Some fractional Hermite–Hadamard-type inequalities for interval-valued coordinated functions

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Abstract

The primary objective of this paper is establishing new Hermite–Hadamard-type inequalities for interval-valued coordinated functions via Riemann–Liouville-type fractional integrals. Moreover, we obtain some fractional Hermite–Hadamard-type inequalities for the product of two coordinated h-convex interval-valued functions. Our results generalize several well-known inequalities.

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1 Introduction

The classical Hermite–Hadamard inequalities state that

\[ f\left(\frac{o + \varsigma}{2}\right) \leq \frac{1}{\varsigma - o} \int_{o}^{\varsigma} f(\chi) d\chi \leq \frac{f(o) + f(\varsigma)}{2}, \]

(1.1)

where \( f : \mathcal{I} \to \mathbb{R} \) is a convex function on the closed bounded interval \( \mathcal{I} \) of \( \mathbb{R} \), and \( o, \varsigma \in \mathcal{I} \) with \( o < \varsigma \). Since they play a crucial role in convex analysis and can be a very powerful tool for measuring and computing errors, many authors have devoted their efforts to generalize inequalities (1.1); see [1–6]. It is worth noting that Sarikaya et al. [7] established new Hermite–Hadamard-type inequalities via the Riemann–Liouville fractional integrals. Since then, many papers have generalized different forms of fractional integrals and presented new and interesting refinements of Hermite–Hadamard-type inequalities using these integrals. Fernandez and Mohammed [8] established some Hermite–Hadamard-type inequalities for the Atangana–Baleanu fractional integral. Mohammed and Abdeljawad [9] proved new Hermite–Hadamard-type inequalities in the context of fractional calculus with respect to functions involving nonsingular kernels. For other related results, we refer the readers to [7–19].

On the other hand, to improve the reliability of the calculation results and automatic operation error analysis, Moore [20] introduced the theory of interval analysis. Interval analysis has a strong model for handling interval uncertainty and has been widely

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applied and stretched in control theory [21], dynamical game theory [22], and many others. Recently, numerous famous inequalities have been extended to interval-valued functions. Chalco-Cano et al. [23] obtained Ostrowski-type inequalities for interval-valued functions by using the Hukuhara derivative. Román-Flores et al. [24] derived the Minkowski and Beckenbach-type inequalities for interval-valued functions. Liu et al. [18] proved Hermite–Hadamard-type inequalities via interval Riemann–Liouville-type fractional integrals for interval-valued functions. Very recently, Zhao et al. [25, 26] established Hermite–Hadamard-type inequalities for interval-valued coordinated functions. Budak et al. [27] gave a definition of Riemann–Liouville-type fractional integrals for interval-valued coordinated functions and presented some new Hermite–Hadamard-type inequalities.

Motivated by Zhao et al. [25, 26] and Budak et al. [27], we present a new class of Hermite–Hadamard-type inequalities for coordinated \( h \)-convex interval-valued functions via Riemann–Liouville-type fractional integrals. We also establish Hermite–Hadamard-type inequalities for the products of two interval-valued coordinated functions.

The paper is organized as follows. Section 2 contains some necessary preliminaries. In Sect. 3, we establish some new Hermite–Hadamard-type inequalities for coordinated \( h \)-convex interval-valued functions via Riemann–Liouville-type fractional integrals. We end with Sect. 4 of conclusions.

2 Preliminaries

In this section, we recall some basic definitions and results on interval analysis. We denote by \( \mathbb{R}_I \) the set of closed bounded intervals of \( \mathbb{R} \) and by \( \mathbb{R}^+ \) and \( \mathbb{R}_I^+ \) the sets of positive real numbers and positive intervals, respectively. We also denote \( \Delta = [o, \varsigma] \times [\rho, \varphi] \). For more notions on interval-valued functions, see [28, 29].

Definition 2.1 ([29]) Let \( h : [0, 1] \to \mathbb{R}^+ \). We say that \( f : [o, \varsigma] \to \mathbb{R}_I^+ \) is an \( h \)-convex interval-valued function if for all \( \chi, \gamma \in [o, \varsigma] \) and \( \tau \in [0, 1] \), we have

\[
f(\tau \chi + (1-\tau)\gamma) \supseteq h(\tau)f(\chi) + h(1-\tau)f(\gamma).
\]

We denote the set of all \( h \)-convex interval-valued functions by \( SX(h, [o, \varsigma], \mathbb{R}_I^+) \).

Definition 2.2 ([26]) A function \( \mathcal{F} : \Delta \to \mathbb{R}_I^+ \) is said to be a coordinated convex interval-valued function if

\[
\mathcal{F}(\tau \chi + (1-\tau)\gamma, \theta \mu + (1-\theta)\nu) \\
\supseteq \tau \theta \mathcal{F}(\chi, \mu) + (1-\theta)\mathcal{F}(\gamma, \mu) + (1-\tau)\theta \mathcal{F}(\gamma, \nu) + \tau (1-\theta) \mathcal{F}(\gamma, \nu)
\]

for all \( (\chi, \gamma), (\mu, \nu) \in \Delta \) and \( \tau, \theta \in [0, 1] \).

Definition 2.3 ([25]) Let \( h : [0, 1] \to \mathbb{R}^+ \). Then \( \mathcal{F} : \Delta \to \mathbb{R}_I^+ \) is called a coordinated \( h \)-convex interval-valued function on \( \Delta \) if the partial mappings

\[
\mathcal{F}_\gamma : [o, \varsigma] \to \mathbb{R}_I^+, \quad \mathcal{F}_\gamma(\chi) = \mathcal{F}(\chi, \gamma), \\
\mathcal{F}_\chi : [\rho, \varphi] \to \mathbb{R}_I^+, \quad \mathcal{F}_\chi(\gamma) = \mathcal{F}(\chi, \gamma)
\]
are \( h \)-convex for all \( y \in [\rho, q] \) and \( \chi \in [o, \varsigma] \). We denote the set of all coordinated \( h \)-convex interval-valued functions on \( \Delta \) by \( SX(ch, \Delta, \mathbb{R}_+^\varphi) \).

