CHARACTERIZING REPRESENTABILITY
BY PRINCIPAL CONGRUENCES
FOR FINITE DISTRIBUTIVE LATTICES
WITH A JOIN-IRREDUCIBLE UNIT ELEMENT

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Abstract. For a finite distributive lattice $D$, let us call $Q \subseteq D$ principal congruence representable, if there is a finite lattice $L$ such that the congruence lattice of $L$ is isomorphic to $D$ and the principal congruences of $L$ correspond to $Q$ under this isomorphism.

We find a necessary condition for representability by principal congruences and prove that for finite distributive lattices with a join-irreducible unit element this condition is also sufficient.

1. Introduction

1.1. Background. For a finite lattice $L$, we denote by $\text{Con}
L$ the congruence lattice of $L$, by $\text{Princ}
L$ the ordered set of principal congruences of $L$, and by $\text{Prime}
L$ the set of prime intervals of $L$. Let $J(L)$ denote the (ordered) set of join-irreducible elements of $L$, and let

\[(1) \quad J^+(L) = \{0, 1\} \cup J(L).\]

Then for a finite lattice $L$,

\[(2) \quad J^+(\text{Con}
L) \subseteq \text{Princ}
L \subseteq \text{Con}
L,\]

since every join-irreducible congruence is generated by a prime interval; furthermore, $0 = \text{con}(x, x)$ for any $x \in L$ and $1 = \text{con}(0, 1)$.

This paper continues G. Grätzer \[9\] (see also \[16\], Section 10-6) and \[11\], Part VI), whose main result is the following statement.

Theorem 1. Let $P$ be a bounded ordered set. Then there is a bounded lattice $K$ such that $P \cong \text{Princ}
K$. If the ordered set $P$ is finite, then the lattice $K$ can be chosen to be finite.

The bibliography lists a number of papers related to this result.

In G. Grätzer and H. Lakser \[15\], we got some preliminary results for the following related problem.

For a finite distributive lattice $D$, let us call $Q \subseteq D$ principal congruence representable (representable, for short), if there is a finite lattice $L$ such that $\text{Con}
L$ is isomorphic to $D$ and $\text{Princ}
L$ corresponds to $Q$ under this isomorphism. Note that by \(1\) and \(2\), if $Q$ is representable, then $J(D) \subseteq Q$ and $0, 1 \in Q$.

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We now state [11, Problem 22.1].

**Problem.** Characterize representable sets for finite distributive lattices.

In this paper, we investigate a combinatorial condition for representability. We prove that this condition is necessary, and for finite distributive lattices with a join-irreducible unit, it is also sufficient.

1.2. Chain representability. A finite chain $C$ is colored by an ordered set $P$, if there is a map $\text{col}$ of $\text{Prime} C$ onto $P$. For $x \leq y \in C$, define the color set, denoted by $\text{colSet}[x, y]$, of an interval $[x, y]$ of $C$ as the set of colors of the prime intervals $p$ in $[x, y]$; in formula,

$$\text{colSet}[x, y] = \{ \text{col} p \mid p \in \text{Prime}[x, y] \}.$$  

Note that if $\text{col} p = p \in P$, then $\text{colSet} p = \{ p \}$.

We use Figure 1 to illustrate coloring. Figure 1 shows an ordered set $P$ and a chain $C = \{ c_1 \prec c_2 \prec c_3 \prec c_4 \prec c_5 \}$ colored by $P$; the color of a prime interval is indicated by a label. Let $D$ be a distributive lattice with $P$ as the ordered set $J(D)$ of join-irreducible elements of $D$; the elements of $J(D)$ are gray-filled in the diagram of $D$ in Figure 1.

For the color sets of prime intervals of this chain $C$, we obtain $\{ a \}$, $\{ b \}$, and $\{ d \}$. Intervals of length 2 of $C$ produce two more color sets $\{ a, b \}$, $\{ b, d \}$; for instance, $\text{colSet}[c_1, c_3] = \{ a, b \}$ and $\text{colSet}[c_2, c_4] = \{ b, d \}$. There are two intervals of length 3, but only $[c_1, c_4]$ yields a new color set: $\text{colSet}[c_1, c_4] = \{ a, b, d \}$. Finally, $C = [c_1, c_5]$ gives the same color set as $[c_1, c_4]$.

