The algebra of two dimensional generalized
Chebyshev - Koornwinder oscillator

Abstract

In the previous works [46, 47] authors have defined the oscillator-like system that
associated with the two variable Chebyshev-Koornwinder polynomials. We call this system
the generalized Chebyshev - Koornwinder oscillator. In this paper we study the properties
of infinite-dimensional Lie algebra that is analogous to the Heisenberg algebra for the
Chebyshev - Koornwinder oscillator. We construct the exact irreducible representation
of this algebra in a Hilbert space $H$ of functions that are defined on a region which
bounded by the Steiner hypocycloid. The functions are square-integrable with respect
to the orthogonality measure for the Chebyshev - Koornwinder polynomials and these
polynomials form an orthonormalized basis in the space $H$. The generalized oscillator
which is studied in the work can be considered as the simplest nontrivial example of
multiboson quantum system that is composed of three interacting oscillators.

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1 INTRODUCTION

The notion of the quantum harmonic oscillator is one of the cornerstones of quantum physics. With this concept closely linked:

1) The Hilbert space $\mathcal{H}$ of oscillator states; the classical Hermite polynomials, which are orthogonal with respect to the Gaussian measure on the real axis and forms the basis in the space $\mathcal{H}$; the fundamental quantum-mechanical operators in $\mathcal{H} —$ the operators of the coordinate, momentum, and quadratic Hamiltonian.

2) The Fock space and main operators in this space — ladder operators, the operator of the number of particles, and identical operator, which are the generators of the oscillator algebra (algebra of dynamical symmetries for the quantum system).

Development of quantum physics at the end of the last century, in particular, the emergence of quantum algebras [1]-[4], has resulted in various generalizations of the quantum harmonic oscillator. The first meaningful generalization was the $q$-oscillator [5]-[9] connected with $q$-deformation of canonical commutation relations of the harmonic-oscillator algebra. The next step in generalization the quantum-mechanical oscillator was the construction of a ladder operators that satisfy certain commutation relations (i.e. construction of an oscillator-like algebra that generalizes the Heisenberg algebra [10]-[12]) connected with some polynomials from the Askey - Wilson scheme [13]-[16]. Analysis of the oscillator algebras related to some known (mostly classical) orthogonal polynomials allowed us to develop a general scheme for construction of the generalized oscillator, i.e. oscillator-like algebra which is a Heisenberg algebra generalization that connected with the given system of orthogonal polynomials on the real axis [17].

In last time increased interest to different applications of orthogonal polynomials in several variables. For general results concerning such polynomials we refer the reader to the monographs [18, 19]. We note also recent works [20, 21], as well as earlier work on two-variable Krall - Schaefer polynomials [22] that associated with integrable systems on the spaces of constant curvature (see also [23, 24]). In this regard, it is natural to extend the construction of the generalized oscillator connected with orthogonal polynomials in one variable to the case of polynomials of several variables. We start by considering a simple but non-trivial generalizations of the classical polynomials in one variable to the case of several variables. A wide class of such polynomials is connected with the root systems of Lie algebras [25]-[28].

One of the first works in this direction was the T. Koornwinder article [29], the main results of which were presented in [18] (see additional details in [30]-[32]). In this work the Koornwinder introduces orthogonal polynomials that are a natural generalization of the classical Chebyshev polynomials associated with the root system of the Lie algebra $sl(3)$. These polynomials are called the Chebyshev - Koornwinder polynomials in the following. In [28] and [32]-[34] these ideas were extended to the case of several variable analogues of the other classical polynomials (see also [35]). More results can be found in the [36, 38]. We note also the works [39]-[45] related to the other generalizations of Chebyshev polynomials on the case of several variables.

The purpose of our work consists in the determination of the algebra of the generalized Chebyshev - Koornwinder oscillator and study some properties of this algebra. Considered oscillator-like system, associated with Chebyshev - Koornwinder polynomials (hereinafter ChK-polynomials) [29], was defined in the our works [46, 47], based on the scheme proposed in [17]. The Chebyshev-Koornwinder oscillator (hereinafter ChK-oscillator) was considered in [46] as a union of three oscillators: sectorial, radial and boundary ones. In [46] the main attention was paid to the quantum-mechanical aspects of the generalized oscillator while the details of the
construction, exact formulas for the ladder operators, and the study of related oscillator algebras were omitted due to space restriction. In [29], were obtained differential expression for the ladder operators of the ChK-oscillator and extended the Koornwinder's algebra of differential operators [29] to Abelian subalgebra of the ChK-oscillator algebra. In this work we give a complete description of the algebras of sectorial, radial, and boundary oscillators, as well as the construction and study of algebra of two-dimensional ChK-oscillator.

The article has the following structure. In Sec.2, we briefly recall some information on ChK-polynomials of the 2nd type together with some details of construction of the oscillator algebras necessary for us in the following. Below are a few comments regarding the content of Sec.2.

Firstly, we note that the choice of the "coordinate operators" $Z$ and $\overline{Z}$ for sectorial oscillator in Sec.2.2.1 (see formulas (2.3a)-(2.3b)) with the help of recurrence relations (2.1) is a natural generalization of the corresponding formula for a one-dimensional coordinate operator (see [14, 15, 17]).

Construction of the corresponding "momentum operators" $P_Z = Z^\dagger$ and $P_{\overline{Z}} = \overline{Z}^\dagger$ is performed with the help of the Poisson kernel (2.4) for the ChK-polynomials and used the method developed in [17]. Ladder operators and quadratic Hamiltonian are constructed for operators $Z, \overline{Z}, Z^\dagger, \overline{Z}^\dagger$ in the standard way. To save space, we omit the unused in this work generalization of the Fourier transform (similar to the transformation in the work [17]), based on the use of the Poisson kernel. Next, note that the correct choice of "coordinate" operator $X$ for radial oscillator in Sec.2.2.2 (see (2.11)) is not as easy as for sectorial oscillator. The mentioned choice demands prior to obtain the recurrence relations for the ChK-polynomials in the "radial direction". The momentum operator $X^\dagger$ is constructed using the corresponding Poisson kernel (2.12). This results in the non-standard form for the ladder operators (2.16) and Hamiltonian (2.18) of the radial oscillator. The additional term in (2.18) can be considered as "interaction energy" of the sectorial and radial oscillators.

Secondly, the introduction in the Subsection 2.2.3 one more "marginal" oscillator due to the fact that according to the (2.2) the union $\mathfrak{A}_{s,r}$ of the sectorial and radial oscillator algebras decomposes into the direct sum $\mathfrak{A}_{s,r} = \mathfrak{A}_{s,r}^{even} \oplus \mathfrak{A}_{s,r}^{odd}$. Thus it is clear that $\mathfrak{A}_{s,r}$ is a subalgebra of the desired algebra of two-dimensional ChK-oscillator. So that we must introduce additional oscillator in Subsection 2.2.3, which we call the "boundary" oscillator, because the generators of the corresponding oscillator algebra are not trivial only in "boundary subspaces" $\mathcal{H}_{0}$ and $\mathcal{H}_{0}$ (see (2.21)).

Further, we note that all constructed in section 2.2 algebras of sectorial, radial and boundary oscillators are associative algebras, which can be (minimally) extended to some (infinite-dimensional) Lie algebras. The construction of such extensions requires the introduction of additional generators satisfying the new commutation relations induced by commutation relations of considered associative algebras. These constructions are the main content of Sec.3—Sec.5.

Finally, in section 6, is determined the Lie algebra of two-dimensional ChK-oscillator. Note that it is more natural to understood as oscillator algebra not the associative algebra but its minimal extension to the Lie algebra. We suppose that this is correct because the considered algebra should be a generalization of the Heisenberg-Lie algebra for the quantum-mechanical oscillator. Some properties of this infinite-dimensional Lie algebra are investigated in Sec.7.
2 Preliminaries

For the reader convenience, we present briefly some information (necessary for further discussions) from the works [46, 47].

2.1 The Chebyshev — Koornwinder polynomials (ChK-polynomials)

The Chebyshev — Koornwinder polynomials (ChK-polynomials) [47] can be defined by recurrent relations

\[ z U_{k,l}(z, \bar{z}) = U_{k+1,l}(z, \bar{z}) + U_{k,l-1}(z, \bar{z}) + U_{k-1,l+1}(z, \bar{z}), \]
\[ \bar{z} U_{k,l}(z, \bar{z}) = U_{k,l+1}(z, \bar{z}) + U_{k-1,l}(z, \bar{z}) + U_{k+1,l-1}(z, \bar{z}), \]

subject to the following conditions

\[ U_{0,-1} = U_{-1,0} = 0, \quad U_{0,0} = 1, \quad U_{k,l}(z, \bar{z}) = U_{l,k}(z, \bar{z}) = U_{k,l}(\bar{z}, z). \]

It is known [46] that ChK-polynomials form an orthonormal basis in the Hilbert space \( \mathcal{H} = L^2(S; \mu(dx dy)) \), where \( S \) — the region within the Steiner hypocycloid (see Fig.1), and \( \mu \) — probability measure on \( S \), given by the formula

\[ \mu(dx, dy) = \frac{1}{2\pi^2} \sqrt{27 - 18z\bar{z} + 4z^3 + 4\bar{z}^3 - z^2\bar{z}^2} dx dy; \quad (z = x + iy). \]

The normalization constant in this formula was calculated in [47] (see also [45, 25]).

