VIOLATION OF S-MATRIX FACTORIZATION IN MASSIVE THIRRING MODEL

T. FUJITA and M. HIRAMOTO
Department of Physics, Faculty of Science and Technology
Nihon University, Tokyo, Japan

ABSTRACT

We present a counter example which shows the violation of the S-matrix factorization in the massive Thirring model. This is done by solving the PBC equations of the massive Thirring model exactly but numerically. The violation of the S-matrix factorization is related to the fact that the crossing symmetry and the factorization do not commute with each other. This confirms that the soliton antisoliton S-matrix factorization picture of the sine-Gordon model is semiclassical and does not lead to a full quantization procedure of the massive Thirring model.

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1. Introduction

Integrable field theories in 1+1 dimensions have been studied in various fields of research. The Heisenberg $xxx$ model is solved by the Bethe ansatz technique [1]. The solution is confirmed most elegantly by the Yang-Baxter equation [2-4]. Also, the Heisenberg $xxx$ model with spin $1/2$ is equivalent to the Thirring model which is solved exactly in the continuum field theory. Since the Yang-Baxter equation corresponds to the S-matrix factorization, there is, therefore, no question about the S-matrix factorization in this case.

Also, the Heisenberg $xyz$ model with spin $1/2$ has been studied quite extensively [5-9]. Since this model is proved to be equivalent to the massive Thirring model in the continuum limit [10], it should be interesting to study these models from different points of view.

The S-matrix factorization is also assumed for the massive Thirring model or sine-Gordon field theory [11]. With this ansatz of S-matrix factorization, the spectrum of the sine-Gordon model has been obtained and is found to be consistent with the spectrum obtained by the WKB method [12-14]. Therefore, it has been believed that the factorization of the S-matrix for the massive Thirring model holds exactly since the bound state spectrum obtained by the WKB method was considered to be exact. However, the recent investigations of the bound state problem of the massive Thirring model from the light cone method as well as from the Bethe ansatz technique have shown that there is only one bound state, on the contrary to the WKB result [15-18]. Indeed, it is proved that the WKB result of the bound state spectrum for the massive Thirring model is not exact [15-17]. This suggests that we should reexamine the S-matrix factorization for the massive Thirring model since the S-matrix factorization has also been believed to hold in an exact fashion.

Until now, this factorization ansatz of the S-matrix for the massive Thirring model has never been questioned seriously. In fact, the factorization of the
S-matrix for the particle particle scattering holds exactly. This can be easily seen by using the Bethe ansatz wave functions. However, it is nontrivial to obtain the S-matrix factorization for the particle-antiparticle scattering case, as we will see it later.

Now, the point is that the factorization of the S-matrix has been obtained for the sine-Gordon field theory with soliton anti-soliton pictures. Therefore, in this case, one does not have to worry about the antiparticle property. However, the soliton and antisoliton are only semiclassical objects and there is no clear way to quantize it. This is related to the fact that the soliton is a solution of the field equation, but not the field itself.

Therefore, even though one can easily write down the particle antiparticle scattering S-matrix with the factorization ansatz starting from the particle particle S-matrix factorization property, the identification of the soliton and antisoliton as the fermion and antifermion cannot be justified quantum mechanically.

In this paper, we present a counter example which shows the violation of the S-matrix factorization for particle hole scattering case in the massive Thirring model. This is based on the Bethe ansatz solutions which we obtain exactly by numerically solving the PBC equations. The structure of the Bethe ansatz solutions is quite similar to the free fermion field theory. The ultraviolet cutoff $\Lambda$ is determined by the fermion number $N$ when we put the theory into the box with the length of $L$. In free field theory, the $\Lambda$ is given as

$$\Lambda = \frac{2\pi N}{L}$$

while, in the massive Thirring model, $\Lambda$ can be obtained as the function of the $N/L$ after we solve the PBC equations properly. Any physical observables can be obtained by letting the $N$ and $L$ infinity. Here, we note that this is the simplest but the best way to define the field theory.

In order to discuss the S-matrix factorization, we have to determine the
vacuum and then create particle hole states as long as we are interested in the charge zero sector. Since the number operator commutes with the hamiltonian of the massive Thirring model, we are only concerned with the charge zero sector.

Here, we show that the S-matrix factorization does not commute with the crossing symmetry. Since the lagrangian of the massive Thirring model is invariant under the charge conjugation, one tends to believe that the crossing symmetry should always hold. Indeed, the crossing symmetry itself holds. However, one must be careful for the order of operations in quantum mechanics. There is no guarantee that one can make use of the crossing symmetry for the S-matrix of the particle hole scattering with the factorization properties together.

In order to check the validity of the S-matrix factorization, we have to solve the equations of the periodic boundary conditions (PBC) and construct the vacuum. However, this is not sufficient if we want to deal with the particle hole scattering. We have to solve the PBC equations for $n-$particle $n-$hole states.

The important point is that the PBC equations for $n-$particle $n-$hole states are different not only from those for the vacuum states but also from each other. Therefore, we have to first solve these PBC equations to determine the rapidities of the vacuum as well as those of the $n-$particle $n-$hole states.

It is always an interesting question how one can construct the particle hole state under the situation where the rapidity distribution of the particle hole states are different from those of the vacuum state. In other words, how can we find or identify the rapidity of the hole state if the rapidities of the negative energy particles for the $n-$particle $n-$hole states are different from those of the vacuum state? Later, we will see that we can find the rapidity of the hole state in the limit of infinitely large $N$ (the number of
particle) and $L$ (the box size). In this limit, we can maintain the particle hole picture.

Here, a question arises. Can we also prove the factorization of the S-matrix even for the particle hole scattering case? As can be seen from the above statement, the hole states carry the information of all the negative energy particles since the rapidities of the particle hole state are different from the vacuum. Therefore, it is highly nontrivial that the S-matrix for the particle hole scattering can be factorized.

Indeed, we will show below that the S-matrix for the particle hole scattering is not factorizable since the factorization and the limit of $N$ and $L$ infinity do not commute with each other. We stress here that, if we show any examples of the violation of the S-matrix factorization, this is sufficient that one cannot rely on the factorization ansatz. On the contrary, if we wanted to show the validity of the S-matrix factorization, then it would have been a very hard work.

