An arithmetic Bernštein–Kušnirenko inequality

César Martínez\textsuperscript{1,4} \cdot Martín Sombra\textsuperscript{2,3}

Received: 23 December 2016 / Accepted: 13 June 2018 / Published online: 6 September 2018
© Springer-Verlag GmbH Germany, part of Springer Nature 2018

Abstract
We present an upper bound for the height of the isolated zeros in the torus of a system of
Laurent polynomials over an adelic field satisfying the product formula. This upper bound is
expressed in terms of the mixed integrals of the local roof functions associated to the chosen
height function and to the system of Laurent polynomials. We also show that this bound is
close to optimal in some families of examples. This result is an arithmetic analogue of the
classical Bernštein–Kušnirenko theorem. Its proof is based on arithmetic intersection theory
on toric varieties.

Keywords  Height of points \cdot Laurent polynomials \cdot Mixed integrals \cdot Toric varieties \cdot
u-Resultants

Mathematics Subject Classification  Primary 14G40; Secondary 14C17 \cdot 14M25 \cdot 52A41

1 Introduction

The classical Bernštein–Kušnirenko theorem bounds the number of isolated zeros of a system
of Laurent polynomials over a field, in terms of the mixed volume of their Newton polytopes

Martínez and Sombra were partially supported by the MINECO research projects MTM2012-38122-C03-02
and MTM2015-65361-P. Martínez was also partially supported by the CNRS project PICS 6381 “Géométrie
diophantienne et calcul formel”. Martínez was also partially supported by the Deutsche Forschungsgemeins-
chaft collaborative research center SFB 1085 “Higher Invariants”.

\textsuperscript{1}  Laboratoire de mathématiques Nicolas Oresme, CNRS UMR 6139, Université de Caen, BP 5186,
14032 Caen Cedex, France
\textsuperscript{2}  ICREA, Passeig Lluís Companys 23, 08010 Barcelona, Spain
\textsuperscript{3}  Departament de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via 585, 08007
Barcelona, Spain
\textsuperscript{4}  Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
This result, initiated by Kušnirenko and put into final form by Bernštein, is also known as the BKK theorem to acknowledge Khovanskiı’s contributions to this subject. It shows how a geometric problem (the counting of the number of solutions of a system of equations) can be translated into a combinatorial, simpler one. It is commonly used to predict when a given system of equations has a small number of solutions. As such, it is a cornerstone of polynomial equation solving and has motivated a large amount of work and results over the past 25 years, see for instance [14,21,24] and the references therein.

When dealing with Laurent polynomials over a field with an arithmetic structure like the field of rationals, it might be also important to control the arithmetic complexity or height of their zero set. In this paper, we present an arithmetic version of the BKK theorem, bounding the height of the isolated zeros of a system of Laurent polynomials over such a field. It is a refinement of the arithmetic Bézout theorem that takes into account the finer monomial structure of the system.

Previous results in this direction were obtained by Maillot [20] and by the second author [23]. Our current result improves these previous upper bounds, and generalizes them to adelic fields satisfying the product formula, and to height functions associated to arbitrary nef toric metrized divisors.

Let \( K \) be a field and \( \overline{K} \) its algebraic closure. Let \( M \simeq \mathbb{Z}^n \) be a lattice and set \( \mathbb{K}[M] \simeq \mathbb{K}[x_1^\pm 1, \ldots, x_n^\pm 1] \) and \( T_M = \text{Spec}(\mathbb{K}[M]) \simeq \mathbb{G}_m^n, \mathbb{K} \) for its group \( \mathbb{K} \)-algebra and algebraic torus over \( \mathbb{K} \), respectively. For a family of Laurent polynomials \( f_1, \ldots, f_n \in \mathbb{K}[M] \), we denote by \( Z(f_1, \ldots, f_n) \) the 0-cycle of \( T_M \) given by the isolated solutions of the system of equations

\[
f_1 = \cdots = f_n = 0
\]

with their corresponding multiplicities (Definition 2.7).

Set \( M_\mathbb{R} = M \otimes \mathbb{R} \simeq \mathbb{R}^n \). Let \( \text{vol}_M \) be the Haar measure on \( M_\mathbb{R} \) normalized so that \( M \) has covolume 1, and let \( MV_M \) be the corresponding mixed volume function. For \( i = 1, \ldots, n \), let \( \Delta_i \subset M_\mathbb{R} \) be the Newton polytope of \( f_i \). The BKK theorem amounts to the upper bound

\[
\text{deg}(Z(f_1, \ldots, f_n)) \leq MV_M(\Delta_1, \ldots, \Delta_n),
\]

which is an equality when the \( f_i \)'s are generic with respect to their Newton polytopes [2,17], see also Theorem 2.9.

Now suppose that \( \mathbb{K} \) is endowed with a set of places \( \mathcal{M} \), so that the pair \( (\mathbb{K}, \mathcal{M}) \) is an adelic field (Definition 3.1). Each place \( v \in \mathcal{M} \) consists of an absolute value \( |\cdot|_v \) on \( \mathbb{K} \) and a weight \( n_v > 0 \). We assume that this set of places satisfies the product formula, namely, for all \( \alpha \in \mathbb{K}^\times \),

\[
\sum_{v \in \mathcal{M}} n_v \log |\alpha|_v = 0.
\]

The classical examples of adelic fields satisfying the product formula are the global fields, that is, number fields and function fields of regular projective curves.

Let \( X \) be toric compactification of \( T_M \) and \( D_0 \) a nef toric metrized divisor on \( X \), see Sects. 4 and 5 for details. This data gives a notion of height for 0-cycles of \( X \), see [3, Chapter 2] or Sect. 4. The height

\[
h_{D_0}(Z(f_1, \ldots, f_n))
\]

is a nonnegative real number, and it is our aim to bound this quantity in terms of the monomial expansion of the \( f_i \)'s.
The toric Cartier divisor $D_0$ defines a polytope $\Delta_0 \subset M_{\mathbb{R}}$. Following [8], we associate to $D_0$ an adelic family of continuous concave functions $\vartheta_{0,v} : \Delta_0 \to \mathbb{R}$, $v \in \mathcal{M}$, called the local roof functions of $D_0$.

For $i = 1, \ldots, n$, write

$$f_i = \sum_{m \in M} \alpha_{i,m} x^m$$

with $\alpha_{i,m} \in \mathbb{K}$. Let $N_{\mathbb{R}} = M_{\mathbb{R}}^\vee \simeq \mathbb{R}^n$ be the dual space and, for each place $v \in \mathcal{M}$, consider the concave function $\psi_{i,v} : N_{\mathbb{R}} \to \mathbb{R}$ defined by

$$\psi_{i,v}(u) = \begin{cases} -\log \left( \sum_{m \in M} |\alpha_{i,m}|_v e^{-\langle m, u \rangle} \right) & \text{if } v \text{ is Archimedean,} \\ -\log \left( \max_{m \in M} |\alpha_{i,m}|_v e^{-\langle m, u \rangle} \right) & \text{if } v \text{ is non-Archimedean.} \end{cases} \quad (1.2)$$

The Legendre–Fenchel dual $\vartheta_{i,v} = \psi_{i,v}^\vee$ is a continuous concave function on $\Delta_i$.

We denote by $\text{MI}_M$ the mixed integral of a family of $n + 1$ concave functions on convex bodies of $M_{\mathbb{R}}$ (Definition 5.6). It is the polarization of $(n + 1)!$ times the integral of a concave function on a convex body. It is a functional that is symmetric and linear in each variable with respect to the sup-convolution of concave functions, see [21, Sect. 8] for details.

The following is the main result of this paper.

**Theorem 1.1** Let $f_1, \ldots, f_n \in \mathbb{K}[M]$, and let $X$ be a proper toric variety with torus $\mathbb{T}_M$ and $D_0$ a nef toric metrized divisor on $X$. Let $\Delta_0 \subset M_{\mathbb{R}}$ be the polytope of $D_0$ and, for $v \in \mathcal{M}$, let $\vartheta_{0,v} : \Delta_0 \to \mathbb{R}$ be $v$-adic roof function of $D_0$. For $i = 1, \ldots, n$, let $\Delta_i \subset M_{\mathbb{R}}$ be the Newton polytope of $f_i$ and, for $v \in \mathcal{M}$, let $\vartheta_{i,v} : \Delta_i \to \mathbb{R}$ be the Legendre–Fenchel dual of the concave function in (1.2). Then

$$h_{D_0}(Z(f_1, \ldots, f_n)) \leq \sum_{v \in \mathcal{M}} n_v \text{MI}(\vartheta_{0,v}, \ldots, \vartheta_{n,v}). \quad (1.3)$$

Using the basic properties of the mixed integral, we can bound the terms in the right-hand side of (1.3) in terms of mixed volumes. From this, we can derive the bound (Corollary 6.8)

$$h_{D_0}(Z(f_1, \ldots, f_n)) \leq \text{MV}_M(\Delta_1, \ldots, \Delta_n) \left( \sum_{v \in \mathcal{M}} \max_{\vartheta_{0,v}} \right)$$

$$+ \sum_{i=1}^n \text{MV}_M(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n) \ell(f_i),$$

where $\ell(f_i)$ denotes the (logarithmic) length of $f_i$ (Definition 6.6). This bound might be compared with the one given by the arithmetic Bézout theorem (Corollary 6.9).

The following example illustrates a typical application of these results. It concerns two height functions applied to the same 0-cycle. Our upper bounds are close to optimal for both of them and, in particular, they reflect their very different behavior on this family of Laurent polynomials.

**Example 1.2** Take two integers $d, \alpha \geq 1$ and consider the system of Laurent polynomials

$$f_1 = x_1 - \alpha, \quad f_2 = x_2 - \alpha x_1^d, \quad \ldots, \quad f_n = x_n - \alpha x_{n-1}^d \in \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

The 0-cycle $Y := Z(f_1, \ldots, f_n)$ of $\mathbb{Q}_m, \mathbb{Q}$ is the single point $(\alpha, \alpha^{d+1}, \ldots, \alpha^{dn-1+d+1})$ with multiplicity one.
Let $\mathbb{P}^n_{\mathbb{Q}}$ be the $n$-dimensional projective space over $\mathbb{Q}$ and $E_{\text{can}}$ the divisor of the hyperplane at infinity, equipped with the canonical metric. Its associated height function is the Weil height. We consider two toric compactifications $X_1$ and $X_2$ of $\mathbb{G}^n_m$. These are given by compactifying the torus via the equivariant embeddings $\iota_i : \mathbb{G}^n_m \hookrightarrow \mathbb{P}^n_{\mathbb{Q}}$, $i = 1, 2$, respectively defined, for $p = (p_1, \ldots, p_n) \in \mathbb{G}^n_m(\mathbb{Q}) = (\mathbb{Q}(X))^n$, by

$$
\iota_1(p) = (1 : p_1 : \cdots : p_n) \quad \text{and} \quad \iota_2(p) = (1 : p_1 : p_2p_1^{-d} : \cdots : p_n p_n^{-d}).
$$

Set $D_i = \iota_i^* E_{\text{can}}$, $i = 1, 2$, which are nef toric metrized divisors on $X_i$. By an explicit computation, we show that

$$
h_{\overline{D}_1}(Y) = \left( \sum_{i=1}^n d_i^{-1} \right) \log(\alpha) \quad \text{and} \quad h_{\overline{D}_2}(Y) = \log(\alpha).
$$

On the other hand, the upper bounds given by Theorem 1.1 are

$$
h_{\overline{D}_1}(Y) \leq \left( \sum_{i=1}^n d_i^{-1} \right) \log(\alpha + 1) \quad \text{and} \quad h_{\overline{D}_2}(Y) \leq n \log(\alpha + 1),
$$

see Example 7.2 for details.

To the best of our knowledge, the first arithmetic analogue of the BKK theorem was proposed by Maillot [20, Corollaire 8.2.3], who considered the case of canonical toric metrics. Another result in this direction was obtained by the second author for the unmixed case and also canonical toric metrics [23, Théorème 0.3]. Theorem 1.1 improves and refines these previous upper bounds, and generalizes them to adelic fields satisfying the product formula and to height functions associated to arbitrary nef toric metrized divisors, see Sect. 7 for details.

The key point in the proof of Theorem 1.1 consists of the construction, for each Laurent polynomial $f_i$, of a nef toric metrized divisor $\overline{D}_i$ on a proper toric variety $X$, such that $f_i$ corresponds to a small section of $\overline{D}_i$ (Proposition 6.2 and Lemma 6.4). The proof then proceeds by applying the constructions and results of [4,8] and basic results from arithmetic intersection theory.

Trying to keep our results at a similar level of generality as those in [8], we faced difficulties to define and study global heights of cycles over adelic fields. This lead us to a more detailed study of these notions. In particular, we give a new notion of adelic field extension that preserves the product formula (Proposition 3.7) and a well-defined notion of global height for cycles with respect to metrized divisors that are generated by small sections (Proposition–Definition 4.15).

As an application of Theorem 1.1, we give an upper bound for the size of the coefficients of the $u$-resultant of the direct image under a monomial map of the solution set of a system of Laurent polynomial equations.

For the simplicity of the exposition, set $\mathbb{K} = \mathbb{Q}$ and $M = \mathbb{Z}^n$. Let $r \geq 0$, $m_0 = (m_{0,0}, \ldots, m_{0,r}) \in (\mathbb{Z}^r)^{r+1}$ and $\alpha_0 = (\alpha_{0,0}, \ldots, \alpha_{0,r}) \in (\mathbb{Z}\setminus\{0\})^{r+1}$, and consider the map $\varphi_{m_0,\alpha_0} : \mathbb{G}^n_m \rightarrow \mathbb{P}^r_{\mathbb{Q}}$ defined by

$$
\varphi_{m_0,\alpha_0}(p) = (\alpha_{0,0} x^{m_{0,0}}(p) : \cdots : \alpha_{0,r} x^{m_{0,r}}(p)). \quad (1.4)
$$

For a 0-cycle $W$ of $\mathbb{P}^r_{\mathbb{Q}}$, let $u = (u_0, \ldots, u_r)$ be a set of $r + 1$ variables and denote by $\text{Res}(W) \in \mathbb{Z}[u_1, \ldots, u_r]$ its set of $r + 1$ variables and denote by $\text{Res}(W)$ its primitive $u$-resultant (Definition 7.4). It is well-defined up a sign. For a vector $\alpha$ with integer entries, we denote by $\ell(\alpha)$ the logarithm of the sum of the absolute values of its entries.
Theorem 1.3. Let \( f_1, \ldots, f_n \in \mathbb{Z} [x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \), \( m_0 \in (\mathbb{Z}^n)^{r+1} \) and \( \alpha_0 \in (\mathbb{Z} \setminus \{0\})^{r+1} \) with \( r \geq 0 \). Set \( \Delta_0 = \text{conv}(m_0, 0, \ldots, m_0, r) \subset \mathbb{R}^n \) and let \( \varphi \) be the monomial map associated to \( m_0 \) and \( \alpha_0 \) as in (1.4). For \( i = 1, \ldots, n \), let \( \Delta_i \subset \mathbb{R}^n \) be the Newton polytope of \( f_i \), and \( \alpha_i \) the vector of nonzero coefficients of \( f_i \). Then

\[
\ell (\text{Res}(\varphi^\ast Z(f_1, \ldots, f_n))) \leq \sum_{i=0}^n \text{MV}_M(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n) \ell (\alpha_i).
\]

The paper is organized as follows. In Sect. 2 we recall some preliminary material on intersection theory and on the algebraic geometry of toric varieties. In Sect. 3 we study adelic fields satisfying the product formula. In Sect. 4 we recall the notions of metrized divisors and its associated measures and heights, with an emphasis on the 0-dimensional case. In Sect. 5 we explain the notation and basic constructions of the arithmetic geometry of toric varieties. In Sect. 6 we prove Theorem 1.1, whereas in Sect. 7 we give examples illustrating the applications of our bounds, and prove Theorem 1.3.

2 Intersection theory and toric varieties

In this section, we recall the proof of the Bernštein–Kušnirenko theorem using intersection theory on toric varieties, which is the model that we follow in our treatment of the arithmetic version of this result. Presenting this proof also allows us to introduce the basic definitions and results on the intersection of Cartier divisors with cycles, and on the algebraic geometry of toric varieties. For more details on these subjects, we refer to [12,19] and to [13].

Let \( K \) be an infinite field and \( X \) a variety over \( K \) of dimension \( n \). For \( 0 \leq k \leq n \), the group of \( k \)-cycles, denoted by \( Z_k(X) \), is the free abelian group on the \( k \)-dimensional irreducible subvarieties of \( X \). Thus, a \( k \)-cycle is a finite formal sum

\[
Y = \sum_V m_V V
\]

where the \( V \)’s are \( k \)-dimensional irreducible subvarieties of \( X \) and the \( m_V \)’s are integers. The support of \( Y \), denoted by \( |Y| \), is the union of the subvarieties \( V \) such that \( m_V \neq 0 \). The cycle \( Y \) is effective if \( m_V \geq 0 \) for every \( V \). Given \( Y, Y' \in Z_k(X) \), we write \( Y' \leq Y \) whenever \( Y - Y' \) is effective.

Let \( Z \) be a subscheme of \( X \) of pure dimension \( k \). For an irreducible component \( V \) of \( Z \), we denote by \( \mathcal{O}_{V,Z} \) the local ring of \( Z \) along \( V \), and by \( l_{\mathcal{O}_{V,Z}}(\mathcal{O}_{V,Z}) \) its length as an \( \mathcal{O}_{V,Z} \)-module. The \( k \)-cycle associated to \( Z \) is then defined as

\[
[Z] = \sum l_{\mathcal{O}_{V,Z}}(\mathcal{O}_{V,Z}) V,
\]

the sum being over the irreducible components of \( Z \).

