Determinants and traces in the algebra of multidimensional discrete periodic operators with defects

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Abstract

As it is shown in previous works, discrete periodic operators with defects are unitarily equivalent to the operators of the form

\[ \mathcal{A}u = A_0u + A_1 \int_0^1 dk_1 B_1 u + ... + A_N \int_0^1 dk_1 ... \int_0^1 dk_N B_N u, \quad u \in L^2([0,1]^N, \mathbb{C}^M), \]

where \((A, B)(k_1, ..., k_N)\) are continuous matrix-valued functions of appropriate sizes. All such operators form a non-closed algebra \(\mathcal{H}_{N,M}\). In this article we show that there exist a trace \(\tau\) and a determinant \(\pi\) defined for operators from \(\mathcal{H}_{N,M}\) with the properties

\[ \tau(\alpha A + \beta B) = \alpha \tau(A) + \beta \tau(B), \quad \tau(AB) = \tau(BA), \quad \pi(AB) = \pi(A)\pi(B), \quad \pi(e^A) = e^{\tau(A)}. \]

The mappings \(\pi, \tau\) are vector-valued functions. While \(\pi\) has a complex structure, \(\tau\) is simple

\[ \tau(A) = \left( \text{Tr} A_0, \int_0^1 dk_1 \text{Tr} A_1 B_1, ... , \int_0^1 dk_1 ... \int_0^1 dk_N \text{Tr} B_N A_N \right). \]

There exists the strong norm under which the closure \(\overline{\mathcal{H}}_{N,M}\) is a Banach algebra, and \(\pi, \tau\) are continuous (analytic) mappings. This algebra contains simultaneously all operators of multiplication by matrix-valued functions and all operators from the trace class. Thus, it generalizes the other algebras for which determinants and traces was previously defined.

Keywords: discrete periodic operators, multidimensional determinants and traces

1. Introduction

Periodic operators with defects play important role in physics and mechanics of waves, see, e.g., discussions in [1]. It is shown in [2] that these operators are unitarily equivalent to some multidimensional integral operators which form a non-closed algebra. In the current paper we try to construct traces and determinants in this algebra, and we try to find some norms under which these mappings are continuous.

Traces and determinants of square matrices are familiar to us from school. The theory of traces and determinants of some classes of operators acting on infinite dimensional Banach
spaces is presented perfectly in the book [3]. Traces and determinants play important role in various fields. They can be used for determining the spectrum (zeroes of determinants) and for deriving various trace formulas, see, e.g. [4], [5]. There is also a general mathematical interest, see, e.g., [6], [7]. Usually, the discussed determinants are scalars because the spectrum of corresponding operators is discrete. In our case we have the operators with discrete and continuous spectral components. This fact leads to vector-valued functional traces and determinants. To define the determinant we factorize the group of invertible elements of our algebra into the product of 
"elementary" subgroups. For each of the subgroup we determine the scalar functional determinant. The vector consisting of all such determinants is the final determinant that we are looking for. The derivative of this determinant at the identity element is exactly the trace. After that we find a norm under which the trace (and hence the determinant) is continuous. Our algebra equipped with this norm becomes a Banach algebra. Let us start with the definition of the space $L^2_{N,M}$ and the integral operators $\langle \cdot \rangle_j$:

Definition 1.1. Let $L^2_{N,M} := L^2([0,1]^N, \mathbb{C}^M)$ be the Hilbert space of all vector-valued (if $M > 1$) square-integrable functions $f(k)$ with $k = (k_1, ..., k_N) \in [0,1]^N$. Define

$$\langle \cdot \rangle_j := \int_{[0,1]^j} \cdot dk_1...dk_j, \quad j \leq N. \quad (1)$$

The algebra of multidimensional periodic operators with defects was introduced in [2] as

Definition 1.2. The algebra of periodic operators with parallel defects

$$\mathcal{H}_{N,M} = \text{Alg}(\{A\cdot\}, \langle \cdot \rangle_1, ..., \langle \cdot \rangle_N)$$

is a minimal non-closed subalgebra of the algebra of continuous linear operators acting on $L^2_{N,M}$, which contains all operators of multiplication by $M \times M$ continuous matrix-valued functions $A\cdot$ and all integral operators $\langle \cdot \rangle_j$.

Usually we will omit indices $N, M$, i.e. we will write $\mathcal{H} := \mathcal{H}_{N,M}$, $L^2 := L^2_{N,M}$. The next theorem proved in [2] give simple representation of the operators from $\mathcal{H}$.

Theorem 1.3. Each operator $A \in \mathcal{H}$ has a following representation

$$Au = A_0u + A_1\langle B_1u \rangle_1 + ... + A_N\langle B_Nu \rangle_N, \quad u \in L^2, \quad (2)$$

where $A$, $B$ are continuous matrix-valued functions on $[0,1]^N$ of sizes

$$\dim(A_0) = M \times M, \quad \dim(B_j) = M_j \times M, \quad \dim(A_j) = M \times M_j, \quad j \geq 1 \quad (3)$$

with some positive integers $M_j$. The set of all operators of the form (2) coincides with $\mathcal{H}$.

For convenience, we often will replace the argument $u$ with $\cdot$ in formulas like (2). For example, it can be proved that the Hermitian adjoint to $A$ (2) is

$$A^* = A_0^* \cdot + B_1^*\langle A_1^* \rangle_1 + ... + B_N^*\langle A_N^* \rangle_N. \quad (4)$$
The main question is to find the explicit procedure that can tell us \( A \in \mathcal{H} \) is invertible or non-invertible. One of such procedures is constructed in [2]. The inverse operator is also constructed. It was shown that if \( A \) is invertible then \( A^{-1} \in \mathcal{H} \). In the current paper, we provide a little modified version of the procedure from [2]:

**Theorem 1.4.** Let \( A \) be of the form (2). Then

**Step 0.** Define

\[
\pi_0 = \det E_0, \quad E_0 = A_0.
\] (5)

If \( \pi_0(k^0) = 0 \) for some \( k^0 \in [0, 1]^N \) then \( A \) is non-invertible else define

\[
A_{j0} = A_0^{-1}A_j, \quad j = 1, ..., N.
\] (6)

**Step 1.** Define

\[
\pi_1 = \det E_1, \quad E_1 = I + \langle B_1A_{10} \rangle_1.
\] (7)

If \( \pi_1(k^1_0) = 0 \) for some \( k^1_0 \in [0, 1]^{N-1} \) then \( A \) is non-invertible else define

\[
A_{j1} = A_{j0} - A_{10}E_1^{-1}\langle B_1A_{j0} \rangle_1, \quad j = 2, ..., N.
\] (8)

**Step 2.** Define

\[
\pi_2 = \det E_2, \quad E_2 = I + \langle B_2A_{21} \rangle_2.
\] (9)

If \( \pi_2(k^2_0) = 0 \) for some \( k^2_0 \in [0, 1]^{N-2} \) then \( A \) is non-invertible else define

\[
A_{j2} = A_{j1} - A_{21}E_2^{-1}\langle B_2A_{j1} \rangle_2, \quad j = 3, ..., N.
\] (10)

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**Step N.** Define

\[
\pi_N = \det E_N, \quad E_N = I + \langle B_NA_{N,N-1} \rangle_N.
\] (11)

If \( \pi_N = 0 \) then \( A \) is non-invertible else \( A \) is invertible.

This Theorem can be used for determining the spectrum of the operator \( A \). Taking \( \pi_{N+1} = 0 \) and \( \lambda I - A_0 \) instead of \( A_0 \) in the scheme (5)-(11) (or, more general, \( A_j(\lambda, k) \) instead of \( A_j(k) \) for all \( j \), see corresponding generalized spectral problems in [8]) we can define the function

\[
D(\lambda) = \min\{j : \pi_j = 0 \text{ for some } k_j \in [0, 1]^{N-j}\}.
\] (12)

Then the spectrum of \( A \) is the following set

\[
\sigma(A) = \{ \lambda : \quad D(\lambda) \leq N \}.
\] (13)

Moreover, the function \( D \) shows the ”degree” of the spectral point. For example, if \( D(\lambda) < N \) then \( \lambda \) belongs to the essential spectrum (in our case this is a continuous part or an eigenvalue of infinite multiplicity), or if \( D(\lambda) = N \) (the maximum value within the spectrum) then \( \lambda \) is
a point of discrete spectrum. If \( D(\lambda) = N + 1 \) (the maximum value) then \( \lambda \) does not belong to the spectrum.