![Figure 1: Steiner hypocycloid](image)

2.2 Chebyshev — Koornwinder oscillator (ChK-oscillator)

In the work [46] we applied the construction [17] of the generalized oscillator connected with a system of orthogonal polynomials on the real axis to the case of ChK-polynomials of two variables. Chebyshev — Koornwinder oscillator (ChK-oscillator) was considered in [46] as a union of three oscillators: sectorial, radial, and boundary ones. To describe them it is convenient
to use the following decompositions of the space $H$ (which we consider as the Fock space), in
the direct sums of subspaces:

$$H = \bigoplus_{N=0}^{\infty} H^{(N)}, \quad H = H_{\text{even}} \bigoplus H_{\text{odd}}, \quad (2.2)$$

where $N$-particle sector $H^{(N)}$ is the closure in $H$ of the linear span of set of basis vectors
$$\{ U_{k,l}(z,\bar{z}) \mid k + l = N \},$$
and subspaces $H_{\text{even}} \cup H_{\text{odd}}$ are the closures of linear spans of the
following sets

$$\{ U_{k,l}(z,\bar{z}) \mid k + l = 2n \}_{n=0}^{\infty} \quad \text{and} \quad \{ U_{k,l}(z,\bar{z}) \mid k + l = 2n + 1 \}_{n=0}^{\infty},$$

respectively. Moreover, we use the following subspaces $H_{\bullet, 0}$ and $H_{0, \bullet}$ which are the closures of
linear spans of the sets

$$\{ U_{k,0}(z,\bar{z}) \}_{k=0}^{\infty} \quad \text{and} \quad \{ U_{0,l}(z,\bar{z}) \}_{l=0}^{\infty},$$

and the spaces $H^{\bullet, 0}$ and $H^{0, \bullet}$ defined by

$$H^{\bullet, 0} = H \ominus H_{\bullet, 0}, \quad H^{0, \bullet} = H \ominus H_{0, \bullet}.$$

As in the case of the standard quantum-mechanical oscillator, ChK-oscillator can be determined
using the ladder operators. The ladder operators $a_{\text{sect}}^{\pm}$ of sectorial oscillator leave invariant the
subspaces $H^{(N)}$, and the ladder operators $a_{\text{rad}}^{\pm}$ of radial oscillator transform the subspaces $H^{(N)}$
into the subspaces $H^{(N+2)}$. In addition, operators $a_{\text{sect}}^{\pm}$ and $a_{\text{rad}}^{\pm}$ leave invariant subspaces $H_{\text{even}}$
and $H_{\text{odd}}$. Finally, the ladder operators $a_{\bullet, 0}^{\pm}$ of boundary oscillator transform the subspace $H^{(N)} \cap H_{\bullet, 0}$
in the subspace $H^{(N+1)} \cap H_{\bullet, 0}$ and equal to zero on $H^{\bullet, 0}$, and the ladder operators $a_{0, \bullet}^{\pm}$ transform the subspace $H^{(N)} \cap H_{0, \bullet}$
in the subspace $H^{(N+1)} \cap H_{0, \bullet}$ and equal to zero on $H^{0, \bullet}$. Action of these operators is schematically shown in figure 2, where the basic elements
$$\{ U_{k,l}(z,\bar{z}) \}_{k,l}^{\infty}$$
are represented by points of the rectangular lattice.
2.2.1 Generalized sectorial oscillator

Taking into account the recurrence relation (2.1), we define the "position operators" $Z$ and $\overline{Z}$ by the relations

\[
Z U_{k,l}(z, \overline{z}) = U_{k+1,l}(z, \overline{z}) + U_{k,l-1}(z, \overline{z}) + U_{k-1,l+1}(z, \overline{z}),
\]

\[
\overline{Z} U_{k,l}(z, \overline{z}) = U_{k,l+1}(z, \overline{z}) + U_{k-1,l}(z, \overline{z}) + U_{k+1,l-1}(z, \overline{z}).
\]

These relations define the operators $Z$ and $\overline{Z}$ on the dense in $\mathcal{H}$ linear span of all ChK-polynomials $U_{k,l}(z, \overline{z})$. After closing of these operators they become bounded operators (which we denote by the same symbols) on the whole space $\mathcal{H}$ and satisfy the relations $Z^* = \overline{Z}$, $\overline{Z}^* = Z$.

Following [17], we define the "momentum operators" as operators that conjugate relatively basis in $\mathcal{H}$ to the above position operators. For this purpose we consider the integral operator $\mathbb{K}$ with the Poisson kernel

\[
K_s(z, \zeta; \xi, \eta) = \sum_{k,l=0}^{\infty} (-1)^{k+l} U_{k,l}(z, \overline{z}) U_{k,l}(\xi, \eta).
\]

The operator $\mathbb{K}$ acts from the Hilbert space $\mathcal{H} = L^2(S, \mu(dx, dy))$ ($z = x + iy$) into the Hilbert space $\mathcal{H}_1 = L^2(S, \mu(d\xi, d\eta))$ ($\zeta = \xi + i\eta$). In the space $\mathcal{H}_1$ we define "position operators" $\zeta$ and $\overline{\zeta}$, using the formulas similar to (2.3a) and (2.3b). For the inverse operator $\mathbb{K}^{-1} : \mathcal{H}_1 \to \mathcal{H}$ is fulfilled the relation $\mathbb{K}^{-1} = \mathbb{K}^*$, where $\mathbb{K}^*$ is the integral operator adjoint to $\mathbb{K}$. Hence $\mathbb{K}$ is an unitary operator.

Using the operator $\mathbb{K}$, we define "momentum operators" $P_Z = Z^\dagger$ and $P_{\overline{Z}} = \overline{Z}^\dagger$, conjugate to $Z$ and $\overline{Z}$ with respect to selected basis of the space $\mathcal{H}$ :

\[
Z^\dagger = \mathbb{K}^{-1} \zeta \mathbb{K}, \quad \overline{Z}^\dagger = \mathbb{K}^{-1} \overline{\zeta} \mathbb{K}.
\]

From (2.5) it follows that operators $P_Z = Z^\dagger$ and $P_{\overline{Z}} = \overline{Z}^\dagger$ are bounded in $\mathcal{H}$ and adjoint to each other

\[
(Z^\dagger)^* = \overline{Z}^\dagger, \quad (\overline{Z}^\dagger)^* = Z^\dagger.
\]

The action of the operators $Z^\dagger$ and $\overline{Z}^\dagger$ on the basic elements of the space $\mathcal{H}$ is given by formulas

\[
Z^\dagger U_{k,l}(z, \overline{z}) = -U_{k+1,l}(z, \overline{z}) - U_{k,l-1}(z, \overline{z}) + U_{k-1,l+1}(z, \overline{z}),
\]

\[
\overline{Z}^\dagger U_{k,l}(z, \overline{z}) = -U_{k,l+1}(z, \overline{z}) - U_{k-1,l}(z, \overline{z}) + U_{k+1,l-1}(z, \overline{z}).
\]

Now we define the quadratic Hamiltonian of sectorial oscillator as

\[
H_s = H_s^{(1)} + H_s^{(2)},
\]

where

\[
H_s^{(1)} = \frac{1}{40} \left( Z\overline{Z} + \overline{Z}Z^\dagger + Z^\dagger \overline{Z} + Z^\dagger Z \right), \quad H_s^{(2)} = \frac{1}{40} \left( \overline{Z}Z + Z\overline{Z}^\dagger + \overline{Z}^\dagger Z + \overline{Z}^\dagger Z \right).
\]

Hamiltonian $H_s$ is a self-adjoint operator in the space $\mathcal{H}$. ChK-polynomials $U_{k,l}(z, \overline{z})$ are the eigenfunctions of $H_s$ with eigenvalues

\[
\lambda_{0,0} = 0; \quad \lambda_{N,0} = \lambda_{0,N} = \frac{1}{10}, \quad (N \geq 1); \quad \lambda_{k,l} = \frac{1}{5}, \quad (k,l \geq 1).
\]
We consider $\mathcal{H}$ as a Fock space and define ladder operators $a_{\text{sect}}^\pm$ by the relations

$$a_{\text{sect}}^+ = \frac{1}{\sqrt{40}}(Z + Z^\dagger), \quad a_{\text{sect}}^- = \frac{1}{\sqrt{40}}(Z - Z^\dagger).$$

These operators acting in "$N$-particle" subspaces $\mathcal{H}^{(N)}$, $N = k + l$ as follows

$$a_{\text{sect}}^\pm U_{k,l}(z, \bar{z}) = \frac{1}{\sqrt{10}} U_{\pm 1, N-(m+1)} \frac{D_{m,N-m}}{m!(N-m)!},$$

where

$$D_{m,N-m} = \frac{\partial^N}{\partial z^m \partial \bar{z}^{N-m}}.$$ 

The ladder operators are adjoint to each other $(a_{\text{sect}}^\pm)^* = a_{\text{sect}}^\mp$, and Hamiltonian in terms of these operators has the form

$$H_s = H_s^{(1)} + H_s^{(2)}, \quad H_s^{(1)} = a_{\text{sect}}^+ a_{\text{sect}}^-, \quad H_s^{(2)} = a_{\text{sect}}^- a_{\text{sect}}^+.$$ 

Self-adjoint "number" operators $N_1$, $N_2$ are defined by their action on the basic elements of the space $\mathcal{H}$:

$$N_1 U_{k,l} = k U_{k,l}, \quad N_2 U_{k,l} = l U_{k,l}.$$ 

We also need two auxiliary operators

$$P_1 = \bigoplus_{N=0}^\infty P_{N,0}, \quad P_2 = \bigoplus_{N=0}^\infty P_{0,N},$$

where

$$P_{N,0} U_{k,l}(z, \bar{z}) = \delta_{k, N} \delta_{l, 0} U_{k,l}(z, \bar{z}), \quad P_{0,N} U_{k,l}(z, \bar{z}) = \delta_{k, 0} \delta_{l, N} U_{k,l}(z, \bar{z}).$$

In [16] the algebra $\mathfrak{A}_s$ of generalized sectorial oscillator was defined as the closure of an associative algebra generated by operators

$$\mathbb{I}, a_{\text{sect}}^\pm, N_1, N_2, P_1, P_2,$$

satisfying the commutation relations

$$[a_{\text{sect}}^+, a_{\text{sect}}^-] = \frac{1}{10} (P_1 - P_2); \quad [N_1, a_{\text{sect}}^\pm] = \mp a_{\text{sect}}^\pm; \quad [N_2, a_{\text{sect}}^\pm] = \pm a_{\text{sect}}^\pm;$$

$$[P_1, P_2] = 0, \quad [N_1, N_2] = 0, \quad [N_i, P_j] = 0, \quad (i, j = 1, 2);$$

$$P_1 a_{\text{sect}}^+ = 0, \quad a_{\text{sect}}^+ P_1 = 0, \quad a_{\text{sect}}^- P_2 = 0, \quad P_2 a_{\text{sect}}^- = 0.$$
2.2.2 Generalized radial oscillator

