Càdlàg semimartingale strategies for optimal trade execution in stochastic order book models

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Abstract
We analyse an optimal trade execution problem in a financial market with stochastic liquidity. To this end, we set up a limit order book model in continuous time. Both order book depth and resilience are allowed to evolve randomly in time. We allow trading in both directions and for càdlàg semimartingales as execution strategies. We derive a quadratic BSDE that under appropriate assumptions characterises minimal execution costs, and we identify conditions under which an optimal execution strategy exists. We also investigate qualitative aspects of optimal strategies such as e.g. appearance of strategies with infinite variation or existence of block trades, and we discuss connections with the discrete-time formulation of the problem. Our findings are illustrated in several examples.

Keywords Optimal trade execution · Continuous-time stochastic optimal control · Limit order book · Stochastic order book depth · Stochastic resilience · Quadratic BSDE · Infinite-variation execution strategy · Semimartingale execution strategy

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1 Introduction
Liquidity in financial markets is not constant but evolves randomly in time. To better understand the effects of stochastic liquidity on trade execution strategies, the
recent research articles Almgren [5], Schied [34], Ankirchner et al. [8], Cheridito and Sepin [15], Ankirchner and Kruse [9], Graewe et al. [20], Kruse and Popier [27], Horst et al. [23], Graewe and Horst [19], Bank and Voß [11], Graewe et al. [21], Horst and Xia [24], Popier and Zhou [32], Ankirchner et al. [7] extend the market impact models of Almgren and Chriss [6] and Bertsimas and Lo [13] by allowing stochastic liquidity parameters. In this article, we take a different route and introduce a variant of the limit order book models of Almgren and Chriss [6] and Bertsimas and Lo [13] by allowing stochastic

To this end, we fix a time horizon \( T \in (0, \infty) \) and consider a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)\) which satisfies the usual conditions and \( \mathcal{F} = \mathcal{F}_T \). Let \( M = (M_t)_{t \in [0,T]} \) be a continuous local martingale. The price impact process \( \gamma \) evolves according to the stochastic dynamics

\[
d\gamma_t = \gamma_t (\mu_t \, d[M]_t + \sigma_t \, dM_t), \quad t \in [0, T], \quad \gamma_0 > 0, \tag{1.1}
\]

where the coefficient processes \( \mu = (\mu_t)_{t \in [0,T]} \) and \( \sigma = (\sigma_t)_{t \in [0,T]} \) are progressively measurable and satisfy appropriate integrability assumptions.

Given an open position \( x \in \mathbb{R} \) of an asset at time \( t \in [0, T] \), an execution strategy is a càdlàg semimartingale \( X = (X_s)_{s \in [t,T]} \) satisfying \( X_{t-} = x \) and \( X_T = 0 \). A positive initial position \( x > 0 \) means that the trader has to sell an amount of \( |x| \) shares, whereas \( x < 0 \) requires to buy an amount of \( |x| \) shares, in both cases having the time interval \([t, T]\) at disposal for trading. The terminal condition \( X_T = 0 \) describes the liquidation constraint that at time \( T \), the position has to be closed. For every \( s \in [t, T] \), the quantity \( X_{s-} \) reflects the remaining position to be closed during \([s, T]\). A jump at time \( s \) (notation: \( \Delta X_s = X_s - X_{s-} \)) is interpreted as a block trade at \( s \). In particular, execution strategies will typically have a block trade at the beginning, i.e., \( X_t \) often differs from the initial position \( X_{t-} = x \). In line with the previous sentence, we always use the convention \( [X]_{t-} = 0 \) and \( [X]_t = (\Delta X)^2 \) concerning the quadratic variation \([X]\) of the process \( X = (X_s)_{s \in [t,T]} \) at the initial time, i.e., the quadratic variation process \([X]\) will typically jump at the initial time as well. We stress that the conventions about possible jumps at the initial time apply also for the initial time \( t = 0 \), i.e., \( X_0 \) can differ from \( X_{0-} = x \) and, consequently, \([X]_0 = (\Delta X_0)^2\) can differ from \([X]_{0-} = 0\).

Market illiquidity implies that trading according to a strategy \( X \) impacts the asset price and induces a price deviation. We thus assume that the actual price of a share is the sum of the unaffected price, which is the price of a share in the absence of trading, plus a deviation that depends on the execution strategy. In our analysis, we focus on the price deviation (see, however, Remark 2.2 for more detail) and model it by associating to each strategy \( X \) a deviation process \( D = (D_s)_{s \in [t,T]} \) which evolves according to

\[
dD_s = -\rho_s D_s \, d[M]_s + \gamma_s \, dX_s + d[\gamma, X]_s, \quad s \in [t, T], \quad D_{t-} = d, \tag{1.2}
\]

where the progressively measurable coefficient process \( \rho = (\rho_s)_{s \in [0,T]} \) is called the resilience process. Notice that a jump of \( X \) at the initial time \( t \) causes a jump of \( D \).
at time $t$. Consequently, $D_t$ can differ from the initial condition $D_{t-} = d$. Typically, the initial price deviation $d \in \mathbb{R}$ is assumed to be zero, but we allow arbitrary values in our formulation and analysis. Given an initial price deviation $d \in \mathbb{R}$, we denote by $\mathcal{A}_t(x, d)$ the set of all càdlàg semimartingale execution strategies for closing an initial position $x \in \mathbb{R}$ during $[t, T]$ which in addition satisfy suitable integrability conditions (see (2.3)–(2.5) below). The expected execution costs for a strategy $X \in \mathcal{A}_t(x, d)$ are given by

$$J_t(x, d, X) = E_t \left[ \int_{[t,T]} D_s - dX_s + \int_{[t,T]} \frac{\gamma_s}{2} [X]_s \right], \quad (1.3)$$

where $E_t[\cdot]$ is a shorthand notation for $E[\cdot|\mathcal{F}_t]$. We follow the convention that for integrals of the forms $\int_{[t,T]} \cdots dX_s$ and $\int_{[t,T]} \cdots d[X]_s$, jumps of the càdlàg integrators $X$ and $[X]$ at time $t$ contribute to the integrals. In contrast, we write $\int_{(t,T]} \cdots dX_s$ and $\int_{(t,T]} \cdots d[X]_s$ when we do not include the jumps at time $t$ into the integrals. In particular, a possible initial block trade at time $t$ contributes to both integrals in the cost functional $J$ in (1.3). Finally, let us mention that in the sequel, the integral $\int_t^T$ is understood as an integral over $(t, T]$. We mostly use this notation with continuous integrators, where the inclusion or not of the endpoints does not play a role.

In this article, we provide a purely probabilistic solution to the stochastic control problem of minimising $J$ over $\mathcal{A}_t(x, d)$. More precisely, under appropriate assumptions, we characterise the value process

$$V_t(x, d) = \text{ess inf}_{X \in \mathcal{A}_t(x, d)} J_t(x, d, X), \quad x, d \in \mathbb{R}, \ t \in [0, T], \quad (1.4)$$

of the control problem in terms of a quadratic backward stochastic differential equation (BSDE). Let $Y = (Y_s)_{s \in [0,T]}$ denote the first solution component of the BSDE (3.2) below; then we show in Theorem 3.4 below that the minimal expected costs amount to

$$V_t(x, d) = \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - d^2 \frac{2}{\gamma_t}.$$  

Moreover, we identify conditions under which an optimal strategy exists and give an explicit representation in terms of the solution of the BSDE.

Observe that the class of execution strategies $\mathcal{A}_t(x, d)$ over which we optimise in (1.4) is a subclass of all càdlàg semimartingales. In particular, the strategies are allowed to have infinite variation. Only few research articles on optimal trade execution so far analyse problems where strategies are not restricted to have finite variation. Infinite variation strategies are considered in Lorenz and Schied [28] in a setting related to ours in order to investigate how optimal execution strategies react to a possible drift in the unaffected price process. The work by Becherer et al. [12] explains how to go beyond finite variation strategies in another model for gains of a large investor. Strategies of infinite variation also appear in Carmona and Webster [14] and Gârleanu and Pedersen [18]. These articles study a generalisation of the self-financing equation and an infinite-horizon portfolio optimisation problem under market frictions, respectively. The recent work by Horst and Kivman [22] considers a model with instantaneous price impact and stochastic resilience from Graewe and Horst [19] and
Horst and Xia [24], and strategies of infinite variation emerge there in the limiting case of vanishing instantaneous price impact.

By allowing strategies of infinite variation, we encounter several interesting effects which we now discuss in more detail. In particular, this extension of the set of admissible controls requires an adjustment of conventional dynamics of the price deviation process and the cost functional as presented, e.g. in Alfonsi and Acevedo [2], Bank and Fruth [10], Fruth et al. [16], Fruth et al. [17], Obizhaeva and Wang [30], Predoiu et al. [33] (see also the references therein). In these papers, the trading is either constrained in one direction or the execution strategies $X$ are assumed to be of finite variation, which de facto translates into the dynamics

$$d\tilde{D}_s = -\rho_s \tilde{D}_s d[M]_s + \gamma_s dX_s$$

(1.5)

for the deviation process and into a cost functional of the form

$$\tilde{J}_t(x,d,X) = Et \left[ \int_{[t,T]} \left( \tilde{D}_s - \frac{\gamma s}{2} \Delta X_s \right) dX_s \right].$$

(1.6)

Tildes in (1.5) and (1.6) are to distinguish these from our setting. Let us recall that for two càdlàg semimartingales $K = (K_s)_{s \in [t,T]}$ and $L = (L_s)_{s \in [t,T]}$, it holds for all $s \in [t,T]$ that $[K, L]_s = \langle K^c, L^c \rangle_s + \sum_{u \in [t,s]} \Delta K_u \Delta L_u$ (see Jacod and Shiryaev [25, Theorem I.4.52]), where $K^c$ and $L^c$ denote the continuous martingale parts of $K$ and $L$ (see [25, Proposition I.4.27]). In particular, we have for all $s \in [t,T]$ that $[X]_s = \langle X^c \rangle_s + \sum_{u \in [t,s]} (\Delta X_u)^2$ and, as $\gamma$ is continuous, $[\gamma, X]_s = \langle \gamma, X^c \rangle_s$. Therefore, if in our setting, an execution strategy $X$ is monotonic or more generally of finite variation, then (1.2) reduces to (1.5), while (1.3) reduces to (1.6) (as $X^c = 0$ in that case). In general, i.e., when $X$ is a càdlàg semimartingale, we have additional terms in the dynamics (1.2) of the deviation process and in the cost functional (1.3) in comparison with the conventional setting of the problem. To explain these additional terms, we now make the following comments:

– Keeping the conventional dynamics (1.5) and the cost functional (1.6) can result in an ill-posed optimisation problem in our setting; see counterexamples in Sects. 5.1 and 4.1, respectively.

– Specifically, using the cost functional (1.6) for strategies $X$ of infinite variation can lead to arbitrarily big negative costs even with a constant (in time) deterministic price impact $\gamma$ (in which case (1.2) and (1.5) are the same); see Sect. 4.1. With the right cost functional (1.3), we recover a well-posed problem; see Sect. 4.2.

– Furthermore, even with the right cost functional (1.3), the dynamics (1.5) for the deviation process can lead to arbitrarily big negative costs; see Sect. 5.1. With the right dynamics (1.2), we again recover a well-posed problem; see Sect. 5.2.

– It is worth noting that the specific form of the corrections to (1.5) and (1.6) when allowing strategies $X$ to have infinite variation can be justified by a limiting procedure from the discrete-time situation; see Appendix A.

The preceding discussion raises the question of whether it is really so necessary to try to include strategies $X$ of infinite variation into the picture. The answer is affirmative: as we do not constrain trading in one direction and allow price impact and
resilience processes $\gamma$ and $\rho$ which are stochastic and can have infinite variation, it is quite natural to expect that strategies of infinite variation come out. The intuition is that an optimal strategy should react to changes in the exogenously given processes $\gamma$ and $\rho$, and if $\gamma$ (or $\rho$) has infinite variation, the optimal strategy should typically have infinite variation as well. But we actually discover a more surprising effect: even when the resilience $\rho$ is a deterministic constant and the price impact process $\gamma$ has $C^\infty$ paths (in particular, all exogenously given processes have finite variation), it can happen that the optimal strategy in our problem (1.4) has infinite variation. This and several other interesting qualitative effects are presented in Sect. 6. We moreover remark that we study a discrete-time version of the problem in Ackermann et al. [1], but that study is concentrated on different questions, as the mentioned effects, being purely continuous-time features, cannot be discussed in the framework of [1].

From another perspective, we should like to mention Carmona and Webster [14] who examine high-frequency trading in limit order books in general (not necessarily related with optimal trade execution). It is very interesting that one of their conclusions is some empirical evidence for the infinite-variation nature of trading strategies of high-frequency traders.

Finally, to complement the above discussion about the necessity of some adjustments in our setting, we discuss related literature from the viewpoint of the adjustments (1.5) $\rightarrow$ (1.2) and (1.6) $\rightarrow$ (1.3):

– Adjustments in the cost functional similar to (1.6) $\rightarrow$ (1.3) already appeared in related settings in Lorenz and Schied [28] and Gârleanu and Pedersen [18]. In both papers, the terms in the cost functional containing the quadratic variation $[X]$ are justified via limiting arguments from discrete time.

– In contrast, the adjustment (1.5) $\rightarrow$ (1.2) in the dynamics of the deviation process is to the best of our knowledge new. It does not emerge in the above papers because they consider constant $\gamma$, in which case $[\gamma, X] \equiv 0$. In order to see the need for the adjustment (1.5) $\rightarrow$ (1.2), it is necessary to consider the price impact itself (i.e., the process $\gamma$) to be of infinite variation.

The remainder of this article is organised as follows. In Sect. 2, we formally introduce the trade execution problem as a stochastic optimisation problem over semimartingales. Section 3 presents the main results. In Sect. 4 resp. Sect. 5, we explain why the cost functional in (1.6) must be adjusted to (1.3) resp. why the dynamics in (1.5) must be adjusted to (1.2) to obtain a well-posed optimisation problem when minimising over semimartingales. We present several examples and describe qualitative effects of optimal strategies in Sect. 6. Section 7 establishes existence results for the BSDE (3.2). Section 8 is devoted to the proofs of the results in Sect. 3. Appendices A and B additionally justify the form of the cost functional in (1.3), the dynamics in (1.2) and the BSDE (3.2) by deriving these objects as continuous-time limits from the corresponding discrete-time problem formulation. Appendix C contains a comparison argument for BSDEs, which is used in Sect. 7.

2 Problem formulation

Throughout, we consider a continuous local martingale $M = (M_t)_{t \in [0, T]}$ and denote by $\mathcal{D}_M$ the Doléans measure on $(\Omega \times [0, T], \mathcal{F} \otimes \mathcal{B}([0, T]))$ associated to $M$, i.e.,
The first two inputs \( \mu \) and \( \sigma \) define the price impact process \( \gamma = (\gamma_s)_{s \in [0,T]} \), which is a positive continuous adapted process satisfying (1.1), i.e.,

\[
\gamma_s = \gamma_0 \exp \left( \int_0^s \left( \mu_u - \frac{\sigma_u^2}{2} \right) d[M]_u + \int_0^s \sigma_u dM_u \right), \quad s \in [0,T],
\]

where \( \gamma_0 > 0 \) is an \( \mathcal{F}_0 \)-measurable random variable. It turns out to be useful also to introduce the process \( \alpha_s = \frac{1}{\gamma_s}, \quad s \in [0,T] \), which then satisfies

\[
d\alpha_s = \alpha_s \left( - (\mu_s - \sigma_s^2) d[M]_s - \sigma_s dM_s \right), \quad s \in [0,T]. \tag{2.1}
\]

The third input \( \rho = (\rho_s)_{s \in [0,T]} \) is called the resilience process and is together with the price impact process \( \gamma \) involved in the following dynamics.

Given \( x \in \mathbb{R} \) and \( t \in [0,T] \), for any execution strategy \( X = (X_s)_{s \in [t,T]} \) with initial position \( X_t^- = x \) (cf. the introduction) and any \( d \in \mathbb{R} \), we define the deviation process \( D = (D_s)_{s \in [t,T]} \) associated to \( X \) as the unique solution of (1.2), i.e., for all \( s \in [t,T] \), it holds that

\[
D_s = e^{-\int_t^s \rho_u d[M]_u} \left( d + \int_{[t,s]} e^{\int_t^r \rho_u d[M]_u} \gamma_r dX_r + \int_{[t,s]} e^{\int_t^r \rho_u d[M]_u} d[\gamma, X]_r \right), \quad D_{t^-} = d. \tag{2.2}
\]

For each \( t \in [0,T] \), we formulate the conditions

\[
E_t \left[ \sup_{s \in [t,T]} \left( \gamma^2_s (X_s - \alpha_s D_s)^4 \right) \right] < \infty \quad \text{a.s.,} \tag{2.3}
\]

\[
E_t \left[ \left( \int_t^T \gamma^2_s (X_s - \alpha_s D_s)^4 \sigma_s^2 d[M]_s \right)^{\frac{1}{2}} \right] < \infty \quad \text{a.s.,} \tag{2.4}
\]

\[
E_t \left[ \left( \int_t^T D_s^4 \alpha_s^2 \sigma_s^2 d[M]_s \right)^{\frac{1}{2}} \right] < \infty \quad \text{a.s.,} \tag{2.5}
\]

where \( E_t[\cdot] \) is shorthand for \( E[\cdot | \mathcal{F}_t] \). Note that if \( E_t[\int_t^T \sigma^2 d[M]_s] < \infty \) a.s., then by the Cauchy–Schwarz inequality, (2.4) follows from (2.3).

For \( x, d \in \mathbb{R} \) and \( t \in [0,T] \), let \( \mathcal{A}_t(x,d) \) be the set of all càdlàg semimartingales \( X = (X_s)_{s \in [t,T]} \) with \( X_{t^-} = x, \ X_T = 0 \) (i.e., execution strategies) satisfying conditions (2.3)–(2.5).

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For every \( x, d \in \mathbb{R} \), \( t \in [0, T] \) and \( X \in \mathcal{A}_t(x, d) \), we define the cost functional \( J \) by (1.3). Conditions under which \( J \) is always well defined are provided in Theorem 3.1 below. The control problem considered in this paper is to minimise the cost functional \( J \) over the execution strategies \( X \in \mathcal{A}_t(x, d) \). The value process \( V \) of the control problem is given by (1.4). If for \( x, d \in \mathbb{R} \), \( t \in [0, T] \), there exists an execution strategy \( X^* = (X^*_s)_{s \in [t, T]} \in \mathcal{A}_t(x, d) \) such that \( V_t(x, d) = J_t(x, d, X^*) \), we call this process \( X^* \) an optimal execution strategy.

**Remark 2.1**

(i) An important special case of our setting is the situation where the continuous local martingale \( M \) is an \( \mathcal{F}_t \)-Brownian motion \( W \), in which case we have \( d[M]_s = ds \) and \( D_W = P \times \text{Leb} \) (Leb denotes the Lebesgue measure).

(ii) More generally, let \( M \) be a continuous local martingale with \( ds \ll d[M]_s \) a.s.

Consider the situation where the dynamics of the price impact process \( \gamma \) is given by \( d\gamma_s = \gamma_s(\mu_s ds + \sigma_s dM_s) \), \( s \in [0, T] \), and the dynamics of the deviation process \( D \) is specified by \( dD_s = -\rho_s D_s - ds + \gamma_s dX_s + d[\gamma, X]_s \), \( s \in [t, T] \), \( D_{t-} = d \)

for \( x, d \in \mathbb{R} \), \( t \in [0, T] \) and \( X \in \mathcal{A}_t(x, d) \). Then the identifications \( \tilde{\mu} \lambda = \mu \) and \( \tilde{\rho} \lambda = \rho \) with \( \lambda_s = ds/d[M]_s \), \( s \in [0, T] \), reduce this situation to our formulation (cf. (1.1) and (1.2)).

**Remark 2.2**

In the problem setting introduced above, we focus on the price deviation only. However, the considerations above also allow to explicitly include an unaffected price into the picture. To this end, assume that the unaffected price is modelled via a càdlàg local martingale \( \mu \). Consider the situation where the dynamics of the price impact process \( \gamma \) is given by \( d\gamma_s = \gamma_s(\tilde{\mu}_s ds + \tilde{\sigma}_s d\mu_s) \), \( s \in [0, T] \), and the dynamics of the deviation process \( D \) is specified by \( dD_s = -\tilde{\rho}_s D_s - ds + \gamma_s dX_s + d[\gamma, X]_s \), \( s \in [t, T] \), \( D_{t-} = d \)

for \( x, d \in \mathbb{R} \), \( t \in [0, T] \) and \( X \in \mathcal{A}_t(x, d) \). Then the identifications \( \tilde{\mu} \lambda = \mu \) and \( \tilde{\rho} \lambda = \rho \) with \( \lambda_s = ds/d[\mu]_s \), \( s \in [0, T] \), reduce this situation to our formulation (cf. (1.1) and (1.2)).
and \( X_T = 0 \), we obtain that the expression in (2.7) equals

\[
X_T S^0_T - X_t - S^0_t - \int_{[t,T]} X_u - dS^0_u = -x S^0_t - \int_{[t,T]} X_u - dS^0_u
\]

\[
= -x S^0_t - \int_{(t,T]} X_u - dS^0_u. \tag{2.8}
\]

Note that for \( t = 0 \), the term \( S^0_0 \) appearing in (2.8) can be an arbitrary \( \mathcal{F}_0 \)-measurable random variable (i.e., we can allow a jump at time 0 in \( S^0 \) like we allow initial jumps in \( X \)). Interestingly, we do not need specific relation between \( S^0_0 \) and \( S^0_0 - \) as the term containing \( S^0_t \) in (2.8) is ultimately reduced with the initial jump in the stochastic integral. It follows from the Burkholder–Davis–Gundy inequality that

\[
E_t \left[ \sup_{u \in [t,T]} (X^*_u - S^0_u)^2 \right] < \infty \text{ a.s.} \tag{2.9}
\]

is satisfied. In particular, under (2.9), the expected (at time \( t \)) costs due to the unaffected price are equal to \(-x S^0_t\) and thus do not depend on the execution strategy; hence the minimisation of the expected (at time \( t \)) total costs in (2.6) reduces to the minimisation of \( J_t(x, d, X) \).

