The pressure of QED from the two-loop 2PI effective action

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Abstract

We compute the pressure of hot quantum electrodynamics from the two-loop truncation of the 2PI effective action. Since the 2PI resummation guarantees gauge-fixing independence only up to the order of the truncation, our result for the pressure presents a gauge dependent contribution of $\mathcal{O}(e^4)$. We numerically characterize the credibility of this gauge-dependent calculation and find that the uncertainty due to gauge parameter dependence is under control for $\xi \lesssim 1$. Our calculation also suggests that the choice of Landau gauge may minimize gauge-dependent effects.
The diagrammatic approach to finite temperature quantum field theories heavily relies on the convergence properties of the used expansion scheme. To cure the poor performance of the perturbative loop expansion, various resummation schemes have been invented [1]. Among these, the loop expansion of the two-particle-irreducible (2PI) effective action implements a ladder resummation, which respects thermodynamical consistency and energy conservation [2, 3]. These features make the 2PI scheme attractive for nonequilibrium field theory applications [4]. A prerequisite for a nonequilibrium method to be credible is, however, its reliability in equilibrium. There, it is important to check the convergence of expansion series of the 2PI effective action. To this aim we calculated the notoriously ill-behaved pressure in a scalar context in Ref. [5], and found a monotonous dependence on the coupling constant as well as a relatively small next-to-leading order correction to the pressure value even at couplings of $O(1)$.

In the framework of gauge theories, however, the implementation of this approximation scheme suffers from various difficulties. One of these is that thermodynamic observables computed within this scheme are gauge fixing independent only up to the order of the truncation. One can illustrate this point by considering the issue of gauge parameter dependence in the covariant gauge. For vanishing background fields, the 2PI effective action is a functional of the fermion, gauge and ghost propagators (respectively denoted by $D$, $G$ and $G_{gh}$) which also depends on the gauge-fixing parameter $\xi$: $\Gamma_{2\text{PI}}[D, G, G_{gh}; \xi]$. The thermal pressure of the system is obtained by evaluating $\Gamma_{2\text{PI}}$ at its stationary point
\[
\begin{align*}
\delta \Gamma_{2\text{PI}}/\delta D = 0, \quad \delta \Gamma_{2\text{PI}}/\delta G = 0 \quad \text{and} \quad \delta \Gamma_{2\text{PI}}/\delta G_{gh} = 0.
\end{align*}
\]
for a given temperature $T$, and by subtracting the same calculation at zero temperature:
\[
\mathcal{P} = - T V \Gamma_{2\text{PI}}[\bar{D}, \bar{G}, \bar{G}_{gh}; \xi] \bigg|_{T=0}.
\]

It is possible to show that the $\xi$-dependence of $\mathcal{P}$ uniquely comes from the explicit $\xi$-dependence of $\Gamma_{2\text{PI}}$ and that it disappears if, in Fourier space,
\[
\sum_{\mu\nu} q_\mu q_\nu \bar{G}_{\mu\nu}(q) = \xi.
\]

This last equation is the BRST identity for the propagator of the exact theory [6], which may break in a truncated resummation. Indeed, within a given truncation of the 2PI effective action, the gauge symmetry does not impose any constraint [7] on the two-point function $\bar{G}$ above the order of

\footnote{The barred propagators denote the solution of the stationarity equations: $\delta \Gamma_{2\text{PI}}/\delta D = 0$, $\delta \Gamma_{2\text{PI}}/\delta G = 0$ and $\delta \Gamma_{2\text{PI}}/\delta G_{gh} = 0$.}
Thus beyond this order Eq. (2) breaks and \( \xi \)-dependent contributions to the pressure appear.

A certain number of strategies can be put forward in order to try to cope with these inconvenient features. The first possibility is to introduce further approximations, on top of the loop expansion. This is the case of the \textit{approximately self-consistent resummations} introduced in Ref. [9]. Using this method, a gauge independent determination of the entropy of QCD has been possible and shows a good agreement with lattice results down to temperatures about 2.5 times the transition temperature. There is however no general understanding on how to systematize this approach and evaluate higher orders in a gauge independent manner.

