THE PI PROPERTY OF GRADED HECKE ALGEBRAS

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Abstract. We show that graded Hecke algebras are PI algebras if and only if they are finitely generated over their centres if and only if the deformation parameters $t_i$ are zero for all $i = 1, \ldots, N$. This generalises a result for symplectic reflection algebras by Etingof - Ginzburg and Brown - Gordon.

1. Introduction

Graded Hecke algebras were first defined by Drinfeld in [Dri86] and then studied in detail by Ram and Shepler in [RS03]. Ram and Shepler show that graded Hecke algebras are generalisations of the graded affine Hecke algebras defined by Lusztig in [Lus89] for real reflection groups. Lusztig’s work on graded affine Hecke algebras was motivated by questions in group representation theory. Drinfeld’s construction of graded Hecke algebras is more general and makes it possible to attach a graded Hecke algebra to any finite subgroup of $GL(V)$, not only the real reflection groups. In [RS03] a full classification of the graded Hecke algebras corresponding to complex reflection groups is achieved. Surprisingly and disappointingly, there are complex reflection groups for which no nontrivial graded Hecke algebras exist. This has inspired other authors to look for further generalisations of graded Hecke algebras, see [SW06].

Our work on graded Hecke algebras is inspired by their connection to geometric questions. Graded Hecke algebras are deformations of skew group algebras $S(V) \ast G$, where $V$ is a finite dimensional vector space over $\mathbb{C}$ and $G$ a finite subgroup of $GL(V)$. The centre of $S(V) \ast G$ is $S(V)^G$, the coordinate ring of the orbit variety $V^*/G$. By studying $S(V) \ast G$ one hopes to understand the $G$-equivariant geometry of $V^*$. If $V$ is a symplectic vector space and the group $G$ preserves the symplectic form, then the symplectic reflection algebras defined in [EG02] appear naturally as special cases of graded Hecke algebras. In this special setting Etingof and Ginzburg are able to find smooth deformations for some of the singular varieties $V^*/G$.

The purpose of this paper is to generalise the first steps in [EG02] to the more general setup of graded Hecke algebras. Our results confirm a claim made by Etingof and Ginzburg in [EG02] Remark (ii), p. 246]. The structure of symplectic reflection algebras displays
a dichotomy depending on the deformation parameter $t$. Namely, a symplectic reflection algebra is finitely generated over its centre if and only if it is a PI algebra if and only if $t = 0$. This is a result of [EG02 Theorem 3.1] and [BG03 Proposition 7.2]. Thus the obvious question to ask is whether graded Hecke algebras display the same dichotomy in their behaviour depending on specialisations of the deformation parameters. As it turns out there is more than one deformation parameter which controls whether or not graded Hecke algebras are PI. We denote these parameters by $t_1, \ldots, t_N$. In the symplectic situation one can reduce to the case where $V$ is a symplectic vector space such that $V$ does not admit any non-degenerate $G$-invariant subspaces. For such a vector space the space of $G$-invariant skew-symmetric bilinear forms on $V$, $((\Lambda^2 V^*)^G)$, is one-dimensional, which gives rise to the one parameter $t$. In general, however, the dimension of the space $((\Lambda^2 V^*)^G)$ can be greater than one, say $N$, which leads to the appearance of $N$ deformation parameters $t_i$. We show that

**Theorem.** A graded Hecke algebra is finitely generated over its centre if and only if it is a PI algebra if and only if $t_i = 0$ for all $i = 1, \ldots, N$.

In Section 2 we begin by defining graded Hecke algebras as in [RS03] and derive some basic ring-theoretic properties of these algebras. The definition of graded Hecke algebras can also be motivated by deformation theory, which is the content of Section 3. This approach gives an explanation for the choice of the construction of graded Hecke algebras and it will be crucial to proving our subsequent results. Our work relies on the results in [EG02] and [BG03 Proposition 7.2] and the techniques developed in [EG02]. We modify their work to account for the fact that we maintain a general setup and do not assume a symplectic structure. Sections 4 and 5 are devoted to providing the details of these adjusted proofs. Finally, in Section 6 we prove our main theorem that tells us for which values of the deformation parameters a graded Hecke algebra has a big centre. As a corollary we deduce the result mentioned above.

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2. Definition and first properties

Let $V$ be a finite dimensional vector space over $\mathbb{C}$ and $G$ a finite subgroup of $GL(V)$. Denote by $\kappa : V \times V \to \mathbb{C}G$ a skew-symmetric bilinear form. Let $T(V) = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus \cdots$
be the tensor algebra of $V$. The action of $G$ on $V$ extends to an action of $G$ on $T(V)$ by $\mathbb{C}$-algebra automorphisms. Construct the skew group algebra $T(V)*G$ and define the following factor algebra

$$A := (T(V)*G)/\langle [v, w] - \kappa(v, w) : v, w \in V \rangle,$$

where $[v, w] = vw - wv$.

The algebra $A$ is a positively filtered algebra, namely

$$F^0_A = \mathbb{C}G, \quad F^1_A = \mathbb{C}G + V \mathbb{C}G, \quad F^i_A = (F^1_A)^i \text{ for } i > 1.$$ 

We form the associated graded algebra $\text{gr}A$ of $A$ under this filtration. From the relations in $A$ it is clear that there exists an epimorphism $S(V) \ast G \twoheadrightarrow \text{gr}A$. Here $S(V)$ denotes the symmetric algebra of $V$, that is $S(V) = T(V)/\langle [v, w] : v, w \in V \rangle$.

**Definition 2.1.** [[RS03, Section 1]] The algebra $A$ is called a graded Hecke algebra if $S(V) \ast G \cong \text{gr}A$ as graded algebras.

Thus a graded Hecke algebra $A$ is isomorphic to $S(V) \otimes \mathbb{C}G$ as a graded vector space, which provides us with a PBW basis for $A$. This PBW basis imposes necessary conditions on the form $\kappa$. For example, $\kappa$ needs to be $G$-invariant in the sense that $\kappa(g(v), g(w)) = g\kappa(v, w)g^{-1}$, because otherwise there exists a nontrivial linear relation in $A$ between elements of $\mathbb{C}G$. In fact, there is a very precise description of the possible choices for $\kappa$. To explain this we need to introduce more notation.

Denote the centraliser of an element $g \in G$ by $Z_G(g)$. Recall that a bireflection in $G$ is an element $s \neq id \in G$ that fixes a subspace of $V$ of codimension 2, that is $\text{rank}_V(id - s) = 2$. Write $V^s = \ker(id - s)$ for the subspace of $V$ fixed by $s$ and observe that $V \cong V^s \oplus V/V^s$, where $V/V^s \cong \text{im}(id - s)$. Let $S$ denote the set of birefections in $G$ and define

$$S' = \{ s \in S \mid \forall g \in Z_G(s), \det(g|_{V/V^s}) = 1 \}$$

It follows directly from the definition that this set is closed under conjugation. Let $S$ denote the normal subgroup of $G$ generated by $S'$. Note that $S \subseteq SL(V)$ and, therefore, $S$ does not contain reflections.

Let us construct specific skew-symmetric bilinear forms on $V$ as follows. Fix $s \in S'$. Since the space $\text{im}(id - s)$ is two-dimensional, there exists a unique - up to scalar multiplication - nonzero skew-symmetric bilinear form on $\text{im}(id - s)$. We can extend this form to all of $V$ by setting $V^s = \ker(id - s)$ to be its radical. Denote the form constructed in this way by $\Omega_s$. Using this form as a starting point we can define new forms: for $g \in G$ and $v, w \in V$,

$$\Omega_{g^{-1}sg}(v, w) := \Omega_s(g(v), g(w)).$$
With some easy calculations one can check that the choice of the subset $S'$ ensures that $\Omega_{g^{-1}sg}$ is well-defined. Moreover, $\Omega_{g^{-1}sg}$ is indeed a nonzero skew-symmetric bilinear form on $\text{im}(id - g^{-1}sg)$ with radical $\ker(id - g^{-1}sg)$. Thus for a fixed element $s \in S'$ the forms corresponding to the elements in the conjugacy class of $s$ are determined by $\Omega_s$.

Finally, let $\Omega$ be any skew-symmetric bilinear form on $V$ which is $G$-invariant, that is $\Omega \in ((\bigwedge^2 V)^*)^G$. The space $((\bigwedge^2 V)^*)^G$ is a finite dimensional vector space over $\mathbb{C}$. Let $\{b_1, \ldots, b_N\}$ denote a basis for $((\bigwedge^2 V)^*)^G$ over $\mathbb{C}$. Then $\Omega = \sum_{i=1}^{N} t_i b_i$ for some $t_i \in \mathbb{C}$.

Now we are ready to state the crucial fact:

**Theorem 2.2.** [Dri86, Section 4], [RS03, Theorem 1.9] With the above notation the algebra $A$ is a graded Hecke algebra if and only if, for all $v, w \in V$,

$$\kappa(v, w) = \Omega(v, w) id + \sum_{s \in S'} c_s \Omega_s(v, w) s,$$

where $\Omega(v, w) = \sum_{i=1}^{N} t_i b_i(v, w)$ for some $t_i \in \mathbb{C}$, and the map $c : S' \to \mathbb{C}$ given by $s \mapsto c_s$ is invariant under conjugation by $G$.

Therefore, a graded Hecke algebra is completely determined by the choice of the complex values for $\{t_i \mid i = 1, \ldots, N\}$ and $\{c_s \mid s \in S'\}$. To express this fact in our notation we henceforth denote a graded Hecke algebra by $A_{t, c}$, where $t$ denotes the $N$-tuple of parameters $t_i$ and $c$ denotes the tuple of parameters $\{c_s \mid s \in S'\}$.

**Examples 2.3.**

1. If $t_i = 0$ for all $i = 1, \ldots, N$ and $c_s = 0$ for all $s \in S'$, then $A_{0, 0} = S(V) * G$. This follows directly from the defining relations of $A_{t, c}$. Thus the graded Hecke algebra $A_{t, c}$ is a deformation of the skew group algebra $S(V) * G$.

