Atom Interference using microfabricated structures

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I. INTRODUCTION

In 1973, Altschuler and Frantz patented an idea for creating an atom interferometer (Altschuler and Frantz, 1973). The beam splitter in their apparatus was a standing-wave optical field. Their ideas were rekindled by Dubetsky et al. (1984), who presented detailed calculations of atomic scattering by standing-wave fields in the context of atom interferometry. It was not until recently, however, that experimentalists were successful in constructing the first atom interferometers. Double-slit interference (Carnal and Mlynek, 1991; Shimizu et al., 1992), Fraunhofer diffraction by microfabricated structures (MS) (Keith et al., 1991; Ekstrom et al., 1995) or by resonant standing wave fields (SW) (Rasel et al., 1995; Giltner et al., 1995), and Fresnel diffraction by one (Chapman et al., 1995) or two (Clauser and Li, 1994) MS have all been observed using atomic beams as matter waves.

Two types of atom-optical elements have been used as beam splitters in these experiments, standing wave fields or microfabricated structures. A SW beam splitter allows one to operate with relatively dense atomic beams, having densities up to $10^{10} \text{ cm}^{-3}$ and flow densities up to $10^{15} \text{ cm}^{-2} \text{ s}^{-1}$. Moreover, by varying the atom field detuning, one can use SW beamsplitters as either amplitude or phase gratings. Additional degrees of freedom are provided by the polarization of the field which can act selectively on targeted magnetic state sublevels. A theory of atom interference in standing wave fields has been developed by Altschuler and Frantz (1973), Dubetsky et al. (1984), Chebotayev et al. (1985), Bordé (1989), Friedberg and Hartmann (1993, 1993a), Dubetsky and Berman (1994) and Janicke and Wilkens (1994). In contrast to SW beam splitters, MS usually scatter atoms in a state independent manner; as a consequence, most experiments involving MS use atoms in their ground (or, possibly, metastable) states. Microfabricated structures provide 100% modulation of the incident atomic beam. They offer the additional advantage that their period and...
duty cycle (ratio of slit opening to period) can be chosen arbitrarily within the limits of current lithographic technology. A theory of atom interference using MS has been developed by Turchette et al. (1992), Clauser and Reinsch (1992) and Carnal et al. (1995).

Both the splitting of an atomic beam into two or more beams coherent with respect to one another and the recombining of the scattered beams are physical processes that are essential to the operation of an atom interferometer. We consider scattering of atoms by an ideal MS, having an infinite number of slits, period $d$, and 100% transmission through the slits. Each MS is normal to the y-axis, and the slits are oriented in the z-direction, so that the axis of the MS is in the x-direction (see Fig. 1).

After scattering from a MS, each in-coming atomic state $\psi$ having $x$-component of center-of-mass momentum $p$ splits into a set of out-going states $\psi_n$, having $x$-components of momenta $p + n\hbar k$ ($k = 2\pi/d, n$ is an integer) which evolve as

$$\psi_n \propto \exp \left[ \frac{i}{\hbar} (p + n\hbar k) x \right] .$$

Interference of two components, such as $\psi_0$ and $\psi_1$, on a screen (see Fig. 1a) leads to an atomic density grating

$$\rho \propto \text{Re}[\psi_0 \psi_1^*] = \cos [kx]$$

having the same period $d$ as the MS.

Observation of this grating in the experiments listed above often has been considered as direct evidence for matter-wave interference. Nevertheless, one can easily see that such a conclusion is not necessarily justified. For particles moving along classical trajectories (Fig. 1b), and for a beam whose angular divergence is sufficiently small to satisfy

$$\theta_b \ll \frac{d}{L},$$

where $L$ is a distance on the order of the distance between the MS and the screen, a shadow of the MS can be seen on the screen at distances $L$ where all matter-wave effects are completely negligible. This example is the simplest manifestation of the classical shadow effect (Chebotayev et al., 1985; Dubetsky and Berman 1994).

To distinguish quantum matter-wave interference from the classical shadow effect, one needs to observe additional features of the phenomena or to choose a scheme where one of the effects is excluded. Young’s double-slit experiment (Carnal and Mlynek, 1991; Shimizu et al., 1992), as well as interference produced by a phase grating created using light that is far-detuned from atomic transition frequencies (Rasel et al., 1995), cannot be explained in terms of atoms moving on classical trajectories. A matter-wave interpretation is also necessary if one observes a shift in the fringe pattern resulting from an index change in one of the arms of an interferometer (Ekstrom et al., 1995). We determine below those particular conditions for which pure quantum interference can be obtained using MS.

Let us estimate a typical distance for which quantum interference effects have to be included. Consider an incident beam which has zero angular divergence. The atoms are assumed to be in a pure state having momentum $p = (0,p_y,0)$ (see Fig. 1b). Localization of the atoms inside each slit leads to an uncertainty in the x-component of atomic momentum $\delta p \sim \frac{h}{d}$, where it is assumed that the slit width $fd$ is comparable with the MS period $d$. A beam passing through the slits acquires an angular divergence $\delta \theta \sim \frac{\delta p}{p_x}$, and atoms passing through a given slit are deposited on the screen with a spot size $\delta x \sim L \delta \theta \sim L \frac{h}{dp_x}$. Interference occurs when spots produced by neighboring slits overlap, i.e., when $\delta x \sim d$.

One finds, therefore, that the characteristic distance for
which matter-wave interference plays an essential role is given by

\[ L \sim L_T, \]  

(4)

where

\[ L_T = 2d^2/\lambda_{dB} \]

(5)

is the so-called Talbot distance and \( \lambda_{dB} = \frac{\hbar}{p_y} \) is the atomic de Broglie wavelength.

![Diagram of MS and Screen with Talbot distance](image)

**FIG. 2.** The incident atomic wave function after scattering from MS transforms into a superposition of the divergent waves emitted from each slit, having angular divergence \( \delta \theta \) inverse to the MS’s period \( d \). (for the given duty cycle). One estimates Talbot distance as a distance between MS and screen large enough to provide overlapping of neighboring divergent waves and their interference.

The manner in which this distance appears in the theory of the optical or atomic Talbot-effect is well known in the context of the Fresnel-Kirchhoff theory of diffraction (see, for example, Patorski, 1989; Winthrop and Worthington, 1965) for optical Talbot-effect theory and (Chapman et al., 1995; Turchette et al., 1992; Clauser and Reinsch, 1992; Carnal et al., 1995) for atomic Talbot-effect theory. One can obtain Eq. (4) using another description. When the number of slits in the MS and the area of the incident atomic beam are infinite, constructive interference occurs only for those directions in which \( p \) is changed by an integral multiple of the recoil momentum \( \hbar k \). In the atomic “rest-frame” (a frame moving along the \( y \)-axis with velocity

\[ u = p_y/M, \]

(6)

where \( M \) is the atomic mass) an out-going state with momentum \( p + n\hbar k \) acquires a phase \( \phi_n = \epsilon_{p+n\hbar k}/\hbar \), where \( \epsilon_{p} = \frac{p^2}{2M} \) is the kinetic energy associated with atomic motion along the \( x \)-axis and \( t \) is the time after scattering from the MS. Comparing this phase with the phase \( \phi_0 \) that an atom would acquire in the absence of the MS, one sees that a dephasing \( \delta \phi = \phi_n - \phi_0 \) occurs for the different out-going state amplitudes as a result of diffraction. The relative dephasing is

\[ \delta \phi = n\phi_D + n^2\phi_t, \]

(7)

where

\[ \phi_D = kv_t, \]  

(7a)

\[ \phi_t = \omega_k t, \]  

(7b)

\[ v = \frac{p_y}{M} \]

is the \( x \) component of atomic velocity, and

\[ \omega_k = \frac{\hbar k^2}{2M} \]

(8)

is a recoil frequency, related to the energy that an atom, having initial momentum \( p = 0 \), acquires as a result of recoil during scattering. The two contributions to the dephasing have different origins. The phase \( \phi_D \) is a Doppler phase that does not disappear in the classical limit \( \hbar \to 0 \); consequently, it must be classical in nature. It is evident from Fig. 2 that, for atoms incident with a nonzero value of \( p \), the shadow moves along the \( x \)-axis together with atoms. The nodes in the shadow are displaced by a distance \( \Delta x = vt \) along the \( x \)-axis, which corresponds to a phase shift of \( 2\pi \frac{\Delta x}{d} \) for the atomic grating, a phase shift that coincides with \( \phi_D \). The phase \( \phi_D \) is analogous to the phase a moving dipole driven by an optical field would acquire in its rest frame as a result of the Doppler frequency shift.

The phase \( \phi_t \) in Eq. (7) is a quantum addition to the dephasing, resulting from recoil. This contribution is responsible for atomic scattering and for matter-wave interference. Quantum effects have to be included when this phase is of order unity, that is, for times of order

\[ t_T = 2\pi/\omega_k. \]

(9)

One finds that length associated with this time in laboratory frame coincides with the Talbot distance

\[ L_T = ut_T. \]

(10)

Thus, we are led to the same conclusion that we reached above by considering the scattering from two adjacent slits; for \( \omega_k t_T \gtrsim 1 \) or \( L \gtrsim L_T \), one must use a quantized description of the atomic center-of-mass motion.

In this chapter we consider the Talbot-effect and other interference phenomena as a consequence of the recoil-effect. In the context of the nonlinear interaction of optical fields with an atomic vapor, the recoil effect was considered by Kol’chenko et al. (1968) and observed by Hall et al. (1976). Quantum structure resulting from the scattering of atoms by a resonant standing wave (resonant Kapitza-Dirac effect), which can be attributed to atomic recoil, was discussed theoretically by Kazantsev et al. (1980) and observed by Moskovitz et al. (1983). Splitting
of optical Ramsey fringes (Baklanov et al., 1976) associated with the resonant Kapitza-Dirac effect was discussed theoretically by Dubetsky and Semibalamut (1978) and observed by Barger et al. (1979). Matter-wave interference resulting from resonant Kapitza-Dirac scattering in a standing wave field has been studied theoretically by Altschuler and Frantz (1973), Dubetsky et al. (1984) and observed by Rasel et al. (1995). The theory of atom interference presented here, based on an interpretation of scattering of atoms by MS in terms of the recoil effect, is a natural extension of the work involving standing-wave fields.

This chapter is organized as follows: in the next Section we discuss conditions necessary for the observation of matter-wave interference in different regimes. Rigorous proof of the equivalence of theories based on Fresnel-Kirchhoff integrals and on the recoil effect is given in Section III, as is a discussion of the atomic gratings that can be produced as a consequence of the Talbot effect. The classical shadow effect is analyzed in Section IV. Section V is devoted to a theory of the Talbot-Lau effect. The Talbot and Talbot-Lau effects for a thermal beam are considered in the Section VI, in the limit where the characteristic length scale in the problem is larger than the Talbot length $L_T$. A discussion of the results is given in Section VII.

II. QUALITATIVE CONSIDERATIONS

The scattering of atoms by gratings can be separated roughly into three categories: classical scattering, Fresnel diffraction, and Fraunhofer diffraction. The limit of Bragg scattering (Martin et al., 1988), in which $\omega k' / u \geq 1$, where $k'$ is the grating thickness, is not discussed in this chapter.

A. Fresnel Diffraction

The Fresnel diffraction limit occurs when $\omega_k t \sim 1$ or $L \sim L_T$. Owing to the angular divergence of the incident beam, it is possible that the diffraction pattern at $L \sim L_T$ will be washed out. To ensure that this does not occur, it is necessary that the spread of Doppler phases, $k u t \theta_b = k L_T \theta_b$, be smaller than unity. This requirement corresponds to inequality (3) when $L \sim L_T$. Using Eq. (3), the condition on $\theta_b$ can be restated as

$$\theta_b \ll \theta_d,$$

where

$$\theta_d = \frac{\hbar k}{p_y} = \frac{\lambda_{dB}}{d}$$

is the angle associated with a single atomic recoil at the MS ($\delta p = \hbar k$). The Talbot effect refers to the self-imaging of a grating in the Fresnel diffraction limit. For self-imaging to occur, the displacement of the scattered atomic beam $L_T \theta_d$ must be much smaller than the beam diameter $D$, which translates into the condition

$$D \gg d.$$  \hspace{1cm} (13)

In this limit, one can consider the beam diameter to be infinite to first approximation; finite beam effects (or, equivalently, gratings with finite slit number) are discussed by Clauser and Reinsch (1992).

Conditions (3, 11, 13) are sufficient to observe the Talbot effect. In this case, the contribution to the wave function’s phase resulting from atomic recoil is of order unity, the scattered beams overlap almost entirely with one another on the screen, and the atomic gratings are not washed out after averaging over atomic velocities $v$ in the incident beam. Matter-wave interference is a critical component of the Talbot effect.

B. Fraunhofer Diffraction

Although the Talbot effect illustrates a matter-wave interference phenomenon, it does not result in an atom interferometer having two arms that are nonoverlapping. We refer to the Fraunhofer diffraction limit as one in which the various diffraction orders are nonoverlapping at a distance $L$ from a single grating. The grating then serves as a beam splitter that physically separates the incident beam into two or more beams. To physically separate the various diffraction orders over a distance $L$, one must require that

$$L \theta_d \gg D.$$  \hspace{1cm} (14)

Using the fact that $\theta_d = \lambda_{dB} / d$ and setting $t = \frac{L}{d}$, one can recast this inequality as

$$\omega_k t \gg D / d; \quad L / L_T \gg D / d,$$  \hspace{1cm} (15)

which requires the quantum phase $\phi_t$ (7) to be larger than $D / d \gg 1$. Consequently, quantum effects play an essential role in an atom interferometer having nonoverlapping beam paths. Note that the Fraunhofer limit cannot be reached for a beam having infinite diameter. We do not consider matter-wave interference in the Fraunhofer limit in this work.

C. Talbot-Lau Regime

To achieve spatial separation of the beams in the Fraunhofer limit and to observe the Talbot effect, the angular divergence $\theta_b$ of the incident beam must be less than $\theta_d$. For typical values $d \sim 200 \text{ nm}$, $M \sim 20$
A.u., $u \sim 10^5 \text{ cm/s}$, the deflection angle $\theta_d \sim 10^{-4} \text{ rad}$. Atomic beams having $\theta_b \ll \theta_d$ have been used to observe atomic scattering by standing waves (Moskovitz et. al., 1983), to build a two-arm atom interferometer (Keith et al., 1991; Rasel et al., 1995; Giltner et al., 1995), and to observe the Talbot-effect (Chapman et al., 1995). Such strong collimation results in a decrease in the atomic flux and a corresponding decrease in signal strength that may be a limiting factor in certain applications of matter-wave interference, such as atom lithography (Timp et al., 1992). Alternatively, one can observe matter-wave interference in beams having larger angular divergence using the atomic Talbot-Lau effect [see for example (Patorski, 1989)]. In the atomic Talbot-Lau effect two or more MS are used. Doppler dephasing following the first MS washes out the normal Talbot effect, but subsequent scattering by a second MS can result in a Doppler rephasing that ultimately leads to a Talbot-like interference pattern. The dephasing-rephasing process is analogous to that occurring in the production of photon echoes (Dubetsky et al., 1984). The atomic Talbot-Lau effect has been observed recently by Clauser and Li (1994) using a K beam scattered by MS. The incident beam is not separated into nonoverlapping beams in the Talbot-Lau effect, but the origin of the interference pattern can still be traced to matter-wave interference since it is related to Fresnel diffraction.

**D. Classical Scattering**

It is worthwhile at this point to return to the classical shadow effect. The shadow effect in a collimated beam is obvious; the atomic grating produced by the MS simply propagates in space over a distance in which diffraction may be a limiting factor in certain applications of matter-wave interference, such as atom lithography (Timp et al., 1992). Alternatively, one can observe matter-wave interference in beams having larger angular divergence using the atomic Talbot-Lau effect [see for example (Patorski, 1989)]. In the atomic Talbot-Lau effect two or more MS are used. Doppler dephasing following the first MS washes out the normal Talbot effect, but subsequent scattering by a second MS can result in a Doppler rephasing that ultimately leads to a Talbot-like interference pattern. The dephasing-rephasing process is analogous to that occurring in the production of photon echoes (Dubetsky et al., 1984). The atomic Talbot-Lau effect has been observed recently by Clauser and Li (1994) using a K beam scattered by MS. The incident beam is not separated into nonoverlapping beams in the Talbot-Lau effect, but the origin of the interference pattern can still be traced to matter-wave interference since it is related to Fresnel diffraction.

$$\phi_D \sim kv t \sim \theta_b L/\theta \gg 1,$$

as they propagate a distance of order $L$ from the MS. The decay of the macroscopic grating is analogous to the free-induction-decay of the macroscopic polarization of a Doppler-broadened atomic vapor following excitation by an optical pulse. In this approach, one can draw on many processes that are well known in the theory of optical coherent transients.

