Helical order in one-dimensional magnetic atom chains and possible emergence of Majorana bound states

Younghyun Kim,1 Meng Cheng,2 Bela Bauer,2 Roman M. Lutchyn,2 and S. Das Sarma3

1Physics Department, University of California, Santa Barbara, CA 93106, USA
2Station Q, Microsoft Research, Santa Barbara, CA 93106-6105, USA
3Condensed Matter Theory Center and Joint Quantum Institute, Department of Physics, University of Maryland, College Park, MD 20742, USA

We theoretically obtain the phase diagram of localized magnetic impurity spins arranged in a one-dimensional chain on top of a one- or two-dimensional electron gas with Rashba spin-orbit coupling. The interactions between the spins are mediated by the Ruderman-Kittel-Kasuya-Yosida (RKKY) mechanism through the electron gas. Recent work predicts that such a system may intrinsically support topological superconductivity when a helical spin-density wave is formed in the spins, and superconductivity is induced in the electron gas. We analyze, using both analytical and numerical techniques, the conditions under which such a helical spin state is stable in a realistic situation in the presence of disorder. We show that it becomes unstable towards the formation of (anti)ferromagnetic domains if the disorder in the impurity spin positions δR becomes comparable with the Fermi wave length. We also examine the stability of the helical state against Gaussian potential disorder in the electronic system using a diagrammatic approach. Our results suggest that in order to stabilize the helical spin state, and thus the emergent topological superconductivity, a sufficiently strong Rashba spin-orbit coupling, giving rise to Dzyaloshinskii-Moriya interactions, is required.

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Magnetism originating from interactions between magnetic atoms mediated by delocalized electrons (the so-called RKKY interaction) represents an important problem in modern condensed matter physics [1] and has been a subject of intense research [2–4]. In this Letter, we consider the specific case of a helical spin density wave (SDW) that might appear in a one-dimensional chain of magnetic impurities that are coupled to a metal or a superconductor. The issue of RKKY-induced magnetism has recently taken on a new and unexpected interesting perspective in the physics of non-Abelian Majorana bound states (Majoranas) [5, 6], with the recent claims of the natural (i.e., self-tuned) emergence of Majorana modes in a chain of Yu-Shiba-Rusinov [7–9] states induced by magnetic atoms at the surface of a superconductor, see Fig. 1. A Majorana-carrying topological superconducting phase should emerge in this system without the tuning of any external parameters due to the existence of an RKKY-stabilized helical order in conjunction with s-wave superconductivity [10–13]. If correct, this is a breakthrough in the prospective realization of non-Abelian topological phases of matter, and hence of great importance. A helical spin texture is a crucial ingredient also in most other proposals for topological superconductivity [14–18].

In the recent Majorana proposals, the presence of the helical order was either assumed a priori [10, 19–24] or shown to exist in rather limited situations [11–13]. In this Letter, we revisit the claims of the emergent self-tuned topological superconductivity in magnetic chains in realistic experimental conditions. Specifically, we address the question whether the helical SDW in the perfectly ordered chain survives in the presence of disorder invariably present in physical systems. We consider two types of disorder: the positional disorder of the magnetic atoms and potential impurity scattering in the substrate. We find that the existence of the SDW necessary for creating Majoranas in the chain becomes severely constrained by disorder, and in fact, the SDW (and therefore, the topological superconductivity) is unlikely to emerge unless a strong spin-orbit (SO) interaction is present in the system.

