A study on the tunneling spectroscopy of an N–pS junction and an N–hS junction

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Abstract
We study the complete tunneling spectroscopy of a normal metal/p-wave superconductor junction (N–pS) and a normal metal/heterostructure superconductor junction (N–hS), using the Blonder–Tinkham–Klapwijk (BTK) method. We find that, for a p-wave superconductor with non-trivial topology, there exists a stable quantized zero-bias conductance peak, and for heterostructure superconductors with non-trivial topology, the emerging zero-bias conductance peak is non-quantized and usually has a considerable gap to the quantized value. Furthermore, the latter is sensitive to parameters, especially to spin–orbit coupling and the s-wave pairing potential. All results of the N–hS junction we obtained suggest that the observation of a small zero-bias conductance peak, instead of a quantized zero-bias conductance peak, in current tunneling experiments is a natural result. Based on the experiments’ parameters, we find that only by varying the strength of the spin–orbit coupling to be several times smaller than the reported one, can the zero–bias conductance peak be as small as the reported one. Furthermore, the results we obtained suggest that both a stronger spin–orbit coupling and proximity s-wave superconductor with a relatively weaker pairing potential can produce a much more striking zero-bias conductance peak (compared to the experiments), even an almost quantized one. As s-wave superconductors are common in nature, this prediction can be verified using current experiments.
1. Introduction

As it hosts exotic non-abelian zero modes [1], which have great potential in topological quantum computations [2–4], the $p$-wave superconductor in either two dimensions [5] or in one dimension [6] has aroused strong and lasting interest for more than a decade. Although there is no definite confirmation of $p$-wave superconductors in solid-state physics [7, 8], several groups [9–14], based on the proximity effect, have proposed a series of heterostructures whose common elements are spin–orbit coupling, the $s$-wave superconducting order and the Zeeman field. They have found, by tuning parameters, that the upper band of the system can be projected away, and the copper pairs formed in the lower band are an ‘effective $p$-wave’. For such an ‘effective $p$-wave’ superconductor, the zero modes known as Majorana bound states emerge at defects or the boundary of the system. Such heterostructures with non-trivial topology are known as topological superconductors.

To detect the Majorana bound states, there are mainly three classes of measurement schemes [15, 16], based on tunneling [17–30], fractional Josephson effects [31–35] and interferometry [36–38]. Recently, several tunneling experiments [39–42] based on the proposals of one-dimensional topological superconductors [11, 12] were carried out, and all these experiments found that a zero-bias conductance peak, which is taken as a signature of Majorana bound states, emerges in the tunneling spectroscopy when the magnetic field along the nanowire exceeds the critical value, i.e. $B > B_c = \sqrt{\Delta^2 + \mu^2}$. However, it is also noticed that the peak height is quite small, having a large discrepancy to the theoretical prediction: a quantized zero-bias peak of height $2e^2/h$ (at zero temperature) [17, 22, 23]. The large discrepancy has raised debate on the origin of the zero-bias conductance peak. To understand the experiments and clarify the origin of the peak, a series of works [43–52] have been carried out, and some of the works point out that the non-quantized zero-bias conductance peaks can have several other origins, such as the Kondo effect [46], smooth end confinement [47], strong disorder [48–50], and suppression of the superconducting pair potential at the end of the heterostructure [52]. Therefore, a definite confirmation of Majorana bound states is still lacking.

As the heterostructures are proposed to be an ‘effective $p$-wave’ superconductor, a comparative study of the tunneling spectroscopy of a normal metal/$p$-wave superconductor ($N-pS$) junction and a normal metal/heterostructure superconductor ($N-hS$) (here we call the heterostructure a ‘heterostructure superconductor’, instead of ‘topological superconductor’, since it can also be topologically trivial) junction to see what extent the ‘effective’ can reach, and whether the $s$-wave pairing still provides a channel (the well-known spin-reversed Andreev reflection) for electrons with energy in-gap to tunnel when the system is in the topological region, is important and valuable, for both understanding of the realized experiments and giving some guidance for future experiments.

In this article, according to the Blonder–Tinkham–Klapwijk (BTK) method [53], we systematically consider the effects due to (i) the length ($L$) of the system, (ii) interface scattering potential ($H$), and (iii) chemical potential mismatch ($\delta\mu$) on the tunneling spectroscopy of both junctions, and only consider the effects due to (iv) spin–orbit coupling ($\alpha$), (v) magnetic field ($B$) along the wire, and (vi) the $s$-wave pairing potential on the tunneling spectroscopy of the
N–hS junction. For the N–pS junction, we obtain a complete tunneling spectroscopy analytically, including the probability of normal reflection (an electron reflected as an electron with the same spin), the probability of an equal-spin Andreev reflection (an electron reflected as a hole with the same spin), and the differential-tunneling conductance and their dependence on system parameters. For the N–hS junction, we obtain the complete tunneling spectroscopy numerically, including the probability of normal reflection, the probability of spin-reversed normal reflection (an electron reflected as an electron with the opposite spin), the probability of spin-reversed Andreev reflection (an electron reflected as a hole with the opposite spin), and the differential-tunneling conductance and their dependence on parameters. Although only the differential-tunneling conductance is observable in experiments, the other reflection coefficients are also very important for us to understand the underlying tunneling process. Among these reflection coefficients, spin-reversed Andreev reflection and equal-spin Andreev reflection are worthy of attention as they can tell us whether the system favors s-wave pairing or p-wave pairing, and give a quantitative estimate of the extent the ‘effective’ has reached.

