New Results on Stability of Linear Discrete-Time Systems With Time-Varying Delay

JUN CHEN¹ AND XIANGYONG CHEN²,³, (Member, IEEE)
¹School of Electrical Engineering and Automation, Jiangsu Normal University, Xuzhou 221116, China
²Key Laboratory of Complex Systems and Intelligent Computing in Universities of ShanDong, School of Automation and Electrical Engineering, Linyi University, Linyi 276005, China
³Hubei Key Laboratory of Advanced Control and Intelligent Automation for Complex Systems, and Engineering Research Center of Intelligent Technology for Geo-Exploration, Ministry of Education, Wuhan 430074, China
Corresponding author: Xiangyong Chen (cxy8305@163.com)

This work was supported in part by NSFC under Grant 61773186 and Grant 11805091, in part by the Science Fundamental Research Project of Jiangsu Normal University under Grant 17XLR045, in part by the Development Plan of Youth Innovation Team of University in Shandong Province under Grant 2019KJN007, and in part by the 111 Project under Grant B17040.

ABSTRACT This paper is concerned with the stability problem for linear discrete-time systems with a time-varying delay. Some relations between two general free-matrix-based summation inequalities are discussed. A novel Lyapunov–Krasovskii functional (LKF) is proposed by modifying the single- and double-summation LKF terms. As a result, a new stability condition is obtained by employing the general free-matrix-based summation inequalities reported recently. Numerical examples are given to show that the obtained stability condition is more relaxed than some of existing results.

INDEX TERMS Discrete-time system, free-matrix-based inequality, Lyapunov functional, stability, time-varying delay.

I. INTRODUCTION

It is well known that time delay, as a natural phenomenon, widely exists in various practical systems such as networked control systems, fuzzy systems and neural networks [1]–[3]. Its existence usually brings oscillation, divergence, and even instability [1]. On the other hand, a proper introduction of small delay can make an unstable system become stable or improve dynamic performances [1]. Therefore, during past decades, linear discrete-time systems with a time-varying delay have attracted extensive attention from academic research [4]–[12] and a number of remarkable results have been reported in the literature [13]–[24].

Nowadays, the LKF method is a powerful tool to deal with the stability problem for linear delayed systems. To achieve a relaxed condition, choosing an appropriate LKF candidate is a key point, which is required to take more information of various state vectors into account. Another important point is to tightly estimate the forward difference of this candidate by employing advanced techniques such as free-weighting-matrix technique [25], summation inequalities [26]–[33], reciprocally convex combination lemmas (RCCLs) [34]–[36] and free-matrix-based (FMB) summation inequalities [37]–[39].

In recent years, instead of free-weighting-matrix technique, summation inequalities are commonly used to deal with summation terms arising in the forward differences of LKFs. For examples, the Wirtinger-based summation inequality is proposed in [26], [27] and [29], which produces a tighter bound than Jensen summation inequality. Later, another kind of summation inequality, called the free-matrix-based summation inequality, is developed by introducing several free matrices [38]. As stated in [38], it covers the Wirtinger-based summation inequality as a special case. In the case that the FMB summation inequality is applied, RCCL is no longer required to further estimate summation terms. Very recently, a general free-matrix-based (GFMB) summation inequality is developed [40], which leads to a relaxed stability condition for linear discrete-time systems with a time-varying delay. However, the interest of the GFMB inequality is not fully utilized [40] when estimating summation terms. It is noted that to analyse the stability of discrete-time neural networks with a time-varying delay [37], another GFMB inequality is proposed by providing more freedom to choose the undetermined augmented vectors. The relations between the two GFMB inequalities deserve to be discussed.

As stated before, constructing an appropriate LKF candidate plays a key role in achieving a relaxed stability condition. To mention a few, to coordinate with the Wirtinger-based summation inequality, a new augmented LKF is proposed...
in [26], in which the quadratic augmented vector contains three vectors \(x(k), \sum_{i=k-h_1}^{k-1} x(i)\) and \(\sum_{i=k-h_2}^{k-h_1-1} x(i)\), where \(x(k)\) is the state, \(h_1\) and \(h_2\) are, respectively, the lower and upper bounds of the discrete-time delay. This proposed LKF is later widely used in the next few years in the literature. Additionally, in order to cooperate with the use of double-summation inequalities, triple-summation LKF terms are included in the LKF proposed in [28]. It is worth pointing out that on the basis of the LKF proposed in [26], a double-summation LKF term \(\sum_{i=k-h_2}^{k-1} x(i)\) is added [40] that is really helpful to reduce the conservatism of the obtained stability condition by employing a GFMB summation inequality. However, there is still some room to develop a new and appropriate LKF candidate. This motivates this research.

