ON THE Riemann Hypothesis

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Abstract. In 1913 Pólya proved that the Riemann hypothesis is equivalent to the Jensen conjecture that all roots of Riemann Xi function are real. In this article we suggest an even entire function associated with the Riemann Xi function. We set up a class for this function with use of its series and product. We prove that it has only purely imaginary zeros. The obtained result implies that both Riemann hypothesis and Jensen conjecture are true.

1. Introduction

In 1859 Riemann paper, he proposed the Riemann zeta function $\zeta(t)$ by the sum [1]

$$\zeta(t) = \sum_{u=1}^{\infty} \frac{1}{u^t},$$

and by Euler product [2]

$$\zeta(t) = \prod_{p} \frac{1}{1 - \frac{1}{p^t}},$$

where $\mathbb{P}$ run prime, $t \in \mathbb{C}$, $u \in \mathbb{N}$ and $\Re(t) > 1$. Here, let us allow that $\mathbb{C}$ and $\mathbb{N}$ are the sets of the complex and natural numbers, respectively, and that $\Re(t)$ and $\Im(t)$ are the real and imaginary parts for the complex variable $t$, respectively.

In fact, (1) can be extended to [1]

$$\xi(t) = (t - 1) \pi^{-\frac{t}{2}} \Gamma \left( \frac{t}{2} + 1 \right) \zeta(t),$$

which can be written as [2]

$$\xi(t) = 4 \int_{1}^{\infty} \frac{d}{dv} \left( v^\frac{3}{2} \varphi^{(1)}(v) \right) v^{-\frac{1}{2}} \cosh \left( \frac{1}{2} (t - \frac{1}{2}) \log v \right) dv,$$

where $t \in \mathbb{C}$ and

$$\varphi(v) = \sum_{y=1}^{\infty} e^{-y^2 \pi v}.$$
Here, (4) is well-known Riemann xi function, which satisfies the following properties:

- (1A) $\xi(t)$ has the functional equation [1]:
  \begin{equation}
  \xi(t) = \xi(1 - t) .
  \end{equation}

- (2A) $\xi(t)$ is an entire function of order $\nu = 1$ (see [3], Theorem 17, p.58; also see [4], Theorem 2.12, p.29).

Following Edwards’ idea (see [2], p.17), we expand (4) as

\begin{equation}
\xi(t) = \sum_{n=0}^{\infty} A_{2n} \left( t - \frac{1}{2} \right)^{2n},
\end{equation}

where

\begin{equation}
A_{2n} = 4 \int_{1}^{\infty} \frac{d}{dv} \left( v^{\frac{3}{2}} \wp^{(1)}(v) \right) v^{-\frac{3}{4}} \cos \left( \frac{x}{2} \log v \right) dv,
\end{equation}

with

\begin{equation}
A_{2n} > 0.
\end{equation}

The Riemann hypothesis for (1) states that real part of all of the nontrivial zeros of (1) is equal to $1/2$ [5]. It is equivalent to the statement that real part of all of the zeros of (4) is equal to $1/2$ (see [6]; also see [7], p.13).

Taking $t = 1/2 + ix$ into (4) implies that [1]

\begin{equation}
\Xi(x) = 4 \int_{1}^{\infty} \frac{d}{dv} \left( v^{\frac{3}{2}} \wp^{(1)}(v) \right) v^{-\frac{3}{4}} \cos \left( \frac{x}{2} \log v \right) dv,
\end{equation}

where $x \in \mathbb{C}$ and $i = \sqrt{-1}$.

Here, (9) is called the Riemann Xi function, which has the following properties:

- (1B) $\Xi(x)$ has the functional equation [1]:
  \begin{equation}
  \Xi(x) = \Xi(x).
  \end{equation}

- (2B) $\Xi(x)$ is an even entire function of order $\nu = 1$ ([4], Theorem 2.12, p.29).

