Undergraduate students’ difficulties in proving mathematics

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Abstract. This study aimed to find out the difficulties of proof experienced by undergraduate students. The type of research used was descriptive case studies. This research was conducted on 12 junior undergraduate students of the Mathematics Education Department at one of the universities in Yogyakarta. Data in this study were obtained through diagnostic tests and interviews. The most dominant difficulty in proving was the difficulty of construction and understanding the proof, difficulties in constructing and evaluating mathematical proof. Those happened because undergraduate students were considered not able to integrate the existing proof by proving that it is done as a complete proof of a case. This difficulty was caused by a lack of accuracy or even prior knowledge that was not sufficient. Lack of relevant knowledge both in the use of theorems and notations was another case of difficulty in proving.

1. Introduction

Proving ability is a fundamental aspect of mathematics education [1]. Proof in mathematical disciplines is the basic knowledge used to determine the truth of a vague mathematical claim [2]. The ability in proving is one of the abilities emphasized in mathematics learning. The student’s ability to prove will provide a frame of development and expression of insight from their lives [3]. Students with good proving abilities can have good analytical thinking and easily find pattern, structures, or regularities in real-world and symbolic situations. Their analytical thinking not only in speculation but presented with tangible proof. A proof is one of the keys to improve mathematical understanding; hence, proof must be intensified in mathematics learning at all levels of education.

Learning mathematics in university is an advanced level of mathematical learning that imposes high standards for their undergraduate students [4]. Mathematical learning at universities requires a reconstruction of old ideas towards formal methods in constructing theories from definitions and proofs of theorems and conjectures in mathematics. There is a huge gap between the point of view of a technical concept and the initial steps of numerical, symbolic and graphical processes towards a deductive mindset from formal to concept and definition that needs to be pressed. Proof is difficult to learn and to teach, and proof has been a special issue discussed in mathematical studies and mathematical education studies [5].

Based on the explanation above, many studies suggest that almost all undergraduate students face difficulties in proving and in constructing their own proofs [6]. There are still many undergraduate students who identify proof by using concrete examples [7] or diagrams [8]. Lestari [9] states that the ability to prove undergraduate students through an inductive-deductive approach has not been able to improve the undergraduate students proving ability. Furthermore, Lestari [9] shows that 42.4% of
undergraduate students have experienced difficulties in reading and understanding mathematical proof, 18.2% of undergraduate students have faced errors in presenting proof of truth, 35.6% undergraduate students have had difficulties in constructing proof directly or indirectly and the level of difficulty in developing mathematical arguments to deny or prove a statement reaches 76.6% of the total undergraduate students.

The ability in proving, misconceptions and the difficulties are interesting topics which have widely subject to discuss in mathematical forums. Thompson [10] identifies that the evidentiary difficulties made by undergraduate students have included difficulties in constructing and understanding proof [11], difficulty in constructing and evaluating proof, confusion in understanding the purpose of proof, errors in providing examples of proven or unproven forms, lack of knowledge of concepts, definitions and notations that are relevant, but not yet familiar with the verification strategy. However, Johnson [12] argued that the proving difficulties of undergraduate students have included difficulties in writing proof, in reading arguments and proof, in finding counter-examples, in making conjectures, in constructing arguments, in evaluating arguments and in improving their arguments.

Furthermore, Stavrou [13] argued that some of the most common mistakes conducted by students make in proving include: a) proving a general statement using a specific example. This case often occurs in implication statements. Weber explained that the use of examples in proving is only a strategy, but it cannot be used as proof of a general statement [14]. Giving a counterexample is just a reinforcement of the evidence made [15], b) assuming the conclusion of a statement in order to conclude the proof to be addressed. This type of error also often occurs in implication statements. Students assume true consequence statements, and then create rotating arguments to say that the consequences are true, c) not proving the bi-conditional statement in two directions but only in one direction, d) not using definitions properly and correctly. Many students do not understand the definition and use correctly, even if students can define the definition properly.

Difficulties encountered by undergraduate students in proving can be caused by several problems, including: 1) proof of the theorem requires steps that are difficult to predict. This can be overcome by planning proof well; 2) the purposes, methods and strategies of proving used are quite complex [16]. Variations of evidentiary difficulties found are concrete reason that it is important for the lecturer to find out the undergraduate student’s difficulties in proving, and then select an effective mathematical learning approach for planning improvements according to the situation of their students [17]. Larsen & Zandieh stated that the improvements needed are the ability of students to modify or clarify a definition, modify existing conjectures, modify conjectures or definitions for conceptualization of proof [18].

Based on the importance of proving both in mathematical discipline and mathematics education, the conditions of the existing evidentiary proving abilities and the difficulties that arise in proving are still prone to occur, so there needs to be a study of the difficulties in further proof. This study aims to describe the difficulties in proving experienced by undergraduate students.

2. Research Method

The type of research used is case study. Case study is a series of scientific activities carried out intensively, in detail and in depth about a program, event, and activity, both at the individual level, a group of people, institutions, or organizations to obtain in-depth knowledge of the event [19]. Case study allowed researchers to scrutinize a particular case. In most cases, case study chooses a small geographical area or a very limited number of individuals as the subject of study. Case studies explore and investigate real and contemporary phenomena of life through detailed contextual analysis covering both events and conditions and their causality [20].

