MATCHED HIGHER ORDER LAGRANGIAN DYNAMICS

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Abstract. We write both the second order and the iterated tangent bundles of a Lie group as cocycle Lie group extensions. For two Lie groups under mutual interactions, matched pair group structures of these higher order frameworks are investigated. On these geometries, by employing symmetries, complete catalogues of Lagrangian dynamics are presented in a hierarchical order for both continuous and discrete dynamics.

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1. Introduction

A symmetry of a dynamical equation is defined by invariance of the system under a (Lie) group action [21,36,43,63]. Reduction of Lagrangian systems under symmetries is one of the main interest of the geometric mechanics [11]. Under a group of symmetries Euler-Lagrange equations on the tangent bundle of configuration space reduces to the Lagrange-Poincaré equations defined over the space of orbits [57]. If, particularly, configuration space of the system is also a Lie group, say $G$, then the Lagrange dynamics on tangent bundle $TG$ reduces to the Euler-Poincaré theory on the Lie algebra $\mathfrak{g}$ under the symmetry [37,54]. Considering a Lagrangian function(al) $\mathcal{L}$ on a Lie algebra $\mathfrak{g}$, the Euler-Poincaré equations are computed to be

$$\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} = - \text{ad}^* \xi \frac{\delta \mathcal{L}}{\delta \xi}$$

where $\text{ad}^* \xi$ is the coadjoint representation of the Lie algebra on its dual $\mathfrak{g}^*$ whereas $\delta \mathcal{L}/\delta \xi$ stands for the Fréchet derivative. Configuration spaces of many physical systems such as rigid bodies, fluid and plasma theories are Lie groups so fits well this Euler-Poincaré framework [1,3,50,72].

Our interest in this work is on coupling (matching) of Lagrangian systems. For this, consider two Lagrangian systems under mutual interactions. Note that, these systems cannot preserve their individual motions in their collective motion. This gives that, to write the equation of motion of the coupled system demands more effort than merely putting together the equations of motions of the individual systems. Such systems have been first studied within the semi-direct product theory [11,51,56], where only one of the systems is allowed to act on the other. Many physical systems fit into this geometry; such as the heavy top [66], and the Maxwell-Vlasov equations [37]. In a recent paper [27], we have addressed the coupling problem under the presence of mutual interactions. The geometrical framework for this generalization has been considered as matched pair Lie groups and Lie algebras. Let us summarize this a bit more detail.

Matched pairs. Consider two Lie groups, say $G$ and $H$. Assume that $G$ acts on $H$ from the right whereas $H$ acts on $G$ from the left. If some compatibility conditions (arising from the associativity property) are
satisfied then their Cartesian product turns out to be a Lie group by itself \cite{46,70}. Here, we denote the product group by \( G \rtimes H \) and called as matched pair of Lie groups. Here, the constituitive Lie groups \( G \) and \( H \) become Lie subgroups of \( G \rtimes H \) under canonical inclusions. Matched pair Lie groups are available in the literature in different names, it is referred as a bicrossedproduct group in \cite{47,50}, the twilled extension in \cite{40}, the double Lie group in \cite{44}, or the Zappa-Szép product in \cite{8}; see also \cite{67,68,69,76}. Conversely, from the decomposition point of view, if a Lie group is isomorphic (as a topological set) to the Cartesian product of two of its subgroups (with trivial intersection), then it is a matched pair Lie group. It is possible to define matched pair of Lie algebras in a similar fashion. For this consider two Lie algebras \( \mathfrak{g} \) and \( \mathfrak{h} \). And assume that \( \mathfrak{g} \) acts on \( \mathfrak{h} \) from the right and \( \mathfrak{h} \) acts on \( \mathfrak{g} \) from the left. Then the Cartesian product of these Lie algebras can be endowed with a Lie algebra structure if some compatibility conditions are guaranteed. In this case, we denote the product Lie algebra by \( \mathfrak{g} \rtimes \mathfrak{h} \) and call it as the matched pair of Lie algebras. A Lie algebra can be decomposed into a matched pair product if it can be stated as a direct sum of its two Lie subalgebras with trivial intersection. Further, it is possible to see that Lie algebra of a matched Lie group is a matched pair Lie algebra.

**Matched Euler-Poincaré equations.** The tangent bundle of a Lie group is a Lie group by itself and called as the tangent group. We have shown in our previous publication \cite{27} that the tangent bundle \( T(G \rtimes H) \) of a matched pair Lie group is a matched pair of Lie groups \( TG \) and \( TH \), that is \( T(G \rtimes H) \cong TG \rtimes TH \). Here, the actions on the tangent bundles are simply tangent lifts of the actions defined on the group level. This framework permitted us to match two Lagrangian dynamics defined on the tangent bundles \( TG \) and \( TH \). Therefore one arrives as a system of Lagrangian equations covering both the Lagrangian dynamics on each tangent bundle \( TG \) and \( TH \) along with some additional terms as a manifestation of the mutual actions. In \cite{27}, the Euler-Poincaré equations have also been studied. In this case, by applying the procedure of Lagrangian reduction to the Lagrangian dynamics on \( T(G \rtimes H) \), the matched Euler-Poincaré equations on the matched pair Lie algebra \( \mathfrak{g} \rtimes \mathfrak{h} \) are derived. To see this system, consider a Lagrangian function(al) \( \mathcal{L} = \mathcal{L}(\xi, \eta) \) depending on \( \xi \) in \( \mathfrak{g} \), and \( \eta \) in \( \mathfrak{h} \), then the matched Euler-Poincaré equations are computed to be

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} = -\text{ad}^*_\xi \frac{\delta \mathcal{L}}{\delta \xi} + \frac{\delta \mathcal{L}}{\delta \xi} \rtimes \mathfrak{h} \mathcal{L} + \frac{\delta \mathcal{L}}{\delta \eta} \rtimes \mathfrak{h} \mathcal{L},
\]

(1.2)

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \eta} = -\text{ad}^*_\eta \frac{\delta \mathcal{L}}{\delta \eta} - \xi \rtimes \mathfrak{g} \mathcal{L} - \frac{\delta \mathcal{L}}{\delta \xi} \rtimes \mathfrak{g} \mathcal{L}.
\]

Even though, we shall define all the terms, and discuss on all the details of this system in the main body of the paper, let us comment on some basic properties. Here, the first terms on the right hand sides of the equations (1.2) are the individual Euler-Poincaré motions on the Lie algebras \( \mathfrak{g} \) and \( \mathfrak{h} \), respectively. To see this, compare those terms with Euler-Poincaré equation (1.1). In (1.2), the second term on the right hand side of the first equation and the third term on the right hand side of the second equation are obtained by dualizing the left action of the Lie algebra \( \mathfrak{h} \) on \( \mathfrak{g} \). So that, if this action is trivial, that is, we have a semi-direct product Lie algebra \( \mathfrak{g} \asymp \mathfrak{h} \), then the Euler-Poincaré dynamics is the one in (1.2) without these terms. On the other hand, the third term on the right hand side of the first line, and the second equation on the right hand side of the second line are manifestation of the action of \( \mathfrak{g} \) on \( \mathfrak{h} \). If, in this case, this action is trivial, that is those terms are identically zero, then we arrive at the Euler-Poincaré equations on the semi-direct product Lie algebra \( \mathfrak{g} \asymp \mathfrak{h} \). In this regard, the matched pair Euler-Poincaré equations (1.2)
involves both of these semi-direct theories so that permits mutual interactions. Besides, we cite publications \[9, 37\] for Lagrangian dynamics on the semi-direct product spaces and their reductions.

Conversely, by introducing matched Euler-Poincaré equations, it has been achieved to represent the equations of motion of a system (if it admits a matched pair decomposition) in terms of the equations of motion of two simpler systems, decorated by the additional terms reflecting the mutual interactions of the subsystems. See also, another work \[27\] where we have studied matched pairs of Hamiltonian and Lie-Poisson dynamics. The matched pair theory has already found applications in plasma theory \[26\] and in thermodynamical processes \[23, 65, 71\]. Even though matched pair dynamics on the matched pair of Lie algebras is an answer to coupling problem of Lagrangian systems, we cannot claim that the problem has been fully solved since there are different Lagrangian theories other then the Euler-Poincaré dynamics. In this work, we particularly interested in the coupling problem of the second order Euler-Poincaré dynamics and discrete Lagrangian dynamics defined on the second order Lie groups. In order to be more explicit about the goals and the novelties of this present work, we now comment one by one on the geometries that we are interested in.

The second order Euler-Poincaré equations. Lagrange equations are second order differential equations. So, in order to recast a higher order differential equation in the Lagrangian framework, one needs to study the system over the higher order tangent bundles of the configuration space. In this case, Lagrangian function should depend on acceleration (and maybe terms of higher order) more than position and velocity \[20\]. This higher order Lagrangian formalism and its Hamiltonian realization is a very old subject goes back to 1850’s \[64\]. More recently, reduction of the higher order Lagrangian systems under symmetries are investigated in \[19\]. In this case, the higher order Euler-Lagrange equation is reduced to the higher order Lagrange-Poincaré equation. If, particularly, configuration space is a Lie group then after the reduction one arrives at the higher order Euler-Poincaré equation \[13, 32\]. For the second order theory, the Lagrangian dynamics is defined on the second order tangent bundle \(T^2 G\). As \(TG\), the second order tangent bundle \(T^2 G\) is also a Lie group. But it is not in the form of a semi-direct product group. It is shown in \[73\] that it can be understood as a cocycle Lie group extension of \(TG\) by a copy of the Lie algebra \(g\). Nevertheless, after the reduction, one arrives at the second order Euler-Poincaré equation, on the product space \(g \times h\), given by

\[
\left( \frac{d}{dt} + ad_{\xi}^* \right) \left( \frac{\delta L}{\delta \xi} - \frac{d}{dt} \left( \frac{\delta L}{\delta \dot{\xi}} \right) \right) = 0.
\]

Notice that, if the Lagrangian function \(L\) does not depend on \(\dot{\xi}\), then the second order Euler-Poincaré equation \(1.3\) turns out to be the first order Euler-Poincaré equation \(1.1\).

Goal I: The second order tangent groups and coupling problem. The first novelty in this paper is to show that the iterated tangent bundle \(TTG := T(TG)\) can be realized as a cocycle Lie group extension of \(T^2 G\) in two different ways (c.f. Subsection 3.3). We address this geometrization while performing the reduction procedures. In the literature, coupling problem of the second order tangent bundles are missing either. Accordingly, we shall show a matched pair decomposition of the second order tangent bundle \(T^2(G \ltimes H)\) as the matched pair of the groups \(T^2 G\) and \(T^2 H\) by proving the Lie group isomorphism \(T^2(G \ltimes H) \cong T^2 G \ltimes T^2 H\) (c.f. Subsection 3.7). Using this identification, we shall also present how cocycle extensions defining \(T^2 G\) and \(T^2 H\) can be matched in order to geometrize \(T^2(G \ltimes H)\) as a cocycle extension of \(T(G \ltimes H)\) with the matched pair Lie algebra \(g \ltimes h\) (c.f. Subsection 3.8). In the realm of
this geometry, we shall present some results on the Lagrangian dynamics as well: It will be exhibited how two second order Euler-Lagrange equations can be matched in the framework of $T^2(G \Rightarrow H)$. Lagrangian reduction will be applied to this dynamics in order to arrive at, what we call, matched second order Euler-Poincaré equations (c.f. Subsection 3.5). This coupling (matching) involves the second order semi-direct product Euler-Poincaré dynamics for the both sides of action as in the one for the (first order) matched pair Euler-Poincaré equations in (1.2).

**Discrete Lagrangian dynamics on Lie groups.** It is possible to study difference equations in the framework of Lagrangian dynamics. The most general geometric realization of discrete systems is given in the Lie groupoid geometry [49, 74]. In this setting, the dynamics is generated by a Lagrangian function defined on a Lie groupoid which is assumed to be the configuration space. Lie groups are particular instances of Lie groupoids, so that discrete dynamics on Lie groups are also available in the literature [41, 52, 53]. In this case, one may study the reduction under symmetry and arrive at reduced discrete systems. There are higher order studies on the discrete systems [4] as well as their reductions [16]. Let us recall basics of discrete dynamics on Lie groups [49, Subsec. 4.1]. A discrete Lagrangian function is defined as a real valued function $\mathcal{L}$ on a Lie group $G$. Then, by applying a discrete version of Hamilton’s principle [74], one arrives at the (2 step) discrete Euler-Lagrange equations (sometimes called as discrete Lie-Poisson equations)

$$
\mu_{k+1} = \text{Ad}^*_{g_k} \mu_k, \quad \mu_k = T^*_G R_{g_k} d\mathcal{L}(g_k),
$$

where $\text{Ad}^*_{g_k}$ is the coadjoint action of $G$ on the dual space $g^*$.

**Matched discrete Lagrangian systems.** As Lie groups and Lie algebras, one can match Lie groupoids and Lie algebroids as well [45, 52]. As a result, the very same algebraic strategy of “matched pairs” may be used once more; this time to study the discrete dynamical systems. We have investigated the matching problem of two discrete Lagrangian systems [29] in the framework of matched pairs of Lie groupoids and Lie algebroids. Being particular instances, one can study matched pair of discrete Lagrangian dynamics on the Lie group level as well. In this case, a Lagrangian function $\mathcal{L} = \mathcal{L}(g, h)$ defined on a matched pair Lie group $G \Rightarrow H$ generates matched discrete Lagrangian equations

$$
\mu_{k+1} = \text{Ad}^*_{g_k} \mu_k \circ h_k + a^*_h T^*_G R_{g_k} \mathcal{L}(g_k, h_k),
$$

where the dual elements are defined by means of the Fréchet derivatives $d_1 \mathcal{L}$ and $d_2 \mathcal{L}$ with respect to the first and second entries that is

$$
T^*_G R_{g_k} \cdot d_1 \mathcal{L}(g_k, h_k) = \mu_k \in g^*, \quad T^*_G R_{h_k} \cdot d_2 \mathcal{L}(g_k, h_k) = \nu_k \in h^*.
$$

We shall define all the terms and action in the system (1.5) detailly in the main body of the paper. But a direct observation following the labels under the terms, one can easily observe the terms induced by the mutual actions. It is needless to remark that by taking one of the actions trivial, one arrives at the semi-direct product theory as in the continuous case.
Goal II: Discrete dynamics on the (second order) tangent groups and coupling problem. As an application of the geometry presented in present paper, or as an addendum to the main body of the paper, we shall compute the discrete Lagrangian dynamics on the tangent group $TG$ (c.f. Subsection 5.2) and the second order tangent group $T^2G$ (c.f. Subsection 5.3). Further, we shall study on coupling problem of these dynamics. We shall first match the discrete dynamics on two tangent groups (c.f. Subsection 5.5) that is $TG \triangleright⊳ TH$, then we shall compute discrete dynamics on $T^2G \triangleright⊳ T^2H$ (c.f. Subsection 5.6).

Another way to categorize the novelties in this present paper can be done in terms of the geometric-algebraic results and the dynamical results. The former results are on the geometry of $T^2G \triangleright⊳ T^2H$, and the latter is matching of discrete and continuous higher order Lagrangian dynamics. We wish to note that one may consider this work as a continuation (or a part) of "matched pairs of Lagrangian and Hamiltonian dynamics" project [27, 28, 29].

Content of this work. The organization of the paper is as follows. In the following Section 2, we shall present the basics on matched pair of Lie groups and Lie algebras, and Lie group extensions via cocyle. In Section 3, the matched pair decompositions of tangent and iterated tangent groups. In this section, matched pairs of cocyle extensions will be also studied. In Section 4, we shall apply the geometric results to the continuous Lagrangian dynamics whereas, in Section 5, we shall focus on the discrete dynamics on the matched pairs.

Notations and Conventions. $G$ is a Lie group and, $\mathfrak{g} = \text{Lie}(G) \cong T_eG$ is its Lie algebra. The dual of $\mathfrak{g}$ is $\mathfrak{g}^* = \text{Lie}^*(G)$. Throughout the work, we shall adapt the letters

\begin{equation}
(1.7)
\begin{aligned}
g, \tilde{g}, \hat{g} \in G, & \quad \xi, \tilde{\xi}, \hat{\xi} \in \mathfrak{g}, & \quad \mu, \tilde{\mu}, \hat{\mu} \in \mathfrak{g}^* \\
\tilde{h}, \hat{h}, \tilde{\tilde{h}} \in H, & \quad \eta, \tilde{\eta}, \hat{\eta} \in \mathfrak{g}, & \quad \nu, \tilde{\nu}, \hat{\nu} \in \mathfrak{h}^*
\end{aligned}
\end{equation}

as elements of the spaces shown. We shall denote left and right multiplications on $G$ by $L_g$ and $R_g$, respectively. The left inner automorphism

\begin{equation}
(1.8) \quad I_g = R_{g^{-1}} \circ L_g
\end{equation}

is a left representation of $G$ on $G$ satisfying $I_g \circ I_h = I_{gh}$. The left adjoint action $Ad_g = T_eI_g$ of $G$ on $\mathfrak{g}$ is defined as the tangent map of $I_g$ at the identity $e \in G$. The infinitesimal left adjoint representation $ad_{\xi} \tilde{\xi}$ is \([\xi, \tilde{\xi}]\) and is defined as derivative of $Ad_g$ over the identity. A left invariant vector field $X_G^G$ generated by $\xi \in \mathfrak{g}$ is

\begin{equation}
(1.9) \quad X_G^G(g) = T_eL_g \xi.
\end{equation}

The identity

\begin{equation}
(1.10) \quad [\xi, \tilde{\xi}] = \left[X_G^G, X_G^G\right]_{JL}
\end{equation}

defines the isomorphism between $\mathfrak{g}$ and the space $\mathfrak{x}^L(G)$ of left invariant vector fields endowed with the Jacobi-Lie bracket. The coadjoint action $Ad^*_g$ of $G$ on the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ is a left representation and is the linear algebraic dual of $Ad_g^{-1}$, namely,

\begin{equation}
(1.11) \quad \left< Ad^*_g, \mu, \xi \right> = \left< \mu, Ad_g \xi \right>
\end{equation}
holds for all $\xi \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. We denote the infinitesimal coadjoint action of $\mathfrak{g}$ on $\mathfrak{g}^*$ by $ad_{\xi}^*$, rather than the linear algebraic dual of infinitesimal adjoint action $ad_{\xi}$. With this choice $ad_{\xi}^*$ turns out to be minus of the linear algebraic dual of $ad_{\xi}$, that is

$$\langle ad_{\xi}^* \mu, \tilde{\xi} \rangle = -\langle \mu, ad_{\xi} \tilde{\xi} \rangle$$

for all $\xi, \tilde{\xi}$ in $\mathfrak{g}$ and $\mu \in \mathfrak{g}^*$. 
2. Matched pair Lie groups, Lie algebras, and extensions

In this subsection we recall the basics on the matched pairs of Lie groups, Lie algebras, and Lie coalgebras. There are extensive literature on these subjects. Some of them are [44, 46, 47, 48, 60, 70, 77] for further details on these subjects. We also refer [27, 28, 29].

2.1. Matched pair Lie groups.

A matched pair Lie group, denoted by \( G \bowtie \bowtie H \), is a Lie group by itself containing two non-intersecting Lie subgroups \( G \) and \( H \) under mutual interactions. Accordingly, we assume that \( H \) acts on \( G \) from the left, and \( G \) acts on \( H \) from the right by

\[
\triangleright: H \times G \rightarrow G, \quad (h, g) \mapsto h \triangleright g, \quad \triangleleft: H \times G \rightarrow H, \quad (h, g) \mapsto h \triangleleft g.
\]

In this case, the multiplication on a matched pair Lie group \( G \bowtie \bowtie H \) is

\[
(g, h) (\tilde{g}, \tilde{h}) = (g (h \triangleright \tilde{g}), (h \triangleleft \tilde{g}) \tilde{h}).
\]

In order to guarantee the associativity of the group operation on \( G \bowtie \bowtie H \), the actions are satisfying the compatibility conditions

\[
h \triangleright (g \tilde{g}) = (h \triangleright g)((h \triangleleft g) \triangleright \tilde{g}), \quad (h \tilde{h}) \triangleleft g = (h \triangleleft (\tilde{h} \triangleright g))(\tilde{h} \triangleleft g).
\]

In case one of the actions in (2.1) is trivial, the matched pair group \( G \bowtie H \) reduces to a semi-direct product group. Particularly, if the action of \( G \) on \( H \) is trivial, then we have semidirect product group \( H \triangleright G \), on the other hand, if the action of \( H \) on \( G \) is trivial, then we arrive at semidirect product group \( H \bowtie G \). The matched pair construction is universal in the following sense. For the proof of the proposition and further details, we cite [48, Prop. 6.2.15].

**Proposition 2.1.** Let \( S \) be a Lie group with subgroups \( G \) and \( H \), such that the group multiplication induces the isomorphism \( S \cong G \times H \) as sets. Then \( S \) is isomorphic to the matched pair Lie group \( G \bowtie H \). In this case, the mutual actions are derived from the relation

\[
hg = (h \triangleright g)(h \triangleleft g),
\]

for any \( g \in G \), and any \( h \in H \). Here, the inclusions of the subgroups \( G \) and \( H \) into \( S \cong G \times H \) are defined to be

\[
G \rightarrow S : g \rightarrow (g, e), \quad H \rightarrow S : h \rightarrow (e, h).
\]

It is possible to define a matched pair Lie group starting with two Lie groups \( G \) and \( H \) under mutual interactions (2.1) satisfying the compatibility conditions (2.3). Then the product group \( G \times H \) is a Lie group with the group multiplication given in (2.2) and it is called as the matched pair of Lie groups \( G \) and \( H \). We use the same notation \( G \bowtie H \) to denote this product group.

Tangent lifts of the mutual group actions on the group level result with infinitesimal actions of the Lie algebras. Let us exhibit these actions for future reference. Consider first the right \( G \) action on \( H \), the infinitesimal right action of the Lie algebra \( g \) of the group \( G \) to \( H \) results with a tangent bundle element \( TH \).
This is defined as
\[ (2.6) \quad H \times \mathfrak{g} \rightarrow TH, \quad (h, \xi) \mapsto h \cdot \xi := \left. \frac{d}{dt} \right|_{t=0} h \cdot x_t, \]
where \(x_t\) is a curve in \(G\) passing through the identity element and tangent to \(\xi\) at that point. Freezing the group element in (2.6) we obtain a representation of the Lie algebra \(\mathfrak{a}\) given by
\[ (2.7) \quad a_h : \mathfrak{g} \mapsto T_h H, \quad a_h(\xi) = h \cdot \xi. \]
Being linear, it is possible to define the dual of this mapping given by
\[ (2.8) \quad a_h^* : T_h^* H \mapsto \mathfrak{g}^*, \quad \langle a_h(\xi), \nu_h \rangle = \langle \xi, a_h^*(\nu_h) \rangle, \]
for any \(\nu_h \in T_h^* H\). Similarly, as for the left action, we define the infinitesimal left action of \(\mathfrak{h}\) of the group \(H\) to \(G\) as follows
\[ (2.9) \quad \mathfrak{h} \times G \rightarrow TG, \quad (\eta, g) \mapsto \eta \triangleright g := \left. \frac{d}{dt} \right|_{t=0} y_t \triangleright g, \]
where \(y_t\) is a curve in \(H\) passing through the identity at \(t = 0\) in the direction of \(\eta\). Freezing the group element in (2.9), we arrive at a linear operator
\[ (2.10) \quad b_g : \mathfrak{h} \mapsto T_g G, \quad b_g(\eta) := \eta \triangleright g. \]
The transpose of this mapping is
\[ (2.11) \quad b_g^* : T_g^* G \mapsto \mathfrak{h}^*, \quad \langle b_g(\eta), \mu_g \rangle = \langle b_g^*(\eta), \mu_g \rangle = \langle \eta, b_g^*(\mu_g) \rangle, \]
for any \(\mu_g \in T_g^* G\).

