Electromagnetic Scattering from Relativistic Bound States

N. K. Devine and S. J. Wallace
Department of Physics and Center for Theoretical Physics
University of Maryland, College Park, MD 20742
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Abstract

The quasipotential formalism for elastic scattering from relativistic bound states is formulated based on the instant constraint in the Breit frame. The quasipotential electromagnetic current is derived from Mandelstam’s five-point kernel and obeys a two-body Ward identity. Breit-frame wave functions are obtained directly by solving integral equations with nonzero total three-momentum, thus accomplishing a dynamical boost. Calculations of electron-deuteron elastic form factors illustrate the importance of the dynamical boost versus kinematic boosts of the rest frame wave functions.

25.30.Bf, 24.10.Jv, 11.10.Qr, 11.10.St
In the study of relativistic bound states and scattering processes for two particles, it is possible to perform a reduction from four dimensions to three dimensions to obtain a quasipotential formalism [1–4]. The quasipotential is the kernel of the three-dimensional equation, which in general may be defined covariantly. A quasipotential reduction is commonly used in studies of the nucleon-nucleon (NN) interaction because it provides a covariant dynamics which is similar to the Schrödinger dynamics. For NN scattering, one generally assumes a quasipotential in the form of meson-exchange interactions with coupling constants selected to provide a realistic description of the NN scattering data and deuteron properties [5].

In this work, a quasipotential reduction procedure is applied to the Mandelstam five-point function to derive a conserved current operator consistent with the quasipotential wave functions. The quasipotential constraint on the five-point kernel and initial and final states must be consistent with conservation of the photon momentum, \( q \). In the Breit frame, where \( q^0 = 0 \), we find that the instant constraint \( p^0 = p'^0 = 0 \) is consistent, where \( p \) is the relative momentum in the initial state and \( p' \) is the relative momentum in the final state. Moreover, the instant constraint leads to a gauge invariant formalism which is symmetric in its treatment of the particles. With the instant constraint in the Breit frame for electromagnetic matrix elements, the initial and final wave functions must be calculated with total three momentum \( P = -\frac{1}{2}q \) for the initial state and \( P = \frac{1}{2}q \) for the final state. Only for \( q = 0 \) are the usual rest frame wave functions used. For \( q \neq 0 \), the required wave functions may be thought of in terms of a boost of the rest frame wave function. In the instant form of relativistic quantum mechanics one encounters a similar form of boost, and it must be dynamical in the sense that the generator of boosts depends on the interaction. In this paper, we formulate the dynamical boost within the instant quasipotential formalism. We present calculations of deuteron electromagnetic form factors to demonstrate the feasibility of calculating the required wave functions in the Breit frame and we illustrate the importance of the dynamical boost.

Formally, the quasipotential is defined such that the results for two-body scattering based on the Bethe-Salpeter equation are reproduced by use of the quasipotential and a three-dimensional propagator. Consider the Bethe-Salpeter t-matrix,\n
\[
T(p, q; P) = K^{BS}(p, q; P) + \int \frac{dp'}{(2\pi)^4} K^{BS}(p, p'; P) G_{0}^{BS}(p'; P) T(p', q; P),
\]

where \( G_{0}^{BS} = iS_{F(1)}(\frac{1}{2}P+P')iS_{F(2)}(\frac{1}{2}P-P') \) is the propagator for two free particles, and \( K^{BS} \) is the kernel consisting of irreducible graphs. This equation may be abbreviated as \( T^{BS} = K^{BS} + K^{BS}G^{BS}T^{BS} \), with implied four-dimensional integration. Note that \( P = p_{1} + p_{2} \) is the total momentum and \( p = \frac{1}{2}(p_{1} - p_{2}) \) is the relative momentum.

A bound state with mass \( M \) gives rise to a pole at \( P^0 = \hat{P}^0 \equiv \sqrt{M^2 + P^2} \):

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T(p, q; P) = -\frac{i}{2P^0} \frac{\Gamma(p; \hat{P})\Gamma(q; \hat{P})}{P^0 - P^0},
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where \( \Gamma = \Gamma_{\uparrow\downarrow}^{0\gamma_{0}} \) and terms regular when \( P^0 = \hat{P}^0 \) are omitted.

