Relaxed Lagrangian duality in convex infinite optimization: reverse strong duality and optimality

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Abstract

We associate with each convex optimization problem posed on some locally convex space with an infinite index set $T$, and a given non-empty family $\mathcal{H}$ formed by finite subsets of $T$, a suitable Lagrangian-Haar dual problem. We provide reverse $\mathcal{H}$-strong duality theorems, $\mathcal{H}$-Farkas type lemmas and optimality theorems. Special attention is addressed to infinite and semi-infinite linear optimization problems.

To Dinh The Luc on the occasion of his 70th anniversary

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1 Introduction

In a recent paper on convex infinite optimization [4], we have provided reducibility, zero duality gap, and strong duality theorems for a new type of Lagrangian-Haar duality associated with families of finite sets of indices. More precisely, given an optimization problem

\[ \inf \ f(x) \ \text{s.t.} \ f_t(x) \leq 0, \ t \in T, \]  

(1.1)
such that $X$ is a locally convex Hausdorff topological vector space, $T$ is an arbitrary infinite index set, and $\{f; f_t, t \in T\}$ are convex proper functions on $X$, as well as a family $\mathcal{H}$ of non-empty finite subsets of the index set $T$, we consider the $\mathcal{H}$-dual problem

$$
\text{(D)}_{\mathcal{H}} \sup_{H \in \mathcal{H}, \mu \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \mu_t f_t(x) \right\},
$$

(1.2)

where $\mu \in \mathbb{R}_+^H$ stands for $(\mu_t)_{t \in H} \in \mathbb{R}_+^H$, with the rule $0 \times (+\infty) = 0$. When $\mathcal{H}$ is the family $\mathcal{F}(T)$ of all non-empty finite subsets of $T$, one gets the standard Lagrangian-Haar dual of $(P)$,

$$
\text{(D)} \sup_{H \in \mathcal{F}(T), \mu \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \mu_t f_t(x) \right\}.
$$

(1.3)

As in [4], this paper pays particular attention to the families $\mathcal{H}_1 := \{\{t\}, t \in T\}$ of singletons and (when $T = \mathbb{N}$) $\mathcal{H}_N := \{\{1, \ldots, m\}, m \in \mathbb{N}\}$ of sets of initial natural numbers. The dual pair $(P) - (\text{D}_{\mathcal{H}_0})$ has been used in [16] in the framework of convex semi-infinite programming (CSIP), where $X = \mathbb{R}^n$. More precisely, [16] gives a sufficient condition for the optimal value of a SIP problem $(P)$ with $T = \mathbb{N}$ to be the limit, as $m \to \infty$, of the optimal values of the sequence of ordinary convex programs $(P_m)_{m \in \mathbb{N}}$ which results of replacing $T$ by $\{1, \ldots, m\}$ in $(P)$. This assumption on $T$ is not as strong as it can seem at first sight as, if $T$ is an uncountable topological space which contains a countable dense subset $S$ and the mapping $t \mapsto f_t(x)$ is continuous on $T$ for any $x \in \bigcap_{t \in T} \text{dom} f_t$, then $(P)$ is equivalent to the countable subproblem which results of replacing $T$ by $S$ in $(P)$. In the particular case of linear semi-infinite programming (LSIP), we can write

$$
\text{(P)} \quad \inf \langle c^*, x \rangle \quad \text{s.t.} \quad \langle a^*_t, x \rangle \leq b_t, \quad t \in T,
$$

(1.4)

with $\{c^*; a^*_t, t \in T\} \subset \mathbb{R}^n$ and $\{b_t, t \in T\} \subset \mathbb{R}$, where, in most applications, $T$ is a convex body (i.e., a compact convex set with non-empty interior) in some Euclidean space and the mapping $t \mapsto (a^*_t, b_t)$ is continuous on $T$. Then, $T$ can be replaced by any finite dense subset $S$ to get an equivalent countable LSIP problem.

There exists a wide literature on the dual pair $(P) - (D)$, see e.g., the works [2], [7], [8], [11], [12], [18], and [19], most of them focused on constraint qualifications and/or duality theorems, some of them making use, in order to get optimality conditions, of suitable versions of the celebrated Farkas’ Lemma that have been reviewed in [5].

The duality theorems for the pair $(P) - (\text{D}_{\mathcal{H}})$ provide conditions guaranteeing a zero duality gap, i.e., that $\inf(P) = \sup(\text{D}_{\mathcal{H}})$ (see, [4] Theorem 6.1]). Other duality theorems in [4] are strong in the sense that the optimal value of $(\text{D}_{\mathcal{H}})$ is attained, situation represented by the equation $\inf(P) = \max(\text{D}_{\mathcal{H}})$ (see, [4] Theorems 5.1-5.3)). Similarly, the reverse duality theorems, in Section 3 of this paper, are duality theorems where the optimal value of $(P)$ is attained, situation represented by the equation $\min(P) = \sup(\text{D}_{\mathcal{H}})$. Reverse (also called converse) duality theorems for the classical Lagrange dual problem, that is, for $\mathcal{H} = \mathcal{F}(T)$, in convex infinite programming (CIP in short)
can be found in [11 Theorem 3.3] and [12 Theorem 3]. Section 4 provides ad hoc Farkas-type results oriented to obtain, in Section 5, optimality conditions which are expressed in terms of multipliers associated to the indices belonging to the elements of $\mathcal{H}$.

2 Preliminaries

Let $X$ be a locally convex Hausdorff topological vector space, and suppose that its topological dual $X^*$, with null element $0_{X^*}$, is endowed with the weak*-topology. We denote by $\overline{A}$ and $\text{ri } A$ the closure and the relative interior of a set $A \subset X$, and by $\text{co } A$ its convex hull. For a set $\emptyset \neq A \subset X$, by the convex cone generated by $A$ we mean $\text{cone } A := \mathbb{R}_+(\text{co } A) = \{\mu x : \mu \in \mathbb{R}_+, x \in \text{co } A\}$, by $\text{span } A$ its linear span, and by $A_* = \text{dom } h$ the recession cone of the closed convex set $epi h$. The negative polar of $\emptyset \neq A \subset X$ is the convex cone $A^- := \{x^* \in X^*: \langle x^*, x \rangle \leq 0, \forall x \in A\}$. The lineality space of a convex set $K \subset X$ is $\text{lin } K = K \cap (-K)$.

The $w^*$-closure of a set $A \subset X^*$ is also denoted by $\overline{A}$. If $A \subset X^* \times \mathbb{R}$, then $\overline{A}$ denotes the closure of $A$ w.r.t. the product topology. A set $A \subset X^* \times \mathbb{R}$ is said to be $w^*$-closed (respectively, $w^*$-closed convex) regarding another subset $B \subset X^* \times \mathbb{R}$ if $\overline{A} \cap B = A \cap B$ (respectively, $(\overline{\text{co } A}) \cap B = \text{co } A \cap B$), see [1] (resp. [6]).

A function $h : X \to \mathbb{R} := \mathbb{R} \cup \{\pm \infty\}$ is proper if its epigraph $\text{epi } h$ is non-empty and never takes the value $-\infty$; it is convex if $\text{epi } h$ is convex; it is lower semicontinuous (lsc, in brief) if $\text{epi } h$ is closed; and it is upper semicontinuous (usc, in brief) if $-h$ is lsc. For a proper function $h$, we denote by $[h \leq 0] := \{x \in X : h(x) \leq 0\}$ its lower level set of 0, and by $\text{dom } h$, $\overline{h}$, $\partial h$, and $h^*$ its domain, its lsc envelope, its Fenchel subdifferential, and its Legendre-Fenchel conjugate, respectively. We also denote by $\Gamma(X)$ the class of lsc proper convex functions on $X$. By $\delta_A$ we denote the indicator function of $A \subset X$, with $\delta_A \in \Gamma(X)$ whenever $A \neq \emptyset$ is closed and convex.