Another possibility is to stick to the loop expansion of the 2PI effective action but play with the freedom in the choice of field representations. Indeed, the exact theory is invariant under reparametrization of the fields and one could exploit this feature in order to define a loop expansion obeying certain properties. This idea has been discussed in Ref. [10] where it has been applied to the linear sigma model in order to define a systematic loop expansion of the 2PI effective action fulfilling Goldstone’s theorem at any order of approximation. Unfortunately no field representation is yet known in gauge theories which would ensure that the BRST identity (2) is fulfilled.

It is finally possible to isolate gauge independent terms in the expression of the pressure by separating contributions from different perturbative orders. This means a re-expansion of the propagators \( \bar{D} \) and \( \bar{G} \) in powers of the coupling. The resulting modified resummation scheme did not show a substantial improvement of convergence [11].

A different point of view is based on the experience that the 2PI loop expansion is known to have good convergence properties [5, 12]. One can expect that contributions above the order of accuracy, and in particular gauge dependences are under control, at least in a large range of coupling values. In this paper we explore this possibility and consider the two-loop truncation of the 2PI effective action using the standard parametrization of the fields. We work in the covariant gauge with arbitrary gauge-fixing parameter \( \xi \), which allows us to study how large gauge dependent contributions can be.

Before embarking on a numerical evaluation, one has however to pay special attention to a second aspect, namely that of renormalization. The difficulty is related to the fact that truncations of the 2PI effective action only resum particular subclasses of perturbative diagrams for which (per-

\footnote{Below this order, the truncated two-point function \( \bar{G} \) coincides with the exact one which fulfills the BRST identity.}

\footnote{More precisely, if one truncates the 2PI effective action at L-loop order, one expects gauge dependences to appear at order \( e^{2L} \).}
turbative) theorems do not apply. Recently a large effort has been put into extending renormalization theorems to the particular classes of diagrams resummed by the loop expansion of the 2PI effective action. This has been first achieved in the framework of scalar theories \[13\] as well as scalar theories coupled to a fermionic field \[14\], and more recently in the framework of QED \[15\] in the covariant gauge. In this latter case, it is important to emphasize that the renormalization procedure differs substantially from the one in perturbation theory. The reason for this is that, for a given loop truncation of the 2PI effective action and in contrast to what happens in perturbation theory, the photon two- and four-point functions develop longitudinal quantum and thermal corrections. Although these contributions are formally of higher order than the order of the truncation, they bring UV divergences which need to be removed before defining a continuum limit. In Ref. \[15\] a renormalization procedure involving a new class of counterterms has been put forward which allows one to deal with this new kind of UV divergences and thus opens the way to practical calculations.

In this letter, we apply these ideas in order to evaluate the pressure of QED from the two-loop 2PI effective action. By solving the stationarity equations for the propagators $\bar{G}$ and $\bar{D}$ and properly determining the counterterms at the temperature of interest we calculate the pressure as given in Eq. 1. This numerical procedure is repeated for several gauge fixing parameters and for various renormalization scales in order to explore how severe the problem of gauge dependence is in the context of thermodynamic calculations within the 2PI framework.

In this work, where we discuss gauge parameter dependence, it is essential that the considered discretization respects gauge symmetry. In this way, the only source for gauge dependences is the particular truncation we use. For numerical purposes it is also convenient to use lattice rather than dimensional regularization. We thus consider QED on a hypercubic lattice of spacing $a$. We denote by $N_β$ the number of points on the time direction and $N$ the number of points on each of the spatial directions. The inverse temperature is $β = N_β/a$ and the spatial volume $V = N^3 a^3$. We decompose the lattice action in three pieces: $S = S_g + S_{gf} + S_f$. As gauge-field action, we consider the non-compact action

$$S_g = \frac{1}{4} a^4 \sum_x \sum_{\mu\nu} F_{\mu\nu}(x) F_{\mu\nu}(x),$$

where the field-strength tensor $F_{\mu\nu}(x) = \Delta^f_{\mu} A_{\nu}(x) - \Delta^f_{\nu} A_{\mu}(x)$ is expressed in terms of the forward derivative\[^4\] $\Delta^f_{\mu} A_{\nu}(x) = a^{-1} [A_{\nu}(x + \hat{\mu}) - A_{\nu}(x)]$. We use

\[^4\]The notation $\hat{\mu}$ stands for the vector of length $a$ along the positive $\mu$ direction.
a discretized covariant gauge fixing term

\[ S_{gf} = \frac{1}{2\xi} a^4 \sum_x \sum_{\mu_\nu} \Delta^b_\mu A_\mu(x) \Delta^b_\nu A_\nu(x) , \]  