**Remark on the Floquet-Bloch dispersion curves.** Almost all papers devoted to the wave propagation through periodic media study the so-called Floquet-Bloch dispersion curves, see, e.g., discussions in [9], [10], and [11]. These curves usually describe the dependence of the spectral parameter (e.g. frequency) of the wave-number \( k \). For the pure periodic media this dependence is well-known and can be expressed as \( \lambda = \lambda_0(k) \) where \( \lambda_0 \) is an implicit function satisfying \( \pi_0 \equiv \pi_0(\lambda, k) = 0 \). The corresponding waves are called ”propagative” because they have no attenuation. In our case we have a lot of defects of different dimensions. So there are the waves which propagate along the defects and exponentially decrease in perpendicular directions. Depending on the dimension of defects and of the context this type of waves is usually called ”guided”, ”surface”, ”local”, ”defect”, Rayleigh waves, Love waves and so on. Our method allows us to obtain dispersion equations for such waves. These are

\[
\lambda = \lambda_j(k_j),
\]

where \( \lambda_j \) are implicit functions satisfying \( \pi_j \equiv \pi_j(\lambda, k_j) = 0 \). Note that while \( \pi_0 \) is a polynomial of \( \lambda \) the other functions \( \pi_j \) are much more complex.

The proof of this theorem gives us an explicit representation of inverse operator:

**Theorem 1.5.** Let \( \mathcal{A} \) be of the form (2). If \( \mathcal{A} \) is invertible then

\[
\mathcal{A} = (A_{0}) \circ (I + A_{1,0}(B_1 \cdot)_1) \circ \ldots \circ (I + A_{N,N-1}(B_N \cdot)_N),
\]

where \( A_{j,j-1} \) are defined in the scheme (5)-(11) and \( I \) is the identity operator. Moreover, the inverse operator is

\[
\mathcal{A}^{-1} = (I - A_{N,N-1}E_N^{-1}(B_N \cdot)_N) \circ \ldots \circ (I - A_{1,0}E_1^{-1}(B_1 \cdot)_1) \circ (A_0^{-1}).
\]

(We will use \( \circ \) to denote the multiplication (composition) of operators.)

Define the following subsets of operators from \( \mathcal{H} \):

\[
\mathcal{H}_0 = \{ A \cdot \}, \quad \mathcal{H}_j = \{ A(B \cdot)_j \}, \quad j = 1, ..., N,
\]

\[
\mathcal{F}_0 = \text{Inv}(\mathcal{H}_0), \quad \mathcal{F}_j = \text{Inv}(\{ I + A : A \in \mathcal{H}_j \}), \quad j = 1, ..., N.
\]

where \( A, B \) denote all possible continuous matrix-valued functions of appropriate sizes (see (3)) and \( \text{Inv} \) means all invertible elements of the set. Let us consider some of their properties:

**Theorem 1.6.** The sets \( \mathcal{F}_j \) are groups \( (\circ \) is a multiplication) and

\[
\mathcal{F}_0 = \{ A : \det A \neq 0 \}, \quad \mathcal{F}_j = \{ I + A(B \cdot)_j : \det(I + (BA)j) \neq 0 \},
\]

where \( I \) is the identity matrix of appropriate size. Because \( \det \ldots \) is a function, the expression \( \neq 0 \) assumes everywhere (for any value of the argument). The inverse operator has the form

\[
(I + A(B \cdot)_j)^{-1} = I - A(I + (BA)j)^{-1}(B \cdot)_j.
\]
The sets (16), (17) are the building blocks for $\mathcal{H}$:

**Theorem 1.7.** The following identities are fulfilled:

\[
\mathcal{H} = \sum_{j=0}^{N} \mathcal{H}_j, \quad \text{Inv}(\mathcal{H}) = \prod_{j=0}^{N} \mathcal{F}_j, \tag{20}
\]

where the order of elements in the product is not important. Moreover, for any $A \in \mathcal{H}$, $B \in \text{Inv}(\mathcal{H})$, and any permutation $(j_0, ..., j_N)$ of the set $(0, ..., N)$ there exist unique representations

\[
A = \sum_{j=0}^{N} A_j, \quad A_j \in \mathcal{H}_j, \quad B = B_{j_0} \circ ... \circ B_{j_N}, \quad B_{j_N} \in \mathcal{F}_{j_N}. \tag{21}
\]

For given operator $A$ of the form (2) we have that $A_0 = A_0 \circ$, $A_j = A_j(B_{j'} \circ)$ in (21). Also, the representations (14), (15) are unique for given order of indices, see (21). To formulate our main result let us introduce the commutative algebras of continuous scalar functions

\[
C_j = \{ f : [0, 1]^{N-j} \to \mathbb{C} | j = 0, ..., N-1; \ C_N = \mathbb{C}; \ C = C_0 \times C_1 \times ... \times C_N. \tag{22}\]

**Theorem 1.8.** The mapping (see definitions of $\pi_j$ in (5)-(11))

\[
\pi = (\pi_0, ..., \pi_N) : \text{Inv}(\mathcal{H}) \to \text{Inv}(\mathcal{C}) \tag{23}
\]

is a group homomorphism. Moreover, $\pi|_{\mathcal{F}_j} = (1, ..., \pi_j, ..., 1)$.

The result (23) of this theorem shows us that $\pi$ is an analogue of the standard determinant of matrices (or matrix-valued functions). The set $\text{Inv}(\mathcal{C})$ has a simple form, it consists of non-zero continuous functions. For the one dimensional case the theory of Fredholm determinants of \{identity + compact operators\} is well developed, see, e.g., [3]. In our case the situation is complicated by the fact that our perturbations are not compact in the usual sense. That is why our construction leads to the vector-valued functional determinant $\pi = (\pi_j)$. Note that by using the different combinations of $\pi_j$ we can construct other homomorphisms such as the product $\pi_0 \cdot ... \pi_N$ but they contain less information than $\pi$. The determinant $\pi(A)$ completely describes the spectrum of the operator $A$. For example, we can define the isospectral set of operators as

\[
\text{Iso}(A) = \{ B : \pi(\lambda I - A) = \pi(\lambda I - B) \text{ for large } \lambda \}. \tag{24}
\]

Along with the vector-valued determinant $\pi$ of invertible operators of the form (2) it is possible to define the vector-valued trace $\tau$ for all operators (invertible and non-invertible) of the form (2):

\[
\tau(A) := \left. \frac{\partial \pi(I + tA)}{\partial t} \right|_{t=0} = \lim_{t \to 0} \frac{\pi(I + tA) - \pi(I)}{t}. \tag{25}
\]

Due to the fact that $\pi$ is a homomorphism the derivative at other points $A \in \mathcal{G}$ can be found as

\[
\left. \frac{\partial \pi(A + tB)}{\partial t} \right|_{t=0} = \pi(A)\tau(A^{-1}B), \tag{26}
\]

where the product of vectors means component-wise product. The next Theorem gives us the explicit formula for $\tau$ and provides its properties.
Theorem 1.9. For any operator $A$ of the form (2) the following identity is fulfilled

$$\tau(A) = (\text{Tr} A_0, \langle \text{Tr} B_1, A_1 \rangle, ..., \langle \text{Tr} B_N, A_N \rangle),$$  \hspace{1cm} (27)

where Tr means the standard trace of square matrices. Moreover, the following properties are fulfilled also

$$\tau(\alpha A + \beta B) = \alpha \tau(A) + \beta \tau(B), \quad \tau(A \circ B) = \tau(B \circ A)$$  \hspace{1cm} (28)

for any $A, B$ of the form (2) and any $\alpha, \beta \in \mathbb{C}$.

