Corners Always Scatter

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Abstract: We study time harmonic scattering for the Helmholtz equation in \( \mathbb{R}^n \). We show that certain penetrable scatterers with rectangular corners scatter every incident wave nontrivially. Even though these scatterers have interior transmission eigenvalues, the relative scattering (a.k.a. far field) operator has a trivial kernel and cokernel at every real wavenumber.

1. Introduction

The diffraction of light around corners and edges, and through slits, provided the first evidence for the wave nature of light. The diffraction patterns caused by plane waves incident on corners or edges were among the first scattered waves to be calculated [15]. Asymptotic expansions based on the geometric theory of diffraction [9] reveal the presence of scattered waves in regions where the simple theory of optics does not. Much of our understanding of classical electromagnetism is based on these patterns. This is why a stealth airplane is built to minimize the scattering from corners and edges.

Although the single frequency inverse scattering problem has a unique solution, the wave scattered from a single incident wave does not contain enough information to determine an obstacle or a penetrable scatterer. In many cases, the same scattered wave might have been scattered by a scatterer supported on a smaller set. Some incident waves may even produce no scattered wave. In this paper we will show that a penetrable scatterer whose support contains a right angle corner as an extreme point of its convex hull will scatter any incident wave nontrivially.

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The same is not true for a compactly supported obstacle. A square with side-length \( \pi \) has Dirichlet eigenfunctions
\[
v(x, y) = 4 \sin(px) \sin(qy) = e^{i(px+qy)} + e^{-i(px+qy)} - e^{i(px-qy)} - e^{-i(px-qy)}
\]
for every pair of integers \( p \) and \( q \), which means that a sum of four time harmonic plane waves incident on this sound soft obstacle produces no scattered wave. Even though the obstacle has corners, it is invisible to this incident pattern.

For a penetrable scatterer, the *interior transmission eigenvalues* play the same role that the Dirichlet eigenvalues play for the sound soft obstacle. Any compactly supported \( L^\infty \) scatterer with positive contrast has infinitely many *interior transmission eigenvalues*. This implies the existence of wavenumbers \( k \) for which there exist \( L^2 \) incident waves defined on the support of the scatterer, which produce no scattered wave.

In the spherically symmetric case, the existence of such wavenumbers has been known for a long time [6,7]. In this case, the corresponding incident waves extend to \( \mathbb{R}^n \) as Herglotz wavefunctions, so the classical *relative scattering operator*\(^1\) has a nontrivial kernel. This is significant because many reconstruction algorithms in inverse scattering theory, such as the linear sampling method of Colton and Kirsch [4], and the factorization method of Kirsch [10], will not work correctly if the kernel and cokernel of the relative scattering operator is non-trivial.

The existence of finitely many interior transmission eigenvalues for general (non-spherically symmetric) scatterers with positive contrast was first shown in [12] in 2008, extended to infinitely many in [3] in 2010, and generalized to higher order operators in [8]. If the support contains a right angle corner, we prove that these incident waves cannot extend to any open neighborhood of the corner. The interpretation is that these incident waves could only be produced by sources located on the boundary of the scatterer, but not by any combination of sources located outside an open neighborhood of the scatterer.

Our analysis relies on two new theorems that are of independent interest. We give a new construction of the so-called complex geometric optics solutions for the Helmholtz equation, combining the techniques of Agmon-Hormander [1] and Ruiz [14] to work in \( L^P \) based Besov spaces. This allows us to improve the local regularity of these solutions without sacrificing the decay as a function of complex frequency.

The second theorem states that the Laplace transform of a harmonic polynomial cannot vanish identically on its complex characteristic variety \( \{ \zeta \mid \zeta \cdot \zeta = 0 \} \). This is a generalization of the well-known fact that the Fourier Transform of the solution to a homogeneous constant coefficient partial differential equation is supported on the real characteristic variety of the differential operator, so that it cannot vanish on that set unless it is identically zero. Although the support statement cannot be true for the Laplace transform because it is an analytic function, we show, in the special case of the Laplacian, that only the zero harmonic polynomial can vanish identically on this variety. A proof of this theorem for a general second order elliptic operator with constant coefficients would remove the restriction of our results to right angle corners.

The classical scattering of time harmonic waves by a penetrable medium can be modeled by the Helmholtz equation
\[
(\Delta + k^2 n(x)^2)u = 0 \quad \text{in } \mathbb{R}^n.
\]

\(^1\) The *relative scattering operator*, also called the *far field operator*, is the usual unitary scattering operator minus the identity.
where $n(x)$ denotes the index of refraction. We assume that the contrast $m(x)$, defined by

$$n(x)^2 = 1 - m(x),$$

is compactly supported. In this model, we seek the total wave as

$$u = v^0 + u^+, \quad v^0 \text{ is the incident wave and } u^+ \text{ the outgoing scattered wave.}$$

This means that

$$\left(\Delta + k^2\right)v^0 = 0 \quad \text{in } \mathbb{R}^n \quad (1)$$

and therefore that

$$\left(\Delta + k^2\right)u^+ = k^2m(v^0 + u^+) \quad (2)$$

The relative scattering operator [5] maps the asymptotics of Herglotz incident waves to the asymptotics of scattered waves. A Herglotz incident wave is defined to be a solution to (1) of the form

$$v^0(x) = \int_{S^{n-1}} g_0(\theta) e^{ikr \cdot \theta} d\sigma(\theta),$$

for some $g_0 \in L^2(S^{n-1})$. The Herglotz incident waves can be characterized as the solutions to (1) whose Fourier transforms belong to the Besov space $B^{-1/2}_{2,\infty}(\mathbb{R}^n)$ [1]. These incident waves have well-defined asymptotics at infinity

$$v^0(r\theta) \sim \frac{e^{ikr}}{(ikr)^{\frac{n-1}{2}}} g_0(\theta) + \frac{e^{-ikr}}{(-ikr)^{\frac{n-1}{2}}} g_0(-\theta).$$

The scattered wave $u^+$ also has asymptotics at infinity

$$u^+(r\theta) \sim \frac{e^{ikr}}{(ikr)^{\frac{n-1}{2}}} \alpha^+(\theta)$$

and the relative scattering operator $S(k)$ maps

$$S(k) : L^2(S^{n-1}) \ni g_0 \mapsto \alpha^+ \in L^2(S^{n-1}).$$

For each $k$, the operator $S(k)$ is compact and normal; it never has a bounded inverse, but a number of methods in inverse scattering require that the kernel, and hence cokernel, of $S(k)$ be trivial. If the contrast $m(x)$ in (2) is compactly supported, then a nontrivial kernel implies that $k^2$ is an interior transmission eigenvalue (ITE) for any domain $\Omega$ that contains the support of $m$ in its interior. This means that there are nontrivial $u^+$ and $v^0$ satisfying

$$\left(\Delta + k^2\right)v^0 = 0 \quad \text{in } \Omega \quad (3)$$

$$\left(\Delta + k^2(1 - m)\right)u^+ = k^2mv^0 \quad \text{in } \Omega \quad (4)$$

$$u^+|_{\partial\Omega} = 0, \quad \frac{\partial u^+}{\partial v}|_{\partial\Omega} = 0. \quad (5)$$

\[2\] We will give these definitions and make use of these norms on page 743. See also [16,18].
The results of [12] and [3] guarantee that, if \( \Omega = \text{supp } m \), and \( m(x)|_{\Omega} > 0 \), the pair \((m, \Omega)\) have infinitely many interior transmission eigenvalues; i.e., there are infinitely many positive values of \( k^2 \) for which there exist nontrivial solutions to \((3, 4, 5)\). These results do not guarantee that any eigenfunction pair \((u^+, v^0)\) extends to \(\mathbb{R}^n\). The scattered wave \(u^+\) belongs to \(H^2_{\text{loc}}(\Omega)\), and therefore always extends to all of \(\mathbb{R}^n\) as a function which is zero outside \(\Omega\). The waves \(v^0\), however, are only guaranteed to satisfy \(v^0 \in L^2(\Omega)\) and \(\Delta v^0 \in L^2(\Omega)\). We will refer to \(v_0\) as an interior incident wave to emphasize that it is only defined inside \(\Omega\). If we pair the same scatterer \(m\) with a slightly larger domain \(\Omega_1\), the condition \(m(x)|_{\Omega_1} > 0\) does not hold, as \(m|_{\Omega_1 \setminus \Omega} \equiv 0\), so we cannot guarantee the existence of ITE’s based on the results of [12] and [3], and the interior incident waves defined in \(\Omega\) may not extend to \(\Omega_1\). Indeed, a consequence of Theorem 2.1 below, is that, if \(m\) satisfies the hypothesis of the theorem, and \(\Omega_1\) contains an open neighborhood of the corner point, then there are no nontrivial solutions to \((3, 4, 5)\), i.e., no interior transmission eigenvalues for the pair \((m, \Omega_1)\).

In the spherically symmetric case \((m = m(|x|))\), every interior incident wave \(v^0\) extends to \(\mathbb{R}^n\) as a spherical harmonic times a Bessel function, which is a Herglotz wavefunction, so that the relative scattering operator genuinely has a nontrivial kernel and cokernel whenever \(k^2\) is an interior transmission eigenvalue. We will say that \(k\) is a non-scattering wavenumber whenever the relative scattering operator \(S(k)\) has a nontrivial kernel. Although all scatterers with positive contrasts have infinitely many real ITE’s (with \(\Omega\) equal to the support of the contrast), no non-spherically symmetric scatterers are currently known to have non-scattering wavenumbers.

In this paper, we show that, if the contrast \(m(x)\) is the characteristic function of an \(n\)-dimensional rectangle times a smooth function which is nonzero at least one corner of the rectangle, any non-scattering interior incident wave \(v^0\) does not extend, as a solution to \((1)\), to any open neighborhood of the corner. In particular, no such scatterer can have non-scattering wavenumbers.

2. All Corners Scatter

The theorem below applies to scatterers whose support contains a corner (a standard right angle corner) as an extreme point of its convex hull (i.e., there exists a hyperplane which touches the support of the scatterer at precisely that corner). We describe this condition in item i) below by stating that \(m\) is the product of a smooth function and the characteristic function of a rectangle.

**Theorem 2.1.** Suppose that \(k \neq 0\), that \(K\) is an \(n\)-dimensional rectangle, and

(i) \(m = \chi_K \varphi(x)\) with \(\varphi \in C^\infty(\mathbb{R}^n)\) and \(\varphi(x_0) \neq 0\) where \(x_0\) is a corner of \(K\)

(ii) the pair \((u^+, v^0)\) are interior transmission eigenfunctions of \(m\) in \(\Omega = \text{supp } m\), i.e.

solutions to \((3, 4, 5)\)

then \(v^0\) cannot be extended as an incident wave (i.e. a solution to \((1)\)) to any open neighborhood of the corner.