We summarise the discussion as follows. Our goal is to minimise \( J_t(x, d, X) \) over strategies \( X \) satisfying (2.3)–(2.5) (i.e., over \( X \in \mathcal{A}_t(x, d) \)). Given a local martingale unaffected price \( S^0 \), a pertinent optimisation problem is to minimise \( J_t(x, d, X) \) over strategies \( X \) satisfying (2.3)–(2.5) as well as (2.9). Given an optimal strategy \( X^* \in \mathcal{A}_t(x, d) \), one thus needs additionally to examine whether \( X^* \) satisfies (2.9). Observe that (2.9) need not be satisfied automatically as it depends on the additional input \( S^0 \), which may have nothing to do with our other inputs (the processes \( M, \rho, \mu \) and \( \sigma \)). However, it is worth noting that if \( S^0 \) is a square-integrable martingale, then under the assumptions of Theorem 3.4 below, the optimal strategy \( X^* \in \mathcal{A}_t(x, d) \) provided in (3.10) satisfies (2.9). Indeed, we shall see in the proof that under the assumptions of Theorem 3.4, for \( X^* \) of (3.10), it holds that

\[
E_t \left[ \sup_{u \in [t,T]} (X^*_u - S^0_u)^2 \right] < \infty \text{ a.s.}
\]

As \( S^0 \) is a square-integrable martingale, we have \( E[|S^0_T|] < \infty \) and therefore also \( E_t[|S^0_T - [S^0]_t|] < \infty \) a.s. Condition (2.9) for \( X^* \) of (3.10) now follows from the Cauchy–Schwarz inequality.

### 3 Main results

We now present the main results, which include an alternative representation of the cost functional, a representation of the value process in terms of solutions to a certain BSDE, a characterisation of existence of an optimal strategy, and an explicit expression for the optimal strategy (when it exists). The proofs are deferred to Sect. 8.

We often make use of the assumption

\[
2\rho + \mu - \sigma^2 > 0 \quad \mathcal{D}_M \text{-a.e.} \tag{3.1}
\]

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If (3.1) is satisfied, we introduce the BSDE
\[
Y_t = \frac{1}{2} + \int_t^T f(s, Y_s, Z_s) d[M]_s
- \int_t^T Z_s dM_s - (M^\perp_T - M^\perp_t), \quad t \in [0, T],
\]
with the driver
\[
f(s, Y_s, Z_s) = -\frac{(\rho_s + \mu_s)Y_s + \sigma_s Z_s}{\sigma^2_s Y_s + \frac{1}{2}(2\rho_s + \mu_s - \sigma^2_s)} + \mu_s Y_s + \sigma_s Z_s
\]
and terminal condition \(\frac{1}{2}\). A solution to the BSDE (3.2) is a triple \((Y, Z, M^\perp)\) of processes where
- \(M^\perp\) is a càdlàg local martingale with \(M^\perp_0 = 0\) and \([M^\perp, M] = 0\),
- \(Z\) is a progressively measurable process such that \(\int_0^T Z_s^2 d[M]_s < \infty\) a.s.,
- \(Y\) is an adapted càdlàg process,

such that (3.2) is satisfied a.s. Notice that \(Y\) is necessarily a special semimartingale (see Jacod and Shiryaev [25, Sect. I.4c]). We now introduce a further assumption:

There exists a solution \((Y, Z, M^\perp)\) to the BSDE (3.2) such that
\[
Y \text{ is } [0, 1/2]\text{-valued, } E\left[[M^\perp]_T\right] < \infty, \text{ and } E\left[\int_0^T Z_s^2 d[M]_s\right] < \infty.
\]

To explain the role of condition (3.1) for the BSDE (3.2) and in (3.4), it is worth noting that under (3.1), the denominator in the first term in (3.3) stays strictly positive whenever \(Y\) is nonnegative. Furthermore, we make the following comments:

- In many situations below (Proposition 3.7, Sects. 4.2, 5.2 and 6), (3.4) is satisfied.
- Two broad subsettings of our general setting which satisfy (3.4) are described in Sect. 7.
- In our general setting, (3.4) is motivated by the discrete-time version of the stochastic control problem (1.4) (see Appendix B).

In Remark 3.5 below, we present an interpretation of the solution component \(Y\) of (3.2) as a saving factor describing the benefits of using an optimal execution strategy compared to an immediate position closure.

If (3.1) and (3.4) hold, we define the process \(\tilde{\beta} = (\tilde{\beta}_s)_{s \in [0, T]}\) pertaining to \((Y, Z)\) from (3.4) by
\[
\tilde{\beta}_s = \frac{(\rho_s + \mu_s)Y_s + \sigma_s Z_s}{\sigma^2_s Y_s + \frac{1}{2}(2\rho_s + \mu_s - \sigma^2_s)}, \quad s \in [0, T].
\]
Theorem 3.1 Let (3.1) and (3.4) be satisfied. For all \( x, d \in \mathbb{R} \), \( t \in [0, T) \) and \( X \in \mathcal{A}_t(x,d) \), the cost functional (1.3) is well defined and admits the a.s. representation

\[
J_t(x,d,X) = Y_t \frac{(d - \gamma_t x)^2}{\gamma_t} - \frac{d^2}{2\gamma_t} + E_t \left[ \int_t^T \frac{1}{\gamma_s} \left( \tilde{\beta}_s (\gamma_s X_s - D_s) + D_s \right)^2 \right. \\
\left. \times \left( \sigma^2_s Y_s + \frac{1}{2} (2\rho_s + \mu_s - \sigma^2_s) \right) d[M]_s \right]. \tag{3.6}
\]

In particular, for all \( x, d \in \mathbb{R} \) and \( t \in [0, T] \), it holds that

\[
V_t(x,d) \geq Y_t \frac{(d - \gamma_t x)^2}{\gamma_t} - \frac{d^2}{2\gamma_t} \quad \text{a.s.} \tag{3.7}
\]

Remark 3.2 Note that (3.4) only postulates existence, but not uniqueness of a solution triple \((Y,Z,M)\) of the BSDE (3.2). By assuming (3.4), we always mean that we fix some solution triple \((Y,Z,M)\) of (3.2) that satisfies the properties in (3.4), and we use this solution in all subsequent statements. In particular, \( \tilde{\beta} \) is then understood as the process defined by (3.5) using \( (Y,Z) \) from this solution. Observe that (3.6) and (3.7) hold for any such solution \((Y,Z,M)\). Using the first part of Theorem 3.4, we provide in Proposition 3.8 a uniqueness result for the BSDE (3.2).

We proceed with describing the solution to our optimisation problem (1.4). The case \( x = \frac{d}{\gamma_t} \) is easy, and we treat it first.

Lemma 3.3 Let (3.1) and (3.4) be satisfied. Fix \( t \in [0, T] \) and \( x, d \in \mathbb{R} \) with \( x = \frac{d}{\gamma_t} \).

Then the value process is \( V_t(x,d) = -\frac{d^2}{2\gamma_t} \), and the strategy \( X^* = (X^*_s)_{s \in [t,T]} \) defined by \( X^*_t = x \), \( X^*_s = 0 \), \( s \in [t,T] \), which closes the position immediately is optimal in \( \mathcal{A}_t(x,d) \). Moreover, this optimal strategy is unique up to \( D[M]|_{[t,T]} \)-nullsets.

To describe the solution beyond the case \( x = \frac{d}{\gamma_t} \), we introduce the condition

\[
E \left[ \exp(c \cdot [M]_T) \right] < \infty \quad \text{for all } c \in (0, \infty). \tag{3.8}
\]

Note that if \( M = W \) is an \((\mathcal{F}_s)\)-Brownian motion, then (3.8) is trivially satisfied.

For a continuous semimartingale \( Q = (Q_s)_{s \in [0,T]} \), we denote its stochastic exponential by \( \mathcal{E}(Q) = (\mathcal{E}(Q)_s)_{s \in [0,T]} \), i.e., \( \mathcal{E}(Q)_s = \exp(Q_s - Q_0 - \frac{1}{2}[Q]_s), s \in [0, T] \). For \( t \leq s \) in \([0, T]\), we also use the notation

\[
\mathcal{E}(Q)_{t,s} = \frac{\mathcal{E}(Q)_s}{\mathcal{E}(Q)_t} = \exp \left( Q_s - Q_t - \frac{1}{2}([Q]_s - [Q]_t) \right).
\]

Theorem 3.4 Let (3.1), (3.4), and (3.8) be satisfied. Suppose furthermore that \( \rho, \mu \) and \( \tilde{\beta} \) (defined by (3.5)) are \( D[M] \)-a.e. bounded.
(i) For all \(x, d \in \mathbb{R}\) and \(t \in [0, T]\), it holds that
\[
V_t(x, d) = \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} \quad a.s.
\]

(ii) Let \(x, d \in \mathbb{R}\) and assume \(x \neq \frac{d}{\gamma_t}\). Then there exists an optimal strategy \(X^* = (X^*_s)_{s \in [0,T]} \in A_t(x, d)\) if and only if there exists a càdlàg semimartingale \(\beta = (\beta_s)_{s \in [0,T]}\) with \(\beta = \beta\ D_M\text{-a.e.}\). In this case, the optimal strategy is unique up to \(D_M\)-nullsets.

(iii) Suppose there exists a càdlàg semimartingale \(\beta = (\beta_s)_{s \in [0,T]}\) with \(\tilde{\beta} = \beta\ D_M\text{-a.e.}\). Define
\[
Q_s = -\int_0^s \beta_r \sigma_r dM_r - \int_0^s \beta_r (\mu_r + \rho_r - \sigma_r^2) d[M]_r, \quad s \in [0, T].
\]
Let \(x, d \in \mathbb{R}\) and \(t \in [0, T]\). Then the optimal strategy \((X^*_s)_{s \in [t,T]} \in A_t(x, d)\) and the associated deviation process \((D^*_s)_{s \in [t,T]}\) (both unique up to \(D_M\)\([t,T]\)-nullsets) are given by the formulas
\[
X^*_s = \left( x - \frac{d}{\gamma_t} \right) \mathcal{E}(Q)_{t,s} (1 - \beta_s), \quad s \in [t, T),
\]
\[
D^*_s = \left( x - \frac{d}{\gamma_t} \right) \mathcal{E}(Q)_{t,s} (-\gamma_s \beta_s), \quad s \in [t, T),
\]
and \(X^*_T = 0, D^*_T = \left( x - \frac{d}{\gamma_T} \right) \mathcal{E}(Q)_{t,T} (-\gamma_T)\).

**Remark 3.5** (a) Let us give an economic interpretation of the process \(Y\). Given a unit open position \(x = 1\) and a deviation \(d = 0\) at time \(t \in [0, T]\), a possible strategy is to close the position immediately at time \(t\) and keep it closed in the remaining period (i.e., \(X_s = 0\) for all \(s \in [t, T]\)). The cost associated to this strategy is given by \(J_t(1, 0, X) = \gamma_t/2\). Theorem 3.4 shows that under appropriate assumptions, the minimal costs in this situation are given by \(V_t(1, 0) = \gamma_t Y_t\). Therefore, we obtain for all \(t \in [0, T]\) that \(2Y_t = V_t(1, 0)/J_t(1, 0, X)\), and thus the random variable \(2Y_t: \Omega \to [0, 1]\) describes to which percentage the costs of selling one unit immediately at time \(t\) can be reduced by executing the position optimally.

(b) Let us discuss the boundedness assumptions in Theorem 3.4. In addition to (3.1) and (3.4), which are already assumed in Theorem 3.1, we need (3.8) and the boundedness of \(\rho, \mu\) (hence, due to (3.1), also \(\sigma^2\)) and \(\tilde{\beta}\). These are strong sufficient conditions for the validity of Theorem 3.4 and can be replaced by appropriate integrability assumptions. For instance, a straightforward generalisation of part (iii) obtained along the lines of our proofs (cf. Remarks 8.5 and 8.8 below) is as follows. Assume

1) (3.1), (3.4),
2) the existence of a càdlàg semimartingale \(\beta = (\beta_s)_{s \in [0,T]}\) with \(\tilde{\beta} = \beta\ D_M\text{-a.e.}\),
3) \(E[\int_0^T \beta_s^4 \sigma^2_s d[M]_s] < \infty\),
4) \( E[\exp(c \int_0^T \chi_s d[M]_s)] < \infty \) for all \( c > 0 \), where \( \chi = (\chi_s)_{s \in [0, T]} \) is defined by
\[
\chi = \sigma^2 + \beta^2 \sigma^2 + (\mu - 2\beta \mu - 2\beta \rho)^+.
\]
Then the value process is given by part (i) of Theorem 3.4, and the unique optimal strategy and the associated deviation process are given by part (iii) of Theorem 3.4. Notice that the integrability conditions 3) and 4) above are satisfied whenever we assume (3.8) and the boundedness of \( \rho, \mu \) and \( \tilde{\beta} \).

It is also possible to get the remaining results of Theorem 3.4 (namely, part (ii) and, if condition 2) above is not satisfied, part (i)) under certain integrability-type conditions instead of the boundedness-type ones. However, these conditions look more cumbersome than 3) and 4) above. Given that in all examples (with some novel effects) discussed below, the boundedness assumptions are satisfied, we formulate Theorem 3.4 under the boundedness assumptions and refrain from further technical discussions of this point.

We now observe that the optimal strategy and the optimal deviation process are dynamically consistent.

**Corollary 3.6** Under the assumptions of Theorem 3.4, suppose there exists a càdlàg semimartingale \( \beta = (\beta_s)_{s \in [0, T]} \) with \( \tilde{\beta} = \beta \) \( \mathcal{D}_M \)-a.e. Define the process \( Q \) as in (3.9). Fix \( x, d \in \mathbb{R} \) and \( t \in [0, T] \). Then for the optimal strategy and deviation process given in (3.10) and (3.11) and for any \( u \in (t, T) \), we have

\[
X^*_s = \left( X^*_u - \frac{D^*_u - \gamma u}{\gamma u} \right) \mathcal{E}(Q)_{u,s} (1 - \beta_s), \quad s \in [u, T),
\]

\[
D^*_s = \left( X^*_u - \frac{D^*_u - \gamma u}{\gamma u} \right) \mathcal{E}(Q)_{u,s} (-\gamma_s \beta_s), \quad s \in [u, T),
\]

and \( X^*_T = 0 \), \( D^*_T = \left( X^*_u - \frac{D^*_u - \gamma u}{\gamma u} \right) \mathcal{E}(Q)_{u,T} (-\gamma_T) \).

In the case of vanishing resilience, it turns out that it is always optimal to close the position immediately and then stay inactive. We formally state this as follows.

**Proposition 3.7** Assume (3.1) and \( \rho \equiv 0 \). Then for all \( t \in [0, T] \) and \( x, d \in \mathbb{R} \), we have \( V_t(x, d) = -x(d - \frac{\gamma}{2} x) \) and there exists a unique (up to \( \mathcal{D}_M|_{[t, T]} \)-nullsets) optimal strategy \( (X^*_s)_{s \in [t, T]} \in \mathcal{A}_t(x, d) \), given by \( X^*_s = x, \ X^*_s = 0, \ s \in [t, T] \).

It is worth noting that we need not assume (3.4) and (3.8) in Proposition 3.7. We shall see, however, that (3.4) is always satisfied in this case.

We close the section with a uniqueness result for the BSDE (3.2).

**Proposition 3.8** Assume (3.1), (3.8) and that the processes \( \rho, \mu \) are \( \mathcal{D}_M \)-a.e. bounded. Let \( (Y^{(i)}, Z^{(i)}, M_{\perp}^{(i)}) \), \( i = 1, 2 \), be solutions of the BSDE (3.2) that both satisfy (3.4) and such that the corresponding processes \( \tilde{\beta}^{(i)} = (\tilde{\beta}^{(i)}_s)_{s \in [0, T]} \) defined by (3.5) are \( \mathcal{D}_M \)-a.e. bounded. Then:

- \( Y^{(1)} \) and \( Y^{(2)} \) are indistinguishable,
- \( Z^{(1)} = Z^{(2)} \) \( \mathcal{D}_M \)-a.e.,
- \( M_{\perp}^{(1)} \) and \( M_{\perp}^{(2)} \) are indistinguishable.
4 Reason for adjusting the cost functional

4.1 Counterexample

In this subsection, we show that minimising the cost functional
\[ \widetilde{J}_t(x, d, X) = \mathbb{E}_t \left[ \int_{[t, T]} \left( D_s - \frac{\gamma_s}{2} \Delta X_s \right) dX_s \right] \] (4.1)
for \( x, d \in \mathbb{R}, t \in [0, T] \) over \( X \in \mathcal{A}_t(x, d) \) (cf. (1.6)) might constitute an ill-posed problem. More precisely, we construct an example where
\[ \text{ess inf}_{X \in \mathcal{A}_t(x, d)} \widetilde{J}_t(x, d, X) = -\infty \quad \text{a.s.} \]

Consider the setting where \( M = W \) is an \((\mathcal{F}_s)\)-Brownian motion and the price impact \( \gamma > 0 \) and the resilience \( \rho > 0 \) are positive deterministic constants (that is, \( \mu = \sigma \equiv 0 \) in terms of our model parameters). We consider the starting time \( t = 0 \) and assume (for simplicity) that the \( \sigma \)-field \( \mathcal{F}_0 \) is trivial. As \( \gamma \) is constant, for all \( X \in \mathcal{A}_0(x, d) \), the associated deviation process \( D \) satisfies
\[ dD_s = -\rho D_s ds + \gamma_s dX_s + d[\gamma_s, X]_s, \quad s \in [0, T). \]
In particular, in this setting the dynamics of \( D \) is of type (1.5) (cf. [17, Eqn. (2)]).

Fix the initial position \( x = 0 \) and the initial deviation \( d = 0 \). For \( \nu \in \mathbb{R} \), consider the execution strategy \( X^{(\nu)} = (X^{(\nu)}_s)_{s \in [0, T]} \) defined by \( X^{(\nu)}_0 = X^{(\nu)}_T = 0, X^{(\nu)}_s = \nu W_s \) for \( s \in [0, T) \) and \( X^{(\nu)}_T = 0 \), i.e., the strategy follows a scaled Brownian motion on \([0, T)\) and has a block trade at time \( T \). For each \( \nu \in \mathbb{R} \), let \( D^{(\nu)} = (D^{(\nu)}_s)_{s \in [0, T]} \) be the deviation process associated to \( X^{(\nu)} \), i.e.,
\[ dD^{(\nu)}_s = -\rho D^{(\nu)}_s ds + \gamma \nu dW_s, \quad s \in [0, T), \]
and \( D^{(\nu)}_0 = 0 \). Note that \( D^{(\nu)} \) is an Ornstein–Uhlenbeck process. One can therefore show that (2.3) is satisfied, and due to \( \sigma \equiv 0 \), (2.4) and (2.5) are satisfied as well, so that \( X^{(\nu)} \in \mathcal{A}_0(0, 0) \) for all \( \nu \in \mathbb{R} \). Since \( \int_0^T D^{(\nu)}_s dW_s \) is a martingale and \( X^{(\nu)} \) is continuous on \((0, T)\) and \( \Delta X^{(\nu)}_T = -X^{(\nu)}_T \), it holds for all \( \nu \in \mathbb{R} \) that
\[ \mathbb{E}_0 \left[ X^{(\nu)}_s D^{(\nu)}_s \right] = -\rho \mathbb{E}_0 \left[ X^{(\nu)}_u D^{(\nu)}_u \right] + \frac{\gamma}{2} (\Delta X^{(\nu)}_T)^2. \]
We have that
\[ d(D^{(\nu)}(X^{(\nu)}))_s = \nu D^{(\nu)}_s dW_s - \rho X^{(\nu)}_s D^{(\nu)}_s ds + \gamma \nu^2 W_s dW_s + \gamma \nu^2 ds, \quad s \in [0, T), \]
and hence
\[ \mathbb{E}[X^{(\nu)}_s D^{(\nu)}_s] = -\rho \int_0^s \mathbb{E}[X^{(\nu)}_u D^{(\nu)}_u] du + \gamma \nu^2 s, \quad s \in [0, T). \]
It follows that
\[ E[X_s^{(v)} D_s^{(v)}] = \frac{\nu v^2}{\rho} (1 - e^{-\rho s}), \quad s \in [0, T). \]

Therefore, we obtain for all \( v \in \mathbb{R} \) that
\[ \tilde{J}_0(0, 0, X^{(v)}) = -\frac{\nu v^2}{\rho} (1 - e^{-\rho T}) + \frac{\nu v^2}{\rho} \left( e^{-\rho T} - 1 + \frac{\rho T}{2} \right). \]

Now we see that if \( \rho > 0 \) is chosen such that \( e^{-\rho T} - 1 + \frac{\rho T}{2} < 0 \) (it is enough to take \( \rho \in (0, 1/T) \)), then
\[ \tilde{J}_0(0, 0, X^{(v)}) \to -\infty \quad \text{as } |v| \to \infty. \]

Thus the cost functional \( \tilde{J} \) leads to an ill-posed optimisation problem.