In this paper, we further study the stability problem for linear discrete-time systems with a time-varying delay via the LKF method. First, some relations between the two GFMB summation inequalities recently proposed in [37] and [40] are discussed. It is pointed out that the GFMB summation inequality proposed in [37] covers that proposed in [40] as a special case. Meanwhile, for convenience of applications, summation inequalities [37] are rewritten in a more compact form. Second, a novel LKF is constructed by modifying the single- and double-summation LKF terms so that more relations among different state vectors are taken into account. Finally, two numerical examples are given to show that the obtained stability condition produces more relaxed results than existing ones, especially one recently reported in [40].

**Notations.** Throughout this paper, \(\text{Sym}(X)\) denotes \(X + X^T\) for any square real matrix \(X\). \(\mathbb{R}^n\) denotes the \(n\)-dimensional Euclidean space and \(\mathbb{R}^{n \times m}\) the set of all \(n \times m\) real matrices. \(I\) and \(0\) denote the identity and zero matrices of appropriate dimension, respectively. \(\mathbb{S}^n_+\) represents the set of symmetric positive-definite matrices of \(\mathbb{R}^{n \times n}\).

**II. PRELIMINARY AND USEFUL LEMMAS**

Consider the following linear discrete-time system with a time-varying delay:

\[
\begin{align*}
    x(k+1) &= Ax(k) + A_dx(k - h(k)), \quad k \geq 0, \\
    x(k) &= \phi(k),
\end{align*}
\]

where \(x(k) \in \mathbb{R}^n\) is the state vector; \(\phi(k)\) is the initial condition; \(A, A_d \in \mathbb{R}^{n \times n}\) are constant matrices; \(h(k)\), abbreviated as \(h_k\), is a time-varying delay, satisfying

\[
0 < h_1 \leq h_k \leq h_2
\]

where \(h_1\) and \(h_2\) are known integers satisfying \(h_1 < h_2\).

This article aims to develop a stability condition that is of less conservatism, compared to some of existing results recently reported. Based on the new stability condition, larger maximum allowable upper bounds (MAUBs) should be obtained for different \(h_1\) so that the considered system (1) is ensured to be stable with the delay \(h_k\) varying within the interval \([h_1, h_2]\) as large as possible. To do so, a novel LKF will be developed and the recently-reported GFMB summation inequalities will be employed.

In the rest of this section, we will present some lemmas that are useful to obtain the main results. Before proceeding, two functions are first defined: \(s_1(h) = h + 1\) and \(s_2(h) = \frac{(h+1)^2 + 2h}{2}\) for \(h \in \mathbb{R}\).

**Lemma 1** [37]: For matrices \(R \in \mathbb{S}^n_+, N_0, N_1, N_2\) with appropriate dimensions, and a vector function \(\{x_a(i)\} \in [0, m]\) where \(x_a(i) = x(a + i)\), the following inequality

\[
- \sum_{i=0}^{m-1} x_a(i) R x_a(i) \leq 2 \sum_{p=0}^{m-1} \frac{m}{2p+1} \omega_p^T N_p R^{-1} N_p^T \omega_p + \sum_{p=0}^{m} \text{Sym} \left[ \omega_p^T N_p \chi_p \right]
\]

holds, where \(\omega_0, \omega_1\) and \(\omega_2\) are any vectors, and

\[
\begin{align*}
    \chi_0 &= \sum_{i=0}^{m-1} x_a(i), \\
    \chi_1 &= -\chi_0 + \frac{2}{m+1} \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} x_a(j), \\
    \chi_2 &= \chi_0 - \frac{6}{m+1} \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} x_a(j) + \frac{12}{(m+1)(m+2)} \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} x_a(i).
\end{align*}
\]

**Remark 1:** If we let \(\omega_0 = \omega_1 = \omega_2 = \omega\) and \(N^T = \begin{bmatrix} N_0 & N_1 & N_2 \end{bmatrix}\), Inequality (3) can be rewritten in a more compact form. So do summation inequalities developed in Lemma 2 proposed in [37].

**Lemma 2:** For matrices \(R \in \mathbb{S}^n_+, N \in \mathbb{R}^{3n \times m}\) and vector functions \(\{x_a(i), y_a(i)\} \in [0, m]\) where \(x_a(i) = x(a + i)\) and \(y_a(i) = x_a(i+1) - x_a(i)\), the inequalities

\[
- \sum_{i=0}^{m-1} x_a(i) R x_a(i) \leq m \omega^T N R^{-1} N \omega + \text{Sym} [\omega^T N R T \chi],
\]

\[
- \sum_{i=0}^{m-1} y_a(i) R y_a(i) \leq \frac{1}{m} \hat{\chi}^T \hat{R} \hat{\chi},
\]

\[
- \sum_{i=0}^{m-1} y_a(i) R y_a(i) \leq m \omega^T N R^{-1} N \omega + \text{Sym} [\omega^T N R T \hat{\chi}],
\]