The following are the main results obtained. For the N–pS junction: (a) when \( \mu > 0 \) and \( L \) are sufficiently long, the equal-spin Andreev reflection is always perfect (injected electrons are completely reflected as a hole with the same spin) at \( E = 0 \) and, consequently, the zero-bias peak is always quantized, of height \( 2e^2/h \) (\( T = 0 \)) and independent of \( H \); however, once \( \mu_s \) crosses zero and becomes negative, i.e. \( \mu_s < 0 \), the zero-bias conductance peak changes into a conductance dip, with a value very close to zero, which indicates that there exists a topological quantum transition when \( \mu_s \) crosses zero. (b) When \( L \) is short, the conductance is very close to zero and no conductance peak appears both for \( \mu_s > 0 \) and \( \mu_s < 0 \). However, when \( L \) is increased to an intermediate value, a non-quantized conductance peak appears at finite energy for \( \mu_s > 0 \), and with the increase of \( L \), the peak moves toward zero-bias voltage with height increasing to the quantized value. For the N–hS junction: (a) with an infinite length, we find that when the magnetic field \( B \) crosses the critical value \( B_c \), a zero-bias peak appears; however, unlike the p-wave case, the peak height is non-quantized, and parameters are dependent. The non-quantization and parameter-dependence suggest that the s-wave pairing always plays a role in the tunneling process. This difference between the N–pS junction and the N–hS junction suggests that the ‘interface electron-Majorana coupling’ model [22, 23] can describe the N–pS junction well, but usually breaks down for the N–hS junction in reality because it neglects the process that two electrons with opposite spin at the interface can tunnel into the heterostructure superconductor as a Cooper pair due to the existence of interband s-wave pairing [13]. (b) Decreasing the spin–orbit coupling will reduce the height of the zero-bias conductance peak, and when the spin–orbit coupling decreases to zero and other parameters remain unchanged, the zero-bias conductance peak disappears, which indicates the breakdown of the usual topological criterion for a heterostructure superconductor. When \( L \) is intermediate, similar to the N–pS junction, there is also a non-quantized conductance peak appearing at finite energy for \( B > B_c \). (c) Adopting the experimental parameters, we study the effects of the s-wave pairing potential and find that a more striking zero-bias conductance peak favors a weaker pairing potential. Furthermore, we find that varying only the strength of the spin–orbit coupling to be several times smaller than the reported one, the zero-bias conductance peak can be as small as the reported one [39]. This suggests that the strength of the spin–orbit coupling may also serve as a possible explanation for the quite small zero-bias conductance observed in the experiment.
The paper is organized as follows: in section 2, we give the theoretical models for the $N-pS$ junction and the $N-hS$ junction explicitly, and based on the BTK method [53], we obtain the tunneling spectroscopy of the $N-pS$ junction and the $N-hS$ junction under different parameter conditions. In section 3, we give a discussion of the tunneling spectroscopy of the $N-pS$ junction and the $N-hS$ junction obtained in section 2. We also conclude the paper at the end of section 3.

2. Theoretical model

2.1. The $N-pS$ junction

We first consider the one-dimensional $N-pS$ junction shown in figure 1(a). Under the representation $\Psi^\dagger(x) = (\psi^\dagger(x), \psi(x))$, the Hamiltonian is given as

$$H = \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} - \mu(x) + V(x) + H\delta(x) \right] \sigma_z + \Delta(x)\sigma_x,$$

where $\sigma^\dagger = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices, $H\delta(x)$ is the scattering potential at the interface, $V(x)$ is potential induced by disorder, external field, etc, for the $N-pS$ junction, and we set $V(x) = 0$. $\mu(x)$ is the chemical potential, and we set $\mu(x) = \mu_n$ for $x < 0$ and $\mu(x) = \mu_s$ for $0 < x < L$, $\delta\mu = \mu_n - \mu_s$ is the chemical mismatch (note $\mu_s$ only has the meaning of chemical potential, when superconductivity is absent). $\Delta(x)$ is the paring potential, which should be determined self-consistently [54]; however, here we assume $\Delta(x) = -i\Delta_p \theta(x)\theta(L - x)\hbar\partial_x$ for simplicity, which means that $\Delta_p$ is a constant in the region $0 < x < L$. For simplicity, in this work we also neglect the effects of disorder, such as impurities that will induce pair-breaking and bound states in the gap [55].