Although the atomic grating is washed out following the interaction with the MS, it is possible to restore the original macroscopic atomic grating by placing a second MS between the first MS and the screen. This effect has been observed recently by Batelaan et al. (1996) using a metastable Ar beam. In the classical trajectory picture, the restoration of the grating corresponds to a Moiré pattern. In the Doppler dephasing picture, the restoration is analogous to the dephasing-rephasing process that occurs for a photon echo. The first MS starts a dephasing process for the different velocity subgroups and the second MS results in a rephasing process (see below, section IV. A). At a particular focal plane, where the rephasing is complete, a macroscopic grating appears. For

$$L \ll L_T$$

effects relating to quantization of the atomic center-of-mass motion play no role.

If $L \sim L_T$, one has to include recoil effects. Gratings appearing in this regime are usually associated with the Talbot-Lau effect, i.e. with the interference of light for the optical case or quantum interference of matter waves in the case of the atomic Talbot-Lau effect. It is shown in Sec. V, however, that the position of the focal planes and gratings’ periods are often the same as in the classical case. From this point of view, the Talbot-Lau effect is a quantum generalization of the shadow effect.

In summary, one can conclude that interference is qualitatively different for collimated beams and beams with large angular divergence. For collimated beams ($\theta_b L \ll \theta$) one has two interesting regimes,
\[ L \sim L_T \quad [\omega_k \ t \sim 1] \quad (20) \]

and

\[ L/L_T \gg D/d \quad [\omega_k \ t \gg D/d] \quad , \quad (20a) \]

corresponding to Fresnel (Talbot effect) and Fraunhofer diffraction (nonoverlapping scattered beams), respectively. For beams having angular divergence \( \theta_b L \gg d \), the Fresnel and Fraunhofer diffraction patterns would wash out following a single MS. Restoration of the atomic gratings in this limit can be achieved using two or more MS. Distances \( L \sim L_T \) correspond to the atomic Talbot-Lau effect and distances

\[ L \ll L_T \quad \omega_k t \ll 1 \quad (21) \]

correspond to the classical shadow effect.

III. TALBOT EFFECT

A. Talbot effect as a recoil effect.

In this section we show that the Talbot effect is a consequence of the recoil an atom undergoes when it passes through a microfabricated structure (MS). We assume that the MS is located in the plane \( y = 0 \), normal to the direction of propagation of the atomic beam. The MS consists of an infinite number of slits oriented in the \( z \)-direction; as such, only the \( x \)-dependence of the atomic wave function changes when atoms pass through the slits. Atomic motion in the \( y \)-direction can be considered as classical in nature provided that \( \lambda_{AB}/d \ll 1 \), but motion along the \( x \)-axis must be quantized. In this section, it is assumed that the incident beam is strongly collimated, \( \theta_b \ll d/L \), where \( d \) is the period of the MS and \( L \) is the distance from the MS to the screen. As such, we can neglect any spread in the transverse velocities in the initial beam and consider all atoms to be incident with transverse momenta \( p = 0 \).

After passing through the MS, the wave function for an atom is given by

\[ \psi(x) = \eta(x), \quad (22) \]

where \( \eta(x) \) is the amplitude transmission function associated with the MS. In the momentum representation, \( \psi(x) \) can be written as a superposition of states having momenta \( p = m \hbar k \), where \( m \) is an integer and \( k = 2\pi/d \). Explicitly, one finds that the Fourier transform of \( \psi(x) \) is given by

\[ \tilde{\psi}(p) = \sqrt{2\pi\hbar} \sum_m \eta_m \delta(p - m\hbar k), \quad (23) \]

where

\[ \eta_m = \int dx e^{-imkx} \eta(x) \quad (24) \]

is a Fourier coefficient. Unless indicated otherwise, all sums run from \(-\infty \) to \(+\infty \). The terms with \( m \neq 0 \) in Eq. (24) can be associated with atomic scattering at angles \( m\hbar k/p_y \), where \( p_y \) is the longitudinal momentum in the atomic beam. It is assumed that \( p_y \) is constant for all atoms in the beam - this restriction is relaxed in Sec. VI. In analogy with electron scattering from a standing wave field [Kapitza-Dirac effect (Kapitza and Dirac, 1933)] or atomic scattering from a resonant standing wave field [resonant Kapitza-Dirac effect (Kazantsev et al., 1980)], the scattering from the MS can be interpreted as arising from the recoil the atoms undergo when they acquire \( m\hbar k \) of momenta by scattering from the MS.

The classical motion of the atoms in the \( y \)-direction associates a distance \( y \) with a time

\[ t = y/u, \quad (25) \]

where \( u = p_y/M \) and \( M \) is the atomic mass. For a given \( u \), the momentum space wave function evolves as

\[ \tilde{\psi}(p, t) = e^{-ip\omega_k t/\hbar} \tilde{\psi}(p) = \sqrt{2\pi\hbar} \sum_m \eta_m \exp[-im^2\omega_k t] \delta(p - m\hbar k), \quad (26) \]

where \( \omega_k = \hbar k^2/2M \) is the recoil frequency. In the coordinate representation

\[ \psi(x, t) = \int_{-\infty}^{\infty} dp e^{ipx/\hbar} \tilde{\psi}(p, t) \quad (27) \]

one finds

\[ \psi(x, t) = \sum_m \eta_m \exp[i mkx - im^2 \phi_t], \quad (28) \]

where the Talbot phase is defined by

\[ \phi_t = \omega_k t. \quad (29) \]

Superposition of the different terms in Eq. (28) leads to a spatial modulation of the atomic density

\[ f(x, t) = |\psi(x, t)|^2. \quad (30) \]

The interference terms in Eq. (30) are a direct manifestation of matter-wave interference. One can see that, as a function of the time of flight \( t = y/u \), the wave function (28) undergoes oscillations on a time scale \( \omega_k^{-1} \). As a consequence, the atomic spatial distribution (30) contains quantum beats at frequencies \((m^2 - n^2)\omega_k\), for integral \( m, n \). Such quantum beats have been predicted by Chebotayev et al. (1985) and observed by Chapman et al. (1995).
It follows from Eq. (23) that the atomic wave function coincides with the amplitude transmission function of the MS when

\[ t = t_T \equiv \frac{2\pi}{\omega_k}. \]  \hspace{1cm} (31)

At this time, atoms are found in the focal plane at

\[ y = L_T = ut_T = 2d^2/\lambda dB, \]  \hspace{1cm} (32)

and a self-image of the MS is produced. In general one finds that the atomic wave function is a periodic function of the Talbot phase \( \phi_t \) having period \( 2\pi \), a periodic function of the time \( t \) having period \( 2\pi/\omega_k \), and a periodic function of the distance \( y \) having period \( L_T \).

The self-imaging of a periodic structure is well known in classical optics as the Talbot effect. To describe this effect, one usually starts from the Fresnel-Kirchhoff equation

\[ \psi(x) = \frac{1}{\sqrt{i\lambda dB y}} \int_{-\infty}^{\infty} dx' \eta(x') \exp \left[ i k dB (x - x')^2/2y \right], \]  \hspace{1cm} (33)

which is written here in the parabolic approximation. To establish an equivalence between Eqs. (23) and (28), one can substitute the Fourier expansion of the function \( \eta(x') \) in Eq. (33), carry out the integration, and express \( y \) in terms of \( t \).

It is possible to derive a useful symmetry property for \( f(x, t) \) when the transmission function \( \eta(x) \) is real, as it is for the MS. For real \( \eta(x) \), there is pure amplitude modulation of the atomic wave function and \( \eta_m = \eta^*_m \). It then follows from Eqs. (23) and (30) that the atomic spatial distribution is invariant under inversion with respect to the plane \( y = L_T/2 \), i.e.

\[ f(x, t)|_{\phi_t = \phi_t} = f(x, t)|_{2\pi - \phi_t}, \]  \hspace{1cm} (34)

\[ f(x, t) = f(x, tT - t) \]  \hspace{1cm} (34a)

\[ f(x, t)|_{y} = f(x, t)|_{L_T - y}. \]  \hspace{1cm} (34b)

Thus, one need calculate \( f(x, t) \) in the range \( 0 \leq y \leq L_T/2 \) to obtain the distribution for all \( y \).

**B. Calculation of the atomic density profile**

We have seen that, for a sufficiently collimated atomic beam, self-imaging of a MS occurs at integral multiples of the Talbot length. To analyze the diffraction pattern for arbitrary \( y \), it is convenient to use Eqs. (28) and (29) to reexpress the atomic wave-function as the convolution (Winthrop and Worthington, 1965)

\[ \psi(x, \phi_t) = (1/d) \int_{x-d}^{x} \eta(x')Z(x - x', \phi_t)dx', \]  \hspace{1cm} (35)

where

\[ Z(x, \phi_t) = \sum_{m} \exp \left[ -im^2\phi_t + imkx \right]. \]  \hspace{1cm} (36)

In the following discussion, we calculate \( \psi(x, \phi_t = 2\pi y/L_T) \) at fractions of the Talbot length, that is, for

\[ y = L_T/n, \]  \hspace{1cm} (37)

or, equivalently, for

\[ \phi_t = 2\pi/n, \]  \hspace{1cm} (38)

where \( n \) is a positive integer.

When \( n = 2 \), one can show that the diffraction pattern is a self-image of the MS, shifted by half a period. For \( n = 2 \), \( \phi_t = \pi \), and

\[ Z(x, \pi) = (1 - e^{ikx}) \sum_{q} e^{2iqkx}. \]  \hspace{1cm} (39)

Using the equality

\[ \sum_{q} e^{iq\alpha} \equiv 2\pi \sum_{s} \delta(\alpha - 2\pi s), \]  \hspace{1cm} (40)

one finds

\[ Z(x, \pi) = d \sum_{odd \ s} \delta \left( x - \frac{d}{2} \right). \]  \hspace{1cm} (41)

For the integration range in Eq. (35), only the \( s = 1 \) term contributes when Eq. (11) is substituted into Eq. (33), leading to

\[ \psi(x, \pi) = \eta \left( x - \frac{d}{2} \right), \]  \hspace{1cm} (42)

i.e. at half-integral multiples of the Talbot length, there are self images of the MS shifted by half a period.

For arbitrary \( n \), it is convenient to write

\[ m = nq + r, \]  \hspace{1cm} (43)

where \( 0 \leq r \leq n - 1 \) and \( q \) and \( r \) are integers. It then follows that

\[ \exp(-im^2\phi_t) = \exp(-2\pi im^2/n) \equiv \exp(-i2\pi r^2/n). \]  \hspace{1cm} (44)

and

\[ Z(x, 2\pi/n) = d \sum_{s} a_s \delta \left( x - \frac{d}{n} \right), \]  \hspace{1cm} (45)

where
\[ a_s(n) = \frac{1}{n} \sum_{r=0}^{n-1} \exp\left[2\pi i r(s - r)/n\right]. \]  

(45a)

The atomic wave function (38) is then given by

\[ \psi(x, 2\pi/n) = \sum_{s=0}^{n-1} a_s(n)\eta \left( x - s \frac{d}{n} \right). \]  

(46)

The meaning of this equation is clear. At distances \( y = L_T/n \), the wave function consists of \( n \) self-images of the amplitude transmission function \( \eta(x) \), having different amplitudes \( a_s(n) \) (some of which might vanish) and spaced from one another by the distance \( d/n \). We refer to such profiles as “higher order atomic gratings.”

\[ z_r(x, 2\pi/n) = \exp\left[i n (k x)^2 / 8\pi\right] \left\{ \exp\left[i n k (q + \frac{1}{2}) + in \frac{\pi}{2}\right], \right. \]  

\[ \left. \text{for } r = 2q + 1, \exp\left[i n k x q\right], \text{ for } r = 2q, \right. \]  

(50)

where \( q \) is integer. Substituting this expression into Eq. (49) and summing over \( r \), one arrives again at Eq. (45), but with an alternative expression for \( a_s(n) \):

\[ a_s(n) = \frac{1}{\sqrt{2n}} \left[ 1 + (-1)^s e^{i n \pi / 2} \right] e^{i s \pi^2 / 2n}. \]  

(51)

### C. Higher-order gratings using the Talbot effect

One can conclude from Eq. (44) that, owing to matter-wave interference, the transmission function \( \eta(x) \) impressed on the atomic wave function by the MS can be copied \( n \) times in the plane \( y = L_T/n \), with each copy separated by \( d/n \). This effect occurs for arbitrary transmission functions and can be used to generate higher order atomic gratings.

Since the \( a_s(n) \) appearing in Eq. (44) are not necessarily equal, the wave function (38) is periodic with period \( d \), but not necessarily with period \( d_g < d \). Moreover, it is possible for the different copies corresponding to different \( s \) to overlap. We refer to a pure, higher order atomic grating as one in which the different grating images do not overlap and for which \( d_g < d \). When the width \( f d \) of the slits in the MS is smaller than spacing \( d/n \), different terms in the wave-function (44) do not overlap with one another and one finds for the atomic density (39):

To simplify the expression for the coefficients \( a_s(n) \), one can use an alternative approach for evaluating \( Z(x, \phi_t) \) (Winthrop and Worthington, 1965). The sum in Eq. (40) can be written in the form

\[ Z(x, \phi_t) = \frac{1}{2\pi} \sum_{q} \int \frac{df}{\sqrt{d}} \exp\left[-i f (q - m) - im^2 \phi_t + im k x\right]. \]  

(48)

Carrying out the integration over \( m \), summation over \( q \) [using Eq. (10)] and integration over \( f \), one arrives at

\[ Z(x, \phi_t) = \sqrt{\frac{\pi}{4\phi_t}} \sum_{r} z_r(x, \phi_t), \]  

(49)

where

\[ z_r(x, \phi_t) = \exp\left[i (k x + 2\pi r)^2 / 4\phi_t\right]. \]  

(49a)

At distances \( y = L_T/n \) one finds

\[ f(x, t) = \sum_{s=0}^{n-1} |a_s(n)|^2 |\eta \left( x - \frac{d}{n} \right)|^2. \]  

(52)

Pure higher order atomic gratings are produced only if the nonvanishing \( |a_s(n)| \) are equal. From Eq. (51) one sees that

\[ |a_s(n)| = \sqrt{\frac{2}{n}} \left\{ \begin{array}{ll} |\cos(n\pi/4)|, & \text{for even } s \vline \vline |\sin(n\pi/4)|, & \text{for odd } s \end{array} \right. \]  

(53)

Three different situations can be distinguished

1. \( n = 2m + 1 \)
2. \( n = 2(2m + 1) \)
3. \( n = 4m \)

for integers \( m \geq 0 \).

In the first case \( |a_s(n)| = \frac{1}{\sqrt{n}} \), independent of \( s \). At distances \( y = L_T/3, L_T/5, \ldots \), pure, higher order atomic gratings having periods \( d_g = d/3, d/5 \ldots \) are produced.

In the second case \( |a_s(n)| = 0 \) for even \( s \) and \( |a_s(n)| = \sqrt{\frac{2}{n}} \), independent of \( s \), for odd \( s \). In the plane \( y = L_T/2 \) the atomic grating is shifted by a half-period \( d/2 \) from the initial grating (as found above); in the plane \( y = L_T/6 \) only terms located at \( x = \frac{d}{6}, \frac{d}{4}, \frac{d}{2} \) in Eq. (52) contribute to the sum (for \( 0 \leq x < d \)), corresponding to an atomic grating having period \( d_g = d/3 \), that is shifted by a distance \( d/6 \) from the initial grating. In general in the focal plane \( y = L_T/[2(2m + 1)] \), one finds an atomic
grating having period \( d_g = d/(2m + 1) \) which is shifted by a distance \( d/[2(2m + 1)] \) from the initial grating.

![Diagram](image)

**FIG. 3.** One period of the atomic density spatial modulation in the planes located at the distances \( y = L_T/m \) \((m = 1, \ldots, 16)\). The case of the atomic beam modulation with microfabricated structure having transmission step-function (54) with relative width \( f = 0.16 \) is shown. While expected period \( d_g \) of the beam self-image greater than slits’ width \( \sigma = df \), one obtains higher-order spatial gratings.

