Coarse-grained Monte Carlo simulations of the phase transition of Potts model on weighted networks

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Developing effective coarse grained (CG) approach is a promising way for studying dynamics on large size networks. In the present work, we have proposed a strength-based CG (s-CG ) method to study critical phenomena of the Potts model on weighted complex networks. By merging nodes with close strength together, the original network is reduced to a CG-network with much smaller size, on which the CG-Hamiltonian can be well-defined. In particular, we make error analysis and show that our strength-based CG approach satisfies the condition of statistical consistency, which demands that the equilibrium probability distribution of the CG-model matches that of the microscopic counterpart. Extensive numerical simulations are performed on scale-free networks, without or with strength-correlation, showing that this s-CG approach works very well in reproducing the phase diagrams, fluctuations, and finite size effects of the microscopic model, while the d-CG approach proposed in our recent work [Phys. Rev. E 82, 011107(2010)] does not.

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I. INTRODUCTION

In the last two decades, we have witnessed dramatic advances in complex networks research, which has been one of the most active topics in statistical physics and closely related disciplines [1,2]. The central issue in this field is to study how the topology of networks influences dynamics, such as phase transition, self-organized criticality and epidemic spreading, etc. Usually, Monte Carlo (MC) simulations have been widely used to study such dynamics. However, the sizes of many real-world networks are very large, such as human brain composed of about 10^{11} neurons and 10^{14} synapses [7], and thereby brute-force simulations are quite expensive and sometimes even become impossible. Phenomenological models, such as mean-field description, may capture certain properties of the system, but often ignore microscopic details and fluctuation effects which may be important near some critical points. Therefore, a promising way to bridge the gap between the microscopic details and system level behaviors is to develop coarse-grained (CG) approaches, aiming at significantly reducing the degree of freedom while properly preserving the microscopic information of interest.

Recently, several CG approaches have been proposed in the literature. Renormalization transformation has been used to reduce the size of self-similar networks while preserving the most relevant topological properties of the original ones [8, 11]. Gfeller and Rios proposed spectral decomposition technique to obtain a CG-network which can reproduce the random walk and synchronization dynamics of the original network [12]. Kevrekidis et al. developed equation-free multiscale computational methods to accelerate simulation using a coarse time-stepper [13], which has been successfully applied to study the CG dynamics of oscillator networks [14], gene regulatory networks [15], and adaptive epidemic networks [16]. Nevertheless, none of the works mentioned above has considered critical phenomena in complex networks, which has been a frontier topic in the context of network science [17].

Very recently, we have proposed a degree-based CG (d-CG) approach to study the critical phenomena of the Ising model and the SIS-epidemic model in unweighted networks [18]. A local mean field (LMF) scheme was introduced to generate the CG network from the microscopic one. Specifically, we have proposed a so-called condition of statistical consistency (CSC) that the CG-model should satisfy to guarantee the validity of the CG-approach. We showed that the CSC can be exactly fulfilled if we merge nodes with the same degree together. Extensive numerical simulations showed that our d-CG approach does work very well to reproduce the phase transition behaviors of the original network, including the critical point and the fluctuation properties, but with much less computational efforts. Our method also makes it feasible to investigate the finite size effects of both models, which should be much more expensive and even forbidden if we use brute-force methods. However, this d-CG approach can only apply to binary networks, i.e., each of the link in the network either exists or not, but with no weight. As we know, many real-world networks are intrinsically weighted, with their links having diverse strengths. Examples include the collaboration networks [19, 21], airport networks [22, 23], metabolic networks [24] predator-prey relationship networks [25], and so on. Therefore, a straightforward question is: Can we use CG approaches to study the critical phenomena in weighted networks?

