PERTURBATIVE EVIDENCE OF NON-UNIVERSALITY IN THE QUANTIZED HALL CONDUCTIVITY OF A DISORDERED RELATIVISTIC 2D ELECTRON GAS

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Abstract

We study a relativistic two-dimensional electron gas in the presence of a uniform external magnetic field and a random static scalar potential. We compute, in first order perturbation theory, the averages of the charge density and of the transverse conductivity for a white-noise potential, and show that, within this treatment, their quantized values are modified by the disorder.

One of the most puzzling features of the quantum Hall effect (QHE) is the apparent insensitivity of the quantization of the Hall conductivity \( \sigma_H \) (in units of \( \frac{e^2}{h} \)) with respect to type of host material, sample geometry, presence of impurities or defects, etc. Being such a high precision phenomenon, it is important to investigate possible deviations from the quantized values of \( \sigma_H \). So far, theoretical investigations have been concerned mostly with the effects of disorder, and they all seem to agree that the QHE is robust with respect to it.

Another possible correction to the QHE is of relativistic origin: if most of the current is carried by few electrons because of localization, or if it is carried mainly by electrons in edge states, there could be relativistic corrections to the QHE at an accuracy level that could be detected experimentally. (It should also be noted that the Dirac Hamiltonian — with the velocity of light replaced with the Fermi velocity — may be regarded as a low energy effective description of an electron-hole system.) In this case, too, no corrections to the quantized values of \( \sigma_H \) have been found.

In this Letter, we study the combined effects of disorder and relativity. We present a first order perturbative calculation of the correction to the charge density \( j_0 \) and the transverse

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conductivity $\sigma_{21}$ due to a random potential with a Gaussian distribution of zero correlation length. Somewhat surprisingly, we find that, although the classical relation between the conductivity and the charge density remains unchanged, their quantized values are modified. The possible implications of this result are discussed at the end of this Letter.

Let us consider an electronic gas in (2 + 1) dimensions in the presence of a uniform magnetic field and a random (scalar) potential. Its Lagrangian is given by ($\hbar = c = 1$)

$$\mathcal{L} = \overline{\psi} (i\partial + eA - m + \mu\gamma^0) \psi.$$  (1)

The field $A = (V, A)$, where $V$ is a static random potential, describing quenched disorder, and the vector potential $A$ accounts for the uniform magnetic field $B$. $\mu$ denotes the chemical potential. The $\gamma$-matrices satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbf{1}$, with $\eta_{\mu\nu} = \text{diag}(1, -1, -1)$ and $\mathbf{1}$ the $2 \times 2$ identity matrix, and we work with the following representation:

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2,$$

where the $\sigma$'s are the usual Pauli matrices.

The charge density $j_0(x)$ is given by

$$j_0(x) = ie \text{Tr} [\gamma_0 S(x,x)],$$  (2)

where $S(x,y)$ is the Feynman propagator of the theory, satisfying

$$(i\partial + eA(x) - m + \mu\gamma^0) S(x,y) = \delta^3(x-y).$$  (3)

If a perturbing electric field $E$ is turned on, a current $J_i(x)$ will be induced which, in the linear response regime, is given by $J_i(x) = \int d^3y \Pi_{i\nu}(x,y) A^\nu(y)$, where $\Pi_{i\nu}(x,y) = -ie^2 \text{Tr} [\gamma_\mu S(x,y) \gamma_\nu S(y,x)]$ is the polarization tensor and $A^\nu(x) = (-E \cdot x, 0)$. Thus, the D.C. conductivity tensor, defined as $\sigma_{ij}(x) = \lim_{E \to 0} \partial J_i(x)/\partial E^j$, is given by

$$\sigma_{ij}(x) = \int d^3y \Pi_{i\mu}(x,y) y^j.$$  (4)

The charge density and the transverse conductivity of the system without disorder ($V = 0$) are well known. They are given by ($m$ and $eB$ are assumed positive)

$$\sigma_{21}|_{V=0} = -\frac{1}{B} j_0|_{V=0} = \frac{e^2}{2\pi} \left\{ \frac{1}{2} - \theta(-\mu - \mu) + \sum_{n=1}^{\infty} \left[ \theta(\mu + \epsilon_n) - \theta(-\mu - \epsilon_n) \right] \right\},$$  (5)

where $\epsilon_n \equiv \sqrt{m^2 + 2neB}$ are the relativistic Landau levels. Aside from an asymmetry, which is characteristic of the relativistic theory, Eq. (5) exhibits an integer quantization of the transverse conductivity in units of $e^2/h$.