Our main results are the following: In the case of an impurity chain coupled to a 1D conductor, we show that the helical SDW emerging due to the well-known $2k_F$ instability is stable provided that the fluctuations of the impurity positions are smaller than the Fermi wavelength of the underlying metallic substrate, which is an important constraint to satisfy since this means that the impurity atoms must form a periodic chain to better than a few angstroms precision. In the opposite regime, where the positional disorder becomes comparable to the Fermi wavelength, the impurity spins form ferro- or antiferromagnetic domains. In the case where the impurity chain is coupled to a 2D conductor (2DEG), see Fig. 1, the effective range of the RKKY interaction becomes smaller and the helical SDW does not form spontaneously. Nevertheless, it is possible to stabilize the helical SDW if the 2D conductor breaks...
inversion symmetry, allowing Dzyaloshinskii-Moriya (D-M) interactions between the impurity spins. The latter favors a helical SDW with a pitch angle that depends on the relative strength of RKKY and D-M interactions. In order to study the robustness of the helical SDW in the 2D setup we consider a minimal model that generates D-M interaction, a 2D electron gas with Rashba SO coupling [25]. We study the magnetic ordering in such a system in the presence of potential impurities in the 2DEG, and show that the helical SDW is stable in such a system provided the interatomic distance between impurity spins is smaller than the effective carrier mean-free path in the conductor.

We consider a 1D chain of magnetic impurity atoms coupled to 1D or 2D conduction electrons with Rashba interactions. The schematic picture of the experimental system is shown in Fig. 1. The corresponding effective Hamiltonian is given by (\( h = 1 \))

\[
H = \int dr \left[ c_\alpha^\dagger \left( \frac{p^2}{2m^*} - \mu + \alpha \frac{\hbar^2}{2m^*} \mathbf{p} \cdot \mathbf{\sigma} \right) c_\beta + J_\mathbf{S}(r) \cdot \mathbf{s}(r) \right] c_\beta^\dagger + \text{J.S.} \cdot \mathbf{s} \tag{1}
\]

where \( c_\alpha \) (\( c_\alpha^\dagger \)) are the conduction electron creation (annihilation) operators with spin \( \alpha \), \( \sigma_i \) are Pauli matrices, \( \mathbf{p} = -i\nabla \) is the momentum operator; \( \mathbf{S} \) and \( \mathbf{s} \) are impurity and electron spin operators. \( m^* \) is the effective mass of electrons and \( J \) is the coupling strength. We assume that magnetic atoms such as \( \text{Co, Gd or Fe} \) have large spin so that one can neglect quantum effects and treat the impurity spins as classical. By integrating out conduction electrons, one arrives at the following Hamiltonian for the impurity spins

\[
H_{\text{RKKY}} = -J^2 \sum_{ij} \sum_{\alpha,\beta} \chi_{\alpha\beta}(R_{ij}) S_i^\alpha S_j^\beta. \tag{2}
\]

Here \( \chi_{\alpha\beta}(R_{ij}) \) is spin-spin susceptibility with \( R_{ij} = |R_i - R_j| \) being the distance between two impurity spins. The real-space spin-spin correlation function \( \chi_{\alpha\beta}(R) \) is given by

\[
\chi_{\alpha\beta}(R) = -\int \frac{d\omega}{2\pi} \text{Tr} \left[ \sigma_\alpha G(\omega, R) \sigma_\beta G(\omega, -R) \right]. \tag{3}
\]

where \( G(\omega, R) \) is the Green’s function for the conduction electrons.

We first consider the model without SO coupling \( \alpha = 0 \), in which case the spin susceptibility is isotropic \( \chi_{\alpha\beta}(R) \propto \delta_{\alpha\beta} F(k_F R) \). Here the range function \( F(k_F R) \) describes RKKY interaction between impurity spins. The function \( F(x) \) for 1D/2D conductor is well-known [26]:

\[
F_{1D}(x) = -\left( \text{Si}(2x) - \frac{\pi}{2} \right), \tag{4}
\]

\[
F_{2D}(x) = -\frac{\pi}{4} \left[ J_0(x) N_0(x) + J_1(x) N_1(x) \right].
\]

Here \( \text{Si}(x) \) is the sine integral function, \( J_0(x) \) and \( N_0(x) \) are Bessel functions of the first and second kind, respectively, and

\[
E_{1D}(q) = \frac{\cos(2\pi q a)}{(2\pi q a)^2} \text{ and } E_{2D}(x) = \frac{\sin(2\pi x)}{(2\pi x)^2}
\]

We consider a 1D chain of magnetic impurity atoms coupled to 1D or 2D conduction electrons with Rashba SO coupling. We study the magnetic ordering in such a system in the presence of potential impurities in the 2DEG, and show that the helical SDW is stable in such a system provided the interatomic distance between impurity spins is smaller than the effective carrier mean-free path in the conductor.

