We propose an intrinsic CFT definition of local bulk operators in AdS$_3$/CFT$_2$ in terms of twisted Ishibashi boundary states. The bulk field $\Phi(X)$ creates a cross cap, a circular hole with opposite edge points identified, in the CFT space-time. The size of the hole is parameterized by the holographic radial coordinate $y$. Our definition is state-independent, non-perturbative, and does not presume or utilize a semi-classical bulk geometry. We argue that, at large $c$, the matrix element between highly excited states satisfies the bulk wave equation in the AdS black hole background.

**Introduction**

AdS/CFT duality has passed many tests. Most checks compare CFT correlation functions with the dependence of AdS quantities on sources at the boundary \[1\]. The CFT construction of local bulk observables \[2\],\[3\], on the other hand, remains underdeveloped. In this note we propose a new representation of local bulk fields in terms of operators that create finite size holes in the space-time of the CFT. We formulate and test our proposal for the case of AdS$_3$/CFT$_2$. In the following, $x = (z, \bar{z})$ and $X = (y, x)$ denote coordinate systems on the 2D space-time and in AdS$_3$, respectively.

Gauge/gravity duality postulates a one-to-one map between the Hilbert space of the CFT and the gravity theory. Local bulk fields should thus have images as suitable non-local operators in the CFT. In particular, if the gravity side is weakly coupled, we can associate to every local CFT operator $\mathcal{O}_h(x)$ an effective field $\Phi_h(X)$ in AdS, such that, as we approach the AdS boundary

$$\lim_{y \to 0} y^{-2h} \Phi_h(y, x) = \mathcal{O}_h(x). \quad (1)$$

To leading order in $1/N$, the linearized bulk field satisfies a free field wave equation

$$\Box_{\text{bulk}} \Phi_h = m_h^2 \Phi_h \quad (2)$$

with $m_h^2 = h(h - d)$. This fact suggests that one can express the bulk field in terms of the associated CFT operator via

$$\Phi_h^{\text{KLL}}(X) = \int d^2x K(X; x) \mathcal{O}_h(x), \quad (3)$$

where $K(X; x)$ denotes a suitable smearing function, that solves the wave equation in the bulk. This is the Kabat, Lifschytz and Lowe prescription \[3\].

The KLL prescription has several shortcomings. The map presumes the existence of a gravity dual: rather than reconstructing the extra dimensional physics, it makes explicit use of the classical bulk geometry. Moreover, since the kernel $K(X; x)$ depends on the geometry, $\Phi_h^{\text{KLL}}(X)$ is a state dependent operator. Finally, there appears to be an obstruction to the existence of a well-defined smearing function for black hole space times \[4\].

Given these issues, it would clearly be desirable to find a definition of local bulk fields, that is (i) inherent to the CFT, (ii) state-independent, and (iii) applicable to black hole space-times. In this note, we will propose such an intrinsic CFT definition for AdS$_3$/CFT$_2$.

Our proposal makes use of Ishibashi boundary states \[5\], twisted via a cross cap identification. Let $|h\rangle$ denote the primary state with (equal left and right) conformal weight $h$: $L_0 |h\rangle = \bar{L}_0 |h\rangle = h |h\rangle$. Algebraically, the cross cap state $|h\rangle_{\ominus}$ is defined as the unique state spanned by descendents of $|h\rangle$ such that

$$(L_n - (-1)^n \bar{L}_n) |h\rangle_{\ominus} = 0. \quad (4)$$

Geometrically, the twisted boundary state $|h\rangle_{\ominus}$ cuts a hole in the surface on which the CFT lives, identifies diametric opposite points on the edge of the hole, and projects onto the Virasoro representation labeled by $h$. The state-operator map associates to the state $|h\rangle_{\ominus}$, viewed as obtained via radial quantization, a local operator $\Phi_h(0, y)$ through the relation

$$\Phi_h(0, y) |0\rangle = y^{L_0 + \bar{L}_0} |h\rangle_{\ominus}. \quad (5)$$

Here $y$ is a scale modulus introduced by the boundary state. Indeed, adding a cross-cap decreases the Euler number of the surface by one, and thus adds three real shape parameters, which we can think of as the location (taken to be the origin in \[5\]) and the size of the hole. By moving the origin to some arbitrary location $(z, \bar{z})$, we thus obtain an operator $\Phi_h(z, \bar{z}, y)$ defined on a 3-dimensional space. This is our proposed CFT definition of the bulk operator associated to the local operator $\mathcal{O}_h(z, \bar{z})$. Note that the state $|h\rangle_{\ominus}$ is normalizable as long as $y < 1$. 

