Electrical Neutrality and Symmetry Restoring Phase Transitions at High Density in a Two-Flavor Nambu-Jona-Lasinio Model∗

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(Dated: )

A general research on chiral symmetry restoring phase transitions at zero temperature and finite chemical potentials under electrical neutrality condition has been conducted in a Nambu-Jona-Lasinio model to describe two-flavor normal quark matter. Depending on that $m_0/\Lambda$, the ratio of dynamical quark mass in vacuum and the 3D momentum cutoff in the loop integrals, is less or greater than 0.413, the phase transition will be second or first order. A complete phase diagram of quark chemical potential versus $m_0$ is given. With the electrical neutrality constraint, the region where second order phase transition happens will be wider than the one without electrical neutrality limitation. The results also show that, for the value of $m_0/\Lambda$ from QCD phenomenology, the phase transition must be first order.

PACS numbers: 11.10.Wx; 11.30.Rd; 11.10.Lm; 11.15.Pg
Keywords: Normal quark matter, electrical neutrality, Nambu-Jona-Lasinio model, high density chiral symmetry restoring, first and second order phase transition

I. INTRODUCTION

Nambu-Jona-Lasinio (NJL) model [1] is a good low energy phenomenological model which can be used to simulate Quantum Chromodynamics (QCD) and is suitable for researching spontaneous breaking of symmetry and its restoring at finite temperature and finite chemical potential in quark matter [2, 3]. In a realistic research of the normal quark matter without color superconductivity, owing to that the quarks carry electrical charges, one must impose electrical neutrality condition on the quark matter. In this paper, we will generally examine symmetry restoring phase transitions at zero temperature and high density in a NJL model with two-flavor normal quark matter under electrical neutrality condition, however, the confinement problem which is beyond the power of a NJL model will not be touched on. This examination will make us get more deep-going understanding of the affect of electrical neutrality on phase transitions and certainly has important theoretical significance. It is noted that such general research has not appeared in present literature. The preceding relevant researches either had completely not touched on electrical neutrality requirement [4] or only limited relevant parameters e.g. the momentum cutoff $\Lambda$ and the dynamical quark mass $m_0$ in vacuum to some given values by phenomenology [5].

It is a well-known fact that in the vacuum of QCD, chiral symmetry is inevitably spontaneously broken [1, 4, 5], so in our model, we will first assume that the quarks in vacuum have acquired a common non-zero dynamical mass from the quark-antiquark condensates, then research that, as increase of the quark chemical potential, the dynamical quark mass could change to zero and the chiral symmetry will finally be restored. In the process, the electrical neutrality condition will always be maintained and the order of the symmetry restoring phase transitions will be the key point of our concern.

For keeping electrical neutrality of the quark matter, we will add the contribution of free electron gas to the effective potential. The discussions will be made in the mean-field approximation. Throughout the paper, a three-dimension momentum cutoff will be used.

The paper is arranged as follows. In Sect.II we will give the effective potential of a NJL model describing two-flavor normal quark matter, its extreme value equation and the electrical neutrality condition. In Sect.III we will discuss the second order phase transition in small dynamical quark mass in vacuum, or equivalently, in large momentum cutoff, and in Sect.IV we expound the second and the first order phase transition in larger ratio of the above two parameters. Finally, in Sect.V we come to our conclusions.

II. EFFECTIVE POTENTIAL, GAP EQUATION AND ELECTRICAL NEUTRALITY CONDITION

The Lagrangian of the NJL model describing the two-flavor normal quark matter can be expressed by

$$\mathcal{L} = \bar{q}i\gamma^\mu \partial_\mu q + G_S[(\bar{q}q)^2 + (\bar{q}i\gamma_5 \vec{\tau} q)^2]$$

with the quark Dirac fields $q$ in the $SU_f(2)$ doublet and the $SU_c(3)$ triplets, i.e.

$$q = \begin{pmatrix} u_i \\ d_i \end{pmatrix} \quad i = r, g, b,$$

where the subscripts $i = r, g, b$ denote the three colors (red, green and blue) of quarks, $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are the Pauli matrices, $G_S$ is the four-fermion coupling

∗The project supported by the National Natural Science Foundation of China under Grant No.10475113.
constants and we have omitted the bare mass of the quarks. Assume that the four-fermion interactions can lead to the scalar quark-antiquark condensates $\langle \bar{q}q \rangle = \phi$, then the chiral $SU_f(2) \otimes SU_f(2)$ flavor symmetry of the Lagrangian (1) will be spontaneously broken down to $SU_f(2)$ and the quarks will get dynamical mass $m = -2G_S \langle \bar{q}q \rangle$. In the mean field approximation [9], we can write the effective potential of the model in the $T \rightarrow 0$ limit by