The same T-matrix and bound state vertex function can be produced with a quasipotential propagator, \( G_{0}^{QP}(p, P) = i\gamma_{0} 2\pi \delta(C(p)) \), where \( C(p) = 0 \) is the constraint which reduces integrations from four to three dimensions:
\[ T = K^{QP} + K^{QP}G_0^{QP}T. \]  

The quasipotential kernel is defined by:

\[ K^{QP} = K^{BS} + K^{BS}(G_0^{BS} - G_0^{QP})K^{QP}. \]  

For the bound state vertex, one has a homogeneous equation, found by substituting Eq. (2) into Eq. (3) and retaining pole terms:

\[ \Gamma(p; \hat{P}) = \int \frac{d^4p'}{(2\pi)^4} K^{QP}(p, p'; \hat{P})G_0^{QP}(p'; \hat{P})\Gamma(p'; \hat{P}). \]  

The two most frequently used quasipotential constraints are the one-particle-on-shell formalism developed by Gross and collaborators [4,5], based on \( C(p) \equiv (p_0^2 - \epsilon_2^2) = 0 \) \( (\epsilon_2 \equiv \sqrt{m_2^2 + p_2^2}) \), and the equally-off-shell formalism in the center-of-mass frame developed by Blankenbecler and Sugar and Logunov and Tavkhelidze [1,2], based on \( C(p) \equiv (p_1^2 - p_2^2)/2\sqrt{P^2} = p \cdot P/\sqrt{P^2} = 0 \). When one particle is on mass shell, there is an inherent and inconvenient asymmetry of the formalism, but this can be overcome [5]. However, a manifestly symmetric treatment is afforded by the equally-off-shell formalism. In the center of mass frame, the equally-off-shell constraint is equivalent to an instant formalism because the constraint causes interactions to have zero time-component of momentum transfer. An extension of the equally-off-shell formalism to the full Dirac space for two fermions has been developed by Mandelzweig and Wallace [3] by incorporating the iterative parts of cross-box Feynman graphs, using a form of the eikonal approximation.

To obtain the quasipotential reduction for electromagnetic interactions, we start from the Mandelstam formalism [6] for a five-point function, \( T_5 \), which has a photon of momentum \( q \) coupled in all possible ways to the two particles and exchanged mesons. The five-point function, \( T_5 \), may be expressed as follows,

\[ T_5 = (1 + TG_0^{BS})K_5^{BS}(1 + G_0^{BS}T), \]  

where four-dimensional integrations are implied. The irreducible five-point kernel, \( K_5^{BS} \), is given by coupling the photon to particles one and two (lowest order impulse contributions) plus coupling the photon to all possible internal lines of the two-body kernel \( K^{BS} \). The five-point function can be written with a single quasipotential constraint applied consistently before, inside, and after the five-point kernel,

\[ T_5 = (1 + TG_0^{QP})K_5^{QP}(1 + G_0^{QP}T). \]  

Equations (6) and (7) imply that,

\[ K_5^{QP} = [1 + K^{QP}(G_0^{BS} - G_0^{QP})] K_5^{BS} [1 + (G_0^{BS} - G_0^{QP})K^{QP}]. \]  

Unlike some previous formulations [4–9], Eqs. (7) and (8) use a single constraint consistent with four-momentum conservation, and contain propagation of negative- as well as positive-energy nucleon states. To extract the electromagnetic matrix element for elastic scattering
from the bound state, one substitutes Eq. (2) into Eq. (7). The desired electromagnetic matrix element is proportional to the residue of the double pole term in the resulting expression, 

$$\langle J^\mu \rangle = -\Gamma G_0^{QP} K_5^{QP} G_0^{QP} \Gamma / 2\sqrt{P_0^2 P^2},$$

with implied four-dimensional integrations.

Instant constraints in the Breit frame may be expressed covariantly as 

$$p \cdot \tilde{P} = p' \cdot \tilde{P} = 0,$$

where $P$ and $P'$ are the initial and final total momenta, and $\tilde{P} = (P' + P) / \sqrt{(P' + P)^2} = (1, 0)$ in the Breit frame. They imply that the wave functions needed are a different slice of the four-dimensional wave function at each different value of $P$. Other choices for the quasipotential constraint are either inconsistent with momentum conservation or with particle-exchange symmetry. For example, the instant constraint in the rest frame of the initial and final bound states is not consistent with conservation of four-momentum of the virtual photon for electromagnetic interactions. Momentum conservation requires $P' = P + q$ and $p' = p + \frac{1}{2} q$, depending on which particle absorbs the virtual photon momentum, $q$. Thus, $p' \cdot P' = p \cdot P + p \cdot q + \frac{1}{2} q \cdot (P + q)$, which is not consistent with both $p' \cdot P' = 0$ and $p \cdot P = 0$ at all values of relative momentum (which is integrated).

