Three-dimensional statistical field theory for density fluctuations in heavy-ion collisions

Hans C. Eggers
Institut für Theoretische Physik, Universität Regensburg, Postfach 10 10 42, W-8400 Regensburg, Germany

Hans-Thomas Elze
CERN-TH, CH-1211 Geneva 23, Switzerland

Ina Sarcevic
Department of Physics, University of Arizona, Tucson AZ 85721, USA

Abstract

A statistical field theory of particle production is presented using a gaussian functional in three dimensions. Identifying the field with the particle density fluctuation results in zero correlations of order three and higher, while the second order correlation function is of a Yukawa form. A detailed scheme for projecting the theoretical three-dimensional correlation onto data of three and fewer dimensions illustrates how theoretical predictions are tested against experimental moments in the different dimensions. An example given in terms of NA35 parameters should be testable against future NA35 data.
1 Introduction

Motivated by the search for a new exotic phase of matter predicted by QCD as well as a better understanding of nuclear matter at high energy densities, much experimental and theoretical effort is being directed towards studying heavy ions at high energies [1]. The fact that tens to hundreds of nucleons collide to produce several hundred particles makes it impossible to calculate analytically exclusive final states. Large-scale Monte Carlo simulations of microscopic processes and statistical methods have so far proven the only viable approaches to the problem.

In the realm of statistical analyses, the measurement and theoretical treatment of one-particle observables such as cross sections, $p_\perp$ distributions etc. have, after a period of dormancy, recently been supplemented by revived interest in correlations. Different theoretical attempts to describe correlations in various regions of phase space, classified under headings such as Bose-Einstein correlations [2], intermittency [3, 4] and effects of collective flow [5], have met with some success but much remains unclear. Some recent experimental work sees the two as connected, saying that correlations of like-sign particles measured in intermittency analyses can be understood as a Bose-Einstein effect [6], while conversely analyzing factorial moments in terms of the correlation integral and the four-momentum transfer $Q^2$ is equivalent to the Bose-Einstein analysis [8]; in hadronic collisions, UA1 has found a power-law dependence in their Bose-Einstein analysis [8].

While the languages may be converging, this does not mean that there is as yet a solid theoretical basis for these correlations, especially those of higher order: it is still most difficult to calculate measurable correlation functions from first principles. It is therefore helpful to explore simple effective field theories which ultimately should be derived from the underlying theory of QCD. Thus the situation bears some resemblance to the pre-BCS era in the research of superconductivity. What we are looking for is a strong interaction analogy to the phenomenologically successful Ginzburg-Landau theory [9].

The present paper attempts to understand correlations in heavy-ion collisions in terms of a statistical field theory based on a gaussian approximation of a suitably defined Ginzburg-Landau theory. This approach was first used by Scalapino and Sugar [10] who identified the field as a pion amplitude. In our case, the field is not an amplitude or particle density but a fluctuation from the mean density, an ansatz used previously [11, 12]. It is motivated, as we shall see, by the fact that it matches experimental findings that in heavy-ion collisions only second order correlations are found while higher order correlations are negligible. In contrast to our previous work which was related to the rapidity variable only, all quantities are here calculated strictly in the three-dimensional phase space of rapidity $y$, azimuthal angle $\phi$ and transverse momentum $p_\perp$ and then projected down for comparison with the data (see, however, the comment in Section VI concerning the proper choice of variables).

The paper is organized as follows. In Section II, we remind the reader of the basic equations for extracting true correlations and present the experimental evidence for neglecting correlations of order higher than two. In Section III, we recapitulate and develop the formalism of statistical field theories in terms of a functional formulation and derive from a three-dimensional functional ansatz a general form of the second order correlation. We show
in Section IV how one integrates out variables to find moments in one, two and three dimensions and how this tests theoretical correlation functions. Using NA35 parameters and $p_{\perp}$ distributions, we show horizontal and vertical moments in all dimensions in Section V. A summary and discussion of some important issues conclude the paper in Section VI.

2 Cumulants in heavy-ion collisions

Traditionally, correlations were measured as a function of the distance between bins in phase space while keeping the bin sizes fixed. Following the proposal to look for a power-law structure in the correlation function [4], a commonly used alternative has become the measurement of normalized factorial moments as a function of decreasing bin size while disregarding all distance information.

The factorial moments $F_q(M)$ are constructed as follows. (For the purposes of this paper we shall stick to the coordinates rapidity, azimuthal angle and transverse momentum, but the formulation is true for all variables.) A given total interval $\Omega_{\text{tot}} = \Delta Y \Delta \phi \Delta P$ is subdivided into $M^3$ bins of side lengths $\left(\Delta Y/M, \Delta \phi/M, \Delta P/M\right)$. With $n_{klm}$ being the number of particles in bin $(k,l,m)$ and $n^{[q]} \equiv n(n-1)\ldots(n-q+1)$, the “vertical” factorial moment is

$$F^v_q(M) \equiv \frac{1}{M^3} \sum_{k,l,m=1}^{M} \frac{\langle n^{[q]}_{klm} \rangle}{\langle n_{klm} \rangle^q} = \frac{1}{M^3} \sum_{k,l,m=1}^{M} \left[ \int_{\Omega_{klm}} \prod_i d^3x_i \rho_q(x_1\ldots x_q) \right] \left[ \int_{\Omega_{\text{tot}}} d^3x \rho_1(x) \right]^q. \quad (1)$$

The second equality illustrates how the factorial moment can be written in terms of integrals of the correlation function $\rho_q$ ($\Omega_{klm}$ is the region of integration over bin $k,l,m$). Because for small bin sizes $n_{klm}$ becomes small and the relative error correspondingly large, an alternative definition is often preferred for three-dimensional analysis, the “horizontal” factorial moment,

$$F^h_q(M) \equiv \frac{1}{M^3} \sum_{k,l,m=1}^{M} \frac{\langle n^{[q]}_{klm} \rangle}{(\langle N \rangle/M^3)^q} = M^{3(q-1)} \sum_{k,l,m=1}^{M} \left[ \int_{\Omega_{klm}} \prod_i d^3x_i \rho_q(x_1\ldots x_q) \right] \left[ \int_{\Omega_{\text{tot}}} d^3x \rho_1(x) \right]^q. \quad (2)$$

The latter form, while being much more stable, has the serious drawback that it is influenced by the shape of the one-particle distribution function $\rho_1$.

