Spin Foams for the $SO(4, C)$ BF theory and the $SO(4, C)$ General Relativity.

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Abstract

The Spin Foam Model for the $SO(4, C)$ BF theory is discussed. The Barrett-Crane intertwiner for the $SO(4, C)$ general relativity is systematically derived. The $SO(4, C)$ Barrett-Crane intertwiner is unique. The propagators of the $SO(4, C)$ Barrett-Crane model are discussed. The asymptotic limit of the $SO(4, C)$ general relativity is discussed. The asymptotic limit is controlled by the $SO(4, C)$ Regge calculus.

1 Introduction

The Spin Foam model of the BF theory [11] for the gauge group $SO(4, C)$ is discussed. The Barrett-Crane model [12] of the $SO(4, C)$ general relativity is systematically derived. The $SO(4, C)$ Barrett-Crane model has been used to develop the concept of reality conditions for the Barrett-Crane models [46].

The asymptotic limit of the $SO(4, C)$ general relativity is discussed. The asymptotic limit [32], [40] is controlled by the $SO(4, C)$ Regge calculus which unifies the Regge calculus theories for all the real general relativity cases for the four dimensional signatures.

2 Spin foam of the $SO(4, C)$ BF model

Consider a four dimensional submanifold $M$. Let $A$ be a $SO(4, C)$ connection 1-form and $B_{ij}$ a complex bivector valued 2-form on $M$. Let $F$ be the curvature 2-form of the connection $A$. Then I define a real continuum BF theory action [46],

$$S_{BF}(A, B_{ij}, \bar{A}, \bar{B}_{ij}) = \text{Re} \int_M B \wedge F,$$

(1)

where $A, B_{ij}$ and their complex conjugates are considered as independent free variables.

The Spin foam model for the $SO(4, C)$ BF theory action can be derived from the discretized BF action by using the path integral quantization as illustrated.
in Ref:[4] for compact groups. Let $\Delta$ be a simplicial manifold obtained by a triangulation of $M$. Let $g_\epsilon \in SO(4, C)$ be the parallel propagators associated with the edges (three-simplices) representing the discretized connection. Let $H_b = \prod_{e \supset b} g_\epsilon$ be the holonomies around the bones (two-simplices) in the four dimensional matrix representation of $SO(4, C)$ representing the curvature. Let $B_b$ be the $4 \times 4$ antisymmetric complex matrices corresponding to the dual Lie algebra of $SO(4, C)$ corresponding to the discrete analog of the $B$ field. Then the discrete BF action is

$$S_d = \text{Re} \sum_{b \in M} \text{tr}(B_b \ln H_b),$$

which is considered as a function of the $B_b$’s and $g_\epsilon$’s. Here $B_b$ the discrete analog of the $B$ field are $4 \times 4$ antisymmetric complex matrices corresponding to dual Lie algebra of $SO(4, C)$. The $\ln$ maps from the group space to the Lie algebra space. The trace is taken over the Lie algebra indices. Then the quantum partition function can be calculated using the path integral formulation as,

$$Z_{BF}(\Delta) = \int \prod b dB_b d\bar{B}_b \exp(iS_d) \prod \epsilon dg_\epsilon = \int \prod b \delta(H_b) \prod \epsilon dg_\epsilon,$$

(2)

where $dg_\epsilon$ is the invariant measure on the group $SO(4, C)$. The invariant measure can be defined as the product of the bi-invariant measures on the left and the right $SL(2, C)$ matrix components. Please see appendix A and B for more details. Similar to the integral measure on the $B$’s an explicit expression for the $dg_\epsilon$ involves product of conjugate measures of complex coordinates.

Now consider the identity

$$\delta(g) = \frac{1}{64\pi^6} \int d\omega \text{tr}(T_\omega(g))d\omega,$$

(3)

where the $T_\omega(g)$ is a unitary representation of $SO(4, C)$, where $\omega = (\chi_L, \chi_R)$ such that $n_L + n_R$ is even, $d\omega = |\chi_L\chi_R|^2$. The details of the representation theory are discussed in appendix B. The integration with respect to $d\omega$ in the above equation is interpreted as the summation over the discrete $n$’s and the integration over the continuous $\rho$’s.

By substituting the harmonic expansion for $\delta(g)$ into the equation (2) we can derive the spin foam partition of the $SO(4, C)$ $BF$ theory as explained in Ref:[11] or Ref:[4]. The partition function is defined using the $SO(4, C)$ intertwiners and the $\{15\omega\}$ symbols.

The relevant intertwiner for the $BF$ spin foam is
The nodes where the three links meet are the Clebsch-Gordan coefficients of $SO(4, C)$. The Clebsch-Gordan coefficients of $SO(4, C)$ are just the product of the Clebsch-Gordan coefficients of the left and the right handed $SL(2, C)$ components. The Clebsch-Gordan coefficients of $SL(2, C)$ are discussed in the references [21] and [36].

The quantum amplitude associated with each simplex $s$ is given below and can be referred to as the $\{15\omega\}$ symbol,

\[
\{15\omega\} = \prod_{j} \omega_j.
\]

The final partition function is

\[
Z_{BF}(\Delta) = \int_{\{\omega_b, \omega_e\}} \prod_b \frac{d\omega_b}{64\pi^3} \prod_s Z_{BF}(s) \prod_b d\omega_b \prod_e d\omega_e,
\]

where the $Z_{BF}(s) = \{15\omega\}$ is the amplitude for a four-simplex $s$. The $d\omega_b = |\chi L\chi_R|^2$ term is the quantum amplitude associated with the bone $b$. Here $\omega_e$ is the internal representation used to define the intertwiners. Usually $\omega_e$ is replaced by $i_e$ to indicate the intertwiner. The set $\{\omega_b, \omega_e\}$ of all $\omega_b$’s and $\omega_e$’s is usually called a coloring of the bones and the edges. This partition function may not be finite in general.

It is well known that the $BF$ theories are topological field theories. A priori one cannot expect this to be true for the case of the $BF$ spin foam models because of the discretization of the $BF$ action. For the spin foam models of the $BF$ theories for compact groups, it has been shown that the partition functions are triangulation independent up to a factor [15]. This analysis is purely based on spin foam diagrammatics and is independent of the group used as long the $BF$ spin foam is defined formally by equation (2) and the harmonic expansion in equation (3) is formally valid. So one can apply the spin foam diagrammatics analysis directly to the $SO(4, C)$ $BF$ spin foam and write down the triangulation independent partition function as

\[
Z'_{BF}(\Delta) = \tau^{n_4-n_3} Z_{BF}(\Delta)
\]

using the result from [15]. In the above equation $n_4, n_3$ is number of four bubbles.
and three bubbles in the triangulation $\Delta$ and

$$\tau = \delta_{SO(4,C)}(I)$$

$$= \frac{1}{64\pi^8} \int d^2\omega d\omega.$$

The above integral is divergent and so the partition functions need not be finite. The normalized partition function is to be considered as the proper partition function because the BF theory is supposed to be topological and so triangulation independent.

