Conformal Invariance and Renormalization Group
in Quantum Gravity Near Two Dimensions

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ABSTRACT
We study quantum gravity in $2 + \epsilon$ dimensions in such a way to preserve the volume preserving diffeomorphism invariance. In such a formulation, we prove the following trinity: the general covariance, the conformal invariance and the renormalization group flow to Einstein theory at long distance. We emphasize that the consistent and macroscopic universes like our own can only exist for matter central charge $0 < c < 25$. We show that the spacetime singularity at the big bang is resolved by the renormalization effect and universes are found to bounce back from the big crunch. Our formulation may be viewed as a Ginzburg-Landau theory which can describe both the broken and the unbroken phase of quantum gravity and the phase transition between them.
1. Introduction

The reconciliation of quantum mechanics and Einstein’s theory of gravitation has been a long standing problem in theoretical physics. It has lead to the modification of Einstein’s general theory of relativity such as string theory. However in two dimensions, this conflict can be resolved because Einstein gravity is power counting renormalizable. The remarkable point is the asymptotic freedom of the gravitational coupling constant near two dimensions. Beyond two dimensions, the gravitational coupling constant acquires the canonical dimension. So it effectively grows at short distance. However this growth is counter balanced by the asymptotic freedom and consistent quantum theories may be constructed even beyond two dimensions. \(2 + \epsilon\) expansion of quantum gravity is base on such an idea[1][2][3]. In view of the exact solution of two dimensional gravity[4], this approach may be developed further[5].

Although this idea works near two dimensions, the draw back is that we have to put \(\epsilon = 1\) or 2 to reach three or four dimensions. One of the motivations to pursue this approach comes from the recent progress in dynamical triangulation method in realistic dimensions. The existence of phase transitions itself is encouraging and the strong coupling phase appears to resemble two dimensional quantum gravity[6]. However physically interesting weak coupling region is afflicted with the conformal mode instability and may require great care to extract physics.

Another motivation to study quantum gravity in \(2 + \epsilon\) dimensions is to regard it as a toy model and try to learn lessons for realistic quantum gravity. In this regard we can cite the questions concerning spacetime singularities which appear at the big bang or the end of the blackhole evaporation. Of course various questions which arise from the existence of event horizons are also interesting to analyze in this context. We can even think of string theory application of this approach by taking \(\epsilon \to 0\) limit.

We use dimensional regularization to compute Feynman diagrams which respects the crucial symmetries of the theory, namely general covariance and confor-
mal invariance. In quantum gravity, the metric $g_{\mu\nu}$ is the fundamental dynamical quantity to integrate. We decompose the metric into the conformal mode $\phi$ and the traceless symmetric tensor $h_{\mu\nu}$. It is also convenient to introduce a reference metric $\hat{g}_{\mu\nu}$. We parametrize the metric of the space-time as $g_{\mu\nu} = \tilde{g}_{\mu\nu}e^{-\phi} = \hat{g}_{\mu\rho}(e^h)_{\rho\nu}e^{-\phi}$ where $h^\mu{}_{\nu}$ field is taken to be traceless $h^\mu{}_{\mu} = 0$, while $h_{\mu\nu}$ is symmetric in $\mu$ and $\nu$. The tensor indices are raised and lowered by the background metric $\hat{g}_{\mu\nu}$.

The dynamics of these two fields turn out to be very different. We need the Einstein action as a counter term to cancel the divergences of correlation functions of $h_{\mu\nu}$ field. However such a counter term is an oversubtraction for the conformal mode. This situation is easy to understand in conformal gravity since the one loop amplitudes do not involve the conformal mode. The Einstein gravity becomes conformally invariant in two dimensions.

The bare Einstein action of the theory which contains $c$ copies of real scalar fields in our parametrization is

$$\frac{1}{G^0} \int d^Dx \sqrt{g} R = \mu^\epsilon \left( \frac{1}{G} - \frac{25 - c}{24\pi \epsilon} \right) \int d^Dx \sqrt{g} e^{-\frac{2}{3}\phi} \left\{ \tilde{R} - \frac{\epsilon(D-1)}{4} \partial_{\mu}\phi \partial_{\nu}\phi \tilde{g}^{\mu\nu} \right\}$$

$$\rightarrow \frac{Q_{eff}^2}{32\pi} \int d^Dx \sqrt{\tilde{g}} (\partial_{\mu}\phi \partial^{\mu}\phi + 2\phi \tilde{R}). \quad (1)$$

In this expression, $G^0$ and $G$ are the bare and the renormalized gravitational coupling constant respectively. $\mu$ is the renormalization scale to define $G$. The effective central charge ($c_{eff}$) which governs the dynamics of the conformal mode can be defined as $Q_{eff}^2 = \frac{25 - c_{eff}}{3} = \frac{25 - c}{3} - \frac{8\pi \epsilon}{G}$. Hence the Liouville theory is obtained for $\phi$ field and the exact gravitational scaling exponents are reproduced by taking $\epsilon \rightarrow 0$ limit in the strong coupling regime. The novel feature beyond two dimensions is that the gravitational coupling constant acts as the effective central charge for the dynamics of the conformal mode $\phi[7][8]$. This approach is further studied in [10][11]. The work which addresses relevant questions to ours is [9].
However the renormalization of $\phi$ field is hard to keep track in a manifestly generally covariant form. We have proposed to formulate the theory in such a way to preserve the manifest volume preserving diffeomorphism invariance[8]. In such a formalism, the general covariance can be recovered by further demanding the conformal invariance on the theory. Another idea to ensure the general covariance on the theory is to consider the renormalization group trajectory in the coupling constant space which leads to the Einstein gravity at long distance. In this paper we study the effectiveness of these ideas in detail by analyzing conformally coupled Einstein gravity. We prove the validity of our proposal to the one loop order and give a concrete prescription to calculate higher order corrections.

It is often thought that we may be in the broken phase of quantum gravity since the expectation value of the metric is certainly nonvanishing in our Universe. We may think of the unbroken phase of quantum gravity in which the metric is fluctuating around the vanishing expectation value. As it turns out that our formalism can describe not only the broken phase of quantum gravity but also the unbroken phase. Furthermore it can study the phase transition between them. In the phase transition theory of statistical systems, the Ginzburg-Landau theory plays an important conceptual role to discuss symmetry breaking and the order of phase transitions and so on. In quantum gravity, it is certainly desirable to have such a description. Our formalism supplies such a conceptually crucial tool to investigate quantum gravity.

We consider Minkowski metric in this paper although Euclidian rotation of the theory is valid at the Feynman perturbation theory level. We perform real calculations in that way. However Euclidian quantum gravity beyond two dimensions is unstable due to the negative sign of the kinetic term of the conformal mode in the weak coupling regime. Presumably such a theory does not possess a stable vacuum. This feature is what we expect for quantum gravity because the Universe has begun with the big bang and it is still expanding.

The organization of this paper is as follows. In section 2, we consider the con-
formally coupled Einstein gravity. In section 3, we renormalize the theory to the one loop level and the renormalization group equations are derived. In section 4, we prove the trinity in our formalism: the general covariance, the conformal invariance and the renormalization group flow to the Einstein gravity at long distance. In section 5, we study the subtractions of the internal loops in preparation for the two loop renormalization of the theory. In section 6, we study the renormalization of the cosmological constant operator. In section 7, we consider the cosmological implications of this model. The last section consists of the conclusions and discussions.