### 2.2. Matched pair Lie algebras.

A matched pair Lie algebra \(\mathfrak{g} \bowtie \mathfrak{h}\) is a Lie algebra by itself containing two non-intersecting Lie algebras \(\mathfrak{g}\) and \(\mathfrak{h}\) under mutual interactions. We write the mutual actions by
\[ (2.12) \quad \triangleright : \mathfrak{h} \otimes \mathfrak{g} \rightarrow \mathfrak{g}, \quad \eta \otimes \xi \mapsto \eta \triangleright \xi, \quad \triangleleft : \mathfrak{h} \otimes \mathfrak{g} \rightarrow \mathfrak{h}, \quad \eta \otimes \xi \mapsto \eta \triangleleft \xi. \]

In this case, the Lie algebra bracket defined on \(\mathfrak{g} \bowtie \mathfrak{h}\) is given by
\[ (2.13) \quad [\langle \xi, \eta \rangle, \langle \hat{\xi}, \hat{\eta} \rangle] = [\langle \xi, \hat{\xi} \rangle + \eta \triangleright \hat{\xi} - \hat{\eta} \triangleright \xi, \langle \eta, \hat{\eta} \rangle + \eta \triangleleft \hat{\xi} - \hat{\eta} \triangleleft \xi]. \]
The Jacobi identity for the matched pair Lie algebra bracket manifests the following compatibility conditions
\[ (2.14) \quad \eta \triangleright [\xi, \hat{\xi}] = [\eta \triangleright \xi, \hat{\xi}] + [\xi, \eta \triangleright \hat{\xi}] + (\eta \triangleleft \xi) \triangleright \hat{\xi} - (\eta \triangleright \xi) \triangleright \hat{\xi}, \quad [\eta, \hat{\eta}] \triangleleft \xi = [\eta, \hat{\eta} \triangleleft \xi] + [\eta \triangleleft \xi, \hat{\eta}] + \eta \triangleleft (\hat{\eta} \triangleright \xi) - \hat{\eta} \triangleleft (\eta \triangleright \xi). \]

Now, we will record here the following proposition which will be useful for the upcoming sections. We refer the reader to [48, Prop. 8.3.2] for further details.

**Proposition 2.2.** Let \(s\) be a Lie algebra with two Lie subalgebras \(\mathfrak{g}\) and \(\mathfrak{h}\) such that \(s\) is isomorphic to the direct sum of \(\mathfrak{g}\) and \(\mathfrak{h}\) as vector spaces thorough the vector addition in \(s\). Then \(s\) is isomorphic to the matched pair \(\mathfrak{g} \bowtie \mathfrak{h}\) as Lie algebras, and the mutual actions are derived from
\[ (2.15) \quad [\eta, \xi] = (\eta \triangleright \xi, \eta \triangleleft \xi). \]
Here, the inclusions of the subalgebras are defined to be

\begin{equation}
\begin{aligned}
g &\hookrightarrow s : \xi \rightarrow (\xi, 0), \\
h &\hookrightarrow s : \eta \rightarrow (0, \eta).
\end{aligned}
\end{equation}

Let \(g\) and \(h\) be two Lie algebras that are mutual interacting as given in (2.12) while satisfying the conditions (2.14). In this case, the product space \(g \times h\) is a Lie algebra with the Lie bracket (2.13), and it is called as the matched pair of Lie algebras \(g\) and \(h\). We denote this structure by \(g \bowtie h\).

If \(G \bowtie H\) is a matched pair Lie group then the Lie algebras \(g\) and \(h\) of the constitutive groups \(G\) and \(H\) forms a matched pair of Lie algebras that is

\begin{equation}
\text{Lie}(G \bowtie H) = \text{Lie}(G) \bowtie \text{Lie}(H),
\end{equation}

where \(\text{Lie}(S)\) denotes the Lie algebra of the Lie group \(S\). In this case, the Lie algebra actions are derived from the mutual group actions (2.1) whereas the compatibility conditions (2.14) are manifestations of the compatibility conditions (2.3) in the group level.

Recall the Lie algebra actions in (2.12). Start with the left action \(\triangleright\) and freeze an element \(\eta\) in \(h\). This results with a linear mapping, denoted by \(\eta \triangleright\), on \(g\). The linear algebraic dual of this mapping is

\begin{equation}
\langle \mu \triangleleft \eta, \xi \rangle = \langle \mu, \eta \triangleright \xi \rangle.
\end{equation}

This is a right representation of \(h\) on \(g^*\). On the other hand, by freezing \(\xi \in g\), we define a linear mapping \(b_\xi\) from \(h\) to \(g\) defined to be

\begin{equation}
b_\xi : h \mapsto g, \quad b_\xi(\eta) = \eta \triangleright \xi.
\end{equation}

The dual of this mapping is

\begin{equation}
b_\xi^*: g^* \mapsto h^*, \quad \langle b_\xi^* \mu, \eta \rangle = \langle \mu, b_\xi \eta \rangle = \langle \mu, \eta \triangleright \xi \rangle.
\end{equation}

Similarly, recall the right action in \(\triangleleft\) and freeze \(\xi\) in \(g\). This reads a linear mapping, denoted by \(\triangleleft \xi\), on the Lie algebra \(h\). Dual of this mapping is

\begin{equation}
\langle \xi \triangleright v, \eta \rangle = \langle v, \eta \triangleleft \xi \rangle.
\end{equation}

This is a left representation of \(h\) on \(g^*\). Now, we freeze an element, say \(\eta\) in \(h\), in the right action \(\triangleleft\). This enables us to define a linear mapping \(a_\eta\) from \(g\) to \(h\) that is

\begin{equation}
a_\eta : g \mapsto h, \quad a_\eta(\xi) = \eta \triangleleft \xi
\end{equation}

along with the dual mapping

\begin{equation}
a_\eta^*: h^* \mapsto g^*, \quad \langle a_\eta^* v, \eta \rangle = \langle v, a_\eta \xi \rangle = \langle v, \eta \triangleleft \xi \rangle.
\end{equation}

Accordingly, the infinitesimal coadjoint action of \(G \bowtie h\) on its dual space \((g \bowtie h)^*\) is computed to be

\begin{equation}
\text{ad}_{\xi, \eta}^*(\mu, v) = (\text{ad}_{\xi}^* \mu - \mu \triangleleft \eta - a_\eta^* v, \text{ad}_{\eta}^* v + \xi \triangleright v + b_\xi^* \mu),
\end{equation}

for all \((\xi, \eta)\) in \(g \bowtie h\) and for all \((\mu, v)\) in \((g \bowtie h)^* \cong g^* \times h^*\).
2.3. **Lie group extensions via cocycles.**

Given a Lie group $\Gamma$, and a continuous (right) $\Gamma$-module $V$, the group cohomology $H^\ast(\Gamma, V)$ of the group $\Gamma$, with coefficients in $V$, is defined to be the homology of the complex

$$
\bigoplus_{n \geq 0} C^n(\Gamma, V), \quad C^n(\Gamma, V) := F(\Gamma \times^n, V),
$$

where the latter is the space of all continuous maps from $\Gamma \times \ldots \times \Gamma$ to $V$, with the differential map $d : C^n(\Gamma, V) \to C^{n+1}(\Gamma, V)$ see, for instance, [35]. Now, given a Lie group $\Gamma$, a (right) $\Gamma$-module $V$, and a function $\varphi \in C^2(\Gamma, V)$, let us define on $\Gamma \times V$ the multiplication

$$
(g, v) \cdot (\tilde{g}, \tilde{v}) := (g \tilde{g}, \varphi(g, \tilde{g}) + v \ast \tilde{g} + \tilde{v}),
$$

where $(g, v)$ and $(\tilde{g}, \tilde{v})$ in $\Gamma \times V$. Accordingly, it follows at once that (2.25) is associative if and only if

$$
d\varphi(g, \tilde{g}, \tilde{g}) = \varphi(\tilde{g}, \tilde{g}) - \varphi(g \tilde{g}, \tilde{g}) + \varphi(g, \tilde{g} \tilde{g}) - \varphi(g, \tilde{g}) \ast \tilde{g} = 0,
$$

that is, $\varphi \in H^2(\Gamma, V)$; a 2-cocycle in the group cohomology of $\Gamma$, with coefficients in $V$. In this case, the group $\Gamma \times V$ is denoted by $\Gamma \rtimes_\varphi V$. Notice that we arrive at the semi-direct product theory if the cocycle $\varphi$ is trivial. This subcase is denoted by $\Gamma \rtimes V$. 
3. Matched (Second Order and Iterated) Tangent Groups

In this section, we present the geometries of tangent groups, second order tangent groups and iterated tangent groups and further investigate the matched pairs of bundles.

3.1. The tangent group of a Lie group.

The group structure on a Lie group $G$ may be lifted to its tangent bundle $TG$ via the (left) trivialization

$$ tr : TG \to G \ltimes g_1, \quad V_g \mapsto (g, T_g L_{g^{-1}}V_g) = (g, \xi^{(1)}), $$

which is both a diffeomorphism, and a group isomorphism. Here, $g_1$ being the Lie algebra of the group $G$, the structure of $G \ltimes g_1 := G \ltimes g_1$ is that of a semidirect product, considering $g_1$ as an abelian group with respect to the addition. It worths to note that this is a particular case of a matched pair; that the action of the (abelian) group $g_1$ on $G$ is trivial. Accordingly, the group operation on $TG$ (identified with its trivialization $G \ltimes g_1$) may be given by the multiplication

$$ \begin{pmatrix} g, \xi^{(1)} \\ \tilde{g}, \tilde{\xi}^{(1)} \end{pmatrix} = \begin{pmatrix} g \tilde{g}, \tilde{\xi}^{(1)} + Ad_{g^{-1}} \xi^{(1)} \end{pmatrix}, $$

where $\begin{pmatrix} g, \xi^{(1)} \end{pmatrix}$ and $\begin{pmatrix} \tilde{g}, \tilde{\xi}^{(1)} \end{pmatrix}$ in $G \ltimes g_1$. Here, the unit element is $(e, 0)$. For further discussions on tangent groups, we list some references [24, 34, 39, 55, 61, 66]. The Lie algebra of the tangent group $TG$ then has the structure of a semidirect sum (a matched pair of Lie algebras with one of the actions being trivial). This is given by

$$ T_{(e,0)}TG = T_{(e,0)}(G \ltimes g_1) =: g_2 \ltimes g_3. $$

Lie algebra bracket on $g_2 \ltimes g_3$ is computed to be

$$ [(\xi^{(2)}, \xi^{(3)}), (\tilde{\xi}^{(2)}, \tilde{\xi}^{(3)})] = ([\xi^{(2)}, \xi^{(3)}], ad_{\xi^{(2)}} \tilde{\xi}^{(3)} - ad_{\tilde{\xi}^{(3)}} \xi^{(2)}), $$

for any $(\xi^{(2)}, \xi^{(3)})$ and $(\tilde{\xi}^{(2)}, \tilde{\xi}^{(3)})$ in $g_2 \ltimes g_3$. Notice that even though we are enumerating the Lie algebras as $g_1, g_2$ and $g_3$, they are isomorphic. We are using this notation to be reader friendly while following elements, spaces and operations.

3.2. The tangent group of a matched pair group.

Let us match two tangent groups, namely $TG$ and $TH$. Notice that, being a tangent bundle of a Lie group, $T(G \bowtie H)$ can be trivialized according to the definition given in (3.1) as

$$ T(G \bowtie H) \longrightarrow (G \bowtie H) \ltimes (b_1) \bowtie \eta_1, \quad (V_g, W_h) \mapsto (g, h; T_g L_{g^{-1}}V_g, T_g L_{h^{-1}}W_h) = (g, h; \xi^{(1)} , \eta^{(1)}). $$

Notice that we identified here $(g \bowtie b_1)1$ with $g_1 \bowtie b_1$. Being a tangent group, $T(G \bowtie H)$ is a Lie group. The following proposition establishes that $T(G \bowtie H)$ is also a match pair, [28 Prop. 2.3].

**Proposition 3.1.** Let $G \bowtie H$ be a matched pair Lie group, then its tangent group $T(G \bowtie H)$ is also a matched pair Lie group and it is isomorphic to the matched pairs of $TG$ and $TH$, that is

$$ T(G \bowtie H) \cong TG \bowtie TH. $$
Proof. We first define embeddings of $TG$ and $TH$ into $T(G\rtimes H)$ and then we derive the mutual group operations. See that, by identifying the total spaces with their left trivializations, we have the followings

\begin{align}
TG &\cong G\ltimes g_1 \rightarrow T(G\rtimes H) \cong (G\rtimes H)\ltimes (g_1 \rtimes h_1), \\
(3.10) \quad TH &\cong H\ltimes h_1 \rightarrow T(G\rtimes H) \cong (G\rtimes H)\ltimes (g_1 \rtimes h_1), \\
(3.11) &\quad (g, \xi^{(1)}) \mapsto (g, e; \xi^{(1)}, 0), \\
&\quad (h, \eta^{(1)}) \mapsto (e; h; 0, \eta^{(1)}).
\end{align}

Following the general tangent group operation character given in (3.2), and after a direct calculation one proves that in the image spaces of the embeddings, the group multiplication can be written as

\begin{equation}
(3.12) \quad (e, h; 0, \eta^{(1)})(g, e; \xi^{(1)}, 0) = (h \triangleright g, h \triangleleft g; (\xi^{(1)}, 0) + \text{Ad}_{(g, e)}^{-1}(0, \eta^{(1)})),
\end{equation}

where $\triangleright$ and $\triangleleft$ are mutual group operations on the group level $G$ and $H$, whereas $\text{Ad}$ is the adjoint action of the tangent group $G\ltimes g_1$ to its Lie algebra $g_2 \ltimes g_3$. Adjoint action can explicitly be computed as

\begin{equation}
\text{Ad}_{(g,e)}^{-1}(0, \eta^{(1)}) = (T_gL_{g^{-1}}(\eta^{(1)} \triangleright g), \eta^{(1)} \triangleleft g).
\end{equation}

Here, we used the notations $\eta^{(1)} \triangleright g$ and $\eta^{(1)} \triangleleft g$ representing the infinitesimal action of $h_1$ on $G$ and the lifted action of $G$ on $h_1$, respectively. More explicitly,

\begin{equation}
\eta^{(1)} \triangleright g : T_{e} \rho_{g}(\eta), \quad \rho_{g} : H \rightarrow G, \quad h \mapsto h \triangleright g,
\end{equation}

\begin{equation}
\eta^{(1)} \triangleleft g : T_{e} \sigma_{g}(\eta), \quad \sigma_{g} : H \rightarrow H, \quad h \mapsto h \triangleleft g.
\end{equation}

Accordingly, we can write the group operation in (3.6) as follows

\begin{equation}
(3.13) \quad (e, h; 0, \eta^{(1)})(g, e; \xi^{(1)}, 0) = (h \triangleright g, h \triangleleft g; \xi^{(1)} + T_gL_{g^{-1}}(\eta^{(1)} \triangleright g), \eta^{(1)} \triangleleft g).
\end{equation}

Now we are ready to derive the mutual actions of $TG$ and $TH$ on each other. This can be done by the universal property presented in Proposition 2.4 and following decomposition of the group multiplication

\begin{equation}
(3.14) \quad ((h, \eta^{(1)}) \triangleright (g, \xi^{(1)}))((h, \eta^{(1)}) \triangleleft (g, \xi^{(1)})) = (e, h; 0, \eta^{(1)})(g, e; \xi^{(1)}, 0).
\end{equation}

So that, we record below the mutual actions of $TG$ and $TH$, which are obtained by some direct calculations,

\begin{equation}
(3.15) \quad (h, \eta^{(1)}) \triangleright (g, \xi^{(1)}) = (h \triangleright g, (h \triangleleft g) \triangleright \xi^{(1)} + (h \triangleright g) \triangleright T_gL_{g^{-1}}(\eta^{(1)} \triangleright g)),
\end{equation}

\begin{equation}
(3.16) \quad (h, \eta^{(1)}) \triangleleft (g, \xi^{(1)}) = \left(h \triangleleft g, \eta^{(1)} \triangleleft g - T_{(h \triangleleft g)}L_{(h \triangleleft g)^{-1}}((h \triangleright g) \triangleleft (\xi^{(1)} + T_gL_{g^{-1}}(\eta^{(1)} \triangleright g))) \right).
\end{equation}

It is immediate to see that the group multiplications in (3.9) and (3.10) satisfy the requirements in (2.3).

Let us record here the group multiplication on $T(G\rtimes H)$ in a more explicit form by referring (3.2). In accordance with this, we compute the multiplication as

\begin{equation}
(3.17) \quad (g, h, \xi, \eta)(\tilde{g}, \tilde{h}, \tilde{\xi}, \tilde{\eta}) = (\tilde{g}, \tilde{h}, \tilde{\xi}, \tilde{\eta})
\end{equation}

where we have

\begin{equation}
\tilde{g} = g(h \triangleright \tilde{g}), \quad \tilde{h} = (h \triangleleft \tilde{g})\tilde{h}, \quad \tilde{\xi} = \xi + \tilde{h}^{-1} \triangleright \left(\text{Ad}_{\tilde{g}}^{-1}\xi + TL_{\tilde{g}^{-1}}(\eta \triangleright \tilde{g})\right),
\end{equation}

\begin{equation}
\tilde{\eta} = \tilde{\eta} + TR_{\tilde{h}}\left(\tilde{h}^{-1} \triangleleft (\text{Ad}_{\tilde{g}}^{-1}\xi + TL_{\tilde{g}^{-1}}(\eta \triangleright \tilde{g}))\right) + \text{Ad}_{\tilde{h}}^{-1}(\eta \triangleleft \tilde{g}).
\end{equation}

The Lie algebras of the constitutive Lie groups $TG$ and $TH$ of a matched tangent group $TG \rtimes TH$ are given by $g_2 \ltimes g_2$ and $h_2 \ltimes h_3$, respectively. In the infinitesimal picture, the group level identification in
\[\text{(3.4)}\] turns out to be
\[
(g_2 \rtimes b_2) \rtimes (g_3 \rtimes b_3) \equiv (g_2 \rtimes g_3) \rtimes (b_2 \rtimes b_3).
\]
Here, the left hand side is a result of the trivialization of the iterated tangent bundle \(TT(G \rtimes H)\) which is assumed to be the tangent bundle of the tangent group \(T(G \rtimes H)\) whereas the right hand side is the manifestation of the discussion, exhibited in \((2.17)\), saying that the Lie algebra of a matched pair group is a matched pair Lie algebra. In the following subsection, we are focusing on the iterated tangent bundles and their group structures.

### 3.3. Iterated tangent group.

In the previous subsection, we have showed that the tangent bundle \(TG\) of a Lie group is a Lie group by itself. Iteratively, considering \(TTG\) as the tangent bundle of the Lie group \(TG\), we can define a Lie group structure on \(TTG\). This can be done directly using the methods presented in Subsection \((3.1)\). For this, first define the following identifications
\[
TTG \equiv T(G \ltimes g_1) \equiv (G \ltimes g_1) \ltimes (g_2 \ltimes g_3)
\]
under iterative application of the left trivialization given by
\[
TTG \to (G \ltimes g_1) \ltimes (g_2 \ltimes g_3), \quad (V_g, V_{\xi^{(1)}}) \to (g, \xi^{(1)}, \xi^{(2)}, \xi^{(3)}):= \left(g, \xi^{(1)}, TL_{g^{-1}}V_g, V_{\xi^{(1)}} - \left[\xi^{(1)}, TL_{g^{-1}}V_g\right]\right),
\]
where we have assumed that to two-tuple \((V_g, V_{\xi^{(1)}})\) is an element of \(T_{(g, \xi)}(G \ltimes g_1)\). Following the previous subsection, the group multiplication is (that of a semidirect product) given by
\[
\left(g, \xi^{(1)}, \xi^{(2)}, \xi^{(3)}\right) \left(g\tilde{g}, \xi^{(1)}, \tilde{\xi}^{(2)}, \tilde{\xi}^{(3)}\right) = \left(g\tilde{g}, \xi^{(1)} + \text{Ad}_{g^{-1}}\xi^{(1)}, \xi^{(2)} + \text{Ad}_{g^{-1}}\xi^{(2)}, \xi^{(3)} + \text{Ad}_{g^{-1}}\xi^{(3)} + [\text{Ad}_{g^{-1}}\xi^{(2)}, \xi^{(1)}]\right),
\]
see, for instance, \([5, 73]\). It is evident that, in this case, the identity element is \((e, 0, 0, 0)\).

### 3.4. Iterated tangent group as a cocycle group extension.

Let us next record two different realizations of \(TTG\), from the point of view of the cocycle group extensions. To this end, we start with the trivialized tangent group \(G \ltimes g_1\) equipped with group operation \((3.2)\). Introduce a right action of this group \(G \ltimes g_1\) to the Lie algebra \(g_2\) given by
\[
(3.17) \quad g_2 \times (G \ltimes g_1) \to g_2, \quad \xi^{(2)} \circ (\tilde{g}, \xi^{(1)}) := \text{Ad}_{\tilde{g}}^{-1}(\xi^{(2)}).
\]
In the light of this group action, we define a semi-direct product group \((G \ltimes g_1) \ltimes g_2\) whose multiplication is given by
\[
(3.18) \quad \left(g, \xi^{(1)}, \xi^{(2)}\right) \left(g\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}\right) = \left(g\tilde{g}, \xi^{(1)} + \text{Ad}_{g^{-1}}(\xi^{(1)}), \xi^{(2)} + \text{Ad}_{g^{-1}}(\xi^{(2)})\right).
\]
Notice that, in the third term, \(\tilde{\xi}^{(1)}\) does not appear. So that we can also write the semi-direct group \((G \ltimes g_1) \ltimes g_2\) as an alternative form \(G \ltimes (g_1 \times g_2)\). Let us record here this observation
\[
(3.19) \quad (G \ltimes g_1) \ltimes g_2 = G \ltimes (g_1 \times g_2).
\]
Here, on the right hand side, we consider that the group $G$ acts on the algebra $g_1 \times g_2$. See that, in this case, there is a diagonal group action of $G$ on the direct product algebra $g_1 \times g_2$ given by

$$\quad (g_1 \times g_2) \times G \rightarrow (g_1 \times g_2), \quad (\xi^{(1)}, \xi^{(2)}) \ast (\tilde{g}, \xi^{(1)}, \xi^{(2)}) := (\text{Ad}_{\tilde{g}}^{-1} \xi^{(1)}, \text{Ad}_{\tilde{g}}^{-1} \xi^{(2)}).$$

(3.20)\(\quad \)

In order to extend the group $G \ltimes (g_1 \times g_2)$ to the iterated tangent group $TTG$, we introduce the following right action of $G \ltimes (g_1 \times g_2)$ to the Lie algebra $g_3$ given by

$$\quad g_3 \times (G \ltimes (g_1 \times g_2)) \rightarrow g_3, \quad \xi^{(3)} \ast (\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}) := \text{Ad}_{\tilde{g}}^{-1} \xi^{(3)}.$$

This action leads to the following $g_3$-valued operation on the group $G \ltimes (g_1 \times g_2)$ in form

$$\phi : \left( G \ltimes (g_1 \times g_2) \right) \times \left( G \ltimes (g_1 \times g_2) \right) \rightarrow g_3, \quad (\langle g, \xi^{(1)}, \xi^{(2)} \rangle \gtrdot \tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}) \rightarrow \left[ \text{Ad}_{\tilde{g}}^{-1} \xi^{(2)}, \xi^{(1)} \right].$$

The mapping $\phi$ is a group 2-cocycle on $G \ltimes (g_1 \times g_2)$. To see this, we need to establish that $\phi$ satisfies the cocycle condition (3.20). Indeed, for arbitrary elements $(g, \xi^{(1)}, \xi^{(2)})$, $(\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)})$ and $(\bar{g}, \bar{\xi}^{(1)}, \bar{\xi}^{(2)})$ in the group $G \ltimes (g_1 \times g_2)$ one computes

$$\phi\left( (\bar{g}, \bar{\xi}^{(1)}, \bar{\xi}^{(2)}), (\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}) \right) - \phi\left( (g, \xi^{(1)}, \xi^{(2)}), (\bar{g}, \bar{\xi}^{(1)}, \bar{\xi}^{(2)}) \right)$$

$$\quad + \phi\left( (g, \xi^{(1)}, \xi^{(2)}), (\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}) \right) - \phi\left( (g, \xi^{(1)}, \xi^{(2)}), (\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}) \right)$$

$$\quad = \phi\left( (\bar{g}, \bar{\xi}^{(1)}, \bar{\xi}^{(2)}), (\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}) \right) - \phi\left( (g, \xi^{(1)}, \xi^{(2)}), (\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}) \right) - \phi\left( (g, \xi^{(1)}, \xi^{(2)}), (\bar{g}, \bar{\xi}^{(1)}, \bar{\xi}^{(2)}) \right)$$

$$\quad + \phi\left( (g, \xi^{(1)}, \xi^{(2)}), (\bar{g}, \bar{\xi}^{(1)}, \bar{\xi}^{(2)}) \right) + \phi\left( (g, \xi^{(1)}, \xi^{(2)}), (\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}) \right)$$

$$\quad - \phi\left( (g, \xi^{(1)}, \xi^{(2)}), (\bar{g}, \bar{\xi}^{(1)}, \bar{\xi}^{(2)}) \right) - \phi\left( (g, \xi^{(1)}, \xi^{(2)}), (\tilde{g}, \tilde{\xi}^{(1)}, \tilde{\xi}^{(2)}) \right)$$

$$\quad = \left[ \text{Ad}_{\tilde{g}}^{-1} \xi^{(2)}, \bar{\xi}^{(1)} \right] - \left[ \text{Ad}_{\tilde{g}}^{-1} \xi^{(2)}, \text{Ad}_{\bar{g}}^{-1} \xi^{(1)} \right] + \left[ \text{Ad}_{\tilde{g}}^{-1} \xi^{(2)}, \bar{\xi}^{(1)} \right] + \left[ \text{Ad}_{\bar{g}}^{-1} \xi^{(2)}, \text{Ad}_{\tilde{g}}^{-1} \xi^{(1)} \right]$$

$$\quad - \text{Ad}_{\tilde{g}}^{-1} \text{Ad}_{\bar{g}}^{-1} \text{Ad}_{\bar{g}}^{-1} \xi^{(2)}, \bar{\xi}^{(1)} = 0.$$

These observations lead to the following proposition where we state that $TTG$ can be recasted as a Lie group extension of $G \ltimes (g_1 \times g_2)$.