3 The $SO(4, C)$ Barrett-Crane Model

3.1 Classical $SO(4, C)$ General Relativity

Consider a four dimensional manifold $M$. Let $A$ be a $SO(4, C)$ connection 1-form and $B^{ij}$ be a complex bivector valued 2-form on $M$. I would like to restrict myself to the non-degenerate general relativity in this section by assuming $b = \frac{1}{4} \epsilon_{abcd} B_{ab} \wedge B_{cd} \neq 0$. The Plebanski action for the $SO(4, C)$ general relativity is obtained by adding a Lagrange multiplier term to impose the Plebanski constraint to the BF theory action given in equation (1). A simple way of writing the action [22] is

$$S_C(A, B_{ij}, \tilde{A}, \tilde{B}_{ij}, \phi) = \text{Re} \left[ \int_M tr(B \wedge F) + \frac{b}{2} \phi^{abcd} B_{ab} \wedge B_{cd} \right],$$

(5)

where $\phi$ is a complex tensor with the symmetries of the Riemann curvature tensor such that $\phi^{abcd} \epsilon^{abcd} = 0$. The physics corresponding to the extrema of the above action has been discussed by me in Ref: [46]. Two important results are

- The Plebanski constraint imposes the condition $B^{ij}_{ab} = \theta^i_a \theta^j_b$ where $\theta^i_a$ is a complex tetrad field [13], [27].
- The field equations correspond to the $SO(4, C)$ general relativity on the manifold $M$ [27].

3.1.1 Relation to Complex Geometry

Let $M$ be a real analytic manifold. Let $M_c$ be the complex analytic manifold which is obtained by analytically continuing the real coordinates on $M$. The analytical continuation of the field equations and their solutions on $M$ to complex $M_c$ can be used to define complex general relativity. Conversely, the field equations of complex general relativity or their solutions on $M_c$ when restricted to $M$ defines the $SO(4, C)$ general relativity. Because of these properties the action $S$ can also be considered as an action for complex general relativity.

Now consider the relation between the complex general relativity on $M_c$ and the $SO(4, C)$ general relativity on $M$. This relation critically depends on
$M$ being a real analytic manifold. It also depends on the fields on it being analytic on some region may be except for some singularities. If the fields and the field equations are discretized we lose the relation to complex general relativity. Thus it is also not meaningful to relate a $SO(4, C)$ Barrett-Crane Model to complex general relativity. If the $SO(4, C)$ Barrett-Crane model has a semiclassical continuum general relativity limit then a relation to complex general relativity may be recovered.

3.2 The $SO(4, C)$ Barrett-Crane Constraints

My goal here is to systematically construct the Barrett-Crane model of the $SO(4, C)$ general relativity. In the previous section I discussed the $SO(4, C)$ BF spin foam model. The basic elements of the BF spin foams are spin networks built on graphs dual to the triangulations of the four simplices with arbitrary intertwiners and the principal unitary representations of $SO(4, C)$ discussed in appendix B. These closed spin networks can be considered as quantum states of four simplices in the BF theory and the essence of these spin networks is mainly gauge invariance. To construct a spin foam model of general relativity these spin networks need to be modified to include the Plebanski Constraints in the discrete form.

A quantization of a four-simplex for the Riemannian general relativity was proposed by Barrett and Crane [12]. The bivectors $B_i$ associated with the ten triangles of a four-simplex in a flat Riemannian space satisfy the following properties called the Barrett-Crane constraints$^1$:

1. The bivector changes sign if the orientation of the triangle is changed.
2. Each bivector is simple.
3. If two triangles share a common edge, then the sum of the bivectors is also simple.
4. The sum of the bivectors corresponding to the edges of any tetrahedron is zero. This sum is calculated taking into account the orientations of the bivectors with respect to the tetrahedron.
5. The six bivectors of a four-simplex sharing the same vertex are linearly independent.
6. The volume of a tetrahedron calculated from the bivectors is real and non-zero.

The items two and three can be summarized as follows:

$$B_i \wedge B_j = 0 \forall i, j,$$

$^1$I would like to refer the readers to the original paper [12] for more details.
where $A \wedge B = \varepsilon_{IJKL} A^{IJ} B^{KL}$ and the $i, j$ represents the triangles of a tetrahedron. If $i = j$, it is referred to as the simplicity constraint. If $i \neq j$ it is referred as the cross-simplicity constraints.

Barrett and Crane have shown that these constraints are sufficient to restrict a general set of ten bivectors $E_i$ so that they correspond to the triangles of a geometric four-simplex up to translations and rotations in a four dimensional flat Riemannian space.

The Barrett-Crane constraints theory can be trivially extended to the $SO(4, C)$ general relativity. In this case the bivectors are complex and so the volume calculated for the sixth constraint is complex. So we need to relax the condition of the reality of the volume.

A quantum four-simplex for Riemannian general relativity is defined by quantizing the Barrett-Crane constraints [12]. The bivectors $B_i$ are promoted to the Lie operators $\hat{B}_i$ on the representation space of the relevant group and the Barrett-Crane constraints are imposed at the quantum level. A four-simplex has been quantized and studied in the case of the Riemannian general relativity before [12]. All the first four constraints have been rigorously implemented in this case. The last two constraints are inequalities and they are difficult to impose. This could be related to the fact that the Riemannian Barrett-Crane model reveal the presence of degenerate sectors [34], [31] in the asymptotic limit [30] of the model. For these reasons here after I would like to refer to a spin foam model that satisfies only the first four constraints as an essential Barrett-Crane model, While a spin foam model that satisfies all the six constraints as a rigorous Barrett-Crane model.

Here I would like to derive the essential $SO(4, C)$ Barrett-Crane model. For this one must deal with complex bivectors instead of real bivectors. The procedure that I would like to use to solve the constraints can be carried over directly to the Riemannian Barrett-Crane model. This derivation essentially makes the derivation of the Barrett-Crane intertwiners for the real and the complex Riemannian general relativity more rigorous.

### 3.2.1 The Simplicity Constraint

The group $SO(4, C)$ is locally isomorphic to $\frac{SL(2, C) \times SL(2, C)}{Z_2}$. An element $B$ of the Lie algebra space of $SO(4, C)$ can be split into the left and the right handed $SL(2, C)$ components,

$$B = B_L + B_R. \quad (6)$$

There are two Casimir operators for $SO(4, C)$ which are $\varepsilon_{IJKL} B^{IJ} B^{KL}$ and $\eta_{IK} \eta_{JL} B^{IJ} B^{KL}$, where $\eta_{IK}$ is the flat Euclidean metric. In terms of the left and right handed split I can expand the Casimir operators as

$$\varepsilon_{IJKL} B^{IJ} B^{KL} = B_L \cdot B_L - B_R \cdot B_R \quad \text{and}$$

$$\eta_{IK} \eta_{JL} B^{IJ} B^{KL} = B_L \cdot B_L + B_R \cdot B_R,$$

where the dot products are the trace in the $SL(2, C)$ Lie algebra coordinates.
The bivectors are to be quantized by promoting the Lie algebra vectors to Lie operators on the unitary representation space of $SO(4, C) \approx SL(2, C) \times SL(2, C)$, $SL(2, C) \times Z_2 \simeq SO(4, C) \approx SL(2, C) / Z_2$. The relevant unitary representations of $SO(4, C)$ labeled by a pair $(\chi_L, \chi_R)$ such that $n_L + n_R$ is even (appendix B). The elements of the representation space $D_{\chi_L} \otimes D_{\chi_R}$ are the eigen states of the Casimirs and on them the operators reduce to the following:

$$\varepsilon_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 - \chi_R^2}{2} \hat{I} \quad \text{and} \quad \eta_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 + \chi_R^2 - 2}{2} \hat{I}. \quad (7)$$

The equation (7) implies that on $D_{\chi_L} \otimes D_{\chi_R}$ the simplicity constraint $B \wedge B = 0$ is equivalent to the condition $\chi_L = \pm \chi_R$. I would like to find a representation space on which the representations of $SO(4, C)$ are restricted precisely by $\chi_L = \pm \chi_R$. Since a $\chi$ representation is equivalent to $-\chi$ representations [21], both cases are equivalent to $\chi_L = -\chi_R$ [21].