$$V(m, \mu, \mu_e) = \frac{m^2}{4G_S} - 6 \int \frac{d^3p}{(2\pi)^3} \{2(E_p - p) + [\theta(\mu_u - E_p)(\mu_u - E_p) + (\mu_u \rightarrow \mu_d)]\} - \frac{\mu_e^4}{12\pi^2}$$

where $E_p = \sqrt{p^2 + m^2}$, $\mu = -\partial V/\partial n$ is the quark chemical potential corresponding to the total quark number density $n$, $\mu_e$ is the chemical potential of electron and

$$\mu_u = \mu - \frac{2}{3}\mu_e, \mu_d = \mu + \frac{1}{3}\mu_e = \mu_u + \mu_e$$

are respectively the chemical potentials of the $u$ and $d$ quarks. The second equality in Eq.(3) is usually refers as beta equilibrium [9]. In vacuum we have $\mu = \mu_e = 0$, thus the effective potential (2) is reduced to

$$V_0(m) = \frac{m^2}{4G_S} - 12 \int \frac{d^3p}{(2\pi)^3}(E_p - p).$$

With a 3D momentum cutoff $\Lambda$, we may find out that the extreme value points of $V_0$ are 1) $m = 0$ and 2) $m = m_0$, where $m_0$ obeys the gap equation

$$\frac{1}{2G_S} = 3{(\sqrt{\lambda^2 + m_0^2} - m_0^2 \ln \frac{\Lambda + \sqrt{\lambda^2 + m_0^2}}{m_0})}. \quad (4)$$

It is easy to verify that if

$$1/2G_S < 3\lambda^2/\pi^2,$$

then $m = 0$ will be a maximum point and simultaneously Eq.(4) will have non-zero solution $m_0$ which is a minimum point. This means spontaneous breaking of chiral symmetry in vacuum. This will assumedly be our presupposition of discussions in this paper. Hence we may replace $1/2G_S$ in the effective potential $V(m, \mu, \mu_e)$ by using Eq.(4). As a result, we obtain from Eq.(2)

$$V(m, \mu, \mu_e) = \frac{3}{4\pi^2} \left\{ m^2 \left( 2\lambda \sqrt{\lambda^2 + m_0^2} - 2m_0^2 \ln \frac{\Lambda + \sqrt{\lambda^2 + m_0^2}}{m_0} \right) + 2\lambda^4 - \lambda \sqrt{\lambda^2 + m^2}(2\lambda^2 + m^2) + m^4 \ln \frac{\Lambda + \sqrt{\lambda^2 + m^2}}{m} \right\} + \frac{3}{4\pi^2} \left\{ \theta(\mu_u - m) \left[ \frac{m^2 \mu_u\sqrt{\mu_u^2 - m^2}}{2} - \frac{\mu_u(\mu_u^2 - m^2)^{3/2}}{3} - \frac{m^4}{2} \ln \frac{\mu_u + \sqrt{\mu_u^2 - m^2}}{m} + (\mu_u \rightarrow \mu_d) \right] \right\} - \frac{\mu_e^4}{12\pi^2} \quad (6)$$

For deriving electrical neutrality condition, it is noted that the electrical charge density in the two-flavor quark matter with electrons is

$$n_Q = \frac{2}{3}n_u - \frac{1}{3}n_d - n_e,$$

from which we may obtain

$$\mu_e = -\frac{\partial V}{\partial n_Q} = -\frac{\partial V}{\partial n_e} = -\mu_Q.$$

Hence the electrical neutrality condition will become $n_Q = -\partial V/\partial \mu_Q = \partial V/\partial \mu_e = 0$ and has the following explicit expression

$$\frac{\partial V}{\partial \mu_e} = \frac{1}{3\pi^2} \left[ 2\theta(\mu_u - m)(\mu_u^2 - m^2)^{3/2} - \theta(\mu_d - m)(\mu_d^2 - m^2)^{3/2} - \mu_e^3 \right] = 0,$$

$$\mu_d = \mu_u + \mu_e. \quad (7)$$

Eq.(7) is a restraint condition about $m, \mu_u$ and $\mu_e$. In fact, in the effective potential (6), instead of $\mu$ and $\mu_e$, we can first consider $\mu_u$ and $\mu_e$ as two chemical potential variables, then by Eq.(7), only one of them, the $u$-quark’s chemical potential $\mu_u$, is left as a single independent one, since $\mu_e$ may be viewed as a function of $\mu_u$ and $m$ by Eq.(7). This treatment will bring about great convenience for the discussions of phase transitions.