**Poking Holes in AdS/CFT:**

**Bulk Fields from Boundary States**

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We propose an intrinsic CFT definition of local bulk operators in AdS$_3$/CFT$_2$ in terms of twisted Ishibashi boundary states. The bulk field $\Phi(X)$ creates a cross cap, a circular hole with opposite edge points identified, in the CFT space-time. The size of the hole is parameterized by the holographic radial coordinate $y$. Our definition is state-independent, non-perturbative, and does not presume or utilize a semi-classical bulk geometry. We argue that, at large $c$, the matrix element between highly excited states satisfies the bulk wave equation in the AdS black hole background.
Testing the proposal

The bulk field \( \Phi_h(z_0, \bar{z}_0, y) \) cuts out a circular hole of radius \( y \) centered around the point \((z_0, \bar{z}_0)\), while gluing together diametric opposite points via the identification

\[
\bar{z} - z_0 = -\frac{y^2}{z - z_0}, \quad (6)
\]

We can write this relation as a global \( SL(2, \mathbb{R}) \) transformation \( \bar{z} = \frac{az + b}{cz + d} \) with \( y = 1/c, z_0 = -d/c \) and \( \bar{z}_0 = a/c \).

The bulk field thus naturally lives on an \( SL(2, \mathbb{R}) \) group manifold, or a subspace thereof. In the following, we will often denote the bulk coordinate \( X \) by the group element \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \).

We wish to verify that the bulk field \( \Phi_h(g) \) has the required properties. Property (1) is evident, since \( h = \lim_{y \to 0} y^{-\Delta + \ell_0 - 2h} \) is a projection onto the primary state \( |h_i = \mathcal{O}_\Delta(0)\rangle \). Property (2) is a less trivial statement, that only holds to leading order at large \( N \). Indeed, since our definition (5) does not presume a free bulk theory, it should automatically incorporate all interactions and \( 1/N \) corrections.

How can a state-independent operator (5) satisfy a state dependent wave equation (2)? A partial answer is that in 2+1-D gravity, and outside of any matter sources, a less trivial statement, that only holds to leading order at large \( N \). Indeed, since our definition (5) does not presume a free bulk theory, it should automatically incorporate all interactions and \( 1/N \) corrections.

For concreteness, consider a matrix element of the bulk field \( \Phi_h(g) \) between two highly excited states

\[
\phi_h \left[ \frac{1}{2} \right] (g) = \langle h_3, h_4 | \Phi_h(g) | h_1, h_2 \rangle \quad (7)
\]

The in-state \( |h_1, h_2\rangle \) is a primary state with conformal weight \( \Delta \gg c/12 \) in the gravity dual, it is a description of a non-rotating BTZ black hole with mass \( M = \Delta - \frac{\ell^2}{2} \). The black hole space-time is obtained from the \( SL(2, \mathbb{R}) \) group manifold by taking the quotient \[ h = \bar{h} \]

\[
g \sim g_\ell g_\omega g_\ell, \quad g_\omega = g_\ell = e^{\pi r_+ \sigma_2}, \quad (8)
\]

Here \( r_+ = \sqrt{8G_NM} = \sqrt{\frac{24 \ell^2}{c} - 1} \) is the black hole radius. (We use \( R_{\text{AdS}} = 1 \) units.) The out-state in (7) represents the black hole with a small extra mass \( \omega \) and angular momentum \( \ell \). It is convenient to parametrize

\[
\bar{z} - \bar{z}_0 = -\frac{y^2}{z - z_0}, \quad (6)
\]

We can write this relation as a global \( SL(2, \mathbb{R}) \) transformation \( \bar{z} = \frac{az + b}{cz + d} \) with \( y = 1/c, z_0 = -d/c \) and \( \bar{z}_0 = a/c \).