Roughly speaking, it can be shown that (27) is in good agreement with the known trace of finite rank operators. In this sense, the definition of $\tau$ (and hence $\pi$) is unique up to elementary combinations of its components. Let us discuss some trace norm under which $\pi$ and $\tau$ are continuous and a completion of $\mathcal{H}$ is a Banach algebra. For an operator $A$ of the form (2) define the functions

$$g_j(k_j) = \sum_{n=1}^{M_j} \sqrt{\lambda_{nj}(k_j)}, \quad k_j = (k_{j+1}, ..., k_N) \in [0,1]^{N-j}, \quad j = 0, ..., N$$  \hspace{1cm} (29)

(for $j = N$ there is no dependence on $k_N$ and $f_N$ is just a number), where $\{\lambda_{nj}\}_{n=1}^{M_j}$ are eigenvalues of $M_j \times M_j$ matrices $C_j$ defined by

$$C_0 := A_0^* A_0, \quad C_j := \langle B_j B_j^* \rangle_j \langle A_j^* A_j \rangle_j.$$  \hspace{1cm} (30)

All $\lambda_{nj}$ are non-negative because they are singular values of the operator $A_j \langle B_j \cdot \rangle_j$. Define the following non-negative function

$$\|A\|_{\text{tr}} = \max_{k_0 \in [0,1]^N} g_0(k_0) + \max_{k_1 \in [0,1]^{N-j}} g_1(k_1) + ... + g_N.$$  \hspace{1cm} (31)

We also denote

$$\|f\|_c = \max_j \max_{k_j \in [0,1]^{N-j}} |f_j(k_j)|, \quad f = (f_0, ..., f_N) \in \mathcal{C},$$  \hspace{1cm} (32)

where $\mathcal{C}$ is a commutative Banach algebra defined in (22) with an element-wise multiplication.

Theorem 1.10. The function $\| \cdot \|_{\text{tr}}$ is a norm on $\mathcal{H}$. The corresponding completion $\overline{\mathcal{H}}$ is a Banach algebra with

$$\|A \circ B\|_{\text{tr}} \leq \|A\|_{\text{tr}} \|B\|_{\text{tr}}, \quad \|A\| \leq \|A\|_{\text{tr}}, \quad \forall A, B \in \overline{\mathcal{H}},$$  \hspace{1cm} (33)

where $\| \cdot \|$ denotes the standard operator norm. The mappings $\tau$ and $\pi$ are continuous and have continuous extensions

$$\tau : \overline{\mathcal{H}} \to \mathcal{C}, \quad \pi : \text{Inv}(\overline{\mathcal{H}}) \to \text{Inv}(\mathcal{C}).$$  \hspace{1cm} (34)
The norm of $\tau$ (as a linear operator) is 1 and $\forall A \in \mathcal{H}$ we have

$$\ln \pi(\lambda I - A) = (\ln \lambda)\tau(I) - \sum_{n=1}^{\infty} \frac{\tau(A^n)}{n\lambda^n}, \quad |\lambda| > \|A\|_{\text{tr}}, \quad (35)$$

where $\ln$ in the left-hand side means element-wise logarithm and $\tau(I) = (M, 0, \ldots, 0)$.

Note that (35) with (27) allow us to obtain more convenient representation of the set (24)

$$\text{Iso}(A) = \{B : \tau(B^n) = \tau(A^n) \text{ for all } n \in \mathbb{N}\}. \quad (36)$$

Another interesting equation $\pi(e^A) = e^{\tau(A)}$ for all $A \in \mathcal{H}$ immediately follows from (35).

Note also that the resolvent (15) allows us to write closed form expressions in the functional calculus, e.g.

$$f(A) = \frac{1}{2\pi i} \oint_{\partial \Omega} f(\lambda)(\lambda I - A)^{-1}d\lambda, \quad \tau(f(A)) = \frac{1}{2\pi i} \oint_{\partial \Omega} f(\lambda)\tau((\lambda I - A)^{-1})d\lambda$$

for analytic functions $f$ defined in some domain $\Omega \supset \sigma(A)$. Let us finish with some exercises: try to extend the results of Theorem 1.7 to the arbitrary subset $\alpha \subset \{0, \ldots, N\}$, i.e. if $0 \in \alpha$ then $\text{Inv}(\sum_{j \in \alpha} \mathcal{H}_j) = \prod_{j \in \alpha} \mathcal{F}_j$; try to prove that $\prod_{r=1}^{N} \mathcal{F}_r \triangleleft \text{Inv}(\mathcal{H})$ is a normal subgroup and $\sum_{r=1}^{N} \mathcal{H}_r \subset \mathcal{H}$ is a two-sided ideal for all $j$. Let us specify the structure of the paper: Section 2 contains the proofs of all theorems; Section 3 provides a simple example of applications of our results to some concrete operator.

2. Proof of Theorems 1.4-1.10

There are a lot of different ways to prove these theorems. We try to make the proof to be more or less elementary, except the last part where we discuss the trace norm. In the last part we intensively use properties of direct integrals [12] and determinants and traces [3] of compact operators. Note that the first four Lemmas repeat the arguments from [2]. We present their Proofs in a short form.

Lemma 2.1. Suppose that two operators of the form (2) are equal

$$A_0 + A_1(B_1^\cdot)_1 + \ldots + A_N(B_N^\cdot)_N = \tilde{A}_0 + \tilde{A}_1(B_1^\cdot)_1 + \ldots + \tilde{A}_N(B_N^\cdot)_N. \quad (37)$$

Then its components are equal too

$$A_0 = \tilde{A}_0 \quad \text{and} \quad A_j(B_j^\cdot)_j = \tilde{A}_j(B_j^\cdot)_j \quad \text{for} \quad j = 1, \ldots, N. \quad (38)$$

Proof. Suppose that $A \neq A_0$. Then there exists some continuous vector-valued function $f$ and $k^0 \in [0, 1]^N$ such that $(A_0 - \tilde{A}_0)f(k^0) = f^0 \neq 0$. Consider some continuous scalar function $\chi(k)$ with properties

$$\chi(k) \leq 1, \quad \chi(k^0) = 1, \quad \chi(k) = 0 \quad \text{for} \quad \|k - k^0\| > \varepsilon. \quad (39)$$
Then the identity (37) along with the continuity of $A$, $B$ and $\tilde{A}$, $\tilde{B}$ leads to

$$f^0 = (A_0 - \tilde{A}_0)(\chi f)(k^0) = \sum_{j=1}^{N} (A_j \langle B_j \chi f \rangle_j - \tilde{A}_j \langle \tilde{B}_j \chi f \rangle_j)(k^0).$$

(40)

The fact that $|\langle \chi \rangle_j| \leq 2\varepsilon$ shows that the norm of the right-hand side of (40) is less than $C\varepsilon$ with some fixed $C$ depending on $A$, $B$ and $\tilde{A}$, $\tilde{B}$ only. This is the contradiction to a fixed norm of the left-hand side of (40). Thus $A_0 = \tilde{A}_0$. Now suppose that we proved (38) up to $r-1$-th component for some $r \geq 1$. So we have the equality

$$A_r \langle B_r \cdot \rangle_r + \ldots + A_N \langle B_{N'} \cdot \rangle_N = \tilde{A}_r \langle \tilde{B}_r \cdot \rangle_r + \ldots + \tilde{A}_N \langle \tilde{B}_{N'} \cdot \rangle_N.$$

(41)

Suppose that $A_r \langle B_r \cdot \rangle_r \neq \tilde{A}_r \langle \tilde{B}_r \cdot \rangle_r$. Then there exists some continuous vector-value function $f$ and $k^0$ such that

$$\langle A_r \langle B_r f \rangle_r - \tilde{A}_r \langle \tilde{B}_r f \rangle_r \rangle(k^0) = f^0 \neq 0.$$

(42)

Let $k_r^0 = (k_{r+1}^0, \ldots, k_N^0)$ be the vector consisting of $N - r$ components of the vector $k^0$. Consider some continuous scalar function $\chi(k)$ with properties

$$\chi(k) \leq 1, \quad \chi([0, 1]^r \times k_r^0) = 1, \quad \chi(k) = 0 \quad \text{for} \quad \|k_r - k_r^0\| > \varepsilon,$$

(43)

where $k_r = (k_{r+1}, \ldots, k_N)$ is the vector consisting of $N - r$ components of the vector $k$. The identities (41), (42) along with the continuity of $A$, $B$ and $\tilde{A}$, $\tilde{B}$ lead to

$$f^0 = (A_r \langle B_r \chi f \rangle_r - \tilde{A}_r \langle \tilde{B}_r \chi f \rangle_r)(k^0) = \sum_{j=r+1}^{N} (A_j \langle B_j \chi f \rangle_j - \tilde{A}_j \langle \tilde{B}_j \chi f \rangle_j)(k^0).$$

(44)

The fact that $|\langle \chi \rangle_j| \leq 2\varepsilon$ for $j \geq r+1$ shows that the norm of the right-hand side of (44) is less than $C\varepsilon$ with some fixed $C$ depending on $A$, $B$ and $\tilde{A}$, $\tilde{B}$ only. This is the contradiction to a fixed norm of the left-hand side of (44). Thus $A_r \langle B_r \cdot \rangle_r = \tilde{A}_r \langle \tilde{B}_r \cdot \rangle_r$ and we finish proof by induction.

