Active Subspace of Neural Networks: Structural Analysis and Universal Attacks

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Abstract. Active subspace is a model reduction method widely used in the uncertainty quantification community. In this paper, we propose analyzing the internal structure and vulnerability and deep neural networks using active subspace. Firstly, we employ the active subspace to measure the number of “active neurons” at each intermediate layer, and reduce the number of neurons from several thousands to several dozens. This motivates us to change the network structure and to develop a new and more compact network, referred to as ASNet, that has significantly fewer model parameters. Secondly, we propose analyzing the vulnerability of a neural network using active subspace and finding an additive universal adversarial attack vector that can misclassify a dataset with a high probability. Our experiments on CIFAR-10 show that ASNet can achieve 23.98× parameter and 7.30× flops reduction. The universal active subspace attack vector can achieve around 20% higher attack ratio compared with the existing approach in all of our numerical experiments. The PyTorch codes for this paper are available online.

Key words. Active Subspace, Deep Neural Network, Network Reduction, Universal Adversarial Perturbation

AMS subject classifications. 90C26, 15A18, 62G35

1. Introduction. Deep neural networks have achieved impressive performance in many applications, such as computer vision [32], nature language processing [55], and speech recognition [23]. Most neural networks use deep structure (i.e., many layers) and a huge number of neurons to achieve a high accuracy and expressive power [41, 19]. However, it is still unclear how many layers and neurons are necessary. Employing an unnecessarily complicated deep neural network can cause huge extra costs in run-time and hardware resources. Driven by resource-constrained applications such as robotics and internet of things, there is an increasing interest in building smaller neural networks by removing network redundancy. Representative methods include network pruning and sharing [17, 25, 27, 36, 35], low-rank matrix and tensor factorization [46, 26, 18, 33, 40], parameter quantization [12, 15], knowledge distillation [28, 43], and so forth. However, most existing methods delete model parameters directly without changing the network architecture [27, 25, 7, 35].

Another important issue of deep neural networks is the lack of robustness. A deep neural
network is desired to maintain a good performance for noisy or corrupted data in order to be deployed in safety-critical applications such as autonomous driving and medical image analysis. However, recent studies have revealed that many state-of-the-art deep neural networks are vulnerable to small perturbations [51]. A substantial number of methods have been proposed to generate adversarial examples. Representative works can be classified into four classes [49], including optimization methods [8, 38, 37, 51], sensitive features [22, 42], geometric transformations [16, 31], and generative models [4]. However, these methods share a fundamental limitation: each perturbation is designed for a given data point, and one has to implement the algorithm again in order to generate the perturbation for a new data sample. Recently, several methods have also been proposed to compute a universal adversarial attack in order to fool a dataset simultaneously (rather than one data sample) in various applications, such as computer vision [37], speech recognition [39], audio [1], and text classifier [5]. However, all the above methods solve a series of data-dependent sub-problems. To our best knowledge, none of the existing algorithms use the universal information extracted from the whole dataset.

This paper investigates the above two issues with the active subspace method [45, 9, 10] that was originally developed for uncertainty quantification. The key idea of the active subspace is to identify the important directions of a multi-variable function, which can contribute significantly to its variance. These directions are constructed by the principal components of the uncentered covariance matrix of gradients instead of the nature coordinates in the parameter space. Afterwards, a response surface can be constructed in this low-dimensional subspace to reduce the number of parameters for partial differential equations [10] and uncertainty quantification [11]. However, the power of active subspace in analyzing and attacking deep neural networks has not been explored.

1.1. Paper Contributions. The contribution of this manuscript is twofold.

- Firstly, we apply the active subspace to some intermediate layers of a deep neural network, and try to answer the following question: how many neurons and layers are important in a deep neural network? Based on active subspace, we propose the definition of “active neurons”. Fig. 1 (a) shows that even though there are tens of thousands of neurons, only dozens of them are important from the active subspace point of view. Fig. 1 (b) further shows that most of the neural network parameters are distributed in the last few layers. This motivates us to cut off the tail layers and replace them with a smaller and simpler new framework called ASNet. ASNet contains three parts: the first few layers of a deep neural network, an active-subspace layer that maps the intermediate neurons to a low-dimensional subspace, and a polynomial chaos expansion layer that projects the reduced variables to the outputs. Our numerical experiments show that the proposed ASNet has much fewer model parameters than the original one. The proposed ASNet can be combined with existing structured re-training methods (e.g., pruning and quantization) to get better accuracy and further fewer model parameters.

- Secondly, we use active subspace to develop a new universal attack method to fool deep neural network on a whole data set. We formulate this problem as a ball-constrained loss maximization problem and propose a heuristic projected gradient descent algorithm to solve it. At each iteration, the ascent direction is the dominant active subspace of the current data points, and the stepsize is decided by the backtracking algorithm. Fig. 1 (c) shows
that the testing accuracy of a deep neural network can decrease to around 20% when it is fooled by our proposed approach.

The rest of this manuscript is organized as follows. In Section 2, we review the key idea of active subspace. Based on the active-subspace method, Section 3 shows how to find the number of active neurons in a deep neural network and further proposes a new and compact network, referred to as ASNet. Section 4 develops a new universal adversarial attack method based on active subspace. The numerical experiments for both ASNet and universal adversarial attacks are presented in Section 5. Finally, we conclude this paper in Section 6.

2. Active Subspace. Active-subspace is an efficient tool for functional analysis and dimension reduction. Its key idea is to construct a low-dimensional subspace for the input variables in which the function value changes dramatically. Given a continuous function $f(x)$ with $x$ described by the probability density function $p(x)$, one can construct an uncentered covariance matrix for the gradient: $C = E[\nabla f(x)\nabla f(x)^T]$. Suppose the matrix $C$ admits the following eigenvalue decomposition,

$$C = V \Lambda V^T,$$

where $V$ includes all orthogonal eigenvectors and

$$\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n), \quad \lambda_1 \geq \cdots \geq \lambda_n \geq 0$$

are the eigenvalues. All eigenvalues are nonnegative because $C$ is semidefinite. One can split the matrix $V$ into two parts,

$$V = [V_1, V_2], \quad \text{where } V_1 \in \mathbb{R}^{n \times r} \text{ and } V_2 \in \mathbb{R}^{n \times (n-r)}.$$

The subspace spanned by matrix $V_1 \in \mathbb{R}^{n \times r}$ is called an active subspace [45], because the value of $f(x)$ is very sensitive to perturbation vectors inside this subspace.