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How can a state-independent operator (5) satisfy a state dependent wave equation (2)? A partial answer is that in 2+1-D gravity, and outside of any matter sources, the background geometry locally looks like AdS3. So the state dependence manifests itself in the form of non-trivial global transition functions on the variable \( g \).

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\[
\bar{z} - \bar{z}_0 = -\frac{y^2}{z - z_0}, \quad (6)
\]
BTZ holonomy \([8]\). We conclude that the Teichmüller space of the hyperbolic cylinder with a cross-cap (shown in fig 1) is isomorphic to the BTZ black hole space-time. This is our first piece of evidence that, at large \(c\), the bulk field \(\Phi_h(g)\) lives on the classical black hole background.

To provide more quantitative support for our proposal, let us set out to compute the amplitude \([7]\) and compare the answer with the bulk mode that solves the wave equation \([2]\) in the BTZ geometry. This bulk mode takes the form

\[
f_{\omega t}(l, \varphi, \rho) = e^{-i\omega t}e^{i\ell\varphi}f_{\omega t}(\rho) \quad \text{with}
\]

\[
f_{\omega t}(\rho) = \rho^h(1-\rho)^{-\frac{1}{2}}F_i(h+i\omega t, h+i\omega t, 2h; \rho)
\]

Here \(F_i(a, b, c; z)\) denotes the ordinary hypergeometric function and \(\rho = r_z^2/r^2\) parametrizes the radial coordinate. In fact, the mode function constitutes an \(SL(2, \mathbb{R})\) matrix element, \(f_{\omega t}(g) = \langle h, \frac{i(\omega + \ell)}{4r_z}; g| h, \frac{i(\omega - \ell)}{4r_z}\rangle\), which evidently satisfies the free bulk wave equation \([2]\).

To compute the CFT matrix element \([\bar{F}]\), it is convenient to start in Euclidean signature, define \(\Phi_h(g)\) as the operator that pokes a hole in the 2D Euclidean space time, and then Wick rotate back to Lorentzian signature. Moreover, we will choose to work in the uniformizing coordinate system \((Z, \bar{Z})\) introduced in \([10]-[11]\). This choice will greatly facilitate our analysis.

A CFT amplitude on a surface with a cross-cap is most conveniently analyzed by introducing the so-called Schottky double, as shown in fig 2. In our case, \(\Sigma\) is a cylinder with a circular hole, and its Schottky double \(\tilde{\Sigma}\) is two cylinders connected via a narrow bridge. \(\Sigma\) admits an (orientation reversing) involution that identifies diametric opposite points on the circular boundary of \(\Sigma\). The reflection symmetry restricts its cross ratio \(Z\) to be real

\[
Z = \bar{Z} \equiv \rho.
\]

A sphere with two punctures and a cross-cap has one real modulus.

Since the boundary reflects left-moving into right-moving modes, the involution interchanges the two chiral halves of the CFT. Moreover, thanks to the projection onto the conformal sector \(h\) in the intermediate channel, the CFT amplitude on the double \(\tilde{\Sigma}\) takes the form of a single non-chiral conformal block \(\bar{F}_i[\frac{12}{34}](Z, \bar{Z})\), which in turn factorizes into the product of two chiral blocks:

\[
\bar{F}_i[\frac{12}{34}](Z, \bar{Z}) = |\Psi_h[\frac{12}{34}](Z)|^2. \quad (13)
\]

Here we absorbed the product of OPE coefficients \(C_{12h}C_{h34}\) into the normalization of the conformal block. The \(Z\)-dependence of the conformal blocks is universal and completely fixed by conformal invariance.

The amplitude \(\phi_h[\frac{12}{34}](g)\) is obtained by taking the square root of the amplitude on the double \(\tilde{\Sigma}\).

\[
\phi_h[\frac{12}{34}](\rho) = \left(\bar{F}_i[\frac{12}{34}](\rho, \rho)\right)^{1/2} \quad (14)
\]

So our task is: (i) compute the conformal block, (ii) take the square root, (iii) compare the result with the mode function \([12]\) in the BTZ black hole background (c.f. \([7]\)).

