Noncompact Gepner Models for Type II Strings on a Conifold and an ALE Instanton

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Abstract

We construct modular invariant partition functions for type II strings on a conifold and a singular Eguchi-Hanson instanton by means of the $SL(2, \mathbb{R})/U(1)$ version of Gepner models. In the conifold case, we find an extra massless hypermultiplet in the IIB spectrum and argue that it may be identified as a soliton. In the Eguchi-Hanson case, our formula is new and different from the earlier result, in particular does not contain graviton. The lightest IIB fields are combined into a six-dimensional $(2,0)$ tensor multiplet with a negative mass square. We give an interpretation to it as a doubleton-like mode.

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1 Introduction

The recognition of the importance of physics near the singularity in the moduli space is one of the highlights in the recent developments of string theory. Well-known examples are the conifold singularity of Calabi-Yau threefolds and the ADE singularities of K3 surfaces. In both cases, type IIA or IIB strings acquire extra massless solitons due to various D-branes wrapped around the vanishing cycles of the singularity[1, 2].

In this contribution we construct modular invariant partition functions for the CFTs which describe those theories on such singularities[1]. At first sight, such an attempt might look like nonsense because one would expect non-perturbative effects near the singularity, where the CFT description of the strings on a compact space breaks down. However, we may still expect some dual perturbative CFT description for those theories[3], where the non-perturbative effects in one theory are studied perturbatively in the other theory. The idea is to “pinpoint” those theories by using an abstract CFT approach, just as we can use ordinary Gepner models to describe some special points of moduli space of ordinary Calabi-Yau compactifications.

This paper is organized as follows. In section 2, we construct a partition function for type II strings on a conifold[4] and argue that the extra massless hypermultiplet that appears in the spectrum may be identified as a soliton. In section 3, we address an issue in the known partition function formulas for the singular ALE spaces. In section 4, we present a new modular invariant partition function for type II strings on a singular Eguchi-Hanson instanton, the simplest (A1) four-dimensional ALE space in the ADE classification. Our formula is new and different from the earlier result[5, 6]. The last section is devoted to some concluding remarks.

\footnote{\textsuperscript{1} We do not consider any D-brane probe.}
2 Conifold

Let us begin with four-dimensional type II theories “compactified” on a conifold. The CFT for the four-dimensional part is a free $N = 2$ SCFT, while the conifold part is the $c = 9$, $SL(2, \mathbb{R})/U(1)$ Kazama-Suzuki model[7]. The clue that leads to the relation between a conifold and the $SL(2, \mathbb{R})/U(1)$ SCFT may be found in the equation of the deformed conifold (See ref. 8 for further explanations and references.):

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = \mu,$$

(1)

where $z_i$ ($i = 1, \ldots, 4$) are the coordinates of $\mathbb{C}^4$. $\mu \to 0$ is the conifold limit. To apply the abstract CFT approach we replace $\mu$ with $\mu z_5^{-1}$, where $z_i$ ($i = 1, \ldots, 5$) are now thought of as the coordinates of $\mathbb{P}^4$. The negative power of $-1$ has been determined by the Calabi-Yau condition, and interpreted as the $SL(2, \mathbb{R})/U(1)$ N = 2 model with level $k = -(1 + 2) = 3$. Its central charge is $c = \frac{3k}{k-2} = 9$. Thus, unlike ordinary Gepner models, the necessary central charge for the internal CFT is supplied by a single $SL(2, \mathbb{R})/U(1)$ CFT.

In fact, if $c > 3$, any unitary representation for the $N = 2$ superconformal algebra can be constructed from the $SL(2, \mathbb{R})/U(1)$ coset[8]. It means that we are allowed to use any $c = 9$ representations. What’s the criteria for the representations to be chosen? Our guideline is modular invariance and spacetime supersymmetry.

The generic (nondegenerate) $N = 2$, $c = 9$ superconformal characters are given by

$$\text{Tr} q^L y^J = q^{h+1/8} y^Q \frac{\vartheta_3(z|\tau)}{\eta^3(\tau)},$$

(2)

(NS sector), where $q = \exp(2\pi i \tau)$ and $y = \exp(2\pi i z)$. To improve the modular behavior of the monomial factor $q^{h+1/8}$, we consider a countable set of infinitely many representations so that the sum of the monomial factors forms a certain theta function. (If the $c = 9$ CFT is realized as an $N = 2$
Liouville × $S^1$ system\[9, 10\], this operation amounts to the momentum summation along $S^1$. It is still not enough to construct a modular invariant combination because the numbers of theta and $\eta$ functions are not the same. (The $\sqrt{\tau}$ factors do not cancel in the modular $S$ transformation.) To cure this problem, we consider \textit{continuously many} representations for each $U(1)$ charge in the above theta function and integrate over them with the Gaussian weight (Liouville momentum integration).