The violation of the factorization ansatz indicates that the bound state spectrum constructed from the S-matrix factorization cannot be justified any more. As mentioned above, this is consistent with recent calculations on the massive Thirring model, which show that the Bethe ansatz PBC equations give only one bound state, on the contrary to the semiclassical result and that of the Bethe ansatz technique with the string hypotheses. In fact, it is shown that there is no string-like solution that satisfies the PBC equations if they are solved exactly for the particle hole configurations [15]. Also, the light cone prescription of the massive Thirring model shows that there is only one bound state [16,17]. This indicates that the S-matrix factorization must correspond to the semiclassical approximation. This point is indeed shown in this paper, that is, the S-matrix factorization can be justified only when one neglects the commutability of the operators. This means that the soliton antisoliton picture is indeed semiclassical. Therefore, in order to make a correct correspondence between the soliton and fermion
in a fully quantized sense, we should not rely on the S-matrix factorizations. This paper is organized as follows. In the next section, we will briefly describe Zamolodchikov’s S-matrix factorization method. Then, in section 3, we discuss the Bethe ansatz solution for the massive Thirring model. In section 4, we present the numerical solutions of the Bethe ansatz PBC equations in order to define the vacuum as well as the $n-$partcile $n-$hole states. Then, section 5 will treat the S-matrix for the particle hole scattering. In particular, we will present the numerical evidences that the S-matrix of the particle hole scattering is not factorizable. Section 6 will summarize what we have understood from this work.

2. Zamolodchikov’s S-matrix factorization

In integrable field theories, there are infinite conservation laws which induce strong selection rules [11]. For example, there is no particle creation or annihilation in the scattering process. Also, there is no change of the momentum before and after the scattering process. This means that the scattering is always elastic.

The S-matrix for the two body scattering case can be defined as

$$S|A_i(\alpha_i)A_j(\alpha_j) >_{in} = \sum_{k,l} S_{ij}^{kl}(\alpha_i, \alpha_j)|A_k(\alpha_i)A_l(\alpha_j) >_{out}$$ \hspace{1cm} (2.1)

where $A_i(\alpha)$’s describe the particle states with the rapidity $\alpha$.

In the same way, we can consider the $N$ body S-matrix

$$S|A_{i_1}(\alpha_{i_1})...A_{i_N}(\alpha_{i_N}) >_{in} = \sum_{j_1...j_N} S_{i_1...i_N}^{j_1...j_N}(\alpha_{i_1}, ..., \alpha_{i_N})|A_{j_1}(\alpha_{i_1})...A_{j_N}(\alpha_{i_N}) >_{out}$$ \hspace{1cm} (2.2)

The basic assumption of the S-matrix factorization is that this S-matrix can be written by the product of the two body S-matrices.
For simplicity, we consider the three body case. One assumes that the S-matrix of the three body scattering can be written as

$$S_{ijk}^{lmn} = \sum_{p_1,p_2,p_3} S_{ij}^{p_1,p_2} (\alpha_1, \alpha_2) S_{p_2k}^{p_3n} (\alpha_3, \alpha_1) S_{p_1l}^{lm} (\alpha_2, \alpha_3)$$  \hspace{1cm} (2.3)

This assumption can be checked by making use of the Bethe ansatz solutions to the massive Thirring model for the particle-particle scattering case.

In this three body scattering case, we can consider the following two different processes. In the first case, the particle 1 scatters with the particle 2, and then the particle 1 scatters with the particle 3, and finally the particle 3 scatters with the particle 2. On the other hand, in the second case, the particle 2 first scatters with the particle 3, and then the particle 3 scatters with the particle 1, and finally the particle 1 scatters with the particle 2.

Since the final state is the same between the above two processes, the two scattering events should give the same scattering process. Therefore, we obtain

$$\sum_{p_1,p_2,p_3} S_{ij}^{p_1,p_2} (\alpha_1, \alpha_2) S_{p_2k}^{p_3n} (\alpha_3, \alpha_1) S_{p_1l}^{lm} (\alpha_2, \alpha_3) = \sum_{q_1,q_2,q_3} S_{jk}^{q_2,q_3} (\alpha_2, \alpha_3) S_{iq_2}^{q_3} (\alpha_1, \alpha_3) S_{q_2q_1}^{mn} (\alpha_1, \alpha_2)$$  \hspace{1cm} (2.4)

This is the Yang-Baxter equation.

Now, Zamolodchikov further assumes that the S-matrix should satisfy the unitarity, the analyticity and the crossing symmetry. The unitarity reads

$$\sum_{j_1,j_2} S_{j_1,j_2}^{i_1,i_2} S_{j_2k_1}^{i_1} S_{k_1k_2}^{j_2} = \delta_{k_1,i_1} \delta_{k_2,i_2}. \hspace{1cm} (2.5)$$

The analyticity is written as

$$S^\dagger (\alpha) = S(-\alpha^*). \hspace{1cm} (2.6)$$

Finally, the crossing symmetry can be stated as

$$S_{ij}^{kl} (\alpha) = S_{ji}^{\tilde{k} \tilde{l}} (i\pi - \alpha) \hspace{1cm} (2.7)$$

where $\tilde{i}$ and $\tilde{l}$ denote the anti-particles of $i$ and $l$ states. Here, the antisoliton of the sine-Gordon model is identified as the antifermion of the
massive Thirring model. Together with the above assumptions, Zamolodchikov obtained the functional equation with the particles and antiparticles included.

However, for the massive Thirring model, it is nontrivial to define the antiparticle state in the nonperturbative sense. This is because one has to first determine the vacuum and then construct the particle hole states. The problem here is that the rapidity distributions for the vacuum and the particle hole states are different from each other since the periodic boundary condition equations are different. Therefore, it is nontrivial to make hole states referring to the vacuum rapidity distribution.

The question we want to address here is whether this crossing symmetry (eq. (2.7)) can be satisfied together with the factorization of the S-matrix or not.

It is obvious that the crossing symmetry itself should hold since the Lagrangian of the massive Thirring model possesses this symmetry. Indeed, this can also be proved by the Bethe ansatz solutions.

However, it is highly nontrivial whether the factorization for the particle hole scattering S-matrix can also hold or not. This is what we want to check in this paper and we prove numerically that the particle hole S-matrices do not satisfy the factorization.

3. Bethe ansatz solution of massive Thirring model

The massive Thirring model is a 1+1 dimensional field theory with current current interactions [19]. Its lagrangian density can be written as

\[
\mathcal{L} = \bar{\psi}(i\gamma_{\mu}\partial^{\mu} - m_0)\psi - \frac{1}{2}g_0 j^{\mu} j_{\mu} \tag{3.1}
\]
where the fermion current \( j_\mu \) is written as
\[
 j_\mu =: \bar{\psi} \gamma_\mu \psi :
\] (3.2)

Choosing a basis where \( \gamma_5 \) is diagonal, the hamiltonian is written as
\[
 H = \int dx \left[ -i(\psi_1^\dagger \frac{\partial}{\partial x} \psi_1 - \psi_2^\dagger \frac{\partial}{\partial x} \psi_2) + m_0(\psi_1^\dagger \psi_2 + \psi_2^\dagger \psi_1) + 2g_0 \psi_1^\dagger \psi_2^\dagger \psi_2 \psi_1 \right].
\] (3.3)

Now, we define the number operator \( N \) as
\[
 N = \int dx \psi^\dagger \psi.
\] (3.4)

This number operator \( N \) commutes with \( H \). Therefore, when we construct physical states, we must always consider physical quantities with the same particle number \( N \) as the vacuum. For different particle number state, the vacuum is different and thus the model space itself is different.

