On the Power of Quantum Algorithms for Vector Valued Mean Computation

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Abstract

We study computation of the mean of sequences with values in finite dimensional normed spaces and compare the computational power of classical randomized with that of quantum algorithms for this problem. It turns out that in contrast to the known superiority of quantum algorithms in the scalar case, in high dimensional $L^p_M$ spaces classical randomized algorithms are essentially as powerful as quantum algorithms.

1 Introduction

The quantum model of computation is a theoretical tool to assess the potential computing power of quantum mechanics. Quantum algorithms have a random component – the measurement process. Thus, there is familiarity to classical (that is, non-quantum) randomized algorithms. On the other hand, they also exploit quantum parallelism due to superpositions. (Details and further references can be found, e.g., in [4].) It is a challenging question to compare the computing power of the quantum with the classical randomized model. The present paper is devoted to a question of such type.

Mean computation of uniformly bounded scalar sequences is by now well understood in both settings. In the scalar case quantum computation yields an improvement of the convergence rate by a factor of order $n^{-1/2}$ (where $n$ is the number of function values used) over the randomized setting. Here we study the vector-valued case. Our results show that for mean computation
in $L^M_p$ spaces of sufficiently high dimension, the speedup of quantum over the randomized setting vanishes. In particular, there is no vector analogue of the quantum summation algorithm of $[1]$ of comparable efficiency.

Our approach also implies, that the many sums problem, which arose in $[13]$, cannot be solved as quickly on a quantum computer as the one sum problem. Or in other words, while classical algorithms can “re-use” function values for computing further components, quantum algorithms fail that property, in general. Details will be discussed in Section $4$.

2 Notation

For $1 \leq p \leq \infty$, $N \in \mathbb{N}$, and a normed space $X$, let

$$L^N_p(X) = \{ f \mid f : \{0, \ldots, N - 1\} \to X \},$$

equipped with the norm

$$\| f \|_{L^N_p(X)} = \left( \frac{1}{N} \sum_{i=0}^{N-1} \| f(i) \|_X^p \right)^{1/p}$$

if $1 \leq p < \infty$, and

$$\| f \|_{L_\infty^N(X)} = \max_{0 \leq i < N} \| f(i) \|_X.$$

The unit ball is denoted as $B^N_p(X) = \{ f \in L^N_p(X) : \| f \| \leq 1 \}$. Furthermore, we write $L^N_p$ for $L^N_p(\mathbb{R})$ and $B^N_p$ for $B^N_p(\mathbb{R})$.

We are concerned with the following problem: Given $f \in B^N_\infty(X)$, accessible through function values (at request $i$ a subroutine returns $f(i) \in X$), compute (approximately)

$$S_N f = \frac{1}{N} \sum_{i=0}^{N-1} f(i) \in X$$

for $f \in B^N_\infty(X)$.

We consider the following error settings, which we present in a general context. Let $D$ and $K$ be nonempty sets and $F$ a nonempty set of $K$-valued functions on $D$. Let $G$ be a normed space and let $S : F \to G$ be a mapping. (In our case $D = \{0, \ldots, N - 1\}$, $K = X$, $F = B^N_\infty(X)$, $G = X$, and $S = S_N$.) Let $n \in \mathbb{N}_0$.

In the classical randomized (also called Monte Carlo) setting we define

$$e^\text{ran}_n(S, F) = \inf_{A_n} \sup_{f \in F} \mathbb{E} \| S(f) - A^*_n(f) \|.$$  \hspace{1cm} (1)
where the infimum is taken over all $A_n = (A^\omega_n)_{\omega \in \Omega}$ of the form

$$A^\omega_n(f) = \varphi^\omega(f(t^\omega_1), \ldots, f(t^\omega_n)).$$

(2)

Here $(\Omega, \Sigma, \mathbb{P})$ is a probability space with $\Omega$ a finite set. For each $\omega \in \Omega$, $t^\omega_i \in D$ $(1 \leq i \leq n)$ and $\varphi^\omega$ is an arbitrary mapping from $K^n$ to $G$. (In the case $n = 0$, the mappings $\varphi^\omega$ are interpreted as not depending on values of $f$, that is, as an element of $G$ depending only on $\omega \in \Omega$.) Consequently, $e_{n,\text{ran}}^\text{ran}(S,F)$ is the best possible error of randomized classical (nonadaptive) algorithms using $\leq n$ (randomized) function values. (Algorithms with $|\Omega| < \infty$ are usually called restricted randomized algorithms \cite{12}. For us this setting is technically convenient, and it entails no loss of generality since we use it only for upper bounds. Comments on lower bounds for the stronger model with arbitrary $\Omega$ are given in Section 4.)