**Proposition 3.2.** Given any group $G$, the iterated tangent group $TTG$ isomorphic to the Lie group extension $(G \ltimes (g_1 \times g_2)) \ltimes g_3$ determined by the cocycle $\phi$ presented in (3.27).
It is immediate to see that \( \chi \) determines a 2-cocyle in the group cohomology of \( g_1 \times g_2 \), with coefficients in \( g_3 \), that is \( \chi \in H^2(g_1 \times g_2, g_3) \). So that, we arrive at the following Lie group extension \((g_1 \times g_2) \ltimes_X g_3 \) where the group operation turns out to be

\[
(\xi(1), \xi(2), \xi(3)) \left( \tilde{\xi}(1), \tilde{\xi}(2), \tilde{\xi}(3) \right) = \left( \tilde{\xi}(1) + \xi(1), \tilde{\xi}(2) + \xi(2), \tilde{\xi}(3) + \xi(3) + [\xi(2), \xi(1)] \right),
\]

where \( (\xi(1), \xi(2), \xi(3)) \) and \( (\tilde{\xi}(1), \tilde{\xi}(2), \tilde{\xi}(3)) \) are two elements in \((g_1 \times g_2) \ltimes_X g_3 \). Further, consider the right action of \( G \) on the extended Lie group \((g_1 \times g_2) \ltimes_X g_3 \) given by

\[
(g_1 \times g_2) \ltimes_X g_3 \times G \to (g_1 \times g_2) \ltimes_X g_3, \quad (\xi(1), \xi(2), \xi(3)) \mapsto (\Ad_{g_1}, \Ad_{g_2}, \xi(1), \xi(2), \xi(3)).
\]

Under the light of these constructions, it becomes immediate now to arrive at the following proposition which states that the iterated tangent group \( TTG \) can be written as a semi-direct product group of \( G \) and extended group \((g_1 \times g_2) \ltimes_X g_3 \). Notice that the group structure on \((g_1 \times g_2) \ltimes_X g_3 \) corresponds to the restiction of the group structure on \( TTG \) where the group variable \( g \) is assumed to be the identity. We denote this as \( TT_e G \equiv (g_1 \times g_2) \ltimes_X g_3 \). It is important to note that this realization of \( TTG \) has already been arrived in [75 Sect. X.9] as a result of a direct calculation without referring to Lie group extensions.

**Proposition 3.3.** Let \( G \) be a Lie group, and \( TTG \) is the iterated tangent group with the multiplication exhibited in (3.16). Then \( TTG \) is a semi-direct product of the group \( G \) and the extended Lie group \( TT_e G = (g_1 \times g_2) \ltimes_X g_3 \), in (3.22), determined by the action (3.23), that is \( TTG \equiv TT_e G \rtimes G \).

### 3.5. The iterated tangent group of a matched pair group.

Let us now consider two Lie groups \( G \) and \( H \) under mutual actions satisfying the compatibility conditions of being a matched pair. In this subsection, our interest to investigate the mutual actions of the iterated bundles \( TTG \) and \( TTH \) induced those from the actions on the group level. So that, we need to iterate the arguments of Subsection 3.1 to the present case. We start with the following observation.

**Proposition 3.4.** For given a matched pair group \( G \rtimes H \), the iterated tangent bundles constitute a matched pair group \( TTG \rtimes TTH \) that is

\[
TT(G \rtimes H) \equiv T(TG \rtimes TH)) \equiv TTG \rtimes TTH.
\]

**Proof.** Let us prove this proposition by starting to analyse the first identity (3.24). The second term \( T(TG \rtimes TH)) \) is the tangent bundle of the matched group \( TG \rtimes TH \). So that, \( T(TG \rtimes TH) \) has the structure of a tangent group. The first term \( TT(G \rtimes H) \) is the iterated tangent bundle of the matched group \( G \rtimes H \). To arrive at the group structure on \( TT(G \rtimes H) \), we only replace \( G \) with \( G \rtimes H \) in (3.16). Accordingly, the first identification in (3.24) can be proved by simply applying tangent functor \( T \) to the both hand sides of (3.4). On the other hand, the second term is, topologically, diffeomorphic to the Cartesian product of two Lie groups \( TTG \) and \( TTH \). But, existence of the third term in (3.24) is also stating that one can match iterated tangent bundles \( TTG \) and \( TTH \) if there is mutual actions between \( G \) and \( H \), and that this matched pair is isomorphic both to \( T(TG \rtimes TH) \) and \( TT(G \rtimes H) \). In order to show these identifications, we start with the following embeddings

\[
TTG \to TT(G \rtimes H), \quad (g, \xi(1), \xi(2), \xi(3)) \mapsto (g, e, (\xi(1), 0), (\xi(2), 0), (\xi(3), 0)),
\]

\[
TTH \to TT(G \rtimes H), \quad (h, \eta(1), \eta(2), \eta(3)) \mapsto (e, h, (0, \eta(1)), (0, \eta(2)), (0, \eta(3))).
\]
Applying Proposition 2.1 to the present case, one can arrive at the following isomorphism, induced by the group multiplication,

\[ TTG \times TTH \to TT(H), \]

\[ \left( (g, \xi^1, \xi^2, \xi^3), (h, \eta^1, \eta^2, \eta^3) \right) \mapsto \left( (g, h), (\xi^1, \eta^1), (\xi^2, \eta^2), (\xi^3, \eta^3) \right). \]

Thus, the mutual actions of \( TTG \) and \( TTH \) are obtained by iterating (3.9) and (3.10). Namely:

\[ \left( h, \eta^1, \eta^2, \eta^3 \right) \triangleright \left( g, \xi^1, \xi^2, \xi^3 \right) = \left( (h, \eta^1) \triangleleft (g, \xi^1), \Delta^2_\triangleright (g, \xi^2), \Delta^3_\triangleright (g, \xi^3) \right) \]

\[ \left( h, \eta^1, \eta^2, \eta^3 \right) \triangleright \left( g, \xi^1, \xi^2, \xi^3 \right) = \left( (h, \eta^1) \triangleleft (g, \xi^1), \tilde{\eta}_2, \tilde{\eta}_3 \right) \]

where we have

\[ (\tilde{\xi}^2, \tilde{\xi}^3) = \left( (h, \eta^1) \triangleleft (g, \xi^1) \right) \triangleright (\xi^2, \xi^3) + \left( (h, \eta^1) \triangleleft (g, \xi^1) \right) \triangleright (gL_{(g, \xi^1)}, (\xi^2, \eta^3)) (g, \xi^1) \]

\[ (\tilde{\eta}^2, \tilde{\eta}^3) = \left( (\eta^1, \eta^2, \eta^3) \triangleleft (g, \xi^1) - TL_{(h, \eta^1) \triangleleft (g, \xi^1)} \right) \left( ((h, \eta^1) \triangleleft (g, \xi^1)) \triangleleft (\xi^2, \xi^3) \right) \]

\[ + TL_{(h, \eta^1) \triangleleft (g, \xi^1)} \left( (\eta^2, \eta^3) \triangleleft (g, \xi^1) \right) \]

Notice that, in these expressions, we have the following notations. The operation \((h, \eta) \triangleright (g, \xi)\) is the one given in (3.9), whereas the operation \((h, \eta) \triangleleft (g, \xi)\) is in (3.10). Similarly, for an element \((h, \eta) \in TH\), the operation \((h, \eta) \triangleright (\xi^1, \xi^2)\) is the lifted action of the group \( TH \) on to the Lie algebra of \( TG \), while \((\eta^1, \eta^2, \eta^3) \triangleleft (g, \xi)\) is the lifted action of the group \( TG \) on the Lie algebra of \( TH \). Finally, for an element \((h, \eta) \in TH\), the operation \((h, \eta) \triangleleft (\xi^1, \xi^2)\) is the right infinitesimal action of the Lie algebra of \( TG \) on the group \( TH \), and the value lies in the vector space \( T_{(h, \eta) \triangleright H} \). Finally, \((\eta^1, \eta^2, \eta^3) \triangleleft (g, \xi)\) denotes the left infinitesimal action of the Lie algebra of \( TH \) on the group \( TG \), and the value lies in \( T_{(h, \eta) \triangleright H} \).

Accordingly, given a matched pair \( G \triangleright H \) of groups, the group structure on the iterated tangent group \( TT(G \triangleright H) \equiv TTG \triangleright TTH \) is given by

\[ (g, h, \xi^1, \eta^1, \xi^2, \eta^2, \xi^3, \eta^3, \tilde{\xi}^1, \tilde{\eta}^1, \tilde{\xi}^2, \tilde{\eta}^2, \tilde{\xi}^3, \tilde{\eta}^3) = (g(h \triangleright \tilde{g}), (h \triangleright \tilde{g})h), \]

\[ \tilde{\xi}^1 + \tilde{h}^{-1} \triangleright \tau(1), \tilde{\eta} + TR_{\tilde{h}^{-1}}(\tilde{h}^{-1} \triangleright \tau(1)) + A_{\tilde{h}^{-1}}(\eta^1 \triangleleft \tilde{g}), \tilde{\xi}^2 + \tilde{h}^{-1} \triangleright \tau(2), \]

\[ \tilde{\eta}^2 + TR_{\tilde{h}^{-1}}(\tilde{h}^{-1} \triangleright \tau(2)) + A_{\tilde{h}^{-1}}(\eta^2 \triangleleft \tilde{g}), \tilde{\xi}^3 + \tilde{h}^{-1} \triangleright \tau(3) + \tau(4), \]

\[ \tilde{\eta}^3 = TR_{\tilde{h}^{-1}}(\tilde{h}^{-1} \triangleright \tau(3)) + A_{\tilde{h}^{-1}}(\eta^3 \triangleleft \tilde{g}) + \tau(5) \]

where we used the following abbreviations

\[ \tau(i) = A_{\tilde{h}^{-1}}(\tilde{\xi}^i) + TL_{\tilde{h}^{-1}}(\eta^1 \triangleleft \tilde{g}), \quad i = 1, 2, 3, \]

wheras

\[ \tau(4) = \left( \tilde{h}^{-1} \triangleright \tau(2), \tilde{\xi}^1 \right) + \left( TR_{\tilde{h}^{-1}}(\tilde{h}^{-1} \triangleleft \tau(2)) + A_{\tilde{h}^{-1}}(\eta^2 \triangleleft \tilde{g}) \right) \triangleright \tilde{\xi}^1 - \tilde{\eta}^1 \triangleright (\tilde{h}^{-1} \triangleright \tau(2)), \]

\[ \tau(5) = \left( TR_{\tilde{h}^{-1}}(\tilde{h}^{-1} \triangleright \tau(2)) + A_{\tilde{h}^{-1}}(\eta^2 \triangleleft \tilde{g}), \tilde{\eta}^1 \right) \triangleright \left( TR_{\tilde{h}^{-1}}(\tilde{h}^{-1} \triangleright \tau(2)) + A_{\tilde{h}^{-1}}(\eta^2 \triangleleft \tilde{g}) \right) \triangleright \tilde{\xi}^1 - \tilde{\eta}^1 \triangleright (\tilde{h}^{-1} \triangleright \tau(2)). \]

Finally, let us remark that given a matched pair \( G \triangleright H \) of Lie groups, and their (matched pair) Lie algebras \( g \triangleright b \), we have the following identification

\[ Lie TT(G \triangleright H) \equiv Lie TTG \triangleright Lie TTH. \]
3.6. Second order tangent group.

In this subsection, we consider the second order tangent bundle $T^2G$ as an embedded submanifold of $TTG$ under the left trivialization. For this, first we consider the left trivialization of $T^2G$ given by

$$T^2G \equiv G \times \mathfrak{g} \times \mathfrak{g}$$

where $\mathfrak{g}$ is isomorphic to $\mathfrak{g}$. Later, we present the following embedding

$$T^2G \rightarrow TTG : G \times \mathfrak{g} \times \mathfrak{g} \rightarrow (G \ltimes \mathfrak{g} \ltimes \mathfrak{g}) \quad \xi, \dot{\xi} \rightarrow (g, \xi, \dot{\xi}).$$

Notice that, in this realization, the first and the second Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ in the trivialization (3.15) of $TTG$ identified. In terms of the elements, we take that $\xi_1(\mathfrak{g}_1) = \xi_2(\mathfrak{g}_2) = \xi$ and $\dot{\xi}_1(\mathfrak{g}_1) = \dot{\xi}$. We refer [2, 14, 15, 17, 31, 73] on the second order tangent bundle of Lie groups. The embedding in (3.29) enables us to compute the group multiplication on the left trivialization of $T^2G$ from the group multiplication of $TTG$ in (3.16) as follows

$$((g, \xi, \dot{\xi})(\tilde{g}, \tilde{\xi}, \dot{\tilde{\xi}}) = (g \tilde{g}, \xi + \text{Ad}_{\tilde{g}}^{-1} \xi, \dot{\xi} + \text{Ad}_{\tilde{g}}^{-1} \dot{\xi} - \text{ad}_{\tilde{\xi}} \text{Ad}_{\tilde{g}}^{-1} \xi),$$

while the unit is $(e, 0, 0) \in G \times \mathfrak{g} \times \mathfrak{g}$. The inversion, therefore, is given by

$$(g, \xi, \dot{\xi})^{-1} = (g^{-1}, -\text{Ad}_g \xi, -\text{Ad}_g \dot{\xi}).$$

Notice that the group structure (3.30) of $T^2G$ can be written as an extension of the tangent group $TG$. To see this, recall the notation and the extension presented in Section 2.3. Take $\Gamma := TG \equiv G \ltimes \mathfrak{g}$, and $V := \mathfrak{g}$, with a right action given by

$$\triangleleft : \mathfrak{g} \times (G \ltimes \mathfrak{g}) \rightarrow \mathfrak{g}, \quad (\hat{\xi}, (g, \xi)) \rightarrow \hat{\xi} \triangleleft (g, \xi) := \text{Ad}_{\hat{g}}^{-1} \xi.$$

One may observe that

$$\varphi : (G \ltimes \mathfrak{g}) \times (G \ltimes \mathfrak{g}) \rightarrow \mathfrak{g}, \quad \varphi((g, \xi), (\tilde{g}, \tilde{\xi})) := -\text{ad}_{\tilde{\xi}} \text{Ad}_{\tilde{g}}^{-1} \xi$$

satisfies the cocycle condition (2.26). As such, we can write the global trivialization in (3.28) as follows

$$T^2G \equiv (G \ltimes \mathfrak{g}) \ltimes \mathfrak{g}.$$

For this extension we also refer to [73].

3.7. Matched pairs of second order groups.

In the rest of this subsection we shall investigate the matched pairs of second order tangent groups, as well as the matched pairs of their Lie algebras.

Proposition 3.5. If $G \rhd H$ is a matched pair Lie group, then so is $T^2(G \rhd H)$. Moreover, there is the isomorphism $T^2(G \rhd H) \equiv T^2G \rhd T^2H$, as Lie groups.
Proof. We are employing Proposition (2.1), exhibiting the universality of matched product, to the present case. So that, let \( G \bowtie H \) be a matched pair of Lie groups, and let us consider the following inclusions

\[
T^2G \rightarrow T^2(G \bowtie H), \quad (g, \xi, \dot{\xi}) \mapsto ((g, e), (\xi, 0), (\dot{\xi}, 0)),
\]

\[
T^2H \rightarrow T^2(G \bowtie H), \quad (h, \eta, \dot{\eta}) \mapsto ((e, h), (0, \eta), (0, \dot{\eta})).
\]

So that we compute the multiplication of these images in the group \( T^2(G \bowtie H) \) as follows, Then, the multiplication in \( T^2(G \bowtie H) \), which may be given by

\[
(g, h, \xi, \eta, \dot{\xi}, \dot{\eta})(\tilde{g}, \tilde{h}, \tilde{\xi}, \tilde{\eta}, \dot{\tilde{\xi}}, \dot{\tilde{\eta}}) = (g(h \triangleright \tilde{g}), (h \triangleright \tilde{g})\tilde{h}, \xi + \tilde{\xi} \triangleright \tau, \eta + TR_h(\tilde{\eta} \triangleright \tau) + Ad_{\tilde{h}^{-1}}(\eta \triangleright \tilde{g}))
\]

where we have used the abbreviations

\[
\tau = Ad_{\tilde{g}^{-1}}\dot{\xi} + TL_{\tilde{g}^{-1}}(\eta \triangleright \tilde{g}), \quad \tilde{\tau} = Ad_{\tilde{g}^{-1}}\dot{\tilde{\xi}} + TL_{\tilde{g}^{-1}}(\eta \triangleright \tilde{g}),
\]

induces a map

\[
T^2G \times T^2H \rightarrow T^2(G \bowtie H),
\]

\[
((g, \xi, \dot{\xi}), (h, \eta, \dot{\eta})) \mapsto ((g, e), (\xi, 0), (\dot{\xi}, 0), (e, h), (0, \eta), (0, \dot{\eta})) =
\]

\[
((g, h), Ad_{(e, h)^{-1}}(\xi, 0) + (0, \eta), \varphi((g, e), (\xi, 0)), ((e, h), (0, \eta))) + Ad_{(e, h)^{-1}}(\dot{\xi}, 0) + (0, \dot{\eta})
\]

\[
= ((g, h), Ad_{(e, h)^{-1}}(\xi, 0) + (0, \eta), -ad_0(\eta)Ad_{(e, h)^{-1}}(\xi, 0) + Ad_{(e, h)^{-1}}(\dot{\xi}, 0) + (0, \dot{\eta})
\]

\[
= ((g, h), Ad_{(e, h)^{-1}}(\xi, 0) + (0, \eta), [Ad_{(e, h)^{-1}}(\xi, 0), (0, \eta)] + Ad_{(e, h)^{-1}}(\dot{\xi}, 0) + (0, \dot{\eta})
\]

\[
= ((g, h), (h^{-1} \triangleright \xi, T_{h^{-1}}R_h(h^{-1} \triangleright \xi) + \eta),
\]

\[
(h^{-1} \triangleright \xi, T_{h^{-1}}R_h(h^{-1} \triangleright \xi), (0, \eta) + (h^{-1} \triangleright \dot{\xi}, T_{h^{-1}}R_h(h^{-1} \triangleright \dot{\xi}) + \dot{\eta}))
\]

\[
= ((g, h), (h^{-1} \triangleright \xi, T_{h^{-1}}R_h(h^{-1} \triangleright \xi) + \eta),
\]

\[
(h^{-1} \triangleright \xi + \eta \triangleright (h^{-1} \triangleright \xi), [T_{h^{-1}}R_h(h^{-1} \triangleright \xi), \eta] - \eta \triangleright (h^{-1} \triangleright \xi) + (h^{-1} \triangleright \dot{\xi}, T_{h^{-1}}R_h(h^{-1} \triangleright \dot{\xi}) + \dot{\eta}))
\]

Accordingly, for the inclusions in (3.32), we may compute the inverse through

\[
T^2(G \bowtie H) \rightarrow T^2G \times T^2H,
\]

\[
((g, h), (\xi, \eta)) \mapsto ((g, e), (A_1, 0), (A_0, 0), (e, h), (0, B_1), (0, B_0)).
\]

Indeed, it follows from

\[
((g, h), (\xi, \eta)) = ((g, e), (A_1, 0)) (e, h), (0, B_1)) = ((g, h), Ad_{(e, h)^{-1}}(A_1, 0) + (0, B_1))
\]

\[
= ((g, h), (h^{-1} \triangleright A_1, T_{h^{-1}}R_h(h^{-1} \triangleright A_1) + B_1))
\]

we arrive at

\[
A_1 = h \triangleright \xi, \quad B_1 = \eta - T_{h^{-1}}r_h(h^{-1} \triangleright \xi) = \eta + T_hL_{h^{-1}}(h \triangleright \xi).
\]
In order to determine $A_0$ and $B_0$ we do the following computation
\[
(\dot{\xi}, \dot{\eta}) = \varphi(((g, e), (A_1, 0)), ((e, h), (0, B_1))) + \text{Ad}_{(e, h)^{-1}}(A_0, 0) + (0, B_0)
\]
\[
= [\text{Ad}_{(e, h)^{-1}}(A_1, 0), (0, B_1)] + \text{Ad}_{(e, h)^{-1}}(A_0, 0) + (0, B_0)
\]
\[
= [[\xi, \eta], (0, B_1)] + (h^{-1} \triangleright A_0, T_{h^{-1}} r_h(h^{-1} \triangleleft A_0) + B_0)
\]
\[
= (\xi, \eta), (0, B_1) + (h^{-1} \triangleright A_0, T_{h^{-1}} r_h(h^{-1} \triangleleft A_0) + B_0)
\]
\[
\]
\[
= -B_1 \triangleright \xi, \eta, B_1 \triangleleft \xi + (h^{-1} \triangleright A_0, T_{h^{-1}} r_h(h^{-1} \triangleleft A_0) + B_0)
\]
\[
= (h^{-1} \triangleright A_0 - B_1 \triangleright \xi, [\eta, B_1] - B_1 \triangleleft \xi + T_{h^{-1}} r_h(h^{-1} \triangleleft A_0) + B_0)
\]
to conclude the terms $A_0$ and $B_0$ in terms of $A_1$ and $B_1$ as follows
\[
(3.34) \quad A_0 = h \triangleright (B_1 \triangleright \xi) + h \triangleright \dot{\xi}, \quad B_0 = \dot{\eta} + B_1 \triangleleft \xi - T_{h^{-1}} R_h(h^{-1} \triangleleft A_0) + [B_1, \eta].
\]
Now we substitute $A_1$ and $B_1$, given in (3.33), into the expressions (3.34) in order to arrive explicitly $A_0$ and $B_0$. These read
\[
(3.35) \quad A_0 = h \triangleright (\eta \triangleleft \xi) + h \triangleright (T_{h^{-1}} L_{h^{-1}}(h \triangleleft \xi) \triangleright \xi) + h \triangleright \dot{\xi},
\]
\[
B_0 = \dot{\eta} + \eta \triangleleft \xi + T_{h^{-1}} L_{h^{-1}}(h \triangleleft \xi) \triangleleft \xi + T_{h^{-1}} L_{h^{-1}}(h \triangleright (\eta \triangleright \xi))
\]
\[
+ T_{h^{-1}} L_{h^{-1}}(h \triangleleft (T_{h^{-1}} L_{h^{-1}}(h \triangleleft \xi) \triangleright \xi)) + T_{h^{-1}} L_{h^{-1}}(h \triangleleft \dot{\xi}) + [T_{h^{-1}} L_{h^{-1}}(h \triangleleft \xi), \eta].
\]
\[
□
\]

