AN $L_q(L_p)$-THEORY FOR SPACE-TIME NON-LOCAL EQUATIONS GENERATED BY LÉVY PROCESSES WITH LOW INTENSITY OF SMALL JUMPS

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Abstract. We investigate an $L_q(L_p)$-regularity ($1 < p, q < \infty$) theory for space-time nonlocal equations of the type $\partial^\alpha_t u = Lu + f$. Here, $\partial^\alpha_t$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$ and $L$ is an integro-differential operator

$$Lu(x) = \int_{\mathbb{R}^d} (u(x) - u(x + y) - \nabla u(x) \cdot y 1_{|y| \leq 1}) j_d(|y|)dy$$

which is the infinitesimal generator of an isotropic unimodal Lévy process. We assume that the jump kernel $j_d(r)$ is comparable to $r^{-d}\ell(r)$, where $\ell$ is a continuous function satisfying

$$C_1 \left( \frac{R}{r} \right)^{\delta_1} \leq \ell(\frac{R}{r}) \leq C_2 \left( \frac{R}{r} \right)^{\delta_2} \text{ for } 1 \leq r \leq R < \infty,$$

where $0 \leq \delta_1 \leq \delta_2 < 2$. Hence, $\ell$ can be slowly varying at infinity. Our result covers $L$ whose Fourier multiplier $\Psi(\xi)$ satisfies $\Psi(\xi) \asymp -\log (1 + |\xi|^\beta)$ for $\beta \in (0, 2]$ and $\Psi(\xi) \asymp -(\log(1 + |\xi|^{\beta/4}))^2$ for $\beta \in (0, 2)$ by taking $\ell(r) \asymp 1$ and $\ell(r) \asymp \log (1 + r^\beta)$ for $r \geq 1$ respectively. In this article, we use the Calderón-Zygmund approach and function space theory for operators having slowly varying symbols.

Contents

1. Introduction 2
2. Main results 5
3. Estimates of the heat kernels and their derivatives 14
4. Estimates of the fundamental solutions 18
5. Estimation of solution: Calderón-Zygmund approach 23
6. Proof of Theorem 2.13 32
7. Further discussions 33
Appendix A. Appendix 34
References 41

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1. Introduction

Equations with space or time nonlocal operators are used to model natural phenomena in various area of science (see e.g. \[21\] [11]). For example, time-fractional heat equation $\partial_t^\alpha u = \Delta u$ ($\alpha \in (0, 1)$) describes subdiffusive aspect of anomalous diffusion caused by particle sticking and trapping effect. Also, when we give relativistic correction to Laplacian, then it becomes relativistic Hamiltonian $-(\sqrt{-\Delta} + m^2 - m)$ which is a nonlocal operator generated by relativistic stable process.

In this article, we consider equations with space-time nonlocal operator

$$\partial_t^\alpha u = \mathcal{L} u + f \quad t > 0,$$

(1.1)

where $\partial_t^\alpha$ is the Caputo fractional derivative of order $\alpha \in (0, 1)$, and $\mathcal{L}$ is an integro-differential operator whose jump kernel can have low intensity of small jumps. Precisely speaking, we have the following representation for $\mathcal{L}$:

$$\mathcal{L} u(x) = \int_{\mathbb{R}^d} (u(x) - u(x + y) - \nabla u(x) \cdot y \chi_{|y| \leq 1} j_d(|y|)) dy,$$

(1.2)

where $j_d(r)$ is decreasing in $r$ and comparable to $r^{-d\ell(r^{-1})}$, and $\ell$ is a positive continuous function which satisfies

$$C_1 \left( \frac{R}{r} \right)^{\delta_1} \leq \ell(R) \leq C_2 \left( \frac{R}{r} \right)^{\delta_2}$$

for $1 \leq r \leq R < \infty$,

where $0 \leq \delta_1 \leq \delta_2 < 2$. We remark that $\ell$ can be slowly varying at infinity (see Definition 2.5 and Assumption 2.7 for detail) due to the possible choice of $\delta_1 = 0$. Therefore, $j_d$ has low intensity of small jumps. Examples of function $\ell$ covered by our result are

$$\ell(r) = r^\beta, \quad \ell(r) = r^\beta/(r^\beta + 1), \quad \ell(r) = \log(1 + r^\beta)$$

for $\beta \in (0, 2)$.

It is known that $\mathcal{L}$ can be considered as a linear operator with symbol $-\psi$, where $\psi$ is the Lévy-Khintchine exponent of Lévy process whose jump kernel is $j_d(|y|)$. Our spatial nonlocal operator is the infinitesimal generator of pure jump Lévy process whose transition density function $p_d(t, x)$ is the fundamental solution to parabolic equation $u_t = \mathcal{L} u$. Under our assumptions, the heat kernel for jump processes which generates $\mathcal{L}$ has different type of estimation for small time and large time (see Section 3). Moreover, since we are dealing with equations with time-fractional derivatives, the fundamental solution to (1.1) is the transition density of time changed process by an inverse subordinator (see Section 4). This implies that our estimation for the fundamental solution needs more exquisite analysis.

In the literature, equations with non-local operators in time variables have been widely studied. See for example, [13] [23] [22] [31]. Regarding (parabolic) equations with spatial nonlocal operator $u_t = Lu + f$, where $L$ is of the form

$$Lu(t, x) = \int_{\mathbb{R}^d} \left( u(t, x + y) - u(t, x) - \nabla_x u(t, x) \cdot y \chi^{(\sigma)}(y) \right) J(t, x, y) dy,$$

an $L_p$-estimation of solution was introduced in [24]. Here, $\chi^{(\sigma)}$ is a function depending on $\sigma \in (0, 2)$ and $J(t, x, y) = a(t, x, y)|y|^{-d-\sigma}$, where $a(x, y)$ is homogeneous of order zero and sufficiently smooth in $y$. Most of the studies focus on $J$ which generalizes $a(t, x, y)$. See e.g. [13] [23] [22] [31]. Quite recently, [9] proved an $L_p$-estimation
of solution to equations
\[ \partial_t^\alpha u = Lu + f, \]
where \( \alpha \in (0, 1] \) (i.e., the result also covers parabolic case) and \( J \) is comparable to
\[ |y|^{-d-\sigma} \]
uniformly in \((t, x)\) and Hölder continuous in \( x \) uniformly in \((t, y)\).

An \( L_q(L_p) \)-regularity result was introduced in [25, Theorem 8.7] and [16]. The result in [25, Theorem 8.7] deals with abstract parabolic Volterra equations of the form
\[ u(t) + \int_0^t a(t - s)Au(s)ds = f(t), \]
where \( a \) is locally integrable function and \( A \) is densely defined closed operator on \( L_p \).

The class of \( A \) is general and it covers operators \(-\phi(-\Delta)\) for Bernstein functions \( \phi \). In [16], the following equation is studied:
\[ \partial_t^\alpha u = -\phi(-\Delta)u + f, \quad t > 0; \quad u(0, \cdot) = u_0, \quad (1.3) \]
where \( \alpha \in (0, 1) \) and \( \phi \) is a Bernstein function satisfying
\[ c \left( \frac{R}{r} \right) \delta_0 \leq \frac{\phi(R)}{\phi(r)} \quad \text{for all} \quad 0 < r \leq R < \infty \quad (c > 0, \quad \delta_0 \in (0, 1]). \quad (1.4) \]

The authors in [16] only used elementary analysis, based on estimation of the heat kernel \( p_d(t, x) \)
\[ |p_d(t, x)| \leq C \left( \phi^{-1}(t^{-1})^{d/2} \wedge \frac{t\phi(|x|^{-2})}{|x|^d} \right), \quad (1.5) \]
(see [14] [16]) which highly depends on scaling condition (1.4).

Although [25, Theorem 8.7] and [16] can cover a large class of operators, it is hard to see if we can apply the results to an integro-differential operator \( \mathcal{L} \) of the form (1.2). To apply [25, Theorem 8.7] or [16], \( \mathcal{L} \) should satisfy the condition in [25, Theorem 8.7] or the symbol \( \psi \) of \( \mathcal{L} \) is written as \( \psi(\xi) = \phi(|\xi|^2) \) for a Bernstein function \( \phi \). However, it seems that it is complicated to check whether the operator \( \mathcal{L} \) (or its symbol \( \psi \)) satisfies the above conditions. In fact, [25, Theorem 8.7] requires a comprehensive background in abstract harmonic analysis to check the conditions therein even for \( A = -\phi(-\Delta) \) (see e.g., [17, Section 3.2]). Moreover, it is difficult to check that \( \psi \) satisfies \( \psi(\xi) = \phi(|\xi|^2) \) for a Bernstein function \( \phi \). To the best of the author’s knowledge, the only known direct relation between \( \psi \) and \( j_d \)
\[ \psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x))j_d(|x|)dx, \]
which follows from Lévy-Khintchine formula, and it is hard to obtain closed form of \( \psi \) from the above integral in general. Even if we know closed form of \( \psi \), it is another problem to check that \( \psi \) can be written as \( \psi(\xi) = \phi(|\xi|^2) \). Thus, it is not easy to see if we can apply [25, Theorem 8.7] or [16] to the \( \mathcal{L} \) in (1.2). In addition, \( \phi \) should satisfy scaling condition (1.4) to apply [16].

Motivated by the above observation, this paper aims to find general conditions on \( \mathcal{L} \) that give \( L_q(L_p) \)-regularity of solutions to (1.1). This extends the results in [25, Theorem 8.7] and [16] in the following two aspects:

(1) Our results require elementary analysis and cover more general operators.

i.e., the symbol \( \psi \) of operator needs not satisfy \( \psi(\xi) = \phi(|\xi|^2) \) for a Bernstein function \( \phi \). Indeed, we do not need neither closed form of \( \psi \) nor smoothness of \( \psi \).
(2) Also, our result does not need scaling condition \(1.4\) even for \(\mathcal{L} = -\phi(-\Delta)\).

To achieve this goal, we give assumptions on \(j_d\) instead of symbol of operator \(\mathcal{L}\) in \(1.2\). Indeed, our assumption on \(j_d\) is quite general so that our results cover all operators of the form \(-\phi(-\Delta)\) for \(\phi(r) = \log(1 + r^\beta)\) with \(\beta \in (0,1]\), and for a Bernstein function \(\phi\) satisfying \(1.3\) and \(\phi(R)/\phi(r) \leq c(R/r)^{\delta_0}\) for all \(0 < r < R < \infty\), where \(0 < \delta_0 \leq \delta_0' < 1\) (see \([2,18,19]\)).

It is worth mentioning that our approach and \([16,25]\) have their own advantages. If the symbol \(\psi\) of an operator \(\mathcal{L}\) is explicitly given as \(\psi(\xi) = \phi(|\xi|^2)\) for a Bernstein function \(\phi\), then \(1.1, 25\) is more accessible since one needs not to know asymptotic behavior of jump kernel of the operator \(\mathcal{L}\). On the contrary, if an operator \(\mathcal{L}\) is given as \(1.2\), then our approach is more accessible since we do not have to obtain exact value of symbol of \(\mathcal{L}\).

In this paper, we investigate maximal \(L_q(L_p)\)-regularity of solutions to (1.1), where \(\mathcal{L}\) is an integro-differential operator generated by an isotropic unimodal pure jump \(\text{Lévy}\) process \(X\) in \(\mathbb{R}^d\). As mentioned above, we assume that the jump kernel of the \(\text{Lévy}\) process \(X\) is comparable to \(|x|^{-d}\ell(|x|^{-1})\) and consider weak scaling conditions on \(\ell\) which cover the case that \(\ell\) is a slowly varying function at infinity. In this case, we may have \(\sup_x p_d(t,x) = \infty\) and thus we cannot have \(1.3\) for our heat kernel.

To obtain our main results, we follow the standard approach in harmonic analysis. Precisely, we control the sharp function of derivative of solution in terms of \(L_\infty\)-norm of free term \(f\), and then use the Fefferman-Stein theorem and the Caldéron-Zygmund theorem. Main difficulty arises here since \(X\) is an isotropic unimodal \(\text{Lévy}\) process which is a more general process than the one in \([16]\). Here we give a short description. Unlike \([16]\), we cannot expect global scaling condition like \(1.3\) to underlying functions related to \(X\). Hence, \(p_d(t,x)\) has different form of estimation comparing to \(1.5\) and thus our proof is much more involved. Indeed, if \(\limsup_{r \to \infty} \ell(r) = \infty\), then \(\ell\) gives the borderline between near and off diagonal estimates. Since the scaling function for parabolic cube is \(\psi\), and the two functions \(\ell\) and \(\psi\) may not be comparable, we need more delicate argument.

We finish the introduction with some notations. We use “:=” or “=:” to denote a definition. The symbol \(\mathbb{N}\) denotes the set of positive integers and \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\). Also, we use \(\mathbb{Z}\) to denote the set of integers. As usual \(\mathbb{R}^d\) stands for the Euclidean space of points \(x = (x^1, \ldots, x^d)\). We set

\[
B_r(x) := \{y \in \mathbb{R} : |x - y| < r\}, \quad \mathbb{R}^{d+1}_+ := \{(t,x) \in \mathbb{R}^{d+1} : t > 0\}.
\]

For \(i = 1, \ldots, d\), multi-indices \(\sigma = (\sigma_1, \ldots, \sigma_d)\), and functions \(u(t,x)\) we set

\[
\partial_{x^i} u = \frac{\partial u}{\partial x^i} = D_{i} u, \quad D^\sigma u = D_1^{\sigma_1} \cdots D_d^{\sigma_d} u, \quad |\sigma| = \sigma_1 + \cdots + \sigma_d.
\]

We also use the notation \(D^m_{\sigma}\) for arbitrary partial derivatives of order \(m\) with respect to \(x\). For an open set \(\mathcal{O}\) in \(\mathbb{R}^d\) or \(\mathbb{R}^{d+1}\), \(C_\infty^c(\mathcal{O})\) denotes the set of infinitely differentiable functions with compact support in \(\mathcal{O}\). By \(\mathcal{S} = \mathcal{S}(\mathbb{R}^d)\) we denote the class of Schwartz functions on \(\mathbb{R}^d\). For \(p > 1\), by \(L_p\) we denote the set of complex-valued Lebesgue measurable functions \(u\) on \(\mathbb{R}^d\) satisfying

\[
\|u\|_{L_p} := \left(\int_{\mathbb{R}^d} |u(x)|^p \, dx \right)^{1/p} < \infty.
\]
Generally, for a given measure space \((M, \mathcal{M}, \mu), L^p(M, \mathcal{M}, \mu; F)\) denotes the space of all \(F\)-valued \(\mathcal{M}^\sigma\)-measurable functions \(u\) so that

\[
\|u\|_{L^p(M, \mathcal{M}, \mu; F)} := \left( \int_M \|u(x)\|^p \mu(dx) \right)^{1/p} < \infty,
\]

where \(\mathcal{M}^\sigma\) denotes the completion of \(\mathcal{M}\) with respect to the measure \(\mu\). If there is no confusion for the given measure and \(\sigma\)-algebra, we usually omit the measure and the \(\sigma\)-algebra. We denote \(a \wedge b := \min\{a, b\}\) and \(a \vee b := \max\{a, b\}\). By \(\mathcal{F}_d\) and \(\mathcal{F}_d^{-1}\) we denote the \(d\)-dimensional Fourier transform and the inverse Fourier transform respectively, i.e.

\[
\mathcal{F}_d(f)(\xi) := \hat{f}(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x)dx, \quad \mathcal{F}_d^{-1}(f)(\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} f(x)dx.
\]

For two real-valued functions \(f, g\) defined on a set \(A\), we write \(f \asymp g\) on \(A\) if there is a constant \(c > 1\) such that \(c^{-1} f(b) \leq g(b) \leq cf(b)\) for all \(b \in A\). Finally if we write \(C = C(\ldots)\), this means that the constant \(C\) depends only on what are in the parentheses. The constant \(C\) can differ from line to line.

2. Main Results

In this section, we introduce our main results. We first present our spatial nonlocal operator \(\mathcal{L}\). Let \(X = (X_t, t \geq 0)\) be a Lévy process on \(\mathbb{R}^d\) with Lévy-Khintchine exponent \(\psi\). Then,

\[
\mathbb{E} e^{i\xi \cdot X_t} = \int_{\mathbb{R}^d} e^{i\xi \cdot x} p_d(t, dx) = e^{-t\psi(\xi)},
\]

where \(p_d(t, dx)\) is the transition probability of \(X_t\). If \(X\) is a pure jump symmetric Lévy process with Lévy measure \(j_d\), then \(\psi\) is of the form

\[
\psi(\xi) = \psi_X(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x)) j_d(dx),
\]

where \(\int_{\mathbb{R}^d} (1 \wedge |x|^2) j_d(dx) < \infty\).

A measure \(\mu(dx)\) is isotropic unimodal if it is absolutely continuous on \(\mathbb{R}^d \setminus \{0\}\) with a radial and radially decreasing density. A Lévy process \(X\) is isotropic unimodal if \(p_d(t, dx)\) is isotropic unimodal for all \(t > 0\). This is equivalent to the condition that the Lévy measure \(j_d(dx)\) of \(X\) is isotropic unimodal if \(X\) is pure jump Lévy process (see [25]).

Throughout this paper, we always assume that \(X\) is a pure jump isotropic unimodal Lévy process with the Lévy-Khintchine exponent \(\psi\). With a slight abuse of notation, we will use the notations \(\psi(|x|) = \psi(x)\) and \(j_d(dx) = j_d(x)dx = j_d(|x|)dx\) for \(x \in \mathbb{R}^d\).

For \(f \in S(\mathbb{R}^d)\), define a linear operator \(\mathcal{L}\) as

\[
\mathcal{L}f(x) = \int_{\mathbb{R}^d} (f(x) - f(x + y) - \nabla f(x) \cdot y 1_{|y| \leq 1}) j_d(dy).
\]

Due to the Lévy-Khintchine representation, \(\psi\) is continuous and negative definite (see [10, Theorem 1.1.5]). Thus, by [10, Proposition 2.1.1] we can understand \(\mathcal{L}\) as the infinitesimal generator of \(X\), and nonlocal operator with Fourier multiplier \(-\psi(|\xi|)\). Precisely speaking, for \(f \in S(\mathbb{R}^d)\), we have the following relation

\[
\mathcal{L}f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E} f(x + X_t) - f(x)}{t} = \mathcal{F}^{-1}(-\psi(|\xi|) \mathcal{F}(f)(\xi))(x). \quad (2.6)
\]
In this context, we also use notations \( L_\psi \) or \( L_X \) instead of \( L \) in this article.

One of well-known examples of isotropic unimodal \( \text{L} \)evy process is subordinate Brownian motion \( Y = Y_t, t \geq 0 \) which is defined by \( Y_t := B_{S_t} \). Here \( B = (B_t, t \geq 0) \) is a \( d \)-dimensional Brownian motion and \( S = (S_t, t \geq 0) \) is a subordinator (i.e., 1-dimensional increasing \( \text{L} \)evy process) independent of \( B \). It is known that there is a Bernstein function \( \phi : \mathbb{R}_+ \to \mathbb{R}_+ \) (i.e., \( (-1)^n \phi^{(n)}(x) \leq 0 \) for all \( n \in \mathbb{N} \), where \( \phi^{(n)} \) is the \( n \)-th derivative of \( \phi \)) satisfying \( \mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)} \). If \( \phi(0^+) = 0 \), then \( \phi \) has the following representation

\[
\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda t})\mu(dt),
\]

where \( b \geq 0 \) and \( \mu \) is a measure satisfying \( \int_{(0,\infty)}(1 + t)\mu(dt) < \infty \). \( \mu \) is called the \( \text{L} \)evy measure of \( \phi \) (see [25]).