\[
- \sum_{i=0}^{m-1} y_a(i) R y_a(i) \leq \frac{1}{m} \hat{\chi}^T \hat{R} \hat{\chi}
\]

hold, where \(\omega \in \mathbb{R}^{3m}\) is any vector, \(\chi_0, \chi_1\) and \(\chi_2\) are defined in Lemma 1, and

\[
\hat{\chi} = \text{diag} [R, 3R, 5R],
\]

\[
\hat{\chi} = \chi_0, \hat{\chi}_1, \hat{\chi}_2,
\]

\[
\hat{x}_0 = x_a(m) - x_a(0), \quad \hat{\chi}_1 = x_a(m) + x_a(0) - 2\tilde{\omega}_0,
\]
\[ \dot{x}_2 = x_a(m) - x_0(0) + 6\bar{\Omega}_0 - 6\bar{\Omega}_2, \]
\[ \bar{\Omega}_0 = \sum_{i=0}^{m} x_a(i) s_1(m), \quad \bar{\Omega}_2 = \sum_{i=0}^{m} \sum_{j=1}^{m} x_a(j) s_2(m). \]

Inequalities (4)-(7) are all induced from Inequality (3). Recently, a novel GFMB summation inequality is proposed in [40]. For convenience of comparison, it is recalled in the following lemma.

Lemma 3 [40]: For a given \( n \times n \)-matrix \( R > 0 \), and three given nonnegative integers \( a, b, k \), satisfying \( a < b \leq k \), a vector function \( x(\cdot) \in \mathbb{R}^n \), taking \( n \times pn \)-matrices \( \Theta_1 \) \((i = 1, 2) \) and a vector \( \xi \in \mathbb{R}^{pn} \) such that
\[ x(k - a) - x(k - b) = \Theta_1 \xi, \]
\[ x(k - a) + x(k - b) - \frac{2}{b - a + 1} \sum_{s=k-b}^{k-a-1} x(s) = \Theta_2 \xi, \]
then, for any \( n \times pn \)-matrices \( N_i \) \((i = 1, 2) \), the following inequality holds:
\[ -\sum_{s=k-b}^{k-a-1} y^T(k) R y(k) \leq (b - a) \xi^T N^T R^{-1} N \xi + \xi^T \text{Sym}[N^T \bar{\Theta}] \xi, \quad (8) \]
where \( \Theta = \begin{bmatrix} \Theta_1 \\ \Theta_2 \end{bmatrix}, N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \), and \( \bar{\Theta} = \begin{bmatrix} R & 0 \\ 0 & 3R \end{bmatrix} \).

Remark 2: If letting \( \chi_0 = \Theta_1 \xi, \quad \chi_1 = \Theta_2 \xi = \Theta_3 \omega \) and \( \omega = \xi \), we get from Inequality (6) that
\[ -\sum_{i=0}^{m-1} y^T(i) R y(i) \leq m \xi^T N^T R^{-1} N \xi + \xi^T \text{Sym}[N^T \bar{\Theta}] \xi, \]
where \( \bar{\Theta} = \text{col}(\Theta_1, \Theta_2, \Theta_3) \). It is not difficult to see that Inequality (6) includes (8) as a special case.

III. MAIN RESULTS
Before proceeding, the following notations are defined:

\[ h_{21} = h_2 - h_1, \quad h_{22} = h_2 - h_k, \]
\[ \bar{h}_1 = h_k - h_1, \quad y(k) = x(k + 1) - x(k), \]
\[ u_1(k) = \sum_{i=k-h_1}^{k} \frac{x(i)}{s_1(h_1)}, \quad u_2(k) = \sum_{i=k-h_2}^{k-h_1} \frac{x(i)}{s_1(h_1)}, \]
\[ u_3(k) = \sum_{i=k-h_2}^{k-h_1} \frac{x(i)}{s_1(h_2)}, \]
\[ v_1(k) = \sum_{i=k-h_1}^{k} \frac{x(i)}{s_2(h_1)}, \]
\[ v_2(k) = \sum_{i=k-h_2}^{k-h_1} \frac{x(i)}{s_2(h_1)}, \]
\[ v_3(k) = \sum_{i=k-h_2}^{k-h_1} \frac{x(i)}{s_2(h_2)}, \]
\[ \xi_0(k) = \text{col}\{x(k), x(k - h_1), x(k - h_k), x(k - h_2)\}, \]
\[ \xi_1(k) = \text{col}\{u_1(k), u_2(k), u_3(k)\}, \]
\[ \xi_2(k) = \text{col}\{v_1(k), v_2(k), v_3(k)\}, \]
\[ \xi(k) = \text{col}\{\xi_0(k), \xi_1(k), \xi_2(k)\}. \]