While the map $\text{colSet}$ assigns a color set to an interval of $C$, the map $\text{Rep}$ assigns an element of $D$ to an interval of $C$:

$$\text{Rep}: [x, y] \mapsto \bigvee \text{colSet}[x, y].$$

Note that $\text{colSet}[x, x] = \emptyset$, so $\text{Rep} [c_1, c_1] = o$; also, $\text{Rep} [c_1, c_2] = a$, and so on, as illustrated in Figure 1.

The set

$$\text{Rep} C = \{ \text{Rep}[x, y] \mid x \leq y \in C \}$$

![Figure 1. Coloring and representation](image-url)
is a subset of $D$; in our example,

$$\text{Rep } C = \{o, a, b, d, b \lor d\} = D - \{a \lor d\}.$$ 

Such a subset of $D$ we call \textit{(colored-)chain representable}. Note that the chain $C$ is not directly related to the finite lattice $L$ representing $D$ as a congruence lattice.

Now we state our first result.

**Theorem 2.** Let $D$ be a finite distributive lattice and let $Q \subseteq D$. If $Q$ is representable, then it is chain representable.

1.3. **Finite distributive lattices with join-irreducible units.** Let $D$ be a finite distributive lattice with a join-irreducible unit. Then we can apply to $D$ the construction in my paper [9] to obtain a finite lattice $L$ with very special properties whose congruence lattice is isomorphic to $D$. Using this as a starting point, we prove our second result.

**Theorem 3.** Let $D$ be a finite distributive lattice with a join-irreducible unit element. Let $Q \subseteq D$. Then $Q$ is representable iff it is chain representable.

1.4. **Outline.** In Section 2 we prove Theorem 2. Section 3 provides a \textit{Proof-by-Picture} of Theorem 3. In view of Theorem 2, to prove Theorem 3, it is sufficient to verify that if $Q \subseteq D$ is chain representable, then there is a lattice $L$ representing it. This lattice $L$ is constructed in Section 4. Finally, in Section 5 we prove Theorem 3. Section 5.6 concludes the paper with a discussion of some very recent results.

I would like to thank the referee for the many improvements he recommended.

1.5. **Notation.** We use the notation as in [11]. You can find the complete \textit{Part I. A Brief Introduction to Lattices and Glossary of Notation} of [11] at tinyurl.com/lattices101

2. **Proving Theorem 2**

Let $D$ be a finite distributive lattice and let $Q \subseteq D$. Let $P$ denote the set of join-irreducible elements of $D$. Finally, let $Q$ be representable by a finite lattice $L$, with bounds 0 and 1, so Con $L = D$ and $Q$ is the set of principal congruences of $L$.

If $C$ is a maximal chain of $D$, then $C$ has a \textit{natural coloring} in $L$: nat$_L$ $p =$ con($p$).

Let $K$ be a finite lattice and let $C$ be a maximal chain of $K$. Then

$$\text{nat}_K(\text{Prime } C) \subseteq \text{nat}_K(\text{Prime } K);$$

note that nat$_K(\text{Prime } C)$ is a proper subset, in general. However, however,

$$\bigvee \text{nat}_K(\text{Prime } K) = \bigvee \text{nat}_K(\text{Prime } C) = 1.$$  

Let $P_1$ and $P_2$ be ordered sets. Recall that $P_1 + P_2$ denotes the (ordinal) \textit{sum} of $P_1$ and $P_2$ ($P_2$ on top of $P_1$). If $P_1$ has a unit, 1$_{P_1}$, and $P_2$ has a zero, 0$_{P_2}$, then we obtain the \textit{glued sum} $P_1 + P_2$ from $P_1 + P_2$ by identifying 1$_{P_1}$ and 0$_{P_2}$.

Now to prove Theorem 2 we enumerate all maximal chains of the lattice $L$: $C_1, C_2, \ldots, C_m$. Let

$$C'_i = \begin{cases} C_i & \text{for } i \text{ odd;} \\ \overline{C_i} & \text{for } i \text{ even,} \end{cases}$$
where $\tilde{C}_i$ denotes the dual of $C_i$ and define the chain $C$ as a glued sum:

$$C = C'_1 \dot{+} \ldots \dot{+} C'_m.$$  

We have a coloring $\text{col}$ for $C$: if $p$ is a prime interval in $C$, then $p$ is a prime interval in exactly one $C'_i$. Let $p' = p$ if $i$ is odd and let $p' = \tilde{p}$ be the dual of $p$ if $i$ is even. Since $C_i$ is a maximal chain in $L$, it follows that $p'$ is a prime interval in $L$. Then $\text{col}(p) = \text{con}(p') \in P$ defines a coloring of $C$. So for $[x, y] \subseteq C$, we obtain the color set $\text{colSet}[x, y] = \{ \text{col}(p) \mid p \in \text{Prime}[x, y] \}$. We define

$$\text{Rep} C = \{ \bigvee (\text{colSet}[x, y]) \mid [x, y] \subseteq C \},$$

a subset of $D$.

