Rigidity of Circle Packings with Flexible Radii

Robert Connelly\textsuperscript{1†} and Zhen Zhang\textsuperscript{2†}

\textsuperscript{1}Department of Mathematics, Cornell University, Ithaca, 14850, New York, USA.
\textsuperscript{2}Center for Applied Mathematics, Cornell University, Ithaca, 14850, New York, USA.

Contributing authors: rc46@cornell.edu; zz628@cornell.edu;
†These authors contributed equally to this work.

Abstract

Circle packings are arrangement of circles satisfying specified tangency requirements. Many problems about packing of circles and spheres occur in nature particularly in material design and protein structure. Surprisingly, little is known about the stability and rigidity of circle packings. In this paper, we study the rigidity of circle packings representing a given planar graph. The radii of circles are flexible with equality and inequality constraints. We provide a dual condition for the packing to be rigid to the first order. This gives us a sufficient condition to show a packing is rigid. Then we will explore the difficulties on rigidity problems beyond the first order.

Keywords: Rigidity, Circle Packing, Planar Graph

1 Introduction

In his 1964 book Regular Figures [15], László Fejes Tóth gave his best guesses of the densest circle packing when all radii are in the interval $[q, 1]$. An interesting observation is triangulated packings often are the densest at given radii. It is well known there exists a triangulated packing of the plane, the hexagonal packing, when all radii are identical. It is, however, not known how big $q$ can be when there are at least two distinct radii in a triangulated circle packing.
An upper bound was given by Gerd Blind, who proved there cannot be any
packing denser than hexagonal packing if \( q > 0.743 \). Since all triangulated
packings are denser than the hexagonal packing, there are no triangulated
packings if \( q > 0.743 \). The current known largest \( q \) from a triangulated packing
with distinct radii is approximately 0.651 given by French computer scientist
Thomas Fernique in 2020[13]. One fundamental region of Fernique’s packing is
shown in Figure 1. The following conjecture was proposed by Robert Connelly:

**Conjecture 1.1.** There exists no circle packing of the triangulated plane with
the ratio of minimal radius to maximal radius greater than that shown in Figure
1 when at least two circles have different sizes, where the ratio is a root of

\[
89x^8 + 1344x^7 + 4008x^6 - 464x^5 - 2410x^4 + 176x^3 + 296x^2 - 96x + 1
\]

with numerical value close to 0.651[12].

We have numerical evidences that suggest this conjecture is true based on
the global rigidity of finite packings. If no finite packing exists within \([q, 1]\)
for a finite graph \( G \), then certainly no infinite packing can contain \( G \) as a
subgraph within radii range \([q, 1]\). For example, the packing in Figure 2 is
optimal for a disk with four neighbors and its ratio is worse than Fernique’s,
hence no packing better than Fernique’s can have a disk of degree 4. We have
candidate packings for some graphs that are believed to be globally rigid when
the minimal disks are only allowed to increase their sizes while maximal disks
are only allowed to decrease their sizes. If these candidates are indeed globally
rigid, then we can eliminate enough finite graphs to force the only possible
infinite graph to be Fernique’s infinite triangulation. This motivates a type of
question that has rarely been studied: determining the rigidity of a packing
with given tangency pattern when the radii are bounded by linear constraints.

This paper discusses the first and second order rigidity of circle packings
representing finite graphs. The result and technique used to prove it are moti-
vated by Connelly, Roth and Whiteley’s similar results in tensegrities[9, 14].
We give a necessary and sufficient condition for first and second order rigid-
ity of circle packings and sufficient conditions for rigidity in general. Global
rigidity is rather mysterious and no technique from tensegrities have yielded
useful results so far. Some special cases can be proved using arguments based
on angle sums proved in [1] and [6].

**1.1 Related Work**

In the special case where all radii are fixed, first order rigidity is identical to
the first order rigidity of a bar framework. Related work on bar and tensegrity
frameworks can be found in [8]. Connelly and Gortler proved if all radii are
fixed and generic then the packing is rigid if and only if there are \( 2n - 3 \)
touching points[10].
In the special case where the graph is triangulated, Bauer, Stephenson and Wegert proved the set of packings is a differentiable manifold determined by radii on the boundary[1]. This result is particularly useful because it gives the global rigidity with certain constraints on the radii. This is one of the few cases where globally rigidity can be proved. Collins and Stephenson[6] also give a fast approximation algorithm to compute interior radii from boundary radii.

The question of whether or not a graph has a circle packing with radius in $[q, 1]$ is known to be hard generally. In particular, Breu and Kirkpatrick proved the problem is NP-complete when $q = 1[4]$. Therefore, we cannot expect a fast algorithm to determine global rigidity of circle packings generally. However, to prove Conjecture 1.1, all we need is solving several dozen special triangulated cases which may not be as hard as general cases.

2 Definitions and Notations

First, we define what a packing is and how it is related to graphs:

**Definition 2.1.** A finite circle packing $p$ in $\mathbb{R}^2$ is a vector in $\mathbb{R}^{3n}$ with coordinates $p = (x_1, y_1, r_1, ..., x_n, y_n, r_n)$, where $n$ is the number of disks, $p_i = (x_i, y_i)$ is the coordinate of the center for disk $i$, and $r_i > 0$ is the radius of the circle $i$. Given a planar embedding $p_0$ of graph $G = (V, E)$, we say $p$ is a packing of $G$ (written as $(G, p)$) if for every edge $(i, j) \in E$, circle $i$ centered at $p_i$ and circle $j$ centered at $p_j$ are externally tangent, i.e. $(r_i + r_j)^2 = (x_i - x_j)^2 + (y_i - y_j)^2$, 

![Fig. 1 A subpacking of Fernique’s packing. Its minimal radius over maximal radius is the largest known for any triangulated packing of the plane with at least 2 distinct radii. The centers of disk 1, 7, 17, 19 form a rectangular fundamental region](image-url)
and for every vertex in $p$, its neighbors are embedded in the same order counterclockwise as in $p_0$. The later condition is known as the orientation of $p$.

In particular, we do not require disks that are not joined by an edge to have disjoint interiors. The reason we do not enforce this common requirement is to avoid global constraints depending both on the graph and the packing. The orientation is to distinguish distinct packings of isomorphic graphs. Although the combinatorics are identical for packings of isomorphic graphs, the geometry can be different. In particular, we want to avoid reflections of a subpacking over lines and circles so the set of packings is a smooth manifold. Throughout this paper, packings of a planar graph will refer to packings with a fixed orientation of that graph.

By the Koebe–Andreev–Thurston (KAT) Theorem, a packing always exists for any planar graph. When the graph $G$ is maximal planar (triangulated triangle), then the packing is unique up to Möbius transformations and reflections (for an elementary and constructive proof, see [7]).

Among all packings, triangulated packings yield simple and interesting results. In particular, Figure 1 is a finite triangulated packing.

**Definition 2.2.** A finite packing is **triangulated** if it’s a packing of a graph that is topologically a triangulated disk. A triangulated packing is **simple** if its interior disks are edge connected and every boundary disk is adjacent to the interior.

It is known that for a simple triangulated packing in the plane, all radii can be determined uniquely and monotonically by the radii on the boundary of the graph (see[6] for details and an algorithm to approximate all radii from boundary radii). Suppose there are $b$ vertices on the boundary of triangulated graph $G$, then the set of packings of $G$ is homeomorphic to $(0,\infty)^b$. An elementary proof based on angle sums around interior disks is given in [1].