2. Conformal Gravity in $2 + \epsilon$ Dimensions

Let us couple a real scalar field $\psi$ to Einstein gravity in a conformally invariant way:

$$\int d^Dx\sqrt{g} \{ R \frac{\epsilon}{8(D-1)} \psi^2 - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi g^{\mu\nu} \}. \quad (2)$$

Since the action is conformally invariant, the conformal mode dependence drops out of the action as,

$$\int d^Dx\sqrt{\hat{\bar{g}}} \{ \tilde{R} \frac{\epsilon}{8(D-1)} \psi^2 - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi \tilde{g}^{\mu\nu} \}, \quad (3)$$

where $\tilde{R}$ is the scalar curvature made out of $\tilde{g}_{\mu\nu}$. By changing the variable as $\sqrt{\frac{\epsilon}{8(D-1)}} \psi = e^{-\frac{\epsilon}{4}\phi}$, we obtain

$$\int d^Dx\sqrt{\bar{g}} e^{-\frac{\epsilon}{4}\phi} \{ \tilde{R} - \frac{\epsilon(D-1)}{4} \partial_\mu \phi \partial_\nu \phi \tilde{g}^{\mu\nu} \}. \quad (4)$$

Note that this is nothing but the Einstein action. Therefore we can regard the conformal mode as a matter field which couples to the Einstein gravity in a conformally invariant way. The Einstein gravity can be quantized from such a view point as long as we maintain the conformal invariance. In fact this strategy is that of D.D.K. in two dimensional quantum gravity [12][13]. We have proposed
to generalize this strategy (or nonlinear sigma model approach to string theory) into $2 + \epsilon$ dimensions[8].

However we point out the crucial difference between (3) and (4). Namely $\psi = 0$ point is mapped to $\phi = \infty$ during the change of the variables. The metric $g_{\mu\nu}$ vanishes at this point. Therefore it is impossible to study the fluctuations of the metric around such a point in the manifestly generally covariant form (4). However in (3), the expansion around $\psi = 0$ point does not present any problems except that the $\tilde{R}$ decouples at this point. As it turns out, this problem can be overcome in the quantum formulation due to the one loop quantum effect. Therefore our formulation has the definite advantage of being capable to study the unbroken phase of quantum gravity in which the metric is fluctuating around the vanishing expectation value.

In our previous work, we have studied Einstein gravity with minimally coupled scalar fields. In this paper we consider even simpler models, namely Einstein gravity with $c$ copies of conformally coupled scalar fields. The action we consider is,

$$ S = \frac{\mu^\epsilon}{G} \int d^D x \sqrt{g} \{ R(1 - \frac{\epsilon}{8(D-1)} \phi_i^2) + \frac{1}{2} \partial_\mu \phi_i \partial_\nu \phi_i g^{\mu\nu} \} $$

$$ = \frac{\mu^\epsilon}{G} \int d^D x \sqrt{\hat{g}} \{ \tilde{R}((1 + \frac{1}{2} \sqrt{\frac{\epsilon}{2(D-1)}} \psi)^2 - \frac{\epsilon}{8(D-1)} \psi_i^2) \} $$

$$ - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi \tilde{g}^{\mu\nu} + \frac{1}{2} \partial_\mu \phi_i \partial_\nu \phi_i \tilde{g}^{\mu\nu} \} $$

We expand $\psi$ around the constant mode in (3): $\sqrt{\frac{\epsilon}{8(D-1)}} \psi \rightarrow \tilde{\mu}_i^\epsilon(1 + \frac{1}{2} \sqrt{\frac{\epsilon}{2(D-1)}} \psi_i).$

We choose the renormalization scale $\mu$ to compensate the scale ($\tilde{\mu}$) of $\psi$ and $\phi_i$ fields. By doing so, we obtain the above action. Therefore in order to probe the beginning of the universe where $\psi$ is small, we need to consider the large renormalization scale $\mu$.

Our strategy is to impose the conformal invariance on the theory while regarding the conformal mode as a matter field. However the conformal invariance is in general broken by quantum effects. Hence we have to cancel the quantum
conformal anomaly by the tree level conformal anomaly. For this purpose, we have proposed to generalize the Einstein gravity as

$$\frac{\mu^\epsilon}{G} \int d^D x \sqrt{\tilde{g}} \{ \tilde{R} L(\psi, \varphi_i) - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi \tilde{g}^{\mu \nu} + \frac{1}{2} \partial_\mu \varphi_i \partial_\nu \varphi_i \tilde{g}^{\mu \nu} \}.$$  \hfill (6)

where $L(\psi, \varphi_i)$ is a function of $\psi$ and $\varphi_i$ with the constraint $L(0,0) = 1$. Note that in conformally coupled Einstein gravity,

$$L(\psi, \varphi_i) = 1 + \sqrt{\frac{\epsilon}{2(D-1)}} \psi + \frac{\epsilon}{8(D-1)}(\psi^2 - \varphi_i^2).$$  \hfill (7)

Since we have generalized the Einstein gravity, the tree action is no longer generally covariant. However our action is still invariant under the volume preserving diffeomorphism. We consider the following gauge transformation of the fields,

$$\delta \tilde{g}_{\mu \nu} = \partial_\mu \epsilon^\rho \tilde{g}_{\rho \nu} + \tilde{g}_{\mu \rho} \partial_\nu \epsilon^\rho + \epsilon^\rho \partial_\rho \tilde{g}_{\mu \nu} - \frac{2}{D} \partial_\rho \epsilon^\rho \tilde{g}_{\mu \nu},$$

$$\delta \psi = \epsilon^\rho \partial_\rho \psi + (D-1) \frac{\partial L}{\partial \psi} \frac{2}{D} \partial_\rho \epsilon^\rho,$$  \hfill (8)

$$\delta \varphi_i = \epsilon^\rho \partial_\rho \varphi_i - (D-1) \frac{\partial L}{\partial \varphi_i} \frac{2}{D} \partial_\rho \epsilon^\rho.$$

If $L$ is given by (7), this gauge invariance is nothing but the general covariance. The last term of each transformation can be viewed as a part of a conformal transformation. Since the action is invariant under the volume preserving diffeomorphism, it is invariant under the gauge transformation (8) if it is invariant under the following conformal transformation with respect to the background metric $\tilde{g}_{\mu \nu}$

$$\delta \tilde{g}_{\mu \nu} = - \tilde{g}_{\mu \nu} \delta \rho,$$

$$\delta h_{\mu \nu} = 0,$$

$$\delta \psi = (D-1) \frac{\partial L}{\partial \psi} \delta \rho,$$  \hfill (9)

$$\delta \varphi_i = -(D-1) \frac{\partial L}{\partial \varphi_i} \delta \rho.$$

Therefore the change of the action under this gauge transformation is nothing
but the conformal anomaly of the theory:

\[
\mu^5 \int d^D x \sqrt{\tilde{g}} \left\{ \frac{1}{2} \left[ \epsilon L - 2(D - 1)\left( \frac{\partial L}{\partial \psi} \right)^2 - \left( \frac{\partial L}{\partial \varphi_i} \right)^2 \right] \tilde{R} \right. \\
- \frac{1}{4} \left\{ \epsilon - 4(D - 1) \frac{\partial^2 L}{\partial \psi^2} \right\} \partial_\mu \psi \partial_\nu \psi \tilde{g}^{\mu \nu} \\
\left. + \frac{1}{4} \left\{ \epsilon + 4(D - 1) \frac{\partial^2 L}{\partial \varphi_i^2} \right\} \partial_\mu \varphi_i \partial_\nu \varphi_i \tilde{g}^{\mu \nu} \right\} \delta \rho, \tag{10}
\]

where \( \delta \rho = \frac{2}{D} \partial_\rho e^\rho \). Furthermore it vanishes for the Einstein gravity. Thus at the tree level, a general covariant theory is obtained from a theory which is invariant under volume preserving diffeomorphism by further demanding the conformal invariance. The advantage of this strategy is that it works at the quantum level. We have proposed to generalize the Einstein gravity to the most general renormalizable form which only possesses the volume preserving diffeomorphism invariance. The general covariance can be recovered by further demanding the conformal invariance. Alternatively we may follow the renormalization group trajectory of the most general renormalizable theory which leads to the Einstein gravity in the weak coupling limit. In this paper we investigate the effectiveness of these ideas in detail.

3. 1-Loop Renormalization and Renormalization Group

In this section we compute the one loop counter terms and derive the renormalization group equations for the coupling constants in conformally coupled Einstein gravity (6). We expand the fields around the classical part (backgrounds) and employ a background gauge. In the background gauge the manifest volume preserving diffeomorphism invariance is maintained. This procedure has been explained in [7][8]. In such a scheme, \( \tilde{g}_{\mu \nu} \) is expanded around the background metric \( \hat{g}_{\mu \nu} \) as \( \tilde{g}_{\mu \nu} = \hat{g}_{\mu \nu}(e^h)^\rho^\nu \). We would like to keep the gauge invariance of (8) as the symmetry of the theory.