For research in ground state of the system, we must consider the extreme value points of $V(m, \mu, \mu_e)$ determined by the equation

$$-\theta(\mu_d - m)(\mu_d^2 - m^2)^{3/2} - \mu_e^3 = 0,$$
\[
\frac{\partial V}{\partial m} = \frac{3}{\pi^2} m \left\{ \left[ \Lambda \sqrt{\Lambda^2 + m_0^2} - m_0^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m_0^2}}{m_0} - (m_0 - m) \right] + \frac{1}{2} \left[ \theta(\mu_u - m) \left( \mu_u \sqrt{\mu_u^2 - m^2} - m^2 \ln \frac{\mu_u + \sqrt{\mu_u^2 - m^2}}{m} + (\mu_u \rightarrow \mu_\ell) \right) \right] \right\} = 0 \quad (8)
\]
and second derivation of \( V(m, \mu_u, \mu_\ell) \) over \( m \) under the constraint given by Eq.(7)

\[
\frac{d^2V}{dm^2} = \frac{\partial^2 V}{\partial m^2} = \left( \frac{\partial^2 V}{\partial m \partial \mu_c} \right)^2 / \partial^2 V / \partial \mu_c^2
\]

with

\[
\frac{\partial^2 V}{\partial m^2} = \frac{3}{\pi^2} \left\{ \Lambda \sqrt{\Lambda^2 + m_0^2} - m_0^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m_0^2}}{m_0} + 3m^2 \left[ \ln \frac{\Lambda + \sqrt{\Lambda^2 + m^2}}{m} - \frac{\Lambda}{\sqrt{\Lambda^2 + m^2}} \right] - \frac{\Lambda^3}{\sqrt{\Lambda^2 + m^2}} 
\]

\[
+ \frac{1}{2} \left[ \theta(\mu_u - m) \left( \mu_u \sqrt{\mu_u^2 - m^2} + 3m^2 \ln \frac{m}{\mu_u + \sqrt{\mu_u^2 - m^2}} + (\mu_u \rightarrow \mu_\ell) \right) \right] \}
\]

(10)
and \( \partial^2 V / \partial m \partial \mu_c \) and \( \partial^2 V / \partial \mu_c^2 \) can be obtained from \( \partial V / \partial \mu_c \) in Eq.(7) when Eq.(3) is taken into account.

Extremal feature of the point \( m = 0 \) is quite important for determination of behavior of the effective potential \( V \). Substituting \( m = 0 \), which is obviously a solution of the extreme value equation (8), into Eq.(7), we will obtain the electrical neutrality condition at \( m = 0 \)

\[ 2\mu_u^3 - (\mu_u + \mu_\ell)^3 - \mu_\ell^3 = 0. \quad (11) \]

Let \( \eta = \mu_\ell / \mu_u \), then from Eq.(11) we may obtain a real number solution \( \eta = 0.256 \). Furthermore, from Eqs.(9) and (10), second derivative of \( V \) over \( m \) at \( m = 0 \) becomes

\[
\frac{d^2V}{dm^2} \bigg|_{m=0} = \frac{\partial^2 V}{\partial m^2} \bigg|_{m=0} = \frac{3}{2\pi^2} \left[ 1 + (1 + \eta)^2 \right] (\mu_u^2 - \mu_{uc}^2)
\]

(12)
where

\[ \mu_{uc}^2 = \frac{2}{1 + (1 + \eta)^2} \left[ \frac{m_0^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m_0^2}}{m_0}}{-\Lambda \sqrt{\Lambda^2 + m_0^2}} \right. + \left. \frac{\sqrt{\Lambda^2 + m_0^2} + \Lambda^2}{\Lambda} \right]. \quad (13) \]

Hence, \( m = 0 \) will be a maximum (minimum) point of \( V \) if \( \mu_u^2 < \mu_{uc}^2 \) (\( \mu_u^2 > \mu_{uc}^2 \)), and the extremal feature of \( m = 0 \) will be determined by derivatives of higher order of \( V \) over \( m \) if \( \mu_u^2 = \mu_{uc}^2 \).

III. SECOND ORDER PHASE TRANSITION IN SMALL \( m_0 / \Lambda \)

It is easy to verify that when \( \mu_u = \mu_\ell = 0 \) (or equivalently, \( \mu = \mu_\ell = 0 \)), the effective potential given by Eq.(6) will reproduce the spontaneous chiral symmetry breaking in vacuum and the quarks get the dynamical mass \( m = m_0 \). Starting from this, we will first analysis possible chiral symmetry restoring by second order phase transition as the chemical potential \( \mu_u \) increases and derive the \( \mu_u - m_0 \) critical curve of the phase transition. The variation of \( V(m, \mu_u, \mu_\ell) \) as increase of \( \mu_u \) will be discussed successively.