4.2 Solution in our framework

In the setting of Sect. 4.1, we recover a well-posed optimisation problem when we use the cost functional (1.3). Let us verify that Theorem 3.4 applies and present an explicit formula for the optimal strategy in \( A_t(x, d) \) for any \( t \in [0, T], x, d \in \mathbb{R}. \)

In this setting, (3.1) and (3.8) are trivially satisfied. The BSDE (3.2) takes the form
\[ dY_s = \rho Y_s^2 ds + Z_s dW_s + dM_s^\perp, \quad s \in [0, T], \quad Y_T = \frac{1}{2}. \]

The solution of (4.2) is
\[ Z \equiv 0, \quad M^\perp \equiv 0, \quad Y_s = \frac{1}{2 + (T - s)\rho}, \quad s \in [0, T]. \]

Observe that \( \tilde{\beta} = Y \) (see (3.5) for the definition of \( \tilde{\beta} \)) in this setting and that \( Y \) is deterministic, increasing, continuous and \((0, 1/2]\)-valued. In particular, \( \tilde{\beta} \) is bounded and a semimartingale. Hence Theorem 3.4 applies, and there exists a unique optimal strategy \( X^* = (X^*_s)_{s \in [t, T]} \in A_t(x, d) \) given by the formulas
\[ X^*_{t-} = x, \]
\[ X^*_T = 0, \]
\[ X^*_s = \left( x - \frac{d}{\gamma} \right) \exp \left( -\int_t^s \frac{\rho}{2 + (T - r)\rho} dr \right) \frac{1 + (T - s)\rho}{2 + (T - s)\rho} \]
\[ = \left( x - \frac{d}{\gamma} \right) \frac{1 + (T - s)\rho}{2 + (T - t)\rho}, \quad s \in [t, T). \]

In the context of optimal trade execution in a limit order book model, this setting is considered in the pioneering work by Obizhaeva and Wang [30], and the optimal strategy \( X^* \) of (4.3) (for \( d = 0 \)) appears in [30, Proposition 3], where the cost functional \( \tilde{J} \) of (4.1) is minimised over strategies of finite variation. We emphasise that we
obtain optimality of (4.3) in this setting as a result of a different optimisation problem
(minimisation of the cost functional $J$ of (1.3) over càdlàg semimartingales).

Notice that the optimal strategy $X^*$ of (4.3) is deterministic, has jumps at times $t$ and $T$ (i.e., block trades in the beginning and in the end) and is continuous on $(t, T)$. It is worth noting that the associated deviation process $D^*$ is constant on $(t, T)$ (but clearly has jumps at times $t$ and $T$). For $d = 0$, the strategy $X^*$ is monotonic. In general, the strategy is monotonic only on $(t, T)$.

In this subsection, we illustrate that the covariation term $[\gamma, X]$ in the definition of
the deviation process $D$ (see (1.2)) can be necessary to obtain a well-posed optimisation problem. We construct an example where $\text{ess inf}_{X \in A_t(x, d)} J_t(x, d, X) = -\infty$ a.s. when the deviation process $D$ associated to $X \in A_t(x, d)$ follows the dynamics

$$dD_t = -\rho_t D_t d[M]_t + \gamma_t dX_t, \quad t \in [t, T],$$

(cf. (1.5) above or Fruth et al. [17, Eqn. (2)]).

Consider the setting where $M = W$ is an $(\mathcal{F}_t)$-Brownian motion, $\mu \equiv 0$, $\sigma > 0$ and $\rho > 0$ are positive deterministic constants with $2\rho - \sigma^2 > 0$, i.e., (3.1) holds. In particular, in our current setting, the price impact process $\gamma$ is a geometric Brownian motion $\gamma_s = \gamma_0 \exp(\sigma W_s - \frac{\sigma^2}{2} s)$, $s \in [0, T]$. We consider the starting time $t = 0$ and assume (for simplicity) that the $\sigma$-field $\mathcal{F}_0$ is trivial. We further fix some initial position $x \in \mathbb{R} \setminus \{0\}$ and the initial deviation $d = 0$. For $\nu \in \mathbb{R}$, define the execution strategy $(X_\nu(s))_{s \in [0, T]}$ by $X_\nu(0-) = X_\nu(0) = x$, $dX_\nu(s) = \nu X_\nu(s) dW_s$ for $s \in [0, T)$ and $X_\nu(T) = 0$, i.e., the strategy follows a geometric Brownian motion on $[0, T]$ and has a block trade at time $T$. For each $\nu \in \mathbb{R}$, let $D_\nu = (D_\nu(s))_{s \in [0, T]}$ be the deviation process associated to $X_\nu$ according to the dynamics (5.1), which is

$$dD_\nu(s) = -\rho_s D_\nu(s) ds + \gamma_s dX_\nu(s) = -\rho D_\nu(s) ds + \nu \gamma_s X_\nu(s) dW_s, \quad s \in [0, T),$$

$$D_\nu(0-) = 0, \quad D_\nu(T) = D_\nu(T-) - \gamma T X_\nu(T).$$

In particular, $D_\nu(s) = \int_0^s \nu e^{-\rho(s-r)} \gamma_r X_\nu(r) dW_r$ for $s \in [0, T)$.

We first verify that $X_\nu \in A_0(x, 0)$ for all $\nu \in \mathbb{R}$. Notice that in the current setting, we have for all $p \in [1, \infty)$ and $\nu \in \mathbb{R}$ that

$$E\left[\sup_{s \in [0, T]} |\gamma_s|^p\right] < \infty, \quad E\left[\sup_{s \in [0, T]} \alpha_s^p\right] < \infty, \quad E\left[\sup_{s \in [0, T]} |X_\nu(s)|^p\right] < \infty \quad (5.2)$$

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This, the Burkholder–Davis–Gundy inequality and the Hölder inequality imply that for all $p \in [2, \infty)$ and $\nu \in \mathbb{R}$, there exists $c \in [1, \infty)$ such that

$$E\left[ \sup_{s \in [0,T]} |D_s^{(\nu)}|^p \right] \leq c E\left[ \left( \int_0^T \nu^2 e^{-2p(T-r)} \gamma_r^2 (X_r^{(\nu)})^2 dr \right)^{p/2} \right] \leq c |\nu| T^{p/2} E\left[ \sup_{r \in [0,T]} \gamma_r^p |X_r^{(\nu)}|^p \right] < \infty.$$  

Furthermore, as $D_T^{(\nu)} = D_T^{(\nu)} + \gamma_T \Delta X_T^{(\nu)} = D_T^{(\nu)} - \gamma_T X_T^{(\nu)}$, we also get $D_T^{(\nu)} \in L^p$, hence

$$E\left[ \sup_{s \in [0,T]} |D_s^{(\nu)}|^p \right] < \infty \quad (5.3)$$

for all $p \in [1, \infty)$ and $\nu \in \mathbb{R}$. It now follows from the Hölder inequality, the Minkowski inequality, (5.2) and (5.3) that (2.3) is satisfied. Since $\sigma^2$ is a deterministic constant, (2.4) then also holds. Furthermore, the Hölder inequality, (5.2), and (5.3) prove that (2.5) is satisfied. Hence we have $X^{(\nu)} \in A_0(x,0)$ for all $\nu \in \mathbb{R}$.

We next consider the cost functional $J$ defined by (1.3) and obtain for any $\nu \in \mathbb{R}$ that

$$J_0(x, 0, X^{(\nu)}) = E \left[ \int_0^T D_s^{(\nu)} dX_s^{(\nu)} + D_T^{(\nu)} \Delta X_T^{(\nu)} + \int_0^T \frac{\gamma_s}{2} \nu^2 (X_s^{(\nu)})^2 ds + \frac{\gamma_T}{2} (\Delta X_T^{(\nu)})^2 \right]$$

$$= \nu E \left[ \int_0^T D_s^{(\nu)} X_s^{(\nu)} dW_s \right] - E[D_T^{(\nu)} X_T^{(\nu)}] + \frac{\nu^2}{2} \int_0^T E[\gamma_s (X_s^{(\nu)})^2] ds$$

$$+ \frac{1}{2} E[\gamma_T \Delta X_T^{(\nu)}]^2]. \quad (5.4)$$

By the Burkholder–Davis–Gundy inequality, the Hölder inequality, (5.2) and (5.3), the stochastic integral $\int_0^T D_s^{(\nu)} X_s^{(\nu)} dW_s$ is a martingale; hence its expectation vanishes. Further, it holds that $d(X_s^{(\nu)})^2 = 2\nu (X_s^{(\nu)})^2 dW_s + \nu^2 (X_s^{(\nu)})^2 ds$, $s \in [0, T)$. This yields that

$$d(\gamma_s (X_s^{(\nu)})^2) = \gamma_s (X_s^{(\nu)})^2 ((\nu^2 + 2\sigma \nu) ds + (2\nu + \sigma) dW_s), \quad s \in [0, T),$$

and hence

$$E[\gamma_s (X_s^{(\nu)})^2] = \gamma_0 x^2 e^{(\nu^2 + 2\sigma \nu) s}, \quad s \in [0, T). \quad (5.5)$$

Besides, we have for all $s \in [0, T)$ that

$$d(D_s^{(\nu)} X_s^{(\nu)}) = -\rho D_s^{(\nu)} X_s^{(\nu)} ds + \nu \gamma_s (X_s^{(\nu)})^2 dW_s + \nu D_s^{(\nu)} X_s^{(\nu)} dW_s$$

$$+ \nu^2 \gamma_s (X_s^{(\nu)})^2 ds. \quad (5.6)$$
Again by the Burkholder–Davis–Gundy inequality, the Hölder inequality, (5.2) and (5.3), one can show that \( \int_0^s \gamma_s(X^{(v)}(s))^2 dW_s \) is a martingale. Therefore, it follows from (5.5) and (5.6) that for all \( v \in \mathbb{R} \setminus \{0\} \) and \( s \in [0, T) \), we have

\[
E[D_s^{(v)}X_s^{(v)}] = -\rho \int_0^s E[D_u^{(v)}X_u^{(v)}] du + v^2 \int_0^s E[\gamma_u(X_u^{(v)})^2] du
\]

\[
= -\rho \int_0^s E[D_u^{(v)}X_u^{(v)}] du + \frac{v^2 \gamma_0 x^2}{\rho + v^2 + 2\sigma v} (e^{(\rho + v^2 + 2\sigma v)s} - 1).
\]

We thus obtain that

\[
E[D_s^{(v)}X_s^{(v)}] = e^{-\rho s} \frac{v^2 \gamma_0 x^2}{\rho + v^2 + 2\sigma v} (e^{(\rho + v^2 + 2\sigma v)s} - 1), \quad s \in [0, T).
\]

Together with (5.5), the cost functional (5.4) becomes

\[
J_0(x, 0, X^{(v)}) = -e^{-\rho T} \frac{v^2 \gamma_0 x^2}{\rho + v^2 + 2\sigma v} (e^{(\rho + v^2 + 2\sigma v)T} - 1)
\]

\[
+ \frac{v^2}{2} \int_0^T \gamma_0 x^2 e^{(v^2 + 2\sigma v)s} ds + \frac{\gamma_0 x^2}{2} e^{(v^2 + 2\sigma v)T}
\]

\[
= \frac{\gamma_0 x^2}{2} \left( e^{(v^2 + 2\sigma v)T} \left( \frac{v^2}{v^2 + 2\sigma v} - \frac{2v^2}{\rho + v^2 + 2\sigma v} + 1 \right)
\]

\[
- \left( \frac{v^2}{v^2 + 2\sigma v} - \frac{2v^2 e^{-\rho T}}{\rho + v^2 + 2\sigma v} \right) \right)
\]

\[
= \frac{\gamma_0 x^2}{2} (I_1(v) - I_2(v)),
\]

where

\[
I_1(v) = e^{(v^2 + 2\sigma v)T} \left( \frac{v^2}{v^2 + 2\sigma v} - \frac{2v^2}{\rho + v^2 + 2\sigma v} + 1 \right),
\]

\[
I_2(v) = \frac{v^2}{v^2 + 2\sigma v} - \frac{2v^2 e^{-\rho T}}{\rho + v^2 + 2\sigma v}.
\]

Observe that

\[
\frac{v^2}{v^2 + 2\sigma v} - \frac{2v^2}{\rho + v^2 + 2\sigma v} + 1 = \frac{1}{1 + \frac{2\sigma}{v}} - \frac{2}{\rho + v^2 + 2\sigma v} + 1
\]

\[
= \frac{2}{v} \left( \frac{\sigma + \rho v}{v^2} + \frac{\rho v}{v^2} + 1 + \frac{2\sigma^2}{v^2} \right),
\]

i.e., this term behaves as \( \frac{2\sigma}{v} \) in the limit \( v \to -\infty \) (in particular, this term is strictly negative provided \( v < 0 \) and \( |v| \) is sufficiently large). It follows that \( I_1(v) \to -\infty \) as
\[ v \to -\infty, \text{ whereas clearly } I_2(v) \to 1 - 2e^{-\rho T} \text{ as } v \to -\infty; \text{ hence} \]
\[ J_0(x, 0, X^{(v)}) \to -\infty \quad \text{ as } v \to -\infty. \]

Thus the dynamics (5.1) leads to an ill-posed optimisation problem.

### 5.2 Solution in our framework

In the setting of Sect. 5.1, we recover a well-posed optimisation problem when we use the dynamics (1.2) instead of (5.1) for the deviation process \( D \). Let us verify that Theorem 3.4 applies and present explicit formulas for the optimal strategy \( X^* \) in \( \mathcal{A}(x, d) \) and the associated deviation process \( D^* \) for any \( t \in [0, T], x, d \in \mathbb{R} \).

In this setting, (3.8) is trivially satisfied, while (3.1) holds true due to our assumption \( 2\rho - \sigma^2 > 0 \). The BSDE (3.2) takes the form
\[
dY_s = \left( \frac{(\rho Y_s + \sigma Z_s)^2}{\sigma^2 Y_s + \rho - \frac{\sigma^2}{2}} - \sigma Z_s \right) ds + Z_s dW_s + dM^s_s, \quad s \in [0, T], \quad Y_T = \frac{1}{2},
\]
and has a deterministic solution
\[
Z \equiv 0, \quad M^t \equiv 0, \quad Y_s = \frac{\rho - \frac{\sigma^2}{2}}{\sigma^2 Y_s + \rho - \frac{\sigma^2}{2}} \mathcal{W}\left( \frac{\rho - \frac{\sigma^2}{2}}{\sigma^2} e^{-\frac{\sigma^2}{2} s} \right)^{-1}, \quad s \in [0, T],
\]
where \( \mathcal{W} \) denotes the Lambert W function, and \( \kappa = \log 2 + \frac{1}{\sigma^2}(2\rho - \sigma^2 + \rho^2 T) \).

Observe that in this setting, \( \tilde{\beta}_s = \frac{\rho Y_s}{\sigma^2 Y_s + \rho - \frac{\sigma^2}{2}}, \quad s \in [0, T] \),
and that both \( Y \) and \( \tilde{\beta} \) are deterministic increasing continuous \((0, 1/2]\)-valued functions. In particular, \( \tilde{\beta} \) is bounded and a semimartingale. Hence Theorem 3.4 applies, and for all \( t \in [0, T], x, d \in \mathbb{R} \), the unique optimal strategy \( X^* = (X^*_s)_{s \in [t, T]} \) in \( \mathcal{A}(x, d) \) and its associated deviation process \( D^* = (D^*_s)_{s \in [t, T]} \) are given by the formulas \( X^*_t = x, D^*_t = d \),
\[
X^*_s = \left( x - \frac{d}{\gamma_t} \right) \mathcal{E}(Q)_t,s (1 - \tilde{\beta}_s), \quad s \in [t, T),
\]
\[
D^*_s = \left( x - \frac{d}{\gamma_t} \right) \mathcal{E}(Q)_t,s (-\gamma_s \tilde{\beta}_s), \quad s \in [t, T),
\]
and \( X^*_T = 0, D^*_T = (x - \frac{d}{\gamma_T})\mathcal{E}(Q)_T (\gamma_T) \), where
\[
Q_s = -\sigma \int_0^s \tilde{\beta}_r dW_r - (\rho - \sigma^2) \int_0^s \tilde{\beta}_r dr, \quad s \in [0, T].
\]

We finally discuss some properties of the optimal strategy in the case \( x \neq \frac{d}{\gamma_t} \) (there is nothing to discuss for \( x = \frac{d}{\gamma_t} \)). Like in the situation of Sect. 4.2, \( X^* \) has jumps at
times \( t \) and \( T \) and is continuous on \((t, T)\). As \( 1 - \tilde{\beta} \) is positive, \( X^* \) has again the same sign as \( x - \frac{d}{\gamma} \) on \((t, T)\). Now in contrast to Sect. 4.2, the associated deviation process \( D^* \) is no longer constant on \((t, T)\). Further, as \( 1 - \tilde{\beta} \) is nonvanishing and has finite variation on \([t, T]\), while \( \mathcal{E}(Q)_t \mathcal{\cdot} \) almost surely has infinite variation on all subintervals of \([t, T]\), we get in contrast to Sect. 4.2 that \( X^* \) almost surely has infinite variation on all subintervals of \([t, T]\) (in particular, \( X^* \) is in no way monotonic on any subinterval). See Figure 1 for an illustration.

### 6 Examples

In this section, we present several interesting qualitative effects that can arise in our framework. For this, we consider several subsettings of the following common setup.

Let \( M = W \) be a Brownian motion and \( \mathcal{F}_s = \mathcal{F}_s^W \) for all \( s \in [0, T] \). Let \( t = 0, x, d \in \mathbb{R} \) with \( x \neq \frac{d}{\gamma} \) (for the case \( x = \frac{d}{\gamma} \), see Lemma 3.3). The resilience here is taken to be a deterministic constant \( \rho \in \mathbb{R} - \{0\} \) (for the case \( \rho = 0 \), see Proposition 3.7). We consider the price impact \( \gamma \) from (1.1) with \( \sigma \equiv 0 \), i.e.,

\[
\gamma_s = \gamma_0 \exp\left( \int_0^s \mu_r dr \right), \quad s \in [0, T].
\]

In particular, \( \gamma \) is continuous and of finite variation. We assume that there exist deterministic constants \( \varepsilon, \nu \in (0, \infty) \) such that

\[
2\rho + \mu \geq \varepsilon \mathcal{D}_W \text{-a.e.} \quad \text{and} \quad \mu \leq \nu \mathcal{D}_W \text{-a.e.} \tag{6.1}
\]

(in particular, (3.1) is satisfied). Our current setup is a special case of the settings considered in Sects. 7.1 and 7.2 below. Therefore, it follows from Proposition 7.1 (alternatively, from Proposition 7.3) that (3.4) is satisfied.

As before, \((Y, Z, M^\perp)\) denotes a solution to the BSDE (3.2) satisfying the requirements in (3.4). We notice that \( M^\perp \equiv 0 \) in our current setup because due to the martingale representation theorem, any local martingale \( M^\perp \) in the Brownian filtration with \( M^\perp_0 = 0 \) and \([M^\perp, W] = 0 \) is indistinguishable from zero. For the process...
\[ \tilde{\beta}_s = \frac{\rho + \mu_s}{2\rho + \mu_s} 2Y_s = \left( 1 - \frac{\rho}{2\rho + \mu_s} \right) 2Y_s, \quad s \in [0, T]. \] (6.2)

Notice that by (6.1), \( \tilde{\beta} \) is bounded. That is, in our current setup, including (6.1), the assumptions of Theorem 3.4 are satisfied.

What varies between the examples in this section is the choice of \( \mu \), i.e., the price impact process \( \gamma \). In the examples below, we distinguish between the following two situations.

**Situation 1:** There exists a càdlàg semimartingale \( \beta = (\beta_s)_{s \in [0, T]} \) with

\[ \tilde{\beta} = \beta \quad D_W\text{-a.e.} \] (6.3)

**Situation 2:** There is no càdlàg semimartingale \( \beta \) such that (6.3) is satisfied.

As we know from Theorem 3.4, in Situation 1, there exists a unique (up to \( D_W \)-null-sets) optimal strategy \( X^* = (X^*_s)_{s \in [0, T]} \in \mathcal{A}_0(x, d) \), given by \( X^*_0 = x, X^*_T = 0 \) and

\[ X^*_s = \left( x - \frac{d}{\gamma_0} \right) \exp \left( - \int_0^s \beta_r(\mu_r + \rho) \, dr \right) (1 - \beta_s), \quad s \in [0, T], \] (6.4)

while in Situation 2, no optimal strategy exists.

Before turning to specific examples, we notice that the multiplier

\[ \left( x - \frac{d}{\gamma_0} \right) \exp \left( - \int_0^s \beta_r(\mu_r + \rho) \, dr \right), \quad s \in [0, T], \]

in (6.4) is a nonvanishing continuous process of finite variation. Therefore, if we want to obtain in Situation 1 the optimal strategy \( X^* \) of infinite variation on \([0, T]\) and/or with jumps inside \((0, T)\), it is enough to construct \( \beta \) (see (6.2) and (6.3)) of infinite variation on \([0, T]\) and/or with jumps inside \((0, T)\).

**Example 6.1** Let \( \mu \) be a continuous process of finite variation satisfying (6.1) such that

the function \( s \mapsto \rho + \mu_s \) is nonvanishing on \([0, T]\) a.s. \( \) (6.5)

Observe that for a fixed \( \omega \in \Omega \), the unique solution to the Bernoulli ODE

\[ d\bar{Y}_s(\omega) = \left( \frac{2(\rho + \mu_s(\omega))^2 \bar{Y}_s(\omega)^2}{2\rho + \mu_s(\omega)} - \mu_s(\omega) \bar{Y}_s(\omega) \right) ds, \quad s \in [0, T], \]

\[ \bar{Y}_T(\omega) = \frac{1}{2}, \]

which is the BSDE (3.2) without the martingale part, is given by the formula

\[ \bar{Y}_s(\omega) = e^{\int_s^T \mu_r(\omega) \, dr} \left( \int_s^T \frac{2(\rho + \mu_r(\omega))^2}{2\rho + \mu_r(\omega)} e^{\int_s^r \mu_u(\omega) \, du} \, dr + 2 \right)^{-1}, \quad s \in [0, T]. \] (6.6)
It follows that it is possible to choose $\mu$ such that $\overline{Y}$ is not adapted. Choosing $\mu$ in such a way, we conclude that the solution $(Y, Z, M^\perp \equiv 0)$ of the BSDE (3.2) satisfies $\mathcal{D}_W[Z \neq 0] > 0$. This yields that with positive probability, $Y$ has infinite variation on $[0, T]$. Define $\varphi_s = \frac{2(\rho + \mu_s)}{2\rho + \mu_s}$, $s \in [0, T]$, which is a nonvanishing (recall (6.5)) continuous process of finite variation. Hence $\tilde{\beta} = \varphi Y$ is a continuous semimartingale that with positive probability has infinite variation on $[0, T]$. Thus we are in Situation 1 with $\beta \equiv \tilde{\beta}$, and the optimal strategy $X^\ast$, which is given by (6.4), has with positive probability infinite variation on $[0, T]$.

