Thermal Green’s Functions
From Quantum Mechanical Path Integrals

D.G.C. McKeon∗
Department of Applied Mathematics, University of Western Ontario,
London, Ontario, CANADA N6A 5B7

A. Rebhan†
Institut für Theoretische Physik der Technischen Universität Wien,
Wiedner Hauptstr. 8–10, A–1040 Vienna, AUSTRIA

In this paper it is shown how the generating functional for Green’s functions
in relativistic quantum field theory and in thermal field theory can be evaluated
in terms of a standard quantum mechanical path integral. With this calculational
approach one avoids the loop-momentum integrals usually encountered in Feynman
perturbation theory, although with thermal Green’s functions, a discrete sum (over
the winding numbers of paths with respect to the circular imaginary time) must be
computed. The high-temperature expansion of this sum can be performed for all
Green’s functions at the same time, and is particularly simple for the static case.
The procedure is illustrated by evaluating the two-point function to one-loop order
in a \( \phi_6^3 \) model.

PACS:11.10.Ef

∗Internet address: tmleafs@uwovax.uwo.ca
†Internet address: rebhana@email.tuwien.ac.at
I. INTRODUCTION

Using Schwinger’s proper-time representation \[^1\], the regulated generating functional for Green’s functions in relativistic quantum field theory and in thermal field theory can be formulated in terms of the matrix elements \( \langle x|e^{-iHt}|y \rangle \) and \( \langle x|e^{-Ht}|y \rangle \), respectively, where \( H \) is generally of the form \( H = \frac{1}{2}(p - A)^2 + V \). Usually these are evaluated using the Hamiltonian formalism leading to Feynman integrals for Green’s functions.

On the other hand, in Ref. \[^1\] Schwinger has demonstrated how in particular the one-loop effective action can be evaluated by using quantum mechanical methods for the matrix elements appearing in the proper-time formulation. In Ref. \[^2\], it has recently been shown by one of the present authors how these matrix elements can be computed by standard quantum mechanical path integrals, and that these can be employed to perform perturbation theory at arbitrary loop order, using the formalism of operator regularization \[^3\]. Independently, Strassler \[^4\], motivated by the techniques developed in string theory, has proposed an alternative formulation based on quantum mechanical path integrals, restricted, however, to one-loop order. First-quantized string theory is usually formulated through path integrals over fields defined on two-dimensional world-sheets. It therefore seems only natural to expect the possibility of rephrasing ordinary field theory in terms of path integrals over functions defined on one-dimensional world lines. Indeed, such an approach has already been outlined by Polyakov in Ref. \[^5\]. The parallel with string theory can be drawn in particular for the one-loop effective action in quantum field theory, where there are no internal vertices. Nevertheless, as shown in Refs. \[^2,6\] in a number of explicit examples, the calculational approach based on quantum mechanical path integrals can be exploited also at higher loop order and for arbitrary fields.

In the present paper, we shall show how this technique can be extended to the evaluation of Green’s functions in the imaginary-time formulation of thermal field theories. Again we find that loop-momentum integrals are obviated, however because of the topological constraint from the circular imaginary time, an infinite sum corresponding to the winding number of paths in imaginary time appears. This sum is equivalent to but not identical with the conventional sum over loop Matsubara frequencies. In fact, we shall show that the representation as a sum over winding numbers suggests a novel method of evaluating the high-temperature expansion of Green’s functions. This expansion can be done for all Green’s functions at the same time, with the coefficients being integrals over the parameters corresponding to the insertion of external vertices. This turns out to be comparatively simple for the static (but momentum-dependent) case. The procedure is worked out for a \( \phi^3 \)-model, and explicit results are derived for the one-loop two-point function.
II. LOOP DIAGRAMS WITHOUT LOOP-MOMENTUM INTEGRALS

If we consider a field theoretical model with an action

\[ S[\phi] = \int d^D x \left[ -\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{3!} \lambda \phi^3 \right], \]

(metric signature \((-,-,\ldots,+))\) then by introducing a background field \(f(x)\) the generating functional for one-particle irreducible Green’s functions can be formulated in terms of the operator

\[ H = \frac{1}{2} \left[ p^2 + m^2 + \lambda f \right], \]

where \(p = -i\partial\).

