LDU factorization

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Abstract. LU-factorization of matrices is one of the fundamental algorithms of linear algebra. The widespread use of supercomputers with distributed memory requires a review of traditional algorithms, which were based on the common memory of a computer. Matrix block recursive algorithms are a class of algorithms that provide coarse-grained parallelization. The block recursive LU factorization algorithm was obtained in 2010. This algorithm is called LEU-factorization. It, like the traditional LU-algorithm, is designed for matrices over number fields. However, it does not solve the problem of numerical instability. We propose a generalization of the LEU algorithm to the case of a commutative domain and its field of quotients. This LDU factorization algorithm decomposes the matrix over the commutative domain into a product of three matrices, in which the matrices L and U belong to the commutative domain, and the elements of the weighted truncated permutation matrix D are the elements inverse to the product of some pair of minors. All elements are calculated without errors, so the problem of instability does not arise.

Introduction

The representation of the matrix $A$ in the form of two factors $A = LU$, where $L$ is the lower triangular matrix and $U$ is the upper triangular matrix, is called the LU decomposition (or LU factorization). This decomposition is considered as the basic algorithm for any library of linear algebra programs [1]. Many algorithms are built on its basis, including solving linear systems, matrix inversion, rank calculation, etc.

With the advent of supercomputers, it became possible to increase the size of matrices in solving applied problems. At the same time, shortcomings of the known matrix algorithms appeared. It became clear that they cannot be applied to large matrices. Problems such as accumulation of rounding errors, loss of accuracy, poor concurrency, loss of sparseness of matrices, high computational complexity, and other problems began to appear (see, for example, [2]).

In 2010, the LEU factorization algorithm was obtained. This is an algorithm that applies to matrices over number fields. It allowed to overcome many of these
shortcomings for finite number fields. It was the first block recursive algorithm with the complexity of matrix multiplication, sparse and highly parallelistic [3].

However, the problem of loss of accuracy, in the case of classical fields, remained as before impossible to overcome. It was required to obtain a generalization of LEU factorization to commutative domains and their fields of quotients. Such an algorithm will allow, for example, to use integers in the calculations instead of approximate rational numbers. Two approaches were proposed to create such an LDU factorization algorithm [5], [6]. The present work continues and completes these studies. We propose the complete dichotomous recursive LDU factorization algorithm for the commutative domain and give its proof.

1 Initial statement of the problem

Let $R$ be a commutative domain, $F$ its field of quotients. Let $A \in R^{n \times n}$ be a matrix that has rank $r$, $r \leq n$. We want to get matrices $L, U \in R^{n \times n}$ of rank $n$, ($L$ is low triangular, $U$ is upper triangular), matrix $D$, that has rank $r$, with $r$ non-zero elements equal $(\det_1)^{-1}, (\det_1\det_2)^{-1}, \ldots, (\det_{r-1}\det_r)^{-1}$, such that

$$A = LDU.$$ 

We denote by $\det_r$ the $r \times r$ nonzero minor of the matrix $A$, whose position is determined by $r$ nonzero rows and columns of the matrix $D$. The determinants of successively nested nondegenerate submatrices of orders $r - 1, r - 2, \ldots, 2.1$ we denote $\det_{r-1}, \det_{r-2}, \ldots, \det_2, \det_1$, respectively.

To solve this problem, we formulate a more general problem, but first give some necessary definitions.

2 Preliminary information

2.1 Semigroup of truncated weighted permutations

The diagram shows the structure of the semigroup of truncated weighted permutations $S_{wp}$:

At the center of the diagram you can see a permutation group $G_p$. Between $S_{wp}$ and $G_p$ there are two more subalgebras: $S_p$ and $G_{wp}$.

If we replace elements with values of 1 in the matrices from the permutation group $G_p$ by arbitrary nonzero elements, then we obtain the group of weighted permutations $G_{wp}$.

If, on the contrary, in the matrices from the permutation group $G_p$, we replace some elements with values 1 with zero elements, we obtain a semigroup of truncated permutations $S_p$.

The semigroup of truncated weighted permutations $S_{wp}$ is obtained from the group $G_{wp}$ if we replace some nonzero elements with zeros.

If we select all diagonal matrices in the semigroup $S_{wp}$, then we obtain a semigroup of (weighted) diagonal matrices $S_{wd}$. Two other subalgebras $S_d$ and
Fig. 1. The structure of the truncated weighted permutations semigroup $S_{wp}$.

$G_{wd}$ are embedded in it. The semigroup $S_d$ is formed by diagonal matrices for which only 1 and 0 can stand on a diagonal. The group $G_{wd}$ is formed by those diagonal matrices from $S_{wp}$ for which there are no zero elements on the diagonal. The identity group $I$ that contains one identity matrix closes this construction.

2.2 Some mappings on semigroups

For matrices from the semigroup $S_{wp}$ we introduce two mappings: unit and extended.

A homomorphism of the multiplicative groups $F^* \rightarrow 1$ (or $R^* \rightarrow 1$) induces a homomorphism of the corresponding subalgebras: $S_{wp} \rightarrow S_p$, $S_{wd} \rightarrow S_{\hat{D}}$, $G_{wp} \rightarrow G_p$, $G_{wd} \rightarrow I$. All nonzero elements of the matrix are replaced by unit elements. On the diagram, they correspond to arrows that are directed down and to the left.

**Definition 1.** The mapping of the matrix $D \in S_{wp}(F)$ induced by the homomorphism $F^* \rightarrow 1$: $D \in S_{wp} \rightarrow D^{-1} \in S_p$

is called unit mapping.

The unit homomorphism of semigroups can be represented by the following commutative diagram:

$$(A, B) \xrightarrow{\rightarrow^{-1}} (A^{-1}, B^{-1})$$

$$(A^{-1}, B^{-1}) \times \downarrow \times \downarrow$$

$$C \xrightarrow{\rightarrow^{-1}} C^{-1}$$

**Definition 2.** The matrix mapping $D \in S_{wp} \rightarrow D^{Ext} \in G_{wp}$ in which every block at the intersection of zero rows and zero columns is replaced by a unit block called extended mapping and is indicated by a "Ext" upper index.

On the diagram, the extended mapping corresponds to 4 arrows that are directed down and to the right: $S_{wp} \rightarrow G_{wp}$, $S_{wd} \rightarrow G_{wd}$, $S_p \rightarrow G_p$, $S_{\hat{D}} \rightarrow I$. As a result of such a transformation, a matrix of full rank is obtained.
**Definition 3.** The mapping of the matrix $D \in S_{wp} \rightarrow \bar{D} = D^{Ext} - D \in S_p$ is called the **complementary mapping** and is denoted by a “bar”.

**Property 1.** The special case of the complementary mapping, when the matrix $D$ belongs to the semigroup $S_p$, is an involution on the semigroup $S_p$. Involution is reversible: $\bar{\bar{D}} = D$.