In contrast to the situation in Sect. 5.2, where the infinite variation in $X^\ast$ was caused by the infinite variation in the exogenous process $\gamma$, all exogenous processes (i.e., $\gamma$ and $\rho$) in this example have finite variation. As easily seen, it is even possible to choose $\mu$ with $C^\infty$ paths. Then $\gamma$ also has $C^\infty$ paths. The process $\rho$ is even constant. The optimal strategy, however, has infinite variation, i.e., “oscillates much more quickly” than the exogenous processes do. This is due to the incoming information that is reflected in the endogenous process $Y$ (which turns out to have infinite variation).

**Example 6.2** Optimal strategies we have seen so far have jumps (block trades) at times 0 and $T$ only. In order to construct an optimal strategy with jumps inside $(0, T)$, it is enough to take

a càdlàg semimartingale $\mu$ satisfying (6.1) that exhibits jumps in $(0, T)$,

i.e., with positive probability, $\{s \in (0, T) : \Delta \mu_s \neq 0\} \neq \emptyset$, and such that

the corresponding process $Y$ is nonvanishing. \hfill (6.7)

Indeed, in this case, $\tilde{\beta}$ is a càdlàg semimartingale; so we are in Situation 1 with $\beta \equiv \tilde{\beta}$. Moreover, as $Y$ is continuous and nonvanishing, we readily see from (6.2) that

$$\Delta \mu_s \neq 0 \iff \Delta \tilde{\beta}_s \neq 0;$$

hence the optimal strategy $X^\ast$, given by (6.4), contains block trades inside $(0, T)$.

To show a specific example of this kind, we consider for some $t_0 \in (0, T)$ the deterministic $\mu$ given by $\mu_s = 1_{[t_0, T]}(s)$, $s \in [0, T]$. Observe that (6.1) is satisfied whenever $\rho > 0$; so we take some $\rho > 0$ in this example. The BSDE (3.2) here takes the form

$$dY_s = \rho Y_s^2 ds + Z_s dW_s + dM_s^\perp, \quad s \in [0, t_0],$$

$$dY_s = \left(\frac{2(\rho + 1)^2 Y_s^2}{2\rho + 1} - Y_s\right) ds + Z_s dW_s + dM_s^\perp, \quad s \in [t_0, T], \quad Y_T = \frac{1}{2},$$

and has a deterministic solution $(Y, Z \equiv 0, M^\perp \equiv 0)$ given by

$$Y_s = (2\rho + 1)(2\rho + 1)^2 - 2\rho^2 e^{s-T})^{-1}, \quad s \in [t_0, T],$$

$$Y_s = \frac{1}{Y_{t_0}^{-1} + (t_0 - s)\rho}, \quad s \in [0, t_0).$$

\hfill (6.8)
Notice that $Y$ is continuous, strictly increasing and $(0, 1/2]$-valued. Hence (6.7) is satisfied, and what is stated after (6.7) applies. Observe that in this specific example,

$$\beta_s = \begin{cases} Y_s, & s \in [0, t_0), \\ Y_s(1 + \frac{1}{2\rho + 1}), & s \in [t_0, T], \end{cases} \tag{6.9}$$

which is a deterministic strictly increasing $(0, 1)$-valued càdlàg function with the only jump at time $t_0$ given by $\Delta \beta_{t_0} = \frac{Y_{t_0}}{2\rho + 1} > 0$. From (6.8) and (6.9), we compute that

$$\exp \left( - \int_0^s \beta_r (\mu_r + \rho) \, dr \right) = \begin{cases} Y_0 Y_s^{-1}, & s \in [0, t_0), \\ e^{t_0 - s} Y_0 Y_s^{-1}, & s \in [t_0, T], \end{cases} \tag{6.10}$$

which together with (6.8) and (6.9) provides the optimal strategy in closed form (see (6.4)). However, the qualitative structure of the optimal strategy $X^*$ in fact follows from (6.4) even without calculating (6.10), as follows.

First, $X^*$ is deterministic, and as $\beta$ is strictly increasing and $(0, 1)$-valued, $X^*$ is monotonic on $(0, T]$. Moreover, the facts that $\beta < 1$, $\Delta \beta_{t_0} > 0$ and $x \neq \frac{d}{\gamma_0}$ together with (6.4) imply that the optimal strategy necessarily has block trades in the end and at time $t_0$. Their signs are opposite to the sign of $x - \frac{d}{\gamma_0}$. Whether or not $X^*$ has a block trade in the beginning depends on the value of the initial deviation $d$. Likewise, we claim the monotonicity of $X^*$ only on $(0, T]$ because whether or not $X^*$ is monotonic on $[0, T]$ also depends on $d$. Namely, $X^*$ has a block trade in the beginning if and only if $x \neq (x - \frac{d}{\gamma_0})(1 - \beta_0)$, i.e., if and only if $d \neq -\frac{\beta_0}{1 - \beta_0} \gamma_0 x$. Likewise, $X^*$ is monotonic on $[0, T]$ if and only if either $x \geq 0$, $d \geq -\frac{\beta_0}{1 - \beta_0} \gamma_0 x$ or $x \leq 0$, $d \leq -\frac{\beta_0}{1 - \beta_0} \gamma_0 x$. (In particular, if $d = 0$, then $X^*$ is monotonic on $[0, T]$.)

Between the block trades, the associated deviation process $D^*$ is constant. Indeed, it follows from (3.11), (6.9) and (6.10) that $D^*_s = (d - \gamma_0 x) Y_0$, $s \in [0, t_0)$, and $D^*_s = (d - \gamma_0 x) Y_0 (1 + \frac{1}{2\rho + 1})$, $s \in [t_0, T]$.

Figure 2 is an illustration for specific parameter values.
Observe that the reaction of the optimal strategy to changes in the price impact is rather sensitive: here only $\mu$ jumps at time $t_0$ (not the price impact $\gamma$ itself), but this already causes a jump in $X^*$ at time $t_0$. Finally, it is worth noting that a model with deterministically time-varying price impact and resilience was considered in Fruth et al. [16, Sect. 8], but examples of the above type are not possible in their framework because the smoothness assumption in [16, Sect. 8] excludes the possibility of block trades inside $(0, T)$ (cf. [16, Theorem 8.4]).

**Example 6.3** In models of price impact that include resilience, it is commonly assumed that the resilience is positive. But a negative resilience also has a natural interpretation, as it models a self-exciting behaviour of the price impact, where trading activities of the large investor stimulate other market participants to trade in the same direction. In this example, we discuss a basic effect of a negative resilience in our model. To this end, we consider some $\rho < 0$ and take a deterministic constant $\mu > -2\rho$ ($> 0$), which ensures (6.1). Here, again, the BSDE (3.2) has a deterministic solution $(Y, Z \equiv 0, M^\perp \equiv 0)$ which is given by

$$Y_s = \frac{1}{2} \mu (2\rho + \mu) \left( (\rho + \mu)^2 - \rho^2 e^{\mu(s-T)} \right)^{-1}, \quad s \in [0, T].$$

It follows that

$$\tilde{\beta}_s = \mu (\rho + \mu) \left( (\rho + \mu)^2 - \rho^2 e^{\mu(s-T)} \right)^{-1}, \quad s \in [0, T],$$

which is a deterministic positive continuous increasing function and in particular a semimartingale. Thus we are in Situation 1 with $\beta \equiv \tilde{\beta}$. Notice that

$$\beta_s > \frac{\mu (\rho + \mu)}{(\rho + \mu)^2} = \frac{\mu}{\rho + \mu} > 1, \quad s \in [0, T],$$

i.e., in contrast to Example 6.2, $\beta$ is now $(1, \infty)$-valued. We set

$$\lambda_s = \left( x - \frac{d}{\gamma_{t_0}} \right) \exp \left( - (\rho + \mu) \int_0^s \beta_r \, dr \right), \quad s \in [0, T],$$

and have by (3.10) and (3.11) that

$$X^*_s = \lambda_s (1 - \beta_s) \quad \text{and} \quad D^*_s = -\lambda_s \gamma_s \beta_s, \quad s \in [0, T]. \quad (6.11)$$

The fact that $\beta$ is $(1, \infty)$-valued makes the factor $1 - \beta$ in (6.11) negative and means that the optimal strategy $X^*$ is not monotonic on $[0, T]$ even for $d = 0$ because of the block trade at time $0$ (this is in contrast to Example 6.2 and Sect. 4.2, where the optimal strategy is monotonic on $[0, T]$ for $d = 0$). Indeed, for the moment, let $d = 0$ and, say, $x > 0$ (the objective is to sell shares). Then in the first block trade, more than $x$ shares are sold and the sell-program is thus changed into a buy-program. This is done to profit from the negative resilience that drives the deviation process $D^*$ associated to $X^*$ down also after the initial block trade and allows to profit from the subsequent buy-program.
It can, however, be shown that $X^*$ is monotonic on $(0, T)$. To this end, we first prove monotonicity on $(0, T)$, as follows. The BSDE (3.2) for $Y$ (just a Bernoulli ODE in this case) and (6.2) imply that $\beta$ satisfies the (Bernoulli) ODE

$$\dot{\beta}_s = (\rho + \mu) \beta_s^2 - \mu \beta_s, \quad s \in [0, T].$$

(6.12)

Also observe that $\dot{\gamma}_s = \mu \gamma_s$ and $\dot{\lambda}_s = -(\rho + \mu) \lambda_s \beta_s$ for all $s \in [0, T]$. It now follows from (6.11) that

$$\dot{X}_s^* = \lambda_s \left( -(\rho + \mu) \beta_s (1 - \beta_s) - (\rho + \mu) \beta_s^2 + \mu \beta_s \right) = -\rho \lambda_s \beta_s, \quad s \in (0, T),$$

which has the same sign as $x - \frac{d}{\gamma_0}$. This shows that $X^*$ is monotonic on $(0, T)$. Further, note that for the final block trade, we have

$$\Delta X_T^* = -X_T^{\ast -} = \lambda_T (\beta_T - 1).$$

Since this also has the same sign as $x - \frac{d}{\gamma_0}$, we conclude that $X^*$ is even monotonic on $(0, T)$.

Moreover, observe that it follows from (6.11) and (6.12) that

$$\dot{D}_s^* = \lambda_s \left( (\rho + \mu) \gamma_s \beta_s^2 - \mu \gamma_s \beta_s - \gamma_s \left( (\rho + \mu) \beta_s^2 - \mu \beta_s \right) \right) = 0, \quad s \in (0, T),$$

i.e., trading is performed in such a way that $D^* \equiv \text{const}$ on $(0, T)$.

The optimal strategy and the corresponding deviation process for specific parameter values are shown in Figure 3.

**Example 6.4** Finally, in order to construct an example of Situation 2, it suffices to take any deterministic càdlàg function $\mu$ such that there exists $\delta \in (0, T)$ with $\mu$ having infinite variation on $[0, T - \delta]$. E.g. one could take $\mu$ to be the Weierstrass function or the function $s \mapsto (s \sin \frac{1}{s})1_{(0, T)}(s)$, $s \in [0, T]$. We also take $\rho \in \mathbb{R} \setminus \{0\}$ such that (6.1) is satisfied.
Notice that in this deterministic framework, the process $Y$ is a deterministic continuous function of finite variation explicitly given by (6.6). In particular, $Y$ is nonvanishing.

To formally prove that we are in Situation 2, assume by way of contradiction that there exists a càdlàg semimartingale $\beta = (\beta_s)_{s \in [0,T]}$ with $\widetilde{\beta} = \beta$ $D_{W}$-a.e. ($\beta$ can be stochastic). Then it follows from (6.2) and the fact that $Y$ is nonvanishing that

$$\frac{\rho}{2\rho + \mu} = 1 - \frac{\beta}{2Y} \quad D_{W}\text{-a.e.} \tag{6.13}$$

Set $S = 1 - \frac{\beta}{2Y}$ and notice that it is a càdlàg semimartingale. As both sides in (6.13) are càdlàg, they are even indistinguishable on $[0, T)$, i.e., almost surely, it holds that

$$\frac{\rho}{2\rho + \mu_s} = S_s, \quad s \in [0, T), \tag{6.14}$$

hence $S \neq 0$ and $S_\cdot \neq 0$ on $[0, T)$, which implies that $\frac{1}{S}$ is also a semimartingale on $[0, T)$. Now (6.14) yields that almost surely,

$$\mu_s = \frac{\rho}{S_s} - 2\rho, \quad s \in [0, T).$$

Thus $\mu$ is itself a semimartingale on $[0, T)$. As $\mu$ is deterministic, this means that $\mu$ has finite variation on each compact subinterval of $[0, T)$, in particular, on $[0, T - \delta]$. The obtained contradiction proves that we are in Situation 2.

This example thus shows that an optimal strategy can fail to exist even when the value process is finite.

### 7 Existence for the BSDE in two subsettings

In this section, we prove in two subsettings existence of a solution $(Y, Z, M^\perp)$ to the BSDE (3.2) with driver (3.3) such that (3.4) holds.

We suppose in both subsettings that the following two conditions are satisfied:

1. There exists $\varepsilon \in (0, \infty)$ such that $2\rho + \mu - \sigma^2 \geq \varepsilon$ $D_M$-a.e., \hspace{1cm} (7.1)
2. There exist $\overline{\rho}, \overline{\mu} \in (0, \infty)$ such that $|\rho| \leq \overline{\rho}$, $|\mu| \leq \overline{\mu}$ $D_M$-a.e. \hspace{1cm} (7.2)

In the first setting, we do not impose restrictions on the filtration, but assume $\sigma \equiv 0$ in order to meet a Lipschitz condition in some place. Subsequently, we consider a second setting with a general $\sigma$, where we assume that $(\mathcal{F}_t)_{t \in [0,T]}$ is a continuous filtration in the sense that any $(\mathcal{F}_t)_{t \in [0,T]}$-martingale is continuous. This condition is for example satisfied for a Brownian filtration.
7.1 General filtration and $\sigma \equiv 0$

Proposition 7.1 Let $\sigma \equiv 0$ and assume that (7.1), (7.2) and (3.8) are satisfied. Then (3.4) holds.

Proof We define the truncation function $L : \mathbb{R} \to [0, 1/2]$ by $L(y) = (y \vee 0) \wedge \frac{1}{2}$ and consider the BSDE (3.2) with truncated driver $\bar{f} : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ given by

$$\bar{f}(s, y) = -\frac{2(\rho_s + \mu_s)^2}{2\rho_s + \mu_s} L(y)^2 + \mu_s L(y), \quad s \in [0, T], y \in \mathbb{R},$$

instead of $f$ defined in (3.3). Our aim is to first obtain a solution $(Y, Z, M^\perp)$ of the BSDE with truncated driver via Papapantoleon et al. [31, Theorem 3.5] and then show that $Y$ is $[0, 1/2]$-valued, i.e., $(Y, Z, M^\perp)$ is also a solution of the BSDE (3.2) with driver (3.3).

Due to (7.1), (7.2) and the definition of $L$, we have for all $y, y' \in \mathbb{R}$, $D_M$-a.e. that

$$|\bar{f}(s, y) - \bar{f}(s, y')| \leq \left(\frac{2(\rho_s + \mu_s)^2}{2\rho_s + \mu_s} + |\mu_s|\right)|y - y'| \leq \left(\frac{6(\rho^2 + \mu^2)}{\varepsilon} + \bar{\mu}\right)|y - y'|.$$

Therefore, [31, Assumption (F3)] is satisfied. It further follows from (3.8) that [31, (F2)] holds true. The fact that $\bar{f}(s, 0) = 0$ for all $s \in [0, T]$ yields [31, (F5)]. Since $M$ is continuous, [31, (F4)] is satisfied for all $\Phi > 0$. Thus by [31, Theorem 3.5] (see also [31, Corollary 3.6]), there exists a solution $(Y, Z, M^\perp)$ of the BSDE (3.2) with driver $\bar{f}$. In particular, the norm in [31, Theorem 3.5] being finite implies that $E[|M^\perp|_T] < \infty$ and $E[\int_0^T Z_s^2 d|M|_s] < \infty$.

In order to show that $Y$ is $[0, 1/2]$-valued, observe that $(\tilde{Y}, \tilde{Z}, \tilde{M}^\perp) = (\frac{1}{2}, 0, 0)$ (resp., $(\tilde{Y}, \tilde{Z}, \tilde{M}^\perp) = (0, 0, 0)$) solves the BSDE

$$d\tilde{Y}_s = \tilde{Z}_s dM_s + d\tilde{M}_s^\perp, \quad s \in [0, T], \quad \tilde{Y}_T = \frac{1}{2} \quad \text{(resp.,} \tilde{Y}_T = 0),$$

with vanishing driver and that

$$\bar{f}\left(s, \frac{1}{2}\right) = -\frac{\rho_s^2}{2(2\rho_s + \mu_s)} \leq 0 \quad \text{(resp.} \bar{f}(s, 0) = 0), \quad s \in [0, T].$$

Finally, it is possible to verify that a comparison principle holds, which yields that $Y \leq \frac{1}{2}$ and $Y \geq 0$. Although the comparison is performed with standard techniques, we could not locate a precise reference that applies in this situation. Therefore, we present the argument in Appendix C. \hfill \Box

Remark 7.2 Note that the setting in Papapantoleon et al. [31] is much more general than ours. Among others, their BSDE may include jumps, and the Lipschitz-continuity of the driver is allowed to be stochastic. For instance, we could replace our conditions (7.1), (7.2), and (3.8) by (3.1) together with the more abstract assumption that there exists a predictable stochastic process $R$ such that for all $y, y' \in \mathbb{R},$
\[ |\tilde{f}(\omega, s, y) - \tilde{f}(\omega, s, y')| \leq R_s(\omega)|y - y'| \quad \mathcal{D}_M \text{-a.e. and for all } c \in (0, \infty), \text{ we have } E[\exp(c \int_0^T R_s d[M_s])] < \infty. \]

Notice, however, that we still need to assume \( \sigma \equiv 0 \) to obtain (possibly stochastic) Lipschitz-continuity. Observe furthermore that the assumptions (7.1), (7.2) and (3.8) in Proposition 7.1 seem reasonable in light of the requirements in our main Theorem 3.4.

### 7.2 General \( \sigma \) and continuous filtration

**Proposition 7.3** Assume that the filtration \( (\mathcal{F}_t)_{t \in [0,T]} \) is continuous in the sense that any \( (\mathcal{F}_t)_{t \in [0,T]} \)-martingale is continuous, and that \( [M]_T \leq c_1 \) a.s. for some deterministic \( c_1 \in (0, \infty) \). Suppose (7.1) and (7.2). Then (3.4) holds.

**Proof** We first consider the BSDE (3.2) with the truncated driver

\[
\tilde{f}(s, y, z) = -\frac{(\rho_s + \mu_s) L(y) + \sigma_s z^2}{\sigma_s^2 L(y) + \frac{1}{2} (2\rho_s + \mu_s - \sigma_s^2)} + \mu_s L(y) + \sigma_s z,
\]

where \( L : \mathbb{R} \to [0, 1/2], L(y) = (y \vee 0) \wedge \frac{1}{2}. \)

Note that (7.1) and (7.2) imply that \( \sigma^2 \leq 2\rho + \mu - \varepsilon \). Moreover, by (7.1) and \( L \geq 0 \), we have \( \sigma^2 L(y) + \frac{1}{2} (2\rho + \mu - \sigma^2) \geq \frac{\varepsilon}{2} \) for all \( y \in \mathbb{R} \). Together with \( 0 \leq L \leq \frac{1}{2} \) and the boundedness of \( \rho, \mu \) and \( \sigma \), it is thus possible to show that there exist deterministic constants \( c_2, c_3 \in (0, \infty) \) such that for all \( y, z \in \mathbb{R} \),

\[
|\tilde{f}(s, y, z)| \leq \frac{1}{2} \mu + \frac{2}{\varepsilon} \left( (\rho_s + \mu_s)^2 L(y)^2 + 2(\rho_s + \mu_s) L(y) \sigma_s z + \sigma_s^2 z^2 + \sigma^3 L(y) \frac{1}{2} (2\rho_s + \mu_s - \sigma^2) \sigma_s z \right) \\
\leq c_2 + \frac{c_3}{2} z^2 \quad \mathcal{D}_M \text{-a.e.}
\]

Furthermore, it holds that \( \int_0^T c_2 d[M_s] \leq c_1 c_2 \). Hence Morlais [29, Assumption (H')\] is satisfied. Observe moreover that \( \tilde{f} \) is continuous in \( (y, z) \). Step 3 and 4 in [29, proof of Theorem 2.5] show that there exists a solution \( (Y, Z, M^\perp) \) of the BSDE (3.2) with driver \( \tilde{f} \), and it satisfies \( E[\int_0^T Z^2_s d[M_s]] < \infty, E[[M^\perp]_T] < \infty \) and that \( Y \) is bounded. In the remainder of this proof, we show that \( Y \in [0, 1/2] \)-valued, which implies that \( (Y, Z, M^\perp) \) is also a solution of the BSDE (3.2) with driver \( f \).