An alternative is to place one particle on-shell [1], say particle two in Figure 4. This is consistent with momentum conservation when the virtual photon is absorbed by particle 1, but not when it is absorbed by particle 2. Thus there is an inconsistency with particle exchange symmetry, although for absorption by the on-shell particle, it has been argued in Ref. [10] that the difficulty can be overcome.

We conclude that conservation of four-momentum and particle exchange symmetry suggest use of the instant constraint in the Breit frame. We turn now to deriving the current operator and wave functions based on Eq. (8).

The impulse-approximation current operator for particle one may be obtained by expanding $K_5^{QP}$ into diagrams, with the leading order terms given by attaching the photon to particle one, together with zero, one, or two exchanged bosons. We have found that these leading diagrams yield an impulse current operator which is consistent with the one-boson-exchange approximation for the quasipotential, 

$$K^{QP} \approx -i v^{OBE}.$$ 

The homogeneous equation for the vertex, 

$$(1 - K^{QP(OBE)} G_0^{QP}) \Gamma = 0,$$

is used to simplify the current with the result that only the box and crossed-box five-point contributions need be evaluated to obtain the current [11]. The propagator that is consistent with this analysis is:

$$G_0^{QP}(p; P) = ig_0(p, P)(2\pi)\delta(p^0),$$

where

$$g_0(p, P) = \sum_{\rho_1, \rho_2} \frac{\Lambda^{\rho_1}_{\rho_2}(p_1)\Lambda^{\rho_2}_{\rho_2}(p_2)}{(\rho_1 + \rho_2)(\epsilon_D/2) - \epsilon_1 - \epsilon_2},$$

$$\rho_i = \pm, \rho_1 \Lambda^{\rho_1}(p_1)\gamma_0^{\rho_1}$$ are projection operators for Dirac spinors with hermitian norm, $\epsilon_i \equiv \sqrt{m_N^2 + p^2} (p_{1,2} = \frac{1}{2} P \pm p)$, and $\epsilon_D \equiv \sqrt{M_D^2 + P^2} = P^0$. Masses are $m_N$ (nucleon) and $M_D$ (deuteron). This propagator is similar to the Dirac two-body propagator of Ref. [3], except for the use of the instant constraint in the Breit frame instead of the equally-off-shell constraint.

The five-point box and crossed-box diagrams are reduced to three dimensional form in the Breit frame using a procedure similar to Ref. [3]. The crossed-box is evaluated using
where the impulse-approximation current for particle one is,

\[
\hat{J}_{IA}^\mu(1) = \Gamma^\mu_1 \gamma_2 \hat{\rho}_1 \left( 1 + \frac{1 - \hat{\rho}_1 \hat{\rho}_2 - 2\epsilon_2 - \hat{\rho}_2 \epsilon_D}{\epsilon_1 + \epsilon_1} \right),
\]

\(\hat{\rho}_1 u^\pm(\pm p_i) = \pm u^\pm(\pm p_i)\), and \(\Gamma^\mu_1\) is the one-body electromagnetic operator. The Breit frame total momenta are \(P' = (\epsilon_D, \frac{1}{2}q)\) and \(P = (\epsilon_D, -\frac{1}{2}q)\).

Using \(K_S^{QP(IA)}\) in \(\langle J^\mu_1 \rangle = \frac{g_2}{\sqrt{g_0}} K_S^{QP} g_0 \Gamma / 2\sqrt{P^0 P}\), and \(\psi \equiv g_0 \Gamma = g_0 v^{OBE} g_0 \Gamma\), the current matrix element for elastic scattering takes the form,

\[
\langle J^\mu_1(1) \rangle = \frac{1}{2\epsilon_D} \int \frac{d^3p}{(2\pi)^3} \psi(p) + \frac{1}{2} q \cdot P' \hat{J}_{IA}^\mu(1) \psi(p; P)
\]