Consider a square integrable function $f(x)$ on the complex sphere $CS^3$ defined by

$$x \cdot x = 1, \forall x \in C^4.$$

It can be Fourier expanded in the representation matrices of $SL(2, C)$ using the isomorphism $CS^3 \simeq SL(2, C)$,

$$f(x) = \frac{1}{8\pi^3} \int Tr(F(\chi)T_\chi(g(x)^{-1})\chi\bar{\chi}d\chi, \quad (9)$$

where the isomorphism $g: CS^3 \rightarrow SL(2, C)$ is defined in the appendix A. The group action of $g = (g_L, g_R) \in SO(4, C)$ on $x \in CS^3$ is given by

$$g(gx) = g_L^{-1}g(x)g_R. \quad (10)$$

Using equation (10) I can consider the $T_\chi(g(x))(z_1, z_2)$ as the basis functions of $L^2$ functions on $CS^3$. The matrix elements of the action of $g$ on $CS^3$ is given by (appendix B)

$$\int \bar{T}_\chi(g(x))(\hat{z}_1, \hat{z}_2)T_\chi(g(gx))(z_1, z_2)dx = T_\chi((g_L)(\hat{z}_1, z_1)T_\chi((g_R)(\hat{z}_2, z_2)d(\hat{\chi} - \chi).$$

I see that the representation matrices are precisely those of $SO(4, C)$ only restricted by the constraint $\chi_L = -\chi_R \approx \chi_R$. So the simplicity constraint effectively reduces the Hilbert space $H$ to the space of $L^2$ functions on $CS^3$. In Ref.[35] the analogous result has been shown for $SO(N, R)$ where the Hilbert space is reduced to the space of the $L^2$ functions on $S^{N-1}$. 

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3.2.2 The Cross-simplicity Constraints

Next let me quantize the cross-simplicity constraint part of the Barrett-Crane constraint. Consider the quantum state space associated with a pair of triangles 1 and 2 of a tetrahedron. A general quantum state that just satisfies the simplicity constraints $B_1 \wedge B_1 = 0$ and $B_2 \wedge B_2 = 0$ is of the form $f(x_1, x_2) \in L^2(CS^3 * CS^3)$, $x_1, x_2 \in CS^3$.

On the elements of $L^2(CS^3 * CS^3)$ the action $B_1 \wedge B_2$ is equivalent to the action of $(B_1 + B_2) \wedge (B_1 + B_2)^2$. This implies that the cross-simplicity constraint $B_1 \wedge B_2 = 0$ requires the simultaneous rotation of $x_1, x_2$ involve only the $\chi_L = \pm \chi_R$ representations. The simultaneous action of $g = (g_L, g_R)$ on the arguments of $f(x_1, x_2)$ is

$$gf(x_1, x_2) = f(g_L^{-1} x_1 g_R, g_L^{-1} x_2 g_R).$$  \hspace{1cm} (11)$$

The harmonic expansion of $f(x_1, x_2)$ in terms of the basis function $T_\chi(g(x))(z_1, z_2)$ is

$$f(x_1, x_2) = \int \int T_\chi^T(\chi_1, \chi_2)(x_1)T_{\chi_1}^T(\chi_2),$$

where I have assumed all the repeated indices are either integrated or summed over for equation only. The rest of the calculations can be understood graphically. The last equation can be graphically written as follows:

$$f(x_1, x_2) = \int \int T_{\chi_1 \chi_2}(x_1, x_2) d\chi_1 d\chi_2,$$

where the box $F$ represents the tensor $F_{\chi_1 \chi_2}^{\chi_1 \chi_2}$. The action of $g \in SO(4, \mathbb{C})$ on $f$ is

$$gf(x_1, x_2) = \int \int T_{\chi_1 \chi_2}(g^{-1}(x_1, x_2)) T_{g \chi_1}^T(g \chi_2) d\chi_1 d\chi_2.$$  \hspace{1cm} (12)

\footnote{Please notice that}

$$
\left(\hat{B}_1 + \hat{B}_2\right) \wedge \left(\hat{B}_1 + \hat{B}_2\right) = \hat{B}_1 \wedge \hat{B}_1 + \hat{B}_1 \wedge \hat{B}_2 + \hat{B}_2 \wedge \hat{B}_2 + 2 \hat{B}_1 \wedge \hat{B}_2.
$$

But since $\hat{B}_1 \wedge \hat{B}_1 = \hat{B}_2 \wedge \hat{B}_2 = 0$ on $f(x_1, x_2)$ we have

$$
\left(\hat{B}_1 + \hat{B}_2\right) \wedge \left(\hat{B}_1 + \hat{B}_2\right) f(x_1, x_2) = \hat{B}_1 \wedge \hat{B}_2 f(x_1, x_2).
$$
Now for any $h \in SL(2, C)$,
\[
T_{a1\chi1}^{b1}(h)T_{a2\chi2}^{b2}(h) = C_{\chi1\chi2}^{a1a2} \bar{C}_{\chi1\chi2}^{a1a2} T_{a3\chi3}^{b3}(h),
\]
where $C$'s are the Clebsch-Gordan coefficients of $SL(2, C)$ \[21\], \[36\]. I have assumed all the repeated indices are either integrated or summed over for the previous and the next two equations. Using this I can rewrite the $g_L$ and $g_R$ parts of the result \[12\] as follows:
\[
T_{a1\chi1}^{b1}(g_L^{-1})T_{a2\chi2}^{b2}(g_L^{-1}) = C_{\chi1\chi2}^{a1a2} \bar{C}_{\chi1\chi2}^{a1a2} T_{a3\chi3}^{b3}(g_L^{-1}) \tag{13}
\]
and
\[
T_{i1\chi1}^{b1}(g_R)T_{i2\chi2}^{b2}(g_R) = C_{\chi1\chi2}^{a1a2} \bar{C}_{\chi1\chi2}^{a1a2} T_{i3\chi3}^{b3}(g_R). \tag{14}
\]
Now we have
\[
gf(x_1, x_2) = \int \cdots \int_{\chi1\chi2\chi L\chi R} \text{F} \, d\chi_1 d\chi_2.
\]