Next let us briefly recall the quantum setting, details and references for further reading can be found in \cite{3}. The notation $\mathbb{Z}[0,M] := \{0, \ldots, M-1\}$ will be convenient. Let $H_1 = \mathbb{C}^2$ be the 2-dimensional complex Hilbert space and let $|0\rangle$ and $|1\rangle$ be the unit vectors in $H_1$. For $m \in \mathbb{N}$ consider the $2^m$-dimensional tensor product space

$$H_m = H_1 \otimes \cdots \otimes H_1.$$

An orthonormal basis is given by the vectors

$$|\ell\rangle = |i_1 \rangle |i_2 \rangle \cdots |i_m \rangle, \quad \ell \in \mathbb{Z}[0,2^m],$$

with $\ell = \sum_{j=1}^{m} i_j 2^m - j$ the binary representation. A quantum query on $F$ is given by a tuple

$$Q = (m, m', m'', Z, \tau, \beta),$$

where $m, m', m'' \in \mathbb{N}$, $m' + m'' \leq m$, $Z$ is a nonempty subset of $\mathbb{Z}[0,2^{m'})$, and

$$\tau : Z \to D$$

and

$$\beta : K \to \mathbb{Z}[0,2^{m''})$$

are any mappings. With a quantum query $Q$ and an input $f \in F$ we associate a unitary operator $Q_f$ on $H_m$ which is defined for $i \in \mathbb{Z}[0,2^m)$, $x \in \mathbb{Z}[0,2^{m'})$, $y \in \mathbb{Z}[0,2^{m-m'-m''})$ on the basis state

$$|i \rangle |x \rangle |y \rangle \in H_m = H_{m'} \otimes H_{m''} \otimes H_{m-m'-m''}$$

3
as
\[
Q_f |i\rangle |x\rangle |y\rangle = \begin{cases} |i\rangle |x \oplus \beta(f(\tau(i)))\rangle |y\rangle & \text{if } i \in \mathbb{Z} \\ |i\rangle |x\rangle |y\rangle & \text{otherwise.} \end{cases}
\]

Here $\oplus$ denotes addition modulo $2^m$. A quantum algorithm with $n$ quantum queries is a tuple
\[
A_n = (Q, (U_i)_{i=0}^n, b, \varphi),
\]
where $Q$ is a quantum query as defined above, $U_i$ are any unitary operators on $H_m$, $b \in \mathbb{Z}[0, 2^m)$, and
\[
\varphi : \mathbb{Z}[0, 2^m) \to G
\]
is an arbitrary mapping. At input $f \in F$ the algorithm produces the state
\[
|\psi\rangle = U_n Q_f U_{n-1} \ldots U_1 Q_f U_0 |b\rangle .
\]
(The symbol $|\psi\rangle$ means any element of the unit sphere of $H_m$, while symbols like $|i\rangle, |x\rangle, |l\rangle$ with $i, x, l$ integers stand for basis states.) Let
\[
|\psi\rangle = \sum_{\ell=0}^{2^m-1} \alpha_{\ell, f} |\ell\rangle .
\]
The state $|\psi\rangle$ is measured, which means that we obtain a random variable $\xi_f(\omega)$ with values in $\{0, \ldots, 2^m - 1\}$, which takes the value $\ell$ with probability $|\alpha_{\ell, f}|^2$. Finally the mapping $\varphi$ is applied, so the output of the algorithm is
\[
A_n(f, \omega) = \varphi(\xi_f(\omega)).
\]
What we described here is a quantum algorithm with one measurement. Such algorithms turn out to be essentially equivalent to algorithms with several measurements (see [3], Lemma 1). Now we define
\[
e_n^3(S, F) = \inf_{A_n} \sup_{f \in F} \inf \{ \varepsilon : \mathbb{P}\{|S(f) - A_n(f, \omega)| \leq \varepsilon\} \geq 3/4\} .
\]
The first infimum is taken over all quantum algorithms $A_n$ which use not more than $n$ quantum queries. Thus, $e_n^3(S, F)$ is the best possible error of quantum algorithms using $\leq n$ quantum queries.
3 Mean Computation

In the case of $X = \mathbb{R}$, the rates are known in both settings. We summarize these previous results in

**Theorem 1.** There are constants $c_0, c_1, c_2 > 0$ such that for $n, N \in \mathbb{N}$ with $n \leq c_0 N$

\[
\begin{align*}
c_1 n^{-1/2} &\leq e_n^{\text{ran}}(S_N, \mathcal{B}_\infty^N) \leq c_2 n^{-1/2} \quad (6) \\
c_1 n^{-1} &\leq e_n^q(S_N, \mathcal{B}_\infty^N) \leq c_2 n^{-1}. \quad (7)
\end{align*}
\]

We often use the same symbol $c, c_1, \ldots$ for possibly different positive constants – also when they appear in a sequence of relations. The estimate (6) is well-known, see e.g. [12], [14], [10], while the upper bound of (7) is due to Brassard, Höyer, Mosca, and Tapp [1], the lower bound to Nayak and Wu [11], see also [3].

Considering the vector valued case, two natural problems arise:

- Does the speedup of the quantum over the randomized setting extend to the vector valued case?
- Can the many sums problem be solved as quickly on a quantum computer as the one sum problem?

Interest in the many sums problem arose in [13]. It is the following problem: Let $M \in \mathbb{N}$ and $X = \mathbb{R}^M$, equipped with any norm. We are given a (completely known to us) $a \in L_N^\infty(X)$, that is, a sequence of vectors

\[ a(j) = (a_{ij})_{i=0}^{M-1} \in X \quad (j = 0, \ldots, N - 1), \]

the task is, given $f \in L_N^\infty$, accessible through function values, compute

\[ T_N a f = \frac{1}{N} \sum_{j=0}^{N-1} f(j) a(j) = \left( \frac{1}{N} \sum_{j=0}^{N-1} a_{ij} f(j) \right)_{i=0}^{M-1} \in X, \]

or, in other words, compute the (weighted) means for $i = 0, \ldots, M - 1$ simultaneously, the error being measured in the norm of $X$.