**Corollary 2.2.** A scatterer \(m\) which satisfies item i) has no non-scattering wavenumbers.

**Proof.** If the kernel of \(S(k)\) is nontrivial, then there is a Herglotz wavefunction \(v^0\) satisfying \((1)\) in \(\mathbb{R}^n\), and an outgoing \(u^+\) satisfying \((2)\) in \(\mathbb{R}^n\) with vanishing far field \(\alpha^+\). Rellich’s lemma and unique continuation [5] guarantee that \(u^+\) vanishes outside the support of \(m\). It follows from the fact that \(m \in L^\infty\) and \(v^0 \in L^2\) that \(u^+ \in H^2_{\text{loc}}(\mathbb{R}^n)\), and
therefore the restriction of \( u^+ \) and its first derivative to \( \partial \Omega \) must vanish. Hence the pair \( (u^+, v^0) \) are interior transmission eigenfunctions in \( \Omega \), but \( v^0 \) extends past the corner, contradicting Theorem 2.1. \( \Box \)

We summarize our proof of Theorem 2.1 in the following paragraph. We will make use of some complex geometric optics solutions to the homogeneous version of (4). Specifically, if we multiply Eq. (4) by any solution \( w \) to

\[
(\Delta + k^2 (1 - m)) w = 0 \tag{6}
\]

and integrate by parts, using the fact that \( u^+ \) and its first derivatives vanish on \( \partial \Omega \), we see that

\[
\int_K w k^2 m v^0 = 0 \tag{7}
\]

Theorem 2.3 below shows that we may choose \( w \) to be exponentially decaying as we move into \( \Omega \) from the corner, so that the main contribution to the integral occurs at the corner. If \( v^0 \) could be extended to a neighborhood of the corner, its Taylor series would necessarily begin with a harmonic polynomial (Lemma 2.4), and the dominant term in the integral would come from the decaying exponential times that harmonic polynomial. This would then imply that the Laplace transform of this harmonic polynomial vanished on the complex characteristic variety associate to the Laplacian, and we devote Sect. 5 to the proof of Theorem 2.5, which says that this cannot be so.

The complex geometric optics solutions we use go back to [17]. There have been many improvements since then, but none provide enough local regularity to show that their contributions to the integral in (7) are dominated by the Laplace transform of the harmonic polynomial. Therefore, we give a new construction in Sect. 4, combining the \( L^p \) techniques in [14] with the geometric \( L^2 \) based constructions in [1] to prove

**Theorem 2.3.** Suppose that \( m(x) \) satisfies i) in Theorem 2.1. For any bounded domain \( D \), and any \( p \in [2, \infty) \), there exist constants \( C \) and \( r \) such that if \( \rho \in \mathbb{C}^n \) and satisfies \( \rho \cdot \rho = 0 \) and \( |\rho| > r \), there exists \( w \) satisfying (6) in \( D \) of the form

\[
w = e^{-x \cdot \rho}(1 + \psi) \tag{8}
\]

with

\[
\|\psi\|_{L^p(D)} \leq \frac{C}{|\rho|} \tag{9}
\]

It is the statement \( 2 \leq p < \infty \) that differentiates Theorem 2.3 from previous constructions. We will need to choose \( p > n \), while maintaining the first power of \( |\rho| \) in the denominator for our proof to succeed.

The simple lemma below notes that the first term in the Taylor series of an incident wave at an interior point is a harmonic polynomial.

**Lemma 2.4.** Suppose that \( v^0 \not\equiv 0 \) and \( x_0 \) is in an open set where \( (\Delta + k^2)v^0 = 0 \). Then the lowest order homogeneous polynomial in the Taylor series for \( v^0 \) at \( x_0 \) is harmonic.
Proof. The function $v^0$ is real analytic at $x_0$, so its Taylor expansion at that point does not vanish identically. We call the lowest order polynomial $P_N$ and the remainder $v^{N+1}$.

$$v^0(x) = P^N(x - x_0) + v^{N+1}(x)$$

$$\Delta v^0(x) = \Delta P^N(x - x_0) + \Delta v^{N+1}(x)$$

$$= Q^{N-2}(x - x_0) + q^{N-1}(x)$$

where $P^N$ and $Q^{N-2}$ are homogeneous polynomials of degree $N$ and $N - 2$ respectively, and

$$|v^{N+1}(x)| \leq c |x - x_0|^{N+1}$$

$$|q^{N-1}(x)| \leq c |x - x_0|^{N-1}.$$  

We may assume that $N \geq 2$ as all polynomials of degree less than two are harmonic. In this case, it follows from

$$\Delta v^0 = -k^2 v_0$$

that

$$|Q^{N-2}(x - x_0)| = |q^{N-1}(x) - k^2 (P^N + v^{N+1})| \leq c |x - x_0|^{N-1},$$

but $Q^{N-2}$ is homogeneous of order $N - 2$, so must be zero.  \□

The final main ingredient, which we will prove in Sect. 5, concerns the Laplace transform of a homogeneous harmonic polynomial, i.e.

$$\widehat{P}(\rho) := \int_{x > 0} e^{-x \cdot \rho} P^N(x) dx$$

where the notation $x > 0$ means that every component of $x$ is greater than 0. We also use the notation $\frac{1}{\rho}$ to denote the vector in $\mathbb{C}^n$ whose components are the reciprocals of the components of $\rho$.

**Theorem 2.5.** The Laplace transform of a nonzero degree $N$ homogeneous harmonic polynomial on $\mathbb{R}^n$ is a degree $N + n$ homogeneous polynomial $Q^{N+n}(\frac{1}{\rho})$ of the reciprocals of the transform variables. If $n \geq 3$, it cannot vanish identically on any open subset of the variety $\rho \cdot \rho = 0$. If $n = 2$, it cannot vanish identically on both an open subset of $\rho_1 = i \rho_2$ and an open subset of $\rho_1 = -i \rho_2$.

Theorem 2.1 is now a fairly direct consequence.

**Proof of Theorem 2.1.** Without loss of generality, we will assume that the rectangle is located in the positive orthant $\{x_j > 0\}$, that $x = 0$ is the corner at which $m$ does not vanish, and that $m(0) = 1$.

If the theorem is false, then we may insert the $w$ of the form (8) in Theorem 2.3, into (7), obtaining

$$0 = \int_{\Omega} e^{-x \cdot \rho} (1 + \psi) m v^0.$$  (10)
In order to insure that the integral in (10) converges, and that the main contribution to the integral comes from the corner, we choose \( \rho \in \mathbb{C}^n \) satisfying \( \rho \cdot \rho = 0 \) and such that \( \min_j \Re \rho_j > \tau |\rho| \), with \( \tau > 0 \). This implies that the real part of the exponent in (10) satisfies

\[
- \Re(x \cdot \rho) = -x \cdot \Re \rho \leq -|x| \min_j \Re \rho_j \leq -|x| \tau |\rho|.
\]

(11)

In order to apply Theorem 2.5, we note that the set

\[
V_\tau := \{ \rho \mid \min_j \Re \rho_j > \tau |\rho| \} \cap \{ |\rho| > 1 \} \cap \{ \rho \mid \rho \cdot \rho = 0 \}
\]

is an open subset of the variety \( \{ \rho \mid \rho \cdot \rho = 0 \} \). It will be nonempty as long as we choose \( \tau < \frac{1}{\sqrt{2n}} \), so we set \( \tau = \frac{1}{4\sqrt{n}} \) for the rest of the proof.

Next, let \( P^N(x) \) be the lowest order homogeneous polynomial in the Taylor expansion for \( v^0 \) in (10), which Theorem 2.4 guarantees is harmonic. For \( n \geq 3 \), a direct application of Theorem 2.5 ensures that the Laplace transform of \( P^N \) does not vanish at least one \( \rho^* \) in \( V_\tau \). For \( n = 2 \), we note that, an open subset of \( \rho \cdot \rho = 0 \) contains an open subset of either \( \rho_1 = i \rho_2 \) or an open subset of \( \rho_1 = -i \rho_2 \). Our harmonic polynomial cannot vanish on both. If it vanishes on one of these, we change \( \rho \) to its complex conjugate \( \rho^* \), which is in the other, and has the same real part, so we may now apply Theorem 2.5 to conclude the existence of a \( \rho^* \) in \( V_\tau \) at which the Laplace Transform doesn’t vanish.

Homogeneity then tells us the value of the Laplace Transform at any scalar multiple of \( \rho^* \), so that for any \( \rho \) of the form

\[
\rho = |\rho| \frac{\rho^*}{|\rho^*|}
\]

(12)

the homogeneity of the transform combines with (11) to provide the lower bound

\[
\left| \int_{x > 0} e^{-x \cdot \rho} P^N \right| \geq \frac{C}{|\rho|^{n+N}}.
\]

(13)

We now return to (10), and prove an upper bound that contradicts (13) when \( |\rho| \) is large. For \( \rho \) of the form (12), the contribution to the integral in (10) from outside a disk \( N_\epsilon \) of radius \( \epsilon \) of the corner is exponentially small, i.e.

\[
\left| \int_{\Omega \setminus N_\epsilon} e^{-x \cdot \rho} (1 + \psi) mv^0 \right| \leq e^{-\epsilon \min_j \Re \rho_j} ||1 + \psi||_2 ||mv^0||_2 \leq Ce^{-\epsilon |\rho|}.
\]

Inserting this estimate in (10), and expanding \( v^0 \) as in Lemma 2.4, we see that

\[
\left| \int_{N_\epsilon} e^{-x \cdot \rho} (1 + \psi) m (P^N(x) + v^{N+1}(x)) \right| \leq Ce^{-\epsilon |\rho|}.
\]

We now rewrite (10) as

\[
\int_{N_\epsilon} e^{-x \cdot \rho} m P^N = -\int_{N_\epsilon} e^{-x \cdot \rho} m Q^{N+1} \tilde{v}^{N+1} - \int_{N_\epsilon} e^{-x \cdot \rho} \psi m (P^N + Q^{N+1} \tilde{v}^{N+1})
\]

\[
-\int_{\Omega \setminus N_\epsilon} e^{-x \cdot \rho} (1 + \psi) mv^0
\]

(14)
where we have factored $v^{N+1} = Q^{N+1} \tilde{v}^{N+1}$ as a homogeneous polynomial times an analytic function $\tilde{v}^{N+1}$. Note that $\tilde{v}^{N+1}$ remains bounded in $N_\epsilon$ because $v^0$ is analytic in a full neighborhood of the corner point. The following lemma tells us how the first two terms on right hand side of (14) decay as $|\rho| \to \infty$.