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Remark 2.1 (Relationships with the Principal Component Analysis). Given a set of data $X = [x_1, \ldots, x^m]$ with each column representing a data sample and each row is zero-mean, the first principal component $w_1$ inherits the maximal variance from $X$, namely,

$$w_1 = \arg \max_{\|w\|_2 = 1} \sum_{i=1}^m (w_1^T x_i)^2 = \arg \max_{\|w\|_2 = 1} w^T X X^T w.$$  

The variance is maximized when $w_1$ is the eigenvector associated with the largest eigenvalue of $XX^T$. The $r$-dimensional principal components are the first $r$ eigenvectors associated with the $r$ largest eigenvalues of $XX^T$. Note that the principal component analysis uses the covariance matrix of input data sets $X$, but the active-subspace method uses the covariance matrix of gradient $\nabla f(x)$. A perturbation along the direction $w_1$ from the principal component analysis does not necessarily cause a significant change on the value of $f(x)$.

The following lemma quantitatively describes that $f(x)$ varies more on average along the directions defined by the columns of $V_1$ than along the directions defined by the columns of $V_2$.

Lemma 2.2. [10] Suppose $f(x)$ is a continuous function and $C$ is obtained from (2.1). For the matrices $V_1$ and $V_2$ generated by (2.3), and the reduced vector $z = V_1^T x$ and $\tilde{z} = V_2^T x$,

$$\mathbb{E}_x[\nabla_z f(x)^T \nabla_z f(x)] = \lambda_1 + \ldots + \lambda_r,$$

$$\mathbb{E}_x[\nabla_{\tilde{z}} f(x)^T \nabla_{\tilde{z}} f(x)] = \lambda_{r+1} + \ldots + \lambda_n.$$

Sketch of proof [10]:

$$\mathbb{E}_x[\nabla_z f(x)^T \nabla_z f(x)] = \text{trace} (\mathbb{E}_x[\nabla_z f(x) \nabla_z f(x)^T]) = \text{trace} (\mathbb{E}_x[V_1^T \nabla_z f(x) \nabla_z f(x)^T V_1]) = \text{trace} (V_1^T C V_1) = \lambda_1 + \ldots + \lambda_r.$$

When $\lambda_{r+1}, \ldots, \lambda_n$ are all zero, Lemma 2.2 implies $\nabla_{\tilde{z}} f(x)$ is zero everywhere. In this case, $f(x)$ is $z$-invariant. We may reduce $x \in \mathbb{R}^n$ to a low-dimensional vector $z = V_1^T x \in \mathbb{R}^r$ and construct a new response surface $g(z)$ to represent $f(x)$. In general, when $\lambda_{r+1}, \ldots, \lambda_n$ are small, we may also construct a response surface $g(z)$ to approximate $f(x)$.

### 2.1. Response Surface.

For a fixed $z$, the best guess for $f$ is the conditional expectation of $f$ given $z$, i.e.,

$$g(z) = \mathbb{E}_x[f(x)|z] = \int f(V_1 z + V_2 \tilde{z}) \rho(\tilde{z}|z) d\tilde{z}.$$  

Based on the Poincaré inequality, the following approximation error bound can be obtained under certain assumptions [10].

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Lemma 2.3. Assume that $f(x)$ is absolutely continuous and square integrable with respect to the probability density function $p(x)$, then the approximation $g(z)$ in (2.7) satisfies:

$$E[(f(x) - g(z))^2] \leq \text{const} \times (\lambda_{r+1} + \ldots + \lambda_n).$$

Sketch of proof [10]:

$$E_x[(f(x) - g(z))^2]$$

$$= E_x[E_\tilde{z}[(f(x) - g(z))^2|z]]$$

$$\leq \text{const} \times E_x[E_\tilde{z}[^\tilde{z} f(x)^T \nabla \tilde{z} f(x)|z]] \quad \text{(Poincaré inequality)}$$

$$= \text{const} \times E_x[\nabla z f(x)^T \nabla z f(x)]$$

$$= \text{const} \times (\lambda_{r+1} + \ldots + \lambda_n). \quad \text{(Lemma 2.2)}$$

In other words, the active-subspace approximation error will be small if $\lambda_{r+1}, \ldots, \lambda_n$ are negligible.

3. Active Subspace for Structural Analysis of Deep Neural Networks. This section applies the active subspace to analyze the internal layers of a deep neural network in order to reveal the number of important neurons at each layer. Afterwards, a new network called ASNet is built to reduce the storage and computational complexity.

3.1. Deep Neural Networks. A deep neural network can be described as

$$f(x_0) = f_L(f_{L-1} \ldots (f_1(x_0))),$$

where $x_0 \in \mathbb{R}^{n_0}$ is an input, $L$ is the total number of layers, and $f_l : \mathbb{R}^{n_{l-1}} \to \mathbb{R}^{n_l}$ is a function representing the $l$-th layer (e.g., combinations of convolution, fully connected, batch normalization, ReLU, or pooling layers). For any $1 \leq l \leq L$, we rewrite the above feed-forward model as a superposition of functions, i.e.,

$$f(x_0) = f_{\text{post}}(f_{\text{pre}}(x_0)),$$

where the pre-model $f_{\text{pre}}(\cdot) = f_L \ldots f_1(\cdot)$ denotes all operations before the $l$-th layer and the post-model $f_{\text{post}}(\cdot) = f_L \ldots f_{l+1}(\cdot)$ denotes all succeeding operations. The intermediate neuron $x = f_{\text{pre}}(x_0) \in \mathbb{R}^{n_l}$ usually lies in a high dimension. We aim to study whether such a high dimensionality is necessary. If not, how can we reduce it?

3.2. The Number of Active Neurons. Denote $\text{loss}(\cdot)$ as the loss function, and

$$c(x) = \text{loss}(f_{\text{post}}(x)).$$

The covariance matrix $C = E[\nabla c(x) \nabla c(x)^T]$ admits the eigenvalue decomposition $C = \Lambda \mathbf{V} \Lambda^T$ with $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. We try to extract the active subspace of $c(x)$ and reduce the intermediate vector $x$ to a small dimension.
Definition 3.1. Suppose $\Lambda$ is computed by (2.2). For any layer index $1 \leq l \leq L$, we define the number of active neurons $n_{l,\text{AS}}$ as follows:

$$n_{l,\text{AS}} = \arg \min \left\{ i : \frac{\sqrt{\lambda_1 + \ldots + \lambda_i}}{\sqrt{\lambda_1 + \ldots + \lambda_{n_l}}} \geq 1 - \epsilon \right\},$$

where $\epsilon > 0$ is a user-defined threshold.

Based on Definition 3.1, the post-model can be approximated by an $n_{l,\text{AS}}$-dimensional function with a high accuracy, i.e.,

$$c(x) \approx g(z) = \mathbb{E}_z[c(x)|z].$$

Here $z = V^T_1 x \in \mathbb{R}^{n_{l,\text{AS}}}$ plays the role of active neurons, $\hat{z} = V^T_2 x \in \mathbb{R}^{n-n_{l,\text{AS}}}$, and $V = [V_1, V_2]$.