To prove Theorem \[2\] we have to establish that $\text{Rep} C = Q$.

To verify that $\text{Rep} C \supseteq Q$, let $x \in Q$. By the definition of $D$, $Q$, and $L$, we can represent $x$ as a principal congruence $\text{con}(a, b)$ in $L$ for some $a \leq b \in L$. Let $C_i$ be one of the maximal chains in $L$ with $a, b \in C_i$. Applying \[5\] to the interval $[a, b]$ of $L$, we obtain that

$$\bigvee (\text{colSet}_{C_i}[a, b]) = \bigvee (\text{colSet}_{C_i}[a, b]) = x.$$

Therefore, $x \in \text{Rep} C$.

Conversely, to verify that $\text{Rep} C \subseteq Q$, let $x = \text{Rep} C$. Then there are $u \leq v \in C$ such that $x = \bigvee (\text{colSet}_{C_i}[u, v])$. We distinguish three cases.

Case 1. $u \leq v \in C_i$ for some $1 \leq i \leq n$. This is easy, just like the converse case, utilizing \[5\].

Case 2. $u \in C'_i$, $v \in C'_j$ for $1 \leq i + 1 < j \leq n$. In this case, $[u, v] \supseteq C'_{i+1}$ and so $x = 1 \in \text{Rep} C$.

Case 3. $u \in C'_i$, $v \in C'_{i+1}$ for some $1 \leq i < n$. Without loss of generality, we can assume that $i$ is odd, so $C'_i = C_i$ and $C'_{i+1} = \tilde{C}_{i+1}$. Then

$$x = \bigvee (\text{colSet}_{C_i}[u, v]) = \bigvee (\text{colSet}_{C'_i}[u, 1C'_j]) \lor \bigvee (\text{colSet}_{C'_{i+1}}[0C'_{i+1}, v])$$

$$= \bigvee (\text{colSet}_{C'_i}[u, 1]) \lor \bigvee (\text{colSet}_{C'_{i+1}}[1, v])$$

$$= \text{con}(u, 1) \lor \text{con}(v, 1) = \text{con}(u \land v, 1) \in Q,$$

which we wanted.

This completes the proof of Theorem \[2\].

3. Finite distributive lattices with join-irreducible units

"Proof-by-Picture"

3.1. A colored chain. Recall that, as in [11], a Proof-by-Picture is not a proof, just an illustration of an idea. We illustrate the proof of Theorem \[3\] with the chain $C$ colored by the ordered set $P = \{ p, q, r, 1 \}$ and the distributive lattice $D$ with a join-irreducible unit satisfying $J(D) = P$, see Figure \[2\]. Note that $1 \in P$; let $P^* = P - \{ 1 \}$.

As in Figure \[1\] we mark an element $z \in D$ with the interval $[x, y]$ of $C$, if

$$z = \text{Rep}[x, y] = \bigvee \text{colSet}[x, y],$$

that is, if the element $z \in D$ is the join of the colors in $[x, y]$. All the elements thus marked form the set $Q = D - \{ v \}$. By definition, $Q$ is chain representable.
We will outline how to construct a finite lattice $L$ such that $\text{Con} L$ is isomorphic to $D$ and $\text{Princ} L$ corresponds to $Q$ under this isomorphism, that is, $Q$ is representable.

For a finite lattice $K$ with zero, $o$ and unit, $i$, we denote by $K^-$ the ordered set obtained by deleting the elements $o$ and $i$ from $K$.

3.2. The frame lattice. For the chain $C$ colored by the ordered set $P$, see Figure 2, we first construct the frame lattice of $C$, Frame $C$, as illustrated in Figure 3, consisting of the following elements:

1. the elements $o, i$, the zero and unit of Frame $C$, respectively;
2. the elements $a_p < b_p$ for every $p \in P$;
3. an element $s_1$, a sectional complement of $a_1$ in $b_1$, that is, $s_1 \land a_1 = o$ and $s_1 \lor a_1 = b_1$;
4. the chain $C$;
5. a universal complement $u$, that is, $u \land x = o$ and $u \lor x = i$ for every $c \in (\text{Frame } C)^-$.

