Abstract

We put a probabilistic framework on the Demographic Prisoner’s Dilemma. In this model, cooperating and defecting individuals are placed on a torus to move and play prisoner’s dilemma game, if they are on the same site. Each individual accumulates its payoff into a quantity called wealth. If an individual becomes wealthy enough, it can have an offspring. If its wealth becomes negative, it disappears.

In this framework we prove that if if the Sucker payoff is far greater than the Reward then for all initial state almost surely all cooperators will die. Moreover if the Temptation payoff (resp. Reward) are far greater than the Punition (resp. Sucker payoff) then for all initial state with positive probability cooperators and defectors live \textit{ad vitam eternam}. We also set a Mean Field model on the demographic prisoner’s dilemma and prove on a linearized version of the Mean Field model that with weaker assumptions with positive probability Cooperators live \textit{ad vitam eternam}.

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1 Introduction

Playing particle systems as a modeling tool in the evolution literature have been introduced by [Smith and Price, 1973]. Since, many ecological models have used playing particle system's approach (see a list of examples in the book of [Hartl and Clark, 1997]). Yet a lot of the literature on the evolutionary games have been focused on random matching over finites games models [Boylan, 1992, Ellison, 1994, Weibull, 1997]. Then motivated by biological applications, spatiality have been introduced in models [Nowak and May, 1993, Szabó and Tőke, 1998, Hauert, 2002]. In those previous models, individuals cannot move and evolution is managed by a Wright-Fisher generation system (i.e. at each step of the evolution one whole generation gives birth to another generation and dies).

[Epstein, 1998] introduced a model where individuals can move and are not synchronized (they die at different times and gives birth at different times). This model is called Demographic Prisoner's dilemma. Before explaining the model and a bit of its probabilistic framework, let us give an intuition inspired by the article of [Turner and Chao, 1999].

You put viruses on a torus. Each virus has embedded in his RNA one of the two following behaviors: either she manufactures diffusible (shared) intracellular product, either she sequesters it and take advantage of the virus producing the product. Each virus can move on a fixed torus. They also have energy such that when they have intracellular product they get energy and when they manufacture it or sequester nothing they lose energy. A virus without energy dies. A virus with a lot of energy can split and bring a child (with its RNA) in the torus.

Notations.

We denote by \( \mathbb{N} \) the set of non negative integers and \( \mathbb{N}^* \) the set of positive integers.

The model is:

1. Let \( (\mathbb{Z}/m\mathbb{Z})^2 \) be a fixed torus with \( m \in \mathbb{N}^* \). It is the space where individuals move.

2. We make the assumption that the torus cannot bear an infinite number of individuals. Let us call \( K \in \mathbb{N}^* \) the maximum number of individuals in the torus.
3. The individuals move according to continuous time independent symmetric simple random walks. At random times (of rate $d \in \mathbb{R}_+^*$) an individual moves from his position to one of its nearest neighbors with equal probability, as shown in the following example of evolution.

![Figure 1: Example of moving of particles](image)

4. Each individual has a wealth. If the wealth becomes non positive, the individual dies (and then stop playing with the other individuals).

5. The wealth changes through games. The game is the prisoner’s dilemma:

- The players have two actions to Cooperate or to Defect.
- If both Cooperate they get a Reward $R$. But if one of the two defects, the Defector gets a Temptation payoff $T$, and this payoff is bigger than the Reward. If a Cooperator is being Defected instead of getting a reward he gets a Sucker payoff $-S$. If both players Defect (let us call them Player 1 and Player 2) the nature flips a coin, if it is head Player 1 gets a Punishment payoff of $-2P$ and Player 2 gets a payoff of 0; if it is tail Player 2 gets a Punishment payoff of $-2P$ and Player 1 gets a payoff of 0.
- Then the payoff satisfies $T > R > 0 > S > P > 0$. This is summarized in the following payoff matrix (action Top and action Left are Cooperate, action Bottom and action Right are Defect).

\[
\begin{pmatrix}
(R, R) & (−S, T) \\
(T, −S) & (P_1, P_2)
\end{pmatrix},
\]

where $(P_1, P_2)$ is a random variable with distribution \(\frac{1}{2}\delta(-2P, 0) + \frac{1}{2}\delta(0, -2P)\).

- Playing Cooperate is strictly dominated by playing Defect.

- To determine the action played by the individuals we place ourself in the framework where: "Each individual plays only one action, either he will Defect every time either he will Cooperate every time". Then a particle has a fixed action (Cooperate or Defect). There are two
kinds of particles: the ones who always Cooperate and the ones who always Defect.

- At the end of each game the wealth are updated, adding the respective payoff of the game played.
- To make the games happen, each couple of individuals is given a Poisson process independent of everything of parameter $v$. When this Poisson process realizes, if the individuals are on the same site and if their wealths are positive the individuals play together. Otherwise nothing happens.

6. Each individual is given a Poisson process of parameter $b > 0$. When this birth Poisson process realizes if the individual’s wealth is more than a given threshold $w_c > 0$ and if there is less than $K$ individuals on the torus then the individual gives birth to an offspring. The offspring has the same strategy as its unique parent (it cooperates if its parent cooperates, defects if its parent defects). Moreover the fixed birth wealth $0 < w_0 < w_c$ of the offspring is given by its parent. That is after the birth (then the parent lose wealth, and the child begins its life with a wealth equal to $w_0$).

[Epstein, 1998] introduced the first Demographic Prisoner’s dilemma model, and explore it doing simulations. The demographic prisoner’s dilemma has been a popular model [Axelrod, 2000] [Ifti et al., 2004] [Ohtsuki and Nowak, 2006] in theoretical biology. Also since he used multi-agent as a way of modeling evolutionary game theory, Epstein made an useful link between computer sciences and biology as presented by [Tumer and Wolpert, 2004].

Yet in conclusion of his article Epstein said: 'The only claims that can be advanced definitely are that this specific complex of assumptions is sufficient to generate cooperative persistence on the timescale explored in the research. [...] Obviously it would be worthwhile to [...] if possible, assess their generality mathematically.'

After this article, [Dorofeenko and Shorish, 2002], assuming statistical independence, proved the convergence of the Epstein model (when the payoff and the step of the grid tend to 0) to a reaction-diffusion process. They also studied via numerical results this reaction diffusion process. [Namekata and Namekata, 2011] add reluctant players (Tit for Tat players where the first action is Defect) in the model. They studied the extended model using simulations.

The main contributions of this article is generalizing Epstein model mathematically that is:

- Putting a probabilistic framework on the Epstein model without doing too strong assumptions (slightly modifying the payoff matrix, making individuals moving following independent simple random walks instead of exclusion process).
- Proving asymptotic (on time) results on the survival of the cooperators.
• Introducing a Mean Field model (or well mixed)

• Proving that we can insure a certain level of cooperation without too strong assumption on the payoff matrix.

Let us introducing the first results.

**Notations.**

• We call configuration the data of the positions, the behaviors, and the wealth of every particles. Then a configuration is an element of $E := ((\mathbb{Z}/m\mathbb{Z})^2 \times \mathbb{R}_+ \times \{-1, 0, 1\})^K$ where $\{-1, 0, 1\}$ indicates the behavior of a particle.

• Since Markov processes are defined by a collection of probability distribution on the space of right continuous, left limited functions from $\mathbb{R}_+$ to $E$ (this space is denoted $D(\mathbb{R}_+, E)$) indexed by $E$. We denote the distributions of the previous model $(P_\sigma)_{\sigma \in E}$ ($\sigma$ represents the initial state).

**Theorem.** There exists a constant $\mu > 0$ depending only on $v, d, K, b, w_0, w_c$ and $m$ such that if:

$$\mu R < S$$

then for each initial configuration $\sigma$ with at least one defector with positive wealth:

$$P_\sigma(\{\text{Eventually all the cooperators will be dead}\}) = 1.$$

**Theorem.** There exists a constant $\nu, \nu' > 0$ depending only on $v, d, K, b, w_0, w_c$ and $m$ such that if:

$$\nu(S + w_0) < R$$

$$\nu'(2P + w_0) < T$$

then for each initial configuration $\sigma$ with at least two cooperators and one defector with positive wealth:

$$P_\sigma(\{\text{The cooperators and defectors present in the beginning never die}\}) > 0.$$
the cooperators will never die.  
The two theorems talk of the red and blue area (or the areas pointed by the arrows) on the following graph. In order to understand better the behavior of this system, we draw some simulations with the following data:

- The size of the torus is $m = 7$.
- There are initially 10 cooperators and 10 defectors.
- Initially all individuals have a wealth of 10.
- The rate of the game Poisson processes is the same than the rate of the moving Poisson processes $d = v = 5$.
- The Temptation payoff is $T = R + 1$ and the Punishment payoff is $P = S - 1$.
- The stopping time of the simulation is 10,000 moves, games or birth Poisson process realizations. (Usually in the previous simulations this density becomes nearly constant after 3,000 realizations of Poisson processes).
- The critical wealth (wealth that allows to give birth) is: $w_c = 10$.
- The birth wealth for a child is: $w_0 = 3$.
- The maximum number of particles $K = 10^m = 10^7$.

**Method**

We make a batch of 100 simulations of the system for each couple of payoff $(R, S)$ ($R, S \in \{0, \ldots, 100\}$). At the end of each simulation we measure the number of cooperators with a positive wealth. After drawing these simulations we take the average of this measure over all these 100 simulations. Finally we make the figure saying that the color red corresponds to a small number of cooperators with a positive wealth at the end of the simulation, and the color blue corresponds to a high number of cooperators with a positive wealth at the end of the simulation (dark red correspond to 0 cooperators with a positive wealth at the end of the simulation).