Lemma 3.1. Suppose the input $x_0$ is bounded, and consider a deep neural network with the following operations: convolution, matrix-vector product, ReLU, batch normalization, and max-pooling. Then for any $l \in \{1, \ldots, L\}$, $x = f_{\text{pre}}(x_0)$, and $c(x) = \text{loss}(f_{\text{post}}(x))$, the reduced function $g(z)$ defined in (3.5) satisfies

$$\mathbb{E}[(c(x) - g(z))^2] = O(\epsilon).$$

Proof. Denote $c(x) = \text{loss}(f_L(\ldots(f_{l+1}(x))))$, where $\text{loss}(y) = -\log \frac{\exp(y_b)}{\sum_{j=1}^b \exp(y_j)}$ is the cross entropy loss function, $b$ is the true label, and $c$ is the total number of classes. We first show $c(x)$ is absolutely continuous and square integrable, and then apply Lemma 2.3 to derive (3.6).

Firstly, all components of $c(x)$ are Lipschitz continuous because (1) the convolution, matrix-vector product, and batch normalization operations are all linear; (2) the max pooling and ReLU functions are non-expansive. Here, a mapping $m$ is non-expansive if $\|m(x) - m(y)\| \leq \|x - y\|$; (3) the cross entropy loss function is smooth and gradient upper bounded, i.e., $\|\nabla \text{loss}(y)\| = \|e_b - \exp(y) / \sum_{j=1}^b \exp(y_j)\| \leq n+1$. The composition of two Lipschitz functions will also be Lipschitz continuous: suppose the Lipschitz constants for $f_1$ and $f_2$ are $\alpha_1$ and $\alpha_2$, respectively, it holds that $\|f_1(f_2(\bar{x})) - f_1(f_2(\bar{x}))\| \leq \alpha_1 \|f_2(\bar{x}) - f_2(\bar{x})\| \leq \alpha_1 \alpha_2 \|ar{x} - \bar{x}\|$ for any vectors $\bar{x}$ and $\bar{x}$. By recursively applying the above rule, $c(x)$ will be Lipschitz continuous:

$$\|c(\bar{x}) - c(\bar{x})\|_2 = \|\text{loss}(f_L(\ldots(f_{l+1}(\bar{x})))) - \text{loss}(f_L(\ldots(f_{l+1}(\bar{x})))))\|_2 \leq \alpha_1 \alpha_2 \|ar{x} - \bar{x}\|_2.$$ 

The intermediate neuron $x$ is in a bounded domain because the input $x_0$ is bounded and all functions $f_j(\cdot)$ are either continuous or non-expansive. Based on the fact that any Lipschitz-continuous function is also absolutely continuous on a compact domain [44], we conclude that $c(x)$ is absolutely continuous.

Secondly, because $x$ is bounded and $c(x)$ is continuous, both $c(x)$ and its square integral will be bounded, i.e., $\int c^2(x) \rho(x) dx < \infty$. 

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Algorithm 3.1 The training procedure of the active subspace network (ASNet)

**Input:** A pretrained deep neural network, the layer index \( l \), and the number of active neurons \( r \).

**Step 1** Initialize the active subspace layer. The active subspace layer is a linear projection where the projection matrix \( V_1 \in \mathbb{R}^{n \times r} \) is computed by Algorithm 3.2. If \( r \) is not given, we use \( r = n_{AS} \) defined in (3.4) by default.

**Step 2** Initialize the polynomial chaos expansion layer. The polynomial chaos expansion layer is a nonlinear mapping from the reduced active subspace to the outputs, as shown in (3.10). The weights \( c_\alpha \) is computed by (3.12).

**Step 3** Construct the ASNet. Combine the pre-model (the first \( l \) layers of the deep neural network) with the active subspace and polynomial chaos expansion layers as a new network, referred to as ASNet.

**Step 4** Fine-tuning. Retrain ASNet for several epochs by the stochastic gradient descent.

**Output:** A new network ASNet.

Finally, by Lemma 2.3, it holds that

\[
\mathbb{E}[(c(x) - g(z))^2] \leq \alpha (\lambda_{n_l,AS+1} + \ldots + \lambda_n) = O(\epsilon).
\]

The proof is completed.

The above lemma shows that the active subspace method can be used to reduce the number of neurons of the \( l \)-th layer from \( n_l \) to \( n_{l,AS} \).

### 3.3. Active Subspace Network (ASNet)

This subsection proposes a new network called ASNet that can reduce both the storage and computational cost. Given a deep neural network, we first choose a proper layer \( l \) and project the high-dimensional intermediate neurons to a low-dimensional vector in the active subspace. Afterwards, the post-model is deleted completely and replaced with a nonlinear model that maps the low-dimensional active feature vector to the output directly. This new network is called as ASNet, and it has three parts:

1. **Pre-model:** the pre-model includes the first \( l \) layers of a deep neural network.
2. **Active subspace layer:** a linear projection from the intermediate neurons to the low-dimensional active subspace.
3. **Polynomial chaos expansion layer:** the polynomial chaos expansion [20, 53] maps the active-subspace variables to the output.

The whole procedure is illustrated in Fig. 2 (b) and Algorithm 3.1.

### 3.4. The Active Subspace Layer

Given a dataset \( D = \{x^1, \ldots, x^m\} \), the empirical covariance matrix is computed by \( \hat{C} = \frac{1}{m} \sum_{i=1}^{m} \nabla c(x^i)\nabla c(x^i)^T \). Instead of calculating the eigenvalue decomposition of \( \hat{C} \), we compute the singular value decomposition of \( \hat{G} \):

\[
(3.7) \quad \hat{G} = [\nabla c(x^1), \ldots, \nabla c(x^m)] = \hat{V} \hat{\Sigma} \hat{U}^T \in \mathbb{R}^{n_l \times m} \text{ with } \hat{\Sigma} = \text{diag}(\hat{\sigma}_1, \ldots, \hat{\sigma}_{n_l}).
\]

The eigenvectors of \( \hat{C} \) is approximated by the left singular vectors \( \hat{V} \) and the eigenvalues of \( \hat{C} \) is approximated by the singular values of \( \hat{G} \), i.e., \( \hat{\Lambda} \approx \hat{\Sigma}^2 \).
Figure 2: (a) The original deep neural network; (b) The proposed ASNet with three parts: a pre-model, an active subspace (AS) layer, and a polynomial chaos expansion (PCE) layer.

Algorithm 3.2 The frequent direction algorithm for computing the active subspace

Input: A dataset with $M_{AS}$ input samples $\{x^i_0\}_{i=1}^{M_{AS}}$, a pre-model $f_{\text{pre}}(\cdot)$, a subroutine for computing $\nabla c(x)$, and the dimension of truncated singular value decomposition $r$.

