Absence of topology in Gaussian mixed states of bosons

Christopher D. Mink,1 Michael Fleischhauer,1 and Razmik Unanyan1
1Department of Physics and Research Center OPTIMAS, University of Kaiserslautern, Germany
(Dated: January 15, 2019)

In a recent paper [Bardyn et al. Phys. Rev. X 8, 011035 (2018)], it was shown that the generalization of the many-body polarization to mixed states can be used to construct a topological invariant which is also applicable to finite-temperature and non-equilibrium Gaussian states of lattice fermions. The many-body polarization defines an ensemble geometric phase (EGP) which is identical to the Zak phase of a fictitious hamiltonian, whose symmetries determine the topological classification. Here we show that in the case of Gaussian states of bosons the corresponding topological invariant is always trivial. This also applies to finite-temperature states of bosons in lattices with a topologically non-trivial band-structure. As a consequence there is no quantized topological charge pumping for translational invariant bulk states of non-interacting bosons.

I. INTRODUCTION

Topological states of matter have fascinated physicists for many decades as they can give rise to phenomena such as protected edge states and edge currents [1], quantized bulk transport in insulating states [2,3] and exotic elementary excitations [8,10]. Recently, several attempts were made to generalize the concept of topology to finite-temperatures and to non-equilibrium steady states of non-interacting fermion systems [12–15]. This has been done for fundamental reasons and because of the intrinsic robustness of steady states of driven-dissipative systems. Integer quantized topological invariants such as the winding of the Berry or Zak phase [19–22] or the Chern number. Famous examples for this are the charge transport in a Thouless pump [4,5,23] or the Hall conductivity in Chern insulators [24–26]. For finite temperatures or under non-equilibrium conditions these quantities are no longer quantized [24]. Furthermore, defining single-particle invariants becomes difficult as the system generally is in a mixed state. While for one-dimensional systems generalizations of geometric phases to density matrices based on the Uhlmann construction [25] can be used [13,15], their application to higher dimensions [14] is faced with difficulties [26].

In a recent paper [18], it was shown that the winding of the many-body polarization introduced by Resta [27] is an alternative and useful many-body topological invariant for Gaussian states of fermions. The polarization of a non-degenerate ground-state |ψ⟩ corresponding to a filled band of a lattice hamiltonian with periodic boundary conditions is the phase (in units of 2π) induced by a momentum shift $\tilde{T}$

$$P = \frac{1}{2\pi} \text{Im} \log \langle \psi | \tilde{T} | \psi \rangle.$$  

$\tilde{T}$ shifts the lattice momentum $p_k = 2\pi k / L$ of all particles by one unit $\tilde{T}^{-1} \hat{c}_{\alpha,k} \tilde{T} = \hat{c}_{\alpha,k+1}$, where L is the number of unit cells and $\alpha$ a band index. As shown by King-Smith and Vanderbilt [28], expression (1) for a filled Bloch band is identical to the geometric Zak phase $\phi_{Zak}$ of this band.

$P$ can straightforwardly be generalized to mixed states $\rho$ and defines the ensemble geometric phase (EGP) $\phi_{\text{EGP}}$:

$$\phi_{\text{EGP}} = \text{Im} \log \text{Tr} \{ \rho \tilde{T} \}.$$  

As shown in [18], the EGP of a Gaussian density matrix is reduced to the ground-state Zak phase of a fictitious Hamiltonian in the thermodynamic limit $L \to \infty$. The symmetries of this fictitious Hamiltonian determine the topological classification [12] following the scheme of Altland and Zirnbauer [29–31]. A phase transition between different topological phases occurs when the gap of the fictitious Hamiltonian closes where $\text{Tr} \{ \rho \tilde{T} \} = 0$. The many-body polarization is a measurable physical quantity [18] and its quantized winding has direct physical consequences. E.g. it can induce quantized transport in an auxiliary system weakly coupled to a finite-temperature or non-equilibrium system [32]. Since in the case of finite temperatures the gapfulness of the many-body state is no longer given, the question arises whether bosonic Gaussian systems can show non-trivial topological properties as well.

In the present paper we show rigorously that topological invariants based on the many-body polarization are always trivial for Gaussian states of bosons. As a consequence there is e.g. no protected quantized charge pump for bosons under periodic, adiabatic variations of system parameters. This also means there is no analogue to the quantization of the Hall conductance for non-interacting bosons.

