Isomorphism of Binary Operations in Differential Geometry

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Abstract: We consider smooth binary operations invariant with respect to unitary transformations that generalize the operations of the Beltrami–Klein and Beltrami–Poincare ball models of hyperbolic geometry, known as Einstein addition and Möbius addition. It is shown that all such operations may be recovered from associated metric tensors that have a canonical form. Necessary and sufficient conditions for canonical metric tensors to generate binary operations are found. A definition of algebraic isomorphism of binary operations is given. Necessary and sufficient conditions for binary operations to be isomorphic are provided. It is proved that every algebraic automorphism gives rise to isomorphism of corresponding gyrogroups. Necessary and sufficient conditions in terms of metric tensors for binary operations to be isomorphic to Euclidean addition are given. The problem of binary operations to be isomorphic to Einstein addition is also solved in terms of necessary and sufficient conditions. We also obtain necessary and sufficient conditions for binary operations having the same function-parameter in the canonical representation of metric tensors to be isomorphic.

Keywords: differential geometry; canonical metric tensor; binary operation; Einstein addition; Euclidean addition; isomorphism

1. Introduction

The theory of the binary operation of the Beltrami–Klein ball model, known as Einstein addition, and the binary operation of the Beltrami–Poincare ball model of hyperbolic geometry, known as Möbius addition has been extensively developed for the last twenty years [1–36]. There appeared a theory that may be called gyrogeometry, based on nice algebraic properties of the aforementioned operations and the concepts of gyrogroups, gyrovector spaces, gyrotrigonometry, and gyrogeometric objects [21–30]. This theory has been extended to the space of matrices with indefinite scalar products, which is closely related to the phenomenon of entanglement in quantum physics [32–34], and to harmonic analysis [5–7].

Recently there appeared a new approach in this theory based on the analysis of local properties of underlying operations, corresponding metric tensors and Riemannian manifolds. It turned out that both Einstein addition and Möbius addition may be recovered from corresponding metric tensors using standard operations of differential geometry: logarithmic mapping, parallel transport, and geodesics [37]. Basic properties of Einstein and Möbius gyrogroups and gyrovector spaces were derived using this approach [37].

Einstein and Möbius additions are special cases of more general binary operations, namely, operations invariant with respect to unitary transformations. A differential geometry approach to the theory of such operations has been developed in [37,38]. The central object in this approach is a metric tensor in a special form, which was called the canonical form. This form depends on two scalar functions, \( m_0 \) and \( m_1 \), which determine the value of the first fundamental form at \( x \) in the directions orthogonal and parallel to \( x \) respectively.
We addressed the following problems in this paper.

1. Is it true that for every binary operation invariant with respect to unitary transformations the metric tensor has a canonical form (8) parametrized by a pair of functions \((m_0, m_1)\)?
2. What are the necessary and sufficient conditions on functions \(m_0, m_1\) such that their canonical metric tensor is smooth and is generated by a binary operation?
3. Find necessary and sufficient conditions under which two binary operations having canonical metric tensors are isomorphic.
4. Show that every algebraic isomorphism of binary operations generates an isomorphism of corresponding gyrogroups.
5. Show that an isomorphism of binary operations is transitive.
6. Find necessary and sufficient conditions for binary operations to be isomorphic to Euclidean addition.
7. Find necessary and sufficient conditions for binary operations to be isomorphic to Einstein addition.
8. Find a binary operation in \(\mathbb{R}^n\) isomorphic to every binary operation isomorphic to Einstein addition in the unit ball.
9. Find necessary and sufficient conditions for two operations having the same function \(m_0\) to be isomorphic.
10. Describe all binary operations isomorphic to Einstein addition and having the same function \(m_0\).
11. Find necessary and sufficient conditions for two operations having the same function \(m_1\) to be isomorphic.

The organization of the paper is as follows. Following the introduction, in Section 2, we prove that all the metric tensors of smooth binary operations invariant with respect to unitary transformations have the canonical form (8). In Section 3, we find the necessary and sufficient conditions on functions \(m_0, m_1\) to generate metric tensors of binary operations. We give the definition of isomorphism of binary operations and necessary and sufficient conditions for binary operations to be isomorphic in terms of the functions \(m_0, m_1\) in Section 4. We also show how to construct binary operations isomorphic to a given binary operation. In Section 5, we show that every algebraic isomorphism between binary operations gives rise to an isomorphism between corresponding gyrogroups. The necessary and sufficient conditions for binary operations to be isomorphic to Euclidean addition are presented in Section 6. The same problems for Einstein addition are solved in Section 7. In this section, a set of binary operations in \(\mathbb{R}^n\) parametrized by positive numbers and isomorphic to Einstein addition are presented. In Section 8 we describe all the binary operations isomorphic to Einstein addition and having the same function \(m_0\) in the representation of their canonical metric tensors. The necessary and sufficient conditions for binary operations, having the same function \(m_0\) in the representation of their canonical metric tensors, to be isomorphic are given in Section 9. In Section 10, we prove that every two binary operations, having the same function \(m_1\) in the representation of their canonical metric tensors, are isomorphic if and only if they are related by the standard scalar multiplication in a corresponding gyrovector space.

2. Binary Operations and Metric Tensors Invariant with Respect to Unitary Transformations

Let \(B_q\) be the open ball in the Euclidean space \(\mathbb{R}^n\) with radius \(q \in (0, \infty)\):

\[
B_q = \{ x \in \mathbb{R}^n : \| x \| < q \}.
\]  

(1)

The set \(B_\infty\) is identified with \(\mathbb{R}^n\).

We consider smooth binary operations \(f : B_q \times B_q \to B_q\) invariant with respect to unitary transformations. More precisely, we make the following assumption.
Assumption 1. The function $f$ is differentiable, and

$$f(Ua, Ub) = Uf(a, b).$$

for all vectors $a, b \in \mathbb{B}_q$ and all $n \times n$ unitary matrices $U$.

For every vector $x \in \mathbb{B}_q$ we consider the matrix

$$g(x) = \frac{\partial f(-x, y)}{\partial y} \bigg|_{y=x},$$

and define the Hermitian positive semi-definite matrix function

$$G(x) = g(x)^\top g(x).$$

Lemma 1. Under Assumption 1 the matrix function $G$ satisfies the following condition: for any vector $x \in \mathbb{B}_q$ and a unitary $n \times n$ matrix $U$ we have

$$G(Ux) = UG(x)U^\top.$$

Proof. We have

$$g(Ux) = \frac{\partial f(-Ux, y)}{\partial y} \bigg|_{y=Ux} = \frac{\partial f(-x, U^{-1}y)}{\partial y} \bigg|_{y=Ux} = g(x)U^{-1} = g(x)U^\top.$$

Therefore

$$G(Ux) = (g(Ux))^\top g(Ux) = Ug(x)^\top g(x)U^\top = UG(x)U^\top.$$

We consider $G$ as a metric tensor in $\mathbb{B}_q$.

Definition 1. The metric tensor $G$ in (4) is said to be associated with the binary operation $f$.

The quadratic form

$$F(x, y) = y^\top G(x)y$$

is called the first fundamental form. For every curve $x: [0, 1] \to \mathbb{B}_q$ the length of $x$ in the space $\mathbb{B}_q$ with the metric tensor $G$ is equal to

$$l(x) = \int_0^1 \sqrt{F(x(t), \dot{x}(t))} dt.$$

We are looking for a general form of symmetric matrix functions $G$ satisfying (5).

Theorem 1. Let condition (5) be satisfied for all $x \in \mathbb{B}_q$ and all unitary matrices $U$. Then there exist functions $m_0, m_1: [0, q^2) \to [0, \infty)$ such that for all non zero $x \in \mathbb{B}_q$ we have

$$G(x) = m_0(\|x\|^2) \left[ I - \frac{xx^\top}{\|x\|^2} \right] + m_1(\|x\|^2) \frac{xx^\top}{\|x\|^2}.$$

Proof. For an arbitrary number $\alpha \in (0, q)$ consider a vector $e = \alpha(1, 0, \ldots, 0)^\top$. For an arbitrary unitary $(n-1) \times (n-1)$ matrix $\hat{U}$ consider the unitary matrix

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \hat{U} \end{pmatrix}.$$
Notice that $Ue = e$, so that Equation (5) implies

$$G(e) = UG(e)U^\top.$$  

We split up the matrix $G(e)$ in a way similar to $U$, obtaining the block matrix

$$G(e) = \begin{pmatrix} a & b^\top \\ b & C \end{pmatrix},$$  

where $a \in \mathbb{R}$, $b \in \mathbb{R}^n$, and $C$ is a symmetric $(n - 1) \times (n - 1)$ matrix. Then

$$G(e) = \begin{pmatrix} a & b^\top \tilde{U}^\top \\ \tilde{U}b & \tilde{U}C\tilde{U}^\top \end{pmatrix}.$$  

Following (5) we have

$$\tilde{U}b = b, \quad \tilde{U}C\tilde{U}^\top = C$$  

for any unitary $(n - 1) \times (n - 1)$ matrix $\tilde{U}$. This means that $b = 0$, and there exists a number $d$ such that $C = dI$. Thus,

$$G(e) = \begin{pmatrix} a & 0 \\ 0 & dI \end{pmatrix}.$$  

For an arbitrary vector $x$ such that $\|x\| = \alpha$ we consider a unitary matrix $U_x$ such that $U_x e = x$. Then, this matrix can be split as follows:

$$U_x = \begin{pmatrix} x/\|x\| & U_x \end{pmatrix},$$

where $U_x$ is an $n \times (n - 1)$ matrix. Since the matrix $U$ is unitary, we have

$$I = U_x U_x^\top = \frac{xx^\top}{\|x\|^2} + \tilde{U}_x \tilde{U}_x^\top.$$  

Equation (5) determines the value of $G(x)$:

$$G(x) = G(U_x e) = U_x G(e) U_x^\top = a \frac{xx^\top}{\|x\|^2} + d \tilde{U}_x \tilde{U}_x^\top = a \frac{xx^\top}{\|x\|^2} + d \left[ I - \frac{xx^\top}{\|x\|^2} \right].$$  

If we set

$$m_0(\alpha^2) = d, \quad m_1(\alpha^2) = a,$$  

then Equation (8) holds. □

**Definition 2.** The representation (8) is called the canonical representation of the metric tensor $G$. We say that the pair of functions $(m_0, m_1)$ parametrizes $G$.

### 3. Binary Operations Invariant with Respect to Unitary Transformations

In this section, we define binary operations with given smooth metric tensors $G$ satisfying condition (5), and conditions on the function $m_1$ necessary and sufficient for the existence of such operations.

In (8) the matrix function $G(x)$ is a metric tensor. Therefore the functions $m_0, m_1$ should be non negative. We impose some smoothness conditions: the matrix function $G$ is differentiable. In terms of the functions $m_0, m_1$ it means that the functions $m_0, m_1$ are differentiable, and $m_0(0) = m_1(0)$.  

If we multiply $G$ by a positive number, then all properties of the object remain the same. Therefore, we further impose the following assumption.

