FINDING A SYSTEM OF ESSENTIAL 2-SUBORBIFOLDS

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Abstract. We make an analogy of Culler-Morgan-Shalen theory. Our main goal is to show that there exists a non-empty system of essential 2-suborbifolds respecting a given splitting of the orbifold fundamental group.

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1. Introduction

For a 3-dimensional manifold $N$, the essential (i.e., incompressible and not boundary parallel) 2-suborbifolds are corresponding to the decompositions of the fundamental group of $N$. If $N$ has an essential and separating 2-subsphere, $\pi_1(N)$ has the free product decomposition which respects its geometric decomposition, and conversely, if $\pi_1(N)$ has a free product decomposition, $N$ has an essential and separating 2-subsphere which realizes its algebraic decomposition. If $N$ has an essential 2-submanifold $S$ which is not a 2-sphere but separating, the above decomposition of $\pi_1(N)$ turns to be an amalgamated free product decomposition, and if $S$ is non-separating, it does to be an HNN extension decomposition. Moreover, [C-S] proved the theorem that if $N$ acts on a simplicial tree nontrivially, $N$ has a non-empty system of essential 2-submanifolds which respects that action.

A similar approach should be considered for 3-orbifolds. If a 3-orbifold $M$ has an essential 2-suborbifold, it is clear that the orbifold fundamental group $\pi_1(M)$ has an amalgamated free product decomposition or an HNN extension decomposition. In [T-Y 2] (respectively, [T-Y 3]) we found an essential non-spherical (respectively, spherical) 2-suborbifold realizing a given algebraic decomposition of the orbifold fundamental group of $M$.

In the present paper we show the following:

Theorem 1.1. Let $M$ be a good, compact, connected, orientable 3-orbifold without non-separating spherical 2-orbifolds. We assume that the fundamental group of each prime component of $M$ is infinite. Suppose that $\pi_1(M)$ has a nontrivial finite splitting. Then there exists a non-empty system of essential 2-suborbifolds $S_1, \ldots, S_n \subset M$ such that for each component $Q$ of $M - \bigcup_{i=1}^n S_i$, $\pi_1(Q)$ is contained in a vertex group.

Boileau, Maillot and Porti showed a related result in [BMP Proposition 7.16], where they treat with the fundamental group of the complement of the set of singular points of a 3-orbifold.

We summarize the contents of the present paper. In Section 2, 3 and 4, we review on the actions on a tree, 3-orbifolds, and OISIBO’s (orbifold identified spaces

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identified along ballic orbifolds) respectively. In Section 5, we prepare an orbifold composition, which is used in Main Theorem as the target space of a b-continuous map. In Section 6, we prove Main Theorem.

2. Preliminaries on the actions on a tree

Throughout this present paper any orbifold is assumed to be good, that is, it is covered by a manifold, and assumed to be compact, connected and orientable unless otherwise stated.

In [Se], some fixed point theorems about group actions on trees are proved. Here we use the following restricted forms of them.

Let $T$ be a simplicial tree, i.e., a connected and simply connected 1-complex, and $G$ a group simplicially acting on $T$.

For $g \in G$, $g$ is called to have an edge inversion if there exists an edge $E$ such that $g(E) = E$ and $g|_E$ is orientation reversing.

The action is called trivial if a vertex of $T$ is fixed by $\Gamma$.

Proposition 2.1. Let $g$ be an element of $G$ with finite order. If $g$ acts on $T$ without edge inversions, then there exists a vertex $p$ of $T$ such that $g(p) = p$.

Proposition 2.2. Let $p_1, p_2 \in T$ be fixed points of $g \in G$ and $\ell$ the unique simple path from $p_1$ to $p_2$. Then any vertex and edge on $\ell$ are fixed by $g$.

Let $n \geq 1$ be an integer. Put $G_n = \langle a_1, \ldots, a_n \mid a_1^{\alpha_1} = \cdots = a_n^{\alpha_n} = (a_i a_j)^{\beta_{i,j}} = 1, \ 1 \leq i < j \leq n \rangle$ where $\alpha_i, \beta_{i,j} \geq 2$ are integers.

Lemma 2.3. If $G_n$ acts on $T$ without edge inversions, then $T$ has a fixed vertex of $G_n$ action.

3. Preliminaries on orbifolds

Definition 3.1. Let $M = (\tilde{M}, p, |M|), \ N = (\tilde{N}, q, |N|)$ be orbifolds. A continuous map $f : M \to N$ is a pair $((|f|, f))$ of continuous maps $|f| : |M| \to |N|$ and $\tilde{f} : \tilde{M} \to \tilde{N}$ which satisfies the following:

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \\
|f| \downarrow & & \downarrow q \\
|M| & \xrightarrow{|f|} & |N|
\end{array}
\]

(i) $|f| \circ p = q \circ \tilde{f}$,
(ii) For each $\sigma \in \text{Aut}(\tilde{M}, p)$ there exists an element $\tau \in \text{Aut}(\tilde{N}, q)$ such that $\tilde{f} \circ \sigma = \tau \circ \tilde{f}$.

A continuous map $f : M \to N$ is b-continuous if there exists a point $x \in |M| - \Sigma M$ such that $|f|(x) \in |N| - \Sigma N$. It was called an orbi-map in [Ta1, T-Y1], etc. A b-continuous map induces a homomorphism between the fundamental groups and local fundamental groups of orbifolds, see [T-Y4, Lemma 3.13] and [Ta1], where the points $x$ and $|f|(x)$ in the above could be base points of the fundamental groups of $M$ and $N$, respectively. The notion of (b-)continuous maps between orbifolds
is naturally generalized for those between OISIBO’s and orbifold compositions in Sections 4 and 5.