**Definition 2.3.** A **flex** (or motion) from packing $p_1$ of $G$ to $p_2$ of $G$ is an analytic path of packings of $G$ from $p_1$ to $p_2$, denoted by $p(t) = \langle x_1(t), y_1(t), r_1(t), ..., x_n(t), y_n(t), r_n(t) \rangle$, where $t \in [0,1]$, $p(0) = p_1$, and $p(1) = p_2$.

By the discussion above it’s not hard to see there’s a flex between any two packings of a simple triangulated disk if all radii are flexible. Hence there must be constraints on the radii in order for rigidity to make sense. Rigidity for packings in general is difficult to determine in this sense. In fact, even rigidity of bar frameworks is difficult to determine beyond the first order. We first inspect the first order rigidity for packings:
Definition 2.4. A infinitesimal flex (or first order flex) is a vector $p' = \langle x'_1, y'_1, r'_1, \ldots, x'_n, y'_n, r'_n \rangle$ satisfying $(p_i - p_j) \cdot (p'_i - p'_j) = (r_i + r_j)(r'_i + r'_j)$ for all edges $(i, j)$.

It’s not hard to see why infinitesimal flex is defined this way by taking derivative on both sides of $(r_i + r_j)^2 = (x_i - x_j)^2 + (y_i - y_j)^2$. This definition is to make sure all tangent pairs remain tangent at least infinitesimally.

In order to make sense of rigidity and infinitesimal rigidity, constraints are needed on the radii of disks. The vertex set $V$ is partitioned into four disjoint sets $V^+, V^-, V^=, V^0$, representing disks that can stay the same or increase their radii, disks that can stay the same or decrease their radii, disks with fixed radii, and free disks with no constraints respectively.

Notation 2.1. For all the figures in this article, disks in $V^-$ are colored blue, disks in $V^+$ are colored red, disks in $V^=$ are colored green, and disks in $V^0$ are colored gray.

Given these constraints it’s possible to define a partial ordering on the set of packings of $G$:

Definition 2.5. Given $p, \tilde{p}$ and a partition of vertices as above, $\tilde{p} \succeq p$ if $\tilde{r}_i \geq r_i$ for all $i \in V^+$, $\tilde{r}_i \leq r_i$ for all $i \in V^-$, and $\tilde{r}_i = r_i$ for all $i \in V^=$.

Three types of rigidity can be defined:

Definition 2.6. A infinitesimal flex $p'$ is proper if $r'_i \geq 0$ for every disk $i$ in $V^+$, $r'_j \leq 0$ for every disk $j$ in $V^-$, and $r'_k = 0$ for every disk $k$ in $V^=$. A disk packing is infinitesimally rigid if the only proper infinitesimal flexes are the derivatives of congruent motions.

Definition 2.7. A flex $p(t)$ with $p(0) = p$ is proper if $p(t) \succeq p$ for $t \in [0, 1]$. A packing is (locally) rigid if every proper flex is a congruent motion.

Definition 2.8. $p$ is globally rigid if $\tilde{p} \succeq p$ implies $\tilde{p}$ is congruent to $p$.

There is no known approach other than brute force to determine the rigidity and global rigidity of a packing in general. The complexity result from [4] suggests no algorithm can determine the general case of global rigidity in polynomial time unless $P = NP$. Infinitesimal rigidity still yields many interesting results. In the next section we will define a matrix that helps us determine infinitesimal rigidity.

3 Packing Rigidity Matrix

Similar to the first order rigidity of tensegrities, the idea is to define a rigidity matrix such that the kernel is the infinitesimal flexes.
Fig. 2 a packing representing the graph of a 4-flower. With boundary disks only allowed to shrink and interior disk only allowed to expand

Definition 3.1. For a packing \( p \) of a planar graph \( G = (V, E) \), where \( \|V\| = n \) and \( \|E\| = m \), we define the **Rigidity Matrix** \( R(p) \) to be the \( m \times 3n \) matrix defined as below:

1. The rows are indexed over the edges
2. The columns are indexed over \( x_i, y_i, \) and \( r_i \) for \( i \in \{1, \ldots, n\} \)
3. For a row indexed by \((i, j)\), if \( k \notin \{i, j\} \), then the entries \( x_k, y_k, r_k \) are 0.
4. For a row indexed by \((i, j)\), if \( k = i \), then the entry \( x_k \) is \( x_i - x_j \), \( y_k \) is \( y_i - y_j \), \( r_k \) is \( -r_i - r_j \).
5. For a row indexed \((i, j)\), if \( k = j \), then the entry \( x_k \) is \( x_j - x_i \), \( y_k \) is \( y_j - y_i \), \( r_k \) is \( -r_i - r_j \).

Any vector \( p' \) in the kernel of \( R(p) \) satisfies \( (p_i - p_j)(p'_i - p'_j) = (r_i + r_j)(r'_i + r'_j) \) for every edge \((i, j) \in E\) by definition of \( R(p) \). The infinitesimal rigidity problem is then a linear programming in the kernel of \( R(p) \). However, the kernel necessarily has proper infinitesimal flexes since the derivative of a congruent motion is always a proper solution.

There are four types of proper infinitesimal flexes that can occur in the kernel:

1. Three dimensions of trivial infinitesimal flexes are generated by translations and rotations which must fix all radii.
2. Proper but nontrivial infinitesimal flexes which do not change any radii.
3. Proper infinitesimal flexes that fix \( V^+, V^- \), and \( V^\pm \) while strictly alter at least one free disk.
4. Proper infinitesimal flexes that strictly alter some disks in \( V^+ \) or \( V^- \).
The first 3 types can be counted by computing the dimension of the kernel with entries on the corresponding radii being 0. To force $r'_i = 0$ in the kernel, simply add a row to $R(p)$ such that the entry on $r_i$ is 1 and the other $3n - 1$ entries are 0. The last type requires more sophisticated approach. If the number of constraints is “just right”, then it might be easier utilize LP duality to work with the cokernel rather than the kernel. However, the following proposition [10, Proposition 2.12] states that the cokernel of $R(p)$ has dimension 0:

**Proposition 3.1.** If $p$ is a packing of a planar embedded graph $G$, then the kernel of $R(p)$ has dimension $3n - m$.

Intuitively, Proposition 3.1 means in a packing of a planar embedded graph, no edge is redundant as a linear constraint. Each edge strictly reduces the dimension of the kernel by 1. Hence the row rank is full and the cokernel has dimension 0.