We adopt the following gauge fixing term,

\[
\frac{1}{2} L(\bar{\psi} + \psi, \bar{\varphi}_i + \varphi_i)(\nabla^\mu h_{\mu \nu} - \frac{\partial_\nu L}{L})(\nabla_\rho h^{\rho \nu} - \frac{\partial^\nu L}{L}), \tag{11}
\]
where the covariant derivative is always taken with respect to the background metric $\hat{g}_{\mu\nu}$. The ghost action follows from the gauge fixing term in a standard way as

$$\nabla_\mu \bar{\eta}_\nu \nabla_\mu \eta'^\nu + \bar{R}_\mu \bar{\eta}_\mu \eta'^\nu - \frac{\partial_\mu L}{L} (\nabla^\mu \bar{\eta}_\mu) \eta'^\nu + \cdots.$$ \hspace{1cm} (12)

The tree action is generalized such that it breaks this gauge invariance at $O(\epsilon)$ because of the conformal anomaly. The tree level conformal anomaly of $O(\frac{\epsilon}{G})$ is required to cancel the one loop level conformal anomaly of $O(1)$ since $G$ is at most $O(\epsilon)$ around the short distance fixed point. The important point is that the one loop divergences of the theory have to respect this symmetry up to finite terms since the symmetry is only ‘softly’ broken at $O(\epsilon)$. Therefore the one loop counter terms can be chosen to possess the volume preserving diffeomorphism invariance. The exact gauge invariance can be recovered later by imposing the conformal invariance on the bare action.

The one loop divergence of this theory is evaluated to be

$$\int d^Dx \sqrt{\hat{g}} \left\{ \frac{26 - (1 + c)}{24\pi \epsilon} \right\} \tilde{R}.$$ \hspace{1cm} (13)

Hence the one loop bare action is

$$\int d^Dx \sqrt{\hat{g}} \frac{\mu^L}{G} \left\{ \tilde{R}L(\psi, \varphi_i) - \frac{1}{2} \partial_\mu \psi \partial_\nu \psi \hat{g}^{\mu\nu} + \frac{1}{2} \partial_\mu \varphi_i \partial_\nu \varphi_i \hat{g}^{\mu\nu} - \frac{A}{\epsilon} G \tilde{R} \right\},$$ \hspace{1cm} (14)

where $A = \frac{25 - c}{24\pi}$. We remark that a Dirac field gives rise to the same divergence with a real conformally coupled scalar field. It is straightforward to substitute a Dirac field for a conformally coupled scalar field.

In what follows, we derive the renormalization group equations for the coupling
constants. For this purpose, we define bare quantities as

\[
\frac{1}{G^0} = \mu \left( \frac{1}{G} - \frac{A}{\epsilon} \right), \\
\frac{1}{G^0} \partial_\mu \psi_0 \partial^\mu \psi_0 = \frac{\mu}{G} \partial_\mu \psi \partial^\mu \psi, \\
\frac{1}{G^0} \partial_\mu \phi^i_0 \partial^\mu \phi^i_0 = \frac{\mu}{G} \partial_\mu \phi^i \partial^\mu \phi^i, \\
L_0(\psi_0, \phi^i_0) = L(\psi, \phi^i) - \frac{A}{\epsilon} G + \frac{G A}{\epsilon} L(\psi, \phi^i).
\]

By solving these equations, we obtain

\[
\psi_0 = \psi(1 + \frac{A G}{2 \epsilon}), \\
\phi^i_0 = \phi^i(1 + \frac{A G}{2 \epsilon}), \\
L_0(\psi_0, \phi^i_0) = L(\psi, \phi^i) - \frac{A G}{\epsilon} + \frac{A G}{\epsilon} L(\psi_0, \phi^i_0) - \frac{A G}{2 \epsilon} \psi \frac{\partial L}{\partial \psi_0} - \frac{A G}{2 \epsilon} \phi^i \frac{\partial L}{\partial \phi^i_0}.
\]

\(\beta\) and \(\gamma\) functions (functionals to be precise) follow by demanding that the bare quantities do not depend on the renormalization scale \(\mu\) as

\[
\beta_G = \epsilon G - A G^2, \\
\beta_L = -G A (L - 1) + \frac{G A}{2} \psi \frac{\partial L}{\partial \psi} + \frac{G A}{2} \phi^i \frac{\partial L}{\partial \phi^i}, \\
\gamma_\psi = -\frac{A G}{2} \psi, \\
\gamma^i_\phi = -\frac{A G}{2} \phi^i.
\]

In this theory, \(L\) contains only finite numbers of couplings. Let us parametrize \(L\):

\[
L = 1 + a \psi + b \psi^2 - d \phi^2.
\]
Then $\beta_L$ simplifies as

$$
\begin{align*}
\beta_a &= -\frac{GA}{2}a, \\
\beta_b &= 0, \\
\beta_d &= 0.
\end{align*}
$$

So only two coupling constants ($G$ and $a$) possess nontrivial $\beta$ functions in this theory. We further remark that we do not need additional couplings to renormalize the theory to the one loop order.

Since we have derived the renormalization group equations for the couplings, we proceed to examine the renormalization group trajectory of the theory in the multidimensional space of the coupling constants. In this theory, we need to consider the two dimensional coupling space of $G$ and $a$. At long distance where the renormalization scale $\mu$ is small, the gravitational coupling is also small. Under such circumstances, the theory is classical and must agree with the conformally coupled Einstein gravity. Therefore we are interested in a particular renormalization group trajectory which starts with the Einstein gravity in the weak coupling limit. We can trace such a trajectory by solving $\beta$ functions. It is clear by inspecting the $\beta$ functions that such a trajectory is attracted to the ultraviolet stable fixed point at short distance as long as $c < 25$. At the fixed point $G^* = \frac{4}{A}$ and $a^* = 0$. Therefore we conclude that the short distance structure of spacetime is described by the following Lagrangian in conformal Einstein gravity near two dimensions.

$$
\int d^D x \sqrt{\tilde{g}} G^* \left\{ \tilde{R}(1 + \frac{\epsilon}{8(D-1)}(\psi^2 - \varphi_i^2)) \\
- \frac{1}{2} \partial_\mu \psi \partial_\nu \psi \tilde{g}^{\mu\nu} + \frac{1}{2} \partial_\mu \varphi_i \partial_\nu \varphi_i \tilde{g}^{\mu\nu} \right\}.
$$

The fixed point of the theory is realized at $\mu \rightarrow \infty$ limit. This point corresponds to the vanishing expectation value of $\psi$ as it is explained in Sec. 2 and in accord with the physical intuition. As we have remarked in Sec. 2, $\tilde{R}$ does not decouple at $\psi = 0$ point due to the one loop quantum effect. Furthermore the theory is
still weak coupling at short distance since $G^* \sim \epsilon$. This theory is thus capable to describe the quantum fluctuations of the metric around the vanishing expectation value. We also note that we can rotate $\psi$ field into the complex direction $\psi \rightarrow i\psi$ in the fixed point action while keeping the reality of it. By doing so, we obtain the Euclidian theory in which the conformal mode becomes indistinguishable from the conformally coupled matter fields. As it is well known the short distance fixed point represents a phase transition point. In the opposite phase of quantum gravity beyond the short distance fixed point, we may use such an Euclidian theory. In the oversubtraction scheme of (1), we recall that the sign of the kinetic term of the conformal mode flips at the fixed point. Our result is consistent with such a picture.

4. General Covariance, Conformal Invariance and Renormalization Group

In this section, we would like to investigate the gauge invariance of the theory in detail. In the weak coupling limit, the theory coincides with conformally coupled Einstein gravity. It has the gauge invariance which is nothing but the general covariance. The central question is whether the general covariance is maintained along the renormalization group trajectory. Since the renormalization group does not change the physics, we expect that the answer to this question is affirmative. In this section we prove that it is indeed the case by utilizing the renormalization group equations.

