HYPERBOLIC P-BARYCENITERS, CIRCUMCENTERS, AND MOEBIUS MAPS

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ABSTRACT. Given a Moebius homeomorphism $f : \partial X \rightarrow \partial Y$ between boundaries of proper, geodesically complete CAT(-1) spaces $X, Y$, and a family of probability measures $\{\mu_x\}_{x \in X}$ on $\partial X$, we describe a continuous family of extensions $\{f_p : X \rightarrow Y \}_{1 \leq p \leq \infty}$ of $f$, called the hyperbolic $p$-barycenter maps of $f$. If all the measures $\mu_x$ have full support then for $p = \infty$ the map $f_\infty$ coincides with the circumcenter map $f$ defined previously in [Bis17]. We use this to show that if $X, Y$ are complete, simply connected manifolds with sectional curvatures $K$ satisfying $-b^2 \leq K \leq -1$, then the circumcenter maps of $f$ and $f^{-1}$ are $\sqrt{b}$-bi-Lipschitz homeomorphisms which are inverses of each other. It follows that closed negatively curved manifolds with the same marked length spectrum are bi-Lipschitz homeomorphic.

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1. INTRODUCTION

The Mostow Rigidity Theorem asserts that for $n \geq 3$ any isomorphism between fundamental groups of closed, hyperbolic $n$-manifolds is induced by an isometry between the manifolds, so such manifolds are determined up to isometry by their fundamental groups. For general closed negatively curved manifolds, one may ask to what extent these manifolds are determined by their fundamental groups. Cheeger showed that if two closed negatively curved manifolds have isomorphic fundamental groups then the total spaces of the two-frame bundles are homeomorphic ([Gro87], 8.2.P), while Gromov showed the unit tangent bundles are homeomorphic via a homeomorphism that preserves the orbits of the geodesic flows ([Gro87], 8.3.E). Farrell and Jones showed that in dimensions $n \geq 5$, the manifolds themselves must be homeomorphic ([FJ89a], but also gave examples of manifolds which are homeomorphic but not diffeomorphic ([FJ89b]).
Burns and Katok conjectured that the data of fundamental group together with lengths of closed geodesics, namely the marked length spectrum, should be enough to determine a closed negatively curved manifold up to isometry. We recall that the marked length spectrum of such a manifold is the function \( l_X : \pi_1(X) \to \mathbb{R}^+ \) which assigns to each based loop the length of the unique closed geodesic in its free homotopy class. Two manifolds \( X, Y \) are said to have the same marked length spectrum if there is an isomorphism \( \phi : \pi_1(X) \to \pi_1(Y) \) such that \( l_X = l_Y \circ \phi \). Otal (\cite{Ota90}) showed that in dimension two this implies that the manifolds \( X, Y \) are isometric. While the marked length spectrum rigidity problem remains open in higher dimensions, Hamenstadt (\cite{Ham92}) showed that equality of the marked length spectrum is equivalent to the geodesic flows of \( X, Y \) being topologically conjugate.

These problems make sense in the more general context of group actions on CAT(-1) spaces. Bourdon showed in \cite{Bou95}, that for a Gromov-hyperbolic group \( \Gamma \) with two quasi-convex actions on CAT(-1) spaces \( X, Y \), the natural \( \Gamma \)-equivariant homeomorphism \( f \) between the limit sets \( \Lambda_X, \Lambda_Y \) is Möbius if and only if there is a \( \Gamma \)-equivariant conjugacy of the abstract geodesic flows on \( \mathcal{G}AX \) and \( \mathcal{G}AY \) compatible with \( f \), where by a Möbius map we mean a map between boundaries which preserves cross-ratios. In particular for \( \tilde{X}, \tilde{Y} \) the universal covers of two closed negatively curved manifolds \( X, Y \) (with sectional curvatures bounded above by \(-1\)), the geodesic flows of \( X, Y \) are topologically conjugate if and only if the induced equivariant boundary map \( f : \partial \tilde{X} \to \partial \tilde{Y} \) is Möbius, both of these conditions being equivalent to equality of the marked length spectra of \( X, Y \).

Bourdon showed (\cite{Bou96}) that if \( X \) is a rank one symmetric space of noncompact type with maximum of sectional curvatures equal to \(-1\) and \( Y \) a CAT(-1) space then any Möbius embedding \( f : \partial X \to \partial Y \) extends to an isometric embedding \( F : X \to Y \). In \cite{Bis15} the problem of extending Möbius maps was considered for general CAT(-1) spaces, where it was shown that any Möbius homeomorphism \( f : \partial X \to \partial Y \) between boundaries of proper, geodesically complete CAT(-1) spaces \( X, Y \) extends to a \((1, \log 2)\)-quasi-isometry \( F : X \to Y \). In \cite{Bis17} an extension of Möbius maps was described which is natural with respect to composition with isometries, called circumcenter extension. The circumcenter extension \( \hat{f} : X \to Y \) of a Möbius map \( f \) was shown to coincide with the \((1, \log 2)\)-quasi-isometric extension described in \cite{Bis15}, and was shown to be locally \(1/2\)-Holder continuous. When \( X, Y \) are complete, simply connected manifolds with sectional curvatures \( K \) satisfying \(-b^2 \leq K \leq -1\) for some \( b \geq 1 \), it was shown in \cite{Bis17} that the circumcenter map \( \hat{f} \) is a \((1, (1 - \frac{1}{b}) \log 2)\)-quasi-isometry.

Our main result is the following:

**Theorem 1.1.** Let \( X, Y \) be complete, simply connected Riemannian manifolds with sectional curvatures satisfying \(-b^2 \leq K \leq -1\) for some constant \( b \geq 1 \). For any Möbius homeomorphism \( f : \partial X \to \partial Y \) with inverse \( g : \partial Y \to \partial X \), the circumcenter extensions \( \hat{f} : X \to Y \) and \( \hat{g} : Y \to X \) are \( \sqrt{b} \)-bi-Lipschitz homeomorphisms which are inverses of each other.
The proof of this theorem relies on the introduction of a continuous family of extensions $\hat{f}_p : X \to Y, 1 \leq p \leq \infty$ of the Moebius map $f$, called hyperbolic $p$-barycenter extensions, such that for $p = \infty$ the map $\hat{f}_\infty$ coincides with the circumcenter extension $\hat{f}$. The definition of these maps relies on the notion of hyperbolic $p$-barycenter of a measure $\mu$ with compact support on a CAT(-1) space $X$, which is defined to be the unique point $x \in X$ minimizing the function

$$y \in X \mapsto \| \cosh(d(., y)) \|_{L^p(\mu)}$$

This leads to a notion of asymptotic hyperbolic $p$-barycenter of a measure $\nu$ with compact support on the space of geodesics $GX$, obtained as a limit of hyperbolic $p$-barycenters of measures $\mu_t$ on $X$ obtained by pushing forward the measure $\nu$ by the geodesic flow $\phi_t : GX \to GX$ and the canonical projection $\pi : GX \to X$. The hyperbolic $p$-barycenter extension $\hat{f}_p$ is then defined using asymptotic hyperbolic $p$-barycenters and the geodesic conjugacy $\phi_f : GX \to GY$ induced by $f$. When $X, Y$ are manifolds, for $1 \leq p < \infty$ the maps $\hat{f}_p$ are $C^1$, and estimates on derivatives of the maps as $p \to \infty$ eventually lead to a proof of the above theorem.

We remark that barycenter maps have been used previously in proving rigidity results, beginning with the work of Besson-Courtois-Gallot ([GB95]) who used it to prove their celebrated entropy rigidity theorem for negatively curved locally symmetric spaces. However the hyperbolic $p$-barycenter maps constructed here do not in general coincide with the barycenter map of Besson-Courtois-Gallot.

As an immediate corollary we have:

**Theorem 1.2.** Let $X, Y$ be closed negatively curved manifolds of dimension $n \geq 2$. If the marked length spectra of $X, Y$ are equal, then $X, Y$ are bi-Lipschitz homeomorphic.

Finally we remark that in [Bis16], it is proved that in certain cases Moebius maps between boundaries of simply connected negatively curved manifolds do extend to isometries (more precisely, local and infinitesimal rigidity results are proved for deformations of the metric on a compact set).

2. Preliminaries

We recall in this section the definitions and facts from [Bis15] and [Bis17] which we will be needing.

2.1. Spaces of Moebius metrics. Let $(Z, \rho_0)$ be a compact metric space with at least four points. For a metric $\rho$ on $Z$ the metric cross-ratio with respect to $\rho$ of a quadruple of distinct points $(\xi, \xi', \eta, \eta')$ of $Z$ is defined by

$$[\xi \xi' \eta \eta']_\rho := \frac{\rho(\xi, \eta)\rho(\xi', \eta')}{\rho(\xi, \eta')\rho(\xi', \eta)}$$

A diameter one metric $\rho$ on $Z$ is antipodal if for any $\xi \in Z$ there exists $\eta \in Z$ such that $\rho(\xi, \eta) = 1$. We assume that $\rho_0$ is diameter one and antipodal. We say two metrics $\rho_1, \rho_2$ on $Z$ are Moebius equivalent if their metric cross-ratios agree:

$$[\xi \xi' \eta \eta']_{\rho_1} = [\xi \xi' \eta \eta']_{\rho_2}$$
for all \((\xi, \xi', \eta, \eta')\). The space of Moebius metrics on \(Z\) is defined to be the set \(\mathcal{M}(Z, \rho_0)\) of antipodal, diameter one metrics \(\rho\) on \(Z\) which are Moebius equivalent to \(\rho_0\). We will write \(\mathcal{M}(Z, \rho_0) = \mathcal{M}\). We have the following from [Bis15]:

**Theorem 2.1.** For any \(\rho_1, \rho_2 \in \mathcal{M}\), there is a positive continuous function \(\frac{d\rho_2}{d\rho_1}\) on \(Z\), called the derivative of \(\rho_2\) with respect to \(\rho_1\), such that the following holds (the "Geometric Mean Value Theorem"):

\[
\rho_2(\xi, \eta)^2 = \frac{d\rho_2}{d\rho_1}(\xi) \frac{d\rho_2}{d\rho_1}(\eta) \rho_1(\xi, \eta)^2
\]

for all \(\xi, \eta \in Z\).

Moreover for \(\rho_1, \rho_2, \rho_3 \in \mathcal{M}\) we have

\[
\frac{d\rho_3}{d\rho_1} = \frac{d\rho_3}{d\rho_2} \frac{d\rho_2}{d\rho_1}
\]

and

\[
\frac{d\rho_2}{d\rho_1} = \frac{1}{d\rho_1(d\rho_2)}
\]

**Lemma 2.2.**

\[
\max_{\xi \in Z} \frac{d\rho_2(\xi)}{d\rho_1} \cdot \min_{\xi \in Z} \frac{d\rho_2(\xi)}{d\rho_1} = 1
\]

Moreover if \(\frac{d\rho_2}{d\rho_1}\) attains its maximum at \(\xi\) and \(\rho_1(\xi, \eta) = 1\) then \(\frac{d\rho_2}{d\rho_1}\) attains its minimum at \(\eta\), and \(\rho_2(\xi, \eta) = 1\).

For \(\rho_1, \rho_2 \in \mathcal{M}\), we define

\[
d_M(\rho_1, \rho_2) := \max_{\xi \in Z} \log \frac{d\rho_2(\xi)}{d\rho_1(\xi)}
\]

From [Bis15] we have:

**Lemma 2.3.** The function \(d_M\) defines a metric on \(\mathcal{M}\). The metric space \((\mathcal{M}, d_M)\) is proper.

### 2.2. Visual metrics on the boundary of a CAT(-1) space.

Let \((X, d_X)\) be a proper CAT(-1) space such that \(\partial X\) has at least four points.

We recall below the definitions and some elementary properties of visual metrics and Busemann functions; for proofs we refer to [Bou95]:

Let \(x \in X\) be a basepoint. The **Gromov product** of two points \(\xi, \xi' \in \partial X\) with respect to \(x\) is defined by

\[
(\xi|\xi')_x = \lim_{(a,a') \to (\xi, \xi')} \frac{1}{2} (d(x, a) + d(x, a') - d(a, a'))
\]

where \(a, a'\) are points of \(X\) which converge radially towards \(\xi\) and \(\xi'\) respectively. The **visual metric** on \(\partial X\) based at the point \(x\) is defined by

\[
\rho_x(\xi, \xi') := e^{-|\xi|_x}
\]
The distance \( \rho_x(\xi, \xi') \) is less than or equal to one, with equality iff \( x \) belongs to the geodesic \( (\xi \xi') \).