Let \( V \) be an irreducible subvariety of \( X \) of codimension one and \( f \) a regular function on an open subset \( U \) of \( X \) such that \( U \cap V \neq \emptyset \). The order of vanishing of \( f \) along \( V \) is defined as

\[
\text{ord}_V(f) = l_{\mathcal{O}_{V,X}(U)}(\mathcal{O}_{V,X}(U))/(f)).
\]

For a Cartier divisor \( D \) on \( X \), the order of vanishing of \( D \) along \( V \) is defined as

\[
\text{ord}_V(D) = \text{ord}_V(g) - \text{ord}_V(h),
\]
with \( g, h \in \mathcal{O}_{V, X}(U) \) such that \( g/h \) is a local equation of \( D \) on an open subset \( U \) of \( X \) with \( U \cap V \neq \emptyset \). This definition does not depend on the choice of \( U, g \) and \( h \). Moreover, \( \text{ord}_V(D) = 0 \) for all but a finite number of \( V \)'s. The support of \( D \), denoted by \( |D| \), is the union of these subvarieties \( V \) such that \( \text{ord}_V(D) \neq 0 \). The Weil divisor associated to \( D \) is then defined as

\[
D \cdot X = \sum_{V} \text{ord}_V(D) V, \quad (2.1)
\]

the sum being over the irreducible components of \( |D| \).

Now let \( W \) be an irreducible subvariety of \( X \) of dimension \( k \). If \( W \not\subset |D| \), then \( D \) restricts to a Cartier divisor on \( W \). In this case, we define \( D \cdot W \) as the Weil divisor of \( W \) obtained by restricting (2.1) to \( W \). This gives a \((k-1)\)-cycle of \( X \). If \( W \subset |D| \), then we set \( D \cdot W = 0 \), the zero element of \( Z_{k-1}(X) \). We extend by linearity this intersection product to a morphism

\[
Z_k(X) \rightarrow Z_{k-1}(X), \quad Y \mapsto D \cdot Y,
\]

with the convention that \( Z_{-1}(X) = 0 \), the zero group.

For \( 0 \leq r \leq n \) and Cartier divisors \( D_i \) on \( X \), \( i = 1, \ldots, r \), we define inductively the intersection product \( \prod_{i=1}^r D_i \in Z_{n-r}(X) \) by

\[
\prod_{i=1}^r D_i = \begin{cases} 
X & \text{if } t = 0, \\
D_1 \cdot \prod_{i=2}^r D_i & \text{if } 1 \leq t \leq r.
\end{cases}
\]

**Definition 2.1** Let \( Y \) be a \( k \)-cycle of \( X \) and \( D_1, \ldots, D_r \) Cartier divisors on \( X \), with \( r \leq k \). We say that \( D_1, \ldots, D_r \) intersect \( Y \) properly if, for every subset \( I \subset \{1, \ldots, r\} \),

\[
\dim \left( |Y| \cap \bigcap_{i \in I} |D_i| \right) = k - |I|.
\]

If \( D_1, \ldots, D_r \) intersect \( X \) properly, then the cycle \( \prod_{i=1}^r D_i \) does not depend on the order of the \( D_i \)'s [12, Corollary 2.4.2]. This conclusion does not necessarily hold if these divisors do not intersect properly.

**Example 2.2** Let \( X = \mathbb{A}^2_K \) and consider the principal Cartier divisors \( D_1 = \text{div}(x_1 x_2) \) and \( D_2 = \text{div}(x_1) \). Then

\[
D_1 \cdot D_2 = 0 \quad \text{and} \quad D_2 \cdot D_1 = (0, 0).
\]

**Proposition 2.3** Let \( X \) be a Cohen-Macaulay variety over \( K \) of pure dimension \( n \) and \( D_1, \ldots, D_n \) Cartier divisors on \( X \). Let \( s_i \) be a global section of \( \mathcal{O}(D_i) \), \( i = 1, \ldots, n \), and write

\[
\prod_{i=1}^n \text{div}(s_i) = \sum_p m_p p \in Z_0(X), \quad (2.2)
\]

where the sum is over the closed points \( p \) of \( X \) and \( m_p \in \mathbb{Z} \). This 0-cycle is effective and, for each isolated closed point \( p \) of the intersection \( \bigcap_{i=1}^n |\text{div}(s_i)| \),

\[
m_p = \dim_K(\mathcal{O}_{p, X}(U)/(f_1, \ldots, f_n)),
\]

where \( U \) is a trivializing neighborhood of \( p \), and \( f_i \) is a defining function for \( s_i \) on \( U \), \( i = 1, \ldots, n \).
Proof The fact that the cycle in (2.2) is effective follows from the hypothesis that the $s_i$’s are global sections. 

For the second statement, by possibly replacing $U$ with a smaller open neighborhood of $p$, we can assume that $\text{div}(s_1), \ldots, \text{div}(s_n)$ intersect $X$ properly on $U$, and so this intersection is of dimension 0. By [12, Proposition 7.1 and Example 7.1.10],

$$m_p = l_{O_p,X(U)}(O_p,X(U)/(f_1, \ldots, f_n)).$$

By [12, Example A.1.1], we have the equality

$$l_{O_p,X(U)}(O_p,X(U)/(f_1, \ldots, f_n)) = \dim_K(O_p,X(U)/(f_1, \ldots, f_n)),$$

completing the proof. \qed

For the rest of this section, we assume that the variety $X$ is projective. With this hypothesis, Chow’s moving lemma allows to construct, given a cycle and a family of Cartier divisors, another family of linearly equivalent Cartier divisors intersecting the given cycle properly, in the sense of Definition 2.1.

Definition 2.4 Let $Y$ be a $k$-cycle of $X$ and $D_1, \ldots, D_k$ Cartier divisors on $X$. The degree of $Y$ with respect to $D_1, \ldots, D_k$, denoted by $\deg_{D_1,\ldots,D_k}(Y)$, is inductively defined by the rules:

1. if $k = 0$, write $Y = \sum p m_p \cdot p$ and set $\deg(Y) = \sum p m_p [K(p):K]$;
2. if $k \geq 1$, choose a rational section $s_k$ of $O(D_k)$ such that $\text{div}(s_k)$ intersects $Y$ properly, and set
   $$\deg_{D_1,\ldots,D_k}(Y) = \deg_{D_1,\ldots,D_{k-1}}(\text{div}(s_k) \cdot Y).$$

The degree of a cycle with respect to a family of Cartier divisors does not depend on the choice of the rational section $s_k$ in (2), see for instance [12, Sect. 2.5] or [19, Sect. 1.1.C].

A Cartier divisor $D$ on $X$ is nef if $\deg_D(C) \geq 0$ for every irreducible curve $C$ of $X$. By Kleiman’s theorem [19, Sect. 1.4.B], for a family of nef Cartier divisors $D_1, \ldots, D_k$ on $X$ and an effective $k$-cycle $Y$ of $X$,

$$\deg_{D_1,\ldots,D_k}(Y) \geq 0.$$  

(2.3)

Proposition 2.5 Let $Y$ be an effective $k$-cycle of $X$ and $D_1, \ldots, D_k$ nef Cartier divisors on $X$. Let $s_k$ be a global section of $O(D_k)$. Then

$$0 \leq \deg_{D_1,\ldots,D_{k-1}}(\text{div}(s_k) \cdot Y) \leq \deg_{D_1,\ldots,D_k}(Y).$$

Proof Since $Y$ is effective and $s_k$ is a global section, $\text{div}(s_k) \cdot Y$ is also effective. Since $D_1, \ldots, D_{k-1}$ are nef, by (2.3) we have that $\deg_{D_1,\ldots,D_{k-1}}(\text{div}(s_k) \cdot Y) \geq 0$, proving the lower bound.

For the upper bound, we reduce without loss of generality to the case when $Y = V$ is an irreducible subvariety of dimension $k$. If $V \subset |\text{div}(s_k)|$, then $\text{div}(s_k) \cdot Y = 0 \in Z_{k-1}(X)$. Hence $\deg(\text{div}(s_k) \cdot Y) = 0$ and the bound follows from the nefness of the $D_i$’s. Otherwise, from the definition of the degree,

$$\deg_{D_1,\ldots,D_{k-1}}(\text{div}(s_k) \cdot V) = \deg_{D_1,\ldots,D_k}(V),$$

which completes the proof. \qed
Corollary 2.6 Let $D_1, \ldots, D_n$ be nef Cartier divisors on $X$ and, for $i = 1, \ldots, n$, let $s_i$ be a global section of $\mathcal{O}(D_i)$. Then

$$0 \leq \deg \left( \prod_{i=1}^{n} \text{div}(s_i) \right) \leq \deg_{D_1, \ldots, D_n}(X).$$

We now turn to toric varieties. Let $M \simeq \mathbb{Z}^n$ be a lattice and set

$$K[M] \simeq K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \quad \text{and} \quad \mathbb{T} = \text{Spec}(K[M]) \simeq \mathbb{C}^n_{\mathbb{m}, K}$$

for its group $K$-algebra and algebraic torus over $K$, respectively. The elements of $M$ correspond to the characters of $\mathbb{T}$ and, given $m \in M$, we denote by $\chi^m \in \text{Hom}(\mathbb{T}, \mathbb{C}_{\mathbb{m}, K})$ the corresponding character. Set also $M_{\mathbb{R}} = M \otimes \mathbb{R}$.

Let $N = M^\vee \simeq \mathbb{Z}^n$ be the dual lattice and set $N_{\mathbb{R}} = N \otimes \mathbb{R}$. Given a complete fan $\Sigma$ in $N_{\mathbb{R}}$, we denote by $X_{\Sigma}$ the associated toric variety with torus $\mathbb{T}$. It is a proper normal variety over $K$ containing $\mathbb{T}$ as a dense open subset. When the fan $\Sigma$ is regular, in the sense that it is induced by a piecewise linear concave function on $N_{\mathbb{R}}$, the toric variety $X_{\Sigma}$ is projective.

Set $X = X_{\Sigma}$ for short. Let $D$ be a toric Cartier divisor on $X$, and denote by $\Psi_D$ its associated virtual support function on $\Sigma$. This is a piecewise linear function $\Psi_D: N_{\mathbb{R}} \to \mathbb{R}$ satisfying that, for each cone $\sigma \in \Sigma$, there exists $m \in M$ such that, for all $u \in \sigma$,

$$\Psi_D(u) = \langle m, u \rangle.$$

The condition that $\Psi_D$ is concave is both equivalent to the conditions that $D$ is nef and that the line bundle $\mathcal{O}(D)$ is globally generated. This line bundle $\mathcal{O}(D)$ is a subsheaf of the sheaf of rational functions of $X$. For each $m \in M$, the character $\chi^m$ is a rational function of $X$, and so it induces a rational section of $\mathcal{O}(D)$ that is regular and nowhere vanishing on $\mathbb{T}$. The rational section corresponding to the point $m = 0$ is called the distinguished rational section of $\mathcal{O}(D)$ and denoted by $s_D$.

The toric Cartier divisor $D$ also determines the lattice polytope of $M_{\mathbb{R}}$ given by

$$\Delta_D = \{ x \in M_{\mathbb{R}} \mid \langle x, u \rangle \geq \Psi_D(u) \text{ for every } u \in N_{\mathbb{R}} \}.$$

A rational section corresponding to a point $m \in M$ is global if and only if $m \in \Delta_D$. The global sections corresponding to the lattice points of $\Delta_D$ form a $K$-basis for the space of global sections of $\mathcal{O}(D)$. Identifying each character $\chi^m$ with the corresponding rational section $\zeta_m$ of $\mathcal{O}(D)$, we have the decomposition

$$\Gamma(X, \mathcal{O}(D)) = \bigoplus_{m \in \Delta_D \cap M} K \cdot \zeta_m.$$  \hspace{1cm} (2.5)

Now let $\Delta_1, \ldots, \Delta_r$ be lattice polytopes in $M_{\mathbb{R}}$. For each $\Delta_i$, we consider its support function, which is the piecewise linear concave function with lattice slopes $\Psi_{\Delta_i}: N_{\mathbb{R}} \to \mathbb{R}$ given by

$$\Psi_{\Delta_i}(u) = \min_{x \in \Delta_i} \langle x, u \rangle.$$  \hspace{1cm} (2.6)

Let $\Sigma$ be a regular complete fan in $N_{\mathbb{R}}$ compatible with the collection $\Delta_1, \ldots, \Delta_r$, in the sense that the $\Psi_{\Delta_i}$’s are virtual support functions on $\Sigma$. Such a fan can be constructed by taking any regular complete fan in $N_{\mathbb{R}}$ refining the complex of cones that are normal to the faces of $\Delta_i$, for all $i$. Let $X$ be the toric variety corresponding to this fan and $D_i$ the toric Cartier divisor on $X$ corresponding to these virtual support functions. By construction, $\Psi_{\Delta_i}$ is concave. Hence $D_i$ is nef and $\mathcal{O}(D_i)$ is globally generated, and its associated polytope coincides with $\Delta_i$.
Let $\text{vol}_M$ be the Haar measure on $M_\mathbb{R}$ such that $M$ has covolume 1, and take $r = n$. The *mixed volume* of $\Delta_1, \ldots, \Delta_n$ is defined as the alternating sum

$$ \text{MV}_M(\Delta_1, \ldots, \Delta_n) = \sum_{j=1}^n (-1)^{n-j} \sum_{1 \leq i_1 < \cdots < i_j \leq n} \text{vol}_M(\Delta_{i_1} + \cdots + \Delta_{i_j}). $$

A fundamental result in toric geometry states that the degree of a toric variety with respect to a family of nef toric Cartier divisors is given by the mixed volume of its polytopes [13, Sect. 5.4]. In our present setting, this amounts to the formula

$$ \deg_{D_1, \ldots, D_n}(X) = \text{MV}_M(\Delta_1, \ldots, \Delta_n). \quad (2.7) $$

We turn to 0-cycles of the torus defined by families of Laurent polynomials.

**Definition 2.7** Let $f_1, \ldots, f_n \in K[M]$ and denote by $V(f_1, \ldots, f_n)_0$ the set of isolated closed points in the variety defined by this family of Laurent polynomials. For each $p \in V(f_1, \ldots, f_n)_0$, let $m_p$ be the maximal ideal of $K[M]$ corresponding to $p$ and set

$$ \mu_p = \dim_K \left( K[M]_{m_p} / (f_1, \ldots, f_n) \right). $$

The 0-cycle associated to $f_1, \ldots, f_n$ is defined as

$$ Z(f_1, \ldots, f_n) = \sum_{p \in V(f_1, \ldots, f_n)_0} \mu_p \ p \in Z_0(\mathbb{T}). $$

Let $f = \sum_{m \in M} \alpha_m x^m \in K[M]$ be a Laurent polynomial. Its *support* is defined as the finite subset of $M$ of the exponents of its nonzero terms, that is $\text{supp}(f) = \{ m | \alpha_m \neq 0 \}$. The *Newton polytope* of $f$ is the lattice polytope in $M_\mathbb{R}$ given by the convex hull of its support, that is $N(f) = \text{conv}(\text{supp}(f))$.

**Proposition 2.8** Let $f_1, \ldots, f_n \in K[M]$. Let $\Sigma$ be a regular complete fan in $N_\mathbb{R}$ compatible with the Newton polytopes of the $f_i$’s and, for $i = 1, \ldots, n$, let $D_i$ be the Cartier divisor on $X_\Sigma$ associated to $N(f_i)$ and $s_i$ the global section of $\mathcal{O}(D_i)$ corresponding to $f_i$ as in (2.5). Write $\prod_{i=1}^n \text{div}(s_i) = \sum v_p \ p$, where the sum is over the closed points of $X_\Sigma$. Then

1. for every $p \in V(f_1, \ldots, f_n)_0$, we have $v_p = \dim_K \left( K[M]_{m_p} / (f_1, \ldots, f_n) \right)$;
2. the inequality $Z(f_1, \ldots, f_n) \leq \prod_{i=1}^n \text{div}(s_i)$ holds.

**Proof** We have that $\prod_{i=1}^n |\text{div}(s_i)| = V(f_1, \ldots, f_n)$. Since $\mathbb{T}$ is Cohen-Macaulay, Proposition 2.3 gives the first statement. Since the sections $s_i$ are global, the 0-cycle $\prod_{i=1}^n \text{div}(s_i)$ is effective. Hence, the second statement follows directly from the first one. \qed

Finally, we prove the version of the Bernstein–Kušnirenko theorem in (1.1).

**Theorem 2.9** Let $f_1, \ldots, f_n \in K[M]$. Then

$$ \deg(Z(f_1, \ldots, f_n)) \leq \text{MV}_M(N(f_1), \ldots, N(f_n)). $$

**Proof** This follows from Proposition 2.8(2), Corollary 2.6 and the formula (2.7). \qed
3 Adelic fields and finite extensions

In this section, we consider adelic fields following [8]. We also give a new notion of adelic field extension that behaves better than the one in loc. cit. With this definition, the product formula is preserved when passing to finite extensions.

**Definition 3.1** Let $K$ be an infinite field and $M$ a set of places. Each place $v \in M$ is a pair consisting of an absolute value $| \cdot |_v$ and a positive real weight $n_v$. We say that $(K, M)$ is an adelic field if

1. for each $v \in M$, the absolute value $| \cdot |_v$ is either Archimedean or associated to a nontrivial discrete valuation;
2. for each $\alpha \in K^\times$, we have that $|\alpha|_v = 1$ for all but a finite number of $v \in M$.

