Cartan-Eilenberg complexes and Foxby equivalence

Ren Wei, Lu Bo, Zhang Chunxia and Liu Zhongkui *
Department of Mathematics, Northwest Normal University, Lanzhou, China.

Abstract

The main aim of this paper is to investigate Cartan-Eilenberg complexes with respect to a semidualizing module, and extend the existed Foxby equivalence. To this end, we define and study Cartan-Eilenberg $W$ complexes and Cartan-Eilenberg $W$-Gorenstein complexes, where $W$ denotes a self-orthogonal class of left $R$-modules.

Key Words: Cartan-Eilenberg complex; Self-orthogonal class; Semidualizing module; Auslander class; Bass class.

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1. Introduction

Recently, Enochs [8] studied Cartan-Eilenberg projective and injective complexes. Recall that a complex $P$ is said to be Cartan-Eilenberg projective [8, Definition 3.1] if $P$, $Z(P)$, $B(P)$ and $H(P)$ are complexes of projective modules, where $Z(P)$, $B(P)$ denote the subcomplexes of cycles and boundaries of the complex $P$ respectively, $H(P) = Z(P)/B(P)$ denotes the homology complex of $P$. Similarly, Cartan-Eilenberg injective complexes are defined. These complexes have origin in [4] to give the definitions of projective and injective resolutions of a complex of modules.

Modules with excellent duality properties have turned out to be a powerful tool. Recall that a semidualizing module over a commutative noetherian ring $R$ is a finite generated module $C$ with $\mathbf{R} \mathbf{H} \mathbf{o} \mathbf{m}_R(C, C) \simeq R$ in the derived category $\mathcal{D}(R)$. There have been abundant results about Auslander categories $\mathcal{A}_C(R)$, $\mathcal{B}_C(R)$ and Foxby equivalence with respect to $C$, see for example [6, 13, 16, 17]. In the following, we use $\tilde{\mathcal{P}}(R)$ (respectively, $\tilde{\mathcal{G}}\mathcal{P}(R)$, $\tilde{\mathcal{P}}_C(R)$, $\tilde{\mathcal{G}}\mathcal{P}_C(R)$) to denote the subcategory of

* Corresponding author.

E-mail: renwei@nwnu.edu.cn (Ren W.), hubo55@sina.com (Lu B.), zhangcx@nwnu.edu.cn (Zhang C.X.), liuzk@nwnu.edu.cn (Liu Z.K.).
the bounded derived category \( \mathcal{D}_b(R) \), consisting of complexes with finite projective (respectively, Gorenstein projective, \( C \)-projective, \( C \)-Gorenstein projective) dimensions. The Foxby equivalences between these categories, see the following diagram, are well known nowadays. Moreover, it is easy to get the inclusions

\[
\text{C-E } \tilde{\mathcal{P}}(R) \subseteq \mathcal{P}(R) \subseteq \tilde{\mathcal{GP}}(R) \cap \mathcal{A}_C(R)
\]

and

\[
\text{C-E } \tilde{\mathcal{P}}(R) \subseteq \text{C-E } (\tilde{\mathcal{GP}}(R) \cap \mathcal{A}_C(R)) \subseteq \tilde{\mathcal{GP}}(R) \cap \mathcal{A}_C(R).
\]

So it is natural to ask: whether there exist subcategories of \( \mathcal{D}_b(R) \) consisting of Cartan-Eilenberg version complexes, for which we have the corresponding Foxby equivalence? We use the symbol “(?)” to denote these subcategories we will seek, and answering these “(?)” is the motivation of this paper.
For a semidualizing module $C$, since the class of $C$-projective modules and the class of $C$-injective modules are known to be self-orthogonal, in Section 3 and 4 of this paper, we in general define and study Cartan-Eilenberg $\mathcal{W}$ complexes and Cartan-Eilenberg $\mathcal{W}$-Gorenstein complexes relative to a self-orthogonal class $\mathcal{W}$ of left $R$-modules. For different choices of $\mathcal{W}$, Cartan-Eilenberg $\mathcal{W}$ complexes encompass aforementioned Cartan-Eilenberg projective and Cartan-Eilenberg injective complexes. Moreover, this covers a wide variety of examples, for instance, let $C$ be a semidualizing module over a commutative noetherian ring $R$, we can define and study Cartan-Eilenberg $C$-projective and Cartan-Eilenberg $C$-injective complexes, which are exactly we interested in. Section 5 is specially devoted to find out “(?))” in the above diagram, more precisely, we study the subcategory of $D_b(R)$ concerning about Cartan-Eilenberg $C$-projective ($C$-injective) and Cartan-Eilenberg $C$-Gorenstein projective ($C$-Gorenstein injective) complexes. This extends the existed Foxby equivalence.

2. Preliminaries

In this section, we shall recall some notations, definitions and results which we need in the later sections.

Throughout this paper, $R$ denotes a ring with unity and all modules are left $R$-modules. An $R$-complex (complex of $R$-modules)

$$X = \cdots \longrightarrow X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots$$

will be denoted $(X, d)$ or $X$. The $n$th cycle module is defined as $\text{Ker} d_n$ and is denoted by $Z_n(X)$, $n$th boundary module is $\text{Im} d_{n+1}$ and is denoted by $B_n(X)$, and $n$th homology module is $H_n(X) = Z_n(X)/B_n(X)$. The complexes of cycles and boundaries, and the homology complex of $X$ is denoted by $Z(X)$, $B(X)$ and $H(X)$ respectively. We use $R$-$\text{Mod}$ to denote the category of $R$-modules, and $\mathcal{C}(R)$ to denote the category of $R$-complexes.

Given an $R$-module $M$, we let $S^n(M)$ denote the complex with all entries 0 except $M$ in degree $n$. We let $D^n(M)$ denote the complex $X$ with $X_n = X_{n-1} = M$ and all other entries 0, and with all maps 0 except $d_n = 1_M$.