The symmetry of the theory becomes evident by investigating the bare action. Under the gauge transformation (8), the bare action (14) changes as,

$$
\delta S^0 = - \int d^D x \sqrt{g} \mu^e \left\{ \frac{1}{2} (\epsilon L - AG - 2(D - 1)) \left( \frac{\partial L}{\partial \psi} \right)^2 - \left( \frac{\partial L}{\partial \phi_i} \right)^2 \right\} \tilde{R} 
$$

$$
- \frac{1}{4} \left\{ \epsilon - 4(D - 1) \frac{\partial^2 L}{\partial \psi^2} \right\} \partial^\mu \psi \partial^\mu \psi 
$$

$$
+ \frac{1}{4} \left\{ \epsilon + 4(D - 1) \frac{\partial^2 L}{\partial \phi_i \partial \phi_j} \right\} \partial^\mu \phi_i \partial^\mu \phi_j \delta \rho, 
$$

where $\delta \rho = \frac{2}{D} \partial^\mu \epsilon^\mu$. Note that this is nothing but the trace anomaly $(T^\rho_\rho)$ of
the theory with respect to the reference (background) metric \( \delta \hat{g}_{\mu\nu} = -\hat{g}_{\mu\nu} \delta \rho \).

The only difference from the classical trace anomaly (10) comes from the one loop counter term. Since the second and third terms in (21) vanish identically, we only need to consider the first term. If we substitute the explicit parametrization of \( L \), the coefficient of \( \tilde{R} \) is

\[
\frac{\mu^\epsilon}{G} \{ \epsilon - GA - 2(D - 1)a^2 + \{ \epsilon - 8(D - 1)b \} a \psi \\
+ \{ \epsilon - 8(D - 1)b \} b \psi^2 - \{ \epsilon - 8(D - 1)d \} d \phi^2 \} = \frac{\mu^\epsilon}{G} \{ \epsilon - GA - 2(D - 1)a^2 \}.
\]

(22)

Here again only the first term is nontrivial. In order to ensure the gauge invariance of the theory, we have to make sure that this quantity vanishes. In fact it vanishes for the classical conformal Einstein gravity. So let us pick a point in the weak coupling region of \( G \) such that the trace anomaly vanishes there. Then we consider the renormalization group trajectory which passes through this point. The Einstein gravity lies on this trajectory in the weak coupling limit. By using the \( \beta \) functions which are obtained in Sec. 3, we can prove the invariance of the trace anomaly under the renormalization group:

\[
\mu \frac{\partial}{\partial \mu} \{ \frac{\mu^\epsilon}{G} (\epsilon - AG - 2(D - 1)a^2) \} = 0.
\]

(23)

Therefore the trace anomaly vanishes along the whole renormalization group trajectory, if it vanishes in the very weak coupling region. Recall that the bare action possesses manifest volume preserving diffeomorphism invariance. This symmetry is promoted to general covariance by further demanding conformal invariance since we can replace \( \tilde{g}_{\mu\nu} \rightarrow g_{\mu\nu} \) in this case. By this replacement, a manifestly volume preserving diffeomorphism invariant action becomes a manifestly generally covariant action. From these considerations, we conclude that the general covariance of the theory is maintained along the renormalization group trajectory which leads to the conformally coupled Einstein gravity in the weak coupling limit.
This conclusion stems from the two points. The first point is that the requirement of general covariance is equivalent to the vanishing of the trace anomaly in our formulation. The second point is the nonrenormalization theorem of the trace anomaly.

The nonrenormalization of the trace anomaly follows from the manifest general covariance with respect to the reference metric in our formulation. Therefore the reference metric is not renormalized. The trace anomaly with respect to the reference metric is defined as

\[ T^\rho_\rho = \hat{g}_{\mu\nu} \frac{\delta S^0}{\delta \hat{g}_{\mu\nu}}, \]  

where \( S^0 \) is the bare action and hence \( T^\rho_\rho \) is a finite operator. It is not renormalized since

\[ \mu \frac{\partial}{\partial \mu} T^\rho_\rho = \mu \frac{\partial}{\partial \mu} (\hat{g}_{\rho\sigma} \frac{\delta S^0}{\delta \hat{g}_{\rho\sigma}}) = 0. \]  

Let us consider more general models defined by the following action to see how these considerations generalize:

\[ \int d^D x \sqrt{\hat{g}} \frac{\mu^\epsilon}{G} \{ L(X^i) \hat{R} + \frac{1}{2} \hat{g}^{\mu\nu} G_{ij}(X^k) \partial_\mu X^i \partial_\nu X^j \}. \]  

We would like to impose the following symmetry analogous to (8),

\[ \delta \hat{g}^{\mu\nu} = \partial_\mu \epsilon^\rho \hat{g}_{\rho\nu} + \hat{g}_{\nu\rho} \partial_\mu \epsilon^\rho + \epsilon^\rho \partial_\rho \hat{g}_{\mu\nu} - \frac{2}{D} \partial_\rho \epsilon^\rho \hat{g}_{\mu\nu}, \]

\[ \delta X^i = \epsilon^\rho \partial_\rho X^i - (D - 1) G^i_j \frac{\partial L}{\partial X^j} \frac{2}{D} \partial_\rho \epsilon^\rho. \]  

To be precise we impose this symmetry on the bare theory as a bare symmetry. At the tree level, we can adopt the gauge fixing condition of (11). The ghost action is also the same form with (12). Note that this is nothing but the well known nonlinear sigma models except the gauge fixing and the ghost terms (BRS trivial sector).
For the BRS nontrivial sector we can employ the knowledge of the nonlinear sigma models[14][15][16]. In the background gauge we need not consider the renormalization of the BRS trivial sector to the one loop level. However beyond one loop we also need to investigate the BRS trivial sector. This problem is investigated in the next section. By this way the one loop counter terms can be evaluated to be

\[-\left\{\frac{26 - N}{24\pi\epsilon}\right\}\tilde{R} + \frac{1}{4\pi\epsilon}\nabla^i\partial_i L\tilde{R} + \frac{1}{4\pi\epsilon}R_{ij}\partial_{i\rho}X^i\partial_{\rho\nu}X^j\tilde{g}^{\mu\nu},\]  

(28)

where $N$ is the dimension of the ‘target’ space.

Here we cite the remarkable connection between the trace anomaly of the bare action and the $\beta$ functions of the theory[16]. Let us denote the couplings and the operators of the theory by $\{\lambda_i\}$ and $\{\Lambda_i\}$. In our case they are $\{L,G_{ij}\}$ and $\{\tilde{R},\partial_{\mu}X^i\partial_{\nu}X^j\tilde{g}^{\mu\nu}\}$. The bare couplings are

$$\lambda_i^0 = \mu^\epsilon\{\lambda_i + \sum_{\nu=1}^{\infty} \frac{a_{ij}^\nu(\lambda_j)}{\epsilon^\nu}\}. \quad (29)$$

The $\beta$ functions follow as

$$\beta_{\lambda_i} = -\epsilon\lambda_i - a_i^1 + \lambda_j \frac{\partial}{\partial \lambda_j} a_i^1. \quad (30)$$

The bare action of the theory is $S_0 = \lambda_i^0\Lambda_i^0$. The renormalized operator $\Lambda_i$ is defined as

$$\Lambda_i = \frac{\partial}{\partial \lambda_i} S_0. \quad (31)$$

In conformally coupled Einstein gravity, there is no operator renormalization to the one loop order. The trace anomaly of the bare action $T^\rho_\rho$ is a finite quantity and hence should be expressed by the linear combinations of the renormalized operators with finite coefficients. In fact these trace anomaly coefficients can be related to the $\beta$ functions of the theory to all orders.
In a general case, the trace anomaly becomes
\[
\mu^\epsilon \frac{\epsilon L - 26 - N}{24 \pi} G + \frac{G}{4 \pi} \nabla^i \partial_i L + 2(D - 1) \partial^i L \partial_i L \tilde{R} + \mu^\epsilon \frac{\epsilon G_{ij}}{2G} R_{ij} + 4(D - 1) \nabla_i \partial_j L \partial_{\mu} X^i \partial_{\nu} X^j \tilde{g}^{\mu \nu}. \tag{32}
\]

As we have explained in the beginning of this section, the one loop counter terms can be chosen to preserve the volume preserving diffeomorphism invariance in general cases since the gauge invariance needs to be only 'softly' broken at \(O(\epsilon)\) by the tree action. The volume preserving diffeomorphism invariance of the bare action can be promoted to full general covariance by imposing conformal invariance. In this way we construct the general covariant theory up to the one loop level.