Now we are in a position to construct a new LKF candidate:
\[ V(k) = V_0(k) + V_1(k) + V_2(k) + V_3(k), \quad (9) \]
where
\[ V_0(k) = y_0^T(k) P y_0(k), \quad (10) \]
\[ V_1(k) = \sum_{i=k-h_1}^{k-h_1-1} y_1^T(k, i) Q_1 y_1(k, i) + \sum_{i=k-h_2}^{k-h_1-1} y_1^T(k, i) Q_2 y_1(k, i), \quad (11) \]
\[ V_2(k) = \sum_{i=k-h_1}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} y_0^T(j) R_1 y(j) + \sum_{i=k-h_2}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} y_0^T(j) R_2 y(j), \quad (12) \]
\[ V_3(k) = \sum_{i=k-h_1}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} y_0^T(j) R_3 x(j) + \sum_{i=k-h_2}^{k-h_1-1} \sum_{j=i}^{k-h_1-1} y_0^T(j) R_4 x(j), \quad (13) \]
with
\[ y_0(k) = \text{col}\{x(k), \sum_{i=k-h_1}^{k-h_1-1} x(i), \sum_{i=k-h_2}^{k-h_1-1} x(i)\}, \]
\[ y_1(k, i) = \text{col}\{x(k), x(i)\}. \]

Theorem 1: For given integers \( h_1 \) and \( h_2 \) satisfying (2), discrete-time system (1) is asymptotically stable if there exist matrices \( P \in \mathbb{S}^m_{+}, Q_1, Q_2 \in \mathbb{S}^p_{+}, R_1, R_2, R_3, R_4 \in \mathbb{R}^p, N_1, N_2, N_3 \in \mathbb{R}^{p \times 10n} \) and \( M_1, M_2, M_3 \in \mathbb{R}^{2n \times 10n} \) such that the following LMIs
\[
\begin{bmatrix}
\Sigma(h_1) & h_1 N_1^T & h_2 N_1^T & h_1 M_1^T & h_2 M_1^T \\
(*) & -h_1 R_1 & 0 & 0 & 0 \\
(*) & (*) & -h_2 R_2 & 0 & 0 \\
(*) & (*) & (*) & -h_1 R_1 & 0 \\
(*) & (*) & (*) & (*) & -h_2 R_2 & 0 \\
(*) & (*) & (*) & (*) & (*) & -h_1 R_3 & 0 \\
(*) & (*) & (*) & (*) & (*) & (*) & -h_2 R_4
\end{bmatrix} < 0,
\]
\[
\begin{bmatrix}
\Sigma(h_2) & h_1 N_2^T & h_2 N_2^T & h_1 M_2^T & h_2 M_2^T \\
(*) & -h_1 R_1 & 0 & 0 & 0 \\
(*) & (*) & -h_2 R_2 & 0 & 0 \\
(*) & (*) & (*) & -h_1 R_1 & 0 \\
(*) & (*) & (*) & (*) & -h_2 R_2 & 0 \\
(*) & (*) & (*) & (*) & (*) & -h_1 R_3 & 0 \\
(*) & (*) & (*) & (*) & (*) & (*) & -h_2 R_4
\end{bmatrix} < 0
\]
hold, where
\[ \Xi(h_1) = \Pi_1^T P \Pi_2 + \Pi_1^T P \Pi_1 + \text{Sym}[(\Pi_2 - \Pi_1)^T P \Pi_0(h_k)], \quad (16) \]
\[ \Pi_1 = \text{col}\{e_1, -e_1, -e_2 - e_3\}, \]
\[ \Pi_2 = \text{col}\{e_s, -e_2 - e_3 - e_4\}. \]
Along the trajectory of system (1), the forward differences of $V_2(k)$ and $V_3(k)$ are, respectively, calculated as follows:

$$
\Delta V_2(k) = y^T(k)(h_1R_1 + h_21R_2)y(k) - \sum_{i=k-h_1}^{k-1} y^T(i)R_1y(i) - \sum_{i=k-h_2}^{k-1} y^T(i)R_2y(i),
$$

$$
\Delta V_3(k) = x^T(k)(h_1R_3 + h_21R_4)x(k) - \sum_{i=k-h_1}^{k-1} x^T(i)R_3x(i) - \sum_{i=k-h_2}^{k-1} x^T(i)R_4x(i).
$$

