A Study of the Time Evolution of Brans-Dicke Parameter and its Role in Cosmic Expansion

Sudipto Roy¹, Soumyadip Chowdhury²

¹Assistant Professor, Department of Physics, St. Xavier’s College, Kolkata, India
²Student of M.Sc. (Physics), Batch of 2015-17, St. Xavier’s College, Kolkata, India
¹E-mail: roy.sudipto@sxccal.edu, roy.sudipto1@gmail.com

Abstract:
The dependence of Brans-Dicke (BD) parameter (ω) upon the scalar field (φ), for different cosmological era of the expanding universe, has been explored. The time dependence of the scalar field has been determined and thereby the explicit time dependence of the BD parameter has been obtained. Experimental observations regarding the time dependence of the gravitational constant has been considered for this study. The scale factor, Hubble parameter, deceleration parameter, matter density, gravitational constant and the density parameters for matter and dark energy have been expressed in terms of the BD parameter and its time derivatives, showing its role in cosmic expansion.

Keywords: Cosmology, Brans-Dicke theory, Scalar Field, Accelerated cosmic expansion, Gravitational Constant, Density parameters for matter and dark energy.

PACS Nos. 04.20.-q, 98.80.Jk, 98.80.-k, 95.30.Sf, 95.30.Ft, 91.10.Op, 95.36.+x

Introduction:
The theory of gravitation, known as Brans-Dicke (BD) theory, is characterized by a dimensionless constant function ω and a scalar field φ. The arbitrary coupling function ω determines the relative importance [1]. In the generalized version of the B-D theory, the dimensionless coupling constant is considered to be a function of time [2]. This time dependence may also be expressed by assuming ω to be explicitly dependent upon the scalar field φ [3]. There are several important reasons for which generalised BD theory has gained so much of important in explaining and analyzing cosmological phenomena. In Kaluza-Klein theories, super gravity theory and in all the known effective string actions, this theory has a natural appearance. It is regarded as the most natural extension of the General Theory of Relativity (GTR), which may justify its applicability in fundamental theories [4]. In the generalized version of Brans-Dicke theory, which is also known as graviton-dilaton theory, ω has been shown to be a function of the scalar field φ (dilaton). Thus, there can be several models depending upon the functional form of the BD parameter. This theory generates the results obtained from GTR, for a constant scalar field and an infinite ω [5, 6]. Using a constant value of ω, BD theory was found to account for almost all important cosmological observations regarding the evolution of the universe. BD theory is capable of explaining the features like inflation [7], early and late time behaviour of the universe [8], cosmic acceleration and structure formation [9], quintessence and the coincidence problem [10], self-interacting potential and cosmic acceleration [11]. For a small negative
value of $\omega$ it correctly explains cosmic acceleration, structure formation and the coincidence problem and, for a large value of $\omega$, BD theory gives the correct amount of inflation and early and late time behaviour. The time dependence of $\omega$ in Brans-Dicke theory has many interesting features. From string and Kaluza-Klein theories it gets a strong corroboration, and in several studies, the dynamics of the universe has been analyzed within its framework. Through these attempts, the phenomena like the evolution of the universe, its accelerated expansion and quintessence, have been explained in a qualitative way without deriving any explicit time dependence of the BD parameter [10, 12-14]. Some recent studies have shown that several models can be formulated based on the concept of a time-varying $\omega$ [6]. Therefore it would be quite natural for the researchers to find an analytical expression of $\omega$ as a function of time, using the field equations of the Brans-Dicke theory. Some attempts were made to derive the time dependence of the BD parameter using very simple empirical forms of scale factor and the scalar field [15, 16]. But the chosen scale factors in these cases produced time independent deceleration parameters, in complete contradiction of observations.

The purpose of the present study is to derive an expression of the BD parameter as an explicit function of the scalar field ($\varphi$) and also an expression showing its explicit time dependence, for different cosmological era of the universe, characterized by different values of the equation of state parameter ($\gamma$). The present matter-dominated universe contains cold matter of negligible pressure (dust), with $\gamma = 0$. For the present study, empirical expressions of the scale factor, scalar field and the BD parameter have been used. Using the field equations of BD theory and also the wave equation for the BD scalar field ($\varphi$), we have determined the values of all parameters involved in these empirical forms. We have also used the experimental results regarding the time variation of the gravitational constant ($G$), to determine the values of these unknown parameters. The scale factor ($a$), Hubble parameter ($H$), deceleration parameter ($q$) matter density ($\rho$) and gravitational constant ($G$) have been expressed as functions of the BD parameter ($\omega$) and its time derivatives. The dependence of density parameters, for matter and dark energy ($\Omega_m, \Omega_d$), upon the BD parameter have also been shown theoretically. These expressions show mathematically, the role played by the dimensionless parameter ($\omega$) in cosmic expansion.