**Lemma 2.4.** If \( X \) is geodesically complete then \( \rho_x \) is a diameter one antipodal metric.

The Busemann function \( B : \partial X \times X \times X \to \mathbb{R} \) is defined by

\[
B(x, y, \xi) := \lim_{a \to \xi} d(x, a) - d(y, a)
\]

where \( a \in X \) converges radially towards \( \xi \).

**Lemma 2.5.** We have \(|B(x, y, \xi)| \leq d(x, y)\) for all \( \xi \in \partial x, x, y \in X \). Moreover \( B(x, y, \xi) = d(x, y) \) iff \( y \) lies on the geodesic ray \( [x, \xi] \) while \( B(x, y, \xi) = -d(x, y) \) iff \( x \) lies on the geodesic ray \( [y, \xi] \).

We recall the following Lemma from [Bou95]:

**Lemma 2.6.** For \( x, y \in X, \xi, \eta \in \partial X \) we have

\[
\rho_y(\xi, \eta)^2 = \rho_x(\xi, \eta)^2 e^{B(x, y, \xi)} e^{B(x, y, \eta)}
\]

An immediate corollary of the above Lemma is the following:

**Lemma 2.7.** The visual metrics \( \rho_x, x \in X \) are Moebius equivalent to each other and

\[
d\rho_y(\xi) = e^{B(x, y, \xi)}
\]

It follows that the metric cross-ratio \([\xi' \eta']_{\rho_x} \) of a quadruple \((\xi, \xi', \eta, \eta')\) is independent of the choice of \( x \in X \). Denoting this common value by \([\xi' \eta']_x\), it is shown in [Bou96] that the cross-ratio is given by

\[
[\xi' \eta']_x = \lim_{(a, a', b, b') \to (\xi', \eta, \eta')} \exp\left(\frac{1}{2}(d(a, b) + d(a', b') - d(a, b') - d(a', b))\right)
\]

where the points \( a, a', b, b' \in X \) converge radially towards \( \xi', \eta, \eta' \in \partial X \).

We assume henceforth that \( X \) is a proper, geodesically complete CAT(-1) space. We let \( M = M(\partial X, \rho_x) \) (this space is independent of the choice of \( x \in X \)). From [Bis15] we have:

**Lemma 2.8.** The map

\[
i_X : X \to M
\]

\[
x \mapsto \rho_x
\]

is an isometric embedding and the image is closed in \( M \).

For \( k > 0 \) and \( y, z \in X \) distinct from \( x \in X \) let \( \angle(-k^2)xyz \in [0, \pi] \) denote the angle at the vertex \( \bar{y} \) in a comparison triangle \( \bar{xyz} \) in the model space \( \mathbb{H}_{-k^2} \) of constant curvature \(-k^2\). From [Bis17] we have:
Lemma 2.9. For \(\xi, \eta \in \partial X\), the limit of the comparison angles \(\angle(-k^2)yxz\) exists as \(y, z\) converge to \(\xi, \eta\) along the geodesic rays \([x, \xi), [x, \eta)\) respectively. Denoting this limit by \(\angle(-k^2)yx\xi\), it satisfies
\[
\sin\left(\frac{\angle(-k^2)yx\xi}{2}\right) = \rho_x(\xi, \eta)\]

Lemma 2.10. For \(x, y \in X, \xi \in \partial X\) and \(k > 0\), the limit of the comparison angles \(\angle(-k^2)yxz\) exists as \(z\) converges to \(\xi\) along the geodesic ray \([x, \xi)\). Denoting this limit by \(\angle(-k^2)yx\xi\), it satisfies
\[
e^{kB(y,x,\xi)} = \cosh(kd(x,y)) - \sinh(kd(x,y)) \cos(\angle(-k^2)yx\xi)
\]

2.3. Conformal maps, Moebius maps and geodesic conjugacies. We recall the definitions of conformal maps, Moebius maps, and the abstract geodesic flow of a CAT(-1) space.

Definition 2.11. A homeomorphism between metric spaces \(f : (Z_1, \rho_1) \to (Z_2, \rho_2)\) with no isolated points is said to be conformal if for all \(\xi \in Z_1\), the limit
\[
df_{\rho_1, \rho_2}(\xi) := \lim_{\eta \to \xi} \frac{\rho_2(f(\xi), f(\eta))}{\rho_1(\xi, \eta)}
\]
exists and is positive. The positive function \(df_{\rho_1, \rho_2}\) is called the derivative of \(f\) with respect to \(\rho_1, \rho_2\). We say \(f\) is \(C^1\) conformal if its derivative is continuous.

Two metrics \(\rho_1, \rho_2\) inducing the same topology on a set \(Z\), such that \(Z\) has no isolated points, are said to be conformal (respectively \(C^1\) conformal) if the map \(id_Z : (Z, \rho_1) \to (Z, \rho_2)\) is conformal (respectively \(C^1\) conformal). In this case we denote the derivative of the identity map by \(\frac{d\rho_2}{d\rho_1}\).

Definition 2.12. A homeomorphism between metric spaces \(f : (Z_1, \rho_1) \to (Z_2, \rho_2)\) (where \(Z_1\) has at least four points) is said to be Moebius if it preserves metric cross-ratios with respect to \(\rho_1, \rho_2\). The derivative of \(f\) is defined to be the derivative \(\frac{df_{\rho_1, \rho_2}}{df_{\rho_2}}\) of the Moebius equivalent metrics \(f_*\rho_2, \rho_1\) as defined in section 2 (where \(f_*\rho_2\) is the pull-back of \(\rho_2\) under \(f\)).

Any Moebius map between compact metric spaces with no isolated points is \(C^1\) conformal, and the two definitions of the derivative of \(f\) given above coincide. Moreover any Moebius map \(f\) satisfies the geometric mean-value theorem,
\[
\rho_2(f(\xi), f(\eta))^2 = \rho_1(\xi, \eta)^2 df_{\rho_1, \rho_2}(\xi)df_{\rho_1, \rho_2}(\xi)
\]

Definition 2.13. Let \((X, d)\) be a CAT(-1) space. The abstract geodesic flow space of \(X\) is defined to be the space of bi-infinite geodesics in \(X\),
\[
\mathcal{G}X := \{\gamma : (-\infty, +\infty) \to X | \gamma \text{ is an isometric embedding}\}
endowed with the topology of uniform convergence on compact subsets. This topology
is metrizable with a distance defined by
\[ d_{\mathcal{G}X}(\gamma_1, \gamma_2) := \int_{-\infty}^{\infty} d(\gamma_1(t), \gamma_2(t)) \frac{e^{-|t|}}{2} \, dt \]

We define also two continuous projections
\[ \pi : \mathcal{G}X \to X \]
\[ \gamma \mapsto \gamma(0) \]

and
\[ p : \mathcal{G}X \to \partial X \]
\[ \gamma \mapsto \gamma(+\infty) \]

It is shown in Bourdon [Bou95] that \( \pi \) is 1-Lipschitz, while \( p \) is an open mapping.

For \( x \in X \), the unit tangent sphere \( T^1_x X \subset \mathcal{G}X \) is defined to be
\[ T^1_x X := \pi^{-1}(x) \]

The abstract geodesic flow of \( X \) is defined to be the one-parameter group of homeomorphisms
\[ \phi_t : \mathcal{G}X \to \mathcal{G}X \]
\[ \gamma \mapsto \gamma_t \]

for \( t \in \mathbb{R} \), where \( \gamma_t \) is the geodesic \( s \mapsto \gamma(s + t) \).

The flip is defined to be the map
\[ \mathcal{F} : \mathcal{G}X \to \mathcal{G}X \]
\[ \gamma \mapsto \bar{\gamma} \]

where \( \bar{\gamma} \) is the geodesic \( s \mapsto \gamma(-s) \).

We observe that for a simply connected complete Riemannian manifold \( X \) with sectional curvatures bounded above by \(-1\), the map
\[ \mathcal{G}X \to T^1X \]
\[ \gamma \mapsto \gamma'(0) \]

is a homeomorphism conjugating the abstract geodesic flow of \( X \) to the usual geodesic flow of \( X \) and the flip \( \mathcal{F} \) to the usual flip on \( T^1X \).

Let \( f : \partial X \to \partial Y \) be a conformal map between the boundaries of CAT(-1) spaces \( X, Y \) equipped with visual metrics. Then \( f \) induces a bijection \( \phi_f : \mathcal{G}X \to \mathcal{G}Y \) conjugating the geodesic flows, which is defined as follows:

Given \( \gamma \in \mathcal{G}X \), let \( \gamma(-\infty) = \xi, \gamma(+\infty) = \eta, x = \gamma(0) \), then there is a unique point \( y \) on the bi-infinite geodesic \( (f(\xi), f(\eta)) \) such that \( d_{\partial X, \partial Y}(\eta) = 1 \). Define \( \phi_f(\gamma) = \).
\( \gamma^* \) where \( \gamma^* \) is the unique geodesic in \( Y \) satisfying \( \gamma^*(-\infty) = f(\xi), \gamma^*(+\infty) = f(\eta), \gamma^*(0) = y \). Then \( \phi_f : GX \to GY \) is a bijection conjugating the geodesic flows. From [Bis15] we have:

**Proposition 2.14.** The map \( \phi_f \) is a homeomorphism if \( f \) is \( C^1 \) conformal. If \( f \) is Moebius then \( \phi_f \) is flip-equivariant.

2.4. Circumcenter extension of Moebius maps. Let \( X \) be a proper, geodesically complete CAT(-1) space. Recall that for any bounded subset \( B \) of \( X \), there is a unique point \( x \) which minimizes the function \( z \mapsto \sup_{y \in B} d(z, y) \). The point \( x \) is called the circumcenter of \( B \), which we will denote by \( x = c(B) \).

Given \( K \leq 0 \), a function \( f : X \to \mathbb{R} \) is said to be \( FK \)-convex if it is continuous and its restriction to any geodesic satisfies \( f'' + Kf \geq 0 \) in the barrier sense. This means that \( f \leq g \) if \( g \) coincides with \( f \) at the endpoints of a subsegment and satisfies \( g'' + Kg = 0 \). We have the following from [AB03]:

**Proposition 2.15.** Let \( y \in X, \xi \in \partial X \). Then:
1. The function \( x \mapsto \cosh(d(x, y)) \) is \( F(-1) \)-convex.
2. The function \( x \mapsto \exp(B(x, y, \xi)) \) is \( F(-1) \)-convex.

From [Bis17] we have the following two propositions:

**Proposition 2.16.** Let \( f \) be a positive, proper, \( F(-1) \)-convex function on \( X \). Then \( f \) attains its minimum at a unique point \( x \in X \).

**Proposition 2.17.** Let \( f_n, f \) be positive, proper, \( F(-1) \)-convex functions on \( X \) such that \( f_n \to f \) uniformly on compacts. If \( x_n, x \) denote the points where \( f_n, f \) attain their minima, then \( x_n \to x \).

Let \( K \) be a compact subset of \( GX \) such that \( p(K) \subset \partial X \) is not a singleton. Define the function \( u_K(z) = \sup_{\gamma \in K} \exp(B(z, \pi(\gamma), \gamma(+\infty))) \). It is shown in [Bis17] that the function \( u_K \) is a proper, positive, \( F(-1) \)-convex function, which hence attains its minimum at a unique \( x \in X \). We call this point the asymptotic circumcenter of \( K \) and denote it by \( x = c_\infty(K) \).

The reason for the name ‘asymptotic circumcenter’ is explained by the following proposition from [Bis17]:

**Proposition 2.18.** Let \( K \) be a compact subset of \( GX \) such that \( p(K) \subset \partial X \) is not a singleton. Define for \( t > 0 \) bounded subsets \( A_t \) of \( X \) by \( A_t = \pi(\phi_t(K)) \), where \( \phi_t \) denotes the geodesic flow on \( GX \). Then
\[
c(A_t) \to c_\infty(K)
\]
as \( t \to +\infty \), i.e. the circumcenters of the sets \( A_t \) converge to the asymptotic circumcenter of \( K \).
Let $f : \partial X \to \partial Y$ be a Moebius homeomorphism between boundaries of proper, geodesically complete CAT(-1) spaces $X, Y$, and let $\phi_f : G_X \to G_Y$ denote the associated geodesic conjugacy.

