Research Article

Strong asymptotics of Jacobi-type kissing polynomials

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ABSTRACT

We investigate asymptotic behaviour of polynomials $p_{\omega}^\alpha(z)$ satisfying varying non-Hermitian orthogonality relations

$$\int_{-1}^{1} x^k p_{\omega}^\alpha(x) h(x) e^{i\omega x} dx = 0, \quad k \in \{0, \ldots, n-1\},$$

where $h(x) = h^*(x) (1-x)^\alpha (1+x)^\beta$, $\omega = \lambda n$, $\lambda \geq 0$ and $h(x)$ is holomorphic and non-vanishing in a certain neighbourhood in the plane. These polynomials are an extension of so-called kissing polynomials ($\alpha = \beta = 0$) introduced in Asheim et al. [A Gaussian quadrature rule for oscillatory integrals on a bounded interval. Preprint, 2012 Dec 6. arXiv:1212.1293] in connection with complex Gaussian quadrature rules with uniform good properties in $\omega$. The analysis carried out here is an extension of what was done in Celsus and Silva [Supercritical regime for the kissing polynomials. J Approx Theory. 2020 Mar 18;225:Article ID: 105408]; Deaño [Large degree asymptotics of orthogonal polynomials with respect to an oscillatory weight on a bounded interval. J Approx Theory. 2014 Oct 1;186:33–63], and depends heavily on those works.

1. Introduction

The purpose of this note is to extend the work done in connection with complex quadrature rules for oscillatory integrals

$$\int_{-1}^{1} f(x) e^{i\omega x} dx.$$

Evaluation of such integrals via the standard Gaussian quadratures can become extremely expensive numerically for large values of $\omega$, motivating the development of new quadrature rules. It was shown in [1] that using the zeros of polynomials $p_{\omega}^\alpha$ which satisfy

$$\int_{-1}^{1} x^k p_{\omega}^\alpha(x) h(x) e^{i\omega x} dx = 0, \quad k \in \{0, \ldots, n-1\}, \quad (1.1)$$

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where \( h(x) = 1 \) identically yields a quadrature rule with ‘good’ properties that naturally reduces to the usual quadrature rule when \( \omega \to 0 \). For more on this and different computational methods, see the monograph by Deaño et al. \[2\]. In this note, we will be interested in the asymptotic analysis of the polynomials \( p_n^\omega \) as \( n, \omega \to \infty \) simultaneously with the slightly more general weight

\[
h(x) = h^*(x)(1 - x)^\alpha (1 + x)^\beta, \quad \alpha, \beta > -1,
\]

and \( h^*(z) \) is holomorphic in a certain region of the plane and non-vanishing along a specific curve.

**Overview of the paper**

Three regimes, separated by the geometry of the zero-attracting curve associated with \( p_n^\omega \) (denoted \( \gamma_\lambda \)), are considered in this work. The main tool for the analysis carried out in all three regimes is the Riemann–Hilbert problem (RHP) for orthogonal polynomials and the Deift-Zhou nonlinear steepest descent method, where the initial RHP is transformed to a normalized RHP with the help of the so-called \( g \)-function, and a global parametrix and a set of local parametrices are constructed. The details of these constructions differ from one regime to the other, and depend on the geometry of \( \gamma_\lambda \). With this in mind, Section 2 serves as a quick reminder of results pertaining to the zero-attracting curve associated with \( p_n^\omega \) for all possible values of \( \lambda \in [0, \infty) \). In Section 3 asymptotic formulas for \( p_n^\omega (z) \) are stated for the subcritical, critical, and supercritical regimes for \( z \in \mathbb{C} \setminus \gamma_\lambda \). Similar formulas can be obtained for \( z \in \gamma_\lambda \), but such calculation is omitted for brevity. Proof of the formula for the supercritical case is provided in Section 4, and sketches of the proofs for the subcritical and critical regimes are provided in Sections 5 and 6, respectively.

This work should be viewed as an extension of the work in \[3,4\]. Some of the main differences include analysing polynomials \( p_n^\omega \) in the critical case, allowing for more general weights, including ones with algebraic singularities at the end-points \( z = 1, z = -1 \), and using a different construction of the global parametrix while analysing the supercritical regime than the one studied in \[3\] (compare leading term in (3.23) below with the one obtained in \[3, Theorem 2.4\]).

**2. Geometry**

Since the weight of orthogonality is complex-valued, it is known that the zeros of \( p_n^\omega \) may not accumulate onto the interval \([-1, 1]\). It turns out that \([-1, 1]\) is the zero-attracting curve in the case where the value \( \omega \) is fixed (see the appendix of \[4\]). When \( \omega \) is allowed to vary with \( n \) as \( \omega = \lambda n, \lambda \geq 0 \), the situation becomes more interesting as we enter the world of varying orthogonality. The work of Gonchar and Rakhmanov \[5\] suggested that one ought to consider a curve \( \gamma_\lambda \) to which \([-1, 1]\) is deformable and satisfies the S-property:

\[
\frac{\partial (U^{\mu_\lambda} + \text{Re}(V))}{\partial n^+}(z) = \frac{\partial (U^{\mu_\lambda} + \text{Re}(V))}{\partial n^-}(z) \quad \forall z \in \gamma_\lambda,
\]

where \( U^{\mu_\lambda}(z) := -\int \log |z - s| \, d\mu_\lambda(s) \) and \( \mu_\lambda \) is the equilibrium measure on \( \gamma_\lambda \) in the external field \( \text{Re}(V) \) (in our setting, \( V(z) = -i\lambda z \)). They further show that such curves are
formed by the trajectories of a quadratic differential $- Q_\lambda(z)(dz)^2$ where $Q_\lambda$ is given by

$$Q_\lambda(z) = \left( \int \frac{d\mu_\lambda(s)}{s - z} + \frac{V'(z)}{2} \right)^2 = \left( \int \frac{d\mu_\lambda(s)}{s - z} - \frac{\lambda}{2} \right)^2. \tag{2.1}$$

To obtain a formula for $Q_\lambda$, it is common to assume something about the support of $\mu_\lambda$ to be proven later on. This was done by Deaño, who showed the following: define

$$\varphi(z) := z + w(z), \quad w(z) = (z^2 - 1)^{1/2}, \quad z \in \mathbb{C} \setminus \gamma_\lambda, \quad w(z) = z + O(z) \text{ as } z \to \infty \tag{2.2}$$

and let $\lambda_{cr}$ be the unique solution of

$$2 \log \left( \frac{2 + \sqrt{\lambda_{cr}^2 + 4}}{\lambda_{cr}} \right) - \sqrt{\lambda_{cr}^2 + 4} = 0 \quad (\lambda_{cr} \approx 1.325 \cdots). \tag{2.3}$$

The following theorem appeared in [4]:

**Theorem 2.1**: Let $V(z) = - i \lambda z$ and $\lambda \in [0, \lambda_{cr})$. Then,

1. there exists a smooth curve $\gamma_\lambda$ connecting $z = 1$ and $z = -1$ that is a part of the level set $\text{Re}(\varphi(z)) = 0$ where

   $$\varphi(z) = 2 \log \varphi(z) + i \lambda w(z). \tag{2.4}$$

2. The measure $d\mu_\lambda(z) = -\frac{1}{2\pi i} \frac{2 + i\lambda z}{w(z)}$ $dz$ is the equilibrium measure on $\gamma_\lambda$ in the external field $\text{Re}(V(z))$.