**Property 2.** $\forall D \in S_p : D + \bar{D} \in G_p$.

Examples of such involution: 
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

### 2.3 Surrounding minors

Hereinafter, we consider matrices over the commutative domain $R$.

**Definition 4.** Let a matrix $\mathcal{M}$ be given and let $A$ be its square submatrix located in the upper left corner. Any submatrix $G$, that is obtained by adding to the block $A$ some row $r$ and some column $c$ of matrix $\mathcal{M}$
\[
G = \begin{pmatrix} A & c \\ r & \omega \end{pmatrix}.
\]

is called the submatrix that surrounds matrix $A$.

**Theorem 1.** Let $A$ be a square matrix, $A^*$ the adjoint matrix for $A$, $\det(A) \neq 0$, $G$ the surrounding matrix for $A$ (1), then
\[
\det(G) = \det(A)\omega - rA^*c.
\]

**Proof.** This equality expresses the decomposition of the determinant $G$ in row $r$ and column $c$.

**Theorem 2.** Let a matrix $\mathcal{M}$ be divided into blocks $\mathcal{M} = \begin{pmatrix} A_k & B \\ C & D \end{pmatrix}$, $A_k$ is a square block of size $k \times k$, its determinant $\det_k$ is non-zero and $A_k^*$ is an adjoint matrix for $A_k$, then the elements of the matrix
\[
A = (\det_k)D - CA_k^*B
\]

are the minors that surround the $A_k$ block.

**Proof.** To prove the theorem, it suffices to apply Theorem 1 to each element of the matrix $A$. 
3 Statement of the problem

3.1 General statement of the problem

Let $R$ be a commutative domain, $F$ its field of quotients. Let a matrix $M \in R^{N \times N}$ ($N = 2^\nu$) be given, and let $\alpha$ be the determinant of the largest nondegenerate (or empty) submatrix located in the upper left corner of the matrix $M$. For the case of an empty submatrix, we set $\alpha = 1$.

Let $n = 2^p$, $A \in R^{n \times n}$ be a matrix of rank $r$, $r \leq n$, and elements of the matrix $A$ be surrounding minors with respect to the minor $\alpha$. In the case of an empty submatrix, we can take $A = M$.

We want to obtain matrices $L, U, M, W \in R^{n \times n}$ of rank $n$, ($L$ lower triangular, $U$ - upper triangular), a matrix $D \in S_{wp}(F)$, of rank $r$, with $r$ non-zero elements equal $(\alpha \det_1)^{-1}, (\det_1 \det_2)^{-1}, \ldots, (\det_{r-1} \det_r)^{-1}$, such that

$$\begin{cases}
\alpha LDU = A \\
L\tilde{D}M = I \\
W\tilde{D}U = I
\end{cases}$$

(1)

We denote by $\det_r$ the $r \times r$ nonzero minor of the matrix $A$, whose position is determined by $r$ nonzero rows and columns of the matrix $D$. The determinants of successively nested nondegenerate submatrices of orders $r-1, r-2, \ldots, 2, 1$ we denote $\det_{r-1}, \det_{r-2}, \ldots, \det_2, \det_1$, respectively.

We denote by $\tilde{D}$ a matrix

$$\tilde{D} = \det_r^{-1}(\alpha D)^{Ext} = \alpha \det_r^{-1} D + \det_r^{-1} \tilde{D},$$

and we denote

$$E = D^{-1}, I = EE^T, J = E^T E, \tilde{I} = I - I, \tilde{J} = I - J,$$

(2)

$E \in S_p$, $I, J \in S_d$, $I$ is the unit matrix.

It should be noted that the matrices $I$ and $D$ have the same nonzero rows, and the matrices $J$ and $D$ have the same nonzero columns:

$$IDJ = D, \quad ID = 0, \quad DJ = 0.$$

We define the properties of the matrices $L$ and $U$ as follows:

$$LI = I, \quad JU = J.$$

(3)

4 Dichotomous Recursive Decomposition Design

We want to describe a procedure that allows you to compute the LDU-factorization

$$(L, D, U, M, \tilde{D}, W, \alpha_r) = \text{LDU}(A, \alpha)$$

in a recursive and dichotomous way.
Let $k\neq 0$, then we assume that $D = I = J = 0$, $M = W = a\mathbf{I}$, $\tilde{D} = a^{-1}\mathbf{I}$, $L = U = \tilde{I} = J = \tilde{D} = \mathbf{I}$, $\alpha_{r} = a$.

(2) If $n = 1$ ($A = [a]$, $a \neq 0$), then we assume that $D = [(a\alpha)^{-1}]$, $\tilde{D} = [a^{-2}]$, $L = U = M = W = [a]$, $I = J = [1]$, $\tilde{I} = J = \tilde{D} = [0]$, $\alpha_{r} = a$.

(3) For $A \neq 0$ and $n > 1$ we divide matrix $A$ into four equal blocks

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \quad (4)$$

We can do the LDU-decomposition for block $A_{11}$:

$$\begin{cases} \alpha L_{11} D_{11} U_{11} = A_{11} \\ L_{11} \tilde{D}_{11} M_{11} = I \\ W_{11} \tilde{D}_{11} U_{11} = I \end{cases} \quad (5)$$

Let $k = \text{rank}(A_{11})$ and $\det_{1}, \det_{2}, \ldots, \det_{k}$ are the non-zero nested leading minors of $A_{11}$, that were found recursively, $\alpha_{k} = \det_{k}$, $\tilde{D}_{11} = \alpha_{k}^{-1}(\alpha D_{11} + \tilde{D}_{11})$ and the non-zero elements of $D_{11}$ equal $(a\det_{1})^{-1}, (a\det_{2})^{-1} \ldots (a\det_{k-1}\alpha_{k})^{-1}$.

Then we can write the equality

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix} \begin{pmatrix} U_{11} & 0 \\ 0 & I \end{pmatrix}. \quad (6)$$

We denote the new blocks:

$$A'_{12} = \alpha_{k} \tilde{D}_{11} M_{11} A_{12}, \quad A'_{21} = \alpha_{k} A_{21} W_{11} \tilde{D}_{11}. \quad (7)$$

The middle matrix can be decomposed as follows

$$\begin{pmatrix} \frac{1}{\alpha} A'_{11} & \frac{1}{\alpha} A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \frac{1}{\alpha} A'_{21} D_{11} \tilde{I} & I \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} A_{11} & \frac{\alpha}{\alpha_{k}} A'_{12} \\ \frac{\alpha}{\alpha_{k}} A_{21} & \frac{1}{\alpha_{k}} A_{22} \end{pmatrix} \begin{pmatrix} I & \frac{1}{\alpha} D_{11}^{-1} A'_{21} \\ 0 & I \end{pmatrix}. \quad (8)$$

We denote

$$E_{11} = D_{11}^{-1}, \quad I_{11} = E_{11} E_{11}^{T}, \quad J_{11} = E_{11}^{T} E_{11}. \quad (9)$$

To denote the generalized inverse matrix, we use the superscript plus. Note that for any matrix $(a_{i,j}) \in S_{wp}$ the generalized inverse matrix coincides with the pseudoinverse matrix:

$$(a_{i,j})^{+} = \{(b_{i,j}) : b_{i,j} = 0 \text{ if } a_{j,i} = 0, \ b_{i,j} = a_{j,i}^{-1} \text{ if } a_{j,i} \neq 0\}.$$ 

A pseudoinverse matrix is obtained by transposing a given matrix and replacing all nonzero elements with inverse elements.