In the third case \( |a_s(n)| = \sqrt{\frac{2}{\pi n}} \), independent of \( s \), for \( s \) even and \( |a_s(n)| = 0 \) for \( s \) odd. In the planes \( y = L_T/4, y = L_T/8 \ldots \), one finds atomic gratings having periods \( d_g = d/2, d/4 \ldots \).

The atomic density profile can no longer be written in the form (2) when

\[
f d > d_g,
\]

In this limit, different components in the wave-function overlap and can interfere with one another in forming the atomic density. Even though the atomic distribution function can still contain narrow peaks having a size of order of \( d_g \), the amplitudes of the peaks are not equal, and the period of the overall diffraction pattern reverts to the period \( d \) of the initial grating.

These different regimes are illustrated in Fig. 3 plotted for an amplitude transmission function defined in the interval \( 0 \leq x < d \) as

\[
\eta(x) = \begin{cases} 
1, & \text{for } 0 \leq x \leq df \\
0, & \text{for } df < x < d 
\end{cases}
\]

The Talbot effect enables one to create pure, higher-order atomic gratings having periods that are limited only by the slit widths in the MS.

**IV. SHADOW EFFECT WITH MICROFABRICATED STRUCTURES**

In the previous section, it was assumed that the angular divergence \( \theta_b \) of the incident beam was less than \( d/L_T \). If this inequality is not satisfied, the diffraction patterns associated with different velocity subgroups in the incident atomic beam result in a washing out of the overall diffraction pattern. For typical beam parameters, this condition restricts \( \theta_b \) to be less than \( 10^{-5} - 10^{-4} \) rad. The restriction on \( \theta_b \) is a limiting factor on the maximum flux of the atomic beam. It is possible to avoid this restriction and increase the atomic flux if echo-like techniques are used.

Using echo techniques that are analogous to those encountered in the study of coherent transients, one can observe matter-wave interference in beams having a large angular divergence (Dubetsky et al., 1984). It turns out, however, that the dephasing and rephasing of the atomic gratings which occurs in such schemes does not depend in any critical manner on quantization of the atomic center-of-mass motion. In other words, the dephasing-rephasing mechanism is the same whether or not \( L < L_T \) (Talbot effect) or \( L < L_T \) (classical limit). As such, it makes sense to consider the limit of classical scattering first, since the analysis is easier and a simple geometric interpretation can be given to the results (Dubetsky and Berman, 1994). Thus, we consider the limit \( L < L_T \) in this section and defer a discussion of the case \( L < L_T \) until Sec. V.

In this and the following section we consider the interaction of an atomic beam with two MS, separated by a distance \( L \). The angular divergence of the incident beam is sufficiently large to satisfy the inequality

\[
\theta_b \gg d/L.
\]

The first MS produces a sum of atomic gratings, one for each velocity subgroup in the initial atomic beam. Immediately following the MS, these gratings overlap and mirror the transmission function of the grating, but downstream from the MS, they dephase relative to one another. As a result, the macroscopic atomic grating is washed out at a distance

\[
\ell \sim d/\theta_b \ll L
\]

from the MS.

Although the macroscopic grating produced by the first MS washes out in a distance of order \( \ell \ll L \), it is possible for the second MS to lead to a restoration of the atomic gratings. For particles moving on classical trajectories, we refer to this process as a shadow effect since it can be interpreted completely by the "shadow" of the incident beam formed by the two MS (Chebotayev et al., 1985) (see Fig. 3).
For a beam having large angular divergence \((\theta_b \sim 1)\), the initial grating produced by the first MS washes out in a distance comparable with the MS’s period, in accordance with Eq. (6). After passing through the second MS, however, macroscopic gratings reappear in specific focal planes. A grating having the same period as the MS is focused in the focal plane \(y = 2L\), while higher-order gratings having periods \(d/n\) (for integer \(n\)) are focused at other locations (to be determined below). The shadow effect can also be demonstrated easily using incoherent light (Chebotayev, 1986).

Although the shadow effect occurs for classical particles, it can be interpreted in terms of a dephasing and rephasing of atomic gratings. The relevant phases are the Doppler phases associated with various Fourier components of the atomic density, as discussed in the Introduction. In such a picture, the final image on the screen depends on a cancellation of Doppler phases in the spatial regions \(0 \rightarrow L\) and \(L \rightarrow 2L\), for example. In other words, the signal is sensitive to the relative Doppler phases in two spatial regions and is a measure of this relative phase. Insofar as interferometers are measures of relative phase, the echo-like rephasing of the atomic gratings can be viewed as a manifestation of atom interferometry. On the other hand, this rephasing is not related to the wave nature of matter. A shadow effect interferometer of this type was used by Batelaan et al. (1996) to measure the displacement of atomic gratings produced by rotation and by gravity.

The same type of Doppler dephasing and rephasing that occurs using MS can also occur when atoms interact with two or more nearly resonant standing wave fields (Baklanov et al., 1976; Barger et al., 1979; Dubetsky, 1976; Chebotayev, 1978; Chebotayev et al., 1978a, LeGouët and Berman, 1979; Mossberg et al., 1979; Dubetsky and Semibalaminut, 1982; Bordé, 1989; Dubetsky and Berman, 1994). When standing wave optical fields are used for modulation of the atomic spatial distribution, the atomic gratings often are monitored by applying a probe pulse in the focal planes that transfers the phase associated with an atomic state population to one associated with an atomic coherence. Atom interferometers of this type have been used for precision measurements of gravitational (Kasevich and Chu, 1991) and inertial (Riehle et al., 1991) phenomena [for a review, see (Müller et al., 1995)]. In these cases, external fields give rise to a displacement of the atomic gratings.

A. Dephasing-rephasing processes using two spatially separated MS

Before calculating the particles’ distribution function, we derive some general properties of grating formation. In this subsection, it is convenient to make a Fourier decomposition of the atomic density profile in the \(x\)-direction. The propagation of each of the Fourier components is then treated separately.

Consider the case when the two MS (1 and 2) have periods \(d_1\) and \(d_2\) and are separated from one another by a distance \(L\). A MS forms a periodic spatial distribution (shadow) which is the same for all atomic velocity subgroups just after passing through the MS. The profile created by the first MS contains a sum of harmonics in the \(x\)-direction having spatial periods

\[
d_{m_1} = d / |m_1|, \tag{59}
\]

where \(m_1\) is an integer. Immediately following the MS, the \(m_1\)th spatial harmonic varies as \(\cos(m_1 k_1 x)\), where \(k_1 = 2\pi / d_1\) is the wave-number associated with the first MS. As the atoms move downstream from the first MS, the \(m_1\)th harmonic acquires a Doppler phase given by

\[
\phi_{m_1}(t) = m_1 k_1 vt, \tag{60}
\]

where \(v\) is the \(x\)-component of atomic velocity and \(t = y/u\) as before. For a time

\[
t_d \sim 1/(k_1 v) \tag{61}
\]

the Doppler phases becomes large, \(\phi_{m_1}(t) \gg 1\), and the macroscopic grating washes out on averaging over
v. Since \( v \sim u \theta_b \), the time \( t_d \) \(^{(61)}\) corresponds to the distance \( \tilde{\ell} \) \(^{(58)}\).

The atoms pass through the second MS at time \( T = L/u \) (\( y = L \)). Downstream from the second MS, each spatial harmonic acquires an additional phase (i.e., a phase in addition to \( \phi_{m_1}(t) \), which, itself, continues to increase following the second MS)

\[
\phi_{m_2}(t) = m_2 k_2 v(t - T),
\]

where \( m_2 \) is another integer. Since the mask created by the second MS is superimposed on the shadow from the first MS, the resulting shadow consists of harmonics having wave numbers

\[
k_h = |m_1 k_1 + m_2 k_2|
\]

and velocity-dependent Doppler phases

\[
\phi(t) = \phi_{m_1}(t) + \phi_{m_2}(t).
\]

The two phases in the rhs of this equation can cancel one another at the so-called echo time \( t_e \) defined by

\[
\phi(t_e) = 0,
\]

corresponding to a focal plane \( y_e = u t_e \). At such times, one produces an harmonic in the atomic density that is independent of \( v \); as a consequence this grating survives any averaging over the velocity distribution in the incident beam. From Eq. \(^{(63)}\), one sees gratings are focused when

\[
t_e/T = y_e/L = \frac{1}{1 + (m_1/m_2)(k_1/k_2)}
\]

where

\[
\frac{m_1}{m_2} = \left( \frac{j}{\ell} \right) \left( 1 - \frac{n}{m} \right),
\]

and

\[
t_e = \frac{m}{n} T
\]

The dephasing-rephasing process is illustrated in Fig. \(^5\).
\[ y_c = \frac{m}{n} L. \]  

(70)

From Eq. (63), one finds that harmonics having
\[ k_h = \frac{|m_1|k_1}{m - 1} \]  

are focused in this plane.

For example, consider the limiting case in which \( d_1 = d_2 \equiv d \) \((j = \ell = 1)\), analogous to the situation studied by Dubetsky and Berman (1994). In the plane \( y = 2L \) \((m/n = 2)\), all harmonics having \( m_2/m_1 = -2 \) \((i.e., \{m_1, m_2\} = \{-1, 2\}; \{-2, 4\}; \{-3, 6\}; \text{etc})\) are focused. As a result [see Eq. (63)], harmonics having \( k_h = k, 2k, 3k; \text{etc} \) are focused in the plane \( y = 2L \).

The period of this atomic grating \( d_g \) corresponds to the smallest value of \( k_h \), namely \( d_g = 2\pi/k = d \). Similarly, in the plane \( y = 3L/2 \) \((m/n = 3/2)\), all harmonics having \( m_2/m_1 = -3 \) \((i.e., \{m_1, m_2\} = \{-1, 3\}; \{-2, 6\}; \{-3, 9\}; \text{etc})\) are focused. As a result [see Eq. (63)], harmonics having \( k_h = 2k, 4k, 6k; \text{etc} \) are focused in the plane \( y = (3/2)L \). The period of this atomic grating is \( d_g = 2\pi/2k = d/2 \). For \( m = (n + 1) \), one finds that atomic gratings having period \( d_g = d/n \) are focused in the plane \( y = [(n + 1)/n]L \).

To treat the case of arbitrary, rational \( d_1/d_2 = j/\ell \), we set
\[ m_1 = -\bar{j}(m - n)q, \quad m_2 = \bar{\ell}mq, \]  

(72)

where \( q \) is an integer,
\[ \bar{j} = j/\mu, \quad \bar{\ell} = \ell/\mu, \]  

(73)

and \( \mu \) is the largest common factor of \( j(m - n) \) and \( \ell m \).

Harmonics having wave numbers (71)
\[ k_h = n\bar{j}k_1 |q|, \]  

(74)

are focused in the plane \( y_c = \frac{m}{n} L \). The minimum possible wave number
\[ k_g = n\bar{j}k_1 \]  

(75)

determines the period of the focused grating
\[ d_g = \frac{2\pi}{k_g} = \frac{d_1}{j n}. \]  

(76)

One concludes that it is possible to create a higher order atomic grating, having a period that is \( \bar{j}n \)-times smaller than that of the first MS, by passing an atomic beam having a large angular divergence through two MS. Although both the Talbot and shadow effects lead to higher order atomic gratings, there is a qualitative difference between the two cases. In the Talbot effect, the structure of the MS is copied \( n \) times in the image plane \( y = LT/n \), giving rise to a profile having period \( d_g = d_1/n \) or \( 2d_1/n \), provided that \( d_g > f_1d_1 \). The minimum period is determined by the slit width. In contrast, the period of the atomic grating produced by the shadow effect in the plane \( y_c = \frac{m}{n} L \) is given by Eq. (76) and is not limited by the slit widths of the MS (although the contrast is determined by the slit widths). The period of the atomic grating is equal to \( d_1/j \) in the focal plane \( y_c = 2L \) and is compressed by a factor \( n \) in the plane \( y_c = \frac{m}{n} L \). This compression lies at the heart of the shadow effect’s application to atomic lithography (Dubetsky and Berman, 1994). Atomic gratings having periods smaller than those of both the MS and even smaller than the slit widths of the MS can be obtained. In this respect the shadow effect has yet an additional advantage over the Talbot-effect, where higher order grating production is not accompanied by compression.

Before proceeding to calculate the atomic density distribution, we should like to estimate the depth of focus of the various gratings. The distances between focal planes are comparable with the distance \( L \) between the MS. One can estimate the depth of focus \( \ell_g \) from the requirement that the phase \( \tilde{\ell} \) be smaller than unity in the region of the focal plane. For \( m_1 \) given by (72) one finds
\[ \phi(t) = qk_g\delta y/u, \]  

(77)

where \( \delta y = (y - \frac{m}{n} L) = (y - y_c) \) in the neighborhood of the focal plane at \( y = \frac{m}{n} L \). Setting \( q = 1 \), \( \phi(t) \sim 1 \) and \( \delta y = \tilde{\ell}_g \), and using Eq. (72) and the fact that \( \theta_h \sim v/u \), one obtains
\[ \tilde{\ell}_g \sim \frac{\tilde{\ell}}{n\bar{j}} \leq \tilde{\ell}, \]  

(78)

where \( \tilde{\ell} \) is given by (78) with \( d = d_1 \). Since \( L \gg \tilde{\ell} \) has been assumed, it is possible to separate the various gratings. The sharpening of depth of focus of the higher order gratings predicted by Eq. (78) is in qualitative agreement with the results shown in Fig. 4.

B. Particles’ distribution profile

We now turn our attention to a calculation of the atomic density profile. In contrast to the Talbot effect, it is not possible to find self-imaging using the shadow effect, since the scattering coefficients for the different spatial harmonics are not the same. Let the transmission functions for the two MS be denoted by \( \chi_s(x) \) \((s = 1 \text{ or } 2) \). \( \chi_s(x) \) is a transmission function for atomic density, while \( \eta_s(x) \) is a transmission function for atomic state amplitudes - for MS having transmission of either 1 or 0, these functions are identical. Atoms are scattered by the MS in the planes \( y = 0 \) and \( y = L \) or, equivalently, at times \( T_1 = 0 \) and \( T_2 = T = L/u \). Calculations are carried out using \( t = y/u \) as a variable. In some sense,
this corresponds to working in the atomic rest frame. As a result of scattering, the atomic density is modified as

\[ f(x, v, T^+_s) = \chi_s(x) f(x, v, T^-_s), \]

where \( T^\pm_s \) are times just after or before a scattering event. Following the scattering event, the distribution evolves as

\[ f(x, v, t) = f(x - v(t - T_s), v, T^+_s). \]  

We assume that, for \( t < 0 \), the atoms are distributed homogeneously in the transverse direction, i.e.

\[ f(x, v, t)|_{t<0} = 1. \]

The assumption of a homogeneous velocity distribution is consistent with a beam having angular divergence since in this limit, the transverse velocity distribution is approximately constant over the range \( d_s/L \). The spatial distribution of the atomic density for \( t > T \) is given by

\[ f(x, t) = \langle \chi_1(x - vt) \chi_2(x - vt - T) \rangle, \]

where \( \langle \ldots \rangle \) represents an average over velocities. Expanding \( \chi_s(x) \) in a Fourier series

\[ \chi_s(x) = \sum_m \chi^{(1)}_m e^{imk_s x}, \]

where \( k_s = 2\pi/d_s \) is the "wave-number" of structure \( s \), one finds

\[ f(x, t) = \sum_{m_1, m_2} \chi^{(1)}_{m_1} \chi^{(2)}_{m_2} \exp[i(k_1 m_1 + k_2 m_2) x] \exp[-i\phi(t)], \]

from which one finds the atomic density at the focal planes given by

\[ f(x, t_e) = \sum_q \chi^{(1)}_{j} \chi^{(2)}_{\ell q} \exp(ik_g x), \]

where \( k_g \) is the wave number of the focused atomic grating given in Eq. (64). Thus, owing to the shadow effect, at the echo-time \( (69) \) an atomic grating is focused having period

\[ d_g = 2\pi/k_g = d_1/(jn). \]

Note that the period of this grating is \( j \) times smaller than the period of the first microfabricated structure.

To reexpress the density function at the focal planes in terms of the transmission functions, we write the Fourier harmonic amplitude \( \chi^{(j)}_s \) as

\[ \chi^{(j)}_s = \int_0^{d_1} dx \chi_j(x) \exp(-ik_j x), \]

where \( \phi(t) \), as defined by Eq. (44), is also a function of \( m \) and \( n \). Owing to condition (54), for \( t \sim T \sim L/u \) and \( v \sim u\phi_0 \), the phase factor in the brackets of Eq. (44) is large, of order \( L\phi_0/d_s \gg 1 \). On averaging over velocities, one finds a nonvanishing contribution only at the particular focal planes or echo-times given by Eq. (65).

