ON REGULAR $\kappa$-BOUNDED SPACES ADMITTING ONLY CONSTANT CONTINUOUS MAPPINGS INTO $T_1$ SPACES OF PSEUDO-CHARACTER $\leq \kappa$

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Abstract. In this paper for each cardinal $\kappa$ we construct an infinite $\kappa$-bounded (and hence countably compact) regular space $R_\kappa$ such that for any $T_1$ space $Y$ of pseudo-character $\leq \kappa$, each continuous function $f : R_\kappa \to Y$ is constant. This result resolves two problems posted by Tzannes in Open Problems from Topology Proceedings [12] and extends results of Ciesielski and Wojciechowski [4] and Herrlich [8].

We shall follow the terminology of [6, 11]. Throughout of this paper all cardinals are assumed to be infinite.

Regular spaces on which every continuous real-valued function (or, more generally, spaces on which every continuous function into a given space) is constant are of particular interest in general topology. Such spaces were constructed and investigated in [1, 3, 4, 5, 7, 8, 9, 10, 13, 14, 15, 16]. For instance, a well-known result of Herrlich [8] states the following:

Theorem 1 ([8, Theorem]). Let $Y$ be a topological space. The following conditions are equivalent:

- $Y$ is a $T_1$-space;
- there exists a regular space $X$ (having at least two points), such that every continuous map from $X$ to $Y$ is constant.

Also, Ciesielski and Wojciechowski in [4] proved the following:

Theorem 2 ([4, Theorem 7]). For any uncountable cardinal $\kappa$ there exists a regular space $Y$ of cardinality $\kappa$ such that any continuous function from $Y$ into any Hausdorff space $Z$ with a countable pseudo-character is constant.

However, all known examples of regular spaces on which every continuous real-valued function is constant are far from being countably compact. In [15] Tzannes constructed a Hausdorff countably compact space $T$ on which every continuous real-valued function is constant. Nevertheless, the space $T$ is strongly non-Urysohn. In particular, no pair of distinct points of $T$ have disjoint closed neighborhoods. In [12] Tzannes posed the following two problems:

Problem 1 ([12, Problem C65]). Does there exist a regular (first countable, separable) countably compact space on which every continuous real-valued function is constant?

Problem 2 ([12, Problem C66]). Does there exist, for every Hausdorff space $R$, a regular (first countable, separable) countably compact space on which every continuous function into $R$ is constant?

Let $\kappa$ be a cardinal. A topological space $X$ is called $\kappa$-bounded if the the closure of each subset $A \subset X$ of cardinality $\leq \kappa$ is compact. It is clear that each $\kappa$-bounded space is countably compact and each $\kappa$-bounded space of density $\leq \kappa$ is compact.

The pseudo-character $\psi(X)$ of a space $X$ is the smallest cardinal $\lambda$ such that each point is the intersection of a family of cardinality $\leq \lambda$ of sets which are open in $X$.

In this paper for each cardinal $\kappa$ we construct an infinite $\kappa$-bounded regular space $R_\kappa$ such that for each $T_1$ space $Y$ of pseudo-character $\leq \kappa$, each continuous function $f : R_\kappa \to Y$ is constant. This result resolves Problems [11, 2] and extends Theorems [1, 2].

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1. Wallman $\kappa$-bounded extension

For a subset $A$ of a topological space $X$ by $\operatorname{cl}_X(A)$ (or simply $\overline{A}$) we denote the closure of $A$ in $X$.

We recall [6, §3.6] that the Wallman extension $W(X)$ of a $T_1$ space $X$ consists of closed ultrafilters, i.e., families $\mathcal{F}$ of closed subsets of $X$ satisfying the following conditions:

- $\emptyset \notin \mathcal{F}$;
- $A \cap B \in \mathcal{F}$ for any $A, B \in \mathcal{F}$;
- a closed set $F \subset X$ belongs to $\mathcal{F}$ if $F \cap A \neq \emptyset$ for every $A \in \mathcal{F}$.

For any $A \subset X$ put

$$\langle A \rangle = \{\mathcal{F} \in W(X) \mid \text{ there exists } F \in \mathcal{F} \text{ such that } F \subset A\}.$$ 

The Wallman extension $W(X)$ of $X$ carries the topology generated by the base consisting of the sets $\langle U \rangle$ where $U$ runs over open subsets of $X$.

