Sampled-Data Consensus of Networked Euler-Lagrange Systems: A Discrete Small-Gain Approach

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ABSTRACT

This paper studies the sampled-data consensus of networked Euler-Lagrange systems. The sampled-data feedback causes infinities at sampling instants in the control input due to the differentiation of the feedback by the conventional control law designed for Euler-Lagrange systems. Although directly removing the differentiation term from the control law may completely avoid the infinity problem, the overall dynamics are also vastly altered. A problem with this method is that its Lyapunov function of the auxiliary variable may increase at sampling instants, making Lyapunov analysis unviable. To address this problem, the interactions between the auxiliary variable and the states are re-analyzed under the new control law, converting the sampled-data consensus problem into the stability of two interconnected subsystems of states and auxiliary variables, respectively. By a modified discrete small gain analysis, the subsystems are proved to be asymptotically stable under a discrete-time small-gain condition, and the consensus of the networked Euler-Lagrange Systems thus follows. It is shown that despite the individual Lyapunov functions for the states and auxiliary variables might not be strictly decreasing, consensus of the overall system is still guaranteed under the small-gain consensus condition.

INDEX TERMS

Euler-Lagrange system, multi-agent system, small-gain theorem, sampled-data control.

I. INTRODUCTION

Multi-agent systems have attracted vast research interest in recent decades. Numerous related problems have been studied, among which the consensus problem is one of the most fundamental and most studied topics [1]–[10]. Dynamic models of individual agents of multi-agent systems remain the central concern in many studies. More complex and accurate agent dynamic models have been employed in the studies to better represent the physical model of the actual systems.

Euler-Lagrange system has advantages in describing a wide range of physical systems [11]–[13]. For this reason, consensus of networked Euler-Lagrange systems has received much research interest [1]–[9], [14]–[18]. Most of the existing works on this subject assume continuous communication.

Modern computer-based systems exclusively rely on sampled-data communication. Though sampled-data consensus of multi-agent systems has been extensively studied, there are still not many results on sampled-data consensus of multiple Euler-Lagrange systems due to the inherent incompatibility between sampled-data control and the complicated nonlinear dynamics of the Euler-Lagrange system. The problem lies in that the typical consensus law for Euler-Lagrange systems contains a term that differentiates the reference velocity, which is discontinuous. As a result, the derivative is infinite and causes the control torque to go to infinity at those discontinuities, i.e., the sampling instants.

An existing method to avoid the infinity problem mentioned above is proposed in [10]. By directly removing the derivative term from the control law, the cause of the infinity problem is altogether eliminated. The consensus analysis for this control law still relies on the conventional proof that
the auxiliary variable $s$ (treated as disturbance) converges to 0, and the networked Euler-Lagrange systems can be treated as linear systems. However, the effect of removing the differentiation term is not accounted for in this analysis. Contrary to the conventional analysis, the decrease of the Lyapunov function of $s$ can only be guaranteed during each sampling interval. The behavior of the auxiliary variable $s$ at the sampling instants is not yet analyzed. Actually, $s$ is found to change abruptly at the sampling instants, and the value of its Lyapunov function may undergo sudden increases. Therefore, the premise of the conventional analysis no longer holds with the control law in [10].

This approach of removing the differentiation term effectively eliminates the infinities in the control inputs, yet it needs a rigorous consensus analysis and a compatible consensus condition. This work aims to establish rigorous proof for this approach and to find a consensus condition based on the discrete-time small-gain theorem. Although continuous-time Lyapunov analysis alone is not enough to prove consensus, the idea of seeing $s$ as a diminishing disturbance still provides insight into the nature of the complicated interactions between $s$ and the agents’ states.