As in the case of sectorial oscillator, we start by defining the "position" operator

\[ X := -5H_s - \frac{1}{4} \left( ZZ^* + Z^*Z + \overline{Z}Z^* + \overline{Z}^*Z \right), \quad \text{Dom}[X] = \mathcal{H}. \tag{2.11} \]

By analogy with (2.5), using the Poisson kernel

\[ K_r(z, \overline{z}; \zeta, \overline{\zeta}) = \sum_{k, l=0}^{\infty} e^{i\frac{\pi}{4}(k+l)} U_{k,l}(z, \overline{z}) U_{k,l}(\zeta, \overline{\zeta}), \tag{2.12} \]

we define the momentum operator \( P_X = X^\dagger \) conjugate relative basis in the space \( \mathcal{H} \) to the coordinate operator. The quadratic Hamiltonian of radial oscillator

\[ H_r = \frac{1}{4} (X^2 + (X^*)^2), \quad \text{Dom}[H_r] = \mathcal{H}, \tag{2.13} \]

is a bounded selfadjoint operator in \( \mathcal{H} \). ChK-polynomials \( U_{k,l}(z, \overline{z}) \) are eigenfunctions of the operator \( H_r \) with eigenvalues

\[ \nu_{00} = \frac{4}{5}, \quad \nu_{N0} = \nu_{0N} = \frac{1}{2} (\text{for } N \geq 1), \quad \nu_{mn} = \frac{1}{5} (\text{for } n, m \geq 1). \tag{2.14} \]

To determine the ladder operators we introduce an auxiliary operator \( I(\alpha, \beta) \) \( (\alpha, \beta \in \mathbb{C}) \) which acts on the basic elements \( U_{k,l}(z, \overline{z}) \) as follows

\[ I_B(\alpha; \beta) U_{k,l}(z, \overline{z}) = \{ \alpha (\delta_{k,N}\delta_{l,0} + \delta_{k,0}\delta_{l,N}) (1 - \delta_{k,0}\delta_{l,0}) + \beta \delta_{k,0}\delta_{l,0} \} U_{k,l}(z, \overline{z}). \tag{2.15} \]

Using this operator, we define the ladder operators of radial oscillator by the relations

\[ a_{\pm}^{\text{rad}} = \frac{1}{\sqrt{10}} \left( X + iX^\dagger \right) - \frac{2}{\sqrt{5}} I_B \left( \frac{1}{4} e^{\pm i\frac{\pi}{4}}, \frac{1}{2} e^{\pm i\frac{\pi}{4}} \right). \tag{2.16} \]

From (2.11), (2.15), (2.16) it follows that the action of \( a_{\pm}^{\text{rad}} \) looks as

\[ a_{\pm}^{\text{rad}} U_{k,l}(z, \overline{z}) = \sqrt{\frac{2}{5}} U_{k+\pm, l\mp 1}(z, \overline{z}). \tag{2.17} \]

In the work [47] we have found the following differential operator representation of the operators \( a_{\pm}^{\text{rad}} \):

\[ a_{\pm}^{\text{rad}} \big|_{\mathcal{H}(N)} = \sqrt{\frac{2}{5}} \sum_{m=0}^{N} U_{m\pm 1, N-m\mp 1} \frac{D_{m,N-m}}{m!(N-m)!}. \]

The (position) operator is given by the relation

\[ a_{-}^{\text{rad}} + a_{+}^{\text{rad}} = \sqrt{\frac{2}{5}} \left\{ (z\overline{z} - 3)I - \sum_{N=1}^{\infty} \left[ \sum_{m=0}^{N} \frac{\alpha_1}{m!(N-m)!} - \sum_{m=0}^{N-1} \frac{\alpha_2}{m!(N-m)!} \right] \right\}, \]

where

\[ \alpha_1 = U_{m-1,N-m+1}D_{m,N-m}\overline{Z} + U_{m+1,N-m-1}D_{m,N-m}Z; \]
\[ \alpha_2 = U_{m,N-m}D_{m,N-m} + U_{N-m,m}D_{N-m,m}, \]

\[ \mathcal{H}(N) \subseteq \mathcal{H}. \]
and $Z$, $\overline{Z}$ are the operators of multiplication by $z$ and $\overline{z}$, respectively. From (2.11)-(2.13), (2.15)-(2.17) we obtain the following expression for the Hamiltonian $H_r$ in terms of the ladder operators

$$H_r = a_{rad}^+a_{rad}^- + a_{rad}^-a_{rad}^+ + I_B \left( \frac{1}{8}, \frac{1}{2} \right).$$

(2.18)

Note that, unlike the Hamiltonian $H_s$ sectorial oscillator, the Hamiltonian $H_r$ radial oscillator contains an extra term $I_B \left( \frac{1}{8}, \frac{1}{2} \right)$ in addition to standard members.

In the work [46] the algebra $\mathfrak{A}_r$ of generalized radial oscillator was defined as a closure of the associative algebra generated by operators

$$a_{rad}^\pm, \ N_1, \ N_2, \ I_B(\frac{1}{2}, \frac{1}{2}), \ \mathbb{I},$$

(2.19)
satisfying the following commutation relations

$$[a_{rad}^-, a_{rad}^+] = \frac{4}{5} I_B(\frac{1}{2}, \frac{1}{2}), \quad a_{rad}^- I_B(\frac{1}{2}, \frac{1}{2}) = 0; \quad I_B(\frac{1}{2}, \frac{1}{2})a_{rad}^+ = 0;$$

$$[N_1, a_{rad}^\pm] = \pm a_{rad}^\pm, \quad [N_2, a_{rad}^\pm] = \pm a_{rad}^\pm; \quad [N_1, I_B(\frac{1}{2}, \frac{1}{2})] = 0, \quad [N_2, I_B(\frac{1}{2}, \frac{1}{2})] = 0. \quad (2.20)$$

### 2.2.3 Generalized boundary oscillator

Recall that $\mathcal{H}_{*,0}$ and $\mathcal{H}_{0,*}$ denote the closures in $\mathcal{H}$ of linear spans of the sets $\{U_k,0\}_{k=0}^\infty$ and $\{U_0,t\}_{t=0}^\infty$. To define the boundary oscillator we use [46] the decomposition of the Hilbert spaces $\mathcal{H}_{*,0}$ and $\mathcal{H}_{0,*}$

$$\mathcal{H}_{*,0} = \bigoplus_{N=0}^\infty (\mathcal{H}^{(N)} \cap \mathcal{H}_{*,0}) \quad \mathcal{H}_{0,*} = \bigoplus_{N=0}^\infty (\mathcal{H}^{(N)} \cap \mathcal{H}_{0,*}) \quad (2.21)$$

and operators $P_{s,t}^{m,n} \bigcirc P_{s,t}^{m,n} U_{k,l} = \delta_{k,m} \delta_{l,n} U_{s,t}$. The ladder operators are defined by decompositions (see Fig.2)

$$a_{*,0}^\pm = \bigoplus_{m=0}^\infty P_{m+1,0}^{m,0}, \quad a_{0,*}^\pm = \bigoplus_{m=0}^\infty P_{0,m+1}^{0,m}, \quad (2.22)$$

and have [47] the following differential operator representation:

$$a_{*,0}^\pm \bigg|_{\mathcal{H}^{(N)}} = U_{N+1,0} \frac{D_{N,0}}{N!}; \quad a_{0,*}^\pm \bigg|_{\mathcal{H}^{(N)}} = U_{0,N+1} \frac{D_{0,N}}{N!}.$$  

For position operators

$$a_{*,0}^+ + a_{*,0}^- = (Z + \overline{Z} - a_{sect} s^+ - a_{sect}^+ Z + (a_{sect}^+)^2) \mathbb{P}_1,$$

$$a_{0,*}^+ + a_{0,*}^- = (Z + \overline{Z} - a_{sect} - a_{sect}^+ Z + (a_{sect}^-)^2) \mathbb{P}_2,$$

this gives [47] the following representation by differential operators:

$$a_{*,0}^+ + a_{*,0}^- = Z + \overline{Z} - \frac{1}{\sqrt{10}} \times \left\{ \bigoplus_{N=1}^\infty \left[ \left( \sum_{m=0}^N U_{m-1,N-m+1} \frac{D_{m,N-m}}{m!(N-m)!} \right) (\mathbb{I} + Z) \right. \right.$$  

$$\left. - \left( \sum_{m=0}^{N-2} U_{m,N-m} \frac{D_{m+2,N-m-2}}{(m+2)!(N-m-2)!} \right) \right] \mathbb{P}_1;$$

$$a_{0,*}^+ + a_{0,*}^- = Z + \overline{Z} - \frac{1}{\sqrt{10}} \times \left\{ \bigoplus_{N=1}^\infty \left[ \left( \sum_{m=0}^N U_{m-1,N-m+1} \frac{D_{m,N-m}}{m!(N-m)!} \right) (\mathbb{I} + Z) \right. \right.$$  

$$\left. - \left( \sum_{m=0}^{N-2} U_{m,N-m} \frac{D_{m+2,N-m-2}}{(m+2)!(N-m-2)!} \right) \right] \mathbb{P}_2;$$

$$a_{*,0}^+ + a_{*,0}^- = (Z + \overline{Z} - a_{sect} s^+ - a_{sect}^+ Z + (a_{sect}^+)^2) \mathbb{P}_1,$$

$$a_{0,*}^+ + a_{0,*}^- = (Z + \overline{Z} - a_{sect} - a_{sect}^+ Z + (a_{sect}^-)^2) \mathbb{P}_2.$$
\[ a_{0,\bullet}^+ + a_{0,\bullet}^- = Z + \mathcal{Z} - \frac{1}{\sqrt{10}} \times \left\{ \sum_{N=1}^{\infty} \left[ \left( \sum_{m=0}^{N} \frac{U_{m+1,N-m-1} D_{m,N-m}}{m!(N-m)!} \right) (\mathbb{I} + \mathcal{Z}) ight] - \left( \sum_{m=0}^{N-2} \frac{U_{m+2,N-m-2} D_{m,N-m}}{m!(N-m)!} \right) \right\} \mathbb{P}_2. \]

The quadratic Hamiltonian of the boundary oscillator has the form

\[ H_0 = \frac{1}{5} \left( H_{\bullet,0} + H_{0,\bullet} - P_{0,0}^{0,0} \right), \]

where

\[ H_{\bullet,0} = (a_{0,0}^+ a_{0,0}^- + a_{0,0}^- a_{0,0}^+), \]

\[ H_{0,\bullet} = (a_{0,0}^+ a_{0,0}^- + a_{0,0}^- a_{0,0}^+). \]