**Theoretical Model:**

In generalized Brans-Dicke theory, the field equations, for a universe filled with a perfect fluid and described by Friedmann-Robertson-Walker space-time, with scale factor $a(t)$ and spatial curvature $k$, are expressed as,

$$\frac{a^2 + k}{a^2} + \frac{\dot{a}}{a} \frac{\dot{\varphi}}{\varphi} - \frac{\omega(\varphi)}{6} \frac{\dot{\varphi}^2}{\varphi^2} = \frac{\rho}{3\varphi}$$  \hspace{1cm} (01)

$$2 \frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2} + \frac{\omega(\varphi)}{2} \frac{\dot{\varphi}^2}{\varphi^2} + 2 \frac{\dot{a}}{a} \frac{\dot{\varphi}}{\varphi} + \frac{\ddot{\varphi}}{\varphi} = - \frac{\rho}{\varphi}$$  \hspace{1cm} (02)

The wave equation for the scalar field ($\varphi$), in generalized Brans-Dicke theory, where $\omega$ is a time dependent parameter, is expressed as,

$$\ddot{\varphi} + 3 \frac{\dot{a}}{a} \frac{\dot{\varphi}}{\varphi} = \frac{\rho - 3P}{2\omega + 3} - \frac{\omega \dot{\varphi}}{2\omega + 3}$$  \hspace{1cm} (03)

The energy conservation for the cosmic fluid is given by [15, 16],

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + P) = 0$$  \hspace{1cm} (04)
The equation of state of the fluid is expressed as,

\[ P = \gamma \rho \] (05)

The values of equation-of-state parameter (\( \gamma \)) are \(-1\) (vacuum energy dominated era), 0 (matter dominated era), \(1/3\) (radiation dominated era), 1 (massless scalar field dominated era).

The solution of equation (4), using equation (5), is obtained as,

\[ \rho = \rho_0 a^{-3(1+\gamma)} \] (06)

For the present study, based on the equations (1), (2) and (3), the following empirical relations of scale factor, scalar field and BD parameter have been chosen.

\[ a = a_0 \left(\frac{t}{t_0}\right)^\varepsilon \exp[\mu(t-t_0)] \] (07)
\[ \varphi = \varphi_0 \left(\frac{a}{a_0}\right)^n \] (08)
\[ \omega = \omega_0 \exp \left[ m(\varphi - \varphi_0) \right] \] (09)

The scale factor (in eqn. 7) has been chosen to ensure a change of sign of the deceleration parameter with time, as per many recent studies showing a transition of cosmic expansion from deceleration to acceleration \([3, 19, 20]\). Here \( \varepsilon, \mu > 0 \) to ensure increase of scale factor with time. The Hubble parameter (\( H \)) and the deceleration parameter (\( q \)), calculated from this scale factor, are written below.

\[ H = \mu - \frac{\varepsilon}{t} \] (10)
\[ q = -1 + \frac{\varepsilon}{(\varepsilon+\mu t)^2} \] (11)

For \( 0 < \varepsilon < 1 \), we get \( q > 0 \) at \( t = 0 \) and, for \( t \rightarrow \infty \), we have \( q \rightarrow -1 \).

Taking \( H = H_0 \) and \( q = q_0 \), at \( t = t_0 \), one gets,

\[ \varepsilon = (H_0 t_0)^2 (q_0 + 1) \] (12)
\[ \mu = H_0 - H_0^2 t_0 (q_0 + 1) \] (13)

The scalar field (in eqn. 8) has been chosen on the basis of some studies on Brans-Dicke theory \([3, 19, 20]\). The value of \( n \) can be determined from the field equations.

The empirical expression of BD parameter (in eqn. 9) has been chosen according to the generalized Brans-Dicke theory, where \( \omega \) is regarded as a function of the scalar field (\( \varphi \)) \([3]\). The values of \( \omega_0 \) and \( m \) have to be determined from the field equations.