When one or more of the arguments of the correlation function become statistically independent it factorizes into lower order parts. To measure true particle correlations, known as cumulants, the trivial background must be subtracted. The first two cumulants are defined as [13]

$$C_2(x_1, x_2) = \rho_2(x_1, x_2) - \rho_1(x_1)\rho_1(x_2), \quad (3)$$
\[ C_3(x_1, x_2, x_3) = \rho_3(x_1, x_2, x_3) - \rho_1(x_1)\rho_2(x_2, x_3) \\
- \rho_1(x_2)\rho_2(x_1, x_3) - \rho_1(x_3)\rho_2(x_1, x_2) \\
+ 2\rho_1(x_1)\rho_1(x_2)\rho_1(x_3). \] (4)

By integrating these well-known relations over each bin, one can derive equations for integrated cumulants \( K^v_q \) in terms of the vertical factorial moments of Eq. (1): the first two are

\[ K^v_2 = F^v_2 - 1, \quad K^v_3 = F^v_3 - 3F^v_2 + 2, \] (5)

and so on for higher orders \([4]\). Whenever there are no true correlations, these cumulants become zero.

Analyses of the \( K_q \)'s is now being carried out routinely in conjunction with the standard factorial moments as a function of bin size. Interestingly, it was found that in the case of heavy-ion collisions, there are only two-particle correlations: while \( K_2 \) is positive for all bin sizes, the values of \( K_3, K_4 \) and \( K_5 \) have all been found to be consistent with zero. While initially this was found in terms of one-dimensional rapidity data, measurements by NA35 in two and three dimensions have confirmed this fact \([3]\). As an example we show in Fig. 1 the third order cumulant derived from NA35 200 A GeV Oxygen-Gold measurements of the factorial moments \([10]\). (This data is given in terms of “flattened variables”.) The higher orders \( K_4 \) and \( K_5 \) were also found to be consistent with zero. Corresponding findings for other nuclei and energies were published before \([12, 17, 18]\).

### 3 Statistical field theory

For the large number of particles produced in high-energy heavy-ion collisions, a statistical theory of particle production is justified. A fruitful starting point has been an analogy first drawn by Feynman and Wilson \([19]\), who pointed out that, by interpreting \((y, \phi, p_\perp)\) as a “spatial” coordinate \(\vec{x}\), the final \(N\)-particle phase space could be treated as if it were a gas, bounded by “walls” made up of the overall kinematic constraints. This analogy then carries over into a correspondence of the total cross section to a (grand canonical) partition function of a gas, while the \(n\)-particle cross section becomes the \(n\)-particle distribution function of the gas. It immediately leads to an identification of the probability of producing a secondary with a certain momentum \((y, \phi, p_\perp)\) with the gas “density” \(\rho_1\) at the corresponding point \(\vec{x}\).

Botke, Scalapino and Sugar \([10, 20]\) utilized these ideas to write down a model for particle production. In analogy to ordinary statistical mechanics, where the density matrix governing the weights of states can be written in terms of the free energy, \(\hat{\rho} \propto \exp(-\beta(\hat{H} - \Omega))\), they defined a functional \(F[\Pi]\) of a random field \(\Pi\), which governs the number of particles produced at point \(\vec{x}\) via

\[
\frac{1}{\sigma} \frac{d^3 \sigma}{d^3 \vec{x}} = \langle \Pi^2(\vec{x}) \rangle = \frac{1}{Z} \int D\Pi e^{-F[\Pi]}\Pi^2(\vec{x}),
\] (6)

where one identifies the particle density with \(\Pi^2\) and \(Z \equiv \int D\Pi \exp(-F[\Pi])\) plays the role of the partition function. Higher order correlation functions can easily be found by the appropriate functional derivatives.
The essence of this ansatz is that all the physics is hidden in the form of the functional $F[\Pi]$ which reduces the greatly redundant information contained in the many microscopic degrees of freedom to just a few phenomenological parameters. (Botke, Scalapino and Sugar showed that their classical functional formalism is isomorphic to a system of quantized Boson fields with normal ordering \cite{20}.)

The analogy with the free energy permits the utilization of the Ginzburg-Landau expansion of the free energy near a second-order phase transition \cite{21}, which uses functionals of the type

$$F[\Pi(\mathbf{x})] = \int d^3x \left[ \alpha (\nabla_x \Pi)^2 + \mu^2 \Pi^2 + \lambda (\Pi^2)^2 \right],$$

giving a minimum expectation value of the order parameter at zero above and nonzero below the phase transition. Applied to low-energy hadronic data by Scalapino and Sugar, it has recently been used for KNO scaling \cite{22} and in an attempt to find a critical index for a phase transition to a quark-gluon plasma \cite{23}. Indeed, recent lattice gauge calculations suggest that, while a first-order transition is likely for the pure glue SU(3) gauge theory and for more than three quark flavors, there is increasing likelihood of a second-order phase transition for two quark flavors \cite{24}, and Wilczek \cite{25} has made first attempts at formulating such a transition in terms of a chiral order parameter.

