Elementary Methods for
Infinite Resistive Networks with Complex Topologies

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Abstract
Finding the equivalent resistance of an infinite ladder circuit is a classical problem in physics. We expand this well-known challenge to new classes of network topologies, in which the unit cells are much more entangled together. The exact analytical results there can still be obtained with elementary methods. These topology classes will add layers of complexity and much more diversity to a very popular kind of physics puzzles for teachers and students.

I. INTRODUCTION

The equivalent resistance $R_{AB}$ of the infinite ladder network composed of identical 1Ω resistors, shown in Fig. 1, can be calculated elegantly by adding one more unit cell (a ladder step) in the front:

$$R_{AB} = R_{\alpha \beta} = 1 + \frac{R_{\alpha \beta}}{1 + R_{\alpha \beta}}.$$  (1)

Here we make the assumption that as the number of ladder steps go to infinity, the equivalent resistance will converge. There are two possible solutions, $R_{AB} = (1 \pm \sqrt{5})/2 \Omega$, and by getting rid of the unphysical one with negative value we arrive at the answer to be uniquely $R_{AB} = (1 + \sqrt{5})/2 \Omega$ which is equal to the golden ratio.

![Fig. 1: (A) The infinite ladder topology network of 1Ω resistors. (B) The trick of adding one more unit cell to the network, where the equal sign is valid under the assumption of convergence.](image)

This problem is addressed in many introductory physics courses in college, and was also introduced to high school students in the very first International Physics Olympiad (Poland 1967). Despite of being widely known for a long time, there aren’t many variations: the unit cells are always linearly linked to the one before or after it via $N = 2$ nodes.

In this paper we will explore new classes of resistive networks with topologies more entangled, where the linear linking between unit cells has $N > 2$ nodes. To our knowledge this kind of electrical circuits is rarely mentioned, though they can still be solved by elementary methods using resistive relations come from superposition theorem and reciprocal theorem. Starting with estimating the results in Section II, we then develop modified methods for infinite networks and consider two twisted ladder topologies as representatives (see Fig. 2). Alongside, we will show some examples of network topologies that can be solved similarly, with all results being found analytically and confirmed numerically. We hope to provide teachers and students with more physics puzzles of this kind to explore.

![Fig. 2: (A) The $N = 3$ infinite asymmetric twisted ladder network of 1Ω resistors. (B) The $N = 4$ infinite symmetric twisted ladder network of 1Ω resistors.](image)
II. ROUGH ESTIMATIONS USING RAYLEIGH’S MONOTONICITY LAW

Even without arriving at the solutions to these problems, students should be encouraged to make guesses about what the resistances might be. We can make some rough estimations for the equivalent resistance values $R^{(asym)}_{AB}$ and $R^{(sym)}_{AB}$ of those infinite twisted ladders, using Rayleigh’s monotonicity law which states that when the resistance in one part of a circuit increases, the effective resistance also increases. By setting some resistors to the extremes of $0\Omega$ and $\infty\Omega$ (see Fig. 3 and Fig. 4), we can achieve within the level of $\pm 10\%$ uncertainty:

$$7/3\Omega \approx 2.33\Omega > R^{(asym)}_{AB} > 13/6\Omega \approx 2.17\Omega \quad (2a)$$
$$17/12\Omega \approx 1.42\Omega > R^{(sym)}_{AB} > 6/5\Omega = 1.20\Omega \quad (2b)$$

III. A TOOLKIT FOR RESISTIVE NETWORKS

In this section we create a toolkit for resistive networks, using only superposition theorem and reciprocal theorem. Define the generalized resistance $G_{\gamma\delta}^{\alpha\beta}$ from the voltage different $U_{\gamma\delta} = V_{\gamma} - V_{\delta}$ between nodes $\gamma$ and $\delta$ when there’s only a current $I_{\alpha\beta}$ comes into node $\alpha$ and goes out from node $\beta$ (see Fig. 5A):

$$G_{\gamma\delta}^{\alpha\beta} = U_{\gamma\delta} / I_{\alpha\beta} \quad (3)$$

The resistance $R_{\alpha\beta}$ between nodes $\alpha$ and $\beta$ is related to this as $R_{\alpha\beta} = G_{\alpha\beta}^{\alpha\beta}$.