These elements are ordered and the lattice operations are formed as in Figure 3 (which shows the construction for the colored chain $C$ of Figure 2).

Note that Frame $C$ is a union of $\{0, 1\}$-sublattices: the chains $C_p = \{o, a_p, b_p, i\}$, for $p \in P$, the chain $C_u = \{o, u, i\}$, and the additional nonchain $\{0, 1\}$-sublattice $S = \{o, a_1, b_1, s_1, i\}$.

The frame lattice Frame $C$ in this paper is based on the idea of the frame lattice in G. Grätzer [9]; the details are different, especially, the inclusion of the chain $C$.

3.3. The ordered set $W$. We are going to construct the lattice $L$ representing $Q \subseteq D$ as an extension of the frame lattice of $C$, Frame $C$. The principal congruence $\text{con}(a_p, b_p)$ of $L$ represents $p \in P$.

We use the lattice $W(p, q)$, for $p < q \in P$, see Figure 4. We add these as sublattices to extend Frame $C$.

The lattice $W(p, q)$ is a variant of the lattice $S(p, q)$ in my paper [9]. The lattice $W(p, q)$ has two more elements than $S(p, q)$, but from a technical point of view it
is much easier to work with. For instance, the crucial formula \([13]\) does not hold if we utilize the lattices \(S(p, q)\).

Since \(p < q\) in \(P\), we want \(\text{con}(a_p, b_p) < \text{con}(a_q, b_q)\) to hold in the extended lattice. We add seven elements to the sublattice \(C_p \cup C_q\) of Frame \(C\), as illustrated.
in Figure 5, to form the sublattice $W(p, q)$. This will ensure that \( \text{con}(a_p, b_p) \leq \text{con}(a_q, b_q) \).

3.4. Flag lattices. Figure 5 shows a flag lattice $\text{Flag}(c_3)$, where $C$ is the colored chain of Figure 2 and $\text{col}[c_3, c_4] = p$.

We add eight elements (black filled in Figure 6) to the sublattice \{o, $a_p, b_p, c_3, c_4, i$\} of Frame $C$, as illustrated in Figure 7, to form the sublattice $\text{Flag}(c_i)$. This extension ensures that \( \text{con}(a_p, b_p) = \text{con}(c_3, c_4) \), where \( \text{col}[c_3, c_4] = p \). Note that if $i \neq j$, then $\text{Flag}(c_i) \cap \text{Flag}(c_j) = C \cup \{o, i\}$.

Similarly, we add $\text{Flag}(c_1), \text{Flag}(c_2), \text{Flag}(c_4)$ to form $L$. We will not draw this extension because even with the diagram the resulting lattice is hard to visualize.

**Figure 6.** The lattice $\text{Flag}(c_3)$

**Figure 7.** Further adding $\text{Flag}(c_3)$
3.5. The role of \( C \). It follows that \( \text{con}(a_p, b_p) \lor \text{con}(a_r, b_r) \) is principal in \( L \). Indeed
\[
\text{con}(a_p, b_p) \lor \text{con}(a_r, b_r) = \text{con}(c_3, c_5).
\]

On the other hand, \( \text{con}(a_p, b_p) \lor \text{con}(a_q, b_q) \) is not principal since there is no interval \([x, y]\) in \( C \) such that \( \text{colSet}[x, y] = \{p, q\} \).

4. Construction

Let \( D \) be a finite distributive lattice with a join-irreducible unit element and let \( P = J(D) \). We can assume that \(|D| > 2\), because Theorem 3 is trivial if \(|D| \leq 2\).

Let \( Q \subseteq D \) be representable by the chain \( C = \{c_1 \prec c_2 \prec \cdots \prec c_n\} \) colored by \( P \). Note that \( 1 \in P \), so there is at least one prime interval in \( C \) colored by \( 1 \).

In this section, we construct a finite lattice \( L \) such that \( \text{Con} L \) is isomorphic to \( D \) and \( \text{Princ} L \) corresponds to \( Q \) under this isomorphism, as required by Theorem 3.

In view of Theorem 2, this construction of the lattice \( L \) and the verification of its properties in Section 5, will complete the proof of Theorem 3.