We need to recall some basic facts about convex analysis recession. Given $h \in \Gamma(X)$, the recession cone of the closed convex set $\text{epi } h$ is the epigraph of the so-called recession function $h_\infty$ of $h$: $(\text{epi } h_\infty) = \text{epi } h_\infty$. The recession function $h_\infty$ coincides with the support function of the domain of the conjugate $h^*$ of $h$ (e.g., [17 Theorem 6.8.5]):

$$h_\infty = (\delta_{\text{dom } h^*})^*.$$  

From (2.1),

$$[h_\infty \leq 0] = (\text{dom } h^*)^- = \{x \in X : \langle x^*, x \rangle \leq 0, \forall x^* \in \text{dom } h^*\},$$  

which is called the recession cone of the function $h$ and provides the common recession cone to all the non-empty sublevel sets $[h \leq r]$. Given $\{h_1, \cdots, h_m\} \subset \Gamma(X)$ such that $\bigcap_{1 \leq k \leq m}\text{dom } h_k \neq \emptyset$, by [13 Proposition 3.2.3] (whose proof is independent of the dimension of $X$), one has for all $\mu \in \mathbb{R}_+$:

$$\left(\sum_{k=1}^m \mu_k h_k\right)_\infty = \sum_{k=1}^m \mu_k (h_k)_\infty.$$  

(2.3)
2.1 Classical Lagrange CIP duality

The support of $\lambda : T \to \mathbb{R}$ is the set $\text{supp} \lambda := \{ t \in T : \lambda_t \neq 0 \}$. Let $\mathbb{R}^{(T)}$ be the space of generalized finite sequences formed by all real-valued functions on $T$ that vanish except on a finite set called support, i.e.,

$$\mathbb{R}^{(T)} := \{ \lambda : T \to \mathbb{R}_+ \text{ such that supp } \lambda \text{ is finite} \}$$

with positive cone $\mathbb{R}_+^{(T)} := \{ \lambda \in \mathbb{R}^{(T)} : \lambda_t \geq 0, \forall t \in T \}$. We can associate to each $\lambda \in \mathbb{R}^{(T)}_+$ the function $\sum_{t \in T} \lambda_t f_t : X \to \mathbb{R} \cup \{ +\infty \}$ such that

$$\left( \sum_{t \in T} \lambda_t f_t \right)(x) = \begin{cases} \sum_{t \in \text{supp } \lambda} \lambda_t f_t(x), & \text{if } \text{supp } \lambda \neq \emptyset, \\ 0, & \text{if } \text{supp } \lambda = \emptyset. \end{cases}$$

So, we can reformulate (D) in (1.3) as

$$(\text{D}) \sup_{\lambda \in \mathbb{R}^{(T)}_+} \inf_{x \in X} \left\{ f(x) + \left( \sum_{t \in T} \lambda_t f_t \right)(x) \right\}.$$

It is known that the function $\varphi : X^* \to \overline{\mathbb{R}}$ such that

$$\varphi(x^*) := \inf_{\lambda \in \mathbb{R}^{(T)}_+} \left( f + \sum_{t \in T} \lambda_t f_t \right)^*(x^*)$$

and the set

$$\mathcal{A} := \bigcup_{\lambda \in \mathbb{R}^{(T)}_+} \text{epi} \left( f + \sum_{t \in T} \lambda_t f_t \right)^* \subset X^* \times \mathbb{R}$$

are both convex, and $\text{epi } \varphi = \mathcal{A}$ (see, for instance, [4], [11], [12]).

We denote the feasible set of (P) by

$$E := \bigcap_{t \in T} [f_t \leq 0].$$

Then,

$$-\infty \leq (f + \delta_E)^*(x^*) \leq \varphi(x^*) \leq f^*(x^*) \leq +\infty, \ \forall x^* \in X^*,$$

and, taking $x^* = 0_{X^*}$, one gets the weak duality for the pair (P) – (D):

$$-\infty \leq \inf_X f \leq \sup(D) \leq \inf(P) \leq +\infty.$$

2.2 Relaxed Lagrange CIP duality

Let $\mathcal{H}$ be a non-empty family of non-empty finite subsets of $T$, that is, $\emptyset \neq \mathcal{H} \subset \mathcal{F}(T)$, with associated dual problem $(\text{D}_{\mathcal{H}})$ as in (1.2). Obviously,

$$\sup(\text{D}_{\mathcal{H}}) \leq \sup(\text{D}_{\mathcal{F}(T)}) = \sup(D) \leq \inf(P). \quad (2.4)$$
Let us define the sets
\[ E_H := \bigcap_{H \in \mathcal{H}, t \in H} [f_t \leq 0], \]
\[ A_H := \bigcup_{H \in \mathcal{H}, \mu \in \mathbb{R}_+^H} \text{epi} \left( f + \sum_{t \in H} \mu_t f_t \right)^*, \]
and the function \( \varphi_H : X^* \to \mathbb{R} \) such that
\[ \varphi_H := \inf_{H \in \mathcal{H}, \mu \in \mathbb{R}_+^H} \left( f + \sum_{t \in H} \mu_t f_t \right)^*. \]
Obviously, \( A_H \subset A \) and \( \varphi_H \geq \varphi \).

**Definition 2.1**

(i) A family \( \mathcal{H} \subset \mathcal{F}(T) \) is said to be covering if \( \bigcup_{H \in \mathcal{H}} H = T \).
(ii) A family \( \mathcal{H} \subset \mathcal{F}(T) \) is said to be directed if for each \( H, K \in \mathcal{H} \) there exists \( L \in \mathcal{H} \) such that \( H \cup K \subset L \).

The families \( \mathcal{F}(T) \) and \( \mathcal{H}_n \) are both covering and directed families, whereas \( \mathcal{H}_1 \) is just covering.

As shown in [4, Proposition 3.2], for each directed covering family \( \mathcal{H} \subset \mathcal{F}(T) \) one has
\[ A_H = A_{\mathcal{F}(T)} = A, \] \hspace{1cm} (2.5)
and, consequently,
\[ \varphi_H = \varphi_{\mathcal{F}(T)} = \varphi, \text{ and } \sup(D_H) = \sup(D_{\mathcal{F}(T)}) \equiv \sup(D). \] \hspace{1cm} (2.6)

Let \( \mathcal{H} \subset \mathcal{F}(T) \) be a covering family. Then, \( E_H = E \) and, according to [4, Lemma 5.2], \( \{f; f_t, t \in T\} \subset \Gamma(X) \) entails
\[ (\varphi_H)^* = f + \delta_E \] \hspace{1cm} (2.7)
and if, additionally, \( E \cap (\text{dom } f) \neq \emptyset \), then
\[ \text{epi}(f + \delta_E)^* = \overline{\text{co}} A_H = \overline{\text{co}} \left( \bigcup_{H \in \mathcal{H}, \mu \in \mathbb{R}_+^H} \text{epi} \left( f + \sum_{t \in H} \mu_t f_t \right)^* \right). \]
Moreover, by [4, Theorem 5.1], \( \mathcal{H} \)-strong duality holds at a given \( x^* \in X^* \), i.e.,
\[ (f + \delta_E)^*(x^*) = \min_{H \in \mathcal{H}, \mu \in \mathbb{R}_+^H} \left( f + \sum_{t \in H} \mu_t f_t \right)^*(x^*), \] \hspace{1cm} (2.8)
if and only if \( A_H \) is \( w^* \)-closed convex regarding \( \{x^*\} \times \mathbb{R} \).
2.3 The $\mathcal{H}$-dual problem as a limit