To satisfy the cross-simplicity constraint the expansion of $gf(x_1, x_2)$ must have contribution only from the terms with $\chi_L = \pm \chi_R$. In the expansion in equation \[13\] and equation \[14\] in the right hand side the terms are defined only up to a sign of $\chi_L$ and $\chi_R$. Let me remove all the terms which does not satisfy $\chi_L = \pm \chi_R$ (say $\pm \chi$). Also let me set $g = 1$. Now we can deduce that the functions denoted by $f(x_1, x_2)$ obtained by reducing $f(x_1, x_2)$ using the cross-simplicity constraints must have the expansion \[5\],
\[
f(x_1, x_2) = 2 \int \int \int_{\chi1\chi2\chi} c_\chi \text{F} \, d\chi_1 d\chi_2 d\chi,
\]
where the coefficients $c_\chi$ are arbitrary. Now the Clebsch-Gordan coefficient terms in the expansion can be re-expressed using the following equation:
\[
C_{\chi1\chi2\chi3}^{\chi1\chi2\chi3} \bar{C}_{\chi1\chi2\chi3}^{\chi1\chi2\chi3} = \frac{8\pi^4}{\chi X} \int_{\chi L(2, C)} T_{i1\chi1}^{z1}(h)T_{i2\chi2}^{z2}(h)T_{i3\chi3}^{z3}(h) dh, \tag{15}
\]
\[3\] I derived this equation explicitly in the appendix of Ref: \[20\].
\[4\] Please see appendix A for the explanation.
\[5\] The factor of 2 has been introduced to include the terms with $\chi_L = -\chi_R$. 

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where \( h, \tilde{h} \in SL(2, C) \) and \( dh \) the bi-invariant measure on \( SL(2, C) \). Using this in the middle two Clebsch-Gordan coefficients of \( \tilde{f}(x_1, x_2) \) we get

\[
\tilde{f}(x_1, x_2) = 2 \int \int \int_{SL(2, C)} \frac{8\pi^4 c\chi}{\chi^4} \chi dh F_{\chi_1\chi_2}(h) d\chi_1 d\chi_2 d\chi.
\]

This result can be rewritten for clarity as

\[
\tilde{f}(x_1, x_2) = 2 \int \int \int_{SL(2, C)} \frac{8\pi^4 c\chi}{\chi^4} \chi dh F_{\chi_1\chi_2}(h) d\chi_1 d\chi_2 d\chi.
\]

Once again applying equation (15) to the remaining two Clebsch-Gordan coefficients we get,

\[
\tilde{f}(x_1, x_2) = 2 \int \int \int_{SL(2, C)} \frac{8\pi^4 c\chi}{\chi^4} \chi dh F_{\chi_1\chi_2}(h) d\chi_1 d\chi_2 d\chi.
\]

By rewriting the above expression, I deduce that a general function \( \tilde{f}(x_1, x_2) \) that satisfies the cross-simplicity constraint must be of the form,

\[
\tilde{f}(x_1, x_2) = \int \int c\chi \int_{SL(2, C) \times SL(2, C)} \left( \frac{8\pi^4}{\chi^4} \right)^2 F_{\chi_1\chi_2}(h) dhd\tilde{h} d\chi_1 d\chi_2 d\chi.
\]

where \( F_{\chi_1\chi_2}(h) \) is arbitrary. Then if \( \Psi(x_1, x_2, x_3, x_4) \) is the quantum state of a tetrahedron that satisfies all of the simplicity constraints and the cross-simplicity
constraints, it must be of the form,
\[
\Psi(x_1, x_2, x_3, x_4) = \int F_{x_1 x_2 x_3 x_4}(h) tr(T_{x_1}(g(x_1) h) tr(T_{x_2}(g(x_2) h))
\]
\[
tr(T_{x_3}(g(x_3) h) tr(T_{x_4}(g(x_4) h)) dh \prod_i d\chi_i.
\]
This general form is deduced by requiring that for every pair of variables with
the other two fixed, the function must be the form of the right hand side of
equation (16).

3.2.3 The \textit{SO}(4, C) Barrett-Crane Intertwiner

Now the quantization of the fourth Barrett-Crane constraint demands that \( \Psi \)
is invariant under the simultaneous complex rotation of its variables. This is
achieved if \( F_{x_1 x_2 x_3 x_4}(h) \) is constant function of \( h \). Therefore the quantum state
of a tetrahedron is spanned by
\[
\Psi(x_1, x_2, x_3, x_4) = \int_{n \in CS^3} \prod_i T_{\chi_i}(g(x_i) g(n)) dn,
\]
where the measure \( dn \) on \( CS^3 \) is derived from the bi-invariant measure on
\( SL(2, C) \). I would like to refer to the functions \( T_{\chi_i}(g(x_i)) \) as the \( T \)-functions
hereafter.

Alternative forms The quantum state can be diagrammatically represented
as follows:
\[
\Psi(x_1, x_2, x_3, x_4) = \int_{n \in CS^3} \prod_i T_{\chi_i}(g(x_i) g(n)) dn.
\]
A unitary representation \( T_\chi \) of \( SL(2, C) \) can be considered as an element of
\( D_\chi \otimes D_\chi^* \) where \( D_\chi^* \) is the dual representation of \( D_\chi \). So using this the Barrett-
Crane intertwiner can be written as an element \( |\Psi\rangle \in \bigotimes_i D_{\chi_i} \otimes D_{\chi_i}^* \) as follows:
\[
|\Psi\rangle = \int_{CS^3} \prod_i T_{\chi_i}(g(x_i) g(n)) dn.
\]
Since $SL(2, C) \approx CS^3$, using the following graphical identity:

$$\int_{SL(2, C)} dg = \int_{SL(2, C)} \chi dx,$$

the Barrett-Crane solution can be rewritten as

$$|\Psi\rangle = \int_{SL(2, C)} \chi dx,$$

which emerges as an intertwiner in the familiar form in which Barrett and Crane proposed it for the Riemannian general relativity. It can be clearly seen that the simple representations for $SO(4, R)$ ($J_L = J_R$) has been replaced by the simple representation of $SO(4, C)$ ($\chi_L = \pm \chi_R$).

**Relation to the Riemannian Barrett-Crane model**: All the analysis done until for the $SO(4, C)$ Barrett-Crane theory can be directly applied to the Riemannian Barrett-Crane theory. The correspondences between the two models are listed in the following table$^6$:

| Property          | $SO(4, R)$ BC model | $SO(4, C)$ BC model |
|-------------------|---------------------|---------------------|
| Gauge group       | $SO(4, R) \cong SL(2, C) \otimes SL(2, C)$ | $SO(4, C) \cong SU(2) \otimes SU(2)$ |
| Representations   | $J_L, J_R$           | $\chi_L, \chi_R$   |
| Simple representations | $J_L = J_R$     | $\chi_L = \pm \chi_R$ |
| Homogenous space  | $S^3 \approx SU(2)$ | $CS^3 \approx SL(2, C)$ |

3.2.4 The Spin Foam Model for the $SO(4, C)$ General Relativity.

The $SO(4, C)$ Barrett-Crane intertwiner derived in the previous section can be used to define a $SO(4, C)$ Barrett-Crane spin foam model. The amplitude

$^6$BC stands for Barrett-Crane. For $\chi_L$ and $\chi_R$ we have $n_L + n_R = \text{even}$. 