4.1. The frame lattice. As in Section 3.2, we first construct the lattice \( \text{Frame} C \), see Figure 3. Also recall that \( \text{Frame} C \) is the union of the chains \( C_p = \{o, a_p, b_p, i\} \), for \( p \in P \), the chain \( C \), and the chain \( C_u = \{o, u, i\} \) with an additional nonchain sublattice, \( S = \{o, a_1, b_1, s_1, i\} \). In formula,
\[
\text{Frame} C = C \cup S \cup C_u \cup \bigcup (C_p \mid p \in P),
\]
where any two distinct components intersect in \( \{o, i\} \).

4.2. The lattice \( L \). We are going to construct the lattice \( L \) (of Theorem 3) as an extension of \( \text{Frame} C \).

We utilize the following lattices, which we shall call component lattices:

\[
\begin{align*}
(8) & \quad W(p, q) & \text{for } p < q \in P; \\
(9) & \quad \text{Flag}(c_i) & \text{for } i < n; \\
(10) & \quad S, \\
(11) & \quad C_u.
\end{align*}
\]

Let \( \text{CompLat} C \) be the set of component lattices associated with the colored chain \( C \).

Recall from Section 3.4 that \( \text{Flag}(c_i) \cap \text{Flag}(c_j) = C \cup \{o, i\} \) for \( i \neq j \).

We start with a simple, but crucial, observation.

Lemma 4. Let \( A \neq B \in \text{CompLat} C \). If either \( A \) or \( B \) is not a flag lattice, then \( A \cap B \) is a chain \( X: \{o, i\}, C_p, \text{ or } C \cup \{o, i\} \). Moreover, the elements of \( X - \{o\} \) are meet-irreducible.

We define the set
\[
L = \bigcup (W(p, q) \mid p < q \in P) \cup \bigcup (\text{Flag}(c_i) \mid i < n) \cup S \cup C_u.
\]

Define the order relation \( \leq \) on \( L \) as follows:
\[
x \leq y \text{ in } L \text{ iff } x \leq y \text{ holds in one of the component lattices.}
\]

In formula,
\[
\leq = \bigcup (\leq_{W(p, q)} \mid p < q \in P) \cup \bigcup (\leq_{\text{Flag}(c_i)} \mid i < n) \cup S \cup \leq_{C_u}.
\]

Lemma 5. The binary relation \( \leq \) on \( L \) is an order relation.
Proof. By (12) and (13), the relation ≤ is reflexive; by definition, it is antisymmetric.

Let \( x \leq y \leq z \) in \( L \). If \( x = y \) or \( y = z \), then \( x \leq z \) trivially holds, so we can assume that \( x < y < z \). By the definition of \( \leq \) in \( L \), see (13), there are component lattices \( A \) and \( B \) so that \( x < y \) in \( A \) and \( y < z \) in \( B \).

If \( A = B \), then \( x < z \) in \( A \), therefore, \( x < z \) in \( L \). So we can assume that \( A \) and \( B \) are distinct lattices and Lemma 4 applies. It follows that \( y = a_p \) or \( y = b_p \) for some \( p \in P \).

If \( y = a_p \), then \( y < z \) in \( B \) implies that \( b_p = y^* \leq z \) in \( B \). So \( z \) is \( b_p \) or \( i \), and \( x < z \) in \( A \) and, therefore, \( x < z \) in \( L \) follows.

If \( y = b_p \), then again \( b_p < z \) in \( B \) and \( x < z \) in \( A \) and, therefore, \( x < z \) in \( L \) follows.

Corollary 6. The ordered set \( L \) is a lattice and each component lattice is a sublattice.

In fact, the union of any number of component lattices is a sublattice.

Let \( A \neq B \in \text{CompLat} \ C \). We call them adjacent, if \( A \cap B \neq \{ o, i \} \).

Corollary 7. Let \( A \) and \( B \) be not adjacent component lattices. Then \( a \in A - \{ o, i \} \) and \( b \in B - \{ o, i \} \) are complementary.

Let \( U \) be a \( \{ o, i \} \)-chain in \( L \). Then for every \( x \in L \), there is a smallest element \( x^U \geq x \) of \( U \) and a largest element \( x_U \leq x \) of \( U \).

If \( A \) and \( B \) are adjacent component lattices, then \( U(A, B) = A \cap B \) is a \( \{ o, i \} \)-chain of \( L \).

Corollary 8. Let \( A \) and \( B \) be adjacent component lattices. Let \( a \in A - \{ o, i \} \) and \( b \in B - \{ o, i \} \). Then \( a \lor b = \max(a^U, b^U) \), where \( U = U(A, B) \).

