The Dirac Field Theory,
some No-Go Theorem

Luca Fabbri
DIPTEM Sez. Metodi e Modelli Matematici dell’Università di Genova &
INFN Sez. di Bologna and Dipartimento di Fisica dell’Università di Bologna

Abstract
We study Dirac field equations generally coupled to electrodynamics with gravity and torsion: we show the incompatibility of the solutions with special symmetries therefore proving in particular that special solutions do not exist; then we address the problem of renormalizability.

Introduction
That nature be successfully represented by theories of such seemingly poor content like those we have at our disposal is a fact that is surprising and amazing at the same time. Yet the universe is apparently described in terms of fermions and bosons alone: the former as spinors with spin-$\frac{1}{2}$ structure only; the latter as tensors with spin having integer values, but still spin-1 and spin-2 structures solely and nothing more. On the other hand, this seemingly poor content is not that easy to study after all if in the entire lifetime of these theories no complete exact solution has been found yet. Clearly, we know solutions for parts of the model; we know solutions for the Dirac field, we know solutions for the Maxwell and for the Einstein-Sciama-Kibble models separately. However if this model is considered in its entirety having matter with electrodynamics plus gravity and torsion all in interaction, then no solution is known. Lacking such comprehensive analytic solution, to describe the exact behaviour of a complete system of fields a strategy is to look for properties of the solutions. In these cases, the techniques used when generality is dropped rely upon the fact that we only want to know special properties of the solution: then one does not need to study them wholly and simplifications may be performed. In doing so however, we have loss of information. In employing these techniques then, a logical consequence is that the positive results are limited by the amount of information that has been lost in the simplification, but negative results still maintains full power even in restricted instances, and so the aim is to look for no-go theorems. In this paper we will study the Dirac field theory with general coupling to electrodynamics in gravitational frameworks finding results in the form of some no-go theorem.

1 Einstein gravity, non-singular dynamics
In this paper, we shall follow [1] for the notations and conventions and for the introduction to the formalism employed: accordingly after that in the most
general system of field equations torsion is isolated and the field equation for the
torsion-spin coupling are inserted back into all other field equations we remain
with the following set of field equations for the geometry

\[ R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} F^2 - \frac{1}{2} g^{\rho\sigma} F_{\rho\sigma} F_{\mu\nu} - \lambda g_{\mu\nu} - \frac{1}{4} m \psi \bar{\psi} g_{\mu\nu} + \]

\[ + \frac{1}{8} (\bar{\psi} \gamma_{\mu} \nabla_{\nu} \psi + \bar{\psi} \gamma_{\nu} \nabla_{\mu} \psi - \nabla_{\nu} \bar{\psi} \gamma_{\mu} \psi - \nabla_{\mu} \bar{\psi} \gamma_{\nu} \psi) \]

\[ \nabla_{\sigma} F^{\sigma\rho} = q (\bar{\psi} \gamma^\alpha \psi) \] (1)

and for the matter field equations

\[ i \gamma^\mu \nabla_{\mu} \psi - \frac{3}{2} \left( \bar{\psi} \gamma_\mu \psi \right) \gamma^\mu \psi - m \psi \equiv \]

\[ \equiv i \gamma^\mu \nabla_{\mu} \psi - \frac{3}{2} \left[ i \left( \bar{\psi} \gamma^\psi \right) \gamma + (\bar{\psi} \psi) I \right] \psi - m \psi = 0 \] (3)

with \( \lambda \) the cosmological constant, \( q \) the charge and \( m \) the mass of the matter
field, and in which for the curvature-energy and gauge-current coupling the
field equations are formally equal to the field equations we have in the torsion-
less case, whereas for the matter field the field equations are equivalent to the
field equations in the torsionless case with non-linear potentials given by self-
interactions of the fields with themselves as \( \bar{\psi} \gamma^\psi \) and \( \bar{\psi} \psi \); in the following we
are going to assume that either \( i \bar{\psi} \gamma^\psi \neq 0 \) or \( \bar{\psi} \psi \neq 0 \) in any instance because if
otherwise we allow the condition \( i \bar{\psi} \gamma^\psi = \bar{\psi} \psi = 0 \) to hold then this would mean
that there is no effect of torsion, whether torsion is present or vanishing itself.

