A_n$, $k = 1$ case dealt with in [11], this theorem completes the classification of the level 1 physical invariants of simple type. All the invariants in eqs.(5.5) are either 0-decoupled, or diagonal-blocks, or products of the two, except for (5.5f). Note further that not all of the physical invariants given in (5.5) are real — i.e. correspond to symmetric coefficient matrices $N$. In particular, each $g = D_{4k}$ has exactly two non-real level 1 physical invariants ($Z_5$ and $Z_6$, or $Z'_5$ and $Z'_6$). However, in terms of restricted characters $\chi(0, \tau)$, all invariants become symmetric.

Proof (i) We will do $B_n$ here; the other algebras in (i) are at least as easy. First, we must find $P_+(B_n, 1)$. This is easy:
\[
P_+(B_n, 1) = \{0, \beta_1, \beta_n\},
\]

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corresponding to the three colabels \(a^\vee_i\) equal to 1. Therefore any invariant will look like
\[
Z = \sum N_{ij} \chi_i \chi_j^*,
\]
where the sum is over \(i, j \in \{0, 1, n\}\).

Invariance under \(\tau \to \tau + 1\) requires that we compute the norms of
\[
p_0 \overset{\text{def}}{=} \rho / \sqrt{h^\vee + 1},
\]
\[
p_1 \overset{\text{def}}{=} (\beta_1 + \rho) / \sqrt{h^\vee + 1}
\]
and
\[
p_n \overset{\text{def}}{=} (\beta_n + \rho) / \sqrt{h^\vee + 1}:
\]
\[
\begin{align*}
p_0^2 &= \frac{4n^2 - 1}{24}, \\
p_1^2 &= \frac{4n^2 + 23}{24}, \\
p_n^2 &= \frac{(n + 1)(2n + 1)}{12}.
\end{align*}
\]

Eqs.(5.6) tell us that any invariant of type \(\{B_n, 1\}\) looks like
\[
Z = N_{00} \chi_0 \chi_0^* + N_{11} \chi_1 \chi_1^* + N_{nn} \chi_n \chi_n^*.
\]

But then any physical invariant of that type must be 0-decoupled. Thm.3 then tells us there must be exactly one 1 in each row and column, so \(N_{00} = N_{11} = N_{nn} = 1\), and
\[
Z = Z_A, \text{ the diagonal invariant.}
\]

(ii) This case can be dealt with similarly. \(P_+(E_6, 1) = \{0, \beta_1, \beta_5\}\); a norm check gives us
\[
Z = N_{00} \chi_0 \chi_0^* + \sum_{i,j \in \{1, 5\}} N_{i,j} \chi_i \chi_j^*.
\]

Therefore any physical invariant of this type will necessarily be 0-decoupled, so there are only two possibilities for these physical invariants: eqs.(5.5a, b). Eq.(5.5b) is indeed an invariant; it corresponds to the outer automorphism \(\sigma\) of \(E_6\).

(iii) The easiest completeness proof for \(C_n\) is to exploit the calculation in [22] of its level 1 \(S^W\)-matrix. An alternate argument, with the advantage of greater generality, will be sketched at the end of this section.

\(P_+(C_n, 1) = \{0, \beta_1, \ldots, \beta_n\}\) and \(P_+(A_1, k) = \{0, 1 \cdot \beta_1', \ldots, k \cdot \beta_1'\}\), using obvious notation. In [22] it is shown that
\[
(S^W(C_n, 1))^{\beta_i \beta_j} = (S^W(A_1, n))_{i^\prime \beta_i, j^\prime \beta_j},
\]
for all \(0 \leq i, j \leq n\) ([23] generalized this duality to between \(S^W(C_n, k)\) and \(S^W(C_k, n)\)). To compare their \(T^W\)-matrices, we need to compute the norms of
\[
p_i \overset{\text{def}}{=} (\beta_i + \rho) / \sqrt{h^\vee + 1}.
\]
We find
\[
p_i^2 = \frac{(n + 1)(2n + 3)}{12} - \frac{(n + 1 - i)^2}{2(n + 2)}.
\]

Therefore from (2.6b) we get
\[
(T^W(C_n, 1))^{\beta_i \beta_j} = \alpha(T^W(A_1, n)^{-1})_{i^\prime \beta_i, j^\prime \beta_j},
\]
for all \(0 \leq i, j \leq n\).
for some \( \alpha \in \mathbb{C} \) of modulus \( |\alpha| = 1 \). From (5.9a,c) we see that the commutants of types \((C_n, 1)\) and \((A_1, n)\) are the same: more precisely, the map defined by \( N_{\beta_i \beta_j} \rightarrow N_{i \beta_i^t, \beta_j^t} \) is a (vector space) isomorphism between \( \Omega_w(C_n, 1) \) and \( \Omega_w(A_1, n) \). Since it also preserves (P2) and (P3), it constitutes a bijection between the respective physical invariants.