![Figure 1](image-url)
For $0 < x < L$, the $p$-wave superconductor, the Hamiltonian in momentum space is given as (in the following, we set $\hbar = m = 1$)

$$H_S = \left[ \frac{k^2}{2} - \mu_x \right] \sigma_z + \Delta_p k \sigma_x,$$

which is the continuum form of the Kitaev model [6]. Its energy spectrum $E_k = \pm \sqrt{\left( \frac{k^2}{2} - \mu_x \right)^2 + (\Delta_p k)^2}$. It is easy to see that the energy gap is only closed at $\mu_x = 0$. The critical point $\mu_x = 0$ separates two physical regimes: the weak-pairing ($\mu_x > 0$) and strong-pairing ($\mu_x < 0$) phases. The two distinct names come from the fact that for $\mu_x > 0$, without pairing, the system is metallic, and therefore the resulting superconducting phase is BCS-like, while for $\mu_x < 0$, without pairing, the system is an insulator and therefore the resulting superconducting phase obviously does not fit the BCS picture.

One of the two phases is a topological superconducting phase with zero-energy Majorana fermions localized at the end of the system. The topology of a system is usually determined by the topology of its momentum space [56]. To determine which phase is a topological superconducting phase, here we show the energy spectrums of these two superconducting phases with open boundaries. Figure 2(a) shows that when the lattice model parameters satisfy $\mu_x > -2t$ (equivalent to $\mu_x > 0$ in the continuum model), there is a pair of zero-energy states (two red points) in the gap, which suggests this phase (the weak-pairing phase) is a topological superconducting phase. When the system is sufficiently long, the two zero modes are well separated and localized at the two ends of the system and, as a result, the wave functions of the two modes do not overlap. Figure 2(b) shows that when $\mu_x < -2t$ (equivalent to $\mu_x < 0$ in the continuum model), no zero energy modes appear in the gap and, as a result, the density profile exhibited is not localized at the end of the system. This phase is the normal superconducting phase. However, when the system is not sufficiently long, the wave functions of the two modes will overlap and, consequently, the energy of the end states will split and no longer be zero. The shorter the system is, the larger the overlap and the split are, as shown in figures 2(c) and (d).

As shown above, when $\mu_x$ goes across the critical point $\mu_x = 0$, the system undergoes a topological phase transition. To study the behavior of tunneling spectroscopy when $\mu_x$ goes across the critical point $\mu_x = 0$, we consider $\Delta_p^2 > 2\mu_x$, which will restrict $\mu_x$ to be very close to $\mu_c$ and the minimum of the energy spectrum is located at $k = 0$. For a given energy $E$ (here we only consider in-gap states, i.e. $E < E_s/2 = |\mu_x|$, as the differential-tunneling conductance at zero-bias voltage), the wave function in the $p$-wave superconductor is given as

$$\psi_S(x) = c(E) \left( \begin{array}{c} iu_+ \\ v_+ \end{array} \right) e^{-k_+ x} + \bar{c}(E) \left( \begin{array}{c} -iu_+ \\ v_+ \end{array} \right) e^{k_+ x} + d(E) \left( \begin{array}{c} iu_- \\ v_- \end{array} \right) e^{-k_- x} + \bar{d}(E) \left( \begin{array}{c} -iu_- \\ v_- \end{array} \right) e^{k_- x},$$

where $k_\pm = \sqrt{2(\Delta_p^2 - \mu_x) \pm 2E^2 + \Delta_p^2 (\Delta_p^2 - 2\mu_x)}$, $u_\pm = \Delta_p k_\pm$, and $v_\pm = E + \frac{k_\pm^2}{2} + \mu$. $c(E)$, $\bar{c}(E)$, $d(E)$ and $\bar{d}(E)$ are energy-dependent coefficients, and when $L = \infty$, $\bar{c}(E) = \bar{d}(E) = 0$, it is obvious that the wave function is localized at the left end of the wire.
For \( x < 0 \), the normal metal lead, the Hamiltonian is

\[
H_N = \left[ \frac{k^2}{2} - \mu_n \right] \sigma_z.
\]  

(4)

here we keep \( \sigma_z \) for convenience. We consider that an electron is injected from the normal lead into the \( p \)-wave superconductor, and the wave function in the normal lead is given as

\[
\psi_N(x) = \begin{pmatrix}
    e^{2i \xi_n x} + b(E) e^{-i \eta_n x} + a(E) \binom{0}{1} \\
    0
\end{pmatrix} e^{i q_n x},
\]

where \( q_{n,h} = \sqrt{2(\mu_n \pm E)} \). \( a(E) \) and \( b(E) \) denote the equal-spin Andreev reflection amplitude and normal reflection amplitude, respectively. (Here we assume the Cooper pairs are composed of electrons with the same spin polarization. With this assumption, the spin-reversed Andreev reflection and spin-reversed normal reflection are naturally absent.)

