DECOMPOSITION OF DIRECT IMAGES OF LOGARITHMIC DIFFERENTIALS

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In [Ko], Kollár proved a remarkable theorem that given a projective map $f$ from a smooth complex algebraic variety $X$ to an arbitrary variety $Y$, the derived direct image of the canonical sheaf $\mathbb{R}f_*\omega_X$ decomposes as a sum $\bigoplus R^if_*\omega_X[-i]$. So in particular, the Leray spectral sequence for $\omega_X$ degenerates. Saito [S] gave a second proof using his theory of Hodge modules, which gives a good conceptual explanation for the theorem. Although the prerequisites for understanding it are rather heavy. A third analytic proof, of the degeneration at least, was given by Takegoshi [T]. The purpose of this note is to give another proof of the full theorem, which is fairly short. The basic idea is as follows. The weak semistable reduction theorem of Abramovich and Karu [AK], can be interpreted as saying that any map can be converted, in the appropriate sense, to a map which is particularly nice from the point of view of logarithmic geometry [K2]. For such maps, we prove that derived direct images of the sheaves of logarithmic differentials decompose as above. It is important that we work with differentials of all degrees simultaneously, because the sum of these direct images carries a Lefschetz operator. The proof of the decomposition theorem ultimately comes down to checking that this sum satisfies the hard Lefschetz theorem and then applying Deligne’s theorem [D]. The final step is to show that this theorem implies Kollár’s.

The core arguments, which take only three pages, are given in the second section. The first section contains some background material on logarithmic geometry, which is included in the interest of making this more accessible. We work over $\mathbb{C}$. The words morphism and map are used interchangeably, according to whim.

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1. Some log geometry

In this expository section, we summarize some facts about log schemes needed later. The details can be found in [K2, O] together with some more specific references given below. For our purposes, the basic example is given by a pair consisting of a smooth scheme $X$ and a reduced divisor $E \subset X$ with normal crossings. We refer to this as a log pair. We can segue to the more flexible notion of log structure by noting that a log pair $(X, E)$, gives rise to the multiplicative submonoid $M \subset \mathcal{O}_X$ of functions invertible outside of $E$. A pre-log structure on a scheme $X$ is a sheaf of commutative monoids $M$ on the étale topology $\mathcal{X}_{et}$ together with a homomorphism $\alpha : M \to \mathcal{O}_X$, where the latter is treated as a monoid with respect to multiplication. It is a log structure if $\alpha^{-1}\mathcal{O}_X^* \cong \mathcal{O}_X^*$. For example, the monoid

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associated to a log pair is a log structure. In general, a pre-log structure can be completed to a log structure in a canonical way. A log scheme is a scheme equipped with a log structure. We will identify log pairs with the associated log scheme. In order to avoid confusion below, we reserve the symbols $X, Y, \ldots$ for log schemes, and use the corresponding symbols $x, y, \ldots$ for the underlying schemes. There are a number of other examples in addition to the one given above. For example, any scheme $X$ can be turned into a log scheme with the trivial log structure $M = \mathcal{O}^*_X$.

There is an important connection between log geometry and toric geometry. Indeed, given a finitely generated saturated submonoid $M \subseteq \mathbb{Z}^n$, the affine toric variety $\text{Spec } \mathbb{C}[M]$ is a log scheme with respect to the pre-log structure induced by $M \to \mathbb{C}[M]$. More generally, recall that a toroidal variety $\text{Spec } \mathcal{O}_X$ is given by a variety $X$ and an open subset $U$, such that the pair $(X, U)$ is étale locally isomorphic to a toric variety with its embedded torus. The toroidal variety $X$ carries a log structure given locally by the one above. The subset $U$ can be understood as the locus where this log structure is trivial. In the log setting, it is convenient to relax the condition on $M$ to allow it to be a finitely generated monoid which is embeddable into an abelian group as a saturated monoid. If $M$ satisfies these conditions, it is called fine and saturated or simply fs. Note that there is a canonical choice for the ambient group, namely the group $M_{	ext{gp}}$ given as a group of fractions.

In general, we want to restrict our attention to log schemes (called fs log schemes) for which the characteristic monoids $M_x = M_x / \mathcal{O}^*_x$ are fs for all geometric points $x$.