2. Let $V$ be a symplectic vector space and $\omega$ the non-degenerate skew-symmetric bilinear form on $V$. Suppose that the group $G$ preserves this form, that is $\omega(g(v), g(w)) = \omega(v, w)$ for all $g \in G$ and $v, w \in V$. Then symplectic reflection algebras, as defined in [EG02], appear as special cases of graded Hecke algebras. Namely, in the study of symplectic reflection algebras one can reduce to the case where $V$ contains no non-degenerate $G$-invariant subspaces. Under this assumption it can be shown that $((\bigwedge^2 V)^*)^G = \mathbb{C}\omega$, see [EG02] Section 2, p. 256]. Thus without loss of generality we set $\Omega = \omega$. Take a bireflection $s$. It is easy to see that the decomposition $V = \ker(id - s) \oplus \text{im}(id - s)$ is in fact $\omega$-orthogonal. If we combine this with the fact that $\omega$ is non-degenerate on $V$, we deduce that $\omega |_{\text{im}(id - s)}$ is non-degenerate. Hence,
without loss of generality we set \( \omega|_{\text{im}(id-s)} = \Omega_s \). Observe that in the symplectic situation \( S = S' \). But we have now arrived at the definition of a symplectic reflection algebra, see [EG02, Theorem 1.3].

**Proposition 2.4.** Let \( A_{t,c} \) be a graded Hecke algebra.

(i) \( A_{t,c} = A_{t,c}(S) \ast' (G/S) \), where \( A_{t,c}(S) \) denotes the graded Hecke algebra corresponding to \( S \) instead of \( G \), and \( \ast' \) denotes some crossed product.

(ii) Let \( \lambda \in \mathbb{C}^* \), then \( A_{t,c} \lambda \ast = A_{t,c} \).

**Proof.** (i) Denote the ideal generated by \( [v,w] = \kappa(v, w) = \Omega(v, w) \text{id} + \sum_{s \in S'} c_s \Omega_s(v, w)s \), for all \( v,w \in V \), by \( I \). Then \( A_{t,c} = (T(V) \ast G)/I \) by definition. By [Pas89, Lemma 1.3], \( (T(V) \ast G)/I = [(T(V) \ast S) \ast' (G/S)]/I \). Since \( \kappa(v, w) \in \mathbb{C}S \) for all \( v,w \in V \), the generators of the ideal \( I \) also generate an ideal of \( T(V) \ast S \), which we denote by \( IS \).

We have \( I = IS \cdot (T(V) \ast G) = (T(V) \ast G) \cdot IS \) and \( A_{t,c} = [(T(V) \ast S) \ast' (G/S)]/I = [(T(V) \ast S)/IS] \ast' (G/S) = A_{t,c}(S) \ast' (G/S) \).

(ii) Define a map \( A_{t,c} \rightarrow A_{t,c} \lambda \) by \( x \mapsto \sqrt{\lambda}x \) and \( g \mapsto g \) for \( x \in T(V), g \in G \). \( \square \)

Because of the close connection between \( A_{t,c} \) and \( A_{t,c}(S) \) exhibited in this proposition we will later restrict ourselves to the case when \( G \) is generated by the elements of \( S \) that it contains, hence to the case \( G = S \).

**Proposition 2.5.** Let \( A_{t,c} \) be a graded Hecke algebra. Then \( A_{t,c} \) is noetherian, prime and has finite global dimension.

**Proof.** We use the fact that \( A_{t,c} \) is filtered such that \( grA_{t,c} \rightarrow T(V) \ast G \). By [MR87, Proposition 1.6.6, Theorem 1.6.9, Corollary 7.6.18] all the properties follow, if one can show that they hold for \( S(V) \ast G \). But \( S(V) \ast G \) is noetherian and prime, because \( G \) is a finite group which acts faithfully on \( S(V) \), see [Pas89, Proposition 1.6, Corollary 12.6]. Furthermore, since we are working over \( \mathbb{C} \), [MR87, Theorem 7.5.6] implies that the global dimension of \( S(V) \ast G \) is finite. \( \square \)

3. PBW DEFORMATION

In this section we show that the graded Hecke algebras \( A_{t,c} \) are precisely a special kind of PBW deformation of the skew group algebra \( R := S(V) \ast G \). This will justify the shape of the form \( \kappa \) which appears in Theorem 2.2. Shepler and Witherspoon also consider graded
Hecke algebras as deformations of $R$ and determine the Hochschild cocycles which appear, see [SW06, Section 8].

The algebra $R = S(V) \ast G$ is naturally a positively graded algebra with $\deg CG = 0$. Let $R^i$ be the $i$th graded part of $R$, for all $i \geq 0$, and note that each $R^i$ is a $CG$-bimodule. Thus we can view $R$ as a graded $CG$-bimodule where the multiplication in $R$ gives a $CG$-bimodule map $R \otimes CG R \to R$. From the relations in $A_{t,e}$ it is clear that - when constructing $A_{t,e}$ as a deformation of $R$ - we do not want to deform the relations in $CG$. Thus to ensure that we do not deform the degree zero part of $R$ we will use the following definition in this section and only in this section.

**Definition 3.1.** Let $B$ be a $CG$-bimodule with a $CG$-bimodule map $B \otimes CG B \to B$. Then we call $B$ a $CG$-algebra.

Therefore, subsequent maps will often be assumed to be $CG$-bimodule maps and we will frequently tensor over $CG$ instead of $C$, which we will clarify by notation. In the literature deformation theory usually happens over a field, but, as mentioned in [EG02, Section 2, p. 256], the theory explained in the following and in particular the results of [BG96] also hold for $CG$-algebras as defined above.

We recall the definition of a graded deformation of $R$. Suppose $R_h$ is an associative unital algebra such that $R_h \cong R \otimes C[\hbar]$ as graded $C$-vector spaces, where $\deg \hbar = 1$. Then $(R_h, \ast)$ is a graded deformation of $R$ if the multiplication $\ast: R_h \times R_h \to R_h$ is a $C[h]$-linear map such that $r_1 \ast r_2 \equiv r_1 r_2 \mod \hbar R_h$, for all $r_1, r_2 \in R$. Thus $R = R_h/\hbar R_h$. In our situation we also require that $R_h$ is a graded $CG[h]$-bimodule and that $\ast$ is a $CG[h]$-bimodule map.

If $R_h$ is a graded deformation of $R$, then the multiplication of two elements $r_1, r_2 \in R$ can be described by

$$r_1 \ast r_2 = r_1 r_2 + \mu_1(r_1, r_2) \cdot \hbar + \mu_2(r_1, r_2) \cdot \hbar^2 + \cdots.$$  

The term $r_1 r_2$ denotes the product in $R$ and the maps $\mu_i : R \times R \to R$ are $CG$-bimodule maps of degree $-i$ with $i \in \mathbb{N}$. These maps completely determine the multiplication in $R_h$ because of $CG[h]$-linearity.

**Remark 3.2.** Graded deformations have the following property: for all $\lambda \in C$ the factor algebra $R_{h,\lambda} := R_h/(\hbar - \lambda)R_h$ is a filtered algebra such that there is a canonical isomorphism $gr R_{h,\lambda} \cong R$ as algebras and also as $CG$-algebras. The filtration on $R_{h,\lambda}$ is induced by the filtration on $R_h$, which in turn is derived from the grading on $R_h$.

Let us now turn to the concept of a PBW deformation. We need to introduce quadratic $CG$-algebras first, see [BG96, 0.1 and 0.2] for the definition of a quadratic algebra over a field.
In our case let $E$ denote a $\mathbb{C}G$-bimodule and let $T_{CG}(E) = \mathbb{C} \oplus E \oplus (E \otimes_{CG} E) \oplus \ldots$ denote the tensor $\mathbb{C}G$-algebra. Let $P$ be a subset of $\mathbb{C}G \oplus E \oplus (E \otimes_{CG} E)$ and also $\mathbb{C}G$-bimodule. If we denote the ideal generated by $P$ by $I(P)$, then the algebra $Q(E, P) := T_{CG}(E)/I(P)$ is called a nonhomogeneous quadratic $\mathbb{C}G$-algebra. If $D$ is a subset of $E \otimes_{CG} E$ and also a $\mathbb{C}G$-bimodule, then we say that the algebra $Q(E, D) := T_{CG}(E)/I(D)$ is a quadratic $\mathbb{C}G$-algebra.

Suppose that $Q(E, P)$ is a nonhomogeneous quadratic $\mathbb{C}G$-algebra. Then there exists a canonical quadratic $\mathbb{C}G$-algebra $Q(E, D)$ associated to $Q(E, P)$. Namely define $\pi : \mathbb{C}G \oplus E \oplus (E \otimes_{CG} E) \mapsto E \otimes_{CG} E$ to be the projection map and set $D = \pi(P)$. The $\mathbb{C}G$-algebra $T_{CG}(E)$ is graded with $\deg\mathbb{C}G = 0$ and $\deg E = 1$. This grading induces a filtration $F_{T_{CG}(E)}^\bullet$ on $T_{CG}(E)$, which in turn induces a filtration on $Q(E, P)$ via the surjection $p : T_{CG}(E) \twoheadrightarrow Q(E, P)$. Namely, $F_{Q(E, P)}^i = p(F_{T_{CG}(E)}^i) = F_{T_{CG}(E)}^i/(F_{T_{CG}(E)}^i \cap I(P))$. The associated graded $\mathbb{C}G$-algebra of $Q(E, P)$ under this filtration, denoted by $grQ(E, P)$, is generated over $\mathbb{C}G$ by $p(E)$. Thus there exists a surjective $\mathbb{C}G$-algebra map $T_{CG}(E) \twoheadrightarrow grQ(E, P)$. Since $\pi(P) = D$, we even have $\psi : Q(E, D) \rightarrow grQ(E, P)$.

**Definition 3.3.** [BG96, Definition 0.3] The nonhomogeneous quadratic $\mathbb{C}G$-algebra $Q(E, P)$ is called a PBW deformation of $Q(E, D)$ if $\psi$ is an isomorphism, that is if $Q(E, D) \cong grQ(E, P)$.

It is clear that any PBW deformation $Q(E, P)$ of $Q(E, D)$ must satisfy the condition $P \cap F^1_{T_{CG}(E)} = 0$. If this condition holds, the $\mathbb{C}G$-bimodule $P$ can be written uniquely in terms of two $\mathbb{C}G$-bimodule maps $\alpha : D \mapsto E$ and $\beta : D \mapsto \mathbb{C}G$ as $P = \{d - \alpha(d) - \beta(d) : d \in D\}$.