It is easy to see that the mapping $\mathcal{F}(T) \supset \mathcal{H} \mapsto \sup(\mathcal{D}_{\mathcal{H}}) \in \mathbb{R}$ is non-decreasing w.r.t. the inclusion $\subset$ in $\mathcal{F}(T)$. Consequently, if the family $\mathcal{H} \subset \mathcal{F}(T)$ is directed, we can express $\sup(\mathcal{D}_{\mathcal{H}})$ as the limit of a net as follows:

$$\sup(\mathcal{D}_{\mathcal{H}}) = \sup_{H \in \mathcal{H}} \sup(\mathcal{D}_{\mathcal{H}}) = \lim_{H \in \mathcal{H}} \sup(\mathcal{D}_{\mathcal{H}}).$$

If, moreover, $\mathcal{H}$ is covering, then

$$\sup(\mathcal{D}) = \lim_{H \in \mathcal{H}} \sup(\mathcal{D}_{\mathcal{H}}). \quad (2.9)$$

In particular, if $T = \mathbb{N}$, we consider the countable program

$$(P_\mathbb{N}) \quad \inf f(x) \text{ s.t. } f_k(x) \leq 0, \ k \in \mathbb{N}, \quad (2.10)$$

and the sequence of finite subproblems

$$(P_m) \quad \inf f(x) \text{ s.t. } f_k(x) \leq 0, \ k \in \{1, \cdots, m\}, \ m \in \mathbb{N}, \quad (2.11)$$

whose ordinary Lagrangian dual problems are

$$(D_m) \quad \sup_{\mu \in \mathbb{R}_+^m} \inf_{x \in X} \left\{ f(x) + \sum_{k=1}^{m} \mu_k f_k(x) \right\}, \ m \in \mathbb{N}. \quad (2.12)$$

From (2.9), the Lagrangian-Haar dual of $(P_\mathbb{N})$,

$$(D_\mathbb{N}) \quad \sup_{\lambda \in \mathbb{R}_+^{(\mathbb{N})}} \inf_{x \in X} \left\{ f(x) + \sum_{k \in \mathbb{N}} \lambda_k f_k(x) \right\}, \quad (2.13)$$

and its $\mathcal{H}_\mathbb{N}$-dual Lagrange problem $(D_{\mathcal{H}_\mathbb{N}})$ can be expressed as limits in this way:

$$\sup(D_\mathbb{N}) = \sup(D_{\mathcal{H}_\mathbb{N}}) = \lim_{m \to \infty} \sup(D_m). \quad (2.14)$$

Corollary 3.3 below provides a sufficient condition for the primal counterpart of (2.12):

$$\inf(P_\mathbb{N}) = \lim_{m \to \infty} \inf(P_m).$$

3 $\mathcal{H}$-reverse strong duality

Let us go back to the general convex infinite optimization problem $(P)$ in (1.1). Along this section we assume that $\{f: f_t, t \in T\} \subset \Gamma(X)$ and $E \cap \text{dom } f \neq \emptyset$, meaning that $\inf(P) \neq +\infty$. 
**Definition 3.1** Given a covering family \( \mathcal{H} \subset \mathcal{F}(T) \), we say that \( \mathcal{H} \)-reverse strong duality holds if

\[
\min(P) = \sup(D_{\mathcal{H}}),
\]
equivalently, that there exists \( \bar{x} \in E \cap \text{dom } f \) such that

\[
f(\bar{x}) = \sup(D_{\mathcal{H}}) \in \mathbb{R}.
\]

We first show that \( \mathcal{H} \)-reverse strong duality can be described in terms of subdifferentiability of the function \( \varphi_{\mathcal{H}} \).

Recall that the subdifferential of a function \( g : X^* \to \mathbb{R} \) at a point \( a^* \in X^* \) is given by

\[
\partial g(a^*) := \begin{cases} 
\{ x \in X : g(x^*) \geq g(a^*) + \langle x^* - a^*, x \rangle, \forall x^* \in X^* \}, & \text{if } g(a^*) \in \mathbb{R}, \\
\emptyset, & \text{if } g(a^*) \notin \mathbb{R}.
\end{cases}
\]

We have

\[
x \in \partial g(a^*) \iff g(a^*) + g^*(x) = \langle a^*, x \rangle.
\]  \hspace{1cm} (3.1)

**Lemma 3.1** Let \( \mathcal{H} \) be a covering family. Then, \( \mathcal{H} \)-reverse strong duality holds if and only if \( \varphi_{\mathcal{H}} \) is subdifferentiable at \( 0_{X^*} \). In such a case one has \( \partial \varphi_{\mathcal{H}}(0_{X^*}) = \text{sol}(P) \), where \( \text{sol}(P) \) is the optimal solution set of \( (P) \).

**Proof** Let \( x \in \partial \varphi_{\mathcal{H}}(0_{X^*}) \). Since we are assuming that \( \mathcal{H} \) is covering, by (2.7) and (3.1), we have

\[
(f + \delta_E)(x) = (\varphi_{\mathcal{H}})^*(x) = -\varphi_{\mathcal{H}}(0_{X^*}) \in \mathbb{R}.
\]

Then \( x \in E \) and

\[
\inf(P) \leq f(x) = -\varphi_{\mathcal{H}}(0_{X^*}) = \sup(D_{\mathcal{H}}) \leq \inf(P).
\]

Consequently, if \( \varphi_{\mathcal{H}} \) is subdifferentiable at \( 0_{X^*} \) then \( \mathcal{H} \)-reverse strong duality holds and

\[
\partial \varphi_{\mathcal{H}}(0_{X^*}) \subset \text{sol}(P).
\]

Assume now that \( \mathcal{H} \)-reverse strong duality holds. There exists \( x \in E \cap (\text{dom } f) \) such that

\[
(\varphi_{\mathcal{H}})^*(x) = f(x) = \sup(D_{\mathcal{H}}) = -\varphi_{\mathcal{H}}(0_{X^*}) \in \mathbb{R},
\]  \hspace{1cm} (3.2)

that means \( x \in \partial \varphi_{\mathcal{H}}(0_{X^*}) \) and the first part of Lemma [3.1] is proved with, in addition, the inclusion \( \partial \varphi_{\mathcal{H}}(0_{X^*}) \subset \text{sol}(P) \). It remains to prove that if \( \mathcal{H} \)-reverse strong duality holds, then \( \text{sol}(P) \subset \partial \varphi_{\mathcal{H}}(0_{X^*}) \). Now for each \( x \in \text{sol}(P) \) we have (3.2). So, \( \varphi_{\mathcal{H}}(0_{X^*}) + (\varphi_{\mathcal{H}})^*(x) = 0 \), that means \( x \in \partial \varphi_{\mathcal{H}}(0_{X^*}) \).  \( \square \)

In favorable circumstances we know that \( \varphi_{\mathcal{H}} \) is a convex function. For instance, when the covering family \( \mathcal{H} \) is also directed, by (2.5) and (2.6), \( A_{\mathcal{H}} = A \) and \( \varphi_{\mathcal{H}} = \varphi \), respectively, implying the convexity of both \( A_{\mathcal{H}} \) and \( \varphi_{\mathcal{H}} \). Another important example is furnished by

\[
\varphi_{\mathcal{H}_1} = \inf_{(t,\mu) \in T \times \mathbb{R}^+} (f + \mu f_I)^*,
\]
which is convex under the assumptions (a), (b), (c) of Corollary 3.1 below (see [4, Remark 5.5]). In order to propose a tractable subdifferentiability criterion when \( \varphi_{\mathcal{H}} \) is convex we need to recall some facts about quasicontinuous convex functions and convex analysis recession.