**Lemma 2.6.** Let $R^N(x)$ be a homogeneous polynomial of degree $N$. Let $\tau > 0$ and $\Re \rho_j > \tau |\rho|$ for all $j = 1, \ldots, n$. Then, for any $f \in L^p$ with $1 \leq p < \infty$.

$$\left| \int_{x>0} e^{-x \cdot \rho} R^N(x) f(x) dx \right| \leq C |\rho|^{-(N+n)n/p} \|f\|_{L^p}$$

**Proof.** Let $\rho = s \theta$, where $\theta \in \mathbb{C}^n$, $|\theta| = 1$, $\Re \theta_j > \tau$, and $s > 0$. Then

$$\int_{x>0} e^{-x \cdot \theta} R^N(x) f(x) dx = \frac{1}{s^{N+n}} \int_{y>0} e^{-y \cdot \theta} R^N(y) f \left( \frac{y}{s} \right) dy \leq \frac{1}{s^{N+n}} \left\| e^{-y \cdot \theta} R^N(y) \right\|_{L^q} \left\| f \left( \frac{y}{s} \right) \right\|_{L^p} = \frac{C_{\tau, n, q, R}}{s^{N+n}} s^{n/p} \|f\|_{L^p}$$

The lemma gives us a bound on the first two terms on the right hand side of (14)

$$\left| \int_{N_\epsilon} e^{-x \cdot \rho} m Q^{N+1} \tilde{v}^{N+1} \right| + \left| \int_{N_\epsilon} e^{-x \cdot \rho} m (P^N + Q^{N+1} \tilde{v}^{N+1}) \right| \leq \frac{C}{|\rho|^{N+n+1}} \left\| m \tilde{v}^{N+1} \right\|_{L^\infty} + \frac{C}{|\rho|^{N+n-n/p}} \left\| \psi m \tilde{v}^{N+1} \right\|_{L^p},$$

which combines with (9) to yield

$$\leq \left\| m \tilde{v}^{N+1} \right\|_{L^\infty} \left( \frac{C}{|\rho|^{N+n+1}} + \frac{C}{|\rho|^{N+n-n/p}} \cdot \frac{C}{|\rho|} \right) \leq \frac{C}{|\rho|^{N+n+(1-n/p)}}$$

Theorem 2.3 allows us to choose any $2 \leq p < \infty$, say $p = 2n$, so the right hand side of (15) is bounded by\(^3\)

$$\leq \frac{C}{|\rho|^{N+n+1/2}}.$$ 

and consequently

$$\left| \int_{N_\epsilon} e^{-x \cdot \rho} m P^N \right| \leq \frac{C}{|\rho|^{N+n+1/2}} + C e^{-\epsilon \tau |\rho|}$$

\(^3\) This is where we make essential use of the $L^p$ estimates with $p > 2$ for $\psi$ in Theorem 2.3. We need $1 - \frac{n}{p}$ to be positive in order to show that these terms are dominated by the Laplace transform of the harmonic polynomial, which is bounded from below by $|\rho|^{-(N+n)}$. 
Because \( m(x) - 1 \) vanishes at \( x = 0 \), we have

\[
\left| \int_{x > 0} e^{-x \cdot \rho} P^N(x)(m(x) - 1)dx \right| = \left| \int_{x > 0} e^{-x \cdot \rho} \tilde{Q}^{N+1}(x) \tilde{m}(x)dx \right| \\
\leq \frac{C}{|\rho|^{N+n+1}} \| \tilde{m} \|_{L^\infty}
\]

which, in combination with Eq. (16), implies

\[
\left| \int_{N_\epsilon} e^{-x \cdot \rho} P^N \right| \leq \frac{C}{|\rho|^{n+N+1/2}} + C e^{-\epsilon|\rho|}
\]  (17)

For \( \rho \) of the form (12), it is straightforward to check that

\[
\left| \int_{x > 0 \cap \{|x| > \epsilon\}} e^{-x \cdot \rho} P^N \right| \leq C e^{-\epsilon|\rho|} \left( 1 + \frac{1}{|\rho|^{N+n}} \right)
\]  (18)

where \( C \) depends only on \( \epsilon, \tau, \) and \( P^N \). Combining (17) and (18) gives

\[
\left| \int_{x > 0} e^{-x \cdot \rho} P^N \right| \leq \frac{C}{|\rho|^{n+N+1/2}} + C e^{-\epsilon|\rho|} \left( 1 + \frac{1}{|\rho|^{N+n}} \right)
\]

which contradicts (13) for large \( |\rho| \) and finishes the proof of the Theorem 2.1. \( \square \)

It remains to prove Theorems 2.3 and 2.5, which are the subjects of Sects. 4 and 5.

3. Estimates for Fundamental Solutions

The proof of Theorem 2.3 will rely on an estimate of the solution to

\[
P_\rho(D)\psi := (\Delta - 2 \rho \cdot \nabla)\psi = f.
\]  (19)

Although we will work in different norms, we follow the outline in [14] and begin by estimating the convolution \( ||\chi_\epsilon * g||_{L^\infty} \) where \( \chi \) is a Schwartz class function,

\[
g(\xi) = \frac{1}{P_\rho(\xi)} \quad \text{and} \quad \chi_\epsilon(\xi) = \frac{1}{\epsilon^n} \chi \left( \frac{\xi}{\epsilon} \right)
\]

We will prove these estimates for a fairly general \( P \), using a geometric approach similar to that in [1]. The key properties of the symbol \( P(\xi) \) are the codimension of its characteristic variety (the set \( \mathcal{M} = P^{-1}(0) \)) and the order to which it vanishes as \( \xi \to \mathcal{M} \). The dimension of \( \mathcal{M} \) tells us the behavior of the solutions to the homogeneous differential equation, while the order of vanishing tells us the behavior of the particular solutions \( G * f \). In the case of Eq. (19), the codimension is 2 and \( P \) vanishes simply on \( \mathcal{M} \).

In the lemma below, the symbol \( \mathcal{S}(\mathbb{R}^n, \mathbb{C}) \) denotes the Schwartz space of smooth rapidly decreasing functions, and \( DP|_{\mathcal{M}} \) denotes the Jacobian of \( P \), restricted to the set \( \mathcal{M} \).

**Theorem 3.1.** Suppose that \( \chi(x) \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}) \) and \( \chi_\epsilon(x) := \epsilon^{-n} \chi \left( \frac{x}{\epsilon} \right) \). If \( P(\xi) \) satisfies

\begin{itemize}
  \item[(i)] \( P : \mathbb{R}^n \to \mathbb{R}^k \) is smooth
  \item[(ii)] \( \mathcal{M} = P^{-1}(0) \) is compact
  \item[(iii)] \( DP|_{\mathcal{M}} \) has constant rank \( k \)
  \item[(iv)] \( \lim \inf_{|\xi| \to \infty} |P(\xi)| \geq B > 0 \)
\end{itemize}
then
(a) $\mathcal{M}$ is a smooth embedded codimension $k$ manifold in $\mathbb{R}^n$
(b) $\|\chi_{\varepsilon} \ast \delta_{\mathcal{M}}\|_{L^\infty} \leq \frac{C}{\varepsilon^k}$
(c) If $P$ is real or complex valued ($k = 1$ or $2$), then
$$\|\chi_{\varepsilon} \ast \frac{1}{P}\|_{L^\infty} \leq \frac{C}{\varepsilon}.$$  
Moreover, if $k \geq 2$ and $F$ is a complex valued function satisfying $|F(P)| \leq \frac{1}{|P|}$ then
$$\|\chi_{\varepsilon} \ast F(P)\|_{L^\infty} \leq \frac{C}{\varepsilon}.$$  

**Remark 3.2.** We define
$$\langle \delta_{\mathcal{M}}, \phi \rangle := \int_{\mathcal{M}} \phi \, d\sigma_{\mathcal{M}}$$  
where $d\sigma_{\mathcal{M}}$ is the natural element of surface area on $\mathcal{M}$.

**Remark 3.3.** If $k \geq 2$ then $\frac{1}{P} \in L^1_{loc}$ is a well defined distribution on the whole of $\mathbb{R}^n$. If $k = 1$ we will use the principal value
$$\left\langle \frac{1}{P}, \phi \right\rangle := \int_{N_{\delta}(\mathcal{M})} (\phi(y) - \phi(m(y))) \frac{dy}{P(y)} + \int_{\mathbb{R}^n \setminus N_{\delta}(\mathcal{M})} \phi(y) \frac{dy}{P(y)},$$  
where $N_{\delta}(\mathcal{M})$ is a neighborhood of $\mathcal{M}$ and $m(y)$ associates with each $y \in N_{\delta}(\mathcal{M})$ the closest point in $\mathcal{M}$. Both of which are described more explicitly in the proposition below.

The following proposition recalls some immediate consequences of the implicit function theorem and the tubular neighborhood theorem (see, for example theorems 7.9 and 10.19 in [11]). We do not include a proof.

**Proposition 3.4.** Suppose that (i), (ii) and (iii) in Theorem 3.1 are satisfied. Then
(A) $\mathcal{M}$ is a smooth compact embedded submanifold of $\mathbb{R}^n$
(B) $\exists \delta > 0$ and a Lipschitz constant $L_{\delta}$ such that writing
$$N_{\delta}(\mathcal{M}) = \{ x \in \mathbb{R}^n \mid d(x, \mathcal{M}) \leq \delta \},$$  
every $x \in N_{\delta}(\mathcal{M})$ has a unique closest point $m(x)$ in $\mathcal{M}$. The map
$$\eta : N_{\delta}(\mathcal{M}) \to \mathcal{M} \times B^k_\delta(0)$$  
defined by
$$\eta(x) = \left( m(x), |x - m(x)| \frac{DP_{m(x)}(x - m(x))}{|DP_{m(x)}(x - m(x))|} \right)$$  
(20)
is a global diffeomorphism from $N_{\delta}(\mathcal{M})$ onto $\mathcal{M} \times B^k_\delta(0)$. Both $\eta$ and $\eta^{-1}$ are Lipschitz with uniform constant $L_{\delta}$.

---

4 Throughout this section, we use $m$ to denote a point in the manifold $\mathcal{M}$, and $m(y)$ to denote the projection of a point $y$ onto that manifold. This $m$ has no relation to the contrast, which we also denote by $m$ in Sects. 2 and 4.
Every point \( m \in \mathcal{M} \) has a \( \delta \)-neighborhood \( U_\delta(m) \subset \mathcal{M} \) that is diffeomorphic to a ball in \( \mathbb{R}^{n-k} \), i.e.

\[
\psi_m : U_\delta(m) := B^m_\delta(m) \cap \mathcal{M} \to B^0_\delta(0).
\]

Both \( \psi_m \) and \( \psi_m^{-1} \) are Lipschitz with uniform constant \( L_\delta \).

Two corollaries (also stated without proof) are:

**Corollary 3.5.** For \( x \in \mathbb{R}^n \),

\[
\text{Area} \left( B^0_r(x) \cap \mathcal{M} \right) := \int_{\mathcal{M} \cap B^0_r(x)} d\sigma_{\mathcal{M}} \leq C \delta r^{n-k}
\]

**Corollary 3.6.** For \( x \in N_\delta(\mathcal{M}) \),

\[
|P(x)| \geq C \delta d(x, \mathcal{M}).
\]

We are going to use diffeomorphisms to rewrite integrals over manifolds as integrals over Euclidean balls, where we can do some explicit calculations. Since our integrals will involve convolutions with Schwartz class functions, we need to describe the properties that the pullbacks of such functions inherit.