A similar term is obtained for photon absorption by particle two. The box-crossed current, \(J^{IA}(1)\), obeys an exact isoscalar, two-body, Ward-Takahashi identity,

\[
qu \cdot \hat{J}_{IA}(1) = g_0^{-1}(p'; P') - g_0^{-1}(p; P),
\]

provided that the one-body current obeys the one-body WT identity, \(q \cdot \Gamma_1 = S_{\langle F \rangle}^{-1}(p') - S_{\langle F \rangle}^{-1}(p_1) = q \cdot \gamma_1\). Thus working in the Breit frame and retaining the five-point crossed-box allows for a gauge invariant quasipotential analysis.

The required wave functions may not in general be obtained by a kinematical boost of rest frame wave functions. Instead they must be calculated directly in the Breit frame. Thus the boost problem within the instant quasipotential formalism involves a change of quasipotential constraint. It is a straightforward matter to prove that the quasipotential must change as follows,

\[
K^{QP2} = K^{QP1} + K^{QP1}(C_0^{QP1} - C_0^{QP2}) K^{QP2},
\]

where, for example, QP1 may refer to the instant constraint in the rest frame, \(C_1(p) = p \cdot P / \sqrt{P^2}\), and QP2 may refer to the Breit frame instant constraint, \(C_2(p) = p \cdot \bar{P}\). Equation (13) defines formally how the interaction kernel changes when a boost from the rest frame to the Breit frame is performed, such that the same underlying covariant kernel applies in the four-dimensional formalism.

Calculations of the instant wave functions in the Breit frame are based on solving,

\[
g_0^{-1}(p'; P) \psi(p'; P) = \int \frac{d^3p}{(2\pi)^3} v(p', p; P) \psi(p; P),
\]

for \(P = \pm q / 2\). An appropriate normalization condition is used. A one-boson-exchange potential is used with scalar, pseudo-vector, and vector mesons (\(\sigma, \delta, \eta, \pi, \omega, \rho\)) to obtain deuteron wave functions using modified Bonn B parameters [11,12].
The Breit frame total angular momentum operator is, \( J = J_1 + J_2 = \vec{L} + S \), where \( \vec{L} = 1 + L \), \( L = r \times p \), \( L = R \times P \), and \( S = \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2) \). Because \( J \) and \( J^z \) commute with \( g_0^{-1} \) and \( v \), solutions of the homogeneous equation (11) are eigenfunctions of \( J \) and \( J^z \).

However, \( 1, \ L, \) and \( S \) separately do not commute with \( g_0^{-1} \). To proceed, we define sixteen Dirac plane-wave basis functions,

\[
\chi^{\rho_1 \rho_2}_{s_1 s_2; M_J}(p, P) \equiv u^{\rho_1}_{1}(\rho_1 p_1)u^{\rho_2}_{2}(\rho_2 p_2)\mathcal{Y}^{M_J}_{s_1 s_2}(\phi),
\]

where \( \mathcal{Y}^{M_J}_{s_1 s_2}(\phi) = e^{i(M_J - s_1 - s_2)\phi}|s_1\rangle|s_2\rangle \),

and \( s_i = \pm \frac{1}{2} \) and \( J^z\mathcal{Y}^{M_J} = M_J\mathcal{Y}^{M_J} \).

Because the usual partial-wave analysis is inapplicable, the homogeneous equation is solved in three dimensions using the basis functions (17) with only \( \phi \) integrations carried out analytically. Radial and polar angle integrations are performed numerically. The homogeneous equation is solved for \( M_J = 0 \) at fixed values of total momentum using the Malfliet-Tjon iteration procedure [13]. Wave functions with polarization states \( M_J = \pm 1 \) are obtained from the \( M_J = 0 \) state by using the raising and lowering operator, \( \psi^{M_J \pm 1} = \sqrt{(J + M_J)(J - M_J + 1)} J^\pm \psi^{M_J} \).

