A classification of CO spaces which are continuous images of compact ordered spaces

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Abstract

A topological space $X$ is called a CO space, if every closed subset of $X$ is homeomorphic to some clopen subset of $X$. Every ordinal with its order topology is a CO space. This work gives a complete classification of CO spaces which are continuous images of compact ordered spaces.

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1 Introduction

A topological space \( X \) is called a CO space if every closed subset of \( X \) is homeomorphic to some clopen subset of \( X \). The simplest example of a compact Hausdorff CO space is a successor ordinal with its order topology.

In this work we characterize the CO spaces which are continuous images of compact interval spaces. There are such spaces which are not ordinals, yet this class is not much bigger than the class of successor ordinals.

So far there has been only one result concerning compact Hausdorff CO spaces which are not continuous images of compact interval spaces. It is due to Bonnet and Shelah [BS]. Assuming \( \Diamond_{\aleph_1} \) they construct a thin tall CO space. The significance of this result is that it indicates that there is no explicit description of general compact Hausdorff CO spaces.

To state the main theorem of this work, we need the following terminology. A space \( \langle X, \tau_X \rangle \) is an interval space, if there is a linear ordering \( < \) of \( X \) such that \( \tau_X \) is the order topology of this linear ordering. That is, a subbase for this topology is the family of sets

\[
\{ \{ x \in X \mid x < a \} \mid a \in X \} \cup \{ \{ x \in X \mid x > a \} \mid a \in X \}.
\]

An interval space \( X \) is called an ordinal space if there is a well ordering of \( X \) such that \( \tau_X \) is the order topology of this well ordering. For infinite cardinals \( \lambda, \mu \), let \( \mu^* \) denote the reverse ordering of \( \mu \) and \( X_{\lambda, \mu} \) denote the interval space of \( \lambda + 1 + \mu^* \). Define \( \alpha(X_{\lambda, \mu}) \) to be the following ordinal: \( \alpha(X_{\lambda, \mu}) := \max(\lambda, \mu) \cdot \omega \). For an infinite cardinal \( \kappa \) let \( X_{\kappa} \) denote the one point compactification of a discrete space of cardinality \( \kappa \) and set \( \alpha(X_{\aleph_1}) := \omega^2 \). The notation \( X \cong Y \) stands for the fact that \( X \) and \( Y \) are homeomorphic, and \( f : X \cong Y \) means that \( f \) is a homeomorphism between \( X \) and \( Y \). The final result of this work is the following theorem.

**Theorem 1.1.** (a) Let \( X \) be a Hausdorff space which is a continuous image of a compact interval space, and assume that \( X \) is a CO space. Then there is a partition \( \{ Z, Y_0, \ldots, Y_{k-1} \} \) of \( X \) into open sets such that
(1) For every $i < k$ either $Y_i \cong X_{\aleph_1}$, or $Y_i \cong X_{\lambda, \mu}$, where $\lambda, \mu$ are some infinite regular cardinals and $\mu > \aleph_0$.

(2) $Z$ is an ordinal space homeomorphic to some successor ordinal $\beta$.

(3) $\beta > \alpha(Y_i)$ for every $i \in I$.

Note that if $X_{\lambda, \mu} \cong X_{\lambda', \mu'}$, then $\{\lambda, \mu\} = \{\lambda', \mu'\}$. So $\alpha(Y_i)$ is well-defined.

(b) If a space $X$ has the above form, then $X$ is a CO space, and $X$ is a continuous image of a compact interval space.

Part (b) of the above theorem is merely an observation. It is Part (a) which is the real subject of this work.

A compact Hausdorff space $X$ is scattered if every nonempty subset of $X$ has an isolated point in its relative topology. Let $K_{CII}$ denote the class of all Hausdorff spaces which are the continuous image of a compact interval space. Section 2 deals with the following intermediate step in the proof of Theorem 1.1.

Theorem 1.2. If $X \in K_{CII}$ and $X$ is a CO space, then $X$ is scattered.

Three main questions arise.

Question 1.3. (a) Is there a non-scattered compact Hausdorff CO space? It is even not known whether it is consistent with ZFC that such a space exists.

(b) The construction of [BS] works only for $\aleph_1$. So we ask whether there is a compact Hausdorff CO space of cardinality $> \aleph_1$ which is not a finite direct sum of a member of the class defined in Theorem 1.1 and a CO space with cardinality $\aleph_1$? It is even not known whether this statement is consistent.

(c) Does it follow from ZFC that there is a compact Hausdorff CO space which does not belong the the class defined in Theorem 1.1?

Let $K_{IVL}$ be the class of 0-dimensional compact interval spaces. The classification those CO spaces which belong to $K_{IVL}$ was dealt with in [BBR].
The classification theorem proved in [BBR] is of course a special case of Theorem 1.1.

After the authors had proved Theorem 1.2, Shelah proved a theorem which turned out to be almost equivalent to 1.2. The statement of this theorem appears in [S] p.355. However, a proof of that theorem has never appeared. That Shelah’s statement is equivalent to a statement about continuous images of interval spaces follows from [H].

**The main steps in the proof of Theorem 1.1.**

In Section 2 we prove that a CO space which is a continuous image of a compact interval space must be scattered (Theorem 1.2). The rest of the sections deals with scattered spaces which are a continuous image of a compact interval space.

Section 3 deals with the question: when a scattered continuous image of a compact interval space is itself an interval space. The characterization uses “obstructions”. We define a class \( \mathcal{O} \) of topological spaces, and prove that every space which is a scattered continuous image of a compact interval space, and which does not embed any member of \( \mathcal{O} \) must be an interval space. This statement appears in Theorem 3.1.

Let \( X \) be a scattered continuous image of a compact interval space, and suppose that \( X \) is a CO space. We shall show that \( X \) is the sum of finitely many copies of \( X_{\aleph_1} \) and a space \( Y \) which omits all members of \( \mathcal{O} \). (See above). Then we use the characterization of CO compact interval spaces from [BBR] to get a description of \( Y \).

Section 4 contains the main technical lemma needed in the proof that the obstructions are omitted (Theorem 4.2). It says that if \( X \) is a scattered CO space then there are no subsets \( M, L, K \subseteq X \) such that \( M \prec L \prec^w K \). (See Definition 4.1). In Theorem 4.2 the CO space \( X \) is assumed to have a very strong Hausdorff property. Because of this assumption we are able to deal only with continuous images of compact interval spaces and not with general compact spaces.
In section 5 we show that the obstructios are omitted and in 6 we obtain the desired characterization.

As a matter of fact, using Theorem 4.2 there is a short clean proof of the characterization of CO scattered compact interval spaces. This has already been done in [BBR], but in a less elegant way. So in Section 7 we prove this characterization. By doing so, this work becomes self-contained and easier to read.

2 Scatteredness of CO spaces which are continuous images of compact interval spaces

In this section we prove Theorem 1.2. The proof is by way of contradiction, but it takes till Theorem 2.29 to reach this contradiction. In two of the intermediate lemmas – Proposition 2.5 and Corollary 2.8, a space $X$ is given, and it is assumed that $X$ is a non-scattered CO space. Since it is not known whether non-scattered CO spaces exist, these lemmas have little or no use once Theorem 1.2 is proved. In addition, Propositions 2.22, 2.23 and 2.24 assume the existence of a CO space which has some extra properties. These assumptions too are likely to be contradictory. See especially 2.24.

We do not prove directly that every CO space which belongs to $K_{CII}$ is scattered. This turns out to be too tedious. Rather, we find certain topological properties of members of $K_{CII}$ which serve as interpolants. For example, in Part 1. below, we prove that every member $X$ of $K_{CII}$ is tightly Hausdorff, and later we use this property of $X$ rather than assuming that $X \in K_{CII}$. There are four other properties of members of $K_{CII}$ which are used as interpolants, and we prove them just before they are used. The class of spaces with these five properties is denoted by $K$. In Theorem 2.29 we prove that every member of $K$ which is a CO space is scattered. Hence the same is true for members of $K_{CII}$.

1. Some Hausdorff-type properties of members of $K_{CII}$. 

5
We start by defining the notion of a tightly Hausdorff space. We shall show that members of $K_{CH}$ are tightly Hausdorff. This property and some of its weaker variants will be used extensively.

**Definition 2.1.** Let $X$ be a topological space.

(a) We denote by $\tau^X$ the topology of $X$. If $A \subseteq X$, then $\text{cl}^X(A)$, $\text{int}^X(A)$ and $\text{acc}^X(A)$ denote respectively the closure, interior and the set of accumulation points of $A$ in $X$. If $x \in X$, then the set of open neighborhoods of $x$ in $X$ is denoted by $\text{Nbr}^X(x)$. Similarly, $\text{Nbr}^X_{cl}(x)$ and $\text{Nbr}^X_{clp}(x)$ denote respectively the the set of closed neighborhoods of $x$ in $X$ and the set of clopen neighborhoods of $x$ in $X$. Superscript $^X$ is omitted when the intended space $X$ can be understood from the context.

(b) A family $A$ of subsets of $X$ is called a *pairwise disjoint family*, if $A \cap B = \emptyset$ for any distinct $A, B \in A$. Let $A$ be pairwise disjoint family of subsets of $X$ and $x \in X$. We say that $x$ is an *accumulation point of $A$*, if every neighborhood of $x$ intersects infinitely many members of $A$. The set of accumulation points of $A$ is denoted by $\text{acc}(A)$. Suppose that $A$ is a pairwise disjoint family of subsets of $X$, such that for every $B, C \subseteq \bigcup A$, if

$$\{A \in A \mid B \cap A \neq \emptyset\} = \{A \in A \mid C \cap A \neq \emptyset\},$$

then

$$\text{acc}(\{B \cap A \mid A \in A\}) = \text{acc}(\{C \cap A \mid A \in A\}).$$

Then $A$ is called a *tight family*.

(c) A subset $A \subseteq X$ is *relatively discrete* if $A$ together with its relative topology is a discrete space. So $A$ is relatively discrete iff $A \cap \text{acc}^X(A) = \emptyset$.

(d) Let $A \subseteq X$. A family $U = \{U_x \mid x \in A\}$ is called a *Hausdorff system* for $A$, if $U$ is a pairwise disjoint family and for every $x \in A$, $U_x \in \text{Nbr}(x)$.

(e) We say that $U$ is a *strong Hausdorff system* for $A$, if $U$ is a Hausdorff system for $A$ and $\text{acc}(U) = \text{acc}(A)$.

(f) Let $X$ be a Hausdorff space. If every relatively discrete subset of $X$ has a Hausdorff system, then $X$ is called a *collectionwise Hausdorff space*. If
every relatively discrete subset of $X$ has a strong Hausdorff system, then $X$ is said to be a strongly Hausdorff space, and if every relatively discrete subset of $X$ has a tight Hausdorff system, then we call $X$ a tightly Hausdorff space.

Note that “tightly Hausdorff” $\Rightarrow$ “strongly Hausdorff” $\Rightarrow$ “collectionwise Hausdorff”.

Lemma 2.2. If $X \in K_{CII}$, then $X$ is tightly Hausdorff.

Proof Let $N$ be a subset of a chain $\langle L, < \rangle$ and $I \subseteq N$ be a convex subset of $L$. We say that $I$ is a convex component of $N$ in $L$ if there is no convex set $I'$ such that $I' \subseteq N$ and $I'$ properly contains $I$. Denote the family of convex components of $N$ in $L$ by $I(N)$. Clearly, $I(N)$ is a partition of $N$, and if $N$ is open in the order topology of $L$, then every member of $I(N)$ is open.

Let $\langle L, < \rangle$ be a compact chain and $f : L \to X$ be a continuous surjective function. Denote the order topology of $\langle L, < \rangle$ by $\tau_L$ and the topology of $X$ by $\tau_X$. Suppose that $A \subseteq X$ is relatively discrete. For every $x \in A$ we define $L_x \in \tau_L$ and $U_x \in \tau_X$. Let $\{x_i \mid i < \alpha\}$ be an enumeration of $A$. We define $L_{x_i}$ and $U_{x_i}$ by induction on $i$. Suppose that $L_{x_j}$ and $U_{x_j}$ have been defined for every $j < i$, set $A_0 = \{x_j \mid j < i\}$, and assume the following induction hypotheses.

1. For every $x \in A_0$ and $I \in I(L_x)$ there is $s_I \in I$ such that $f(s_I) = x$.
2. $L_x \cap L_y = \emptyset$ for every distinct $x, y \in A_0$.
3. $f^{-1}(U_x) \subseteq L_x$ for every $x \in A_0$.
4. $f(\text{cl}(L_x)) \cap A = \{x\}$ for every $x \in A_0$.

Claim 1  (i) If $s \in \text{acc}(\{L_x \mid x \in A_0\})$, then $f(s) \in \text{acc}(A_0)$. (ii) If $s \in \text{cl}(\bigcup\{L_x \mid x \in A_0\})$, then either $f(s) \in \text{acc}(A_0)$ or for some $x \in A_0$, $s \in \text{cl}(L_x)$.

Proof Statement (ii) follows trivially from (i). Let $s$ be as in the (i) and $J$ be an open interval containing $s$. Then for every finite set $\sigma \subseteq A_0$ there
are distinct $x, y, z \in A_0 - \sigma$ such that $L_x, L_y, L_z$ intersect $J$. Then there is $I \in \mathcal{I}(L_x) \cup \mathcal{I}(L_y) \cup \mathcal{I}(L_z)$ such that $I \subseteq J$. Assume that $I \in \mathcal{I}(L_x)$. Then $f(s_I) = x$. This implies that for every neighborhood $N$ of $s$, $f(N) \cap A_0$ is infinite. So if $U \in \text{Nbr}(f(s))$, then $f^{-1}(U) \in \text{Nbr}(s)$, so $f(f^{-1}(U))$ contains an infinite subset of $A_0$. Now, $f(f^{-1}(U)) = U$. Hence $U$ contains an infinite subset of $A_0$. So $f(s) \in \text{acc}(A_0)$. Claim 1 is proved.

Denote $x_i$ by $y$, and set $K = \text{cl}(\bigcup\{L_x \mid x \in A_0\})$. Then $y \notin f(K)$. This relies on the following three facts.

- $A \cap \bigcup_{x \in A_0} f(\text{cl}(L_x)) = A_0$.
- If $s \in \text{cl}(\bigcup\{L_x \mid x \in A_0\}) - \bigcup_{x \in A_0} f(\text{cl}(L_x))$, then $f(s) = \text{acc}(A_0)$.
- $A$ is relatively discrete.

Hence $V_y := X - f(K) \in \text{Nbr}(y)$. Choose $W_y \in \text{Nbr}(y)$ such that $\text{cl}(W_y) \cap A = \{y\}$ and define $M_y = f^{-1}(V_y \cap W_y)$ and $L_y = \bigcup\{I \in \mathcal{I}(M_y) \mid y \in f(I)\}$.

Clearly, $\mathcal{I}(L_y) = \{I \in \mathcal{I}(M_y) \mid y \in f(I)\}$ and $f^{-1}(y) \subseteq L_y$.

**Claim 2** There is $U_y \in \text{Nbr}(y)$ such that $f^{-1}(U_y) \subseteq L_y$.

**Proof** Suppose that Claim 2 is false. Then for every $F \in \text{Nbr}_{\text{cl}}(y)$, $f^{-1}(F) \cap (L - L_y) \neq \emptyset$. So $H := \bigcap\{f^{-1}(F) \cap (L - L_y) \mid F \in \text{Nbr}_{\text{cl}}(y)\} \neq \emptyset$.

Let $a \in H$. Then for every $F \in \text{Nbr}_{\text{cl}}(y)$, $f(a) \in F$. So $f(a) = y$. But $a \notin L_y$. A contradiction. This proves Claim 2.

Let $U_y \in \text{Nbr}(y)$ be such that $f^{-1}(U_y) \subseteq L_y$. We check that the induction hypotheses (1) - (4) hold for $L_y$ and $U_y$. The definition of $L_y$ implies that (1) holds, and the definition of $U_y$ implies that (3) holds.

$$L_y \subseteq M_y \subseteq f^{-1}(X - f(K)) \subseteq f^{-1}(X - f(\bigcup_{x \in A_0} L_x)) \subseteq L - \bigcup_{x \in A_0} L_x.$$ 

So (2) holds.

We prove (4). Certainly, $y \in f(L_y)$. Recall that $L_y \subseteq M_y \subseteq f^{-1}(W_y)$. So $\text{cl}(L_y) \subseteq \text{cl}(f^{-1}(W_y))$. Also, $\text{cl}(f^{-1}(W_y)) \subseteq f^{-1}(\text{cl}(W_y))$. So $\text{cl}(L_y) \subseteq f^{-1}(\text{cl}(W_y))$ and hence

$$f(\text{cl}(L_y)) \cap A \subseteq f(f^{-1}(\text{cl}(W_y)) \cap A) = \text{cl}(W_y) \cap A = \{y\}.$$
The first equality follows from the surjectiveness of \( f \). This shows that (4) is fulfilled. We have completed the inductive construction.

We show that \( \mathcal{U} := \{ U_x \mid x \in A \} \) is a tight Hausdorff system for \( A \). Let \( x, y \in A \) be distinct. Then \( f^{-1}(U_x) \subseteq L_x \) and \( f^{-1}(U_y) \subseteq L_y \). Since \( L_x, L_y \) are disjoint, so are \( U_x \) and \( U_y \). Observe the following fact.

(*) Let \( W, Z \) be compact Hausdorff spaces, \( h : W \to Z \) be continuous and \( \mathcal{C} \subseteq \mathcal{P}(W) \). If \( \{ h(C) \mid C \in \mathcal{C} \} \) is a pairwise disjoint family, then

\[
\text{acc}(h(C)) = \text{acc}(h(C)).
\]

To see this, note that the fact \( h(\text{acc}(C)) \subseteq \text{acc}(h(C)) \) holds even without assuming that \( W \) and \( Z \) are compact. Now, the sets \( h(\text{acc}(C)) \) and \( \text{acc}(h(C)) \) are closed, and it is easy to see that \( h(\text{acc}(C)) \) is dense in \( \text{acc}(h(C)) \). So these sets must be equal and hence (*) holds.

Let \( A' \subseteq A \) and suppose that \( B = \{ y_x \mid x \in A' \} \), where \( y_x \in U_x \) for every \( x \in A' \). We show that \( \text{acc}(B) = \text{acc}(A') \). For every \( x \in A' \) let \( w_x \in L_x \cap f^{-1}(y_x) \). Such a choice is possible since \( y_x \in U_x \subseteq f(L_x) \). Let \( I_x \in \mathcal{I}(L_x) \) be such that \( w_x \in I_x \). Then by the definition of \( L_x \) there is \( z_x \in I_x \) such that \( f(z_x) = x \). Set \( M = \{ z_x \mid x \in A' \} \) and \( N = \{ w_x \mid x \in A' \} \). Then \( f(M) = A' \) and \( f(N) = B \). Applying (\( * \)) to \( \mathcal{C} := \{ \{ m \} \mid m \in M \} \), we conclude that (i) \( \text{acc}(A') = f(\text{acc}(M)) \). Similarly, (ii) \( \text{acc}(B) = f(\text{acc}(N)) \). It is also clear that (iii) \( \text{acc}(M) = \text{acc}(N) \). To see this let \( z \in \text{acc}(M) \). Then, without loss of generality, there is a strictly increasing sequence \( \{ z_{x_i} \mid i < \mu \} \subseteq M \) which converges to \( z \). This implies that \( \{ I_{x_i} \mid i < \mu \} \) is a strictly increasing sequence converging to \( z \), and so \( \{ w_{x_i} \mid i < \mu \} \) converges to \( z \). That is, \( z \in \text{acc}(N) \). We have shown that \( \text{acc}(M) \subseteq \text{acc}(N) \), and the same argument proves that \( \text{acc}(N) \subseteq \text{acc}(M) \). So (iii) holds.

From (i) - (iii) it follows that \( \text{acc}(B) = \text{acc}(A') \).

We prove that if \( A' \subseteq A \), then \( \text{acc}(\{ U_x \mid x \in A' \}) \subseteq \text{acc}(A') \). For \( x \in A' \) let \( V_x = f^{-1}(U_x) \). By (\( * \)), \( f(\text{acc}(\{ V_x \mid x \in A' \})) = \text{acc}(\{ U_x \mid x \in A' \}) \).

Let \( y \in \text{acc}(\{ U_x \mid x \in A' \}) \). So there is \( z \in \text{acc}(\{ V_x \mid x \in A' \}) \) such that \( y = f(z) \). There are a 1–1 sequence \( \{ x_i \mid i < \mu \} \subseteq A' \) and a strictly monotonic
sequence \( \{ z_i | i < \mu \} \) such that \( z_i \in V_{x_i} \) and \( \lim_{i<\mu} z_i = z \). By the construction, \( V_x = f^{-1}(U_x) \subseteq L_x \), so for every \( i < \mu \) there is \( I_i \in \mathcal{I}(L_{x_i}) \) such that \( z_i \in I_i \). Let \( w_i \in f^{-1}(x_i) \cap I_i \). (The definition of the \( L_x \)'s assures the existence of \( w_i \)). Since the \( I_i \)'s are pairwise disjoint and since \( z_i, w_i \in I_i \), it follows that \( \lim_{i<\mu} w_i = \lim_{i<\mu} z_i \). Hence \( \lim_{i<\mu} x_i = \lim_{i<\mu} f(w_i) = \lim_{i<\mu} f(z_i) = f(z) = y \). So \( y \in \text{acc}(A') \). We have proved the following facts.

1. If \( A' \subseteq A \) and \( \{ y_x | x \in A' \} \) is such that \( y_x \in U_x \) for every \( x \in A' \), then \( \text{acc}(A') = \text{acc}(\{ y_x | x \in A' \}) \).

2. For every \( A' \subseteq A \), \( \text{acc}(\{ U_x | x \in A' \}) \subseteq \text{acc}(A') \).

Facts (1) and (2) imply that \( \{ U_x | x \in A \} \) is a tight family. So \( A \) has a tight Hausdorff system.

For a Hausdorff space \( X \) denote by \( \text{Is}(X) \) the set of isolated points of \( X \) and set \( D(X) = X - \text{Is}(X) \). Now define the \( \alpha \)'s derivative of \( X \) as follows. \( D_0(X) = X, D_{\alpha+1}(X) = D(D_{\alpha}(X)) \) and \( D_{\delta}(X) = \bigcap_{\alpha<\delta} D_{\alpha}(X) \) when \( \delta \) is a limit ordinal. Suppose now that \( X \) is a compact Hausdorff space. The rank of \( X \) is the first ordinal \( \alpha \) such that \( D_{\alpha}(X) \) is finite or perfect. (A set is perfect if it does not have isolated points in its relative topology). Denote the rank of \( X \) by \( \text{rk}(X) \). Define \( \ker(X) = D_{\text{rk}(X)+1}(X) \) and call \( \ker(X) \) the perfect kernel of \( X \). Hence \( \ker(X) \) is either the empty set or an infinite perfect set. It is easy to check that \( X \) is scattered iff \( \ker(X) = \emptyset \).

Let \( \text{Clop}(X) \) and \( \text{Clsd}(X) \) denote respectively the set of clopen subsets, and the set of closed subsets of a general Hausdorff space \( X \) and set \( \text{Po}(X) = \{ x \in X | \text{ there is } U \in \text{Nbr}(x) \text{ such that } \text{Is}(U) = \emptyset \} \). For a compact Hausdorff space define \( \mathcal{S}(X) = \{ F \in \text{Clsd}(\ker(X)) | F \text{ is scattered} \} \) and \( \Omega(X) = \sup(\{ \text{rk}(F) | F \in \mathcal{S}(X) \}) \).

The proof of Theorem 1.2 is by way of contradiction. In the end of this section we assume that \( X \) is a counter-example to the theorem, and conclude that \( (2^{\Omega(X)})^+ < |\ker(X)| \), which turns out to be a contradiction. The proof is divided to a series of subclaims, the first of which is the following statement about members \( X \) of \( K_{CH} \).
2. If $X$ is a CO space, then $\Omega(X)$ is not attained by any member of $\mathcal{S}(X)$.

Let $F \in \mathcal{S}(X)$ and $A \subseteq \text{Is}(F)$ be such that $D(F) = \text{acc}(A) = \text{acc}((\text{Is}(F) - A)$.

Let $\{U_x \mid x \in \text{Is}(F)\}$ be a Hausdorff system for $\text{Is}(F)$ and define $F_1 = \text{cl}(F \cup \bigcup\{U_x \cap \ker(X) \mid x \in A\})$. The set $F_1$ is called a fattening of $F$. The precise definition of a fattening is given below.

**Definition 2.3.** Let $X$ be a Hausdorff compact space, $F \in \mathcal{S}(X)$, $F_1 \in \text{Clsd}(\ker(X))$ and $F \subseteq F_1$. We call $F_1$ a fattening of $F$ if the following holds.

(F1) $\text{Is}(F) = \text{Is}(F_1) \cup (\text{Is}(F) \cap \text{Po}(F_1))$.

(F2) $D(F) = \text{acc}(\text{Is}(F_1)) = \text{acc}(\text{Is}(F) \cap \text{Po}(F_1))$.

**Proposition 2.4.** If $X$ is collectionwise Hausdorff and compact and $F \in \mathcal{S}(X)$, then $F$ has a fattening.

**Proof** Let $\{U_x \mid x \in \text{Is}(F)\}$ and $\{V_y \mid y \in \text{Is}(D(F))\}$ be Hausdorff systems for $\text{Is}(F)$ and $\text{Is}(D(F))$ respectively. For every $y \in \text{Is}(D(F))$ let $A_y$ be an infinite subset of $\text{Is}(F) \cap V_y$ such that $(\text{Is}(F) \cap V_y) - A_y$ is also infinite. Then $\text{acc}(A_y) = \text{acc}((\text{Is}(F) \cap V_y) - A_y) = \{y\}$. Let

$$F_1 = \text{cl}\left(F \cup \bigcup\left\{U_x \cap \ker(X) \mid x \in \bigcup_{y \in \text{Is}(D(F))}A_y\right\}\right).$$

It is left the reader to check that $F_1$ is as required. 

**Proposition 2.5.** Let $X$ be a collectionwise Hausdorff CO space and $E \in \mathcal{S}(X) - \{\emptyset\}$. Suppose that $\text{rk}(E) = \alpha$. Then for every $n \in \omega$ there is $F \in \mathcal{S}(X)$ such that $|D_\alpha(F)| = n$.

**Proof** For $\alpha = 0$ the claim of the proposition follows from the fact that $\ker(X)$ is infinite, so we assume that $\alpha > 0$. The proof is by induction on $n$. We may assume that $|D_\alpha(E)| = 1$. Suppose that $F \in \mathcal{S}(X)$ and $|D_\alpha(F)| = n$. We show that ($\ast$) there is $G \in \mathcal{S}(X)$ such that $|D_\alpha(G)| = 2n$. 