II. THE BOSONIC RICE-MELE MODEL

Bloch hamiltonians with a topologically non-trivial band structure can lead to non-trivial many-body invariants of non-interacting fermions if all single-particle states of the corresponding band(s) are filled. In such states the many-body polarization can show e.g. a non-trivial winding under cyclic parameter variations. Surprisingly, the latter property survives at finite temperatures, i.e. even if the considered band is no longer fully occupied. Therefore one may ask if the many-body polarization can show non-trivial behavior in the case of
non-interacting bosons?
To illustrate what happens in such a case let us consider one of the simplest 1D lattice models with single-particle topological properties, the Rice-Mele model (RMM) [23]. It has a unit cell consisting of two lattice sites with different on-site energies $\pm \Delta$ and describes the hopping of particles with alternating hopping amplitudes $t_{1/2}$ (see inset of Fig.1). The Hamiltonian reads

$$H = -t_1 \sum_j \hat{a}_j^\dagger \hat{b}_j - t_2 \sum_j \hat{a}_{j+1}^\dagger \hat{b}_j + h.a.$$  \quad (3)

where $\hat{a}_j, \hat{b}_j$ are particle annihilation operators at the two sites of the $j$th unit cell and we assume periodic boundary conditions. This model is well-known to have a non-trivial winding of the Zak-phase [21]

$$\phi_{\text{Zak}} = \int_{BZ} dk \langle u_n(k)|\partial_k|u_n(k)\rangle$$ \quad (4)

of any of the two subbands $n = 1, 2$ upon cyclic variations of the parameters $\Delta, t_1 - t_2$ encircling the origin ($\Delta = 0, t_1 = t_2$) where the band gap closes. Here $|u_n(k)\rangle$ are the single-particle Bloch states of the $n$th band at lattice momentum $k$. Performing such a loop adiabatically, one can induce bulk transport if one subband is filled with fermions. At the same time also the many-body polarization shows a non-trivial winding which, as shown by King-Smith and Vanderbilt, is strictly connected to the winding of $\phi_{\text{Zak}}$ [28].

Let us now consider the bosonic analogue of the RMM. To be specific, let us assume the parameterization $t_{1/2} = (t_0 \pm \delta(t))/2$, $\Delta(t) = A \cos(2\pi t/T)$, $\delta(t) = B \sin(2\pi t/T)$ and $t_0 = B$, where $T$ is the time of one cycle. If initially only one unit cell is occupied, the center of mass of the wavepacket moves by exactly one unit cell after a full cycle. This is because this special initial state has equal amplitudes in all momentum eigenmodes of the band. The situation is very different however, when we consider a translationally invariant system, where the many-body state returns to itself after a full cycle modulo a phase factor. For $t = 0$ and $A \gg B$ such a translationally invariant state of lowest energy with average occupation of one per unit cell is the product of coherent states

$$|\psi(0)\rangle = \prod_j |\alpha = 0\rangle_j |\beta = 1\rangle_j$$ \quad (5)

corresponding to the $a$ and $b$ sites respectively. Since the Hamiltonian is quadratic, the state remains a product of coherent states at all times

$$|\psi(t)\rangle = \prod_j |\alpha(t)\rangle_j |\beta(t)\rangle_j, \quad |\alpha(t)|^2 + |\beta(t)|^2 = 1.$$ \quad (6)

Let us now consider the particle transport after a full period $T$ of the topological pump. In Fig.1 we have shown the integrated particle flux through the $n$th unit cell

$$\Phi_n = -i \int_0^T dt \langle \psi(t)|t_1\rangle(t)(\hat{a}_n \hat{b}_n^\dagger - \hat{b}_n \hat{a}_n^\dagger)|\psi(t)\rangle$$ \quad (7)

for different values of the rescaled cycle time $RT$ where $R = A = B$. One notices that there is a net particle transport but it can take on vastly different values and is not quantized after an integer number of cycles. Thus quantized transport requires fine tuning of the initial state and is completely absent when an initial state is considered that is invariant under lattice translations.