Virasoro conformal blocks are uniquely determined by the conformal Ward identity. An explicit expression is not available yet, however, though exact or semi-classical properties are known. Known exact results are

(a) Zamolodchikov’s recursion formula \([8]\) relating Virasoro and global conformal blocks and

(b) the modular ‘fusion’ matrices, obtained by Ponsot and Teschner from Liouville CFT and quantum Teichmüller theory \([9]\).

Semi-classical expressions have recently been obtained in \([10]\). Unfortunately, none of the known results allow us to read off the specific answer that we need.

Given this state of affairs, it’s a reasonable strategy at this point to invert the sequence of step (i)-(iii), and first deduce the desired expression for the 2D conformal block that we need in order to find a precise match

\[
\phi_h[\frac{12}{34}](\rho) = f_{\omega t}(\rho) \quad (15)
\]

between the CFT amplitude and the bulk mode function. Imposing this match, we deduce that the conformal block should take the following form

\[
\Psi_h[\frac{34}{12}](Z) = Z^h F_i(1 + \frac{1}{2r_z} h_{13}, h + \frac{1}{2r_z} h_{24}, 2h; Z) \quad (16)
\]

To see this, note that taking the chiral conjugate of the conformal block \(\bar{F}_i[\frac{34}{12}](Z)\) amounts flipping the sign of \(\frac{1}{2r_z} h_{13}\) and \(\frac{1}{2r_z} h_{24}\) inside the argument of the hypergeometric function. Then, using the standard identity

\[
F_i(a, b, c; Z) = (1 - Z)^{-a - b} F_i(c - a, c - b, c; Z),
\]

direct inspection shows that the CFT amplitude \([13]-[14]\) reproduces the expression \([12]\).
Formula (16) for the conformal blocks is a conjecture. We will now present two pieces of supporting evidence.

The uniformizing coordinate system (11) has the special property that the semi-classical Virasoro conformal blocks effectively reduce to global conformal blocks (10). Consider the definition \( \Psi_h|z\rangle = \langle O_4(\infty)|O_2(1)|P_h|O_3(Z)O_1(0) \rangle \) of the chiral block. Here \( P_h \) is the projection operator onto the sector spanned by all descendents of \( |h\rangle \). From the commutator \([L_n, L_{-n}] = 2nL_0 + \frac{c}{12}n(n^2 - 1)\) we read off that the norm of the descendant states grows linear with \( c \), except for the ‘global descendents’ of the form \( L_{-1}^n|h\rangle \). Due to the special choice of the coordinate system (11), the \( L_{-n} \) operators do not produce any other large factors proportional to the conformal weight \( \Delta \). So, as argued in (10), we may replace the projection operator \( P_h \) by a restricted sum over global descendents only.

Global conformal blocks are known to satisfy a differential equation of the form \( L^2_\text{tot} F_h(z, \bar{z}) = 2m_h^2 F_h(z, \bar{z}) \) with \( L^2_\text{tot} = L^2 + \bar{L}^2 \) the Casimir of the global conformal algebra \( so(2,2) \) acting on the intermediate channel. The solution factorizes into chiral global conformal blocks given in terms of the hypergeometric function via \( \Psi_h = z^h \tilde{F}_h(h+h_{13}, h+h_{24}, 2h; z) \), with \( h_{ij} = h_i - h_j \). In our setting, we also need to take into account that in the \((Z, \bar{Z})\) system, the generator of scale transformations \( \frac{\partial}{\partial z} = \frac{i}{2} \frac{\partial}{\partial h} \) is rescaled by a factor \( \frac{i}{2} \) relative to the standard \( L_0 \)-generator. This renormalizes \( h_{ij} \) to \( \frac{i}{2} h_{ij} \). This is precisely what we need to recover formula (16).

