Decomposition of Third-Order Linear Time-Varying Systems into Its Second- and First-Order Commutative Pairs

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Abstract
Decomposition is a common tool for the synthesis of many physical systems. It is also used for analyzing large-scale systems which are then known as tearing and reconstruction. On the other hand, commutativity of cascade-connected systems has gained a great deal of interest, and its possible benefits have been pointed out on the literature. In this paper, the necessary and sufficient conditions for decomposition of any third-order linear time-varying system as a commutative pair of first- and second-order systems of which parameters are also explicitly expressed, are investigated. Further, additional requirements in case of nonzero initial conditions are derived. This paper highlights the direct formulas for realization of any third-order linear time-varying systems as a series (cascade) connection of first- and second-order subsystems. This series connection is commutative so that it is independent from the sequence of subsystems in the connection. Hence, the convenient sequence can be decided by considering the overall performance of the system when the sensitivity, disturbance, and robustness effects are considered. Realization covers transient responses as well as steady-state responses.

Keywords Differential equations · Initial conditions · Analogue control · Equivalent circuits · Physical systems

1 Introduction
Differential equations arise as common models in the physical, mathematical, biological and engineering sciences, and most real physical processes are governed by...
differential equations. The fundamental laws governing many physical process are known relationships between various quantities and their derivatives. In general, most real physical processes involve more than one independent variable and the corresponding differential equations. In particular, differential equations are used for modeling problems in electric–electronics engineering, the touchstone and largest branch of engineering technology, and includes a diverse range of subdisciplines, such as embedded systems, control systems, telecommunications, and power systems. For instance, in system and control theory, analog systems are modeled as continuous time-varying systems and described by ordinary differential equations. In the system, the transfer function, also known as the system function or network function, is a mathematical representation of the relation between the input and output based on the differential equations describing the system such as cascade and feedback connections. When the cascade connection in system design is considered, the commutativity concept places an prominent role to improve different system performances.

Cascade connection of subsystems is a commonly used method for designing many engineering systems, especially electrical and electronic devices [5,21,25,28]. For example, cascade connection is used for connecting the server module located in another subnetwork via an intermediate computer that has two network interfaces for two subnetworks. The order of connection is important for achieving more reliable systems which are less sensitive and more robust to internal and external disturbances, and it may depend on many criteria such as the used design technique, engineering ingenuity, and traditional habits. Therefore, the change of the order of connection may be thought for the possibility of obtaining better performances without spoiling the main function of the total system (commutativity). Hence, commutativity is important from engineering point of view. For a more detail of the importance and usefulness of cascade-connected systems, we operate the reader to read the reference [18].

When two simple systems are connected in cascade, that is, the output of the former acts as the input of the later [6,26], if the order of connection does not change the input–output relation of the combined system, then we say that these systems are commutative.

There are a great deal of literature about the commutativity of continuous-time linear time-varying systems [7–12,16,19,24] though there are a few works on the discrete time-varying systems [14,15]. The first paper on the commutativity in the literature has been studied by Marshall in [19], and it is proved that a time-varying system can be commutative with another time-varying system. Then, commutativity conditions of second-order, third-order, and fourth-order systems were studied in [7,9–11,24], respectively. In [8], the most general necessary and sufficient conditions for the commutativity of systems of any order but without initial conditions were studied. This study also includes results concerning the commutativity properties of feedback control systems and Euler differential systems. Moreover, the previous results for commutativity conditions of first-order, second-order, third-order and fourth-order systems were shown to be deduced from the main theorem of [8].

More than two decades later, the explicit commutativity conditions for linear time-varying differential systems with nonzero initial conditions [16] and the explicit commutativity conditions for the fifth-order systems were derived for the first time in [16].
In [12], necessary and sufficiently conditions for the decomposition of a second-order linear time-varying system into two cascade-connected commutative first-order linear time-varying subsystems were studied. Further, explicit formulas describing these subsystems were presented by illustrative examples and simulations. Final study on the commutativity of analogue systems was studied in [13] covering inverse commutativity conditions expressed in terms of the coefficients of the differential equation describing system.

Digital systems are modeled as discrete time-varying systems and described by difference equations. Even though there are many papers on the theory and application of discrete time-varying systems (see [1,2,17,20,23,27] and the references therein) for modeling, analyzing and solving real engineering problems because of the fact that the trend in the modern technology is sliding to the digital world from the analog world, there are only two references [14,15] debuted to investigation of commutativity of discrete-time systems. The concept of commutativity for digital systems was defined in [14] for the first time. Then, the possible benefits of commutativity such as noise disturbance, effects, and parameter sensitivity are outlined in [15]. In unpublished work, the transitivity property is examined and it holds for analog systems. For digital systems, however, it has not been reported anywhere.

Commutativity of linear time-invariant systems has simply been handled by some other authors as well. For example, Richard G. Lyons in his book [18], after defining commutativity as the property “swapping the order of two cascaded systems does not alter the final output,” has pointed out that two linear time-invariant systems obey this useful commutative property; although this fact is simple to show for time-invariant systems either by using transfer function approach or unit impulse response, it is not always valid for time-invariant systems with nonzero initial conditions and time-varying systems with or without initial conditions. But the following points indicated by R. G. Lyons about commutativity of cascaded systems are important: Although different sequences of commutative systems have identical output values for the same input, their intermediate data will usually not be equal. This commutative characteristic comes in handy for designers of digital filters, as shown in Chapters 5 and 6 [18]. Namely, he has shown how IIR filters with cascaded structures can be improved: IIR filter stability and quantization noise problems by building high-performance filters by implementing combinations of cascaded lower-performance filters. As a rule of thumb in the design of cascaded filters, it is prudent to specify their individual passband ripple values to be roughly half of the desired ripple specification for the combined filter.