5. Proving Theorem 3

5.1. Two lemmas. In this section, under the same assumptions as in Section 4, we describe the congruences of the lattice \( L \) we constructed in the previous section. We verify that the finite distributive lattice \( D \) is isomorphic to the congruence lattice of \( L \) and under this isomorphism, the elements of \( Q \) correspond to the principal congruences.

We start the proof with two easy lemmas.

Lemma 9. For every \( x \in L \), there is an \( \{ o, i \} \)-sublattice \( A \) of \( L \) containing \( x \) and isomorphic to \( M_3 \).

Proof. For \( x \in \{ o, i, u, a_1 \} \), take \( A = \{ u, a_1, c_1, o, i \} \). If \( x = s \), then \( A = \{ u, s, c_1, o, i \} \) is such a sublattice. Otherwise, let \( A = \{ u, a_1, x, o, i \} \). \( \square \)

An internal congruence of \( L \) is a congruence \( \alpha > 0 \), such that \( \{ o \} \) and \( \{ i \} \) are congruence blocks of \( \alpha \).

Lemma 10. Let us assume that \( \alpha \) is not an internal congruence of \( L \). Then \( \alpha = 1 \).

Proof. Indeed, if \( \alpha \) is not an internal congruence of \( L \), then there is an \( x \in L - \{ o, i \} \) such that \( x \equiv o \pmod{\alpha} \) or \( x \equiv i \pmod{\alpha} \). Using the sublattice \( A \) provided by Lemma 9 we conclude that \( \alpha = 1 \), since \( A \) is a simple \( \{ o, i \} \)-sublattice. \( \square \)
5.2. The congruences of a W lattice. We start with the congruences of the lattice $W(p, q)$ with $p < q \in P$, see Figure 8.

**Lemma 11.** The lattice $W(p, q)$ has two internal congruences:

$$\text{con}(a_p, b_p) < \text{con}(a_q, b_q).$$

**Proof.** An easy computation.

First, check that Figure 8 correctly describes the two join-irreducible internal congruences $\text{con}(a_p, b_p)$ and $\text{con}(a_q, b_q)$.

Then, check all 19 prime intervals $[x, y]$ and show that $\text{con}(x, y)$ is either not an internal congruence or equals $\text{con}(a_p, b_p)$ or $\text{con}(a_q, b_q)$. For instance,

$$\text{con}(e_{p,q}, f_{p,q}) = \text{con}(a_p, b_p)$$

and $\text{con}(h_{p,q}, b_p)$ is not an internal congruence because

$$f_{p,q} \equiv i \pmod{\text{con}(h_{p,q}, b_p)}.$$ 

The other 17 cases are similar.

Finally, note that the two join-irreducible internal congruences we found are comparable, so there are no other internal congruences. □

5.3. The congruences of flag lattices. There is only one important congruence of a flag lattice. It is $\text{con}(a_p, b_p) = \text{con}(c_i, c_{i+1})$. It identifies $\text{con}(a_p, b_p)$ with $\text{con}(c_i, c_{i+1})$, where $[c_i, c_{i+1}]$ is of color $p$.

Note that there may be many flag lattices containing a given $a_p, b_p$.

5.4. The congruences of L. Let $\alpha$ be an internal congruence of $L$. Then for every $A \in \text{CompLat} C$, we associate with $\alpha$ the internal congruence $\alpha_A$ of $A$, the restriction of $\alpha$ to $A$. These congruences are compatible, in the following sense.

Let $A, B \in \text{CompLat} C$, let $\beta$ be internal congruences on $A$, and let $\gamma$ be internal congruence on $B$. We call the congruences $\beta$ and $\gamma$ compatible, if either $A$ and $B$ are not adjacent or they are adjacent and $\beta_{A \cap B} = \gamma_{A \cap B}$. 

![Figure 8. The internal congruences of $W(p, q)$ for $p < q \in P$](image-url)
Lemma 12. Let $\alpha(A)$ be an internal congruence for every $A \in \text{CompLat } C$. Let us assume that the congruences $\alpha(A)$ are compatible. Then there is an internal congruence $\alpha$ of $L$ such that $\alpha(A) = \alpha_A$ for $A \in \text{CompLat } C$. This congruence $\alpha$ is unique.