(4)
given in terms of the backward derivative \( \Delta^b_\mu A_\mu(x) = a^{-1}[A_\mu(x) - A_\mu(x - \hat{\mu})] \) for latter convenience. Finally, the fermionic action is taken to be the naive chiral action

\[ S_f = -\frac{1}{2a} a^4 \sum_x \left[ \bar{\psi}(x + \hat{\mu}) \gamma_\mu U_\mu(x) \psi(x) - \bar{\psi}(x) \gamma_\mu U_\mu^+(x) \psi(x + \hat{\mu}) \right] , \]  

(5)

where \( U_\mu(x) = \exp(iaeA_\mu(x)) \) represents a link variable.

Normally, the interacting two-point function \( \bar{D}(x,y) \) or \( \bar{G}(x,y) \) corresponds to the correlator of two operators at \( x \) and \( y \). On the lattice however, where the fundamental objects are link variables, it is more convenient to introduce these two-point functions as

\[ \bar{D}(x,y) = \langle \bar{\psi}(x)\psi(y) \rangle_c \quad \text{and} \quad \bar{G}_{\mu\nu}(x,y) = \langle A_\mu(x)A_\nu(y - (1 - \delta_{\mu\nu})\hat{\nu}) \rangle_c . \]  

(6)

This definition maintains the usual translation and reflection symmetries of \( G \) and \( D \), as well as the identity \( G_{\mu\nu}(x,y) = G_{\nu\mu}(x,y) \). Notice also that the discretization we consider here, respects the chiral symmetry of our massless fermion: \( \bar{D}(x,y) = \sum_\mu \gamma_\mu \bar{D}_\mu(x,y) \). We shall thus consider the 2PI effective action as a functional of \( D_\mu \) rather than a functional of \( D \).

Since the pressure (1) cannot be determined exactly, we consider the loop expansion of the 2PI effective action as obtained from the Cornwall-Jackiw-Tomboulis formula [3] (a trivial term stemming from the ghosts is included in our numerics but not written explicitly here):

\[ \Gamma_{\text{2PI}}[D,G] = -N_f \text{Tr} \left[ \log D^{-1} + D_0^{-1} D \right] + \frac{1}{2} \text{Tr} \left[ \log G^{-1} + G_0^{-1} G \right] + \Gamma_{\text{int}}[D,G] \]  

(7)

where we have defined \( \text{Tr} O \equiv a^4 \sum_x \sum_i O_{ii}(x,x) = \beta V \sum_i O_{ii}(x = 0) \) and we have included the possibility of an arbitrary number of fermionic flavors \( N_f \). The functional \( \Gamma_{\text{int}}[D,G] \) is given – up to an overall sign – by all 0-leg 2PI diagrams that one can draw using the two-point functions \( D \) and \( G \) and the tree level vertices generated by the lattice action. These arise from the

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\(^5\) The parameter \( N_f \) will be used in what follows in order to eliminate the doublers which appear as a result of discretizing the fermionic action. Since our discretization generates 16 fermion tastes (one pair in each direction) we shall set \( N_f \) to 1/16. This is similar, for instance, to the fourth root taken on the staggered fermion determinant in the context of lattice gauge field theory [16].
expansion of the link variable $U_\mu(x)$, and in turn of $S_f$, in powers of $A_\mu(x)$. To make sure that the pressure we calculate is correct up to $O(e^3)$, we have to expand $U_\mu(x)$ to $O(e^2)$. The vertex obtained from expanding $U_\mu(x)$ to $O(e^3)$ brings no contribution to the pressure in the case of vanishing background fields, which we assume throughout this work.