Now applying Equation (6) in Lemma 2 to estimate summation terms in (22) yields:

$$
- \sum_{i=k-h_1}^{k-1} y^T(i)R_1y(i) \leq \xi^T(k)(\text{Sym}[N_1^T\Gamma_1]) + h_1N_1^T\tilde{R}_1^{-1}N_1\xi(k),
$$

$$
- \sum_{i=k-h_2}^{k-1} y^T(i)R_2y(i) \leq \xi^T(k)(\text{Sym}[N_2^T\Gamma_2]) + h_1N_2^T\tilde{R}_2^{-1}N_2\xi(k),
$$

$$
- \sum_{i=k-h_2}^{k-1} y^T(i)R_2y(i) \leq \xi^T(k)(\text{Sym}[N_2^T\Gamma_3]) + h_2N_3^T\tilde{R}_3^{-1}N_3\xi(k).
$$

Combining (22) with (24), (25) and (26) yields:

$$
\Delta V_2(k) \leq \xi^T(k)(\mathcal{Z}_{21} + \mathcal{Z}_{22}(h_1))\xi(k),
$$

where

$$
\mathcal{Z}_{22}(h_1) = h_1N_1^T\tilde{R}_1^{-1}N_1 + h_1N_2^T\tilde{R}_2^{-1}N_2 + h_2N_3^T\tilde{R}_3^{-1}N_3.
$$

Now we apply Equation (4) in Lemma 2 to estimate summation terms in (23):

$$
- \sum_{i=k-h_1}^{k-1} x^T(i)R_3x(i) \leq \xi^T(k)(\text{Sym}[M_1^T\Upsilon_1]) + h_1M_1^T\tilde{R}_3^{-1}M_1\xi(k),
$$

$$
- \sum_{i=k-h_2}^{k-1} x^T(i)R_4x(i) \leq \xi^T(k)(\text{Sym}[M_2^T\Upsilon_2]) + h_1M_2^T\tilde{R}_4^{-1}M_2\xi(k),
$$

$$
- \sum_{i=k-h_2}^{k-1} x^T(i)R_4x(i) \leq \xi^T(k)(\text{Sym}[M_3^T\Upsilon_3]) + h_2M_3^T\tilde{R}_4^{-1}M_3\xi(k).
$$

Then, combining (23) with (28), (29) and (30) leads to:

$$
\Delta V_3(k) \leq \xi^T(k)(\mathcal{Z}_{31} + \mathcal{Z}_{32}(h_1))\xi(k).
$$
where
\[ \mathcal{Z}_{32}(h_k) = h_1 M_1^T \tilde{R}_5^{-1} M_1 + h_k M_2^T \tilde{R}_4^{-1} M_2 + h_2 k M_3^T \tilde{R}_3^{-1} M_3. \] (32)

According to the above discussions, the forward difference of \( V(k) \) is estimated as follows:
\[ \Delta V(k) \leq \xi^T(k) \left( \dot{\Sigma}(h_k) + \mathcal{Z}_{32}(h_k) + \mathcal{Z}_{22}(h_k) \right) \xi(k), \] (33)
where \( \dot{\Sigma}(h_k) = \mathcal{Z}_0(h_k) + \mathcal{Z}_1(h_k) + \mathcal{Z}_{21} + \mathcal{Z}_{31} \).

Define \( \Omega(h_k) = \dot{\Sigma}(h_k) + \mathcal{Z}_{22}(h_k) + \mathcal{Z}_{32}(h_k) \). It is noted that \( \dot{\Sigma}(h_k) \) is affine with the delay \( h_k \). So, \( \Omega(h_k) < 0 \) for any \( h_k \neq [h_1, h_2] \) is naturally ensured by the endpoint constraints \( \Omega(h_1) < 0 \) and \( \Omega(h_2) < 0 \). From Schur complement, Inequalities (14) and (15), respectively, respond to \( \Omega(h_1) < 0 \) and \( \Omega(h_2) < 0 \). Hence, there is a sufficiently small scalar \( \epsilon > 0 \) such that \( \Delta V(k) \leq -\epsilon \| x(k) \| \) for any \( x(k) \neq 0 \), which ensures the asymptotical stability of system (1). This completes the proof.

**Remark 3:** Compared to Theorem 2 proposed in [40], there exist three points in Theorem 1 that may lead to the reduction of conservatism. One is to include the LKF term \( \sum_{i=1}^{n-1} x_i^2(j) R x(j) \) in the LKF candidate. The second is to utilize Inequality (5) to estimate summation terms in (22), instead of Inequality (8), which could take more relations among the double-summation LKF terms into account. The third is to construct the augmented vector \( \eta_j(k) = \text{col}(x(k), x(i)) \), instead of the state \( x(i) \), which could make the coupling information between the state \( x(k) \) and its single-summation terms considered.

**IV. NUMERICAL EXAMPLES**

**Example 1:** Consider system (1) with
\[ A = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix}. \]

To compare the conservatism of stability conditions, MAUBs for different \( h_1 \) are obtained by Theorem 1 proposed in this paper and other conditions reported recently in the literature and listed in Table 1. From Table 1, it is seen that Theorem 1 produces more relaxed results than conditions proposed in [17], [26]–[28], [30], [34] and [38] and the same results as those proposed in [40]. This means that Theorem 1 is not more conservative than other conditions, including one proposed in [40].