**Definition 3.7.** A family of \( \varepsilon \)-mollifiers, \( \chi_\varepsilon(x, y) \), defined on \( \Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n \) satisfies

(i) \( \sup_{x \in \Omega_1} \int_{\Omega_2} |\chi_\varepsilon(x, y)| \, dy \leq C \)

(ii) \( |\chi_\varepsilon(x, y)| \leq \frac{C N}{\varepsilon^n} \left( \frac{\varepsilon}{|x-y|} \right)^N \) for all \( N \in \mathbb{N} \)

(iii) \( |\nabla_y \chi_\varepsilon(x, y)| \leq \frac{C N}{\varepsilon^{n+1}} \left( \frac{\varepsilon}{|x-y|} \right)^N \) for all \( N \in \mathbb{N} \)

**Lemma 3.8.** If \( \chi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}) \), then

\[
\chi_\varepsilon(x, y) := \frac{1}{\varepsilon^n} \chi \left( \frac{x-y}{\varepsilon} \right)
\]

is a family of \( \varepsilon \)-mollifiers defined on \( \Omega_1 \times \Omega_2 = \mathbb{R}^n \times \mathbb{R}^n \).

**Definition 3.9.** The pullback of a family of \( \varepsilon \)-mollifiers is defined\(^5\) to be

\[
\psi^* \chi_\varepsilon(x, y) := \chi_\varepsilon(\psi(x), \psi(y)). \tag{21}
\]

The next lemma explains why we need to work with general \( \varepsilon \)-mollifiers.

**Lemma 3.10.** If \( \psi \) and \( \psi^{-1} \) are uniformly Lipschitz diffeomorphisms, then the pullback of a family of \( \varepsilon \)-mollifiers is a family of \( \varepsilon \)-mollifiers.

---

\(^5\) It seems natural to include a factor of \( \det(D\psi) \) in (21), to treat \( \chi_\varepsilon dx_1 \wedge \cdots \wedge dx_n \) as an \( n \)-form. We do not add the factor because it makes the proof of Lemma 3.10 slightly longer.
Proof. Let \( L_1 \) and \( L_2 \) be the Lipschitz constants for \( \psi \) and \( \psi^{-1} \), respectively. For (i), we estimate

\[
\sup_{x \in \psi^{-1}(\Omega_1)} \int_{\psi^{-1}(\Omega_2)} \chi_\varepsilon(x, \psi(y)) dy = \sup_{x \in \Omega_1} \int_{\Omega_2} \chi_\varepsilon(x, y) \frac{dy}{\det(D\psi(y))} \leq \sup_{x \in \Omega_1} L_2^n \int_{\Omega_2} \chi_\varepsilon(x, y) dy
\]

Next

\[
|\chi_\varepsilon(x, \psi(y))| \leq \frac{C_N}{\varepsilon^n} \left( \frac{\varepsilon}{|\psi(x) - \psi(y)|} \right)^N \leq \frac{CNL_2^n}{\varepsilon^n} \left( \frac{\varepsilon}{|x-y|} \right)^N.
\]

Finally, for (iii),

\[
|\nabla_y \chi_\varepsilon(x, \psi(y))| = |D\psi \cdot \nabla_v \chi_\varepsilon(u, v)| \big|_{u=\psi(x)} \big|_{v=\psi(y)} \leq L_1 \frac{C_N}{\varepsilon^{n+1}} \left( \frac{\varepsilon}{|\psi(x) - \psi(y)|} \right)^N \leq \frac{CNL_1L_2^n}{\varepsilon^{n+1}} \left( \frac{\varepsilon}{|x-y|} \right)^N.
\]

\[\Box\]

Proposition 3.11. Let \( \chi_\varepsilon \) be a family of \( \varepsilon \)-mollifiers defined on \( \Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n \) and \( \mathcal{M} \) a compact embedded submanifold of \( \mathbb{R}^n \) of codimension \( k \). Then

\[
\sup_{x \in \Omega_1} \int_{\mathcal{M} \cap \Omega_2} |\chi_\varepsilon(x, m)| d\sigma_{\mathcal{M}}(m) \leq \frac{C}{\varepsilon^k}
\]

for small \( \varepsilon \).

Proof. We may assume that \( \mathcal{M} \subset \Omega_2 \). Let \( \delta \) be the uniform constant in Proposition 3.4. Fix \( x \in \Omega_1 \) and assume that \( \varepsilon < \delta \). According to (ii) in Definition 3.7 we have

\[
\int_{\mathcal{M} \cap \{m | |x-m| \geq \delta \}} \chi_\varepsilon(x, m) d\sigma_{\mathcal{M}}(m) \leq \frac{C_N}{\varepsilon^n} \left( \frac{\varepsilon}{\delta} \right)^N \text{area}(\mathcal{M}).
\]

On the other hand

\[
\int_{\mathcal{M} \cap \{m | |x-m| \leq \varepsilon \}} \chi_\varepsilon(x, m) d\sigma_{\mathcal{M}}(m) \leq \frac{C_0}{\varepsilon^n} \text{area}(\mathcal{M} \cap B^n_\varepsilon(x)) \leq \frac{C_0L^n_{\delta}}{\varepsilon^k},
\]

where \( L_\delta \) is the Lipschitz constant. To estimate the remaining part of the integral, we use local coordinates \( \psi \), based at \( m(x) \), the point on \( \mathcal{M} \) closest to \( x \), as described in Proposition 3.4 (D). Let \( \Psi = \psi^{-1} \). Then

\[
\int_{\mathcal{M} \cap \{m | |x-m| < \delta \}} \chi_\varepsilon(x, m) d\sigma_{\mathcal{M}}(m) = \int_{B^n_\delta(0) \setminus B^{n-k}_\varepsilon(0)} \Psi^* \chi_\varepsilon \Psi^* d\sigma_{\mathcal{M}}.
\]
Because

\[ |\chi_\varepsilon(x, m)| \leq \frac{C_N}{\varepsilon^n} \left( \frac{\varepsilon}{(|x - m(x)|^2 + |m(x) - m|^2)^{1/2}} \right)^N \]

\[ \leq \frac{C_N}{\varepsilon^n} \left( \frac{\varepsilon}{|m(x) - m|} \right)^N = \frac{C_N}{\varepsilon^n} \left( \frac{\varepsilon}{\rho} \right)^N, \]

where \( \rho = |m(x) - m| \), we may use polar coordinates centered at \( m(x) \) to see that

\[
\int_{B^\delta_n(0) \setminus B^\varepsilon_n(0)} |\Psi^* \chi_\varepsilon \Psi^* d\sigma| \leq L^\delta_n \int \frac{d\sigma}{S^{n-k-1}} \int_\varepsilon^{\delta} \frac{N}{\varepsilon^n} \left( \frac{\varepsilon}{\rho} \right)^N \rho^{n-k-1} d\rho
\]

\[ \leq L^\delta_n \omega_{n-k-1} C N^N \varepsilon^{n-n-1} \frac{\delta^n-1-N - \varepsilon^{n-k-N}}{|n-k-N|} \]

\[ \leq L^\delta_n \omega_{n-k-1} C N^N \varepsilon^{n-n-1} \frac{\varepsilon^{-k}}{|n-k-N|} \]

where \( S_{n-k-1} \) is the unit sphere in \( \mathbb{R}^{n-k} \) and \( \omega_{n-k-1} \) its surface measure. The claim follows by taking \( N > n - k \). \( \square \)

Remark 3.12. In the proof of Proposition 3.11, when considering \( x \in N^\delta(\mathcal{M}) \), we only required that the mollifier satisfy

\[ |\chi_\varepsilon(x, y)| \leq \frac{C_N}{\varepsilon^n} \left( \frac{\varepsilon}{|m(x) - m(y)|} \right)^N. \]

We will use this observation in the proof of Proposition 3.13 below, which will finish the proof of Theorem 3.1.

**Proposition 3.13.** Let \( \chi_\varepsilon \) be a family of \( \varepsilon \)-mollifiers, \( \mathcal{M}, P \) and \( k \geq 2 \) satisfy the conditions in Theorem 3.1, and \( F : \mathbb{R}^k \rightarrow \mathbb{C} \) satisfy \( |F(P)| \leq \frac{C}{|P|} \). Then, for sufficiently small \( \varepsilon \),

\[
\left| \int_{\mathbb{R}^n} \chi_\varepsilon(x, y) F(P(y)) dy \right| \leq \frac{C}{\varepsilon}.
\]

If \( k = 1 \) then

\[
\left| \int_{\mathbb{R}^n} \frac{\chi_\varepsilon(x, y)}{P(y)} dy \right| \leq \frac{C}{\varepsilon},
\]

where \( \frac{1}{P} \) is defined by principal value as in Remark 3.3.

**Proof.** We assume that \( \varepsilon < \frac{\delta}{2} \), with \( \delta \) the constant in Proposition 3.4 (C). Because

\[ |F(P)| \leq \frac{C}{|P|} \leq \frac{C}{\varepsilon} \text{ on } N^\delta(\mathcal{M}) \setminus N^\varepsilon(\mathcal{M}) \text{ and } \leq C_\delta \text{ outside } N^\delta(\mathcal{M}), \]

\[
\int_{\mathbb{R}^n \setminus N^\varepsilon(\mathcal{M})} |\chi_\varepsilon F(P)| \, dy \leq \sup_{y \in \mathbb{R}^n \setminus N^\varepsilon(\mathcal{M})} |F(P(y))| \|\chi_\varepsilon\|_{L^1} \leq \frac{C}{\varepsilon}.
\]
For the moment, we restrict to the case that $k = \text{codim}(\mathcal{M}) \geq 2$, so that $F(P) \in L^1(\mathbb{R}^n)$. If $x \notin N_\delta(\mathcal{M})$, then

$$
\sup_{y \in N_\delta(\mathcal{M})} |\chi_\varepsilon(x, y)| \leq \left(\frac{\varepsilon}{\delta/2}\right)^N \frac{C_N}{\varepsilon^n}
$$

so that

$$
\sup_{x \notin N_\delta(\mathcal{M})} \int_{N_\delta(\mathcal{M})} |\chi_\varepsilon F(P)\big| \, dy \leq \int_{N_\delta(\mathcal{M})} |F(P)| \, dy \left(\frac{\varepsilon}{\delta/2}\right)^N \frac{C_N}{\varepsilon^n}.
$$

and choosing $N \geq n - 1$ shows that this bounded by a constant over $\varepsilon$.

