Gauged maximal supergravities and hierarchies of nonabelian vector-tensor systems *

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Abstract: We describe generalizations of the manifestly E\textsubscript{6(6)} covariant formulation of five-dimensional gauged maximal supergravity with regard to the structure of the vector and tensor fields. We indicate how the group-theoretical structures that we discover seem to play a role in gauged supergravities in various space-time dimensions.

1 Introduction

As is well known, gaugings are the only known supersymmetric deformations of maximal supergravity. A gauging is obtained by coupling the abelian vector fields, which arise in toroidally compactified eleven-dimensional or IIB supergravity, to charges assigned to the elementary fields. The resulting gauge group must be a subgroup of the duality group G and is encoded in these charges; supersymmetry severely restricts the possible gauge groups. We have developed a general group-theoretical method for determining which gaugings are consistent [1]. It is based on the so-called embedding tensor which defines how the gauge field charges are embedded into the duality group G. Treating the embedding tensor as a spurionic object that transforms covariantly under G, the Lagrangian and transformation rules remain formally G-covariant. The embedding tensor can then be characterized group-theoretically. When freezing it to a constant, the G-invariance is broken. It turns out that admissible embedding tensors must be subject to a quadratic and a linear group-theoretical constraint. The quadratic constraint ensures that one is dealing with a proper subgroup of G and the linear one is required by the supersymmetry of the action upon switching on the gauge interactions. The linear constraints are shown in table 1 for dimensions $d = 3, 4, 5, 6, 7$.

However, there is a subtle issue related to the field configuration. Because antisymmetric tensor fields of rank $p$ are dual to antisymmetric tensors of rank $d - p - 2$ in the usual setting of a quadratic Lagrangian, the field configuration is not unique. It is well known that this feature is of relevance for certain gaugings. For instance, in the standard framework only the gauge fields belonging to the adjoint representation of the gauge group can carry the corresponding charges. Problems with charged vector fields in other

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Table 1: Decomposition of the embedding tensor $\Theta$ for maximal supergravities in various space-time dimensions in terms of irreducible $G$ representations. Only the underlined representations are allowed according to the representation constraint. The R-symmetry group $H$ is the maximal compact subgroup of $G$.

| $d$ | $G$ | $H$ | $\Theta$ |
|-----|-----|-----|--------|
| 7   | SL(5) | USp(4) | $10 \times 24 = 10 + 15 + 40 + 175$ |
| 6   | SO(5,5) | USp(4) $\times$ USp(4) | $16 \times 45 = 16 + 144 + 560$ |
| 5   | $E_6(6)$ | USp(8) | $27 \times 78 = 27 + 351 + 1728$ |
| 4   | $E_7(7)$ | SU(8) | $56 \times 133 = 56 + 912 + 6480$ |
| 3   | $E_8(8)$ | SO(16) | $248 \times 248 = 1 + 248 + 3875 + 27000 + 30380$ |

representations are usually circumvented by dualizing these fields, after which the gauging can still proceed [2, 3, 4]. However, the fact that the field representation must first be carefully adapted before switching on the gauge coupling, hampers a general analysis of the gaugings. Moreover, the manifest $G$-covariance is affected by the fact that the fields will no longer constitute $G$ representations. In [5] a novel system of vector-tensor fields was adopted for five-dimensional supergravity, in which the tensor and the vector fields each constitute a complete $G$ representation. Because of the presence of additional gauge transformations, also depending on the embedding tensor, the number of degrees of freedom remains, however, unchanged. This extended field configuration can thus accommodate any possible gauging and the group-theoretical analysis proceeds in a uniform way. The viability of this formulation can be established a posteriori by rederiving all the supersymmetry transformations and the Lagrangian in the new formulation; indeed, apart from the two types of group-theoretical constraints on the embedding tensor, no other conditions are necessary.

Motivated by this result we have analyzed whether similar results can also be derived for other space-time dimensions. Here we report on the outcome of this analysis, which indicates that hierarchies of vector-tensor systems do appear in other dimensions, in a similar group-theoretical setting. We first review the results of [5] relevant for the vector-tensor couplings and their generalization. Finally we discuss their application in various dimensions.