The families of all Riemann-integrable real-valued functions on \([0, \varsigma]\), interval-valued functions on \([o, \varsigma]\) and on \( \Delta \) are denoted by \( R_{[(0, \varsigma)]}, \mathcal{IR}_{[(o, \varsigma)]} \), and \( \mathcal{ID}_{(\Delta)} \). We have the following:

**Theorem 2.4** ([30]) Let \( f : [o, \varsigma] \rightarrow \mathbb{R}_\varphi \) be such that \( f = \{f, \tilde{f}\} \). Then \( f \in \mathcal{IR}_{[(o, \varsigma)]} \) iff \( f, \tilde{f} \in \mathcal{IR}_{[(o, \varsigma)]} \) and

\[
(\mathcal{IR}) \int_0^\varsigma f(s) \, ds = \left[ (\mathcal{R}) \int_0^\varsigma f(s) \, ds, (\mathcal{R}) \int_0^\varsigma \tilde{f}(s) \, ds \right].
\]

**Theorem 2.5** ([31]) Let \( F : \Delta \rightarrow \mathbb{R}_\varphi \). If \( F \in \mathcal{ID}_{(\Delta)} \), then

\[
(\mathcal{ID}) \int_\Delta \mathcal{F}(t, s) \, ds = (\mathcal{IR}) \int_0^\varsigma dt (\mathcal{IR}) \int_\rho^q \mathcal{F}(t, s) \, ds.
\]

**Definition 2.6** ([16]) Let \( f : [o, \varsigma] \rightarrow \mathbb{R}_\varphi \) and \( f \in \mathcal{IR}_{[(o, \varsigma)]} \). Then the interval Riemann–Liouville-type integrals of \( f \) are defined by

\[
\mathcal{3}^o_\alpha f(\vartheta) = \frac{1}{\Gamma(\alpha)} \int_0^\vartheta (\vartheta - \chi)^{\alpha - 1} f(\chi) \, d\chi, \quad \vartheta > o,
\]

\[
\mathcal{3}^\varsigma_\alpha f(\vartheta) = \frac{1}{\Gamma(\alpha)} \int_\varsigma^\vartheta (\vartheta - \chi)^{\alpha - 1} f(\chi) \, d\chi, \quad \vartheta < \varsigma,
\]

where \( \alpha > 0 \), and \( \Gamma \) is the gamma function.

**Theorem 2.7** ([32]) Let \( f : [o, \varsigma] \rightarrow \mathbb{R}_\varphi^+ \), \( f \in \mathcal{IR}_{[(o, \varsigma)]} \), and \( h : [0, 1] \rightarrow \mathbb{R}_\varphi^+ \). Iff \( \in SX(h, [o, \varsigma]), \mathbb{R}_\varphi^+ \), then

\[
\frac{1}{\varrho h(\frac{\varrho}{2})} f \left( \frac{\varrho + \varsigma}{2} \right) \geq \frac{\Gamma(\alpha)}{(\varsigma - o)^\alpha} \left[ \mathcal{3}^o_\alpha f(\varsigma) + \mathcal{3}^\varsigma_\alpha f(o) \right]
\]

\[
\geq [f(o) + f(\varsigma)] \int_0^1 t^{\alpha - 1} [h(t) + h(1 - t)] \, dt
\]

with \( \alpha > 0 \).

The Riemann–Liouville-type fractional integrals of interval-valued coordinated functions \( F(t, s) \) are given as follows.

**Definition 2.8** ([27]) Let \( F : \Delta \rightarrow \mathbb{R}_\varphi^+ \) and \( F \in \mathcal{ID}_{(\Delta)} \). The Riemann–Liouville-type integrals \( \mathcal{3}^{\varrho, \alpha}_{\varrho, \beta}, \mathcal{3}^\varsigma_{\alpha, \beta}, \mathcal{3}^\varsigma_{\varsigma, \beta}, \mathcal{3}^\varrho_{\varsigma, \beta} \) of \( F \) of order \( \alpha, \beta > 0 \) are defined by

\[
\mathcal{3}^{\varrho, \alpha}_{\varrho, \beta} F(\chi, \gamma) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^\chi \int_\gamma^{\chi - t} (\chi - t)^{\alpha - 1} (\gamma - s)^{\beta - 1} F(t, s) \, ds \, dt, \quad \chi > o, \gamma > \rho,
\]

\[
\mathcal{3}^\varsigma_{\alpha, \beta} F(\chi, \gamma) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_\varsigma^\chi \int_\gamma^{\varsigma - t} (\varsigma - t)^{\alpha - 1} (\gamma - s)^{\beta - 1} F(t, s) \, ds \, dt, \quad \chi > o, \gamma < q,
\]

\[
\mathcal{3}^\varsigma_{\varsigma, \beta} F(\chi, \gamma) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_\varsigma^\chi \int_\gamma^{\varsigma - t} (\varsigma - t)^{\alpha - 1} (\gamma - s)^{\beta - 1} F(t, s) \, ds \, dt, \quad \chi > o, \gamma > q,
\]

\[
\mathcal{3}^\varrho_{\varsigma, \beta} F(\chi, \gamma) = \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_\gamma^{\chi - t} (\chi - t)^{\alpha - 1} (\gamma - s)^{\beta - 1} F(t, s) \, ds \, dt, \quad \chi > o, \gamma < q.
\]
\[ \tilde{J}_{\tilde{x}^{-\beta},\tilde{\gamma}^{-\alpha}}^\beta(t, s) \propto \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^1 \int_0^t (t - \chi)^{\alpha-1}(s - \gamma)^{\beta-1} F(t, s) \, ds \, dt, \quad \chi < \tilde{\gamma}, \tilde{\gamma} > 0, \]

\[ \tilde{J}_{\tilde{x}^{-\beta},\tilde{\gamma}^{-\alpha}}^\beta(t, s) \propto \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_0^1 \int_0^t (t - \chi)^{\alpha-1}(s - \gamma)^{\beta-1} F(t, s) \, ds \, dt, \quad \chi < \tilde{\gamma}, \tilde{\gamma} < q. \]

### 3 Main results

In this section, we prove some new Hermite–Hadamard-type inequalities for coordinated \( h \)-convex interval-valued functions via the interval Riemann–Liouville-type integrals.

**Theorem 3.1** Let \( \mathcal{F}: \Delta \to \mathbb{R}_+^+ \) be such that \( \mathcal{F} = [\mathcal{F}, \mathcal{F}] \) and \( \mathcal{F} \in \mathcal{ID}(\Delta) \), and let \( h: [0, 1] \to \mathbb{R}^+ \). If \( \mathcal{F} \in SX(ch, \Delta, \mathbb{R}_+^+ \mathbb{R}) \), then

\[ \frac{1}{\alpha \beta h^2(\tilde{\gamma})} \mathcal{F} \left( \frac{\alpha + \tilde{\gamma} + \rho + q}{2}, \frac{\mu + \nu}{2} \right) \geq \mathcal{F}(h(\tilde{\gamma})) + \mathcal{F}(h(\tilde{\gamma} + \rho)) + \mathcal{F}(h(\tilde{\gamma} + \rho + q)). \]

Proof Since \( \mathcal{F} \in SX(ch, \Delta, \mathbb{R}_+^+ \mathbb{R}) \), we have

\[ \frac{1}{h^2(\tilde{\gamma})} \mathcal{F} \left( \frac{\alpha + \tilde{\gamma} + \rho + q}{2}, \frac{\mu + \nu}{2} \right) \geq \mathcal{F}(\tilde{\gamma} + \rho + q). \]

Let \( \tilde{\gamma} = \tau o + (1 - \tau)\tilde{\gamma}, \, \gamma = (1 - \tau)\rho + \tau \tilde{\gamma}, \, \mu = \theta \rho + (1 - \theta)q, \, w = (1 - \theta)\rho + \theta q, \, \tau, \theta \in [0, 1]. \)

Then

\[ \frac{1}{h^2(\tilde{\gamma})} \mathcal{F} \left( \frac{\alpha + \tilde{\gamma} + \rho + q}{2}, \frac{\mu + \nu}{2} \right) \geq \mathcal{F}(\tau o + (1 - \tau)\tilde{\gamma} + \theta q) + \mathcal{F}((1 - \tau)\rho + \theta q) + \mathcal{F}((1 - \tau)\rho + \theta q). \]