An adelic field $(K, M)$ satisfies the product formula if, for every $\alpha \in K^\times$,

$$\prod_{v \in M} |\alpha|_v^{n_v} = 1.$$ 

Let $(K, M)$ be an adelic field. For each place $v \in M$, we denote by $K_v$ the completion of $K$ with respect to the absolute value $| \cdot |_v$. By a theorem of Ostrowski, if $v$ is Archimedean, then $K_v$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ [10, Chapter 3, Theorem 1.1]. In particular, an adelic field has only a finite number of Archimedean places.

**Example 3.2** Let $M_\mathbb{Q}$ be the set of places of $\mathbb{Q}$ consisting of the Archimedean and $p$-adic absolute values of $\mathbb{Q}$, normalized in the standard way, and with all the weights equal to one. The adelic field $(\mathbb{Q}, M_\mathbb{Q})$ satisfies the product formula.

**Example 3.3** Let $K(C)$ denote the function field of a regular projective curve $C$ over a field $\kappa$. To each closed point $v \in C$ we associate the absolute value and weight given, for $f \in K(C)^\times$, by

$$|f|_v = c_\kappa^{-\text{ord}_v(f)} \quad \text{and} \quad n_v = [K(v) : \kappa],$$

where $\text{ord}_v(f)$ denotes the order of vanishing of $f$ at $v$ and

$$c_\kappa = \begin{cases} e & \text{if } \#\kappa = \infty, \\ \#\kappa & \text{if } \#\kappa < \infty. \end{cases}$$

The set of places $M_{K(C)}$ is indexed by the closed points of $C$, and consists of these absolute values and weights. The pair $(K(C), M_{K(C)})$ is an adelic field which satisfies the product formula.

**Lemma 3.4** Let $F$ be a finite extension of $K$ and $v \in M$. Then

$$F \otimes_K K_v \simeq \bigoplus_{w} E_w,$$  

where the sum is over the absolute values $| \cdot |_w$ on $F$ whose restriction to $K_v$ coincides with $| \cdot |_v$, and where the $E_w$’s are local Artinian $K_v$-algebras with maximal ideal $p_w$. For each $w$, we have $E_w/p_w \simeq F_w$.

**Proof** Since $K \hookrightarrow F$ is a finite extension, the tensor product $F \otimes K_v$ is an Artinian $K_v$-algebra. By the structure theorem for Artinian algebras,

$$F \otimes_K K_v \simeq \bigoplus_{i \in I} E_i,$$
where $I$ is a finite set and the $E_i$’s are local Artinian $\mathbb{K}_v$-algebras. Let $p_i$ be the maximal ideal of $E_i$, for each $i$. These are the only prime ideals of $\mathbb{F} \otimes \mathbb{K}_v$, and so $\operatorname{rad}(\mathbb{F} \otimes \mathbb{K}_v) = \bigcap_{i \in I} p_i$.

Each $w$ in the decomposition (3.3) corresponds to an absolute value $| \cdot |_w$ on $\mathbb{F}$ extending $| \cdot |_v$, and there is a natural inclusion $\mathbb{F} \hookrightarrow \mathbb{F}_w$. The diagonal morphism $\mathbb{F} \to \bigoplus_w \mathbb{F}_w$ extends to a map of $\mathbb{K}_v$-vector spaces $\mathbb{F} \otimes \mathbb{K}_v \to \bigoplus_w \mathbb{F}_w$.

By [5, Chapitre VI, Sect. 8.2 Proposition 11(b)], this morphism is surjective and its kernel is the radical ideal of $\mathbb{F} \otimes \mathbb{K}_v$. Therefore

$$\bigoplus_{i \in I} E_i/p_i = \left( \bigoplus_{i \in I} E_i \right) / \operatorname{rad}(\mathbb{F} \otimes \mathbb{K}_v) \cong \bigoplus_w \mathbb{F}_w.$$  

(3.4)

The summands in both extremes of (3.4) are fields over $\mathbb{K}_v$, and so local Artinian $\mathbb{K}_v$-algebras.

By the uniqueness of the decomposition in the structure theorem for Artinian algebras, there is a bijection between the elements in $I$ and the $w$’s, identifying each $i \in I$ with the unique $w$ such that $E_i/p_i \cong \mathbb{F}_w$. \hfill \Box

The following definition was introduced by Gubler in the context of $M$-fields, see [15, Remark 2.5].

**Definition 3.5** Let $(\mathbb{K}, M)$ be an adelic field and $\mathbb{F}$ a finite extension of $\mathbb{K}$. For every place $v \in M$, we denote by $\mathcal{N}_v$ the set of absolute values $| \cdot |_w$ on $\mathbb{F}$ that extend $| \cdot |_v$ with weight given by

$$n_w = \dim_{\mathbb{K}_v}(E_w) \left[ \frac{[\mathbb{F} : \mathbb{K}]}{[\mathbb{F}_w : \mathbb{K}_v]} \right] n_v,$$

where the $E_w$’s are the local Artinian $\mathbb{K}_v$-algebras in the decomposition of $\mathbb{F} \otimes \mathbb{K}_v$ from Lemma 3.4. Set $\mathcal{N} = \bigcup_{v \in M} \mathcal{N}_v$. The pair $(\mathbb{F}, \mathcal{N})$ is an adelic field. The adelic fields of this form are called *adelic field extensions* of $(\mathbb{K}, M)$.

**Remark 3.6** With notation as in Lemma 3.4,

$$\dim_{\mathbb{K}_v}(E_w) = l_{E_w}(E_w)[\mathbb{F}_w : \mathbb{K}_v],$$

where $l_{E_w}(E_w)$ is the length of $E_w$ as a module over itself. This follows from [12, Lemma A.1.3] applied to the morphism $\mathbb{K}_v \to E_w$. Hence, the weights in Definition 3.5 can be alternatively written as

$$n_w = l_{E_w}(E_w) \left[ \frac{[\mathbb{F}_w : \mathbb{K}_v]}{[\mathbb{F} : \mathbb{K}]} \right] n_v.$$

**Proposition 3.7** Let $(\mathbb{K}, M)$ be an adelic field and $(\mathbb{F}, \mathcal{N})$ an adelic field extension of $(\mathbb{K}, M)$. Then

(1) the equality $\sum_{w \in \mathcal{N}_v} n_w = n_v$ holds for every place $v \in M$;
(2) if $(\mathbb{K}, M)$ satisfies the product formula, then $(\mathbb{F}, \mathcal{N})$ also does.

**Proof** From the definition of adelic field extension and Lemma 3.4,

$$\sum_{w \in \mathcal{N}_v} n_w = \sum_{w \in \mathcal{N}_v} \dim_{\mathbb{K}_v}(E_w) \left[ \frac{[\mathbb{F} : \mathbb{K}]}{[\mathbb{F}_w : \mathbb{K}_v]} \right] n_v = \dim_{\mathbb{K}_v}(\mathbb{F} \otimes \mathbb{K}_v) \left[ \frac{[\mathbb{F} : \mathbb{K}]}{[\mathbb{F} : \mathbb{K}]} \right] n_v = n_v,$$
which proves statement (1). To prove the second statement, let \( \alpha \in \mathbb{F}^\times \) and consider the multiplication map \( \eta_\alpha : \mathbb{F} \to \mathbb{F} \) given by \( \eta_\alpha(x) = \alpha x \). The norm \( N_{\mathbb{F}/\mathbb{K}}(\alpha) \in \mathbb{K}^\times \) is defined as the determinant of this \( \mathbb{K} \)-linear map. Moreover, \( \eta_\alpha \) extends to the \( \mathbb{K}_v \)-linear map

\[
\eta_\alpha \otimes 1_{\mathbb{K}_v} : \mathbb{F} \otimes \mathbb{K}_v \to \mathbb{F} \otimes \mathbb{K}_v,
\]

which has the same determinant. Using the decomposition in (3.3), write \( \alpha \otimes 1_{\mathbb{K}_v} = (\alpha_w)_v \) with \( \alpha_w \in E_w \). Hence \( \eta_\alpha \otimes 1_{\mathbb{K}_v} = \bigoplus_v \eta_\alpha \) and

\[
N_{\mathbb{F}/\mathbb{K}}(\alpha) = \det(\eta_\alpha \otimes 1_{\mathbb{K}_v}) = \prod_{w \in \mathcal{N}_v} N_{E_w/\mathbb{K}_v}(\alpha_w).
\]

By [6, Chapitre III, Sect. 9.2, Proposition 1], \( N_{E_w/\mathbb{K}_v}(\alpha_w) = N_{E_w/\mathbb{K}_v}(\alpha_w)^{l_{E_w}}(E_w) \). Moreover, by [18, VI Proposition 5.6],

\[
N_{E_w/\mathbb{K}_v}(\alpha_w) = \prod_{\sigma} (\alpha_w)^{l_{E_w}}(E_w)\mathbb{K}_v,
\]

where the product is over the different embeddings \( \sigma \) of \( E_w \) in an algebraic closure of \( \mathbb{K}_v \), and \( [E_w : \mathbb{K}_v]_\sigma \) denotes the inseparability degree of the extension \( \mathbb{K}_v \hookrightarrow E_w \). Furthermore, the number of such embeddings is equal to the separability degree \( [E_w : \mathbb{K}_v]_F \). For every embedding \( \sigma \), we have \( |\sigma(\alpha_w)|_v = |\alpha|_w \) because the base field \( \mathbb{K}_v \) is complete. Since \( [E_w : \mathbb{K}_v]_F = [E_w : \mathbb{K}_v] \), we get

\[
|N_{E_w/\mathbb{K}_v}(\alpha_w)|^w_v = \prod_{w \in \mathcal{N}_v} |\sigma(\alpha_w)^{l_{E_w}}(E_w)\mathbb{K}_v|_v = \prod_{w \in \mathcal{N}_v} |\alpha|_v^{[E_w : \mathbb{K}_v]}.
\]

Since \( N_{E_w/\mathbb{K}_v}(\alpha) \in \mathbb{K}^\times \), if \( (\mathbb{K}, \mathfrak{M}) \) satisfies the product formula, then

\[
\prod_{w \in \mathfrak{M}} |\alpha|_w^{n_w} = \left( \prod_{v \in \mathfrak{N}_v} |N_{E_w/\mathbb{K}_v}(\alpha)|^w_v \right)^{\frac{1}{[\mathbb{F} : \mathbb{K}]}} = 1,
\]

concluding the proof. \( \Box \)

**Example 3.8** Let \( \mathbb{F} \) be a number field. This is a separable extension of \( \mathbb{Q} \). By [5, Chapitre VI, Sect. 8.5, Corollaire 3], we have that \( \mathbb{F} \otimes \mathbb{Q}_v \simeq \bigoplus_{w \in \mathcal{N}_v} \mathbb{F}_w \) for all \( v \in \mathfrak{M}_Q \). Therefore, the weight associated to each place \( w \in \mathfrak{N}_v \) is

\[
n_w = \frac{[\mathbb{F}_w : \mathbb{Q}_v]}{[\mathbb{F} : \mathbb{Q}]}.
\]

**Example 3.9** Let \( (\mathbb{K}(\mathbb{C}), \mathfrak{M}_{\mathbb{K}(\mathbb{C})}) \) be the function field of a regular projective curve \( C \) over a field \( \kappa \) with the structure of adelic field as in Example 3.3. The places of \( \mathbb{K}(\mathbb{C}) \) correspond to the closed points of \( C \) with absolute values and weights given by (3.1). Let \( \mathbb{F} \) be a finite extension of \( \mathbb{K}(\mathbb{C}) \) and \( \mathfrak{N} \) the set of places of \( \mathbb{F} \) as in Definition 3.5. There is a regular projective curve \( B \) over \( \kappa \) and a finite map \( \pi : B \to C \) such that the extension \( \mathbb{K}(\mathbb{C}) \hookrightarrow \mathbb{F} \) identifies with the morphism \( \pi^* : \mathbb{K}(\mathbb{C}) \hookrightarrow \mathbb{K}(B) \). For each place \( v \in \mathfrak{M}_{\mathbb{K}(\mathbb{C})} \), the absolute values of \( \mathbb{F} \) that extend \( |\cdot|_v \) are in bijection with the fiber \( \pi^{-1}(v) \).

For each closed point \( v \in C \), the integral closure in \( \mathbb{K}(B) \) of \( \mathcal{O}_{v,C} \) coincides with \( \mathcal{O}_{\pi^{-1}(v),B} \), the local ring of \( B \) along the fiber \( \pi^{-1}(v) \). The ring \( \mathcal{O}_{\pi^{-1}(v),B} \) is of finite type over \( \mathcal{O}_{v,C} \). With notation as in Lemma 3.4, by [5, Chapter VI, Sect. 8.5, Corollaire 3], we have \( E_w \simeq \mathbb{F}_w \) for all \( w \in \mathfrak{N}_v \). Hence, the weight of \( w \) is given by

\[
n_w = \frac{[\mathbb{F}_w : \mathbb{K}(\mathbb{C})]}{[\mathbb{F} : \mathbb{K}(\mathbb{C})]}[\mathbb{K}(v) : \kappa].
\]
Let \( e(w/v) \) denote the ramification index of \( w \) over \( v \). By [5, Chapter VI, Sect. 8.5, Corollaire 2], we have that \( [\mathbb{F}_w : \mathbb{K}(C)_v] = e(w/v) [\mathbb{K}(w) : \mathbb{K}(v)] \). Therefore, for each place \( w \in \mathfrak{M}_v \), the weight of \( w \) can also be expressed as
\[
n_w = \frac{e(w/v) [\mathbb{K}(w) : \kappa]}{[\mathbb{F} : \mathbb{K}(C)]}.
\]

Following [8], a global field is a finite extension of the field of rational numbers or of the function field of a regular projective curve, with the structure of adelic field described in Examples 3.8 and 3.9. For these fields, Proposition 3.7 is already a known result, see for instance [7, Proposition 2.1].

By a result of Artin and Whaples, global fields can be characterized as the adelic fields having an absolute value that is either Archimedean or associated to a discrete valuation whose residue field has finite order over the field of constants [1, Theorems 2 and 3].

Function fields of varieties of higher dimension provide examples of adelic fields satisfying the product formula, and that are not global fields.

**Example 3.10** Let \( \mathbb{K}(S) \) be the function field of an irreducible normal variety \( S \) over a field \( \kappa \) of dimension \( s \geq 1 \), and \( E_1, \ldots, E_{s-1} \) nef Cartier divisors on \( S \). Set \( S^{(1)} \) for the set of irreducible hypersurfaces of \( S \). For each \( V \in S^{(1)} \), the local ring \( \mathcal{O}_{V,S} \) is a discrete valuation ring. We associate to \( V \) the absolute value and weight given, for \( f \in \mathbb{K}(S) \), by
\[
|f|_V = c_{E_1, \ldots, E_{s-1}} \text{ord}_V(f) \quad \text{and} \quad n_V = \deg_{E_1, \ldots, E_{s-1}}(V),
\]
with \( c_E \) as in (3.2). The set of places \( \mathfrak{M}_{\mathbb{K}(S)} \) is indexed by \( S^{(1)} \), and consists of these absolute values and weights. For \( f \in \mathbb{K}(S)^\times \),
\[
\sum_{V \in S^{(1)}} n_V \log |f|_V = \log(c_E) \sum_{V \in S^{(1)}} \deg_{E_1, \ldots, E_{s-1}}(V) \text{ord}_V(f) = \deg_{E_1, \ldots, E_{s-1}}(\text{div}(f)) = 0,
\]
because the Cartier divisor \( \text{div}(f) \) is principal. Hence \( (\mathbb{K}(S), \mathfrak{M}_{\mathbb{K}(S)}) \) satisfies the product formula.

### 4 Height of cycles

In this section, we introduce a notion of global height for cycles of a variety over an adelic field, with respect to a family of metrized divisors generated by small sections. We also recall the notion of local height of cycles from [8, Chapter 1] and give a more explicit description of this construction in the 0-dimensional case.

Let \( (\mathbb{K}, \mathfrak{M}) \) be an adelic field satisfying the product formula, and \( X \) a normal projective variety over \( \mathbb{K} \). For each place \( v \in \mathfrak{M} \), we denote by \( X^\text{an}_v \) the \( v \)-adic analytification of \( X \). In the Archimedean case, if \( \mathbb{K}_v \simeq \mathbb{C} \), then \( X^\text{an}_v \) is an analytic space over \( \mathbb{C} \) whereas, if \( \mathbb{K}_v \simeq \mathbb{R} \), then \( X^\text{an}_v \) is an analytic space over \( \mathbb{R} \), that is, an analytic space over \( \mathbb{C} \) together with an antilinear involution, as explained in [8, Remark 1.1.5]. In the non-Archimedean case, \( X^\text{an}_v \) is a Berkovich space over \( \mathbb{K}_v \) as in [8, Sect. 1.2].