**Proposition 2.1.** ([14, Lemma 3.1]) For any $R$-module $M$ and any $R$-complex $X$, we have the following natural isomorphisms:

1. $\text{Hom}_{\mathcal{C}(R)}(D^n(M), X) \cong \text{Hom}_R(M, X_n)$.
2. $\text{Hom}_{\mathcal{C}(R)}(S^n(M), X) \cong \text{Hom}_R(M, Z_n(X))$.
3. $\text{Hom}_{\mathcal{C}(R)}(X, D^n(M)) \cong \text{Hom}_R(X_{n-1}, M)$.
We will frequently consider complexes $X$ with $d^X = 0$. Such a complex is completely determined by its family of terms $(X_n)_{n \in \mathbb{Z}}$, i.e., by the underlying structure of $X$ as a graded module with the grading over $\mathbb{Z}$. So by the term graded module we will mean a complex $X$ with $d^X = 0$.

Throughout the paper, we use both the subscript notation for degrees of complex and the superscript notation to distinguish complexes: for example, if $(C^i)_{i \in \mathbb{Z}}$ is a family of complexes, then $C^i_n$ denotes the degree-$n$ term of the complex $C^i$.

The left derived functor of the tensor product functor of $R$-complexes is denoted by $- \otimes^L_R -$, and $R\text{Hom}_R(-, -)$ denotes the right derived functor of the homomorphism functor of complexes. The symbol “$\simeq$” is used to designate quasi-isomorphisms in the category $C(R)$ and isomorphisms in the derived category $D(R)$.

A semidualizing module over a commutative noetherian ring $R$ is a finite generated $R$-module $C$ with $R\text{Hom}_R(C, C) \simeq R$ in the derived category $D(R)$, or equivalently, $\text{Ext}^i_R(C, C) = 0$ for $i \geq 1$ and the homothety map $R \to \text{Hom}_R(C, C)$ is an isomorphism. An $R$-module is $C$-projective if it has the form $C \otimes_R P$ for some projective $R$-module $P$. An $R$-module is $C$-injective if it has the form $\text{Hom}_R(C, E)$ for some injective $R$-module $E$. Let $\mathcal{P}_C = \{C \otimes_R P \mid P \text{ is a projective } R\text{-module}\}$ and $\mathcal{I}_C = \{\text{Hom}_R(C, E) \mid E \text{ is an injective } R\text{-module}\}$ denote the class of $C$-projective and $C$-injective modules, respectively.

Regarded as a complex concentrated in degree zero, a semidualizing module $C$ is a semidualizing complex in the sense of [6, Definition 2.1]. Its related Auslander categories $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ are defined as

$$\mathcal{A}_C(R) = \left\{ X \in D_b(R) \left| C \otimes^L_R X \in D_b(R) \text{ and the canonical map } X \to R\text{Hom}_R(C, C \otimes^L_R X) \text{ is an isomorphism in } D(R) \right. \right\}$$

and

$$\mathcal{B}_C(R) = \left\{ Y \in D_b(R) \left| R\text{Hom}(C, Y) \in D_b(R) \text{ and the canonical map } C \otimes^L_R R\text{Hom}(C, Y) \to Y \text{ is an isomorphism in } D(R) \right. \right\},$$

where $D_b(R)$ is the full subcategory of $D(R)$ consisting of homological bounded complexes. There is an equivalence of categories which is called Foxby equivalence:

\[ \mathcal{A}_C(R) \xrightarrow{C \otimes^L_R -} \mathcal{B}_C(R). \]
3. Cartan-Eilenberg \( \mathcal{W} \) complexes

For the rest of the paper we will use the abbreviation C-E for Cartan-Eilenberg. Let \( \mathcal{W} \) be a class of \( R \)-modules. \( \mathcal{W} \) is called self-orthogonal if it satisfies the following condition:

\[
\text{Ext}^i_R(W, W') = 0 \text{ for all } W, W' \in \mathcal{W} \text{ and all } i \geq 1.
\]

In the following, \( \mathcal{W} \) always denotes a self-orthogonal class of \( R \)-modules which is closed under extensions, finite direct sums and direct summands. Geng and Ding in [13, Remark 2.3] enumerated a variety of interesting examples of self-orthogonal classes.

This section is devoted to define and study C-E \( \mathcal{W} \) complexes. Recall that \( X \) is called a \( \mathcal{W} \) complex [18] if \( X \) is acyclic and \( Z_n(X) \in \mathcal{W} \) for any \( n \in \mathbb{Z} \). We will denote the class of \( \mathcal{W} \) complexes by \( \widetilde{\mathcal{W}} \).

**Definition 3.1.** A complex \( X \) is said to be a C-E \( \mathcal{W} \) complex if \( X, Z(X), B(X) \) and \( H(X) \) are complexes each of whose terms belongs to \( \mathcal{W} \).

**Remark 3.2.**

1. For any module \( M \in \mathcal{W} \) and any \( n \in \mathbb{Z} \), \( D^n(M) \) and \( S^n(M) \) are C-E \( \mathcal{W} \) complexes.

2. In particular, if \( \mathcal{W} \) denotes the class of all projective (respectively, injective) modules, then C-E \( \mathcal{W} \) complexes are precisely C-E projective (respectively, C-E injective) complexes.

3. If we put \( \mathcal{W} = \mathcal{P}_C \) (respectively, \( \mathcal{W} = \mathcal{I}_C \)), then a C-E \( \mathcal{W} \) complex above is particularly called C-E C-projective (respectively, C-E C-injective) complex.

4. The following will show that every complex \( X \in \widetilde{\mathcal{W}} \) is a C-E \( \mathcal{W} \) complex.

**Proposition 3.3.** \( X \) is a C-E \( \mathcal{W} \) complex if and only if \( X \) can be divided into direct sums \( X = X' \oplus X'' \) where \( X' \in \widetilde{\mathcal{W}} \) and \( X'' \) is a graded module with all items in \( \mathcal{W} \).

