ON ATKIN AND SWINNERTON-DYER CONGRUENCE RELATIONS (2)

A.O.L. ATKIN, WEN-CHING WINNIE LI, AND LING LONG

ABSTRACT. In this paper we give an example of a noncongruence subgroup whose three-dimensional space of cusp forms of weight 3 has the following properties. For each of the four residue classes of odd primes modulo 8 there is a basis whose Fourier coefficients at infinity satisfy a three-term Atkin and Swinnerton-Dyer congruence relation, which is the $p$-adic analogue of the three-term recursion satisfied by the coefficients of classical Hecke eigenforms. We also show that there is an automorphic $L$-function over $\mathbb{Q}$ whose local factors agree with those of the $l$-adic Scholl representations attached to the space of noncongruence cusp forms.

1. Introduction

The cusp forms of weight $k$ for congruence subgroups of the classical modular group $SL_2(\mathbb{Z})$ form a vector space of finite dimension, and have a basis of forms which are simultaneous eigenfunctions of almost all the Hecke operators. In terms of the series expansions at the cusp infinity, we can write such a form as $\sum_{n\geq 1} a(n) w^n$, where $w = e^{2\pi i z/\mu}$ is the local uniformizer, and the Fourier coefficients $a(n)$ satisfy

$$a(np) - A(p)a(n) + B(p)a(n/p) = 0 \quad (1)$$

for all $n \geq 1$ and almost all primes $p$; as usual the number-theoretic function $a(x)$ is defined to be zero if $x$ is not a rational integer. The values of $a(n)$, $A(p)$, and $B(p)$ lie in an algebraic number field, and

$$|A(p)| \leq 2p^{(k-1)/2}$$

and

$$|B(p)| = p^{k-1} \quad (2)$$

for almost all $p$. If we normalize our basis by setting $a(1) = 1$, it is clear that we have $A(p) = a(p)$ for all relevant $p$; there is also a Dirichlet character $\chi$ such that $B(p) = \chi(p)p^{k-1}$.

It has been known for a long time that noncongruence subgroups of the modular group exist, and the finite-dimensional vector space of their cusp forms of weight $k$ will still have a basis of forms with series expansions and Fourier coefficients $a(n)$. However the classical theory of the Hecke operators collapses so that no three-term relation like (1) above can be expected to hold identically.

The research of the second author was supported in part by an NSA grant #MDA904-03-1-0069 and an NSF grant #DMS-0457574. Part of the research was done when she was visiting the National Center for Theoretical Sciences in Hsinchu, Taiwan. She would like to thank the Center for its support and hospitality. The third author was supported in part by an NSF-AWM mentoring travel grant for women. She would further thank the Pennsylvania State University and the Institut des Hautes Études Scientifiques for their hospitality.
The first systematic investigation of noncongruence forms was carried out by Atkin and Swinnerton-Dyer [ASD71]. First they evolved an effective technique for constructing forms and functions on noncongruence subgroups. Then for a small number of explicit examples they found an appropriate analogy of (1), without making any formal conjectures, and without proof except for the isolated case of weight 2 and dimension 1.

For almost all $p$ they replaced the equality in (1) by congruence modulo a suitable power of $p$ (which we call a 3-term ASD congruence here), with a $p$-adic basis of forms (so that the $a(n)$ were $p$-adic); the $A(p)$ were however apparently algebraic in general, although in different number fields for each $p$ (and of course could no longer be identified as $a(p)$), and the $B(p)$ were $p^{k-1}$ times a root of unity. Using $p$-adic Lie groups, Cartier [Car71] established the 3-term ASD congruence for noncongruence weight-2 forms.

A major advance was made by Scholl some fifteen years later [Sch85]. To the $d$-dimensional space of cusp forms of weight $k \geq 3$ of a noncongruence subgroup whose modular curve was defined over $\mathbb{Q}$, Scholl associated a family of $2d$-dimensional $l$-adic representations of the Galois group over $\mathbb{Q}$ such that the characteristic polynomials $P_p(T)$ of the Frobenius elements at almost all $p$ were degree $2d$ polynomials over $\mathbb{Z}$, independent of $l$, and all the cusp forms in this space with algebraic Fourier coefficients satisfied a $(2d + 1)$-term congruence relation given by $P_p(T)$. Moreover, all zeros of $P_p(T)$ had absolute value equal to $p^{(k-1)/2}$. In particular, he showed that the 3-term ASD congruence was valid for the case of a 1-dimensional space. When $d > 1$, while Scholl did not refine the $(2d+1)$-term congruence to establish the 3-term ASD congruences, he did point out in Sections 5.6 and 5.8 of [Sch85] that if the Frobenius matrices of the Scholl’s representation are diagonalizable (which are expected conjecturally) and the primes are mostly ordinary (which are expected for forms with no CM), then the 3-term ASD congruences will follow.

A related problem is the interpretation of the algebraic numbers $A(p)$ and $B(p)$. In the case $d = 1$, these are of course rational integers, and Scholl established a number of examples (in [Sch88] and [Sch04]) where $A(p)$ were the Fourier coefficients of a congruence modular form of weight $k$ and $B(p) = p^{k-1}$. This also holds for weight 3 noncongruence cusp forms associated to K3 surfaces over $\mathbb{Q}$, as shown in [LLY05].

More recently Li, Long, and Yang [LLY05] have given an example with dimension $d = 1$ and weight 3, where the eigenforms are the same for all $p$ (so that the $a(n)$ in (1) are in a number field and not merely $p$-adic), and the numbers $A(p)$ are the Fourier coefficients of two congruence weight-3 forms of character $\chi$, and $B(p) = \chi(p)p^2$. This is as much as could possibly be hoped for, and is clearly exceptional. Another example in the same vein is obtained by Fang et al in [F5].

In the present paper we give an example with dimension 3 which can be decomposed into a 1-dimensional space and a 2-dimensional space where the basis of the 1-dimensional space satisfies the 3-term ASD congruence with the $A(p)$ being coefficients of an explicit weight-3 cusp form. In the 2-dimensional space,

1) for each of the four residue classes of odd primes modulo 8 there is a basis (again over a number field and not merely $p$-adic) of two forms, for each of which the $a(n)$ enjoy the 3-term ASD congruence;
2) the corresponding $A(p)$ are again the Fourier coefficients of congruence modular forms, and $B(p) = \pm p^2$.

Our main result is recorded in Theorem 8.4. While not as simple as the example in [LLY05], it still exhibits far more structure than the general situation. It may be significant that both these examples arise from noncongruence groups which are normal subgroups of congruence subgroups of genus zero, with a cyclic factor group.

We shall show that there is an automorphic $L$-function over $\mathbb{Q}$ whose local factors agree with those of the $l$-adic Scholl representations attached to the space of weight-3 cusp forms of our noncongruence group. This exemplifies the Langlands philosophy, which predicts that $l$-adic representations that are motivic and ramified at finitely many places should relate to representations of $GL_n$. When this happens, we say that the $l$-adic representations are modular. It is worth pointing out that the modularity is proved by first restricting both Scholl’s $l$-adic representation and the $l$-adic representation attached to the automorphic form to the Galois group of $\mathbb{Q}(i)$ and then comparing them using Livné’s criterion [Liv87], since both residual representations have even trace. This is different from the method in [LLY05], where the Galois group was over $\mathbb{Q}$ and Serre’s criterion [Ser] was used. To the best of our knowledge, the modularity technique introduced by Wiles is not yet applicable when the base field is $\mathbb{Q}(i)$.

This paper is organized as follows. In Sec. 2 we introduce the normal subgroups $\Gamma_n$ of $\Gamma_1(5)$ and study properties of the operators involved. Using an explicit model of an elliptic surface over the modular curve of interest to us, we construct in Sec. 3 a family of $l$-adic representations of the Galois group over $\mathbb{Q}$, which are isomorphic to Scholl representations. Taking advantage of the concrete model, we compute the traces of the Frobenius elements, and obtain the determinants of the representations. The case of $\Gamma_2$, studied in [LLY05] using geometry, is reviewed in Sec. 4.

The rest of the paper concerns the group $\Gamma_4$, whose 3-dimensional space of weight-3 cusp forms is to be analyzed. As shown by Scholl [Sch85], there is a 7-term ASD congruence at $p$, obtained by comparing the $l$-adic theory and the $p$-adic theory, more precisely, from the agreement of the characteristic polynomials of the Frobenius element at $p$ in both theories. In order to obtain 3-term ASD congruences, we need to decompose the vector space of each side into three 2-dimensional invariant subspaces and prove the agreement of the characteristic polynomials on the 2-dimensional subspaces. This is achieved by taking eigenspaces of suitably chosen operators which commute with the action of the Frobenius at $p$. These operators, which play the role of the Hecke operators on congruence forms, are constructed using the symmetries of the elliptic surface.

The details are carried out as follows. In Sec. 5 we decompose each side into the direct sum of a 2-dimensional $(+)$-space and a 4-dimensional $(-)$-space. On the $l$-adic side, this gives the decomposition of the Scholl representation $\rho_l$ into the sum of $\rho_{l,+}$ and $\rho_{l,-}$. The $(+)$-spaces can be easily identified as arising from the group $\Gamma_2$, and hence the agreement of the characteristic polynomials. The operators used to further decompose the $(-)$-spaces into the sum of two 2-dimensional subspaces are introduced in Sec. 6. On the $(-)$-space of the $l$-adic side, these operators yield, for each odd prime $p \neq l$, a factorization of the characteristic polynomial of the Frobenius element at $p$ as a product of two degree-2 polynomials of a specific type. Aided by Magma, we are able to obtain these factors explicitly for $p = 3, 7, 13$. 
By comparing the actions of these operators on both \((-\cdot)\)-spaces, we show in Sec.
7 that there is a basis of the \((-\cdot)\)-space satisfying the 3-term ASD congruence at
\(p\) given by the two degree-2 factors of the characteristic polynomial at \(p\) obtained
in Sec. 6. Finally in Sec. 8 we identify a cusp form \(f'\) of weight 3 and an idele
class character \(\chi\) of \(\mathbb{Q}(i)\) of order 4, which yields a cusp form \(h(\chi)\) for \(GL_2(\mathbb{Q})\), and
prove that the \(l\)-adic representation attached to \(f' \times h(\chi)\) and \(\rho_{l,-}\) have the same
semi-simplification. This establishes the modularity of \(\rho_{l,-}\) and gives explicit ASD
congruences at the same time.