If $x \in N_\delta(\mathcal{M})$, we can use the diffeomorphism $\eta$ and its inverse $H$, described in (C) of Proposition 3.4 to obtain

$$
\sup_{x \in N_\delta(\mathcal{M})} \int_{N_\delta(\mathcal{M})} |\chi_\varepsilon(x, y)F(P(y))\big| \, dy
$$

$$
= \sup_{u \in \mathcal{M} \times B_\varepsilon^k(0)} \int_{\mathcal{M} \times B_\varepsilon^k(0)} |H^* \chi_\varepsilon(u, v)F(P(H(v)))\big| \frac{d\sigma_{\mathcal{M}}(m)ds}{|\det(D\eta)|}
$$

where $v = (m, s) \in \mathcal{M} \times B_\varepsilon^k(0)$. Because $|F(P(H(s)))| \leq \frac{C}{|P(y)|} \leq \frac{C}{|s|}$ here and $|\det(D\eta)|$ is bounded from below by the $n$-th power of the Lipschitz constant $L_2$, this is bounded by

$$
\leq CL_2^{-n} \int_{B_\varepsilon^k(0)} \left(\sup_{u \in \mathcal{M} \times B_\varepsilon^k(0)} \int_{\mathcal{M}} |H^* \chi_\varepsilon| \, d\sigma_{\mathcal{M}}\right) \frac{1}{|s|} \, ds.
$$

For each fixed $s$,

$$
|H^* \chi_\varepsilon| \leq \frac{C_N}{\varepsilon^n} \left(\frac{\varepsilon}{|u - (m, s)|}\right)^N,
$$

so according to Remark 3.12 we can apply Proposition 3.11 to the manifold $\mathcal{M} \times \{s\}$ to show that the quantity in brackets in (22) satisfies

$$
\sup_{u \in \mathcal{M} \times B_\varepsilon^k(0)} \int_{\mathcal{M}} |H^* \chi_\varepsilon| \, d\sigma_{\mathcal{M}} \leq \frac{C}{\varepsilon^k}.
$$

This implies the estimate

$$
\sup_{x \in N_\delta(\mathcal{M})} \int_{N_\delta(\mathcal{M})} |\chi_\varepsilon(x, y)F(P(y))\big| \, dy \leq \int_{B_\varepsilon^k(0)} \frac{C \, ds}{\varepsilon^k |s|} = \frac{C}{\varepsilon^k} \cdot \varepsilon^{k-1},
$$

which completes the proof in the codimension 2 case.

If $\mathcal{M}$ is of codimension one we have the definition

$$
\langle \frac{1}{P}, \phi \rangle = \int_{N_\delta(\mathcal{M})} \frac{\big(\phi(y) - \phi(m(y))\big)}{P(y)} \, dy + \int_{\mathbb{R}^n \setminus N_\delta(\mathcal{M})} \phi(y) \frac{dy}{P(y)}
$$
Corners Always Scatter

and note that this agrees with \( \int_{\mathbb{R}^n} \phi \frac{dy}{P} \) for all \( \phi \in C^\infty_0(\mathbb{R}^n \setminus \mathcal{M}) \). With this definition,

\[
\frac{1}{P} \star \chi_\varepsilon = \int_{\mathbb{R}^n \setminus N_\varepsilon} \chi_\varepsilon \frac{dy}{P} + \int_{N_\varepsilon(\mathcal{M})} \frac{(\chi_\varepsilon(x, y) - \chi_\varepsilon(x, m(y)))}{P(y)} dy.
\]

We estimate the first integral as we did in the codimension \( \geq 2 \) case, and rewrite the second as

\[
\int_{\mathcal{M}} \left[ \int_{-\varepsilon}^{\varepsilon} \chi_\varepsilon(m(x), v(x), m(y), v(y)) - \chi_\varepsilon(m(x), v(x), m(y), 0) \frac{d\nu(y)}{P(m(y), v(y))} \right] d\sigma_{\mathcal{M}}.
\]

where \( m(x) \) again denotes the closest point on \( \mathcal{M} \), and \( \nu(x) \) is the normal coordinate, given explicitly by the second component on the right hand side of Eq. (20). The numerator of the innermost integrand can be estimated by \( |\nabla \chi_\varepsilon| \varepsilon \), and Corollary 3.6 guarantees that the denominator satisfies \( |P| > C\varepsilon|v(y)| \). We may now employ item (iii) in Definition 3.7 to conclude that the integral in brackets satisfies \( |\tilde{\chi}_\varepsilon| \leq \frac{C N}{\varepsilon^n} \left( \frac{\varepsilon}{|m(x) - m(y)|} \right)^N \)

so that Theorem 3.11 and Remark 3.12 apply here, and we may conclude that \( |\int_{\mathcal{M}} \tilde{\chi}_\varepsilon d\sigma_{\mathcal{M}}| \leq \frac{C_{\varepsilon}}{\varepsilon^k} \) with \( k = 1 \) in this case.

We need only one application of Theorem 3.1 for our proof of Theorem 2.3. We return to (19) and set

\[
g(\xi) = \frac{1}{-\xi \cdot \xi - 2i\rho \cdot \xi}
\]

**Proposition 3.14.** There is a constant \( C \), depending only on the dimension \( n \) and \( \chi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}) \), so that

\[
\| \chi_\varepsilon \ast g \|_\infty \leq \frac{C}{\varepsilon \rho}
\]

**Proof.** Let \( \rho = s\Theta \) where \( \Theta \in \mathbb{C}^n \) has unit norm and \( s = |\rho| \). We will apply the estimate in item c) from Theorem 3.1, but first we need do some scaling

\[
\chi_\varepsilon \ast g(s\eta) = -\int \chi \left( \frac{s\eta - \xi}{\varepsilon} \right) \frac{1}{\xi \cdot \xi + 2i s\Theta \cdot \xi} \frac{d^n \xi}{\varepsilon^n}
\]

letting \( \sigma = s\xi \) gives

\[
= -\frac{1}{s^2} \int \chi \left( \frac{\eta - \sigma}{\varepsilon} \right) \frac{1}{\sigma \cdot \sigma + 2i\Theta \cdot \sigma} \frac{d^n \sigma}{\varepsilon^n}
= \frac{1}{s^2} \tilde{\chi}_\varepsilon \ast \frac{1}{P}
\]

where \( \tilde{P} = -\xi \cdot \xi - 2i\Theta \cdot \xi \). According to Theorem 3.1,

\[
\| \chi_\varepsilon \ast g(s\eta) \|_\infty \leq \frac{1}{s^2} \frac{C}{\varepsilon} \leq \frac{C}{\varepsilon \rho}
\]

Recalling that \( s = |\rho| \), and that \( \| \chi_\varepsilon \ast g(s\eta) \|_\infty = \| \chi_\varepsilon \ast g \|_\infty \) gives

\[
\| \chi_\varepsilon \ast g \|_\infty \leq \frac{C}{\varepsilon |\rho|}
\]
4. Proof of Theorem 2.3

In order to prove Theorem 2.3, we insert the ansatz (8) into (6) to see that $\psi$ must satisfy

$$ (\Delta - 2\rho \cdot \nabla)\psi = -k^2(1 - m(x))(1 + \psi) \quad \text{in } D. \quad (24) $$

We replace the right hand side of (24) using

$$ Q = -k^2(1 - m(x))\Phi_D $$

where $\Phi_D$ is smooth, compactly supported, and identically equal to one on the bounded domain $D$. We seek $\psi$ satisfying

$$ (\Delta - 2\rho \cdot \nabla)\psi = Q(1 + \psi) \quad \text{in } \mathbb{R}^n \quad (25) $$

noting that a solution to (25) in $\mathbb{R}^n$ will satisfy (24) in an open neighborhood of $D$. We will construct $\psi$ by summing the series

$$ \psi = \sum_{N=0}^{\infty} \psi^N \quad (26) $$

where $\psi^0 = 0$ and the remaining $\psi^N$ satisfy

$$ (\Delta - 2\rho \cdot \nabla)\psi^N = Q\psi^{N-1} \quad (27) $$

The existence of solutions to (27) and the convergence of the sum will follow from an estimate of solutions to the constant coefficient differential equation

$$ P_{\rho}(D)\psi := (\Delta - 2\rho \cdot \nabla)\psi = f. \quad (28) $$

The simplest estimate would follow from taking the Fourier transform of both sides and dividing by the symbol $P_{\rho}(\xi)$. If we use the letter $g(\xi)$ to denote the reciprocal of $P_{\rho}$, we want to estimate

$$ \hat{\psi} = g \hat{f} $$

or equivalently

$$ \psi = G * f \quad (29) $$

where $*$ denotes convolution and $G$ is the inverse Fourier transform of $(2\pi)^{-n/2}g$. A simple $L^\infty$ estimate for $g$ does not hold because of the zeros of $P$, but these affect the behavior of $\psi$ for large $x$, and our goal is to prove a strong local estimate. We are willing to prove an estimate that allows $\psi$ to grow as $x \to \infty$ in exchange for a good local estimate, i.e. $L^q$ for large $q$ on compact sets. We will separate the local and global behavior by writing $G$, the inverse Fourier transform of $g$, as a sum of functions $G_j$ with compact support, and estimating each separately.

We introduce a dyadic partition of unity. Let

$$ 1 = \phi_0(s) + \sum_{j=1}^{\infty} \phi_j(s) \quad (30) $$
where $\phi_0$ and $\phi$ are $C^\infty$ even functions of $s \in \mathbb{R}$, and

$$\text{supp } \phi_0 \subset [-2, 2]$$

$$\text{supp } \phi \subset [\frac{1}{2}, 2]$$

and

$$\Phi_j(x) := \phi\left(\frac{|x|}{2^j}\right) \text{ for } j \geq 1, \quad \Phi_0(x) = \phi_0(|x|),$$

so that

$$\text{supp } \Phi_j \subset B_{2^{j+1}}(0) \setminus B_{2^{j-1}}(0), \quad \text{supp } \Phi_0 \subset B_2(0).$$

We will make use of the fact that

$$\hat{\Phi}_j(\xi) = 2^n j \hat{\Phi}(2^j \xi) = \frac{\hat{\Phi}(\frac{\xi}{\epsilon})}{\epsilon^n}$$

which makes the \{\hat{\Phi}_j\} a family of $\epsilon$-mollifiers with $\chi = \hat{\Phi}$ and with $\epsilon = 2^{-j}$.