By Theorem [6, 3.6.21], the Wallman extension $W(X)$ is $T_1$ and compact. By Theorem [6, 3.6.22] a $T_1$-space $X$ is normal if and only if $W(X)$ is Hausdorff.

Consider the map $j_X : X \to W(X)$ assigning to each point $x \in X$ the principal closed ultrafilter consisting of all closed sets $F \subset X$ containing the point $x$. It is clear that the image $j_X(X)$ is dense in $W(X)$. Since the space $X$ is $T_1$, Theorem 3.6.21 from [6] provides that the map $j_X : X \to W(X)$ is a topological embedding. So, $X$ can be identified with the subspace $j_X(X)$ of $W(X)$.

**Lemma 1.** Let $\mathcal{F} \in W(X)$ and $A \subset X$. Then $\mathcal{F} \in \operatorname{cl}_{W(X)}(A)$ if and only if $\operatorname{cl}_X(A) \in \mathcal{F}$.

**Proof.** $(\Rightarrow)$ To derive a contradiction assume that $\mathcal{F} \in \operatorname{cl}_{W(X)}(A)$, but $\operatorname{cl}_X(A) \notin \mathcal{F}$. Using the maximality of $\mathcal{F}$ we can find an element $F \in \mathcal{F}$ such that $F \subset X \setminus \operatorname{cl}_X(A)$. Then $(X \setminus \operatorname{cl}_X(A))$ is an open neighborhood of $\mathcal{F}$ such that $(X \setminus \operatorname{cl}_X(A)) \cap A = \emptyset$ which implies a contradiction.

$(\Leftarrow)$ Assume that $\operatorname{cl}_X(A) \in \mathcal{F}$ and fix any open neighborhood $\langle U \rangle$ of $\mathcal{F}$. Then there exists an element $H \in \mathcal{F}$ such that $H \subset U$. Observe that $\operatorname{cl}_X(A) \cap H \in \mathcal{F}$ and $\operatorname{cl}_X(A) \cap H \subset U$. Hence $A \cap (U) \neq \emptyset$ witnessing that $\mathcal{F} \in \operatorname{cl}_{W(X)}(A)$.

Given a cardinal $\kappa$, in the Wallman extension $W(X)$ of a $T_1$-space $X$, consider the subspace

$$W_\kappa X = \bigcup \{ \operatorname{cl}_{W(X)}(C) \mid C \subset X \text{ and } |C| \leq \kappa \}$$

of $W(X)$. The space $W_\kappa(X)$ is called the **Wallman $\kappa$-bounded extension** of $X$. The Wallman $\kappa$-bounded extension was introduced and investigated in [2]. In particular, there it was proved the following:

**Proposition 1 ([2] Proposition 3.2).** For any $T_1$ space $X$, the space $W_\kappa(X)$ is $\kappa$-bounded.

For a topological space $X$ by $2^X$ we denote the family of all closed subsets of $X$. Let $C$ be any subfamily of $2^X$. Following [2], a topological space $X$ is called **totally $C$-normal** if for any disjoint sets $A \in C$ and $B \in 2^X$ there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.

Let $\kappa$ be a cardinal. A topological space $X$ is called **totally $\kappa$-normal** if it is totally $\mathcal{C}$-normal for the family $\mathcal{C}$ of closed subsets of the closures of subsets of cardinality $\leq \kappa$ in $X$.

**Proposition 2 ([2] Proposition 2.9).** Each $\kappa$-bounded regular space $X$ is totally $\kappa$-normal.

**Proposition 3.** The Wallman $\kappa$-bounded extension $W_\kappa(X)$ of $X$ is regular iff $X$ is totally $\kappa$-normal.

**Proof.** $(\Rightarrow)$ Assume that the space $W_\kappa(X)$ is regular. By Proposition 3 the space $W_\kappa(X)$ is $\kappa$-bounded. To show that the space $X$ is totally $\kappa$-normal, take any subset $C \subset X$ of cardinality $|C| \leq \kappa$ and two disjoint closed subsets $F, E$ of $X$ such that $F \subset \operatorname{cl}_X(C)$. Lemma 4 implies that $\operatorname{cl}_{W_\kappa(X)}(F) \cap \operatorname{cl}_{W_\kappa(X)}(E) = \emptyset$. Since $\operatorname{cl}_{W_\kappa(X)}(F) \subset \operatorname{cl}_{W_\kappa(X)}(C)$ and $|C| \leq \kappa$, the set $\operatorname{cl}_{W_\kappa(X)}(F)$ is compact. Since $W_\kappa(X)$ is regular the sets $\operatorname{cl}_{W_\kappa(X)}(F)$ and $\operatorname{cl}_{W_\kappa(X)}(E)$ have disjoint open neighborhoods $U_1$ and $U_2$, respectively, in $W_\kappa(X)$. Put $V_1 = U_1 \cap X$ and $V_2 = U_2 \cap X$. Then $V_1$ and $V_2$ are disjoint open neighborhoods (in $X$) of $F$ and $E$, respectively. Hence $X$ is totally $\kappa$-normal.