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bility is to consider a magnetic spin chain coupled to a quasi-
1D superconductor with an open Fermi surface where tunneling
matrix elements along the chain $t_{ij}$ are much stronger than
perpendicular to the chain $t_{\perp}$, i.e., $|t_{ij}| \gg |t_{\perp}|$.

We now investigate the stability of the helical SDW in 1D
structures against disorder in realistic experimental condi-
tions, e.g., taking into account positional disorder of the
magnetic impurities. The relevant length scale to which the
positional disorder must be compared is the Fermi wave-
length. This may be particularly short in a metal, typically
few angstroms, potentially making this problem a crucial one.
In this case even small deviations of the impurity positions
from perfect periodicity may lead to frustration of the mag-
netic interaction which would ultimately destroy the helical
SDW. We study this question numerically using a simulated
annealing procedure which allows us to identify the ground
state spin configuration of the system in the presence of po-
ositional disorder. In our simulations, positional disorder is
characterized by a scale $\delta R$, and atom positions are chosen as
$R_i = a_i + r$, where $r \in [-\delta R, \delta R]$ is uniformly chosen.
We consider 50 disorder realizations for each parameter set, and for each disorder realization perform annealing with a local-update Monte Carlo procedure followed by a gradient-
based energy optimization for 400 initial configurations. We
analyze the lowest-energy configurations we obtain using the
pitch angle of adjacent spins,

$$\theta_i = \arccos(S_i \cdot S_{i+1}).$$

Furthermore, we confirm that all spins lie in a plane by
calculating $T_{ij} = (S_i \times S_{i+1}) \cdot S_{i+2}$ and confirming that
$|\langle|T_{ij}|\rangle| = 0$ within error bars; this justifies the ansatz chosen
above.

In the main panel of Fig. 3, we show the pitch angle av-
eraged over position within each system and different disor-
der realizations for various values of $k_F$ and $\delta R$. Our results
confirm that the effect of disorder is governed by the product
$k_F \cdot \delta R$, as indicated by the collapse of all curves in the figure.

To understand in more detail how positional disorder af-
facts the low-energy configurations, consider the histograms
shown in the inset of Fig. 3. For $k_F \cdot \delta R \leq 0.2$, the histogram
is sharply peaked around the pitch angle of the clean case. For
larger values of $k_F \cdot \delta R$, the peak height is drastically reduced and the peak is broadened. Upon further increasing disorder,
the peak is rapidly split into two peaks at $\theta = 0$ and $\theta = \pi$,
where the peak at $\theta = 0$ is higher. This is indicative of con-
figurations with ferromagnetic clusters separated by domain
walls. As the size of these clusters is reduced, the peaks at
$\theta = 0, \pi$ become more balanced and ultimately have the same
height. This is reflected by the mean pitch angle approaching
$\pi/2$. A key observation is that already for small values of the
disorder $k_F \cdot \delta R \ll 1$, where the mean value is well away from
$\pi/2$, the system is not in a helical phase but instead is
composed of ferromagnetic clusters. This is different from
the heuristic disorder model assumed in Ref. 19.

Our calculation for positional disorder establishes posi-
tional disorder to be a severe constraint restricting the sponta-
neous emergence of a helical SDW in the system of impurity
spins coupled to a 1D metal. Even assuming that one might
be able to reduce disorder in the positions of the magnetic
impurities through very careful sub-nm control of the impu-
ity placement, there are further effects such as thermal fluc-
tuations and potential disorder in the 1D metal itself which
generically suppress helical magnetic ordering. Furthermore,
all Majorana proposals ultimately require a sizeable coupling
to a bulk s-wave superconductor. However, the interaction of
the impurity spins with superconducting electrons in 2D or
3D favors an (anti-)ferromagnetic ground state and competes
with helical ordering. Thus, we conclude that the observation
of a spontaneously formed helical SDW in 1D systems is quite
challenging and unlikely unless special care is taken in reduc-
ing all types of disorder in the system—the self-tuned helical
SDW is only possible in an ideal theoretical model, not in the
laboratory where disorder is inherently present.