Follow from the definition, switching the voltage’s nodes and the current’s nodes gives the permutation rule:

$$G_{\alpha\beta}^{\gamma\delta} = -G_{\alpha\beta}^{\delta\gamma} = -G_{\beta\alpha}^{\gamma\delta} \quad (4)$$

For an arbitrary node $\sigma$, using the additivity of voltage difference and superposition theorem for current, we have the summing rule:

$$G_{\gamma\delta}^{\alpha\beta} = G_{\gamma\delta}^{\alpha\sigma} + G_{\alpha\beta}^{\sigma\delta}, \quad G_{\alpha\beta}^{\gamma\delta} = G_{\alpha\beta}^{\gamma\sigma} + G_{\beta\alpha}^{\sigma\delta} \quad (5)$$

With reciprocal theorem (which can be derived from superposition theorem and uniqueness theorem), we arrive at the flipping rule:

$$G_{\alpha\beta}^{\gamma\delta} = G_{\gamma\delta}^{\alpha\beta} \quad (6)$$

Some students might be more familiar with source transformation, which can also be used to obtain the above result and indeed is a direct application of reciprocal theorem.
From equations (5) and (6), every generalized resistance $G_{\alpha\beta}$ can be written in terms of resistances as:

$$G_{\alpha\beta} = \frac{R_{\alpha\delta} + R_{\beta\gamma} - R_{\alpha\gamma} - R_{\beta\delta}}{2}.$$ (7)

For a “black box” of resistors with $n$ terminals $\alpha_1, \alpha_2, \ldots, \alpha_n$, this implies that knowing the values of $n(n-1)/2$ resistances $\{R_{\alpha_i\alpha_j}\}$ is enough to determine all characteristics $\{G_{\alpha_i\alpha_j}\}$ of the “black box”, thus knowing the global property of the network. In other words, internal resistor networks with identical sets of $\{R_{\alpha_i}\}$ are equivalent even though their topologies are different.

For $n = 3$, $\Delta$-$Y$ are the simplest topologies possible, and it is known that local transformation from $\Delta$ topology to $Y$ topology (see Fig. 6A) does not change any global property. An extended version of $\Delta$-$Y$ transformation exists (see Fig. 6B): an arbitrary internal network can be simplified to either $\Delta$ or $Y$ topology, since there always exists a star network that satisfies (7) for any $G_{\alpha\beta}$ with $\{\alpha, \beta, \gamma, \delta\} \subset \{\alpha_1, \alpha_2, \alpha_3\}$. This simplification, illustrated in Fig 6, will be applied directly in Section IV.A.

**FIG. 6:** (A) The $\Delta$-$Y$ transformation. (B) An extended version of $\Delta$-$Y$ transformation, where $\Delta$ can be generalized to any internal topology.

As it will be shown below, this method is easy to be done for $n = 3$. When $n > 3$, as the simplest topologies of which an arbitrary internal network can be simplified down to are either a mesh or specific combinations of many stars, it becomes highly convoluted to analyze even the simplified topologies in relation to the new unit cell. In this case, a method independent of internal topology becomes preferable.

**IV. CALCULATING THE EQUIVALENT RESISTANCES**

In this section, using the two methods above, we will show that $R_{AB}^{(sym)} = \sqrt{2}\sqrt{7}$ and $R_{AB}^{(asym)} = \sqrt{-1 + 2\sqrt{7}}$, along with analytical results for other $N > 2$ entangled topologies.