To answer this question, in the present work, we have

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considered the critical phenomena of the Potts model in weighted complex networks. The Potts model is related to a number of important topics in statistical and mathematical physics [26, 27] and was successfully applied to neural networks, multiclass classification problems, graph coloring problem, and so on. It contains a system of coupled nodes, each of which has $p$ possible states. Only when two nodes are in the same state, they have pairwise interactions. With the increment of temperature, the Potts model undergoes an order-disorder phase transition at some critical temperature. For $p = 2$, Potts model is equivalent to the well-known Ising model. Instead of the $d$-CG scheme, we have proposed a strength-based CG ($s$-CG) approach, where those nodes with similar strength are merged together to form a CG-node. Note that in weighted networks, it is unpractical to merge nodes with exactly the same strength together. By detailed analysis of the discrepancy between the Hamiltonian of a CG configuration and that of its corresponding microscopic configurations, we show that the $s$-CG approach can approximately satisfy the CSC defined on weighted networks. Extensive numerical simulations are performed on scale-free (SF) networks, without or with strength-correlation, showing that our $s$-CG approach works very well in reproducing the phase diagrams, fluctuations, and finite size effects of the microscopic model, while the simple $d$-CG does not. Compared to our previous work [18], the present study step forward several important steps. First of all, we should note that $s$-CG is a brand new method compared to $d$-CG and the latter cannot apply to weighted networks, although they share some similar ideas. Second, weighted networks are of more ubiquitous importance than binary unweighted ones, thus the $s$-CG approach should find more applications. What is more, we have extended the study from the simple two-state Ising model to a more general one, the multi-state Potts model. In addition, we have performed error analysis in the present study, which clearly demonstrates the robustness of our approaches.

II. COARSE GRAINING PROCEDURE

A. CG Potts Model

In this paper, we consider the $p$-states Potts model on a weighted network consisted of $N$ nodes, whose Hamiltonian is given by

$$ H = - \sum_{i<j} w_{ij} \delta_{\alpha_i, \alpha_j}, $$

(1)

where $w_{ij}$ is the weight on the edge connecting a pair of nodes $i$ and $j$ ($w_{ij} = 0$ if the nodes $i$ and $j$ are not connected). $\alpha_i (1, \ldots, p)$ denotes the state of node $i$, $\delta_{\alpha_i, \alpha_j} = 1$ if $\alpha_i = \alpha_j$ and 0 otherwise.

To setup the CG-Potts model, one needs to obtain the CG-Hamiltonian defined on the CG-network, followed by CG-MC simulations to study the dynamic behaviors. The CG-network is simply obtained by node-merging, i.e., $q_\mu$ nodes within the original micro-network are merged into a single CG-node $C_\mu$, where $\mu = 1, \ldots, N^c$ labels the CG-node and $N^c$ is size of the CG-network. Following the LMF scheme used in Ref. [18], the weight of link between two CG nodes $\mu$ and $\nu$ reads,

$$ w_{\mu\nu} = \begin{cases} \frac{2}{q_\mu(q_\mu-1)} \sum_{i,j \in C_\mu, i < j} w_{ij} & \text{for } \mu = \nu, \\ \frac{1}{q_\mu q_\nu} \sum_{i \in C_\mu, j \in C_\nu} w_{ij} & \text{for } \mu \neq \nu. \end{cases} $$

(2)

The CG-Hamiltonian $\bar{H}$ can be readily obtained,

$$ \bar{H} = \bar{H}_1 + \bar{H}_2 $$

where

$$ \bar{H}_1 = - \sum_{\mu} \bar{w}_{\mu\mu} \sum_{\alpha} \eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1) $$

(3a)

$$ \bar{H}_2 = - \sum_{\mu,\nu(\mu > \nu)} \bar{w}_{\mu\nu} \sum_{\alpha} \eta_{\mu,\alpha} \eta_{\nu,\alpha} $$

(3b)

Herein, $\bar{H}_1 (\bar{H}_2)$ denote CG interactions inside(among) the CG-nodes, respectively. $\eta_{\mu,\alpha}$ stands for the number of $\alpha$-state micro-nodes inside $C_\mu$. Since there are $\frac{(\eta_{\mu,1} - 1)}{2}$ possible distinct pairs of $\alpha$-state micro-nodes inside $C_\mu$, and each pair has a weighted coupling $\bar{w}_{\mu\nu}$, the CG-interactions among all the $\alpha$-state nodes inside $C_\mu$ is given by

$$ \bar{H}_{\mu,1}^{(\alpha)} = - \bar{w}_{\mu\mu} \eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1). $$

Summation this over all CG-nodes $\mu$ and states $\alpha$ gives the result in Eq.(3a). Eq.(3b) can be interpreted in a similar way. Note that Eq.(3) are closed at the CG level, i.e., as long as one has constructed the CG-network, $\bar{w}$ and $\bar{H}$ are then both well defined, based on which one can perform CG-MC simulations without going back to the micro-level.