3. $\gamma_\lambda$ has the S-property in the field $\text{Re}(V(z))$.

**Remark 2.1**: In fact, Deaño’s proof shows that for $\lambda = \lambda_{cr}$, $\gamma_\lambda$ is a union of two smooth curves that meet at $2i/\lambda_{cr}$.

**Remark 2.2**: Observe that with this theorem in mind, one can calculate the function $Q_\lambda(z)$ via Privalov’s lemma and (2.1) and find

$$Q_\lambda(z) = \frac{1}{4} \frac{(2 + i\lambda z)^2}{z^2 - 1}. \tag{2.5}$$

As for the supercritical case $\lambda \in (\lambda_{cr}, \infty)$, Celsus and Silva showed in [3] that (most of) the zeros of $p_n^{\omega}$ accumulate on two disconnected arcs, $\gamma_1, \gamma_2$ (which depend on $\lambda$) that appear as trajectories of the quadratic differential $- Q_\lambda(z; x_\ast)(dz)^2$ where

$$Q_\lambda(z; x) := \frac{\lambda^2 (z - z_\lambda(x)) (z + z_\lambda(x))}{4} \frac{z^2 - 1}{z^2 - 1}, \quad \text{and} \quad z_\lambda(x) = x + \frac{2i}{\lambda}, \tag{2.6}$$

and $x_\ast$ is some special value. More precisely, combining their work with Gonchar and Rakhmanov’s, we deduce that the counting measure associated with $p_n^{\omega}$ weakly converges to $\mu_\lambda$. The density of $\mu_\lambda$ is given in the following theorem, due to Celsus and Silva [3]:

**Theorem 2.2**: Let $\lambda > \lambda_{cr}$ and define $Q_\lambda(z) := Q_\lambda(z, x_\ast)$, where $x_\ast(\lambda) \in (0, 1)$ is the unique value for which $\text{Re} \int_{z_\lambda(x_\ast)}^1 Q_\lambda(s) \, ds = 0$ and $\lim_{\lambda \to \infty} x_\ast(\lambda) = 1$. Then, there exist
3. Statement of results

3.1. Asymptotics: one-cut case

Let \( \lambda_{cr} \) be as in (2.3). In the non-critical case \( \lambda < \lambda_{cr} \), the situation was described completely for \( h(x) = 1 \) identically in [4]. To extend this result to \( h(x) \) as in (1.2), we need the following Szegö function

\[
S_h(z) := \exp \left\{ \frac{w(z)}{2\pi i} \int_{\gamma_{\lambda}} \frac{\log[(w_+ h)(x)]}{z - x} \frac{dx}{w_+(x)} \right\}, \quad z \in \mathbb{C} \setminus \gamma_{\lambda},
\]

where \( w \) is as in (2.2) and \( h^*(z) \) is holomorphic in a neighbourhood containing the compact set delimited by \( \gamma_{\lambda} \cup [-1, 1] \) and non-vanishing on \( \gamma_{\lambda} \). Properties of \( S_h \) will be discussed in Section 5.

**Theorem 3.1 (Subcritical Case \( \lambda < \lambda_{cr} \))**: Let \( 0 \leq \lambda < \lambda_{cr} \) and \( h(z) \) be as above. Then for \( n \) large enough, polynomials \( p_n^\lambda \) have degree exactly \( n \) and locally uniformly for \( z \in \mathbb{C} \setminus \gamma_{\lambda} \)

\[
p_n^\lambda(z) = \left( \frac{\varphi(z)}{2} \right)^n \exp \left( -\frac{\ln \lambda}{2\varphi(z)} \right) \left( \frac{S_h(\infty)}{S_h(z)} + O \left( n^{-\frac{1}{2}} \right) \right) \quad \text{as} \quad n \to \infty.
\]
When $\lambda = \lambda_{cr}$, the geometry of $\gamma_\lambda$ changes. More precisely, $\gamma_\lambda$ is no longer an analytic arc, but rather a union of two analytic arcs, see [4]. However, by slightly changing the analysis, we may still write an asymptotic formula for $p_n^{\alpha}$.

**Theorem 3.2 (Critical Case $\lambda = \lambda_{cr}$):** Let $\lambda = \lambda_{cr}$ and $h(z)$ be as above. Then for $n$ large enough, polynomials $p_n^{\alpha}$ have degree exactly $n$ and locally uniformly for $z \in \mathbb{C} \setminus \gamma_\lambda$

\[
p_n^{\alpha}(z) = \left(\frac{\varphi(z)}{2}\right)^n \exp \left(\frac{-in\lambda}{2\varphi(z)} \left(\frac{S_h(\infty)}{S_h(z)} + O\left(n^{-1/2}\right)\right)\right) \text{ as } n \to \infty. \quad (3.3)
\]

We sketch a proof of Theorems 3.1, 3.2 in Sections 5, 6, respectively.

### 3.2. Asymptotics: two-cut case

To present the results when $\lambda > \lambda_{cr}$, we construct the main term of the asymptotics using the approach of [6] relying on Theta functions, instead of the meromorphic differential approach taken in [3, Theorem 2.4]. We introduce those here. Let $z_s = z_s(x_s)$ (see Theorem 2.2) and

\[
\gamma(z) := \left(\frac{z + z_s z - 1}{z - z_s z + 1}\right)^{1/4}, \quad z \in \mathbb{C} \setminus (\gamma_1 \cup \gamma_2), \quad (3.4)
\]

where $\gamma(z)$ is holomorphic off $\gamma_1 \cup \gamma_2$ and the branch is chosen so that $\gamma(\infty) = 1$. Further, set

\[
A(z) = \frac{\gamma(z) + \gamma^{-1}(z)}{2} \quad \text{and} \quad B(z) := \frac{\gamma(z) - \gamma^{-1}(z)}{-2i}. \quad (3.5)
\]

The functions $A(z)$ and $B(z)$ are holomorphic in $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$, $A(\infty) = 1$, $B(\infty) = 0$, and

\[
A_{\pm}(s) = \pm B_{\mp}(s), \quad s \in (\gamma_1 \cup \gamma_2) \setminus \{\pm 1, z_s, -\overline{z_s}\}. \quad (3.6)
\]

#### 3.2.1. Riemann surface

Let $\mathfrak{R}$ be the Riemann surface associated with the algebraic equation $\gamma^2 = Q_\lambda(z)$, with $Q_\lambda$ as in Theorem 2.2. This surface is realized as two copies of $\mathbb{C}$ cut along $\gamma_{1,2}$ and glued together in such a way that the right side of $\gamma_i$ on $\mathfrak{R}^{(0)}$, the first sheet, is connected with the left side of the same arc on the second sheet, $\mathfrak{R}^{(1)}$. Furthermore, $\pi : \mathfrak{R} \to \mathbb{C}$ be the natural projection. We will denote points on the surface with boldface symbols $z, t, s$ and their projections by regular script $z, s, t$ and $F^{(i)}(z)$, $i \in \{0, 1\}$, stands for the pull-back under $\pi(z)$ of a function $F(z)$ from $\mathfrak{R}^{(i)}$ into $\mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$. Note that for a fixed $z \in \mathbb{C} \setminus (\gamma_1 \cup \gamma_2)$, the set $\pi^{-1}(z)$ contains exactly two elements, one on each sheet, and we denote by $z^{(k)}$ the unique point satisfying $\pi^{-1}(z) \cap \mathfrak{R}^{(k)}$.