The case study used was a descriptive case study regarding the analysis of the difficulties in proving. This research was conducted on undergraduate students of the mathematics education department at one of the universities in Yogyakarta. This study involved 12 junior undergraduate students who have finished first calculus and basic algebra course.

2.1 Research Instruments and Data Analysis
The data were obtained from the work on the diagnostic test and interviews with both the undergraduate students and the lecturer. The diagnostic test was used to recover how undergraduate students did in proving mathematical arguments. Meanwhile, interviews were conducted to obtain data regarding confirmation of undergraduate students’ answers. The interview was basically used to verify difficulties undergraduate students’ proving and to complete the information that was not obtained from the analysis of the verification capability diagnostic test.

The difficulty of proving in this study was analyzed based on the indicators of difficulties in proving argued by Thompson [10]. Indicators used in this study were: 1) difficulties in construction and proof evaluation; 2) difficulties in reading and understanding proof; 3) confusion in understanding the purpose of proof; 4) error in providing examples of proven and not proven concepts; 5) lack of knowledge regarding concepts, definitions, and relevant notations

3. Results and Discussion

3.1. Undergraduate students were given two types of tasks of proving mathematics. The first task was a sequence solution of an inequality given to the undergraduate students in order for them to criticize it. For the second task, an incomplete sequence of proof was given to the undergraduate students in order for them to complete it coherently.

Constructing and Evaluating Proof Difficulties

In Figure 1, undergraduate students multiply both segments with negative numbers which makes proving more complicated.

In Figure 1, undergraduate students chose to multiply inequality with negative. Even though the steps taken are legal, it makes more difficult to prove by adding steps and changing the direction of inequality. Besides, the undergraduate students did not show the reason for the falsification of solution set $x > 7$ for interval $-1 \leq x < 7$, while for interval $x < -1$, the inequality is solved. So, the solution is $\{x | x > 7 \text{ and } x < -1, x \text{ is real element}\}$.
Undergraduate students evaluate proof with less relevant arguments by giving example.

Figure 2. Undergraduate students present an evaluation of proof that was less relevant to the problem. The solution set offered by the case was a real number of more than 7, but the undergraduate students chose -2 to evaluate the proof so that the activity of evaluating the proof presented by the undergraduate students became irrelevant to the problem. The results of the interview indicate that this error is caused by the lack of thoroughness and even prior knowledge that is not sufficient for the given case. In the third picture, undergraduate students are considered good enough in the concept of real numbers, but cannot evaluate the proof from the previous section. This causes the conclusion to be taken inappropriately and the proof does not arrive at the goal.

The difficulty of constructing proof by undergraduate students is prompted by the lack of knowledge of the undergraduate students about the concepts discussed, as well as the difficulty of starting proof. This is consistent with Weber's explanation [14] of the difficulty of proof which often occurs; one of them is because the undergraduate students find it difficult to initiate proof [21]. Selden, Selden, & Benkhalti [22] stated that this difficulty is also caused by the fact that the undergraduate students lack understanding about the law and basic logic in mathematics and the misuse of definitions.

3.2. Difficulties in Reading and Understanding Proof

Here is a sample case of difficulty in reading and understanding mathematical proof:

Figure 3. Undergraduate students make mistakes in evaluating the proof-taking conclusion.

Translate: 8 is not more than 1. Hence, if desired \( x > 7 \) as the solution, the domain of \( x \) must be real positive.

In Figure 2, undergraduate students presented an evaluation of proof that was less relevant to the problem. The solution set offered by the case was a real number of more than 7, but the undergraduate students chose -2 to evaluate the proof so that the activity of evaluating the proof presented by the undergraduate students became irrelevant to the problem. The results of the interview indicate that this error is caused by the lack of thoroughness and even prior knowledge that is not sufficient for the given case. In the third picture, undergraduate students are considered good enough in the concept of real numbers, but cannot evaluate the proof from the previous section. This causes the conclusions to be taken inappropriately and the proof does not arrive at the goal.

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3.2. Difficulties in Reading and Understanding Proof

Here is a sample case of difficulty in reading and understanding mathematical proof:
Figure 4. Undergraduate students are not able to integrate the existence proof with the proof that will be carried out so that the evidentiary objectives cannot meet. Figure 4 shows that the undergraduate students could not present the further proof of the theorem and assumed that the theorem had been proven by showing proof for a kind of triangle. They could not integrate the existing proof with their own proof due to the lack of concepts of proof. This was caused by the difficulty in reading and understanding proof so that the undergraduate students were not able to construct further proof due to the lack of understanding of the existing proof. Undergraduate students who have difficulty in reading or understanding previous proof will have difficulties in the construction of the further proof. This can trigger a gap between the statements of one another so that the proof of the purpose cannot be fulfilled.