Next, we proceed with the idea of separating the first 3 types infinitesimal flexes from the last type by forcing some radii to be fixed in the kernel of $R(p)$. Let $e_k$ be the unit row vector in $\mathbb{R}^{3n}$ that is 1 on the location indexed by $r_k$. Let $E_S$ be matrix with rows $e_k$ for $k \in S$. We can define the new matrix:

**Definition 3.2.** The Extended Packing Rigidity Matrix, $R_e(p)$, of a given configuration $p$ is defined as the block matrix

$$
\begin{bmatrix}
R(p) \\
E_{V^-} \\
E_{V^+} \\
E_{V^+} - E_{V^-}
\end{bmatrix}
$$

The kernel of $R_e(p)$ has the first 3 types of infinitesimal flexes. If the dimensional of the kernel is greater than 3, then either the bar framework is infinitesimally flexible, or the radius of at least one free disk is infinitesimally flexible. To distinguish the second type from the third type, we can further compute the dimension of kernel for the matrix $R'(p) = \begin{bmatrix} R(p) \\ E_V \end{bmatrix}$ that fixes all radii. If the kernel of $R'(p)$ has dimension 3, then it cannot have any nontrivial infinitesimal flex fixing all radii. The difference between the dimension of the kernel of $R'(p)$ and that of $R_e(p)$ tells us if there are any infinitesimal flexes altering the radii of a free disk.

**Example:** Consider the packing in Figure 2 where each large disk has radius 1 and disk 5 is centered at origin. Suppose we have a constraint on each disk according to the coloring: disks 1 to 4 can only decrease their sizes and disk 5 can only increase its size.

The extended rigidity matrix is
Rigidity of Circle Packings with Flexible Radii

In this case, the matrix has a 3-dimensional kernel, and 1-dimensional cokernel. Theorem 4.1 from next section tells us this packing is infinitesimally rigid since the only vector in its cokernel satisfy some linear constraints. It is useful to first define the concept of a stress when talking about the cokernel of $R_e(p)$:

**Definition 3.3.** A stress is a real function $\omega : E \to \mathbb{R}$. If the net force defined in equation (1) on each vertex is zero vector, then it is an equilibrium stress.

Intuitively, we think a stress on an edge as a “force density per length”. If the stress on edge $(i, j)$ is $\omega_{ij}$, then the force acting on disk $i$ from disk $j$ is vector $\omega_{ij}(p_i - p_j)$ in the direction from $p_j$ to $p_i$. The force balance condition on vertex $i$ can be written as a vector sum:

$$\sum_{j|(i,j) \in E} \omega_{ij}(p_i - p_j) = 0$$  \hspace{1cm} (1)

For the constraints on the radii, it is helpful to determine whether the forces acting on a disk are pushing the boundary towards its center or pulling it away from its center. This quantity can be calculated as the following force sum:

$$\omega_i = \sum_{j|(i,j) \in E} \omega_{ij}(r_i + r_j)$$  \hspace{1cm} (2)

The cokernel of $R(p)$ are equilibrium stresses where the the radial force sum defined in (2) is 0 for every $i$. The cokernel of $R_e(p)$ are equilibrium stresses with the radial force sum defined in (2) being 0 only for $i \in V^0$. In next section we prove a theorem showing equivalence between infinitesimal rigidity and the existence of a specific stress.

### 4 First order rigidity of circle packings

Our main result for the first order theory is the following theorem:

**Theorem 4.1.** A disk packing is infinitesimally rigid if and only if the following conditions hold:
1. (Fixed Radius Condition) The packing is infinitesimally rigid when the radii of disks in $V^+$, $V^-$, and $V^0$ are fixed.

2. (Stress Existence Condition) There exists an equilibrium stress $\omega$ such that the the radial force sum defined in (2) is positive on $V^-$, negative on $V^+$, and $0$ on $V^0$.

Proof Intuitively, the first condition eliminates infinitesimal motions that are possible even if you fix all constrained disks. The second condition eliminates the proper infinitesimal motions where some disk in $V$ even if you fix all constrained disks. The second condition eliminates the proper infinitesimal motion where some disk in $V^+$ or $V^-$ changes its radius.

$\Leftarrow$: Let $\omega = (\ldots \omega_{ij} \ldots)$ be a desired stress and $p'$ be any proper infinitesimal motion. Extend our $\omega$ as $\omega_c = (\ldots \omega_{ij} \ldots \omega_k \ldots)$ which must be in the cokernel of $R_c(p)$, where $(i, j)$ are indexed over edges and $k$ is indexed over disks in $V^+$, $V^-$, and $V^0$.

We have

$$\omega_c R_c(p) p' = \sum_{(i, j)} \omega_{ij} ((p_i - p_j) - (p'_i - p'_j)) - (r_i + r_j)(r'_i + r'_j)) + \sum_{k \notin V^0} \omega_k r'_k \quad (3)$$

The first part is $0$ by definition of an infinitesimal motion. The second term is none-positive because we assumed $\omega_k$ and $r'_k$ either have opposite signs or $r'_k = 0$. Hence any $\omega_k r'_k \neq 0$ would imply $\omega_c R_c(p) p' < 0$, contradicting $\omega_c$ being in the cokernel of $R_c(p)$. Therefore, all $r'_k = 0$, and by condition 1, the packing is infinitesimally rigid.

$\Rightarrow$: If a packing is infinitesimally rigid, then (i) is automatically true. To prove (ii), we employ the Farkas’ Alternative:

Lemma 4.1. (Farkas’ Alternative): Either $Ax = b$ has a solution and $x \geq 0$(in every coordinate), or there exists $y$ such that $A^T y \leq 0$ but $y^T b > 0$.

The idea is to construct a matrix $A$ so that $A^T y \leq 0$ holds if and only if $y$ is a proper infinitesimal motion. Then, $y^T b \neq 0$ suggests the infinitesimal motion is nontrivial.

Let $A^T = \begin{bmatrix} R(p) \\ -R(p) \\ -E_{V^+} \\ E_{V^-} \\ -E_{V^0} \end{bmatrix}$. Let $b$ be any vector in $\mathbb{R}^{3n}$ positive on radii of $V^+$, negative on radii of $V^-$, and $0$ everywhere else. $A^T y \leq 0$ forces $y$ to be an infinitesimal motion because $R(p)y \leq 0$ and $-R(p)y \leq 0$ implies $R(p)y = 0$. $y$ is a proper infinitesimal motion because all the entries of $y$ indexed by the radii have the correct signs. $y^T b > 0$ implies the infinitesimal motion $y$ cannot fix all radii.

Since our packing is infinitesimally rigid, such a $y$ must not exist. As a result, the other case $Ax = b$ and $x \geq 0$ must be true. Suppose $x = [\omega^+, \omega^-, \omega^{E_x}]$ is such a solution where the 3 components correspond to $R(p), -R(p)$, and the remaining rows. Let $\omega_i$ be the entry in $\omega^{E_x}$ corresponding to radius of disk $i$ for $r$ in $V^+$ or $V^-$. First observe $\omega = \omega^+ - \omega^-$ is an equilibrium stress on the edges because the $x, y$ terms in $Ax = b$ satisfy:

$$\sum_{j|ij \in E} (\omega_{ij}^+ - \omega_{ij}^-)(p_i - p_j) = 0$$
The radius terms, using \( i \in V^- \) as an example, in \( Ax = b \) has the form

\[
\sum_{j|ij \in E} (\omega_{ij}^+ - \omega_{ij}^-)(-r_i - r_j) + \omega_i = b_i
\]

It can be re-written as

\[
\sum_{j|ij \in E} (\omega_{ij}^+ - \omega_{ij}^-)(r_i + r_j) = -b_i + \omega_i
\]

Since \( i \in V^- \), \( b_i < 0 \). By our assumption \( \omega_i \geq 0 \). As a result the radial sum of forces around \( i \) for stress \( \omega^+ - \omega^- \) satisfies

\[
\sum_{j|ij \in E} (\omega_{ij}^+ - \omega_{ij}^-)(r_i + r_j) > 0
\]

For \( i \in V^+ \), \( \omega_i \) is replaced by \( -\omega_i \) and \( b_i \) is positive, the proof then is identical. For \( i \in V^0 \), the \( \omega_i \) term vanishes and \( b_i = 0 \).