However, our aim here is more modest in that we do not specify the dependence of the constants $\alpha$, $\mu$ and $\lambda$ entering the free energy functional in Eq. (7) on the dynamical parameters of the heavy-ion reaction under consideration such as total center-of-mass energy, mass numbers and impact parameter. Instead, we treat them as phenomenological parameters to be determined from the experimental data and to be interpreted later on theoretically with the help of a more microscopic model as discussed in Section VI. Also we do not consider the definition of the functional of Eq. (8) below an expansion valid for small values of the field $\Phi$ only, and we cannot draw any conclusions whether the system undergoes a phase transition or not. An exception would arise when drastic and systematic variations in the phenomenological parameters as deduced from one experiment compared to another are observed (see Section V).

We define a random field $\Phi$ as a function in the three-dimensional space spanned by $(y, \phi, p_\perp)$. Throughout, $p_\perp$ will be implicitly divided by a constant scale $P$ so that it is dimensionless. Since we are not looking for a phase transition, we omit the quartic term, $\lambda \approx 0$, and start with the functional \cite{12}

$$F[\Phi] = \int_0^P dy \int_{-P/2}^{P/2} d^2p_\perp \left[ a^2 \left( \frac{\partial \Phi}{\partial y} \right)^2 + a^2 (\nabla_{\mathbf{p}_\perp} \Phi)^2 + \mu^2 \Phi^2 \right].$$

We note that $F$ is not rotationally invariant but rather boost-invariant in the direction of the collision axis, i.e. tailored to the specific symmetry of the collision. The integration bound $P$ is chosen the same for $y$ and $p_\perp$ for simplicity but does not have to be the same in general.

The expectation value of any function of $\Phi$ is found by taking functional derivatives with respect to a source term, $J \cdot \Phi$, added into the integrand of $F$ in Eq. (8) and thus into the exponential entering the functional integral for the partition function $Z$ (take $\vec{x}_i \equiv$
\[ (y_i, \phi_i, p_{\perp i}), \]
\[
\langle \Phi(\vec{x}_1) \ldots \Phi(\vec{x}_k) \rangle = \frac{1}{Z} \int D\Phi e^{-F[\Phi]} \Phi(\vec{x}_1) \ldots \Phi(\vec{x}_k)
\]
\[
= \frac{1}{Z} \frac{\delta^k Z[F, J]}{\delta J(\vec{x}_1) \ldots \delta J(\vec{x}_k)} \bigg|_{J=0}.
\]

For the functional (8), we find the three-dimensional form of the two-point function
\[
\langle \Phi(\vec{x}_1)\Phi(\vec{x}_2) \rangle = \frac{1}{8\pi a^2} \frac{e^{-R/\xi}}{R},
\]
where \( \xi = a/\mu \), and with \( p_i \equiv p_{\perp i}, \)
\[
R \equiv \sqrt{(y_1 - y_2)^2 + p_1^2 + p_2^2 - 2p_1 p_2 \cos(\phi_1 - \phi_2)}.
\]

So far, we have not defined the field \( \Phi \) in terms of physical observables. For reasons that will become apparent shortly, we define \( \Phi(\vec{x}) \) as the fluctuation at the point \( \vec{x} \) of the particle density for a given event, \( \hat{\rho}_1(\vec{x}) \), above/below the mean single particle distribution \( \rho_1 \) at that point:
\[
\Phi(\vec{x}) \equiv \frac{\hat{\rho}_1(\vec{x})}{\rho_1(\vec{x})} - 1.
\]

Experimentally, \( \Phi \) is the (normalized) difference between the event histogram and the event-averaged one-particle distribution for a given bin.

This definition was previously used by Dremin and Nazirov [11] and in a one-dimensional context by Elze and Sarcevic [12]. The rationale in the present and previous case is the same: all cumulants except the second order cumulant become exactly zero. To see how this comes about, we note that \( \rho_q(\vec{x}_1, \ldots, \vec{x}_q) = \langle \hat{\rho}_1(\vec{x}_1) \ldots \hat{\rho}_1(\vec{x}_q) \rangle \), and with the definition [12] and the relations between the \( \rho_q \) and cumulants \( C_q \) one finds that the expectation values of \( \Phi \) can be written in terms of the reduced cumulants \( k_q(\vec{x}_1, \ldots, \vec{x}_q) \equiv C_q(\vec{x}_1, \ldots, \vec{x}_q)/\rho_1(\vec{x}_1) \ldots \rho_1(\vec{x}_q) \) as
\[
\langle \Phi(\vec{x}_1)\Phi(\vec{x}_2) \rangle = k_2(\vec{x}_1, \vec{x}_2),
\]
\[
\langle \Phi(\vec{x}_1)\Phi(\vec{x}_2)\Phi(\vec{x}_3) \rangle = k_3(\vec{x}_1, \vec{x}_2, \vec{x}_3),
\]
\[
\langle \Phi(\vec{x}_1)\Phi(\vec{x}_2)\Phi(\vec{x}_3)\Phi(\vec{x}_4) \rangle = k_4(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4)
\]
\[
+ \sum_{(3)} k_2(\vec{x}_1, \vec{x}_2) k_2(\vec{x}_3, \vec{x}_4),
\]
the sum running over 3 permutations in the arguments. On the other hand, the form of the functional (8) ensures that all expectation values of an odd number of \( \Phi \)'s are zero, while for the fourth order,
\[
\langle \Phi(\vec{x}_1)\Phi(\vec{x}_2)\Phi(\vec{x}_3)\Phi(\vec{x}_4) \rangle = \sum_{(3)} \langle \Phi(\vec{x}_1)\Phi(\vec{x}_2) \rangle \langle \Phi(\vec{x}_3)\Phi(\vec{x}_4) \rangle,
\]
so that \( k_4 = 0 \) also, and similarly for the higher even cumulants. These equations are actually just the relations between cumulants and central moments, see Eq. (3.38) in [13].

