Casimir Invariants for Systems Undergoing Collective Motion

C. Allen Bishop\textsuperscript{1}, Mark S. Byrd\textsuperscript{1,2}, and Lian-Ao Wu\textsuperscript{3,4}

\textsuperscript{1}Physics Department, Southern Illinois University, Carbondale, Illinois 62901-4401
\textsuperscript{2}Computer Science Department, Southern Illinois University, Carbondale, Illinois 62901
\textsuperscript{3}Department of Theoretical Physics and History of Science, The Basque Country University (EHU/UPV), P.O. Box 644, 48080 Bilbao, Spain and
\textsuperscript{4}IKERBASQUE–Basque Foundation for Science, 48011, Bilbao, Spain

(Dated: January 20, 2010)

Dicke states are states of a collection of particles which have been under active investigation for several reasons. One reason is that the decay rates of these states can be quite different from a set of independently evolving particles. Another reason is that a particular class of these states are decoherence-free or noiseless with respect to a set of errors. These noiseless states, or more generally subsystems, can avoid certain types of errors in quantum information processing devices. Here we provide a method for calculating invariants of systems of particles undergoing collective motions. These invariants can be used to determine a complete set of commuting observables for a class of Dicke states as well as identify possible logical operations for decoherence-free/noiseless subsystems.

Our method is quite general and provides results for cases where the constituent particles have more than two internal states.

PACS numbers: 03.67.Pp,03.65.Yz,11.30.-j,34.80.Pa

I. INTRODUCTION

Decoherence-free/noiseless subsystems (DFS) are now part of an arsenal of weapons used to prevent errors in quantum information processing and storage \cite{16}. (For reviews see \cite{17,18}.) DFS are subsystems which are immune to certain types of errors. The most common type found in the literature is a DFS which is immune to collective errors. These types of quantum systems were studied earlier by Dicke in a different context \cite{9}.

There are several types of states which are now called Dicke states. One such set corresponds to a set of particles which undergo a collective motion, are distinguishable, and do not interact with each other. These states are unchanged by particle interchange, or more generally, the interchange of particular constituents \cite{10}. One particularly clear example is a gas interacting with an external field which has a wavelength significantly longer than the container confining the particles. These are also conditions for collective motion, i.e., the external field interacts in the same way with each particle. In this case, if the size of the container \( \sim R \) and the wavelength of the field is \( \lambda \), then the “Dicke limit” \( \lambda \gg R \) is said to be satisfied. In this limit, when the external influence gives rise to errors in a quantum computing device, the errors are called collective, whether they describe an evolution of each particle which is unitary or not.

Since errors are the greatest obstacle to building a fully functional quantum computing device, any method which aids in the prevention of errors is quite important. However, for the practical use of a DFS/NS for quantum information processing one requires the ability to perform universal computing on these states. This requires finding evolutions which do not take the states out of the protected subspace during gating operations \cite{3}. We refer to such operations as being compatible with the DFS structure. In the physical systems considered by Dicke, one could imagine evolutions of the states which do not change the essential features of the state (energy or total angular momentum quantum numbers), but are indeed nontrivial evolutions. In the case of quantum information processing, these enable quantum computing in a DFS.

In both the early analysis of Dicke states and also quantum computing applications, primarily only two internal states of the constituents were considered. However, three or more internal states of an atom could certainly become important in various experiments and could also arise in particle physics where more than two degrees of freedom are associated with both flavor and color symmetries. Recent experiments \cite{11,12} and proposed experiments \cite{14,15} have provided explicit constructions for these so-called Dicke states using a variety of physical systems.

Here we carry the consideration of Dicke states to the extreme. We consider collections of particles undergoing some collective motions, for example collective errors, and ask the following question. What Hamiltonians give rise to evolutions which are compatible with these motions? Our results are not restricted to any particular number of internal states for each of the constituents, nor are they restricted to any number of particles. We then answer the question by using a construction of invariants analogous to Casimir’s construction of invariants for Lie algebras and Lie groups.