The inspiration for the solution is simple: suppose $s$ converges very rapidly during a sampling interval, then its effects as a disturbance on the states would be very small, then, the system can be seen as a linear system with very small disturbance. If this linear system is convergent, its states will be smaller (with a slight abuse of description) at the next sampling instant. Then, the update to $s$ would also be smaller. This would mean an even smaller disturbance to the states caused by $s$. The whole process resembles that of the discrete small-gain theorem. The difference is that the subsystems are continuous-time, and $s$ exerts influence on the states at sampling instants and during the intervals. This interaction between the states and auxiliary variables resembles the small-gain theorem.

The small-gain theorem is a powerful tool for studying the stability of interconnected systems [19] and has been extended to the study of multi-agent systems [20]–[22]. In this work, we have demonstrated that the problem investigated fits into the frameworks of the small-gain theorem. The remainder of the paper is organized as follows. In Section II, preliminaries on the problem investigated are presented. Detailed analysis of the system and the derivation of the small-gain condition is given in Section III. In Section IV, the effectiveness of our approach is illustrated through a numerical example.

## II. PRELIMINARIES AND PROBLEM FORMULATION

Communication links among the agents can be described by a weighted directed graph (digraph) $G = (V, E, A)$, where $V = \{1, 2, \ldots, N\}$ is the set of nodes, $E = V \times V$ is the set of edges, and $A = (a_{ij})_{N \times N}$ is the weighted adjacency matrix. $(i, j) \in E$ indicates that agent $i$ receives information from agent $j$, $a_{ij} > 0$ if and only if $(j, i) \in E$, otherwise $a_{ij} = 0$.

Assume that there is no self-loop, i.e. $a_{ii} = 0$, $i \in V$. Let $deg(i) = \sum_{j=1}^{N} a_{ij}, D = diag (deg(1), \ldots, deg(n))$. The Laplacian matrix is $L = (l_{ij})_{N \times N} = D - A$. All eigenvalues of $L$ are in the open right half plane except for the one zero eigenvalue: $0 = \lambda_1 \leq Re(\lambda_2) \leq \cdots \leq Re(\lambda_N)$ where $\lambda_i \in C$ ($i = 1, 2, \ldots, N$) are the eigenvalues of $L$.

Consider $N$ networked Euler-Lagrange systems that are fully actuated and have the following dynamics:

$$M_i (q_i) \ddot{q}_i + C_i (q_i, \dot{q}_i) \dot{q}_i + g_i (q_i) = \tau_i, \quad i = 1, 2, \ldots, N \tag{1}$$

where $q_i = [q_{i1}, q_{i2}, \ldots, q_{im}]^T \in \mathbb{R}^m$ is the generalized position, $M_i (q_i) = M_i^T (q_i) \in \mathbb{R}^{m \times m}$ is the inertia matrix, $C_i (q_i, \dot{q}_i) \in \mathbb{R}^{m \times m}$ is the Coriolis and centripetal matrix, $g_i (q_i) \in \mathbb{R}^m$ is the gravitational torque, $\tau_i \in \mathbb{R}^m$ is the control input, and the following general assumptions hold for the Euler-Lagrange system (1):

1) There exist positive-definite parameters $k_c$ and $k_d$ such that $0 < k_c I_m \leq M_i (q_i) \leq k_d I_m$.

2) $M_i (q_i) - 2C_i (q_i, \dot{q}_i)$ is skew-symmetric, i.e. for any $r \in \mathbb{R}^m, r^T (M_i (q_i) - 2C_i (q_i, \dot{q}_i)) r = 0$.

The networked Euler-Lagrange systems in (1) are sampled at $t_k, k = 0, 1, \ldots, \text{ where } 0 = t_0 < t_1 < \cdots < t_k < \cdots \text{, and } t_k \rightarrow \infty \text{ as } t \rightarrow \infty$. The sampling intervals can be time-varying: $h_k = t_{k+1} - t_k$. The control objective is to drive the networked Euler-Lagrange systems in (1) to achieve consensus, i.e., $\forall i, j \in V$:

$$\lim_{t \rightarrow \infty} (q_i - q_j) = 0$$

The following sampled-data consensus control law is designed in [10]:

$$\tau_i = -K_s s_i + C_i (q_i, \dot{q}_i) \ddot{q}_i + g_i (q_i) \tag{2}$$
where $K_i$ is a positive-definite matrix.

$$\dot{q}_{r,i} = -\rho \sum_{j \in N_i} a_{ij} (q_i(t_k) - q_j(t_k)), \quad t \in [t_k, t_{k+1})$$  \hspace{1cm} (3)$$

is the reference quantity and

$$s_i = \dot{q}_i - \dot{q}_{r,i}$$  \hspace{1cm} (4)$$

is the auxiliary variable.