Let $\hat{H}$ be a fattening of $F$, let $H \in \text{Clop}(X)$ and $\varphi$ be such that $\varphi : \hat{H} \cong H$. Set $\hat{H}^0 = \text{cl}(\text{Is}(F) \cap \text{Po}(\hat{H}))$ and $H^0 = \varphi(\hat{H}^0)$. Clearly,

$$D(\hat{H}^0) = D(\text{cl}(\text{Is}(F) \cap \text{Po}(\hat{H}))) = \text{acc}(\text{Is}(F) \cap \text{Po}(\hat{H})),$$

and by (F2),

$$\text{acc}(\text{Is}(F) \cap \text{Po}(\hat{H})) = \text{acc}(\text{Is}(\hat{H})).$$

So

$$D(\hat{H}^0) = \text{acc}(\text{Is}(\hat{H})).$$

The same holds for $H^0$ and $H$, namely,

$$D(H^0) = \text{acc}(\text{Is}(H)).$$

Since $H$ is clopen in $X$, it follows that $\text{Is}(H) = \text{Is}(X) \cap H$. So

$$D(H^0) = \text{acc}(\text{Is}(X) \cap H),$$

and hence

$$D(H^0) \cap \text{Po}(X) = \emptyset.$$

Let $K_0$ be a clopen subset of $X$ homeomorphic to $\ker(X)$. Then there is $F^0 \subseteq K_0$ such that $F^0 \cong F$. We shall show that $F^0 \cup H^0$ is the set $G$ required in $(\ast)$. Since $F^0 \subseteq K_0$ and $K_0$ is open and perfect, we have that $F^0 \subseteq \text{Po}(X)$. So $D(H^0) \cap F^0 = \emptyset$. This implies that $H^0 \cap F^0$ is a finite subset of $\text{Is}(H^0)$, and hence

1. $D(F^0 \cup H^0)$ is the disjoint union of $D(F^0)$ and $D(H^0)$.

2. $H^0 \subseteq \ker(X)$.

Recall that $\hat{H}^0 = \text{cl}(\text{Is}(F) \cap \text{Po}(\hat{H}))$. So $\text{Is}(\hat{H}^0) = \text{Is}(F) \cap \text{Po}(\hat{H})$ and hence $\text{Is}(\hat{H}^0) \subseteq \text{Po}(\hat{H})$. It follows that $\text{Is}(\hat{H}^0) \subseteq \ker(\hat{H})$, and this implies that $\text{cl}(\text{Is}(\hat{H}^0)) \subseteq \ker(\hat{H})$. But $\text{cl}(\text{Is}(\hat{H}^0)) = \hat{H}^0$, so $\hat{H}^0 \subseteq \ker(\hat{H})$. Since $\varphi$ takes $\hat{H}$ to $H$ and $\hat{H}^0$ to $H^0$, it follows that $H^0 \subseteq \ker(H)$. So $H^0 \subseteq \ker(X)$.

Now we show that
Clearly, \( D(\hat{H}^0) = \text{acc}(\text{Is}(\hat{H}^0)) = \text{acc}(\text{Po}(\hat{H}) \cap \text{Is}(F)) \). By (F2), \( \text{acc}(\text{Po}(\hat{H}) \cap \text{Is}(F)) = D(F) \). So \( D(\hat{H}^0) = D(F) \). Since \( \alpha > 0 \), it follows that \( D_\alpha(\hat{H}^0) = D_\alpha(F) \). So
\[
|D_\alpha(H^0)| = |D_\alpha(\hat{H}^0)| = |D_\alpha(F)| = n.
\]
We have proved (3).

Recall that \( F_0 \subseteq \text{Po}(X) \). So
\[
(4) \quad F_0 \subseteq \ker(X).
\]
From (2) and (4) it follows that, \( F_0 \cup H^0 \in S(X) \). Since \( F_0 \cong F \), \( |D_\alpha(F_0)| = n \). Hence by Facts (1) and (3), \( |D_\alpha(F_0 \cup H^0)| = 2n \).

Any continuous image of a sequentially compact space is sequentially compact. So we have the following fact.

**Proposition 2.6.** Every member of \( K_{\text{CH}} \) is sequentially compact.

For scattered spaces \( F \) and \( G \) define \( F \preceq G \), if either \( \text{rk}(F) < \text{rk}(G) \) or \( \text{rk}(F) = \text{rk}(G) \) and \( |D_{\text{rk}(F)}(F)| < |D_{\text{rk}(F)}(G)| \). Let \( X \) be a Hausdorff space. We say that \( X \) is strongly Hausdorff for convergent sequences if every 1–1 convergent sequence in \( X \) has a strong Hausdorff system.

**Proposition 2.7.** Let \( X \) be a sequentially compact space which is strongly Hausdorff for convergent sequences. Suppose that \( F_0 \preceq F_1 \preceq \ldots \) is a sequence of members of \( S(X) \). Then there is \( F \in S(X) \) such that \( F_i \preceq F \) for every \( i \in \omega \).

**Proof** We may assume that \( |D_{\text{rk}(F)}(F_i)| \geq i \). More precisely, there is a subsequence \( \{m_i \mid i \in \omega \} \) and for every \( i \) there is a closed subset \( \hat{F}_i \subseteq F_{m_i} \) with the property that \( F_i \preceq \hat{F}_{i+1} \) and \( |D_{\text{rk}(\hat{F}_i)}(\hat{F}_i)| \geq i \) for every \( i \in \omega \). To see this we distinguish between the cases: (i) \( \{\text{rk}(F_i) \mid i \in \omega \} \) is eventually constant, and (ii) \( \{\text{rk}(F_i) \mid i \in \omega \} \) is not eventually constant. If (i) happens we define \( \hat{F}_i = F_{i+n_0} \), where \( n_0 \) is such that for every \( i,j \geq n_0 \), \( \text{rk}(F_i) = \text{rk}(F_j) \). Suppose that (ii) happens. Then we take a subsequence \( \{m_i \}_{i \in \omega} \) such that for every \( i \), \( \text{rk}(F_{m_i+1}) > \text{rk}(F_{m_i}) + 1 \). Let \( \{\alpha_i \mid i \in \omega \} \) be such that
\[ \lim_{i} \alpha_i = \lim_{i} \text{rk}(F_i) \] and for every \( i \), \( \text{rk}(F_{m_i}) < \alpha_i < \text{rk}(F_{m_{i+1}}) \). Let \( \widehat{F}_i \) be a closed subset of \( F_{m_i} \) such that \( |D_{\alpha_i}(\widehat{F}_i)| = i \). Hence \( \{\widehat{F}_i \mid i \in \omega \} \) is as desired.

It follows that there is a 1–1 sequence \( \{x_i \mid i \in \omega \} \) such that \( x_i \in D_{\text{rk}((\widehat{F}_i))} \) for every \( i \in \omega \). We may assume that \( \{x_i\} \) is a convergent sequence. So by the sequential strong Hausdorff property of \( X \), \( \{x_i\} \) is a convergent sequence. Let \( \widehat{F}_m = \text{cl}(\bigcup_{i \in \omega} \widehat{F}_i \cap U_i) \). Let \( x = \lim_{i \in \omega} x_i \) and \( \alpha = \sup_{i \in \omega} \text{rk}(\widehat{F}_i) \). It easy to see that \( D_{\alpha}(\widehat{F}_i) = \{x\} \) and clearly, \( F \subseteq S(X) \). So for every \( i \in \omega \), \( F_i \prec \widehat{F}_i + 1 \prec F \).

**Corollary 2.8.** Let \( X \) be a CO space with a nonempty kernel. Suppose also that \( X \) is collectionwise Hausdorff, strongly Hausdorff for convergent sequences and sequentially compact. Then for every \( F \in S(X) \), \( \text{rk}(F) < \Omega(X) \).

**Proof** Suppose by contradiction that \( F \in S(X) \) and \( \text{rk}(F) = \Omega(X) \). By Proposition 2.5, there is a sequence \( F_1, F_2, \ldots \) of members of \( S(X) \) such that \( |D_{\Omega(X)}(F_i)| = i \). By Proposition 2.7, there is \( H \in S(X) \) such that \( \text{rk}(H) > \Omega(X) \). A contradiction, so \( \text{rk}(F) < \Omega(X) \) for every \( F \in S(X) \).

If \( \ker(X) \neq \emptyset \) and \( \text{rk}(F) < \Omega(X) \) for every \( F \in S(X) \), then we say that \( \Omega(X) \) is not attained in \( X \).

We next define the notion of a good point and prove the following statement for non-scattered members \( X \) of \( K_{\text{CH}} \).

**3. If \( \Omega(X) \) is not attained in \( X \), then the set of good points of \( X \) is perfect.**

Let \( X \) be a compact Hausdorff space. A member \( x \in X \) is called a **good point** of \( X \) if for every \( \alpha < \Omega(X) \) and \( U \in \text{Nbr}(x) \) there is \( F \in S(X) \) such that \( D_{\alpha}(F) \cap U \neq \emptyset \). Note that if \( \Omega(X) \) is not attained, then it is a limit ordinal.

We shall show that if \( \Omega(X) \) is not attained, then the set of good points is a nonempty perfect set. The existence of a good point is a trivial consequence of the compactness of \( X \). It is also trivial that the set of good points is closed. So we have the following fact.
**Proposition 2.9.** Let $X$ be a compact Hausdorff space with a nonempty kernel. Then the set of good points of $X$ is closed and nonempty.

The following proposition is well-known and easy to prove. Recall that according to our definition of scatteredness, a scattered space is compact Hausdorff.

**Proposition 2.10.** Let $Y$ be a scattered space, $X$ be a Hausdorff space and $g : Y \to X$ be a continuous surjective function. Then $X$ is scattered.

There is another property of members of $K_{CH}$ that we now establish. Let $\lambda$ be an infinite cardinal, $A \subseteq X$ and $x \in X$. Call $x$ a $\lambda$-accumulation point of $A$ if $|U \cap A| = \lambda$ for every $U \in \text{Nbr}(x)$. A linear ordering $\langle L, < \rangle$ is $\lambda$-dense, if $|L| > 1$ and for every $a < b$ in $L$, $|(a, b)| = \lambda$.

**Proposition 2.11.** (a) Let $\lambda$ be an infinite regular cardinal and $\langle L, < \rangle$ be a linear ordering of power $\lambda$. Then either $L$ has a subset of order type $\lambda$ or $\lambda^*$, or $L$ has a $\lambda$-dense subset.

(b) Let $\alpha$ be a successor ordinal equipped with its order topology and $g : \alpha \to X$ be a continuous surjective function. Then $|\text{Is}(X)| = |X|$.

(c) Let $X \in K_{CH}$. Suppose that $A \subseteq X$ and $\lambda := |A|$ is an infinite regular cardinal. Then either there is $B \subseteq A$ such that $|B| = \lambda$, $B$ is relatively discrete and $\text{cl}(B)$ is scattered, or $A$ has at least two $\lambda$-accumulation points.

**Proof** (a) Define an equivalence relation on $L$ as follows: $a \sim b$ if the open interval whose endpoints are $a$ and $b$ has cardinality $\lambda$. If there is an equivalence class of cardinality $\lambda$, then that equivalence class contains an increasing or decreasing sequence of type $\lambda$. If every equivalence class has cardinality $\lambda$, then the chain of equivalence classes is $\lambda$-dense.

(b) For every $x \in \text{Is}(X)$ let $\beta_x \in g^{-1}(x)$. Define $B = \{\beta_x \mid x \in \text{Is}(X)\}$ and $C = \text{cl}(B)$. Then

$$g(\text{cl}(B)) = \text{cl}(g(B)).$$

As $X$ is a continuous image of a scattered space, $X$ must be scattered. In
particular, Is($X$) is dense in $X$. So from (1) we conclude that $g(C) = X$. Either $C$ or $C$ minus its maximum have the same order type as $B$, so in particular, $|C| = |B|$. We thus have that $|X| \leq |C| = |B| = |\text{Is}(X)|$.

(c) Suppose that $X$, $\lambda$ and $A$ are as in Part (c), and let $\langle L, < \rangle$ be a compact linear ordering and $h : L \to X$ be continuous and surjective. For every $a \in A$ let $\ell_a \in h^{-1}(a)$ and let $M = \{ \ell_a \mid a \in A \}$. By Part (a), either (i) $M$ contains an increasing or decreasing sequence of type $\lambda$ or (ii) $M$ contains a $\lambda$-dense subset. Assume first that (i) happens. We may then assume that $M$ is an increasing sequence of type $\lambda$. Let $N = \text{cl}^L(M)$. Then $N$ is a compact interval space, and $N$ is scattered. Let $C = h(N)$. By Proposition 2.10, $C$ is a scattered space. If $b \in C - A$, then there is $n \in \text{cl}(M) - M$ such that $h(n) = b$. Hence since $h|_M$ is $1$–$1$, $b \in \text{acc}(A)$. It follows that $\text{Is}(C) \subseteq A$. By Part (b), $|\text{Is}(C)| = \lambda$. Hence $\text{Is}(C)$ is a relatively discrete subset of $A$ of cardinality $\lambda$ and its closure is scattered. That is, $B := \text{Is}(C)$ fulfills the requirements of the proposition.

Suppose next that (ii) happens. We may assume that $M$ is $\lambda$-dense. It is trivial that if $n \in \text{acc}(M)$, then $n$ is a $\lambda$-accumulation point of $M$. Since $h|_M$ is $1$–$1$, $h(n)$ is then a $\lambda$-accumulation point of $h(M) = A$. We have thus verified that

(1) for every $n \in \text{acc}(M)$, $h(n)$ is a $\lambda$-accumulation point of $A$.

Next we notice that

(2) $\text{acc}(A) \subseteq h(\text{acc}(M))$.

Let $a \in \text{acc}(A)$. Set $A' = A - \{ a \}$ and $M' = M - h^{-1}(\{ a \})$. Then $h(M') = A'$ and hence $\text{cl}(A') = h(\text{cl}(M'))$. Clearly, $a \in \text{acc}(A') \subseteq \text{cl}(A')$ and thus $a \in h(\text{cl}(M'))$. But $a \not\in h(M')$. So $a \in h(\text{acc}(M')) \subseteq h(\text{acc}(M))$.

It follows from (1) and (2) that every accumulation point of $A$ is a $\lambda$-accumulation point. If $A$ has at least two accumulation points, then the requirements of the proposition are fulfilled. Otherwise $A$ has exactly one accumulation point, which means that $\text{cl}(A)$ is homeomorphic to the one
point compactification of a discrete space of cardinality $\lambda$. If this happens, then $A - \text{acc}(A)$ is relatively discrete and $\text{cl}(A - \text{acc}(A))$ is scattered. So the requirements of the proposition are again fulfilled.

For a compact Hausdorff space $X$, let $\text{Good}(X)$ denote the set of good points of $X$. If $X$ is a scattered space and $x \in X$, then the rank of $x$ in $X$ is denoted by $\text{rk}^X(x)$. Note that if $F \in \text{Nbr}_X(x)$, then $\text{rk}^F(x) = \text{rk}^X(x)$.

**Proposition 2.12.** Let $X$ be a strongly Hausdorff compact space. Suppose also that $(\ast)$ for every $A \subseteq X$, if $\lambda := |A|$ is an infinite regular cardinal, then either $A$ has at least two $\lambda$-accumulation points, or there is $B \subseteq A$ such that $|B| = \lambda$, $B$ is relatively discrete and $\text{cl}(B)$ is scattered.

Assume further that $\ker(X) \neq \emptyset$ and that $\Omega(X)$ is not attained. Then $\text{Good}(X)$ is a nonempty perfect set.

**Proof** By Proposition 2.9, $\text{Good}(X) \neq \emptyset$. So suppose by contradiction that $x$ is a good point of $X$, $U' \in \text{Nbr}(x)$ and $\text{Good}(X) \cap U' = \{x\}$. Let $U \in \text{Nbr}(x)$ be such that $\text{cl}(U) \subseteq U'$. Since $\Omega(X)$ is not attained, $\Omega(X)$ is a limit ordinal. Let $\lambda = \text{cf}(\Omega(X))$ and $\{\alpha_i \mid i < \lambda\}$ be a strictly increasing sequence of ordinals converging to $\Omega(X)$. For every $i < \lambda$ let $F_i \in S(X)$ be such that $\text{rk}(F_i) = \alpha_i$ and $F_i \subseteq U$ and choose $x_i \in D_{\alpha_i}(F_i)$. It may happen that $|\{x_i \mid i < \lambda\}| < \lambda$. In that case we show that $\{x_i \mid i < \lambda\}$ can be replaced by another sequence $\{x'_i \mid i < \lambda\}$ such that $|\{x'_i \mid i < \lambda\}| = \lambda$. So suppose that $|\{x_i \mid i < \lambda\}| < \lambda$. Then by taking a subsequence we may assume that $x_i = x_j$ for every $i, j < \lambda$. Then $x_0$ is a good point, and since $x$ is the only good point in $U$, $x_0 = x$. For every $i < \lambda$ let $x'_i \in D_{\alpha_i}(F_{i+1}) - D_{\alpha_{i+1}}(F_{i+1})$.

So $x'_i \neq x_{i+1} = x$. It therefore follows from the above argument that for every $i$, $|\{j \mid x'_j = x'_i\}| < \lambda$. So $|\{x'_i \mid i < \lambda\}| = \lambda$. We may thus assume that $\{x_i \mid i < \lambda\}$ is 1–1. We apply $(\ast)$ to $A := \{x_i \mid i < \lambda\}$. Every $\lambda$-accumulation point of $A$ is a good point and it belongs to $U'$. So since
$|U \cap \text{Good}(X)| = 1$, it follows that $A$ has at most one $\lambda$-accumulation point. So by $(\ast)$ there is $B \subseteq A$ such that $|B| = \lambda$, $B$ is relatively discrete and $\text{cl}(B)$ is scattered. Let $B = \{x_{i(j)} \mid j < \lambda\}$ and $U = \{U_j \mid j < \lambda\}$ be a strong Hausdorff system for $B$. We may assume that $\text{cl}(U_j) \cap \text{cl}(U_{j'}) = \emptyset$ for every $j \neq j'$. For every $j < \lambda$ define $\hat{F}_j = F_{i(j)} \cap \text{cl}(U_j)$ and define $F = \text{cl}(\bigcup_{j<\lambda} \hat{F}_j)$. Clearly $F \subseteq \text{ker}(X)$. Also, since $\text{cl}(B)$ is scattered and $U$ is a strong Hausdorff system, it follows that $F$ is scattered. So $F \in S(X)$. For every $j < \lambda$, $\text{rk}^F(x_{i(j)}) = \text{rk}^{\hat{F}_j}(x_{i(j)}) = \text{rk}^{F_{i(j)}}(x_{i(j)}) = \alpha_{i(j)}$. It follows that $\text{rk}(F) \geq \Omega(X)$. A contradiction to the non-attainment of $\Omega(X)$.

4. Sets which code ordinals.

We shall define the notion of a code of an ordinal. Codes are certain compact Hausdorff spaces which code ordinals. Two other notions are to be defined: the “perfect end” of a compact Hausdorff space $F$ — this is a certain nonempty closed subset of $F$, and the notion of a “demonstrative subset” of a compact Hausdorff space $X$. Two facts about codes are important:

- If $F$ and $H$ are codes for two different ordinals, $x$ belongs to the perfect end of $F$, and $U \in \text{Nbr}(x)$, then $U$ is not homeomorphic to any open subset of $H$, (Proposition 2.17).

- If $X \in K_{CH}$, $\text{ker}(X) \neq \emptyset$, $\Omega(X)$ is not attained and $0 < \alpha < \Omega(X)$, then any neighborhood of a member of $\text{Good}(X)$ contains a demonstrative set which is an $(\alpha + 1)$-code, (Lemma 2.20).

We first verify the following property of members of $K_{CH}$.

**Proposition 2.13.** Let $X \in K_{CH}$ and $A \subseteq X$ be relatively discrete. Suppose that $\text{cl}(A)$ is not scattered. Then there is $B \subseteq A$ such that

$$\text{acc}(B) = \text{ker}(\text{cl}(A)).$$

**Proof** Let $\langle L, < \rangle$ be a linear ordering and $g : L \to X$ be continuous and surjective. For every $a \in A$ let $\ell_a \in g^{-1}(a)$ and define $L_0 = \text{cl}(\{\ell_a \mid a \in A\})$ and $g_0 = g|L_0$. So $g_0 : L_0 \to \text{cl}(A)$ and $g_0^{-1}(A)$ is topologically dense in $L_0$. 

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We may thus assume that \(g : L \to \text{cl}(A)\) and \(g^{-1}(A)\) is topologically dense in \(L\).

Let \(U = L - g^{-1}(\ker(\text{cl}(A)))\). Then \(U\) is an open subset of \(L\). Let \(\mathcal{I}\) be the partition of \(U\) into maximal convex subsets of \(L\). Then every member of \(\mathcal{I}\) is an open interval of \(L\). Since \(g^{-1}(A)\) is topologically dense in \(L\), \(g^{-1}(A) \cap I \neq \emptyset\) for every \(I \in \mathcal{I}\). Choose \(m_I \in I \cap g^{-1}(A)\) and set \(M = \{m_I \mid I \in \mathcal{I}\}\) and \(B = g(M)\). Clearly, \(\text{acc}(M) \cap U = \emptyset\). So \(\text{cl}(M) \subseteq M \cup g^{-1}(\ker(\text{cl}(A)))\).

Since \(L\) is compact, \(g(cl(M)) = cl(B)\). Hence \(cl(B) \subseteq B \cup \ker(\text{cl}(A))\). Since \(\text{acc}(B) \cap B = \emptyset\), it follows that \(\text{acc}(B) \subseteq \ker(\text{cl}(A))\).

Let \(x \in \ker(\text{cl}(A))\), and assume by contradiction that \(x \notin \text{acc}(B)\). Since \(A\) is relatively discrete, we have \(x \notin A\), and in particular, \(x \notin B\). So \(x \notin \text{cl}(B)\). This implies that \(g^{-1}(x) \cap \text{acc}(M) = \emptyset\). Then every \(y \in g^{-1}(x)\) has an open neighborhood \(W_y\) such that \(\{I \in \mathcal{I} \mid I \cap W_y \neq \emptyset\}\) is finite. Note that \(g^{-1}(A) \subseteq U = \bigcup \mathcal{I}\). So \(\bigcup \mathcal{I}\) is dense in \(L\). Using the facts that \(\bigcup \mathcal{I}\) is dense in \(L\) and that \(y \notin \bigcup \mathcal{I}\) we conclude that there are \(I_y, J_y \in \mathcal{I}\) such that \(y\) is the right endpoint of \(I_y\) and the left endpoint of \(J_y\). Let \(V_y = I_y \cup \{y\} \cup J_y\). Define \(V = \bigcup \{V_y \mid y \in g^{-1}(x)\}\). Since \(g^{-1}(x) \subseteq V\) and \(L\) is compact, it follows that \(g(V)\) is a neighborhood of \(x\). But \(V \subseteq (L - g^{-1}(\ker(\text{cl}(A)))) \cup g^{-1}(x)\), so \(g(V) \cap \ker(\text{cl}(A)) = \{x\}\). This means that \(x\) is an isolated point of \(\ker(\text{cl}(A))\), a contradiction. It follows that \(x \in \text{acc}(B)\), so \(\ker(\text{cl}(A)) \subseteq \text{acc}(B)\).

**Proposition 2.14.** Let \(X\) be a compact Hausdorff perfect space. Then there is a relatively discrete subset \(A \subseteq X\) such that \(\text{cl}(A)\) contains a nonempty perfect set.

**Proof** We show that \([0, 1]\) is a continuous image of \(X\). If \(X\) is not 0-dimensional let \(x, y\) be two distinct points in the same connected component of \(X\) and \(g : X \to [0, 1]\) be a continuous function such that \(g(x) = 0\) and \(g(y) = 1\). Then \(\text{Rng}(g) = [0, 1]\). If \(X\) is 0-dimensional and perfect, then the Cantor set is a continuous image of \(X\) and \([0, 1]\) is a continuous image of the Cantor set. So there is a continuous surjective \(g\) from \(X\) to \([0, 1]\).
Let \( B \subseteq [0, 1] \) be a relatively discrete set such that \( \text{cl}(B) - B \) is perfect. For every \( b \in B \) choose \( x_b \in g^{-1}(b) \) and define \( A = \{ x_b \mid b \in B \} \). It follows from the relative discreteness of \( B \) that \( A \) is also relatively discrete. We have shown in Proposition 2.10 that a continuous image of a scattered space is scattered. As \( \text{cl}(B) \) is not scattered, \( \text{cl}(A) \) cannot be scattered. \( \square \)

In order to define codes, we introduce the notion of the perfect derivative of a compact Hausdorff space \( X \). For a compact Hausdorff space \( X \) define

\[
\text{PD}(X) := X - \text{Is}(X) - \text{Po}(X),
\]

\[
\text{PD}_0(X) := X, \quad \text{PD}_{\alpha+1}(X) := \text{PD}(\text{PD}_\alpha(X))
\]

and if \( \delta \) is a limit ordinal, then define

\[
\text{PD}_\delta(X) := \bigcap_{\alpha<\delta} \text{PD}_\alpha(X).
\]

The perfect rank of \( X \) is defined by \( \text{prk}(X) = \max(\{ \alpha \mid \text{PD}_\alpha(X) \neq \emptyset \}) \) and the perfect end is defined by \( \text{Pend}(X) = \text{PD}_{\text{prk}(X)}(X) \). Note that \( \text{Pend}(X) \) is the union of a finite set of isolated points and a perfect set. Each may be empty but not both.

For \( x \in X \), the property: “\( x \) belongs to \( \text{PD}_\alpha(X) \)” is a local property. This is expressed in the next observation which is trivial and is not proved.

**Proposition 2.15.** Let \( Z \) be a compact Hausdorff space. Suppose that \( U \subseteq G \subseteq Z \), \( U \) is open and \( G \) is closed. Then \( U \cap \text{PD}_\alpha(G) = U \cap \text{PD}_\alpha(Z) \) for every ordinal \( \alpha \).

**Definition 2.16.** Let \( \alpha \geq 2 \). A compact Hausdorff space \( F \) is called an \( \alpha \)-code if

- (C1) \( \text{prk}(F) = \alpha \),

- (C2) \( \text{Po}(\text{PD}_\beta(F)) = \emptyset \) for every \( 0 < \beta < \alpha \),

- (C3) \( \text{Pend}(F) \) is perfect.

A set which is an \( \alpha \)-code for some ordinal \( \alpha \), is called a code. \( \blacksquare \)
Proposition 2.17. Suppose that $F$ is an $\alpha$-code, $H$ is a $\beta$-code and $\alpha \neq \beta$. If $x \in \text{Pend}(F)$, then there are no $U \in \text{Nbr}^F(x)$ and $V \in \tau^H$ such that $U \cong V$.

Proof Suppose by contradiction that $F$ is an $\alpha$-code, $H$ is a $\beta$-code, $\alpha \neq \beta$ and there are $x \in \text{Pend}(F)$, $U' \in \text{Nbr}^F(x)$, $V' \in \tau^H$ and $\varphi$ such that $\varphi : U' \cong V'$. Note that by the definition of codes, $\alpha > 1$. Let $U$ be an open subset of $U'$ such that $F_1 := \text{cl}(U) \subseteq U'$, and set $V = \varphi(U)$ and $H_1 = \varphi(F_1)$. Note that $\text{Pend}(F) = \text{PD}_\alpha(F)$ and hence $\text{PD}_\alpha(F)$ is a nonempty perfect set. By Proposition 2.15, $\text{PD}_\alpha(F_1) \cap U = \text{PD}_\alpha(F) \cap U$. So $\text{PD}_\alpha(F_1) \cap U \neq \emptyset$ and $\text{PD}_\alpha(F_1) \cap U$ has no isolated points. Now, since $\varphi \upharpoonright F_1$ is a homeomorphism between $F_1$ and $H_1$ which takes $U$ to $V$, we have that $\text{PD}_\alpha(H_1) \cap V = \varphi(\text{PD}_\alpha(F_1) \cap U)$. It follows that

(†) $\text{Po}(\text{PD}_\alpha(H_1) \cap V) \neq \emptyset$.