Considering \( \omega \) as a function of \( \varphi \), equation (3) can be written as,

\[ \ddot{\varphi} + 3 \frac{\dot{\varphi}}{a} = \frac{\rho - 3P}{2\omega + 3} - \frac{\dot{\varphi}^2}{2\omega + 3} \frac{d\omega}{d\varphi} \] (14)

Combining the equations (5), (8), (9) with (14) one gets,

\[ \omega = \frac{1}{f H^2} \left[ \frac{\rho(1-3\gamma)}{\varphi} - 3H^2 \left(n^2 + 2n - nq\right) \right] \] (15)
Where,

\[ f(t) = 2(n^2 + 2n - nq) + mn^2 \varphi \]
\[ H(t) = \mu + \frac{e}{t} \]
\[ q(t) = -1 + \frac{e}{(e+\mu t)^2} \]
\[ \rho(t) = \frac{\rho_0}{a_0^{2(1+y)}} \left( \frac{t}{t_0} \right)^{-3e(1+y)} \exp[-3(1+y)\mu(t-t_0)] \]
\[ \varphi(t) = \varphi_0 \left( \frac{t}{t_0} \right)^{n\varphi} \exp[n\mu(t-t_0)] \]

The two above equations, for \( \rho \) and \( \varphi \), are obtained by substituting equation (7) into the equations (6) and (8) respectively.

Taking \( \gamma = 0 \) in equation (15) for the present matter dominated era and writing all parameter values for the present time \( (t = t_0) \), one obtains,

\[ m = \frac{\rho_0}{n^2 \omega_0 H_0^2 \varphi_0^2} - \frac{(2\omega_0 + 3)(n^2 + 2n - nq_0)}{n^2 \omega_0 \varphi_0} \]  
(16)

Using the equations (1), (2) and (8), and taking \( k = 0, P = 0 \), we get,

\[ \omega = \frac{1}{n^2} \left[ 2 + 2q + 2n - n^2 + nq - \frac{\rho}{\varphi H^2} \right] \]  
(17)

Writing all parameter values for \( t = t_0 \) in equation (17), one gets,

\[ \omega_0 = \frac{1}{n^2} \left[ 2 + 2q_0 + 2n - n^2 + nq_0 - \frac{\rho_0}{\varphi_0 H_0^2} \right] \]  
(18)

Using equation (18) in (16) one obtains,

\[ m = \frac{\rho_0}{\left[ 2(1 + q_0 + n) - n^2 + nq_0 - \frac{\rho_0}{\varphi_0 H_0^2} \right] H_0^2 \varphi_0^2} - \frac{\frac{1}{n^2}(2(1 + q_0 + n) - n^2 + nq_0 - \frac{\rho_0}{\varphi_0 H_0^2}) + 3(n^2 + 2n - nq_0)}{2(1 + q_0 + n) - n^2 + nq_0 - \frac{\rho_0}{\varphi_0 H_0^2} \varphi_0} \]  
(19)

The above expression (eqn. 19) should be used as the value of \( m \) in the expression of \( f(t) \), which is a part of the expression for \( \omega \) in equation (15). Eliminating \( \omega \) from the equations (1) and (2), taking \( k = 0 \) and \( P = \gamma \rho \), one obtains,

\[ 2 \ddot{a} + 4 \frac{\dot{a}^2}{a^2} + 5 \frac{\dot{a} \varphi}{a \varphi} + \frac{\dot{\varphi}}{\varphi} = \frac{\rho}{\varphi} (1 - \gamma) \]  
(20)

Using equation (8) in (20) for \( \gamma = 0 \) and writing all parameter values for \( t = t_0 \) one gets,

\[ n^2 + (4 - q_0)n + \left( 4 - 2q_0 - \frac{\rho_0}{\varphi_0 H_0^2} \right) = 0 \]  
(21)

Equation (21) is quadratic in \( n \). Its two roots are given by,
\[ n_{\pm} = \frac{1}{2} \left[ q_0 - 4 \pm \left( q_0^2 + \frac{4\rho_0}{\varphi_0 H_0^2} \right)^{1/2} \right] \]  \hspace{1cm} (22)