**Definition 3.2** A convex function \( g : X^* \to \overline{\mathbb{R}} \) is said to be \( \tau(X^*, X) \)-quasicontinuous ([14], [15]), where \( \tau \) is the Mackey topology on \( X^* \), if the following four properties are satisfied:

1. \( \text{aff}(\text{dom } g) \) is \( \tau(X^*, X) \)-closed (or \( \sigma(X^*, X) \)-closed),
2. \( \text{aff}(\text{dom } g) \) is of finite codimension,
3. the \( \tau(X^*, X) \)-relative interior of \( \text{dom } g \), say \( \text{ri}(\text{dom } g) \), is non-empty,
4. the restriction of \( g \) to \( \text{aff}(\text{dom } g) \) is \( \tau(X^*, X) \)-continuous on \( \text{ri}(\text{dom } g) \).

Lemmas 3.2, 3.3, 3.4 below will be used in the sequel.

**Lemma 3.2 ([14, Proposition 5.4])** Let \( h \in \Gamma(X) \). The conjugate function \( h^* \) is \( \tau(X^*, X) \)-quasicontinuous if and only if \( h \) is weakly inf-locally compact; that is to say \( [h \leq r] \) is weakly locally compact for each \( r \in \mathbb{R} \).

**Lemma 3.3 ([20, Theorem II.4])** A convex function \( g : X^* \to \overline{\mathbb{R}} \) majorized by a \( \tau(X^*, X) \)-quasicontinuous one is \( \tau(X^*, X) \)-quasicontinuous, too.

**Lemma 3.4 ([20, Theorem III.3])** Let \( g : X^* \to \overline{\mathbb{R}} \) be a \( \tau(X^*, X) \)-quasicontinuous convex function such that \( g(0_{X^*}) \neq -\infty \) and \( \text{cone } \text{dom } g \) is a linear subspace of \( X^* \). Then \( \partial g(0_{X^*}) \) is the sum of a non-empty weakly compact convex set and a finite dimensional linear subspace of \( X \).

We define the recession cone of \( (P) \) by setting

\[
(P)_{\infty} := \bigcap_{t \in T} [(f_t)_\infty \leq 0] \cap [f_\infty \leq 0].
\]

For the next theorem and the corollaries below, recall that \( \inf (P) \neq +\infty \) as \( E \cap \text{dom } f \neq \emptyset \).

**Theorem 3.1 (\( \mathcal{H} \)-reverse strong duality)** Let \( \mathcal{H} \) be a covering family such that \( \varphi_{\mathcal{H}} \) is convex \( \tau(X^*, X) \)-quasicontinuous and \( (P)_{\infty} \) is a linear subspace of \( X \). Then \( \mathcal{H} \)-reverse strong duality holds:

\[
\min(P) = \sup(D_{\mathcal{H}}) \in \mathbb{R}.
\]

Moreover, \( \text{sol}(P) \) is the sum of a weakly compact convex set and a finite dimensional linear subspace of \( X \).
Proof: One has $\varphi_\mathcal{H}(0_{X^*}) = -\sup(D_{\mathcal{H}}) \geq -\inf(P) > -\infty$ (the last strict inequality holds as $E \cap \text{dom } f \neq \emptyset$). In order to apply Lemma 3.4 to the convex function $\varphi_\mathcal{H}$, we have to prove that $\text{cone} \ \text{dom } \varphi_\mathcal{H}$ is a linear subspace. We have

$$\text{cone} \ \text{dom } \varphi_\mathcal{H} = (\text{dom } \varphi_\mathcal{H})^- = \{x^* \in X^* : \langle x^*, x \rangle \leq 0, \forall x \in (\text{dom } \varphi_\mathcal{H})^-\}.$$ 

Therefore, $\text{cone} \ \text{dom } \varphi_\mathcal{H}$ is a linear subspace if and only if $(\text{dom } \varphi_\mathcal{H})^-$ is a linear subspace. Now,

$$\text{dom } \varphi_\mathcal{H} = \bigcup_{H \in \mathcal{H}} \bigcup_{\mu \in \mathbb{R}^H_+} \text{dom } \left(f + \sum_{t \in H} \mu_t f_t\right)^*$$

and we can write

$$(\text{dom } \varphi_\mathcal{H})^- = \bigcap_{H \in \mathcal{H}} \bigcap_{\mu \in \mathbb{R}^H_+} \left(\text{dom } \left(f + \sum_{t \in H} \mu_t f_t\right)^*\right)^-$$

$$= \bigcap_{H \in \mathcal{H}} \bigcap_{\mu \in \mathbb{R}^H_+} \left[\left(f + \sum_{t \in H} \mu_t f_t\right)^* \leq 0\right] \quad \text{(by (2.2))}$$

$$= \bigcap_{H \in \mathcal{H}} \bigcap_{\mu \in \mathbb{R}^H_+} \left[\left(f_\infty + \sum_{t \in H} \mu_t (f_t)_\infty\right) \leq 0\right] \quad \text{(by (2.3))}$$

$$= \bigcap_{H \in \mathcal{H}} \left[\left(\sup_{\mu \in \mathbb{R}^H_+} \left(f_\infty + \sum_{t \in H} \mu_t (f_t)_\infty\right)\right) \leq 0\right]$$

$$= \bigcap_{H \in \mathcal{H}} \left[\left(f_\infty + \sup_{\mu \in \mathbb{R}^H_+} \sum_{t \in H} \mu_t (f_t)_\infty\right) \leq 0\right]$$

$$= \bigcap_{H \in \mathcal{H}} \left[\left(f_\infty + \delta_{[\sup_{H \in \mathcal{H}} (f_t)_\infty \leq 0]}\right) \leq 0\right] = \bigcap_{H \in \mathcal{H}} \bigcap_{t \in H} [(f_t)_\infty \leq 0] \cap [f_\infty \leq 0]$$

the penultimate equality coming from the fact that the family $\mathcal{H}$ is covering. We conclude the proof of Theorem 3.1 with Lemmas 3.1 and 3.4.

Remark 3.1: Note that if $X = X^* = \mathbb{R}^n$ then the function $\varphi_\mathcal{H}$, when convex, is automatically $\tau(X^*, X)$-quasicontinuous since any extended real-valued convex function on $\mathbb{R}^n$ with non-empty domain is quasicontinuous (e.g., [21, Theorem 10.1]).

Corollary 3.1 ($\mathcal{H}_1$-reverse strong duality) Assume that (P) satisfies the following conditions:

(a) $\text{dom } f \subset \bigcap_{t \in T} \text{dom } f_t$,

(b) $T$ is a convex and compact subset of some locally convex topological vector space,

(c) $T \ni t \mapsto f_t(x)$ is concave and usc on $T$ for each $x \in \bigcap_{t \in T} \text{dom } f_t$,
(d) There exists \((\bar{t}, \bar{\mu}) \in T \times \mathbb{R}_+\) such that \(f + \bar{\mu}f_\bar{t}\) is weakly inf-locally compact,

(e) \((P)_\infty\) is a linear subspace.