Retaining contributions from only those \( m_i \) given by (72) corresponding to the various focal planes, one finds from Eq. (44) that the atomic density in the focal planes is given by

\[ f(x, t_e) = \sum_q \chi^{(1)}_{j} \chi^{(2)}_{\ell q} \exp(ik_g x), \]

where \( k_g \) is the wave number of the focused atomic grating given in Eq. (64). Thus, owing to the shadow effect, at the echo-time \( (69) \) an atomic grating is focused having period

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from which one finds the atomic density at the focal planes given by

\[ f(x, t_e) = \sum_q \chi^{(1)}_{j} \chi^{(2)}_{\ell q} \exp(ik_g x), \]

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\[ f(x, t_e) = \sum_q \chi^{(1)}_{j} \chi^{(2)}_{\ell q} \exp(ik_g x), \]

where \( k_g \) is the wave number of the focused atomic grating given in Eq. (64). Thus, owing to the shadow effect, at the echo-time \( (69) \) an atomic grating is focused having period

\[ d_g = 2\pi/k_g = d_1/(jn). \]
\[ f(x, t_e) = \frac{1}{j(m-n)} \sum_{s=0}^{j(m-n)-1} \int_0^{d_2} \frac{dx_2}{d_2} \chi_2(x_2) \chi_1 \left( \frac{d_1}{j(m-n)} \left( s + \left\{ \frac{m \ell f_2 - x}{d_2} \right\}_F \right) \right) . \] (93)

Consider in detail the case when the microfabricated structures have duty cycles (ratio of slit openings to periods) \( f_j \) and

\[ \chi_j(x) = \begin{cases} 1, & \text{for } \left\{ \frac{x}{d_j} \right\}_F < f_j \\ 0, & \text{for } \left\{ \frac{x}{d_j} \right\}_F > f_j \end{cases} \] (94)

Introducing dimensionless variables

\[ w = x/d_j, \quad z = x_2/(d_2 f_2) \] (95)

and taking into account that the argument of function \( \chi_1 \) in Eq. (93) is positive, one obtains

\[ f(x, t_e) = \frac{f_2}{j(m-n)} \sum_{s=0}^{\lfloor s \rfloor} h_s(w), \] (96)

where

\[ h_s(w) = \int_0^1 dz \theta \left( \beta - (s + \{ \alpha z - w \}_F) \right), \] (96a)

\[ \alpha = m \ell f_2, \quad \beta = \frac{1}{j(m-n)} f_1, \] (96b)

and \( \theta(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x < 0 \end{cases} \) is the Heaviside step function. It is sufficient to consider only the range

\[ 0 \leq w \leq 1. \] (97)

For \( 0 \leq s \leq \lfloor \beta \rfloor - 1 \), the integrand in Eq. (96a) is equal to unity. Therefore,

\[ f(x, t_e) = \frac{f_2}{j(m-n)} \left[ \lfloor \beta \rfloor + h_{\beta F}(w) \right], \] (98)

and one needs to evaluate the expression (96a) only for \( s = \lfloor \beta \rfloor \),

\[ h_{\beta F}(w) = \int_0^1 dz \theta \left( \{ \beta \}_F - \{ \alpha z - w \}_F \right). \] (99)

The first term in the Eq. (98) brackets is independent of \( w = \frac{x}{d_j} \); consequently, it is only the second term which corresponds to the atomic gratings.

A method for evaluating the integral (99) is given in the Appendix. Using this method one finds

\[ h_{\beta F}(w) = \left[ \{ \beta \}_F [\alpha]_I + S(w) \right] / \alpha, \] (100)

where the function \( S(w) \) is given by

\[ S(w) = \begin{cases} \{ \alpha \}_F - w, & \text{for } 0 \leq w \leq 1 - \{ \beta \}_F \\ \{ \beta \}_F + \{ \alpha \}_F - 1, & \text{for } 1 - \{ \beta \}_F \leq w \leq \{ \alpha \}_F \\ \{ \beta \}_F + w - 1, & \text{for } \{ \alpha \}_F \leq w \leq 1 + \{ \alpha \}_F - \{ \beta \}_F \\ \{ \alpha \}_F, & \text{for } 1 + \{ \alpha \}_F - \{ \beta \}_F \leq w \leq 1 \end{cases} , \] (101)

\[ S(w) = \begin{cases} \{ \alpha \}_F - w, & \text{for } 0 \leq w \leq \{ \alpha \}_F \\ 0, & \text{for } \{ \alpha \}_F \leq w \leq 1 - \{ \beta \}_F \\ \{ \beta \}_F + w - 1, & \text{for } 1 - \{ \beta \}_F \leq w \leq 1 + \{ \alpha \}_F - \{ \beta \}_F \\ \{ \alpha \}_F, & \text{for } 1 + \{ \alpha \}_F - \{ \beta \}_F \leq w \leq 1 \end{cases} , \] (101a)
\[ S(w) = \begin{cases} \{\beta\}_F, & \text{for } 0 \leq w \leq \{\alpha\}_F - \{\beta\}_F \\ \{\alpha\}_F - w, & \text{for } \{\alpha\}_F - \{\beta\}_F \leq w \leq 1 - \{\beta\}_F \\ \{\beta\}_F + \{\alpha\}_F - 1, & \text{for } 1 - \{\beta\}_F \leq w \leq \{\alpha\}_F \\ \{\beta\}_F + w - 1, & \text{for } \{\alpha\}_F \leq w \leq 1 \end{cases}, \text{ if } 1 - \{\alpha\}_F \leq \{\beta\}_F \leq \{\alpha\}_F, \quad (101b) \]

\[ S(w) = \begin{cases} \{\beta\}_F, & \text{for } 0 \leq w \leq \{\alpha\}_F - \{\beta\}_F \\ \{\alpha\}_F - w, & \text{for } \{\alpha\}_F - \{\beta\}_F \leq w \leq \{\alpha\}_F \\ 0, & \text{for } \{\alpha\}_F \leq w \leq 1 - \{\beta\}_F \\ \{\beta\}_F + w - 1, & \text{for } 1 - \{\beta\}_F \leq w \leq 1 \end{cases}, \text{ if } \{\beta\}_F \leq \text{min}(\{\alpha\}_F,1 - \{\alpha\}_F). \quad (101c) \]

Substituting Eq. (100) in the rhs of Eq. (98) one finds that the atomic beam profile at a given focal plane is equal to

\[ f(x,t_e) = [\alpha[\beta]_I + \{\beta\}_F \{\alpha\}_I \]

\[ + S(w = x/d_g)]/[m(m - n)j]. \quad (102) \]

C. Main features

All dependencies in (101) coincide if

\[ \{\alpha\}_F = \{\beta\}_F = 1/2, \quad (103) \]

when

\[ S(w) = \left| w - \frac{1}{2} \right|. \quad (104) \]

In this case the grating amplitude

\[ A = f(x,t_e)_{\max} - f(x,t_e)_{\min}, \quad (105) \]

for given \( m,\ell,j,n \), achieves a maximum value \( A = (2m\ell)(m - n)^{-1} \). To maximize this quantity for a given grating period, one has to choose \( \ell = j = 1 \) and \( m = n + 1 \) \((\ell = \bar{j} = 1)\), which corresponds to the focal planes \( y = 2L \ (n = 1), y = \frac{3}{2}L \ (n = 2), y = \frac{4}{3}L \ (n = 3) \ldots \)

where gratings having periods \( d_g = d_1/n \) are focused. To satisfy condition (103), one can choose

\[ f_1 = 1/2, \ f_2 = 1/[2(n + 1)], \quad (106) \]

for which

\[ A = 1/[2(n + 1)]. \quad (107) \]

The constant background term of \( f(x,t_e) \) [first two terms in the numerator of Eq. (102)] vanishes since

\[ \alpha = \beta = 1/2. \quad (108) \]

To achieve this maximum signal, one must use slits in the second microfabricated structure whose width is smaller than the atomic grating period \( (d_1 = d_2) \) and duty cycles \( f_1 = \frac{5}{7}, f_2 = \frac{5}{12} \)

at the different focal planes. Chosen case is optimum for 5-order grating focused at the plane \( y_c = \frac{5}{7}L \). This grating has amplitude \( A = \frac{1}{7} \) and twice large background term. The amplitude of this grating is not less than amplitudes of the lower-order gratings focused at the other planes for given duty cycles.

For illustration we plot in Fig. 6 the grating profiles at different focal planes \( y_c = \frac{2n+1}{7}L \ (n = 1, \ldots 5) \)

for MS having the same periods \( d_1 = d_2, \) and duty cycles \( f_1 = \frac{5}{7}, f_2 = \frac{5}{12} \), such that the parameters \( \alpha \) and

![Graph showing grating profiles](image-url)
The amplitudes of the gratings that are focused in the planes $y = \frac{n + 1}{n}L$ having maximum amplitude. When the amplitude of the 5th order grating is optimized and $f_2 \approx 0.5$, the amplitudes of the gratings that are focused in the planes $y = \frac{n + 1}{n}L$ $(n = 1, \ldots, 4)$ are less than or equal to the amplitude of the 5th order grating. This feature is seen in Fig. 7.

The geometric simulation introduced above allows one to obtain the positions of the atomic gratings. It can also be used to provide some quantitative results. For example, one finds from Eqs. (101, 102) that the shadow effect disappears at the focal plane $y = 2L$ if both MS have the same period $(d_1 = d_2 = d)$ and duty cycles, $f_1 = f_2 = 0.5$. In this case, $m = 2$, $\alpha = j = \ell = 1$, and, from Eq. (96b), one finds that $\alpha = 1$, $\beta = 0.5$. From Eq. (101a) one finds that $S(w) \equiv 0$; there is no atomic grating. The reason for the absence of the grating under these conditions is evident from Fig. 8.

![Microfabricated structures](image)

**FIG. 7.** To explain an absence of shadow-effect at the focal plane $y = 2L$ for half-open MSs having the same periods. An illumination of an arbitrary point $O$ at this focal plane from given slit $AB$ of the second MS is determined by the number of particles moving into the point $O$ inside angle $AOB$, i.e. it is proportional to the length of the bold part of $A'B'$, given by $s = |A'B'| - \frac{d}{2}$. Since $|A'B'| = 2|AB| = d$, $s$ is always equal to $\frac{d}{2}$ independently on the point $O$ $x$-coordinate. Consequently, any variation of the particles’ distribution at the focal plane $y = 2L$ disappears.

The geometric picture can also be used to explain the absence of background terms at the focal planes $y = \frac{n + 1}{n}L$ produced by MS having duty cycles $\{106\}$ and equal periods $(d_1 = d_2 = d)$. One can see from Eq. (102, 104) that the background disappears because there are no particles at the points $x_q = \frac{d}{2} (q + \frac{1}{2})$, where $q$ is integral. A geometric interpretation of this result is presented in Fig. 8.

![Microfabricated structures](image)

**FIG. 8.** To prove that two MS having equal periods $(d_1 = d_2 = d)$ and duty cycles $f_1 = \frac{1}{2}$ and $f_2 = \frac{q}{d(n + 1)}$ (case $n = 3$ is shown) produce a background-free atomic grating at the focal plane $y = \frac{n + 1}{n}L$, one can notice that points $x_q = d_q (q + \frac{1}{2})$ are not achievable for particles $(d_q = \frac{d}{2} L \equiv 0)$ is a grating period). It follows from the fact that an angle, built from any point $x_q$ and arbitrary slit of the second MS, meets the closed part of the first MS.

V. TALBOT-LAU EFFECT

When the spatial separation of the MS is increased to the point where

$$L \sim L_T = 2d^2/\lambda dB,$$

(109)

it is no longer possible to neglect quantization of the atomic center-of-mass in calculating the transverse motion of the atoms. Just as in the Talbot effect, the recoil an atom undergoes on scattering from a grating must be taken into account. It turns out, however, that the Doppler dephasing and rephasing encountered in analyzing the problem of classical scattering by MS still can be given a classical interpretation when $L \sim L_T$, provided that the angular divergence of the incident beam is sufficiently large, $\theta_b \gg d/L_T = \lambda dB/d = \theta_d$. In other words, even though we must account for quantization of the atoms’ center-of-mass motion, effects related to Doppler dephasing (which are automatically included in a quantized motion approach) are unchanged from the classical case.

We have already alluded to this result in the Introduction. Recall that matter-wave interference results from the overlap on the screen of atomic wave functions associated with states having center-of-mass momenta $p$ and $p + nhk$. The relative dephasing between these states $\{i\}$ contains a Doppler part $\{14\}$ and a quantum part...
Since the washing out and restoration of the macroscopic atomic gratings is connected with an averaging over atomic velocities \(v\), one expects that the Doppler part, proportional to \(v\), is responsible for the dephasing-rephasing effect. This contribution is actually classical in nature (i.e., it does not vanish in the limit \(\hbar \to 0\)) and enters the calculations whether or not the quantum contribution to the phase has to be considered. As a consequence, the dephasing-rephasing process is the same for the classical shadow effect and the quantum Talbot-Lau effect. It turns out, however, that for separations of the MS equal to a rational multiple of the Talbot length, the Talbot effect can actually result in a decrease of the period of the atomic gratings from those periods which would result from the classical shadow effect result. The decrease in period occurs for MS consisting of open slits and opaque strips; it would not occur, for example, in resonant standing wave fields.

Since the Doppler dephasing determines the position of the focal planes and the period of the atomic grating, one can carry over the results Eqs. (109, 110) obtained in Sec. IV for the shadow effect. In this section, we are interested in the variation of the atomic gratings in a given focal plane as a function of the separation of the MS. In other words, we look for those separations \(L\) for which the Talbot effect significantly modifies the gratings that would have been produced by the shadow effect alone. This is analogous to photon echo studies of atomic relaxation in which the echo amplitude is monitored as a function of the separation between the excitation pulses.

It should be noted that the Talbot-Lau effect has been studied using light by Clauser and Reinsch (1992) for the parameters

\[
\frac{d_1}{d_2} = 3; \quad y_e = 3L, \quad (110)
\]

accompanying to \(\{m, n, j, j', \ell, \ell'\} = \{3, 1, 3, 1, 1, 1/3\}\) [recall that \(j = j/\mu; \ell = \ell/\mu\), where \(\mu\) is the largest common factor of \(j(m - n)\) and \(ml\)] and a grating period \(d_g = d_1\). The atomic Talbot-Lau effect was demonstrated by Clauser and Li (1994) using K atoms for the parameters

\[
\frac{d_1}{d_2} = 2; \quad y_e = 2L, \quad (111)
\]

\(\{m, n, j, \ell, j, \ell'\} = \{2, 1, 2, 1, 1, 1/2\}\), \(d_g = d_1\). A theoretical study of the atomic Talbot-Lau effect was also carried out for the parameters (111) by Carnal et al. (1995). The conclusion as to the period and location of the atomic gratings follows from purely classical considerations in this case; there is no need to invoke arguments related to the wave nature of matter (Clauser and Reinsch, 1992; Clauser and Li, 1994; Carnal et al., 1995).