In order to understand better the area in green and orange (intermediate area between the two pointed areas) we will consider the following Mean Field model:

- particles stop giving birth.
- there is no spatial condition.
- instead of considering a linear interaction with many particles, we consider one defector and one cooperator with non linear interaction (the evolution depends also on the distribution of the process).
- all particles begin with the same distribution of wealth.
The Mean Field system is the Markov process \((\mathcal{C}(t), \mathcal{D}(t))_t\). We will look at the distribution of \((\mathcal{C}(t))_t\). The intuition of \(\mathbb{P}(\mathcal{C}(t) = w) = p\) is: at time \(t\), \(p\%\) of the population of cooperators has a wealth \(w\). We define \((\mathcal{C}(t), \mathcal{D}(t))_t\) by: (with \(\beta_0 + \rho_0 = 1\) fixed) and an initial measure \(m_0\)

\[
\begin{align*}
\mathcal{D}(t_{n+1}) - \mathcal{D}(t_n) &= 1_{\mathcal{D}(t_n) > 0} U_\mathcal{D}(t_{n+1}) \\
\mathcal{C}(t_{n+1}) - \mathcal{C}(t_n) &= 1_{\mathcal{C}(t_n) > 0} U_\mathcal{C}(t_{n+1})
\end{align*}
\]

where:

- \((t_n^D)\) and \((t_n^C)\) are sequences of Poisson times of intensity \(\frac{v}{2}\) (i.e., for example with \(\mathcal{N}^D\) a Poisson process of parameter \(\frac{v}{2}\) for all \(n \in \mathbb{N}\) \(t_n^D\) is defined by \(t_n^D = \inf(t > 0/\mathcal{N}_t^D \geq n))\).

- for \(I = \{t_n^C, n \in \mathbb{N}\} \cup \{t_n^D, n \in \mathbb{N}\}\), all \(t \in I\) and \(U_\mathcal{D}(t)\) and \(U_\mathcal{C}(t)\) are random variables independent from everything such that:

\[
U_\mathcal{D}(t) = \begin{cases} 
-2\mathbb{P} \text{ with probability } \frac{1}{2}\rho \mathbb{P}(\mathcal{D}(t) > 0) \\
T \text{ with probability } \beta \mathbb{P}(\mathcal{C}(t) > 0) \\
0 \text{ with probability } 1 - \beta \mathbb{P}(\mathcal{C}(t) > 0) - \frac{1}{2}\rho \mathbb{P}(\mathcal{D}(t) > 0)
\end{cases}
\]
and

\[ U_{C}(t) = \begin{cases} 
- S & \text{with probability } \rho^0 P(D(t) > 0) \\
R & \text{with probability } \beta^0 P(C(t) > 0) \\
0 & \text{with probability } 1 - \beta^0 P(C(t) > 0) - \rho^0 P(D(t) > 0)
\end{cases} \]

The intuition behind these formulas is that we update (for example \( C(t) \)) following a Poisson process of intensity \( \tilde{v} \). If \( C > 0 \) we update it doing:

\[ \begin{cases}
C(t) - S & \text{with probability } \rho^0 P(D(t) > 0) \\
C(t) + R & \text{with probability } \beta^0 P(C(t) > 0) \\
C(t) & \text{with probability } 1 - \beta^0 P(C(t) > 0) - \rho^0 P(D(t) > 0)
\end{cases} \]

An important remark is that the law of the wealth of any cooperator, resp.
any defector, (as a stochastic process) in the spatial model converges in finite
dimensional distribution to \( C \), resp \( D \) when \( m^2 = N \) and \( d \to +\infty \) then when
\( N \to +\infty \).

This distribution has three principles:

- A drift (positive or negative)
- A diffusion
- An absorbing bound on 0 causing non linearity.

Hence this process is non linear. We consider a simpler process instead : a
linearized process \((\tilde{C}(t), \tilde{D}(t)) \).

\[ \begin{align*}
\tilde{C}(t_{n+1}) &= \tilde{C}(t_n) - S \quad \text{with probability } \rho^0 \\
\tilde{D}(t_{n+1}) &= \tilde{D}(t_n) + R \quad \text{with probability } \beta^0 \\
\tilde{C}(t_{n+1}) &= \tilde{C}(t_n) \quad \text{with probability } 1 - \rho^0 - \beta^0
\end{align*} \]

avec \((\tilde{U}_{C}(n))_n \) et \((\tilde{U}_{D}(n))_n \) two independent sequences of i.i.d. random variables
such that

\[ \begin{cases} 
- S & \text{with probability } \rho^0 \\
R & \text{with probability } \beta^0 \\
0 & \text{with probability } 1 - \rho^0 - \beta^0
\end{cases} \]

et

\[ \begin{cases} 
- 2P & \text{with probability } \frac{\rho}{2} \\
T & \text{with probability } \beta^0 \\
0 & \text{with probability } 1 - \rho^0 - \frac{\beta^0}{2}
\end{cases} \]

On this linearized system we have the following results:

**Theorem 1.1.** Let us suppose that:

\[ \beta^0 R - \rho^0 S > 0 \]
Then we have: for any fixed $q_0 > 0$

$$\overline{\mathbb{C}}(t) \to +\infty \quad \mathbb{P}_{q_0} \text{a.s.}$$

$$\mathbb{P}_{q_0}(\forall t \geq 0, \overline{\mathbb{C}}(t) > 0) > 0$$

We also have the following proposition about the concentration of wealth.

**Proposition.** For all $q_0 \in \mathbb{R}_+$

We denote :

$$m := \frac{v}{2}(\beta^0 R - \rho^0 S), \quad \sigma^2 := \frac{v}{2}(\beta^0 R^2 + \rho^0 S^2).$$

Then we have : $\forall \eta > 0, t \geq 0$

$$\mathbb{P}_{q_0}(\overline{\mathbb{C}}(t) \in \left[q_0 + mt - \eta \sqrt{\sigma^2 t}, q_0 + mt + \eta \sqrt{\sigma^2 t}\right]) \geq 1 - \eta^{-2}.$$

Moreover for all $\tau$ such that:

$$\tau < q_0 - \frac{\eta^2 \sigma^2}{4m},$$

then

$$\mathbb{P}(\overline{\mathbb{C}}(t) > \tau) \geq 1 - \eta^{-2}.$$
This article is in two parts. In the first one, we introduce the spatial model with his probabilistic framework. We also state and give the sketch of the proofs of the two qualitative theorems. In the second part, we give some reminders on Continuous Time Markov processes (in particular the theory of infinitesimal generators). We do that in order to describe and justify the Mean Field model. Then we state Mean field theorem. The proofs of the theorems are in the appendix.

2 Spatial Model

Remark. We call particle, player or individual, the agent defined in the following.

2.1 Model

Let us give a probabilistic framework for the Demographic prisoner’s dilemma of [Epstein, 1998]. Let \((\Omega, \mathcal{T}, \mathbb{P})\) be a probability space.

1. Let \(m \in \mathbb{N}^*\). Let \(K \in \mathbb{N}^*\).

2. For a particle \(i \in \{1, \ldots, K\}\), let us denote \(X^i(t)\) its position at time \(t \in \mathbb{R}^+\). Then \(\forall i \in \{1, \ldots, K\} \ X^i\) is a continuous simple symmetric random walk on \((\mathbb{Z}/m\mathbb{Z})^2\) of rate \(d \in \mathbb{R}^+\).

3. We denote the wealth of particle \(i \in \{1, \ldots, K\}\) at time \(t \in \mathbb{R}^+\) by \(Y^i(t) \in \mathbb{R}\).

4. The action played by \(i \in \{1, \ldots, K\}\) is coded in a process \(Z^i\). Particle \(i\) plays \textit{Cooperate} if \(Z^i = 0\) and plays \textit{Defect} if \(Z^i = 1\). If \(Z^i = -1\) the particle does not play.

5. The wealth changes through games.

- To make the games happen, each couple of individuals \((i, j)\) is given a Poisson process independent of everything of parameter \(v\). When this Poisson process realizes (for example at a time \(t\)), if the individuals are on the same site \(i.e.\) if \(X^i(t) = X^j(t)\), if their wealths are positive \(i.e.\) if \(Y^i(t) > 0\) and \(Y^j(t) > 0\) and if \(Z^i, Z^j\) are both not equal to -1 the individuals play together. Otherwise nothing happens. Then if an individual has a negative wealth, he can’t play with the other individuals.

- A point to notice is that two players cannot lose wealth simultaneously. As a consequence, two individuals cannot kill each other.

- At the end of each game (between for example player \(i\) and \(j\)) \(Y^i\) and \(Y^j\) are updated, adding the payoff of the game played by \(i\) and \(j\).

6. For \(i \in \{1, \ldots, K\}\) when \(Y^i > w_c\), and when the birth Poisson process of \(i\) realizes, he gives birth to an offspring. The index \(j\) of the offspring
is drawn uniformly randomly from $\{j \in \{1, \ldots, K\}, Z^j = -1\}$, and then
$X^j, Y^j, Z^j$ are updated doing $X^j \leftarrow X^i, Y^j \leftarrow w_0$ and $Z^j \leftarrow Z^i$.

We have that $(X(t), Y(t), Z(t))_t$ is a Continuous Time Markov Chain, we
denote by $(\mathcal{F}^d_t)_t$ its filtration such that $(X(t), Y(t), Z(t))_t$ is adapted to $(\mathcal{F}^d_t)_t$.

**Definition 2.1.** We call set of configuration $((\mathbb{Z}/m\mathbb{Z})^2 \times \mathbb{R} \times \{-1, 0, 1\})^K := E$. A configuration is an element usually denoted $\sigma$ of $((\mathbb{Z}/m\mathbb{Z})^2 \times \mathbb{R} \times \{-1, 0, 1\})^K$.

We denote the non trivial configurations by:

- $C$ the set of configurations with at least 2 cooperators and 1 defector with positive wealths.
- $C'$ the set of configurations with at least 1 cooperator and 1 defector with positive wealths.

A particle system is a Continuous Time Markov Chain with state space: the
space of configuration. It is usually denoted $(X(t), Y(t), Z(t))_t := (\sigma_t)_t$.

Before stating the theorems, let us remind the data of the model.