Then,

\[
\min(P) = \sup_{(t, \mu) \in T \times \mathbb{R}_+} \inf_{x \in X} \{ f(x) + \mu f_t(x) \} \in \mathbb{R}.
\]

**Proof** From the first three assumptions and [4, Remark 5.5] we get that \(\varphi_{\mathcal{H}_1}\) is convex. Moreover, \(\varphi_{\mathcal{H}_1} = \inf_{(t, \mu) \in T \times \mathbb{R}_+} (f + \mu f_t)^*\) is majorized by the function \((f + \bar{\mu} f_\bar{t})^*\), which is \(\tau(X^*, X)\)-quasicontinuous by Lemma 3.2 as, by (d), \(f + \bar{\mu} f_\bar{t} \in \Gamma(X)\) is weakly inf-locally compact. So, by Lemma 3.3 \(\varphi_{\mathcal{H}_1}\) is \(\tau(X^*, X)\)-quasicontinuous, and we conclude the proof by applying Theorem 3.1 with \(\mathcal{H} = \mathcal{H}_1\) thanks to (e). □

The next result recovers a variant of the reverse duality theorem of [11, Theorem 3.3].

**Corollary 3.2 (\(\mathcal{F}(T)\)-reverse strong duality)** Assume that \(E \cap \text{dom } f \neq \emptyset\) and that the two following conditions are satisfied:

(f) \(\exists \lambda \in \mathbb{R}_+^{(T)}\) such that \(f + \sum_{t \in T} \lambda_t f_t\) is weakly inf-locally compact.

(e) \((P)_\infty\) is a linear subspace.

Then we have

\[
\min(P) = \sup(D) = \sup_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{x \in X} \left\{ f(x) + \sum_{t \in T} \lambda_t f_t(x) \right\} \in \mathbb{R}.
\]

**Proof** Condition (f) amounts to

\[\exists H \in \mathcal{F}(T), \exists \mu \in \mathbb{R}_+^H \text{ such that } f + \sum_{t \in H} \mu_t f_t \text{ weakly inf-locally compact.}\]

Moreover, \(\varphi_{\mathcal{F}(T)}\) is majorized by \((f + \sum_{t \in H} \mu_t f_t)^*\) which is \(\tau(X^*, X)\)-quasicontinuous by Lemma 3.2. By Lemma 3.3 \(\varphi_{\mathcal{F}(T)}\) is then \(\tau(X^*, X)\)-quasicontinuous. Taking \(\mathcal{H} = \mathcal{F}(T)\) in Theorem 3.1 we obtain, by (2.5) and (2.6),

\[
\min(P) = \sup(D_H) = \sup(D),
\]

and the proof is complete. □

We finally consider the countable case when \(T = \mathbb{N}\). Let \((P_N), (P_m), (D_N),\) and \((D_m)\) be as in (2.10), (2.11), (2.12), and (2.13), respectively.

**Corollary 3.3 (\(\mathcal{H}_N\)-reverse strong duality)** Assume \(\inf(P_N) \neq +\infty\) and the two conditions below are satisfied:

(g) \(\exists (N, \mu) \in \mathbb{N} \times \mathbb{R}_+^N\) such that \(f + \sum_{k=1}^N \mu_k f_k\) is weakly inf-locally compact,

(e) \((P)_\infty\) is a linear subspace.
Then we have
\[ \min(P_N) = \lim_{m \to \infty} \inf(P_m) = \lim_{m \to \infty} \sup(D_m) = \sup(D_N). \]

Moreover, the optimal solution set of \( (P_N) \) is the sum of a weakly compact convex set and a finite dimensional linear subspace.

**Proof** Since the covering family \( \mathcal{H}_N \) is directed we know that \( \varphi_{\mathcal{H}_N} \) is a convex function. Moreover, \( \varphi_{\mathcal{H}_N} \) is majorized by \( \left( f + \sum_{k=1}^{N} \mu_k f_k \right)^* \) which is \( \tau(X^*, X) \)-quasicontinuous by Lemma 3.2. By Lemma 3.3 \( \varphi_{\mathcal{H}_N} \) is then \( \tau(X^*, X) \)-quasicontinuous and, by [4, Formula (5.6)], \( \sup(D_N) = \lim_{m \to \infty} \sup(D_m) \). Applying Theorem 3.1 with \( \mathcal{H} = \mathcal{H}_N \) we obtain,
\[ \min(P_N) = \sup(D_N) = \sup_{m \in \mathbb{N}} \sup(D_m) = \lim_{m \to \infty} \sup(D_m) \leq \lim_{m \to \infty} \inf(P_m) \leq \min(P_N), \]
and the proof is complete. \( \square \)

**Remark 3.2** We now comment conditions (a) – (g) when \( X = \mathbb{R}^n \), that is, in CSIP. Conditions (d), (f), and (g) are obviously satisfied while condition (e) is equivalent [10, Exercise 8.15] to
\[ (h) \quad f_\infty(x) > 0, \forall x \in [(0^+ E) \cap M^\perp] \setminus \{0_n\}, \]
where \( M = \{ x \in \text{lin}(0^+ E) : f_\infty(x) = 0 = f_\infty(-x) \} \). So, Corollary 3.2 is, in the CSIP setting, equivalent to [10, Theorem 3.2] (see also [10, Theorem 8.8(ii)]). Analogously, [10, Corollary 4.2] is the CSIP version of Corollary 3.3.

If \( (P) \) is the LSIP problem in \( (1.4) \), we can write \( f(x) = \langle c^*, x \rangle \) and \( f_t(x) = \langle a^*_t, x \rangle - b_t, \ t \in T \). Then since all functions have full domain, (a) trivially holds. Moreover, since \( (P)_\infty = \bigcap_{t \in T} \{ a^*_t \leq 0 \} \cap \{ c^* \leq 0 \} \), condition (e) can be expressed as follows:
\[ (e') \quad \{ x \in X : \langle c^*, x \rangle \leq 0; \langle a^*_t, x \rangle \leq 0, \forall t \in T \} \]
\[ \{ x \in X : \langle c^*, x \rangle = 0 = \langle a^*_t, x \rangle, \forall t \in T \}. \]

Moreover, condition (e') can be reformulated in terms of the data as
\[ (e'') \quad \text{The pointed cone of } \text{cone} \{ \{ c^*; a^*_t, t \in T \} \times \mathbb{R}^+ \} \text{ (i.e., its intersection with the orthogonal subspace to its lineality) is a half-line in } \mathbb{R}^{n+1} \text{ [10, Theorem 5.13(ii)] (or, more precisely, the half-line } \mathbb{R}^+ (0_n, 1) \text{ [7, page 155]).} \]

In the same vein, since \( \text{dom } f = \mathbb{R}^n, f_\infty = \langle c^*, \cdot \rangle, 0^+ E = \bigcap_{t \in T} \{ a^*_t \leq 0 \} \), and
\[ M^\perp = \{ x : \langle c^*, x \rangle = 0 = \langle a^*_t, x \rangle, \forall t \in T \}^{\perp} = \text{span} \{ c^*; a^*_t, t \in T \}, \]
condition (h) can be expressed as
\[ (h') \quad \langle c^*, x \rangle > 0, \forall x \in \bigcap_{t \in T} \{ a^*_t \leq 0 \} \cap \text{span} \{ c^*; a^*_t, t \in T \} \setminus \{0_n\}. \]