Now we consider higher loop amplitudes. In order to renormalize the two loop amplitudes, we need to subtract all subdivergences. If we do so, all divergences are guaranteed to be local. This point will be investigated in the next section. The two loop conformal anomaly is \(O(G) \sim O(\epsilon)\) around the ultraviolet fixed point which we are most interested in. Recall that we have balanced the tree and the one loop conformal anomaly of \(O(1)\) around the ultraviolet fixed point. So we can cancel the two loop trace anomaly by slightly deforming \(L\) and \(G_{ij}\) by \(O(\epsilon^2)\). The two loop divergences in the theory with the symmetry breaking of this magnitude should be subtracted again by the counter terms which is invariant under the volume preserving diffeomorphism because at most \(\frac{1}{\epsilon^2}\) poles can appear at the two loop level. At still higher orders, we repeat finer and finer deformations of \(L\) and \(G_{ij}\). This procedure defines the \(2 + \epsilon\) expansion of general quantum gravity to all orders. The argument for the two loop case can be repeated inductively for higher loop cases. We hope a rigorous proof for the renormalizability of \(2 + \epsilon\) dimensional quantum gravity can be constructed along this line of arguments.

One of the technical subtleties of the renormalization program concerns the BRS trivial sector. However the renormalization of this sector must be done away with by a judicious choice of gauge since there are infinite degrees of freedom in
gauge fixing. This point will be underscored by a concrete calculation in the next section. Therefore we expect no difficulty to renormalize this sector.

In conclusion we claim that the following trinity holds in our formulation of quantum gravity near two dimensions:

The theory is generally covariant.

The theory is conformally invariant with respect to the reference metric $\hat{g}_{\mu\nu}$.

The theory is on the renormalization group trajectory which leads to the Einstein gravity in the weak coupling limit.

5. Renormalization of quantum fields

In the background gauge we decompose the fields into the classical (background) fields and the quantum fields. We functionally integrate the quantum fields. The background fields may be viewed as the external fields and the quantum fields as the internal fields. It is at the heart of the BPHZ renormalization scheme that we subtract all subdivergences to make the remaining overall divergence of a particular diagram local. We need to renormalize the quantum fields in general to subtract subdivergences. In nongauge theories there is little difference between the classical and quantum fields and the renormalization of the quantum fields is the same with that of the classical fields. However in gauge theories the renormalization of the quantum fields is in general different from that of the background fields due to the gauge fixing. However this is essentially a technical problem and we expect no difficulty in the renormalization of the quantum fields if we can renormalize the effective action of the background fields. Since there are infinite degrees of freedom to choose a gauge, it must be possible to choose a gauge in which there is no need to renormalize the quantum fields. So formally we can argue that we need not worry about the renormalization of the quantum fields.

Let us try to renormalize the theory to the two loop level in the background gauge. In this case we need to subtract all one loop subdivergences. All such subdivergences can be viewed as the quantum two point functions. So we only
need to make the quantum two point functions finite to the one loop order in order to ensure the two loop renormalizability of the theory in the background gauge. Since we have already made the effective action for the background fields finite to the one loop order, we expect that the quantum two point functions can be renormalized to the one loop order also.

Since we are interested in the renormalization of the quantum fields, we can use the flat background metric ($\hat{g}_{\mu\nu} = \eta_{\mu\nu}$). The background dependence can be recovered by the manifest covariance with respect to the background metric. We adopt the following gauge fixing term

$$\frac{1}{2} L (\partial^\mu (h_{\mu\nu} + \lambda h_{\mu\rho} h_{\rho\nu}^\sigma) - \frac{\partial_\nu L}{L})^2, \quad (33)$$

where we introduce a gauge fixing parameter $\lambda$. The ghost sector is

$$\partial_\mu \bar{\eta}_\mu \partial^\mu \eta^\nu + \partial_\mu \bar{\eta}_\mu \partial_\rho h^{\mu\nu} \eta^\rho + \frac{1}{2} \partial_\mu \bar{\eta}_\mu \partial_\rho h^{\mu\nu} h_{\rho\nu} - \frac{1}{2} \lambda \partial_\mu \bar{\eta}_\mu \partial_\rho h^{\mu\nu} h_{\rho\nu} - \frac{1}{2} \frac{1}{2} - \lambda \partial_\mu \bar{\eta}_\mu \partial_\nu h_{\mu\rho} + \frac{1}{2} + \lambda \partial_\mu \bar{\eta}_\mu \partial_\nu h_{\mu\rho} - 2\lambda \partial_\mu \bar{\eta}_\mu \partial_\rho h_{\mu\nu} + \cdots. \quad (34)$$

First we calculate the two point functions of $\psi$ and $\varphi_i$ fields. Just like the background gauge calculation, we find no divergences of $\partial_\mu \psi \partial^\mu \psi$ and $\partial_\mu \varphi_i \partial^\mu \varphi_i$ type. Next we consider the two point functions of $h_{\mu\nu}$. There are only two types of them $\partial_\mu h_{\mu\nu} \partial^\rho h^{\mu\nu}$ and $\partial_\mu h^{\mu\rho} \partial_\nu h_{\nu\rho}$. The particular combination of them appears in the quadratic part of $\tilde{R}$ as

$$\tilde{R}_2 = \frac{1}{4} (\partial_\rho h_{\mu\nu} \partial^\rho h^{\mu\nu}) - \frac{1}{2} (\partial_\mu h^{\mu\rho} \partial_\nu h_{\nu\rho}). \quad (35)$$

$\psi$ and $\varphi_i$ fields give the standard divergence of

$$-\frac{(1 + c)}{24\pi \epsilon} \tilde{R}_2. \quad (36)$$
The ghost contribution in this gauge is found to be

\[-\left(\frac{1+12\lambda+4\lambda^2}{24\pi\epsilon}\right)\bar{R}_2 + \frac{\lambda^2}{2\pi\epsilon}\partial_\mu h^{\mu\rho}\partial^\nu h_{\nu\rho}.\]  

(37)

When we evaluate $h_{\mu\nu}$ field contribution, it is convenient to use the background field method. Namely we expand $\tilde{g}_{\mu\nu} = (e^h)_{\mu\nu}$ around the expectation value of $h_{\mu\nu} = \hat{h}_{\mu\nu} + h'_{\mu\nu}$:

\[\tilde{g}_{\mu\nu} = (e^{\hat{h}+h'})_{\mu\nu} = (e^{\hat{h}(1-\frac{G}{6\pi\epsilon L})})_{\mu\rho}(e^{h})^\rho_{\nu}.\]  

(38)

Here we have to be careful about the one particle irreducibility of the fields which are related nonlinearly. If we parametrize $\tilde{h}^\mu_\nu$ as in [7][8]

\[\tilde{h}^\mu_\nu = \hat{g}^{\mu\rho}H_{\rho\nu} - \frac{1}{D}\delta^\mu_\nu\hat{g}^{\rho\sigma}H_{\rho\sigma},\]  

(39)

then $h'_{\mu\nu}$ and $H_{\mu\nu}$ are related nonlinearly

\[h'_{\mu\nu} = H_{\mu\nu} - \frac{1}{2}(H\hat{h} + \hat{h}H)_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\hat{h}^{\rho\sigma}H_{\rho\sigma} + \left(\frac{1}{12}\hat{h}H^2 - \frac{1}{6}H\hat{h}H + \frac{1}{12}H^2\hat{h} + \frac{G}{6\pi\epsilon L}\hat{h}\right)_{\mu\nu}.\]  

(40)

The one particle irreducibility of $h'_{\mu\nu}$ and $H_{\mu\nu}$ ($<h'_{\mu\nu}>=<H_{\mu\nu}>=0$) can be made consistent only after the subtraction (the last term on the right hand side). This is the reason that the such a singular term appears in (38). However this term can be cancelled by the renormalization of the quantum field.