Following the BTK methods [53], the two wave functions (3) and (5) need to satisfy the boundary conditions,
Figure 3. (a) Parameters: $L = \infty$, $Z = 0$, $\mu_n = 1$, $\mu_s = 0.004$, $\Delta_p = 0.1$. $A(eV) = |a(eV)|^2 q_h / q_e$ is the equal-spin Andreev reflection probability (blue line). $B(eV) = |b(eV)|^2$ is normal reflection probability (purple line). $G(eV) ((e^2 / h)$ is its unit) is the differential-tunneling conductance (red line). The black line is the normalization: $A(eV) + B(eV) = 1$. (b) $\mu_s = -0.004$, the inset is an enlarged exhibition of the lower two lines: equal-spin Andreev reflection probability and tunneling conductance. (c) $Z = 2$, (d) $Z = 10$, (e) $L = 40, Z = 2$, (f) $L = 60, Z = 2$, (g) $L = 40, Z = 5$, (h) $L = 60, Z = 5$. The unmentioned parameters for (b)–(h) are the same as (a). In this subsection, we adopt the same color-labeling scheme for all figures.
\[ ψ_S(x = L) = 0; ψ_S(x = 0) = ψ_N(x = 0); \]
\[ v_i ψ_S(x = 0^+) - v_n ψ_N(x = 0^-) = -iZ \sigma_z ψ_N(x = 0), \]

where \( Z = 2H \), \( v_i = \partial H_S/\partial k \) and \( v_n = \partial H_N/\partial k \), two \( 2 \times 2 \) matrices, are the velocity operators [57]. From equation (6), we can obtain \( a(E) \) and \( b(E) \) directly, and according to [53], the zero-temperature differential-tunneling conductance is given as

\[ G(eV) = \frac{e^2}{h} [1 + A(eV) - B(eV)], \]

where \( A(E) = |a(E)|^2 q_h/q_e \), \( (E = eV) \), is the equal-spin Andreev reflection probability, and \( B(E) = |b(E)|^2 \) is the normal reflection probability. Here, as we only consider in-gap states, the waves expressed in equation (3) do not carry current, and \( A(eV) \) and \( B(eV) \) satisfy the normalization condition, i.e. \( A(eV) + B(eV) = 1 \). For different parameters \( \mu_s \), \( L \) and \( Z \) (we set \( \mu_n = 1 \) as the unit), the results are shown in figure 3.

Figures 3(a) and (b) show the important difference of differential-tunneling conductance between \( \mu_s > 0 \) (topologically non-trivial) and \( \mu_s < 0 \) (topologically trivial). For \( \mu_s > 0 \), a quantized zero-bias conductance peak of \( 2e^2/h \) stably exists (as shown in figures 3(a), (b) and (d)). A stable and quantized zero-bias conductance peak is a manifestation of perfect equal-spin Andreev reflection and indicates a stable topological phase. This result is the same as the result obtained by an ‘interface electron-Majorana coupling’ model [22, 23]; therefore, even though we do not assume the existence of the Majorana-bound states at the wire end, the non-trivial topology of the \( p \)-wave superconductor manifests itself in the tunneling spectroscopy obtained by matching the wave functions of the two bulks directly. For \( \mu_s < 0 \), the in-gap differential-tunneling conductance almost vanishes. The important difference indicates that, for the \( p \)-wave superconductor, the tunneling experiments can detect the topological quantum transition at \( \mu_s = 0 \) effectively. Figures 3(c) and (d) show that increasing the interface scattering potential only narrows the width of the peak; it neither changes the quantized height (an analytical proof for this is given in the appendix) nor the position of the peak. The wire length has a strong impact on the conductance for \( \mu_s > 0 \), as shown in figures 3(e) and (f), for \( L = 40 \) and \( L = 60 \), the conductance peaks appear at finite-bias voltage and the peak will move left (zero-bias voltage) with increasing \( L \) (this agrees with the usual two end-Majorana coupled picture, and also agrees with the results shown in figures 2(c) and (d)). The effect of the interface-scattering potential for finite length is similar to that with infinity length mentioned above, as shown in figures 3(g) and (h). For sufficiently short \( L \), for example, \( L < 10 \), there is no conductance peak and the in-gap differential-tunneling conductance almost vanishes. For \( \mu_s < 0 \), the topologically trivial phase, the wire length \( L \) and the interface scattering potential \( Z \) has little effect, the in-gap differential-tunneling conductance remains very small.