2. The groups and operators

2.1. The groups \(\Gamma_n\). Listed below are the cusps of \(\pm \Gamma^1(5)\) and a choice of generators
of their stabilizers in \(SL_2(\mathbb{Z})\):

| cusps of \(\pm \Gamma^1(5)\) | cusp widths | generators of stabilizers |
|-----------------------------|-------------|---------------------------|
| \(\infty\)                  | 5           | \(\zeta\) = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix} |
| 0                           | 1           | \(\theta\) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} |
| \(-2\)                      | 5           | \(A\zeta A^{-1}\) = \begin{pmatrix} 11 & 20 \\ -5 & -9 \end{pmatrix} |
| \(-\frac{5}{2}\)           | 1           | \(A\theta A^{-1}\) = \begin{pmatrix} 11 & 25 \\ -4 & -9 \end{pmatrix} |

Here \(A = \begin{pmatrix} -2 & -5 \\ 1 & 2 \end{pmatrix}\). Therefore the group \(\Gamma^1(5)\) is generated by \(\zeta, \theta, A\zeta A^{-1}, A\theta A^{-1}\) with the relation

\((A\theta A^{-1})(A\zeta A^{-1})\theta \zeta = I_2\).

Let \(E_1(z)\) and \(E_2(z)\) be two weight-3 Eisenstein series of \(\Gamma^1(5)\) with integral
coefficients, vanishing at all cusps except at \(\infty\) and \(-2\), respectively; these series are
constructed explicitly in [LLY05]. The function \(t = \frac{E_1(z)}{E_2(z)}\) generates the field
of meromorphic functions on \(X^1(5)\), the modular curve associated to \(\Gamma^1(5)\).

For \(n \geq 2\), let \(\varphi_n\) denote a homomorphism from \(\Gamma^1(5)\) to \(\mathbb{C}^\times\) which sends
\(\zeta, A\zeta A^{-1}\) to \(\omega_n, \omega^{-1}_n\), respectively, and the other generators to 1, where \(\omega_n = e^{2\pi i/n}\). Denote the kernel of \(\varphi_n\) by \(\Gamma_n\); it is an index-\(n\) subgroup of \(\Gamma^1(5)\), generated
by

\(\zeta^n, A\zeta^n A^{-1}, \zeta^i \theta \zeta^{-i}, \zeta^i A\theta A^{-1} \zeta^{-i}\), for \(i = 0, \cdots, n - 1\),
subject to the relation

\((A\theta A^{-1})(A\zeta^n A^{-1})\theta(\zeta A\theta A^{-1} \zeta^{-1})(\zeta \theta \zeta^{-1})\cdots
(\zeta^{n-1} A\theta A^{-1} \zeta^{-(n-1)}) (\zeta^{n-1} \theta \zeta^{-(n-1)}) \zeta^n = I_2\).

The modular curve of \(\Gamma_n\), denoted by \(X_n\), is of genus 0. It is an \(n\)-fold cover
of the modular curve of \(\Gamma^1(5)\), ramified only at the cusps \(\infty\) and \(-2\), both with
ramification degree \(n\). A generator for the field of meromorphic functions of \(X_n\) is

\(t_n = \sqrt[n]{\frac{E_1}{E_2}}\). \hspace{1cm} (3)

A cusp form is called \(n\)-integral if its Fourier coefficients are algebraic and integral
outside the places dividing \(n\).
Proposition 2.1.  
(1) $\Gamma_n$ is a normal subgroup of $\Gamma^1(5)$ with the quotient $\Gamma^1(5)/\Gamma_n$ cyclic of order $n$ generated by $\zeta$. The characters of this quotient are $\varphi_n^j$, $1 \leq j \leq n$, where $\varphi_n$ is as above.
(2) $\dim S_3(\Gamma_n) = n - 1$.
(3) An $n$-integral basis of $S_3(\Gamma_n)$ consists of $\sqrt{E_1^{n-j}(z)E_2^j(z)}$, $1 \leq j \leq n - 1$.

Proof. Since $\Gamma_n$ is the kernel of the homomorphism $\varphi_n$, it is a normal subgroup of $\Gamma^1(5)$. It follows from the definition of $\varphi_n$ that the quotient $\Gamma^1(5)/\Gamma_n$ is the cyclic group generated by $\zeta$, and hence the characters are as described in (1).

The group $\Gamma_n$ has no elliptic elements and has $2 + 2n$ cusps, of which two have width $5n$, and $2n$ have width 1. Moreover $-I \notin \Gamma_n$. Using the dimension formula in [Shi71] Theorem 2.25, one concludes $\dim S_3(\Gamma_n) = n - 1$.

It is easy to verify that $\sqrt{E_1^{n-j}(z)E_2^j(z)}$, $1 \leq j \leq n - 1$, are all weight-3 cusp forms on $\Gamma_n$ and are linearly independent, as seen from their Fourier expansions, which we normalize as

$$\sqrt{E_1^{n-j}(z)E_2^j(z)} = e^{(2\pi i z)j/5n}(1 + \sum_{r \geq 1} c_r e^{2\pi i rz/5}),$$

where the coefficients $c_r$ are $n$-integral, since $E_1, E_2 \in \mathbb{Z}[[e^{2\pi iz/5}]]$. \qed

2.2. Operators. The matrices $A = \begin{pmatrix} -2 & -5 \\ 1 & 2 \end{pmatrix}$ and $\zeta = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ normalize $\Gamma^1(5)$ and all the $\Gamma_n$. Note that $A$ is the diamond operator (2). The elliptic modular surface $E_{\Gamma^1(5)}$ over $\Gamma^1(5)$ in the sense of Shioda [Shi72] has a model given by

$$y^2 = t(x^3 - \frac{1 + 12t + 14t^2 - 12t^3 + t^4}{48t^2}x + \frac{1 + 18t + 75t^2 + 75t^4 - 18t^5 + t^6}{864t^3}), \quad (4)$$

where $t$ is the parameter for the genus zero modular curve $X^1(5)$. The action of $A$ on this elliptic surface is

$$A: (x, y, t) \mapsto (-x, y/t, -1/t).$$

It is a morphism of order 4 defined over $\mathbb{Q}$. The operator $\zeta$ acts trivially on $E_{\Gamma^1(5)}$. The actions of $A$ and $\zeta$ on the modular curve $X_n$ for $\Gamma_n$ are described in terms of the function $t_n$ as follows:

$$A(t_n) = \frac{\omega_{2n}}{t_n}, \quad \zeta(t_n) = \omega_{2n}^{-2}, \quad (5)$$

where $\omega_{2n} = e^{2\pi i/2n}$ is a primitive $2n$-th root of unity. For even $n$, an equation of the elliptic modular surface $E_n$-th root of unity.
3. \( l \)-adic Representations

3.1. Two \( l \)-adic representations. For any field \( K \), write \( G_K \) for \( \text{Gal}(\overline{K}/K) \). As in [LLY05], for a prime \( l \), we introduce a family of \( l \)-adic representations \( \rho_{n,l}^* \) of \( G_{\mathbb{Q}} \) attached to the space of weight 3 cusp forms of \( \Gamma_n \) by using the explicit model \( (6) \) of the elliptic surface \( \mathcal{E}_n \) over \( X_n \) as follows.

Denote by \( X_0^2 \) the modular curve \( X_n \) with the cusps removed. Let \( h : \mathcal{E}_n \rightarrow X_n \) be the natural elliptic fibration endowed with \( \mathcal{E}_n \). Let \( h^0 : \mathcal{E}_n \rightarrow X_0^0 \) be its restriction to \( X_0^0 \), which is a smooth map. For any prime \( l \), one obtains a sheaf

\[
\mathcal{F}_l = R^1h^0\mathbb{Q}_l
\]

on \( X_0^0 \). Here \( \mathbb{Q}_l \) is the constant sheaf on the elliptic surface \( \mathcal{E}_n \) and \( R^1 \) is the derived functor. The inclusion map \( i : X_0^0 \rightarrow X_n \) then transports the sheaf on \( X_0^0 \) to a sheaf \( i_*\mathcal{F}_l \) on \( X_n \). The action of \( G_{\mathbb{Q}} \) on the \( \mathbb{Q}_l \)-space

\[
W_{n,l} = H^1(X_n \otimes \overline{\mathbb{Q}}, i_*\mathcal{F}_l)
\]

defines an \( l \)-adic representation, denoted by \( \rho_{n,l}^* \), of the Galois group \( G_{\mathbb{Q}} \).

Scholl’s representation \( \rho_{n,l} \) of \( G_{\mathbb{Q}} \) attached to \( S_3(\Gamma_n) \) in [Sch85] is constructed by first choosing an auxiliary modular curve \( X(N) \) and pulling the universal elliptic curve on \( X(N) \) to the fibre product of \( X(N) \) with \( X_n \), going through the same process as above with \( X_n \) replaced by its fibre product with \( X(N) \), and at the end taking the \( SL(\mu_n \times \mathbb{Z}/N) \)-invariant part of the cohomology space to be the representation space of \( \rho_{n,l} \) (cf. (2.1.2) of [Sch85]). Since \( [1] \) is an algebraic model for the universal elliptic curve \( \mathcal{E}_{\Gamma^1(5)} \) of \( \Gamma^1(5) \) on \( X_{\Gamma^1(5)} \), we may use \( X_{\Gamma^1(5)} \) as the auxiliary curve in Scholl’s construction. Then \( \mathcal{E}_n \) is the pullback of \( \mathcal{E}_{\Gamma^1(5)} \) to the fibre product of \( X_n \) with \( X_{\Gamma^1(5)} \), which is isomorphic to \( X_n \). Therefore the two representations are isomorphic; we shall remove * from now on.

When \( n = 2 \), it is shown in [SB85] that the K3 surface

\[
\mathcal{G} : t_2^2 = uv(1-u)(v+1)(u-v)
\]

has an elliptic fibration

\[
y^2 + (1-t_2^2)xy - t_2^2y = x^3 - t_2^2s^2
\]

over \( X_2 \). Furthermore, the fibration defined by \([1]\) is birationally isomorphic (over \( \mathbb{Q} \)) to the fibration over \( \mathcal{E}_2 \) defined by \([5]\). The space \( W_{2,l} \), constructed above can be embedded into \( H^2(\mathcal{G}, \mathbb{Q}_l) \), which shows that the representation \( \rho_{2,l} \) is unramified at 5 if \( l \neq 5 \). Since \( \mathcal{G} \) is obtained as a double cover of \( \mathbb{P}^2 \) branched over six lines with branch locus shown in [SB85] Fig. 1], we see that the zero and pole locus of \( t_2 = 0 \) is a divisor on \([5]\) with normal crossings, which extends to a divisor with relative normal crossing over \( \mathbb{Z}_5 \). When \( n = 4 \), \( \mathcal{E}_4 \) is obtained from \( \mathcal{E}_2 \) by replacing \( t_2 \) by \( t_2^4 \). Therefore \( \rho_{4,l} \) is also unramified at 5.

3.2. Traces. Let \( q = p^r \) where \( r \) is a positive integer. Denote by \( \text{Frob}_p \) the action of the Frobenius element at \( p \) on an elliptic surface.