In summary, the specific form of the functional (8) and the definition of \( \Phi \) as a fluctuation field ensure that all cumulants of order 3 and higher are zero, while \( k_2(\vec{x}_1, \vec{x}_2) \) has the Yukawa form (10).

We conclude this section with some comments:

1. Of course with better statistics it may eventually become clear that there is some small residual cumulant of higher order; the presently large error bars would permit that. Were this to happen, the theory in its present form would have to be modified by, for example, the inclusion of interaction terms of higher order in \( \Phi \), with or without additional implications for the existence of a phase transition.

2. The present model, where the higher order cumulants are exactly zero, is incompatible with the linking ansatz of Carruthers and Sarcevic [26] used for hadronic collisions, in which higher order cumulants are products of \( k_2 \) according to \( k_q = A_q k_2^{q-1} \), an ansatz which for certain values of the constants \( A_q \) yields the negative binomial distribution. So far, there is no direct evidence for linking in heavy-ion collisions [17]; however, the present size of error bars does not permit any conclusions on this point.

3. Thirdly, there has been some discussion whether cumulants of higher order can be identically zero consistently. Since this is a technical question with no consequences for our further development, we defer this point to the Appendix.

4 Projecting down to lower dimensions

Apart from the fact that all higher order cumulants are exactly zero by construction, the main result of our model is the Yukawa form for the second reduced cumulant, \( k_2 \propto e^{-R/\xi} / R \). This can be compared to data only after a suitable integration over its variables. In its present form, \( k_2 \) is not a power law of any of its variables. Setting \( \cos \theta = 1 \), one can show analytically that \( \int_0^{\delta y} d|y_1 - y_2| \int_0^{\delta \phi} dp_1 dp_2 R^{-1} \) becomes linear in \( \delta y \delta \phi \delta p \), while numerical integration over all variables yields \( K_2 \propto (\delta y \delta \phi \delta p)^{-\alpha} \), with \( \alpha = 1 \) within error, i.e. the integrated version resembles a power law.

We now derive the detailed form of integrated cumulants \( K_2 \) for various dimensions in terms of the three-dimensional \( k_2(\vec{x}_1, \vec{x}_2) \). For this purpose, the “vertically” and “horizontally” normalized versions Eqs.(1) and (2) have to be treated separately.

The second order vertical cumulant is, in three dimensions, (always taking \( \vec{x} \equiv (y, \phi, p_{\perp}) \))

\[
K_v^2(\delta y, \delta \phi, \delta p) = F_v^2 - 1 = \frac{1}{M^3} \sum_{k,l,m=1}^M K_v^2(k, l, m), \quad (17)
\]

with

\[
K_v^2(k, l, m) = \frac{\langle n_{klm} (n_{klm} - 1) \rangle - \langle n_{klm} \rangle^2}{\langle n_{klm} \rangle^2} = \frac{\int_{\Omega_{klm}} d^3\vec{x}_1 d^3\vec{x}_2 C_2(\vec{x}_1, \vec{x}_2)}{\left[ \int_{\Omega_{klm}} d^3\vec{x} \rho_1(\vec{x}) \right]^2}
\]
\[ \int_{\Omega_{klm}} d^3 \vec{x}_1 d^3 \vec{x}_2 \frac{\rho_1(\vec{x}_1)\rho_1(\vec{x}_2)}{\left[ \int_{\Omega_{klm}} d^3 \vec{x} \rho_1(\vec{x}) \right]^2} = 0. \] (18)

In other words, the integration of \( k_2 \) to compare with data involves a correction due to the shape of the one-particle three-dimensional distribution function \( \rho_1(\vec{x}) \). Equation (18) as it stands is exact: a knowledge of the theoretical reduced cumulant \( k_2 \) can only be translated into a measurable factorial cumulant \( K_2 \) when the full three-dimensional one-particle distribution is known and taken into account (of course the same is true for \( r_2 \equiv \rho_2/\rho_1 \rho_1 \) versus \( F_2 \)).

Knowledge of \( k_2(\vec{x}_1, \vec{x}_2) \) and \( \rho_1(\vec{x}) \) in three dimensions can also immediately be used to compare to factorial cumulant data of lower dimensions. For example, in \((y, \phi)\), the cumulant is \( K_2^v(\delta y, \delta \phi) = M^{-2} \sum_{lm} K_2^v(l, m) \) with the transverse momentum integrated over the whole window \( \Delta P \) (cf. Section VI),

\[
K_2^v(l, m) = \frac{\langle n_{lm}(n_{lm} - 1) \rangle - \langle n_{lm} \rangle^2}{\langle n_{lm} \rangle^2} = \int_{\Omega_l} dy_1 dy_2 \int_{\Omega_\phi} d\phi_1 d\phi_2 \int_{\Delta P} dp_1 dp_2 \frac{k_2(\vec{x}_1, \vec{x}_2) \rho_1(\vec{x}_1)\rho_1(\vec{x}_2)}{\left[ \int_{\Omega_{lm}} dy \int_{\Omega_\phi} d\phi \int_{\Delta P} dp \rho_1(\vec{x}) \right]^2}, \] (19)

and the cumulant for the rapidity only is \( K_2^v(\delta y) = M^{-1} \sum_m K_2^v(m) \), with \( K_2^v(m) \) an integral just like Eq. (15) but with \( \int_{\Omega_l} \) being replaced by \( \int_{\Delta \phi} \). Cumulants of other variable combinations are obtained analogously.