1) \( 0 < \mu_u < m_0 < \mu_{uc} \). Here we impose the limitation \( m_0 < \mu_{uc} \) which, by Eq.(13), implies that \( \Lambda / m_0 > 2.865 \), i.e. we are confined to the region \( m_0 / \Lambda < 0.349 \). In this case, from Eqs.(7)-(12), the effective potential \( V \) will have the minimum point \( m = m_0 \) with \( \mu_u = 0 \) which comes from the electrical neutrality condition for \( \mu_u < m \), and the maximum point \( m = 0 \) with the electrical neutrality condition \( \mu_\ell = \eta \mu_u \) for \( \mu_u > m \). A following question is that, for the case of \( \mu_u > m \), whether there is any other non-zero solution satisfying the electrical neutrality condition (7) for the gap equation \( (\partial V / \partial m) / m = 0 \) coming from Eq.(8)? To answer this question, we note that in this case the above two equations may be reduced to

\[ 2(\mu_u^2 - m^2)^{3/2} - (\mu_u^2 - m^2)^{3/2} = \mu_\ell^3 \quad (14) \]

and

\[ m_0^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m_0^2}}{m_0} - \Lambda \sqrt{\Lambda^2 + m_0^2} = \]

\[ \frac{1}{2} m_0^2 \ln \left( \frac{\mu_u + \sqrt{\mu_u^2 - m_0^2}}{\mu_u + \sqrt{\mu_u^2 - m_0^2}} \right)^2 \]

\[ + \frac{1}{2} (\mu_u \sqrt{\mu_u^2 - m_0^2} + \mu_\ell \sqrt{\mu_\ell^2 - m_0^2} - \Lambda \sqrt{\Lambda^2 + m_0^2}). \quad (15) \]
Denote that
\[ a = \Lambda/m_0, \ x = \mu_u/m_0, \ \beta = m/\mu_u, \ \alpha = \mu_c/\mu_u \]
then Eqs.(14) and (15) can be changed into
\[ 2(1 - \beta^2)^{3/2} - [(1 + \alpha)^2 - \beta^2]^{3/2} = \alpha^3 \]  
(16)
and
\[ \ln(a + \sqrt{a^2 + 1} - a\sqrt{a^2 + 1} = \frac{x^2\beta^2}{2}\ln\frac{(a + \sqrt{a^2 + x^2\beta^2})^2}{x^2(1 + \sqrt{1 - \beta^2})(1 + \alpha + \sqrt{(1 + \alpha)^2 - \beta^2})} \]
\[ + \frac{x^2}{2} \frac{\partial}{\partial \mu_u} \ln(\mu_u) \]
(17)

It is easy to check that when \( a \geq 2.865 \), Eqs.(16) and (17) have no solution with \( x < 1 \) and \( \beta < 1 \). In other words, when \( a^{-1} = m_0/\Lambda \leq 0.349 \), the gap equation \((\partial V/\partial m)/m = 0\) and the electrical neutrality equation (7) have no solution with \( \mu_u < m_0 \) and \( m < \mu_u \) indeed.

To sum up, when \( \mu_u < m_0 < \mu_{uc} \), the effective potential will have only a maximum point \( m = 0 \) and a minimum point \( m = m_0 \) accompanied with \( \mu_c = 0 \). The latter corresponds to ground state of the system which is similar to the case of vacuum. This shows that spontaneous chiral symmetry breaking in vacuum will be maintained in the case of \( \mu_u < m_0 \) and \( \mu_c = 0 \). The dynamical quark mass \( m_0 \) could be changed only if \( \mu_u > m_0 \).

2) \( m_0 < \mu_u < \mu_{uc} \). In this case, \( m = 0 \) is still a maximum point of \( V \) by Eq.(12). On the other hand, it is not difficult to see that now Eqs.(7) and (8) have no solution with \( \mu_u < m \). Thus we are left only the case of \( \mu_u > m \) and the gap equation \((\partial V/\partial m)/m = 0\) and the electrical neutrality condition will take the forms of Eqs.(14) and (15). In view of the definition of \( \mu_{uc}^2 \), given by Eq.(13) and the constraint \( \mu_u^2 < \mu_{uc}^2 \), it may be deduced that Eqs.(14) and (15) have non-zero solution \((m_1, \mu_{e1})\). From Eq.(9) we obtain that

\[ \left. \frac{\partial^2 V}{\partial m^2} \right|_{(m_1, \mu_{e1})} = \left. \frac{\partial^2 V}{\partial m^2} \right|_{(m_1, \mu_{e1})} - \left. \frac{\partial^2 V}{\partial m^2} \right|_{(m_1, \mu_{e1})} \]

(18)