The values of different cosmological parameters used in this article are:

\[ H_0 = \frac{72 Km}{Mpc} = 2.33 \times 10^{-18} \text{ sec}^{-1}, \quad q_0 = -0.55, \quad \rho_0 = 2.83 \times 10^{-27} \text{ Kg m}^{-3} \]
\[ \varphi_0 = \frac{1}{\varpi_0} = 1.498 \times 10^{10} \text{ Kg s}^2 \text{ m}^{-3}, \quad t_0 = 1.4 \times 10^{10} \text{ Years} = 4.42 \times 10^{17} \text{ s} \]

Using these values in equation (22) one gets,

\[ n_+ = -1.94 \quad \text{and} \quad n_- = -2.61 \]  \hspace{1cm} (23)

Since \( \varphi = \varphi_0 a^n \) was chosen empirically (and not obtained as a solution of the field equations) the parameter \( n \) can also take values other than the two values in equation (23). The important fact about these two values is that both of them are negative, indicating a decrease of \( \varphi \) and an increase of \( G(\equiv \frac{1}{\varpi}) \) with time. There are studies that show a decrease of \( \varphi \) with time [3, 15].

According to a study by Banerjee and Pavon [10], \( -3/2 < \omega_0 < 0 \).

Using equation (18), the ranges of \( n \) values, satisfying this requirement, are found to be,

\[ n < -2.061, \quad -0.838 < n < -0.455 \quad \text{and} \quad n > 1.905 \]  \hspace{1cm} (24)

The sign of \( n \) determines whether \( \varphi \) increases or decreases with time, as per equation (8) where the scale factor \( a(t) \) always increases with time in an expanding universe. The larger the value of \( |n| \), greater would be its rate of change with time. Using equation (8), the fractional rate of change of the gravitational constant, at the present epoch, is given by,

\[ \left( \frac{\dot{G}}{G} \right)_{t=t_0} = \left[ \frac{1}{1/\varpi} \frac{d}{dt} \left( \frac{1}{\varphi} \right) \right]_{t=t_0} = - \left( \frac{\varphi}{\varphi} \right)_{t=t_0} = - nH_0 \quad \text{or}, \quad n = - \frac{1}{H_0} \left( \frac{\dot{G}}{G} \right)_{t=t_0} \]  \hspace{1cm} (25)

Using equation (25), the values of \( n \) can be more reliably determined from the experimental findings of \( \left( \frac{\dot{G}}{G} \right)_{t=t_0} \). Its sign is found to be both positive and negative experimentally [17].

Combining equation (8) with (20), one obtains,

\[ \varphi = \frac{\rho(1-\gamma)}{H^2(2+n)(2+n-q)} \]  \hspace{1cm} (26A)

Using equation (26A), one obtains the following expression for \( \varphi_0 \) by writing all parameter values for \( t = t_0 \) and taking \( \gamma = 0 \) for the present matter dominated era of the universe.

\[ \varphi_0 = \frac{\rho_0}{H_0^2(2+n)(2+n-q_0)} \]  \hspace{1cm} (26B)

To express the dependence of \( \omega \) upon \( \gamma \), equations (26A, B) and (18) are substituted into equation (9), leading to the following equation.
\[
\omega = \frac{1}{n^2} \left[ 2 + 2q_0 + 2n - n^2 + nq_0 - \frac{\rho_0}{\varphi_0 H_0^2} \right] 
\times \exp \left[ m \left( \frac{-\rho(1-\gamma)}{H^2(2+n)(2+n-\gamma)} - \frac{\rho_0}{H^0_0^2(2+n)(2+n-q_0)} \right) \right]
\] (27)

To express the explicit time dependence of \( \omega \), substitutions have to be made from the equations (6), (7), (10), (11) into equation (27), leading to the following form.

\[
\omega = A \exp \left[ -m \rho \left( \frac{t-t_0}{\tau_0} \right)^\varepsilon \exp \left[ \frac{\mu(1-\varepsilon)}{(2+n)^2} \right] \right]
\] (28)

where \( A = \omega_0 \exp \left[ -m \rho_0 \right] \).

Here, \( \varepsilon, \mu, \omega_0, m \) and \( n \) are obtained from equations (12), (13), (18), (19) and (25) respectively.