1: Select $r$ samples $x^i_0$, compute $x^i = f_{\text{pre}}(x^i_0)$, and construct an initial matrix $S \leftarrow [\nabla c(x^1), \ldots, \nabla c(x^r)]$.

2: for $t=1, 2, \ldots, \text{do}$

3: Compute the singular value decomposition $V\Sigma U^T \leftarrow \text{svd}(S)$, where $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r)$.

4: If the maximal number of samples is reached, stop.

5: Update $S$ by the soft-thresholding (3.8).

6: Get a new sample $x^\text{new}_0$, compute $x^\text{new} = f_{\text{pre}}(x^\text{new}_0)$, and replace the last column of $S$ (now all zeros) by the gradient vector $S(:, r) \leftarrow \nabla c(x^\text{new})$.

7: end for

Output: The projection matrix $V \in \mathbb{R}^{n \times r}$ and the singular values $\Sigma \in \mathbb{R}^{r \times r}$.

We use the memory-saving frequent direction method [21] to compute the $r$ dominant singular value components, i.e., $G \approx \hat{V}_r \hat{\Sigma}_r \hat{U}_r^T$. Here $r$ is much smaller than the total number of samples. The frequent direction approach only stores an $n \times r$ matrix $S$. In the beginning, each column of $S \in \mathbb{R}^{n \times r}$ is initialized by a gradient vector. Then the randomized singular value decomposition [24] is used to generate $S = U\Sigma V^T$. Afterwards, $S$ is updated in the following way,

\begin{equation}
S \leftarrow V \sqrt{\Sigma^2 - \sigma_r^2}.
\end{equation}

Now the last column of $S$ are all zeros and will be replaced with the gradient vector of a new sample. By repeating this process, $SS^T$ will approximate $GG^T$ with a high accuracy and $V$ will approximate the left singular vectors of $G$. The algorithm framework is presented in Algorithm 3.2.

After obtaining $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_r)$, we can approximate the number of active neurons...
as

\[
\hat{n}_{l,\text{AS}} = \arg \min_i \left\{ i : \frac{\sqrt{\sigma_i^2 + \ldots + \sigma_r^2}}{\sqrt{\sigma_i^2 + \ldots + \sigma_r^2}} \geq 1 - \epsilon \right\}.
\]  

Under the condition that \( \sigma_i^2 \rightarrow \lambda_i \) for \( i = 1, \ldots, r \) and \( \lambda_r \rightarrow 0 \) for \( i = r + 1, \ldots, n_l \), (3.9) can approximate \( n_{l,\text{AS}} \) in (3.4) with a high accuracy. Further, the projection matrix \( \tilde{V}_1 \) is chosen as the first \( \hat{n}_{l,\text{AS}} \) columns of \( V \).

### 3.5. Polynomial Chaos Expansion Layer

After obtaining the reduced variables \( z \), we construct a new surrogate model to approximate the post-model of a deep neural network. This problem can be regarded as an uncertainty quantification problem if we see \( z \) as a random vector.

By the polynomial chaos expansion [52], the network output \( y \in \mathbb{R}^r \) is approximated by a linear combination of some orthogonal polynomial basis functions:

\[
\hat{y} \approx \sum_{|\alpha| = 0}^{p} c_{\alpha} \phi_{\alpha}(z), \text{ where } |\alpha| = \alpha_1 + \ldots + \alpha_d.
\]  

Here \( \phi_{\alpha}(z) \) is a multivariate polynomial basis function chosen based on the probability density function of \( z \). When the parameters \( z = [z_1, \ldots, z_r]^T \) are independent, both the joint density function and the multi-variable basis function can be decomposed into products of one-dimensional functions, i.e., \( \rho(z) = \rho_1(z_1) \ldots \rho_r(z_r) \), \( \phi_{\alpha}(z) = \phi_{\alpha_1}(z_1) \phi_{\alpha_2}(z_2) \ldots \phi_{\alpha_r}(z_r) \).

The marginal basis function \( \phi_{\alpha_j}(z_j) \) is uniquely determined by the marginal density function \( \rho_i(z_i) \). The scatter plot in Fig. 3 shows that the marginal probability density of each variable \( z_i \) in a deep neural network is close to a Gaussian distribution. Suppose \( \rho_i(z_i) \) follows a Gaussian distribution, then \( \phi_{\alpha_j}(z_j) \) will be a Hermite polynomial [34], i.e.,

\[
\phi_0(z) = 1, \quad \phi_1(z) = z, \quad \phi_2(z) = 4z^2 - 2, \quad \phi_{p+1}(z) = 2z\phi_p(z) - 2p\phi_{p-1}(z).
\]

In a general setting, the elements in \( z \) can be non-Gaussian correlated. In this case, the basis functions \( \{\phi_{\alpha}(z)\} \) can be built via the Gram-Schmidt approach described in [13].

The coefficient vector \( c_{\alpha} \) can be computed by a linear least-square optimization. Denote \( z_j^* = \tilde{V}_1^T f_{\text{pre}}(x_j^*) \) as the random samples and \( y_j^* \) as the network output for \( j = 1, \ldots, M_{\text{PCE}} \). The coefficient vector \( c_{\alpha} \) can be computed by

\[
\min_{c_{\alpha}} \sum_{j=1}^{M_{\text{PCE}}} \left( y_j^* - \sum_{|\alpha| = 0}^{p} c_{\alpha} \phi_{\alpha}(z_j^*) \right)^2.
\]

Based on the Nyquist-Shannon sampling theorem, the number of samples to train \( c_{\alpha} \) needs to satisfy \( M_{\text{PCE}} \geq 2N_{\text{basis}} = 2(r+p) \). However, this number can be reduced to a smaller set of “important” samples by the D-optimal design [56] or the sparse regularization approach [14].

The polynomial chaos expansion builds a surrogate model to approximate the deep neural network output \( y \). This idea is similar to the knowledge distillation [28], where a pre-trained
teacher network teaches a smaller student network to learn the output feature. However, our polynomial-chaos layer uses one nonlinear projection whereas the knowledge distillation uses a series of layers. Therefore, the polynomial chaos expansion is more efficient in terms of computational and storage cost.