\textbf{Lemma 2.2.} Consider the operator $A$ of the form (2). Suppose that $\det A_0(k^0) = 0$ at some $k^0 \in [0, 1]^N$. Then $A$ is non-invertible.

\textbf{Proof.} Let $f^0$ be the corresponding null-vector $A_0(k^0)f^0 = 0$ with Hilbert norm $\|f^0\| = 1$. Without loss of generality we may assume that $k^0 \in (0, 1)^N$. For all sufficiently small $\varepsilon > 0$ define scalar functions

$$\chi_{\varepsilon}(k) = \begin{cases} \varepsilon^{-\frac{N}{2}}, & \max_{j=1, \ldots, N} |k_j - k_j^0| < \varepsilon/2, \\ 0, & \text{otherwise}. \end{cases} \quad (45)$$

The Hilbert norm of functions $f_{\varepsilon}(k) = \chi_{\varepsilon}(k)f_0$ is equal to 1 but the norm of

$$Af_{\varepsilon} = A_0f_{\varepsilon} + \sum_{j=1}^{N} A_j \langle B_j \chi f_{\varepsilon} \rangle_j$$

(46)
tends to 0 for \( \varepsilon \to 0 \) because the support of \( f_\varepsilon \) tends to \( k_0 \). All matrix-valued functions \( A, B \) are continuous and the Hilbert norm of \( |\chi_j| \) is equal to \( \varepsilon^2 \) and tends to 0 for \( \varepsilon \to 0 \). The Banach Theorem about continuous inverse operators shows us that \( \mathcal{A} \) is non-invertible.

Lemma 2.3. Consider the operator (2) of the special form

\[ \mathcal{A} = \mathcal{I} + A_r \langle B_r \rangle_r + \sum_{j=r+1}^{N} A_j \langle B_j \rangle_j \]  

with some \( r \geq 1 \). If

\[ \det(I + \langle B_r A_r \rangle_r)(k^0) = 0 \]  

at some \( k^0 \in [0, 1]^{N-r} \) (the determinant does not depend on the first \( r \) components of \( k \)) then \( \mathcal{A} \) is non-invertible.

Proof. Let \( f_0 \) be a null-vector of the matrix \((I + \langle B_r A_r \rangle_r)(k^0)\) with the Hilbert norm \( \|f_0\| = 1 \). Without loss of generality we may assume that \( k^0 \in (0, 1)^N \). For all sufficiently small \( \varepsilon > 0 \) define scalar functions

\[ \chi_\varepsilon(k) = \begin{cases} \varepsilon^{-\frac{N-r}{2}}, & \max_{j=r+1, \ldots, N} |k_j - k^0_j| < \varepsilon/2, \\ 0, & \text{otherwise} \end{cases} \]  

and vector-valued functions \( f_\varepsilon = \chi_\varepsilon A_r f_0 \). Suppose that the Hilbert norm of functions \( \|f_\varepsilon\| \) tends to zero for \( \varepsilon \to 0 \). Then the fact that \( \|\chi_\varepsilon f_0\| = 1 \) gives us

\[ \|\chi_\varepsilon f_0 + \langle \chi_\varepsilon B_r A_r f_0 \rangle_r\| = \|\chi_\varepsilon f_0 + \langle B_r f_\varepsilon \rangle_r\| = 1 + o(1), \quad \varepsilon \to 0. \]  

On the other hand

\[ \|\chi_\varepsilon f_0 + \langle \chi_\varepsilon B_r A_r f_0 \rangle_r\| = \|\chi_\varepsilon (I + \langle B_r A_r \rangle_r) f_0\| = o(1), \quad \varepsilon \to 0 \]  

because \( f_0 \) is a null-vector of \((I + \langle B_r A_r \rangle_r)(k^0)\), the function \( \chi_\varepsilon \) does not depend on the first \( r \) components of \( k \) and its support tends to \([0, 1]^r \times k^0 \) for \( \varepsilon \to 0 \). The formulas (50) and (51) contradict each other, which means that our assumption is not clear and we have that

\[ \|f_\varepsilon\| \not\to 0 \quad \text{for} \quad \varepsilon \to 0. \]  

The identity (47) and the definition of \( f_\varepsilon \) give us

\[ \mathcal{A} f_\varepsilon = \chi_\varepsilon A_r (I + \langle B_r A_r \rangle_r) f_0^0 + \sum_{j=r+1}^{N} A_j \langle \chi_\varepsilon B_j A_r f_0 \rangle_j, \]  

which leads to

\[ \|\mathcal{A} f_\varepsilon\| \to \varepsilon \quad \text{for} \quad \varepsilon \to 0, \]  

since we have arguments (51), continuity of \( A, B \) and \( \|\langle \chi_\varepsilon \rangle_j\| = \varepsilon^\frac{2r}{2r} \) tends 0 for \( \varepsilon \to 0 \) and \( j > r \). The formulas (52), (54) and the Banach Theorem about continuous inverse operators show us that \( \mathcal{A} \) is non-invertible. \( \blacksquare \)
Lemma 2.4. The set $\mathcal{H}_j$ (10) is an algebra.

Proof. It follows from the following identities:
\[
A_j \langle B_j \rangle_j + \tilde{A}_j \langle \tilde{B}_j \rangle_j = \tilde{C}_j \langle \tilde{D}_j \rangle_j,
\]
where
\[
\tilde{C}_j = \begin{pmatrix} A_j & \tilde{A}_j \end{pmatrix}, \quad \tilde{D}_j = \begin{pmatrix} B_j \\ \tilde{B}_j \end{pmatrix}
\]
and
\[
\left( A_j \langle B_j \rangle_j \right) \circ \left( \tilde{A}_j \langle \tilde{B}_j \rangle_j \right) = A_j \langle B_j \tilde{A}_j \tilde{B}_j \rangle_j = \tilde{C}_s \langle \tilde{D}_s u \rangle_s,
\]
where
\[
s = \max\{j, r\} \quad \text{and} \quad \begin{cases} \tilde{C}_s = A_j \langle B_j \tilde{A}_j \rangle_j, & j \leq r \\ \tilde{D}_s = \tilde{B}_r, & j > r \end{cases} \quad \blacksquare
\]

Lemma 2.5. The set $\mathcal{F}_j$ (17) is a group for any $j = 0, \ldots, N$. If $j \geq 1$ then the element $\mathcal{A} = I + A \langle B \rangle_j$ belongs to $\mathcal{F}_j$ if and only if the determinant of $E = I + \langle BA \rangle_j$ is non-zero everywhere. In this case the inverse operator is
\[
\mathcal{A}^{-1} = I - AE^{-1} \langle B \rangle_j.
\]

Proof. For $j = 0$ the statement is trivial. Consider the case $j \geq 1$. If $\mathcal{A}, \mathcal{B} \in \mathcal{F}_j$ then by (57), (58) the element $\mathcal{C} = \mathcal{A} \circ \mathcal{B}$ has the form $\mathcal{C} = I + C \langle D \rangle_j$ with some continuous matrix-valued functions $C, D$ and hence it belongs to $\mathcal{F}_j$ because it is invertible like $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{A} = I + A \langle B \rangle_j$ be some element of $\mathcal{F}_j$. If $\det E = 0$ at some point then by Lemma 2.3 the operator $\mathcal{A}$ is non-invertible, which is impossible because $\mathcal{A} \in \mathcal{F}_j$. Then $\det E \neq 0$ everywhere and hence $E^{-1}$ is a continuous matrix-valued function. Define $\mathcal{B} = I - AE^{-1} \langle B \rangle_j$. Then
\[
\mathcal{A} \circ \mathcal{B} = I + A \langle B \rangle_j - AE^{-1} \langle B \rangle_j - A \langle BA E^{-1} \rangle_j \langle B \rangle_j =
\]
\[
I + (A - AE^{-1}) - A \langle BA \rangle E^{-1} \langle B \rangle_j = I + (A - AE^{-1}) \langle B \rangle_j = I,
\]
where we used the fact that $E$ does not depend on the first $j$ components of the vector $k$. \blacksquare