The one-loop generating functional is given by

\[ i\Gamma^{(1)}[f] = \frac{1}{2} \int dx \langle x | \ln H | y \rangle, \]

whereas at higher loop orders \(\langle x | H^{-1} | y \rangle\) is the basic building block.

In operator regularization \([3]\) both \(\ln H\) and \(H^{-1}\) are defined through derivatives of \(H^{-s}\),

\[ \ln H = \lim_{s \to 0} -\frac{d}{ds} H^{-s}, \quad (4a) \]

\[ H^{-1} = \lim_{s \to 1} \frac{d}{ds} (s - 1) H^{-s}, \quad (4b) \]

which is represented as

\[ H^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt (it)^{s-1} e^{-iHt}. \quad (5) \]

In this way the regulated generating functional (which turns out to be finite upon taking the limit in \(s\)) is expressed entirely in terms of matrix elements \(\langle x | \exp -iHt | y \rangle\); the one-loop generating functional in particular is derived from the \(\zeta\)-function \([8]\)

\[ \zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt (it)^{s-1} \times \int dx dy \delta^D (x - y) \langle x | \exp -iHt | y \rangle, \quad (6) \]

according to

\[ i\Gamma^{(1)}[f] = \frac{1}{2} \lim_{s \to 0} \frac{d}{ds} \zeta_H(s). \quad (7) \]

(Operator regularization at higher loop orders is discussed in Ref. \([9]\).)

The perturbative evaluation can proceed by using the Schwinger expansion \([1]\)
\[ e^{-i(H_0+H_1)t} = e^{-iH_0 t} - it \int_0^t du e^{-i(1-u)H_0 t} H_1 e^{-iuH_0 t} + \ldots \]  

(8)

and inserting complete sets of states, which involves having to compute loop-momentum integrals.

An alternative \[2\] for the evaluation of the matrix elements \[\langle x| \exp -iHt|y \rangle\] is to use the standard quantum mechanical path integral \[10]\]

\[ \langle x| \exp -i [\frac{1}{2} (p - A)^2 + V] t|y \rangle = \int Dq(\tau) \exp i \int_0^t d\tau \left[ \frac{1}{2} \dot{q}^2(\tau) + \dot{q}(\tau) \cdot A(q(\tau)) - V(q(\tau)) \right] \]  

(9)

with \(q(0) = y, q(t) = x\). Using a perturbative expansion of the right hand side of Eq. (9), we can write for the matrix element \[\langle x| \exp -iHt|y \rangle\] with \(H\) given by Eq. (2)

\[ M_{xy} \equiv \langle x| \exp -i \left[ \frac{1}{2} p^2 + \lambda f \right] t|y \rangle = \int Dq(\tau) \sum_{N=0}^{\infty} \left( -\frac{i\lambda}{2} \right)^N \prod_{i=1}^{N} \int_0^t d\tau_i f(q(\tau_i)) \times \exp i \int_0^t d\tau \frac{\dot{q}^2(\tau)}{2}, \]  

(10)

where we have omitted the trivial factor \(\exp -i(m^2/2)t\). Upon Fourier decomposition of the background field,

\[ f(q(\tau_i)) = \frac{1}{(2\pi)^{D/2}} \int d^D k_i \tilde{f}(k_i) \exp -ik_i \cdot q(\tau_i), \]  

(11)

this matrix element can be expressed in terms of the standard result \[10,11\] for a particle subject to an external force

\[ \int Dq(\tau) \exp i \int_0^t d\tau \left[ \frac{1}{2} \dot{q}^2(\tau) - \gamma_N(\tau) \cdot q(\tau) \right] \]

\[ = \frac{1}{(2\pi it)^{D/2}} \exp i \left[ \frac{(x-y)^2}{2t} \right] \]

\[ - \frac{1}{t} \int_0^t d\tau \gamma_N(\tau) \cdot [x\tau + y(t - \tau)] \]

\[ + \frac{1}{2} \int_0^t d\tau' d\tau'' \gamma_N(\tau') \cdot \gamma_N(\tau'') G(\tau', \tau'') \]  