B. CSC: Condition of statistical consistency

The above procedure tells us how to calculate the CG-Hamiltonian if we already have the CG-network. However, which $q_\mu$ nodes are merged together to form a CG-node $C_\mu$ is yet not determined. Generally speaking, one may construct the CG-network deliberately, for instance, one may simply generate $N^c$ values, $q_\mu$ obeying $\sum_{\mu=1}^{N^c} q_\mu = N$ and then just randomly merge $q_\mu$ micro-nodes to form $C_\mu$. Therefore, an important question arises: How to guarantee that the CG-model can reproduce the dynamics of the corresponding microscopic model correctly?

We address this problem by extending the so-called CSC as proposed in [18]. We introduce $\bar{\eta}_\mu = \sum_{\mu=1}^{N^c} q_\mu = N$. Since the above procedure tells us how to calculate the CG-Hamiltonian if we already have the CG-network.
\{\eta_{\mu,\alpha}\}_{\alpha=1,...,p} to denote the state of \( C_\mu \) and \( \vec{\eta} = \{\eta_{\mu}\}_{\mu=1,...,N} \) to denote the configuration of the CG-network. Note that a given CG configuration \( \vec{\eta} \) corresponds to many microscopic configurations, which defines the degeneracy factor \( g(\vec{\eta}) \). In the equilibrium state of the CG-model, the probability of finding a given CG-configuration \( \vec{\eta} \) is given by the canonical distribution, i.e.,

\[
p_{\text{CG}}(\vec{\eta}) = g(\vec{\eta})e^{-\mathcal{H}/k_BT}/\mathcal{Z},
\]

where \( \mathcal{Z} = \sum_{\vec{\eta}} p_{\text{CG}}(\vec{\eta}) \) is the CG partition function. It is important to note, however, that \( p(\vec{\eta}) \) can be calculated exactly from the equilibrium distribution of the micro-model,

\[
p_{\text{micro}}(\vec{\eta}) = \sum' e^{-\mathcal{H}/k_BT}/\mathcal{Z},
\]

where \( \mathcal{Z} \) is the partition function of the micro-model, and the prime means summation over all the microscopic configurations that contribute to \( \vec{\eta} \). Since we are interested in the equilibrium phase transition behavior of the Potts model, we thus assert that for the CG-model to be statistically consistent with the micro-model, \( p_{\text{CG}}(\vec{\eta}) \) and \( p_{\text{micro}}(\vec{\eta}) \) must be equal, i.e., the CSC reads

\[
g(\vec{\eta})e^{-\mathcal{H}/k_BT}/\mathcal{Z} = \sum' e^{-\mathcal{H}/k_BT}/\mathcal{Z}. \tag{4}
\]

C. \textit{s-CG Scheme and error analysis}

In the present work, we propose a \( s \)-CG scheme to construct the CG-network, i.e., nodes with same or similar strengths are merged together to form a CG-node, where the strength \( s_i \) of node \( i \) is defined as \( s_i = \sum_j w_{ij} \) \cite{22, 28}. In the following, we will show that if nodes inside each CG-node have same strengths, the CSC will hold exactly within the ANA. In addition, if the strengths within \( C_\mu \) are nearly the same, the CSC can also hold approximately.

In the literature, ANA \cite{17, 22, 31} has been widely used to study the ensemble averaged dynamics of complex networks and proved to be successful. ANA assumes that one can replace the dynamics on a given network by that on a weighted fully connected graph with connectivity \( A_{ij} = d_id_j/(DN) \), where \( d_i \) (\( d_j \)) denotes the degree of node \( i \) (\( j \)) and \( D \) is the mean degree of the network. Analogously, in weighted networks link weight can be expressed as

\[
w_{ij} = s_is_j/(SN) \tag{5}
\]

where \( S \) is the mean strength of the network. Substituting Eq. (5) into Eq. (2), the adjacency matrix of the CG-network now reads,

\[
\bar{w}_{\mu\nu} = \frac{2}{q_\mu(q_\mu - 1)} \sum_{i,j \in C_\mu} (s_\mu + \delta s_i)(s_\mu + \delta s_j)/SN \tag{6a}
\]

\[
\bar{w}_{\mu\nu} = \frac{1}{q_\mu q_\nu} \sum_{i \in C_\mu, j \in C_\nu} (s_\mu + \delta s_i)(s_\nu + \delta s_j)/SN \tag{6b}
\]