Denote by $\alpha$ a cycle on $\mathfrak{R}$ that passes through $\pi^{-1}(-\overline{z_s})$ and $\pi^{-1}(z_s)$ and whose natural projection is the arc $\hat{\gamma}$ that smoothly meets $\gamma_1, \gamma_2$ at $z_s, -\overline{z_s}$, belongs to the region delimited by infinite trajectories in Figure 1, and agrees with the orthogonal trajectory of $-Q(z)(dz)^2$ in a small neighbourhood of $z_s, -\overline{z_s}$. We assume that $\pi(\alpha) \cap (\gamma_1 \cup \gamma_2) = \{z_s, -\overline{z_s}\}$ and orient $\alpha$ towards $-\overline{z_s}$ within $\mathfrak{R}^{(0)}$. Similarly, we define $\beta$ to be a cycle on $\mathfrak{R}$ that passes
Furthermore, we have

\[ \alpha \] is normalized so that \( H(z) = 1 \), and under this normalization, Riemann showed that

\[
\text{Im}(B) > 0, \quad \text{where} \quad B := \oint_{\beta} H.
\] (3.10)

Given this normalized differential, we can define the Abel map \( A(z) := \int_{\pi^{-1}(1)} H \) where the path of integration is chosen to lie in \( \mathcal{A}_{\alpha, \beta} := \mathcal{A} \setminus \{\alpha, \beta\} \). This function is holomorphic on \( \mathcal{A}_{\alpha, \beta} \) that satisfies

\[
(A_+ - A_-)(z) = \begin{cases} 
1, & z \in \beta \setminus \pi^{-1}(-1), \\
-B, & z \in \alpha \setminus \pi^{-1}(-1).
\end{cases}
\] (3.11)

### 3.2.2. Szegő function

Let

\[
\tilde{S}_h(z^{(k)}) := \exp \left\{ \frac{1}{4\pi i} \oint_{\pi^{-1}(1)} \log(h) \Omega(z^{(k)}, x^{(1-k)}) \right\}
\] (3.12)

where \( w \) is as in (3.8) and \( \Omega(z^{(k)}, x^{(1-k)}) \) is the meromorphic differential on \( \mathcal{A} \) with simple pole at \( z^{(k)}, x^{(1-k)} \) with residues 1, -1, respectively and \( \oint_{\alpha} \Omega = 0 \).

**Proposition 3.3.** Let \( \tilde{S}_h \) be as above and \( h(z) = h^*(z)(1 - z)^\gamma(1 + z)^\beta \) where \( h^*(z) \) is holomorphic, non-vanishing in a neighbourhood of \( \gamma_1 \cup \gamma_2 \cdot \tilde{\gamma} \) and \( h(z) \) is holomorphic in a neighbourhood of each point of \( (\gamma_1 \cup \gamma_2) \setminus \{\pm 1, z_\ast, -\overline{z_\ast}\} \). Define

\[
c_h = c_h(\lambda) := \frac{1}{2\pi i} \oint_{\pi^{-1}(1)} \log(h) \mathcal{H}.
\] (3.13)

Then \( \tilde{S}_h \) is holomorphic and non-vanishing on \( \mathcal{A} \setminus \{\alpha, \pi^{-1}(1) \} \) and satisfies the relation \( \tilde{S}_h(z^{(k)}) \cdot \tilde{S}_h(z^{(1-k)}) = 1 \) identically. Furthermore, \( \tilde{S}_h \) possesses continuous traces on \( \alpha \cup \pi^{-1}(1) \setminus \{\pi^{-1}(\pm 1), \pi^{-1}(z_\ast), \pi^{-1}(-\overline{z_\ast})\} \) that satisfy

\[
\tilde{S}_{h_+}(s) = \tilde{S}_{h_-}(s) \begin{cases} 
e{e}^{2\pi i ch}, & s \in \alpha \setminus \{z_\ast, -\overline{z_\ast}\}, \\
1/h(s), & s \in \pi^{-1}(1) \setminus \{\pi^{-1}(\pm 1)\}.
\end{cases}
\] (3.14)

Furthermore, we have \( \tilde{S}_h(z^{(0)}) = |z - e|^{-\alpha_e^2/2}, \) \( e \in \{\pm 1, z_\ast, -\overline{z_\ast}\} \), where \( \alpha_e = 0 \) for \( e = z_\ast, -\overline{z_\ast}, \alpha_e = \alpha \) when \( e = 1 \) and \( \alpha_e = \beta \) when \( e = -1 \).

For a proof of this, see [6, Section 6.1].
Proposition 3.4: Let $z$ be given by (3.6) and choice of the branch choices in Definition of $\mathcal{A}$ and $\mathcal{B}$, and define $z_{n,k}$ by the equation
\[ \tilde{A}(z_{n,k}) = \tilde{A}(p^{(k)}) + c_h + n\left(\frac{1}{2} + B\tau\right) + j_{n,k} + m_{n,k}B, \quad j_{n,k}, m_{n,k} \in \mathbb{Z}, \tag{3.16} \]
where $p = i\text{Im}(z_\ast)/(1 - \text{Re}(z_\ast))$ and
\[ \tau := -\frac{1}{\pi i} \int_{\tilde{p}} Q_\gamma^{1/2}(s) \, ds. \tag{3.17} \]
Since $\mathcal{R}$ is of genus one, $\mathcal{A}$ is bijective and Equation (3.16) defines $z_{n,k}$ uniquely. In fact, by considering the branch choices in the definition of $\mathcal{A}$, $\mathcal{B}$, the following holds.

**Proposition 3.4:** Let $\tau$ be given by (3.17), $z_{n,k} = z_{n,k}(\lambda)$ as in (3.16), and $p$ as above. Then for any subsequence $\mathbb{N}_+$ the point $\infty^{(0)}$ is a topological limit point of $\{z_{n,1}\}_{n \in \mathbb{N}_+}$ if and only if $\infty^{(1)}$ is a topological limit point of $\{z_{n,0}\}_{n \in \mathbb{N}_+}$.