As far as $I_{11}\tilde{D}_{11} = \tilde{D}_{11} = \tilde{D}_{11} J_{11}$, $D_{11}^{+} I_{11} = 0$ and $D_{11}^{+} \tilde{D}_{11} = J_{11}$ we get

$$A'_{12} = \frac{1}{\alpha \alpha_{k}} \tilde{I}_{11} A'_{12} = \frac{1}{\alpha} \tilde{D}_{11} M_{11} A_{12}, \quad A'_{21} = \frac{1}{\alpha \alpha_{k}} A'_{21} \tilde{J}_{11} = \frac{1}{\alpha} A_{21} W_{11} \tilde{D}_{11}. \quad (9)$$
For the lower right block we get
\[ A''_{22} = (A'_{21}J_{11} + A''_{21})D''_{11}A'_{12} + A'_{21}D''_{11}A''_{12} + \alpha\alpha_kA''_{22} \]
\[ A'_{22} = \frac{1}{\alpha\alpha_k}(\alpha\alpha_k^2A_{22} - A'_{21}D_{11}^+A'_{12}). \] (10)

Matrices $A''_{12}$ and $A''_{21}$ are matrices of surrounding minors with respect to minor $\alpha_k$. See the prove in Theorem 4.

Let \[ \begin{cases} \alpha_kL_{21}D_{21}U_{21} = A''_{21} \\ L_{21}d_{21}M_{21} = I \end{cases} \quad \text{and} \quad \begin{cases} \alpha_kL_{12}D_{12}U_{12} = A''_{12} \\ L_{12}d_{12}M_{12} = I \end{cases} \]
be LDU decomposition of blocks $A''_{21}$ and $A''_{12}$.

Let $A'$ be submatrix of $A$ which is fixed with non zero rows of matrix $\text{diag}(I_{11}, I_{12}, I_{21})$ and non zero columns of matrix $\text{diag}(J_{11}, J_{21}, J_{12})$. Submatrices $A''_{12}, A''_{21}$ and the submatrix, which corresponds to minor $\alpha_k$, do not have common nonzero rows and columns, so the sequence of nested non zero minors of submatrix $A'$ can be selected in different ways.

Let $\text{rank}(A''_{21}) = l_1$ and $\text{rank}(A''_{12}) = m_1$, then the rank of submatrix $A'$ is equal $s = k + m_1 + l_1$. Suppose that we have obtained the following sequences of nested minors: $\det, ..., \det_k, ..., \det_l, (l = l_1 + k)$, for the block $(A_{11}, A_{21})^T$ and $\det_k, ..., \det_l, \det_k, ..., \det_m, m = m_1 + k$, for the block $(A_{11}, A_{12})$.

For the matrix $A'$ we can set the following sequences of nested minors: $\det_1, ..., \det_k, ..., \det_1, \det_{i+1}, ..., \det_s,$
with $\det_{i+1} = \lambda\det_{i+1}^\prime, \quad i = 1, 2, ..., m_1, \quad (\lambda = \det_1/\det_k),$
in particular, $\det_s = \lambda\det_m^\prime$.

We denote $\alpha_t = \det_t, \quad \alpha_s = \det_s, \quad J_{12} = \lambda J_{12} + \bar{J}_{12}, \quad I_{12} = \lambda I_{12} + \bar{I}_{12},$
\[ L_{12} = \lambda L_{12}^\prime, \quad U_{12} = \lambda U_{12}, \quad D_{12} = \lambda^{-2}D_{12}, \] (11)
\[ \hat{D}_{12} = \lambda^{-1}I_{12}^\prime \hat{D}_{12}, \quad M_{12} = \lambda M_{12}, \quad W_{12} = \lambda W_{12}. \] (12)

Then the last system can be written as follows:
\[ \begin{cases} (\lambda\alpha_k)L_{12}D_{12}U_{12}^\prime = \hat{\lambda}A''_{12} \\ L_{12}^\prime \hat{D}_{12}^\prime M_{12}^\prime = I \end{cases}, \]
\[ W_{12}^\prime D_{12}U_{12}^\prime = I \]
So we can write the following matrix equation:
\[ \begin{pmatrix} \alpha D_{11} & \alpha D_{11} \alpha_k^\prime A''_{12} \\ \alpha_k A''_{21} & \alpha_k A''_{22} \end{pmatrix} = \begin{pmatrix} L_{12}^\prime & 0 \\ 0 & L_{21} \end{pmatrix} \begin{pmatrix} \alpha D_{11} & \alpha D_{11} \alpha_k^\prime A''_{12} \\ \alpha_k A''_{21} & \alpha_k A''_{22} \end{pmatrix} \begin{pmatrix} U_{12} & 0 \\ 0 & U_{12}^\prime \end{pmatrix}, \] (13)
where we denote
\[ A''_{22} = \hat{D}_{21}M_{21}A'_{22}W_{12}D_{12}^\prime = \hat{D}_{21}M_{21}A'_{22}W_{12}I_{12}^\prime \hat{D}_{12} \] (14)
and use the following equation: \( I_{21}J_{11} = 0 \), \( J_{11}J_{21} = 0 \) and \( L_{12}D_{11}U_{21} = D_{11} \). To check the last equation we can write each matrices as follows: \( L_{12} = L_{12}I_{12} + I_{12} \), \( D_{11} = I_{11}D_{11}J_{11} \), \( U_{21} = J_{21}U_{21} + \dot{J}_{21} \).