- $m \in \mathbb{N}$ the size of the torus, state space of the position.
- $K \in \mathbb{N}^*$ is the maximum number of players on the torus.
- We fix the payoff initially $T > R > 0 > S > P > 0$. The payoff
  matrix is
  $\begin{pmatrix} (R, R) & (-S, T) \\ (T, -S) & (P_1, P_2) \end{pmatrix}$,
  where $(P_1, P_2)$ is a random variable with distribution $\frac{1}{2}\delta(-2P, 0) + \frac{1}{2}\delta(0, -2P)$
- The position is denoted by $X$, the wealth by $Y$ and the strategy by $Z$.
- If $Z = 0$ the particle is a cooperator, if $Z = 1$ the particle is defector. If $Z = -1$ the particle cannot play (because she is not born yet).
- If $Y \leq 0$, the particle stop playing. If $Y > w_c > 0$ the particle can give
  birth to a particle with initial wealth $w_0 < w_c$.
- $d \in \mathbb{R}_+$ is the rate of the Poisson process making a particle move. $v \in \mathbb{R}_+$
  is rate of the Poisson process making a couple of particles plays together.
- $b \in \mathbb{R}_+$ is the rate of the Poisson process making the player trying to have an offspring.
2.2 Almost sure extinction

In this subsection we will show a first qualitative theoretic result: when the sucker payoff \( S \) is far greater than the reward payoff \( R \) then almost surely for any initial configuration (with at least one defector) cooperators will die.

**Theorem 2.2.** There exists a constant \( \mu > 0 \) depending only on \( v, d, K, b, w_0, w_c \) and \( m \) such that if:

\[
\mu R < S \tag{1}
\]

then for each initial configuration with at least one defector \( \sigma \):

\[
P_\sigma(\{ \text{Eventually all the cooperators will be dead} \}) = 1.
\]

**Sketch of the proof.** The aim of the proof is showing that the total wealth of cooperators goes to 0 almost surely. The sketch of the proof is the following:

1. The only way the total cooperator wealth decreases is via games with defectors (giving birth does not change the total wealth of cooperators). Since \( (\mathbb{Z}/m\mathbb{Z})^2 \) is finite, going from a configuration \( \sigma_0 \), the first game between a defector and a cooperator arrives at a finite random time \( \tau \). At this time the variation of total wealth is less than \( -S + RN_\tau \) where \( N_\tau \) is the Poisson process counting the games.

2. Then to apply a Law of Large Number argument we want a condition such that:

\[
\mathbb{E}(-S + RN_\tau) < 0.
\]

3. To find this condition we have to upper bound uniformly in the configurations the first time for a defector to play with a cooperator. This will be one of the main difficulty of the proof.

   (a) We remark that there exists a minimum number \( m \) of realizations of Poisson processes such that for each non trivial configuration there is a Cooperator-Defector game in less than \( m \) realizations.

   (b) Since the evolution is managed by Poisson processes, for every configuration \( \sigma \) there is a Cooperator-Defector game in less than \( m \) realizations of Poisson processes with probability \( \varepsilon(\sigma) > 0 \).

   (c) Since \( \varepsilon(\sigma) \) is the probability of a particle moving and a game happening, \( \varepsilon \) does not depend on the wealth of the individuals. Since if we don’t look at the wealth of the individuals there is a finite number of configurations, we take \( \varepsilon \) the minimum over these configurations. This part is really detailed in the proof.

   (d) The probability that a Cooperator-Defector game happens after \( km \) realizations of Poisson processes is bigger than \( (1 - \varepsilon)^k \).

4. We finish the proof using Borel Cantelli’s Lemma.
2.3 Coexistence \textit{ad vitam eternam}

In this subsection we will show a second qualitative result that is: when the reward is far greater than the sucker payoff and the initial offspring wealth then with positive probability the cooperators live until the end of times.

\textbf{Theorem 2.3.} There exists a constant $\nu, \nu' > 0$ depending only on $d, v, w_c, w_0, K$ and $m$ such that if:

\begin{align*}
\nu(S + w_0) &< R \quad (2) \\
\nu'(2P + w_0) &< T \quad (3)
\end{align*}

then for each configuration $\sigma$ with at least two cooperators and one defector:

\[ \mathbb{P}_\sigma(\{\text{the cooperators and defectors of } \sigma \text{ never die}\}) > 0 \]

\textbf{Sketch of the proof.} The main ideas are to consider a \textit{Ghost system} and then to extend the result of the \textit{Ghost system} using Burkholder-Davis-Gundy inequalities and Borel-Cantelli Lemma. We call the system introduced in the first section \textit{True system}.

1. In the \textit{Ghost system}, everybody plays even with negative wealth. Also whenever something in the system happens all individual’s wealth decrease of $w_0$ (except if an individual gives birth, this particular individual does not lose additional wealth). If in the \textit{Ghost system} we can prove that every individual always have positive wealth then this \textit{Ghost system} is actually the \textit{True system}.

2. Using a Burkholder-Davis-Gundy argument with Borel Cantelli’s Lemma, we prove on the \textit{Ghost system} that the minimum wealth over all individuals $C_{\min}^t \to +\infty$ a.s. when $t \to +\infty$.

3. We just have to consider an event (of positive probability) such that no individual has non positive wealth until they accumulated enough wealth to make the coupling between the \textit{Ghost system} and the \textit{True system}.

Theorem 2.2 and Thm 2.3 are qualitative results. In order to have quantitative results we consider in the following a Mean Field model. In simulation we notice that birth are really useful for maintaining the cooperation (see for example Run 3 and Run 4 of Epstein [Epstein, 1998]), in order to guarantee some result on the survival of cooperators in the Mean Field model we won’t consider births.
3 Mean Field model

3.1 Reminders on Markov process and infinitesimal generators

Let us firstly give some reminders about Markov processes.

One good way to study Markov processes is via their distribution. For a complete separable state space $F$, a process $X = (X(t))_{t \geq 0}$ has value in the canonical space.

**Definition 3.1 (Canonical space).** The canonical space $D(\mathbb{R}_+, F)$ is the space of right continuous functions from $[0, +\infty)$ to $F$ with left limits, endowed with the Skohorod topology associated to its usual metric. With this metric, $D(\mathbb{R}_+, F)$ is complete and separable (see the book of Ethier Kurtz [Ethier and Kurtz, 1986] for more details).

We denote by $(\mathcal{T}_t)_t$ the natural filtration associated to $D(\mathbb{R}_+, F)$. We define the canonical process $X := (X(t))_{t \geq 0}$ by:

$$\forall t \geq 0, \quad X(t) : \omega \in D(\mathbb{R}_+, F) \mapsto \omega(t) \in F$$

To define a Markov process in a discrete countable state space $F$ (which will be the case here), we need an initial measure $\nu$ on $F$ and a matrix called rate matrix $(A(x,y))_{x,y \in F}$ with $A(x,x) = -\sum_{y \in F, y \neq x} A(x,y)$. For $x \neq y$, $A(x,y) \geq 0$ gives the rates of transition of the future process from $x$ to $y$. With this matrix we define the generator (which is one of the main tool in the study of Markov processes) of the following Markov process such that: for each $f$ bounded and measurable (for the Borel sets of $F$) from $F$ to $\mathbb{R}$ (this set is denoted $\mathcal{M}_b(F)$), and for all $x \in F$:

$$A f(x) = \sum_{y \in F} A(x,y)[f(y) - f(x)]$$

The generator is a bounded linear operator on the bounded functions from $F$ to $\mathbb{R}$. We note in the same way a rate matrix $A = (A(x,y))_{x,y \in F}$ and the associated Markov generator $A : \mathcal{M}_b(F) \to \mathcal{M}(F)$ (with $\mathcal{M}(F)$ the space of measurable functions of $F$). We will always denoting the matrix with two entries (for example $A(.,.)$) and the generator with no entry (for example $A$).

Conversely giving a generator we can construct the rate matrix looking at the rates (terms that will be before the $[f(y) - f(x)]$ in the expression of the generator.

**Definition 3.2.** Let $A$ be a rate matrix and an initial measure $\nu$ we can construct a Markov process $X$ as follow:

1. Let $(Y(n))_n$ be a Markov chain in $F$ with initial distribution $\nu$ and with transition matrix $\left(\frac{A(x,y)}{|A(x,x)|}\right)_{x,y \in F}$. We allow $|A(x,x)| = 0$ by taking for
one $y_0 \in F$, $A(x,y_0) = 1$ and for all $y \neq y_0$, $A(x,y) = 0$. We put $y_0$ only to have $(Y_n)_{n}$ well defined, $y_0$ doesn’t have any interest to $X$ as we will see in the following.

2. Let $\Delta_0, \Delta_1, \ldots$, be independent and exponentially distributed with parameter 1 (and independent of $Y(.)$) random variables.

3. We define the Markov process $(X(t))_t$ in $F$ with initial distribution $\nu$ and generator $A$ by:

\[
X(t) = \begin{cases}
Y(0), & 0 \leq t < \frac{\Delta_0}{A(Y(0),Y(0))} \\
Y(k), & \sum_{j=0}^{k-1} \frac{\Delta_j}{A(Y(j),Y(j))} \leq t < \sum_{j=0}^{k} \frac{\Delta_j}{A(Y(j),Y(j))}
\end{cases}
\]

Note that we allow $A(x,x) = 0$ taking $\Delta/0 = \infty$.

Since it will be constant in the evolution, we denote $N$ the (initial) number of particles. Let us firstly introduce the generator of the spatial system (but without birth). Let us begin by some notations

**Notations**

Let $(\Omega, T, \mathbb{P})$ be a probability space. Let $(e_1, \ldots, e_N)$ be the canonical basis of $\mathbb{R}^N$ (with $N \in \mathbb{N}^*$). Let $(e_1^1, e_1^2, \ldots, e_N^1, e_N^2)$ be the canonical basis of $(\mathbb{R}^2)^N$. We denote $E = ((\mathbb{Z}/m\mathbb{Z})^2 \times \mathbb{R} \times \{0,1\})^N$. We identify $((\mathbb{Z}/m\mathbb{Z})^2 \times \mathbb{R} \times \{0,1\})^N$ to $((\mathbb{Z}/m\mathbb{Z})^2 \times \mathbb{R} \times \{0,1\})^N$, then we can write $(x,y,z) \in ((\mathbb{Z}/m\mathbb{Z})^2 \times \mathbb{R} \times \{0,1\})^N$ with $x \in ((\mathbb{Z}/m\mathbb{Z})^2)^N$, $y \in \mathbb{R}^N$, $z \in \{0,1\}^N$.