A morphism of log schemes consists of a morphism of schemes and a compatible morphisms of monoids. Of course, any morphism of schemes can be regarded as a morphisms of log schemes when equipped with the trivial log structures. Here are some more interesting examples.

**Example 1.1.** Given log pairs $\mathcal{X} = (X, E)$ and $\mathcal{Y} = (Y, D)$. A semistable morphism of log schemes $f : \mathcal{X} \to \mathcal{Y}$ is a morphism $f$ of schemes which is given étale locally (or analytically) by

\[
y_1 = x_1 x_2 \ldots x_{r_1} \\
y_2 = x_1 x_{r_1+1} x_2 \ldots x_{r_2} \\
\vdots \\
y_{d+1} = x_{r_d+1} \\
\vdots
\]

such that $x_1 \ldots x_{r_d}$ and $y_1 \ldots y_d$ are the local equations for $E$ and $D$ respectively.

**Example 1.2.** A homomorphism of fs monoids $P \to Q$ induces a morphism of fs log schemes $\text{Spec } \mathbb{C}[Q] \to \text{Spec } \mathbb{C}[P]$. In particular, toric morphisms of toric varieties are morphisms of log schemes.

Log structures give rise to logarithmic differentials in a rather general way. Given a log pair $\mathcal{X} = (X, E)$, set

\[
\Omega^k_{\mathcal{X}} = \Omega^k(X, E)
\]

which is generated locally by $k$-fold wedge products of

\[
\frac{dx_1}{x_1} \ldots \frac{dx_{r_d}}{x_{r_d}}, dx_{r_d+1}, \ldots
\]
More generally, given a log scheme $X$ over $\mathbb{C}$, we define the $\mathcal{O}_X$-module $\Omega^1_X = \Omega^1_{X/\mathbb{C}}$ as the universal sheaf which receives a $\mathbb{C}$-linear derivation $d : \mathcal{O}_X \to \Omega^1_X$ and homomorphism $d \log : M \to \Omega^1_X$ satisfying $m \cdot d \log m = dm$. Of course, by the universal property of ordinary Kähler differentials, we have a map $\Omega^1_X \to \Omega^1_X$ taking $df$ to $df$, but this is generally not an isomorphism. There is a relative version of differentials $\Omega^1_{X/Y}$ for a morphism $f : X \to Y$ of log schemes. This fits into an exact sequence

$$f^* \Omega^1_Y \to \Omega^1_X \to \Omega^1_{X/Y} \to 0$$

There is a notion of smoothness in this setting, called log smoothness, which can be defined using a variant of the usual infinitesimal lifting condition [K2, §3.3]. However, it is weaker than the name suggests. For instance, while smoothness implies flatness, the corresponding statement for log smoothness is false. Nevertheless, some expected properties do hold. For a log smooth map $\Omega^1_{X/Y}$ and therefore its exterior powers $\Omega^m_{X/Y}$, they are locally free. Kato [K2, thm 3.5] gives a criterion which allows us to verify some basic examples:

**Example 1.3.** Toric (and more generally toroidal) varieties are log smooth over $\text{Spec} \mathbb{C}$ with its trivial log structure.

**Example 1.4.** A semistable map between log pairs is a log smooth morphism.

To rectify some of the defects of log smoothness just alluded to, we need a few more conditions. A morphism of fs monoids $h : P \to Q$ is exact if satisfies $(h^{gp})^{-1}(Q) = P$; it is integral if $\mathbb{Z}[P] \to \mathbb{Z}[Q]$ is flat; it is saturated if for any morphism $P \to P'$ to an fs monoid the push out $Q \oplus_P P'$ is fs; and it is vertical if the image of $P$ does not lie in any proper face of $Q$. For example, the diagonal embedding $\mathbb{N} \subset \mathbb{N}^n$ satisfies all of these conditions. A morphism of $f : (X, M) \to (Y, N)$ of fs log schemes, is respectively exact, integral, saturated or vertical, if the map of characteristic monoids $(f^{-1} N)_P \to N_x$ has the same property for each geometric point $x$. Integral maps are exact and also flat as a map of schemes. Saturated morphisms are integral with reduced fibres [IT, §3.6] Here are the key examples for us.