Proof of Theorem 1.6. It follows from Lemmas 2.2 and 2.5 \blacksquare

Proof of Theorem 1.4

Step 0. If $\pi_0 = \det \mathcal{A}_0(k_0) = 0$ at some point $k_0 \in [0, 1]^N$ then by Lemma 2.2 the operator $\mathcal{A}$ is non-invertible. Suppose that $\det \mathcal{A}_0 \neq 0$ everywhere. Then $\mathcal{A}_0^{-1}$ is a continuous matrix-valued function and we may define the operator (see (5))
\[
\mathcal{A}_0 = \mathcal{A}_0^{-1} \mathcal{A} = I + A_1 \langle B_1 \rangle_1 + \ldots + A_N \langle B_N \rangle_N.
\]

Step 1. If $\pi_1 = \det \mathcal{E}_1(k_1) = 0$ at some point $k_1 \in [0, 1]^{N-1}$ then by Lemma 2.3 the operator $\mathcal{A}_0$ and hence $\mathcal{A}$ (see (60)) are non-invertible. Suppose that $\det \mathcal{E}_1 \neq 0$ everywhere.
Then $E_1^{-1}$ is a continuous matrix-valued function and we may define the operator (see (59) and (8))
\[ A_1 = (\mathcal{I} + A_{10} \langle B_1 \cdot \rangle_1)^{-1} \circ A_0 = (\mathcal{I} - A_{10} E_1^{-1} \langle B_1 \cdot \rangle_1) \circ A_0 = \]
(61)
\[ \mathcal{I} + A_{21} \langle B_2 \cdot \rangle_2 + \ldots + A_{N1} \langle B_N \cdot \rangle_N. \quad (62) \]

Step 2. If $\pi_2 = \det E_2(k_2^0) = 0$ at some point $k_2^0 \in [0,1]^{N-2}$ then by Lemma 2.3 the operator $A_1$ and hence $A_0$ and $A$ (see (60)-(62)) are non-invertible. Suppose that $\det E_2 \neq 0$ everywhere. Then $E_2^{-1}$ is a continuous matrix-valued function and we may define the operator (see (59) and (10))
\[ A_2 = (\mathcal{I} + A_{21} \langle B_2 \cdot \rangle_2)^{-1} \circ A_1 = (\mathcal{I} - A_{21} E_2^{-1} \langle B_2 \cdot \rangle_2) \circ A_1 = \]
(63)
\[ \mathcal{I} + A_{32} \langle B_3 \cdot \rangle_3 + \ldots + A_{N2} \langle B_N \cdot \rangle_N. \quad (64) \]

Repeating this procedure up to the step $N$ we finish the proof. Note that we also obtain the identities (14) and (15).

**Proof of Theorem 1.5.** It follows immediately from the proof of Theorem 1.4 and from the identity for inverse operators (59).

**Definition 2.6.** For $j = 1, \ldots, N$ and for any two continuous matrix-valued functions $A$ and $B$ of sizes $M \times M_1$ and $M_1 \times M$ ($M_1$ is any positive integer) defined on $[0,1]^N$ introduce the following scalar function
\[ \tilde{\pi}_j(A, B) = \det(I + \langle BA \rangle_j). \quad (65) \]

**Lemma 2.7.** Let $\mathcal{A}_i = \mathcal{I} + A_i \langle B_i \cdot \rangle_j$, $i = 1, 2$ be two arbitrary operators of the form (2). Then there exist continuous matrix-valued functions $A_3, B_3$ satisfying
\[ A_1 \circ A_2 = \mathcal{I} + A_3 \langle B_3 \cdot \rangle_j, \quad \tilde{\pi}_j(A_1, B_1) \tilde{\pi}_j(A_2, B_2) = \tilde{\pi}_j(A_3, B_3). \quad (66) \]

**Proof.** Consider the composition
\[ A_1 \circ A_2 = (\mathcal{I} + A_1 \langle B_1 \cdot \rangle_j) \circ (\mathcal{I} + A_2 \langle B_2 \cdot \rangle_j) = \]
\[ \mathcal{I} + A_1 \langle B_1 \cdot \rangle_j + A_2 \langle B_2 \cdot \rangle_j + A_1 \langle B_1 A_2 \rangle_j \langle B_2 \rangle_j = \mathcal{I} + A_3 \langle B_3 \cdot \rangle_j \]
with
\[ A_3 = \begin{pmatrix} A_1 & A_2 + A_1 \langle B_1 A_2 \rangle_j \\ B_1 & B_2 \end{pmatrix}, \quad B_3 = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}. \]

Then
\[ \tilde{\pi}_j(A_3, B_3) = \det(I + \langle B_3 A_3 \rangle_j) = \]
\[ \det \left( I + \langle B_1 A_1 \rangle_j, \quad I + \langle B_2 A_2 \rangle_j + \langle B_2 A_1 \rangle_j \langle B_1 A_2 \rangle_j \right) = \]
\[ \det \left( I + \langle B_1 A_1 \rangle_j, \quad I + \langle B_2 A_2 \rangle_j \right) \det \left( I + \langle B_1 A_2 \rangle_j \right) = \]
\[ \det(I + \langle B_1 A_1 \rangle_j) \det(I + \langle B_2 A_2 \rangle_j) = \tilde{\pi}_j(A_1, B_1) \tilde{\pi}_j(A_2, B_2). \quad \Box \]
Lemma 2.8. Suppose that \( \mathcal{A} = \mathcal{I} + \mathcal{A} \langle \mathbf{B} \rangle_j = \mathcal{I} \) is an identity operator. Then \( \pi_j(\mathcal{A}, \mathcal{B}) = 1 \).

Proof. Acting \( \mathcal{A} \) on each column of the matrix \( \mathcal{A} \) and after that multiplying by \( \mathcal{B} \) and integrating we deduce that
\[
\mathcal{A} \langle \mathbf{B} \mathbf{A} \rangle_j = 0 \Rightarrow \langle \mathbf{B} \mathbf{A} \rangle_j^2 = 0.
\]
Then
\[
1 = \det \mathcal{I} = \det(\mathcal{I} - t^2 \langle \mathbf{B} \mathbf{A} \rangle_j^2) = \det(\mathcal{I} + t \langle \mathbf{B} \mathbf{A} \rangle_j) \det(\mathcal{I} - t \langle \mathbf{B} \mathbf{A} \rangle_j) = f(t)f(-t),
\]
where \( t \in \mathbb{C} \) and \( f(t) = \det(\mathcal{I} + t \langle \mathbf{B} \mathbf{A} \rangle_j) \) is a polynomial in \( t \). Then \( f(t) \) is a constant and \( f(t) = f(0) = 1 \). At the same time \( \pi_j(\mathcal{A}, \mathcal{B}) = f(1) = 1 \). ■

Lemma 2.9. The following implication is fulfilled
\[
\mathcal{I} + \mathcal{A}_1 \langle \mathbf{B}_1 \rangle_j = \mathcal{I} + \mathcal{A}_2 \langle \mathbf{B}_2 \rangle_j \in \mathcal{F}_j \Rightarrow \pi_j(\mathcal{A}_1, \mathcal{B}_1) = \pi_j(\mathcal{A}_2, \mathcal{B}_2).
\]  