Let us see how the skew group algebra $R = S(V) \ast G$ and the graded Hecke algebras $A_{t,c}$ fit into this picture. Take $E := V \otimes_{\mathbb{C}} \mathbb{C}G$ and observe that $E$ is a free right $\mathbb{C}G$-module by multiplication and a free left $\mathbb{C}G$-module by $g(v \otimes g') = g(v) \otimes g \cdot g'$, for $v \in V$ and $g, g' \in \mathbb{C}G$. Let $D \subseteq E \otimes_{\mathbb{C}G} E$ be the $\mathbb{C}$-span of

$$\{ (v \otimes 1) \otimes_{\mathbb{C}G} (w \otimes g) - (w \otimes 1) \otimes_{\mathbb{C}G} (v \otimes g) \},$$

for $v, w \in V$ and $g \in \mathbb{C}G$. Note that $D$ is a $\mathbb{C}G$-bimodule with the actions

$$g'[(v \otimes 1) \otimes_{\mathbb{C}G} (w \otimes g) - (w \otimes 1) \otimes_{\mathbb{C}G} (v \otimes g)] =$$

$$(g'(v) \otimes 1) \otimes_{\mathbb{C}G} (g'(w) \otimes g'g) - (g'(w) \otimes 1) \otimes_{\mathbb{C}G} (g'(v) \otimes g'g),$$

$$[(v \otimes 1) \otimes_{\mathbb{C}G} (w \otimes g) - (w \otimes 1) \otimes_{\mathbb{C}G} (v \otimes g)]g' = (v \otimes 1) \otimes_{\mathbb{C}G} (w \otimes gg') - (w \otimes 1) \otimes_{\mathbb{C}G} (v \otimes gg').$$

We claim that $S(V) \ast G \cong Q(E, D)$ as $\mathbb{C}G$-algebras. To see this we construct an isomorphism $\theta : T_{CG}(E) \rightarrow T(V) \ast G$, where $T(V)$ denotes the usual tensor algebra of $V$ over $\mathbb{C}$, as the
Let $A_{t,c}$ be a graded Hecke algebra and let $Q(E,D) \cong S(V) \ast G$ with $E$ and $D$ defined as above. Then there exists $P \subseteq CG \oplus E \oplus (E \otimes CG E)$ such that $A_{t,c} = Q(E,P)$, a nonhomogeneous quadratic $CG$-algebra. Moreover, the quadratic $CG$-algebra associated to $Q(E,P)$ under the projection $\pi : CG \oplus E \oplus (E \otimes CG E) \mapsto E \otimes CG E$ is $S(V) \ast G$ and so $A_{t,c}$ is a PBW deformation of $S(V) \ast G$.

**Proof.** A graded Hecke algebra is defined as $A_{t,c} = (T(V) \ast G)/I$, where $I$ denotes the ideal generated by $[v_1,v_2] - \kappa(v_1,v_2)$, for all $v_1,v_2 \in V$. Equivalently one could choose as generators of the ideal $I$ the elements $([v_1,v_2] - \kappa(v_1,v_2))g$, for all $v_1,v_2 \in V$ and $g \in CG$. We have seen on the previous page that $T_{CG}(E) \cong T(V) \ast G$ via an isomorphism which we labelled $\theta$. The map $\theta$ induces an isomorphism $A_{t,c} \cong T_{CG}(E)/I(P) = Q(E,P)$, where $P$ is the $\mathbb{C}$-span of

$$\{(v_1 \otimes 1) \otimes_{CG} (v_2 \otimes g) - (v_2 \otimes 1) \otimes_{CG} (v_1 \otimes g) - \kappa(v_1,v_2)g\},$$

for $v_1,v_2 \in V, g \in CG$. Note that we can extend the $CG$-action on the subset $D$ to make $P$ into a $CG$-bimodule, because $\kappa$ is $G$-invariant. The quadratic $CG$-algebra naturally associated to $A_{t,c} \cong Q(E,P)$ is clearly $S(V) \ast G \cong Q(E,D)$. Now, by definition $grA \cong S(V) \ast G$, and hence graded Hecke algebras are PBW deformations of $S(V) \ast G$. Therefore, we can write $P$ in terms of $CG$-bimodule maps $\alpha$ and $\beta$. Namely, let $\alpha = 0$ and $\beta[(v_1 \otimes 1) \otimes_{CG} (v_2 \otimes g) - (v_2 \otimes 1) \otimes_{CG} (v_1 \otimes g)] = \kappa(v_1,v_2)g$. \qed

We now want to show that, conversely, all PBW deformations of $S(V) \ast G$ with certain properties are graded Hecke algebras. Observe that the $CG$-algebra $S(V) \ast G$ is Koszul, which can be seen from tensoring the Koszul resolution of the trivial $S(V)$-module $\mathbb{C}$ on the right by $CG$. We have the following result

**Theorem 3.5.** Let $Q(E,D)$ be a quadratic Koszul $CG$-algebra, where $E$ is a free $CG$-module from either side. Assume that we are given $Q(E,P)$ in terms of $CG$-bimodule maps $\alpha : D \mapsto E$, $\beta : D \mapsto CG$, and $P = \{d - \alpha(d) - \beta(d) : d \in D\}$. Then $Q(E,D) \mapsto grQ(E,P)$ is an isomorphism if and only if the following are satisfied.
where the domain of all these maps is \((D \otimes_{CG} E) \cap (E \otimes_{CG} D)\).

Remark 3.6. In fact, in [BG96, Theorem 4.1] it is proved that the conditions (i) - (iii) on the maps \(\alpha\) and \(\beta\) and the fact that \(Q(E, D)\) is Koszul allow one to construct a graded deformation \((Q(E, D)_h, \ast)\). Let \(Q(E, D)_{h,1} = Q(E, D)_h/( (h - 1)Q(E, D)_h )\). One then obtains \(CG\)-bimodule maps

\[
Q(E, D) \xrightarrow{\psi} grQ(E, P) \xrightarrow{\rho} grQ(E, D)_{h,1} \xrightarrow{\varphi} Q(E, D),
\]

where \(\psi\) is the natural surjection from above and, by Remark 3.2 in this section, \(\varphi\) is an isomorphism. The map \(\rho\) comes from the \(CG\)-bimodule map that includes \(E\) in \(Q(E, D)_h\) and then projects onto \(Q(E, D)_{h,1}\). This map extends uniquely to an algebra and \(CG\)-bimodule map \(T_{CG}(E) \rightarrow Q(E, D)_{h,1}\), which factors through \(Q(E, P)\). The map \(\rho\) is then the associated graded map of this map \(Q(E, P) \rightarrow Q(E, D)_{h,1}\). Finally, one checks that the composition \(\varphi \circ \rho \circ \psi\) is the identity map on elements of degrees zero and one in \(Q(E, D)\).

Since \(Q(E, D)\) is generated by those elements, the composition is just the identity map which implies that \(Q(E, D) \cong grQ(E, P)\) and \(\text{gr}Q(E, P) \cong \text{gr}Q(E, D)_{h,1}\).

Corollary 3.7. Let \(Q(E, D) \cong S(V) * G\) with \(E\) and \(D\) as before. Suppose \(Q(E, P)\) is a nonhomogeneous quadratic \(CG\)-algebra and a PBW deformation of \(Q(E, D)\). Then the \(CG\)-bimodule \(P\) is given by \(CG\)-bimodule maps \(\alpha : D \mapsto E\), \(\beta : D \mapsto CG\), such that \(P = \{ d - \alpha(d) - \beta(d) : d \in D \}\). Assume that \(\alpha = 0\). Then \(Q(E, P)\) is a graded Hecke algebra.

Proof. See [EG02, p.257] for the first part of this proof. More details are taken from [Gor03] and are given here for the reader’s convenience.

Since \(\beta\) is a right \(CG\)-module map, it is determined by some antisymmetric mapping, say \(\kappa : V \times V \rightarrow CG\). Namely, \(\beta[(v \otimes 1) \otimes_{CG} (w \otimes 1) - (w \otimes 1) \otimes_{CG} (v \otimes 1)] = \kappa(v, w)\). The fact that \(\beta\) is a \(CG\)-bimodule map translates into \(\kappa\) being \(G\)-invariant in the sense that \(\kappa(g(v), g(w)) = g\kappa(v, w)g^{-1}\) for all \(g \in G\). Only condition (ii) of the previous theorem is non-trivial and it reduces to \(0 = id \otimes_{CG} \beta - \beta \otimes_{CG} id\) on \((D \otimes_{CG} E) \cap (E \otimes_{CG} D)\). We can use the isomorphism \(\theta : T_{CG}(E) \rightarrow T(V) * G\) again to identify \(D\) with \(C \otimes CG\), where \(C\) denotes the space of commutators in \(V^\otimes\). Since \(\beta\) is a right \(CG\)-module map, condition (ii) reduces to \(id \otimes \kappa - \kappa \otimes id = 0\) on \((C \otimes V) \cap (V \otimes C)\). In [EG02, p.257] it is proved that this condition implies that, for all \(v, w \in V\), we have \(\kappa(v, w) = a_1(v, w) id + \sum_{s \in S} a_s(v, w)s\),
for some skew-symmetric bilinear forms \( a_1, a_s : V \times V \to \mathbb{C} \). Furthermore, it follows that \( a_1 \in ((\wedge V)^*)^G \) and that \( V^s \subseteq \ker a_s \) for all \( s \in S \).

Now suppose \( s \in S \). Then the form \( a_s \) must be proportional to the skew-symmetric bilinear form \( \Omega_s \) on \( V \), which we constructed in Section 2. Thus assume without loss of generality that \( a_s = \Omega_s \) for all \( s \in S \). Denote a basis of the 2-dimensional subspace \( \text{im}(id - s) \) by \( v_1, v_2 \in V \) and extend this to a basis \( v_1, \ldots, v_n \) of \( V \). Suppose \( g \in Z_G(s) \), then the \( G \)-invariance of \( \kappa \) implies that \( \Omega_s(v, w) = \Omega_s(g(v), g(w)) \) for all \( v, w \in V \). Since \( g(V^s) = V^s \), this reduces to the statement \( \Omega_s(v_1, v_2) = \Omega_s(g(v_1), g(v_2)) \). Write \( g(v_1) = av_1 + cv_2 \) and \( g(v_2) = bv_1 + dv_2 \) for some \( a, b, c, d \in \mathbb{C} \). Then

\[
\Omega_s(g(v_1), g(v_2)) = \Omega(av_1 + cv_2, bv_1 + dv_2) = (ad - bc)\Omega_s(v_1, v_2),
\]

which implies \( ad - bc = 1 \). Hence for all \( g \in Z_G(s) \) we have \( \det(g|_{V/V^s}) = 1 \), so \( s \in S' \). Thus \( \kappa \) must have precisely the form described in Theorem 2.2.