To investigate the effect of disorder on the system, we average physical observables over all possible (static) configurations of $V(x)$, with a suitable weight. Here we choose an uncorrelated Gaussian probability distribution:

$$P[V] = \exp \left\{ -\frac{1}{2g} \int d^2x \, V^2(x) \right\}.$$  (6)

Since the charge density $j_0$ and the conductivity tensor $\sigma_{ij}$ are highly non-local functionals of $V$, we shall perform the averaging perturbatively. This is done by expanding the propagator...
$S$ in a power series in $V$ and using the fact that $\langle V(x) \rangle = 0$ and $\langle V(x)V(y) \rangle = g \delta^2(x - y)$ for the distribution ( averaging of products of three or more $V$’s can be obtained using Wick’s theorem, but we shall not need them in what follows). In matrix notation ($\Gamma(x, x') \equiv e\gamma_0 V(x) \delta^3(x - x')$):

$$S = S_0 \sum_{n=0}^{\infty} (-1)^n (\Gamma S_0)^n. \tag{7}$$

Here $S_0$ is the unperturbed Feynman propagator, satisfying Eq. (3) with $V = 0$. It is given by

$$S_0(x, y) = M(x, y) \int dp^0 e^{-ip^0(x^0 - y^0)} \Sigma(p^0, x - y), \tag{8}$$

where $\Sigma \equiv \Sigma_0 \gamma^0 + \Sigma_1 \gamma^1 + \Sigma_2 \gamma^2 + \Sigma_3 1$, with ($\xi \equiv eB (x - y)^2/2$)

$$\Sigma_0(p^0, x - y) = \frac{eB}{8\pi^2} e^{-\xi/2} \sum_{n=0}^{\infty} \left[ \frac{p^0 + \mu + m}{(p^0 + \mu)^2 - \epsilon_{n+1}^2} + \frac{p^0 + \mu - m}{(p^0 + \mu)^2 - \epsilon_n^2} \right] L_n^0(\xi) e^{-\alpha n}, \tag{9a}$$

$$\Sigma_1(p^0, x - y) = -\frac{ie^2 B^2}{4\pi^2} (x^1 - y^1) e^{-\xi/2} \sum_{n=0}^{\infty} \frac{L_n^1(\xi) e^{-\alpha n}}{(p^0 + \mu)^2 - \epsilon_{n+1}^2}, \tag{9b}$$

$$\Sigma_2(p^0, x - y) = -\frac{ie^2 B^2}{4\pi^2} (x^2 - y^2) e^{-\xi/2} \sum_{n=0}^{\infty} \frac{L_n^1(\xi) e^{-\alpha n}}{(p^0 + \mu)^2 - \epsilon_{n+1}^2}, \tag{9c}$$

$$\Sigma_3(p^0, x - y) = \frac{eB}{8\pi^2} e^{-\xi/2} \sum_{n=0}^{\infty} \left[ \frac{p^0 + \mu + m}{(p^0 + \mu)^2 - \epsilon_{n+1}^2} - \frac{p^0 + \mu - m}{(p^0 + \mu)^2 - \epsilon_n^2} \right] L_n^0(\xi) e^{-\alpha n}, \tag{9d}$$

where $L_n^a(z)$ ($a = 0, 1$) are Laguerre polynomials and $M(x, y)$ is a gauge dependent factor, given by

$$M(x, y) = \exp \left\{ ie \int_y^x A_\mu(z) dz^\mu \right\}, \tag{10}$$

where the integral is performed along a straight line connecting $y$ to $x$. The integral over $p^0$ in Eq. (8) must be performed along the contour depicted in Fig. 1 (A UV regulator $e^{-\alpha m}$ is explicitly displayed in Eq. (1); the limit $\alpha \to 0^+$ must be taken at the end of the calculation of physical quantities.)

Now, we turn to the computation of the first order perturbative correction to the charge density due to disorder. It is given by

$$\langle j_0(x) \rangle^{(1)} = ie^3 \int d^3 y d^3 z \text{Tr} \left[ \gamma_0 S_0(x, y) \gamma_0 S_0(y, z) \gamma_0 S_0(z, x) \right] \langle V(y)V(z) \rangle \tag{11}$$

$$= 4\pi^2 ie^3 g \int d^2 y \int dp^0 \text{Tr} \left[ \gamma_0 \Sigma(p^0, x - y) \gamma_0 \Sigma(p^0, 0) \gamma_0 \Sigma(p^0, y - x) \right].$$

Evaluating the trace and performing the integral over $y$, one finds

$$\langle j_0 \rangle^{(1)} = \frac{ie^5 g B^2}{8\pi^3} \lim_{\alpha \to 0^+} \sum_{\ell=0}^{\infty} \sum_{n=0}^{\infty} e^{-\alpha(\ell+2n)} \left[ I_{\ell+1,n+1}^+ + I_{\ell,n}^- + J_{\ell+1,n+1}^+ + J_{\ell,n+1}^- \right]. \tag{12}$$
where

\[
I_{\ell,n}^\pm = \int \frac{(p^0 + \mu \pm m)^3 dp^0}{[(p^0 + \mu)^2 - \epsilon_n^2][p^0 + \mu - \epsilon_n^2]^2},
\]

\[
J_{\ell,n}^\pm = \int \frac{2n eB (p^0 + \mu \pm m) dp^0}{[(p^0 + \mu)^2 - \epsilon_n^2][p^0 + \mu - \epsilon_n^2]^2}.
\]