By Proposition 2.15, $\text{PD}_\alpha(H_1) \cap V = \text{PD}_\alpha(H) \cap V$. The only ordinal $\gamma > 1$ for which $\text{Po}(\text{PD}_\gamma(H))$ is nonempty is $\beta$ but $\alpha > 1$, and is different from $\beta$. So $\text{Po}(\text{PD}_\alpha(H_1) \cap V) = \emptyset$. This contradicts (†), hence the Proposition is proved.

Definition 2.18. Let $\Omega$ be a limit ordinal and $F$ be a compact Hausdorff space. We say that $F$ is $\Omega$-demonstrative if

(D1) $\Omega(F) = \Omega$ and $\Omega$ is not attained in $F$;

(D2) $\text{Pend}(F) \subseteq \text{Good}(F)$.

In the next lemma we use the following properties of members of $K_{\text{CIH}}$.

(TH1) $X$ is tightly Hausdorff.

(TH2) For every relatively discrete subset $A$ of $X$, if $\text{cl}(A)$ is not scattered, then there is $B \subseteq A$ such that $\text{acc}(B)$ is a nonempty perfect set.
(TH3) For every $A \subseteq X$, if $\lambda := |A|$ is an infinite regular cardinal, then either $A$ has at least two $\lambda$-accumulation points, or there is $B \subseteq A$ such that $|B| = \lambda$, $B$ is relatively discrete and $\text{cl}(B)$ is scattered.

Observe the following fact.

**Proposition 2.19.** For any of the properties (TH1) - (TH3), if $Y$ is a closed subspace of a space having the property, then $Y$ has the same property.

We shall later use two other properties of members of $K_{\text{CH}}$.

- $X$ is sequentially compact.

This property is used in showing that $\Omega(X)$ is not attained – a fact which is assumed in the next lemma. See Proposition 2.7 and Corollary 2.8. Another (and last) property to be used is

- For every infinite cardinal $\lambda$ and a closed subset $F \subseteq X$: if $|F| = 2^{\lambda^+}$, then there is a scattered subspace $H \subseteq F$ such that $|H| = \lambda^+$.

This is proved in Proposition 2.28 and it is used at the end of the proof of Theorem 1.2 where it is shown that $|\text{Good}(X)|$ cannot be much larger than $|\Omega(X)|$.

Let $K_{\text{TH}}$ be the class of all compact Hausdorff spaces that have Properties (TH1) - (TH3). Note that by Lemma 2.2, Proposition 2.13 and Proposition 2.12, $K_{\text{CH}} \subseteq K_{\text{TH}}$.

**Lemma 2.20.** Let $X \in K_{\text{TH}}$. Suppose that ker $(X) \neq \emptyset$ and that $\Omega(X)$ is not attained in $X$. Let $g \in \text{Good}(X)$ and $V \in \text{Nbr}(g)$. Then for every $\alpha \in \Omega(X) - \{0\}$, $V$ contains an $\Omega(X)$-demonstrative $(\alpha + 1)$-code.

**Proof** Let $H$ be a closed neighborhood of $g$ such that $H \subseteq V$. Clearly, we may replace $V$ by $H$. Also, $H \in K_{\text{TH}}$ (this follows from 2.19), ker $(H) \neq \emptyset$, $\Omega(H) = \Omega(X)$, and thus $\Omega(H)$ is not attained in $H$. So we may replace $H$ by $X$ and prove that $X$ contains an $\Omega(X)$-demonstrative $(\alpha + 1)$-code.
Property (TH3) is just \((*)\) of Proposition 2.12 and (TH1) is stronger than being strongly Hausdorff. So 2.12 implies that \(\text{Good}(X)\) is a nonempty perfect set.

By Proposition 2.14, there is a relatively discrete subset \(A \subseteq \text{Good}(X)\) such that \(\text{cl}(A)\) contains a nonempty perfect set, and by (TH2) we may assume that \(\text{acc}(A)\) is perfect and nonempty.

By (TH1), \(A\) has a tight Hausdorff system \(U = \{U_a \mid a \in A\}\). For every \(a \in A\) let \(F_a\) be a subset of \(U_a\) such that \(F_a\) is compact and scattered, \(a \not\in F_a\) and \(\text{rk}(F_a) = \alpha\). The existence of \(F_a\) is assured by the goodness of \(a\). Also, choose a closed neighborhood \(H'_a\) of \(a\) such that \(H'_a \subseteq U_a\) and \(H'_a \cap F_a = \emptyset\), and define \(H_a = H'_a \cap \ker(X)\). Set \(S = \bigcup_{a \in A} F_a\) and \(T = \bigcup_{a \in A} H_a\) and define \(C = \text{cl}(S \cup T)\). We shall show that \(C\) is an \(\Omega(X)\)-demonstrative \((\alpha + 1)\)-code.

We start with the fact that \(C\) is a code. Clearly,

(1) \(F_a, H_a \in \text{Clop}(C)\) for every \(a \in A\).

Since \(U\) is tight and \(F_a \cup H_a \subseteq U_a\) for every \(a \in A\), it follows that

(2) \(\text{acc}\left(\{F_a \cup H_a \mid a \in A\}\right) = \text{acc}\left(\{F_a \mid a \in A\}\right) = \text{acc}\left(\{H_a \mid a \in A\}\right) = \text{acc}(A)\).

Also recall that

(3) \(\text{acc}(A)\) is perfect.

and

(4) \(F_a\) is scattered with rank \(\alpha\) and \(H_a\) is perfect.

From (2) and the tightness of \(U\) it follows that

\[C = (\bigcup_{a \in A} F_a) \cup (\bigcup_{a \in A} H_a) \cup \text{acc}(A),\]

and Facts (1) - (4) imply that \(\text{Po}(C) = \bigcup_{a \in A} H_a\) and that \(\text{Is}(C) = \bigcup_{a \in A} \text{Is}(F_a)\).

It thus follows that

\[\text{PD}(C) = (\bigcup_{a \in A} \text{D}(F_a)) \cup \text{acc}(A).\]

From the tightness of \(U\) and Fact (4) it now follows that
(5) for every $0 < \beta \leq \alpha$, $PD_\beta(C) = (\bigcup_{a \in A} D_\beta(F_a)) \cup acc(A)$ and $PD_{\alpha+1}(C) = acc(A)$.

Since $acc(A)$ is perfect, we conclude that $PD_{\alpha+2}(C) = \emptyset$ and hence (i) $prk(C) = \alpha + 1$. From the first part of (5) and from (4) and (1) it follows that (ii) for every $0 < \beta < \alpha + 1$, $Po(PD_\beta(C)) = \emptyset$. Finally, from the second part of (5) and from (i) we have (iii) $Pend(C) = PD_{\alpha+1}(C) = acc(A)$, and from (3) we conclude that (iv) $Pend(C)$ is perfect. Facts (i), (ii) and (iv) are Clauses (C1)-(C3) in the definition of an $(\alpha + 1)$-code. We have thus proved that $C$ is an $(\alpha + 1)$-code.

We show that $C$ is $\Omega(X)$-demonstrative. Let $a \in A$, then $H_a$ is the intersection of a closed neighborhood of $a$ with $ker(X)$. Since $a \in Good(X)$, it follows that $\Omega(H_a) = \Omega(X)$, and that $a \in Good(H_a)$. Since $H_a \subseteq C \subseteq X$, we also have that $\Omega(H_a) \leq \Omega(C) \leq \Omega(X)$. Hence, $\Omega(C) = \Omega(X)$, and since $\Omega(X)$ is not attained in $X$, it is not attained in $C$. We have shown that $\Omega(C) = \Omega(X)$ and $\Omega(X)$ is not attained in $C$.

That is, $C$ fulfills Clause (D1) in the definition of demonstrative sets.

We have also shown above that for every $a \in A$, $a \in Good(C)$. That is, $A \subseteq Good(C)$. In Fact (iii) we proved that $Pend(C) = acc(A)$. So $Pend(C) \subseteq acc(Good(C))$. Since Good($C$) is closed, $Pend(C) \subseteq Good(C)$. So Clause (D2) holds. We have shown that $C$ is $\Omega(X)$-demonstrative.

5. Coding subsets of $\Omega(X)$ and proliferation systems.

The assumption “$X$ is a non-scattered CO space” is contradictory. To reach this contradiction, we show that $|Good(X)|$ is much larger than $|\Omega(X)|$. First we code subsets of $\Omega(X)$ by subsets of $Good(X)$. This coding implies that $|Good(X)| \geq 2^{\Omega(X)}$. Next we code sets of subsets of $\Omega(X)$ by subsets of $Good(X)$. This leads to the conclusion that $|Good(X)| \geq 2^{2^{\Omega(X)}}$. We repeat this procedure twice more and then reach a contradiction. The above three steps use an identical argument which in the first case is applied to the set of $\alpha$-codes, and in the second, to the set of codes of subsets of $\Omega(X)$. 

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The notion which provides the unified argument is called a “proliferation system”, and the conclusion of the iterated use of this argument is stated in Corollary 2.25. It will be evident that for any limit ordinal \( \Omega \) the set of \( \Omega \)-demonstrative codes is a proliferation system, and that if \( X \) is a non-scattered CO space, then all these codes are realized in \( X \). This makes Corollary 2.25 applicable to \( X \).

A pair \( \hat{X} = \langle X, e \rangle \) consisting of a topological space \( X \) and a point \( e \in X \) is called a pointed space. Let \( P = \{ \hat{X}_t \mid t \in T \} \) be an indexed family such that for every \( t \in T \), \( \hat{X}_t \) is a class of pointed spaces closed under homeomorphisms. Then \( P \) is called a type system. By “closed under homeomorphisms” we mean that if \( \langle X, e \rangle \in \hat{X}_t \) and \( \varphi \) is a homeomorphism between \( X \) and \( Y \), then \( \langle Y, \varphi(e) \rangle \in \hat{X}_t \). Denote \( T \) by \( T_P \) and set \( \hat{X}_P := \bigcup_{t \in T} \hat{X}_t \).

Let \( \mathcal{X} \) be a class of topological spaces and \( Y \) be a topological space. We say that \( \mathcal{X} \) occurs in \( Y \) if there is \( X \subseteq Y \) such that \( X \in \mathcal{X} \). We say that a class \( \hat{X} \) of pointed spaces occurs in \( Y \) if there are \( X \subseteq Y \) and \( e \in X \) such that \( \langle X, e \rangle \in \hat{X} \).

**Definition 2.21.** (a) A type system \( P = \{ \hat{X}_t \mid t \in T \} \) is called a proliferation system (P-system) if the following hold.

(P1) \( T \) is infinite and every member of \( \mathcal{X}_P \) is compact Hausdorff.

(P2) Suppose that \( s, t \in T \) are distinct, \( \langle F, d \rangle \in \hat{X}_s \), \( H \in \mathcal{X}_t \) and \( V \in \text{Nbr}^F(d) \). Then there is no \( U \in \tau^H \) such that \( V \simeq U \).

(P3) For every \( \langle F, d \rangle \in \hat{X}_P \), \( V \in \text{Nbr}^F(d) \) and \( t \in T \) there is \( Y \in \mathcal{X}_t \) such that \( Y \subseteq V \).

Note the definition of a proliferation system does not exclude the possibility that for every \( t \in T_P \), \( \hat{X}_t = \emptyset \). However, by (P3), if for some \( t \in T_P \), \( \hat{X}_t \neq \emptyset \), then for all \( t \in T_P \), \( \hat{X}_t \neq \emptyset \).

(b) Let \( P \) be a P-system and \( X \) be a compact Hausdorff space. We say that \( P \) occurs in \( X \) if \( \mathcal{X}_P \) occurs in \( X \).
(c) Let \( \mathcal{P} = \{ \hat{X}_t \mid t \in T \} \) be a P-system and \( \emptyset \neq \Gamma \subseteq T_{\mathcal{P}} \). A pointed compact Hausdorff space \( \langle F, d \rangle \) is called a \( \Gamma \)-marker if there is a family \( \mathcal{F} \) of subsets of \( F \) and \( \{ d_{F'} \mid F' \in \mathcal{F} \} \) such that

\begin{align*}
\text{(M1)} & \quad \mathcal{F} \subseteq \text{Clop}(F), \\
\text{(M2)} & \quad \mathcal{F} \text{ is a tight family,} \\
\text{(M3)} & \quad \text{for every } F' \in \mathcal{F}, \langle F', d_{F'} \rangle \in \bigcup_{t \in \Gamma} \hat{X}_t, \\
\text{(M4)} & \quad \text{for every } V \in \text{Nbr}(d) \text{ and } t \in \Gamma \text{ there is } F' \in \mathcal{F} \cap \hat{X}_t \text{ such that } d_{F'} \in V, \\
\text{(M5)} & \quad F = \text{cl}(\bigcup \mathcal{F}).
\end{align*}

\( \mathcal{F} \) is called a filler for \( \langle F, d \rangle \).

\((c)\) Denote the powerset of a set \( A \) by \( P(A) \). Suppose that \( P = \{ \hat{X}_t \mid t \in T \} \) is a P-system. For every \( \Gamma \in P(T) - \{ \emptyset \} \) define

\[ \hat{\mathcal{M}}^P_\Gamma = \{ \langle X, d \rangle \mid \langle X, d \rangle \text{ is a } \Gamma \text{-marker} \}. \]

For \( \Gamma \subseteq P(T) - \{ \emptyset \} \) define

\[ Q^P_\Gamma = \{ \hat{\mathcal{M}}^P_\Gamma \mid \Gamma \in \Gamma \}. \]

In general \( Q^P_\Gamma \) need not be a P-system, but we shall see that if \( \Gamma \) is an infinite set of pairwise incomparable subsets of \( T \) with the same cardinality, then \( Q^P_\Gamma \) is a P-system.

**Proposition 2.22.** (a) Let \( \mathcal{P} \) be a P-system and \( \Gamma, \Delta \in P(T_{\mathcal{P}}) - \{ \emptyset \} \), and assume that \( \Gamma \nsubseteq \Delta \). Suppose that \( \langle F, d \rangle \) and \( \langle H, e \rangle \) are respectively a \( \Gamma \)-marker and a \( \Delta \)-marker. Then there do not exist \( U \in \text{Nbr}^F(d) \) and \( V \in \tau^H \) such that \( U \cong V \).

(b) Let \( \mathcal{P} \) be a P-system and \( \Gamma \subseteq P(T_{\mathcal{P}}) - \{ \emptyset \} \) be infinite. Suppose that for every distinct \( \Gamma, \Delta \in \Gamma \), \( |\Gamma| = |\Delta| \) and \( \Gamma \nsubseteq \Delta \). Then \( Q^P_\Gamma \) is a P-system.
Proof (a) Let $\Gamma$, $\Delta$, $\langle F, d \rangle$ and $\langle H, e \rangle$ be as specified in (a). Suppose by way of contradiction that there are $U \in \text{Nbr}^F(d)$ and $V \in \tau^H$ such that $U \cong V$. We may assume that $U = V$. So for every $A \subseteq U$, $\tau^F|A = \tau^U|A = \tau^H|A$, and $A$ is open in $F$ iff $A$ is open in $U$ iff $A$ is open in $H$.

Let $F$ and $H$ be fillers for $\langle F, d \rangle$ and $\langle H, e \rangle$ respectively. Let $t \in \Gamma - \Delta$. Let $F' \in \mathcal{X}_t \cap F$ be such that $d_{F'} \in U$ and $F_0 = F' \cap U$. So $d_{F'} \in F_0$, and since $F'$ is open in $F$, it follows that $F_0$ is open in $U$, and hence in $H$. That is, $F_0 \in \text{Nbr}^H(d_{F'})$. Suppose by contradiction that $d_{F'} \in \bigcup \mathcal{H}$. Let $H' \in \mathcal{H}$ be such that $d_{F'} \in H'$. There is $s \in \Delta$ such that $H' \in \mathcal{X}_s$. Then $s \neq t$. Since $H'$ is open in $H$, we have that $H' \cap F_0$ is open in $H$, and hence it is open in $F$. This implies that $H' \cap F_0$ is open in $F'$. Hence

$$H' \cap F_0 \in \text{Nbr}^F(d_{F'}).$$

On the other hand,

$$H' \cap F_0 \text{ is open in } H'.$$

Recall that $\langle F', d_{F'} \rangle \in \mathcal{X}_t$ and that $H' \in \mathcal{X}_s$. The last four mentioned facts contradict Property (P2) of $\mathcal{P}$.

It follows that $d_{F'} \in H - \bigcup \mathcal{H}$. Since every member of $\mathcal{H}$ is clopen in $H$, $d_{F'} \in H - \bigcup \{\text{cl}^H(H') \mid H' \in \mathcal{H}\}$. Now, $\text{cl}^H(\bigcup \mathcal{H}) = H$, hence $d_{F'} \in \text{acc}^H(\mathcal{H})$. From the tightness of $\mathcal{H}$ it follows that $d_{F'} \in \text{acc}^H(\{d_{H'} \mid H' \in \mathcal{H}\})$. Recall that $F_0 \in \text{Nbr}^H(d_{F'})$. So there is $H' \in \mathcal{H}$ such that $d_{H'} \in F_0$. Clearly, $F_0 \cap H'$ is open in $H'$ and hence

$$H' \cap F_0 \in \text{Nbr}^{H'}(d_{H'}).$$

$H' \cap F_0$ is open in $H$ and it is a subset of $U$. So it is open in $F$. It follows that

$$H' \cap F_0 \text{ is open in } F'.$$

There is $s \neq t$, such that $\langle H', d_{H'} \rangle \in \mathcal{X}_s$, and on the other hand, $F' \in \mathcal{X}_t$. These facts contradict Property (P2) of $\mathcal{P}$. This proves (a).
(b) Denote $Q_\mathcal{P}$ by $Q$. So $T_Q = \Gamma$ and for every $\Gamma \in \Gamma$, $\hat{X}_\Gamma = \hat{M}_\mathcal{P}$. By definition, every $\Gamma$-marker is compact Hausdorff. So every member of $X_Q$ is compact Hausdorff. Since also, $\Gamma$ is infinite, $Q$ fulfills (P1).

That $Q$ fulfills (P2), was indeed proved in Part (a).

We prove (P3). Let \langle F, d \rangle \in \hat{X}_Q$ and $\Delta \in \Gamma$. There is $\Gamma \in \Gamma$ such that $\langle F, d \rangle \in \hat{X}_\Gamma$. Suppose that $F$ is a filler for $\langle F, d \rangle$. Let $f : \Gamma \to \Delta$ be a bijection. If $F' \in F \cap X_t$, choose $H_{F'} \subseteq F'$ such that $H_{F'} \in X_{f(t)}$ and define $\mathcal{H} = \{H_{F'} \mid F' \in F\}$ and $H = \text{cl}(\bigcup \mathcal{H})$. That $H_{F'}$ exists follows from (P3) applied to $\mathcal{P}$. The tightness of $F$ implies that $d \in \text{cl}(\bigcup \mathcal{H})$. It now follows trivially from the definition of $H$ and $\mathcal{H}$ that $\langle H, d \rangle \in \hat{X}_\Delta$. So $Q$ fulfills (P3).

Note that if $\mathcal{P}$ is a P-system and $\mathcal{P}$ occurs in $X$, then for every $t \in T_\mathcal{P}$, $X_t$ occurs in $X$. This follows from (P3). Suppose that $X$ is a tightly Hausdorff compact CO space and $\mathcal{P}$ is a P-system occurring in $X$. We shall show that $\mathcal{M}_{\mathcal{P}}$ occurs in $X$ for every $\Gamma \in \mathcal{P}(T_\mathcal{P}) - \{\emptyset\}$. In order to show this, we first establish the existence of so-called $\mu$-special $\{t\}$-markers.

Let $X$ be a topological space, $A \subseteq X$, $x \in X$ and $\mu$ be an infinite cardinal. Denote the set of $\mu$-accumulation points of $A$ in $X$ by $\text{acc}_\mu^X(A)$. We use $\text{acc}_\mu(A)$ as an abbreviation of the above. Let $\mathcal{P}$ be a P-system, $t \in T_\mathcal{P}$ and $\langle F, e \rangle$ be a $\{t\}$-marker with a filler $\mathcal{F}$. For every $F' \in \mathcal{F}$ choose $e_{F'}$ such that $\langle F', e_{F'} \rangle \in \hat{X}_t$. We call $\langle F, e \rangle$ a $\mu$-special $\{t\}$-marker if $e \in \text{acc}_\mu(\{e_{F'} \mid F' \in \mathcal{F}\})$.

**Proposition 2.23.** Let $X$ be a compact Hausdorff CO space and $\mathcal{P}$ be a P-system such that $\mathcal{P}$ occurs in $X$.

(a) There is a set $\{\langle G_t, g_t \rangle \mid t \in T_\mathcal{P}\}$ such that

1. for every distinct $s, t \in T_\mathcal{P}$, $g_s \neq g_t$,
2. for every $t \in T_\mathcal{P}$, $\langle G_t, g_t \rangle \in \hat{X}_t$ and $G_t \subseteq X$,
3. $\{g_t \mid t \in T_\mathcal{P}\}$ is relatively discrete.
(b) Suppose that in addition to the above, $X$ is tightly Hausdorff. Then for every $t \in T_P$ there is $K \subseteq X$ and $c \in K$ such that $\langle K, c \rangle$ is a $|T_P|$-special $\{t\}$-marker.

**Proof** Suppose that $P = \{\hat{X}_t \mid t \in T\}$.

(a) Let $\langle F, d \rangle \in \hat{X}_P$ be such that $F \subseteq X$. By (P3), for every $t \in T$ there is $\langle H_t, e_t \rangle \in \hat{X}_t$ such that $H_t \subseteq F$. Let $G_t$ be a clopen subset of $X$ homeomorphic to $H_t$ and let $g_t \in G_t$ be the image of $e_t$ under the homeomorphism between $H_t$ and $G_t$. Suppose by way of contradiction that for some distinct $s, t \in T$, $g_s \in G_t$. Then $G_s \cap G_t \in \text{Nbr}^{G_s}(g_s)$ and $G_s \cap G_t$ is open in $G_t$. This contradicts (P2), so for every distinct $s, t \in T$, $g_s \notin G_t$. We thus have that for every $t \in T$, $G_t$ is open and $G_t \cap \{g_s \mid s \in T\} = \{g_t\}$. This means that $\{g_t \mid t \in T\}$ is relatively discrete and that for every distinct $s, t \in T$, $g_s \neq g_t$. So $\{\langle G_t, g_t \rangle \mid t \in T\}$ is as required.

(b) Denote $\mu = |T|$ and let $A = \{g_t \mid t \in T\}$ be as assured in (a). So $A$ is relatively discrete, $|A| = \mu$ and for every $a \in A$ there is $G \subseteq X$ such that $\langle G, a \rangle \in \hat{X}_P$. It is trivial that in a compact space every set of cardinality $\mu$ has a $\mu$-accumulation point. So let $c$ be a $\mu$-accumulation point of $A$.

Fix $t \in T$. We construct a set $K \subseteq X$ such that $\langle K, c \rangle$ is a $\{t\}$-marker. Let $\{U_a \mid a \in A\}$ be a tight Hausdorff system for $A$. If $a \in A$, then for some $s \in T$, $a = g_s$. Since $g_s \in G_s$ and $\langle G_s, g_s \rangle \in \hat{X}_s$, we may apply (P3) to $\langle G_s, g_s \rangle$. Now, $U_a \in \text{Nbr}(g_s)$, so there is $\langle E_a, e_a \rangle \in \hat{X}_s$ such that $E_a \subseteq U_a \cap G_s$. Let $\mathcal{E} = \{E_a \mid a \in A\}$. Define $K = \text{cl}(\bigcup \mathcal{E})$. In the definition of a $\Gamma$-marker we need to have a choice function $\{d_{F'} \mid F' \in \mathcal{F}\}$. So define $d_{E_a}$ to be $e_a$ for every $a \in A$. It follows trivially from the construction that (M1) - (M5) hold for $\langle K, c \rangle, \mathcal{E}, \{e_a \mid a \in A\}$ and $\{t\}$. That is, $\langle K, c \rangle$ is a $\{t\}$-marker. Recall that $c$ was chosen to be a $\mu$-accumulation point of $A$. Since $\{U_a \mid a \in A\}$ is a tight family, and $a, e_a \in U_a$ for every $a \in A$, it follows that $c$ is a $\mu$-accumulation point of $\{e_a \mid a \in A\}$. This assures that $\langle K, c \rangle$ is $\mu$-special. □

**Lemma 2.24.** Let $X$ be a tightly Hausdorff compact CO space and $P$ be a $P$-system such that $P$ occurs in $X$. Then for every $\Gamma \in \mathcal{P}(T_P) - \{\emptyset\}$, there
are $H \subseteq X$ and $c \in H$ such that $\langle H, c \rangle$ is a $\Gamma$-marker.

**Proof** Let $\mathcal{P} = \{ \mathcal{X}_t | t \in T \}$ and denote $|T|$ by $\mu$. Choose a countable subset $T_0$ of $T$, and for every $t \in T_0$ let $\langle K_t, c_t \rangle$ be a $\mu$-special $\{t\}$-marker such that $K_t$ is a clopen subset of $X$. The existence of a $\mu$-special $\langle K_t, c_t \rangle$ was proved in Proposition 2.23(b), and that $K_t$ may be a clopen set follows from the fact that $X$ is a CO space.

Let $t \in T_0$. Since $\langle K_t, c_t \rangle$ is $\mu$-special, there are a filler $\mathcal{F}_t$ for $\langle K_t, c_t \rangle$ and a set $\{ e_{F'} | F' \in \mathcal{F}_t \}$ such that

1. for every $F' \in \mathcal{F}_t$, $\langle F', e_{F'} \rangle \in \hat{\mathcal{X}}_t$,
2. $c_t \in \text{acc}(\{ e_{F'} | F' \in \mathcal{F}_t \})$.

By Proposition 2.22(a), for every distinct $s, t \in T_0$, $c_s \notin K_t$. This implies that $\{ c_t | t \in T_0 \}$ is infinite and relatively discrete. For every $t \in T_0$ choose $V_t \in \text{Nbr}(c_t)$ in such a way that

3. $\{ V_t | t \in T_0 \}$ is a tight Hausdorff system for $\{ c_t | t \in T_0 \}$.

Define $\mathcal{F}_t' = \{ F' \in \mathcal{F}_t | e_{F'} \in V_t \}$ and $\mathcal{E}_t = \{ e_{F'} | F' \in \mathcal{F}_t' \}$. Also choose $c \in \text{acc}(\{ c_t | t \in T_0 \})$. So

4. $c \in \text{acc}(\{ V_t | t \in T_0 \})$.

Let $\Gamma \in \mathbf{P}(T \cap \mathcal{P}) - \{ \emptyset \}$. We construct $H$ such that $\langle H, c \rangle$ is a $\Gamma$-marker. For every $t \in T_0$ let $f_t : \mathcal{F}_t' \rightarrow \Gamma$ be a surjection. Let $t \in T_0$ and $F' \in \mathcal{F}_t'$. Then $V_t \in \text{Nbr}(e_{F'})$. We use (P3) and the fact that $\langle F', e_{F'} \rangle \in \hat{\mathcal{X}}_F$ in order to conclude that there are $H_{F'}$ and $d_{F'}$ such that

5. $\langle H_{F'}, d_{F'} \rangle \in \hat{\mathcal{X}}_{f_t(F')}$, and $H_{F'} \subseteq V_t \cap F'$.