(12a)

with

\[ \gamma_N(\tau) = \sum_{i=1}^{N} \delta(\tau - \tau_i) k_i \]  

(12b)
and $G$ being the one-dimensional Green’s function

$$G(\tau, \tau') = \frac{1}{2} |\tau - \tau'| - \frac{1}{2} (\tau + \tau') + \frac{\tau \tau'}{t}. \quad (12c)$$

By Eq. (12a), the contribution to Eq. (10) that is proportional to $\lambda^N$ when $x = y$, which determines the one-loop $N$-point Green’s function, is

$$M_{xx}^{(N)} = \frac{1}{N!} \left( \frac{-i\lambda}{2(2\pi)^{D/2}} \right)^N \int \prod_{i=1}^{N} d^D k_i \tilde{f}(k_i)$$

$$\times \int_0^t d\tau_1 \ldots d\tau_N \frac{1}{(2\pi i)^{D/2}}$$

$$\times \exp i \left[ \sum_{i=1}^{N} k_i \cdot x + \frac{1}{2} \sum_{i,j=1}^{N} k_i \cdot k_j G(\tau_i, \tau_j) \right]. \quad (13)$$

When substituted into Eq. (11), the integral over $x$ implements momentum conservation ($\sum_i k_i = 0$), and the integration variables $\tau_i$, appropriately combined and rescaled, turn out to correspond to the usual Feynman parameters. At no stage, however, does one encounter any loop-momentum integral.

This procedure can be applied beyond one-loop order [6] and to models involving spinors and vectors [6,4]. An advantage of using this formalism when dealing with a non-Abelian vector field theory is that no complicated three or four point vertices need be handled, as can be seen from Ref. [3].

### III. THERMAL GREEN’S FUNCTIONS

We now show how this technique can be used to evaluate thermal Green’s functions. If one were to determine these quantities associated with the model of Eq. (1), a starting point is the partition function [12]

$$Z = \int_{\text{periodic}} D\phi \exp \left\{ \int_0^\beta dx_0 \int d^{D-1} x$$

$$\times \left( -\frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial x_0} \right)^2 + (\nabla \phi)^2 + m^2 \phi^2 \right] - \frac{\lambda}{3!} \phi^3 \right) \right\}, \quad (14)$$

where “periodic” denotes the constraint that $\phi$ is periodic in imaginary time $x_0$,

$$\phi(x_0, \mathbf{x}) = \phi(x_0 + \beta, \mathbf{x}). \quad (15)$$

The regulated generating functional for one-loop Green’s functions is again determined by a $\zeta$-function analogous to Eq. (3),

$$\zeta_H(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{s-1}$$

$$\times \int dx \ dy \ \delta^D(x - y) \langle x | \exp -Ht | y \rangle, \quad (16)$$
where the $p^2$ contained in $H$ is contracted with a Euclidean metric.

The central matrix element to be evaluated and its path integral representation now are

$$M_{x y} = \langle x | \exp \left( -\frac{1}{2} [p^2 + \lambda f] t \right) | y \rangle$$

$$= \int Dq(\tau) \exp \left( -\int_0^t d\tau' \frac{1}{2} \left[ q^2(\tau') + \lambda f(q(\tau')) \right] \right)$$

with $q(0) = y$, $q(t) = x$, and with the identification

$$q_0 \equiv q_0 + n\beta, \quad n \text{ integer}. \quad (18)$$

Paralleling the steps of the previous section, one can again perform a perturbative expansion in $\lambda$ and restrict oneself to plane waves

$$\sqrt{\beta} (2\pi)^{D-1} f(q(\tau_i)) = \exp \left( -i (\omega_n q_0(\tau_i) + k_i \cdot q(\tau_i)) \right), \quad (19)$$

where the periodicity condition $\text{(15)}$ restricts $\omega_n$ to the discrete set of Matsubara frequencies

$$\omega_n = \frac{2\pi n}{\beta}. \quad (20)$$

An $N$-point Green’s function is thus determined by the expression

$$M_{x y}^{(N)} = \frac{1}{N!} \left( \frac{-\lambda}{2(2\pi)^{(D-1)/2} \sqrt{\beta}} \right)^N \int_0^t d\tau_1 \ldots d\tau_N P_{x y}^{\beta} [\gamma_N], \quad (21a)$$