Since the polarization $\Phi_n$ is the phase of a complex function (modulo $2\pi$) it can only change by an integer valued amount upon a full cycle of evolution, provided there are no transitions to other states. The latter can be guaranteed by an adiabatic evolution. Since for any finite system there is a finite-size gap, adiabatic evolution is possible by a sufficiently large cycle time. One finds that the polarization winding of the bosonic Rice-Mele model always vanishes. In fact one can easily calculate the polarization at any time $t$ exactly. Fixing the gauge, i.e. fixing the origin of the spatial coordinate on the circle...
of length $L$, one finds
\[
P = \frac{1}{2\pi} \text{Im} \ln \prod_j [\alpha(t)\beta(t)] \exp\left\{ -\frac{2\pi i}{L} \left( j\hat{a}_j\hat{a}_j + \left( j + \frac{1}{2} \right) \hat{b}_j^\dagger \hat{b}_j \right) \right\} \alpha(t)\beta(t) \tag{8}
\]
\[= \frac{1}{2\pi} \text{Im} \ln \prod_j \left\{ \exp\left\{ (e^{\frac{2\pi i}{L} j} - 1) |\alpha(t)|^2 \right\} \times \exp\left\{ (e^{\frac{2\pi i}{L} (j + \frac{1}{2})} - 1) |\beta(t)|^2 \right\} \right\} = 0,
\]
where we have evaluated the unitary operator $\hat{T}$ in normally ordered form given by
\[
\hat{T} = \prod_{r,s} \exp\left\{ \left( e^{\frac{2\pi i}{L} (r + s/n)} - 1 \right) \hat{a}_{r,s}^\dagger \hat{a}_{r,s} \right\} \tag{9}
\]
The polarization is therefore constant in time. Clearly, there is no connection between the particle flux and the change of the many-body polarization. But it is even more surprising that the latter does not wind irrespective of the path taken in parameter space. We will show that the absence of polarization winding is a generic feature of Gaussian bosonic systems which is in sharp contrast to the fermionic analogue.

III. POLARIZATION FOR BOSONS

The goal of this section is to calculate the expectation value of the unitary operator
\[
\hat{T} = \exp\left( \frac{2\pi i}{L} \sum_{r,s} \left( r + \frac{s}{n} \right) \hat{a}_{r,s}^\dagger \hat{a}_{r,s} \right) \tag{10}
\]
for an arbitrary Gaussian state. Here $a_{r,s}^\dagger, a_{r,s}$ are bosonic creation and annihilation operators respectively, where $r = 0, \ldots, L-1$ labels unit cells and $s = 0, \ldots, n-1$ internal sites in the unit cell. $0 \leq \frac{s}{n} < 1$ and we have set the lattice constant equal to unity. The results of the following discussion do also not depend on the dimension of the system nor the total number of particles. We note that the operator $\hat{T}$ is not gauge invariant because it changes under an arbitrary shift of the origin of the spatial coordinate system. Throughout this paper we choose a coordinate system in which $\exp\left( \frac{2\pi i}{L} (r + \frac{s}{n}) \right) \neq 1$ for any $r, s$.

We consider a general bosonic Gaussian state $\rho$ which can be formally expressed in diagonal form (Glauber-Sudarshan representation) in terms of multi-mode coherent states
\[
\rho = \int d^2 \alpha \, \mathcal{P}(\alpha) \, |\alpha\rangle \langle \alpha|,
\]
where $d^2 \alpha = d\alpha_x d\alpha_y$, with $\alpha_\epsilon = (\alpha + \alpha^*)/2$ and $\alpha_i = (\alpha - \alpha^*)/(2i)$ being the real and imaginary parts of the coherent amplitude
\[
\mathcal{P}(\alpha) = N \int d^2 \eta \exp\left\{ -\frac{1}{2} \eta^T (\mathbf{V} - \mathbf{I}) \eta - i (2\alpha + \alpha^*)^T \eta \right\}.
\]
Here $\mathbf{I}$, $\alpha = ((\alpha_1, r, \alpha_1, i), (\alpha_2, r, \alpha_2, i), \ldots)$ and $\eta = ((\eta_1, r, \eta_1, i), (\eta_2, r, \eta_2, i), \ldots)$ represent the identity matrix and real vectors respectively with dimension $2nL$ (note that $nL$ is the number of bosonic modes of the problem). $N$ is a normalization constant ensuring that $\int d^2 \alpha \, \mathcal{P}(\alpha) = 1$. The explicit form of $N$ is not relevant for our purposes. $\alpha_0 = (\langle \hat{a} + \hat{a}^\dagger \rangle, \langle i(\hat{a} - \hat{a}^\dagger) \rangle^T$ encodes the expectation values of the mode operators and $\mathbf{V}$ is the $2nL \times 2nL$ covariance matrix of the system, which for a single mode and $n = 1$ reads
\[
\mathbf{V} = \left( \begin{array}{cc}
\langle \langle \hat{q}\hat{q} \rangle \rangle & \frac{1}{2} \langle \langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle \rangle \\
\frac{1}{2} \langle \langle \hat{p}\hat{q} + \hat{q}\hat{p} \rangle \rangle & \langle \langle \hat{p}\hat{p} \rangle \rangle 
\end{array} \right).
\]
Here $\hat{q} = \hat{a} + \hat{a}^\dagger$ and $\hat{p} = -i(\hat{a} - \hat{a}^\dagger)$, and $\langle \langle xy \rangle \rangle = \langle \langle x \rangle \langle y \rangle \rangle$. $\mathbf{V}$ is a real and symmetric matrix by construction and is also positive definite due to the Heisenberg uncertainty principle. $P$ is positive and well defined if furthermore $\mathbf{V} > \mathbf{I}$. In this case the state is a statistical mixture of coherent states i.e. is a classical state. A quantum state is considered to be nonclassical if it cannot be written as a statistical mixture of coherent states. In this paper we consider more general bosonic Gaussian states (A good introduction to bosonic Gaussian states can be found, for example, in [33]).