$(\Leftarrow)$ Assume that $X$ is totally $\kappa$-normal. Given any closed ultrafilter $\mathcal{F} \in W_\kappa(X)$ and open neighborhood $\langle U \rangle$ of $\mathcal{F}$ find a closed set $F \in \mathcal{F}$ such that $F \subset U$. With no loss of generality we
can assume that there exists a subset $C \subset X$ such that $|C| \leq \kappa$ and $F \subset \text{cl}_X(C)$. By the total $\pi$-normality of $X$, there exists an open subset $V$ of $X$ such that $F \subset V \subset \text{cl}_X(V) \subset U$. Then Lemma 1 implies that
\[ F \in \langle V \rangle \subset \text{cl}_{W_\kappa(X)}(\langle V \rangle) = \langle \text{cl}_X(V) \rangle \subset \langle U \rangle \]
witnessing that the space $W_\kappa(X)$ is regular. \qed

2. Herrlich Extension

In this section we briefly recall a part of famous construction due to Herrlich [8] and establish a few important properties of it.

Let $X$ be a topological space and $\rho$ be an equivalence relation on $X$. Then for any $x \in X$ by $[x]$ we denote the equivalence class of the relation $\rho$ which contains $x$ and for any subset $A \subset X$ put $[A] = \{ [x] \mid x \in A \}$. Also, we agree to denote $[x]$ ($[A]$, resp.) simply as $x$ ($A$, resp.) if $[x] = \{ x \}$ ($[x] = \{ x \}$ for each $x \in A$, resp.).

For any distinct elements $a, b$ of a topological space $X$ by $\text{Const}(X)_{a, b}$ we denote the class of $T_1$ spaces such that $f(a) = f(b)$ for any continuous map $f : X \to Y \in \text{Const}(X)_{a, b}$. Following [8], for any distinct points $a, b$ of a space $X$ it can be constructed a space $H_{a, b}(X)$ such that for any $Y \in \text{Const}(X)_{a, b}$, each continuous function $h : H_{a, b}(X) \to Y$ is constant. It can be done in two steps.

Step 1. For each topological space $Z$ by $P(Z)$ we denote the set $Z \times Z$ endowed with the topology $\tau$ which satisfies the following conditions:

- if $(z, x) \in U \in \tau$, then there exists an open set $V \subset X$ such that $x \in V$ and $\{ z \} \times V \subset U$;
- if $(z, a) \in U \in \tau$, then there exists an open set $W \subset Z$ such that $z \in W$ and $W \times \{ a \} \subset U$.

By $X(Z)$ we denote the quotient space $P(Z)/\rho$ where $\rho$ is the smallest equivalence relation satisfying $(z_1, b)\rho(z_2, b)$ for any $z_1, z_2 \in Z$. Observe that for any $z \in Z$ the vertical fiber $\{ z \} \times Z$ is homeomorphic to $X$. Since the map $h(z) = (z, a)$ is a canonical embedding of $Z$ into $X(Z)$, we can identify $Z$ with the subspace $\{ z \} \times X$. Observe that for each $Y \in \text{Const}(X)_{a, b}$, each continuous function $f : X(Z) \to Y$, $f((z, a)) = f([z, b])$ where $z$ is an arbitrary element of $Z$. Hence for any $Y \in \text{Const}(X)_{a, b}$ each continuous map $f : X(Z) \to Y$ is constant on $Z$.

Step 2. Put $H_1 = X$ and for each $n \in \mathbb{N}$ let $H_{n+1} = X(H_n)$. For any $n \in \mathbb{N}$ by $b_n$ we denote the element $([z, b]) \in H_n$ where $z$ is any element of $H_{n-1}$. Recall that we identify $H_n$ with a subspace $H_n \times \{ a \}$ of $H_{n+1}$ which implies that $H_n \subset H_{n+1}$ for each $n \in \mathbb{N}$. Finally, by $H_{a, b}(X)$ we denote the set $\bigcup_{n \in \mathbb{N}} H_n$ endowed with the topology $\tau$ which satisfies the following condition: a subset $U \subset H_{a, b}(X)$ is open in $(H_{a, b}(X), \tau)$ if the set $U \cap H_n$ is open in $H_n$ for each $n \in \mathbb{N}$. The space $H_{a, b}(X)$ is called a Herrlich extension of $X$.