Proof. By compatibility, we can define a binary relation $\alpha$ as the union of the $\alpha(A)$, that is,

$$\alpha = \bigcup \{ \alpha(A) \mid A \in \text{CompLat } C \}.$$  

By definition, $\alpha$ is reflexive and transitive. To prove that $\alpha$ is a congruence, it is sufficient to verify the Substitution Properties. The Meet Substitution Property is trivial. By utilizing the Technical Lemma, see [8, Lemma 11] and [11, Theorem 3.1], we only have to do the Join Substitution Property for two comparable elements.

So let $a \leq b \in A - \{o, i\}$, where $A$ is a component lattice of $L$, let $c \in L - \{o, i\}$, and let $a \equiv b \pmod{\alpha}$. By Corollaries [7] and [8], we can assume that there is a component lattice $B$ of $L$ such that $c \in B$. Let $U = A \cap B$ be the chain shared by $A$ and $B$. We assume also that $a \lor c, b \lor c \in L - \{o, i\}$. Since $b \lor c \neq i$, it follows that $a^U, b^U, c^U < i$. Therefore, by Corollary [8] $a \lor c, b \lor c \in U - \{i\} \subseteq A$, and the congruence $a \equiv b \pmod{\alpha}$ now follows because it holds in $A$ for $\alpha(A)$. (By utilizing the Technical Lemma for Congruences of Finite Lattices, see [10], we could assume that $a < b, c$ in the last paragraph.)

Now we are ready to describe the join-irreducible congruences of $L$. Of course, the unit congruence, $1$, is join-irreducible, generated by any nontrivial interval $[0, x]$ and $[x, 1]$, as well as by $[a_1, b_1]$.

Definition 13. Let $r \in P$ with $r < 1$. For $A \in \text{CompLat } C$, define $g(A)$ as follows.

(i) Let $A = W(p, q)$ for $p < q \in P$. Define

$$g(W(p, q)) = \begin{cases} \text{con}_{W(p, q)}(a_q, b_q) & \text{for } q \leq r; \\ \text{con}_{W(p, q)}(a_p, b_p) & \text{for } q \notin r \text{ and } p \leq r; \\ 0 & \text{for } q \notin r \text{ and } p \notin r. \end{cases}$$

(ii) Let $A = \text{Flag}(c_i)$ for $i < n$ with $\text{col}[c_i, c_{i+1}] = p$. Now we define the congruence $g(\text{Flag}(c_i))$ by enumerating the prime intervals it collapses in $\text{Flag}(c_i)$:

(a) the five prime intervals of $\text{con}(a_p, b_p)$ in $\text{Flag}(c_i)$ provided that $p \leq r$, see Figure 9.

(b) the prime intervals $[c_j, c_{j+1}] \in C$ satisfying $\text{con}(c_j, c_{j+1}) \leq r$;

(c) the prime intervals $[c_j', c_{j+1}']$ with $j < i$ satisfying $\text{con}(c_j, c_{j+1}) \leq r$.

(iii) Let $A = S$. Then $g(S) = 0$.

(iv) Let $A = C_u$. Then $g(C_u) = 0$.

Lemma 14. The congruences $\{ g(A) \mid A \in \text{CompLat } C \}$ are compatible.

Proof. Let $A \neq B \in \text{CompLat } C$. Let $[x, y]$ be a prime interval of $L$ satisfying

$$0 < x < y < i,$$
$$[x, y] \subseteq A \cap B,$$
$$x \equiv y \pmod{g(A)}.$$ 

It follows from (16) that both $A$ and $B$ are $W$ lattices or flag lattices. We distinguish two cases.
Case 1. Let $A$ be the flag lattice $\text{Flag}(c_i)$, where $i < n$ and $\text{col}[c_i, c_{i+1}] = p$. The only prime interval in $A$ with (15) that can be shared with another W lattice or flag lattice is $[a_p, b_p]$ and it only can be shared with the W lattice $B = W(p, q)$ for $p < q \in P$ or with the W lattice $B = W(q, p)$ for $q < p \in P$.

$\varrho(A)$ restricted to $[x, y]$ is 0 if $p \not\leq r$ and 1 if $p \leq r$ by Definition 13(i). Similarly, $\varrho(B)$ restricted to $[x, y]$ is 0 if $p \not\leq r$ and 1 if $p \leq r$ by Definition 13(ii). So we get compatibility.