2 Some results for $d = 5$ with a slight generalization

In maximal five-dimensional supergravity, the (abelian) vector fields $A^M_\mu$ transform in a representation $27$ of $G = E_6(6)$; $\delta A^M_\mu = -\Lambda^\alpha(t_\alpha) N^M_\mu A^N_\mu$, where the 78 independent $E_6(6)$ generators are denoted by $(t_\alpha) M^N$. Because the gauge group is a subgroup of $E_6(6)$, its generators $X_M$ are decomposable in terms of the $t_\alpha$, i.e.,

$$X_M = \Theta^\alpha M t_\alpha,$$

where $\alpha = 1, 2, \ldots, 78$ and $M = 1, 2, \ldots, 27$. The gauging is thus encoded in a real embedding tensor $\Theta^\alpha M$ assigned to the $27 \times 78$ representation of $E_6(6)$. The embedding tensor acts as a projector whose rank equals the dimension of the gauge group (up to abelian gauge fields corresponding to a possible central extension of the gauge algebra). The $X_M$ generate a group and thus define a Lie algebra, $[X_M, X_N] = f^P_{MN} X_P$, with $f^P_{MN}$ the as yet unknown structure constants of the gauge group. Hence the embedding tensor must satisfy the closure condition,

$$\Theta^\alpha M \Theta^\beta N f_{\alpha \beta \gamma} = f^P_{MN} \Theta^\gamma P,$$
were \( f_{\alpha\beta\gamma} \) denotes the structure constants of \( E_{6(6)} \), according to
\[
[t_\alpha, t_\beta] = f_{\alpha\beta\gamma} t_\gamma.
\]
Consequently the structure constants \( f_{MN^P} \) satisfy the Jacobi identities, but only in the subspace projected by the embedding tensor!

Once the gauge group is specified, covariant derivatives are introduced, 
\[
D_\mu = \partial_\mu - g A_\mu M X_M,
\]
where \( g \) denotes the gauge coupling constant. They lead to covariant field strengths,
\[
\Theta_M^{\alpha} F_{\mu\nu}^M = \Theta_M^{\alpha}(\partial_\mu A_\nu^M - \partial_\nu A_\mu^M - \partial_\mu X_M^{NP} A_\nu^N A_\nu^P).
\]
(3)
Gauge field transformations are given by
\[
\Theta_M^{\alpha} \delta A_\mu^M = \Theta_M^{\alpha}(\partial_\mu \Lambda^M - g f_{NP^M} A_\mu^N A_\nu^P).
\]
(4)

Because of the contraction with the embedding tensor, the above results apply to only a subset of the gauge fields; the remaining ones do not appear in the covariant derivatives and are not directly involved in the gauging. The gauge fields that do appear in the covariant derivatives, are only determined up to additive contributions by gauge fields that vanish upon contraction with \( \Theta_M^{\alpha} \).

While the gauge fields involved in the gauging should transform in the adjoint representation of the gauge group, the gauge field charges should also coincide with \( X_M \) in the \( 27 \) representation. Therefore \( X_{MN^P} \equiv (X_M)_N^P \) must decompose into the adjoint representation of the gauge group plus possible extra terms which vanish upon contraction with the embedding tensor, \( \Theta_M^{\alpha} \).

The tensor \( X_{MN^P} \) transforms in the \( 351 \) representation of \( E_{6(6)} \), just as the embedding tensor itself. Exploiting the symmetric \( E_{6(6)} \)-invariant tensors \( d_{MNP} \) and \( d^{MNP} \), one constructs an antisymmetric tensor \( Z^{MN} \equiv X_{PQ}^M d^{NPQ} \), which thus also belongs to the \( 351 \) representation. For what follows the inverse equation is relevant,
\[
X_{MN}^{P^Q} = d_{MNQ} Z^{PQ}.
\]
(7)

According to \( [5] \) one can derive the following relations,
\[
Z^{MN} \Theta_N^{\alpha} = 0, \quad Z^{MN} X_N = 0, \quad X_{MN}^{[P} Z^{Q]N} = 0,
\]
where, in the second equation, \( X_M \) is taken in an arbitrary representation. The third equation implies that \( Z^{MN} \) is invariant under the gauge group.