Which theta functions should we use? To determine them we require spacetime supersymmetry. We now have three theta functions: one from the free complex fermion for the two transverse spacetime dimensions, one from the Jacobi theta function in the $N = 2$ characters, and one from the character summation above. They must be GSO projected in an appropriate way to give a vanishing partition function for the theory to be supersymmetric. Therefore, we need some identities among the products of three theta functions. One solution has been known for a long time\[11\]:

\[
\Lambda_1(\tau) \equiv \Theta_{1,1}(\tau, 0) \left( \vartheta_3^2(0|\tau) + \vartheta_4^2(0|\tau) \right) - \Theta_{0,1}(\tau, 0) \vartheta_2^2(0|\tau) = 0, \quad (3)
\]

where

\[
\Theta_{m,1}(\tau, z) = \sum_{n \in \mathbb{Z}} q^{(n+m/2)^2} y^{(n+m/2)}, \quad (4)
\]

$(m = 0, 1)$ are the level-1 $SU(2)$ theta functions. There is another solution\[11\]:

\[
\Lambda_2(\tau) \equiv \Theta_{0,1}(\tau, 0) \left( \vartheta_3^2(0|\tau) - \vartheta_4^2(0|\tau) \right) - \Theta_{1,1}(\tau, 0) \vartheta_2^2(0|\tau) = 0, \quad (5)
\]

which is nothing but the modular $S$ transform of (3). Their modular transformations are

\[
\Lambda_1(\tau + 1) = i\Lambda_1(\tau), \quad \Lambda_2(\tau + 1) = -\Lambda_2(\tau), \quad (6)
\]

and

\[
\Lambda_1\left( -\frac{1}{\tau} \right) = \frac{e^{-\frac{3\pi i}{4}\tau^2}}{\sqrt{2}} \left( -\Lambda_1(\tau) + \Lambda_2(\tau) \right),
\]

\[
\Lambda_2\left( -\frac{1}{\tau} \right) = \frac{e^{-\frac{3\pi i}{4}\tau^2}}{\sqrt{2}} \left( \Lambda_1(\tau) + \Lambda_2(\tau) \right). \quad (7)
\]

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Figure 1: $N = 2, c = 9$ representations used as the internal CFT (NS-sector).

Thus $|\Lambda_1(\tau)/\eta^3(\tau)|^2 + |\Lambda_2(\tau)/\eta^3(\tau)|^2$ is modular invariant. Including the other contributions as well, we obtain the following modular invariant partition function:

$$Z_{\text{conifold}} = \int \frac{d^2\tau}{(\text{Im}\tau)^2} \frac{1}{(\text{Im}\tau)^2 |\eta(\tau)|^6} \left[ |\Lambda_1(\tau)/\eta^3(\tau)|^2 + |\Lambda_2(\tau)/\eta^3(\tau)|^2 \right]. \quad (8)$$

One may easily read off from $Z_{\text{conifold}}$ which $N = 2, c = 9$ representations are used (Figure 1; see ref.\[4\] for the R-sector.).

The following observations support that $Z_{\text{conifold}}$ is really the partition function for type II strings on a conifold:

- First of all, the spectrum exhibits a continuum. This is due to the integration over the $N = 2$ representations (Liouville momentum integration), which is a consequence of the requirement from modular invariance. Therefore, spacetime, which was initially supposed to be four-dimensional, becomes
five-dimensional effectively. Another related observation is that the graviton, the dilaton and the $B$-field correspond to the $(h, Q) = (1/4, 0)$ point, and hence are massive. This is a generic $N = 2$ representation and corresponds to a non-normalizable (“principal unitary series”) $SL(2, \mathbb{R})$ representation. This is certainly different from the ordinary CFT description of the Calabi-Yau “compact”ifications.