It is well-known that the \( \text{L} \)evy-Khintchine exponent of \( Y \) is \( \xi \mapsto \phi(|\xi|^2) \) and the \( \text{L} \)evy measure of \( Y \) has the density \( J_d(x) = J_d(|x|) \), where

\[
J_d(r) = \int_{(0,\infty)} (4\pi t)^{-d/2}e^{-\lambda^2/4t}\mu(dt).
\]

Hence, \( L_Y \) also has representation (2.6) with \( J_d \) and \( \phi(|\cdot|^2) \) in place of \( j_d \) and \( \psi \).

For example, by taking \( \phi(\lambda) = \lambda^{\beta/2} (\beta \in (0,2)) \), we obtain the fractional Laplacian \( \Delta^{\beta/2} = (-\Delta)^{\beta/2} \), which is the infinitesimal generator of a rotationally symmetric \( \beta \)-stable process in \( \mathbb{R}^d \).

In order to describe the regularity of solution, we introduce Sobolev space related to the operator \( L \). For \( \gamma \in \mathbb{R} \), and \( u \in S(\mathbb{R}^d) \), define linear operators

\[
(-L)^{\gamma/2} = (-L_\psi)^{\gamma/2}, \quad (1-L)^{\gamma/2} = (1 - L_\psi)^{\gamma/2}
\]

as follows

\[
\mathcal{F}\{(-L)^{\gamma/2}u\} = (\psi(|\xi|))^{\gamma/2}\mathcal{F}(u)(\xi), \quad \mathcal{F}\{(1-L)^{\gamma/2}u\} = (1 + \psi(|\xi|))^{\gamma/2}\mathcal{F}(u)(\xi).
\]

For \( 1 < p < \infty \), let \( H_p^{\psi;\gamma} \) be the closure of \( S(\mathbb{R}^d) \) under the norm

\[
\|u\|_{H_p^{\psi;\gamma}} := \|\mathcal{F}^{-1}\{(1 + \psi(|\cdot|))^{\gamma/2}\mathcal{F}(u)(\cdot)\}\|_{L_p} < \infty.
\]

Then from the definition of \( H_p^{\psi;\gamma} \) the operator \( (1 - L)^{\gamma/2} \) can be extended from \( S(\mathbb{R}^d) \) to \( L_p \). Throughout this article, we use the same notation \( (1 - L)^{\gamma/2} \) for this extension. For more information, see e.g. [10]. Also note that if \( \psi(|\xi|) = |\xi|^2 \), then \( H_p^{\psi;\gamma} \) is a standard Bessel potential space \( H_p^\gamma \) and \( H_p^{\psi;0} = L_p \) due to the definition.

The following lemma is a collection of useful properties of \( H_p^{\psi;\gamma} \).

**Lemma 2.1.** Let \( 1 < p < \infty \) and let \( \gamma \in \mathbb{R} \).

(i) The space \( H_p^{\psi;\gamma} \) is a Banach space.

(ii) For any \( \mu \in \mathbb{R} \), the map \( (1-L)^{\mu/2} \) is an isometry from \( H_p^{\psi;\gamma} \) to \( H_p^{\psi;\gamma - \mu} \).

(iii) If \( \mu > 0 \), then we have continuous embeddings \( H_p^{\psi;\gamma + \mu} \subset H_p^{\psi;\gamma} \) in the sense that

\[
\|u\|_{H_p^{\psi;\gamma}} \leq C\|u\|_{H_p^{\psi;\gamma + \mu}},
\]

where the constant \( C \) is independent of \( u \).

(iv) For any \( u \in H_p^{\psi;\gamma + 2} \), we have

\[
(\|u\|_{H_p^{\psi;\gamma}} + \|Lu\|_{H_p^{\psi;\gamma}}) \asymp \|u\|_{H_p^{\psi;\gamma + 2}}. \tag{2.8}
\]
Proof. The first and second assertions are direct consequences of the definition. Recall that \( \psi \) is a continuous negative definite function. Hence, the third assertion comes from [10] Theorem 2.3.1. Finally, the last assertion can be obtained by using the second assertion and [10] Theorem 2.2.7. \( \square \)

Now we introduce our non-local operator in time variable and related definitions. For \( \alpha > 0 \) and \( \varphi \in L_1((0, T)) \), the Riemann-Liouville fractional integral of the order \( \alpha \) is defined as

\[
I^\alpha_t \varphi := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \varphi(s) \, ds, \quad 0 \leq t \leq T.
\]

We also define \( I^0_0 \varphi := \varphi \). Take \( n \in \mathbb{N} \) such that \( \alpha \in [n-1, n) \). If \( \varphi(t) \) is \( (n-1) \)-times differentiable and \( \left(\frac{d}{dt}\right)^{n-1} I^{n-\alpha}_t \varphi \) is absolutely continuous on \([0, T]\), then the Riemann-Liouville fractional derivative \( D^\alpha_t \varphi \) and the Caputo fractional derivative \( \partial^\alpha_t \varphi \) are defined as

\[
D^\alpha_t \varphi := \left(\frac{d}{dt}\right)^n (I_t^{n-\alpha} \varphi), \quad (2.9)
\]

and

\[
\partial^\alpha_t \varphi = D^\alpha_t \varphi \left(\varphi(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \varphi^{(k)}(0)\right).
\]

Using Fubini's theorem, we see that for any \( \alpha, \beta \geq 0 \),

\[
I^\alpha_t I^\beta_t \varphi = I^{\alpha+\beta}_t \varphi, \quad (a.e.) \quad t \leq T. \quad (2.10)
\]

Note that \( D^\alpha_t \varphi = \partial^\alpha_t \varphi \) if \( \varphi(0) = \varphi^{(1)}(0) = \cdots = \varphi^{(n-1)}(0) = 0 \). By (2.10) and (2.9), if \( \alpha, \beta \geq 0 \),

\[
D^\alpha_t D^\beta_t \varphi = D^{\alpha+\beta}_t \varphi, \quad D^\alpha_t I^\beta_t \varphi = D^{\alpha-\beta}_t \varphi,
\]

where we define \( D^\alpha_t \varphi := I^{-\alpha}_t \varphi \) for \( \alpha < 0 \). Also if \( \varphi(0) = \varphi^{(1)}(0) = \cdots = \varphi^{(n-1)}(0) = 0 \) then by definition of \( \partial^\alpha_t \),

\[
I^\alpha_t \partial^\alpha_t u = I^\alpha_t D^\alpha_t u = u.
\]

For \( p, q \in (1, \infty), \gamma \in \mathbb{R} \) and \( T < \infty \), we denote

\[
H^\gamma_{q,p}(T) := L_q \left((0, T); H^\gamma_{q,p}\right), \quad L_{q,p}(T) := H^0_{q,p}(T).
\]

We write \( u \in C^{\alpha,\infty}_p([0, T] \times \mathbb{R}^d) \) if \( D^m_x u, \partial_t^\alpha D^m_x u \in C([0, T]; L_p) \) for any \( m \in \mathbb{N}_0 \). Also \( C^{\infty}_p(\mathbb{R}^d) = C^\infty_p \) denotes the set of functions \( u_0 = u_0(x) \) such that \( D^m_x u_0 \in L_p \) for any \( m \in \mathbb{N}_0 \).

**Definition 2.2.** Let \( \alpha \in (0, 1), 1 < p, q < \infty, \gamma \in \mathbb{R}, \) and \( T < \infty \).

(i) We write \( u \in H^\gamma_{q,p,\alpha,\gamma+2}(T) \) if there exists a sequence \( u_n \in C^{\alpha,\infty}_p([0, T] \times \mathbb{R}^d) \) satisfying

\[
\|u - u_n\|_{H^\gamma_{q,p,\alpha,\gamma+2}(T)} \to 0 \quad \text{and} \quad \|\partial^\alpha_t u_n - \partial^\alpha_t u_m\|_{H^\gamma_{q,p}(T)} \to 0
\]
as \( n, m \to \infty \). We call this sequence \( u_n \) a defining sequence of \( u \), and we define

\[
\partial^\alpha_t u = \lim_{n \to \infty} \partial^\alpha_t u_n \text{ in } H^\gamma_{q,p}(T).
\]

The norm in \( H^\gamma_{q,p,\alpha,\gamma+2}(T) \) is naturally given by

\[
\|u\|_{H^\gamma_{q,p,\alpha,\gamma+2}(T)} = \|u\|_{H^\gamma_{q,p}(T)} + \|\partial^\alpha_t u\|_{H^\gamma_{q,p}(T)}.
\]
Assumption 2.6. Let $\lambda > 0$ such that the jumping kernel $j$ satisfies
\[ \|u\|_{H^2_p} \leq C\|u\|_{H^2_p}. \]
Continuing the above argument, we obtain the desired result.

Remark 2.3. (i) Obviously, $H^{q,p,\gamma+2}(T)$ is a Banach space.

(ii) By following the argument in [23, Remark 3], we can show that the embedding $H^{q,\gamma}_p \subset H^{q,\gamma+2}_p$ is continuous for any $n \in \mathbb{N}$. This and Lemma 2.1 (iv) imply that $\|u\|_{H^{q,\gamma+2}_p} \leq C\|u\|_{H^{q,\gamma}_p}$. Hence, by following the proof, we can obtain all the claims.

Lemma 2.4. Let $\alpha \in (0,1), 1 < p, q < \infty, \gamma \in \mathbb{R}$, and $T < \infty$.

(i) The space $H^{\alpha,p,\gamma+2}(T)$ is a closed subspace of $H^{q,p,\gamma+2}(T)$.

(ii) $C_c(\mathbb{R}^{d+1})$ is dense in $H^{q,p,\gamma+2}(T)$.

(iii) For any $\gamma, \nu \in \mathbb{R}$, $(1-L)\nu/2 : H^{q,p,\gamma+2}(T) \rightarrow H^{q,p,\gamma-\nu+2}(T)$ is an isometry and for any $u \in H^{q,p,\gamma+2}(T)$, we have
\[ \partial^\alpha u = (1-L)^{\nu/2}u. \]

Proof. If $\mathcal{L} = -\phi(-\Delta)$ for a Bernstein function $\phi$, then the lemma is a consequence of [10, Lemma 2.7]. Here we emphasize that the proof of [10, Lemma 2.7] is only based on Lemma 2.1 and Remark 2.3 (ii). Hence, by following the proof, we can obtain all the claims.

Definition 2.5. A function $f : (a, \infty) \rightarrow [0, \infty)$, for some $a > 0$, is called slowly varying at infinity if for each $\lambda > 0$
\[ \lim_{x \to \infty} \frac{f(\lambda x)}{f(x)} = 1. \]

Now, we introduce our assumption on the Lévy density $j_d$.

Assumption 2.6. Let $\ell : (0, \infty) \rightarrow (0, \infty)$ be a continuous function. We assume that the jumping kernel $j_d : (0, \infty) \rightarrow (0, \infty)$ is differentiable and satisfies

(i) The function $-\frac{1}{r} \frac{d}{dr} (\frac{j_d}{\ell}) (r)$ is decreasing;

(ii) There exist constants $\kappa_1, \kappa_2 > 0$ such that for any $m = 0, 1$ and $r > 0$,
\[ \kappa_1 r^{-d-m} \ell(r^{-1}) \leq (1-m) \frac{d^m}{dr^m} j_d(r) \leq \kappa_2 r^{-d-m} \ell(r^{-1}). \]

The function $\ell$ gives the intensity of jumps of corresponding Lévy process. Since $j_d(r) \asymp r^{-d} \ell(r^{-1})$ for $r > 0$, the behavior of $\ell$ at infinity (resp. 0) shows the behavior of small (resp. large) jumps. We impose the following condition on $\ell$ in Assumption 2.6.

Assumption 2.7. We assume that $\ell$ in Assumption 2.6 is independent of $d$ and satisfies
\[ C_1 \left( \frac{R}{r} \right)^{\delta_1} \leq \frac{\ell(R)}{\ell(r)} \leq C_2 \left( \frac{R}{r} \right)^{\delta_2} \quad \text{for} \ 1 \leq r \leq R < \infty; \]
with constants $C_1, C_2 > 0$ and $0 \leq \delta_1 \leq \delta_2 < 2$. We also assume that there exists $\delta_3 > 0$ such that
\[ C_1 \left( \frac{R}{r} \right)^{\delta_3} \leq \frac{h(r)}{h(R)} \quad \text{for} \ 1 \leq r \leq R < \infty, \]
where

\[ h(r) := K(r) + L(r) \quad \text{for} \quad r > 0, \]
\[ K(r) := r^{-2} \int_{0}^{r} s(s^{-1})ds, \quad L(r) := \int_{r}^{\infty} s^{-1}(s^{-1})ds \quad \text{for} \quad r > 0. \quad (2.14) \]

In the rest of the article, we use a vector notation \( \delta = (\delta_1, \delta_2, \delta_3) \in \mathbb{R}^3 \) instead of listing \( \delta_i \) (i = 1, 2, 3) for notational convenience.

**Remark 2.8.** (i) By (2.12) we obtain

\[ \liminf_{r \to \infty} \ell(r) > 0, \quad (2.15) \]

which is essential to our approach. If \( \delta_1 < 0 \), (2.15) may not hold. For example, if \( \ell(r) = \frac{1}{\log(1 + r)} \), then \( \delta_1 \) should be negative. It is easy to see that (2.15) does not hold in this case. Thus, we only consider the case that \( \delta_1 \geq 0 \). On the other hand, (2.11) (with \( m = 0 \)) implies that \( \ell \) bounded near zero since \( j_d(|y|)dy \) is a Lévy measure.

(ii) Since the constant \( \delta_1 \) in (2.12) can be 0, the function \( \ell \) can be a slowly varying function at infinity. We can see the relation between the characteristic exponent and the jump kernel of isotropic unimodal Lévy process with low intensity of small jumps in [18, 12].

(iii) Note that \( K, L \) and \( h \) are independent of the dimension \( d \). Clearly, \( L \) is strictly decreasing in \( r \). Under (2.11), for any \( r > 0 \), we have

\[ c(d)\kappa_2^{-1}r^{-2} \int_{|y| \leq r} |y|^2 j_d(|y|)dy \leq K(r) \leq c(d)\kappa_1^{-1}r^{-2} \int_{|y| \leq r} |y|^2 j_d(|y|)dy, \]
\[ c(d)\kappa_2^{-1} \int_{|y| \geq r} j_d(|y|)dy \leq L(r) \leq c(d)\kappa_1^{-1} \int_{|y| \geq r} j_d(|y|)dy, \]
\[ c(d)\kappa_2^{-1}r^{-2} \int_{\mathbb{R}^d} (r^2 \wedge |y|^2) j_d(|y|)dy \leq h(r) \leq c(d)\kappa_1^{-1}r^{-2} \int_{\mathbb{R}^d} (r^2 \wedge |y|^2) j_d(|y|)dy. \]

Since \( h'(r) = K'(r) + L'(r) = -2r^{-1}K(r) < 0 \), \( h \) is strictly decreasing in \( r \). Thus, the inverse function \( h^{-1} \) of \( h \) is well-defined and (2.13) makes sense. The function \( \psi \) and \( h \) are related as

\[ C_0 h(r) \leq \psi(r^{-1}) \leq 2h(r), \quad \text{for} \quad r > 0 \quad (2.16) \]

where the constant \( C_0 \) (we may take \( C_0 < 1 \)) only depends on \( d \) (see inequalities (6) and (7) in [24]). Moreover, for any \( 0 < c < 1 \),

\[ h(cr) \leq c(d)\kappa_1^{-1}c^{-2}r^{-2} \int_{\mathbb{R}^d} (c^2r^2 \wedge |y|^2) j_d(|y|)dy \leq \kappa_2\kappa_1^{-1}c^{-2}h(r). \quad (2.17) \]

(iv) The condition (2.13) is about the behavior of \( \ell \) near zero. Similar to (2.12), we could give weak scaling conditions to \( \ell \) near zero instead of (2.13). However, to cover general decay rates for jump kernel, we choose to give condition (2.13). Indeed, the behavior of \( h \) near infinity is determined by the behavior of \( \ell \) near zero only. To see this, let \( r > 1 \). Then we have

\[ K(r) \asymp r^{-2} \int_{|y| \leq 1} |y|^2 j_d(|y|)dy + r^{-2} \int_{1}^{r} s\ell(s^{-1})ds \]
\[ \asymp r^{-2} + r^{-2} \int_{1}^{r} s\ell(s^{-1})ds. \quad (2.18) \]
Thus, for $r > 1$

$$h(r) = K(r) + L(r) \approx r^{-2} + r^{-2} \int_1^r s\ell(s^{-1})ds + \int_r^\infty s^{-1}\ell(s^{-1})ds.$$  

This shows that the behavior of $h$ near infinity depends only on the behavior of $\ell$ near zero. The following are some conditions on $\ell$ that give (2.13): for $0 < r \leq R < 1$

- $\ell(R)/\ell(r) \geq c(R/r)^{\delta_1}$ with $\delta_1 > 2$;
- $c_1(R/r)^{\delta_1} \leq \ell(R)/\ell(r) \leq c_2(R/r)^{\delta_2}$ with $0 < \delta_2 \leq \delta_1 < 2$;
- $\ell(r) \asymp r^2$.

(v) Assumption 2.6 is used to obtain upper bounds for derivative of heat kernel. Using these upper bounds, we obtain upper bounds for derivative of our fundamental solution. If $X_t$ is a subordinate Brownian motion, then

$$j_{d+2}(r) = -\frac{1}{2\pi^{1/2}} \frac{d}{dr} j_d(r) \quad \text{for} \quad r > 0$$

holds due to (2.27). Thus, Assumption 2.6 (i) holds for all subordinate Brownian motions. Suppose that for any $d \geq 1$, the jump kernel $j_d$ of $d$-dimensional subordinate Brownian motion satisfies $j_d(r) \asymp r^{-d}\ell(r^{-1})$ for $r > 0$ with a function $\ell$ independent of $d$. Then, Assumption 2.6 (ii) also holds true.

**Remark 2.9.** Due to Assumption 2.6 we can apply [20] Theorem 1.5] to obtain $(d+2)$-dimensional isotropic unimodal Lévy process $\tilde{X}_t$ with the same characteristic exponent $\psi(|\xi|) = \psi_X(|\xi|)$, whose transition density $p_{d+2}(t,x) = p_{d+2}(t,|x|)$ is radial, radially decreasing in $x$ and satisfies

$$p_{d+2}(t,r) = \frac{1}{2\pi^{1/2}} \frac{d}{dr} p_d(t,r) \quad \text{for} \quad r > 0.$$  

This implies that for $t > 0$ and $x \in \mathbb{R}^d$,

$$|D_x p_d(t,x)| \leq 2\pi |x| p_{d+2}(t,|x|).$$  

(2.19)

By inspecting the proof of [20] Theorem 1.5], we can also find that the jumping kernel $\tilde{j}_{d+2}$ of $\tilde{X}_t$ is given by

$$\tilde{j}_{d+2}(r) = -\frac{1}{2\pi^{1/2}} \frac{d}{dr} j_d(r) \quad \text{for} \quad r > 0.$$  

(2.20)

By Assumption 2.6 $\tilde{j}_{d+2}$ satisfies (2.11) for $m = 0$ with $d + 2$ in place of $d$. Thus, we can obtain the upper bounds for $|D_x p_d(t,x)|$ by using upper bounds of $p_{d+2}$.

**Remark 2.10.** (i) Since $j_d(r)$ is decreasing in $r$, we have that for $r > 0$

$$\ell(r^{-1}) \leq \kappa_1^{-1} r^d j_d(r) \leq (d + 2)\kappa_1^{-1} r^{-2} \int_0^r s^{d+1} j_d(s)ds \leq CK(r) \leq Ch(r),$$

where the constant $C$ depends only on $\kappa_1, d$.