The fact that \( A_{t,c} \) is a PBW deformation of \( Q(E, D) \cong S(V) \ast G = R \) will become crucial in the proof of our main theorem. In Remark 3.6 in this section we observed that there must exist a graded deformation \( (R_h, \ast) \) of \( R \) such that \( R_h/((h-1)R_h) = A_{t,c} \). For \( v, w \in V \subset R \) we have \( v \ast w = vw + \mu_1(v, w) \cdot h + \mu_2(v, w) \cdot h^2 + \ldots \) in \( R_h \), for some \( \mathbb{C}G \)-bimodule maps \( \mu_i : R \times R \to R \) of degree \( -i \). Since \( v \ast w \) has degree 2, we must have \( \mu_i(v, w) = 0 \) for \( i > 2 \), hence \( v \ast w = vw + \mu_1(v, w) \cdot h + \mu_2(v, w) \cdot h^2 \). Moreover, in the factor \( R_h/((h-1)R_h) \) we have \( v \ast w = vw + \mu_1(v, w) + \mu_2(v, w) \). On the other hand, \( R_h/((h-1)R_h) = A_{t,c} \) and, therefore, \( v \ast w = w \ast v + \kappa(v, w) \). We deduce that \( [\mu_1(v, w) - \mu_1(w, v)] + [\mu_2(v, w) - \mu_2(w, v)] = \kappa(v, w) \).

Thus we must have \( \mu_1(v, w) - \mu_1(w, v) = 0 \), because \( \deg \mu_1(v, w) = \deg \mu_1(w, v) = 1 \) and \( \kappa(v, w) \in \mathbb{C}G \) has degree zero. In summary, for all \( v, w \in V \), we have

\[
\mu_2(v, w) - \mu_2(w, v) = \mu_1(v, w) = \left[ \sum_{i=1}^{N} t_i b_i(v, w) \right] \cdot id + \sum_{s \in S'} c_s \Omega_s(v, w)s.
\]

Similarly, one can see that for all \( p, p' \in S(V) \) the difference \( \mu_2(p, p') - \mu_2(p', p) \) depends linearly on the parameters in \( t \) and \( c \).

4. The spherical subalgebra

Let \( A_{t,c} \) denote a graded Hecke algebra as defined in Section 2. Recall that the symmetrizing idempotent in \( \mathbb{C}G \subseteq A_{t,c} \) is given by \( e = \frac{1}{|G|} \sum_{g \in G} g \). The spherical subalgebra of \( A_{t,c} \) is defined as \( eA_{t,c}e \). It is easy to see that the filtration on \( A_{t,c} \) intersects with the spherical subalgebra to induce a filtration \( F^* eA_{t,c}e \) on \( eA_{t,c}e \). We have graded algebra isomorphisms

\[
gr(eA_{t,c}e) = e(gr A_{t,c}e) \cong e(S(V) \ast G)e \cong S(V)^G,
\]
where the inverse of the last isomorphism is given by the map \( p \mapsto pe \) for \( p \in S(V)^G \). Observe that \( S(V)^G = Z(S(V) * G) = Z(A_{0,0}) \).

The space \( A_{t,e} \) has a left \( A_{t,e} \)-module structure and a right \( eA_{t,e} \)-module structure, both given by multiplication. Again the filtration of \( A_{t,e} \) induces a filtration \( F_{A_{t,e}} \) on the module \( A_{t,e} \). We have \( gr(A_{t,e}) \cong S(V) \cong A_{0,0}e \), which can be deduced by using the same isomorphisms as above for \( gr(eA_{t,e}) \).

**Lemma 4.1.**

(i) \( eA_{t,e} \) is a finitely generated \( \mathbb{C} \)-algebra and a noetherian domain.

(ii) \( A_{t,e} \) is finitely generated as right \( eA_{t,e} \)-module.

**Proof.** This follows directly from associated graded techniques as in [MR87, Lemma 7.6.11] and the Hilbert-Noether Theorem, see [Ben93, Theorem 1.3.1]. □

Recall that \( S \) denotes the subgroup of \( G \) generated by the elements of the set \( S' \), as defined in Section 2.

**Lemma 4.2.** Assume that \( G = S \). Then \( A_{t,e} \cong \End_{eA_{t,e}}(A_{t,e}) \) as algebras.

**Proof.** In large parts we use the proof of [EG02, Theorem 1.5 (iv)]. For the reader’s convenience we give the full details.

Left multiplication by elements of \( A_{t,e} \) induces an algebra map \( \eta : A_{t,e} \to \End_{eA_{t,e}}(A_{t,e}) \) by \( a \mapsto (l_a : a'e \mapsto aa') \), for \( a,a' \in A_{t,e} \). This map is in fact filtration preserving, where a filtration on \( \End_{eA_{t,e}}(A_{t,e}) \) is defined as follows. Denote the generators of \( gr(A_{t,e}) \) as \( gr(eA_{t,e}) \)-module by \( \overline{u}_1, \ldots, \overline{u}_n \) and let \( \deg(\overline{u}_i) = d_i \). Then \( A_{t,e} \) is generated as \( eA_{t,e} \)-module by representatives of the \( \overline{u}_1, \ldots, \overline{u}_n \) denoted by \( u_i \in A_{t,e} \), see the proof of [MR87, Lemma 7.6.11]. Now take an element \( f \in \End_{eA_{t,e}}(A_{t,e}) \). We can find \( m \in \mathbb{Z} \) such that \( f(u_i) \in F_{A_{t,e}}^{d_i + m} \) for all \( i = 1, \ldots, n \). Therefore, \( f(F_{A_{t,e}}^j) \subseteq F_{A_{t,e}}^{j + m} \) for all \( j \geq 0 \). Thus we have an increasing \( \mathbb{Z} \)-filtration on \( \End_{eA_{t,e}}(A_{t,e}) \):

\[
F_{\End}^m = \{ f \in \End_{eA_{t,e}}(A_{t,e}) \mid f(F_{A_{t,e}}^m) \subseteq F_{A_{t,e}}^{m + n} \forall n \in \mathbb{Z} \}.
\]

Since \( \eta \) is filtration preserving, we can construct the algebra map \( gr(\eta) \). It now suffices to show that \( gr(\eta) \) is an algebra isomorphism, see [MR87, Corollary 7.6.14]. To this end we consider the composite

\[
gr A_{t,e} \xrightarrow{gr(\eta)} gr\left(\End_{eA_{t,e}}(A_{t,e})\right) \xrightarrow{j} \End_{gr(eA_{t,e})}(gr(A_{t,e}))
\]

where the map \( j \) is given by \( f + F_{\End}^{m-1} \mapsto [ae + F_{A_{t,e}}^{k-1} \mapsto f(ae) + F_{A_{t,e}}^{k+m-1}] \), for \( a \in F_k \). The map \( j \) is clearly injective and we have reduced the problem to showing that the composite
on a maximal order in its quotient field if and only if $G$ divides $j$. Let us show that $\varphi$ is an algebra isomorphism. We tensor on the left with the quotient field of $S(V)^G$, $Q(S(V)^G)$:

$$Q(S(V)^G) \otimes_{S(V)^G} (S(V) \star G) \xrightarrow{id \otimes (j \circ gr(\eta))} Q(S(V)^G) \otimes_{S(V)^G} \text{End}_{S(V)^G}(S(V)).$$

Let us show that $\varphi := id \otimes (j \circ gr(\eta))$ is an algebra isomorphism. We have the following isomorphisms as $S(V)^G$-modules, see [Eis95, Lemma 2.4, Proposition 2.10], which imply algebra isomorphism:

$$Q(S(V)^G) \otimes_{S(V)^G} (S(V) \star G) \cong \left[ Q(S(V)^G) \otimes_{S(V)^G} S(V) \right] \star G \cong Q(S(V)) \star G,$$

$$Q(S(V)^G) \otimes_{S(V)^G} \text{End}_{S(V)^G}(S(V)) \cong \text{End}_{Q(S(V)^G)}[Q(S(V)^G) \otimes_{S(V)^G} S(V)] \cong \text{End}_{Q(S(V)^G)}[Q(S(V))],$$

where $Q(S(V)^G) \otimes_{S(V)^G} S(V) \cong Q(S(V))$, because $S(V)$ is a finitely generated $S(V)^G$-module. The map $\varphi$ is given by $\sum_{g \in G} p_g g \mapsto [x \mapsto \sum_{g \in G} p_g \cdot g(x)]$, for $p_g \in Q(S(V))$. First observe that $\varphi$ is clearly not the zero map. Then note that $Q(S(V)) \star G$ is a simple ring, since $Q(S(V))$ is a simple ring and $G$ acts faithfully on $Q(S(V))$, see [MR87, Proposition 7.8.12]. Thus $\ker \varphi = 0$. Now count the dimensions of the $Q(S(V)^G)$-vector spaces on each side of the map $\varphi$. We have $\dim_{Q(S(V)^G)}[Q(S(V)) \star G] = |G|^2 = \dim_{Q(S(V)^G)}[\text{End}_{Q(S(V)^G)}[Q(S(V))]]$, since $Q(S(V))$ is a Galois extension of $Q(S(V)^G)$ and $[Q(S(V)) : Q(S(V)^G)] = |G|$.

The fact that $\varphi = id \otimes (j \circ gr(\eta))$ is an isomorphism now implies that $j \circ gr(\eta)$ is injective, because of the following commutative diagram:

$$\begin{array}{ccc}
Q(S(V)) \star G & \xrightarrow{\varphi} & \text{End}_{Q(S(V)^G)}[Q(S(V))] \\
\uparrow & & \uparrow \\
S(V) \star G & \xrightarrow{j \circ gr(\eta)} & \text{End}_{S(V)^G}(S(V))
\end{array}$$

where the vertical map on the left is an embedding, since the elements of $S(V)$ are nonzero divisors of $S(V) \star G$.