Applying controller (2) to the Euler-Lagrange system (1) yields

$$M_i(q_i) \dot{q}_i + C_i(q_i, \dot{q}_i) s_i = -K_i s_i, \quad i = 1, 2, \ldots, N$$  \hspace{1cm} (5)$$

Since $\dot{q}_{r,i}$ is 0 during the sampling interval, the above dynamics can be written as

$$M_i(q_i) \dot{s}_i + C_i(q_i, \dot{q}_i) s_i = -K_i s_i$$  \hspace{1cm} (6)$$

Choose the Lyapunov function:

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} s_i^T(t) M_i(q_i) s_i(t)$$  \hspace{1cm} (7)$$

Its derivative along the trajectory of (10) is

$$\dot{V}(t) = \frac{1}{2} \sum_{i=1}^{N} (s_i^T(t) M_i(q_i) s_i(t) + s_i^T(t) M_i(q_i) \dot{s}_i(t))$$

$$+ \frac{1}{2} \sum_{i=1}^{N} (-2s_i^T C_i(q_i, \dot{q}_i) s_i - 2s_i^T K_i s_i + s_i^T(t) \dot{M}_i(q_i) s_i(t))$$

$$- \sum_{i=1}^{N} s_i^T(t) K_i s_i \leq 0$$  \hspace{1cm} (8)$$

It is claimed in [10] that (8) implies the strict decaying of $V(t)$, therefore $s_i \to 0$ as $t \to \infty$, and the networked Euler-Lagrange systems are thus linearized.

However, the assumption that $\dot{q}_{r,i} = 0$ only holds during the sampling intervals. At the sampling instants, $\dot{q}_{r,i}$ is updated, i.e., its value jumps. Combining with the definition of $s_i$, we see that the value of $s_i$ also jumps, i.e.

$$s_i(t_{k+1}) = s_i(t_{k+1}^-) + \rho \sum_{j \in N_i} a_{ij} (q_i(t_{k+1}) - q_j(t_{k+1}))$$

$$- (q_i(t_k) - q_j(t_k))$$  \hspace{1cm} (9)$$

At the sampling instants, $\dot{q}_{r,i} \neq 0$, and the jump behavior of $s_i$ may interrupt the decaying of $V(t)$. Thus the Lyapunov analysis does not apply to the entire time domain, and any results established upon it is invalid. The jump behavior and interruption of the decaying are illustrated in Section IV.

The goal of this work is to find a new way to deal with the consensus problem for the closed-loop networked Euler-Lagrange systems and develop a correct consensus condition.

### III. MAIN RESULTS

In this section, we analyze in detail the interactions between the auxiliary variable $s$ and the generalized states to establish their interconnected subsystems. Then, through proper bounding, we develop ISS-Lyapunov functions for the subsystems. Finally, a consensus condition is obtained through discrete small-gain theorem.