Variation of the quasipotential with total momentum is approximated in a very simple manner, \( \hat{V}(p' - p, P = \pm \frac{1}{2}q) = \hat{V}(p' - p)/\lambda(q^2) \) where \( \lambda(q^2) \) is fit to produce the correct deuteron total energy, \( \epsilon_D = (M_D^2 + q^2/4)^{1/2} \). Figure 2 shows that the required change of the potential is modest, with \( \lambda \) varying linearly over a wide range of \( q^2 \) values. When \( \lambda(q^2) = 1 \) is used, the potential is too attractive and the binding energy of the deuteron increases from \( 2m_N - M_D \approx 2.2 MeV \) at \( q^2 = 0 \) to \( 2m_N - M_D \approx 4.2 MeV \) at \( q^2 = 200 fm^{-2} \). The difference in the deuteron form factors produced by setting \( \lambda(q^2) = 1 \) is minor in comparison with the ambiguity in the boost of rest frame wave functions. Note that with the instant constraint \( v^{(OBE)} \) as well as \( g_0 \) and \( J_{IA} \) are non-singular.

Results for the deuteron magnetic form factor are shown in Figure 3. The solid line result includes the impulse approximation plus \( \rho \pi\gamma, \omega\sigma\gamma, \) and \( \omega\eta\gamma \) meson-exchange-currents, calculated with the instant wave functions and current operators. We use the same meson-exchange-current operators as Hummel and Tjon [11,14,15] with \( g_{\rho\pi\gamma} = .563, g_{\omega\sigma\gamma} = -.4, g_{\omega\eta\gamma} = -.206 \). To illustrate the importance of the dynamical boost, we compare the impulse approximation contributions based on solving Eq. (16) (dotted line) and two approximations based on using the rest frame wave functions with a kinematical boost as follows,

\[
\psi(p, P) = \Lambda_1(\mathcal{L})\Lambda_2(\mathcal{L})\psi(p_{rest}, P_{rest}),
\]

where \( P_{rest} = (m_D, 0) \), \( P = (\epsilon_D, \pm q/2) \), and \( \mathcal{L}P_{rest} = P \). The Lorentz transform of the relative momenta, \( \mathcal{L}p_{rest} = p \), is ambiguous since it is not possible to simultaneously satisfy both constraints: \( (p^0 = 0) \) and \( (p^0_{rest} = 0) \). In Figure 3, the dashed line is the result of satisfying \( (p^0 = 0) \), while the dash-dotted line is the result of satisfying \( (p^0_{rest} = 0) \).

The instant quasipotential formalism in the Breit frame provides in principal a solution to the long-standing problem of how to boost three-dimensional wave functions with the instant constraint. The significant differences in the results based on instant wave functions
calculated directly in the Breit frame and the two approximations to them suggest the importance of the formalism of this paper.

We wish to thank Dr. A. Delfino for helpful discussions on the Malfliet-Tjon procedure. Support for this work by the U.S. Department of Energy under grant DE-FG02-93ER-40762 is gratefully acknowledged.
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FIGURES

FIG. 1. Lowest order impulse-approximation diagram with initial and final vertex functions $\Gamma_i(p, P)$ and $\Gamma_f(p', P')$.

FIG. 2. The scaling of the potential, $\hat{V}(p' - p, P = \pm \frac{1}{2}q) = \hat{V}(p' - p)/\lambda(q^2)$, that produces constant deuteron mass, $M_D = (c_D^2 - P^2)^{1/2} = 2m_N - 2.22464 MeV$: full propagator (solid), ++ states only (dotted).

FIG. 3. Elastic e-d magnetic form factor: consistent calculation with IA+MEC (solid). Impulse approximation only: consistent calculation (dotted), boost approximations with $p^0(Breit) = 0$ (dashed) and $p^0(cm) = 0$ (dash-dotted). See text.
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Unlike some previous formulations [7–9], Eqs. (7) and (8) use a single constraint consistent with four-momentum conservation, and contain propagation of negative- as well as positive-energy nucleon states. To extract the electromagnetic matrix element for elastic scattering.
from the bound state, one substitutes Eq. (2) into Eq. (7). The desired electromagnetic matrix element is proportional to the residue of the double pole term in the resulting expression,
\[ \langle J^\mu \rangle = -i G_0^{QF} K_0^{QP} F_0^{QP} \Gamma /2 \sqrt{P_0 P_0^0}, \]
with implied four-dimensional integrations.