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\(Z_{BC}(s)\) of a four-simplex \(s\) is given by the \(\{10\chi\}_{SO(4,C)}\) symbol given below:

\[
\{10\chi\}_{SO(4,C)} = ...
\]

where the circles are the Barrett-Crane intertwiners. The integers represent the tetrahedra and the pairs of integers represent triangles. The intertwiners use the four \(\chi\)'s associated with the links that emerge from it for its definition in equation (18). In the next subsection, the propagators of this theory are defined and the \(\{10\chi\}\) symbol is expressed in terms of the propagators in the subsubsection that follows it.

The \(SO(4,C)\) Barrett-Crane partition function of the spin foam associated with the four dimensional simplicial manifold with a triangulation \(\Delta\) is

\[
Z(\Delta) = \sum_{\{\chi_b\}} \left( \prod_b \frac{d_{\chi_b}^2}{64\pi^3} \right) \prod_s Z(s),
\]

where \(Z(s)\) is the quantum amplitude associated with the 4-simplex \(s\) and the \(d_{\chi_b}\) adopted from the spin foam model of the \(BF\) theory can be interpreted as the quantum amplitude associated with the bone \(b\).

### 3.2.5 The Features of the \(SO(4,C)\) Spin Foam

- **Areas:** The squares of the areas of the triangles (bones) of the triangulation are given by \(\eta_{IK}\eta_{JL}B^{IJ}_{b}B^{KL}_{b}\). The eigen values of the squares of the areas in the \(SO(4,C)\) Barrett-Crane model from equation (18) are given by

\[
\eta_{IK}\eta_{JL}\hat{B}^{IJ}_{b}\hat{B}^{KL}_{b} = (\chi^2 - 1) \hat{I}
\]

\[
= \left( \frac{n^2}{2} - \rho^2 - 1 + i\rho \right) \hat{I}.
\]

One can clearly see that the area eigen values are complex. The \(SO(4,C)\) Barrett-Crane model relates to the \(SO(4,C)\) general relativity. Since in the \(SO(4,C)\) general relativity the bivectors associated with any two dimensional flat object are complex, it is natural to expect that the areas defined in such a theory are complex too. This is a generalization of the concept of the space-like and the time-like areas for the real general relativity models: Area is imaginary if it is time-like and real if it is space-like.

- **Propagators:** Laurent and Freidel have investigated the idea of expressing simple spin networks as Feynman diagrams [37]. Here we will apply this
idea to the $SO(4, C)$ simple spin networks. Let $\Sigma$ be a triangulated three surface. Let $n_i \in CS^3$ be a vector associated with the $i^{th}$ tetrahedron of the $\Sigma$. The propagator of the $SO(4, C)$ Barrett-Crane model associated with the triangle $ij$ is given by

$$G_{\chi_{ij}}(n_i, n_j) = Tr(T_{\chi_{ij}}(g(n_i))T_{\chi_{ij}}^\dagger(g(n_j))) = Tr(T_{\chi_{ij}}(g(n_i)g^{-1}(n_j))),$$

where $\chi_{ij}$ is a representation associated with the triangle common to the $i^{th}$ and the $j^{th}$ tetrahedron of $\Sigma$. If $X$ and $Y$ belong to $CS^3$ then

$$tr(g(X)g(Y)^{-1}) = 2X.Y,$$

where $X.Y$ is the Euclidean dot product and $tr$ is the matrix trace. If $\lambda = e^t$ and $\frac{1}{\lambda}$ are the eigen values of $g(X)g(Y)^{-1}$ then,

$$\lambda + \lambda^{-1} = 2X.Y$$

$$X.Y = \cosh(t).$$

From the expression for the trace of the $SL(2, C)$ unitary representations, (appendix A, [21]) I have the propagator for the $SO(4, C)$ Barrett-Crane model calculated as

$$G_{\chi_{ij}}(n_i, n_j) = \frac{\cos(p_{ij}\eta_{ij} + \nu_{ij}\theta_{ij})}{|\sinh(\eta_{ij} + i\theta_{ij})|^2},$$

where $\eta_{ij} + i\theta_{ij}$ is defined by $n_i.n_j = \cosh(\eta_{ij} + i\theta_{ij})$. Two important properties of the propagators are listed below.

1. Using the expansion for the delta on $SL(2, C)$ I have

$$\delta_{CS^3}(X, Y) = \delta_{SL(2, C)}(g(X)g^{-1}(Y)) = \frac{1}{8\pi^2} \int \bar{\chi}\chi Tr(T\chi(g(X)g^{-1}(Y)))d\chi,$$

where the suffix on the deltas indicate the space in which it is defined. Therefore

$$\int \bar{\chi}\chi G\chi(X, Y)) = 8\pi^2\delta_{CS^3}(X, Y).$$

2. Consider the orthonormality property of the principal unitary representations of $SL(2, C)$ given by

$$\int_{CS^3} T_{z_1\chi_1}^{\dagger}(g(X))T_{z_2\chi_2}(g(X))dX = \frac{8\pi^4}{\chi_1\chi_1}\delta(\chi_1 - \chi_2)\delta(z_1 - \hat{z}_1)\delta(z_2 - \hat{z}_2),$$

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where the delta on the $\chi$’s is defined up to a sign of them. From this I have

$$\int_{CS^3} G_{\chi_1}(X, Y) G_{\chi_2}(Y, Z) dY = \frac{8\pi^4}{\chi_1 \bar{\chi}_1} \delta(\chi_1 - \chi_2) G_{\chi_1}(X, Z).$$

• The $\{10\chi\}$ symbol can be defined using the propagators on the complex three sphere as follows:

$$Z(s) = \int_{x_k \in CS^3} \prod_{i < j} T_{\chi_{ij}}(g(x_i) g(x_j)) \prod_k dx_k,$$

$$= \int_{x_k \in CS^3} \prod_{i < j} G_{\chi_{ij}}(x_i, x_j) \prod_k dx_k,$$

where $i$ denotes a tetrahedron of the four-simplex. For each tetrahedron $k$, a free variable $x_k \in CS^3$ is associated. For each triangle $ij$ which is the intersection of the $i$’th and the $j$’th tetrahedron, a representation of $SL(2, \mathbb{C})$ denoted by $\chi_{ij}$ is associated.

• Discretization Dependence and Local Excitations: It is well known that the BF theory is discretization independent and is topological. The spin foam for the $SO(4, \mathbb{C})$ general relativity is got by imposing the Barrett-Crane constraints on the BF Spin foam. After the imposition of the Barrett-Crane constraints the theory loses the discretization independence and the topological nature. This can be seen in many ways.

- The simplest reason is that the $SO(4, \mathbb{C})$ Barrett-Crane model corresponds to the quantization of the discrete $SO(4, \mathbb{C})$ general relativity which has local degrees of freedom.

- After the restriction of the representations involved in BF spin foams to the simple representations and the intertwiners to the Barrett-Crane intertwiners, various important identities used in the spin foam diagrammatics and proof of the discretization independence of the BF theory spin foams in Ref: [15] are no longer available.