Using the equations (6-9), one obtains the following expressions for different cosmological quantities of interest in terms of \( \omega \) and its derivatives.

\[
a = a_0 \left[ 1 + \frac{1}{m \varphi_0} \ln \left( \frac{\omega}{\omega_0} \right) \right]^{1/n}
\] (29)

\[
H = \frac{\dot{a}}{a} = \frac{1}{m \varphi_0} \left[ 1 + \frac{1}{m \varphi_0} \ln \left( \frac{\omega}{\omega_0} \right) \right]^{-1} \frac{\dot{\omega}}{\omega}
\] (30)

\[
q = -\frac{\ddot{a}}{a^2} = -1 + n \left[ 1 + \left( m \varphi_0 + \ln \left( \frac{\omega}{\omega_0} \right) \right) \left( 1 - \frac{\dot{\omega} \omega}{\dot{\omega}^2} \right) \right]
\] (31)

\[
\rho = \rho_0 a_0^{-3(1+\gamma)} \left[ 1 + \frac{1}{m \varphi_0} \ln \left( \frac{\omega}{\omega_0} \right) \right]^{-3(1+\gamma)}
\] (32)

\[
G = \frac{1}{\varphi} = \frac{1}{\varphi_0 + (1/m) \ln(\omega/\omega_0)}
\] (33)

In the above expressions (29-33), the parameters \( \omega_0 \) and \( m \) (eqns. 18, 19) are functions of the parameter \( n \) which controls the change of the scalar field (\( \varphi \)) with time.

Using equation (32), the density parameter for all matter (dark + baryonic) can be written as,

\[
\Omega_m = \frac{\rho}{\rho_c} = \frac{\rho_0}{\rho_c} a_0^{-3(1+\gamma)} \left[ 1 + \frac{1}{m \varphi_0} \ln \left( \frac{\omega}{\omega_0} \right) \right]^{-3(1+\gamma)}
\] (34)

Here, \( \rho_c \) is the critical density of matter-energy of the universe. \( \rho_c \equiv 10^{-26} Kg \; m^{-3} \)

The density parameter for dark energy is given by,

\[
\Omega_d = 1 - \Omega_m = 1 - \frac{\rho_0}{\rho_c} a_0^{-3(1+\gamma)} \left[ 1 + \frac{1}{m \varphi_0} \ln \left( \frac{\omega}{\omega_0} \right) \right]^{-3(1+\gamma)}
\] (35)

Equations (34) and (35) show the dependence of density parameters, of matter and dark energy respectively, upon the Brans-Dicke parameter (\( \omega \)).
The present era of the universe is matter dominated with negligible pressure \( (\gamma = 0) \). From the equations (6) and (7), we thus have,

\[
\rho(t) = \frac{\rho_0}{a(t)^3} \left( \frac{t}{t_0} \right)^{-3e} \text{Exp}[-3\mu(t - t_0)] \tag{36}
\]

For the present universe, with \( \gamma = P/\rho = 0 \), the expressions of \( \omega \), from the equations (15) and (28), are respectively obtained as the following two equations, (37) and (38).

\[
\omega = \frac{1}{f H^2} \left[ \frac{\rho}{\varphi} - 3H^2 (n^2 + 2n - nq) \right] \tag{37}
\]

\[
\omega = A \text{Exp} \left[ \frac{m\rho_0[a_0(t/t_0)^e \text{Exp}[\mu(t-t_0)]]^{-3}}{(\mu+\frac{e}{2})^2(2+n)(2+n-1+\frac{e}{e+\mu})} \right] \tag{38}
\]

with \( A = \omega_0 \text{Exp} \left[ \frac{-m\rho_0}{H_0^2(2+n)(2+n-q_0)} \right] \)

In equation (37), \( f(t) = 2(n^2 + 2n - nq) + mn^2\varphi \) and \( \rho(t) \) is given by equation (36).

Combining the equations (8) and (9) one gets,

\[
\omega = \omega_0 \text{Exp} \left[ m\varphi_0((a/a_0)^n - 1) \right] \tag{39}
\]

Using equations (7) and (18) in (39) one obtains,

\[
\omega = \frac{1}{n^2} \left[ 2 + 2q_0 + 2n - n^2 + nq_0 - \frac{\rho_0}{\varphi_0 H_0^2} \right] \times \text{Exp} \left[ m\varphi_0(t/t_0)^n \text{Exp}[n\mu(t-t_0)] - 1 \right] \tag{40}
\]

Equation (40) is an expression of \( \omega \) without any explicit dependence on \( \gamma \) and thus applicable for the entire span of cosmological time of the expanding universe. Here, equations (12), (13), (19) and (25) can be used to obtain the values of \( \varepsilon, \mu, m, n \) respectively.