For the horizontal normalization, the three-dimensional cumulant \( K_2^h = M^{-3} \sum_{klm} K_2^h(l, m, k) \) consists of

\[
K_2^h(l, m, k) = \frac{\langle n_{klm}(n_{klm} - 1) \rangle - \langle n_{klm} \rangle^2}{\langle \langle N \rangle_\Omega / M^3 \rangle^2}, \] (20)

where \( \langle N \rangle_\Omega \) is by definition the number of particles within the experimentally defined total volume \( \Omega_{tot} = Y \Delta \phi \Delta P \). Care must be taken to define theoretical quantities such that they are normalized to this experimental domain. Keeping this in mind, we can write

\[
K_2^h(l, m, k) = M^6 \int_{\Omega_{klm}} d^3 \vec{x}_1 d^3 \vec{x}_2 k_2(\vec{x}_1, \vec{x}_2) \rho_1(\vec{x}_1)\rho_1(\vec{x}_2) \left[ \int_{\Omega_{tot}} d^3 \vec{x} \rho_1(\vec{x}) \right]^2. \] (21)

Projections onto two dimensions \((y, \phi)\) are then of the form of Eq. (19) but with the prefactor \( M^4 \) and with \( \Omega_{tot} \) replacing the bin region integrals \( \Omega_l, \Omega_m, \Delta P \) in the denominator. For one-dimensional data in \( y \), the prefactor is \( M^2 \) and \( \Omega_l \) becomes \( \Delta \phi \) in the numerator and \( \Omega_{tot} \) in the denominator integrals. Corresponding versions can be written down for \( K_2^v(\delta \phi) \) and \( K_2^h(\delta p) \).

For horizontal factorial moments, it is important to remember that the simple relations between factorial moments and cumulants, Eq. (14), are not valid but that rather

\[
F_2^h = K_2^h + \sum_{k,l,m=1}^{M} \langle n_{klm} \rangle^2 / \langle N \rangle_\Omega^2 \] (22)
for three dimensions, with corresponding relations for lower dimensions. This has to be taken into account in an eventual comparison with data.

With these relations it is thus possible, given any three-dimensional theoretical function $k_2$ (or $r_2$), to compute factorial cumulants and moments for any combination of its variables. Doing this for different variables can serve as a strong test of the theoretical function as the moments probe its different regions.

5 Comparison with the data

So far, there are only two published sets of heavy-ion factorial moment data spanning several dimensions: one for 200 A GeV Sulfur-Gold collisions measured by the EMU01 collaboration [27, 28] and one- and two-dimensional data of the KLM collaboration [29]. Both are unfortunately not suitable for our analysis, the former because of probable contamination of the $(\eta, \phi)$ moment by gamma conversion [30], the latter because the data is binned with different $M$ values in each variable. Both, being emulsion experiments, can measure only up to two dimensions. (An example of EMU01 analysis can be found in Ref. [31].)

An analysis by NA35 of 200 A GeV Oxygen-Gold moments in all dimensions is in preparation but not yet available; also, prospective Sulfur-Sulfur moments will provide further tests of the model [32]. In anticipation of such data, we have made a detailed study of moments for NA35 experimental parameters and their measured O+Au transverse momentum distribution. Lacking the appropriate data, we have set the parameters to arbitrary but plausible values $a = 2.0$, $\xi = 1.0$, hoping that exact fits can be made when experimental moments become available.

Since all the equations of the previous section are exact, they can be used in their unabridged forms whenever the full three-dimensional $\rho_1(\vec{x})$ is known and enough CPU time is available. In making comparisons with data, it will however usually be necessary to approximate these exact forms, partly because three-dimensional one-particle distributions are not available, partly to save on computer time. We have therefore made the following approximations in our simulation:

1. We factorize the three-dimensional one-particle distribution into its separate variables:

$$\rho_1(\vec{x}) = \langle N \rangle_\Omega g(y) h(\phi) f(p_\perp) ,$$

where the factor $\langle N \rangle_\Omega$ ensures that the three distributions $g$, $h$ and $f$ are separately normalized over their respective total intervals $\Delta Y$, $\Delta \Phi$ and $\Delta P$. (This factorization is known not to be true for some cases [33].)

2. The azimuthal distribution is taken as flat, $h(\phi) = 1/\Delta \Phi$.

3. We use the full experimental parametrization for $f(p_\perp)$ provided by NA35 [34];

4. For the rapidity distribution, we use a gaussian parametrization with $\sigma = 1.32$ [33], and transform $g(y_1)g(y_2)$ in the numerator with $Y = (y_1 + y_2)/2$, $y = y_2 - y_1$, to

$$g(y_1)g(y_2) = \frac{1}{2\pi \sigma^2} e^{-Y^2/\sigma^2} e^{-y^2/4\sigma^2} ,$$

(24)
and since $k_2(\vec{x}_1, \vec{x}_2)$ is a function of $|y_1 - y_2|$ only, we have $R(\Delta Y) \equiv \sum_m \int_{\Omega_m} dY \exp(-Y^2/\sigma^2) = \int_{\Delta Y} dY \exp(-Y^2/\sigma^2)$ as a constant prefactor, while the remaining gaussian in $y$ is included in the numerical integration.

5. As $k_2$ depends on the differences $|y_1 - y_2|$ and $|\phi_1 - \phi_2|$ only, we can transform to relative coordinates and integrate out the center-of-mass coordinate in these variables (the “strip approximation” [26]).

With these approximations, we can derive the complete behavior of all cumulant moments both for the vertical and horizontal normalizations. In Fig. 2, we show the vertical cumulants for three-dimensional binning in $(\delta y, \delta \phi, \delta p)$, for the two-dimensional $(\delta y, \delta \phi)$ and all three one-dimensional cumulants $K_2(\delta y)$, $K_2(\delta \phi)$ and $K_2(\delta p_{\perp})$. The pseudo-power-law behavior is clearly visible for the three-dimensional case, while $K_2$ for two and one dimension show the familiar saturation caused by the projection process [36]. Fig. 3 shows the corresponding horizontal cumulants $K_2^h$ for the same set of variables. All curves of Figs. 2 and 3 are for $a = 2.0$, $\xi = 1.0$.