A b-continuous map \( f : M \to N \) is an embedding if \( f(M) \) is a suborbifold of \( N \) and \( f : M \to f(M) \) is an isomorphism of orbifolds.

For other terminologies, see [10].

4. Preliminaries on OISIBO’s

**Definition 4.1.** Let \( I, J \) be countable sets, \( X_i (i \in I) \) n-orbifolds, and \( B_j (j \in J) \) ballic n-orbifolds. Let \( f_j : B_j \to X_{i(j, \epsilon)} \) be embeddings (as orbifolds) such that \( f_j^\epsilon(B_j) \subset \text{Int } X_{i(j, \epsilon)} \) and \( f_j^1(B_j) \) are mutually disjoint, where \( j \in J, i(j, \epsilon) \in I, \epsilon = 0, 1 \). Then we call \( X = \left( X_i, B_j, f_j^\epsilon \right) \) \( i \in I, j \in J, \epsilon = 0, 1 \) an n-orbifold identified space identified along ballic orbifolds (n-OISIBO). The maps \( f_j^1 \circ (f_j^\epsilon)^{-1} \) and their inverses are called the identifying maps of \( X \). Each \( X_i, B_j \) are called a particle of \( X \), and an identifying ballic orbifold, respectively. We define the equivalence relation \( \sim \) in \( \coprod_{j \in J} (|X_i| \cup |B_j|) \) to be generated by

\[
y \sim |f_j^\epsilon(y), \quad \epsilon = 0, 1, \quad y \in |B_j|, \quad j \in J.
\]

We call the identified space \( \coprod_{j \in J} (|X_i| \cup |B_j|)/ \sim \) the underlying space of \( X \), denoted by \( |X| \), and call the identified space \( \{(\cup_{i \in I} \Sigma X_i) \cup (\cup_{j \in J} \Sigma B_j)\}/ \sim \) the singular set of \( X \), denoted by \( \Sigma X \).

**Definition 4.2.** Let \( X = \left( X_i, B_j, f_j^\epsilon \right) \) \( i \in I, j \in J, \epsilon = 0, 1 \) and \( X' = \left( X'_i, B'_j, g_j^\epsilon \right) \) \( k \in K, \ell \in L, \epsilon = 0, 1 \) be OISIBO’s. We say that \( X \) and \( X' \) are isomorphic if there exists a set of maps \( \{\varphi_i, \psi_j\}_{i \in I, j \in J} \) and bijections \( \eta : I \to K, \xi : J \to L\) such that the following (i) and (ii) hold:

(i) For each \( i \in I \), \( \varphi_i \) is an isomorphism (of orbifolds) from \( X_i \) to \( X'_i \), and

(ii) For each \( j \in J \), \( \psi_j \) is an isomorphism (of orbifolds) from \( B_j \) to \( B'_j \).

The system \( h = \{(\varphi_i, \psi_j)_{i \in I, j \in J}, \eta, \xi\} \) is called an isomorphism from \( X \) to \( X' \).

**Definition 4.3.** Let \( X = \left( X_k, B_l, f_k^\ell \right) \) \( k \in K, \ell \in L, \epsilon = 0, 1 \) and \( X' = \left( X'_i, B'_j, f'_j^\ell \right) \) \( i \in I, j \in J, \epsilon = 0, 1 \) be OISIBO’s. We say that \( X \) is a covering of \( X' \) if there exists a set of maps \( \{\varphi_i, \psi_j\}_{i \in I, j \in J} \) and surjections \( \eta : I \to K, \xi : J \to L\) such that the following (i) and (ii) hold:

(i) Each \( \varphi_i \) is a covering map (of orbifolds) from \( X'_i \) to \( X_{\eta(i)} \), where \( \eta(i) \in K \), and each \( \psi_j \) is a covering map (of orbifolds) from \( B'_j \) to \( B_{\xi(j)} \), where \( \xi(j) \in L \).

(ii) For each \( j \in J \) and \( \epsilon = 0, 1 \), \( \varphi_{i(j, \epsilon)} \circ f_j^\epsilon = f_{\xi(j)}^\epsilon \circ \psi_j \).

Note that the continuous map \( |p| : |X'| \to |X| \) naturally induced by \( \{\varphi_i, \psi_j\}_{i \in I, j \in J} \) is surjective, and induces the usual covering map from \( |X'| - |p|^{-1}(\Sigma X) \) to \( |X| - \Sigma X \).

We call the system \( p = \langle |p|, \{\varphi_i, \psi_j\}_{i \in I, j \in J} \rangle \) a covering map from \( X' \) to \( X \).

**Definition 4.4.** Let \( \tilde{X}, X \) be OISIBO’s, and \( p : \tilde{X} \to X \) a covering. We call \( p \) a universal covering if for any covering \( p' : X' \to X \), there exists a covering \( q : \tilde{X} \to X' \) such that \( p = p' \circ q \). As the usual covering theory, for any OISIBO \( X \), there exists a unique universal covering \( p : \tilde{X} \to X \).
Definition 4.5. Let $X', X$ be OISIBO’s, and $p : X' \to X$ a covering. We define the deck transformation group $\text{Aut}(X', p)$ of $p$ by

$$(4.2) \quad \text{Aut}(X', p) = \{ h : X' \to X' \mid h \text{ is an isomorphism such that } p \circ h = p \}.$$ 

We sometimes denote an OISIBO $X$ by $(\tilde{X}, p, |X|)$, where $p : \tilde{X} \to X$ is the universal covering and $|X|$ is the underlying space of $X$. Any orbifold is considered as a special case of an OISIBO.