$\mathcal{P}(\alpha)$ can be used to evaluate the expectation value of any normally ordered operator function $f(\{\hat{a}_{\mu}^\dagger, \hat{a}_{\mu}\})$ by the replacement $\langle \hat{a}^\dagger \rightarrow \alpha^* \rangle$ and $\langle \hat{a} \rightarrow \alpha \rangle$ and integration. The $\mathcal{P}$ function may be singular and can attain negative values. All integration with $\mathcal{P}(\alpha)$ must therefore be understood in the distributional sense.

Using eq. (10) we find
\[
\langle \hat{T} \rangle = N^i \int d^2 \eta \int d^2 \alpha \exp\left\{ -\frac{1}{2} \eta^T (\mathbf{V} - \mathbf{I}) \eta - i\alpha_0^T \eta \right\} \times \exp\left\{ -2i\alpha_0^T \mathbf{U} - \alpha_0^T (\mathbf{I} - \mathbf{U}) \alpha \right\},
\]
where $\mathbf{U}$ is a unitary operator
\[
\langle \mathbf{U} \rangle_{r_1,s_1; r_2,s_2} = \exp\left( \frac{2\pi i}{L} \left( r_1 + \frac{s_1}{n} \right) \right) \delta_{r_1 r_2} \delta_{s_1 s_2}.
\]
According to our assumption (exp $\left( \frac{2\pi i}{L} (r + \frac{s}{n}) \right) \neq 1$), $\mathbf{I} - \mathbf{U}$ is an invertible symmetric complex matrix. In addition, its real part $\mathbf{I} - \mathbf{U}^* \mathbf{U}$ is positive definite. In this case the Gaussian integral (14) over $\alpha$ is well-defined and is proportional to $|\det(\mathbf{I} - \mathbf{U})|^{-1/2}$. We note that when the matrix is complex, the calculation of the square root requires some special care. However, one can show that
any symmetric complex matrix has a unique symmetric square root whose real part is positive definite \cite{37}. After successive integration over $\alpha$ and then over $\eta$ we eventually obtain
\[
\langle \hat{T} \rangle = N_2 \left[ \det (\mathbf{V} + \mathbb{I}) \det \left( \mathbf{I} - \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \mathbf{U} \right) \right]^{-1/2} 
\times \exp \left( -\frac{1}{2} \alpha_0^T \mathbf{M}^{-1} \alpha_0 \right),
\]
where $N_2$ is a positive number and $\mathbf{M} = \mathbf{V} - \mathbb{I} + 2(1 - \mathbf{U})^{-1}$. Substituting this expectation value into the expression of the many-body polarization \cite{1} one obtains
\[
P = -\frac{1}{4\pi} \text{Im} \ln \left[ \det (\mathbf{V} + \mathbb{I}) \det \left( \mathbf{I} - \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \mathbf{U} \right) \right] 
\times \exp \left( -\frac{1}{2} \alpha_0^T \mathbf{M}^{-1} \alpha_0 \right),
\]
where
\[
\mathbf{W} = \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \mathbf{U}.
\]
One can show that the second term in eq. (17) is a single valued function of system parameters and therefore does not contribute to the change of polarization. In the next section we will show that the first term in eq. (17) vanishes in the thermodynamic limit of infinite system size $L \to \infty$.