First we show how the many sums problem can be reduced to vector-valued mean computation.

**Lemma 1.** For all $n \in \mathbb{N}_0$, $N \in \mathbb{N}$,

\[ \sup_{a \in \mathcal{B}_\infty^N(X)} e_n^{\text{ran}}(T_N a, \mathcal{B}_\infty^N) \leq e_n^{\text{ran}}(S_N, \mathcal{B}_\infty^N(X)), \]
and
\[ \sup_{a \in \mathcal{B}_\infty^N(X)} e_2^q(T_N^a, \mathcal{B}_\infty^N) \leq e_1^q(S_N, \mathcal{B}_\infty^N(X)). \]

Proof. Let \( a \in \mathcal{B}_\infty^N(X) \) and define
\[ V^a : \mathcal{B}_\infty^N \rightarrow \mathcal{B}_\infty^N(X) \]
by setting for \( f \in \mathcal{B}_\infty^N \) and \( 0 \leq j < N \)
\[(V^a f)(j) = f(j)a(j) \in X.\]
Then \( T_N^a = S_NV^a \), and the result for \( e_{ran}^q \) follows immediately from the definition.

In the quantum case we need some technicalities connected with the special finite representation in the form of the query. Let \( m^* \in \mathbb{N} \) be arbitrary and define \( \beta : \mathbb{R} \rightarrow \mathbb{Z}_{[0, 2^{m^*})} \) for \( z \in \mathbb{R} \) by
\[ \beta(z) = \begin{cases} 0 & \text{if } z < -1 \\ \lfloor 2^{m^*-1}(z + 1) \rfloor & \text{if } -1 \leq z < 1 \\ 2^{m^*} - 1 & \text{if } z \geq 1. \end{cases} \tag{8} \]
Furthermore, let \( \gamma : \mathbb{Z}_{[0, 2^{m^*})} \rightarrow \mathbb{R} \) be defined for \( y \in \mathbb{Z}_{[0, 2^{m^*})} \) as
\[ \gamma(y) = 2^{-m^*+1}y - 1. \tag{9} \]
It follows that for \( -1 \leq z \leq 1 \),
\[ -1 \leq \gamma(\beta(z)) \leq z \leq \gamma(\beta(z)) + 2^{-m^*+1} \leq 1. \tag{10} \]
Define \( \Gamma : \mathcal{B}_\infty^N \rightarrow \mathcal{B}_\infty^N(X) \) by
\[ (\Gamma f)(i) = (\gamma \circ \beta \circ f)(i) a(i). \tag{11} \]
This mapping is of the form (13) of [3] with \( \kappa = 1 \), \( \eta : D \rightarrow D \) the identity on \( D = \mathbb{Z}_{[0, N)} \), \( \beta \) as above, and \( \varrho : D \times \mathbb{Z}_{[0, 2^{m^*})} \rightarrow X \) being defined as
\[ \varrho(i, z) = \gamma(z)a(i). \]
Hence, by Corollary 1 of [3],
\[ e_2^q(S_N\Gamma, \mathcal{B}_\infty^N) \leq e_1^q(S_N, \mathcal{B}_\infty^N(X)). \tag{12} \]
By (11),
\[ S_N\Gamma f = S_NV^a(\gamma \circ \beta \circ f) = T_N^a(\gamma \circ \beta \circ f). \]
Because of (10) we have
\[
\sup_{f \in B_N^\infty} \| T_N^a f - S_N \Gamma f \| = \sup_{f \in B_N^\infty} \| T_N^a f - T_N^a (\gamma \circ \beta \circ f) \| \\
\leq 2^{-m^*+1} \| T_N^a \| \leq 2^{-m^*+1}.
\]

This together with Lemma 6 (i) of [3] and (12) above implies
\[
e_{2n}(T_N^a, B_\infty^N) \leq e_{2n}(S_N \Gamma, B_\infty^N) + 2^{-m^*+1} \\
\leq e_n(S_N, B_\infty^N(X)) + 2^{-m^*+1}.
\]

Since \(m^*\) can be made arbitrarily large, the desired result follows.

To derive upper bounds for the randomized setting, we need some concepts and results from the theory of Banach space valued random variables, which can be found in [8]. Let \(1 < p \leq 2\). For a Banach space \(X\) the type \(p\) constant of \(X\) is the smallest \(c > 0\) such that for all \(m\) and all \(x_1, \ldots, x_m \in X\)
\[
E \left\| \sum_{i=1}^m \varepsilon_i x_i \right\|_X^p \leq c^p \sum_{i=1}^m \| x_i \|_X^p,
\]
where \((\varepsilon_i)_{i=1}^m\) are independent, centered, \([-1, 1]\) valued Bernoulli variables. We put \(T_p(X) = +\infty\) if there is no such \(c\). It is known that for \(1 \leq p \leq \infty\) there is a constant \(c > 0\) such that for all \(M \in \mathbb{N}\)
\[
T_p(L_p^M) \leq c \quad \text{if} \quad 1 < p < 2 \quad (13)
\]
\[
T_2(L_p^M) \leq c \quad \text{if} \quad 2 \leq p < \infty \quad (14)
\]
\[
T_2(L_\infty^M) \leq c (\log(M + 1))^{1/2}. \quad (15)
\]
(Throughout this paper \(\log\) stands for \(\log_2\).) Using the concepts above, the upper bounds for the classical randomized setting, first observed by Mathé [9], are easily derived:

**Proposition 1.** Let \(1 \leq p \leq 2\). Then for \(n \in \mathbb{N}\) and any Banach space \(X\),
\[
e_n^{ran}(S_N, B_\infty^N(X)) \leq 4 T_p(X) n^{1/p-1}.
\]

**Proof.** We use the vector valued Monte Carlo method
\[
S_N f \approx A_n(f) = \frac{1}{n} \sum_{l=1}^n f(\xi_l).
\]
where $\xi_l$ are uniformly distributed on $\{0, \ldots, N - 1\}$ independent random variables. By Proposition 9.11 of [8]

$$
\mathbb{E} \| S_N f - A_n(f) \|_X \leq (\mathbb{E} \| S_N f - A_n(f) \|_X^p)^{1/p} \\
= n^{-1} (\mathbb{E} \left\| \sum_{l=1}^{n} (S_N f - f(\xi_l)) \right\|_X^p)^{1/p} \\
\leq 2T_p(X)n^{-1} (\sum_{l=1}^{n} \mathbb{E} \| S_N f - f(\xi_l) \|_X^p)^{1/p} \\
\leq 4T_p(X)n^{1/p-1} \max_{0 \leq i < N} \| f(i) \|_X.
$$

Next we show that these upper bounds can be carried over to the quantum setting. Indeed, we shall show that one can transform any randomized algorithm $A_{ran}^n$ into a quantum algorithm $A_{q}^n$ with $n$ queries and essentially the same error behaviour. We consider this in the general context in which we defined the error quantities.

**Lemma 2.** Let $n \in \mathbb{N}_0$, let $A_{ran}^n = (A_{ran}^\omega)_{\omega \in \Omega}$ be any randomized algorithm on $F$ using $n$ function values (recall that we assume a finite $\Omega$). Suppose further that $\theta : K \rightarrow K$ is a mapping such that $\theta(K)$ is a finite set and for all $f \in F$ we have $\theta \circ f \in F$. Then there is a quantum algorithm $A_{q}^n$ using $n$ queries such that for all $f \in F$ the distribution of $A_{q}^n(f)$ is the same as that of $A_{ran}^n(\theta \circ f)$. Moreover,

$$
e_{q}^n(S, F) \leq 4e_{ran}^n(S, F) + \sup_{f \in F} \| S(f) - S(\theta \circ f) \|. \quad (16)
$$

**Proof.** Choose any $m'_1, m'_2, m'' \in \mathbb{N}$ satisfying

$$n \leq 2^{m'_1}, \quad |\Omega| \leq 2^{m'_2}, \quad |\theta(K)| \leq 2^{m''}.
$$

Put $m' = m'_1 + m'_2$ and $m = m' + nm''$. It is no loss of generality to assume $\Omega \subseteq \mathbb{Z}[0, 2^{m'_2}]$ and $\{\omega\} \in \Sigma$ for all $\omega \in \Omega$.

Let $\sigma$ be any bijection of $\theta(K)$ onto a subset $Z_0$ of $\mathbb{Z}[0, 2^{m''}]$. Setting $\beta = \sigma \circ \theta : K \rightarrow \mathbb{Z}[0, 2^{m''}]$ and letting $\gamma : \mathbb{Z}[0, 2^{m''}] \rightarrow K$ be any extension of $\sigma^{-1} : Z_0 \rightarrow \theta(K)$ to all of $\mathbb{Z}[0, 2^{m''}]$, we get

$$
\gamma \circ \beta = \theta. \quad (17)
$$

8
We identify \( \mathbb{Z}[0, 2^{m'1}] \) with \( \mathbb{Z}[0, 2^{m'}] \times \mathbb{Z}[0, 2^{m''}] \). Let

\[
Z = \{(i, \omega) : 0 \leq i \leq n - 1, \omega \in \Omega \} \subseteq \mathbb{Z}[0, 2^{m'}],
\]

and define \( \tau : Z \to D \) as \( \tau(i, \omega) = t_{i+1}^\omega \). The query of \( A_n^0 \) is given by

\[
Q = (m, m', m'', Z, \tau, \beta),
\]

The quantum algorithm \( A_n^0 \) is defined by

\[
A_n^0 = (Q, (U_i)_{i=0}^n, 0, \varphi),
\]

where \( U_i \) and \( \varphi \) will be specified in the sequel. As \( U_0 \) we choose any unitary operator on \( H_m \) mapping

\[
|0\rangle |0\rangle |0\rangle \in H_{m'_1} \otimes H_{m'_2} \otimes H_{m'-m'} = H_m
\]

to

\[
\sum_{\omega \in \Omega} \mathbb{P}(\{\omega\})^{1/2} |n - 1\rangle |\omega\rangle |0\rangle.
\]

(18)