Rather than continuing this presentation of \( d = 5 \) results, we consider a slight generalization which will contain the \( d = 5 \) results as a special case. Namely, we replace \( Z^{MN} \)
by a tensor $Z^{M,I}$ belonging to a product representation of $G$, where $G$ will be kept arbitrary. Here indices $M, N, \ldots$ remain, as before, related to the representation to which the gauge fields have been assigned, but the indices $I, J, \ldots$ belong to some other irreducible representation of $G$. The new tensor $Z^{M,I}$ now appears in a modification of (7),

$$X_{(MN)}^P = d_{I,MN} Z^{P,I}, \quad (9)$$

where $d_{I,MN}$ is a $G$-invariant tensor, i.e., $t_{\alpha M}^P d_{I,PN} + t_{\alpha N}^P d_{I,PM} + t_{\alpha I}^J d_{J,MN} = 0$, which is symmetric in $M$ and $N$. Obviously (7) will now apply to the two representations separately, with generators $X_{MN}^P$ and $X_{MI}^J$, respectively. It then follows that $X_{MN}^P$, $X_{MI}^J$ and $Z^{M,I}$ are gauge invariant tensors which take their values in representations belonging to the embedding tensor. Note, however, that we are not insisting that the embedding tensor belongs to an irreducible representation. Furthermore it follows that

$$Z^{M,I} X_{MN}^P = 0 = Z^{M,I} X_{MJ}^R, \quad (10)$$

We expect that (11) and (11) are equivalent versions of the quadratic constraint (of course, upon the condition that the linear constraint on the embedding tensor holds).

It then follows that the tensor $Y_{IM}^J$, defined by

$$Y_{IM}^J \equiv X_{MI}^J + 2 d_{I,MN} Z^{N,J}, \quad (11)$$

satisfies an orthogonality equation

$$Z^{M,I} Y_{IN}^J = 0. \quad (12)$$

Furthermore one can derive the following property,

$$X_{[MN]}^P X_{[QP]}^R + X_{[QM]}^P X_{[NP]}^R + X_{[NQ]}^P X_{[MP]}^R = -Z^{R,I} d_{I,P[Q} X_{MN]}^P. \quad (13)$$

This equation is relevant in the next section where we will employ $X_{[MN]}^P$ as an extension of the gauge group structure constants $f_{MN}^P$, which satisfies the Jacobi identity up to terms proportional to $Z$. As the right-hand side vanishes upon contraction with the embedding tensor $\Theta^\alpha_R$, we see that the $X_{[MN]}^P$ satisfy the Jacobi identity in the subspace projected by the embedding tensor, just as the gauge group structure constants.

The above results are directly compatible with the results for $d = 5$ by removing the distinction between the indices $M, \ldots$ and $I, \ldots$. Note that $Y_{IM}^J$ becomes equal to $-X_{IM}^J$, so that (9) and (12) will coincide.