IV. POLARIZATION IN THE THERMODYNAMIC LIMIT

In Ref. \cite{18} it was shown that the polarization of a general Gaussian mixed state $\rho$ of lattice fermions at commensurate filling can be written as a sum of the polarization of a pure state $|\psi\rangle$ plus a term that vanishes in the thermodynamic limit of infinite system size $L \to \infty$.
\[
P(\rho) = P(|\psi\rangle\langle\psi|) + \mathcal{O}(L^{-n}), \quad \alpha > 0.
\]
Here $|\psi\rangle$ is the many-body ground state of the so-called fictitious Schrödinger's equation iteratively. This yields a determinant in eq. (17) can thus be written as
\[
\det \left( \mathbf{I} - \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \mathbf{U} \right) = \det \left( \mathbf{I} - \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \right) = \det \left( \mathbf{I} - \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \right).
\]
As a consequence of the periodic boundary conditions the covariance matrix $\mathbf{V}$ is block-circulant. Since the model has lattice translational invariance, the covariance matrix is diagonalized by the Fourier transform and we can write:
\[
\mathbf{U}_{FT} \mathbf{V} - \mathbb{I}_{4L} \mathbf{U}_{FT}^{\dagger} = \bigoplus_{k=0}^{L-1} \mathbf{v}_k - \mathbb{I}_4 \mathbf{v}_k + \mathbb{I}_4,
\]
where $\oplus$ denotes the direct sum which constructs a block diagonal matrix. Since $\mathbf{V}$ is positive definite, the resulting $k$-dependent $4 \times 4$ blocks have eigenvalues $\lambda \left( \frac{\mathbf{v}_k - \mathbb{I}_4}{\mathbf{v}_k + \mathbb{I}_4} \right)$ with absolute values obeying
\[
|\lambda \left( \frac{\mathbf{v}_k - \mathbb{I}_4}{\mathbf{v}_k + \mathbb{I}_4} \right)| < 1 \quad \forall k = 0, \ldots, L - 1.
\]
The maximum absolute eigenvalue will let us introduce an upper bound:
\[
\lambda_{\text{max}} \equiv \max_k \left| \lambda \left( \frac{\mathbf{v}_k - \mathbb{I}_4}{\mathbf{v}_k + \mathbb{I}_4} \right) \right|.
\]
The transformed unitary matrix $\mathbf{U}$, given by eq. (15), is:
\[
\left( \mathbf{U}_{FT} \mathbf{U}_{FT}^{\dagger} \right)_{j,s_1;k,s_2} = \delta_{j,k+1}\delta_{s_1,s_2} \exp \left( \frac{2\pi i s_1}{L} \frac{\mathbf{v}_k + \mathbb{I}_4}{\mathbf{v}_k + \mathbb{I}_4} \right).\]
To make the following expressions more compact, we furthermore introduce $\mathbf{m}_k \equiv \frac{\mathbf{v}_k + \mathbb{I}_4}{\mathbf{v}_k + \mathbb{I}_4} (\mathbb{I}_2 \oplus \exp \left( \frac{2\pi i}{L} \right) \mathbb{I}_2)$. The determinant in eq. (17) can thus be written as
\[
\det \left( \mathbf{I} - \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \right) = \det \left( \mathbf{I} - \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \right).
\]
This block determinant can be reduced by applying Schur’s identity iteratively. This yields a determinant of dimension $4 \times 4$:
\[
\det \left( \mathbf{I} - \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \right) = \det \left( \mathbf{I} - \frac{\mathbf{V} - \mathbb{I}}{\mathbf{V} + \mathbb{I}} \right).
\]
According to eq. (24), a single matrix $\mathbf{m}_k$ is bounded and thus the product of matrices must be bounded as well, i.e. $\|\prod_k \mathbf{m}_k\| = \mathcal{O}(L^{-n})$. We split off the maximum absolute eigenvalues $\Lambda \equiv \lambda_{\text{max}} \prod_k \mathbf{m}_k$ such that $|\text{Tr}(\Lambda)| \leq 4$ and define a small parameter $\epsilon \equiv 4\lambda_{\text{max}}^L$. We can then express the polarization $P$ by expanding the determinant and
logarithm in this small parameter:

\[
\ln \det \left( \mathbb{1}_4 - \sum_k m_k \right) = \ln \det \left( \mathbb{1}_4 - \frac{\epsilon}{4} \mathbf{A} \right) \\
= \ln \left( 1 - \frac{\epsilon}{4} \text{Tr}(\mathbf{A}) + O(\epsilon^2) \right) \\
= -\frac{\epsilon}{4} \text{Tr}(\mathbf{A}) + O(\epsilon^2). \tag{28}
\]

With this we find the following system-size scaling of the polarization for Gaussian bosonic states

\[
4\pi |P| \leq \frac{\epsilon}{4} |\text{Tr}(\mathbf{A})| + O(\epsilon^2) \\
\leq \epsilon + O(\epsilon^2). \tag{29}
\]

Since we know that \(0 < \lambda_{\text{max}} < 1\), the small parameter \(\epsilon\) vanishes exponentially in \(L\),

\[
\alpha = -\ln(\lambda_{\text{max}}) > 0 \implies \epsilon = 4e^{-\alpha L}.
\]

Therefore, as the system approaches the thermodynamic limit, the first term of the many-body polarization in eq. (15) vanishes exponentially and only the topologically trivial second term remains. This is not only true for the strict limit \(L \to \infty\), but can be extended to finite system sizes as well. We can always find a sufficiently large but finite \(L'\) where the exponential bound yields \(-\frac{1}{2} < P < +\frac{1}{2}\). This condition prohibits a winding of the polarization for all \(L \geq L'\) and since the polarization must be quantized over one adiabatic loop, its change is zero.