**Proof:** It follows from (3.6) and choice of the branch of $(\cdot)^{1/4}$ that $\gamma(p) = 1$ and
\[ \begin{cases} (B/A)(z), & z \in \mathcal{R}^{(0)}, \\ -(A/B)(z), & z \in \mathcal{R}^{(1)}, \end{cases} \tag{3.18} \]
is a rational function on $\mathcal{R}$ with two simple zeros $\infty^{(0)}$ and $p^{(0)}$ and two simple poles $\infty^{(1)}$ and $p^{(1)}$ (if it happens that $p \in (\gamma_1 \cup \gamma_2) \setminus \{\pm 1, z_\ast, -z_\ast\}$, then we choose $p^{(0)} \in \mathcal{R}$ precisely in such a way that it is a zero of (3.18) and $p^{(1)}$ so it is a pole of (3.18); it is, of course, still true that these points are distinct and $\pi(p^{(k)}) = p$). Therefore, Abel’s theorem yields that
\[ \int_{p^{(0)}}^{\infty^{(1)}} \mathcal{H} = \int_{p^{(1)}}^{\infty^{(0)}} \mathcal{H} \mod \mathbb{Z} + B\mathbb{Z}, \tag{3.19} \]
while the relations (3.16), in particular, imply that
\[ \int_{p^{(0)}}^{z_{n,0}} \mathcal{H} = \int_{p^{(1)}}^{z_{n,1}} \mathcal{H} \mod \mathbb{Z} + B\mathbb{Z}. \tag{3.20} \]
Let $z_k$ be a topological limit of a subsequence $\{z_{n,i}\}$. Holomorphy of the differential $\mathcal{H}$ implies that
\[ \int_{p^{(k)}}^{z_{n,k}} \mathcal{H} = \int_{p^{(k)}}^{z_{n,k}} \mathcal{H} \mod \mathbb{Z} + B\mathbb{Z}, \tag{3.21} \]
as $i \to \infty$, where the integral from $z_k$ to $z_{n,k}$ is taken along the path that projects into a segment joining $z_k$ and $z_{n,k}$. The desired claim now follows from (3.19), (3.20), and the unique solvability of the Jacobi inversion problem on $\mathcal{R}$.
Now, we define

$$\Theta_{n,k}(z) = \exp \left\{ -2\pi i (m_{n,k} + \tau n) A(z) \right\} \frac{\theta \left( A(z) - \tilde{A}(z_{n,k}) - \frac{B+1}{2} \right)}{\theta \left( A(z) - \tilde{A}(p^{(k)}) - \frac{B+1}{2} \right)}. \quad (3.21)$$

The functions $\Theta_{n,k}(z)$ are meromorphic on $\mathcal{R}_{\alpha,\beta}$ with exactly one pole, which is simple and located at $p^{(k)}$, and exactly one zero, which is also simple and located at $z_{n,k}$ (observe that the functions $\Theta_{n,k}(z)$ can be analytically continued as multiplicatively multivalued functions on the whole surface $\mathcal{R}$; thus, we can talk about simplicity of a pole or zero regardless whether it belongs to the cycles of a homology basis or not). Moreover, according to (3.11), (3.16), and periodicity properties of $\theta$, they possess continuous traces on $\alpha, \beta$ away from $\pi^{-1}(-1)$ that satisfy

$$\Theta_{n,k-}(s) = \Theta_{n,k+}(s) \left\{ \begin{array}{ll}
\exp \left\{ -\pi i (n + 2c_i) \right\}, & s \in \alpha \setminus \{\pi^{-1}(-1)\}, \\
\exp \left\{ -2\pi i \tau n \right\}, & s \in \beta \setminus \{\pi^{-1}(-1)\}.
\end{array} \right. \quad (3.22)$$

### 3.2.4. Subsequences $N(\lambda, \varepsilon)$

It will be important for our analysis (see Section 4.5) that $\Theta_{n,1}(z; \lambda)$, defined in (3.21), does not vanish near $\infty^{(0)}$. Hence, we will consider subsequences $N(\lambda, \varepsilon)$ which are defined as

$$N(\lambda, \varepsilon) := \left\{ n \in \mathbb{N} : z_{n,1} \notin \mathcal{R}^{(0)} \cap \pi^{-1}(\{|z| \geq 1/\varepsilon\}) \right\}.$$

Then there exists a constant $c(\lambda, \varepsilon) > 0$ such that $|\Theta_{n,1}^{(1)}(\lambda; t)| \geq c(\lambda, \varepsilon)$ for $n \in N(\lambda, \varepsilon)$. Note that $N(\lambda, \varepsilon)$ contains $n$ or $n-1$ for all $n \geq 1$. To prove this, suppose to the contrary that for any $\varepsilon > 0$, there exists $n_\varepsilon$ such that $n_\varepsilon$, $n_\varepsilon - 1 \notin N(\lambda, \varepsilon)$. By the very definition of $N(\lambda, \varepsilon)$, it then holds that $z_{n_\varepsilon-1,1}$, $z_{n_\varepsilon,1} \to \infty^{(0)}$ as $\varepsilon \to 0$. This implies $1/2 + B\tau = m + nB$ for some $m, n \in \mathbb{Z}$, which is false. We are ready to state the asymptotic formula for $p_{n}^{\omega}(z)$.

### Theorem 3.5 (Supercritical Case ($\lambda > \lambda_{cr}$))

Let $\lambda > \lambda_{cr}$, $V(z) = -i\lambda z$, $h(z)$ as in Proposition 3.3, and $\phi_1(z) = \int_{1}^{z} Q^{1/2}_{\lambda}(s) ds$. Then, there exists a constant $\ell^*$ (defined in (4.2)) so that

$$p_{n}^{\omega}(z) = e^{n(V(z) - \ell^* + \phi_1(z))} \left( A\Theta_{n,1}^{(0)}(z) + O(n^{-1}) \right) \quad \text{for } n \to \infty, \ n \in N(\lambda, \varepsilon)$$

locally uniformly for $z \in \mathbb{C} \setminus \gamma_{\lambda}$.

As was discussed in the introduction, both one- and two-cut cases require the same analysis in spirit. Hence, we will start with the proof of Theorem 3.5 in Section 4, and sketch the proofs Theorems 3.1, 3.2 in Sections 5, 6, respectively.
4. Proof of Theorem 3.5

4.1. $g$-function

Before we begin our analysis of polynomials $p_n^p$, we will require a collection of functions and their properties, which we list here for convenience. Let

$$g(z) := \int \log(z - s) \, d\mu_\lambda(s), \quad z \in \mathbb{C} \setminus (-\infty, -1) \cup \gamma_\lambda$$

(4.1)

where $\log(\cdot - s)$ is holomorphic outside of $(-\infty, -1] \cup \gamma_\lambda [-1, s)$, where $\gamma_\lambda(z_1, z_2)$, $z_1z_2 \in \gamma_\lambda$ is the segment of $\gamma_\lambda$ that proceeds from $z_1$ to $z_2$. Then it follows from (2.1) that there is $\ell^* \in \mathbb{C}$ so that

$$g(z) = \frac{V(z) - \ell^*}{2} + \phi_1(z) \quad \text{and} \quad \phi_e(z) := 2\int_z^\infty Q^{1/2}_\lambda(s) \, ds, \quad e \in \{\pm 1, z_*, -\bar{z}_*\},$$