(ii) From (2.13) and the fact that $h$ is strictly decreasing, we have the following: for any $0 < A < \infty$, there exists a constant $c = c(A)$ such that

$$c(A) \left( \frac{r^{-1}}{R^{-1}} \right)^{\delta_3} \leq \frac{h(r)}{h(R)}, \quad \text{for} \quad A < r < R < \infty.$$  

(2.21)
Therefore, we have (put \(r = h^{-1}(R), \ R = h^{-1}(r)\))
\[
\frac{h^{-1}(r)}{h^{-1}(R)} \leq c(A) \left( \frac{r}{R} \right)^{1/\delta_3}, \quad \text{for } 0 < r < R < h(A). \tag{2.22}
\]
Also, from (2.17), we can obtain
\[
\frac{R}{r} \leq \left( \frac{h^{-1}(r)}{h^{-1}(R)} \right)^{2}, \quad \text{for } 0 < r < R < \infty. \tag{2.23}
\]

Now we introduce second assumption on \(\ell\). Depending on whether \(\ell\) is bounded or not, we have two different type of heat kernel upper bounds. Recall the due to Remark 2.10 (i), \(\ell(r^{-1}) \leq Ch(r)\) for all \(r > 0\).

**Assumption 2.11.** The function \(\ell\) in Assumption 2.7 satisfies

(i) either \(\limsup_{r \to \infty} \ell(r) < \infty\);

(ii) or \(\limsup_{r \to \infty} \ell(r) = \infty\) and \(\ell(r) \asymp \sup_{s \leq r} \ell(s)\).

If \(\ell\) satisfies (ii), then we further assume that

(ii)–(1) either
\[
\limsup_{r \to 0} \frac{h(r)}{\ell(r^{-1})} < \infty. \tag{2.24}
\]

(ii)–(2) or
\[
\limsup_{r \to 0} \frac{h(r)}{\ell(r^{-1})} = \infty
\]
and for any \(a > 0\) there is a constant \(C(a) > 0\) such that
\[
\sup_{0 < r < 1} h(r) \exp \left( -a \frac{h(r)}{\ell(r^{-1})} \right) \leq C(a). \tag{2.25}
\]

**Remark 2.12.** (i) If \(0 < \delta_1 \leq \delta_2 < 2\) and (2.12) holds for all \(0 < r < R < \infty\), then we see that (2.24) holds.

(ii) Since \(h\) is decreasing, (2.25) is equivalent to \(\sup_{r > 0} h(r)e^{-ah(r)/\ell(r^{-1})} \leq C(a)\).

(iii) By Lemma 2.1, there exists \(c > 0\) such that
\[
L(r) \leq h(r) \leq cL(r) \quad \text{for } 0 < r \leq 1. \tag{2.26}
\]
By the change of variable, we see that \(L(r) = \int_0^{r^{-1}} s^{-1}\ell(s)ds\) for \(r > 0\). From Remark 2.8 (iii) and the fact that \(j_a(\{y\})dy\) is a Lévy measure, we see that \(L(1) = \int_0^1 s^{-1}\ell(s)ds < \infty\). Since \(L\) is strictly decreasing,
\[
L(r^{-1}) = L(1) + \int_1^r s^{-1}\ell(s)ds \leq \frac{L(1)}{L(1/2)}L(r^{-1}) + \int_1^r s^{-1}\ell(s)ds \quad \text{for } r > 2.
\]
Thus, \(L(r^{-1}) \asymp \int_1^r s^{-1}\ell(s)ds\) for \(r > 2\). Using this with (2.26) and (ii), it follows that (2.25) is equivalent to the following: for any \(a > 0\), there exists \(C(a) > 0\) such that
\[
\sup_{r > 1} \int_1^r \frac{\ell(s)}{s} ds \cdot \exp \left( -a \frac{\ell(r)}{\ell(r)} \int_1^r \frac{\ell(s)}{s} ds \right) \leq C(a). \tag{2.27}
\]
It is known that for a slowly varying function \(\ell : (b, \infty) \to (0, \infty)\), it holds that \(\frac{1}{\ell(r)} \int_b^r \frac{\ell(s)}{s} ds \to \infty\) as \(r \to \infty\) (see [1] Proposition 1.5.9a).
Let $\mathcal{G}$ be the set of functions $\ell$ satisfying (2.27). We can check that $(\log (1 + r))^k \in \mathcal{G}$ for any $k \in \mathbb{N}$. See Lemma A.1 for more properties of $\mathcal{G}$.

The following theorem is main result of this article. Note that to prove the parabolic ($\alpha = 1$) counterpart of our main result, it seems that we need more differentiability to $j_\ell$. See Remark 5.2 for detail.

**Theorem 2.13.** Let $\alpha \in (0, 1)$, $p, q \in (1, \infty)$, $\gamma \in \mathbb{R}$, and $T \in (0, \infty)$. Suppose Assumption 2.6 Assumption 2.7 and Assumption 2.11 hold. Then for any $f \in \mathbb{H}_{q,p}^{\psi,\gamma}(T)$, the equation

$$\partial_t^\alpha u = \mathcal{L}u + f, \quad t > 0; \quad u(0, \cdot) = 0 \quad (2.28)$$

has a unique solution $u$ in the class $\mathbb{H}_{q,p}^{\alpha,\psi,\gamma+2}(T)$, and for the solution $u$ it holds that

$$\|u\|_{\mathbb{H}_{q,p}^{\alpha,\psi,\gamma+2}(T)} \leq C \|f\|_{\mathbb{H}_{q,p}^{\psi,\gamma}(T)}, \quad (2.29)$$

where $C > 0$ depends only on $\alpha, d, \kappa_1, \kappa_2, p, q,\gamma, \ell, C_0, C_1, C_2, T$ and $\delta$. Furthermore, we have

$$\|\mathcal{L}u\|_{\mathbb{H}_{q,p}^{\psi,\gamma}(T)} \leq C \|f\|_{\mathbb{H}_{q,p}^{\psi,\gamma}(T)}, \quad (2.30)$$

where $C > 0$ depends only on $\alpha, d, \kappa_1, \kappa_2, p, q,\gamma, \ell, C_0, C_1, C_2$ and $\delta$.

**Remark 2.14.** In Section 7, we will discuss Lévy processes such that $\ell$ depends on $d$. One of such examples is $\mathcal{L} = -\log (1 - \Delta)$.

**Remark 2.15.** In [16, 4.3] with non-zero initial data is considered. By using our result and following the approach in [16], we obtain the following: Let $\psi(r) = \phi_k(r^2)$, where $\phi_k(r) = \log (1 + r^2)^k$ with $\beta \in (0, 1)$ and $k \in \mathbb{N}$. Then, the solution $u$ to (1.3) satisfies that

$$\|u\|_{\mathbb{H}_{q,p}^{\alpha,\psi,\gamma+2}(T)} \leq C \left(\|u_0\|_{\mathbb{H}_{q,p}^{\psi,\gamma+2}(T)} + \|f\|_{\mathbb{H}_{q,p}^{\psi,\gamma}(T)}\right).$$

Compared to the result in [16], we need the additional regularity $2/kq^\ell$ for initial data $u_0$ since $\phi_k$ does not satisfy (1.3) (see [16, Definition 2.3] for $\mathbb{H}_{q,p}^{\psi,\gamma}$). It is nontrivial to remove this additional regularity for $u_0$.

**Example 2.16.** The following are examples of $\ell, h$ and $j_\ell$ satisfying Assumption 2.6 Assumption 2.7 and Assumption 2.11

(i) Let $\beta \in (0, 2)$, $\ell(r) = r^\beta$, and let $j_\ell(r) = r^{-\beta} \ell(r^{-1})$. Then $\ell$ satisfies (2.12) with $\delta_1 = \delta_2 = \beta$ and $K(r) \asymp L(r) \asymp h(r) \asymp r^{-\beta}$. Hence, we can easily check that Assumption 2.6 and Assumption 2.7 hold. Also, $\ell$ satisfies Assumption 2.11(i)-(1).

Recall that the fractional Laplacian $-(-\Delta)^{\beta/2}$ is an example of operators covered by $\ell$ with explicit form.

(ii) Let $\beta \in (0, 2)$, $\ell(r) = r^\beta / (1 + r^\beta)$, and let $j_\ell(r) = r^{-\beta} \ell(r^{-1})$. Then $\ell$ satisfies (2.12) with $\delta_1 = \delta_2 = 0$. Also, by a direct computation, we see that

$$-\frac{d}{dr} j_\ell(r) = dr^{-\beta - 1} \frac{r^{-\beta}}{1 + r^{-\beta}} = r^{-\beta} \frac{r^{-\beta}}{1 + r^{-\beta}} = r^{-\beta} \ell(r^{-1}) \quad \text{for} \quad r > 0,$$

and thus $-\frac{1}{2} (\frac{d}{dr} j_\ell) (r)$ is a decreasing function (recall $\beta < 2$). Since $\ell(s^{-1}) \asymp (1 \wedge s^{-\beta})$, it is easy to see that $L(r) \asymp r^{-\beta}$ for $r \geq 1$, and $K(r) \asymp r^{-\beta} + r^{-\beta} \asymp r^{-\beta}$ by (2.18). Thus $h(r) \asymp r^{-\beta}$ for $r \geq 1$ and (2.13) follows with $\delta_1 = \beta$. Hence, $\ell, h$ and $j_\ell$ satisfy Assumption 2.6 and Assumption 2.7. Note that, $\ell$ obviously satisfies Assumption 2.11(i) since $\lim_{r \to \infty} \ell(r) = 1 < \infty$. 
An operator $L$ covered by the above function $\ell$ with explicit form is $L = -\log (1 + (-\Delta)^{3/2})$. In this case, the characteristic exponent is given as $\psi(r) =: \phi(r^2) := \log (1 + r^\beta)$. Then, it is known that $\phi(r) = \log (1 + r^\beta)$ is a Bernstein function and for each $d \geq 1$, the corresponding $d$-dimensional process $X_t$ is a subordinate Brownian motion, with jumping kernel $J_d$ satisfying (see e.g. [18, 19])

$$J_d(r) \asymp r^{-d}(1 \wedge r^{-\beta}) \asymp r^{-d} \frac{r^{-\beta}}{1 + r^{-\beta}} \quad \text{for } r > 0.$$ 

Hence, we deduce that $J_d$ satisfies Assumption 2.6 with $\ell(r) = r^\beta/(1 + r^\beta)$.

(iii) Let $\beta > 0$, $\ell(r) = \log (1 + r^\beta)$, and let $j_d(r) = r^{-d}\ell(r^{-1})$. Then $\ell$ satisfies (2.12) with $\delta_1 = \delta_2 = 0$. Using the argument in (i) we can check that for $r \geq 1$

$$h(r) \asymp \begin{cases} 
    r^{-\beta} & \text{if } \beta \in (0, 2) \\
    r^{-2}\log r & \text{if } \beta = 2 \\
    r^{-2} & \text{if } \beta > 2.
\end{cases}$$

Hence, Assumption 2.7 follows. Observe that

$$-\frac{1}{r} \frac{d}{dr} j_d(r) = dr^{-d-2} \log (1 + r^{-\beta}) + \beta r^{-d-2} \frac{1}{1 + r^\beta}.$$ 

Thus $-\frac{1}{r} \frac{d}{dr} j_d(r)$ is a decreasing function. Moreover, using

$$(1 + r)(1 + r^{-1}) \geq \frac{1}{2} \quad \text{for } r > 0,$$ 

(2.31)

we can check that $j_d$ satisfies Assumption 2.6. Also, using Lemma A.1 (ii), (iv) and Remark 2.12 (iii), we see that $\ell$ satisfies Assumption 2.11 (ii)-(2).

The operator $L = -(\log (1 + (-\Delta)^{3/4}))^2$ with $\beta \in (0, 2)$ can be covered by $\ell(r) = \log (1 + r^\beta)$. In this case, $\psi(r) =: \phi(r^2) =: (\log (1 + r^{3/2}))^2$, and by using theory of Bernstein functions in [26, Chapter 7], we see that $\phi$ satisfies conditions (A-2) and (A-3) in [18]. Hence, by [18, Proposition 4.1], we can check that for each $d \geq 1$, jumping kernel $J_d$ of corresponding $d$-dimensional subordinate Brownian motion satisfies

$$J_d(r) \asymp r^{-d-2}\phi'(r^2) \asymp r^{-d}\log (1 + r^{-\beta}) \quad \text{for } r \leq 1.$$ 

Since $\phi(r)$ satisfies condition (H2) in [19], we use [19, Lemma 3.3(a)] to obtain that for each $d \geq 1$ we have

$$J_d(r) \asymp r^{-d}\phi(r^{-2}) \asymp r^{-d}\log (1 + r^{-\beta/2}) \asymp r^{-d}\log (1 + r^{-\beta}) \quad \text{for } r \geq 1.$$ 

Using these estimates for $J_d$, for each $d \geq 1$, we have $J_d(r) \asymp r^{-d}\ell(r^{-1})$ for all $r > 0$ and hence it satisfies Assumption 2.7 Assumption 2.6 and Assumption 2.11.

(iv) For $b \in (0, 1/2)$, let $\ell(r) := \exp\{\log(1 + r)^b\} - 1$. Then, by [11, Theorem 1.3.1] (put $c(x) = 1$ and $\varepsilon(x) = (\log x)^{b-1}$ therein), we can check that $\ell$ is a slowly varying function at infinity. Moreover, $\ell(r) \asymp r^b$ for $r < 1$. Thus, we see that $\ell$ and $h$ satisfy (2.12) and (2.13) respectively. Let $j_d(r) := r^{-d}\ell(r^{-1})$. Then, using (2.31) and the same argument as in (ii), it is easy to check that $j_d(r)$ satisfies Assumption 2.6 with $\ell$. Finally, due to Lemma A.1 (v), $\ell$ satisfies Assumption 2.11 (ii)-(2).

The above examples can be summarized as follows.

\[ \text{\begin{center} \begin{tabular}{|c|c|c|} \hline
\end{center} \end{tabular}} \]
3. Estimates of the heat kernels and their derivatives

In this section, we obtain sharp bounds of the heat kernel and its derivative for equation \( \partial_t u = Lu \) under the Assumption 2.6, Assumption 2.7 and Assumption 2.11. For the rest of this section, we suppose that the Assumption 2.6 and Assumption 2.7 hold.

Let \( p_d(t, x) = p_d(t, |x|) \) be the transition density of \( X_t \). Then it is well-known that for any \( t > 0, x \in \mathbb{R}^d \)

\[
p_d(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{-t\psi(|\xi|)} d\xi. \tag{3.1}
\]

Since \( X_t \) is isotropic, \( p_d(t, x) \) is rotationally invariant in \( x \) (i.e. \( p_d(t, x) = p_d(t, |x|) \)). We put \( p_d(t, r) := p_d(t, x) \) if \( r = |x| \) for notational convenience. Since \( X_t \) is unimodal, \( r \mapsto p(t, r) \) is a decreasing function. Moreover, \( p(t, x) \leq p(t, 0) \in (0, \infty) \) for \( t > 0 \) and \( x \in \mathbb{R}^d \).

The first part of this section consists of estimates for the heat kernel \( p_d \) to the equation

\[
\partial_t u = Lu. \tag{3.2}
\]

The following lemma gives off diagonal type upper bound for the heat kernel. The result holds for all isotropic unimodal Lévy processes.

**Proposition 3.1** ([12, Theorem 5.4]). For any \((t, x) \in (0, \infty) \times \mathbb{R}^d\), we have

\[
p_d(t, x) \leq Ct|x|^{-d} K(|x|),
\]

where the constant \( C > 0 \) depends only on \( \kappa_2, d \).

Next, we will give the sharp upper bound for heat kernel. Depending on whether \( \ell \) is bounded or not, we have two different type of the upper bounds. If \( \ell \) satisfies the condition in Assumption 2.11(ii), we define

\[
\bar{\ell}(r) := \sup_{s \leq r} \ell(s) \quad \text{for} \quad r > 0.
\]

Then, \( \bar{\ell} : (0, \infty) \rightarrow (0, \infty) \) is increasing continuous function and \( \ell(r) \asymp \bar{\ell}(r) \) for all \( r > 0 \). Also since \( \ell \) satisfies Assumption 2.7 and is bounded near zero (recall Remark 2.8(i)), we can apply Lemma A.2 to obtain a strictly increasing continuous function \( \ell^* \) satisfying

\[
\ell^* \asymp \bar{\ell} \asymp \ell.
\]

Let \( \ell^{-1} \) be the inverse function of \( \ell^* \). For any \( a > 0 \) and \( r, t > 0 \), we define

\[
\theta(a, r, t) = \theta_a(r, t) := r \vee (\ell^{-1}(a/t))^{-1}. \tag{3.3}
\]

The following heat kernel upper bounds are proved in [7, Proposition 2.9, Corollary 2.13].
Proposition 3.2. (i) Suppose \( \ell \) satisfies the condition in Assumption 2.11 (i). For any \( T > 0 \), there exist constants \( c_1, b_1 > 0 \) depending only on \( \kappa_1, \kappa_2, d \) and \( T \) such that

\[
p_d(t, x) \leq c_1 t \frac{K(|x|)}{|x|^d} \exp(-b_1 th(|x|)) \quad (3.4)
\]
holds for all \( (t, x) \in (0, T] \times (\mathbb{R}^d \setminus \{0\}) \).

(ii) Suppose \( \ell \) satisfies the condition in Assumption 2.11 (ii) and \( \theta \) is given by (3.3). Then, there exist constants \( a_0, b_2 > 0 \) such that the following holds: for any \( T > 0 \), there exist \( c_2 > 0 \) depending only on \( \kappa_1, \kappa_2, d \) and \( T \) such that

\[
p_d(t, x) \leq c_2 t \frac{K(\theta_{a_0}(|x|, t))}{[\theta_{a_0}(|x|, t)]^d} \exp(-b_2 th(\theta_{a_0}(|x|, t))) \quad (3.5)
\]
holds for all \( (t, x) \in (0, T] \times \mathbb{R}^d \).

Proof. If \( \ell \) satisfies Assumption 2.11 (i), then (3.4) is a direct consequence of [7, Corollary 2.13]. If we define

\[
\theta'(a, r, t) = \theta'_a(r, t) := r \lor (\bar{\ell}^{-1}(a/t))^{-1},
\]
where \( \bar{\ell}^{-1} \) is the right continuous inverse of \( \bar{\ell} \), then (3.5) holds with \( \theta' \) in place of \( \theta \) due to [7, Corollary 2.9]. Since \( \ell^* \simeq \bar{\ell} \), by inspecting the proof of [7, Corollary 2.9], we can check \( \theta' \) in [7, Corollary 2.9] can be replaced by \( \theta_a \). The lemma is proved. \( \Box \)

We will only use the upper bound of \( p_d(t, x) \) in (3.4) for the proofs of our results. The following lemma will be used several times.

Lemma 3.3. Let \( a, b > 0 \) and \( d \in \mathbb{N} \).

