The Nonperturbative Gauge Coupling of $N = 2$ Supersymmetric Theories

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Abstract

We argue that the topology of the quantum coupling space and the low energy effective action on the Coulomb branch of scale invariant $N = 2$ $SU(n)$ gauge theories pick out a preferred nonperturbative definition of the gauge coupling up to non-singular holomorphic reparametrizations.

June 1998
The quantum coupling space of scale invariant $N = 2$ supersymmetric gauge theories is a subset of the classical one obtained by discrete identifications under the action of the S-duality group. These S-duality identifications imply an exact quantum equivalence between classically inequivalent theories. However, while the topology of the quantum coupling space has an invariant meaning, its parametrization is unambiguous only at weak coupling. Different parametrizations can give rise to different S-duality group actions on the gauge couplings.

In one nonperturbative definition of the coupling, the S-duality group was found to be $\Gamma^0(2) \subset SL(2, \mathbb{Z})$ for the scale invariant $SU(n) \ N = 2$ SQCD theories [1]. This S-duality group is generated by $T: \tau \rightarrow \tau + 2$ and $S: \tau \rightarrow -1/\tau$ subject to the single constraint $S^2 = 1$. (We take the gauge coupling to be $\tau = \frac{2}{\pi} + i \frac{8a g^2}{\pi}$, differing by a factor of two from the usual definition.) A different nonperturbative parameterization of the $SU(3)$ gauge coupling proposed in [2] gives an S-duality group with the same $T$ generator, but a different $S: \tau \rightarrow -4/(3\tau)$. In the first case the fundamental domain of the S-duality group is the region $|\tau| \geq 1$ and $|\text{Re}\tau| \leq 1$ which has cusps (zero opening angle) at $\tau = i\infty$ and $\tau = 1$ and a $\mathbb{Z}_2$ orbifold point at $\tau = i$. By contrast, in the second case the fundamental domain is $|\tau| \geq 2/\sqrt{3}$ and $|\text{Re}\tau| \leq 1$ with cusp at weak coupling, $\mathbb{Z}_2$ orbifold at $\tau = 2i/\sqrt{3}$ and $\mathbb{Z}_6$ orbifold at $\tau = 1 + i/\sqrt{3}$. Although there exists a unique conformal map between these two fundamental domains mapping the strong coupling cusp of the first to the $\mathbb{Z}_6$ point of the second, the map is singular there since the opening angle changes. This is reflected in the fact that the corresponding S-duality groups are not isomorphic as abstract groups.

In this letter we argue that the topology of the quantum coupling space together with the low energy effective action on the Coulomb branch pick out a preferred nonperturbative definition of the gauge coupling up to non-singular holomorphic reparametrizations. We explicitly compute the S-duality group associated with this definition of the coupling, finding agreement with the results of [3] which were obtained by a different method. We find for the finite $N = 2$ SQCD with $SU(n)$ gauge group that the S-duality group can be presented as the subgroup of $SL(2, \mathbb{R})$ generated by $T: \tau \rightarrow \tau + 2$ and

$$S: \tau \rightarrow \begin{cases} \frac{-1}{\sec^2(\pi/2n)} \tau & n \text{ even} \\ \frac{-1}{\sec^2(\pi/2n)} \tau & n \text{ odd} \end{cases}$$

(1)

acting on the classical coupling space $\{\text{Im}\tau > 0\}$.