(In the case \( n = 0 \) we replace \( n - 1 \) by 1.) Applying \( Q_f \) maps a state of the form

\[
|n - 1\rangle |\omega\rangle |0\rangle \ldots |0\rangle |0\rangle \in H_{m'_1} \otimes H_{m'_2} \otimes H_{m''} \otimes \cdots \otimes H_{m''} = H_m
\]

with \( \omega \in \Omega \) to

\[
|n - 1\rangle |\omega\rangle |\beta(f(t_n^\omega))\rangle \ldots |0\rangle |0\rangle.
\]

Now \( U_1 \) is any unitary mapping that decreases the first component by one and interchanges the third with the last, that is, maps the state above to

\[
|n - 2\rangle |\omega\rangle |0\rangle \ldots |0\rangle |\beta(f(t_n^\omega))\rangle
\]

(for all \( \omega \in \Omega \)). The next application of \( Q_f \), followed by a \( U_2 \) which decreases the first component by one and interchanges the third with the last but one, results in

\[
|n - 3\rangle |\omega\rangle |0\rangle \ldots |\beta(f(t_{n-1}^\omega))\rangle |\beta(f(t_n^\omega))\rangle.
\]

Continuing this way we reach, after the \( n \)-th application of \( Q_f \), the state

\[
|0\rangle |\omega\rangle |\beta(f(t_1^\omega))\rangle \ldots |\beta(f(t_{n-1}^\omega))\rangle |\beta(f(t_n^\omega))\rangle.
\]

9
The last $U_n$ does not change anything, that is, it is the identity on $H_m$. Finally, $\varphi : \mathbb{Z}[0,2^m) \to G$ is defined for

$$(i, \omega, z_1, \ldots, z_n) \in \mathbb{Z}[0,2^{m_1}) \times \mathbb{Z}[0,2^{m_2}) \times \cdots \times \mathbb{Z}[0,2^{m_n}) = \mathbb{Z}[0,2^m)$$

by

$$\varphi(i, \omega, z_1, \ldots, z_n) = \begin{cases} \varphi^\omega(\gamma(z_1), \ldots, \gamma(z_n)) \in G & \text{if } \omega \in \Omega \\ 0 \in G & \text{otherwise.} \end{cases}$$

Taking into account (17) and (18), it follows readily that $A^q_n(f)$ takes the value

$$\varphi^\omega(\gamma \circ \beta \circ f(t_1^\omega), \ldots, \gamma \circ \beta \circ f(t_n^\omega)) = \varphi^\omega(\theta \circ f(t_1^\omega), \ldots, \theta \circ f(t_n^\omega))$$

with probability $|\mathbb{P}\{\{\omega\}\}|^{1/2} = \mathbb{P}(\{\omega\})$. So does $A^{\text{ran}}_n(\theta \circ f)$, by definition, which proves the first statement of the lemma.

To show (16), we fix any $\delta > 0$ and assume that $A^{\text{ran}}_n$ satisfies

$$\sup_{f \in F} \mathbb{E} \|S(f) - A^\omega_n(f)\| \leq e^{\text{ran}}_n(S,F) + \delta.$$ 

By Chebyshev’s inequality, for any $f \in F$,

$$\mathbb{P}\{\omega : \|S(\theta \circ f) - A^\omega_n(\theta \circ f)\| \leq 4 \mathbb{E} \|S(\theta \circ f) - A^\omega_n(\theta \circ f)\|\} \geq 3/4.$$ 

Hence, with probability at least $3/4$, we also have

$$\|S(\theta \circ f) - A^\omega_n(f)\| \leq 4 \mathbb{E} \|S(\theta \circ f) - A^\omega_n(\theta \circ f)\| \leq 4 e^{\text{ran}}_n(S,F) + 4\delta,$$

and consequently,

$$\|S(f) - A^\omega_n(f)\| \leq \|S(f) - S(\theta \circ f)\| + \|S(\theta \circ f) - A^\omega_n(f)\|$$

$$\leq \sup_{f \in F} \|S(f) - S(\theta \circ f)\| + 4 e^{\text{ran}}_n(S,F) + 4\delta.$$ 

Since $\delta > 0$ was arbitrary, this proves (16).

**Corollary 1.** If $n \in \mathbb{N}_0$, $N \in \mathbb{N}$, $F = \mathcal{B}^N_\infty(X)$ with $X$ a finite dimensional normed space, and $S : F \to G$ is any continuous mapping to a normed space $G$, then

$$e^q_n(S,F) \leq 4 e^{\text{ran}}_n(S,F).$$
Proof. Since $X$ is finite dimensional, the unit ball $B_X$ is compact. Using finite $(1/k)$-nets in $B_X$, we can choose $\theta_k : X \to B_X$ ($k \in \mathbb{N}$) in such a way that $\theta_k(X)$ is finite for all $k$ and

$$\lim_{k \to \infty} \sup_{x \in B_X} \|x - \theta_k(x)\|_X = 0.$$ 

This implies

$$\lim_{k \to \infty} \sup_{f \in B_{L^\infty}^N(X)} \|f - \theta_k \circ f\|_{L^\infty(X)} = 0.$$ 

The set $F = B_{L^\infty}^N(X)$ is also compact. Therefore, $S$ is uniformly continuous on $F$. It follows that

$$\lim_{k \to \infty} \sup_{f \in F} \|S(f) - S(\theta_k \circ f)\|_G = 0,$$

and the result is a consequence of Lemma 2.