**Example 2:** Consider system (1) with
\[ A = \begin{bmatrix} 0.648 & 0.04 \\ 0.12 & 0.654 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1512 & -0.0518 \\ 0.0259 & -0.1091 \end{bmatrix}. \]

For Example 2, MAUBs are obtained by Theorem 1 proposed in this paper and other conditions proposed in [26]–[28], [30], [34], [38] and [40]. It is seen from Table 2 that as the value of \( h_1 \) takes values of 5, 7, 11, 13 and 20, MAUBs obtained by Theorem 1 are 23, 25, 31 and 38 that are all larger than other conditions, especially including that proposed in [40]. This clearly shows that Theorem 1 is more relaxed than all of other conditions, although the number of decision variables is larger than those involved in other conditions. Generally speaking, less conservatism is usually achieved at the cost of more computational complexity. The proposed condition is no exception.

**V. CONCLUSION**

This article has investigated the stability problem for linear discrete-time systems with a time-varying delay. Some relations between two general free-matrix-based summation inequalities have been discussed and meanwhile, their compact versions have been presented. By modifying the single- and double-summation LKF terms, a novel Lyapunov–Krasovskii functional has been constructed. By employing newly-developed summation inequalities, a new stability condition has been obtained, which is of less conservatism compared to some existing results.

The proposed method can be used to address the stability problem for other discrete-time delayed systems such as discrete-time Takagi–Sugeno fuzzy systems with time-varying delay [41], [42]. Especially, the idea of introducing free matrices to improve freedoms may be extended to other control fields like delayed conic-type nonlinear systems [14]–[16].

**REFERENCES**

[1] K. Gu, V. L. Kharitonov, and J. Chen, Stability of Time-Delay Systems. Boston, MA, USA: Birkhäuser 2003.
[2] J. H. Park, T. H. Lee, Y. Liu, and J. Chen, Dynamic Systems With Time Delays: Stability and Control. Singapore, Springer-Nature, 2019.
[3] X.-M. Zhang, Q.-L. Han, A. Seuret, F. Gouaisbaut, and Y. He, “Overview of recent advances in stability of linear systems with time-varying delays,” *IET Control Theory Appl.*, vol. 13, no. 1, pp. 1–16, Jan. 2019.
[4] X. Jin, L. Wang, D. Yu, Y. Geng, and R. Xu, “Pulse train controlled single-input dual-output buck converter with coupled inductors,” *IEEE Access*, vol. 6, pp. 41504–41517, Jul. 2018.
[5] T. H. Lee, “Geometry-based conditions for a quadratic function: Application to stability of time-varying delay systems,” *IEEE Access*, vol. 8, pp. 92462–92468, May 2020.

**TABLE 1.** The MAUBs \( h_2 \) for different \( h_1 \) in Example 1 (\( x \) means infeasibility).

| \( h_1 \) | 2 | 4 | 6 | 9 | 11 | NVs |
|---|---|---|---|---|---|---|
| Thm. 5 [26] | 20 | 21 | 21 | 22 | 23 | 10.5n^2 + 3.5n |
| Thm. 1 [27] | 20 | 21 | 21 | 22 | 23 | 29.5n^2 + 12.5n |
| Thm. 1 [38] | 21 | 22 | 22 | 23 | 23 | 78.5n^2 + 12.5n |
| Thm. 1 [34] | 21 | 22 | 22 | 23 | 24 | 10.5n^2 + 3.5n |
| Thm. 2 [30] | 21 | 21 | 21 | 22 | 23 | 24n^2 + 5n |
| Thm. 1 [28] | 20 | 21 | 21 | 22 | 23 | 32.5n^2 + 6.5n |
| Thm. 1 [40] | 22 | 22 | 22 | 23 | 24 | 97n^2 + 4n |

**TABLE 2.** The MAUBs \( h_2 \) for different \( h_1 \) in Example 2.

| \( h_1 \) | 5 | 7 | 11 | 13 | 20 | NVs |
|---|---|---|---|---|---|---|
| Thm. 5 [26] | 20 | 22 | 25 | 27 | 34 | 10.5n^2 + 3.5n |
| Thm. 1 [27] | 20 | 22 | 26 | 28 | 34 | 29.5n^2 + 12.5n |
| Thm. 1 [38] | 21 | 22 | 26 | 27 | 34 | 78.5n^2 + 12.5n |
| Thm. 1 [34] | 21 | 22 | 26 | 28 | 35 | 10.5n^2 + 3.5n |
| Thm. 2 [30] | 20 | 22 | 26 | 27 | 34 | 24n^2 + 5n |
| Thm. 1 [28] | 20 | 22 | 26 | 28 | 34 | 32.5n^2 + 6.5n |
| Thm. 2 [40] | 22 | 24 | 28 | 30 | 37 | 97n^2 + 4n |
| Thm. 1 | 23 | 23 | 29 | 31 | 38 | 160.5n^2 + 5.5n |
[6] T. H. Lee, J. H. Park, and S. Xu, “Relaxed conditions for stability of time-varying delay systems,” *Automatica*, vol. 75, pp. 11–15, Jan. 2017.