Before turning to the nonperturbative definition of the gauge coupling, we wish to point out some properties [3] of the result [1]. The S-duality group is not freely
generated by $T$ and $S$, but is subject to the constraints

$$S^2 = 1$$

and

$$(ST^{-1})^{2n} = 1, \quad \text{for } n \text{ odd.}$$

For the $SU(n)$ with $n$ even the S-duality group is isomorphic to $\Gamma^0(2)$, though this duality group is naturally enlarged to $SL(2, \mathbb{Z})$ in the $SU(2)$ case $[\text{II}, \text{III}]$. For $n$ odd the fundamental domain is defined by $|\text{Re}\tau| \leq 1$ and $|\tau| \geq \sec(\pi/2n)$ with edges identified. This fundamental domain has three special points: a weak coupling singularity at $\tau = i\infty$, a $\mathbb{Z}_2$ orbifold point at $\tau = i\sec(\pi/2n)$, and a $\mathbb{Z}_{2n}$ orbifold point at $\tau = \pm 1 + i \tan(\pi/2n)$. It is an open question whether there is an alternative characterization of the physics at the $\mathbb{Z}_{2n}$ orbifold points of the odd $n$ theories or the strong coupling cusp points of the even $n$ theories.

We now turn to the topological definition of the coupling parameter in the $SU(n)$ scale invariant theories.

The quantum coupling space of scale invariant $N = 2$ SQCD has isolated singularities at special couplings where the whole Coulomb branch is singular. Just as traversing paths around the singularities on the Coulomb branch generate elements of the low energy electric-magnetic (EM) duality group (reflected in monodromies of the BPS spectrum), we argue that monodromies of the BPS spectrum around the singularities of the quantum coupling space, $\mathcal{F}$, generate the S-duality group.

Consider the scale invariant $N = 2$ supersymmetric gauge theory with $SU(n)$ gauge group and $2n$ hypermultiplets in the fundamental representation. At a generic point on the Coulomb branch the gauge group is broken to $U(1)^{n-1}$ whose effective couplings $\tau_{ij}$ form a section of an $Sp(2n-2, \mathbb{Z})$ bundle on the Coulomb branch reflecting the EM duality identifications of the low energy effective description. The matrix of the effective couplings was identified in $[\text{IV}]$ with the the period matrix of the genus $n-1$ hyperelliptic curve $\Sigma_n$

$$y^2 = P^2(x) - f x^{2n}$$

where $P(x) = x^n - \sum_{\ell=0}^{n-1} u_\ell x^{n-\ell}$. The moduli $u_\ell$ parametrize the Coulomb branch and $f$ is a function of the gauge coupling $\tau$. At weak coupling $\tau \to i\infty$, $f \sim e^{i\pi\tau}$. It is important to note that the coupling $f$, being a parameter specifying the quantum field theory and not an order parameter (vev) specifying a vacuum, cannot depend on the $u_\ell$. Thus a “total parameter space” including both the coupling parameter and the
Coulomb branch vevs has the structure of a fiber bundle with the Coulomb branch as its fibers and the coupling space $\mathcal{F}$ as the base.

The complex structure of $\Sigma_n$ degenerates whenever the discriminant of (4) vanishes. At fixed coupling $f$ these singularities of the low-energy effective action are resolved by including in the effective description all states that become massless there. The charge vectors of BPS states which are massive in the vicinity of a singularity undergo $Sp(2n-2,\mathbb{Z})$ EM duality group monodromies upon traversing closed paths in the Coulomb branch around the singularity.

Equivalently, we can think about the same singularities as singularities in the coupling parameter space, $\mathcal{F}$, at a fixed vacuum $u_\ell$. We call such singularities “specific singularities” since their locations depend on the specific values of vacuum moduli. Specific singularities are not the only singularities in $\mathcal{F}$. The complex structure of (4) also degenerates at the “special singularities” $f = f_s \equiv \{0, 1, \infty\}$. These values of the coupling parameter are special in that the whole Coulomb branch becomes singular whenever $f = f_s$. Thus we can think of the quantum coupling space $\mathcal{F}$ as a three punctured sphere. More generally, in $N = 2$ scale invariant theories with simple gauge group, the gauge coupling $\tau$ is a section of a holomorphic line bundle over a three punctured sphere whose structure group is identified with the S-duality group. One of the punctures corresponds to weak coupling and the other two to special strongly coupled theories. We will later identify the monodromies around $f = 0$ and $\infty$ as the $T$ and $S$ duality generators, respectively; see Fig. 1.