Now we can expand the action using our standard parametrization $\tilde{g}_{\mu\nu} = (\hat{g})_{\mu\rho}(e^{h})^\rho_{\nu}$ just like the background gauge calculation. The gauge fixing term can
be expanded around the mean field and it is found to be
\[ \frac{1}{2}(\partial^\mu(\hat{h} + H + (\lambda - \frac{1}{2})H\hat{h} + \hat{h}H) + \frac{1}{2}\eta\hat{h}_{\rho\sigma}H^{\rho\sigma})_{\mu\nu})^2. \] (41)

In this way we find the contribution of \( h_{\mu\nu} \) field as
\[ \frac{27 + 60\lambda + 4\lambda^2}{24\pi\epsilon}\tilde{R}_2 - \frac{\lambda^2}{2\pi\epsilon}\partial_\mu h^{\mu\rho}\partial_\nu h_{\nu\rho} + \frac{1}{3\pi\epsilon}\tilde{R}_2, \] (42)
where the last term is due to the change of the variables we have just explained.

In this way the total divergence is found to be
\[ \frac{25 - c + 48\lambda + 8}{24\pi\epsilon}\tilde{R}_2. \] (43)

Just like the Yang-Mills theory, the divergence of the two point functions of \( h_{\mu\nu} \) is of the form \( \tilde{R}_2 \). The ghost contribution is essential for this result. We also conclude that there is no renormalization of the gauge fixing term for \( h_{\mu\nu} \) fields. Since the first term \((25-c)\) can be absorbed in the gravitational coupling constant renormalization, we need the wave function renormalization as
\[ h^0_{\mu\nu} = h_{\mu\nu}(1 - \frac{1 + 6\lambda G}{6\pi\epsilon L}). \] (44)

Lastly we evaluate the divergent part of the ghost two point function to be
\[ \frac{G}{12\pi\epsilon L}\partial_\mu \tilde{\eta}_\nu \partial^\mu \eta^\nu. \] (45)

It is clear that this divergence can be subtracted by the ghost wave function renormalization.

Furthermore we can check the consistency of the result by calculating three point functions such as \( \partial_\mu \psi \partial_\nu \psi h^{\mu\nu} \). The one loop divergence of this type is found to be
\[ \frac{1 + 6\lambda}{12\pi\epsilon L}\partial_\mu \psi \partial_\nu \psi h^{\mu\nu}, \] (46)
which is consistent with the wave function renormalization of \( h_{\mu\nu} \).
In fact there are other types of the quantum two point functions which contain the derivatives of $L$ such as $\partial_{\mu}L \partial_{\nu}h^{\mu\nu}$. Such terms are $O(a \sim \sqrt{\epsilon})$ in generic cases. They have to be dealt with since $\frac{1}{\epsilon}$ pole can arise in the one loop integration. It is straightforward to calculate these divergences and we expect no difficulty to subtract them since the tree level symmetry breaking is $O(\epsilon)$. However they become $O(\epsilon)$ at the short distance fixed point which we are most interested in since $a$ vanishes there. So there is no subdivergence of this type at short distance and we are content not to discuss them in this paper.

Note that the wave function renormalization for $h_{\mu\nu}$ field is not necessary if we choose $\lambda = -\frac{1}{6}$. This result underscores our expectation that the renormalization of the quantum fields can be done away with by a judicious gauge fixing. In this section we have performed all the necessary wave function renormalization of the quantum fields to ensure that the two loop counter terms are local at short distance. Due to the manifest volume preserving diffeomorphism invariance in the background gauge, these divergences can be renormalized with our action. So we are certain that the theory can be renormalized to the two loop order and hope to report the results of such a calculation in the near future.

6. Renormalization of Composite Operators

In this section we consider the renormalization of the composite operators such as the cosmological constant operator. In the classical theory the cosmological constant operator is

$$\int d^D x \sqrt{\hat{g}} = \int d^D x \sqrt{\hat{g}}(1 + \frac{1}{2} \sqrt{\frac{\epsilon}{2(D-1)}} \psi)^{\frac{2D}{\epsilon}}$$

$$= \int d^D x \sqrt{\hat{g}} \exp(\sqrt{\frac{2}{\epsilon} \psi - \frac{1}{4} \psi^2 + \cdots}).$$

(47)

In quantum theory, this operator is renormalized. We consider the infinitesimal perturbation of the theory by the cosmological constant operator. The renormalized cosmological constant operator can be determined by requiring that the
operator is invariant under the gauge transformation (8). Since the gravitational dressing of the spinless operators involve only $\psi$ field, we require the invariance with respect to the following transformation

$$
\delta \psi = (a' + \frac{\epsilon}{4}\psi)\delta \rho,
\delta \hat{g}_{\mu\nu} = -\hat{g}_{\mu\nu}\delta \rho,
$$

where $a' = (D - 1)a$. This requirement is equivalent to impose the background independence of the theory including the cosmological constant operator.

Let the cosmological constant operator to be $\int d^Dx \sqrt{\hat{g}}\Lambda(\psi)$. We parametrize $\Lambda(\psi) = exp(\alpha\psi + \frac{\beta}{2}\psi^2 + \cdots)$. If we vary the background metric as in (48), this operator varies due to the one loop quantum effect as

$$
\delta \Lambda(\psi) = \frac{G}{8\pi} \frac{\partial^2}{\partial \psi^2} \Lambda(\psi) \delta \rho.
$$

The variation due to $\psi$ field transformation as in (48) is

$$
\delta \Lambda(\psi) = (a' + \frac{\epsilon}{4}\psi) \frac{\partial}{\partial \psi} \Lambda(\psi) \delta \rho.
$$

The sum of the above must cancel the variation of $\delta(\sqrt{\hat{g}})\Lambda = -\frac{D}{2}\sqrt{\hat{g}}\Lambda \delta \rho$.

The coefficients $\alpha, \beta, \cdots$ are determined in this way as

$$
\alpha = \frac{4\pi a}{G} (-1 \pm \sqrt{1 + \frac{G}{2\pi a^2}}),
\beta = -\frac{\epsilon \alpha}{4a + \frac{\alpha a}{\pi}},
\cdots.
$$

We choose the $+$ sign out of the two possible branches in the above expression since it agrees with the classical expression in the weak coupling limit. This strategy
determines the renormalized cosmological constant operator around the ultraviolet fixed point:

\[ \int d^D x \sqrt{g} \exp \left( \frac{Q}{\sqrt{\epsilon}} \left( 1 + \frac{5}{16} \epsilon \right) \psi - \frac{Q^2}{16} \psi^2 + \cdots \right), \]

where \( Q^2 = \frac{25 - c}{3} \).

Recall that at the short distance fixed point, the theory is invariant under \( \psi \to -\psi \). In this context we point out the analogy with the Ising model which also possesses the \( Z_2 \) discrete symmetry. The cosmological constant operator is analogous to the spin operator since it is not invariant under the \( Z_2 \) symmetry. In fact \( \Lambda(\psi) \to \Lambda(-\psi) \) under this transformation. Physically speaking, we are considering the fluctuations of the metric around the vanishing expectation value. Under such a circumstance, the cosmological constant operator may fluctuate into the negative (or complex) value.