The surface \( \mathcal{E}_{\Gamma^1(5)} \) is a rational surface with vanishing \( H^2(\mathcal{E}_{\Gamma^1(5)}, \mathbb{Z}) \) (i.e., \( S_3(\Gamma^1(5)) = 0 \)). Hence \( \dim W_{1,l} = 0 \). So for all odd prime powers \( p^r \) we have

\[
\text{Tr} \rho_{1,l}(\text{Frob}_{p^r}) = 0.
\]
We proceed to compute $\text{Tr}_{p_n,l}(\text{Frob}_q)$ for $n = 2, 4$. The method is the same as that used in Section 5 of [LLY05]. By the Lefschetz fixed point theorem,

$$\text{Tr}_{p_n,l}(\text{Frob}_q) = - \sum_{t \in X_n(F_q)} \text{Tr}_n t,$$

where $\text{Tr}_n t$ is the trace of the $\text{Frob}_q$ restricted to the stalk at $t$ of $i_*\mathcal{F}_l$. Similarly,

$$\text{Tr}_{p_1,l}(\text{Frob}_q) = - \sum_{t \in X^1(5)(F_q)} \text{Tr} t.$$

Recall that all the modular curves $X_n$ have genus zero, hence $X_n(F_q) = \mathbb{P}^1(F_q)$ can be identified as $F_q \cup \{\infty\}$. The curve $X_n$ is an $n$-fold cover of $X^1(5)$ and so is $\mathcal{E}_n$ over $\mathcal{E}_{1,5}$ under the map $(x, y, t_n) \mapsto (x, y, t_n^2)$. As such, the fibre over $\infty$ of $X_n(F_q)$ is the same as the fibre over $\infty$ of $X^1(5)(F_q)$ since $\mathcal{E}_{1,5}$ is semistable, and hence has the same trace. Moreover, $\text{Tr}_n t_n$ at $t_n \in X_n(F_q)$ is equal to $\text{Tr} t_n^2$ at $t_n^2 \in X^1(5)(F_q)$. The operator $A$ on $X^1(5)$ is $\mathbb{Q}$-rational; it induces a bijection between the rational points on the fibre at $t \neq 0, \infty$ and those at $-1/t$.

**Lemma 3.1.** When $n = 2, 4$, for all prime powers $q = p^r$ with $q \equiv 3 \mod 4$,

$$\text{Tr}_{p_n,l}(\text{Frob}_q) = 0.$$

**Proof.** Note that $-1$ is not a quadratic residue in $F_q$. So $A$ induces a bijection between the $F_q$-rational points on the fibres at the points of $X^1(5)(F_q)$ parametrized by the quadratic residues of $F_q$ and those parameterized by the non-residues. Hence

$$\text{Tr}_{p_2,l}(\text{Frob}_q) = - \sum_{t_2 \in X_2(F_q)} \text{Tr}_2 t_2 = - \sum_{t \in X^1(5)(F_q)} \text{Tr} t^2$$

$$= - \sum_{t \in X^1(5)(F_q)} \text{Tr} t = \text{Tr}_{p_1,l}(\text{Frob}_q) = 0.$$ 

When $n = 4$, note that $(\mathbb{F}_q^\times)^4 = (\mathbb{F}_q^\times)^2$ since $4 \nmid q - 1$. So

$$\text{Tr}_{p_4,l}(\text{Frob}_q) = - \sum_{t_4 \in X_4(F_q)} \text{Tr}_4 t_4 = - \sum_{t_2 \in X_2(F_q)} \text{Tr}_2 t_2 = \text{Tr}_{p_2,l}(\text{Frob}_q) = 0.$$ 

□

**Lemma 3.2.** For all prime powers $q = p^r$ with $q \equiv 5 \mod 8$,

$$\text{Tr}_{p_4,l}(\text{Frob}_q) = \text{Tr}_{p_2,l}(\text{Frob}_q).$$

**Proof.** The operator $A$ induces a bijection between $F_q$-rational points on the fibres at the points of $X^1(5)(F_q)$ parametrized by $(\mathbb{F}_q^\times)^4$ and those parameterized by $-(\mathbb{F}_q^\times)^4$. Observe that $(\mathbb{F}_q^\times)^2 = (\mathbb{F}_q^\times)^4 \cup -(\mathbb{F}_q^\times)^4$ when $q \equiv 5 \mod 8$ since $-1$ is a quadratic residue, but not a 4-th power residue. Therefore

$$\text{Tr}_{p_4,l}(\text{Frob}_q) = - \sum_{t \in X^1(5)(F_q)} \text{Tr} t^4 = - \sum_{t \in X^1(5)(F_q)} \text{Tr} t^2 = \text{Tr}_{p_2,l}(\text{Frob}_q).$$

□

**Lemma 3.3.** For all prime powers $q = p^r$ with $q \equiv 1 \mod 8$,

$$\text{Tr}_{p_4,l}(\text{Frob}_q) - \text{Tr}_{p_2,l}(\text{Frob}_q) \equiv 0 \mod 4.$$
Proof. In this case, the field \( \mathbb{F}_q \) contains 8th roots of unity. Denote by \( i \) an element in \( \mathbb{F}_q \) of order 4. The desired statement is equivalent to

\[
\sum_{t \in X^1(5)(\mathbb{F}_q), \ t \neq 0, \infty} (\text{Tr} t^2 - \text{Tr} t^4) \equiv \sum_{t \in X^1(5)(\mathbb{F}_q), \ t \neq 0, \infty} (\text{Tr} t^2) \equiv 0 \mod 4.
\]

Owing to the symmetry given by \( A \), we have \( \text{Tr} t^2 = \text{Tr} (-t)^2 = \text{Tr}(-1/t^2) \). This implies that four distinct values of \( t \in \mathbb{F}_q^\times \) give rise to the same trace, except when \( t^2 = -1/t^2 \), that is, \( t^4 = -1 \). Thus the statement is reduced to

\[
\sum_{t \in X^1(5)(\mathbb{F}_q), \ t^4 = -1} \text{Tr} t^2 \equiv 0 \mod 4,
\]

which is equivalent to \( \text{Tr} i - \text{Tr}(-i) \) being even. In terms of the model for the elliptic surface over \( X^1(5) \), this means checking the difference of the number of \( \mathbb{F}_q \)-rational points on \( Y^2 = X^3 + \frac{13+2i}{4}X \) and \( Y^2 = X^3 + \frac{1-2i}{4}X \), which is obviously even. \( \square \)

By running a Magma program calculating the traces of \( \text{Frob}_p \) on each fibre for small values of \( r \) and \( p = 3, 7, 13, 17 \), we obtain the following characteristic polynomials, independent of \( l \neq p \):

| \( p \) | char. poly. for \( \rho_{2,l}(\text{Frob}_p) \) | char. poly. for \( \rho_{4,l}(\text{Frob}_p) \) |
|---|---|
| 3 | \( x^2 - 3^2 \) | \( (x^2 - 3^2)(x^4 - 10x^2 + 3^4) \) |
| 7 | \( x^2 - 7^2 \) | \( (x^2 - 7^2)(x^4 + 30x^2 + 7^4) \) |
| 13 | \( x^2 - 10x + 13^2 \) | \( (x^2 - 10x + 13^2)(x^4 + 62x^2 + 13^4) \) |
| 17 | \( x^2 + 30x + 17^2 \) | \( (x^2 + 30x + 17^2)(x^4 - 10x + 17^4)^2 \) |

(\text{Table 1})

3.3. Determinants.

Lemma 3.4. Let \( \sigma_1, \sigma_2 : G_{\mathbb{Q}} \to \mathbb{Z}_2^\times \) be two characters unramified outside 2. If they agree on \( \text{Frob}_p \) for \( p = 3 \) and \( p = 13 \), then they are equal.

Proof. We follow the same argument as in the proof of lemma 5.2 in \([LLY05]\). Notice that the only quadratic extensions of \( \mathbb{Q} \) which are unramified outside 2 are \( \mathbb{Q}(i), \mathbb{Q}(\sqrt{2}), \) and \( \mathbb{Q}(\sqrt{-2}) \), and \( p = 3 \) is inert in the first two fields, while \( p = 13 \) is inert in the third. Hence the assumption implies that \( \sigma_1 = \sigma_2 \). \( \square \)

Now apply the above lemma to the determinants of \( \rho_{2,l}(\text{Frob}_p) \) and \( \rho_{4,l}(\text{Frob}_p) \) for \( p = 3, 13 \) which can be read off from Table 1. Since the determinant is independent of the choice of \( l \), we conclude

**Corollary 3.5.** For all odd primes \( p \neq l \), we have

\[
\det(\rho_{2,l}(\text{Frob}_p)) = \chi_{-4}(p)^2 \quad \text{and} \quad \det(\rho_{4,l}(\text{Frob}_p)) = \chi_{-4}(p)p^6,
\]

where \( \chi_{-4}(p) = \left( \frac{-4}{p} \right) \) is the character attached to the extension \( \mathbb{Q}(i) \).
4. The case $n = 2$

The space $S_2(\Gamma_2)$ is 1-dimensional, spanned by $h_2 = \sqrt{E_1 E_2}$. It was shown in [Sch85] (corresponding to the $A(2)$ surface) and [LLY05] that Scholl's $l$-adic representation $\rho_{2,l}$ is modular, arising from a weight-3 newform $g_2 = \eta(4z)^6$ with complex multiplication, resulting from the fact that $E_2$ is a K3 surface with Picard number 20. Here $\eta(z)$ is the Dedekind eta function. It then follows from Scholl’s theorem that $h_2$ satisfies a 3-term ASD congruence with $A(p)$ and $B(p)$ coming from $g_2$ since $\rho_{2,l}$ has degree 2.

5. Decomposition of spaces for the case $n = 4$

5.1. Weight-3 modular forms of $\Gamma_4$. Note that $\Gamma_4$ is an index-2 subgroup of $\Gamma_2$ with $t_4 = \sqrt{E_1 E_2}$. We know $S_3(\Gamma_4) = \{h_1, h_2, h_3\}$, where

$$h_1 = \frac{E_1}{t_4}, \quad h_2 = \sqrt{E_1 E_2}, \quad h_3 = E_2 \cdot t_4.$$ 

5.2. Actions. The operators $A$ and $\zeta$ defined in section 2.2 induce actions on the cohomology space $W_{4,l}$, denoted by $A^*$ and $\zeta^*$ respectively, and on the space of cusp forms $S_3(\Gamma_4)$, again denoted by $A$ and $\zeta$. We examine these actions.

As discussed before, 

$$A(t_4) = t_4 \big|_A = \frac{\omega_8}{t_4}.$$ 

Therefore, the action of $A$ is defined over $\mathbb{Q}(\omega_8) = \mathbb{Q}(i, \sqrt{2})$. It acts on $h_1, h_2, h_3$ as follows:

$$A \cdot h_j = h_j \big|_A = \omega_8^{4-j} \cdot h_{4-j}, \quad \text{for } j = 1, 2, 3. \quad (10)$$

On the surface $E_4$, $A$ induces the map given by

$$A : (x, y, t_4) \mapsto (-x, iy, \frac{\omega_8}{t_4}).$$

It follows that $A^2 : (x, y, t_4) \mapsto (x, -y, t_4)$, which maps every point $P = (x, y)$ on the fibre at $t_4$ to its additive inverse $-P = (x, -y)$. Since $A^2$ is the identity on $X_n$ and is $-1$ on the sheaf $\mathcal{F}_1$, the action of $(A^*)^2$ on $W_{4,l}$ is multiplication by $-1$.