**Assumption 2.** The functions $m_0, m_1$ in (5) are differentiable, $m_0(s) > 0$, $m_1(s) > 0$ for all $s \in [0, q)$, and $m_0(0) = m_1(0) = 1$.

A binary operation $\oplus$ in $\mathbb{B}_q$ is defined as follows [37]. Consider a manifold with metric tensor $G$. Denote by $exp_a(v)$ the exponential mapping from a point $a$ at vector $v$, denote by $log$ the logarithmic mapping: $log(v) = w$ such that $exp_0(w) = v$, and $P_{0 \to a}(v)$ is the parallel transport of a vector $v$ from zero to $a$. Then

$$a \oplus b = exp_a(P_{0 \to a}(log(b))).$$

In [37] it is proved that if the metric tensor $G$ is given by (3) and (4), then the operations $\oplus$ and $f$ coincide,

$$f(a, b) = a \oplus b \quad (9)$$

for all $a, b \in \mathbb{B}_q$.

Further we need a procedure for calculating $a \oplus b$ for various functions $m_0$ [37].

If $b = 0$, then $a \oplus b = a$. If $a = 0$, the $a \oplus b = b$. If $a, b \in \mathbb{B}_q \setminus \{0\}$, then we obtain $a \oplus b$ by the following three steps [37]:

1. **Calculate a vector**
   $$X_0 = \frac{b}{\|b\|} \int_0^{\|b\|} \sqrt{m_1(s^2)} ds.$$

2. **Calculate a vector**
   $$X_1 = \frac{1}{\sqrt{m_0(||a||^2)}} \left[ I - \frac{aa^\top}{\|a\|^2} \right] X_0 + \frac{1}{\sqrt{m_1(||a||^2)}} \frac{aa^\top}{\|a\|^2} X_0.$$

3. **Solve the following initial value problem:**
   $$\begin{align*}
   \ddot{x} + & \frac{m_0'(u)}{m_0(u)} \left|_{u = ||x||^2} \right. 2(x^\top \dot{x}) \dot{x} + \left[ \frac{m_1'(u)}{m_1(u)} - \frac{2m_0'(u)}{m_0(u)} \right] \left|_{u = ||x||^2} \right. \frac{(x^\top \dot{x})^2}{\|x\|^2} \\
   + & \frac{m_1(u) - m_0(u) - um_0'(u)}{um_1(u)} \left|_{u = ||x||^2} \right. \left[ \|x\|^2 - \frac{(x^\top \dot{x})^2}{\|x\|^2} \right] x = 0, \\
   x(0) &= a, \quad \dot{x}(0) = X_1.
   \end{align*} \quad (10)$$

The value $a \oplus b$ is equal to $x(1)$,

$$a \oplus b = x(1).$$

The value of $a \oplus b$ is well-defined if and only if the vector $x(1)$ belongs to the ball $\mathbb{B}_q$, that is, a solution of the initial value problem (10) exists on $[0, 1]$, and $\|x(t)\| < q$ for all $t \in [0, 1]$.

**Theorem 2.** The value of $a \oplus b$ is well-defined for all $a, b \in \mathbb{B}_q$ if and only if

$$\int_0^q \sqrt{m_1(s^2)} ds = \infty. \quad (11)$$

**Proof.** Assume the right hand side of (11) is finite. Denote it by $A$. Find a number $r \in (0, q)$ such that

$$\int_0^r \sqrt{m_1(s^2)} ds > \frac{A}{2}. \quad (12)$$
Pick a vector \( a \) such that \( \|a\| = r \). Let us try to calculate \( a \oplus a \). The first two steps of the procedure give
\[
X_0 = \frac{a}{\|a\|} \int_0^{\|a\|} \sqrt{m_1(s^2)} ds, \quad X_1 = \frac{1}{\sqrt{m_1(\|a\|^2)}} X_0.
\]

In step 3, the values of \( x(0) \) and \( \dot{x}(0) \) are parallel. Therefore, a solution \( x(\cdot) \) has the form \( x(t) = a\varphi(t) \), where \( \varphi \) is a scalar function. In this case
\[
\frac{(x^\top x)^2}{\|x\|^2} = \|a\|^2 (\varphi')^2,
\]
and Equation (10) take the form
\[
\dot{\varphi} + \frac{m_1'}{m_1} \|a\|^2 \varphi^2 \varphi = 0, \quad \varphi(0) = 1,
\]
(13)

The differential equation in (13) is separable. Integrating of (13) yields
\[
\varphi = \frac{C_1}{\sqrt{m_1(\|a\|^2)\varphi^2}}, \quad C_1 = \frac{1}{\|a\|} \int_0^{\|a\|} \sqrt{m_1(s^2)} ds.
\]

Therefore
\[
\int_0^{\|a\|} \sqrt{m_1(s^2)} ds = C_1,
\]
and
\[
\int_0^{\|a\|} \varphi(0) \sqrt{m_1(s^2)} ds = \int_0^{\|a\|} \sqrt{m_1(s^2)} ds.
\]

Since \( \|x(1)\| = \|a\| \varphi(1) \) and \( \varphi(0) = 1 \), we have
\[
\int_0^{\|x(1)\|} \sqrt{m_1(s^2)} ds = \int_0^{\|a\|} \sqrt{m_1(s^2)} ds,
\]
and
\[
\int_0^{\|x(1)\|} \sqrt{m_1(s^2)} ds = 2 \int_0^{\|a\|} \sqrt{m_1(s^2)} ds > A.
\]

Inequality (12) implies that a solution \( x(\cdot) \) of the initial value problem (10) on the interval \([0,1]\) such that \( \|x(t)\| < q \) for all \( t \in [0,1] \) does not exist. Hence \( a \oplus a \) is not defined.

Conversely, assume that equality (11) holds. Consider arbitrary vectors \( a \in \mathbb{B}_q, X_1 \in \mathbb{R}^n \) and a solution \( x \) of the initial value problem (10). We need to show that \( x(t) \in \mathbb{B}_q \) for all \( t \in [0,1] \). Since \( x(\cdot) \) is a geodesic in \( \mathbb{B}_q \) with metric tensor \( G \), the function \( x(t)^\top G(x(t)) \dot{x}(t) \) is constant for \( t \in [0,1] \). The constant is equal to \( X_1^\top G(a) X_1 \). At the same time,
\[
x(t)^\top G(x(t)) \dot{x}(t) \geq \|x(t)\|^2 m_1(\|x(t)\|^2) \geq \left( \frac{d\|x(t)\|}{dt} \right)^2 m_1(\|x(t)\|^2).
\]

Therefore, for all \( t \in [0,1] \) we have
\[
t \sqrt{X_1^\top G(a) X_1} \geq \int_0^{\|x(t)\|} \sqrt{m_1(s^2)} ds.
\]

Owing to (11) we have \( \|x(t)\| < q \) for all \( t \in [0,1] \). \( \square \)
Then for every unitary \( n \times (1–3) \) is generated by \( G \), or generated by the pair \( x \) where the infimum is taken over the set of smooth curves such that

\[
\inf \left\{ \int_0^q \sqrt{m_1(s^2)} ds \right\} = \infty.
\]

We now define a scalar multiplication \( t \otimes a \) in \( B_q \) associated with the operation \( \oplus \) [37] such that

\[
(t_1 \otimes a) \oplus (t_2 \otimes a) = (t_1 + t_2) \otimes a, \quad (t_1 \otimes (t_2 \otimes a)) = (t_1 t_2) \otimes a
\]

(14)

for all \( t_1, t_2 \in \mathbb{R} \) and \( a \in B_q \). Let \( h \) be the function \([0, q) \rightarrow [0, \infty), \) given by

\[
h(p) = \int_0^p \sqrt{m_1(s^2)} ds.
\]

Owing to Assumptions 2 and 3, the function \( h \) is an increasing bijection \([0, q) \rightarrow [0, \infty)\). Therefore it has an inverse, which is also an increasing bijection, and which is denoted by \( h^{-1} \). The multiplication by a number is given as follows: for all \( t \in \mathbb{R}^2 \) we have \( t \otimes a = 0 \) if \( a = 0 \), and for non zero \( a \in B_q \)

\[
t \otimes a = \frac{a}{\|a\|} h^{-1}(t h(\|a\|)).
\]

(15)

It is straightforward that this operation satisfies conditions (14) [37]. Notice that \( \|t \otimes a\| \rightarrow q \) as \( |t| \rightarrow \infty \) for every \( a \in B_q \setminus \{0\} \) owing to Assumption 3.

We denote by \( \text{dist}(a, b) \) the distance between points \( a \) and \( b \) in \( B_q \) with a canonical metric tensor \( G \). That is,

\[
\text{dist}(a, b) = \inf \left\{ \int_0^1 \sqrt{x(s)^T G(x(s)) x(s)} ds \right\},
\]

where the infimum is taken over the set of smooth curves \( x \) such that \( x(0) = a \) and \( x(1) = b \).

The relation between the distance function and the function \( h \) is given as follows [37].

**Theorem 3.** For all \( a, b \in B_q \)

\[
\text{dist}(a, b) = h(\|(-a) \oplus b\|).
\]

In particular, for both Einstein addition and Möbius addition we have [37]

\[
\text{dist}(a, b) = \text{atanh}(\|(-a) \oplus b\|).
\]

**Definition 3.** Let \( m_0, m_1: [0, 1) \rightarrow (0, \infty) \) be functions that satisfy Assumptions 2 and 3, and let \( G \) be the canonical metric tensor (8) parametrized by \( (m_0, m_1) \). Then we say that the binary operation \( \oplus \), defined in steps 1–3, is generated by \( G \), or generated by the pair \( (m_0, m_1) \).

**Lemma 2.** Let \( a \oplus b \) be an operation generated by a pair of functions \( (m_0, m_1) \) satisfying Assumptions 2 and 3. Then for every unitary \( n \times n \) matrix \( U \) and all \( a, b \in B_q \) we have

\[
(Ua) \oplus (Ub) = U(a \oplus b).
\]

(16)

**Proof.** We apply the steps 1–3 of the procedure defining \( a \oplus b \). Assume \( X_0, X_1, \) and \( x(\cdot) \) are the vectors and the function that are calculated on the steps 1–3 for \( a \oplus b \). Assume \( X_{0,U}, X_{1,U} \) and \( x_U(\cdot) \) are the corresponding values for the sum \((Ua) \oplus (Ub)\). Then, clearly,

\[
X_{0,U} = UX_0, \quad X_{1,U} = UX_1, \quad x_U = Ux.
\]

(17)
Therefore,
\[(Ua) \oplus (Ub) = x_U(1) = Ux(1) = U(a \oplus b). \quad (18)\]
\[\blacklozenge\]

4. Isomorphic Operations

In this section, we consider necessary and sufficient conditions under which binary operations in the balls \(B_q, B_p\) with \(q, p \in (0, \infty)\) are isomorphic.

Assume \(q, p \in (0, \infty)\). Let \(G, G\) be canonical metric tensors in \(B_q, B_p\) generated by the pairs of functions \((m_0, m_1)\) and \((\tilde{m}_0, \tilde{m}_1)\) respectively. Assume the pairs \((m_0, m_1)\) and \((\tilde{m}_0, \tilde{m}_1)\) satisfy Assumptions 2 and 3. Denote by \(\oplus, \hat{\oplus}\) the binary operations generated by the pairs of functions \((m_0, m_1)\) and \((\tilde{m}_0, \tilde{m}_1)\) respectively.