In 1913 Jensen [8] proved that the Riemann hypothesis is equivalent to the statement that $\Xi(x)$ has only real zeros. This is called the Jensen conjecture [9]. In 1927 Pólya [10, 11] showed that the idea that $\Xi(x)$ is expressed by its series and product, which is considered in the Laguerre-Pólya class [12]. The work of Csordas and Varga [12] said that the Turán inequalities for $\Xi(x)$ are necessary and sufficient conditions for the Riemann hypothesis. The history of the Turán inequalities to test the Riemann hypothesis was investigated by Conrey and Ghosh in [13]. However, Conrey said that the Turán inequalities give a necessary but not sufficient condition for the truth of the Jensen conjecture (see [14], p.129). In 2019 Griffin, Ono, Rolan and Zagier proposed the idea to test the Riemann hypothesis by Jensen polynomials for the Riemann xi-function constructed from certain Taylor coefficients [15].
Similarly, putting $t = 1/2 + \tau$ into (1) gives

$$
\Theta (\tau) = 4 \int_1^{\infty} \frac{d}{dv} \left( v^{\frac{3}{2}} g_1(v) \right) v^{-\frac{3}{4}} \cosh \left( \frac{\tau}{2} \log v \right) dv,
$$

where $x \in \mathbb{C}$. An analogy of (11) was considered by Csordas and Varga in [13].

Clearly, (11) has the functional equation

$$
\Theta (\tau) = \Theta (-\tau).
$$

Taking

$$
\tau = ix
$$

into (11), we show

$$
\Xi (x) = \Theta (ix).
$$

This implies that the Riemann hypothesis is the statement that all zeros of $\Theta (\tau)$ are purely imaginary.

Motivated by the idea of Pólya [10], we consider a class of $\Theta (\tau)$, which be structured by its series and product. Based on the result, the main target of this paper is to prove:

**Theorem 1.** All of the zeros of $\Theta (\tau)$ lie on the critical line $\Re (\tau) = 0$.

The structure of this paper is designed as follows. In Section 2 we consider the series, order and product of (11). In Section 3 we propose a class of (11) by using its series and product representations to prove Theorem 1. We also discuss the products of (4) and (9).

2. The series, order and product for (11)

Assume the notations above, we have the followings:

**Proposition 1.** There is

$$
\Theta (\tau) = \sum_{n=0}^{\infty} A_{2n} \tau^{2n}.
$$

*Proof.* Putting $t = 1/2 + \tau$ into (11), we obtain the required result. \hfill \Box

**Proposition 2.** $\Theta (\tau)$ is an even entire function of order $\upsilon = 1$.

*Proof.* By suing (6) and (17), we see that $\xi (t)$ and $\Theta (\tau)$ has the same coefficients $A_{2n}$.

By (2A), we have (see [16] p.4)

$$
\lim_{n \to \infty} \sqrt[n]{A_{2n}} = 0
$$

such that $\Theta (\tau)$ is an entire function.

Making use of (2A), Theorem 2.2.2 in Boas’ book (see [17], p.9) implies that

$$
\upsilon = \limsup_{n \to \infty} \frac{n \ln n}{\ln (1/\ln |A_{2n}|)} = 1.
$$

From (12) it implies that $\Theta (\tau)$ is an even function.
Thus, we see that $\Theta (\tau)$ is an even entire function of order $\nu = 1$.

We hence complete the proof.

We now adopt Proposition 2 to set up the product of (11) as follows:

**Theorem 2.** There exists

\begin{equation}
\Theta (\tau) = \Theta (0) \prod_{\Im (\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right),
\end{equation}

where the product takes over the all zeros $\tau_k$ of $\Theta (\tau)$.

**Proof.** Because $\Theta (\tau)$ is an even entire function of order $\nu = 1$, and

\begin{equation}
\Theta (0) = A_0 = 4 \int_1^\infty \frac{d}{dv} \left( v^\frac{1}{2} \wp^{(1)} (v) \right) v^{-\frac{1}{4}} \frac{1}{(2n)!} dv > 0,
\end{equation}

the Hadamard’s factorization theorem (see [16], p.24; also see [17], p.22) gives

\begin{equation}
\Theta (\tau) = \Theta (0) e^{\varphi \tau} \prod_{\tau_k} \left( 1 - \frac{\tau}{\tau_k} \right) e^{\frac{\tau}{\tau_k}},
\end{equation}

where $\tau \in \mathbb{C}$ and $\varphi$ is a constant.