The map $\zeta : t_4 \mapsto \omega_8^{1/2} \cdot t_4$ induces an action on $h_j$ as follows:

$$\zeta \cdot h_j = \omega_8^{2j} \cdot h_j \quad \text{for } j = 1, 2, 3. \quad (11)$$

On the surface $E_4$, it acts via

$$\zeta : (x, y, t_4) \mapsto (x, y, \omega_8^{1/2} \cdot t_4).$$

5.3. Decomposition of spaces. With 4 and $l$ fixed, write $W$ for $W_{4,l}$. Following Scholl [Sch85], consider the associated $p$-adic Scholl space $V$ which contains $S_3(\Gamma_4, \mathbb{Q}_p)$ as a subspace and $S_3(\Gamma_4, \mathbb{Q}_p) \vee$ as a quotient. The map $\zeta^2$ sends $(x, y, t_4)$ to $(x, y, -t_4)$, which is defined over $\mathbb{Q}$ and of order 2 on $V, W$. Denote by $V_{\pm}$ the eigenspaces of $\zeta^2$ on $V$ with eigenvalues $\pm 1$. It is easy to verify that $V_-$, which contains $h_1$ and $h_3$, is a 4-dimensional vector space over $\mathbb{Q}_p$. The space $V_+$, which contains $h_2$, is fixed by $\zeta^2$. It is the $p$-adic space attached to $S_3(\Gamma_2)$. The map $A$ acts on $V$ with $A^2$ being multiplication by $-1$.

On the $l$-adic side, $W$ decomposes similarly into a 2-dimensional $\mathbb{Q}_l$ space $W_+$ and a 4-dimensional space $W_-$. Likewise, $(\zeta^*)^2$ acts on $W_{\pm}$ as multiplication by...
We denote by $\rho_{\pm,l}$, or $\rho_\pm$ if there is no ambiguity, the representation $\rho_{4,l}$ of $G_\mathbb{Q}$ restricted to $W_\pm$.

Corollary 3.5 implies

**Corollary 5.1.** For all odd $p \neq l$, we have

$$\det \rho_+(\text{Frob}_p) = \left(\frac{-4}{p}\right)p^2 \quad \text{and} \quad \det \rho_-(\text{Frob}_p) = p^4. \quad (12)$$

**Lemma 5.2.** $\zeta^* A^* \zeta = A$ on $W$ and $\zeta A \zeta = A$ on $V$.

**Proof.** On $W$ this follows from their actions on the surface level:

$$\zeta A \zeta(x, y, t_4) = \zeta A(x, y, -it_4) = \zeta(-x, iy, \frac{\omega_8}{-it_4}) = (-x, iy, \frac{\omega_8}{t_4}) = A(x, y, t_4).$$

On $V$ this follows from the actions of $\zeta$ and $A$ on $S_3(\Gamma_4)$.

Hence $\zeta^*$ and $A^*$ (resp. $\zeta$ and $A$) generate a copy of the quaternion group $H_8$ acting on the space $W_-$ (resp. $V_-$).

As we shall be comparing the actions of the Frobenius element at $p$ on $V$ and $W$, we write $F$ for its action on $V$ and $F_p$ for its action on $W$, keeping in mind that on the $p$-adic side there is only one Frobenius action, while on the $l$-adic side there are plenty. In general, for an operator $B$ acting on a finite-dimensional vector space $X$, denote by $\text{Char}(X, B)(T)$ the characteristic polynomial of $B$ in variable $T$. As shown in [Sch85], the congruences at $p$ will follow from equality of the characteristic polynomials of $F$ and $F_p$ for the relevant subspaces. Our argument is a refinement and generalization of that used in [LLY05].

To begin with, Scholl’s theorem in [Sch85] gives

$$\text{Char}(W, F_p)(T) = \text{Char}(V, F)(T).$$

Applying the argument following [Sch85 4.4] to the matrix $\zeta^2$, we obtain

$$\text{Char}(W_+, F_p)(T) = \text{Char}(V_+, F)(T) \in \mathbb{Z}[T], \quad (13)$$

$$\text{Char}(W_-, F_p)(T) = \text{Char}(V_-, F)(T) \in \mathbb{Z}[T]. \quad (14)$$

Since the congruences resulting from the 1-eigenspaces are for $h_2 \in S_3(\Gamma_2)$, whose congruence relations have been established in [LLY05] and reviewed in the previous section, we shall concentrate on the $(-1)$-eigenspaces $V_-$ and $W_-$.

Under the action of $\zeta$, the space $V_-$ further decomposes into eigenspaces $V_-, \pm i$ with eigenvalues $\pm i$ respectively. In particular, $h_1 \in V_-, i$ and $h_3 \in V_-, -i$.

**Corollary 5.3.** The constant terms of $\text{Char}(V_-, F)(T)$ and $\text{Char}(W_-, F_p)(T)$ are $p^4$.

6. **Factorizing local $L$-factors**

In this section we shall confine ourselves to the spaces $V_-$ and $W_-$. Recall that $\zeta^2$, $A^2$ on $V_-$ and $(\zeta^*)^2$, $(A^*)^2$ on $W_-$ all act as multiplication by $-1$. 


6.1. **Operators on** $V_-$ **and** $W_-$. In addition to $\zeta$ and $A$, consider also the operators

$$B_{-2} := (1 + \zeta) A \quad \text{and} \quad B_2 := (1 - \zeta) A$$
onumber

on $V_-$. Likewise, we introduce the operators

$$B_{-2}^* := A^*(1 + \zeta^*) \quad \text{and} \quad B_2^* := A^*(1 - \zeta^*)$$
onumber

on $W_-$. It is straightforward to check that

$$(B_{-2})^2 = -2I = (B_2)^2 \quad \text{and} \quad (B_{-2}^*)^2 = -2I = (B_2^*)^2.$$  

Lemma 5.2 implies

$$B_{-2}B_2 = -B_2B_{-2} \quad \text{and} \quad B_{-2}^*B_2^* = -B_2^*B_{-2}^*.$$  

**Lemma 6.1.** Let $p$ be an odd prime not equal to $l$. On $W_-$ we have

\begin{enumerate}
\item \[\zeta^* F_p = F_p(\zeta^*)^p = \left(\frac{-1}{p}\right) F_p \zeta^*, \quad A^* F_p = F_p A^*(\zeta^*)^{(p-1)/2},\]
\item \[B_{-2}^* F_p = \left(\frac{\pm 2}{p}\right) F_p B_{-2}^* .\]
\end{enumerate}

**Proof.** (1) Since $F_p$ on $W_-$ is the geometric Frobenius in the Galois group, the action can be computed as the pullback via Frobenius in the reduction mod $p$. Hence we examine the actions modulo $p$ on the elliptic surface $E_4$:

$$\text{Frob}_p \zeta(x, y, t_4) = \text{Frob}_p(x, y, -\omega_4 t_4) = (x^p, y^p, (-\omega_4)^p t_4^p) = \zeta^p \text{Frob}_p(x, y, t_4),$$

$$\text{Frob}_p A(x, y, t_4) = \text{Frob}_p(-x, iy, \frac{\omega_8}{t_4}) = (-x^p, iy^p, \frac{\omega_8^p}{t_4^p}),$$

$$\zeta^{(p-1)/2} A \text{Frob}_p(x, y, t_4) = \zeta^{(p-1)/2} A(x^p, y^p, t_4^p) = \zeta^{(p-1)/2}(-x^p, iy^p, \frac{\omega_8^p}{t_4^p})$$

$$= \zeta^{(p-1)/2}(-x^p, iy^p, \omega_4^{1-p} \frac{\omega_8^p}{t_4^p})$$

$$= (-x^p, iy^p, \omega_4^{1-p} \frac{\omega_8^p}{t_4^p}).$$

When $p \equiv 1 \mod 4$, this gives $\text{Frob}_p A = \zeta^{(p-1)/2} A \text{Frob}_p$. When $p \equiv 3 \mod 4$, we have $\text{Frob}_p A = A^2 \zeta^2 (\zeta^{(p-1)/2} A \text{Frob}_p)$. On the cohomology space, the order of the operators are reversed. Since the actions of $(A^*)^2$ and $(\zeta^*)^2$ on $W_-$ are both $-1$, we obtain the desired conclusion.

(2) By a straightforward computation using (1), we have

$$B_{-2}^* F_p = A^*(1 + \zeta^*) F_p = A^* F_p(1 + (\zeta^*)^p)$$

$$= F_p A^*(\zeta^*)^{(p-1)/2}(1 + (\zeta^*)^p)$$

$$= \left(\frac{-2}{p}\right) F_p A^*(1 + \zeta^*) = \left(\frac{-2}{p}\right) F_p B_{-2}^*.$$  

The other statement is proved in a similar way.
On the $p$-adic side, we have $\zeta h_j = \omega_i^j h_j$ and $A(h_j) = \omega_8^{4-j} h_{4-j}$. Thus $\zeta$ on $V_-$ is defined over $\mathbb{Q}_p$ whenever $\sqrt{-1} \in \mathbb{Q}_p$, that is, $p \equiv 1 \mod 4$, and $A$ on $V_-$ is defined over $\mathbb{Q}_p$ for $p \equiv 1 \mod 8$. Since $B_2(h_j) = (1 - \zeta)\omega_8^{4-j} h_{4-j} = (1 - \omega_4^{4-j})\omega_8^{4-j} h_{4-j} = (j - 2)\sqrt{2} h_{4-j}$, hence $B_2$ on $V_-$ is defined over $\mathbb{Q}_p$ whenever $\sqrt{2} \in \mathbb{Q}_p$, in particular, when $p \equiv 7 \mod 8$. Similarly, $B_{-2}$ on $V_-$ is defined over $\mathbb{Q}_p$ whenever $\sqrt{-2} \in \mathbb{Q}_p$, in particular, $p \equiv 3 \mod 8$. We record this discussion in

Lemma 6.2. On $V_-$ we have $\zeta F = F\zeta$ when $\sqrt{-1} \in \mathbb{Q}_p$, $AF = FA$ when $\omega_8 \in \mathbb{Q}_p$, $B_2 F = FB_2$ when $\sqrt{2} \in \mathbb{Q}_p$, and $B_{-2} F = FB_{-2}$ when $\sqrt{-2} \in \mathbb{Q}_p$.

6.2. Factorizing local $L$-factors. The aim of this subsection is to factor, for each odd prime $p$, the characteristic polynomial $\text{Char}(W_-, \mathbb{Q}_p)(T)$ as a product of two quadratic characteristic polynomials arising from a suitable restriction of $p_-$.

Proposition 6.3. Let $\delta \in \{-1, -2, 2\}$ and let $\sigma_1$ and $\sigma_2$ be two 1-dimensional representations of $G_{\mathbb{Q}(\sqrt{p})}$ over a totally ramified extension $F$ of $\mathbb{Q}_p$, unramified outside the place dividing 2. Then $\sigma_1 = \sigma_2$ if they agree at $\text{Frob}_v$ for $v$ dividing 3, 13 if $\delta = -1$ or $-2$, and for $v$ dividing 3, 7, 13 if $\delta = 2$.