**Definition 4.** Operations \(\oplus\) in \(B_q\) and \(\hat{\oplus}\) in \(B_p\) are isomorphic if there exists a smooth bijection \(\varphi : B_q \to B_p\) such that for all \(a, b \in B_q\)
\[\varphi(a \oplus b) = \varphi(a) \hat{\oplus} \varphi(b) \quad (19)\]
and
\[\varphi(Ux) = U\varphi(x) \quad (20)\]
for all unitary matrices \(U\) and all vectors \(x \in B_q\).

The function \(\varphi\) is said to be the bijection of the isomorphism between the operations \(\oplus, \hat{\oplus}\).

**Remark 1.** Assume \(\varphi\) is an isomorphism of operations \(\oplus\) and \(\hat{\oplus}\). Then the following conditions on \(\varphi\) hold.

1. \(\varphi(-x) = -\varphi(x)\) for all \(x \in B_q\).
2. \(\varphi(0) = 0\).
3. The function \(-\varphi\) is also an isomorphism between the operations \(\oplus\) and \(\hat{\oplus}\).

**Proof.** Condition (20) with \(U = -I\) implies
\[\varphi(-x) = -\varphi(x). \quad (21)\]

Therefore, the functions \(\varphi\) satisfying (20) are odd. Besides, \(\varphi(0) = U\varphi(0)\) for all unitary matrices \(U\). Therefore, \(\varphi(0) = 0\). The last property follows from the fact that \((-a) \oplus (-b) = -(a \oplus b)\), for all \(a, b \in B_q,\) and \((-a) \hat{\oplus} (-b) = -(a \hat{\oplus} b)\) for all \(a, b \in B_p\). \(\Box\)

**Remark 2.** The relation of isomorphisms is symmetric: if operations \(\oplus\) and \(\hat{\oplus}\) are isomorphic with a bijection \(\varphi\), then operations \(\hat{\oplus}\) and \(\oplus\) are isomorphic with the bijection \(\varphi^{-1}\).

**Proof.** Since \(\varphi\) is a bijection, there exists an inverse mapping \(\varphi^{-1} : B_p \to B_q\). Equations (3) and (19) are equivalent to the equations:
\[\varphi^{-1}(c \hat{\oplus} d) = \varphi^{-1}(c) \oplus \varphi^{-1}(d),\]
\[\varphi^{-1}(Uy) = U\varphi^{-1}(y) \quad (22)\]
for all \(c, d, y \in B_p\) and unitary matrices \(U\). \(\Box\)

The following theorem asserts that the relation of isomorphism is transitive.

**Theorem 4.** Assume operations \(\oplus_1\) in \(B_{q_1}\), \(\oplus_2\) in \(B_{q_2}\) are isomorphic with an isomorphism \(\varphi_1 : B_{q_1} \to B_{q_2}\), and operations \(\oplus_2\) and \(\oplus_3\) in \(B_{q_2}\) are isomorphic with an isomorphism \(\varphi_2 : B_{q_2} \to B_{q_3}\). Then operations \(\oplus_1\) and \(\oplus_3\) are isomorphic with an isomorphism \(\varphi_2 \circ \varphi_1 : B_{q_1} \to B_{q_3}\).
Proof. For all $a, b \in \mathbb{B}_q$ we have 
\[ \varphi_2(\varphi_1(a \oplus_1 b)) = \varphi_2(\varphi_1(a) \oplus_2 \varphi_1(b)) = \varphi_2(\varphi_1(a) \oplus_3 \varphi_2(\varphi_1(b)), \]
and for all unitary matrices $U$ and all $x \in \mathbb{B}_q$ we have 
\[ \varphi_2(\varphi_1(Ux)) = U\varphi_2(\varphi_1(x)). \]

\[\square\]

Owing to property (20) the isomorphisms have the following representation.

**Lemma 3.** Assume $\varphi$ is a bijection of an isomorphism between an operation in $\mathbb{B}_q$ generated by a pair of functions $(m_0, m_1)$ satisfying Assumptions 2 and 3 and an operation in $\mathbb{B}_p$ generated by the pair of functions $(\tilde{m}_0, \tilde{m}_1)$ also satisfying Assumptions 2 and 3. Then there exists a smooth bijection $\tilde{\varphi}: [0, q) \to [0, p)$ and a number $s \in \{-1, 1\}$ such that for all $x \in \mathbb{B}_q$ we have 
\[ \varphi(x) = s\tilde{\varphi}(\|x\|) \frac{x}{\|x\|}. \]

**Proof.** Let $a \in (0, q)$ and a vector $e$ be the first unit vector: $e = (1, \ldots, 0)^\top$. Consider an arbitrary unitary $(n - 1) \times (n - 1)$ matrix $U_1$ and an $n \times n$ unitary matrix $U = \text{blockdiag}\{1, U_1\}$. Then $Ue = e$. Apply condition (20): 
\[ \varphi(ae) = \varphi(Uae) = U\varphi(ae). \]

If $\varphi(ae)$ is not parallel to $e$, then there exists a matrix $U_1$ such that Equation (24) is wrong. Therefore the vectors $ae$ and $\varphi(ae)$ are parallel: there exists a number $\lambda = \lambda(a) \in (0, p)$ such that $\varphi(ae) = s\lambda(a)e$, where $s = \text{sign}[e^\top \varphi(ae)]$. From (19) it follows that $\varphi(0) = 0$. Therefore $\varphi(\alpha e) \neq 0$. Due to continuity of the bijection $\varphi$ the number $s$ is the same for all $\alpha$. Set 
\[ \tilde{\varphi}(\alpha) = \lambda(a). \]

Then 
\[ \varphi(\alpha e) = s\tilde{\varphi}(\|ae\|) \frac{ae}{\|ae\|}. \]

For an arbitrary vector $x \in \mathbb{B}_q$ we find a unitary matrix $U$ such that 
\[ x = U\|x\|e. \]

Then 
\[ \varphi(x) = \varphi(U\|x\|e) = U\varphi(\|x\|e) = Us\tilde{\varphi}(\|x\|) \frac{\|x\|e}{\|x\|} = s\tilde{\varphi}(\|x\|) \frac{x}{\|x\|}. \]

The function $\tilde{\varphi}$ is smooth since the function $\varphi$ is smooth. \[\square\]

**Remark 3.** Assume operations $\oplus$ and $\oplus^2$ are isomorphic. Then there exists an isomorphism $\varphi$ between these operations that has the representation (23) with $s = 1$.

**Proof.** Assume $\varphi$ is an isomorphism, which satisfies condition (23) with $s = -1$. Owing to statement 2 of Remark 1 the function $-\varphi$ is also an isomorphism that satisfies condition (23) with $s = 1$. \[\square\]

Further, without loss of generality, we assume that in the representation (23) of isomorphisms $\varphi$ we have $s = 1$.

The following definition is useful.
**Definition 5.** We say that a function $\varphi : \mathbb{B}_q \rightarrow \mathbb{B}_p$ is determined by an increasing bijection $\tilde{\varphi} : [0, q) \rightarrow [0, p)$ if for all $x \in \mathbb{B}_q$
\[
\varphi(x) = \tilde{\varphi}(\|x\|) \frac{x}{\|x\|}. \tag{25}
\]

Assume $\varphi$ is a bijection of an isomorphism between $\oplus$ and $\otimes$. How can the functions $\tilde{m}_0, \tilde{m}_1$ be found in terms of $\varphi, m_0$ and $m_1$?

**Theorem 5.** Assume binary operations $\oplus$ in $\mathbb{B}_p$ and $\otimes$ in $\mathbb{B}_q$ with $p, q \in (0, \infty]$ are generated by the pairs of functions $(m_0, m_1)$ and $(\tilde{m}_0, \tilde{m}_1)$ respectively.

Then the operations $\oplus, \otimes$ are isomorphic if and only if there exists an increasing smooth bijection $\varphi : [0, q) \rightarrow [0, p)$ such that $\tilde{\varphi}'(0) > 0$, and for all $r \in (0, q)$
\[
\tilde{m}_0(r^2) = m_0(\tilde{\varphi}(r)^2) \left[ \frac{\varphi(r)}{\varphi(0)r} \right]^2, \tag{26}
\]
\[
\tilde{m}_1(r^2) = m_1(\tilde{\varphi}(r)^2) \left[ \frac{\varphi'(r)}{\varphi'(0)} \right]^2. \tag{27}
\]

If conditions (26) and (27) are satisfied, then the function $\varphi : \mathbb{B}_q \rightarrow \mathbb{B}_p$ determined by $\tilde{\varphi}$ is an isomorphism between $\oplus$ and $\otimes$.

**Proof.** First we prove the necessity. Assume the operations $\tilde{\oplus}$ and $\oplus$ are isomorphic, and $\varphi$ is the corresponding isomorphism satisfying condition (25). For every non zero vector $x \in \mathbb{B}_q$ consider an $n \times n$ matrix $g(x)$ such that
\[
(-x) \oplus (x + \Delta x) = g(x)\Delta x + o(\|\Delta x\|).
\]

Then
\[
G(x) = g(x)^\top g(x)
\]
is the metric tensor generated by the pair $(m_0, m_1)$. A similar relation concerns the operation $\otimes$: there exists an $n \times n$ matrix $\tilde{g}(x)$ such that
\[
(-x) \tilde{\otimes} (x + \Delta x) = \tilde{g}(x)\Delta x + o(\|\Delta x\|),
\]
and
\[
\tilde{G}(x) = \tilde{g}(x)^\top \tilde{g}(x)
\]
is the metric tensor generated by the pair $(\tilde{m}_0, \tilde{m}_1)$. Then
\[
\varphi([(-x) \tilde{\otimes} (x + \Delta x)]) = \varphi(\tilde{g}(x)\Delta x + o(\|\Delta x\|)) = \varphi(\tilde{g}(x)\Delta x + o(\|\Delta x\|)), \tag{28}
\]
and
\[
\varphi(-x) \oplus \varphi(x + \Delta x) = (-\varphi(x)) \oplus (\varphi(x) + \varphi'(x)\Delta x + o(\|\Delta x\|)) = g(\varphi(x))\varphi'(x)\Delta x + o(\|\Delta x\|). \tag{29}
\]

By means of (19), (28) and (29) we have
\[
\varphi'(0)\tilde{g}(x) = g(\varphi(x))\varphi'(x),
\]
and therefore
\[ \phi'(0)^2 \hat{G}(x) = \phi'(x) G(\phi(x)) \phi'(x)^\top. \]  
(30)

Since \( G(\phi(x)) > 0 \), and \( \phi'(x) \neq 0 \), we get
\[ \phi'(0) \neq 0. \]