Making use of (12) and (20), we have

\begin{equation}
\Theta (\tau) = \Theta (0) e^{\varphi \tau} \prod_{\Im (\tau_k) > 0} \left( 1 - \frac{\tau}{\tau_k} \right) e^{\frac{\tau}{\tau_k}},
\end{equation}

such that

\begin{equation}
\Theta (\tau) = \Theta (0) e^{\varphi \tau} \prod_{\Im (\tau_k) > 0} \left( 1 - \frac{\tau}{\tau_k} \right) \left( 1 + \frac{\tau}{\tau_k} \right) e^{\left( \frac{\tau}{\tau_k} - \frac{\tau}{\tau_k} \right)}. \tag{22}
\end{equation}

To simply (22), we carry out

\begin{equation}
\Theta (\tau) = \Theta (0) e^{\varphi \tau} \prod_{\Im (\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right). \tag{23}
\end{equation}

Consider

\begin{equation}
\Theta (-\tau) = \Theta (0) e^{-\varphi \tau} \prod_{\Im (\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right). \tag{24}
\end{equation}

By (12), we have

\begin{equation}
\Theta (\tau) = \Theta (-\tau)
\end{equation}

such that the combination of (23) and (24) gives

\begin{equation}
\Theta (0) e^{\varphi \tau} \prod_{\Im (\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) = \Theta (0) e^{-\varphi \tau} \prod_{\Im (\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right). \tag{26}
\end{equation}

From (26) we have $\varphi = 0$ such that (23) can be rewritten as

\begin{equation}
\Theta (\tau) = \Theta (0) \prod_{\Im (\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right). \tag{27}
\end{equation}
Theorem 3. There exits any positive number $\varepsilon > 0$ such that the series
\begin{equation}
\Phi (\varepsilon) = \sum_{\tau_k} \frac{1}{|\tau_k|^{1+\varepsilon}}
\end{equation}
is convergent.

Proof. Since $\Theta (\tau)$ is an even entire function of order $\nu = 1$, and (27) is uniformly convergent for $\tau \in \mathbb{C}$. Following Levin ([16], p.8), it implies that there is any positive number $\varepsilon > 0$ such that the series (28) is convergent. It is well known that $\{\tau_k\}_{k=1}^{\infty}$ is a sequence of complex numbers, numbered in order of modulus $|\tau_k| < |\tau_{k+1}|$ for $k \in \mathbb{N}$, with $\tau_k \neq 0$ (see [17], p.18).

Therefore, we complete the proof. $\square$

3. The proof of Theorem 1

At first, we recall that
\begin{equation}
\Theta (\tau) = \sum_{n=0}^{\infty} A_{2n} \tau^{2n},
\end{equation}
where $A_{2n} > 0$, and
\begin{equation}
\Theta (\tau) = \Theta (0) \prod_{\Im(\tau_k)>0} \left(1 - \frac{\tau^2}{\tau_k^2}\right).
\end{equation}

Combining (29) and (30) gives a class of $\Theta (\tau)$, that is,
\begin{equation}
\Theta (\tau) = \sum_{n=0}^{\infty} A_{2n} \tau^{2n} = \Theta (0) \prod_{\Im(\tau_k)>0} \left(1 - \frac{\tau^2}{\tau_k^2}\right).
\end{equation}

Let $\overline{\Theta (\tau)}$ and $\overline{\tau}$ denote the complex conjugates of $\Theta (\tau)$ and $\tau$. From (29) we arrive at
\begin{equation}
\overline{\Theta (\tau)} = \left(\sum_{n=0}^{\infty} A_{2n} \tau^{2n}\right) = \sum_{n=0}^{\infty} (A_{2n} \tau^{2n}) = \sum_{n=0}^{\infty} A_{2n} \overline{\tau}^{2n}.
\end{equation}

In view of (8), we have
\begin{equation}
A_{2n} > 0
\end{equation}
such that (32) becomes
\begin{equation}
\overline{\Theta (\tau)} = \sum_{n=0}^{\infty} A_{2n} \overline{\tau}^{2n} = \sum_{n=0}^{\infty} A_{2n} \overline{\tau}^{2n}.
\end{equation}

It follows from (34) that
\begin{equation}
\overline{\Theta (\tau)} = \Theta (\overline{\tau}).
\end{equation}

Combining (30) and (35), we may give
\begin{equation}
\Theta (\overline{\tau}) = \Theta (0) \prod_{\Im(\tau_k)>0} \left(1 - \frac{\tau^2}{\tau_k^2}\right).
\end{equation}
Here, (36) is the first product of $\Theta (\tau )$. We now directly find $\Theta (\tau )$ in (30) to show

$$(37) \quad \Theta (\tau ) = \left[ \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) \right] = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right).$$