Experienced filter designers routinely partition high-order IIR filters into a string of second- or first-order filters arranged in cascade. This is because these lower-order filters are easier to design and are less susceptible to coefficient quantization errors and stability problems. Further, this implementation allows easier data word scaling to reduce the potential overflow effects of data word size growth. Although optimization the partitioning of a high-order filter into multiple second- or first-order filter sections is a challenging task even for time-invariant case, one simple method for arranging cascaded subsections has been proposed [22]. For time-varying analog systems, however, such a decomposition process has not been studied, and the purpose of this paper is to fulfill this vacancy to some extent.
In this paper, after deriving some mathematical preliminaries in Sect. 2, the basic equations that must be satisfied for commutativity are derived for in Sect. 3. These equations are solved in Sect. 4. In Sect. 5, the coefficients of the second- and first-order components are explicitly expressed in terms of those of the original third-order system. Section 5.1 covers a few illustrative examples. And finally, the paper ends with Section 6 conclusion.

2 Mathematical Preliminaries

Let \( C \) be a third-order linear time-varying analog system described by

\[
c_3(t)y'''(t) + c_2(t)y''(t) + c_1(t)y'(t) + c_0(t)y(t) = x(t),
\]

with the input \( x(t) \) and output \( y(t) \), where \( c_i(t) \) are time-varying coefficients which are piecewise continuous on \([t_0, \infty)\); this set of functions are denoted by \( P[t_0, \infty) \); also assume the initial conditions \( y(t_0), y'(t_0), y''(t_0) \) at the initial time \( t_0 \in \mathbb{R} \), where the number of overhead dots represents the order of derivatives. Due to its order of 3, \( c_3(t) \neq 0 \).

It is well known that such a system has a unique solution for all \( x(t) \in P[t_0, \infty) \). Consider the decomposition of \( C \) as the cascade connection of a first-order system \( A \) and second-order \( B \) described by

\[
A : a_1(t)y_A'(t) + a_0(t)y_A(t) = x_A(t),
\]

\[
B : b_2(t)y_B''(t) + b_1(t)y_B'(t) + b_0(t)y_B(t) = x_B(t),
\]

with the initial conditions

\[
y_A(t_0),
\]

\[
y_B(t_0), \ y_B'(t_0).
\]

Due to their orders \( a_1(t) \neq 0, b_2(t) \neq 0 \). Further, assume \( a_i, b_i, x_A, x_B \in P[t_0, \infty) \). Moreover, assume that \( a_i \)'s are differentiable up to second order and \( b_i \)'s are differentiable up to first order. Assume also that the cascade connection of \( A \) and \( B \), denoted by \( AB \) or \( BA \) according to their order of connection as shown in Fig. 1a, b, respectively, is commutative. That is \( AB \) and \( BA \) have the same input–output relation.

Due to the connection in Fig. 1a, it is obvious that

\[
x_A(t) = x(t),
\]

\[
y_A(t) = x_B(t),
\]

\[
y_B(t) = y(t).
\]

Differentiating Eq. (2.3), we obtain

\[
b_2' y_B'' + b_2 y_B''' + b_1' y_B' + b_1 y_B'' + b_0' y_B + b_0 y_B' = x'._B.
\]
Fig. 1 Cascade connection of $A$ and $B$: $a$ $AB$, $b$ $BA$

From Eq. (2.7), $x_B' = y_A'$, and then solving Eq. (2.2) for $y_A'$, finally using Eq. (2.7), we again obtain

$$x_B' = y_A' = \frac{x_A - a_0 y_A}{a_1} = \frac{x_A - a_0 x_B}{a_1}$$

$$= \frac{1}{a_1} \left[ x_A - a_0 (b_2 y''_B + b_1 y'_B + b_0 y_B) \right], \quad (2.10)$$

where the last equality is obtained by using the expression Eq. (2.3) for $x_B$. Finally, inserting Eq. (2.10) in Eq. (2.9) and replacing $y_B \rightarrow y$, $x_A \rightarrow x$ due to Eqs. (2.8) and (2.6), respectively, we obtain the following third-order differential system for the connection $AB$

$$a_1 b_2 y''' + (a_1 b'_2 + a_1 b_1 + a_0 b_2)y'' + (a_1 b'_1 + a_1 b_0 + a_0 b_1)y' + (a_1 b'_0 + a_0 b_0)y = x, \quad (2.11)$$

$$y(t_0) = y_B(t_0), \quad (2.12)$$

$$y'(t_0) = y'_B(t_0), \quad (2.13)$$

$$y''(t_0) = y''_B(t_0) = \frac{y_A(t_0) - b_0(t_0)y_B(t_0) - b_1(t_0)y'_B(t_0)}{b_2(t_0)}. \quad (2.14)$$

Equations (2.12) and (2.13) are obvious due to Eq. (2.8). Equation (2.14) is obtained as follows: Due to Eq. (2.8), $y(t_0) = y''_B(t_0)$ which is computed from Eq. (2.3) and inserting $x_B(t_0) = y_A(t_0)$ due to Eq. (2.7).