Consequently,

\[ \frac{1}{\alpha \beta h^2(\tilde{\gamma})} \mathcal{F} \left( \frac{\alpha + \tilde{\gamma} + \rho + q}{2}, \frac{\mu + \nu}{2} \right) = \frac{1}{h^2(\tilde{\gamma})} \mathcal{F} \left( \frac{\alpha + \tilde{\gamma} + \rho + q}{2}, \frac{\mu + \nu}{2} \right) \int_0^1 \int_0^1 \tau^\alpha \rho^\beta - 1 d\theta d\tau \]

\[ \geq \left[ \int_0^1 \int_0^1 \tau^\alpha \rho^\beta - 1 \mathcal{F}(\tau o + (1 - \tau)\tilde{\gamma} + \theta q) d\theta d\tau \right]. \]
\[ + \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}((1-\tau)\sigma + \tau \zeta, \theta \rho + (1-\theta)\rho) \, d\theta \, d\tau \]

\[ + \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}(\tau \sigma + (1-\tau)\zeta, (1-\theta)\rho + \theta \rho) \, d\theta \, d\tau \]

\[ + \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}((1-\tau)\sigma + \tau \zeta, (1-\theta)\rho + \theta \rho) \, d\theta \, d\tau \]

\[ = \left[ \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}(\tau \sigma + (1-\tau)\zeta, \theta \rho + (1-\theta)\rho) \right] \, d\theta \, d\tau \]

\[ + \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}((1-\tau)\sigma + \tau \zeta, \theta \rho + (1-\theta)\rho) \, d\theta \, d\tau \]

\[ + \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}((1-\tau)\sigma + \tau \zeta, (1-\theta)\rho + \theta \rho) \, d\theta \, d\tau \]

\[ + \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}(\tau \sigma + (1-\tau)\zeta, (1-\theta)\rho + \theta \rho) \, d\theta \, d\tau \]

\[ = \left[ \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}(\tau \sigma + (1-\tau)\zeta, \theta \rho + (1-\theta)\rho) \right] \, d\theta \, d\tau \]

\[ \mathcal{F}(\tau \sigma + (1-\tau)\zeta, \theta \rho + (1-\theta)\rho) \, d\theta \, d\tau \]

\[ + \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}((1-\tau)\sigma + \tau \zeta, \theta \rho + (1-\theta)\rho) \, d\theta \, d\tau \]

\[ + \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}((1-\tau)\sigma + \tau \zeta, (1-\theta)\rho + \theta \rho) \, d\theta \, d\tau \]

\[ + \int_0^1 \int_0^1 \tau^{-1} \theta^{-1} \mathcal{F}(\tau \sigma + (1-\tau)\zeta, (1-\theta)\rho + \theta \rho) \, d\theta \, d\tau \]

\[ = \left[ \Gamma(\alpha) \Gamma(\beta) \left( \frac{\zeta}{\zeta - \sigma}, \zeta(\gamma) = \frac{\zeta}{\zeta - \sigma} \right) \mathcal{F}(\zeta(\gamma), \sigma) + \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) \right] \, d\theta \, d\tau \]

\[ + \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) + \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) \]

\[ + \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) + \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) \]

\[ + \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) + \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) \]

\[ = \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) + \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) \]

\[ \left( \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) + \frac{\zeta}{\zeta - \sigma} \mathcal{F}(\zeta(\gamma), \rho) \right) \]

\[ \text{where } \eta(\chi) = \frac{\zeta}{\zeta - \sigma}, \zeta(\gamma) = \frac{\zeta}{\zeta - \sigma}. \]
Similarly, since $\mathcal{F} \in SX(ch, \triangle, \mathbb{R}^+)_{\alpha}$,

$$
\mathcal{F}(\tau o + (1 - \tau)\varsigma, \theta\rho + (1 - \theta)q) + \mathcal{F}((1 - \tau)o + \tau\varsigma, \theta\rho + (1 - \theta)q) + \mathcal{F}(\tau o + (1 - \tau)\varsigma, (1 - \theta)\rho + \theta q) + \mathcal{F}((1 - \tau)o + \tau\varsigma, (1 - \theta)\rho + \theta q) \geq [h(\tau)h(\theta) + h(1 - \tau)h(\theta) + h(\tau)h(1 - \theta) + h(1 - \tau)h(1 - \theta)]
$$

$$
\geq [\mathcal{F}(o, \rho) + \mathcal{F}(o, q) + \mathcal{F}(\varsigma, \rho) + \mathcal{F}(\varsigma, q)].
$$

Multiplying both sides of (3.4) by $\tau^{a-1}\theta^{b-1}$ and integrating on $[0,1] \times [0,1]$, we have

$$
\frac{\Gamma(a)\Gamma(\beta)}{(\varsigma - o)^{(a+1)}(q - \rho)^{(b-1)}} \left[ \mathcal{J}^{(a+1)}_{\alpha,\beta}\mathcal{F}(\varsigma, q) + \mathcal{J}^{(a+1)}_{\beta,\alpha}\mathcal{F}(\varsigma, \rho) + \mathcal{J}^{(b-1)}_{\alpha,\beta}\mathcal{F}(q, \rho) + \mathcal{J}^{(b-1)}_{\beta,\alpha}\mathcal{F}(q, q) \right] 
$$

$$
= \left[ \int_0^1 \int_0^1 \tau^{a-1}\theta^{b-1} \left[ h(\tau)h(\theta) + h(1 - \tau)h(\theta) + h(\tau)h(1 - \theta) + h(1 - \tau)h(1 - \theta) \right] d\tau d\theta \right] 
$$

Using inequalities (3.3) and (3.5) completes the proof. \hfill \square

**Example 3.2** Let $\Delta = [0,2] \times [0,2]$. Let $h(\theta) = \theta$, $\alpha = \beta = \frac{1}{2}$, and $\mathcal{F}(\chi, \gamma) = [(2 - \sqrt{2})(2 - \sqrt{3}), (2 + \sqrt{2})(2 + \sqrt{3})]$. Then

$$
\mathcal{F}(o, \rho) + \mathcal{F}(o, q) + \mathcal{F}(\varsigma, \rho) + \mathcal{F}(\varsigma, q) 
$$

$$
= [72 - 32\sqrt{2}, 72 + 32\sqrt{2}].
$$

Therefore

$$
[16,144] \geq \left[ 66 - 16\sqrt{2} - 8\sqrt{2}\pi + 2\pi + \frac{\pi^2}{2}, 66 + 16\sqrt{2} + 8\sqrt{2}\pi + 2\pi + \frac{\pi^2}{2} \right] 
$$

$$
\geq [72 - 32\sqrt{2}, 72 + 32\sqrt{2}].
$$

Consequently, Theorem 3.1 is verified.
Remark 3.3 If $\mathcal{F} = \mathcal{F}$ and $h(\theta) = \theta$, then we get Theorem 3 of [33]. If $\mathcal{F} = \mathcal{F}$, $h(\theta) = \theta$ and $\alpha = \beta = 1$, then we get Theorem 1 of [34].