Let $h$ be the variable sampling interval $q = [q_1^T, \ldots, q_N^T]^T$, $s = [s_1^T, \ldots, s_N^T]^T$, $K = \text{diag} \{K_1, \ldots, K_N\}$, $M = \text{diag} \{M_1, \ldots, M_N\}$, $R_i^T R_i = M_i$ and $R_i^T R = M$. Then (3) and (4) give

$$q(t_{k+1}) = q(t_k) - \rho (L \otimes I_m) q(t_k) + \int_{t_k}^{t_{k+1}} s d\tau$$

$$= (1 - \rho (L \otimes I_m)) q(t_k) + \int_{t_k}^{t_{k+1}} s d\tau$$  \hspace{1cm} (10)$$

and (9) gives

$$s(t_{k+1}) = s(t_{k+1}^-) + \rho (L \otimes I_m) (q(t_{k+1}) - q(t_k))$$

$$= s(t_{k+1}^-) - \rho \sigma^2 (L \otimes I_m)^2 q(t_k)$$

$$+ \rho (L \otimes I_m) \int_{t_k}^{t_{k+1}} s d\tau$$  \hspace{1cm} (11)$$

We call (10) and (11) the $q$-subsystem and the $s$-subsystem, respectively, and this notion is used throughout the rest of the paper.

**Remark 1:** The closed-loop system can be seen as two subsystems: the $q$-subsystem and the $s$-subsystem as shown in (10-11). The $q$-subsystem is a linear first-order multi-agent system, and the $s$-subsystem is all the auxiliary variables combined and is treated as a disturbance. Under the control laws (2-3), each Euler-Lagrange agent has its $q$-subsystem and $s$-subsystem. The overall $q$-subsystem and $s$-subsystem is just the totality of the individual $q$-subsystems and $s$-subsystems. Analysis can be carried out only under such settings.

Equation (8) can be rewritten into

$$\dot{V}(t) = -s^T K s \leq -2dV(t) \leq 0$$  \hspace{1cm} (12)$$

for any $d > 0$ such that $dM < K$. Then the following decay characteristic holds:

$$V(s(t_{k+1})) \leq e^{-2dh} V(s(t_k))$$  \hspace{1cm} (13)$$

From (11) we know

$$V(s(t_{k+1})) = \frac{1}{2} \left\| Rs(t_{k+1}^-) - h \rho \sigma^2 (L \otimes I_m)^2 q(t_k) \right\|^2$$

$$+ \rho (L \otimes I_m) \int_{t_k}^{t_{k+1}} s d\tau$$

$$\leq \frac{1}{2} \left\| Rs(t_{k+1}^-) + \rho (L \otimes I_m) \int_{t_k}^{t_{k+1}} s d\tau \right\|^2$$

156550

Y. Wang et al.: Sampled-Data Consensus of Networked Euler-Lagrange Systems

VOLUME 9, 2021
\[ + \left\| h \rho^2 R (L \otimes I_m)^2 q (t_k) \right\|^2 \]
\[
\leq \frac{1}{2} \left( 1 + \frac{1}{r} \right) \left\| Rs (t_{k+1}^-) + \rho R (L \otimes I_m) \int_{t_k}^{t_{k+1}} sd \tau \right\|^2 \\
+ \frac{1}{2} (1 + r) \left\| h \rho^2 R (L \otimes I_m)^2 q (t_k) \right\|^2 \tag{14} \]
for any \( r > 0 \). Next, we bound and transform the first term in (14) into the form of Lyapunov function.
\[
\left\| Rs (t_{k+1}^-) + \rho R (L \otimes I_m) \int_{t_k}^{t_{k+1}} sd \tau \right\|^2 
\leq \left( \left\| Rs (t_{k+1}^-) \right\| + \left\| \rho R (L \otimes I_m) \int_{t_k}^{t_{k+1}} sd \tau \right\| \right)^2
\leq \left( \left\| Rs (t_{k+1}^-) \right\| + \left\| \rho R (L \otimes I_m) \int_{t_k}^{t_{k+1}} \|s\| d \tau \right\| \right)^2
\]