3.6. Structured Re-training of ASNet. The pre-model can be further compressed by various techniques such as network pruning and sharing [25], low-rank factorization [40, 33, 18], or data quantization [15, 12]. Denote $\theta$ as the weights in ASNet and $(x^i_0, y^i)$ as the training dataset. We re-train the network by solving the following regularized optimization problem:

\begin{equation}
\theta^* = \arg \min_\theta \frac{1}{m} \sum_{i=1}^m \text{loss}(f(\theta; x^i_0, y^i)) + \lambda R(\theta).
\end{equation}

Here $(x^i_0, y^i)$ is a training sample, $m$ is the total number of training samples, $\text{loss}(\cdot)$ is the cross-entropy loss function, $R(\theta)$ is a regularization function, and $\lambda$ is a regularization parameter. Different regularization functions can result in different model structures. For instance, an $\ell_1$ regularizer $R(\theta) = \|\theta\|_1$ [2, 47, 54] will return a sparse weight, an $\ell_{1,2}$-norm regularizer will result in a column-wise sparse weights, a nuclear norm regularizer will result in low-rank weights. At each iteration, we solve (3.13) by a stochastic proximal gradient decent algorithm [50]

\begin{equation}
\theta^{k+1} = \arg \max_\theta (\theta - \theta^k)^T g^k + \frac{1}{2\alpha_k} \|\theta - \theta^k\|_2^2 + \lambda R(\Theta).
\end{equation}

Here $g^k = \frac{1}{|B_k|} \sum_{i \in B_k} \nabla_\theta \text{loss}(f(\theta; x^i_0, y^i))$ is the stochastic gradient, $B_k$ is a batch at the $k$-th step, and $\alpha_k$ is the stepsize.

In this work, we chose the $\ell_1$ regularization to get sparse weight matrices. In this case, problem (3.14) has a closed-form solution:

\begin{equation}
\theta^{k+1} = S_{\alpha_k \lambda}(\theta^k - \alpha_k g^k),
\end{equation}

where $S_{\lambda}(x) = x \max(0, 1 - \lambda/|x|)$ is a soft-thresholding operator.
4. Active-Subspace for Universal Adversarial Attacks. This section investigates how to generate a universal adversarial attack by the active-subspace method. Given a smooth function \( f(x) \), the maximal perturbation direction is defined by

\[
\mathbf{v}_\delta^* = \arg\max_{\|\mathbf{v}\|_2 \leq \delta} \mathbb{E}_x [(f(x + \mathbf{v}) - f(x))^2].
\]

Here, \( \delta \) is a user-defined perturbation upper bound. By the first order Taylor expansion, we have

\[
f(x + \mathbf{v}) \approx f(x) + \nabla f(x)^T \mathbf{v},
\]

and problem (4.1) can be reduced to

\[
(4.2) \quad \mathbf{v}_{AS} = \arg\max_{\|\mathbf{v}\|_2 = 1} \mathbb{E}_x [(\nabla f(x)^T \mathbf{v})^2] = \arg\max_{\|\mathbf{v}\|_2 = 1} \mathbf{v}^T \mathbb{E}_x [\nabla f(x) \nabla f(x)^T] \mathbf{v}.
\]

The vector \( \mathbf{v}_{AS} \) is exactly the dominant eigenvector of the covariance matrix of \( \nabla f(x) \). The solution for (4.1) can be approximated by \( +\delta \mathbf{v}_{AS} \) or \( -\delta \mathbf{v}_{AS} \). Here, both \( \mathbf{v}_{AS} \) and \( -\mathbf{v}_{AS} \) are solutions of (4.2) but their effect on (4.1) are different.

**Example 4.1.** Consider a two-dimensional function \( f(x) = \mathbf{a}^T x - b \) with \( \mathbf{a} = [1, -1]^T \) and \( b = 1 \) in a two-dimensional square domain \( x \in [0, 1]^2 \), as shown in Fig. 4 (a). It follows from \( \nabla f(x) = \mathbf{a} \) that the covariance matrix \( \mathbf{C} = \mathbf{a} \mathbf{a}^T \). The dominant eigenvector of \( \mathbf{C} \) or the active-subspace direction is \( \mathbf{v}_{AS} = \mathbf{a}/\|\mathbf{a}\|_2 = [1/\sqrt{2}, -1/\sqrt{2}] \). We apply \( \mathbf{v}_{AS} \) to perturb \( f(x) \) and plot \( f(x + \delta \mathbf{v}_{AS}) \) in Fig. 4 (b), which shows a significant difference even for a small permutation \( \delta = 0.3 \). Furthermore, we plot the perturbed function along the first principal component direction \( \mathbf{w}_1 = [1/\sqrt{2}, 1/\sqrt{2}]^T \) in Fig. 4 (c). However, \( \mathbf{w}_1 \) does not result in any perturbation because \( \mathbf{a}^T \mathbf{w}_1 = 0 \). This example indicates the difference between the active-subspace and principal component analysis: the active-subspace direction can capture the sensitivity information of \( f(x) \) whereas the principal component is independent of \( f(x) \).

4.1. Universal Perturbation of Deep Neural Networks. Given a dataset \( \mathcal{D} \) and a classification function \( j(x) \) that maps an input sample to an output label. The universal perturbation seeks for a vector \( \mathbf{v}^* \) whose norm is upper bounded by \( \delta \), such that the class label can be per-
The backtracking line search approach \cite{3} is employed to choose $s$ with a high probability, i.e.,

$$v^* = \arg\max_{\|v\| \leq \delta} \text{prob}_{x \in \mathcal{D}}[j(x + v) \neq j(x)] = \arg\max_{\|v\| \leq \delta} \mathbb{E}_x[1_{j(x+v) \neq j(x)}],$$

where $1_d$ equals one if the condition $d$ is satisfied and zero otherwise. Solving problem (4.3) directly is challenging because both $1_d$ and $j(x)$ are discontinuous. By replacing $j(x)$ by with the loss function $c(x) = \text{loss}(f(x))$ and the indicator function $1_d$ with a quadratic function, we reformulate problem (4.3) as

$$\max_{v} \mathbb{E}_x[(c(x + v) - c(x))^2] \quad \text{s.t.} \quad \|v\|_2 \leq \delta.$$ 

The ball-constrained optimization problem (4.4) can be solved by various numerical techniques such as the spectral gradient descent method \cite{6} and the limited-memory projected quasi-Newton \cite{48}. However, these methods can only guarantee convergence to a local stationary point. Instead, we are interested in computing a direction that can achieve a better objective value by a heuristic algorithm.