### 3 Nonabelian vector-tensor systems

In the previous section we presented the transformations (7) and the field strengths (3) of the vector fields, which were the conventional ones except that these expressions were contracted with the embedding tensor and thus apply to only part of the fields. One way to make them exact is to remove the fields that are projected to zero by the embedding tensor, for instance, by dualizing them into tensor fields. Because this would affect the duality invariance, we will set up an alternative formulation where vectors and tensor gauge fields constitute full representations of the duality group; at the same time, we will introduce extra gauge invariances so that the total number of degrees of freedom remains the same, irrespective of the embedding tensor that has been adopted.
To see how this works let us follow [5] and consider the gauge transformations of the vector fields,
\[ \delta A_\mu^M = \partial_\mu \Lambda^M - g X_{\{PQ\}^M} L^P A_\mu^Q - g Z^{M,I} \Xi_{\mu I}, \]
where \( \Lambda^M \) is the usual gauge transformation parameter and the transformations proportional to \( \Xi_{\mu I} \) allow us to gauge away those vector fields that are perpendicular to the embedding tensor. This equation is thus an extension of (13). The usual field strength is not covariant and we define a modified field strength,
\[ \mathcal{H}_{\mu \nu}^M = F_{\mu \nu}^M + g Z^{M,I} B_{\mu \nu}^I, \]
which contains tensor fields \( B_{\mu \nu}^I \). Here we use the notation,
\[ F_{\mu \nu}^M = \partial_\mu A_\nu^M - \partial_\nu A_\mu^M + g X_{[NP]^M} A_\mu^N A_\nu^P. \]
By making a suitable choice for the transformation rule of the tensor field \( B_{\mu \nu}^I \),
\[ \delta B_{\mu \nu}^I = 2 \partial_{[\mu} \Xi_{\nu]} - g X_{M}^{J} A_{[\mu}^{M} \Xi_{\nu]}^{J} + g \Lambda^{M} X_{M}^{J} B_{\mu \nu}^{J} \]
\[ + \Lambda^{M} \left[ - d_{I,MN} F_{\mu \nu}^{N} + g d_{I,N[M} X_{PQ]}^{N} A_{\mu}^{P} A_{\nu}^{Q} \right] - g Y_{IM}^{J} \Phi_{\mu \nu}^{J M}, \]
the modified field strength transforms covariantly, \( \delta \mathcal{H}_{\mu \nu}^M = -g X_{PN}^M \Lambda^P \mathcal{H}_{\mu \nu}^N. \) Here we introduced a new tensor gauge transformation with parameter \( \Phi_{\mu \nu}^{J M} \), which does not interfere with the field strength (13) in view of (12).2 In \( d = 5 \) dimensions this option is not relevant, as the tensor fields appeared only in the combination \( Z^{M,I} B_{\mu \nu}^I \).

In the general case we are about to discover the need for yet another tensor field \( S_{\mu \nu \rho \tau}^M \), and in this way we should expect to generate a whole hierarchy of tensor fields. Before continuing, let us make a number of comments. First of all, it is important to note that the covariant derivative does not transform under tensor gauge transformations, by virtue of (8), and also the Ricci identity, \( [D_\mu, D_\nu] = -g F_{\mu \nu}^M X_M, \) will still hold. Secondly, to verify the consistency of the transformations obtained so far, one may consider the commutator algebra. While the tensor transformations commute,
\[ [\delta(\Xi_1), \delta(\Xi_2)] = [\delta(\Phi_1), \delta(\Phi_2)] = [\delta(\Lambda), \delta(\Xi)] = 0, \]
the commutator of a vector and a tensor gauge transformation gives rise to tensor gauge transformations,
\[ [\delta(\Lambda), \delta(\Xi)] = \delta(\Xi) + \delta(\Phi), \]
with
\[ \Xi_{\mu I} = - \frac{1}{2} g X_{MI}^{J} \Xi_{\mu J} \Lambda^{M}, \]
\[ \Phi_{\mu \nu}^{J M} = (\partial_{[\mu} \Xi_{\nu]} - \frac{1}{2} g X_{NI}^{J} A_{[\mu}^{N} \Xi_{\nu]}^{J}) \Lambda^{M} - \frac{1}{2} g X_{[NP]}^{M} \Lambda^{N} A_{[\mu}^{P} \Xi_{\nu]}^{I} + \frac{1}{2} A_{[\mu}^{M} \Xi_{\nu]}^{I}. \]
Likewise,
\[ [\delta(\Lambda), \delta(\Phi)] = \delta(\Phi), \quad \Phi_{\mu \nu}^{J M} = g \Lambda^{P}(X_{PN}^{M} \Phi_{\mu \nu}^{I N} - X_{PI}^{J} \Phi_{\mu \nu}^{J M}). \]
The remaining commutator of two vector gauge transformations gives rise to a combination of vector and tensor gauge transformations,
\[ [\delta(\Lambda_1), \delta(\Lambda_2)] = \delta(\Lambda) + \delta(\Xi) + \delta(\Phi) , \]