The second term in the right-handed side of Eq.(18) is positive since \( \partial^2 V/\partial \mu_{uc}^2 \) derived by Eq.(7) is always negative. The first term may be calculated with the result

\[ \left. \frac{\partial^2 V}{\partial m^2} \right|_{(m_1, \mu_{e1})} = \frac{6}{\pi^2} \left\{ m_0^2 \ln \frac{(\Lambda + \sqrt{\Lambda^2 + m_1^2})^2}{(\mu_u + \sqrt{\mu_u^2 - m_1^2})(\mu_u + \sqrt{\mu_u^2 - m_1^2})} \right\} \mu_{e1} \]
\[ = \frac{6}{\pi^2} \left\{ m_0^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m_0^2}}{m_0} - \frac{m_1^2 \Lambda}{\sqrt{\Lambda^2 + m_1^2}} \right\} \mu_{e1} \]
\[ > \frac{6}{\pi^2} \left\{ \frac{1}{2} (\mu_u^2 + \mu_{uc}^2) - \frac{1}{2} (\mu_u \sqrt{\mu_u^2 - m_1^2} + \mu_{uc} \sqrt{\mu_{uc}^2 - m_1^2}) - \Lambda^2 \left( 1 - \frac{\Lambda}{\sqrt{\Lambda^2 + m_1^2}} \right) \right\} \mu_{e1} > 0, \]
where we have used Eq.(15) with \( m \) replaced by \( m_1 \) and the condition \( \mu_u^2 < \mu_{uc}^2 \), or equivalently,

\[ \frac{1}{2} (1 + (1 + \eta)^2) \mu_{uc}^2 > \frac{1}{2} (1 + (1 + \eta)^2) \mu_u^2 > \frac{1}{2} (\mu_u^2 + \mu_{uc}^2) \]

In this way, it is proven that

\[ \frac{\partial^2 V}{\partial m^2} \] at the minimum point of \( V \), so it will correspond to the ground state of the system satisfying electrical neutrality.

It may be found by examining Eq.(15) that when \( \mu_u = \)
expressed by Eq.(14) may be found out to be positive, then the first term has to decrease so that the original \( m_0 \) will change to \( m < m_0 \), and going up of the fourth term \(- \Lambda \sqrt{\Lambda^2 + m^2}\) is consistent with the reduction of \( m \) in the first term. This means that \( m \) will decrease from \( m_0 \) and finally it may continuously reduce to zero, thus we come to a critical point of second order phase transition at which the broken chiral symmetry will be restored. The second order \( \mu_u - m_0 \) critical curve will be denoted by \( C_2(m_0) \) whose equation can be obtained by setting \( m = 0 \) in Eqs.(14) and (15) and has the explicit expression

\[
\mu_u = \mu_{uc} = \left\{ \begin{array}{ll}
\frac{2}{1 + (1 + \eta)^2} \left[ m_0^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m_0^2}}{m_0} - \Lambda \sqrt{\Lambda^2 + m_0^2 + \Lambda^2} \right]^{1/2} &= C_2(m_0).
\end{array} \right.
\]

3) \( m_0 < \mu_u = \mu_{uc} \). In this case, the extreme value equation (8) has the only solution \( m = 0 \) and the electrical neutrality condition (7) gives \( \mu_e = \eta \mu_u \) with \( \eta = 0.256 \). At the only extreme point \( m = 0 \) of \( V \), the n-th derivative of \( V \) over \( m \) under the electrical neutrality condition expressed by Eq.(14) may be found out to be

\[
d^nV \bigg|_{m=0} = \left\{ \begin{array}{ll}
0, & \text{when } n = 2, 3, 5 \\
\frac{9}{\pi^2} \ln \frac{\Lambda^2}{\mu_{uc}^2 G(\eta)}, & \text{when } n = 4 \\
\frac{15}{\pi^2} \left( 9 \frac{\Lambda^2}{\mu_{uc}^2} + 1.744 \right), & \text{when } n = 6
\end{array} \right.
\]

(20)

where we have used the denotation

\[
G(\eta) = (1 + \eta) \exp \left( \frac{9 + 6\eta + 7\eta^2}{5 + 2\eta + 4\eta^2} \right).
\]

Eq.(20) implies that when

\[
\mu_{uc}^2 \leq \Lambda^2 G^{-1}(\eta),
\]

(21)

\( m = 0 \) will be the only minimum point of \( V \) and the broken chiral symmetry will be restored in a second order phase transition. In view of Eq.(19), the condition (21) also means the constraint

\[
\frac{m_0}{\Lambda} \leq 0.342,
\]

(22)
i.e. in the region \( 0 < m_0/\Lambda \leq 0.342, \mu_u = \mu_{uc} \) is a curve of second order phase transition. It is noted that the constraint (22) is consistent with the presupposition \( m_0 < \mu_{uc} \), i.e. \( m_0/\Lambda < 0.349 \).