Combining the $O(e)$ and $O(e^2)$ vertices into two-loop 2PI diagrams, performing the relevant traces and making use of the properties of $D$, we obtain the following contributions to the interacting part $\Gamma_{\text{int}}$ of the 2PI effective action:

$$\frac{1}{\beta V} \Gamma_{\text{int}}^a = e^2 N_f a^4 \sum_{x, \mu \neq \nu} G_{\mu \nu}(x) \left[ D_\mu(x) D_\nu(x + \hat{\mu} + \hat{\nu}) + D_\nu(x) D_\mu(x + \hat{\nu} + \hat{\nu}) \right]$$

$$+ e^2 N_f a^4 \sum_{x, \mu} G_{\mu \mu}(x) \left[ 2D_\mu(x - \hat{\mu}) D_\mu(x + \hat{\mu}) + 2D_\mu(x) D_\mu(x) \right]$$

$$+ \sum_\nu \left[ D_\nu(x - \hat{\mu}) D_\nu(x + \hat{\mu}) + D_\nu(x) D_\nu(x) \right],$$

$$\frac{1}{\beta V} \Gamma_{\text{int}}^b = a e^2 N_f \sum_{\mu} G_{\mu \mu}(x = 0) \left[ D_\mu(x = \hat{\mu}) - D_\mu(x = -\hat{\mu}) \right].$$

The contribution $\Gamma_{\text{int}}^a$ is the usual fermion loop with a somewhat peculiar photon line \((6)\). The lattice spacing $a$ in $\Gamma_{\text{int}}^b$ manifests that this diagram is a lattice artefact. Both fermion loops are individually quadratically divergent, they together make sure that at the lowest perturbative level the photon receives no mass renormalization.

Together with a counterterm contribution $\delta \Gamma_{\text{int}}$ (see below), the expressions \((8)\) and \((9)\) provide the full $O(e^2)$ interaction part of the 2PI effective action: $\Gamma_{\text{int}} = \Gamma_{\text{int}}^a + \Gamma_{\text{int}}^b + \delta \Gamma_{\text{int}}$. If we now introduce the self-energies

$$\Sigma_\mu(x) = D^{-1}_\mu(x) - D^{-1}_{0,\mu}(x) \quad \text{and} \quad \Pi_{\mu \nu}(x) = G^{-1}_{\mu \nu}(x) - G^{-1}_{0,\mu \nu}(x),$$

and use the explicit formula \((7)\) for the 2PI effective action, we can write the stationarity equations defining the interacting two-point functions $\bar{D}$ and $\bar{G}$ as

$$4N_f \bar{\Sigma}_\mu(x) = \frac{1}{a^4 \beta V} \frac{\partial \Gamma_{\text{int}}}{\partial D_\mu(x)} \quad \text{and} \quad \bar{\Pi}_{\mu \nu}(x) = \frac{2}{a^4 \beta V} \frac{\partial \Gamma_{\text{int}}}{\partial G_{\mu \nu}(x)}.$$

The interacting two-point functions $\bar{D}$ and $\bar{G}$ are thus obtained after simultaneously solving Eqs. \((10)\)-(\(11\)). As it can easily be checked, in the two-loop
approximation that we consider here, Eqs. (11) do not involve any discretized integral in direct space. On the other hand, Eqs. (10) can be conveniently solved in momentum space. To this order of the truncation we can thus completely avoid calculating loops by simply Fourier transforming the propagators back and forth in every step of the iterative procedure. We define the Fourier transforms of a generic fermionic \((D)\) or gauge \((G)\) two-point function, respectively, as

\[
i^{-1}D_{\mu}(k) = a^4 \sum_x e^{-ik \cdot x} D_{\mu}(x) \quad \text{and} \quad \alpha_{\mu\nu}^{-1}(k)G_{\mu\nu}(k) = a^4 \sum_x e^{-ik \cdot x} G_{\mu\nu}(k) ,
\]

where \(\alpha_{\mu\nu}(k) = 1\) if \(\mu = \nu\) and \(\alpha_{\mu\nu}(k) = \exp(-ia(k_{\mu} + k_{\nu})/2)\) otherwise. This particular definition of the Fourier transform of \(G\) is connected to the fact that the gauge field has to be thought as attached to the midpoints of the links. Even this unusual variant of the fast Fourier transformation is available as legacy code [17]. Solving Eqs. (10) in Fourier space, needs that we determine the Fourier transforms of the free inverse propagators. After inspection of the free (quadratic) contribution to \(S\), one obtains