Now we are computing the mutual actions of $T^2G$ and $T^2H$. For this, we are referring to Proposition (2.1) where the universal property of the matched pair construction. In accordance with this, we first embed the groups $T^2G$ and $T^2H$ into the matched pair as determined in (3.32) then we perform the group multiplication to find the mutual actions. That is
\[
(3.36) \quad (e, h, (0, \eta), (0, \dot{\eta}))(g, e, (\xi, 0), (\dot{\xi}, 0)) = \left( h, \eta, \dot{\eta} \right) \triangleright \left( g, \xi, \dot{\xi} \right) \left( h, \eta, \dot{\eta} \right) \triangleleft \left( g, \xi, \dot{\xi} \right).
\]
Here, the multiplication is the induced group multiplication in $T^2(G \cong H)$. After a direct calculation we arrive at the left action of $T^2H$ on $T^2G$ given by
\[
\]
\[
(3.37) \quad \left( h, \eta, \dot{\eta} \right) \triangleright \left( g, \xi, \dot{\xi} \right) = \left( h \triangleright g, (h \triangleleft g) \triangleright (\xi + T_{g} L_{g^{-1}}(\eta \triangleright g)),
\]
\[
(h \triangleleft g) \triangleright (\eta \triangleleft g) \triangleright (\xi + T_{g} L_{g^{-1}}(\eta \triangleright g))
\]
\[
+ (h \triangleright g) \triangleright \left( T_{h \triangleleft g} L_{g \cdot h^{-1}}(h \triangleleft g) \triangleleft (\eta \triangleleft g) \triangleright (\xi + T_{g} L_{g^{-1}}(\eta \triangleright g)) \right) \triangleright (\xi + T_{g} L_{g^{-1}}(\eta \triangleright g))
\]
\[
+ (h \triangleright g) \triangleright (\dot{\xi} + T_{g} L_{g^{-1}}(\eta \triangleright g) - \text{ad}_{\xi} (T_{g} L_{g^{-1}}(\eta \triangleright g)) + (\eta \triangleleft g) \triangleleft \dot{\xi}).
\]
whereas the right action of $T^2G$ on $T^2H$ is

\[
(\bar{h}, \eta, \tilde{\eta}) \mathbin{\prec} (g, \xi, \tilde{\xi}) = \left( h \prec g, \eta \circ g + T_{h \circ g} L_{(h \circ g)^{-1}} \left( (h \prec g) \mathbin{\prec} (\xi + T_g L_{g^{-1}}(\eta \circ g)) \right), \right.
\]
\[
\left. \tilde{\eta} \circ g + \left( (\eta \circ g) \mathbin{\prec} \xi + (\eta \circ g) \mathbin{\prec} (\xi + T_g L_{g^{-1}}(\eta \circ g)) \right) + T_{h \circ g} L_{(h \circ g)^{-1}} \left( (h \prec g) \mathbin{\prec} \left( (\eta \circ g) \mathbin{\prec} (\xi + T_g L_{g^{-1}}(\eta \circ g)) \right) \mathbin{\prec} (\xi + T_g L_{g^{-1}}(\eta \circ g)) \right) \right)
\]
\[
+ T_{h \circ g} L_{(h \circ g)^{-1}} \left( (h \prec g) \mathbin{\prec} \left( (\eta \circ g) \mathbin{\prec} (\xi + T_g L_{g^{-1}}(\eta \circ g)) \right) \mathbin{\prec} (\xi + T_g L_{g^{-1}}(\eta \circ g)) \right) \right)
\]
\[
+ T_{h \circ g} L_{(h \circ g)^{-1}} \left( (h \prec g) \mathbin{\prec} \left( (\eta \circ g) \mathbin{\prec} (\xi + T_g L_{g^{-1}}(\eta \circ g)) \right) \mathbin{\prec} (\xi + T_g L_{g^{-1}}(\eta \circ g)) \right) \right).
\]

3.8. Matching of cocycle extensions.

In Section 3.6, we have shown that the second order $T^2G$ can be written as a cocycle extension of the group $TG$. This reads that $T^2G \approx T^2H$ being isomorphic to the second order tangent bundle $T^2(G \approx H)$ can be written as an extension of $T(G \approx H)$ with an isomorphic copy of the Lie algebra $\mathfrak{g} \approx \mathfrak{h}$. So that, in view of the isomorphism $T^2(G \approx H) \cong T^2G \approx T^2H$ of groups, where all three groups are cocycle group extensions, the relation between these cocycles deserves an investigation. This, we shall do in the following proposition. As for the proposition, we shall need the following analogue of [46, Lemma 3.2].

Lemma 3.6. Let $G \approx H$ be a matched pair Lie group and $\mathfrak{g} \approx \mathfrak{h}$ be their (matched pair) Lie algebra. Then, the left action of $H$ on $\mathfrak{g}$ has the following distribution law

\[
h \prec [\xi, \tilde{\xi}] = (h \prec \xi, (h \prec \xi) \mathbin{\prec} \tilde{\xi} - (h \prec \tilde{\xi}) \mathbin{\prec} \xi
\]

for any $h \in H$, and any $\xi, \tilde{\xi} \in \mathfrak{g}$, where we employ the notations

\[
(h \mathbin{\prec} \xi) \mathbin{\prec} \tilde{\xi} := \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} (h \mathbin{\prec} e^t \mathbin{\prec} e^s \tilde{\xi}), \quad (h \mathbin{\prec} \tilde{\xi}) \mathbin{\prec} \xi := \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} (h \mathbin{\prec} e^t \mathbin{\prec} e^s \xi)
\]

for any (1-parameter) curves given by $e^t, e^s \in G$ so that $e^0 = e = e^0_0$, that $\frac{d}{dt} \big|_{t=0} e^1_1 = \xi$, and that $\frac{d}{ds} \big|_{s=0} e^1_2 = \tilde{\xi}$.

Proof. For any $h \in H$, and any $\xi, \tilde{\xi} \in \mathfrak{g}$ we compute the following

\[
h \mathbin{\prec} [\xi, \tilde{\xi}] = \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} (h \mathbin{\prec} e^t \mathbin{\prec} e^s \tilde{\xi} - (h \mathbin{\prec} \xi) \mathbin{\prec} \tilde{\xi})
\]

\[
= \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} (h \mathbin{\prec} e^t \mathbin{\prec} e^s \tilde{\xi}) - (h \mathbin{\prec} \xi) \mathbin{\prec} \tilde{\xi}
\]

\[
= \left. \frac{d}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} (h \mathbin{\prec} e^t \mathbin{\prec} e^s \tilde{\xi}) - (h \mathbin{\prec} \xi) \mathbin{\prec} \tilde{\xi}
\]

where we note that

\[
(h \mathbin{\prec} e^1) \mathbin{\prec} e^{-1}_1 = (h \mathbin{\prec} e^1)^{-1}.
\]
So that, one can observe
\[
\begin{align*}
\frac{d}{dt}\bigg|_{t=0} & \frac{d}{ds}\bigg|_{s=0} (h \triangleright e_1^i)((h \triangleleft e_1^i) \blacktriangleright e_2^j)((h \triangleleft e_1^i) \blacktriangleright e_3^k) (h \triangleright e_1^i)^{-1} = \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} (h \triangleright e_1^i)((h \triangleright e_1^i) \blacktriangleright e_2^j)(h \triangleright e_1^i)^{-1} \\
&= \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} (h \triangleright e_1^i)(h \triangleright e_2^j)(h \triangleright e_1^i)^{-1} + \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} ((h \triangleright e_1^i) \blacktriangleright e_2^j) \\
&= [h \triangleright \xi, h \triangleright \xi] + (h \triangleright \xi) \triangleright \xi.
\end{align*}
\]

On the other hand,
\[
\begin{align*}
\frac{d}{dt}\bigg|_{t=0} & \frac{d}{ds}\bigg|_{s=0} (h \triangleright e_1^i)((h \triangleleft e_1^i) \blacktriangleright e_2^j)(h \triangleleft e_1^i)^{-1} = \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} ((h \triangleright e_1^i) \blacktriangleright e_2^j) = -(h \triangleright \xi) \triangleright \xi.
\end{align*}
\]

The claim thus follows.

Now, we recall the following mappings, defining the Lie group extensions,
\[
\begin{align*}
\varphi : (G \ltimes \mathfrak{g}) \times (G \ltimes \mathfrak{g}) & \to \mathfrak{g}, \quad \varphi((g, \xi); (\tilde{g}, \tilde{\xi})) := -\text{ad}_\xi \text{Ad}_{\tilde{g}}^{-1} \eta \\
\psi : (H \ltimes \mathfrak{h}) \times (H \ltimes \mathfrak{h}) & \to \mathfrak{h}, \quad \psi((h, \eta); (\tilde{h}, \tilde{\eta})) := -\text{ad}_\eta \text{Ad}_{\tilde{h}}^{-1} \eta,
\end{align*}
\]
where $\varphi$ is a Lie algebra valued 2-cocycle on the group $G \ltimes \mathfrak{g}$ whereas $\psi$ is a $\mathfrak{h}$ valued 2-cocycle on the group $H \ltimes \mathfrak{h}$. We relate these cocycles with the one defined for the matched pair
\[
\Phi : (G \bowtie H) \ltimes (\mathfrak{g} \bowtie \mathfrak{h}) \to \mathfrak{g}, \quad \Phi((g, h, \xi, \eta); (\tilde{g}, \tilde{h}, \tilde{\xi}, \tilde{\eta})) := -\text{ad}_\xi \text{Ad}_{\tilde{g}}^{-1} \text{Ad}_{\tilde{h}}^{-1} (\xi, \eta),
\]
where $\text{ad}_\xi(\eta)$ is the infinitesimal adjoint representation of $\mathfrak{g} \bowtie \mathfrak{h}$ on itself whereas $\text{Ad}_{\tilde{g},\tilde{h}}^{-1}$ is the adjoint representation of $G \bowtie H$ on $\mathfrak{g} \bowtie \mathfrak{h}$. The following proposition relates the cocycles $\varphi$, $\psi$ and $\Phi$.

**Proposition 3.7.** Let $G \bowtie H$ be a matched pair Lie group, and let $T^2(G \bowtie H) \cong T(G \bowtie H \ltimes \mathfrak{g} \bowtie \mathfrak{h}) \cong T^2G \cong TG \ltimes \mathfrak{g}$, and $T^2H \cong TH \ltimes \mathfrak{h}$. Then,
\[
\varphi\left(\left(\left(g, h, \xi, \eta\right); (\tilde{g}, \tilde{h}, \tilde{\xi}, \tilde{\eta}) \right)\right) = \left(\tilde{h}^{-1} \triangleright \phi((g, \xi); (\tilde{g}, \tilde{h} \triangleright \tilde{\xi}))
\right.
\]
\[
+ \left( (\tilde{h}^{-1} \triangleright (\tilde{h} \triangleright \tilde{\xi})) \triangleright \text{Ad}_{\tilde{g}}^{-1} \xi - (\tilde{h}^{-1} \triangleright \text{Ad}_{\tilde{g}}^{-1} \xi) \triangleright (\tilde{h} \triangleright \tilde{\xi}) - \text{ad}_\xi(\tilde{h}^{-1} \triangleright T_\tilde{g}L_{\tilde{g}}(\eta \triangleright \tilde{g})) \right) \\
+ \left( T_{\tilde{h}^{-1}}R_{\tilde{g}}(\tilde{h}^{-1} \triangleright \zeta) + \text{Ad}_{\tilde{g}}^{-1}(\eta \triangleright \tilde{g}) \right) \triangleright \tilde{\xi} - \tilde{\eta} \triangleright (\tilde{h}^{-1} \triangleright \zeta). \\
\psi\left((h, \eta \triangleright \tilde{g}); (\tilde{h}, \tilde{\eta})\right) = -\text{ad}_\eta\left( T_{\tilde{g}}R_{\tilde{g}}(\tilde{h}^{-1} \triangleright \zeta) + \text{Ad}_{\tilde{g}}^{-1}(\eta \triangleright \tilde{g}) \right) \triangleright \tilde{\xi} - \tilde{\eta} \triangleright (\tilde{h}^{-1} \triangleright \zeta),
\]
where
\[
(\triangleright \tilde{g}) \triangleright \tilde{\xi} := \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} (h \triangleright e_1^i) \triangleright e_2^j, \quad (h \triangleright \xi) \triangleright \xi := \frac{d}{dt}\bigg|_{t=0} \frac{d}{ds}\bigg|_{s=0} (h \triangleright e_1^i) \triangleright e_1^i
\]
for any (1-parameter) curves given by $e_1^i, e_2^j \in G$ so that $e_1^i = e = e_2^j$, that $\frac{d}{dt}\bigg|_{t=0} e_1^i = \xi_1$, and that $\frac{d}{ds}\bigg|_{s=0} e_2^j = \tilde{\xi}$. 
Proof. It follows from [28, (2.14)] that
\[
\Phi\left((g, h), (\xi, \eta); (\tilde{g}, \tilde{h}), (\tilde{\xi}, \tilde{\eta})\right) = \left[ \text{Ad}_{(g, h)^{-1}}(\xi, \eta); (\tilde{\xi}, \tilde{\eta}) \right]
\]
\[
= \left( \tilde{h}^{-1} \triangleright (\text{Ad}_{g^{-1}} \xi + T_{\tilde{g}} L_{g^{-1}}(\eta \triangleright \tilde{g})), T_{\tilde{h}^{-1}} R_{\tilde{g}}(\tilde{h}^{-1} \triangleleft \zeta) + \text{Ad}_{\tilde{g}^{-1}}(\eta \triangleright \tilde{g})), (\tilde{\xi}, \tilde{\eta}) \right)
\]
\[
= \left( \tilde{h}^{-1} \triangleright \text{Ad}_{\tilde{g}^{-1}} \xi, \tilde{\xi} \right) - \text{ad}_{\tilde{g}}(\tilde{h}^{-1} \triangleright T_{\tilde{g}} L_{g^{-1}}(\eta \triangleright \tilde{g})) + (T_{\tilde{h}^{-1}} R_{\tilde{g}}(\tilde{h}^{-1} \triangleleft \zeta)
\]
\[\quad + \text{Ad}_{\tilde{g}^{-1}}(\eta \triangleright \tilde{g})) \triangleright \tilde{\xi} - \tilde{\eta} \triangleright (\tilde{h}^{-1} \triangleright \zeta); \text{Ad}_{\tilde{g}^{-1}}(\eta \triangleright \tilde{g}), \tilde{\eta} \right]
\]
\[\quad - \text{ad}_{\eta}(T_{\tilde{h}^{-1}} R_{\tilde{g}}(\tilde{h}^{-1} \triangleleft \zeta)) + \text{Ad}_{\tilde{g}^{-1}}(\eta \triangleright \tilde{g})) \triangleright \tilde{\xi} - \tilde{\eta} \triangleright (\tilde{h}^{-1} \triangleright \zeta) \right)
\]
where \( \zeta \) is in (3.42) and we used [28, (2.13)] in the forth equation. Next, following Lemma 3.6, we see that
\[
\tilde{h}^{-1} \triangleright \left[ \text{Ad}_{\tilde{g}^{-1}} \xi, \tilde{h} \triangleright \tilde{\xi} \right] = \left( \tilde{h}^{-1} \triangleright \text{Ad}_{\tilde{g}^{-1}} \xi, \tilde{\xi} \right) + (\tilde{h}^{-1} \triangleleft \text{Ad}_{\tilde{g}^{-1}} \xi) \triangleright (\tilde{h} \triangleright \tilde{\xi}) - (\tilde{h}^{-1} \triangleleft \tilde{h} \triangleright \tilde{\xi}) \triangleright \text{Ad}_{\tilde{g}^{-1}} \xi,
\]
in other words,
\[
\left[ \tilde{h}^{-1} \triangleright \text{Ad}_{\tilde{g}^{-1}} \xi, \tilde{\xi} \right] = \tilde{h}^{-1} \triangleright \left[ \text{Ad}_{\tilde{g}^{-1}} \xi, \tilde{h} \triangleright \tilde{\xi} \right] + (\tilde{h}^{-1} \triangleleft \text{Ad}_{\tilde{g}^{-1}} \xi) \triangleright (\tilde{h} \triangleright \tilde{\xi}) - (\tilde{h}^{-1} \triangleleft \text{Ad}_{\tilde{g}^{-1}} \xi) \triangleright (\tilde{h} \triangleright \tilde{\xi})
\]
\[\quad = \tilde{h}^{-1} \triangleright \phi((g, \xi); (\tilde{g}, \tilde{h} \triangleright \tilde{\xi})) + (\tilde{h}^{-1} \triangleleft \text{Ad}_{\tilde{g}^{-1}} \xi) \triangleright (\tilde{h} \triangleright \tilde{\xi}) - (\tilde{h}^{-1} \triangleleft \text{Ad}_{\tilde{g}^{-1}} \xi) \triangleright (\tilde{h} \triangleright \tilde{\xi}).
\]
The claim, then, follows from
\[
\left[ \text{Ad}_{\tilde{g}^{-1}}(\eta \triangleright \tilde{g}), \tilde{\eta} \right] = \psi((h, \eta \triangleright \tilde{g}); (\tilde{h}, \tilde{\eta})).
\]

Let us note also that if the mutual actions of the groups (\( G, H \)) are trivial, then (3.41) reduces all the way down to
\[
\Phi\left((g, h), (\xi, \eta); (g, \tilde{h}), (\tilde{\xi}, \tilde{\eta})\right) = \left( \phi((g, \xi); (g, \tilde{\xi})); \psi((h, \eta); (\tilde{h}, \tilde{\eta})) \right).
\]
This section contains Euler-Lagrange and Euler-Poincaré equations for the higher order frameworks presented in the previous section. That is, we present Euler-Lagrange equations on both $TG \rightrightarrows TH$ and $T^2G \rightrightarrows T^2H$. Then we employ Lagrangian reduction, under the group of symmetries, to these dynamics in order to arrive at the Euler-Poincaré equations on matched pair Lie algebras $g \rightrightarrows h$ and $(g \times \dot{g}) \rightrightarrows (h \times \dot{h})$.

4.1. First order Lagrangian dynamics. Recall the identification (3.1) of the tangent group $TG$ with its trivialization $G \ltimes g$. So that for a Lagrangian function(al) defined on $TG$ there exists a unique Lagrangian function(al) on $G \ltimes g$. In order to arrive at the equation of motion generated by such a Lagrangian function(al) $L = L(g, \xi)$ we compute the variation of the action integral

$$\delta \int_a^b L(g, \xi,\dot{\xi}) \, dt = \int_a^b \left( \frac{\delta L}{\delta g} \frac{d g}{d t} + \frac{\delta L}{\delta \xi} \frac{d \xi}{d t} \right) dt.$$  \hspace{1cm} (4.1)

By applying Hamilton’s principle to the variations of the base (group) component and the reduced variational principle

$$\delta \xi = \dot{\eta} + [\xi, \eta]$$  \hspace{1cm} (4.2)

to the fiber (Lie algebra) component, the trivialized Euler-Lagrange equation [7, 14, 15, 22, 25] is computed to be

$$\frac{d}{d t} \frac{\delta L}{\delta \xi} = T^* L(g) \frac{\delta L}{\delta g} - ad^* \xi \frac{\delta L}{\delta \xi}.$$  \hspace{1cm} (4.3)

For the reduced variational principle we refer to [10, 24, 54] and for the Lagrangian dynamics on semidirect product spaces to [9, 11, 37, 56]. If $L$ is independent of the group variable $g$ that is, $L(g, \xi) = L(\xi)$, then the trivialized Euler-Lagrange equations (4.3) reduce to the Euler-Poincaré equations

$$\frac{d}{d t} \frac{\delta L}{\delta \xi} = -ad^* \xi \frac{\delta L}{\delta \xi}$$  \hspace{1cm} (4.4)
on $g$. Such a reduction occurs if the Lagrangian $L$ is left invariant, under the left group action of $G$ on $G \ltimes g$. This procedure is called as Euler-Poincaré reduction.

4.2. First order Lagrangian dynamics on iterated bundles.

Being a tangent bundle $TTG$ over $TG$, it is possible to study Lagrangian dynamics on it and its trivialization $(G \ltimes g_1) \ltimes (g_2 \ltimes g_3)$. We denote the iterated tangent bundle and its trivialization by the same notation $TTG$. Accordingly, consider a Lagrangian function(al) $L = L(g, \xi(1), \xi(2), \xi(3))$ on $TTG$ and the variation

$$\delta \int \mathcal{L} \, dt = \int \left( \frac{\delta L}{\delta (g, \xi(1), \xi(2), \xi(3))} \delta \left(g, \xi(1), \xi(2), \xi(3)\right) \right) dt$$  \hspace{1cm} (4.5)

of the associated action integral to obtain trivialized Euler-Lagrange equations [30]. To formulate variational principle on $TTG$ we proceed as follows. For the variations $(\delta g, \delta \xi(1))$ in the first and the second integrals...
Proposition 4.1. Now it is a direct calculation to arrive at that all these variations lead to the following set of equations.

\[
\left( \xi^{(2)}, \xi^{(3)} \right) = T_{e,0} L_{g,e^{(1)}} \delta \left( g, e^{(1)} \right) = \left( TL_{g^{-1}} \delta g, \delta e^{(1)} + [TL_{g^{-1}} \delta g, e^{(1)}]_{g} \right),
\]

where \( \left( \xi^{(2)}, \xi^{(3)} \right) \in g \ltimes g \), and obtain

\[
\delta g = TL_{g} \xi^{(2)}, \quad \delta e^{(1)} = \xi^{(3)} + [\xi^{(1)}, \xi^{(2)}]_{g},
\]

for arbitrary choices of \( \left( \xi^{(2)}, \xi^{(3)} \right) \). To obtain the variations \( \delta \xi^{(2)}, \delta \xi^{(3)} \) in the third and the fourth integrals in Eq. (4.5) we consider the reduced variational principle

\[
\delta \left( \xi^{(2)}, \xi^{(3)} \right) = \frac{d}{dt} \left( \xi^{(2)}, \xi^{(3)} \right) + \frac{d}{dt} \left( \xi^{(2)}, \xi^{(3)} \right) \bigg|_{g_{2} \ltimes g_{3}},
\]

on the Lie algebra \( g_{2} \ltimes g_{3} \) of \( G \ltimes g_{1} \) for arbitrary choice of \( \left( \xi^{(2)}, \xi^{(3)} \right) \) in \( g_{2} \ltimes g_{3} \). Here, the Lie algebra bracket on \( g_{2} \ltimes g_{3} \) is

\[
\left( \left[ \xi^{(2)}, \xi^{(3)} \right], \xi^{(2)}, \xi^{(3)} \right) \bigg|_{g_{2} \ltimes g_{3}} = \left( ad_{\xi^{(2)}} \xi^{(2)}, ad_{\xi^{(3)}} \xi^{(3)} - ad_{\xi^{(2)}} \xi^{(3)} \right).
\]

Note that, the variational principle in Eq. (4.6) is the same as the one in Eq. (4.2) but this time Lie algebra elements are two-tuples and the Lie algebra bracket is (4.7). So, we have

\[
\delta \xi^{(2)} = \delta \xi^{(2)} + \left[ \xi^{(2)}, \xi^{(2)} \right], \quad \delta \xi^{(3)} = \delta \xi^{(3)} + \left[ \xi^{(2)}, \xi^{(3)} \right] - \left[ \xi^{(2)}, \xi^{(3)} \right] + \left[ \xi^{(3)}, \xi^{(3)} \right].
\]

Now it is a direct calculation to arrive at that all these variations lead to the following set of equations.