Similarly, due to the connection in Fig. 1b, it is obvious that

$$x_B(t) = x(t), \quad (2.15)$$

$$y_B(t) = x_A(t), \quad (2.16)$$

$$y_A(t) = y(t). \quad (2.17)$$
Differentiating (2.2) two times and ordering the terms, we obtain
\[ a_1y'''_A + (2a'_1 + a_0)y''_A + (a''_1 + 2a'_0)y'_A + a''_0y_A = x''_A. \]

(2.18)

Since \( x''_A(t) = y''_B(t) \) due to Eq. (2.16), finding \( y''_B \) from Eq. (2.3), and using Eq. (2.16) again, we have

\[ x''_A = y''_B = \frac{x_B - b_1y'_B - b_0y_B}{b_2}. \]

(2.19)

Next inserting in the value of \( x_A \) from Eq. (2.2) and the value of \( x'_A \) from derivative of Eq. (2.2) into the above equation, we obtain

\[ x''_A = \frac{x_B - b_1(a'_1y'_A + a_1y''_A + a'_0y'_A + a_0y'_A) - b_0(a_1y'_A + a_0y_A)}{b_2}. \]

(2.20)

Inserting Eqs. (2.20) in (2.18) and noting \( y_A = y \) [Eq. (2.17)] and \( x_B = x \), [Eq. (2.15)], we obtain the third-order differential equation describing \( BA \) as

\[ a_1b_2y''' + (2a'_1b_2 + a_0b_2 + a_1b_1)y'' + (a''_1b_2 + 2a'_0b_2 + a'_1b_1 + a_0b_1 + a_1b_0)y' + (a''_0b_2 + a'_0b_1 + a_0b_0)y = x, \]

(2.21)

\[ y(t_0) = y_A(t_0), \]

(2.22)

\[ y'(t_0) = y'_A(t_0) = \frac{y_B(t_0) - a_0(t_0)y_A(t_0)}{a_1(t_0)}, \]

(2.23)

\[ y''(t_0) = y''_A(t_0) = \frac{1}{a_1(t_0)}y'_B(t_0) - \frac{a_0(t_0) + a'_1(t_0)}{a_1^2(t_0)}y_B(t_0) + \left[ \frac{a''_1(t_0) + a'_0(t_0)a_0(t_0)}{a_1^2(t_0)} - \frac{a''_0(t_0)}{a_1(t_0)} \right] y_A(t_0). \]

(2.24)

The derivative of the initial conditions in Eqs. (2.22)–(2.24) is done as follows: Equation (2.22) follows from Eq. (2.17). To find Eq. (2.22), we start from Eq. (2.17) and write \( y'(t) = y'_A(t_0) \), from Eq. (2.2)

\[ y(t) = y'_A(t) = \frac{x_A(t) - a_0(t)y_A(t)}{a_1(t)} = \frac{y_B(t) - a_0(t)y_A(t)}{a_1(t)}. \]

(2.25)

Inserting \( t = t_0 \) yields Eq. (2.23). To find Eq. (2.24), we start from Eq. (2.17), take derivative of Eq. (2.2), and solve result for \( y''_A \)

\[ y'' = y''_A = \frac{y_B - (a'_1 + a_0)y'_A - a'_0y_A}{a_1}. \]

(2.26)

Using the expression Eq. (2.25) for \( y'_A \) in Eq. (2.26), ordering the terms and evaluating at \( t = t_0 \) yield the initial conditions in Eq. (2.24).
3 Commutativity Requirements

For the commutativity of subsystems $A$ and $B$, their combinations $AB$ and $BA$ must have the same outputs for general values of the same input and the same initial conditions. This is due to the existence of unique equal solutions of differential equations derived in Eqs. (2.11)–(2.14) and (2.21)–(2.24) for the same input and initial conditions. Hence, equating the coefficients of these differential equations, collecting the like terms we result with

\[
\begin{align*}
    a_1 b'_2 &= 2a'_1 b_2 \\
    a_1 b'_1 &= a'_1 b_1 + (a''_1 + 2a'_0) b_2 \\
    a_1 b'_0 &= a''_0 b_2 + a'_0 b_1 \\
    y &= y_B = y_A \\
    y' &= y'_B = \frac{y_B - a_0 y_A}{a_1} \\
    y'' &= \frac{y_A - b_0 y_B - b_1 y'_B}{b_2} = \frac{1}{a_1} y'_B - \frac{a_0 + a'_1}{a^2_1} y_B + \left( \frac{a''_0 + a_0 a'_1 - a'_0 a_1}{a^3_1} \right) y_A
\end{align*}
\]

Note that Eqs. (3.4)–(3.6) [so should (3.7)–(3.10)] should be valid at the initial time $t = t_0$ which is not shown explicitly. Before proceeding further, we simplify Eqs. (3.4)–(3.6) to obtain simpler set of constraints.