We expand $\psi$, $G$, and $f$ with respect to this partition, i.e.,

$$\psi = \sum \psi_j = \sum \Phi_j \psi$$

$$f = \sum f_j = \sum \Phi_j f$$

$$G = \sum G_j = \sum \Phi_j G$$

so that (29) becomes

$$\psi_m = \Phi_m \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} G_j \ast f_k$$

(31)

If we recall that the support of the convolution is a subset of the sum of the supports, we see that if $r_1 = 2^{k-1} - 2^{j+1} > 0$ or $r_2 = 2^{j-1} - 2^{k+1} > 0$, the support of $G_j \ast f_k$ is contained outside the ball of radius $r_1$ or $r_2$, respectively. In particular, this means that

$$\Phi_m G_j \ast f_k = 0$$

if

$$2^{m+1} < 2^{j-1} - 2^{k+1}$$

which will always be the case if

$$j > 3 + \max(k, m)$$

so that the second sum in (31) is finite

$$\psi_m = \Phi_m \sum_{k=0}^{\infty} \sum_{j=0}^{\max(k, m)+3} G_j \ast f_k.$$
Taking the Fourier transform gives

\[ \hat{\psi}_m = (2\pi)^{-n} \Phi_m * \sum_{k=0}^{\infty} \sum_{j=0}^{\max(k,m)+3} g_j \hat{f}_k \]

where \( g_j = \hat{\Phi}_j * g = (2\pi)^n \hat{G}_j \), so that

\[ \|\hat{\psi}_m\|_p \leq (2\pi)^{-n} \|\Phi_m\|_1 \sum_{k=0}^{\infty} \sum_{j=0}^{\max(k,m)+3} \|g_j\|_{\infty} \|\hat{f}_k\|_p \]

We may now estimate the convolution \( \|\hat{\Phi}_j * g\|_{L^\infty} \) using (23) of Proposition 3.14 with \( \chi = \Phi \) and \( \epsilon = 2^{-j} \) to establish that

\[ \|g_j\|_{\infty} \leq \frac{C}{|\rho|} 2^j \]

for \( |\rho| \) sufficiently large, so that

\[ \|\hat{\psi}_m\|_p \leq \|\Phi_m\|_1 \sum_{k=0}^{\infty} \sum_{j=0}^{\max(k,m)+3} \frac{C}{|\rho|} 2^j \|\hat{f}_k\|_p \]

Because \( \hat{\Phi}_m(\xi) = 2^{nm} \hat{\Phi}(2^m \xi) \), its \( L^1 \) norm is the same as the \( L^1 \) norm of \( \hat{\Phi} \), which does not depend on \( m \), so

\[ \|\hat{\psi}_m\|_p \leq \frac{C}{|\rho|} \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\max(k,m)+3} 2^j \right) \|\hat{f}_k\|_p \]

\[ \leq \frac{C}{|\rho|} \sum_{k=0}^{\infty} 2^{\max(k,m)+4} \|\hat{f}_k\|_p \]

which we rewrite as

\[ \sup_m 2^{-m} \|\hat{\psi}_m\|_p \leq \frac{C}{|\rho|} \sum_{k=0}^{\infty} 2^k \|\hat{f}_k\|_p \] \hspace{1cm} (32)

with a new constant \( C \) that is \( 2^4 \) times the old one.

Our goal is to estimate the \( L^q(D) \) norm of \( \psi \) on a compact set \( D \) for \( q > 2 \), and this is bounded by the left hand side of (32) if we choose \( p < 2 \) to be the dual exponent. In our application, \( f \) will be the right hand side of (27) which will have its support in \( D \), so the sum on the right hand side of (32) will also be a finite sum, bounded by a constant times \( \|\hat{f}\|_p \). We will have the desired bound for \( \psi \) as long as we can guarantee that the Fourier transform of \( f \) is in \( L^p \) for all \( p \leq 2 \).

In the special case that \( p = 2 \), the Plancherel inequality tells us that (32) is equivalent to

\[ \sup_m 2^{-m} \|\psi_m\|_2 \leq \frac{C}{|\rho|} \sum_{k=0}^{\infty} 2^k \|f_k\|_2 \]
This kind of estimate was used in [1] to study constant coefficient PDE’s with simple characteristics, including, as the principal example, the free Helmholtz equation. The norms defined there were:

\[ \| f \|_{B_{k/2}^*} := \sup_{0 \leq j < \infty} \frac{1}{2^{jk/2}} \| f_j \|_2 = \| \hat{f} \|_{B_{2,\infty}^{-k/2}} \]

\[ \| f \|_{B_{k/2}^*} := \sum_{j=0}^{\infty} 2^{jk/2} \| f_j \|_2 = \| \hat{f} \|_{B_{2,1}^{-k/2}} \]

The authors showed, in particular, that the incident waves for the Helmholtz equation in \( B_{1/2}^* \) were exactly the Herglotz wave functions. The spaces \( B_{k/2} \) and \( B_{k/2}^* \) are Fourier transforms of Besov-spaces, which are defined in [2, 16, 18] using the partition of unity in (30). For \( s \in \mathbb{R}, 0 < p \leq \infty \) and \( 0 < q < \infty \),

\[ B_{p,q}^s = B_{p,q}^s(\mathbb{R}^n) = \{ f \in \mathscr{S}'(\mathbb{R}^n) \mid \| f \|_{B_{p,q}^s} < \infty \} \]

where \( \mathscr{S}' \) denotes the space of tempered distributions, \( \mathcal{F}^{-1} \) in the line below means the inverse Fourier transform, and the norms are defined as

\[ \| f \|_{B_{p,q}^s} := \left( \sum_{j=0}^{\infty} \left( R_j \| \mathcal{F}^{-1}(\Phi_j f) \|_{L^p} \right)^q \right)^{1/q} \]

with \( R_j = 2^j \), and with the usual modification for \( q = \infty \).

**Definition 4.1.** We say that \( f \in \widehat{B}_{p,q}^s \) if \( \hat{f} \in B_{p,q}^s \) and write \( \| f \|_{\widehat{B}_{p,q}^s} = \| \hat{f} \|_{B_{p,q}^s} \). Note that

\[ \| f \|_{\widehat{B}_{p,q}^s} = \left( \sum_{j=0}^{\infty} \left( R_j \| \mathcal{F}^{-1}(\Phi_j f) \|_{L^p} \right)^q \right)^{1/q} = \left( \sum_{j=0}^{\infty} \left( R_j \| \Phi_j \hat{f} \|_{L^p} \right)^q \right)^{1/q} \]

The fact that \( \widehat{B}_{p,q}^s \) is a Banach space follows from the corresponding fact for \( B_{p,q}^s \) [18, 2.3.3]. We simply note that the Fourier transform, acting on tempered distributions, is one to one, and that convergence in the \( B_{p,q}^s \) norm implies convergence as tempered distributions.

Our estimate in (32) may be written as

\[ \| \psi \|_{\widehat{B}_{p,\infty}^s} \leq \frac{C}{|\rho|} \| f \|_{\widehat{B}_{p,1}^s}. \] (33)

We have proved

**Proposition 4.2.** For every \( f \in \widehat{B}_{p,1}^s \), there exists a \( \psi \in \widehat{B}_{p,\infty}^s \) satisfying (28) and the estimate (33).
We now return to (27). We will use Lemma 4.3 below to show that

\[ Q \in \hat{B}_{1,1}^{1,1} \quad \text{and} \quad \|Qg\|_{B_{p,1}^1} \leq C_Q \|g\|_{B_{p,\infty}^1}, \]  

(34)

where \( C_Q \) denotes a constant depending on \( Q \). Combining (34) with (33) shows that

\[ \|\psi\|_{L^q(D)} \leq R \|\psi\|_{B_{p,\infty}^1} \leq \frac{C}{|\rho|} \|Q\|_{B_{p,1}^1} \]

and hence that the series (26) converges when \(|\rho| > C_Q\) and therefore that, for \( 2 < q = \frac{p}{p-1} \) and \( D \) contained in a ball of radius \( R \), the sum \( \psi \) satisfies

\[ \|\psi\|_{L^q(D)} \leq R \|\psi\|_{B_{p,\infty}^1} \leq \frac{C}{|\rho|} \|Q\|_{B_{p,1}^1} \]

and establishes (9) for all \( q > 2 \) (and, because \( D \) is bounded, for \( q < 2 \) as well) and therefore proves Theorem 2.3.

It remains to prove (34). The function \( Q \) satisfies

\[ Q = \prod_{i=1}^n (H^+(x_i) - H^+(x_i - 1))q(x), \]

where \( q \) is smooth and supported in a ball of radius \( R \), and \( H^+(t) \) is the Heaviside function, the indicator function of the positive half line.

\[ \|q\|_{B_{p,1}^1} = \sum_{j=0}^{\log_2 R} 2^j \|\hat{\Phi}_j \* \hat{q}\|_p \leq 2R \sup_j \|\hat{\Phi}_j\|_1 \|\hat{q}\|_p = 2R \|\hat{\Phi}\|_1 \|\hat{q}\|_p \]

so \( q \in B_{p,1}^1 \). The lemma below tells us that multiplication by the Heaviside function preserves \( B_{p,1}^1 \) and that multiplication by smooth compactly supported \( q \) maps \( B_{p,\infty}^{-1} \) to \( B_{p,1}^1 \). This is enough to establish (34) and finish this section.

**Lemma 4.3.** Suppose that \( q \) is smooth and supported in the ball of radius \( R \), and \( \Theta \) a unit vector in \( \mathbb{R}^n \). Then

\[ \|qg\|_{B_{p,1}^1} \leq 2R^2 \|\hat{q}\|_1 \|\hat{g}\|_{B_{p,\infty}^1} \]

(35)

\[ \| (H_+(x \cdot \Theta) - H^+(x \cdot \Theta - 1))g(x) \|_{B_{p,1}^1} \leq C_p \|g\|_{B_{p,1}^1} \]

(36)

for \( 1 < p < \infty \).

**Proof.**

\[ \|qg\|_{B_{p,1}^1} = \sum_{j=0}^{\infty} 2^j \left\| (qg\Phi_j) \right\|_p \]
Because $q$ has compact support, the sum is finite, i.e.,

$$= \sum_{j=0}^{\log_2 R} 2^j \| \hat{q} \ast \hat{g}_j \|_p,$$

where $g_j$ denotes $g \ast \Phi_j$

$$\leq \sum_{j=0}^{\log_2 R} 2^{2j} \| \hat{q} \|_1 \left( 2^{-j} \| \hat{g}_j \|_p \right)$$

$$\leq \left( \sum_{j=0}^{\log_2 R} 2^{2j} \right) \| \hat{q} \|_1 \| g \|_{B_{p,1}^{-1}}$$

$$\leq 2R^2 \| \hat{q} \|_1 \| g \|_{B_{p,1}^{-1}}$$

which establishes (35). To prove (36)

$$\| H^+(x \cdot \Theta)g \|_{B_{p,1}^1} = \sum_{j=0}^{\infty} 2^j \| \hat{H}^+ \hat{g}_j \|_p$$

$$= \sum_{j=0}^{\infty} 2^j \| \hat{H}^+ \ast \hat{g}_j \|_p$$

but convolution with $\hat{H}^+(\xi \cdot \Theta)$ is just a one dimensional Hilbert transform in the direction $\Theta$, which is bounded from $L^p$ to $L^p$ for all $1 < p < \infty$, so that

$$\leq \sum_{j=0}^{\infty} 2^j C_p \| \hat{g}_j \|_p$$

$$\leq C_p \| g \|_{B_{p,1}^1}.$$

The same estimate holds for $H^+(x \cdot \Theta - 1)$ because rigid motions induce bounded maps from $\overline{B_{p,1}^1}$ to itself. □

5. Proof of Theorem 2.5

In this section, the letters $i, k, m,$ and $n$ will represent integers. We will use what are sometimes called array, or componentwise operations, as well as standard multi-index notation. If $\eta$ is a vector in $\mathbb{C}^n$, and $\alpha$ is a multi-index (i.e. also a vector), we will use $\eta^\alpha$ to mean the product

$$\eta^\alpha = \eta_1^{\alpha_1} \cdots \eta_n^{\alpha_n}.$$

When $a$ is a scalar, $\eta^a$ will denote the vector

$$\eta^a = (\eta_1^a, \ldots, \eta_n^a).$$
Similarly, we will use a scalar divided by a vector, or a vector divided by a vector, to denote componentwise division, e.g.

\[ \frac{1}{\eta} = \left( \frac{1}{\eta_1}, \ldots, \frac{1}{\eta_n} \right), \]

We let \( \sigma_k(\eta) \) denotes the \( k \)’th elementary symmetric function of \((\eta_1, \ldots, \eta_n)\). The two symmetric functions we will make use of are

\[ \sigma_n(\eta) = \prod_{i=1}^{n} \eta_i, \]
\[ \sigma_{n-1}(\eta) = \sum_{i=1}^{n} \prod_{j \neq i} \eta_j. \]

In this section, we will use the superscript \( \hat{\cdot} \) to indicate that an index does not occur, so that

\[ \eta_{\hat{i}} = (\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_n) \]

means the \( n - 1 \)-dimensional vector that omits the \( i \)’th component of \( \eta \). We will use the notation \( P_{\hat{i}} \) and \( P(\eta_{\hat{i}}) \) interchangeably to denote a polynomial which does not depend on the \( i \)’th variable.