$$P_{x y}^{\beta} [\gamma_N] = \int Dq(\tau) \exp \left( -\int_0^t d\tau \left[ \frac{1}{2} q^2(\tau) - \gamma_N(\tau) \cdot q(\tau) \right] \right), \quad (21b)$$

where

$$i\gamma_0^N(\tau) = \sum_{i=1}^N \omega_n \delta(\tau - \tau_i),$$

$$i\gamma_N(\tau) = \sum_{i=1}^N k_i \delta(\tau - \tau_i). \quad (21c)$$

However, because of the topological constraint $\text{(18)}$, it is no longer possible to perform a shift of variable

$$q(\tau) = \bar{q}(\tau) - \int_0^t d\tau' G(\tau, \tau') \gamma_N(\tau')$$

with $G$ given by Eq. $\text{(12c)}$ to bring the path integral $\text{(21b)}$ in Gaussian form. Instead of a free particle subject to an external force, the quantum mechanical problem described by the path integral $\text{(21b)}$ is now that of a particle on a circle (with respect to $q_0$). Its path integral treatment has been given in Ref. $\text{[13]}$, and the solution, adapted to our case, reads
\[ P_{xy}^\beta = \sum_{n=-\infty}^{\infty} P_{(x_0+n\beta,x)y} \]  

This corresponds to summing over the paths from \( y \) to \( x \) differing in winding number \( n \) around the circular imaginary time.

We consequently need only compute a zero-temperature \((\beta = \infty)\) path integral with \( q(0) = y \), \( q(t) = x \), and \( q_0(t) = x_0 + n\beta \), then perform the sum over \( n \), in order to determine the finite-temperature path integral (21b). With \( \beta = \infty \), the shift of variable (22) is legitimate.

Upon substituting Eq. (22) into Eq. (21b) at \( \beta = \infty \), we find that

\[
P_{xy}^{\infty}[\gamma_N] = \exp \left[ \frac{1}{T} \int_0^T d\tau \left[ x\tau + y(t - \tau) \right] \right] e^{\int_0^T d\tau' \gamma_N(\tau')} e^{\int_0^T d\tau'' \gamma_N(\tau'')} G(\tau', \tau'') \times \int Dq(\tau) \exp - \int_0^T d\tau \frac{1}{2} q^2(\tau). \tag{24}
\]

The remaining path integral is given by

\[
\int Dq(\tau) \exp - \int_0^T d\tau \frac{1}{2} q^2(\tau) = \langle x | \exp - \frac{p^2}{2} t | y \rangle = \int \frac{d^Dk}{(2\pi)^D} e^{-ik(x-y) - k^2t/2} = \frac{1}{(2\pi)^{D/2}} e^{-\frac{(x-y)^2}{2t}}. \tag{25}
\]

By Eqs. (23), (24), and (25), we obtain

\[
M_{xy}^{\beta(N)} = \frac{1}{N!} \left( \frac{-\lambda t}{2(2\pi)^{(D-1)/2} \sqrt{\beta}} \right)^N \int_0^1 d\sigma_1 \ldots d\sigma_N \sum_{n=-\infty}^{\infty} \frac{1}{(2\pi t)^{D/2}} \times \exp \left\{ - \left[ (x-y)^2 + 2n\beta(x_0 - y_0) + n^2 \beta^2 \right] / 2t \right\}

-i(x-y) \cdot \sum_{i=1}^N k_i \sigma_i - iy \cdot \sum_{i=1}^N k_i - in\beta \sum_{i=1}^N \omega_{n_i} \sigma_i

+ \frac{1}{2} \sum_{i,j=1}^N \left( k_i \cdot k_j + \omega_{n_i} \omega_{n_j} \right) G(\sigma_i t, \sigma_j t) \right\}, \tag{26}
\]

where we have rescaled \( \tau_i = \sigma_i t \).