Definition 4.6. Let $X = (\tilde{X}, p, |X|)$, $Y = (\tilde{Y}, q, |Y|)$ be OISIBO’s. A continuous map $f : X \to Y$ is a pair $(\tilde{f}, \tilde{f})$ of continuous maps $|f| : |X| \to |Y|$ and $\tilde{f} : \tilde{X} \to \tilde{Y}$ which satisfies the same property as (i) and (ii) in Definition 3.1. A continuous map $f : X \to Y$ is $b$-continuous if there exists a point $x \in |X| - \Sigma X$ such that $|f|(x) \in |Y| - \Sigma Y$.

We define a homotopy of OISIBO’s by using of continuous maps of OISIBO’s as the usual homotopy. If the continuous maps at 0 and 1 levels of the homotopy are $b$-continuous, this homotopy is called a $b$-homotopy. See [1-Y-2].

We define a path in an OISIBO $X$ by using of a $b$-continuous map $\alpha = (|\alpha|, \tilde{\alpha}) : [0,1] \to X$ with $|\alpha|(0) \in |X| - \Sigma X$. If a path $\alpha$ in $X$ satisfies that $|\alpha|(0) = |\alpha|(1)$, it is called a loop in $X$.

By using of loops in an OISIBO $X$, we define the fundamental group of $X$ as the usual theory. A $b$-continuous map $f : X \to Y$ between OISIBO’s $X$ and $Y$ induces a homomorphism between the fundamental groups and local fundamental groups of $X$ and $Y$, where the points $x \in |X| - \Sigma X$ and $|f|(x) \in |Y| - \Sigma Y$ in the definition of $b$-continuous map are the base points of the fundamental groups of $X$ and $Y$, respectively.

As usual covering theory, various similar results holds such as the following:

Proposition 4.7. Let $X$ be an OISIBO and let $x, y$ be any two points of $|X| - \Sigma X$. Then the fundamental groups $\pi_1(X, x)$ and $\pi_1(X, y)$ are isomorphic.

We often denote $\pi_1(X, x_0)$ by $\pi_1(X)$ dropping a base point if not necessary.

Proposition 4.8. Let $X$ be an OISIBO and $p : \tilde{X} \to X$ the universal covering of $X$. Then the fundamental group $\pi_1(X)$ is isomorphic to the deck transformation group $\text{Aut}(\tilde{X}, p)$.

Proposition 4.9. Let $X$ be an OISIBO. For each subgroup $H$ of $\pi_1(X)$, there exists a covering $p : \tilde{X} \to X$ such that the OISIBO $\tilde{X}$ has a fundamental group which is isomorphic to $H$.

5. Orbifold compositions

Definition 5.1. Let $I$, $J$ be countable sets, $X_i \ (i \in I)$ and $Y_j \ (j \in J)$ be $n$-OISIBO’s. Let $f_j^x : Y_j \times \varepsilon \to X_{i(j, \varepsilon)}$ be $b$-continuous-maps, $f_j^x = (|f_j^x|, \tilde{f}_j^x)$, such that $(f_j^x)_\varepsilon$ are monic, where $j \in J$, $i(j, \varepsilon) \in I$, $\varepsilon = 0, 1$. Then we call $X = (X_i, Y_j \times [0,1], f_j^x)_{i \in I, j \in J, \varepsilon = 0,1}$ an $n$-dimensional orbifold composition of type III. The maps $f_j^x$ are called the attaching maps of $X$, which may have intersections and self-intersections. Each $X_i$, $Y_j \times [0,1]$ is called a component of $X$. The equivalence relation $\sim$ in $\bigsqcup_{i \in I, j \in J} (|X_i| \cup (|Y_j| \times [0,1]))$ is defined to be generated by

$$(5.1) \quad (y, \varepsilon) \sim |f_j^x|(y, \varepsilon), \quad \varepsilon = 0, 1, \quad y \in |Y_j|, \quad j \in J.$$
We call the identified space \([\bigcup_{i,j \in I} (|X_i| \cup (|Y_j| \times [0,1]))]/ \sim\) the underlying space of \(X\), denoted by \(|X|\), and call the identified space \((\cup_{i \in I} \Sigma X_i) \cup (\cup_{j \in J} \Sigma (Y_j \times [0,1]))\)/ \sim the singular set of \(X\), denoted by \(\Sigma X\).

In the definition of an orbifold composition, each \((f_i^\epsilon)_\epsilon\) is monic, so that we can obtain the unique lift of any path \(\ell\) in \(X\) such that \(\ell[0,1] \cap \Sigma X = \emptyset\).

We define a covering of an orbifold composition as similar as that of an OISIBO. We sometimes denote an orbifold composition \(X\) by \((\tilde{X}, p, |X|)\), where \(p : \tilde{X} \to X\) is the universal covering and \(|X|\) is the underlying space of \(X\). Any orbifold and any OISIBO are considered as special cases of an orbifold composition.

**Definition 5.2.** Let \(X = (\tilde{X}, p, |X|), Y = (\tilde{Y}, q, |Y|)\) be orbifold compositions. A continuous map \(f : X \to Y\) is a pair \((|f|, \tilde{f})\) of continuous maps \(|f| : |X| \to |Y|\) and \(\tilde{f} : \tilde{X} \to \tilde{Y}\) which satisfies the same property as (i) and (ii) in Definition 5.1.

A continuous map \(f : X \to Y\) is \(b\)-continuous if there exists a point \(x \in |X| - \Sigma X\) such that \(|f|(x) \in |Y| - \Sigma Y\).

By using of loops in an orbifold composition \(X\), we define the fundamental group of \(X\) as the usual theory. A \(b\)-continuous map \(f : X \to Y\) between orbifold compositions \(X\) and \(Y\) induces a homomorphism between the fundamental groups and local fundamental groups of \(X\) and \(Y\), where the points \(x \in |X| - \Sigma X\) and \(|f|(x) \in |Y| - \Sigma Y\) in the definition of \(b\)-continuous map are the base points of the fundamental groups of \(X\) and \(Y\), respectively.