### A. $N = 3$ Infinite Asymmetric twisted Ladder

To calculate $R_{AB}$ for the asymmetric twisted ladder, we apply the extended $\Delta$-$Y$ transformation mentioned in Section III (see Fig. 7A). Assume that the equivalent resistance between nodes A, B and C converge in the infinity limit, the trick of adding one more unit cell can be used as shown in Fig. 7B, which gives us a set of three consistency equations:

$$R_{AB} = r_\alpha + r_\beta = 1 + r_\beta + \frac{(1 + r_\alpha)(1 + r_\gamma)}{2 + r_\alpha + r_\gamma},$$

$$R_{BC} = r_\beta + r_\gamma = \frac{1 + r_\alpha + r_\gamma}{2 + r_\alpha + r_\gamma},$$

$$R_{CA} = r_\gamma + r_\alpha = 1 + r_\beta + \frac{r_\alpha(2 + r_\gamma)}{2 + r_\alpha + r_\gamma}.$$ (8)

Three equations for three real positive unknowns $r_\alpha, r_\beta, r_\gamma$ can be solved analytically:

$$r_\alpha = \frac{-1 + \sqrt{7} + \sqrt{2}\sqrt{7}}{2},$$

$$r_\beta = \frac{1 - \sqrt{7} + \sqrt{2}\sqrt{7}}{2},$$

$$r_\gamma = \frac{-1 + \sqrt{-7} + 4\sqrt{7}}{2}.$$ (9)

Thus we arrive at

$$R_{AB}^{(asym)} = \sqrt{2}\sqrt{7} \approx 2.30\Omega,$$ (10)

which is in agreement with (10). We also confirm this results with numerical evaluation (see Fig. 8).

The method can also be applied to many different topologies, such as those in Fig. 9 which are linear linking with number of nodes $N = 3$ between unit cells. See Table I for the list of equivalent resistances $R_{AB}$. 

![Diagram](image-url)
Equivalent Resistance

\[ \Delta(N) = \frac{R_{AB}(N)}{R_{AB}} - 1 \]

\[ \Delta = R_{AB}(N)/R_{AB} - 1, \] which approaches 0 exponentially fast.

TABLE I: Results for \( R \) given in Fig. 9.

| Circuit          | Equivalent Resistance \( R_{AB} \) (in \( \Omega \)) |
|------------------|--------------------------------------------------|
| A                | \(-1 - \sqrt{7 + \sqrt{8 + 4\sqrt{5}}}/2 \approx 0.77\) |
| B                | \(-1 + \sqrt{7 + 2\sqrt{5}} / 4 \approx 0.72\) |
| C                | \(-1 + \sqrt{11 + \sqrt{45}} / 2 \approx 0.70\) |
| D                | \(-1 - \sqrt{7 + 2\sqrt{5}} / 2 \approx 0.65\) |

B. \( N = 4 \) Infinite Symmetric Twisted Ladder

We represent the infinite resistive network between nodes A, B, C, and D by a “black box” with four terminals \( \alpha, \beta, \gamma \) and \( \delta \) (see Fig. 10A). Assume that the equivalent resistances between nodes A, B, C, and D converge in the infinity limit, the trick of adding one more unit cell can be used as shown in Fig. 10B.