### 4.2. Recursive Projection Method

Using the first order Taylor expansion $c(x + v) \approx c(x) + v^T \nabla c(x)$, we reformulate problem (4.4) as a ball constrained quadratic problem

$$\max_{v} v^T \mathbb{E}_x[\nabla c(x) \nabla c(x)^T] v \quad \text{s.t.} \quad \|v\|_2 \leq \delta.$$ 

Problem (4.5) is easy to solve because its closed-form solution is exactly the dominant eigenvector of the covariance matrix $\mathbf{C} = \mathbb{E}_x[\nabla c(x) \nabla c(x)^T]$ or the first active-subspace direction. However, the dominant eigenvector in (4.5) may not be efficient because $c(x)$ is nonlinear. Therefore, we compute $v$ recursively by

$$v^{k+1} = \text{proj}(v^k + s^k d v^k),$$

where $\text{proj}(v) = v \times \min(1, \delta/\|v\|_2)$, $s^k$ is the stepsize, and $d v^k$ is approximated by

$$d v^k = \arg\max_{d v} d v^T \mathbb{E}_x \left[ \nabla c \left( x + v^k \right) \nabla c \left( x + v^k \right)^T \right] d v \quad \text{s.t.} \quad \|d v\|_2 \leq 1.$$ 

Namely, $d v^k$ is the dominant eigenvector of $C^k = \mathbb{E}_x \left[ \nabla c \left( x + v^k \right) \nabla c \left( x + v^k \right)^T \right]$. Because $d v^k$ is the direction that maximizes the changes in $\mathbb{E}_x[(c(x+v) - c(x))^2]$, we expect that the attack ratio keeps increasing, i.e., $r(v^{k+1}; \mathcal{D}) \geq r(v^k; \mathcal{D})$, where

$$r(v; \mathcal{D}) = \frac{1}{|\mathcal{D}|} \sum_{x \in \mathcal{D}} 1_{j(x^t + v) \neq j(x^t)}.$$

The backtracking line search approach \cite{3} is employed to choose $s^k$ such that the attack ratio of $v^k + s^k d v^k$ is higher than the attack ratio of both $v^k$ and $v^k - s^k d v^k$, i.e.,

$$s^k = \min_i \left\{ s^k_{i,t} : r(v^{k+1}_{i,t}; \mathcal{D}) > \max(r(v^{k+1}_{i,t-1}; \mathcal{D}), r(v^k; \mathcal{D})) \right\},$$

where $s^k_{i,t} = (-1)^t s_0^k \gamma^t$, $t \in \{1, -1\}$, $s_0$ is the initial stepsize, $\gamma < 1$ is the decrease ratio, and $v^{k+1}_{i,t} = \text{proj}(v^k + s^k_{i,t} d v^k)$. If such a stepsize $s^k$ exists, we update $v^k$ by (4.6) and repeat the process. Otherwise, we record the number of failures and stop the algorithm when the failure time is greater than a threshold.

The overall flow is summarized in Algorithm 4.1.
Algorithm 4.1 Recursive Active Subspace Universal Attack

Input: A pre-trained deep neural network denoted as $c(x)$, a classification oracle $j(x)$, a training dataset $D^0$, an upper bound for the attack vector $\delta$, an initial stepsize $s_0$, a decrease ratio $\gamma < 1$, and the parameter in the stopping criterion $\alpha$.

1: Initialize the attack vector as $v^0 = 0$.
2: for $k = 0, 1, \ldots$ do
3: Select the training dataset as $D = \{x^j + v^k : x^j \in D^0$ and $j(x^j + v^k) = j(x^j)\}$, then compute the dominate active subspace direction $d_v$ by Algorithm 3.2.
4: for $i = 0, 1, \ldots, I$ do
5: Let $s^k_{i, \pm} = (-1)^t s_0 \gamma^t$ and $v^{k+1}_{i, \pm} = \text{proj}(v^k + s^k_{i, \pm} d_v^k)$ . Compute the attack ratios $r(v^{k+1}_{i, 1})$ and $r(v^{k+1}_{i, -1})$ by (4.8).
6: If either $r(v^{k+1}_{i, 1})$ or $r(v^{k+1}_{i, -1})$ is greater than $r(v^k)$, stop the process. Return $s^k = (-1)^t s^k_{i, 1}$, where $t = 1$ if $r(v^{k+1}_{i, 1}) \geq r(v^{k+1}_{i, -1})$ and $t = -1$ otherwise.
7: end for
8: If no stepsize $s^k$ is returned, let $s^k = s_0 \gamma^I$ and record this step as a failure. Compute the next iteration $v^{k+1}$ by the projection (4.6).
9: end for

Output: The universal active adversarial attack vector $v_{AS}$.

5. Numerical Experiments. In this section, we show the power of active-subspace in revealing the number of active neurons, compressing neural networks, and computing the universal adversarial perturbation direction. All codes are implemented in PyTorch and are available online\(^2\).

5.1. Structural Analysis and Compression. We test the ASNet constructed by Algorithm 3.1, where the polynomial order is $p = 2$, the number of active neurons is set as $r = 50$, and the threshold in Equation (3.4) is set as $\epsilon = 0.05$. We re-train ASNet by the knowledge distillation [28] if necessary and seek for the sparse weights in ASNet-s by the convex optimization method in Section 3.6. In all figures and tables, the numbers in the bracket of ASNet($\cdot$) or ASNet-s($\cdot$) indicate the index of a cut-off layer. We report the performance for different cut-off layers in terms of accuracy, storage, and computational complexities.

5.1.1. Efficiency of Active-subspace. We first show the effectiveness of ASNet constructed by Steps 1-3 of Algorithm 3.1 without fine-tuning. We investigate the following three properties. (1) Redundancy of neurons. The distributions of the first 200 singular values of the matrix $\hat{G}$ [defined in (3.7)] are plotted in Fig. 5 (a). The singular values decrease almost exponentially for layers $l \in \{4, 5, 6, 7\}$. Although the total numbers of neurons are 8192, 16384, 16384, and 16384, the numbers of active neurons are only 105, 84, 54, and 36, respectively. (2) Redundancy of the layers. We cut off the deep neural network at an intermediate layer and replace the subsequent layers with one simple logistic regression [29]. As shown by the red bar in Fig. 5 (b), the logistic regression can achieve relatively high accuracy.

\(^2\)https://github.com/chunfengc/ASNet
Figure 5: Structural analysis of VGG-19 on the CIFAR-10 dataset. (a) The first 200 singular values for layers $4 \leq l \leq 7$; (b) The accuracy (without any fine-tuning) obtained by active-subspace (AS) and polynomial chaos expansions (PCE) compared with principal component analysis (PCA) and logistic regression (LR).

Table 1: Accuracy and storage of ASNet on VGG-19 for CIFAR-10.

| Network      | Accuracy | Storage (MB) | Flops ($10^6$) |
|--------------|----------|--------------|----------------|
| VGG-19       | 93.28%   | 76.45 MB     | 398.14         |
| ASNet(5)     | 91.46%   | (23.41×)     | (340.11×)      |
| ASNet-s(5)   | 90.40%   | (36.33×)     | (527.91×)      |
| ASNet(6)     | 92.87%   | (22.70×)     | (22.70×)       |
| ASNet-s(6)   | 91.08%   | (39.73×)     | (515.98×)      |
| ASNet(7)     | 93.31%   | (21.99×)     | (249.41×)      |
| ASNet-s(7)   | 90.87%   | (36.64×)     | (415.68×)      |

This verifies that the features trained from the first few layers already have a high expression power since replacing all subsequent layers with a simple expression loses little accuracy.