Above we have restricted \( \mu_s \) to be close to \( \mu_c \), and there exists a significant mismatch between \( \mu_n \) and \( \mu_s \). Canceling this restriction, we find, for \( \mu_s > 0 \), increasing \( \mu_s \) greatly widens the width of the zero-bias peak of the in-gap tunneling spectroscopy; however, the peak’s quantization behavior does not change and, therefore, the main physics does not change, as shown in figure 4(a). For \( \mu_s < 0 \), compared figures 4(b) to 3(b), it is easy to see that decreasing \( \mu_s \) has little effect on the in-gap tunneling spectroscopy.
2.2. \(N-hS\) junction

The one-dimensional \(N-hS\) junction is shown in figure 1(b). For \(x<0\), the Hamiltonian is a generalized 4 \(\times\) 4 matrix form of equation (4). For \(x>0\), the Hamiltonian is now given as [11] (in momentum space, under representation \(\Psi = (\psi_1^\dagger, \psi_2^\dagger, \psi_3^\dagger, \psi_4^\dagger)\))

\[
H_{hS} = \xi_k \tau_x + B \sigma_x + a k \sigma_y \tau_z + \Delta_x \tau_x, \tag{8}
\]

where \(\xi_k = k^2/2 - \mu_s\), \(B\) is the in-plane magnetic field along the wire, \(\alpha\) is the spin–orbit coupling strength and \(\Delta_s\) is the \(s\)-wave pair potential. \(\tau = (\tau_x, \tau_y, \tau_z)\) and \(\sigma = (\sigma_x, \sigma_y, \sigma_z)\) are Pauli matrices in particle-hole space and spin space, respectively. The heterostructure described by this Hamiltonian is the one that is realized in the experiment [39]. The quasiparticle energy spectrum is given as

\[
E_k^\pm = \sqrt{\xi_k^2 + (ak)^2 + \Delta_s^2 + B^2} \pm 2 \sqrt{\xi_k^2 a^2 k^2 + B^2 (\xi_k^2 + \Delta_s^2)},
\]

where the energy gap \(\Delta_g \equiv \{E_k^\pm\}_{\text{min}}\) is closed (\(\Delta_g = 0\)) when the magnetic field \(B\) reaches the critical value \(B_c = \sqrt{\mu_s^2 + \Delta_s^2}\), which separates \(B < \sqrt{\mu_s^2 + \Delta_s^2}\), the topologically trivial phase, from \(B > \sqrt{\mu_s^2 + \Delta_s^2}\), the topologically non-trivial phase. The energy–momentum relation here is much more complicated than that of the \(p\)-wave superconductor. This complication makes it impossible to write down the analytical form of the wave function \(\psi_{hS}(x)\) directly, so we must seek help from numerical tools.

As the Hamiltonian (8) is a 4 \(\times\) 4 matrix, the wave function in the normal metal is a four-component vector, which takes the form

\[
\psi_N(x) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & e^{i q_x x} & e^{-i q_x x} & 0 \\
0 & b_1 & b_1 & 0 \\
0 & 0 & a_\uparrow & a_\downarrow
\end{pmatrix}
\]

where \(b_1(E)\) denotes the normal reflection amplitude, \(b_\uparrow(E)\) denotes the spin-reversed reflection amplitude, \(a_\uparrow(E)\) denotes the equal-spin Andreev reflection amplitude, and \(a_\downarrow(E)\) denotes the...
spin-reversed Andreev reflection amplitude. Now, the boundary conditions take the form

\[ \psi_{\mathcal{H}_S}(x = L) = 0; \]
\[ \psi_{\mathcal{H}_S}(x = 0) = \psi_N(x = 0); \]
\[ v_{\mathcal{H}_S}\psi_{\mathcal{H}_S}(x = 0^+) - v_n\psi_N(x = 0^-) = -iZ\sigma_0\tau_z\psi_N(x = 0), \] (10)

where \( v_{\mathcal{H}_S} = \partial H_{\mathcal{H}_S}/\partial k \) is a 4 × 4 matrix, and \( v_n = -i\partial_x\sigma_0\tau_Z \) is also generalized to 4 × 4 matrix.

Based on equation (10), we can obtain \( b_{\uparrow,\downarrow}(E) \) and \( a_{\uparrow,\downarrow}(E) \), and the differential-tunneling conductance is given as

\[ G(eV) = \frac{e^2}{h} \left[ 1 + A_{\uparrow}(eV) + A_{\downarrow}(eV) - B_{\uparrow}(eV) - B_{\downarrow}(eV) \right]. \] (11)

where \( A_{\uparrow,\downarrow}(eV) = |a_{\uparrow,\downarrow}(eV)|^2q_e/q_n \) and \( B_{\uparrow,\downarrow}(eV) = |b_{\uparrow,\downarrow}(eV)|^2 \). \( A_{\uparrow,\downarrow}(eV) \) and \( B_{\uparrow,\downarrow}(eV) \) in the gap region should satisfy the normalization condition:

\[ A_{\uparrow}(eV) + A_{\downarrow}(eV) + B_{\uparrow}(eV) + B_{\downarrow}(eV) = 1. \]

For different parameters, the results are shown in figure 5.