The middle matrix can be decomposed in two ways:

\[
\begin{pmatrix}
\alpha D_{11} & \alpha D_{12} \\
\alpha D_{21} & \frac{1}{\alpha_k} A_{22}'
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
\frac{1}{\alpha_k} A_{22}' D_{12}' & I
\end{pmatrix}
\begin{pmatrix}
\alpha D_{11} & \alpha D_{12} \\
\alpha D_{21} & \frac{1}{\alpha_k} A_{22}'
\end{pmatrix}
\begin{pmatrix}
I & \frac{1}{\alpha_k} D_{21}' A_{22}' \dot{J}_{12} \\
0 & I
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
\alpha D_{11} & \alpha D_{12} \\
\alpha D_{21} & \frac{1}{\alpha_k} A_{22}'
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
\frac{1}{\alpha_k} \tilde{I}_{21} A_{22}' D_{12}' & I
\end{pmatrix}
\begin{pmatrix}
\alpha D_{11} & \alpha D_{12} \\
\alpha D_{21} & \frac{1}{\alpha_k} A_{22}'
\end{pmatrix}
\begin{pmatrix}
I & \frac{1}{\alpha_k} D_{21}' A_{22}' \\
0 & I
\end{pmatrix}
\]

(15)

We use the following equations \( D_{12}' D_{11} = 0 \), \( D_{11} D_{21}' = 0 \), and for the lower right block we get

\[
\frac{1}{\alpha_k} A_{22}'' = \frac{1}{\alpha_k} A_{22}' \tilde{J}_{12} + \frac{1}{\alpha_k} \tilde{I}_{21} A_{22}' \tilde{J}_{12} + \frac{\alpha}{\alpha_s} A_{22}''
\]

or

\[
\frac{1}{\alpha_k} A_{22}'' = \frac{1}{\alpha_k} \tilde{I}_{21} A_{22}' + \frac{\alpha}{\alpha_s} A_{22}''
\]

Both of these expressions give the same value for \( A_{22}'' \):

\[
A_{22}'' = \frac{\alpha_s}{\alpha_k} \tilde{I}_{21} A_{22}' \tilde{J}_{12} = \frac{1}{\alpha_k} \tilde{D}_{21} M_{21} A_{22}' W_{12} \tilde{D}_{12}.
\]

(16)

Further we will use the second decomposition.

The matrix \( A_{22}'' \) is the matrix of surrounding minors of \( A \) with respect to minor \( \alpha_s \). See prove in Theorem 4.

Let

\[
\begin{cases}
\alpha_s L_{22} D_{22} U_{22} = A_{22}''
L_{22} D_{22} M_{22} = I \\
W_{22} D_{22} U_{22} = I
\end{cases}
\]

be a decomposition of the matrix \( A_{22}'' \), then we can write the equality

\[
\begin{pmatrix}
D_{11} & D_{12}' \\
D_{21} & \frac{1}{\alpha_s} A_{22}''
\end{pmatrix}
= \begin{pmatrix}
I & 0 \\
0 & L_{22}
\end{pmatrix}
\begin{pmatrix}
D_{11} & D_{12}' \\
D_{21} & \frac{1}{\alpha_s} A_{22}''
\end{pmatrix}
\begin{pmatrix}
I & 0 \\
0 & U_{12}
\end{pmatrix}
\]

(17)

To prove this equality we can write matrices \( L_{22}, U_{22}, D_{12}, D_{12} \) as follows

\( L_{22} = L_{22} I_{22} + \tilde{I}_{22}, \ U_{22} = J_{22} U_{22} + \tilde{J}_{22}, \ D_{21} = I_{21} D_{21} J_{21}, \ D_{12} = I_{12} D_{12} J_{12}, \)

and check the equations

\( L_{22} D_{21} = D_{21} \) and \( D_{12}' U_{22} = D_{12}' \).

As a result of the sequence (6), (8), (13), (15), (17) of decompositions we obtain the LDU-decomposition of the matrix \( A \) in the form

\[
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
= \alpha LDU, \quad D = \begin{pmatrix}
D_{11} & D_{12}' \\
D_{21} & D_{22}
\end{pmatrix}
= \begin{pmatrix}
D_{11} & \lambda^{-2} D_{12} \\
D_{21} & D_{22}
\end{pmatrix}
\]

(18)
with $D_{12}^r = \lambda^{-2} D_{12}$ and such matrices $L$ and $U$:

$$L = \begin{pmatrix} L_{11} & 0 \\ L_3 & L_{12} \end{pmatrix}, \quad U = \begin{pmatrix} I & 0 \\ 0 & U_{22} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ L_3 & L_{12} \end{pmatrix}, \quad M = \begin{pmatrix} U_1 & U_2 \\ 0 & U_4 \end{pmatrix} \quad \text{with}$$

$$U_2 = \alpha_k^{-1} U_{21} D_{11}^r A_{12}^r + \alpha_k^{-1} \alpha_k^{-1} D_{21}^r A_{22}^r U_{12}^r =$$

$$= \alpha_k^{-1} J_{11} M_{11} A_{12} + \alpha_k^{-1} \alpha_k^{-1} J_{21} M_{21} A_{22}, \quad L_3 = \alpha_k^{-1} A_{12}^r L_{11}^r + \alpha_k^{-1} \alpha_k^{-1} L_{21}^r \tilde{A}_{22}^r D_{12}^r =$$

$$= \alpha_k^{-1} A_{21} W_{11} I_{11} + \alpha_k^{-1} \alpha_k^{-1} \tilde{D}_{21} M_{21} A_{22}^r W_{12} I_{12} \quad \tilde{D} = \alpha_r^{-1} (\alpha D + D) \quad M = \tilde{D}^{-1} L_1^{-1}, = \tilde{D}^{-1} \begin{pmatrix} L_1^{-1} & 0 \\ -L_4^{-1} L_3 L_1^{-1} & L_4^{-1} \end{pmatrix}, \quad \tilde{D}^{-1} =$$

$$\left( \begin{array}{ccc} \lambda^{-1} \tilde{D}_{12} M_{12} \tilde{D}_{11} M_{11} & 0 \\ -\tilde{D}_{22} M_{22} \tilde{D}_{21} M_{21} L_3 \lambda^{-1} \tilde{D}_{12} M_{12} \tilde{D}_{11} M_{11} \tilde{D}_{22} M_{22} \tilde{D}_{21} M_{21} & \end{array} \right),$$

$$W = U^{-1} \tilde{D}^{-1} = \begin{pmatrix} U_1^{-1} - U_1^{-1} U_2 U_4^{-1} \\ 0 \\ U_4^{-1} \end{pmatrix} \tilde{D}^{-1} =$$

$$\begin{pmatrix} W_{11} \tilde{D}_{11} W_{21} \tilde{D}_{21} & -W_{11} \tilde{D}_{11} W_{21} U_3 W_{12} \tilde{D}_{12} J_{12}^{-1} W_{22} \tilde{D}_{22} \\ 0 & W_{12} \tilde{D}_{12} J_{12}^{-1} W_{22} \tilde{D}_{22} \end{pmatrix} \tilde{D}^{-1}. \quad \text{(24)}$$

In the expressions (18) - (24), the $LDU$ decomposition of the matrix $A$ are given and the matrices $W$ and $M$ that satisfy the conditions $LdM = I$ and $WdU = I$ are obtained.

We proved the correctness of the following recursive algorithm.