The experimental observation of helical ordering in Fe
chains deposited on Ir (001) surface [27] indicates that D-M
interaction is necessary to stabilize the helical SDW. The mi-
croscopic origin of the D-M interaction is complex and often
associated with the presence of SO coupling. Therefore, we
now consider an impurity spin chain coupled to a 2D conduc-
tor with Rashba SO interaction $\alpha \neq 0$, see Eq. (1). After
integrating out the conduction electrons that mediate the in-
teractions between impurity spins, we arrive at the anisotropic
model for impurity spins [25]

$$H = -J^2\frac{2m^*q^2_F}{\pi^2} \sum_{i,j} F_{2D}(q_F R_{ij})(\cos(2k_F R_{ij})S_i \cdot S_j
+ \sin(2k_F R_{ij})y_i \cdot (S_i \times S_j) + [1 - \cos(2k_F R_{ij})]S_i^x S_j^x),$$

(7)

Here the function $F_{2D}(R_{ij})$ is defined in Eq. (4); $q_F =
\sqrt{k_F^2 + k_F^2}$ with $k_F = m^*\alpha$. In order to understand
ground state properties of the Hamiltonian (7), it is instructive
to perform the following local transformation: \( \hat{S}_i^{x/z} = \cos(2k_FR_i)S_i^{x/z} \pm \sin(2k_FR_i)S_i^{z/x} \) and \( \hat{S}_i^y = S_i^y \) which is simply an SO(3) rotation around the \( y \) axis by the angle \( 2k_FR_i \). In the new rotated basis, the Hamiltonian (7) contains only the RKKY interaction. As argued above, the ground state of an impurity chain coupled to a 2D conductor is ferromagnetic. Thus, one can unwind the rotation to obtain the actual spin ordering. A simple calculation indicates that the ground states in this case corresponds to a helical SDW with a pitch angle given by \( \tan(\theta_E) \). This result should be contrasted with the spontaneous helical SDW with the pitch angle given by \( 2k_Fa \). Thus, strong SO coupling is essential for the helical RKKY Majorana proposals to be realized in magnetic impurity chains.

We now analyze the effect of potential disorder scattering, which is relevant for the RKKY Majorana proposals involving disordered superconductors. As previously mentioned, potential disorder scattering randomizes magnetic interactions and therefore affects ordering of magnetic atoms. Before considering the case with SO coupling, it is useful to first discuss the RKKY case which has been extensively studied in the literature [3, 4, 28–35]. It is well-known that the disorder-averaged spin susceptibility \( \chi(r) \) decays exponentially with the decay length \( l_c \). At large distances \( r \gg l_c \), however, the susceptibility \( \chi_{\alpha\beta}(r) \) does not represent interactions between impurity spins in a given sample. Indeed, the fluctuations of the interaction are considerably larger than its typical value. Thus, one has to consider sample-specific interactions which decay much slower than \( \chi(r) \), i.e. as a power law. As shown below, the situation is qualitatively similar in disordered metals with Rashba interactions. Therefore, at small distances between magnetic atoms (\( \alpha \ll l_c \)) the short range nature of spin-spin interactions dominates and the system forms a helical spin density wave, similar to the clean case, whereas at large distances between magnetic impurity atoms (\( \alpha \gg l_c \)) random RKKY interactions cause frustration and destroy magnetic order. In this sense, substrate random disorder scattering is similar to the positional disorder in the magnetic chain itself which drives the system into a paramagnetic phase, see Fig. 3.