We will also use the superscript \( \hat{\cdot} \) to denote the Laplace transform, \( \hat{P} \), of a degree \( N \) homogeneous polynomial \( P(x) \), given by

\[ \hat{P}(\rho) = \int_{x > 0} e^{-\rho \cdot x} P(x) dx \] (37)

where \( x > 0 \) means that each component \( x_i > 0 \). Making the substitutions \( y_i = \rho_i x_i \), with \( \rho \) real and \( \rho > 0 \) (for the moment) we have

\[ \hat{P}(\rho) = \int_{y > 0} e^{-1 \cdot y} \left( \frac{y}{\rho} \right) \sigma_n \left( \frac{1}{\rho} \right) dy. \]

If \( P = \sum_{|\alpha| = N} p_\alpha x^\alpha \), then

\[ \hat{P}(\rho) = \sum_{|\alpha| = N} p_\alpha \frac{1}{\rho^{\alpha+1}} \int_{y > 0} e^{-1 \cdot y} y^\alpha dy = \sum_{|\alpha| = N} p_\alpha \left( \frac{1}{\rho} \right)^{\alpha+1} \alpha! \]

where \( \alpha + 1 \) is the multi-index with components \( \alpha_i + 1 \). Thus

\[ \hat{P}(\rho) = Q^{N+n} \left( \frac{1}{\rho} \right) \]

where \( Q = Q^{N+n} \) is the homogeneous polynomial of degree \( N + n \) with coefficients \( q_{\alpha+1} = \alpha! p_\alpha \). The main assertion of Theorem 2.5 is that \( \hat{P}(\rho) \) does not vanish on any open subset of the variety \( \rho \cdot \rho = 0 \). This is equivalent to the assertion that the polynomial \( Q(\eta) \) does not vanish identically on any open subset of

\[ \left\{ \frac{1}{\eta}, \frac{1}{\eta} = 0 \right\} \]
where

\[ \rho \cdot \rho = \frac{1}{\eta} \cdot \frac{1}{\eta} = \frac{\sigma_{n-1}(\eta^2)}{\sigma_n(\eta^2)} = \frac{\sigma_{n-1}(\eta^2)}{\sigma_n^2(\eta)}. \]

(38)

If \( P \) is harmonic, \( Q(\eta) = \hat{P}(\frac{1}{\eta}) \) has an additional representation.

**Lemma 5.1.** If \( P \) is harmonic and homogeneous, then

\[
Q(\eta) = \hat{P}(\frac{1}{\eta}) = \frac{\sigma_n(\eta)}{\sigma_{n-1}(\eta^2)} \sum_{i=1}^{n} \left( P_i + \eta_i Q_i \right)
\]

(39)

where \( P_i \) and \( Q_i \) are homogeneous polynomials of degree \( N + 2n - 2, \ N + 2n - 3 \), respectively, which do not depend on \( \eta_i \).

**Proof.** We will prove (39) on the open set where \( \Re \rho > 0 \) and that \( \rho \cdot \rho \neq 0 \). Because \( Q \) is a polynomial in \( \eta \), the right hand side of (39) is also a polynomial, and the identity must hold everywhere.

We start with (37), integrate by parts, and recall that \( P \) is harmonic,

\[
\hat{P}(\rho) = \int_{x>0} \frac{\Delta e^{-\rho \cdot x}}{\rho \cdot \rho} P(x) dx \\
= \frac{1}{\rho \cdot \rho} \sum_{i=1}^{n} \left( \int_{x_i>0} e^{-\rho \cdot x} \left( \rho_i P + \frac{\partial}{\partial x_i} P \right) + \int_{x_i>0} e^{-\rho \cdot x} \Delta P \right) \\
= \frac{1}{\rho \cdot \rho} \sum_{i=1}^{n} \left( \rho_i \int_{x_i>0} e^{-\rho_i \cdot x} P \bigg|_{x_i=0} + \int_{x_i>0} e^{-\rho_i \cdot x} \frac{\partial}{\partial x_i} P \bigg|_{x_i=0} \right) \\
= \frac{1}{\rho \cdot \rho} \sum_{i=1}^{n} \left( \rho_i \tilde{P}_i \left( \frac{1}{\rho_i^2} \right) + \tilde{Q}_i \left( \frac{1}{\rho_i^2} \right) \right),
\]

where \( \tilde{P}_i \) and \( \tilde{Q}_i \) simply denote polynomials in \( n - 1 \) variables.

Recalling (38), the polynomial \( Q \) then satisfies

\[
Q(\eta) = \frac{(\sigma_n(\eta))^2}{\sigma_{n-1}(\eta^2)} \sum_{i=1}^{n} \left( \frac{1}{\eta_i} \tilde{P}_i(\eta_i^2) + \tilde{Q}_i(\eta_i^2) \right) \\
= \frac{\sigma_n(\eta)}{\sigma_{n-1}(\eta^2)} \sum_{i=1}^{n} \left( \frac{\sigma_n(\eta)}{\eta_i} \tilde{P}_i(\eta_i^2) + \sigma_n(\eta) \tilde{Q}_i(\eta_i^2) \right) \\
= \sigma_n(\eta) \sum_{i=1}^{n} \left( P_i(\eta_i^2) + \eta_i Q_i(\eta_i^2) \right),
\]

so that \( P_i(\eta_i^2) = \sigma_{n-1}(\eta_i^2) \tilde{P}_i(\eta_i^2) \) and \( Q_i(\eta_i^2) = \sigma_{n-1}(\eta_i^2) \tilde{Q}_i(\eta_i^2) \) are the polynomials \( P_i \) and \( Q_i \) of (39), and the proof is finished. \( \Box \)

The irreducibility of the denominator, \( \sigma_{n-1}(\eta^2) \) in (39) will play a role in several parts of our proof, so we prove this fact here:
Lemma 5.2. If \( n \geq 3 \), \( \sigma_{n-1}(\eta^2) \) is an irreducible polynomial. If \( n = 2 \), then \( \sigma_{n-1}(\eta^2) = \eta_1^2 + \eta_2^2 = (\eta_1 - i \eta_2)(\eta_1 + i \eta_2) \).

Proof. The statement for \( n = 2 \) is obvious. We will prove this lemma for \( n \geq 3 \) by induction, making use of the identity

\[
\sigma_{n-1}(\eta^2) = \eta_1^2 \sigma_{n-2}(\eta_1^2) + \sigma_{n-1}(\eta_1^2)
\]

for \( \eta \in \mathbb{C}^n \). If \( \sigma_{n-1}(\eta^2) \) factors, and one factor does not depend on \( \eta_1 \), then we must have

\[
\eta_1^2 \sigma_{n-2}(\eta_1^2) + \sigma_{n-1}(\eta_1^2) = \sigma_{n-1}(\eta^2) = p\bar{\gamma}(\eta_1^2 \bar{q}\bar{\gamma} + r\bar{\gamma})
\]

where \( p\bar{\gamma}, q\bar{\gamma}, \) and \( r\bar{\gamma} \) are non-constant polynomials independent of \( \eta_1 \). Equating coefficients of \( \eta_1^2 \) gives

\[
\sigma_{n-2}(\eta_1^2) = p\bar{\gamma} \bar{q}\bar{\gamma},
\]

which contradicts the induction hypothesis because \( \sigma_{n-2}(\eta_1^2) = \sigma_{m-1}(\xi^2) \) with \( m = n - 1 \) and \( \xi = \eta_1^2 \in \mathbb{C}^m \).

On the other hand, if both factors depend on \( \eta_1 \), i.e.,

\[
\eta_1^2 \sigma_{n-2}(\eta_1^2) + \sigma_{n-1}(\eta_1^2) = (\eta_1 p\bar{\gamma} + r\bar{\gamma})(\eta_1 q\bar{\gamma} + s\bar{\gamma}).
\]

Equating coefficients of \( \eta_1^2 \) again gives

\[
\sigma_{n-2}(\eta_1^2) = p\bar{\gamma} \bar{q}\bar{\gamma}
\]

and contradicts the induction hypothesis.

We finish the induction by verifying irreducibility in the case \( n = 3 \). In this case,

\[
\sigma_2(\eta^2) = \eta_1^2 (\eta_2 + i \eta_3)(\eta_2 - i \eta_3) + \eta_2^2 \eta_3.
\]

If

\[
p\bar{\gamma}(\eta_1^2 \bar{q}\bar{\gamma} + r\bar{\gamma}) = \sigma_2(\eta^2) = \eta_1^2 (\eta_2 + i \eta_3)(\eta_2 - i \eta_3) + \eta_2^2 \eta_3
\]

equating the coefficients of \( \eta_1^2 \), we see again that \( p\bar{\gamma} \) must divide \( (\eta_2 + i \eta_3)(\eta_2 - i \eta_3) \).

Equating the coefficients of the terms that do not involve \( \eta_1^2 \), tells us that \( p\bar{\gamma} \) also divides \( \eta_2^2 \eta_3 \), but this is impossible because the two have prime factorizations without common factors.

If, on the other hand, \((\eta_1 p\bar{\gamma} + r\bar{\gamma})(\eta_1 q\bar{\gamma} + s\bar{\gamma}) = \sigma_2(\eta^2)\), expanding both sides of the equation shows that

\[
\eta_1^2 p\bar{\gamma}q\bar{\gamma} + \eta_1(q\bar{\gamma} r\bar{\gamma} + p\bar{\gamma}s\bar{\gamma})+ = \eta_1^2 (\eta_2 + i \eta_3)(\eta_2 - i \eta_3) + \eta_2^2 \eta_3^2 + r\bar{\gamma}s\bar{\gamma}
\]

then

\[
p\bar{\gamma}q\bar{\gamma} = (\eta_2 + i \eta_3)(\eta_2 - i \eta_3)
\]
which implies that $p_1$ must be a constant multiple of either $(\eta_2 + i \eta_3)$ or $(\eta_2 - i \eta_3)$ and $q_1$ must be a constant multiple of the other. Also

$$q_1 r_1 = -p_1 s_1$$

so that $p_1$ must divide $r_1$ because it does not divide $q_1$. However

$$r_1 s_1 = \eta_2^2 \eta_3^2$$

does not have $(\eta_2 \pm i \eta_3)$ as a factor, so this is also impossible and the proof is complete. □

We will need two more propositions for the proof of Theorem 2.5. The first follows easily from the previous lemma.