The bound of \( s \) can be obtained through its Lyapunov function:
\[
\|s\| \leq \sqrt{\frac{V}{k_m}} \leq \sqrt{\frac{V (S (t_k))}{k_m}} e^{-d (\tau - t_k)}
\]
Then,
\[
\int_{t_k}^{t_{k+1}} sd \tau \leq \int_{t_k}^{t_{k+1}} \|s\| d \tau 
\leq \sqrt{\frac{V (S (t_k))}{k_m}} \int_{t_k}^{t_{k+1}} e^{-d (\tau - t_k)} d \tau 
\leq \frac{(1 - e^{-dh})}{d} \sqrt{\frac{V (S (t_k))}{k_m}} \tag{15}
\]
By (13) we get
\[
\| Rs (t_{k+1}^-) \|^2 = s^T (t_{k+1}^-) M s (t_{k+1}^-) \leq e^{-2dh} V (s (t_k))
\]
so
\[
\left( \left\| Rs (t_{k+1}^-) \right\| + \left\| \rho R (L \otimes I_m) \int_{t_k}^{t_{k+1}} \|S\| d \tau \right\| \right)^2 
\leq \left( e^{-dh} \sqrt{V (S (t_k))} + \left\| \rho R (L \otimes I_m) \int_{t_k}^{t_{k+1}} \|S\| d \tau \right\| \right)^2
\leq \left( e^{-dh} \sqrt{V (s (t_k))} + \left\| \rho R (L \otimes I_m) \right\| \right.
\times \left. \frac{1 - e^{-dh}}{d} \frac{1}{\sqrt{k_m}} \sqrt{V (s (t_k))} \right)^2
\leq \epsilon^2 V (S (t_k))) \]
Here we have denoted
\[
\epsilon = e^{-dh} + \rho \| R (L \otimes I_m) \| \frac{1 - e^{-dh}}{d} \frac{1}{\sqrt{k_m}} 
\leq e^{-dh} + \frac{\lambda_N \rho}{d} \sqrt{k_M} \left( 1 - e^{-dh} \right) \tag{16}
\]
Then, (14) can be written in ISS-Lyapunov form:
\[
V (s (t_{k+1})) - V (s (t_k)) 
\leq - \left( 1 - e^{-2} - \frac{e^2}{r} \right) V (t_k) 
+ (1 + r) \left( \rho^2 h \right)^2 q^T (t_k) (L \otimes I_m) q (t_k)
\times M (L \otimes I_m)^2 q (t_k) \tag{17}
\]
For (16) to be a valid ISS-Lyapunov function, the following inequality have to be satisfied:
\[
0 < 1 - e^{-2} - \frac{e^2}{r} < 1 \tag{18}
\]
Since \( r > 0 \) can arbitrarily chosen after \( \epsilon \), it is only required that \( \epsilon < 1 \) for (18) to be possible. From (16) we see that in order to ensure \( \epsilon < 1 \), parameters \( d \) and \( \rho \) should be selected to satisfy the following inequality when designing the controller:
\[
\frac{\lambda_N \rho}{d} \sqrt{k_M} < 1 \tag{19}
\]
note that \( d \) is determined by the selection of \( K_i \) in (2).
Next, we focus on the q-subsystem (10). It is common knowledge that when \( h \) is sufficiently small, the linear system
\[
q (t_{k+1}) = (I - h p (L \otimes I_m)) q (t_k)
\]
can reach consensus, and it is possible to find \( 0 < \gamma < 1 \) and a positive semi-definite matrix \( Q \) such that
\[
q^T (t_{k+1}) Q q (t_{k+1}) \leq \gamma^2 q^T (t_k) Q q (t_k)
\]
i.e. \( (I - h p (L \otimes I_m))^T Q (I - h p (L \otimes I_m)) \leq \gamma^2 Q \). Define the Lyapunov function for the q-subsystem:
\[
V_q (q) = q^T Q q \tag{20}
\]
and \( Q = U^T U \), \( z \) is an arbitrary positive scalar, then the Lyapunov function for the next sampling instant in the q-subsystem can be rendered in the form of ISS-Lyapunov function:
\[
V_q (q (t_{k+1})) \]
\[
= \left( (1 - h p (L \otimes I_m)) q (t_k) + \int_{t_k}^{t_{k+1}} S d \tau \right)^T 
* Q \left( (1 - h p (L \otimes I_m)) q (t_k) + \int_{t_k}^{t_{k+1}} S d \tau \right)
\leq \left\| (1 - h p (L \otimes I_m)) U q (t_k) + U \int_{t_k}^{t_{k+1}} S d \tau \right\|^2
\]
\[ \begin{align*}
\leq & \gamma^2 V_q(q(t_k)) + \frac{(1 - e^{-dh})^2}{d^2} \|Q\| \frac{V(S(t_k))}{k_m} \\
&+ 2 \|1 - h \rho (L \otimes I_m) Uq(t_k)\| T \left\| \int_{t_k}^{t_{k+1}} Sd\tau \right\| \\
&\leq \left(1 + \frac{1}{\varepsilon}\right) \gamma^2 V_q(q(t_k)) \\
&+ (1 + z) \|Q\| \frac{(1 - e^{-dh})^2}{d^2} \frac{V(S(t_k))}{k_m} \\
&\leq - \left(1 - \frac{e^{-2}}{r}\right) V(t_k) \\
&+ (1 + r) \left(\rho^2 h^2\right) q^T(t_k) (L \otimes I_m)^2 (L \otimes I_m)^2 q(t_k) \\
&\leq - \left(1 - \frac{e^{-2}}{r}\right) V(t_k) \\
&+ (1 + r) \left(\rho^2 h^2\right) k_m v^T q(t_k) \\
&\leq - \left(1 - \frac{e^{-2}}{r}\right) V(t_k) \\
&+ (1 + r) \left(\rho^2 h^2\right) k_m v^T q(t_k)
\end{align*} \]