\[
\begin{align*}
R_{AB} &= R_{\alpha\beta} = \left\{ \begin{array}{c}
1 + G_{\alpha\beta}^\gamma - (1 + G_{\alpha\beta}^\delta) \\
1 + G_{\alpha\beta}^\alpha + G_{\alpha\beta}^\gamma \\
- (1 + G_{\alpha\beta}^\gamma) - (1 + G_{\alpha\beta}^\delta) \\
2 + R_{\gamma\delta}
\end{array} \right\}, \\
R_{CD} &= R_{\alpha\delta} = \left\{ \begin{array}{c}
G_{\beta\delta}^\alpha G_{\gamma\delta}^\beta R_{\alpha\beta} \\
2 + G_{\alpha\beta}^\alpha G_{\beta\alpha}^\gamma G_{\beta\gamma}^\alpha G_{\beta\gamma}^\beta \\
2 + G_{\gamma\delta}^\gamma G_{\beta\delta}^\gamma G_{\gamma\delta}^\beta G_{\gamma\delta}^\gamma
\end{array} \right\}, \\
R_{AD} &= R_{\alpha\delta} = \left\{ \begin{array}{c}
1 + R_{\alpha\gamma} G_{\alpha\delta}^\beta \\
1 + G_{\alpha\gamma}^\alpha + G_{\alpha\gamma}^\beta + G_{\alpha\gamma}^\delta + G_{\alpha\gamma}^\gamma \\
0
\end{array} \right\}, \\
R_{BC} &= R_{\beta\gamma} = \left\{ \begin{array}{c}
1 + R_{\beta\delta} G_{\gamma\delta}^\alpha \\
1 + G_{\delta\beta}^\gamma + G_{\delta\beta}^\gamma + 2 + G_{\delta\beta}^\gamma + G_{\delta\beta}^\gamma \\
0
\end{array} \right\}, \\
R_{AC} &= R_{\alpha\gamma} = \left\{ \begin{array}{c}
1 + G_{\alpha\gamma}^\alpha G_{\alpha\gamma}^\beta G_{\alpha\gamma}^\gamma G_{\alpha\gamma}^\delta \\
-1 - G_{\alpha\gamma}^\alpha - G_{\alpha\gamma}^\beta - 1 + R_{\gamma\delta} \\
1 + G_{\alpha\gamma}^\alpha + G_{\alpha\gamma}^\beta + G_{\gamma\delta}^\gamma \\
2 + G_{\alpha\gamma}^\alpha + G_{\alpha\gamma}^\beta + G_{\gamma\delta}^\gamma
\end{array} \right\}, \\
R_{BD} &= R_{\alpha\delta} = \left\{ \begin{array}{c}
1 + G_{\beta\delta}^\gamma G_{\beta\delta}^\gamma G_{\beta\delta}^\gamma \\
1 + G_{\delta\beta}^\gamma G_{\delta\beta}^\gamma + G_{\delta\beta}^\gamma + 1 + R_{\alpha\delta} \\
-1 - G_{\delta\beta}^\gamma - G_{\delta\beta}^\gamma - 1 + R_{\alpha\delta} \\
1 + G_{\delta\beta}^\gamma + 2 + G_{\delta\beta}^\gamma + G_{\delta\beta}^\gamma
\end{array} \right\}, \end{align*}\]
with
\[
\begin{bmatrix}
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2 \\
a_3 & b_3 & c_3 \\
\end{bmatrix} = \sum_{i,j,k=1}^{3} \epsilon_{ijk} a_i b_j c_k, \quad (12)
\]
where \( \epsilon_{ijk} \) is the Levi-Civita symbol. From the set of equations, the real positive \( R_{\alpha\beta}, R_{\alpha\gamma}, R_{\alpha\delta}, R_{\gamma\delta} \) values can be found analytically:
\[
\begin{align*}
R_{\alpha\beta} &= \sqrt{-1 + 2\sqrt{2}} , \\
R_{\alpha\gamma} &= \frac{-4 + 2\sqrt{2} + \sqrt{5} + 5 + 4\sqrt{2}}{4}, \\
R_{\alpha\delta} &= \frac{\sqrt{5} - 2\sqrt{2} + \sqrt{5} + 4\sqrt{2}}{4}, \\
R_{\gamma\delta} &= -1 + \sqrt{4\sqrt{2} - 2} .
\end{align*}
\]
Thus we arrive at
\[
R_{AB} = \sqrt{-1 + 2\sqrt{2}} \approx 1.35\Omega , \quad (14)
\]
which is in agreement with [20]. We also confirm this result with numerical evaluation (see Fig. 11). This same method of using voltage path summation and current conservation to obtain a set of consistency equations can also be applied to many different topologies, such as those in Fig. 12 which are linear linking with number of nodes \( N \geq 4 \) between unit cells. See Table II for the list of equivalent resistances \( R_{AB} \).
inductors and capacitors. Those applications will be discussed elsewhere[8].

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