Proof. Consider $\sigma = \sigma_1/\sigma_2 : G_{\mathbb{Q}(\sqrt{p})} \rightarrow F^\times$. Since $F^\times$ is a pro-$2$-group, so if $\sigma \neq 1$ then its fixed field contains a quadratic extension $K$ over $\mathbb{Q}(\sqrt{p})$ unramified away from 2.

For $\delta = -1$, the possible fields $K$ are $\mathbb{Q}(i, \sqrt{1 + i})$, $\mathbb{Q}(i, \sqrt{1 - i})$ in which $\pi = 3$ is inert, and $\mathbb{Q}(i, \sqrt{2})$ in which $\pi = 3 + 2i$ (dividing 13) is inert.

For $\delta = -2$, the possible fields $K$ are $\mathbb{Q}(\sqrt{-2}, i)$, $\mathbb{Q}(i\sqrt{-2})$ in which $\pi = 1 + \sqrt{-2}$ (dividing 3) is inert, and $\mathbb{Q}(\sqrt{-2})$ in which $\pi = 13$ is inert.

For $\delta = 2$, let $\varepsilon = \sqrt{2^2 - 1}$. The possibilities are

$$\mathbb{Q}(\sqrt{2}, \varepsilon, \sqrt{2}), \mathbb{Q}(\sqrt{2}, i\varepsilon, \sqrt{2}), \mathbb{Q}(\sqrt{2}, \varepsilon), \mathbb{Q}(\sqrt{2}, i\varepsilon)$$

in which $\pi = 3$ is inert, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(i\sqrt{3})$ in which $\pi = 13$ is inert, and $\mathbb{Q}(\sqrt{-2}, i)$ in which $\pi = 3 + \sqrt{2}$ (dividing 7) is inert.

We proceed to choose a quadratic character of $G_{\mathbb{Q}(\sqrt{p})}$ unramified outside the unique place $v$ dividing 2 which will be needed for our purpose. Denote by $\theta_3$ the quadratic character attached to the extension $\mathbb{Q}(i, \sqrt{2})$ over $\mathbb{Q}(\sqrt{3})$. When $\delta = -1$, the primes $p \equiv 3, 7 \mod 8$ are inert in $\mathbb{Q}(i)$ with residue field $\mathbb{F}_{p^2}$ containing a square root of 2, hence they split in $\mathbb{Q}(i, \sqrt{2})$. The primes $p \equiv 1 \mod 8$ split in $\mathbb{Q}(i)$ with residue field $\mathbb{F}_p$ containing a square root of 2, hence these places also split in $\mathbb{Q}(i, \sqrt{2})$. We have $\theta_{-1}(\text{Frob}_v) = 1$ at places $v$ dividing $p \equiv 1 \mod 8$. The primes $p \equiv 5 \mod 8$ split in $\mathbb{Q}(i)$ with residue field $\mathbb{F}_p$ in which 2 is not a square, thus $\theta_{-1}(\text{Frob}_v) = -1$ only at places $v$ above $p \equiv 5 \mod 8$. Similarly, $\theta_{-2}(\text{Frob}_v) = -1$ only at places $v$ dividing $p \equiv 3 \mod 8$ and $\theta_2(\text{Frob}_v) = -1$ only at places $v$ dividing $p \equiv 7 \mod 8$.

Write $B_{-1}^*$ for $\zeta$. Recall that $B_{-1}^*, B_{-2}^*$ and $B_2^*$ act on the 4-dimensional space $W_-$ over $\mathbb{Q}_i$, and satisfy $(B_{-1}^*)^2 = -I$, $(B_{-2}^*)^2 = (B_2^*)^2 = -2I$. The commuting
relations between these operators and the Frobenius elements described in Lemma 6.1 show that for $\delta = -1, 2, -2$, $B_\delta^t$ is defined over $\mathbb{Q}(\sqrt{\delta})$. For each $\delta$, choose a prime $l$ so that $\mathbb{Q}_l(B_\delta^t)$ is a quadratic extension of $\mathbb{Q}_l$. Regarding $W_-$ as a 2-dimensional vector space over $\mathbb{Q}_l(B_\delta^t)$, we may restrict the representation $\rho_-$ to $H_\delta := G_{\mathbb{Q}(\sqrt{\delta})}$, obtaining a 2-dimensional representation $\rho_{-\delta}$ of $H_\delta$ on the space $W_-^\prime$ over $\mathbb{Q}_l(B_\delta^t)$.

**Theorem 6.4.** The $L$-factor of the 4-dimensional representation $\rho_-$ at an odd prime $p \neq l$ is equal to the product of the $L$-factors of the 2-dimensional representation $\rho_{-\delta}$ over the places of $\mathbb{Q}(\sqrt{\delta})$ dividing $p$ for $\delta = -1, -2, 2$.

**Proof.** Extend scalars to $\mathbb{Q}_l$. Let $J_{-1} = B_{-1}^t$, $J_{-2} = B_{-2}^t/\sqrt{2}$, and $J_2 = B_2^t/\sqrt{2}$ as endomorphisms of $W_-$. By Lemmas 6.2 and 6.3, we have

(i) $J_2^2 = -I$ for $\delta = -1, 2, -2$, and $J_{-1}J_2 = J_{-2} = J_2J_{-1}$.

(ii) If $\varepsilon_\delta : G_{\mathbb{Q}} \rightarrow \{\pm 1\}$ are the characters corresponding to the fields $\mathbb{Q}(\sqrt{\delta})$, then $\rho_-(g)J_5 = \varepsilon_\delta(g)J_5\rho_-(g)$ for all $g \in G_{\mathbb{Q}}$ and $\delta = -1, 2, -2$.

So the $J_5$ generate a subalgebra of $\text{End}(W_-)$ isomorphic to $M_2(\mathbb{Q}_l)$. Hence one can find a basis of $W_-$ with respect to which the $J_5$ are represented by

\[
J_{-1} = \begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \quad J_{-2} = \begin{pmatrix} 0 & iI_2 \\ iI_2 & 0 \end{pmatrix}.
\] (15)

For $g \in H_\delta$, $\rho_-(g)$ commutes with $J_5$, hence is of the form

\[
\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix}, \quad \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}, \quad \begin{pmatrix} P & Q \\ Q & P \end{pmatrix},
\]

according as $\delta = -1, 2, -2$. Setting $N = \cap_\delta H_\delta = G_{\mathbb{Q}(i, \sqrt{2})}$, we get

\[
\rho_-(N) = \left\{ \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \right\}, \quad \rho_-(H_{-1} \setminus N) = \left\{ \begin{pmatrix} P & 0 \\ 0 & -P \end{pmatrix} \right\},
\]

\[
\rho_-(H_2 \setminus N) = \left\{ \begin{pmatrix} 0 & Q \\ -Q & 0 \end{pmatrix} \right\}, \quad \rho_-(H_{-2} \setminus N) = \left\{ \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} \right\}.
\]

Let $\sigma_{-1} : H_{-1} \rightarrow GL_2(\mathbb{Q}_l)$ be the representation mapping $g \in H_{-1}$ to the matrix $P$ in the expression of $\rho_-(g)$ above. Identifying the character $\theta_{-1}$ of $\mathbb{Q}(i, \sqrt{2})/\mathbb{Q}(i)$ discussed above with the character on $H_{-1}/N$, we have

\[
\rho_{-1}|_{H_{-1}} = \begin{pmatrix} \sigma_{-1} & 0 \\ 0 & \sigma_{-1} \otimes \theta_{-1} \end{pmatrix} = \text{Ind}_{H_{-1}}^G(\sigma_{-1})|_{H_{-1}}.
\]

This shows that $\sigma_{-1}$ is $\rho_{-1}$.

For $\delta = \pm 2$ we can choose a basis so that $J_{5\delta}$ is represented by $\begin{pmatrix} iI_2 & 0 \\ 0 & -iI_2 \end{pmatrix}$, $J_{-1}$ by $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ and the third matrix determined by property (i). A similar argument then shows that

\[
\rho_{-\delta}|_{H_\delta} = \begin{pmatrix} \rho_{-\delta} & 0 \\ 0 & \rho_{-\delta} \otimes \theta_\delta \end{pmatrix} = \text{Ind}_{H_{\delta}}^G(\rho_{-\delta})|_{H_\delta} \quad \text{for } \delta = 2, -2 \text{ (and } -1). \quad (16)
\]

Therefore the local $L$-factors attached to $\rho_-$ have the asserted property. \qed
6.3. The determinants of the restricted representations. Using Table 1 in Sec. 3.2 we obtain the following:

\[ \text{Char}(W_-, F_3)(T) = (T^2 - 2\sqrt{-2}T - 3^2)(T^2 + 2\sqrt{-2}T - 3^2), \]
\[ \text{Char}(W_-, F_7)(T) = (T^2 - 8\sqrt{-2}T - 7^2)(T^2 + 8\sqrt{-2}T - 7^2), \]
\[ \text{Char}(W_-, F_{13})(T) = (T^2 + 20iT - 13^2)(T^2 - 20iT - 13^2). \]

As this is the unique way to factor \( \text{Char}(W_-, F_p)(T) \) for \( p = 3, 7, 13 \) into a product of two degree two polynomials with opposite coefficients for \( T \), applying Theorem 6.4 we obtain the following table of the values of \( \det \rho_{-, \delta}(\text{Frob}_v) \):

|  \( \delta \)  | \( v \) above 3 | \( v \) above 7 | \( v \) above 13 |
|----|---|---|---|
| -1 | 3 \( ^4 \) | 7 \( ^4 \) | -13 \( ^2 \) |
| -2 | -3 \( ^2 \) | 7 \( ^4 \) | 13 \( ^4 \) |
|  2 | 3 \( ^4 \) | -7 \( ^2 \) | 13 \( ^4 \) |

Combined with Proposition 6.3 we conclude

Corollary 6.5. For \( \delta = -1, -2, 2 \), at a place \( v \) of \( \mathbb{Q}(\sqrt{\delta}) \) dividing an odd prime \( p \neq l \), we have

\[ \det \rho_{-, \delta}(\text{Frob}_v) = \theta_\delta(\text{Frob}_v)Nv^2, \]

where \( Nv \) is the norm of \( v \).

As remarked earlier, for each odd prime \( p \neq l \) and not \( \equiv 1 \mod 8 \), there is a quadratic extension \( \mathbb{Q}(\sqrt{\delta}) \) in which \( p \) splits and \( \theta_\delta(\text{Frob}_v) = -1 \) at places \( v \) dividing \( p \), hence we can combine Theorem 6.4 with the above corollary to give a more detailed description of the factorization of local factors of \( \rho_- \).