(i) There exists \( C = C(b, d) > 0 \) such that for all \( t > 0 \)

\[
\int_{\mathbb{R}^d} t \frac{K(|x|)}{|x|^d} \exp(-b th(|x|)) \, dx \leq C.
\]

(ii) Suppose \( \ell \) satisfies the condition in Assumption 2.11 (ii) and \( \theta \) is given by (3.3). There exists \( C = C(a, b, d) > 0 \) such that for all \( t > 0 \)

\[
\int_{\mathbb{R}^d} t \frac{K(\theta_a(|x|, t))}{[\theta_a(|x|, t)]^d} \exp(-b th(\theta_a(|x|, t))) \, dx \leq C.
\]

Proof. (i) It is easy to see that

\[
\int_{\mathbb{R}^d} t|x|^{-d} K(|x|) e^{-b th(|x|)} \, dx = C(d) \int_0^\infty \rho \bar{\ell}^{-1}(\rho) e^{-b th(\rho)} \, d\rho
\]

\[
= C \int_{h^{-1}(t^{-1})}^{h^{-1}(t^{-1})} \rho \bar{\ell}^{-1}(\rho) e^{-b th(\rho)} \, d\rho + C \int_{h^{-1}(t^{-1})}^\infty \rho \bar{\ell}^{-1}(\rho) e^{-b th(\rho)} \, d\rho. \quad (3.6)
\]
Note that for any \( a > 0 \),
\[
\int_a^\infty \rho^{-1} K(\rho) d\rho = \int_a^\infty \rho^{-3} \int_0^\rho s \ell(s^{-1}) ds d\rho \\
= \int_0^a \int_a^\infty \rho^{-3} ds \rho \ell(s^{-1}) ds + \int_a^\infty \int_a^\infty \rho^{-3} ds \rho \ell(s^{-1}) ds \\
= \frac{a - 2}{2} \int_a^\infty s \ell(s^{-1}) ds + \frac{1}{2} \int_a^\infty a \ell(s^{-1}) ds \\
= \frac{1}{2} (C(a) + L(a)) \leq h(a).
\]

Also, by the change of variable \( h(\rho) \to \rho \) and the fact that \( h'(\rho) = -2\rho^{-1} K(\rho) \), we have
\[
\int_0^a t_0 \rho^{-1} K(\rho) e^{-bt_0(\rho)} d\rho \leq \frac{1}{2} t \int_{h(a)}^\infty e^{-bt_0 \rho} d\rho = Ce^{-bt_0(a)}.
\]

Applying (3.7) and (3.8) to (3.6) with \( a = h^{-1}(t^{-1}) \), we obtain the first assertion.

(ii) By the definition of \( \theta_a(r, t) \), we see that
\[
\int_{|x|^d} K(\theta_a(|x|, t)) [\theta_a(|x|, t)]^d \exp \left( -b t h(\theta_a(|x|, t)) \right) dx \\
= \int_{|x| \geq (t^{-1}(a/t))^{-1}} K(|x|) t \frac{K(|x|)}{|x|^d} \exp \left( -b t h(|x|) \right) dx \\
+ \int_{|x| < (t^{-1}(a/t))^{-1}} t e^{-1}(a/t)^d K(t^{-1}(a/t)^{-1}) \exp \left( -b t h(t^{-1}(a/t)^{-1}) \right) dx \\
\leq \int_{|x| \geq (t^{-1}(a/t))^{-1}} K(|x|) \frac{t}{|x|^d} \exp \left( -b t h(|x|) \right) dx \\
+ \int_{|x| < (t^{-1}(a/t))^{-1}} t e^{-1}(a/t)^d K(t^{-1}(a/t)^{-1}) \exp \left( -b t h(t^{-1}(a/t)^{-1}) \right) dx.
\]

The first term on the right hand side is bounded above by a constant \( C \) due to (i). The second term can be easily handled by using the relations \( K \leq h \) and \( se^{-s} \leq 1 \). Thus, we obtain the desired result.

The following lemmas give upper bound of \( p_d(t, x) \) for sufficiently large \( t > 0 \) without Assumption 2.11.

**Lemma 3.4.** There exist \( t_1 = t_1(d, \kappa_1, \kappa_2, t, C_0, C_1, C_2, \delta) > 0 \) and \( C > 0 \) depending only on \( t_1 \) such that for all \( t \geq t_1 \) and \( x \in \mathbb{R}^d \),
\[
p_d(t, x) \leq C \left( (h^{-1}(t^{-1}))^{-d} \wedge t \frac{K(|x|)}{|x|^d} \right).
\]

**Proof.** First, we show that there is \( t_0 > 0 \) such that \( e^{-C_0 h(|\cdot|^{-1})} \in L_1 \) for \( t \geq t_0 \), where \( C_0 \) is a constant from (2.16). By (2.16) and (2.26), we see that \( \psi(r^{-1}) \asymp L(r) \) for \( r < 1 \). Hence, if \( |\xi| \geq 1 \), then we have
\[
\int_{|\xi|^{-1}}^1 s^{-1} \ell(s^{-1}) ds \leq L(|\xi|^{-1}) \leq c_1 \psi(|\xi|).
\]

Thus, by (2.15) and the above inequality, there exists \( C > 0 \) such that
\[
C \log |\xi| \leq \psi(|\xi|) \quad \text{for} \quad |\xi| > 1.
\]
Thus if \( t > 0 \) satisfies \( Ct \geq 2d \), then we have
\[
e^{-t\psi(\xi)} \leq e^{-Ct\log|\xi|} \leq |\xi|^{-2d} \quad \text{for} \quad |\xi| > 1.
\]
This and (2.19) show that there is \( t_0 > 0 \) such that \( e^{-C_0t\theta(|\xi|^{-1})} \in L_1 \) for \( t \geq t_0 \).

On the other hand, by (2.21) we obtain that (recall \( C_0 \) comes from (2.16))
\[
c_2 \left( \frac{|\xi|}{(h^{-1}(t^{-1}))^{-1}} \right)^{d_3} \leq C_0 \theta(h(|\xi|^{-1})) \quad \text{for} \quad (h^{-1}(t^{-1}))^{-1} \leq |\xi| \leq 1.
\]

Take \( t_1 \geq 2t_0 \) satisfying \( 1 \leq h^{-1}(t_1^{-1}) \). Then, using the above observation, we have that for \( t \geq t_1 \) and \( |\xi| \geq 1 \),
\[
e^{-C_0th(|\xi|^{-1})} \leq e^{-C_0th(|\xi|^{-1})/2} \leq e^{-C_0th(|\xi|^{-1})/3} e^{-c_2(h^{-1}(t_1^{-1}))^{d_3}}.
\]

Using the above inequalities, (3.1), and (2.16), for \( t \geq t_1 \), we obtain
\[
|p_d(t, x)| \leq \int_{\mathbb{R}^d} e^{-C_0th(|\xi|^{-1})} d\xi
\]
\[
\leq \int_{|\xi| \leq (h^{-1}(t_1^{-1}))^{-1}} 1 d\xi + \int_{(h^{-1}(t_1^{-1}))^{-1} \leq |\xi| \leq 1} e^{-c_2|\xi|^{d_3}(h^{-1}(t_1^{-1}))^{d_3}} d\xi
\]
\[
+ \int_{|\xi| \geq 1} e^{-C_0th(|\xi|^{-1})} e^{-c_2(h^{-1}(t_1^{-1}))^{d_3}} d\xi
\]
\[
\leq c_3(h^{-1}(t_1^{-1}))^{-d} + (h^{-1}(t_1^{-1}))^{-d} \int_{|\xi| \leq 1} e^{-c_2|\xi|^{d_3}} d\xi
\]
\[
+ c_3(h^{-1}(t_1^{-1}))^{-d} \int_{|\xi| \geq 1} e^{-C_0th(|\xi|^{-1})} d\xi
\]
\[
\leq c_4(h^{-1}(t_1^{-1}))^{-d},
\]
where for the third and last inequalities, we used that there exists \( c_5(c_2, \delta_3) > 0 \) such that \( e^{-c_2x^{d_3}} \leq c_5x^{-d} \) for all \( x \geq 1 \) and integrability of \( e^{-C_0th(|\xi|^{-1})} \), respectively. This completes the proof. \( \square \)

**Remark 3.5.** (i) In Lemma 3.4 we use (2.13) to show that \( e^{-C_0th(|\xi|^{-1})} \) is integrable for \( t \geq t_0 \). If we assume that \( \ell \) satisfies Assumption 2.11 (ii), then we obtain the same near diagonal upper bound without (2.15). Indeed, if \( \ell \) satisfies Assumption 2.11 (ii), then by Proposition 3.2 (ii), we see that for fixed \( T > 0 \)
\[
p_d(t, 0) = \int_{\mathbb{R}^d} e^{-t\psi(|\xi|)} d\xi \leq C_{\ell^{-1}}(a_0/t)^d e^{-crh(\ell^{-1}(a_0/t)^{-1})} \quad \text{for} \quad t \leq T. \quad (3.9)
\]
This implies that
\[
\int_{\mathbb{R}^d} e^{-t\psi(|\xi|)} d\xi \leq \int_{\mathbb{R}^d} e^{-T\psi(|\xi|)} d\xi \leq C_{\ell^{-1}}(a_0/T)^d e^{-crh(\ell^{-1}(a_0/T)^{-1})} \quad \text{for} \quad t > T.
\]

Thus, \( e^{-\psi(|\xi|)} \) is integrable for all \( t > 0 \).

(ii) Suppose that the function \( \ell \) satisfies Assumption 2.11 (ii)–(1). Then by using Lemma 3.4 and (3.9), we can check that
\[
|p_d(t, x)| \leq |p_d(t, 0)| \leq C(h^{-1}(a_0/t))^{-d} \leq C(h^{-1}(t_1^{-1}))^{-d} \quad \text{for} \quad t > 0,
\]
where the last inequality follows from (2.23). Hence, if \( \ell \) satisfies Assumption 2.11 (ii)–(1), then

\[
p_d(t, x) \leq C \left( (h^{-1}(t^{-1}))^{-d} \wedge t \frac{K(|x|)}{|x|^d} \right),
\]

holds for all \((t, x) \in (0, \infty) \times \mathbb{R}^d\).

4. Estimates of the fundamental solutions

Let \( Q_t \) be an increasing Lévy process independent of \( X_t \) having the Laplace transform

\[
\mathbb{E} \exp(-\lambda Q_t) = \exp(-t \lambda^\alpha),
\]

and let

\[
R_t := \inf\{ s > 0 : Q_s > t \}
\]

be the inverse process of the subordinator \( Q_t \). Denote \( \varphi(t, r) \) denote the probability density function of \( R_t \) and define

\[
q_d(t, x) := \int_0^\infty p_d(r, x) d_r \mathbb{P}(R_t \leq r) = \int_0^\infty p_d(r, x) \varphi(t, r) dr.
\]

We will show in Lemma 4.5 that the function \( q_d(t, x) \) becomes the fundamental solution to

\[
\partial_\alpha^\alpha u = Lu \quad t > 0, \quad u(0, \cdot) = u_0.
\]

In this section, we also provide a sharp estimation of \( q_d \) and its derivatives. Note that we can find a result for subordinate Brownian motion in [16, Lemma 5.1] (see also [5, Theorem 1.1]).

For \( \beta \in \mathbb{R} \), denote

\[
\varphi_{\alpha, \beta}(t, r) := D_t^{\beta-\alpha} \varphi(t, r) := (D_t^{\beta-\alpha} \varphi(\cdot, r))(t).
\]

It is known (see e.g. [10, Lemma 3.7 (ii)]) that

\[
|\varphi_{\alpha, \beta}(t, r)| \leq C t^{-\beta} e^{-c(rt-\alpha)^{1/(1-\alpha)}}
\]

for \( rt^{-\alpha} \geq 1 \), and

\[
|\varphi_{\alpha, \beta}(t, r)| \leq \begin{cases} C r t^{-\alpha-\beta} & \beta \in \mathbb{N} \\ C t^{-\beta} & \beta \notin \mathbb{N} \end{cases}
\]

for \( rt^{-\alpha} \leq 1 \), where the constants \( c, C > 0 \) depending only on \( \alpha, \beta \).

For \((t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\} \) define

\[
q_d^{\alpha, \beta}(t, x) := \int_0^\infty p_d(r, x) \varphi_{\alpha, \beta}(t, r) dr.
\]

Note that since \( \varphi_{\alpha, \beta}(t, r) \) is integrable in \( r \in (0, \infty) \), we can check that \( q_d^{\alpha, \beta} \) is well-defined.

Lemma 4.1. Let \( \alpha \in (0, 1) \) and \( \beta \in \mathbb{R} \).

(i) For any \((t, x) \in (0, \infty) \times \mathbb{R}^d \setminus \{0\}\),

\[
D_t^{\beta-\alpha} q_d(t, x) = q_d^{\alpha, \beta}(t, x).
\]

(ii) For any \( t > 0, \xi \in \mathbb{R}^d \),

\[
\mathcal{F} \{ q_d^{\alpha, \beta}(t, \cdot) \}(\xi) = t^{\alpha-\beta} E_{\alpha, 1-\beta + \alpha}(-t^{\alpha} \psi(|\xi|)),
\]
where $E_{a,b}$ be the two-parameter Mittag-Leffler function defined as

$$E_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + b)}, \quad \text{for } z \in \mathbb{C}, a > 0, b \in \mathbb{C},$$

with the convention $E_{a,1}(0) = 1$.

**Proof.** (i) By Proposition 3.1 we have

$$|p_d(r,x)| \leq C(d, \kappa_2, x, K)r.$$

Hence, we can prove the assertion by following the proof of [16 Lemma 3.7 (iii)].

(ii) We can directly prove this by following the proof of [16 Lemma 3.7 (iv)] with $\psi(|\xi|)$ in place of $\phi(|\xi|^2)$ since we need no property of $\phi$.

The lemma is proved. \(\square\)

For the rest of the paper, we assume that Assumption 2.7, Assumption 2.6 and Assumption 2.11 hold.

**Lemma 4.2.** Let $\alpha \in (0,1)$, $\beta \in \mathbb{R}$, and $m = 0, 1$. Then there exists a constant $C$ depending only on $\alpha, \beta, \kappa_2$ and $m$ such that for any $t > 0$ and $x \in \mathbb{R}^d \setminus \{0\}$,

$$|D_x^m g_{d,\alpha,\beta}(t,x)| \leq Ct^{2\alpha-\beta} \frac{K(|x|)}{|x|^{d+m}}.$$  \hspace{1cm} (4.6)

**Proof.** Using (2.19) and Proposition 3.1 we have that for any $x \in \mathbb{R}^d \setminus \{0\}$, $r > 0$ and for any $y \in B_x(x)$ for sufficiently small $\varepsilon > 0$,

$$|D_x^s p_d(r,y)| \leq C(K_2, d) r \frac{K(|y|)}{|y|^{d+|\sigma|}} \leq C(K_2, d, x, \varepsilon, K)r \quad \text{for } |\sigma| \leq 1.$$  \hspace{1cm} (4.7)

Hence, by the dominated convergence theorem, we have

$$D_x^m g_{d,\alpha,\beta}(t,x) = \int_0^\infty D_x^m p_d(r,x)\varphi_{\alpha,\beta}(t,r)dr.$$  \hspace{1cm} (4.8)

By (4.7), (4.2) and (4.3)

$$|D_x^m g_{d,\alpha,\beta}(t,x)| \leq C \int_0^{t^\alpha} rK(|x|)|x|^{-d-m}t^{-\beta} dr + C \int_{t^\alpha}^\infty rK(|x|)|x|^{-d-m}t^{-\beta}e^{-c(rt^{-\alpha})^{1-\alpha}} dr \leq Ct^{2\alpha-\beta} \frac{K(|x|)}{|x|^{d+m}},$$

where for the second integral, we used the change of variable $rt^{-\alpha} \rightarrow r$. Thus, the lemma is proved. \(\square\)

**Lemma 4.3.** Let $\alpha \in (0,1)$, $\beta \in \mathbb{R}$, and $m = 0, 1$, and let $t_1 > 0$ be taken from Lemma 3.4. Then for any $(t, x) \in (0,\infty) \times \mathbb{R}^d$ satisfying $t^\alpha h(|x|) \geq 1$, we have

$$\int_{t_1}^\infty |D_x^m p_d(r,x)||\varphi_{\alpha,\beta}(t,r)|dr \leq C \int_{(h(|x|))^{-1}}^{2t_1} (h^{-1}(r^{-1}))^{-d-m}rt^{-\alpha-\beta} dr$$

for $\beta \in \mathbb{N}$, and

$$\int_{t_1}^\infty |D_x^m p_d(r,x)||\varphi_{\alpha,\beta}(t,r)|dr \leq C \int_{(h(|x|))^{-1}}^{2t_1} (h^{-1}(r^{-1}))^{-d-m}t^{-\beta} dr$$

for $\beta \in \mathbb{N}$. \hspace{1cm} (4.9)
Hence, we obtain

$$\int_{t_1}^{\infty} |D_x^m p_d(r,x)||\varphi_{a,\beta}(t,r)|dr \leq C (I_1 + I_2 + I_3),$$

where

$$I_1 = \int_{t_1}^{(h(|x|))^{-1}} r \frac{K(|x|)}{|x|^{d+m}} t^{-\alpha-\beta} dr,$$

$$I_2 = \int_{(h(|x|))^{-1}}^{r_0} \frac{|x|^m (h^{-1}(r^{-1}))^{-d-m} r^{-\alpha-\beta} dr}{h^{-1}(r^{-1})^{-1}},$$

$$I_3 = \int_{r_0}^{\infty} \frac{|x|^m (h^{-1}(r^{-1}))^{-d-m} t^{-\beta} e^{-c(rt^{-\alpha})} dr}{r_0}.$$

(4.9)

and the constant $C$ depends only on $\alpha, \beta, d, \kappa_1, \kappa_2, \delta, \ell, C_0, C_1, C_2$ and $m$. We first estimate $I_1$. It is easy to see that

$$I_1 \leq (h(|x|))^{-2} |x|^{-d-m} t^{-\alpha-\beta} = C \int_{(h(|x|))^{-1}}^{2(h(|x|))^{-1}} |x|^{-d-m} t^{-\alpha-\beta} dr.$$

For $t_1 < (h(|x|))^{-1} < r < 2(h(|x|))^{-1}$, we have (recall that $h^{-1}$ is decreasing)

$$h^{-1}(2r^{-1}) < |x| < h^{-1}(r^{-1}).$$

Hence, we obtain

$$I_1 \leq C \int_{(h(|x|))^{-1}}^{2(h(|x|))^{-1}} h^{-1}(2r^{-1})^{-d-m} t^{-\alpha-\beta} dr.$$

By (2.22), we see that $h^{-1}(r^{-1}) \leq Ch^{-1}(2r^{-1})$ for $r^{-1} < t_1^{-1}$. This and the above estimate for $I_1$ give

$$I_1 \leq C \int_{(h(|x|))^{-1}}^{2(h(|x|))^{-1}} h^{-1}(r^{-1})^{-d-m} t^{-\alpha-\beta} dr \leq \int_{(h(|x|))^{-1}}^{2r_0} h^{-1}(r^{-1})^{-d-m} t^{-\alpha-\beta} dr.$$

Note that for $r \geq (h(|x|))^{-1}$, we have $|x| \leq h^{-1}(r^{-1})$. This gives that

$$I_2 \leq C \int_{(h(|x|))^{-1}}^{2r_0} (h^{-1}(r^{-1}))^{-d-m} t^{-\alpha-\beta} dr \leq C \int_{(h(|x|))^{-1}}^{2r_0} (h^{-1}(r^{-1}))^{-d-m} t^{-\alpha-\beta} dr.$$

For $I_3$, since $e^{-s} \leq c(\alpha)s^{1-\alpha}$ for $s \geq 1$ and $s \mapsto (h^{-1}(s^{-1}))^{-d-m}$ is decreasing, we have (recall $|x| \leq h^{-1}(r^{-1})$)

$$I_3 \leq C \int_{r_0}^{2r_0} (h^{-1}(r^{-1}))^{-d-m} t^{-\alpha-\beta} dr + C \int_{2r_0}^{\infty} (h^{-1}(r^{-1}))^{-d-m} t^{-\beta} e^{-c(rt^{-\alpha})} t^{-\alpha} dr \leq C \int_{(h(|x|))^{-1}}^{2r_0} (h^{-1}(r^{-1}))^{-d-m} t^{-\alpha-\beta} dr + C \left( h^{-1} \left( \frac{t^{-\alpha}}{2} \right) \right)^{-d-m} t^{-\alpha}. $$
Thus, we have

\[
\left(h^{-1}\left(\frac{1}{2} t^{-\alpha}\right)\right)^{-d-m} t^{\alpha-\beta} = C \int_{t_1}^{2t} \left(h^{-1}\left(\frac{t^{-\alpha}}{2}\right)\right)^{-d-m} r t^{-\alpha-\beta} dr
\]

\[
\leq C \int_{t_1}^{2t} \left(h^{-1}(r^{-1})\right)^{-d-m} r t^{-\alpha-\beta} dr
\]

\[
\leq C \int_{(h(|x|))^{-1}}^{2t} \left(h^{-1}(r^{-1})\right)^{-d-m} r t^{-\alpha-\beta} dr.
\]

Thus, we have

\[
I_1 + I_2 + I_3 \leq C \int_{(h(|x|))^{-1}}^{2t} \left(h^{-1}(r^{-1})\right)^{-d-m} r t^{-\alpha-\beta} dr,
\]

and this proves the claim for \( \beta \in \mathbb{N} \) and \( h(|x|) \leq t_1^{-1} \).