In Section \textsuperscript{[III]} we review the standard Casimir construction for single-particle invariants. In Section \textsuperscript{[IV]} we extend the construction to sets of \( N \) particles each with \( d \) internal states. Section \textsuperscript{[V]} discusses the physical implications of our results. In particular, we discuss the use of these invariants for Dicke state identification as well as the manipulation of decoherence-free or noiseless subsystem. Section \textsuperscript{[VI]} concludes.
II. IDENTIFYING INVARIANTS

A Casimir Operator is a member of the center of the universal enveloping algebra meaning such an operator will commute with every element of the universal enveloping algebra. For matrix representations of quantum evolutions, which we will consider here, the universal enveloping algebra is the algebra of all products of Lie algebra basis elements. It is most important for our purposes that the Casimir operators commute with every generator of the Lie algebra and the collective errors form a representation of the Lie algebra (which is the algebra of Hermitian matrices). Once we find such invariants, we will have set of Hamiltonians which commute with collective errors and are therefore compatible transformations. We begin by reviewing the construction of Casimir invariants.

Let a basis for the Lie algebra of SU(d) (hereafter denoted \( \mathcal{L}(SU(d)) \)) be given by a set \( \{ \lambda_i \} \) with the normalization and properties described in the Appendix. The Casimir operators of SU(d) are known. The most familiar, the quadratic Casimir, is proportional to the sum of the squares of the elements,

\[
C_2 \propto \sum_i \lambda_i \lambda_i. \tag{II.1}
\]

This along with all other Casimir operators can be obtained using the formula \[17\] \[18\]

\[
I_n = \text{Tr}(a_{\lambda_1} \circ \text{ad}_{\lambda_2} \circ \cdots \circ \text{ad}_{\lambda_n}) \lambda_{\lambda_1} \lambda_{\lambda_2} \cdots \lambda_{\lambda_n}. \tag{II.2}
\]

For example,

\[
C_2 = \sum_{a_1, a_2, b_1, b_2} f_{a_1, b_1, a_2, b_2} a_{a_1} b_{a_2} \lambda_{a_1} \lambda_{a_2}. \tag{II.3}
\]

which reduces to Eq. \([\text{II.1}]\) using Eq. \([\text{A.17}]\). It turns out that the formula given in Eq. \([\text{II.2}]\) does not produce independent invariants for the collective errors. However, the independent invariants can be obtained \[17\] and may be written in terms of the totally symmetric \(d\)-tensor. For example, the cubic Casimir invariant is

\[
C_3 = \sum_{ijk} d_{ijk} \lambda_i \lambda_j \lambda_k. \tag{II.4}
\]

Higher order Casimir operators can be constructed using the general formulation

\[
C_n = \sum_{i_1, i_2, \ldots, i_n} d_{i_1, i_2, \ldots, i_n} \ldots d_{i_{n-4}, i_{n-3}, i_{n-2}} \times d_{i_{n-2}, i_{n-1}, i_n} \lambda_{i_1} \lambda_{i_2} \lambda_{i_3} \cdots \lambda_{i_{n-1}} \lambda_{i_n}. \tag{II.5}
\]

The sum is over all elements of the algebra.

To show independence, one may begin with Eq. \([\text{II.2}]\) and reduce the expressions using the identities in the appendix. Here our objective is to find a set of operators which commute with the set of collective motions. A basis for these motions is given by the set of operators of the form

\[
S_j = \sum_\alpha \lambda_j^{(\alpha)}, \tag{II.6}
\]

where the sum is taken over the particles in the system. These types of operators also form a basis for the collective errors acting on a DFS/NS and linear combinations give the stabilizer elements. (See Sec. \[\text{IV D} \] for the definition and discussion.) An element of the algebra (with real coefficients) which commutes with these provides the Hamiltonians which are compatible with a DFS/NS.

III. EXPLICIT FORMS FOR THE INVARIANTS

In this section we will find a set of independent operations for which each element of the set commutes with all members of the algebra formed by the \( S_j \). Denote the algebra of the \( S_j \) by \( \mathcal{A} \).