Theorem 3.4 Let $\mathcal{F} : \Delta \rightarrow \mathbb{R}^+_\mathcal{F}$ be such that $\mathcal{F} = [\mathcal{F}, \overline{\mathcal{F}}]$ and $\mathcal{F} \in \mathcal{TD}_{\Delta}$, and let $h : [0, 1] \rightarrow \mathbb{R}^+$. If $\mathcal{F} \in \mathcal{SX}(ch, \Delta, \mathbb{R}^+_\mathcal{F})$, then

\[
\frac{1}{h^2(\Delta)} \mathcal{F}\left(\frac{\alpha+\beta}{2}, \frac{\rho+q}{2}\right) \geq \frac{\Gamma(\alpha+1)}{2h(\Delta)(\alpha+\beta)} \left[ \mathcal{G}_{\alpha, \beta}^{\alpha, \beta} + \mathcal{G}_{\rho, \beta}^{\alpha, \beta} + \mathcal{K}_{\rho, \beta}^{\alpha, \beta} \right] \\
+ \frac{\Gamma(\beta+1)}{2h(\Delta)(\alpha+\beta)} \left[ \mathcal{G}_{\rho, \beta}^{\alpha, \beta} + \mathcal{K}_{\rho, \beta}^{\alpha, \beta} \right] \\
\geq \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(\alpha+\beta)^2} \mathcal{F}(\alpha, \rho) \mathcal{F}(\beta, \rho) \mathcal{F}(\alpha, q) \mathcal{F}(\beta, q) \mathcal{F}(\alpha, q) \mathcal{F}(\beta, q) \tag{3.6}
\]

Proof Using Theorem 2.7 and $\mathcal{F} \in \mathcal{SX}(ch, \Delta, \mathbb{R}^+_\mathcal{F})$, we get

\[
\frac{1}{\beta h(\Delta)} \mathcal{F}\left(\frac{\beta+q}{2}\right) \geq \frac{\Gamma(\beta)}{(q-\rho)^2} \left[ \mathcal{G}_{\rho, \beta}^{\beta, \beta} + \mathcal{K}_{\rho, \beta}^{\beta, \beta} \mathcal{F}(\rho) \right] \\
\geq \left[ \mathcal{F}(\rho) + \mathcal{F}(q) \mathcal{F}(\rho) \right] \int_0^1 \theta^{-1} \left[ h(\theta) + h(1-\theta) \right] \, d\theta,
\]

that is,

\[
\frac{1}{\beta h(\Delta)} \mathcal{F}\left(\frac{\chi+q}{2}\right) \geq \frac{1}{(q-\rho)^2} \left[ \int_\rho^q (q-\gamma)^{-1} \mathcal{F}(\chi, \gamma) \, d\gamma + \int_\rho^q (\gamma-\rho)^{-1} \mathcal{F}(\chi, \gamma) \, d\gamma \right] \\
\geq \left[ \mathcal{F}(\chi, \rho) + \mathcal{F}(\chi, q) \right] \int_0^1 \theta^{-1} \left[ h(\theta) + h(1-\theta) \right] \, d\theta
\]
for all $\chi \in [\alpha, \zeta]$. Moreover, we have

$$\frac{1}{h(\frac{1}{2})} \int_0^{\zeta} (\zeta - \chi)^{\alpha-1} F\left(\chi, \frac{\rho + q}{2}\right) d\chi$$

$$\geq \frac{1}{(\zeta - \alpha)\alpha(q - \rho)^{\beta}} \left[ \int_0^{\zeta} \int_0^{q} (\zeta - \chi)^{\alpha-1}(q - \gamma)^{\beta-1} F(\chi, \gamma) d\gamma d\chi \
+ \int_0^{\zeta} \int_0^{q} (\zeta - \chi)^{\alpha-1}(\gamma - \rho)^{\beta-1} F(\chi, \gamma) d\gamma d\chi \right]$$

(3.7)

and

$$\frac{1}{h(\frac{1}{2})} \int_0^{\zeta} (\chi - \alpha)^{\alpha-1} F\left(\chi, \frac{\rho + q}{2}\right) d\chi$$

$$\geq \frac{1}{(\zeta - \alpha)\alpha(q - \rho)^{\beta}} \left[ \int_0^{\zeta} \int_0^{q} (\chi - \alpha)^{\alpha-1}(q - \gamma)^{\beta-1} F(\chi, \gamma) d\gamma d\chi \
+ \int_0^{\zeta} \int_0^{q} (\chi - \alpha)^{\alpha-1}(\gamma - \rho)^{\beta-1} F(\chi, \gamma) d\gamma d\chi \right]$$

(3.8)

Similarly, we have

$$\frac{1}{h(\frac{1}{2})} \int_0^{q} (q - \gamma)^{\beta-1} F\left(\frac{\rho + q}{2}, \gamma\right) d\gamma$$

$$\geq \frac{1}{(\zeta - \alpha)\alpha(q - \rho)^{\beta}} \left[ \int_0^{\zeta} \int_0^{q} (\zeta - \gamma)^{\alpha-1}(q - \gamma)^{\beta-1} F(\gamma, \gamma) d\gamma d\chi \
+ \int_0^{\zeta} \int_0^{q} (\zeta - \gamma)^{\alpha-1}(\gamma - \rho)^{\beta-1} F(\gamma, \gamma) d\gamma d\chi \right]$$

(3.9)

and

$$\frac{1}{h(\frac{1}{2})} \int_0^{q} (q - \gamma)^{\beta-1} F\left(\frac{\rho + q}{2}, \gamma\right) d\gamma$$

$$\geq \frac{1}{(\zeta - \alpha)\alpha(q - \rho)^{\beta}} \left[ \int_0^{\zeta} \int_0^{q} (\zeta - \gamma)^{\alpha-1}(q - \gamma)^{\beta-1} F(\gamma, \gamma) d\gamma d\chi \
+ \int_0^{\zeta} \int_0^{q} (\zeta - \gamma)^{\alpha-1}(\gamma - \rho)^{\beta-1} F(\gamma, \gamma) d\gamma d\chi \right]$$