For the upper bound, let \( \hat{Y} = \frac{1}{2} - Y, \hat{Z} = -Z \) and \( \hat{M}^\perp = -M^\perp \). Then we have

\[
d\hat{Y}_t = -\tilde{f}(t, \hat{Y}_t, \hat{Z}_t) d[M]_t + \hat{Z}_t dM_t + d\hat{M}^\perp_t, \quad t \in [0, T], \quad \hat{Y}_T = 0,
\]

\( \hat{M}^\perp = -M^\perp \).
where for $t \in [0, T]$,

$$
\hat{f}(t, \hat{Y}_t, \hat{Z}_t) = \frac{(\rho_t + \mu_t) L(Y_t) + \sigma_t Z_t}{\sigma_t^2 L(Y_t) + \frac{1}{2} (2 \rho_t + \mu_t - \sigma_t^2)} - \frac{1}{2} \mu_t + \hat{Y}_t \mu_t \frac{L(Y_t) - \frac{1}{2}}{Y_t - \frac{1}{2}} + \sigma_t \hat{Z}_t
$$

$$
\hat{Y}_t \frac{L(Y_t) - \frac{1}{2}}{Y_t - \frac{1}{2}} \left( \frac{\sigma_t^2 \mu_t}{2} - (2 \rho_t + \mu_t) \mu_t (L(Y_t) + \frac{1}{2}) \right)
$$

$$
\hat{Z}_t \sigma_t (\rho_t + \mu_t) L(Y_t) + \mu_t \frac{L(Y_t) + \frac{1}{2}}{\sigma_t^2 L(Y_t) + \frac{1}{2} (2 \rho_t + \mu_t - \sigma_t^2)} + \sigma_t
$$

$$
\frac{\rho_t^2 L(Y_t)^2 + \sigma_t^2 Z_t^2}{\sigma_t^2 L(Y_t) + \frac{1}{2} (2 \rho_t + \mu_t - \sigma_t^2)}
$$

with the convention that $0/0 := 0$. Denote $\hat{f}(t, y, z) = y \psi_t + z \eta_t + \varphi_t$ for $t \in [0, T]$, $y, z \in \mathbb{R}$, and observe that $\psi$ and $\eta$ are bounded. Since $\rho^2 L(Y)^2 + \sigma^2 Z^2 \geq 0$ and $\sigma^2 L(Y) + \frac{1}{2} (2 \rho + \mu - \sigma^2) \geq \frac{\varepsilon}{2} > 0$, we have $\varepsilon \geq 0$ $\mathcal{D}_M$-a.e. Define the process $\Gamma = (\Gamma_t)_{t \in [0, T]}$ by $d\Gamma_t = \Gamma_t \psi_t d[M]_t + \Gamma_t \eta_t dM_t$, $t \in [0, T]$, $\Gamma_0 = 1$. One can then show that $\hat{Y}$ has the representation

$$
\hat{Y}_t = \Gamma_t^{-1} E_t \left[ \int_t^T \Gamma_s \varphi_s d[M]_s \right], \quad t \in [0, T],
$$

and hence $\hat{Y} \geq 0$, i.e., $Y \leq \frac{1}{2}$.

Next, we show that $\hat{Y}$ is nonnegative. To this end, we first choose $\delta \in (0, \infty)$ such that $\frac{\varepsilon}{2} \geq \frac{2 \sigma^2}{2 \rho + \mu - \sigma^2} \mathcal{D}_M$-a.e. Let $h: \mathbb{R} \to \mathbb{R}$ be the function $h(y) = 1 - e^{-\delta y}$ and let $\tilde{Y} = (\tilde{Y}_t)_{t \in [0, T]}$ be the process $\tilde{Y}_t = h(Y_t)$. Then for all $t \in [0, T]$, we have

$$
d\tilde{Y}_t = h'(Y_t) dY_t + \frac{1}{2} h''(Y_t) dY_t
$$

$$
= -\left( f(t, Y_t, Z_t) h'(Y_t) - \frac{Z_t^2 h''(Y_t)}{2} \right) d[M]_t
$$

$$
+ \frac{1}{2} h''(Y_t) d[M]^t + h'(Y_t) Z_t dM_t + h'(Y_t) dM_t^t. \quad (7.3)
$$
Let \( \tilde{Z} = (\tilde{Z}_t)_{t \in [0,T]} \), \( \tilde{M}^\perp = (\tilde{M}^\perp_t)_{t \in [0,T]} \) and \( A = (A_t)_{t \in [0,T]} \) be the processes \( \tilde{Z}_t = h'(Y_t) Z_t, \tilde{M}^\perp_t = \int_0^t h'(Y_s) dM^\perp_s \) and \( A_t = -\frac{1}{2} \int_0^t h''(Y_s) d[M^\perp]_s \). Observe that \( h'(y) = \delta e^{-\delta y} = \delta (1 - h(y)) \) for all \( y \in \mathbb{R} \) and \( h''(y) = -\delta^2 e^{-\delta y} = -\delta h'(y) \) for all \( y \in \mathbb{R} \). In particular, the process \( A \) is nondecreasing. We obtain from (7.3) that for all \( t \in [0,T] \),

\[
d\tilde{Y}_t = -\left( -\frac{\delta(\rho_t + \mu_t)^2 L(Y_t)^2(1-Y_t)}{\sigma_t^2 L(Y_t)} + 2\sigma_t(\rho_t + \mu_t) L(Y_t) \tilde{Z}_t \right)
+ \delta \mu_t L(Y_t)(1-\tilde{Y}_t) + \sigma_t \tilde{Z}_t
+ Z_t^2 \frac{dH(Y_t)}{\sigma_t^2 L(Y_t) + \frac{1}{2}(2\rho_t + \mu_t - \sigma_t^2)} \big) d[M]_t
- dA_t + \tilde{Z}_t dM_t + d\tilde{M}^\perp_t
\]

Denote the coefficients of \( \tilde{Y} \) resp. \( \tilde{Z} \) by

\[
\tilde{\psi}_t = \frac{\delta L(Y_t)(1-Y_t)}{\tilde{Y}_t} \left( \mu_t - \frac{(\rho_t + \mu_t)^2 L(Y_t)}{\sigma_t^2 L(Y_t) + \frac{1}{2}(2\rho_t + \mu_t - \sigma_t^2)} \right), \quad t \in [0,T],
\]

\[
\tilde{\eta}_t = \frac{\sigma_t}{\left( \sigma_t^2 L(Y_t) + \frac{1}{2}(2\rho_t + \mu_t - \sigma_t^2) \right)}, \quad t \in [0,T],
\]

and define \( \tilde{\Gamma} \) by \( \tilde{\Gamma}_t = \tilde{\Gamma}_t \tilde{\psi}_t d[M]_t + \tilde{\Gamma}_t \tilde{\eta}_t dM_t, t \in [0,T], \tilde{\Gamma}_0 = 1 \). Note that the process \( \frac{\delta L(Y_t)(1-Y_t)}{\tilde{Y}_t} \) is bounded. Together with (7.1), (7.2) and \( 0 \leq L \leq \frac{1}{2} \), it follows that \( \tilde{\psi} \) and \( \tilde{\eta} \) are bounded. One can then show that

\[
\tilde{Y}_t \tilde{\Gamma}_t = E_t \left[ \int_t^T \tilde{\Gamma}_s Z_s^2 h'(Y_s) \left( \frac{\delta}{2} - \frac{\sigma_s^2}{\sigma_s^2 L(Y_s) + \frac{1}{2}(2\rho_s + \mu_s - \sigma_s^2)} \right) d[M]_s \right.
+ \tilde{Y}_T \tilde{\Gamma}_T + \int_t^T \tilde{\Gamma}_s dA_s, \quad t \in [0,T].
\]
Due to the choice of $\delta$, we have $Z^2 h'(Y) \left( \frac{\delta}{2} - \frac{\sigma^2}{\sigma^2 L(Y) + \frac{1}{2} (2\rho + \mu - \sigma^2)} \right) \geq 0$ $\mathcal{D}_M$-a.e. Since furthermore $A$ is nondecreasing and $\tilde{Y}$ has the terminal value $1 - e^{-\frac{\delta}{2}} \geq 0$, it follows from (7.4) that $\tilde{Y} \geq 0$ and hence $Y \geq 0$. $\square$

8 Proofs of results from Sect. 3

We first present a technical lemma that is used in the proof of Theorem 3.1.

**Lemma 8.1** Let $(Y, Z, M^{\perp})$ be a solution of the BSDE (3.2) which satisfies (3.4). Then $Y_{T^-} = \frac{1}{2} a.s.$, i.e., $Y$ does not jump at the terminal time.

**Proof** We have, with $f$ defined in (3.3), that

$$Y_t = \frac{1}{2} + E_t \left[ \int_t^T f(s, Y_s, Z_s) d[M]_s \right] = \frac{1}{2} + E_t[A_T] - A_t, \quad t \in [0, T],$$

where $A_t = \int_0^t f(s, Y_s, Z_s) d[M]_s, t \in [0, T]$. As $A = (A_t)_{t \in [0, T]}$ is a continuous process, it holds that $\lim_{t \uparrow T} A_t = A_T$; hence $A_T$ is $\mathcal{F}_{T^-}$-measurable. Therefore,

$$\lim_{t \uparrow T} E_t[A_T] = E[A_T | \mathcal{F}_{T^-}] = A_T \quad a.s.$$ The result now follows from (8.1). $\square$

We furthermore introduce the following lemma that we employ in the proofs of Theorem 3.1, Lemmas 8.3 and 8.6 and Theorem 3.4. It provides helpful representations for the dynamics of the process $A = X - \alpha D$, where $X$ is an execution strategy and $D$ its deviation.

**Lemma 8.2** Let $x, d \in \mathbb{R}$ and $t \in [0, T]$. Suppose that $X = (X_s)_{s \in [t, T]}$ is a càdlàg semimartingale with $X_{t^-} = x$ and $X_T = 0$, and let $D = (D_s)_{s \in [t, T]}$ be the associated deviation process given by (1.2). It then holds for $A = (A_s)_{s \in [t, T]}$ defined by $A_s = X_s - \alpha_s D_s, s \in [t, T]$, that

$$dA_s = -D_s d\alpha_s + \alpha_s \rho_s D_s d[M]_s = (A_s - X_s) \left( \frac{d\alpha_s}{\alpha_s} - \rho_s d[M]_s \right), \quad s \in [t, T].$$

**Proof** It follows from (1.1), (2.1) and (1.2) that $d[\alpha, D]_s = -\alpha_s d[\gamma, X]_s$ for all $s \in [t, T]$. By integration by parts and (1.2), it then holds for all $s \in [t, T]$ that

$$dA_s = dX_s - D_s d\alpha_s - \alpha_s dD_s - d[\alpha, D]_s$$

$$= dX_s - D_s d\alpha_s - \alpha_s \rho_s D_s d[M]_s - \alpha \gamma_s dX_s - \alpha_s d[\gamma, X]_s + \alpha_s d[\gamma, X]_s$$

$$= -D_s d\alpha_s + \alpha_s \rho_s D_s d[M]_s.$$

The second equality in the claim now follows from the fact that $-D_s = \frac{A_s - X_s}{\alpha_s}, \quad s \in [t, T]$, by the definition of $A$. $\square$
Proof of Theorem 3.1 We fix \( x, d \in \mathbb{R}, \ t \in [0, T] \) and \( X \in \mathcal{A}_t(x, d) \) throughout the proof. First observe that it follows from (1.2) and the fact that \( \gamma = 1/\alpha \) that for all \( s \in [t, T] \), we have \( d[\alpha, D]_s = -\alpha_s d[\gamma, X]_s \) and \( d[D]_s = \frac{1}{\alpha_s^2} d[X]_s \). This shows that

\[
\alpha_T D_T^2 = \alpha_t d^2 + \int_{[t, T)} \alpha_s d D_s^2 + \int_{[t, T)} D_s^2 d\alpha_s + \int_{[t, T]} d[\alpha, D^2]_s
\]

\[
= \alpha_t d^2 + 2 \int_{[t, T]} \alpha_s D_s d D_s + \int_{[t, T]} \alpha_s d[D]_s + \int_{[t, T]} D_s^2 d\alpha_s
\]

\[
+ 2 \int_{[t, T]} D_s d[\alpha, D]_s
\]

\[
= \alpha_t d^2 - 2 \int_{[t, T]} \rho_s \alpha_s D_s^2 d[M]_s + 2 \int_{[t, T]} D_s dX_s
\]

\[
+ 2 \int_{[t, T]} \alpha_s D_s d[\gamma, X]_s + \int_{[t, T]} \frac{1}{\alpha_s} d[X]_s + \int_{[t, T]} D_s^2 d\alpha_s
\]

\[
- 2 \int_{[t, T]} \alpha_s D_s d[\gamma, X]_s
\]

\[
= \alpha_t d^2 - \int_t^T \alpha_s D_s^2 (2\rho_s + \mu_s - \sigma_s^2) d[M]_s + 2 \int_{[t, T]} D_s dX_s
\]

\[
+ \int_{[t, T]} \gamma_s d[X]_s - \int_t^T \alpha_s D_s^2 \sigma_s dM_s. \tag{8.2}
\]

The first equality in Lemma 8.2 and (2.1) prove that for all \( s \in [t, T] \)

\[
d(X_s - \alpha_s D_s) = \alpha_s D_s (\rho_s + \mu_s - \sigma_s^2) d[M]_s + \sigma_s \alpha_s D_s dM_s.
\]

In particular, the process \( X - \alpha D \) has continuous sample paths. Moreover, it follows that for all \( s \in [t, T] \)

\[
d(X_s - \alpha_s D_s)^2 = 2(X_s - \alpha_s D_s) d(X_s - \alpha_s D_s) + d[X - \alpha D]_s
\]

\[
= 2(X_s - \alpha_s D_s) (\alpha_s D_s (\rho_s + \mu_s - \sigma_s^2) d[M]_s + \sigma_s \alpha_s D_s dM_s)
\]

\[
+ \sigma_s^2 \alpha_s^2 D_s^2 d[M]_s
\]

\[
= \alpha_s D_s (2(X_s - \alpha_s D_s)(\rho_s + \mu_s - \sigma_s^2) + \sigma_s^2 \alpha_s D_s) d[M]_s
\]

\[
+ 2 \sigma_s \alpha_s D_s (X_s - \alpha_s D_s) dM_s. \tag{8.3}
\]
Next observe that (1.1) and (3.2) imply for all $s \in [t, T]$ that
\[
d(\gamma_s Y_s) = \gamma_s Y_s (\mu_s d[M]_s + \sigma_s dM_s) + \gamma_s Z_s dM_s + \gamma_s dM^\perp_s + \gamma_s \sigma_s Z_s d[M]_s
\]
\[
= \gamma_s \left( \frac{((\rho_s + \mu_s)Y_s + \sigma_s Z_s)^2}{\sigma_s^2 Y_s + \frac{1}{2}(2\rho_s + \mu_s - \sigma^2_s)} - \mu_s Y_s - \sigma_s Z_s \right) d[M]_s
\]
\[
= \frac{\gamma_s((\rho_s + \mu_s)Y_s + \sigma_s Z_s)^2}{\sigma_s^2 Y_s + \frac{1}{2}(2\rho_s + \mu_s - \sigma^2_s)} d[M]_s + \gamma_s (\sigma_s Y_s + Z_s) dM_s + \gamma_s dM^\perp_s.
\]
This and (8.3) prove that
\[
\gamma_T Y_T^- (X_T^- - \alpha_T D_T^-)^2
\]
\[
= \gamma_t Y_t (x - \alpha_t d)^2 + \int_{(t, T)} \gamma_s Y_s^- d(X_s^- - \alpha_s D_s)^2
\]
\[
+ \int_{(t, T)} (X_s^- - \alpha_s D_s)^2 d(\gamma_s Y_s) + [\gamma Y_s (X - \alpha D_s)]_{T^-}
\]
\[
= \gamma_t Y_t (x - \alpha_t d)^2
\]
\[
+ \int_t^T \left( D_s Y_s (2(X_s^- - \alpha_s D_s)(\rho_s + \mu_s - \sigma^2_s) + \sigma_s^2 \alpha_s D_s)ight)
\]
\[
+ (X_s^- - \alpha_s D_s)^2 \frac{(\rho_s + \mu_s)Y_s + \sigma_s Z_s)^2}{\sigma_s^2 Y_s + \frac{1}{2}(2\rho_s + \mu_s - \sigma^2_s)}
\]
\[
+ 2\sigma_s (\sigma_s Y_s + Z_s) D_s (X_s^- - \alpha_s D_s) d[M]_s
\]
\[
+ \int_t^T (2\sigma_s D_s Y_s (X_s^- - \alpha_s D_s) + \gamma_s (X_s^- - \alpha_s D_s)^2 (\sigma_s Y_s + Z_s)) dM_s
\]
\[
+ \int_{(t, T)} \gamma_s (X_s^- - \alpha_s D_s)^2 dM^\perp_s.
\]
(8.4)

Since $Y_T^- = \frac{1}{2}$ by Lemma 8.1, it holds that
\[
-\left(D_T^- - \frac{1}{2\alpha_T} X_T^- \right) X_T^- = \gamma_T Y_T^- (X_T^- - \alpha_T D_T^-)^2 - \frac{\alpha_T D^2_T^-}{2}.
\]

This and the fact that $X_T = 0$ show that
\[
\int_{[t, T]} D_s^- dX_s + \int_{[t, T]} \frac{\gamma_s}{2} d[X]_s = \int_{[t, T]} D_s^- dX_s + \int_{[t, T]} \frac{\gamma_s}{2} d[X]_s
\]
\[
+ \gamma_T Y_T^- (X_T^- - \alpha_T D_T^-)^2 - \frac{\alpha_T D^2_T^-}{2}.
\]
It then follows from (8.2) and (8.4) that
\[
\int_{[t,T]} D_s dX_s + \int_{[t,T]} \frac{\gamma_s}{2} d[X]_s
\]
\[= \gamma_t Y_t (x - \alpha_t d)^2 - \frac{\alpha_t d^2}{2}
\]
\[+ \int_t^T \left( D_s Y_s (2(X_s - \alpha_s D_s) (\rho_s + \mu_s - \sigma_s^2) + \sigma_s^2 \alpha_s D_s)
\]
\[+ (X_s - \alpha_s D_s)^2 \gamma_s \frac{((\rho_s + \mu_s) Y_s + \sigma_s Z_s)^2}{\sigma_s^2 Y_s + \frac{1}{2} (2\rho_s + \mu_s - \sigma_s^2)}
\]
\[+ 2\sigma_s (\sigma_s Y_s + Z_s) D_s (X_s - \alpha_s D_s) + \alpha_s D_s^2 \frac{2\rho_s + \mu_s - \sigma_s^2}{2} d[M]_s
\]
\[+ \int_t^T \left( 2\sigma_s D_s Y_s (X_s - \alpha_s D_s) + \gamma_s (X_s - \alpha_s D_s)^2 (\sigma_s Y_s + Z_s) + \frac{\sigma_s \alpha_s D_s^2}{2} \right) dM_s
\]
\[+ \int_{(t,T)} \gamma_s (X_s - \alpha_s D_s)^2 dM_s^\perp.
\]
We thus have, with \(\tilde{\beta}\) defined by (3.5) and \(J\) given by (1.3), that
\[
J_t(x, d, X)
\]
\[= \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t}
\]
\[+ E_t \left[ \int_t^T \frac{1}{\gamma_s} \left( \tilde{\beta}(\gamma_s X_s - D_s) + D_s \right)^2 \left( \sigma_s^2 Y_s + \frac{1}{2} (2\rho_s + \mu_s - \sigma_s^2) \right) d[M]_s
\]
\[+ \int_t^T \left( 2\sigma_s D_s Y_s (X_s - \alpha_s D_s)
\]
\[+ \gamma_s (X_s - \alpha_s D_s)^2 (\sigma_s Y_s + Z_s) + \alpha_s D_s^2 \frac{\sigma_s}{2} \right) dM_s
\]
\[+ \int_{(t,T)} \gamma_s (X_s - \alpha_s D_s)^2 dM_s^\perp \right].
\]
It therefore remains to show that
\[
E_t \left[ \int_t^T \left( 2\sigma_s D_s Y_s (X_s - \alpha_s D_s) + \gamma_s (X_s - \alpha_s D_s)^2 (\sigma_s Y_s + Z_s) + \frac{\alpha_s D_s^2 \sigma_s}{2} \right) dM_s \right] = 0
\]
(8.5)
and
\[
E_t \left[ \int_{(t,T)} \gamma_s (X_s - \alpha_s D_s)^2 dM_s^\perp \right] = 0.
\]
(8.6)
Consider first the integral \( \int_t^T \gamma_s(X_s - \alpha_s D_s)^2 Z_s dM_s \). By the Burkholder–Davis–Gundy inequality, we have for some constant \( c \in (0, \infty) \) that

\[
E_t \left[ \sup_{r \in [t,T]} \left| \int_t^r \gamma_s(X_s - \alpha_s D_s)^2 Z_s dM_s \right| \right] 
\leq c E_t \left[ \left( \int_t^T \gamma_s^2(X_s - \alpha_s D_s)^4 Z_s^2 d[M]_s \right)^{1/2} \right].
\]

Due to \( E_t[\int_t^T Z_s^2 d[M]_s] < \infty \) and (2.3), the Cauchy–Schwarz inequality implies that

\[
E_t \left[ \sup_{r \in [t,T]} \left| \int_t^r \gamma_s(X_s - \alpha_s D_s)^2 Z_s dM_s \right| \right] 
\leq c \left( E_t \left[ \sup_{s \in [t,T]} (\gamma_s^2(X_s - \alpha_s D_s)^4) \right] \right)^{1/2} \left( E_t \left[ \int_t^T Z_s^2 d[M]_s \right] \right)^{1/2} < \infty.
\]

Therefore, \( \int_t^T \gamma_s(X_s - \alpha_s D_s)^2 Z_s dM_s \) is a true martingale, and hence

\[
E_t \left[ \int_t^T \gamma_s(X_s - \alpha_s D_s)^2 Z_s dM_s \right] = 0.
\]

Similarly, \( E_t[[M^\perp]_T] < \infty \) and (2.3) imply (8.6). Furthermore, we obtain from (2.4) and the fact that \( Y \) is bounded that

\[
E_t \left[ \left( \int_t^T \gamma_s^2(X_s - \alpha_s D_s)^4 \sigma_s^2 Y_s^2 d[M]_s \right)^{1/2} \right] < \infty
\]

and hence

\[
E_t \left[ \int_t^T \gamma_s(X_s - \alpha_s D_s)^2 \sigma_s Y_s dM_s \right] = 0.
\]

In order to show that

\[
E_t \left[ \int_t^T 2\sigma_s D_s Y_s(X_s - \alpha_s D_s) dM_s \right] = 0,
\]

note that Young’s inequality implies for all \( s \in [t, T] \) that

\[
D_s^2(X_s - \alpha_s D_s)^2 \leq \frac{1}{2} (D_s^4 \alpha_s^2 + \gamma_s^2(X_s - \alpha_s D_s)^4).
\]
This together with $0 \leq Y \leq \frac{1}{2}$, (2.5) and (2.4) yields
\[
E_t \left[ \left( \int_t^T 4\sigma_s^2 D_s^2 Y_s^2 (X_s - \alpha_s D_s)^2 d[M]_s \right)^{1/2} \right] \\
\leq \frac{1}{\sqrt{2}} E_t \left[ \left( \int_t^T \sigma_s^2 D_s^4 \alpha_s^2 d[M]_s \right)^{1/2} \right] \\
+ \frac{1}{\sqrt{2}} E_t \left[ \left( \int_t^T \sigma_s^2 \gamma_s^2 (X_s - \alpha_s D_s)^4 d[M]_s \right)^{1/2} \right] < \infty.
\]

Moreover, it follows from (2.5) that also
\[
E_t \left[ \int_t^T \alpha_s D_s^2 \sigma_s d[M]_s \right] = 0.
\]

We thus have established (8.5) and (8.6), which completes the proof. □

The following uniqueness result for optimal strategies, which relies on the representation of the cost functional (3.6), is applied in the proofs of Lemma 3.3, Theorem 3.4 and Proposition 3.7.