2Observe that the decomposition of (17) is ambiguous, as we can redefine the parameter \( \Phi_{\mu \nu}^{I M} \) with terms proportional to \( \Xi_{\mu I} \) and \( \Lambda^{M} \).
where
\[
\tilde{\Lambda}^M = g \Lambda_1^N \Lambda_2^P X_{[NP]}^M,
\]
\[
\tilde{\tilde{\epsilon}}_{\mu I} = -g \Lambda_1^M \Lambda_2^N d_{I,Q[M} X_{NP]}^Q A^P_{\mu},
\]
\[
\tilde{\Phi}_{\mu I}^M = \Lambda_1^M \Lambda_2^N \left[ -\frac{2}{3} d_{I,NP} \mathcal{F}_{\mu \nu}^P + g d_{I,R[N} X_{PQ]}^R A^P_{\mu} A^Q_{\nu} \right].
\] (23)

We can now continue and introduce a rank-3 antisymmetric gauge field \(S_{\mu \nu \rho} I^M\). Just as before we propose a covariant field strength, but now for the tensor field \(B_{\mu \nu \rho}\),
\[
\mathcal{H}_{\mu \nu \rho} I \equiv 3 \left[ D_{[\mu} B_{\nu \rho] I} + 2 d_{I,MN} A_{[\mu}^M (\partial_{\nu} A_{\rho]}^N + \frac{1}{3} g X_{[PQ]}^N A_{\nu}^P A_{\rho]}^Q) \right]
+ g Y_{JM} S_{\mu \nu \rho} I^M,
\] (24)
with the covariant derivative \(D_{[\mu} B_{\nu \rho]} I = \partial_{[\mu} B_{\nu \rho]} I - g X_{M[I}^J A_{[\mu}^M B_{\nu \rho] J}\). This field strength should remain invariant under the tensor and transform covariantly under the vector gauge transformations, i.e., \(\delta \mathcal{H}_{\mu \nu \rho} I = g \Lambda^N X_{M[I}^J \mathcal{H}_{\mu \nu \rho} J\). This can be achieved provided we assign the following transformations to \(S_{\mu \nu \rho} I^M\),
\[
\delta S_{\mu \nu \rho} I^M = g \Lambda^N X_{NT}^J S_{\mu \nu \rho} J^M - g \Lambda^N X_{NP} M S_{\mu \rho} \mu^P
+ 3 D_{[\mu} \tilde{\Phi}_{\nu \rho]} I^M + 3 A_{[\mu}^M D_{\nu} \tilde{\Xi}_{\rho]} I - 3 \partial_{[\mu} A_{\nu}^M \Xi_{\rho]} I - 2 g d_{I,NP} Z^{P,J} A_{[\mu}^M A_{\nu}^N \Xi_{\rho]} J
+ 4 d_{I,NP} \Lambda^{[M} A_{]N}^\mu \partial_{\rho} A_{\rho]}^P + 2 g X_{NT}^J d_{I,PQ} \Lambda^Q A_{[\mu}^M A_{\nu}^N A_{\rho]}^P. \] (25)

This procedure can in principle be continued to higher-rank tensor fields. Note, for instance, the additional orthogonality relation, \(Y_{JM} d_{I,(NP} \delta_Q)^M = 0\).

4 Vector-tensor couplings in various dimensions

So far, our arguments have been rather abstract without giving any indication as to why the vector-tensor hierarchy that we exhibited should have a role to play in gauged maximal supergravities (apart from the five-dimensional theory, which inspired this analysis). In this last section we will briefly discuss this issue. As it turns out, the group-theoretical setting for generating the vector-tensor hierarchies leads to results that fit in very naturally with what is known about the ungauged theories and with what one expects for the gauged versions. The tensor representations tend to come out such that tensors that are dual to each others transform also in representations of \(G\) that are mutually conjugate. Furthermore, the embedding tensors always seem to enable the covariant construction of the relevant topological couplings. There remain open questions, in particular for even space-time dimensions. In any case, our conclusion is that the group-theoretical structure of the vector-tensor hierarchies is such that they clearly play a role in gauged supergravities. The scope of this paper does not allow us to present the couplings in any detail but we hope to report on this elsewhere. We close with a brief perusal of gauged maximal supergravities in various dimensions.