For the system of Dirac field equations the solutions are expected to display
localization, and accordingly stable compact solutions must find place in static
spherically symmetric spacetimes: in the frame at rest with respect to the origin

\[ A \]

of coordinates \( t, r, \theta, \varphi \) the metric is given in terms of two functions of the radial
coordinate \( A(r) \) and \( B(r) \) in the following form

\[ g_{tt} = A^2 \quad g_{rr} = -B^2 \quad g_{\theta\theta} = -r^2 \quad g_{\varphi\varphi} = -r^2 (\sin \theta)^2 \] (4)

while the tetrads are given by

\[ e^0_i = A \quad e^1_i = B \quad e^2_i = r \quad e^3_i = r \sin \theta \] (5)

where \( g_{\mu\nu} = e^a_{\mu} g^{a\nu} \eta_{ab} \) in terms of the Minkowskian matrix for which

\[ \{ \gamma_a, \gamma_b \} = 2 \eta_{ab} \] (6)

define the gamma matrices; the Levi-Civita connection is

\[ \Lambda^a_{\mu\nu} = \frac{\partial}{\partial x^\mu} \quad \Lambda^a_{\nu\mu} = \frac{\partial}{\partial x^\nu} \quad \Lambda^a_{\lambda\varphi} = -\frac{1}{2} \cot \theta (\sin \theta)^2 \quad \Lambda^a_{\varphi\varphi} = -\frac{1}{2} \sin \theta (\sin \theta)^2 \] (7)

while the spin connection is

\[ \omega^a_{\mu\nu} = \frac{\partial}{\partial x^\nu} \quad \omega^a_{2\theta} = -\frac{1}{2} \sin \theta \quad \omega^a_{3\varphi} = -\frac{1}{2} \sin \theta \quad \omega^a_{3\varphi} = -\cos \theta \] (8)

with \( \omega^a_{\mu\nu} = \partial_{\nu} e^a_{\mu} - \partial_{\mu} e^a_{\nu} + e^c_{\mu} e^a_{\nu} \Lambda^c_{\nu\mu} \) and for which the spinorial connection is

\[ \Omega_\alpha = \frac{\partial}{\partial x^\nu} \gamma_{\alpha \nu} \quad \Omega_\varphi = 0 \quad \Omega_\theta = \frac{1}{2 \sin \theta} \gamma_{1 \gamma_2} \quad \Omega_{\varphi} = \left( \frac{1}{2 \sin \theta} \gamma_{1 \gamma_2} + \frac{1}{2} \cot \theta \gamma_2 \right) \sin \theta \gamma_3 \] (9)
known as the Fock-Ivanenko coefficients. With the same symmetry consideration it is possible to see that the electrodynamic potential has to produce an electrostatic radial configuration for which \( A_t = \phi \) with \( \phi = \phi(r) \) as usual.

Now in the literature, in references [2, 3, 4, 5] Dirac fields in such a background have been considered while in papers [6, 7] Dirac fields with gauge potentials in this background have been studied, in [8] Dirac fields with electrodynamics near black holes whose exterior is given by this background have been discussed in terms of no-go theorems, while in the present paper we consider Dirac fields with electrodynamics and gravitation with torsion in this background given as

\[
i (m + \frac{3}{16} \overline{\psi} \psi) \left( \sqrt{Ar^2 \sin \theta} \overline{\psi} \right) - \gamma \left( \frac{3}{16} i \overline{\psi} \gamma \psi \right) \left( \sqrt{Ar^2 \sin \theta} \overline{\psi} \right) + \\
+ \gamma_0 \frac{1}{A} (i \gamma \phi + \frac{\alpha}{A}) \left( \sqrt{Ar^2 \sin \theta} \overline{\psi} \right) - \gamma_1 \frac{1}{A} \frac{\alpha}{A} \left( \sqrt{Ar^2 \sin \theta} \overline{\psi} \right) - \\
- \gamma_2 \frac{1}{A} \frac{\partial}{\partial r} \left( \sqrt{Ar^2 \sin \theta} \overline{\psi} \right) - \gamma_3 \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \left( \sqrt{Ar^2 \sin \theta} \overline{\psi} \right) = 0 \quad (10)
\]

and these are the field equations we are going to employ in the following study.