**Definition 5.3.** Let \(X\) be an orbifold composition. Define \(O_i(X), i = 1, 2, 3\) as follows:

- \(O_1(X) = \{f : S^1 \to X, \text{ a } b\)-continuous map \(|f| \text{ is of finite order } (\neq 1) \text{ in } \pi_1(X)\}\),
- \(O_2(X) = \{f : S \to X, \text{ a } b\)-continuous map \(|S| \text{ is a spherical } 2\)-orbifold\},
- \(O_3(X) = \{f : \mathcal{D}B \to X, \text{ a } b\)-continuous map \(|\mathcal{D}B| \text{ is the double of a ballic } 3\)-orbifold } B\}.

A \(b\)-continuous map \(f : S^1 \to X \in O_1(X)\) is **trivial** if there exists a \(b\)-continuous map \(g\) from a discal \(2\)-orbifold \(D\) to \(X\) such that \(g|D = f\) and the index of \(D\) equals to the order of \(|f|\). \(O_1(X)\) is trivial if every element of \(O_1(X)\) is trivial. We call \(f : S \to X \in O_2(X)\) trivial if there exists a \(b\)-continuous map \(g : c* S \to X\) such that \(g|S = f\), where \(c* S\) is the cone on \(S\). \(O_2(X)\) is trivial if every element of \(O_2(X)\) is trivial. We define the triviality of \(O_3(X)\) similarly.

Note that if \(O_1(X)\) is trivial, then any covering \(\tilde{X}\) of \(X\) inherits the triviality.

**Remark 5.4.** Let \(X\) be an orbifold composition, and \(\tilde{X}\) the universal cover of \(X\). If \(O_2(X)\) is trivial, then \(\pi_2(\tilde{X}) = 0\).

**Proposition 5.5.** Let \(M\) be a compact \(3\)-orbifold, and \(X\) an orbifold composition. If \(O_1(X)\) and \(O_2(X)\) are trivial, then for any homomorphism \(\varphi : \pi_1(M) \to \pi_1(X)\), there exists a \(b\)-continuous map \(f : M \to X\) such that \(f_* = \varphi\).

**Proof.** Let \(M_0 = |M| - \text{ Int } U(\Sigma M)\), where \(U(\Sigma M)\) is the small regular neighborhood of \(\Sigma M\). We can construct a \(b\)-continuous map from the 1-skeleton of \(M_0\) to \(X\) associated with \(\varphi\). Since \(O_1(X)\) and \(O_2(X)\) are trivial, this \(b\)-continuous map is extendable to \(M_0\) and furthermore to \(M\), that is, we have obtained the desired \(b\)-continuous map. \(\square\)
Proposition 5.6. Let $X$ be an orbifold composition, and $f : S^1 \to X$ a $b$-continuous map. If $\text{Fix}([f]_A) \neq \phi$, then $f$ is extendable to a $b$-continuous map from a discal 2-orbifold $D$ to $X$ where $D = D^2(n)$, and $n$ is the order of $[f]_A$.

Proof. Let $q : D^2 \to D$ be the universal covering. Choose a point $x \in \text{Fix}([f]_A)$. We can construct the structure map of the desired $b$-continuous map by mapping the cone point of $D^2$ to $x$ and performing the skeletonwise and equivariant extension. □

Let $S$ be a spherical 2-orbifold and let $q : (\tilde{S}, \tilde{x}_0) \to (S, x_0)$, $x_0 \notin \Sigma S$, be the universal covering. Let $\tau$ be an element of $\pi_1(S, x_0)$, $\tilde{x}_\tau$ one of the two points of $\text{Fix}(\tau_A)$, and $x_\tau = q(\tilde{x}_\tau)$. By the symbol $\mu(x_\tau)$, we shall mean the local normal loop around $x_\tau$. Let $\ell$ be a path in $|S| - \Sigma S$ from $\mu(x_\tau)(0)$ to $x_0$ such that $\tau = ([\ell^{-1} \cdot \mu(x_\tau) \cdot \ell])^k$, $k \in \mathbb{Z}$.

Proposition 5.7. Let $S$ be a spherical 2-orbifold, $X$ an orbifold composition, and $f : S \to X$ a $b$-continuous map. Suppose that there exists a point $\tilde{d} \in \text{Fix}(f_\ast \pi_1(S))_A$, and for any $\tau \in \pi_1(S, x_0)$ there exists an interval $\ell_\tau$ including $\tilde{d}$ and $f(\tilde{x}_\tau)$ which is fixed by $\sigma_A$, where $\sigma = f_\ast(\tau)$. If $\pi_2$ of the universal cover $\tilde{X}$ of $X$ is 0, then $f$ is extendable to a $b$-continuous map from the cone on $S$ to $X$.

Proof. The proof is similar to that of [T-Y 3 Proposition 5.6]. □

Proposition 5.8. Let $DB$ be the double of a ballic 3-orbifold $B$, $X$ an orbifold composition, and $f : DB \to X$ a $b$-continuous map. Suppose that $\text{Fix}(f_\ast \pi_1(DB))_A$ is connected, and for $\tau \in \pi_1(DB, x_0)$, $\pi_1(\text{Fix}(f_\ast(\tau)))_A = 1$ and there exists an interval $\ell_\tau$ including $\tilde{d}$ and $f(\tilde{x}_\tau)$ which is fixed by $\sigma_A$, where $\sigma = f_\ast(\tau)$. If $\pi_2$ and $\pi_3$ of the universal cover $\tilde{X}$ of $X$ is 0, then $f$ is extendable to a $b$-continuous map from the cone on $DB$ to $X$.