Let us denote $C_b(E)$ the set of continuous bounded functions from $E$ to $\mathbb{R}$. We denote by $\Psi_N = \{(i,j) \in \{1, \ldots, N\}/i \neq j\}$ all the individuals couples which can play together.

The infinitesimal generator $A$ with domain $C_b(E)$ of $(X,Y,Z)$ is: for all $f \in C_b(E)$ and for all $(x,y,z) \in E$ 

\[
Af(x,y,z) = A_d f(x,y,z) + v A_g f(x,y,z).
\]

The part $A_d$ is the generator representing the motion of the individuals. The individuals move following independent random walks, then $A_d$ is defined by: for all $f \in C_b(E)$:

\[
A_d f(x,y,z) = \sum_{i=1}^{N} \sum_{c=-1}^{2} \sum_{\epsilon=\pm 1}^{2} \frac{d}{2 \times 2} [f(x + c\epsilon_i, y, z) - f(x, y, z)].
\]

The generator $A_g$ is the generator representing the evolution of the wealth of individuals through games. Let us give firstly $A_g$ then the explication of how it
works. We have for each function $f \in C_b(E)$:

$$A_g f(x, y, z) = \sum_{(i, j) \in \mathcal{P}} \frac{1}{2} \mathbb{1}_{x_i = x_j} \mathbb{1}_{y_i > 0, y_j > 0} \begin{bmatrix} \mathbb{1}_{z_i = 0, z_j = 0} (f(x, y + Re_i + Re_j, z) - f(x, y, z)) + \\
\mathbb{1}_{z_i = 1, z_j = 0} (f(x, y + Te_i - Se_j, z) - f(x, y, z)) + \\
\mathbb{1}_{z_i = 1, z_j = 0} (f(x, y + Te_j - Se_i, z) - f(x, y, z)) + \\
\mathbb{1}_{z_i = 0, z_j = 1} \left( \frac{1}{2} (f(x, y - 2Pe_i, z) - f(x, y, z)) + \\
\frac{1}{2} (f(x, y - 2Pe_j, z) - f(x, y, z)) \right) \end{bmatrix}$$

(7)

Let us explain a bit this generator:

- $\mathbb{1}_{x_i = x_j}$ represents the spatial structure meaning that a game makes the wealth change only if the two individuals are on the same site.
- $\mathbb{1}_{y_i > 0, y_j > 0}$ checks if both individuals have a positive wealth (i.e. their wealth are positive). To have the generator of the ghost system, the only change is: replacing this indicator function by 1.
- the big bracket represents the change of wealth through the game of the two individuals. If $z = 0$ the individual is a cooperator, if $z = 1$ the individual is a defector. The term $\mathbb{1}_{z_i = 1, z_j = 0}$ looks at the strategies of the players and choose the right payoff to give to the individuals. For example if individual 1 and 2 are playing (both have a positive wealth and on the same site) if individual 1 cooperates and 2 defects then individual 1 gets a payoff of $-S$ and individual 2 gets $T$, the term in the generator representing this type of transition is:

$$\mathbb{1}_{z_i = 0, z_j = 1} [f(x, y - Sc_1 + Te_2, z) - f(x, y, z)].$$

Notations. The distribution of the previous $(X, Y, Z)$ defined by the generator $A_g$ is: $\mu^{d, N}$.

### 3.2 Mean Field Model

#### 3.2.1 Definition and intuition

The Mean Field process is a non linear Markov process. That means that the temporal marginals of the distribution $\mu$ of the process is described by a generator also depending on these time marginals $\mu$. For that purpose we call this kind of generator non-linear generator. The change between a non linear Markov process and a homogeneous Markov process is: when determining the transitions in an homogeneous Markov process you have to look where the process is, while in the non linear case to determine the transition you look both where the process is and where it could be.

The Mean Field system is the non linear Markov process $(\mathcal{C}(t), \mathcal{D}(t))$: for $\beta^0 + \rho^0 = 1$ fixed (representing the initial density of cooperators and defectors)
defined by: an initial probability measure of $\mathbb{R}_+$, $m_0$ representing the initial distribution of wealth of all individuals and an increment relation:

$$
\mathcal{D}(t_{n+1}^D) - \mathcal{D}(t_n^D) = 1_{\mathcal{D}(t_n^D) > 0} U_D(t_{n+1}^D) \\
\mathcal{C}(t_{n+1}^C) - \mathcal{C}(t_n^C) = 1_{\mathcal{C}(t_n^C) > 0} U_C(t_{n+1}^C)
$$

(8) (9)

where:

- $(t_n^D)$ and $(t_n^C)$ are sequences of Poisson times of intensity $v$.

- for $I = \{t_n^C, n \in \mathbb{N}\} \cup \{t_n^D, n \in \mathbb{N}\}$, all $t \in I$ and $U_D(t)$ and $U_C(t)$ are random variables independent from everything such that:

$$
U_D(t) = \begin{cases} 
-2P & \text{with probability } \frac{1}{2}\rho^0 \mathbb{P}(\mathcal{D}(t) > 0) \\
T & \text{with probability } \beta^0 \mathbb{P}(\mathcal{C}(t) > 0) \\
0 & \text{with probability } 1 - \beta^0 \mathbb{P}(\mathcal{C}(t) > 0) - \frac{1}{2}\rho^0 \mathbb{P}(\mathcal{D}(t) > 0)
\end{cases}
$$

and

$$
U_C(t) = \begin{cases} 
-S & \text{with probability } \rho^0 \mathbb{P}(\mathcal{D}(t) > 0) \\
R & \text{with probability } \beta^0 \mathbb{P}(\mathcal{C}(t) > 0) \\
0 & \text{with probability } 1 - \beta^0 \mathbb{P}(\mathcal{C}(t) > 0) - \rho^0 \mathbb{P}(\mathcal{D}(t) > 0)
\end{cases}
$$

$(\mathcal{C}(t), \mathcal{D}(t))_t$ represents the wealth of a typical cooperator and a typical defector in an infinite population of individuals moving at a high speed.

### 3.2.2 Justification of the mean field model

To arrive to the Mean Field model from the spatial model we have to assume that:

- There is no birth anymore. As a consequence $Z$ is constant over time. We also set $N \in \mathbb{N}$ the number of particles in the system.

- The collection $(Z_i)_{1 \leq i \leq N}$ are i.i.d with distribution $\beta^0 \delta_0 + \rho^0 \delta_1$ (with $\beta^0 + \rho^0 = 1$ previously fixed).

- The moving rate $d$ belongs to $\mathbb{N}$ instead of $\mathbb{R}_+$. This is done in order to have an homogenization result.

- The size of the torus depends on the initial number of particles. We set: $m^2 = N$. (We can also not link the size of the torus and the initial number of particles but in that case we have to slow the time of evolution replacing $(X(t), Y(t), Z)_t$ by $(X(t/N), Y(t/N), Z)_t$).

- Initially the distribution of wealth of all individuals is $m_0$ probability measure of $\mathbb{R}_+$. 

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The generator of the spatial system is $A$ with domain $C_b(E)$: for all $(x, y, z) \in E$

$$Af(x, y, z) = A_d f(x, y, z) + v A_g f(x, y, z).$$

with $A_d$ defined in (6) and $A_g$ in (7).

Using Theorem 4.2 of [Gibaud, 2016], when $d$ goes to $+\infty$, we have that there exists $\mu$ probability measure of $D([\mathbb{R}^+, \mathbb{R} \times \{0, 1\})$ such that the particle system described by the generator $A$ is $\mu$-chaotic. The analytic way of describing $\mu$ is done in the following. The useful consequence of that is the following convergence in finite dimensional distribution: with $(Y, z)$ random variable taking value in $D([\mathbb{R}^+, \mathbb{R} \times \{0, 1\})$ and with distribution $\mu$

$$\forall i \in \mathbb{N}, \quad (Y^i, Z^i) \overset{D}{\underset{d \to +\infty}{\rightarrow}} (Y, z). \quad (10)$$

Let us describe analytically $\mu$. In order to make the notations lighter we use the following:

- $\rho_t := \mathbb{P}(Y(t) > 0 \mid z = 1)$,
- $\beta_t := \mathbb{P}(Y(t) > 0 \mid z = 0)$.

To have a good description of the distribution of $(Y(t), z)_t$ we use evolution equations given in Corollary 4.3 of [Gibaud, 2016]. The distribution of $(Y(t), z)_t$ verifies the following evolution equations: with $\mathcal{L}(Y(0)) = m_0$ and $\mathcal{L}(z) = \rho_0 \delta_0 + (1 - \rho) \delta_1$:

- for all $y \in \mathbb{R}$
  $$\partial_t \mathbb{P}(Y(t) = y \mid z = 0) = \begin{aligned} & v \mathbf{1}_{y > R} \beta^0 \beta_t \mathbb{P}(Y(t) = y - R \mid z = 0) \hfill \\
  & + v \beta^0 \rho_t \mathbb{P}(Y(t) = y + S \mid z = 0) \hfill \\
  & - v(\rho^0 \rho_t + \beta^0 \beta_t) \mathbb{P}(Y(t) = y \mid z = 0) \end{aligned} \quad (11)$$

- for all $y \in \mathbb{R}$
  $$\partial_t \mathbb{P}(Y(t) = y \mid z = 1) = \begin{aligned} & v \mathbf{1}_{y > T} \beta^0 \beta_t \mathbb{P}(Y(t) = y - T \mid z = 1) \hfill \\
  & + v \frac{T}{2} \rho_t \mathbb{P}(Y(t) = y + 2 \mid z = 1) \hfill \\
  & + v \frac{T}{2} \rho_t \mathbb{P}(Y(t) = y \mid z = 1) \hfill \\
  & - v(\rho^0 \rho_t + \beta^0 \beta_t) \mathbb{P}(Y(t) = y \mid z = 1). \end{aligned} \quad (12)$$