**Lemma 8.3** Suppose (3.1) and (3.4) and fix a solution $(Y, Z, M^\perp)$ of the BSDE (3.2) that satisfies the properties in (3.4). Let $\tilde{\beta}$ be the process defined by (3.5) pertaining to $(Y, Z)$. Let $x, d \in \mathbb{R}$ and $t \in [0, T]$. Assume that $V_t(x, d) = \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t}$ and that there exists an optimal strategy in $A_t(x, d)$. Then the optimal strategy is unique up to $\mathcal{D}_M[t, T]$-nullsets.

**Proof** Let $X^*, X \in A_t(x, d)$ be two optimal strategies with associated deviation processes $D^*$ and $D$, respectively. Combine the assumption that the value process satisfies $V_t(x, d) = \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t}$ with Theorem 3.1 to obtain that a.s.,
\[
E_t \left[ \int_t^T \frac{1}{\gamma_s} \left( \tilde{\beta}_s (\gamma_s X_s^* - D_s^*) + D^*_s \right)^2 \left( \sigma_s^2 Y_s + \frac{1}{2} (2\rho_s + \mu_s - \sigma_s^2) \right) d[M]_s \right] = 0
\]

By taking expectations, it follows that
\[
E \left[ \int_t^T \frac{1}{\gamma_s} \left( \tilde{\beta}_s (\gamma_s X_s^* - D_s^*) + D^*_s \right)^2 \left( \sigma_s^2 Y_s + \frac{1}{2} (2\rho_s + \mu_s - \sigma_s^2) \right) d[M]_s \right] = 0.
\]

This implies
\[
\tilde{\beta}(\gamma X^* - D^*) + D^* = 0 \quad \mathcal{D}_M[t, T]-a.e. \tag{8.7}
\]

By Lemma 8.2, the process $A^* = (A^*_s)_{s \in [t, T]}$ defined by $A^*_s = X^*_s - \alpha_s D^*_s$ therefore satisfies
\[
dA^*_s = \tilde{\beta}_s A^*_s \left( \frac{d\alpha_s}{\alpha_s} - \rho_s d[M]_s \right), \quad s \in [t, T].
\]
For $X$, $D$ and $A = X - \alpha D$, we analogously obtain (8.7) and

$$dA_s = \tilde{\beta}_s A_s \left( \frac{d\alpha_s}{\alpha_s} - \rho_s d[M]_s \right), \quad s \in [t, T].$$

Hence $A$ and $A^*$ have the same starting point $A_t = x - \alpha_t d = A_t^*$ and satisfy the same dynamics. It follows that $A$ and $A^*$ are indistinguishable. Together with (8.7), this yields that $D = -\tilde{\beta} \gamma A = -\tilde{\beta} \gamma A^* = D^* \mathcal{D}_M|_{[t, T]}$-a.e. Finally, it follows from the definition of $A$ and $A^*$ that $X = X^* \mathcal{D}_M|_{[t, T]}$-a.e. \(\square\)

**Proof of Lemma 3.3** Suppose that $x = \frac{d}{\gamma_t}$. Let $X^* = (X^*_s)_{s \in [t, T]}$ be defined by $X^*_{t-} = x$, $X^*_s = 0$, $s \in [t, T]$. Then $X^*$ is a càdlàg semimartingale with $X^*_{t-} = x$ and $X^*_T = 0$. The associated deviation process $D^* = (D^*_s)_{s \in [t, T]}$ satisfies

$$D^*_t = d + \Delta D^*_s = d + \gamma_t \Delta X^*_s = d - \gamma_t x = 0,$$

and hence $D^*_s = 0$ for all $s \in [t, T]$. It follows that $X^*_s - \alpha_s D^*_s = 0$, $s \in [t, T]$, and thus conditions (2.3)–(2.5) are satisfied, i.e., $X^* \in \mathcal{A}_p(x, d)$. Since $D^*_s = 0$ and $\gamma_s X^*_s - D^*_s = 0$ for all $s \in [t, T]$, Theorem 3.1 yields that $X^*$ is optimal and that $V_t(x, d) = \frac{\gamma_t}{\gamma_t}(d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} = -\frac{d^2}{2\gamma_t}$. Uniqueness up to $\mathcal{D}_M|_{[t, T]}$-nullsets follows from Lemma 8.3. \(\square\)

For the proof of Lemma 8.6, we need the following technical result. It provides conditions which ensure that the conditional expectation of the supremum of a process with a certain exponential structure is a.s. finite (see also Remark 8.5 below).

**Lemma 8.4** Suppose (3.8) is satisfied. Let $\eta = (\eta_s)_{s \in [0, T]}$ and $\nu = (\nu_s)_{s \in [0, T]}$ be progressively measurable processes such that $|\eta| \leq c_\eta$ and $|\nu| \leq c_\nu \mathcal{D}_M$-a.e. for some constants $c_\eta, c_\nu \in (0, \infty)$. Let $t \in [0, T]$ and define $N = (N_s)_{s \in [t, T]}$ by

$$N_s = \exp \left( \int_t^s \eta_r dM_r + \int_t^s \nu_r d[M]_r \right), \quad s \in [t, T].$$

It then holds a.s. that

$$E_t \left[ \sup_{s \in [t, T]} N_s \right] \leq 2 \left( E_t \left[ e^{6c_\eta^2 ([M]_T - [M]_t)} \right] \right)^{\frac{1}{2}} \left( E_t \left[ e^{(2c_\nu + c_\eta^2) ([M]_T - [M]_t)} \right] \right)^{\frac{1}{2}} < \infty.$$

**Proof** We introduce the process $L = (L_s)_{s \in [t, T]}$ defined by

$$L_s = \exp \left( \int_t^s \eta_r dM_r - \frac{1}{2} \int_t^s \eta_r^2 d[M]_r \right), \quad s \in [t, T].$$

We then have

$$N_s = L_s \exp \left( \int_t^s \left( \nu_r + \frac{\eta_r^2}{2} \right) d[M]_r \right), \quad s \in [t, T],$$

and thus by the Cauchy–Schwarz inequality that
\[
E_t \left[ \sup_{s \in [t, T]} N_s \right] \leq \left( E_t \left[ \sup_{s \in [t, T]} L_s^2 \right] \right)^{1/2} \left( E_t \left[ \sup_{s \in [t, T]} \exp \left( \int_t^s (2\nu_r + \eta_r^2) d[M_r] \right) \right] \right)^{1/2}.
\]  
(8.8)

Since \(2\nu + \eta^2\) is bounded by \(2c\nu + c^2\eta\), it holds that
\[
E_t \left[ \sup_{s \in [t, T]} \exp \left( \int_t^s (2\nu_r + \eta_r^2) d[M_r] \right) \right] \leq E_t [e^{(2c\nu + c^2\eta)((M)_T - (M)_t)}].
\]  
(8.9)

Next, observe that
\[
E \left[ \exp \left( \frac{1}{2} \int_0^T \eta_r^2 d[M_r] \right) \right] < \infty
\]
because \(\eta^2\) is bounded and (3.8) is assumed to hold. Therefore, Novikov’s criterion implies that \(L\) is a true martingale. Thus it follows from Doob’s maximal inequality that
\[
\left( E_t \left[ \sup_{s \in [t, T]} L_s^2 \right] \right)^{1/2} \leq 2 (E_t [L_T^2])^{1/2}.
\]  
(8.10)

We define \(\tilde{L} = (\tilde{L}_s)_{s \in [t, T]}\) by
\[
\tilde{L}_s = \exp \left( \int_t^s 4\eta_r dM_r - \frac{1}{2} \int_t^s (4\eta_r)^2 d[M_r] \right), \quad s \in [t, T],
\]
and observe that by the Cauchy–Schwarz inequality, it holds that
\[
E_t [\tilde{L}_T^2] = E_t \left[ \exp \left( \int_t^T 2\eta_r dM_r - \int_t^T 4\eta_r^2 d[M_r] \right) \exp \left( \int_t^T 3\eta_r^2 d[M_r] \right) \right]
\leq (E_t [\tilde{L}_T])^{1/2} \left( E_t \left[ \exp \left( \int_t^T 6\eta_r^2 d[M_r] \right) \right] \right)^{1/2}.
\]  
(8.11)

As a nonnegative local martingale, \(\tilde{L}\) is a supermartingale, hence
\[
E_t [\tilde{L}_T] \leq \tilde{L}_t = 1.
\]
Together with the fact that \(\eta^2\) is bounded by \(c^2\eta\), we obtain from (8.11) that
\[
E_t [\tilde{L}_T^2] \leq (E_t [e^{6c^2\eta((M)_T - (M)_t)})]^{1/2}.
\]  
(8.12)

It follows from (8.8)–(8.10) and (8.12) that
\[
E_t \left[ \sup_{s \in [t, T]} N_s \right] \leq 2 (E_t [e^{6c^2\eta((M)_T - (M)_t)})]^{1/4} (E_t [e^{(2c\nu + c^2\eta)((M)_T - (M)_t)})]^{1/2}.
\]

This is a.s. finite due to (3.8).
Remark 8.5 The same computations also yield $E[\sup_{s \in [0,T]} N_s] < \infty$ (and hence also for all $t \in [0, T]$ that $E_t[\sup_{s \in [t,T]} N_s] < \infty$ a.s.) whenever we replace (3.8) and the boundedness assumption on $\eta$ and $\nu$ by
\[-E[\exp(\int_0^T 6\eta_s^2 d[M_s])] < \infty \quad \text{and} \quad -E[\exp(\int_0^T (2\nu_s + \eta_s^2)^+ d[M_s])] < \infty.

This observation is used in Remark 8.8 to justify the claim in part (b) of Remark 3.5.

In the next lemma, we show how to construct (see below) from a $\mathcal{D}_M$-a.e. bounded sequence $(\beta^n)_{n \in \mathbb{N}}$ of càdlàg semimartingales (8.13) a sequence of admissible strategies $(X^n)_{n \in \mathbb{N}}$ with the additional properties (8.14) and (8.16). We use this result in the proof of Theorem 3.4.

Lemma 8.6 Suppose (3.1) and (3.8) are satisfied. Assume $\rho$ and $\mu$ are $\mathcal{D}_M$-a.e. bounded. Let $(\beta^n)_{n \in \mathbb{N}}$ be a sequence of càdlàg semimartingales $\beta^n = (\beta^n_s)_{s \in [0,T]}$ that are $\mathcal{D}_M$-a.e. bounded uniformly in $n$. Let $t \in [0,T]$ and $x, d \in \mathbb{R}$. Define for each $n \in \mathbb{N}$ the process $X^n = (X^n_s)_{s \in [t,T]}$ by $X^n_t = x$,
\[ X^n_s = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t,s} (1 - \beta^n_s), \quad s \in [t,T], \tag{8.13} \]
and $X^n_T = 0$, where
\[ Q^n_s = -\int_0^s \beta^n_r \sigma_r dM_r - \int_0^s \beta^n_r (\mu_r + \rho_r - \sigma_r^2) d[M]_r, \quad s \in [0,T]. \]

Then we have:

(i) $X^n \in A_t(x,d)$ for all $n \in \mathbb{N}$.

(ii) For all $n \in \mathbb{N}$, the associated deviation process $D^n$ satisfying (1.2) with $X$ replaced by $X^n$ a.s. has the representations
\[ D^n_s = -\beta^n_s (\gamma_s X^n_s - D^n_s), \quad s \in [t,T], \tag{8.14} \]
and
\[ D^n_s = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t,s} (-\gamma_s \beta^n_s), \quad s \in [t,T], \tag{8.15} \]
and for the terminal value $D^n_T$, we have $D^n_T = (x - \frac{d}{\gamma_T})\mathcal{E}(Q^n)_{t,T} (-\gamma_T)$.

(iii) It holds that
\[ \sup_{n \in \mathbb{N}} E_t \left[ \sup_{s \in [t,T]} \left(\gamma_s^4 (X^n_s - \alpha_s D^n_s)^8\right) \right] < \infty \quad \text{a.s.} \tag{8.16} \]

Proof Denote the constants that bound $|\rho|$ and $|\mu|$ $\mathcal{D}_M$-a.e. by $c_\rho$ and $c_\mu$, respectively. Note that due to (3.1), $|\sigma|$ is $\mathcal{D}_M$-a.e. bounded by $c_\sigma = \sqrt{2c_\rho + c_\mu}$. Let $b \in (0, \infty)$ be such that $|\beta^n| \leq b \mathcal{D}_M$-a.e. for all $n \in \mathbb{N}$. Now fix $n \in \mathbb{N}$. Since $\beta^n$ is
a càdlàg semimartingale, $X^n$ defined by (8.13) is also a càdlàg semimartingale. Note that moreover $X^n_{t-} = x$ and $X^n_T = 0$. We denote by $D^n = (D^n_s)_{s \in [t, T]}$ the associated deviation process. Let $\hat{A}^n = (\hat{A}^n_s)_{s \in [t, T]}$ be the process defined by

$$\hat{A}^n_s = \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t, s}, \quad s \in [t, T].$$

Observe that $X^n_s = \hat{A}^n_s (1 - \beta^n_s)$ for all $s \in [t, T)$. Together with (2.1), it follows that

$$d\hat{A}^n_s = \beta^n_s \hat{A}^n_s \left(\frac{d\alpha_s}{\alpha_s} - \rho_s d[M]_s\right) = (X^n_s - \hat{A}^n_s) \left(-\frac{d\alpha_s}{\alpha_s} + \rho_s d[M]_s\right), \quad s \in [t, T].$$

Let $A^n = (A^n_s)_{s \in [t, T]}$ be the process defined by $A^n_s = X^n_s - \frac{D^n_s}{\gamma_t}$. Then by Lemma 8.2, $\hat{A}^n$ and $A^n$ satisfy the same dynamics and start in the same point $\hat{A}^n_t = x - \frac{d}{\gamma_t} = A^n_t$ at time $t$. Consequently, they are indistinguishable. This implies that

$$D^n_s = \gamma_s (X^n_s - A^n_s) = \gamma_s (X^n_s - \hat{A}^n_s) = -\beta^n_s \gamma_s \hat{A}^n_s, \quad s \in [t, T), \quad (8.17)$$

and, proceeding further,

$$D^n_s = -\beta^n_s \gamma_s A^n_s = -\beta^n_s (\gamma_s X^n_s - D^n_s), \quad s \in [t, T).$$

This shows (8.14), while (8.15) follows from (8.17). For the terminal value $D^n_T$, we have

$$D^n_T = D^n_{T-} + \gamma_T \Delta X^n_T = D^n_{T-} - \gamma_T X^n_{T-}$$

$$= \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t, T} \left(1 - \gamma_T (1 - \beta^n_{T-})\right)$$

$$= \left(x - \frac{d}{\gamma_t}\right)\mathcal{E}(Q^n)_{t, T} (-\gamma_T).$$

Furthermore, it follows from $A^n_s = \hat{A}^n_s$, $s \in [t, T]$, that

$$\gamma_s^4 (X^n_s - \alpha_s D^n_s)^8 = \gamma_s^4 (x - \alpha_t d)^8 \mathcal{E}(Q^n)_{t, s}^8, \quad s \in [t, T].$$

Note that

$$\gamma_s = \gamma_t \exp \left(\int_t^s \mu_r d[M]_r + \int_t^s \sigma_r dM_r - \frac{1}{2} \int_t^s \sigma_r^2 d[M]_r\right), \quad s \in [t, T].$$

Therefore, we have

$$E_t\left[\sup_{s \in [t, T]} (\gamma_s^4 (X^n_s - \alpha_s D^n_s)^8)\right]$$

$$= \gamma_t^4 (x - \alpha_t d)^8$$

$$\times E_t\left[\sup_{s \in [t, T]} e^{\int_t^s (4\sigma_r - 8\beta^n_s \sigma_r) dM_r + \int_t^s (-8\beta^n_s (\mu_r + \rho_r - \sigma^2) - 4(\beta^n_s)^2 \sigma^2 + 4\mu_r - 2\sigma^2) d[M]_r}\right].$$
Define \( n^n = 4 \sigma - 8 \beta^n \sigma \) and \( v^n = -8 \beta^n (\mu + \rho - \sigma^2) - 4 (\beta^n)^2 \sigma^2 + 4 \mu - 2 \sigma^2. \) Since (3.8) is assumed to hold true and since we have \( |n^n| \leq 4c_\sigma + 8bc_\sigma \) as well as \( |v^n| \leq 8b (c_\mu + c_\rho + c_\sigma^2) + 4b^2 c_\sigma^2 + 4c_\mu + 2c_\sigma^2, \) we obtain (8.16) from Lemma 8.4. Observe furthermore that by Jensen’s inequality, it follows that (2.3) holds true. To show (2.4), note that (3.8) and (2.3) and the Cauchy–Schwarz inequality prove that (2.4) is satisfied. It follows from (8.14) that \( \alpha^2_s (D^n_s)^4 = (\beta^n_s)^4 \gamma^2_s (X^n_s - \alpha_s D^n_s)^4, \) \( s \in [t, T). \) Since \( \beta^n \) is \( \mathcal{D}_{M-}\)a.e. bounded, the fact that (2.4) is satisfied hence already implies that (2.5) holds true as well. We have thus shown that \( X^n \in \mathcal{A}_t(\omega, d). \)

In the following lemma, we obtain the existence of an approximating càdlàg semimartingale sequence \( (\beta^n)_{n \in \mathbb{N}} \) for any progressively measurable, \( \mathcal{D}_{M-}\)a.e. bounded process \( \beta. \) This enables us to exploit Lemma 8.6 for the proof of the representation of the value process in Theorem 3.4.

**Lemma 8.7** Assume that \( E[[M]_T] < \infty \) and let \( \beta = (\beta_s)_{s \in [0, T]} \) be a progressively measurable process that is bounded \( \mathcal{D}_{M-}\)a.e. Then there exists a sequence \( (\beta^n)_{n \in \mathbb{N}} \) of càdlàg semimartingales \( \beta^n = (\beta^n_s)_{s \in [0, T]} \) that are \( \mathcal{D}_{M-}\)a.e. bounded uniformly in \( n \) and such that for all \( p \in [1, \infty), \) we have \( E[\int_0^T |\beta_s - \beta^n_s|^p d[M]_s] \to 0 \) as \( n \to \infty. \)

**Proof** By Karatzas and Shreve [26, Lemma 3.2.7], there exists a sequence \( (\hat{\beta}^n)_{n \in \mathbb{N}} \) of (càdlàg) simple processes \( \hat{\beta}^n = (\hat{\beta}^n_s)_{s \in [0, T]} \) such that \( E[\int_0^T |\beta_s - \hat{\beta}^n_s|^2 d[M]_s] \to 0 \) as \( n \to \infty. \) Define \( \hat{\beta}^n_s (\omega) = \lim_{n \to \infty} \hat{\beta}^n_s (\omega), \) \( s \in [0, T], \) \( \omega \in \Omega, \) \( n \in \mathbb{N}. \) Let \( b \in (0, \infty) \) be such that \( |\beta| \leq b \) \( \mathcal{D}_{M-}\)a.e. and define for each \( n \in \mathbb{N} \) the process \( \beta^n \) by \( \beta^n_s (\omega) = (\beta^n_s (\omega) \wedge b) \vee (-b), \) \( s \in [0, T], \) \( \omega \in \Omega. \) Since \( |\beta^n_s (\omega)| \leq b \) for all \( s \in [0, T], \) \( \omega \in \Omega, \) \( n \in \mathbb{N}, \) and \( \beta^n \) is càdlàg for all \( n \in \mathbb{N}, \) it follows that \( (\beta^n)_{n \in \mathbb{N}} \) is a sequence of càdlàg semimartingales that are \( \mathcal{D}_{M-}\)a.e. bounded uniformly in \( n. \) Furthermore, since \( |\beta - \beta^n| \leq |\beta - \hat{\beta}^n| \) and \( \beta^n = \hat{\beta}^n \mathcal{D}_{M-}\)a.e., we have that

\[
E \left[ \int_0^T |\beta_s - \beta^n_s|^2 d[M]_s \right] \leq E \left[ \int_0^T |\beta_s - \hat{\beta}^n_s|^2 d[M]_s \right] = E \left[ \int_0^T |\beta_s - \hat{\beta}^n_s|^2 d[M]_s \right] \to 0 \quad \text{as } n \to \infty.
\]

For \( p \in [1, 2), \) the convergence \( E[\int_0^T |\beta_s - \beta^n_s|^p d[M]_s] \to 0 \) follows from Jensen’s inequality. For \( p \in (2, \infty), \) the convergence holds due to \( |\beta - \beta^n| \leq 2b \) \( \mathcal{D}_{M-}\)a.e. \( \square \)