In fact, a stronger statement holds even when we no longer insist that the disks stay tangent. The following corollary is obvious from the proof above:

**Corollary 4.2.** If the stress existence condition holds with stress \( \omega \), then any infinitesimal flex \( p' \) ignoring tangency conditions while satisfying the following conditions must preserve all radii in \( V^+ \cup V^- \) and all tangent relations between disk \( i \) and disk \( j \) if \( \omega_{ij} \neq 0 \):

1. If \( \omega_{ij} < 0 \), disk \( i \) and disk \( j \) remain tangent or become separated
2. If \( \omega_{ij} > 0 \), disk \( i \) and disk \( j \) remain tangent or become overlapped
3. All radii move in the desired direction or remain the same.

**Proof** Consider \( \omega R_e(p)p' = \sum_{(i,j)} \omega_{ij}((p_i - p_j) \cdot (p'_i - p'_j) - (r_i + r_j)(r'_i + r'_j)) + \sum_{k \in V^0} \omega_k r'_k \). If the above conditions hold, then each term is non-positive. However, \( \omega \) is in the cokernel of \( R_e(p) \), hence all terms are 0.

A useful fact about infinitesimal rigidity is that it implies rigidity. This is a consequence of the product rule for derivative:

**Proposition 4.3.** If a disk packing is infinitesimally rigid, it is rigid.

**Proof** If a packing is infinitesimally rigid, then there is no nontrivial proper infinitesimal motion \( p' \) that satisfies

\[
(p_i - p_j) \cdot (p'_i - p'_j) - (r_i + r_j)(r'_i + r'_j) = 0
\]

Consider the \( n^{th} \) derivative of the tangency condition,
0 = \left(\frac{\partial}{\partial t}\right)^n [(p_i(t) - p_j(t)) \cdot (p_i(t) - p_j(t)) - (r_i(t) + r_j(t))(r_i(t) + r_j(t))]

= \sum_{i=0}^{n} \binom{n}{i} \left(\frac{\partial}{\partial t}\right)^i (p_i(t) - p_j(t)) \cdot \left(\frac{\partial}{\partial t}\right)^{n-i} (p_i(t) - p_j(t)) - \left(\frac{\partial}{\partial t}\right)^i (r_i(t) + r_j(t)) \left(\frac{\partial}{\partial t}\right)^{n-i} (r_i(t) + r_j(t))

(4)

Without loss of generality, we assume the trivial motion up to its highest non-trivial order is zero. Now if the first \(n-1\) degree derivatives are all 0, then we are left with:

\((p_i(t) - p_j(t)) \cdot \left(\frac{\partial}{\partial t}\right)^n (p_i(t) - p_j(t)) - (r_i(t) + r_j(t)) \left(\frac{\partial}{\partial t}\right)^n (r_i(t) + r_j(t)) = 0\)

Observe that the \(n^{th}\) derivative \(\left(\frac{\partial}{\partial t}\right)^n p(t)\) satisfies the same expression as the one we need for infinitesimal rigidity. Therefore, by induction, if there is no nontrivial proper first order flex, there cannot be any nontrivial proper flex with desired signs on its lowest nonzero order. \(\square\)

The proof of theorem 4.1 does not rely on the dimension, so the result holds for sphere packings in any dimension. In particular, we can determine whether or not a 3D ball packing is infinitesimally rigid given its combinatorial structure.

Now we know that infinitesimal rigidity implies rigidity. Is the reverse true? The answer is negative. It’s possible for a packing to be rigid locally with a nontrivial infinitesimal flex. In these cases, infinitesimal rigidity gives us no information on whether or not the structure is rigid. One such example is the packing in Figure 3, where disks with identical color have the same size.

If we use the boundary radii as a basis to determine interior radii, then due to symmetry disk 2 and 5 must have identical partial derivatives respect to the interior. This means increasing the size of disk 2 while decreasing the size of disk 5 at the same rate will be a valid infinitesimal motion. This can be easily verified by computing the dimension of the null space of the extended rigidity matrix of this packing, which is 4.

There are essentially two constraints that are determined only by the radii - the angle sums around disk 4 and 7 must be \(2\pi\). We define the following function that maps 3 radii of a triangle to the angle:

\(\alpha(x, y, z) = \cos^{-1}\left(\frac{(x + y)^2 + (x + z)^2 - (y + z)^2}{2(x + y)(x + z)}\right)\)

It is well known \(\alpha(x, y, z)\) is monotonically increasing with \(y\) and \(z\) while monotonically decreasing with \(x\)[6]. Using these monotonic relations, we can bound one of \(r_2\) and \(r_5\) given the other one. Given radius of disk 2, the radius of disk 5 is minimal when disk 1,3,4, and 7 are fixed in size. Similarly, disk 5 is maximal when disk 4,7,6,8,9, and 10 are fixed in size. As a result, we can
compute the minimal and maximal value of $r_5$ given $r_2$. The minimal value of $r_5$ satisfies:

$$\alpha(r_4, 1, 1) + 2\alpha(r_4, r_5, 1) + 2\alpha(r_4, r_2, 1) = 2\pi$$

Similarly, the maximal value of $r_5$ must satisfy:

$$3\alpha(1, r_4, r_4) + 2\alpha(1, r_4, r_2) + 2\alpha(1, r_4, r_5) = 2\pi$$

The two contours are shown in Figure 4.

Denote the radius $r_4$ as $q$. Solving these equations we have the following lower and upper bound for $r_5$:

$$\min r_5 = \frac{-2q(q + 1) \cos^2 \left( \frac{1}{4} \right) \left( \cos^{-1} \left( \frac{q^2 + 2q + 1}{(q+1)^2} \right) + 2 \cos^{-1} \left( \frac{(q-1)r_2 + q(q+1)}{(q+1)(q+r_2)} \right) \right)}{(q + 1) \cos \left( \frac{1}{2} \cos^{-1} \left( \frac{q^2 + 2q - 1}{(q+1)^2} \right) + \cos^{-1} \left( \frac{(q-1)r_2 + q(q+1)}{(q+1)(q+r_2)} \right) \right) + q - 1}$$

$$\max r_5 = \frac{-2(q + 1) \cos^2 \left( \frac{1}{4} \right) \left( 3 \cos^{-1} \left( \frac{q^2 - 2q + 1}{(q+1)^2} \right) + 2 \cos^{-1} \left( \frac{q(-r_2) + q + r_2 + 1}{q_2 + q + r_2 + 1} \right) \right)}{(q + 1) \cos \left( \frac{3}{2} \cos^{-1} \left( \frac{-q^2 + 2q + 1}{(q+1)^2} \right) + \cos^{-1} \left( \frac{q(-r_2) + q + r_2 + 1}{q_2 + q + r_2 + 1} \right) \right) - q + 1}$$

Figure 5 gives a closer view of $(\min r_5 - \max r_5)$ near the intersection point. We can see the interval $[r_{\min}, r_{\max}]$ is not empty at one point when the two contours intersect at the configuration in Figure 3. After plugging in the value for radii given by Figure 3 into $(\min r_5 - \max r_5)$, the zeroth and first order derivatives are 0 while the second order derivative is positive. Showing
Fig. 4 The contour plot for minimal and maximal $r_5$ at given $r_2$

Fig. 5 The value for $\min r_5 - \max r_5$ at given $r_2$
(min $r_5 - \max r_5$) has exactly one non-positive point means Figure 3 is globally rigid given its coloring.