Fix \( v \in \mathfrak{M} \) and set
\[
X_v = X \times \text{Spec}(\mathbb{K}_v).
\]
Given a 0-cycle \( Y \) of \( X_v \), a usual construction in Arakelov geometry associates a signed measure on \( X^\text{an}_v \), denoted by \( \delta_Y \), that is supported on \( |Y|_v \) and has total mass equal to \( \deg(Y) \),
Let \( q \) be a closed point of \( X_v \). The function field \( K(q) \) is a finite extension of \( \mathbb{K}_v \) and \( \deg(q) = [K(q) : \mathbb{K}_v] \). If \( v \) is Archimedean, then \( \deg(q) \) is either equal to one or two. In the first case, the analytification of \( q \) is a point of \( X^an_v \), whereas, in the second case, it is a pair of conjugate points. If \( v \) is non-Archimedean, choose an affine open neighborhood \( U = \text{Spec}(A) \) of \( q \) and \( A \to K(q) \) the corresponding morphism of \( \mathbb{K}_v \)-algebras. The analytification of \( q \) is the point \( q^an \in U^an \subset X^an_v \) corresponding to the multiplicative seminorm given by the composition

\[
A \longrightarrow K(q) \xrightarrow{|\cdot|} \mathbb{R}_{\geq 0},
\]

where \(| \cdot |\) is the unique extension to \( K(q) \) of the absolute value \(| \cdot |_v\).

Since the measure \( \delta_q \) is supported on the point \( q^an \) and has total mass \( \deg(q) \), it follows that

\[
\delta_q = [K(q) : \mathbb{K}_v] \delta_{q^an}, \tag{4.1}
\]

where \( \delta_{q^an} \) denotes the Dirac delta measure on \( q^an \). For an arbitrary 0-cycle \( Y \) of \( X_v \), the signed measure \( \delta_Y \) is obtained from (4.1) by linearity. It is a discrete signed measure of total mass equal to \( \deg(Y) \).

Let \( D \) be a Cartier divisor on \( X \). A metric on the analytic line bundle \( \mathcal{O}(D)^an_v \) is an assignment that, to each open subset \( U \subset X^an_v \) and local section \( s \) on \( U \), associates a continuous function

\[
\|s(\cdot)\|_v : U \longrightarrow \mathbb{R}_{\geq 0}
\]

that is compatible with restrictions to open subsets, vanishes only when the local section does, and respects multiplication of local sections by analytic functions, see \([8, \text{Definitions 1.1.1 and 1.3.1}]\). This notion allows to define local heights of 0-cycles.

**Definition 4.1** Let \( D \) be a Cartier divisor on \( X \), and \( \| \cdot \|_v \), a metric on \( \mathcal{O}(D)^an_v \). For a 0-cycle \( Y \) of \( X_v \) and a rational section \( s \) of \( \mathcal{O}(D) \) that is regular and non-vanishing on the support of \( Y \), the local height of \( Y \) with respect to the pair \( (\| \cdot \|_v, s) \) is defined as

\[
h_{\| \cdot \|_v}(Y; s) = -\int_{X^an_v} \log \|s\|_v \delta_Y.
\]

We now study the behavior of these objects with respect to adelic field extensions. Let \((\mathbb{F}, \mathfrak{f})\) be an extension of the adelic field \((\mathbb{K}, \mathfrak{m})\) (Definition 3.5) and fix a place \( w \in \mathfrak{m}_v \), so that \( \mathbb{F}_w \) is a finite extension of the local field \( \mathbb{K}_v \). Let \( q \) be a closed point of \( X_v \) and consider the subscheme \( q_w \) of \( X_w = X \times \text{Spec}(\mathbb{F}_w) \) obtained by base change. Decompose

\[
K(q) \otimes_{\mathbb{K}_v} \mathbb{F}_w = \bigoplus_{j \in I} G_j
\]

as a finite sum of local Artinian \( \mathbb{F}_w \)-algebras and, for each \( j \in I \), denote by \( q_j \) the corresponding closed point of \( X_w \). Thus the associated cycle is given by \( [q_w] = \sum_{j \in I} 1_{G_j}(G_j) \cdot q_j \). Hence, by (4.1) and Remark 3.6,

\[
\delta_{[q_w]} = \sum_{j \in I} \dim_{\mathbb{F}_w}(G_j) \cdot \delta_{q_j^an}.
\]

The inclusion \( \mathbb{K}_v \hookrightarrow \mathbb{F}_w \) induces a map of the corresponding analytic spaces

\[
\pi : X^an_w \longrightarrow X^an_v. \tag{4.2}
\]
In the non-Archimedean case, this map of Berkovich spaces is defined locally by restricting seminorms.

The following proposition gives the behavior of the measure associated to a 0-cycle with respect to field extensions.

**Proposition 4.2** With notation as above, let $Y$ be a 0-cycle of $X_v$ and set $Y_w$ for the 0-cycle of $X_w$ obtained by base change. Then

$$\pi_\ast \delta_Y = \delta_Y.$$  

**Proof** By the compatibility of the map $\pi$ with restriction to subschemes, we have that $\pi(q^\text{an}_j) = q^\text{an}$ for all $j \in I$. It follows that

$$\pi_\ast \delta[q_w] = \sum_{j \in I} \dim_F(G_j) \pi_\ast \delta q^\text{an}_j = \left( \sum_{j \in I} \dim_F(G_j) \right) \delta q^\text{an} = [K(q) : K_v] \delta q^\text{an} = \delta_q.$$  

$\square$

Let $D$ be a Cartier divisor on $X$ and $\| \cdot \|_v$ a metric on $O(D)_{\text{an}}^v$. The extension of this metric to a metric $\| \cdot \|_w$ on the analytic line bundle $O(D)_{\text{an}}^w$ on $X_{\text{an}}^w$ is obtained by taking the inverse image with respect to the map $\pi$ in (4.2), that is

$$\| \cdot \|_w = \pi_\ast \| \cdot \|_v.$$  

(4.3)

Proposition 4.2 implies directly the invariance of the local height with respect to adelic field extensions.

**Proposition 4.3** With notation as above, let $Y$ be a 0-cycle of $X_v$ and $s$ a rational section of $O(D)_{\text{an}}^v$ that is regular and non-vanishing on the support of $Y$. Set $Y_w$ and $s_w = \pi_\ast s$ for the 0-cycle and rational section obtained by base extension. Then

$$h_{\| \cdot \|_w}(Y_w, s_w) = h_{\| \cdot \|_v}(Y, s).$$

To define global heights of cycles over an adelic field, we consider adelic families of metrics on the Cartier divisor $D$ satisfying a certain compatibility condition.

**Definition 4.4** An (adelic) metric on $D$ is a collection $\| \cdot \|_v$ of metrics on $O(D)_{\text{an}}^v$, $v \in \mathcal{M}$, such that, for every point $p \in X(\overline{K})$ and a choice of a rational section $s$ of $O(D)$ that is regular and non-vanishing at $p$ and of an adelic field extension $(\mathcal{F}, \mathcal{N})$ such that $p \in X(\mathcal{F})$,

$$\| s(p^\text{an}_w) \|_w = 1$$  

(4.4)

for all but a finite number of $w \in \mathcal{N}$. We denote by $\overline{D} = (D, (\| \cdot \|_v)_{v \in \mathcal{M}})$ the corresponding (adelically) metrized divisor on $X$.

In addition, $\overline{D}$ is semipositive if each of its $v$-adic metrics is semipositive in the sense of [8, Definition 1.4.1].

The condition (4.4) does not depend on the choice of the rational section $s$ and of the adelic field extension $(\mathcal{F}, \mathcal{N})$.

**Remark 4.5** When $K$ is a global field, the classical notion of compatibility for a collection of metrics $\| \cdot \|_v$ on $O(D)_{\text{an}}^v$, $v \in \mathcal{M}$, is that of being quasi-algebraic, in the sense that there is an integral model that induces all but a finite number of these metrics [8, Definition 1.5.13].

By Proposition 1.5.14 in loc. cit., a quasi-algebraic metrized divisor $\overline{D}$ is adelic in the sense of Definition 4.4. The converse is not true, as it is easy to construct toric adelic metrized divisors that are not quasi-algebraic (Remark 5.4).
For a 0-cycle $Y$ of $X$ and a place $v \in \mathcal{M}$, we denote by $Y_v$ the 0-cycle of $X_v$ defined by base change. When $Y = p$ is a closed point of $X$, by Lemma 3.4 applied to the finite extension $K(p)$ of $K$, the 0-dimensional subscheme $p_v = p \times \text{Spec}(K_v)$ of $X_v$ decomposes as

$$p_v = \text{Spec}(K(p) \otimes_K K_v) \simeq \bigsqcup_{w \in \mathfrak{N}_v} \text{Spec}(E_w),$$

where the $E_i$'s are the local Artinian $K_v$-algebras in (3.3). Let $q_w$, $w \in \mathfrak{N}_v$, be the irreducible components of this subscheme. Then, the associated 0-cycle of $X_v$ writes down as

$$[p_v] = \sum_{w \in \mathfrak{N}_v} l_{E_w}(E_w) q_w$$

and, for each $w \in \mathfrak{N}_v$, we have $K(q_w) \simeq K(p)_w$. For an arbitrary $Y$, the 0-cycle $Y_v$ is obtained by linearity.

Let $D = (D, (\| \cdot \|_v)_{v \in \mathcal{M}})$ be a metrized divisor on $X$, $Y$ a 0-cycle of $X$ and $s$ a rational section of $O(D)$ that is is regular and non-vanishing on the support of $Y$. For each place $v \in \mathcal{M}$, we set

$$h_{D,v}(Y; s) = h_{\| \cdot \|_v}(Y_v; s),$$

where $Y_v$ is the 0-cycle of $X_v$ obtained by base change. The condition that $D$ is adelic implies that $h_{D,v}(Y; s) = 0$ for all but a finite number of places.

If $s'$ is another rational section of $O(D)$ that is is regular and non-vanishing on $|Y|$, then $s' = fs$ with $f \in K(X)^\times$ and, for $v \in \mathcal{M}$,

$$h_{D,v}(Y; s') = h_{D,v}(Y; s) - \log |\gamma|_v$$

where $Y = \sum_p \mu_p p$ and $\gamma = \prod_p f(p)^{\mu_p} \in K^\times$.

**Definition 4.6** Let $D$ be a metrized divisor on $X$ and $Y$ a 0-cycle of $X$. The *global height* of $Y$ with respect to $D$ is defined as

$$h_D(Y) = \sum_{v \in \mathcal{M}} n_v h_{D,v}(Y; s),$$

with $s$ a rational section of $O(D)$ that is is regular and non-vanishing on $|Y|$.

The local heights in (4.6) are zero for all but a finite number of places, and so this sum is finite. The equality (4.5) together with the product formula imply that this sum does not depend on the rational section $s$.

Given a metrized divisor $D$ on $X$ and an adelic field extension $(\mathbb{F}, \mathcal{M})$, we denote by $D_{\mathbb{F}}$ the metrized divisor on $X_{\mathbb{F}}$ obtained by extending the $v$-adic metrics of $D$ as in (4.3).

**Proposition 4.7** Let $D$ be a metrized divisor on $X$, $Y$ a 0-cycle of $X$ and $(\mathbb{F}, \mathcal{M})$ an adelic field extension of $(K, \mathcal{M})$. Then

$$h_{D_{\mathbb{F}}}(Y_{\mathbb{F}}) = h_D(Y).$$

**Proof** Let $s$ be a rational section of $O(D)$ that is is regular and non-vanishing on $|Y|$ and $v \in \mathcal{M}$. By Propositions 4.3 and 3.7(1),

$$\sum_{w \in \mathfrak{N}_v} n_w h_{D_{\mathbb{F}},w}(Y_{\mathbb{F}}, s) = \sum_{w \in \mathfrak{N}_v} n_w h_{D,v}(Y, s) = n_v h_{D,v}(Y, s).$$

The statement follows by summing over all the places of $K$. 

$\square$
Since the global height is invariant under field extension, it induces a notion of global height for algebraic points, that is, a well-defined function

\[ h_{\mathcal{F}} : X(\mathbb{K}) \rightarrow \mathbb{R}. \]

When \( \mathbb{K} \) is a global field, this notion coincides with the one in [7, Definition 2.2].

Now we turn to cycles of arbitrary dimension. Let \( V \) be a \( k \)-dimensional irreducible subvariety of \( X \) and \( D_0, \ldots, D_{k-1} \) a family of \( k \) semipositive metrized divisors on \( X \). For each place \( v \in M \), we can associate to this data a measure on \( X_{v}^{an} \) denoted by

\[ c_1(D_0) \wedge \cdots \wedge c_1(D_{k-1}) \wedge \delta_{Y_{v}^{an}} \]

and called the \( v \)-adic Monge-Ampère measure of \( V \) and \( D_0, \ldots, D_{k-1} \), see [11, Définition 2.4] or [8, Definition 1.4.6]. For a \( k \)-cycle \( Y \) of \( X \), this notion extends by linearity to a signed measure on \( X_{v}^{an} \), denoted by \( c_1(D_0) \wedge \cdots \wedge c_1(D_{k-1}) \wedge \delta_{Y_{v}^{an}} \). It is supported on \(|Y_v|_{v}^{an} \) and has total mass equal to the degree \( \deg_{D_0,\ldots,D_{k-1}}(Y) \).

We recall the notion of local height of cycles from [8, Definition 1.4.11].

**Definition 4.8** Let \( Y \) be a \( k \)-cycle of \( X \) and, for \( i = 0, \ldots, k \), let \((D_i, s_i)\) be a semipositive metrized divisor on \( X \) and a rational section of \( \mathcal{O}(D_i) \) such that \( \text{div}(s_0), \ldots, \text{div}(s_k) \) intersect \( Y \) properly (Definition 2.1). For \( v \in M \), the local height of \( Y \) with respect to \((D_0, s_0), \ldots, (D_k, s_k)\) is inductively defined by the rule

\[
\begin{align*}
\hbar_{D_0,\ldots,D_{k-1},v}(Y; s_0, \ldots, s_k) &= \hbar_{D_0,\ldots,D_{k-1},v}(\text{div}(s_k) \cdot Y; s_0, \ldots, s_{k-1}) \\
&- \int_{X_v^{an}} \log \|s_k\|_v \, c_1(D_0) \wedge \cdots \wedge c_1(D_{k-1}) \wedge \delta_{Y_{v}^{an}}
\end{align*}
\]

and the convention that the local height of the cycle \( 0 \in Z_{-1}(X) \) is zero.

**Remark 4.9** (1) The local height is linear with respect to the group structure of \( Z_k(X) \). In particular, the local heights of the cycle \( 0 \in Z_k(X) \) are zero.

(2) For a \( 0 \)-cycle \( Y \) of \( X \) and \( v \in M \), the \( v \)-adic Monge-Ampère measure coincides with the measure associated to the \( 0 \)-cycle \( Y_v \) of \( X_v \) at the beginning of this section. Hence, Definition 4.8 applied to a \( 0 \)-cycle coincides with Definition 4.1.

The following notion is the arithmetic analogue of global sections of a line bundle, and Proposition 4.11 below is an analogue for local heights of Proposition 2.5.

**Definition 4.10** Let \( \overline{D} = (D, (\|\cdot\|_v)_{v \in M}) \) be a metrized divisor on \( X \). A global section \( s \) of \( \mathcal{O}(D) \) is \( \overline{D} \)-small if, for all \( v \in M \),

\[ \sup_{q \in X_{v}^{an}} \|s(q)\|_v \leq 1. \]

**Proposition 4.11** Let \( Y \) be an effective \( k \)-cycle of \( X \) and, for \( i = 0, \ldots, k \), let \((\overline{D}_i, s_i)\) be a semipositive metrized divisor on \( X \) and a rational section of \( \mathcal{O}(D_i) \) such that \( \text{div}(s_0), \ldots, \text{div}(s_k) \) intersect \( Y \) properly and such that \( s_k \) is \( \overline{D}_k \)-small. Then, for all \( v \in M \),

\[ \hbar_{D_0,\ldots,D_{k-1},v}(\text{div}(s_k) \cdot Y; s_0, \ldots, s_{k-1}) \leq \hbar_{D_0,\ldots,D_k,v}(Y; s_0, \ldots, s_k). \]

**Proof** Since the cycle \( Y \) is effective and the metrized divisors \( \overline{D}_i \) are semipositive, their \( v \)-adic Monge-Ampere measure is a measure, that is, it takes only nonnegative values. Since the global section \( s_k \) is \( \overline{D}_k \)-small, \( \log \|s_k(q)\|_{k,v} \leq 0 \) for all \( q \in X_{v}^{an} \). The inequality follows then from the inductive definition of the local height. \( \square \)
Our next step is to define global heights for cycles over an adelic field. We first state an auxiliary result specifying the behavior of local heights with respect to change of sections, extending (4.5) to the higher dimensional case. The following lemma and its proof are similar to [15, Corollary 3.8].

**Lemma 4.12** Let \( Y \) be a \( k \)-cycle of \( X \) and \( \overline{D}_0, \ldots, \overline{D}_k \) semipositive metrized divisors on \( X \). Let \( s_i, s_i' \) be rational sections of \( \mathcal{O}(D_i), i = 0, \ldots, k \), such that both \( \text{div}(s_0), \ldots, \text{div}(s_k) \) and \( \text{div}(s_0'), \ldots, \text{div}(s_k') \) intersect \( Y \) properly. Then there exists \( \gamma \in \mathbb{K}^\times \) such that, for all \( v \in \mathfrak{M} \),

\[
\log h_{\overline{D}_0, \ldots, \overline{D}_k, v}(Y; s_0', \ldots, s_k') = \log h_{\overline{D}_0, \ldots, \overline{D}_k, v}(Y; s_0, \ldots, s_k) - \log |\gamma|_v. \tag{4.7}
\]

**Proof** Let \( s_i'' \) be rational sections of \( \mathcal{O}(D_i), i = 0, \ldots, k \), such that \( (s_0'', \ldots, s_k'') \) is generic enough so that, for every subset \( J \subset \{0, \ldots, k\} \), the family of divisors

\[
\{\text{div}(s_j) \mid j \in J\} \cup \{\text{div}(s'_j) \mid j \notin J\}
\]

intersects \( Y \) properly.