Herein, we have written \( s_i = s_\mu + \delta s_i \), with \( s_\mu = \frac{1}{q_\mu} \sum_{i \in C_\mu} s_i \) being the mean strength within \( C_\mu \). \( \Omega_\mu = \frac{S_\mu^2}{SN}(1 - \Omega_\mu) \) where \( \langle \delta s^2 \rangle = \frac{1}{q_\mu} \sum_{i \in C_\mu} (\delta s_i)^2 \) is the variance of strength within \( C_\mu \). In the first equation, we have used the fact that \( \sum_{i,j \in C_\mu} \delta s_i \delta s_j = 2(\sum_{i \in C_\mu} \delta s_i)\sum_{j \in C_\nu} \delta s_j + \sum_{i,j \in C_\nu} (\delta s_i)^2 = 0 \). The second equation holds simply because \( \sum_{i \in C_\mu, j \in C_\nu} \delta s_i \delta s_j = \sum_{i \in C_\mu} \delta s_i \sum_{j \in C_\nu} \delta s_j = 0 \).

Substituting Eq. (6) into Eq. (3), we can get

\[
H_1 = -\frac{1}{SN} \sum_\mu S_\mu^2 (1 - \Omega_\mu) \sum_\alpha \eta_{\mu,\alpha} (\eta_{\mu,\alpha} - 1)/2 \tag{7a}
\]

\[
H_2 = \frac{1}{SN} \sum_{\mu,\nu(\neq \mu)} S_\mu S_\nu \sum_\alpha \eta_{\mu,\alpha} \eta_{\nu,\alpha} \tag{7b}
\]

To compare the CG-Hamiltonian with the microscopic one, we now group the micro-nodes with same state \( \alpha \) inside \( C_\mu \) as \( C_{\mu,\alpha} \). Clearly, the size of \( C_{\mu,\alpha} \) is \( \eta_{\mu,\alpha} \). As in Eq. (3), we can also split the micro-Hamiltonian \( H \) into two parts,

\[
H = H_1 + H_2 \tag{8}
\]

where \( H_1 \) and \( H_2 \) denote energy contributions from intra and inter the CG-nodes respectively. With ANA, and noting the fact only nodes with same states have interactions at the micro-level, one has

\[
H_1 = -\sum_\mu \sum_\alpha \sum_{i,j \in C_{\mu,\alpha}} s_is_j \eta_{\mu,\alpha}(\eta_{\mu,\alpha} - 1)/SN \tag{9a}
\]

\[
H_2 = -\sum_{\mu,\nu(\neq \mu)} \sum_\alpha \sum_{i \in C_{\mu,\alpha}, j \in C_{\nu,\alpha}} s_is_j \eta_{\mu,\alpha} \eta_{\nu,\alpha} \tag{9b}
\]

Following similar steps to obtain Eq. (7), we may also write \( s_i = s_{\mu,\alpha} + \delta s_i \) (here node \( i \) belongs to the group
Here, we note that the nodes with $\alpha$ mainly depends on the difference between $\mu, \alpha$ and $H$.

In the equilibrium state, one expects $C_{\mu, \alpha}$ may scan throughout $C_{\mu}$ for many times, such that $S_{\mu, \alpha}$ averaged over time is close to $S_{\mu}$. Hence $(H - \bar{H})/H$ averaged over long time could be small. Note that if we merge nodes randomly, $\Omega_{\mu} \ll 1$ and $\Omega_{\mu, \alpha} \ll 1$ will be violated and the above reasoning should fail. We thus conclude that the practical s-CG approach, by merging nodes with similar strength together, can satisfy the CSC approximately.

III. NUMERICAL RESULTS

To show the validity of our s-CG approach, we perform extensive simulations on weighted SF networks. SF networks are much heterogeneous and serve as better candidates to test our method than other homogeneous networks, such as small-world or random networks (other types of complex networks have also been investigated, the qualitative results are the same and not shown here).

We first generate a regular (unweighted) SF network by using the Barabási–Albert (BA) model with power-law degree distribution $P(k) \sim k^{-3}$. To convert this unweighted SF network into a weighted one, we use the algorithm as proposed in Ref. [33]: The weight of a link between node $i$ and $j$ ($1 \leq i, j \leq N$) is given by $w_{ij} = (\Theta + \theta)^{\mu}/2$, where $\theta$ is a tunable parameter. Note that $\theta = 0$ corresponds to an unweighted network.