\(d = 4\) supergravity: Here the duality group is \(G = E_7(7)\) and the vector fields transform in the 56 representation. An obvious subtlety (just as in \(d = 6\) dimensions) is, that the duality group applies to the equations of motion and not to the Lagrangian, whereas the 56 representation of the vector fields covers both the electric and the magnetic potentials. We do not view this as a negative feature, but rather we expect that
Eventually the observations of this paper will have some positive impact on this question. As shown in table 1, the embedding tensor $\Theta_M^\alpha$ belongs to the 912 representation of $E_7(7)$. Consequently we know that the fully symmetric part of $X_{MN}^Q \Omega_{PQ}$ must vanish, where $\Omega$ denotes the invariant skew-symmetric tensor of $E_7(7)$. Obviously there exists an $E_7(7)$-invariant tensor, $d_{\alpha\beta, MN} = t_{\alpha\beta} M^P \Omega_{NP}$. Upon multiplication by $\Theta_M^\alpha$ it follows that $X_{(MN)}^P = -\Theta_M^\alpha t_{\alpha\beta} M^P \Omega_{NP}$, so that (12) holds with $Z^{M,\alpha} = 1/2 \epsilon^{MN} \Theta_N^\alpha$. This implies that rank-2 tensor fields $B_{\mu\nu\alpha}$ will exist in the 133 representation of $E_7(7)$. Note that $Z^{M,\beta} \Theta_M^\alpha = 0$ by virtue of the quadratic constraint on the embedding tensor.

At this point we have no complete Lagrangian, but we note that the embedding tensor $\Theta_M^\alpha$ has precisely the right structure to construct an $E_7(7)$-invariant topological coupling $H \wedge B$ of the vector field strengths $H_{\mu\nu}^M$ and the rank-2 tensor fields $B_{\mu\nu\alpha}$.

d = 5 supergravity: We have already discussed the characteristic features of this case (see also, [5]), with the vector fields transforming in the 27 representation and the rank-2 tensors in the 27 representation of $E_6(6)$. The $d$-symbol is the $E_6(6)$-invariant symmetric tensor $d_{MNP} = d_{MNP}$. The tensor $Z_{MN}$ is antisymmetric and belongs to the 351 representation of $E_6(6)$, just as the embedding tensor. It enables the construction of an $E_6(6)$-invariant topological term $B \wedge DB$, which is known to appear in the $d = 5$ theories on a par with the Chern-Simons term for the vector gauge fields.

The tensor $Y$ from (11) reduces to $Y_{MN}^P = -X_{MN}^P = -\Theta_M^\alpha t_{\alpha N}^P$, so that rank-3 tensors appear only in the form $t_{\alpha N}^P S_{\mu\nu\rho N} \equiv S_{\mu\nu\rho\alpha}$ and thus belong to the adjoint 78 representation of $E_6(6)$. These fields do not appear in the Lagrangian.

d = 6 supergravity: The duality group equals $G = E_5(5) = SO(5, 5)$ with vector fields transforming in the 16 representation of SO(5, 5). The embedding tensor must belong to the 144 representation. This is a vector-spinor representation so that the embedding tensor can be written as $\Theta_M^{[mn]} = \theta_{\overline{\alpha\beta}}^{[mn]} \Gamma_{\overline{\alpha\beta}}$. Here $\Gamma_{\overline{\alpha\beta}}$ is symmetric in the spinor indices $\alpha, \beta$ and corresponds to the SO(5, 5) gamma matrices restricted to the chiral subspace (after contraction with the charge conjugation matrix). The obvious choice for the $d$-symbol is $d_{\alpha N}^{\alpha \beta} = \Gamma_{\alpha \beta \alpha N}$. The 45 generators of the duality group in the 16 representation are proportional to $\Gamma_{mn}$, so that we derive the following relation for the gauge generators,

$$X_{(\alpha \beta)} = \theta_{\overline{\alpha\beta}}^{mn} (\Gamma_{mn})_{\overline{\alpha\beta}} = \theta_{\overline{\alpha\beta}}^{mn} \Gamma_{\overline{\alpha\beta}}$$

From (14) we can then read off the tensor $Z_{\overline{\alpha\beta}} = \theta_{\overline{\alpha\beta}}^{mn}$. Hence, we expect rank-2 tensors $B_{\mu\nu \alpha}$ in the 10 representation, which are indeed present in the ungauged theory.