Corollary 6.6. For each odd prime \( p \neq l \), there is a constant \( a_p \), not depending on \( l \), such that \( \text{Char}(W_-, F_p)(T) = (T^2 - a_pT + p^2)^2 \) if \( p \equiv 1 \mod 8 \), and \( \text{Char}(W_-, F_p)(T) = (T^2 - a_pT - p^2)(T^2 + a_pT - p^2) \) otherwise.

7. ASD congruences for \( S_3(\Gamma_4) \)

A cusp form \( h(z) \) in \( S_3(\Gamma_4) \) with 2-integral Fourier coefficients \( a(n) \) is said to satisfy a 3-term Atkin-Swinnerton-Dyer congruence relation at an odd prime \( p \) if there exist algebraic integers \( A_p \) and \( B_p \) with \( |\sigma(A_p)| \leq 2p \) and \( |\sigma(B_p)| = p^2 \) for all embeddings \( \sigma \) so that for all \( n \geq 1 \),

\[ a(np) - A_pa(n) + B_pa(n/p) \equiv (pm)^2 \]

is integral at some place above \( p \). For brevity, we refer to this as \( h \) satisfying an ASD congruence at \( p \) given by \( T^2 - A_pT + B_p \).

In this section we shall prove the following two main results.

Theorem 7.1. For an odd prime \( p \neq l \), \( \text{Char}(W_-, F_p)(T) \) has the following factorization for some \( A_p \in \mathbb{Z} \).

1. If \( p \equiv 1 \mod 8 \), \( \text{Char}(W_-, F_p)(T) = (T^2 - A_pT + p^2)^2 \);
2. If \( p \equiv 5 \mod 8 \), \( \text{Char}(W_-, F_p)(T) = (T^2 - iA_pT - p^2)(T^2 + iA_pT - p^2) \);
3. If \( p \equiv 3 \) or \( 7 \mod 8 \), \( \text{Char}(W_-, F_p)(T) = (T^2 - \sqrt{-2}A_pT - p^2)(T^2 + \sqrt{-2}A_pT - p^2) \).


**Theorem 7.2** (ASD congruence for \( S_3(\Gamma_4) \)). The ASD congruence holds on the space \( S_3(\Gamma_4) = \langle h_1, h_2, h_3 \rangle \). More precisely, \( h_2 \) lies in \( S_3(\Gamma_2) \) and it satisfies the ASD congruence relations with the congruence form \( g_2(z) = \eta(4z)^6 \). For each odd prime \( p \), the subspace \( \langle h_1, h_3 \rangle \) has a basis depending on the residue of \( p \) modulo \( 8 \) satisfying a 3-term ASD congruence at \( p \) as follows.

1. If \( p \equiv 1 \mod 8 \), then both \( h_1 \) and \( h_3 \) satisfy the 3-term ASD congruence at \( p \) given by \( T^2 - A_p T + p^2 \);
2. If \( p \equiv 5 \mod 8 \), then \( h_1 \) (resp. \( h_3 \)) satisfies the 3-term ASD-congruence at \( p \) given by \( T^2 - iA_p T - p^2 \) (resp. \( T^2 + iA_p T - p^2 \));
3. If \( p \equiv \pm 3 \mod 8 \) (resp. \( p \equiv \pm 7 \mod 8 \)), then \( h_1 \pm h_3 \) (resp. \( h_1 \pm ih_3 \)) satisfy the 3-term ASD congruence at \( p \) given by \( T^2 \pm \sqrt{-2}A_p T - p^2 \), respectively.

The polynomials above are factors of \( \text{Char}(W_-, F_p)(T) \) as shown in Theorem 7.1.

**Corollary 7.3.** For all primes \( p \) we have

\[ \text{Char}(W_-, F_p)(T) \equiv T^4 + 1 \mod 2. \]

**Proof.** When \( p \equiv 3 \mod 4 \), the verification is straightforward. When \( p \equiv 1 \mod 8 \), \( \text{Tr}_{p-}(\text{Frob}_p) = 2A_p \equiv 0 \mod 4 \) by Lemma 6.3 hence \( A_p \) is even. When \( p \equiv 5 \mod 8 \), Lemma 6.3 implies \( \text{Tr}_{p-}(\text{Frob}_p^2) = 2(2p^2 - A_p^2) \equiv 0 \mod 4 \), thus \( A_p \) is even.

**7.1. Proof of Theorems 7.1 and 7.2.** The space \( S_3(\Gamma_4) \) is spanned by \( h_1, h_2, \) and \( h_3 \). We know that \( h_2 \) generates the space \( S_3(\Gamma_2) \) and it satisfies ASD congruence relations as proved in [LLY05] and reviewed in section 4. So it remains to prove the theorem for the subspace \( \langle h_1, h_3 \rangle \) as stated. The general strategy is to find suitable operators \( B \) and \( B^* \) acting on \( V_- \) and \( W_- \) respectively, commuting with the action of the Frobenius at \( p \), such that the characteristic polynomials of \( F \) and \( F_p \) on the respective eigenspace of \( B \) and \( B^* \) with the same eigenvalue agree. Since the characteristic polynomial of \( F_p \) is independent of the choice of \( l \neq p \), we shall choose \( l \equiv p \mod 8 \) so that \( \mathbb{Q}_l \) always contains the eigenvalues of \( B^* \).

**7.1.1. Case I.** \( p \equiv 1 \mod 8 \). The eigenspaces \( V_{-, \pm i} \) of \( \zeta \) on \( V_- \) (resp. \( W_{-, \pm i} \)) of \( \zeta^* \) on \( W_- \) with eigenvalue \( \pm i \) are \( F \)- (resp. \( F_p \)-) invariant. Further, since \( A^* \) commutes with the action of \( F \), and it maps \( W_{-, i} \) to \( W_{-, -i} \) isomorphically, we get

\[ \text{Char}(W_{-, i}, F_p)(T) = \text{Char}(W_{-, -i}, F_p)(T) = T^2 - A_p T + B_p \]

for some constants \( A_p \) and \( B_p \). That \( A_p \in \mathbb{Z} \) and \( B_p = p^2 \) follows from 110 and Corollary 6.3.

By [Sch85], we know that \( F_p \) on \( W_- \) and \( F \) on \( V_- \) have the same characteristic polynomials, and the same is true for \( \zeta^* F_p \) and \( \zeta F \). This implies

\[ \text{Char}(W_{-, \pm i}, F_p)(T) = \text{Char}(V_{-, \pm i}, F)(T). \]

Combined with \( h_1 \in V_{-, i} \) and \( h_3 \in V_{-, -i} \), this proves the asserted ASD-congruence.

To prove the remaining cases, we shall need the following Lemma. Let \( \delta = -1, 2, -2 \) and \( B_p^\delta \) be as in the previous section so that \( (B_p^\delta)^2 = \lambda \), where \( \lambda = -1, 2, -2 \) according as \( \delta = -1, 2, -2 \). Let \( p \) and \( l \) be primes such that \( \mathbb{Q}_p \) and \( \mathbb{Q}_l \) contain \( \sqrt{\delta} \). Then \( W_- \) (resp. \( V_- \)) decomposes into a direct sum of eigenspaces.
that the traces of $B_h$ (resp. $B_{\mp h}$) of $B_3^\pm$ (resp. $B_3$), which are invariant under the action of the Frobenius at $p$ by Lemmas 6.1 and 6.2.

**Lemma 7.4.** With the above notation, if

$$\text{Char}(W_{-\mp h}, B_3^\pm F_p)(T) = \text{Char}(W_{-\mp h}, B_3^\pm F_p)(T) = T^2 - a_p T + b_p$$

(19)

for some constants $a_p$ and $b_p$, then $a_p$ and $b_p$ lie in $\lambda \mathbb{Z}$, and

$$\text{Char}(W_{-\mp h}, F_p)(T) = \text{Char}(W_{-\mp h}, F)(T).$$

**Proof.** Write $B_3 = \sum k_i A_i$ as a linear combination of $A_i \in SL_2(\mathbb{Z})$ with coefficients $k_i \in \mathbb{Z}$. The traces of $B_3 F$ on $V$ and $V_+$ are equal to $\sum k_i \text{Tr}(A_i F)$ on the respective spaces, and likewise for $B_3^\pm F_p$ on $W$ and $W_\mp$. We conclude from Sec. 4.4 of [Sch85] that the traces of $B_3 F$ on $V_-$ and $B_3^\pm F_p$ on $W_\mp$ agree and they are in $\mathbb{Z}$.

On the $p$-adic side, using $\zeta(h_j) = \omega_j^2 h_j$ and $A(h_j) = \omega_j^{4-j} h_{4-j}$, one finds that each $(\pm \sqrt{\lambda})$-eigenspace of $B_3$ on $V_-$ contains a linear combination $h_{\pm \lambda}$ of $h_1$ and $h_3$; further, the dual of $h_{\pm \lambda}$ appears in the quotient of $(\pm \sqrt{\lambda})$-eigenspace modulo $h_{\pm \lambda}$. This shows that each eigenspace of $B_3$ on $V_-$ is 2-dimensional.

Let $\sqrt{\lambda} \alpha_1$ and $\sqrt{\lambda} \alpha_2$ (resp. $-\sqrt{\lambda} \alpha_3$ and $-\sqrt{\lambda} \alpha_4$) be the eigenvalues of $B_3^\pm F_p$ on $W_{-\mp h}$ (resp. $W_{-\mp h}$) so that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the eigenvalues of $F_p$ on $W_-$. In view of (19), we may assume $\alpha_3 = -\alpha_1$ and $\alpha_4 = -\alpha_2$. Thus $T^2 - a_p T + b_p = (T - \sqrt{\lambda} \alpha_1)(T - \sqrt{\lambda} \alpha_2)$ and

$$\text{Char}(W_{-\mp h}, F_p)(T) = (T^2 - \alpha_1^2)(T^2 - \alpha_2^2) \in \mathbb{Z}[T].$$

Therefore $b_p = \lambda \alpha_1 \alpha_2 = \pm \lambda \alpha_2^2 \in \lambda \mathbb{Z}$ and $a_p^2 = \lambda(\alpha_1 + \alpha_2)^2 = \lambda(\alpha_1^2 + \alpha_2^2) + 2b_p \in \lambda \mathbb{Z}$.

This combined with $2a_p = \text{Tr} B_3^\pm F_p \in \mathbb{Z}$ implies $a_p \in \lambda \mathbb{Z}$ since $\lambda$ is square-free.