We proceed to prove the formula (4.7) with the \( s_i'' \)’s in the place of the \( s_i' \)’s. Hence, we want to prove that there is \( \tilde{\gamma} \in \mathbb{K}^\times \) such that, for every \( v \in \mathfrak{M} \),

\[
\log h_{\overline{D}_0, \ldots, \overline{D}_k, v}(Y; s_0'', \ldots, s_k'') = \log h_{\overline{D}_0, \ldots, \overline{D}_k, v}(Y; s_0, \ldots, s_k) - \log |\tilde{\gamma}|_v. \tag{4.9}
\]

To this end, consider first the particular case when \( s_i = s_i'' \), \( i = 0, \ldots, k - 1 \). Set \( s_k' = f s_k \) with \( f \in \mathbb{K}(X)^\times \), and \( \left( \prod_{i=0}^{k-1} \text{div}(s_i) \right) \cdot Y = \sum_p \mu_p p \). By [8, Theorem 1.4.17(3)], the equality (4.7) holds with \( \tilde{\gamma}_k = \prod_p f(p)^{\mu_p} \).

By [8, Theorem 1.4.17(1)], the local height is symmetric in the pairs \((\overline{D}_i, s_i)\). By the hypothesis (4.8), we can reorder the metrized line bundles and rational sections, and iterate the above construction for every \( i = 0, \ldots, k \). This proves (4.9) with \( \tilde{\gamma} = \prod_{i=0}^k \tilde{\gamma}_i \).

Assuming that the \( s_i'' \)’s are generic enough so that the condition in (4.8) also holds with the \( s_i' \)’s instead of the \( s_i'' \)’s, similarly there exists \( \tilde{\gamma}' \in \mathbb{K}^\times \) such that, for every \( v \in \mathfrak{M} \),

\[
\log h_{\overline{D}_0, \ldots, \overline{D}_k, v}(Y; s_0'', \ldots, s_k'') = \log h_{\overline{D}_0, \ldots, \overline{D}_k, v}(Y; s_0', \ldots, s_k') - \log |\tilde{\gamma}'|_v. \tag{4.10}
\]

The statement follows by combining (4.9) and (4.10). \( \square \)

We consider the following notions of positivity of metrized divisors.

**Definition 4.13** Let \( \overline{D} \) be a metrized divisor on \( X \).

1. \( \overline{D} \) is nef if \( D \) is nef, \( \overline{D} \) is semipositive, and \( \log h_{\overline{D}}(p) \geq 0 \) for every closed point \( p \) of \( X \).
2. \( \overline{D} \) is generated by small sections if, for every closed point \( p \in X \), there is a \( \overline{D} \)-small section \( s \) such that \( p \notin |\text{div}(s)| \).

**Lemma 4.14** Let \( Y \) be an effective \( k \)-cycle of \( X \) and \((\overline{D}_i, s_i)\) semipositive metrized divisors on \( X \) together with rational sections of \( \mathcal{O}(D_i), i = 0, \ldots, k \), such that \( \text{div}(s_0), \ldots, \text{div}(s_k) \) intersect \( Y \) properly. Suppose that \( \overline{D}_i, i = 1, \ldots, k \), are generated by small sections. Then there exists \( \zeta \in \mathbb{K}^\times \) such that, for all \( v \in \mathfrak{M} \),

\[
\log h_{\overline{D}_0, \ldots, \overline{D}_k, v}(Y; s_0, \ldots, s_k) \geq \log |\zeta|_v + \log h_{\overline{D}_0, v}\left(\prod_{i=1}^k \text{div}(s_i)\right) \cdot Y, s_0). \]
Proof For $k = 0$, the statement is obvious, so we only consider the case when $k \geq 1$. By Lemma 4.12, it is enough to prove the statement for any particular choice of rational sections $s_i$, provided that their associated Cartier divisors intersect $Y$ properly.

We can also reduce without loss of generality to the case when $Y = V$ is an irreducible variety of dimension $k$. We can then choose rational sections $s_i$, $i = 0, \ldots, k$, such that $s_i$ is $D_i$-small. By Proposition 4.11,

$$h_{\{D_0, \ldots, D_k\}, v}(V; s_0, \ldots, s_k) \geq h_{\{D_0, \ldots, D_{k-1}\}, v}(\text{div}(s_k) \cdot V; s_0, \ldots, s_{k-1}).$$

Since $\text{div}(s_k) \cdot V$ is an effective $(k-1)$-cycle, the statement follows by induction on $k$. \qed

Proposition-Definition 4.15 Let $Y$ be an effective $k$-cycle of $X$, and $D_0, \ldots, D_k$ semipositive metrized divisors on $X$ such that $D_1, \ldots, D_k$ are generated by small sections. Let $s_i$ be rational sections of $\mathcal{O}(D_i)$, $i = 0, \ldots, k$, such that $\text{div}(s_0), \ldots, \text{div}(s_k)$ intersects $Y$ properly. The global height of $Y$ with respect to $D_0, \ldots, D_k$ is defined as the sum

$$h_{\{D_0, \ldots, D_k\}, v}(Y) = \sum_{v \in \mathfrak{M}} n_v h_{\{D_0, \ldots, D_k\}, v}(Y; s_0, \ldots, s_k). \tag{4.11}$$

This sum converges to an element in $\mathbb{R} \cup \{+\infty\}$, and its value does not depend on the choice of the $s_i$'s.

Proof The existence of rational sections $s_i$ such that $\text{div}(s_0), \ldots, \text{div}(s_k)$ intersects $Y$ properly follows from the moving lemma, with the hypothesis that $X$ is projective.

By Lemma 4.14 and the fact that the local heights of $0$-cycles are zero for all but a finite number of places, the local heights in (4.11) are nonnegative, except for a finite number of $v$'s. Hence, the sum converges to an element in $\mathbb{R} \cup \{+\infty\}$. Lemma 4.12 and the product formula imply that the value of this sum does not depend on the choice of the $s_i$'s. \qed

This definition generalizes the notion of global height of cycles of varieties over global fields in [8, Sect. 1.5], to cycles of varieties over an arbitrary adelic field, in the case when the considered metrized divisors are generated by small sections.

In principle, the sum in (4.11) might contain an infinite number of nonzero terms. Nevertheless, we are not aware of any example where this phenomenon actually happens. Moreover, for varieties over global fields, the local heights of a given cycle are zero for all but a finite number of places [8, Proposition 1.5.14], and so their global height is a real number given as a weighted sum of a finite number local heights.

In this context, we propose the following question.

Question 4.16 Let $Y$ be a $k$-cycle of $X$ and, for each $i = 0, \ldots, k$, let $(D_i, s_i)$ be a semipositive metrized divisor on $X$ and a rational section of $\mathcal{O}(D_i)$ such that $\text{div}(s_0), \ldots, \text{div}(s_k)$ intersect $Y$ properly. Is it true that

$$h_{\{D_0, \ldots, D_k\}, v}(Y; s_0, \ldots, s_k) = 0$$

for all but a finite number of $v \in \mathfrak{M}_{K}$?

A positive answer would imply that, for a variety over an adelic field and a family of semipositive metrized divisors, the global height of a cycle is a well-defined real number, given as a weighted sum of a finite number local heights.

The following results are arithmetic analogues of Proposition 2.5 and Corollary 2.6.
Proposition 4.17 Let $Y$ be an effective $k$-cycle of $X$, and $\overline{D}_0, \ldots, \overline{D}_k$ semipositive metrized divisors on $X$ such that $\overline{D}_0$ is nef and $\overline{D}_1, \ldots, \overline{D}_k$ are generated by small sections. Let $s_k$ be a $\overline{D}_k$-small section. Then

$$0 \leq h_{\overline{D}_0, \ldots, \overline{D}_{k-1}}(\text{div}(s_k) \cdot Y) \leq h_{\overline{D}_0, \ldots, \overline{D}_k}(Y).$$

Proof We reduce without loss of generality to the case when $Y = V$ is an irreducible subvariety of dimension $k$. If $V \subset |\text{div}(s_k)|$, the first inequality is clear. For the second inequality, we choose rational sections $s_i, i = 0, \ldots, k-1$, and $s'_k$ such that $\text{div}(s_0), \ldots, \text{div}(s_{k-1}), \text{div}(s'_k)$ intersect $Y$ properly. Using Lemmas 4.12 and 4.14, the product formula and the fact that $\overline{D}_0$ is nef, we deduce that $h_{\overline{D}_0, \ldots, \overline{D}_k}(Y) \geq 0$.

Otherwise, $V \not\subset |\text{div}(s_k)|$ and we choose rational sections $s_i, i = 0, \ldots, k-1$, such that $\text{div}(s_0), \ldots, \text{div}(s_k)$ intersect $Y$ properly. The first inequality follows by applying the argument above to $\text{div}(s_k) \cdot Y$, whereas the second one is given by Proposition 4.11.

Corollary 4.18 Let $\overline{D}_0, \ldots, \overline{D}_n$ be semipositive metrized divisors on $X$ such that $\overline{D}_0$ is nef and $\overline{D}_1, \ldots, \overline{D}_n$ are generated by small sections. Let $s_i$ be $\overline{D}_i$-small sections, $i = 1, \ldots, n$. Then

$$0 \leq h_{\overline{D}_0, \ldots, \overline{D}_n} \left( \prod_{i=1}^n \text{div}(s_i) \right) \leq h_{\overline{D}_0, \ldots, \overline{D}_n}(X).$$

5 Metrics and heights on toric varieties

In this section, we recall the necessary background on the arithmetic geometry of toric varieties following [4,8]. In the second part of Sect. 2, we presented elements of the algebraic geometry of toric varieties over a field. In the sequel, we will freely use the notation introduced therein.

Let $(\mathbb{K}, \mathcal{M})$ be an adelic field satisfying the product formula. Let $M \simeq \mathbb{Z}^n$ be a lattice and $\mathbb{T} \simeq \mathbb{G}_{m, \mathbb{K}}^n$ its associated torus over $\mathbb{K}$ as in (2.4). For $v \in \mathcal{M}$, we denote by $\mathbb{T}_{v, an}$ the $v$-adic analytification of $\mathbb{T}$, and by $\mathbb{S}_v$ its compact torus. In the Archimedean case, $\mathbb{S}_v$ is isomorphic to the polycircle $(S^1)^n$, whereas in the non-Archimedean case, it is a compact analytic group, see [8, Sect. 4.2] for a description. Moreover, there is a map $\text{val}_v : \mathbb{T}_{v, an} \to \mathbb{N}_{\mathbb{R}}$ defined, in a given splitting, as

$$\text{val}_v(x_1, \ldots, x_n) = (-\log |x_1|_v, \ldots, -\log |x_n|_v).$$

This map does not depend on the choice of the splitting, and the compact torus $\mathbb{S}_v$ coincides with its fiber over the point $0 \in \mathbb{N}_{\mathbb{R}}$.

Let $X$ be a projective toric variety with torus $\mathbb{T}$ given by a regular complete fan $\Sigma$ on $\mathbb{N}_{\mathbb{R}}$, and $D$ a toric Cartier divisor on $X$ given by a virtual support function $\Psi_D$ on $\Sigma$. Recall that $X$ contains $\mathbb{T}$ as a dense open subset. Let $\| \cdot \|_v$ be a toric $v$-adic metric on $D$, that is, a metric on the analytic line bundle $\mathcal{O}(D)_{v, an}$ that is invariant under the action of $\mathbb{S}_v$. The associated $v$-adic metric function is the continuous function $\psi_{\| \cdot \|_v} : \mathbb{N}_{\mathbb{R}} \to \mathbb{R}$ given by

$$\psi_{\| \cdot \|_v}(u) = \log \|s_D(p)\|_v, \quad (5.1)$$

for any $p \in \mathbb{T}_{v, an}$ with $\text{val}_v(p) = u$ and where $s_D$ is the distinguished rational section of $\mathcal{O}(D)$. This function satisfies that $|\psi_{\| \cdot \|_v} - \Psi_D|$ is bounded on $\mathbb{N}_{\mathbb{R}}$ and moreover, this difference
extends to a continuous function on $N$, the compactification of $N$ induced by the fan $\Sigma$. Indeed, the assignment

$$\| \cdot \|_v \mapsto \psi_{\| \cdot \|_v}$$

(5.2)
is a one-to-one correspondence between the set of toric $v$-adic metrics on $D$ and the set of such continuous functions on $N$ [8, Proposition 4.3.10]. In particular, the toric $v$-adic metric on $D$ associated to the virtual support function $\Psi_D$ is called the canonical $v$-adic toric metric of $D$ and is denoted by $\| \cdot \|_{v,\text{can}}$.

Furthermore, when $\| \cdot \|_v$ is semipositive, $\psi_{\| \cdot \|_v}$ is a concave function and it verifies that $|\psi_{\| \cdot \|_v} - \Psi_D|$ is bounded on $N$, and the assignment in (5.2) gives a one-to-one correspondence between the set of semipositive toric $v$-adic metrics on $D$ and the set of such concave functions on $N$.

When $\| \cdot \|_v$ is semipositive, we also consider a continuous concave function on the polytope $\vartheta_{\| \cdot \|_v} : \Delta_D \to \mathbb{R}$ defined as the Legendre-Fenchel dual of $\psi_{\| \cdot \|_v}$, that is

$$\vartheta_{\| \cdot \|_v}(x) = \inf_{u \in N} \langle x, u \rangle - \psi_{\| \cdot \|_v}(u).$$

The assignment $\| \cdot \|_v \mapsto \vartheta_{\| \cdot \|_v}$ is a one-to-one correspondence between the set of semipositive toric $v$-adic metrics on $D$ and that of continuous concave functions on $\Delta_D$. Under this assignment, the canonical $v$-adic toric metric of $D$ corresponds to the zero function on $\Delta_D$.

**Definition 5.1** An (adelic) toric metric on $D$ is a collection of toric $v$-adic metrics $(\| \cdot \|_v)_{v \in M}$ such that $\| \cdot \|_v = \| \cdot \|_{v,\text{can}}$ for all but a finite number of $v \in M$. We denote by $\overline{D} = (D, (\| \cdot \|_v)_{v \in M})$ the corresponding (adelic) toric metrized divisor on $X$.

**Example 5.2** The collection $(\| \cdot \|_{v,\text{can}})_{v \in M}$ of $v$-adic toric metrics on $D$ is adelic in the sense of Definition 5.1. We denote by $\overline{D}$ the corresponding canonical toric metrized divisor on $X$.

Let $\overline{D}$ be a toric metrized divisor on $X$. For each $v \in M$, we set

$$\psi_{\overline{D},v} = \psi_{\| \cdot \|_v} \quad \text{and} \quad \vartheta_{\overline{D},v} = \vartheta_{\| \cdot \|_v}$$

for the associated $v$-adic metric function and $v$-adic roof function, respectively.

**Proposition 5.3** Let $\overline{D} = (D, (\| \cdot \|_v)_{v \in M})$ be toric divisor together with a collection of toric $v$-adic metrics. If $\overline{D}$ is adelic in the sense of Definition 5.1, then it is also adelic in the sense of Definition 4.4. Moreover, both definitions coincide in the semipositive case.

**Proof** Let $p \in X(\overline{K})$ and choose an adelic field extension $(\mathbb{F}, \mathfrak{M})$ such that $p \in X(\mathbb{F})$. Then $p_{\mathbb{F}}$ is a rational point of $X_\mathbb{F}$ and the inclusion

$$\iota : p_\mathbb{F} \hookrightarrow X_\mathbb{F}$$

is an equivariant map. Hence the inverse image $\iota^* \overline{D}$ is an adelic toric metric on $p_\mathbb{F}$ and so, for $w \in \mathfrak{M}$,

$$\log \| p_\mathbb{F} \|_w = \psi_{\iota^* \overline{D},w}(0),$$

and this quantity vanishes for all but the finite number of $w \in \mathfrak{M}$ such that $\| \cdot \|_w$ is not the canonical metric. Since this holds for all $p \in X(\overline{K})$, we conclude that $\overline{D}$ is adelic in the sense of Definition 4.4.

For the second statement, assume that $\overline{D}$ is semipositive and adelic in the sense of Definition 4.4. Let $x_i \in M$, $i = 1, \ldots, s$, be the vertices of the lattice polytope $\Delta_D$. By [8,
Example 2.5.13], for each $i$ there is an $n$-dimensional cone $\sigma_i \in \Sigma$ corresponding to $x_i$ under the Legendre-Fenchel correspondence, $i = 1, \ldots, s$. Each of these $n$-dimensional cones corresponds to a 0-dimensional orbit $p_i$ of $X$. Denote by $i_i : p_i \hookrightarrow X$ the inclusion of this orbit.

Fix $1 \leq i \leq s$. Modulo a translation, we can assume without loss of generality that $x_i = 0$. By [8, Proposition 4.8.9], for $v \in \mathcal{M}$,

$$\vartheta_{D, v}(x_i) = \vartheta_{i_i}^{-1}(0) = - \log \| s_D(p_i) \|_v.$$  

Hence $\vartheta_{D, v}(x_i) = 0$ for all but a finite number of $v$’s.

On the other hand, let $x_0$ be the distinguished point of $X$, which coincides with the neutral element of $\mathbb{T}$, and denote by $i_0 : x_0 \hookrightarrow X$ its inclusion. By [8, Proposition 4.8.10],

$$\max_{x \in \Delta_D} \vartheta_{D, v}(x) = \vartheta_{i_0}^{-1}(0) = - \log \| s_D(x_0) \|_v.$$  

Hence $\max_{x \in \Delta_D} \vartheta_{D, v}(x) = 0$ for all but a finite number of $v$’s.