The Hamiltonian \( H_0 \) is a bounded selfadjoint operator in \( \mathcal{H} \). ChK-polynomials \( U_{k,l}(z, \bar{z}) \) are eigenfunctions for \( H_0 \) with eigenvalues

\[ \mu_{0,0} = \frac{1}{5}; \quad \mu_{0,N} = \mu_{N,0} = \frac{2}{5}, \quad \text{(for} \ N \geq 1), \quad \mu_{k,l} = 0, \quad \text{(for} \ k, l \geq 1). \]

Then the algebra \( \mathfrak{A}_0 \) of generalized boundary oscillator was defined in [40] as a closure of the associative algebra with generators

\[ \mathbb{I}, \ N_1, \ N_2, \ P^{s,0}_{m,0}, \ P^{0,s}_{0,m}, \ P^{k,0}_{0,m}, \ P^{0,k}_{m,0}, \] (2.23)

satisfying the commutation relations

\[ \begin{align*}
[N_1, N_2] &= 0, \quad [N_1, P^{k,0}_{m,0}] = (m - k)P^{k,0}_{m,0}, \quad [N_1, P^{0,k}_{0,m}] = 0, \\
[N_1, P^{0,0}_{0,n}] &= -kP^{0,0}_{0,n}, \quad [N_1, P^{0,l}_{m,0}] = mP^{0,l}_{m,0}; \\
[N_2, P^{k,0}_{m,0}] &= 0, \quad [N_2, P^{0,k}_{0,m}] = (m - k)P^{0,k}_{0,m}, \\
[N_2, P^{0,0}_{0,n}] &= nP^{0,0}_{0,n}, \quad [N_2, P^{0,l}_{m,0}] = -nP^{0,l}_{m,0}; \\
[P^{k,0}_{m,0}, P^{l,0}_{n,0}] &= \delta_{k,n}P^{l,0}_{m,0} - \delta_{m,0}P^{k,0}_{n,0}, \\
[P^{k,0}_{m,0}, P^{0,l}_{0,n}] &= \delta_{k,0}\delta_{n,0}P^{0,l}_{m,0} - \delta_{m,0}\delta_{l,0}P^{k,0}_{n,0}; \\
[P^{k,0}_{m,0}, P^{0,l}_{0,n}] &= \delta_{k,0}\delta_{n,0}P^{0,l}_{m,0} - \delta_{m,0}\delta_{l,0}P^{k,0}_{n,0}; \\
[P^{k,0}_{0,m}, P^{l,0}_{n,0}] &= \delta_{k,n}P^{l,0}_{0,m} - \delta_{m,0}\delta_{l,0}P^{k,0}_{n,0}; \\
[P^{0,k}_{0,m}, P^{l,0}_{n,0}] &= \delta_{k,0}\delta_{n,0}P^{0,l}_{0,m} - \delta_{l,0}\delta_{n,0}P^{0,k}_{0,m}; \\
[P^{0,k}_{0,m}, P^{0,l}_{0,n}] &= \delta_{k,0}\delta_{n,0}P^{0,l}_{0,m} - \delta_{l,0}\delta_{n,0}P^{0,k}_{0,m}. \end{align*} \] (2.24)
3 Algebra of generalized sectorial oscillator

In this section, we define the infinite-dimensional Lie algebra $\mathfrak{A}_s$ as a closure of the sectorial oscillator algebra which is the (minimal) expansion of an associative algebra determined (2.9) and (2.10). To do this we need some auxiliary operators entered below. Recall that

$$P_{s,t}^{m,n}U_{k,l}(z, \overline{z}) = \delta_{k,m}\delta_{l,n}U_{s,t}(z, \overline{z}),$$  \hspace{1cm} (3.1)

if $m, n, s, t, k, l \geq 0$, and

$$P_{s,t}^{m,n}U_{k,l}(z, \overline{z}) = 0,$$

if at least one of the indices is negative.

Using (3.1), (2.7) and decomposition (2.2), we have

$$a_{\text{sect}}^\pm = \frac{1}{\sqrt{10}} \bigoplus_{m,n=0}^{\infty} P_{m\pm1, n\pm1}^{m,n}. \hspace{1cm} (3.2)$$

We introduce the operators

$$P_{\bullet,k}^{(n)} = \bigoplus_{m=k}^{\infty} P_{m-k,n}^{m-n,n}, \hspace{1cm} P_{p,\bullet}^{(q)} = \bigoplus_{l=p}^{\infty} P_{q,l-q}^{p,l-p}, \hspace{1cm} k, n, q, p \geq 0. \hspace{1cm} (3.3)$$

The relations (2.8) and (3.3) imply that

$$P_1 = \bigoplus_{N=0}^{\infty} P_{N,0}^{N,0} = P_{\bullet,0}^{(0)}, \hspace{1cm} P_2 = \bigoplus_{N=0}^{\infty} P_{0,N}^{0,N} = P_{0,\bullet}^{(0)}.$$

The action of all operators defined above shown in the following diagram.

![Figure 3: The action of operators (3.1), (3.3)](image)

We will start building algebra $\mathfrak{A}_s$ of the sectorial oscillator with the definition of algebras $\mathfrak{A}_s^{(N)}$, for $N \geq 0$, as associative algebras with generators

$$\mathbb{I}, \mathbb{N}_1, \mathbb{N}_2, P_{k\mp1, l\pm1}^{k,l}, P_{k,l}^{k,l}, \hspace{0.5cm} (k + l = N, k, l \geq 0), \hspace{1cm} (3.4)$$
satisfying the commutation relations

\[ [N_1, N_2] = 0, \quad [N_1, P_{k,l}^m] = [N_2, P_{k,l}^m] = 0, \quad [P_{k,l}^m, P_{m,n}^{l,n}] = 0, \]
\[ [N_1, P_{k+1,l+1}^{k,l}] = \mp P_{k+1,l+1}^{k,l}, \quad [N_2, P_{k+1,l+1}^{k,l}] = \pm P_{k+1,l+1}^{k,l}, \quad (3.5) \]
\[ [P_{m+1,n+1}^{m,n}, P_{k,l}^{k,l}] = \delta_{m,k} \delta_{n,l} P_{m+1,n+1}^{k,l} - \delta_{k,m+1} \delta_{l,n+1} P_{m,n}^{m,n}, \]
\[ [P_{m-1,n+1}^{m,n}, P_{k,l}^{k,l}] = \delta_{m,k+1} \delta_{n,l-1} \left( P_{k,l}^{k,l} - P_{k+1,l-1}^{k,l} \right). \]

It is easy to check the validity of the Jacobi identities. Therefore, the algebras \( \mathfrak{a}_s^{(N)} \), \( N \geq 0 \), generated by operators (3.4) satisfying the relations (3.5), are Lie algebras with dimension \( 3(N + 2) \). Using the decomposition (2.2), we determine the infinite-dimensional Lie algebra \( \tilde{\mathfrak{a}}_s \)

\[ \tilde{\mathfrak{a}}_s = \bigoplus_{N=0}^{\infty} \mathfrak{a}_s^{(N)}. \]

We define the Lie algebra \( \tilde{\mathfrak{a}}_s \) as algebra, obtained from \( \tilde{\mathfrak{a}}_s \) by addition to generators (3.4) of all formal series composed of these generators, including operators (3.2) and (3.3). The commutation relations of the algebra \( \tilde{\mathfrak{a}}_s \) are induced by the relations (3.5).

Let \( I_s \) be an ideal of the algebra \( \tilde{\mathfrak{a}}_s \) generated by commutation relations:

\[ [a_{s \text{ sect}}^-, a_{s \text{ sect}}^+] = \frac{1}{10} \left( P_{0,0}^{(0)} - P_{0,0}^{(0)} \right); \]
\[ [N_1, a_{s \text{ sect}}^\pm] = \mp a_{s \text{ sect}}^\pm, \quad [N_2, a_{s \text{ sect}}^\pm] = \pm a_{s \text{ sect}}^\pm, \quad [N_1, N_2] = 0; \]
\[ [a_{s \text{ sect}}^\pm, P_{\bullet,k}^{(n)}] = \frac{1}{\sqrt{10}} \left( P_{\bullet,k}^{(n+1)} - P_{\bullet,k+1}^{(n)} \right); \]
\[ [a_{s \text{ sect}}^\pm, P_{k,l}^{k,l}] = \frac{1}{\sqrt{10}} \left( P_{k+1,l+1}^{k,l} - P_{k+1,l-1}^{k,l} \right); \]
\[ [P_{k,l}^{k,l}, P_{m,n}^{m,n}] = \left( P_{k,l}^{k,l} - P_{m,n}^{m,n} \right); \]
\[ [P_{\bullet,k}^{(n)}, P_{\bullet,l}^{(m)}] = P_{\bullet,l}^{(m)} - P_{\bullet,k}^{(n)}, \]
\[ [P_{k,l}^{(n)}, P_{k,l}^{(m)}] = P_{l,m}^{(m)} - P_{l,m}^{(m)}; \]
\[ [N_1, P_{k,l}^{(n)}] = (k-n) P_{k,l}^{(n)}, \quad [N_2, P_{k,l}^{(n)}] = (n-k) P_{k,l}^{(n)}; \]
\[ [N_1, P_{k,l}^{(n)}] = (n-k) P_{k,l}^{(n)}, \quad [N_2, P_{k,l}^{(n)}] = (k-n) P_{k,l}^{(n)}; \]
\[ [P_{k+1,l+1}^{k,l}, P_{k,l}^{(n)}] = P_{k+1,l-1}^{k,l} - P_{k,l}^{(n)}; \]
\[ [P_{k+1,l+1}^{k,l}, P_{k,l}^{(n)}] = P_{k+1,l-1}^{k,l} - P_{k,l}^{(n)}; \]
\[ [P_{k+1,l+1}^{k,l}, P_{k,l}^{(n)}] = P_{k+1,l-1}^{k,l} - P_{k,l}^{(n)}; \]
\[ [P_{k+1,l+1}^{k,l}, P_{k,l}^{(n)}] = P_{k+1,l-1}^{k,l} - P_{k,l}^{(n)}; \]
From (3.1), (3.3) and (2.15) should be the following equality

\[ [P^{k,l}_{k-t,t+l}, P^{(n)}_{m,\bullet}] = P^{m,k+l-m}_{k-t,t+l} - P^{k,l}_{n,k+l-n}; \]  

\[ [P^{k,l}_{k-t,t+l}, P^{m,n}_{m+s,n+s}] = \delta_{k,m+s} \delta_{l,n+s} P^{m,n}_{k+t,l+t} - \delta_{m,k+t} \delta_{n,l+t} P^{k,l}_{m+s,n+s}; \]  

\[ [P^{k,l}_{k-t,t+l}, P^{m,n}_{m-s,n-s}] = \delta_{k,m-s} \delta_{l,n-s} P^{m,n}_{k+t,l+t} - \delta_{m,k+t} \delta_{n,l+t} P^{k,l}_{m-s,n-s}; \]  

\[ [N_1, P^{k,l}_{k+t,l+t}] = \mp t P^{k,l}_{k+t,l+t}; \quad [N_2, P^{k,l}_{k+t,l+t}] = \pm t P^{k,l}_{k+t,l+t}. \]  