Proposition 4.1. Trivialized Euler-Lagrange equations generated by a Lagrangian function(al) \( \mathcal{L} \) given by \( \mathcal{L} \left( g, e^{(1)}, e^{(2)}, e^{(3)} \right) \) on TTG are

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi^{(2)}} = T^{e} L_{g} \frac{\delta \mathcal{L}}{\delta g} - ad_{e^{(1)}}(\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta e^{(1)}} + \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta e^{(2)}} + \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta e^{(3)}}) - ad_{e^{(2)}} \frac{\delta \mathcal{L}}{\delta e^{(2)}} - ad_{e^{(3)}} \frac{\delta \mathcal{L}}{\delta e^{(3)}}.
\]

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi^{(3)}} = \frac{\delta \mathcal{L}}{\delta \xi^{(1)}} - ad_{e^{(2)}} \frac{\delta \mathcal{L}}{\delta \xi^{(2)}} - ad_{e^{(3)}} \frac{\delta \mathcal{L}}{\delta \xi^{(3)}}.
\]

It is possible to write the equations (4.9) as a single equation by substituting the second equation into the first one. This reads the following formulation of the trivialized Euler-Lagrange equations

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi^{(2)}} = T^{e} L_{g} \frac{\delta \mathcal{L}}{\delta g} - ad_{e^{(1)}} \left( \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta e^{(1)}} + \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta e^{(2)}} + \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta e^{(3)}} \right) - ad_{e^{(2)}} \frac{\delta \mathcal{L}}{\delta e^{(2)}} - ad_{e^{(3)}} \frac{\delta \mathcal{L}}{\delta e^{(3)}}.
\]

Reductions. Recall the decomposition of TTG presented in Proposition 3.3. In accordance with this, being a subgroup, \( G \) has a well defined action on TTG. So that, if the Lagrangian is independent of the group variable \( \mathcal{L} = \mathcal{L} \left( e^{(1)}, e^{(2)}, e^{(3)} \right) \), we arrive at the equation

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi^{(2)}} = -ad_{e^{(1)}} \left( \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta e^{(1)}} + \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta e^{(2)}} + \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta e^{(3)}} \right) - ad_{e^{(2)}} \frac{\delta \mathcal{L}}{\delta e^{(2)}} - ad_{e^{(3)}} \frac{\delta \mathcal{L}}{\delta e^{(3)}}
\]

on the extended Lie group \( (g_{1} \times g_{2}) \ltimes g_{3} \) given in (3.22). If the Lagrangian \( \mathcal{L} \) is independent of the Lie algebra variable \( e^{(1)} \), that is for \( \mathcal{L} = \mathcal{L} \left( g, e^{(2)}, e^{(3)} \right) \) we obtain the equations

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi^{(2)}} = -ad_{e^{(2)}} \frac{\delta \mathcal{L}}{\delta e^{(2)}} - ad_{e^{(3)}} \frac{\delta \mathcal{L}}{\delta e^{(3)}}.
\]

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi^{(3)}} = -ad_{e^{(2)}} \frac{\delta \mathcal{L}}{\delta e^{(2)}} - ad_{e^{(3)}} \frac{\delta \mathcal{L}}{\delta e^{(3)}}.
\]
Further reduction is possible if the Lagrangian does not depend on the group variable either. This reads Euler-Poincaré equations on $\mathfrak{g}_2 \ltimes \mathfrak{g}_3$.

**Proposition 4.2.** The Euler-Poincaré equations generated by a Lagrangian function(al) $\mathcal{L} = \mathcal{L}(\xi^{(2)}, \xi^{(3)})$ defined on the Lie algebra $\mathfrak{g}_2 \ltimes \mathfrak{g}_3$ of the group $G \ltimes \mathfrak{g}_1$ are

\begin{align}
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi^{(2)}} &= -ad_{\xi^{(3)}}^* \frac{\delta \mathcal{L}}{\delta \xi^{(2)}} - ad_{\xi^{(3)}}^* \frac{\delta \mathcal{L}}{\delta \xi^{(3)}}
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi^{(3)}} &= -ad_{\xi^{(2)}}^* \frac{\delta \mathcal{L}}{\delta \xi^{(3)}}
\end{align}

for $\left(\xi^{(2)}, \xi^{(3)}\right) \in \mathfrak{g}_2 \ltimes \mathfrak{g}_3$.

**Proof.** There are several ways to arrive at this Euler-Poincaré equations. The first is to consider $TTG$ as a semidirect product of the group $G \ltimes \mathfrak{g}_1$ and the Lie algebra $\mathfrak{g}_2 \ltimes \mathfrak{g}_3$, and use directly the reduced variational principle in (4.8) for a Lagrangian function $\mathcal{L} = \mathcal{L}(\xi^{(2)}, \xi^{(3)})$. The second way is to reduce the Euler-Lagrange dynamics (4.12) under the invariance of $G$. This is called reduction by stages in [11, 37]. Note finally, that the dependence $\mathcal{L} = l(\xi^{(2)})$ reduces all Lagrangian dynamics of this subsection to the Euler-Poincaré equation (4.4) on $\mathfrak{g}$.

\[\square\]

### 4.3. Second order Euler-Lagrange equations.

Recalling the discussions done in Subsection 3.6 in order to arrive the trivialized second order Euler-Lagrange equations on $T^2G$, we first address the dynamics (4.12) with the labellings $\xi^{(2)} = \xi$ and $\xi^{(3)} = \dot{\xi}$. We refer [2, 14, 15, 17, 31] on the second order Lagrangian dynamics on Lie groups.

**Proposition 4.3.** On the image space of canonical immersion $T^2G$ as a cocycle extension $(G \ltimes \mathfrak{g}) \ltimes \dot{\mathfrak{g}}$ given in (3.31), the trivialized second order Euler-Lagrange equation is

\begin{align}
\left(\frac{d}{dt} + ad_{\xi}^*\right) \left( \frac{\delta \mathcal{L}}{\delta \xi} - \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\xi}} \right) &= T^*L_{\mathfrak{g}_1} \frac{\delta \mathcal{L}}{\delta \mathfrak{g}_1}
\end{align}

If the Lagrangian $\mathcal{L} = \mathcal{L}(\xi, \dot{\xi})$ does not depend on the group variable, we arrive at the second order Euler-Poincaré equations

\begin{align}
\left(\frac{d}{dt} + ad_{\xi}^*\right) \left( \frac{\delta \mathcal{L}}{\delta \xi} - \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\xi}} \right) &= 0,
\end{align}

on the reduced space $\mathfrak{g} \times \dot{\mathfrak{g}}$.

**Proof.** The choices $\xi^{(2)} = \xi$ and $\xi^{(3)} = \dot{\xi}$ reduce the trivialized Euler-Lagrange equations (4.12) to the set of equations

\begin{align}
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} &= T^*L_{\mathfrak{g}_1} \frac{\delta \mathcal{L}}{\delta \mathfrak{g}_1} + ad_{\xi}^* \frac{\delta \mathcal{L}}{\delta \xi} + ad_{\xi} \frac{\delta \mathcal{L}}{\delta \xi},
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\xi}} &= -ad_{\xi} \frac{\delta \mathcal{L}}{\delta \xi}
\end{align}
on the image space of the immersion in TTG. To replace the last term on the right hand side of the first equation we proceed to compute

\[
\begin{align*}
\frac{ad^*}{dt} \frac{\delta L}{\delta \xi} &= \frac{d}{dt} \left( \frac{ad^*}{dt} \frac{\delta L}{\delta \xi} \right) - \frac{ad^*}{dt} \frac{d}{dt} \left( \frac{\delta L}{\delta \xi} \right) \\
&= -\frac{d^2}{dt^2} \left( \frac{\delta L}{\delta \xi} \right) - \frac{ad^*}{dt} \frac{d}{dt} \left( \frac{\delta L}{\delta \xi} \right) \\
&= -\left( \frac{d}{dt} + ad^* \right) \frac{d}{dt} \left( \frac{\delta L}{\delta \xi} \right)
\end{align*}
\]

(4.17)

where we used the second equation. After rearranging terms and substituting the identity in Eq.(4.17), the first equation in (4.16) reads

\[
\begin{align*}
\frac{d}{dt} \frac{\delta L}{\delta \xi} + \frac{ad^*}{dt} \frac{\delta L}{\delta \xi} &= T^* \mathcal{L}_g \frac{\delta L}{\delta g} + \left( \frac{d}{dt} + ad^* \right) \frac{d}{dt} \left( \frac{\delta L}{\delta \xi} \right) \\
&= -\frac{d^2}{dt^2} \left( \frac{\delta L}{\delta \xi} \right) - \frac{ad^*}{dt} \frac{d}{dt} \left( \frac{\delta L}{\delta \xi} \right) \\
&= -\left( \frac{d}{dt} + ad^* \right) \frac{d}{dt} \left( \frac{\delta L}{\delta \xi} \right)
\end{align*}
\]

Alternatively, we may derive the second order Euler-Poincaré equations directly from the Euler-Poincaré equations (4.13) on the semidirect product Lie algebra \( g_2 \ltimes g_3 \) by choosing \( \xi^{(2)} = \xi \) and \( \xi^{(3)} = \dot{\xi} \).

We summarize the discussions done in Subsections [4.1, 4.2 and 4.3] in the following diagram. Here, EPR. will denote the Euler-Poincaré reductions.

\[
\begin{array}{ccc}
TTG & \xrightarrow{\text{EL in }(4.10)} & (G \ltimes g) \xrightarrow{\varphi} \hat{g} \\
& \xrightarrow{\text{2nd EL in }(4.14)} & G \ltimes g \xrightarrow{\text{EL in }(4.15)} \\
& \xrightarrow{\text{EPR.}} & (g_1 \times g_2) \ltimes h_\chi g_3 \\
(4.18) & \xrightarrow{\text{EPR.}} & g \times \hat{g} \xrightarrow{\text{2nd EP in }(4.15)} g \xrightarrow{\text{EP in }(4.13)}
\end{array}
\]

4.4. Matched first order Lagrangian dynamics. Let us consider the tangent bundle \( T(G \ltimes H) \), with its trivialization \( (G \ltimes H) \ltimes (g \ltimes h) \), and consider a Lagrangian function \( \mathcal{L} \). We refer [28] for a detailed discussion on the matched Lagrangian dynamics of the first order. The matched Euler-Lagrange equations generated by \( \mathcal{L} = \mathcal{L}(g, h, \xi, \eta) \) is computed to be

\[
\begin{align*}
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi} &= T^* \mathcal{L}_g \frac{\delta \mathcal{L}}{\delta g} + \left( \frac{d}{dt} + ad^* \right) \frac{d}{dt} \left( \frac{\delta \mathcal{L}}{\delta \xi} \right) \\
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \eta} &= T^* \mathcal{L}_h \frac{\delta \mathcal{L}}{\delta h} + \left( \frac{d}{dt} + ad^* \right) \frac{d}{dt} \left( \frac{\delta \mathcal{L}}{\delta \eta} \right)
\end{align*}
\]

(4.19)

where the dual operations are those stated in (2.18)-(2.23), and the mapping \( \sigma_h : G \to H \) is defined by \( g \to h \ltimes g \).
If Lagrangian function is independent of the group variables, that is \( \mathcal{L} = \mathcal{L}(\xi, \eta) \), then the matched Euler-Lagrange equations reduce to the matched Euler-Poincaré equations \( g \ltimes h \) given by

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\xi}} = -\text{ad}^*_\xi \frac{\delta \mathcal{L}}{\delta \xi} + \frac{\delta \mathcal{L}}{\delta \xi} \cdot \eta + \frac{\delta \mathcal{L}}{\delta \eta} \cdot \dot{\eta},
\]

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\eta}} = -\text{ad}^*_\eta \frac{\delta \mathcal{L}}{\delta \eta} - \frac{\delta \mathcal{L}}{\delta \xi} \cdot \dot{\xi} + \frac{\delta \mathcal{L}}{\delta \dot{\eta}} \cdot \dot{\eta}.
\]

(4.20)

Here, the first terms on the right hand sides of the equations (4.20) are the individual Euler-Poincaré motions on the Lie algebras \( g \) and \( h \), respectively. The rest are the manifestations of the mutual actions. So that, if the action of \( h \) on \( g \) is trivial, the matched pair Lie algebra reduces to a semi-direct product Lie algebra \( g \ltimes h \), and the semi-direct product Euler-Poincaré equations

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\xi}} = -\text{ad}^*_\xi \frac{\delta \mathcal{L}}{\delta \xi} + \frac{\delta \mathcal{L}}{\delta \xi} \cdot \eta - \xi \cdot \frac{\delta \mathcal{L}}{\delta \eta} - b^*_\xi \frac{\delta \mathcal{L}}{\delta \xi},
\]

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\eta}} = -\text{ad}^*_\eta \frac{\delta \mathcal{L}}{\delta \eta} - \xi \cdot \frac{\delta \mathcal{L}}{\delta \xi} - b^*_\eta \frac{\delta \mathcal{L}}{\delta \xi}.
\]

are obtained. If only the action of \( g \) on \( h \) is taken to be trivial, then matched pair Lie algebra reduces to \( g \ltimes h \).

In this case, the Euler-Poincaré dynamics is turned out to be

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\xi}} = -\text{ad}^*_\xi \frac{\delta \mathcal{L}}{\delta \xi} + \frac{\delta \mathcal{L}}{\delta \xi} \cdot \eta,
\]

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\eta}} = -\text{ad}^*_\eta \frac{\delta \mathcal{L}}{\delta \eta} - \xi \cdot \frac{\delta \mathcal{L}}{\delta \xi} - b^*_\eta \frac{\delta \mathcal{L}}{\delta \xi}.
\]

If, both of the actions are trivial, we arrive at the Euler-Poincaré dynamics on the direct product \( g \times h \) given by

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\xi}} = -\text{ad}^*_\xi \frac{\delta \mathcal{L}}{\delta \xi},
\]

\[
\frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \dot{\eta}} = -\text{ad}^*_\eta \frac{\delta \mathcal{L}}{\delta \eta}.
\]

4.5. Matched second order Euler-Lagrange equations. In this section, we shall elaborate the problem of matching two 2nd order Euler-Lagrange and Euler-Poincaré systems under mutual interaction. For this, first recall 2nd order Euler-Lagrange equation (4.14) defined on \( T^2G \) and 2nd order Euler-Poincaré equation (4.15) defined on \( g \ltimes h \). Let us now consider a Lagrangian function \( \mathcal{L} = \mathcal{L}(g, h, \xi, \eta, \xi, \eta) \) on the second order tangent bundle \( T^2(G \ltimes H) \) of the matched pair Lie group \( G \ltimes H \). A direct calculation reads the following proposition.

**Proposition 4.4.** Matched 2nd order Euler-Lagrange equations, generated by a Lagrangian function \( \mathcal{L} = \mathcal{L}(g, h, \xi, \eta, \xi, \eta) \), defined on \( T^2(G \ltimes H) \) are computed to be

\[
\begin{align*}
\left( \frac{d}{dt} + \text{ad}^*_\xi \eta \right) (D_{\xi} \mathcal{L}) & - (D_{\xi} \mathcal{L}) \cdot \dot{\xi} \cdot \eta - \text{ad}^*_\eta \mathcal{L}(D_{\eta} \mathcal{L}) = T^*_{\xi G} L_g \left( \frac{\delta \mathcal{L}}{\delta g} \right) \cdot h + T^*_{\xi G} \mathcal{L}_h \left( \frac{\delta \mathcal{L}}{\delta h} \right), \\
\left( \frac{d}{dt} + \text{ad}^*_\eta \mathcal{L} \right) (D_{\eta} \mathcal{L}) & + \dot{\xi} \cdot (D_{\eta} \mathcal{L}) + b^*_\xi \mathcal{L}(D_{\xi} \mathcal{L}) = T^*_{\eta H} L_h \left( \frac{\delta \mathcal{L}}{\delta h} \right).
\end{align*}
\]

(4.21)

where the dual operations are those stated in (2.18)-(2.23), and the mapping \( \sigma_h : G \rightarrow H \) is defined by \( g \rightarrow h \ltimes g \), whereas we use the abbreviations

\[
D_{\xi} \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \xi} - \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \xi}, \quad D_{\eta} \mathcal{L} = \frac{\delta \mathcal{L}}{\delta \eta} - \frac{d}{dt} \frac{\delta \mathcal{L}}{\delta \eta}.
\]

**Proof.** Starting with what we have already obtained in Proposition 4.3 namely

\[
\left( \frac{d}{dt} + \text{ad}^*_\xi \eta \right) \left( \left( \frac{\delta \mathcal{L}}{\delta \xi} \frac{\delta \mathcal{L}}{\delta \eta} - \frac{d}{dt} \left( \frac{\delta \mathcal{L}}{\delta \dot{\xi}} \frac{\delta \mathcal{L}}{\delta \dot{\eta}} \right) \right) = T^* L_{(g, h)} \left( \frac{\delta L}{\delta g} \frac{\delta L}{\delta h} \right),
\]

the claim follows from (2.24) and (28 (3.6)).
It is immediate now to observe that if Lagrangian does not depend on \((\dot{\xi}, \dot{\eta})\), that is it is a first order Lagrangian then matched 2nd order Euler-Lagrange equations \((4.21)\) turns out to be matched Euler-Lagrange equations \((4.19)\). Several reductions of the system \((4.21)\) are available under the existence of group symmetries. The most important one is the symmetry due to the action of the group \(G \ltimes H\) on \(T^2(G \rightleftarrows H)\). In this case, one arrives the dynamics on \((g \times \hat{g}) \ltimes (h \times \hat{h})\). This occurs if the Lagrangian is invariant under the group action. Accordingly, we arrive at matched second order Euler-Poincaré equations generated by \(L = L(\xi, \eta, \dot{\xi}, \dot{\eta})\). Obeying the notations presented in Proposition \((4.4)\), we record this result in the following Corollary.

**Corollary 4.5.** Matched 2nd order Euler-Poincaré equations, generated by a Lagrangian function \(L = L(\xi, \eta, \dot{\xi}, \dot{\eta})\) defined on \((g \times \hat{g}) \ltimes (h \times \hat{h})\), are computed to be

\[
\left(\frac{d}{dt} + ad^r_\xi\right)(D_\xi L) - \left(D_\xi L\right)^* \triangleleft \eta - a^*_\eta (D_\eta L) = 0, \\
\left(\frac{d}{dt} + ad^r_\eta\right)(D_\eta L) + \xi \triangleright (D_\eta L) + b^*_\xi (D_\xi L) = 0,
\]

where we use the notations and abbreviations exhibited in Proposition \((4.4)\). Some particular cases of the system \((4.22)\) are of great interest in the semi-direct product theory. See that if the action \(G\) on \(H\) is trivial, then matched pair product turns out to be a semi-direct product \((h \times \hat{h}) \ltimes (g \times \hat{g})\) as such the equations of motions reduce to

\[
\left(\frac{d}{dt} + ad^r_\xi\right)(D_\xi L) - a^*_\eta (D_\eta L) = 0, \\
\left(\frac{d}{dt} + ad^r_\eta\right)(D_\eta L) + \xi \triangleright (D_\eta L) = 0,
\]

where we obey the notations in \((4.4)\). Similarly, if this time the action of \(H\) on \(G\) is trivial, then matched 2nd order Euler-Poincaré equations \((4.22)\) becomes semi-direct product theory on \((h \times \hat{h}) \ltimes (g \times \hat{g})\). In this case the dynamics is given by

\[
\left(\frac{d}{dt} + ad^r_\xi\right)(D_\xi L) - \triangleleft \eta (D_\eta L) = 0, \\
\left(\frac{d}{dt} + ad^r_\eta\right)(D_\eta L) + \xi \triangleright (D_\eta L) = 0.
\]

If both of the actions are trivial then we arrive at the dynamics on Cartesian product space with no mutual interaction. This reads the following set of equations

\[
\left(\frac{d}{dt} + ad^r_\xi\right)(D_\xi L) = 0, \\
\left(\frac{d}{dt} + ad^r_\eta\right)(D_\eta L) = 0.
\]
In the following diagram we summarize the equations derived in Subsections 4.4 and 4.5 and compare them with the dynamics presented in Subsections 4.1 and 4.3.

\[(4.26)\]

\[
\begin{align*}
T^2 G \\
\text{EL in (4.14)}
\end{align*}
\]

\[
\begin{align*}
T^2(G \Leftrightarrow H) \\
\text{mEL in (4.21)}
\end{align*}
\]

\[
\begin{align*}
\mathfrak{g} \times \dot{\mathfrak{g}} \\
\text{EP in (4.15)}
\end{align*}
\]

\[
\begin{align*}
\mathfrak{g} \Rightarrow \mathfrak{h} \\
\text{mEP in (4.20)}
\end{align*}
\]

\[
\begin{align*}
\mathfrak{g} \times \dot{\mathfrak{g}} \Leftrightarrow (\mathfrak{h} \times \dot{\mathfrak{h}}) \\
\text{mEP in (4.22)}
\end{align*}
\]

\[
\begin{align*}
\mathfrak{g} \\
\text{EP in (4.4)}
\end{align*}
\]

\[
\begin{align*}
\mathfrak{g} \Rightarrow \mathfrak{h} \\
\text{mEP in (4.20)}
\end{align*}
\]
5. Matched Discrete (Second Order) Lagrangian Dynamics

This section is reserved for the (two step) discrete dynamics on the Lie groups, and to the coupling problem. We first recall the discrete dynamics on the Lie group level then we present the discrete Lagrangian dynamics on the tangent group $T G$ and the second order tangent group $T^2 G$. We also address matching problems of these systems. Following our previous work [29], we exhibit discrete Euler-Lagrange equations on the matched pair group $G \rtimes H$, then extend this discussion to the matched tangent groups $T G \rtimes TH$ and $T^2 G \rtimes T^2 H$.

5.1. Discrete dynamics on Lie groups. Let us recall basics of discrete dynamics on Lie groups [49, Subsect. 4.1]. A discrete Lagrangian function is a real valued function $L$ defined on a Lie group $G$. The discrete action sum associated to it is given by

$$
S_L : G^N \to \mathbb{R},
\quad (g_1, \ldots, g_N) \mapsto \sum_{k=1}^N L(g_k),
$$

where $G^N$ is the Cartesian product of $G$ by itself $N$ times. Now the discrete Hamilton’s principle may be recalled, from [74], as follows. A sequence $(g_1, \ldots, g_N)$ is a solution of the Lagrangian system if and only if it is a critical point of (5.1). So, one arrives at the discrete Euler-Lagrange equations

$$
\sum_{k=1}^{N-1} \left[ \vec{\xi}_k(g_k)(L) - \xi_{k+1}(g_k+1)(L) \right] = 0,
$$

where $\vec{\xi}$ is the left invariant vector field generated by $\xi$ in $g$ whereas $\xi$ is the right invariant vector field. In particular, for $N = 2$, the discrete Euler-Lagrange equations are given by

$$
\vec{\xi}(g_k)(L) - \vec{\xi}(g_{k+1})(L) = 0
$$

for every $\xi$ in $g$. The left hand side of this equation can be written as

$$
\langle dL(g_k), \vec{\xi}(g_k) \rangle - \langle dL(g_{k+1}), \vec{\xi}(g_{k+1}) \rangle = \langle dL(g_k), T_e L_{g_k} \xi \rangle - \langle dL(g_{k+1}), T_e R_{g_{k+1}} \xi \rangle
= \langle T_e^* L_{g_k} dL(g_k) - T_e^* R_{g_{k+1}} dL(g_{k+1}), \xi \rangle,
$$

for any $\xi \in g$, and any $g_k, g_{k+1} \in G$. As such, the discrete Euler-Lagrange equations may be written by

$$
T_e^* L_{g_k} dL(g_k) - T_e^* R_{g_{k+1}} dL(g_{k+1}) = 0.
$$

In the light of these computations, and using the coadjoint action, we record the discrete Euler-Lagrange equations in the following Proposition, [6, 49, 52, 53]. It has fundamental importance for the rest of the section.