V. POLARIZATION WINDING

In one-dimensional lattice systems with a Hamiltonian or a Liouvillian which depend on an external parameter \(\lambda\) in a cyclic way, the winding of the EGP or the many-body polarization with \(\lambda\) defines a topological invariant:

\[
w = \Delta P = \oint d\lambda \frac{\partial P(\lambda)}{\partial \lambda}. \tag{30}
\]

In two-dimensional translational invariant lattice models a similar construction defines a Chern number. E.g. introducing particle number operators in mixed real and momentum space by performing a discrete Fourier-transformation in one direction, \(\hat{a}_y(k_y) \sim \sum_j \hat{a}_j \exp(2\pi i k_y j / L)\), one can define a momentum-dependent polarization (where we have suppressed band indices for simplicity)

\[
P_z(k_y) = \frac{1}{2\pi i} \text{Im} \ln \left\langle \exp \left( \frac{2\pi i}{L} \sum_j \hat{a}_j^\dagger(k_y) \hat{a}_j(k_y) \right) \right\rangle. \tag{31}
\]

The winding of \(P(k)\) when going through the Brillouin zone in \(k\) defines a Chern number

\[
C = \int_{\text{BZ}} dk_y \frac{\partial P_z(k_y)}{\partial k_y} = \int_{\text{BZ}} dk_x \frac{\partial P_y(k_x)}{\partial k_x}. \tag{32}
\]

If we consider the polarization in a Gaussian mixed state of bosons \(\rho(\lambda)\), which is uniquely defined along a closed path of the parameter \(\lambda\) in parameter space, we can argue from eq. (29) that the winding of the many-body polarization vanishes for a sufficiently large but finite system size \(L\). In the following we will explicitly show that this holds true independently of the system size.

Let us assume that the polarization is a function of two real parameters which change cyclically in time from 0 to \(T\). Then the change of the polarization between times \(t = 0\) and \(t = T\) can be described as a loop along a closed path \(C\) in parametric space. The two parameters can be combined into a complex variable \(\chi\). Thus the change of polarization can be written as

\[
\Delta P = -\frac{1}{4\pi} \text{Im} \oint_C d\chi \frac{\partial}{\partial \chi} \ln \det \left[ \mathbb{1}_{2n_L} - \mathbf{W}(\chi) \right]. \tag{33}
\]

Moreover, using

\[
\frac{\partial}{\partial \chi} \ln \det \left[ \mathbb{1} - \mathbf{W}(\chi) \right] = \text{Tr} \left[ (\mathbb{1} - \mathbf{W}(\chi))^{-1} \frac{\partial (\mathbb{1} - \mathbf{W}(\chi))}{\partial \chi} \right]. \tag{34}
\]

we derive the following expression for \(\Delta P\)

\[
\Delta P = \frac{1}{4\pi} \text{Im} \text{Tr} \oint_C d\chi \left[ (\mathbb{1} - \mathbf{W}(\chi))^{-1} \frac{\partial (\mathbb{1} - \mathbf{W}(\chi))}{\partial \chi} \right]. \tag{35}
\]

Now we are ready to proof that the change of polarization vanishes for any bosonic Gaussian state. For that we first review some facts about zeros of determinants of holomorphic matrix-valued functions (for more details see [38]).