A comparison of the two Figures shows that there is virtually no effect of the non-flat rapidity distribution, while the steep $p_{\perp}$ distribution has a large effect in raising both $K_2^h(\delta y, \delta \phi, \delta p)$ and $K_2^h(\delta p)$ above their vertical counterparts. All cumulants must of course have the same value for $M = 1$, independent of their dimension, variable or normalization. This is also clearly illustrated in the Figures.

Comparing to “real” experimental data, the procedure would be as follows: first, one takes the $F_2$ data of highest dimension and finds from these the corresponding $K_2$ from Eqs.(5) or (22), depending on the normalization used. To this the free parameters $a$ and $\xi$ are fitted using the appropriate formula of Section III. If the fit is disastrous, that of course is the end of the game. If it is reasonably good, further tests of the proposed cumulant $k_2(\vec{x}_1, \vec{x}_2)$ are performed by plotting on experimental $K_2$’s of lower dimension and different variables the relevant theoretical formulae of Section III keeping $a$ and $\xi$ fixed to their higher-dimensional best fit values. Of course, one can start out with lower-dimension best fits and find the corresponding curves for higher dimensions. This procedure is a nontrivial test of the theory because, for example, $K_2(\delta y)$ integrates over the entire regions $\Delta \Phi$ and $\Delta P$ and hence probes the long-distance behavior of the theoretical function $k_2$, while the three-dimensional experimental $K_2$ tests only the short-range behavior of the theoretical $k_2$.

6 Conclusions

To summarize our work, we studied integrated two-particle and higher order correlations in the form of factorial moments and cumulants and made predictions as to the general behavior of horizontal and vertical moments for the NA35 O+Au data. An exact comparison will depend on a fit of two parameters, after which everything else is fixed for all dimensions. It will be interesting to see how well our model will do with upcoming data from NA35 and hopefully other experiments.

In particular, we applied a simple statistical field theory model with a gaussian “free energy” functional motivated by analogy to the Ginzburg-Landau theory of superconductivity,
but without explicitly implying a (second order) phase transition. Our model, which is defined in terms of the field of density fluctuations, was first advocated in Ref. [12] to describe one-dimensional rapidity correlations as observed in high-energy heavy-ion collisions. Here, we extended the study to cover one-, two-, and three-dimensional correlations as measured through the intermittency analysis in the full phase space spanned by the variables rapidity, azimuthal angle, and transverse momentum of secondary particles.

In our present formulation, the intrinsic scale $P$ for the momentum cannot be deduced from the integrated dimensionless moments being compared with experiment; only direct measurements of the corresponding correlation functions, e.g. $k_q$, would give enough detailed information for this purpose. Thus, also the mass parameter and correlation length of our model can only be determined as dimensionless constants.

It should be stressed that (by construction) our model yields vanishing cumulant correlation functions of higher order by construction, $k_{q \geq 3} = 0$, which agrees with all presently available heavy-ion data (cf. Section II).

Working towards a more microscopic foundation of the Ginzburg-Landau type model, a comment about the right choice of kinematic variables seems appropriate: Since any attempt at deriving the effective three-dimensional statistical field theory from a more fundamental theory necessarily begins with a four-dimensional space-time formulation, i.e. leaving the Feynman-Wilson gas analogy [19] behind, three- or four-momenta conjugate to space-time coordinates are the natural variables. We therefore urge experimentalists to present their “intermittency” or correlation data in terms of three-momenta and with particle identification whenever this is feasible. Assuming azimuthal symmetry as before, the relevant variables for a more refined theoretical analysis seem to be $p_\parallel$, $p_\perp$, $\phi$.

Of course, the difficult problem how to incorporate the effects of the very asymmetric initial conditions in $p_\parallel$ and $p_\perp$ into the theoretical description remains as disturbing as before. Presently we attempted to include all available experimental information here by properly folding in at least the relevant one-particle distributions into the projection integrals for the lower-dimensional cumulant moments. An important consequence of this procedure is that, if our model fits a three-dimensional data set, then the projections onto two- and one-dimensional ones are predicted with no further freedom.

It is also important to point out that functionals defined in different dimensions produce different results: $K_2$ calculated from a one-dimensional functional does not correspond to $K_2$ obtained from deriving $k_2$ from a three-dimensional functional and then projecting down onto one dimension. Thus the formula obtained analytically in Ref. [12] from the one-dimensional functional, $K_2 = 2\gamma \xi^2[(\delta y/\xi) - 1 + e^{-\delta y/\xi}] / \delta y$, does not hold in our present formulation (even though it may fit the data).

Concerning the application of a statistical approach to energies higher than in current heavy-ion collisions, we have to deal with the expected increasing importance of hard partonic scattering events (minijets) [1]. They will help to populate more and more the high-$p_\perp$ tails of the one-particle distributions, where a priori we cannot expect our model to apply. Whereas presently high momenta are effectively cut off by the $p_\perp$ distributions, this may become a problem at higher center-of-mass energies. Eventually a momentum cut-off has to be introduced to limit the analysis to the same region of soft physics implicitly investigated
here. A thorough treatment of surface effects in phase space, together with the asymmetric initial conditions [10], then becomes mandatory.

Finally, we point out how a Ginzburg-Landau-type model may be based on a more fundamental field theory. Assuming that the observed multiparticle correlations and fluctuations arise in the dense hadronic phase of matter (after a possible hadronization phase transition), the first step of a derivation for this conservative scenario consists in choosing a phenomenologically satisfactory effective field theory of hadronic interactions such as the sigma model. (This presents, of course, an alternative to the view that the relevant fluctuations stem from partonic showering processes which might seem more natural for high-energy \(e^+e^\)-collisions.) Keeping the high excitation energy of the compressed matter in mind, one tentatively begins with a field theory at finite temperature, neglecting for simplicity the difficult non-equilibrium aspects of heavy-ion collisions. Aiming at a three-dimensional Ginzburg-Landau statistical picture one is led to integrate out the (imaginary) time-dependent modes from the path-integral representation of the (sigma model) field theory. This has so far been achieved in a one-loop approximation for an arbitrary scalar field theory yielding a temperature-dependent three-dimensional effective action\(^4\). It represents the infrared limit of the originally chosen theory in accordance with ideas on dimensional reduction and can be used as a starting point to calculate the correlation functions studied in the present paper (see Section III).