Proof. Taking the inverse operator (see (59) in Lemma 2.8) and using (67) we have two identities
\[
(\mathcal{I} + \mathcal{A}_1 \langle \mathbf{B}_1 \rangle_j) \circ (\mathcal{I} - \mathcal{A}_1 \mathbf{E}_1^{-1} \langle \mathbf{B}_1 \rangle_j) = \mathcal{I},
\]
\[
(\mathcal{I} - \mathcal{A}_1 \mathbf{E}_1^{-1} \langle \mathbf{B}_1 \rangle_j) \circ (\mathcal{I} + \mathcal{A}_2 \langle \mathbf{B}_2 \rangle_j) = \mathcal{I},
\]
where \( \mathbf{E}_1 = \mathcal{I} + \langle \mathbf{B}_1 \mathcal{A}_1 \rangle_j \). Then Lemmas 2.7 and 2.8 give us
\[
\pi_j(\mathcal{A}_1, \mathcal{B}_1)\pi_j(-\mathcal{A}_1 \mathbf{E}_1^{-1}, \mathcal{B}_1) = 1 = \pi_j(-\mathcal{A}_1 \mathbf{E}_1^{-1}, \mathcal{B}_1)\pi_j(\mathcal{A}_2, \mathcal{B}_2),
\]
which leads to \( \pi_j(\mathcal{A}_1, \mathcal{B}_1) = \pi_j(\mathcal{A}_2, \mathcal{B}_2) \). ■

Definition 2.10. For any \( j = 1, \ldots, N \) and any \( \mathcal{A} \in \mathcal{F}_j \) define the mapping \( \pi_j(\mathcal{A}) = \pi_j(\mathcal{A}, \mathcal{B}) \), where \( \mathcal{A} = \mathcal{I} + \mathcal{A} \langle \mathbf{B} \rangle_j \) is some representation of \( \mathcal{A} \). By Lemma 2.9 this definition of \( \pi_j(\mathcal{A}) \) is correct. Also define \( \pi_0(\mathcal{A} \cdot) = \det \mathcal{A} \) for any \( \mathcal{A} \cdot \in \mathcal{F}_0 \).

Lemma 2.11. The mapping \( \pi_j : \mathcal{F}_j \rightarrow \mathcal{C}_j \) given by the definition 2.10 is a group homomorphism (see also definition of \( \mathcal{C}_j \) after (15)).

Proof. Now this result follows from Lemma 2.7. ■

Lemma 2.12. Suppose that \( \mathcal{A} = \mathcal{A}_j \mathcal{A}_r \) for some \( \mathcal{A}_j \in \mathcal{F}_j \) and \( \mathcal{A}_r \in \mathcal{F}_r \) and \( j \neq r \). Then there exists unique representation \( \mathcal{A} = \tilde{\mathcal{A}}_j \tilde{\mathcal{A}}_r \) with \( \tilde{\mathcal{A}}_j \in \mathcal{F}_j \) and \( \tilde{\mathcal{A}}_r \in \mathcal{F}_r \). The identities \( \pi_j(\mathcal{A}_j) = \pi_j(\tilde{\mathcal{A}}_j) \) and \( \pi_r(\mathcal{A}_r) = \pi_r(\tilde{\mathcal{A}}_r) \) are fulfilled. Moreover, if \( j < r \) then \( \tilde{\mathcal{A}}_j = \mathcal{A}_j \), if \( r < j \) then \( \tilde{\mathcal{A}}_j = \mathcal{A}_r \).
Proof. Consider the case $1 \leq j < r$ (other cases can be proved similarly). Take some representations of $A_j$ and $A_r$

$$A_j = \mathcal{I} + A_j \langle B_j \rangle, \ A_r = \mathcal{I} + A_r \langle B_r \rangle.$$  

Then the following identities are fulfilled

$$A = A_j \circ A_r = (\mathcal{I} + A_j \langle B_j \rangle) \circ (\mathcal{I} + A_r \langle B_r \rangle) =$$

$$\mathcal{I} + A_j \langle B_j \rangle + A_r \langle B_r \rangle + A_j \langle B_j A_r \rangle \langle B_r \rangle =$$

$$(\mathcal{I} + (A_r + A_j \langle B_j A_r \rangle \langle B_r \rangle) \circ (I + A_j \langle B_j \rangle) \circ (I + A_j \langle B_j \rangle) =$$

$$(\mathcal{I} + (A_r + A_j \langle B_j A_r \rangle \langle B_r \rangle) \circ (I - A_j E^{-1} \langle B_j \rangle) \circ (I + A_j \langle B_j \rangle) =$$

$$(\mathcal{I} + (A_r + A_j \langle B_j A_r \rangle \langle B_r \rangle) \circ (I - B_r A_r E^{-1} \langle B_j \rangle) \circ A_j =$$

$$\mathcal{A}_r \circ A_j = \mathcal{A}_r \circ A_j,$$

where $E_j = I + \langle B_j A_r \rangle$ (see Lemma 2.5), $\mathcal{A}_j = A_j$ and $\mathcal{A}_r = \mathcal{I} + \mathcal{A}_r \langle B_r \rangle$ with

$$\mathcal{A}_r = A_r + A_j \langle B_j A_r \rangle, \ B_r = B_r - \langle B_r A_r \rangle E^{-1} \langle B_j \rangle.$$

Thus, we have $\pi_j(A_j) = \pi_j(\mathcal{A}_j)$ and

$$\pi_r(A_r) = \det(I + \langle B_r A_r \rangle) = \det(I + \langle B_r A_r \rangle) +$$

$$\langle B_r A_r \rangle \langle B_r A_r \rangle - \langle B_r A_r \rangle E^{-1} \langle B_r A_r \rangle = \det(I + \langle B_r A_r \rangle) +$$

$$\det(I + \langle B_r A_r \rangle) = \pi_r(A_r).$$

Suppose that we have two different representations $A = \mathcal{A}_r \mathcal{A}_j = \mathcal{A}_r \mathcal{A}_j$ with $\mathcal{A}_r, \mathcal{A}_j \in \mathcal{F}_r$ and $\mathcal{A}_r, \mathcal{A}_j \in \mathcal{F}_j$. Then $\mathcal{F}_r \ni \mathcal{A}_r^{-1} \mathcal{A}_r = \mathcal{A}_j^{-1} \mathcal{A}_j \in \mathcal{F}_r$, which gives us $A_r = \mathcal{A}_r$ and $A_j = \mathcal{A}_j$ because by Lemma 2.1 we have that $\mathcal{F}_r \cap \mathcal{F}_j = \{I\}$ for $r \neq j$. 

Lemma 2.13. The set $\mathcal{G}$ defined in Theorem 1.8 is a group. For any $A \in \mathcal{G}$ there exists unique representation

$$A = A_0 \circ A_1 \circ ... \circ A_N \ with \ A_j \in \mathcal{F}_j.$$  

(71)

The mapping $\pi$ defined in (5)-(11) and (23) has the form

$$\pi(A) = (\pi_0(A_0), \pi_1(A_1), ..., \pi_N(A_N)).$$  

(72)
Proof. If \( A, B \in \mathcal{J} \) then \( A \circ B \) is invertible and by Lemma 2.14 it belongs to \( \mathcal{J} \). The decomposition (71) follows from the steps of Theorem 1.4, see also its Proof and (11). The formula (15) discussed in the Proof of Theorem 1.4 along with Lemma 2.4 gives us that \( A^{-1} \in \mathcal{J} \) and hence \( \mathcal{J} \) is a group. Suppose that we have two decompositions

\[
A = A_0 \circ A_1 \circ \ldots \circ A_N = \tilde{A}_0 \circ \tilde{A}_1 \circ \ldots \circ \tilde{A}_N.
\]

Then using \( A_j^{-1} \in \mathcal{F}_j \) and (52), (53) we obtain

\[
\tilde{A}_0^{-1} A_0 = \tilde{A}_1 \circ \ldots \circ \tilde{A}_N \circ (A_1 \circ \ldots \circ A_N)^{-1} = I + \{\text{integral operators}\},
\]

which gives us \( \tilde{A}_0^{-1} A_0 = I \) by Lemma 2.1. Repeating these arguments we deduce that \( \tilde{A}_j = A_j \) for all \( j \). The identity (72) follows from the definition of \( \pi_j \) given in Theorem 1.4, its Proof and Definitions 2.6 and 2.10.