In order to estimate the characteristic decay length \( l_c \), we calculate the spin-spin susceptibility tensor \( \chi_{\alpha\beta}(r) \) in the presence of SO coupling. We employ a standard disorder diagrammatic technique, see Supplementary Material for details. We consider a model with Gaussian random disorder, where the disorder potential \( V(r) \) is \( \delta \)-correlated: \( \langle V(r)V(r') \rangle = (2\pi\nu_F\tau)^{-1} \delta(r-r') \) where \( \nu_F \) and \( \tau \) are the density of states at the Fermi level and impurity scattering time, respectively. Our main results are summarized in Fig. 4. Since SO coupling breaks the SU(2) symmetry, we now have four non-zero spin-spin correlation functions \( \chi_{xx}(q) \), \( \chi_{yy}(q) \), \( \chi_{zz}(q) \) and \( \chi_{zz}(q) \). Their spatial dependence is characterized by an oscillatory pre-factor and an exponentially-decaying envelope function, see Fig. 4. The pre-factor has a spatial dependence which is very similar to the spin-spin interaction in the clean limit, see Eq. (7). The characteristic decay length \( l_c \) can be obtained by fitting the envelope function. In the limit of \( k_R \ll k_F, l_c \) is very weakly dependent on the SO interaction strength, and is determined by the mean-free path \( l \) (up to a numerical prefactor of order one).

Having established the limitations on the stability of the helical SDW, we now discuss the helical RKKY Majorana proposals [10–13, 23] and compare them with the semiconductor nanowire ones [16, 17] which have recently been studied extensively experimentally [36–41]. As shown above, spontaneous formation of the helical SDW with a pitch angle \( 2k_Fa \) critically relies on one-dimensionality. One of the systems that has been put forward involves nuclear spins coupled to 1D semiconductor electrons [11, 12]. However, the crossover temperature \( T^* \) above which helical order disappears is very low in this system \( (T^* \sim 1\text{mK}) \) due to the small hyperfine coupling. When coupled to a higher-dimensional conductor, the ground state magnetic ordering is ferromagnetic in the absence of D-M interaction. Therefore, large SO coupling is necessary for the realization of the helical SDW in realistic experimental conditions, in which case the pitch angle of helical order is set by the SO wave length. The only evidence for helical order in chains of magnetic atoms comes from the experiment [27] involving Fe atoms placed on an Ir(001) surface. This supports our conclusion, since the D-M interaction is very large in this experiment. The difference of the pitch angle from the “sweet spot” \( (2k_Fa) \) has implications for Majorana proposals. Indeed, as shown in Refs. [10, 23, 24], some tuning is generally necessary in order to drive the system into the topological phase which might be quite challenging in magnetic impurity chains. This is to be contrasted with the semiconductor Majorana proposals where one can tune the mag-
netic field to drive the system into a topological superconducting phase. We emphasized the role of random disorder scattering on the stability of the helical order; however, it also has an effect on the effective Hamiltonian for the Majorana wire. Indeed, it has been shown that disorder has a detrimental effect on the stability of the topological superconducting state [42–45]. Our finding that magnetic impurities form ferromagnetic domains has implications for helical RKKY Majorana proposals since such ordering affects proximity-induced superconducting pairing and therefore suppresses the topological phase. On the positive side, we believe that a big advantage of the RKKY Majorana proposal is the ability to detect zero-energy bound states directly using STM rather than tunneling transport measurements as suggested originally in Ref. [18].

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Supplemental Material for “Helical order in one-dimensional magnetic atom chains and possible emergence of Majorana bound states”

SPIN SUSCEPTIBILITY IN DISORDERED 2DEG WITH RASHBA SPIN-ORBIT COUPLING

In this supplementary material we present the details of the calculation for the spin-spin susceptibility in two-dimensional electron systems with Rashba spin-orbit coupling. The expressions for the clean case have been obtained in Ref. [25]. In this paper, we focus on the dirty system and take into account potential impurity scattering.