Clearly, much more can be done here to include non-perturbative effects if one wants to investigate, for example, effects of a phase transition possibly contained in the chosen field theory. Formally integrating out the fields of the three-dimensional effective action in favor of the fields squared (“densities”), one obtains a Ginzburg-Landau “free energy” functional with its parameters, cf. Eq. (7), given as temperature-dependent functions of the renormalized coupling constants of the original field theory.

Having outlined these further steps in the development of our model, we conclude that the study of correlations and fluctuations in dense hadronic matter seems a promising approach to further our understanding of strong interactions. In particular, the investigation of one-particle observables alone, which are nicely reproduced by a wide selection of event generators [1], in principle cannot provide a sufficiently detailed knowledge of the most interesting (and least understood) soft physics aspects of QCD.

Acknowledgements:

We thank I. Derado, J. Grote, J. Schukraft, D. Skelding, E. Stenlund and R.J. Wilkes for supplying us with the data and helping us to understand it in numerous discussions. We are indebted to P. Carruthers, A. Kühnichel and P. Seyboth for helpful remarks and discussions. Special thanks to P. Lipa and E.A. de Wolf for helpful comments pertaining to the appendix. HCE thanks the Alexander von Humboldt Foundation for support. HThE gratefully acknowledges support by the Heisenberg program of the Deutsche Forschungsgemeinschaft. This work was supported in part by the Department of Energy, Contract No. DE-F602-88ER40456 and DE-F602-85ER40213.

\(^4\) Preliminary results of this study were reported by one of us (H.-Th.E.) at the 1992 Workshop on Finite Temperature Field Theory at ZfF, Bielefeld, Germany.
References

[1] Quark Matter '91, Gatlinburg, Tennessee, November 11–15, 1991, Nucl. Phys. A544 (1992).

[2] W. Bauer, C.K. Gelbke and S. Pratt, preprint MSUCL-824 (1992), (to be published).

[3] *Proceedings of the Ringberg Workshop on Multiparticle Production*, 1991, edited by R.C. Hwa and W. Ochs and N. Schmitz (World Scientific, 1992).

[4] A. Bialas and R. Peschanski, Nucl. Phys. B273, 703 (1986); Nucl. Phys. B308, 857 (1988).

[5] U. Mayer, E. Schnedermann and U. Heinz, preprint TPR-92-13 (to be published).

[6] I. Derado, G. Jancso and N. Schmitz, preprint MPI-PhE/92-07 (1992), (to be published).

[7] P. Lipa, P. Carruthers, H.C. Eggers and B. Buschbeck, Phy. Lett. 285B, 300 (1992); H.C. Eggers, P. Lipa, P. Carruthers and B. Buschbeck, (submitted to Phys. Rev. D).

[8] UA1 Minimum Bias Collaboration, B. Buschbeck *et al.*, in *Proceedings of the XXII International Symposium on Multiparticle Dynamics, Santiago de Compostela, 1992*, edited by C. Pajares, (to be published).

[9] A.L. Fetter and J.D. Walecka, *Quantum theory of many particle systems*, McGraw Hill (1971).

[10] D. J. Scalapino and R. L. Sugar, Phys. Rev. D 8, 2284 (1973).

[11] I. Dremin and M.T. Nazirov, in 3.

[12] H.-Th. Elze and I. Sarcevic, Phys. Rev. Lett. 68, 1988 (1992).

[13] A. Stuart and J.K. Ord, *Kendall’s Advanced Theory of Statistics*, Vol.1, 5th edition, (Oxford University Press, New York 1987).

[14] P. Carruthers, H.C. Eggers, and I. Sarcevic, Phys. Lett. 254B, 258 (1991).

[15] NA35 Collaboration, I. Derado, in 3; P. Seyboth, in 4, p. 293C.

[16] K. Kadija and P. Seyboth, Phys. Lett. 287B, 363 (1992).

[17] H.C. Eggers, Ph.D. thesis, University of Arizona (1991).

[18] P. Carruthers, H.C. Eggers and I. Sarcevic, Phys. Rev. C 44, 1629 (1991).

[19] K.G. Wilson, Cornell preprint CLNS-131 (1970), published in *Proceedings of the 14th Scottish Universities Summer School in Physics*, 1973, edited by R. L. Crawford and R. Jennings, (Academic Press, 1974); R.P. Feynman, unpublished.
[20] J.C. Botke, D.J. Scalapino and R.L. Sugar, Phys. Rev. D 9, 813 (1974); Phys. Rev. D 10, 1604 (1974).

[21] L.D. Landau and E.M. Lifschitz, *Statistische Physik*, 6th edition, (Akademie Verlag, Berlin, 1984).

[22] P. Carruthers and I. Sarcevic, Phys. Lett. 189B, 442 (1987).

[23] R.C. Hwa and M.T. Nazirov, preprint OITS-490 (1992); R.C. Hwa and J. Pan, preprint OITS-496 (1992), (to be published).

[24] For a review, see D. Toussaint, invited talk presented at the Int. Symp. LATTICE-91, Tsukuba, Japan, (1991), Arizona preprint AZPH-TH/92-1 (to be published).

[25] F. Wilczek, preprint IASSNS-HEP-92/23 (1992), to be published in the Proceedings of the IFT Conference on Dark Matter, 1992; Int. J. Mod. Phys. A7, 3911 (1992).