**Proof.** Since \( X' \in \widetilde{\mathcal{W}} \) is acyclic, \( B_n(X') = Z_n(X') \in \mathcal{W} \), \( H_n(X') = 0 \) for all \( n \in \mathbb{Z} \). Then \( X' \) is a C-E \( \mathcal{W} \) complex. It is easy to see \( X'' \) is a C-E \( \mathcal{W} \) complex. Then such direct sum is a C-E \( \mathcal{W} \) complex.

Conversely, suppose that \( X \) is a C-E \( \mathcal{W} \) complex. We have the exact sequences of \( R \)-modules

\[
0 \to B_n(X) \to Z_n(X) \to H_n(X) \to 0, \quad 0 \to Z_n(X) \to X_n \to B_{n-1}(X) \to 0.
\]

Since \( Z_n(X), B_n(X), H_n(X) \in \mathcal{W} \) for all \( n \in \mathbb{Z} \), each sequence splits. This allows us to write \( X_n = B_n(X) \oplus H_n(X) \oplus B_{n-1}(X) \). Then

\[
d_n : X_n = B_n(X) \oplus H_n(X) \oplus B_{n-1}(X) \rightarrow X_{n-1} = B_{n-1}(X) \oplus H_{n-1}(X) \oplus B_{n-2}(X)
\]
is the map \((x, y, z) \to (z, 0, 0)\). Let \(X' = \bigoplus_{n \in \mathbb{Z}} D^n(B_{n-1}(X))\) and \(X'' = \bigoplus_{n \in \mathbb{Z}} S^n(H_n(X))\). Then \(X = X' \oplus X''\), and we obtain the desired direct sum decomposition.

Here, we let \(C\) be a semidualizing module over a commutative noetherian ring \(R\), set \(\mathcal{W} = \mathcal{P}_C\) and \(\mathcal{W} = \mathcal{I}_C\), respectively. Then we have the following results.

**Corollary 3.4.** \(X\) is a \(C\)-E\(C\)-projective complex if and only if \(X\) can be divided into direct sums \(X = X' \oplus X''\) where \(X' \in \widehat{\mathcal{P}}_C\) and \(X''\) is a graded module with all items in \(\mathcal{P}_C\).

**Corollary 3.5.** \(X\) is a \(C\)-E\(C\)-injective complex if and only if \(X\) can be divided into direct sums \(X = X' \oplus X''\) where \(X' \in \widehat{\mathcal{I}}_C\) and \(X''\) is a graded module with all items in \(\mathcal{I}_C\).

4. **Cartan-Eilenberg \(\mathcal{W}\)-Gorenstein complexes**

Recall that an \(R\)-module \(M\) is said to be \(\mathcal{W}\)-Gorenstein [13, Definition 2.2] if there exists an exact sequence

\[W_\bullet = \cdots \to W_1 \to W_0 \to W_{-1} \to W_{-2} \to \cdots\]

de modules in \(\mathcal{W}\) such that \(M = \text{Ker}(W_{-1} \to W_{-2})\) and \(W_\bullet\) is \(\text{Hom}_R(\mathcal{W}, -)\) and \(\text{Hom}_R(-, \mathcal{W})\) exact. In this case, \(W_\bullet\) is called a complete \(\mathcal{W}\)-resolution of \(M\). This covers a various of examples by different choices of \(\mathcal{W}\), for instance, Gorenstein projective and Gorenstein injective modules. Naturally, in this section, we shall focus on complexes \(G\), where \(G, Z(G), B(G), H(G)\) are complexes consisting of \(\mathcal{W}\)-Gorenstein modules.

It is an important question to establish relationships between a complex \(X\) and the modules \(X_n, n \in \mathbb{Z}\). If \(R\) is an \(n\)-Gorenstein ring, Enochs and Garcia Rozas in [9] (also, see [12]) showed that a complex \(X\) is Gorenstein projective (respectively, Gorenstein injective) if and only if \(X_n\) is a Gorenstein projective (respectively, Gorenstein injective) module for all \(n \in \mathbb{Z}\). This has been further developed by Liu and Zhang [19] and Yang [21], and now we know that the same result holds over any associated ring.

In [18, Section 5.1], the author defined \(\tilde{\mathcal{W}}\)-Gorenstein complexes, similar to the definition of \(\mathcal{W}\)-Gorenstein module, by replacing the modules in \(\mathcal{W}\) with complexes in \(\tilde{\mathcal{W}}\). It is proved that: a complex \(X\) is \(\tilde{\mathcal{W}}\)-Gorenstein if and only if \(X_n\) is a \(\mathcal{W}\)-Gorenstein module for all \(n \in \mathbb{Z}\) [18, Sect. 5.1, Theorem A].

We now want to define \(\text{C-E} \mathcal{W}\)-Gorenstein complexes. For such a definition we have two options. One is to modify Definition 3.1 and the other is to define such
a complex in terms of appropriated resolutions of complexes. The result below will show that the two definitions are equivalent.

Firstly, we need to recall a notion:

**Definition 4.1.** ([9, Definition 5.3]) A complex of complexes
\[ C = \cdots \to C^2 \to C^1 \to C^0 \to C^{-1} \to \cdots \]
is said to be C-E exact if the following sequences are all exact:

1. \[ \cdots \to C^1 \to C^0 \to C^{-1} \to \cdots \]
2. \[ \cdots \to Z(C^1) \to Z(C^0) \to Z(C^{-1}) \to \cdots \]
3. \[ \cdots \to B(C^1) \to B(C^0) \to B(C^{-1}) \to \cdots \]
4. \[ \cdots \to C^1/Z(C^1) \to C^0/Z(C^0) \to C^{-1}/Z(C^{-1}) \to \cdots \]
5. \[ \cdots \to C^1/B(C^1) \to C^0/B(C^0) \to C^{-1}/B(C^{-1}) \to \cdots \]
6. \[ \cdots \to H(C^1) \to H(C^0) \to H(C^{-1}) \to \cdots \]

**Definition 4.2.** A complex \( G \) is said to be a C-E W-Gorenstein complex, if there exists a C-E exact sequence
\[ \mathbb{W} = \cdots \to W^1 \to W^0 \to W^{-1} \to W^{-2} \to \cdots \]
such that

1. each \( W^i \) is a C-E \( W \) complex;
2. \( G = \text{Ker} (W^{-1} \to W^{-2}) \);
3. the sequence remains exact when \( \text{Hom}_{C(R)}(V, -) \) and \( \text{Hom}_{C(R)}(-, V) \) are applied to it for any C-E \( W \) complex \( V \).