5 The Combinatorics

5.1 The dimension count

If a planar packing $(G, p)$ has $n$ vertices and $m$ edges, by Proposition 3.1 the kernel of $R(p)$ always has dimension $3n - m[10]$. The trivial motions generate a tangent space of dimension 3 unless there is only a single disk. In order to make a packing infinitesimally rigid, we need to reduce the degree of freedom by $3n - m - 3$. Since we need $k$ equalities or $k + 1$ inequalities to reduce the degree of freedom by $k$, we need at least $3n - m - 3$ fixed disks, or $3n - m - 2$ constraints including inequalities, to make a packing infinitesimally rigid.

In the special case where the packing is triangulated and simple, we can use Euler’s formula to show

$$3n - m - 3 = b$$

where $b$ is the number of disks on the boundary. Euler’s characteristics gives

$$n - m + f = 1$$

where $f$ is the number of triangular faces. Other than the boundary edges, every other edge is used in two triangular faces. This gives

$$3f = 2m - b$$

Plugging equation (7) into (6) gives (5). This is consistent with the result proved in [1] that such a packing is globally uniquely determined by the boundary radii. In Figure 3, there are 8 disks on the boundary and 8 inequality constraints. Therefore, it is clear Figure 3 cannot be infinitesimally rigid simply by counting.

In summary, the first order degree of freedom (dimension of the kernel of $R(p)$) is always $3n - m - 3$. If the packing is simple and triangulated, then this number equal to the number of disks on the boundary $b$. For infinitesimal rigidity, at least $b$ equality or $b + 1$ inequality constraints are needed. In next subsection, we determine some combinatorial properties of the set of disks that infinitesimally rigidify a given packing.

5.2 Maximal independent set

To understand the structure of disks that can rigidify a given packing, the concept of an independent set can be helpful. Intuitively, if the radius $r_i$ is determined by the radii in some set of disks $S$, then $r_i$ gives no more information once all the radii in $S$ are known. Since we focus on the set of disks that rigidify the packing in the first order, linear independence makes sense.
Definition 5.1. Given a packing \((G, p)\), a set \(S\) of disks is **linearly independent** if no radii in a proper subset \(A \subset S\) determine any radius in its complement \(A^c\) to the first order. Otherwise we say \(S\) is linearly dependent. An independent set \(S\) is **maximal** if there does not exist disk \(d_i\) such that \(S \cup \{d_i\}\) is linearly independent.

In terms of our matrices, \(S\) being independent means the row rank of 
\[
\begin{bmatrix}
  R(p) \\
  E_S
\end{bmatrix}
\]
is \(m + S\), where \(m\) is the number of edges. To see this, suppose the row rank is not maximal, i.e. \(e_i\) where \(i \in S\) is in the row space of 
\[
\begin{bmatrix}
  R(p) \\
  E_{S - \{i\}}
\end{bmatrix},
\]
then there is a vector \(\omega\) in the left null space of 
\[
\begin{bmatrix}
  R(p) \\
  E_S
\end{bmatrix}.
\]
According to equation 3, we can assign \(V^+, V^-, \) and \(V^=\) based on the signs of \(\omega\) such that no infinitesimal flex can vary the radii of these disks in a proper way. However, since we do not have the fixed radius condition in theorem 4.1, the packing may not be rigid. In this case, only the radii in the subset is rigid while rest of the packing can flex.

An independent set \(S\) is maximal if radii in \(S\) determines all radii up to the first order, i.e. 
\[
\begin{bmatrix}
  R(p) \\
  E_S
\end{bmatrix}\]
and 
\[
\begin{bmatrix}
  R(p) \\
  E_V
\end{bmatrix}
\]
have the same rank where \(V\) is the vertex set. Observe that 
\[
\begin{bmatrix}
  R(p) \\
  E_V
\end{bmatrix}
\]
has the same kernel as the submatrix produced by removing all columns corresponding to radii. Hence the kernel are infinitesimal motions of the bar framework. An example is given in Figure 6. The set \(S\) of
green disks is linearly independent but not maximal. \( S \cup \{7\} \) is neither maximal nor linearly independent. \( S \cup \{1\} \) is both maximal and linearly independent.

Next we focus on the packings with infinitesimally rigid bar frameworks. In these packings \( \begin{bmatrix} R(p) \\ E_V \end{bmatrix} \) has a 3 dimensional kernel. \( R(p) \) has a \( 3n - m \) dimensional kernel. Therefore, any maximal linearly independent set must have \( 3n - m - 3 \) disks. The following should now be obvious:

**Proposition 5.1.** Given a packing \((G, p)\) with infinitesimally rigid bar framework, if \( S = V^= \) is a maximal set and \( S^C = V^0 \), then the packing is infinitesimally rigid.

*Proof* Since the bar framework is infinitesimally rigid, the kernel of \( \begin{bmatrix} R(p) \\ E_V \end{bmatrix} \) has dimension 3. Since \( S \) is maximal, no radii can be added to \( S \) to further reduce the dimension of its kernel, so the kernel of \( \begin{bmatrix} R(p) \\ E_S \end{bmatrix} \) also has dimension 3. The three dimensions in the kernel can only be trivial infinitesimal motions. □

The maximal linearly independent sets of a packing \((G, p)\) have an obvious structure which guarantees many optimization problems can be solved efficiently over these sets:

**Proposition 5.2.** Given a packing \((G, p)\), the linearly independent sets form a matroid.

*Proof* There are many equivalent ways to define a matroid. Here we use the independent sets to define it. The empty set is independent. The subset of an independent set is clearly independent. Suppose \( A \) and \( B \) are independent sets such that \( A > B \), then \( \begin{bmatrix} R(p) \\ E_A \end{bmatrix} \) has a row \( e_k \) not in the row space of \( \begin{bmatrix} R(p) \\ E_B \end{bmatrix} \). Then \( B \cup \{k\} \) is independent. □

Given a packing \((G, p)\) with infinitesimally rigid bar framework and a non-negative real cost function \( f : V \to \mathbb{R}^+ \) of rigidifying the radius of each disk, then the minimum cost to infinitesimally rigidify the packing can be computed quickly. It is well known an independent set with minimum cost can always be computed through greedy algorithm if and only if the independent sets form a matroid[11]. Many types of optimization problems can be solved quickly on a matroid[5].

### 6 Generic Rigidity

In [10], it was proved that if a packing of \( G \) has the properties \( V = V^=, m = 2n - 3 \), and all radii being generic (algebraically independent), then it is infinitesimally rigid. This result can be generalized to packings with more than
Rigidity of Circle Packings with Flexible Radii

2n − 3 edges and with fewer fixed radii. The following semi-algebraic version of Sard’s Theorem is proved in [10]:

**Theorem 6.1.** Let \( X, Y \) be smooth semi-algebraic manifolds of dimension \( d_1 \) and \( d_2 \) over \( \mathbb{Q} \) and \( f : X \to Y \) a rational map. Then the critical value of \( f \) are a semi-algebraic subset of \( Y \), defined over \( \mathbb{Q} \), and of dimension strictly less than \( d_2 \).