For \( \beta \in \mathbb{N} \) and \( h(|x|) \geq t_1^{-1} \), if we use the relation \(|x| \leq h^{-1}(r^{-1})\) for \( r \geq (h(|x|))^{-1} \), we have

\[
\int_{t_1}^{\infty} |D_x^m p_d(r, x)||\varphi_{\alpha, \beta}(t, r)| dr \leq C \left(I_4 + I_3\right),
\]

where

\[
I_4 = \int_{t_1}^{t} \left(h^{-1}(r^{-1})\right)^{-d-m} r t^{-\alpha-\beta} dr,
\]

and \( I_3 \) comes from (4.9) which is bounded by

\[
C \int_{(h(|x|))^{-1}}^{2t} \left(h^{-1}(r^{-1})\right)^{-d-m} r t^{-\alpha-\beta} dr.
\]

Since \( (h(|x|))^{-1} \leq t_1 \), it is easy to see that

\[
I_4 \leq C \int_{(h(|x|))^{-1}}^{2t} \left(h^{-1}(r^{-1})\right)^{-d-m} r t^{-\alpha-\beta} dr.
\]

Thus, we prove the lemma for the case \( \beta \in \mathbb{N} \).

Finally, for the case \( \beta \notin \mathbb{N} \), by following all of the above computations with

\[
|\varphi_{\alpha, \beta}(t, r)| \leq Ct^{-\beta}\quad \text{for} \quad r \leq t^\alpha,
\]

we have

\[
\int_{t_1}^{\infty} |D_x^m p_d(r, x)||\varphi_{\alpha, \beta}(t, r)| dr \leq C \int_{(h(|x|))^{-1}}^{2t} \left(h^{-1}(r^{-1})\right)^{-d-m} r^{-\beta} dr.
\]

Thus, we obtain the first assertion.

Now suppose that \( \ell \) satisfies Assumption 2.11 (ii)–(1). Then using (6.10) we have

\[
\int_{0}^{\infty} |D_x^m p_d(r, x)||\varphi_{\alpha, \beta}(t, r)| dr \leq C \left(I'_1 + I_2 + I_3\right),
\]

where

\[
I'_1 = \int_{0}^{(h(|x|))^{-1}} |D_x^m p_d(r, x)||\varphi_{\alpha, \beta}(t, r)| dr.
\]

Since \( I'_1 \) can be handled like \( I_1 \), the lemma is proved. \( \square \)
Corollary 4.4. Let $\alpha \in (0,1)$ and $\beta \in \mathbb{R}$.

(i) There exists a constant $C = C(\alpha, \beta, d, \kappa_1, \kappa_2, \delta, \ell, C_0, C_1, C_2)$ such that

$$\int_{\mathbb{R}^d} |q_d^{\alpha,\beta}(t,x)| dx \leq Ct^{\alpha-\beta}, \quad t > 0.$$  

(ii) For any $0 < \varepsilon < T < \infty$,

$$\int_{\mathbb{R}^d} \sup_{[\varepsilon,T]} |q_d^{\alpha,\beta}(t,x)| dx < \infty.$$  

Proof. (i) Due to the similarity, we only consider the case $\beta \in \mathbb{N}$. By Lemma 4.2,

$$\int_{\mathbb{R}^d} |q_d^{\alpha,\beta}(t,x)| dx = \int_{|x| \geq h^{-1}(t-\alpha)} |q_d^{\alpha,\beta}(t,x)| dx + \int_{|x| < h^{-1}(t-\alpha)} |q_d^{\alpha,\beta}(t,x)| dx$$

$$\leq C \int_{|x| \geq h^{-1}(t-\alpha)} r^{2\alpha-\beta} K(|x|) |x|^d dx$$

$$+ C \int_{|x| < h^{-1}(t-\alpha)} \int_0^{t_1} p_d(r,x) \varphi_{\alpha,\beta}(t,r) dr dx$$

$$+ C \int_{|x| < h^{-1}(t-\alpha)} \int_{(h(|x|))^{-1}}^{h^{-1}(r-1))} (h^{-1}(r-1))^{-d} r t^{-\alpha-\beta} dr dx.$$  

Using that $\int_{\mathbb{R}^d} p_d(r,x) dx = 1$ for all $r > 0$ and $\int_0^\infty |\varphi_{\alpha,\beta}(t,r)| dr \leq Ct^{\alpha-\beta}$, we have

$$\int_0^\infty \int_{\mathbb{R}^d} p_d(r,x) dx |\varphi_{\alpha,\beta}(t,r)| dr \leq Ct^{\alpha-\beta}.$$  

Hence, by Fubini’s theorem and (3.7), we obtain

$$\int_{\mathbb{R}^d} |q_d^{\alpha,\beta}(t,x)| dx \leq C \int_{r \geq h^{-1}(t-\alpha)} r^{2\alpha-\beta} K(r) \frac{1}{r} dr + C t^{\alpha-\beta}$$

$$+ C \int_0^{2\alpha} \int_{(h(|x|))^{-1}}^{h^{-1}(r-1))} (h^{-1}(r-1))^{-d} r t^{-\alpha-\beta} dx dr$$

$$\leq C t^{\alpha-\beta} + C \int_0^{2\alpha} r t^{-\alpha-\beta} dr \leq C t^{\alpha-\beta}.$$  

(ii) Like (i), we only prove the case $\beta \in \mathbb{N}$. Let $0 < \varepsilon < T < \infty$. Since $t^{2\alpha-\beta} \leq C(\varepsilon, T, \alpha, \beta)$ for $t \in [\varepsilon, T]$, by Lemma 4.2

$$|q_d^{\alpha,\beta}(t,x)| \leq C(\alpha, \beta, d, \kappa_1, \kappa_2, \delta, \ell, C_0, \varepsilon, T) \frac{K(|x|)}{|x|^d}, \quad t \in [\varepsilon, T].$$  

Also, if $\varepsilon^\alpha h(|x|) \geq 1$, and $t \in [\varepsilon, T]$, then using Lemma 4.2 again, we obtain

$$|q_d^{\alpha,\beta}(t,x)| \leq C \left( \int_0^{t_1} p_d(r,x) \varphi_{\alpha,\beta}(t,r) dr + \int_{(h(|x|))^{-1}}^{h^{-1}(r-1))} (h^{-1}(r-1))^{-d} r dr \right).$$
By Fubini’s theorem and \( \int_{\mathbb{R}^d} p_d(r,x)dx = 1 \) for all \( r > 0 \), we have
\[
\int_{\mathbb{R}^d} \sup_{t \in [\varepsilon,T]} \int_0^{t_1} p_d(r,x)dx \left( \sup_{t \in [\varepsilon,T]} |\varphi_{\alpha,\beta}(t,r)| \right) dr \\
\leq C \int_0^{\infty} p_d(r,x)dx \left( \sup_{t \in [\varepsilon,T]} |\varphi_{\alpha,\beta}(t,r)| \right) dr \\
\leq C \int_0^{T^\alpha} r \varepsilon^{-\alpha-\beta} dr + C \int_0^{\infty} \varepsilon^{-\beta} e^{-\varepsilon(rT^{-\alpha})} dr \leq C.
\]

Hence, like the proof of (i),
\[
\int_{\mathbb{R}^d} \sup_{t \in [\varepsilon,T]} |q_d^{\alpha,\beta}(t,x)| dx \\
= \int_{|x| \geq h^{-1}(\varepsilon^{-\alpha})} |q_d^{\alpha,\beta}(t,x)| dx + \int_{|x| < h^{-1}(\varepsilon^{-\alpha})} |q_d^{\alpha,\beta}(t,x)| dx \\
\leq C \int_{|x| \geq h^{-1}(\varepsilon^{-\alpha})} \frac{K(|x|)}{|x|^d} dx + C \int_{\mathbb{R}^d} \int_0^{t_1} p_d(r,x)\varphi_{\alpha,\beta}(t,r) dr dx \\
+ C \int_{|x| < h^{-1}(\varepsilon^{-\alpha})} \int_{(h(|x|))^{-1}}^{2T^\alpha} (h^{-1}(r^{-1}))^{-d} r dr dx \\
\leq C + C \int_0^{2T^\alpha} \int_{(h(|x|))^{-1}}^{r} (h^{-1}(r^{-1}))^{-d} r dr dx \leq C + C \int_0^{2T^\alpha} r dr < \infty.
\]

The corollary is proved. \hfill \Box

Recall that \( C_p^\infty(\mathbb{R}^d) \) is the set of functions \( u_0 = u_0(x) \) such that \( D_x^m u_0 \in L_p \) for any \( m \in \mathbb{N}_0 \).

**Lemma 4.5.** Let \( u_0 \in C_p^\infty(\mathbb{R}^d) \), and define \( u \) as
\[
u(t,x) := \int_{\mathbb{R}^d} q_d(t,x-y)u_0(y)dy.
\]

(i) As \( t \downarrow 0 \), \( u(t,\cdot) \) converges to \( u_0(\cdot) \) uniformly on \( \mathbb{R}^d \) and also in \( H_p^n \) for any \( n \in \mathbb{N}_0 \).

(ii) \( u \in C_p^{\infty}( [0,T] \times \mathbb{R}^d ) \) and \( u \) satisfies (4.1).

**Proof.** We can prove the lemma by following proof of [16, Lemma 5.1]. \hfill \Box

### 5. Estimation of solution: Calderón-Zygmund approach

In this section we prove some a priori estimates for solutions to the equation with zero initial condition
\[
\partial_t^\alpha u = Lu + f, \quad t > 0; \quad u(0,\cdot) = 0.
\]

We first provide the representation formula.
Lemma 5.1. (i) Let \( u \in C_c^\infty(\mathbb{R}^{d+1}_+) \) and denote \( f := \partial_t^\alpha u - Lu \). Then
\[
 u(t, x) = \int_0^t \int_{\mathbb{R}^d} q_d^{\alpha,1}(t - s, x - y) f(s, y) dy ds. \tag{5.2}
\]

(ii) Let \( f \in C_c^\infty(\mathbb{R}^{d+1}_+) \) and define \( u \) as in (5.2). Then \( u \) satisfies equation (5.1) for each \((t, x)\).

Proof. The lemma is an extension of [16, Lemma 4.1] which treats the case \( \psi(|\xi|) = \phi(|\xi|^2) \), where \( \phi \) is a Bernstein function. The proof of [16, Lemma 4.1] only uses (4.5) with \( \phi(|\xi|^2) \) in place of \( \psi(|\xi|) \) and Corollary 4.4. Since no property of \( \phi \) is used, we can prove all the claims by repeating the same argument. For more detail, see also [15, Lemma 3.5]. \( \square \)

For \( f \in C_c^\infty(\mathbb{R}^{d+1}_+) \), we define
\[
 L_0 f(t, x) := \int_{-\infty}^t \int_{\mathbb{R}^d} q_d^{\alpha,1}(t - s, x - y) f(s, y) dy ds,
\]
\[
 L f(t, x) := \int_{-\infty}^t \int_{\mathbb{R}^d} q_d^{\alpha,1}(t - s, x - y) L f(s, y) dy ds, \tag{5.3}
\]
where \( L f(s, y) := \mathcal{L}(f(s, \cdot))(y) \).

Note that \( L f \) is bounded for any \( f \in C_c^\infty(\mathbb{R}^{d+1}_+) \). Thus, the operator \( L \) is well defined on \( C_c^\infty(\mathbb{R}^{d+1}_+) \). For each fixed \( s \) and \( t \) such that \( s < t \), define
\[
 \Lambda_{t,s}^0 f(x) := \int_{\mathbb{R}^d} q_d^{\alpha,1}(t - s, x - y) L f(s, y) dy,
\]
\[
 \Lambda_{t,s} f(x) := \int_{\mathbb{R}^d} q_d^{\alpha,1}(t - s, x - y) f(s, y) dy.
\]

Note that, by Corollary 4.4 \( \Lambda_{t,s} f \) and \( \Lambda_{t,s}^0 f \) are square integrable. Moreover, using (4.5) and the recurrence relation \( E_{a,b}(z) = \frac{1}{1(z)} + z E_{a,a+b}(z) \) (\( z \in \mathbb{C} \)) we have
\[
 \mathcal{F}_d \{ q_d^{\alpha,1}(t - s, \cdot) \}(\xi) = (t - s)^{-1} E_{a,0}(-(t - s)^\alpha \psi(|\xi|)) = -(t - s)^{\alpha - 1} \psi(|\xi|) E_{a,1}(-(t - s)^\alpha \psi(|\xi|)) = \psi(|\xi|) \mathcal{F}_d \{ q_d^{\alpha,1}(t - s, \cdot) \}(\xi).
\]

Hence,
\[
 \mathcal{F}_d \{ \Lambda_{t,s}^0 f \}(\xi) = \psi(|\xi|) \mathcal{F}_d \{ q_d^{\alpha,1} \}(t - s, \xi) \hat{f}(s, \xi) = \mathcal{F}_d \{ \Lambda_{t,s} f \}(\xi).
\]

Thus, we have
\[
 \mathcal{L} L_0 f(t, x) = L f(t, x) = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{t-\varepsilon} \left( \int_{\mathbb{R}^d} q_d^{\alpha,1}(t - s, x - y) f(s, y) dy \right) ds. \tag{5.4}
\]

We give a short description of our strategy to obtain the estimation of solution. By (5.2), and (5.4) if we let \( u \) be a solution to equation (5.1), then, we have \( \mathcal{L} u = L f \). Hence to obtain the estimation of solution, it is enough to show the \( L_q(L_p) \)-boundedness of the linear operator \( L \) (Theorem 5.12).

We prove Theorem 5.12 step by step by using the following lemmas;
(i) We prove \( L_2 \)-boundedness of \( L \) in Lemma 5.4
(ii) From Lemma 5.4 to Lemma 5.9 we control the mean oscillation of \( L \) in terms of \( ||f||_{L_\infty(\mathbb{R}^{d+1}_+)} \).
(iii) We prove Theorem 5.12 by using the result of (ii) and Theorem 5.11

**Remark 5.2.** For parabolic (i.e. \( \alpha = 1 \)) equations, we see from (5.4) that the above argument requires estimation of \( L^p_d \). To obtain estimation on \( L^p_d \) we need more differentiability on \( j_d \) and corresponding conditions in Assumption 2.7 (i), (ii). Thus, to cover more general \( j_d \), we only consider the case \( \alpha \in (0, 1) \) in this article.

The following two lemmas will be used for step (ii). For notational convenience, we define a strictly increasing function \( \kappa : (0, \infty) \rightarrow (0, \infty) \) as
\[
\kappa(r) := (h(r))^{-1/\alpha}
\]
Recall that we assume that Assumption 2.7, Assumption 2.6 and Assumption 2.11.

**Lemma 5.3.** Let \( \alpha \in (0, 1) \). There exists \( C = C(\alpha, d, \kappa_1, \kappa_2, \delta, \ell, C_0, C_1, C_2) \) such that for any \( b > 0 \)

\[
\int_{\kappa(b)}^{\infty} \int_{|y| \geq b} |D_x q_d^{\alpha, \alpha+1}(s, y)| dy ds \leq C b^{-1}, \quad (5.5)
\]

\[
\int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} |q_d^{\alpha, \alpha+1}(s, y)| dy ds \leq C. \quad (5.6)
\]

**Proof.** Let \( t_1 > 0 \) be the constant in Lemma 3.4. We will consider two cases separately.  

**Case 1** \( \ell \) satisfies Assumption 2.11 (i).  
We first show (5.5). By (4.6),

\[
\int_{\kappa(b)}^{\infty} \int_{|y| \geq b} |D_x q_d^{\alpha, \alpha+1}(s, y)| dy ds \\
\leq \int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s^{-\alpha})} \int_0^\infty |D_x p_d(r, y)| |\varphi_{\alpha, \alpha+1}(s, r)| dr dy ds \\
+ C \int_{\kappa(b)}^{\infty} \int_{|y| \geq h^{-1}(s^{-\alpha})} s^{\alpha-1} K(|y|) \frac{1}{|y|^{d+1}} dy ds \\
= \int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s^{-\alpha})} \int_0^{t_1} |D_x p_d(r, y)| |\varphi_{\alpha, \alpha+1}(s, r)| dr dy ds \\
+ \int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s^{-\alpha})} \int_{t_1}^\infty |D_x p_d(r, y)| |\varphi_{\alpha, \alpha+1}(s, r)| dr dy ds \\
+ C \int_{\kappa(b)}^{\infty} \int_{|y| \geq h^{-1}(s^{-\alpha})} s^{\alpha-1} K(|y|) \frac{1}{|y|^{d+1}} dy ds \\
=: I_1 + I_2 + CI_3. \quad (5.7)
\]
By [A.2], Lemma [4.3] and Lemma [A.4] we obtain $I_1 + I_2 \leq Cb^{-1}$. By (3.7) and the change of variable ($s^\alpha \to s$),

$$I_3 = \int_{(h(b))^{-1/\alpha}}^\infty \int_{h^{-1}(s^{-\alpha})}^\infty s^{\alpha-1} K(\rho) \frac{d\rho}{\rho^2} ds$$

$$\leq C \int_{(h(b))^{-1/\alpha}}^\infty h^{-1}(s^{-\alpha})^{-1} s^{\alpha-1} \int_{h^{-1}(s^{-\alpha})}^\infty \frac{K(\rho)}{\rho} d\rho ds$$

$$\leq C \int_{(h(b))^{-1/\alpha}}^\infty h^{-1}(s^{-\alpha})^{-1} s^{-1} ds$$

$$= C \int_{(h(b))^{-1}}^\infty h^{-1}(s^{-1})^{-1} s^{-1} ds. \quad (5.8)$$

Thus, by Lemma [A.3] with $f(r) = h(r^{-1})$ and $k = 1$, we obtain $I_3 \leq Cb^{-1}$.