And in this case, \( \mathbb{W} \) is called a complete C-E \( W \)-resolution of \( G \).

**Theorem 4.3.** For a complex \( G \), the following are equivalent:

1. \( G \) is a C-E \( W \)-Gorenstein complex.
2. \( G, Z(G), B(G) \) and \( H(G) \) are complexes consisting of \( W \)-Gorenstein modules.
3. \( G, Z(G), B(G) \) and \( H(G) \) are \( \mathbb{W} \)-Gorenstein complexes.

**Proof.** We start by noting that (2) \( \iff \) (3) is immediate from [8, Section 5.1, Theorem A].

(1) \( \implies \) (2). Suppose that
\[ \mathbb{W} = \cdots \to W^1 \to W^0 \to W^{-1} \to W^{-2} \to \cdots \]
is a complete C-E \( W \)-resolution such that \( G = \text{Ker} (W^{-1} \to W^{-2}) \). Then, there is an exact sequence of modules in \( W \)
\[ \cdots \to W_n^1 \to W_n^0 \to W_n^{-1} \to W_n^{-2} \to \cdots \]
such that \( G_n = \text{Ker}(W^{-1}_n \to W^{-2}_n) \) for all \( n \in \mathbb{Z} \). For any module \( M \in \mathcal{W} \), \( D^n(M) \) is a C-E \( \mathcal{W} \) complex for all \( n \in \mathbb{Z} \). It follows from Proposition 2.1 that there is a natural isomorphism \( \text{Hom}_{\mathcal{C}(R)}(D^n(M), W^i) \cong \text{Hom}_R(M, W^i_n) \) for all \( i \in \mathbb{Z} \). Then the following commutative diagram with the first row exact

\[
\begin{array}{ccccccc}
\cdots & \xrightarrow{-} & \text{Hom}_{\mathcal{C}(R)}(D^n(M), W^1) & \xrightarrow{-} & \text{Hom}_{\mathcal{C}(R)}(D^n(M), W^0) & \xrightarrow{-} & \text{Hom}_{\mathcal{C}(R)}(D^n(M), W^{-1}) & \xrightarrow{-} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \xrightarrow{-} & \text{Hom}_R(M, W^1_n) & \xrightarrow{-} & \text{Hom}_R(M, W^0_n) & \xrightarrow{-} & \text{Hom}_R(M, W^{-1}_n) & \xrightarrow{-} & \cdots 
\end{array}
\]

will yield that the lower row is exact. Similarly, there is a natural isomorphism \( \text{Hom}_{\mathcal{C}(R)}(W^i, D^{n+1}(M)) \cong \text{Hom}_R(W^i_n, M) \) for all \( i \in \mathbb{Z} \). By applying \( \text{Hom}_{\mathcal{C}(R)}(-, D^{n+1}(M)) \) to the sequence \( \mathcal{W} \), we get an exact sequence

\[
\cdots \to \text{Hom}_R(W^{-1}_n, M) \to \text{Hom}_R(W^0_n, M) \to \text{Hom}_R(W^1_n, M) \to \cdots .
\]

Hence, \( G_n \) is a \( \mathcal{W} \)-Gorenstein module.

For any \( n \in \mathbb{Z} \), consider the sequence

\[
Z_n(\mathcal{W}) = \cdots \to Z_n(W^2) \to Z_n(W^1) \to Z_n(W^0) \to Z_n(W^{-1}) \to \cdots .
\]

It is exact (as \( \mathcal{W} \) is C-E exact) with each module \( Z_n(W^i) \in \mathcal{W} \) for all \( i \in \mathbb{Z} \). Note that \( Z_n(G) = \text{Ker}(Z_n(W^{-1}) \to Z_n(W^{-2})) \). In order to prove \( Z_n(G) \) is a \( \mathcal{W} \)-Gorenstein module, it suffices to prove that \( Z_n(\mathcal{W}) \) is both \( \text{Hom}_R(\mathcal{W}, -) \) exact and \( \text{Hom}_R(-, \mathcal{W}) \) exact.

For any module \( M \in \mathcal{W} \), \( S^n(M) \) is a C-E \( \mathcal{W} \) complex for all \( n \in \mathbb{Z} \), and there is a natural isomorphism \( \text{Hom}_{\mathcal{C}(R)}(S^n(M), W^i) \cong \text{Hom}_R(M, Z_n(W^i)) \) for all \( i \in \mathbb{Z} \). By applying \( \text{Hom}_{\mathcal{C}(R)}(S^n(M), -) \) to the sequence \( \mathcal{W} \), an argument similar to the above shows that \( \text{Hom}_R(M, Z_n(\mathcal{W})) \) is exact.

Since \( H_n(W^i) = Z_n(W^i)/B_n(W^i) \), \( B_{n-1}(W^i) \cong W^i_n/Z_n(W^i) \) are modules in \( \mathcal{W} \), the exact sequence \( 0 \to H_n(W^i) \to W^i_n/B_n(W^i) \to B_{n-1}(W^i) \to 0 \) yields that the module \( W^i_n/B_n(W^i) \) belongs to \( \mathcal{W} \). Consider the exact sequence of complexes

\[
0 \to B_n(\mathcal{W}) \to \mathcal{W}_n \to \mathcal{W}_n/B_n(\mathcal{W}) \to 0,
\]

where

\[
B_n(\mathcal{W}) = \cdots \to B_n(W^2) \to B_n(W^1) \to B_n(W^0) \to B_n(W^{-1}) \to \cdots .
\]