Formula (16) receives additional support from known results in Liouville conformal field theory (9), or equivalently, from the identification (12) between Virasoro conformal blocks and the Hilbert states obtained by quantizing Teichmüller space (13). The central result of (9) is that the monodromy properties of Virasoro blocks are identical to those of the invariant tensors obtained by gluing together two Clebsch-Gordan coefficients of the quantum group \( U_q(sl(2)) \). Using this abstract tensor categorical definition of conformal blocks, it was found that the 4-point conformal blocks have (in a suitable normalization) an algebraic representation in terms of b-deformed hypergeometric functions

\[
\Psi_h^{|L_\Delta \rangle}(x) = e^{2\pi\alpha_h x} F_b(\alpha_h + \alpha_{13}, \alpha_h + \alpha_{24}, 2\alpha_h; -ix)
\]

(17)

where \( \alpha_i \) are the Liouville momenta \( h_i = \alpha_i(Q - \alpha_{-i}) \), and \( q, b \) and \( Q \) are related to \( c \) via \( q = e^{\pi b^2}, \ c = 1 + 6Q^2 \) and \( Q = b + b^{-1} \). The variable \( Z_b = e^{2\pi b x} \) defines a quantum coordinate \( Z_b \), on which \( U_q(sl(2)) \) acts via suitable \( q \)-deformed \( SL(2, \mathbb{R}) \) transformations.

The expression (17) is an eigen function of the \( q \)-deformed Casimir operator \( C_{12} \) acting on the intermediate channel (9). Hence, in the small \( b \) = large \( c \) limit, it is reasonable to identify the quantum group coordinate \( Z_b \) with our uniformizing coordinate \( Z \). We further have

\[
\alpha \simeq bh \quad \alpha_{13} \simeq \frac{ib}{2r_+} h_{13}, \quad \alpha_{24} \simeq \frac{ib}{2r_+} h_{24}
\]

and the b-deformed hypergeometric function reduces to

\[
F_b(\alpha, \beta, \gamma; -ix) \rightarrow {}_2F_1(A, B; C; Z)
\]

\[
\alpha = bA, \quad \beta = bB, \quad \gamma = bC.
\]

So in the limit of large central charge \( c \) we again recover the desired result (16).

2-point function

What about the higher point functions? Consider the 2-point amplitude between two excited states

\[
G_h(g_1, g_2) = \langle \Delta | \Phi_h(g_1) \Phi_h(g_2) | \Delta \rangle, \quad (18)
\]

of equal conformal weight \( \Delta \gg \frac{c}{12} \). We will assume that the bulk locations \( g_1 \) and \( g_2 \) are slightly smeared, so that the highest frequencies that contribute to the 2-point function remain sub-Planckian. In this regime, the 2-point function is expected to reduce to the Hartle-Hawking propagator of a massive scalar field of mass \( m_h \) in the BTZ black hole background. On the CFT side, this result arises as follows.

Inserting a complete set of states factorizes the 2-point function into a product of 1-point matrix elements

\[
\sum_{h_3, h_4} \langle \Delta | \Phi_h(g_1) | h_3, h_4 \rangle \langle h_3, h_4 | \Phi_h(g_2) | \Delta \rangle \quad (19)
\]

By our low energy assumption, the contribution of the descendant states is subleading in \( 1/N \). Using our result for the matrix elements of \( \Phi_h(g) \) between energy-momentum eigenstates, we find that (after analytic continuation to Minkowski signature) the 2-point function takes the form

\[
\sum_{\ell} \int_0^\infty d\omega \left( n_+(\omega) f^\omega_{\ell}(g_1) f^\omega_{\ell}(g_2) + n_-(\omega) f^\omega_{\ell}(g_1) f^\omega_{\ell}(g_2) \right)
\]

where \( n_\pm(\omega) \) denote the level densities of intermediate states that contribute in the sum (19).