[26] P. Carruthers and I. Sarcevic, Phys. Rev. Lett. 63, 1562 (1989).

[27] EMU01 Collaboration, R. J. Wilkes et al., in *Proceedings of the XXII International Cosmic Ray Conference*, vol. 4 p. 21, (Trinity College, Dublin, 1991).

[28] EMU01 Collaboration, M.I. Adamovich et al., Phys. Rev. Lett. 65, 412 (1990).

[29] KLM Collaboration, R. Holynski et al., Phys. Rev. Lett. 62, 733 (1989); Phys. Rev. C 40, 2449 (1990).

[30] EMU01 Collaboration, M.I. Adamovich et al., preprints UWSEA-PUB-92-07 and LUIP 9202 (to be published).

[31] H.C. Eggers et al., in Proceedings of the XXII International Symposium on Multiparticle Dynamics, Santiago de Compostela, 1992, edited by C. Pajares, (to be published).

[32] I. Derado and A. Kühmichel, private communication.

[33] J. Schukraft, preprint CERN-PPE/91-04, in Proceedings of the International Workshop on Quark Gluon Plasma Signatures, 1990 (to be published).

[34] NA35 Collaboration, H. Ströbele et al., Z. Phys. C38, 89 (1988).

[35] NA35 Collaboration, W. Heck et al., Z. Phys. C38, 19 (1988).

[36] A. Bialas and J. Seixas, Phys. Lett. 250B, 161 (1989); W. Ochs, Phys. Lett. 247B, 101 (1990).

[37] A.H. Mueller, Phys. Rev. D 4, 150 (1971).
Appendix:
Finding multiplicities from cumulants

When all factorial cumulants are known, it is usually possible to derive from them the multiplicity distribution $P_n$. This is done via the factorial moment generating function

$$Q(\lambda) = \sum_n (1 - \lambda)^n P_n,$$

which can also be expanded in terms of the unnormalized cumulants $f_q \equiv \int C_q$ as

$$Q(\lambda) = \exp \left[ \sum_q \frac{(-\lambda)^q}{q!} f_q \right].$$

For our model, only the first two cumulants are nonzero, $f_1 = \bar{n}$, the average total multiplicity, and $f_2 = K_2 \bar{n}^2$. With appropriate transformations and the identity $P_n = (1/n!)(-\partial/\partial \lambda)^n Q(\lambda)|_{\lambda=1}$, one can derive the multiplicity distribution in terms of Hermite polynomials

$$P_n = \frac{\bar{n}^n}{n!} (f_2/2)^{n/2} e^{-f_1 + f_2/2} H_n \left[i(f_2 - f_1)/\sqrt{2f_2}\right],$$

from which, for example, $P_1 = (f_1 - f_2)P_0$. Clearly $P_1$ becomes negative for $f_2 > f_1$, rendering the multiplicity distribution invalid. In terms of normalized factorial moments, this requires $K_2 < \bar{n}^{-1}$ for Eq. (27) to be valid. All odd $P_n$’s are similarly dependent on the sign of the factor $(f_1 - f_2)$, while the even ones are positive. For most heavy-ion data, the average multiplicity is very large, meaning that the multiplicity distribution (27) cannot be trusted for such cases.

Clearly, something is amiss. The resolution of this dilemma is found on returning to the identification of the normalized cumulant functions $k_q$ with the expectation values of the fluctuation fields, Eqs.(13)ff. In this identification, the difference between factorial cumulants and ordinary cumulants is neglected, which for the large multiplicities of heavy ion collisions is negligible. Mathematically, however, there is a difference, and this shows up in the above dilemma for the derivation of $P_n$ from the factorial cumulants.

If we work with ordinary cumulants, however, we can unambiguously derive the multiplicity distribution as follows: the characteristic function

$$\phi(t) = \sum_{n=0}^\infty e^{itn} P_n$$

is also the sum of ordinary cumulants $\kappa_q$ (e.g. $\kappa_1 = \langle n \rangle = \bar{n}$; $\kappa_2 = \langle n^2 \rangle - \langle n \rangle^2$),

$$\phi(t) = \exp \left[ \sum_{q=1}^{\infty} \frac{(it)^q \kappa_q}{q!} \right],$$

which for our truncated cumulant set is

$$\phi(t) = \exp \left[ it\bar{n} - t^2 \kappa_2/2 \right]$$
and since (up to a normalization constant) the multiplicity distribution is

\[ P_n = \int_{-\infty}^{\infty} e^{-itn\phi(t)} \, dt , \tag{31} \]

we get

\[ P_n \propto \exp \left[ -\frac{(n - \bar{n})^2}{2\kappa_2} \right] , \tag{32} \]

just an ordinary gaussian multiplicity distribution which is defined perfectly well.

Figure captions:

Figure 1: Third order cumulant \( K_3 \) as a function of the number of bins \( M \) for NA35 OAu data in \((y, \phi, p_\perp)\) \[16\]. Cumulants of higher order are also compatible with zero. This fact is confirmed in analyses in terms of other variables and different colliding nuclei.

Figure 2: One function \( k_2(\vec{x}_1, \vec{x}_2) \) determines all: Theoretical vertical cumulant moments \( K_2^v \) for various dimensions, for fixed parameters \( a = 2.0, \xi = 1.0 \), incorporating the experimental NA35 rapidity and \( p_\perp \) distributions for 200 A GeV O+Au.

Figure 3: Theoretical horizontal cumulant moments \( K_2^h \) for the same fixed parameters and NA35 distributions as in Fig. 3. The effect of the \( p_\perp \) distribution is clearly visible. Comparison with NA35 data requires conversion to horizontal factorial moments \( F_2^h \) and a fit of \( a \) and \( \xi \).