In quantum gravity the cosmological constant operator may serve the order parameter just like the spin operator in the Ising model. We can think of two distinctive phases of quantum gravity which possess vanishing and nonvanishing expectation values of the cosmological constant operator. Obviously we are in the phase which possesses the nonvanishing expectation value of the cosmological constant operator since our Universe is certainly large. In the other phase the macroscopic spacetime does not exist and it might be a topological phase. In a manifestly covariant subtraction scheme (1), the sign of the kinetic term of the conformal mode \( (Q_{eff}^2) \) flips. In this sense, we may rotate \( \psi \to i\psi \) in the ‘topological’ phase. In doing so, the cosmological constant operator becomes a complex operator as it is the case in two dimensional quantum gravity with the central charge \( 1 < c < 25 \). Then the analogy with a spin model can be drawn and the vanishing expectation value of the cosmological constant operator is conceivable.

One of the great mysteries in Nature is the cosmological constant problem. We have to explain why the observed cosmological constant is so small compared to the planck scale (or any other subatomic scale). One of the possible explanations
is to invent a symmetry. We observe that this $Z_2$ symmetry can prohibit the cosmological constant. Therefore we speculate that the resolution of the cosmological constant problem in Nature might be due to such a discrete symmetry.

At the fixed point, the coefficient of $\tilde{R}$ is $\frac{1}{G^*}(1 + \frac{\epsilon}{8(D-1)}\psi^2)$. This acts as the effective inverse coupling at the scale set by $\psi$. Let $G = G^* - \delta G$, then the $\beta_G$ function in (17) becomes $\mu \frac{\partial}{\partial \mu} \delta G = -\epsilon \delta G$ around the fixed point. We can identify $\frac{\delta G}{G^*} = \frac{\epsilon}{8(D-1)}\psi^2$. Since $\mu \frac{\partial}{\partial \mu} \psi = -\frac{\epsilon}{2} \psi$ around the fixed point, the right hand side satisfies the same renormalization group equation with the left hand side. So such an identification is consistent with the renormalization group considerations in sec. 3.

In the classical limit, the coefficient of $\tilde{R}$ is $\frac{1}{G^*}e^{-\frac{D\phi}{2}}$ as in (4). This is the effective inverse gravitational coupling at long distance. The cosmological constant operator in the classical limit is $\Lambda = e^{-\frac{D\phi}{2}}$. The classical scaling relation is

$$\Lambda^\epsilon \sim \left(\frac{1}{G}\right)^D.$$  

(53)

Now let us consider how this scaling behavior changes at short distance. Around the short distance fixed point, $\delta \Lambda(\psi) \sim \frac{Q}{\sqrt{\epsilon}} \psi$. We find the following scaling relation between $\delta(\frac{1}{G})$ and $\delta \Lambda$.

$$\left(\delta \Lambda\right)^2 \sim \delta(\frac{1}{G}).$$

(54)

Remarkably the cosmological constant operator is no longer the most relevant operator at the short distance fixed point. The quantum renormalization effect also works in the right direction to resolve the cosmological constant problem.

7. Quantum Cosmology in $2 + \epsilon$ Dimensions

In this section, we discuss the cosmological implications of this model. Let us consider the Robertson-Walker spacetime in $D$ dimensions

$$ds^2 = -dt^2 + r(t)^2 \tilde{g}_{ij} dx^i dx^j,$$

$$\tilde{g}_{ij} = \delta_{ij} + \frac{kx_i x_j}{1 - k|\vec{x}|^2}.$$  

(55)
The Ricci curvature is

\[ R_{tt} = \frac{\dddot{r}}{r}(D - 1), \quad R_{ij} = -(r\dddot{r} + \epsilon(\dot{r})^2 + \epsilon k)\tilde{g}_{ij}, \tag{56} \]

where \( \dot{r} = \frac{d}{dt}r \). Let the energy momentum tensor of the matter as

\[ T_{\mu\nu} = p g_{\mu\nu} + (p + \rho) U_\mu U_\nu, \tag{57} \]

with \( U^t = 1 \) and \( U^i = 0 \). The equation of the energy conservation is

\[ \frac{d}{dr}(\rho r^{D-1}) = -(D - 1)p r^{D-2}. \tag{58} \]

For the conformal matter where \( p = \frac{\rho}{D-1} \), we find \( \rho \sim r^{-D} \). The Einstein’s field equation is

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi GT_{\mu\nu}. \tag{59} \]

Alternatively we may use the Wheeler-DeWitt equation[17], which can be obtained from the action

\[ S = 2 \int_{\partial M} \sqrt{h} K dS + \int_M d^Dx \sqrt{g} R + (\text{matter}). \tag{60} \]

The second term is integrated over spacetime and the first over its boundary. \( K \) is the trace of the extrinsic curvature \( K_{ij} \) of the boundary. By taking the variation with respect to the lapse, we obtain the Hamiltonian constraint

\[ \sqrt{h}(K^2 - K_{ij} K^{ij} - R^{(D-1)} - 16\pi G \rho) = 0. \tag{61} \]

For the Robertson-Walker metric, we find:

\[ K_{ij} = rr\delta_{ij}, \quad K = \frac{\dot{r}}{r}(D - 1), \tag{62} \]

\[ R^{(D-1)} = -\epsilon(D - 1)\frac{k}{r^2}. \]
Either from the Einstein equation or from the Wheeler-DeWitt equation, we obtain
\[ \dot{r}^2 + k = \frac{16\pi G}{\epsilon(D-1)} \rho r^2. \] (63)

The right hand side of this equation possesses the two dimensional limit if \( G \sim O(\epsilon) \) and we denote it by \( \frac{8(\Delta_0-1)}{|Q^2|} \).

We point out the similarity of this equation with \( k = 1 \) (closed universe) to the Wheeler-DeWitt equation of the Liouville theory. This similarity is not surprising since the Liouville theory can be obtained from the Einstein action in \( 2 + \epsilon \) dimensions by taking \( \epsilon \to 0 \) limit as it is explained in (1). Recall that the weak coupling region of \( 2 + \epsilon \) dimensional quantum gravity is related to the Liouville theory with \( c > 25 \). The Wheeler-DeWitt equation in the minisuperspace approximation is[18]:
\[ \frac{Q^2}{8} \left(-\left(\frac{\partial}{(\frac{Q}{2})^2 \partial \phi}\right)^2 + 1\right) + \Delta_0 - 1)\Psi = 0, \] (64)
where \( \Psi = e^{Q(\beta+\frac{Q}{2})\phi} \sim e^{\pm i\sqrt{2\Delta_0}|Q|\phi} \). This solution in the minisuperspace approximation possesses the complex Liouville exponents. This wave function is (delta function) normalizable with the standard \( L^2 \) norm:
\[ \| \Psi \|^2 = \int_{-\infty}^{\infty} d\phi |\Psi(\phi)|^2. \]

In order to represents a macroscopic universe like our own, we need to form a wave packet with nontrivial matter contents:
\[ \int dp a(p)e^{ip\phi} \Phi_{\text{matter}}(p, x_0) \] (65)
where \( \Phi_{\text{matter}}(p, x_0) \) is a time \( (x_0) \) dependent matter wave function and \( a(p) \) is a certain weight to form a wave packet. If we draw the analogy with the particle quantum mechanics, we think of the following wave packet:
\[ \int \frac{dp}{\sqrt{2\pi}} e^{-\frac{1}{2}(p-p_0)^2} e^{ip\phi} e^{i\frac{Q^2}{2}x_0} \]
\[ \sim \frac{1}{(1 + x_0^2)^{\frac{D}{2}}} \exp \left( -\frac{1}{2} \frac{1}{1 + x_0^2} (\phi - x_0p_0)^2 \right), \] (66)
where we have suppressed the phase factor. Since \( \phi \) is the scale factor of the
universe, this wave packet represents an expanding (or contracting) universe.

From these considerations it is clear that the classical solutions of the Einstein theory correspond to the wave functions with complex Liouville exponents. We emphasize that the states with complex Liouville exponents (macroscopic states) must occur in the theory in order to ensure the existence of spacetime itself. It is certainly not a trivial requirement since we know that all physical states possess real Liouville exponents in two dimensional quantum gravity with $c \leq 1$. In this sense the existence of macroscopic spacetime itself can serve as the order parameter in quantum gravity. From the viewpoint of quantum gravity the $c = 1$ barrier of two dimensional quantum gravity is not a problem but a blessing because macroscopic states can exist for $c > 1$. In fact the weak coupling region of $2 + \epsilon$ dimensional quantum gravity is very similar to two dimensional quantum gravity with $c > 25$.