The Hamiltonian for the 2DEG with Rashba spin-orbit coupling is given by Eq. (1) of the main text. We now include effects of impurity scattering. We consider a Gaussian disorder potential defined by the correlation function $V(r)V(r') = (2\pi\nu_F)^{-1} \delta(r - r')$. The disorder-averaged imaginary time single-particle Green’s function is given by

$$\overline{G}(\omega_n, \mathbf{k}) = \frac{i(\omega_n + \text{sgn}(\omega_n)\frac{k^2}{2}) - k_R^2 + \mu}{\left[i(\omega_n + \text{sgn}(\omega_n)\frac{k^2}{2}) - k_R^2 + \mu\right]^2 - k_R^2 k^2},$$

(S1)

where $k^2 = k_x^2 + k_y^2$, $\omega_n = (2n + 1)\pi T$, with $T$ being the temperature; $\sigma_i$ are Pauli matrices, $\mu$ is the chemical potential and $k_R = m^* \alpha$, with $\alpha$ being the Rashba spin-orbit coupling. We set $h = k_B = m = 1$ henceforth. After integrating out the electronic degrees of freedom in Hamiltonian (1), we arrive at the following classical model for the impurity spins:

$$H_J = -J^2 \sum_{a,b} \sum_{i,j} \overline{\chi}_{ij}(\mathbf{R}_a - \mathbf{R}_b) S_a^i S_b^j.$$  

(S2)

Here $\overline{\chi}_{ij}(\mathbf{R})$ is the disordered-averaged spin-spin susceptibility defined as

$$\overline{\chi}_{ij}(\mathbf{R}) = -T \sum_{n=1}^{\infty} \int \frac{d\mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q} \cdot \mathbf{R}} \chi_{ij}(\omega_n, \mathbf{q})$$

(S3)

$$\chi_{ij}(\omega_n, \mathbf{q}) = \sum_{m=1}^{\infty} \frac{1}{(2\pi\nu_F)^{m-1}} \text{Tr} \left[ \prod_{l=1}^{m} \frac{d\mathbf{k}_l}{(2\pi)^2} \sigma_i \overline{G}(\omega_n, \mathbf{k}_1) \cdots \overline{G}(\omega_n, \mathbf{k}_m) \sigma_j \overline{G}(\omega_n, \mathbf{k}_m - \mathbf{q}) \cdots \overline{G}(\omega_n, \mathbf{k}_1 - \mathbf{q}) \right]$$

(S4)

with the sum over $m$ representing ladder diagrams. We assume here $kFL \gg 1$. In order to simplify notations, we will drop the Matsubara frequency index $n$ from now on. Next, we define a $4 \times 4$ matrix $\Gamma_D(\omega, \mathbf{q})$ as

$$[\Gamma_D(\omega, \mathbf{q})]_{\mu\nu} = \frac{1}{4\pi \nu_F} \int \frac{d\mathbf{k}}{(2\pi)^2} \text{Tr}[\sigma_\mu \overline{G}(\omega, \mathbf{k}) \sigma_\nu \overline{G}(\omega, \mathbf{k} - \mathbf{q})].$$

(S5)

FIG. S1. Normal scale (up) and log scale (down) Plot of $\overline{\chi}_{ii}(\mathbf{R})$ and $\chi_{ii}(\mathbf{R})$ for $k_FR = 10$. Extracted values for $l_0^c$ and $l_c$ are 9.9 and 8.2, respectively.