The sum over \( n \) can be expressed in terms of the Jacobi function

\[
\theta_3(\nu | \tau) = \sum_{n=-\infty}^{\infty} \exp i\pi \left[ \tau n^2 + 2\nu n \right], \tag{27}
\]

so that

\[
M_{xy}^{\beta(N)} = \frac{1}{N!} \left( \frac{-\lambda t}{2(2\pi)^{(D-1)/2} \sqrt{\beta}} \right)^N \int_0^1 d\sigma_1 \ldots d\sigma_N \frac{1}{(2\pi t)^{D/2}} \times \exp \left\{ - \frac{(x-y)^2}{2t} - i \sum_{i=1}^N \left[ \sigma_i (x-y) + y \right] \cdot k_i + \frac{1}{2} \sum_{i,j=1}^N k_i \cdot k_j G(\sigma_i t, \sigma_j t) \right\}

\times \theta_3 \left( \frac{i\beta x_0 - y_0}{2\pi t} + i \sum_{i=1}^N \sigma_i \omega_{n_i} \right) \left| \frac{i\beta^2}{2\pi t} \right|. \tag{28}
\]
At one-loop order, only the matrix element with \( x = y \) is needed, and integration over \( x \) again enforces \( \sum_i k_i = 0 \). In particular, in \( D = 6 \) dimensions, the two-point function with \( k_1 = k = -k_2 \) and \( \omega_{n_1} = \omega_n = -\omega_{n_2} \) is determined by

\[
\zeta^{(2)}(s) = \frac{1}{\Gamma(s)} \frac{\lambda^2}{64\pi^2\beta} \int_0^\infty dt \int_0^1 d\sigma_1 d\sigma_2 \\
\times \exp \left\{ \left[ (\omega_n^2 + k^2)(-|\sigma_1 - \sigma_2| + (\sigma_1 - \sigma_2)^2) - m^2 \right] \frac{t}{2} \right\} \\
\times \theta_3 \left( \frac{-\beta}{2\pi} (\sigma_1 - \sigma_2) \omega_n \frac{|i\beta^2|}{2\pi t} \right). \tag{29}
\]

The symmetric form of the integrand together with the periodicity of the \( \theta_3 \)-function allows us to simplify \( \int_0^1 d\sigma_1 d\sigma_2 \rightarrow \int_0^1 d(\sigma_1 - \sigma_2) \) in Eq. (29). At zero temperature (where \( \theta_3 \rightarrow 1 \)), \( \nu_1 = \sigma_1 - \sigma_2 \) would exactly correspond to the usual Feynman parameter for combining two propagators. The sum defining \( \theta_3 \), however, does not correspond to the conventional sum over loop Matsubara frequencies. The latter are recovered after employing Jacobi’s imaginary transformation

\[
\theta_3 \left( \frac{\nu}{\tau} \right) - \frac{1}{\tau} = (-i\tau)^{\frac{3}{2}} e^{i\pi\nu^2/\tau} \theta_3 (\nu/\tau), \tag{30a}
\]

which in our case reads

\[
\theta_3 \left( \frac{-\beta}{2\pi} \sum_i \sigma_i \omega_{n_i} \frac{|i\beta^2|}{2\pi t} \right) \\
= \frac{\sqrt{2\pi t}}{\beta} \theta_3 \left( \frac{i}{\beta} \sum_i \sigma_i \omega_{n_i} t \frac{2\pi it}{\beta^2} \right) \\
\times \exp \left\{ -\frac{1}{2} \left( \sum_i \sigma_i \omega_{n_i} \right)^2 t \right\}. \tag{30b}
\]

With Eq. (30b), Eq. (29) coincides with the conventional expression for the finite-temperature two-point Green’s function where the loop-momentum integration has been done, identifying \( 2\pi n/\beta \) with the usual loop Matsubara frequency. It should be noticed that no loop momentum integral was ever encountered in the derivation of Eq. (29); the sum over loop frequencies is replaced by the equivalent but conceptually different sum over winding numbers in Eq. (23).

**IV. HIGH-TEMPERATURE EXPANSION**

The standard way to evaluate the finite-temperature Green’s functions in the imaginary-time formalism is to do the sum over loop Matsubara frequencies first, keeping the loop-momentum integrals as such, without combining denominators by Feynman parametrization.