With (19), we have

$$(38) \quad \Theta (0) > 0$$

such that (37) is

$$(39) \quad \Theta (\tau ) = \left[ \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) \right] = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right).$$

Further,

$$(40) \quad \Theta (\tau ) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right)$$

$$= \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right).$$

To simply (40), we get

$$(41) \quad \Theta (\tau ) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right).$$

Similarly, (40) is the second product of $\Theta (\tau )$. On combination of (36) and (40), we obtain

$$(42) \quad \Theta (\tau ) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right).$$

By using (35), we have

$$(43) \quad \Theta (\tau ) = \Theta (\tau )$$

such that

$$(44) \quad \Theta (\tau ) = \Theta (\tau ) = \Theta (\tau ).$$

Using (44) to find the complex conjugate of (42), we discover that

$$(45) \quad \Theta (\tau ) = \Theta (\tau ) = \left[ \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) \right] = \left[ \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) \right].$$
Since
\[
\Theta (\tau) = \Theta (\tau) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right)
\]
(46)
\[
= \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right)
\]
and
\[
\Theta (\tau) = \Theta (\tau) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right)
\]
(47)
\[
= \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right),
\]
the identity (45) is simplified as
\[
\Theta (\tau) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{\tau^2}{\tau_k^2} \right).
\]
(48)

We now consider the special value of \( \Theta (\tau) \) as follows.

Putting \( \tau = 1 \) into (48) implies that
\[
\Theta (1) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{1}{\tau_k^2} \right) = \Theta (0) \prod_{\Im(\tau_k) > 0} \left( 1 - \frac{1}{\tau_k^2} \right) > 0
\]
(49)
since (11) gives
\[
\Theta (1) = 4 \int_{\frac{1}{2}}^{\infty} \frac{d}{dv} \left( v^2 \wp (1) (v) \right) v^{-\frac{1}{2}} \cosh \left( \frac{1}{2} \log v \right) dv
\]
with
\[
0 < \Theta (1) < \infty.
\]

To investigate the convergence of (49), we consider the following:

By Theorem 3, the series
\[
\Phi (\varepsilon) = \sum_{\tau_k} \frac{1}{|\tau_k|^{1+\varepsilon}}
\]
(50)
converges for any positive number \( \varepsilon > 0 \).
Taking $\varepsilon = 1$ into (50) yields that the series
\begin{equation}
\Phi(1) = \sum_{\tau_k} \frac{1}{|\tau_k|^2}
\end{equation}
is convergent.

Following Knopp’s idea (see [18], p.9), it follows from (51) that
\begin{equation}
\sum\frac{1}{\tau_k^2}
\end{equation}
and
\begin{equation}
\sum\frac{1}{\tau_k^2}
\end{equation}
are absolutely convergent.

By using the fact that (52) and (53) are absolutely convergent, it is easy to see that (52) and (53) are convergent (see Theorem 3 in [18], p.10).

Thus, we show form (49) that
\begin{equation}
\sum\frac{1}{\tau_k^2} = \sum\frac{1}{\tau_k^2},
\end{equation}
which yields that
\begin{equation}
\sum\left(\frac{1}{\tau_k^2} - \frac{1}{\tau_k^2}\right) = 0.
\end{equation}

In view of (55), we obtain
\begin{equation}
\frac{1}{\tau_k^2} - \frac{1}{\tau_k^2} = 0
\end{equation}
which leads to
\begin{equation}
\tau_k^2 - \tau_k^2 = 0.
\end{equation}

In fact, (57) can be written as
\begin{equation}
\tau_k^2 - \tau_k^2 = (\tau_k - \tau_k)(\tau_k + \tau_k) = -4\Im(\tau_k)\Re(\tau_k) = 0.
\end{equation}

From (49) we see
\begin{equation}
\Im(\tau_k) > 0
\end{equation}
such that (58) leads to
\begin{equation}
\Re(\tau_k) = 0.
\end{equation}

Substituting (60) into (54) and taking
\begin{equation}
\Im(\tau_k) = \omega_k > 0,
\end{equation}
we find that
\[ \sum_{\tau_k} \frac{1}{\tau_k^2} = \sum_{\tau_k} \frac{1}{\overline{\tau_k^2}} = - \sum_{\tau_k} \frac{1}{|\tau_k|^{2(1+\omega_k^2)}} = - \sum_{\tau_k} \frac{1}{\omega_k^2} \]
is convergent.