5 Algorithm of LDU-decomposition

$$(L, D, U, M, \tilde{D}, W, \alpha_r) = \text{LDU}(A, \alpha).$$

(1) If $(A = 0)$ then

\{ $\alpha_r = \alpha; \quad D = I = J = 0; \quad L = U = I = J = \tilde{D} = I; \quad M = W = \tilde{D} = \alpha I; \}$

(2) If $(n = 1 \& A = [a] \& a \neq 0)$ then

\{ $\alpha_r = a; \quad L = U = M = W = [a]; \quad D = [(\alpha \ast a)^{-1}]; \quad \tilde{D} = [a^{-2}];$
\( J = I = [1] \); \( \bar{I} = \bar{J} = \bar{D} = [0] \);

(3) If \((n \geq 2 \& \ A \neq 0)\) then

\[
\{ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \}
\]

(3.1)

\[
(L_{11}, D_{11}, U_{11}, M_{11}, d_{11}, W_{11}, \alpha_k) = LDU(A_{11}, \alpha),
\]

\[
A_{12}^0 = M_{11} * A_{12};
\]

\[
A_{12}^1 = a_k * \hat{D}_{11} * A_{12}^0;
\]

\[
A_{12}^2 = \hat{D}_{11} * A_{12}^0 / \alpha;
\]

\[
A_{21}^0 = A_{21} * W_{11};
\]

\[
A_{21}^1 = a_k * A_{21}^0 * \hat{D}_{11};
\]

\[
A_{21}^2 = A_{21}^0 * D_{11} / \alpha;
\]

(3.2)

\[
(L_{21}, D_{21}, U_{21}, M_{21}, d_{21}, W_{21}, \alpha_l) = LDU(A_{21}^2, \alpha_k),
\]

(3.3)

\[
(L_{12}, D_{12}, U_{12}, M_{12}, d_{12}, W_{12}, \alpha_m) = LDU(A_{12}^2, \alpha),
\]

\[
\lambda = \frac{a_k}{a_k} * a_s = \lambda * a_m;
\]

\[
A_{12}^0 = A_{12}^1 * D_{11}^+ * A_{12}^0;
\]

\[
A_{12}^1 = (aa_k^2 * A_{22} - A_{22}^0) / (aa_k);
\]

\[
A_{22}^0 = \hat{D}_{21} * M_{21} * A_{22}^1 * W_{12} * \hat{D}_{12};
\]

\[
A_{22}^1 = A_{22}^0 / (a_k^2 \alpha);
\]

(3.4)

\[
(L_{22}, D_{22}, U_{22}, M_{22}, d_{22}, W_{22}, \alpha_r) = LDU(A_{22}^3, a_s),
\]

\[
J_{12}^\lambda = \lambda * J_{12} + \bar{J}_{12}; \quad I_{12}^\lambda = \lambda I_{12} + \bar{I}_{12};
\]

\[
L_{12}^\lambda = L_{12} * I_{12}^\lambda; \quad U_{12} = \bar{U}_{12};
\]

\[
U_2 = J_{11} * M_{11} * A_{12} / a_k + J_{21} * M_{21} * A_{22}^1 / (a_l \alpha);
\]

\[
L_3 = A_{21} * W_{11} * I_{11} / a_k + \bar{D}_{21} * M_{21} * A_{22}^1 * W_{12} * I_{12} / (a_{m} * a_k \alpha);
\]

\[
L = \begin{pmatrix} L_{11} & 0 \\ L_{3} & L_{21} \end{pmatrix}, \quad D = \begin{pmatrix} D_{11} & \lambda^2 D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \quad U = \begin{pmatrix} U_{21} U_{11} & U_2 \\ 0 & U_{22} U_{12} \end{pmatrix},
\]

\[
\hat{D} = \alpha(\alpha_r)^{-1} D + (\alpha_r)^{-1} \bar{D},
\]

\[
M = \hat{D}^{-1} \begin{pmatrix} I_{12}^{\lambda^{-1}} \tilde{D}_{12} M_{12} \tilde{D}_{11} M_{11} \\ -\tilde{D}_{22} M_{22} \tilde{D}_{21} M_{21} L_3 I_{12}^{\lambda^{-1}} \tilde{D}_{12} M_{12} \tilde{D}_{11} M_{11} \tilde{D}_{22} M_{22} \tilde{D}_{21} M_{21} \end{pmatrix},
\]

\[
W = \begin{pmatrix} W_{11} \tilde{D}_{11} W_{21} \tilde{D}_{21} -W_{11} \tilde{D}_{11} W_{21} U_2 W_{12} \tilde{D}_{12} I_{12}^{\lambda^{-1}} W_{22} \tilde{D}_{22} \\ 0 \end{pmatrix} \begin{pmatrix} W_{12} \tilde{D}_{12} I_{12}^{\lambda^{-1}} W_{22} \tilde{D}_{22} \end{pmatrix} \hat{D}^{-1}.
\]
5.1 Auxiliary Theorems

The following statements prove the factorization algorithm.

**Theorem 3.** Let $M_k$ and $M_s$ be corner blocks of size $k \times k$ and $s \times s$, ($s > k$, $s = k + t$) of the matrix $M$, their determinants $\det_k$ and $\det_s$ not equal to zero. Let the matrix $A$ be formed by the surrounding minors of the block $M_k$ and let it be divided into blocks $A = \begin{pmatrix} A_1^1 & A_2^1 \\ A_3^1 & A_4^1 \end{pmatrix}$, wherein $A_1^1$ is a square block of size $t \times t$ and $A_1^*$ is its adjoint matrix, then elements of matrix

$$
\frac{1}{\det_k} (\det_s A - \frac{1}{\det_s} A_1^* A A_1^T A^2)
$$

are the minors of the matrix $M$ that surround the block $M_s$.

**Proof.** Proof can be found in ([7], Theorem 2) or in ([8] pp. 23-25). In [8], this theorem is called the “Determinant Identity of Descent”.

You can see that Theorem 3 generalizes Theorem 2 if we assume that the block $A_k$ can have size 0 and the determinant of such an empty block is 1. And Theorem 2 is a special case of Theorem 3 if we consider each element of the original matrix as a surrounding minor for an empty block.

**Theorem 4.** Let the matrix $A$ be formed by the surrounding minors of the upper left corner block $\alpha$ of the matrix $M$. Let matrix $A$ be divided into blocks (4) and all equalities of system (5) are true, $\text{rank}(A_{11}) = k$ and $\alpha_k$ is the largest non-zero minor of $A_{11}$. Then matrices $A_{12}^{(1)}$ and $A_{21}^{(2)}$ (9) are matrices of surrounding minors with respect to minor $\alpha_k$, $A_{22}^{(2)}$ (16) is the matrix of surrounding minors with respect to minor $\alpha_s$.

**Proof.**

To simplify writing the proof, we consider the case when the non-zero block $D^1$ of matrix $D_{11}$ be in the upper left corner of $D_{11}$ and we denote by $A^1$ a non-degenerate block of size $k \times k$ in upper left corner of the matrix $A_{11}$.