Fix any $Y \in \text{Const}(X)_{a, b}$. To see that each continuous function $h : H_{a, b}(X) \to Y$ is constant take any distinct points $x, y \in H_{a, b}(X)$ and observe that there exists $n \in \mathbb{N}$ such that $\{ x, y \} \subset H_n$. Recall that any continuous function $g : H_{n+1} \to Y$ is constant on $H_n$. Hence the restriction of $h$ on the set $H_{n+1}$ is constant on $H_n$ witnessing that $h(x) = h(y)$.

Remark 1. The definition of the topology on $H_{a, b}(X)$ (see also [8]) implies the following:

- $H_n$ is a closed subset of $H_{a, b}(X)$ for each $n \in \mathbb{N}$;
- a subset $A \subset H_{a, b}(X)$ is closed iff $A \cap H_n$ is closed in $H_n$ for each $n \in \mathbb{N}$;
- if $X$ is regular, then so is $H_{a, b}(X)$.

Let $\kappa$ be a cardinal. A space $X$ is called $\kappa$-accumulative if $|A| \leq \kappa$ for each subset $A \subset X$ of cardinality $\leq \kappa$.

Proposition 4. For a cardinal $\kappa$ the following statements hold:

1) if $X$ is $\kappa$-accumulative, then so is $H_{a, b}(X)$;
2) if $X$ is $\kappa$-accumulative and totally $\pi$-normal, then $H_{a, b}(X)$ is totally $\pi$-normal.

Proof. Consider statement 1. Observe that $H_1 = X$ is $\kappa$-accumulative by the assumption. Assume that $H_{k-1}$ is $\kappa$-accumulative and consider a subset $A \subset H_k$ of cardinality $\leq \kappa$. For each $h \in H_{k-1}$ by $X_h$ we denote the vertical fiber $\{ h \} \times (X \setminus \{ b \}) \cup \{ b \} \subset H_k$ which is homeomorphic to $X$ and hence it is $\kappa$-accumulative. Put $A_1 = \{ h \in H_{k-1} \mid X_h \cap A \setminus \{ b \} \neq \emptyset \}$. Since $|A| \leq \kappa$ the set
$A_1$ has cardinality $\leq \kappa$ as well. Put $A_2 = \cup \{\text{cl}_{X_h}(A \cap X_h) \mid h \in A_1\}$. Since for each $h \in H_{n-1}$ the subspace $X_h$ is $\kappa$-accumulative, the set $A_2$ has cardinality $\leq \kappa$. Let $A_3 = \text{cl}_{H_{n-1}}(A_1)$. Since $H_{k-1}$ is $\kappa$-accumulative and $|A_1| \leq \kappa$, the set $A_2$ has cardinality $\leq \kappa$. Finally, the definition of the topology on $H_k$ implies that $\text{cl}_{H_k}(A) \subset A_2 \cup A_3 \cup \{b_k\}$ witnessing that $H_k$ is $\kappa$-accumulative. Hence for each $n \in \mathbb{N}$ the space $H_n$ is $\kappa$-accumulative.

To see that $H_{a,b}(X)$ is $\kappa$-accumulative fix any subset $B \subset H_{a,b}(X)$ of cardinality $\leq \kappa$. By Remark 1 $\text{cl}_{H_{a,b}}(X)(B) = \cup_{n \in \mathbb{N}} \text{cl}_{H_n}(B \cap H_n)$. Since each $H_n$ is $\kappa$-accumulative $|\text{cl}_{H_n}(B \cap H_n)| \leq \kappa$. Hence $|\text{cl}_{H_{a,b}}(X)(B)| \leq \kappa$ witnessing that $H_{a,b}(X)$ is $\kappa$-accumulative.