- $\overline{\chi}_{ii}(\mathbf{R})$ is the disordered-averaged spin-spin susceptibility defined as

$$\overline{\chi}_{ij}(\mathbf{R}) = -T \sum_{n=1}^{\infty} \int \frac{d\mathbf{q}}{(2\pi)^2} e^{-i\mathbf{q} \cdot \mathbf{R}} \chi_{ij}(\omega_n, \mathbf{q})$$

(S3)

$$\chi_{ij}(\omega_n, \mathbf{q}) = \sum_{m=1}^{\infty} \frac{1}{(2\pi\nu_F)^{m-1}} \text{Tr} \left[ \prod_{l=1}^{m} \frac{d\mathbf{k}_l}{(2\pi)^2} \sigma_i \overline{G}(\omega_n, \mathbf{k}_1) \cdots \overline{G}(\omega_n, \mathbf{k}_m) \sigma_j \overline{G}(\omega_n, \mathbf{k}_m - \mathbf{q}) \cdots \overline{G}(\omega_n, \mathbf{k}_1 - \mathbf{q}) \right]$$

(S4)

with the sum over $m$ representing ladder diagrams. We assume here $kFL \gg 1$. In order to simplify notations, we will drop the Matsubara frequency index $n$ from now on. Next, we define a $4 \times 4$ matrix $\Gamma_D(\omega, \mathbf{q})$ as

$$[\Gamma_D(\omega, \mathbf{q})]_{\mu\nu} = \frac{1}{4\pi \nu_F} \int \frac{d\mathbf{k}}{(2\pi)^2} \text{Tr}[\sigma_\mu \overline{G}(\omega, \mathbf{k}) \sigma_\nu \overline{G}(\omega, \mathbf{k} - \mathbf{q})].$$

(S5)
Therefore, we can rewrite disorder-averaged susceptibility
\[ \chi_{ij}(q) = \frac{1}{2} \left[ \sum_{\mu=0}^{3} [\sigma_{\mu}]_{ac} [\sigma_{\mu}]_{db} \right] \]
into terms of Green’s functions, i.e. by choosing an appropriate contour of integration. In the present case, one has to take into account a branch cut around \( \omega_n = 0 \). Using the relation between Matsubara and retarded and advanced Green’s functions,
\[ G(i\omega_n, q) = \theta(\omega_n)G_R(i\omega_n, q) + \theta(-\omega_n)G_A(i\omega_n, q), \]
one finds that the spin susceptibility \( \chi^R_{ij}(R) \) without vertex corrections (i.e. ignoring ladder diagrams) is given by
\[ \chi^R_{ij}(q) = -T \sum_{n=-\infty}^{\infty} \Gamma_D(\omega_n, q) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{dk}{(2\pi)^2} n_F(\omega) \times \left[ \text{Tr}[\sigma_i \bar{G} R(\omega, q) \sigma_j \bar{G} R(\omega, k - q)] - (R \rightarrow A) \right] \]
(S10)

As one can see the static susceptibility \( \chi^R_{ij}(q) \) does not contain the diffusion ladder between retarded and advanced Green’s function. Thus, in general there is no long-range (i.e. power-law decaying) contribution to the susceptibility \( \chi^R_{ij}(q) \) originating from the diffusive modes. Therefore, the disorder-averaged susceptibility still decays exponentially even when one takes into account ladder diagrams. An analytical calculation for the \( \alpha = 0 \) case shows that in the limit \( k_F l \gg 1 \) the difference between \( l_0^R \) and \( l_c \) is small. Similarly, the same relationship between \( \chi^R_{ij}(R) \) and \( \chi^R_{ij}(q) \) holds in the presence of spin-orbit \( \alpha \neq 0 \) provided that \( k_F \gg k_B \). Thus, for practical purposes it is sufficient to consider \( \chi^R_{ij}(R) \). The results for the numerical integration of Eq. (S10) for \( q = q_x \) are shown in Fig. S2. One can see that \( \chi^R_{ij}(q) \) becomes quickly suppressed as \( k_F l \) approaches one.
FIG. S2. Plot of $\chi_{0xx}(q)$ and $\text{Im}[\chi_{0xz}(q)]$ for different values of the disorder mean-free path $l$. Here Rashba spin-orbit coupling $k_R/k_F = 0.1$. In the absence of spin-orbit coupling, $\chi_{0xx}(q)$ can be reduced to the density-density correlation function which was first calculated S. Das Sarma in Ref.[48] in the context of the static screening properties in 2D semiconductor systems.