(21)

We focus again on the ISS-Lyapunov function (17), it can be further bounded for further small-gain analysis.

\[ V(s(t_{k+1})) - V(s(t_k)) \leq - \left(1 - \frac{e^{-2}}{r}\right) V(t_k) \]

(22)

where \((L \otimes I_m)^2 (L \otimes I_m) \leq vQ\). Next we present the generalized small-gain theorem:

**Lemma 1:** Consider two interconnected system, namely \(x_1\)-subsystem and \(x_2\)-subsystem. Assume the subsystems possess ISS-Lyapunov functions such that

\[ V_1(x_1(t_{k+1})) - V_1(x_1(t_k)) \leq -\sigma_1 V_1(x_1(t_k)) + \rho_1 V_2(x_2(t_k)) \]

(23)

\[ V_2(x_2(t_{k+1})) - V_2(x_2(t_k)) \leq -\sigma_2 V_2(x_2(t_k)) + \rho_2 V_1(x_1(t_k)) \]

(24)

with \(\sigma_i\) satisfying that \(id - \sigma_i \in \mathcal{K}\). If the following small gain condition holds

\[ \sigma^{-1}_1 \circ \rho_1 \circ \sigma^{-1}_2 \circ \rho_2 < id \]

(25)

then

\[ \lim_{k \to \infty} V_1(x_1(t_k)) = \lim_{k \to \infty} V_2(x_2(t_k)) = 0 \]

(26)

**Proof:** The proof is identical to Corollary 4.2 of [22]. \(\square\)

**Remark 2.** The proof of Lemma 1 does not depend on any specific type of subsystems, but rather solely on the ISS-Lyapunov functions. In other words, the convergence result (26) holds for any system if only (23), (24) and (25) are satisfied. This means the generalized small-gain theorem Lemma 1 also applies to subsystems (10) and (11).