(a) Bethe ansatz wave functions

The hamiltonian eq.(3.3) can be diagonalized by the Bethe ansatz wave functions. The Bethe ansatz wave function \( \Psi(x_1, ..., x_N) \) for \( N \) particles can be written as [13-14]
\[
 \Psi(x_1, ..., x_N) = \exp(im_0 \sum x_i \sinh \beta_i) \prod_{1 \leq i < j \leq N} [1 + i\lambda(\beta_i, \beta_j) \epsilon(x_i - x_j)]
\] (3.5)

where \( \beta_i \) is related to the momentum \( k_i \) and the energy \( E_i \) of the \( i \)-th particle as
\[
 k_i = m_0 \sinh \beta_i.
\] (3.6a)
\[
 E_i = m_0 \cosh \beta_i.
\] (3.6b)

Here, \( \beta_i \)'s are complex variables.

\( \epsilon(x) \) is a step function and is defined as
\[
 \epsilon(x) = \begin{cases} 
 -1 & x < 0 \\
 1 & x > 0.
\end{cases}
\] (3.7)
\[ \lambda(\beta_i, \beta_j) \] is related to the phase shift function \( \phi(\beta_i - \beta_j) \) as

\[
\exp (i\phi(\beta_i - \beta_j)) = \frac{1 + i\lambda(\beta_i, \beta_j)}{1 - i\lambda(\beta_i, \beta_j)}. \tag{3.8}
\]

The phase shift function \( \phi(\beta_i - \beta_j) \) can be explicitly written as

\[
\phi(\beta_i - \beta_j) = -2 \tan^{-1} \left[ \frac{1}{2} g_0 \tanh \frac{1}{2} (\beta_i - \beta_j) \right]. \tag{3.9}
\]

In this case, the eigenvalue equation becomes

\[
H | \beta_1...\beta_N > = (\sum_{i=1}^{N} m_0 \cosh \beta_i) | \beta_1...\beta_N > \tag{3.10}
\]

where \( | \beta_1...\beta_N > \) is related to \( \Psi(x_1, ..., x_N) \) as

\[
| \beta_1...\beta_N > = \int dx_1...dx_N \Psi(x_1, ..., x_N) \prod_{i=1}^{N} \psi^\dagger(x_i, \beta_i) | 0 >. \tag{3.11}
\]

Also, \( \psi(x, \beta) \) can be written in terms of \( \psi_1(x) \) and \( \psi_2(x) \) as,

\[
\psi(x, \beta) = e^{\beta \frac{\alpha}{2}} \psi_1(x) + e^{-\beta \frac{\alpha}{2}} \psi_2(x). \tag{3.12}
\]

From the definition of the rapidity variable \( \beta_i \)'s, one sees that for positive energy particles, \( \beta_i \)'s are real while for negative energy particles, \( \beta_i \) takes the form \( i\pi - \alpha_i \) where \( \alpha_i \)'s are real. Therefore, in what follows, we denote the positive energy particle rapidity by \( \beta_i \) and the negative energy particle rapidity by \( \alpha_i \).

(b) Periodic Boundary Conditions

The Bethe ansatz wave functions satisfy the eigenvalue equation [eq.(3.10)]. However, they still do not have proper boundary conditions. The simplest way to define field theoretical models is to put the theory in a box of length \( L \) and impose periodic boundary conditions (PBC) on the states.

Therefore, we demand that \( \Psi(x_1, ..., x_N) \) be periodic in each argument \( x_i \). This gives the boundary condition

\[
\Psi(x_i = 0) = \Psi(x_i = L). \tag{3.13}
\]
This leads to the following PBC equations,

\[
\exp(im_0 L \sinh \beta_i) = \exp(-i \sum_j \phi(\beta_i - \beta_j)). \tag{3.14}
\]

Taking the logarithm of eq. (3.14), we obtain

\[
m_0 L \sinh \beta_i = 2\pi n_i - \sum_j \phi(\beta_i - \beta_j) \tag{3.15}
\]

where \(n_i\)'s are integers. These are equations which we should now solve.

4. Numerical Solutions

The parameters we have in the PBC equations are the box length \(L\) and the particle number \(N\). In field theory, we introduce the ultraviolet cutoff parameter. In the PBC equations, the cutoff parameter is the particle number \(N\). This is quite similar to the free field theory where the cutoff parameter is defined by the particle number \(N\). Once we determine the particle number \(N\) and the box length \(L\), then we determine all the rapidity values necessary to obtain any physical quantity. In this sense, the Bethe ansatz solution can be well defined as a field theory. Any physical observables can be obtained by letting the \(N\) and \(L\) infinity.

There is also an important quantity which is the density of the system \(\rho\). It becomes

\[
\rho = \frac{N}{L}. \tag{4.1a}
\]

Here, the system is fully characterized by the density \(\rho\). For later convenience, we define the effective density \(\rho_0\) as

\[
\rho_0 = \frac{N_0}{L_0} \tag{4.1b}
\]
where $L_0$ and $N_0$ are defined as $L_0 = m_0 L$ and $N_0 = \frac{1}{2} (N - 1)$, respectively.

It is important to note that the beta function of the massive Thirring model vanishes to all orders [20], and thus there is no need to worry about the renormalization of the coupling constant. However, one has to be careful for the coupling constant normalization arising from the way one makes the current regularization. Here, we employ the Schwinger’s normalization throughout this paper.

In what follows, we solve the PBC equations numerically in the same manner as in ref.[15]. The numerical method to solve the PBC equations is explained in detail in ref.[15]. We note that the errors involved in the numerical calculations are very small. For example, the numerical uncertainty for the rapidity values will appear at the $10^{-6}$ level, and therefore, this does not generate any problem in the present discussion.

(a) Vacuum state

The PBC equations for the vacuum which is filled with negative energy particles ( $\beta_i = i\pi - \alpha_i$ ), become

$$\sinh \alpha_i = \frac{2\pi n_i}{L_0} - \frac{2}{L_0} \sum_{j \neq i} \tan^{-1} \left( \frac{1}{2} g_0 \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right), \quad (i = 1, \ldots, N)$$

where $n_i$ runs as

$$n_i = 0, \pm 1, \pm 2, \ldots, \pm N_0.$$ 

There is no ambiguity to determine the vacuum. The vacuum is indeed constructed uniquely [15].