Nearly 90% of the proof difficulties in this study refer to the difficulty in reading and understanding proofs. Most of the undergraduate students left the paper blank due to the difficulty in the initiation of further proof they found. In line with Stavrou [13], the interview reveals that the undergraduate students cannot understand the existing proof. According to the lecturer, this case is caused by the lack of reading other sequences of proof. This explanation is also in harmony with Weber [14]. Undergraduate students must have a lot of experiences in comparing methods of proof that are widely used with all the evidence that has been found; one of the ways is to read a lot of evidence. This mathematical insight means that proof not only needs to be understood, but also needs to be explored [23]. Hence, the more evidence they study, the more they can explore. This also means that proof is an open statement to be accessed and explored furthermore.

3.3. Confusion in Understanding the Purpose of Proof
Here is the sample undergraduate student’s answer sheet:
Figure 5. Undergraduate students are not able to integrate existence proof with proof that will be carried out so that the evidentiary objectives cannot meet.

Figure 5 shows that the undergraduate students were not able to draw a conclusion from the proof. The undergraduate students falsified that the solution set given was not really meant to the inequality, so the undergraduate students searched for another solution, but actually the finding was just the same. The question required undergraduate students to designate an inappropriate step, but the undergraduate students evaluated the answer with an x value that was not relevant. Therefore, in the end, the undergraduate students provided an alternative solution that was precisely contradictory. In line with Stavrou [13], the undergraduate students assure that the example is right and shows the turning proof that has been falsified before. The immature understanding causes undergraduate students to be confused about what the evidentiary goals, and lack of the mastery concept is one of the backgrounds from the most common proof difficulties, not least the concepts and cases raised by the researcher.

3.4. Error in Providing Examples of Proven and Not Proven Concepts
The first task asked the undergraduate students to test the validity of the proof that was given and to find the error. It means that the undergraduate students had to be able to show contradictions and to provide counterexamples. However, the undergraduate students provided an example which was not relevant with the case so that the construction of the proof offered became blurred and created new contradictions that were not needed. In line with Komatsu and Jones [8], undergraduate students have difficulty in providing the right counterexample for a case and many undergraduate students find contradictions, but do not revise the conjecture that they have worked before.

3.5. Lack of Knowledge Regarding Concepts, Definitions, and Relevant Notations
The type of difficulty that often occurs is their lack of knowledge about concepts, definitions, and relevant notations. Almost 90% of the difficulties experienced indicate the lack of knowledge about concepts, definitions and relevant notations. Based on the undergraduate students' answers, the concept of fractional inequality settlement was still very minimal; one of which was indicated by the concept of
"moving segments" brought by undergraduate students as well as the multiplication of the two sections with \(x + 1\). In the concept, there was no term "move segment", and operating two segments with non-zero numbers was misinterpreted as the concept of "moving segments". The limitations of understanding concepts, definitions, and relevant notations are the most difficult challenges and types of proof difficulties that are carried out by most undergraduate students. Fukawa-Connelly [24] has stated earlier that most undergraduate students do not understand the definition well, so they cannot apply it in proofs.

Figure 6. Undergraduate students do not understand well the verification strategies that can be done.

In general, undergraduate students still have difficulties in proofs because they are not familiar with the available proof strategies. This is also supported by the fact that all undergraduate students have not been able to apply the concept of proofs of the theorem. Based on the undergraduate students' answers, the proof flow has not been able to be identified. Undergraduate students are not able to examine the implications offered by the case. Almost all undergraduate students still do not use a proof strategy. This is represented by the fact that no undergraduate student has completely proven the given theorem, even many undergraduate students let the answer sheet remain empty.

Stavrou [13] reviewed the possibility that undergraduate students chose not to prove and to leave the answer sheet blank because they understood that the proof to be presented actually added complexity, so they preferred to leave their answer sheets blank. In line with this, the results of interviews with undergraduate students informing the advanced proof design had a gap so that the objective of proof could not be fulfilled, and even the statement given further added the complexity of the proof being carried out. Actually undergraduate students have understood if a triangle is not a right triangle, then the
triangle is an obtuse or acute triangle. The undergraduate students have already known about this fact, but they still failed to prove the theorem [25]. This causes the function of proof to verify and shows the logical structure of an idea that is useless, and the deductive chain breaks up [26]. Some undergraduate students think that in proving, a general statement can sufficiently be proven in some cases [25].

4. Conclusion
The difficulty in constructing proof faced by undergraduate students appears because of their lack of prior knowledge and their difficulty in starting proof [14], [21], [22]. Undergraduate students feel difficult to initiate [13] because they cannot understand the existing proofs. This case is also caused by their lack of reading other sequences toward proofs [14]. Moreover, undergraduate students feel difficult because they falsify the statement and then try to prove it, but actually the real finding is the same. This happened when undergraduate students were given the case [8]. They also said the types of difficulty that often occurs are their lack of knowledge about concepts, definitions, and relevant notations that have been done by almost 90% of the undergraduate students. This case happens because the undergraduate students also do not understand the definition well, so they cannot apply it in proofs [24]. Another difficulty shown is because the undergraduate students are not familiar with the existing proof strategies. Therefore, they cannot determine the best strategy to prove the cases given to them.

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