**Results:**

Table-1 has a list of values of \( n, \omega_0 \) and \( m \) for different values of \( \left( \frac{\dot{a}}{a} \right)_{t=t_0} \), whose range has been chosen from experimental findings [17-20]. Both \( \omega_0 \) and \( m \) depend upon \( n \) (eqns 18, 19). The signs of the parameters, \( n \) and \( m \), determine whether \( \omega \) increases or decreases with time, as per equations (8) and (9). When they have the same sign, \( \omega \) increases with time if \( \omega_0 \) is positive. When they have opposite signs, \( \omega \) decreases with time if \( \omega_0 \) is positive. For these two cases, the reverse happens if \( \omega_0 \) is negative. Table-1 shows that for extremely low values of \( \left( \frac{\dot{a}}{a} \right)_{t=t_0} \), one gets very large values of \( |\omega_0| \). It clearly indicates that larger values of \( \omega \) causes smaller rate of change of the gravitational constant \( (G = 1/\varphi) \). Equation (15) shows a linear relation between \( \frac{d\omega}{dt} \) and \( \frac{d\varphi}{dt} \) and equation (6) suggests that \( \rho \) decreases faster for larger values of \( \gamma \). Thus, one concludes that \( \omega \) changes faster for greater values of \( \gamma \).
Conclusions:

In the present study we have determined the nature of dependence of the Brans-Dicke parameter upon the scalar field and also the dependence of the scalar field upon time. Therefore, this study has enabled us to determine the time dependence of BD parameter which plays a very important role in the accelerated expansion of the universe. The signs of the parameters $n$, $m$ and $\omega_0$ determine whether the BD parameter increases or decreases with time. Actually, both $\omega_0$ and $m$ are functions of $n$ (eqns 18, 19), which determines the time dependence of the gravitational constant (eqn. 25). According to a study by Banerjee and Pavon [10], we must have $-\frac{3}{2} < \omega_0 < 0$. The range of values for $n$, satisfying this requirement, have been determined. The values of $n$ can be best determined by the experimentally measured values of $\left(\frac{\dot{G}}{G}\right)_{t=t_0}$, reported by several researchers [17-20]. For a wide range of values of this quantity, we have determined the values of the parameters $n$, $m$ and $\omega_0$ and listed them in Table-1. In deriving the expression of $\omega_0$ (eqn. 18), equal weightages have been given to the equations (1) and (2). Instead of doing this, one may calculate the value of $\omega_0$ separately from the equations (1) and (2) and determine their weighted average with unequal weights assigned to these two values. For this purpose, a new parameter can be introduced to represent their relative weightage, allowing us to determine the value of $\omega_0$ correctly, making it more consistent with more advanced studies in this regard. Using the same set of empirical relations (eqns. 8 and 9) for the scalar field and the BD parameter, we have determined three expressions of $\omega(t)$, represented by the equations (15), (28) and (40), showing clearly its dependence upon time and the equation of state parameter ($\gamma$). The form of these expressions for the present matter dominated state of the universe, which is considered to be a pressureless dust with $\gamma = 0$, are shown by the equations (37) and (38). A limitation of this study is that, it is based upon empirical forms of the scale factor ($a$) and the scalar field ($\varphi$), which are not the solutions of the Brans-Dicke field equations. An improvement over this method can be made by choosing one of these parameters empirically and determining the other from the field equations. This is our plan for a completely new study to be carried out in future.