Case 2. Let $A$ be the W lattice $W(p, q)$ for $p < q \in P$. The only prime interval in $A$ with (15) and (17) that can be shared with another W lattice is $[a_p, b_p]$ and it only can be shared with the W lattice $B = W(p, q)$ for $p < q \in P$ or with the W lattice $B = W(q, p)$ for $q < p \in P$. (Sharing it with a flag lattice was discussed in Case 1.) In both cases we apply (14) to get compatibility: if $p \leq r$, then we get $[a_p, b_p]$ collapsed by both $\varrho_A$ and $\varrho_B$; if $p \not\leq r$, then $\varrho_A = 0$ and $\varrho_B = 0$. □

By Lemma 12 we have the congruence $\varrho = \text{con}(a_r, b_r)$ on $L$ so that $\varrho_A = \varrho(A)$ for every component lattice $A$ of $L$.

Corollary 15. The join-irreducible congruence of $L$ are the congruences $\text{con}(a_r, b_r)$ for $r \in P$.

Corollary 16. The map

$$r \mapsto \text{con}(a_r, b_r) \text{ for } r \in P$$

uniquely extends to an isomorphism $\varphi$ between $D$ and $\text{Con} L$. 
5.5. **Principal congruences of** $L$. To complete the proof of Theorem 3, it remains to prove the following two results.

**Lemma 17.** Under the isomorphism $\phi$ of Corollary 16 if $q \in Q \subseteq D$, then $\phi q$ is a principal congruence of $L$.

**Proof.** Let $q \in Q$. Since $Q$ is represented by the chain $C$ colored by $P = J(D)$, there is an interval $[x, y]$ of $C$ such that $q = \bigvee \text{colSet}[x, y]$. So $\phi$ maps $q \in Q$ to a principal congruence $\text{con}(x, y)$ of $L$. □

**Lemma 18.** Let $d \in D$ and let $\phi d$ be a principal congruence of $L$. Then $d \in Q$.

**Proof.** Let $d \in D$ and let $\phi d = \text{con}(x, y)$, where $x \leq y \in L$. If $x = y$, then $d = 0 \in Q$. If $x \prec y$ in $L$, then $\phi d = \text{con}(a_p, b_p)$, where $p \in P \subseteq Q$.

Finally, let $[x, y]$ be of length at least 2, that is, $x = z_0 \prec z_1 \prec \cdots \prec z_k = y$, where $2 \leq k$. By the construction of $L$, there is a component lattice $A$ of $L$, such that $x, y \in A$. If $A$ is a $W$ lattice, or $S$, or $C_u$, then $\text{con}(z_i, z_{i+1}) = 1$, for some $i < k$, so $d = 1 \in P \subseteq Q$.

Finally, let $A$ be a flag lattice $\text{Flag}(c_i)$ for $i < n$. By inspecting the diagram of $\text{Flag}(c_i)$ (Figure 6 and Figure 9), we conclude that one of the following three cases occurs:

(i) $\text{con}(z_i, z_{i+1}) = 1$ for some $i < k$;
(ii) $x, y \in C$;
(iii) there are elements $u, v \in C$ so that $x = u'$ and $y = v'$.

In Case (i), it follows that $d = 1 \in P \subseteq Q$.

In Case (ii), we conclude that $d = \bigvee \text{colSet}[x, y]$ and so $d \in Q$ by the definition of chain representability, see Section 1.2.

In Case (iii), we argue as in Case (ii) with the interval $[u, v]$. □

5.6. **Discussion.** Problem 22.1 of my book [11] (see Section 1.1) remains unsolved. In G. Grätzer and H. Lakser [15], we proved some relevant results:

(i) For a finite distributive lattice $D$, the set $Q = D$ is representable.
(ii) If a finite distributive lattice $D$ has a join-irreducible unit element, then $Q = J(D) \cup \{0, 1\}$ is representable.
(iii) Let $D$ be the eight-element Boolean lattice with atoms $a_1, a_2, a_3$. Then the set $Q = \{0, a_1, a_2, a_3, 1\} \subseteq D$ is not representable.

We also introduced in G. Grätzer and H. Lakser [15] the following concept. Let us call a finite distributive lattice $D$ **fully representable**, if every $Q \subseteq D$ is representable provided that $\{0, 1\} \cup J(D) \subseteq Q$. In G. Grätzer and H. Lakser [15], we observe that every fully representable finite distributive lattice is planar.

G. Czédli [6] and [7] combine to give a deep characterization of fully representable finite distributive lattices as follows:

A finite distributive lattice $D$ is fully principal congruence representable iff $D$ is planar and it has at most one join-reducible dual atom.

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