**Example 1.5.** A semistable map between log pairs is vertical and saturated (and therefore integral and therefore exact). This is because the maps of characteristic monoids are sums of diagonal embeddings $\mathbb{N}^m \subset \mathbb{N}^{n_1+\cdots+n_m}$.

**Example 1.6.** Abramovich and Karu [AK, def 0.1] define a map from a scheme to a regular scheme to be weakly semistable if it is toroidal and equidimensional with reduced fibres. Such a map is log smooth vertical and saturated when the schemes are endowed with the log structures induced from the toroidal structures, cf [IT, rmk 3.6.6].

We note that there is a parallel theory of log analytic spaces given by an analytic space $(X, \mathcal{O}_X)$ and a homomorphism of monoids $\alpha : M \to \mathcal{O}_X$ satisfying conditions similar to those above. Most of the basic definitions and constructions carry over as before. In addition, there is a new construction often called the real blow up [KN]. Given an fs log analytic space $X$, we can define a new topological space $X^{\text{log}}$ and a continuous map $\lambda : X^{\text{log}} \to X$. As a set

$$X^{\text{log}} = \left\{ (x, h) \mid x \in X, h \in \text{Hom}(M^{gp}_x, S^1), \forall f \in \mathcal{O}_x, h(f) = \frac{f(x)}{|f(x)|} \right\}$$
and $\lambda$ is given by the evident projection. For example, when $X$ is given by the log pair $(\mathbb{C}^n, \{z_1 \ldots z_n = 0\})$, $X^{log} = ([0, \infty) \times S^1)^n$ as a topological space, with $\lambda((r_1, u_1, r_2, u_2, \ldots)) \mapsto (r_1 u_1, r_2 u_2, \ldots)$. The construction is functorial so that a morphism $f : X \rightarrow Y$ gives a continuous map $f^{log} : X^{log} \rightarrow Y^{log}$ compatible with the $\lambda$'s. We note also that $X^{log}$ can be made into a ringed space with structure sheaf $\mathcal{O}_{X^{log}}$ such that $\lambda$ becomes a morphism. One can picture $(X, E)^{log}$ as adding an ideal boundary to $X - E$ which is homeomorphic to the boundary of a tubular neighbourhood about $E$, and this picture is compatible with what is happening on $Y$. This leads to the remarkable fact, due to Usui, that when $f$ is proper and semistable $f^{log}$ is topologically a fibre bundle. A generalization of this due to Nakayama and Ogus [NO] will be used below.

2. Main theorems

**Theorem 2.1.** Let $f : X \rightarrow Y$ be a projective exact vertical log smooth map of fs log schemes and suppose that $Y$ is the log scheme associated to a log pair. Then

$$\mathbb{R}f_\ast \Omega^k_{X/Y} \cong \bigoplus_i R^i f_\ast \Omega^k_{X/Y}[-i]$$

for all $k$.

Readers dismayed by this long string of adjectives should keep in mind that a projective semistable map satisfies all of the above assumptions, and therefore the conclusion. The semistable case, however, is not strong enough to imply Kollár’s theorem. We really need the version just stated. Before giving the proof, we record the following elementary fact, which can be proved by induction on the length of the filtrations.

**Lemma 2.2.** Suppose that $(V_i, F_i^\bullet)$ are two filtered finite dimensional vector spaces such that $\dim \text{Gr}^p(V_1) = \dim \text{Gr}^p(V_2)$ for all $p$. Then a linear isomorphism $L : V_1 \rightarrow V_2$ which is compatible with the filtrations will induce an isomorphism of associated graded spaces.

**Proof of theorem.** Let $d$ denote the relative dimension of $f$. Let $Y$ be defined by the log pair $(Y, D)$. The verticality assumption implies that the log structure of $X$ is trivial on $U = X - f^{-1}D$. The restriction $f|_U : U \rightarrow Y - D$ is smooth and projective in the usual sense. So it is topologically a fibre bundle. In fact a theorem of Nakayama and Ogus [NO] shows that $f|_U$ prolongs naturally to a fibre bundle $f^{log} : X^{log} \rightarrow Y^{log}$ over $Y^{log}$.