**Definition 2.19.** The circumcenter extension of the Moebius map $f$ is the map $\hat{f} : X \to Y$ defined by

$$\hat{f}(x) := c_{\infty}(\phi_f(T^1_x X)) \in Y$$

(note that $p(\phi_f(T^1_x X)) = \partial Y$ is not a singleton so the asymptotic circumcenter of $\phi_f(T^1_x X)$ exists).

In [Bis15], a $(1, \log 2)$-quasi-isometric extension $F : X \to Y$ of the Moebius map $f$ is constructed as follows. Since $f$ is Moebius, push-forward by $f$ of metrics on $\partial X$ to metrics on $\partial Y$ gives a map between the spaces of Moebius metrics $f_* : \mathcal{M}(\partial X) \to \mathcal{M}(\partial Y)$, which is easily seen to be an isometry. For each $\rho \in \mathcal{M}(\partial Y)$, we can choose a nearest point to $\rho$ in the subspace of visual metrics $i_Y(Y) \subset \mathcal{M}(\partial Y)$. This defines a nearest-point projection $r_Y : \mathcal{M}(\partial Y) \to Y$. The extension $F$ is then defined by

$$F = r_Y \circ f_* \circ i_X$$

In [Bis17] it is shown that if $\rho \in \mathcal{M}(\partial Y)$ is the push-forward of a visual metric on $\partial X$, $\rho = f_* \rho_x$ for some $x \in X$, then in fact there is a unique visual metric $\rho_y \in \mathcal{M}(\partial Y)$ nearest to $\rho$, given by $y = \hat{f}(x)$, the asymptotic circumcenter of $\phi_f(T^1_x X)$. It follows that the extension $F$ defined above is uniquely determined and equals the circumcenter extension $\hat{f}$.

**Proposition 2.20.** Let $x \in X$ and let $\rho = f_* \rho_x \in \mathcal{M}(\partial Y)$. Then $y = \hat{f}(x)$ is the unique minimizer of the function $z \in Y \mapsto d_M(\rho, \rho_z)$. In particular, $\hat{f} = F$, so $\hat{f}$ is a $(1, \log 2)$-quasi-isometry.

The circumcenter extension has the following naturality properties with respect to composition with isometries:

**Proposition 2.21.** Let $f : \partial X \to \partial Y$ be a Moebius homeomorphism.

1. If $f$ is the boundary map of an isometry $F : X \to Y$ then $\hat{f} = F$.
2. If $G : X \to X, H : Y \to Y$ are isometries with boundary maps $g, h$, then

$$h \circ \hat{f} \circ g = H \circ \hat{f} \circ G$$

3. **Hyperbolic p-barycenters in CAT(-1) spaces**

Let $X$ be a proper, geodesically complete CAT(-1) space.

**Proposition 3.1.** Let $f$ be a positive $\mathcal{F}(-1)$-convex function on $X$. Then for any $p \geq 1$, the function $f^p$ is $\mathcal{F}(-1)$-convex.
Proposition 3.2. Suppose the support of \( f \) and \( g \) satisfy
\[
\mu, p \text{ is a positive function.}
\]

\( g \frac{1}{\sinh(2a)} ((\sinh a)(d + c) \cosh x + (\cosh a)(d - c) \sinh x) \]
and
\[
h \frac{1}{\sinh(2a)} ((\sinh a)(\delta^p + c^p) \cosh x + (\cosh a)(\delta^p - c^p) \sinh x) \]
satisfy \( g'' - g = 0, h'' - h = 0 \), and \( f \) agrees with \( g \) at the endpoints of \([-a,a]\), while \( f^p \) agrees with \( h \) at the endpoints of \([-a,a]\). Since \( f \) is \( F(-1) \)-convex, \( f(x) \leq g(x) \) on \([-a,a]\), hence for \( x \in [-a,a] \),
\[
f(x)^p \leq g(x)^p
\]
so \( f(x)^p \leq h(x) \), hence \( f^p \) is \( F(-1) \)-convex.

For \( x \in X \) let \( v_x : X \to \mathbb{R} \) denote the function
\[
v_x(y) = \cosh d(x,y).
\]
Given \( 1 \leq p \leq \infty \) and a probability measure \( \mu \) on \( X \) with compact support, define a function \( u_{\mu,p} \) on \( X \) by
\[
u_{\mu,p}(x) := ||v_x||_{L^p(\mu)}
\]
Proposition 3.2. Suppose the support of \( \mu \) is not a singleton. Then for \( 1 \leq p < \infty \), the function \( u_{\mu,p} \) is a positive, proper, \( F(-1) \)-convex function, while for \( p = \infty \), the function \( u_{\mu,\infty} \) is a positive, proper, \( F(-1) \)-convex function.

Proof: Let \( 1 \leq p \leq \infty \). If \( u_{\mu,p}(x) = 0 \) for some \( x \), then \( v_x = 0 \) \( \mu \)-a.e., but \( v_x(y) > 0 \) for \( y \neq x \), so we must have \( \mu(\{x\}) = 1 \) and \( \operatorname{supp}(\mu) = \{x\} \), a contradiction. Thus \( u_{\mu,p} \) is a positive function.

If \( x_n \to x \) in \( X \), then clearly \( v_{x_n} \to v_x \) uniformly on compacts, hence \( u_{\mu,p}(x_n) \to u_{\mu,p}(x) \) since \( \mu \) has compact support, so the functions \( u_{\mu,p} \) are continuous.

Moreover if the support of \( \mu \) is contained in a ball \( B(x_0,R) \), then for \( x \) outside \( B(x_0,R) \) we have \( v_x(y) \geq \cosh(d(x,x_0) - R) \) for all \( y \) in \( B(x_0,R) \), hence
\[
u_{\mu,p}(x) \geq \cosh(d(x,x_0) - R) \to \infty
\]
as \( d(x,x_0) \to \infty \), so \( u_{\mu,p} \) is proper.
For \( p < \infty \), by the previous proposition the function \( x \mapsto \cosh^p d(x, y) \) is \( \mathcal{F}(-1) \)-convex for each \( y \in Y \), from which it follows easily that the function

\[
u^p_{\mu, p}(x) = \int_X \cosh^p d(x, y) d\mu(y)
\]
is \( \mathcal{F}(-1) \)-convex. For \( p = \infty \), since \( v_x \) is continuous we have

\[u_{\mu, \infty}(x) = \|v_x\|_{L^\infty(\mu)} = \sup_{y \in \text{supp}(\mu)} \cosh d(x, y)\]
so \( u_{\mu, \infty} \) is \( \mathcal{F}(-1) \)-convex since a supremum of functions satisfying the \( \mathcal{F}(-1) \)-convexity inequality also satisfies the \( \mathcal{F}(-1) \)-convexity inequality. ⋄

As a consequence of the previous proposition, we can make the following definition:

**Definition 3.3.** For a probability measure \( \mu \) on \( X \) with compact support, the hyperbolic \( p \)-barycenter of \( \mu \) is defined to be the unique minimizer of the function

\[x \mapsto \int_X \cosh^p d(x, y) d\mu(y)\]

if \( p < \infty \), while for \( p = \infty \) it is defined to be the unique minimizer of the function

\[x \mapsto \sup_{y \in \text{supp}(\mu)} \cosh d(x, y)\]

Note that the existence of a unique minimizer is obvious if \( \text{supp}(\mu) \) is a singleton, while if \( \text{supp}(\mu) \) is not a singleton the existence is guaranteed by Propositions 2.10 and 2.12. We denote the hyperbolic \( p \)-barycenter by \( \mathcal{c}^p(\mu) \).

Note that for \( p = \infty \) it is easy to see that the hyperbolic \( \infty \)-barycenter equals the circumcenter of the support of \( \mu \),

\[c^\infty(\mu) = c(\text{supp}(\mu))\]

For \( x \in X \), define a positive, continuous function \( w_x \) on \( G X \) by

\[w_x(\gamma) := \exp(B(x, \pi(\gamma), \gamma(+\infty)))\]

Now let \( \nu \) be a probability measure on \( G X \) with compact support \( K \) such that \( p(K) \subset \partial X \) is not a singleton. For \( 1 \leq p \leq \infty \), define a function \( U_{\nu, p} : X \to \mathbb{R} \) by

\[U_{\nu, p}(x) := \|w_x\|_{L^p(\nu)}\]

**Proposition 3.4.** For \( 1 \leq p < \infty \), the function \( U_{\nu, p}^p \) is a positive, proper, \( \mathcal{F}(-1) \)-convex function. For \( p = \infty \), the function \( U_{\nu, \infty} \) is a positive, proper, \( \mathcal{F}(-1) \)-convex function.

**Proof:** Let \( 1 \leq p \leq \infty \). If \( x_n \to x \) in \( X \), then for \( \gamma \in G X \),

\[| \log w_{x_n}(\gamma) - \log w_x(\gamma) | = | B(x_n, \pi(\gamma), \gamma(+\infty)) - B(x, \pi(\gamma), \gamma(+\infty)) | \leq d(x_n, x)\]

so \( \log w_{x_n} \to \log w_x \) uniformly on \( G X \), hence \( w_{x_n} \to w_x \) uniformly on compacts in \( G X \), thus \( U_{\nu, p}(x_n) \to U_{\nu, p}(x) \) since \( \nu \) has compact support. Thus \( U_{\nu, p} \) is continuous.

Suppose \( 1 \leq p < \infty \). Then

\[U_{\nu, p}^p(x) = \int_{G X} \exp(pB(x, \pi(\gamma), \gamma(+\infty))) d\nu(\gamma)\]
For each \( \gamma \in \mathcal{G}X \), by Propositions 2.15 and 3.1 the function

\[
x \in X \mapsto \exp(pB(x, \pi(\gamma), \gamma(+\infty)))
\]

is \( \mathcal{F}(-1) \)-convex, so it follows from the above expression that \( U^{p}_{\nu,p} \) is \( \mathcal{F}(-1) \)-convex.

Now let \( x_{n} \) be a sequence in \( X \) tending to infinity, and suppose \( U^{p}_{\nu,p}(x_{n}) \) does not tend to \(+\infty\). Passing to a subsequence, we may assume \( U^{p}_{\nu,p}(x_{n}) \leq C \) for all \( n \) for some \( C > 0 \), and \( x_{n} \to \xi \in \partial X \). Since \( p(K) \subset \partial X \) is not a singleton, we can choose \( \eta \in p(K) \) such that \( \eta \neq \xi \). Let \( N \subset \partial X \) be a compact neighbourhood of \( \eta \) not containing \( \xi \), and let \( A = p^{-1}(N) \subset \mathcal{G}X \), so \( \nu(A) > 0 \) since \( A \) is a neighbourhood of a point in \( K = \text{supp}(\nu) \). Since the angles \( \angle (-1)xyz \) depend continuously on \( x \in X \) and \( y, z \in X \cap \partial X \) distinct from \( x \), there is an \( \epsilon > 0 \) and an \( M \geq 1 \) such that \( \angle (-1)x_{n}\pi(\gamma)(+\infty) \geq \epsilon \) for all \( n \geq M \) and all \( \gamma \in A \). Then for \( n \geq M \) and \( \gamma \in A \cap \text{supp}(\nu) \), by Lemma 2.10 we have

\[
\exp(B(x_{n}, \pi(\gamma), \gamma(+\infty))) = \cosh(d(x_{n}, \pi(\gamma))) - \sinh(d(x_{n}, \pi(\gamma))) \cos(\angle(-1)x_{n}\pi(\gamma)(+\infty)) \\
= e^{-d(x_{n}, \pi(\gamma))} + 2 \sinh(d(x_{n}, \pi(\gamma))) \sin^{2}\left(\frac{\angle(-1)x_{n}\pi(\gamma)(+\infty)}{2}\right) \\
\geq 2 \sinh(d(x_{n}, x_{0}) - R) \sin^{2}(\epsilon/2)
\]

where \( x_{0} \in X, R > 0 \) are chosen such that \( \pi(K) \subset B(x_{0}, R) \) and \( M \) is large enough so that \( d(x_{n}, x_{0}) > R \) for \( n \geq M \). It follows that for \( n \geq M \)

\[
U^{p}_{\nu,p}(x_{n}) \geq (2 \sinh(d(x_{n}, x_{0}) - R) \sin^{2}(\epsilon/2))^{p} \nu(A) \to +\infty
\]

This contradicts \( U^{p}_{\nu,p}(x_{0}) \leq C \). Thus \( U^{p}_{\nu,p} \) is proper.