The sequence is split exact at modules levels, so we have an exact sequence of complexes of \( \mathbb{Z} \)-modules

\[
0 \to \text{Hom}_R(\mathcal{W}_n/B_n(\mathcal{W}), M) \to \text{Hom}_R(\mathcal{W}_n, M) \to \text{Hom}_R(B_n(\mathcal{W}), M) \to 0.
\]
Note that $\text{Hom}_R(\mathbb{W}_n, M)$ is exact. Also, by applying $\text{Hom}_{C(R)}(-, S^n(M))$ to the sequence $\mathbb{W}$, we get that $\text{Hom}_R(\mathbb{W}_n/B_n(\mathbb{W}), M)$ is exact by an argument analogous to the above. Thus $\text{Hom}_R(B_n(\mathbb{W}), M)$ is exact. Similarly, the exact sequence of complexes

$$0 \to Z_n(\mathbb{W}) \to \mathbb{W}_n \to B_{n-1}(\mathbb{W}) \to 0$$

yields an exact sequence of complexes of $\mathbb{Z}$-modules

$$0 \to \text{Hom}_R(B_{n-1}(\mathbb{W}), M) \to \text{Hom}_R(\mathbb{W}_n, M) \to \text{Hom}_R(Z_n(\mathbb{W}), M) \to 0,$$

where $\text{Hom}_R(B_{n-1}(\mathbb{W}), M)$ and $\text{Hom}_R(\mathbb{W}_n, M)$ are exact. So $\text{Hom}_R(Z_n(\mathbb{W}), M)$ is exact. This implies that $Z_n(G)$ is a $\mathbb{W}$-Gorenstein module.

There is an exact sequence of modules $0 \to Z_n(G) \to G_n \to B_{n-1}(G) \to 0$ for all $n \in \mathbb{Z}$ where $Z_n(G), G_n$ are $\mathbb{W}$-Gorenstein modules. As above, $\text{Hom}_R(B_{n-1}(\mathbb{W}), M)$ is exact for any $M \in \mathbb{W}$, which implies that $B_{n-1}(G) = \text{Ker}\left(B_{n-1}(W^{-1}) \to B_{n-1}(W^{-2})\right) \in \mathbb{W}$, where $\mathbb{W} = \{N \mid \text{Ext}^1_R(N, M) = 0$ for all $M \in \mathbb{W}$ and $i \geq 1\}$. It follows from [13, Corollary 2.6 (1)] that $B_{n-1}(G)$ is $\mathbb{W}$-Gorenstein for all $n \in \mathbb{Z}$. Similarly, from the exact sequence of modules $0 \to B_n(G) \to Z_n(G) \to H_n(G) \to 0$ we can get that $H_n(G)$ is $\mathbb{W}$-Gorenstein. This completes the proof.

(2) $\implies$ (1). For any $n \in \mathbb{Z}$, consider the exact sequence of modules

$$0 \to B_n(G) \to Z_n(G) \to H_n(G) \to 0,$$

where $B_n(G)$ and $H_n(G)$ are $\mathbb{W}$-Gorenstein. Suppose $W^{B_n(G)}$ and $W^{H_n(G)}$ are complete $\mathbb{W}$-resolutions of $B_n(G)$ and $H_n(G)$, respectively. By the Horseshoe Lemma, we can construct a complete $\mathbb{W}$ resolution of $Z_n(G)$: $W^{Z_n(G)} = W^{B_n(G)} \oplus W^{H_n(G)}$. Similarly, consider the exact sequence of modules

$$0 \to Z_n(G) \to G_n \to B_{n-1}(G) \to 0,$$

and we can construct a complete $\mathbb{W}$-resolution of $G_n$: $W^{G_n} = W^{Z_n(G)} \oplus W^{B_{n-1}(G)} = W^{B_n(G)} \oplus W^{H_n(G)} \oplus W^{B_{n-1}(G)}$. Set $W^{i}_n = W_{i}^{B_n(G)} \oplus W_{i}^{H_n(G)} \oplus W_{i}^{B_{n-1}(G)}$ and $d^{W^i}_n : W^{i}_n \to W^{i-1}_n$ which maps $(x, y, z)$ to $(z, 0, 0)$ for all $i, n \in \mathbb{Z}$. Then $(W^i, d^{W^i})$ is a complex such that $G_n = \text{Ker}(W^{i-1}_n \to W^{i-2}_n)$.

It is easily seen that $W^i$ is a $C$-$\mathbb{W}$ complex for all $i \in \mathbb{Z}$ and $G = \text{Ker}(W^{-1} \to W^{-2})$. For any $n \in \mathbb{Z}$, $\cdots \to Z_n(W^1) \to Z_n(W^0) \to Z_n(W^{-1}) \to \cdots$ is a complete $\mathbb{W}$-resolution of $Z_n(G)$, and $\cdots \to W^1_n \to W^0_n \to W^{-1}_n \to \cdots$ is a complete $\mathbb{W}$-resolution of $G_n$, so they both are exact. Hence, we can get that

$$\mathbb{W} = \cdots \to W^1 \to W^0 \to W^{-1} \to W^{-2} \to \cdots$$

is $C$-$\mathbb{W}$ exact. It remains to prove that, for any $C$-$\mathbb{W}$ complex $V$, $\mathbb{W}$ is still exact when $\text{Hom}_{C(R)}(V, -), \text{Hom}_{C(R)}(-, V)$ applied to it.
However, it suffices to prove that the assertion holds when we pick $V$ particularly as $V = D^n(M)$ and $V = S^n(M)$ for any module $M \in \mathcal{W}$ and all $n \in \mathbb{Z}$ by Proposition 3.3. Note that $\mathcal{W}_n, Z_n(\mathcal{W})$ and $\mathcal{W}_n/B_n(\mathcal{W})$ are complete $\mathcal{W}$-resolutions, hence from Proposition 2.1 the desired result follows.

In particular, if we set $\mathcal{W}$ to be the class of injective modules $\mathcal{I}$, then

**Corollary 4.4.** ([9, Theorem 8.5]) For an $R$-complex $G$, the following are equivalent:

1. $G$ has a complete C-E injective resolution.
2. $G, Z(G), B(G)$ and $H(G)$ are complexes consisting of Gorenstein injective modules.
3. $G, Z(G), B(G)$ and $H(G)$ are Gorenstein injective complexes.