(4.2)

where the domain of holomorphy for $\phi_e$ is $\mathbb{C} \setminus ((-\infty, -1) \cup \gamma_\lambda)$ for $e = 1$, $\mathbb{C} \setminus (\gamma_\lambda \cup [1, \infty))$ for $e = -1$, and $\mathbb{C} \setminus ((-\infty, -1) \cup \gamma_\lambda \cup (-1, -\bar{z}_*) \cup \gamma_\lambda (z_*, 1) \cup [1, \infty))$ for $e \in \{z_*, -\bar{z}_*\}$. From Figure 1, we immediately deduce that $\tau \in \mathbb{R}$ (see (3.17)) and

$$\phi_{1, \pm}(s) = \begin{cases} \pm 2\pi i \mu_\lambda(\gamma_\lambda[s, 1]), & s \in \gamma_2, \\ \pm 2\pi i \mu_\lambda(\gamma_\lambda[s, 1]) + 2\pi i \tau, & s \in \gamma_1. \end{cases}$$

(4.3)

Furthermore, using the fact that $\mu_\lambda$ is a probability measure and definition (3.17) yields

$$\phi_1(z) = \begin{cases} \phi_{z_+}(z) \pm \pi i, & z \in \mathbb{C} \setminus (-\infty, -1) \cup \gamma_\lambda \cup (1, \infty) \\ \phi_{\bar{z}_-}(z) \pm \pi i + 2\pi i \tau, & z \in \mathbb{C} \setminus (-\infty, -1) \cup \gamma_\lambda \cup (-1, \infty) \end{cases}$$

(4.4)

and $+$ (resp. $-$) is chosen when $z$ belongs to the left (resp. right) of $(-\infty, -1) \cup \gamma_\lambda \cup (1, \infty)$, oriented from $-\infty$ to $\infty$, and we use the fact that

$$\frac{1}{2} = -\frac{1}{\pi i} \int_{\gamma_1} Q^{1/2}_{\lambda, +}(s) \, ds$$

(4.5)

which follows from a residue calculation and the reflection symmetry of $\gamma_1, \gamma_2$, see [3, Proposition 3.5]. In fact, it follows from the general theory of quadratic differentials along with the choice of branch of $Q^{1/2}_\lambda$ that $\phi_1(z)$ is a conformal map from the domain in the complement of the critical graph of $Q_\lambda(z)(dz)^2$ containing $z = 0$ (see Figure 1) to the right half plane. Similarly, $\phi_{z_+}(z)$ maps the domain not containing $z = 0$ to the left half plane. For more on this, see [3, Section 4]. With this, (4.3), and (4.2) in mind, we can write

$$(g_+ - g_-)(s) = \begin{cases} 0, & s \in (1, \infty), \\ \pm \phi_{1, \pm}(s), & s \in \gamma_2, \\ \pi i, & s \in \gamma, \\ \pm (\phi_{1, \pm}(s) - 2\pi i \tau), & s \in \gamma_1, \\ 2\pi i, & s \in (-\infty, -1). \end{cases}$$

(4.6)
Furthermore,
\[
(g_+ + g_- - V + e^s)(s) = \begin{cases}
\phi_1(s), & s \in (1, \infty), \\
0, & s \in \gamma_2, \\
\phi_{z_0}(s), & s \in \gamma, \\
2\pi i\tau, & s \in \gamma_1, \\
\phi_{-1}(s) + 2\pi i\tau, & s \in (-\infty, -1).
\end{cases}
\] (4.7)

For \( e \in \{ \pm 1 \} \), \( \phi_e(z) \sim |z - e|^{1/2} \) as \( z \to e \). Hence, it follows from (4.3), (4.4) that \( (\phi_e(z))^2 \) is well-defined and conformal in a small enough neighbourhood of \( e \), which we will denote \( U_e \). Furthermore, it follows from (4.3) that \( (\phi_1(z))^2 \) maps \( \gamma_1 \cap U_1 \) into \( (-\infty, 0) \) and \( (\phi_{-1}(z))^2 \) does the same to \( \gamma_2 \cap U_{-1} \). In a similar vein, for \( e \in \{ z_\alpha, -\overline{z}_\alpha \} \), \( \phi_e(z) \sim |z - e|^{3/2} \) as \( z \to e \). In a small neighbourhood of \( z = e \), (4.4) allows us to write
\[
\phi_{z_0, \pm}(s) = \mp 2\pi i\mu_\lambda(\gamma_\lambda[z_\alpha, s]), \quad \phi_{-\overline{z}_\alpha, \pm}(s) = \pm 2\pi i\mu_\lambda(\gamma_\lambda[s, -\overline{z}_\alpha]).
\] (4.8)

Hence, an analytic branch of \( (-\phi_e)^{2/3} \) can be chosen and \( (-\phi_e)^{2/3} \) is conformal in a neighbourhood of \( z = e \). By the choice of \( \gamma_\lambda \) (see Figure 1 and the second paragraph of Section 3.2.1), both \( (\phi_e(z))^2 \), \( e \in \{ \pm 1 \} \) and \( (-\phi_e)^{2/3}, e \in \{ z_\alpha, -\overline{z}_\alpha \} \) map the segments of \( \gamma_1, \gamma_2 \) within \( U_e \) into \( (-\infty, 0) \).

### 4.2. Initial Riemann–Hilbert problem

We first deform \([-1, 1]\) to the curve \( \gamma_\lambda \) from Theorem 2.2. To arrive at asymptotics of \( p_n^\alpha(z) \), we will use the Riemann–Hilbert approach along with Deift-Zhou nonlinear steepest descent method. The connection between the RHP below and orthogonal polynomials was first observed in the work of Fokas et al. [7,8], while the nonlinear steepest descent method was developed by Deift and Zhou in [9]. More precisely, we seek a matrix \( Y \) that solves the following RHP (denoted RHP-Y)

(a) \( Y \) is analytic in \( \mathbb{C} \setminus \gamma_\lambda \), and \( \lim_{z \to \infty} Y(z)z^{-n\sigma_3} = I \), where \( \sigma_3 := \text{diag}(1, -1) \).

(b) \( Y \) has continuous traces as \( z \to \gamma_\lambda \setminus \{ \pm 1 \} \) and
\[
Y_+(s) = Y_-(s) \begin{pmatrix} 1 & w_n(s) \\ 0 & 1 \end{pmatrix}
\] for \( s \in \gamma_\lambda \setminus \{ \pm 1 \} \),

where \( \gamma_\lambda \) is oriented from \(-1\) to \(1\) and \( w_n(z) = h(z) e^{i\beta n z} \).

(c) As \( z \to 1 \), the first column of \( Y \) is bounded while the second behaves like \( \mathcal{O}(|z - 1|^\alpha) \), \( \mathcal{O}(\log |z - 1|) \), \( \mathcal{O}(1) \), for \( \alpha \in (0, 1) \), \( \alpha = 0 \), \( \alpha > 0 \), respectively. Similar behaviour holds as \( z \to -1 \) (replace \( \alpha \to \beta \) and \( 1 \to -1 \)).

It was observed in [7,8] that under the assumption that
\[
\deg p_n^\alpha = n \quad \text{and} \quad \mathcal{C}(p_n^\alpha w_n)(z) \sim z^{-(n+1)} \quad \text{as} \quad z \to \infty,
\] (4.9)

where \( (\mathcal{C}f)(z) = \frac{1}{2\pi i} \int_{\gamma_\lambda} [f(s)/(s-z)]ds \), this problem is solved by the matrix
\[
Y(z) := \begin{pmatrix} p_n^\alpha(z) & \mathcal{C}(p_n^\alpha w_n)(z) \\ -2\pi i k_{n-1}^2 p_n^\alpha(z) & -2\pi i k_{n-1}^2 \mathcal{C}(p_n^\alpha w_n)(z) \end{pmatrix},
\] (4.10)
where \( \kappa_n \) is the leading of the orthonormal polynomials associated with \( w_n(z) \), so that \( \kappa_n^{-1}C(p_n^{\omega}w_n)(z) = z^{-n}[1 + o(1)] \) as \( \rightarrow \infty \). Moreover, any solution of RHP-\( Y \) must take the form in (4.10) (see, for example, [6]).