\(^3\)The difference of these two sums would have been more conspicuous, had we considered fermionic fields: there we would have had an alternating sum over integer winding numbers on the one hand, and on the other a sum over Matsubara frequencies determined by half-integers.
On the other hand, in our calculational approach, the loop-momentum integrals are circumvented, leaving us with integrals equivalent to Feynman parameter integrals and either with the sum over loop Matsubara frequencies or the sum over winding numbers, which are Jacobi transforms of each other.

It has recently been shown in Ref. [15] how to obtain a systematic high-temperature expansion of the sum over Matsubara frequencies after the other integrals are all performed. In the following we shall demonstrate that the representation as a sum over winding numbers, which naturally emerges from the computational scheme considered here, is an interesting alternative as concerns the task of a high-temperature expansion. Keeping the integrations over the vertex insertion parameters $\sigma_i$ to the end, one can in fact aim at deriving a generic form of the high-temperature expansions of all (one-loop) $N$-point Green’s functions at the same time. This will turn out to be feasible and comparatively simple, too, for the static case $\omega_n = 0 (k_i \neq 0)$.

Inserting the matrix element (28) into the $\zeta$-function representation of a one-loop $N$-point Green’s function, in the static case we are led to compute the proper-time integral

$$I_N = \int_0^\infty dt \, t^{s-1+N-D/2} \theta_3 \left( 0 \left| \frac{\beta^2}{2\pi t} \right| \right) \exp \left[ -\frac{1}{2} Q_N t \right],$$

where

$$Q_N \equiv m^2 - \frac{1}{2} \sum_{i,j=1}^N k_i \cdot k_j \left( |\sigma_i - \sigma_j| - (\sigma_i + \sigma_j) + 2\sigma_i \sigma_j \right)$$

(32)

contains the dependence on external momenta and vertex insertion parameters. Integration of the latter is postponed for the moment.

The integral in Eq. (31) is readily performed [16], and the sum naturally splits into a zero-temperature contribution (the term with $n = 0$) and temperature contributions,

$$I_N = I_N^\infty + I_N^\beta,$$

$$= (Q_N/2)^{\nu-s} \Gamma(s-\nu)$$

$$+ 4 \sum_{n=1}^{\infty} \left( \frac{\sqrt{Q_N}}{n\beta} \right)^\nu K_\nu(n\beta\sqrt{Q_N}),$$

(33)

with $\nu \equiv D/2 - N$ and $K_\nu$ being a modified Bessel function of integer (half-integer) order for even (odd) dimension $D$. Since the temperature contribution is free of ultraviolet divergences, we have put the regulating parameter $s = 0$ there; the otherwise necessary differentiation with respect to $s$ in Eq. (10) just amounts to dropping the overall factor $1/\Gamma(s)$ in $\zeta_H$, Eq. (10).

The low-temperature expansion of $I_N^\beta$ in the form (33) can be obtained from the asymptotic formula $K_\nu(z) \sim \sqrt{\pi/2z} \, e^{-z}$; the sum over $n$ is, however, slowly converging for high temperatures. A series expansion in powers (and logarithm) of $\beta$ can be found by the Mellin summation technique [17], leading to
\[ I_N^{\beta} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \zeta(z) \times 4 \int_0^\infty \frac{dn}{n} \beta^{-\nu} Q_n^{\nu/2} n^{z-\nu-1} K_\nu(\beta n \sqrt{Q_N}) \]

\[ = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dz \zeta(z) 2^{z-\nu} \beta^{-z} Q_N^{\nu-1/2} \Gamma(z)^2 \Gamma(\frac{z}{2} - \nu), \]

(34)

where \( c \) is a real number with \( c > \max(1, 2\nu) \) and \( \zeta(z) \) is the Riemann \( \zeta \)-function. Whereas the series representation (33) is defined only for \( \nu > -\frac{1}{2} \), the integral representation (34) allows us to analytically continue to general \( \nu \).