\[
D_{0,\mu}^{-1}(k) = -\hat{k}_{\mu} \quad \text{and} \quad G_{0,\mu\nu}^{-1}(k) = \hat{k}^2 \delta_{\mu\nu} - (1 - \xi^{-1})\hat{k}_{\mu}\hat{k}_{\nu} ,
\]

with the usual short-hand notations \(\bar{k}_{\mu} a = \sin(k_{\mu} a)\) and \(\hat{k}_{\mu} a = 2\sin(k_{\mu} a/2)\).

Eqs. (10)-(11) can be solved for any non-vanishing lattice spacing \(a\), leading to perfectly finite two-point functions \(D\) and \(G\). In order to define a proper continuum limit of the latter, as \(a \to 0\), one needs however to absorb UV divergences. Renormalization of Eqs. (10) and (11) was considered in Ref. [15] in the context of dimensional regularization and at zero temperature. There, renormalization was achieved by adding a contribution \(\delta \Gamma_{\text{int}}\) to the functional \(\Gamma_{\text{int}}\). This contribution carries the counterterms needed for renormalization. In extending this result to lattice regularization, one has to pay attention to the presence of new vertices originating from the expansion of the link variable \(U_\mu(x)\) in powers of the field \(A_\mu(x)\) (see above). In our present calculation, in addition to the usual vertex coupling \(A\) to \(\bar{\psi}\) and \(\psi\) (which leads to \(\Gamma_{\text{int}}^a\)), there is a new vertex coupling \(A^2\) to \(\bar{\psi}\) and \(\psi\) (which leads to \(\Gamma_{\text{int}}^b\)). This new vertex brings an extra factor of \(a\), which is such that the superficial degree of divergence of a given diagram is the same as in dimensional regularization. It follows that we can here apply the same type of analysis of UV divergences as the one used in Ref. [15].

\[\delta = 4 - E_A - (3/2)E_\psi,\] where \(E_A\) and \(E_\psi\) respectively denote the number of external photon and fermion legs of the diagram at hand.
At two-loop order, the shift $\delta \Gamma_{\text{int}}$ is given in lattice regularization by

$$
\delta \Gamma_{\text{int}} = \frac{\delta g_1}{8} \frac{1}{\beta V} \sum_{k, \mu} G_{\mu\mu}(k) \sum_q G_{\nu\nu}(q) + \frac{\delta g_2}{4} \frac{1}{\beta V} \sum_{k} \sum_{\mu} G_{\mu\nu}(k) \sum_q G_{\mu\nu}(q)
$$

$$
+ \frac{1}{2} \sum_k \sum_{\mu\nu} G_{\mu\nu}(k) \left[ \delta Z_3 k^2 \delta_{\mu\nu} - (\delta Z_3 - \delta \lambda) \hat{k}_\mu \hat{k}_\nu + \delta M^2 \delta_{\mu\nu} \right]
$$

$$
- 4N_f \delta Z_2 \sum_{k, \mu} \bar{k}_\mu D_\mu(k).
$$

(14)

It leads to additional contributions at the level of the self-energies, in particular a longitudinal wave function renormalization ($\delta \lambda$) as well as a photon mass counterterm ($\delta M^2$).

The counterterms $\delta g_1$ and $\delta g_2$ allow to remove subdivergences hidden in Eqs. (11) and involving four photon legs (see below). After these have been removed, there only remain temperature independent overall divergences that need to be absorbed in the counterterms $\delta Z_2$, $\delta Z_3$, $\delta \lambda$ and $\delta M^2$. Although the exact O(4) symmetry is broken on the lattice, the tensor structure of the self energies at a fixed scale $k$ is restored in the continuum limit of an isotropic lattice theory. This allows us to use the renormalization conditions introduced in the context of the continuum theory [18]. In particular, one can show that the overall divergences have the structure

$$
\bar{\Sigma}^{\text{div}}(k) = -\sigma \bar{k}_\mu \quad \text{and} \quad \bar{\Pi}^{\text{div}}(k) = \pi_M \delta_{\mu\nu} + \pi_T (\delta_{\mu\nu} \hat{k}_\mu \hat{k}_\nu) + \pi_L \hat{k}_\mu \hat{k}_\nu
$$