### 4.3. First transformation

Let \( T(z) := e^{nt^*\sigma_3}Y(z) e^{-n(g(z) + t^*/2)\sigma_3} \). Then, using the properties discussed in Section 4.1, \( T \) satisfies the following RHP, denoted RHP-to  

(a) \( T(z) \) is holomorphic in \( \mathbb{C} \setminus \gamma_\lambda \) and \( \lim_{z \to \infty} T = I \),

(b) \( T(z) \) has continuous traces on \( \gamma_\lambda \setminus \{ \pm 1, z_*, -\bar{z}_* \} \) that satisfy

\[
T_+(s) = T_-(s) \begin{pmatrix}
1 & 0 \\
\mp e^{-n\phi_1(s)/h(s)} & 1
\end{pmatrix}, \quad s \in \gamma_1,
\]

\[
\begin{pmatrix}
e^{-n\phi_1(s)} & h(s) e^{2n\pi i} \\
0 & e^{-n\phi_1(s)}
\end{pmatrix}, \quad s \in \hat{\gamma},
\]

\[
\begin{pmatrix}
e^{-n\phi_1(s)} & h(s) e^{-2n\pi i} \\
0 & e^{-n\phi_1(s)}
\end{pmatrix}, \quad s \in \gamma_2,
\]

(c) \( T \) behaves the same as \( Y \) as \( z \to \pm 1 \).

### 4.4. Opening the lenses

Let \( \gamma_{i,\pm} \) be as in Figure 2, and let \( \Gamma_{i,\pm} \) denote the open sets delimited by \( \gamma_{i,\pm} \) and \( \gamma_i \). Set

\[
X(z) := T(z) \begin{pmatrix} 1 & 0 \\ \mp e^{-n\phi_1(z)/h(z)} & 1 \end{pmatrix}, \quad z \in \Gamma_{i,\pm},
\]

\[
I, \quad \text{otherwise.}
\]

Then \( X \) solves the following RHP (RHP-\( X \))

(a) \( X \) is analytic in \( \mathbb{C} \setminus (\gamma_\lambda \cup \gamma_{i,\pm}) \), \( \lim_{z \to \infty} X = I \),

(b) \( X \) has continuous traces on \( \gamma_\lambda \setminus \{ \pm 1, -\bar{z}_*, z_* \} \) that satisfy RHP-to \( \hat{\gamma} \), as well as

\[
X_+(s) = X_-(s) \begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix}, \quad s \in \gamma_j, j = 1, 2,
\]

\[
\begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix}, \quad s \in \gamma_{i,\pm}, i = 1, 2,
\]

(c) as \( z \to 1 \) from outside [inside],
Figure 2. Opening the lenses in the supercritical regime for kissing polynomials.

\[ X(z) = \begin{cases} 
\mathcal{O} \left( \frac{|z - 1|^\alpha}{1} \right), & -1 < \alpha < 0 \\
\mathcal{O} \left( \frac{\log |z - 1|}{1} \right), & \alpha = 0 \\
\mathcal{O} \left( \frac{1}{1} \right), & \alpha > 0 
\end{cases} \]

with similar behaviour for \( z \to -1 \) where \( \beta \) replaces \( \alpha \).

### 4.5. Global parametrix

From the discussion before (4.6), we see that the jumps on \( \gamma_{i,\pm} \) and the off diagonal entry in the jump on \( \hat{\gamma} \) are exponentially small. Removing these quantities, we now seek a matrix \( N \) satisfying the following RHP (RHP-N)

(a) \( N \) is analytic off of \( \gamma_{j} \), satisfying \( \lim_{z \to \infty} N = I \)
(b) \( N \) possesses continuous traces on \( \gamma_{j} \setminus \{ \pm 1, -\bar{u}, z_{*} \} \) that satisfy

\[
N_{\pm}(s) = N_{-}(s) \begin{cases}
e^{(2j-1)2\pi i \sigma_{3}} \begin{pmatrix} 0 & h(s) \\ -1/h(s) & 0 \end{pmatrix}, & s \in \gamma_{j}, j = 1, 2, \\
e^{n\pi i \sigma_{3}}, & s \in \hat{\gamma}.
\end{cases}
\]

Since the weight of orthogonality in (1.1) is complex-valued, \( \deg p_{n}^{\alpha} \leq n \) in general. This can be seen as the underlying reason for considering asymptotics along subsequences. Hence, we shall solve RHP-N only for \( n \in \mathbb{N}(\lambda, \epsilon) \) from Section 3.2.3. To this end, let

\[
M_{n,0}(z) = \Theta_{n,0}(z) \begin{cases}
B(z), & z \in \mathcal{R}^{(0)} (\alpha), \\
A(z), & z \in \mathcal{R}^{(1)} (\alpha),
\end{cases} \quad \text{and} \quad M_{n,1}(z) = \Theta_{n,1}(z) \begin{cases}
A(z), & z \in \mathcal{R}^{(0)} (\beta), \\
-B(z), & z \in \mathcal{R}^{(1)} (\beta),
\end{cases}
\]

where functions \( A(z), B(z) \) are defined in (3.5). These functions are holomorphic on \( \mathcal{R} \setminus \{ \alpha \cup \beta \cup \pi^{-1}(\gamma_{j}) \} \) since the pole of \( \Theta_{n,k}(z) \) is cancelled by the zero of \( B(z) \). Each function
has exactly two zeros, namely, \( z_{n,k} \) and \( \infty^{(k)} \). It follows from (3.6) and (3.22) that

\[
\begin{align*}
M_{n,k}^{(0)}(s) &= \mp M_{n,k+}^{(1)}(s), \\
M_{n,k}^{(0)}(s) &= \mp e^{-2\pi i n} M_{n,k+}^{(1)}(s), \\
M_{n,k}^{(i)}(s) &= e^{(-1)^i \pi i(n+2\epsilon_k)} M_{n,k+}^{(i)}(s), \quad s \in \gamma_1, \\
M_{n,k}^{(i)}(s) &= M_{n,k-}^{(0)}(s), \quad s \in \gamma_2.
\end{align*}
\]

Then, with \( \tilde{S}_h \) as defined by (3.12), a solution of RHP-\( N \) is given by

\[
N(z) = M^{-1}(\infty)M(z), \quad M(z) := \begin{pmatrix} M_{n,1}^{(0)}(z) & M_{n,1}^{(1)}(z) \\ M_{n,0}^{(0)}(z) & M_{n,0}^{(1)}(z) \end{pmatrix} \tilde{S}_h^{\sigma_3}(z^{(0)}).
\]

Indeed, RHP-\( N(\alpha) \) follows from holomorphy of \( \tilde{S}_h(z) \) and \( M_{n,k}(z) \) discussed in Proposition 3.3 and right after (4.12). RHP-\( N(\beta) \) can be checked by using (3.14) and (4.13). It will be important for our analysis that \( N \) is invertible, which it is. Indeed, since the jump matrices for \( N \) all have determinant 1 and \( \lim_{z \to \infty} N(z) = I \), the function \( \det(N(z)) \) is holomorphic in \( \mathbb{C} \setminus \{ \pm 1, -\overline{\epsilon}, \epsilon \} \), with at most square root singularities there, and hence is a constant. The normalization at infinity yields \( \det(N(z)) = 1 \) identically.