Now, we show (5.6). By (A.3) and Lemma 4.3,

$$\int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} q_d^{\alpha,\alpha+1}(s, y) dy ds$$

$$\leq \int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_{0}^{\kappa_1} p_d(r, y)|\varphi_{\alpha,\alpha+1}(s, r)| dr dy ds$$

$$+ \int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_{\kappa_1}^{\infty} p_d(r, y)|\varphi_{\alpha,\alpha+1}(s, r)| dr dy ds$$

$$\leq C + C \int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_{(h(y))^{-1}}^{2s^\alpha} (h^{-1}(r^{-1}))^{-d} s^{-\alpha-1} dr dy ds$$

$$= C + C I_4,$$

Note that if $\kappa(4b) < s$, then by Fubini’s theorem

$$I_4 = \int_{\kappa(4b)}^{\infty} \int_{0}^{4b} \int_{(h(\rho))^{-1/\alpha}}^{2s^\alpha} (h^{-1}(r^{-1}))^{-d} s^{-\alpha-1} \rho^{d-1} dr d\rho ds$$

$$= \int_{(h(4b))^{-1/\alpha}}^{\kappa(4b)} \int_{0}^{4b} \int_{(h(\rho))^{-1/\alpha}}^{2s^\alpha} (h^{-1}(r^{-1}))^{-d} s^{-\alpha-1} \rho^{d-1} dr d\rho ds$$

$$+ \int_{(h(4b))^{-1/\alpha}}^{\kappa(4b)} \int_{0}^{4b} \int_{0}^{2s^\alpha} (h^{-1}(r^{-1}))^{-d} s^{-\alpha-1} \rho^{d-1} dr d\rho ds$$

$$\leq C \int_{(h(4b))^{-1/\alpha}}^{\kappa(4b)} \int_{0}^{4b} \int_{0}^{2s^\alpha} (h^{-1}(r^{-1}))^{-d} s^{-\alpha-1} \rho^{d-1} d\rho dr ds$$

$$+ C b^{d} \int_{(h(4b))^{-1/\alpha}}^{\kappa(4b)} \int_{(h(4b))^{-1/\alpha}}^{2s^\alpha} (h^{-1}(r^{-1}))^{-d} s^{-\alpha-1} dr ds.$$

We can easily check

$$\int_{(h(4b))^{-1/\alpha}}^{\kappa(4b)} \int_{0}^{4b} (h^{-1}(r^{-1}))^{-d} s^{-\alpha-1} \rho^{d-1} d\rho dr ds$$

$$\leq \int_{(h(4b))^{-1/\alpha}}^{\kappa(4b)} s^{-\alpha-1} dr ds \leq C. \quad (5.10)$$
By Fubini’s theorem and Lemma A.3 with \( f(r) = h(r^{-1}) \) and \( k = d \), we have

\[
\begin{align*}
\int_0^\infty \int_0^{s(t,x)} & \left( h^{-1}(r^{-1}) \right)^{-d} s^{-\alpha-1} \, dr \\
& = \int_0^\infty \int_{(h(4b))^{-1}}^{s(t,x)} \left( h^{-1}(r^{-1}) \right)^{-d} s^{-\alpha-1} \, dr \\
& \leq C b^d \int_0^\infty r^{-\alpha} \left( h^{-1}(r^{-1}) \right)^{-d} dr = C.
\end{align*}
\]

(\textbf{Case 2}) \( \ell \) satisfies the Assumption 2.11 (ii).

Suppose that \( \ell \) satisfies Assumption 2.11 (ii)–(2). We first show (5.5). Using (4.8) and (4.6), we have

\[
\int_{(h(4b))^{-1}}^{s(t,x)} \left( h^{-1}(r^{-1}) \right)^{-d} s^{-\alpha-1} \, dsdr
\]

By (5.10) and (5.11),

\[
\int_0^\infty \int_{(h(4b))^{-1}}^{s(t,x)} \left( h^{-1}(r^{-1}) \right)^{-d} s^{-\alpha-1} \, drds.
\]

Using (5.8) and (A.6) we have (5.5). For (5.6), we use (A.7) and Lemma 4.3 for (5.9). Then, by (5.10) and (5.11), we have

\[
\int_{(h(4b))^{-1}}^{s(t,x)} \left( h^{-1}(r^{-1}) \right)^{-d} s^{-\alpha-1} \, drds.
\]

Now suppose that \( \ell \) satisfies Assumption 2.11 (ii)–(1). Then by Lemma 4.3 for all \( (t, x) \in (0, \infty) \times \mathbb{R}^d \) such that \( h^\alpha |x| \geq 1 \) we have,

\[
|D_x q_d^{\alpha+1}(t, x)| \leq C \int_{(h(\alpha|x|))^{-1}}^{2\alpha} \left( h^{-1}(r^{-1}) \right)^{-d-mt^{-\alpha-1}} dr.
\]

This implies that when we show (5.5), from (5.7), we only need to handle \( I_3 \) and

\[
\int_{(h(4b))^{-1}}^{s(t,x)} \left( h^{-1}(r^{-1}) \right)^{-d} s^{-\alpha-1} \, drds
\]

Hence, using Lemma A.4 with \( f(r) = h(r^{-1}) \) and (A.8), we have (5.5). For (5.6), we can check that from (5.9), we only need to handle \( I_4 \). Since \( I_4 \leq C \), we prove all claims. The lemma is proved.

Recall that the operator \( L \) defined in (5.3) satisfies (5.4).
Lemma 5.4. For any $f \in C_c^\infty(\mathbb{R}^{d+1})$, we have
\[
\|Lf\|_{L_2(\mathbb{R}^{d+1})} \leq C(\alpha,d)\|f\|_{L_2(\mathbb{R}^{d+1})}.
\]
Consequently, the operators $L$ can be continuously extended to $L_2(\mathbb{R}^{d+1})$.

Proof. Like Lemma 5.1, we can prove the lemma by following the proof of Lemma 4.2. Again note that the only difference between our lemma and Lemma 4.2 is that we just replace $\psi(|\xi|)$ in place of $\phi(|\xi|^2)$ since we do not need any property of $\phi$. \hfill $\Box$

Recall that $\kappa(b) = (h(b))^{-1/\alpha}$. For $(t, x) \in \mathbb{R}^{d+1}$ and $b > 0$, denote
\[
Q_b(t, x) = (t - \kappa(b), t + \kappa(b)) \times B_b(x),
\]
and
\[
Q_b = Q_b(0, 0), \quad B_b = B_b(0).
\]
For measurable subsets $Q \subset \mathbb{R}^{d+1}$ with finite measure and locally integrable functions $f$, define
\[
f_Q = \int_Q f(s, y)dyds = \frac{1}{|Q|} \int_Q f(s, y)dyds,
\]
where $|Q|$ is the Lebesgue measure of $Q$.

Lemma 5.5. Let $b > 0$, and $f \in C_c^\infty(\mathbb{R}^{d+1})$ have a support in $(-3\kappa(b), 3\kappa(b)) \times B_{3b}$. Then,
\[
\int_{Q_b} |Lf(t, x)|dxdt \leq C(\alpha, d)\|f\|_{L_\infty(\mathbb{R}^{d+1})}.
\]

Proof. By Hölder’s inequality and Lemma 5.4,
\[
\int_{Q_b} |Lf(t, x)|dxdt \leq \|Lf\|_{L_2(\mathbb{R}^{d+1})}|Q_b|^{-1/2} \leq C\|f\|_{L_2(\mathbb{R}^{d+1})}|Q_b|^{-1/2}
\]
\[
= \left(\int_{-3\kappa(b)}^{3\kappa(b)} \int_{B_{3b}} |f(t, x)|^2dydt\right)^{1/2} |Q_b|^{-1/2} \leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})}.
\]
The lemma is proved. \hfill $\Box$

Lemma 5.6. Let $b > 0$, and $f \in C_c^\infty(\mathbb{R}^{d+1})$ have a support in $(-3\kappa(b), \infty) \times \mathbb{R}^d$. Then,
\[
\int_{Q_b} |Lf(t, x)|dxdt \leq C(\alpha, d, \kappa_1, \kappa_2, \delta, \ell, C_0, C_1, C_2)\|f\|_{L_\infty(\mathbb{R}^{d+1})}.
\]

Proof. Take $\eta = \eta(t) \in C_c(\mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta(t) = 1$ for $t \leq 2\kappa(b)$, and $\eta(t) = 0$ for $t \geq 5\kappa(b)/2$. Since $Lf = L(f\eta)$ on $Q_b$ and $|f\eta| \leq |f|$, it is enough to assume $f(t, x) = 0$ if $|t| \geq 3\kappa(b)$ to prove the lemma.

Choose a function $\zeta = \zeta(x) \in C_c(\mathbb{R}^d)$ such that $\zeta = 1$ in $B_{7b/3}$, $\zeta = 0$ outside of $B_{8b/3}$ and $0 \leq \zeta \leq 1$. Set $f_1 = \zeta f$ and $f_2 = (1 - \zeta)f$. Then due to the linearity, $Lf = Lf_1 + Lf_2$. Also, since $Lf_1$ can be estimated by Lemma 5.3 to prove the lemma, we may further assume that $f(t, y) = 0$ if $y \in B_{2b}$. Hence, for any $x \in B_b$,
\[
\int_{\mathbb{R}^d} \left|q_d^{\alpha, \alpha+1}(t-s, x-y)f(s, y)\right|dy = \int_{|y-x| \geq 2b} \left|q_d^{\alpha, \alpha+1}(t-s, y)f(s, x-y)\right|dy
\]
\[
\leq \int_{|y| \geq b} \left|q_d^{\alpha, \alpha+1}(t-s, y)f(s, x-y)\right|dy.
\]
Using Lemma 4.12 and (3.7),
\[
\int_{|y| \geq b} |q_d^{\alpha+1}(t-s, y)f(s, x-y)| \, dy \\
\leq \|f\|_{L_\infty(\mathbb{R}^{d+1})} 1_{|s| \leq 2\kappa(b)} \int_{|y| \geq b} |q_d^{\alpha+1}(t-s, y)| \, dy \\
\leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})} 1_{|s| \leq \kappa(b)} \int_0^\infty |t-s|^{-\alpha-1} \frac{K(\rho)}{\rho^d} \rho^{d-1} d\rho \\
\leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})} 1_{|s| \leq \kappa(b)} (t-s)^{-\alpha-1} h(b).
\]
Note that if $|t| \leq \kappa(b)$ and $|s| \leq \kappa(b)$ then $|t-s| \leq 4\kappa(b)$. Thus we have
\[
|Lf(t, x)| \leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})} h(b) \int_{|t-s| \leq 4\kappa(b)} |t-s|^{-\alpha-1} \, ds \leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})}.
\]
This implies the desired estimate. The lemma is proved. \qed

**Lemma 5.7.** Let $b > 0$, and $f \in C_c^\infty(\mathbb{R}^{d+1})$ have a support in $(-\infty, -2\kappa(b)) \times \mathbb{R}^d$. Then there is $C = C(\alpha, d, \kappa_1, \kappa_2, \ell, \epsilon, C_0, C_1, C_2)$ such that for any $(t_1, x), (t_2, x) \in Q_b$,
\[
\int_{Q_b} \int_{Q_b} |Lf(t_1, x) - Lf(t_2, x)| \, dx dt_1 \, dx dt_2 \leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})}.
\]

**Proof.** We will show that
\[
|Lf(t_1, x) - Lf(t_2, x)| \leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})},
\]
and this certainly proves the lemma.

Without loss of generality, we assume that $t_1 > t_2$. Recall $f(s, y) = 0$ if $s \geq -2\kappa(b)$. Thus, if $t > -\kappa(b)$, by applying the fundamental theorem of calculus and (4.4), we have
\[
|Lf(t_1, x) - Lf(t_2, x)| = \left| \int_{-\infty}^{-2\kappa(b)} \int_{\mathbb{R}^d} q_d^{\alpha, \alpha+2} (t-s, x-y) f(s, y) \, dt dy ds \right|.
\]
Using Corollary 4.4 (i),
\[
\int_{t_2}^{t_1} \int_{\mathbb{R}^d} |q_d^{\alpha, \alpha+2} (t-s, x-y) f(s, y)| \, dt dy \\
\leq C\|f\|_{L_\infty(\mathbb{R}^d)} \int_{t_2}^{t_1} \int_{\mathbb{R}^d} |q_d^{\alpha, \alpha+2} (t-s, y)| \, dy dt \\
\leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})} \int_{t_2}^{t_1} (t-s)^{-2} \, dt.
\]
Thus, if $-\kappa(b) \leq t_2 < t_1 \leq \kappa(b),$
\[
\left| \int_{-\infty}^{-2\kappa(b)} \int_{\mathbb{R}^d} q_d^{\alpha, \alpha+2} (t-s, x-y) f(s, y) \, dt dy ds \right| \\
\leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})} \left( \int_{t_2}^{t_1} (t-s)^{-2} \, ds dt \right) \\
\leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})} \left( \int_{t_2}^{t_1} \frac{1}{\kappa(b)} \, dt \right) \leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})}.
\]
This completes the proof. \qed
Lemma 5.8. Let $b > 0$, and $f \in C_c^\infty(\mathbb{R}^{d+1})$ have a support in $(-\infty, -2\kappa(b)) \times B_{2b}$. Then there is $C = C(\alpha, d, \kappa_1, \kappa_2, \delta, \ell, C_0, C_1, C_2)$ such that for any $(t, x_1), (t, x_2) \in Q_b$,
\[ \int_{Q_b} \int_{Q_b} |Lf(t, x_1) - Lf(t, x_2)| \, dx_1 \, dt \, dx_2 \, dt \leq C(\alpha, d, \kappa_1, \kappa_2, \delta, \ell, C_0) \|f\|_{L_\infty(\mathbb{R}^{d+1})}. \]

Proof. Like the previous lemma it is enough to show that
\[ |Lf(t, x_1) - Lf(t, x_2)| \leq C\|f\|_{L_\infty(\mathbb{R}^{d+1})}. \]
Recall $f(s, y) = 0$ if $s \geq -2\kappa(b)$ or $|y| \leq 2b$. Thus, if $t > -\kappa(b)$, by the fundamental theorem of calculus,
\[ |Lf(t, x_1) - Lf(t, x_2)| = \int_{\infty}^{-2\kappa(b)} \int_{|y| \geq 2b} (q_d^{\alpha+1}(t - s, x_1 - y) - q_d^{\alpha+1}(t - s, x_2 - y)) f(s, y) \, dy \, ds \]
\[ = \int_{\infty}^{-2\kappa(b)} \int_{|y| \geq 2b} \nabla_x q_d^{\alpha+1}(t - s, \lambda(x_1, x_2, u) - y) \cdot (x_1 - x_2) \, du \, f(s, y) \, dy \, ds, \]
where $\lambda(x_1, x_2, u) = ux_1 + (1 - u)x_2$. Since $x_1, x_2 \in B_b$ and $|y| \geq 2b$, we can check that $|\lambda(x_1, x_2, u) - y| \geq b$. Thus, by the change of variable $(\lambda(x_1, x_2, u) - y) \rightarrow y$,
\[ |Lf(t, x_1) - Lf(t, x_2)| \leq Cb\|f\|_{L_\infty(\mathbb{R}^{d+1})} \int_{\infty}^{-2\kappa(b)} \int_{|y| \geq b} |\nabla_x q_d^{\alpha+1}(t - s, y)| \, dy \, ds \]
\[ \leq Cb\|f\|_{L_\infty(\mathbb{R}^{d+1})} \int_{\kappa(b)}^{\infty} \int_{|y| \geq b} |q_d^{\alpha+1}(s, y)| \, dy \, ds. \]
Thus, by (5.5), we obtain the desired result. \qed

Lemma 5.9. Let $b > 0$, and $f \in C_c^\infty(\mathbb{R}^{d+1})$ have a support in $(-\infty, -2\kappa(b)) \times B_{3b}$. Then for any $(t, x) \in Q_b$
\[ \int_{Q_b} |Lf(t, x)| \, dx \, dt \leq C(\alpha, d, \kappa_1, \kappa_2, \delta, \ell, C_0, C_1, C_2) \|f\|_{L_\infty(\mathbb{R}^{d+1})}. \]

Proof. For $(t, x) \in Q_b$,
\[ |Lf(t, x)| \leq \int_{\infty}^{-2\kappa(b)} \int_{B_{3b}} |q_d^{\alpha+1}(t - s, x - y)| \, f(s, y) \, dy \, ds \]
\[ \leq C\|f\|_{L_\infty} \int_{\infty}^{-2\kappa(b)} \int_{B_{3b}} |q_d^{\alpha+1}(t - s, x - y)| \, dy \, ds \]
\[ \leq C\|f\|_{L_\infty} \int_{\kappa(b)}^{\infty} \int_{B_{3b}} |q_d^{\alpha+1}(s, y)| \, dy \, ds =: C\|f\|_{L_\infty}(I + II), \]
where
\[ I = \int_{\kappa(b)}^{\kappa(4b)} \int_{B_{3b}} |q_d^{\alpha+1}(s, y)| \, dy \, ds, \quad II = \int_{\kappa(4b)}^{\infty} \int_{B_{3b}} |q_d^{\alpha+1}(s, y)| \, dy \, ds. \]
By Corollary 4.3 (i) and the scaling properties of $\kappa$ or $\ell$
\[ I = \int_{\kappa(b)}^{\kappa(4b)} \int_{B_{3b}} |q_d^{\alpha+1}(s, y)| \, dy \, ds \leq C \int_{\kappa(b)}^{\kappa(4b)} s^{-1} \, ds \leq C \log \left( \frac{\kappa(4b)}{\kappa(b)} \right) \leq C \log (16). \]
By [5.9], we also have \( II \leq C \). Thus, \( I \) and \( II \) are bounded by a constant independent of \( b \). Hence, we have
\[
|Lf(t,x)| \leq C\|f\|_{L^\infty(\mathbb{R}^{d+1})},
\]
and the lemma is proved. □

**Corollary 5.10.** There is \( C = C(\alpha, d, \kappa_1, \kappa_2, \delta, \ell, C_0, C_1, C_2) \) such that for any \( f \in C^\infty(\mathbb{R}^{d+1}) \) and \( b > 0 \)
\[
\int_{Q_b} \int_{Q_b} |Lf(t,x) - Lf(s,y)| dtdyds \leq C\|f\|_{L^\infty(\mathbb{R}^{d+1})}.
\]

**Proof.** Take a function \( \eta = \eta(t) \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \eta \leq 1, \eta = 1 \) on \((-\infty, -8\kappa(b)/3)\) and \( \eta(t) = 0 \) for \( t \geq -7\kappa(b)/3 \). Then for any \((t,x), (s,y)\)
\[
|Lf(t,x) - Lf(s,y)| \leq |Lf(t,x) - Lf(s,x)| + |Lf(s,x) - Lf(s,y)|
\]
\[
\leq \sum_{i=1}^{2} (|Lf_i(t,x) - Lf_i(s,x)| + |Lf_i(s,x) - Lf_i(s,y)|),
\]
where \( f_1 = f\eta \) and \( f_2 = f(1 - \eta) \). By Lemma 5.6 it is easy to see that
\[
\int_{Q_b} \int_{Q_b} (|Lf_2(t,x) - Lf_2(s,x)| + |Lf_2(s,x) - Lf_2(s,y)|) dtdyds
\]
\[
\leq 4 \int_{Q_b} |Lf_2(t,x)| dtdx \leq C\|f\|_{L^\infty(\mathbb{R}^{d+1})}.
\]

Note that due to Lemma 5.4
\[
\int_{Q_b} \int_{Q_b} |Lf_1(t,x) - Lf_1(s,x)| dtdyds \leq C\|f\|_{L^\infty(\mathbb{R}^{d+1})}.
\]

Hence, it only remains to consider \( |Lf_1(s,x) - Lf_1(s,y)| \). For this, take \( \zeta \in C^\infty_c(\mathbb{R}^d) \) such that \( 0 \leq \zeta \leq 1, \zeta = 1 \) on \( B_{7b/3} \) and \( \zeta = 0 \) outside of \( B_{5b/3} \). Then denoting \( f_3 = f_1(1 - \zeta) \), and \( f_4 = f_1\zeta \)
\[
|Lf_2(s,x) - Lf_2(s,y)| \leq \sum_{i=3}^{4} |Lf_i(s,x) - Lf_i(s,y)|
\]
Applying Lemma 5.8 and Lemma 5.9 to \( f_3 \) and \( f_4 \) respectively, we have
\[
\int_{Q_b} \int_{Q_b} |Lf_2(s,x) - Lf_2(s,y)| dtdyds \leq C\|f\|_{L^\infty(\mathbb{R}^{d+1})},
\]
and the corollary is proved. □

For locally integrable functions \( f \) on \( \mathbb{R}^{d+1} \), we define the BMO semi-norm of \( f \) on \( \mathbb{R}^{d+1} \) as
\[
\|f\|_{BMO(\mathbb{R}^{d+1})} = \sup_{Q} \int_{Q} |f(t,x) - f_Q| dt dx
\]
where \( f_Q = \int_{Q} f(t,x) dt dx \) and
\[
Q := \{Q_b(t_0,x_0) : b > 0, (t_0,x_0) \in \mathbb{R}^{d+1}\}.
\]

For measurable functions \( f \) on \( \mathbb{R}^{d+1} \), we define the sharp function
\[
f^\#(t,x) = \sup_{Q_b} \int_{Q_b} |f(s,y) - f_Q| ds dy,
\]
where the supremum is taken over all $Q_b(r, z) \in \mathbb{Q}$ containing $(t, x)$.