Let \(\mathbf{F}(\chi)\) be a matrix-valued function that is analytic in a domain \(C\). Under the assumption that all values of \(\mathbf{F}(\chi)\) on the boundary \(C\) of \(C\) are invertible operators it is possible to show [38] that

\[
\mathcal{M} = \frac{1}{2\pi i} \text{Tr} \oint_C d\chi \left[ (\mathbf{F}(\chi))^{-1} \frac{d\mathbf{F}(\chi)}{d\chi} \right]
\]

is the number of zeros of \(\det \mathbf{F}(\chi)\) inside \(C\) (including their multiplicities). Combining this with equation (35), we obtain

\[
\Delta P = \frac{1}{2} \mathcal{M}, \tag{36}
\]

where \(\mathcal{M}\) is the number of solutions (zeros) of

\[
\det \left[ \mathbb{1} - \mathbf{W}(\chi) \right] = 0
\]

inside the closed path \(C\) in parametric space. In order to estimate \(\mathcal{M}\), we use a generalization of Rouche’s theorem for the matrix valued complex function [38], which states:

**Rouche’s Theorem:** Let \(C\) be a closed contour bounding a domain \(C\). If \(\|\mathbf{F}(\chi)\| < 1\) on \(C\) then

\[
\frac{1}{2\pi i} \text{Tr} \oint_C d\chi \left[ (\mathbf{I} + \mathbf{F}(\chi))^{-1} \frac{d\mathbf{F}(\chi)}{d\chi} \right] = 0.
\]
Applying Rouche’s theorem to our problem, where

\[ \| F(\chi) \| = \| W(\chi) \| = \left\| \frac{V - I}{V + I} \right\| < 1 \]

we see that for any \( V > 0 \), i.e. for any Gaussian bosonic state

\[ \| W(\chi) \| < 1. \] (37)

Therefore the change of polarization is equal to zero, irrespective of the system size.

\[ \Delta P = 0. \] (38)

This proofs that for any bosonic Gaussian state the total change of the many-body polarization along a closed path in parametric space is zero. This is in sharp contrast to free fermion systems in which the winding of the many-body interactions is topologically non-trivial, i.e. possesses bands with a non-vanishing Chern number. As a consequence there is e.g. no topologically protected quantized charge transport of Gaussian states of bosons and the latter requires strong interactions \(^{[39]}\).

VI. CONCLUSION

We have shown that the many-body polarization of translationally invariant Gaussian states of bosons consists of two contributions. The first one is always topologically trivial, whereas the second one approaches zero in the thermodynamic limit of infinite system size. Its winding upon a cyclic change of the state, which in the case of fermions defines a many-body topological invariant, vanishes for any system size. Thus many-body topological invariants based on the polarization are always trivial in finite-temperature states or Gaussian non-equilibrium states of non-interacting bosons. This is also the case if the band structure of the underlying lattice hamiltonian is topologically non-trivial, i.e. possesses bands with a non-vanishing Chern number. As a consequence there is e.g. no topologically protected quantized charge transport of Gaussian states of bosons and the latter requires strong interactions \(^{[39]}\).

acknowledgment

Financial support from the DFG through SFB TR 185 is gratefully acknowledged.