We will not state here, but there are dual results about C-E Gorenstein projective complexes if $\mathcal{W}$ is the class of projective modules $\mathcal{P}$.

In particular, set $\mathcal{W} = \mathcal{P}_C$ and $\mathcal{W} = \mathcal{I}_C$ respectively, where $C$ is a semidualizing module over a commutative noetherian ring $R$. In [13], the authors called $\mathcal{P}_C$-Gorenstein and $\mathcal{I}_C$-Gorenstein modules $C$-Gorenstein projective and $C$-Gorenstein injective modules respectively. Accordingly, C-E $\mathcal{P}_C$-Gorenstein (respectively, C-E $\mathcal{I}_C$-Gorenstein) complexes are called C-E $C$-Gorenstein projective (respectively, C-E $C$-Gorenstein injective) complexes. We have the following results.

**Corollary 4.5.** For a complex $G$, the following are equivalent:

1. $G$ is a C-E $C$-Gorenstein projective (respectively, C-E $C$-Gorenstein injective) complex.
2. $G, Z(G), B(G)$ and $H(G)$ are complexes consisting of $C$-Gorenstein projective (respectively, $C$-Gorenstein injective) modules.

5. C-E complexes and Foxby equivalence

Throughout this section, $R$ is a commutative noetherian ring, $C$ is a semidualizing module over $R$. We always regard $C$ as a complex concentrated in degree zero.

It is well known that $\mathcal{P}_C$ and $\mathcal{I}_C$ are self-orthogonal classes. In this section, we shall in particular focus on C-E complexes relative to $\mathcal{P}_C, \mathcal{I}_C$, and related Foxby equivalence.

Let $\tilde{\mathcal{P}}(R)$ denote the subcategory of $D_b(R)$ defined by specifying its objects as complexes with finite projective dimension, that is, $X \in \tilde{\mathcal{P}}(R)$ if and only if $X \simeq \mathcal{P}(R)$.
where $P$ is a bounded complex of projective modules. Then C-E $\tilde{\mathcal{P}}(R)$ is the subcategory of $\mathcal{D}(R)$ defined as

$$C-E\ \tilde{\mathcal{P}}(R) = \{ X \in \mathcal{D}_b(R) \mid X \simeq Q \text{ with } Q, Z(Q), B(Q), H(Q) \in \tilde{\mathcal{P}}(R) \}.$$ 

Similarly, $\tilde{\mathcal{P}}_C(R)$ and C-E $\tilde{\mathcal{P}}_C(R)$ can be defined.

Foxby [11] studied modules in Auslander class $\mathcal{A}^0_C(R)$ and Bass class $\mathcal{B}^0_C(R)$, where

$$\mathcal{A}^0_C(R) = \left\{ M \in R\text{-Mod} \mid \text{Tor}_i^R(C, M) = 0 = \text{Ext}_i^R(C, C \otimes_R M), \text{ and the canonical map } M \rightarrow \text{Hom}_R(C, C \otimes_R M) \text{ is a canonical isomorphism} \right\},$$

and

$$\mathcal{B}^0_C(R) = \left\{ N \in R\text{-Mod} \mid \text{Ext}_i^R(C, N) = 0 = \text{Tor}_i^R(C, \text{Hom}_R(C, N)), \text{ and the canonical map } C \otimes_R \text{Hom}_R(C, N) \rightarrow N \text{ is a canonical isomorphism} \right\}.$$ 

By [6, Observation 4.10], $\mathcal{A}^0_C(R)$ and $\mathcal{B}^0_C(R)$ coincide with the subcategories of $\mathcal{A}_C(R)$ and $\mathcal{B}_C(R)$ consisting of $R$-complexes concentrated in degree zero, respectively. It is well known that $\mathcal{A}^0_C(R)$ contains all flat (projective) $R$-modules, and $\mathcal{B}^0_C(R)$ contains all injective $R$-modules.

It is easily seen that the following holds by the definition of each subcategory.

**Proposition 5.1.** There are inclusions of categories:

$$C-E\ \tilde{\mathcal{P}}(R) \subseteq \tilde{\mathcal{P}}(R) \subseteq \mathcal{A}_C(R), \ C-E\ \tilde{\mathcal{P}}_C(R) \subseteq \tilde{\mathcal{P}}_C(R) \subseteq \mathcal{B}_C(R).$$

**Proposition 5.2.** There is an equivalence of categories:

$$C-E\ \tilde{\mathcal{P}}(R) \xrightarrow{\sim} C-E\ \tilde{\mathcal{P}}_C(R).$$

**Proof.** Let $X \in C-E\ \tilde{\mathcal{P}}(R)$. Then there exists an isomorphism $X \simeq P$ in $\mathcal{D}(R)$, where $P$ is a bounded C-E projective complex, that is, $P$, $Z(P)$, $B(P)$, $H(P)$ are bounded complexes of projective modules. Assume that

$$P = 0 \rightarrow P_s \rightarrow P_{s-1} \rightarrow \cdots \rightarrow P_1 \rightarrow 0.$$ 

Then $C \otimes_R P \in \tilde{\mathcal{P}}_C(R)$. It remains to prove that $Z(C \otimes_R P)$, $B(C \otimes_R P)$ and $H(C \otimes_R P)$ all belong to $\mathcal{P}_C(R)$.