Let $\mathcal{H} = \{ H_{F'} | t \in T_0$ and $F' \in \mathcal{F}_t' \}$. For $H' = H_{F'} \in \mathcal{H}$ denote $d_{F'}$ by $b_{H'}$. Define $H = \text{cl}(\bigcup \mathcal{H})$. We verify that $\langle H, c \rangle$ is a $\Gamma$-marker, that $\mathcal{H}$ is a filler for $\langle H, c \rangle$ and that $\{ b_{H'} | H' \in \mathcal{H} \}$ is the choice function required in the definition of a $\Gamma$-marker.

That (M1), (M3) and (M5) hold is trivial. We check that (M2) holds. We have to show that $\mathcal{H}$ is a tight family. For $t \in T_0$ set $\mathcal{H}_t = \{ H_{F'} | F' \in \mathcal{F}_t' \}$. Then
(i) \( \mathcal{H} = \bigcup_{t \in T_0} \mathcal{H}_t \).

(ii) For every \( t \in T_0 \), \( \mathcal{H}_t \) is a tight family.

That \( \mathcal{H}_t \) is tight follows from the facts: \( \mathcal{F}'_t \) is tight and \( H_{F'} \subseteq F' \) for every \( F' \in \mathcal{F}'_t \). By (5),

(iii) \( H' \subseteq V_t \) for every \( t \in T_0 \) and \( H' \in \mathcal{H}_t \),

(iv) \( \{ V_t \mid t \in T_0 \} \) is tight.

Facts (i)-(iv) easily imply that \( \mathcal{H} \) is tight. So \( \mathcal{H} \) satisfies (M2).

We next verify (M4). Let \( s \in \Gamma \) and \( W \in \text{Nbr}(c) \). For every \( t \in T_0 \) choose \( F'_t \in \mathcal{F}'_t \) such that \( f_t(F'_t) = s \). Set \( H^0_t = H_{F'_t} \) and \( a_t = b_{H^0_t} \). So \( a_t \in V_t \) and \( \langle H^0_t, a_t \rangle \in \hat{X}_s \).

From the facts:

- \( c \) is an accumulation point of \( \{ V_t \mid t \in T_0 \} \),
- \( a_t \in V_t \) for every \( t \in T_0 \),
- \( \{ V_t \mid t \in T_0 \} \) is tight,

we conclude that \( c \in \text{acc}(\{ a_t \mid t \in T_0 \}) \). So there is \( t_0 \in T_0 \) such that \( a_{t_0} \in W \). Recall that \( \langle H^0_{t_0}, a_{t_0} \rangle \in \hat{X}_s \). Also \( \langle H^0_{t_0}, a_{t_0} \rangle \in \mathcal{H} \). We have thus found \( H' \in \mathcal{H} \) such that \( \langle H', b_{H'} \rangle \in \hat{X}_s \) and \( b_{H'} \in W \). This shows that \( H, \mathcal{H} \) and \( \{ b_{H'} \mid H' \in \mathcal{H} \} \) fulfill (M4).

\[ \square \]

Let \( X \) be a compact Hausdorff space and \( \mathcal{P} \) be a P-system. Define

\[ \text{End}_\mathcal{P}(X) = \{ e \in X \mid \text{There is } F \subseteq X \text{ such that } \langle F, e \rangle \in \hat{X}_\mathcal{P} \}, \]

and \( \text{Good}_\mathcal{P}(X) = \text{cl}(\text{End}_\mathcal{P}(X)) \). For an infinite cardinal \( \mu \) set \( \beth_0(\mu) = \mu \), for every \( n \in \omega \), \( \beth_{n+1}(\mu) = 2^{\beth_n(\mu)} \) and \( \beth_\omega(\mu) = \bigcup_{n \in \omega} \beth_n(\mu) \).

**Corollary 2.25.** Let \( X \) be a tightly Hausdorff compact CO space, \( \mathcal{P} \) be a P-system and \( \mu = |T_\mathcal{P}| \). If \( \mathcal{P} \) occurs in \( X \), then \( |\text{Good}_\mathcal{P}(X)| \geq \beth_4(\mu) \).

**Proof** We prove that \( |\text{Good}_\mathcal{P}(X)| \geq \beth_\omega(\mu) \). Denote \( \mathcal{P} \) and \( T_\mathcal{P} \) by \( \mathcal{P}_0 \) and \( T_0 \) respectively. We define by induction a P-system \( \mathcal{P}_n \). Suppose that \( \mathcal{P}_n \) has been defined. For simplicity, denote \( \mathcal{P}_n \) by \( \mathcal{R} \) and \( T_{\mathcal{P}_n} \) by \( T \). We assume by induction that \( \mathcal{R} \) occurs in \( X \), that \( |T| = \beth_n(\mu) \) and that \( \text{Good}_{\mathcal{P}_n}(X) \subseteq \text{Good}_\mathcal{P}(X) \). Let \( \Gamma \subseteq \mathcal{P}(T) \) be such that

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(1) \(|\Gamma| = 2^{|T|}\), 
(2) for any distinct \(\Gamma, \Delta \in \Gamma\), \(|\Gamma| = |T|\) and \(\Gamma \nsubseteq \Delta\).

Then by Proposition 2.22(b), \(P_{n+1} := Q_{\Gamma_n}^P\) is a P-system. Denote \(P_{n+1}\) by \(S\). By the induction Hypothesis, \(R\) occurs in \(X\), and hence by Lemma 2.24, \(S\) occurs in \(X\). Note that \(|TS| = 2^{|T|}\). By Proposition 2.23(a), 

\[|End_S(X)| \geq |TS| = 2^{|T|} = \beth_n(\mu)\]

By the definition of markers, \(End_S(X) \subseteq Good_R(X)\). Since \(Good_S(X) = \Cl(End_S(X))\) and \(Good_R(X)\) is closed, it follows that \(Good_S(X) \subseteq Good_R(X)\). Hence by the induction hypothesis,

\[Good_S(X) \subseteq Good_P(X)\]

This concludes the inductive construction.

Since for every \(n\), \(Good_{P_n}(X) \subseteq Good_P(X)\) and \(|Good_{P_n}(X)| \geq \beth_n(\mu)\), it follows that \(|Good_P(X)| \geq \beth_\omega(\mu)\). \(\Box\)

We shall apply Corollary 2.25 to the class of all \(\Omega\)-demonstrative codes. To this end we show that this class forms a P-system.

For a limit ordinal \(\Omega\) and \(\alpha < \Omega\) define

\[\widehat{X}_\alpha^\Omega = \{ \langle F, e \rangle \mid F \in K_{TH}, F \text{ is an } \Omega\text{-demonstrative } (\alpha + 1)\text{-code and } e \in \Pend(F) \}\]

Now define \(P^\Omega = \{ \widehat{X}_\alpha^\Omega \mid 1 \leq \alpha < \Omega \}\).

**Proposition 2.26.** If \(\Omega\) is a limit ordinal, then \(P^\Omega\) is a proliferation system.

**Proof** By the definitions, \(P^\Omega\) fulfills (P1). That (P2) holds is proved in Proposition 2.17

We prove (P3). Let \(\langle F, e \rangle \in \widehat{X}_\alpha^\Omega\). Then \(F\) is an \(\Omega\)-demonstrative \((\alpha + 1)\)-code and \(e \in \Pend(F)\). Let \(V \in \Nbr^F(e)\) and \(\beta < \Omega\). By definition,
$F \in K_{TH}$, ker$(F) \neq \emptyset$, $\Omega(F) = \Omega$ and $\Omega(F)$ is not attained. By Clause (D2) of 2.18, $e \in \text{Good}(F)$. So the assumptions of Lemma 2.20 are fulfilled by $F$, $e$ and $V$. So by 2.20, $V$ contains an $\Omega$-demonstrative $(\beta + 1)$-code. 

6. The conclusion of the proof.

We next see that if $\Omega(X)$ is not attained, then $\mathcal{P}^{\Omega(X)}$ occurs in $X$.

**Proposition 2.27.** Let $X \in K_{TH}$ be such that ker$(X) \neq \emptyset$ and $\Omega(X)$ is not attained. Then $\mathcal{P}^{\Omega(X)}$ occurs in $X$ and Good$_{\mathcal{P}(X)}(X) \subseteq \text{Good}(X)$.

**Proof** The fact “$\mathcal{P}^{\Omega(X)}$ occurs in $X$” is part of Lemma 2.20. Denote $\mathcal{P}^{\Omega(X)}$ by $\mathcal{P}$. We verify that Good$_{\mathcal{P}}(X) \subseteq \text{Good}(X)$. Recall that by definition, Good$_{\mathcal{P}}(X) = \text{cl}(\text{End}_{\mathcal{P}}(X))$ and that Good$(X)$ is closed. So it suffices to show that End$_{\mathcal{P}}(X) \subseteq \text{Good}(X)$. Let $e \in \text{End}_{\mathcal{P}}(X)$. This means that there is $F \subseteq X$ such that $\langle F, e \rangle \in \hat{X}_{\mathcal{P}}$. By the definition of $\mathcal{P}$, we have that $e \in \text{Pend}(F)$, and from $\Omega(X)$-demonstrativeness of $F$ it follows that $e \in \text{Good}(F)$. The fact $\Omega(F) = \Omega(X)$ implies that Good$(F) \subseteq \text{Good}(X)$. So $e \in \text{Good}(X)$. That is, End$_{\mathcal{P}}(X) \subseteq \text{Good}(X)$ and hence Good$_{\mathcal{P}}(X) \subseteq \text{Good}(X)$.

We need one last property of spaces which are a continuous image of a compact interval space.

**Proposition 2.28.** Let $X \in K_{CI}$. Then for every infinite cardinal $\lambda$ and a closed subset $F \subseteq X$: If $|F| \geq (2^\lambda)^+$, then $F$ contains a scattered subspace $H$ such that $|H| = \lambda^+$.

**Proof** (a) A closed subspace of an interval space is an interval space. This implies that a closed subspace of a member of $K_{CI}$ is a member of $K_{CI}$. So it suffices to show that if $X \in K_{CI}$ and $|X| \geq (2^\lambda)^+$, then $X$ contains a scattered subspace $H$ such that $|H| = \lambda^+$.

Let $\langle L, \prec \rangle$ be a compact linear ordering and $g : L \to X$ be continuous and surjective. There is $A \subseteq L$ such that $|A| = (2^\lambda)^+$ and $g \upharpoonright A$ is $1$–$1$. By Erdős Rado Theorem, there is $B \subseteq A$ such that $B$ is order isomorphic to
\( \lambda^+ \) or to the reverse ordering of \( \lambda^+ \). Let \( C = \text{cl}(B) \) and \( H = g(C) \). Then \( |C| = \lambda^+ \). It is obvious that \( C \) is homeomorphic to \( \lambda^+ + 1 \) with is order topology, so \( C \) is scattered. Hence \( H = g(C) \) is scattered. Since \( g|B \) is 1–1 and \( |B| = \lambda^+ \), it follows that \( |H| = \lambda^+ \). So \( H \) is as desired.  

Let \( K \) be the class of all compact Hausdorff spaces \( X \) such that

1. \( X \in K_{\text{TH}} \).
2. \( X \) is sequentially compact.
3. For every infinite cardinal \( \lambda \) and a closed subset \( F \subseteq X \): If \( |F| \geq (2^\lambda)^+ \), then \( F \) contains a scattered subspace \( H \) such that \( |H| = \lambda^+ \).

The class \( K_{\text{CH}} \) is contained in \( K \), and indeed, this has been already shown. So the following statement implies Theorem 1.2.

**Theorem 2.29.** For every \( X \in K \): if \( X \) is a CO space, then \( X \) is scattered.

**Proof** Suppose by contradiction that \( X \in K \), \( X \) is a CO space and \( X \) is not scattered. Since \( X \) is tightly Hausdorff, it is collectionwise Hausdorff and strongly Hausdorff for convergent sequences. \( X \) is also sequentially compact. Hence by Corollary 2.8, \( \Omega(X) \) is not attained in \( X \).

By Proposition 2.26, \( \mathcal{P}^{\Omega(X)} \) is a P-system, and by 2.27, \( \mathcal{P}^{\Omega(X)} \) occurs in \( X \).

Denote \( \mathcal{P}^{\Omega(X)} \) by \( \mathcal{P} \) and \( |\Omega(X)| \) by \( \mu \). Note that \( |T_{\mathcal{P}}| = \mu \). Then by Corollary 2.25, \( |\text{Good}_{\mathcal{P}}(X)| \geq (\beth_3(\mu))^+ \). By 2.27, \( \text{Good}_{\mathcal{P}}(X) \subseteq \text{Good}(X) \). Hence \( |\text{Good}(X)| \geq (\beth_3(\mu))^+ \).

By Clause (3) in the definition of \( K \), there is a scattered subspace \( F \subseteq \text{Good}(X) \) such that \( |F| = (\beth_2(\mu))^+ \). From the scatteredness of \( F \) it follows that \( A := \text{Is}(F) \) is dense in \( F \). So if \( |A| \leq \mu \), then \( |F| \leq \beth_2(\mu) \). It follows that \( |A| \geq \mu^+ \).

Since \( A \) is relatively discrete and \( X \in K_{\text{TH}} \), there is a tight Hausdorff system for \( A \). Denote it by \( U = \{ U_a \mid a \in A \} \). Let \( \gamma : A \rightarrow \Omega(X) \) be a surjection. Recall that \( A \subseteq F \subseteq \text{Good}(X) \). So for every \( a \in A \) there is
Proof of Theorem [1.2] It suffices to check that $K_{CII} \subseteq K$. Let $X \in K_{CII}$. By Lemma [2.2] $X$ is tightly Hausdorff, that is, Clause (TH1) of $K_{TH}$ is fulfilled by $X$. Clause (TH2) is implied by Proposition [2.13], and Clause (TH3) is implied by Proposition [2.12]. By Proposition [2.6] $X$ is sequentially compact, and Proposition [2.28] implies that $X$ fulfills Clause (3) in the definition of $K$. So $X \in K$.  

3 Orderability of continuous images of interval spaces

In this section we consider the following question. Suppose that $X$ is a continuous image of a compact interval space, and $X$ is scattered. Is $X$ an interval space? The answer to this question is in terms of obstructions. That is, we define a class of spaces $O$, and prove that $X$ is an interval space if and only if $X$ has no subspace homeomorphic to a member of $O$.

For infinite cardinals $\kappa, \lambda$ and $\mu$ we define the topological space $X_{\kappa, \lambda, \mu}$ as follows. $X_{\kappa, \lambda, \mu}$ is the quotient of the disjoint union of the interval spaces $\kappa + 1, \lambda + 1$ and $\mu + 1$, where the points $\kappa, \lambda$ and $\mu$ are identified. Say that $\langle \kappa, \lambda, \mu \rangle$ is a legal triple if $\kappa, \lambda$ and $\mu$ are regular cardinals and $\lambda, \mu > \aleph_0$. Let $T = \{X_{\kappa, \lambda, \mu} | \langle \kappa, \lambda, \mu \rangle$ is a legal triple$\}$.  

Let $\lambda$ be an uncountable regular cardinal and $S \subseteq \lambda$. Let $\vec{\mu} : S \rightarrow \text{On}$ be such that $\vec{\mu}(\alpha)$ is an uncountable regular cardinal for every $\alpha \in S$. Denote $\vec{\mu}(\alpha)$ by $\mu_\alpha$. Define the space $X = X_{\lambda, \vec{\mu}}$ as follows. Let $\hat{\omega} = \{-i - 1 | i \in \omega\}$ be the set of negative integers. The universe of $X$ is

$$(\lambda + 1) \cup \left( \bigcup_{\alpha \in S} \{\alpha\} \times (\mu_\alpha \cup \hat{\omega}) \right).$$
An open base $\mathcal{B}$ of the topology of $X$ consists of the following sets.

(1) For every $\alpha \in S$, an open set $U \subseteq \mu_\alpha$ and a subset $V$ of $\hat{\omega}$, $\{\alpha\} \times U, \{\alpha\} \times V \in \mathcal{B}$.

(2) Let $W$ be an open subset of $\lambda+1$ and $\sigma \subseteq W \cap S$ be finite. For every $i \in \sigma$ let $F_i$ be a closed subset of $\mu_i + 1$ not containing $\mu_i$ and $G_i$ be a finite subset of $\hat{\omega}$. Then
\[
W \cup \bigcup_{\alpha \in W \cap S} \{\alpha\} \times (\mu_\alpha \cup \hat{\omega}) - \bigcup_{i \in \sigma} \{i\} \times (F_i \cup G_i) \in \mathcal{B}.
\]

Denote $\mathcal{B}$ by $\mathcal{B}_{\lambda,\vec{\mu}}$. It is left to the reader to check that $X_{\lambda,\vec{\mu}}$ is compact. Let $S = \{X_{\lambda,\vec{\mu}} \mid \text{Dom}(\vec{\mu}) \text{ is a stationary subset of } \lambda\}$ and $O = T \cup S \cup \{X_{\aleph_1}\}$. (Recall that $X_{\aleph_1}$ is the one point compactification of a discrete space of cardinality $\aleph_1$).

**Theorem 3.1.** Let $X$ be a scattered continuous image of a compact interval space. Then $X$ is an interval space iff no subset of $X$ is homeomorphic to a member of $O$.

**Proposition 3.2.** (a) Let $X$ be a closed subspace of a scattered continuous image of a compact interval space. Then $X$ is a continuous image of a compact interval space.

(b) If $X$ is a scattered continuous image of a compact interval space, then there is a 0-dimensional compact interval space $Y$ such that $X$ is a continuous image of $Y$.

**Proof** (a) Suppose that $X$ is a subspace of $Y$, and $Y$ that is a continuous image of $Z$. Then the preimage of $X$ in $Z$ is a closed subset of $Z$, and thus it is an interval space.

(b) Let $\langle L, < \rangle$ be a compact chain and $f : L \to X$ be continuous and onto. Let $L' = \{0, 1\} \times L$ and $<'$ be the lexicographic order of $L'$. Then $L'$ is compact and 0-dimensional. Define $f' : L' \to X$ by $f'(\langle i, a \rangle) = f(a)$. Then $f'$ is continuous and $\text{Rng}(f') = X$.

We need the following theorem from $\text{[B]}$.  

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Theorem 3.3. The following conditions are equivalent.

(1) $X$ is a scattered continuous image of a compact interval space.

(2) There is a family $\{U_x \mid x \in X\}$ of clopen subsets of $X$ such that

\begin{enumerate}
\item[(2.1)] For every $x \in X$, $\{x\} = D_{\text{rk}(X)}(U_x)$.
\item[(2.2)] For every $x, y \in X$: if $y \in U_x$, then $U_y \subseteq U_x$.
\item[(2.3)] For every $x, y \in X$: if $y \notin U_x$ and $x \notin U_y$, then $U_x \cap U_y = \emptyset$.
\end{enumerate}

Proof In [B] Theorem 1.5 it is proved that for every topological space $X$: $X$ is a scattered continuous image of a compact 0-dimensional interval space, iff $X$ satisfies Clause (2). But by Proposition 3.2(b), $X$ is a scattered continuous image of a compact interval space iff $X$ is a scattered continuous image of a compact 0-dimensional interval space. So Theorem 3.3 follows.

Let $U = \{U_x \mid x \in X\}$ be as in the above theorem. Then we call $U$ a tree-like clopen system for $X$. It is easy to see that if $X$ is scattered and compact and $U$ is a tree-like clopen system for $X$, then $U \cup \{X - U \mid U \in U\}$ is a subbase for the topology of $X$. Let $X$ be a scattered compact space. We say that $X$ is unitary if for some $e \in X$, $D_{\text{rk}(X)}(X) = \{e\}$. If $X$ is unitary, then the above $e$ is denoted by $e^X$. Every scattered compact space $X$ is a finite union of pairwise disjoint clopen sets $U$ such that $U$ is unitary and $\text{rk}(U) = \text{rk}(X)$.

It is clear that if $X$ is a finite union of pairwise disjoint clopen sets which are interval spaces, then $X$ is an interval space.

Let $\langle P, < \rangle$ be a poset and $x \in P$. We define $P^{<x} = \{y \in P \mid y < x\}$. The sets $P^{\leq x}$ etc. are defined analogously. Suppose that $\langle L, < \rangle$ is a linear ordering and $a \in L$. We denote the cofinality of $L^{<a}$ by $\text{cf}_{\langle L, < \rangle}(a)$ and the cointiality of $L^{>a}$ by $\text{cf}_{\langle L, < \rangle}^+(a)$.

We shall also need the following well-known facts.

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Proposition 3.4. Let $X$ be a compact Hausdorff space and $\leq$ be a partial ordering of $X$ such that $\leq$ is a closed subset of $X \times X$.

(a) If $C \subseteq X$ is a chain, then $C$ has a supremum and an infimum.

(b) Suppose that $X$ has the following property. (H1) For every $x, y \in X$: if $x \nsucceq y$, then there are open sets $U$ and $V$ such that $U$ is an initial segment of $\langle X, \leq \rangle$ and $x \in U$, $V$ is a final segment of $\langle X, \leq \rangle$ and $y \in V$, and $U \cap V = \emptyset$. Then for every chain $C \subseteq X$, $\sup(C), \inf(C) \in \text{cl}^X(C)$.

(c) Suppose that $X$ satisfies (H1) and let $C \subseteq X$ be a chain such that for every nonempty $A \subseteq C$, $\sup(A), \inf(A) \in C$. Then $C$ is closed in $X$, and the order topology of $C$ coincides with the induced topology of $C$.

Proof of Theorem 3.1. We first prove that an interval space does not have a subspace homeomorphic to a member of $\mathcal{O}$. Since a closed subspace of an interval space is necessarily an interval space, it suffices to show that for every $X \in \mathcal{O}$, $X$ is not an interval space.

The proofs that $X_{\kappa_1}$ is not an interval space, and that for a legal triple $\langle \kappa, \lambda, \mu \rangle$ the space $X_{\kappa, \lambda, \mu}$ is not an interval space are left to the reader.

We show that a space of the type $X_{\lambda, \vec{\mu}}$, where $\text{Dom}(\vec{\mu})$ is stationary in $\lambda$ is not an interval space.

So let $X_{\lambda, \vec{\mu}} \in \mathcal{S}$ and suppose by way of contradiction that $\prec$ is a linear ordering of $X_{\lambda, \vec{\mu}}$ which induces the topology of $X_{\lambda, \vec{\mu}}$. Let $S = \text{Dom}(\vec{\mu})$ and denote $\vec{\mu}(\alpha)$ by $\mu_\alpha$. Also denote $X_{\lambda, \vec{\mu}}$ by $X$. Let $<^{<\omega_1}$ denote the linear ordering of the ordinals. For an ordinal $\alpha$ let $\tau^\alpha$ be the order topology of $\langle \alpha, <^{<\omega_1} \upharpoonright \alpha \rangle$. Consider the sets $(\lambda + 1) \cap X^{<\lambda}$ and $(\lambda + 1) \cap X^{<\lambda}$. One of them must be of cardinality $\lambda$ and the other of cardinality $< \lambda$. This is so since in the interval space $\langle \lambda + 1, \tau^{\lambda+1} \rangle$ every two closed sets of cardinality $\lambda$ intersect in a set of cardinality $\lambda$. So we may assume that $|(\lambda + 1) \cap X^{<\lambda}| = \lambda$ and $|(\lambda + 1) \cap X^{<\lambda}| < \lambda$. Hence for some $\alpha_0 < \lambda$, $[\alpha_0, \lambda]^{<\omega_1} \subseteq X^{<\lambda}$.

Let $X_{\kappa, \vec{\nu}} \in \mathcal{S}$ and $C \subseteq \kappa$ be a club. Define $X_{\kappa, \vec{\nu}} \upharpoonright C$ as follows.

$$X_{\kappa, \vec{\nu}} \upharpoonright C = C \cup \{\kappa\} \cup \bigcup_{\alpha \in S \cap C} \{\alpha\} \times (\vec{\nu}(\alpha) \cup \hat{\omega}).$$
Then for some $\vec{\nu}'$, $X_{\kappa,\vec{\nu}} \upharpoonright C \cong X_{\kappa,\vec{\nu}'}$ and $\text{Dom}(\vec{\nu}')$ is stationary in $\kappa$. So if $X_{\kappa,\vec{\nu}}$ is counter-example, then $X_{\kappa,\vec{\nu}} \upharpoonright C$ too is a counter-example. We may thus replace $X_{\lambda,\vec{\nu}}$ by $X_{\lambda,\vec{\nu}} \upharpoonright [\alpha_0, \lambda]^<_{\text{on}}$, and assume that $\lambda \subseteq X^{<\lambda}$.

We say that $\alpha \in \lambda$ is a bad if there is $\beta = \beta_\alpha <_{\text{on}} \alpha$ such that $\alpha < \beta$. Suppose by way of contradiction that the set $B$ of bad points is stationary. Then the function taking every $\alpha \in B$ to $\beta_\alpha$ is constant on an unbounded set. Let $\gamma$ be this constant value. Then $\gamma \supseteq \lambda$. A contradiction, so $B$ is non-stationary. Let $C$ be a club disjoint from $B$ and $X_0 = X \upharpoonright C$. For some $\vec{\mu}'$, $X_0 \cong X_{\lambda,\vec{\mu}'}$, and $\text{Dom}(\vec{\mu}')$ is stationary in $\lambda$. Replacing $X$ by $X_0$, we may assume that $<_{\text{on}} \upharpoonright \lambda = <_{\text{on}} \upharpoonright \lambda$.

Let $\alpha \in S$ be a limit ordinal. We show that there is $\gamma_\alpha \in \mu_\alpha$ such that $\{\alpha\} \times [\gamma_\alpha, \mu_\alpha) \subseteq X^{>\alpha}$. The subspace $Y = \{\alpha\} \cup \{\alpha\} \times \mu_\alpha$ of $X$ is homeomorphic to $\langle \mu_{\alpha+1}, \tau_{\mu_{\alpha+1}} \rangle$, and the subsets $A_1 := \{\alpha\} \times \mu_\alpha \cap X^{=\alpha}$ and $A_2 := \{\alpha\} \times \mu_\alpha \cap X^{=\alpha}$ of $Y$ have only one common point in their closures. So one of these sets must have cardinality $\mu_\alpha$ and the other must have cardinality $< \mu_\alpha$. Suppose by contradiction that $|A_1| = \mu_\alpha$. Then $A_1 \cup \{\mu_\alpha\}$ is a closed subset of $Y$ of cardinality $\mu_\alpha$. Hence $A_1 \cup \{\mu_\alpha\} \cong Y$. It follows that for every $B \subseteq A_1$: if $|B| = \mu_\alpha$, then $\mu_\alpha \in \text{cl}(B)$. This implies that $\text{cf}(X^{<\mu_\alpha}, <_{\text{on}} X^{<\mu_\alpha}) = \mu_\alpha > \aleph_0$. The sets $A_1$ and $\mu_\alpha \cap X^{<\mu_\alpha}$ are closed unbounded subsets of $X^{<\mu_\alpha}$, and they are disjoint. A contradiction. So $|A_1| < \mu_\alpha$, and hence there is $\gamma_\alpha \in \mu_\alpha$ such that $\{\alpha\} \times [\gamma_\alpha, \mu_\alpha) \subseteq X^{>\alpha}$.