It remains to prove the last assertion. Since $F$ on $V_-$ has the same eigenvalues as $F_p$ on $W_{-\mp}$, first consider the situation that one of $\alpha_1, \alpha_2$ is an eigenvalue of $F$ on $V_{-\mp \sqrt{\lambda}}$. Due to the symmetry on $\alpha_1$ and $\alpha_2$, we may assume it is $\alpha_1$. Then there are three possibilities for the second eigenvalue of $F$ on $V_{-\mp \sqrt{\lambda}}$: (i) $\alpha_2$, (ii) $-\alpha_1$, and (iii) $-\alpha_2$. If it is case (i), then we are done. If it is case (ii), then the eigenvalues of $B_3 F$ are $\sqrt{\lambda} \alpha_1$, $-\sqrt{\lambda} \alpha_1$, $-\sqrt{\lambda} \alpha_2$, and $-\sqrt{\lambda} \alpha_2$ so that $B_3 F$ has zero trace. As $B_3 F$ and $B_3^\pm F_p$ have the same trace, we conclude that $\alpha_1 = -\alpha_2$ and hence the assertion also holds. If it is case (iii), then the eigenvalues for $B_3 F$ are $\sqrt{\lambda} \alpha_1$, $-\sqrt{\lambda} \alpha_2$, $\sqrt{\lambda} \alpha_1$, and $-\sqrt{\lambda} \alpha_2$. Since the traces of $B_3 F$ and $B_3^\pm F_p$ are the same, one concludes that $\alpha_2 = 0$, which contradicts $|\alpha_2| = p$. So this case cannot occur.

Finally we note that case (ii) also takes care of the situation that one of $-\alpha_1$ and $-\alpha_2$ is an eigenvalue of $F$ on $V_{-\mp \sqrt{\lambda}}$. This completes the proof of the Lemma. \qed

7.1.2. **Case II.** $p \equiv 5 \mod 8$. In this case $\zeta$ is defined over $\mathbb{Q}_p$. Since $\zeta^* F_p = F_p \zeta^*$ by Lemma 6.1, $F_p$ leaves invariant the eigenspaces $W_{-\mp}$ and $W_{-\pm}$ of $\zeta^*$ on $W_-$. Further, by Lemma 6.1, $A^*$ commutes with $\zeta^* F_p$ and it maps $W_{-\mp}$ isomorphically to $W_{-\pm}$, therefore

$$\text{Char}(W_{-\mp i}, \zeta^* F_p)(T) = \text{Char}(W_{-\mp i}, \zeta^* F_p)(T) = T^2 - a_p T + b_p$$

for some constants $a_p$ and $b_p$. By Lemma 7.4, we have $a_p, b_p \in \mathbb{Z}$ and

$$\text{Char}(W_{-\mp i}, \zeta F_p)(T) = \text{Char}(W_{-\pm i}, \zeta F)(T).$$

Combining with (16) and Corollary 6.3 and noticing $h_1 \in V_{-\mp i}$ and $h_3 \in V_{-\pm i}$, we obtain the desired assertions for this case.
7.1.3. Case III. \( p \equiv 3 \mod 8 \). Denote by \( V_{-\pm \sqrt{-2}} \) (resp. \( W_{-\pm \sqrt{-2}} \)) the eigenspaces of \( B_{-2} \) on \( V_{-} \) (resp. \( B_{-2}^* \) on \( W_{-} \)) with eigenvalue \( \pm \sqrt{-2} \). By Lemma 6.1, the eigenspaces of \( B_{-2}^* \) are invariant under \( F_p \), \( B_{-2}^* \) commutes with \( B_{-2}^* F_p \) and it maps \( W_{-\sqrt{-2}} \) isomorphically to \( W_{-,-\sqrt{-2}} \). So there are \( a_p \) and \( b_p \) such that

\[
\text{Char}(W_{-\sqrt{-2}}, B_{-2}^* F_p)(T) = \text{Char}(W_{-,-\sqrt{-2}}, B_{-2}^* F_p)(T) = T^2 - a_p T + b_p.
\]

It follows from \([10]\), Corollary [6.6] and Lemma [7.4] that the characteristic polynomial of \( F_p \) has the asserted form and

\[
\text{Char}(W_{-\pm \sqrt{-2}}, B_{-2}^* F_p)(T) = \text{Char}(V_{-\pm \sqrt{-2}}, B_{-2}^* F)(T).
\]

Finally it is a straightforward computation, using the actions of \( A \) and \( \zeta \) given by \([10]\) and \([11]\), to check that \( h_1 \pm h_3 \) are eigenfunctions of \( B_{-2} \) on \( V_{-} \) with eigenvalues \( \pm \sqrt{-2} \), respectively.

7.1.4. Case IV. \( p \equiv 7 \mod 8 \). The detailed analysis parallels the previous case with the roles of \( B_{-2}^* \) and \( B_{-2}^* \) interchanged; the eigenvalues of \( B_{-2} \) on \( V_{-} \) (resp. \( B_{-2}^* \) on \( W_{-} \)) are \( \pm \sqrt{-2} \) with eigenspaces \( V_{-\pm \sqrt{-2}} \) (resp. \( W_{-\pm \sqrt{-2}} \)). In this case one checks that \( h_1 \pm ih_3 \in V_{-\pm \sqrt{-2}} \).

This completes the proof of Theorem [7.1] and Theorem [7.2].

8. Modularity of \( \rho_{4,i} \)

8.1. An automorphic representation. Let \( K = \mathbb{Q}(i, 2^{1/4}) \), which is a Galois extension over \( \mathbb{Q} \) with Galois group \( \text{Gal}(K/\mathbb{Q}) \) dihedral of order 8. It is a degree 4 extension over \( \mathbb{Q}(i) \) with Galois group \( \text{Gal}(K/\mathbb{Q}(i)) \) cyclic of order 4. The Artin reciprocity map from the idele class group of \( \mathbb{Q}(i) \) to \( \text{Gal}(K/\mathbb{Q}(i)) \) followed by an isomorphism from \( \text{Gal}(K/\mathbb{Q}(i)) \) to the group \( < i > \) generated by \( i \in \mathbb{C}^\times \) yields an idele class character of \( \mathbb{Q}(i) \) of order 4, denoted by \( \chi \). It is ramified only at the place \( 1 + i \) (above 2) of \( \mathbb{Q}(i) \). Its values at the places above the odd primes \( p \) of \( \mathbb{Q} \) are as follows:

1. For \( p \equiv 3 \mod 4 \), it remains a prime in \( \mathbb{Q}(i) \). We know that \( 2 \) (resp. \( -2 \)) is a square in \( \mathbb{Z}/p\mathbb{Z} \) when \( p \equiv 7 \) (resp. \( 3 \)) (mod 8), and is thus a fourth power in the residue field of \( \mathbb{Q}(i) \) at \( p \). Consequently, \( p \) splits completely in \( K \) and we have \( \chi(p) = 1 \).
2. For \( p \equiv 1 \mod 8 \), it splits into two places \( v_1, v_2 \) of \( \mathbb{Q}(i) \). Since 2 is a square modulo 8, we have \( \chi(v_1) = \chi(v_2) = \pm 1 = 2^{(p-1)/4} \mod p \).
3. For \( p \equiv 5 \mod 8 \), it splits into two places \( v_1, v_2 \) of \( \mathbb{Q}(i) \). We have \( \chi(v_1) = \chi(v_2)^{-1} = \pm i \), again determined by \( 2^{(p-1)/4} \mod v_1 \) (resp. \( v_2 \)) since 2 is not a square modulo \( p \). The opposite sign comes from the fact that \( v_1 \) and \( v_2 \) are complex conjugates, and \( 2^{(p-1)/4} \) is a fourth root of unity, which is congruent to \( i \) modulo one prime and \( -i \) modulo the other.

Thus \( \chi^2 \) is a quadratic character of the idele class group of \( \mathbb{Q}(i) \) which is equal to \( -1 \) only at the places above \( p \equiv 5 \mod 8 \). In other words, \( \chi^2 = \epsilon_{\mathbb{Q}(i)} \) in section 6.
Let 
\[ f_1(z) = \frac{\eta(2z)^{12}}{\eta(z)^{\eta(4z)^5}} = q^{1/8}(1 + q - 10q^2 + \cdots) = \sum_{n \geq 1} a_1(n)q^{n/8}, \]
\[ f_3(z) = \eta(z)^{5}\eta(4z) = q^{3/8}(1 - 5q + 5q^2 + \cdots) = \sum_{n \geq 1} a_3(n)q^{n/8}, \]
\[ f_5(z) = \frac{\eta(2z)^{12}}{\eta(z)^{5}\eta(4z)^5} = q^{5/8}(1 + 5q + 8q^2 + \cdots) = \sum_{n \geq 1} a_5(n)q^{n/8}, \]
\[ f_7(z) = \eta(z)^{\eta(4z)^5} = q^{7/8}(1 - q - q^2 + \cdots) = \sum_{n \geq 1} a_7(n)q^{n/8}, \] \hspace{1cm} (20)
and define
\[ f' = f'(z) = f_1(z) + 4f_3(z) + 2\sqrt{-2}(f_3(z) - 4f_7(z)) = \sum_{n \geq 1} a(n)q^{n/8}. \] \hspace{1cm} (21)

It is easy to verify that \( f'(8z) \) is a classical cuspform of level 256, weight 3, and quadratic character \( \chi_{-4} \) associated to \( \mathbb{Q}(i) \), and that it is an eigenform of the Hecke operators at odd primes. The twists of \( f' \) by the three quadratic characters of \( (\mathbb{Z}/8\mathbb{Z})^\times \) also have the same property.

Let \( \rho' \) be the \( \lambda \)-adic representation of \( G_{\mathbb{Q}} \) associated to \( f' \). Then \( L(s, f') = \prod_{p \neq 2} 1/(1 - a(p)p^{-s} + (\frac{1}{4})p^{2s-2}) \) is equal to \( L(s, \rho') \) if we remove the factor at \( l \) divisible by \( \lambda \). As the Fourier coefficients of \( f' \) lie in \( \mathbb{Z}[\sqrt{-2}] \), the \( \lambda \)-adic representation \( \rho' \) yields an action of \( G_{\mathbb{Q}} \) on a 2-dimensional vector space over \( \mathbb{Q}(\sqrt{-2}) \).

Denote by \( \rho \) the restriction of \( \rho' \) to the index-2 subgroup \( G_{\mathbb{Q}(i)} \) so that it is a degree two \( \lambda \)-adic representation of \( G_{\mathbb{Q}(i)} \). Corresponding to \( \rho \) is the cusp form \( f \) for \( GL_2(\mathbb{Q}(i)) \), which is the lifting of \( f' \) to a form over \( \mathbb{Q}(i) \) under the base change by Langlands (c.f. [Lan80]). Since \( f' \) and \( f' \) twisted by \( \chi_{-4} \) both lift to the same form \( f \), corresponding to the representation \( \rho \otimes \chi \) is the cusp form \( f_{\chi} \), called \( f \) twisted by \( \chi \), for \( GL_2(\mathbb{Q}(i)) \), whose \( L \)-function is
\begin{align*}
L(s, f_{\chi}) & = \prod_{p \equiv 3 \mod 4} \left( 1 - a(p)p^{-s} - p^{2-2s} \right)/\left( 1 - a(p)p^{-s} + p^{2-2s} \right) \\
& \times \prod_{p \equiv 1 \mod 8} \left( 1 - \chi(v_1)a(p)p^{-s} + p^{2-2s} \right)/\left( 1 - \chi(v_2)a(p)p^{-s} + p^{2-2s} \right) \\
& \times \prod_{p \equiv 5 \mod 8} \left( 1 - a(p)i(p^{-s} - p^{2-2s}) \right)/\left( 1 + a(p)i(p^{-s} - p^{2-2s}) \right),
\end{align*}
which is \( L(s, \rho \otimes \chi) \) after removing the factors at the places dividing \( l \). In the formula above, when \( p \equiv 1 \mod 8, v_1 \) and \( v_2 \) are two places of \( \mathbb{Q}(i) \) dividing \( p \), and \( \chi(v_1)\chi(v_2) = \pm 1 \equiv 2(p-1)/4 \mod p \), as discussed above. Observe that while there are two choices for \( \chi \), the \( L \)-function above is independent of the choice. Moreover, the \( L \)-function remains the same if \( f' \) is twisted by any quadratic character of \( (\mathbb{Z}/8\mathbb{Z})^\times \). Finally, \( \rho \otimes \chi \) can be realized as a 2-dimensional representation of \( G_{\mathbb{Q}(i)} \) over \( \mathbb{Q}(i) \).