(3.10)
Summing inequalities (3.7)–(3.10), we have

\[
\frac{\Gamma(\alpha + 1)}{2h(\frac{1}{2})(\varsigma - o)^{\alpha}} \left[ \zeta^\alpha_{\nu}, \mathcal{F}\left(\varsigma, \frac{\rho + q}{2}\right) + \zeta^\alpha_{\varsigma}, \mathcal{F}\left(o, \frac{\rho + q}{2}\right) \right] \\
+ \frac{\Gamma(\beta + 1)}{2h(\frac{1}{2})(q - \rho)^{\beta}} \left[ \zeta^\beta_{\nu}, \mathcal{F}\left(\frac{\alpha + \varsigma}{2}, q\right) + \zeta^\beta_{\varsigma}, \mathcal{F}\left(\frac{\alpha + \varsigma}{2}, q\right) \right] \\
\geq \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{(\varsigma - o)^{\alpha}(q - \rho)^{\beta}} \left[ \zeta^{\alpha,\beta}_{\nu,\nu}, \mathcal{F}(\varsigma, \rho) + \zeta^{\alpha,\beta}_{\nu,\varsigma}, \mathcal{F}(\varsigma, \rho) + \zeta^{\alpha,\beta}_{\varsigma,\nu}, \mathcal{F}(o, \rho) + \zeta^{\alpha,\beta}_{\varsigma,\varsigma}, \mathcal{F}(o, \rho) \right] \\
\geq \frac{\beta\Gamma(\alpha + 1)}{(\varsigma - o)^{\alpha}} \left[ \zeta^\alpha_{\nu}, \mathcal{F}(\varsigma, \rho) + \zeta^\alpha_{\varsigma}, \mathcal{F}(\varsigma, \rho) + \zeta^\beta_{\nu}, \mathcal{F}(o, \varsigma) + \zeta^\beta_{\varsigma}, \mathcal{F}(o, \varsigma) \right] \\
\times \int_0^1 \theta^{\beta - 1}[h(\theta) + h(1 - \theta)] d\theta \\
+ \frac{\alpha\Gamma(\beta + 1)}{(q - \rho)^{\beta}} \left[ \zeta^\alpha_{\nu}, \mathcal{F}(o, \rho) + \zeta^\alpha_{\varsigma}, \mathcal{F}(o, \rho) + \zeta^\beta_{\nu}, \mathcal{F}(\varsigma, \rho) + \zeta^\beta_{\varsigma}, \mathcal{F}(\varsigma, \rho) \right] \\
\times \int_0^1 \tau^{\alpha - 1}[h(\tau) + h(1 - \tau)] d\tau,
\]

which gives the second and third inequalities in (3.6).

Using the first inequality in (2.1), we get

\[
\frac{1}{h^2(\frac{1}{2})} \mathcal{F}\left(\frac{\alpha + \varsigma}{2}, \frac{\rho + q}{2}\right) \geq \frac{\Gamma(\alpha + 1)}{h(\frac{1}{2})(\varsigma - o)^{\alpha}} \left[ \zeta^\alpha_{\nu}, \mathcal{F}\left(\frac{\alpha + \varsigma}{2}, \frac{\rho + q}{2}\right) \right] \tag{3.11}
\]

and

\[
\frac{1}{h^2(\frac{1}{2})} \mathcal{F}\left(\frac{\alpha + \varsigma}{2}, \frac{\rho + q}{2}\right) \geq \frac{\Gamma(\beta + 1)}{h(\frac{1}{2})(q - \rho)^{\beta}} \left[ \zeta^\beta_{\nu}, \mathcal{F}\left(\frac{\alpha + \varsigma}{2}, \frac{\rho + q}{2}\right) \right] \tag{3.12}
\]

Summing inequalities (3.11) and (3.12), we get the first inequality in (3.6).

Using the second inequality in (2.1), we also state

\[
\frac{\Gamma(\alpha)}{(\varsigma - o)^{\alpha}} \left[ \zeta^\alpha_{\nu}, \mathcal{F}(o, \rho) + \zeta^\alpha_{\varsigma}, \mathcal{F}(\varsigma, \rho) \right] \geq \left[ \mathcal{F}(o, \rho) + \mathcal{F}(\varsigma, \rho) \right] \int_0^1 \tau^{\alpha - 1}[h(\tau) + h(1 - \tau)] d\tau,
\]

\[
\frac{\Gamma(\alpha)}{(\varsigma - o)^{\alpha}} \left[ \zeta^\alpha_{\nu}, \mathcal{F}(o, \rho) + \zeta^\alpha_{\varsigma}, \mathcal{F}(\varsigma, \rho) \right] \geq \left[ \mathcal{F}(o, \rho) + \mathcal{F}(\varsigma, \rho) \right] \int_0^1 \theta^{\alpha - 1}[h(\theta) + h(1 - \theta)] d\theta,
\]

and

\[
\frac{\Gamma(\beta)}{(q - \rho)^{\beta}} \left[ \zeta^\beta_{\nu}, \mathcal{F}(\varsigma, \rho) + \zeta^\beta_{\varsigma}, \mathcal{F}(\varsigma, \rho) \right] \geq \left[ \mathcal{F}(\varsigma, \rho) + \mathcal{F}(\varsigma, \rho) \right] \int_0^1 \theta^{\beta - 1}[h(\theta) + h(1 - \theta)] d\theta,
\]

which gives the last inequality in (3.6). This completes the proof. \(\square\)

**Remark 3.5** If \(\mathcal{F} = \mathcal{F}\) and \(h(\theta) = \theta\), then we get Theorem 4 of [33]. If \(\alpha = \beta = 1\), then we get Theorem 3.5 of [25]. If \(\alpha = \beta = 1\) and \(h(\theta) = \theta\), then we get Theorem 7 of [26].
Theorem 3.6 Let $\mathcal{F}, \mathcal{G} : \Delta \to \mathbb{R}^+_2$ be such that $\mathcal{F} = [\mathcal{F}, \mathcal{F}], \mathcal{G} = [\mathcal{G}, \mathcal{G}]$, and $\mathcal{F} \mathcal{G} \in \mathcal{TD}(\Delta)$, and let $h : [0,1] \to \mathbb{R}^+$. If $\mathcal{F} \in \mathcal{SX}(ch_1, \Delta, \mathbb{R}^+_2)$ and $\mathcal{G} \in \mathcal{SX}(ch_2, \Delta, \mathbb{R}^+_2)$, then

$$\begin{align*}
\Gamma(\alpha)\Gamma(\beta) \left[ \sum_{\gamma}^{\alpha, \beta} \mathcal{F}(o,q)\mathcal{G}(o,q) + \sum_{\epsilon}^{\alpha, \beta} \mathcal{F}(o,\rho)\mathcal{G}(o,\rho) \\
+ \sum_{\theta}^{\alpha, \beta} \mathcal{F}(\xi,\gamma)\mathcal{G}(\xi,\gamma) + \sum_{\nu}^{\alpha, \beta} \mathcal{F}(\xi,\rho)\mathcal{G}(\xi,\rho) \right]
\geq M(o,\xi,\rho,q) \int_0^1 \int_0^1 \tau^{\alpha-1}\beta^{\beta-1} \left[ h_1(1-\tau)h_1(1-\beta)h_2(1-\tau)h_2(1-\beta) \\
+ h_1(1-\tau)h_1(1-\beta)h_2(1-\tau)h_2(1-\beta) \\
+ h_1(1-\tau)h_1(1-\beta)h_2(1-\tau)h_2(1-\beta) \right] d\tau d\theta
\end{align*}$$