According to (8) and (25)
\[ \phi'(x) = \left[ \phi(\|x\|) \frac{x}{\|x\|} \right]' = \phi'(\|x\|) \frac{xx^\top}{\|x\|^2} + \phi'(\|x\|) \left[ I - \frac{xx^\top}{\|x\|^2} \right], \]

\[ G(\phi(x)) = m_0(\phi(\|x\|)^2) \left[ I - \frac{xx^\top}{\|x\|^2} \right] + m_1(\phi(\|x\|)^2) \frac{xx^\top}{\|x\|^2}. \]

Hence,
\[ \phi'(x) G(\phi(x)) \phi'(x)^\top = m_0(\phi(\|x\|)^2) \frac{\tilde{\phi}(\|x\|)^2}{\phi'(0)^2} \phi'(\|x\|)^2 \frac{xx^\top}{\|x\|^2}. \]
\[ = m_0(\phi(\|x\|)^2) \left[ I - \frac{xx^\top}{\|x\|^2} \right] + m_1(\phi(\|x\|)^2) \frac{xx^\top}{\|x\|^2}. \]

Since \( \hat{G}(x) \) is the metric tensor generated by the pair of functions \((m_0, m_1)\), we have
\[ \hat{G}(x) = m_0(\phi(\|x\|)^2) \left[ I - \frac{xx^\top}{\|x\|^2} \right] + m_1(\phi(\|x\|)^2) \frac{xx^\top}{\|x\|^2}. \]
(32)

Equations (30) and (31) imply
\[ \hat{G}(x) = \frac{m_0(\phi(\|x\|)^2) \phi'(|x|)^2}{\phi'(0)^2} \left[ I - \frac{xx^\top}{\|x\|^2} \right] + \frac{m_1(\phi(\|x\|)^2) \phi'(|x|)^2}{\phi'(0)^2} \frac{xx^\top}{\|x\|^2}. \]
(33)

Comparing Equations (32) and (33), we get
\[ m_0(|x|^2) = m_0(\phi(\|x\|)^2) \left[ \frac{\phi(|x|)}{\phi'(0)|x|} \right]^2, \]
\[ m_1(|x|^2) = m_1(\phi(\|x\|)^2) \left[ \frac{\phi'(|x|)}{\phi'(0)} \right]^2. \]

The necessity is thus proved.

We now prove the sufficiency. Assume \( \tilde{\phi} : [0, q) \to [0, p) \) is a smooth bijection, \( \phi'(0) > 0 \), and conditions (26) and (27) hold. Define a function \( \phi \) by Equation (25). Then \( \phi \) is a smooth bijection \( \mathbb{B}_q \to \mathbb{B}_p \). Denote by \( \oplus \) the binary operation in \( \mathbb{B}_p \) given by
\[ a \oplus b = \phi^{-1}(\phi(a) \oplus \phi(b)). \]
(34)

for all \( a, b \in \mathbb{B}_q \). The theorem will be proved if we show that \( \oplus = \oplus \). To this end it is sufficient from [37] to prove that the metric tensors associated with these two operations are equal. We have
\[ \phi(-x) \oplus \phi(x + \Delta x) \]
\[ = \left[ \phi(|x + \Delta x|) \right] x + \phi'(|x + \Delta x|) \Delta x \]
\[ = \left[ \phi(|x|) \right] \left( I - \frac{xx^\top}{\|x\|^2} \right) + \phi'(|x|) \frac{xx^\top}{\|x\|^2} \Delta x + o(|x|). \]
(35)
Therefore

\[ (-x) \hat{\oplus} (x + \Delta x) = \]

\[ = \varphi^{-1} \left[ g(\varphi(x)) (\varphi(x + \Delta x) - \varphi(x)) + o(\|\Delta x\|) \right] \]

\[ = \frac{1}{\varphi'(0)} g(\varphi(x)) \left[ \frac{\varphi(\|x\|)}{\|x\|} \left( I - \frac{xx^\top}{\|x\|^2} \right) + \varphi'(\|x\|) \frac{xx^\top}{\|x\|^2} \right] \Delta x + o(\|\Delta x\|). \]

Taking into account that the metric tensor \( G \) associated with the operation \( \oplus \) is equal to

\[ G(x) = g(x)^\top g(x) = m_0(\|x\|^2) \left[ I - \frac{xx^\top}{\|x\|^2} \right] + m_1(\|x\|^2) \frac{xx^\top}{\|x\|^2} , \]

we can calculate the metric tensor \( \hat{G} \) associated with the operation \( \hat{\varphi} \):

\[ \hat{G}(x) = h(x)^\top G(\varphi(x)) h(x), \]

where

\[ h(x) = \left[ \frac{\varphi'(\|x\|)}{\varphi'(0)\|x\|} \left( I - \frac{xx^\top}{\|x\|^2} \right) + \frac{\varphi'(\|x\|)}{\varphi'(0)} \frac{xx^\top}{\|x\|^2} \right] . \]

It is straightforward to check that the metric tensor \( \hat{G} \) has the form (8) parametrized by the functions

\[ m_0(r^2) = m_0(\varphi(r)^2) \left[ \frac{\varphi'(r)}{\varphi'(0)r} \right] ^2, \]

\[ m_1(r^2) = m_1(\varphi(r)^2) \left[ \frac{\varphi'(r)}{\varphi'(0)} \right] ^2. \]

These functions coincide with the functions \( \bar{m}_0 \) and \( \bar{m}_1 \) respectively. Hence, the metric tensors associated with the operations \( \oplus \) and \( \hat{\oplus} \) coincide. According to the uniqueness of binary operations associated with the same metric tensor [37], we have \( \oplus = \hat{\oplus} \).

Following definition (34) of the operation \( \hat{\oplus} \), this operation is isomorphic to the operation \( \oplus \) with the isomorphism \( \varphi \). \( \square \)

Assume \( \varphi_t \) is a family of smooth increasing bijections \([0, q] \to [0, p] \) with \( t \in (0, \infty) \) and such that \( \varphi'(0) = t \). Assume there exists a limit in \( C^1[0, q] \): \( \lim_{t \to 0} \varphi_t \), which we denote by \( \varphi_0(s) \). Assume the function \( \varphi_0 \) is a smooth increasing bijection \([0, q] \to [0, p] \). Assume an operation \( \oplus \) is generated by a pair \((m_0, m_1)\). For all \( t \geq 0 \) consider the functions

\[ \varphi_t(x) = \varphi_t(\|x\|) \frac{x}{\|x\|} \]

and operations \( \oplus_t \) defined as follows: for all \( a, b \in B_q \)

\[ a \oplus_t b = \varphi_t^{-1}(\varphi_t(a) \oplus \varphi_t(b)). \]

Then the operations \( \oplus_t \) are isomorphic to the operation \( \oplus \) for all \( t > 0 \). Further, we show that the limiting operation \( \oplus_0 \) may not be isomorphic to the operation \( \oplus \).

5. Gyrogroups

In this section we consider a simple way to get gyrogroups via bijections. We use the following definitions [34].
Definition 6. A set $S$ with a binary operation $\oplus$ is called a groupoid. A groupoid $(S, \oplus)$ is called a gyrogroup if its binary operation satisfies the following axioms.

1. There is at least one element, $0$, in $S$ such that

$$0 \oplus a = a$$

for all $a \in S$.

2. There is an element $0 \in S$ satisfying (36) such that for every $a \in S$ there is an element $-a \in S$ (called a left inverse of $a$) such that

$$(-a) \oplus a = 0.$$  (37)

3. For every $a,b \in S$ there is an automorphism $\text{gyr}[a,b]: S \rightarrow S$ of the groupoid $S$ such that for every $c \in S$ we have

$$a \oplus (b \oplus c) = (a \oplus b) \oplus (\text{gyr}[a,b]c).$$  (38)

4. The operator $\text{gyr}: S \times S \rightarrow \text{Aut}(S, \oplus)$ possesses the following property:

$$\text{gyr}[a \oplus b, b] = \text{gyr}[a, b]$$  (39)

for all $a,b \in S$.

Notice [30] that the left identity $0$ is also a right identity, and the left inverse of $a$ is also a right inverse of $a$: $a \oplus 0 = a, a \oplus (-a) = 0$ for all $a \in S$.

Theorem 6. Assume $(S_1, \oplus_1)$ is a gyrogroup, $S_2$ is a set of elements, and $\varphi: S_2 \rightarrow S_1$ is a bijection. Then for the binary operation given by

$$a \oplus_2 b = \varphi^{-1}(\varphi(a) \oplus_1 \varphi(b))$$

the groupoid $(S_2, \oplus_2)$ is a gyrogroup.

Proof. We have to show that axioms 1–4 in Definition 6 are satisfied.

1. Pick an element $0 \in S_1$ satisfying condition 2. For every $a \in S_2$ we have

$$a \oplus_2 \varphi^{-1}(0) = \varphi^{-1}(\varphi(a) \oplus_1 0) = a.$$  (41)

Hence, $\varphi^{-1}(0)$ is an element satisfying axiom 1 for the groupoid $(S_2, \oplus_2)$.

2. For every $a \in S_2$ we have

$$a \oplus_2 \varphi^{-1}(-\varphi(a)) = \varphi^{-1}(\varphi(a) \oplus_1 (-\varphi(a))) = \varphi^{-1}(0).$$  (42)

Therefore, axiom 2 is satisfied for the groupoid $(S_2, \oplus_2)$.

3. For every $x,y \in S_1$ let $\text{gyr}_1[x,y]$ be the automorphism of $S_1$ from the property 3. For every $a,b,c \in S_2$ set

$$\text{gyr}_2[a,b]c = \varphi^{-1}(\text{gyr}_1[\varphi(a), \varphi(b)]\varphi(c)).$$  (43)
Theorem 7. Assume the gyroproup \( S \). Then for the binary operation \( \oplus \):

\[
\text{Definition 7.} \quad \text{A gyroproup} \quad (S, \oplus)_\text{gyrocommutative} \quad \text{if for all} \quad a, b \in S
\]

\[
a \oplus b = \text{gyr}[a, b](b \oplus a).
\]

Theorem 7. Assume \((S_1, \oplus_1)\) is a gyrocommutative gyrogroup, \(S_2\) is a set of elements, and \(\varphi: S_2 \to S_1\) is a bijection. Then for the binary operation \(\oplus_2\), given by

\[
a \oplus_2 b = \varphi^{-1}(\varphi(a) \oplus_1 \varphi(b)),
\]

the gyrogroup \((S_2, \oplus_2)\) is gyrocommutative.

Proof. For all \(a, b \in S_2\) we have

\[
a \oplus_2 b &= \varphi^{-1}(\varphi(a) \oplus_1 \varphi(b)) \\
&= \varphi^{-1}(\text{gyr}_1[\varphi(a), \varphi(b)](\varphi(b) \oplus_1 \varphi(a))) \\
&= \varphi^{-1}(\text{gyr}_1[\varphi(a), \varphi(b)](b \oplus_2 a)) \\
&= \text{gyr}_2[a, b](b \oplus_2 a).
\]

Hence, axiom (47) is satisfied.
We can apply these results to balls $B_q$ with $q \in (0, \infty]$. Notice that for Euclidean addition (and hence for all additions isomorphic to Euclidean addition) the operation $gyr[a, b]$ is equal to identity. For Einstein addition (and hence for all additions isomorphic to Einstein addition) the operation $gyr[a, b]$ is a multiplication by a unitary matrix.