It remains to show that $j \circ gr(\eta)$ is surjective, hence that im$(j \circ gr(\eta)) = \text{End}_{S(V)^G}(S(V))$. To do this we will use the concept of a maximal order. By [Mar95, Theorem 4.6], $S(V) \star G$ is a maximal order in its quotient field if and only if $G$ does not contain reflections in its action on $S(V)$. But we have assumed that $G = S \subseteq SL(V)$ and so $S$ does not contain reflections.
Moreover, the map \( \varphi \) shows that the quotient rings of \( S(V) \ast G \) and \( \text{End}_{S(V)G}(S(V)) \) are isomorphic. Note that the quotient ring of \( \text{End}_{S(V)G}(S(V)) \) is indeed \( \text{End}_{Q(S(V))G}[Q(S(V))] \).

Now we use the commutative diagram from above again. As \( S(V) \ast G \) is a maximal order in its quotient ring \( Q(S(V)) \ast G \) and \( j \circ \text{gr}(\eta) \) is injective, \( \text{im}(j \circ \text{gr}(\eta)) \) is also a maximal order in the quotient ring \( \text{End}_{Q(S(V))G}[Q(S(V))] \). But \( \text{End}_{S(V)G}(S(V)) \supseteq S(V) \ast G \) via the embedding \( j \circ \text{gr}(\eta) \) and \( \text{End}_{S(V)G}(S(V)) \) is finitely generated over \( S(V) \ast G \), since it is finitely generated over \( S(V)G \). Thus \( \text{End}_{S(V)G}(S(V)) \) is an order in its quotient ring \( \text{End}_{Q(S(V))G}[Q(S(V))] \) equivalent to the maximal order \( \text{im}(j \circ \text{gr}(\eta)) \). Now the maximality of \( \text{im}(j \circ \text{gr}(\eta)) \) implies that \( \text{im}(j \circ \text{gr}(\eta)) = \text{End}_{S(V)G}(S(V)) \). \( \square \)

**Proposition 4.3.** Assume that \( G = S \). Then \( Z(eA_{t,e}) \cong Z(A_{t,e}) \) as \( \mathbb{C} \)-algebras.

**Proof.** We adapt the proof in [EG02, Theorem 3.1] very slightly and mention it for completeness.

Define a \( \mathbb{C} \)-algebra map \( \psi : Z(A_{t,e}) \to Z(eA_{t,e}) \) by \( z \mapsto ze = eze \) for \( z \in Z(A_{t,e}) \). We want to construct an inverse algebra map to \( \psi \) denoted by \( \varphi : Z(eA_{t,e}) \to Z(A_{t,e}) \). Say \( eae \in Z(eA_{t,e}) \) and let \( r_{eae} \) be right multiplication by \( eae \). Then \( r_{eae} \) is an element of \( \text{End}_{eA_{t,e}e}(A_{t,e}) \). By the isomorphism \( \eta : A_{t,e} \to \text{End}_{eA_{t,e}e}(A_{t,e}) \) of the previous lemma we have \( r_{eae} = \eta(x(a)) = l_{x(a)} \) for some \( x(a) \in A_{t,e} \), where \( l_{x(a)} \) denotes left multiplication by \( x(a) \). Since left multiplication commutes with right multiplication in \( \text{End}_{eA_{t,e}e}(A_{t,e}) \), \( r_{eae} = l_{x(a)} \) is central in \( \text{End}_{eA_{t,e}e}(A_{t,e}) \). Now the isomorphism \( \eta \) implies that \( x(a) \in Z(A_{t,e}) \). Thus define \( \varphi : eae \mapsto x(a) \). This is an algebra map because the isomorphism \( \eta \) is an algebra map. It remains to show that \( \psi \) and \( \varphi \) are inverse to each other. We have \( \varphi \circ \psi : z \mapsto eze \mapsto x(z) \). As \( z \) is central, we have \( r_{eze} = rz \) and \( rz = l_z \). This implies that \( l_{x(z)} = l_z \), that is \( \eta(x(z)) = \eta(z) \), which implies \( x(z) = z \), because \( \eta \) is an isomorphism. On the other hand \( \psi \circ \varphi : eae \mapsto x(a) \mapsto ex(a)e \). For all \( y \) in \( Z(eA_{t,e}) \) we have \( l_{x(a)}(y \cdot e) = r_{eae}(y \cdot e) \). But \( l_{x(a)}(y \cdot e) = x(a) \cdot y \cdot e = y \cdot x(a) \cdot e = y \cdot ex(a)e \), because \( x(a) \in Z(A_{t,e}) \), and \( r_{eae}(y \cdot e) = y \cdot eae \cdot e = y \cdot eae \). Thus \( y \cdot ex(a)e - y \cdot eae = 0 \) and \( y[ex(a)e - eae] = 0 \). Since \( eA_{t,e}e \) does not contain zero divisors by Lemma 4.1, this implies \( ex(a)e = eae \) as required. \( \square \)

### 5. Preliminary results

Recall from Section 2 that \( A_{t,e}(S) \) is defined as the subalgebra of \( A_{t,e} \) constructed with the subgroup \( S \) of \( G \). We can reduce to the case \( G = S \) without loss of generality for our main theorem because of the following result:
Lemma 5.1. If $A_{t,c}(S)$ is a finitely generated module over its centre $Z(A_{t,c}(S))$, then $A_{t,c}$ is a finitely generated module over its centre and a PI algebra.

Proof. In Proposition 2.4 we saw that $A_{t,c}$ is a finitely generated module over its centre and a PI algebra. It now suffices to show that $A_{t,c}(S)$ is finitely generated over $Z(A_{t,c}(S))$. By the initial assumption it only remains to show that $Z(A_{t,c}(S))$ is finitely generated over $[Z(A_{t,c}(S))]^{G/S}$. We have $C \subseteq Z(A_{t,c}(S)) \subseteq A_{t,c}(S)$, and $A_{t,c}(S)$ is an affine $C$-algebra, which is a finite $Z(A_{t,c}(S))$-module. Thus the Artin-Tate lemma, see [MR87, Lemma 13.9.11], implies that $Z(A_{t,c}(S))$ is an affine $C$-algebra as well. But $G/S$ acts as a group of automorphisms on $Z(A_{t,c}(S))$ and we can use the Hilbert-Noether theorem, see [Ben93, Theorem 1.3.1], to deduce that $Z(A_{t,c}(S))$ is a finite $[Z(A_{t,c}(S))]^{G/S}$-module. Now $A_{t,c}$ is finitely generated over a commutative subalgebra and hence a PI algebra by [MR87, Corollary 13.1.13].

Conversely, if $A_{t,c}$ is a PI algebra then its subalgebra $A_{t,c}(S)$ is also a PI algebra, see [MR87, Lemma 13.1.7]. In general, this does not imply that $A_{t,c}(S)$ is finitely generated over its centre $Z(A_{t,c}(S))$. However, we will derive this implication as a consequence of our main theorem in the last section.

Throughout the remainder of this section we assume that $G = S$.

Recall that a Poisson bracket on a commutative $C$-algebra, say $S(V)^G$, is a bilinear map $\{-,-\} : S(V)^G \times S(V)^G \to S(V)^G$ such that $S(V)^G$ is a Lie algebra under the bracket $\{-,-\}$ and the Leibniz identity holds. In particular, $\{-,-\}$ satisfies the Jacobi identity. Moreover, a Poisson bracket on $S(V)^G$ can be identified with an element of $\text{Hom}_{S(V)^G}(\bigwedge^2 D_{S(V)^G/C}, S(V)^G)$, where $D_{S(V)^G/C}$ denotes the module of Kähler differentials of $S(V)^G$ over $C$. The module $D_{S(V)^G/C}$ is an $S(V)^G$-module and the generators of $D_{S(V)^G/C}$ are denoted by $dp$ for $p \in S(V)^G$. The identification of a bracket $\{-,-\}$ with $\alpha \in \text{Hom}_{S(V)^G}(\bigwedge^2 D_{S(V)^G/C}, S(V)^G)$ is as follows: for $p, p' \in S(V)^G$, $\{p, p'\} \mapsto (\alpha : dp \wedge dp' \mapsto \{p, p'\})$. The Jacobi identity on $\{-,-\}$ imposes a relation on the map $\alpha$.

The algebra $S(V)^G$ is graded using the usual grading on $S(V)$. Denote the $i$th graded part of $S(V)^G$ by $S^i(V)^G$ and observe that $S^i(V)^G = 0$ for $i < 0$. A Poisson bracket $\{-,-\}$ on $S(V)^G$ is said to have degree $d$ if $\{-,-\} : S^i(V)^G \times S^j(V)^G \to S^{i+j+d}(V)^G$. Note that each element $\omega$ of $((\bigwedge^2 V)^*)^G$ induces a Poisson bracket on $S(V)$ by extending $\omega$ linearly and using the Leibniz rule. Let us denote this bracket by $\{-,-\}_\omega$. The fact that $\omega$ is $G$-invariant
forces \{-, -\}_\omega to be a $G$-invariant bracket on $S(V)$ as well. Thus \{-, -\}_\omega restricts to a Poisson bracket on $S(V)^G$. Furthermore, the bracket \{-, -\}_\omega on $S(V)^G$ has degree $-2$.

**Lemma 5.2.** Any Poisson bracket on $S(V)^G$ of degree $-2$ is induced by an element of $((\Lambda^2 V^*)^G)$. Any Poisson bracket of degree less than $-2$ is zero.

*Proof.* We proceed as in the proof of [EG02, Lemma 2.23], but do not assume that the vector space $V$ comes equipped with a symplectic form. Full details are given for the convenience of the reader.

Let \{-, -\} denote a Poisson bracket on $S(V)^G = \mathcal{O}(V^*/G)$ of degree $d$. In the proof of this lemma we will extend the bracket \{-, -\} on $S(V)^G$ to a $G$-invariant Poisson bracket of degree $d$ on $S(V) = \mathcal{O}(V^*)$, denoted by \{\{-, -\}\}. We are then able to prove that such a bracket on $S(V)$ is zero for $d < -2$ and that it has to be induced by an element of $((\Lambda^2 V^*)^G)$ for $d = -2$.