4) \( \mu_u > \mu_{uc} \). We still confine ourselves to the case of \( \mu_{uc} \leq \Lambda^2 G^{-1}(\eta) \). Obviously, by Eq.(12) and \( \mu_u > \mu_{uc} \), \( m = 0 \) is now a minimum point of \( V \). But one can raise such a question that in the above condition, whether \( \partial V / \partial \mu_e = 0 \) and \( (\partial^2 V / \partial m^2) / m = 0 \) also have some non-zero \( m \) solutions with \( m < \mu_u \)? To answer this query, we can rewrite the above equations, i.e. Eqs.(14) and (15) by

\[
2(1 - \beta^2)^{3/2} - [(1 + \alpha)^2 - \beta^2]^{3/2} = \alpha^3
\]

(23)

and

\[
\frac{1}{2}[1 + (1 + \eta)^2] = \frac{\beta^2 \gamma^2}{2} \ln \frac{(b + \sqrt{b^2 + \beta^2 \gamma^2})^2}{\gamma^2(1 + \sqrt{1 - \beta^2})(1 + \alpha + \sqrt{(1 + \alpha)^2 - \beta^2})} + \gamma^2 \frac{2}{2} \left[ \sqrt{1 - \beta^2} + (1 + \alpha) \sqrt{(1 + \alpha)^2 - \beta^2} \right] + b^2 - b \sqrt{b^2 + \beta^2 \gamma^2},
\]

(24)

where we have used the denotations

\[
\alpha = \mu_e / \mu_u, \quad \beta = m / \mu_u, \quad \gamma = \mu_e / \mu_{uc}
\]

\[
b^2 = \Lambda^2 / \mu_{uc}^2 = [1 + (1 + \eta)^2] a^2 / 2 \left[ \ln(a + \sqrt{a^2 + 1}) - a \sqrt{a^2 + 1 + a^2} \right], \quad a = \Lambda / m_0,
\]

(25)

It turns out by numerical calculation that Eqs.(23) and (24) may have solutions with \( \beta < 1 \) \((m < m_0)\) and \( \gamma > 1 \) \((\mu_u > \mu_{uc})\) except that \( a = \Lambda / m_0 \) is quite large (e.g. \( a \geq 5 \)). In other words, when \( \mu > \mu_{uc} \), the effective potential \( V \) could have extreme value point with \( m \neq 0 \). However, it can be proven that \( m = 0 \) is always the least
FIG. 1: The $\mu_u - m_0$ phase diagram of the model. Here $\mu_u$ and $m_0$ are both scaled by the momentum cutoff $\Lambda$. The straight line through the origin and the point $C$ is $\mu_u = m_0$. The critical curve $\mu_u = C_2(m_0)$ of second order phase transitions starts from the origin, through the points $A$ and $C$, ends at the point $B$. The critical curve $\mu_u = C_1(m_0)$ of first order phase transition begins from the point $B$ then extends to the right in the region above $\mu = \mu_{uc}$. $B$ is a tricritical point.

minimum point. In fact, by a direct calculation, we may obtain the difference between the values of $V$ at the extreme value points $m \neq 0$ and $m = 0$ expressed by

$$V|_{m \neq 0} - V|_{m = 0} = \frac{3}{4\pi^2} \left\{ 2\Lambda^4 - 2\Lambda^3 \sqrt{\Lambda^2 + m^2} + \Lambda^2 m^2 - \frac{1}{2} \left[ 1 + (1 + \eta)^2 \right] \mu_{uc}^2 m^2 \right\}$$

$$- \frac{1}{12\pi^2} \left\{ 3\mu_u (\mu_u - m_0)^{3/2} + 3(\mu_u + \mu_e) [(\mu_u + \mu_e)^2 - m_0^2^{3/2} + \mu_e^2] \right\} |_{\mu_e = \alpha \mu_u}$$

$$+ \frac{1}{12\pi^2} \left[ 3\mu_u^4 + 3(\mu_u + \mu_e)^4 + \mu_e^4 \right] |_{\mu_e = \eta \mu_u}$$

$$\geq \frac{\mu_{uc}^4}{4\pi^2} \left\{ 3 \left[ 2b^4 - 2b^3 \sqrt{b^2 + \beta^2 \gamma^2} + b^2 \beta^2 \gamma^2 - \frac{1 + (1 + \eta)^2}{2} \beta^2 \gamma^2 \right] - (1 - \beta^2)^{3/2} 

- (1 + \alpha) \left[ (1 + \alpha)^2 - \beta^2 \gamma^2 - \alpha^2/3 + 1 + (1 + \eta)^4 \right] + \eta^4/3 \right\}, \text{ when } \mu_u > \mu_{uc}.$$  \tag{26}

In the condition $\mu_{uc}^2 \leq \Lambda^2 G^{-1}$ or $b^2 \geq G(\eta)$, substituting all the possible solutions of $\alpha, \beta$ and $\gamma$ obtained from Eqs.(23) and (24) into Eq.(26), we will always obtain that $V|_{m \neq 0} - V|_{m = 0} > 0$. This indicates that $m = 0$ is indeed the least minimal value point of $V$ and when $\mu_u > \mu_{uc}$, the chiral symmetry has been restored through a second order phase transition in the case of $\mu_{uc}^2 \leq \Lambda^2 G^{-1}$. FIG. 1 is the complete $\mu_u - m_0$ phase diagram of the model. In this diagram, the discussed second order phase transition above will correspond to the segment of the curve $\mu_u = C_2(m_0)$ from the origin to the point $A$ whose location is determined by the equality $C_2(m_0) = \Lambda G^{-1/2}(\eta)$. 