\[
\begin{align*}
    y &= y_B = y_A, \\
    y' &= y'_B = \frac{1 - a_0}{a_1} y_A, \\
    y'' &= \left[ \frac{1 - b_0}{b_2} - \frac{b_1 (1 - a_0)}{b_2 a_1} \right] y_A = \frac{(1 - a_0)(1 - a_0 - a'_1) - a'_0 a_1}{a^2_1} y_A. \\
\end{align*}
\]

Hence, Eq. (3.9) requires.

\[
\left[ \frac{1 - b_0}{b_2} - \frac{b_1 (1 - a_0)}{b_2 a_1} - \frac{(1 - a_0)(1 - a_0 - a'_1) - a'_0 a_1}{a^2_1} + \frac{a''_0}{a^3_1} \right] y_A = 0. \tag{3.10}
\]

Equation (3.1) has a solution for $b_2$ in terms of $a_i$’s as

\[
b_2 = e_2 a^2_1, \tag{3.11}
\]

where $e_2$ is an arbitrary nonzero constant. Using this solution in (3.2) and taking integral, we proceed

\[
a_1 b'_1 = a'_1 b_1 + (a''_1 + 2a'_0)e_2 a^2_1
\]
\[
\frac{a_1 b_0' - a_1' b_1}{a_1^2} = e_2 (a_1'' + 2a_0') \\
\frac{d}{dt} \left( \frac{b_1}{a_1} \right) = e_2 (a_1'' + 2a_0') \\
b_1 \frac{a_1'}{a_1} = e_2 (a_1' + 2a_0) + e_1 \\
b_1 = e_2 (a_1' + 2a_0) a_1 + e_1 a_1. \tag{3.12}
\]

Inserting values of \( b_2 \) in Eq. (3.11) and \( b_1 \) in Eq. (3.12) into Eq. (3.3), we proceed

\[
a_1 b_0' = a_1'' e_2 a_1^2 + a_0' \left[ e_2 (a_1' + 2a_0) a_1 + e_1 a_1 \right] \\
b_0' = a_1'' e_2 a_1 + a_0' \left[ e_2 (a_1' + 2a_0) + e_1 \right] \\
= e_2 \left( a_1'' + a_0' a_1' + 2a_0 a_0' \right) + e_1 a_0' = e_2 \frac{d}{dt} (a_0 a_1 + a_0^2) + e_1 a_0', \\
b_0 = e_2 (a_0 a_1 + a_0^2) + e_1 a_0' + e_0. \tag{3.13}
\]

where \( e_0 \) and \( e_1 \) are an arbitrary constants. In the matrix form

\[
\begin{bmatrix}
b_2 \\
b_1 \\
b_0
\end{bmatrix} =
\begin{bmatrix}
a_1^2 & 0 & 0 \\
a_0' + 2a_0 & 1 & 0 \\
a_0' a_1 + a_0^2 & a_0' & 1
\end{bmatrix}
\begin{bmatrix}
e_2 \\
e_1 \\
e_0
\end{bmatrix}. \tag{3.14}
\]

Hence, Eqs. (3.1)–(3.3) are equivalently replaced by Eq. (3.14). Inserting values of \( b_2, b_1, b_0 \) computed in Eqs. (3.11)–(3.13) in Eq. (3.10), after simplification, we result with

\[
(e_2 + e_1 + e_0 - 1) y(t_0) = 0. \tag{3.15}
\]

Since \( t_0 \) is any initial state for nonzero initial conditions \( y_A(t_0) = y_B(t_0) = y(t_0) \neq 0 \), \( (y_B'(t_0)) \) may be zero if \( a_0 = 1 \) due to Eq. (3.8), Eq. (3.15) implies that

\[
e_2 + e_1 + e_0 = 1. \tag{3.16}
\]

If commutativity with nonzero initial conditions is to be satisfied, Eq. (3.10) can be replaced by

\[
y_B(t_0) = y_A(t_0) \neq 0 \tag{3.17}
\]
\[
y_B'(t_0) = \frac{1 - a_0}{a_1} y_A(t_0), \tag{3.18}
\]
\[
e_2 + e_1 + e_0 = 1, \tag{3.19}
\]
\[
y'' = \frac{(1 - a_0)(1 - a_0 - a_1') - a_0' a_1}{a_1^2} y_A(t_0). \tag{3.20}
\]
4 Decomposition Formulas

We now express the coefficients of the decompositions $A$ and $B$ in terms of these of the decomposed system $C$. Comparing Eqs. (2.1) and (2.11), equating the coefficients of third derivatives, and using Eq. (3.14), we have

$$a_1b_2 = c_3 = a_1e_1a_1^2 \rightarrow a_1 = \left(\frac{c_3}{e_2}\right)^{1/3}. \quad (4.1)$$