Taking as $X$ finite dimensional $L_p$-spaces, we get from Proposition 1, (13)–(15), and Corollary 1, Corollary 2.

**Corollary 2.** Let $1 \leq p \leq \infty$. Then there is a constant $c > 0$ such that for $n, M, N \in \mathbb{N}$, and $X = L_p^M$

$$e_n^q(S_N, B_{L^\infty}^N(X)) \leq 4e_n^{ran}(S_N, B_{L^\infty}^N(X))$$

$$\leq \begin{cases} 
n^{-1/2} & \text{if } 2 \leq p < \infty \\
n^{-1/2}\log(M + 1)^{1/2} & \text{if } p = \infty \\
n^{-1+1/p} & \text{if } 1 \leq p < 2. \end{cases}$$

To derive lower bounds, we need two simple technical lemmas.

**Lemma 3.** For all $n \in \mathbb{N}_0$, $N, N_1 \in \mathbb{N}$ with $N_1 \leq N$ and normed spaces $X$ the following holds: Let $N = kN_1 + l$ for some $k \in \mathbb{N}$, $l \in \mathbb{N}_0$ with $0 \leq l < N_1$. Then

$$\sup_{a \in B_{L^\infty}^N(X)} e_n^q(T_N^a, B_{L^\infty}^{N_1}(X)) \leq \frac{N}{kN_1} \sup_{a \in B_{L^\infty}^N(X)} e_n^q(T_N^a, B_{L^\infty}^N(X)) < 2 \sup_{a \in B_{L^\infty}^N(X)} e_n^q(T_N^a, B_{L^\infty}^N(X)).$$

Proof. The second inequality is obvious. To prove the first, let $a \in B_{L^\infty}^N(X)$. Define $\bar{a} \in B_{L^\infty}^N(X)$ by

$$\bar{a}(i) = \begin{cases} a(i \mod N_1) & \text{if } i < kN_1 \\
0 & \text{if } kN_1 \leq i < N. \end{cases}$$
Define $\Gamma : \mathcal{B}_{\infty}^{N_1} \to \mathcal{B}_{\infty}^{N}$ by setting for $f \in L_{\infty}^{N_1}$

$$(\Gamma f)(i) = f(i \mod N_1).$$

Then $\Gamma$ is of the form (12) of [3]. For $f \in \mathcal{B}_{\infty}^{N_1}$,

$$T_{N}^{\tilde{a}} f = \frac{1}{N} \sum_{i=0}^{N-1} (\Gamma f)(i) \tilde{a}(i) = \frac{1}{N} \sum_{j=0}^{N_1-1} k f(j) a(j) = \frac{k N_1}{N} T_{N}^{a} f.$$ 

By [3], Corollary 1 and Lemma 6 (ii), the result follows. \quad \square

**Lemma 4.** For all $n \in \mathbb{N}_0$, $N, M, M_1 \in \mathbb{N}$ with $M_1 \leq M$, and $1 \leq p \leq \infty$,

$$\sup_{a \in \mathcal{B}^N_{\infty}(L_p)} e_n^q(T_{N}^{a}, \mathcal{B}_{\infty}^{N}) \leq \sup_{a \in \mathcal{B}^N_{\infty}(L_p)} e_n^q(T_{N}^{a}, \mathcal{B}_{\infty}^{N}).$$

**Proof.** Define $J : L_p^{M_1} \to L_p^{M}$ by setting for $g \in L_p^{M_1}$

$$(Jg)(i) = \begin{cases} \left( \frac{M}{M_1} \right)^{\frac{1}{p}} g(i) & \text{if } i < M_1 \\ 0 & \text{if } M_1 \leq i < M, \end{cases}$$

and $P : L_p^{M} \to L_p^{M_1}$ for $g \in L_p^{M}$ by

$$(Pg)(i) = \left( \frac{M_1}{M} \right)^{\frac{1}{p}} g(i) \quad (0 \leq i < M_1).$$

Clearly,

$$\|P\| = \|J\| = 1.$$ 

Let $a \in \mathcal{B}^N_{\infty}(L_p^{M_1})$. Define $\tilde{a} \in \mathcal{B}^N_{\infty}(L_p^{M})$ by

$$\tilde{a}(j) = J a(j) \quad (0 \leq j < N).$$

Then for $f \in \mathcal{B}^N_{\infty}$

$$PT_{N}^{\tilde{a}} f = \frac{1}{N} \sum_{j=0}^{N-1} f(j) P \tilde{a}(j) = \frac{1}{N} \sum_{j=0}^{N-1} f(j) a(j) = T_{N}^{a} f.$$ 

By Lemma 1 of [5],

$$e_n^q(T_{N}^{a}, \mathcal{B}_{\infty}^{N}) = e_n^q(PT_{N}^{\tilde{a}}, \mathcal{B}_{\infty}^{N}) \leq \|P\| e_n^q(T_{N}^{a}, \mathcal{B}_{\infty}^{N}) = e_n^q(T_{N}^{a}, \mathcal{B}_{\infty}^{N}).$$ 

\quad \square
We also need the following result which is contained in Proposition 6 of [5]. Here $J_{\infty,p}^N : L_\infty^N \to L_p^N$ denotes the identical embedding. For brevity we set for $N \in \mathbb{N}$ with $N > 4$

$$\lambda(N) := (\log \log N)^{-3/2}(\log \log \log N)^{-1}.$$

**Proposition 2.** Let $1 \leq p \leq \infty$. There are constants $c_0, c_1 > 0$ such that for all $n, N \in \mathbb{N}$ with $N > 4$ and $n \leq c_0N$

$$e_n^q(J_{\infty,p}^N, B_{\infty}^N) \geq c_1 \begin{cases} 1 & \text{if } 2 < p \leq \infty, \\ \lambda(N) & \text{if } p = 2, \\ (\log N)^{-2/p+1} & \text{if } 1 \leq p < 2. \end{cases}$$

Note that, necessarily, $c_0 < 1$, since $e_n^q(J_{\infty,p}^N, B_{\infty}^N) = 0$ (this is easy to check, for an argument of this type see [6], relation (12) and its proof).