### A. Grating formation

The geometry is the same as that considered for the classical shadow effect, except that \(L\) is no longer restricted to be less than \(L_T\). Again, it is convenient to work in the atomic rest frame defined by \(t = y/u\). As discussed above, it is necessary to quantize the atomic motion only in the \(x\)-direction. The atoms undergo scattering at the MS at times \(T_s = L_s/u\) \((T_1 = 0, T_2 = T = L/\mu u)\). For thin gratings, the atomic wave function \(\psi(x, t)\) undergoes jumps at the MS given by

\[
\psi(x, T_s^+) = \eta_s(x)\psi(x, T_s^-), \quad (112)
\]

where \(\psi(x, T_s^\pm)\) is the wave function on either side of grating \(s\), \(\eta_s(x)\) is the amplitude transmission function of grating \(s\), and

\[
\chi_s(x) = |\eta_s(x)|^2 \quad (113)
\]

is the transmission function of grating \(s\) [for MS consisting of a series of slits, \(\eta_s(x) = \chi_s(x)\)]

To characterize the atomic beam using a quantized center-of-mass description, one can use the Wigner distribution function defined by

\[
f(x, p, t) = \int \frac{d\hat{x} dp}{2\pi\hbar} \exp(-ip\hat{x}/\hbar)\psi(x + \frac{\hat{x}}{2}, t)\psi^*(x - \frac{\hat{x}}{2}, t). \quad (114)
\]

For scattering at a MS, one finds

\[
f(x, p, T_s^+) = \sum_{n_+, n_s} \exp(im_s k_s x) \eta_{n_s}(x) \left[\eta_{n_s}^s(x) \right]^* f \left[x, p - \frac{\hbar k}{2}(n_s + n_s'), T_s^-\right], \quad (115)
\]

where

\[
m_s = n_s - n'_s, \quad (116a)
\]

and

\[
d_1 = 3; \quad y_e = 3L, \quad (110)
\]
\[ \eta^{(s)}_n = \int_0^{d_s} \frac{dx}{d_x} e^{-ink_x} \eta_s(x) \]  

(116b)

is a Fourier component of \( \eta_s(x) \), having period \( d_s \) and wave number \( k_s = 2\pi/d_s \). For times other than \( T_s \), the Wigner distribution function evolves freely as

\[ f(x, p, t) = f(x - v(t - T_s), p, T_s^+) \],

(117)

where \( v = p/M \).

Applying Eqs. (116, 117) one obtains the atomic distribution function for times \( t > T \) (\( y > L \)) to be

\[ f(x, p, t) = \sum_{n, n'} \eta_{n_1}^{(1)} \eta_{n_2}^{(2)} \exp \left\{ im_1k_1 \left[ x - v(t - T) - \left( v - \frac{\hbar k_2}{2M} (n_2 + n_2') \right) T \right] \right\} \]

\[ + im_2k_2 \left[ x - v(t - T) \right] \}

\[ f \left\{ x - \left[ v - \frac{\hbar k_2}{2M} (n_2 + n_2') \right] T - v(t - T), \right\} \]

\[ p - \frac{\hbar}{2} (k_1 (n_1 + n_1') + k_2 (n_2 + n_2')) \right\}, \]

(118)

where \( f(x, p) \) is the Wigner distribution function of the incoming atomic beam. The atomic spatial distribution is given by

\[ f(x, t) = \int dp f(x, p, t) \]

(119)

which can be obtained from Eq. (118) as

\[ f(x, t) = \sum_{n, n'} \eta_{n_1}^{(1)} \eta_{n_2}^{(2)} \exp \left\{ i (m_1k_1 + m_2k_2) \left[ x - \frac{\hbar k_2}{2M} (n_2 + n_2') \right] T \right\} \]

\[ - \left[ v + \frac{\hbar k_1}{2M} (n_1 + n_1') T \right) \}

\[ \left\{ x - \left[ v + \frac{\hbar k_2}{2M} (n_2 + n_2') \right] T - v(t - T), \right\} \]

\[ -i \left[ v + \frac{\hbar k_1}{2M} (n_1 + n_1') \right] [m_1k_1 t + m_2k_2 (t - T)] \}

(120)

In this expression terms having \( (m_1k_1 + m_2k_2) \neq 0 \) contribute to the atomic gratings. Owing to the assumption of an incident beam having large angular divergence \( \theta_b \sim v/u \gg d/L \), the Doppler phases associated with these terms oscillate rapidly as a function of \( p \), except in the echo focal planes. As a consequence, the positions and periods of the atomic gratings are the same as those in the classical shadow effect [see Eqs. (123)]. In the remainder of this section, we calculate the atomic density in the focal planes \( y_e \) or, equivalently, at times \( t_e = y_e/u \) given in Eqs. (123).

It is possible to simplify Eq. (120) if we assume that the angular divergence \( \theta_b \) of the incident beam is less than \( \theta_D = D/L \), such that a freely propagating beam would undergo negligible diffraction over a distance of order \( L \). For \( p \sim Mu \theta_b \ll MD/T \), one can neglect the dependence on \( n_i \) and \( n_i' \) of the distribution function appearing in Eq. (120). Then the sum over \( n_i \) can be carried out using the formula

\[ \sum_n e^{-ina} \eta^{(s)}_n \eta_{n-\nu}^{(s)} = e^{-i\nu\alpha/2} \eta_s \left( x - \frac{\alpha}{2k_s} \right) \eta_s^* \left( x + \frac{\alpha}{2k_s} \right), \]

(121)

where

\[ [F(x)]_\nu = \int_0^{d_s} \frac{dx}{d_x} e^{-i k_x x} F(x) \]

(122)

is a Fourier component of the function \( F(x) \). As a result one finds that the atomic density in the echo focal planes is given by

\[ f(x, t_e) = f(x) \sum_q e^{iqk_s x} \chi^{(1)}_{m-n} \frac{\phi_f(m, n)}{2\pi} \frac{\phi_f(m, n)}{2\pi} \]

(123)
where $\chi_1^{(1)}$ is a Fourier component of the transmission function $\chi_1(x)$,
\[
\phi_T(m, n) = \frac{j_2(m-n)}{\ell} \omega_k T \tag{124}
\]
is a Talbot phase associated with a specific focal plane, and
\[
f(x) = \int dp f(x, p) \tag{125}
\]
is the initial spatial distribution in the atomic beam. Since the beam diameter is much larger than the period of the gratings,
\[
D \gg d_g \tag{126}
\]
one can neglect the variation of $f(x)$ and set $f(x) = 1$ in Eq. (123).

The distribution function (123) is identical with the shadow effect result [83], except for the presence of the Talbot phases. The main features of the dependence of the atomic density on the Talbot phase in the Talbot-Lau effect are the same as those for the Talbot effect considered in Sec. III. The density (123) is an oscillating function of $\phi_T(m, n)$ having period $2\pi$. If $\phi_T(m, n)$ is increased by $2\pi$, or, equivalently, if the separation between the MS is increased by $L_T(m, n)$, the density distribution in the corresponding echo plane is unchanged. The Talbot distance associated with a given focal plane is defined here as
\[
L_T(m, n) = \frac{2d_1^2}{\lambda_d B} j^2(m-n). \tag{127}
\]
In terms of $L_T(m, n)$, the Talbot phase [124] is equal to
\[
\phi_T(m, n) = 2\pi[L/L_T(m, n)]. \tag{128}
\]
In our notation, the Talbot phase is a function of $L$, while the Talbot distance is independent of $L$. Note that, as defined by Eq. (127), there is a different Talbot length associated with the signal for different focal planes, $y_c = (m/n)L$. We wish to examine the signal in a given focal plane as a function of the separation $L$ of the MS, or, equivalently, as a function of $\phi_T(m, n)$. When $L = L_T(m, n)$ [with $\phi_T(m, n) = 2\pi$] the atomic density (123) is the same as that of the shadow effect [83].

Using arguments similar to those leading to Eqs. (54), one can prove that, for pure amplitude modulation of the wave functions, i.e., for real amplitude transmission functions $\eta_j(x) = \eta_j^*(x)$, the dependence of the particles’ distribution on the Talbot phase is symmetric with respect to the point $\phi_T(m, n) = \pi [L = L_T(m, n)/2],
\[
f(x, t_c)|_{\phi_T(m, n)} = f(x, t_c)|_{2\pi - \phi_T(m, n)}, \tag{129}
\]
\[
f(x, t_c)|_L = f(x, t_c)|_{L_T-L}. \tag{130}
\]

The question arises as to what values of $\phi_T(m, n)$ lead to especially interesting results, i.e., atomic gratings that differ significantly from the gratings that would be produced by the shadow effect. We have found that the atomic gratings are significantly modified by the Talbot effect when the Talbot phase is a rational multiple of $2\pi$,
\[
\phi_T(m, n) = 2\pi \frac{mT}{nT}. \tag{131}
\]
where $m_T$ and $n_T$ are positive integers having no common factors. We proceed to analyze the atomic density function in the focal planes for separations of the MS corresponding to Eq. (131), that is, for $L = L_T(m, n)$.

For Talbot phases given by Eq. (131), the sum in Eq. (123) can be divided into $n_T - 1$ independent sums having
\[
q = n_T q' + r \tag{132}
\]
where $0 \leq r \leq n_T$. For Talbot phases given by Eq. (131), any dependence on $q'$ disappears in the last factor of Eq. (123), allowing one to rewrite Eq. (123) as
\[
f(x, t_c) = \sum_{r=0}^{n_T-1} \sum_{q'} \int_0^{d_1} \int_0^{d_2} dx_1 dx_2 \exp \left\{ i (q' n_T + r) \left[ k_g x + \bar{j} (m-n) k_1 x_1 - mT k_2 x_2 \right] \right\} \times \chi_1(x_1) \eta_2 \left( x_2 - r \frac{mT}{nT} d_2 \right) \eta_2^* \left( x_2 + r \frac{mT}{nT} d_2 \right), \tag{133}
\]
to sum over $q'$ using Eq. (84), and to integrate over $x_1$. The calculations are similar to those used to obtain Eq. (93) from Eq. (88), and one can obtain
\[
f(x, t_c) = \sum_{r=0}^{n_T-1} \sum_{s=0}^{j(m-n)n_T-1} \int_0^{d_2} dx_2 \frac{d_1}{d_2} \exp \left\{ \frac{2\pi i r}{n_T} \left[ s + \left[ n_T \left( \frac{mT x_2}{d_2} - \frac{x}{d_g} \right) \right] \right] \right\} \times \eta_2 \left( x_2 - r \frac{mT}{nT} d_2 \right) \eta_2^* \left( x_2 + r \frac{mT}{nT} d_2 \right) \chi_1 \left\{ \frac{d_1}{j(m-n)n_T} \left[ s + \left[ n_T \left( \frac{mT x_2}{d_2} - \frac{x}{d_g} \right) \right] \right] \right\} \cdot \left(134 \right)
\]
B. Higher-order gratings using the Talbot-Lau technique

Equation (134) is the basic result of this section. It gives the atomic density function in the focal plane for separation of the MS that corresponds to Talbot phases which are rational multiples of $2\pi$. For specified transmission functions, it can be evaluated numerically in focal planes defined by $t_e = y_e/u = (m/n)T = (m/n)L/u$ for arbitrary $(j/\ell) = d_1/d_2$ (recall that $j = j/\mu; \ell = \ell/\mu$, where $\mu$ is the largest common factor of $j(m-n)$ and $\ell m$), $m_T$ and $d_T$ [$L = L_T(m, n)\frac{2\pi}{\eta_T}$]. In this subsection, we are interested primarily in showing that, owing to the Talbot effect, periodic atomic density gratings can be produced whose periods $d_T$ are smaller than the corresponding periods $d_g$ which would have been produced by the shadow effect.

The first thing to note is that the function $\chi_1$ in Eq. (134), considered as a function of $x$, is periodic with period

$$d_T = d_g/n_T,$$  \hspace{1cm} (135)

$n_T$-times smaller than the period $d_g$ of the shadow-effect grating. Unfortunately, this does not guarantee that $f(x, t_e)$ is periodic with period $d_T$, owing to the exponential term in Eq. (134). Under the transformation $x \rightarrow x + d_T$, the exponential term is multiplied by the phase-factor

$$\exp(2\pi ir/n_T)$$  \hspace{1cm} (136)

which is a function of the summation index $r$. If the summation over $r$ in Eq. (134) somehow was restricted to $r = 0$, the atomic grating would have period $d_T$. Restricting the summation to $r = 0$ can be accomplished by choosing the amplitude transmission function such that the product $\eta_2\left(x_2 - \frac{r}{\eta_T}d_2\right)\eta_2^*(x_2 + \frac{r}{\eta_T}d_2)$ is nonvanishing only for $r = 0$. To simplify the discussion, we have taken $m_T = 1$ [$L = L_T(m, n)/n_T$].

For MS consisting of slits and opaque strips, both the amplitude transmission functions $\eta_j(x)$ and transmission functions $\chi_j(x)$ are equal to the Heaviside step-function

$$\eta_j(x) = \chi_j(x) = \theta \left( f_j - \left\{ \frac{x}{d_j} \right\}_F \right),$$  \hspace{1cm} (137)

where $f_j$ is the duty cycle of MS $j$. The functions $\left\{ \frac{x}{d_j} \pm \frac{r}{\eta_T} \right\}_F$ shown in Fig. 1 represent the profile of the second MS displaced by $\pm \frac{r}{\eta_T}d_2$.

![FIG. 9. Plots of the functions $\{ \pm a \}$. In the shaded areas only transmissions $\phi_2(x_2 \pm ad_2) \neq 0$. When slits relative width $f_2 < \min \{(a), 1 - (a)\}$ there are no shaded areas in the vicinity of the point $x_2 = 0$. If also $f_2 < |1 - 2(a)|$, then shaded areas have no common points and product $\phi_2(x_2 + ad_2)\phi_2(x_2 - ad_2) = 0$ for any $x_2$.

In the range $0 \leq x_2 < d_2$, the product $\eta_2\left(x_2 - \frac{r}{\eta_T}d_2\right)\eta_2^*(x_2 + \frac{r}{\eta_T}d_2) = \theta \left( f_2 - \left\{ \frac{x}{\eta_T} \pm \frac{r}{\eta_T} \right\}_F \right)$ is sufficient small and product $\eta_2\left(x_2 - \frac{r}{\eta_T}d_2\right)\eta_2^*(x_2 + \frac{r}{\eta_T}d_2)$ vanishes for $r \neq 0$ provided that $f_2$ is sufficiently small and provided that $\frac{r}{\eta_T} \neq \frac{x_2}{d_2}$ (if $\frac{r}{\eta_T} = \frac{x_2}{d_2}$, the gratings are displaced by $\frac{d_2}{\eta_T}$ and overlap for any $f_2$).

If

$$f_2 \leq \min \left( \frac{r}{\eta_T}, 1 - \frac{r}{\eta_T} \right),$$  \hspace{1cm} (138)

the only regions where $\eta_2(x_2 \pm \frac{r}{\eta_T}d_2)$ does not vanish are

$$\frac{x_2}{d_2} \in \left[ 1 - \frac{r}{\eta_T}, f_2 + 1 - \frac{r}{\eta_T} \right] \text{ and } \frac{x_2}{d_2} \in \left[ \frac{r}{\eta_T}, f_2 + \frac{r}{\eta_T} \right],$$  \hspace{1cm} (139)

respectively. These two intervals have no common regions if $f_2 + 1 - \frac{r}{\eta_T} \leq \frac{r}{\eta_T}$ or $f_2 + \frac{r}{\eta_T} \leq 1 - \frac{r}{\eta_T}$, i.e.

$$f_2 \leq 1 - \frac{2r}{\eta_T}.$$  \hspace{1cm} (140)

Inequality (140) must hold for all $r \neq 0$ to guarantee that the atomic grating has period $d_T = d_g/n_T$. Clearly, inequality (140) does not hold for $r = \frac{r}{\eta_T}$ when $n_T$ is even. While this does not preclude the possibility of higher order gratings for $n_T$ even, it does suggest that we consider the cases of even and odd $n_T$ separately.
1. \( n_T \) odd

In this case, we write

\[ n_T = 2n' + 1, \quad (141) \]

where \( n' \) is a positive integer or zero. For the summation range \( 0 \leq r \leq n_T - 1 \) in Eq. (134), the minimum value of the rhs of both inequalities (138) and (140) is \( 1/n_T \), which occurs for \( r = 1 \) or \( r = 2n' \) in (138) and for \( r = n' \) or \( r = n' + 1 \) in (140). Thus, provided that

\[ f_2 \leq \frac{1}{2n' + 1} \quad (142) \]

one can produce atomic gratings having period \( d_T = d_o/(2n' + 1) \) in the focal plane \( y = \frac{n}{2}L \) for separations between the MS equal to \( L = \frac{2n}{2n+1} \) or, equivalently, for a Talbot phase (124) equal to \( 2\pi m/n \). Under these conditions, one omits terms having \( r \neq 0 \) in Eq. (134) and finds

\[ f(x, t_o) = \frac{f_2}{j(m-n)n_T} \sum_{s=0}^{[\beta']_t} h_s(w), \quad (143) \]

\[ h_s(w) = \int_0^1 dz \theta (\beta' - (s + \{\alpha'z - w\}_F)), \quad (143a) \]

\[ \alpha' = n_T \alpha = m\ell (2n' + 1)f_2, \]

\[ \beta' = n_T \beta = j(m-n)(2n' + 1)f_1, \quad (143b) \]

where \( \alpha \) and \( \beta \) are given by (161), and dimensionless variables

\[ w = \frac{x}{d_T}, \quad z = \frac{x_2}{d_2f_2} \quad (144) \]

have been introduced. Note that the ratio

\[ \alpha/\beta = \alpha'/\beta' = \frac{m}{m-n} \frac{f_2 \ell}{f_1 j} = \frac{m - f_2d_2}{m - f_1d_1} \quad (145) \]

depends on the focal plane and ratio of slit widths. In a manner similar to arriving at Eq. (102), one can obtain

\[ f(x, t_o) = \{\alpha' \{\beta'\}_t + \{\beta'\}_F \{\alpha'\}_t + S(w) \}
\]

\[ \times \left[ m(m-n)\ell \right] (2n' + 1)^{-2} \quad (146) \]

where \( S(w) \) is given by Eqs. (101) and (116) with the replacements \( \alpha \rightarrow \alpha', \ \beta \rightarrow \beta' \).