(3.13)

where

$$\begin{align*}
M(o,\xi,\rho,q) &= \mathcal{F}(o,\rho)\mathcal{G}(o,\rho) + \mathcal{F}(\xi,\rho)\mathcal{G}(\xi,\rho) + \mathcal{F}(o,q)\mathcal{G}(o,q) + \mathcal{F}(\xi,q)\mathcal{G}(\xi,q), \\
N(o,\xi,\rho,q) &= \mathcal{F}(o,\rho)\mathcal{G}(o,q) + \mathcal{F}(\xi,\rho)\mathcal{G}(\xi,q) + \mathcal{F}(o,q)\mathcal{G}(o,\rho) + \mathcal{F}(\xi,q)\mathcal{G}(\xi,\rho), \\
\mathcal{P}(o,\xi,\rho,q) &= \mathcal{F}(o,\rho)\mathcal{G}(\xi,\rho) + \mathcal{F}(\xi,\rho)\mathcal{G}(o,\rho) + \mathcal{F}(o,q)\mathcal{G}(\xi,q) + \mathcal{F}(\xi,q)\mathcal{G}(o,q), \\
\mathcal{Q}(o,\xi,\rho,q) &= \mathcal{F}(o,\rho)\mathcal{G}(\xi,q) + \mathcal{F}(\xi,\rho)\mathcal{G}(o,q) + \mathcal{F}(o,q)\mathcal{G}(\xi,\rho) + \mathcal{F}(\xi,q)\mathcal{G}(o,\rho).
\end{align*}$$

Proof Since $\mathcal{F} \in \mathcal{SX}(ch_1, \Delta, \mathbb{R}^+_2)$ and $\mathcal{G} \in \mathcal{SX}(ch_2, \Delta, \mathbb{R}^+_2)$, we have

$$\begin{align*}
\mathcal{F}(\tau o + (1-\tau)\xi, \theta \rho + (1-\theta)\rho)
\geq h_1(\tau)h_1(\theta)\mathcal{F}(o,\rho) + h_1(\tau)h_1(1-\theta)\mathcal{F}(o,q) \\
+ h_1(1-\tau)h_1(\theta)\mathcal{F}(\xi,\rho) + h_1(1-\tau)h_1(1-\theta)\mathcal{F}(\xi,q),
\end{align*}$$

and

$$\begin{align*}
\mathcal{F}(\tau o + (1-\tau)\xi, (1-\theta)\rho + \theta q)
\geq h_1(\tau)h_1(1-\theta)\mathcal{F}(o,\rho) + h_1(\tau)h_1(\theta)\mathcal{F}(o,q) \\
+ h_1(1-\tau)h_1(1-\theta)\mathcal{F}(\xi,\rho) + h_1(1-\tau)h_1(\theta)\mathcal{F}(\xi,q),
\end{align*}$$

and
\( F((1 - \tau)\sigma + \tau \varsigma, \theta \rho + (1 - \theta)q) \)

\[ \geq h_1(1 - \tau)h_1(\theta)F(\sigma, \rho) + h_1(1 - \tau)h_1(1 - \theta)F(\sigma, q) + h_1(\tau)h_1(\theta)F(\varsigma, \rho) + h_1(\tau)h_1(1 - \theta)F(\varsigma, q), \]

\( G((1 - \tau)\sigma + \tau \varsigma, (1 - \theta)\rho + (1 - \theta)q) \)

\[ \geq h_2(1 - \tau)h_2(\theta)G(\sigma, \rho) + h_2(1 - \tau)h_2(1 - \theta)G(\sigma, q) + h_2(1 - \tau)h_2(\theta)G(\varsigma, \rho) + h_2(1 - \tau)h_2(1 - \theta)G(\varsigma, q), \]

\( G((1 - \tau)\sigma + \tau \varsigma, \theta \rho + (1 - \theta)q) \)

\[ \geq h_2(1 - \tau)h_2(\theta)G(\sigma, \rho) + h_2(1 - \tau)h_2(1 - \theta)G(\sigma, q) + h_2(1 - \tau)h_2(\theta)G(\varsigma, \rho) + h_2(1 - \tau)h_2(1 - \theta)G(\varsigma, q), \]

\( G((1 - \tau)\sigma + \tau \varsigma, (1 - \theta)\rho + (1 - \theta)q) \)

\[ \geq h_2(1 - \tau)h_2(\theta)G(\sigma, \rho) + h_2(1 - \tau)h_2(1 - \theta)G(\sigma, q) + h_2(1 - \tau)h_2(\theta)G(\varsigma, \rho) + h_2(1 - \tau)h_2(1 - \theta)G(\varsigma, q). \]

Since \( F, G \in \mathbb{R}_T^+ \), we have

\[ F((1 - \tau)\sigma + \tau \varsigma, \theta \rho + (1 - \theta)q)G((1 - \tau)\sigma + \tau \varsigma, (1 - \theta)\rho + (1 - \theta)q) \]

\[ + F((1 - \tau)\sigma + \tau \varsigma, (1 - \theta)\rho + (1 - \theta)q)G((1 - \tau)\sigma + \tau \varsigma, (1 - \theta)\rho + (1 - \theta)q) \]

\[ + F((1 - \tau)\sigma + \tau \varsigma, \theta \rho + (1 - \theta)q)G((1 - \tau)\sigma + \tau \varsigma, (1 - \theta)\rho + (1 - \theta)q) \]

\[ + F((1 - \tau)\sigma + \tau \varsigma, (1 - \theta)\rho + (1 - \theta)q)G((1 - \tau)\sigma + \tau \varsigma, (1 - \theta)\rho + (1 - \theta)q) \]

\[ \geq M(\sigma, \varsigma, \rho, \theta) \]

\[ \times [h_1(1 - \tau)h_1(1 - \theta)h_2(1 - \tau)h_2(1 - \theta) + h_1(1 - \tau)h_1(1 - \theta)h_2(1 - \tau)h_2(\theta) + h_1(1 - \tau)h_1(1 - \theta)h_2(\tau)h_2(1 - \theta) + h_1(1 - \tau)h_1(1 - \theta)h_2(\tau)h_2(\theta)] \]

\[ + N(\sigma, \varsigma, \rho, \theta)\left[ h_1(1 - \tau)h_1(1 - \theta)h_2(1 - \tau)h_2(1 - \theta) + h_1(1 - \tau)h_1(1 - \theta)h_2(1 - \tau)h_2(\theta) + h_1(1 - \tau)h_1(1 - \theta)h_2(\tau)h_2(1 - \theta) + h_1(1 - \tau)h_1(1 - \theta)h_2(\tau)h_2(\theta) \right] \]

\[ + P(\sigma, \varsigma, \rho, \theta)\left[ h_1(1 - \tau)h_1(1 - \theta)h_2(1 - \tau)h_2(1 - \theta) + h_1(1 - \tau)h_1(1 - \theta)h_2(\tau)h_2(1 - \theta) + h_1(1 - \tau)h_1(1 - \theta)h_2(\tau)h_2(\theta) \right] \]

\[ + h_1(1 - \tau)h_1(\theta)h_2(1 - \tau)h_2(\theta) + h_1(1 - \tau)h_1(\theta)h_2(\tau)h_2(\theta). \]
\[ + Q(o, \zeta, \rho, q)[h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta) + h_1(\tau)h_1(1-\theta)h_2(1-\tau)h_2(\theta) \\
+ h_1(1-\tau)h_1(\theta)h_2(\tau)h_2(1-\theta) + h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)] \]