We can write the LDU decomposition of matrix $A_{11}$ and the equalities $W_{11} d_{11} U_{11} = I$ and $L_{11} d_{11} M_{11} = I$ in such block shape:

$$
A_{11} = \begin{pmatrix} A_{11}^1 & A_{11}^2 \\ A_{11}^3 & A_{11}^4 \end{pmatrix} = \alpha \begin{pmatrix} L_{11}^1 & 0 \\ 0 & L_{11}^2 \end{pmatrix} \begin{pmatrix} D_{11}^1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_{11}^1 & U_{11}^2 \\ 0 & I \end{pmatrix},
$$

$$
\begin{pmatrix} W_{11}^1 & 0 \\ 0 & \alpha_k I \end{pmatrix} \begin{pmatrix} \alpha_k^{-1} D_{11}^1 & 0 \\ 0 & \alpha_k^{-1} I \end{pmatrix} \begin{pmatrix} U_{11}^1 & U_{11}^2 \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},
$$

$$
\begin{pmatrix} L_{11}^1 & 0 \\ 0 & L_{11}^2 I \end{pmatrix} \begin{pmatrix} \alpha_k^{-1} D_{11}^1 & 0 \\ 0 & \alpha_k^{-1} I \end{pmatrix} \begin{pmatrix} M_{11}^1 & 0 \\ 0 & M_{11}^2 \alpha_k I \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
$$

The determinant of the $M$ submatrix, which is defined by all rows and columns of the minors $\alpha$ and the block $A_{11}^1$ is equals $\alpha_k$. Due to the Sylvester determinant identity (see [1]) we can write the equality:

$$
\det(A_{11}^1) = \alpha_k \alpha_k^{-1}.
$$
From the first equality and Sylvester determinant identity we get

\[ \alpha L_{11}^1 D_{11}^1 U_{11}^1 = A_{11}^1, \quad \alpha L_{11}^1 D_{11}^1 U_{11}^2 = A_{11}^2, \]

\[ \alpha k \alpha^{-2} (U_{11}^1)^{-1} (D_{11}^1)^{-1} (L_{11}^1)^{-1} = A_{11}^{1^*}, \quad \alpha k \alpha^{-2} (U_{11}^2)^{-1} (D_{11}^2)^{-1} = A_{11}^{1^*} L_{11}^1. \]

The matrix \(A_{11}^{1^*}\) is the adjoin matrix for \(A_{11}^1\). From the second equality we get

\[ \alpha W_{11}^1 D_{11}^1 U_{11}^1 = \alpha_r I, \quad \alpha W_{11}^1 = \alpha_k (U_{11}^1)^{-1} (D_{11}^1)^{-1}, \quad W_{11}^2 = -\alpha W_{11}^1 D_{11}^1 U_{11}^2. \]

The consequence is the expressions for the blocks \(W_{11}^1\) and \(W_{11}^2\):

\[ W_{11}^1 = \alpha^{1-k} A_{11}^1 L_{11}^1, \quad W_{11}^2 = -\alpha^{1-k} A_{11}^{1^*} A_{11}^2, \quad W_{11} = \begin{pmatrix} W_{11}^1 & W_{11}^2 \\ 0 & \alpha_k I \end{pmatrix}. \quad (26) \]

From the third equality we obtain the expressions \(\alpha L_{11}^1 D_{11}^1 M_{11}^1 = \alpha_k I, M_{11}^2 = -\alpha L_{11}^2 D_{11}^1 M_{11}^1\), so

\[ M_{11}^1 = \alpha^{1-k} U_{11}^1 A_{11}^{1^*}, \quad M_{11}^3 = -\alpha^{1-k} A_{11}^{1^*} A_{11}^1, \quad M_{11} = \begin{pmatrix} M_{11}^1 & 0 \\ -M_{11}^3 & \alpha_k I \end{pmatrix}. \quad (27) \]

Let the matrix \(A_{21}\) be divided into two blocks \(A_{21} = (A_{211}, A_{212})\) = \(\begin{pmatrix} A_{211} & A_{212} \\ A_{212}^* & A_{212}^* \end{pmatrix}\), then matrix \(A_{21}''\) can be written as follows:

\[ A_{21}'' = \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} \alpha^{1-k} A_{211}^{1^*} L_{21}^1 - \alpha^{1-k} A_{11}^{1^*} A_{212}^2 \\ 0 \alpha_k I \end{pmatrix} = \]

\[ (0, \frac{1}{\alpha} (\alpha_k A_{211}^2 - \frac{1}{\alpha^{k-1}} A_{2121}^{1^*} A_{11}^{1^*} A_{211}^2)). \]

According to Theorem 3, \(A_{21}''\) is a matrix of surrounding minors with respect to the block \(A_{11}^1\).

Let the matrix \(A_{12}\) be divided into two blocks \(A_{12} = \begin{pmatrix} A_{121} & A_{122} \\ A_{122}^* & A_{122}^* \end{pmatrix}\), then matrix \(A_{12}''\) can be written as follows:

\[ A_{12}'' = \frac{1}{\alpha} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} \alpha^{1-k} U_{11}^1 A_{121}^{1^*} & 0 \\ -\alpha^{1-k} A_{112}^{1^*} A_{122}^1 \alpha_k I \end{pmatrix} = \]

\[ \begin{pmatrix} \frac{1}{\alpha} (\alpha_k A_{122}^2 - \frac{1}{\alpha^{k-1}} A_{1221}^{1^*} A_{121}^2) \end{pmatrix}. \]

According to Theorem 3, \(A_{12}''\) is a matrix of surrounding minors with respect to the block \(A_{11}^1\).

Matrices \(W_{11}, D_{11}\) and \(M_{11}\) have such block shape:

\[ W_{11} D_{11}^1 M_{11} = \begin{pmatrix} W_{11} L_{11}^1 & 0 \\ 0 & \alpha_k I \end{pmatrix}, \quad \begin{pmatrix} D_{11}^1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} M_{11} & 0 \\ 0 & \alpha_k I \end{pmatrix} = \begin{pmatrix} W_{11} D_{11}^1 M_{11}^1 & 0 \\ 0 & 0 \end{pmatrix}. \]

\[ W_{11} D_{11}^1 M_{11} = \alpha^{2-k} A_{11}^{1^*} L_{11}^1 D_{11}^1 U_{11}^1 A_{11}^{1^*} = \alpha_k \alpha^{-k} A_{11}^{1^*}. \]
By definition (10) and expression (7) we have

\[ A'_{22} = \frac{1}{\alpha \alpha_k}(\alpha \alpha_k^2 A_{22} - A'_{21} D_{11}^+ A'_{12}) = \frac{1}{\alpha}(\alpha \alpha_k A_{22} - A_{21} W_{12} \tilde{D}_{12} J_1 \frac{\alpha}{\alpha_k} M_{11} A_{12}) = \]

\[ \alpha_k A_{22} - \frac{\alpha}{\alpha_k^2} A_{21}(W_{11} D_{11} M_{11}) A_{12} = \]

\[ \alpha_k A_{22} - \alpha^{1-k} A_{21} \begin{pmatrix} A_{11}' & 0 \\ 0 & 0 \end{pmatrix} A_{12} = \begin{pmatrix} A_{22}^{i*} A_{22}^{i} \\ A_{22}^{j} A_{22}^{j*} \end{pmatrix}. \]

According to Theorem 3, \( \alpha^{-1} A'_{22} \) is a matrix of surrounding minors with respect to the minor \( \alpha_k \).