(b) $1p - 1h$ configuration

One particle-one hole ($1p - 1h$) state can be made by taking out one negative energy particle (the $n_{i_0}$ particle) and putting it into a positive
energy state. In this case, the PBC equations become

\[ n_i \neq n_{i_0} \]

\[
\sinh \alpha_i = \frac{2\pi n_i}{L_0} - \frac{2}{L_0} \tan^{-1} \left[ \frac{1}{2} g_0 \coth \frac{1}{2} (\alpha_i + \beta_{i_0}) \right] \\
- \frac{2}{L_0} \sum_{j \neq i,i_0} \tan^{-1} \left[ \frac{1}{2} g_0 \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right] \tag{4.3a}
\]

\[ n_i = n_{i_0} \]

\[
\sinh \beta_{i_0} = \frac{2\pi n_{i_0}}{L_0} + \frac{2}{L_0} \sum_{j \neq i_0} \tan^{-1} \left[ \frac{1}{2} g_0 \coth \frac{1}{2} (\beta_{i_0} + \phi j) \right] \tag{4.3b}
\]

where \( \beta_{i_0} \) can be a complex variable as long as it can satisfy eqs.(4.3).

It is important to note that the momentum allowed for the positive energy state must be determined by the PBC equations. Also, the momenta occupied by the negative energy particles are different from the vacuum case as long as we keep the values of \( N \) and \( L \) finite, as can be seen from eqs.(4.2) and (4.3).

Note that the lowest configuration one can consider is the case in which one takes out \( n_i = 0 \) particle and puts it into the positive energy state. This must be the first excited state since it has a symmetry of \( \alpha_i = -\alpha_{-i} \). Indeed, as discussed in ref.[15], this state corresponds to the only bound state (the boson state) in this model.

Next, we consider the following configurations in which we take out \( n_i = \pm 1, \pm 2, \ldots \) particles and put them into the positive energy state. These configurations of one particle-one hole states turn out to be the scattering states [15].

(c) \( 2p - 2h \) configurations

In the same way as above, we can make two particle-two hole \((2p - 2h)\) states. Here, we take out the \( n_{i_1} \) and \( n_{i_2} \) particles and put them into positive energy states. The PBC equations for the two particle-two
hole states become

\[ n_i \neq n_{i1}, n_{i2} \]

\[ \sinh \alpha_i = \frac{2\pi n_i}{L_0} - \frac{2}{L_0} \tan^{-1} \left( \frac{1}{2} g_0 \tanh \frac{1}{2} (\alpha_i + \beta_i) \right) \]

\[ -\frac{2}{L_0} \tan^{-1} \left[ \frac{1}{2} g_0 \coth \frac{1}{2} (\alpha_i + \beta_{i2}) \right] \]

\[ -\frac{2}{L_0} \sum_{j \neq i, i_1, i_2} \tan^{-1} \left[ \frac{1}{2} g_0 \tanh \frac{1}{2} (\alpha_i - \alpha_j) \right] \] (4.4a)

\[ n_i = n_{i1} \]

\[ \sinh \beta_{i1} = \frac{2\pi n_{i1}}{L_0} + \frac{2}{L_0} \tan^{-1} \left[ \frac{1}{2} g_0 \tan \frac{1}{2} (-\beta_{i1} + \beta_{i2}) \right] \]

\[ +\frac{2}{L_0} \sum_{j \neq i_1, i_2} \tan^{-1} \left[ \frac{1}{2} g_0 \coth \frac{1}{2} (\beta_{i1} + \alpha_j) \right] \] (4.4b)

\[ n_i = n_{i2} \]

\[ \sinh \beta_{i2} = \frac{2\pi n_{i2}}{L_0} + \frac{2}{L_0} \tan^{-1} \left[ \frac{1}{2} g_0 \tan \frac{1}{2} (\beta_{i2} - \beta_{i1}) \right] \]

\[ +\frac{2}{L_0} \sum_{j \neq i_1, i_2} \tan^{-1} \left[ \frac{1}{2} g_0 \coth \frac{1}{2} (\beta_{i2} + \alpha_j) \right] \] (4.4c)

All the configurations one can construct for the two particle two hole states turn out to be the scattering states in this model [15].

5. Factorization of S-matrix

Since the vacuum state is denoted by the rapidities \( \alpha_1, \ldots, \alpha_N \),

\[ |\text{vac} > = |\alpha_1, ..., \alpha_N > \]

we can define the S-matrix of the vacuum to vacuum transition. In this case, we can write it as

\[ < \text{vac}|S|\text{vac} > = \prod_{i > j} S_0(\alpha_i, \alpha_j) \] (5.1)
where \( S_0(\alpha_i, \alpha_j) \) denotes the two body S-matrix between \( i \) and \( j \) particles to make a transition from \( x_i < x_j \) to \( x_i > x_j \) states. It can be written as

\[
S_0(\alpha_i, \alpha_j) = \exp(i\phi(\alpha_i - \alpha_j))
\]

where \( \phi(\alpha_i - \alpha_j) \) can be written explicitly as

\[
\phi(\alpha_i - \alpha_j) = 2 \tan^{-1}\left[\frac{1}{2} g_0 \tanh \frac{1}{2}(\alpha_i - \alpha_j)\right].
\] (5.2)

Note that these particles here are negative energy ones. Obviously, the vacuum to vacuum transition must be unity,

\[
<\text{vac}|S|\text{vac}> = 1.
\] (5.3)

Therefore, we have always the constraint

\[
\prod_{i>j}^N S_0(\alpha_i, \alpha_j) = 1.
\] (5.4)

We also give a correspondence between \( S_0(\alpha_i, \alpha_j) \) as defined here and \( S_{pp}^0(\alpha_i, \alpha_j) \) as defined in section 2.

\[
S_{pp}^0(\alpha_i, \alpha_j) = S_0(\alpha_i, \alpha_j)
\] (5.5)

where \( pp \) denotes the particle particle scattering. If it is for particle hole scattering case, the S-matrix can be written as \( S_{ph}^{hp}(\alpha_i, \beta_j^h) \) as will be soon explained below. It should be noted that, in the massive Thirring model, the states are completely specified by the rapidity variables \( \alpha \) with the index of particle or hole. These are the quantum numbers that can specify the states.