The goal is to prove the following theorem:

**Theorem 6.2.** Let \( (G = (V, E), p) \) be a planar packing with \( n \) vertices and \( m \geq 2n - 3 \) edges. If there exists \( S \subset V \) such that \( S = 3n - m - 3 \) and the radii of disks in \( S \) are generic, then the packing is infinitesimally rigid by fixing the radii in \( S \) and freeing the radii not in \( S \).

**Proof** The packing manifold \( X_G \) is defined by tangency relations \((x_i - x_j)^2 + (y_i - y_j)^2 = (r_i + r_j)^2\) of dimension \( 3n - m \)[1]. It is clearly a semi-algebraic manifold. Consider the following projection:

\[
f : X_G \to \mathbb{R}^S
\]

which maps a packing \( p \) to the radii in \( S \). \( f \) is a polynomial map. Theorem 6.1 states the critical values of \( f \) cannot be generic. Therefore, a generic point must be regular. In particular, \( f(p) \) is a regular value. Consider the tangent map:

\[
Df : TX_G \to T\mathbb{R}^S
\]

Since \( f(p) \) is a regular value, \( p \) is a regular point. As a result, \( Df \) is surjective at \( p \). The dimension of the tangent space of \( X_G \) at \( p \), \( TX_G \), is \( 3n - m \) and the dimension of \( T\mathbb{R}^S \) is \( S = 3n - m - 3 \). Therefore, the kernel has dimension \( 3n - m - (3n - m - 3) = 3 \). There are always three dimensions of infinitesimal motions at \( p \), which can only be generated by the trivial infinitesimal motions. Therefore, the packing \( (G, p) \) is infinitesimally rigid.

This result is not too surprising. Since being generic is stronger than linear independence, \( 3n - m - 3 \) generic radii are sufficient for infinitesimal rigidity just as the same number of linearly independent radii is sufficient in proposition 5.1.

Take the set of red and blue disks in Figure 3 as an example. If the radii in \( V^+ \cup V^- \) are generic in a different packing of \( G \), fixing these radii would be infinitesimally rigid. The nontrivial first order motion is caused by linear dependency in a special packing. In this case it’s the reflection symmetry along the line joining \( p_4 \) and \( p_7 \). In the next section we will present a method to deal with the special packing that generically should be rigid but is first order flexible due to it being a special arrangement.
7 Second Order Rigidity and Prestress Stability

Suppose a packing \((G, p)\) has a nontrivial infinitesimal flex, it can still be rigid. One such example is given in Figure 3 discussed in Section 4. Our approach in section 4 is not practical for larger packings because finding explicit relationships among radii becomes very difficult as the number of disks grow. This section introduces the idea of second order rigidity and prestress stability. Taking the second derivative of the tangency constraint we have:

\[
(p_i - p_j)(p''_i - p''_j) + (p'_i - p'_j)^T(p''_i - p''_j) - (r'_i + r'_j)(r''_i + r''_j) - (r'_i + r'_j)(r''_i + r''_j) = 0
\]

Rearranging the terms gives the following equality:

\[
(p_i - p_j)(p''_i - p''_j) - (r'_i + r'_j)(r''_i + r''_j) = (r''_i + r''_j)^2 - (p'_i - p'_j)^T(p'_i - p'_j)
\] (8)

Intuitively, the packing would be second order rigid if no nontrivial proper first order motion can be extended to a “proper” second order motion. In order to define “proper” for \(p''\), the constraints on the radii in \(p''\) need to be modified based on a given proper first order flex \(p'\). If disk \(i\) is in \(V^+\) and \(r'_i > 0\), then \(r''_i\) should not be constrained because locally \(r_i\) increases due to \(r'_i > 0\) even if \(r''_i < 0\). Therefore, we need to move disks that strictly change their radii in the first order from \(V^+\) and \(V^-\) to \(V^0\). Let the new partition be...
V = \tilde{V}^+ \sqcup \tilde{V}^- \sqcup \tilde{V}^0$, and $R'_e(p)$ be the extended rigidity matrix with the new partition, $[R(p)
abla \tilde{V}^+ \sqcup \tilde{V}^- \sqcup \tilde{V}^0]$

**Definition 7.1.** Given a packing $(G, p)$ and $p'$ is a proper infinitesimal motion, $(p', p'')$ is proper if (8) holds and $r''_i \geq 0$ for all $i \in \tilde{V}^+$, $r''_i \leq 0$ for all $i \in \tilde{V}^-$, $r''_i = 0$ for all $i \in \tilde{V}^0$. $p'$ is extendable if there exists a $p''$ such that $(p', p'')$ is proper. $(G, p)$ is second order rigid if all nontrivial infinitesimal motions $p'$ are not extendable.

One should be careful that the partition of the vertices now depends on $p'$. Now we are ready to prove the following proposition:

**Proposition 7.1.** Given a packing $(G, p)$ and $p'$ is a proper infinitesimal motion, $p'$ is not extendable if there exists an equilibrium stress $\omega$ such that the radial force sum defined in (2) is non-negative on $\tilde{V}^-$, non-positive on $\tilde{V}^+$, 0 on $\tilde{V}^0$, and satisfies the following inequality:

$$\sum_{(i,j) \in E} \omega_{ij}[(p'_i - p'_j)(p''_i - p''_j) - (r'_i + r'_j)^2] > 0$$  \hspace{1cm} (9)

**Proof** First extend the stress as $\omega_e = (\ldots \omega_{ij} \ldots, \omega_k)$ where $(i, j)$ is the stress indexed over $E$ and $\omega_k$ is radial force sum defined in (2) for disk $k$ indexed over $\tilde{V}^+ \sqcup \tilde{V}^- \sqcup \tilde{V}^0$. $\omega_e$ is in the cokernel of $R'_e(p)$ be definition. If a proper $p'$ is extendable, then the following equation must hold:

$$0 = \omega_e R'_e(p)p''$$

$$= \sum_{(i,j) \in E} \omega_{ij}[(p'_i - p'_j)(p''_i - p''_j) - (r'_i + r'_j)(r''_i + r''_j)] + \sum_{k \in \tilde{V}^+ \sqcup \tilde{V}^- \sqcup \tilde{V}^0} \omega_k r''_k$$  \hspace{1cm} (10)

Substituting (8) into (10) we have:

$$0 = \sum_{(i,j) \in E} -\omega_{ij}[(p'_i - p'_j)(p'_i - p'_j) - (r'_i + r'_j)^2] + \sum_{k \in \tilde{V}^+ \sqcup \tilde{V}^- \sqcup \tilde{V}^0} \omega_k r''_k$$  \hspace{1cm} (11)

The first term is strictly negative by (9), and the second term is non-positive. Since we reached a contradiction if $p''$ exists, $p'$ is not extendable. \qed

**Definition 7.2.** If a stress $\omega$ described in proposition 7.1 exists, we say $\omega$ blocks $p'$. $(G, p)$ is prestress stable if some stress blocks every nontrivial proper infinitesimal motion.