To this extent, we work in the chiral representation where the field \( \psi \) is taken in the following form

\[
\psi_{\pm} = \left( \begin{array}{c}
\rho e^{i \frac{\alpha}{2}} \\
\pm \rho e^{i \frac{\beta}{2}} \\
\eta e^{i \frac{\alpha}{2}} \\
\pm \eta e^{i \frac{\beta}{2}}
\end{array} \right) \quad (11)
\]

because this is the form for which the corresponding bilinear fields

\[
\overline{\psi}_{\pm} \gamma^0 \psi_{\pm} = \pm \overline{\psi}_{\pm} \gamma^1 \gamma \psi_{\pm} = 2 (\eta^2 + \rho^2) \quad (12)
\]

\[
\overline{\psi}_{\pm} \gamma^1 \psi_{\pm} = \pm \overline{\psi}_{\pm} \gamma^0 \gamma \psi_{\pm} = \pm 2 (\eta^2 - \rho^2) \quad (13)
\]

\[
i \overline{\psi} \gamma \psi = 4 \eta \rho \sin \left( \frac{\alpha - \beta}{2} \right) \quad (14)
\]

\[
\overline{\psi} \psi = 4 \eta \rho \cos \left( \frac{\alpha - \beta}{2} \right) \quad (15)
\]

are the most general to be compatible with the rotational invariance: from these it is also possible to construct additional scalar and pseudo-scalar bilinear fields

\[
\overline{\psi} \gamma^0 \psi \gamma^a \psi = - \overline{\psi} \gamma^a \psi \gamma^0 \psi = = \left( \overline{\psi} \psi \right)^2 + \left( i \overline{\psi} \gamma \psi \right)^2 = 16 \eta^2 \rho^2 \quad (16)
\]

\[
\left( \frac{i \overline{\psi} \gamma \psi}{\overline{\psi} \psi} \right) = \tan \left( \frac{\alpha - \beta}{2} \right) \quad (17)
\]

which being scalars depend on the radial coordinate only, and because we have that the angular and radial function have been separated we know that the product \( \eta \rho \) and the difference \( \alpha - \beta \) depend on the radial coordinate alone.

Working in the chiral representation we have that we can write the previous field equations given by (10) in terms of the chiral representation of the field solutions given by (11) and after this substitution, it is possible to employ the
identities $i\gamma_2\gamma_3\psi_\pm = \pm\psi_\pm$ and $i\gamma_1\gamma_2\psi_\pm = \psi_\mp$ to write the field equations
\[
\left( m + \frac{3}{4}\eta\rho \cos \left( \frac{\alpha - \beta}{2} \right) \right) (\gamma_3\psi_\pm) - \gamma i \left( \frac{3}{4}\eta\rho \sin \left( \frac{\alpha - \beta}{2} \right) \right) (\gamma_3\psi_\pm) - \gamma i \frac{\sqrt{A_2}}{\sqrt{R^2}} \frac{\partial}{\partial r} \left( \sqrt{A_2} \psi_\pm \right) = 0
\]
from which upon comparison of its components it is possible to separate
\[
\left( m + \frac{3}{4}\eta\rho \cos \left( \frac{\alpha - \beta}{2} \right) \right) (\gamma_3\psi_\pm) - \gamma i \left( \frac{3}{4}\eta\rho \sin \left( \frac{\alpha - \beta}{2} \right) \right) (\gamma_3\psi_\pm) - \gamma i \frac{\sqrt{A_2}}{\sqrt{R^2}} \frac{\partial}{\partial r} \left( \sqrt{A_2} \psi_\pm \right) = 0 \tag{18}
\]
from (16-17) we know that $\eta\rho$ and $\alpha - \beta$ do not depend on any angle then the previous constraints (25-26) furnish in particular
\[
\frac{\partial}{\partial \theta} \ln \left( \frac{\pi}{\rho} \right) = 0 \tag{23}
\]
\[
\frac{\partial}{\partial \theta} \ln \left( \frac{\pi}{\rho} \right) = 0 \tag{24}
\]
which have to hold for any possible value of the angular coordinates.