Finally, we define the algebra \( \mathfrak{A}_s \) of sectorial oscillator as quotient algebra

\[ \mathfrak{A}_s = \hat{\mathfrak{A}}_s / I_s. \]  

In other words, algebra \( \mathfrak{A}_s \) is Lie algebra with generators \((3.2), (3.3), (3.4)\) satisfy the commutation relations \((3.6)-(3.23)\). Clearly, that \( \mathfrak{A}_s \) is a closure of the associative algebra generated by operators \((2.9)\) and relations \((2.10)\).

### 4 Algebra of generalized radial oscillator

In this section, we define the infinite-dimensional Lie algebra \( \mathfrak{A}_r \) of the radial oscillator just as for sectorial oscillator. Namely, we describe the (minimal) extension of an associative algebra determined by generators \((2.19)\) with commutation relations \((2.20)\).

Using relations \((3.1)\) and \((2.17)\), we obtain

\[ a^{\pm}_{rad} = \sqrt{\frac{2}{5}} \bigoplus_{k \geq 0} P^{k,l}_{k \pm 1, l \pm 1}. \]  

From \((3.1), (3.3)\) and \((2.15)\) should be the following equality

\[ I_B \left( \frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2} \left( P^{(0)}_{\bullet,0} + P^{(0)}_{0,\bullet} - P^{0,0}_{0,0} \right). \]  

Let’s define algebras \( \mathfrak{A}^{(even)}_r \) and \( \mathfrak{A}^{(odd)}_r \) as associative algebras with generators

\[ \mathbb{I}, N_1, N_2, P_{k \pm t,l \pm t}, \quad k, l, t \geq 0, \]  

(\( k = 2n, l = 2m, k = 2n + 1, l = 2m + 1 \), respectively) satisfying commutation relations

\[ [P^{k,l}_{k+t,l+t}, P^{m,n}_{m+s,n+s}] = \delta_{k,m+s} \delta_{l,n+s} P^{m,n}_{k+t,l+t} - \delta_{m,k+t} \delta_{n,l+t} P^{k,l}_{m+s,n+s}; \]  

\[ [P^{k,l}_{k+t,l+t}, P^{m,n}_{m-s,n-s}] = \delta_{k,m-s} \delta_{l,n-s} P^{m,n}_{k+t,l+t} - \delta_{m,k+t} \delta_{n,l+t} P^{k,l}_{m-s,n-s}; \]  

\[ [N_1, P^{k,l}_{k+t,l+t}] = \pm t P^{k,l}_{k+t,l+t}; \quad [N_2, P^{k,l}_{k+t,l+t}] = \pm t P^{k,l}_{k+t,l+t}; \quad [N_1, N_2] = 0. \]

Checking that the Jacobi identity true, we prove the following lemma:

**Lemma 4.1.** The algebras \( \mathfrak{A}^{(even)}_r \) and \( \mathfrak{A}^{(odd)}_r \) are (infinite-dimensional) Lie algebras.

Using the decomposition \((2.22)\), we introduce algebra \( \mathfrak{A}_r \)

\[ \mathfrak{A}_r = \mathfrak{A}^{(even)}_r \bigoplus \mathfrak{A}^{(odd)}_r. \]
We define the Lie algebra $\hat{\mathfrak{A}}_r$ as an algebra obtained from $\hat{\mathfrak{A}}_r$ by adding of all formal series in generators (4.3), including operators (4.1), (4.2), and also the operators of the algebra \[ (4.8)-(4.19). \] The algebra $\mathfrak{A}$ is a closure of the associative algebra generated by the operators (2.19) as factor algebra (2.19) which satisfy the commutation relations (2.20). Finally, we determine the Lie algebra $\mathfrak{A}_r$ of radial oscillator as factor algebra \[ \mathfrak{A}_r = \hat{\mathfrak{A}}_r/I_r. \] Algebra $\mathfrak{A}_r$ is the Lie algebra with generators (4.1)-(4.3), (4.15)-(4.17) satisfying commutation relations (1.8)-(1.19). The algebra $\mathfrak{A}_r$ is a closure of the associative algebra generated by the operators (2.19) which satisfy the commutation relations (2.20).
5 Algebra of generalized boundary oscillator

In this section we mainly use the same scheme of reasoning as in the previous two sections. Therefore, we give here only a sketch of our construction. We define the infinite-dimensional Lie algebra $\mathfrak{A}_0$ as the minimal extension of the associative algebra determined by the generators (2.23) with the relations (2.24)-(2.29). Thus $\mathfrak{A}_0$ is the closure of the boundary oscillator algebra.

We introduce the algebra $\hat{\mathfrak{A}}_0$, as an associative algebra which generators (2.23) satisfy the commutation relations (2.24) - (2.29). Then we define infinite-dimensional Lie algebra $\hat{\mathfrak{A}}_0$ as algebra obtained from $\mathfrak{A}_0$ by addition of all formal series in generators (2.23), including operators (2.22). The commutation relations of the algebra $\hat{\mathfrak{A}}_0$ are induced by the relations (2.24) - (2.29), as well as

\[
[N_1, a_{0,0}^±] = ±a_{0,0}^±, \quad [N_2, a_{0,0}^±] = 0, \quad [a_{0,0}^±, a_{0,0}^±] = -P_{0,0}^{0,0},
\]

\[
[P_{m,0}^{k,0}, a_{0,0}^±] = P_{m,0}^{k±1,0} - P_{m±1,0}^{k,0}, \quad [P_{0,m}^{k,0}, a_{0,0}^±] = δ_{k,0}P_{0,m}^{±1,0} - δ_{m,0}P_{0,0}^{0,k}, \quad (5.1)
\]

\[
[P_{0,m}^{k,0}, a_{0,0}^±] = P_{0,m}^{k±1,0} - δ_{m,0}P_{0,0}^{k,0}, \quad [P_{n,0}^{0,l}, a_{0,0}^±] = δ_{l,0}P_{n,0}^{±1,0} - P_{n±1,0}^{0,l}, \quad (5.2)
\]

\[
[a_{0,0}^±, a_{0,0}^±] = P_{0,0}^{0,0}, \quad [a_{0,0}^+, a_{0,0}^-] = P_{0,0}^{0,0}, \quad [a_{0,0}^+, a_{0,0}^+] = P_{0,0}^{0,0}, \quad (5.3)
\]

generate the ideal $I_0$ of the algebra $\hat{\mathfrak{A}}_0$. Finally, we define the Lie algebra $\mathfrak{A}_0$ as the factor algebra $\mathfrak{A}_0 = \hat{\mathfrak{A}}_0/I_0$.

Thus algebra $\mathfrak{A}_0$ is the Lie algebra generated operators (2.22)-(2.23) satisfying the commutation relations (2.24)-(2.29), (5.1)-(5.3). This algebra is the closure of the associative algebra with generators (2.23) satisfying the commutation relations (2.24)-(2.29).

6 Algebra of generalized Chebyshev-Koornwinder oscillator

In this section we construct and investigate the main object of our work — the algebra $\mathfrak{A}$ of two-dimensional generalized Chebyshev-Koornwinder oscillator that is a union of sectorial, radial, and boundary oscillators. In the work [40], the algebra $\mathfrak{A}$ was defined as a closure of the associative algebra $\hat{\mathfrak{A}}$ generated by operators

\[
\mathbb{I}, \ N_1, \ N_2, \ a_{\text{sect}}^±, \ a_{\text{rad}}^±, \ a_{0,0}^±, \ P_{m,n}^{k,l}, \ P_{m,n}^{k,l}, \ P_{m,n}^{k,l}. \quad (6.1)
\]

It is assumed that indexes $k, l, m, n \geq 0$ are fulfilled one of the following conditions

1) \quad \quad m + n = k + l, \quad m - n = k - l \pm 2t;
2) \quad \quad \quad m - n = k - l, \quad m + n = k + l \pm 2t; \quad (6.2)
3) \quad k l = 0, \quad m n = 0;
and the generators \((6.1)\) satisfy the commutation relations \((3.6) - (3.23), (4.8) - (4.19), (2.24) - (2.29), (5.1) - (5.3)\).