For any $k \in \mathbb{Z}$, consider the exact sequence

$$0 \rightarrow Z_k(P) \rightarrow P_k \xrightarrow{d^P_k} B_{k-1}(P) \rightarrow 0,$$

which is split since $B_{k-1}(P)$ is projective. Then we get an exact sequence

$$0 \rightarrow C \otimes_R Z_k(P) \rightarrow C \otimes_R P_k \xrightarrow{C \otimes_R d^P_k} C \otimes_R B_{k-1}(P) \rightarrow 0.$$
So

\[ C \otimes_R Z_k(P) = \ker(C \otimes_R d_k^P) = Z_k(C \otimes_R P), \]

and

\[ C \otimes_R B_{k-1}(P) = \operatorname{im}(C \otimes_R d_k^P) = B_{k-1}(C \otimes_R P). \]

Similarly, applying \( C \otimes_R - \) to the split exact sequence \( 0 \to B_k(P) \to Z_k(P) \to H_k(P) \to 0 \), we get a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & C \otimes_R B_k(P) & \longrightarrow & C \otimes_R Z_k(P) & \longrightarrow & C \otimes_R H_k(P) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_k(C \otimes_R P) & \longrightarrow & Z_k(C \otimes_R P) & \longrightarrow & H_k(C \otimes_R P) & \longrightarrow & 0,
\end{array}
\]

which yields that \( H_k(C \otimes_R P) = Z_k(C \otimes_R P)/B_k(C \otimes_R P) \cong C \otimes_R H_k(P) \). Since \( Z(P), B(P), H(P) \) are complexes of projective modules, the desired result follows.

Conversely, for a complex \( Y \in \text{C-E} \widetilde{\mathcal{P}}_C(R) \), assume that \( Y \simeq Q \) with \( Q \) a bounded C-E \( C \)-projective complex, that is, \( Q, Z(Q), B(Q), H(Q) \) are bounded complexes of \( C \)-projective modules. Suppose that \( \cdots \to F_1 \to F_0 \to C \to 0 \) is a projective resolution of the semidualizing module \( C \). Then there is a quasi-isomorphism \( \alpha : F \to C \), where

\[ F = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0. \]

Thus \( \mathbf{R}\hom_R(C, Y) \simeq \hom_R(F, Y) \simeq \hom_R(F, Q) \). For any \( C \)-projective module \( C \otimes_R P \), where \( P \) is a projective module, \( \operatorname{Ext}_R^i(C, C \otimes_R P) \cong \operatorname{Ext}_R^i(C, C) \otimes_R P = 0 \) for any \( i \geq 1 \). This implies that \( \operatorname{Ext}_R^i(C, Q_k) = 0 \), and hence \( \hom_R(\alpha, Q_k) : \hom_R(C, Q_k) \to \hom_R(F, Q_k) \) is a quasi-isomorphism. Therefore, it follows from [17, Proposition 2.6] that \( \hom_R(\alpha, Q) : \hom_R(C, Q) \to \hom_R(F, Q) \) is a quasi-isomorphism. Hence \( \mathbf{R}\hom_R(C, Y) \simeq \hom_R(C, Q) \).

As \( Q \) is a bounded complex of \( C \)-projective modules, it follows from [17, Theorem 1] that \( \hom_R(C, Q) \in \mathcal{P}(R) \). For any \( k \in \mathbb{Z} \), since \( Z_k(Q), B_k(Q), H_k(Q) \) are \( C \)-projective, the following sequences of \( R \)-modules

\[
\begin{align*}
0 & \longrightarrow Z_k(Q) \longrightarrow Q_k \xrightarrow{d_k^Q} B_{k-1}(Q) \longrightarrow 0, \\
0 & \longrightarrow B_k(Q) \longrightarrow Z_k(Q) \longrightarrow H_k(Q) \longrightarrow 0
\end{align*}
\]

are split exact. Applying \( \hom_R(C, -) \) to these sequences, an argument analogous to the above shows that

\[
\begin{align*}
B_{k-1}(\hom_R(C, Q)) & = \operatorname{im}(\hom_R(C, d_k^Q)) = \hom_R(C, B_{k-1}(Q)), \\
Z_k(\hom_R(C, Q)) & = \ker(\hom_R(C, d_k^Q)) = \hom_R(C, Z_k(Q)), \\
H_k(\hom_R(C, Q)) & = Z_k(\hom_R(C, Q))/B_k(\hom_R(C, Q)) \cong \hom_R(C, H_k(Q)),
\end{align*}
\]
which are all projective. Hence $\text{Hom}_R(C, Q) \in \text{C-E} \widetilde{\mathcal{P}}(R)$.

Finally, from the inclusions of categories $\text{C-E} \widetilde{\mathcal{P}}(R) \subseteq \mathcal{A}_C(R)$ and $\text{C-E} \widetilde{\mathcal{P}}_C(R) \subseteq \mathcal{B}_C(R)$, the equivalence of categories is easily seen by [6, Theorem 4.6].

We denote by $\mathcal{GP}$ the class of Gorenstein projective modules, and $\mathcal{GP}_C$ the class of $C$-Gorenstein projective modules. Note that we refer the notion of $C$-Gorenstein projective modules in the sense of Geng and Ding [13], which is different from that of Holm and Jørgensen [16, Definition 2.7]. Moreover, we can define the subcategories $\widetilde{\mathcal{GP}}(R)$ and $\widetilde{\mathcal{GP}}_C(R)$ of $\mathcal{D}(R)$ clearly as we do for $\widetilde{\mathcal{P}}(R)$ and $\widetilde{\mathcal{P}}_C(R)$ above.

**Proposition 5.3.** There are inclusions of categories:

$$
\text{C-E} \widetilde{\mathcal{P}}(R) \subseteq \text{C-E} \left( \widetilde{\mathcal{GP}}(R) \cap \mathcal{A}_C(R) \right) \subseteq \mathcal{A}_C(R),
$$

$$
\text{C-E} \widetilde{\mathcal{P}}_C(R) \subseteq \text{C-E} \widetilde{\mathcal{GP}}_C(R) \subseteq \mathcal{B}_C(R),
$$

where

$$
\text{C-E} \left( \widetilde{\mathcal{GP}}(R) \cap \mathcal{A}_C(R) \right) = \{ X \in \mathcal{D}(R) | X \simeq G \text{ with } G, Z(G), B(G), H(G) \in \widetilde{\mathcal{GP}}(R) \cap \mathcal{A}_C(R) \},
$$

$$
\text{C-E} \widetilde{\mathcal{GP}}_C(R) = \{ Y \in \mathcal{D}(R) | Y \simeq E \text{ with } E, Z(E), B(E), H(E) \in \widetilde{\mathcal{GP}}_C(R) \}.
$$