IV. SECOND AND FIRST ORDER PHASE TRANSITIONS IN LARGER $m_0/\Lambda$

It may be seen from Eq.(20) that, when going along the curve $\mu_u = \mu_{uc}$ toward the region with $\mu_{uc}^2 > \Lambda^2 G^{-1}(\eta)$, or equivalently, in view of Eq.(22), $m_0/\Lambda > 0.342$, one will get $m = 0$ becoming a maximum point of $V$, however when $\mu_u > \mu_{uc}$, $m = 0$ is again a minimum point. Such change of minimax property of $m = 0$ could lead to two possibilities: either a second order phase transition will continue or a first order phase transition will happen. For examining a concrete realization of the above two possibilities, we will start from the equations to determine the critical curve of a first order phase transition. In electrical neutrality condition, these equations read

$$V(m = 0) = V(m = m_1), \quad \frac{\partial V}{\partial \mu_e} \bigg|_{m = 0} = 0, \quad \frac{\partial V}{\partial \mu_e} \bigg|_{m = m_1} = 0,$$

$$\frac{\partial V}{\partial m} \bigg|_{m = m_1} = 0, \quad \frac{\partial^2 V}{\partial m^2} \bigg|_{m = 0} > 0. \quad (27)$$

Since $\partial V/\partial \mu_e |_{m = 0} = 0$ determines only the ratio $\eta = \mu_e/\mu_{uc}$ at $m = 0$, Eqs.(27) will have the following explicit expressions:

$$\frac{\mu^4}{3} \left[ 1 + (1 + \eta)^4 + \eta^4 / 3 \right] = m_1 \left[ 1 + (1 + \eta)^2 \right] - 2\Lambda^4 + 2\Lambda^3 \sqrt{\Lambda^2 + m_1^2}
+ \theta(\mu_u - m_1)\mu_u(\mu_u^2 - m_1^2)^{3/2} / 3 + (\mu_u \rightarrow \mu_d)] + \mu_e^4 / 9, \quad (28)$$

$$2\theta(\mu_u - m_1)(\mu_u^2 - m_1^2)^{3/2} - \theta(\mu_d - m_1)(\mu_d^2 - m_1^2)^{3/2} - \mu_e^3 = 0, \quad (29)$$

$$\frac{1 + (1 + \eta)^2}{2} \mu_{uc}^2 = \Lambda^2 - \Lambda \sqrt{\Lambda^2 + m_1^2} + m_1^2 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m_1^2}}{m_1}
+ \frac{1}{2} \left\{ \theta(\mu_u - m_1) \left[ \mu_u \sqrt{\mu_u^2 - m_1^2} \ln \frac{\mu_u + \sqrt{\mu_u^2 - m_1^2}}{m_1} \right] + (\mu_u \rightarrow \mu_d) \right\} \quad (30)$$

and

$$\mu_e^2 > \mu_{uc}^2. \quad (31)$$

We will discuss respectively the two cases of $\mu_{uc} > m_0$ and $\mu_{uc} < m_0$ which could appear when $\mu_{uc}^2 > \Lambda^2 G^{-1}(\eta)$.

1) $\mu_u > \mu_{uc} > m_0$. Since $\mu_{uc} > m_0$ can be satisfied only if $m_0/\Lambda < 0.349$ and in view of Eqs.(21) and (22),

$$\frac{\gamma^4}{3} \left[ 1 + (1 + \eta)^4 + \eta^4 / 3 \right] = \beta^2 \gamma^2 \left[ 1 + (1 + \eta)^2 - b^2 \right] - 2\beta^4 + 2\beta^3 \sqrt{b^2 + \beta^2 \gamma^2}
+ \frac{\gamma^4}{3} \left\{ (1 - \beta^2)^{3/2} + (1 + \alpha)[(1 + \alpha)^2 - \beta^2]^{3/2} + \gamma^4 / 3 \right\}, \quad (32)$$

Eqs.(29) and (30) will be separately identical to Eqs.(23) and (24), and Eq.(31) will simply become $\gamma > 1$, where we have again used the denotations given by Eq.(25). By numerical solution of Eqs. (23), (24) and (32), we have found that in the whole region $0.342 < m_0/\Lambda < 0.349$, there is no solution with $m_1 \neq 0$. This means that in this region a first order phase transition could not happen. In fact, the obtained solution is $m_1 = 0$, which makes Eq.(32) becomes a trivial identity and Eqs.(23) and (24) are reduced to $\mu_e/\mu_u = \eta$ and $\mu = \mu_{uc}$. The last two
equalities are precisely the equation of the curve $C_2(m_0)$. Hence we can conclude that in this region, the phase transition remains to be second order.