Let $\alpha \in S$ be a limit ordinal. It follows that $\text{cf}(X^{\alpha}, <_{\text{on}} \upharpoonright \alpha) = \mu_\alpha > \aleph_0$. However, $\{\alpha\} \times \dot{\omega}$ is an $\omega$-sequence converging to $\alpha$. We conclude that $(\{\alpha\} \times \dot{\omega}) \cap X^{>\alpha}$ is finite. Since $\prec$ and $<_{\text{on}}$ coincide on $\lambda$, the subset $\alpha$ of $X$ is cofinal in $X^{>\alpha}$. So there are $\gamma_\alpha < \alpha$ and $a_\alpha \in \{\alpha\} \times \dot{\omega}$ such that $a_\alpha \prec \gamma_\alpha$. Then $\alpha \mapsto \gamma_\alpha$ is a regressive function defined on a stationary subset of $\lambda$. Let $\gamma$ be such that $\gamma_\alpha = \gamma$ for an unbounded set of $\alpha$'s. Denote this unbounded set by $D$. Then by the definition of the topology of $X_{\lambda,\vec{\mu}}$, $\lambda \in \text{acc}(\{a_\alpha \mid \alpha \in D\})$. But $a_\alpha \prec \gamma < \lambda$ for every $\alpha \in D$. A contradiction. So $X_{\lambda,\vec{\mu}}$ is not homeomorphic to an interval space.
We prove the other direction of the theorem by induction on \( \text{rk}(X) \). The statement of the induction hypothesis requires some preparation.

Let \( X \) be a scattered continuous image of a compact interval space and \( U = \{ U_x \mid x \in X \} \) be a tree-like clopen system for \( X \). For \( x, y \in X \) define \( x \leq_U y \) if \( x \in U_y \). Clearly \( \leq_U \) is a partial ordering of \( X \). We say that \( \langle X, U \rangle \) is simple if \( \langle X, \leq_U \rangle \) has a maximum \( e^U \), and there are an uncountable regular cardinal \( \lambda \) and a strictly increasing sequence \( \{ x_\alpha \mid \alpha < \lambda \} \) in \( \langle X, \leq_U \rangle \) such that \( X - \{ e^U \} = \bigcup_{\alpha < \lambda} U_{x_\alpha} \).

We shall prove by induction on \( \alpha \) the following statement.

\((*)_\alpha\) If \( X \) is a scattered continuous image of a compact interval space, \( \text{rk}(X) = \alpha \) and no subspace of \( X \) is homeomorphic to a member of \( \mathcal{O} \), then \( X \) is an interval space. If in addition, \( \langle X, U \rangle \) is simple, then there is a linear ordering \( \leq_X \) of \( X \) such that the order topology of \( \leq_X \) is the topology of \( X \) and \( e^U \) is the maximum of \( \langle X, \leq_X \rangle \).

Denote \((*)_{<\alpha} \equiv \bigwedge_{\beta<\alpha}(*)_\beta \). It is trivial that \((*)_0 \) holds. We shall prove that if \( \alpha > 0 \), and for every \((*)_{<\alpha} \) holds, then \((*)_\alpha \) holds.

A poset \( \langle P, \leq \rangle \) is called a reverse tree if \( P^{>x} \) is a chain for every \( x \in X \). A subset \( D \) of a poset \( P \) is directed, if for every \( a, b \in D \) there is \( c \in D \) such that \( a, b \leq c \). We say that \( D \) is principal if for some \( d \in P \), \( D = P^{\leq_d} \). We say that \( D \) is generated by \( A \) if \( D = \{ p \in P \mid \text{ there is } a \in A \text{ such that } p \leq a \} \). For \( x \in P \) we set \( D_x = \{ D \mid D \text{ is a maximal directed subset of } P^{<x} \} \). We leave it to the reader to check that if \( P \) is a reverse tree, \( x \in P \) and \( D_1, D_2 \in D_x \) are distinct, then \( D_1 \cap D_2 = \emptyset \). Also, if \( P \) is a reverse tree, \( x \in P \) and \( D \in D_x \), then there is a chain which generates \( D \). Let \( D \in D_x \). If \( D \) has a maximum, then the cofinality of \( D \) is defined to be 1. Otherwise, there is unique regular cardinal \( \nu \) such that \( D \) is generated by a chain of type \( \nu \). We denote this \( \nu \) by \( \text{cf}(D) \). A subset \( A \) of a poset \( P \) is unbounded, if there is no \( p \in P \) such that \( p \geq a \) for every \( a \in A \).

Let \( X \) be a scattered continuous image of a compact interval space. Let \( U = \{ U_x \mid x \in X \} \) be a tree-like clopen system for \( X \). Clearly, \( \langle X, \leq_U \rangle \) is
a reverse tree. Note also that $\leq_U$ is a closed subset of $X \times X$. This is so, since $\not\leq = \bigcup_{x \in X} (X - U_x) \times U_x$. It is trivial that $\langle X, \tau^X, \leq_U \rangle$ satisfies Property (H1) from Proposition 3.4. The set of maximal points in $\langle X, \leq_U \rangle$ is finite. Suppose otherwise. Let $y$ be an accumulation point of the set of maximal points. Since $U_y$ is a neighborhood of $y$ it contains a maximal point $z \neq y$. But then $z <_U y$. A contradiction. It follows that the set of maximal points is finite. Also, $X = \bigcup \{ U_x : x \text{ is a maximal point of } X \}$. If $V$ is a clopen subset of $X$, then $\{ V \cap U_x : x \in V \}$ is a tree-like clopen system for $V$. It follows that if $X$ is a scattered continuous image of a compact interval space, then there are $\langle X_1, U_1 \rangle, \ldots, \langle X_n, U_n \rangle$ such that $\{ X_1, \ldots, X_n \}$ is a partition of $X$ into clopen sets, and $\langle X_i, \leq_{U_i} \rangle$ has a maximum for every $i \leq n$. It is trivial that if $X$ has a maximum $e^{U}$, then $X$ is unitary and that $e^U = e^X$.

Claim 1 Let $X$ be a scattered continuous image of a compact interval space such that no subspace of $X$ is homeomorphic to a member of $O$. Let $x \in X$. Then the following facts hold.

1. $|D_x| \leq \aleph_0$.

2. If there are distinct $D_0, D_1 \in D_x$ such that $\text{cf}(D_0), \text{cf}(D_1) \geq \aleph_1$, then $D_x$ is finite and every member of $D_x$ other than $D_0$ and $D_1$ has a maximum.

Proof (1) The proof relies on the fact that $X_{\aleph_1}$ is not embeddable in $X$. Let $A \subseteq U_x$ be infinite, and assume that for every $D \in D_x$, $|A \cap D| \leq 1$. We show that $A$ is discrete and that $A \cup \{ x \}$ is the one point compactification of $A$. If $y, z \in A$ are distinct, then $z \not\leq_U y$. That is, $z \notin U_y$. So $U_y$ is a neighborhood of $y$ disjoint from $A - \{ y \}$. We show that every neighborhood $V$ of $x$ contains all but finitely many members of $A$. We may assume that $V$ has the form $U_x - \bigcup_{y \in \sigma} U_y$, where $\sigma$ is a finite subset of $U_x$. If $y \in \sigma$, then $y \leq_U x$ and so $|U_y \cap A| \leq 1$. So $A - V$ is finite. Hence $x \in \text{acc}(A)$. Let $y \in X - \{ x \}$. We show that $y \notin \text{acc}(A)$. If $y \notin U_x$, then $U_y \cap U_x = \emptyset$. Hence
\[ U_y \cap A = \emptyset. \] If \( y \in U_x \), then \(|U_y \cap A| \leq 1\). So \( y \not\in \text{acc}(A)\). We have shown that \( \text{acc}(A) = \{x\} \). So \( A \cup \{x\} \) is the one point compactification of \( A \). Since \( X_{\aleph_1} \) is not embeddable in \( X \), \(|A| \leq \aleph_0 \), and since \( \mathcal{D}_x \) is a pairwise disjoint family, \(|\mathcal{D}_x| = \aleph_0\).

(2) The proof relies on the fact that no member of \( T \) is embeddable in \( X \). Suppose that \( D_0, D_1 \in \mathcal{D}_x \), \( D_0 \neq D_1 \) and \( \text{cf}(D_0), \text{cf}(D_1) \geq \aleph_1 \). For \( i = 0, 1 \) let \( E_i \) be a chain which generates \( D_i \) and such that the order type of \( E_i \) is a regular cardinal \( \lambda_i \). We may assume that for every nonempty bounded \( B \subseteq E_i \), \( \sup(B) \in E_i \). Suppose by way of contradiction that \( D_x \) is infinite. Since \( D_x \) is a pairwise disjoint family, it follows that there is a countably infinite set \( A \subseteq U_x \) such that \(|A \cap D| \leq 1\) for every \( D \in \mathcal{D}_x \).

We show that \( Y := A \cup E_1 \cup E_2 \cup \{x\} \) is homeomorphic to \( X_{\aleph_0, \lambda_1, \lambda_2} \). For \( i = 1, 2 \), \( E_i \cup \{x\} \) is a chain in \( X \) closed under infima and suprema. So by Proposition 3.4(c), its induced topology \( \tau_i \) coincides with its order topology. That is, \( \langle E_i \cup \{x\}, \tau_i \rangle \cong \lambda_i + 1 \). It is also clear that \( A \cup \{x\} \) with its induced topology is homeomorphic to \( \omega + 1 \). These facts imply that \( \langle Y, \tau^X | Y \rangle \cong X_{\aleph_0, \lambda_1, \lambda_2} \in T \). A contradiction.

Suppose by contradiction that \( D \in \mathcal{D}_x - \{D_0, D_1\} \) and \( D \) is nonprincipal. Let \( E \) be unbounded chain in \( D \) such that the order type of \( E \) is a regular cardinal \( \mu \). We may assume that for every nonempty bounded \( B \subseteq E \), \( \sup(B) \in E \). Just as in the previous argument we conclude that \( E \cup E_1 \cup E_2 \cup \{x\} \cong X_{\mu, \lambda_1, \lambda_2} \in T \). A contradiction. We have proved Claim 1.

Suppose that \((*)_{<\alpha_0}\) holds, and we prove \((*)_{\alpha_0}\). Let \( X \) be a scattered continuous image of a compact interval space, suppose that \( \text{rk}(X) = \alpha_0 \), and that no subspace of \( X \) is homeomorphic to a member of \( \mathcal{O} \). Let \( \mathcal{U} \) be a tree-like clopen system for \( X \). Since \( X \) can be partitioned into finitely many clopen sets \( X_i \) with tree-like systems \( \mathcal{U}_i \) such that each \( \langle X_i, \leq_{\mathcal{U}_i} \rangle \) has a maximum, we may assume that \( X \) has a maximum. We deal separately with three cases.

Case 1 Assume that \( \langle X - \{e\}, \leq_{\mathcal{U}} | (X - \{e\}) \rangle \) contains a chain \( C \) with
uncountable cofinality, such that $X - \{e\} = \bigcup_{x \in C} U_x$. In such a situation it is required that we prove that there is a linear ordering $\leq_X$ of $X$ which induces the topology of $X$ and in which $e = \max(\langle X, \leq_X \rangle)$.

We may assume that $C$ is order isomorphic to an uncountable regular cardinal $\lambda$. We may further assume that for every nonempty bounded $A \subseteq C$, $\sup(A) \in C$. Obviously, $C \cup \{e\}$ is order isomorphic to $\lambda + 1$. So by proposition 3.4, the induced topology on $C \cup \{e\}$ coincides with the order topology of $C \cup \{e\}$. So the order isomorphism between $\lambda + 1$ and $C \cup \{e\}$ is a homeomorphism. Let

$$\alpha \mapsto y_\alpha, \; \alpha \leq \lambda,$$

be the isomorphism between $\lambda + 1$ and $C$.

Let $J_\alpha = \bigcup_{\beta < \alpha} U_{y_\beta}$. We check that $J_\alpha \cup \{y_\alpha\}$ is closed. If $\alpha = \beta + 1$, then $J_\alpha = U_\beta$. So $J_\alpha \cup \{y_\alpha\}$ is closed. Suppose $\alpha$ is a limit. Since $U_{y_\alpha}$ is closed, $\cl(J_\alpha) \subseteq U_{y_\alpha}$. Let $x \in U_{y_\alpha} - J_\alpha - \{y_\alpha\}$, $U_x$ is an open neighborhood of $x$. Suppose by contradiction that $U_x \cap J_\alpha \neq \emptyset$. Then for some $\beta < \alpha$, $U_x \cap U_\beta \neq \emptyset$, and hence for every $\gamma \geq \beta$, $x$ and $y_\gamma$ are comparable in $\leq_U$. But $x <_U y_\alpha$ and hence $x \not\geq_U y_\alpha$. Since $y_\alpha = \sup^{<_U} \{y_\gamma | \beta \leq \gamma < \alpha\}$, there is $\delta < \alpha$ such that $x \not\leq_U y_\delta$. So $x <_U y_\delta$. So $x \in U_\delta \subseteq J_\alpha$. A contradiction. Hence $U_x \cap U_\beta = \emptyset$. So $J_\alpha \cup \{y_\alpha\}$ is closed.

Let $\alpha < \lambda$. We say that $\alpha$ is inconvenient (with respect to the sequence $\{y_\alpha | \alpha < \lambda\}$), if $\alpha$ is a limit ordinal, and there are an uncountable regular cardinal $\mu_\alpha$ and disjoint sets $Y_\alpha, Z_\alpha \subseteq U_{y_\alpha} - J_\alpha - \{y_\alpha\}$ such that

1. $Y_\alpha$ is discrete, $Y_\alpha \cup \{y_\alpha\}$ is homeomorphic to $\omega + 1$,
2. $Z_\alpha$ is homeomorphic to $\mu_\alpha$ and $Z_\alpha \cup \{y_\alpha\}$ is homeomorphic to $\mu_\alpha + 1$.

Let $S$ be the set of inconvenient ordinals, $\vec{\mu} = \{\mu_\alpha | \alpha \in S\}$ and

$$Y = \{y_\alpha | \alpha \leq \lambda\} \cup \bigcup_{\alpha \in S} Y_\alpha \cup \bigcup_{\alpha \in S} Z_\alpha.$$

Claim 2 $Y \cong X_{\lambda, \vec{\mu}}$.  

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**Proof** At first we verify that $Y$ is closed. Let $x \in \text{cl}(Y)$. Let $\alpha$ be the first ordinal such that $x \in U_{y_{\alpha}}$. So $x \notin J_{\alpha}$. Hence $U_x$ is a neighborhood of $x$ disjoint from $J_{\alpha} \cup (X - U_{y_{\alpha}})$. The complement of this set in $X$ is $U_{y_{\alpha}} - J_{\alpha}$ and $Y \cap (U_{y_{\alpha}} - J_{\alpha}) = \{y_{\alpha}\} \cup Y_{\alpha} \cup Z_{\alpha}$. So $x \in \text{cl}(\{y_{\alpha}\} \cup Y_{\alpha} \cup Z_{\alpha})$. But $\{y_{\alpha}\} \cup Y_{\alpha} \cup Z_{\alpha}$ is closed. So $x \in \{y_{\alpha}\} \cup Y_{\alpha} \cup Z_{\alpha} \subseteq Y$. Hence $Y$ is closed.

For $\alpha \in S$ let $\{z_{\alpha,i} \mid i < \mu_{\alpha}\}$ be an enumeration of $Z_{\alpha}$ such that the function $i \mapsto z_{\alpha,i}$, $i < \mu_{\alpha}$, is a homeomorphism between $\mu_{\alpha}$ and $Z_{\alpha}$, and let $\{y_{\alpha,i} \mid i < \omega\}$ be a 1–1 enumeration of $Y_{\alpha}$. Define $\psi : X_{\lambda,\vec{\mu}} \rightarrow Y$ as follows:

1. $\psi(\alpha) = y_{\alpha}$, $\alpha \leq \lambda$;
2. $\psi(\langle \alpha, i \rangle) = y_{\alpha,i}$, $\alpha \in S$, $i \in \mu_{\alpha}$;
3. $\psi(\langle \alpha, -i - 1 \rangle) = z_{\alpha,i}$, $\alpha \in S$, $i \in \omega$.

Clearly, $\psi$ is a bijection. We prove that $\psi$ is a homeomorphism between $X_{\lambda,\vec{\mu}}$ and $Y$. Since both $X_{\lambda,\vec{\mu}}$ and $Y$ are compact, it suffices to show that for every $B \in \mathcal{B}_{\lambda,\vec{\mu}}$, $\psi(B)$ is open in $Y$.

We first show that $Y_{\alpha}$ and $Z_{\alpha}$ are open in $Y$. Note that $Y_{\alpha} \cup Z_{\alpha} = Y \cap (U_{y_{\alpha}} - (J_{\alpha} \cup \{y_{\alpha}\}))$. Since $U_{y_{\alpha}} - (J_{\alpha} \cup \{y_{\alpha}\})$ is open in $X$, it follows that $Y_{\alpha} \cup Z_{\alpha}$ is open in $Y$. Both $Z_{\alpha} \cup \{y_{\alpha}\}$ and $Y_{\alpha} \cup \{y_{\alpha}\}$ are compact and hence closed in $X$. So they are closed in $Y$. Since $Y_{\alpha} = (Y_{\alpha} \cup Z_{\alpha}) - (Z_{\alpha} \cup \{y_{\alpha}\})$, it follows that $Y_{\alpha}$ is open in $Y$. Similarly, $Z_{\alpha}$ is open in $Y$, because $Z_{\alpha} = (Y_{\alpha} \cup Z_{\alpha}) - (Y_{\alpha} \cup \{y_{\alpha}\})$.

Let $B = \{\alpha\} \times V \in \mathcal{B}_{\lambda,\vec{\mu}}$, where $V$ is an open subset of $\mu_{\alpha}$. Since $\psi(\{\alpha\}) \times \mu_{\alpha}$ is a homeomorphism onto $Z_{\alpha}$, $\psi(B)$ is open in $Z_{\alpha}$. So it is open in $Y$. Similarly, if $B = \{\alpha\} \times V \in \mathcal{B}_{\lambda,\vec{\mu}}$, where $V \subseteq \hat{\omega}$, then $\psi(B)$ is open in $Y_{\alpha}$. So it is open in $Y$.

Let $W$ be an open subset of $\lambda + 1$ and $\sigma \subseteq W \cap S$ be finite. For every $i \in \sigma$ let $F_i$ be a closed subset of $\mu_i + 1$ not containing $\mu_i$ and $G_i$ be a finite subset of $\hat{\omega}$. Let $B = W \cup \bigcup_{\alpha \in W \cap S} \{\alpha\} \times (\mu_{\alpha} \cup \hat{\omega}) - \bigcup_{i \in \sigma} \{i\} \times (F_i \cup G_i)$. It remains to show that when $B$ has this form, then $\psi(B)$ is open in $Y$. 

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For \( i \in \sigma \), \( F_i \) is compact in \( \mu_i \). So \( \{z_{i,j} \mid j \in F_i\} \) is compact in \( Z_i \). So it is closed in \( Y \). But \( \psi(\{i\} \times F_i) = \{z_{i,j} \mid j \in F_i\} \). Hence \( \psi(\{i\} \times F_i) \) is closed in \( Y \). Also, \( \psi(\{i\} \times G_i) \) is finite and hence closed in \( Y \). So \( \psi(\bigcup_{i \in \sigma}(F_i \cup G_i)) \) is closed in \( Y \).

Set \( W' = W \cup \bigcup_{\alpha \in W \cap S} \{\alpha\} \times (\mu_\alpha \cup \hat{\omega}) \). Then \( B = W' - \bigcup_{i \in \sigma}(F_i \cup G_i) \). We have already shown that \( \psi(\bigcup_{i \in \sigma}(F_i \cup G_i)) \) is closed in \( Y \). So it remains to show that \( \psi(W') \) is open in \( Y \).

We may assume that \( W \) is an open convex subset of \( \lambda + 1 \). Let us first deal with the case that \( W = (\beta, \gamma) \), where \( \beta, \gamma \in \lambda + 1 \). Then

\[
\psi(W') = \{y_\alpha \mid \alpha \in (\beta, \gamma)\} \cup \bigcup_{\alpha \in (\beta, \gamma) \cap S} (Y_\alpha \cup Z_\alpha) = Y' \cap (U_{y_\beta} - U_{y_\gamma}).
\]

So \( \psi(W') \) is open in \( Y \).

If \( W \) has the form \( (\beta, \lambda) \). Then \( \psi(W') = Y - U_{y_\beta} \). If \( W = [0, \gamma) \), then \( \psi(W') = Y \cap U_{y_\gamma} \), and finally, if \( W = \lambda + 1 \), then \( \psi(W') = Y \). In all cases \( \psi(W') \) is open in \( Y \). Hence \( \psi(B) \) is open in \( Y \).

Since \( \psi \) takes all members of an open base of \( X_{\lambda,\vec{\mu}} \) to open subsets of \( Y \) and both \( X_{\lambda,\vec{\mu}} \) and \( Y \) are compact Hausdorff, \( \psi \) is a homeomorphism between \( X_{\lambda,\vec{\mu}} \) and \( Y \). This proves Claim 2.

We found that \( Y \) is homeomorphic to \( X_{\lambda,\vec{\mu}} \). However, no subspace of \( X \) is homeomorphic to a member of \( S \). So \( X_{\lambda,\vec{\mu}} \notin S \). This implies that the set \( S \) of inconvenient ordinals is non-stationary. Let \( A \) be a closed and unbounded subset of \( \lambda \) disjoint from \( S \), and let \( \{x_\alpha \mid \alpha < \lambda\} \) be a strictly increasing enumeration of \( \{y_\beta \mid \beta \in A\} \). Denote \( e \) by \( x_\lambda \). The function \( \alpha \mapsto x_\alpha, \alpha \leq \lambda \) is again a homeomorphism.

We claim that there are no inconvenient ordinals with respect to \( \{x_\alpha \mid \alpha < \lambda\} \). Let \( \alpha \in \lambda \) be a limit ordinal. There is \( \beta \) such that \( x_\alpha = y_\beta \). Clearly, \( \beta \) is a limit ordinal. Hence \( U_{x_\alpha} - \bigcup_{\gamma < \alpha} U_{x_\gamma} = U_{y_\beta} - \bigcup_{\gamma < \beta} U_{y_\gamma} \). Since \( \beta \) is not inconvenient with respect to \( \{y_\gamma \mid \gamma < \lambda\} \) it follows that \( \alpha \) is not inconvenient with respect to \( \{x_\gamma \mid \gamma < \lambda\} \).
We shall define by induction on \( \alpha \leq \lambda \) linear orderings \( \leq_{\alpha} \) of \( U_x \). Since \( e = x_{\lambda} \) and \( U_e = X \), the ordering \( \leq_{\lambda} \) is an ordering of \( X \). This will be the ordering required in the theorem.

Denote \( U_x \) and \( D_x \) by \( U_\alpha \) and \( D_\alpha \) respectively, and define \( I_\alpha = \bigcup_{\beta < \alpha} U_\beta \).

We need the following facts.

(3) Suppose that \( \alpha \) is a limit ordinal. Then \( I_\alpha \in D_\alpha \). Also, there do not exist distinct \( D_1, D_2 \in D_\alpha - \{ I_\alpha \} \) such that \( \text{cf}(D_1) \geq \aleph_0 \) and \( \text{cf}(D_2) \geq \aleph_1 \).

(4) Suppose that \( \alpha \) is a limit ordinal. If \( D_\alpha \) is infinite, then for every \( D \in D_\alpha - \{ I_\alpha \} \), \( \text{cf}(D) \leq \aleph_0 \).

The proof of (3) relies on the fact that \( \alpha \) is not inconvenient. For suppose by contradiction that \( D_1, D_2 \in D_\alpha - \{ I_\alpha \} \), \( \text{cf}(D_1) \geq \aleph_0 \), \( \text{cf}(D_2) \geq \aleph_1 \). Let \( E_1, E_2 \) be chains which generate \( D_1 \) and \( D_2 \) respectively, and such that the order types of \( E_1 \) and \( E_2 \) are regular cardinals. Suppose further that \( E_i \cup \{ x_\alpha \} \) is closed under suprema in \( \langle X, \leq \rangle \) for \( i = 1, 2 \). Let \( E \) be a closed and unbounded subset of \( \{ x_\beta \mid \beta < \alpha \} \) with order type which is a regular cardinal. It follows that if \( \text{cf}(D_1), \text{cf}(D_2) \geq \aleph_1 \), then \( E \cup E_1 \cup E_2 \cup \{ x_\alpha \} \in \mathcal{T} \). If \( \text{cf}(D_1) = \aleph_0 \) and \( \text{cf}(D_2) \geq \aleph_1 \), then setting \( Y = E_1 \) and \( Z = E_2 \) shows that \( \alpha \) is inconvenient. We have proved (3).

The proof of (4) relies on the fact that \( \alpha \) is not inconvenient. Suppose by contradiction that \( D_\alpha \) is infinite and for some \( D \in D_\alpha - \{ I_\alpha \} \), \( \text{cf}(D) \geq \aleph_1 \). Let \( Y \subseteq U_\alpha - I_\alpha - D \) be a countably infinite set such that for every \( D' \in D_\alpha \), \( |Y \cap D'| \leq 1 \). Let \( Z \) be a chain which generates \( D \) whose order type is a regular cardinal and such that \( Z \cup \{ x_\alpha \} \) is closed under suprema. Then the pair \( Y, Z \) is an evidence that \( \alpha \) is inconvenient. We have proved (4).

If \( \langle W, \zeta \rangle \) is a topological space and \( A \subseteq W \), denote the relative topology that \( A \) inherits from \( \langle W, \zeta \rangle \) by \( \zeta \upharpoonright A \). If \( \preceq \) is a linear ordering of \( A \), denote by \( \tau^{A} \) the order topology of \( \langle A, \preceq \rangle \). Let \( \rho \) denote the topology \( \tau^{X} \) of \( X \).
We now define by induction on \( \alpha \leq \lambda \) the linear ordering \( \leq_\alpha \) of \( U_\alpha \). We assume by induction that

\[(I1) \quad \tau^{\leq_\alpha} = \rho|U_\alpha.\]

\[(I2) \quad \text{If } \beta < \gamma, \text{ then } \leq_\beta \subseteq \leq_\gamma.\]

\[(I3) \quad \text{If } \beta < \gamma, \text{ then } U_\beta \text{ is an initial segment of } \langle U_\gamma, \leq_\gamma \rangle.\]

Since \( \text{rk}(x_0) < \text{rk}(e) = \alpha_0 \), we may apply the induction hypothesis that \((\ast)_{<\alpha_0} \) holds to \( x_0 \). Let \( \leq_0 \) be a linear ordering of \( U_0 \) which induces the relative topology of \( U_0 \). Suppose that \( \leq_\beta \) has been defined. Since \( \text{rk}(U_{\beta+1} - U_\beta) < \text{rk}(e) = \alpha_0 \), by the induction hypothesis, there is a linear ordering \( \leq' \) of \( U_{\beta+1} - U_\beta \) which induces the relative topology of \( U_{\beta+1} - U_\beta \). Then \( \leq_\beta \cup \leq' \cup U_\beta \times (U_{\beta+1} - U_\beta) \) is a linear ordering of \( U_{\beta+1} \) which satisfies the induction hypotheses.

Suppose that \( \delta \) is a limit ordinal and \( \leq_\beta \) has been defined for every \( \beta < \delta \). Let \( \leq'_\delta = \bigcup_{\beta<\delta} \leq_\beta \). So \( \leq'_\delta \) is a linear ordering of \( I_\delta \).