**Example 3.1** Consider the linear semi-infinite programming problem
\[ (P) \quad \inf_{x \in \mathbb{R}^2} \quad f(x) = \langle c^*, x \rangle \]
\[ \text{s.t.} \quad -tx_1 + (t - 1)x_2 + t^2 \leq 0, \ t \in [0, 1], \]
with \( c^* \in \mathbb{R}_+^2 \setminus \{ (0, 0) \} \) (see [4, Example 3.1]). According to Remark 3.2, (a), (d), (f), and (g) hold independently of the data. Condition (b) holds because \([0, 1] \subset \mathbb{R}\) is compact and convex and (c) because \( t \mapsto -tx_1 + (t - 1)x_2 + t - t^2\) is concave on \(\mathbb{R}\) for any \(x \in \mathbb{R}^2\). Regarding (e), the set in \((e')\)

\[
\{ x \in \mathbb{R}^2 : \langle c', x \rangle \leq 0; -tx_1 + (t - 1)x_2 \leq 0, \forall t \in [0, 1] \} = \{ x \in \mathbb{R}_+^2 : \langle c', x \rangle \leq 0 \}
\]

is \(\{ (0, 0) \}\) when \(c^*\) belongs to the interior \(\mathbb{R}_+^2\) of \(\mathbb{R}_+^2\) and a positive axis when \(c^*\) belongs to its boundary. Hence, \((e)\) only holds for \(c^* \in \mathbb{R}_+^2\). Observe that the cone in \((e'')\) is

\[
\text{cone} \left\{ \begin{pmatrix} c_1^* \\ c_2^* \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\} \times \mathbb{R}_+,
\]

and its pointed cone is

\[
\mathbb{R}_+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{resp., cone} \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \text{cone} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}
\]

when \(c^* \in \mathbb{R}_+^2\) \((c^* \in \mathbb{R}_+^2 \setminus \{ 1, 1 \}, c^* \in \mathbb{R}_+^2 \setminus \{ 1, 0 \}, \) respectively). So, we get again that \((e)\) only holds for \(c^* \in \mathbb{R}_+^2\). Regarding condition (h), since \(\cap_{t \in [0, 1]} [a_t^* \leq 0] = \mathbb{R}_+\), and span \(\{ c^*; a_t^*, t \in T \} = \mathbb{R}^2\) if \(c^* \in \mathbb{R}_+^2\), so that \((h)\) holds while span \(\{ c^*; a_t^*, t \in T \}\) is a positive axis and \((h)\) fails, otherwise. Thus, \((e)\) and \((h)\) hold or not simultaneously.

In conclusion, by Corollary 3.1, \(H_1\)-reverse strong duality holds whenever \(c^* \in \mathbb{R}_+^2\) while, by Corollary 3.2, \(F(T)\)-reverse strong duality holds whenever \(c^* \in \mathbb{R}_+^2\). Observe that, from the direct computations carried out in [4, Example 3.1], \(H_1\)-reverse strong duality actually holds for all \(c^* \in \mathbb{R}_+^2 \setminus \{ (0, 0) \}\).

**Example 3.2** The countable linear semi-infinite programming problem

\[
(P_N) \quad \inf_{x \in \mathbb{R}^2} x_2 \\
\text{s.t.} \quad x_1 + k(k + 1)x_2 \geq 2k + 1, \quad k \in \mathbb{N},
\]

violates the assumptions of Corollaries 3.1, 3.2 and 3.3 as \((b)\) and \((c)\) obviously fail, as well as \((e)\) and \((h)\). In fact, \((e')\) and \((e'')\) fail because

\[
\{ x \in \mathbb{R}^2 : x^2 \leq 0, -x_1 - k(k + 1)x_2 \leq 0, k \in \mathbb{N} \} = \mathbb{R}_+ \times \{ 0 \}
\]

is not a linear subspace and the pointed cone of

\[
\text{cone} \{ (0, 1); (-1, -k(k + 1)), k \in \mathbb{N} \} \times \mathbb{R}_+ = \{ x \in \mathbb{R}^3 : x_1 \leq 0, x_3 \geq 0 \}
\]

is not a half-line, respectively, while \((h)\) fails because \(x_2\) vanishes on an edge of

\[
(0^+ E) \cap M^\perp = 0^+ E \cap \mathbb{R}^2 = \text{cone} \{ (2, 1), (1, 0) \}.
\]

So, we cannot apply the mentioned corollaries to conclude that \(H\)-reverse strong duality holds for \(H = H_1, H_2, F(T)\). Actually, \(H\)-reverse strong duality does not hold for these three families because the feasible set of \((P_N)\) is

\[
E = \text{co} \left\{ \left\{ \left( k, \frac{1}{k} \right), k \in \mathbb{N} \right\} \cup \{ x \in \mathbb{R}^2 : x_1 + 2x_2 = 3, x_1 \leq 1 \} \right\},
\]

which implies \(\inf(P_N) = 0\) with \(\text{sol}(P_N) = \emptyset\), while \(\sup(D) = -\infty\), which in turn implies \(\sup(D_H) = -\infty\) for any \(H\) such that \(\emptyset \neq H \subset F(T)\), by \(2.4\).
4 \textit{H}–Farkas lemma

We now establish some new versions of Farkas lemma relative to a given family \( H \subset \mathcal{F}(T) \). These results assert the equivalence between some inclusion (i) of the solution set \( E \) of \( \{ f_t(x) \leq 0, t \in T \} \) into certain set involving \( f \) and some condition (ii) involving \( \{ f; f_t, t \in T \} \) and \( H \). We first provide a Farkas-type result relative to the family \( H_1 \) without assuming the lower semicontinuity of the involved functions. Stronger results (characterizations of Farkas lemma) will be then obtained under the lower semicontinuity (or even continuity) assumption.

**Proposition 4.1 (\( H_1 \)-Farkas lemma)** Assume that conditions (a), (b), (c) in Corollary 3.1 altogether with the generalized Slater condition:

\[ \exists \bar{x} \in \text{dom } f : f_t(\bar{x}) < 0, \forall t \in T. \]

Then, for any \( \alpha \in \mathbb{R} \), the following statements are equivalent:

(i) \[ f_t(x) \leq 0, \forall t \in T \] \( \Rightarrow \) \( f(x) \geq \alpha \).

(ii) There exist \( \bar{t} \in T \) and \( \bar{\mu} \in \mathbb{R}_+ \) such that

\[ f(x) + \bar{\mu} f_{\bar{t}}(x) \geq \alpha, \forall x \in X. \] (4.1)

**Proof** We observe first that (i) is equivalent to \( \inf(P) \geq \alpha \), where \( \inf(P) = \max(D_{H_1}) \); i.e., (i) is equivalent to

\[ \max_{(t,\mu) \in T \times \mathbb{R}_+} \inf_{x \in \text{dom } f} \{ f(x) + \mu f_t(x) \} \geq \alpha. \]

In other words, there exists \( (\bar{t}, \bar{\mu}) \in T \times \mathbb{R} \) satisfying (4.1), which is (ii), and we are done. \( \square \)

Observe that statement (i) means that \( E \) is contained in the reverse convex set \( \{ x \in X : f(x) \geq \alpha \} \) while (ii) would be the same replacing the infinite family \( \{ f_t, t \in T \} \) by the singleton one \( \{ f_t \} \), so that Lemma 4.1 characterizes when an inequality \( f(x) \geq \alpha \) is consequence of some single constraint \( f_t(x) \leq 0 \).

The next two propositions provide, under the lower semicontinuity assumption, a characterization in terms of \( A_H \) (statement (I)) of the Farkas lemma (statement (II)) relative to an arbitrary non-empty covering family \( H \subset \mathcal{F}(T) \).

**Proposition 4.2 (Characterization of \( H \)-Farkas lemma)** Let \( H \subset \mathcal{F}(T) \) be a covering family. Assume that \( \{ f; f_t, t \in T \} \subset \Gamma(X), E \cap (\text{dom } f) \neq \emptyset \), and consider the following statements:

(I) \( A_H \) is \( w^* \)-closed convex regarding \( \{ 0_{X^*} \} \times \mathbb{R} \).