Since the integrands of the above integrals go to zero at infinity at least as fast as \((p^0)^{-3}\), one can close the contour depicted in Fig. 1 with a semicircle of infinite radius in the upper half-plane, and evaluate the integral using residues. The complete evaluation is very tedious, but, because of the analytic structure of the integrands, the result can be written as

\[
\langle j_0 \rangle^{(1)} = I_{\text{vac}} + I_0 \theta(-\mu - m) + \sum_{n=1}^{\infty} [I_n \theta(\mu - \epsilon_n) + I_{-n} \theta(-\mu - \epsilon_n)].
\]

Here we shall evaluate \(I_0\) explicitly; the other \(I\)'s can be evaluated similarly. \(I_0\) results from the contribution of the pole in \(p^0 = -\mu - m\) to the integral in Eq. (13). The terms containing such pole are \(I_{\ell,n}\) \((\ell = 0\) or \(n = 0\)) and \(J_0, J_{n+1}\) \((n = 0, 1, 2, \ldots)\); calculating the residues in \(p^0 = -\mu - m\) and collecting the results, we obtain

\[
\langle j_0 \rangle^{(1)}_{-\mu-m} = -\frac{e^4gB^2}{4\pi^2} \lim_{\alpha \to 0^+} \sum_{n=1}^{\infty} \left[ -\frac{e^{-\alpha n}}{2neB} - \frac{m^2 e^{-\alpha n}}{n^2 e^2 B^2} + \frac{m^2 e^{-2\alpha n}}{n^2 e^2 B^2} + \frac{e^{-2\alpha(n-1)}}{2neB} \right] \theta(\mu + m) = -\frac{e^4gB}{8\pi^2} \lim_{\alpha \to 0^+} \left[ \ln(1 - e^{-\alpha}) - e^{2\alpha} \ln(1 - e^{-2\alpha}) + \frac{2m^2}{eB} \sum_{n=1}^{\infty} \frac{e^{-2\alpha n} - e^{-\alpha n}}{n^2} \right] \theta(\mu + m).
\]

(The factor \(\theta(\mu + m)\) assures that the pole \(-\mu - m\) contribute to the integrals only if it is inside the contour of integration.) Taking the limit \(\alpha \to 0^+\), the first two terms combine to yield

\[
\langle j_0 \rangle^{(1)}_{-\mu-m} = \frac{e^4gB \ln 2}{8\pi^2} [1 - \theta(-\mu - m)],
\]

whereas the third term vanishes (one can take the limit inside the sum because the latter is uniformly convergent for \(\alpha \geq 0\)). It follows that

\[
I_0 = -\frac{e^4gB \ln 2}{8\pi^2}.
\]

Performing an analogous calculation for the poles in \(p^0 = -\mu \pm \epsilon_n\) \((n = 1, 2, \ldots)\), one finally obtains

\[
\langle j_0 \rangle^{(1)} = \frac{e^4gB \ln 2}{8\pi^2} \left\{ \frac{1}{2} - \theta(-\mu - m) + \sum_{n=1}^{\infty} [\theta(\mu - \epsilon_n) - \theta(-\mu - \epsilon_n)] \right\}.
\]

(The condition that the pole in \(-\mu - \epsilon_n\) \((n = 0, 1, 2, \ldots)\) must be negative to contribute to the integrals in Eq. (12) gives rise to a factor \(\theta(\mu + \epsilon_n) = 1 - \theta(-\mu - \epsilon_n)\). Thus, such a pole contributes also to \(I_{\text{vac}}\).)
The first order term in the perturbative expansion of $\langle \sigma_{21}(x) \rangle$ can be obtained with the help of the following identity, valid provided the chemical potential is in an energy gap:

$$\langle \sigma_{21} \rangle = -\frac{\partial}{\partial B} \langle j_0 \rangle. \quad (19)$$

From Eqs. (18) and (19) one immediately obtains

$$\langle \sigma_{21} \rangle^{(1)} = -\frac{e^4 g \ln 2}{8\pi^2} \left\{ \frac{1}{2} - \theta(-\mu - m) + \sum_{n=1}^{\infty} \left[ \theta(\mu - \epsilon_n) - \theta(-\mu - \epsilon_n) \right] \right\}. \quad (20)$$

What is surprising in this result is that it does not vanish in the low energy limit, although a similar calculation performed with the non-relativistic propagator gives no correction to the transverse conductivity. Therefore, contrary to what one would expect, the average over disorder does not commute with the non-relativistic limit, at least for the particular probability distribution we have considered in this paper.

On the other hand, if one performs the calculation presented here using

$$P[V] = \exp \left\{ -\frac{1}{2gM^2} \int d^2x \left[ (\nabla V)^2 + M^2V^2 \right] \right\} \quad (21)$$

as the probability distribution for the random potential, one obtains no correction to the transverse conductivity, regardless of the value of $M$. In particular, one does not recover the results obtained using (16) if one first averages with (21) and then takes the limit $M \to \infty$, even though the former distribution corresponds to the $M \to \infty$ limit of the latter. This seems to indicate that our results are somewhat pathological. It should be noted, however, that they emerge from a combination of disorder, relativity and perturbation theory. It would be interesting to investigate if they survive a non-perturbative treatment.

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Figure Caption:

**Figure 1:** Integration contour in the complex $p^0$-plane used in the definition of the Feynman propagator.