Proposition 5.1. The discrete Euler-Lagrange equations generated by a Lagrangian function $L$ on a Lie group $G$ is

$$
\mu_{k+1} = Ad_{g_k}^* \mu_k, \quad \mu_k = T_e R_{g_k} dL(g_k).
$$
5.2. Discrete dynamics on tangent groups.

In this case, we present the discrete Lagrangian dynamics (where \( N = 2 \)) on the tangent group \( T G \) identified with its left trivialization \( G \ltimes g \). For this, consider a Lagrangian function \( \mathcal{L} \) defined on the semi-direct group \( G \ltimes g \). We directly mimic the equation (5.4). For this aim, we replace the group element in \( G \) and the dual element in \( g^* \) with a pair group element in \( G \ltimes g \) and a pair of dual elements in the dual of the Lie algebra of \( G \ltimes g \), respectively. The Lie algebra of the group \( G \ltimes g \) is the semi-direct product Lie algebra \( g_1 \ltimes g_2 \). So that the dual of this space is \( g_1^* \times g_2^* \). An element of the dual space is given by a two-tuple \((\mu^1, \mu^2)\). The following Proposition determines the dynamical equations. It is easy to see that it reduces to Proposition 5.1 if the Lagrangian function \( \mathcal{L} \) does not depend on the Lie algebra variable \( \xi \).

**Proposition 5.2.** The discrete Euler-Lagrange equations on the group \( T G \), identified with its trivialization \( G \ltimes g \), generated by a Lagrangian function \( \mathcal{L} = \mathcal{L}(g, \xi) \) is

\[
\begin{aligned}
\mu^{(1)}_{k+1} &= \text{Ad}^*_{g_{k+1}^{-1}} \mu^{(1)}_k - \text{ad}^*_{\xi_k} \text{Ad}^*_{g_{k+1}^{-1}} \mu^{(2)}_k, \\
\mu^{(2)}_{k+1} &= \text{Ad}^*_{g_{k+1}^{-1}} \mu^{(2)}_k
\end{aligned}
\]

along with the following definitions

\[
\mu^{(1)}_k = T^*e R_{g_k} \frac{\delta \mathcal{L}}{\delta g_k}, \quad \mu^{(2)}_k = \text{Ad}^* g_k \frac{\delta \mathcal{L}}{\delta \xi_k}.
\]

**Proof.** Recall the discrete equation (5.4). Replace the group element \( g_k \) and the dual element \( \mu_k \) with \((g_k, \xi_k)\) in \( G \ltimes g \) and \((\mu^{(1)}_k, \mu^{(2)}_k)\) in \( g_1^* \times g_2^* \), respectively. So that we write (5.4) as

\[
(\mu^{(1)}_{k+1}, \mu^{(2)}_{k+1}) = \text{Ad}^*_{(g_{k+1}, \xi_{k+1})} (\mu^{(1)}_k, \mu^{(2)}_k), \quad (\mu^{(1)}_k, \mu^{(2)}_k) = T^*_{(e,0)} R_{(g_k, \xi_k)} \cdot d\mathcal{L}(g_k, \xi_k).
\]

Let us now compute the terms in this equation in most explicit way. For this we start with the coadjoint action of \( G \ltimes g \) on the dual space, and the cotangent lift of the right translation as

\[
\text{Ad}^*_{(g_k, \xi_k)} (\mu^{(1)}_k, \mu^{(2)}_k) = \left( \text{Ad}^*_{g_k^{-1}} \mu^{(1)}_k - \text{ad}^*_{\xi_k} \text{Ad}^*_{g_k^{-1}} \mu^{(2)}_k, \text{Ad}^*_{g_k^{-1}} \mu^{(2)}_k \right),
\]

\[
T^*_{(e,0)} R_{(g_k, \xi_k)} \Theta_{g_k \xi_k} = \left( T^* R_{g_k} \Theta_{g_k}, \text{Ad}^*_{g_k} \Theta_{\xi_k} \right),
\]

respectively. Here, \((\Theta_{g_k}, \Theta_{\xi_k})\) is in the cotangent space \( T^*_{(g_k, \xi_k)} (G \ltimes g) \). Eventually, we arrive at the following equations

\[
\mu^{(1)}_{k+1} = \text{Ad}^*_{g_{k+1}^{-1}} \mu^{(1)}_k - \text{ad}^*_{\xi_k} \text{Ad}^*_{g_{k+1}^{-1}} \mu^{(2)}_k, \quad \mu^{(2)}_{k+1} = \text{Ad}^*_{g_{k+1}^{-1}} \mu^{(2)}_k,
\]

with the identifications (5.6). The proof of Proposition follows if the second equation is substituted into the first one. \(\square\)

5.3. Discrete dynamics on second tangent groups.

In this case, as a particular case, we study the discrete dynamics on the second order tangent bundle \( T^2G \). As always, we identify \( T^2G \) with its left trivialization \((G \ltimes g) \ltimes \hat{g} \) then, accordingly, consider the group operation on it given in (5.30). In this case, we assume that the discrete dynamics is generated by a Lagrangian function is defined on group \((G \ltimes g) \ltimes \hat{g} \). Lie algebra of \( T^2G \) is given by a three-tuple \( g_1^* \times g_2^* \times g_3^* \). Accordingly, the dual space is assumed to be \( g_1^* \times g_2^* \times g_3^* \). Following the notations of this present work, the
following Proposition states the discrete dynamical equations on the second order tangent group $T^2G$. It is needless to say that the dynamical equation in the following proposition reduces to the ones in Proposition 5.2 if the Lagrangian function $\mathcal{L}$ does not depend on the second Lie algebra element $\dot{\xi}$.

**Proposition 5.3.** Consider the second order tangent bundle group $T^2G$ with operation (3.30). A Lagrangian function $\mathcal{L}$ generates the discrete Euler-Lagrange equations on $T^2G$ as

$$
\begin{align*}
\mu^{(1)}_{k+1} &= \text{Ad}^*_{g_k} \mu^{(1)}_k - \text{ad}_{\xi_k}^* \text{Ad}^*_{g_k^{-1}} \mu^{(2)}_k - \text{ad}_{\xi_k}^* \text{Ad}^*_{g_k^{-1}} \mu^{(3)}_k + \left(\text{ad}_{\xi_k}^*\right)^2 \text{Ad}^*_{g_k^{-1}} \mu^{(3)}_k, \\
\mu^{(2)}_{k+1} &= \text{Ad}^*_{g_k} \mu^{(2)}_k - 2 \text{ad}_{\xi_k}^* \text{Ad}^*_{g_k^{-1}} \mu^{(3)}_k, \\
\mu^{(3)}_{k+1} &= \text{Ad}^*_{g_k} \mu^{(3)}_k,
\end{align*}
$$

with the following abbreviations

$$
\mu^{(1)}_k = T_{\xi_k, \xi_k} \frac{\delta \mathcal{L}}{\delta g_k}, \quad \mu^{(2)}_k = \text{Ad}^*_{g_k} \frac{\delta \mathcal{L}}{\delta \xi_k} + \text{Ad}_{g_k} \text{ad}_{\xi_k}^* \frac{\delta \mathcal{L}}{\delta \xi_k}, \quad \mu^{(3)}_k = \text{Ad}^*_{g_k} \frac{\delta \mathcal{L}}{\delta \xi_k}.
$$

**Proof.** We once more recall Proposition 5.1 and the discrete Euler-Lagrange equations (5.4) given for an arbitrary Lie group $G$. In this equation, we replace the group element $g_k$ and the dual element $\mu_k$ with a group element $(g_k, \xi_k, \dot{\xi}_k)$ and a dual element $(\mu^{(1)}_k, \mu^{(2)}_k, \mu^{(3)}_k)$, respectively. In a closed form, (5.4) can be written as

$$
\left(\mu^{(1)}_{k+1}, \mu^{(2)}_{k+1}, \mu^{(3)}_{k+1}\right) = \text{Ad}^*_{(g_k, \xi_k, \dot{\xi}_k)^{-1}} \left(\mu^{(1)}_k, \mu^{(2)}_k, \mu^{(3)}_k\right),
$$

$$
\left(\mu^{(1)}_k, \mu^{(2)}_k, \mu^{(3)}_k\right) = T_{(\xi_k, \dot{\xi}_k)} \cdot d\mathcal{L}(g_k, \xi_k, \dot{\xi}_k).
$$

In order to write these equations in most explicit form, we first compute the coadjoint action of the group $T^2G$ to the dual of its Lie algebra. Define a differentiable curve $\left(\gamma_k(t), \xi_k(t), \dot{\xi}_k(t)\right) \in T^2G$ passes through $(g, 0, 0) \in T^2G$, and

$$
\frac{d}{dt} \bigg|_{t=0} \left(\gamma_k(t), \xi_k(t), \dot{\xi}_k(t)\right) = (\xi^{(1)}_k, \xi^{(2)}_k, \xi^{(3)}_k).
$$

As for the adjoint action we compute the following

$$
\text{Ad}_{(g_k, \xi_k, \dot{\xi}_k)} \left(\xi^{(1)}_k, \xi^{(2)}_k, \xi^{(3)}_k\right) = \frac{d}{dt} \bigg|_{t=0} \left(\xi_k(t), \dot{\xi}_k(t), \xi_k(t)\right) = \left(\text{Ad}_{g_k} \xi^{(1)}_k, \text{Ad}_{g_k} \xi^{(2)}_k + \text{Ad}_{g_k} \text{ad}_{\xi_k} \xi^{(3)}_k, \right.
\left.\text{Ad}_{g_k} \xi^{(3)}_k + \text{Ad}_{g_k} \text{ad}_{\xi_k} \xi^{(1)}_k + 2 \text{Ad}_{g_k} \text{ad}_{\xi_k} \xi^{(2)}_k + \text{Ad}_{g_k} \text{ad}_{\xi_k} \xi^{(1)}_k \right).
$$

Obeying the definition of the coadjoint action in (1.11), we can write

$$
\left(\text{Ad}^*_{(g_k, \xi_k, \dot{\xi}_k)^{-1}} \left(\mu^{(1)}_{k+1}, \mu^{(2)}_{k+1}, \mu^{(3)}_{k+1}\right), \left(\xi^{(1)}_k, \xi^{(2)}_k, \xi^{(3)}_k\right)\right) = \left(\mu^{(1)}_{k+1}, \mu^{(2)}_{k+1}, \mu^{(3)}_{k+1}\right), \text{Ad}_{(g_k, \xi_k, \dot{\xi}_k)} \left(\xi^{(1)}_k, \xi^{(2)}_k, \xi^{(3)}_k\right).
$$
Accordingly, a direct computation results with the following expression

\[
\text{Ad}_{g_k}^* \left( \mu^{(1)}_{k-1}, \mu^{(2)}_{k-1}, \mu^{(3)}_k \right) = \left( \text{Ad}_{g_k}^* \mu^{(1)}_{k-1} - \text{ad}^*_k \text{Ad}_{g_k}^* \mu^{(2)}_{k-1} - \text{ad}^*_k \text{Ad}_{g_k}^* \mu^{(3)}_k + \text{ad}^*_k \text{Ad}_{g_k}^* \mu^{(3)}_k, \right.
\]

\[
\left. \text{Ad}_{g_k}^* \mu^{(2)}_k - 2 \text{ad}^*_k \text{Ad}_{g_k}^* \mu^{(3)}_k, \text{Ad}_{g_k}^* \mu^{(3)}_k \right) .
\]

In order to compute the transposed infinitesimal right action, we proceed similarly. We begin with the tangent lift of the right translation on the second order groupoid \( T^2 \Gamma \) given by

\[
T^*_e g_k \left( \xi^{(1)}_k, \xi^{(2)}_k, \xi^{(3)}_k \right) = \frac{d}{dt} \bigg|_{t=0} \left( \gamma^{(1)}_k(t), \gamma^{(2)}_k(t), \gamma^{(3)}_k(t) \right) \left( g_k, \xi_k, \dot{\xi}_k \right)
\]

\[
= \left( T^*_e g_k \xi^{(1)}_k, \text{Ad}_{g_k} \xi^{(2)}_k, \text{Ad}_{g_k} \xi^{(3)}_k - \text{ad} \xi_k \text{Ad}_{g_k} \xi^{(2)}_k \right)
\]

and proceed through the dualization of it determined by

\[
\left( \left( T^*_e g_k \delta \Theta^{(1)}_k, \text{Ad}_{g_k} \delta \Theta^{(2)}_k, \text{Ad}_{g_k} \delta \Theta^{(3)}_k \right) \right) = \left( \left( \delta \Theta^{(1)}_k, \frac{\delta \Theta^{(2)}_k}{\delta \xi_k}, \frac{\delta \Theta^{(3)}_k}{\delta \xi_k} \right) \right) .
\]

This reads the cotangent lift of the right translation map on the groupoid \( T^2 \Gamma \)

\[
T^*_e g_k \left( \Theta^{(1)}_k, \Theta^{(2)}_k, \Theta^{(3)}_k \right) = \left( T^*_e g_k \Theta^{(1)}_k, \text{Ad}_{g_k} \Theta^{(2)}_k + \text{Ad}_{g_k} \text{Ad}_{g_k} \Theta^{(3)}_k \right)
\]

for an arbitrary element \( (\Theta_{g_k}, \Theta_{\xi_k}, \Theta_{\dot{\xi}_k}) \) in the cotangent space \( T^*_e g_k \). As a result, the former equation of \((5.4)\) reads \((5.10)\) while the latter becomes

\[
\left( \mu^{(1)}_k, \mu^{(2)}_k, \mu^{(3)}_k \right) = T^*_e g_k \left( \frac{\delta \Theta^{(1)}_k}{\delta g_k}, \frac{\delta \Theta^{(2)}_k}{\delta \xi_k}, \frac{\delta \Theta^{(3)}_k}{\delta \xi_k} \right) .
\]

\[
(5.13)
\]

\[
\square
\]

### 5.4. Discrete dynamics on matched pair Lie groups.

In this subsection, we will exhibit discrete Lagrangian dynamics on matched pair Lie groups. These equations have been obtained as a particular case of matching of discrete dynamics on the groupoid level in \([29]\). For this end, consider a matched pair Lie group \( G \Rightarrow H \) as defined in Subsection \([21]\) and, particularly, consider the group operation \((2.2)\). We start to our investigation by resenting the tangent lifts of the left and right translations on the groupoid \( G \Rightarrow H \) given in \([28]\) \((2.54)\&\((2.55)\) as

\[
T_{(g_1, h_1)} L_{(g_2, h_2)} \left( U_{g_2}, V_{h_2} \right) = \left( T_{(h_1 g_2, h_2)} L_{g_1} (h_1 \ast U_{g_2}), T_{h_1 g_2} R_{h_2} (h_1 \ast U_{g_2}) + T_{h_2} L_{(h_1 g_2)} V_{h_2} \right),
\]

\[
T_{(g_1, h_1)} R_{(g_2, h_2)} \left( U_{g_1}, V_{h_1} \right) = \left( T_{g_1} R_{(h_1 g_2)} U_{g_1} + T_{h_1 g_2} L_{g_1} (V_{h_1} \ast g_2), T_{h_1 g_2} R_{h_2} (V_{h_1} \ast g_2) \right).
\]

We can thus compute the left and right invariant vector fields generated by a Lie algebra element \( (\xi, \eta) \in \mathfrak{g} \Rightarrow \mathfrak{h} \) as

\[
(\mathfrak{g}, \mathfrak{h}) (g, h) = T_{(g, h)} L_{(g_1, h_1)} (\xi, \eta) = (h \ast \xi (g), h \ast \xi + \dot{\eta} (h)),
\]

\[
(\mathfrak{g}, \mathfrak{h}) (g, h) = T_{(g_2, h_2)} R_{(g_1, h_1)} (\xi, \eta) = (\dot{\xi} (g) + \eta \ast g, \dot{\eta} \ast \dot{g} (h)).
\]

\[
(5.14)
\]

\[
(5.15)
\]
Recalling the discrete Euler-Lagrange equations (5.2), discrete dynamics on $G \triangleright H$ generated by a Lagrangian function $L : G \triangleright H \to \mathbb{R}$ is then given by

\[(5.16)\]

\[
\ell(\xi, \eta)(g_k, h_k)(\mathcal{U}) - \ell(\xi, \eta)(g_{k+1}, h_{k+1})(\mathcal{U}) = 0.
\]

Let now the exterior derivative of the Lagrangian $\mathcal{U} : G \triangleright H \to \mathbb{R}$ be a two-tuple $(d_1 \mathcal{U}, d_2 \mathcal{U})$, where $d_1 \mathcal{U}$ denotes the derivative with respect to group variable $g \in G$ whereas $d_2 \mathcal{U}$ denotes the derivative with respect to group variable $h \in H$. Then, in view of the left and right invariant vector fields (5.14) - (5.15), we arrive at

\[
\ell(\xi, \eta)(g_k, h_k)(\mathcal{U}) + \Delta \mathcal{U}(\mathcal{L}) - \ell(\xi, \eta)(g_{k+1}, h_{k+1})(\mathcal{U}) = 0.
\]

It is possible to single out $\xi \in g$ and $\eta \in h$ from these equations, that is,

\[
\ell(\xi, (T^* L_{g_k} \cdot d_1 \mathcal{U}(g_k, h_k)) \ast h_k + a_{g_k}^* h_k d_2 \mathcal{U}(g_k, h_k) - T^* R_{g_{k+1}} \cdot d_1 \mathcal{U}(g_{k+1}, h_{k+1}))
\]

\[
+ \eta, T^* L_{h_k} \cdot d_2 \mathcal{U}(g_k, h_k) - b_{g_{k+1}}^* d_1 \mathcal{U}(g_{k+1}, h_{k+1}) - g_{k+1} \triangleright \xi^* \mathcal{L}^* R_{h_{k+1}} \cdot d_2 \mathcal{U}(g_{k+1}, h_{k+1}) = 0,
\]

where we have used the definitions $a_{g_k}^*$ in (2.8) and $b_{g_{k+1}}^*$ in (2.11). So that we summarize the discussion done in this subsection in the following Proposition by stating the dynamical equation.

**Proposition 5.4.** In particular, taking the covectors

\[(5.17)\]

\[
T^* R_{g_k} \cdot d_1 \mathcal{U}(g_k, h_k) = \mu_k \in g^*, \quad T^* R_{h_k} \cdot d_2 \mathcal{U}(g_k, h_k) = \nu_k \in h^*,
\]

the discrete Euler-Lagrange equations on the matched pair Lie group $G \triangleright H$ can be written as

\[(5.18)\]

\[
\mu_{k+1} = \text{Ad}_{g_k}^* \mu_k \ast h_k + a_{g_k}^* h_k d_2 \mathcal{U}(g_k, h_k) - T^* R_{h_{k+1}} v_k
\]

\[
g_{k+1} \triangleright v_{k+1} = \text{Ad}_{h_k}^* v_k - b_{g_{k+1}}^* T^* R_{h_{k+1}} \mu_k.
\]

It is possible to reduce the matched discrete Lagrange equations by taking one of the actions trial. Let us depict this geometrizations. If the (right) action of $G$ on $H$ is trivial, we have the discrete Euler-Lagrange equation

\[(5.19)\]

\[
\text{Ad}_{g_k}^* \mu_k \ast h_k = \mu_{k+1}
\]

\[
\text{Ad}_{h_k}^* v_k - b_{g_{k+1}}^* d_1 \mathcal{U}(g_{k+1}, h_{k+1}) = v_{k+1},
\]

on the semidirect product Lie group $G \triangleright H$ with the definition (5.17). On the other extreme, assuming the (left) action of $H$ on $G$ to be trivial, we arrive at the equation

\[(5.20)\]

\[
\text{Ad}_{g_k}^* \mu_k \ast h_k + a_{h_k}^* h_k d_2 \mathcal{U}(g_k, h_k) = \mu_{k+1}
\]

\[
\text{Ad}_{h_k}^* v_k = g_{k+1} \triangleright v_{k+1}.
\]

on the semidirect product Lie group $G \triangleright H$ with the definition (5.17). If both actions are trivial, then the equations reduce all the way down to

\[(5.21)\]

\[
\text{Ad}_{g_k}^* \mu_k = \mu_{k+1}, \quad \text{Ad}_{h_k}^* v_k = v_{k+1},
\]

with the definition (5.17). Here is the result and the proof.
5.5. Discrete dynamics on matched tangent groups.

In this Subsection, we present discrete dynamics on a matched pair of two tangent groups. Consider \(TG\) and \(TH\), and the matched pair \(T(G \bowtie H)\) or its isomorphic copy \(TG \bowtie TH\). The latter is the proper one since, in this case, we can directly address the proposition \(5.4\) to the present case.

**Proposition 5.5.** The matched pair discrete dynamics on \(T(G \bowtie H) \equiv TG \bowtie TH\), generated by a Lagrangian \(\mathcal{L} = \mathcal{L}(g, \xi; h, \eta)\), is given by the equations

\[
\begin{align*}
\mu_{k+1}^{(1)} &= (\tau_k^{(1)} - \text{ad}^*_{\xi_k} \mu_{k}^{(2)}) \ast h_k + \left(\mu_k^{(2)} \ast h_k\right) \ast \eta_k + a_{\eta_k}^{\ast} \theta_k^{(2)} + \alpha_{h_k}^* (T^* L_{h_k}^\ast \theta_k^{(2)}) \ast \eta_k, \\
\mu_{k+1}^{(2)} &= \tau_k^{(2)} \ast h_k + \alpha_{h_k}^* (T^* R_{h_k}^{-1} \nu_k^{(1)}) + \alpha_{h_k}^* (T^* L_{h_k}^\ast \theta_k^{(2)}), \\
g_{k+1} \ast \gamma_k^{(1)} - g_k \ast \left(\xi_k \ast \gamma_k^{(2)}\right) &= \theta_k^{(1)} - \text{ad}_{\eta_k}^* \theta_k^{(2)} - b_{g_{k+1}}^* (T^* R_{h_k}^{-1} \mu_k^{(1)}) - b_{g_{k+1}}^* b_{\xi_k} \ast \theta_k^{(2)}, \\
g_{k+1} \ast \gamma_k^{(2)} &= \text{Ad}_{h_k}^* \gamma_k^{(2)} - b_{g_{k+1}}^* (T^* L_{g_{k+1}}^{-1} \tau_k^{(2)}) \ast h_k,
\end{align*}
\]

where we have used the abbreviations

\[
\begin{align*}
\mu_k^{(1)} &= T_k^* R_{g_k} \frac{\delta \mathcal{L}}{\delta g_k}, \\
\mu_k^{(2)} &= \text{Ad}_{h_k}^* \frac{\delta \mathcal{L}}{\delta \xi_k}, \\
\gamma_k^{(1)} &= T_k^* R_{h_k} \frac{\delta \mathcal{L}}{\delta h_k}, \\
\gamma_k^{(2)} &= \text{Ad}_{h_k}^* \frac{\delta \mathcal{L}}{\delta \eta_k},
\end{align*}
\]

and

\[
\begin{align*}
\tau_k^{(i)} &= \text{Ad}_{h_k}^* \mu_k^{(i)}, \\
\theta_k^{(i)} &= \text{Ad}_{h_k}^* \gamma_k^{(i)}
\end{align*}
\]

for \(i = 1, 2\).