**Theorem 1:** The networked Euler-Lagrange systems (1) under the sampled-data control input (2) can reach consensus if the communication graph contains a directed spanning tree and the following inequality holds:

\[ \frac{(\rho^2 h^2)^2 (1 - e^{-dh})^2 v \|Q\| k_m}{(1 - \varepsilon)^2 (1 - \gamma)^2 d^2 k_m} < 1 \]

(27)

**Proof:** Comparing the ISS-Lyapunov functions (21) & (22) with (23) & (24) in Lemma 1, we get

\[ \sigma_1 (x) = \left(1 - \frac{e^{-2}}{r}\right) x \]

\[ \rho_1 (x) = (1 + r) \left(\rho^2 h^2\right) k_m v x \]

\[ \sigma_2 (x) = \left(1 - \frac{e^{-2}}{r}\right) x \]

\[ \rho_2 (x) = (1 + z) \|Q\| \frac{(1 - e^{-dh})^2}{d^2} \frac{1}{k_m} x \]

Putting the above into (25) yields

\[ (1 + r) \left(\rho^2 h^2\right) k_m v \|Q\| \frac{(1 - e^{-dh})^2}{d^2} \frac{1}{k_m} < 1 \]

(28)

Note that the above inequality can be further simplified, and its conservatism can be reduced by choosing parameters \(r\) and \(z\) that minimize the left-hand side. Take the following function for example,

\[ f(r) = \frac{1 + r}{1 - e^{-2} - \frac{\varepsilon^2}{r}} \]

find the extremum where \(\frac{df(r)}{dr} = 0\), we get

\[ \frac{df}{dr} (1 + r) \left(1 - e^{-2} - \frac{\varepsilon^2}{r}\right) - (1 + r) \frac{df}{dr} \left(1 - e^{-2} - \frac{\varepsilon^2}{r}\right)^2 = 0 \]

and find the extremum at \(r = \frac{\varepsilon^2}{1 - e^{-2}}\) and \(f(r) = \frac{1}{(1 - e^{-2})^2}\). Then, the minimum of the left-hand side of (28) for all \(r > 0\) and \(z > 0\) is

\[ \frac{(\rho^2 h^2)^2 (1 - e^{-dh})^2 v \|Q\| k_m}{(1 - e^{-2} - \frac{\varepsilon^2}{r})^2} = \frac{(\rho^2 h^2)^2 (1 - e^{-dh})^2 v \|Q\| k_m}{(1 - \varepsilon)^2 (1 - \gamma)^2 d^2 k_m} \]

therefore, it can be concluded from Lemma 1 that

\[ \lim_{k \to \infty} V(s(t_k)) = \lim_{k \to \infty} V_y(q(t_k)) = 0 \]

if (27) holds. Consensus result of the system follows the diminishing of \(V_y(q(t_k))\). \(\square\)

**Remark 3:** Condition (27) is consistent with some qualitative facts of the subsystems. Smaller \(\rho\) means smaller jump of \(s\), which is beneficial to the convergence. Smaller \(\rho h\) helps ensure consensus of the sampled-data linear systems. \(\frac{1 - e^{-dh}}{d} \) is proportional to the influence of \(s\) on the \(q\)-subsystem, which should be kept small for the convergence.
IV. NUMERICAL EXAMPLES

In this section, we first show the jump behavior i.e., the abrupt change of the auxiliary variable \(s\) (specifically, the auxiliary variable \(s_1 \in \mathbb{R}^2\) of the 1st agent), as well as the increase in its Lyapunov function \(V_1(s) = s_1^T M_i(q_i) s_1\). Then, an extreme case with divergent states is provided to show that \(s\) is not only non-strictly decreasing but also can be divergent when the states are divergent. Lastly, an example is given to verify the small-gain consensus condition.

![FIGURE 1. The communication graph.](image1)

Consider a network of five 2-DOF Euler-Lagrange systems connected by the communication network shown in Fig. 1. The sampling period is set as 0.4s. The initial positions are chosen as \(q_1(0) = [0.2, 0.2]^T\), \(q_1(0) = [0.0, 0.1]^T\), \(q_1(0) = [0.1, 0]^T\), \(q_1(0) = [0, 0]^T\), \(q_1(0) = [-0.1, 0.1]^T\), and the initial velocities are all 0. When applying the consensus control law (2-3), we can observe the behavior of the auxiliary variable \(s_1\) of the 1st agent.