We want to prove that $(\mathcal{C}, \mathcal{D})$ is a good description of $(Y(t), z)_t$. Firstly let us denote $\beta'_t = \mathbb{P}(\mathcal{C}(t) > 0)$ and $\rho'_t = \mathbb{P}(\mathcal{D}(t) > 0)$. Let’s verify that $(\mathcal{C}(t))_t$.
and \((\mathfrak{D}(t))_t\) verifies the ordinary differential equation (11), (12). Because for all \(y \in \mathbb{R}_+\), \(\forall y' \notin \{y + x : x \in \{-R, S\}\}\), \(\mathbb{P}(\mathcal{C}(t + h) = y|\mathcal{C}(t) = y') = o(h)\) we have:

\[
\mathbb{P}(\mathcal{C}(t + h) = y) = \mathbb{P}(\mathcal{C}(t + h) = y \cap \mathcal{C}(t) = y) + \mathbb{P}(\mathcal{C}(t + h) = y \cap \mathcal{C}(t) \neq y)
\]

\[
= \mathbb{P}(\mathcal{C}(t + h) = y|\mathcal{C}(t) = y) \mathbb{P}(\mathcal{C}(t) = y) + \mathbb{P}(\mathcal{C}(t + h) = y \cap \mathcal{C}(t) \in \{y - R, y + S\}) + o(h)
\]

\[
= (1 - vh(\beta^0 \beta'_1 + \rho^0 \rho'_1)) \mathbb{P}(\mathcal{C}(t) = y)
\]

\[
+ vh\beta^0 \beta'_1 1_{y>R} \mathbb{P}(\mathcal{C}(t) = y - R) + vh\rho^0 \rho'_1 \mathbb{P}(\mathcal{C}(t) = y + T) + o(h).
\]

\[
\mathbb{P}(\mathcal{C}(t + h) = y) \text{ is equal to: }
\]

\[
\mathbb{P}(\mathcal{C}(t) = y) vh(1 - \rho_i - \beta_i) + 1_{k>R} vh\beta_i \mathbb{P}(\mathcal{C}(t) = y - R) + vh\rho_i \mathbb{P}(\mathcal{C}(t) = y + S) + o(h).
\]

Dividing by \(h\) and making \(h \rightarrow 0\) we get the evolution equations (11) and (12) by saying that for all \(y \in \mathbb{R}\) \(\mathbb{P}(Y(t) = y|z = 0) = \mathbb{P}(\mathcal{C}(t) = y)\) and \(\mathbb{P}(Y(t) = y|z = 1) = \mathbb{P}(\mathfrak{D}(t) = y)\).

Since this process is non linear, we analyze a linearized version \((\mathfrak{T}(t), \mathfrak{D}(t))_t\). The linearized system is a system where cooperators and defectors can play even if their wealth are negative. It is a Mean field ghost version. The wealth of cooperators in the linearized \((\mathfrak{T}(t))_t\) is defined by an initial measure \(m_0\) and an increment relation:

\[
\mathfrak{T}(t^\xi_{n+1}) - \mathfrak{T}(t^\xi_n) = \mathfrak{U}_\xi(t^\xi_n)
\]

(13)

where:

- \((t^\xi_n)_n\) are defined in [8].
- for all \(t \in \{t^\xi_n, n \in \mathbb{N}\}\) and \(\mathfrak{U}_\xi(t)\) are random variables independent from everything such that:

\[
\mathfrak{U}_\xi(t) = \begin{cases} 
-S & \text{with probability } \rho^0 \\
R & \text{with probability } \beta^0 \\
0 & \text{with probability } 1 - \beta^0 - \rho^0
\end{cases}
\]

The consequence of this linearization is the following theorem that gives us a simple condition to have "survival" of cooperators ad vitam eternam with positive probability.

**Theorem 3.3.** Let us suppose that:

\[
\beta^0 R - \rho^0 S > 0
\]

(14)

Then we have: for a fixed \(q_0 > 0\) when \(t \rightarrow +\infty\) \(\mathbb{P}_{q_0}\) almost surely

\[
\mathfrak{T}(t) \rightarrow +\infty
\]

and

\[
\mathbb{P}_{q_0}(\forall t \geq 0, \mathfrak{T}(t) > 0) > 0.
\]

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We also have a result about the concentration of wealth of cooperators.

**Proposition 3.4.** For all $q_0 \in \mathbb{R}_+$

We denote:

$$m := v (\beta^0 R - \rho^0 S), \quad \sigma^2 := v (\beta^0 R^2 + \rho^0 S^2).$$

Then we have: $\forall \eta > 0, t \geq 0$

$$
\mathbb{P}_{q_0} \left( \bar{C}(t) \in \left[ q_0 + mt - \eta \sqrt{\sigma^2 t}, q_0 + mt + \eta \sqrt{\sigma^2 t} \right] \right) \geq 1 - \eta^{-2}. \quad (15)
$$

Moreover for all $\tau$ such that:

$$
\tau < q_0 - \frac{\eta^2 \sigma^2}{4m},
$$

then

$$
\mathbb{P}(\bar{C}(t) > \tau) \geq 1 - \eta^{-2}.
$$

4 Proofs on the Spatial Model

Before beginning the proofs let us introduce some useful tools. In Section 2.1, every movement, game, birth is managed by independent Poisson processes with rates $d, v$ or $b$. Instead of considering this collection of Poisson processes we consider:

1. A unique Poisson process with rate $K(b + d) + K(K-1) v$ i.e. the sum of the rates of the previous Poisson processes. This Poisson process gives a sequence of Poisson times $(t_n)_n$

2. A sequence (independent of everything) of independent random variables $(C_n)_n$ selects if a movement, game or birth happens. An element of this sequence choose a movement with probability $\frac{Kd}{K(b+d)+\frac{K(K-1)}{2} v}$, a game with probability $\frac{K(K-1)v}{K(b+d)+\frac{K(K-1)}{2} v}$ and a birth with probability $\frac{Kb}{K(b+d)+\frac{K(K-1)}{2} v}$.

3. 3 sequences $(E^m_n)_n, (E^g_n)_n, (E^b_n)_n$, (independent of everything) of i.i.d random variables uniformly choosing the agents. $(E^m_n)_n$ and $(E^b_n)_n$ are made of uniform on $\{1, \ldots, K\}$ random variables. $(E^g_n)_n$ is made of uniform on $\left\{1, \ldots, \frac{K(K-1)}{2}\right\}$ random variables.

4. At the $n$-th realization of the unique Poisson process, $C_n$ chooses the type of evolution (movement, game, or birth). If $C_n = \text{movement}$ (resp birth) then $E^m_n$ (resp $E^b_n$) gives the particle that will move (resp will give birth (if she is wealthy enough and if there are less than $K$ particles
on the torus)). If the particle moves, we draw \( V_n \) an other uniform on \{Top, Bottom, Left, Right\} random variable to decide where the particle moves. If \( C_n = \text{game} \) then \( E_n^a \) gives the couple of particles that will play (if their \( Z \neq -1 \), if they are on the same site and if their wealths are positive).

In this section, instead of considering continuous time Markov process \((X(t), Y(t), Z(t))_t\) we consider the induced Markov chain \((X_n, Y_n, Z_n)_n := (\sigma_n)_n \) where \( \forall n \in \mathbb{N} \) \((X_n, Y_n, Z_n) = (X(t_n), Y(t_n), Z(t_n))\).

**Definition 4.1.** We call the photograph of a configuration \( \sigma \), the data of the positions of individuals and their strategies. The photograph of \( \sigma \) can be seen as the equivalent class of \( \sigma \) for the following equivalence relation: \( \sigma \sim \tilde{\sigma} \) if and only if \( \forall x \in (\mathbb{Z}/m\mathbb{Z})^2, \forall z \in \{0, 1\} \)

\[
\text{Card} \{i \in \{1, \ldots K\}, X^i = x, Y^i > 0, Z^i = z\} = \text{Card} \{i \in \{1, \ldots K\}, \tilde{X}^i = x, \tilde{Y}^i > 0, \tilde{Z}^i = z\}.
\]

In words, two configurations are equivalent if and only if on each position \( x \in (\mathbb{Z}/m\mathbb{Z})^2 \) there is the same number of cooperators with positive wealth and defectors with positive wealth.

We denote by \( p(\bullet) \) the canonical projection on the space of photographs.

**Remark.** There are less than \( \prod_{N=0}^K (2m^2)^{N} = (2m^2)^{N}^{(K+1)/2} \) photographs. We call \( q \) the number of photographs. Those photographs are denoted \( p_1, \ldots, p_q \).

**Notations.** For simplicity, we will denote the wealth of cooperator \( i \) (with \( Z^i = 0 \)), \( C^i \) and the wealth of defector \( j \) (with \( Z^j = 1 \)), \( D^j \).

For a configuration \( \sigma \) we denote \( N_C(\sigma) \) (resp. \( N_D(\sigma) \)) the set of the indexes of the cooperators (resp. defectors) with positive wealth.

### 4.1 Almost sure extinction

In this section we prove Theorem [I]

**Proof.**

Firstly let us define the first cooperator-defector time. For that let us denote for all \( n \in \mathbb{N} \), \( C^{tot}_n := \sum_{i=1}^{K} 1_{Z^i=0} Y^i_n \) the sum of wealth of cooperators.

**Definition 4.2.** We define \( \tau_1 \) the first time when a cooperator-defector game happens that is:

\[
\tau_1 = \inf \{ n \in \mathbb{N}^*/C^{tot}_n - C^{tot}_{n-1} < 0 \}
\]

Let us denote \( (\tau_i)_i \) the sequence of these cumulated stopping time with \( \tau_0 = 0 \). If there is no cooperator left after \( \tau_i \) we define for all \( k \in \mathbb{N}^* \) \( \tau_{i+k} = +\infty \) a.s. 

That is the time of the first Cooperator-Defector game is \( \tau_1 \), the time of the second one is \( \tau_2 \) et so on.
Firstly let us prove the uniform (on the configurations) upper bounding lemma.

**Lemma 4.3.** There is $m \in \mathbb{N}$ and $\varepsilon > 0$ such that for each configuration $\sigma \in \mathcal{C}'$ and all $s \in \mathbb{R}$ we have:

$$\mathbb{P}_\sigma(\tau_1 > s) \leq (1 - \varepsilon)[s/m]$$

**Proof.** Firstly let us notice, if there is at least one defector initially, defectors cannot be extinct. Indeed the only way that a defector dies is via Defector-Defector game yet since the payoff of such a game is a random variable with distribution $\frac{1}{2}\delta_{(0,-2P)} + \frac{1}{2}\delta_{(-2P,0)}$ then, when they play together two defectors cannot die at the same time. As a consequence there will be at least one defector left at any time. Moreover since $((\mathbb{Z}/m\mathbb{Z})^2$ is finite we have for each $\sigma \in \mathcal{C}'$ $\tau_1 < +\infty$ $\mathbb{P}_\sigma$ a.s.