Note that the Casimir operators formed from the elements \( S_j \) form a representation of \( \mathcal{L}(SU(d)) \) if the \( \lambda_i \) do \[19\]. Therefore these are invariants of the algebra \( \mathcal{A} \), i.e. they commute with elements of this algebra. However, this is not an irreducible algebra. Thus the construction must rely on the identification of the irreducible components.

To proceed, we first calculate the Casimir invariants of \( \mathcal{L}(SU(d)) \). Then, noting that linear combinations of these invariants are also invariants, we extract reducible components of the invariants. From a physical perspective, this means identifying \( n \)-body interactions which are contained within the \( m \)-body interactions where \( n \leq m \).

The quadratic Casimir operator for the algebra \( \mathcal{A} \) is

\[
J_2 = \sum_{i,j,k,l} f_{ijkl} S_j S_i \propto \sum_j S_j S_j. \tag{III.1}
\]

Expanding this in terms of the basis elements \( \{ \lambda_i \} \) gives

\[
J_2 \propto \sum_i \left( \sum_\alpha \lambda_i^{(\alpha)} \right)^2 = \sum_i \left( \sum_\alpha (\lambda_i^{(\alpha)})^2 + 2 \sum_{\alpha<\beta} \lambda_i^{(\alpha)} \lambda_i^{(\beta)} \right). \tag{III.2}
\]

Note that the first term of the last expression is the sum of single-particle Casimir invariants. This allows us to infer that the second term in Eq. \([\text{III.2}]\) is also an invariant quantity. Furthermore, the only nontrivial contributions appearing in the commutator \([\sum_i \lambda_i^{(\alpha)} \lambda_i^{(\beta)}], S_i \) have the form

\[
[\lambda_i^{(\alpha)} \lambda_i^{(\beta)}, \lambda_i^{(\gamma)} + \lambda_i^{(\alpha)} \lambda_i^{(\beta)}], \tag{III.3}
\]

with all other terms vanishing. Since this can be rewritten as

\[
2i f_{ijk} (\lambda_k^{(\alpha)} \lambda_i^{(\beta)} - \lambda_k^{(\alpha)} \lambda_i^{(\beta)}) = 0, \tag{III.4}
\]

we conclude that

\[
\sum_{\alpha<\beta} \lambda_i^{(\alpha)} \lambda_i^{(\beta)} \text{ is an invariant.} \tag{III.5}
\]
we find that
\[ I_2^{(\alpha, \beta)} = \sum_i \lambda_i^{(\alpha)} \lambda_i^{(\beta)} \]  
(III.5)
is also an independent invariant for each pair \((\alpha, \beta)\).

Now consider
\[ J_3 = \sum f_{ijk} f_{klm} S_j S_k S_m \]
\[ = \sum f_{ijk} f_{klm} \left( \sum_{\alpha} \lambda_j^{(\alpha)} \left( \sum_{\beta} \lambda_i^{(\beta)} \right) \left( \sum_{\gamma} \lambda_k^{(\gamma)} \right) \right). \]
(III.6)

Expanding the sums over the particle (Greek) indices, and reducing the results, three types of terms are obtained. First, if all three superscripts are the same, for example \(\lambda_i^{(\alpha)} \lambda_j^{(\alpha)} \lambda_k^{(\alpha)}\), the term reduces to the quadratic Casimir invariant for particle \(\alpha\). Since any linear combination of invariants is invariant, the sum of all terms having this form is also invariant. Second, if two are the same, e.g. \(\lambda_i^{(\alpha)} \lambda_j^{(\alpha)} \lambda_k^{(\beta)}\), then the result reduces to \(I_2^{(\alpha, \beta)}\), thus terms of this form are also invariant quantities. Third, if all three are different, we obtain
\[ I_3^{(\alpha, \beta, \gamma)} = \sum_{ijk} f_{ijk} \lambda_i^{(\alpha)} \lambda_j^{(\beta)} \lambda_k^{(\gamma)}, \]  
(III.7)
as an independent invariant. Notice this case is different from the ordinary Casimir construction where no such independent invariant arises for a term of the form of \(J_3\).