What this means is that the completeness proof for \( A_1 \) level \( n \), carries over to \( C_n \) level \( 1 \), and the list of \( C_n \) physical invariants can be read off from that of \( A_1 \). The result is eqs.(5.5c-g).

(iv) Note that \( P_1(D_n, 1) = \{0, \beta_1, \beta_{n-1}, \beta_n\} \), and the norms of \( p_0 \overset{\text{def}}{=} \rho/\sqrt{h^v + 1} \) and \( p_i = (\beta_i + \rho)/\sqrt{h^v + 1} \) for \( i = 1, n-1, n \) are:

\[
p_0^2 = \frac{n(n-1)}{6}, \quad p_1^2 = p_0^2 + 1, \quad p_{n-1}^2 = p_n^2 = p_0^2 + \frac{n}{4}. \tag{5.10}
\]

Therefore, for \( n \not\equiv 0 \mod 4 \), \( n \equiv 4 \mod 8 \), and \( n \equiv 0 \mod 8 \) respectively, an invariant will look like

\[
Z = N_{00} \chi_0 \chi_0^* + N_{11} \chi_1 \chi_1^* + N_{n-1,n-1} \chi_{n-1} \chi_{n-1}^* + N_{nn} \chi_n \chi_n^* + N_{n-1,n} \chi_{n-1} \chi_n^* + N_{n,n-1} \chi_n \chi_{n-1}^*
\]

\[
Z = N_{00} \chi_0 \chi_0^* + N_{11} \chi_1 \chi_1^* + N_{n-1,n-1} \chi_{n-1} \chi_{n-1}^* + N_{nn} \chi_n \chi_n^* + N_{n-1,n} \chi_{n-1} \chi_n^* + N_{n,n-1} \chi_n \chi_{n-1}^* + N_{nn} \chi_n \chi_n^* + N_{n,n} \chi_n \chi_n^* \tag{5.11a}
\]

\[
Z = N_{00} \chi_0 \chi_0^* + N_{11} \chi_1 \chi_1^* + N_{n-1,n-1} \chi_{n-1} \chi_{n-1}^* + N_{nn} \chi_n \chi_n^* + N_{n-1,n} \chi_{n-1} \chi_n^* + N_{n,n-1} \chi_n \chi_{n-1}^* + N_{nn} \chi_n \chi_n^* + N_{n,n} \chi_n \chi_n^* + N_{n,0} \chi_n \chi_0^* \tag{5.11b}
\]

Then any physical invariant for \( n \not\equiv 0 \mod 4 \) will be 0-decoupled, so by Thm.3 the only possibilities for those invariants are given in eqs.(5.5a, h-l). That all these invariants are realized can be most easily seen using the list of physical invariants compiled in e.g. [9]. (5.5h) is his invariant \( \mathcal{M}^{[\nu]} \), while (5.5i) is his \( \mathcal{M}^{[\nu]i} \). Eqs.(5.5j, k, l) can now be obtained by multiplication: e.g. (5.5i) multiplied on the right and left by (5.5h) gives (5.5j).

That completes the proof for \( n \not\equiv 0 \mod 8 \). The case \( n \equiv 0 \mod 8 \) is more difficult. The following argument is only one of many that can be used.

As before, [9] can be used to show (5.5m-p) are all invariants. (Incidently, [9] missed 2 of the physical invariants for each \( n \equiv 0 \mod 4 \).) That (5.5a, h) are the only permutation invariants is easy to see from (5.11c). So we may assume \( N \) is non-0-decoupled.

Using (5.1) we can easily show \( S_{00}^W = S_{01}^W = S_{0,n-1}^W = S_{0n}^W \); since their squares must sum to 1 we know they must all equal \( \frac{1}{2} \). Then we can read off from (5.2) that

\[
4N_{00} = \sum_{i,j=0,1,n-1,n} N_{ij} \quad \tag{5.12a}
\]

for any invariant \( N \).

Let \( N \) be any non-0-decoupled physical invariant. By (5.12a), \( \sum N_{ij} = 4 \). Now, multiplying \( N \) on the right by the coefficient matrix of (5.5m) and using (5.12a) tells us

\[
N_{00} + N_{0,n-1} = N_{n-1,0} + N_{n-1,n-1} + N_{n,0} + N_{n,n-1}. \quad \tag{5.12b}
\]
Similar calculations give formulas for $N_{00} + N_{0,n}$, etc. From these it follows that one of $N_{0,n-1}, N_{0n}$ must be 0 and the other 1, and similarly for $N_{n-1,0}, N_{n0}$. Each one of the resulting 4 possible assignments of 0,1 to $N_{0,n-1}, N_{0n}, N_{n-1,0}$ and $N_{n0}$ turns out to fix all other $N_{ij}$, and gives us one of (5.5m-p).