**Proof.** Recall the matched pair discrete dynamics from \(5.17\) and \(5.18\). Accordingly, we replace \(G\) and \(H\) with \(TG\) and \(TH\), respectively. In terms of elements, this reads the replication of the group elements \(g_k \in G\), \(h_k \in H\), and the dual space elements \(\mu_k\) and \(\nu_k\) with the group elements \((g_k, \xi_k), (h_k, \eta_k)\) and the dual space elements \((\mu_k^{(1)}, \mu_k^{(2)})\) and \((\nu_k^{(1)}, \nu_k^{(2)})\), respectively. These substitutions lead us to the following equations

\[
\begin{align*}
(\mu_k^{(1)}, \mu_k^{(2)}) &= \text{Ad}_{(g_k, \xi_k)}^* \left(\mu_k^{(1)} \ast \mu_k^{(2)}\right) \ast (h_k, \eta_k) + a_{(h_k, \eta_k)}^* d_2 \mathcal{L}((g_k, \xi_k), (h_k, \eta_k)), \\
(g_{k+1}, \xi_{k+1}) \ast (\nu_k^{(1)}, \nu_k^{(2)}) &= \text{Ad}_{(h_k, \eta_k)}^* \left(\nu_k^{(1)} \ast \nu_k^{(2)}\right) - b_{(g_{k+1}, \xi_{k+1})}^* d_1 \mathcal{L}((g_{k+1}, \xi_{k+1}), (h_{k+1}, \eta_{k+1})),
\end{align*}
\]

where

\[
T^* R_{(g_k, \xi_k)} \cdot d_1 \mathcal{L}((g_k, \xi_k), (h_k, \eta_k)) = (\mu_k^{(1)}, \mu_k^{(2)}),
\]

and

\[
T^* R_{(h_k, \eta_k)} \cdot d_2 \mathcal{L}((g_k, \xi_k), (h_k, \eta_k)) = (\nu_k^{(1)}, \nu_k^{(2)}).
\]

Let us first note that

\[
\text{Ad}_{(g_k, \xi_k)}^* \left(\xi_k^{(1)} \ast \xi_k^{(2)}\right) = \left. \frac{d}{dt} \right|_{t=0} (g_k, \xi_k) \left(\gamma_k^{(1)}(t), t \xi_k^{(2)}(t)\right) \left(g_k^{-1} \cdot - \text{Ad}_{g_k} \xi_k\right) =
\]

\[
\text{Ad}_{g_k} \xi_k^{(1)} + \text{Ad}_{g_k} \xi_k^{(2)} + \text{Ad}_{g_k} \text{ad}_{\xi_k} \xi_k^{(1)}.
\]

Thus, from the pairing

\[
\left\langle \text{Ad}_{(g_k, \xi_k)}^* (\mu_k^{(1)}, \mu_k^{(2)}), (\xi_k^{(1)} \ast \xi_k^{(2)}) \right\rangle = \left\langle (\mu_k^{(1)}, \mu_k^{(2)}), \text{Ad}_{(g_k, \xi_k)} (\xi_k^{(1)}, \xi_k^{(2)}) \right\rangle
\]
we obtain
\[ \text{Ad}^*_{(g_k, \xi_k)}(\mu^{(1)}_k, \mu^{(2)}_k) = \left( \text{Ad}^*_{g_k} \mu^{(1)}_k, \text{Ad}^*_{\xi_k} \mu^{(2)}_k, \text{Ad}^*_{g_k} \mu^{(1)}_k \right). \]
As for the transposed right action we proceed through
\[ \left( (\mu^{(1)}_k, \mu^{(2)}_k) \ast (h_k, \eta_k), (\xi^{(1)}_k, \xi^{(2)}_k) \right) = \left( (\mu^{(1)}_k, \mu^{(2)}_k), (h_k, \eta_k) \ast (\xi^{(1)}_k, \xi^{(2)}_k) \right), \]
where
\[ (h_k, \eta_k) \ast (\xi^{(1)}_k, \xi^{(2)}_k) = \frac{d}{dt} \bigg|_{t=0} (h_k, \eta_k) \ast (\gamma^{(1)}(t), t \xi^{(2)}_k) = (h_k \ast \xi^{(1)}_k, h_k \ast \xi^{(2)}_k + h_k \ast (\eta_k \ast \xi^{(1)}_k)). \]
As such,
\[ (\mu^{(1)}_k, \mu^{(2)}_k) \ast (h_k, \eta_k) = (\mu^{(1)}_k \ast h_k + (\mu^{(2)}_k \ast h_k)) \ast \eta_k, (\mu^{(2)}_k \ast h_k), \]
and hence
\[ \text{Ad}^*_{(g_k, \xi_k)}(\mu^{(1)}_k, \mu^{(2)}_k) \ast (h_k, \eta_k) = ((\text{Ad}^*_{g_k}) \mu^{(1)}_k - \text{ad}^*_{\xi_k} \text{Ad}^*_{\mu^{(2)}_k} \ast h_k + (\text{Ad}^*_{g_k}) \text{Ad}^*_{h_k} \text{Ad}^*_{(g_k, \xi_k)}(\mu^{(1)}_k, \mu^{(2)}_k) \ast h_k). \]
Finally, the pairing
\[ \left( \alpha_{(h_k, \eta_k)}(\Theta^{(1)}_k, \Theta^{(2)}_k), (\xi^{(1)}_k, \xi^{(2)}_k) \right) = \left( (\Theta^{(1)}_k, \Theta^{(2)}_k), \alpha_{(h_k, \eta_k)}(\xi^{(1)}_k, \xi^{(2)}_k) \right) = \left( (\Theta^{(1)}_k, \Theta^{(2)}_k), (h_k, \eta_k) \ast (\xi^{(1)}_k, \xi^{(2)}_k) \right) \]
and the calculation
\[ (h_k, \eta_k) \ast (\xi^{(1)}_k, \xi^{(2)}_k) = \frac{d}{dt} \bigg|_{t=0} (h_k, \eta_k) \ast (\gamma^{(1)}(t), t \xi^{(2)}_k) = (h_k \ast \xi^{(2)}_k, \eta_k \ast \xi^{(1)}_k + T_{h_k} L_{\eta_k}^{-1}(h_k \ast (\xi^{(2)}_k + (\eta_k \ast \xi^{(1)}_k))) \bigg) \]
yields
\[ \alpha_{(h_k, \eta_k)}(\Theta^{(1)}_k, \Theta^{(2)}_k) = \left( \alpha_{\eta_k}(\Theta^{(2)}_k) + \alpha_{h_k}(T_{h_k} L_{\eta_k}^{-1}(\Theta^{(2)}_k)) \ast \eta_k, \alpha_{h_k}(\Theta^{(1)}_k) + \alpha_{h_k}(T_{h_k} L_{\eta_k}^{-1}(\Theta^{(2)}_k)) \right). \]
As a result, in view of
\[ (\Theta^{(1)}_{k+1}, \Theta^{(2)}_{k+1}) = d_2 \Theta((g_k, \xi_k), (h_k, \eta_k)) = T^{(c, 0)} R_{h_k, \eta_k}^{-1} \left( \gamma^{(1)}_k, \gamma^{(2)}_k \right) = \left( T^{(c, 0)} R_{h_k, \eta_k}^{-1} \gamma^{(1)}_k, \text{Ad}^*_{h_k} \gamma^{(2)}_k \right), \]
the equation (5.28) takes the form of (5.22) and (5.23).

As for (5.29), we begin with
\[ (\eta^{(1)}_{k+1}, \eta^{(2)}_{k+1}) \ast (g_{k+1}, \xi_{k+1}) = \frac{d}{dt} \bigg|_{t=0} (\beta^{(1)}_{k+1}(t), \eta^{(2)}_{k+1}) \ast (g_{k+1}, \xi_{k+1}) = \left( \eta^{(1)}_{k+1} \ast g_{k+1}, (\eta^{(1)}_{k+1} \ast g_{k+1}) \ast \xi_{k+1} + T_{g_{k+1}} L_{\eta^{(2)}_{k+1}}^{-1}(\eta^{(2)}_{k+1} \ast g_{k+1}) \right). \]
Thus, we conclude from the pairing
\[ \left( b^{*}_{(g_{k+1}, \xi_{k+1})}(\Theta^{(1)}_{k+1}, \Theta^{(2)}_{k+1}), (\eta^{(1)}_{k+1}, \eta^{(2)}_{k+1}) \right) = \left( (\Theta^{(1)}_{k+1}, \Theta^{(2)}_{k+1}), (\eta^{(1)}_{k+1}, \eta^{(2)}_{k+1}) \ast (g_{k+1}, \xi_{k+1}) \right) \]
that

\[ b^*_k (g_{k+1}, \xi_{k+1}) (\Theta^{(1)}_{k+1}, \Theta^{(2)}_{k+1}) = \left( b^{*,2}_{g_{k+1}} (\Theta^{(1)}_{k+1}), b^{*,2}_{g_{k+1}} (\Theta^{(2)}_{k+1}), g_{k+1} \triangleleft b^{*,1}_{g_{k+1}} (T^*_k L_{g_{k+1}} (\Theta^{(2)}_{k+1})) \right). \]

On the next step, we compute

\[
\left( \eta^{(1)}_{k+1}, \eta^{(2)}_{k+1} \right) \triangleleft (g_{k+1}, \xi_{k+1}) = \left. \frac{d}{dt} \right|_{t=0} (\beta^{(1)}_{k+1} (t), \eta^{(2)}_{k+1}) \triangleleft (g_{k+1}, \xi_{k+1}) =
\left( \eta^{(1)}_{k+1} \triangleleft g_{k+1}, \eta^{(2)}_{k+1} \triangleleft g_{k+1} + (\eta^{(1)}_{k+1} \triangleleft g_{k+1}) \triangleleft \xi_{k+1} \right).
\]

Hence, in view of

\[
\left( (g_{k+1}, \xi_{k+1}) \triangleleft \nu (v^{(1)}_{k+1}, v^{(2)}_{k+1}), (\eta^{(1)}_{k+1}, \eta^{(2)}_{k+1}) \right) = \left( (v^{(1)}_{k+1}, v^{(2)}_{k+1}), (\eta^{(1)}_{k+1}, \eta^{(2)}_{k+1}) \right) \triangleleft (g_{k+1}, \xi_{k+1})
\]

we obtain

\[
(g_{k+1}, \xi_{k+1}) \triangleright (v^{(1)}_{k+1}, v^{(2)}_{k+1}) = (g_{k+1} \triangleright v^{(1)}_{k+1} + g_{k+1} \triangleright (\xi_{k+1} \triangleright v^{(2)}_{k+1}), g_{k+1} \triangleright v^{(2)}_{k+1}).
\]

Finally, taking

\[
\left( \Theta^{(1)}_{k+1}, \Theta^{(2)}_{k+1} \right) = d_1 \mathcal{L}((g_{k+1}, \xi_{k+1}), (h_{k+1}, \eta_{k+1})) =
T^*_R (g_{k+1}, \xi_{k+1})^{-1} \left( \mu^{(1)}_{k+1}, \mu^{(2)}_{k+1} \right) = \left( T^*_R \mathcal{L}_{k+1} \mu^{(1)}_{k+1}, \text{Ad}_{g_{k+1}}^{*} \mu^{(2)}_{k+1} \right).
\]

the equation (5.29) yields (5.24) and (5.25). The abbreviations (5.26) follow from

\[
\left( \mu^{(1)}_{k}, \mu^{(2)}_{k} \right) = T^*_R \mathcal{L}_{k+1} \left( \mu^{(1)}_{k+1}, \mu^{(2)}_{k+1} \right) \triangleleft \mathcal{L}((g_k, \xi_k), (h_k, \eta_k)) = \left( T^*_R \mathcal{L}_{k} \mu^{(1)}_{k}, \text{Ad}_{g_{k}}^{*} \mu^{(2)}_{k} \right),
\]

and

\[
(v^{(1)}_{k}, v^{(2)}_{k}) = T^*_R \mathcal{L}_{k+1} \left( v^{(1)}_{k+1}, v^{(2)}_{k+1} \right) \triangleleft \mathcal{L}((g_k, \xi_k), (h_k, \eta_k)) = \left( T^*_R \mathcal{L}_{k+1} \mu^{(1)}_{k}, \text{Ad}_{g_{k}}^{*} \mu^{(2)}_{k} \right).
\]

\[\square\]

5.6. Discrete dynamics on matched second tangent groups.

In this part, we introduce discrete dynamics on matched second order tangent groups \( T^2(G \bowtie H) \).
We consider \( T^2G \) and \( T^2H \) and matched second order tangent group \( T^2(G \bowtie H) \) or \( T^2G \bowtie T^2H \) which is isomorphic copy of \( T^2(G \bowtie H) \) by Proposition 5.5. In this Subsection, we shall imitate the equation (5.4) once more for the case of matched second order tangent groups. Let us determine our setting. We denote the Lie algebra of the group \( T^2G \) by a three tuple \( g_1 \times g_2 \times g_3 \) whereas the Lie algebra of the group \( T^2H \) by \( h_1 \times h_2 \times h_3 \), respectively. Notice that, even though we shall consider the semi-direct product structures and the cocycle terms while making calculations we omit to label them in the present notations of the product spaces. In accordance with this, the dual of spaces are given by \( g^*_1 \times g^*_2 \times g^*_3 \) and \( h^*_1 \times h^*_2 \times h^*_3 \), respectively. it is seen as the swap of \( g_k \in G, h_k \in H, \mu_k \in g^* \) and \( v_k \in h^* \) with \((g_k, \xi_k, \eta_k) \in T^2G, (h_k, \eta_k, \hat{\eta}_k) \in T^2H, (\mu^{(1)}_{k}, \mu^{(2)}_{k}, \mu^{(3)}_{k}) \in g^*_1 \times g^*_2 \times g^*_3 \) and \((v^{(1)}_{k}, v^{(2)}_{k}, v^{(3)}_{k}) \in h^*_1 \times h^*_2 \times h^*_3 \), respectively. We replace \( G \) and \( H \) with \( T^2G \) and \( T^2H \), respectively. In terms of elements, it is seen as the swap of \( g_k \in G, h_k \in H, \mu_k \in g^* \) and \( v_k \in h^* \) with \((g_k, \xi_k, \eta_k) \in T^2G, (h_k, \eta_k, \hat{\eta}_k) \in T^2H, (\mu^{(1)}_{k}, \mu^{(2)}_{k}, \mu^{(3)}_{k}) \in g^*_1 \times g^*_2 \times g^*_3 \) and \((v^{(1)}_{k}, v^{(2)}_{k}, v^{(3)}_{k}) \in h^*_1 \times h^*_2 \times h^*_3 \), respectively.
Proposition 5.6. The matched pair discrete dynamics on $T^2(G \bowtie H) \cong T^2G \bowtie T^2H$, generated by a Lagrangian $\mathcal{L} = \mathcal{L}(g, \xi, \dot{h}, \eta, \dot{\eta})$, is given by the equations

\begin{align}
(5.32) \quad \mu^{(1)}_{k+1} &= \left( \tau^{(1)}_k - \text{ad}^*_{\xi_k} \tau^{(2)}_k - \text{ad}^*_{\xi_k} \tau^{(3)}_k + \text{ad}^*_{\xi_k} \tau^{(3)}_k \right) \ast h_k + \left( \left( \tau^{(2)}_k - 2 \text{ad}^*_{\xi_k} \tau^{(3)}_k \right) \ast h_k \right) \ast \eta_k \\
&+ \left( \left( \tau^{(3)}_k \ast h_k \right) \ast \eta_k + \left( \tau^{(3)}_k \ast h_k \right) \ast \dot{\eta}_k + \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right) + \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \ast \eta_k \\
&+ \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right) \ast \eta_k + \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right) \ast \eta_k, \\
(5.33) \quad \mu^{(2)}_{k+1} &= \left( \tau^{(2)}_k - \text{ad}^*_{\xi_k} \tau^{(3)}_k \right) \ast h_k + 2 \left( \tau^{(3)}_k \ast h_k \right) \ast \eta_k + \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right) + 2 \text{ad}^*_{\eta_k} \tau^{(3)}_k \ast \eta_k \\
&+ \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right), \\
(5.34) \quad \mu^{(3)}_{k+1} &= \tau^{(3)}_k \ast h_k + \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right), \\
(5.35) \quad g_{k+1} \ast \left( v^{(1)}_k + \left( \xi_{k+1} \ast v^{(2)}_k + \left( \xi_{k+1} \ast v^{(3)}_k \right) \right) \right) &= \hat{\theta}^{(1)}_k - \text{ad}^*_{\eta_k} \theta^{(2)}_k - \text{ad}^*_{\eta_k} \theta^{(3)}_k + \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \theta^{(3)}_k \right) - \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \theta^{(3)}_k \right), \\
(5.36) \quad g_{k+1} \ast \left( v^{(2)}_k + 2 \text{ad}^*_{\eta_k} \theta^{(3)}_k - \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right) \right) &= \hat{\theta}^{(1)}_k - \text{ad}^*_{\eta_k} \theta^{(2)}_k - \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right), \\
(5.37) \quad g_{k+1} \ast v^{(3)}_k &= \hat{\theta}^{(1)}_k - \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right),
\end{align}

where we have used the abbreviations

\begin{align}
(5.38) \quad \mu^{(1)}_k &= T^r_{\dot{h}_k} \frac{\delta \mathcal{L}}{\delta \dot{h}_k}, \quad \mu^{(2)}_k = \text{Ad}^*_{\dot{h}_k} \frac{\delta \mathcal{L}}{\delta \dot{h}_k} - \text{Ad}^*_{\xi_k} \frac{\delta \mathcal{L}}{\delta \xi_k}, \quad \mu^{(3)}_k = \text{Ad}^*_{\eta_k} \frac{\delta \mathcal{L}}{\delta \eta_k}, \\
(5.39) \quad v^{(1)}_k &= T^r_{\dot{h}_k} \frac{\delta \mathcal{L}}{\delta \dot{h}_k}, \quad v^{(2)}_k = \text{Ad}^*_{\dot{h}_k} \frac{\delta \mathcal{L}}{\delta \dot{h}_k} - \text{Ad}^*_{\xi_k} \frac{\delta \mathcal{L}}{\delta \xi_k}, \quad v^{(3)}_k = \text{Ad}^*_{\eta_k} \frac{\delta \mathcal{L}}{\delta \eta_k},
\end{align}

for $i = 1, 2, 3$.

Proof. Once again, keeping (5.17) and (5.18) in mind, this time we consider the groups $T^2G$ and $T^2H$. The group elements shall take the form $(\xi_k, \dot{\xi}_k, \xi_{k+1} \ast \xi_{k+1}) \in T^2G$, $(h_k, \eta_k, \dot{\eta}_k) \in T^2H$, while the dual space elements being $(\mu^{(1)}_k, \mu^{(2)}_k, \mu^{(3)}_k) \in \text{Lie}(T^2G)^*$ and $(v^{(1)}_k, v^{(2)}_k, v^{(3)}_k) \in \text{Lie}(T^2H)^*$. Accordingly, (5.18) will transform into

\begin{align}
(5.40) \quad (\mu^{(1)}_k, \mu^{(2)}_k, \mu^{(3)}_k) &= \text{Ad}^*_{\xi_k} \mu^{(1)}_k, \mu^{(2)}_k, \mu^{(3)}_k \ast (h_k, \eta_k, \dot{\eta}_k), \quad (\mu^{(1)}_k, \mu^{(2)}_k, \mu^{(3)}_k) \ast (h_k, \eta_k, \dot{\eta}_k) = d_2 \mathcal{L}, \\
(5.41) \quad (g_{k+1} \ast \dot{\xi}_k, \dot{g}_{k+1} \ast \dot{\xi}_k, \dot{g}_{k+1} \ast \dot{\xi}_k, \dot{g}_{k+1} \ast \dot{\xi}_k, \dot{g}_{k+1} \ast \dot{\xi}_k) &= \text{Ad}^*_{\xi_k} \left( v^{(1)}_k, v^{(2)}_k, v^{(3)}_k \right) - \text{ad}^*_{\tau^{(3)}_k} \left( \text{ad}^*_{\dot{h}_k} \text{ad}^*_{\dot{h}_k} \tau^{(3)}_k \right),
\end{align}

while (5.17) will be incarnated as

\begin{align}
(5.42) \quad T^r_{\dot{h}_k} \ast d_1 \mathcal{L}((g_k, \dot{\xi}_k, \dot{\xi}_k), (h_k, \eta_k, \dot{\eta}_k)) &= (\mu^{(1)}_k, \mu^{(2)}_k, \mu^{(3)}_k), \\
(5.43) \quad T^r_{\dot{h}_k} \ast d_2 \mathcal{L}((g_k, \dot{\xi}_k, \dot{\xi}_k), (h_k, \eta_k, \dot{\eta}_k)) &= (v^{(1)}_k, v^{(2)}_k, v^{(3)}_k).
\end{align}
Let us begin with (5.40). We have
\[
\text{Ad}_{(g_k, \xi_k, \eta_k)}(\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}) = \frac{d}{dt} \bigg|_{t=0} (g_k, \xi_k, \eta_k) (\gamma_k^{(1)}(t), \tau_k^{(2)}, \tau_k^{(3)}) (s_k^{-1}, -\text{Ad}_{g_k} \xi_k, -\text{Ad}_{g_k} \eta_k)
\]
\[
= \left( \text{Ad}_{g_k} \xi_k^{(1)}, \text{Ad}_{g_k} \xi_k^{(2)} + \text{Ad}_{g_k} \text{ad}_{\xi_k} \xi_k^{(1)}, \text{Ad}_{g_k} \xi_k^{(3)} + \text{Ad}_{g_k} \text{ad}_{\xi_k} \xi_k^{(1)} + 2 \text{Ad}_{g_k} \text{ad}_{\xi_k} \xi_k^{(2)} + \text{Ad}_{g_k} \text{ad}_{\xi_k} \xi_k^{(1)} \right).
\]

Then, in view of the pairing
\[
\left\{ \text{Ad}_{(g_k, \xi_k, \eta_k)}^{-1}(\mu_k^{(1)}, \mu_k^{(2)}, \mu_k^{(3)}), (\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}) \right\} = \left\{ (\mu_k^{(1)}, \mu_k^{(2)}, \mu_k^{(3)}), \text{Ad}_{(g_k, \xi_k, \eta_k)}(\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}) \right\},
\]
we obtain
\[
\text{Ad}_{(g_k, \xi_k, \eta_k)}^{-1}(\mu_k^{(1)}, \mu_k^{(2)}, \mu_k^{(3)}) = \left( \text{Ad}_{g_k}^{-1} \mu_k^{(1)} - \text{ad}_{\xi_k} \text{Ad}_{g_k}^{-1} \mu_k^{(2)} - \text{ad}_{\xi_k} \text{Ad}_{g_k}^{-1} \mu_k^{(3)} \right)
\]
\[
+ \text{ad}_{\xi_k}^{-2} \text{Ad}_{g_k}^{-1} \mu_k^{(3)} \text{Ad}_{g_k}^{-1} \mu_k^{(2)} - 2 \text{ad}_{\xi_k}^{-1} \text{Ad}_{g_k}^{-1} \mu_k^{(3)} \text{Ad}_{g_k}^{-1} \mu_k^{(3)} \right).
\]

We next calculate
\[
(h_k, \eta_k, \eta_k) \triangleright (\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}) = \frac{d}{dt} \bigg|_{t=0} (h_k, \eta_k, \eta_k) \triangleright (\gamma_k^{(1)}(t), \tau_k^{(2)}, \tau_k^{(3)})
\]
\[
= \left(h_k \triangleright \xi_k^{(1)}, h_k \triangleright \xi_k^{(2)} + h_k \triangleright (\eta_k \triangleright \xi_k^{(1)}), h_k \triangleright \xi_k^{(3)} + 2h_k \triangleright (\eta_k \triangleright \xi_k^{(2)} + h_k \triangleright (\eta_k \triangleright (\eta_k \triangleright \xi_k^{(1)}))
\]
\[
+ h_k \triangleright (\eta_k \triangleright \xi_k^{(1)}) \right).
\]

Therefore, since
\[
\left\{ (\mu_k^{(1)}, \mu_k^{(2)}, \mu_k^{(3)}) \triangleright (h_k, \eta_k, \eta_k), (\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}) \right\} = \left\{ (\mu_k^{(1)}, \mu_k^{(2)}, \mu_k^{(3)}), (h_k, \eta_k, \eta_k) \triangleright (\xi_k^{(1)}, \xi_k^{(2)}, \xi_k^{(3)}) \right\},
\]
we have
\[
(\mu_k^{(1)}, \mu_k^{(2)}, \mu_k^{(3)}) \triangleright (h_k, \eta_k, \eta_k) = (\mu_k^{(1)} \triangleright h_k + (\mu_k^{(2)} \triangleright h_k) \triangleright \eta_k + ((\mu_k^{(3)} \triangleright h_k) \triangleright \eta_k) \triangleright \eta_k
\]
\[
+ (\mu_k^{(3)} \triangleright h_k) \triangleright \eta_k, \mu_k^{(2)} \triangleright h_k + 2(\mu_k^{(3)} \triangleright h_k) \triangleright \eta_k, \mu_k^{(3)} \triangleright h_k \right).
\]