Because from (16-17) we know that $\eta\rho$ and $\alpha - \beta$ do not depend on any angle then the previous constraints (25-26) furnish in particular
\[
\frac{\partial}{\partial \theta} \ln \left( \frac{\pi}{\rho} \right) = 0 \tag{27}
\]
\[
2 \cos \theta \frac{\partial}{\partial \theta} (\alpha + \beta) = 0 \tag{28}
\]
so that the derivative of (27) with respect to the azimuthal angle and the derivative of (28) with respect to the elevation angle can be compared giving
\[
\sin \theta = 0 \tag{29}
\]
which can not be valid in general, and therefore we have gotten a contradiction. In this derivation, we have used no other hypothesis than assuming the conditions $\eta \neq 0$ and $\rho \neq 0$ because if otherwise $\eta = 0$ or $\rho = 0$ then we would have had for the scalars $i\psi\gamma_\psi = \psi\psi = 0$ against our initial assumption of having
torsional effects present in the Dirac field equation, and therefore the intuitive idea that for the Dirac field equation with torsion no solution can fit into a static spherically symmetric spacetime has here been proven rigorously.

We also remark that these results have been obtained in the case of the torsional interactions present for Dirac field equations of the least-order derivative, which is the simplest but not the only scheme that can be followed, and in fact, the inclusion of torsion in Dirac field equations of higher-order derivative can be achieved, for instance by postulating that the gravitational action is not the simplest Ricci scalar but a general function $F$ of the Ricci scalar, as it has recently been discussed in [9]: in this reference it has been shown that the matter field equations are

\[
i\gamma^\mu \nabla_\mu \psi - \frac{3}{16} F' \left[ i \left( \bar{\psi} \gamma \psi \right) \gamma + \left( \bar{\psi} \psi \right) I \right] \psi - m \psi = 0
\]

(30)

with $m$ the mass of the matter field, equivalent to the field equations in the torsionless case obtained before with non-linear potentials given by self-interactions of the fields with themselves as $i\bar{\psi} \gamma \psi$ and $\bar{\psi} \psi$ with running coupling given by the energy-dependent scale factor $F'$ at denominator; so we are going to assume that either $i\bar{\psi} \gamma \psi \neq 0$ or $\bar{\psi} \psi \neq 0$ and $F' \neq 1$ in any case. With these assumptions, we have again the result for which for the Dirac field equation with torsion no solution can fit into a static spherically symmetric spacetime whatever.

This is a no-go theorem stating that Dirac matter field equations in the most general coupling are not compatible with rotationally invariant backgrounds.

Moreover, we know that a particular case of static spherically symmetric solution is the rotationally invariant Dirac delta distribution $\delta(r)$ and therefore, a particular application of this theorem is that the Dirac field equations with torsion have no Dirac delta distribution among the solutions at all.

And this is a no-go corollary stating that Dirac matter field equations in the most general coupling admit no Dirac delta distributions as possible solutions.

Thus we have proven that, for Dirac field equations with torsion, static spherically symmetric spacetimes are not allowed and in particular Dirac delta distributions are not solutions: this fact has a natural interpretation when we consider that for Einstein geometrical equations the torsion field is due to the spin density of the massive field, whose helicity structure induces a twist of the spacetime incompatible with the rotational invariance of the background and in particular the spinorial self-interactions produce centrifugal barriers preventing the fields to collapse onto singular point distributions.