(II) For \( \alpha \in \mathbb{R} \), the next two conditions are equivalent:

(i) \[ f_t(x) \leq 0, \forall t \in T \] \( \Rightarrow \) \( f(x) \geq \alpha \),
(ii) there exist $H \in \mathcal{H}$ and $\mu \in \mathbb{R}^H_+$ such that
\[ f(x) + \sum_{t \in H} \mu_t f_t(x) \geq \alpha, \forall x \in X. \tag{4.2} \]

Then, \([(I) \implies (II)], and the converse implication, \([(II) \implies (I)], holds when \( \inf(P) \in \mathbb{R} \).

**Proof**  By the characterization of $\mathcal{H}$-strong duality at a point in (2.8), applied to $x^* = 0_{X^*}$, one gets that (I) is equivalent to
\[ \inf(P) = \max(D_H), \tag{4.3} \]
which is itself equivalent to the existence of $H \in \mathcal{H}$ and $\mu \in \mathbb{R}^H_+$ such that
\[ \inf(P) = \inf_{x \in X} \left( f(x) + \sum_{t \in H} \mu_t f_t(x) \right). \]

Since (i) is equivalent to $\inf(P) \geq \alpha$, it now follows that \([(I) \implies (II)].

Conversely, if $\inf(P) \in \mathbb{R}$ and (II) holds, then just take $\alpha = \inf(P)$. As (II) holds, it follows that there are $H \in \mathcal{H}$ and $\mu \in \mathbb{R}^H_+$ such that (4.2) holds, and
\[ \sup(D_H) \geq \inf_{x \in X} \left( f(x) + \sum_{t \in H} \mu_t f_t(x) \right) \geq \alpha = \inf(P). \]

In other words, $\sup(D_H) = \inf(P)$, $\sup(D_H)$ is attained at $H \in \mathcal{H}$ and $\mu \in \mathbb{R}^H_+$, meaning that (4.3) holds, which is (I), and the proof is complete. \(\square\)

**Remark 4.1**  In the special case when $\mathcal{H} = \mathcal{F}(T)$ the condition (ii) in Proposition 4.2 reads as
\[(ii') \text{ there exists } \lambda \in \mathbb{R}^T_+ \text{ such that } f(x) + \sum_{t \in T} \lambda_t f_t(x) \geq \alpha, \text{ for all } x \in X,\]
and Proposition 4.2 goes back to the Farkas lemma given in [2, Theorem 2] under a slightly different qualification condition. So, Proposition 4.2 is a variant of [2, Theorem 2].

Let us get back to the linear case, where
\[ f(x) = \langle c^*, x \rangle, \ f_t(x) = \langle a^*_t, x \rangle - b_t, t \in T, \tag{4.4} \]
with $\{c^*, a^*_t, t \in T\} \subset X^*$, and $\{b_t, t \in T\} \subset \mathbb{R}$. Then, $\mathcal{A}_H = \{(c^*, 0)\} + \mathcal{K}_H$ (see [4 (4.4)]), where
\[ \mathcal{K}_H = \bigcup_{H \in \mathcal{H}} \text{cone } \{(a^*_t, b_t), \ t \in H\} + \{0_{X^*}\} \times \mathbb{R}_+. \]
In particular,
\[ \mathcal{K}_{H_t} = \bigcup_{t \in T} \text{cone } \{(a^*_t, b_t + \varepsilon) : \varepsilon \geq 0\}. \]
and, by [1] Proposition 4.1,
\[ K_{F(T)} = \text{cone}\left(\{(a_t^*, b_t) : t \in T\} \times \mathbb{R}_+\right). \]

For instance, for the LSIP problem in Example 3.1,
\[ K_{H_1} = \bigcup_{t \in [0,1]} \text{cone}\left\{(-t, t - 1, t^2 - t + \varepsilon) : \varepsilon \geq 0\right\} \]
while \( K_{F(T)} \) is (see [1] Example 4.1) the union of the origin with the epigraph of the convex function
\[ \psi(x) := \begin{cases} \frac{x_1 x_2}{x_1 + x_2}, & x \in \mathbb{R}_2^2 \setminus \{0_2\}, \\ +\infty, & \text{else}. \end{cases} \]

We finish this section with a characterization, in terms of \( K_{H} \), of the Farkas lemma (statement (II) below) relative to an arbitrary non-empty covering family \( H \subset F(T) \).

**Proposition 4.3 (\( H \)-Farkas lemma for linear infinite systems)** Consider the linear functions \( \{f_t : t \in T\} \) defined in (4.4), and suppose that \( \inf(P) \) is finite and that \( H \) is a covering family. Given \( c^* \in X^* \), the following statements are equivalent:

(I) \( \text{co}(K_H) \cap \{-c^* \times \mathbb{R}_+\} = K_H \cap \{-c^* \times \mathbb{R}_+\}, \)

(II) For \( \alpha \in \mathbb{R} \), the following statements are equivalent:

(i) \( \langle a_t^*, x \rangle \leq b_t, \forall t \in T \implies \langle c^*, x \rangle \geq \alpha. \)

(ii) There exist \( H \in H \) and \( \mu \in \mathbb{R}^H_+ \) such that \( \sum_{t \in H} \mu_t a_t^* = -c^* \) and \( -\sum_{t \in H} \mu_t b_t \geq \alpha. \)

**Proof.** When \( H \) is a covering family and \( E \neq \emptyset \), according to [1] Corollary 5.3], one has
\[ \left( \inf(P) = \max(D_H) \right) \iff \left( \text{co}(K_H) \cap \{-c^* \times \mathbb{R}_+\} = K_H \cap \{-c^* \times \mathbb{R}_+\} \right). \quad (4.5) \]
The rest of the proof is similar to that of Proposition 4.2 using (2.8) and (4.5). □

### 5 \( H \)-optimality conditions

In this section we establish optimality conditions for the problem (P) associated with some family \( H \subset F(T) \). We shall represent by \( \text{sol}(D_H) \) the set of optimal solutions of (D_H). In particular, when \( H = F(T) \), one obtains the classical KKT conditions involving finitely many multipliers and, when \( H = H_1 \), optimality conditions involving a unique multiplier.

**Theorem 5.1 (Primal-dual \( H \)-optimality condition)** Let \( \bar{x} \in E \cap (\text{dom } f) \), \( H \in H \) and \( \mu \in \mathbb{R}^H_+ \). Then, the following statements are equivalent:

(i) \( \bar{x} \in \text{sol}(P) \), \( (H, \mu) \in \text{sol}(D_H) \), and \( \inf(P) = \sup(D_H) \).

(ii) \( f(\bar{x}) = \inf_X \left( f + \sum_{t \in H} \mu_t f_t \right) \), and \( \mu_t f_t(\bar{x}) = 0 \), for all \( t \in H \).

(iii) \( 0_{X^*} \in \partial \left( f + \sum_{t \in H} \mu_t f_t \right)(\bar{x}) \), and \( \mu_t f_t(\bar{x}) = 0 \), for all \( t \in H \).
Proof. [(i) ⇒ (ii)] We have
\[ \inf_X \left( f + \sum_{t \in H} \mu_t f_t \right) = \sup(D_H) = \inf(P) = f(\bar{x}), \]
and
\[ f(\bar{x}) = \inf_X \left( f + \sum_{t \in H} \mu_t f_t \right) \leq f(\bar{x}) + \sum_{t \in H} \mu_t f_t(\bar{x}) \leq f(\bar{x}). \]
Hence, \( \sum_{t \in H} \mu_t f_t(\bar{x}) = 0 \) and (ii) holds.