Defining and expanding \(J_4\) produces one new invariant,
\[ I_4^{(\alpha, \beta, \gamma)} = \sum_{ijk} d_{ijk} \lambda_i^{(\alpha)} \lambda_j^{(\beta)} \lambda_k^{(\gamma)}. \]  
(III.8)

Continuing with this will iteratively produce a set of independent invariants for collective motions of particles. For four qutrits this set, \(I_2, I_3, I_4\) is complete [20].

**IV. PHYSICAL IMPLICATIONS**

After the motivation in the introduction and the construction of the invariants, we now consider more explicitly the implications of our findings.

**A. Motion of Dicke States**

In Ref. [1] Dicke examined the spontaneous radiation of photons emitted from a gas consisting of two-level particles. Gasses of both small and large extent were treated separately, the scale being determined relative to the wavelength \(\lambda\) of an externally applied field. Taking \(R\) to be the spatial extent of the container, the two cases correspond to \(\lambda \gg R\) or \(\lambda \ll R\). In both cases it was assumed that there was insufficient overlap of the wave functions of separate particles to require symmetrization of the states. It was also assumed that each particle coupled to the common radiation field via an electric dipole interaction. In general, the interaction energy of the \(n\)th particle with the field can be written as
\[ H_i^{(n)} = -\mathbf{A}(r_n) \cdot \left( e_1 \sigma_i^{(x)} + e_2 \sigma_i^{(y)} \right), \]  
(IV.1)

for some constant real vectors \(e_1\) and \(e_2\).

In the case of a gas confined to a small region of space the vector potential can effectively be considered an independent function of the spatial coordinates \(r_n\). In this approximation the total interaction energy becomes
\[ H_I = c_1 \sum_{\alpha} \sigma_i^{(x)} + c_2 \sum_{\alpha} \sigma_i^{(y)}, \]  
(IV.2)

where \(c_1\) and \(c_2\) denote constants. There are two degrees of freedom associated with the internal energy of any given particle. The energy eigenvalues of the \(j\)th particle, corresponding to the diagonal operator \(\sigma_i^{(j)}\), take on the values \(\pm \hbar \omega / 2\). The sum of all internal particle energies, together with the translational energy of the gas \(H_0\) and the interaction with the field, provides a complete description of a gaseous system consisting of mutually noninteracting particles.

The Hamiltonian for this system can be broken up into two parts,
\[ H = H_0 + \left( c_1 \sum_{\alpha} \sigma_i^{(x)} + c_2 \sum_{\alpha} \sigma_i^{(y)} + \hbar \omega / 2 \sum_{\alpha} \sigma_i^{(z)} \right), \]  
(IV.3)

where the first part describes the translational energy of the system and thus depends solely on the spatial positions \(r_n\), while the second is a quantity independent of these coordinates. As a result, these two parts commute implying the existence of simultaneous eigenfunctions of the two contributions. Let us denote these energy eigenstates
\[ \psi_{pq} = U_p(r_1, r_2, \ldots, r_N) \Phi_q, \]  
(IV.4)

where \(U_p\) depends on the spatial coordinates and \(\Phi_q\) is a function of the internal coordinates. The operators \(S_i = \sum_{\alpha} \sigma_i^{(\alpha)}\) (\(i = x, y, z\)) not only individually commute with the spatially independent quantity \(S^2 = S_x^2 + S_y^2 + S_z^2\), but also satisfy the same commutation relations (up to a multiplicative scaling factor) as the three components of angular momentum. In other words, they form a representation of the \(SO(3)\) algebra. Stationary states of this system can therefore be identified with those eigenstates that conserve the square of the total angular momentum operator, i.e., \(\Phi_j \equiv \Phi_{jm}\), with \(S^2\Phi_{jm} = j(j+1)\Phi_{jm}\) and \(|m| \leq j \leq N/2\). Consequently, the stationary states of a gaseous system confined to a small region of space can be expressed as
\[ \psi_{jm} = U_j(r_1, r_2, \ldots, r_N) \Phi_{jm}. \]  
(IV.5)
Since the individual particles which form the gas all experience a common interaction with the radiation field, the system as a whole evolves in a collective manner. However, while this collective motion is occurring on these states, they may still undergo other non-trivial evolutions. Such operations conserve energy and angular momentum. Hamiltonians corresponding to these non-trivial evolutions commute with the collective operators and thus can be constructed from the previously derived invariants. Furthermore, the number of internal states is not restricted to two, but can be arbitrary. Many internal states may be undergoing simultaneous transitions to other internal states, collectively, while still undergoing this evolution.