Recall that the representation \( \rho_{-1} \), the restriction of \( \rho_- \) to \( G_{\mathbb{Q}(i)} \), can be viewed as a representation of \( G_{\mathbb{Q}(i)} \) to the 2-dimensional vector space \( W_- \) over \( \mathbb{Q}(i (\zeta)) \), and its associated \( L \)-function, written as an Euler product over the odd primes, agrees with the \( L \)-function attached to \( \rho_- \), as shown in Theorem 6.3. Take \( l = 2 \) so that
we may identify $\mathbb{Q}_2(\zeta)$ with $\mathbb{Q}_2(i)$ such that $\rho_{-,-1}$ and $\rho \otimes \chi$ have the same local $L$-factors at the place $3+2i$ and hence also at $3-2i$. This is possible from comparing (14) and the Fourier coefficient $a(13)$ of $f'$.

We want to show that $\rho_-$ and $\rho \otimes \chi$ have the same local $L$-factors over the odd primes, and are hence isomorphic. This will follow from

**Theorem 8.1.** The two representations $\rho_{-,-1}$ and $\rho \otimes \chi$ of $G_{\mathbb{Q}(i)}$ have isomorphic semi-simplifications.

**Proof.** Note that both representations are unramified outside the place $1+i$. In view of Theorem 7.1, Corollary 6.5 and the explicit expression of the $L$-function attached to $f_\chi$, the two representations have the same determinants and the same local $L$-factors at $3, 3+2i, 3-2i$, and 7. Moreover, by Corollary 7.3 and the definition of $f'$, for both representations, the characteristic polynomials of the Frobenius elements at places outside $1+i$ are all congruent to $T^2 + 1$ modulo 2.

To proceed, we use the following result of Livn´e [Liv87], which is an extension of Serre’s method [Ser] applied to the case of representations with even trace.

**Theorem 8.2 (Livné).** Let $K$ be a global field, $S$ a finite set of places of $K$, and $E$ a finite extension of $\mathbb{Q}_2$. Denote the maximal ideal in the ring of integers of $E$ by $\mathcal{P}$ and the compositum of all quadratic extensions of $K$ unramified outside $S$ by $K_S$. Suppose $\rho_1, \rho_2 : G_K \rightarrow GL_2(E)$ are continuous representations, unramified outside $S$, and furthermore satisfying

1. $\text{Tr} \rho_1 \equiv \text{Tr} \rho_2 \equiv 0 \mod \mathcal{P}$ and $\det \rho_1 \equiv \det \rho_2 \mod \mathcal{P}$;
2. There exists a set $T$ of places of $K$, disjoint from $S$, for which
   a. The image $T'$ of the set $\{\text{Frob}_t\}_{t \in T}$ in the $\mathbb{Z}/2\mathbb{Z}$-vector space $\text{Gal}(K_S/K)$ has the property that any cubic homogeneous polynomial on $\text{Gal}(K_S/K)$ which vanishes on $T'$ vanishes on the vector space $\text{Gal}(K_S/K)$;
   b. $\text{Tr} \rho_1(\text{Frob}_t) = \text{Tr} \rho_2(\text{Frob}_t)$ and $\det \rho_1(\text{Frob}_t) = \det \rho_2(\text{Frob}_t)$ for all $t \in T$.

Then $\rho_1$ and $\rho_2$ have isomorphic semi-simplifications.

Apply the above theorem to the case $\rho_1 = \rho_{-,-1}$, $\rho_2 = \rho \otimes \chi$ with $K = \mathbb{Q}(i)$, $E = \mathbb{Q}_2(i, \sqrt{2}) = \mathbb{Q}_2(\omega_8)$ and $S = \{1+i\}$. Then $K_S = \mathbb{Q}(i, \sqrt{2}, \sqrt{1+i})$ is a biquadratic extension of $K$ with the third quadratic extension being $\mathbb{Q}(i, \sqrt{1+i})$. Choose the set $T$ to consist of the three places $3, 3+2i$ and $3-2i$ of $\mathbb{Q}(i)$, which split in $\mathbb{Q}(i, \sqrt{2}), \mathbb{Q}(i, \sqrt{1-i})$, and $\mathbb{Q}(i, \sqrt{1+i})$, respectively, and are inert in the other two quadratic extensions of $\mathbb{Q}(i)$. Thus the Frobenius elements at the places in $T$ are precisely the three nontrivial elements of $\text{Gal}(K_S/K)$. Further, $\rho_1$ and $\rho_2$ have the same local $L$-factors at these three places, as observed before. Therefore all the conditions are satisfied, and hence $\rho_1$ and $\rho_2$ have isomorphic semi-simplifications. $\square$

**Corollary 8.3.** Representations $\rho_-$ and $\rho \otimes \chi$ have the same local $L$-factors over all odd primes $p$.

Combined with Theorem 7.1, we obtain an explicit description of the three-term ASD congruence relation in Theorem 7.2.

**Theorem 8.4.** [ASD congruence for the space $< h_1, h_3 >$] For each odd prime $p$, the subspace $< h_1, h_3 >$ has a basis depending on the residue of $p \mod 8$ satisfying a 3-term ASD congruence at $p$ as follows.
(1) If \( p \equiv 1 \mod 8 \), then both \( h_1 \) and \( h_3 \) satisfy the 3-term ASD congruence at \( p \) given by \( T^2 - \text{sgn}(p)a_1(p)T + p^2 \), where \( \text{sgn}(p) = \pm 1 \equiv 2^{(p-1)/4} \mod p \); 
(2) If \( p \equiv 5 \mod 8 \), then \( h_1 \) (resp. \( h_3 \)) satisfies the 3-term ASD-congruence at \( p \) given by \( T^2 + 4ia_5(p)T - p^2 \) (resp. \( T^2 - 4ia_5(p)T - p^2 \)); 
(3) If \( p \equiv 3 \mod 8 \), then \( h_1 \pm h_3 \) satisfy the 3-term ASD congruence at \( p \) given by \( T^2 \pm 2\sqrt{-2a_3(p)}T - p^2 \), respectively; 
(4) If \( p \equiv 7 \mod 8 \), then \( h_1 \pm ih_3 \) satisfy the 3-term ASD congruence at \( p \) given by \( T^2 \pm 8\sqrt{-2a_7(p)}T - p^2 \), respectively.

Here \( a_1(p) \), \( a_3(p) \), \( a_5(p) \), \( a_7(p) \) are given by (17).

Finally we remark that the character \( \chi \) of \( G_{\mathbb{Q}(i)} \) may be viewed as an idele class character of \( \mathbb{Q}(i) \) by class field theory. Thus there is a cuspform \( h(\chi) \) of \( GL_2(\mathbb{Q}) \) whose local \( L \)-factors agree with those of \( \chi \). The local \( L \)-factors of \( \rho \otimes \chi \) are in fact the local factors of the form \( f' \times h(\chi) \) on \( GL_2(\mathbb{Q}) \times GL_2(\mathbb{Q}) \). We summarize this discussion in

**Theorem 8.5.** There are two cusp forms \( f' \) and \( h(\chi) \) of \( GL_2(\mathbb{Q}) \) such that \( \rho_- \) and \( f' \times h(\chi) \) have the same local \( L \)-factors over primes \( p \neq l \).

Together with the fact that \( L(s, \rho_+) \) is automorphic, we have shown

**Corollary 8.6.** There is an automorphic \( L \)-function over \( \mathbb{Q} \) whose local factors agree with those of the \( l \)-adic Scholl representation attached to the space \( S_3(\Gamma_4) \) at all primes \( p \neq l \).

9. Acknowledgements

The authors are deeply indebted to the referee whose comments and suggestions led to significant improvements of several proofs. Special thanks are due to William A. Stein for facilitating part of the computational results in this paper.

**References**

[ASD71] A. O. L. Atkin and H. P. F. Swinnerton-Dyer, *Modular forms on noncongruence subgroups*, Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1971, pp. 1–25.

[Car71] P. Cartier, *Groupes formels, fonctions automorphes et fonctions zeta des courbes elliptiques*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, Gauthier-Villars, Paris, 1971, pp. 291–299.

[F5] L. Farg, J. W. Hoffman, B. Linowitz, A. Rupinski, and H. Verrill, *Modular forms on noncongruence subgroups and Atkin-Swinnerton-Dyer relations*, preprint, 2005.

[Lau80] R. P. Langlands, *Base change for GL(2)*, Annals of Mathematics Studies, vol. 96, Princeton University Press, Princeton, N.J., 1980.

[LLY05] W.-C. W. Li, L. Long, and Z. Yang, *On Atkin and Swinnerton-Dyer congruence relations*, Journal of Number Theory 113 (2005), no. 1, 117–148.

[Liv87] R. Livné, *Cubic exponential sums and Galois representations*, Contemporary Math. 67 (1987), 247–261.

[Sch85] A. J. Scholl, *Modular forms and de Rham cohomology: Atkin-Swinnerton-Dyer congruences*, Invent. Math. 79 (1985), no. 1, 49–77.

[Sch88] A. J. Scholl, *The l-adic representations attached to a certain noncongruence subgroup*, J. Reine Angew. Math. 392 (1988), 1–15.

[Sch04] A. J. Scholl, *On some l-adic representations of galois group attached to noncongruence subgroups*, [http://arXiv.org/abs/math/0402111](http://arXiv.org/abs/math/0402111) (2004).

[Ser] J. P. Serre, *Résumé de cours 1983-4*, Collège de France.
[Shi71] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971, Kanô Memorial Lectures, No. 1.

[Shi72] T. Shioda, *On elliptic modular surfaces*, J. Math. Soc. Japan 24 (1972), 20–59.

[SB85] J. Stienstra and F. Beukers, *On the Picard-Fuchs equation and the formal Brauer group of certain elliptic K3-surfaces*, Math. Ann. 271 (1985), no. 2, 269–304.

Department of Mathematics, University of Illinois at Chicago, Chicago, IL 60637, USA

E-mail address: aolatkin@math.uic.edu

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

E-mail address: wli@math.psu.edu

Department of Mathematics, Iowa State University, Ames, IA 50011, USA

E-mail address: linglong@iastate.edu