(15)

where $\sigma$, $\pi_T$, $\pi_L$ and $\pi_M$ represent quantities which diverge as $a \to 0$ (quadratically for $\pi_M$ and logarithmically for the rest of them). Comparing these expressions to those for the counterterms, we find that all divergences can be absorbed by setting

$$
\delta Z_2 = -\sigma, \quad \delta Z_3 = -\pi_T, \quad \delta \lambda = -\pi_L \quad \text{and} \quad \delta M^2 = -\pi_M.
$$

(16)

The set of Eqs. (16) does not fix the finite parts of the counterterms. In order to do so, we fix $\delta Z_2$, $\delta Z_3$, $\delta \lambda$ and $\delta M^2$ through the renormalization conditions

$$
\frac{\partial \Sigma_{\text{div}}^*}{\partial k_3} \bigg|_{k^*} = 0, \quad \frac{\partial \Pi_{22}^{\text{div}}}{\partial k_3^2} \bigg|_{k^*} = 0, \quad \frac{\partial \Pi_{33}^{\text{div}}}{\partial k_3^2} \bigg|_{k^*} = 0, \quad \text{and} \quad \Pi_{33}^{\text{div}} \bigg|_{k^*} = 0,
$$

(17)

where $k^* = (0, 0, \mu, 0)$ and $\mu$ denotes our renormalization scale. The star on the self-energies means that these are considered at a reference temperature $T^*$. The first two renormalization conditions are similar to those which are
used in perturbation theory and completely determine the counterterms $\delta Z_2$ and $\delta Z_3$. In perturbation theory, where the (lattice) Ward identity for $\bar{\Pi}(k)$ prevents the appearance of longitudinal corrections to the self energy, the third and fourth conditions in Eq. (17) are trivially satisfied. In our case, however, we need to fix two counterterms ($\delta \lambda$ and $\delta M^2$) that cancel UV divergences of $O(e^4)$. A natural way to fix these is to impose the Ward identity on $\bar{\Pi}$ at the renormalization point $k^* = (0, 0, \mu, 0)$ and for a given temperature $T^*$. We do so at $k^*$ and in a small neighborhood of $k^*$. In this way, we obtain the third and fourth renormalization conditions in Eq. (17). The arbitrariness of this condition introduces an ambiguity of order $O(e^4)$.

As already discussed in Ref. [15], when renormalizing the two-point function $\bar{G}$, one has not only to pay attention to longitudinal overall divergences but also to longitudinal subdivergences which involve four-photon legs. Again, if no truncation is considered, these subdivergences automatically cancel since they reproduce the exact four-photon function which is transverse. However, for a given truncation of the 2PI effective action, this cancellation of divergences is only true up to the order of the truncation. Above, new divergences appear which need to be absorbed by means of the counterterms $\delta g_1$ and $\delta g_2$. The particular structure of these divergences has been worked out in Ref. [15] for the case of dimensional regularization. The result is that, in order to absorb the four-photon divergences, one needs to impose, at the renormalization point, the transversality of a four-point function defined by means of a set of Bethe-Salpeter equations. Here, we extend this result to the case of lattice regularization.

The Bethe-Salpeter equations can be written as a closed set of equations for a four-point function $\bar{V}_{\mu\nu,\sigma\rho}(p, k)$ involving four photon legs and a four-point function $\bar{W}_{ij,\sigma\rho}(p, k)$ involving two photon and two fermion legs [15]. Similarly to what we did with the propagator $\bar{D}$, we turn the Dirac indices $i, j$ into one Lorentz index $\mu$: $\sum_\mu \bar{W}_{\mu,\sigma\rho} \gamma_{\mu ij} = \bar{W}_{ij,\sigma\rho}$. Given that $k^* = (0, 0, \mu, 0)$, the renormalization conditions fixing $\delta g_1$ and $\delta g_2$, as given in Ref. [15], read

$$\bar{V}_{2233}(k^*, k^*) = 0 \text{ and } \bar{V}_{3333}(k^*, k^*) = 0. \quad (19)$$