Moreover, we have

\[
\int_0^1 \int_0^1 \tau^{\alpha-1} \theta^{\beta-1} F(\tau \omega + (1-\tau) \zeta, \theta \rho + (1-\theta) q) \\
\times G(\tau \omega + (1-\tau) \zeta, \theta \rho + (1-\theta) q) \, d\tau \, d\theta \\
+ \int_0^1 \int_0^1 \tau^{\alpha-1} \theta^{\beta-1} F(\tau \omega + (1-\tau) \zeta, (1-\theta) \rho + \theta q) \\
\times G(\tau \omega + (1-\tau) \zeta, (1-\theta) \rho + \theta q) \, d\tau \, d\theta \\
+ \int_0^1 \int_0^1 \tau^{\alpha-1} \theta^{\beta-1} F((1-\tau) \omega + \tau \zeta, \theta \rho + (1-\theta) q) \\
\times G((1-\tau) \omega + \tau \zeta, \theta \rho + (1-\theta) q) \, d\tau \, d\theta \\
+ \int_0^1 \int_0^1 \tau^{\alpha-1} \theta^{\beta-1} F((1-\tau) \omega + \tau \zeta, (1-\theta) \rho + \theta q) \\
\times G((1-\tau) \omega + \tau \zeta, (1-\theta) \rho + \theta q) \, d\tau \, d\theta \\
\geq \mathcal{M}(o, \zeta, \rho, q) \int_0^1 \int_0^1 \tau^{\alpha-1} \theta^{\beta-1} [h_1(1-\tau)h_1(1-\theta)h_2(1-\tau)h_2(1-\theta) \\
+ h_1(1-\tau)h_1(\theta)h_2(1-\tau)h_2(\theta) + h_1(\tau)h_1(1-\theta)h_2(\tau)h_2(1-\theta) \\
+ h_1(\tau)h_1(\theta)h_2(2\tau)h_2(1-\theta)] \, d\tau \, d\theta \\
+ \mathcal{N}(o, \zeta, \rho, q) \int_0^1 \int_0^1 \tau^{\alpha-1} \theta^{\beta-1} [h_1(1-\tau)h_1(1-\theta)h_2(1-\tau)h_2(1-\theta) \\
+ h_1(1-\tau)h_1(1-\theta)h_2(1-\tau)h_2(\theta) + h_1(\tau)h_1(1-\theta)h_2(\tau)h_2(\theta) \\
+ h_1(\tau)h_1(\theta)h_2(\tau)h_2(1-\theta)] \, d\tau \, d\theta \\
+ \mathcal{P}(o, \zeta, \rho, q) \int_0^1 \int_0^1 \tau^{\alpha-1} \theta^{\beta-1} [h_1(\tau)h_1(1-\theta)h_2(1-\tau)h_2(1-\theta) \\
+ h_1(1-\tau)h_1(1-\theta)h_2(\tau)h_2(1-\theta) + h_1(\tau)h_1(\theta)h_2(\tau)h_2(1-\theta) \\
+ h_1(1-\tau)h_1(\theta)h_2(\tau)h_2(1-\theta)] \, d\tau \, d\theta \\
+ \mathcal{Q}(o, \zeta, \rho, q) \int_0^1 \int_0^1 \tau^{\alpha-1} \theta^{\beta-1} [h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta) \\
+ h_1(\tau)h_1(1-\theta)h_2(1-\tau)h_2(\theta) + h_1(1-\tau)h_1(\theta)h_2(\tau)h_2(1-\theta) \\
+ h_1(\tau)h_1(\theta)h_2(1-\tau)h_2(1-\theta)] \, d\tau \, d\theta .
\]

By Definition 2.8 we get

\[
\int_0^1 \int_0^1 \tau^{\alpha-1} \theta^{\beta-1} F(\tau \omega + (1-\tau) \zeta, \theta \rho + (1-\theta) q) \\
\times G(\tau \omega + (1-\tau) \zeta, \theta \rho + (1-\theta) q) \, d\tau \, d\theta
\]
Remark 3.7 If \( \alpha = \beta = 1 \) and \( h(\theta) = \theta \), then we get Theorem 8 of [26]. If \( F = \tilde{F} \), \( h(\theta) = \theta \), and \( \alpha = \beta = 1 \), then we get Theorem 4 of [35].

**Theorem 3.8** Let \( F, G : [0, 1] \times [\rho, q] \to \mathbb{R}^+ \) be such that \( F = [F, \tilde{F}], G = [G, \tilde{G}] \), and \( F, G \in \mathcal{ID}_{(\Delta)} \), and let \( h : [0, 1] \to \mathbb{R}^+ \). If \( F \in SX(ch_1, \Delta, \mathbb{R}^+ \) and \( G \in SX(ch_2, \Delta, \mathbb{R}^+ \), then

\[
\frac{1}{2\alpha \beta h^2 \left( \frac{1}{2} \right) h^2 \left( \frac{1}{2} \right)} \mathcal{F} \left( \frac{\alpha + \xi}{2}, \frac{\rho + q}{2} \right) \mathcal{G} \left( \frac{\alpha + \xi}{2}, \frac{\rho + q}{2} \right)
\]

\[
\geq \frac{\Gamma(\alpha) \Gamma(\beta)}{2(\xi - \rho)^\alpha (q - \rho)^\beta} \left[ \mathcal{M}(\alpha, \rho, q) \int_0^1 \tau^{\alpha - 1} d\tau \int_0^1 \theta^{\beta - 1} \left[ h_1(\tau) h_1(\theta) [h_2(\tau) h_2(1 - \theta)] + h_2(1 - \tau) h_2(\theta) + h_2(1 - \tau) h_2(1 - \theta) \right] d\theta \right.
\]

\[
+ h_1(\tau) h_1(1 - \theta) \left[ h_2(\tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) \right] \int_0^1 \theta^{\beta - 1} \left[ h_1(\tau) h_1(\theta) [h_2(\tau) h_2(1 - \theta)] + h_2(1 - \tau) h_2(\theta) + h_2(1 - \tau) h_2(1 - \theta) \right] d\theta \]

\[
+ h_1(\tau) h_1(1 - \theta) \left[ h_2(\tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) \right] \int_0^1 \theta^{\beta - 1} \left[ h_1(\tau) h_1(\theta) [h_2(\tau) h_2(1 - \theta)] + h_2(1 - \tau) h_2(\theta) + h_2(1 - \tau) h_2(1 - \theta) \right] d\theta \]

\[
+ \mathcal{N}(\alpha, \rho, q) \int_0^1 \tau^{\alpha - 1} d\tau \int_0^1 \theta^{\beta - 1} \left[ h_1(\tau) h_1(\theta) [h_2(\tau) h_2(1 - \theta)] + h_2(1 - \tau) h_2(\theta) + h_2(1 - \tau) h_2(1 - \theta) \right] d\theta \]

\[
+ h_1(\tau) h_1(1 - \theta) \left[ h_2(\tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) \right] \int_0^1 \theta^{\beta - 1} \left[ h_1(\tau) h_1(\theta) [h_2(\tau) h_2(1 - \theta)] + h_2(1 - \tau) h_2(\theta) + h_2(1 - \tau) h_2(1 - \theta) \right] d\theta \]