We determine the algebra \(\mathfrak{A}\) of generalized Chebyshev - Koornwinder oscillator as infinite-dimensional Lie algebra which is a minimal extension of the algebra \(\mathfrak{A}\). To define the algebra \(\mathfrak{A}\) we consider first the algebra \(\hat{\mathfrak{A}}\) obtained from \(\mathfrak{A}\) by addition of all formal series in generators \((6.1)\). The commutation relations of the algebra \(\hat{\mathfrak{A}}\) induced by the relations \((3.6) - (3.23), (4.8) - (4.19), (2.24) - (2.29), (5.1) - (5.3)\). It is helpful to introduce the following notation for some of such series,

\[
P^{[k], l}_{[m], n} = \bigoplus_{s} P^{s, k, l}_{s + m, n} \quad \hat{P}^{k, l}_{m, n} = \bigoplus_{s} P^{k, s + l}_{m, s + n}.
\]

\textbf{Remarks} 1) Recall that \(P^{m, n}_{k, l} = 0\), if at least one of the indices is negative;

2) Operators \((6.3)\), \((6.4)\), \((6.5)\) can be expressed by the operators \((6.3)\) as follows:

\[
P^{(n)} = \hat{P}^{[n], k}_{[k], n}, \quad P^{(n)} = \hat{P}^{k, n}_{k, [k]},
\]

\[
P^{+ l}_{+, l} = \hat{P}^{[k], l}_{[l], k}, \quad P^{l, - l}_{+, l} = \hat{P}^{k, l}_{[l], k},
\]

\[
P^{k}_{l, +} = \hat{P}^{k, [l]}_{m, [n]}, \quad P^{k}_{l, -} = \hat{P}^{k, [l]}_{m, [n]}, \quad a_{+, 0} = \hat{P}^{0, 0}_{[\pm 1], 0}, \quad a_{-, 0} = \hat{P}^{0, 0}_{[\pm 1], 0}.
\]

The following commutation relations

\[
\begin{align*}
[N_1, N_2] &= 0, & [N_1, a_{\pm \text{sect}}] &= \mp a_{\pm \text{sect}}, & [N_1, a_{\pm \text{rad}}] &= \pm a_{\pm \text{rad}}; \\
[N_1, \hat{P}^{k, l}_{[m], n}] &= (m - k) \hat{P}^{k, l}_{[m], n}; \\
[N_1, \hat{P}^{k, [l]}_{m, [n]}] &= (m - k) \hat{P}^{k, [l]}_{m, [n]}; \\
[N_1, P^{k, l}_{n, m}] &= (n - k) P^{k, l}_{n, m}; \\
[N_2, a_{\pm \text{sect}}] &= \pm a_{\pm \text{sect}}, & [N_2, a_{\pm \text{rad}}] &= \pm a_{\pm \text{rad}}, & [N_2, P^{k, l}_{n, m}] &= (m - l) P^{k, l}_{n, m}; \\
[N_2, \hat{P}^{k, l}_{[m], n}] &= (n - l) \hat{P}^{k, l}_{[m], n}; \\
[N_2, \hat{P}^{k, [l]}_{m, [n]}] &= (n - l) \hat{P}^{k, [l]}_{m, [n]};
\end{align*}
\]

\[
\begin{align*}
[a_{\pm \text{sect}}, a_{\pm \text{sect}}] &= \frac{1}{10} \left( \hat{P}^{0, 0}_{[0], 0} - \hat{P}^{0, 0}_{0, [0]} \right), \\
[a_{\pm \text{sect}}, a_{\pm \text{rad}}] &= \pm \frac{1}{5} \hat{P}^{0, 0}_{[\pm 1], 0}, \\
[a_{\pm \text{sect}}, a_{\pm \text{rad}}] &= \pm \frac{1}{5} \hat{P}^{0, 0}_{[\pm 1], 0}, \\
[a_{\pm \text{sect}}, \hat{P}^{[k], l}_{[m], n}] &= \frac{1}{\sqrt{10}} \left( \hat{P}^{[k], l}_{[m+1], n \pm 1} - \hat{P}^{[k+1], l}_{[m], n} \right), \\
[a_{\pm \text{sect}}, \hat{P}^{k, [l]}_{m, [n]}] &= \frac{1}{\sqrt{10}} \left( \hat{P}^{k, [l]}_{m+1, [n] \pm 1} - \hat{P}^{k+1, [l]}_{m, [n]} \right), \\
[a_{\pm \text{sect}}, P^{k, l}_{m, n}] &= \frac{1}{\sqrt{10}} \left( P^{k, l}_{m+1, n \pm 1} - P^{k+1, l}_{m, n} \right).
\end{align*}
\]
\[
\left\{
\begin{align*}
[a_{\text{rad}}^+, a_{\text{rad}}^-] &= \frac{2}{5} \left( \hat{P}^{[0], 0}_{[0], 0} - \hat{P}^{0, [0]}_{0, [0]} - P_{0, 0}^0 \right); \\
[a_{\text{rad}}^\pm, \hat{P}^{[k], l}_{[m], n}] &= \sqrt{\frac{2}{5}} \left( \hat{P}^{[k], l}_{[m \pm 1], n \pm 1} - \hat{P}^{[k \mp 1], l \mp 1}_{[m], n} \right) \\
[a_{\text{rad}}^\pm, \hat{P}^{k, [l]}_{m, [n]}] &= \sqrt{\frac{2}{5}} \left( \hat{P}^{k, [l]}_{m \pm 1, [n \pm 1]} - \hat{P}^{k \pm 1, l \mp 1}_{m, [n]} \right); \\
[a_{\text{rad}}^\mp, P^{k, l}_{m, n}] &= \sqrt{\frac{2}{5}} \left( P^{k, l}_{m \mp 1, n \pm 1} - P^{k \mp 1, l \pm 1}_{m, n} \right).
\end{align*}
\]
(6.7)

\[
\left\{
\begin{align*}
\hat{P}^{[k], l}_{[m], n}, \hat{P}^{[q], s}_{[u], v} &= (\hat{P}^{[u+v-k-l-q], s}_{[m], n} - \hat{P}^{[m+n-q-s-k], l}_{[u], v}) \\
\hat{P}^{[k], l}_{[m], n}, \hat{P}^{q, [s]}_{u, [v]} &= (P^{q, l-v+s}_{u-k+m, n} - P^{q-m+k, l}_{u, n-s+v}) \\
\hat{P}^{k, [l]}_{m, [n]}, \hat{P}^{q, [s]}_{u, [v]} &= (\hat{P}^{q, [s] \mp k-l+u+v}_{m, [n]} - \hat{P}^{k, [s] \mp l+m+n-q-s}_{u, [v]}) \\
\hat{P}^{[k], l}_{[m], n}, P^{q, s}_{u, v} &= (P^{q, s}_{u+v+m-k-l, n} - P^{q+s+k-m-n, l}_{u, v}); \\
\hat{P}^{k, [l]}_{m, [n]}, P^{q, s}_{u, v} &= (P^{q, s}_{u+v+m-k-l, n} - P^{k, q+s+l-m-n}_{u, v}).
\end{align*}
\]
(6.8)

\[
\left\{
\begin{align*}
\left[ P^{k, l}_{m, n}, P^{s, t}_{u, v} \right] &= \delta_{k, u} \delta_{l, v} P^{s, t}_{m, n} - \delta_{m, s} \delta_{n, t} P^{k, l}_{u, v}.
\end{align*}
\]
(6.10)

generate the ideal \( I \) of the algebra \( \hat{\mathfrak{A}} \).

We define algebra \( \mathfrak{A} \) as the factor algebra

\[ \mathfrak{A} = \hat{\mathfrak{A}} / I. \]

The algebra \( \mathfrak{A} \) is the infinite-dimensional associative algebra generated by the operators

\[ I, N_1, N_2, a_{\text{sect}}^\pm, a_{\text{rad}}^\pm, \hat{P}^{k, [l]}_{m, [n]}, \hat{P}^{[k], l}_{[m], n}, P^{k, l}_{m, n}, \]
(6.11)
satisfying commutation relations (6.4)-(6.10). (Recall that it is assumed that the indexes \( k, l, m, n \geq 0 \), and if at least one of the indices of operators \( P \) and \( \hat{P} \) is negative, then the corresponding operator is equal to zero).

The algebras \( \hat{\mathfrak{A}} \) and \( \mathfrak{A} \) are Lie algebras because one can check that for generators (6.11) are fulfilled Jacobi identities.

### 7 Investigation of algebra \( \mathfrak{A} \)

#### 7.1 Commutation relations between generators of the algebra \( \mathfrak{A} \)

For the convenience of further considerations we shall divide the set of generators (6.11) of the algebra \( \mathfrak{A} \) on 4 subset.
| type | generators | symbolic notation |
|------|------------|------------------|
| I    | $I, N_1, N_2$ | $A_I = A$ |
| II   | $a_{sect}^\pm, a_{rad}^\pm$ | $A_{II} = B$ |
| III$_{left}$ | $\hat{P}_m^k, l$ | $A_{III}^{left} = C^l$ |
| III$_{right}$ | $\hat{P}_m^k, l$ | $A_{III}^{right} = C^r$ |
| IV   | $P_{m,n}^k, l$ | $A_{IV} = D$ |

From relations (6.4) - (6.10) it follows that

$[A_1, A_2] = 0, [A, B_1] = B_2, [A, C_1^l] = C_2^l, [A, C_1^r] = C_2^r, [A, D_1] = D_2$;  

$[B_1, B_2] = C^l + C^r + D, [B, C_1^l] = C_2^l + C_3^l, [B, C_1^r] = C_2^r + C_3^r, [B, D_1] = D_2 + D_3$;  

$[C_1^l, C_2^l] = C_3^l + C_4^l, [C_1^r, C_2^r] = C_3^r + C_4^r, [C^l, C^r] = D_1 + D_2, [C^l, D_1] = D_2 + D_3$;  

$[D_1, D_2] = D_3 + D_4$.  

(7.1)  

(7.2)  

(7.3)  

(7.4)

### 7.2 Center $Z(\mathfrak{A})$ of the algebra $\mathfrak{A}$

An element $z$ belongs to the center $Z(\mathfrak{A})$ of the algebra $\mathfrak{A}$ if and only if this element commutes with all generators $X$ (6.11) of the algebra $\mathfrak{A}$:

$[z, X] = 0$.  