Now combining (5.44) and (5.45), we arrive at
\[
\text{Ad}_{(g_k, \xi_k, \eta_k)}^{-1}(\mu_k^{(1)}, \mu_k^{(2)}, \mu_k^{(3)}) \triangleright (h_k, \eta_k, \eta_k)
\]
\[
= \left( \left( \text{Ad}_{g_k}^{-1} \mu_k^{(1)} - \text{ad}_{\xi_k} \text{Ad}_{g_k}^{-1} \mu_k^{(2)} - \text{ad}_{\xi_k} \text{Ad}_{g_k}^{-1} \mu_k^{(3)} + (\text{ad}_{\xi_k}^{-2})_\xi \text{Ad}_{g_k}^{-1} \mu_k^{(3)} \right) \triangleright h_k
\]
\[
+ \left( \text{Ad}_{g_k}^{-1} \mu_k^{(2)} - 2 \text{ad}_{\xi_k} \text{Ad}_{g_k}^{-1} \mu_k^{(3)} \right) \triangleright \eta_k + ((\text{ad}_{\xi_k}^{-2})_\xi \text{Ad}_{g_k}^{-1} \mu_k^{(3)} \triangleright h_k)
\]
\[
+ (\text{Ad}_{g_k}^{-1} \mu_k^{(3)} \triangleright h_k) \triangleright \eta_k, \text{Ad}_{g_k}^{-1} \mu_k^{(2)} - 2 \text{ad}_{\xi_k} \text{Ad}_{g_k}^{-1} \mu_k^{(3)} \right) \triangleright h_k
\]
\[
+ 2(\text{Ad}_{g_k}^{-1} \mu_k^{(3)} \triangleright h_k) \triangleright \eta_k, \text{Ad}_{g_k}^{-1} \mu_k^{(3)} \triangleright h_k \right).
\]
On the next move, we proceed towards the latter term on the right hand side of (5.40). To this end, we calculate
\[
\alpha_{(h_k, \eta_k, \eta_k)}(\xi^{(1)}_k, \xi^{(2)}_k, \xi^{(3)}_k) = \frac{d}{dt} \bigg|_{t=0} (h_k, \eta_k, \eta_k) \ast (\gamma^{(1)}_k(t), T\xi^{(2)}_k, T\xi^{(3)}_k)
\]
\[
= \left( \alpha_{h_k} \xi^{(1)}_k, \alpha_{h_k} \xi^{(2)}_k, \alpha_{h_k} \xi^{(3)}_k \right) + T_{h_k} L_{\eta_k}^{-1} \left( h_k \ast \xi^{(2)}_k \right) + T_{h_k} L_{\eta_k}^{-1} \left( h_k \ast \xi^{(3)}_k \right)
\]
\[
\alpha_{h_k} \xi^{(1)}_k + 2 \alpha_{h_k} \xi^{(2)}_k + \alpha_{h_k} (\eta_k \ast \xi^{(2)}_k) + 2 T_{h_k} L_{\eta_k}^{-1} \left( h_k \ast (\eta_k \ast \xi^{(2)}_k) \right)
\]
\[
+ T_{h_k} L_{\eta_k}^{-1} \left( h_k \ast (\eta_k \ast \xi^{(1)}_k) \right) + T_{h_k} L_{\eta_k}^{-1} \left( h_k \ast \xi^{(3)}_k \right) + T_{h_k} L_{\eta_k}^{-1} \left( h_k \ast (\eta_k \ast \xi^{(1)}_k) \right)
\]
\[
- \text{ad}_{\eta_k} T_{h_k} L_{\eta_k}^{-1} \left( h_k \ast (\eta_k \ast \xi^{(1)}_k) \right).
\]
Then, the pairing
\[
\left\langle \alpha_{(h_k, \eta_k, \eta_k)}(\Theta^{(1)}_k, \Theta^{(2)}_k, \Theta^{(3)}_k), (\xi^{(1)}_k, \xi^{(2)}_k, \xi^{(3)}_k) \right\rangle = \left\langle (\Theta^{(1)}_k, \Theta^{(2)}_k, \Theta^{(3)}_k), \alpha_{(h_k, \eta_k, \eta_k)}(\xi^{(1)}_k, \xi^{(2)}_k, \xi^{(3)}_k) \right\rangle
\]
implies
\[
\alpha_{(h_k, \eta_k, \eta_k)}^* \left( \Theta^{(1)}_k, \Theta^{(2)}_k, \Theta^{(3)}_k \right) = \left( \alpha_{h_k}^* \Theta^{(1)}_k + \alpha_{h_k}^* \Theta^{(2)}_k + \alpha_{h_k}^* (T_{h_k}^* L_{\eta_k}^{-1} \Theta^{(2)}_k) \ast \eta_k + \alpha_{h_k}^* \Theta^{(3)}_k
\]
\[
+ \alpha_{h_k}^* \Theta^{(3)}_k \ast \eta_k + \alpha_{h_k}^* (T_{h_k}^* L_{\eta_k}^{-1} \Theta^{(3)}_k) \ast \eta_k
\]
\[
+ \alpha_{h_k}^* (T_{h_k}^* L_{\eta_k}^{-1} (\text{ad}_{\eta_k} \Theta^{(3)}_k)) \ast \eta_k, \alpha_{h_k}^* (T_{h_k}^* L_{\eta_k}^{-1} \Theta^{(2)}_k) + 2 \Theta^{(3)}_k \ast \eta_k + \text{ad}_{\eta_k} (T_{h_k}^* L_{h_k}^{-1} \Theta^{(3)}_k) \ast \eta_k
\]
\[
+ \text{ad}_{\eta_k} (T_{h_k}^* L_{h_k}^{-1} \Theta^{(3)}_k)) \right).
\]
Finally, we put together (5.46) and (5.47), in view of
\[
\left( \Theta^{(1)}_k, \Theta^{(2)}_k, \Theta^{(3)}_k \right) = d_2^\circ (g_k, \xi_k, \xi_k), (h_k, \eta_k, \eta_k))
\]
\[
= T_{(e, 0, 0)} R_{h_k, \eta_k, \eta_k}^{-1} \left( v^{(1)}_k, v^{(2)}_k, v^{(3)}_k \right) = \left( T_{h_k}^* R_{h_k}^{-1} v^{(1)}_k, \text{Ad}_{h_k}^* v^{(2)}_k - \text{ad}_{\eta_k} \text{Ad}_{h_k}^* v^{(3)}_k, \text{Ad}_{h_k}^* v^{(3)}_k \right)
\]
to obtain (5.32), (5.33), and (5.34).

Let us next focus on (5.41) explicitly. The former term on the right hand side is readily by (5.44), that is,
\[
\text{Ad}_{(h_k, \eta_k, \eta_k)}^* (v^{(1)}_k, v^{(2)}_k, v^{(3)}_k) = \left( \text{Ad}_{h_k}^* v^{(1)}_k - \text{ad}_{\eta_k} \text{Ad}_{h_k}^* v^{(2)}_k - \text{ad}_{\eta_k} \text{Ad}_{h_k}^* v^{(3)}_k + (\text{ad})_{\eta_k}^2 \text{Ad}_{h_k}^* v^{(3)}_k, \text{Ad}_{h_k}^* v^{(2)}_k - 2 \text{ad}_{\eta_k} \text{Ad}_{h_k}^* v^{(3)}_k, \text{Ad}_{h_k}^* v^{(3)}_k \right).
\]
As for the latter term, on the right hand side of (5.41), we calculate

\[
(\eta^{(1)}_{k+1}, \eta^{(2)}_{k+1}, \eta^{(3)}_{k+1}) \succ (g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1}) = \frac{d}{dt}_{t=0} \left( \beta^{(1)}_{k+1}(t), \eta^{(2)}_{k+1}, \eta^{(3)}_{k+1} \right) \succ (g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1})
\]

\[
= \left( \eta^{(1)}_{k+1} \succ g_{k+1}, (\eta^{(1)}_{k+1} \prec g_{k+1}) \succ \xi_{k+1} + T_{g_{k+1}} L_{g_{k+1}}^{-1} (\eta^{(2)}_{k+1} \succ g_{k+1}), 2(\eta^{(2)}_{k+1} \prec g_{k+1}) \succ \dot{\xi}_{k+1}
\right.
\]

\[
+ \left( \eta^{(1)}_{k+1} \prec g_{k+1} \prec \xi_{k+1} \right) \succ \xi_{k+1} + \left( \eta^{(1)}_{k+1} \prec g_{k+1} \prec \dot{\xi}_{k+1} \right) \succ \eta^{(3)}_{k+1} \prec g_{k+1} + T_{g_{k+1}} L_{g_{k+1}}^{-1} (\eta^{(3)}_{k+1} \prec g_{k+1})
\]

\[
- \text{ad}_{\dot{\xi}_{k+1}} \left( T_{g_{k+1}} L_{g_{k+1}}^{-1} (\eta^{(3)}_{k+1} \prec g_{k+1}) \right)
\].

Then, in view of the pairing

\[
\left\langle b^{*}_{(g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1})} \left( \Theta^{(1)}_{k+1}, \Theta^{(2)}_{k+1}, \Theta^{(3)}_{k+1} \right), (\eta^{(1)}_{k+1}, \eta^{(2)}_{k+1}, \eta^{(3)}_{k+1}) \right\rangle
\]

we obtain

\[
b^{*}_{(g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1})} \left( \Theta^{(1)}_{k+1}, \Theta^{(2)}_{k+1}, \Theta^{(3)}_{k+1} \right) = \left( b^{*}_{(g_{k+1}, \Theta^{(1)}_{k+1})} + g_{k+1} \succ b^{*}_{(g_{k+1}, \Theta^{(2)}_{k+1})} + g_{k+1} \succ \left( \xi_{k+1} \succ b^{*}_{(g_{k+1}, \Theta^{(3)}_{k+1})} \right) \right)
\]

\[
+ g_{k+1} \succ b^{*}_{(g_{k+1}, \Theta^{(1)}_{k+1})} \succ (T_{g_{k+1}} L_{g_{k+1}}^{-1} (\Theta^{(2)}_{k+1})) + g_{k+1} \succ b^{*}_{(g_{k+1}, \Theta^{(3)}_{k+1})}
\]

\[
+ b^{*}_{g_{k+1}} \left( T_{g_{k+1}} L_{g_{k+1}}^{-1} \left( \text{ad}^{*}_{g_{k+1}} (\Theta^{(3)}_{k+1}) \right) \right), b^{*}_{g_{k+1}} \left( T_{g_{k+1}} L_{g_{k+1}}^{-1} (\Theta^{(3)}_{k+1}) \right) \right)
\].

On the other hand, for the left hand side of (5.41), we compute

\[
(\eta^{(1)}_{k+1}, \eta^{(2)}_{k+1}, \eta^{(3)}_{k+1}) \prec (g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1}) = \frac{d}{dt}_{t=0} \left( \beta^{(1)}_{k+1}(t), \eta^{(2)}_{k+1}, \eta^{(3)}_{k+1} \right) \prec (g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1})
\]

\[
= \left( \eta^{(1)}_{k+1} \prec g_{k+1}, \eta^{(2)}_{k+1} \prec g_{k+1} + (\eta^{(1)}_{k+1} \prec g_{k+1}) \prec \xi_{k+1}, \eta^{(3)}_{k+1} \prec g_{k+1}
\right.
\]

\[
+ 2(\eta^{(2)}_{k+1} \prec g_{k+1}) \prec \xi_{k+1} + \left( \eta^{(1)}_{k+1} \prec g_{k+1} \prec \xi_{k+1} \right) \prec \xi_{k+1} + \left( \eta^{(1)}_{k+1} \prec g_{k+1} \prec \xi_{k+1} \right) \prec \xi_{k+1} \right).
\]

Accordingly, the pairing

\[
\left\langle (g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1}) \prec (v^{(1)}_{k+1}, v^{(2)}_{k+1}, v^{(3)}_{k+1}), (\eta^{(1)}_{k+1}, \eta^{(2)}_{k+1}, \eta^{(3)}_{k+1}) \prec (g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1}) \right\rangle
\]

implies

\[
(g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1}) \prec (v^{(1)}_{k+1}, v^{(2)}_{k+1}, v^{(3)}_{k+1}) = \left( v^{(1)}_{k+1} \prec g_{k+1} \prec (\xi_{k+1} \prec v^{(2)}_{k+1}) \prec (g_{k+1} \prec v^{(3)}_{k+1})
\right.
\]

\[
+ g_{k+1} \prec (\xi_{k+1} \prec v^{(3)}_{k+1}), g_{k+1} \prec (\xi_{k+1} \prec v^{(3)}_{k+1}) + g_{k+1} \prec (\xi_{k+1} \prec v^{(3)}_{k+1}) + g_{k+1} \prec (\xi_{k+1} \prec v^{(3)}_{k+1})
\].

At last, taking

\[
\left( \Theta^{(1)}_{k+1}, \Theta^{(2)}_{k+1}, \Theta^{(3)}_{k+1} \right) = d_{t} \mathcal{V}((g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1}), (h_{k+1}, \eta_{k+1}, \eta_{k+1}))
\]

\[
= T_{(\epsilon,0,0)} R_{(g_{k+1}, \xi_{k+1}, \dot{\xi}_{k+1})}^{-1} \left( \mu^{(1)}_{k+1}, \mu^{(2)}_{k+1}, \mu^{(3)}_{k+1} \right)
\]

\[
= (T_{t} R_{g_{k+1}})^{\epsilon} \mu^{(1)}_{k+1}, \text{Ad}^{*}_{g_{k+1}} \mu^{(2)}_{k+1} - \text{ad}_{g_{k+1}}^{*} \mu^{(3)}_{k+1}, \text{Ad}^{*}_{g_{k+1}} \mu^{(3)}_{k+1}
\]

(5.41) assumes the form of (5.36)-(5.37).
The abbreviations $(5.38)$ is merely a result of

\[
(\mu_k^{(1)}, \mu_k^{(2)}, \mu_k^{(3)}) = T^*_e R_{g_k, \xi_k, \dot{\xi}_k} \cdot d_1 \mathcal{L}((g_k, \xi_k, \dot{\xi}_k, (h_k, \eta_k, \dot{\eta}_k)) =
\left( T^*_e R_{g_k} \frac{\delta \mathcal{L}}{\delta g_k} \cdot \text{Ad}^*_g \frac{\delta \mathcal{L}}{\delta \xi_k} + \text{Ad}^*_g \frac{\delta \mathcal{L}}{\delta \dot{\xi}_k} \right),
\]

and

\[
(\nu_k^{(1)}, \nu_k^{(2)}, \nu_k^{(3)}) = T^*_e R_{h_k, \eta_k, \dot{\eta}_k} \cdot d_2 \mathcal{L}((g_k, \xi_k, \dot{\xi}_k, (h_k, \eta_k, \dot{\eta}_k)) =
\left( T^*_e R_{h_k} \frac{\delta \mathcal{L}}{\delta h_k} \cdot \text{Ad}^*_h \frac{\delta \mathcal{L}}{\delta \eta_k} + \text{Ad}^*_h \frac{\delta \mathcal{L}}{\delta \dot{\eta}_k} \right).
\]

\[
\square
\]

Here, we present the following diagram to summarize the discrete dynamics obtained in this section. The left column is for the uncoupled systems derived in Subsections 5.1, 5.2, and 5.3, whereas the right column stands for the matched discrete dynamical systems in Subsection 5.4, 5.5, and 5.6.

\[
\begin{array}{ccc}
\text{T}^2G & \xrightarrow{\text{2nd dEL in (5.10)}} & \text{T}^2(G \bowtie H) \\
\downarrow & & \downarrow \\
\text{T}G & \xrightarrow{\text{dEL in (5.5)}} & \text{T}(G \bowtie H) \\
\downarrow & & \downarrow \\
G & \xrightarrow{\text{dEL in (5.4)}} & G \bowtie H
\end{array}
\]

6. Discussions

In this work, we have matched two Lie groups under mutual interactions. Further, we have studied the geometry of the second order and the iterated tangent bundles of matched pair Lie groups. In accordance with this, we have studied continuous and discrete dynamical equations of higher order. There are diagrams in the manuscript summarizing the discussions. In diagram 4.18, we exhibited the (unmatched) Lagrangian dynamics on $T G$, $T^2 G$, $TT G$ as well as their Euler-Poincaré reductions in a hierarchical way. In 4.26 we have presented both the matched Lagrangian dynamics on the second order and iterated tangent bundles along with pointing out the canonical immersions of the unmatched dynamics. The last diagram 5.48 summarizes of whole Section 5 by determining the all discrete dynamical equation both in unmatched and matched forms.

As it is well-known a generalization of the classical Lagrangian dynamics is available on the Lie algebroid framework, see, for example, [18, 33, 42, 58, 74]. It is also possible to study higher order Lagrangian dynamics in this framework [12, 38, 59]. As a future work, we plan to generalize the discussion
presented in this work to the realms of Lie algebroids both for the first order and the higher order dynamics. That is, we wish to carry the matched pair discussions to the prolonged Lie algebroids, and give matched Euler-Lagrange equations on this geometry.

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References

[1] R. Abraham and J. E. Marsden. Foundations of mechanics. Benjamin/Cummings Publishing Company Reading, Massachusetts, 1978.
[2] L. Abrunheiro, M. Camarinha, and J. Clemente-Gallardo. Cubic polynomials on Lie groups: reduction of the Hamiltonian system. Journal of Physics A: Mathematical and Theoretical, 44(35):355203, 2011.
[3] V. I. Arnold. Mathematical methods of classical mechanics, volume 60. Springer Science & Business Media, 1989.
[4] R. Benito, M. de León, and D. M. de Diego. Higher order discrete Lagrangian mechanics. 2006.
[5] W. Bertram. Differential geometry. Lie groups and symmetric spaces over general base fields and rings. American Mathematical Soc., 2008.
[6] A. I. Bobenko and Y. B. Suris. Discrete Lagrangian reduction, discrete Euler-Poincaré equations, and semidirect products. Lett. Math. Phys., 49(1):79–93, 1999.
[7] N. Bou-Rabee and J. E. Marsden. Hamilton-Pontryagin integrators on Lie groups part I: Introduction and structure-preserving properties. Foundations of Computational Mathematics, 9(2):197–219, 2009.
[8] M. G. Brin. On the Zappa-Szép product. Comm. Algebra, 33(2):393–424, 2005.
[9] H. Cendra, D. D. Holm, J. E. Marsden, and T. S. Ratiu. Lagrangian reduction, the Euler-Poincaré equations, and semidirect products. Translations of the American Mathematical Society-Series 2, 186:1–26, 1998.
[10] H. Cendra, J. E. Marsden, S. Pekarsky, and T. S. Ratiu. Variational principles for Lie-Poisson and Hamilton-Poincaré equations. Moscow Math. J, 3(3):833–867, 2003.
[11] H. Cendra, J. E. Marsden, and T. S. Ratiu. Lagrangian reduction by stages, volume 722. American Mathematical Soc., 2001.
[12] L. Colombo. Second-order constrained variational problems on lie algebroids: Applications to optimal control. Journal of Geometric Mechanics, 9(1), 2017.
[13] L. Colombo and D. de Diego. Higher-order variational problems on Lie groups and optimal control applications. Journal of Geometric Mechanics, 6(4), 2014.
[14] L. Colombo and D. M. de Diego. On the geometry of higher-order variational problems on Lie groups. arXiv:1104.3221, 2011.
[15] L. Colombo and D. M. de Diego. Optimal control of underactuated mechanical systems with symmetries. Dynamical Systems, pages 149–158, 2013.
[16] L. Colombo, F. Jiménez, and D. M. de Diego. Discrete second-order Euler-Poincaré equations: applications to optimal control. International Journal of Geometric Methods in Modern Physics, 9(04):1250037, 2012.
[17] L. Colombo and P. D. Prieto-Martínez. Unified formalism for higher-order variational problems and its applications in optimal control. International Journal of Geometric Methods in Modern Physics, 11(04):1450034, 2014.
[18] M. de León, J. C. Marrero, and E. Martínez. Lagrangian submanifolds and dynamics on Lie algebroids. Journal of Physics A: Mathematical and General, 38(24):R241, 2005.
[19] M. de León, P. Pitaiga, and P. R. Rodrigues. Symplectic reduction of higher order Lagrangian systems with symmetry. Journal of Mathematical Physics, 35(12):6546–6556, 1994.
[20] M. de León and P. R. Rodrigues. Generalized Classical Mechanics and Field Theory: a geometrical approach of Lagrangian and Hamiltonian formalisms involving higher order derivatives, volume 112. Elsevier, 2011.
[54] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1999.

[55] J. E. Marsden, T.S. Ratiu, and G. Raugel. Symplectic connections and the linearisation of Hamiltonian systems. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 117(3-4):329–380, 1991.

[56] J. E. Marsden, T.S. Ratiu, and A. Weinstein. Semidirect products and reduction in mechanics. *Trans. Amer. Math. Soc.*, 281(1):147–177, 1984.

[57] J. E. Marsden and J. Scheurle. Lagrangian reduction and the double spherical pendulum. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 44(1):17–43, 1993.

[58] E. Martínez. Lagrangian mechanics on lie algebroids. *Acta Applicandae Mathematica*, 67(3):295–320, 2001.

[59] E. Martínez. Higher-order variational calculus on lie algebroids. *arXiv preprint arXiv:1501.06520*, 2015.

[60] W. Michaelis. Lie coalgebras. *Adv. Math.*, 38(1):1–54, 1980.

[61] P. W. Michor. *Topics in differential geometry*, volume 93. American Mathematical Soc., 2008.

[62] T. Mokri. Matched pairs of lie algebroids. *Glasgow Math. J.*, 39(2):167–181, 1997.

[63] P. J. Olver. *Applications of Lie groups to differential equations*, volume 107 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993.

[64] M. Ostrogradsky. Mémoires sur les équations différentielles, relatives au problème des isopérimètres. *Mem. Acad. St. Petersbourg*, 6:385–517, 1850.

[65] M. Pavelka, V. Klika, and M. Grmela. *Multiscale Thermo-Dynamics: Introduction to GENERIC*. Walter de Gruyter GmbH & Co KG, 2018.

[66] T.S. Ratiu. The motion of the free n-dimensional rigid body. *Indiana Univ. Math. J.*, 29(4):609–629, 1980.

[67] J. Szép. On factorisable, not simple groups. *Acta Univ. Szeged. Sect. Sci. Math.*, 13:239–241, 1950.

[68] J. Szép. über eine neue Erweiterung von Ringen. I. *Acta Sci. Math. Szeged*, 19:51–62, 1958.

[69] J. Szép. Sulle strutture fattorizzabili. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat.(8)*, 32:649–652, 1962.

[70] M. Takeuchi. Matched pairs of groups and bismash products of Hopf algebras. *Comm. Algebra*, 9(8):841–882, 1981.

[71] P. Vagner and M. Pavelka. Multiscale thermodynamics of charged mixtures. *arXiv preprint arXiv:1903.01274*, 2019.

[72] C. Vizman. Geodesic equations on diffeomorphism groups. *Symmetry, Integrability and Geometry: Methods and Applications*, 4(0):30–22, 2008.

[73] C. Vizman. The group structure for jet bundles over Lie groups. *J. Lie Theory*, 23(3):885–897, 2013.

[74] A. Weinstein. Lagrangian mechanics and groupoids. In *Mechanics day (Waterloo, ON, 1992)*, volume 7 of *Fields Inst. Commun.*, pages 207–231. Amer. Math. Soc., Providence, RI, 1996.

[75] K. Yano and S. Ishihara. *Tangent and cotangent bundles: differential geometry*. Marcel Dekker, Inc., New York, 1973. Pure and Applied Mathematics, No. 16.

[76] G. Zappa. Sulla costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro. In *Atti Secondo Congresso Un. Mat. Ital.*, Bologna, 1940, pages 119–125. Edizioni Cremonense, Rome, 1942.

[77] T. Zhang. Double cross biproduct and bi-cycle bicrossproduct Lie bialgebras. *J. Gen. Lie Theory Appl.*, 4:Art. ID S090602, 16, 2010.

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