The MC simulation at the microscopic level follows standard Metropolis dynamics: At each step, a micro-node is randomly selected and its state is randomly updated with an acceptance probability $\min(1, e^{-\Delta H/k_B T})$, where $\Delta H$ is the associated change of the micro-Hamiltonian, $k_B$ is the Boltzmann constant, $T$ is the temperature. In the present work, we set $k_B = 1$. Similarly, during each CG-MC step, a CG-node $C_{\mu}$ is randomly chosen with probability proportional to its size $g_{\mu}$. The probability for the process that an $\alpha$-node changes to a $\beta$-node, with correspondingly $\eta_{\mu, \alpha} \rightarrow \eta_{\mu, \beta} - 1$ and $\eta_{\mu, \beta} \rightarrow \eta_{\mu, \beta} + 1$, is given by $\eta_{\mu, \alpha} \min(1, e^{-\Delta H/k_B T})$, where $\Delta H$ is the change of CG-Hamiltonian during this process. Since $N_c$ can be much smaller than $N$, the CG-MC is expected to be much faster and memory-saving than the micro-level MC simulation.

The collective state of the system is described by the total magnetic moment $M = \frac{1}{2\pi} \sum_{\mu, \alpha} |M_{\mu, \alpha}|$, where $M_{\mu, \alpha} = \frac{p_{\mu, \alpha} - 1}{p - 1}$ ($\mu = 1, \cdots, N_c$) denotes the $\alpha$-component of the magnetic moment within $C_{\mu}$. With increasing temperature $T$, the Potts model undergoes a phase transition at some critical temperature $T_c$ from an ordered state, where $M \sim O(1)$ is strictly nonzero, to a disordered state with $M \approx 0$. We use the similar s-CG approach to construct the CG-network with different $N_c$ and compare the results obtained from CG-MC simulations with those of micro-MC simulations.

To begin, we show the results in Fig.1 for $\theta = 0$, where the network are essentially unweighted and the s-CG approach is identical to the $d$-CG. Fig.1(a) and 1(b) show the moment $M$ and susceptibility $\chi = \beta N((M^2) - (M)^2)$.
as functions of $T$, respectively. The susceptibility is related to the variance of the total magnetization according to the fluctuation-dissipation theorem. Apparently, our results (empty squares and solid circles) are in excellent agreements with the micro-level counterparts (solid lines). As comparisons, we have also shown the results obtained by a random-merging (RM) CG-model (dotted lines) and the heterogeneous mean field theories (HMFT)\cite{52} (empty triangles). Here, the RM model means that one simply merge $N/N^c$ randomly selected nodes to form a CG-node. Evidently this random scheme fails to reproduce the microscopic behaviors at all. The results of the HMFT are obtained by numerically solving the self-consistent equations of order parameter\cite{34}. We find that the HMFT can predict the curve of $M \sim T$ quite well, however, it fails to predict the curve of $\chi \sim T$. Strikingly, even when the original network is reduced to one with only 16 CG-nodes, the CG model still faithfully reproduces the phase transition curves and fluctuation properties. Since $N^c$ is largely reduced compared to $N$, a considerable speed-up of CPU time can be achieved which makes it feasible to study system size effects. Fig.1(c) plots $T_c$ as a function of $\ln N$, obtained by our CG method with $N^c = 64$. $T_c$ is determined as the location of the peak in the $\chi \sim T$ curve, see Fig.1(b). The dependence is linear with a slope $\simeq 1.68$, which agrees rather well with a theoretical prediction $T_c/\ln N = \frac{S}{\theta} \simeq 1.67$\cite{34}, where $S$ is the average node strength in the network.

For $\theta \neq 0$, the networks are weighted. Here we take $\theta = 2.4$ as an example to ensure the heterogeneity of the link weights. Figure2(a) and (2b) show $M$ and $\chi$ as functions of $T$ respectively. As in Fig.1(b), the peak in $\chi$ locates the critical point $T_c$. Clearly, the s-CG results (solid circle) are still in excellent agreements with the MC results (solid lines), however, the d-CG (solid squares)\cite{18} and RM-CG (dotted lines) both fails. For such weighted networks, the dynamic equations of HMFT is not available either. Thus, for such weighted networks, our s-CG approach is the only promising CG approach so far. In Fig.2(c), we have also shown the dependence of $T_c$ on the network size. Apparently, there is also a linear dependence between $T_c$ and $\ln N$ with the slope being about 1.288. As mentioned in the last paragraph, this slope depends on the average strength $S$. For a weighted network, one may estimate $S$ by $\langle w_{ij} \rangle$, where $\langle w_{ij} \rangle \simeq \int_0^1 x^\theta/4dx = \frac{\theta+1}{2\theta+3}$. Substituting $D = 20$, $\theta = 2.4$ and $p = 3$ to these formula, we obtain $T_c/\ln N = \frac{S}{\theta} \simeq 1.293$, which is consistent with the simulation value.