Proof of Theorem 1.8. Let \( A, B \in \mathcal{J} \) be two operators. Consider their decompositions

\[
A = A_0 \circ A_1 \circ \ldots \circ A_N, \quad B = B_0 \circ B_1 \circ \ldots \circ B_N, \quad A_j, B_j \in \mathcal{F}_j.
\]

By Lemma 2.12 we can rearrange the terms in the product \( A \circ B \) to obtain

\[
A \circ B = A_0 \circ A_1 \circ \ldots \circ A_N \circ B_0 \circ B_1 \circ \ldots \circ B_N = \tilde{A}_0 \circ \tilde{B}_0 \circ \ldots \circ \tilde{A}_N \circ \tilde{B}_N \tag{73}
\]

with

\[
\tilde{A}_j, \tilde{B}_j \in \mathcal{F}_j \quad \text{and} \quad \tilde{\pi}_j(\tilde{A}_j) = \tilde{\pi}_j(A_j), \quad \tilde{\pi}_j(\tilde{B}_j) = \tilde{\pi}_j(B_j). \tag{74}
\]

Denoting \( C_j = \tilde{A}_j \circ \tilde{B}_j \) we obtain the unique representation for the product (see Lemma 2.13)

\[
A \circ B = C_0 \circ \ldots \circ C_N. \tag{75}
\]

Using (74) along with Lemma 2.11 we deduce that

\[
\tilde{\pi}_j(C_j) = \tilde{\pi}_j(\tilde{A}_j)\tilde{\pi}_j(\tilde{B}_j) = \tilde{\pi}_j(A_j)\tilde{\pi}_j(B_j), \tag{76}
\]

which with (72) give us

\[
\pi(A \circ B) = \pi(A)\pi(B). \tag{77}
\]

Proof of Theorem 1.7. In general, these results are similar to the results of Lemma 2.13 and can be obtained in the same manner.

Proof of Theorem 1.9. i) First note that \( \pi(I + O(t)) = \pi(I) + O(t) \) for \( t \to 0 \), where \( O \) and \( O \) are standard \( O \)-notations for bounded operators and vectors. Now for any operator \( A \) of the form (2) we have that

\[
I + tA = (I + tA_0) \circ (I + tA_1) \circ \ldots \circ (I + tA_N) \circ (I + O(t^2)), \tag{78}
\]

which leads to

\[
\pi(I + tA) = \pi(I + tA_0)\pi(I + tA_1)\ldots\pi(I + tA_N)\pi(I + O(t^2)) = \pi(I + O(t^2)). \tag{79}
\]
\[
\left(\det(I + tA_0), 1, \ldots, 1\right) \left(1, \det(I + t(B_1A_1))_1, 1, \ldots, 1\right) \ldots
\]

\[
\ldots\left(1, \ldots, 1, \det(I + t(B_NA_N)_N)\right) \left(\pi(I) + O(t^2)\right) =
\]

\[
\pi(I) + t(\text{Tr} A_0, \langle \text{Tr} B_1A_1 \rangle_1, \ldots, \langle \text{Tr} B_NA_N \rangle_N) + O(t^2),
\]

which give us (27). Note that in (79)-(82) we use the standard asymptotics of \(\det\) and the fact that \(\pi(I) = (1, ..., 1)\). The identities

\[
\pi(I + tA + tB) = \pi\left((I + tA) \circ (I + tB) \circ (I + O(t^2))\right) =
\]

\[
\pi(I + tA)\pi(I + tB)(\pi(I) + O(t^2)) = \pi(I) + tA(\pi(A) + tB(\pi(B) + O(t^2))
\]

lead to the first formula in (28). The identities

\[
\pi(I - t^2B \circ A - t^2A^2 - t^2B^2) = \pi\left((I + tA) \circ (I + tB) \circ (I - tA - tB) \circ (I + O(t^3))\right) =
\]

\[
\pi(I + tB) \circ (I + tA) \circ (I - tA - tB) \circ (I + O(t^3)) =
\]

lead to

\[
\tau(A^2 + B^2 + B \circ A) = \tau(A^2 + B^2 + A \circ B),
\]

which with the first identity gives us the second identity in (28). □

**Lemma 2.14.** Consider an operator \(A = A(B\cdot)_j : L^2 \to L^2\). Then the spectrum of \(A\) consists of eigenvalues. All non-zero eigenvalues of \(A\) coincide with non-zero eigenvalues of the matrix \(C := (BA)_j\). The algebraic multiplicities of these eigenvalues are the same for \(A\) and \(C\).

**Proof.** Without loss of generality we assume \(j \neq 0, N\). The direct integral representation

\[
A = \int_{k_j \in [0,1]^{N-j}} A(k_j), \ k_j = (k_{j+1}, ..., k_N).
\]

gives us that the spectrum \(\sigma(A)\) consists of eigenvalues \(\lambda(k_j)\) of the finite rank operators

\[
A(k_j) = A(k_{\overline{j}}, k_j)(B(k_{\overline{j}}, k_j)_j, k_{\overline{j}} = (k_1, ..., k_{j-1}).
\]
Now, it is not difficult to verify the following statements

\[ A(k_j)u_0(k_\gamma, k_j) = \lambda(k_j)u_0(k_\gamma, k_j) \Rightarrow \begin{cases} 
C(k_j)\bar{u}_0(k_j) = \lambda(k_j)\bar{u}_0(k_j), \\
\bar{u}_0(k_j) = (B(k_\gamma, k_j)u_0(k_\gamma, k_j))_j,
\end{cases} \]

\[ A(k_j)u_1(k_\gamma, k_j) = \lambda(k_j)u_1(k_\gamma, k_j) + u_0(k_\gamma, k_j) \Rightarrow \begin{cases} 
C(k_j)\bar{u}_1(k_j) = \lambda(k_j)\bar{u}_1(k_j) + \bar{u}_0(k_j), \\
\bar{u}_1(k_j) = (B(k_\gamma, k_j)u_1(k_\gamma, k_j))_j,
\end{cases} \]

\[ C(k_j)\bar{u}_0(k_j) = \lambda(k_j)\bar{u}_0(k_j) \Rightarrow \begin{cases} 
A(k_j)u_0(k_\gamma, k_j) = \lambda(k_j)u_0(k_\gamma, k_j), \\
u_0(k_j, k_\gamma) = A(k_\gamma, k_j)\bar{u}_0(k_j),
\end{cases} \]

\[ C(k_j)\bar{u}_1(k_j) = \lambda(k_j)\bar{u}_1(k_j) + \bar{u}_0(k_j) \Rightarrow \begin{cases} 
A(k_j)u_1(k_\gamma, k_j) = \lambda(k_j)u_1(k_\gamma, k_j) + u_0(k_\gamma, k_j), \\
u_1(k_j, k_\gamma) = A(k_\gamma, k_j)\bar{u}_1(k_j)
\end{cases} \]

These statements show the one-to-one correspondence between eigenvalues and eigenvectors (including adjoint eigenvectors which belong to Jordan blocks) of \( A(k_j) \) and \( C(k_j) \).

**Proof of Theorem 1.10.** Due to Lemma 2.11 and to the fact that each summand of \( A \in \mathcal{H} \) \((2)\) is a direct integral of finite rank operators (see \((89),(90)\)) we may write the following isomorphism of linear spaces

\[ \mathcal{H} \cong \int_{k \in [0,1]^N} \mathcal{I}_0 dk \oplus \int_{k_1 \in [0,1]^{N-1}} \mathcal{I}_1 dk_1 \oplus \ldots \oplus \mathcal{I}_N, \quad (91) \]

where \( \mathcal{I}_j \) is an algebra of finite rank operators acting on \( L_j^2 \). Taking for each \( \mathcal{R} \in \mathcal{I}_j \) the trace norm \( \| \mathcal{R} \|_{TR} = Tr(\mathcal{R}^*\mathcal{R})^{\frac{1}{2}} \) (see \([3], \text{Theorem 5.1}\) we obtain the norm on the direct integral \( \int_{k_j \in [0,1]^{N-j}} \mathcal{I}_j dk_j \):

\[ \| \int_{k_j \in [0,1]^{N-j}} \mathcal{R}(k_j)dk_j \|_{tr} = \max_{k_j \in [0,1]^{N-j}} \| \mathcal{R}(k_j) \|_{TR}. \]

The sum of these norms for all \( j \) coincides with the norm \( \| \cdot \|_{tr} \) \((31)\) on \( \mathcal{H} \) (we also use \((91)\) and Lemma 2.14 which allows us to compute the trace norm explicitly).