**Case 1.1** There is \( D \in D_\delta - \{I_\delta\} \) such that \( \text{cf}(D) \geq \aleph_1 \). By (4), \( D_\delta \) is finite, and by (3) every \( D' \in D_\delta - \{I_\delta, D\} \) has a maximum. Let \( \sigma = \{\max(D') \mid D' \in D_\delta - \{I_\delta, D\}\} \). For every \( x \in \sigma \), \( \text{rk}(x) < \alpha_0 \), so by the induction hypothesis, there is a linear ordering \( \leq_x \) of \( U_x \) which induces the topology of \( U_x \). Also, let \( \leq_\sigma \) be a linear ordering of \( \sigma \). We claim that \( \delta \neq \lambda \). This is so since \( I_\delta \) and \( D \) are distinct maximal directed sets in \( D_\delta \), so there cannot be a chain \( I \) in \( X^{<\alpha_0} \) such that \( \bigcup_{x \in I} U_x = X^{<\alpha_0} \), and we assumed that such an \( I \) exists for \( x_\lambda \). Let \( Z = D \cup \{x_\delta\} \). It is easy to check that \( Z \) is closed in \( X \). So \( Z \) is a scattered continuous image of a compact interval space. Since \( Z \subseteq U_\delta \) and \( \text{rk}(U_\delta) = \text{rk}(x_\delta) < \alpha_0 \), it follows that \( \text{rk}(Z) < \alpha_0 \). Let \( U^Z = \{U^Z_x \mid x \in Z\} \), where \( U^Z_x \) is defined as follows: if \( x \neq x_\delta \), then \( U^Z_x = U_x \) and \( U^Z_{x_\delta} = Z \). Then \( U^Z \) is a tree-like clopen system for \( Z \). Let \( J \) be an unbounded chain in \( D \). Then \( Z - \{x_\delta\} = D = \bigcup_{x \in J} U^Z_x \). We assumed that \((\ast)_{<\alpha_0} \) holds. So there is a linear ordering \( \leq_Z \) of \( Z \) such that \( \tau^{\leq_Z} = \rho|Z \) and such that \( x_\delta = \min(\langle Z, \leq_Z \rangle) \). We define the required linear ordering \( \leq_\delta \) of \( U_\delta \) as follows.

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(1) \( I_{\delta} \leq_{\delta} Z \). (This means: for every \( a \in I_{\delta} \) and \( b \in Z \), \( a \leq_{\delta} b \)).

(2) For every \( x \in \sigma \), \( Z \leq_{\delta} U_{x} \).

(3) For every \( x, y \in \sigma \): if \( x \leq_{\sigma} y \), then \( U_{x} \leq_{\delta} U_{y} \).

(4) \( \leq_{\delta} \mid I_{\delta} = \leq'_{\delta}, \leq_{\delta} \mid Z = \leq_{Z} \) and \( \leq_{\delta} \mid U_{x} = \leq_{x} \) for every \( x \in \sigma \).

Clearly, \( \leq_{\delta} \) is a linear ordering of \( U_{\delta} \). Recall that \( \tau^{\leq_{\delta}} \) denotes the order topology of \( \langle U_{\delta}, \leq_{\delta} \rangle \). We show that \( \tau^{\leq_{\delta}} = \rho \mid U_{\delta} \).

Let \( W = I_{\delta} \cup \{x_{\delta}\} \) and \( \mathcal{X} := \{W, Z\} \cup \{U_{x} \mid x \in \sigma\} \). Clearly, \( \mathcal{X} \) is a finite cover of \( U_{\delta} \). We shall argue as follows. At first we check that every member of \( \mathcal{X} \) is closed in both \( \tau^{\leq_{\delta}} \) and \( \rho \mid U_{\delta} \). Then we show (*) For every \( T \in \mathcal{X} \), \( \tau^{\leq_{\delta}} \mid T = \rho \mid T \).

Notice that if \( \mathcal{F} \) is a finite cover of a space \( \langle S, \eta \rangle \) consisting of closed sets, then for every \( V \subseteq S \): \( V \in \eta \) iff \( V \cap F \in \eta \mid F \) for every \( F \in \mathcal{F} \). That is \( \{\eta \mid F \mid F \in \mathcal{F}\} \) determines \( \eta \). Hence (*) implies that \( \tau^{\leq_{\delta}} = \rho \).

We show that for every \( x \in \sigma \), \( \tau^{\leq_{\delta}} \mid U_{x} = \rho \mid U_{x} \). Recall that \( \leq_{x} \) is a linear ordering of \( U_{x} \) such that \( \tau^{\leq_{x}} = \rho \mid U_{x} \). Also, \( U_{x} \) is a closed interval in \( \langle U_{\delta}, \leq_{\delta} \rangle \) and \( \leq_{\delta} \mid U_{x} = \leq_{x} \). So

\[
\tau^{\leq_{\delta}} \mid U_{x} = \tau^{\leq_{\delta} \mid U_{x}} = \tau^{\leq_{x}} = \rho \mid U_{x}.
\]

An identical argument shows that \( \tau^{\leq_{\delta}} \mid Z = \rho \mid Z \).

We show that \( \tau^{\leq_{\delta}} \mid W = \rho \mid W \). By (I3) and the definition of \( \leq_{\delta} \), for every \( \alpha < \delta \), \( U_{\alpha} \) is an initial segment of \( \langle I_{\delta}, \leq'_{\delta} \rangle \) and \( \leq_{\delta} \mid U_{\alpha} = \leq_{\alpha} \). Since \( I_{\delta} \) is an initial segment of \( \langle U_{\delta}, \leq_{\delta} \rangle \) and \( \leq_{\delta} \mid I_{\delta} = \leq'_{\delta} \), it follows that \( U_{\alpha} \) is an initial segment of \( \langle U_{\delta}, \leq_{\delta} \rangle \) and \( \leq_{\delta} \mid U_{\alpha} = \leq_{\alpha} \) for every \( \alpha < \delta \). Since \( U_{\alpha} \) is an initial segment of \( \langle U_{\delta}, \leq_{\delta} \rangle \), the order topology of \( \langle U_{\alpha}, \leq_{\alpha} \mid U_{\alpha} \rangle \) is equal to the relative topology it inherits from \( \langle U_{\delta}, \tau^{\leq_{\delta}} \rangle \). And hence \( \tau^{\leq_{\alpha}} = \tau^{\leq_{\delta}} \mid U_{\alpha} \).

By the induction hypothesis, \( \tau^{\leq_{\alpha}} = \rho \mid U_{\alpha} \). So \( \tau^{\leq_{\delta}} \mid U_{\alpha} = \rho \mid U_{\alpha} \). Hence for every \( \alpha < \delta \), \( U_{\alpha} \) is compact in the topology \( \tau^{\leq_{\delta}} \). \( \{U_{\alpha} \mid \alpha < \delta\} \) is an increasing sequence of initial segments of \( \langle U_{\delta}, \leq_{\delta} \rangle \) and \( \sup^{\leq_{\delta}}(\bigcup_{\alpha < \delta} U_{\alpha}) = x_{\delta} \).
So $\bigcup_{\alpha<\delta} U_\alpha \cup \{x_\delta\}$ is compact in $\langle U_\delta, \tau \rangle$. Recall that $\bigcup_{\alpha<\delta} U_\alpha \cup \{x_\delta\} = W$. So $W$ is compact in $\langle U_\delta, \tau^{\leq \delta} \rangle$. It is easy to see that $\text{cl}^{(X,\rho)}(\bigcup_{\alpha<\delta} U_\alpha) = W$. So $W$ is compact in $\langle X, \rho \rangle$.

In order to show that $\rho \upharpoonright W = \tau^{\leq \delta} \upharpoonright W$ it thus suffices to prove that $\tau^{\leq \delta} \upharpoonright W \subseteq \rho \upharpoonright W$. Let $V$ be open in $\langle W, \tau^{\leq \delta} \upharpoonright W \rangle$. Suppose that $x_\delta \notin V$. Then $V = \bigcup_{\alpha<\delta} (V \cap U_\alpha)$. Take $\alpha < \delta$. Then $V \cap U_\alpha$ is open in $\langle U_\alpha, \tau^{\leq \delta} \upharpoonright U_\alpha \rangle$. Since $U_\alpha$ is an initial segment of $\langle U_\delta, \leq \delta \rangle$, $\tau^{\leq \delta} \upharpoonright U_\alpha = \tau^{\leq \delta} \upharpoonright U_\alpha$. So $V \cap U_\alpha$ is open in $\langle U_\alpha, \leq \delta \upharpoonright U_\alpha \rangle$. But $\leq \delta \upharpoonright U_\alpha = \leq \alpha$. So $V \cap U_\alpha$ is open in $\langle U_\alpha, \leq \alpha \rangle$. By (II), $V \cap U_\alpha$ is open in $\langle U_\alpha, \rho \upharpoonright U_\alpha \rangle$. Since $U_\alpha \in \rho$, $V \cap U_\alpha \in \rho$. So $V \in \rho$.

Suppose next that $x_\delta \in V$. Then $V - \{x_\delta\}$ is open in $\langle W, \leq \delta \upharpoonright W \rangle$. By the previous paragraph, $V - \{x_\delta\}$ is open in $\langle W, \rho \upharpoonright W \rangle$. It remains to show that $V$ contains a ($\rho \upharpoonright W$)-neighborhood of $x_\delta$. Clearly, $V$ contains an open final segment of $\langle W, \leq \delta \upharpoonright W \rangle$. Hence for some $\alpha < \delta$, $V \supseteq W - U_\alpha$. But $W - U_\alpha = (U_\delta - U_\alpha) \cap W$. Obviously, $(U_\delta - U_\alpha) \cap W$ is a ($\rho \upharpoonright W$)-neighborhood of $x_\delta$. So $V$ is open in $\langle W, \rho \upharpoonright W \rangle$. This implies that $\tau^{\leq \delta} \upharpoonright W = \rho \upharpoonright W$.

It follows that $\rho \upharpoonright U_\delta = \tau^{\leq \delta}$.

**Case 1.2** $\mathcal{D}_\delta - \{I_\delta\} \neq \emptyset$ and there is no $D \in \mathcal{D}_\delta - \{I_\delta\}$ such that $\text{cf}(D) \geq \aleph_1$. For every $D \in \mathcal{D}_\delta - \{I_\delta\}$ let $\beta_D \in \{1, \omega\}$ and $\{x_{D,i} \mid i < \beta_D\}$ be a strictly increasing unbounded sequence in $D$. For every $D \in \mathcal{D}_\delta - \{I_\delta\}$ let $V_{D,0} = U_{x_{D,0}}$ and for $0 < i < \beta_D$ let $V_{D,i} = U_{x_{D,i}} - U_{x_{D,i-1}}$. Let $\gamma \leq \omega$ and $\{V_i \mid i < \gamma\}$ be a $1$-$1$ enumeration of $\{V_{D,i} \mid D \in \mathcal{D}_\delta - \{I_\delta\} \text{ and } i < \beta_D\}$. Then $V_i$ is a scattered continuous image of a compact interval space with rank $< \alpha_0$. By the induction hypothesis there is a linear ordering $\leq_i$ of $V_i$ such that $\tau^{\leq_i} = \rho \upharpoonright V_i$. Let $\leq_i$ be defined as follows.

1. $I_\delta \leq \delta \leq \ldots \leq \delta \ V_n \leq \delta \leq \ldots \leq \delta \ V_1 \leq \delta \ V_0$.

2. $\leq_i \mid I_\delta = \leq_i'$, and $\leq_i \mid V_i = \leq_i$ for every $i \in \omega$.

It is left to the reader to check that $\tau^{\leq \delta} = \rho \upharpoonright U_\delta$.

**Case 1.3** $\mathcal{D}_\delta = \{I_\delta\}$. Define $\leq_\delta$ as follows: $I_\delta \leq \delta \ x_\delta$ and $\leq_\delta \mid I_\delta = \leq'_\delta$. Note that in this case $x_\delta = \max(\langle U_\delta, \leq_\delta \rangle)$. So if $\delta = \lambda$, then the second part
of \((\ast)_{\alpha_0}\) is fulfilled. It is left to the reader to check that \(\tau_{\leq s} = \rho \upharpoonright U_\delta\).

Case 2 Assume that \(\langle X - \{e\}, \leq_U \rangle (X - \{e\})\) contains an unbounded chain \(I\) with uncountable cofinality, and that \(X - \{e\} - \bigcup_{x \in I} U_x \neq \emptyset\). Let \(\lambda\) be an uncountable cardinal and \(\{x_\alpha \mid \alpha < \lambda\}\) be an unbounded strictly increasing sequence in \(X^< \). Let \(D_0 = \bigcup_{\alpha < \lambda} U_{x_\alpha}\) and \(W = D_0 \cup \{e\}\). Then \(D_0 \in \mathcal{D}_e\). By Facts (1) and (2), there are two possibilities.

1. There is \(D_1 \in \mathcal{D}_e - \{D_0\}\) such that \(\text{cf}(D_1) \geq \aleph_1\), \(\mathcal{D}_e\) is finite and every member of \(\mathcal{D}_e - \{D_0, D_1\}\) has a maximum.

2. \(|\mathcal{D}_e| \leq \aleph_0\) and for every \(D \in \mathcal{D}_e - \{D_0\}\), \(\text{cf}(D) \leq \aleph_0\).

Case 2.1 (1) happens. Let \(\sigma = \{\max(D) \mid D \in \mathcal{D}_e - \{D_0, D_1\}\}\). For every \(x \in \sigma\), \(\text{rk}(U_x) < \alpha_0\). So by the induction hypothesis \(U_x\) is homeomorphic to an interval space. Since \(U_x\) is clopen for every \(x \in \sigma\), it suffices to show that \(Z := X - \bigcup_{x \in \sigma} U_x\) is homeomorphic to an interval space. For \(i = 0, 1\) let \(Z_i = D_i \cup \{e\}\). Then \(Z_i\) fulfill the assumptions of Case 1. Let \(\leq_0\) be a linear ordering of \(Z_0\) such that \(e = \max(\langle Z_0, \leq_0\rangle)\) and \(\tau_{\leq 0} = \rho \upharpoonright Z_0\). Let \(\leq_1\) be a linear ordering of \(Z_1\) such that \(e = \min(\langle Z_1, \leq_1\rangle)\) and \(\tau_{\leq 1} = \rho \upharpoonright Z_1\). Clearly, \(Z = Z_0 \cup Z_1\). Define the relation \(\leq\) on \(Z\) as follows:

1. \(\leq|Z_0 = \leq_0\) and \(\leq|Z_0 = \leq_0\).

2. \(Z_0 \leq Z_1\).

It is left to the reader to check that \(\leq\) is a linear ordering of \(Z_0 \cup Z_1\) and that \(\tau_{\leq} = \rho \upharpoonright Z\).

Case 2.2 (2) happens. This case is similar to Case 2.1. For every \(D \in \mathcal{D}_e - \{D_0\}\) let \(\beta_D \in \{1, \omega\}\) and \(\{x_{D,i} \mid i < \beta_D\}\) be a strictly increasing unbounded sequence in \(D\). For every \(D \in \mathcal{D}_e - \{D_0\}\) let \(V_{D,0} = U_{x_{D,0}}\) and for \(0 < i < \beta_D\) let \(V_{D,i} = U_{x_{D,i}} - U_{x_{D,i-1}}\). Let \(\gamma \leq \omega\) and \(\{V_i \mid i < \gamma\}\) be a 1–1 enumeration of \(\{V_{D,i} \mid D \in \mathcal{D}_e - \{D_0\}\}\) and \(i < \beta_D\). Then \(V_i\) is a scattered continuous image of a compact interval space with rank \(< \alpha_0\).
By the induction hypothesis there is a linear ordering $\leq_i$ of $V_i$ such that $\tau^{\leq_i} = \rho|V_i$. Let $Z = D_0 \cup \{e\}$. Then $Z$ fulfills the assumptions of Case 1. So there is a linear ordering $\leq'$ of $Z$ such that $e = \max(\langle Z, \leq' \rangle)$ and $\tau^{\leq'} = \rho|Z$. Let $\leq$ be defined as follows.

(1) $D_0 \leq e \leq \ldots \leq V_n \leq \ldots \leq V_1 \leq V_0$.

(2) $\leq|Z = \leq'$, and $\leq|V_i = \leq_i$ for every $i \in \omega$.

It is left to the reader to check that $\tau^{\leq} = \rho$.

**Case 3** Assume that $X - \{e\}$ does not contain an unbounded chain with uncountable cofinality. This case too is similar to Case 1.2. For every $D \in \mathcal{D}_e$ let $\beta_D \in \{1, \omega\}$ and $\{x_{D,i} | i < \beta_D\}$ be a strictly increasing unbounded sequence in $D$. For every $D \in \mathcal{D}_e$ let $V_{D,0} = U_{x_{D,0}}$ and for $0 < i < \beta_D$ let $V_{D,i} = U_{x_{D,i}} - U_{x_{D,i-1}}$. Let $\gamma \leq \omega$ and $\{V_i | i < \gamma\}$ be a 1–1 enumeration of $\{V_{D,i} | D \in \mathcal{D}_e \text{ and } i < \beta_D\}$. Then $V_i$ is a scattered continuous image of a compact interval space with rank $< \alpha_0$. By the induction hypothesis there is a linear ordering $\leq_i$ of $V_i$ such that $\tau^{\leq_i} = \rho|V_i$. Let $\leq$ be defined as follows.

(1) $e \leq \ldots \leq V_n \leq \ldots \leq V_1 \leq V_0$.

(2) $\leq|V_i = \leq_i$ for every $i \in \omega$.

It is left to the reader to check that $\tau^{\leq} = \rho$.

### 4 A lemma about CO spaces

**Definition 4.1.** Let $K$ and $L$ be unitary scattered compact spaces.

(a) $K$ and $L$ are *almost homeomorphic* ($K \approx L$) if there are clopen neighborhoods $U$ and $V$ of $e^K$ and $e^L$ respectively such that $U \cong V$.

(b) We define the relation $K \prec^w L$ as follows. $K \prec^w L$ if for some $K' \approx K$, $K' \subseteq L$, $e^{K'} = e^L$ and $K \not\cong L$.

We also define the relation $K \prec L$. Say that $K \prec L$ if for some $K' \approx K$, $K' \subseteq L$, $\text{rk}(K) = \text{rk}(L)$ and $K \not\cong L$. Note that this implies that $e^{K'} = e^L$. 51
(c) Let $X$ be a compact space and $D \subseteq X$. For every $d \in D$ let $V_d$ be an open neighborhood of $d$. The family $\mathcal{V} := \{V_d \mid d \in D\}$ is called a strong Hausdorff system for $D$ if for every distinct $d, e \in D$, $V_d \cap V_e = \emptyset$ and

$$\text{cl}(\bigcup \{V_d \mid d \in D\}) = \bigcup \{\text{cl}(V_d) \mid d \in D\} \cup \text{cl}(D).$$

$\mathcal{V}$ is called a clopen strong Hausdorff system for $D$ if every $V_d$ is clopen.

**Theorem 4.2.** Let $X$ be a scattered compact space, and assume that for every subset $S$ of $X$ with regular cardinality there is $D \subseteq S$ such that $|D| = |S|$, and $D$ has a clopen strong Hausdorff system.

(a) Suppose that there are unitary scattered compact spaces $L$ and $M$ and a family $\{L_i \mid i \in \omega\}$ of subsets of $X$ such that $M \prec L$ and for every $i < j < \omega$, $L_i \approx L$ and $e^{L_i} \neq e^{L_j}$. Then $X$ is not a CO space.

(b) Suppose that there are unitary scattered compact spaces $K$, $L$ and $M$ such that $M \prec L \prec^w K \subseteq X$, then $X$ is not a CO space.

**Definition 4.3.** Let $Y$ be a scattered compact space.

(a) For an ordinal $\theta$ define $R_\theta(Y) := \{z \in Y \mid \text{rk}^Y(z) = \theta\}$.

(b) Let $K$ be a unitary space with rank $\theta$. We say that $Y$ is $K$-based if $\text{rk}(Y) \geq \theta + 1$, and there are $\mathcal{U}, \mathcal{V} \subseteq \text{Clop}(Y)$ such that the following holds.

1. $\mathcal{U}$ is a pairwise disjoint family, and $\mathcal{V}$ is a pairwise disjoint family.

2. For every $U \in \mathcal{U}$, $U \approx K$, and for every $V \in \mathcal{V}$, $V$ is unitary and $\text{rk}(V) = \theta + 1$.

3. $R_\theta(Y) \subseteq \bigcup \mathcal{U}$ and $R_{\theta+1}(Y) \subseteq \bigcup \mathcal{V}$.

(c) Suppose that $K, L$ are unitary spaces with the same rank $\theta$. We say that $Y$ is $\{K, L\}$-based if $\text{rk}(Y) \geq \theta + 1$, and there are $\mathcal{U}, \mathcal{V} \subseteq \text{Clop}(Y)$ such that the following holds.

1. $\mathcal{U}$ is a pairwise disjoint family, and $\mathcal{V}$ is a pairwise disjoint family.
(2) For every $U \in \mathcal{U}$, $U \approx K$ or $U \approx L$. For every $V \in \mathcal{V}$, $V$ is unitary and $\text{rk}(V) = \theta + 1$.

(3) $R_\theta(Y) \subseteq \bigcup \mathcal{U}$ and $R_{\theta+1}(Y) \subseteq \bigcup \mathcal{V}$.

(4) For every $W \in \text{Clop}(Y)$, if $\text{rk}(W) \geq \theta + 1$, then there are $U,V \in \text{Clop}(Y)$ such that $U \subseteq W$, $U \approx K$ and $V \approx L$.

Note that a space $Y$ is $K$-based iff it is $\{K,K\}$-based. Suppose that $Y$ is $\{K,L\}$-based and $\mathcal{U}, \mathcal{V}$ are families assured by the $\{K,L\}$-basedness of $Y$. We denote $\mathcal{U}, \mathcal{V}$ by $\mathcal{U}_Y$ and $\mathcal{V}_Y$ respectively.

The trivial proof of the following proposition is left to the reader.

**Proposition 4.4.** Let $Y, Z, K, L$ be a compact scattered spaces, and assume that $K$ and $L$ are unitary with the same rank $\theta$.

(a) Suppose that $Y$ is $\{K,L\}$-based and $U \in \text{Clop}(Y)$. If $\text{rk}(U) \geq \theta + 1$, then $U$ is $\{K,L\}$-based.

(b) Assume that $K \not\approx L$ and that $Y$ is $K$-based and $Z$ is $\{K,L\}$-based. Then $Y \not\approx Z$.

(c) Assume that $K \not\approx L$ and that $Y$ is $K$-based and $Z$ is $\{K,L\}$-based. Assume further that $Y, Z$ are clopen unitary subspaces of $X$. Then $e^Y \neq e^Z$.

(d) Assume that $K \not\approx L$ and that $Y$ is $\{K,L\}$-based. Let $Z = Y - \bigcup \{U \in \mathcal{U}_Y \mid U \approx L\}$. Then $Z$ is $K$-based, $\text{rk}(Z) = \text{rk}(Y)$, $e^Z = e^Y$, $\mathcal{U}_Z = \{U \in \mathcal{U}_Y \mid U \approx K\}$ and $\mathcal{V}_Z = \{V \cap Z \mid V \in \mathcal{V}_Y\}$.

**Proposition 4.5.** Let $X$ be a scattered compact space, and assume that for every infinite subset $S$ of $X$ there is an infinite subset $D \subseteq S$ such that $D$ has a clopen strong Hausdorff system. Let $L$ be a unitary scattered compact space, and $\{L_i \mid i \in \omega\}$ be a family of subsets of $X$ such that

(1) $L_i \approx L$ for every $i \in \omega$ and for $i < j < \omega$, $e^{L_i} \neq e^{L_j}$.

or
(2) for every $i$, $L_i$ is unitary and $L$-based, for every $i, j$, \( \text{rk}(L_i) = \text{rk}(L_j) \), and for $i < j < \omega$, $e^L_i \neq e^L_j$.

Then $X$ has a unitary $L$-based subset $F$ such that $\text{rk}(F) = \text{rk}(L_i) + 1$.

**Proof** Let $X$, $L$ and $\{L_i \mid i \in \omega \}$ be as in the hypotheses of the proposition. Denote $\text{rk}(L)$ by $\theta$ and $\text{rk}(L_i)$ by $\alpha$. Let $A = \{e^L_i \mid i \in \omega \}$. For every $a \in A$, if $a = e^L_i$, denote $L_i$ by $L_a$. Let $x$ be an isolated point of $\text{acc}(A)$ and let $U \in \text{Nbr}_{clp}^X(x)$ be such that $U \cap \text{acc}(A) = \{x\}$. Let $B = A \cup U$. Hence $\text{cl}(B) = B \cup \{x\}$. There are an infinite subset $C \subseteq B$ and a family $\mathcal{T} = \{T_c \mid c \in C\}$ such that $\mathcal{T}$ is a clopen strong Hausdorff system for $C$. So $\text{cl}^X(\bigcup_{c \in C} T_c) = (\bigcup_{c \in C} T_c) \cup \{x\}$. For every $c \in C$ let $L'_c = L_c \cap U \cap T_c$, and let $F = \text{cl}^X(\bigcup_{c \in C} L'_c)$. Then $F = (\bigcup_{c \in C} L'_c) \cup \{x\}$. Clearly, $R_{\alpha}(F) \cap (\bigcup_{c \in C} L'_c) = C$ and $\text{acc}(C) = \{x\}$. So $R_{\alpha+1}(F) = \{x\}$, $R_{\alpha}(F) = C$ and $\text{rk}(F) = \alpha + 1$. Hence $F$ is unitary. We now distinguish between the two cases.

**Case 1** For every $i \in \omega$, $L_i \approx L$.

Hence $\alpha = \theta$. Let $\mathcal{U} = \{L'_c \mid c \in C\}$. Then $\mathcal{U} \subseteq \text{Clop}(F)$, $\mathcal{U}$ is a pairwise disjoint family and $R_{\theta}(F) = C \subseteq \bigcup \mathcal{U}$. Also, for every $c \in C$, $L'_c \approx L_c \approx L$.

We now distinguish between the two cases. Define $\mathcal{U}_F$ to be $\mathcal{U}$ and $\mathcal{V}_F$ to be $\{F\}$. Then $\mathcal{U}_F$ and $\mathcal{V}_F$ demonstrate that $F$ is $L$-based.

**Case 2** For every $i \in \omega$, $L_i$ is $L$-based.

Recall that $L'_c = L_c \cap U \cap T_c$. So $L'_c \in \text{Clop}(L_c)$. So by Proposition 4.4(a), $L'_c$ is $L$-based. Let $\mathcal{U} = \bigcup \{\mathcal{U}_c \mid c \in C\}$ and $\mathcal{V} = \bigcup \{\mathcal{V}_c \mid c \in C\}$. Clearly, $\mathcal{U}$ and $\mathcal{V}$ are pairwise disjoint families. (This is so because $\{L'_c \mid c \in C\}$ is a pairwise disjoint family). Also, for every $U \in \mathcal{U}$, $U \approx L$ and for every $V \in \mathcal{V}$, $V$ is unitary and $\text{rk}(V) = \theta + 1$. Since $\text{rk}^F(x) = \alpha + 1 > \theta + 1$ and $F - \{x\} = \bigcup_{c \in C} L'_c$, it follows that $R_{\theta}(F) = \bigcup_{c \in C} R_{\theta}(L'_c)$ and $R_{\theta+1}(F) = \bigcup_{c \in C} R_{\theta+1}(L'_c)$. Recall that $R_{\theta}(L'_c) \subseteq \bigcup \mathcal{U}_c$ and $R_{\theta+1}(L'_c) \subseteq \bigcup \mathcal{V}_c$. Hence $R_{\theta}(F) \subseteq \bigcup \mathcal{U}$ and $R_{\theta+1}(F) \subseteq \bigcup \mathcal{V}$. So $\mathcal{U}$ and $\mathcal{V}$ demonstrate that $F$ is $L$-based.