For \( p = \infty \), since \( w_{x} \) is continuous

\[
U_{\nu,\infty}(x) = \|w_{x}\|_{L^{\infty}(\nu)} = \sup_{\gamma \in K} \exp(B(x, \pi(\gamma), \gamma(+\infty)))
\]

so \( U_{\nu,\infty} \) coincides with the function \( u_{K} \) defined in the previous section, and \( u_{K} \) is \( \mathcal{F}(-1) \)-convex and proper since \( p(K) \) is not a singleton.

In light of the previous proposition, we can make the following definition:

**Definition 3.5.** Let \( \nu \) be a probability measure on \( \mathcal{G}X \) with compact support \( K \), such that \( p(K) \) is not a singleton. For \( 1 \leq p \leq \infty \), the asymptotic hyperbolic \( p \)-barycenter of \( \nu \) is defined to be the unique minimizer of the function

\[
x \in X \mapsto \|\exp(B(x, \pi(.), p(.)))\|_{L^{p}(\nu)}
\]

We denote it by \( c_{\nu}^{p}(\nu) \).

Note that for \( p = \infty \), the asymptotic hyperbolic \( \infty \)-barycenter of \( \nu \) coincides with the asymptotic circumcenter of \( K = \text{supp}(\nu) \) (since \( U_{\nu,\infty} = u_{K} \)). Asymptotic hyperbolic \( p \)-barycenters can be described as limits of hyperbolic \( p \)-barycenters of measures on expanding sets:
Proposition 3.6. Let $1 \leq p \leq \infty$. Let $\nu$ be a probability measure on $\mathcal{G}X$ with compact support $K$, such that $\nu(K)$ is not a singleton. For $t > 0$, consider the probability measures $\mu_t$ on $X$ defined by $\mu_t = (\pi \circ \phi_t)_* \nu$ (where $\phi_t$ denotes the geodesic flow on $\mathcal{G}X$). Then

$$c^p(\mu_t) \rightarrow c^p_\infty(\nu)$$

as $t \rightarrow \infty$.

Proof: For $p = \infty$, this follows from Proposition 2.18 since $\text{supp} \mu_t = \pi \circ \phi_t(K)$ (because $\nu$ has compact support and $\pi \circ \phi_t : \mathcal{G}X \rightarrow X$ is continuous and surjective). Let $1 \leq p < \infty$. For $t > 0$, consider the proper, $\mathcal{F}(-1)$-convex functions

$$u^p_{\mu_t, \nu}(x) \cdot 2^p e^{-pt} = \int_X \cosh^p(d(x, y)) \cdot 2^p e^{-pt} d\mu_t(y)$$

$$= \int_{\mathcal{G}X} \cosh^p(d(x, \pi \circ \phi_t(\gamma))) \cdot 2^p e^{-pt} d\nu(\gamma)$$

$$= \int_{\mathcal{G}X} \cosh^p(d(x, \gamma(t))) \cdot 2^p e^{-pt} d\nu(\gamma)$$

Since $p(K) \subset \partial X$ is not a singleton, for $t > 0$ large enough $\pi \circ \phi_t(K)$ is not a singleton so the functions $u^p_{\mu_t, \nu}$ are positive. Given a ball $B \subset X$, as $t \rightarrow \infty$

$$d(x, \gamma(t)) - t \rightarrow B(x, \pi(\gamma), \gamma(\infty))$$

uniformly for $x \in B$ and $\gamma \in K$ (this is a standard consequence of exponential convergence of asymptotic geodesics in $\text{CAT}(-1)$ spaces), and hence

$$\cosh^p(d(x, \gamma(t))) \cdot 2^p e^{-pt} \rightarrow \exp(pB(x, \pi(\gamma), \gamma(\infty)))$$

uniformly for $x \in B$ and $\gamma \in K$. It follows easily that as $t \rightarrow \infty$

$$u^p_{\mu_t, \nu}(x) \cdot 2^p e^{-pt} = \int_{\mathcal{G}X} \cosh^p(d(x, \gamma(t))) \cdot 2^p e^{-pt} d\nu(\gamma)$$

$$\rightarrow \int_{\mathcal{G}X} \exp(pB(x, \pi(\gamma), \gamma(\infty))) d\nu(\gamma)$$

$$= U^p_{\nu, \mu}(x)$$

uniformly for $x \in B$. Since $u^p_{\mu_t, \nu} \cdot 2^p e^{-pt}$ and $U^p_{\nu, \mu}$ are positive, proper, $\mathcal{F}(-1)$-convex functions with unique minimizers $c^p(\mu_t)$ and $c^p_\infty(\nu)$ respectively, it follows from Proposition 2.18 that $c^p(\mu_t) \rightarrow c^p_\infty(\nu)$ as $t \rightarrow \infty$.

Proposition 3.7. Let $\nu$ be a probability measure on $\mathcal{G}X$ with compact support $K$ such that $p(K) \subset \partial X$ is not a singleton. Then

$$c^p_\infty(\nu) \rightarrow c^\infty_\infty(\nu) = c_\infty(K)$$

as $p \rightarrow \infty$.

Proof: Given a sequence $p_n$ tending to $\infty$, let $x_n = c^\infty_\infty(\nu), x = c^\infty_\infty(\nu)$. Suppose the sequence $\{x_n\}$ is unbounded. Passing to a subsequence we may assume $x_n \rightarrow \xi \in \partial X$. As in the proof of Proposition 3.3 choosing $\eta \in p(K)$ distinct from $\xi$, and
$N \subset \partial X$ a compact neighbourhood of $\eta$ disjoint from $\xi$, letting $A = p^{-1}(N)$, there are $\epsilon > 0$ and $M \geq 1$ such that for all $n \geq M$ we have

$$U_{\nu,p_n}(x_n) \geq (2 \sinh(d(x_n, x_0) - R) \sin^2(\epsilon/2))^{p_n} \nu(A)$$

where $x_0 \in X, R > 0$ are such that $B(x_0, R)$ contains $\pi(K)$. Thus

$$U_{\nu,p_n}(x) \geq U_{\nu,p_n}(x_n)$$

$$\geq 2 \sinh(d(x_n, x_0) - R) \sin^2(\epsilon/2) \nu(A)^{1/p_n}$$

$$\to +\infty$$

contradicting the fact that

$$U_{\nu,p_n}(x) = \|w_x\|_{L^p(\nu)} \to \|w_x\|_{L^\infty(\nu)} = U_{\nu,\infty}(x)$$

This proves that $\{x_n\}$ is bounded. Let $y$ be a limit point of the sequence $\{x_n\}$. Passing to a subsequence we may assume $x_n \to y$. Then the functions $w_{x_n}$ on $GX$ converge uniformly to $w_y$ on the compact $K$, hence

$$U_{\nu,p_n}(x_n) = \|w_{x_n}\|_{L^p(\nu)} \to \|w_y\|_{L^\infty(\nu)} = U_{\nu,\infty}(y)$$

while

$$U_{\nu,p_n}(x_n) \leq U_{\nu,p_n}(x) \to U_{\nu,\infty}(x)$$

so it follows that

$$U_{\nu,\infty}(y) \leq U_{\nu,\infty}(x)$$

which implies $y = x$ since $x$ is the unique minimizer of the function $U_{\nu,\infty}$. Thus the only limit point of the bounded sequence $\{x_n\}$ is $x$, hence $x_n \to x$ as required. \hfill \Box

4. Hyperbolic $p$-barycenter extension of Moebius maps

**Definition 4.1.** Let $X, Y$ be proper, geodesically complete $\text{CAT}(-1)$ spaces, and let $f : \partial X \to \partial Y$ be a Moebius homeomorphism. Let $\phi = \phi_f : GX \to GY$ be the associated geodesic conjugacy. Given $1 \leq p \leq \infty$, and a family of probability measures $\mathcal{M} = \{\mu_x\}_{x \in X}$ on $GX$ such that $\text{supp}(\mu_x) = T^1_x X$ for all $x$, the hyperbolic $p$-barycenter extension of $f$ with respect to the family $\mathcal{M}$ is the map $\hat{f}_{p,\mathcal{M}} : X \to Y$ defined by

$$\hat{f}_{p,\mathcal{M}}(x) := \text{proj}_{G}(\phi_* \mu_x)$$

(note that $\text{supp}(\phi_* \mu_x) = \phi(T^1_x X)$, and $p(\phi(T^1_x X)) = \partial Y$, so the asymptotic hyperbolic $p$-barycenter of $\phi_* \mu_x$ exists).

Since $\text{supp}(\phi_* \mu_x) = \phi(T^1_x X)$, for $p = \infty$ the point $\hat{f}_{\infty,\mathcal{M}}(x)$ is the unique minimizer of the function

$$z \in Y \mapsto \sup_{\gamma \in \phi(T^1_x X)} \exp(B(z, \pi(\gamma), p(\gamma)))$$

hence

$$\hat{f}_{\infty,\mathcal{M}} = \hat{f}(x)$$

where $\hat{f} : X \to Y$ is the circumcenter extension of $f$. It follows easily from Proposition 3.7 that for all $x \in X$,

$$\hat{f}_{p,\mathcal{M}}(x) \to \hat{f}(x)$$

as $p \to \infty$. 
The following proposition is straightforward, we omit the proof:

**Proposition 4.2.** If $\Gamma$ is a group acting by isometries on $X$ and $Y$, and $f : \partial X \to \partial Y$ is $\Gamma$-equivariant, and the family $\mathcal{M}$ is $\Gamma$-equivariant, i.e. $\gamma_* \mu_x = \mu_{\gamma x}$ for all $x$, then the extension $\hat{f}_{p,\mathcal{M}} : X \to Y$ is $\Gamma$-equivariant.

For the rest of this article, we will only consider the case when $X, Y$ are complete, simply connected manifolds with sectional curvatures $K$ satisfying $-b^2 \leq K \leq -1$. We fix two such manifolds $X, Y$ and a Moebius homeomorphism $f : \partial X \to \partial Y$. We introduce some notation:

We identify as usual $G X$ with $T^1 X$, and the map $p : G X \to \partial X$ with a map $p : T^1 X \to \partial X$. We identify the map $\phi_f : G X \to G Y$ with a map $\phi : T^1 X \to T^1 Y$ conjugating the geodesic flows. For $x \in X, \xi \in \partial X$, we denote by $x\xi \in T^1 X$ the tangent vector $\gamma'(0)$ where $\gamma$ is the unique unit speed geodesic such that $\gamma(0) = x, \gamma(+\infty) = \xi$. Denote by $q_x : \partial X \to T^1 X$ the map $\xi \mapsto \overrightarrow{x\xi}$. Then $q_x$ is a homeomorphism with inverse given by the restriction to $T^1 X$ of $p : T^1 X \to \partial X$.

For $x \in X$ and $y, z \in \partial X$ distinct from $x$, we denote by $\angle yxz \in [0, \pi]$ the Riemannian angle between the geodesic rays $[xy)$ and $[xz)$ at $x$. We note that the upper and lower bounds on sectional curvatures imply upper and lower bounds on Riemannian angles by comparison angles,

$$\angle(-b^2) yxz \leq \angle yxz \leq \angle(-1) yxz$$

For the rest of this article, we fix a probability measure $\mu$ on $\partial X$ such that $\text{supp}(\mu) = \partial X$. Define and fix a family $\mathcal{M} = \{\mu_x, x \in X\}$ of probability measures on the unit tangent spheres $T^1_x X$ by $\mu_x := (q_x)_* \mu$, so that $\text{supp}(\mu_x) = T^1_x X$. For $1 \leq p \leq \infty$, we will denote simply by $F_p : X \to Y$ the hyperbolic $p$-barycenter extension $\hat{f}_{p,\mathcal{M}} : X \to Y$ of $f : \partial X \to \partial Y$ with respect to this family $\mathcal{M}$, and we will denote by $F : X \to Y$ the circumcenter extension $\hat{f} : X \to Y$ of $f$. We note that then by Proposition 4.1, $F_p \to F$ pointwise on $X$ as $p \to \infty$.