(a) S-matrix

Now, we first define the one particle one hole state. This can be denoted as

\[
|1p1h> = |\beta_1, \alpha_2^\dagger, ..., \alpha_N^\dagger >\equiv |\beta_1, \beta_1^h>
\] (5.6)
where \( \beta_1^h \) is the rapidity of the hole state when \( N \) and \( L \) become infinity. But for the moment, it is only symbolically written since \( \alpha_i^\dagger \)'s differ from \( \alpha_i \)'s of the vacuum state. Therefore, the S-matrix for the \( 1p - 1h \) state can be written as

\[
S_{1p1h}(\beta_1, \alpha_2^\dagger, ..., \alpha_N^\dagger) = \prod_{i>j>1}^N S_0(\alpha_i^\dagger, \alpha_j^\dagger) \prod_{k=2}^N S_0(\beta_1, \alpha_k^\dagger). \tag{5.7}
\]

Here, we note that, in the large \( N \) and \( L \) limits, we find that

\[
\alpha_i^\dagger \rightarrow \alpha_i
\]
as will be shown later numerically in Table 1.

In what follows, we only discuss those particle hole scattering processes where the particle states are always involved. We do not consider here particle particle scattering or hole hole scattering processes.

We note that this particular example is sufficient since we are only interested in finding an example of the violation of the factorization. Therefore, we define the S-matrix for the particle-hole scattering in the following way,

\[
S_{ph}^{hp}(\beta_1, \beta_1^h) \equiv \prod_{k=2}^N S_0(\beta_1, \alpha_k^\dagger).
\tag{5.8}
\]

Next, we want to define the two particle two hole state. This can be denoted as

\[
|2p2h> = |\beta_1, \beta_2, \alpha_3^\dagger, ..., \alpha_N^\dagger > \equiv |\beta_1, \beta_1^h, \beta_2, \beta_2^h >. \tag{5.9}
\]

Again, \( \beta_1^h \) and \( \beta_2^h \) are written symbolically.

Therefore, the S-matrix for the \( 2p - 2p \) state can be written as

\[
S_{2p2h}(\beta_1, \beta_2, \alpha_3^\dagger, ..., \alpha_N^\dagger) = S_0(\beta_1, \beta_2) \prod_{i>j>2}^N S_0(\alpha_i^\dagger, \alpha_j^\dagger) \prod_{k=3}^N S_0(\beta_1, \alpha_k^\dagger) \prod_{l=3}^N S_0(\beta_2, \alpha_l^\dagger). \tag{5.10}
\]
Here again, we note that, in the large $N$ and $L$ limits, we find that

$$\alpha_i^\dagger \to \alpha_i.$$  

In the same way as $S^{hp}_{ph}$, we define the $2p-2h$ scattering process where the particle states are involved.

$$S^{hphp}_{phph}(\beta_1, \beta_1^h, \beta_2, \beta_2^h) \equiv \prod_{k=3}^{N} S_0(\beta_1, \alpha_k^\dagger) \prod_{l=3}^{N} S_0(\beta_2, \alpha_l^\dagger). \tag{5.11}$$

Now that the $\alpha_i^\dagger$ and $\alpha_i^\dagger$ approach to the $\alpha_i$ for the large $N$ and $L$ limits, one may think that one can replace the $\alpha_i^\dagger$ and $\alpha_i^\dagger$ by the $\alpha_i$. We denote these S-matrices by $S^{hp}_{ph}(\beta_1, \beta_1^h)(V)$ and $S^{hphp}_{phph}(\beta_1, \beta_1^h, \beta_2, \beta_2^h)(V)$. Therefore, if we take the large $N$ and $L$ limits at this stage without care for the difference between $\alpha_i$, $\alpha_i^\dagger$ and $\alpha_i^\dagger$, then we obtain that the S-matrix $S^{hphp}_{phph}(\beta_1, \beta_1^h, \beta_2, \beta_2^h)(V)$ for the $2p-2h$ state can be factorized into $1p-1h$ S-matrices. That is,

$$S^{hphp}_{phph}(\beta_1, \beta_1^h, \beta_2, \beta_2^h)(V) = S^{hp}_{ph}(\beta_1, \beta_1^h)(V)S^{hp}_{ph}(\beta_2, \beta_2^h)(V). \tag{5.12}$$

Eq.(5.12) shows that the S-matrix of the two-particle two-hole scattering seems indeed factorized. This is the main reason why people believe that the factorization ansatz for the particle hole scattering should hold in the same way as the massless case or in other words as the particle particle scattering case.

Based on the factorization ansatz of the S-matrix, Zamolodchikov and Zamolodchikov obtained the Yang-Baxter type functional equation for the S-matrix which can determine the shape of the S-matrix itself by the simple algebraic manner [11]. However, the derivation of the functional equation implicitly assumes the factorization of the particle hole scattering S-matrix. Even though the factorization of the particle hole S-matrix is valid in terms of their rapidity variables, it is nontrivial to show that this factorization can hold even for the common rapidity variables with the vacuum.
Therefore, one must be careful for the treatment of the rapidity variables when one calculates the S-matrix. In particular, one should carefully treat the order of the operations when one makes the large $N$ and $L$ limit in quantum mechanics. The large limits of $N$ and $L$ should be taken as late as possible.

(b) Large $L$ and $N$ limits

In what follows, we show that, in the large $N$ and $L$ limits, eq.(5.12) does not hold. In order to see it, we first rewrite the S-matrix of the $1p - 1h$ state in terms of the vacuum rapidities.

\[ S_{ph}^{hp}(\beta_1, \beta_1^h) = \prod_{k=2}^{N} S_0(\beta_1, \alpha_k + \delta_k) \]  

(5.13)

where $\delta_i$ is defined as the difference between $\alpha_i$ and $\alpha_i^\dagger$, that is, $\delta_i = \alpha_i^\dagger - \alpha_i$.

Since $\delta_i$ is quite small for the large values of $N$ and $L$, we can expand $S_0(\beta_1, \alpha_j + \delta_j)$ in terms of $\delta_i$. Here, we expand the phase shift function

\[ S_0(\beta_1, \alpha_j + \delta_j) = \exp \left[ i\phi(\beta_1 + \alpha_j - i\pi) + i\delta_j \frac{\partial}{\partial \alpha_j} \phi(\beta_1 + \alpha_j - i\pi) \right. \]

\[ + i \frac{\delta_j^2}{2} \frac{\partial^2}{\partial \alpha_j^2} \phi(\beta_1 + \alpha_j - i\pi) + \ldots \]  

(5.14)

The first terms of eqs.(5.14) correspond to the S-matrices which are factorizable.

Now, the important point is that the $\delta_i$ itself becomes zero as the values of $N$ and $L$ become infinity but the summation of $\delta_i$ stays finite in this limit. We will discuss it in detail below. Therefore, it is important to keep the second term for the S-matrix evaluation.