| $\left(\frac{\dot{G}}{G}\right)_{t=t_0}$ $(Yr^{-1})$ | $n$ | $\omega_0$ | $m$ |
|---------------------------------|------|-----------|------|
| -1.00E-09                       | 13.60935278 | -8.88784E-01 | 1.09005E-10 |
| -7.50E-10                       | 10.20701459 | -8.49636E-01 | 1.27703E-10 |
| -5.00E-10                       | 6.804676391 | -7.68226E-01 | 1.74769E-10 |
| -2.50E-10                       | 3.402338196 | -4.99082E-01 | 4.68038E-10 |
| -1.00E-10                       | 1.360935278 | 5.32569E-01 | -1.46195E-09 |
| -7.50E-11                       | 1.020701459 | 1.25104E+00 | -1.02528E-09 |
| -5.00E-11                       | 0.680467639 | 2.99939E+00 | -9.49143E-10 |
| -2.50E-11                       | 0.34023382  | 1.07358E+01 | -1.29075E-09 |
| -1.00E-11                       | 0.136093528 | 5.63670E+01 | -2.70303E-09 |
| -7.50E-12                       | 0.102070146 | 9.62505E+01 | -3.52075E-09 |
| -5.00E-12                       | 0.068046764 | 2.07159E+02 | -5.17152E-09 |
In this table, the symbol \( \mathbf{E} \pm \mathbf{y} \) denotes \( 10^{\pm y} \).

### References:

1. Brans C and Dicke RH. Mach's principle and a relativistic theory of gravitation. Phys. Rev. 1961; 124: 925-935.
2. Nordtvedt K. Post-Newtonian metric for a general class of scalar-tensor gravitational theories and observational quintessence. Astrophys. J. 1970; 161: 1059-1067.
3. Banerjee N and Ganguly K. Generalised Scalar-Tensor theory and cosmic acceleration. Int. J. Mod. Phys. D 2009; 18 (3): 445-451.
4. Will CM. Theory and Experiment in Gravitational Physics, Cambridge University Press 1993; Chapter-5: 123-126.
5. Benkestein JD and Meisels A. General relativity without general relativity. Phys. Rev. D 1978; 18: 4378-4386.
6. Alimi JM and Serna A. Scalar-tensor Cosmological Models. Phys. Rev. D 1996; 53: 3074-3087.
7. Mathiazhagan C and Johri VB. An Inflationary Universe in Brans-Dicke Theory: A Hopeful Sign of Theoretical Estimation of the Gravitational Constant. Classical and Quantum Gravity 1984; 1 (2): L29- L32.
9. Daile L, Steinhardt PJ and Bertschinger EW. Prescription for successful extended inflation. Physics Letters B 1989; 231 (3): 231-236.
10. Bertolami O. and Martins PJ. Nonminimal coupling and quintessence. Phys. Rev. D 2000; 61: 064007-064012.
11. Benerjee N and Pavon D. Cosmic acceleration without quintessence. Phys. Rev. D 2001; 63: 043504-043510.
12. Sen S and Seshadri TR. Self interacting Brans Dicke Cosmology and Quintessence. Int. J. Mod. Phys. D 2003; 12: 445-460.
13. Rama SK and Ghosh S. Short Distance Repulsive Gravity as a Consequence of Non Trivial PPN Parameters β and γ. Phys. Lett. B 1996; 383: 31-38.
14. Rama SK. Some Cosmological Consequences of Non Trivial PPN Parameters β and γ. Phys. Lett. B 1996; 373: 282-288.
15. Rama SK. Singularity Free (Homogeneous Isotropic) Universe in Graviton-Dilaton Models. Phys.Rev.Lett. 1997; 78: 1620-1623.
16. Sahoo BK and Singh LP. Time Dependence of Brans-Dicke Parameter ω for an Expanding Universe. Mod. Phys. Lett. A 2002; 17 (36): 2409-2415.
17. Jamil M and Momeni D. Evolution of the Brans-Dicke Parameter in Generalized Chameleon Cosmology. Chin. Phys. Lett. 2011; 28 (9): 099801, 1-4.
18. Ray S, Mukhopadhyay U and Dutta Choudhury SB. Dark energy models withm time-dependent gravitational constant. Int. J. Mod. Phys. D 2007; 16: 1791 - 1802.
19. Roy S and Islam M. A study of cosmic expansion generated by the non-conservation of matter in the framework of Brans-Dicke theory. International Journal of Physics and Mathematical Sciences 2016; 6 (2): 1-10.
20. Roy S. Time Evolution of Various Cosmological Parameters and Their Inter Dependence in the Framework of Brans-Dicke Theory. IOSR Journal of Mathematics 2016; 12 (3) Version VII: 27-35.
21. Roy S. A Theoretical Study of the Cosmic Expansion in the Framework of Brans-Dicke Theory. IOSR Journal of Applied Physics (IOSR-JAP) 2016; 8(3) Version III: 4-12.