\[
+ h_1(\tau) h_1(1 - \theta) \left[ h_2(\tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) \right] \int_0^1 \theta^{\beta - 1} \left[ h_1(\tau) h_1(\theta) [h_2(\tau) h_2(1 - \theta)] + h_2(1 - \tau) h_2(\theta) + h_2(1 - \tau) h_2(1 - \theta) \right] d\theta \]

\[
+ h_1(\tau) h_1(1 - \theta) \left[ h_2(\tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) + h_2(1 - \tau) h_2(1 - \theta) \right] \int_0^1 \theta^{\beta - 1} \left[ h_1(\tau) h_1(\theta) [h_2(\tau) h_2(1 - \theta)] + h_2(1 - \tau) h_2(\theta) + h_2(1 - \tau) h_2(1 - \theta) \right] d\theta
\]
\[ + h_2(\tau)h_3(1 - \theta) + h_3(\tau)h_2(\theta) \]
\[ + h_1(\tau)h_3(1 - \theta)[h_2(1 - \tau)h_3(1 - \theta) + h_3(\tau)h_2(\theta) + h_2(\tau)h_2(1 - \theta)] \]
\[ \int_{\theta}^{1} d\theta. \]

**Proof.** Since \( F \in SX(ch_1, \Delta, R_+^\ast) \) and \( G \in SX(ch_2, \Delta, R_+^\ast) \), we have
\begin{align*}
\mathcal{F} & \left( \frac{o + \zeta}{2}, \frac{\rho + q}{2} \right) \mathcal{G} \left( \frac{o + \zeta}{2}, \frac{\rho + q}{2} \right) \\
& = \mathcal{F} \left( \frac{\tau o + (1 - \tau)\zeta}{2}, \frac{(1 - \tau)\rho + (1 - \theta)q}{2} \right) + \mathcal{F} \left( (1 - \tau)\rho + (1 - \theta)q \right) \\
& \times \mathcal{G} \left( \frac{(1 - \tau)\rho + (1 - \theta)q}{2}, \frac{(1 - \tau)\rho + (1 - \theta)q}{2} \right) \\
& \geq h_1^2 \left( \frac{1}{2} \right) h_2^2 \left( \frac{1}{2} \right) \\
& \times \left[ \mathcal{F} \left( \tau o + (1 - \tau)\zeta, \theta \rho + (1 - \theta)q \right) + \mathcal{F} \left( (1 - \tau)\rho + (1 - \theta)q \right) \mathcal{G} \left( (1 - \tau)\rho + (1 - \theta)q \right) \right] \\
& \times \left[ \mathcal{F} \left( (1 - \tau)\rho + (1 - \theta)q \right) + \mathcal{G} \left( (1 - \tau)\rho + (1 - \theta)q \right) \mathcal{G} \left( (1 - \tau)\rho + (1 - \theta)q \right) \right] \\
& + h_1^2 \left( \frac{1}{2} \right) h_2^2 \left( \frac{1}{2} \right) \\
& \times \left[ h_1(\tau)h_1(\theta)[h_2(1 - \theta)h_3(1 - \theta) + h_3(1 - \tau)h_3(1 - \theta) + h_3(1 - \tau)h_2(\theta)] \right] \\
& + h_1(\tau)h_3(1 - \theta)[h_2(\tau)h_3(1 - \theta) + h_3(1 - \tau)h_3(1 - \theta) + h_3(1 - \tau)h_2(\theta)] \\
& + h_1(1 - \tau)h_1(\theta)[h_2(1 - \tau)h_3(1 - \theta) + h_3(1 - \tau)h_3(1 - \theta) + h_3(1 - \tau)h_2(\theta)] \\
& + h_1(1 - \tau)h_1(1 - \theta)[h_2(1 - \tau)h_3(1 - \theta) + h_3(1 - \tau)h_3(1 - \theta) + h_3(1 - \tau)h_2(\theta)] \]
\[ \times \left[ h_1(\tau)h_1(\theta) \left[ h_2(1-\tau)h_2(1-\theta) + h_2(\tau)h_2(\theta) + h_2(\tau)h_2(1-\theta) \right] \right] \]
\[+ h_1(\tau)h_1(1-\theta) \left[ h_2(1-\tau)h_2(\theta) + h_2(\tau)h_2(1-\theta) + h_2(\tau)h_2(\theta) \right] \]
\[+ h_1(1-\tau)h_1(\theta) \left[ h_2(\tau)h_2(1-\theta) + h_2(1-\tau)h_2(\theta) + h_2(1-\tau)h_2(1-\theta) \right] \]
\[+ h_1(1-\tau)h_1(1-\theta) \]
\[\times \left[ h_2(\tau)h_2(\theta) + h_2(1-\tau)h_2(1-\theta) + h_2(1-\tau)h_2(\theta) \right] P(o, \zeta, \rho, \varphi) \]
\[+ h_1(\tau)h_1(1-\theta) \left[ h_2(1-\tau)h_2(\theta) + h_2(\tau)h_2(1-\theta) + h_2(\tau)h_2(\theta) \right] \]
\[+ h_1(1-\tau)h_1(\theta) \left[ h_2(\tau)h_2(\theta) + h_2(1-\tau)h_2(1-\theta) + h_2(1-\tau)h_2(\theta) \right] \]
\[+ h_1(1-\tau)h_1(1-\theta) \]
\[\times \left[ h_2(\tau)h_2(1-\theta) + h_2(1-\tau)h_2(1-\theta) + h_2(1-\tau)h_2(\theta) \right] Q(o, \zeta, \rho, \varphi) \]
\[ + 2h_1^2 \left( \frac{1}{2} \right) h_2^2 \left( \frac{1}{2} \right) Q(\alpha, \zeta, \rho, q) \]
\[ \times \int_0^1 \tau^{\alpha-1} d\tau \int_0^1 \theta^{\beta-1} \left[ h_1(\tau) h_1(\theta) \left[ h_2(1-\tau) h_2(\theta) \right. \right. \]
\[ + h_2(\tau) h_1(1-\theta) + h_2(\tau) h_2(\theta) \left. \left] + h_1(\tau) h_1(1-\theta) \left[ h_2(1-\tau) h_2(1-\theta) + h_2(\tau) h_2(\theta) + h_2(\tau) h_2(1-\theta) \right. \right. \right] d\theta, \]

which rearranges to the required result. \( \square \)

**Remark 3.9** If \( \alpha = \beta = 1 \) and \( h(\theta) = \theta \), then we get Theorem 9 of [26]. If \( F = F_1, h(\theta) = \theta \), and \( \alpha = \beta = 1 \), then we get Theorem 5 of [35].

### 4 Conclusion

In this paper, we proved some new Hermite–Hadamard-type inequalities for coordinated \( h \)-convex interval-valued functions via Riemann–Liouville-type fractional integrals. The results generalize the previous results given in [25–27, 33, 35]. Moreover, in the future investigation, these results may be extended for different kinds of convexities and fractional integrals.

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### Availability of data and materials

Not applicable.

### Competing interests

The authors declare that they have no competing interests.

### Authors' contributions

Each of the authors contributed to each part of this study equally, all authors read and approved the final manuscript.

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