In the same way, we rewrite the S-matrix of the $2p - 2h$ state.

\[ S_{phph}^{hp}(\beta_1, \beta_1^h, \beta_2, \beta_2^h) = \prod_{k=3}^{N} S_0(\beta_1, \alpha_k + \epsilon_k) \prod_{l=3}^{N} S_0(\beta_2, \alpha_l + \epsilon_l) \]  

(5.15)
where \( \epsilon_i \) is defined as \( \epsilon_i = \alpha_i^+ - \alpha_i \). Since \( \epsilon_i \) is very small for the large values of \( N \) and \( L \), we can expand \( S_0(\beta_1, \alpha_k + \epsilon_k) \) and \( S_0(\beta_2, \alpha_l + \epsilon_l) \) in terms of \( \epsilon_i \) in the same way as eqs. (5.14)

\[
S_0(\beta_1, \alpha_k + \epsilon_k) = \exp \left[ i \phi(\beta_1 + \alpha_k - i\pi) + i \epsilon_k \frac{\partial}{\partial \alpha_k} \phi(\beta_1 + \alpha_k - i\pi) + \frac{1}{2} \epsilon_k^2 \frac{\partial^2}{\partial \alpha_k^2} \phi(\beta_1 + \alpha_k - i\pi) + ... \right]
\]

(5.16a)

\[
S_0(\beta_2, \alpha_l + \epsilon_l) = \exp \left[ i \phi(\beta_2 + \alpha_l - i\pi) + i \epsilon_l \frac{\partial}{\partial \alpha_l} \phi(\beta_2 + \alpha_l - i\pi) + \frac{1}{2} \epsilon_l^2 \frac{\partial^2}{\partial \alpha_l^2} \phi(\beta_2 + \alpha_l - i\pi) + ... \right]
\]

(5.16b)

We can also check later that the \( \epsilon_i \) itself becomes zero when \( N \) and \( L \) become infinity. However, the summation of \( \epsilon_i \) is finite and therefore, we have to keep them when we want to show the S-matrix factorization.

Now, we rewrite eqs. (5.13) and (5.15) in terms of eqs. (5.14) and (5.16).

As mentioned above, we are only concerned with the particle hole scattering S matrix factorization.

Therefore, eqs. (5.13) and (5.15) can be written as

\[
S_{ph}^{hp}(\beta_1, \beta_1^b) = \exp \left[ i \sum_{m=2}^{N} \phi(\beta_1 + \alpha_m - i\pi) \right]
\]

\[
\times \exp \left[ i \sum_{k=2}^{N} \left( \delta_k \frac{\partial \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k} + \frac{1}{2} \delta_k^2 \frac{\partial^2 \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k^2} \right) + ... \right] \]

(5.17)

\[
S_{phhp}^{hp}(\beta_1, \beta_1^b, \beta_2, \beta_2^b) = \exp \left[ i \sum_{m=3}^{N} \phi(\beta_1 + \alpha_m - i\pi) + i \sum_{m=3}^{N} \phi(\beta_2 + \alpha_m - i\pi) \right]
\]

\[
\times \exp \left[ i \sum_{k=3}^{N} \left( \epsilon_k \frac{\partial \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k} + \frac{1}{2} \epsilon_k^2 \frac{\partial^2 \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k^2} \right) \right.
\]

\[
+ \left. i \sum_{k=3}^{N} \left( \epsilon_k \frac{\partial \phi(\beta_2 + \alpha_k - i\pi)}{\partial \alpha_k} + \frac{1}{2} \epsilon_k^2 \frac{\partial^2 \phi(\beta_2 + \alpha_k - i\pi)}{\partial \alpha_k^2} \right) + ... \right] \]

(5.18)
Clearly from eqs. (5.17) and (5.18), the S-matrix factorization cannot hold unless the exponential terms except the first one vanish altogether.

The important point is that $\sum \delta_k$ contains all the information of the $1p-1h$ state and therefore depends on $\beta_1$ and $\beta^h_1$ which are the quantum numbers that characterize the $1p-1h$ state. In the same manner, the summation $\sum \epsilon_k$ depends on $\beta_1$, $\beta^h_1$, $\beta_2$, and $\beta^h_2$ which characterize the $2p-2h$ state.

(c) Numerical results

Now, we want to present the numerical check of the various quantities which appear in eqs. (5.17) and (5.18). Here, we first solve numerically the PBC equations (4.2), (4.3) and (4.4). We vary the number of particle $N$ and the box size $L_0$.

First, we want to check how the rapidity values of the $1p-1h$ and $2p-2h$ configurations converge into those of the vacuum state when $N$ becomes large. In Table 1, we plot several values of the rapidity as the function of $N$ with the fixed value of the density $\rho_0$. This clearly shows that the rapidities of the $1p-1h$ and $2p-2h$ states approach to those of the vacuum state when $N$ becomes very large. Therefore, eq.(5.12) indeed holds. However, we must be careful when we treat any physical quantities which depend on the sum of the rapidity difference between the vacuum state and the $1p-1h$ or $2p-2h$ configurations.

The important point is that, even though each rapidity of the $1p-1h$ and $2p-2h$ states approaches to that of the vacuum, the sum of the rapidity differences stays finite.

In order to see the effect mentioned above, we define the following quantities

$$D_{1p1h}(n_1) \equiv D_{1p1h}(\beta_1, \beta^h_1) = \sum_{k=2}^{N} \delta_k \frac{\partial \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k}$$  (5.19a)
\[ E_{1p1h}(n_1) \equiv E_{1p1h}(\beta_1, \beta_1^h) = \frac{1}{2} \sum_{k=2}^{N} \delta_k^2 \frac{\partial^2 \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k^2} \] (5.19b)

\[ D_{2p2h}(n_1, n_2) \equiv D_{2p2h}(\beta_1, \beta_2, \beta_1^h, \beta_2^h) = \sum_{k=3}^{N} \epsilon_k \frac{\partial \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k} \] (5.19c)

\[ E_{2p2h}(n_1, n_2) \equiv E_{2p2h}(\beta_1, \beta_2, \beta_1^h, \beta_2^h) = \frac{1}{2} \sum_{k=3}^{N} \epsilon_k \frac{\partial^2 \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k^2} \] (5.19d)

where \( \beta_1 \) and \( \beta_2 \) correspond to the rapidity values for the \( n_1 \) and \( n_2 \) states.