**Theorem 5.11** (Fefferman-Stein Theorem). For any $1 < p < \infty$, and $f \in L_p(\mathbb{R}^{d+1})$,

$$C^{-1} \| f \|_{L_p(\mathbb{R}^{d+1})} \leq \| f \|_{L_\infty(\mathbb{R}^{d+1})} \leq C \| f \|_{L_p(\mathbb{R}^{d+1})},$$

where $C > 1$ depends on $\alpha, d, p, \kappa_1, \kappa_2$.

**Proof.** See [28, Theorem I.3.1, Theorem IV.2.2]. We only remark that due to (2.17), the balls $Q_b(s, y)$ satisfy the conditions (i)–(iv) in [28, Section 1.1].

Recall that linear operators $L$ is given by

$$Lf(t, x) = \lim_{\varepsilon \downarrow 0} \int_{-\infty}^{t-\varepsilon} \left( \int_{\mathbb{R}^d} q^{n, \alpha+1}_d(t-s, x-y) f(s, y) dy \right) ds,$$

The following theorem is our main result in this section. The proof is quite standard.

**Theorem 5.12.** (i) For any $f \in L_2(\mathbb{R}^{d+1}) \cap L_\infty(\mathbb{R}^{d+1})$,

$$\| Lf \|_{BMO(\mathbb{R}^{d+1})} \leq C(\alpha, d, \kappa_1, \kappa_2, \delta, \ell, C_0, C_1, C_2) \| f \|_{L_\infty(\mathbb{R}^{d+1})}.$$

(ii) For any $p, q \in (1, \infty)$ and $f \in C^\infty_c(\mathbb{R}^{d+1})$,

$$\| Lf \|_{L_q(\mathbb{R}; L_p(\mathbb{R}^d))} \leq C(\alpha, d, p, q, \kappa_1, \kappa_2, \delta, \ell, C_0, C_1, C_2) \| f \|_{L_q(\mathbb{R}; L_p(\mathbb{R}^d))}.$$

**Proof.** If $L_X = -\phi(-\Delta)$ for Bernstein function $\phi$ satisfying (1.4), then the theorem is a direct consequence of [16, Theorem 4.10]. Since we have all the necessary results (i.e. Corollary 4.4 and Corollary 5.10) we can prove theorem by following the proof of [16, Theorem 4.10].

6. **Proof of Theorem 2.13**

In this section, we will prove Theorem 2.13.

**Proof of Theorem 2.13** Due to Lemma 2.4, it suffices to prove case $\gamma = 0$.

**Step 1 (Existence and estimation of solution).**

First assume $f \in C^\infty_c(\mathbb{R}^{d+1})$, and let $u(t, x)$ be a function with representation (5.2). Using Remark 2.3 and the integrability of $q^{n, \alpha}_d$, we can easily check $D^\alpha u$, $LD^\alpha u \in C([0, T]; L_p)$, and thus $u \in C^\infty(\mathbb{R}; L_p(\mathbb{R}^d))$. Also, by Lemma 5.1, $u$ satisfies equation (3.2) with $u(0, \cdot) = 0$.

Now we show estimation (2.29) and (2.30). Take $\eta_k = \eta_k(t) \in C^\infty(\mathbb{R})$ such that $0 \leq \eta_k \leq 1$, $\eta_k(t) = 1$ for $t \leq T + 1/k$ and $\eta_k(t) = 0$ for $t > T + 2/k$. Since $f\eta_k \in L_q(\mathbb{R}; L_p(\mathbb{R}^d))$, and $f(t) = f\eta_k(t)$ for $t \leq T$, By Theorem 5.12 (ii), we have

$$\| L u \|_{L_{q,p}(T)} = \| L f \|_{L_{q,p}(T)} = \| L(f\eta_k) \|_{L_{q,p}(T)} \leq \| L(f\eta_k) \|_{L_q(\mathbb{R}; L_p(\mathbb{R}^d))} \leq C \| f \|_{L_q(\mathbb{R}; L_p(\mathbb{R}^d))}.$$

Hence, by the dominated convergence theorem, taking $k \to \infty$, we have

$$\| L u \|_{L_{q,p}(T)} \leq C \| f \|_{L_{q,p}(T)}.$$

Also, by Corollary 4.4 and Minkowski’s inequality, we can easily check that

$$\| u \|_{L_{q,p}(T)} \leq C(\gamma) \| f \|_{L_{q,p}(T)}.$$

Therefore, using the above inequalities and (2.8) we prove estimation (2.29) and (2.30). For general $f$, we take a sequence of functions $f_n \in C^\infty_c(\mathbb{R}^{d+1})$ such that $f_n \to f$ in $L_{q,p}(T)$. Let $u_n$ denote the solution with representation (5.2) with $f_n$
Lemma 5.1, \( C_u \)

**Step 1**

In the following.

\( u(2.28) \) and estimation \( (2.29) \) and \( (2.30) \) also holds for \( u \).

Since \( \ell \) (Uniqueness of solution).

**Step 2**

in place of \( f \) is

\( (2.29) \) applied to \( u_m - u_n \) shows that \( u_n \) is a Cauchy sequence in \( \mathbb{H}^{q,p,2}(T) \). By taking \( u \) as the limit of \( u_n \) in \( \mathbb{H}^{q,p,2}(T) \), we find that \( u \) satisfies \( (2.28) \) and estimation \( (2.29) \) and \( (2.30) \) also holds for \( u \).

**Step 2 (Uniqueness of solution).**

Let \( u \in \mathbb{H}^{q,p,2}(T) \) be a solution to equation \( (2.28) \) with \( f = 0 \). Take \( u_n \in C_c^\infty (\mathbb{R}^{d+1}_+) \) which converges to \( u \) in \( \mathbb{H}^{q,p,2}(T) \), and let \( f_n := \partial^\alpha u_n - Lu_n \). Then by Lemma 5.1, \( u_n \) satisfies representation \( (5.2) \) with \( f_n \). Hence, by the argument in \( \text{Step 1} \), we have

\[
\|u_n\|_{\mathbb{H}^{q,p,2}(T)} \leq C(T) \leq \|f_n\|_{L^q,p(T)}.
\]

Since \( f_n = \partial^\alpha u_n - Lu_n \) converges to 0 due to the choice of \( u_n \), we conclude that \( u = 0 \) in \( \mathbb{H}^{q,p,2}(T) \). The theorem is proved.

\[\square\]

7. **FURTHER DISCUSSIONS**

The purpose of this section is to handle the case \( \ell = \ell_d \) depends on \( d \). A typical example is \( \mathcal{L} = -\log (1 - \Delta) \). In this case, the corresponding jumping kernel \( j_d \) satisfies

\[
j_d(r) \asymp r^{-d}\ell_d(r^{-1}) \quad \text{for } r > 0,
\]

where

\[
\ell_d(r) := r^{(1-d)/2}e^{-r^{-1}} + e^{-r}1_{r \geq 1}(7.1)
\]

(see \[27\]). To handle operators whose jump intensity \( \ell_d \) depend on \( d \), we impose the following.

**Assumption 7.1.**

(i) The function \(-\frac{1}{r} \left( \frac{d}{r^d} j_d \right) (r) \) is decreasing and there exist continuous functions \( \ell_d \) and \( \ell_{d+2} \) such that

\[
\kappa_1 r^{-d}\ell_d(r^{-1}) \leq j_d(r) \leq \kappa_2 r^{-d}\ell_d(r^{-1}) \quad \text{for } r > 0,
\]

and

\[
\kappa_1 r^{-d-2}\ell_{d+2}(r^{-1}) \leq \frac{1}{r} \frac{d}{r^d} j_d(r) \leq \kappa_2 r^{-d-2}\ell_{d+2}(r^{-1}) \quad \text{for } r > 0. (7.3)
\]

Moreover, \( \ell_d \) and \( \ell_{d+2} \) satisfy \( (2.12) \) and lim sup \( r \to \infty (\ell_d(r) \lor \ell_{d+2}(r)) < \infty \).

(ii) \( h_d \) satisfies \( (2.13) \) with \( \delta_1 > 0 \) and \( \delta_2 > 0 \) and \( K_d(r) \asymp K_{d+2}(r) \), \( h_d(r) \asymp h_{d+2}(r) \) for \( r > 0 \), where \( K_{d+i} \) and \( h_{d+i} \) are define as in \( (2.14) \) with \( \ell_{d+i} \) (i = 0, 2).

**Remark 7.2.**

(i) When the given process \( X_t \) is a subordinate Brownian motion, then \( (7.2) \) implies \( (7.3) \). See Remark 2.8 (v).

(ii) It is easy to see that \( \ell_d(r) = r^{(1-d)/2}e^{-r^{-1}} + e^{-r}1_{r \geq 1} \) from \( (7.1) \) satisfies \( (2.12) \) with \( \delta_1 = \delta_2 = 0 \). Also, we can check that (recall \( (2.16) \))

\[
K_d(r) \asymp (1 \lor r^{-2}) \asymp K_{d+2}(r), \quad h_d(r) \asymp (1 + r^{-2}) \asymp h_{d+2}(r) \quad \text{for } r > 0.
\]

In this case, the corresponding process \( X_t \) is a subordinate Brownian motion since \( \phi(r) = \log (1 + r) \) is a Bernstein function. Hence, Assumption \( 7.1 \) holds with \( \ell_d(r) = r^{(1-d)/2}e^{-r^{-1}} + e^{-r}1_{r \geq 1} \).

(iii) If \( \limsup_{r \to \infty} \ell(r) = \infty \), then by Proposition 8.2 (ii) we have

\[
p_d(t, x) \leq c_2 t \frac{K(\theta_{\alpha\theta}(\|x\|, t))}{[\theta_{\alpha\theta}(\|x\|, t)]^d} \exp (-b_2 t \theta(\theta_{\alpha\theta}(\|x\|, t)))
\]
In this case \( \theta_{a_0}(r,t) = r \vee (\ell_1^{-1}(a_0/t))^{-1} \) may depend on \( d \), and we need additional assumption to follow our approach.

**Theorem 7.3.** Let \( \alpha \in (0,1) \), \( p,q \in (1,\infty) \), \( q' = q/(q-1) \), \( \gamma \in \mathbb{R} \), and \( T \in (0,\infty) \). Suppose Assumption \( 7.1 \) holds. Then for any \( f \in H^{\psi,\gamma}_{q,p}(T) \), the equation

\[
\partial_t^\alpha u = Lu + f, \quad t > 0; \quad u(0,\cdot) = 0
\]

has a unique solution \( u \) in the class \( H^{\alpha,\psi,\gamma+2}_{q,p,0}(T) \), and for the solution \( u \) it holds that

\[
\| u \|_{H^{\alpha,\psi,\gamma+2}_{q,p,0}(T)} \leq C \| f \|_{H^{\psi,\gamma}_{q,p}(T)},
\]

where \( C > 0 \) depends only on \( \alpha,d,\kappa_1,\kappa_2,p,q,\gamma,\ell,C_0,C_1,C_2,T, \) and \( \delta \). Furthermore, we have

\[
\| Lu \|_{H^{\psi,\gamma}_{q,p}(T)} \leq C \| f \|_{H^{\psi,\gamma}_{q,p}(T)},
\]

where \( C > 0 \) depends only on \( \alpha,d,\kappa_1,\kappa_2,p,q,\gamma,\ell,C_0,C_1,C_2, \) and \( \delta \).

**Proof.** Under Assumption \( 7.1 \) (i) we have Proposition \( 3.2 \) (i) and Lemma \( 3.4 \). Also, by Assumption \( 7.1 \) (ii) we may assume that \( K \) and \( h \) are independent of \( d \). Hence, by following the argument in Section 3, Section 4 and Section 5 we obtain the theorem. \( \square \)

**Appendix A. Appendix**

**Lemma A.1.**

(i) Suppose that \( \ell_1 \in G \) and let \( \ell_2 \) be a function which satisfies \( \ell_2 \in (a,b) \) for some \( 0 < a < b < \infty \). Then \( \ell_1 \ell_2 \in G \).

(ii) Suppose \( \ell \in G \). Then for any \( b > 0 \), \( r \mapsto \ell(r^b) \) belongs to \( G \).

(iii) Let \( \ell_1 \in G \) and \( \ell_2 \) be an increasing function satisfying \( \ell_2 \geq c \) on \([1,\infty) \) for some \( c > 0 \). Then, \( \ell_1/\ell_2 \in G \).

(iv) Let \( \ell_1(r) = \log(1+r) \) and let \( \ell_{k+1} = \ell_k \circ \ell_1(r) \) for \( k \in \mathbb{N} \). Then for any \( n \in \mathbb{N} \), \( k_1,\ldots,k_n \in \mathbb{N} \) and \( b_1,\ldots,b_n > 0 \) we have

\[
\Lambda(r) = \prod_{i=1}^n (\ell_{k_i}(r))^{b_i} \in G.
\]

(v) For \( b \in (0,1/2) \), \( \ell(r) = (e^{(\log(1+r))^b} - 1) \in G \).

**Proof.** (i)&(ii) Trivial.

(iii) Let \( \ell = \ell_1/\ell_2 \). Using that \( \ell_2 \geq c, \ell_1 \in G \), we see that for any \( a > 0 \)

\[
\sup_{r > 1} \int_1^r \frac{\ell_1(s)}{s\ell_2(s)} ds \cdot \exp \left( -a(\frac{\ell_2(r)}{\ell_1(r)}) \int_1^r \frac{\ell_1(s)}{s\ell_2(s)} ds \right)
\]

\[
\leq \sup_{r > 1} \frac{1}{c} \int_1^r \frac{\ell_1(s)}{s} ds \cdot \exp \left( -a \int_1^r \frac{\ell_1(s)}{s} ds \right) \leq C.
\]

(iv) Let \( \tilde{\Lambda} = (\ell_1)^2 \Lambda \). Then \( \tilde{\Lambda}(r) = \prod_{i=1}^{n+1} (\ell_{k_i}(r))^{b_i} \), and we may set \( k_1 = 1, b_1 = 2 \). If we show that \( \tilde{\Lambda} \in G \), then due to (iii), it follows that \( \Lambda \in G \). Thus we will show that \( \tilde{\Lambda} \in G \). Set \( \ell_0(s) = s \). By using the change of variable,

\[
\int_1^r \frac{\ell_1(s)}{s} ds \approx \int_{\log 2}^{\log(1+r)} \prod_{i=1}^{n+1} (\ell_{k_i-1}(s))^{b_i} ds.
\]
It is easy to see that
\[
\int_{\log 2}^{\log (1+r)} \prod_{i=1}^{n+1} \ell_{k_i-1}(s)^{b_i} \, ds \leq C \log (1+r) \tilde{\Lambda}(r).
\]
Moreover, by the integration by parts and \(((\ell_{k_i-1})^{b_i})'(s) \leq C(b_i, k_i),
\[
\int_{\log 2}^{\log (1+r)} \prod_{i=1}^{n+1} \ell_{k_i-1}(s)^{b_i} \, ds
\]
\[
= \log (1 + r) \tilde{\Lambda}(r) - \log 2 \tilde{\Lambda}(1) - \int_{\log 2}^{\log (1+r)} \left( \prod_{i=1}^{n+1} \ell_{k_i-1}(s)^{b_i} \right)'(s) \, s \, ds
\]
\[
\geq \log (1 + r) \tilde{\Lambda}(r) - \log 2 \tilde{\Lambda}(1) - C \int_{\log 2}^{\log (1+r)} s \, ds
\]
\[
\geq \log (1 + r) \tilde{\Lambda}(r) - \log 2 \tilde{\Lambda}(1) - C(\log (1 + r))^2.
\]
Since \(k_1 = 1\) and \(b_1 = 2\) (for \(\tilde{\Lambda}\)), there exists \(M > 1\) such that
\[
\int_{1}^{r} \tilde{\Lambda}(s) s^{-1} \, ds \geq \frac{1}{2} \log (1 + r) \tilde{\Lambda}(r), \quad \text{for } r > M.
\]
Thus, we have
\[
\sup_{r > M} \int_{1}^{r} \tilde{\Lambda}(s) s \, ds \cdot \exp \left( - \frac{a}{\tilde{\Lambda}(r)} \int_{1}^{r} \tilde{\Lambda}(s) \, ds \right)
\]
\[
\leq C \sup_{r > M} \log (1 + r) \tilde{\Lambda}(r) \cdot \exp (-c \log (1 + r)) \leq C.
\]
The above inequality implies that \(\Lambda \in \mathcal{G}\) since it is bounded on \((1, M]\).

(v) By the change of variable, we have
\[
\int_{1}^{r} \ell(s) s^{-1} \, ds \asymp \int_{\log 2}^{\log (1+r)} (e^{sb} - 1) \, ds \leq C e^{((\log (1+r))^b) \log (1+r)}.
\]
Also, by the integration by parts, we obtain
\[
\int_{\log 2}^{\log (1+r)} (e^{sb} - 1) \, ds = \ell(r) \log (1 + r) - \ell(1) \log 2 - b \int_{\log 2}^{\log (1+r)} s^b e^{sb} \, ds
\]
\[
\geq \ell(r) \log (1 + r) - \ell(1) \log 2
\]
\[
- b(\log (1 + r))^b \int_{\log 2}^{\log (1+r)} (e^{sb} - 1) \, ds - b(\log (1 + r))^{b+1}.
\]
This shows that there exists \(M > 1\) such that for \(r > M\)
\[
\frac{1}{\ell(r)} \int_{\log 2}^{\log (1+r)} (e^{sb} - 1) \, ds \geq (\log (1 + r))^{1-b} - C.
\]
Hence, we have (recall that \(b \in (0, 1/2)\))
\[
\sup_{r > M} \int_{1}^{r} \frac{\ell(s)}{s} \, ds \cdot \exp \left( - \frac{a}{\ell(r)} \int_{1}^{r} \frac{\ell(s)}{s} \, ds \right)
\]
\[
\leq C \sup_{r > M} \left( e^{((\log (1+r))^b) \log (1+r) e^{(-c(\log (1+r))^{1-b})}} \right) \leq C.
\]
Thus \(\ell \in \mathcal{G}\). The lemma is proved. \(\square\)
Lemma A.2. Let \( f : (0, \infty) \to (0, \infty) \) be an increasing continuous function satisfying \( \sup_{0 < r < 1} f(r) < \infty, \lim_{r \to \infty} f(r) = \infty \) and

\[
\frac{f(R)}{f(r)} \leq c_1 \left( \frac{R}{r} \right) ^\delta \quad \text{for } 1 \leq r \leq R < \infty
\]

for some \( c_1, \delta > 0 \). Then there is a strictly increasing continuous function \( \tilde{f} : (0, \infty) \to (0, \infty) \) satisfying

\[
f(r) \leq \tilde{f}(r) \leq Cf(r) \quad \text{for } r > 0,
\]

where the constant \( C \) does not depend on \( r \).