Proof. The proof is similar to that of [T-Y 3 Proposition 5.7]. □

Proposition 5.9. [T-Y 3 Proposition 5.8] Let $X$ be a 3-OISIBO whose particles are irreducible. Let $p : \tilde{X} \to X$ be the universal covering and $\sigma \in \text{Aut}(\tilde{X}, p)$ be any nontrivial element of finite order. Suppose that each particle of $\tilde{X}$ is non-compact. Then the following holds:

(i) $\text{Fix}(\sigma) \neq \phi$ and is homeomorphic to a tree.
(ii) $O_i(X)$ is trivial.

Since we assume that any orbifold is orientable, the restriction of $\sigma$ to each particle is orientation preserving, and each identifying ballic orbifold is orientable.

Proposition 5.10. [T-Y 3 Proposition 5.9] Let $X$ be a 3-OISIBO, each particle of which is irreducible, and $p : \tilde{X} \to X$ the universal covering. Let $G$ be any subgroup of $\text{Aut}(\tilde{X}, p)$, which is isomorphic to the orbifold fundamental group of a spherical 2-orbifold $S$. Suppose that each particle of $\tilde{X}$ is non-compact. Then the following holds:

(i) $\text{Fix}(G)$ is either a point or a tree.
(ii) $\pi_2(\tilde{X}) = \pi_3(\tilde{X}) = 0$.
(iii) $O_i(X)$'s are trivial, $i = 1, 2, 3$. 

Proposition 5.11. Let $X = (\tilde{X}, p, |X|)$, $Y = (\tilde{Y}, q, |Y|)$ be orbifold compositions, and $f = ([f], \tilde{f}) : X \to Y$ a $b$-continuous map. Then for $[\alpha] \in \pi_1(X, x)$,

$$f \circ [\alpha] = (f_*([\alpha]))_A \circ \tilde{f}.$$ \hfill (5.2)

Proof. Let $\tilde{x} \in p^{-1}(x)$ be the base point of $\tilde{X}$. Note that $[\alpha]_A$ is characterized as the element of $\text{Aut}(\tilde{X}, p)$ which transforms $\tilde{a}(0) = \tilde{x}$ to $\tilde{a}(1)$, and $(f_*([\alpha]))_A = [f \circ \alpha]_A$ is characterized as the element of $\text{Aut}(\tilde{Y}, q)$ which transforms $\tilde{f}(\tilde{a}(0))$ to $\tilde{f}(\tilde{a}(1))$. By the definition of $b$-continuous map, there exists an element $\tau \in \text{Aut}(\tilde{Y}, q)$ such that $\tilde{f} \circ [\alpha] = \tau \circ \tilde{f}$. On the other hand, $\tau(f(\tilde{a}(0))) = (\tau \circ \tilde{f})(\tilde{a}(0)) = (f \circ [\alpha])(\tilde{a}(0)) = \tilde{f}(\tilde{a}(1))$. Hence $\tau = (f_*([\alpha]))_A$. \hfill $\square$

Proposition 5.12. Let $X = (X_i, Y_j \times [0, 1], f_\xi)$ be an orbifold composition, where each particle of each $X_i$ and $Y_j$ is an orientable and irreducible 2-orbifold whose universal covering is noncompact. Then $O_i(X)$’s are trivial, $i = 1, 2, 3$.

Proof. Let $p : \tilde{X} \to X$ be the universal covering. From the uniqueness of the universal covering, we may assume that $\tilde{X}$ is the orbifold composition constructed in a similar method described in [17, Y2].

Claim: Let $G$ be any subgroup of $\text{Aut}(\tilde{X}, p)$, which is isomorphic to the fundamental group of a spherical 2-orbifold. Consider the associated 1-complex $C(\tilde{X})$ of $\tilde{X}$. Then, there exists a vertex OISIBO $\tilde{Z}$ of $\tilde{X}$ with respect to $C(\tilde{X})$ such that $G(\tilde{Z}) = \tilde{Z}$.

Indeed, $G$ is finite and acts on the tree $C(\tilde{X})$ without edge inversions. By Lemma 2.8 we have the claim.

Take any element $f \in O_1(X)$. By the claim, there exists an OISIBO $\tilde{Z}$ of $\tilde{X}$ such that $[f]_A(\tilde{Z}) = \tilde{Z}$. Then by Proposition 5.9 $\text{Fix}([f]_A) \neq \emptyset$ in $\tilde{Z}$. Thus $\text{Fix}(f) \neq \emptyset$ in $\tilde{X}$. By using of Proposition 5.6 $f$ is extendable to a $b$-continuous map from $D^2(n)$ to $X$.

Take any element $f \in O_2(X)$, $f : S \to X$. Let $q : \tilde{S} \to S$ be the universal covering and $\tilde{f} : \tilde{S} \to \tilde{X}$ the structure map of $f$. Let $B = c \ast S$ be the cone on $S$ and $c$ the cone point of $B$. Let $\tilde{q} : \tilde{B} = \tilde{c} \ast \tilde{S} \to \tilde{B}$ be the universal covering, $\tilde{c} = \tilde{q}^{-1}(c)$ and $\tilde{q}(\tilde{t}x + (1 - t)\tilde{c}) = tq(\tilde{x}) + (1 - t)c$, $\tilde{x} \in \tilde{S}$. Note that $f_*\pi_1(S)$ is isomorphic to a spherical 2-orbifold group. By the claim, there exists a vertex OISIBO $\tilde{Z}$ of $\tilde{X}$ such that $f_*\pi_1(S)(\tilde{Z}) = \tilde{Z}$. By Proposition 5.10 $\text{Fix}(f_*\pi_1(S)) \neq \emptyset$. Thus there exists a point $\tilde{d} \in \tilde{Z}$ such that $(f_*\pi_1(S))_A\tilde{d} = \tilde{d}$.