(7.5)

The element $X$ can be represented in the following form

$z = d_0 I + d_1 N_1 + d_2 N_2 + d_s^+ a_{sect}^+ + d_s^- a_{sect}^- + d_r^+ a_{rad}^+ + d_r^- a_{rad}^- + 

\sum_{k,l,m,n \geq 0} (d_k(L)^{[k], l} P_{m,n}^k, l + d_k(R)^{[k], l} P_{m,n}^k, l) + \sum_{k,l,m,n \geq 0} d_k(D)^{[k], l} P_{m,n}^k, l$  

(7.6)
Choose $X = P_{s,t}^{s,t}$ in (7.3). From the commutation relations (6.4)-(6.10) we get

$$[I, P_{s,t}^{s,t}] = 0, \quad [N_1, P_{s,t}^{s,t}] = 0, \quad [N_2, P_{s,t}^{s,t}] = 0; \quad (7.7a)$$

$$[a_{\text{sec}}^{\pm}, P_{s,t}^{s,t}] = \frac{1}{\sqrt{10}} (P_{s+1,t+1}^{s+1,t+1} - P_{s,t}^{s+1,t+1}) \neq 0; \quad (7.7b)$$

$$[a_{\text{rad}}^{\pm}, P_{s,t}^{s,t}] = \sqrt{\frac{2}{5}} (P_{s+1,t+1}^{s+1,t+1} - P_{s,t}^{s+1,t+1}) \neq 0; \quad (7.7c)$$

$$[\hat{P}_{[m,n]}^{[k,l]} P_{s,t}^{s,t}] = \left( P_{s+t+m-k-l, n}^{s+t+m-k-l, n} - P_{s,t}^{s+t+m-k-l, n} \right) = \begin{cases} 0, & \text{as } m = k, n = l = t \\ \neq 0, & \text{otherwise}; \end{cases} \quad (7.7d)$$

$$[\hat{P}_{m,[n]}^{k,l} P_{s,t}^{s,t}] = \left( P_{m+s+t-n-k-l}^{m+s+t-n-k-l} - P_{s,t}^{m+s+t-n-k-l} \right) = \begin{cases} 0, & \text{as } n = l, m = k = s \\ \neq 0, & \text{otherwise}; \end{cases} \quad (7.7e)$$

$$[P_{m,n}^{k,l}, P_{s,t}^{s,t}] = \begin{cases} P_{m,n}^{s,t}, & \text{as } (k,l) = (s,t) \neq (m,n) \\ -P_{s,t}^{k,l}, & \text{as } (m,n) = (s,t) \neq (k,l) \\ 0, & \text{otherwise}; \end{cases} \quad (7.7f)$$

We remind that all generators (6.11) are linearly independent. Then, from (7.3) it follows that all coefficients at nonzero commutators must be equal to zero. This allows us to simplify the (7.6) as follows

$$z = d_0 I + d_1 N_1 + d_2 N_2 + \sum_{k \geq 0} d_{k,t;k,t}^{(L)} \hat{P}_{[k],t}^{[k],t} + \sum_{l \geq 0} d_{s,t;s,t}^{(R)} \hat{P}_{s,[l]}^{s,[l]} +$$

$$\sum_{k,l \geq 0} d_{k,t,m,n}^{(D)} P_{m,n}^{k,l} + \sum_{k,l \geq 0} d_{k,t,k,t}^{(D)} P_{k,k}^{k,k}. \quad (7.8)$$

We can put $(s,t) = (k,l)$ or $(s,t) = (m,n)$, because $s, t$ are arbitrary integer numbers. Then we can prove that all coefficients $d_{k,t,m,n}^{(D)}$ in the penultimate sum in the right-hand side of (7.8) equals to zero. Thus (7.8) become simpler and takes the form

$$z = d_0 I + d_1 N_1 + d_2 N_2 + \sum_{k \geq 0} d_{k,t;k,t}^{(L)} \hat{P}_{[k],t}^{[k],t} + \sum_{l \geq 0} d_{s,t;s,t}^{(R)} \hat{P}_{s,[l]}^{s,[l]} + \sum_{k,l \geq 0} d_{k,t,k,t}^{(D)} P_{k,k}^{k,k}. \quad (7.9)$$

Let $X$ be one of ladder operators, for example $a_{\text{sec}}^{+}$. Then, from (7.7b), (6.6) and

$$[N_1, a_{\text{sec}}^{+}] = -a_{\text{sec}}^{+} \neq 0, \quad [N_2, a_{\text{sec}}^{+}] = a_{\text{sec}}^{+} \neq 0,$$

we conclude that all coefficients in the right-hand side of the equality (7.9), excluding $d_0$, are equal to zero. So, we have $z = d_0 I$ that is,

$$\mathfrak{Z}(\mathfrak{A}) = d_0 I,$$

which means that the center $\mathfrak{Z}(\mathfrak{A})$ of the algebra $\mathfrak{A}$ is one-dimensional.
7.3 Maximal Abelian subalgebra of the algebra $\mathfrak{A}$

We consider the commutative subalgebra $\mathfrak{L}$ in $\mathfrak{A}$, generated by the operators $\{P_{m,n}\}_{m,n \geq 0}$. We denote by $\mathfrak{M}$ Abelian subalgebra in $\mathfrak{A}$, obtained from $\mathfrak{L}$ by addition to generators $P_{m,n}$ of all formal series in these generators. Note that

$$N_1 = \bigoplus_{m,n \geq 0} mP_{m,n}, \quad N_2 = \bigoplus_{m,n \geq 0} nP_{m,n}, \quad I = \bigoplus_{m,n \geq 0} P_{m,n},$$

which means that an arbitrary element $u \in \mathfrak{M}$ can be written as

$$u = \bigoplus_{k,l \geq 0} c_{k,l}P_{k,l}. \quad (7.11)$$

We show that $\mathfrak{M}$ is a maximal Abelian subalgebra in Lie algebra $\mathfrak{A}$. Indeed, suppose that there is some element $z$ of $\mathfrak{A}$ such that

$$[z, X] = 0,$$

for all $X \in \mathfrak{M}$. Choosing $X = P_{s,t}^s$ (see reasons above), we obtain for $z$ the equality $\{7.9\}$, i.e.

$$z = d_0 I + d_1 N_1 + d_2 N_2 + \sum_{k \geq 0} d_{k, t}^{(L)} \hat{P}_{k, t}^{[k], t} + \sum_{l \geq 0} d_{s, l}^{(R)} \hat{P}_{s, l}^{[s], l} + \sum_{k, l \geq 0} d_{k, l}^{(D)} kP_{k, l}.$$  \(7.12\)

Taking into account $\{6.3\}$ and $\{7.10\}$, the relation $\{7.12\}$ can be rewritten as

$$z = \bigoplus_{k, l \geq 0} \left( k + l + d_{k, l}^{(D)} \right) P_{k, l}.$$  \(7.11\)

In view of $\{7.11\}$, this means that $z \in \mathfrak{M}$. Hence it follows that subalgebra $\mathfrak{M}$ is the maximal Abelian subalgebra in the Lie algebra $\mathfrak{A}$.

Note that $\mathfrak{M}$ contains the commutative subalgebra constructed in work $\{47\}$. This subalgebra is an extension of Koornwinder algebra $\{29\}$ consisting of all differential operators of the variables $z, \overline{z}$, for which Chebyshev-Koornwinder polynomials are eigenfunctions.

### 7.4 Subalgebras and ideals in $\mathfrak{A}$. Representation $\mathfrak{A}$ in the form of semidirect sum

Let us consider the following vector subspace of the algebra $\mathfrak{A}$:

$$\mathfrak{A}_1 = \text{Span}\{I, N_1, N_2\};$$
$$\mathfrak{A}_2 = \text{Span}\{a_\pm \text{sect}, a_\pm \text{rad}\};$$
$$\mathfrak{A}_3^L = \text{Span}\{\hat{P}_{k, l}^{[k], l}\}, \quad \mathfrak{A}_3^R = \text{Span}\{\hat{P}_{m, n}^{[m], n}\}, k, l, m, n \geq 0;$$
$$\mathfrak{A}_3 = \mathfrak{A}_3^L + \mathfrak{A}_3^R;$$
$$\mathfrak{A}_4 = \text{Span}\{P_{m, n}\}, k, l, m, n \geq 0;$$
$$\mathfrak{B}_1 = \mathfrak{A}_1 + \mathfrak{A}_3^L; \quad \mathfrak{B}_1^R = \mathfrak{A}_2 + \mathfrak{A}_3^R;$$
$$\mathfrak{B}_1 = \mathfrak{B}_1^L + \mathfrak{B}_1^R;$$

20
\[ B_2 = A_2 \oplus A_3 \oplus A_4, \quad B_2 = A_2 \oplus A_3 \oplus A_4, \]
\[ B_2 = B_2 \oplus B_2. \]
\[ B_2 = A_2 \oplus A_3 \oplus A_4, \quad B_2 = A_3 \oplus A_4, \]
\[ B_3 = B_3 \oplus B_3. \]

Of the commutation relation (7.1)–(7.4), it follows that

• \( A_1 \) — Abelian subalgebra of infinite-dimensional algebra \( \mathfrak{A} \);
• \( B_2^L, B_2^R, B_2^L \oplus B_2^R \) — two-sided ideals in \( \mathfrak{A} \), where \( \oplus \) — direct sum of ideals;
• \( B_3^L \) — two-sided ideals in \( \mathfrak{A} \) and therefore two-sided ideals in \( B_2 \) and in \( B_3^L \);
• \( B_3^R \) — two-sided ideals in \( \mathfrak{A} \) and therefore two-sided ideals in \( B_2 \) and in \( B_3^R \);
• \( A_4 \) — two-sided ideals in \( \mathfrak{A} \) and therefore two-sided ideals in \( B_2 \);

Then \( A_4 \) is two-sided ideals in \( B_2^L \) and in \( B_2^R \), as well in \( B_3, B_3^L \) and in \( B_3^R \) too. We construct derived series for the algebra \( \mathfrak{A} \)
\[ \mathfrak{A}^{(0)} = \mathfrak{A}, \quad \mathfrak{A}^{(1)} = [\mathfrak{A}^{(0)}, \mathfrak{A}^{(0)}], \quad \mathfrak{A}^{(2)} = [\mathfrak{A}^{(1)}, \mathfrak{A}^{(1)}], \quad \ldots. \]

It can be checked that
\[ \mathfrak{A}^{(1)} = B_2, \quad \mathfrak{A}^{(n)} = \mathfrak{A}^{(2)} = \mathfrak{B}_3, \quad \forall n \geq 2. \quad (7.13) \]

From (7.13) it follows that \( \mathfrak{A} \) is not solvable and therefore also not nilpotent algebra.

Denote by \( \mathfrak{N} \) a radical (i.e. maximal solvable ideal) of algebra \( \mathfrak{A} \). It is known that \( \mathfrak{Z}(\mathfrak{A}) \subseteq \mathfrak{N} \). Then, taking into account (7.13), we have
\[ \mathfrak{Z}(\mathfrak{A}) \subseteq \mathfrak{N} \subset \mathfrak{A}. \]

To verify that the generalization of the theorem Levi-Maltsev [48] true for algebra \( \mathfrak{A} \), we must find the radical \( \mathfrak{N} \) of the algebra \( \mathfrak{A} \).