Here we encounter the following open problem. Does there exist a binary operation generated by a pair of functions $(m_0, m_1)$ satisfying Assumptions 2 and 3, which is not isomorphic to Einstein addition and not isomorphic to Euclidean addition?

6. Operations Isomorphic to Euclidean Addition

The metric tensor of Euclidean addition in $\mathbb{R}^n = \mathbb{B}_\infty$ is equal to the identity. This means that $m_0 \equiv 1, m_1 \equiv 1$ in the canonical representation (8) of the Euclidean metric tensor.

The following theorem gives necessary and sufficient conditions for a binary operation generated by a pair of functions $(m_0, m_1)$ to be isomorphic to Euclidean addition.

**Theorem 8.** Let $G$ be a canonical metric tensor in $B_q$ parametrized by a pair of functions $(m_0, m_1)$ satisfying Assumptions 2 and 3. Let $\oplus$ be a binary operation generated by $G$. Then the operation $\oplus$ is isomorphic to Euclidean addition if and only if for all $u \in [0, q^2)$ we have

$$m_1(u) = \frac{[(um_0(u))']^2}{m_0(u)}. \quad (50)$$

If an operation $\oplus$ is isomorphic to Euclidean addition, then there exists a positive number $t$ such that the bijection $\varphi: B_q \rightarrow \mathbb{R}^n$ of the isomorphism between $\oplus$ and Euclidean addition is given by

$$\varphi(x) = \sqrt{m_0(\|x\|^2)}tx. \quad (51)$$

**Proof.** According to Theorem 5 the operation $\oplus$ is isomorphic to Euclidean addition if and only if there exists a differentiable bijection $\tilde{\varphi}: [0, q) \rightarrow [0, \infty)$ such that $\tilde{\varphi}'(0) > 0$, and for all $r \in [0, q)$ we have

$$m_0(r^2) = \left[ \frac{\phi'(r)}{\tilde{\varphi}'(0)r} \right]^2 \quad (52)$$

and

$$m_1(r^2) = \left[ \frac{\phi'(r)}{\tilde{\varphi}'(0)} \right]^2. \quad (53)$$

Assume such a function $\phi$ exists, and let $t = \phi'(0)$. Solving Equation (52) for $\phi$ yields

$$\phi(r) = \sqrt{m_0(r^2)}.$$  

Therefore

$$\phi'(r) = t \frac{d}{dr}(\sqrt{r^2m_0(r^2)}) = t \frac{d}{dr}(r^2m_0(r^2)) \frac{1}{\sqrt{m_0(r^2)}}.$$  

Denote $u = r^2$. Then (53) implies

$$m_1(u) = \left[ \frac{1}{\sqrt{m_0(u)}} \frac{d}{du}(um_0(u)) \right]^2,$$

which coincides with (50).
Now assume that (50) holds. Then for all \( r \in [0, q) \) we have

\[
\sqrt{m_1(r^2)} = \frac{1}{\sqrt{m_0(r^2)}} \frac{d(r^2m_0(r^2))}{d(r^2)} = \frac{d(r\sqrt{m_0(r^2)})}{dr}.
\]

For an arbitrary positive number \( t \) define

\[
\tilde{\phi}(r) = tr\sqrt{m_0(r^2)}.
\]

Since \( m_0(0) = 1 \), we have \( t = \tilde{\phi}'(0) \). Therefore

\[
m_0(r^2) = \left[ \frac{\phi(r)}{\phi'(0)} \right]^2,
\]

\[
m_1(r^2) = \left[ \frac{\phi'(r)}{\phi'(0)} \right]^2,
\]

and Equations (52) and (53) are satisfied.

According to Theorem 5 the isomorphism \( \phi: \mathbb{B}_q \to \mathbb{R}^n \) is given by

\[
\phi(x) = \tilde{\phi}(\|x\|) \frac{x}{\|x\|} = \sqrt{m_0(\|x\|^2)}tx.
\]

\[\square\]

**Example 1.** Let \( \oplus \) be a binary operation with a canonical metric tensor \( G \) parametrized by a pair of functions \((m_0, m_1)\) satisfying Assumptions 2 and 3. Let \( \oplus \) be isomorphic to Euclidean addition in \( \mathbb{R}^n \). If

\[
m_0(u) = \frac{1}{1 - u'}, \tag{54}
\]

then, according to Theorem 8, we have

\[
m_1(u) = \frac{1}{(1 - u')^2}. \tag{55}
\]

If

\[
m_0(u) = \frac{1}{(1 - u')^2}, \tag{56}
\]

then, according to Theorem 8, we have

\[
m_1(u) = \frac{(1 + u')^2}{(1 - u')^4}. \tag{57}
\]

The pairs of functions \((m_0, m_1)\) in (54)–(57) appear [38] when we study the sets of cogyrolines for Einstein addition and Möbius addition.

7. Operations Isomorphic to Einstein Addition

The metric tensor of Einstein addition \( \oplus_E \) in \( \mathcal{B}_1 \) is canonical, having the form (8) with parameters

\[
m_{0,E}(u) = \frac{1}{1 - u'}, \quad m_{1,E}(u) = \frac{1}{(1 - u')^2}.
\]

Assume \( q \in (0, \infty] \) and a pair of functions \((m_0, m_1)\) satisfies Assumptions 2 and 3. Consider a canonical metric tensor \( G \) parametrized by this pair. Let \( \oplus \) be the binary operation generated by \((m_0, m_1)\). Then we raise the following question.
What are the conditions on a pair of functions \((m_0, m_1)\) necessary and sufficient for the operation \(\oplus\) to be isomorphic to Einstein addition \(\oplus_E\)?

**Theorem 9.** An operation \(\oplus\) generated by a pair of functions \((m_0, m_1)\) satisfying Assumptions 2 and 3 is isomorphic to Einstein addition if and only if \(\lim_{u \to q^2} um_0(u) = \infty\) and there exists a positive number \(C\) such that

\[
m_1(u) = \frac{[(um_0(u))^2]}{m_0(u)(1 + C um_0(u))}
\]

for all \(u \in [0, q^2]\).

If Equation (58) holds, then the corresponding isomorphism between \(\oplus\) and \(\oplus_E\) is given by a function \(\phi\) having representation (25) with

\[
\phi(r) = \frac{\sqrt{Cr^2m_0(r^2)}}{\sqrt{1 + Cr^2m_0(r^2)}}.
\]

**Proof.** First we prove the necessity. Assume the operation \(\oplus\) is isomorphic to Einstein addition. By Theorem 5 with \(m_0 = m_{0,E}, m_1 = m_{1,E}\), there exists a differentiable bijection \(\tilde{\phi}: [0, q) \to [0, 1)\) such that \(\tilde{\phi}'(0) > 0\) and

\[
m_0(r^2) = \frac{1}{1 - \tilde{\phi}(r)^2} \left[ \frac{\tilde{\phi}(r)}{\tilde{\phi}'(0)} \right]^2,
\]

\[
m_1(r^2) = \frac{1}{(1 - \tilde{\phi}(r)^2)^2} \left[ \frac{\tilde{\phi}'(r)}{\tilde{\phi}'(0)} \right]^2
\]

for all \(r \in [0, q)\). Since \(\tilde{\phi}\) is a bijection \([0, q) \to [0, 1)\), we have \(\lim_{r \to q} \tilde{\phi}(r) = 1\), and therefore

\[
\lim_{u \to q^2} um_0(u) = \lim_{u \to q^2} \frac{1}{1 - \tilde{\phi}(\sqrt{u})^2} \left[ \frac{\tilde{\phi}(\sqrt{u})}{\tilde{\phi}'(0)} \right]^2 = \infty.
\]

Let \(t = \tilde{\phi}'(0)\) and \(u = r^2\). Then

\[
m_0'(u) = \frac{\tilde{\phi}(\sqrt{u})\tilde{\phi}'(\sqrt{u})}{(1 - \tilde{\phi}(\sqrt{u})^2)^{3/2}} - \frac{\tilde{\phi}(\sqrt{u})^2}{(1 - \tilde{\phi}(\sqrt{u})^2)^{3/2}},
\]

\[
m_0(u) + um_0'(u) = \frac{\tilde{\phi}(\sqrt{u})\tilde{\phi}'(\sqrt{u})}{(1 - \tilde{\phi}(\sqrt{u})^2)^{3/2}} + \frac{\tilde{\phi}(\sqrt{u})^2}{(1 - \tilde{\phi}(\sqrt{u})^2)^{3/2}},
\]

\[
1 + t^2 um_0(u) = \frac{1}{1 - \tilde{\phi}(\sqrt{u})^2},
\]

and

\[
\frac{[(um_0(u))^2]}{m_0(u)(1 + t^2 um_0(u))} = \frac{\tilde{\phi}(\sqrt{u})^2}{(1 - \tilde{\phi}(\sqrt{u})^2)^2} = m_1(u)
\]

with \(C = t^2\). Equation (62) coincides with Equation (58). The necessity is proved.

Now we prove the sufficiency. According to Theorem 5, it suffices to show the existence of a bijection \(\hat{\phi}: [0, q) \to [0, 1)\) such that Equations (60) and (61) hold. Set \(t = \sqrt{C}\). Define

\[
\hat{\phi}(r) = \frac{tr\sqrt{m_0(r^2)}}{\sqrt{1 + t^2 r^2 m_0(r^2)}}.
\]
Owing to Assumption 2 we have \( m_0(0) = 1 \) and \( m_1(u) > 0 \) for all \( u \in [0, q^2) \). Then \( \bar{\phi}'(0) = t > 0 \) and (58) implies that the function \( um_0(u) \) is strictly increasing. Since \( \bar{\phi}(0) = 0 \) and \( \lim_{u \to q^2} um_0(u) = \infty \), the function \( \bar{\phi} \) is an increasing bijection \([0, q^2) \to [0, 1)\). Now we check Equation (60):

\[
\frac{1}{1 - \bar{\phi}(r)} \left[ \frac{\bar{\phi}(r)}{r\bar{\phi}'(0)} \right]^2 = m_0(r^2),
\]

and Equation (61):

\[
\frac{1}{(1 - \bar{\phi}(r))^2} \left[ \frac{\bar{\phi}'(r)}{\bar{\phi}'(0)} \right]^2 = \frac{1}{1 + t^2r^2 m_0(r^2)} \left[ \frac{d(r\sqrt{m_0(r^2)})}{dr} \right]^2
\]

\[
= \frac{1}{1 + t^2r^2 m_0(r^2)} \frac{1}{m_0(r^2)} \left[ \frac{d(u m_0(u))}{du} \right]_{u=r^2} = m_1(r^2).
\]

The sufficiency is proved. The isomorphism is given by (63).