Consider operators \( A, B \in \mathcal{H} \) and \( C = A \circ B \in \mathcal{H} \). They have unique representations

\[ A = \sum_{j=0}^{N} A_j, \quad B = \sum_{j=0}^{N} B_j, \quad C = \sum_{j=0}^{N} C_j, \quad A_j, B_j, C_j \in \int_{k_j \in [0,1]^{N-j}} \mathcal{I}_j. \]

The operators \( C_j \) are of the form (see \((57)\))

\[ C_j = A_j \circ B_j + \sum_{r=0}^{j-1} (A_r \circ B_j + A_j \circ B_r). \]

Denoting the standard operator norm of operators acting on some Hilbert space as \( \| \cdot \| \) and using the fact that the standard operator norm is weaker than the trace norm and the fact
that the trace norm is sub-multiplicative (see [3], Theorem (5.1) and Eq. (2.6) on p. 51) we obtain (we use also the fact that the norm of direct integrals is a maximum of integrands)

\[ \|C_j\|_{\text{tr}} \leq \|A_j\|_{\text{tr}}\|B_j\|_{\text{tr}} + \sum_{r=0}^{j-1} (\|A_rB_j\|_{\text{tr}} + \|A_jB_r\|_{\text{tr}}) \leq \|A_j\|_{\text{tr}}\|B_j\|_{\text{tr}} + \sum_{r=0}^{j-1} (\|A_r\|\|B_j\|_{\text{tr}} + \|A_j\|_{\text{tr}}\|B_r\|_{\text{tr}}) \]

\[ \leq \|A_j\|_{\text{tr}}\|B_j\|_{\text{tr}} + \sum_{r=0}^{j-1} (\|A_r\|\|B_j\|_{\text{tr}} + \|A_j\|_{\text{tr}}\|B_r\|_{\text{tr}}), \]

which lead to \(\|A \circ B\|_{\text{tr}} \leq \|A\|_{\text{tr}}\|B\|_{\text{tr}}\) because \(\|A \circ B\|_{\text{tr}} = \|C\|_{\text{tr}} = \sum_{j=0}^{N} \|C_j\|_{\text{tr}}\). Due to Lemma 2.14 and [3], Corollary 3.4 we also obtain that \(\|\tau(A)\|_c \leq \|A\|_{\text{tr}}\) and then \(\|\tau\| = 1\) since \(\|\tau(I)\|_c = \|I\|_{\text{tr}}\). Using (26) and the first identity of (28) we obtain that

\[ \frac{\partial \pi(\lambda I - A)}{\partial \lambda} = \pi(\lambda I - A)\tau((\lambda I - A)^{-1}) = \pi(\lambda I - A) \sum_{n=0}^{\infty} \frac{\tau(A^n)}{\lambda^{n+1}}, \quad (92) \]

which after integration by \(\lambda\) becomes (35). The continuity of \(\pi\) follows from the continuity of \(\tau\), (35) and the identity

\[ \|\pi(A + B) - \pi(A)\|_c \leq \|\pi(A)\|_c\|\pi(I + A^{-1}B) - \pi(I)\|_c, \]

which tends to 0 for \(\|B\|_{\text{tr}} \rightarrow 0\) because \(\|\cdot\|_{\text{tr}}\) is a sub-multiplicative norm. ■

3. Example

In this section we apply our method to some synthetic example of integral operator. Let \(N = 2\) and \(M = 1\). Consider the following self-adjoin operator acting on \(L^2_{1,2}\)

\[ Au = -\int_{0}^{1} udk_1 - f \int_{0}^{1} fudk_1 - \int_{0}^{1} \int_{0}^{1} udk_1dk_2, \quad u \in L^2_{2,1}, \quad (93) \]

where \(f\) is some real continuous scalar function with \(\int_{0}^{1} fdk_1 = 0\) (for convenience). Taking \(\lambda I - A, \lambda \in \mathbb{C}\) and using notations (11) we have

\[ \lambda I - A = \lambda \cdot \langle \cdot \rangle_1 + f\langle f \cdot \rangle_1 + \langle \cdot \rangle_2. \quad (94) \]

The spectrum of \(A\) is

\[ \sigma(A) = \{ \lambda : \lambda I - A \text{ is non-invertible} \}. \quad (95) \]

Using our scheme (5)-(13) we will calculate this spectrum explicitly and with the "degree" (essential or discrete). In our case the matrices \(A, B\) (some of them are scalars, see (2)) are

\[ A_0 = \lambda, \quad B_0 = 1, \quad A_1 = \begin{pmatrix} 1 & f \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 \\ f \end{pmatrix}, \quad A_2 = 1, \quad B_2 = 1. \quad (96) \]
On the Step 0 of Theorem 1.4 we have
\[ \pi_0 = \lambda, \quad E_0 = \lambda, \quad A_{10} = \lambda^{-1} \begin{pmatrix} 1 & f \end{pmatrix}, \quad A_{20} = \lambda^{-1}. \] (97)

On the Step 1 of Theorem 1.4 we have
\[ \pi_1 = \frac{(\lambda + 1)(\lambda + \langle f^2 \rangle_1)}{\lambda^2}, \quad E_1 = \begin{pmatrix} 1 + \lambda^{-1} & 0 \\ 0 & 1 + \lambda^{-1} \langle f^2 \rangle_1 \end{pmatrix}, \quad A_{21} = \frac{1}{\lambda + 1}. \] (98)

On the last Step 2 of Theorem 1.4 we have
\[ \pi_2 = \frac{\lambda + 2}{\lambda + 1}, \quad E_2 = \frac{\lambda + 2}{\lambda + 1}. \] (99)

Thus the vector-valued determinant (23) of our operator \( \lambda I - A \) is
\[ \pi \left( \lambda \cdot + \langle \cdot \rangle_1 + f \langle \cdot \rangle_1 + \langle \cdot \rangle_2 \right) = \left( \lambda, \frac{(\lambda + 1)(\lambda + \langle f^2 \rangle_1)}{\lambda^2}, \frac{\lambda + 2}{\lambda + 1} \right). \] (100)

Due to Theorem 1.4 the condition \( \lambda I - A \) is non-invertible follows from the presence of zeroes \( \pi_j \) (components of our determinant). Thus, in our case the spectrum is
\[ \sigma(A) = \{0\} \cup \{-1\} \cup \{\lambda : \lambda = -(f^2)_1 \text{ for some } k_2\} \cup \{-2\}. \] (101)

The ”degree” of spectral points can be calculated with the function (12)
\[ D(\lambda) = \begin{cases} 
0, & \lambda = 0, \\
1, & \lambda = -1 \text{ or } \lambda = -(f^2)_1 \neq 0, \\
2, & \lambda = -2 \neq -(f^2)_1, \\
3, & \text{otherwise}. 
\end{cases} \] (102)

In particular \( \lambda = -2 \) is an isolated eigenvalue of \( A \) iff \( \langle f^2 \rangle_1 \neq 2 \) for all \( k_2 \in [0,1] \). The Floquet-Bloch dispersion curves (see remark before Theorem 1.8) are of the form
\[ \begin{cases} 
\lambda_0(k) = 0, & k \in [0,1]^2, \\
\lambda_{1a}(k_2) = -1, & k_2 \in [0,1], \\
\lambda_{1b}(k_2) = -(f^2)_1, & k_2 \in [0,1], \\
\lambda_2 = -2.
\end{cases} \] (103)

For all \( \lambda \not\in \sigma(A) \) the resolvent has the form (see (15))
\[ (\lambda I - A)^{-1} = \lambda^{-1} \left( I - \frac{\langle f^2 \rangle_1}{\lambda + 1} + \frac{f \langle f \cdot \rangle_1}{\lambda + 2} \right). \] (104)

Due to (27) the trace of \( A \) is
\[ \tau \left( \lambda \cdot + \langle \cdot \rangle_1 + f \langle f \cdot \rangle_1 + \langle \cdot \rangle_2 \right) = (\lambda, 1 + \langle f^2 \rangle_1, 1). \] (105)
Due to (29)-(31) the trace norm of $\mathcal{A}$ is 
\[ \|\lambda I - \mathcal{A}\|_{tr} = |\lambda| + 2 + \max_{k_2} f^2_{k_2}. \]  
(106)

Taking component-wise logarithm of (100) and using (35) we obtain 
\[ \tau(\mathcal{A}^n) = (-1)^n(0, 1 + \langle f^2 \rangle^n, 2^n - 1). \]  
(107)

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