- The BF partition function is simply gauge invariant measure of the volume of space of flat connections. Consider the following harmonic expansion of the delta function which was used in the derivation of the $SO(4, \mathbb{C})$ BF theory:

$$\delta(g) = \frac{1}{8\pi^4} \int d\omega tr(T_\omega(g)) d\omega.$$

Imposition of the Barrett-Crane constraints on the BF theory spin foam, suppresses the terms corresponding to the non-simple representations. If only the simple representations are allowed in the right hand side, it is no longer peaked at the identity. This means that
the partition function for the $SO(4, C)$ Barrett-Crane model involves contributions only from the non-flat connections which has local information.

– In the asymptotic limit study of the $SO(4, C)$ spin foams in section four the discrete version of the $SO(4, C)$ general relativity (Regge calculus) is obtained. The Regge calculus action is clearly discretization dependent and non-topological.

4 The Asymptotic Limit of the $SO(4, C)$ Barrett-Crane models.

The asymptotic limit of the real Barrett-Crane models has been studied before 31, 30, 32, 34 to a certain degree. Here I will discuss the asymptotic limit of the $SO(4, C)$ Barrett-Crane model. For the first time I show here that we can extract bivectors which satisfy the essential Barrett-Crane constraints from the asymptotic limit. Consider the amplitude of a four-simplex given by Eq. 18 with a real scale parameter $\lambda$,

$$Z_\lambda = \int_{n_k \in CS^3} \prod_{i<j} G_{\lambda \chi_{ij}}(n_i, n_j) \prod_k dm_k,$$

$$= \int_{n_k \in CS^3} \prod_{i<j} \frac{\cos(\lambda \rho_{ij} \eta_{ij} + \lambda n_{ij} \theta_{ij})}{|\sinh(\lambda \eta_{ij} + i \lambda \theta_{ij})|^2} \prod_k dx_k,$$

$$= \int_{n_k \in CS^3} \prod_{i<j} \sum_{\epsilon_{ij} = \pm 1} \frac{\exp(i \epsilon_{ij} \lambda (\rho_{ij} \eta_{ij} + n_{ij} \theta_{ij}))}{2 |\sinh(\lambda \eta_{ij} + i \lambda \theta_{ij})|^2} \prod_k dx_k,$$

where the $\eta_{ij} + i \theta_{ij}$ is defined by $n_i \cdot n_j = \cosh(\eta_{ij} + i \theta_{ij})$. Here the $\zeta_{ij} = \eta_{ij} + i \theta_{ij}$ is the complex angle between $n_i$ and $n_j$. The asymptotic limit of $Z_\lambda(s)$ under $\lambda \to \infty$ is controlled by

$$S(\{n_i, \bar{n}_i\}, \{\chi_{ij}, \bar{\chi}_{ij}\})$$

$$= \sum_{i<j} \epsilon_{ij} (\rho_{ij} \eta_{ij} + n_{ij} \theta_{ij}) + \text{Re} \left( \sum_i q_i (n_i, n_i - 1) \right)$$

$$= \text{Re} \left( \sum_{i<j} \epsilon_{ij} \bar{\chi}_{ij} \zeta_{ij} + \sum_i q_i (n_i, n_i - 1) \right),$$

where the $q_i$ are the Lagrange multipliers to impose $n_i, \bar{n}_i = 1, \forall i$. My goal now is to find stationary points for this action. The stationary points are determined by

$$\sum_{i \neq j} \epsilon_{ij} \bar{\chi}_{ij} \frac{\partial \zeta_{ij}}{\partial n_i} + q_j n_j = 0, \forall j,$$  

(20a)
and \( n_j, n_j = 1, \forall j \) where the \( j \) is a constant in the summation.

\[
\frac{\partial \zeta_{ij}}{\partial n_i} = \frac{n_j}{\sinh(\zeta_{ij})}. \tag{21}
\]

Using equation (21) in equation (20a) and taking the wedge product of the equation with \( n_j \) we have,

\[
\sum_{i \neq j} \varepsilon_{ij} \bar{x}_{ij} \frac{n_i \wedge n_j}{\sinh(\zeta_{ij})} = 0, \forall j.
\]

If

\[
\bar{E}_{ij} = i\varepsilon_{ij} \bar{x}_{ij} \frac{n_i \wedge n_j}{\sinh(\zeta_{ij})},
\]

then the last equation can be simplified to

\[
\sum_{i \neq j} E_{ij} = 0, \forall j. \tag{22}
\]

We now consider the properties of \( E_{ij} \):

- Each \( i \) represents a tetrahedron. There are ten \( E_{ij} \)'s, each one of them is associated with one triangle of the four-simplex.

- The square of \( E_{ij} \):

\[
\bar{E}_{ij} \cdot \bar{E}_{ij} = \frac{-\bar{x}_{ij}^2}{\sinh^2(\zeta_{ij})} (n_j^2 n_i^2 - (n_i \cdot n_j)^2)
\]

\[
= \frac{-\bar{x}_{ij}^2}{\sinh^2(\zeta_{ij})} (1 - (\cosh(\zeta_{ij})^2)
\]

\[
= \bar{x}_{ij}^2.
\]

- The wedge product of any two \( E_{ij} \) is zero if they are equal to each other or if their corresponding triangles belong to the same tetrahedron.

- Sum of all the \( E_{ij} \) belonging to the same tetrahedron are zero according to equation (22).

It is clear that these properties contain the first four Barrett-Crane constraints. So we have successfully extracted the bivectors corresponding to the triangles of a general flat four-simplex in \( SO(4,C) \) general relativity and the \( n_i \) are the normal vectors of the tetrahedra. The \( \chi_{ij} \) are the complex areas of the triangle as one would expect. Since we did not impose any non-degeneracy conditions, it is not guaranteed that the tetrahedra or the four-simplex have non-zero volumes.
The asymptotic limit of the partition function of the entire simplicial manifold with triangulation $\Delta$ is

$$S(\Delta, \{n_{is} \in CS^3, \chi_{ij}, \bar{\chi}_{ij}, \varepsilon_{ij}\}) = \text{Re} \sum_{i<j,s} \varepsilon_{ij}s\bar{\chi}_{ij}\zeta_{ij}s,$$

where I have assumed variable $s$ represents the four simplices of $\Delta$ and $i, j$ represents the tetrahedra. The $\varepsilon_{ij}s$ can be interpreted as the orientation of the triangles. Each triangle has a corresponding $\chi_{ij}$. The $n_{is}$ denote the unit complex vector associated with the side of the tetrahedron $i$ facing the inside of a simplex $s$. Now there is one bivector $E_{sij}$ associated with each side facing inside of a simplex $s$ of a triangle $ij$ defined by

$$\tilde{E}_{ij}s = i\varepsilon_{ij}s\bar{\chi}_{ij}\frac{n_i \wedge n_{js}}{\sinh(\zeta_{ij}s)}.$$ 