[7] C.-K. Zhang, F. Long, Y. He, W. Yao, L. Jiang, and M. Wu, “A relaxed quadratic function negative-determination lemma and its application to time-delay systems,” *Automatica*, vol. 113, Mar. 2020, Art. no. 108764.

[8] H. Shen, Y. Zhu, L. Zhang, and J. H. Park, “Extended dissipative state estimation for Markov jump neural networks with unreliable links,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 28, no. 2, pp. 346–358, Feb. 2017.

[9] C. Chen, D. Zou, and C. Li, “Improved Jaya algorithm for economic dispatch considering valve-point effect and multi-fuel options,” *IEEE Access*, vol. 8, pp. 84981–84995, May 2020.

[10] W. Duan, Y. Li, and J. Chen, “Further stability analysis for time-delayed neural networks based on an augmented Lyapunov functional,” *IEEE Access*, vol. 7, pp. 104655–104666, Jul. 2019.

[11] C. Gong, G. Zhu, and L. Wu, “New weighted integral inequalities and its application to exponential stability analysis of time-delay systems,” *IEEE Access*, vol. 4, pp. 6231–6237, Sep. 2016.

[12] D. Liao, S. Zhong, J. Cheng, L. Luo, and Q. Zhong, “New stability criteria of discrete systems with time-varying delays,” *IEEE Access*, vol. 7, pp. 1677–1684, Nov. 2019.

[13] J. Chen, J. H. Park, and S. Xu, “Stability analysis of systems with time-varying delay: A quadratic-partitioning method,” *IET Control Theory Appl.*, vol. 13, no. 18, pp. 3184–3189, Dec. 2019.

[14] S. He, W. Lyu, and F. Liu, “Robust $H_\infty$ sliding mode controller design of a class of time-delayed discrete conic-type nonlinear systems,” *IEEE Trans. Syst., Man, Cybern., Syst.*, early access, Dec. 24, 2019, doi: 10.1109/TSMC.2018.2884491.

[15] S. He, Q. Ai, C. Ren, J. Dong, and F. Liu, “Finite-time resilient controller design of a class of uncertain nonlinear systems with time-delays under asynchronous switching,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 49, no. 2, pp. 281–286, Feb. 2019.

[16] S. He, J. Song, and F. Liu, “Robust finite-time bounded controller design of time-delay conic nonlinear systems using sliding mode control strategy,” *IEEE Trans. Syst., Man, Cybern., Syst.*, vol. 48, no. 11, pp. 1863–1873, Nov. 2018.

[17] O. M. Kwon, M. J. Park, J. H. Park, S. M. Lee, and E. J. Cha, “Stability and stabilization for discrete-time systems with time-varying delays via augmented Lyapunov-Krasovskii functional,” *J. Franklin Inst.*, vol. 350, no. 3, pp. 521–540, Apr. 2013.

[18] Z. Feng, J. Lam, and G.-H. Yang, “Optimal partitioning method for stability analysis of continuous/discrete delay systems,” *Int. J. Robust Nonlinear Control*, vol. 25, no. 4, pp. 559–574, Jan. 2015.

[19] J. Chen, J. H. Park, and S. Xu, “Stability analysis for neural networks with time-varying delay via improved techniques,” *IEEE Trans. Cybern.*, vol. 49, no. 12, pp. 4495–4500, Dec. 2019.

[20] J. Chen, J. H. Park, and S. Xu, “Stability analysis for delayed neural networks with an improved general free-matrix-based integral inequality,” *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 31, no. 2, pp. 675–684, Feb. 2020.

[21] J. Chen and X. Chen, “An insight into stability conditions of discrete-time systems with time-varying delay,” *IEEE Access*, vol. 7, pp. 155818–155824, Oct. 2019.

[22] X.-M. Zhang, Q.-L. Han, A. Seuret, and F. Gouaisbaut, “An improved reciprocally convex inequality and an augmented Lyapunov–Krasovskii functional for stability of linear systems with time-varying delay,” *Automatica*, vol. 84, pp. 221–226, Oct. 2017.

[23] J.-H. Kim, “Further improvement of Jensen inequality and application to stability of time-delayed systems,” *Automatica*, vol. 64, pp. 121–125, Feb. 2016.