Consider statement 2. By statement 1, the space $H_{a,b}(X)$ is $\kappa$-accumulative. Then it is clear that for each $n \in \mathbb{N}$ the space $H_n$ is $\kappa$-accumulative as well. Observe that $H_1 = X$ is totally $\pi$-normal by the assumption. Assume that $H_{k-1}$ is totally $\pi$-normal and consider a closed subset $A \subset H_k$ of cardinality $\leq \kappa$. Fix any open set $U \subset H_k$ such that $A \subset U$. We will show that there exists an open set $W \subset H_k$ such that $A \subset W \subset \text{cl}_{H_k}(W) \subset U$ which would provide that $H_k$ is totally $\pi$-normal. Since $H_{k-1}$ is totally $\pi$-normal there exists an open subset $V$ of $H_{k-1}$ such that $A \cap H_{k-1} \subset V \subset \text{cl}_{H_{k-1}}(V) \subset U \cap H_{k-1}$. Moreover, since $U$ is open in $H_k$ for each $h \in V$ there exists an open neighborhood $V_h(a)$ of $a$ in $X_h$ such that $(h) \times V_h(a) \subset U$. Since $X_h$ is regular we can assume that $\text{cl}_{X_h}(V_h(a)) \subset U \cap X_h$. Then $V^* = \cup \{\text{cl}_{X_h}(V_h(a)) \mid h \in V\}$. Let $A_1 = \{h \in H_{k-1} \mid A \cap X_h \neq \emptyset\}$. Since the set $X_h$ is closed in $H_k$ for each $h \in H_{k-1}$, the set $A \cap X_h$ is closed in $H_k$ as well. Since $X_h$ is totally $\pi$-normal for each $h \in A_1$ there exists an open subset $W_h$ of $X_h$ such that $A \cap X_h \subset W_h \subset \text{cl}_{X_h}(W_h) \subset U \cap X_h$. Since $X_h$ is regular we can also assume that for each $h \in A_1, W_h$ satisfies the following condition: if $(h,a) \notin A \cap X_h$, then $(h,a) \notin \text{cl}_{X_h}(W_h)$. Let $W = \cup \{W_h \mid h \in A_1\} \cup V^*$. One can check that $W$ is an open subset of $H_k$ and $A \subset W \subset \text{cl}_{H_k}(W) \subset U$ witnessing that $H_k$ is totally $\pi$-normal. Hence $H_n$ is totally $\pi$-normal for each $n \in \mathbb{N}$.

To see that $H_{a,b}(X)$ is totally $\pi$-normal fix any closed subset $B \subset H_{a,b}(X)$ of cardinality $\leq \kappa$ and an open subset $U \subset H_{a,b}(X)$ such that $B \subset U$. By Remark 1 for each $n \in \mathbb{N}$ the set $B_n = B \cap H_n$ is closed in $H_n$. Since each $H_n$ is totally $\pi$-normal there exists an open subset $V_n$ of $H_n$ such that $B_n \subset V_n \subset \text{cl}_{H_n}(V_n) \subset U \cap H_n$ for each $n \in \mathbb{N}$. Then $V = \cup_{n \in \mathbb{N}} V_n$ is an open (in $H_{a,b}(X)$) set which contains $B$ and, by Remark 1 $\text{cl}_{H_{a,b}}(X)(V) = \cup_{n \in \mathbb{N}} \text{cl}_{H_n} V_n \subset U$ witnessing that $H_{a,b}(X)$ is totally $\pi$-normal. □

3. MAIN RESULT

For any ordinals $\alpha, \beta$ by $[\alpha, \beta]$ ([\alpha, \beta]$ resp.) we denote the set of all ordinals $\gamma$ such that $\alpha \leq \gamma < \beta$ $[\alpha \leq \gamma \leq \beta$, resp.]. Further if some ordinal $\alpha$ is considered as a topological space, then $\alpha$ is assumed to carry the order topology.

Before formulating the main result of this paper we prove two auxiliaries lemmas.

**Lemma 2.** Let $\kappa$ be a cardinal, $\xi$ be a regular cardinal $> \kappa^+$ and $Y$ be a $T_1$ space of pseudo-character $\psi(Y) \leq \kappa$. Then for each continuous map $f : \xi \to Y$ there exist $y \in Y$ and $\mu \in \xi$ such that $[\mu, \xi) \subset f^{-1}(y)$.