Let $1 \leq p < \infty$. For any $x \in X$, the point $F_p(x) \in Y$ is the unique minimizer of the function

$$z \in Y \mapsto \int_{T^1_y Y} \exp(p B(z, \pi(w), p(w))) d(\phi_* \mu_x)(w)$$

$$= \int_{\partial X} \exp(p B(z, \pi(\overrightarrow{x\xi}), f(\xi))) d\mu(\xi)$$

(the equality above following from $\phi_* \mu_x = (\phi \circ q_x)_* \mu$, while the point $F(x) \in Y$ is the unique minimizer of the function

$$z \in Y \mapsto \sup_{\xi \in \partial X} \exp(B(z, \pi(\overrightarrow{x\xi}), f(\xi)))$$

We recall some facts about Busemann functions. Given $y \in Y$ and $\eta \in \partial Y$, the Busemann function $z \in Y \mapsto B(z, y, \eta)$ is a $C^2$ convex function on $Y$. We denote its gradient vector field by $\nabla B^\eta$ (it is independent of the choice of the point $y \in Y$), which is given at a point $z \in Y$ by $\nabla B^\eta(z) = \nabla B^\eta$. The Hessian of the Busemann function,

$$w \in T^1 Y \mapsto d^2 B^\eta(w, w) := \nabla_w \nabla B^\eta, w >$$
is a nonnegative definite quadratic form on $T_zY$ which can be described in terms of unstable Jacobi fields as follows:

Let $\gamma$ denote the unique geodesic such that $\gamma'(0) = -\bar{z}\eta$. Recall that an unstable Jacobi field is a Jacobi field $J$ along $\gamma$ such that $||J(t)||$ is bounded for $t \leq 0$. For any $w \in T_zY$, there exists a unique unstable Jacobi field $J_w$ along $\gamma$ such that $J_w(0) = w$. We then have

$$d^2B^n_2(w, w) =< J'_w(0), J_w(0) > = \frac{1}{2} \frac{d}{dt} ||J(t)||^2$$

The upper and lower bounds of $-1$ and $-b^2$ on sectional curvatures give bounds on the growth of unstable Jacobi fields, which together with the above expression gives

$$||w||^2 \leq d^2B^n_2(w, w) \leq b||w||^2$$

for $w \in \bar{z}\eta^\perp$, while for $w = \bar{z}\eta$ we have for any $v \in T_zY$

$$d^2B^n_2(w, v) =< \nabla_w \nabla B^n, v > = 0$$

because the integral curves of $\nabla B^n$ are geodesics (backward asymptotic to $\eta$). It follows that for any $w \in T_zY$, if $w^{1, \eta}$ denotes the orthogonal projection of $w$ to $\bar{z}\eta^\perp$, then

$$||w^{1, \eta}||^2 \leq d^2B^n_2(w, w) \leq b||w^{1, \eta}||^2$$

**Proposition 4.3.** Let $1 \leq p < \infty$. The function

$$u : z \in Y \mapsto \int_{\partial X} \exp(pB(z, \pi(\phi(xz))), f(\xi)))d\mu(\xi)$$

is $C^2$ and strictly convex, i.e. the Hessian $d^2u_z$ is positive definite for all $z \in Y$.

**Proof:** For each $\xi \in \partial X$, the function $u_\xi : z \mapsto \exp(pB(z, \pi(\phi(xz))), f(\xi)))$ is $C^2$, with gradient given by $\nabla u_\xi(z) = p\nabla B^{f(\xi)}(z) \exp(pB(z, \pi(\phi(xz))), f(\xi)))$, and Hessian given by

$$(d^2u_\xi)_z(w, w)$$

$$= (p < \nabla_w \nabla B^{f(\xi)}, w > +p^2 < \nabla B^{f(\xi)}(z), w >^2) \exp(pB(z, \pi(\phi(xz))), f(\xi)))$$

$$\geq (p||w^{1, f(\xi)}||^2 + p^2 < z\eta f(\xi), w >^2) \exp(pB(z, \pi(\phi(xz))), f(\xi)))$$

$$> 0$$

for $w \neq 0$ since either $||w^{1, f(\xi)}||^2 > 0$ or $< z\eta f(\xi), w >^2 > 0$. It follows that $u$ is $C^2$ with Hessian given by

$$d^2u_z(w, w) = \int_{\partial X} (d^2u_\xi)_z(w, w)d\mu(\xi) > 0$$

for $w \neq 0$ since $(d^2u_\xi)_z(w, w) > 0$ for all $\xi \in \partial X$. $\diamond$

**Proposition 4.4.** For $x \in X$, the point $F_p(x) \in Y$ is the unique $z \in Y$ such that

$$\int_{\partial X} < z\eta f(\xi), w > \exp(pB(z, \pi(\phi(xz))), f(\xi)))d\mu(\xi) = 0$$

for all $w \in T_zY$. 

**Proof:** Since $F_p(x)$ is the unique minimizer of the function $u$ of the previous proposition, for any $w \in T_{F_p(x)}Y$

$$0 = <\nabla u(F_p(x)), w>$$

$$= \int_{\partial X} p <\nabla B^p(\xi)(F_p(x)), w> \exp(pB(z, \pi(\phi(x\xi)), f(\xi)))d\mu(\xi)$$

$$= -p \int_{\partial X} <\nabla F_p(x)f(\xi), w> \exp(pB(z, \pi(\phi(x\xi)), f(\xi)))d\mu(\xi).$$

Moreover since $u$ is strictly convex with unique minimizer $F_p(x)$, if $\nabla u(z) = 0$ for some $z \in Y$ then $z = F_p(x)$. ⋄

**Definition 4.5.** A probability measure $\nu$ on $\partial Y$ is said to be balanced at $z \in Y$ if the vector-valued integral

$$\int_{\partial Y} \vec{z}\vec{\eta}d\nu(\eta) = 0,$$

or equivalently if for all $w \in T_zY$

$$\int_{\partial Y} < \vec{z}\vec{\eta}, w> d\nu(\eta) = 0,$$

For $1 \leq p < \infty$ and $x \in X$, define probability measures $\mu_p^x$ on $\partial X$ by

$$d\mu_p^x(\xi) = c_{x,p}^{-1} \cdot \exp(pB(F_p(x), \pi(\phi(x\xi)), f(\xi)))d\mu(\xi)$$

where $c_{x,p} > 0$ is the constant defined by

$$c_{x,p} = \int_{\partial X} \exp(pB(F_p(x), \pi(\phi(x\xi)), f(\xi)))d\mu(\xi)$$

Then the previous proposition says that the measure $f_\ast \mu_p^x$ on $\partial Y$ is balanced at the point $F_p(x) \in Y$. The following characterization of the circumcenter map $F : X \to Y$ in terms of balanced measures will be useful:

**Proposition 4.6.** Let $x \in X$ and $y \in Y$. Let $K \subset \partial X$ be the set where the function

$$\xi \in \partial X \mapsto \log \frac{df_\ast \rho_x}{d\rho_y}(f(\xi))$$

attains its maximum value. Then the following are equivalent:

(1) $y = F(x)$.

(2) For any $w \in T_yY$, there exists $\xi \in K$ such that $<y\vec{f}(\xi), w> \leq 0$.

(3) The convex hull in $T_yY$ of the compact $\{y\vec{f}(\xi) | \xi \in K\} \subset T_yY$ contains the origin of $T_yY$.

(4) There exists a probability measure $\nu$ on $\partial X$ with support contained in $K$ such that $f_\ast \nu$ is balanced at $y$. 

**Proof:** (1) $\Rightarrow$ (2): Given that $y = F(x)$, suppose there exists $w \in T^1_y Y$ such that $\langle y f(\xi), w \rangle > 0$ for all $\xi \in K$. Then we can choose a neighbourhood $N$ of $K$ in $\partial X$ and $\epsilon, \delta > 0$ such that $\langle y f(\xi), w \rangle > \epsilon$ for all $\xi \in N$, and

$$\log \frac{df^*_\rho_x}{d\rho_y}(f(\xi)) \leq M - \delta$$

for $\xi \in \partial X - N$, where $M = d_M(f_*\rho_x, \rho_y)$ is the maximum value of the function $\xi \in \partial X \mapsto \log \frac{df^*_\rho_x}{d\rho_y}(f(\xi))$.

Let $y_t = \exp_y(tw)$. As $t \to 0$, for $\xi \in N$ we have

$$\log \frac{df^*_\rho_x}{d\rho_y}(f(\xi)) = \log \frac{df^*_\rho_x}{d\rho_y}(f(\xi)) - \log \frac{d\rho_w}{d\rho_y}(f(\xi))$$

$$\leq M + B(y_t, y, f(\xi))$$

$$= M - t < y f(\xi), w > + o(t)$$

$$\leq M - ct + o(t)$$

$$< M$$

for $t > 0$ small enough depending only on $\epsilon$, while for $\xi \in \partial X - N$ we have

$$\log \frac{df^*_\rho_x}{d\rho_y}(f(\xi)) \leq (M - \delta) + B(y_t, y, f(\xi))$$

$$\leq M - \delta + t$$

$$< M$$

for $0 < t < \delta$. Thus for $t > 0$ small enough,

$$d_M(f_*\rho_x, \rho_y) = \sup_{\xi \in \partial X} \log \frac{df^*_\rho_x}{d\rho_y}(f(\xi)) < M = d_M(f_*\rho_x, \rho_y)$$

contradicting the fact that $y$ is the unique minimizer of the function $p \in Y \mapsto d_M(f_*\rho_x, \rho_p)$. This proves (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Let $z \in Y$ be distinct from $y$, and let $w \in T^1_y Y$ be the initial velocity of the geodesic joining $y$ to $z$. By hypothesis, there exists $\xi \in K$ such that $\langle y f(\xi), w \rangle > 0$. Since $y f(\xi)$ is the inward pointing normal to the boundary of the horoball $H = \{p \in Y : B(p, y, f(\xi)) \leq 0\}$ which is strictly convex, it follows that $z \notin H$, so $B(z, y, f(\xi)) > 0$, hence

$$d_M(f_*\rho_x, \rho_z) = \sup_{\eta \in \partial X} \log \frac{df^*_\rho_x}{d\rho_z}(f(\eta))$$

$$\geq \log \frac{df^*_\rho_x}{d\rho_z}(f(\xi))$$

$$= \log \frac{df^*_\rho_x}{d\rho_y}(f(\xi)) - \log \frac{d\rho_z}{d\rho_y}(f(\xi))$$

$$= d_M(f_*\rho_x, \rho_y) + B(z, y, f(\xi))$$

$$> d_M(f_*\rho_x, \rho_y)$$
thus $y$ minimizes the function $p \in Y \mapsto d_M(f_*\rho_x, \rho_p)$, so $y = F(x)$.

(2) $\Rightarrow$ (3): Suppose the convex hull $L \subset T_y Y$ of the compact $\{ yf(\xi) | \xi \in K \} \subset T_y Y$ does not contain the origin of $T_y Y$, then there is a hyperplane in $T_y Y$ separating $L$ from the origin, so if $w \in T_y Y$ is a unit normal vector to this hyperplane, then $< v, w > 0$ for all $v \in L$ (after possibly replacing $w$ by $-w$ if necessary), in particular $< yf(\xi), w > 0$ for all $\xi \in K$, a contradiction to our hypothesis.

(3) $\Rightarrow$ (4): A convex combination $\sum_{i=1}^k \alpha_i yf(\xi)$ where $\xi_1, \ldots, \xi_k \in K$ can be written as a vector-valued integral $\int_{\partial X} yf(\xi) d\lambda(\xi)$ where $\lambda$ is the probability measure supported on the finite set $\{ \xi_1, \ldots, \xi_k \}$ with masses $\alpha_1, \ldots, \alpha_k$. Any point in the convex hull of the compact $\{ yf(\xi) | \xi \in K \}$ is a limit of such convex combinations, in particular by our hypothesis we have $\int_{\partial X} yf(\xi) d\lambda_n(\xi) \to 0$ for some sequence of probability measures $\lambda_n$ supported on $K$, taking a weak limit of these measures gives a probability measure $\nu$ supported on $K$ such that $\int_{\partial X} yf(\xi) d\nu(\xi) = 0$.