Let us denote by \( A''_{21} \) and \( A''_{12} \) (\( i = 1..4 \)) the blocks of the matrices \( A'_{21} \) and \( A'_{12} \), correspondingly, and denote by \( A''_{22} \) and \( A''_{22} \) the upper and lower blocks. of the matrix \( \alpha^{-1} A'_{22} \). \( \alpha^{-1} A'_{22} = [A''_{22} A''_{22}^T] \). Similarly to expressions (26) and (27), we obtain the following expressions:

\[ \tilde{D}_{21} M_{21} = \begin{pmatrix} 0 & 0 \\ \bar{A}_{21}^m A_{21}^s & \alpha_l I \end{pmatrix}, \quad W_{12} \tilde{D}_{12} = \begin{pmatrix} 0 & -\alpha_k^{1-m} A_{12}^{m} A_{12}^{m*} \\ 0 & \alpha_m I \end{pmatrix}. \]

According to Theorem 3,

\[ A''_{22} = \alpha^{1-l} \tilde{D}_{21} M_{21} (\alpha^{-1} A'_{22}) = \alpha^{1} \begin{pmatrix} 0 & 0 \\ -\alpha_k^{1-l} A_{21}^m A_{21}^s & \alpha_l I \end{pmatrix} \begin{pmatrix} A''_{22}^{j} & A''_{22}^{j*} \\ A''_{22}^{i*} & A''_{22}^{i} \end{pmatrix} = \]

\[ \begin{pmatrix} \frac{1}{\alpha_k}(\alpha_l A_{22}^{j} - \alpha_m A_{21}^m A_{21}^s A_{22}^{j}) \\ \frac{1}{\alpha_m}(A_{22}^{i*} A_{22}^{i}) \end{pmatrix} \]

is a matrix of surrounding minors with respect to the minor \( \alpha_l \).

Finally, let us turn to the matrix (16):

\[ A''_{22} = \frac{1}{\alpha_k} \left( \begin{pmatrix} A''_{22}^{j} & A''_{22}^{j*} \\ A''_{22}^{i*} & A''_{22}^{i} \end{pmatrix} \begin{pmatrix} 0 & -\alpha_k^{1-m} A_{12}^{m} A_{12}^{m*} \\ 0 & \alpha_m I \end{pmatrix} \right) = \]

\[ \begin{pmatrix} 0 & \frac{1}{\alpha_m}(A_{22}^{i*} A_{22}^{i}) - \frac{1}{\alpha_k}(A_{22}^{i*} A_{22}^{i}) \end{pmatrix} \]

\[ \begin{pmatrix} 0 & \frac{1}{\alpha_m}(A_{22}^{i*} A_{22}^{i}) - \frac{1}{\alpha_k}(A_{22}^{i*} A_{22}^{i}) \end{pmatrix} \]

Here we introduced notations \( A_{22}^{j} \) and \( A_{22}^{i} \) for the left and right blocks of matrix \( A_{22}^{j} \), used definitions (11), (12) and Sylvester determinant identity. According to Theorem 3, \( A''_{22} \) is a matrix of surrounding minors with respect to the minor \( \alpha_s \).
Example

We give below an example of a LDU-decomposition of a matrix in the form of three identities $A = LDU$, $\hat{L}D\hat{M} = I$, $W\hat{D}\hat{U} = I$:

$$
\begin{pmatrix}
0 & 2 & 3 & 0 \\
0 & 0 & 0 & -3 \\
5 & 3 & 2 & 1 \\
0 & -1 & 0 & 0 \\
\end{pmatrix}
= 
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & -30 & 0 & 0 \\
3 & 0 & 10 & 0 \\
-1 & 0 & 0 & -45 \\
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
10 & 0 & -5 & 2 \\
0 & 2 & 3 & 0 \\
0 & 0 & -45 & 0 \\
0 & 0 & 0 & -30 \\
\end{pmatrix},
$$

$$
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & -30 & 0 & 0 \\
3 & 0 & 10 & 0 \\
-1 & 0 & 0 & -45 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 & -\frac{1}{90} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{1350} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{60750} \\
\end{pmatrix}
\begin{pmatrix}
135 & 0 & -90 & 0 \\
-45 & 0 & 0 & 0 \\
675 & 0 & 0 & 1350 \\
0 & 0 & -45 & 0 \\
\end{pmatrix} = I,
$$

$$
\begin{pmatrix}
0 & 90 & -90 & 0 \\
-45 & 0 & -2025 & 0 \\
0 & 0 & 1350 & 0 \\
0 & 0 & -450 & 0 \\
\end{pmatrix}
= 
\begin{pmatrix}
0 & -\frac{1}{90} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{1350} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{60750} \\
\end{pmatrix}
\begin{pmatrix}
10 & 0 & -5 & 2 \\
0 & 2 & 3 & 0 \\
0 & 0 & -45 & 0 \\
0 & 0 & 0 & -30 \\
\end{pmatrix} = I.
$$

The products of matrices $\hat{D}\hat{M}$ and $W\hat{D}$ can be reduced to a triangular form by inserting the product of the permutation matrix and the inverse permutation matrix between the factors:

$$
\hat{D}\hat{M} = 
\begin{pmatrix}
-\frac{1}{90} & 0 & 0 & 0 \\
0 & \frac{1}{1350} & 0 & 0 \\
0 & 0 & -\frac{1}{900} & 0 \\
0 & 0 & 0 & -\frac{1}{60750} \\
\end{pmatrix}
\begin{pmatrix}
-45 & 0 & 0 & 0 \\
0 & -450 & 0 & 0 \\
135 & 0 & -90 & 0 \\
675 & 0 & 0 & 1350 \\
\end{pmatrix},
$$

$$
W\hat{D} = 
\begin{pmatrix}
-90 & 0 & 675 & 90 \\
0 & -45 & -2025 & 0 \\
0 & 0 & 1350 & 0 \\
0 & 0 & 0 & -450 \\
\end{pmatrix}
= 
\begin{pmatrix}
-\frac{1}{90} & 0 & 0 & 0 \\
0 & -\frac{1}{90} & 0 & 0 \\
0 & 0 & -\frac{1}{900} & 0 \\
0 & 0 & 0 & -\frac{1}{60750} \\
\end{pmatrix}.
$$

Conclusion

A dichotomous $LDU$ factorization algorithm was proposed. It is applied to matrices in which the size is some power of 2. Such an algorithm is well parallelized and efficient for a supercomputer with distributed memory due to the presence of a coarse-grained block structure. If you want to find the decomposition of an arbitrary rectangular matrix, you must first arrange it arbitrarily inside a square matrix of a suitable size, perform the decomposition, and in the resulting factors, you need to discard the extra zero parts of the matrices.