Efficiency of the active-subspace and polynomial chaos expansion. We compare the proposed active-subspace layer with the principal component analysis [30] in projecting the high-dimensional neuron to a low-dimensional space, and also compare the polynomial chaos expansion layer with logistic regression in terms of their efficiency to extract class labels from the low-dimensional variables. Fig. 5 (b) shows that the combination of active-subspace and polynomial chaos expansion can achieve the best accuracy.

5.1.2. CIFAR-10. We continue to present the results of ASNet and ASNet-s on CIFAR-10 by two widely used networks: VGG-19 and ResNet-110 in Tables 1 and 2, respectively. The second column shows the testing accuracy for the corresponding network. We report
Table 2: Accuracy and storage on ResNet-110 for CIFAR-10.

| Network       | Accuracy | Storage (MB) | Flops ($10^6$) |
|---------------|----------|--------------|----------------|
| ResNet-110    | 93.78%   | 6.59         | 252.89         |
| ASNet(61)     | 89.56%   | 1.15         | 252.89         |
|               |          | (3.37×)      | (1.79×)        |
| ASNet-s(61)   | 89.26%   | 1.23         | 252.89         |
|               |          | (4.41×)      | (2.42×)        |
| ASNet(67)     | 90.16%   | 1.61         | 141.24         |
|               |          | (3.19×)      | (2.38×)        |
| ASNet-s(67)   | 89.69%   | 1.22         | 141.24         |
|               |          | (2.97×)      | (1.63×)        |
| ASNet(73)     | 90.48%   | 1.61         | 155.40         |
|               |          | (3.11×)      | (2.21×)        |
| ASNet-s(73)   | 90.02%   | 1.61         | 155.40         |
|               |          | (4.41×)      | (3.37×)        |

Table 3: Accuracy and storage on VGG-19 for CIFAR-100.

| Network       | Top1   | Top5   | Storage (MB) | Flops ($10^6$) |
|---------------|--------|--------|--------------|----------------|
| VGG-19        | 71.90% | 89.57% | 76.62        | 398.18         |
| ASNet(7)      | 70.77% | 91.05% | 10.26        | 190.51         |
|               |        |        | (19.23×)     | (249.41×)      |
| ASNet-s(7)    | 70.20% | 90.90% | 8.44         | 144.81         |
|               |        |        | (21.56×)     | (244.57×)      |
| ASNet(8)      | 69.50% | 90.15% | 10.17        | 228.26         |
|               |        |        | (52.50×)     | (779.04×)      |
| ASNet-s(8)    | 69.17% | 89.73% | 8.09         | 172.69         |
|               |        |        | (55.36×)     | (593.92×)      |
| ASNet(9)      | 72.00% | 90.61% | 15.46        | 247.14         |
|               |        |        | (30.49×)     | (577.10×)      |
| ASNet-s(9)    | 71.38% | 90.28% | 11.32        | 183.78         |
|               |        |        | (32.49×)     | (296.74×)      |

the storage and computational costs for the pre-model, post-model (i.e., active-subspace plus polynomial chaos expansion for ASNet and ASNet-s), and overall results, respectively. For both examples, ASNet and ASNet-s can achieve a similar accuracy with the teacher network yet with much smaller storage and computational cost. For VGG-19, ASNet achieves 14.43× storage savings and 3.44× computational reduction; ASNet-s achieves 23.98× storage savings and 7.30× computational reduction. For most ASNet and ASNet-s networks, the storage and computational costs of the post-models achieve significant performance boosts by our proposed network structure changes. It is not surprising to see that increasing the layer index (i.e., cutting off the deep neural network at a later layer) can produce a higher accuracy.

5.1.3. CIFAR-100. Next, we present the results of VGG-19 and ResNet-110 on CIFAR-100 in Tables 3 and 4, respectively. On VGG-19, ASNet can achieve 7.45× storage savings.
Table 4: Accuracy and storage on ResNet-110 for CIFAR-100.

| Network     | Top-1     | Top-5     | Storage (MB) | Flops (10^6) |
|-------------|-----------|-----------|--------------|--------------|
|             |           |           | Pre-M         | AS+PCE Overall |
| ResNet-110  | 71.94%    | 91.71%    | 6.61         | 252.89       |
| ASNet(75)   | 63.01%    | 88.55%    | 1.79 (3.73×) | 172.67 (367.88×) |
| ASNet-s(75) | 63.16%    | 88.65%    | 1.47 (3.99×) | 143.11 (254.69×) |
| ASNet(81)   | 65.82%    | 90.02%    | 2.64 (3.07×) | 186.83 (302.96×) |
| ASNet-s(81) | 65.73%    | 89.95%    | 2.20 (1.20×) | 155.61 (208.38×) |
| ASNet(87)   | 67.71%    | 90.17%    | 3.48 (2.41×) | 200.98 (238.04×) |
| ASNet-s(87) | 67.65%    | 90.10%    | 2.91 (1.20×) | 166.50 (163.50×) |

and 2.08× computational reduction, and ASNet-s can achieve 9.06× storage savings and 2.73× computational reduction. The accuracy loss is negligible for VGG-19 but larger for ResNet-110. Note that the performance boost of ASNet is obtained by just changing the network structures and without any model compression (e.g., pruning, quantization, or low-rank factorization).

5.2. Universal Adversarial Attacks. This subsection demonstrate the effectiveness of active-subspace in identifying a universal adversarial attack vector. We denote the result generated by Algorithm 4.1 as “AS” and compare it with the “UAP” method in [37] and with “random” Gaussian distribution vector. The parameters in Algorithm 4.1 are set as \( \alpha = 10 \) and \( \delta = 5, \ldots, 10 \). The default parameters of UAP are applied except for the maximal iteration. In the code implementation of [37], the maximal iteration is set as infinity, which is time-consuming when the training dataset or the number of classes is large. In our experiments, we set the maximal iteration as 10. In all figures and tables, we report the average attack ratio and CPU time in training out of ten repeated experiments with different training datasets. A higher attack ratio means the corresponding algorithm is better in fooling the given deep neural network. The datasets are chosen in two ways. We firstly test data points from one class (e.g., trousers in Fashion-MNIST) because these data points share lots of common features and have a higher probability to be attacked by a universal perturbation vector. We then conduct experiments on the whole dataset to show our proposed algorithm can also provide good results even if the dataset has very diverse features.

5.2.1. Fashion-MNIST. Firstly, we present the adversarial attack result on Fashion-MNIST by a 4-layer neural network. There are two convolutional layers with kernel size equals 5×5. The size of output channels for each convolutional layer is 20 and 50, respectively. Each convolutional layer is followed by a ReLU activation layer and a max-pooling layer with a kernel size of 2×2. There are two fully connected layers. The first fully connected layer has an input feature 800 and output feature 500.
Fig. 6 presents the attack ratio of our active-subspace method compared with the baselines UAP method [37] and a Gaussian random vector. The top figures show the results for just one class of data (i.e., trouser), and the bottom figures show the results for all ten classes. For all perturbation norms, the active-subspace method can achieve around 30% higher attack ratio than UAP while more than 10 times faster. This verifies that the active-subspace method has better universal representation ability compared with UAP because the active-subspace can find a universal direction while UAP solves data-dependent subproblems independently.