As an illustration of the difference between specific and special singularities, consider the scale invariant $SU(3)$ theory. The Coulomb branch of this theory is described by a curve $\Sigma_3$:

$$y^2 = (x^3 - u_2x - u_3)^2 - fx^6.$$  \hspace{1cm} (5)

The complex structure of $\Sigma_3$ degenerates when the discriminant of the right hand side of (5) vanishes,

$$u_3^{10}(f-1)f^3\left(f-\left[1 - \frac{4u_2^3}{27u_3^2}\right]^2\right) = 0,$$  \hspace{1cm} (6)

and for $f \to \infty$ with $u_\ell$ kept finite. (In the latter case by an appropriate rescaling of $x$ and $y$, $\Sigma_3$ reduces to the singular curve $y^2 = x^6$.) Clearly, $f = \{0, 1, \infty\}$ are always singularities of the low-energy effective action irrespective of the choice of the vacuum moduli $u_\ell$. These are the special singularities of the coupling parameter space $\mathcal{F}$. The fourth singularity in $\mathcal{F}$ is at $f = [1 - (4u_2^3/27u_3^2)]^2$. This “specific” singularity differs from the previous three in that its position in $\mathcal{F}$ depends on the choice of the
Figure 1: The basis of the S-duality generators $T$ and $ST^{-1}$ is associated with the noncontractable loops $\gamma_0$ and $\gamma_1$. The filled dots represent “special” singularities in $\mathcal{F}$, and the empty ones represent “specific” singularities. A third special singularity is at $f = \infty$.

Coulomb branch vacuum. While there are always three special singularities at fixed positions in $\mathcal{F}$ for any rank of the gauge group, the number of specific singularities is rank dependent. For example, the scale invariant $SU(2)$ theory does not have specific singularities at all. Note that the $SU(3)$ vacuum with $u_2 = 0$ and $u_3 \neq 0$ is special in that the specific singularity coincides with the $f_s = 1$ special singularity.

The monodromies of the BPS spectrum or low energy $U(1)^{n-1}$ couplings around singularities on the coupling space $\mathcal{F}$ encode information about the S-duality group. But while S-duality transformations on the gauge coupling $\tau$ must be the same for any choice of vacuum moduli and hypermultiplet mass parameters, the monodromies in $\mathcal{F}$ around $f_s$ actually depend on this choice. This presents a puzzle: How can the S-duality group information which is invariant under changes in the vacuum moduli be extracted from these monodromies? The key to solving this puzzle is to realize that, in principle, noncontractable loops in $\mathcal{F}$ can be generators of both the S-duality and the low energy EM duality groups. In fact, nontrivial loops around the specific singularities in $\mathcal{F}$ have nothing to do with the S-duality group: any such loop can be deformed in the combined Coulomb branch and quantum coupling space to a loop around a singularity on the Coulomb branch at a fixed value of the gauge coupling parameter $f$. This follows simply from the fiber bundle structure of the combined vev plus parameter space pointed out earlier, and the fact that by definition the specific singularities move in the coupling space $\mathcal{F}$ as we move in the Coulomb branch. We will
call the monodromies around loops that encircle only specific singularities in \( \mathcal{F} \) “pure EM monodromies”.

So an S-duality group (and hence the coupling \( \tau \) it acts on) can be defined as the subgroup of the \( Sp(2n-2, \mathbb{Z}) \) EM duality group generated by the monodromies around the special singularities \( f_s \) in \( \mathcal{F} \). More precisely, those monodromies will be representations of the S-duality group generators in the low energy EM duality group \( Sp(2n-2, \mathbb{Z}) \) modulo the pure EM monodromies. In this way the S-duality group does not suffer from any ambiguity in the choice of basis cycles around the special singularities in \( \mathcal{F} \) since how those cycles are chosen to go around the specific singularities can only change the monodromies by pure EM monodromies. We must also mod out the representation of the S-duality group generators in \( Sp(2n-2, \mathbb{Z}) \) by its \( GL(n-1, \mathbb{Z}) \) subgroup generated by simple (integral) change of bases of the \( U(1)^{n-1} \) charge lattice. This is because the definition of the microscopic coupling in the classical theory is already insensitive to any choice of basis of \( U(1)^{n-1} \) generators.