[24] M. J. Park, S. H. Lee, O. M. Kwon, and J. H. Ryu, “Enhanced stability criteria of neural networks with time-varying delays via a generalized free-weighting matrix integral inequality,” *J. Franklin Inst.*, vol. 355, no. 14, pp. 6531–6548, Sep. 2018.

[25] S. Xu and J. Lam, “On equivalence and efficiency of certain stability criteria for time-delay systems,” *IEEE Trans. Autom. Control*, vol. 52, no. 1, pp. 95–101, Jan. 2007.

[26] A. Seuret, F. Gouaisbaut, and E. Fridman, “Stability of discrete-time systems with time-varying delays via a novel summation inequality,” *IEEE Trans. Autom. Control*, vol. 60, no. 10, pp. 2740–2745, Oct. 2015.

[27] P. T. Nam, P. N. Pathirana, and H. Trinh, “Discrete Wirtinger-based inequality and its application,” *J. Franklin Inst.*, vol. 352, no. 5, pp. 1893–1905, May 2015.

[28] J. Chen, S. Xu, Q. Ma, Y. Li, Y. Chu, and Z. Zhang, “Two novel general summation inequalities to discrete-time systems with time-varying delay,” *J. Franklin Inst.*, vol. 354, no. 13, pp. 5537–5558, Sep. 2017.

[29] X.-M. Zhang and Q.-L. Han, “Abel lemma-based finite-sum inequality and its application to stability analysis for linear discrete time-delay systems,” *Automatica*, vol. 57, pp. 199–202, Jul. 2015.

[30] X.-G. Liu, F.-X. Wang, and M.-L. Tang, “Auxiliary function-based summation inequalities and their applications to discrete-time systems,” *Automatica*, vol. 78, pp. 211–215, Apr. 2017.

[31] S. Y. Lee, J. Park, and P. Park, “Bessel summation inequalities for stability analysis of discrete-time systems with time-varying delays,” *Int. J. Robust Nonlinear Control*, vol. 29, no. 2, pp. 473–491, Jan. 2019.

[32] P. Park, S. Y. Lee, and W. I. Lee, “New stability analysis for discrete time-delay systems via auxiliary-function-based summation inequalities,” *J. Franklin Inst.*, vol. 353, no. 18, pp. 5068–5080, Dec. 2016.

[33] E. Gyurkovics, K. Kiss, I. Nagy, and T. Takács, “Multiple summation inequalities and their application to stability analysis of discrete-time delay systems,” *J. Franklin Inst.*, vol. 354, no. 1, pp. 123–144, Jan. 2017.

[34] C.-K. Zhang, Y. He, L. Jiang, and M. Wu, “An improved summation inequality to discrete-time systems with time-varying delay,” *Automatica*, vol. 74, pp. 10–15, Dec. 2016.

[35] C.-K. Zhang, Y. He, L. Jiang, M. Wu, and Q.-G. Wang, “An extended reciprocally convex matrix inequality for stability analysis of systems with time-varying delay,” *Automatica*, vol. 85, pp. 481–485, Nov. 2017.

[36] P. Park, J. W. Ko, and C. Jeong, “Reciprocally convex approach to stability of systems with time-varying delays,” *Automatica*, vol. 47, no. 1, pp. 235–238, Jan. 2011.

[37] J. Chen, J. H. Park, and S. Xu, “Stability analysis of discrete-time neural networks with an interval-like time-varying delay,” *Neurocomputing*, vol. 329, pp. 248–254, Feb. 2019.

[38] J. Chen, S. Xu, and J. Lu, “Summation inequality and its application to stability analysis for time-delayed systems,” *IET Control Theory Appl.*, vol. 10, no. 4, pp. 391–395, Feb. 2016.

[39] J. Chen, S. Xu, X. Jia, and B. Zhang, “Novel summation inequalities and their applications to stability analysis for systems with time-varying delay,” *IEEE Trans. Autom. Control*, vol. 62, no. 5, pp. 2470–2475, May 2017.

[40] S. B. Qiu, X. G. Liu, F. X. Wang, and Q. Chen, “Stability and passivity analysis of discrete-time linear systems with time-varying delay,” *Syst. Control Lett.*, vol. 357, pp. 6951–6967, Jul. 2020.

[41] X. Su, P. Shi, L. Wu, and Y. Song, “A novel control design on discrete-time Takagi-Sugeno fuzzy systems with time-varying delays,” *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 4, pp. 655–671, Aug. 2013.

[42] X. Su, P. Shi, L. Wu, and M. V. Basin, “Reliable filtering with strict dissipativity for T-S fuzzy time-delay systems,” *IEEE Trans. Cybern.*, vol. 44, no. 12, pp. 2470–2483, Dec. 2014.