Using the work of Illusie, Kato and Nakayama [IKN, cor 7.2], we may conclude that sheaves $R^i f_\ast \Omega^k_{X/Y}$ are locally free and that the relative Hodge to de Rham spectral sequence degenerates. Let $\eta' \in H^1(X, \Omega^1_X)$ denote $c_1$ of a relatively ample line bundle, and let $\eta \in H^1(\Omega^1_{X/Y}) \cong \text{Hom}_{D(Y)}(\mathcal{O}_Y, \mathbb{R}f_\ast \Omega^1_{X/Y}[1])$ denote the image of $\eta'$. Let $V^k = \mathbb{R}f_\ast \Omega^k_{X/Y}$ and $V = \bigoplus_k V^k[-k]$. We have a Lefschetz operator $L : V^k \rightarrow V^{k+1}$ given by cup product with $\eta$. By adding these, we get a map $L : V \rightarrow V[1]$. Our goal is to establish the hard Lefschetz property that $L'$ induces an isomorphism

$$L' : \mathcal{H}^{d-i}(V) \cong \mathcal{H}^{d+i}(V)$$
and Nakayama's lemma, it suffices to check the fibres. For this we may appeal to a theorem of Fujisawa.

Note that $C$ is stable under base change to $R$. The hard Lefschetz property for $(R, R)$ coincide with the ranks of $R^{d-i-k}f_*\Omega^k_{X/Y}$ and $R^{d-k}f_*\Omega^{k+i}_{X/Y}$ respectively, and therefore with each other. Therefore by lemma 2.2 and Nakayama’s lemma, it suffices to check the hard Lefschetz property for $(R^i, \Omega^i_{X/Y})$ and all $y \in Y$. By [K1], we have a canonical identification

$$\lambda^*\mathbb{R}f_*\Omega^i_{X/Y} \cong \mathbb{R}f^*_i\Omega^i_C \otimes \mathcal{O}_{Y/\log}$$

Note that $\mathbb{R}f^*_i\log C$ also has an action by $L$, and (3) respects these actions. Since the stalk $(R^i f^*_i \Omega^i_{X/Y})_y$ is just $H^i(X_y, \mathbb{C})$, when $y \notin D$, and $R^i f^*_i \log C$ is a local system, we can conclude that we have a hard Lefschetz theorem everywhere, i.e. $L^i : R^{d-i} f^*_i \log C \cong f^{d-i} f^*_i \log C$. By the previous remarks, this implies (1) and consequently the theorem.

**Remark 2.3.** The above argument can be pushed a bit to imply the same decomposition for a proper holomorphic semistable map of analytic log pairs provided that there is a 2-form on $X$ which restricts to a Kahler form on the components of all the fibres. For this we may appeal to a theorem of Fujisawa [F, thm 6.10] to conclude $R^i f_* \Omega^i_{X/Y}$ are locally free and that the relative Hodge to de Rham spectral degenerates. The rest of the argument is the same as above.

Before turning to Kollár’s theorem, we need the following:

**Lemma 2.4.** Suppose that $Y = (Y, D)$ is a log pair, $X$ is a variety with normal Gorenstein singularities, and that $f : X \to Y$ is map such that the log scheme $X$ defined by $f^{-1}D$ is fs. We suppose furthermore that $f : X \to Y$ is log smooth and saturated. Then $\Omega^d_{X/Y} \equiv \omega_{X/Y}$, where $d = \dim X - \dim Y$ and $\omega_{X/Y}$ is the relative dualizing sheaf.

**Proof.** The line bundle

$$\Omega^d_{X/Y} \otimes \omega_{X/Y}^{-1} \cong \mathcal{O}_X(E)$$

where $E = \sum n_i E_i$ is a Cartier divisor supported on $f^{-1}D$. In fact, we can make the choice of $E$ canonical. The divisor $E$ is determined by its restriction to the regular locus of $U \subseteq X$, since the complement of $U$ has codimension at least 2. So we may replace $X$ by $U$. Then $E$ is the divisor of the canonical map