Figures 5(a) and (b) show that when \( B > \sqrt{\mu^2 + \Delta_s^2} \), in the topological region, a zero-bias conductance peak is formed. However, contrary to the quantized zero-bias conductance peak of an \( N-pS \) junction, here the zero-bias peak is non-quantized and sensitive to parameters. Increasing the interface-scattering potential not only narrows the width of the peak, but also generally increases the height of the peak, as shown in figures 5(c) and (d). From figures 5(a) and (b), we can also see that the increase of the peak height is due to a suppression of the spin-reversed Andreev reflection and a simultaneous increase of the equal-spin Andreev reflection by the interface-scattering potential. However, with a further increase of the interface-scattering potential, this corresponding increasing effect will finally become saturated, and the zero-bias conductance peak will still have a gap to the quantized value, as shown in figure 5(d). For comparison, the inset in figure 5(d) shows that when \( B < \sqrt{\mu^2 + \Delta_s^2} \), the normal phase region, the zero-bias conductance monotonously decreases with increasing scattering potential.

For comparison, figures 5(e) and (f) shows that when \( B < \sqrt{\mu^2 + \Delta_s^2} \), the normal phase, no zero-bias peak appears, and by increasing the interface-scattering potential, the normal reflection is greatly enhanced and the spin-reversed Andreev reflection and the conductance are greatly reduced, which is a familiar phenomenon in \( N-S \) junctions [53]. Comparing figures 5(a) and 5(b) to (c) and (d), it is not hard to find that when the system goes from the normal phase to the topological phase, the probability of an equal-spin Andreev reflection is greatly enhanced (but still has a considerable gap to the perfect equal-spin Andreev reflection) and the spin-reversed Andreev reflection amplitude is greatly reduced, which indicates the equal-spin pairing (\( p \)-wave pairing) becomes much more favored in the topological region than in the normal region.

Spin–orbit coupling also has a strong impact on tunneling spectroscopy. As shown in figure 6(a), when decreasing the spin–orbit coupling and fixing other parameters, the height of the zero-bias peak and the probability of an equal-spin Andreev reflection occurring are significantly reduced. A further inspection shows that once the spin–orbit coupling decreases to a critical value (named \( \alpha_g \)) not only does the peak height keep decreasing to a smaller value but the energy gap \( \Delta_s \) also begins to depend on spin–orbit coupling (when \( \alpha < \alpha_g \), we say the spin–orbit coupling is weak), with a dependence that monotonically decreases as the spin–orbit
When the spin–orbit coupling is decreased to zero and the gap closes, the system becomes gapless and the zero-bias conductance peak disappears, which indicates a breakdown of the topological criterion.

Although we have shown that decreasing the spin–orbit coupling (from the figure 5 parameter, $\alpha = 0.5$) lowers the peak, we find that increasing the spin–orbit coupling does not correspond to a monotonic increase of the peak height. As shown in figures 6(b) and (c), the zero-
bias conductance peak first increases and then decreases with increasing spin–orbit coupling, the optimal spin–orbit coupling \( \alpha_c \) under the parameters given in figure 6(c) is about 0.6. In the following, when \( \alpha < \alpha_c \), we say that the spin–orbit coupling is in the intermediate region, and when \( \alpha > \alpha_c \), we say that the spin–orbit coupling is strong. From figure 6(c), we also find when \( \alpha \) goes beyond \( \alpha_c \), a larger \( \alpha \) corresponds to a lower peak and the reduction effect due to the increase of spin–orbit coupling is significant. However, we also find, almost simultaneously, that when \( \alpha \) goes beyond \( \alpha_c \), the saturation effect of the interface scattering potential for weak spin–orbit coupling is absent, and a stronger interface potential will induce a higher peak, as shown in figure 6(d). When the interface scattering potential goes to infinity, the peak goes to the quantized value, i.e. \( 2e^2/h \), and the width of the peak goes to zero. A quantized peak located at zero-bias voltage with vanishing width is apparently a manifestation of the Majorana end states.