Comparing Eqs. (2.1) and (2.11), equating the coefficients of second derivatives, and using Eq. (3.14), we obtain

$$a_1b_2' + a_1b_1 + a_0b_2 = c_2 \rightarrow a_0 = \frac{1}{b_2} (c_2 - a_1b_2' - a_1b_1)$$

$$= \frac{1}{e_2a_1^2} \left\{ c_2 - a_1 \frac{d}{dt} (e_2a_1^2) - a_1e_2[(a_1' + 2a_0)a_1 + e_1a_1] \right\}$$

$$= \frac{1}{e_2a_1^2} \left[ c_2 - 2e_2a_1^2a_1' - e_2a_1^2(a_1' + 2a_0) - a_1^2 \right]$$

$$= \frac{1}{e_2} \left( \frac{c_2}{a_1^2} - 2e_2a_1' - e_2a_1' - 2e_2a_0 - e_1 \right) = \frac{c_2}{e_2a_1^2} - 3a_1' - 2a_0 - \frac{e_1}{e_2}$$

$$3a_0 = \frac{c_2}{e_2a_1^2} - 3a_1' - \frac{e_1}{e_2}.$$

Dividing by 3 and using Eq. (4.1), we proceed as

$$a_0 = \frac{c_2}{e_2a_1^2} - a_1' - \frac{e_1}{3e_2} = \frac{(e_2)^{1/3}c_2}{3e_2(c_3)^{2/3}} - \frac{1}{3} \left( \frac{c_3}{e_2} \right)^{-2/3} \frac{c_3'}{e_2} - \frac{e_1}{3e_2}$$

$$= \frac{c_2}{3e_2^{1/3}c_3^{2/3}} - \frac{c_3'}{3e_2^{2/3}c_3^{1/3}} - \frac{e_1}{3e_2} = \frac{c_2 - c_3'}{3e_2^{1/3}c_3^{2/3}} - \frac{e_1}{3e_2}. \quad (4.2)$$

Having computing $a_1$ and $a_0$ in Eqs. (4.1) and (4.2), inserting those values in Eq. (3.14), we compute $b_2$, $b_1$, $b_0$ and the results:

$$b_2 = 3e_2^{1/3}c_3^{2/3}, \quad (4.3)$$

$$b_1 = \frac{1}{3} \left[ \left( \frac{e_2}{c_3} \right)^{1/3}(2c_2 - c_3') + e_1 \left( \frac{c_3}{e_2} \right)^{1/3} \right], \quad (4.4)$$

$$b_0 = \frac{1}{9} \left[ \left( \frac{e_2}{c_3} \right)^{1/3}(3c_2' - 3e_3'') + \frac{c_3^2}{c_3} \left( \frac{c_3'}{c_3} \right)^2 - 4c_2'c_3' \right]$$

$$+ \frac{1}{9} \left[ \frac{e_1(c_2 - c_3')}{e_2^{1/3}c_3^{2/3}} - \frac{2e_1}{e_2} \right] + b_0. \quad (4.5)$$
Comparing Eq. (2.1) with Eq. (2.11), two additional equations should be satisfied for the equivalence of $C$ and $AB$ (or $BA$, since $AB$ is a commutative pair). These are

$$c_1 = a_1b_1' + a_1b_0 + a_0b_1,$$

$$c_0 = a_1b_0' + a_0b_0.$$  \hspace{1cm} (4.6)

Inserting the values of $a_1, a_0$ in Eqs. (4.1), (4.2) and $b_1, b_0$ as computed in Eqs. (4.4), (4.5) into Eqs. (4.6) and (4.7) and making a great deal of computations, we obtain the additional conditions to be satisfied;

$$c_1 = \left( c_2' - \frac{2}{3} c_3'' \right) \frac{1}{c_3} \left[ \frac{5}{9} \left( c_3' \right)^2 - c_2 c_3' + \frac{c_2^2}{3} \right] + c_3^{1/3} \frac{1}{e_2^{1/3}} \left( e_0 - \frac{e_1^2}{3e_2} \right),$$ \hspace{1cm} (4.8)

$$c_0 = \frac{1}{3} \left( c_2'' - c_3''' \right) + \frac{1}{3c_3} (c_2 - 2c_3')(c_2' - c_3'')$$

$$+ \frac{1}{27c_3^2} \left[ 15 \left( c_3' \right)^2 - 8c_3' c_2 - 6c_3''c_2 + c_2^2 \right] (c_2 - c_2')$$

$$+ \frac{c_2 - c_3'}{3c_3^2/3b_2^{1/3}} \left( e_0 - \frac{e_1^2}{3e_2} \right) + \frac{1}{3} \left( \frac{2e_1^2}{9} - \frac{e_0}{e_2} \right).$$ \hspace{1cm} (4.9)

In light of the result obtained so far, we now express the main theorem about the decomposition of a third-order linear time-varying system into its commutative first- and second-order linear time-varying components.

**Theorem 1** The necessary and sufficient conditions that a third-order linear time-varying system described by Eq. (2.1) into its cascade-connected linear time-varying commutative pairs of first order and second order are that

(i) The coefficients $c_1$ and $c_0$ are expressible in terms of $c_3$ and $c_2$ through formulas Eqs. (4.8) and (4.9) where $e_2, e_1, e_0$ are some constants.