(4) $\Rightarrow$ (2): Suppose there is $w \in T_y Y$ such that $< yf(\xi), w > 0$ for all $\xi \in K$, then since $\nu$ is supported on $K$ we have $\int_{\partial X} < yf(\xi), w > d\nu(\xi) > 0$, a contradiction to the fact that $f_*\nu$ is balanced at $y$. \hfill \Box

**Proposition 4.7.** The hyperbolic $p$-barycenter map $F_p : X \to Y$ is $C^1$ and its derivative satisfies

$$
\int_{\partial X} d^2B^{\xi(\xi)}_{F_p(x)}(DF_p(v), DF_p(v)) d\mu_p^x(\xi) + p \int_{\partial X} \frac{1}{2} \frac{< DF_p(v), F_p(x)f(\xi) >^2}{d\mu_p^x(\xi)}
$$

$$
= p \int_{\partial X} \frac{1}{2} \frac{< DF_p(v), F_p(x)f(\xi) >^2}{d\mu_p^x(\xi)}
$$

for all $x \in X, v \in T_x X$.

**Proof:** Note that for $x_1, x_2 \in X$ and $\xi \in \partial X$, the geodesic conjugacy $\phi : T^1 X \to T^1 Y$ satisfies

$$
B(\pi(\phi(x_1\xi)), \pi(\phi(x_2\xi)), f(\xi)) = B(x_1, x_2, \xi)
$$

hence, fixing a basepoint $x_0 \in X$, the function $(x, y) \in X \times Y \mapsto B(y, \pi(\phi(x_0\xi)), f(\xi))$ can be written as

$$
B(y, \pi(\phi(x_0\xi)), f(\xi)) = B(y, \pi(\phi(x_0\xi)), f(\xi)) + B(\pi(\phi(x_0\xi)), \pi(\phi(x_0\xi)), f(\xi))
$$

$$
= B(y, \pi(\phi(x_0\xi)), f(\xi)) + B(x_0, x, \xi)
$$

and is hence $C^2$ as a function of $(x, y)$ (even though $\phi$ is not necessarily even $C^1$). Thus letting $e_1(y), \ldots, e_n(y), y \in Y$ be a smooth orthonormal frame field on $Y$, we can define a $C^1$ function $H = (H_1, \ldots, H_n) : X \times Y \to \mathbb{R}^n$ by

$$
H_i(x, y) = \int_{\partial X} < \nabla B^{\xi(\xi)}(y), e_i(y) > \exp(pB(y, \pi(\phi(x_0\xi)), f(\xi))) d\mu(\xi)
$$

Then by Proposition 4.4, $F_p(x) \in Y$ is defined implicitly by the equation $H(x, F_p(x)) = 0$. For $y = F_p(x)$ and $w \in T_y Y$, let $\nabla_a e_i = \sum_j \eta_{ij} e_j$, then, using $H(x, y) = 0$, the
partial derivative \((D_yH_i)(w)\) is given by

\[
(D_yH_i)(w) = \int_{\partial X} <\nabla_x \nabla B^f(\xi), e_i(y)> \exp(pB(y, \pi(\phi(x_\xi))), f(\xi)) d\mu(\xi) + \sum_j \eta_{ij} H_j(x, y) \\
+ p \int_{\partial X} <\nabla B^f(\xi)(y), e_i(y)> <\nabla B^f(\xi)(y), w> \exp(pB(y, \pi(\phi(x_\xi))), f(\xi)) d\mu(\xi) \\
= \int_{\partial X} <\nabla_x \nabla B^f(\xi), e_i(y)> \exp(pB(y, \pi(\phi(x_\xi))), f(\xi)) d\mu(\xi) \\
+ p \int_{\partial X} <\nabla B^f(\xi)(y), e_i(y)> <\nabla B^f(\xi)(y), w> \exp(pB(y, \pi(\phi(x_\xi))), f(\xi)) d\mu(\xi) \\
= \frac{1}{p} (d^2u)_y(w, e_i(y))
\]

where \(u\) is the strictly convex function of Proposition 4.3. Since \(d^2u_y\) is positive-definite, it follows that \(D_yH\) is invertible, hence by the Implicit Function Theorem \(F_p\) is \(C^1\).

Given \(v \in T_xX\), let \((x_t)_{|t|<\epsilon}\) be the geodesic with initial velocity \(v\), then, as above we can write

\[
B(F_p(x_t), \pi(\phi(x_\xi)), f(\xi)) = B(F_p(x_t), \pi(\phi(x_\xi)), f(\xi)) + B(\pi(\phi(x_\xi)), \pi(\phi(x_\xi)), f(\xi)) \\
= B(F_p(x_t), \pi(\phi(x_\xi)), f(\xi)) + B(x, x_t, \xi)
\]

so

\[
\frac{d}{dt} B(F_p(x_t), \pi(\phi(x_\xi)), f(\xi)) = \langle x_\xi, v > - < F_p(x)f(\xi), DF_p(v) >
\]

So differentiating the equality \(H_i(x_t, F_p(x_t)) = 0\) at \(t = 0\) gives, writing \(\tau(\xi) = \exp(pB(F_p(x), \pi(\phi(x_\xi)), f(\xi)))\),

\[
0 = \int_{\partial X} d^2B^f_{F_p(x)}(DF_p(v), e_i(F_p(x))) \tau(\xi) d\mu(\xi) + \sum_j \eta_{ij} H_j(x, F_p(x)) \\
+ p \int_{\partial X} < -F_p(x)f(\xi), e_i(F_p(x)) > \langle x_\xi, v > - < F_p(x)f(\xi), DF_p(v) > \rangle \eta(\xi) d\mu(\xi) \\
= \int_{\partial X} d^2B^f_{F_p(x)}(DF_p(v), e_i(F_p(x))) \tau(\xi) d\mu(\xi) \\
+ p \int_{\partial X} < -F_p(x)f(\xi), e_i(F_p(x)) > \langle x_\xi, v > - < F_p(x)f(\xi), DF_p(v) > \rangle \eta(\xi) d\mu(\xi)
\]
for $i = 1, \ldots, k$. It follows that for any $w \in T_{F_p(x)}$,

\[
0 = \int_{\partial X} d^2 B_{F_p(x)}(DF_p(v), w) > \tau(\xi)d\mu(\xi)
\]

\[
+ p \int_{\partial X} < - F_p(x)f(\xi), w > (\int_{\partial X} x_\xi - x_\xi, v > - < F_p(x)f(\xi), DF_p(v) >)\tau(\xi)d\mu(\xi)
\]

so the formula stated in the proposition follows by putting $w = DF_p(v)$. ∎

**Proposition 4.8.** For any $x \in X$, $v \in T_x X$,

\[
\frac{1}{p} \int_{\partial X} \|DF_p(v)^f(\xi)\|^2 d\mu_p^x(\xi) \leq \int_{\partial X} < \overrightarrow{x_\xi}, v >^2 d\mu_p^x(\xi)
\]

(where $DF_p(v)^f(\xi) \in T_{F_p(x)}Y$ denotes the orthogonal projection of $DF_p(v)$ to $F_p(x)f(\xi)^\perp$).

**Proof:** By the previous proposition,

\[
\int_{\partial X} < \overrightarrow{F_p(x)f(\xi)}, DF_p(v) >^2 d\mu_p^x(\xi)
\]

\[
= \int_{\partial X} < \overrightarrow{F_p(x)f(\xi)}, DF_p(v) > < \overrightarrow{x_\xi}, v > d\mu_p^x(\xi)
\]

\[
- \frac{1}{p} \int_{\partial X} d^2 B_{F_p(x)}(DF_p(v), DF_p(v))d\mu_p^x(\xi)
\]

\[
\leq \int_{\partial X} < \overrightarrow{F_p(x)f(\xi)}, DF_p(v) > < \overrightarrow{x_\xi}, v > d\mu_p^x(\xi)
\]

\[
\leq \left( \int_{\partial X} < \overrightarrow{F_p(x)f(\xi)}, DF_p(v) >^2 d\mu_p^x(\xi) \right)^{1/2} \left( \int_{\partial X} < \overrightarrow{\xi}, v >^2 d\mu_p^x(\xi) \right)^{1/2}
\]

hence

\[
\int_{\partial X} < \overrightarrow{F_p(x)f(\xi)}, DF_p(v) >^2 d\mu_p^x(\xi) \leq \int_{\partial X} < \overrightarrow{\xi}, v >^2 d\mu_p^x(\xi)
\]

We then have

\[
\frac{1}{p} \int_{\partial X} \|DF_p(v)^f(\xi)\|^2 d\mu_p^x(\xi) \leq \frac{1}{p} \int_{\partial X} d^2 B_{F_p(x)}(DF_p(v), DF_p(v))d\mu_p^x(\xi)
\]

\[
= \int_{\partial X} < \overrightarrow{F_p(x)f(\xi)}, DF_p(v) > < \overrightarrow{\xi}, v > d\mu_p^x(\xi) - \int_{\partial X} < \overrightarrow{F_p(x)f(\xi)}, DF_p(v) >^2 d\mu_p^x(\xi)
\]

\[
\leq \left( \int_{\partial X} < \overrightarrow{F_p(x)f(\xi)}, DF_p(v) >^2 d\mu_p^x(\xi) \right)^{1/2} \left( \int_{\partial X} < \overrightarrow{\xi}, v >^2 d\mu_p^x(\xi) \right)^{1/2}
\]

\[
\leq \int_{\partial X} < \overrightarrow{\xi}, v >^2 d\mu_p^x(\xi)
\]

∎
Proposition 4.9. For any $x, y \in X$,

$$\cosh(d(F_p(x), F_p(y))) \leq \int_{\partial X} \exp(B(F_p(y), F_p(x), f(\xi)))d\mu_p^x(\xi)$$

and

$$\cosh(bd(F_p(x), F_p(y))) \geq \int_{\partial X} \exp(bB(F_p(y), F_p(x), f(\xi)))d\mu_p^x(\xi)$$

Proof: Let

$$\theta(\xi) = \angle F_p(y)F_p(x)f(\xi), \theta_1(\xi) = \angle(-1)F_p(y)F_p(x)f(\xi), \theta_0(\xi) = \angle(-b^2)F_p(y)F_p(x)f(\xi),$$

then

$$\theta_b(\xi) \leq \theta(\xi) \leq \theta_1(\xi)$$

and by Proposition 4.10,

$$\int_{\partial X} \cos(\theta(\xi))d\mu_p^x(\xi) = 0$$

since the measure $f_*\mu_p^x$ is balanced at the point $F_p(x)$. Thus

$$\int_{\partial X} \cos(\theta_1(\xi))d\mu_p^x(\xi) \leq 0 \leq \int_{\partial X} \cos(\theta_0(\xi))d\mu_p^x(\xi)$$

so the proposition follows from Lemma 2.4.

We now fix a point $z \in X$. Then the following second-order Taylor expansion holds for $w \in T_{F_p(z)}Y$, uniformly in $\xi \in \partial X$:

$$B(\exp_{F_p(z)}(w), F_p(z), f(\xi)) = -\langle w, F_p(z)f(\xi) \rangle + \frac{1}{2}d^2B^f(\xi)(w, w) + o(||w||^2)$$

as $w \to 0$.

We fix a unit tangent vector $v \in T^1_zX$, and consider two points $x = \exp_z(tv), y = \exp_z(-tv)$, with $t > 0$ small.