**Remark.** The first step in finding \( \mathfrak{N} \) is to construct solvable ideal \( \mathfrak{L} \subseteq \mathfrak{N} \) such that
\[ [\mathfrak{L}, \mathfrak{L}] = \mathfrak{Z}(\mathfrak{A}) \Rightarrow \mathfrak{Z}(\mathfrak{A}) \subseteq \mathfrak{L}. \quad (7.14) \]

There are two alternatives
\[ \mathfrak{L} \quad \text{does not exist} \quad \Rightarrow \quad \mathfrak{N} = \mathfrak{Z}(\mathfrak{A}), \quad (7.15a) \]
\[ \mathfrak{L} \quad \text{exist} \quad \Rightarrow \quad \mathfrak{Z}(\mathfrak{A}) \subset \mathfrak{N}. \quad (7.15b) \]

Our hypothesis is true (7.15a), i.e \( \mathfrak{A} \) reductive [49], but at present we cannot give the proof of validity of this hypothesis.

Recall that a Lie algebra \( \mathfrak{L} \) is the semi-direct sum of Lie subalgebras \( \mathfrak{T} \) and \( \mathfrak{M} \) (\( \mathfrak{L} = \mathfrak{M} \oplus \mathfrak{T} \)) if
\[ [\mathfrak{T}, \mathfrak{T}] \subset \mathfrak{T}, \quad [\mathfrak{M}, \mathfrak{M}] \subset \mathfrak{M}, \quad [\mathfrak{M}, \mathfrak{T}] \subset \mathfrak{T}. \quad (7.17) \]
As seen from (7.17), $\mathcal{S}$ is an ideal in $\mathcal{L}$.

Since $\mathcal{A} = \mathcal{A}_1 + \mathcal{B}_2$, where $\mathcal{A}_1$ is a subalgebra in $\mathcal{A}$ and $\mathcal{B}_2$ is an ideal in $\mathcal{A}$, then from (7.16), (7.17) we have

\[ \mathcal{A} = \mathcal{A}_1 \oplus \mathcal{B}_2. \]

We introduce notations

\[ \mathcal{A}^L = \mathcal{A}_1 \oplus \mathcal{B}_2^L, \quad \mathcal{A}^R = \mathcal{A}_1 \oplus \mathcal{B}_2^R, \]

where $\mathcal{A}_1$ is a subalgebra in $\mathcal{A}^L$ and in $\mathcal{A}^R$; $\mathcal{B}_2^L$ is an ideal in $\mathcal{A}^L$; $\mathcal{B}_2^R$ is an ideal in $\mathcal{A}^R$. Then, from (7.16), (7.17) implies that

\[ \mathcal{A}^L = \mathcal{A}_1 \oplus \mathcal{B}_2^L, \quad \mathcal{A}^R = \mathcal{A}_1 \oplus \mathcal{B}_2^R. \]

In addition, as

\[ \mathcal{B}_2^L = \mathcal{B}_1^L + \mathcal{A}_4, \quad \mathcal{B}_2^R = \mathcal{B}_1^R + \mathcal{A}_4, \]

where $\mathcal{B}_1^L$ is a subalgebra in $\mathcal{B}_2^L$; $\mathcal{B}_1^R$ is a subalgebra in $\mathcal{B}_2^R$; $\mathcal{A}_4$ is an ideal in $\mathcal{B}_2^L$ and in $\mathcal{B}_2^R$, then

\[ \mathcal{B}_2^L = \mathcal{B}_1^L \oplus \mathcal{A}_4, \quad \mathcal{B}_2^R = \mathcal{B}_1^R \oplus \mathcal{A}_4. \]

However note that, $\mathcal{B}_2 \neq \mathcal{B}_1 \oplus \mathcal{A}_4$, as $\mathcal{B}_1^L + \mathcal{B}_1^R = \mathcal{B}_1$ is not an subalgebra in $\mathcal{B}_2$.

### 7.5 Proof simplicity of the ideal $\mathcal{A}_4$

We prove that $\mathcal{A}_4$ is the simple ideal. To do this, we suppose that there exists a non-zero two-sided ideal $I \subset \mathcal{A}_4$ and show that

\[ I = \mathcal{A}_4. \]  

(7.18)

To begin with, we choose the standard basis in $\mathcal{A}_4$

\[ \{ P_{m,n}^{k,l}; (k, l) \neq (m, n); h_{m,0}, h_{0,n}, h_{m,n} \}, \]

where $h_{m,0} = P_{m,0}^0$, $h_{0,n} = P_{0,n}^0$, $h_{m,n} = P_{m+1,n+1}^m - P_{m,n}^m$.

To prove that this set is really forms the basis of the $\mathcal{A}_4$, we should check that any element $P_{m,n}^m (m \geq 0, n \geq 0)$ belongs to the linear span of this set. But this assertion follows from the obvious equality

\[ P_{m,n}^m = \sum_{k=0}^{\min(m,n)} h_{m-k,n-k}. \]

Further, if $I$ is ideal distinct from zero, then there exists a nonzero element $z \in I$. We present $z$ in the form of a linear combination of the basis elements

\[ z = \sum_{k, l, m, n \geq 0} \alpha_{m,n}^{k,l} P_{m,n}^{k,l} + \sum_{m,n \geq 0} \beta_{m,n} h_{m,n}. \]  

(7.19)

Let $X = P_{u,v}^{s,t}, ((u, v) \neq (s, t))$ is an element of $\mathcal{D}$. Then

\[ (\text{ad}^2_X)z = -2\alpha_{s,t}^{u,v} X. \]

Since $z \in I$, it follows that $\alpha_{s,t}^{u,v} X \in I$, and if $\alpha_{s,t}^{u,v} \neq 0$, then $X = P_{u,v}^{s,t} \in I$. Doing so for all elements $P_{u,v}^{s,t}$ with $(u, v) \neq (s, t)$, we get two alternatives
1. There is a nonzero element \( P_{s,t}^{u,v} \in I \).

2. All the coefficients \( \alpha_{m,n}^{k,l} \) in the expansion (7.19) are equal to zero, and then
\[
z = \sum_{m,n \geq 0} \beta_{m,n} h_{m,n} \in I.
\]

We have to show that in both cases, equation (7.18) is true, i.e. \( I = \mathfrak{A}_4 \).

In the first case to check the validity of equality (7.18) it is enough to show that if \( X = P_{s,t}^{u,v} \in I \) under the condition \( (s, t) \neq (u, v) \), then any basic element \( P_{k,l}^{m,n} \) (with \( (k, l) \neq (m, n) \)) and any basic element \( h_{m,n} \) belong \( I \). Using commutation relations
\[
[P_{u,v}^{s,t}, P_{s,t}^{k,l}] = P_{u,v}^{k,l} \in I, \quad (k, l) \neq (u, v);
\]
we obtain
\[
[P_{u,v}^{k,l}, P_{m,n}^{u,v}] = -P_{m,n}^{k,l} \in I.
\]
If \( (k, l) = (u, v) \), then we have
\[
[P_{k,l}^{s,t}, P_{s,t}^{m,n}] = P_{k,l}^{m,n} \in I, \quad (k, l) \neq (m, n).
\]
Further, it is straightforward to show that if \( (k, l) \neq (m, n) \) from \( P_{k,l}^{m,n} \in I \) implies that \( P_{m,n}^{k,l} \in I \).
It remains to show that any element \( h_{m,n} \in I \). Since for \( (s, t) \neq (u, v) \),
\[
P_{s,t}^{u,v} \in I \Rightarrow P_{u,v}^{s,t} \in I,
\]
then
\[
[P_{u,v}^{s,t}, P_{s,t}^{m,n}] = (P_{u,v}^{m,n} - P_{s,t}^{s,t}) \in I.
\]
For \( u = m + 1, v = n + 1, s = m, t = n \) from (7.20) it follows that
\[
h_{m,n} = (P_{m+1,n+1}^{m,n} - P_{m,n}^{m,n}) \in I.
\]
(7.21)
Note that (7.21) is true for both \( h_{m,0} \) and \( h_{0,n} \). So, the first variant considered fully.

Let us now consider the second variant. In this case there exists at least one non-null element \( h_{m,n} \in I \). Then if \( (s, t) \neq (m, n) \) and \( (s, t) \neq (m + 1, n + 1) \), we have for any \( P_{s,t}^{m,n} \)
\[
[h_{m,n}, P_{m,n}^{s,t}] = [(P_{m+1,n+1}^{m,n} - P_{m,n}^{m,n}), P_{m,n}^{s,t}] = -P_{m,n}^{s,t} \in I.
\]
Using the facts proved in considering the first variant, we get \( I = \mathfrak{A}_4 \). So, the relation (7.18) is proven. Since \( \mathfrak{A}_4 \) has no nonzero ideals other than \( \mathfrak{A}_4 \), then \( \mathfrak{A}_4 \) is simple algebra.

8 Conclusion

1. One can consider presented in the paper ChK-oscillator as the simplest nontrivial example of a quantum system composed of three interacting one-dimensional oscillators. Note that in all known to the authors papers on generalized oscillators associated with orthogonal polynomials in several variables these oscillators form a system of independent one-dimensional oscillators, because the related oscillator algebras splits in direct sum of classical Lie algebras.
2. An interesting question is under which conditions the oscillator algebra is finite-dimensional. The answer to this question was given in the work [50] for an one-dimensional generalized oscillator related to a system of polynomials orthogonal with respect to symmetric measure on real axis. In the cited work were given some consideration of the oscillator algebras associated with multi-boson systems.

The results of this work suggest that in the nontrivial case (i.e. when a multi-dimensional oscillator describes a system of interacting particles) the corresponding oscillator algebra is infinite-dimensional.

3. In conclusion, we note that in our work was made only the first step in study of Lie algebra of two-dimensional ChK-oscillator. In particular, it was necessary to investigate the possibility of constructing the root system for considered infinite-dimensional Lie algebra. For such investigation one must to find the radical subalgebra of oscillator algebra and to check the validity of Levi-Maltsev decomposition. The authors intend to investigate this question in subsequent publications.

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