**Proof of Theorem 3.4** (i) We first prove the representation of the value process.

Let \( t \in [0, T) \) and \( x, d \in \mathbb{R}. \) Since \( \tilde{\beta} \) is \( \mathcal{D}_{M-}\)a.e. bounded and we assume (3.8), there exists by Lemma 8.7 a sequence \( (\beta^n)_{n \in \mathbb{N}} \) of càdlàg semimartingales \( \beta^n = (\beta^n_s)_{s \in [0, T]} \) that are \( \mathcal{D}_{M-}\)a.e. bounded uniformly in \( n \) and such that for all \( p \in [1, \infty), \) it holds that

\[
E_t \left[ \int_t^T |\tilde{\beta}_s - \beta^n_s|^p d[M]_s \right] \to 0 \quad \text{in } L^1(P) \text{ as } n \to \infty. \quad (8.18)
\]
In particular, by passing to a subsequence, we can obtain almost sure convergence in (8.18). We further obtain from Lemma 8.6 that for each \( n \in \mathbb{N} \), there exists \( X^n \in \mathcal{A}_t(x, d) \) such that \( D^n_s = -\beta^n_s (\gamma^n_s X^n_s - D^n_s) \), \( s \in [t, T) \), where \( D^n \) denotes the deviation process associated to \( X^n \), and that

\[
\sup_{n \in \mathbb{N}} E_t \left[ \sup_{s \in [t, T)} \left( \gamma^n_s (X^n_s - \alpha_s D^n_s)^8 \right) \right] < \infty \quad \text{a.s.} \quad (8.19)
\]

It then holds that

\[
\tilde{\beta}_s (\gamma^n_s X^n_s - D^n_s) + D^n_s = (\tilde{\beta}_s - \beta^n_s) (\gamma^n_s X^n_s - D^n_s), \quad s \in [t, T).
\]

Together with Theorem 3.1 and \( X^n \in \mathcal{A}_t(x, d) \), this implies for all \( n \in \mathbb{N} \) that a.s.,

\[
V_t(x, d) \leq J_t(x, d, X^n) = Y_t \gamma_t (d - \gamma tx)^2 - \frac{d^2}{2\gamma_t} + E_t \left[ \int_t^T \frac{1}{\gamma^n_s} \left( \tilde{\beta}_s - \beta^n_s \right)^2 (\gamma^n_s X^n_s - D^n_s)^2 \right.
\]

\[
\times \left( \alpha_s^2 Y_s + \frac{1}{2} (2\rho_s + \mu_s - \sigma_s^2) \right) d[M]_s \right]. \quad (8.20)
\]

By the Cauchy–Schwarz inequality we have for all \( n \in \mathbb{N} \) that

\[
E_t \left[ \int_t^T \frac{1}{\gamma^n_s} (\tilde{\beta}_s - \beta^n_s)^2 (\gamma^n_s X^n_s - D^n_s)^2 d[M]_s \right]
\]

\[
= E_t \left[ \int_t^T \gamma^n_s (\tilde{\beta}_s - \beta^n_s)^2 (X^n_s - \alpha_s D^n_s)^2 d[M]_s \right]
\]

\[
\leq \left( E_t \left[ \int_t^T \gamma^n_s^2 (X^n_s - \alpha_s D^n_s)^4 d[M]_s \right] \right)^{1/2} \left( E_t \left[ \int_t^T (\tilde{\beta}_s - \beta^n_s)^4 d[M]_s \right] \right)^{1/2}. \quad (8.21)
\]

Moreover, we have for all \( n \in \mathbb{N} \) that

\[
E_t \left[ \int_t^T \gamma^n_s^2 (X^n_s - \alpha_s D^n_s)^4 d[M]_s \right]
\]

\[
\leq E_t \left[ \sup_{s \in [t, T]} \left( \gamma^n_s^2 (X^n_s - \alpha_s D^n_s)^4 \right) ([M]_T - [M]_t) \right]
\]

\[
\leq \left( E_t \left[ \sup_{s \in [t, T]} \left( \gamma^n_s^4 (X^n_s - \alpha_s D^n_s)^8 \right) \right] \right)^{1/2} \left( E_t \left[ ([M]_T - [M]_t)^2 \right] \right)^{1/2}. \quad (8.22)
\]

Because \( \rho, \mu, \sigma \) and \( Y \) are all bounded, it follows from (3.8), (8.19), (8.22), (8.18) and (8.21) that along a subsequence, the right-hand side of (8.20) converges to the
quantity \( \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} \) a.s. as \( n \to \infty \). We obtain the inequality

\[
V_t(x, d) \leq \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} \quad \text{a.s.}
\]

The converse inequality is provided in Theorem 3.1.

(ii) Let \( x \neq \frac{d}{\gamma_0} \). In this step, we prove that if there exists an optimal strategy, then there is a càdlàg semimartingale \( \beta = (\beta_s)_{s \in [0, T]} \) with \( \tilde{\beta} = \beta \ D_M\text{-a.e.} \) For the other implication and for the uniqueness statement, consider \( t = 0 \) in step (iii) below.

Assume that there exists an optimal strategy \( X^* = (X^*_s)_{s \in [0, T]} \in \mathcal{A}_0(x, d) \). It then follows from \( V_0(x, d) = \frac{Y_0}{\gamma_0} (d - \gamma_0 x)^2 - \frac{d^2}{2\gamma_0} \) and Theorem 3.1 that

\[
\tilde{\beta}(\gamma X^* - D^*) + D^* = 0 \quad D_M\text{-a.e.,} \tag{8.23}
\]

where \( D^* = (D^*_s)_{s \in [0, T]} \) denotes the deviation process associated to \( X^* \) via (1.2). This yields for the process \( A^* = (A^*_s)_{s \in [0, T]} \) defined by \( A^*_s = X^*_s - \alpha_s D^*_s \) that, by Lemma 8.2,

\[
dA^*_s = \tilde{\beta}_s A^*_s \left( \frac{d\alpha_s}{\alpha_s} - \rho_s d[M]_s \right), \quad s \in [0, T],
\]

and \( A^*_0 = x - \frac{d}{\gamma_0} \). It follows that for all \( s \in [0, T] \), we have

\[
A^*_s = \left( x - \frac{d}{\gamma_0} \right) \mathcal{E} \left( \int_0^s \tilde{\beta}_r \alpha_r d\alpha_r - \int_0^s \rho_r \beta_r d[M]_r \right)_s = \left( x - \frac{d}{\gamma_0} \right) \mathcal{E}(\tilde{Q})_s,
\]

where \( \tilde{Q}_s = \int_0^s \frac{\tilde{\beta}_r}{\alpha_r} d\alpha_r - \int_0^s \rho_r \beta_r d[M]_r \), \( s \in [0, T] \). Since \( \tilde{Q} = (\tilde{Q}_s)_{s \in [0, T]} \) is a continuous semimartingale, its stochastic exponential \( \mathcal{E}(\tilde{Q}) \) is strictly positive. Together with the assumption \( x \neq \frac{d}{\gamma_0} \), it follows that \( A^* \) is nonvanishing. Consequently, \( \beta = -\frac{D^*}{\gamma A^*} \) is a càdlàg semimartingale, whereas (8.23) proves that \( \tilde{\beta} = \beta \ D_M\text{-a.e.} \)

(iii) Suppose that there exists a càdlàg semimartingale \( \beta = (\beta_s)_{s \in [0, T]} \) with \( \tilde{\beta} = \beta \ D_M\text{-a.e.} \), and let \( t \in [0, T] \). It then follows from Lemma 8.6 that (3.10) defines a strategy \( X^* \in \mathcal{A}_t(x, d) \) such that for the associated deviation process \( D^* \), we have the representation (3.11) and, moreover,

\[
D^* = -\beta(\gamma X^* - D^*) = -\tilde{\beta}(\gamma X^* - D^*) \quad D_M|_{[t, T]}\text{-a.e.}
\]

Then Theorem 3.1 implies that \( J_t(x, d, X^*) = \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} \), and since \( V_t(x, d) = \frac{Y_t}{\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} \), the strategy \( X^* \) is optimal. The uniqueness up to \( D_M|_{[t, T]}\)-nullsets follows from Lemma 8.3. \( \Box \)

**Remark 8.8** To justify the claim in part (b) of Remark 3.5, it now remains to verify that for any \( x, d \in \mathbb{R} \) and \( t \in [0, T] \), the strategy \( X^* \) in (3.10) belongs to \( \mathcal{A}_t(x, d) \), i.e., satisfies (2.3)–(2.5), under conditions 1)–4) in part (b) of Remark 3.5 only. (2.3)
\( (2.4) \) are verified essentially the same argument (use Remark 8.5 and condition 4) in part (b) of Remark 3.5. For \( (2.5) \), we notice that 
\[
D^* = -\beta (\gamma X^* - D^*) \quad \text{\( D^M|_{[t,T]} \)-a.e.}
\]
implies 
\[
(D^*)^4 = \gamma^2 (X^* - \alpha D^*)^4 \beta^4 \quad \text{\( D^M|_{[t,T]} \)-a.e.}
\]
Hence \( (2.5) \) follows via the Cauchy–Schwarz inequality from \( (2.3) \) and condition 3) in part (b) of Remark 3.5.

**Proof of Corollary 3.6** The assumptions allow to apply Theorem 3.4. First notice that for any strategy \( X \), the associated process \( X - \frac{D_u}{\gamma_u} \) is continuous (although both \( X \) and \( D \) can have jumps), which follows from \( (1.2) \). Together with \( (3.10) \) and \( (3.11) \), this yields for any \( u \in (t,T) \) that
\[
X_u - \frac{D_u}{\gamma_u} = X_u - \frac{D_u}{\gamma_u} = (x - \frac{d}{\gamma_t}) \mathcal{E}(Q)_{t,u}.
\]
(8.24)

All statements of the corollary now follow from (8.24) and the trivial fact that for all \( s \in [u, T) \), we have \( \mathcal{E}(Q)_{t,u} \mathcal{E}(Q)_{u,s} = \mathcal{E}(Q)_{t,s} \).

**Proof of Proposition 3.7** If \( \rho \equiv 0 \), the driver \( (3.3) \) of the BSDE \( (3.2) \) equals 0 for \( (Y,Z) = (\frac{1}{2},0) \). Hence \( (Y,Z,M^\perp) = (\frac{1}{2},0,0) \) solves the BSDE \( (3.2) \), and \( (3.4) \) is clearly satisfied. We then obtain that \( \tilde{\beta}_s = 1 \) for all \( s \in [0, T] \). By Theorem 3.1, it holds for all \( x, d \in \mathbb{R} \), \( t \in [0, T] \) and \( X \in \mathcal{A}_t(x,d) \) that
\[
J_t(x, d, X) = \frac{1}{2\gamma_t} (d - \gamma_t x)^2 - \frac{d^2}{2\gamma_t} + E_t \left[ \int_t^T \frac{1}{2} \gamma_s \mu_s X_s^2 d[M_s] \right].
\]
(8.25)

Notice that due to \( (3.1) \) and \( \rho \equiv 0 \), the process \( \mu \) is positive. The optimality of closing the position immediately and the formula for the value process now follow from (8.25). The uniqueness up to \( D^M|_{[t,T]} \)-nullsets follows from Lemma 8.3.

**Proof of Proposition 3.8** By Theorem 3.4, we have \( \gamma_t Y_t(1) = V_t(1,0) = \gamma_t Y_t(2), \) \( t \in [0, T] \). Therefore \( Y(1) \) and \( Y(2) \) are indistinguishable. Comparing the canonical decompositions (see Jacod and Shiryaev [25, Sect. I.4c]) of the special semimartingale \( Y = Y(1) = Y(2) \), where \( f \) denotes the driver \( (3.3) \) of the BSDE \( (3.2) \), gives
\[
Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s(1)) d[M_s] + \int_0^t Z_s(1) dM_s + M^\perp_t(1)
\]
\[
= Y_0 - \int_0^t f(s, Y_s, Z_s(2)) d[M_s] + \int_0^t Z_s(2) dM_s + M^\perp_t(2), \quad t \in [0, T].
\]
For the local martingale parts, we have
\[
\int_0^t Z_s(1) dM_s + M^\perp_t(1) = \int_0^t Z_s(2) dM_s + M^\perp_t(2).
\]
(8.26)
This implies for all \( t \in [0, T] \) that
\[
[M^\perp,(1) - M^\perp,(2)]_t = \left[ M^\perp,(1) - M^\perp,(2), \int_0^t (Z_s^{(2)} - Z_s^{(1)})dM_s \right]_t
\]
\[
= \int_0^t (Z_s^{(2)} - Z_s^{(1)})d[M^\perp,(1) - M^\perp,(2), M]_s = 0.
\]

Thus \( M^\perp,(1) - M^\perp,(2) \) is a local martingale with \( [M^\perp,(1) - M^\perp,(2)] = 0 \) and starting in 0. It follows from the Burkholder–Davis–Gundy inequality that \( M^\perp,(1) \) and \( M^\perp,(2) \) are indistinguishable. Then (8.26) implies further that \( \int_0^t (Z_s^{(2)} - Z_s^{(1)})dM_s = 0 \) and hence
\[
\int_0^t (Z_s^{(2)} - Z_s^{(1)})^2d[M]_s = \left[ \int_0^t (Z_s^{(2)} - Z_s^{(1)})dM_s \right]^2 = 0.
\]

It follows that \( Z^{(1)} = Z^{(2)} \) \( D_M \)-a.e. \( \square \)

**Appendix A: Once again on the cost functional and the dynamics of the deviation process**

Here, we motivate the dynamics (1.2) of the deviation process and the cost functional (1.3) via a limiting procedure from a discrete-time setting.

Without loss of generality, we consider the starting time \( t = 0 \). We fix an initial position \( x \in \mathbb{R} \) and an initial deviation \( d \in \mathbb{R} \) and consider a continuous-time execution strategy \( X \in A_0(x, d) \). For any (large) \( N \in \mathbb{N} \), we set \( h = \frac{T}{N} \) and consider trading in discrete time at points of the grid \( \{kh: k = 0, \ldots, N\} \). More precisely, the continuous-time strategy \( X \) is approximated by the discrete-time strategy that consists of trades \( \xi_{kh}, k \in \{0, \ldots, N\} \), at the grid points, where
\[
\xi_0 = X_0 - x, \quad \xi_{kh} = X_{kh} - X_{(k-1)h}, \quad k \in \{1, \ldots, N\}.
\]

Notice that \( \xi_{kh} \) is \( \mathcal{F}_{kh} \)-measurable, \( k = 0, \ldots, N \). Further, for \( k \in \{1, \ldots, N\} \), we introduce \( \beta_{kh} = \exp(-\int_{(k-1)h}^{kh} \rho_s d[M]_s) \) and the notations \( \eta_r = \exp(-\int_0^r \rho_s d[M]_s) \) and \( \nu_r = \gamma_r \exp(\int_0^r \rho_s d[M]_s), r \in [0, T] \).

In the discrete-time setting of Ackermann et al. [1], the deviation process (now denoted by \( \tilde{D}^{(h)} \)) is defined by
\[
\tilde{D}^{(h)}_{0-} = d, \quad \tilde{D}^{(h)}_{(kh)-} = (\tilde{D}^{(h)}_{((k-1)h)-} + \gamma_{(k-1)h} \xi_{(k-1)h}) \beta_{kh}, \quad k \in \{1, \ldots, N\}.
\]

The minus in the subscript of \( \tilde{D}^{(h)}_{(kh)-} \) is purely notational (this is a discrete-time process); the meaning of \( \tilde{D}^{(h)}_{(kh)-} \) is that this is the deviation at time \( kh \) directly prior to the trade \( \xi_{kh} \) at time \( kh \), and we preserve the minus sign in order to make the notation consistent with [1]. A straightforward calculation shows that
\[
\tilde{D}^{(h)}_{(kh)-} = d \prod_{l=1}^k \beta_{lh} + \sum_{i=1}^k \gamma_{(i-1)h} \xi_{(i-1)h} \prod_{l=i}^k \beta_{lh}, \quad k \in \{1, \ldots, N\}.
\]
Substituting the definition of $\beta_{kh}$, we obtain that for all $k \in \{1, \ldots, N\}$,

$$
\tilde{D}_{(kh)}^{(h)} = e^{-\int_0^{kh} \rho(s)d[M](s)}d + \sum_{i=1}^{k} \gamma(i-1)h \xi(i-1)he^{-\int_0^{(i-1)h} \rho(s)d[M](s)}
= \eta_{kh}\left(d + \sum_{i=1}^{k} \nu(i-1)h \xi(i-1)h\right) = \eta_{kh}L_{(kh)}^{(h)}, \tag{A.1}
$$

where for $k \in \{0, \ldots, N\}$, we set

$$
L_{kh}^{(h)} = d + \sum_{j=0}^{k} \nu_{jh} \xi_{jh} = d + \gamma(X_0 - x) + \sum_{j=1}^{k} \nu_{jh}(X_{jh} - X_{(j-1)h})
= d + \gamma(X_0 - x) + \sum_{j=1}^{k} \nu_{(j-1)h}(X_{jh} - X_{(j-1)h})
+ \sum_{j=1}^{k} (\nu_{jh} - \nu_{(j-1)h})(X_{jh} - X_{(j-1)h}).
$$

The last expression shows that the continuous-time limit, for $N \to \infty$ (and hence $h = \frac{T}{N} \to 0$) of the processes $(L_{kh}^{(h)})_{k \in \{0, \ldots, N\}}$, is the process $(L_s)_{s \in [0, T]}$ given by

$$
L_s = d + \int_{[0,s]} \nu_r dX_r + \int_{[0,s]} d[v, X]_r, \quad s \in [0, T]
$$

(apply Jacod and Shiryaev [25, Proposition I.4.44 and Theorem I.4.47]). Combining this with (A.1) and the definition of $\nu_r, r \in [0, T]$, recovers that the continuous-time limit of the processes $(\tilde{D}_{(kh)}^{(h)})_{k \in \{0, \ldots, N\}}$ is the process $(D_s)_{s \in [0, T]}$ given by

$$
D_s = \eta_s L_s, \quad s \in [0, T]
$$

(and $D_{0-} = d$), which is nothing else but (2.2) or, equivalently, (1.2).

We now turn to the cost functional. In the discrete-time setting, the cost is given by

$$
\sum_{j=0}^{N} \left( \tilde{D}_{(jh)}^{(h)} + \frac{\gamma_{jh}}{2} \xi_{jh} \right) \xi_{jh}.
$$

Set $X_{-h} = X_{0-} (= x)$. Then it holds that

$$
\sum_{j=0}^{N} \left( \tilde{D}_{(jh)}^{(h)} + \frac{\gamma_{jh}}{2} \xi_{jh} \right) \xi_{jh}
= \sum_{j=0}^{N} \tilde{D}_{(jh)}^{(h)}(X_{jh} - X_{(j-1)h}) + \sum_{j=0}^{N} \frac{\gamma(j-1)h}{2}(X_{jh} - X_{(j-1)h})^2
+ \sum_{j=0}^{N} \frac{1}{2}(\gamma_{jh} - \gamma_{(j-1)h})(X_{jh} - X_{(j-1)h})^2.
$$

$$
\tag{A.2}
$$
For the first term on the right-hand side of (A.2), we have

\[ N \sum_{j=0}^{N} \tilde{D}^{(h)}_{(jh)}(X_{jh} - X_{(j-1)h}) = \sum_{j=0}^{N} \eta_{jh}L^{(h)}_{(j-1)h}(X_{jh} - X_{(j-1)h}) \]

\[ = \sum_{j=0}^{N} \eta_{(j-1)h}L^{(h)}_{(j-1)h}(X_{jh} - X_{(j-1)h}) \]

\[ + \sum_{j=0}^{N} L^{(h)}_{(j-1)h}(\eta_{jh} - \eta_{(j-1)h})(X_{jh} - X_{(j-1)h}), \]

which has the continuous-time limit

\[ \int_{[0,T]} \eta_{s}L_{s}^{-} dX_{s} + \int_{[0,T]} L_{s}^{-} d[\eta, X]_{s} = \int_{[0,T]} D_{s}^{-} dX_{s}, \]

as \( \eta \) is a continuous process of finite variation. Further, the second term on the right-hand side of (A.2) tends to \( \int_{[0,T]} \frac{\gamma_{s}}{2} d[X]_{s} \), and the third term to \( \frac{1}{2} \gamma_{T}^{2} [X]_{T} = 0 \) because \( \gamma \) is continuous. As the continuous-time limit of the discrete-time cost, we thus obtain

\[ \int_{[0,T]} D_{s}^{-} dX_{s} + \int_{[0,T]} \frac{\gamma_{s}}{2} d[X]_{s}, \]

which motivates our form of the cost functional in continuous time.

**Appendix B: Heuristic derivation of the BSDE (3.2)**

We have seen that the BSDE (3.2) plays a central role both in our results and in the proofs. But where does it come from? In this appendix, we motivate the BSDE (3.2) via a heuristic limiting procedure from discrete time.