Proposition 4.10. As $t \to 0$, we have

$$\int_{\partial X} \exp(bB(F_p(y), F_p(x), f(\xi)))d\mu_p^x(\xi)$$

$$\geq 1 + \left(2b^2\int_{\partial X} <DF_p(v), F_p(z)f(\xi)>^2 d\mu_p^x(\xi) \right)t^2$$

$$+ p^2 \left(\int_{\partial X} <v, z\xi > d\mu_p^x(\xi) \right)^2 t^2 + o(t^2)$$

Proof: Since $x = \exp_z(tv)$, we can write $F_p(x) = \exp_{F_p(z)}(tDF_p(v) + w)$ where $w = o(t)$ as $t \to 0$. For any $\xi \in \partial X$, we have

$$B(F_p(x), \pi(\phi(\xi)))$$

$$= B(F_p(x), F_p(z), f(\xi)) + B(F_p(z), \pi(\phi(\xi)), f(\xi)) + B(\pi(\phi(\xi)), \pi(\phi(\zeta)), f(\xi))$$

$$= B(F_p(x), F_p(z), f(\xi)) + B(F_p(z), \pi(\phi(\xi)), f(\xi)) - B(x, z, \xi),$$
so, using the second-order Taylor expansion of Busemann functions,

\[
B(F_p(x), \pi(\phi(x))) - B(F_p(z), \pi(\phi(z))) = B(F_p(x), F_p(z), f(\xi)) - B(x, z, \xi)
\]

\[
= \langle v, z \xi \rangle - \langle DF_p(v), F_p(z)f(\xi) \rangle t - \langle w, F_p(z)f(\xi) \rangle + \frac{1}{2} \left( d^2 B_f(\xi)(DF_p(v) + w, tDF_p(v) + w) - d^2 B^\xi(v, v)t^2 \right) + o(t^2)
\]

\[
= a(v, \xi)t - \langle w, F_p(z)f(\xi) \rangle + b(v, \xi)t^2 + o(t^2)
\]

where

\[
a(v, \xi) = \langle v, z \xi \rangle - \langle DF_p(v), F_p(z)f(\xi) \rangle,
\]

\[
b(v, \xi) = \frac{1}{2} \left( d^2 B_f(\xi)(DF_p(v), DF_p(v)) - d^2 B^\xi(v, v) \right).\]

It follows that

\[
\exp(pB(F_p(x), \pi(\phi(x))), f(\xi)))
\]

\[
= \exp \left( p \left( a(v, \xi)t - \langle w, F_p(z)f(\xi) \rangle + b(v, \xi)t^2 + o(t^2) \right) \right)
\]

\[
= 1 + pa(v, \xi)t - p < w, F_p(z)f(\xi) > + \left( pb(v, \xi) + \frac{1}{2} p^2 a(v, \xi)^2 \right) t^2 + o(t^2)
\]

and hence

\[
\frac{c_{x,p}}{c_{z,p}} = \int_{\partial X} \exp(pB(F_p(x), \pi(\phi(x))), f(\xi)))d\mu(\xi) \cdot c_{z,p}^{-1}
\]

\[
= \int_{\partial X} \left( 1 + pa(v, \xi)t - p < w, F_p(z)f(\xi) > + \left( pb(v, \xi) + \frac{1}{2} p^2 a(v, \xi)^2 \right) t^2 + o(t^2) \right)d\mu_p^z(\xi)
\]

\[
= \int_{\partial X} \left( 1 + pa(v, \xi)t + \left( pb(v, \xi) + \frac{1}{2} p^2 a(v, \xi)^2 \right) t^2 + o(t^2) \right)d\mu_p^z(\xi)
\]

(where in the last line above we have used the fact that \( f^* \mu_p^z \) is balanced at \( F_p(z) \)).
Thus, letting \( c(v, \xi) = ph(v, \xi) + \frac{1}{2}p^2a(v, \xi)^2 \), we have

\[
\frac{d\mu^z_p}{d\mu^p}(\xi) = \frac{\exp(pB(F_p(x), \pi(\phi(x^0)), f(\xi))) e_{x,p}}{\exp(pB(F_p(z), \pi(\phi(z^0)), f(\xi))) e_{x,p}}
\]

\[
= \frac{1 + pa(v, \xi)t - p < w, F_p(z)f(\xi) > + (pb(v, \xi) + \frac{1}{2}p^2a(v, \xi)^2) t^2 + o(t^2)}{\int_{\partial X} (1 + pa(v, \xi)t - p < w, F_p(z)f(\xi) > + c(v, \xi)t^2 + o(t^2)) \, d\mu^z_p(\xi)}
\]

\[
= \left(1 + pa(v, \xi)t - p < w, F_p(z)f(\xi) > + c(v, \xi)t^2 + o(t^2)\right)
\]

\[
\cdot \left(1 - p \left(\int_{\partial X} a(v, \xi)d\mu^z_p(\xi)\right) \cdot t + \left(p^2 \left(\int_{\partial X} a(v, \xi)d\mu^z_p(\xi)\right) - \int_{\partial X} c(v, \xi)d\mu^z_p(\xi)\right) \cdot t^2 + o(t^2)\right)
\]

\[
= 1 + p \left(a(v, \xi) - \int_{\partial X} a(v, \xi)d\mu^z_p(\xi)\right) \cdot t - p < w, F_p(z)f(\xi) >
\]

\[
+ \left(c(v, \xi) - \int_{\partial X} c(v, \xi)d\mu^z_p(\xi) + p^2 \left(\int_{\partial X} a(v, \xi)d\mu^z_p(\xi)\right) + o(t^2)\right)
\]

Letting \( F_p(y) = \exp(F_y(z)(tDF_p(-v) + w') \) where \( w' = o(t) \) as \( t \to 0 \), we have also

\[
\exp(bB(F_p(y), F_p(x), f(\xi))) = \exp(b(B(F_p(y), F_p(z), f(\xi)) - B(F_p(x), F_p(z), f(\xi))))
\]

\[
= \exp(b(2 < DF_p(v), F_p(z)f(\xi) > t < w - w', F_p(z)f(\xi) >
\]

\[
+ \frac{1}{2}d^2B^f(\xi)(tDF_p(-v) + w', tDF_p(-v) + w') - \frac{1}{2}d^2B^f(\xi)(tDF_p(v) + w, tDF_p(v) + w)
\]

\[
+ o(t^2))
\]

\[
= \exp\left(b \left(2 < DF_p(v), F_p(z)f(\xi) > t < w - w', F_p(z)f(\xi) > + o(t^2)\right)\right)
\]

\[
= 1 + b \left(2 < DF_p(v), F_p(z)f(\xi) > t < w - w', F_p(z)f(\xi) >\right)
\]

\[
+ 2b^2 < DF_p(v), F_p(z)f(\xi) > t^2 + o(t^2)
\]

It follows that when computing the integral

\[
\int_{\partial X} \exp(bB(F_p(y), F_p(x), f(\xi))) d\mu^z_p(\xi)
\]

\[
= \int_{\partial X} \exp(bB(F_p(y), F_p(x), f(\xi))) \frac{d\mu^z_p}{d\mu^p}(\xi) d\mu^z_p(\xi),
\]

after multiplying out the above expansions for \( \exp(bB(F_p(y), F_p(x), f(\xi))) \) and \( \frac{d\mu^z_p}{d\mu^p}(\xi) \), when we integrate with respect to \( \mu^z_p \), then neglecting terms which are \( o(t^2) \), the other terms involving \( w, w' \) (which are \( o(t) \)) vanish because the measure \( f_*\mu^z_p \) is balanced at \( F_p(z) \), while the integrals of the terms \( p(a(v, \xi)) \int_{\partial X} a(v, \xi)d\mu^z_p(\xi) \).
and \( (c(v, \xi) - \int_{\partial X} c(v, \xi) d\mu_p^z(\xi)) t^2 \) vanish since \( \mu_p^z \) is a probability measure. Also
\[
\int_{\partial X} a(v, \xi) d\mu_p^z(\xi) = \int_{\partial X} < v, z \xi > d\mu_p^z(\xi)
\]
because \( f_*\mu_p^z \) is balanced at \( F_p(z) \). Thus we are finally left with
\[
\int_{\partial X} \exp(bB(F_p(y), F_p(x), f(\xi))) d\mu_p^z(\xi)
= 1 + p^2 \left( \int_{\partial X} < v, z \xi > d\mu_p^z(\xi) \right)^2 t^2 \\
+ 2b \cdot p \cdot \left( \int_{\partial X} < v, z \xi > < DF_p(v), F_p(z) f(\xi) > - < DF_p(v), F_p(z) f(\xi) >^2 d\mu_p^z(\xi) \right) t^2 \\
+ 2b^2 \left( \int_{\partial X} < DF_p(v), F_p(z) f(\xi) >^2 d\mu_p^z(\xi) \right) t^2 + o(t^2)
\]
Now by Proposition 4.7,
\[
p \cdot \left( \int_{\partial X} < v, z \xi > < DF_p(v), F_p(z) f(\xi) > - < DF_p(v), F_p(z) f(\xi) >^2 d\mu_p^z(\xi) \right)
= \int_{\partial X} d^2 B(f(\xi), DF_p(v), DF_p(v)) d\mu_p^z(\xi)
\geq 0
\]
and hence
\[
\int_{\partial X} \exp(bB(F_p(y), F_p(x), f(\xi))) d\mu_p^z(\xi)
\geq 1 + p^2 \left( \int_{\partial X} < v, z \xi > d\mu_p^z(\xi) \right)^2 t^2 \\
+ 2b^2 \left( \int_{\partial X} < DF_p(v), F_p(z) f(\xi) >^2 d\mu_p^z(\xi) \right) t^2 + o(t^2)
\]
as required. \( \diamond \)

**Proposition 4.11.** We have
\[
\frac{p^2}{2b^2} \left( \int_{\partial X} < v, z \xi > d\mu_p^z(\xi) \right)^2 \leq \int_{\partial X} ||DF_p(v)^{-1} f(\xi)||^2 d\mu_p^z(\xi)
\]
Proof: It follows from Propositions 4.10 and 4.9 that
\[
\cosh(b d(F_p(x), F_p(y))) \\
\geq \int_{\partial X} \exp(b B(F_p(y), F_p(x), f(\xi))) d\mu_p^\xi(\xi) \\
\geq 1 + p^2 \left( \int_{\partial X} <v, \overrightarrow{x\xi}> d\mu_p^\xi(\xi) \right)^2 t^2 \\
+ 2b^2 \left( \int_{\partial X} <DF_p(v), F_p(z)f(\xi)^{\rightarrow}>^2 d\mu_p^\xi(\xi) \right) t^2 + o(t^2)
\]

Now as \( t \to 0 \) we have \( d(F_p(x), F_p(y)) = 2t\|DF_p(v)\| + o(t) \), thus
\[
\cosh(b d(F_p(x), F_p(y))) = 1 + 2t^2\|DF_p(v)\|^2 + o(t^2).
\]

It follows from the preceding inequality that
\[
2b^2\|DF_p(v)\|^2 \\
\geq p^2 \left( \int_{\partial X} <v, \overrightarrow{x\xi}> d\mu_p^\xi(\xi) \right)^2 \\
+ 2b^2 \left( \int_{\partial X} <DF_p(v), F_p(z)f(\xi)^{\rightarrow}>^2 d\mu_p^\xi(\xi) \right).
\]

Using
\[
\|DF_p(v)\|^2 = \int_{\partial X} <DF_p(v), F_p(z)f(\xi)^{\rightarrow}>^2 + \|DF_p(v)^{\perp f(\xi)}\|^2 d\mu_p^\xi(\xi)
\]
and the previous inequality, we obtain the inequality stated in the proposition. □

For each \( z \in X \), we now let \( \mu_p^\xi \) be a weak limit as \( p \to \infty \) of the probability measures \( \mu_p^\xi \) on \( \partial X \).

Proposition 4.12. For all \( v \in T_z X \), we have
\[
\int_{\partial X} <v, z\overrightarrow{\xi}> d\mu^\xi_{\infty}(\xi) = 0,
\]
i.e. the probability measure \( \mu^\xi_{\infty} \) on \( \partial X \) is balanced at \( z \in X \).

Proof: From Propositions 4.8 and 4.11, for any \( v \in T^1 z X \), we have
\[
\frac{p^2}{2b^2} \left( \int_{\partial X} <v, z\overrightarrow{\xi}> d\mu_p^\xi(\xi) \right)^2 \\
\leq \int_{\partial X} \|DF_p(v)^{\perp f(\xi)}\|^2 d\mu_p^\xi(\xi) \\
\leq p \int_{\partial X} <v, z\overrightarrow{\xi}>^2 d\mu_p^\xi(\xi)
\]
and hence
\[
\left( \int_{\partial X} <v, z\xi > d\mu_p^z(\xi) \right)^2 \leq \frac{2\theta^2}{p} \int_{\partial X} <v, z\xi >^2 d\mu_p^z(\xi) \\
\leq \frac{2\theta^2}{p}.
\]

Since \( \mu_p^z \to \mu_\infty^z \) weakly as \( p \to \infty \) along some sequence, passing to the limit above gives
\[
\int_{\partial X} <v, z\xi > d\mu_\infty^z(\xi) = 0
\]
as required. \( \diamondsuit \)

**Lemma 4.13.** For \( x \in X, y \in Y, \xi \in \partial X \) we have
\[
\frac{df^*_p}{d\rho_y}(f(\xi)) = \exp(B(y, \pi(\phi(x\xi)), f(\xi)))
\]

**Proof:** Let \( z = \pi(\phi(x\xi)) \in Y \), then by definition of \( \phi \) we have \( \frac{df^*_p}{d\rho_z}(f(\xi)) = 1 \), so by the Chain Rule
\[
\frac{df^*_p}{d\rho_z}(f(\xi)) = \frac{df^*_p}{d\rho_x}(f(\xi)) \cdot \frac{d\rho_x}{d\rho_y}(f(\xi)) \\
= 1 \cdot \exp(B(y, z, f(\xi)))
\]
\( \diamondsuit \)

**Proposition 4.14.** Let \( K \subset \partial X \) be the set where the function
\[
\xi \in \partial X \mapsto \frac{df^*_p}{d\rho_{F(z)}}(f(\xi))
\]
attains its maximum value. Then the support of the measure \( \mu_\infty^z \) is contained in \( K \).