**Proposition 5.3.** If $P$ is harmonic and homogeneous, and $\hat{P}(\rho)$ vanishes identically on $\{\rho \cdot \rho = 0\}$, then $\sigma_{n-1}(\eta^2)$ divides the polynomial $Q(\eta) = \hat{P}(\frac{1}{\eta})$.

**Proof.** Because $\hat{P}$ vanishes on $\{\rho \cdot \rho = 0\}$, it follows from (38) that $Q$ vanishes on the set $\{\sigma_{n-1}(\eta^2) = 0\} \setminus \{\sigma_n(\eta^2) = 0\}$. Therefore, the product $\sigma_n(\eta^2) Q(\eta^2)$ vanishes on the entire variety $\{\sigma_{n-1}(\eta^2) = 0\}$, and hence must be divisible by $\sigma_{n-1}(\eta^2)$ by Hilbert’s Nullstellensatz. For $n \geq 3$, $\sigma_{n-1}(\eta^2)$ is irreducible and does not divide $\sigma_n(\eta^2)$, so it must divide $Q$. In the case $n = 2$, $\sigma_{n-1}(\eta^2)$ has two factors; neither factor divides $\sigma_n(\eta^2)$, so both divide $Q$. □

The proof of the next proposition will not be so easy.

**Proposition 5.4.** $\sigma_{n-1}(\eta^2)$ cannot divide any polynomial $Q$ of the form (39).

but the proof of Theorem 2.5 is an immediate consequence.

**Proof of Theorem 2.5.** If $n \geq 3$, the hypothesis of Theorem 2.5 is that $\hat{P}$ vanishes on an open subset of $\{\rho \cdot \rho = 0\}$, which means that $Q$, vanishes on an open subset of the irreducible variety $\sigma_{n-1}(\eta^2) = 0$. But this means that $Q$ vanishes on the whole variety by [13, p.91] or [19] and that $\sigma_{n-1}(\eta^2)$ divides $Q$, contradicting Proposition 5.4.

If $n = 2$, we have the same hypothesis for each of the irreducible factors, $\rho_1 - i \rho_2$ and $\rho_1 + i \rho_2$, so we may conclude that each divides $Q$, and therefore that $Q$ is divisible by their product $\sigma_{n-1}(\eta^2)$. □

**Proof of Proposition 5.4.** We will make essential use of the fact that $\sigma_{n-1}(\eta^2)$ is even in each component $\eta_j$ of $\eta$.

**Lemma 5.5.** Every polynomial $R(\eta)$ has a unique decomposition into a sum

$$R(\eta) = \sum_{\tau \in \{0,1\}^n} \eta^\tau R_\tau(\eta^2)$$

where $\tau$ is a multi-index with each component equal to 0 or 1. If $R$ has the special form $R = \sum_i (\eta_i P_i + Q_\tau)$, then each of the coefficients $R_\tau(\eta^2)$ has the special form

$$R_\tau = \sum_i S_\tau(\eta_i^2)$$

(40)
Proof. We express $R$ as a sum of monomials,
\[ R(\eta) = \sum_{\alpha} p_{\alpha} \eta^\alpha \]
group the terms that are even or odd for each $\eta_i$ together
\[ = \sum_{\tau \in \{0,1\}^n} \left( \sum_{\alpha \equiv 2 \tau} (p_{\alpha} \eta^\alpha) \right) \]
and remove a single power of $\eta_i$ from each monomial that is odd in $\eta_i$
\[ = \sum_{\tau \in \{0,1\}^n} \left( \sum_{\alpha \equiv 2 \tau} (p_{\alpha} \eta^{\alpha - \tau}) \right) \eta^\tau \] (41)
so that the summands in the parentheses contains only even powers
\[ = \sum_{\tau \in \{0,1\}^n} R_\tau (\eta^2) \eta^\tau \] (42)
The explicit formula for each $R_\tau$ in (41) implies that the decomposition is unique. Suppose now that $R$ has the special form $\sum_i (\eta_i P_i + Q_i)$, we can first decompose each of the $Q_i$ and the $P_i$,
\[
\eta_i P_i + Q_i = \eta_i \left( \sum_{\tau_i \in \{0,1\}^{n-1}} P_{\tau_i} (\eta_i^2) \eta_i^{\tau_i} \right) + \sum_{\tau_i \in \{0,1\}^{n-1}} Q_{\tau_i} (\eta_i^2) \eta_i^{\tau_i}
\]
\[ = \sum_{\tau_i \in \{0,1\}^{n-1}} P_{\tau_i} (\eta_i^2) \left( \eta_i^{\tau_i} \eta_i^1 \right) + \sum_{\tau_i \in \{0,1\}^{n-1}} Q_{\tau_i} (\eta_i^2) \left( \eta_i^{\tau_i} \eta_i^0 \right) \]
which shows that each summand $\eta_i P_i + Q_i$ has a decomposition where the coefficients of $\eta^\tau$ are independent of $\eta_i$. Thus the sum has coefficients which are sums of such functions. \qed

Lemma 5.6. If a polynomial $S(\eta^2)$ divides $R(\eta) = \sum_{\tau \in \{0,1\}^n} \eta^\tau R_\tau (\eta^2)$, then $S$ divides each $R_\tau$.

Proof. Suppose that
\[ R(\eta) = S(\eta^2) C(\eta) \]
expand both $R$ and $C$ as in Lemma 5.5
\[ \sum_{\tau \in \{0,1\}^n} \eta^\tau R_\tau (\eta^2) = S(\eta^2) \sum_{\tau \in \{0,1\}^n} \eta^\tau C_\tau (\eta^2) \]
\[ = \sum_{\tau \in \{0,1\}^n} \eta^\tau S(\eta^2) C_\tau (\eta^2) \]
and now use the uniqueness of the expansion to equate the coefficients of each monomial $\eta^\tau$. \qed
The last ingredient necessary for the proof of Proposition 5.4 is

**Proposition 5.7.** $\sigma_{n-1}^2(s)$ does not divide any polynomial of the form $T(s) = \sum T_i$ unless $T$ is identically zero.

Before giving its proof, we use it to finish the proof of Proposition 5.4. If, as in the hypothesis of Proposition 5.4, $\sigma_{n-1}^2(\eta^2)$ divides $R = \sum (\eta_i P_i + Q_i)$, then, according to Lemma 5.6, $\sigma_{n-1}^2(\eta^2)$ divides each of the $R_i$ in the expansion of Lemma 5.5, and each $R_i$ has the special form (40). Proposition 5.7 says that this is impossible (the variable $s$ replaces $\eta^2$) and thus finishes the proof of Proposition 5.4. \(\square\)

**Proof of Proposition 5.7.** We will prove the proposition by induction on the number of independent variables. We will expand all polynomials as polynomials in the single variable $s_1$ with coefficients that depend on the other variables. We begin with

$$\sigma_{n-1}(s) = s_1 \sigma_{n-2}(s_1) + \sigma_{n-1}(s_1)$$

$$\sigma_{n-1}^2(s) = s_1^2 \sigma_{n-2}^2(s_1) + s_1 2 \sigma_{n-2}(s_1)\sigma_{n-1}(s_1) + \sigma_{n-1}^2(s_1)$$

If a general polynomial $T(s)$ has $\sigma_{n-1}^2(s)$ as a factor, then expanding the equality $T = \sigma_{n-1}^2 C$ in powers of $s_1$ gives

$$\sum_{k=0}^{N} s_1^k T^k(s_1) = \left( s_1^2 \sigma_{n-2}^2(s_1) + s_1 2 \sigma_{n-2}(s_1)\sigma_{n-1}(s_1) + \sigma_{n-1}^2(s_1) \right) \left( \sum_{k=0}^{N-2} s_1^k C^k(s_1) \right)$$

Equating coefficients of powers of $s_1$ gives

$$T^N(s_1) = \sigma_{n-2}^2(s_1)C^{N-2}(s_1) \quad (43)$$

and, for $j = 1 \ldots (N - 2)$,

$$T^{N-j}(s_1) = \sigma_{n-2}^2(s_1)C^{N-2-j}(s_1) + \ldots \quad (44)$$

where the . . . indicate terms involving $C^k$ for $k > N - 2 - j$. We will not need to use the equations for $T^1$ and $T^0$.

Now, if $T$ has the special form $T = \sum T_i$, with the $T_i$ independent of $s_i$, then each of the $T^k$, except $T^0$, will have the special form

$$T^k = \sum_{i=2}^{n} T_{1,i}^k,$$

where the subscripts indicate that $T_{1,i}^k$ is independent of both $\eta_1$ and $\eta_i$. Thus Eq. (43) becomes

$$\sum_{i=2}^{n} T_{1,i}^N = \sigma_{n-2}^2(s_1)C^{N-2}(s_1) \quad (45)$$

but this is exactly the hypothesis of the proposition for one fewer dimension. If we let $\beta = s_1$ and $m = n - 1$, then (45) becomes

$$\sum_{i=1}^{m} T_{i}^N(\beta) = \sigma_{m-1}^2(\beta)C^{N-2}(\beta)$$
and the induction hypothesis guarantees that $C_{N-2}^N$ and $T_N^N$ are both identically zero. Once we know that $C_{N-2}^N$ is zero, we may conclude that the term represented by the ... in Eq. (44) for $T_{N-1}^N$ is zero, and repeat the argument to conclude that $C_{N-3}^N$ and $T_{N-2}^N$ are zero. We continue in this manner to conclude that all the $C_k$, and therefore all the $T^k$, are zero.

Finally, we verify the proposition in the case $n = 2$. In this case, we must check that the equality below

$$p_Nx^N + q_Ny^N = (x + y)^2 \sum_{k=0}^{N-2} c_kx^ky^{N-2-k}$$

is only possible if $p_N$, $q_N$, and all the $c_k$ are zero. Equating powers of $x$ and $y$ give

$$p_N = c_{N-2}$$
$$0 = 2c_{N-2} + c_{N-1}$$
for $j = 2 \ldots (N - 2)$

$$0 = c_{N-(j+2)} + 2c_{N-(j+1)} + c_{N-j}$$
and

$$0 = c_1 + 2c_0$$
$$q_N = c_0.$$ 

Discarding the first and last equations gives the invertible tridiagonal system

$$\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} =
\begin{pmatrix}
2 & 1 & 0 & \hdots & 0 \\
1 & 2 & 1 & \hdots & 0 \\
0 & 1 & 2 & \hdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \hdots & 0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
c_{N-2} \\
c_{N-1} \\
\vdots \\
c_1 \\
c_0
\end{pmatrix}$$

whence we conclude that all the $c_k$ are zero. This finishes the proof of the proposition.

$\Box$

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