In the next section we consider a particular type of Dicke states which is actually invariant under these collective motions. Although the argument follows the usual Dicke state which is actually invariant under these collective motions. Thus it applies to the present case as well since DFS/NS states suitable for quantum information processing correspond to degenerate Dicke states.

B. DFS-Compatible Hamiltonians

Let us suppose that the Dicke states corresponding to a collective DFS/NS are spanned by the set \{ |λ⟩ ⊗ |μ⟩ \}, with \( λ = 1, \ldots, d \) and \( μ = 1, \ldots, n \). Here the |λ⟩’s distinguish a particular basis state of an encoded d-state system and the |μ⟩’s label the \( n \) orthogonal elements which span each qudit dimension. When acted upon by the collective errors \( S_j \) these DFS/NS states have the property that

\[
S_j \langle λ | ⊗ |μ⟩ = \sum_{μ' = 1}^{n} M_{μμ',j} |λ⟩ ⊗ |μ'⟩ . \tag{IV.6}
\]

In other words, these encoded qudit states remain unaffected by the presence of such noise since they map every |λ⟩ to itself. One can parameterize the collective errors using a set of time-independent complex numbers \{ \( v_j \) \},

\[
D(v_1, v_2, ...) = \exp \left[ \sum_j v_j S_j \right] . \tag{IV.7}
\]

The DFS/NS states are not the only accessible states inherent to a system. There are some orthogonal to these which cannot protect against collective noise. When information is leaked into these regions of the systems Hilbert space it may be permanently lost. Gates which are used to manipulate the state of an encoded qudit should therefore operate in a manner such that they map DFS/NS states to other DFS/NS states. It can be shown that a sufficient condition for a transformation \( U = \exp(-iHt) \) to satisfy this compatibility requirement is that

\[
UD(v_1, v_2, ...)U^\dagger = D(v'_1, v'_2, ...) , \tag{IV.8}
\]

or, equivalently

\[
\sum_j v_j U S_j U^\dagger = \sum_j v'_j S_j . \tag{IV.9}
\]

Taking the derivative of both sides of this equation with respect to time yields a sufficient condition for a Hamiltonian to generate a compatible transformation

\[
[H, S_j] = 0, \ \forall S_j . \tag{IV.10}
\]

Since the Casimir operators for the algebra \( \mathcal{A} \) satisfy this condition, they can be used to generate nondissipative transformations of a DFS/NS encoding. We will discuss the implications of these results for the case of a three qudit encoding next, with a particular emphasis on the ability of these operations to generate universal quantum computation.

C. Three Qudits

As mentioned earlier, a basis for the collective errors is given by the set

\[
S_i = \sum_α \lambda_i^{(α)} \tag{IV.11}
\]

where the subscript indicates the type of error and the superscript labels the particle on which the operator acts. The invariants \( I_2, I_3, \) and \( I_4 \) not only commute with every element of this set, but can also be used to form a representation of the Lie algebra of \( SU(2) \) \[20\]. It has been shown that the encoded, or logical analogues of the Pauli matrices acting on an encoded qubit can be given in terms of these invariants by the relations

\[
\vec{X} = \frac{1}{2\sqrt{3}} \left[ I_2^{(2,3)} - I_2^{(1,3)} \right], \quad \vec{Y} = \frac{I_3}{2\sqrt{3}} , \tag{IV.12}
\]

and

\[
\vec{Z} = \left[ I_2^{(2,3)} + I_2^{(1,3)} - 2I_2^{(1,2)} \right]/6 . \tag{IV.13}
\]

All three of these generators can be expressed in terms of two body interactions since \( I_3 \) can be decomposed into products of \( I_2 \). In fact, the invariant \( I_2 \) alone suffices to perform universal computation using encoded qubits that are comprised of three physical qudits since they are able to generate any single qubit rotation, and can also be combined in such a way as to implement an entangling CNOT gate as well. This is due to the fact that the states which were used in Ref. \[21\] for the CNOT are also present in the expansions of the logical states encoded into qudits having \( d \geq 3 \).