**Proof:** Let \( M = \sup_{\xi \in \partial X} \frac{df^*_p}{d\rho_{F(z)}}(f(\xi)) \). Given \( \xi \in \partial X - K \), we can choose \( \epsilon > 0 \) and a neighbourhood \( U \) of \( \xi \) such that \( \frac{df^*_p}{d\rho_{F(z)}}(f(\eta)) \leq M(1 - \epsilon) \) for all \( \eta \in U \). Let \( \psi \) be a continuous function on \( \partial X \) with support contained in \( U \) such that \( 0 \leq \psi \leq 1 \) on \( \partial X \) and such that \( \psi = 1 \) on a neighbourhood \( N \) of \( \xi \).

For \( \eta \in \partial X \),
\[
|B(F_p(z), \pi(\phi(\overline{z}\eta)), f(\eta)) - B(F(z), \pi(\phi(\overline{z}\eta)), f(\eta))| \leq d(F_p(z), F(z)) \to 0
\]
as \( p \to \infty \), so we can choose \( p_0 > 1 \) such that for \( p \geq p_0 \), by Lemma 4.13 for all \( \eta \in U \) we have
\[
\exp(B(F_p(z), \pi(\phi(\overline{z}\eta)), f(\eta))) \leq \exp(B(F(z), \pi(\phi(\overline{z}\eta)), f(\eta)))(1 + \epsilon) \\
= \frac{df^*_p}{d\rho_{F(z)}}(f(\eta))(1 + \epsilon) \\
\leq M(1 - \epsilon^2)
\]
while convergence of \( L^p(\mu) \) norms to the \( L^\infty(\mu) \) norm as \( p \to \infty \) implies that for \( \mu_0 \) chosen large enough, for \( p \geq \mu_0 \) we have

\[
e_{z,p} = \left\| \frac{df^\star \rho_{z}}{d\rho_{F(z)}}(f(\cdot)) \right\|_{L^p(\mu)}^p \geq (M(1 - \epsilon^2/2))^p
\]

hence 

\[
\int_{\partial X} \psi(\eta) d\mu^x_\infty(\eta) \leq \int_U \exp(pB(F_p(z), \pi(\phi(\overline{z}\eta))), f(\eta))) d\mu(\eta) \cdot c^{-1}_{z,p} \\
\leq \left( \frac{M(1 - \epsilon^2)}{M(1 - \epsilon^2/2)} \right)^p \\
\to 0
\]

as \( p \to \infty \). Letting \( p \to \infty \) gives \( \int_{\partial X} \psi(\eta) d\mu^x_\infty(\eta) = 0 \), so \( \mu^x_\infty(N) = 0 \) since \( \psi = 1 \) on \( N \). It follows that \( \xi \) is not in the support of \( \mu^x_\infty \).

**Proposition 4.15.** Let \( g : \partial Y \to \partial X \) be the inverse of the Moebius map \( f : \partial X \to \partial Y \). Then the circumcenter extensions \( F : X \to Y \) and \( G : Y \to X \) of the maps \( f, g \) are inverses of each other.

**Proof:** Let \( x \in X \) and \( y = F(x) \in Y \). Let \( i_x : \partial X \to \partial X \) denote the antipodal map of \( \partial X \) centered at the point \( x \), i.e. the conjugate of the flip map \( T^1_x \to T^1_x, v \mapsto -v \), by the natural map \( p : T^1_x X \to \partial X \). Let \( K \subset \partial X \) be the set where the function

\[
\xi \in \partial X \mapsto \log \frac{df^\star \rho_x}{d\rho_y}(f(\xi)) = \log \frac{d\rho_x}{df^\star \rho_y}(\xi)
\]

attains its maximum value. Then by Lemma 2.2, \( i_x(K) \) is contained in the set where the same function attains its minimum value. By the Chain Rule, for \( \xi \in \partial X \) and \( \eta = f(\xi) \in \partial Y \), we have

\[
\log \frac{df^\star \rho_x}{d\rho_y}(f(\xi)) + \log \frac{dg^\star \rho_y}{d\rho_x}(g(\eta)) = 0.
\]

Letting \( K' = f(i_x(K)) \subset \partial Y \), it follows that \( K' \) is contained in the set \( J \) say where the function

\[
\eta \in \partial Y \mapsto \log \frac{dg^\star \rho_y}{d\rho_x}(g(\eta))
\]

attains its maximum value. Let \( \nu_1 \) be the probability measure \( \nu_1 := (i_x)_\star \mu^x_\infty \) on \( \partial X \) and let \( \nu_2 \) be the probability measure \( \nu_2 = f_{\star} \nu_1 \) on \( \partial Y \). By Proposition 4.14 the support of \( \mu^x_\infty \) is contained in \( K \), hence the support of \( \nu_2 \) is contained in \( K' \).
Now for any $\xi \in \partial X$, we have $\overrightarrow{x_i x}(\xi) = -\overrightarrow{x_\xi}$, and hence, for any $v \in T_x X$,
\[
\int_{\partial X} <v, \overrightarrow{x_\xi}> d\nu_2(\xi) = -\int_{\partial X} <v, \overrightarrow{x_\xi}> d\mu_\infty(\xi) = 0
\]
by Proposition 4.12. It follows that the measure $g_*\nu_2$ is balanced at $x \in X$, therefore, since the support of $\nu_2$ is contained in the set $K' \subset J$, by (4) of Proposition 4.6 applied to the Moebius map $g : \partial Y \to \partial X$ we have $x = G(y)$. ◯

Proposition 4.16. The circumcenter extension $F : X \to Y$ is $\sqrt{b}$-Lipschitz.

Proof: Let $x, y \in X$. For $\xi \in \partial X$, let
\[
u_1(\xi) = \log \frac{df_*\rho_x}{d\rho_F(x)}(f(\xi)) = \exp(B(F(x), \overrightarrow{x_\xi}, f(\xi)))
\]
and let $K_x, K_y$ be the subsets of $\partial X$ where the functions $u_x, u_y$ attain their maximum values respectively, and let the maximum values be $r(x), r(y)$ respectively. Then for $\xi \in K_x, \eta \in K_y$, we have
\[
B(F(y), F(x), f(\xi)) = B(F(y), \pi(\phi(y_\xi)), f(\xi)) + B(\pi(\phi(y_\xi)), \pi(\phi(x_\xi)), f(\xi)) + B(\pi(\phi(x_\xi)), F(x), f(\xi))
\]
and similarly
\[
B(F(x), F(y), f(\eta)) = u_x(\eta) + B(x, y, \eta) - r(x)
\]
and
\[
B(F(x), F(y), f(\eta)) = u_x(\eta) + B(x, y, \eta) - r(x)
\]
thus
\[
B(F(y), F(x), f(\xi)) + B(F(x), F(y), f(\eta))
\]
\[
= (B(y, x, \xi) + B(x, y, \eta)) + (u_y(\xi) - r(y)) + (u_x(\eta) - r(x))
\]
\[
\leq B(y, x, \xi) + B(x, y, \eta).
\]
Since the measures $f_*\mu_x^p, f_*\mu_y^p$ are balanced at the points $F_p(x), F_p(y)$ respectively, and $F_p(x) \to F(x), F_p(y) \to F(y)$ as $p \to \infty$, it is easy to see that the measures
Now for \( t > 0 \), \( f, \mu^x_\infty, f, \mu^y_\infty \) are balanced at the points \( F(x), F(y) \) respectively, so an argument similar to the proof of Proposition 4.9 (using Lemma 2.10) gives
\[
cosh(d(F(x), F(y))) = \int_{\partial X} \exp(B(F(y), F(x), f(\xi))) d\mu^x_\infty(\xi)
\]
\[
cosh(d(F(y), F(x))) = \int_{\partial X} \exp(B(F(x), F(y), f(\eta))) d\mu^y_\infty(\eta).
\]
The measures \( \mu^x_\infty, \mu^y_\infty \) are also balanced at the points \( x, y \) respectively, so again an argument similar to the proof of Proposition 4.9 gives
\[
cosh(bd(x, y)) = \int_{\partial X} \exp(bB(y, x, \xi)) d\mu^x_\infty(\xi)
\]
\[
cosh(bd(y, x)) = \int_{\partial X} \exp(bB(x, y, \eta)) d\mu^y_\infty(\eta).
\]
Now using the fact that the supports of \( \mu^x_\infty, \mu^y_\infty \) are contained in \( K_x, K_y \) respectively, we have
\[
cosh^2(d(F(x), F(y)))
\leq \left( \int_{\partial X} \exp(B(F(y), F(x), f(\xi))) d\mu^x_\infty(\xi) \right) \left( \int_{\partial X} \exp(B(F(x), F(y), f(\eta))) d\mu^y_\infty(\eta) \right)
\]
\[
= \int_{\partial X} \int_{\partial X} \exp(B(F(y), F(x), f(\xi)) + B(F(x), F(y), f(\eta))) d\mu^x_\infty(\xi) d\mu^y_\infty(\eta)
\]
\[
\leq \int_{\partial X} \int_{\partial X} \exp(B(y, x, \xi) + B(x, y, \eta)) d\mu^x_\infty(\xi) d\mu^y_\infty(\eta)
\]
\[
= \left( \int_{\partial X} \exp(bB(y, x, \xi)) d\mu^x_\infty(\xi) \right) \left( \int_{\partial X} \exp(bB(x, y, \eta)) d\mu^y_\infty(\eta) \right)
\]
\[
\leq \left( \int_{\partial X} \exp(bB(y, x, \xi)) d\mu^x_\infty(\xi) \right)^{1/b} \left( \int_{\partial X} \exp(bB(x, y, \eta)) d\mu^y_\infty(\eta) \right)^{1/b}
\]
\[
\leq (\cosh^2(bd(x, y)))^{1/b}
\]
thus
\[
cosh^b(d(F(x), F(y))) \leq \cosh(bd(x, y)).
\]
Now for \( t \geq 0 \), \( \cosh^b(t) \geq 1 + bt^2/2 \), and there is a universal constant \( C > 0 \) such that \( \cosh bt \leq 1 + b^2t^2 \) for \( 0 \leq t \leq C \), so \( d(x, y) \leq C \) implies
\[
\frac{1}{2} d(F(x), F(y))^2 \leq bd(x, y)^2,
\]
thus \( F \) is locally Lipschitz. It follows that \( F \) is differentiable almost everywhere. At a point \( x \) of differentiability of \( F \), for \( v \in T^*_x X \) letting \( y = \exp_x(tv) \), a Taylor expansion of both sides of the inequality \( \cosh^b(d(F(x), F(y))) \leq \cosh(bd(x, y)) \) up to second order in \( t \) easily gives \( ||DF(v)|| \leq \sqrt{b} \). Now it is a standard fact that if \( F \) is a locally Lipschitz map between complete Riemannian manifolds such that \( ||DF|| \leq C \) almost everywhere then \( F \) is \( C \)-Lipschitz, hence in our case \( F \) is \( \sqrt{b} \)-Lipschitz.
Proof of Theorem 1.1 Since the maps $F : X \to Y, G : Y \to X$ are inverses of each other, it follows from Proposition 4.16 above that they are $\sqrt{5}$-bi-Lipschitz homeomorphisms. ♦

Proof of Theorem 1.2 Given $X, Y$ closed, negatively curved $n$-manifolds, after rescaling the metrics on $X, Y$ by the same constant $C > 0$, we may assume by choosing $C$ appropriately that both manifolds have sectional curvatures bounded above by $-1$. Then the universal covers $\tilde{X}, \tilde{Y}$ are CAT(-1) spaces and equality of the marked length spectra of $X, Y$ implies existence of an equivariant M"obius homeomorphism $f : \partial \tilde{X} \to \partial \tilde{Y}$. By the naturality of the circumcenter extension, the circumcenter extension $\hat{f} : \tilde{X} \to \tilde{Y}$ is equivariant, and is a bi-Lipschitz homeomorphism by the previous theorem, hence induces a bi-Lipschitz homeomorphism $F : X \to Y$. ♦

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