**Proposition 4.6.** Let $X$ be a compact CO space and $L \prec^w K \subseteq X$. Then
there is a family \( \{L_i \mid i \in \omega \} \) of subsets of \( X \) such that for every \( i, L_i \approx L \) and for every \( i \neq j, e^{L_i} \neq e^{L_j} \).

**Proof** We may assume that \( K \) is clopen in \( X \). Let \( L' \subseteq K \) be such that \( L' \approx L \) and \( e^{L'} = e^K \). We define by induction clopen sets \( L_n \) and \( L_{n,i} \), \( i \leq n \). We assume by induction on \( n \) that for every \( i \leq n \), \( L_{n,i} \approx L' \), \( L_n = \bigcup_{i \leq n} L_{n,i} \) and that for every \( i \neq j \), \( e^{L_{n,i}} \neq e^{L_{n,j}} \).

Let \( L_0 = L_{0,0} \in \text{Clop}(X) \) be homeomorphic to \( L' \). Then the the induction hypotheses hold for \( n = 0 \). Suppose that \( L_n \) and \( L_{n,i} \), \( i \leq n \) have been defined. Let \( L_{n+1} \) be a clopen set homeomorphic to \( L' \cup L_n \) and \( \psi : L' \cup L_n \approx L_{n+1} \). For \( i \leq n \) define \( L_{n+1,i} = \psi(L_{n,i}) \) and let \( L_{n+1,n+1} = \psi(L') \). We check that the induction hypotheses hold. The only fact that needs to be verified is that for every distinct \( i, j \leq n+1 \), \( e^{L_{n+1,i}} \neq e^{L_{n+1,j}} \). If \( i, j \leq n \) then \( e^{L_{n+1,i}} = \psi(e^{L_{n,i}}) \) and \( e^{L_{n+1,j}} = \psi(e^{L_{n,j}}) \). So since \( e^{L_{n,i}} \neq e^{L_{n,j}} \) and \( \psi \) is \( 1-1 \), it follows that \( e^{L_{n+1,i}} \neq e^{L_{n+1,j}} \). Suppose that \( i \leq n \) and \( j = n+1 \). Note that \( L_{n,i} \in \text{Nbr}_c(e^{L_{n,i}}) \). However, there is no \( U \in \text{Nbr}_c(e^{L'}) \) such that \( U \approx L_{n,i} \). Suppose by contradiction that such a \( U \) exists. Since \( K \in \text{Nbr}_c(e^{L'}) \), it follows that \( K \approx K \cap U \approx U \approx L' \approx L \). A contradiction. So \( U \) does not exist, and hence \( e^{L'} \neq e^{L_{n,i}} \). It follows that

\[
e^{L_{n+1,n+1}} = \psi(e^{L'}) \neq \psi(e^{L_{n,i}}) = e^{L_{n+1,i}}.
\]

So the induction hypotheses hold for \( n+1 \). This completes the inductive construction.

It follows that \( \{e^{L_{n,i}} \mid n \in \omega, i \leq n \} \) is infinite. □

**Proof of Theorem 4.2 (a)** Assume by way of contradiction that \( X \) is a CO space and that \( L, M \) and \( \{L_i \mid i \in \omega \} \) are as in the hypotheses of [4.2 (a)]. Denote \( \text{rk}(L) \) by \( \theta \).

We prove by induction on \( \alpha \geq \theta + 1 \) that there is \( X_\alpha \subseteq X \) such that \( X_\alpha \) is \( L \)-based, \( X_\alpha \) is unitary and \( \text{rk}(X_\alpha) = \alpha \). By the first case of Proposition 4.3, there is a subspace \( F \subseteq X \) such that \( F \) is unitary of rank \( \theta + 1 \) and \( F \) is \( L \)-based. That is, \( X_{\theta+1} \) exists. Suppose that \( X_\alpha \) exists. We may assume
that $X_\alpha$ is clopen in $X$. Denote $U_{X_\alpha}, V_{X_\alpha}$ by $\mathcal{U}$ and $\mathcal{V}$ respectively. For every $x \in R_\theta(X_\alpha)$ there is a unique $L_x \in \mathcal{U}$ such that $x \in L_x$. So $L_x \approx L_x$. Let $x \in R_{\theta+1}(X_\alpha)$. Then there is a unique $V_x \in \mathcal{V}$ such that $x \in V_x$. Choose $A_x \subseteq R_\theta(V_x)$ such that $A_x$ and $R_\theta(V_x) - A_x$ are infinite. Note that $\text{acc}(A_x) = \text{acc}(R_\theta(V_x) - A_x) = \{x\}$. For every $y \in A_x$ choose $M_y \subseteq L_y$ such that $M_y \approx M_x$. Define

$$
\mathcal{U}' = \{M_y \mid x \in R_{\theta+1}(X_\alpha) \text{ and } y \in A_x\} \\
\cup \{L_y \mid x \in R_{\theta+1}(X_\alpha) \text{ and } y \in R_\theta(V_x) - A_x\} \\
\cup \{L_y \mid y \in R_\theta(X_\alpha) - \bigcup \mathcal{V}\}.
$$

Let $Y = \text{cl}(\bigcup \mathcal{U}')$ and $\mathcal{V}' = \{V \cap Y \mid V \in \mathcal{V}\}$.

We shall see that $Y$ is $\{L, M\}$-based, and that $\mathcal{U}_Y$ and $\mathcal{V}_Y$ can be taken to be $\mathcal{U}'$ and $\mathcal{V}'$. Let $y \in R_\theta(X_\alpha)$. Define $N_y = M_y$ if for some $x \in R_{\theta+1}(X_\alpha)$, $y \in A_x$, and otherwise let $N_y = L_y$. Then for every $y \in R_\theta(X_\alpha)$, $y \in N_y \subseteq Y$ and $N_y \in \text{Clop}(Y)$. So $R_\theta(X_\alpha) \subseteq Y$ and for every $y \in R_\theta(X_\alpha)$, $\text{rk}_Y(y) = \text{rk}_{N_y}(y) = \theta$. That is, $R_\theta(X_\alpha) \subseteq R_\theta(Y)$. Suppose by way of contradiction that $R_\theta(Y) - R_\theta(X_\alpha) \neq \emptyset$, and let $y \in R_\theta(Y) - R_\theta(X_\alpha)$. Since $\text{rk}_Y(y) \leq \text{rk}_{X_\alpha}(y)$, $\text{rk}_{X_\alpha}(y) > \theta$. Hence every neighborhood of $y$ intersects $R_\theta(X_\alpha)$. Since $R_\theta(X_\alpha) \subseteq R_\theta(Y)$, every neighborhood of $y$ intersects $R_\theta(Y)$. This contradicts the fact that $\text{rk}_Y(y) = \theta$. So $R_\theta(X_\alpha) = R_\theta(Y)$. Clearly, $\text{cl}(R_\theta(X_\alpha)) = D_\theta(X_\alpha)$ and $\text{cl}(R_\theta(Y)) = D_\theta(Y)$. So $D_\theta(X_\alpha) = D_\theta(Y)$. So for every $\eta \geq \theta$, $D_\eta(X_\alpha) = D_\eta(Y)$. It follows that $\text{rk}(Y) = \text{rk}(X_\alpha)$, $Y$ is unitary and $e^Y = e^{X_\alpha}$.

We next show that one can take $U_Y$ to be $\mathcal{U}'$. Clearly, $\mathcal{U}'$ is a pairwise disjoint family, every member of $\mathcal{U}'$ is almost homeomorphic to either $L$ or $M$. For every space $Z$, $R_{\theta+1}(Z) = D_{\theta+1}(Z) - D_{\theta+2}(Z)$. So since $D_\eta(X_\alpha) = D_\eta(Y)$ for every $\eta \geq \theta$, $R_{\theta+1}(X_\alpha) = R_{\theta+1}(Y)$. By the construction, for every $x \in R_{\theta+1}(X_\alpha)$ and a neighborhood $W$ of $x$ there are $U, V \in \text{Clop}(Y)$ such that $U, V \subseteq W$, $U \approx L$ and $V \approx M$. So for every $x \in R_{\theta+1}(Y)$ and a neighborhood $W$ of $x$ there are $U, V \in \text{Clop}(Y)$ such that $U, V \subseteq W$, $U \approx L$ and $V \approx M$. By the construction, $R_\theta(X_\alpha) \subseteq \bigcup \mathcal{U}'$. So since
$R_\theta(Y) = R_\theta(X_\alpha)$, it follows that $R_\theta(Y) \subseteq \bigcup \mathcal{U}'$.

We check that $\mathcal{V}_Y$ can be taken to be $\mathcal{V}'$. Clearly, $\mathcal{V}'$ is a pairwise disjoint family. Since $R_{\theta+1}(X_\alpha) \subseteq \bigcup \mathcal{V}$, it follows that $R_{\theta+1}(X_\alpha) \cap Y \subseteq (\bigcup \mathcal{V}) \cap Y$. But $R_{\theta+1}(X_\alpha) = R_{\theta+1}(Y) = R_{\theta+1}(Y) \cap Y$ and $(\bigcup \mathcal{V}) \cap Y = \bigcup \mathcal{V}'$. So $R_{\theta+1}(Y) \subseteq \bigcup \mathcal{V}'$. Let $V \in \mathcal{V}$. Denote $e^V$ by $x$. So $V = V_x$. Also, $x \in R_{\theta+1}(X_\alpha) = R_{\theta+1}(Y)$. So $x \in V \cap Y$ and indeed $(V \cap Y) \cap R_{\theta+1}(Y) = \{x\}$. So $V \cap Y$ is unitary and $rk^Y(V) = \theta + 1$. We have shown everything that is required in order to conclude that $Y$ is $\{L, M\}$-based.

We verify that $Y \prec X_\alpha$. We have already seen that $rk(Y) = rk(X_\alpha)$ and that $Y$ is unitary. Also, $Y \subseteq X_\alpha$ and $X_\alpha$ is unitary. It remains to show that $Y \not\prec X_\alpha$. Recall that $X_\alpha$ is $L$-based and that $Y$ is $\{L, M\}$-based. Also, $M \prec L$ and hence $M \not\prec L$. Then by Proposition 4.5(b), $Y \not\prec X_\alpha$. So $Y \prec X_\alpha$.

It follows that $Y \prec^\omega X_\alpha$. By Proposition 4.6 there is a family $\{Y_i \mid i \in \omega\}$ of subsets of $X$ such that for every $i$, $Y_i \approx Y$ and for every $i \neq j$, $e^{Y_i} \neq e^{Y_j}$.

For every $i \in \omega$ let $Z_i = Y_i - \bigcup\{U \in \mathcal{U}_{Y_i} \mid U \approx M\}$. By Proposition 4.4(d), $Z_i$ is $L$-based, $e^{Z_i} = e^{Y_i}$ and $rk(Z_i) = \alpha$. By the second case of Proposition 4.3, there is $Z \subseteq X$ such that $rk(Z) = \alpha + 1$ and $Z$ is $L$-based. So $X_{\alpha+1} := Z$ is as required.

Let $\delta$ be a limit ordinal, and suppose that for every $\alpha < \delta$, $X_\alpha$ exists. Set $\theta = rk(L)$. Let $\lambda = cf(\delta)$ and $\{\alpha_i \mid i < \lambda\}$ be a strictly increasing sequence converging to $\delta$ such that $\alpha_0 > \theta + 1$. For $i < \lambda$ let $Y_i \subseteq X$ be a clopen unitary $L$-based set with rank $\alpha_i$. Hence for every $i < \lambda$, $rk^X(e^{Y_i}) = \alpha_i$. It follows that $e^{Y_i} \neq e^{Y_j}$ for every $i \neq j$. So $|\{e^{Y_i} \mid i < \lambda\}| = \lambda$. Hence there are $A \subseteq \{e^{Y_i} \mid i < \lambda\}$ and $\mathcal{W} = \{W_a \mid a \in A\}$ such that $|A| = \lambda$ and $\mathcal{W}$ is a clopen strong Hausdorff system for $A$. That is, $\mathcal{W}$ is a pairwise disjoint family consisting of clopen sets, $a \in W_a$ for every $a \in A$, and $cl^X(\bigcup \mathcal{W}) = (\bigcup \mathcal{W}) \cup acc^X(A)$. For $a = e^{Y_i} \in A$ set $Y^a = Y_i$.

Note the following fact. (*) If $\mathcal{V} = \{V_b \mid b \in B\}$ is a clopen strong Hausdorff system for $B$, and $\mathcal{F} = \{F_b \mid b \in B\}$ is a family of closed sets such
that \( b \in F_b \subseteq V_b \) for every \( b \in B \). Then \( \text{cl}^X(\bigcup F) = (\bigcup F) \cup \text{acc}^X(B) \).

For every \( a \in A \) let \( Y_0^a = Y^a \cap W_a \), and let \( \hat{Y} = \text{cl}^X(\bigcup \{Y_0^a \mid a \in A\}) \). So by (\ref{eq:Y0}), \( \hat{Y} = (\bigcup \{Y_0^a \mid a \in A\}) \cup \text{acc}^X(A) \). Note that for every \( a \in A \), \( Y_0^a \) is clopen in \( \hat{Y} \), \( Y_0^a \approx Y^a \) and \( a = e^{Y_0^a} \). Hence \( \text{rk}(\hat{Y})(a) = \text{rk}^{Y_0^a}(a) = \text{rk}Y^a(a) \). It follows that \( \sup_{a \in A} \text{rk}(\hat{Y})(a) \geq \delta \). So \( \text{rk}(\hat{Y}) \geq \delta \).

**Claim 1** For every \( y \in \hat{Y} - \bigcup_{a \in A} Y_0^a \), \( \text{rk}(\hat{Y})(y) > \theta + 1 \).

**Proof** Let \( y \in \hat{Y} - \bigcup_{a \in A} Y_0^a \). Recall that \( \hat{Y} = (\bigcup_{a \in A} Y_0^a) \cup \text{acc}^X(A) \). So \( y \in \text{acc}^X(A) \). We show that for every \( V \in \text{Nbr}^X(y) \) there is \( z \in V \cap \hat{Y} \) such that \( \text{rk}(\hat{Y})(z) > \theta + 1 \). We may assume that \( V \) is clopen. Since \( y \in \text{acc}^X(A) \), it follows that \( V \cap A \neq \emptyset \). Let \( a \in V \cap A \). Then \( \text{rk}^{Y_0^a \cap V}(a) = \text{rk}^{Y_0^a}(a) = \text{rk}Y^a(a) > \theta + 1 \). Clearly, \( Y_0^a \cap V \subseteq \hat{Y} \). So \( \text{rk}(\hat{Y})(a) > \theta + 1 \). It follows that \( \text{rk}(\hat{Y})(y) > \theta + 1 \). So Claim 1 is proved.

**Claim 2** \( \hat{Y} \) is \( L \)-based.

**Proof** Recall that for every \( a \in A \), \( Y_0^a = Y^a \cap W_a \) and \( \text{rk}(Y_0^a) = \text{rk}(Y^a) > \theta + 1 \). Hence by Proposition \((\ref{prop:4.4}a)\), \( Y_0^a \) is \( L \)-based. Let \( U_a, V_a \) demonstrate that \( Y_0^a \) is \( L \)-based. Set \( U = \bigcup_{a \in A} U_a \) and \( V = \bigcup_{a \in A} V_a \). We show that \( U, V \) demonstrate that \( \hat{Y} \) is \( L \)-based. Since \( \{Y_0^a \mid a \in A\} \) is a pairwise disjoint family, and for every \( a \in A \), \( U_a \) is a pairwise disjoint family, it follows that \( U \) is a pairwise disjoint family. Similarly, \( V \) is a pairwise disjoint family. It is also trivial that for every \( U \in U \), \( U \approx L \) and that for every \( V \in V \), \( \text{rk}(V) = \theta + 1 \).

Let \( y \in R^\theta(\hat{Y}) \). By Claim 1, there is \( a \in A \) such that \( y \in Y_0^a \). Since \( Y_0^a \) is clopen in \( \hat{Y} \), \( \text{rk}^{Y_0^a}(\hat{Y}) = \theta \). So there is \( U \in U_a \) such that \( y \in U \). But \( U \in U \). It follows that \( y \in U \cup V \). That is, \( R^\theta(\hat{Y}) \subseteq \bigcup U \).

An identical argument shows that \( R^{\theta+1}(\hat{Y}) \subseteq \bigcup V \). We have shown that \( \hat{Y} \) is \( L \)-based. So Claim 2 is proved.

Let \( x \in \hat{Y} \) be such that \( \text{rk}(\hat{Y})(x) = \delta \), and let \( T \in \text{Clop}(X) \) be such that \( T \cap D_0(\hat{Y}) = \{x\} \). Set \( Y = T \cap \hat{Y} \). Hence \( Y \) is unitary of rank \( \delta \). By Proposition \((\ref{prop:4.4}a)\), \( Y \) is \( L \)-based. Define \( X_\delta = Y \). Then \( X_\delta \) is as required.

We have proved that for every ordinal \( \alpha \), \( X \) contains a subset with rank \( \alpha \).
A contradiction. So \( X \) is not a CO-space.

(b) Let \( M \prec L \prec wK \subseteq X \) be as in Part (b). By Proposition 4.6, there is a family \( \{ L_i \mid i \in \omega \} \) such for every \( i \in \omega \), \( L_i \cong L \) and for every \( i \neq j \), \( e^{L_i} \neq e^{L_j} \). By Part (a) of this theorem, and since \( M \prec L \), \( X \) is not a CO space.

5 CO spaces must omit the obstructions

The existence of strong Hausdorff systems is used in this section. However, the full strength of Lemma 2.2 is not needed and only the following fact is used.

**Corollary 5.1.** Let \( X \) be a continuous image of a compact interval space. Let \( A \subseteq X \) be such that \( |A| \) is regular and \( \text{cl}(A) \) is scattered. Then there is \( B \subseteq A \) such that \( |B| = |A| \) and \( B \) has a strong Hausdorff system.

**Proposition 5.2.** Let \( X \) be a scattered continuous image of an interval space, and assume that \( X \) is a CO space.

(a) If \( \langle \kappa, \lambda, \mu \rangle \) is a legal triple, then \( X_{\kappa,\lambda,\mu} \) is not embeddable in \( X \).

(b) The set \( \{ e^F \mid F \subseteq X \text{ and } F \cong X_{\aleph_1} \} \) is finite.

**Proof** (a) Suppose by way of contradiction that \( \langle \kappa, \lambda, \mu \rangle \) is a legal triple and \( X_{\kappa,\lambda,\mu} \) is embeddable in \( X \). We may assume that \( \kappa \leq \lambda \leq \mu \). It is then obvious that \( \mu + 1 \prec \mu + 1 + \lambda^* \prec X_{\kappa,\lambda,\mu} \). So by Theorem 4.2(b) and Corollary 5.1, \( X \) is not a CO space. A contradiction, so \( X_{\kappa,\lambda,\mu} \) is not embeddable in \( X \).

(b) Assume by contradiction that \( \{ e^F \mid F \subseteq X \text{ and } F \cong X_{\aleph_1} \} \) is finite. Clearly, \( X_{\aleph_0} \prec \aleph \aleph_1 \), so by Theorem 4.2(a) and Corollary 5.1, \( X \) is not a CO space. A contradiction, so \( \{ e^F \mid F \subseteq X \text{ and } F \cong X_{\aleph_1} \} \) is finite.

We also have to prove that obstructions of the type \( X_{\lambda,\mu} \) are not embeddable in \( X \). In order to show this we consider the following space. Let \( \lambda \) be a cardinal and \( S \subseteq \lambda \). For \( \alpha \in S \) let \( L_{\alpha} = 1 + \omega^* \) and for every \( \alpha \in \lambda - S \) let
\( L_\alpha = 1 \). Define \( X_{\lambda,S} \) to be the topological space with universe \( \sum_{\alpha < \lambda} L_\alpha + 1 \) and with the order topology as its topology.

**Proposition 5.3.** (a) Let \( \lambda \) be an uncountable regular cardinal and \( S_1, S_2 \subseteq \lambda \) be subsets of \( \lambda \) such that \( S_1 - S_2 \) is stationary. Then \( X_{\lambda,S_1} \ncong X_{\lambda,S_2} \). Also, \( X_{\lambda,S_1} \) is not homeomorphic to an ordinal.

(b) Let \( X \) be a scattered continuous image of a compact interval space, and assume that \( X \) is a CO space. Let \( Y = X_{\lambda,\vec{\mu}} \) and assume that \( \text{Dom}(\vec{\mu}) \) is a stationary set in \( \lambda \). Then \( Y \) is not embeddable in \( X \).

**Proof** (a) Suppose by way of contradiction that \( f : X_{\lambda,S_1} \cong X_{\lambda,S_2} \). For \( S \subseteq \lambda \) represent \( X_{\lambda,S} \) in the following way. \( X_{\lambda,S} = (\lambda + 1) \cup (S \times \omega) \), where for \( \beta \in S \), \( L_\beta = \{\beta\} \cup (\{\beta\} \times \omega) \) and for \( \beta \notin S \), \( L_\beta = \{\beta\} \). For \( \alpha < \lambda \) denote \( X_{\lambda,S} \restriction [\alpha, \lambda] = [\alpha, \lambda] \cup ([S \cap [\alpha, \lambda]) \times \omega) \).

Note that \( \text{acc}(\lambda + 1) \subseteq \text{acc}(X_{\lambda,S_1}) \subseteq \lambda + 1 \). This implies that \( f[\text{acc}(\lambda)] \subseteq \lambda + 1 \). It follows that there is a club \( C \) in \( \lambda \) such that \( f \restriction C = \text{Id} \). For every \( \beta \in S_1 \), \( \beta \in \text{acc} X_{\lambda,S_1}(\{\beta\} \times \omega) \) and for every \( \beta \in S_2 \), \( \beta \notin \text{acc} X_{\lambda,S_2}(X_{\lambda,S_2} \restriction [\beta+1, \lambda]) \). Hence for every \( \beta \in (S_1 - S_2) \cap C \), there are \( \gamma_\beta < \beta \) and \( m_\beta, n_\beta \in \omega \) such that \( f(\langle \beta, m_\beta \rangle) = \gamma_\beta \) or \( f(\langle \beta, m_\beta \rangle) = \langle \gamma_\beta, n_\beta \rangle \)\. There is a stationary subset \( S \subseteq (S_1 - S_2) \cap C \) such that for every \( \alpha, \beta \in S \), \( m_\alpha = m_\beta \) and \( n_\alpha = n_\beta \). By Fodor’s Lemma, for some distinct \( \alpha, \beta \in S \), \( \gamma_\alpha = \gamma_\beta \). So \( f(\langle \beta, n_\beta \rangle) = f(\langle \gamma, n_\gamma \rangle) \). So \( f \) is not 1–1. A contradiction, so \( f \) does not exist.

We show that \( X_{\lambda,S_1} \) is not homeomorphic to an ordinal space. Since \( \lambda + 1 = X_{\lambda,\emptyset} \), \( X_{\lambda,S_1} \ncong \lambda + 1 \). But \( X_{\lambda,S_1} \) is a unitary space with rank \( \lambda \), and upto a homeomorphism, the only ordinal space which is unitary with rank \( \lambda \) is \( \lambda + 1 \). So \( X_{\lambda,S_1} \) is not homeomorphic to an ordinal space.

(b) Suppose by way of contradiction that \( Y = X_{\lambda,\vec{\mu}} \subseteq X \). Let \( S = \text{Dom}(\vec{\mu}) \). Let \( Y_0 \) be a clopen unitary subspace of \( Y \) such that \( e^{Y_0} = \lambda + 1 \). Then \( Y_0 \) contains a clopen subspace \( Y_1 \) homeomorphic to \( X_{\lambda,\vec{\mu}_1} \), where \( \text{Dom}(\vec{\mu}_1) \) is a final segment of \( S \). Clearly \( e^{Y_1} = \lambda \). Now, replace \( Y \) by \( Y_1 \). Then we may assume that \( e^Y = \lambda \).
By the easy direction of Theorem 3.1, $X_{\lambda,\vec{\mu}}$ is not homeomorphic to an interval space. Moreover, if $U \in \text{Nbr}_{\text{clp}}^Y(\lambda)$, then $U$ contains a space homeomorphic to $X_{\lambda,\vec{\mu}'}$, where $\text{Dom}(\vec{\mu}')$ is a final segment of $S$. So $U$ is not homeomorphic to an interval space. Let $Z$ be the subspace of $Y$ whose universe is $(\lambda + 1) \cup (S \times \hat{\omega})$. Then $Z$ is homeomorphic to $X_{\lambda,S}$ which is an interval space. So $Y \not\approx Z$. Also, $e^Z = \lambda$. It follows that $Z \prec^w Y$.

Clearly, $\lambda + 1$ is a closed unitary subspace of $Z$, $e^{\lambda+1} = \lambda = e^Z$ and $\text{rk}^Z(\lambda) = \text{rk}^{\lambda+1}(\lambda)$. For every $U \in \text{Nbr}^Z_{\text{clp}}(\lambda)$ there is a final segment $S'$ of $S$ such that $U$ contains a subspace homeomorphic to $X_{\lambda,S'}$. By Part (a), $U$ is not homeomorphic to an ordinal. So $Z \not\approx \lambda + 1$. It follows that $\lambda + 1 \prec Z$. We thus have $\lambda + 1 \prec Z \prec^w Y$. By Corollary 5.1 and Theorem 4.2(b), $X$ is not a CO space. A contradiction so $X_{\lambda,\vec{\mu}}$ is not embeddable in $X$.  

6 The characterization

In this section we prove Theorem 1.1.

**Theorem 6.1.** Let $X$ be a scattered continuous image of a compact interval space, and assume that $X$ is a CO space. Then there is a finite family of pairwise disjoint spaces $\{Y_i \mid i \in I\}$ and an ordinal $\alpha + 1$ disjoint from the $Y_i$’s such that $X \cong (\alpha + 1) \cup \bigcup_{i \in I} Y_i$ and

1. For every $i \in I$ either $Y_i \cong X_{\lambda,\mu}$, where $\lambda, \mu$ are infinite regular cardinals and $\mu > \aleph_0$ or $Y_i \cong X_{\aleph_1}$.

2. $\alpha \geq \alpha(Y_i)$ for every $i \in I$.

We quote the following Theorem from [BBR]

**Theorem 6.2.** Let $X$ be a compact interval space, and assume that $X$ is a CO space. Then there is a finite family of pairwise disjoint spaces $\{Y_i \mid i \in I\}$ and an ordinal $\alpha + 1$ disjoint from the $Y_i$’s such that $X \cong (\alpha + 1) \cup \bigcup_{i \in I} Y_i$ and
(1) For every $i \in I$, $Y_i \cong X_{\lambda, \mu}$, where $\lambda, \mu$ are infinite regular cardinals and $\mu > \aleph_0$.

(2) $\alpha \geq \alpha(Y_i)$ for every $i \in I$.