The S-duality group constraints (2) and (3) can now be explicitly computed as follows. We first note that there is a choice of \( SU(n) \) vacua on the Coulomb branch, namely \( u_\ell = 0, \ell = 2, \ldots, n-1 \), and \( u_n \equiv u \neq 0 \), for which the step of modding out by EM monodromies is done automatically. This is simply because for this choice of vacuum all the singularities in \( \mathcal{F} \) are “special singularities”. (This Coulomb branch submanifold was used in [6, 7, 3] where S-duality identifications were analyzed from an algebraic point of view; here we just use these vacua for calculational convenience.) In these special vacua there is an unbroken global \( \mathbb{Z}_n \) discrete subgroup of the anomaly-free \( U(1)_R \) which implies that the branch points of the curve (4) are distributed in pairs around the \( n \)-th roots of unity. Choose an overcomplete basis of cycles \( \{ \beta^a, \gamma^a \} \), \( a = 1, \ldots, n \), as in Fig. 2; note that \( \sum_a \beta^a = \sum_a \gamma^a = 0 \).

As \( f \to e^{2\pi i} f \) for \( |f| \ll 1 \) (thus performing the \( T \) monodromy around \( f = 0 \)) one can show by contour dragging that

\[
T : \begin{cases} 
\beta_a \to \beta_a \\
\gamma^a \to \gamma^a + \beta_a - \beta_{a-1}
\end{cases}
\]

(7)

(where \( \beta_0 \equiv \beta_n \)). Repeated \( T \) monodromies clearly do not close on the identity. Now take \( f \) close to 1. The arrangement of cuts and cycles in Fig. 2 stays the same except that the branch points further from the origin move off towards \( x = \infty \). As \( (f - 1) \to e^{2\pi i} (f - 1) \) for \( |f - 1| \ll 1 \) (thus performing the \( ST^{-1} \) monodromy around \( f = 1 \)) we find

\[
ST^{-1} : \begin{cases} 
\beta_a \to -\beta_a + \beta_{a-1} + \gamma^a \\
\gamma^a \to \beta_{a-1}
\end{cases}
\]

(8)
Figure 2: Contours for a basis of cycles for the $SU(n)$ curve $(n = 3)$ on one sheet of the $x$-plane. The thick wavy lines represent the cuts and the $\gamma^a$ contours close on the second sheet.

It is a simple exercise to show that for $n$ odd (3) is satisfied, while for $n$ even there is no relation. Finally, from (7) and (8) we derive

$$S^2 : \begin{cases} \beta_a \to \beta_{a-1} \\ \gamma^a \to \gamma^{a-1} \end{cases}$$

(9)

This is one of the $GL(n-1, \mathbb{Z})$ monodromies which implement a change of $U(1)^{n-1}$ basis and are invisible to the microscopic coupling $\tau$. (9) therefore implies the constraint (4), thus completing the derivation of the S-duality group of the $SU(n)$ scale invariant theory.

It would be interesting to extend this construction to theories with other simple and semi-simple gauge groups. In particular, extension to the elliptic models of [8] may permit a comparison with the $SL(2, \mathbb{Z})$ S-duality group of $N = 4$ gauge theories.

Acknowledgments

It is a pleasure to thank T.-M. Chiang, J. Diller, B. Greene, Z. Kakushadze, J. Minahan, A. Shapere, G. Shiu, H. Tye, E. Witten, and P. Yi for helpful comments and discussions. PCA thanks the University of Cincinnati Physics Department for its hospitality. This work is supported in part by NSF grant PHY-9513717. The work of PCA is supported in part by an A.P. Sloan fellowship.
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