Instant constraints in the Breit frame may be expressed covariantly as \( p \cdot \vec{P} = p' \cdot \vec{P} = 0 \), where \( P \) and \( P' \) are the initial and final total momenta, and \( \vec{P} = (P' + P) / \sqrt{(P' + P)^2} = (1, 0) \) in the Breit frame. They imply that the wave functions needed are a different slice of the four-dimensional wave function at each different value of \( P \). Other choices for the quasipotential constraint are either inconsistent with momentum conservation or with particle-exchange symmetry. For example, the instant constraint in the rest frame of the initial and final bound states is not consistent with conservation of four-momentum of the virtual photon for electromagnetic interactions. Momentum conservation requires \( P' = P + q \) and \( p' = p \pm \frac{1}{2}q \), depending on which particle absorbs the virtual photon momentum, \( q \). Thus, \( p' \cdot P' = p \cdot P + p \cdot q \pm \frac{1}{2}q \cdot (P + q) \), which is not consistent with both \( p' \cdot P' = 0 \) and \( p \cdot P = 0 \) at all values of relative momentum (which is integrated).

An alternative is to place one particle on-shell [5], say particle two in Figure 1. This is consistent with momentum conservation when the virtual photon is absorbed by particle 1, but not when it is absorbed by particle 2. Thus there is an inconsistency with particle exchange symmetry, although for absorption by the on-shell particle, it has been argued in Ref. [10] that the difficulty can be overcome.

We conclude that conservation of four-momentum and particle exchange symmetry suggest use of the instant constraint in the Breit frame. We turn now to deriving the current operator and wave functions based on Eq. (8).

The impulse-approximation current operator for particle one may be obtained by expanding \( K_0^{QP} \) into diagrams, with the leading order terms given by attaching the photon to particle one, together with zero, one, or two exchanged bosons. We have found that these leading diagrams yield an impulse current operator which is consistent with the one-boson-exchange approximation for the quasipotential, \( K^{QP} \approx -i v^{OBE} \). The homogeneous equation for the vertex, \((1 - K^{QP(OBE)}) F_0^{QP}) \Gamma = 0 \), is used to simplify the current with the result that only the box and crossed-box five-point contributions need be evaluated to obtain the current [11]. The propagator that is consistent with this analysis is:

\[ G_0^{QP}(p, P) = ig_0(p, P)(2\pi) \delta(p^0), \]  

where

\[ g_0(p, P) = \sum_{\rho_1, \rho_2} \frac{\Lambda_{\rho_1}^{\rho_1}(p_1) \Lambda_{\rho_2}^{\rho_2}(p_2)}{(\rho_1 + \rho_2)(\epsilon_B/2) - \epsilon_1 - \epsilon_2}, \]
\[ \rho_i = \pm \rho_i \Lambda^0_i(\mathbf{p}_i)\gamma^0_i \] are projection operators for Dirac spinors with hermitian norm, \( \epsilon_i \equiv \sqrt{m_N^2 + \mathbf{P}_i^2} \) for particle 1, and \( \epsilon_D \equiv \sqrt{M_D^2 + \mathbf{P}^2} = P^0 \). Masses are \( m_N \) (nucleon) and \( M_D \) (deuteron). This propagator is similar to the Dirac two-body propagator of Ref. [3], except for the use of the instant constraint in the Breit frame instead of the equally-off-shell constraint.

The five-point and crossed-box diagrams are reduced to three dimensional form in the Breit frame using a procedure similar to Ref. [3]. The crossed-box is evaluated using a form of the eikonal approximation, and the relative energy of both five-point diagrams is integrated with the retardation of the potentials neglected. The result is,

\[ K_{5}^{QP(IA)}(k', P'; k, P) = \int \frac{d^3p}{(2\pi)^3} v^{OBE}(k', p') \; g_0(p'; P') \; \hat{J}_{IA}(1) \; g_0(p; P) \; v^{OBE}(p, k) \]  

(11)

where the impulse-approximation current for particle one is,

\[ \hat{J}_{IA}(1) \equiv \Gamma^\mu_1 \gamma_2 \rho_2 \left( 1 + \frac{1 - \rho_1 \rho_2 2 \epsilon_2 - \rho_2 \epsilon_D}{\epsilon_1 + \epsilon_1} \right), \]  

(12)

\[ \rho_i \pm \Delta_i(\mathbf{p}_i) = \pm \Delta_i(\mathbf{p}_i), \] and \( \Gamma^\mu_1 \) is the one-body electromagnetic operator. The Breit frame total momenta are \( P' = (\epsilon_D, \frac{1}{2} \mathbf{q}) \) and \( P = (\epsilon_D, -\frac{1}{2} \mathbf{q}) \).