(ii) If the condition $y(t_0)$ of $C$ is different from zero, additional necessary and sufficient conditions are expressed as

$$e_2 + e_1 + e_0 = 1,$$  \hspace{1cm} (4.10)

$$y'(t_0) = \left[ \left( \frac{e_2}{c_3} \right)^{1/3} \left( 1 + \frac{e_1}{3e_2} \right) - \frac{c_2 - c_3'}{3c_3} \right] y(t_0) \text{ at } t = t_0,$$  \hspace{1cm} (4.11)

$$y'(t_0) = \left[ \left( \frac{e_2}{c_3} \right)^{1/3} \left( 1 + \frac{e_1}{3e_2} \right) - \frac{c_2 - c_3'}{3c_3} \right]^2 y_A(t_0)$$

$$+ \frac{d}{dt} \left[ \left( \frac{e_2}{c_3} \right)^{1/3} \left( 1 + \frac{e_1}{3e_2} \right) - \frac{c_2 - c_3'}{3c_3} \right] y_A(t_0).$$  \hspace{1cm} (4.12)

**Proof** Part (i) is simply re-expressing of Eqs. (4.8) and (4.9). Equation (4.10) is the repetition of Eq. (3.19). Equation (4.11) is obtained from Eq. (3.18) by inserting in
the values of $a_1$ and $a_0$ in Eqs. (4.1) and (4.2), respectively. So,

$$y'(t_0) = \frac{1 - a_0}{a_1} y(t_0) = \left( \frac{e_2}{c_3} \right)^{1/3} \left( 1 + \frac{e_1}{3e_2} - \frac{c_2 - c_3}{3c_3} \right) y(t_0)$$

$$= \left[ \left( \frac{e_2}{c_3} \right)^{1/3} \left( 1 + \frac{e_1}{3e_2} \right) - \frac{c_2 - c_3}{3c_3} \right] y(t_0).$$

Finally, Eq. (4.12) is obtained from Eq. (3.19) by inserting in values of $a_1$ and $a_0$ in Eqs. (4.1) and (4.2), respectively, as follows:

$$y''(t_0) = \frac{(1 - a_0)(1 - a_0 - a_1') - a_0a_1}{a_1^2} y_A(t_0)$$

$$= \frac{(1 - a_0)^2 - a_1' + a_0a_1'}{a_1^2} y_A(t_0)$$

$$= \left[ \frac{(1 - a_0)^2}{a_1^2} + \frac{d}{dt} \frac{1}{a_1} - \frac{d}{dt} \frac{a_0}{a_1} \right] y_A(t_0) = \left[ \frac{(1 - a_0)^2}{a_1^2} + \frac{d}{dt} \frac{1 - a_0}{a_1} \right] y_A(t_0)$$

$$= \left[ \left( \frac{e_2}{c_3} \right)^{1/3} \left( 1 + \frac{e_1}{3e_2} \right) - \frac{c_2 - c_3}{3c_3} \right] y_A(t_0)$$

$$+ \frac{d}{dt} \left[ \left( \frac{e_2}{c_3} \right)^{1/3} \left( 1 + \frac{e_1}{3e_2} \right) - \frac{c_2 - c_3}{3c_3} \right] y_A(t_0).$$

The corollary expresses how to obtain the commutative pairs of decomposition $A$ and $B$. \hfill \Box

**Corollary 2**  For a third-order linear time-varying system $C$ described by Eq. (2.1) with the conditions of Theorem I satisfied, the decomposed commutative pairs $A$ and $B$ are found by Eqs. (4.1) and (4.2) for the coefficients $a_1$ and $a_0$ of $A$, and by Eqs. (4.3), (4.4), (4.5) for the coefficients of $b_2$, $b_1$, $b_0$ of $B$, all, respectively. Further, for the commutative decompositions with nonzero initial condition $y(t_0) \neq 0$, Eq. (4.10) relating the constants $e_2$, $e_1$, $e_0$ must be satisfied, and $y_B'(t_0) = y'(t_0)$ and $y''(t_0)$ must be expressible in terms of $y(t_0) = y_B(t_0) = y_A(t_0)$ as in Eqs. (4.11) and (4.12), respectively.

**Proof**  The proof follows from the development of the mentioned equation. Equality of $y(t_0) = y_B(t_0) = y_A(t_0)$ is a result of Eqs. (2.12) and (2.15); equality of $y'(t_0) = y'_A(t_0)$ is already expressed in Eq. (2.12). \hfill \Box

### 5 Examples

In this section, four examples are considered to illustrate the results of the paper. The simulations are conducted by MATLAB R2012a and obtained by a PC Intel Core i3 CPV, 2.13 GHz, 3.86 GB of RAM well verify the results.
5.1 Example 1

Let $C$ be the third-order linear time-varying system defined by

$$y'''(t) + (t + 1)y''(t) + \frac{1}{3}(t^2 + 2t)y'(t) + \frac{1}{27}(t^3 + 3t^2 + 9)y(t) = x(t), \quad (5.1)$$

which the coefficients are

$$c_3 = 1, \quad c_2 = (t + 1), \quad c_1 = \frac{1}{3}(t^2 + 2t), \quad c_0 = \frac{1}{27}(t^3 + 3t^2 + 9), \quad (5.2)$$

with the constants

$$e_2 = e_1 = 1, \quad e_0 = -1, \quad (5.3)$$

which satisfies Eq. (4.10), it is true the conditions (i) of Theorem I are satisfied; that is $c_1$ and $c_0$ satisfy Eqs. (4.8) and (4.9), respectively. For the validity of the decomposition with nonzero initial condition $y(t_0) \neq 0$, condition (4.10) of (ii) is satisfied by the constant chosen in Eq. (5.3). Further, Eqs. (4.11) and (4.12) of (ii) together with Corollary 2 yield