For every $v \in \mathcal{M}$ such that $\vartheta_{D, v}(x_i) = 0$ for all $i$ and $\max_{x \in \Delta_D} \vartheta_{D, v}(x) = 0$, we have that $\vartheta_{D, v} \equiv 0$ because this local roof function is a concave function on $\Delta_D$. Hence, $\| \cdot \|_v$ coincides with the $v$-adic canonical metric of $D$ for all these places.

\begin{remark}
In the general non-semipositive case, Definitions 5.1 and 4.4 do not coincide. For instance, when $X = \mathbb{P}^1_{\mathbb{K}}$, a collection of metrics $\| \cdot \|_v$, $v \in \mathcal{M}$, satisfies Definition 4.4 if and only if its associated metric functions satisfy that

$$\psi_{D, v}(0) = 0 \quad \text{and} \quad \lim_{u \to \pm \infty} \psi_{D, v}(u) - \Psi_D(u) = 0$$

for all but a finite number of places. In the absence of convexity, these conditions do not imply that $\psi_{D, v} = \Psi_D$ for all but a finite number of places.
\end{remark}

A classical example of toric metrized divisors are those given by the inverse image of an equivariant map to a projective space equipped with the canonical metric on its universal line bundle. Below we describe this example and we refer to [8, Example 5.1.16] for the technical details.

Let $m = (m_0, \ldots, m_r) \in M^{r+1}$ and $\alpha = (\alpha_0, \ldots, \alpha_r) \in (\mathbb{K}^\times)^{r+1}$, with $r \geq 0$. The monomial map associated to this data is defined as

$$\varphi_{m, \alpha} : \mathbb{T} \longrightarrow \mathbb{P}^r_{\mathbb{K}}, \quad p \longmapsto (\alpha_0 \chi^{m_0}(p) : \cdots : \alpha_r \chi^{m_r}(p)).$$  

For a toric variety $X$ with torus $\mathbb{T}$ corresponding to fan that is compatible with the polytope $\Delta = \text{conv}(m_0, \ldots, m_r) \subset M_{\mathbb{R}}$, this extends to an equivariant map $X \to \mathbb{P}^r_{\mathbb{K}}$, also denoted by $\varphi_{m, \alpha}$.

\begin{example}
With notation as above, let $E_{\text{can}}$ be the divisor of the hyperplane at infinity of $\mathbb{P}_\mathbb{K}^r$, equipped with the canonical metric at all places. Then $D = \varphi_{m, \alpha}^* E$ is the nef toric Cartier divisor on $X$ corresponding to the translated polytope $\Delta - m_0$. We consider the semipositive toric metrized divisor $\overline{D} = \varphi_{m, \alpha}^* E$ on $X$.

For each $v \in \mathcal{M}$, the $v$-adic metric function of $\overline{D}$ is given by

$$\psi_{\overline{D}, v} : \mathbb{N}_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad u \longmapsto \min_{0 \leq j \leq r} \left( (m_j - m_0, u) - \log \left| \frac{\alpha_j}{\alpha_0} \right|_v \right).$$

\end{example}
The polytope corresponding to $D$ is $\Delta - m_0$ and, for each $v \in \mathcal{M}$, the $v$-adic roof function of $D$ is given by

$$\vartheta_{D,v}(x) = \max_{\lambda} \sum_{j=0}^{r} \lambda_j \log |a_j|_v - \log |a_0|_v,$$

the maximum being over the vectors $\lambda = (\lambda_0, \ldots, \lambda_r) \in \mathbb{R}^{r+1}_{\geq 0}$ with $\sum_{j=0}^{r} \lambda_j = 1$ such that $\sum_{j=0}^{r} \lambda_j (m_j - m_0) = x$. In other words, this is the piecewise affine concave function on $\Delta - m_0$ parametrizing the upper envelope of the extended polytope

$$\text{conv} \left( (m_j - m_0, \log |a_j/a_0|_v)_{0 \leq j \leq r} \right) \subset \mathbb{M}_R \times \mathbb{R}.$$

**Definition 5.6** For $i = 0, \ldots, n$, let $g_i : \Delta_i \to \mathbb{R}$ be a concave function on a convex body $\Delta_i \subset \mathbb{M}_R$. The mixed integral of $g_0, \ldots, g_n$ is defined as

$$\text{MI}_M(g_0, \ldots, g_n) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{0 \leq i_0 < \cdots < i_j \leq n} \int_{\Delta_{i_0} + \cdots + \Delta_{i_j}} g_{i_0} \Box \cdots \Box g_{i_j} \, d\text{vol}_M,$$

where $\Delta_{i_0} + \cdots + \Delta_{i_j}$ denotes the Minkowski sum of polytopes, and $g_{i_0} \Box \cdots \Box g_{i_j}$ the sup-convolution of concave functions, which is the function on $\Delta_{i_0} + \cdots + \Delta_{i_j}$ defined as

$$g_{i_0} \Box \cdots \Box g_{i_j}(x) = \sup (g_{i_0}(x_{i_0}) + \cdots + g_{i_j}(x_{i_j})), $$

where the supremum is taken over $x_{i_l} \in \Delta_{i_l}$, $l = 0, \ldots, j$, such that $x_{i_0} + \cdots + x_{i_j} = x$.

The mixed integral is symmetric and additive in each variable with respect to the sup-convolution. Moreover, for a concave function $g : \Delta \to \mathbb{R}$ on a convex body $\Delta$, we have $\text{MI}_M(g, \ldots, g) = (n+1)! \int_{\Delta} g \, d\text{vol}_M$, see [21, Sect. 8] for details.

The following is a restricted version of a result by Burgos Gil, Philippon and the second author, giving the global height of a toric variety with respect to a family of semipositive toric metrized divisors in terms of the mixed integrals of the associated local roof functions [8, Theorem 5.2.5].

**Theorem 5.7** Let $D_i$, $i = 0, \ldots, n$, be semipositive toric metrized divisors on $X$ such that $D_1, \ldots, D_n$ are generated by small sections. Then

$$h_{D_0, \ldots, D_n}(X) = \sum_{v \in \mathcal{M}} n_v \text{MI}_M(\vartheta_{D_0,v}, \ldots, \vartheta_{D_n,v}). \tag{5.4}$$

**Remark 5.8** The result in [8, Theorem 5.2.5] is more general. Given semipositive toric metrized divisors $D_i$, $i = 0, \ldots, n$, and rational sections $s_i$ such that div$(s_0), \ldots, \text{div}(s_n)$ intersect $X$ properly, the corresponding local heights are zero except for a finite number of places, and the formula (5.4) holds without any extra positivity assumption.

### 6 Proof of Theorem 1.1

Let $(\mathbb{K}, \mathcal{M})$ be an adelic field satisfying the product formula. Let $f \in \mathbb{K}[M]$ be a Laurent polynomial and $\Delta \subset \mathbb{M}_R$ its Newton polytope. Let $X$ be a projective toric variety over $\mathbb{K}$ given by a fan on $\mathbb{N}_R$ that is compatible with $\Delta$, and $D$ the Cartier divisor on $X$ given by this polytope. To prove Theorem 1.1, we first construct a toric metric on $D$ such that the associated toric metrized divisor $D$ is semipositive and generated by small sections, and the
global section of $\mathcal{O}(D)$ associated to $f$ is $\overline{D}$-small. We obtain this metrized divisor as the inverse image of a metrized divisor on a projective space.

For $r \geq 0$, let $\mathbb{P}^r_K$ be the $r$-dimensional projective space over $K$ and $E$ the divisor of the hyperplane at infinity. We denote by $\overline{E}$ this Cartier divisor equipped with the $\ell^1$-norm at the Archimedean places, and the canonical one at the non-Archimedean ones. This metric is defined, for $p = (p_0 : \cdots : p_r) \in \mathbb{P}^r_K(K_v)$ and a global section $s$ of $\mathcal{O}(E)$ corresponding to a linear form $\rho_s \in K[x_0, \ldots, x_r]$, by

$$
\|s(p)\|_v = \begin{cases} 
\frac{|\rho_s(p_0, \ldots, p_r)|_v}{\sum_j |p_j|_v} & \text{if } v \text{ is Archimedean}, \\
\frac{\max_j |p_j|_v}{\sum_j |p_j|_v} & \text{if } v \text{ is non-Archimedean},
\end{cases}
$$

(6.1)

The projective space $\mathbb{P}^r_K$ has a standard structure of toric variety with torus $\mathbb{G}^r_{m,K}$, included via the map $(z_1, \ldots, z_r) \mapsto (1 : z_1 : \cdots : z_r)$. Thus $\overline{E}$ is a toric metrized divisor. It is a particular case of the weighted $\ell^p$-metrized divisors on toric varieties studied in [9, Sect. 5.2].

The following result summarizes the basic properties of this toric metrized divisor and its combinatorial data.

**Proposition 6.1** The toric metrized divisor $\overline{E}$ on $\mathbb{P}^r_K$ is semipositive and generated by small sections. For $v \in \mathcal{M}$, its $v$-adic metric function is given, for $u = (u_1, \ldots, u_r) \in \mathcal{M}$, by

$$
\psi_{\overline{E},v}(u) = \begin{cases} 
- \log \left(1 + \sum_{j=1}^r e^{-u_j}\right) & \text{if } v \text{ is Archimedean,} \\
\min(0, u_1, \ldots, u_r) & \text{if } v \text{ is non-Archimedean.}
\end{cases}
$$

(6.2)

The polytope corresponding to $E$ is the standard simplex $\Delta^r$ of $\mathbb{R}^r$. For $v \in \mathcal{M}$, the $v$-adic roof function of $\overline{E}$ is given, for $x = (x_1, \ldots, x_r) \in \Delta^r$, by

$$
\vartheta_{\overline{E},v}(x) = \begin{cases} 
- \sum_{j=0}^r x_j \log(x_j) & \text{if } v \text{ is Archimedean,} \\
0 & \text{if } v \text{ is non-Archimedean,}
\end{cases}
$$

with $x_0 = 1 - \sum_{j=1}^r x_j$.

**Proof** The distinguished rational section of the line bundle $\mathcal{O}(E)$ corresponds to the linear form $x_0 \in K[x_0, \ldots, x_r]$. Hence, for an Archimedean place $v$ and a point $z = (z_1, \ldots, z_r) \in \mathbb{G}^r_{m,K}(K_v)$,

$$
\psi_{\overline{E},v}(\text{val}_v(z)) = \log \|s_E(z)\|_v = - \log \left(1 + \sum_{j=1}^r |z_j|\right),
$$

which gives the expression in (6.2) for this case. The non-Archimedean case is done similarly. We can easily check that these metric functions are concave. In the Archimedean case, this can be done by computing its Hessian and verifying that it is nonpositive and, in the non-Archimedean case, it is immediate from its expression. Hence, $\overline{E}$ is semipositive.

Set $s_j$ for the global sections corresponding to the linear forms $x_j \in K[x_0, \ldots, x_r]$, $j = 0, \ldots, r$. We have that $\bigcap_{j=0}^r \text{div}(s_j) = \emptyset$, and so this is a set of generating global sections. It follows from the definition of the metric in (6.1) that these global sections are $\overline{E}$-small. Hence, $\overline{E}$ is generated by small sections.
The fact that the polytope corresponding to $E$ is the standard simplex is classical, see for instance [13, page 27]. When $v$ is Archimedean, the $v$-adic roof function can be computed similarly as the one for the Fubini-Study metric in [8, Example 2.4.3]. When $v$ is non-Archimedean, $v$-adic roof function is zero, because the metric $\| \cdot \|_v$ is canonical. \qed

Set $r \geq 0$. Take $m = (m_0, \ldots, m_r) \in M^{r+1}$ and $\alpha = (\alpha_0, \ldots, \alpha_r) \in (\mathbb{K}^\times)^{r+1}$, and consider the polytope $\Delta = \text{conv}(m_0, \ldots, m_r) \subset M_\mathbb{R}$. Let $X$ be a projective toric variety over $\mathbb{K}$ given by a fan on $\mathcal{N}_\mathbb{R}$ that is compatible with $\Delta$. Let $\varphi_{m, \alpha} : \mathbb{T} \to \mathbb{P}_{\mathbb{K}}^r$ be the monomial map in (5.3) and set

$$D_m = \text{div}(\frac{1}{E} - \mathfrak{m}_0) + \varphi_{m, \alpha}^* E,$$

which coincides with the Cartier divisor on $X$ corresponding to $\Delta$. For each $v \in \mathfrak{M}$, we consider the metric on $\mathcal{O}(D_m)^\text{an}_v \simeq \mathcal{O}(\varphi_{m, \alpha}^* E)^\text{an}_v$ defined by

$$\| \cdot \|_{m, \alpha, v} = |\alpha_0|_v^{-1} \varphi_{m, \alpha}^* \| \cdot \|_{\mathbb{E}, v},$$

(6.3)

the homothecy by $|\alpha_0|_v$ of the inverse image by $\varphi_{m, \alpha}$ of the $v$-adic metric of $\mathbb{E}$. We then set

$$\overline{D}_{m, \alpha} = (D_m, (\| \cdot \|_{m, \alpha, v})_{v \in \mathfrak{M}}).$$

(6.4)

Since $\varphi_{m, \alpha}$ is an equivariant map and $\mathbb{E}$ is toric, this is a toric metrized divisor on $X$.

**Proposition 6.2** The toric metrized divisor $\overline{D} = \overline{D}_{m, \alpha}$ on $X$ is semipositive and generated by small sections. For $v \in \mathfrak{M}$, its $v$-adic metric is given, for $p \in \mathbb{T}(\mathbb{K}_v)$, by

$$\| s_D(p) \|_v = \begin{cases} \left( \sum_{j=0}^r |\alpha_j x^{m_j}(p)|_v \right)^{-1} & \text{if } v \text{ is Archimedean}, \\ \left( \max_{0 \leq j \leq r} |\alpha_j x^{m_j}(p)|_v \right)^{-1} & \text{if } v \text{ is non-Archimedean}. \end{cases}$$

(6.5)

The $v$-adic metric function of $\overline{D}$ is given, for $u \in \mathcal{N}_\mathbb{R}$, by

$$\psi_{\overline{D}, v}(u) = \begin{cases} -\log \left( \sum_{j=0}^r |\alpha_j|_v e^{-(m_j, u)} \right) & \text{if } v \text{ is Archimedean}, \\ \min_{0 \leq j \leq r} (m_j, u) - \log |\alpha_j|_v & \text{if } v \text{ is non-Archimedean}, \end{cases}$$

(6.6)

and the $v$-adic roof function of $\overline{D}$ is given, for $x \in \Delta$, by

$$\vartheta_{\overline{D}, v}(x) = \begin{cases} \max_{j} \sum_{j=0}^r \lambda_j \log |\alpha_j|_v \lambda_j & \text{if } v \text{ is Archimedean}, \\ \max_{j} \sum_{j=0}^r \lambda_j \log |\alpha_j|_v & \text{if } v \text{ is non-Archimedean}, \end{cases}$$

(6.7)

the maximum being over the vectors $\lambda = (\lambda_0, \ldots, \lambda_r) \in \mathbb{R}_{\geq 0}^{r+1}$ with $\sum_{j=0}^r \lambda_j = 1$ such that $\sum_{j=0}^r \lambda_j m_j = x$.

**Proof** Set $\overline{D} = \varphi_{m, \alpha}^* \mathbb{E}$ for short. This is a toric metrized divisor on $X$ that is semipositive and generated by small sections, due to Proposition 6.1 and the preservation of these properties under inverse image. Since the $v$-adic metrics of $\overline{D}$ are homothecies of those of $\overline{D}$, it follows that $\overline{D}$ is semipositive too. Moreover, a global section $\zeta$ of $\mathcal{O}(\overline{D}) \simeq \mathcal{O}(D)$ is $\overline{D}$-small if and only if the global section $\alpha_0 \zeta$ is $\overline{D}$-small. It follows that $\overline{D}$ is also generated by small sections.
Using (6.1) and the definition of the monomial map $\varphi_{m,\alpha}$, for $v \in \mathfrak{M}$, the $v$-adic metric of $\overline{D'}$ is given, for $p \in \mathbb{T}(\mathbb{K}_v)$, by

$$
\|s_{D'}(p)\|_v = \begin{cases}
\left( \sum_{j=0}^{r} \left| \frac{\alpha_j}{\alpha_0} \chi^{m_j-m_0}(p) \right|_v \right)^{-1} & \text{if } v \text{ is Archimedean,} \\
\max_{0 \leq j \leq r} \left| \frac{\alpha_j}{\alpha_0} \chi^{m_j-m_0}(p) \right|_v^{-1} & \text{if } v \text{ is non-Archimedean.}
\end{cases}
$$

Since $D = \text{div}(\chi^{-m_0}) + D'$, their distinguished rational sections are related by $s_D = \chi^{-m_0} s_{D'}$. It follows from (6.3) that

$$
\|s_D(p)\|_v = |\alpha_0|_v^{-1} |\chi^{-m_0}(p)|_v \|s_{D'}(p)\|_v,
$$

which implies the formulae in (6.5). As a consequence, we obtain also the expressions for the $v$-adic metric functions of $\overline{D}$.