- Consider type IIB theory. We have one chiral- and one anti-chiral primary fields of $h = 1/2$ in the NS-sector (Figure 1). They (together with their spectral flows) give rise to a massless $N = 2$, $U(1)$ vector multiplet and a hypermultiplet. The vector multiplet agrees with the Hodge number $h_{2,1} = 1$ of the deformed conifold, while the hypermultiplet is an extra one. We will argue that this hypermultiplet may be identified as the famous massless soliton\[1\] coming from the wrapped D3 brane. The extra massless fields are due to the second chiral primary field $(h, Q) = (1, 2)$, which exhibit a gap above. Technically, the representations on the boundary of the unitarity region are degenerate representations, and hence are smaller than the generic ones. The irreducible character for $(h, Q) = (1/2, 1)$ is given by

$$
\text{Tr} q^{L_0} y^{J_0}|_{(h, Q) = (1/2, 1)} = \frac{q^{1+1/8} y^2 \vartheta_3(z|\tau)}{1 + q^{1/8} y^{\frac{3}{4}}} \eta^3(\tau),
$$

and not in the generic form like (2). In fact, their difference is $(h, Q) = (1/2, 1))$

$$
(2) - (9) = \frac{q^{1+1/8} y^2 \vartheta_3(z|\tau)}{1 + q^{1/8} y^{\frac{3}{4}}} \eta^3(\tau),
$$

which is precisely the irreducible character for $(h, Q) = (1, 2)$\[12\]. (And similarly for the anti-chiral primary fields; they are shown by the dots in

\footnote{This is neither the universal hypermultiplet, nor the one coming from the Kähler form on the conifold; the dilaton is paired with the $(h, Q) = (1/4, 0)$ representation, while the Kähler form does not have a compact support and hence will correspond to a continuous representation of $SL(2, \mathbb{R})$.}

\footnote{Degenerate representations on the boundary of the unitarity region correspond to discrete series of $SL(2, \mathbb{R})$\[8\].}
Therefore, a generic representation with \( Q = 1 \) splits into two chiral primaries on reaching the boundary of the unitarity region. This implies that the massless fields made out of the \( (h, Q) = (1/2, \pm 1) \) (anti-)chiral primary fields cannot acquire masses (= Liouville momenta) alone, without “eating” the fields from the \( (h, Q) = (1, \pm 2) \) representations, and both are trapped near the singularity. This will support the identification of the extra hypermultiplet as the massless soliton.

- An \( N = 2, U(1) \) vector multiplet and a hypermultiplet are also the massless excitations of the intersecting M5-branes\(^{[13]}\). This will agree with the duality of a conifold to the intersecting NS5-brane system proposed in refs.\(^{[3, 14]}\).

### 3 ALE Instantons: a Puzzle

Let us next consider the “blow-down” limit of the ALE instantons. A similar guess from the defining equations suggests that the corresponding CFT will be a certain Landau-Ginzburg orbifold of a tensor product of \( N=2 \) minimal models and a \( SL(2, \mathbb{R})/U(1) \) Kazama-Suzuki model with total central charge \( 6 \). For example, the equations for the \( A_{n-1} \)-series are given by

\[
z_1^n + z_2^2 + z_3^2 = \mu z_4^{-n},
\]

where \( z_i \) \((i = 1, \ldots, 4)\) are now the coordinates of \( \mathbb{P}^3 \). Thus the corresponding CFT is the product of the minimal model of level \( n - 2 \) and the \( SL(2, \mathbb{R})/U(1) \) CFT of level \( n + 2 \), orbifolded by \( \mathbb{Z}_n \).

We will now clarify what is the puzzle in the known formulas for the partition functions obtained in the previous works\(^{[6]}\). (See also ref.\(^{[15]}\).) The standard argument goes as follows: To construct a modular invariant, three different \( N = 2 \) SCFTs must be taken into account: the free \( N = 2 \) SCFT for the four transverse spacetime dimensions, the level-\((n-2)\) minimal

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\(^{4}\) This is a phenomenon reminiscent of the BPS saturation or the Higgs mechanism. We thank T. Eguchi and S.-K. Yang for comments.
model, and the level-\((n+2)\) \(SL(2,\mathbb{R})/U(1)\) SCFT. Since the \(SL(2,\mathbb{R})/U(1)\) construction scans the whole \(N=2\) unitarity region\([8]\), we may consider any \(U(1)\) charge lattice and sum over the representations in the third \(c = 3 + \frac{6}{n}\) SCFT. Thus the building blocks must consist of three Jacobi theta functions (two from the transverse dimensions and one from \(SL(2,\mathbb{R})/U(1)\)), an \(N=2\) minimal character, and a certain theta function.