Choose any $\tau \in \pi_1(S)$. We put $\sigma = f_*(\tau)$. Since $\sigma_A \in (f_*\pi_1(S))_A$, $\sigma_A(\tilde{d}) = \tilde{d}$. Moreover, by the fact $\tau_A(\tilde{x}_\tau) = \tilde{x}_\tau$ and Proposition 5.11 $\sigma_A(\tilde{f}(\tilde{x}_\tau)) = f_*(\tau)_A(\tilde{f}(\tilde{x}_\tau)) = \tilde{f} \circ \tau_A(\tilde{x}_\tau) = \tilde{f}(\tilde{x}_\tau)$.

Let $\tilde{Z}_1$ be the OISIBO in which $\tilde{f}(\tilde{x}_\tau)$ is included and let $\tilde{Z}_k$ be the OISIBO in which $\tilde{d}$ is included. Since $\sigma_A$ acts on the tree $C(\tilde{X})$, $\tilde{Z}_1$ and $\tilde{Z}_k$ are invariant by $\sigma_A$. In addition, since $\sigma_A$ is of finite order, we can apply Proposition 2.2 and get that any vertex OISIBO $\tilde{Z}_i$ and any edge OISIBO $\tilde{Z}_j$ between $\tilde{Z}_1$ and $\tilde{Z}_k$ are invariant by $\sigma$. By Proposition 5.9 for each $\tilde{Z}_i$ and each $\tilde{Z}_j$, $\text{Fix} \sigma_A(\tilde{Z}_i)$ is a tree and $\text{Fix} (\sigma_A|\tilde{Z}_j)$ is (a tree × $[0, 1]$).

Note that since the structure map of each attaching map from $\tilde{Z}_{i_0}$ to $\tilde{Z}_{i_1}$ and the restriction of $\sigma_A$ to $\tilde{Z}_{i_0}$ and $\tilde{Z}_{i_1}$ commute, any point of $\text{Fix} (\sigma_A|\tilde{Z}_{i_0})$ is mapped to a point of $\text{Fix} (\sigma_A|\tilde{Z}_{i_1})$. Hence $\text{Fix} \sigma_A$ in $\tilde{X}$ is connected. Thus we can find an interval which is fixed by $\sigma_A$, and connecting $\tilde{f}(\tilde{x}_\tau)$ and $\tilde{d}$. 
We will show that \( \pi_2(\tilde{X}) = 0 \). Take any continuous map \( \varphi : S^2 \to \tilde{X} \). Since \( \varphi(S^2) \) is compact, there exists a connected and compact subset \( P \) of \( \tilde{X} \) which contains a finite number of OISIBO’s \( \tilde{Z}_i \) of \( \tilde{X} \) such that \( \varphi(S^2) \subset P \). By Proposition \[.10\](ii), for each \( \tilde{Z}_i \), \( \pi_2(\tilde{Z}_i) = H_2(\tilde{Z}_i) = 0 \). Dividing \( P = P_1 \cup P_2 \) such that \( P_1 \) consists of \( k \) vertex OISIBO’s, \( (k - 1) \) edge OISIBO’s, and \( \tilde{Y} \times [0, \frac{1}{2} + \varepsilon] \) and \( P_2 \) consists of a vertex OISIBO and \( \tilde{Y} \times [\frac{1}{2} - \varepsilon, 1] \), \( P_1 \cap P_2 = \tilde{Y} \times [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon] \) where \( \tilde{Y} \times [0, 1] \) is an edge OISIBO of \( \tilde{X} \), we can show \( H_2(P) = 0 \) by the induction on \( k \) and Mayer-Vietoris exact sequences. Thus \( \pi_2(P) = 0 \), which gives the fact \( [\varphi] = 0 \) in \( \pi_2(\tilde{X}) \), so is in \( \pi_2(\tilde{X}) \). Then the triviality of \( f \) follows from Proposition \[5.7\].

The triviality of \( O_3(X) \) is derived from showing \( \pi_3(\tilde{X}) = 0 \) and Proposition \[5.8\] in the similar manner.

Let \( M \) be a 3-orbifold, and \( X \) an orbifold composition. We say that two b-continuous maps \( f, g : M \to X \) are \( C \)-equivalent if there is a sequence of b-continuous maps \( f = f_0, f_1, \ldots, f_n = g \) from \( M \) to \( X \) with either \( f_i \) is b-homotopic to \( f_{i-1} \) or \( f_i \) agrees with \( f_{i-1} \) on \( M - B \) for a certain ballic 3-orbifold \( B \subset M \) with \( B \cap \partial M \) a discal orbifold or \( |B \cap |\partial M| = 0 \). Note that C-equivalent b-continuous maps induce the same homomorphisms \( \pi_1(M) \to \pi_1(X) \) up to choices of base points and inner automorphisms.

**Remark 5.13.** Let \( f, g \) be \( C \)-equivalent maps from a 3-orbifold \( M \) to an orbifold composition \( X \). If \( O_i(X) \)'s are trivial, \( i = 2, 3 \), then \( f \) and \( g \) are b-homotopic.