By the active-subspace approach, almost 100% of data points in the first class dataset can be attacked and around 75% for the whole dataset. This coincides with our intuition that the data points in one class have higher similarity than data points from different classes.

In Fig. 7, we plot one image from Fashion-MNIST and its perturbation by the active-subspace attack vector. The attacked image in Fig. 7 (c) still looks like a trouser for a human. However, the deep neural network misclassifies it as a t-shirt/top.

5.2.2. CIFAR-10. Next, we show the numerical results of attacking VGG-19 on the dataset CIFAR-10. Fig. 8 compares the active-subspace method compared with the baseline UAP and a Gaussian random vector. The top figures show the results by the dataset in the first class (i.e., automobile), and the bottom figures show the results for all ten classes. For both two cases, the proposed active-subspace attack can achieve 20% higher attack ratios and is three times faster than UAP. This is similar to the results in Fashion-MNIST because the active-subspace has a better ability to capture the global information.

We further show the effects of different number of training samples in Fig. 9. When the number of samples is increased, the testing attack ratio is getting better and better. In our
Figure 7: The effect of our attack method on one data sample in the Fashion-MNIST dataset. (a) A trouser from the original dataset. (b) An active-subspace perturbation vector with the $\ell_2$ norm equals to 5. (c) The perturbed sample is misclassified as a t-shirt/top by the deep neural network.

Figure 8: Universal adversarial attacks of VGG-19 on CIFAR-10 with respect to different $\ell_2$-norm perturbations. (a)-(c): The training attack ratio, the testing attack ratio, and the CPU time in seconds for attacking one class dataset. (d)-(f): The results for attacking ten classes dataset together.
Figure 9: Adversarial attack of VGG-19 on CIFAR-10 with different number of training samples. The $\ell_2$-norm perturbation is fixed as 10. (a) The results of attacking the dataset from the first class; (b) The results of attacking the whole dataset with 10 classes.

Table 5: Cross-model performance for CIFAR-10

| Model     | ResNet-20 | ResNet-44 | ResNet-56 | ResNet-110 | VGG-19 |
|-----------|-----------|-----------|-----------|-------------|--------|
| ResNet-20 | 91.35%    | 87.74%    | 86.28%    | 87.38%      | 81.16% |
| ResNet-44 | 84.75%    | 92.28%    | 87.03%    | 85.44%      | 83.44% |
| ResNet-56 | 83.63%    | 86.67%    | 90.15%    | 87.39%      | 84.38% |
| ResNet-110| 71.02%    | 77.58%    | 74.19%    | 92.77%      | 77.32% |
| VGG-19    | 53.61%    | 59.74%    | 61.49%    | 66.29%      | 80.02% |

shows that ResNet-20 is easier to attack than other models, which agrees with our intuition that the structure of ResNet-20 is simple. On the contrary, VGG-19 is the most robust. The success of cross-model attacks indicate that these neural networks could find a similar feature.

5.2.3. CIFAR-100. Finally, we show the results on CIFAR-100 for both the first class (i.e., dolphin) and all classes. Similar to Fashion-MNIST and CIFAR-10, Fig. 10 shows that active-subspace can achieve higher attack ratios than both UAP and a random Gaussian vector. Further, compared with CIFAR-10, CIFAR-100 is easier to be attacked partially because it has more classes.

We summarize the results for different datasets in Table 6. The second column shows the number of classes in the dataset. In terms of testing attack ratio for the whole dataset, active-subspace achieves 24.2%, 15%, and 6.1% higher attack ratios than UAP for Fashion-MNIST, CIFAR-10, and CIFAR-100, respectively. In terms of the CPU time, active-subspace achieves $42 \times$, $5 \times$, and $14 \times$ speedup than UAP on the Fashion-MNIST, CIFAR-10, and CIFAR-100, respectively.

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Figure 10: Results for universal adversarial attack for CIFAR-100 with respect to different $\ell_2$-norm perturbations. (a)-(c): The results for attacking the dataset from the first class. (d)-(f): The results for attacking ten classes dataset together.

Table 6: Summary of the universal attack for different datasets by the active-subspace compared with UAP and the random vector. The norm of perturbation is equal to 10.

|                  | Training Attack ratio | Testing Attack ratio | CPU time (s) |
|------------------|-----------------------|----------------------|--------------|
|                  | # Class | AS     | UAP    | Rand | AS     | UAP    | Rand | AS     | UAP   |
| Fashion-MNIST    | 1      | 100.0% | 93.6%  | 1.8% | 98.0%  | 91.3%  | 3.0% | 0.15   | 5.49  |
|                  | 10     | 79.2%  | 51.5%  | 8.0% | 73.3%  | 49.1%  | 12.3%| 1.40   | 58.85 |
| CIFAR-10         | 1      | 94.7%  | 79.8%  | 8.0% | 84.5%  | 57.9%  | 10.6%| 8.18   | 52.83 |
|                  | 10     | 86.5%  | 65.9%  | 10.2%| 74.9%  | 59.9%  | 17.0%| 37.01  | 181.72|
| CIFAR-100        | 1      | 97.2%  | 87.9%  | 19.7%| 92.1%  | 84.3%  | 37.9%| 13.32  | 248.78|
|                  | 100    | 93.7%  | 86.5%  | 38.7%| 83.5%  | 77.4%  | 52.0%| 14.32  | 204.50|

6. Conclusions. This paper has proposed to analyze deep neural networks by the active subspace method originally developed for dimensionality reduction and uncertainty quantification. We have investigated two problems: how many neurons and layers are necessary (or important) in a deep neural network, and how to generate a universal adversarial attack vector that can be applied to a set of testing data? Firstly, we have presented a definition of “the number of active neurons” and have shown its theoretical error bounds for model reduction. Our numerical study has shown that many neurons and layers are not needed. Based on this observation, we have proposed a new network called ASNet by cutting off the whole neural network at a proper layer and replacing all subsequent layers with an active subspace layer.
and a polynomial chaos expansion layer. The numerical experiments show that the proposed deep neural network structural analysis method can produce a new network with significant storage savings and computational speedup yet with little accuracy loss. Our methods can be combined with existing model compression techniques (e.g., pruning, quantization and low-rank factorization) to develop compact deep neural network models that are more suitable for the deployment on resource-constrained platforms. Secondly, we have applied the active subspace to generate a universal attack vector that is independent of a specific data sample and can be applied to a whole dataset. Our proposed method can achieve a much higher attack ratio than the existing work [37] and enjoys a lower computational cost.

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