The fact that for the spinorial massive field equations with torsion, spinorial massive solutions are not point particles is also discussed using the Papapetrou method by Popławski in [10].

2 Einstein and Weyl geometries, non-point solutions and conformal kinematics

So far we have seen that, for the Dirac matter field equations with torsional contributions, static spherically symmetric spacetimes arising from the full rotational invariance of the background are not permitted and in particular material fields are not singular delta distributions representing point particles, this
latter point discussed also in reference \[10\]: these results imply that stationary cylindrically symmetric spacetimes arising from axial rotational invariance are necessary and in particular material fields are described by regular functions representing extended objects, as discussed in references \[11\] and \[12\], and where this last issue is discussed in detail especially in reference \[12\]; further analyses on the subject have been performed in reference \[13\]. These considerations have implications for the problem of non-renormalizability in quantum field theory.

In fact, the non-renormalizability of point-like particles at high energies disappears if extended objects are considered at small scales: non-renormalizable interactions between fields as zero-range forces between point-like particles at high energies disappear if contact forces between extended objects act at short distances; the range of the contact forces between extended objects is the size of the extended object, while short range means compact objects.

Further we notice that these results have been obtained in the case of Dirac matter field equations in the framework in which the gravitational action was a function of the Ricci scalar, which is a special but not the only path that can be pursued, and in fact, the construction of Dirac massless field equations in a framework of another special type can be accomplished, for instance by postulating that the gravitational action is conformally invariant, as it has lately been discussed in \[14\]: in this reference it has been shown that the massless field equations are written in the usual form

\[
\hat{i}\gamma^\mu \nabla_\mu \psi - \frac{1}{2} Q^{\alpha\sigma\nu} \varepsilon_{\alpha\sigma\nu\rho} \gamma^\rho \psi = 0
\]

because the completely antisymmetric part of torsion cannot be further substituted with the spin density, since neither this nor any other irreducible decomposition of torsion is explicitly given by the spin density in the same way in which there is no irreducible decomposition of the curvature that is given by the energy density while instead both torsion and curvature are coupled to both spin and energy densities; thus we have that in general for gravity in conformally-invariant schemes particles may still be point-like. However due to the property of the background to be scale-invariant we have that all the problems the fields had concerning non-renormalizability do not appear in the first place.

Indeed, at a fundamental level the renormalizability properties are essentially linked to the conformal properties of any field theory as a simple analysis based on dimensional arguments can demonstrate straightforwardly.

We would like to conclude by summarizing what we have seen about the Dirac field equations, which have been considered in two types of gravitational dynamics described in terms of either the Einstein or the Weyl curvature: in the first circumstance, torsion turned into non-linear self-interactions of the fields with themselves and massive states were possible, in the second circumstance, torsion remained an independent degree of freedom but still linearly interacting with fields in massless configuration: in the former situation, the field solutions had to be extended objects, therefore renormalizable as extended objects have finite size; in the latter situation, there could be point particles, but still renormalization due to their conformal properties. So we have that for the Dirac fields all problems about non-renormalizability are not present, although in the two gravitational models we have discussed this property is due to two different reasons: on the one hand non-renormalization is circumvented, because of the finite size of the solutions; on the other hand renormalization is ensured,
because of the conformal properties of the model. Dirac field equations seems renormalizable no matter what gravitational model we postulate.

As a last remark, we would like to mention that an extensive discussion about renormalization in the case of extended objects and for conformal field theories can be found respectively in [15] and in [16].

Conclusion

In the present paper, we studied Dirac matter field equations in the most general coupling: we have shown that for massive states, stationary cylindrically symmetric spacetimes arising from the axial rotational invariance of the background are necessary and in particular the material fields are described by regular functions representing extended objects; in massless configurations, static spherically symmetric spacetimes arising from the full rotational invariance of the background are allowed and in particular the fields may be described by delta distributions representing point particles. In both cases, problems about non-renormalizability of specific models do not appear.

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