[1] Y. Hatsugai, Chern Number and Edge States in the Integer Quantum Hall Effect, Phys. Rev. Lett. 71, 3697 (1993)
[2] K. V. Klitzing, G. Dorda, and M. Pepper, New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance, Phys. Rev. Lett. 45, 494 (1980).
[3] D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Quantized Hall Conductance in a Two-Dimensional Periodic Potential, Phys. Rev. Lett. 49, 405 (1982).
[4] D. J. Thouless, Quantization of particle transport, Phys. Rev. B 27, 6083 (1983).
[5] Q. Niu, and D.J. Thouless, Quantised adiabatic charge transport in the presence of substrate disorder and many-body interactions, J. Phys. A, 17, 2453 (1984).
[6] S. Nakajima, T. Tomita, S. Taie, T. Ichinose, H. Ozawa, L. Wang, M. Troyer, and Y. Takahashi, Topological Thouless Pumping of Ultracold Fermions, Nat. Phys. 12, 296 (2016).
[7] D. C. Tsui, H. L. Störmer, and A. C. Gossard, Two-Dimensional Magnetotransport in the Extreme Quantum Limit, Phys. Rev. Lett. 48, 1559 (1982).
[8] R. B. Laughlin, Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations, Phys.Rev.Lett. 50, 1395 (1983).
[9] D. Arovas, J. R. Schrieffer, and F. Wilczek, Fractional Statistics and the Quantum Hall Effect, Phys. Rev. Lett. 53, 722 (1984).
[10] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Non-Abelian anyons and topological quantum computation, Rev. Mod. Phys. 80, 1083 (2008).
[11] J. E. Avron, M. Fraas, G. M. Graf, and O. Kenneth, Quantum response of dephasing open systems, New J. Phys. 13 053042 (2011).
[12] C.-E. Bardyn, M. A. Baranov, C. V. Kraus, E. Rico, A. Imamoglu, P. Zoller, and S. Diehl, Topology by dissipation, New J. Phys. 15, 085001 (2013).
[13] O. Viyuela, A. Rivas, and M. A. Martin-Delgado, Uhlmann Phase as a Topological Measure for One-Dimensional Fermion Systems, Phys. Rev. Lett. 112, 130401 (2014).
[14] O. Viyuela, A. Rivas, and M. A. Martin-Delgado, Two-Dimensional Density-Matrix Topological Fermionic Phases: Topological Uhlmann Numbers, Phys. Rev. Lett. 113, 076408 (2014).
[15] Z. Huang and D. P. Arovas, Topological Indices for Open and Thermal Systems via Uhlmann’s Phase, Phys. Rev. Lett. 113, 076407 (2014).
[16] E. P. L. van Nieuwenburg and S. D. Huber, Classification of mixed-state topology in one dimension, Phys. Rev. B 90, 075141 (2014).
[17] D. Linzner, L. Wawer, F. Grusdt, M. Fleischhauer, Reservoir-induced Thouless pumping and symmetry protected topological order in open quantum chains, Phys. Rev. B (R) 94, 201105 (2016).
[18] C. E. Bardyn, L. Wawer, A. Altland, M. Fleischhauer, S. Diehl, Probing the topology of density matrices, Phys. Rev. X 8, 011035 (2018).
[19] M. V. Berry, Quantal Phase Factors Accompanying Adiabatic Changes, Proc. R. Soc. A 392, 45 (1984).
[20] F. Wilczek and A. Zee, Appearance of Gauge Structure in Simple Dynamical Systems, Phys. Rev. Lett. 52, 2111 (1984).
[21] J. Zak, Berry’s Phase for Energy Bands in Solids, Phys. Rev. Lett. 62, 2747 (1989).
[22] D. Xiao, M. Chang, Q. Niu, Berry phase effects on electronic properties, Rev. Mod. Phys. 82, 1959 (2010).
[23] M. J. Rice and E. J. Mele, Elementary Excitations of a Linearly Conjugated Diatomic Polymer, Phys. Rev. Lett. 49, 1455 (1982).
[24] L. Wang, M. Troyer, and X. Dai, Topological Charge Pumping in a One-Dimensional Optical Lattice, Phys. Rev. Lett. 111, 026802 (2013).
[25] A. Uhlmann, Parallel Transport and "Quantum Holonomy" along Density Operators, Rep. Math. Phys. 24, 229 (1986).
[26] J.C. Budich and S. Diehl, Topology of density matrices, Phys. Rev. B 91, 165140 (2015).
[27] R. Resta Quantum Mechanical Position Operator in Extended Systems, Phys. Rev. Lett. 80, 1800 (1998)
[28] R. D. King-Smith and David Vanderbilt Theory of polarization of crystalline solids, Phys. Rev. B 47, 1651 (1993)
[29] A. Altland, and M. Zirnbauer Nonstandard symmetry classes in mesoscopic normal-superconducting hybrid structures, Phys. Rev. B 55, 1142 (1997)
[30] Andreas P. Schnyder, Shinsei Ryu, Akira Furusaki, and Andreas W. W. Ludwig Classification of topological insulators and superconductors in three spatial dimensions, Phys. Rev. B 78, 195125 (2008)
[31] Shinsei Ryu, Andreas P Schnyder, Akira Furusaki and Andreas W W Ludwig, Topological insulators and superconductors: ten-fold way and dimensional hierarchy, New J. of Phys. (2010)
[32] L. Wawer, R. Li, M. Fleischhauer (in preparation)
[33] Ch. Weedbrook, S. Pirandola, R. Garcia-Patron, T.C. Ralph, J. H. Shapiro, S. Lloyd, Gaussian quantum information, Rev. Mod. Phys. 84, 621 (2012).
[34] A. S. Holevo, M. Sohma and O. Hirota, Capacity of quantum Gaussian channels, Phys. Rev. A 59, 1820 (1999).
[35] E. C. G. Sudarshan, Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, Phys. Rev. Lett. 10, 277 (1963).
[36] R. J. Glauber, The Quantum Theory of Optical Coherence Phys. Rev. 130, 2529 (1963).
[37] T. Kato, Perturbation Theory for Linear Operators, Springer (1995).
[38] I. Gohberg, S. Goldberg, and M. A. Kaashoek, Classes of Linear Operators, Vol. I, Operator Theory: Adv. Appl., Vol. 49, Birkhäuser, Basel, (1990).
[39] M. Lohse, C. Schweizer, O. Zilberberg, M. Aidelsburger, and I. Bloch, A Thouless quantum pump with ultracold bosonic atoms in an optical superlattice, Nat. Phys. 12, 350 (2015).