FIG. 1: (color online). (a)-(b) $M$ and $\chi$ as functions of $T$ for the ferromagnetic Potts model on a weighted scale-free network with mean degree $D = 20$. $N = 1024$, $p = 3$, $\theta = 2.4$ and $N^c = 16$. (c) Dependence of $T_c$ on the network size $N$. The error bars are omitted for clarity since they are smaller than the symbol sizes.

In real-world networks, correlation is an ubiquitous feature. For instance, social networks show that nodes with large degrees tend to connect together, a property referred to as “assortative mixing”\cite{37}. In contrast, many technological and biological networks show “disassortative mixing”, i.e., connections between high-degree and low-degree nodes are more probable\cite{36,37}. Previous studies showed that correlations may play important roles in network dynamics\cite{36,99}. In the present work, we have used our s-CG method to study the phase transition of Potts model on correlated networks, which can not be studied by the HMFT which assumes no degree correlation. To characterize the assortative property of the weighted network, a strength correlation coefficient $r$, an extension of the degree correlation\cite{37}, can be defined as

$$r = \frac{\langle s_is_j \rangle - \langle s_i \rangle \langle s_j \rangle}{\langle s_i^2 \rangle - \langle s_i \rangle^2}. \tag{12}$$

Here $s_i$ and $s_j$ are the strengths of the two end-nodes of an edge. $r$ is zero for networks with no strength-correlation, such as BA-SF networks, and positive or negative for assortative or disassortative mixing networks, respectively.

Figure 3(a) shows $T_c$ as a function of $r$, obtained from our s-CG approach and micro-MC simulations for $\theta = 0$. Again, the fits between CG-MC and MC are good. Figure 3(b) shows the effects of correlated network size on $T_c$. Interestingly, we find that the linear dependence between $T_c$ and $\ln N$ is lost for correlated networks. For assortative(disassortative) networks $T_c$ grows monotonically much faster(slower) than $\ln N$, respectively. In other words, the ordered state in an assortative(disassortative) network is harder(easier) to be destroyed with increasing temperature than in an un-correlated network. This is understandable since a ‘hub’-node in the network is...
more difficult to change its state than a ‘leaf’-node due to larger energy barrier. In an assortative network, hub-nodes are connected together, such that they tend to freeze into a local ordered state which is stable to thermal fluctuations. For a disassortative network, a hub-node is usually connected to many leaf-nodes. Since leaf-nodes can change state easily, the ‘alone’ hub-node is more likely to change state with the help of their ‘boiling’ neighbors. Therefore, assortative correlations tend to increase $T_c$ as observed here.

In Fig.(4), the magnetization $M$ and susceptibility $\chi$ of the ferromagnetic Potts model on weighted networks are plotted as functions of temperature $T$ at different correlation coefficient $r$, obtained from our $s$-CG approach and micro-MC simulations. Again, the agreements between CG-MC and MC are excellent, further demonstrating the validity of our method.

IV. CONCLUSIONS

In summary, we have developed a strength-based $s$-CG approach for coarse-graining study of the phase transition of the Potts model on weighted networks. We have utilized a mean-field scheme to generate the connectivity of the CG-network and derived the CG-Hamiltonian. To address the problem how to guarantee the validity of the CG-model, we have proposed the so-called CSC, which requires that the probability to find a given CG-configuration in the equilibrium state, calculated from the CG-model, should be the same as that calculated from the original microscopic model. We show, by performing error analysis, that our $s$-CG approach, by merging nodes with close strengths together, holds the CSC approximately with ANA. Detailed numerical simulations demonstrate clearly that our $s$-CG approach can reproduce the microscopic MC simulation results very well, not only for the onset of phase transition, but also for the fluctuations and system size effects.