$$y_A(t_0) = y_B(t_0) = y(t_0), \quad (5.4)$$

$$y'_B(t_0) = y'(t_0) = \left[\left(1 + \frac{1}{3}\right) - \frac{t_0 + 1}{3}\right]y(t_0)$$

$$= \left(1 - \frac{t_0}{3}\right)y(t_0) = y(t_0) \text{ for } t_0 = 0, \quad (5.5)$$

$$y''(t_0) = \left[\left(1 + \frac{1}{3} - \frac{t_0 + 1}{3}\right)^2 + \frac{d}{dt}\left(1 + \frac{1}{3} - \frac{t + 1}{3}\right)\right]y(t_0)$$

$$= \left[\left(1 - \frac{t_0}{3}\right)^2 + \frac{d}{dt}\left(1 - \frac{t}{3}\right)\right]y(t_0)$$

$$= \left[\left(1 - \frac{t_0}{3}\right)^2 - \frac{1}{3}\right]y(t_0) = \frac{2}{3}y(t_0) \text{ for } t_0 = 0. \quad (5.6)$$

Corollary 2 yields the following coefficients for decomposed subsystems $A$ and $B$:

$$A : y'_A(t) + \frac{t}{3}y_A(t) = x_A(t), \quad (5.7)$$

$$B : y''_B(t) + \frac{2t + 3}{3}y'_B(t) + \frac{t^2 + 3t - 6}{9}y_B(t) = x_B(t). \quad (5.8)$$

Simulations are carried out with a sinusoidal input of amplitude 10, bias $-5$ and frequency 3. Fixed step length of 0.01 is used by ode(Bogacki–Shampine). Simulink
Decomposition of $C$ into its commutative pairs $A$ and $B$ ($AB$, $BA$, $C$); some of the conditions of decomposition are not
results of MATLAB R2012 are shown in Fig. 2. The initial time $t_0$ is assumed 1, and the initial states are taken as $y(1) = y_A(1) = y_B(1) = 1$. When $y_B'(1) = y'(1) = 1$ and $y''(1) = 2/3$ as implied by (5.4), (5.5), (5.6) all the decomposition conditions are satisfied and $AB$, $BA$, and $C$ give the same responses as indicated by the figure legend. But when $y''(1)$ is changed to 2 which does not satisfy (5.6), the response $C1$ becomes different from those of $AB$ and $BC$, that is the decomposition get spoiled; although $A$ and $B$ are commutative, they are not the correct decomposition of $C$. On the other hand, when $y_B'(1)$ is made $-1$, that is (5.5) is not satisfied, the response of $AB$ (indicated by $AB3$) gets different from those of $BA$ and $C$, so commutative decomposition of $C$ into $A$ and $B$ is not valid again.

### 5.2 Example 2

Consider $C$ defined by

$$t^3 y'''(t) + 7t^2 y''(t) + 9ty'(t) + y(t) = x(t), \quad (5.9)$$

which satisfies the condition of Theorem I with $e_2 = e_1 = 1$, $e_0 = -1$. Hence, with $c_3 = t^3$, $c_2 = 7t^2$, the initial conditions should satisfy Eqs. (4.11) and (4.12):

$$y'(t_0) = \left[ \frac{1}{t_0} \left( 1 + \frac{1}{3} \right) - \frac{7t_0^2 - 3t_0^2}{3t_0^3} \right] y(t_0) = 0, \quad (5.10)$$
The decompositions $A$ and $B$ are found by using Eqs. (4.1), (4.2) and (4.3)–(4.5) as

\begin{align*}
  A & : t y''_A(t) + y_A(t) = x_A(t), \quad y_A(t_0) = y(t_0), \quad y'_A(t_0) = y'(t_0) = 0, \\
  B & : t^2 y''_B(t) + 4t y'_B(t) + y_B(t) = x_B(t), \quad y_B(t_0) = y(t_0), \quad y'_B(t_0) = y'(t_0) = 0.
\end{align*}

(5.12, 5.13)

Note that $y'(t_0) = y'_B(t_0)$ and $y''(t_0)$ are zero for all initial times $t_0$. The simulations are carried out with a sinusoidal input of amplitude 100, frequency 100 Hz and phase $\pi/3$ rad; the initial time $t_0$ is taken as 0.01 and stop time is 0.15; ode(Bogacki–Shampine) solver is used with step length of 0.001. The initial values are assumed as $y_A(t_0) = y_B(t_0) = y(t_0) = -4$. As it is seen in Fig. 3, $C$ and its commutative decompositions $AB$ and $BA$ yield the same responses (see $AB = BA = C$). When the decomposition requirement on initial condition gets spoiled, that is $y'(0.01) = y'_B(0.01) \neq 0$ and

$$
\begin{align*}
  y''(t_0) &= \left[ \frac{1}{t_0} \left( 1 + \frac{1}{3} \right) - \frac{7t_0^2 - 3t_0^3}{3t_0^3} \right]^2 \\
  &+ \frac{d}{dt} \left[ \frac{1}{t} \left( 1 + \frac{1}{3} \right) - \frac{7t^2 - 3t^3}{3t^3} \right]_{t=t_0} y(t_0) = 0.
\end{align*}
$$

(5.11)
taken as $-100$, the decomposition is not valid at all as seen from plots $AB_1$, $BA_1$, $C_1$ in the figure. It is important to note that the cascade connection $BA$ is least affected from this change. Hence, it is preferable decomposition or synthesis of $C$ when compared with $AB$ as far as sensitivity to initial conditions is concerned.