Here \( \frac{\partial \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k} \) and \( \frac{\partial^2 \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k^2} \) can be written explicitly using eq.(3.9) as

\[ \frac{\partial \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k} = g_0 \frac{1}{\frac{g_0^2}{4} + (1 + \frac{g_0^2}{4}) \sinh^2 \frac{1}{2}(\beta_1 + \alpha_k)}. \] (5.20)

\[ \frac{\partial^2 \phi(\beta_1 + \alpha_k - i\pi)}{\partial \alpha_k^2} = \frac{-g_0}{2} \frac{(1 + \frac{g_0^2}{4}) \sinh \frac{1}{2}(\beta_1 + \alpha_k) \cosh \frac{1}{2}(\beta_1 + \alpha_k)}{\left(\frac{g_0^2}{4} + (1 + \frac{g_0^2}{4}) \sinh^2 \frac{1}{2}(\beta_1 + \alpha_k)\right)^2}. \] (5.21)

Since we have solved the PBC equations numerically, we know all the rapidity values, and thus we can calculate eqs.(5.19) together with eq.(5.20) and eq.(5.21).

In Table 2, we plot the values of \( D_{1p1h}, E_{1p1h}, D_{2p2h} \) and \( E_{2p2h} \) as the function of \( N \) and \( L_0 \) for the lowest scattering state. As can be seen, the calculated values of \( D_{1p1h} \) and \( D_{2p2h} \) do not depend very much on the \( N \) and \( L_0 \) as long as we keep the same density \( \rho_0 \).

However, as can be seen from Table 2, the \( E_{1p1h} \) and the \( E_{2p2h} \) decrease as \( N \) increases. They behave as

\[ E_{1p1h}(n_1) \sim \frac{1}{N} \]

\[ E_{2p2h}(n_1, n_2) \sim \frac{1}{N}. \]

Therefore, these quantities do not survive at the large \( N \) limit.
This clearly shows that the sum of the rapidities times the derivative of the S-matrix is finite even though each rapidity value becomes zero when \( N \) and \( L \) become infinity. Therefore, we prove that these second terms do not vanish.

In order to see more clearly the above effects, we rewrite the \( S_{ph}(\beta_1, \beta^h_1) \) and \( S_{phph}(\beta_1, \beta^h_1, \beta_2, \beta^h_2) \) in terms of \( D_{1p1h}(n_1) \) and \( D_{2p2h}(n_1, n_2) \).

\[
S_{ph}(\beta_1, \beta^h_1) = S_{ph}(\beta_1, \beta^h_1)(V) \exp[iD_{1p1h}(n_1)] \quad (5.22)
\]

\[
S_{phph}(\beta_1, \beta^h_1, \beta_2, \beta^h_2) = S_{ph}(\beta_1, \beta^h_1)(V) S_{ph}(\beta_2, \beta^h_2)(V) \times \exp[iD_{2p2h}(n_1, n_2) + iD_{2p2h}(n_2, n_1)] \quad (5.23)
\]

Now, it is obvious that the \( D_{1p1h}(n_1) \) and \( D_{2p2h}(n_1, n_2) \) are quite different from each other since they carry the information of the many body nature of the particle hole states. Indeed, the \( D_{2p2h}(n_1, n_2) \) depends on \( \beta_1, \beta_2, \beta^h_1 \) and \( \beta^h_2 \). Therefore, the factorization of the particle hole S-matrix cannot be justified. In fact, as we see below, the numerical results confirm the difference between the \( D_{1p1h}(n_1) \) and \( D_{2p2h}(n_1, n_2) \).

(d) Field theory limit

Now, we should take the field theory limit since this is the only way to compare our results with Zamolodchikov’s factorization ansatz. For the field theory limit, we have to let \( \rho \to \infty \) [15]. In the case of the bound state problem, when we let \( \rho \to \infty \), then we should take the limit of \( m_0 \to 0 \), keeping the excited state energy finite. In order to see it more concretely, we repeat here how it is done for the bound state problem. The excited state energy \( \Delta E \) is written as [15]

\[
\Delta E = m_0 \left( A + B \left( \frac{\rho}{m_0} \right)^{\alpha} \right)
\]

where \( A \) and \( B \) are some constants which depend on the coupling constant. Here, \( \alpha \) is a constant with positive value which is smaller.
than unity. In this case, when we make $\rho \to \infty$, keeping $\Delta E$ finite, we can make a fine tuning of $m_0$ such that

$$m_0^{1-\alpha} \rho^\alpha = \text{finite.}$$

This means that we should let $m_0 \to 0$.

However, in the present case, the situation is a bit different and even simpler since the correction factor itself does not have any dimensions. Therefore, the field theory limit is just that we let the $\rho_0$ infinity. In fig.1, we show the calculated absolute values of $D_{1p1h}$ and $D_{2p2h}$ as the function of $\rho_0^{-1}$. In this case, they can be well parameterized by the following functions as

$$D_{1p1h}(n_1) = C^{(1)} \rho_0^{-1} + D^{(1)}$$  \hspace{1cm} (5.24a)$$

$$D_{2p2h}(n_1, n_2) = C^{(2)} \exp \left( -\frac{\kappa_2}{\rho_0} \right) + D^{(2)}$$  \hspace{1cm} (5.24b)$$

where $C^{(i)}$, $D^{(i)}$ and $\kappa_2$ are some constants which depend on the coupling constant and the rapidity values of the particle hole states.

As can be seen, when the $\rho_0$ goes to infinity, then the $D_{1p1h}$ and $D_{2p2h}$ approach to some finite numbers. Namely, they become

$$D_{1p1h}(n_1) \to D^{(1)}$$  \hspace{1cm} (5.25a)$$

$$D_{2p2h}(n_1, n_2) \to C^{(2)} + D^{(2)}$$  \hspace{1cm} (5.25b)$$

This means that we have made the field theory limit in a correct way. The values of $C^{(i)}$, $D^{(i)}$ and $\kappa_2$ for several cases of the particle hole excited states are shown in Table 3.

From the fig.1, we know that the breaking of the factorization of the S-matrix is the order of $|D_{1p1h} - D_{2p2h}|$ which is smaller than unity. We do not know whether this value of $|D_{1p1h} - D_{2p2h}|$ can be said to be large or small. In any case, the S-matrix factorization is violated at the elementary level.
Here, we again note that the S-matrix factorization for the particle particle scattering must hold. This is essentially the same as the vacuum case. It is interesting to observe that the energy of the vacuum state can be obtained analytically, though the vacuum energy alone is not physically very meaningful.

6. Conclusions

We have presented a numerical proof that the S-matrix factorization for the particle hole scattering in the massive Thirring model is not satisfied. This is mainly because the factorization of the S-matrix and the crossing symmetry do not commute with each other.

It is always simpler to find a counter example of the violation of the S-matrix factorization than to prove the validity of the factorization. This is clear since we only have to find out any type of example which shows the violation of the factorization. This is indeed the point we have worked out in this paper. The present calculations present a convincing evidence that the S-matrix factorization for the particle hole scattering does not hold exactly.