![FIG. 10. Atomic spatial distribution created by two microfabricated structures, having the same spacings \( d_1 = d_2 \), relative widths \( f_1 = \frac{1}{q} \), \( f_2 = \frac{1}{q} \) and separated on the distance \( L \). At the focal plane \( y = \frac{1}{2}L \) (m = 3, n = 2), where owing to the shadow effect, one expects gratings with period \( d_o = \frac{d}{n_T} \), the wave-matter interference leads to the higher order gratings if one chooses distance between fields coinciding with the integer fraction of the Talbot distance \( L = \frac{2n}{n_T} \), \( n_T = 1 \), . . . . \)].

The amplitude of the grating (146) is maximum when \( m = n + 1 \), \( j = \ell = 1 \), and

\[ \{\alpha'\}_F = \{\beta'\}_F = \frac{1}{2}, \quad (147) \]

for which

\[ f_1 = \frac{2q_1 + 1}{2(2n' + 1)}, \quad f_2 = \frac{2q_2 + 1}{2(n + 1)(2n' + 1)}. \quad (148) \]

where \( 0 \leq q_1 \leq 2n' \), \( 0 \leq q_2 \leq n \) are integers. Under these conditions, one finds

\[ f(x, t_o) = \frac{\frac{q_1}{2}(q_1 + q_2) + \left\{ \frac{2x(2n' + 1)}{d_2} \right\}_F}{(n + 1)(2n' + 1)^2} = -\frac{1}{2}. \quad (149) \]

This grating has amplitude
\[
A = \frac{1}{2(n+1)(2n'+1)^2} \quad (150)
\]

and a background term whose amplitude is \((2q_1q_2 + q_1 + q_2)\) times larger than \(A\). Talbot-Lau gratings for different values of the Talbot phase are shown in Fig. 10.

2. \(n_T\) even

The atomic density patterns in Fig. 10 have been drawn for both even and odd values of \(n_T\). From this figure one sees that qualitatively new features appear for even values of \(n_T\). When the Talbot phase \(\phi_T(m,n) = \pi, \pi/3, \pi/5\) \((n_T = 2, 6, 10)\) gratings having period \(d_g, d_g/3, d_g/5\), or 

\[
d_T = 2d_g/n_T = d_g/(n_T/2) \quad (151)
\]

\[
f(x, t_e) = \frac{1}{2\pi n'(m-n)} \sum_{r'=0}^{n'-1} \sum_{s=0}^{2j(m-n)n'-1} \int_{0}^{d_g} \frac{d_r}{d_2} \left\{ 1 + (-1)^{r'} [\frac{n' \pi}{m'} + s + 2n' (m' \bar{x}_r - \frac{x}{d_2})] \right\} \eta_2 (x_2 - r' \frac{m_2}{n'})
\]

\[
\times \exp \left\{ \frac{\pi i}{n'} [s + 2n' (m' \bar{x}_r - \frac{x}{d_2})] \right\} \eta_2 \chi_1 \left\{ \frac{d_1}{(m-n)n_T} \left[ s + 2n' (m' \bar{x}_r - \frac{x}{d_2}) \right] \right\} . \quad (154)
\]

The transmission function \(\chi_1\) still has period \(d_g/n_T = d_g/(2n')\), but the first factor in the integrand has twice this period, \(d_T = 2d_g/n_T = d_g/n'\). As in the case of odd \(n_T\), one must choose \(f_2\) sufficiently small to eliminate all but the \(r' = 0\) terms in the sum to ensure that the atomic grating has period \(d_T\). Inequalities (138) and (140) are satisfied if

\[
f_2 \leq \frac{1}{2n'}, \quad (155)
\]

which is a sufficient condition for neglect of terms with \(r' \neq 0\) in Eq. (154). As a result one arrives at

\[
f(x, t_e) = \frac{f_2}{2\pi n'(m-n)} \sum_{s=0}^{[\beta']_1} h_s(w), \quad (156)
\]

\[
h_s(w) = \int_{0}^{1} dz \left( 1 + (-1)^{n'm_Tm\bar{\ell} + s + [\alpha'z - w]} \right) \times \theta \left\{ \beta' - \left[ s + \alpha'z - w \right] \right\} . \quad (156a)
\]

\[
\alpha' = n_T \alpha = 2n' \bar{m} f_2; \quad \beta' = n_T \beta = 2n' \bar{j} (m-n) f_1; \quad (156b)
\]

\[
w = \frac{\bar{x}}{d_T}. \quad (156c)
\]

This expression can be evaluated in the same manner used to evaluate Eq. (43), but the evaluation is more complicated owing to two factors: (i) contributions \(h_s(w)\) with \(s < [\beta']_1\) are not independent of \(w\), and (ii) for \(s = [\beta']_1\) one has to consider separately contributions from odd and even \([\alpha'z - w]_F\). The situation simplifies for integer \(\beta'\), when the \(\theta\)-factor equals 1 for \(s < \beta'\) and 0 for \(s = \beta'\), independent of the values of \(z\), \(w\), and \(\alpha'\). This is the only limit considered in the subsection. For integer \(\beta'\), one can carry out the summation over \(s\) in Eq. (154).

For even \(\beta'\), when \(\sum_{s=0}^{[\beta']_1} (-1)^s = 0\), the gratings are washed out and

\[
f(x, t) = f_1 f_2. \quad (157)
\]

This result is consistent with the vanishing of the atomic gratings in Fig. 10 for Talbot phases equal to \(\pi/2, \pi/4\) and \(\pi/6\), corresponding to values of \(\beta'\) equal to 2, 4, 6.

When \(\beta'\) is odd one finds

\[
f(x, t) = f_1 f_2 \left( 1 + \frac{(-1)^{n'm_Tm\bar{\ell}}}{\alpha' \beta'} h'(w) \right), \quad (158)
\]

\[
h'(w) = \alpha' \left\{ 2 \int_{0}^{1} dz \theta \left\{ \frac{1}{2} - \left\{ \frac{\alpha'z - w}{2} \right\}_F \right\} - 1 \right\}, \quad (158a)
\]

where the equality

\[
n_T = 2n', \quad (152)
\]

where \(n'\) is a positive integer, one divides the sum over \(r\) in Eq. (134) into two parts having

\[
r = n' q + r', \quad (153)
\]

with \(q = 0\) or 1 with \(r'\) restricted to the range \(0 \leq r' \leq n' - 1\). In the second part \((q = 1)\) one shifts the integration variable from \(x_2\) to \((x_2 + \frac{2}{n'}d_2)\), which leads to the same factors \(\eta_2, \eta_2, \chi_1\) in both the \(q = 0\) and \(q = 1\) terms. In this manner, one arrives at the expression

\[
I = \int_{0}^{1} dz \theta \left\{ 1 - \left\{ \frac{\alpha'z - w}{2} \right\}_F \right\} \left\{ 1 - \left\{ \frac{\alpha'z - w}{2} \right\}_F \right\} . \quad (159)
\]
\[-1\]^{[x]_{j}} = 2\theta \left[ \frac{1}{2} - \left\{ \frac{x}{2} \right\}_F \right] - 1 \quad (159)

has been used. Equation (158) can be reduced to

\[ k'(w) = 2 \begin{cases} 
\left\{ \frac{\alpha'}{T} \right\}_F - w, & 0 < w < 2 \left\{ \frac{\alpha'}{T} \right\}_F, \\
- \left\{ \frac{\alpha'}{T} \right\}_F, & 2 \left\{ \frac{\alpha'}{T} \right\}_F < w < 1, \\
w - 1 - \left\{ \frac{\alpha'}{T} \right\}_F, & 1 < w < 1 + 2 \left\{ \frac{\alpha'}{T} \right\}_F, \\
\left\{ \frac{\alpha'}{T} \right\}_F, & 1 + 2 \left\{ \frac{\alpha'}{T} \right\}_F < w < 2 \\
1 - \left\{ \frac{\alpha'}{T} \right\}_F, & 0 < w < 2 \left\{ \frac{\alpha'}{T} \right\}_F - 1, \\
\left\{ \frac{\alpha'}{T} \right\}_F - w, & 2 \left\{ \frac{\alpha'}{T} \right\}_F - 1 < w < 1, \\
\left\{ \frac{\alpha'}{T} \right\}_F - 1, & 1 < w < 2 \left\{ \frac{\alpha'}{T} \right\}_F, \\
w - 1 - \left\{ \frac{\alpha'}{T} \right\}_F, & 1 + 2 \left\{ \frac{\alpha'}{T} \right\}_F < w < 2 
\end{cases} \quad (160) \]

Eq. (159) with the replacements \( \beta, \alpha, w \rightarrow \frac{1}{2}, \frac{\alpha'}{T}, \frac{\alpha'}{T} \). Using these values in Eqs. (101a, 101b), one obtains

\[ f(x, t_e) = \frac{1}{12} \left[ 1 - \frac{2}{n''^2} \left[ 2 \left\{ \frac{m'n}{d_1} \right\}_F - 1 \right] \right]. \quad (161) \]

Since the atomic density is a periodic function of the distance between the MS having period \( L_T \), the dependence of the grating amplitude \( A \) at a given focal plane must also be a periodic function of \( L \) having period \( L_T \) for a fixed value of the ratio \( y_e/L \). One period of \( A(L) \) is shown in Fig. 11 at the focal planes \( y_e = 2L, 3L/2 \). This dependence is plotted for values of \( L \) equal to rational multiples of the Talbot length \( L = \frac{\pi n}{\alpha'} L_T \). One can not expect the dependence of \( A(L) \) to be smooth, because the transmission function (137) is discontinuous; even small changes in the ratio \( \frac{n'n}{d_1} \) can lead to dramatic changes in the atomic density (134).

C. Comparison of the Talbot and Talbot-Lau effects

Qualitatively, the transition from the shadow effect to the Talbot-Lau effect for a beam having a large angular divergence that is scattered by two MS parallels the transition from spatial modulation to Talbot self-imaging of a collimated beam that is scattered by a single MS. Similarities and differences of these transitions, which occur when the characteristic length scale in the problem changes from \( L \ll L_T \) to \( L \sim L_T \), can be summarized as follows:

![Graph showing the dependence of the grating amplitude A(L) on the distance L at the focal planes y = 2L and y = 3/2 L as a function of the distance L between microfabricated structures. Two separated microfabricated structures have the same period (d_1 = d_2) and relative width f_1 = \frac{1}{2}, f_2 = \frac{1}{3}.](image)
Transition from shadow to Talbot-like profile (collimated beam)

Atomic density is a periodic function of the distance $L$ between the MS and the screen having period $L_T = \frac{2\lambda_D}{\lambda_{AB}}$.

Higher order gratings (with respect to the MS-grating) can be obtained at distances $L = \frac{k\lambda_D}{n_T}$; if, for example, $n_T$ is odd, an atomic grating having period $\frac{d}{n_T}$ is produced if the MS’s duty cycle $f < \frac{1}{n_T}$.

The atomic grating’s profile is the same as MS’s profile, no compression occurs.

D. Additional examples including a quantum Talbot-Lau effect

To make some connection with previous work, we analyze the atomic density for the parameters of Eq. (11), corresponding to the Talbot-Lau effect studied theoretically by Carnal et al. (1995) and realized experimentally by Claser and Li (1994). The appropriate parameters are $\{m, n, j, \ell, d\} = \{2, 1, 2, 1, 1/2\}$.

$$d_g = d_1, \quad L_T(2, 1) = \frac{d_1^2}{\lambda_{AB}}, \quad d_1 = 2d_2$$

When the distance between the MS is $L = \frac{\lambda_D}{2\lambda_{AB}}$, corresponding to $\phi_T = \pi$ [the case analyzed by Carnal et al. (1995)], one has $n_T = 2n' = L_T(2, 1)/L = 2$, and the corresponding values of $\alpha'$ and $\beta'$ obtained from Eqs. (156) are

$$\alpha' = 2f_2, \quad \beta' = 2f_1.$$}

As in (Carnal et al., 1995), we choose $f_2 = \frac{1}{2}$ and $f_1 = \frac{1}{2}$ or $f_1 = \frac{1}{4}$. In order to apply the results of Sec. V.B.2, it is necessary that $f_2 \leq 1/n_T = 1/2$; clearly, this requirement is met.

When $f_1 = \frac{1}{2}$, the parameter $\beta'$ is an integer ($\beta' = 1$) and one can use Eqs. (158, 160) to obtain the atomic density

$$f(x, t_e) = \frac{1}{2} \left(1 - 2 \left\{ \frac{x}{d_1} \right\}_F - 1 \right),$$

coinciding with the profile obtained by Carnal et al. (1995). Since in this case both parameters $\alpha'$ and $\beta'$ are integers, the shadow effect does not lead to any atomic grating (see below). The grating (164) arises entirely as a result of matter-wave interference.

When $f_1 = \frac{1}{4}$, the parameter $\beta' = \frac{1}{4}$, and one has to return to Eq. (156), in which only the $s = 0$ term in the sum contributes. Carrying out the integration in Eq. (156), one finds

$$f(x, t_e) = \left\{ \begin{array}{ll}
0, & \text{for } \left\{ \frac{x}{d_1} \right\}_F \leq \frac{1}{4}, \\
\frac{x}{d_1} - \frac{1}{4}, & \text{for } \frac{1}{4} \leq \left\{ \frac{x}{d_1} \right\}_F \leq \frac{1}{2}, \\
\frac{1}{4}, & \text{for } \frac{1}{2} \leq \left\{ \frac{x}{d_1} \right\}_F \leq \frac{3}{4}, \\
1 - \left\{ \frac{x}{d_1} \right\}_F, & \text{for } \frac{3}{4} \leq \left\{ \frac{x}{d_1} \right\}_F \leq 1.
\end{array} \right.$$ (165)

coinciding with the distribution calculated by Carnal et al. (1995). To compare this profile with that caused by the shadow effect, one finds from Eqs. (102, 101), that the shadow effect distribution function is given by $f(x, t_e)_{\text{shadow}} = f(x - \frac{d_1}{2}, t_e)$. Thus, owing to matter-wave interference, the atomic grating (164) is shifted by a half-period from the grating that would have been produced by the shadow effect alone.

In general, the atomic gratings produced when a beam scatters from two, separated MS cannot be attributed entirely to quantum effects since the classical shadow effect contributes to grating formation. If the parameters are chosen in such a way, however, that the classical shadow effect vanishes, then any atomic gratings that are formed can be attributed solely to quantum matter-wave interference. We have already alluded to this result above. Returning to Eqs. (101, 102), one finds that the shadow effect grating $S(w)$ disappears if the parameters
\[ \alpha = mf_2 \text{ or } \beta = j(m-n)f_1 \text{ are integers. One can guarantee that } \alpha \text{ is integral by choosing} \]
\[ f_1 = f_2 = \frac{1}{2}, n = 1, m = 2, j = 1, \ell = 1, \quad (166) \]
which corresponds to \( y_e = 2L, d_1 = d_2, \beta = \ell = 1, \alpha = 1, \beta = \frac{1}{2} \) and a Talbot phase \( \phi_T = 2\pi \frac{L}{L_T} \). The atomic density in this focal plane as a function of Talbot phase is shown in Fig. 12.

![Figure 12: An evolution of the atomic spatial distribution at the echo-point \( t = 2T \) (focal plane \( y = 2L \)], induced by two separated microfabricated structures having the same spacings and 50\% relative widths \( f_1 = f_2 = \frac{1}{2} \), with an increase of the distance \( L \) between structures on the scale of the Talbot distance \( L_T \). Chosen case allows exclude any influence of the classical shadow effect and interpret the atomic grating as an exact consequence of the wave-matter interference. The atomic grating as a function of \( L \) is monitored for \( L < \frac{1}{2}L_T \) with step \( \frac{1}{2}L_T \) starting from the point where one has no influence of the atom interference. Gratings at \( \frac{1}{2}L_T < L \leq L_T \) can be obtained using the relation (131).