[(ii) ⇒ (iii)] We have
\[ \left( f + \sum_{t \in H} \mu_t f_t \right)(\bar{x}) = f(\bar{x}) = \inf_X \left( f + \sum_{t \in H} \mu_t f_t \right). \]
Thus, \( \bar{x} \in \arg\min \left( f + \sum_{t \in H} \mu_t f_t \right) \) or, equivalently, \( 0_{X^*} \in \partial \left( f + \sum_{t \in H} \mu_t f_t \right)(\bar{x}) \).

[(iii) ⇒ (i)] Now we write
\[ \inf(P) \leq f(\bar{x}) = \left( f + \sum_{t \in H} \mu_t f_t \right)(\bar{x}) = \inf_X \left( f + \sum_{t \in H} \mu_t f_t \right) \leq \sup(D_H) \leq \inf(P), \]
and (i) holds. \( \square \)

Corollary 5.1 (1st \( \mathcal{H} \)-optimality condition for \( P \)) Assume that \( \inf(P) = \max(D_H) \) and let \( \bar{x} \in E \cap (\text{dom } f) \). Then, the following statements are equivalent:
(i) \( \bar{x} \in \text{sol}(P) \).
(ii) For each \( (H, \mu) \in \text{sol}(D_H) \), we have
\[ 0_{X^*} \in \partial \left( f + \sum_{t \in H} \mu_t f_t \right)(\bar{x}), \text{ and } \mu_t f_t(\bar{x}) = 0, \forall t \in H. \] (5.1)
(iii) There exists \( (H, \mu) \in \text{sol}(D_H) \) such that (5.1) is fulfilled.

Proof. [(i) ⇒ (ii)] is just [(i) ⇒ (iii)] in Theorem 5.1
[(ii) ⇒ (iii)] is due to the assumption \( \text{sol}(D_H) \neq \emptyset \).
[(iii) ⇒ (i)] follows from [(iii) ⇒ (i)] in Theorem 5.1. \( \square \)

Corollary 5.2 (2nd \( \mathcal{H} \)-optimality condition for \( P \)) Let \( \mathcal{H} \subset \mathcal{F}(T) \) be a covering family. Assume that \( \{f; f_t, t \in T\} \subset \Gamma(X) \) and \( E \cap (\text{dom } f) \neq \emptyset \). Assume further that \( \mathcal{A}_H \) is \( w^* \)-closed convex regarding \( \{0_{X^*}\} \times \mathbb{R} \). Then \( \bar{x} \in \text{sol}(P) \) if and only if there exist \( H \in \mathcal{H} \) and \( \mu \in \mathbb{R}^H_+ \) such that (5.1) holds.

Proof. Taking \( x^* = 0_{X^*} \) in (2.8) one has \( \inf(P) = \max(D_H) \). Corollary 5.1 concludes the proof. \( \square \)
Remark 5.1 When $\mathcal{H} = \mathcal{F}(T)$, the conclusion of Corollary 5.3 is that $\bar{x} \in \text{sol}(P)$ if and only if there exist $\lambda \in \mathbb{R}_+^T$ such that

$$0_{X^*} \in \partial \left( f + \sum_{t \in T} \lambda_t f_t \right) (\bar{x}) \text{ and } \lambda_t f_t(\bar{x}) = 0, \forall t \in T,$$

which recalls us about the optimality condition given in [2, Theorem 3] under the assumptions that both the sets $K_{\mathcal{F}(T)}$ and $\text{epi} f^* + \overline{\mathcal{K}_{\mathcal{F}(T)}}$ are $w^*$-closed.

Corollary 5.3 ($\mathcal{H}$–optimality condition for linear (P)) Let $(P)$ be linear with $E \neq \emptyset$. Let $\mathcal{H}$ be a covering family. Assume that $K_{\mathcal{H}}$ is weak*-closed convex regarding $\{-c^*\} \times \mathbb{R}$. Then $\bar{x} \in \text{sol}(P)$ if and only if there exist $H \subset \mathcal{H}$ and $\mu \in \mathbb{R}_+^H$ such that

$$\sum_{t \in H} \mu_t a^*_t = -c^* \text{ and } \sum_{t \in H} \mu_t b_t = -\langle c^*, \bar{x} \rangle. \quad (5.2)$$

Proof. By [3, Corollary 5.3] one has $\inf (P) = \max(D_{\mathcal{H}})$. In the linear case one has (5.1) $\iff$ (5.2). We conclude the proof with Corollary 5.1.\hfill \square

Corollary 5.4 (Optimality condition for $(D_{\mathcal{H}})$) Assume that $\min(P) = \sup(D_{\mathcal{H}}) \neq +\infty$, and let $H \subset \mathcal{H}$ and $\mu \in \mathbb{R}_+^H$. Then, the following statements are equivalent:

(i) $(H, \mu) \in \text{sol}(D_{\mathcal{H}})$.

(ii) For each $\bar{x} \in \text{sol}(P)$, (5.1) holds.

(iii) There exists $\bar{x} \in \text{sol}(P)$ such that (5.1) is fulfilled.

Proof. [(i) $\Rightarrow$ (ii)] follows from [(i) $\Rightarrow$ (iii)] in Theorem 5.1.

[(ii) $\Rightarrow$ (iii)] is due to the assumption $\text{sol}(P) \neq \emptyset$.

[(iii) $\Rightarrow$ (i)] follows from [(iii) $\Rightarrow$ (i)] in Theorem 5.1.\hfill \square

We finish by revisiting again Example 3.1, with $\mathcal{H} = \mathcal{H}_1$. For $c^* \in \mathbb{R}_+^2$, let us check the fulfilment of (5.2) at $\bar{x} = \left( \left( \frac{c_1^*}{c_1^* + c_2^*} \right)^2, \left( \frac{c_2^*}{c_1^* + c_2^*} \right)^2 \right)$. Taking $H = \{ \overline{t} \}$, with $\overline{t} = \frac{c_1^*}{c_1^* + c_2^*} \in ]0, 1[,$ and $\mu \in \mathbb{R}_+^{(0,1)}$ such that $\mu_t = c_1^* + c_2^* > 0$ and $\mu_{\overline{t}} = 0$ for all $t \in [0, 1] \setminus \{ \overline{t} \}$, one has

$$\sum_{t \in H} \mu_t a^*_t = (c_1^* + c_2^*) \left( -\frac{c_1^*}{c_1^* + c_2^*}, -\frac{c_2^*}{c_1^* + c_2^*} \right) = -c^*$$

and

$$\sum_{t \in H} \mu_t b_t = (c_1^* + c_2^*) \left( \left( \frac{c_1^*}{c_1^* + c_2^*} \right)^2 - \frac{c_1^*}{c_1^* + c_2^*} \right) = -\frac{c_1^* c_2^*}{c_1^* + c_2^*} = -\langle c^*, \bar{x} \rangle,$$

so that $\bar{x} \in \text{sol}(P)$ (recall that $K_{\mathcal{H}_1}$ is closed). Moreover, $(H, \mu) \in \text{sol}(D_{\mathcal{H}})$ by Corollary 5.4 as

$$\partial \left( c^* + \sum_{t \in H} \mu_t a^*_t \right) = \left\{ c^* + (c_1^* + c_2^*) \left( -\frac{c_1^*}{c_1^* + c_2^*}, -\frac{c_2^*}{c_1^* + c_2^*} \right) \right\} = \{(0, 0)\}$$. \hfill \square
and the complementarity condition $\mu_t f_t(\bar{x}) = 0$, for all $t \in T$, holds.

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