Denote

\[ m_2(u) = um_0(u). \]

**Lemma 4.** Identity (58) holds if and only if for all \( r \in (0, q) \) we have

\[
\int_0^r \sqrt{m_1(s^2)} ds = \frac{1}{\sqrt{C}} \text{atanh} \left( \sqrt{\frac{C m_2(r^2)}{1 + C m_2(r^2)}} \right). \tag{64}
\]

**Proof.** We use the definition of the function \( m_2 \) to write Equation (58) in the following equivalent form:

\[
\sqrt{m_1(s^2)} = \frac{s m_2'(s)}{\sqrt{m_2(s^2)(1 + C m_2(s^2))}} \tag{65}
\]

for all \( s \in (0, q) \). Equation (64) follows from integration of Equation (65) over \([0, r]\) using the following two formulas,

\[
\frac{d}{ds} m_2(s^2) = 2sm_2'(s^2),
\]

and

\[
\int_0^y \frac{dx}{\sqrt{x(1+Cx)}} = \frac{1}{\sqrt{C}} \sqrt{\frac{Cy}{1+Cy}}.
\]

We recall that the function

\[ h(r) = \int_0^r \sqrt{m_1(s^2)} ds \]

is used in the definition of the operation \( \otimes \) of multiplication by numbers [37]. In particular, for all numbers \( t \) and vectors \( x \in \mathbb{B}_q \) we have

\[ t \otimes x = h^{-1}(t h(\|x\|)) \frac{x}{\|x\|}. \]

**Lemma 5.** Condition (64) is equivalent to the condition

\[
m_2(r^2) = \frac{\tanh^2(\sqrt{C}h(r))}{C(1 + \tanh^2(\sqrt{C}h(r)))} \tag{66}
\]

for all \( r \in [0, q) \).
Proof. Solving (64) for \( m_2(r^2) \) yields (66). □

**Corollary 1.** Let \( \oplus \) be an operation generated by a pair of functions \( (m_0, m_1) \) satisfying Assumptions 2 and 3. Then the operation \( \oplus \) is isomorphic to Einstein addition \( \oplus_E \) if and only if \( \lim_{u \to q} m_0(u) = \infty \) and there exists a positive number \( C \) such that condition (65) holds or condition (66) holds.

**Proof.** The statement follows from Lemmas 4 and 5, and Theorem 9. □

**Example 2.** Let

\[
m_1(u) = \frac{1}{(1-u)^2}, \quad C = 1.
\]  

Then

\[
\int_0^r \sqrt{m_1(s^2)}ds = \tanh(r),
\]

and Equation (64) implies

\[
r^2 = \frac{m_2(r^2)}{1 + m_2(r^2)},
\]

which leads to

\[
m_2(r^2) = \frac{r^2}{1-r^2}, \quad m_0(u) = \frac{1}{1-u}.
\]  

The functions \( m_0 \) and \( m_1 \) in (67) and (68) generate Einstein addition.

**Example 3.** Let

\[
m_1(u) = \frac{1}{(1-u)^2}, \quad C = 4.
\]  

Then

\[
\int_0^r \sqrt{m_1(s^2)}ds = \tanh(r),
\]

and Equation (66) implies

\[
m_2(r^2) = \frac{\tanh^2(2 \tanh(r))}{4(1 - \tanh^2(2 \tanh(r)))} = \frac{r^2}{(1-r^2)^2}, \quad m_0(u) = \frac{1}{(1-u)^2}.
\]  

The functions \( m_0 \) and \( m_1 \) in (69) and (70) generate Möbius addition.

**Theorem 10.** Let \( \oplus \) be a binary operation in \( \mathbb{B}_q \) generated by a pair of functions \( (m_0, m_1) \) satisfying Assumptions 2 and 3. Then the operation \( \oplus \) is isomorphic to Einstein addition \( \oplus_E \) in \( \mathbb{B}_1 \) if and only if the operation \( \oplus \) is isomorphic to an operation \( \tilde{\oplus} \) in \( \mathbb{R}^n \) generated by the functions

\[
\tilde{m}_0(u) = 1, \quad \tilde{m}_1(u) = \frac{1}{1+u}.
\]

The operation \( \oplus_E \) is isomorphic to the operation \( \tilde{\oplus} \) with an isomorphism \( \phi: \mathbb{B}_q \to \mathbb{B}_1 \) determined by the scalar function

\[
\phi(r) = \frac{r}{\sqrt{1+r^2}}.
\]

**Proof.** First, we show that the operation \( \tilde{\oplus} \) is isomorphic to Einstein addition \( \oplus_E \) in \( \mathbb{B}_1 \), which is generated by the functions

\[
m_{0,E}(u) = \frac{1}{1-u}, \quad m_{1,E}(u) = \frac{1}{(1-u)^2}.
\]
We define a bijection $\tilde{\phi}: [0, \infty) \rightarrow [0, 1)$ as follows. Let

$$\phi(r) = \frac{r}{\sqrt{1 + r^2}}.$$  

Then

$$m_{0,E}(\tilde{\phi}(r)^2) \left[ \frac{\phi(r)}{\phi'(0)r} \right]^2 = 1 = \tilde{m}_0(r^2),$$

$$m_{1,E}(\tilde{\phi}(r)^2) \left[ \frac{\phi'(r)}{\phi'(0)} \right]^2 = \frac{1}{1 + r^2} = \tilde{m}_1(r^2).$$

According to Theorem 5 the operations $\oplus$ and $\oplus_E$ are isomorphic. Following Theorem 4 the operations $\oplus$ and $\oplus_E$ are isomorphic if and only if the operations $\tilde{\oplus}$ and $\oplus$ are isomorphic. 

Notice that the operation $\tilde{\oplus}$ is isomorphic to Einstein addition with an isomorphism $\phi$ determined by the function

$$\phi^{-1}(r) = \frac{r}{\sqrt{1 - r^2}}.$$  

**Theorem 11.** The operation $\tilde{\oplus}$ in $\mathbb{R}^n$ generated by the functions

$$\tilde{m}_0(u) = 1, \quad \tilde{m}_1(u) = \frac{1}{1 + u}$$

is isomorphic to the operations $\tilde{\oplus}_t$ in $\mathbb{R}^n$ generated by the functions

$$\tilde{m}_{0,t}(u) = 1, \quad \tilde{m}_{1,t}(u) = \frac{1}{1 + t^2u},$$

for all $t > 0$. The corresponding isomorphism $\tilde{\phi}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is determined by the function

$$\tilde{\phi}(r) = tr.$$  

**Proof.** We have

$$\tilde{m}_0(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}(r)}{\tilde{\phi}'(0)r} \right]^2 = 1 = \tilde{m}_{0,t}(r^2),$$

$$\tilde{m}_1(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}'(r)}{\tilde{\phi}'(0)} \right]^2 = \frac{1}{1 + t^2r^2} = \tilde{m}_{1,t}(r^2).$$

The statement of the theorem follows from Theorem 5. 

**Theorem 12.** The operation $\oplus$ in $\mathbb{R}^n$ generated by the functions

$$\tilde{m}_0(u) = 1, \quad \tilde{m}_1(u) = \frac{1}{1 + t^2u}$$

is isomorphic to Einstein addition $\oplus_E$ with an isomorphism $\phi: \mathbb{R}^n \rightarrow \mathbb{B}_1$ determined by the scalar function

$$\phi_t(r) = \frac{tr}{\sqrt{t^2r^2 + 1}}.$$  

**Proof.** The proof follows from Theorems 4, 10 and 11. 

For every positive number $t$ let the function $\phi_t: \mathbb{R}^n \rightarrow \mathbb{B}_1$ be given by

$$\phi_t(w) = \frac{tw}{\sqrt{1 + t^2\|w\|^2}}.$$
Then for all $a, b \in \mathbb{R}^n$

$$a \oplus_t b = \phi_t^{-1}(\phi_t(a) \oplus_E \phi_t(b)).$$

Direct calculations show that

$$a \oplus_t b = \left[\sqrt{1 + t^2\|b\|^2} + \frac{t^2a^\top b}{1 + \sqrt{1 + t^2\|a\|^2}}\right] a + b. \ (71)$$

The operations $\oplus_t$ with $t > 0$ are isomorphic to each other and to Einstein addition. Thus, any study of hyperbolic geometry in $B_1$ with Einstein addition is equivalent to a corresponding study of hyperbolic geometry in $\mathbb{R}^n$ with the binary operation

$$a \oplus E b = \left[\sqrt{1 + \|b\|^2} + \frac{a^\top b}{1 + \|a\|^2}\right] a + b.$$

The limit $t \to 0$ in (71) yields Euclidean addition, which is not isomorphic to Einstein addition (since, for instance, the fact that Einstein addition is not commutative).

The operation of scalar multiplication is defined according to general rules [37]. Since

$$h_t(p) = \int_0^p \sqrt{m_{1,t}(s^2)} ds = \int_0^p \frac{ds}{\sqrt{1 + t^2s^2}} = \frac{1}{t} \sinh(tp),$$

we get

$$r \otimes_t a = \frac{a}{\|a\|} h_t^{-1}(r h_t(\|a\|)) = \frac{a}{\|a\|} \frac{1}{t} \sinh(r \sinh(t\|a\|)).$$

For $a, b \in \mathbb{R}^n$, $a$ not parallel to $b$, the co-gyroline

$$Q(a, b) = \{(sa) \oplus_t b : s \in \mathbb{R}\}$$

is a Euclidean line in $\mathbb{R}^n$:

$$Q(a, b) = \{x \in P(a, b) : (a_\perp)^\top x = (a_\perp)^\top b\}$$

where $a_\perp$ is a vector orthogonal to $a$, and $P(a, b)$ is a two dimensional plane containing both $a$ and $b$.

For $a, b \in \mathbb{R}^n$, $b \neq 0$, the gyroline

$$S(a, b) = \{a \oplus_t (sb) : s \in \mathbb{R}\}$$

is a hyperbola in $\mathbb{R}^n$ that lies on a plane containing $a$ and $b$. In particular, if $a^\top b = 0$, $a \neq 0$, then

$$a \oplus_t (sb) = \sqrt{1 + s^2t^2\|b\|^2} a + sb.$$

The ratio of the coefficients of $a$ and $b$ tends to $\pm t\|b\|$ as $s \to \pm \infty$:

$$\lim_{s \to \pm \infty} \frac{\sqrt{1 + s^2t^2\|b\|^2}}{s} = \pm t\|b\|.$$

These hyperbolas tend to lines as $t \to 0$. 
Theorem 13. Let $\oplus$ be a binary operation in $\mathbb{R}^n$ generated by a pair of functions $(m_0, m_1)$ satisfying Assumptions 2 and 3 and such that $m_1(u) = 1$ for all $u$. Then the operation $\oplus$ is isomorphic to Einstein addition if and only if $\lim_{u \to \infty} um_0(u) = \infty$ and there exists a positive number $t$ such that

$$m_0(r^2) = \frac{\tanh^2(tr)}{1 - \tanh^2(tr) r^2 t^2}$$

(72)

for all $r \in (0, \infty)$.