**VI. TALBOT AND TALBOT–LAU EFFECTS IN A THERMAL ATOMIC BEAM.**

Up to this point, all effects related to a distribution of longitudinal velocities \( \bar{u} \) in the atomic beam have been neglected. Averaging over \( \bar{u} \) is not important for the shadow effect since the focal planes are located at \( y_e = (m/n)L \), independent of \( u \). In both the Talbot and Talbot-Lau effects, the Talbot phase depends on the Talbot length \( L_T = 2\bar{d}^2/\lambda_{dB} \), which, in turn, is proportional to \( \bar{u} \) owing to the presence of the De Broglie wavelength. To achieve the maximum contrast in the Talbot and Talbot-Lau effects, it is necessary to longitudinally cool the atomic beam (Clauser and Li, 1994).

The results of Secs. IV and VI must be averaged over \( \bar{u} \) once changes in the Talbot phase originating from the distribution of longitudinal velocities becomes of order unity. For a thermal beam having a Maxwellian distribution over longitudinal velocities, the averaging can be carried out using the function tabulated by Kruse and Ramsey (1951). For other distributions numerical integration is needed. Such calculations are not included in this contribution.

![Figure 13: An origin of the Talbot effect in the thermal beam at an asymptotic distance \( L > L_T \). When incident beam of atoms having momenta \( p \) splits in a set of scattered beams having momenta \( p \pm n\hbar k \). The wave functions of the states associated with momenta \( p + \hbar k \) and \( p - \hbar k \), \( p + 2\hbar k \) and \( p - 2\hbar k \), \ldots \( p + n\hbar k \) and \( p - n\hbar k \) acquire the same Talbot phase and therefore no dephasing between these states occurs, independently on the distance \( L \) and atomic velocity \( u \). Interference of these states remains at the asymptotic distance, leads to the 2nd, 4th ... harmonics in the atomic distribution which form the 2-order atomic grating on the screen.

Instead, we examine the role of the longitudinal velocity distribution when the width \( \bar{u} \) of the longitudinal velocity distribution is of order of the average velocity,

\[ \bar{u} \sim u \quad (167) \]

for distances

\[ (D/d)L_T \gg y \gg L_T \quad (168) \]

in the Talbot effect and separations \( L \) between the MS

\[ (D/d)L_T \gg L \gg L_T \quad (168a) \]

in the Talbot-Lau effect. We want to examine whether or not it is possible under these conditions to obtain atomic gratings having periods smaller than the MS producing the scattering.
To understand how the gratings can survive the average over longitudinal velocities $u$, one should note that it is the atomic density $\rho(\mathbf{r},\mathbf{v})$ and not the wave function amplitude that is averaged. The phase factors in the atomic density depending on the Talbot phase can be unity for specific combinations of the spatial harmonics in the atomic wave functions. The way in which this can be achieved is illustrated in Fig. 13. When one combines on the screen components of the scattered atomic states associated with momenta $\pm h\mathbf{k}$ and $-h\mathbf{k}$, $2h\mathbf{k}$ and $-2h\mathbf{k}$, ... etc., the amplitudes of the combining states acquire the same Talbot phases since the energy of the scattered atoms does not depend on the direction of scattering. The interference from these pairs of states leads to a superposition of atomic gratings having period $d/2$, $d/4$, etc.; the overall period of the grating is $d/2$. Gratings originated from Talbot-Lau effect can survive in a similar manner.

### A. Atomic density profile for a thermal beam.

#### 1. Talbot effect

Consider first the Talbot effect, i.e., the atomic grating produced when an atomic beam having negligible angular divergence, but a finite spread of longitudinal velocities, is scattered by a MS having period $d$. For a given $u$, the atomic density in the plane $y=ut$ is given by [see Eqs. (25), (30)]

$$f_u(x,t) = \sum_{n,n'} \eta_n \eta_{n'} \exp \left[ \pm i (n-n') kx - i \left( n^2 - n'^2 \right) \phi_t(u) \right],$$

(169)

where the Talbot phase $\phi_t(u) = \omega_k t = \omega_k y/u$, as given by Eq. (29), is a function of $u$ for fixed $y$. Recall that $\eta_n$ is a Fourier component of the amplitude transmission function $\eta(x)$. Since the Talbot phase is much greater than unity in the asymptotic region (168), the distribution (169) oscillates rapidly as a function of $u$. After averaging over $u$, a non-zero result arises from only those terms having

$$n' = \pm n.$$  

(170)

There is a constant background term $\bar{f}$ (for a MS consisting of an array of slits, $\bar{f}$ is equal to the duty cycle $f$ of the MS) corresponding to contributions with $n' = n$ and an interference term $\tilde{f}(x,t)$ corresponding to contributions from $n' = -n$. Neglecting all other terms, one finds

$$f(x,t) = \bar{f} + \tilde{f}(x,t),$$

(171)

where

$$\bar{f} = \sum_n |\eta_n|^2 = \int_0^d \frac{dx'}{d} \chi(x'),$$

(171a)

and

$$\tilde{f}(x,t) = \int_0^d \frac{dx'}{d} \frac{dx''}{d} \eta(x') \eta^*(x'') \times \sum_{n \neq 0} \exp \left[ \pm i nk (2x-x'-x'') \right].$$

(171b)

The atomic density profile has a period given by

$$d_y = \frac{d}{2}.$$  

(172)

Note that the density profile is independent of $t$ for the times $t \gg L_T/u$ under consideration.

To evaluate the atomic distribution (171), it is convenient to introduce new variables

$$\bar{x} = \frac{1}{2} (x' + x''), \quad \hat{x} = x' - x''.$$

(173)

After adding and subtracting a term having $n=0$ in Eq. (171) one can carry out the summation to obtain

$$\frac{d}{2} \sum_n \delta (\bar{x} - \frac{d}{2}) \delta (\hat{x} - \frac{n}{2}),$$

making use of Eq. (10). Switching integration variables from $(x',x'')$ to $(\bar{x},\hat{x})$ one sees that the $\delta$–functions having $s = 0$ and $1$ are the only ones that contribute to the sum, and it follows from Eq. (171) that

$$f(x,t) = \bar{f} - \int_0^d \frac{dx'}{d} \eta(x') \left[ F(x) + F \left( x + \frac{d}{2} \right) \right],$$

(174)

where

$$F(\bar{x}) = \int_{|\bar{x}|<\min(\bar{x},d-\bar{x})} \frac{d\hat{x}}{d} \eta^* \left( \bar{x} + \frac{\hat{x}}{2} \right).$$

(175)

For a transmission function corresponding to a periodic array of slits having duty cycle $f$, one finds $\bar{f} = \int_0^d \frac{dx'}{d} \chi(x') = f$, and

$$F(x) = \left\{ \begin{array}{ll} 2 \left( f - 2 \left| \frac{x}{2} \right| - \frac{d}{2} \right), & \text{for } \left| \frac{x}{2} \right| < f < \frac{d}{2} \\ 0, & \text{for } \left| \frac{x}{2} \right| > f \end{array} \right.$$

(176)

When $f < \frac{d}{2}$, the two $F$–functions in Eq. (174) do not overlap with one another, and the atomic density is given by

$$f(x,t) = f(1-f) + 2 \left\{ \begin{array}{ll} w, & \text{for } 0 < w < \frac{f}{2} \\ f-w, & \text{for } \frac{f}{2} < w < f \\ 0, & \text{for } f < w < \frac{d}{2} \end{array} \right.$$

(177)
where \( w = \frac{1}{2} \left\{ \frac{2x}{d} \right\} f \). This function has period \( \frac{d}{2} \). For \( f > \frac{1}{2} \) one arrives at the distribution

\[
f(x, t) = f(1 - f) + 2 \left\{ \begin{array}{l}
f - \frac{1}{2}, \text{ for } 0 < w < f - \frac{1}{2}; \\
w, \text{ for } f - \frac{1}{2} < w < \frac{1}{2}; \\
f - w, \text{ for } \frac{1}{2} < w < f \end{array} \right.
\]

(178)

which also has period \( \frac{d}{2} \). The manner in which the atomic density profile changes as \( y \) varies from \( y \ll L_T \) to \( y \gg L_T \) is shown in Fig. 14.

The manner in which the atomic density profile changes as \( y \) varies from \( y \ll L_T \) to \( y \gg L_T \) is shown in Fig. 14.

Fig. 14. Talbot effect in the thermal beam. The initial atomic distribution (dashed lines), created just after passing through the microfabricated structure with the slits’ relative width \( f \) and spacing \( d \), transforms into the second order grating (having spacing \( \frac{d}{2} \)) on the distance much larger than the Talbot distance owing to the wave-matter interference.

To evaluate the Talbot-Lau density profile in the asymptotic limit \( \frac{d}{2} \), one must return to Eq. (123) and average it over longitudinal velocities. Using the Fourier expansion of the amplitude transmission functions in Eq. (123) and setting the smooth envelope function \( f(x) \) equal to unity, one finds

\[
f(x, t_e) = \sum_{q,n_2} \exp \left( iq \left[ y - \phi_T(m, n; u) \left( 2n_2 - m\ell q \right) \right] \right)
\]

\[
\times \chi_{j(m-n)q}^{(1)} \left( \frac{d_2}{d_2} \right) \left( n_{-2-m\ell q}^{(2)} \right)^*,
\]

(180)

where \( \phi_T(m, n; u) \) is given by (124).

On averaging over \( u \) for \( \phi_T(m, n; u) \gg 1 \), one finds that only those terms in the sum having \( q = 0 \) or

\[
n_2 = m\ell q, \quad q \neq 0
\]

(181)

contribute to the density. The \( q = 0 \) term is a background, given by \( \tilde{f}_1 \tilde{f}_2 \). \( \tilde{f} \) is defined in terms of \( \chi_{1} \) in the same way that \( \bar{f} \) is defined in terms of \( \chi \) in Eq. (171a), while the terms satisfying Eq. (181) lead to the atomic grating. Explicitly, one finds

\[
f(x, t_e) = \tilde{f}_1 \left[ \tilde{f}_2 - \int \frac{dx_2}{d_2} \eta_2(x_2) \right] + \tilde{f}(x, t_e),
\]

(182)

\[
\tilde{f}(x, t_e) = \sum_{q} \exp \left( iqk_g x \right) \chi_{1}^{(1)} \left( \frac{d_2}{d_2} \right) \left( \eta_{-2-m\ell q}^{(2)} \right)^*.
\]

(182a)

where the summation over \( q \) includes \( q = 0 \) and other values of \( q \) leading to integral \( n_2 \) in Eq. (181). The function \( F^{(2)}_{\ell} \) is a Fourier component of the function \( F_2(x) \) defined in terms of \( \eta_2(x) \) in the same way that \( F \) is defined in terms of \( \eta \) in Eq. (177). The density is independent of \( t_e \) for spatial separations \( L \gg L_T \) of the MS.

When \( m\ell \) is even, all \( q \) are allowed according to Eq. (181). When \( m\ell \) is odd, only even values \( q \) contribute, which means that the grating \( \chi_{1}^{(1)} \) has a period equal to \( d_2/2 \). Equation (182a) has the same structure as Eq. (92). Repeating the calculations leading to Eq. (92) one arrives at the formula

\[
\tilde{f}(x, t_e) = \frac{1}{\bar{f}(m-n)} \sum_{s=0}^{j(m-n)-1} \int_0^{d_2} \frac{dx_2}{d_2} F_2(x_2)
\]

\[
\times \chi_{1} \left\{ \frac{d_1}{j(m-n)} \left[ s + \left( \frac{m\ell d_2^2}{d_2} - \frac{x}{d_2} \right) \right] \right\}_{\bar{f}},
\]

(183)

where

\[
\xi = \begin{cases} 1, & \text{for } m\ell \text{ even} \\ 0, & \text{for } m\ell \text{ odd} \end{cases}
\]

(184)

Consider now the case of MS having transmission functions (14), when the function \( F_2(x) \) is given by the rhs
of Eq. (176) with $d$ and $f$ replaced by $d_2$ and $f_2$. Using dimensionless variables

$$w = \xi x/d_y, \quad z = x_2/f_2 d_2,$$  \hfill (185)  

one arrives at equations that are the analogues of Eqs. (176), namely

$$f(x, t_c) = f_1 f_2 (1 - f_2) + \tilde{f}(x, t_c),$$  \hfill (186)  

$$\tilde{f}(x, t_c) = \frac{f_2^2}{\tilde{f}(m-n)} \sum_s h_s(w),$$  \hfill (186a)  

$$h_s(w) = \int_0^1 dz \tilde{F}(z) \theta (\beta - (s + \{\alpha z - w\} F)), \hfill (186b)$$  

$$\tilde{F}(z) = 4 \left\{ \begin{array}{ll}
\frac{1}{2} z, & \text{for } z < \frac{1}{2}, \\
\frac{1}{2} - z, & \text{for } z > \frac{1}{2},
\end{array} \right. \hfill (186c)$$  

$$\alpha = \xi m \tilde{f}_2, \quad \beta = \tilde{f}(m-n) f_1. \hfill (186d)$$  

Omitting further calculations which are essentially the same as those used to derive Eqs. (101, 102), one finds

$$\tilde{f}(x, t_c) = \frac{f_2^2}{\tilde{f}(m-n)} \left[ \beta I + \sum_{s=1}^{[\beta]} F(a_s, b_s) + f'(w) \right],$$  \hfill (187)  

where

$$F(a, b) = \int_a^b dz \tilde{F}(z)$$  \hfill (188)  

is given by

$$F(a, b) = \left\{ \begin{array}{ll}
2(b^2 - a^2) & \text{max}(a, b) \leq \frac{1}{2}, \\
4b - 2(3a^2 + b^2) - 1 & a \leq \frac{1}{2} \leq b,
\end{array} \right. \hfill (189)$$  

the quantities $a_s$ and $b_s$ are given in the Appendix by Eqs. (A3, A3), and

$$f'(w) = \left\{ \begin{array}{ll}
F(a_{[\alpha]} + 1, 1), & \text{for } 0 \leq w \leq 1 - \{\beta\}_F, \\
F(a_{[\alpha]} + 1, 1) + F(0, b_0), & \text{for } 1 - \{\beta\}_F \leq w \leq 1 + \{\alpha\}_F - \{\beta\}_F, \\
F(0, b_0), & \text{for } 1 + \{\alpha\}_F - \{\beta\}_F \leq w \leq 1,
\end{array} \right. \hfill (190)$$  

$$f'(w) = \left\{ \begin{array}{ll}
F(a_{[\alpha]} + 1, 1), & \text{for } 0 \leq w \leq \{\alpha\}_F, \\
F(0, b_0), & \text{for } \{\alpha\}_F \leq w \leq 1 - \{\beta\}_F, \\
F(0, b_0) + F(b_{[\alpha]} + 1), & \text{for } 1 + \{\alpha\}_F - \{\beta\}_F \leq w \leq 1,
\end{array} \right. \hfill (190a)$$  

$$f'(w) = \left\{ \begin{array}{ll}
F(a_{[\alpha]} + 1, b_{[\alpha]} + 1), & \text{for } 0 \leq w \leq \{\alpha\}_F - \{\beta\}_F, \\
F(a_{[\alpha]} + 1, 1), & \text{for } \{\alpha\}_F - \{\beta\}_F \leq w \leq 1 - \{\beta\}_F, \\
F(0, b_0) + F(a_{[\alpha]} + 1), & \text{for } 1 - \{\beta\}_F \leq w \leq \{\alpha\}_F, \\
F(0, b_0), & \text{for } \{\alpha\}_F \leq w \leq 1,
\end{array} \right. \hfill (190b)$$  

$$f'(w) = \left\{ \begin{array}{ll}
F(a_{[\alpha]} + 1, b_{[\alpha]} + 1), & \text{for } 0 \leq w \leq \{\alpha\}_F - \{\beta\}_F, \\
F(a_{[\alpha]} + 1, 1), & \text{for } \{\alpha\}_F - \{\beta\}_F \leq w \leq 1 - \{\beta\}_F, \\
F(0, b_0), & \text{for } \{\alpha\}_F \leq w \leq 1,
\end{array} \right. \hfill (190c)$$  