**Lemma 5.14.** Let \( M \) be an orbifold, and \( X \) an orbifold-composition. Let \( p : M \to \tilde{M} \) and \( q : \tilde{X} \to X \) be the universal coverings. Suppose \( \dim M = n \) and \( \pi_{n-1}(|\tilde{X}|) = 0 \). Let \( \tilde{g} : |\tilde{M}| \to |\tilde{X}| \) be a continuous map which satisfies the condition that there exists a homomorphism \( \varphi : \text{Aut}(\tilde{M}, p) \to \text{Aut}(\tilde{X}, q) \) such that for each \( \sigma \in \text{Aut}(\tilde{M}, p) \), \( \tilde{g} \circ \sigma = \varphi(\sigma) \circ \tilde{g} \). Then there exists a continuous map \( \tilde{f} : |\tilde{M}| \to |\tilde{X}| \) which satisfies the following:

(1) There exists a point \( \tilde{x} \in |\tilde{M}| - p^{-1}(\Sigma M) \) such that \( \tilde{f}(\tilde{x}) \in |\tilde{X}| - q^{-1}(\Sigma X) \).

(2) There exists an \( n \)-ball \( B^n \subset |\tilde{M}| - p^{-1}(\Sigma M) \) such that \( B^n \cap \sigma(B^n) = \emptyset \) for each \( \sigma \in \text{Aut}(\tilde{M}, p) \), \( \sigma \neq \text{id} \), and

\[
\tilde{f} \left( |\tilde{M}| - \bigcup_{\sigma \in \text{Aut}(\tilde{M}, p)} \sigma(B^n) \right) = \tilde{g} \left( |\tilde{M}| - \bigcup_{\sigma \in \text{Aut}(\tilde{M}, p)} \sigma(B^n) \right).
\]

(3) For each \( \sigma \in \text{Aut}(\tilde{M}, p) \), \( \tilde{f} \circ \sigma = \varphi(\sigma) \circ \tilde{f} \).

**Proof.** The proof is similar to that of \[ [5.14]\] Lemma 5.4. \( \square \)

**Theorem 5.15** (Transversality Theorem). Let \( M \) be a good, compact, connected, orientable 3-orbifold, and \( X \) a 3-orbifold composition. Suppose that there exists an edge OISIBO \( X \times [0, 1] \) of \( X \), the core \( Y \) satisfies that \( O_2(X - Y) \) and \( O_2(Y) \) are trivial. Then, for any b-continuous map \( f : M \to X \), there exists a b-continuous map \( g = ([g], \tilde{g}) : M \to \tilde{X} \) which satisfies the following:

(i) \( g \) is \( C \)-equivalent to \( f \).

(ii) Each component of \( g^{-1}(Y) \) is a compact, properly embedded, 2-sided, incompressible 2-suborbifold in \( M \).

(iii) For properly chosen product neighborhoods \( Y \times [-1, 1] \) of \( Y = Y \times 0 \) in \( X \), and \( g^{-1}(Y) \times [-1, 1] \) of \( g^{-1}(Y) = g^{-1}(Y) \times 0 \) in \( M \), \( |g| \) maps each
fiber $x \times \{ [-1,1] \}$ homeomorphically to the fiber $|g|(x) \times \{ [-1,1] \}$ for each $x \in |g^{-1}(Y)|$.

Proof. The proof is similar to that of [Ta1] Theorem 5.5. \qed

Corollary 5.16. In Theorem 5.14 if $O_i(X)$ are trivial, $i = 2, 3$, then $f$ and $g$ are b-homotopic by Remark 5.13

6. GROUP REPRESENTATIONS AND SPLITTINGS OF GROUPS

For the contents of the present section we refer to the original paper [CS].

An isomorphism from a group $\Pi$ to $\pi_1(G, \mathcal{G})$ is called a splitting of $\Pi$ where $\pi_1(G, \mathcal{G})$ is the fundamental group of a graph of groups $(G, \mathcal{G})$. A splitting is trivial if there exists a vertex group which is isomorphic to the whole $\pi_1(G, \mathcal{G})$.

Let $\Pi$ be a finitely generated group. Take a system of generators $g_1, \ldots, g_n$ for $\Pi$. We define a set $R(\Pi) = \{ (\rho(g_1), \ldots, \rho(g_n)) \mid a representation \rho : \Pi \to SL_2(\mathbb{C}) \}$. The points of $R(\Pi)$ correspond to the representations of $\Pi$ in $SL_2(\mathbb{C})$ bijectively, and we often identify such 1-1 corresponding points. For each $g \in \Pi$, we may define a map $\tau_g : R(\Pi) \to \mathbb{C}$ by $\tau_g(\rho) = \text{tr}(\rho(g))$. Let $T$ be the ring generated by all the functions $\tau_g$, $g \in \Pi$. It is finitely generated ([CS Proposition 1.4.1]). Thus we can take and fix $\gamma_1, \ldots, \gamma_m \in \Pi$ such that $\tau_{\gamma_1}, \ldots, \tau_{\gamma_m}$ generate $T$. With those elements we define a map $t : R(\Pi) \to \mathbb{C}^m$ by $t(\rho) = (\tau_{\gamma_1}(\rho), \ldots, \tau_{\gamma_m}(\rho))$, and set $X(\Pi) = t(R(\Pi))$. For each $g \in \Pi$, there exists a function $I_g : X(\Pi) \to \mathbb{C}$ which maps $t(\rho)$ to $\text{tr}(\rho(g))$. It is a regular function.

Theorem 6.1. ([CS Theorem 2.1.1]) If the group $\Pi$ acts without edge inversions on the tree $T$, then $\Pi$ is isomorphic to $\pi_1(T/\Pi, \mathcal{G})$ where $(T/\Pi, \mathcal{G})$ is defined in [CS] pp.123-124.

Theorem 6.2. ([CS Theorem 2.2.1]) Let $C$ be an affine curve contained in $X(\Pi)$ and $\mathcal{C}$ be a non-singular projective curve uniquely determined by $C$. To each ideal point $\hat{x}$ of $\mathcal{C}$ one can associate a splitting of $\Pi$ with the property that an element $g$ of $\Pi$ is contained in a vertex group if and only if $I_g$ does not have a pole at $\hat{x}$. Thus, in particular, the splitting is non-trivial.