To this end, we consider a discrete-time version of the stochastic control problem (1.4). For \( h > 0 \) such that \( h = \frac{T}{N} \) for some \( N \in \mathbb{N} \), \( t \in [0, T] \) and \( x, d \in \mathbb{R} \), let \( \mathcal{A}^{h}_{t}(x, d) \) be the subset of all processes \( X = (X_{s})_{s \in [t,T]} \in \mathcal{A}_{t}(x, d) \) of the form

\[ X_{s} = \sum_{k=0}^{N} X_{(kh) \vee t} 1_{[kh,(k+1)h]}(s) \]

for all \( s \in [t, T] \). Moreover, let

\[ V_{t}^{h}(x, d) = \text{ess inf}_{X \in \mathcal{A}^{h}_{t}(x, d)} J_{t}(x, d, X) \]

for all \( x, d \in \mathbb{R}, t \in [0, T], h > 0 \) with \( h = \frac{T}{N} \) for some \( N \in \mathbb{N} \). Then it follows from Ackermann et al. [1] that for each \( h > 0 \) with \( h = \frac{T}{N} \) for some \( N \in \mathbb{N} \), there exists a process \( Y^{h} = (Y_{t}^{h})_{t \in [0,h,...,T]} \) such that

\[ V_{t}^{h}(x, d) = \frac{\gamma_{h}}{\gamma_{t}} (d - \gamma_{t} x)^{2} - \frac{d^{2}}{2 \gamma_{t}}, \]

\( x, d \in \mathbb{R}, t \in [0, h, \ldots, T] \). The discrete-time process \( Y^{h} \) is given by the backward recursion

\[ Y_{T}^{h} = \frac{1}{2} \text{ and, for } t \in [0, h, \ldots, T - h], \]

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\[ Y_t^h = E_t \left[ \frac{Y_{t+h}^h}{Y_t} \right] \]

\[ - \frac{\left( E_t[Y_{t+h}^h \left( e^{-\int_t^{t+h} \rho_s d[M]_s} - \frac{Y_{t+h}}{Y_t} \right)} \right)^2}{E_t[Y_{t+h}^h \frac{Y_{t+h}}{Y_t} \left( e^{-\int_t^{t+h} \rho_s d[M]_s} - \frac{Y_{t+h}}{Y_t} \right)^2 + \frac{1}{2} (1 - \frac{Y_{t+h}}{Y_t} e^{-2 \int_t^{t+h} \rho_s d[M]_s})]} . \] (B.1)

We aim at deriving — at least heuristically — the dynamics of the continuous-time limit \( Y = (Y_t)_{t \in [0, T]} \) of \( Y^h \). To this end, we suppose that \( Y \) can be decomposed as

\[ dY_t = a_t d[M]_t + Z_t dM_t + dM_t^\perp, \quad t \in [0, T], \] (B.2)

where \((a_t)_{t \in [0, T]}, (Z_t)_{t \in [0, T]}\) are progressively measurable processes \( (a_t)_{t \in [0, T]} \) is still to be determined) and \( M^\perp = (M^\perp_t)_{t \in [0, T]} \) is a local martingale orthogonal to \( M \). From (B.2), we deduce that \( a_t \) should be identified as the limit

\[ a_t = \lim_{h \to 0} \frac{E_t[Y_{t+h}^h] - Y_t}{E_t[[M]_{t+h} - [M]_t]}, \quad t \in [0, T]. \]

Assume that replacing \( Y^h \) with \( Y \) in (B.1) introduces an error only of the magnitude \( o(E_t[[M]_{t+h} - [M]_t]) \). Then for all \( t \in [0, T] \), we can get the expression for \( a_t \) by evaluating the limit, for \( h \to 0 \), of

\[ \frac{1}{E_t[[M]_{t+h} - [M]_t]} \times \left( E_t[Y_{t+h}^h] - E_t \left[ \frac{Y_{t+h}^h}{Y_t} Y_{t+h} \right] \right) + \frac{\left( E_t[Y_{t+h}^h \left( e^{-\int_t^{t+h} \rho_s d[M]_s} - \frac{Y_{t+h}}{Y_t} \right)} \right)^2}{E_t[Y_{t+h}^h \frac{Y_{t+h}}{Y_t} \left( e^{-\int_t^{t+h} \rho_s d[M]_s} - \frac{Y_{t+h}}{Y_t} \right)^2 + \frac{1}{2} (1 - \frac{Y_{t+h}}{Y_t} e^{-2 \int_t^{t+h} \rho_s d[M]_s})]} . \] (B.3)

For the remainder of this section, we fix \( t \in [0, T] \) and assume that all stochastic integrals with respect to \( dM \) and \( dM^\perp \) that appear are true martingales. We define the process \( \Gamma = (\Gamma_s)_{s \in [t, T]} \) by \( \Gamma_s = \frac{Y_s}{Y_t} \). Since for all \( s \in [t, T] \),

\[ d(\Gamma_s Y_s) = (Y_s \Gamma_s \mu_s + \Gamma_s \sigma_s Z_s) d[M]_s + (Y_s \Gamma_s \sigma_s + \Gamma_s Z_s) dM_s + \Gamma_s dM^\perp_s, \]

it holds for all \( h \in (0, T - t) \) that

\[ E_t[\Gamma_{t+h} Y_{t+h}] = Y_t + E_t \left[ \int_t^{t+h} (Y_s \Gamma_s \mu_s + \Gamma_s \sigma_s + \Gamma_s Z_s) d[M]_s \right] . \] (B.4)

Together with

\[ E_t[Y_{t+h}] = Y_t + E_t \left[ \int_t^{t+h} a_s d[M]_s \right], \quad h \in (0, T - t), \]
we obtain heuristically that

\[
\frac{E_t[Y_{t+h}] - E_t[\Gamma_{t+h}Y_{t+h}]}{E_t[[M]_{t+h}] - [M]_t} = \frac{E_t[\int_t^{t+h} (a_s(1 - \Gamma_s) - Y_s\Gamma_s\mu_s - \Gamma_s\sigma_s Z_s) d[|M|]_s]}{E_t[\int_t^{t+h} d[|M|]_s]}
\]

\[
\rightarrow -Y_t\mu_t - \sigma_t Z_t \quad \text{as } h \rightarrow 0. \quad \text{(B.5)}
\]

Furthermore, it holds for all \( h \in (0, T - t) \) that

\[
Y_{t+h}e^{-\int_t^{t+h} \rho_s d[|M|]_s} = Y_t + \int_t^{t+h} (a_s - \rho_s Y_s) e^{-\int_t^s \rho_r d[|M|]_r} d[|M|]_s
\]

\[
+ \int_t^{t+h} Z_s e^{-\int_t^s \rho_r d[|M|]_r} dM_s + \int_{(t,t+h]} e^{-\int_t^t \rho_r d[|M|]_r} dM^I_s. \quad \text{(B.6)}
\]

From (B.4) and (B.6), we derive heuristically that

\[
\frac{E_t[Y_{t+h}(e^{-\int_t^{t+h} \rho_s d[|M|]_s} - \Gamma_{t+h})]}{E_t[[M]_{t+h}] - [M]_t}
\]

\[
= \frac{E_t[\int_t^{t+h} ((a_s - \rho_s Y_s) e^{-\int_t^s \rho_r d[|M|]_r} - (Y_s\Gamma_s\mu_s + \Gamma_s\sigma_s Z_s)) d[|M|]_s]}{E_t[\int_t^{t+h} d[|M|]_s]}
\]

\[
\rightarrow -\rho_t Y_t - Y_t\mu_t - \sigma_t Z_t \quad \text{as } h \rightarrow 0. \quad \text{(B.7)}
\]

Recall that \( \Gamma_s^{-1} = \frac{\alpha_s}{\alpha_t}, s \in [t, T] \), with

\[
d\Gamma_s^{-1} = \Gamma_s^{-1} (-(\mu_s - \sigma_s^2) d[|M|]_s - \sigma_s dM_s), \quad s \in [t, T].
\]

Therefore, it holds that

\[
d(Y_s\Gamma_s^{-1}) = \left( -Y_s\Gamma_s^{-1}(\mu_s - \sigma_s^2) + \Gamma_s^{-1} \alpha_s - Z_s\Gamma_s^{-1}\sigma_s \right) d[|M|]_s
\]

\[
+ (\Gamma_s^{-1} Z_s - Y_s\Gamma_s^{-1}\sigma_s) dM_s + \Gamma_s^{-1} dM^I_s, \quad s \in [t, T]. \quad \text{(B.8)}
\]

Moreover, we have for all \( h \in (0, T - t) \) that

\[
(e^{-\int_t^{t+h} \rho_s d[|M|]_s} - \Gamma_{t+h})^2
\]

\[
= -2 \int_t^{t+h} \left( e^{-\int_t^s \rho_r d[|M|_r]} - \Gamma_s \right) \Gamma_s\sigma_s dM_s
\]

\[
+ \int_t^{t+h} \left( \Gamma_s^2 \sigma_s^2 - 2(e^{-\int_t^s \rho_r d[|M|_r]} - \Gamma_s)(\rho_s e^{-\int_t^s \rho_r d[|M|_r]} + \Gamma_s\mu_s) \right) d[|M|]_s. \quad \text{(B.9)}
\]
It follows from (B.8) and (B.9) that

\[
Y_{t+h} \Gamma_{t+h}^{-1} (e^{-f_t^{t+h}} \rho_s d[M]_s - \Gamma_{t+h})^2 \\
= \int_{t}^{t+h} \left( Y_s \Gamma^{-1}_s \left( \Gamma^{-2}_s \sigma^2_s - 2(e^{-f_t^s \rho_r d[M]_r} - \Gamma_s)(\rho_s e^{-f_t^s \rho_r d[M]_r} + \Gamma_s \mu_s) \right) \\
+ (e^{-f_t^s \rho_r d[M]_r} - \Gamma_s)^2 \Gamma^{-1}_s \left( -Y_s (\mu_s - \sigma^2_s) + \alpha_s - Z_s \sigma_s \right) \left( -2\sigma_s (Z_s - Y_s \sigma_s) (e^{-f_t^s \rho_r d[M]_r} - \Gamma_s) \right) d[M]_s \\
+ \int_{t}^{t+h} \left( \Gamma^{-1}_s \left( -2\sigma_s (Z_s - Y_s \sigma_s) (e^{-f_t^s \rho_r d[M]_r} - \Gamma_s) \right) d[M]_s \\
+ \int_{(t,t+h]} \left( e^{-f_t^s \rho_r d[M]_r} - \Gamma_s \right) \Gamma^{-1}_s \left( e^{-f_t^s \rho_r d[M]_r} - \Gamma_s \right) dM_s, \quad h \in (0, T - t),
\]

and hence

\[
\begin{align*}
E_t [Y_{t+h} \Gamma_{t+h}^{-1} (e^{-f_t^{t+h}} \rho_s d[M]_s - \Gamma_{t+h})^2 ] \\
& = E_t \left[ \int_{t}^{t+h} \left( Y_s \Gamma^{-1}_s \left( \Gamma^{-2}_s \sigma^2_s - 2(e^{-f_t^s \rho_r d[M]_r} - \Gamma_s)(\rho_s e^{-f_t^s \rho_r d[M]_r} + \Gamma_s \mu_s) \right) \\
+ (e^{-f_t^s \rho_r d[M]_r} - \Gamma_s)^2 \Gamma^{-1}_s \left( -Y_s (\mu_s - \sigma^2_s) + \alpha_s - Z_s \sigma_s \right) \left( -2\sigma_s (Z_s - Y_s \sigma_s) (e^{-f_t^s \rho_r d[M]_r} - \Gamma_s) \right) d[M]_s \right], \quad h \in (0, T - t).
\end{align*}
\]

Therefore, we obtain heuristically that

\[
\frac{E_t [Y_{t+h} \Gamma_{t+h}^{-1} (e^{-f_t^{t+h}} \rho_s d[M]_s - \Gamma_{t+h})^2 ]}{E_t ([M]_{t+h} - [M]_t)} \rightarrow Y_t \sigma_t^2 \quad \text{as } h \rightarrow 0. \quad \text{(B.10)}
\]

From

\[
\Gamma_{t+h}^{-1} e^{-2f_t^{t+h}} \rho_s d[M]_s = 1 - \int_{t}^{t+h} \Gamma^{-1}_s e^{-2f_t^s \rho_r d[M]_r} (2\rho_s + \mu_s - \sigma^2_s) d[M]_s \\
- \int_{t}^{t+h} e^{-2f_t^s \rho_r d[M]_r} \Gamma^{-1}_s \sigma_s dM_s, \quad h \in (0, T - t),
\]
we derive heuristically that
\[
E_t\left[\frac{1}{2}(1 - \Gamma_{t+h}^{-1} e^{-2\int_t^{t+h} \rho_r d[M]_s})\right] \\
E_t[\{M\}_{t+h} - [M]_t]
\]
\[
= E_t\left[\int_t^{t+h} \frac{1}{2}(\Gamma_s^{-1} e^{-2\int_t^s \rho_r d[M]_r}(2\rho_r + \mu_r - \sigma_r^2))d[M]_s\right] \\
E_t[\int_t^{t+h} d[M]_s]
\]
\[
\rightarrow \frac{1}{2}(2\rho_t + \mu_t - \sigma_t^2) \quad \text{as } h \rightarrow 0.
\] (B.11)

We conclude from (B.5), (B.7), (B.10), and (B.11) that the limit for \( h \rightarrow 0 \) of (B.3) equals
\[
a_t = -Y_t \mu_t - \sigma_t Z_t + \frac{(-\rho_t Y_t - Y_t \mu_t - \sigma_t Z_t)^2}{Y_t \sigma_t^2 + \frac{1}{2}(2\rho_t + \mu_t - \sigma_t^2)} = -f(t, Y_t, Z_t)
\]
with \( f \) given in (3.3). Finally, the fact (which is proved in [1]) that the discrete-time processes \( Y^h, h \in (0, T - t) \), are \((0, 1/2)-valued explains the requirement in (3.4) that \( Y \) is \([0, 1/2]-valued.

**Appendix C: Comparison argument for Sect. 7.1**

Here we justify via a comparison argument that in Proposition 7.1, we get that \( Y \) is \([0, 1/2]-valued.

In the following result, we are interested in a BSDE with driver \( f \) and terminal value \( \xi \) of the form
\[
dY_s = -f(s, Y_s)d[M]_s + Z_s dM_s + dM^1_s, \quad s \in [0, T], \quad Y_T = \xi, \quad (C.1)
\]
and denote such a BSDE by BSDE\((f, \xi)\). Recall that in Proposition 7.1, the driver does not depend on \( Z \). Therefore, we do not consider a dependence on \( Z \) in (C.1).

**Proposition C.1** Assume (3.8). Let \( f \) and \( \tilde{f} \) be progressively measurable and \( f \) Lipschitz-continuous, i.e., there exists some \( L \in (0, \infty) \) such that for all \( y, y' \in \mathbb{R} \), it holds that \( |f(s, y) - f(s, y')| \leq L|y - y'| \) \( \mathcal{P}_M \)-a.e. Moreover, let \( \xi \) and \( \tilde{\xi} \) be \( \mathcal{F}_T \)-measurable random variables. Let \( (Y, Z, M^\perp) \) be a solution of the BSDE\((f, \xi)\) and \((\tilde{Y}, \tilde{Z}, \tilde{M}^\perp)\) a solution of the BSDE\((\tilde{f}, \tilde{\xi})\) such that \( E[\int_0^T Z^2_s d[M]_s] < \infty \), \( E[[M^\perp]_T] < \infty \), \( E[\int_0^T \tilde{Z}^2_s d[M]_s] < \infty \) and \( E[[\tilde{M}^\perp]_T] < \infty \). Set \( \delta Y_t = Y_t - \tilde{Y}_t \) and \( \delta f_t = f(t, Y_t) - \tilde{f}(t, \tilde{Y}_t) \) for \( t \in [0, T] \). Furthermore, define
\[
b_t = 1_{\{Y_t \neq \tilde{Y}_t\}}(f(t, Y_t) - \tilde{f}(t, \tilde{Y}_t))(Y_t - \tilde{Y}_t)^{-1}, \quad t \in [0, T],
\]
and introduce the process \( \Gamma = (\Gamma_t)_{t \in [0, T]} \) given by \( \Gamma_t = \exp(\int_0^t b_s d[M]_s), t \in [0, T] \). Then \( \delta Y \) admits the representation
\[
\delta Y_t = \Gamma_t^{-1} E_t\left[\Gamma_T \delta Y_T + \int_t^T \Gamma_s \delta f_s d[M]_s\right], \quad t \in [0, T]. \quad (C.2)
\]
In particular:
(i) If $\xi \geq \tilde{\xi}$ a.s. and $f(s, \tilde{Y}_s) \geq \tilde{f}(s, \tilde{Y}_s)$ $\mathcal{D}_M$-a.e., then $Y_t \geq \tilde{Y}_t$ a.s. for all $t \in [0, T]$.
(ii) If $\xi \leq \tilde{\xi}$ a.s. and $f(s, \tilde{Y}_s) \leq \tilde{f}(s, \tilde{Y}_s)$ $\mathcal{D}_M$-a.e., then $Y_t \leq \tilde{Y}_t$ a.s. for all $t \in [0, T]$.

Proof It holds for all $t \in [0, T]$ that
\[
\delta Y_t = Y_T - \tilde{Y}_T + \int_t^T \left( f(s, Y_s) - \tilde{f}(s, \tilde{Y}_s) \right) d[M]_s
- \int_t^T Z_s dM_s - (M^\perp_T - M^\perp_t) + \int_t^T \tilde{Z}_s dM_s + (\tilde{M}_T^\perp - \tilde{M}_t^\perp).
\]

Since we have for all $s \in [0, T]$ that
\[
f(s, Y_s) - \tilde{f}(s, \tilde{Y}_s) = f(s, Y_s) - f(s, \tilde{Y}_s) + f(s, \tilde{Y}_s) - \tilde{f}(s, \tilde{Y}_s) = b_s \delta Y_s + \delta f_s,
\]
it follows that
\[
d(\delta Y_s) = -(b_s \delta Y_s + \delta f_s) d[M]_s + Z_s dM_s - \tilde{Z}_s dM_s + dM^\perp_s - d\tilde{M}^\perp_s, \quad s \in [0, T].
\]
Together with $d\Gamma_s = \Gamma_s b_s d[M]_s$, $s \in [0, T]$, we obtain by integration by parts that
\[
\Gamma_t \delta Y_T = \Gamma_t \delta Y_t - \int_t^T \Gamma_s (b_s \delta Y_s + \delta f_s) d[M]_s + \int_t^T \Gamma_s Z_s dM_s - \int_t^T \Gamma_s \tilde{Z}_s dM_s
+ \int_{(t, T]} \Gamma_s dM^\perp_s - \int_{(t, T]} \Gamma_s d\tilde{M}^\perp_s + \int_t^T \Gamma_s \delta Y_s \Gamma_s b_s d[M]_s, \quad t \in [0, T].
\]
If the local martingales $S = \int_0^T \Gamma_s Z_s dM_s$, $\tilde{S} = \int_0^T \Gamma_s \tilde{Z}_s dM_s$, $U = \int_{[0, \cdot]} \Gamma_s dM^\perp$ and $\tilde{U} = \int_{[0, \cdot]} \Gamma_s d\tilde{M}^\perp$ are true martingales, then it follows that
\[
\Gamma_t \delta Y_t = E_t \left[ \Gamma_T \delta Y_T + \int_t^T \Gamma_s \delta f_s d[M]_s \right], \quad t \in [0, T].
\]
which yields the representation \((C.2)\) of $\delta Y$.

To show that $S$ is a martingale, note first that due to the Lipschitz-continuity of $f$, the process $b$ is bounded $\mathcal{D}_M$-a.e. by the corresponding Lipschitz constant. By the Cauchy–Schwarz inequality, it holds that
\[
E \left[ \left( \int_0^T \Gamma_s^2 Z_s^2 d[M]_s \right)^{\frac{1}{2}} \right] \leq E \left[ \left( \sup_{t \in [0, T]} \Gamma_t^2 \right)^{\frac{1}{2}} \left( \int_0^T Z_s^2 d[M]_s \right)^{\frac{1}{2}} \right]
\leq \left( E \left[ \sup_{t \in [0, T]} \Gamma_t^2 \right] \right)^{\frac{1}{2}} \left( E \left[ \int_0^T Z_s^2 d[M]_s \right] \right)^{\frac{1}{2}}.
\]
Since $b$ is bounded and (3.8) holds, we have $E[\sup_{t \in [0, T]} \Gamma_t^2] < \infty$. We also have by assumption that $E[\int_0^T Z_t^2 d[M]_s] < \infty$. Therefore, it follows from (C.3) and the Burkholder–Davis–Gundy inequality that $E[\sup_{t \in [0, T]} |S_t|] < \infty$. Thus $S$ is a martingale. A similar reasoning applies also to $\tilde{S}, U$ and $\tilde{U}$.

Finally, the claims (i) and (ii) are straightforward consequences of (C.2). □

We now apply Proposition C.1 to obtain $0 \leq Y \leq \frac{1}{2}$ in the proof of Proposition 7.1. Observe that $(\tilde{Y}, \tilde{Z}, \tilde{M}^\perp) = (\frac{1}{2}, 0, 0)$ is a solution of the BSDE $(0, \frac{1}{2})$, which obviously satisfies $E[|\tilde{M}^\perp|_T] < \infty$ and $E[\int_0^T \tilde{Z}_t^2 d[M]_s] < \infty$. Moreover, with $\tilde{f}$ as defined in the proof of Proposition 7.1, it holds that

$$\tilde{f}(s, \frac{1}{2}) = \frac{-\rho_s^2}{2(2\rho_s + \mu_s)} \leq 0, \quad s \in [0, T],$$

and both BSDEs have the same terminal value $\frac{1}{2}$. Therefore Proposition C.1 applies and yields $Y \leq \tilde{Y} = \frac{1}{2}$.

For the other bound, note that $(\tilde{Y}, \tilde{Z}, \tilde{M}^\perp) = (0, 0, 0)$ is a solution of the BSDE $(0, 0)$ with $E[|\tilde{M}^\perp|_T] < \infty$ and $E[\int_0^T \tilde{Z}_t^2 d[M]_s] < \infty$. Since $\tilde{f}(s, 0) = 0$ for all $s \in [0, T]$ and $Y_T = \frac{1}{2} \geq 0 = \tilde{Y}_T$, it follows from Proposition C.1 that $Y \geq \tilde{Y} = 0$.

Remark C.2 Notice that in the proof of Proposition C.1, we need (3.8) and the Lipschitz-continuity of $f$ only to show that $E[\sup_{t \in [0, T]} \Gamma_t^2]$ is finite. Replace these two conditions by the assumption that there exists a predictable process $R$ such that for all $y, y' \in \mathbb{R}$, $|f(\omega, s, y) - f(\omega, s, y')| \leq R_s(\omega)|y - y'| \mathcal{D}_M$-a.e. and for all $c \in (0, \infty)$, $E[\exp(c \int_0^T R_s d[M]_s)] < \infty$. Then we still have that

$$E\left[ \sup_{t \in [0, T]} \Gamma_t^2 \right] = E\left[ \sup_{t \in [0, T]} \exp\left( 2 \int_0^t b_s d[M]_s \right) \right] \leq E\left[ \exp\left( 2 \int_0^T R_s d[M]_s \right) \right] < \infty.$$  

Hence the claim of Proposition C.1 also applies to the setting mentioned in Remark 7.2.

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