**Proposition 3.** Let $1 \leq p \leq \infty$. Then there are constants $c_0, c_1 > 0$ such that for $n, M, N \in \mathbb{N}$ with $4 < n \leq c_0 \min(M, N)$, and $X = L_p^M$

$$\sup_{a \in B_{\infty}^N(X)} e_n^q(T_a^N, B_{\infty}^N) \geq c_1 \begin{cases} n^{-1/2}\lambda(n) & \text{if } 2 \leq p \leq \infty, \\ n^{-1+1/p}(\log n)^{-2/p+1} & \text{if } 1 \leq p < 2. \end{cases}$$

**Proof.** Assume

$$4 < n \leq (c_0/2) \min(M, N), \tag{19}$$

where the constant $c_0$ is that from Proposition 2. Define

$$k = \lceil \log_2(c_0^{-1}n) \rceil, \quad N_1 = 2^k.$$ 

It follows that

$$c_0^{-1}n \leq N_1 \leq 2c_0^{-1}n,$$

which implies $n \leq c_0 N_1$, and, because of $c_0 < 1$ and (19), also

$$4 < N_1 \leq \min(M, N).$$

From this and Lemmas 3 and 4 we obtain

$$\sup_{a \in B_{\infty}^{N_1}(L_p^N)} e_n^q(T_a^{N_1}, B_{\infty}^{N_1}) \leq 2 \sup_{a \in B_{\infty}^{N_1}(L_p^N)} e_n^q(T_a^N, B_{\infty}^N) \leq 2 \sup_{a \in B_{\infty}^{N_1}(L_p^N)} e_n^q(T_a^{N_1}, B_{\infty}^{N_1}). \tag{20}$$
For $1 \leq p < 2$ define $a \in B_{\infty}^{N_1}(L_p^{N_1})$ by

$$a(j) = N_1^{1/p} e_j \quad (0 \leq j < N_1 - 1),$$

where $e_j = (\delta_{ij})_{i=0}^{N_1-1}$ are the unit vectors in $\mathbb{R}^{N_1}$. Then

$$T_{N_1}^a = N_1^{1/p-1} J_{\infty,p}^{N_1}.$$

Thus, from Proposition 2,

$$c_n^q(T_{N_1}^a, B_{\infty}^{N_1}) \geq c_1 N_1^{1/p-1} (\log N_1)^{-2/p+1} \geq c n^{1/p-1} (\log n)^{-2/p+1},$$

which, together with (20) implies the required statement.

For $2 \leq p \leq \infty$ we recall that $N_1 = 2^k$ and let $W_{N_1}$ be the Walsh matrix, defined by

$$W_{N_1} = ((-1)^{i \cdot j})_{i,j=0}^{N_1-1}.$$ 

Here

$$i \cdot j := \sum_{l=1}^{k} i_l j_l$$

with $i = \sum_{l=1}^{k} i_l 2^{k-l}$ and $j = \sum_{l=1}^{k} j_l 2^{k-l}$ the binary representations. Note that

$$W_{N_1}^2 = N_1 I_{N_1}, \quad (21)$$

where $I_{N_1}$ is the respective identity matrix. Let $w(j)$ denote the $j$-th column vector of $W_{N_1}$ and define $a \in B_{\infty}^{N_1}(L_p^{N_1})$ by

$$a(j) = w(j) \quad (0 \leq j < N_1 - 1).$$

Let $W$ denote the operator from $L_p^{N_1}$ to $L_2^{N_1}$ with matrix $W_{N_1}$, that is,

$$Wf = \sum_{j=0}^{N_1-1} f(j)w(j) \quad (f \in L_p^{N_1}).$$

It follows from (21) that for $f \in L_p^{N_1}$,

$$WT_{N_1}^a f = \frac{1}{N_1} W \sum_{i=0}^{N_1-1} f(j)w(j) = \sum_{i=0}^{N_1-1} f(j) e_j,$$

and consequently

$$WT_{N_1}^a = J_{\infty,2}^{N_1}. \quad (21)$$
On the other hand, since $W_{N_1}$ is an orthogonal matrix,

$$\|W : L_p^{N_1} \rightarrow L_2^{N_1}\| \leq \|I_{N_1} : L_p^{N_1} \rightarrow L_2^{N_1}\| \leq \|W : L_2^{N_1} \rightarrow L_2^{N_1}\| = N_1^{1/2}.$$  