In the form (34), \( I_N^{\beta} \) can be evaluated by closing the contour to the left and computing a sum over residues. There is no contribution from the contour integral over the large arc if \( \beta \sqrt{Q_N} \ll 1 \), which is fulfilled in the limit of high temperature, \( \beta \to 0 \). For even dimension (integer \( \nu \)) we find

\[ I_N^{\beta} = \sum_{j=0}^{\nu-1} (-1)^j 2^{\nu-2j+1} \frac{\Gamma(\nu-j)}{\Gamma(j+1)} \zeta(2\nu-2j) \beta^{-2\nu+2j} Q_N^j + 2^{1-\nu} \Gamma(\frac{1}{2}) \Gamma(\frac{1}{2} - \nu) \beta^{-1} Q_N^{\nu-1/2} \]

\[ + \frac{2(-Q_N/2)\nu}{\Gamma(\nu+1)} \left[ \ln \frac{\beta \sqrt{Q_N}}{4\pi} - \frac{1}{2} (\psi(1) + \psi(1+\nu)) \right] \]

\[ + 2(-Q_N/2)\nu \sum_{j=1}^{\infty} (-1)^j \frac{\Gamma(2j+1)\zeta(2j+1)}{\Gamma(j+1)\Gamma(j+1+\nu)} \left( \frac{\beta \sqrt{Q_N}}{4\pi} \right)^{2j}, \]

(35)

where \( \psi \) is the Digamma function. For nonpositive \( \nu \) the first sum does not contribute; for \( \nu < 0 \), the logarithmic term disappears, too, and the last sum starts at \( j = |\nu| \).

A particular \( N \)-point Green’s function is determined by

\[ \int_0^1 d\sigma_1 \ldots d\sigma_N I_N. \]

For example, the two-point function in six dimensions (\( \nu = 1 \)) is given by (dropping the zero-temperature part)

\[ \Pi^\beta(\omega_n = 0, k) = \frac{\lambda^2}{64\pi^3} \int_0^1 d\sigma_1 d\sigma_2 I_2^{\beta}, \]

(36)

which in the massless case involves only elementary integrations, yielding

\[ \Pi^\beta(\omega_n = 0, k) \]

\[ = \frac{\lambda^2}{128\pi^3} \left[ \frac{4\pi^2}{3} \beta^{-2} - \frac{1}{2} \beta^2 k^2 \right] \left[ \ln \frac{\beta k}{4\pi} + \gamma - \frac{4}{3} \right] \]

\[ - k^2 \sum_{j=1}^{\infty} (-1)^j \frac{\zeta(2j+1)}{(2j+1)(2j+3)} \left( \frac{\beta k}{4\pi} \right)^{2j} \]

(37)

Our results are a direct generalization of the high-temperature expansions obtained in Refs. [18-19] for space-time independent quantities such as thermodynamic potentials. In fact, the same kind of infinite sums occurs there, the essential difference being that here we
still have to evaluate the integrals over vertex insertion (or Feynman) parameters. Indeed, the effective potential is contained in our results as the special case \( k_i = 0 \), where \( Q_N \equiv m^2 \), independent of the parameters \( \sigma_i \), rendering the remaining integrals trivial.

In the nonstatic case, however, where one could derive a generalized version of Eq. (34), the comparative simplicity of the above derivation is lost. There the condition \( \beta \sqrt{Q_N} \ll 1 \), which allowed us to close the integration contour in Eq. (34) to obtain Eq. (35) is not fulfilled because then \( \sqrt{Q} \sim \omega_n \sim 1/\beta \). It would be wrong to invoke analytic continuation at this point and to substitute \( i \omega_n \rightarrow k_0 + i \epsilon \), with \( k_0 \) a continuous variable independent of \( \beta \), thereby dropping the contributions from the large arc after all. This supposed transition to the real-time formulation is problematic because Eq. (34) has still to be integrated over the Feynman parameters implicit in \( Q_N \). It has recently been shown by Weldon [20] that although in the imaginary-time formulation Feynman-parametrized Green’s functions are well-defined, they have to be modified in the real-time theory.

Let us emphasize that this presents no problem in principle as our formalism is the imaginary-time one, and we have verified equivalence to the usual results above. One has to be careful, however, in performing the analytic continuation to real frequencies. The latter is straightforward only with the final result at hand, i.e. after all the integrals have been evaluated.