Proof. Following Theorem 5, it is sufficient to show that there exists a differentiable increasing bijection $\tilde{\phi}: (0, 1) \to (0, \infty)$ such that for all $p \in (0, 1)$

$$\frac{1}{1 - p^2} = m_0(\tilde{\phi}(p))^2 \left[ \tilde{\phi}'(0) p \right]^2,$$

(73)

$$\frac{1}{(1 - p^2)^2} = \left[ \tilde{\phi}'(0) \right]^2$$

(74)

if and only if the function $m_0$ has the form (72). Let $t = (\tilde{\phi}'(0))^{-1}$. Then Equation (74) can be integrated,

$$\tilde{\phi}(p) = t^{-1} \tanh(p).$$

(75)

If $r = \tilde{\phi}(p)$, then

$$p = \tanh(rt).$$

(76)

Substituting the value of $p$ from (76) into (73) and solving it for $m_0$, we get (72).

Remark 4. Operations in $\mathbb{R}^n$ generated by the functions $m_1 = 1$ and $m_0$ satisfying condition (72) are isomorphic for different positive numbers $t$ to the operation with $t = 1$. The isomorphism is a simple stretching: $x \to t^{-1} x$.

8. Operations Isomorphic to Einstein Addition and Having the Same Function $m_0$

Set $m_0(u) = (1 - u)^{-1}$. Consider functions $m_1$ satisfying Equation (58):

$$m_1(u) = \frac{[(um_0(u))]^2}{m_0(u) [1 + \mu^2 uu_0(u)]}, \quad \forall u \in [0, 1),$$

with positive numbers $\mu$. According to (59) we set

$$\tilde{\phi}_\mu(r) = \frac{r \mu}{\sqrt{1 - r^2 + \mu^2 r^2}}.$$

Then

$$\left[ \frac{\tilde{\phi}_\mu(r)}{\tilde{\phi}_\mu'(0) \sqrt{1 - \tilde{\phi}_\mu(r)^2}} \right]^2 = \frac{1}{1 - r^2} = m_0(r^2) \quad \forall r \in [0, 1),$$

(77)

and

$$\left[ \frac{\tilde{\phi}_\mu'(r)}{\tilde{\phi}_\mu'(0)(1 - \tilde{\phi}_\mu(r)^2)} \right]^2 = m_1(r^2) \quad \forall r \in [0, 1).$$

(78)
Let $\oplus_\mu$ be the binary operation generated by the canonical metric tensor $G$ parametrized by the functions $m_0, m_1$ given in (77) and (78). Following Theorem 9, the operations $\oplus_\mu$ are isomorphic to Einstein addition for all $\mu > 0$. Moreover, for the function

$$\varphi_\mu(x) = \phi_\mu(||x||) \frac{x}{||x||} \tag{79}$$

we have

$$a \oplus_\mu b = \varphi_\mu^{-1}(\varphi_\mu(a) \oplus_E \varphi_\mu(b)),$$

where $\oplus_E$ is Einstein addition. The goal of this section is to find a closed form for the operation $\oplus_\mu$.

From (79) we get

$$\varphi_\mu(x) = \frac{\mu x}{\sqrt{1 + (\mu^2 - 1)||x||^2}} \quad \forall x \in B_1,$$

$$\varphi_\mu^{-1}(x) = \frac{x}{\sqrt{\mu^2 + ||x||^2(1 - \mu^2)}} \quad \forall x \in B_1.$$

For every vector $x \in B_1$ let

$$\lambda_{\mu,x} = \frac{1}{\sqrt{1 + (\mu^2 - 1)||x||^2}}.$$

Then

$$\varphi_\mu(x) = \mu \lambda_{\mu,x} x.$$

Taking into account that

$$||\varphi_\mu(a) \oplus_E \varphi_\mu(b)||^2 = 1 - \frac{(1 - ||\varphi_\mu(a)||^2)(1 - ||\varphi_\mu(b)||^2)}{(1 + \varphi_\mu(a) \cdot \varphi_\mu(b))^2},$$

and

$$1 - ||\varphi(x)||^2 = 1 - \frac{\mu^2||x||^2}{1 + (\mu^2 - 1)||x||^2} = (1 - ||x||^2)\lambda_{\mu,x}^2,$$

we get

$$a \oplus_\mu b = \frac{\varphi_\mu(a) \oplus_E \varphi_\mu(b)}{\sqrt{\mu^2 + (1 - \mu^2)||\varphi_\mu(a) \oplus_E \varphi_\mu(b)||^2}} \tag{80}$$

$$= \frac{\varphi_\mu(a) \oplus_E \varphi_\mu(b)}{\sqrt{\mu^2 + (1 - \mu^2)(1 - ||\varphi_\mu(a) \oplus_E \varphi_\mu(b)||^2)}}$$

$$= \frac{(1 + \varphi_\mu(a) \cdot \varphi_\mu(b))(\varphi_\mu(a) \oplus_E \varphi_\mu(b))}{\sqrt{(1 + \varphi_\mu(a) \cdot \varphi_\mu(b))^2 - (1 - \mu^2)(1 - ||\varphi_\mu(a)||^2)(1 - ||\varphi_\mu(b)||^2)}}$$

$$= \frac{(\lambda_{\mu,a} + \lambda_{\mu,b} a \cdot b/||b||^2)a + \lambda_{\mu,a} \lambda_{\mu,b} \sqrt{1 - ||a||^2(b - a \cdot b/||b||^2)}a}{\sqrt{\lambda_{\mu,a} a^2 + \lambda_{\mu,b} b^2 + \lambda_{\mu,a} \lambda_{\mu,b} (a \cdot b)^2 - \mu^2 ||a||^2 ||b||^2 + (1 - ||a||^2)(1 - ||b||^2)}}$$

For the case $\mu = 1$ we have $\lambda_{1,x} = 1$ and $\oplus_1$ is Einstein addition:

$$a \oplus_E b = a \oplus_1 b = \frac{(1 + a \cdot b/||b||^2)a + \sqrt{1 - ||a||^2}(b - a \cdot b/||b||^2)}{1 + a \cdot b}.$$

According to Theorem 9, the operations $\oplus_\mu$ are isomorphic to Einstein addition $\oplus_E$ for all $\mu > 0$. 


In the limit of (80) as \( \mu \to 0 \), we have the operation \( \oplus_{coE} \):

\[
a \oplus_{coE} b = \frac{a\gamma_a + b\gamma_b}{\sqrt{1 + \|a\gamma_a + b\gamma_b\|^2}},
\]

where \( \gamma_x = (1 - \|x\|^2)^{-1/2} \) for all \( x \in \mathbb{B}_1 \). The operation \( \oplus_{coE} \) is isomorphic to Euclidean addition with an isomorphism \( \varphi: \mathbb{B}_1 \to \mathbb{R}^n \), given by

\[
\varphi(x) = x\gamma_x.
\]

9. Isomorphic Operations with the Same Function \( m_0 \)

Let \( q \in (0, \infty] \). As in Section 8, we consider

(i) a pair of functions \((m_0, m_1)\) satisfying Assumptions 2 and 3, a canonical metric tensor \( G \) and a binary operation \( \oplus \) in \( \mathcal{B}_q \) generated by \( (m_0, m_1) \), and

(ii) a pair of functions \((\tilde{m}_0, \tilde{m}_1)\) satisfying Assumptions 2 and 3, a canonical metric tensor \( \tilde{G} \) and a binary operation \( \tilde{\oplus} \) in \( \mathcal{B}_q \) generated by \( (\tilde{m}_0, \tilde{m}_1) \).

Our goal is to answer the following question. How to characterize metric tensors \( G, \tilde{G} \) if the operations \( \oplus, \tilde{\oplus} \) are isomorphic, and \( m_0 = \tilde{m}_0 \)? Further in this section we assume \( m_0 = \tilde{m}_0 \).

According to Theorem 5 the operations \( \oplus \) and \( \tilde{\oplus} \) are isomorphic, if and only if there exists an increasing smooth bijection \( \tilde{\phi}: [0, q] \to [0, q] \) such that \( \tilde{\phi}'(0) > 0 \), and

\[
m_0(r^2) = m_0(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}(r)}{\tilde{\phi}'(0)r} \right]^2,
\]

\[
\tilde{m}_1(r^2) = m_1(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}'(r)}{\tilde{\phi}'(0)} \right]^2
\]

for all \( r \in [0, 1) \). We can solve Equation (81) for the function \( \tilde{\phi} \), and substitute this function into Equation (82) to find a relation between the functions \( m_1 \) and \( \tilde{m}_1 \). Thus, Theorem 5 for our case when \( m_0 = \tilde{m}_0 \) has the following form.

**Theorem 14.** Assume the pairs of functions \((m_0, m_1)\), and \((\tilde{m}_0, \tilde{m}_1)\) satisfy Assumptions 2 and 3. Denote by \( \oplus \) and \( \tilde{\oplus} \) the binary operations in \( \mathcal{B}_q \) generated by the pairs of functions \((m_0, m_1)\) and \((\tilde{m}_0, \tilde{m}_1)\) respectively. Assume a differentiable bijection \( \tilde{\phi}: [0, 1] \to [0, 1] \) is such that \( \tilde{\phi}'(0) > 0 \), and

\[
m_0(r^2) = m_0(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}(r)}{\tilde{\phi}'(0)r} \right]^2
\]

for all \( r \in [0, 1) \).

Then the operations \( \oplus \) and \( \tilde{\oplus} \) are isomorphic if and only if

\[
\tilde{m}_1(r^2) = m_1(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}'(r)}{\tilde{\phi}'(0)} \right]^2
\]

for all \( r \in [0, 1) \).

The positive number \( t = \tilde{\phi}'(0) \) can be chosen arbitrarily. For each such \( t \) Equation (83) defines a function \( \tilde{\phi} \). Then from (84) we get the corresponding function \( m_1 \). Hence, we get a one parameter set of functions \( m_1 \) such that the binary operations generated by pairs \((m_0, m_1)\) are isomorphic. It is remarkable that in the limit \( t \to 0 \) we get operations isomorphic to Euclidean addition. Let us consider two special cases.
Special case 1. Assume
\[ m_0(u) = \frac{1}{1 - u}. \]
Denote \( t = \dot{\phi}'(0) \). Then Equation (83) may be solved for \( \dot{\phi} \):
\[ \dot{\phi}(r) = \frac{tr}{\sqrt{1 + (t^2 - 1)r^2}}. \] (85)

Theorem 14 states that the operations \( \oplus \) and \( \hat{\oplus} \) are isomorphic if and only if
\[ m_1(r^2) = m_1(\dot{\phi}(r)^2) \frac{1}{(1 + (t^2 - 1)r^2)^3}. \] (86)

Assume, as for Einstein addition,
\[ m_1(u) = \frac{1}{(1 - u)^2}. \]
Then
\[ \hat{m}_1(r^2) = \frac{1}{(1 - r^2)^2(1 + (t^2 - 1)r^2)}. \]
If \( t = 1 \), then we obviously get \( \dot{\phi}(r) = r \) and \( \hat{m}_1 = m_1 \). If \( t \to 0 \), then \( \dot{\phi}(r) \to 0 \) for all \( r \in [0, 1) \), and for the corresponding limit \( \hat{m}_1 = \lim_{t \to 0} \hat{m}_1 \) we have:
\[ \hat{m}_1(r^2) = \lim_{t \to 0} \hat{m}_1(r^2) = \frac{1}{(1 - r^2)^3}. \] (87)

The functions \( m_0, \hat{m}_1 \) satisfy condition (50). According to Theorem 8, a binary operation generated by the canonical metric tensor (8) with the functions \( m_0, \hat{m}_1 \) is isomorphic to Euclidean addition.