Using \( K_{5}^{QP(IA)} \) in \( \langle J^\mu \rangle = \Gamma g_0 K_{5}^{QP} g_0 \Gamma / 2 \sqrt{P^0 P_0}, \) and \( \psi \equiv g_0 \Gamma = g_0 v^{OBE} g_0 \Gamma \), the current matrix element for elastic scattering takes the form,

\[ \langle J_{IA}^\mu(1) \rangle = \frac{1}{2\epsilon_D} \int \frac{d^3p}{(2\pi)^3} \; \overline{\psi}(\mathbf{p}) + \frac{1}{2} \mathbf{q} \cdot \mathbf{P'} \; \hat{J}_{IA}^\mu(1) \psi(\mathbf{p}; P) \]  

(13)

A similar term is obtained for photon absorption by particle two. The box-crossed current, \( \hat{J}_{IA}(1) \), obeys an exact isoscalar, two-body, Ward-Takahashi identity,

\[ q \cdot \hat{J}_{IA}(1) = g_0^{-1}(p'; P') - g_0^{-1}(p; P), \]  

(14)

provided that the one-body current obeys the one-body WT identity, \( q \cdot \Gamma_1 = S_1^{-1}((p'_1) - S_1^{-1}(p_1)) = q \cdot \gamma_1 \). Thus working in the Breit frame and retaining the five-point crossed-box allows for a gauge invariant quasipotential analysis.

The required wave functions may not in general be obtained by a kinematical boost of rest frame wave functions. Instead they must be calculated directly in the Breit frame. Thus the boost problem within the instant quasipotential formalism involves a change of quasipotential constraint. It is a straightforward matter to prove that the quasipotential must change as follows,

\[ K^{QP2} = K^{QP1} + K^{QP1}(C_1^{QP1} - C_0^{QP2})K^{QP2}, \]  

(15)

where, for example, QP1 may refer to the instant constraint in the rest frame, \( C_1(p) = p \cdot P / \sqrt{P^2} \), and QP2 may refer to the Breit frame instant constraint, \( C_2(p) = p \cdot \bar{P} \). Equation (15) defines formally how the interaction kernel changes when a boost from the rest frame to the Breit frame is performed, such that the same underlying covariant kernel applies in the four-dimensional formalism.
Calculations of the instant wave functions in the Breit frame are based on solving,

\[ g_0^{-1}(p'; p) \psi(p'; P) = \int \frac{d^3p}{(2\pi)^3} v(p', p; P) \psi(p; P), \]

for \( P = \pm q/2 \). An appropriate normalization condition is used. A one-boson-exchange potential is used with scalar, pseudo-vector, and vector mesons (\( \sigma, \delta, \eta, \pi, \omega, \rho \)) to obtain deuteron wave functions using modified Bonn B parameters [11,12].

The Breit frame total angular momentum operator is, \( \mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 = \mathbf{L} + \mathbf{S} \), where \( \mathbf{L} = 1 + \mathbf{L}, \mathbf{L} = \mathbf{r} \times \mathbf{p}, \mathbf{L} = \mathbf{R} \times \mathbf{P} \), and \( \mathbf{S} = \frac{1}{2}(\mathbf{\sigma}_1 + \mathbf{\sigma}_2) \). Because \( \mathbf{J} \) and \( J^z \) commute with \( g_0^{-1} \) and \( v \), solutions of the homogeneous equation (16) are eigenfunctions of \( \mathbf{J} \) and \( J^z \). However, \( \mathbf{L}, \mathbf{S} \) separately do not commute with \( g_0^{-1} \). To proceed, we define sixteen Dirac plane-wave basis functions,

\[ \chi_{p_1, p_2; M_J}(p, P) \equiv u_{p_1}^i(p_1 p_1)u_{p_2}^j(p_2 p_2)\mathcal{Y}_{s_1, s_2}^{M_J}(\phi), \]

where

\[ \mathcal{Y}_{s_1, s_2}^{M_J}(\phi) = e^{i(M_J-s_1-s_2)\phi}|s_1\rangle|s_2\rangle, \]

and \( s_i = \pm \frac{1}{2} \) and \( J^z \mathcal{Y}_{s_1, s_2}^{M_J} = M_J \mathcal{Y}_{s_1, s_2}^{M_J} \).