The reason we care about second order rigidity is the following:

**Proposition 7.2.** If a packing $(G, p)$ is second order rigid, then it is rigid.
Proof Without loss of generality we assume the trivial motion is 0 in a proper motion \((G, p(t))\). Suppose the lowest order nonzero derivative is of order \(n\), in the proof of proposition 4.3 we showed the \(n^{th}\) derivative must be an infinitesimal motion. The goal is to show the second order motion will appear at the \(2n^{th}\) derivative.

\[
0 = \left( \frac{\partial}{\partial t} \right)^{2n} \left[ (p_i(t) - p_j(t)) \cdot (p_i(t) - p_j(t)) - (r_i(t) + r_j(t))(r_i(t) + r_j(t)) \right]
\]

\[
= \sum_{i=0}^{2n} \binom{2n}{i} \left( \frac{\partial}{\partial t} \right)^{n-i} (p_i(t) - p_j(t)) \cdot \left( \frac{\partial}{\partial t} \right)^{2n-i} (p_i(t) - p_j(t))
- \left( \frac{\partial}{\partial t} \right)^{n} (r_i(t) + r_j(t)) \left( \frac{\partial}{\partial t} \right)^{2n-i} (r_i(t) + r_j(t))
\]

Now if the first \(n - 1\) degree derivatives are all 0, then we are left with:

\[
2 \left[ (p_i(t) - p_j(t)) \cdot \left( \frac{\partial}{\partial t} \right)^{2n} (p_i(t) - p_j(t))
- (r_i(t) + r_j(t)) \left( \frac{\partial}{\partial t} \right)^{2n} (r_i(t) + r_j(t)) \right]
\]

\[
= - \left( \frac{2n}{n} \right) \left[ \left( \left( \frac{\partial}{\partial t} \right)^{n} (p_i(t) - p_j(t)) \right) \cdot \left( \left( \frac{\partial}{\partial t} \right)^{n} (p_i(t) - p_j(t)) \right)
+ \left( \left( \frac{\partial}{\partial t} \right)^{n} (r_i(t) + r_j(t)) \right)^{2} \right]
\]
Observe these are the same relations as in equation (8) only off by a constant factor. Therefore, $2/(2^n)p^{2n}$ will extend $p^n$ as a proper second order motion if $p^n$ is moved to the first order. Since $(G, p)$ is second order rigid, no such nontrivial infinitesimal flex can exist on order $n$. As a result, $p(t)$ must be a trivial motion.

Figure 7 to Figure 9 are some examples of packings that have a one dimensional nontrivial infinitesimal motion $p'$, and the stress blocking $p'$ is labeled on the edges. Figure 7 is a case of a small packing that shows the idea behind prestress stability. Disk 5 is moving away from disk 7 in the only nontrivial first order motion, but disk 4 and disk 6 are “pulling” it back as soon as disk 5 moves infinitesimally. Disk 5 cannot resist the force from disk 4 and 6 because it cannot increase its radii. Figure 8 and 9 are far more complex.

One observation is that switching $V^+$ and $V^-$ does not preserve the prestress stability. In Figure 7, if disk 5 can increase its radius, then pulling disk 5 away from disk 7 while increasing its radius is a valid motion. Suppose we switch the elements in $V^+$ and $V^-$, then the first order motion would be $-p'$ with stress $-\omega$. This causes the first term of equation (11) to be positive. Therefore, the stress would no longer block the infinitesimal motion.

At the end of this section, we prove the converse of proposition 7.1 to establish the duality result for second order rigidity:

**Theorem 7.3.** Given a packing $(G, p)$ and $p'$ a nontrivial proper infinitesimal motion, exactly one of the following statements is true:
1. \( p' \) is extendable
2. There exists an equilibrium stress \( \omega \) blocking \( p' \).

**Proof** Proposition 7.1 proved one direction. We prove the other here. First, notice the statement is similar to theorem 4.1. Therefore, an argument based on Farkas’ Alternative should be expected. We use the following version of Farkas’ alternative with mixed equalities and inequalities from [9]:

**Lemma 7.1.** Let \( A_0, A_+ \) be two matrices with \( n \) columns and \( m_0, m_1 \) rows, respectively. \( b_0 \in \mathbb{R}^{m_0}, b_+ \in \mathbb{R}^{m_1} \). Then there is a vector \( x \in \mathbb{R}^n \) such that

\[
A_0 x = b_0
\]

\[
A_+ x \leq b_+
\]

if and only if for all vectors \( y_0 \in \mathbb{R}^{m_0}, y_+ \in \mathbb{R}^{m_1} \) such that

\[
y_0 A_0 + y_+ A_+ = 0
\]

\[
y_+ \geq 0
\]

we have \( y_0 b_0 + y_+ b_+ \geq 0 \).

Let \( A_0 = R(p), b_0 = \{..., -(p'_i - p'_j)^T (p'_i - p'_j) + (r'_i + r'_j)^2, ..., \}, A_+ = \begin{bmatrix} -E \tilde{V}_+ \\ -E \tilde{V}_- \\ -E \tilde{V}_- \end{bmatrix} \), \( b_+ = 0, x = p'', y_0 = \omega \). Then the lemma states \( p' \) extends to \( p'' \) if and only if for all equilibrium stress \( \omega \) satisfying the constraints on radial force sums must have

\[
\sum_{(i,j) \in E} \omega_{ij} \left[ -(p'_i - p'_j)^T (p'_i - p'_j) + (r'_i + r'_j)^2 \right] \geq 0 \tag{13}
\]

which can be re-written as

\[
\sum_{(i,j) \in E} \omega_{ij} \left[ (p'_i - p'_j)^T (p'_i - p'_j) - (r'_i + r'_j)^2 \right] \leq 0 \tag{14}
\]

Therefore, if \( p' \) is not extendable, some \( \omega \) satisfying the radial force sum condition must violate equation (14).

This establishes the duality for second order rigidity. The following corollary is now obvious:

**Corollary 7.4.** A packing \((G, p)\) is second order rigid if and only if every nontrivial first order flex \( p' \) is blocked by some stress \( \omega \).

In the next section, we list some questions based on observation from a fairly large number of finite packings. In particular, if the \( G \) is a simple triangulated disk, many good properties seem to hold. Recall that the set of boundary radii globally and monotonically determines the packing[1][6] if \((G, p)\) is triangulated and simple. It is quite likely some type of “convexity” exists in this case that allow many global optimization problems to be solved by a local optimum.
8 Optimization Problems and Open Problems

Our paper proves the duality result in first and second order rigidity. For a general rigidity problem that is over constrained, it is likely the stress problem is as hard as the radius problem. However, in the context of local optimization over some function of the radii, it is almost guaranteed the result packing only has one stress unless the graph is highly symmetric. This is because the optimization process halts as soon as the first stress that asserts local optimality exists. It’s unlikely multiple stresses occur exactly at the same time without symmetry.

**Example 8.1.** Maximizing $\frac{r_{\text{min}}}{r_{\text{max}}}$ for the corresponding planar embedded graphs yields Figure 2, 3, and 9.