Let $(T_n)_n$ be the sequence of events $T_n = \{\tau_1 = n\}$. Let $m$ the maximum number of realizations of Poisson process for a defector to play with a cooperator with positive probability in all configurations i.e.

$$m = \max_{\sigma \in \mathcal{C}'} \min_n \{n \in \mathbb{N}/\mathbb{P}_\sigma(T_n) > 0\}.$$

Because there is a finite number of photographs the existence of $m$ is insured. Then there is $\varepsilon > 0$ such that for each configuration:

$$\mathbb{P}_\sigma(\tau_1 \leq m) \geq \varepsilon \quad (16)$$

For each configuration $\sigma^0 \in \mathcal{C}'$ and all $k \in \mathbb{N}^*$ we have:

$$\mathbb{P}_{\sigma^0}(\tau_1 > km) = \mathbb{P}_{\sigma^0}(\tau_1 > m) \mathbb{P}_{\sigma^0}(\tau_1 > km | \tau_1 > m) \leq (1 - \varepsilon) \sum_{\sigma^1 \in \mathcal{C}'} \mathbb{P}_{\sigma^0}(\tau_1 > km, \sigma_m = \sigma^1 | \tau_1 > m)$$

Moreover by the Strong Markov property, we have:

$$\mathbb{P}_{\sigma^0}(\tau_1 > km) \leq (1 - \varepsilon) \sum_{\sigma^1 \in \mathcal{C}'} \mathbb{P}_{\sigma^0}(\tau_1 > (k - 1)m | \sigma_m = \sigma^1 | \tau_1 > m) \mathbb{P}_{\sigma^0}(\sigma_m = \sigma^1 | \tau_1 > m)$$

$$\leq (1 - \varepsilon)^k \prod_{i=1}^{k} \sum_{\sigma^i \in \mathcal{C}'} \mathbb{P}_{\sigma^{i-1}}(\sigma_m = \sigma^i | \tau_1 > m) \mathbb{P}_{\sigma^{i-1}}(\sigma_m = \sigma^i | \tau_1 > m) \mathbb{P}_{\sigma^{i-1}}(\sigma_m = \sigma^i | \tau_1 > m)$$

$$\leq (1 - \varepsilon)^k$$

$\square$
Now that the lemma is proved let us use it to prove the theorem.

$$E_{\sigma}(c_{\tau_n+1}^{\text{tot}}|\exists i \in N_{\xi}(\tau_n) c_{\tau_n}^i > 0) = \sum_{\sigma' \in C'} E_{\sigma'} \left( \left( c_{\tau_n+1}^{\text{tot}} - c_{\tau_n}^{\text{tot}} \right) 1_{\sigma_{\tau_n} = \sigma'| \exists i \in N_{\xi}(\tau_n) c_{\tau_n}^i > 0} \right)$$

$$= \sum_{\sigma' \in C'} E_{\sigma'} \left( (c_{\tau_1}^{\text{tot}} - c_0^{\text{tot}}) P_{\sigma} (\sigma_{\tau_n} = \sigma'| \exists i \in N_{\xi}(\tau_n) c_{\tau_n}^i > 0) \right)$$

$$\leq \sum_{\sigma' \in C'} (E_{\sigma'} (\tau_1 R - S) P_{\sigma} (\sigma_{\tau_n} = \sigma'| \exists i \in N_{\xi}(\tau_n) c_{\tau_n}^i > 0)$$

$$\leq \mu R - S.$$

The hypothesis of the theorem is:

$$\nu = \mu R - S < 0$$

We have: for all \( n \in \mathbb{N} \)

$$c_{\tau_n+1}^{\text{tot}} = \sum_{k=0}^{n} c_{\tau_k+1}^{\text{tot}} - c_{\tau_k}^{\text{tot}} + c_0^{\text{tot}}$$

Let \( \sigma \in C' \) we get:

$$E_{\sigma}(c_{\tau_n+1}^{\text{tot}}) = E_{\sigma}(c_0^{\text{tot}}) + \sum_{k=0}^{n} E_{\sigma}(c_{\tau_k+1}^{\text{tot}} - c_{\tau_k}^{\text{tot}})$$

$$= E_{\sigma}(c_0^{\text{tot}}) + \sum_{k=0}^{n} E_{\sigma}(c_{\tau_k+1}^{\text{tot}} - c_{\tau_k}^{\text{tot}}|\exists i \in N_{\xi}(\tau_k) c_{\tau_k}^i > 0) P_{\sigma}(\exists i \in N_{\xi}(\tau_k) c_{\tau_k}^i > 0)$$

$$+ \sum_{k=0}^{n} E_{\sigma}(c_{\tau_k+1}^{\text{tot}} - c_{\tau_k}^{\text{tot}}|\forall i \in N_{\xi}(\tau_k) c_{\tau_k}^i \leq 0) P_{\sigma}(\forall i \in N_{\xi}(\tau_k) c_{\tau_k}^i \leq 0)$$

$$\leq E_{\sigma}(c_0^{\text{tot}}) + \nu \sum_{k=0}^{n} P_{\sigma}(\exists i \in N_{\xi}(\tau_k) c_{\tau_k}^i > 0)$$

But \( \nu < 0 \) and \( \forall n \in \mathbb{N} \) \( E_{\sigma}(c_{\tau_n}^{\text{tot}}) > -KS \) then we have:

$$\sum_{k=0}^{+\infty} P_{\sigma}(\exists i \in N_{\xi}(\tau_k) c_{\tau_k}^i > 0) < +\infty.$$

Applying Borel Cantelli’s Lemma we have: \( P_{\sigma} \) almost surely eventually all co-operators will be dead.

4.2 Coexistence ad vitam eternam

Before proving Theorem 2.3 let us prove an almost sure divergence lemma.
Lemma 4.4. Let \((X_n)_n\) a Markov chain on \(\mathbb{R}\) such that:

- \(\exists X\) random variable in \(\mathbb{R}_+\) such that: \(\forall x_0 \in \mathbb{R}, \mathbb{E}_{x_0}(X) \leq \kappa, \mathbb{E}_{x_0}(X^2) \leq \kappa'\) and \(\mathbb{E}_{x_0}(X^4) \leq \kappa''\) and such that: \(\forall x_0 \in \mathbb{R} \forall n \in \mathbb{N}\)
  \[|X_{n+1} - X_n| \leq X \quad \mathbb{P}_{x_0} \text{ a.s.}\]

- \(\exists \tilde{X}\) random variable from \((\Omega, \mathcal{A}, \mathbb{P})\) to \(\mathbb{R}\) with positive expectation \(\mathbb{E}(X) = \alpha > 0\) such that:
  \[\forall s \in \mathbb{R}, \forall x_0 \in \mathbb{R}, \quad \mathbb{P}_{x_0}(X_1 - X_0 > s) \leq \mathbb{P}(\tilde{X} > s)\]

Then we have: for a fixed \(0 < \delta < \alpha \mathbb{P}_\sigma\) almost surely: \(\exists N_0(\sigma) \in \mathbb{N}, \forall N > N_0(\sigma)\)

\[X_N > \delta N \rightarrow +\infty.\]

Corollary 4.5. Let \((\sigma_n)_n\) a Markov chain of a space \(E\) with countable dimension. Let \((X_n)_n\) be one of its coordinates. Let us suppose

- \(\exists X\) random variable in \(\mathbb{R}_+\) such that: \(\forall x_0 \in \mathbb{R}, \mathbb{E}_{x_0}(X) \leq \kappa, \mathbb{E}_{x_0}(X^2) \leq \kappa'\) and \(\mathbb{E}_{x_0}(X^4) \leq \kappa''\) and such that: \(\forall x_0 \in \mathbb{R} \forall n \in \mathbb{N}\)
  \[|X_{n+1} - X_n| \leq X \quad \mathbb{P}_{x_0} \text{ a.s.}\]

- \(\exists \tilde{X}\) random variable from \((\Omega, \mathcal{A}, \mathbb{P})\) to \(\mathbb{R}\) with positive expectation \(\mathbb{E}(X) = \alpha > 0\) such that:
  \[\forall s \in \mathbb{R}, \forall x_0 \in \mathbb{R}, \quad \mathbb{P}_{x_0}(X_1 - X_0 > s) \leq \mathbb{P}(\tilde{X} > s)\]

Then we have: for a fixed \(0 < \delta < \alpha \mathbb{P}_\sigma\) almost surely: \(\exists N_0(\sigma) \in \mathbb{N}, \forall N > N_0(\sigma)\)

\[X_N > \delta N \rightarrow +\infty.\]

The proof of the corollary is the same as the proof of the lemma except that the initial condition is for example with \(\sigma \in E \{\sigma_0 = \sigma\}\) and the Strong Markov property are done on \((\sigma_n)_n\). Let us begin the proof of the lemma.

Proof. To prove that \(X_n \rightarrow +\infty\) almost surely we firstly prove that \((X_n)_n\) is a sub-martingale and use Doob decomposition, then we upper bound the expectation of the quadratic variation of the martingale part of \(X_n\) in order to finally use Borel Cantelli’s Lemma with Burkholder-Davis-Gundy inequality.

1. Let us prove that \((X_n)_n\) is a sub-martingale. Let \(x \in \mathbb{R}\).

\[
\mathbb{E}_x(X_{n+1}|X_n) = \mathbb{E}_x(X_n + X_{n+1} - X_n|X_n) \\
= X_n + \mathbb{E}_{X_n}(X_1 - X_0) \\
\geq X_n + \alpha \\
\geq X_n.
\]
Then using a Doob decomposition we get: for all \( x \in \mathbb{R} \)

\[
X_n = M_n + \sum_{k=0}^{n-1} \mathbb{E}_x(X_{k+1} - X_k | X_k)
\]

with \( M_0 = 0 \) and \( (M_n)_n = \left( \sum_{k=0}^{n-1} X_{k+1} - \mathbb{E}_x(X_{k+1} | X_k) \right)_n \) a martingale.