For its roof function, consider first the linear map $H : N_{\mathbb{R}} \to \mathbb{R}^{r+1}$ given, for $u \in N_{\mathbb{R}}$, by $H(u) = (\langle m_0, u \rangle, \ldots, \langle m_r, u \rangle)$. For each place $v$, consider the concave function $g_v : \mathbb{R}^{r+1} \to \mathbb{R}$ given, for $v \in \mathbb{R}^{r+1}$, by

$$
g_v(v) = \begin{cases}
-\log \left( \sum_{j=0}^{r} |\alpha_j|_v^{-1} e^{-v_j} \right) & \text{if } v \text{ is Archimedean,} \\
\min_{0 \leq j \leq r} v_j - \log |\alpha_j|_v & \text{if } v \text{ is non-Archimedean.}
\end{cases}
$$

Notice that $\psi_{\overline{D},v} = H^* g_v$. The domain of the Legendre-Fenchel dual of $g_v$ is the simplex $S$ given as the convex hull of the vectors in the standard basis of $\mathbb{R}^{r+1}$. This Legendre-Fenchel dual is given, for $\lambda \in S$, by

$$
g^\vee_v(\lambda) = \begin{cases}
\sum_{j=0}^{r} \lambda_j \log \left( \frac{\alpha_j}{\alpha_0} \right) & \text{if } v \text{ is Archimedean,} \\
\max_{\lambda} \sum_{j=0}^{r} \lambda_j \log |\alpha_j|_v & \text{if } v \text{ is non-Archimedean.}
\end{cases}
$$

For the Archimedean case, this formula follows from [9, Proposition 5.8], whereas in the non-Archimedean case, it is given by Example 5.5.

By [8, Proposition 2.3.8(3)], the $v$-adic roof function $\vartheta_{\overline{D},v}$ is the direct image under the dual map $H^\vee$ of the Legendre-Fenchel dual $g^\vee_v$, which gives the stated formulae in (6.7). □

**Definition 6.3** Let $f \in \mathbb{K}[M]$ be a Laurent polynomial and $X$ be a projective toric variety over $\mathbb{K}$ given by a fan on $N_{\mathbb{R}}$ that is compatible with the Newton polytope $N(f)$. Write $f = \sum_{j=0}^{r} \alpha_j \chi^{m_j}$ with $m_j \in M$ and $\alpha_j \in \mathbb{K}^\times$. The toric metrized divisor on $X$ associated to $f$ is defined as

$$
\overline{D}_f = \overline{D}_{m,\alpha},
$$

the toric metrized divisor in (6.4) for the data $m = (m_0, \ldots, m_r) \in M^{r+1}$ and $\alpha = (\alpha_0, \ldots, \alpha_r) \in (\mathbb{K}^\times)^{r+1}$. It does not depend on the ordering of the terms of $f$. For $v \in \mathfrak{M}$, we denote by $\psi_{f,v}$ and $\vartheta_{f,v}$ the $v$-adic metric and roof functions of $\overline{D}_f$, respectively.

**Lemma 6.4** With notation as in Definition 6.3, the global section of $\mathcal{O}(D_f)$ associated to $f$ is $\overline{D}_f$-small.

**Proof** Set $\overline{D} = \overline{D}_f$ for short, and let $s = fs_D$ be the global section of $\mathcal{O}(D)$ associated to $f$. For $v \in \mathfrak{M}$ and $p \in \mathbb{T}(\mathbb{K}_v)$,

$$
\|s(p)\|_v = |f(p)|_v \|s_D(p)\|_v = \left| \sum_{j=0}^{r} \alpha_j \chi^{m_j}(p) \right|_v \|s_D(p)\|_v.
$$

Springer
It follows from (6.5) that \( \|s\|_v \leq 1 \) on \( T(\mathbb{R}_v) \), and so \( s \) is \( D \)-small. \( \square \)

The following result corresponds to Theorem 1.1 in the introduction.

**Theorem 6.5** Let \( f_1, \ldots, f_n \in \mathbb{K}[M] \), and let \( X \) be a proper toric variety with torus \( \mathbb{T}_M \) and \( D_0 \) a nef toric metrized divisor on \( X \). Let \( \Delta_0 \subset M_{\mathbb{R}} \) be the polytope of \( D_0 \) and, for \( v \in \mathfrak{M} \), let \( \vartheta_{0,v} : \Delta_i \rightarrow \mathbb{R} \) be the \( v \)-adic roof function of \( D_0 \). For \( i = 1, \ldots, n \), let \( \Delta_i \subset M_{\mathbb{R}} \) be the Newton polytope of \( f_i \) and, for \( v \in \mathfrak{M} \), let \( \vartheta_{i,v} : \Delta_i \rightarrow \mathbb{R} \) be the \( v \)-adic roof function on the metric associated to \( f_i \). Then

\[
\vartheta_{D_0} (Z(f_1, \ldots, f_n)) \leq \sum_{v \in \mathfrak{M}} n_v \text{MI}_M (\vartheta_{0,v}, \ldots, \vartheta_{n,v}).
\]

**Proof** Let \( \Sigma \) be the complete fan corresponding to the proper toric variety \( X \). By taking a refinement, we can assume without loss of generality that \( \Sigma \) is regular and compatible with the Newton polytopes \( \Delta_i, i = 1, \ldots, n \). Hence \( X \) is a projective toric variety and \( D_0 \) a nef toric metrized divisor, and there are nef toric Cartier divisors \( D_i, i = 1, \ldots, n \), corresponding to these Newton polytopes.

For \( i = 1, \ldots, n \), we denote by \( D_i \) the toric metrized divisor associated to \( f_i \) (Definition 6.3). By Proposition 6.2, each \( D_i \) is semipositive and generated by small sections and, by Lemma 6.4, the global sections \( s_i \) of \( O(D_i) \) corresponding to \( f_i \) are \( D_i \)-small. Applying Corollary 4.18 and Theorem 5.7,

\[
\vartheta_{D_0} (Z(f_1, \ldots, f_n)) \leq \vartheta_{D_0, \ldots, D_n} (X) = \sum_{v \in \mathfrak{M}} n_v \text{MI}_M (\vartheta_{D_0,v}, \ldots, \vartheta_{D_n,v}).
\]

Due to Proposition 2.8(2), the inequality \( Z(f_1, \ldots, f_n) \leq \prod_{i=1}^n \text{div}(s_i) \) holds. By the linearity of the global height and the nefness of \( D_0 \),

\[
\vartheta_{D_0} (Z(f_1, \ldots, f_n)) \leq \vartheta_{D_0} \left( \prod_{i=1}^n \text{div}(s_i) \right),
\]

which concludes the proof. \( \square \)

**Definition 6.6** Let \( \alpha = (\alpha_0, \ldots, \alpha_r) \in (\mathbb{K}^\times)^r \) with \( r \geq 1 \). For \( v \in \mathfrak{M} \), the \( v \)-adic (logarithmic) length of \( \alpha \) is defined as

\[
\ell_v (\alpha) = \begin{cases} 
\log (\sum_{j=0}^r |\alpha_j|_v) & \text{if } v \text{ is Archimedean,} \\
\log (\max_{0 \leq j \leq r} |\alpha_j|_v) & \text{if } v \text{ is non-Archimedean.}
\end{cases}
\]

The (logarithmic) length of \( \alpha \) is defined as \( \ell (\alpha) = \sum_{v \in \mathfrak{M}} n_v \ell_v (\alpha) \).

For a Laurent polynomial \( f \in \mathbb{K}[M] \), we define its \( v \)-adic (logarithmic) length, denoted by \( \ell_v (f) \), as the \( v \)-adic length of its vector of coefficients, \( v \in \mathfrak{M} \). We also define its (logarithmic) length, denoted by \( \ell (f) \), as the length of its vector of coefficients.

**Lemma 6.7** Let \( \vartheta_i : \Delta_i \rightarrow \mathbb{R} \) be concave functions on convex bodies, \( i = 0, \ldots, n \). Then

\[
\text{MI}_M (\vartheta_0, \ldots, \vartheta_n) \leq \sum_{i=0}^n \left( \max_{x \in \Delta_i} \vartheta_i (x) \right) \text{MV}_M (\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n)
\]
Proof Set $c_i = \max_{x \in \Delta_i} \vartheta_i(x)$ for short. By the monotonicity of the mixed integral [21, Proposition 8.1]
\[
\text{MI}_M(\vartheta_0, \ldots, \vartheta_n) \leq \text{MI}_M(c_0|_{\Delta_0}, \ldots, c_n|_{\Delta_n}),
\]
where $c_i|_{\Delta_i}$ denotes the constant function $c_i$ on the convex body $\Delta_i$. By [21, formula (8.3)]
\[
\text{MI}_M(c_0|_{\Delta_0}, \ldots, c_n|_{\Delta_n}) = \sum_{i=0}^{n} c_i \text{MV}_M(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n),
\]
giving the stated inequality. \qed

Corollary 6.8 With notation as in Theorem 6.5,
\[
\text{h}_{D_0}(Z(f_1, \ldots, f_n)) \leq \left( \sum_{v \in \mathfrak{M}} n_v \max_{x \in \Delta_0} \vartheta_{0,v}(x) \right) \text{MV}_M(\Delta_1, \ldots, \Delta_n)
\]
\[
+ \sum_{i=1}^{n} \ell(f_i) \text{MV}_M(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n).
\]

In particular, for the canonical metric on $D_0$ (Example 5.2),
\[
\text{h}_{D_0^\text{can}}(Z(f_1, \ldots, f_n)) \leq \sum_{i=1}^{n} \ell(f_i) \text{MV}_M(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n).
\]

Proof For $1 \leq i \leq n$ and $v \in \mathfrak{M}$, let $\vartheta_{i,v}$ be the $v$-adic roof function of the toric semipositive metric associated to $f_i$. Using (6.7), we compute the value of $\psi_{i,v}(0) = -\vartheta_{i,v}(0)$ to obtain
\[
\max_{x \in \Delta_i} \vartheta_{i,v}(x) = \ell_v(f_i).
\]
The first statement follows then from Theorem 6.5 and Lemma 6.7. The second statement is a particular case of the first one, using the fact that the $v$-adic roof functions of $D_0^\text{can}$ are the zero functions on $\Delta_0$. \qed

We readily derive from the previous corollary the following version of the arithmetic Bézout theorem.

Corollary 6.9 Let $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$ and let $D_0^\text{can}$ be the divisor at infinity of $\mathbb{P}_\mathbb{K}^n$ equipped with the canonical metric. Then
\[
\text{h}_{D_0^\text{can}}(Z(f_1, \ldots, f_n)) \leq \sum_{i=1}^{n} \left( \prod_{j \neq i} \deg(f_j) \right) \ell(f_i),
\]
where $\deg(f_i)$ denotes the total degree of the polynomial $f_i$.

7 Comparisons, examples and applications

In this section, we first compare our main results (Theorem 6.5 and Corollary 6.8) with the previous ones. Next, we compute the bounds given by these results in two families of examples, and compare them with the actual height of the 0-cycles. The first family of examples illustrates a case in which these bounds do approach the height of the 0-cycle, while the second one shows a situation where the bound of Theorem 6.5 is sharp and that of...
Corollary 6.8 is not. Finally, we present an application bounding the height of the resultant of a 0-cycle defined by a system of Laurent polynomials.

The first arithmetic analogue of the BKK theorem was proposed by Maillot [20, Corollaire 8.2.3]. With notations as in Theorem 6.5, suppose that \( f_1, \ldots, f_n \in \mathbb{Z}[M] \) and that \( D_0 \) is the nef toric divisor corresponding to the polytope \( \Delta_0 = \sum_{i=1}^n \Delta_i \). Then Maillot’s result amounts to the upper bound

\[
h_{D_0}^{\text{can}}(Z(f_1, \ldots, f_n)) \leq \sum_{i=1}^n (m(f_i) + L(\Delta_i)) \text{MV}_M(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n),
\]

(7.1)

where \( m(f_i) \) denotes the logarithmic Mahler measure of \( f_i \), and \( L(\Delta_i) \) a constant associated to the polytope \( \Delta_i \).

This result is similar to Corollary 6.8 specialized to a system of Laurent polynomials with integer coefficients, and the toric divisor \( D_0 \) associated to the polytope given by the Minkowski sum \( \sum_{i=1}^n \Delta_i \), equipped with the canonical metric. The factors \( m(f_i) + L(\Delta_i) \) in (7.1) and \( \ell(f_i) \) in (6.8) are comparable, albeit the fact that the constant \( L(\Delta_i) \) is not effective, see [23, Remark 4.2] for a discussion on this point.

Another previous result in this direction was obtained by the second author [23, Théorème 0.3]. Using again the notation in Theorem 6.5, suppose that \( f_1, \ldots, f_n \in \mathbb{Z}[M] \) and that the polytope \( \Delta_0 \) associated to the nef toric divisor \( D_0 \) contains \( \Delta_i \), \( i = 1, \ldots, n \). Then

\[
h_{D_0}^{\text{can}}(Z(f_1, \ldots, f_n)) \leq n! \text{vol}_M(\Delta_0) \sum_{i=1}^n \ell(f_i).
\]

This result is equivalent to the specialization of the upper bound in (6.8) to a system of Laurent polynomials with integer coefficients and Newton polytopes contained in the polytope \( \Delta_0 \).

We next turn to the computation of the bounds given by Theorem 6.5 and Corollary 6.8 in two families of examples.

We keep the notation of Sect. 6. We need the the following auxiliary computation of mixed volumes. For its proof, we recall that the mixed volume function associated to the Lebesgue measure of \( \mathbb{R}^n \) is \( \text{MV}_n(\Delta_1, \ldots, \Delta_n) = -\sum_{u \in S^{n-1}} \Psi_{\Delta_1}(u) \text{MV}_{n-1}(\Delta_2^u, \ldots, \Delta_n^u), \)

(7.2)

where \( S^{n-1} \) is the unit sphere of \( \mathbb{R}^n \), \( \Psi_{\Delta_1} \) is the support function of \( \Delta_1 \) as in (2.6), \( \Delta_i^u \) is the unique face of \( \Delta_i \) that minimizes the functional \( u \) on this polytope, and \( \text{MV}_n \) and \( \text{MV}_{n-1} \) denote the mixed volume functions associated to the Lebesgue measure of \( \mathbb{R}^n \) and \( u^\perp \simeq \mathbb{R}^{n-1} \), respectively. In fact, the sum ranges through the normal vectors of the facets of each polytope. We refer to [22, formula (5.1.22)] for more details.

**Lemma 7.1** Let \( \Delta \subset M_{\mathbb{R}} \) be a lattice polytope, and \( m_i \in M, i = 2, \ldots, n \), linearly independent lattice points. Denote by \( \overline{0m_i} \) the segment between 0 and \( m_i \), and \( u \in N \) the smallest lattice point orthogonal to all the \( m_i \)'s, which is unique up to a sign. Let \( P = \sum_{i=2}^n \mathbb{Z}m_i \subset M \) be the sublattice generated by the \( m_i \)'s, and \( P^{\text{sat}} \) its saturation. Then

\[
\text{MV}_M(\Delta, \overline{0m_2}, \ldots, \overline{0m_n}) = [P^{\text{sat}} : P] \text{vol}_\mathbb{Z}(\Delta, u),
\]

where \( \langle \Delta, u \rangle \) is the image of \( \Delta \) under the functional \( u : M_{\mathbb{R}} \to \mathbb{R} \).
Proof Choosing a basis, we identify $\mathcal{M} = \mathbb{Z}^n$. With this identification, $\text{MV}_M = \text{MV}_n$, the mixed volume associated to the Lebesgue measure of $\mathbb{R}^n$. The formula in (7.2) applied to the polytopes $\Delta, \overline{0m_2}, \ldots, \overline{0m_n}$ implies that

$$\text{MV}_n(\Delta, \overline{0m_2}, \ldots, \overline{0m_n}) = -\frac{1}{\|u\|}(\Psi_\Delta(u) + \Psi_\Delta(-u)) \text{MV}_{n-1}(\overline{0m_2}, \ldots, \overline{0m_n}),$$

(7.3)

where $\|u\|$ is the Euclidean norm. We have that

$$\Psi_\Delta(u) + \Psi_\Delta(-u) = \min_{x \in \Delta} \langle x, u \rangle + \min_{x \in \Delta} \langle x, -u \rangle = -\text{vol}_\mathbb{Z}(\Delta, u)$$

(7.4)

By the Brill-Gordan duality theorem [16, Lemma 1], we have the equality $\|u\| = \text{vol}_{n-1}(P_{\mathbb{R}}/P_{\text{sat}})$, where $\text{vol}_{n-1}$ denotes the Lebesgue measure of $u^\perp$. Hence

$$\frac{1}{\|u\|} \text{MV}_{n-1}(\overline{0m_2}, \ldots, \overline{0m_n}) = \text{MV}_{P_{\text{sat}}}(\overline{0m_2}, \ldots, \overline{0m_n}) = [P_{\text{sat}} : P].$$

(7.5)

The result follows then from (7.3), (7.4) and (7.5).