Using the corresponding generators in the 10 representation, $(t_{mn})_p^q = 4 \eta_{pq} \delta_n^q$ one finds from (14) for the tensor $Y_{\overline{\alpha\beta}}^{mn}$:

$$Y_{\overline{\alpha\beta}}^{mn} = 2 \theta_{\overline{\alpha\beta}}^{dp} \Gamma_{\overline{\alpha\beta}} \eta_{dp} \delta_n^q - 2 \theta_{\overline{\alpha\beta}}^{dp} \Gamma_{\overline{\alpha\beta}} \eta_{dp} \delta_n^q + 2 \theta_{\overline{\alpha\beta}}^{dp} \Gamma_{\overline{\alpha\beta}} \eta_{dp} \delta_n^q.$$ 

This shows that three-forms are generated in the combination, $\Gamma_{\overline{\alpha\beta}}^{m} S_{\mu\nu\rho \alpha} \equiv S_{\mu\nu\rho \alpha}$. Indeed, in six dimensions, three-form tensor fields are dual to vector fields and thus appear in the conjugate representation. The validity of the orthogonality relations (12) is ensured by the quadratic constraint on the embedding tensor (c.f. eq. (6.16) of [5]). The embedding tensor $\theta_{\overline{\alpha\beta}}^{mn}$ is of the right structure to write down a topological coupling $H \wedge S$ between the rank-2 tensor field strengths $H_{\mu\nu}^M$ and the new rank-3 tensor fields $S_{\mu\nu\rho \alpha}$.

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3This is discussed in a forthcoming paper [6].
$d=7$ supergravity: The duality group equals $G = \text{E}_{4(4)} = \text{SL}(5)$ with the vector fields transforming in the $\mathbf{10}$ representation. The embedding tensor must take its values in the $\mathbf{15} + \mathbf{40}$ representation, so that it can be written as $\Theta_{[mn]p}^q = \delta_{[m}^q Y_{n]}^p - 2 \varepsilon_{mnrs} Z^{rsq}$. Where $Y_{nm}$ is a symmetric tensor and $Z^{mn,p}$ is antisymmetric in $[m,n]$ and satisfies the irreducibility constraint $Z^{[mn,p]} = 0$. The gauge generators in the $\mathbf{10}$-representation then take the form

$$\left( X_{mn} \right)_{pq}^{rs} = -2 \delta_{[m[p} Y_{q]}^n - 4 \varepsilon_{mntu} Z^{u[r} \delta_{q]s} ,$$

from which we find

$$\left( X_{mn} \right)_{pq}^{rs} + \left( X_{pq} \right)_{mn}^{rs} = -4 \varepsilon_{tunpq} Z^{t[r,s]} = 2 Z^{rs,t} d_{t, \{mn\} [pq]} ,$$

where the $d$-symbol was defined by $d_{r, \{mn\} [pq]} = \varepsilon_{rnmnpq}$. This establishes that $Z$ is the same tensor that appears in (11). Hence we expect tensors $B_{\mu\nu\rho}$ transforming in the $\mathbf{5}$ representation, which, indeed, are known to be present in the ungauged theory.

Likewise one can evaluate the combination (11) and show that the tensor $Y_{[mn]}^q$ is given by $Y_{[mn]}^q = \delta^q_{[m} Y_{n]}^p$. Therefore there will be rank-3 gauge fields appearing in the combination, $\delta^q_{[m} Y_{n]}^p S_{\mu\nu\rho}^{mn} = -Y_{pm} S_{\mu\nu\rho}^{mn} \equiv -Y_{pq} S_{\mu\nu\rho}^{q}$. The rank-3 tensors thus transform in the $\mathbf{5}$ representation and are thus dual to the rank-2 tensors.

Finally the $Y$-component of the embedding tensor is of the right structure to write down a topological coupling $S \wedge DS$ of these rank-three tensor fields $S_{\mu\nu\rho}$. The embedding tensors with $Z = 0$ encode all the CSO($p, q, r$) gaugings (where $p + q + r = 5$), of which a subclass was constructed in [3]. In this subclass all rank-2 tensors can be gauged away, and one is left with only rank-3 tensor fields.

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4. In [1] we incorrectly omitted the $\mathbf{40}$ representation. Its presence was deduced in [5].