### 4.6. Local parameters

Let \( U_e, \ e \in \{ \pm 1 \} \) be an open disk centred at \( e \) with fixed radius \( \delta \) small enough so that it is in the domain of holomorphy of \( h^*(z) \). We seek a matrix \( P_e \), that solves the following RHP-\( P_e \):

(a) \( P_e \) satisfies RHP-\( X(a, b, c) \) within \( U_e \),

(b) \( P_e(z) = N(z)(I + \mathcal{O}(n^{-1})) \) uniformly on \( \partial U_e \) as \( n \to \infty \).

Denote \( \Psi_{-1}(\xi) := \sigma_3 \Psi_\alpha(\xi) \sigma_3, \Psi_1(\xi) := \Psi_\beta(\xi) \), where \( \Psi_\alpha \) is as in [10, Equations (6.23)–(6.25)]. Furthermore, \( \Psi_e := \sigma_3 A \sigma_3 \) for \( e = z_\epsilon, \Psi_e = A \) for \( e = -\overline{\epsilon} \) and \( A \) is the Airy matrix that appears in [11, Section 7.6]. Define

\[
J_e = \begin{cases}
I, & e = 1, \\
\begin{cases}
I, & e = -1, \\
\begin{cases}
e^{\pm \pi i n \sigma_3/2}, & e = z_\epsilon, \\
e^{\pi i (-\pm 1/2) n \sigma_3}, & e = -\overline{\epsilon}.
\end{cases}
\end{cases}
\end{cases}
\]

where the ‘+’ is used for \( z \) to the left of \( (-\infty, -1) \cup \gamma_\lambda \cup (1, \infty) \) and the ‘−’ sign is used otherwise. Next, let \( r_1(z) = \sqrt{h^*(z)}(z + 1)^{\beta}(z - 1)^{\alpha/2}, \ z \in U \setminus \gamma_\lambda \) and \((z - 1)^{-\alpha/2} \) is principal, with \( r_{-1} \) is defined similarly, and \( r_e = \sqrt{h(z)} \) be a holomorphic branch in \( U_e \) for \( e \in \{ z_\epsilon, -\overline{\epsilon} \} \). Finally, let

\[
\zeta_e(z) := \left( \frac{1}{4} \phi_e(z) \right)^2, \quad e \in \{ \pm 1 \}, \\
\zeta_e(z) := \left( -\frac{3}{4} \phi_e(z) \right)^{2/3}, \quad e \in \{ z_\epsilon, -\overline{\epsilon} \},
\]

where \( \phi_e \) is defined in (4.2) and the branches are chosen as in Section 4.1. We now require that \( \gamma_{e,\pm} \) be preimages of \( I_\pm := \{ z : \arg(z) = \pm 2\pi/3 \} \).
It now follows by the definition of $J$, $\Psi_1$, $r_\epsilon$ and (4.2), (4.2), (4.15), and (4.4) that

$$P_\epsilon(z) = E_\epsilon(z) \Psi_1(e^{n^2 \zeta_\epsilon(z)}) r_\epsilon^{-\sigma_3} e^{-n\phi_\epsilon(z)\sigma_3/2} J_\epsilon$$

(4.17)
satisfies RHP-$P_\epsilon(a, b)$. The choice of $E_\epsilon$ to ensure RHP-$P_\epsilon(c)$ holds is made below. To satisfy the matching condition RHP-$P_\epsilon(c)$, we simply need to choose

$$E_\epsilon(z) := N(z) J_\epsilon^{-1} r_\epsilon^{\sigma_3}(z) S_\epsilon^{-1}(n^2 \zeta_\epsilon(z)),$$

(4.18)

where $S_\epsilon = \sigma_3 S \sigma_3$ for $e = -1$ and $S_\epsilon = S$ for $e = 1$, and $S(\zeta) := \frac{\zeta}{\sqrt{2}} \left( \frac{1}{i} \right)$ and we take the principal branch of $\zeta^{1/4}$. Holomorphy in $U_\epsilon \{ e \}$ follows from RHP-$N(b)$, definition of $S$, while the behaviour of $N$ near $e \in \{ \pm 1 \}$, the behaviour of $r_\epsilon$ near $e$, and the fact that $\zeta_\epsilon(z)$ possesses a simple zero at $e$ yield holomorphy in $U_\epsilon$.

4.7. Final Riemann–Hilbert problem

We now define

$$R(z) := X(z) \begin{cases} N^{-1}(z), & z \in \mathbb{C} \setminus \left( \cup_{\gamma_\lambda} U_\epsilon \cup \gamma_\lambda \cup \gamma_{i,\pm} \right), \\ P^{-1}_\epsilon(z), & z \in U_\epsilon \setminus \left( \gamma_\lambda \cup \gamma_{i,\pm} \right), \end{cases}$$

(4.19)

where $\partial U_\epsilon$ are oriented clockwise. Then, $R(z)$ is analytic in $\mathbb{C} \setminus (\gamma_{i,\pm} \cup (\cup_e \partial U_\epsilon))$ and

$$R_+(s) = R_-(s) \begin{cases} I + \mathcal{O}(e^{-cn}) & \text{for } s \in (\gamma_\lambda \cup \gamma_{i,\pm}) \setminus U_\epsilon, \\ I + \mathcal{O}(n^{-1}) & \text{for } s \in \cup_e \partial U_\epsilon. \end{cases}$$

(4.20)

The first equality follows from the discussion before (4.6), while the second equality holds by boundedness of $N$ with $n$ and construction of $P_\epsilon$, see RHP-$P_\epsilon(c)$. It now follows from [11, Corollary 7.108] that

$$R(z) = I + \mathcal{O}(n^{-1}) \quad \text{as } n \to \infty,$$

(4.21)

uniformly for $z \in \mathbb{C} \setminus (\gamma_{i,\pm} \cup (\cup_e \partial U_\epsilon))$. The asymptotic formula of $p_n^{\omega}(z)$ outside the lenses and away from endpoints follows by undoing the above transformations as was done in [4].