7. MAIN THEOREM

Theorem 7.1. Let $M$ be a good, compact, connected, orientable 3-orbifold without non-separating spherical 2-orbifolds. We assume that the fundamental group of each prime component of $M$ is infinite. Suppose that $\pi_1(M)$ has a nontrivial finite splitting. Then there exists a non-empty system of essential 2-suborbifolds $S_1, \ldots, S_n \subset M$ such that for each component $Q$ of $M - \bigcup_{i=1}^n S_i$, $\pi_1(Q)$ is contained in a vertex group.

Proof. From the hypotheses, $\pi_1(M)$ is isomorphic to $\pi_1(G, \mathcal{G})$, the fundamental group of a graph of groups $(G, \mathcal{G})$. Along the splitting of $\pi_1(M)$, we construct an orbifold composition (of general type) $X$ as follows:

Step a. We may assume that $M$ has no spherical boundary components. Take a base point $y_0 \in |M| - \Sigma M$ of $M$.

Step b. By the hypotheses, we can take a prime decomposition of $M$, each component of which has an infinite fundamental group. Gluing back the prime
components by b-continuous maps along the ballic 3-orbifolds attached in capping off, we obtain an OISIBO $W$ with $\pi_1(W) \cong \pi_1(M)$, each particle of which is an irreducible 3-orbifold.

**Step c**  Let $G_i$, $i \in I$ and $H_j$, $j \in J$ be the vertex groups and the edge groups of $(G, \mathcal{G})$, respectively. Take the covering OISIBO $X_i$ of $W$ associated with each vertex group $G_i$ of $(G, \mathcal{G})$, and the covering OISIBO $Y_j$ of $W$ associated with each edge group $H_j$ of $(G, \mathcal{G})$. If an edge $e_j$ of $G$ has vertices $v_{j_0}, v_{j_1}$, then $H_j < G_{j_t}, t = 0, 1$. Thus there exist covering maps $p^j_t : Y_j \to X_{j_t}$ which induce monomorphisms $(p^j_t)_* : H_j \to G_{j_t}, t = 0, 1$.

**Step d**  The system $X = (X_i, Y_j \times [0, 1], p^j_t)_{t=0,1,i\in I,j\in J}$ is a desired orbifold composition with $\pi_1(X) \cong \pi_1(T/\pi_1(M), \mathcal{G}) \cong \pi_1(M)$. Take a base point $x_0 \in [Y_1 \times \{t\} - \Sigma(Y_1 \times \{t\})$ of $X$.

By Proposition 5.12 $O_i(X)$ are trivial, $i = 1, 2, 3$. By Proposition 5.7 we can construct a b-continuous map $f : M \to X$ which induces an isomorphism $\varphi : \pi_1(M, y_0) \to \pi_1(X, x_0)$.

Note that the set $J$ is finite. For all $j \in J$, take $f^{-1}(Y_j \times \{0\})$. By applying Proposition 5.12 for $X - Y_j$, we obtain the fact that $O_2(X - Y_j)$ is trivial. And applying Proposition 5.10 for $Y_j$, we obtain the fact that $O_2(Y_j)$ is trivial. We have already shown that $O_i(X)$’s are trivial, $i = 2, 3$. With modifications through b-homotopies if necessary, by Theorems 5.13 and Corollary 5.16 each component of $f^{-1}(\bigcup_{j \in J} Y_j \times \{0\})$ is a compact, properly embedded, 2-sided, incompressible 2-suborbifold in $M$. If one of such components is boundary parallel, we can reduce the number of components by modifications through b-homotopies. Note that we can make those modifications fixing on some neighborhood of each component of $\bigcup_{j \in J} f^{-1}(Y_j)$, which is already incompressible. After those modifications, we obtain a system of essential 2-suborbifolds $S_1, \ldots, S_n$ as (not necessarily all) components of $f^{-1}(Y_j \times \{0\}), j \in J$.

By the construction of $S_1, \ldots, S_n$, for each component $Q$ of $M - \bigcup_{j=1}^n S_j$, $\pi_1(Q)$ is contained in a vertex group of $(G, \mathcal{G})$. \hfill \Box

**Corollary 7.2.** Let $M$ be a good, compact, connected, orientable 3-orbifold without non-separating spherical 2-orbifolds. We assume that the fundamental group of each prime component of $M$ is infinite. Suppose that $\pi_1(M)$ acts on a simplicial tree $T$ nontrivially without edge inversions such that $T/\pi_1(M)$ is finite. Then there exists a non-empty system of essential 2-suborbifolds $S_1, \ldots, S_n \subset M$ such that for each component $Q$ of $M - \bigcup_{i=1}^n S_i$, $\pi_1(Q)$ fixes a vertex of the tree.

**Proof.** By Theorems 6.1 and 7.1 \hfill \Box

**Corollary 7.3.** Let $M$ be a good, compact, connected, orientable 3-orbifold without non-separating spherical 2-orbifolds. We assume that the fundamental group of each prime component of $M$ is infinite. Let $C$ be an affine curve contained in $\mathcal{X}(\pi_1(M))$. To each idel point $\hat{x}$ of $C$ one can associate a splitting of $\pi_1(M)$ with the property that an element $g$ of $\pi_1(M)$ is contained in a vertex group if and only if $\hat{I}_g$ does not have a pole at $\hat{x}$. Then there exists a non-empty system of essential 2-suborbifolds
$S_1, \ldots, S_n \subset M$ such that for each component $Q$ of $M - \bigcup_{i=1}^{n} S_i$, $\pi_1(Q)$ is contained in a vertex group.

**Proof.** By Theorems 6.2 and 7.1. □

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