The form of \( n_\pm(\omega) \) is determined by taking the limit (11) where \( \Phi_h(g_1) \) and \( \Phi_h(g_2) \) both approach the AdS boundary, where they reduce to local CFT operators. In this limit, it has already been shown (14) that the 2-point function (19) receives its dominant contribution from the identity conformal block and reproduces the boundary-to-boundary propagator in the black hole background. This result can be viewed as a confirmation of the eigenvalue thermalization hypothesis (ETH): the 2-point function (19) for \( g_1 \) and \( g_2 \) close to the boundary behaves
as the CFT 2-point function $\text{tr}(\rho_\beta O_h(x_1)O_h(x_2))$ in the thermal state $\rho_\beta = e^{-\beta H}$ with temperature $\beta = \frac{\Delta}{2\pi}$ equal to a BTZ black hole of mass $M = \Delta - \frac{\ell^2}{2\pi}$. This shows that the spectral densities $n_\pm(\omega)$ take the form of thermal probability distributions with

$$n_+(\omega) = e^{-\beta\omega} n_-(\omega). \quad (20)$$

Via the found match with the mode functions, we can now extend this result to the bulk and confirm that (in the low energy regime in which the expansion $19$ is valid) the bulk 2-point function $18$ coincides with the Hartle-Hawking propagator.

**Concluding comments**

Our results have some bearing on the firewall puzzle. After analytic continuation to Lorentzian signature, the individual mode functions $f_{\omega}(g)$ exhibit a $(\rho - 1)\omega/\gamma^r$ branch-cut at the location of the black hole horizon. This behavior is as expected, given that the mode functions carry a definite energy as seen from outside. However, since the spectral densities $n_\pm(\omega)$ satisfy $20$, the 2-point function exhibits perfectly smooth behavior at the black hole horizon. This is also no surprise, since we’re simply reversing Hawking’s original derivation and its arrow of implication. Indeed, it is natural to propose that the two basic characteristics of a black hole imply each other

$$\text{Smoothness of} \quad \text{the event horizon} \quad \iff \quad \text{Thermality & analyticity of the 2-point function}$$

Analyticity of the 2-point function is a natural consequence of our geometric CFT definition of bulk fields. Mode functions are identified with conformal blocks, which are analytic functions of their arguments. Thermality is a manifestation of the ETH, which asserts that energy eigenstates behave like thermal states for few point functions that only probe a small subsystem of the total quantum system. The two properties combined appear sufficient to conclude that the 2-point function $18$ is smooth across the horizon.

There is some fine-print, however. In the expansion $19$, we omitted the contribution of descendant states. In space-time language, this presumes that the field operator $\Phi_h(g)$ produces only low energy modes, and does not excite any boundary gravitons. So our bulk fields have the properties of low energy effective quantum fields provided they are restricted to act within a ‘code subspace’ of the total CFT Hilbert space, spanned by states that are compatible with a given semi-classical geometry.

Finally, it should be noted that the definition $11$ of the bulk field $\Phi(y, z, \bar{z})$ is in fact less unique that it appears. Whereas Ishibashi states $| h \rangle_\beta$ are invariant under reparametrizations that leave the location of the circle $11$ fixed, it is not invariant under reparametrizations that deform the location of the circle. For every closed curve $C$ surrounding a point $x$, one can find a coordinate system for which $C$ looks like a circle centered around $x$. So Ishibashi states and our bulk operators should in fact be labeled by arbitrary closed curves $C$. The matrix element $\langle \Delta + \omega | \Phi_h(C) | \Delta \rangle$ between some given initial and final primary state, however, only depends on the 3 moduli $(y, z, \bar{z})$ associated with the hole created by $\Phi_h(C)$. Most of the shape parameters of $C$ can be removed by using that the initial and final states are Virasoro highest weight states. So a bulk space-time location is *neither* uniquely specified by the operator $\Phi_h(C)$ nor state-independent: it is determined by the relation between the curve $C$ of the bulk operator, and the choice of initial and final state.

This redundancy is likely to play a key role in establishing an effective form of bulk locality, which dictates that space-like separated bulk fields should, to a high degree of accuracy, commute with each other. In particular, a bulk field $\Phi_h(X)$ should commute with any given local operator $O(x)$ on the boundary. The argumentation here mirrors the ‘secret sharing protocol’ put forward in $15$: since for a given initial and final state, there are many curves $C$ that map to the space-time point $X$, it should always be possible to choose a representative curve $C$ that is space-like separated from some given $x$. With this choice, locality of the boundary theory ensures that $\Phi_h(C)$ commutes with $O(x)$.

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