In order to construct the bracket \{\{-, -\}\}, we first pick a smooth open subset of $V^*/G$ as follows. Let $Y$ be the set of points in $V^*$ that are fixed by some nontrivial element of $G$, so $Y = \cup_{g \in G, g \neq 1} (V^*)^g$. Note that $(V^*)^g$ is the zero set of the ideal $I_g < S(V)$ given by $I_g = \langle gv - v : v \in V \rangle$. Since the action of $G$ on $V$ is faithful, $I_g \neq 0$. Hence $(V^*)^g$ is a proper closed subset of $V^*$ for all $g \in G$. Then $Y$ is the zero set of $I := \cap_{g \in G, g \neq 1} I_g$ and a proper closed subset of $V^*$. Therefore, the open set $X := V^* \setminus Y$ is a quasi-affine variety. Furthermore, the action of $G$ on $X$ is free in a set-theoretic sense, that is for all $x \in X$ the stabiliser of $x$ in $G$, denoted by $G_x$, is trivial. Now [Dré04, Proposition 4.12] says that the quotient map $\pi : V^* \to V^*/G$ is étale at $x \in V^*$ if and only if $G_x$ is trivial. A consequence of $\pi$ being étale at all $x \in X$ is an isomorphism between the completions of the local rings $\mathcal{O}(V^*/G)_{\pi(x)}$ and $\mathcal{O}(V^*)_x$, that is $\mathcal{O}(V^*/G)_{\pi(x)} \cong \widehat{\mathcal{O}(V^*)_x}$, for all $x \in X$, see [Dré04, Proposition 4.2]. Since $V^*$ is a smooth variety, the local ring $\mathcal{O}(V^*)_x$ is regular for all $x \in V^*$, which implies that $\widehat{\mathcal{O}(V^*)_x}$ is regular for all $x \in V^*$, see [Har77, Theorem I.5.1, Theorem I.5.4A]. Thus, using the same results, we deduce that $\mathcal{O}(V^*/G)_{\pi(x)}$ is regular for all $\pi(x) \in V^*/G$ such that $x \in X$. But $\pi|_X : X \to X/G$ is surjective, hence $X/G$ is a smooth variety.

Now take the given Poisson bracket \{-, -\} on $S(V)^G$ of degree $d$. For any open subset $U$ of $V^*/G$, \{-, -\} defines a map $\mathcal{O}(U) \times \mathcal{O}(U) \to \mathcal{O}(U)$. Since the quotient map $\pi$ is closed, it takes the open subset $X \subseteq V^*$ to an open subset $X/G$ of $V^*/G$. Hence the bracket \{-, -\} restricts to a Poisson bracket of degree $d$ on the sheaf of regular functions $\mathcal{O}_{X/G}$ of the smooth variety $X/G$. We now observe that we can lift this bracket on $\mathcal{O}_{X/G}$ to a $G$-invariant Poisson bracket of degree $d$ on $\mathcal{O}_X$. The reason for this is that the action of
$G$ on $X$ is free in a set-theoretic sense, so the quotient map $\pi|_X : X \to X/G$ is not only étale but also a Galois cover, see [Mil98, Definition 6.1]. We saw that a Poisson bracket $\{-,-\}$ on $\mathcal{O}_{X/G}$ can be identified with an element of $\text{Hom}_{\mathcal{O}_{X/G}}(\bigwedge^2 D_{\mathcal{O}_{X/G}/\mathbb{C}}, \mathcal{O}_{X/G})$, which is the set of global sections of the second exterior power of the tangent sheaf on $X/G$, see [Har77, Definition, p.180] for the definition of a tangent sheaf. Now the theory on étale sheaves and Galois coverings, as outlined in [Mil98, Section 6], allows one to identify $\text{Hom}_{\mathcal{O}_{X/G}}(\bigwedge^2 D_{\mathcal{O}_{X/G}/\mathbb{C}}, \mathcal{O}_{X/G})$ with $\left[\text{Hom}_{\mathcal{O}_X}(\bigwedge^2 D_{\mathcal{O}_X/\mathbb{C}}, \mathcal{O}_X)\right]^G$. Let us denote the resulting $G$-invariant Poisson bracket of degree $d$ on $\mathcal{O}_X$ by $\{-,-\}_X$.

Recall that $X = V^* \setminus Y$. The next step is to extend the bracket $\{-,-\}_X$ on $\mathcal{O}_X$ to a $G$-invariant Poisson bracket on $\mathcal{O}(V^*) = S(V)$. Since the group $G = S \subseteq SL(V)$ does not contain reflections, each non-identity element in $G$ has at least two eigenvalues different from 1. This implies that the codimension of $V^g$ is at least 2 for all $g \in G$, which translates into the corresponding ideal $I_g$ having height at least 2. Hence the height of $I$ and the codimension of $Y$ in $V$ is at least 2 as well, see [Kum85, Section II.1.3]. This enables us to apply [FSR05, Theorem 1.5.14] to extend a regular element $x \in \mathcal{O}(X) = \mathcal{O}_X(X)$ to a regular element $\tilde{x}$ in $\mathcal{O}(V^*)$ such that $\tilde{x}|_X = x$. Note that this is well-defined: say $x = x' \in \mathcal{O}(X)$. Then we must have $\tilde{x} = \tilde{x}' \in \mathcal{O}(V^*)$, because $\tilde{x}$ and $\tilde{x}'$ agree on the non-empty and therefore dense open subset $X$ of $V^*$. Furthermore, the map $\mathcal{O}(V^*) \to \mathcal{O}(X)$ given by restriction is a surjection. Thus we construct a Poisson bracket on $\mathcal{O}(V^*) = S(V)$ denoted by $\{-,-\}$ as follows: for $\tilde{x}, \tilde{x}' \in \mathcal{O}(V^*)$, define $\langle\{\tilde{x}, \tilde{x}'\}\rangle := \langle\{x, x'\}\rangle_X$. Since the bracket $\{-,-\}_X$ is $G$-invariant and of degree $d$, the new Poisson bracket $\{-,-\}$ is $G$-invariant and of degree $d$ as well.

The bracket $\{-,-\}$ on $S(V)$ corresponds to a $G$-invariant element of degree $d$ in $\text{Hom}_{S(V)}(\bigwedge^2 D_{S(V)/\mathbb{C}}, S(V))$. On the other hand we have, $\bigwedge^2 \text{Hom}_{S(V)}(D_{S(V)/\mathbb{C}}, S(V)) \cong \text{Hom}_{S(V)}(\bigwedge^2 D_{S(V)/\mathbb{C}}, S(V))$. Furthermore, $\text{Hom}_{S(V)}(D_{S(V)/\mathbb{C}}, S(V)) \cong \text{Der}_C(S(V))$, the latter being the algebra of $C$-derivations on $S(V)$, see [MRS7, Proposition 15.1.10]. In summary, $\text{Hom}_{S(V)}(\bigwedge^2 D_{S(V)/\mathbb{C}}, S(V)) \cong \bigwedge^2 \text{Der}_C(S(V))$. It is now easy to see that an element of degree $d < -2$ in $\bigwedge^2 \text{Der}_C(S(V))$ is zero. Hence if the degree of $\{-,-\}$ is less than $-2$, then the bracket $\{-,-\}$ is zero. Furthermore, an element of degree $-2$ in $\bigwedge^2 \text{Der}_C(S(V))$ must be an element of $\bigwedge^2 V^*$. If we assume in addition that this element is $G$-invariant, then it must be an element of $(\bigwedge^2 V^*)^G \cong ((\bigwedge^2 V^*)^G)^G$.

\begin{lemma}
Suppose the parameters in $t$ and $c$ are such that $eA_{t,c}e$ is commutative. Let $	ext{MaxSpec}(eA_{t,c}e)$ denote the set of maximal ideals of $eA_{t,c}e$. Then there exists a non-empty Zariski-open subset $\mathcal{M}$ of $\text{MaxSpec}(eA_{t,c}e)$ such that, if $\mathfrak{m} \in \mathcal{M}$ and if we let $T_{\mathfrak{m}} := A_{t,c}e \otimes_{eA_{t,c}e} (eA_{t,c}e/\mathfrak{m})$ denote the corresponding induced $A_{t,c}$-module, then $T_{\mathfrak{m}} \cong CG$ as $G$-module.
\end{lemma}
Theorem 2.2. We continue to assume for now that $G$ is determined by the values chosen for the parameters \{\(t_i | i = 1, \ldots, N\)\} and \{\(c_s | s \in S'\)\}, see Theorem 6.1. We continue to assume for now that $G$ is generated by the elements in $S'$, hence $G = S$.

Remark 6.1. It is probably possible to obtain the following result for all finite groups $G \subseteq GL(V)$, that is to drop the assumption $G = S$. However, it is not trivial to prove this and we do not need this version for our purposes.

Theorem 6.2. Assume $G = S$. Then $eA_{t,c}e$ is commutative if and only if $t_i = 0$ for all $i = 1, \ldots, N$.

Proof. The proof is the same as the one for [EG02, Lemma 2.24].

6. Proof of the main theorem

Let $A_{t,c}$ be a graded Hecke algebra as defined in Section 2 and recall that $A_{t,c}$ is completely determined by the values chosen for the parameters $\{t_i | i = 1, \ldots, N\}$ and $\{c_s | s \in S'\}$, see Theorem 6.2. We continue to assume for now that $G$ is generated by the elements in $S'$, hence $G = S$.

Remark 6.1. It is probably possible to obtain the following result for all finite groups $G \subseteq GL(V)$, that is to drop the assumption $G = S$. However, it is not trivial to prove this and we do not need this version for our purposes.

Theorem 6.2. Assume $G = S$. Then $eA_{t,c}e$ is commutative if and only if $t_i = 0$ for all $i = 1, \ldots, N$.

Proof. The proof is the same as the one for [EG02, Lemma 2.24].