Figure 7(a) shows that when the chemical potentials are not mismatched between the normal lead and the heterostructure superconductor, and the magnetic field is absent, there exists a sharp peak corresponding to a resonant spin-reversed Andreev reflection at the gap boundary, similar to the N–S junction [53]. Note that in the absence of the magnetic field, there are no spin-reversed normal reflections and equal-spin Andreev reflections, even with strong spin–orbit coupling. This indicates that the magnetic field is a necessity to induce equal-spin pairing. As shown in figure 7(b), increasing the magnetic field will drive the peak left (zero-bias voltage), and the finite magnetic field also drives the peak away from the gap boundary. However, when there is a big mismatch between the chemicals, which is usually needed to guarantee that the magnetic field satisfying the topological criterion is still not large enough to break down the superconductivity, such an interesting phenomenon is absent (no
finite-bias peak appears in figures 5(e) to (f)). Once the magnetic field reaches the critical value 
\[ \mu \Delta_c = B_c \] 
s a zero-bias conductance peak is formed; however, the peak that is formed has a significantly reduced height as shown in figure 7(c). A sudden reduction of the peak height may imply that the zero-bias conductance peak and the finite-bias peak have different origins. For weak or intermediate spin–orbit coupling, further increasing the magnetic field has little effect on the zero-bias conductance peak; however, when the spin–orbit coupling is strong enough, the peak height monotonically increases to a parameter-dependent saturation value (here we do not consider the breakdown of superconductivity due to a strong magnetic field), as shown in figure 7(d). A further study in the stronger spin–orbit coupling region shows that the absence of a potential chemical mismatch makes the approach of zero-bias conductance peak to the quantized value even more difficult.

To discuss the effects of the pairing potential on the tunneling potential, here we adopt the experimental parameters: 
\[ m = 0.015 m_e, \quad \Delta_s = 0.25 \text{ meV}, \quad \mu_s = 0 \] 
and we choose \( \mu = 20 \text{ meV} \). Figure 8(a) shows the tunneling spectroscopy at a temperature of zero. Comparing the zero-bias conductance peak with the experiment’s measured value of \( \sim 0.1 e^2/h \), the result should still be several times larger, even if we consider the temperature’s smearing effect. However, as discussed previously, decreasing spin–orbit coupling (intermediate region) greatly reduces the peak height. In figure 8(b), it is shown that halving the spin–orbit coupling almost corresponds to halving the peak height. Therefore, if the spin–orbit coupling is several times smaller than the reported one, the height of the zero-bias conductance peak decreases.
peak will decrease to be comparable with the experiment’s measured value. Figure 8(b) also shows that by decreasing the pairing potential, the peak height is greatly increased. To achieve a sufficiently small-pairing potential, the zero-bias conductance peak is found to be almost quantized. This result seems counterintuitive, as the pairing potential is a necessity to induce the topological superconductor. This confusion can be clarified by the fact that when $\Delta < \frac{V_c}{2} \approx \frac{m \alpha^2}{2}$, the upper band’s effect is negligible, and as a result, the system becomes an ‘effective $p$-wave’ superconductor [13]. This suggests that to observe a more striking peak during experiments, it is better to choose a relatively weaker pairing potential proximity superconductor.

The effects of finite length $L$ for the $N-hS$ junction are similar to the $N-pS$ junction; that is, a conductance peak will locate at a finite-bias voltage when $L$ is not sufficiently long. With the increase of $L$, the peak moves toward the zero-bias voltage; see figure 8(c). For a sufficiently long wire, we find that by increasing the magnetic field, the width of the zero-bias conductance peak will be greatly widened; see figure 8(d).

3. Discussion and conclusion

In this work, based on the BTK method, we give a thorough study of the tunneling spectroscopy of the $N-pS$ junction and the $N-hS$ junction. Comparing the tunneling spectroscopy of the
N – pS junction with the N – hS junction, we find that a zero-bias conductance peak appears in both systems when their topological criterions are satisfied. However, contrary to the stable quantized zero-bias peak of the N – pS junction, the zero-bias conductance peak of the N – hS junction is non-quantized and sensitive to parameters. The non-quantization of the zero-bias conductance peak does not mean that the Majorana end state is absent, but rather that its existence is guaranteed by the non-trivial topology of the bulk and the bulk-edge correspondence [58]. The non-quantization only indicates that there are more transport channels compared to the N – pS junction. For small and intermediate spin-orbit coupling, we find the additional channels make important contributions to the transport. This indicates that the perfect equal-spin Andreev reflection is absent, the heterostructure superconductor has a considerable gap to a truly p-wave superconductor, and the s-wave pairing still plays a role in the tunneling process. Therefore, even if the topological criterion is satisfied, using a Majorana chain to denote the heterostructure superconductor and then based on the ‘interface electron-Majorana coupling’ model [22, 23], expecting a quantized zero-bias conductance peak to emerge is in fact usually unjustified.

For strong spin-orbit coupling, we find that, although for the weak interface-scattering potential the zero-bias conductance peak is very small and monotonically decreasing with increasing spin-orbit coupling, a very strong interface-scattering potential can suppress the additional transport channels effectively and make the equal-spin Andreev reflection become very close to the perfect level, with the zero-bias conductance peak approaching the quantized value. However, strong spin-orbit coupling is difficult to realize, and a very strong interface-scattering potential also makes the width of the zero-bias conductance peak very narrow, which will make detection difficult. Contrary to the combination of strong spin–orbit coupling and a strong interface-scattering potential, decreasing the pairing potential is a much more effective way to suppress the additional channels and enhance the zero-bias conductance peak to the quantized value.