2) $\mu_{uc} < \mu_e < m_0$. In FIG. 1 this corresponds to the right-handed side of the point $C$. For $m < \mu_{uc}$, it is easy to verify that the electrical neutrality equation (23) and the gap equation (24) have non-zero solution $m_1$ for $\mu > \mu_{uc}$. For $m > \mu_{uc}$, by solving Eqs.(29) and (30), it is obtained that $m = m_0$ with $\mu_e = 0$ is a minimum point. Hence, with $m_1 = m_0$ and $\mu_e = 0$ being taken, Eq.(28) which determines first order phase transition point may be changed into

$$\mu = \left[\frac{3}{1 + (1 + \eta)^4 + \eta^4/3} \left(m_0^4 \ln \frac{\Lambda + \sqrt{\Lambda^2 + m_0^2}}{m_0} - m_0^2 \Lambda \sqrt{\Lambda^2 + m_0^2 - 2\Lambda^2 + 2\Lambda^3 + m_0^2}}\right)^{1/4} \equiv C_1(m_0). \quad (33)$$

Eq.(33) expresses the equation of the curve $C_1(m_0)$. $C_1(m_0)$ may becomes a first order phase transition curve only if $\mu_u = C_1(m_0) > \mu_{uc}$. From this constraint we obtain

$$m_0/\Lambda \geq 0.413 \quad (34)$$

which is obviously in the right-handed side of the point $C$. At $m_0/\Lambda = 0.413$, the curve $\mu_u = C_1(m_0)$ and the curve $\mu_u = C_2(m_0)$ intersects. The intersection point denoted by $B$ becomes the starting point of the curve $\mu_u = C_1(m_0)$ above $\mu > \mu_{uc}(m_0)$.

In the region $0.349 < m_0/\Lambda < 0.413$, i.e. in the segment between the points $C$ and $B$, similar to the case of $\mu_{uc} > m_0$, it can be proven that Eqs.(23), (24) and (32) have only the solution $m_1 = 0$, $\mu_e/\mu_u = \eta$ and $\mu = \mu_{uc}$, i.e. the solutions are actually reduced to the second order phase transition curve $\mu_u = C_2(m_0)$. Therefore, in the $C-B$ segment, we still have second order phase transition represented by the critical curve $\mu_u = C_2(m_0)$.

In summary, in the region with $\mu_{uc}^2 > \Lambda^2 G^{-1}(\eta)$ or $m_0/\Lambda > 0.342$, the critical curve $\mu_u = \mu_{uc} = C_2(m_0)$ of second order phase transition may be extended to the point $B$ where $m_0/\Lambda = 0.413$, then a critical curve $\mu_u = C_1(m_0)$ of first order phase transition will start from the point $B$ in the region of $\mu_u > \mu_{uc}$. So the point $B$ is a tricritical point.

It should be indicated that in the case without and with electrical neutrality constraint, the feature of phase transition of the NJL model is different. Without electrical neutrality condition, as was discussed in Ref.[4], the second order phase transition curve will end at a similar point $A$, then from $A$ through the point $C$ straight to the right-handed side of the point $B$, one will always have a first order phase transition curve. However, in present case with electrical neutrality requirement, from $A$ to $B$, one continue to get a second order phase transition, instead of a first order one.

The total conclusions of this paper come from a general research of the used NJL model, i.e. the parameters of the model, the 3D momentum cutoff $\Lambda$ and the dynamical quark mass $m_0$ in vacuum (correspondingly, the four-fermion coupling constant $G_S$) have been considered as arbitrary ones. If the discussed NJL model is used to simulate QCD for normal quark matter and the conventional phenomenological values of the parameters $\tilde{F}$

$$G_S = 5.0163 GeV^{-2}, \Lambda = 0.6533 GeV$$

are taken, then by Eq.(4) we will obtain $m_0/\Lambda = 0.48$. Based on the results of present paper, chiral symmetry restoring at high density in this model must be first order phase transition.

V. CONCLUSIONS

In this paper, we have generally analyzed chiral symmetry restoring phase transitions at zero temperature and high density in a NJL model to describe two-flavor normal quark matter under electrical neutrality condition. It has been found that the feature of phase transitions is decided by the ratio $m_0/\Lambda$, where $m_0$ is the dynamical quark mass in vacuum and $\Lambda$ is the 3D momentum cutoff of the loop integrals. Depending on $m_0/\Lambda$ is less or greater than 0.413, the phase transition will be second or first order. As a comparison, the resulting region where a second order phase transition happens is wider than the one in the case without electrical neutrality constraint. For the value of $m_0/\Lambda$ based on QCD phenomenology, the phase transition must be first order.

The present discussions of normal quark matter based on a NJL model can be generalized to the case of color superconducting quark matter where one must also consider diquark condensates, besides the quark-antiquark condensates, and at the same time, also impose color neutrality condition, besides electrical neutrality one.
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