In these examples there are two cases. For Figure 2, the result is infinitesimally rigid and so there are $3n - m - 2$ inequality constraints. This is the case generally for most graphs that have no symmetry. For Figure 3 and 9, they are flexible in the first order with only $3n - m - 3$ inequality constraints. However, the only stress produced in the optimization process is able to block the first order motion. The following question is critical for proving conjecture 1.1:

**Question 8.1.** If a packing has locally maximal $\frac{r_{\text{min}}}{r_{\text{max}}}$, is it also globally maximal? If not, is it at least true for simple triangulated packings?

The following question is based on observation:

**Question 8.2.** Does local rigidity imply first or second order rigidity for simple triangulated packings?

Generally, packings that are rigid but not rigid to the first or second order can be generated from examples of such bar frameworks. However, the bar framework is necessarily rigid to the first order for simple triangulated packings. Therefore counterexamples are much harder to find even if the answer is negative.

**Example 8.2.** Maximizing the sum of radii with $0 \leq r_i \leq 1$ yields the packing in Figure 10.

In Figure 10, there are 27 vertices and 61 edges. In this case, $3n - m - 3 = 17$ is exactly the number of disks with radius 1. Together with the radius sum constraint there are $3n - m - 2$ constraints. Adding an extra row of 1 on every radii corresponding to radius sum to $R_e(p)$ gives exactly one stress. This is generally the result of such optimization process. Unlike the radii ratio function, the radius sum function does not seem to preserve symmetry. As a result, even rigid first order flexible packings are hard to find. This motivates the following question:
Fig. 10 The result of local optimization over the sum of radius

**Question 8.3.** If a simple triangulated packing \( m \geq 2n - 3 \) with no radius greater than 1 has maximal radius sum locally, does it always have at least \( 3n - m - 3 \) disks of radius 1?

Next we give an answer to a more general version of above questions: If a packing is locally rigid, is it globally rigid? The answer is unfortunately false given the following counter example:

**Example 8.3.** Figure 11 has two packings that are reflections of each other. Both are rigid locally. The packing on the right is not globally rigid because of the packing on the left.

Figure 11 shows the answer to the question is negative. The green disks with the same label have identical radii in both packings. They form a maximal independent set in the first order, and therefore must locally determine the packing.

Another observation is that global rigidity depends on the specific constraints on the radii. Even though both packings in Figure 11 are infinitesimally rigid and mirror reflections of each other, the figure on the left is globally rigid because disk 2 is in \( V^+ \) so the packing on the right is not valid, while the figure on the right is not globally rigid since the figure on the left is valid for its constraints.

There are also cases where local rigidity clearly implies global rigidity. If the packing is triangulated and simple, results from \([1, 6]\) show the boundary radii determine interior radii strictly and monotonically. Therefore, if all the
boundary radii are bounded above with at least one interior radius bounded below or vice versa, then local rigidity implies global rigidity. One such example is Figure 2.

Another example is Figure 3, which was shown to be globally rigid using arguments based on angle sums in section 4. Our observation is that when the symmetries of the graph are preserved in the packing, local rigidity appears to imply global rigidity.

Next question concerns existence of generic packings and rigidity when removing an edge.

**Question 8.4.** Given a planar embedded generically rigid graph \( G \) with \( n \) vertices and \( m \geq 2n - 3 \) edges, is there a circle packing of \( G \) with \( 3n - m - 3 \) generic radii? If so, is the set of such packings dense in the set of packings of \( G \)?

Intuitively, the answer to both of these questions should be affirmative if the graph is “nice” enough. We give a heuristic argument on why the answers might be yes. For simple triangulated packings, the answers are obvious given there are \( 3n - m - 3 \) boundary radii and that boundary radii can be arbitrary[1, 6]. For graphs that are not triangulated, the idea is to remove edges from a triangulated one. Based on Lemma 3.1, we know removing an edge will always free a radius in the first order. The question then becomes whether or not it’s always possible to extend this first order flex to a real flex.

Using Figure 7 with an extra edge \((5, 7)\) as an example, removing \((5, 7)\) still keeps the packing rigid in the second order. However, this seems to require the packing to be not generic before we remove the edge. When disk 5 is not on the same line as disk 4 and disk 6, removing the edge 5, 7 does create an extra degree of freedom. Notice here we need \( G \) to be planar in order for Lemma 3.1 to hold. Without this assumption, it’s possible to have duplicated disks sharing 3 neighbors that can never have generic radii.
References

[1] Bauer, D., Stephenson, K., Wegert, E.: Circle packings as differentiable manifolds. Beiträge zur Algebra und Geometrie / Contributions to Algebra and Geometry 53(2), 399–420 (2011). https://doi.org/10.1007/s13366-011-0078-y

[2] Blind, G.: Über unterdeckungen der ebene durch kreise. Journal für die reine und angewandte Mathematik (Crelles Journal) (236), 145–173 (1969). https://doi.org/10.1515/crll.1969.236.145

[3] Blind, G.: Unterdeckung der ebene durch inkongruente kreise. Archiv der Mathematik 26(1), 441–448 (1975). https://doi.org/10.1007/bf01229765

[4] Breu, H., Kirkpatrick, D.G.: Unit disk graph recognition is np-hard. Computational Geometry 9, 3–24 (1998). https://doi.org/10.1016/S0925-7721(97)00014-X

[5] Calinescu, G., Chekuri, C., Pál, M., Vondrák, J.: Maximizing a monotone submodular function subject to a matroid constraint. SIAM Journal on Computing 40(6), 1740–1766 (2011). https://doi.org/10.1137/080733991

[6] Collins, C., Stephenson, K.: A Circle Packing Algorithm. Computational Geometry 25(3), 233–256 (2003). https://doi.org/10.1016/S0925-7721(02)00099-8

[7] Connelly, R., Gortler, S.J.: Packing disks by flipping and flowing. Discrete & Computational Geometry (2020). https://doi.org/10.1007/s00454-020-00242-8

[8] Connelly, R., Guest, S.: Frameworks, Tensegrities, and Symmetry. Cambridge University Press, Cambridge (2022). https://doi.org/10.1017/9780511843297

[9] Connelly, R., Whiteley, W.: Second-order rigidity and prestress stability for tensegrity frameworks. SIAM J. Discret. Math. 9(3), 453–491 (1996). https://doi.org/10.1137/S0895480192229236

[10] Connelly, R., Gortler, S.J., Theran, L.: Rigidity for sticky discs. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 475(2222), 20180773 (2019). https://doi.org/10.1098/rspa.2018.0773

[11] Edmonds, J.: Matroids and the greedy algorithm. Mathematical Programming volume 1, 127–136 (1971). https://doi.org/10.1007/BF01584082

[12] Fernique, T.: A Densest ternary circle packing in the plane. arXiv
Rigidity of Circle Packings with Flexible Radii

(2019). https://doi.org/10.48550/ARXIV.1912.02297. https://arxiv.org/abs/1912.02297

[13] Fernique, T., Hashemi, A., Sizova, O.: Compact packings of the plane with three sizes of discs. Discrete & Computational Geometry, 613–635 (2020). https://doi.org/10.1007/s00454-019-00166-y

[14] Roth, B., Whiteley, W.: Tensegrity Frameworks. Transactions of The American Mathematical Monthly 265(2), 419–446 (1991). https://doi.org/10.2307/1999743

[15] Tóth, L.F.: Regular Figures. Macmillan, New York (1964)