**Proof of Theorem 6.1** Let $\sigma = \{e^F \mid F \subseteq X \text{ and } F \cong X_{\aleph_1} \}$. Then by Proposition 5.2(b), $\sigma$ is finite. For every $x \in \sigma$ let $F_x \subseteq X$ be such that $F_x \cong X_{\aleph_1}$ and $x = e^{F_x}$. Let $\widehat{F} = \bigcup_{x \in \sigma} F_x$, $Z$ be a clopen subset of $X$ homeomorphic to $\widehat{F}$ and $\psi$ be a homeomorphism between $\widehat{F}$ and $Z$. Clearly, $\psi[\sigma] \subseteq \sigma$ and $|\psi[\sigma]| = |\sigma|$. So $\psi[\sigma] = \sigma$. That is, $\sigma \subseteq Z$.

Let $Y = X - Z$. Since $Y \cap \sigma = \emptyset$, it follows that $Y$ does not contain a subspace homeomorphic to $X_{\aleph_1}$. Since $X$ is a CO space and by Propositions 5.2(a) and 5.3(b), $X$ does not contain a subspace homeomorphic to $X_{\kappa, \lambda, \mu}$, where $\langle \kappa, \lambda, \mu \rangle$ is a legal triple, and $X$ does not contain a subspace homeomorphic to $X_{\lambda, \vec{\mu}}$, where $\text{Dom}(\vec{\mu})$ is stationary in $\lambda$. So the same holds for $Y$. By Theorem 3.1, $Y$ is homeomorphic to an interval space. We claim that $Y$ is a CO space. If $Y$ is countable, then $Y$ is homeomorphic to an ordinal. So $Y$ is CO. Assume that $Y$ is uncountable. Let $F$ be a closed subset of $Y$. There is $U \in \text{Clop}(X)$ such that $U \cong F$. Let $V = U \cap Z$ and $W = U \cap Y$. If $V = \emptyset$, then $U \in \text{Clop}(Y)$, so there is nothing more to do. Suppose that $V \neq \emptyset$. Recall that $V \subseteq U \cong F \subseteq Y$. So $V$ is a closed subspace of an interval space, and hence $V$ too is a compact interval space. The only compact interval spaces embeddable in $Z$ are finite spaces and spaces which are a disjoint union of finitely many copies of $X_{\aleph_0}$. So for some $n \in \omega$, $V \cong \omega \cdot n + 1$, or $V$ is finite. Since $Y$ is uncountable and $Y$ is a scattered compact interval space, it contains a clopen set homeomorphic to $\omega^2 + 1$. So for every $n \in \omega$, $Y \cong Y \cup (\omega \cdot n + 1)$, where the union is disjoint. It thus suffices to find a clopen subset of $Y \cup (\omega \cdot n + 1)$ which is homeomorphic to $U = V \cup W$. $W$ is a clopen subset of $Y$ and either $V$ is finite or it is homeomorphic to $\omega \cdot n + 1$. In either case $V$ is homeomorphic to a clopen subset of $\omega \cdot n + 1$. So $Y$ is a CO space.
By Theorem 6.2, there is a finite family of pairwise disjoint spaces \( \{Y_i \mid i \in I\} \) and an ordinal \( \alpha + 1 \) disjoint from the \( Y_i \)'s such that:

1. \( Y \cong (\alpha + 1) \cup \bigcup_{i \in I} Y_i \); 
2. for every \( i \in I \), \( Y_i \cong X_{\lambda, \mu} \), where \( \lambda, \mu \) are infinite regular cardinals and \( \mu > \aleph_0 \); 
3. \( \alpha \geq \alpha(Y_i) \) for every \( i \in I \).

If \( \sigma = \emptyset \), then the above description of \( Y \) fulfills the requirements of the theorem. Suppose that \( \sigma \neq \emptyset \). Then it remains to show that \( \alpha \geq \omega^2 \). This is certainly true if \( I \neq \emptyset \). So suppose that \( I = \emptyset \). Note that \( X_{\aleph_0} \prec X_{\aleph_1} \). So by Proposition 4.6, there is a family \( \{L_i \mid i \in \omega\} \) of subsets of \( X \) such that for every \( i \), \( L_i \approx X_{\aleph_0} \), and for every \( i \neq j \), \( e^{L_i} \neq e^{L_j} \). That is, \( R_1(\alpha + 1) \) is infinite. This implies that \( \alpha \geq \omega^2 \).

\[ \square \]

**Proof of Theorem 1.1** Combine Theorems 1.2 and 6.1. \[ \square \]

7 **Characterization of CO compact interval spaces.**

In the previous section we quoted without proof Theorem 6.2 from [BBR]. However, Theorem 6.2 follows easily from the previous sections. So for completeness, we include a proof of Theorem 6.2.

The following proposition is an addition to Theorem 4.2(a).

**Proposition 7.1.** Let \( X \) be a scattered compact space, and assume that for every subset \( S \) of \( X \) with regular cardinality there is \( D \subseteq S \) such that \( |D| = |S| \), and \( D \) has a clopen strong Hausdorff system.

Suppose that there is a family \( \{L_i \mid i \in \omega\} \) of compact subsets of \( X \) such that (1) for every \( i \in \omega \), \( L_i \) is unitary, and (2) for every distinct \( i, j \in \omega \), \( U \in \text{Nbr}_{\text{clp}}(e^{L_i}) \) and \( V \in \text{Clop}(L_j) \), \( U \not\sim V \). Then \( X \) is not a CO space.

**Proof** Suppose by contradiction that \( X \) is a CO space. Denote \( e^{L_i} \) by \( e_i \).

We may assume that for every \( i \in \omega \), \( L_i \) is clopen in \( X \). Then by (2), for every distinct \( i, j \in \omega \), \( e_i \neq e_j \). We may further assume that acc\( \{e_i \mid i \in \omega\} \)
is a singleton. Denote it by \( x \). We may also assume that for every \( i < j \in \omega \), \( \text{rk}^X(e_i) \leq \text{rk}^X(e_j) \). There is an infinite subset \( \sigma \subseteq \omega \) such that \( \{ e_i | i \in \sigma \} \) has a clopen strong Hausdorff system. We may assume \( \sigma = \omega \). Let \( \{ U_i | i \in \omega \} \) be a clopen strong Hausdorff system for \( \{ e_i | i \in \omega \} \). Let \( L_i' = L_i \cap U_i \) and \( K = (\bigcup_{i \in \omega} L_i') \cup \{ x \} \). Then \( K \) is closed and unitary, \( e^K = x \) and \( \text{rk}^K(x) = \text{Sup}(\{ \text{rk}^X(e_i) | i \in \omega \}) \). We define \( M, L \) such that \( M \subseteq L \subseteq K \). Let \( \tau \subseteq \sigma \subseteq \omega \) be such that \( \tau, \sigma - \tau \) and \( \omega - \sigma \) are infinite. Let \( L = (\bigcup_{i \in \sigma} L_i') \cup \{ x \} \) and \( M = (\bigcup_{i \in \tau} L_i') \cup \{ x \} \). It is obvious that \( M, L \) are closed and unitary, that \( e^L = e^M = x \) and that \( \text{rk}(L) = \text{rk}(M) = \text{rk}(K) \).

We show that \( L \prec K \). We already know that \( L, K \) are unitary, \( L \subseteq K \) and that \( \text{rk}(L) = \text{rk}(K) \). So it remains to show that \( L \not\cong K \). Suppose by contradiction that \( U \in \text{Nbr}^{K}_{\text{clp}}(x) \), \( V \in \text{Nbr}^{L}_{\text{clp}}(x) \) and \( U \cong V \). Let \( f : U \cong V \). Since \( x \) is the only accumulation point of \( \{ e_i | i \in \omega \} \), it follows that \( \{ e_i | i \in \omega \} - U \) is finite. So there is \( i \in \omega - \sigma \) such that \( e_i \in U \). Clearly, \( f(e_i) \neq x \), so there is \( j \in \sigma \) such that \( f(e_i) \in F^i_j \).

We consider the sets \( S = f^{-1}(L'_j \cap V) \cap (L'_i \cap U) \) and \( T = (L'_j \cap V) \cap f[L'_i \cap U] \). Then \( e_i \in S \subseteq L_i, T \subseteq L_j \) and \( f \upharpoonright S : S \cong T \). We check that \( S \in \text{Clop}(L_i) \) and \( T \in \text{Clop}(L_j) \). Since \( L_i', U \in \text{Clop}(K) \), it follows that \( L_i' \cap U \in \text{Clop}(K) \). Also, since \( L_i' \cap V \in \text{Clop}(V) \), it follows that \( f^{-1}(L'_j \cap V) \in \text{Clop}(U) \). Hence \( f^{-1}(L'_j \cap V) \in \text{Clop}(K) \). So \( S \in \text{Clop}(K) \). Now, \( S \subseteq K \cap L_i \subseteq L_i \) and \( K \cap L_i \in \text{Clop}(L_i) \). Hence \( S \in \text{Clop}(L_i) \). A similar calculation shows that \( T \in \text{Clop}(L_j) \). So \( S \in \text{Nbr}^{L_i}_{\text{clp}}(e_i) \) and \( f[S] \in \text{Clop}(L_j) \). That is, \( S \in \text{Nbr}^{L_i}_{\text{clp}}(e_i), T \in \text{Clop}(L_j) \) and \( S \cong T \). But \( i \in \omega - \sigma \) and \( j \in \sigma \). So \( i \neq j \). These facts contradict (2). Hence \( K \not\cong L \). We have shown that \( L \prec K \).

A similar argument shows that \( M \prec L \).

We have shown that \( M \prec L \prec K \). So by Theorem 4.2(b), \( X \) is not a CO space. A contradiction. So \( X \) is not a CO space.

Let \( X \) be a space and \( x \in X \). Then \( x \) is called a **double-limit point** of \( X \), if there are infinite cardinals \( \lambda, \mu \) and an embedding \( f : X_{\lambda, \mu} \to X \) such that
\( \text{cf}(\mu) \geq \aleph_1 \) and \( f(e^{X_{\lambda,\mu}}) = x \).

We represent \( X_{\lambda,\mu} \) as \((\lambda + 1) \cup \{0\} \times \mu\). The subspace \( \lambda + 1 \) of \( X_{\lambda,\mu} \) is denoted by \( X^0_{\lambda,\mu} \) and the subspace \( \{\lambda\} \cup \{0\} \times \mu \) of \( X_{\lambda,\mu} \) is denoted by \( X^1_{\lambda,\mu} \).

**Proposition 7.2.** Let \( X \) be a CO compact interval space.

(a) Let \( x \in X \) be a double-limit point. Then there are regular cardinals \( \lambda, \mu \) and \( U \in \text{Nbr}_{\text{clp}}(x) \) such that \( \mu \geq \aleph_1 \) and \( U \cong X_{\lambda,\mu} \).

(b) The set of double-limit points of \( X \) is finite.

**Proof** (a) Note that the following facts.

(1) There is a subset \( F \subseteq X_{\lambda,\mu} \) such that \( F \cong X_{\text{cf}(\lambda),\text{cf}(\mu)} \) and \( e^F = e^{X_{\lambda,\mu}} \).

(2) If \( U \in \text{Nbr}_{\text{clp}}(e^{X_{\lambda,\mu}}) \), then \( U \cong X_{\lambda,\mu} \).

Let \( x \) be a double-limit point of \( X \), and let \( f, \lambda \) and \( \mu \) be as in the definition of a double-limit point. By Fact (1), we may assume that \( \lambda, \mu \) are regular cardinals. We may also assume that \( \mu \geq \lambda \). Let \( F = \text{Rnf}(f) \), and let \( V \in \text{Nbr}_{\text{clp}}(x) \) be a unitary subspace such that \( e^V = x \). Then either \( F \cong V \) or \( F \prec_w V \). Suppose by contradiction that \( F \prec_w V \). Clearly, \( f[X^1_{\lambda,\mu}] \prec F \).

By Corollary 5.1, Theorem 4.2 applies to \( X \). So since \( f[X^1_{\lambda,\mu}] \prec F \prec_w V \), it follows from 4.2(b) that \( X \) is not a CO space. A contradiction, so \( V \cong F \).

By Fact (2), there is \( W \in \text{Nbr}_{\text{clp}}(x) \) such that \( W \cong X_{\lambda,\mu} \).

(b) It follows from Part (a) that if \( X \) is a CO compact interval space and \( x \in X \) is a double-limit point of \( X \), then there is a unique pair \( \langle \lambda, \mu \rangle = \langle \lambda_x, \mu_x \rangle \) which satisfies:

(1) \( \mu \geq \aleph_1 \) and \( \mu \geq \lambda \).

(2) There is an embedding \( f : X_{\lambda,\mu} \rightarrow X \) such that \( f(e^{X_{\lambda,\mu}}) = x \).

Also, \( \lambda_x, \mu_x \) are regular cardinals.

Suppose by contradiction that \( X \) contains infinitely many double-limit points.

**Case 1** There are \( \lambda, \mu \) and an infinite set \( A \) of double-limit points of \( X \) such that for every \( x \in A \), \( \langle \lambda_x, \mu_x \rangle = \langle \lambda, \mu \rangle \). Let \( L = X_{\lambda,\mu} \) and \( M = X^1_{\lambda,\mu} \).
Note that $X_{\lambda,\mu}^1 \cong \mu + 1$. So since $\text{rk}(L) = \text{rk}(M) = \mu$, it follows that $M \prec L$. Then there is a family $\{L_i \mid i \in \omega\}$ such that for every $i \in \omega$, $L_i \subseteq X$ and $L_i \cong L$, and for every distinct $i, j \in \omega$, $e^{L_i} \neq e^{L_j}$. By Theorem 4.2(a), $X$ is not a CO space. A contradiction.

Case 2 The set $\{\langle \lambda, x \rangle \mid x \text{ is a double-limit point of } X\}$ is infinite. Note that if $\langle \lambda, \mu \rangle \neq \langle \kappa, \nu \rangle$, then for every $U \in \text{Nbr}_{\text{clp}}(X_{\lambda,\mu})$ and $V \in \text{Clop}(X_{\kappa,\nu})$, $U \not\sim V$. So $X$ satisfies the conditions of Proposition 7.1. Hence $X$ is not a CO space. A contradiction.

It follows that the set of double-limit points of $X$ is finite. □

Proposition 7.3. Let $X$ be a CO compact interval space. Then there are no cardinal with uncountable cofinality $\lambda$ and a stationary subset $S \subseteq \lambda$ such that $X_{\lambda,S}$ is embeddable in $X$.

Proof Suppose by contradiction that $X_{\lambda,S}$ is embeddable in $X$. Without loss of generality $\lambda$ is regular. Let $T \subseteq S$ be a stationary subset of $\lambda$ such that $S - T$ is stationary in $\lambda$. By Proposition 5.3(a), $\lambda + 1 \prec X_{\lambda,T} \prec X_{\lambda,S}$. So by Theorem 4.2(b), $X$ is not a CO space. A contradiction. □

Theorem 7.4. Let $X$ be a compact scattered interval space. Suppose that $X$ does not have double-limit points, and there are no cardinal $\lambda$ with uncountable cofinality and a stationary subset $S \subseteq \lambda$ such that $X_{\lambda,S}$ is embeddable in $X$. Then $X$ is homeomorphic to an ordinal space.

Proof The proof is by induction on the rank of $X$. If $X$ is a scattered compact interval space with countable rank, then $X$ is countable. Hence $X$ is homeomorphic to the interval space of a countable ordinal. So the claim is true for every space with countable rank.

Suppose that the claim is true for every space with rank $< \alpha$. Let $X$ be a compact scattered interval space with rank $\alpha$, and suppose that $X$ does not have double-limit points, and there are no uncountable cardinal $\lambda$ and a stationary subset $S \subseteq \lambda$ such that $X_{\lambda,S}$ is embeddable in $X$. We may assume
that $X$ is unitary. Let $<$ be a linear ordering of $X$ such that $\tau^X = \tau^\prec$. We may assume that $e^X \in \text{acc}(X^{<e^X})$.

**Case 1** $\text{cf}_{(X,\prec)}^-(e^X) = \omega$. Assume first that $e^X \in \text{acc}(X^{>e^X})$. Then since $e^X$ is not a double-limit point of $X$, it follows that $\text{cf}_{(X,\prec)}^+(e^X) = \omega$. Let $\{x_i \mid i \in \omega\}$ be a strictly increasing sequence converging to $e^X$ such that for every $i \in \omega$, $x_i$ has a successor in $(X,\prec)$. Similarly let $\{y_i \mid i \in \omega\}$ be a strictly decreasing sequence converging to $e^X$ such that for every $i \in \omega$, $y_i$ has a predecessor in $(X,\prec)$. Let $U_0 = X^{\leq x_0}$, and for every $i > 0$ let $U_i = (x_{i-1}, x_i]$. Similarly, let $V_0 = X^{\geq y_0}$, and for every $i > 0$ let $V_i = [y_i, y_{i-1})$. So for every $i \in \omega$, $U_i$ and $V_i$ are clopen subsets of $X$ and $\text{rk}(U_i), \text{rk}(V_i) < \alpha$. By the induction hypothesis, for every $i \in \omega$ there are well orderings $<_U$, of $U_i$ and $<_V$ of $V_i$ which induce the topologies of $U_i$ and of $V_i$. Define a new linear ordering $<'$ on $X$.

$$U_0 <' V_0 <' U_1 <' V_1 <' \ldots <' e^X$$

and for every $x, y \in X$: if for some $i$, $x, y \in U_i$, then $x <' y$ iff $x < U_i y$, and if for some $i$, $x, y \in V_i$, then $x <' y$ iff $x < V_i y$.

It is obvious that $<'$ is a well ordering of $X$ and that $\tau^<' = \tau^X$.

The case that $e^X \not\in \text{acc}(X^{>e^X})$ is similar but simpler.

**Case 2** $\text{cf}_{(X,\prec)}^-(e^X) > \omega$. Denote $\text{cf}_{(X,\prec)}^-(e^X)$ by $\lambda$. Let $\{x_i \mid i \in \lambda\}$ be a strictly increasing continuous sequence converging to $e^X$. Let

$$S = \{i \in \lambda \mid i \text{ is a limit ordinal}, \text{ and } x_i \in \text{acc}^X(X^{>x_i})\}.$$

For every $i \in S$, $\text{cf}_{(X,\prec)}^-(x_i) = \omega$, for otherwise $x_i$ is a double-limit point. It follows that $X_{\lambda,S}$ is embeddable in $X$. So $S$ is not stationary. Let $C$ be a club of $\lambda$ such that every point of $C$ is a limit point and such that $C \cap S = \emptyset$. So for every $i \in C$, $x_i$ has a successor in $X$. For $i \in C$ let $i + C 1$ be the successor of $i$ in $C$. Hence for every $i \in C$, $X_i := (x_i, x_i + c_1]^{(X,\prec)}$ is a clopen subset of $X$. Also, $X_0 := [\text{min}(X), \text{min}(C)]^{(X,\prec)}$ is clopen in $X$. Clearly, $e^X = \text{max}(X)$, for otherwise, $e^X$ is a double-limit point in $X$. Hence $X = (\bigcup_{i \in C \cup (0)} X_i) \cup \{e^X\}$.
For every \( i \in C \cup \{0\} \), \( \text{rk}(X_i) < \text{rk}(X) \). So by the induction hypothesis there is a well-ordering \( <_i \) of \( X_i \) such that \( \tau^{<_i} = \tau^X \upharpoonright X_i \). Define the linear ordering \( <' \) of \( X \) such that \( \langle X, <' \rangle \) is the lexicographic sum \( \sum_{i \in C \cup \{0\}} \langle X_i, <_i \rangle + \langle \{e^X\}, \emptyset \rangle \).

It is easy and left to the reader to check that \( \tau^{<_i} = \tau^X \). \( \square \)

**Proof of Theorem 6.2** Let \( X \) be a CO compact interval space. By Theorem 7.2, \( X \) is scattered.

By Proposition 7.3, there is no cardinal \( \lambda \) with uncountable cofinality and a stationary set \( S \subseteq \lambda \) such that \( X_{\lambda,S} \) is embeddable in \( X \).

By Proposition 7.2(a) and (b), there are \( k \in \omega \), clopen sets \( U_i \subseteq X \), and regular infinite cardinals \( \lambda_i, \mu_i \) \( i < k \), such that

1. \( \mu_i \geq \aleph_1 \) and \( U_i \cong X_{\lambda_i,\mu_i} \).
2. \( X - \bigcup_{i < k} U_i \) has no double-limit points.

By Proposition 7.4, \( X - \bigcup_{i < k} U_i \) is homeomorphic to an ordinal space \( \alpha + 1 \).

Let \( \mu = \max(\mu_0, \ldots, \mu_{k-1}) \). Then \( \mu + 1 < X_{\lambda,\mu} \). Hence by Proposition 4.6, there is a family \( \{L_i \mid i \in \omega\} \) of subsets of \( X \) such that for every \( i, L_i \cong \mu + 1 \) and for every \( i \neq j \), \( e^{L_i} \neq e^{L_j} \). Clearly, for every \( i \in \omega \), \( e^{L_i} \notin \bigcup_{i < k} U_i \). This implies that \( \mu \cdot \omega + 1 \) is embeddable in \( \alpha + 1 \). Hence \( \mu \cdot \omega \leq \alpha \). \( \square \)
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Notation index

$X_{\lambda,\mu}$. The interval space of $\lambda + 1 + \mu^*$

$\alpha(X_{\lambda,\mu}) = \max(\lambda, \mu) \cdot \omega$. 1

$K_{\mathrm{CH}}$. The class of all Hausdorff spaces which are a continuous

image of a compact interval space 2

$\mathrm{cl}^X(A)$. Closure of $A$ in $X$ 4

$\mathrm{int}^X(A)$. Interior of $A$ in $X$ 4

$\mathrm{acc}^X(A)$. Set of accumulation points of $A$ in $X$ 4

$\mathrm{Nbr}^X(x)$. Set of open neighborhoods of $x$ in $X$ 4

$\mathrm{Nbr}_{\mathrm{cl}}^X(x)$. Set of closed neighborhoods of $x$ in $X$ 4

$\mathrm{Nbr}_{\mathrm{clp}}^X(x)$. Set of clopen neighborhoods of $x$ in $X$ 4

$\mathrm{acc}(\mathcal{A})$. The set of accumulation points of a family of sets $\mathcal{A}$ 4

$I(L)$. The family of convex components of $N$ in a linear ordering $L$ 5

$\mathrm{Is}(X)$. The set of isolated points of $X$ 8

$D(X) = X - \mathrm{Is}(X)$ 8

$D_{\alpha}(X)$. The $\alpha$’s derivative of $X$ 8

$rk(X)$. The rank of $X$ 8

$\ker(X) = D_{rk(X)}(X)$. The maximal perfect subset of $X$ 8

$\mathrm{Clop}(X)$. The set of clopen subsets of $X$ 8

$\mathrm{Clsd}(X)$. The set of closed subsets of $X$ 8

$\mathrm{Po}(X) = \{x \in X \mid \text{there is } U \in \mathrm{Nbr}(x) \text{ such that } \mathrm{cl}(U) \text{ is perfect}\}$ 8

$\mathcal{S}(X) = \{F \in \mathrm{Clsd}(\ker(X)) \mid F \text{ is scattered}\}$ 8

$\Omega(X) = \sup\{\{\mathrm{rk}(F) \mid F \in \mathcal{S}(X)\}\}$ 8

$\mathrm{Good}(X)$. The set of good points of $X$ 15

$rk^X(x) = \max\{\{\alpha \mid x \in D_{\alpha}(X)\}\}$ 15

$\mathrm{PD}(X) = X - \mathrm{Is}(X) - \mathrm{Po}(X)$. The perfect derivative of $X$ 18

$\mathrm{PD}_{\alpha}(X)$. The $\alpha$’s perfect derivative of $X$ 18

$\mathrm{prk}(X) = \max\{\{\alpha \mid \mathrm{PD}_{\alpha}(X) \neq \emptyset\}\}$. The perfect rank of $X$ 18

$\mathrm{Pend}(X) = \mathrm{PD}_{\mathrm{prk}(X)}(X)$. The perfect end of $X$. 18

$K_{\mathrm{TH}}$. The class of all compact Hausdorff spaces that have
| Properties (TH1) - (TH3) | 20 |
|--------------------------|----|
| $\hat{X}_t$ | The family of pointed spaces associated with $t$ in a P-system $\mathcal{P}$ | 23 |
| $T_\mathcal{P}$ | The index set of a type system $\mathcal{P}$ | 23 |
| $X_t$ | The family of spaces associated with $t$ in a P-system $\mathcal{P}$ | 23 |
| $\hat{X}_\mathcal{P}$ | The class of all pointed spaces of a type system $\mathcal{P}$ | 23 |
| $X_\mathcal{P}$ | The class of all spaces of a type system $\mathcal{P}$ | 23 |
| $\mathcal{P}(A)$ | Power set of $A$ | 24 |
| $\mathcal{Q}_\Gamma$ | 24 |
| $\hat{\mathcal{M}}_\mathcal{P}^\Gamma = \{ \langle X, d \rangle \mid \langle X, d \rangle \text{ is a } \Gamma\text{-marker} \}$ | 24 |
| $\mathcal{Q}_\mathcal{P}^\Gamma = \{ \hat{\mathcal{M}}_\mathcal{P}^\Gamma \mid \Gamma \in \text{End}_\mathcal{P}(X) \}$ | 24 |
| $\text{acc}_\mu(A)$ | The set of $\mu$-accumulation points of $A$ | 26 |
| $\text{End}_\mathcal{P}(X)$ | 29 |
| $\text{Good}_\mathcal{P}(X)$ | 29 |
| $\hat{X}_\alpha^\Omega$ | The class of all pointed spaces which are $\Omega$-demonstrative $(\alpha+1)$-codes with a member of $\text{Pend}(X)$ as their distinguished point | 30 |
| $\mathcal{P}_\Omega$ | The P-system of $\Omega$-demonstrative codes | 30 |
| $\mathcal{B}_{\lambda,\mu}$ | A base of $X_{\lambda,\mu}$ | 34 |
| $P^{<x} = \{ y \in P \mid y < x \}$ | 36 |
| $\text{cf}_{\langle L,\prec \rangle}^-(a)$ and $\text{cf}_{\langle L,\prec \rangle}^+(a)$ | The cofinality of $a$ from the left and the cofinality of $a$ from the right | 36 |
| $X_{\lambda,S}$ | 58 |
| $V_a$ | For a member $a$ of a BA $B$, $V_a = \{ x \in \text{Ult}(B) \mid a \in x \}$ | 68 |
Definition index

accumulation point of $\mathcal{A}$ 4
accumulation point: $\lambda$-accumulation point. A point $x$ is a $\lambda$-accumulation point of $\mathcal{A}$ if $|U \cap \mathcal{A}| = \lambda$ for every $U \in \text{Nbr}(x)$ 13
attained: $\Omega(X)$ is not attained in $X$ 12
code: $\alpha$-code 18
code 18
collectionwise Hausdorff space. $X$ is collectionwise Hausdorff if every relatively discrete subset of $X$ has a Hausdorff system 5
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marker: $\Gamma$-marker 24
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pairwise disjoint set of elements of a Boolean algebra 68
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perfect end 18
perfect kernel 8
perfect rank 18
perfect set. A set which does not have isolated points in its relative topology 8
pointed space. A pair $\langle X, x \rangle$, where $X$ is a topological space and $x \in X$ 23
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| Term                                                                 | Page |
|----------------------------------------------------------------------|------|
| proliferation system                                                | 23   |
| rank of \( x \) in \( X \)                                          | 15   |
| rank. The rank of \( X \), the first ordinal \( \alpha \) such that \( D_{\alpha}(X) \) is finite or perfect | 8    |
| relatively discrete. \( A \) is relatively discrete if \( A \cap \text{acc}(A) = \emptyset \) | 4    |
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