If the $n_{is}$ are chosen such that they satisfy stationary conditions

$$\sum_{i \neq j} E_{ij}s = 0, \forall j, s,$$

and if

$$\theta_{ij} = \left(\sum_s \varepsilon_{ij}s\zeta_{ij}s\right),$$

then

$$S(\Delta, \{\chi_{ij}, \bar{\chi}_{ij}, \varepsilon_{ij}\}) = \text{Re} \sum_{i<j,s} \varepsilon_{ij}s\bar{\chi}_{ij}\zeta_{ij}s,$$

$$= \text{Re} \sum_{i<j} \bar{\chi}_{ij}\theta_{ij}$$

can be considered to describe the Regge calculus for the $SO(4, C)$ general relativity. The angle $\theta_{ij}$ are the deficit angles associated with the triangles and the $n_{is}$ are the complex vector normals associated with the tetrahedra. From the analysis that has been done in this section, it is easy see that the $SO(4, C)$ Regge calculus contains the Regge calculus theories for all the signatures. The Regge calculus for each signature can be obtained by restricting the $n_{is}$ and the $\chi_{ij}$ to the corresponding homogenous space and representations [10]. Also by the properly restricting the $n_{is}$ and the $\chi_{ij}$ we can derive the Regge calculus corresponding to the mixed Lorentzian and multi-signature Barrett-Crane models described in the previous subsections. The details of the relation between the $SO(4, C)$ Regge Calculus and the real Regge Calculus for different signature will be studied elsewhere.
A Unitary Representations of SL(2, C)

The Representation theory of SL(2, C) was developed by Gelfand and Naimarck [21]. Representation theory of SL(2, C) can be developed using functions on $C^2$ which are homogenous in their arguments\(^7\). The space of functions $D_\chi$ is defined as functions $f(z_1, z_2)$ on $C^2$ whose homogeneity is described by

$$f(az_1, az_2) = a^{\chi_1}a^{\chi_2}f(z_1, z_2),$$

for all $a \neq 0$, where $\chi$ is a pair $(\chi_1, \chi_2)$. The linear action of $SL(2, C)$ on $C^2$ defines a representation of $SL(2, C)$ denoted by $T_\chi$. Because of the homogeneity of functions of $D_\chi$, the representations $T_\chi$ can be defined by its action on the functions $\phi(z)$ of one complex variable related to $f(z_1, z_2) \in D_\chi$ by

$$\phi(z) = f(z, 1).$$

There are two qualitatively different unitary representations of $SL(2, C)$: the principal series and the supplementary series, of which only the first one is relevant to quantum general relativity. The principal unitary irreducible representations of $SL(2, C)$ are the infinite dimensional. For these $\chi_1 = -\chi_2 = \frac{n+it}{2}$, where $n$ is an integer and $t$ is a real number. In this article I would like to label the representations by a single complex number $\chi = \frac{n}{2} + iT$, wherever necessary.

The $T_\chi$ representations are equivalent to $T_{-\chi}$ representations [21]. Let $g$ be an element of $SL(2, C)$ given by

$$g = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},$$

where $\alpha, \beta, \gamma$ and $\delta$ are complex numbers such that $\alpha\delta - \beta\gamma = 1$. Then the $D_\chi$ representations are described by the action of a unitary operator $T_\chi(g)$ on the square integrable functions $\phi(z)$ of a complex variable $z$ as given below:

$$T_\chi(g)\phi(z) = (\beta z_1 + \delta)\chi^{-1}(\beta \bar{z}_1 + \bar{\delta})^{-1}\phi(\frac{\alpha z + \gamma}{\beta z + \delta}),$$  \hspace{1cm} (23)

This action on $\phi(z)$ is unitary under the inner product defined by

$$(\phi(z), \eta(z)) = \int \bar{\phi}(z)\eta(z)d^2z,$$

where $d^2z = \frac{i}{2}dz \wedge d\bar{z}$ and I would like to adopt this convention everywhere. Completing $D_\chi$ with the norm defined by the inner product makes it into a Hilbert space $H_\chi$.

Equation (23) can also be written in kernel form [17],

$$T_\chi(g)\phi(z_1) = \int T_\chi(g)(z_1, z_2)\phi(z_2)d^2z_2,$$

\(^7\)These functions need not be holomorphic but infinitely differentiable may be except at the origin $(0, 0)$. 

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Here $T_{\chi}(g)(z_1, z_2)$ is defined as

$$T_{\chi}(g)(z_1, z_2) = (\beta z_1 + \delta)\chi^{-1}(\beta z_1 + \delta) - \chi^{-1}(\beta z_1) - \chi^{-1}(\beta z_1 - g(z_1)),$$

(24)

where $g(z_1) = \frac{\alpha z_1 + \gamma}{\beta z_1 + \delta}$. The Kernel $T_{\chi}(g)(z_1, z_2)$ is the analog of the matrix representation of the finite dimensional unitary representations of compact groups. An infinitesimal group element, $a$, of $SL(2, \mathbb{C})$ can be parameterized by six real numbers $\varepsilon_k$ and $\eta_k$ as follows [44]:

$$a \approx I + \frac{i}{2} \sum_{k=1}^{3} (\varepsilon_k \sigma_k + \eta_k i \sigma_k),$$

where the $\sigma_k$ are the Pauli matrices. The corresponding six generators of the $\chi$ representations are the $H_k$ and the $F_k$. The $H_k$ correspond to rotations and the $F_k$ correspond to boosts. The bi-invariant measure on $SL(2, \mathbb{C})$ is given by

$$dg = \left(\frac{i}{2}\right)^3 \frac{d^2 \beta d^2 \gamma d^2 \delta}{|\delta|^2} = \left(\frac{i}{2}\right)^3 \frac{d^2 \alpha d^2 \beta d^2 \gamma}{|\alpha|^2}.$$

This measure is also invariant under inversion in $SL(2, \mathbb{C})$. The Casimir operators for $SL(2, \mathbb{C})$ are given by

$$\hat{C} = \text{det} \left[ \begin{array}{cc} \hat{X}_3 & \hat{X}_1 - i \hat{X}_2 \\ \hat{X}_1 + i \hat{X}_2 & -\hat{X}_3 \end{array} \right]$$

and its complex conjugate $\overline{C}$ where $X_i = F_i + iH_i$. The action of $C$ ($\overline{C}$) on the elements of $D_{\chi}$ reduces to multiplication by $\chi_1^2 - 1$ ($\chi_2^2 - 1$). The real and imaginary parts of $C$ are another way of writing the Casimirs. On $D_{\chi}$ they reduce to the following

$$\text{Re}(\overline{C}) = \left(-\rho^2 + \frac{n^2}{4} - 1\right) \hat{I},$$

$$\text{Im}(\overline{C}) = \rho \hat{I}.$$
where the $\int d\chi$ indicates the integration over $\rho$ and the summation over $n$. From the expressions for the Fourier transforms, I can derive the orthonormality property of the $T_\chi$ representations,

$$
\int_{SL(2, \mathbb{C})} T_{z_1 \chi_1}^\dagger (g) T_{z_2 \chi_2} (g) dg = \frac{8\pi^4}{\chi_1 \chi_1} \delta(\chi_1 - \chi_2) \delta(z_1 - \z_1) \delta(z_2 - \z_2),
$$

where $T_\chi^\dagger$ is the Hermitian conjugate of $T_\chi$.