Proof. We prove the lemma by constructing \( \tilde{f} \). Extend \( f \) to \( \mathbb{R}_+ \cup \{0\} \) by

\[
f(0) = \lim_{r \downarrow 0} f(r).
\]

Now let \( A = \{ r \geq 0 : \exists s \geq 0, s \neq r \text{ such that } f(r) = f(s) \} \). Then, we can check that \( A = \bigcup_{k=1}^\infty [r_k, l_k] \), where \( [r_k, l_k] \) are pairwise disjoint closed intervals.

Case 1. Assume that \( f(0) = 0 \) or \( f(r) \) is strictly increasing for \( r \leq 1 \). Then, there is a positive number \( a > 0 \) such that \( A \subset [0, a)^c \). Note that for \( [r_k, l_k] \) we can choose \( \varepsilon_k \) in \( (0, 1) \) such that \( [l_k, l_k + \varepsilon_k] \subset A^c \). Now define \( \tilde{f}_k \) on \( [r_k, l_k + \varepsilon_k] \) as

\[
\tilde{f}_k(r) = \frac{f(l_k + \varepsilon_k) - f(r_k)}{l_k + \varepsilon_k - r_k} (r - r_k) + f(r_k).
\]

Then \( \tilde{f}_k \) is continuous, strictly increasing on \( [r_k, l_k + \varepsilon_k] \) and it satisfies \( \tilde{f}_k(r_k) = f(r_k) \), and \( \tilde{f}_k(l_k + \varepsilon_k) = f(l_k + \varepsilon_k) \). Moreover, on \( [r_k, l_k + \varepsilon_k] \), \( \tilde{f}_k \) satisfies

\[
1 \leq \frac{\tilde{f}_k(r)}{f(r)} \leq \frac{f(l_k + \varepsilon_k)}{f(r_k)} \leq c_1 \left( \frac{l_k + \varepsilon_k}{l_k} \right) ^\delta \leq C(a)
\]

since \( f(r_k) = f(l_k), l_k > r_k > a \). Now define \( \hat{A} = \bigcup_{k=1}^\infty [r_k, l_k + \varepsilon_k] \) and

\[
\tilde{f}(r) = 1_{\hat{A}}(r) f(r) + \sum_{k=1}^\infty 1_{[r_k, l_k + \varepsilon_k]} \tilde{f}_k(r).
\]

Then \( \tilde{f} \) is a desired function.

Case 2. Now assume that \( f(0) \neq 0 \) and \( f(r) \) is not strictly increasing for \( r \leq 1 \). Then there exists \( b \geq 1 \) such that \([0, b] = [r_1, l_1] \). Take \( \varepsilon_1 \) and \( \tilde{f}_1 \) for \([r_1, l_1] \) corresponding to \( \varepsilon_k \) and \( f_k \) in above case. Then on \([r_1, l_1] \), we have

\[
1 \leq \frac{\tilde{f}_1(r)}{f(r)} \leq \frac{f(b + \varepsilon_1)}{f(0)} \leq c_1 \left( \frac{b + \varepsilon_1}{b} \right) ^\delta \leq C(b)
\]

since \( f(0) = f(b) > 0 \). For other \( k \), we have the same result by following Case 1.

Hence, by taking \( \tilde{f} \) as in (A.1), the lemma is proved.  \( \square \)

Lemma A.3. Let \( f : (0, \infty) \to (0, \infty) \) be a strictly increasing continuous function and \( f^{-1} \) be its inverse. Suppose that there exist \( c, \gamma > 0 \) such that \( (f(R)/f(r)) \leq c(R/r)^\gamma \) for \( 0 < r \leq R < \infty \). Then, for any \( k > 0 \), there exists \( C > 0 \) such that for any \( b > 0 \)

\[
\int_{(f(b^{-1}))^{-1}}^{\infty} s^{-1} f^{-1}(s^{-1})^k ds \leq C b^{-k}.
\]
Lemma A.5. By the change of variable and Fubini’s theorem and Lemma A.3, we see that
\[ \frac{f(b^{-1})}{s^{-1}} \leq c \left( \frac{b^{-1}}{f^{-1}(s^{-1})} \right)^{\gamma} \text{ for } s > (f(b^{-1}))^{-1}. \]

Thus,
\[ \int_{(f(b^{-1}))^{-1}}^{\infty} \frac{s^{-1}}{f^{-1}(s^{-1})} ds \leq C b^{-k} (f(b^{-1}))^{-k/\gamma} \int_{(f(b^{-1}))^{-1}}^{\infty} s^{-1-k/\gamma} ds = C b^{-k}. \]

\[ \square \]

We use the following lemma with \( f(r) = h(r^{-1}) \). Note that by using (2.17), one can check that \( h(r^{-1}) \) is a strictly increasing function satisfying \( h(R^{-1}) \leq c(R/r)^2 h(r^{-1}) \) for any \( 0 < r < R \).

Lemma A.4. Let \( f : (0, \infty) \to (0, \infty) \) be a strictly increasing function and \( f^{-1} \) be its inverse. Suppose that there exist \( c, \gamma > 0 \) such that \((f(R)/f(r)) \leq c(R/r)^{\gamma} \) for \( 0 < r \leq R < \infty \). Then, there exists \( C > 0 \) such that for any \( b > 0 \)
\[ \int_{(f(b^{-1}))^{-1/\alpha}}^{\infty} \int_{b \leq |y| \leq (f^{-1}(s^{-\alpha}))^{-1}} \int_{(f(|y|^{-1})^{-1}}^{2s^{\alpha}} (f^{-1}(r^{-1}))^{d+1}s^{-\alpha-1} drdyds \leq C b^{-1}. \]

Proof. By the change of variable and Fubini’s theorem and Lemma A.3
\[ \int_{(f(b^{-1}))^{-1/\alpha}}^{\infty} \int_{b \leq |y| \leq (f^{-1}(s^{-\alpha}))^{-1}} \int_{(f(|y|^{-1})^{-1}}^{2s^{\alpha}} (f^{-1}(r^{-1}))^{d+1}s^{-\alpha-1} drdyds \]
\[ \leq \int_{(f(b^{-1}))^{-1/\alpha}}^{\infty} \int_{(f(|y|^{-1})^{-1}}^{2s^{\alpha}} (f^{-1}(r^{-1}))^{d+1}s^{-\alpha-1} ddrds \]
\[ \leq \int_{(f(b^{-1}))^{-1/\alpha}}^{\infty} \int_{(f(b^{-1}))^{-1}}^{2s^{\alpha}} (f^{-1}(r^{-1}))^{d+1}s^{-\alpha-1} drds \]
\[ \leq \int_{(f(b^{-1}))^{-1/\alpha}}^{\infty} f^{-1}(r^{-1})s^{-\alpha-1} drds \]
\[ \leq \int_{(f(b^{-1}))^{-1}}^{\infty} f^{-1}(r^{-1})s^{-\alpha-1} dr \leq C b^{-1}. \]

\[ \square \]

Lemma A.5. Let \( \alpha \in (0, 1) \). Suppose the function \( \ell \) satisfies Assumption (2.17) (i) let \( \kappa(b) = (h(b))^{-1/\alpha} \) and let \( t_1 > 0 \) be taken from Lemma A.7. Then, there exists \( C > 0 \) depending only on \( \alpha, \kappa_1, \kappa_2, d, \ell, C_0, C_1, C_2, \) and \( \delta \) such that for any \( b > 0 \)
\[ \int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s^{-\alpha})} \int_{0}^{t_1} |D_x p_d(r, y)||\varphi_{\alpha, \alpha+1}(s, r)| drdyds \leq C b^{-1}, \]
\[ \int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_{0}^{t_1} p_d(r, y)||\varphi_{\alpha, \alpha+1}(s, r)| drdyds \leq C. \]
Proof. By \([2.19], [2.20]\), Proposition \(3.2\) (i), \((4.2)\) and \((4.3)\),
\[
\int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s^{-\alpha})} \int_{0}^{t_1} \left| D_x p_d(r, y) \right| |\varphi_{\alpha, \alpha+1}(s, r)| \, dr \, dy \, ds \\
\leq \int_{\kappa(b)}^{\infty} \int_{b}^{h^{-1}(s^{-\alpha})} \int_{0}^{\infty} rK(r)\rho^{-2} e^{-Crh(r)} s^{-\alpha-1} \, dr \, dpds \\
\leq C \int_{\kappa(h(b))^{-1/\alpha}}^{\infty} \int_{b}^{h^{-1}(s^{-\alpha})} \rho^{-2} \frac{1}{h(\rho)} s^{-\alpha-1} \, dp \, ds,
\]
where we used the relations \(se^{-s} \leq Ce^{-s/2} (s > 0)\) and \(K(\rho) \leq h(\rho)\) for the last inequality. By Fubini’s theorem we have
\[
\int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s^{-\alpha})} \int_{0}^{t_1} \left| D_x p_d(r, y) \right| |\varphi_{\alpha, \alpha+1}(s, r)| \, dr \, dy \, ds \\
\leq C \int_{\kappa(h(b))^{-1/\alpha}}^{\infty} \int_{b}^{h^{-1}(s^{-\alpha})} \rho^{-2} \frac{1}{h(\rho)} s^{-\alpha-1} \, dp \, ds
\]
which shows \((A.2)\).

Now we prove \((A.3)\). Using Proposition \(3.2\) (i), \((4.2)\) and \((4.3)\), we see that
\[
\int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_{0}^{t_1} p_d(r, y) \varphi_{\alpha, \alpha+1}(s, r) \, dr \, dy \, ds \\
\leq C \int_{\kappa(4b)}^{\infty} \int_{0}^{4b} \int_{0}^{\infty} rK(\rho)\rho^{-1} e^{-C^{-1}rh(\rho)} s^{-\alpha-1} \, dr \, dpds \\
\leq C \int_{\kappa(4b)}^{\infty} \int_{0}^{\infty} \int_{\mathbb{R}^d} rK(|y|)|y|^{-d} e^{-C^{-1}rh(|y|)} e^{-C^{-1}rh(4b)} s^{-\alpha-1} \, dy \, drds \tag{A.5} \\
\leq C \int_{\kappa(4b)}^{\infty} \int_{0}^{\infty} e^{-C^{-1}rh(4b)/2} s^{-\alpha-1} \, drds \\
\leq C \int_{\kappa(h(4b))^{-1/\alpha}}^{\infty} (h(4b))^{-1} s^{-\alpha-1} \, ds \leq C,
\]
where for the third inequality we use Lemma \(3.3\) (i). \(\square\)

The following lemma is counter part of Lemma \(A.5\). The proof is more delicate than that of Lemma \(A.5\) due to the fact that \(h(r)\) and \(\ell(r^{-1})\) may not be comparable for \(0 < r \leq 1\).

**Lemma A.6.** Let \(\alpha \in (0, 1)\). Suppose the function \(\ell\) satisfies Assumption \(2.11\) (ii)–(ii). Let \(\kappa(b) = (h(b))^{-1/\alpha}\) and let \(t_1 > 0\) be taken from Lemma \(3.4\). Then, there exists \(C > 0\) depending only on \(\alpha, \kappa_1, \kappa_2, d, \ell, C_0, C_1, C_2,\) and \(\delta\) such that for any \(b > 0\)
\[
\int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s^{-\alpha})} \int_{0}^{t_1} \left| D_x p_d(r, y) \right| |\varphi_{\alpha, \alpha+1}(s, r)| \, dr \, dy \, ds \leq Cb^{-1}, \tag{A.6} \\
\int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_{0}^{t_1} p_d(r, y) |\varphi_{\alpha, \alpha+1}(s, r)| \, dr \, dy \, ds \leq C. \tag{A.7}
\]
Proof. Note that \( p(t,0) \) is well-defined on \((0,t_1)\). We first show (A.7). We split the integral into two parts.

\[
\int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_0^{t_1} p_d(r,y) |\varphi_{\alpha,\alpha+1}(s,r)| dr dy ds
\]

\[
\leq \int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_0^{a_0(\ell^*(|y|^{-1})^{-1})} p_d(r,y) |\varphi_{\alpha,\alpha+1}(s,r)| dr dy ds
\]

\[
+ \int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} 1_{a_0(\ell^*(|y|^{-1})^{-1}) \leq t_1} \int_{a_0(\ell^*(|y|^{-1})^{-1})}^{t_1} p_d(r,y) |\varphi_{\alpha,\alpha+1}(s,r)| dr dy ds
\]

\[
= I + II.
\]

We can obtain \( I \leq C \) by using Proposition 3.2 (ii) and the same argument in the proof of (A.3) (see (A.5)). Thus, we will show \( II \leq C \) for some constant \( C \) for the rest of the proof of (A.7). Observe that

\[
II \leq \int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_0^{a_0(\ell^*(|y|^{-1})^{-1})} p_d(r,0) |\varphi_{\alpha,\alpha+1}(s,r)| dr dy ds
\]

\[
\leq \int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} \int_0^{a_0(\ell^*((4b)^{-1})^{-1})} p_d(r,0) |\varphi_{\alpha,\alpha+1}(s,r)| dr dy ds
\]

\[
+ \int_{\kappa(4b)}^{\infty} \int_{|y| \leq 4b} 1_{a_0(\ell^*((4b)^{-1})^{-1}) \leq t_1} \int_{a_0(\ell^*((4b)^{-1})^{-1})}^{t_1} p_d(r,0) |\varphi_{\alpha,\alpha+1}(s,r)| dr dy ds
\]

\[
= II_1 + II_2.
\]

Since \( r \mapsto h(r) \) is decreasing, we see that \( h((\ell^{-1}(a_0/r))^{-1}) \geq h(4b) \) for \( r \leq a_0(\ell^*((4b)^{-1})^{-1})^{-1} \). Hence, by Proposition 3.2 (ii) and Fubini’s theorem

\[
II_1 = \int_{\kappa(4b)}^{\infty} \int_0^{4b} \int_0^{a_0(\ell^*((4b)^{-1})^{-1})} ((\ell^{-1}(a_0/r))^d e^{-crh((\ell^{-1}(a_0/r))^{-1})} s^{-\alpha-1} \rho^{-d-1} dr ds \]

\[
\leq \int_{\kappa(4b)}^{\infty} \int_0^{a_0(\ell^*((4b)^{-1})^{-1})} \int_0^{(\ell^{-1}(a_0/r))^{-1}} ((\ell^{-1}(a_0/r))^d e^{-crh(4b)} s^{-\alpha-1} \rho^{-d-1} d\rho dr ds
\]

\[
\leq C \int_{\kappa(4b)}^{\infty} \frac{1}{h(4b)} s^{-\alpha-1} ds \leq C.
\]

Also for \( II_2 \), by using Proposition 3.2 (ii) and relation \( K \leq h \) and \( se^{-s} \leq ce^{-s/2} \), we have

\[
II_2 \leq Cb^d \int_{\kappa(4b)}^{\infty} \int_0^{\ell^*_1} p_d(a_0(\ell^*((4b)^{-1})^{-1}),0) s^{-\alpha-1} dr ds
\]

\[
\leq Cb^d \int_{\kappa(4b)}^{\infty} \int_0^{\ell^*_1} b^d ((\ell^*(|y|^{-1})^{-1})^{-1} K(4b) e^{-c\rho h(4b)} s^{-\alpha-1} dr ds
\]

\[
\leq C \int_{\kappa(4b)}^{\infty} e^{-c\rho h(4b)} s^{-\alpha-1} dr ds
\]

\[
\leq Ch(4b) e^{-c\rho h(4b)} s^{-\alpha-1} \leq Ch(4b) e^{-c\rho h(4b)} s^{-\alpha-1} \leq C.
\]
Thus, we obtain $II \leq C$.

Now, we show (A.6). First, we see that

\[
\int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s - \alpha)} \int_{0}^{t_1} |D_x p_d(r, y)| \varphi_{\alpha, \alpha + 1}(s, r) \, dr \, dy \, ds \\
\leq \int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s - \alpha)} III(s, y) + 1_{a_0(t^*(|y|^{-1}))^{-1} \leq t_1} IV(s, y) \, dy \, ds,
\]

where

\[
III(s, y) = \int_{0}^{a_0(t^*(|y|^{-1}))^{-1}} |D_x p_d(r, y)| \varphi_{\alpha, \alpha + 1}(s, r) \, dr,
\]

\[
IV(s, y) = \int_{a_0(t^*(|y|^{-1}))^{-1}}^{t_1} |D_x p_d(r, y)| \varphi_{\alpha, \alpha + 1}(s, r) \, dr.
\]

Like (A.4), we have

\[
\int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s - \alpha)} III(s, y) \, dy \, ds \\
\leq C \int_{\kappa(b)}^{\infty} \int_{b}^{h^{-1}(s - \alpha)} \int_{0}^{\infty} \rho^{-2} e^{-crh(\rho)} s^{-\alpha - 1} \, dr \, ds \, dy \\
\leq C \int_{b}^{\infty} \int_{(h(\rho))^{-1/\alpha}}^{\infty} \int_{0}^{\infty} \rho^{-2} e^{-crh(\rho)} s^{-\alpha - 1} \, dr \, ds \\
\leq C \int_{b}^{\infty} \int_{(h(\rho))^{-1/\alpha}}^{\infty} \frac{1}{h(\rho)} \rho^{-2} s^{-\alpha - 1} \, ds \, d\rho \leq C \int_{b}^{\infty} \rho^{-2} d\rho \leq Cb^{-1}.
\]

Hence, we only need to control IV. Observe that by Remark 3.5 (i), Remark 2.9 and (2.25)

\[
\int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s - \alpha)} IV(s, y) \, dy \, ds \\
\leq \int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s - \alpha)} \int_{0}^{t_1} |y| p_{d+2}(r, 0) s^{-\alpha - 1} \, dr \, dy \, ds \\
\leq C \int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s - \alpha)} \int_{a_0(t^*(|y|^{-1}))^{-1}}^{t_1} |y| p_{d+2}(a_0(t^*(|y|^{-1}))^{-1}, 0) s^{-\alpha - 1} \, dr \, dy \\
\leq C \int_{\kappa(b)}^{\infty} \int_{b \leq |y| \leq h^{-1}(s - \alpha)} \int_{a_0(t^*(|y|^{-1}))^{-1}}^{t_1} |y|^{-d-1} e^{-\frac{h(\rho)}{h(\rho)} \frac{h(\rho)}{h(\rho)}} s^{-\alpha - 1} \, dr \, dy \\
\leq C \int_{\kappa(b)}^{\infty} \int_{b}^{h^{-1}(s - \alpha)} \rho^{-2} \frac{h(\rho)}{h(\rho)} e^{-\frac{h(\rho)}{h(\rho)}} s^{-\alpha - 1} \, ds \, d\rho \\
\leq C \int_{\kappa(b)}^{\infty} \int_{b}^{h^{-1}(s - \alpha)} \rho^{-2} \frac{1}{h(\rho)} s^{-\alpha - 1} \, ds \, d\rho \\
\leq C \int_{b}^{\infty} \int_{(h(\rho))^{-1/\alpha}}^{\infty} \rho^{-2} \frac{1}{h(\rho)} s^{-\alpha - 1} \, ds \, d\rho \leq Cb^{-1},
\]

Thus, we obtain (A.6). The lemma is proved.
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