To summarize, we have shown that the recently proposed method of evaluating loop diagrams in field theory by the use of quantum mechanical path integrals can be extended to the evaluation of thermal Green’s functions in the imaginary-time formalism. This alternative approach leads to a representation of the latter in terms of sums over winding numbers of paths in place of the usual loop Matsubara frequencies. This different but equivalent representation was shown to also suggest alternative methods in performing high-temperature expansion, and we have been able to derive a universal expansion for static one-loop Green’s functions.

The extension of our analysis to more interesting field theories is mostly straightforward. In Refs. [2,6] it has been shown how for example Yang-Mills theory can be reformulated in terms of quantum mechanical path integrals, if a Feynman-type background field gauge is used, leading to a very economical calculational scheme for the evaluation of Green’s functions. It is not equally straightforward, though, to employ gauges that lead to a more complicated kinetic term. Further, with operator regularization it is almost mandatory to use a background-covariant gauge fixing [22]. A somewhat more involved case is Green’s functions with external fermionic lines, because the different boundary conditions of fermions and bosons have then to be accommodated in one path integral expression. We intend to return to this subject in a forthcoming publication.

---

\[^4\] The uncorrected formulae of Feynman parametrization in the real-time theory as in Refs. [21] correspond to an incorrect analytic continuation which has a deceptive analytic behavior at \( k_0 = 0 = k \), but possesses branch points at unphysical locations, and is singular at infinity [20]. The same phenomenon would occur by dropping the contribution from the large arc after closing the contour in Eq. (34).
ACKNOWLEDGMENTS

One of us (D.G.C.M.) would like to thank NSERC for financial support. R. and D. MacKenzie provided stimulus for this investigation.
REFERENCES

[1] J. Schwinger, Phys. Rev. **82**, 664 (1951).
[2] D.G.C. McKeon, Can. J. Phys. (in press).
[3] D.G.C. McKeon and T.N. Sherry, Phys. Rev. **D35**, 3854 (1987).
[4] M. Strassler, Nucl. Phys. **B385**, 145 (1992).
[5] A.M. Polyakov, *Gauge Fields and Strings*, p. 166ff (Harwood Academic Publishers, Chur, 1987)
[6] D.G.C. McKeon, Ann. Phys. (N.Y.) (in press).
[7] B.S. DeWitt, in *Quantum Gravity II*, edited by C.J. Isham, R. Penrose, and D.W. Sciama (Oxford University Press, New York, 1981); L.F. Abbott, Nucl. Phys. **B185**, 189 (1981).
[8] S. Hawking, Comm. Math. Phys. **55**, 133 (1977).
[9] L. Culumovic, D.G.C. McKeon and T.N. Sherry, Ann. Phys. (N.Y.) **197**, 94 (1990).
[10] R.P. Feynman, Rev. Mod. Phys. **20**, 367 (1948); Phys. Rev. **80**, 440 (1950).
[11] C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, p. 432 (McGraw-Hill, New York, 1980).
[12] R.P. Feynman, Phys. Rev. **91**, 1291 (1953); J. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, 1989).
[13] W.K. Burton and A.H. DeBorde, Nuovo Cim. **2**, 197 (1955); L.S. Schulman, *Techniques and applications of path integration*, ch. 23 (Wiley, New York, 1981); H. Kleinert, *Path integrals in quantum mechanics, statistics, and polymer physics*, ch. 6 (World Scientific, Singapore, 1990).
[14] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher transcendental functions*, vol. II (McGraw-Hill, New York, 1953)
[15] F.T. Brandt, J. Frenkel and J.C. Taylor, Phys. Rev. **D44**, 1801 (1991).
[16] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press, New York, 1980).
[17] B. Davies, *Integral Transforms and Their Applications* (Springer, New York, 1978).
[18] H.W. Braden, Phys. Rev. **D25**, 1028 (1982).
[19] H.E. Haber and H.A. Weldon, J. Math. Phys. **23**, 1852 (1982).
[20] H.A. Weldon, Phys. Rev. **D47**, 594 (1993).
[21] P.S. Gribosky and B.R. Holstein, Z. Phys. **C47**, 205 (1990); P.F. Bedaque and A. Das, Phys. Rev. **D45**, 2906 (1992).
[22] A. Rebhan, Phys. Rev. **D39**, 3101 (1989).