**Proof.** Let $B = \{y \in Y \mid f^{-1}(y) \text{ is a cofinal subset of } \xi\}$. We claim that $|B| \leq 1$. Indeed, if there exist distinct elements $y_1, y_2 \in B$, then the sets $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are closed and unbounded in $\xi$ providing that $f^{-1}(y_1) \cap f^{-1}(y_2) \neq \emptyset$ which yields a contradiction. Hence there are two cases to consider:

1) the set $B$ is empty;
2) the set $B$ is singleton;

Consider case 1. Put $\alpha_0 = 0$ and for each $\beta \leq \kappa^+$ put $\alpha_\beta = \sup(f^{-1}(f(\alpha_\beta - 1))) + 1$ if $\beta$ is a successor ordinal and $\alpha_\beta = \sup(\alpha_\delta \mid \delta < \beta)$ if $\beta$ is a limit ordinal. Since $\psi(\xi) > \kappa^+$ the sequence $\{\alpha_\beta \mid \beta \leq \kappa^+\}$ is a subset of $\xi$. Observe that for each distinct $\beta_1, \beta_2 \in \kappa^+ + 1$, $f(\alpha_{\beta_1}) \neq f(\alpha_{\beta_2})$. Put $y = f(\alpha_{\kappa^+})$ and observe that $\psi(\alpha_{\kappa^+}) = \kappa^+$. Since $\psi(Y) \leq \kappa$ there exists a family $\mathcal{U}$ of cardinality $\leq \kappa$ of open neighborhoods of $y$ such that $\bigcap \mathcal{U} = \{y\}$. Then for each $U \in \mathcal{U}$, $f^{-1}(U)$ is an open neighborhood of $\alpha_{\kappa^+}$ providing that there exists $\mu_U \in \alpha_{\kappa^+}$ such that $[\mu_U, \alpha_{\kappa^+}] \subset f^{-1}(U)$. Since $\psi(\alpha_{\kappa^+}) > \kappa$ the ordinal $\mu = \sup\{\mu_U \mid U \in \mathcal{U}\}$ belongs to $\alpha_{\kappa^+}$. Hence
\[ [\mu, \alpha_\kappa] \subset \bigcap \{ f^{-1}(U) \mid U \in \mathcal{U} \} = f^{-1}(y). \]

Then for each ordinal \( \alpha_\beta \in [\mu, \alpha_\kappa] \), \( f(\alpha_\beta) = f(\alpha_\kappa) \)
which implies a contradiction. Hence case 1 is not possible.

Consider case 2. Let \( y \in B \). Observe that \( f^{-1}(y) \) is closed and unbounded in \( \xi \). Fix any family \( \mathcal{U} \) of cardinality \( \leq \kappa \) of open neighborhoods of \( y \) such that \( \cap \mathcal{U} = \{ y \} \). Since for each \( U \in \mathcal{U}, f^{-1}(U) \)
is an open set which contains \( f^{-1}(y) \), for each \( \alpha \in f^{-1}(y) \) there exists an ordinal \( \mu_\alpha \) \( \in \alpha \) such that \( [\mu_\alpha, \alpha] \subset f^{-1}(U) \). For each \( U \in \mathcal{U} \) define the map \( h_U : f^{-1}(y) \rightarrow \xi \) by \( h_U(\alpha) = \mu_\alpha, \alpha \in f^{-1}(y) \).

Since \( \xi \) is a regular cardinal, for each \( U \in \mathcal{U} \) the map \( h_U \) satisfies conditions of Fodor’s Lemma [11] Lemma III.6.14. Hence for each \( U \in \mathcal{U} \) there exists an ordinal \( \mu_U \in \xi \) and an unbounded (even stationary) in \( \xi \) subset \( A \subset f^{-1}(y) \) such that \( h_U(\alpha) = \mu_U \) for each \( \alpha \in A \). At this point it is clear that \( [\mu_U, \xi] \subset f^{-1}(U) \) for any \( U \in \mathcal{U} \). Since \( cf(\xi) > \kappa \) the ordinal \( \mu = \sup \{ \mu_U \mid U \in \mathcal{U} \} \)
belongs to \( \xi \). Hence \( [\mu, \xi] \subset \bigcap \{ f^{-1}(U) \mid U \in \mathcal{U} \} = f^{-1}(y) \).

For any distinct cardinals \( \alpha, \beta \), by \( T_{\alpha, \beta} \) we denote the punctured Tychonoff \((\alpha, \beta)\)-plank, i.e.,
the subspace \((\alpha + 1) \times (\beta + 1) \setminus \{ (\alpha, \beta) \} \) of the Tychonoff product \((\alpha + 1) \times (\beta + 1) \).