$$\omega_{X/Y} \cong \Omega^d_X \otimes f^*\omega_Y^{-1} \to \Omega^d_{X/Y}$$

Our goal is to show that $E$ is in fact trivial as a divisor. Since this is now a local problem, there is no loss of generality in assuming that $Y$ is affine. The log smoothness and saturation assumptions imply that $f$ is flat. Therefore all of the irreducible components of $f^{-1}D$, and in particular $E$, must map onto components of $D$. Thus the preimage of a general curve $C \subset Y$ will meet all the components of $E$. The conditions of log smoothness and saturation and the isomorphism (4) are stable under base change to $C$. Therefore we may assume that $Y$ is a curve and that $D$ is a point with local parameter $y$. The fibre $f^{-1}D$ is reduced because $f$ is
saturated. Choose a general point \( p \) of an irreducible component \( E_i \) of \( E \). Then \( E_i \) is smooth in neighbourhood of \( p \) and \( f^*y \) gives a defining equation for it. Thus we may choose coordinates \( x_1 = f^*y, x_2, \ldots, x_n \) at \( p \). Then a local generator of \( \Omega^d_{X/Y} \) at \( p \) is given by

\[
(d \log x_1 \wedge dx_2 \wedge \ldots \wedge dx_n) \otimes (d \log y)^{-1} = (dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n) \otimes (dy)^{-1}
\]

which coincides with a generator of \( \omega_{X/Y} \). Thus \( E \) is trivial.

\[\Box\]

**Theorem 2.5** (Kollár). If \( X \) smooth and \( f : X \to Y \) is projective then

\[
\mathbb{R}f_*\omega_X \cong \bigoplus_i R^i f_*\omega_X[-i]
\]

**Proof.** The first step is to apply the log version of the weak semistable reduction theorem of Abramovich and Karu [AK]. Although Illusie and Temkin [IT, thm 3.10] have given such a version, it is a bit simpler to use the original form of the theorem, and then translate the conclusion. The theorem yields a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow{f'} & & \downarrow{f} \\
D \subset Y' & \xrightarrow{p} & Y
\end{array}
\]

where \( p \) is generically finite, \( Y' \) is smooth, \( D \) is a divisor with simple normal crossings, \( X' \) is birational to the fibre product, and \( f' \) is weakly semistable. Then, as we explained in example 1.6, \( f' \) is log smooth exact vertical and saturated with respect to the log schemes \( \mathcal{X}' \) defined by \( f'^{-1}D \) and \( \mathcal{Y}' \) defined by \( D \). Furthermore, it is known that \( X' \) has rational Gorenstein singularities [AK, lemma 6.1].

By lemma 2.4 and theorem 2.1, we obtain

\[
\mathbb{R}f'_*\omega_{X'/Y'} = \bigoplus R^i f'_*\omega_{X'/Y'}[-i]
\]

and therefore

\[
(5) \quad \mathbb{R}f'_*\omega_{X'/Y'} \cong \bigoplus R^i f'_*\omega_{X'}[-i].
\]

Fix a resolution of singularities \( g : \tilde{X} \to X' \). Then \( \mathbb{R}g_*\omega_{\tilde{X}} = g_*\omega_{\tilde{X}} = \omega_{X'} \), where the first equality follows from the Grauert-Riemenschneider vanishing theorem, and the second from the fact that \( X' \) has rational singularities. This together with (5) shows that

\[
(6) \quad \mathbb{R}(f' \circ g)_*\omega_{\tilde{X}} = \bigoplus R^i(f' \circ g)_*\omega_{\tilde{X}}[-i].
\]

We have an inclusion \((\pi \circ g)^*\omega_X \subset \omega_{\tilde{X}}\) which gives an injection

\[
\sigma : \omega_X \to (\pi \circ g)^*\omega_X \hookrightarrow (\pi \circ g)_*\omega_{\tilde{X}}
\]

The map \( \sigma \) splits the normalized Grothendieck trace

\[
\tau = \frac{1}{\deg X'/X} tr : \mathbb{R}(\pi \circ g)_*\omega_{\tilde{X}} \cong (\pi \circ g)_*\omega_{\tilde{X}} \to \omega_X
\]

It follows that \( \omega_X \) is a direct summand of \((\pi \circ g)_*\omega_{\tilde{X}}\), and that this relation persists after applying a direct image functor. Therefore applying \( \tau \) to (6) yields

\[
\mathbb{R}f_*\omega_X = \bigoplus R^i f_*\omega_X[-i].
\]
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