Because the usual partial-wave analysis is inapplicable, the homogeneous equation is solved in three dimensions using the basis functions (17) with only \( \phi \) integrations carried out analytically. Radial and polar angle integrations are performed numerically. The homogeneous equation is solved for \( M_J = 0 \) at fixed values of total momentum using the Malfliet-Tjon iteration procedure [13]. Wave functions with polarization states \( M_J = \pm 1 \) are obtained from the \( M_J = 0 \) state by using the raising and lowering operator, \( \psi_{M_J \pm 1} = \sqrt{(J+M_J)(J-M_J+1)}J^\pm \psi_{M_J} \).

Variation of the quasipotential with total momentum is approximated in a very simple manner, \( \tilde{V}(p' - p, P = \pm q/2) = \tilde{V}(p' - p)/\lambda(q^2) \) where \( \lambda(q^2) \) is fit to produce the correct deuteron total energy, \( \epsilon_D = (M_D^2 + q^2/4)^{1/2} \). Figure 2 shows that the required change of the potential is modest, with \( \lambda \) varying linearly over a wide range of \( q^2 \) values. When \( \lambda(q^2) = 1 \) is used, the potential is too attractive and the binding energy of the deuteron increases from \( 2m_N - M_D \approx 2.2MeV \) at \( q^2 = 0 \) to \( 2m_N - M_D \approx 4.2MeV \) at \( q^2 = 200fm^{-2} \). The difference in the deuteron form factors produced by setting \( \lambda(q^2) = 1 \) is minor in comparison with the ambiguity in the boost of rest frame wave functions. Note that with the instant constraint \( \psi^{(OBE)} \) as well as \( g_0 \) and \( J_{IA} \) are non-singular.

Results for the deuteron magnetic form factor are shown in Figure 3. The solid line result includes the impulse approximation plus \( \rho \pi \gamma, \omega \sigma \gamma, \) and \( \omega \eta \gamma \) meson-exchange-currents, calculated with the instant wave functions and current operators. We use the same meson-exchange-current operators as Hummel and Tjon [11,14,15] with \( g_{\rho \pi \gamma} = .563, g_{\omega \sigma \gamma} = -.4, g_{\omega \eta \gamma} = -.206 \). To illustrate the importance of the dynamical boost, we compare the impulse approximation contributions based on solving Eq. (16) (dotted line) and two approximations based on using the rest frame wave functions with a kinematical boost as follows,

\[ \psi(p, P) = \Lambda_1(\mathcal{L})\Lambda_2(\mathcal{L})\psi(p_{rest}, P_{rest}), \]
FIG. 2. The scaling of the potential, \( \hat{V}(p' - p, \mathbf{P} = \pm \frac{1}{2} \mathbf{q}) = \hat{V}(p' - p)/\lambda(q^2) \), that produces constant deuteron mass, \( M_D = (c_D^2 - \mathbf{P}^2)^{1/2} = 2m_N - 2.22464 \text{MeV} \); full propagator (solid), \( \uparrow \uparrow \) states only (dotted).

where \( P_{\text{rest}} = (m_D, 0) \), \( P = (c_D, \pm q/2) \), and \( \mathcal{L}P_{\text{rest}} = P \). The Lorentz transform of the relative momenta, \( \mathcal{L}P_{\text{rest}} = p \), is ambiguous since it is not possible to simultaneously satisfy both constraints: \( (p^0 = 0) \) and \( (p^0_{\text{rest}} = 0) \). In Figure 3, the dashed line is the result of satisfying \( (p^0 = 0) \), while the dash-dotted line is the result of satisfying \( (p^0_{\text{rest}} = 0) \).

The instant quasipotential formalism in the Breit frame provides in principal a solution to the long-standing problem of how to boost three-dimensional wave functions with the instant constraint. The significant differences in the results based on instant wave functions calculated directly in the Breit frame and the two approximations to them suggest the importance of the formalism of this paper.

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FIG. 3. Elastic e-d magnetic form factor: consistent calculation with IA+MEC (solid). Impulse approximation only: consistent calculation (dotted), boost approximations with $p^0(Breit) = 0$ (dashed) and $p^0(cm) = 0$ (dash-dotted). See text.

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