Since $A_{t,c}$ is a PBW deformation of $R = S(V) \ast G$, as seen in Corollary 6.7 there exists a graded deformation $(R_h, \ast)$ of $R$ such that $R_h/(h - 1)R_h \cong A_{t,c}$. In order to describe such a deformation $R_h$ explicitly we introduce the auxiliary variable $h$ and set $T(V)[h] := T(V) \otimes \mathbb{C}[h]$. Let the degree of $h$ be 1 and assume that the group $G$ acts trivially on $h$. Define

$$R_h := (T(V)[h] \ast G)/(\langle [v, w] - \kappa(v, w)h^2 : v, w \in V \rangle).$$

The algebra $R_h$ is indeed a graded deformation of $R$. Namely, since the relation $[v, w] = \kappa(v, w)h^2$ is now homogeneous, $R_h$ is an associative unital graded algebra. It is easy to see that $R_h/hR_h = S(V) \ast G = R$ and that $R_h/(h - 1)R_h = A_{t,c}$. If we pick a vector space basis $v_1, \ldots, v_n$ of $V$, we obtain a vector space basis of $S(V)$ consisting of ordered monomials in the $v_i$. Now we can think of $p \in S(V)$ as an element of $T(V)$. We can then use the projection $T(V) \otimes \mathbb{C}G \rightarrow R_h$ to obtain an epimorphism of $\mathbb{C}$-vector spaces $\pi: (S(V) \otimes \mathbb{C}G)[h] \rightarrow R_h$ given by $\sum_{i=0}^m p_i h^i \mapsto \sum_{i=0}^m p_i h^i$, where $p_i \in S(V) \otimes \mathbb{C}G$, that is $p_i = \sum_{g \in G} p_{i,g} g$ and each $p_{i,g}$ is a linear combination of ordered monomials in the $v_i$. We want to show that $\pi$ is an isomorphism, hence that the underlying vector space of $R_h$ is $R \otimes \mathbb{C}[h]$. Thus we need to prove that $\pi(\sum_{i=0}^m p_i h^i) = 0$ implies $p_i = 0$ for all $i = 0, \ldots, m$. The map $\pi$ is a homogeneous map of degree zero. Hence we can assume without loss of generality that $\sum_{i=0}^m p_i h^i$ is a homogeneous element of degree $k$. So $p_i \in S(V) \otimes \mathbb{C}G$ has degree $k - i$. Denote the projection $R_h \rightarrow R_h/(h - 1)R_h = A_{t,c}$ by $\varrho$. If $\pi(\sum_{i=0}^m p_i h^i) = 0$,
then \( \rho(\pi(\sum_{i=0}^{m} p_i h^i)) = \sum_{i=0}^{m} p_i + (h-1)Rh = 0 \). This holds if and only if \( \sum_{i=0}^{m} p_i \in (h-1)R_h \) which is the case if and only if \( \sum_{i=0}^{m} p_i = 0 \in R_h \). But the elements \( p_i \in R_h \) have distinct degrees, which means that we must have \( p_i = 0 \) for all \( i = 0, \ldots, m \) as required.

The multiplication \( \ast \) in \( R_h \) is given by the multiplication in \( T(V) \ast G \) and the additional relations \( v \ast w - w \ast v = \kappa(v, w)h^2 \) for all \( v, w \in V \), which are extended by \( \mathbb{C}[h] \)-linearity. We have seen during the proof of Corollary \( \mathcal{S} \) that \( \kappa(-, -) \) is a \( \mathbb{C}G \)-bimodule map, which makes \( \ast \) into a \( \mathbb{C}G[h] \)-bimodule map. Thus \( R_h \) is indeed a graded \( \mathbb{C}G[h] \)-bimodule. We can express multiplication in \( R_h \) in terms of \( \mathbb{C}G \)-bimodule maps \( \mu_i : R \times R \to R \) of degree \( -i \). At the end of Section \( \mathcal{S} \) we observed that \( \kappa(v, w) = \mu_2(v, w) - \mu_2(w, v) \) and \( \mu_2(v, w) - \mu_1(v, w) = 0, \) for all \( v, w \in V \). Furthermore, \( p \ast p' - p' \ast p = \mu_2(p, p')h^2 - \mu_2(p', p)h^2 \) for all \( p, p' \in S(V) \).

Let us form the spherical subalgebra \( eR_h e \) of \( R_h \). Clearly, \( (eR_h e, \ast) \) is a graded deformation of \( e(S(V) \ast G)e \cong S(V)^G \), that is \( eR_h e/heR_h e = e(S(V) \ast G)e \). This is because we chose the maps \( \mu_i \) to be \( \mathbb{C}G \)-invariant. As vector spaces, \( eR_h e \cong S(V)^G[h] \). Also, \( eR_h e/(h-1)eR_h e = eA_{t,c}e \). Given \( p, p' \in S(V)^G \cong eR_h e/heR_h e \) let \( \tilde{p}, \tilde{p}' \) denote lifts of these elements to \( eR_h e \). We define a Poisson bracket \( \{-,-\} \) on \( S(V)^G \) by \( \{p, p'\} := h^{-2}(\tilde{p} \ast \tilde{p}' - \tilde{p}' \ast \tilde{p}) \mod (heR_h e) \). It is easy to check that this indeed defines a Poisson bracket and that \( \{p, p'\} = \mu_2(p, p') - \mu_2(p', p), \) for all \( p, p' \in S(V)^G \). We claim that

\[
(1) \quad eA_{t,c}e \text{ commutative } \iff eR_h e \text{ commutative } \iff \{-,-\} \equiv 0
\]

Let us first show the equivalence on the right hand side. From the last description of the Poisson bracket it becomes obvious that, if \( eR_h e \) is commutative, then \( \{-,-\} \equiv 0 \). Conversely, if \( \{p, p'\} = 0 \) for all \( p, p' \in S(V)^G \), then \( \mu_2(p, p') = \mu_2(p', p) \) for all \( p, p' \in S(V)^G \). Since \( eR_h e \) is a deformation of \( S(V)^G \), the multiplication \( \ast \) in \( eR_h e \) is determined by the multiplication \( S(V)^G \ast S(V)^G \subset eR_h e \ast eR_h e \) and then extended by \( \mathbb{C}[h] \)-linearity. But we now have, for all \( p, p' \in S(V)^G \), \( p \ast p' - p' \ast p = [\mu_2(p, p') - \mu_2(p', p)]h^2 = 0 \). Hence \( eR_h e \) is commutative. For the equivalence on the left hand side we observe that, if \( eR_h e \) is commutative, the factor algebra \( eR_h e/(h-1)eR_h e = eA_{t,c}e \) is certainly also commutative. Conversely, assume \( eR_h e/(h-1)eR_h e \) is commutative, but \( eR_h e \) is not. Then, by the above, the Poisson bracket is nonzero and so there exist \( p, p' \in eR_h e/heR_h e \) such that \( \{p, p'\} = \neq 0 \). Choose representatives \( \tilde{p}, \tilde{p}' \in eR_h e \) of \( p, p' \). We can assume without loss of generality that \( \tilde{p}, \tilde{p}' \) are homogeneous elements of \( eR_h e \). Then \( h^{-2}(\tilde{p} \ast \tilde{p}' - \tilde{p}' \ast \tilde{p}) = \tilde{f} \) such that \( \tilde{f} \equiv \tilde{f} \mod (heR_h e) \) and \( \tilde{f} \) is a nonzero homogeneous element of \( eR_h e \). Now consider \( [\tilde{p}, \tilde{p}'] \) mod \( ((h-1)eR_h e) \). Since \( eR_h e/(h-1)eR_h e = eA_{t,c}e \) is assumed to be commutative, \( [\tilde{p}, \tilde{p}'] \equiv 0 \mod ((h-1)eR_h e) \). But \( [\tilde{p}, \tilde{p}'] = h^2\tilde{f} \), i.e \( [\tilde{p}, \tilde{p}'] \equiv \tilde{f} \mod ((h-1)eR_h e) \). Thus
\( \tilde{f} \in ((h - 1)eR_h e) \), which means that \( \tilde{f} \) is divisible by \( (h - 1) \). We conclude that \( \tilde{f} \) is not homomorphic, a contradiction.

It now remains to prove that the Poisson bracket \( \{-, -\} \) on \( S(V)^G \) vanishes if and only if \( t_i = 0 \) for all \( i = 1, \ldots, N \). Since the degree of the map \( \mu_2 \) is \(-2\), the degree of the Poisson bracket is also \(-2\). Hence Lemma 5.2 implies that the bracket is induced by some element \( \omega \) of \( ((\wedge^2 V)^*)^G \). In terms of the basis \( \{b_1, \ldots, b_N\} \) of \( ((\wedge^2 V)^*)^G \) write \( \omega = \sum_{i=1}^N \lambda_i b_i \), for some \( \lambda_i \in \mathbb{C} \). Let \( \{-, -\}_i \) denote the Poisson bracket induced by \( b_i \). From the explanations preceding Lemma 5.2 it is easy to see that we must have \( \{-, -\} = \sum_{i=1}^N \lambda_i \{-, -\}_i \). Furthermore, at the end of Section 3 we observed that the difference \( \mu_2(p, p') - \mu_2(p', p) \) depends linearly on the parameters \( t \) and \( c \) for all \( p, p' \in S(V)^G \). Thus the Poisson bracket \( \{-, -\} \) depends linearly on the parameters \( t \) and \( c \), and so do the scalars \( \lambda_i \). Let \( f_i : \mathbb{C}^N \times \mathbb{C}^{[S]} \to \mathbb{C} \) denote linear functions and write \( \{-, -\} = \sum_{i=1}^N f_i(t, c) \{-, -\}_i \). Now the Poisson bracket vanishes if and only if \( f_i(t, c) = 0 \) for all \( i = 1, \ldots, N \), since the brackets \( \{-, -\}_i \) are linearly independent by the linear independency of the basis elements \( b_i \in (\wedge^2 V)^* \). We need to show that this is the case if and only if \( t_i = 0 \) for all \( i = 1, \ldots, N \).

The equations \( f_i(t, c) = 0, i = 1, \ldots, N \), form a system of homogeneous linear equations of rank \( r \leq N \). Thus the solution space \( V(f_i) \subset \mathbb{C}^N \oplus \mathbb{C}^{[S]} \) of these equations has dimension \( (N + |S|) - r \geq (N + |S|) - N = |S| \). On the other hand, the system of linear equations given by \( t_i = 0, i = 1, \ldots, N \), has rank \( N \) and, therefore, its solution space, \( V(t_i) \), is \(|S|\)-dimensional. Thus \( \dim(V(f_i)) \geq \dim(V(t_i)) \). We will show that \( V(f_i) \subsetneq V(t_i) \) which implies the result by containment and equality of dimensions.

To show that \( V(f_i) \subsetneq V(t_i) \) we assume that the parameters \( t, c \) are such that \( f_i(t, c) = 0 \) for all \( i = 1, \ldots, N \). Then the Poisson bracket on \( S(V)^G \) vanishes and \( eA_{t, c} e \) is commutative. We can now use Lemma 5.3

**Corollary 6.3.**

(i) \( eA_{0, c} e \cong Z(A_{0, c}) \) as \( \mathbb{C} \)-algebras.