**Lemma 3.** Let \( \kappa \) be a cardinal and \( Y \) be a \( T_1 \) space such that \( \psi(Y) \leq \kappa \). Then for each regular cardinals \( \lambda, \xi \) such that \( \kappa^+ < \lambda < \xi \) and for each continuous map \( f : T_{\lambda, \xi} \rightarrow Y \) there exist \( y \in Y \), \( \alpha \in \lambda \) and \( \mu \in \xi \) such that \( [\alpha, \lambda] \times [\mu, \xi] \setminus \{ (\lambda, \xi) \} \subset f^{-1}(y) \).

**Proof.** By Lemma [2] there exist \( y \in Y \) and \( \alpha \in \lambda \) such that \( [\alpha, \lambda] \times [\xi] \subset f^{-1}(y) \). Since \( \psi(Y) \leq \kappa \) there exists a family \( \mathcal{U} \) of cardinality \( \leq \kappa \) of open neighborhoods of \( y \) such that \( \cap \mathcal{U} = \{ y \} \). The continuity of \( f \) implies that for each \( U \in \mathcal{U} \) the set \( f^{-1}(U) \) is open and contains the set \( [\alpha, \lambda] \times [\xi] \).

Since \( \lambda < cf(\xi) \) for each \( U \in \mathcal{U} \) there exists \( \mu_U \in \xi \) such that \( [\alpha, \lambda] \times [\mu_U, \xi] \subset f^{-1}(U) \). Since \( \kappa < cf(\xi) \) the ordinal \( \mu = \sup \{ \mu_U \mid U \in \mathcal{U} \} \) belongs to \( \xi \). Then \( [\alpha, \lambda] \times [\mu, \xi] \subset \bigcap \{ f^{-1}(U) \mid U \in \mathcal{U} \} = f^{-1}(y) \).

The following Theorem resolves Problem [1] and Problem [2] and is the main result of this paper.

**Theorem 3.** For each cardinal \( \kappa \) there exists a regular infinite \( \kappa \)-bounded space \( R_\kappa \) such that for any \( T_1 \) space \( Y \) of pseudo-character \( \psi(Y) \leq \kappa \) each continuous map \( f : R_\kappa \rightarrow Y \) is constant.

**Proof.** Fix any cardinal \( \kappa \). By \( T \) we denote the punctured Tychonoff plank \( T_{\kappa^+, \kappa^+} \). Observe that \( T \) is \( \kappa \)-bounded and \( \kappa \)-accumulative. Let \( \mathbb{Z} \) be a discrete set of integers and \( a, b \) be distinct points which do not belong to \( T \times \mathbb{Z} \). By \( R \) we denote the set \( T \times \mathbb{Z} \cup \{ a, b \} \) endowed with the topology \( \tau \) which satisfies the following conditions:

- the Tychonoff product \( T \times \mathbb{Z} \) is open in \( R \);
- if \( a \in U \in \tau \), then there exists \( n \in \mathbb{N} \) such that \( \{ (t, -k) \mid t \in T \text{ and } k > n \} \subset U \);
- if \( b \in U \in \tau \), then there exists \( n \in \mathbb{N} \) such that \( \{ (t, k) \mid t \in T \text{ and } k > n \} \subset U \).

One can check that the space \( R \) is regular, \( \kappa \)-accumulative and \( \kappa \)-bounded. For convenience, we denote the subset \( T \times \{ n \} \subset R \) by \( T_n \) for each \( n \in \mathbb{Z} \).

On the space \( R \) consider the smallest equivalence relation \( \sim \) such that
\[
( x, \kappa^{++} + 2n ) \sim ( x, \kappa^{++} + 2n + 1 ) \quad \text{and} \quad ( \kappa^{++}, y, 2n ) \sim ( \kappa^{++}, y, 2n - 1 )
\]
for any \( n \in \mathbb{Z}, x \in \kappa^{++} \) and \( y \in \kappa^{++} \).