In addition, the invariant \( I_2^{(α,β)} \) can also be used to perform the generalized exchange interaction between the states \( |p⟩^{(α)} |q⟩^{(β)} \) associated with particles \( α \) and \( β \) since
it has been shown in Ref. \[20\] that

\[
\exp \left[ -i(\pi/4) \sum_j \lambda_j \otimes \lambda_j \right] |\alpha\beta\rangle = -i \exp(\pi i/2d) |\beta\alpha\rangle ,
\]  

\text{(IV.14)}

for \(\alpha, \beta = 1, 2, \ldots, d\).

Clearly these are linear combinations of the two-body interactions which are comprised of the invariants \(I^{(\alpha,\beta)}_2\). Three-body and higher order interactions are less often experimentally controllable, but are also, in principle, viable candidates for quantum gates. For example the logical \(Y\) interaction for qudits is proportional to \(I_3\).

\section{V. Conclusions}

For quantum systems containing many particles, each having a number of internal states, the system could be in a vast array of possible states corresponding to a large Hilbert space dimension. The evolution of such states can be fairly simple however, as in the case of a system undergoing collective motion. Such motions occur, for example, when \(\lambda \gg R\) so that each particle feels the same field. If states, or subsystems, of a collection of particles are invariant under collective motions, they are decoherence-free, or noiseless with respect to any collective operation, unitary or not. This leads to the promising method for error prevention–encoding in one of these subspaces to avoid collective errors. To take advantage of such an encoding for the purposes of quantum information processing, one requires a complete set of logical operations to be performed on these subsystems which is compatible with the encoding. We have provided a way in which to find the set of Hamiltonians for this purpose.

However, we also note that since collective motions commute with the invariant operators we have presented here, the invariants may be measured while the system undergoes these collective errors. This allows one to describe the system by the values of these operators. Indeed one of the original motivations for studying these invariants was to find a complete set of commuting observables to completely specify a quantum system. (See for example Ref. \[22\] and references therein.) Not all of the invariants presented here will commute with each other, but they each commute with the collective motions. A subset of these invariant operators which also mutually commute will help provide a complete set of commuting operators along with the energy and total angular momentum.

Our work is quite general and can be applied to any set of \(d\)-state systems undergoing collective motions. Therefore, we have extended the Dicke-state description explicitly to the general case leading the way to the description of sets of particles undergoing collective motions and their manipulation when the particles have more than two internal states.

\textbf{Acknowledgments}

This material is based upon work supported by the National Science Foundation under Grant No. 0545798.

\appendix

\section{Appendix A: The algebra of \(SU(d)\)}

We have chosen the following convention for the normalization of the algebra of Hermitian matrices which are generators of \(SU(d)\).

\[
\text{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}.
\]  

\text{(A.1)}

The commutation and anticommutation relations of the matrices representing the basis for the Lie algebra can be summarized using the following equation:

\[
\lambda_i \lambda_j = \frac{2}{d} \delta_{ij} + i f_{ijk} \lambda_k + d_{ijk} \lambda_k,
\]  

\text{(A.2)}

where here, and throughout this appendix, a sum over repeated indices is understood. The sums are written explicitly for clarity only in a few cases.

As with any Lie algebra we have the Jacobi identity:

\[
f_{ilm} f_{jkl} + f_{jlm} f_{kil} + f_{klm} f_{jil} = 0.
\]  

\text{(A.3)}

There is also a Jacobi-like identity,

\[
f_{ilm} d_{jkl} + f_{jlm} d_{kil} + f_{klm} d_{jil} = 0,
\]  

\text{(A.4)}

which was given by Macfarlane, et al. \[23\].