As with all previous recursive algorithms, its complexity (up to a constant) is equal to the complexity ($n^{\omega}$) of matrix multiplication. Like other $LU$ algorithms, it gives a gain of $r/n$ times when applied to matrices of small rank $r$. But it is
also efficient for full rank sparse matrices. An example of this type of matrices that is important in applications is considered in the work [4].

Since the decomposition of the upper right block and the lower left block will be performed simultaneously, it is desirable that these two blocks have more nonzero elements than the other two blocks. If there is an almost-diagonal, tridiagonal or ribbon matrix, then it must first be multiplied by a permutation matrix so that the main diagonal is located in these two blocks.

It should be noted that this algorithm is a generalization of the LEU algorithm [3] to the case of a commutative domain. Therefore, it can be looked at as another proof of the LEU algorithm. We specifically emphasized the non-unity of the decomposition due to the fact that Eq. (15) can be applied in either of two versions. The complexity of the proof of the LEU algorithm was the reason that some authors proposed their own proofs, in which they stated the uniqueness of the decomposition and even came up with a new name for the matrix $E$. Matrix $E$ is called the “Bruhat Permutation Matrix”, since it first appeared in the works of Bruhat (see [9], [10]).

This LDU factorization algorithm is another step in creating a common library of block-recursive linear algebra algorithms. The first in this area were A.A. Karatsuba [11] and W. Strassen [12]. The understanding of the importance of recursive algorithms for supercomputer computing came only in recent decades and led to a program for creating decentralized dynamic control technology for the supercomputer’s computing process (see [16], [13], [17]). Other examples of recursive algorithms in the commutative domain, such as computing the inverse and adjoint matrices, the kernels of a linear operator, can be found in [14], [15], [13]. It is expected that new recursive algorithms should appear in the class of problems of orthogonal matrix factorization.

It is important to note that this algorithm does not accumulate errors and all computations take place in the commutative domain. The application of the Chinese remainder theorem and the transition to finite fields provides a way to reduce the total number of operations and very efficient parallelization on a supercomputer. Note that it is not necessary to search for original images for the elements of matrix $D$ by their images in finite fields, since they are easily found by the diagonal elements of matrices $L$ and $U$. Thus, the upper bound for the maximum minor of matrix $A$ can be used to estimate the largest element which appear at the end of the computational process.

The discussed algorithms are used in the cloud computer algebra Math Partner [18]. You can find this system at: mathpar.ukma.edu.ua.

References

1. Bosilca G. et al.: Flexible Development of Dense Linear Algebra Algorithms on Massively Parallel Architectures with DPLASMA. In: 2011 IEEE International Symposium on Parallel and Distributed Processing Workshops and PhD Forum, Shanghai, 1432-1441 (2011) doi: 10.1109/IPDPS.2011.299
2. Dongarra J.: With Extrim Scale Computing the Rules Have Changed, in Mathematical Software. In: ICMS 2016, 5th International Congress, Proceedings (G.-M. Greuel, T. Koch, P. Paule, A. Sommese, eds.), Springer, LNCS, 9725, 3-8 (2016)

3. Malaschonok G.I.: Fast Generalized Bruhat Decomposition. In: Ganzha, V.M., Mayr, E.W., Vorozhtsov, E.V. (eds.) 12th International Workshop on Computer Algebra in Scientific Computing (CASC 2010), LNCS 6244. Springer, Berlin Heidelberg, 194-202 (2010)

4. Pernet C., Storjohann A.: Time and space efficient generators for quasiseparable matrices. Journal of Symbolic Computation, 85, (2), 224-246 (2018)

5. Malashonok G.: Generalized Bruhat decomposition in commutative domains. In: Computer Algebra in Scientific Computing, CASC’2013, LNCS 8136, Springer, Heidelberg, 231-242 (2013)

6. Malashonok G., Scherbinin A.: Triangular Decomposition of Matrices in a Domain. In: Computer Algebra in Scientific Computing, LNCS 9301, Springer, Switzerland, 290-304 (2015)

7. Malashonok G.I.: Effective Matrix Methods in Commutative Domains. In: Formal Power Series and Algebraic Combinatorics. Springer, Berlin, 506-517 (2000)

8. Malashonok G.I.: Matrix computational methods in commutative rings. Monograph. Tambov, Tambov University Publishing House, 214 p. (2002)

9. Bruhat F.: Repr´ eSENTATIONS INDUCTES DES GROUPES DE LIE SEMI-SIMPLES R´ EELS, C.R. Acad. Sci. Paris, 238, 550-553 (1954)

10. Manthey W. and Helmke U.: Bruhat canonical form for linear systems. Linear Algebra and its Applications, 425, 261-282 (2007)

11. Karatsuba A., Ofman Yu.: Multiplication of multivalued numbers on automata. Reports of the Academy of Sciences of the USSR, 145, (2), 293-294 (1962)

12. Strassen V.: Gaussian Elimination is not optimal. Numerische Mathematik, 13, 354-356 (1969)

13. G. Malashonok G., Ilchenko E.: Recursive Matrix Algorithms in Commutative Domain for Cluster with Distributed Memory. In: 2018 Ivannikov Memorial Workshop (IVMEM), Yerevan, Armenia, 40-46 (2018) doi: 10.1109/IVMEM.2018.00015, arXiv:1903.04394

14. Akritas A.G., Malashonok G.I.: Computation of Adjoint Matrix. In: Computational Science, ICCS 2006, LNCS 3992, Springer, Berlin, 486-489, (2006)

15. Malashonok G.: On computation of kernel of operator acting in a module Tambov University Reports. Ser. Natural and Technical Sciences, 13, (1), 129-131 (2008)

16. Malashonok G.I.: Management of parallel computing process, Tambov University Reports. Ser. Natural and Technical Sciences, 14, (1) 269-274 (2009)

17. Malashonok G.I., Sidko A.A.: Parallel computer algebra: a new scheme for controlling the parallelization of matrix recursive algorithms. In: Fifth International Conference on High Performance Computing (HPCUA 2018) being held October 22-23, 2018 in Kyiv, Ukraine, 77-85 (2018)

18. Malashonok G.I.: MathPartner Computer Algebra, Programming and Computer Software, 43,(2), 112-118 (2017)