In order to impose these renormalization conditions, we do not need to solve the set of Bethe-Salpeter equations for arbitrary values of the momenta and arbitrary configurations of Lorentz indices. Indeed, the set of equations remains closed if we fix one of the momenta to $k = k^*$ and two
of the Lorentz indices to $\sigma = \rho = 3$. We shall thus consider equations for $\bar{V}_{\mu\nu}(p) = \bar{V}_{\mu33}(p, k^*)$ and $\bar{W}_\alpha(p) = \bar{W}_{\alpha33}(p, k^*)$. Introducing the notations

$$A_{\sigma\rho}(p) = \delta(p - k^*)\delta_{\sigma3}\delta_{\rho3},$$

$$V_{\mu\nu}(p) = \bar{G}_{\mu\alpha}(p)\bar{V}_{\alpha\beta}(p)\bar{G}_{\beta\nu}(p),$$

$$W_\mu(p) = -2\bar{D}_\mu(p)\sum_\rho\bar{W}_\rho(p)\bar{D}_\rho(p) + \bar{W}_\mu(p)\sum_\rho\bar{D}_\rho(p)\bar{D}_\rho(p),$$

we may write the corresponding set of Bethe-Salpeter equations as

$$\bar{V}_{\mu\nu}(p) = -\frac{\delta_{\mu\nu}}{2\beta V}\sum_{q,\rho}[V_{\rho\rho}(q) - 2A_{\rho\rho}(q)] - \frac{\delta_{g2}}{\beta V}\sum_q[V_{\mu\nu}(q) - 2A_{\mu\nu}(q)]$$

$$-\sum_{q,\rho}\frac{\partial \Pi_{\mu\nu}(p)}{\partial D_\rho(q)}W_\rho(q),$$

$$\bar{W}_\mu(p) = -\sum_{q,\rho}\frac{\partial \Sigma_\mu(k)}{\partial G_{\rho\sigma}(q)}[V_{\rho\sigma}(q) - 2A_{\rho\sigma}(q)] - \sum_{q,\rho}\frac{\partial \Sigma_\mu(p)}{\partial D_\rho(q)}W_\rho(q)$$

We solved this pair of equations iteratively by adjusting $\delta g_1$ and $\delta g_2$ after each step so that Eq. (19) is always fulfilled. As expected, the numerical values of these counterterms scale as $\sim e^4 \log(a)$. Once the counterterms have been fixed according to the renormalization conditions (17) and (19), we can solve for the physical two-point functions $\bar{D}$ and $\bar{G}$ which admit a proper continuum limit. Plugging this values into the CJT formula (7) truncated at two-loop order gives us a non-perturbative approximation to the QED pressure, compatible with perturbation theory up to order $O(e^3)$. Notice that, even with all our counterterms, there is a quartic divergence remaining in the pressure. This divergence is temperature independent and can be removed by a ‘cosmological constant’ renormalization. The renormalization condition is usually given by the requirement of zero vacuum pressure. Here we do not renormalize or evaluate the model at zero temperature. We determine the counterterms in the equations of motion at $T^*$. Then, using these counterterms we evaluate the pressure at $T^*$ and $T^*/2$. Assuming a $\sim T^4$ scaling with the temperature, we determine the pressure as the difference of the divergent pressure values as obtained from the formula of the effective action, divided by $(15/16)(T^*)^4$. The assumed scaling of temperature is broken due to the presence of the renormalization scale. This effect introduces an error of $O(e^4)$ which is above the actual accuracy of our calculation.

In order to improve numerical stability, we took into account the following points. Calculating the pressure difference involves the subtraction of two quartically divergent contributions. Instead, we carried out the spatial part of the trace in $\Gamma_{2PI}$ after performing the subtraction. An other
Figure 1: Two-loop QED pressure as a function of the coupling $e$ and for different values of the gauge-fixing parameter $\xi$. The plain line corresponds to $\xi = 0$ (Landau gauge), long-dashed lines to $\xi = 1$ (Feynman gauge) and short-dashed lines to $\xi = 2$. The sensitivity with respect to the renormalization scale $\mu$ is illustrated in the case of the Feynman gauge. We also plot the perturbative $O(e^2)$ result for comparison.

important alteration to the equations above was the exclusion of the spatially homogeneous lattice mode on the level of the 2PI effective action. This was necessary to avoid instabilities as $e \to 0$, since the finite photon mass contribution behaves as $\sim e^4$.