Assume
\[ m_1(u) = \frac{1}{(1 - u)^4}. \]
Then
\[ \hat{m}_1(r^2) = \frac{1 + (t^2 - 1)r^2}{(1 - r^2)^4}. \]
The binary operations generated by pairs \((m_0, \hat{m}_1)\) are isomorphic to Einstein addition, and not isomorphic to Euclidean addition.

Special case 2. Assume
\[ m_0(u) = \frac{1}{(1 - u)^2}. \]
Solve Equation (83) for \( \dot{\phi} \):
\[ \dot{\phi}(r) = \frac{\sqrt{2tr}}{\sqrt{2t^2r^2 + (1 - r^2)^2 + (1 - r^2) \sqrt{(1 - r^2)^2 + 4t^2r^2}}}. \]
Calculate the factor on the right hand side of (84):
\[ \left[ \frac{\dot{\phi}(r)^2}{t} \right]^2 = \frac{2(1 + r^2)^2}{[2t^2r^2 + (1 - r^2)^2 + (1 - r^2) \sqrt{(1 - r^2)^2 + 4t^2r^2}][(1 - r^2)^2 + 4t^2r^2]} \].
According to Theorem 14 an operation generated by the pair \((m_0, m_1)\) is isomorphic to an operation generated by the pair \((m_0, \tilde{m}_1)\) if and only if there exists a positive number \(t\) such that

\[
\tilde{m}_1(r^2) = m_1(\varphi(r))^2 \left(\frac{\varphi'(r)}{t}\right)^2.
\]

for all \(r \in [0,1)\).

For Möbius addition we have

\[
m_{1,M}(r^2) = \frac{1}{(1-r^2)^2}.
\]

Direct calculations show that the binary operation generated by a pair \((m_0, \tilde{m}_1)\) is isomorphic to the binary operation generated by the pair \((m_0, m_{1,M})\) if and only if there exists a non-zero number \(t\) such that

\[
\tilde{m}_1(r^2) = (1 + r^2)^2 \left(1 - r^2 + \frac{2t^2r^2}{1 - r^2} + \frac{2t^2r^2}{1 - r^2 + \sqrt{(1-r^2)^2 + 4t^2r^2}}\right).
\]

If \(t = 1\) then obviously \(\tilde{\phi}(r) = r\) and \(\tilde{m}_1 = m_{1,M}\). The corresponding binary operations are isomorphic to Möbius addition (and hence isomorphic to each other and isomorphic to Einstein addition) for all \(t \neq 0\). Therefore these operations are not isomorphic to Euclidean addition in \(\mathbb{R}^n\). We denote the limit \(t \to 0\) by \(\hat{m}_1\):

\[
\hat{m}_1(r^2) = \lim_{t \to 0} \tilde{m}_1(r^2) = \left(1 + r^2\right)^2 \left(1 - r^2 + \frac{2t^2r^2}{1 - r^2 + \sqrt{(1-r^2)^2 + 4t^2r^2}}\right).
\]

According to Theorem 8, the operation generated by the pair \((m_0, m_1)\) is isomorphic to Euclidean addition, since

\[
\hat{m}_1(u) = \frac{[(um_0(u))^2]^2}{m_0(u)}
\]

for all \(u \in [0,1)\).

### 10. Isomorphic Operations with the Same Function \(m_1\)

Let \((m_0, m_1)\) and \((\tilde{m}_0, m_1)\) be pairs of functions satisfying Assumptions 2 and 3. Let \(\oplus, \tilde{\oplus}\) be binary operations in \(\mathbb{B}_q\) generated by the pairs \((m_0, m_1)\) and \((\tilde{m}_0, m_1)\) respectively. Notice that the second function in both pairs is the same. The distance functions in the space with operations \(\oplus\) and \(\tilde{\oplus}\) are completely determined by the same function \(m_1\), and therefore they coincide.

Now we consider the problem of finding a relation between \(\oplus\) and \(\tilde{\oplus}\).

**Theorem 15.** Canonical metric tensors of binary operations \(\oplus\) and \(\tilde{\oplus}\) have the same function \(m_1\) and the binary operations \(\oplus\) and \(\tilde{\oplus}\) are isomorphic if and only if there exists a positive number \(t\) such that

\[
a \tilde{\oplus} b = \left(\frac{1}{t}\right) \oplus [(t \oplus a) \oplus (t \oplus b)]
\]

for all \(a, b \in \mathbb{B}_q\), where \(\otimes\) is a scalar multiplication in the space with a binary operation \(\oplus\).

**Proof.** Assume that identity (88) holds for a positive number \(t\). Let \(h\) be the function given by

\[
h(p) = \int_0^p \sqrt{m_1(s^2)} ds.
\]
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Since \( m_1(0) = 1 \), we have \( 1 = h'(0) = (h^{-1})'(0) \), where \( h^{-1} \) is the inverse of \( h \). According to the definition of multiplication by numbers [37] we have

\[ t \otimes x = \frac{x}{\|x\|} h^{-1}(t \cdot h(\|x\|)) \]

for all \( x \in B_q \). Let

\[ \tilde{\phi}_t(p) = h^{-1}(t \cdot h(p)) \].

Then \( t = \tilde{\phi}_t'(0) \) and

\[ t \otimes x = \frac{x}{\|x\|} \tilde{\phi}_t(\|x\|) \].

Direct calculations show that

\[ \frac{1}{t} \otimes [(t \otimes (-x)) \oplus (t \otimes (x + \Delta x))] = o(\|\Delta x\|) \]

\[ + \left( \sqrt{m_0(\tilde{\phi}_t(\|x\|)^2)} \tilde{\phi}_t(\|x\|) \left[ I - \frac{xx^T}{\|x\|^2} \right] + \sqrt{m_1(\tilde{\phi}_t(\|x\|)^2)} \frac{\tilde{\phi}_t(\|x\|) xx^T}{t \|x\|^2} \right) \Delta x. \]  

(89)

Therefore the metric tensor of the operation \( \oplus \) has the form (8) parametrized by the functions

\[ \tilde{m}_0(r^2) = m_0(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}(\|x\|)}{\|x\|} \right]^2, \]

\[ \tilde{m}_1(r^2) = m_1(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}(\|x\|)}{\|x\|} \right]^2. \]  

(90)

According to Theorem 14 the operations \( \oplus \) and \( \oplus \tilde{\phi} \) are isomorphic. By definition,

\[ \int_0^p \sqrt{m_1(s^2)} ds = h(p) = \frac{1}{t} \tilde{h}(\tilde{\phi}_t(p)) = \frac{1}{t} \int_0^p \tilde{\phi}_t(p) \sqrt{m_1(s^2)} ds \]

\[ = \frac{1}{t} \int_0^p \sqrt{m_1(\tilde{\phi}(u)^2) \tilde{\phi}'(u)} du = \int_0^p \sqrt{\tilde{m}_1(u^2)} du. \]  

(91)

Here we employed the change of variable: \( s = \tilde{\phi}(u) \). Now differentiate identity (91) with respect to \( p \) to get

\[ m_1(p^2) = \tilde{m}_1(p^2) \]

for all \( p \in [0, q] \).

Conversely, assume that the operations \( \oplus \) and \( \oplus \tilde{\phi} \) are isomorphic, and that the functions \( m_1 \) and \( \tilde{m}_1 \) coincide. Then according to Theorem 5, Definition 4 and Remarks 1 and 2 there exists a differentiable bijection \( \tilde{\phi}: [0, q) \to [0, q) \) such that \( \tilde{\phi}'(0) > 0 \),

\[ \tilde{m}_0(r^2) = m_0(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}(\|x\|)}{\|x\|} \right]^2, \]

\[ m_1(r^2) = \tilde{m}_1(r^2) = m_1(\tilde{\phi}(r)^2) \left[ \frac{\tilde{\phi}(\|x\|)}{\|x\|} \right]^2. \]  

(92)

for all \( r \in [0, q) \) and

\[ a \oplus \tilde{\phi}^{-1}(\phi(a) \oplus \phi(b)) \]  

(93)

for all \( a, b \in B_q \), where

\[ \phi(x) = \tilde{\phi}(\|x\|) \frac{x}{\|x\|} \]

for all \( x \in B_q \). Set \( t = \tilde{\phi}'(0) \). Define the function

\[ h(p) = \int_0^p \sqrt{m_1(s^2)} ds. \]
Then
\[ h(p) = \frac{1}{t} \int_0^p \sqrt{m_1(\hat{\varphi}(u)^2)\hat{\varphi}'(u)} \, du = \frac{1}{t} \int_0^{\hat{\varphi}(p)} \sqrt{m_1(s^2)} \, ds = \frac{1}{t} h(\hat{\varphi}(p)). \]

Therefore,
\[ \hat{\varphi}(p) = h^{-1}(t h(p)) \]
for all \( p \in [0, q) \). Hence,
\[ t \otimes a = h^{-1}(t h(||x||)) \frac{x}{||x||} = \hat{\varphi}(||x||) \frac{x}{||x||} = \varphi(x) \]
for all \( x \in B_q \). Since the function \( \varphi \) is invertible, we have
\[ x = \varphi^{-1}(t \otimes x) \]
for all \( x \in B_q \). Now we apply Equation (93) obtaining
\[ a \hat{\oplus} b = \]
\[ = \varphi^{-1}[\varphi(a) \oplus \varphi(b)] \]
\[ = \varphi^{-1}\left[ t \otimes \left( \frac{1}{t} \right) \oplus \left[ (t \otimes a) \oplus (t \otimes b) \right] \right] \]
\[ = \left( \frac{1}{t} \right) \oplus \left[ (t \otimes a) \oplus (t \otimes b) \right] \] (94)
for all \( a, b \in B_q \).

Example. If \( \oplus \) is Einstein addition, and \( \hat{\oplus} \) is Möbius addition, then
\[ m_0(u) = \frac{1}{1 - u}, \quad m_1(u) = \bar{m}_1(u) = \bar{m}_0(u) = \frac{1}{(1 - u)^2}. \]
In this case Equation (88) holds with \( t = 2 \).

11. Conclusions

In this paper we consider smooth binary operations invariant with respect to unitary transformations. We prove that such binary operations have tensors (8) parametrized by non-negative functions \( m_0, m_1 \). It is shown that binary operations are well-defined iff a special additional constraint (11) on functions \( m_1 \) hold. We present necessary and sufficient conditions for two operations to be isomorphic. We give necessary and sufficient conditions for binary operations to be isomorphic to Einstein addition, and, separately, to Euclidean addition. We also pointed out necessary and sufficient conditions for two operations having the same function \( m_0 \), or, separately, the same function \( m_1 \), to be isomorphic.

Future research may concern a development of calculus (integration, differentiation, differential equations) in gyrospaces, finding necessary and sufficient conditions for binary operations to give rise of gyrogroups, an integration of geodesic equations in the polar coordinates, and a description of binary operations with respect to which the Maxwell equations are invariant.

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