As a matter of fact, we put (60) into (49) to show that the series
\[ (63) \quad \Theta (1) = \Theta (0) \prod_{k=1}^{\infty} \left( 1 + \frac{1}{\omega_k^2} \right) \]
always converges.

In a similar manner, we substitute (60) and (61) into (48) to get
\[ (64) \quad \Theta (\tau) = \Theta (0) \prod_{k=1}^{\infty} \left( 1 + \frac{\tau^2}{\omega_k^2} \right). \]

With the aid of (60), we indeed discover that all of the zeros of \( \Theta (\tau) \) lie on the critical line \( \Re (\tau) = 0. \)

Therefore, we complete the proof of Theorem 2.

**Remark.** Combining (11) and (64) yields that
\[ (65) \quad \Theta (\tau) = 4 \int_1^{\infty} \frac{d}{dv} \left( \frac{3}{2} \wp^{(1)} (v) \right) v^{-\frac{3}{2}} \cosh \left( \frac{\tau}{2} \log v \right) dv \]
\[ = \Theta (0) \prod_{k=1}^{\infty} \left( 1 + \frac{\tau^2}{\omega_k^2} \right). \]

With use of (13) and (65), (14) can be written as
\[ (66) \quad \Xi (x) = \Theta (ix) = 4 \int_1^{\infty} \frac{d}{dv} \left( \frac{3}{2} \wp^{(1)} (v) \right) v^{-\frac{3}{2}} \cos \left( \frac{x}{2} \log v \right) dv \]
\[ = \Theta (0) \prod_{k=1}^{\infty} \left( 1 - \frac{x^2}{\omega_k^2} \right), \]
which leads to
\[ (67) \quad \Xi (x) = \Xi (0) \prod_{k=1}^{\infty} \left( 1 - \frac{x^2}{\omega_k^2} \right) \]
because of the fact that taking \( x = 0 \) into (65) gives
\[ (68) \quad \Xi (0) = \Theta (0). \]

Eq. (67) agrees with the result of Riemann [1].

Indeed, all of the zeros of \( \Xi (x) \) are real. This implies the Jensen conjecture is true.
Taking $\tau = t - 1/2$ into (65), we obtain

$$
\Theta \left( t - \frac{1}{2} \right) = 4 \int_{1}^{\infty} \frac{d}{dv} \left( v^{\frac{3}{2}} \phi^{(1)}(v) \right) v^{-\frac{1}{4}} \cosh \left[ \frac{1}{2} \left( t - \frac{1}{2} \right) \log v \right] dv
$$

(69)

$$
= \Theta (0) \prod_{k=1}^{\infty} \left[ 1 + \frac{(t - \frac{1}{2})^{2}}{\omega_{k}^{2}} \right].
$$

From (4) and (69) we have

$$
\Theta \left( t - \frac{1}{2} \right) = \xi (t)
$$

(70)

$$
= 4 \int_{1}^{\infty} \frac{d}{dv} \left( v^{\frac{3}{2}} \phi^{(1)}(v) \right) v^{-\frac{1}{4}} \cosh \left( \frac{1}{2} \left( t - \frac{1}{2} \right) \log v \right) dv
$$

such that

$$
\xi (t) = \Theta (0) \prod_{k=1}^{\infty} \left[ 1 + \frac{(t - \frac{1}{2})^{2}}{\omega_{k}^{2}} \right].
$$

(71)

Taking $t = 1/2$ into (71), we obtain

$$
\xi \left( \frac{1}{2} \right) = \Theta (0),
$$

(72)

which reduces from (71) to

$$
\xi (t) = \xi \left( \frac{1}{2} \right) \prod_{k=1}^{\infty} \left[ 1 + \frac{(t - \frac{1}{2})^{2}}{\omega_{k}^{2}} \right].
$$

(73)

Thus, (73) reduces to $u_{k} = 1/2 \pm i\omega_{k}$, which implies that the Riemann Hypothesis is true. The present work is considered as a special case reported in [19]. Compared with the technology [20], this is more easier to work on it.

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