Besides a zero-bias peak appearing when the system is driven into the topological phase, there are two other common features appearing in the figures. The first feature occurs when the system goes from the normal phase to the topological phase, and the equal-spin Andreev reflection amplitude at the neighborhood of zero-bias voltage always has a sudden and comparatively large increment. This indicates that in the topological region, the equal-spin pairing becomes important, and the zero-bias conductance peak must be related to the equal-spin pairing at zero-bias voltage. From figure 5, we also see that the larger $A_1(eV = 0)$ is, the higher the peak is. A quantized zero-bias conductance peak always corresponds to a perfect equal-spin Andreev reflection, i.e. $A_1(eV = 0) = 1$. This correspondence indicates that $A_1(eV = 0)$ can be used as a measure of the ‘effective’. A larger $A_1(eV = 0)$ implies that the equal-spin pairing becomes more favored, and this indicates that the heterostructure turns out to be a more ‘effective $p$-wave superconductor’. The second feature is that when the magnetic field is turned on, the discontinuity of the tunneling spectroscopy at the induced-gap boundary is greatly softened. Such a ‘soft effect’ will make the position of the induced-gap boundary difficult to detect and, as a result, the gap closure also becomes difficult to detect. Therefore, the ‘soft effect’ induced by the magnetic field can be supplied as a possible explanation for the missing observation of the gap closure. Soft gaps caused by other factors were discussed in detail in [59].

In conclusion, we found that the observation of a non-quantized value under experimental parameters is a natural result, even without considering effects due to disorder, sub-bands and other inhomogeneities. Furthermore, a spin–orbit coupling several times smaller than that reported in the experiment can be taken as a possible explanation for the quite small zero-bias
conductance observed in experiments. We suggest that, to observe a more striking zero-bias conductance peak in future experiments, a weaker pairing potential proximity superconductor would probably be a better choice.

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Appendix. Quantized zero-bias conductance peak of the N–pS junction

For the N–pS junction, the velocity operators $v_s$ and $v_n$ are given as ($\hbar = m = 1$),

$$
v_s = \left( \begin{array}{cc} k & \Delta \\ \Delta & -k \end{array} \right) = -i \left( \begin{array}{cc} \partial_x & \text{i} \Delta \\ \text{i} \Delta & -\partial_x \end{array} \right),
$$

$$
v_n = \left( \begin{array}{cc} k & 0 \\ 0 & -k \end{array} \right) = -i \left( \begin{array}{cc} \partial_x & 0 \\ 0 & -\partial_x \end{array} \right). \tag{A.1}
$$

For $E = 0$, the wave function in the $p$-wave superconductor (here we consider the length is infinity) takes a simpler form,

$$
\psi_S(x) = c \left( \begin{array}{c} 1 \\ i \end{array} \right) e^{-k_x x} + d \left( \begin{array}{c} i \\ 1 \end{array} \right) e^{-k_\perp x}, \tag{A.2}
$$

where $k_\pm = \sqrt{2(\Delta^2 - \mu_s) \pm 2\sqrt{\Delta^2 (\Delta^2 - 2\mu_s)}}$. The wave function in the normal lead takes the form

$$
\psi_N(x) = \left( \begin{array}{c} e^{2i\nu x} + b \\ 0 \end{array} \right) e^{-i\nu x} + a \left( \begin{array}{c} 0 \\ 1 \end{array} \right) e^{i\nu x}, \tag{A.3}
$$

where $\nu = q_n = \sqrt{2\mu_n}$, and we use $\nu$ to denote both of them. By matching the two wave functions according to the boundary conditions (6), we obtain

$$
1 + b = i(c + d),
a = (c + d),
(\Delta - k_+ c + (\Delta - k_-)d - q(1 - b) = -iZ(1 + b),
i(\Delta - k_+ c + i(\Delta - k_-)d + qa = iZ a. \tag{A.4}
$$

A direct calculation gives

$$
a = -i,
b = 0,
c = - \frac{q + i(\Delta - k_- - Z)}{k_+ - k_-},
d = \frac{q + i(\Delta - k_+ - Z)}{k_+ - k_-}. \tag{A.5}
$$
\( b = 0 \) indicates no normal reflection and \( a = -i \) indicates a perfect equal-spin Andreev reflection. According to the formula (7), the differential-tunneling conductance at \( E = 0 \) is

\[
G(0) = \frac{e^2}{h} [1 + A(0) - B(0)]
\]

\[
= \frac{e^2}{h} \left[ 1 + |a|^2 - |b|^2 \right]
\]

\[
= 2 \frac{e^2}{h},
\]

and the zero-bias conductance is quantized independent of the interface-scattering potential.

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