The crucial observation is that the following theta identity holds\([5, 6\) (See also ref.\([16]\), eq.(2.24).):

\[
\sum_{m \in \mathbb{Z}^2} \Theta_{m,k+2}(\tau, \frac{kz}{k+2}) \left[\vartheta_{3}^{3}(0) ch_{l,m}^{(NS)}(\tau, z) - \vartheta_{3}^{4}(0) ch_{l,m}^{(NS)}(\tau, z) - \vartheta_{2}^{2}(0) ch_{l,m}^{(R)}(\tau, z)\right] = \chi_{l}^{(k)}(\tau, z) \left(\vartheta_{3}^{4}(0) - \vartheta_{2}^{4}(0) - \vartheta_{2}^{2}(0)\right),
\]

(12)

where \(ch_{l,m}(\tau, z)\) and \(\chi_{l}^{(k)}(\tau, z)\) are the \(N=2\) minimal and the affine \(SU(2)\) characters, respectively. This means that if we take \(\Theta_{m,k+2}(\tau, \frac{kz}{k+2})\) as the theta function for the \(c = 3 + \frac{6}{n}\), \(SL(2,\mathbb{R})/U(1)\) SCFT and consider the combinations of operators specified in the LHS of (12), we get a product of a level-\(k\) \(SU(2)\) WZW model and a free fermion theory. Modular invariants are easily constructed from those of the \(SU(2)\) WZW models. This fact has been used to rationalize the duality between the singular K3 and the system of NS5-branes at the partition function level\([3, 7].\)

Now here comes the puzzle: Since the lightest fields always come from the identity operator of \(SU(2)\) while Jacobi’s identity is independent of the type of the singularity, the ground-state degeneracy does not change whatever the singularity is! To look into the situation more closely, let us consider the simplest \(A_1 \ (n = 2)\) case, the blow-down limit of the Eguchi-Hanson instanton. In this case the partition function is simply given by

\[
Z_{EH}^{(\text{graviton})} = \int \frac{d^2\tau}{(\text{Im}\tau)^2} \frac{1}{(\text{Im}\tau)^{\frac{7}{2}} |\eta(\tau)|^{18}} \cdot \left(\frac{1}{2} \left(\vartheta_{3}^{4} - \vartheta_{2}^{4} - \vartheta_{2}^{2} + \vartheta_{1}^{4}\right)\right)^2.
\]

(13)
This is almost identical to the partition function for $D = 10$ type II(B) strings, but the essential difference is the number of $\eta$ functions; since the free $c = 6$, $N = 2$ SCFT for the four-dimensional flat space is replaced by the $SL(2, \mathbb{R})/U(1)$ SCFT, the partition function (13) has only nine (rather than twelve) $\eta$ functions in the denominator. Therefore, not the whole ghost ground-state energy $-1 + \frac{1}{2} = -\frac{1}{2}$ is absorbed in the $\eta$ functions, but the $-\frac{1}{8}$ due to the shortage of three $\eta$s is canceled by the “Liouville ground-state energy” of $+\frac{1}{8}$ from the $SL(2, \mathbb{R})/U(1)$ internal SCFT. (Otherwise $q^{1/8}$ has no where to go and the modular invariance is lost. In the conifold case this was $q^{1/4}$ and can be seen as the gap above the origin (Figure 1).) Thus the graviton is again paired with a non-normalizable state of the internal SCFT, and hence acquires a (mass)$^2$ of $\frac{1}{8}$. This is always the case for other ADE singularities, being a common feature of non-critical strings formulated as such Gepner-like models. No other field is lighter than the graviton in this partition function. This implies that the modular invariant constructed from the identity (12) does not respect the geometry of the ALE spaces, nor can it see the zeromodes of the dual NS5-branes. This is the puzzle.

Is there any other orbit than (12) that closes under modular transformations? In the next section we will construct a new modular invariant partition function\footnote{The work done in collaboration with M. Naka.} for the Eguchi-Hanson instanton by choosing a different initial condition in the $\beta$-method.

## 4 New Partition Function for the Eguchi-Hanson Instanton

A general method to obtain a modular invariant combination from a set of theta functions has been given in Gepner’s original paper\footnote{We thank M. Bando, H. Kawai and T. Kugo for discussions on this point.} and was called “$\beta$-method”. We first note that the quartic polynomial of the theta functions
in (13) can be written as follows:

\[(\vartheta_3 + \vartheta_4)^2(\vartheta_2^2 - \vartheta_4^2) - (\vartheta_2 + \vartheta_1)^2(\vartheta_2^2 - \vartheta_1^2)\]

\[+(\vartheta_3 - \vartheta_4)^2(\vartheta_3^2 - \vartheta_4^2) - (\vartheta_2 - \vartheta_1)^2(\vartheta_2^2 - \vartheta_1^2)\]

\[= 2(\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 + \vartheta_1^4). \quad (14)\]