The Fourier analysis on $SL(2, \mathbb{C})$ can be used to study the Fourier analysis on the complex three sphere $CS^3$. If $x = (a, b, c, d) \in CS^3$ then the isomorphism $g: CS^3 \rightarrow SL(2, \mathbb{C})$ can be defined by the following:

$$
g(x) = \begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}.
$$

Then, the Fourier expansion of $f(x) \in L^2(CS^3)$ is given by

$$
f(x) = \frac{1}{8\pi^4} \int Tr(F(\chi)T_\chi(g^{-1})\chi \bar{\chi} d\chi
$$

and its inverse is

$$
F(\chi) = \int f(g)T_\chi(g(x)) dx,
$$

where the $dx$ is the measure on $CS^3$. The measure $dx$ is equal to the bi-invariant measure on $SL(2, \mathbb{C})$ under the isomorphism $g$.

The expansion of the delta function on $SL(2, \mathbb{C})$ from equation (26) is

$$
\delta(g) = \frac{1}{8\pi^4} \int \frac{1}{n^2 + \rho^2} \frac{\cos(n\eta + n\theta)}{\sinh(\eta + i\theta)} d\rho.
$$

Let me calculate the trace $tr [T_\chi(g)]$. If $\lambda = e^{\rho + i\theta}$ and $1/\lambda$ are the eigen values of $g$ then

$$
tr [T_\chi(g)] = \frac{\lambda^{\chi_1} \bar{\lambda}^{\chi_2} + \lambda^{-\chi_1} \bar{\lambda}^{-\chi_2}}{|\lambda - \lambda^{-1}|^2},
$$

which is to be understood in the sense of distributions \[21\]. The trace can be explicitly calculated as

$$
br [T_\chi(g)] = \frac{\cos(n\eta + n\theta)}{2 |\sinh(\eta + i\theta)|^2}.
$$

Therefore, the expression for the delta on $SL(2, \mathbb{C})$ explicitly is

$$
\delta(g) = \frac{1}{8\pi^4} \sum_n \int d\rho(n^2 + \rho^2) \frac{\cos(n\eta + n\theta)}{\sinh(\eta + i\theta)^2}.
$$

Let us consider the integrand in equation (26). Using equation (25) in it we have

$$
Tr(F(\chi)T_\chi(g^{-1})\chi \bar{\chi} = \chi \bar{\chi} \int f(\hat{g})Tr(T_\chi(\hat{g})T_\chi(g^{-1})) d\hat{g}
$$

$$
= \chi \bar{\chi} \int f(\hat{g})Tr(T_\chi(\hat{g}g^{-1})) d\hat{g}.
$$

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But, since the trace is insensitive to an overall sign of \( \chi \), so are the terms of the Fourier expansion of the \( L^2 \) functions on \( SL(2, C) \) and \( CS^3 \).

**B Unitary Representations of \( SO(4, C) \)**

The group \( SO(4, C) \) is related to its universal covering group \( SL(2, C) \times SL(2, C) \) by the relationship \( SO(4, C) \approx \frac{SL(2, C) \times SL(2, C)}{Z_2} \). The map from \( SO(4, C) \) to \( SL(2, C) \times SL(2, C) \) is given by the isomorphism between complex four vectors and \( GL(2, C) \) matrices. If \( X = (a, b, c, d) \) then \( G : C^4 \rightarrow GL(2, C) \) can be defined by the following:

\[
G(X) = \begin{bmatrix}
a + ib & c + id \\
-c + id & a - ib
\end{bmatrix}.
\]

It can be easily inferred that \( \det G(X) = a^2 + b^2 + c^2 + d^2 \) is the Euclidean norm of the vector \( X \). Then, in general a \( SO(4, C) \) rotation of a vector \( X \) to another vector \( Y \) is given in terms of two arbitrary \( SL(2, C) \) matrices \( g_L A_B, g_R A'_B, g_L A_B' \in SL(2, C) \) by

\[
G(Y)^{A_A'} = g_L A_B g_R A'_B G^{AB}(X),
\]

where \( G^{AB}(X) \) is the matrix elements of \( G(X) \). The above transformation does not differentiate between \( (L_B^A, R_B^A) \) and \( (-L_B^A, -R_B^A) \) which is responsible for the factor \( Z_2 \) in \( SO(4, C) \approx \frac{SL(2, C) \times SL(2, C)}{Z_2} \).

The unitary representation theory of the group \( SL(2, C) \times SL(2, C) \) is easily obtained by taking the tensor products of two Gelfand-Naimarck representations of \( SL(2, C) \). The Fourier expansion for any function \( f(g_L, g_R) \) of the universal cover is given by

\[
f(g_L, g_R) = \frac{1}{64\pi^8} \int \chi_L \bar{\chi}_L \chi_R \bar{\chi}_R F(\chi_L, \chi_R) T_\chi(g_L^{-1}) T_\chi(g_R^{-1}) d\chi_L d\chi_R,
\]

where \( \chi_L = \frac{n_L + i\eta}{2} \) and \( \chi_R = \frac{n_R + i\eta}{2} \). The Fourier expansion on \( SO(4, C) \) is given by reducing the above expansion such that \( f(g_L, g_R) = f(-g_L, -g_R) \).

From equation [28] I have

\[
tr [T_\chi(-g)] = (-1)^n tr [T_\chi(-g)],
\]

where \( \chi = \frac{n+\eta}{2} \). Therefore

\[
f(-g_L, -g_R) = \frac{1}{8\pi^4} \int \chi_L \bar{\chi}_L \chi_R \bar{\chi}_R F(\chi_L, \chi_R)(-1)^{n_L+n_R} T_\chi(g_L^{-1}) T_\chi(g_R^{-1}) d\chi_L d\chi_R.
\]

This implies that for \( f(g_L, g_R) = f(-g_L, -g_R) \), I must have \( (-1)^{n_L+n_R} = 1 \). From this, I can infer that the representation theory of \( SO(4, C) \) is deduced from the representation theory of \( SL(2, C) \times SL(2, C) \) by restricting \( n_L + n_R \) to be even integers. This means that \( n_L \) and \( n_R \) should be either both odd
numbers or even numbers. I would like to denote the pair \((\chi_L, \chi_R)\) \((n_L + n_R \text{ even})\) by \(\omega\).

There are two Casimir operators available for \(SO(4, C)\), namely \(\varepsilon_{IJKL} \hat{B}^{IJ} \hat{B}^{KL}\) and \(\eta_{IK} \eta_{JL} \hat{B}^{IJ} \hat{B}^{KL}\). The elements of the representation space \(D_{\chi_L} \otimes D_{\chi_R}\) are the eigen states of the Casimirs. On them, the operators reduce to the following:

\[
\varepsilon_{IJKL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_L^2 - \chi_R^2}{2} \quad \text{and} \quad (31)
\]

\[
\eta_{IK} \eta_{JL} \hat{B}^{IJ} \hat{B}^{KL} = \frac{\chi_R^2 + \chi_L^2 - \chi_R}{2} \quad (32)
\]

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