2. Now let us upper bound for all \( x \in \mathbb{R} \) and all \( N \in \mathbb{N} \), \( \mathbb{E}_x([M_N]^2) = \mathbb{E}_x\left( \sum_{k=0}^{N-1} (M_{k+1} - M_k)^2 \right)^2 \)

- Let us prove that \( \exists c > 0 \) such that \( \forall k \in \mathbb{N} \) \( \forall x \in \mathbb{R} \) \( \mathbb{E}_\sigma((M_{k+1} - M_k)^2) \leq c \). First let us rewrite \( M_{k+1} - M_k \) in function of the variation of \( X_k \).

\[
M_{k+1} - M_k = X_{k+1} - X_k - \mathbb{E}_x(X_{k+1} - X_k | X_k).
\]

We get using Strong Markov property:

\[
\mathbb{E}_x(|X_{k+1} - X_k| | X_k) = \mathbb{E}_{X_k}(|X_1 - X_0|) \in [-\kappa, \kappa]
\]

Then we get:

\[
|X_{k+1} - X_k - \mathbb{E}_x(X_{k+1} - X_k | X_k)| \leq |X_{k+1} - X_k| + \kappa
\]

Then using Strong Markov property we have: \( \forall x \in \mathbb{R} \)

\[
\mathbb{E}_\sigma((M_{k+1} - M_k)^2) \leq \mathbb{E}_\sigma\left( \mathbb{E}_{X_k} (X_1 - X_0)^2 \right) + 3\kappa^2 \leq \kappa' + 3\kappa^2 := c
\]

- Let us prove that \( \exists c' > 0 \) such that \( \forall N \in \mathbb{N} \) \( \forall x \in \mathbb{R} \) \( \mathbb{E}_x([M_N]^2) \leq N^2c^2 + Nc' \)

Using the previous arguments we have: \( \exists c' > 0 \) such that for all \( k \in \mathbb{N} \):

\[
\mathbb{E}_x((M_{k+1} - M_k)^4) \leq c'
\]

We have:

\[
\mathbb{E}_x([M_N]^2) = \sum_{0 \leq k, \ell \leq N-1} \mathbb{E}_x((M_{k+1} - M_k)^2(M_{\ell+1} - M_\ell)^2)
\]

\[
= \sum_{0 \leq k \neq \ell \leq N-1} \mathbb{E}_x((M_{k+1} - M_k)^2(M_{\ell+1} - M_\ell)^2) + \sum_{k=0}^{N-1} \mathbb{E}_x((M_{k+1} - M_k)^4)
\]

Using Strong Markov property we have for all \( k < \ell \):

\[
\mathbb{E}_x((M_{k+1} - M_k)^2(M_{\ell+1} - M_\ell)^2) = \mathbb{E}_x\left((M_{k+1} - M_k)^2\mathbb{E}_x\left((M_{\ell+1} - M_\ell)^2|F_{\tau_i}\right)\right)
\]

\[
= \mathbb{E}_x\left((M_{k+1} - M_k)^2\mathbb{E}_{X_{\tau_i}}(M_{\ell-k+1} - M_{\ell-k})\right)
\]

\[
\leq c \mathbb{E}_x((M_{k+1} - M_k)^2) \leq c^2.
\]
Finally we have:

$$E_x(|M_N|^2) \leq N^2c^2 + Nc' \quad (17)$$

3. Firstly let us notice that 

$$E_x((-M_N)^2) \leq N^2c^2 + Nc'. $$

To finish the divergence of $\langle X_n \rangle_n$ we need Burkholder-Davis-Gundy inequality [Beiglböck et al., 2015] which says that for $1 \leq p < +\infty$ there exists $a_p < +\infty$ such that for every $N \in \mathbb{N}$ and every $(M_n)_n$ martingale we have:

$$E \left( \left( \max_{1 \leq n \leq N} M_n \right)^p \right) \leq a_p E \left( [M_N]^{p/2} \right)$$

Then we have:

$$0 < \delta < \alpha$$

$$\mathbb{P}_x \left( \min_{0 \leq n \leq N} M_n < -N(\alpha + \delta) \right) = \mathbb{P}_x \left( \max_{0 \leq n \leq N} -M_n > N(\alpha + \delta) \right) \leq \mathbb{P}_x \left( \max_{0 \leq n \leq N} -M_n \right)^4 > N^4(\alpha + \delta)^4$$

$$\leq \frac{1}{N^4(\alpha + \delta)^4} \mathbb{E}_x \left( \left( \max_{0 \leq n \leq N} -M_n \right)^4 \right)$$

$$\leq a_4 \frac{1}{N^4(\alpha + \delta)^4} \mathbb{E}_x (|M_N|^2) \leq a_4 \frac{N^2c^2 + Nc'}{N^4(\alpha + \delta)^4}$$

which is summable.

Applying Borel Cantelli’s lemma we have $\mathbb{P}_x$ almost surely: $\exists N_0(\sigma) \in \mathbb{N}, \forall N > N_0(x)$

$$X_N \geq M_N + \alpha N > \delta N \longrightarrow +\infty.$$ 

\[ \square \]

Let us now prove Theorem 2.3.

Proof. The configuration $\sigma^0$ is fixed for the sequence of the proof.

Definition 4.6. For all configuration $\sigma \in \mathcal{C}$, for a cooperator $i \in N_{\mathcal{C}}(\sigma)$ we denote $\tau_i^1$ the first time where there is a game between $i$ and a cooperator.

$$\tau_i^1 = \inf \left\{ n \in \mathbb{N}^+ / \mathcal{E}_n^i - \mathcal{E}_{n-1}^i > 0 \right\}$$

Let us denote $(\tau_i^1)_n$ the sequence of these cumulated stopping times (by the same way we define the cumulative stopping times in the previous proof) with $\tau_i^0 = 0$ for all cooperators.

The issue here is that we can have extinction of the cooperators. Hence we don’t have $\forall \sigma \in \mathcal{C} \forall i \in N_{\mathcal{C}}(\sigma) \tau_i^1 < +\infty \mathbb{P}_\sigma$ a.s.

Hence we have to consider another system and couple them.
1. We call the system introduced in Section 2.1 the **True System**. Let $C^i_n$ be the wealth of cooperator $i \in N_E(\sigma)$ at time $n \in \mathbb{N}$ in the **True System**.

2. The other system is called the **Ghost system**. In this system we have that:
   - cooperators and defectors can play even if they have negative wealth,
   - cooperators cannot give birth,
   - at each step of time, there is a decrease of $w_0$ of cooperator wealths,
   - when defectors give birth they don’t loose wealth (this part is not necessary to make the proof but make the proof easier).

We denote the probability measure associated to the **Ghost system** $G$. For example probability that the wealth of cooperator $i \in N_E(\sigma_0)$ is 3 in the **Ghost system** is denoted $G_{\sigma_0}(C^i_n = 3)$.

It is the following system $(\tilde{\sigma}_t)$ that we couple to $(\sigma_t)$. Since the cooperators won’t die in the **Ghost system** we have $\forall \sigma \in C \forall i \in N_E(\sigma) \tau^i_1 < +\infty \ \mathbb{G}_\sigma$.

We have the following upper bounding lemma. The proof is the same as Lemma 4.3, the only modification is to change $T_n = \{\tau_1 = n\}$ by $T_n = \left\{ \max_{i \in N_E(\sigma)} \tau^i_1 = n \right\}$.

**Lemma 4.7.** There is $m \in \mathbb{N}$ and $\varepsilon > 0$ such that for each configuration $\sigma \in E$ with at least two cooperators and all $s \in \mathbb{R}$ we have:

$$G_{\sigma} \left( \max_{i \in N_E(\sigma)} \tau^i_1 > s \right) \leq (1 - \varepsilon)^{\lfloor s/m \rfloor}.$$

We denote $G$ the expectation using the probability measure $G$. We look at the wealth of one individual then the birth it gives, make its wealth decrease, then the upper bounding of the variation of wealth has to take it into account.

Using the same argument as those in the previous theorem, we prove that: $\exists \nu > 0$ such that: $\forall n \in \mathbb{N} \forall \sigma \in E$ with at least two cooperators

$$G_{\sigma} \left( C^i_{\tau^i_n+1} - C^i_{\tau^i_n} \right) \geq R - \nu(S + w_0) := \alpha > 0$$

Also for all $n \in \mathbb{N}$ we have: (because $R < S + w_0$)

$$|C^i_{\tau^i_n+1} - C^i_{\tau^i_n}| \leq (S + w_0)\tau^i$$

Then using Lemma 4.7 we can apply Corollary 4.5 to get: for a fixed $0 < \delta < \alpha$ $\forall i \in N_E(\sigma) \mathbb{G}_{\sigma}$ almost surely:

$$C^i_{\tau^i_n} > \delta \tau^i_n \rightarrow +\infty$$

Then intersecting those events we have for a fixed $0 < \delta < \alpha$, $\mathbb{G}_{\sigma}$ almost surely:

$$\forall i \in N_E(\sigma), \quad C^i_{\tau^i_n} > \delta \tau^i_n \rightarrow +\infty. \quad (18)$$
Now let us prove that \(\min_{i\in N_{\sigma}(\sigma)} C^i_n\) cannot go under 0 infinitely often. Because we have \(\forall n \in \mathbb{N} \forall k \in \{\tau^i_n, \ldots, \tau^i_{n+1}\}\) \(C^i_n > C^i_{\tau^n_k} - (S + w_0)(\tau^i_{n+1} - \tau^i_n)\) and because \(C^i_{\tau^i_{n+1}} > \delta \tau^i_n\geq n\) we have for all \(n \in \mathbb{N}\):

\[
\exists k \in \{\tau^i_n, \ldots, \tau^i_{n+1}\}, C^i_k \leq 0 \subseteq \{\delta \tau^i_n \leq (S + w_0)(\tau^i_{n+1} - \tau^i_n)\}
\]

\[
\subseteq \left\{ \min_{j \in N_{\sigma}(\sigma)} \delta \tau^j_n \leq (S + w_0) \max_{j \in N_{\sigma}(\sigma)} (\tau^j_{n+1} - \tau^j_n) \right\}
\]

\[
\subseteq \left\{ \delta n \leq (S + w_0) \max_{j \in N_{\sigma}(\sigma)} (\tau^j_{n+1} - \tau^j_n) \right\}
\]