In Fig. 1 we plot the QED pressure in the two-loop 2PI approximation, for a wide range of coupling values ($0 \leq e \leq 2.4$) and for various values of the gauge-fixing parameter. As discussed in Ref. [8] the higher the gauge-fixing parameter is, the less convergent the 2PI loop expansion becomes. It is thus meaningless to consider our calculation for too high values of $\xi$ and, as suggested in Ref. [8], we restricted our calculations to values of the gauge-fixing parameter ranging from $\xi = 0$ (Landau gauge) to $\xi = 2$. For small values of the coupling, our results are almost insensitive to the gauge fixing parameter and nicely reproduce the perturbative result to order $O(e^2)$. This comes as no surprise since the two-loop 2PI approximation contains all diagrams contributing to order $O(e^3)$.\footnote{Our calculation reproduces the $O(e^3)$ result in the infinite volume limit only.} Numerically we find a good agreement with perturbation theory up to $e \sim 1$ which is precisely where the perturbative
expansion usually breaks down.

For large values of the coupling, our calculation becomes a priori sensitive to two types of uncertainties. First of all, renormalization is done by imposing renormalization conditions at a certain momentum $k^* = (0, 0, \mu, 0)$ which introduces an artificial dependence on the scale $\mu$. Moreover the truncation of the 2PI effective action introduces gauge parameter dependences starting at order $O(e^4)$. These two types of uncertainties can be taken as a way to estimate the error of the calculation.

The dependence with respect to the renormalization scale $\mu$ is illustrated in Fig. 1 for the case $\xi = 1$ (Feynman gauge) where $\mu$ is varied in the interval $\pi T \leq \mu \leq 4\pi T$ as it is usually done in calculations at finite temperature. A study of the $\mu$-dependence as the gauge-fixing parameter is varied and for a given value of the coupling is depicted in Fig. 2. Notice that, at fixed gauge-fixing parameter $\xi$, the $\mu$-dependence is not monotonous. However $\mu = 2\pi T$ roughly represents the value at which the pressure reaches its maximum value, in this range. We notice that the uncertainty due to scale dependence is not particularly severe, which indicates the good convergence behavior of the 2PI approach. Moreover this uncertainty is $\sim 1\%$ for $\xi = 2$ and decreases considerably down to its minimum value reached for $\xi = 0$, which makes the Landau gauge a particularly interesting choice among all possible gauges. We also notice that, in general, choosing a higher renormalization scale flattens the gauge dependence towards the Landau gauge value.

The second source of uncertainties is gauge dependence. As already mentioned a calculation for high values of the gauge fixing parameter makes little sense. In the considered range of gauge parameter values, the error due to gauge dependence is of the order of or less than $1 - 1.5\%$. The Landau gauge
plays again a special role since it corresponds to the value of $\xi$ for which the pressure is the less sensitive to gauge parameter dependence. Indeed, independently of the value of the coupling, one has $p_\xi - p_{\xi=0} \sim \xi^2$ as $\xi \to 0$, as it is clear on the logarithmic plot of Fig. 3.

![Figure 3: Gauge-fixing parameter dependence of the two-loop pressure.](image)

In conclusion, our calculation shows, in covariant gauge, a relatively small error coming from gauge parameter dependence. The parametric suppression of the gauge parameter dependence has already been shown in Ref. [8]. We have now established that the so far unknown coefficients of this parametric dependence do not spoil this behavior. The gauge dependence can also be regarded as a bonus feature, which opens a way to error estimates without the need for considering higher order diagrams. We think, that the 2PI effective action can be regarded as an efficient resummation technique for gauge theories, where the actual choice of gauge fixing has an impact on the quality of the resummation. As for the particular calculation presented here, the Landau gauge is the preferred choice. If this result persists in QCD, it could serve as a justification for the exclusive use of Landau gauge in the QCD Schwinger-Dyson equations [19].

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