It follows from Lemma 1 of [5] that

$$e_n(q(J_{N_1}^N, 2^{\infty}) = e_n^q(WT_{N_1}^a, 2^{\infty}) \leq \|W\|e_n^q(T_{N_1}^a, 2^{\infty}) \leq N_1^{1/2}e_n^q(T_{N_1}^a, 2^{N_1}).$$

Hence, by Proposition 2

$$e_n^q(T_{N_1}^a, 2^{\infty}) \geq c_1 N_1^{-1/2} \lambda(N_1) \geq c n^{-1/2} \lambda(n).$$

Now the result follows from (20). \qed

From Proposition 3, Lemma 1, and Corollary 1 we get

**Corollary 3.** Let $1 \leq p \leq \infty$. Then there are constants $c_0, c_1 > 0$ such that for $n, M, N \in \mathbb{N}$ with $n \leq c_0 \min(M, N)$, and $X = L_p^M$

$$e_n^{\text{ran}}(S_N, 2^{\infty}(X)) \geq 4^{-1} e_n^{\text{ran}}(S_N, 2^{\infty}(X)) \geq c_1 \begin{cases} n^{-1/2} \lambda(n) & \text{if } 2 \leq p \leq \infty \\ n^{-1+1/p}(\log n)^{-2/p+1} & \text{if } 1 \leq p < 2. \end{cases}$$

We summarize the main results (contained in Corollaries 2 and 3) in the following theorem, in which we suppress logarithmic factors.

**Theorem 2.** Let $1 \leq p \leq \infty$. Then there is a constant $c_0$ such that for $n, M, N \in \mathbb{N}$ with $n \leq c_0 \min(M, N)$, and $X = L_p^M$

$$e_n^{\text{ran}}(S_N, 2^{\infty}(X)) \asymp \log e_n^{\text{ran}}(S_N, 2^{\infty}(X)) \asymp \begin{cases} n^{-1/2} & \text{if } 2 \leq p \leq \infty \\ n^{-1+1/p} & \text{if } 1 \leq p < 2. \end{cases}$$

### 4 Comments

In the scalar case, by Theorem 1 there is a quantum algorithm with rate $n^{-1}$, while no classical randomized algorithm can be better than of the order $n^{-1/2}$. Theorem 2 shows that vector quantum summation fails to give any speedup over the classical randomized setting for any $1 \leq p \leq \infty$, provided the dimension is high ($M \geq N$). Under this assumption there is, in particular, no vector-valued version of the quantum summation algorithm.
of \([1]\) which is more efficient than the classical vector-valued Monte Carlo algorithm.

Let us also discuss the many sums problem. This is best done for the case \(p = \infty\), that is, \(X = L_\infty^M\). Let \(a \in B^N_\infty(L_\infty^M)\), thus \(|a_{ij}| \leq 1\) for all \(i, j\). We can compute one component of the result,

\[
\left( \sum_{j=0}^{N-1} a_{ij} f(j) \right)^{M-1}_{i=0}
\]
say, the first, by the scalar-valued Monte Carlo method:

\[
\frac{1}{n} \sum_{l=1}^{n} a_{1,\xi_l} f(\xi_l),
\]

where \(\xi_l\) are uniformly distributed on \(\{0, \ldots, N - 1\}\) independent random variables. This has error rate \(n^{-1/2}\). Once the function values \(f(\xi_l)\) have been obtained, they can be re-used in the computation of all other components of the solution vector. All we have to take care is that the probability of having the desired precision in all components is large enough – this way we just lose a logarithmic factor (see Corollary \([2]\)). Now, can we do so in the quantum setting, that is, can we re-use query results? If so, we should be able to obtain the same rate \(n^{-1}\) (maybe, again, up to a logarithmic factor) as in the scalar case. It turned out that this is not the case, that is, there are matrices \(a \in B^N_\infty(L_\infty^M)\) such that the best rate is (up to logarithms) \(n^{-1/2}\) (Proposition \([3]\)).

Let us finally mention that the lower bounds for the randomized setting (Corollary \([4]\), obtained here as a byproduct of the analysis of the quantum case, can be slightly improved and extended to the case of randomized algorithms with general (i.e., also infinite) \(\Omega\). Let us denote the respective minimal error by \(\tilde{e}_{n}^{\text{ran}}\) (trivially, \(e_{n}^{\text{ran}} \geq \tilde{e}_{n}^{\text{ran}}\)). The following holds for \(1 \leq p \leq \infty\): There are constants \(c_0, c_1 > 0\) such that for \(n, M, N \in \mathbb{N}\) with \(n \leq c_0 \min(M, N)\), and \(X = L_\infty^M\)

\[
\tilde{e}_{n}^{\text{ran}}(S_N, B^N_\infty(X)) \geq c_1 \begin{cases} 
    n^{-1/2} & \text{if } 2 \leq p < \infty \\
    \min(n^{-1/2}(\log(M + 1))^{1/2}, 1) & \text{if } p = \infty \\
    n^{-1+1/p} & \text{if } 1 \leq p < 2.
\end{cases}
\]

For \(2 \leq p < \infty\) this follows from the scalar case \(X = \mathbb{R}\) (Theorem \([5]\)). The case \(1 \leq p < 2\) is a direct consequence of standard lower bound techniques \([12], [14], [2]\). So is the case \(p = \infty\), except that, in addition, Lemma 5.3 of \([7]\) has to be used. We omit further details.
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