Here, we want to examine the implication and consequence of this result. As stressed in this paper, it is important to realize that the S-matrix factorization should hold for the particle particle scattering even for the massive Thirring model. This can be easily seen from the Bethe ansatz solution which is eq.(3.5). Indeed, if one looks at the vacuum to vacuum transition eq.(5.1), then one sees that the S-matrix is factorized into the product of the two body S-matrices. This is the consequence of the particle particle scattering. Clearly, the massless case must have this nice feature of the
S-matrix factorization since, in this case, one can treat all the scattering processes as two species of fermion scattering without going to the particle hole scattering. Therefore, those models which are equivalent to the six vertex models should receive no effects from the present investigation.

However, those models which are equivalent to the eight vertex models should be carefully treated if one is concerned with the bound state spectrum with S-matrix factorization ansatz. At least, we now know that the massive Thirring model does not have the S-matrix factorization property for the particle hole scattering. This result is consistent with recent calculations for the bound state of the massive Thirring model [15-17]. The Bethe ansatz as well as the light cone calculations show that there is only one bound state on the contrary to the prediction by the S-matrix factorization equation.

Here, we want to discuss the implication of the present result in connection with the Yang-Baxter equation. The Yang-Baxter equation is a matrix equation which is obtained by imposing certain conditions. The present study does not intend to check the Yang-Baxter equation itself. Instead, we have only examined the factorization property of the S-matrix at the elementary level. Therefore, we have never questioned whether the requirement of the Yang-Baxter equation is physically reasonable or not. We think that the requirement of the Yang-Baxter equation is indeed reasonable if the S-matrix is factorizable at the elementary level.

Now, the question is how we can interpret the results (the bound state spectrum) predicted by the Yang-Baxter equation when the S-matrix factorization is violated at the elementary level. In the massive Thirring / sine-Gordon model, the situation is now clear. Namely, the spectrum obtained by the Yang-Baxter equation with a violation of the S-matrix factorization at the elementary level is semiclassical.

However, this point is only proved for the massive Thirring model. We do
not know whether there might be some examples in which the Yang-Baxter equation gives exact spectrum even though the S-matrix factorization is violated at the elementary level.

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Figure captions :

Fig.1a : We show the calculated values of $D_{1p1h}$ and $D_{2p2h}$ as the function of $\rho_0^{-1}$. This is the excited state with $n_{i0} = 1$, $n_{i1} = 1$ and $n_{i2} = -1$. The solid line is drawn with eqs. (5.24) with the parameters in Table 3.

Fig.1b : The same as Fig.1a. This is the excited state with $n_{i0} = 2$, $n_{i1} = 1$ and $n_{i2} = 2$.

Fig.1c : The same as Fig.1a. This is the excited state with $n_{i0} = 2$, $n_{i1} = 1$ and $n_{i2} = -2$. 
## Table 1

Rapidities

|       | N     | $\alpha_{n_i}$ | $\alpha^\dagger_{n_i}$ | $\alpha^\dagger_{n_i}$ |
|-------|-------|----------------|------------------------|------------------------|
| $n_i = 5$ |       |                 |                        |                        |
| 101   | 0.859 | 0.801          | 1.142                  |                        |
| 401   | 0.223 | 0.202          | 0.247                  |                        |
| 1601  | 0.0559| 0.0504         | 0.0574                 |                        |
| 6401  | 0.0140| 0.0126         | 0.0141                 |                        |
| 12801 | 0.0070| 0.0063         | 0.0070                 |                        |
| $n_i = 10$ |      |                 |                        |                        |
| 101   | 1.578 | 1.546          | 1.863                  |                        |
| 401   | 0.443 | 0.424          | 0.486                  |                        |
| 1601  | 0.1118| 0.1064         | 0.1147                 |                        |
| 6401  | 0.0280| 0.0266         | 0.0282                 |                        |
| 12801 | 0.0140| 0.0133         | 0.0140                 |                        |
| $n_i = 15$ |      |                 |                        |                        |
| 101   | 2.147 | 2.128          | 2.390                  |                        |
| 401   | 0.656 | 0.640          | 0.713                  |                        |
| 1601  | 0.1676| 0.1623         | 0.1718                 |                        |
| 6401  | 0.0420| 0.0406         | 0.0422                 |                        |
| 12801 | 0.0210| 0.0203         | 0.0210                 |                        |
| $n_i = 20$ |      |                 |                        |                        |
| 101   | 2.603 | 2.592          | 2.806                  |                        |
| 401   | 0.862 | 0.848          | 0.926                  |                        |
| 1601  | 0.2232| 0.2181         | 0.2288                 |                        |
| 6401  | 0.0559| 0.0546         | 0.0563                 |                        |
| 12801 | 0.0280| 0.0273         | 0.0281                 |                        |

We plot the rapidity values for several cases of $n_i$ for the vacuum, $1p-1h$ state with $n_{i_0} = 0$ and $2p-2h$ state with $n_{i_1} = 1$, $n_{i_2} = -1$. The density $\rho_0$ is fixed to $\rho_0 = 20$. 

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Table 2

\((g_0 = 6.28)\)

| \(N\)  | \(D_{1p1h}(n_{i_0})\) | \(E_{1p1h}(n_{i_0})\) | \(D_{2p2h}(n_{i_1}, n_{i_2})\) | \(E_{2p2h}(n_{i_1}, n_{i_2})\) |
|-------|---------------------|---------------------|---------------------|---------------------|
| 101   | 0.277               | \(1.00 \times 10^{-2}\) | 0.767               | \(7.31 \times 10^{-3}\) |
| 401   | 0.261               | \(2.40 \times 10^{-3}\) | 0.764               | \(1.73 \times 10^{-3}\) |
| 1601  | 0.257               | \(5.94 \times 10^{-4}\) | 0.762               | \(4.25 \times 10^{-4}\) |
| 6401  | 0.256               | \(1.48 \times 10^{-4}\) | 0.763               | \(1.06 \times 10^{-4}\) |
| 12801 | 0.256               | \(7.39 \times 10^{-5}\) | 0.762               | \(5.28 \times 10^{-5}\) |

We plot the values of \(D_{1p1h}, E_{1p1h}, D_{2p2h}\) and \(E_{2p2h}\) for several cases of the number of the particles \(N\) for the lowest scattering states \((n_{i_0} = 1, n_{i_1} = 1, n_{i_2} = -1)\). The density \(\rho_0\) is fixed to \(\rho_0 = 20\).
We plot the values of the parameters $C^{(1)}$, $D^{(1)}$. The numbers in parenthesis indicate error bars in the fitting.
We plot the values of the parameters $C^{(2)}$, $D^{(2)}$ and $\kappa_2$. The numbers in parenthesis indicate error bars in the fitting.
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