Hence we get: for all \(n \in \mathbb{N}\)

\[
\mathbb{G}_{\sigma} \left( \min_{i \in N_{\sigma}(\sigma)} C^i_n \leq 0 \right) \leq K \mathbb{G}_{\sigma} \left( \max_{i \in N_{\sigma}(\sigma)} (\tau^i_{n+1} - \tau^i_n)(S + w_0) > \delta n \right)
\]

\[
\leq K(S + w_0)^2 \frac{1}{\delta^2 n^2} \mathbb{G}_{\sigma} \left( \max_{i \in N_{\sigma}(\sigma)} (\tau^i_{n+1} - \tau^i_n)^2 \right)
\]

which is summable using Lemma 4.7 Then we have with (18)

\[
\mathbb{G}_{\sigma} \left( \min_{i \in N_{\sigma}(\sigma)} C^i_n \leq 0 \ i.o. \right) = 0. \tag{19}
\]

We do the same reasoning with the defectors. We have \(\exists \nu' > 0\) and for all \(i \in N_{\bar{D}}(\sigma)\) there exists \((\tilde{\tau}^i_n)\) such that there is \(\bar{m} \in \mathbb{N}\) and \(\bar{\varepsilon} > 0\) such that for each configuration \(\sigma \in E\) with at least one cooperator and one defector and all \(s \in \mathbb{R}\) we have:

\[
\mathbb{G}_{\sigma} \left( \max_{i \in N_{\bar{D}}(\sigma)} \tilde{\tau}^i_n > s \right) \leq (1 - \bar{\varepsilon})^{\lfloor s/\bar{m} \rfloor}
\]

Then we get for a fixed \(0 < \delta' < T - \nu'(2P + w_0), \mathbb{G}_{\sigma}\) almost surely:

\[
\forall i \in N_{\bar{D}}(\sigma), \quad \bar{D}^i_{\tilde{\tau}^i_n} > \delta' \tilde{\tau}^i_n \longrightarrow +\infty \tag{20}
\]

and also we have:

\[
\mathbb{G}_{\sigma} \left( \min_{i \in N_{\bar{D}}(\sigma)} \bar{D}^i_n \leq 0 \ i.o. \right) = 0 \tag{21}
\]

Let \(D = \min \left( \min_{i \in N_{\sigma}(\sigma)} \min_{n \in \mathbb{N}} C^i_n, \min_{i \in N_{\bar{D}}(\sigma)} \min_{n \in \mathbb{N}} \bar{D}_n^i \right)\), using (18), (19), (20) and (21) we have for all \(\sigma \in \mathcal{C}, \mathbb{G}_{\sigma}(D = -\infty) = 0\). A a consequence for \(0 < p \leq 1\) fixed, \(\exists L \in \mathbb{R}\) such that:

\[
\mathbb{G}_{\sigma}(D \leq L) \leq p
\]

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Let us denote \( \sigma + L \) the set of configurations such that for all \( \sigma \in \sigma + L \) \( p(\sigma) = p(\bar{\sigma}) \) and the wealth of every individual with positive wealth in \( \sigma \) is increased by at least \( |L| \) in \( \bar{\sigma} \). Then if in \( \sigma \) an individual has a wealth of \( k > 0 \), in configurations of \( \sigma + L \) he has a wealth of \( k + |L| \) in \( \bar{\sigma} \). As a consequence we have \( N_C(\bar{\sigma}) = N_C(\sigma) \)

and \( N_D(\bar{\sigma}) = N_D(\sigma) \).

Since in the Ghost system, the evolution of wealth only depends on the photographs we have: for all \( \bar{\sigma} \in \sigma + L \)

\[
G_{\sigma}(\forall n \in \mathbb{N} \forall i \in N_C(\sigma) \cup N_D(\sigma), Y^i_n > 0) \geq 1 - p
\]

Let \( n_0 \in \mathbb{N} \) such that:

\[
\exists N \in \mathbb{N} \text{ such that } t_n \leq t < t_{n+1}
\]

\[
\frac{1}{N} \bar{C}(t) = \frac{q_0}{N} + \frac{1}{N} \sum_{n=1}^{N} \bar{U}_C(t_n^\epsilon) \xrightarrow{N \to +\infty} \beta^0 R - \rho^0 S
\]

Then if \( \beta^0 R - \rho^0 S > 0 \) we have \( \mathbb{P}_{q_0} \) a.s. \( \bar{C}(t) \xrightarrow{t \to +\infty} +\infty \) and since the increasing are bounded, as a consequence for all \( \eta > 0 \) there exists \( M \in \mathbb{RZ} + \mathbb{SZ} \) such that:

\[
\mathbb{P}_{q_0} \left( \min_{t \in \mathbb{R}_+} \bar{C}(t) > M \right) \geq 1 - \eta
\]
Since $\mathcal{C}$ is a Levy process i.e. since the increments of $\mathcal{C}$ don’t depend on where $\mathcal{C}$ is, we have:

$$
P_{q_0 + |\mathcal{M}|} \left( \min_{t \in \mathbb{R}^+} \mathcal{C}(t) > \mathcal{M} + |\mathcal{M}| \right) \geq 1 - \eta
$$

Let $T > 0$ be such that

$$
P_{q_0} \left( \min_{t \leq T} \mathcal{C}(t) > 0, \mathcal{C}(T) = q_0 + |\mathcal{M}| \right) > 0
$$

Hence we get using Markov property:

$$
P_{q_0} \left( \min_{t \in \mathbb{R}^+} \mathcal{C}(t) > 0 \right) \geq P_{q_0} \left( \min_{t \leq T} \mathcal{C}(t) > 0, \mathcal{C}(T) = q_0 + |\mathcal{M}| \right) P_{q_0 + |\mathcal{M}|} \left( \min_{t \in \mathbb{R}^+} \mathcal{C}(t) > 0 \right) \geq 1 - \eta
$$

5.2 Proof of Proposition 3.4

Firstly let us notice that $(\mathcal{C}(t))_t$ only take values in $q_0 + \mathbb{R} \mathcal{Z} + \mathcal{S} \mathcal{Z}$. Then we get:

$$
\partial_t \mathbb{E}(\mathcal{C}(t)) = \sum_{k, \ell \in \mathbb{Z}} (q_0 + k\mathcal{R} - \ell \mathcal{S}) \partial_t P(\mathcal{C}(t) = q_0 + k\mathcal{R} - \ell \mathcal{S})
= v(\beta^0 \mathcal{R} - \rho^0 \mathcal{S})
$$

Let us compute for all $t > 0$ the variance of $\mathcal{C}(t)$.

$$
\partial_t \mathbb{E}(\mathcal{C}(t))^2 = \sum_{k, \ell \in \mathbb{Z}} (q_0 + k\mathcal{R} - \ell \mathcal{S})^2 \partial_t P(\mathcal{C}(t) = q_0 + k\mathcal{R} - \ell \mathcal{S})
= \sum_{k, \ell \in \mathbb{Z}} (q_0 + (k - 1)\mathcal{R} - \ell \mathcal{S} + \mathcal{R})^2 \beta^0 P(\mathcal{C}(t) = q_0 + (k - 1)\mathcal{R} - \ell \mathcal{S})
+ \sum_{k, \ell \in \mathbb{Z}} (q_0 + k\mathcal{R} - \ell \mathcal{S} - \mathcal{S})^2 \rho^0 P(\mathcal{C}(t) = q_0 + k\mathcal{R} - (\ell - 1)\mathcal{S})
- \mathbb{E}(\mathcal{C}(t))^2
= \beta^0 \mathcal{R}^2 + \rho^0 \mathcal{S}^2 + 2\mathbb{E}(\mathcal{C}(t)) \partial_t \mathbb{E}(\mathcal{C}(t))
= \beta^0 \mathcal{R}^2 + \rho^0 \mathcal{S}^2 + \partial_t \mathbb{E}(\mathcal{C}(t))^2
$$

Then we have: \(\forall t > 0\)

$$
\forall(\mathcal{C}(t)) = (\beta^0 \mathcal{R}^2 + \rho^0 \mathcal{S}^2) t
$$

Then using the Chebyshev inequality we have the following concentration inequality: for all $\varepsilon_t > 0$
\[ P(\overline{C}(t) - \mathbb{E}(\overline{C}(t)) > \varepsilon_t) \leq \frac{v}{\varepsilon_t^2} (\beta^0 R^2 + \rho^0 S^2). \]  \hspace{1cm} (23)

We now take for all \( \eta > 0 \), \( \varepsilon_\eta_t = \eta \sqrt{v \beta^0 R^2 + \rho^0 S^2} \) and obtain (15).

Moreover we notice that
\[ \mathbb{E}(\overline{C}^\eta(t)) - \varepsilon_\eta_t = q_0 + t(\beta^0 R - \rho^0 S) - \eta \sqrt{v(\beta^0 R^2 + \rho^0 S^2)} \sqrt{t}. \]

The minimum of \( t \mapsto q_0 + t(\beta^0 R - \rho^0 S) - \eta \sqrt{v(\beta^0 R^2 + \rho^0 S^2)} \sqrt{t} \) is reached in
\[ t = \frac{v \eta^2 (\beta^0 R^2 + \rho^0 S^2)}{4(\beta^0 R - \rho^0 S)^2}. \]

This minimum is \( q_0 - \frac{v \eta^2 ((\beta^0 R^2 + \rho^0 S^2))}{4(\beta^0 R - \rho^0 S)} \). Hence for all \( \tau \) such that:
\[ \tau + q_0 - \frac{\eta^2 v (\beta^0 R^2 + \rho^0 S^2)}{4(\beta^0 R - \rho^0 S)} > 0 \]

We have:
\[ P_{q_0} (\overline{C}(t) \geq \tau) \geq 1 - \eta^{-2}. \]

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