5.3 Example 3

Let $C$ be the third-order Euler system defined by

$$t^3 y''' + 9t^2 y'' + \frac{53}{3} ty' + \frac{155}{27} y = x. \quad (5.14)$$

Comparing it with (2.1), its coefficients are

$$c_3 = t^3, \ c_2 = 9t^2, \ c_1 = \frac{53}{3} t, \ c_0 = \frac{155}{27}. \quad (5.15)$$

It is true that the choice $e_2 = e_1 = 1$, $e_0 = -1$ satisfy the conditions of Theorem I; that is Eqs. (4.8), (4.9) and (4.10) are satisfied. Hence, the decomposition into first- and second-order commutative pairs with nonzero initial condition $y(t_0) \neq 0$ is possible. The initial condition of $A$ and $B$ as well as those of $C$ is found by using...
The decompositions $A$ and $B$ are found by using the coefficients given in Eqs. (4.1), (4.2) and (4.3)–(4.5). With the above initial conditions, $A$ and $B$ are defined by

$$y_A(t_0) = y_B(t_0) = y(t_0) \neq 0,$$

$$y'_B(t_0) = y'(t_0) = \left[ \left( \frac{1}{t_0^3} \right)^{1/3} \left( 1 + \frac{1}{3} \right) - \frac{9t_0^2 - 3t_0^2}{3t_0^2} \right]$$

$$y(t_0) = -\frac{2}{3t_0} y(t_0),$$

$$y''(t_0) = \begin{cases} \left[ \frac{1}{t_0} \left( 1 + \frac{1}{3} \right) - \frac{3t_0^2 - 3t_0^2}{3t_0^2} - 1 \right]^2 \\ + \frac{d}{dt} \left( \frac{4}{3} \frac{t_0^2 - 3t_0^2}{3t_0^2} \right) \bigg|_{t=t_0} \end{cases} y(t_0)$$

$$= \left( \frac{4}{9t_0^2} + \frac{2}{3t_0^2} \right) y(t_0) = \frac{10}{9t_0^2} y(t_0).$$

The decompositions $A$ and $B$ are found by using the coefficients given in Eqs. (4.1), (4.2) and (4.3)–(4.5). With the above initial conditions, $A$ and $B$ are defined by

Fig. 5 Outputs of the original system $C$ and its cascade decompositions $AB$ and $BA$ when disturbance exists at the
\[ A : t y_A'(t) + \frac{5}{3} y_A(t) = x_A(t); \quad y_A(t_0) = y(t_0), \quad (5.19) \]

\[ B : t^2 y_B''(t) + \frac{16}{3} t y_B'(t) + \frac{31}{9} y_B(t) = x_B(t); \]

\[ y_B(t_0) = y(t_0); \quad y_B'(t_0) = y'(t_0) = -\frac{2}{3t_0^2} y(t_0). \quad (5.20) \]

The simulations are done for sinusoidal input of amplitude 10 and frequency 1. The initial time \( t_0 = 1 \); ode3(Bogacki–Shampine) solver is used with a fixed step-length of 0.01; simulations are stopped at \( t = 10 \). When the initial conditions \( y''(1) = 10/9 \), \( y'(1) = y_B'(1) = -2/3 \) are chosen in accordance with \( y_A(1) = y_B(1) = y(1) = 1 \) as to satisfy the decomposition above-mentioned conditions, \( AB, BA, C \) give the same response as shown in Fig. 4 (see \( AB = BA = C \)). In the same figure, zero-input responses (\( AB_1 = BA_1 = C_1 \)) and zero-state responses (\( AB_2 = BA_2 = C_2 \)) are also potted. Obviously, decomposition is valid for unexited–unrelaxed and exited–relaxed cases as well.

### 5.4 Example 4

This example is the same as the first one except all the initial conditions are taken as zero and a noise signal is added between the junction of subsystems \( A \) and \( B \). The noise is a pulse sequence with amplitude 4, % 50 pulse with, and a bias of \(-2.3\). The simulation results are shown in Fig. 5. Obviously, the interconnection \( AB \) is less affected by this noise than \( BA \) connection when compared with the output of the original system \( C \). Hence, the cascade synthesis \( AB \) should be preferred rather than \( BA \).

### 6 Conclusions

In this paper, the decomposition of any third-order linear time-varying system into its first- and second-order commutative pairs is investigated. Explicit decomposition formulas are derived for the case of zero and nonzero initial conditions. The results are validated by computer simulations. The work is original and appears for the first time in the literature. It is important from the synthesis and/or design point of views of engineering systems. Many design methods are based on tearing and reconstruction, which is combining simple components to obtain an assembly. Further, it is shown that some combinations may be better than the others when sensitivity to initial conditions and noise disturbance at the interconnection is taken into account. On the other hand, commutativity of cascade-connected systems has gained a great deal of interest and its possible benefits have been pointed out on the literature. Hence, the results of this paper can be used readily for beneficial synthesis of third-order linear time-varying systems.

The results of this work can be extended for discrete-time systems and even for fractional-order systems \([3,4]\) as well.
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