![FIGURE 2. The auxialiary vairable \(s_1\).](image2)

As shown in the above figures, the auxiliary variable \(s_1\) changes abruptly, and its Lyapunov function \(V_1(s)\) suddenly increase at some sampling instants. This confirms our theoretical prediction.

![FIGURE 3. The lyapunov function \(V\).](image3)

Another implication of the small-gain analysis is that the auxiliary variables are also divergent when the states are divergent. The following is an extreme example where the states, i.e., q-subsystem is divergent, the Lyapunov function \(V_1(s)\) also increases with time. This shows that the auxiliary variables do not even converge, as is claimed in [10].

Next, we show the effectiveness of Theorem 1 and how it guides the selection of controller parameters. There are several quantities to be determined before applying the small-gain condition (27):

1. \(k_M\) and \(k_m\) are the physical properties of the Euler-Lagrange system. In this paper, \(k_M = 8.5296\), \(k_m = 0.1304\).
2. \(d\) is the decaying factor of \(s\), and is determined by the matrix \(K_i\) in (2) and calculated with (12), i.e. \(K > dM\). In this case, \(K\) is chosen as \(K = 10I\), so \(d = 1.1724\).
3. \(\rho\) is the scaling parameter to be chosen to satisfy (19). It is clear in (27) that smaller \(\rho\) is extremely beneficial to the condition, but small \(\rho\) also slows the consensus. In this case, the maximum value of \(\rho\) is 0.0342. We choose \(\rho = 0.03\).
4. \(h\) is the length of the sampling interval, we choose \(h = 0.4\).
5. Matrix \(Q\) is determined using LMI tools.

![FIGURE 4. The lyapunov function \(V\) with divergent q-subsystem.](image4)
6. $\gamma < 1$ is obtained together with $Q$, $\gamma = 0.9923$, $\|Q\| = 7.2965$.
7. Parameter $\epsilon$ is calculated with (16), $\epsilon = 0.9572$.
8. Parameter $v$ is determined with $(L \otimes I_m)^T (L \otimes I_m) \leq vQ$, in this case, $v = 0.033$.

Substitute the above into (27) to calculate the small-gain condition:

$$
\begin{align*}
(\rho^2 h)^2 \left(1 - e^{-dh}\right)^2 v \|Q\| k_M \\
(1 - e)^2 (1 - \gamma)^2 d^2 k_m = 0.9694 < 1
\end{align*}
$$

As shown in Fig. 5, the networked Euler-Lagrange systems reach consensus.

**FIGURE 5.** Consensus of the two components of the generalized positions, respectively.

It is clearly revealed in the calculation of the parameters that the decay rate $d$ of the $s$-subsystem plays an important role: with larger $d$, larger values can be assigned to $\rho$ for faster convergence; larger $d$ also makes it easier to satisfy the small-gain condition (27). This fact is consistent with the inspiration mentioned in Section I, that when viewing $s$ as a disturbance to the $q$-subsystem, the faster $s$ decays, the closer the system is to a linear one.

Lastly, it is worthwhile to mention that when using this small-gain condition, the Lyapunov function of $s$ is not required to decay strictly across all sampling instants. This again shows the advantage of our approach over the existing one.

**V. CONCLUSION**

In this paper, we have investigated the consensus problem of networked Euler-Lagrange systems under sampled-data communication. One approach that avoids the infinity problem caused by discontinuous feedback is thoroughly studied. It is found that this approach introduces complicated interactions between the states and auxiliary variables and renders the conventional consensus analysis incompatible. To solve this problem, the interactions are analyzed, and a model of two interconnected subsystems is created to describe the dynamics of the states and auxiliary variables, respectively. Discrete-time small-gain theorem is employed to analyze the interconnected systems, and small-gain theorem-based consensus conditions are obtained for the overall dynamics. Finally, numerical examples are provided to illustrate the problem we pointed out and the effectiveness of the discrete-time small-gain consensus condition.

Future efforts may incorporate more common scenarios and concerns as time delay, data quantization etc.

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