In principle one can use Eqs. (182, 187, 190) to derive a general analytical expression for the atomic density distribution, but, given the large number of cases, such an expression is of limited use. Instead, one can write a computer code based on Eqs. (182, 187, 190) to obtain the atomic density profile. Using this code, we varied the duty cycles of the MS to optimize the atomic grating amplitude in the focal planes $y = \frac{n+1}{2} L$ for $n = 1 - 4$ and equal periods of the MS, $j = \ell = 1$. Calculations show that one has to choose

$$f_1 = f_2 = \frac{1}{2} \hfill (191)$$  

in all cases except $n = 3$, where the optimal duty cycles
are given by

\[ f_1 = \frac{1}{2}, \quad f_2 = \frac{1}{4}. \]  

(192)

For these optimal values of the duty cycles, it is a simple matter to obtain analytical expressions for the atomic density profile in a given focal plane. For example, at the echo plane \( y = 2L \) \((n = 1, \ m = 2)\) the parameters \( \alpha \) and \( \beta \) are equal 1 and \( \frac{1}{2} \), respectively, and the atomic density is given by

\[
f(x, t_e) = \frac{1}{8} + \frac{1}{4} \{ F(a_1, b_1) \\
+ \theta \left( w - \frac{1}{2} \right) [F(0, b_0) + F(b_1, 1)] \},
\]

(193)

where \( w = \left\{ \frac{\pi x}{2L} \right\}_F \), \( b_{0,1} = w + \frac{1}{2} \), \( a_1 = w \). Using Eq. \[189\] one arrives at the atomic grating profile

\[
f(x, t_e) = \frac{1}{4} \left\{ 1 + 2w(1 - 2w), \text{ for } w < \frac{1}{2} \right\} \\
- \left\{ 3 - 2w(3 - 2w), \text{ for } w > \frac{1}{2} \right\}.
\]

(194)

Similar calculations leads to the atomic grating profiles

\[
f(x, t_e) = \frac{1}{36} \left\{ 9 + 2w(1 - 2w), \text{ for } w < \frac{1}{2} \right\} \\
\frac{1}{12} \left\{ 11 - 2w(3 - 2w), \text{ for } w > \frac{1}{2} \right\},
\]

(194a)

at the focal plane \( y = \frac{3}{2}L \), where \( w = \left\{ \frac{3x}{2L} \right\}_F \); \n
\[
f(x, t_e) = \frac{1}{8} \left\{ 1 + w(1 - 2w), \text{ for } w < \frac{1}{2} \right\} \\
\frac{1}{2} \left\{ 2 - w(3 - 2w), \text{ for } w > \frac{1}{2} \right\},
\]

(194b)

at the focal plane \( y = \frac{5}{2}L \), where \( w = \left\{ \frac{5x}{2L} \right\}_F \); and

\[
f(x, t_e) = \frac{1}{100} \left\{ 25 + 2w(1 - 2w), \text{ for } w < \frac{1}{2} \right\} \\
\frac{1}{50} \left\{ 27 - 2w(3 - 2w), \text{ for } w > \frac{1}{2} \right\}.
\]

(194c)

at the focal plane \( y = \frac{5}{2}L \), where \( w = \left\{ \frac{5x}{2L} \right\}_F \). These atomic density profiles are shown in Fig. \[15\]. Since the optimal duty cycles \[191, 192\] correspond to the limit where the shadow effect vanishes, the density profiles \[194\] cannot be vestiges of the shadow effect. They must originate from matter-wave interference. One can compare the density profile \[194a\], valid for distances \( L \) between the MS larger than the Talbot distance, with that of Fig. \[10\] for \( L \sim L_T \) (Talbot-Lau effect).

VII. CONCLUSION

Atom interferometry is an emerging field of atomic, molecular and optical physics. In this review, we have focused on the scattering of atoms by one or more microfabricated structures (MS). We have seen that the scattering can be described in purely classical terms for characteristic length scales \( L \ll L_T \), where \( L_T \) is the so-called Talbot length. For \( L \gtrsim L_T \), a classical description of the atomic center-of-mass motion is no longer adequate. Our approach has relied on an interpretation of the phenomena in terms of the recoil an atom acquires when it is scattered from a MS. With this approach, we could make a connection with the theory of coherent transients, for which a rich literature has already been developed. We have considered both collimated beams (Talbot effect) and beams having large angular divergence (shadow effect, Talbot-Lau effect) and have allowed for a broad distribution of longitudinal velocities in the atomic beam (Talbot and Talbot-Lau effects in a thermal beam). The next step would be to extend our considerations to regimes corresponding to Bragg scattering and Fraunhofer diffraction, allowing for an analysis of atom interferometers which split atomic wave functions into nonoverlapping paths.

Scattering of atoms by MS shares both similarities and differences with scattering of atoms by standing-wave optical fields (SW). Similarities include a periodical recovery of the atomic interference pattern at multiples of the
Talbot distance \( b \) [or \( 2b \) for the Talbot-Lau effect], a compression of the atomic gratings with respect to the periods of the MS or SW, spatial separation of the higher order atomic gratings in different focal planes, and splitting of the incident beam into an infinite set of scattered beams having momenta \((p \pm n\hbar k)\). The differences are due in large part to the nature of the scattering. The MS produce a piecewise continuous atomic density profile while the SW produce a smooth atomic density profile. As a result, the decrease in period relative to that of the classical shadow effect observed in the Talbot-Lau effect using MS does not occur for standing-wave fields. Moreover, the possibility of observing a Talbot-Lau effect using MS does not occur for scattering by standing-wave fields. Therefore, the shadow effect does not give rise to atomic gratings in the focal plane of the incident beam. When one observes the Talbot-Lau effect caused entirely by matter-wave interference (see Fig. 12) does not occur for the smooth amplitude modulations by SW (Dubetsky and Berman, 1994). In the case of scattering by MS, the fact that the shadow effect does not occur for the smooth amplitude modulation by SW (Dubetsky and Berman, 1994). In the case of scattering by MS, the qualitative nature of the atomic density profile depends on the properties of the incident atomic beam. When one observes the Talbot effect using a monovelocity beam, the atomic gratings are discontinuous functions (see Fig. 3), but when one averages these gratings over longitudinal velocities they are piecewise continuous, but they are transformed into profiles which are discontinuous only in the second derivative when averaged over transverse velocities (compare Figs. 10 and 11). These examples show that averaging over transverse or longitudinal velocities tends to smooth out the atomic density profiles.

It is clear that many of the situations analyzed in this chapter have direct applications to atom lithography. We can expect that future developments in this emerging field will incorporate many of the basic ideas which have been encountered in our discussion.

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**APPENDIX:**

A method for calculating the integral

\[
F_{\alpha z-w}(w) = \int_{0}^{1} dz \theta \{ \{ \beta \}_{F} - \{ \alpha z - w \}_{F} \},
\]

with \(0 \leq w \leq 1\) is presented here. The function \(\theta\) is the Heaviside step function and \([a]\) and \(\{a\}\) refer to the integral and fractional parts of \(a\). Depending on the values of the parameters \(\alpha\), \(\beta\) and \(w\) the integrand in this equation can jump from 0 to 1 several times inside the range \(z \in [0,1]\). The value of the integral is the total length of the intervals in this range where

\[
\zeta(z) = \{ \alpha z - w \}_{F} \leq \{ \beta \}_{F}.
\]

To determine this length one needs to find the values of \(z\) for which the function \(\{ \alpha z - w \}_{F}\) equals 0 and where it equals \(\{ \beta \}_{F}\). This function is shown in Fig. 16.

![Fig. 16. Function \(\{ \alpha z - w \}_{F}\) undergoes jumps at the points \(a_{r} = (r - 1 + w)/\alpha\) and is equal to \(\{ \beta \}_{F}\) at the points \(b_{r} = (r - 1 + w + \{ \beta \})/\alpha\). For \([a] > 1\) and \(1 \leq r \leq [a]\), \(a_{r} < 1\), i.e. contribution to Eq. (A1) from the range \([a_{1}, a_{[a]}\) is \((a_{a} - 1)\) \(b_{r} - a_{r}\); when \(w > 1 - \{ \beta \}\) one gets term \(b_{0}\) from \([0, a_{1}]\); term from \(z \in [a_{[a]}, b_{[a]}]\) is equal to \(\{ \beta \}_{F}/\alpha\) or \(1 - a_{[a]}\) for \(b_{[a]} < 1\) or \(b_{[a]} > 1\); in the same manner one calculates term from range \([a_{[a]}+1, b_{[a]}+1]\) in different possible cases.](image-url)
\[ \zeta(z) = \alpha (z - a_r) \]  
(A4)

and equals \( \{\beta\}_F \) at the point

\[ z = b_r = a_r + \frac{\{\beta\}_F}{\alpha} = (r - 1 + w + \{\beta\}_F)/\alpha. \]  
(A5)

Since contributions to the integral \( (A1) \) vanish unless \( a_r \leq 1 \) and \( b_r \geq 0 \) and since \( 0 \leq w \leq 1 \), it follows from Eqs. \( (A3) \) \( (A5) \) that \( 0 \leq r \leq (1 + |\alpha|) \). For the time being, we assume that \( \alpha > 1 \).

All intervals \([a_r, b_r]\) totally or partially within the range \([0, 1]\) contribute to the integral \( (A1) \). Let us denote the contribution from the range \([a_r, b_r]\) by \( A_r \) and the total value of the integral by

\[ h_{[\beta]}(w) = \sum_{r=0}^{1+|\alpha|} A_r. \]  
(A6)

When \( a_r > 0 \) and \( b_r < 1 \) the interval \([a_r, b_r]\) lies entirely to the range \([0, 1]\) and

\[ A_r = b_r - a_r = \frac{\{\beta\}_F}{\alpha}; \quad a_r > 0 \text{ and } b_r < 1 \]  
(A7)

When \( 1 > a_r > 0 \) and \( b_r \geq 1 \), the maximum value of \( z \) contributing to the integral \( (A1) \) is \( z = 1 \) and

\[ A_r = 1 - a_r; \quad 1 > a_r > 0 \text{ and } b_r \geq 1 \]  
(A8)

Similarly, for \( a_r < 0 \) and \( 0 < b_r < 1 \),

\[ A_r = b_r; \quad a_r < 0 \text{ and } 0 < b_r < 1 \]  
(A8a)

and for \( b_r \leq 0 \),

\[ A_r = 0; \quad b_r \leq 0 \]  
(A8b)

For given \( r, \alpha, \) and \( \beta \), the values of \( a_r \) and \( b_r \) can depend on \( w \), giving rise to a dependence of \( h_{[\beta]} \) on \( w \). Note that \( (b_r - a_r) \leq 1 \), which follows from Eqs. \( (A3) \) \( (A5) \) and the assumption that \( \alpha > 1 \).

We first consider the range \( 1 \leq r \leq |\alpha| - 1 \), for which \( b_r < 1 \) and \( a_r > 0 \) for any \( w \in [0, 1] \). It then follows from Eq. \( (A5) \) that the total contribution \( A' \) to the integral \( (A1) \) from the region \([a_1, b_{|\alpha|-1}]\) is given by

\[ A' = \sum_{r=1}^{|\alpha|-1} A_r = \{\beta\}_F (|\alpha|-1)/\alpha. \]  
(A9)

This contribution is independent of \( w \) and represents a constant background term.

Since \( r \leq |\alpha| + 1 \), the only remaining contributions to the integral can come from \( A_0, A_{|\alpha|}, \) and \( A_{|\alpha|+1} \). These terms depend on \( w \) and represent the atomic gratings. Let us first consider \( A_0 \). If \( r = 0, b_0 = -1 + w + \{\beta\}_F)/\alpha < 1 \) and \( a_0 = (-1 + w)/\alpha < 0 \). It then follows from Eqs. \( (A3) \) \( (A5) \) \( (A8) \) that

\[ A_0 = \begin{cases} 
(\{\beta\}_F - 1)/\alpha, & \text{for } w \geq (1 - \{\beta\}_F) \equiv \bar{w} \\
0, & \text{for } w \leq \bar{w}.
\end{cases} \]  
(A10)

We now turn our attention to \( A_{|\alpha|} \) and \( A_{|\alpha|+1} \). It follows from Eq. \( (A3) \) that \( a_{|\alpha|} \in [0, 1] \). Only the points \( b_{|\alpha|}, a_{|\alpha|+1}, b_{|\alpha|+1} \) can lie to the right of the range \([0, 1]\), which occurs when

\[ w \geq (\{\alpha\}_F + 1 - \{\beta\}_F) = w_1, \]  
(A11)

\[ w \geq \{\alpha\}_F = w_2, \]  
(A11a)

\[ w \geq (\{\alpha\}_F - \{\beta\}_F) = w_3, \]  
(A11b)

respectively. From Eqs. \( (A8) \) \( (A8a) \) \( (A8b) \), one finds that \( A_{|\alpha|} \), and \( A_{|\alpha|+1} \) are given by

\[ A_{|\alpha|} = \begin{cases} 
b_{|\alpha|} - a_{|\alpha|} = \{\beta\}_F/\alpha & w < w_1 \\
1 - a_{|\alpha|} = \frac{\alpha F - w}{\alpha} & w > w_1
\end{cases} \]  
(A12)

\[ A_{|\alpha|+1} = \begin{cases} 
0 & w > w_2 \\
1 - a_{|\alpha|+1} = \frac{\alpha F - w}{\alpha} & w_3 < w < w_2 \\
b_{|\alpha|+1} - a_{|\alpha|+1} = \{\beta\}_F/\alpha & w < w_3
\end{cases} \]  
(A12a)

In order to sum \( A_0, A_{|\alpha|}, \) and \( A_{|\alpha|+1} \) it is convenient to separate regions of \( \alpha \) and \( \beta \) according to the relative values of \( w_1, w_2, w_3 \) and \( \bar{w} \). Since

\[ w_3 \leq w_2 \leq w_1, \]  
(A13)

\[ w_3 \leq \bar{w} \leq w_1, \]  
(A13a)

and \( w_2 = \{\alpha\}_F \geq 0 \) one can distinguish four cases

\[ w_3 \leq 0 \leq w_2; \quad \{(\beta\}_F \geq \max \{\{\alpha\}_F, 1 - \{\alpha\}_F\}, \]  
(A14)

\[ w_3 \leq 0 \leq w_2; \quad \{(\alpha\}_F \leq \{\beta\}_F \leq 1 - \{\alpha\}_F\}, \]  
(A14a)

\[ 0 \leq w_3 \leq w_2 \leq w_1; \quad \{(\alpha\}_F \leq \{\beta\}_F \leq \{\alpha\}_F\}, \]  
(A14b)

\[ 0 \leq w_3 \leq w_2 \leq \bar{w}; \quad \{(\beta\}_F \leq \min \{\{\alpha\}_F, 1 - \{\alpha\}_F\}. \]  
(A14c)

Consider, for example, the case \( (A14) \) for the range of \( w_1 \) given by

\[ 0 \leq w \leq \bar{w}. \]  
(A15)
For integer $\beta$ this corresponds to the entire range of allowed $w$, $0 \leq w \leq 1$. From Eqs. (A10, A12, A12a) one finds $A_0 = 0$, $A_{[\alpha]} = \{\beta\}_F / \alpha$, and $A_{[\alpha]+1} = (\{\alpha\}_F - w) / \alpha$. As a result, one finds that $A(w) = A_0 + A_{[\alpha]} + A_{[\alpha]+1}$ is given by

$$A(w) = (\{\beta\}_F + \{\alpha\}_F - w) / \alpha. \quad (A16)$$

By combining Eqs. (A9, A16), one obtains

$$h_{[\beta]}(w) = (\{\beta\}_F [\alpha]_I + \{\alpha\}_F - w) / \alpha. \quad (A17)$$

Even though Eq. (A17) have been derived for $\alpha \geq 1$, one can verify that it holds for arbitrary $\alpha$.

Other values of $w$ and other cases (A14a-A14c) can be considered in the same manner. As a result, one arrives at Eqs. (100, 101) of the text.

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[1] The scattering of atoms by standing-wave fields, rather than MS, is a bit more subtle. For resonant standing wave fields, which can act as amplitude gratings, the situation is unchanged. On the other hand, off-resonant fields act as phase gratings for the atoms; as such, they produce no effect on classically moving particles. Strictly speaking, therefore, one must quantize the center-of-mass motion to calculate the scattering of the atoms by the fields. Nevertheless, in a manner analogous to the normal photon echo, it is possible to assign phases to the atoms while they are freely evolving between the MS and the screen and still consider the motion as classical in these regions.