(ii) \( grZ(A_{0, c}) \cong S(V)^G \).

(iii) \( A_{0, c} \) is a finitely generated module over \( Z(A_{0, c}) \) and \( A_{0, c} \) is a PI-algebra.
Proof. (i) Follows from Proposition \[13\] and the previous theorem.

(ii) In Proposition \[4.3\] we found an isomorphism $\psi : Z(A_{0,e}) \to Z(eA_{0,e}) = eA_{0,e}$ given by $z \mapsto ze$, for $z \in Z(A_{0,e})$. The map $\psi$ is filtration preserving since $e \in F^0$. Thus we have $\psi(F^i_{Z(A_{0,e})}) \subseteq \psi(Z(A_{0,e})) \cap F^i_{eA_{0,e}}$ for all $i \geq 0$. But if $z \in Z(A_{0,e})$ and $\psi(z) = ze \in F^i_{eA_{0,e}}$, then we can easily see that $z \in F^i_{A_{0,e}} \cap Z(A_{0,e})$, because $F^i_{eA_{0,e}} = eF^i_{A_{0,e}}$. Now the surjectivity of $\psi$ implies that $\psi(F^i_{Z(A_{0,e})}) = F^i_{eA_{0,e}}$ for all $i \geq 0$. Hence $\psi$ is a strict map, see [MR87, 7.6.12]. Then [MR87] Corollary 7.6.14 implies that the induced map $grZ(A_{0,e}) \mapsto gr(eA_{0,e})$ is bijective. But $gr(eA_{0,e}) \cong S(V)^G$ as we saw at the beginning of Section \[4\]

(iii) It is enough to show that $grA_{0,e}$ is finitely generated over $grZ(A_{0,e})$, because we can then use associated graded arguments as in Lemma \[4.1\]. Denote the isomorphism $\gamma : S(V) \ast G \to grA_{0,e}$. Since $S(V) \ast G$ is finitely generated over $S(V)^G = Z(S(V) \ast G)$, $\gamma(S(V) \ast G) = grA_{0,e}$ is finitely generated over $\gamma(Z(S(V) \ast G)) = Z(\gamma(S(V) \ast G)) = Z(grA_{0,e})$. Thus it remains to prove that $grZ(A_{0,e}) = Z(grA_{0,e})$. We have the following maps: $grZ(A_{0,e}) \to gr(eA_{0,e}) = e(grA_{0,e})$ given by $z \mapsto ze$ for all $z \in grZ(A_{0,e})$ as seen in Part (ii) of this corollary. And a map $S(V)^G \to e(grA_{0,e})$ given by $p \mapsto \gamma(p)e$. Both of these maps are isomorphisms as observed in Part (ii) of this corollary and at the beginning of Section \[4\]. Thus for each $\gamma(p)e$ there exists a unique $z \in grZ(A_{0,e})$ such that $ze = \gamma(p)e$. Since $\gamma(S(V)^G) = Z(grA_{0,e})$, we can now define a map $grZ(A_{0,e}) \to Z(grA_{0,e})$ by $z \mapsto \gamma(p)$. It is obvious that this map is bijective. Now $A_{0,e}$ is finitely generated over a commutative subalgebra and hence a PI algebra by [MR87] Corollary 13.1.13].

Corollary 6.4. Let $A_{t,e}$ be a graded Hecke algebra. Assume $G = S$. Then $A_{t,e}$ is a PI algebra if and only if $A_{t,e}$ is a finitely generated module over its centre if and only if $t_i = 0$ for all $i \in \{1, \ldots, N\}$.

Proof. From Theorem 6.2 and the subsequent corollary we know

$A_{t,e}$ is a PI algebra $\iff A_{t,e}$ is a finite $Z(A_{t,e})$-module $\iff t_i = 0 \forall i \in \{1, \ldots, N\}$.

Thus it remains to prove that if $A_{t,e}$ is a PI algebra then $t_i = 0$ for all $i \in \{1, \ldots, N\}$. To reach a contradiction assume that $t_i \neq 0$ for some $i = 1, \ldots, N$. This implies that the form $\Omega = \sum_{i=1}^{N} t_i b_i$ is a nonzero $G$-invariant skew-symmetric form on $V$. We claim that in this situation there exists a subalgebra of $A_{t,e}$ which is a symplectic reflection algebra. Existing results on symplectic reflection algebras will provide us with the necessary contradiction.

Let $U := \{ u \in V \mid \Omega(u,v) = 0 \text{ for all } v \in V \}$, the radical of $\Omega$. Then $U$ is a $G$-invariant subspace of $V$, because the form $\Omega$ is $G$-invariant. By Maschke’s theorem, we can find a $G$-invariant complement $W$ such that $V = U \oplus W$. Take $v, v' \in V$. We can write $v = u + w$ and
\( v' = u' + w' \) for some \( u, u' \in U, w, w' \in W \). We have \( \Omega(v, v') = \Omega(u + w, u' + w') = \Omega(w, w') \).

Therefore, the form \( \Omega \) is determined by its restriction to \( W \), denoted by \( \Omega|_W : W \times W \to \mathbb{C} \).

Moreover, by construction, the form \( \Omega|_W \) is not only a nonzero \( G \)-invariant skew-symmetric form on \( W \), but also non-degenerate. In other words, \( W \) is a symplectic vector space.

Let \( G' \) denote the subgroup of \( G \) which is generated by those elements that are bireflections in their action on the subspace \( W \). It is clear that \( G' \) is closed under conjugation by elements of \( G \). We claim that the elements in \( W \) and the elements in \( G' \) generate a subalgebra of \( A_{t,c} \) which is a symplectic reflection algebra. Obviously \( T(W) \ast G' \) is a subalgebra of \( T(V) \ast G \).

In order to prove our claim we need to examine the relations

\[
\kappa(w, w') = \Omega(w, w') \ id + \sum_{s \in S'} c_s \Omega_s(w, w') s,
\]

for all \( w, w' \in W \). In particular, we need to show that \( \Omega_s|_{W \times W} = 0 \) for all elements \( s \in S' \) that are not bireflections in their action on \( W \). Indeed, take \( s \in S' \). Since \( \dim(\text{im}(id - s)) = 2 \), we have \( \dim(\text{im}(id - s) \cap W) \leq 2 \). Assume that \( \dim(\text{im}(id - s) \cap W) = 0 \), then \( s \) fixes \( W \).

But by construction, see Section 2, the subspace \( V^s = \ker(id - s) \) lies in the radical of \( \Omega_s \).

Thus we deduce \( \Omega_s|_{W \times W} = 0 \) for this situation. Assume that \( \dim(\text{im}(id - s) \cap W) = 1 \). Say \( \text{im}(id - s) \cap W = \mathbb{C} x \). Since \( \Omega_s|_{W \times W} \) is a skew-symmetric form, \( \Omega_s|_{W \times W}(\lambda x, \mu x) = 0 \) for all \( \lambda, \mu \in \mathbb{C} \). But \( W = (\text{im}(id - s) \cap W) \oplus (\ker(id - s) \cap W) \) and \( \ker(id - s) \) is again in the radical of \( \Omega_s \). Thus in this situation we also have \( \Omega_s|_{W \times W} = 0 \).

Denote the subalgebra of \( A_{t,c} \) generated by \( W \) and \( G' \) by \( A(W, G') \). Note that the action of \( G' \) on \( W \) is faithful. Namely, the decomposition \( V = U \oplus W \) is \( G' \)-invariant. Take a generator \( s \) of \( G' \subseteq G \). Then \( s \) is a bireflection on \( V \), because \( s \in G \), but \( s \) is also a bireflection on \( W \).

We deduce that \( \dim(\text{im}(id - s) \cap U) = 0 \). So the group \( G' \) acts trivially on \( U \). Now if \( g \in G' \) is such that \( g|_W = id \), then \( g|_V = id \). But because \( G \subseteq GL(V) \) acts faithfully on \( V \), this implies that \( g = id \). Therefore \( G' \hookrightarrow GL(W) \) and the subalgebra \( A(W, G') \) is a symplectic reflection algebra.

Since \( A(W, G') \) is a subalgebra of the PI algebra \( A_{t,c} \), it is also a PI algebra, see \([\text{MR87}], \text{Lemma 13.1.7}\). In \([\text{BG03}, \text{Proposition 7.2}]\) it is shown that if \( \Omega|_W \neq 0 \), then the centre of the symplectic reflection algebra \( A(W, G') \) is just \( \mathbb{C} \). The fact that \( A(W, G') \) is also prime, see Proposition 2.5, together with \([\text{MR87}], \text{Proposition 13.6.11}]\) now implies that \( A(W, G') \) is a finite dimensional \( \mathbb{C} \)-vector space. But this is a contradiction to the fact that \( A(W, G') \cong S(W) \otimes \mathbb{C} G' \) as a \( \mathbb{C} \)-vector space.

We now drop the assumption \( G = S \) and finish with the general result:
Corollary 6.5. Let $A_{t,c}$ be a graded Hecke algebra. Then $A_{t,c}$ is a PI algebra if and only if $A_{t,c}$ is a finitely generated module over its centre if and only if $t_i = 0$ for all $i \in \{1, \ldots, N\}$.

Proof. We have $S \triangleleft G$ and we denote the graded Hecke algebra constructed with $S$ instead of $G$ by $A_{t,c}(S)$. We have the implications:

$$
\begin{array}{c}
A_{t,c} \text{ PI} \\
\downarrow \\
A_{t,c}(S) \text{ PI} \\
\downarrow \\
A_{t,c}(S) \text{ a finite } Z(A_{t,c}(S)) \text{-module} \\
\downarrow \\
A_{t,c} \text{ a finite } Z(A_{t,c}) \text{-module}
\end{array}
$$

where the vertical implications are [MR87, Lemma 13.1.7] and [MR87, Corollary 13.1.13]. The horizontal implications are the corollary above and Lemma 5.1. Thus we know now that $A_{t,c}(S)$ is a finite $Z(A_{t,c}(S))$-module if and only if $A_{t,c}$ is a finite $Z(A_{t,c})$-module. But, by the corollary above, $A_{t,c}(S)$ is a finite $Z(A_{t,c}(S))$-module if and only if $t_i = 0$ for all $i \in \{1, \ldots, N\}$. □

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