The level 1 cases are sufficiently simple that several different proofs are possible. An example presumably is an explicit computation of the commutant based on the calculations of level 1 $S^W$-matrices made in [22]. The above arguments have the big advantage of being applicable to the more complicated types — e.g. $(A_2, k)$ (see [24]). An important tool in these cases is (5.3e) and the consequence for positive invariants when $\epsilon_x \cdot \epsilon_{\ell x} = -1$. To illustrate this, an alternate proof of $(C_n, 1)$ will now be provided for $n$ odd. Of course, it can also be used to prove completeness for $A_1$, odd levels.

Our goal is to prove that the diagonal invariant is the only physical invariant of these levels. We will accomplish this by first showing that the only permutation invariant is the diagonal invariant (5.5). Then (5.3e) tells us that any physical invariant must necessarily be 0-decoupled. Thm.3 then completes the argument.

Let $S_{ij} = S^W(C_n, 1)_{\beta_i, \beta_j}$. We get from (5.1), by computing $S_{0,\ell+1}/S_{0\ell}$, that

$$S_{0\ell} = \beta_n \sin \pi \frac{\ell + 1}{n + 2}$$

(5.13a)

for some constant $\beta_n$. Let $N_{ij} = \delta_{j, \sigma_i}$ be a permutation invariant. Then (2.7b) tells us $S_{ij} = S_{\sigma_i, \sigma_j}$. Since $\sigma(0) = 0$, (5.13a) now implies

$$\sigma(i) \in \{i, n - i\}.$$  

(5.13b)

But (2.7a) tells us $T_{ij} = T_{\sigma_i, \sigma_j}$, i.e. $p_i^2 \equiv p_{\sigma_i}^2$ (mod 2). These norms are computed in (5.9b). For odd $n$, $(n + 1 - i)^2 \equiv (i + 1)^2$ (mod 4(n + 2)) has no solutions for $i$, so for $n$ odd, $\sigma(i) = i$ and the only permutation invariant is the diagonal invariant (5.5a).

Now let $N_{ij}$ be any positive invariant. Suppose $N_{0k} > 0$. We want to show this can only happen when $k = 0$. By $[m]$ we will mean the unique number congruent to $m$ (mod $2(n + 2)$) lying in $0 \leq [m] < 2(n + 2)$. Let $x = (p_0; p_k)$. Then $\epsilon_x = +1$ (see (5.3c)). Let $\ell$ be relatively prime to $2(n + 2)$ the order of $x$. Then a simple calculation we will not include here shows

$$\epsilon_{\ell x} = \{(-1)^{(n+1-k)(\ell+1)} \epsilon_{\ell, k, \ell} \}, \{(-1)^{(n+1-k)(\ell+1)} \epsilon_{k, \ell} \} = \epsilon_{0,\ell} \cdot \epsilon_{k,\ell},$$

(5.14a)

where

$$\epsilon_{i,j} = \begin{cases} +1 & \text{if } 0 \leq [(n + 1 - i)j] < n + 2 \\ -1 & \text{if } [(n + 1 - i)j] > n + 2 \end{cases},$$

(5.14b)

and $\epsilon''_{i,j}$ is a sign depending only on $\ell$. Then (5.3c), together with $N_{ij}$ being a positive invariant, and $N_{0k} > 0$, forces $\epsilon_{0,\ell} = \epsilon_{k,\ell}$. Consider $\ell_m = 2^m n + 2$, for $m > 0$. Each $\ell_m$ is relatively prime to $2(n + 2)$, since $n + 2$ is odd. (2.7a) implies $x^2 \equiv 0$ (mod 2), which by (5.9b) implies $k$ must be even. Therefore $\epsilon_{0,\ell_m} = \epsilon_{0,2^m}$ and $\epsilon_{k,\ell_m} = \epsilon_{k,2^m}$. Now let $M$ be defined by $2^M < n + 2 < 2^{M+1}$. Then $\epsilon_{0,2^m} = -1 \forall m = 1, \ldots, M$. Hence $\epsilon_{k,2^m} = -1$ for those $m$. Since in addition $0 < n + 1 - k < n + 2$, these force $n + 1 - k = n + 1$, i.e. $k = 0$. 

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Similarly, \( N_{k0} > 0 \) would also imply \( k = 0 \). Therefore the only positive invariants for odd \( n \) are 0-decoupled. This completes the proof that, for odd \( n \), the only physical invariant of \( C_n \) level 1, or \( A_1 \) level \( n \), is the diagonal invariant.

6. Conclusion

We begin the paper by generalizing the analysis of [14] to any semi-simple algebra. We then prove that the Roberts-Terao-Warner lattice method is complete, and make the method more efficient by limiting the class of lattices that need be considered. By making use of the additional multiplicative structure the commutant has, which was not exploited by [14], we obtain some results in Sec.5 which help us to completely classify all level 1 physical invariants — see eqs.(5.5). The most attractive feature of the arguments in Sec.5 is their generality.