5. Sketch of proof of Theorem 3.1

The starting point for this analysis is the same initial problem RHP-$Y$, with $\gamma_\lambda$ as in Theorem 2.1. We highlight only the main steps here:

(a) Using the same $g$-function as in [4] and $\phi$ as in Theorem 2.1, we make the transformation $T(z) = 2^{n\sigma_3} Y(z) e^{-n[g(z) + \log 2]\sigma_3}$. The main difference to highlight is that the jump of $T$ are slightly different:

$$T_+(s) = T_-(s) \begin{pmatrix} e^{-n\phi_+(s)} & h(s) \\ 0 & e^{n\phi_+(s)} \end{pmatrix} \quad \text{for } s \in \gamma_\lambda \setminus \{ \pm 1 \}.$$
(b) We ‘open the lenses’ in a similar fashion as well

\[
X(z) = \begin{cases} 
T(z) & \text{z outside the lens,} \\
T(z) \begin{pmatrix} 1 & 0 \\
-e^{-n\phi(z)/h(z)} & 1 \end{pmatrix} & \text{z on the upper lens,} \\
T(z) \begin{pmatrix} 1 & 0 \\
e^{-n\phi(z)/h(z)} & 1 \end{pmatrix} & \text{z on the lower lens,}
\end{cases}
\tag{5.1}
\]

where the ‘upper’ and ‘lower’ lips refer to Figure 3

(c) To account for \(h(z)\) in the weight of orthogonality, we define a different Szegő function, which is given in (3.1). Observe that \(S_h\) is analytic and non-vanishing in \(\mathbb{C} \setminus \gamma_\lambda\) and satisfies

\[
S_{h,+}(s)S_{h,-}(s) = (w+h)(s) \quad \text{for } s \in \gamma_\lambda \setminus \{\pm 1\}.
\tag{5.2}
\]

Using this, we construct the global parametrix, \(N\) (here, \(w, \varphi\) are as in (2.2))

\[
N(z) := (S_h(\infty))^{\sigma_3} \begin{pmatrix} 1 & 1/w(z) \\
1/2\varphi(z) & \varphi(z)/2w(z) \end{pmatrix} S_h^{-\sigma_3}(z),
\tag{5.3}
\]

(d) The local parametrices needed near \(z = \pm 1\) are as in [10] to allow for a general \(\alpha, \beta\) in the weight \(h(z)\). Similar local analysis was done in Section 4.6

(e) The final RHP is defined in a similar fashion to what was done in Section 4.7

6. Sketch of proof of Theorem 3.2

In the case \(\lambda = \lambda_{cr}\) curve \(\gamma_\lambda\) seizes to be smooth, and we must modify the lenses as shown in Figure 4. In this setting, we will define matrices \(T, X, N, R\) in the same way as was done in the sub-critical case. However, we will need to perform some local analysis at the midpoint of \(\gamma_\lambda\), which lies at \(2i/\lambda_{cr}\).
6.1. **Local parametrix around** $2i/\lambda_{cr}$

Let $U_c$ be a disk centred at $z^* = 2i/\lambda_{cr}$ small enough so that $h(z)$ (see the second line of Section 4) is holomorphic in $\overline{U}_c$, and let $\phi$ be defined as in Theorem 2.1. We seek a matrix $P_c(z)$ to solve the following RHP (RHP-$P_c$):

1. $P_c(z)$ satisfies RHP-X(a, b) within $U_c$.
2. $P_c(z)$ is bounded as $z \to 2i/\lambda_{cr}$ and $N^{-1}(z)P_c(z) = I + \mathcal{O}(n^{-1/2})$ uniformly for $z \in \partial U_c$.

We will need a new conformal map near the point $2i/\lambda_{cr}$. To this end, let $\phi_c(z) = \pm \phi(z)$, $z \in U_{c,\pm}$, where $U_{c,+}$ (resp., $U_{c,-}$) is the component of $U_c$ to the left (resp., right) of $\gamma_\lambda$. Then, $\phi_c$ is holomorphic in $U_c$ and since $z^*$ is a simple zero of $Q_{\lambda_{cr}}^{1/2}$, we have that $|\phi_c(z) - \phi_c(z^*)| \sim |z - z^*|^2$ as $z \to z^*$. Furthermore, by Theorem 2.1, we have that $\phi_{\pm}(s) = \pm 2\pi i \mu_\lambda([s,1])$ for $s \in \gamma_\lambda$, and we can see that $\phi_c(z)$ is purely imaginary and positive on $\gamma_\lambda(-1, z^*)$ and negative purely imaginary on $\gamma_\lambda(z^*, 1)$. With this in mind, we can define a branch of $(\phi_c(z) - \phi_c(z^*))^{1/2}$ that is holomorphic and, WLOG (up to restricting $U_c$ to a smaller neighbourhood) conformal in $U_c$ and maps $\gamma_\lambda(-1, z^*) \cap U_c$ to $\{ z \mid \arg(z) = \pi/4 \}$, $\gamma_\lambda(z^*, 1) \cap U_c$ to $\{ z \mid \arg(z) = 3\pi/4 \}$. Using this branch, the map $\zeta(z) := -(\phi_c(z) - \phi_c(z^*))^{1/2}$ is conformal, maps $\gamma_\lambda(-1, z^*) \cap U_c$ into $\{ z \mid \arg(z) = 5\pi/4 \}$ and $\Gamma_+$ into $\mathbb{R}$.

Since $h(z)$ is holomorphic and non-vanishing in $U_c$, we can define a holomorphic branch of $r(z) := \sqrt{h(z)}$. Furthermore, let

$$J(z) := \begin{cases} 
\begin{pmatrix} 0 & -1 \\
1 & 0
\end{pmatrix}, & z \in U_{c,+}, \\
I, & z \in U_{c,-}.
\end{cases} \quad (6.1)$$

**Figure 4.** Curves $\Gamma_{\pm}$ and $\gamma_\lambda$. 

Finally, let $C$ be the matrix given in [12, Section 7.5.3] explicitly in terms of exponentials and erf(z). $C$ is holomorphic in $\mathbb{C} \setminus \mathbb{R}$, satisfies the jump relation

$$C_+(s) = C_-(s) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

and has the asymptotic expansion $C(\zeta) \sim (I + \sum_{k=0}^{\infty} \begin{pmatrix} 0 & b_k \\ 0 & 0 \end{pmatrix} \zeta^{-(2k+1)}) e^{-\zeta^2 \sigma_3}$. Let

$$P_c(z) := E_c(z) C \left( \sqrt{\frac{n}{2}} \cdot \zeta_c(z) \right) J^{-1}(z) r^{-\sigma_3} e^{-n \phi(z) \sigma_3 / 2}, \quad E_c(z) := N(z) r^{\sigma_3} (z) J(z).$$

Then, $P_c$ satisfies RHP-$P_c(a, b)$ for any $E_c(z)$ holomorphic in $U_c$. Furthermore, by the very definition of $C, J, r$, it follows that $P_c$ is bounded as $z \to z^*$. Since the matrices involved in its definition are holomorphic in $U_c$, $E_c(z)$ is holomorphic in $U_c$. RHP-$P_c(d)$ follows from the behaviour of $C(\zeta)$ as $\zeta \to \infty$ [12, Equation (7.19)], that $\phi_c(z^*) \in i\mathbb{R}$, and the relation

$$e^{-n \phi(z) \sigma_3 / 2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e^{n \phi(z) \sigma_3 / 2}.$$

Acknowledgments

The author is grateful to Maxim Yattselev for his guidance and the many useful discussions, suggestions, and comments. The author would also like to thank Alfredo Deaño and Guilherme Silva for their help, support and encouragement.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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