The LHS of the equation clearly shows how the GSO projection should be done in each sector; for each term, the first factor is the state sum of the transverse spacetime dimensions, while the second is the one coming from the internal $N = 2$ SCFT. The first and the third terms are in the NS sector, and the second and the fourth ones are in the R sector, for both spacetime and internal SCFTs. We now consider a variant of this equation:

\[(\vartheta_3 + \vartheta_4)^2(\vartheta_2 + \vartheta_1)^2 - (\vartheta_2 + \vartheta_1)^2(\vartheta_3 - \vartheta_4)^2\]

\[+(\vartheta_3 - \vartheta_4)^2(\vartheta_2 - \vartheta_1)^2 - (\vartheta_2 - \vartheta_1)^2(\vartheta_3 + \vartheta_4)^2\]

\[= 16\vartheta_1\vartheta_2\vartheta_3\vartheta_4. \quad (15)\]

Each term gives the same phase in the modular $T$ transformation, and their sum returns to itself up to a phase by the $S$ transformation. Using this, we write a new modular invariant:

\[Z^{(\text{doubleton})}_{\text{EH}} = \int \frac{d^2 \tau}{(\text{Im}\tau)^2(\text{Im}\tau)^2|\eta(\tau)|^{18}} \cdot \frac{1}{16^2}\]

\[\cdot \left| (\vartheta_3 + \vartheta_4)^2 \left((\vartheta_2 + \vartheta_1)^2 + (\vartheta_2 - \vartheta_1)^2\right) \right.\]

\[\left. - (\vartheta_2 + \vartheta_1)^2 \left((\vartheta_3 - \vartheta_4)^2 + (\vartheta_3 + \vartheta_4)^2\right) \right.\]

\[+ (\vartheta_3 - \vartheta_4)^2 \left((\vartheta_2 - \vartheta_1)^2 + (\vartheta_2 + \vartheta_1)^2\right)\]

\[\left. - (\vartheta_2 - \vartheta_1)^2 \left((\vartheta_3 + \vartheta_4)^2 + (\vartheta_3 - \vartheta_4)^2\right)\right|^2. \quad (16)\]

In fact, the alternating sum inside $| \cdots |$ vanishes trivially; however, it can be interpreted as a consequence of the cancellation between the NS- and the R-sectors. Again, the first and the third terms are in the NS-sector for the
transverse spacetime SCFT, but they are now paired with the R-sector of the internal SCFT (and similarly for the transverse R-sector)\footnote{Note that $Z_{EH}^{(\text{graviton})}$ and $Z_{EH}^{(\text{doubleton})}$ are separately modular invariant.}.

Remarkably, $Z_{EH}^{(\text{doubleton})}$ does not contain any graviton because of its peculiar GSO projection. The lightest IIB fields are combined into a (2,0) tensor multiplet in six dimensions, which coincides with the zeromode excitations on the IIA NS5-brane\cite{18}; this is a manifestation of T-duality. In fact, its $(mass)^2$ is negative ($= -\frac{1}{8}$)! It does not necessarily mean the instability of our vacuum because there is no reason to believe that the six-dimensional spacetime is flat any more. Perhaps it might be understood as a doubleton-like mode. The doubleton\cite{19} is known to be the lowest Kaluza-Klein mode with a negative $(mass)^2$ in (say) the $AdS_7 \times S^4$ compactification of $D = 11$ supergravity. It is a pure gauge mode in the bulk but has a holographic dual on the six-dimensional $AdS$ boundary, on which an M5-brane sits. If the T-duality relation between a singular ALE and NS5-branes persists even in strong coupling, and if the abstract CFT approach can consistently describe physics near the singularity (as it seemed to be in the conifold case), the CFT must “see” some alternative dual to the strongly coupled type IIA theory; the one-loop CFT calculation for the latter itself is certainly inconsistent. Thus if the NS5-branes could be replaced by M5-branes, the mysterious tachyonic fields would then have an explanation as a doubleton-like mode. For the full understanding of this, we will need to generalize our formula to other ADE cases.

5 Concluding Remarks

We have constructed partition functions for type II strings on a conifold and a singular Eguchi-Hanson instanton by using a Gepner-like abstract CFT approach.

\footnote{The change of the fermion boundary condition was discussed in ref.\cite{17}, but they were led to a different result.}
In the conifold case, we found an extra massless hypermultiplet and argued that this might be identified as the massless soliton. The appearance of the massless soliton in the spectrum comes as a surprise; to confirm the identification further, the nature of the extra states must be clarified in the boundary-state formulation of D-branes.

The doubleton-like mode in the Eguchi-Hanson case is also quite unexpected. The relation to the M-theory dual is still mysterious. It would be interesting to compare (reconcile?) our result with the works of refs.[10].

Acknowledgments

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