Example 7.2 Let $d, \alpha \geq 1$ be integers and consider the system of Laurent polynomials given by

$$f_1 = x_1 - \alpha, \quad f_2 = x_2 - \alpha x_1^d, \quad \ldots, \quad f_n = x_n - \alpha x_{n-1}^d \in \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}].$$

Its zero set in $\mathbb{T}^{\mathbb{Z}_n} = \mathbb{G}^n_{\mathbb{m}, \mathbb{Q}}$ consists of the rational point

$$p = (\alpha, \alpha^{d+1}, \ldots, \alpha^{d^{n-1}+d^{n-2}+\cdots+1}) \in \mathbb{T}^{\mathbb{Z}_n}(\mathbb{Q}) = (\mathbb{Q}^\times)^n.$$

Let $X$ be a proper toric variety over $\mathbb{Q}$, and $D_0^{\text{can}}$ a nef toric Cartier divisor on $X$ equipped with the canonical metric. Let $\Delta_0 \subset \mathbb{R}^n$ be the polytope corresponding to $D_0$ and, for $i = 1, \ldots, n$, set

$$u_i = e_i + de_{i+1} + \cdots + d^{n-i}e_n \in \mathbb{Z}^n,$$

where the $e_j$’s are the vectors in the standard basis of $\mathbb{Z}^n$. The height of $p$ with respect to $D_0^{\text{can}}$ is

$$h_{D_0^{\text{can}}}(p) = \left(\text{vol}_\mathbb{Z}\left(\Delta_0, \sum_{i=1}^n u_i\right)\right) \log(\alpha).$$

(7.6)

To prove this, let $v \in \mathfrak{M}_\mathbb{Q}$. By (5.1), the local height of $p$ with respect to the pair $(D_0^{\text{can}}, s_{D_0})$ is given by

$$h_{D_0^{\text{can}}, v}(p, s_{D_0}) = -\log \|s_{D_0}(p)\|_{v, \text{can}} = -\Psi_{\Delta_0}(\text{val}_v(p)).$$

Set $u = \sum_{i=1}^n u_i$ for short. Since $\text{val}_v(p) = -\log |\alpha|_v u$,

$$-\Psi_{\Delta_0}(\text{val}_v(p)) = \begin{cases} \log |\alpha|_v \max_{m \in \Delta_0 \cap \mathbb{Z}^n} \langle m, u \rangle & \text{if } v = \infty, \\ \log |\alpha|_v \min_{m \in \Delta_0 \cap \mathbb{Z}^n} \langle m, u \rangle & \text{if } v \neq \infty. \end{cases}$$

By adding these contributions,

$$h_{D_0^{\text{can}}}(p) = \log(\alpha) \left( \max_{m \in \Delta_0 \cap \mathbb{Z}^n} \langle m, u \rangle - \min_{m \in \Delta_0 \cap \mathbb{Z}^n} \langle m, u \rangle \right).$$
An arithmetic Bernštein–Kušnirenko inequality

Next we compare the value of the height of \( p \) with the bounds given by Corollary 6.8. We have \( \ell(f_i) = \log(\alpha + 1) \) for all \( i \). Consider the dual basis of the \( u_i \)'s, given by

\[ m_1 = e_1, m_2 = e_2 - de_1, \ldots, m_n = e_n - de_{n-1} \in \mathbb{Z}^n. \]

For \( i = 1, \ldots, n \), the Newton polytope \( \Delta_i \) of \( f_i \) is a translate of the segment \( 0m_i \), and \( u_i \) is the smallest lattice point in the line \((\sum_{j \neq i} \mathbb{R}m_j)^\perp \). Moreover the sublattice \( \sum_{j \neq i} \mathbb{Z}m_i \) is saturated. By Lemma 7.1

\[ \text{MV}_{\mathbb{Z}^n}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n) = \text{vol}_{\mathbb{Z}}(\Delta_0, u_i). \]

Therefore, the bound given by Corollary 6.8 is

\[ h_{D_0}^\text{can}(p) \leq \left( \sum_{i=1}^{n} \text{vol}_{\mathbb{Z}}(\Delta_0, u_i) \right) \log(\alpha + 1). \]

Example 1.2 in the introduction consists of the particular cases corresponding to the polytopes \( \Delta_0 = \Delta^n \), the standard simplex of \( \mathbb{R}^n \), and \( \Delta_0 = \text{conv}(0, m_1, \ldots, m_n) \).

In the following example, we exhibit a situation where the difference between the bounds given by the results in Sect. 6 is noticeable. Recall that passing from Theorem 6.5 to Corollary 6.8 amounts to replacing the local roof functions by constant functions on the polytope bounding them from above. Hence, to maximize the discrepancy between these two concave functions, we look for local roof functions that are tent-shaped, which is the situation where the difference between the mean value and the maximum value of these functions is the greatest possible.

**Example 7.3** Let \( \alpha \geq 1 \) be an integer and consider the system of Laurent polynomials

\[ f_i = x_i - \alpha \in \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \quad i = 1, \ldots, n, \]

Its zero set in \( \mathbb{G}^n_{m, \mathbb{Q}} \) is the rational point \( p = (\alpha, \ldots, \alpha) \in (\mathbb{Q}^\times)^n \). Let \( X = \mathbb{P}^n_{\mathbb{Q}} \) and let \( E^\text{can} \) be the divisor of the hyperplane at infinity equipped with the canonical metric. Then the height of \( p \) with respect to \( E^\text{can} \) is

\[ h_{E^\text{can}}(p) = \log(\alpha). \]

Next we compare the value of this height with the bound given by Theorem 6.5. Since the explicit computation of the mixed integrals appearing in this bound is somewhat involved, instead of giving its exact value we are going to approximate them with an upper bound that is easier to compute.

The polytope associated to the toric Cartier divisor \( E \) is \( \Delta_0 = \Delta^n \), the standard simplex of \( \mathbb{R}^n \). For each \( v \in \mathbb{M}_{\mathbb{Q}} \), the \( v \)-adic roof function \( \vartheta_{0,v} \) of \( E^\text{can} \) is the zero function on this simplex.

For each \( i = 1, \ldots, n \), let \( \Delta_i = \text{N}(f_i) \subset \mathbb{R}^n \) be the Newton polytope of \( f_i \), which coincides with the segment \( 0e_i \). For \( v \in \mathbb{M}_{\mathbb{Q}} \), let \( \vartheta_{i,v} \) be the \( v \)-adic roof function associated to \( f_i \) (Definition 6.3). This function is given, for \( t e_i \in \Delta_i = 0e_i \), by

\[ \vartheta_{i,\infty}(t e_i) = \begin{cases} (1 - t) \log(\alpha) - t \log t - (1 - t) \log(1 - t) & \text{if } v = \infty, \\ (1 - t) \log |\alpha|_v & \text{if } v \neq \infty. \end{cases} \]
For the Archimedean place, the $v$-adic roof functions are nonnegative, and so their mixed integral can be expressed as a mixed volume

$$MI_{\mathbb{Z}^n}(\vartheta_{0,\infty}, \ldots, \vartheta_{n,\infty}) = MV_{\mathbb{Z}^{n+1}}(\tilde{\Delta}_0, \ldots, \tilde{\Delta}_n), \quad (7.7)$$

with $\tilde{\Delta}_i = \text{conv}(\text{graph}((\vartheta_i, \infty), \Delta_i \times \{0\}) \subset \mathbb{R}^n \times \mathbb{R}$. Consider the concave function $\vartheta : \Delta^n \to \mathbb{R}$ defined by

$$x = (x_1, \ldots, x_n) \mapsto \log(2) + \log(\alpha)\left(1 - \sum_{i=1}^n x_i\right),$$

and set $\tilde{\Delta} = \text{conv}\left(\text{graph}(\vartheta), \Delta^n \times \{0\}\right) \subset \mathbb{R}^n \times \mathbb{R}$. Notice that $\vartheta_{i,\infty} \leq \vartheta$ on $\Delta_i$, and so $\tilde{\Delta}_i \subset \tilde{\Delta}, i = 0, \ldots, n$. By the monotony of the mixed volume,

$$MV_{\mathbb{Z}^{n+1}}(\tilde{\Delta}_0, \ldots, \tilde{\Delta}_n) \leq MV_{\mathbb{Z}^{n+1}}(\tilde{\Delta}, \ldots, \tilde{\Delta}) = (n+1)! \int_{\Delta^n} \vartheta \, dx = (n+1)! \left(\log(2) \text{vol}(\Delta^n) + \log(\alpha) \int_{\Delta^n} \sum_{i=1}^n x_i \, dx\right) = (n+1) \log(2) + \log(\alpha). \quad (7.8)$$

When $v$ is non-Archimedean, we have that $|\alpha|_v \leq 1$ because $\alpha$ is an integer. Hence $\vartheta_{i,v} \leq 0$, and so the mixed integral of these concave functions is nonpositive. Theorem 6.5 together with (7.7) and (7.8) gives the upper bound

$$h_{E,\text{an}}(p) \leq (n+1) \log(2) + \log(\alpha).$$

To conclude the example, we compute the bound given by Corollary 6.8. For $i = 1, \ldots, n$, we have that $\ell(f_i) = \log(\alpha + 1)$ and $MV_{\mathbb{Z}^n}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n) = 1$. Hence, this bound reduces to

$$h_{E,\text{an}}(p) \leq n \log(\alpha + 1),$$

concluding the study of this example.

As an application of our results, we bound the size of the coefficients of the $u$-resultant of the direct image under an equivariant map of the 0-cycle defined by a family of Laurent polynomials. As in the previous sections, let $(\mathbb{K}, \mathfrak{M})$ be an adelic field satisfying the product formula, $\overline{\mathbb{K}}$ an algebraic closure of $\mathbb{K}$, and $M \simeq \mathbb{Z}^n$ a lattice.

**Definition 7.4** Let $W \in Z_0(\mathbb{P}^r_{\mathbb{K}})$ be a 0-cycle of a projective space over $\mathbb{K}$ and $u = (u_0, \ldots, u_r)$ a set of $r+1$ variables. Write $W_{\mathbb{K}} = \sum_{q} \mu_q q \in Z_0(\mathbb{P}^r_{\overline{\mathbb{K}}})$ for the 0-cycle obtained from $W$ by the base change $\mathbb{K} \hookrightarrow \overline{\mathbb{K}}$. The $u$-resultant (or Chow form) of $W$ is defined as

$$\text{Res}(W) = \prod_{q}(q_0 u_0 + \cdots + q_r u_r)^{\mu_q} \in \mathbb{K}(u)^{\times},$$

the product being over the points $q = (q_0 : \cdots : q_r) \in \mathbb{P}^r_{\overline{\mathbb{K}}}(\overline{\mathbb{K}})$ in the support of $W_{\overline{\mathbb{K}}}$. It is well-defined up to a factor in $\mathbb{K}^{\times}$.

The length of a Laurent polynomial (Definition 6.6) is invariant under adelic field extensions and multiplication by scalars. It is also submultiplicative, in the sense that it satisfies the inequality

$$\ell(fg) \leq \ell(f) + \ell(g),$$

for $f, g \in \mathbb{K}[M]$. The following result corresponds to Theorem 1.3 in the introduction.
Theorem 7.5 Let \( f_1, \ldots, f_n \in \mathbb{K}[M] \), \( m_0 \in M^{r+1} \) and \( \alpha_0 \in (\mathbb{K}^\times)^{r+1} \) with \( r \geq 0 \). Set \( \Delta_0 = \text{conv}(m_0, \ldots, m_{0,r}) \subset M_\mathbb{R} \) and let \( \varphi : T_M \to \mathbb{P}_\mathbb{K}^r \) be the monomial map associated to \( m_0 \) and \( \alpha_0 \) as in (5.3). For \( i = 1, \ldots, n \), let \( \Delta_i \subset M_\mathbb{R} \) be the Newton polytope of \( f_i \), and \( \alpha_i \) the vector of nonzero coefficients of \( f_i \). Then

\[
\ell(\text{Res}(\varphi_*Z(f_1, \ldots, f_n))) \leq \sum_{i=0}^n \text{MV}_M(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n) \ell(\alpha_i).
\]

Proof Write \( Z(f_1, \ldots, f_n) = \sum \mu_p p \), the sum being over the points \( p \in T_M(\mathbb{K}) \). Since the length is invariant under adelic field extensions and submultiplicative, we deduce that

\[
\ell(\text{Res}(\varphi_*Z(f_1, \ldots, f_n))) \leq \sum_p \ell(\text{Res}(\varphi_*Z(f_1, \ldots, f_n))) \leq \sum_p \ell(\alpha_{0,0}) \chi_{m_0,0}(p) u_0 + \cdots + \alpha_{0,r} \chi_{m_0,r}(p) u_r).
\] (7.9)

Let \( X \) be a proper toric variety over \( \mathbb{K} \) defined by a fan that is compatible with \( \Delta_i, \) \( i = 0, \ldots, n \), and let \( D_0 \) be the toric metrized divisor on \( X \) associated to \( m_0 \) and \( \alpha_0 \) as in (6.4). Given a point \( p \in T_M(\mathbb{K}) \), we deduce from (6.5) that

\[
\ell(\alpha_{0,0}) \chi_{m_0,0}(p) u_0 + \cdots + \alpha_{0,r} \chi_{m_0,r}(p) u_r) = h_{D_0}(p).
\] (7.10)

By Proposition 6.2, the toric metrized divisor is semipositive and generated by small sections. In particular, it is nef. Similarly as in (6.9), we also get from Proposition 6.2 that the \( v \)-adic roof functions of \( D_0 \) satisfy \( \sum_{v \in M} n_v \max \vartheta_{0,v} = \ell(\alpha_0) \). Hence, Corollary 6.8 implies that

\[
\sum_p \mu_p h_{D_0}(p) \leq \sum_{i=0}^n \ell(\alpha_i) \text{MV}(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n).
\] (7.11)

The statement follows then from (7.9), (7.10) and (7.11).

Acknowledgements We thank José Ignacio Burgos, Roberto Gualdi and Patrice Philippon for useful discussions. We also thank Walter Gubler, Philipp Habegger and the referee for their helpful comments and suggestions for improvement on a previous version of this paper. Part of this work was done while the authors met at the Universitat de Barcelona and at the Université de Caen. We thank both institutions for their hospitality.

References

1. Artin, E., Whaples, G.: Axiomatic characterization of fields by the product formula for valuations. Bull. Amer. Math. Soc. 51, 469–492 (1945)
2. Bernstein, D.N.: The number of roots of a system of equations. Funkcional. Anal. i Priložen. 9, 1–4 (1975). (English translation: Functional Anal. Appl. 9, 183–185, (1975))
3. Bombieri, E., Gubler, W.: Heights in diophantine geometry, New Math. Monogr., vol. 4. Cambridge Univ. Press, Cambridge (2006)
4. Burgos Gil, J.I., Moriwaki, A., Philippon, P., Sombra, M.: Arithmetic positivity on toric varieties. J. Algebraic Geom. 25, 201–272 (2016)
5. Bourbaki, N.: Eléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations. (French) Actualités Scientifiques et Industrielles, No. 1308 Hermann, Paris (1964)
6. Bourbaki, N.: Eléments de mathématique. Algèbre. Chapitres 1 à 3. (French) Hermann, Paris (1970)
7. Burgos Gil, J.I., Philippon, P., Rivera-Letelier, J., Sombra, M.: The distribution of Galois orbits of points of small height in toric varieties, e-print arXiv:1509.01011v1 (2015)
8. Burgos Gil, J.I., Philippon, P., Sombra, M.: Arithmetic geometry of toric varieties. Metrics, measures and heights, Astérisque, vol. 360. Société Mathématique de France, Paris (2014)
9. Burgos Gil, J.I., Philippon, P., Sombra, M.: Successive minima of toric height functions. Ann. Inst. Fourier (Grenoble) 65, 2145–2197 (2015)
10. Cassels, J.W.S.: Local fields, London Math. Soc. Stud. Texts, vol. 3. Cambridge Univ. Press, Cambridge (1986)
11. Chambert-Loir, A.: Mesures et équidistribution sur les espaces de Berkovich. J. Reine Angew. Math. 595, 215–235 (2006)
12. Fulton, W.: Intersection theory, Ergeb. Math. Grenzgeb. (3), vol. 2. Springer-Verlag, Berlin (1984)
13. Fulton, W.: Introduction to toric varieties, Ann. of Math. Stud., vol. 131. Princeton Univ. Press, Princeton (1993)
14. Gelfand, I.M., Kapranov, M.M., Zelevinsky, A.V.: Discriminants, resultants, and multidimensional determinants. In: Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston (1994)
15. Gubler, W.: Heights of subvarieties over $\mathbb{M}$-fields, arithmetic geometry (Cortona, 1994), sympois. math., pp. 190–227. Cambridge Univ. Press, Cambridge (1997)
16. Heath-Brown, D.R.: Diophantine approximation with square-free numbers. Math. Z. 187, 335–344 (1984)
17. Kušnirenko, A.G.: Polyèdres de Newton et nombres de Milnor. Invent. Math. 32, 1–31 (1976)
18. Lang, S.: Algebra. Revised third edition. In: Graduate Texts in Mathematics, vol. 211. Springer, New York (2002)
19. Lazarsfeld, R.: Positivity in algebraic geometry. I. Classical setting: line bundles and linear series, Ergeb. Math. Grenzgeb. (3), vol. 48. Springer-Verlag, Berlin (2004)
20. Maillot, V.: Géométrie d’Arakelov des variétés toriques et fibrés en droites intégrables. In: Mémoires de la Société Mathématique de France, vol. 80, pp. III1–VI129 (2000)
21. Philippon, P., Sombra, M.: A refinement of the Bernstein–Kušnirenko estimate. Adv. Math. 218, 1370–1418 (2008)
22. Schneider, R.: Convex bodies: the Brunn-Minkowski theory, Encyclopedia Math. Appl., vol. 44. Cambridge Univ. Press, Cambridge (1993)
23. Sombra, M.: Minimums successifs des variétés toriques projectives. J. Reine Angew. Math. 586, 207–233 (2005)
24. Sturmfels, B.: Solving systems of polynomial equations. In: CBMS Regional Conference Series in Mathematics, 97. Published for the Conference Board of the Mathematical Sciences, Washington. American Mathematical Society, Providence (2002)