Let \( X \) be the quotient space \( R/\sim \) of \( R \) by the equivalence relation \( \sim \). Being the continuous image of the \( \kappa \)-bounded space \( R \) the space \( X \) is \( \kappa \)-bounded as well. It is straightforward to check that \( X \) is regular and hence, by Proposition [2] \( X \) is totally \( \pi \)-normal. Also, one can check that \( X \) is \( \kappa \)-accumulative. We claim that for each \( T_1 \) space \( Y \) of pseudo-character \( \psi(Y) \leq \kappa \) and for each continuous map \( f : X \rightarrow Y \), \( f(a) = f(b) \). Indeed, fix any space \( Y \) such that \( \psi(Y) \leq \kappa \) and let \( f : X \rightarrow Y \) be any continuous map. Observe that for any \( n \in \mathbb{Z} \) the subspace \( [T_n] = \{ x \mid x \in T_n \} \subset X \) is homeomorphic to \( T_n \). By Lemma [3] for each \( n \in \mathbb{Z} \) there exist \( y_n \in Y \) and ordinals \( \alpha_n \in \kappa^{++}, \beta_n \in \kappa^{++} \) such that
\[
\{ ( x, \kappa^{++} + n ) \mid x \in [\alpha_n, \kappa^{++}] \} \subset f^{-1}(y_n) \quad \text{and} \quad \{ ( \kappa^{++} + y, n ) \mid y \in [\beta_n, \kappa^{++}] \} \subset f^{-1}(y_n).
\]
Recall that \([x, \kappa^+, 2n] = [x, \kappa^+, 2n + 1]\) and \([\kappa^+, y, 2n] = [\kappa^+, y, 2n - 1]\) for any \(n \in \mathbb{Z}, x \in \kappa^+\) and \(y \in \kappa^+\), which implies that \(y_{2n - 1} = y_{2n} = y_{2n + 1}, n \in \mathbb{Z}\). Put \(\alpha = \sup\{\alpha_n \mid n \in \mathbb{Z}\}\) and \(\beta = \sup\{\beta_n \mid n \in \mathbb{Z}\}\). Hence there exists a unique \(y \in Y\) such that

\[
D = \{[(x, \kappa^+, n), \alpha, \kappa^+] \mid x \in \alpha, \kappa^+, n \in \mathbb{Z}\} \cup \{[(\kappa^+, y, n), \beta, \kappa^+] \mid y \in \beta, \kappa^+, n \in \mathbb{Z}\} \subset f^{-1}(y).
\]

Observe that \((a, b) \subset \mathcal{F} \subset f^{-1}(y)\). Hence \(f(a) = f(b)\).

Next consider the Herrlich extension \(H_{a,b}(X)\) of \(X\). Recall that for any space \(Y\) of pseudo-character \(\psi(Y) \leq \kappa\), each continuous function \(f : H_{a,b}(X) \to Y\) is constant. Since \(X\) is totally \(\pi\)-normal and \(\kappa\)-accumulative, Proposition 3 implies that \(H_{a,b}(X)\) is totally \(\pi\)-normal.

Finally, let \(R_\kappa\) be the Wallman \(\kappa\)-bounded extension \(W_\kappa(H_{a,b}(X))\) of \(H_{a,b}(X)\). Recall that \(H_{a,b}(X)\) is a dense subspace of \(R_\kappa\). Hence for any \(T\) space \(Y\) of pseudo-character \(\psi(Y) \leq \kappa\), each continuous function \(f : R_\kappa \to Y\) is constant on the dense subspace \(H_{a,b}(X) \subset R_\kappa\), witnessing that \(f\) is constant on the whole \(R_\kappa\). By Proposition 1 the space \(R_\kappa\) is \(\kappa\)-bounded. Proposition 3 implies that \(R_\kappa\) is regular.

The next theorem shows the main result never holds if the space \(Y\) is not \(T_1\).

**Theorem 4.** Let \(X\) be any space and \(Y\) be a space which is not \(T_1\). Then for any proper open set \(W \subset X\) there exists a continuous map \(f : X \to Y\) such that \(f(a) \neq f(b)\) for each points \(a \in W, b \in X \setminus W\).

**Proof.** Since the space \(Y\) is not \(T_1\) it contains points \(p_1\) and \(p_2\) such that \(p_2 \in \{p_1\}\). Fix any proper open subset \(W\) of \(X\). For each \(a \in W\) put \(f(a) = p_1\) and for each \(b \in X \setminus W\) put \(f(b) = p_2\). It is clear that the map \(f\) is continuous and \(f(a) \neq f(b)\) for any \(a \in W\) and \(b \in X \setminus W\).

**Corollary 1.** Let \(X\) be a non-anti-discrete space and \(Y\) be a space which is not \(T_1\). Then there exists a continuous non-constant map \(f : X \to Y\).

Theorem 3 and Theorem 4 provide the following analogue of Theorem 1.

**Theorem 5.** A space \(Y\) is \(T_1\) if and only if for any cardinal \(\kappa\) there exists an infinite regular \(\kappa\)-bounded space \(R_\kappa\) such that any continuous map \(f : R_\kappa \to Y\) is constant.

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