For small ranks and levels, the lattice method for generating invariants is computationally speaking quite practical, as is shown by the work in [19]. But its greatest value may be that it provides a convenient theoretical description for the commutant. A natural first step for classifying all physical invariants in a given class (see e.g. [10,14]) involves understanding the commutant, and lattices could provide a valuable geometrical tool for that. An exciting recent development was the translation given in [25] of the \( A_1 \) completeness proof into the lattice language (another proof, for odd levels, is included at the end of Sec.5). The resulting argument was surprisingly simple, suggesting that the Roberts-Terao-Warner approach may be a particularly fruitful one for the search for the other, more elusive completeness proofs (e.g. \( A_2 \)). This direction is being actively pursued in [24]. Of course the proof in this paper that lattice partition functions span the commutant, is a necessary first step for this program.

A generalization of this lattice approach, for use in finding heterotic invariants, was given in [6]. Among other things, it does not require the self-dual lattices to be even. Included there was a proof that it generates all heterotic invariants. Using this generalization, it may also be possible to apply lattices to coset models (see [25]), but work in that direction has not yet been completed.

Another avenue suggested by this paper lies in investigating the structure of the commutant as an algebra. More work in this direction is currently underway.

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References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, \textit{Nucl. Phys.} B\textbf{241} (1984) 333
[2] D. Gepner and E. Witten, \textit{Nucl. Phys.} B\textbf{278} (1986) 493
[3] V.G. Kac, *Infinite Dimensional Lie Algebras*, 3rd ed. (Cambridge University Press, Cambridge, 1990)

[4] S. Kass, R.V. Moody, J. Patera and R. Slansky, *Affine Lie Algebras, Weight Multiplicities, and Branching rules* Vol.1 (University of California Press, Berkeley, 1990)

[5] A.N. Schellekens, *Classification of ten-dimensional heterotic strings* (CERN preprint TH.6325, 1991)

[6] T. Gannon, *Partition functions for heterotic WZW conformal field theories* (Carleton preprint, 1992)

[7] S. Bais and P. Bouwknegt, *Nucl. Phys.* B279 (1987) 561; A.N. Schellekens and N.P. Warner, *Phys. Rev.* D34 (1986) 3092

[8] A.N. Schellekens and S. Yankielowicz, *S. Nucl. Phys.* B327 (1989) 673

[9] D. Bernard, *Nucl. Phys.* B288 (1987) 628

[10] A. Cappelli, C. Itzykson and J.-B. Zuber, *Nucl. Phys.* B280 [FS18] (1987) 445; A. Cappelli, C. Itzykson and J.-B. Zuber, *Commun. Math. Phys.* 113 (1987) 1; D. Gepner and Z. Qui, *Nucl. Phys.* B285 (1987) 423

[11] C. Itzykson, *Nucl. Phys. (Proc. Suppl.*)* 5B (1988) 150

P. Degiovanni *Commun. Math. Phys.* 127 (1990) 71

[12] N.P. Warner, *Commun. Math. Phys.* 130 (1990) 205

[13] P. Roberts and H. Terao, *Int. J. Mod. Phys.* A7 (1992) 2207

[14] M. Bauer and C. Itzykson, *Commun. Math. Phys.* 127 (1990) 617

[15] J.H. Conway and N.J.A. Sloane, *Sphere packings, Lattices and Groups* (Springer-Verlag, New York, 1988)

[16] B. Schoenberg, *Elliptic Modular Functions* (Springer-Verlag, Berlin, 1974)

[17] Ph. Ruelle, *Commun. Math. Phys.* 133 (1990) 181

[18] T. Gannon and C.S. Lam, *Rev. Math. Phys.* 3 (1991) 331

[19] Q. Ho-Kim and T. Gannon, *The low level modular invariants of rank 2 algebras* (work in progress)

[20] Ph. Ruelle, E. Thiran and J. Weyers, *Comm. Math. Phys.* 133 (1990) 305

[21] E. Verlinde, *Nucl. Phys.* B300 [FS22] (1988) 360; G. Moore and N. Seiberg, *Phys. Lett.* B212 (1988) 451

[22] V.G. Kac and M. Wakimoto, *Adv. Math.* 70 (1988) 156

[23] D. Versteegen, *Comm. Math. Phys.* 137 (1991) 567

[24] T. Gannon and P. Roberts, *The modular invariant partition functions of SU(3)* (work in progress)

[25] P. Roberts, *Whatever Goes Around Comes Around: Modular Invariance in String Theory and Conformal Field Theory*, Ph.D. Thesis (Institute of Theoretical Physics, Goteborg, 1992)