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[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. 74, 47 (2002).
[2] S. N. Dorogovtsev and J. F. F. Mendes, Adv. Phys. 51, 1079 (2002).
[3] M. Newman, SIAM Review 45, 167 (2003).
[4] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez, and D.-U. Hwang, Phys. Rep. 424, 175 (2006).
[5] A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, and C. Zhou, Phys. Rep. 469, 93 (2008).
[6] D. P. Landau and K. Binder, A Guide to Monte Carlo Simulations in Statistical Physics (Cambridge University Press, Cambridge, 2000).
[7] M. I. Rabinovich, P. Varona, A. I. Selverston, and H. D. I. Abarbanel, Rev. Mod. Phys. 78, 1213 (2006).
[8] B. J. Kim, Phys. Rev. Lett. 93, 168701 (2004).
[9] C. Song, S. Havlin, and H. A. Makse, Nature 433, 392 (2005).
[10] K.-I. Goh, G. Salvi, B. Kahng, and D. Kim, Phys. Rev. Lett. 96, 018701 (2006).
[11] F. Radicchi, J. J. Ramasco, A. Barrat, and S. Fortunato, Phys. Rev. Lett. 101, 148701 (2008).
[12] D. Gfeller and P. D. L. Rios, Phys. Rev. Lett. 100, 174104 (2008).
[13] I. G. Kevrekidis, C. W. Gear, J. M. Hyman, P. G. Kevrekidis, O. Runborg, and C. Theodoropoulos, Comm. Math. Sci. 1, 715 (2003).
[14] S. J. Moon, R. Ghanem, and I. G. Kevrekidis, Phys. Rev. Lett. 96, 144101 (2006).
[15] R. Erbana, I. G. Kevrekidis, D. Adalsteinsson, and T. C. Elston, J. Chem. Phys. 124, 084106 (2006).
[16] T. Gross and I. G. Kevrekidis, Eur. Phys. Lett. 82, 38004 (2008).
[17] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. 80, 1275 (2008).
[18] H. S. Chen, Z. H. Hou, H. W. Xin, and Y. J. Yan, Phys. Rev. E 82, 011107 (2010).
[19] M. E. J. Newman, Phys. Rev. E 64, 016131 (2001).
[20] M. E. J. Newman, Phys. Rev. E 64, 016132 (2001).
[21] A. L. Barabsi, H. Jeong, Z. Nda, E. Ravasz, A. Schubert, and T. Vicsek, Phys. Rev. E 311, 590 (2002).
[22] A. Barrat, M. Barthlemy, R. Pastor-Satorras, and A. Vespignani, Proc. Natl. Acad. Sci. USA 101, 3747 (2004).
[23] E. Almaas, P. L. Krapivsky, and S. Redner, Phys. Rev. E 71, 036124 (2005).
[24] E. Almaas, B. Kovcs, T. Vicsek, Z. N. Oltval, and A. L. Barabasi, Nature (London) 427, 839 (2004).
[25] S. L. Pimm, Food Webs, 2nd ed. (The University of Chicago Press, Chicago, 2002).
[26] F. Y. Wu, Rev. Mod. Phys. 54, 235 (1982).
[27] M. A. Garey and D. S. Johnson, Computers and intractability (Freeman, New York, 1979).
[28] S. H. Yook, H. Jeong, and A.-L. Barabsi, Phys. Rev. Lett. 86, 5835 (2001).
[29] M. Bognar and R. Pastor-Satorras, Phys. Rev. E 68, 036112 (2003).
[30] G. Caldarelli, A. Capocci, P. D. L. Rios, and M. A. M. noz, Phys. Rev. Lett. 89, 258702 (2002).
[31] R. Pastor-Satorras and A. Vespignani, Phys. Rev. E 65, 035108R (2002).
[32] A.-L. Barabasi and R. Albert, Science 286, 509 (1999).
[33] K. Park, Y. C. Lai, and N. Ye, Phys. Rev. E 70, 026109 (2004).
[34] S. Dorogovtsev, A. Goltsev, and J. Mendes, Eur. Phys. J. B 38, 177 (2004).
[35] M. E. J. Newman, Phys. Rev. Lett. 89, 208701 (2002).
[36] R. Pastor-Satorras, A. Vazquez, and A. Vespignani, Phys. Rev. Lett. 87, 258701 (2001).
[37] S. Maslov and K. Sneppen, Science 296, 910 (2002).
[38] A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Phys. Rev. E 78, 051105 (2008).
[39] M. Boguna and R. P. Satorras, Phys. Rev. E 66, 047104 (2002).