**Proof.** It is immediate by the definition of each subcategory and by [13, Remark 3.13].

**Proposition 5.4.** There is an equivalence of categories:

$$
\text{C-E} \left( \widetilde{\mathcal{GP}}(R) \cap \mathcal{A}_C(R) \right) \cong \text{C-E} \widetilde{\mathcal{GP}}_C(R).
$$

**Proof.** Let $X \in \text{C-E} \left( \widetilde{\mathcal{GP}}(R) \cap \mathcal{A}_C(R) \right)$. Then there exists an isomorphism $X \simeq G$ in $\mathcal{D}(R)$, where $G, Z(G), B(G), H(G)$ belong to $\mathcal{A}_C(R)$ and also are bounded complexes of Gorenstein projective modules. Assume that

$$
G = 0 \longrightarrow G_s \longrightarrow G_{s-1} \longrightarrow \cdots \longrightarrow G_i \longrightarrow 0,
$$

where $G_k \in \mathcal{GP} \cap \mathcal{A}_C^0(R)$ for $i \leq k \leq n$. Let $\alpha : F \to C$ be a projective resolution of the semidualizing module $C$, where

$$
F = \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0.
$$

Then $\alpha$ is a quasi-isomorphism, and $C \otimes^L_R X \simeq F \otimes_R X \simeq F \otimes_R G$. For any $k \in \mathbb{Z}$, it follows from $\text{Tor}^i_R(C, G_k) = 0$ that $\alpha \otimes_R G_k : F \otimes_R G_k \to C \otimes_R G_k$ is a quasi-isomorphism. Then by [7, Proposition 2.14], $\alpha \otimes_R G : F \otimes_R G \to C \otimes_R G$ is a quasi-isomorphism. Hence, it follows from [13, Theorem 3.11] that $C \otimes^L_R X \simeq$
\( C \otimes_R G \in \widehat{\mathcal{GP}}_C(R) \). It remains to prove that \( Z(C \otimes_R G) \), \( B(C \otimes_R G) \), \( H(C \otimes_R G) \) belong to \( \widehat{\mathcal{GP}}_C(R) \).

For any \( k \in \mathbb{Z} \), consider the exact sequence

\[
0 \rightarrow Z_k(G) \rightarrow G_k \xrightarrow{d^G_k} B_{k-1}(G) \rightarrow 0.
\]

Since \( B_{k-1}(G) \in \mathcal{A}_C^0(R) \), it follows that \( \text{Tor}^R_i(C, B_{k-1}(G)) = 0 \). Then we get an exact sequence

\[
0 \rightarrow C \otimes_R Z_k(G) \rightarrow C \otimes_R G_k \xrightarrow{C \otimes_R d^G_k} C \otimes_R B_{k-1}(G) \rightarrow 0.
\]

So

\[
C \otimes_R Z_k(G) = \text{Ker}(C \otimes_R d^G_k) = Z_k(C \otimes_R G),
\]

\[
C \otimes_R B_{k-1}(G) = \text{Im}(C \otimes_R d^G_k) = B_{k-1}(C \otimes_R G).
\]

By applying \( C \otimes_R - \) to the exact sequence \( 0 \rightarrow B_i(G) \rightarrow Z_i(G) \rightarrow H_i(G) \rightarrow 0 \), a similar argument yields that \( H_k(C \otimes_R G) = Z_k(C \otimes_R G)/B_k(C \otimes_R G) \cong C \otimes_R H_k(G) \). Since \( Z(G), B(G), H(G) \) are complexes of modules in \( \mathcal{GP} \cap \mathcal{A}_C^0(R) \), the desired result follows by [13, Theorem 3.11].

Conversely, assume that \( Y \simeq E \) with \( E \) a bounded complex of \( C \)-Gorenstein projective modules. Since \( \text{Ext}^i_R(C, E_k) = 0 \), an analogous argument shows that \( \text{RHom}_R(C, Y) \simeq \text{Hom}_R(C, E) \). It follows from [13, Theorem 3.11] that \( \text{Hom}_R(C, E) \) is in \( \mathcal{GP}(R) \cap \mathcal{A}_C(R) \). It remains to prove that \( Z(\text{Hom}_R(C, E)), B(\text{Hom}_R(C, E)) \) and \( H(\text{Hom}_R(C, E)) \) all belong to \( \mathcal{GP}(R) \cap \mathcal{A}_C(R) \).

Moreover, as \( \text{Ext}^i_R(C, Z_k(E)) = 0 = \text{Ext}^i_R(C, B_k(E)) \), a similar argument yields that

\[
B_{k-1}(\text{Hom}_R(C, E)) = \text{Im}(\text{Hom}_R(C, d^E_k)) = \text{Hom}_R(C, B_{k-1}(E)),
\]

\[
Z_k(\text{Hom}_R(C, E)) = \text{Ker}(\text{Hom}_R(C, d^E_k)) = \text{Hom}_R(C, Z_k(E)),
\]

\[
H_k(\text{Hom}_R(C, E)) = Z_k(\text{Hom}_R(C, E))/B_k(\text{Hom}_R(C, E)) \cong \text{Hom}_R(C, H_k(E)),
\]

which are all modules in \( \mathcal{GP} \cap \mathcal{A}_C^0(R) \). Hence \( \text{Hom}_R(C, E) \in \text{C-E (} \mathcal{GP}(R) \cap \mathcal{A}_C(R) ) \) by [13, Theorem 3.11]. Now the desired equivalence of categories follows.

\[ \square \]

**Remark 5.5.** Dually, the results concerning with injectivity, \( C \)-injectivity, Gorenstein injectivity, \( C \)-Gorenstein injectivity and Bass class hold.

Now, we are in a position to obtain the main result of this section, which extends the existed Foxby equivalence.

**Theorem 5.6.** There are equivalences of categories:
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