The following identities, also provided in \[23\], are useful

\[
d_{ijk} = 0,
\]  

\text{(A.5)}

\[
d_{ijk} f_{ijk} = 0,
\]  

\text{(A.6)}

\[
f_{ijk} f_{ijk} = d \delta_{il},
\]  

\text{(A.7)}

\[
d_{ijk} d_{ijk} = \frac{d^2 - 4}{d} \delta_{il},
\]  

\text{(A.8)}

and

\[
f_{ijm} f_{klm} = \frac{2}{d} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (d_{ikm} d_{jlm} - d_{jkm} d_{ilm})
\]  

\text{(A.9)}

and finally

\[
d_{piq} d_{qjr} f_{rkm} = \frac{d^2 - 4}{2d} f_{ijk},
\]  

\text{(A.10)}

\[
d_{piq} d_{qjr} d_{rkm} = \frac{d^2 - 12}{2d} d_{ijk}.
\]  

\text{(A.11)}

The proofs of these are fairly straight-forward, but we omit them here.
[1] P. Zanardi and M. Rasetti, Phys. Rev. Lett. **79**, 3306 (1997).
[2] L.-M Duan and G.-C. Guo, Phys. Rev. A **57**, 737 (1998).
[3] D.A. Lidar, I.L. Chuang and K.B. Whaley, Phys. Rev. Lett. **81**, 2594 (1998).
[4] E. Knill, R. Laflamme and L. Viola, Phys. Rev. Lett. **84**, 2525 (2000).
[5] J. Kempe, D. Bacon, D.A. Lidar, and K.B. Whaley, Phys. Rev. A **63**, 042307 (2001).
[6] D.A. Lidar, D. Bacon, J. Kempe, and K.B. Whaley, Phys. Rev. A **63**, 022306 (2001).
[7] D.A. Lidar and K.B. Whaley, in *Irreversible Quantum Dynamics* (Springer-Verlag, Berlin, 2003).
[8] M. S. Byrd, L.-A. Wu and D. A. Lidar, J. Mod. Optics **51**, 2449 (2004).
[9] R. Dicke, Phys. Rev. **93**, 99 (1954).
[10] We will refer to the constituents as particles although we emphasize that the constituents need not be individual particles. They could be sets of particles, collections of subsystems, etc.
[11] R. Prevedel, G. Cronenberg, M. S. Tame, M. Paternostro, P. Walther, M. S. Kim, and A. Zeilinger, Phys. Rev. Lett. **103**, 020503 (2009).
[12] W. Wieczorek, R. Krischek, N. Kiesel, P. Michelberger, G. Tóth, and H. Weinfurter, Phys. Rev. Lett. **103**, 020504 (2009).
[13] K. Härkönen, F. Plastina, and S. Maniscalco, Phys. Rev. A **80**, 033841 (2009).
[14] J. E. M. Kiffner and C. Keitel, Phys. Rev. A **75**, 032313 (2007).
[15] C.A. Bishop and M.S. Byrd, Phys. Rev. A **77**, 012314 (2008).
[16] T. D. B. Hume, C.W. Chou and D. Wineland, Phys. Rev. A **80**, 052302 (2009).
[17] B. Gruber and L. O’Raifeartaigh, J. Math. Phys. 5, 1796 (1964).
[18] Jürgen Fuchs and Christoph Schweigert, *Symmetries, Lie Algebras and Representations* (Cambridge University Press, 1997).
[19] M.S. Byrd, Phys. Rev. A **73**, 032330 (2006).
[20] C.A. Bishop and M.S. Byrd, J. Phys. A: Math. Theor. **42**, 055301 (2009).
[21] D.P. DiVincenzo, Science **270**, 255 (1995).
[22] A. Bohm, *Quantum Mechanics: Foundations and Applications, 3rd Ed., Chapter 5* (Springer-Verlag, New York, New York, 1993).
[23] A.J. Macfarlane, A. Sudbery and P.H. Weisz, Commun. Math. Phys. **11**, 77 (1968).