In the remaining part of this section we study the singularity of the spacetime which we have to face at the beginning (or end) of the Universe. Since we have constructed the consistent theory of quantum gravity near two dimensions, we must be able to answer this question. Furthermore our theory is still weak coupling at the short distance fixed point since $G^{*} \sim \epsilon$. So we must be able to answer this question by solving the classical equations of the renormalized action:

\[
\int d^{D}x \sqrt{\tilde{g}} \frac{\mu^{\epsilon}}{G} \{ \bar{R}(1 + a\psi + \frac{\epsilon}{8(D-1)}(\psi^{2} - \varphi_{i}^{2})) \\
- \frac{1}{2} \partial_{\mu}\psi\partial_{\nu}\bar{\psi}\bar{g}^{\mu\nu} + \frac{1}{2} \partial_{\mu}\varphi_{i}\partial_{\nu}\varphi_{i}\bar{g}^{\mu\nu} \}.
\]

(67)

By taking the variations with respect to $h_{\mu\nu}$, $\psi$ and $\varphi_{i}$ around $\bar{R} = 0$, we obtain the field equations
\[-a\partial_\mu\partial_\nu\psi\]
\[-\frac{\epsilon}{8(D-1)}\partial_\mu\partial_\nu\psi^2 + \frac{1}{2}\partial_\mu\psi\partial_\nu\psi\]
\[+\frac{\epsilon}{8(D-1)}\partial_\mu\partial_\nu\varphi_i^2 - \frac{1}{2}\partial_\mu\varphi_i\partial_\nu\varphi_i = 0,\]
\[\partial_\mu\partial^\mu\psi = 0,\]
\[\partial_\mu\partial^\mu\varphi_i = 0.\]

(68)

The cosmological solution is

\[
\psi = \frac{1}{\sqrt{c}}\varphi_i = x_0,\]
\[h_{\mu\nu} = 0.\]  

(69)

In the classical limit the conformal factor is \(e^{-\phi} \sim x_0^{\frac{D}{2}}\). By equating the line elements

\[
ds^2 = e^{-\phi}(-dx_0^2 + \tilde{g}_{ij}dx^idx^j)\]
\[= -dt^2 + r^2\tilde{g}_{ij}dx^idx^j,\]

(70)

we find \(t \sim x_0^{\frac{D}{2}}\) and \(r \sim t^{\frac{2}{D}}\). Note that this solution is consistent with (63) if \(k = 0\) since we have assumed \(\tilde{R} = 0\).

As we trace the history back to the big bang, the Lagrangian itself is renormalized. This solution remains to be valid not only in the classical limit but also throughout the renormalization group trajectory up to the short distance fixed point where \(a = 0\). At the short distance fixed point, the Lagrangian possesses the discrete symmetry \((\psi \rightarrow -\psi, \varphi_i \rightarrow -\varphi_i)\) which may be called as ‘time reversal’ symmetry. If we trace the history of the universe before the big bang \((\psi < 0)\), the universe looks like just the time reversal of what happens for \(\psi > 0\). So eventually it reaches the classical region again where \(\psi^2\) is very large and the gravitational coupling is very small. Thus we conclude that the universe bounces back from the big crunch due to the quantum effect in quantum gravity near two dimensions.
Now we consider \( k = \pm 1 \) case at the short distance fixed point where \( a = 0 \). By the way the equations of motion are identical to those in the classical limit. We consider symmetric solutions between \( \psi \) and \( \varphi_i \) just as \( k = 0 \) case. Then the only difference turns out to be the introduction of mass terms for \( \psi \) and \( \varphi_i \) fields. We find that \( \psi \) and \( \varphi_i \) fields are governed by the following effective Lagrangian

\[
L = \frac{1}{2} \dot{\psi}^2 - \frac{k}{8} \epsilon^2 \psi^2.
\]  

(71)

A point particle which moves according to this Lagrangian represents the scale of the universe. If \( k = 1 \), the potential is harmonic and the universe cannot expand forever. On the other hand, the \( k = -1 \) case corresponds to the inverted harmonic potential and the universe expands forever. At short distance the difference of \( k \) is irrelevant as we expected.

If we assume that the universe is described by a wave function, then it is not surprising to find that the spacetime singularity is resolved quantum mechanically[17]. However in quantum gravity near two dimensions we can derive such a conclusion from the first principle since a consistent theory can be constructed in this case.

If the universe is open, the universe is symmetric around the big bang and it expands in both the future and the past directions. In such a situation the direction of time might be reversed before the big bang. On the other hand, the closed universes inevitably face the big crunch. So we must conclude that the closed universes near two dimensions cannot escape the karma of infinite cycles of death and rebirth. The only possibility to escape this infinite cycle is presumably the quantum tunneling mechanism (topology change).

8. Conclusions and Discussions

In this paper we have studied the conformally coupled Einstein gravity. We have formulated quantum gravity in \( 2 + \epsilon \) dimensions in such a way to preserve the
volume preserving diffeomorphism invariance. We have proposed such a formulation in order to keep track the dynamics of the conformal mode in a renormalizable fashion. We have used the conformal Einstein gravity as a testing ground to demonstrate the effectiveness of our formulation.

In such a formulation, the prescription to enforce the general covariance is crucial. We have identified the gauge transformation which ensures the general covariance of the theory. In our formulation, this gauge invariance holds if the theory is conformally invariant with respect to the background metric. It is also shown that such a requirement is equivalent to choose theories on the renormalization group trajectory which leads to the Einstein gravity at long distance.

We also discussed the physical implications of the conformally coupled Einstein gravity. As it is explained in the introduction, the dynamics of the conformal mode can be understood by the oversubtraction (1). In conformally couple Einstein gravity, there is no other factors which influence the dynamics of the conformal mode unlike the minimal coupling case[8]. So the sign of the kinetic term of the conformal mode is expected to flip at precisely the same point where \( \beta_G = 0 \). This is not so in the minimal coupling case. Our results in the formulation which avoids the oversubtraction is consistent with such a picture.

Therefore the short distance fixed point we have studied in this paper is not an ordinary second order phase transition point which appears in the conventional field theory. At this point, the signature of the conformal mode changes from spacelike to timelike. Recall that the conformal mode is identified as the macroscopic ‘time’ in the wave packet construction (63). Therefore we may say that the time is born in this phase transition. The cosmological constant operator may serve the order parameter as we have discussed in Sec. 5.

We have also discussed the implications of this model for the spacetime singularity at the big bang or the end of the blackhole evaporation. We have found that the universe bounces back from the big crunch due to the quantum renormalization effect. In this context we may wonder what happens to the Ricci curvature
singularity at short distance. We expect that the spacetime becomes conformally invariant and selfsimilar at the fixed point. Then the singularity may be resolved in a simple scaling. In order to answer this question, we need to renormalize such generally covariant operators. We expect that the physical operators are covariant under the volume preserving diffeomorphism. We presumably need to require that the physical operators must scale under the conformal transformation. It is not at all obvious that such operators can be constructed and this problem requires more investigations.

We also hope that our approach sheds light on the blackhole physics. The singularity which appears in the final stage of blackholes is very similar to the singularity we have discussed. So we expect that the blackholes also bounce back in the final stage into the original universe. However the riddles of blackhole physics arise largely from the presence of the event horizons. Therefore we need to investigate the event horizons also. We expect to find no uncontrollable divergences in physical quantities in this domain since we have constructed a consistent theory.

As we have emphasized, our formulation can be viewed as a Ginzburg-Landau type theory which is capable to study the phase transitions in quantum gravity. In the conventional approach, the metric is expanded around the nonvanishing expectation value. In fact it is impossible to expand around the vanishing expectation value of the metric in a generally covariant formulation. Our formulation allows us to do so by relaxing the manifest symmetry. In fact the short distance fixed point we have found can be interpreted as such a point. We may draw the analogy with the nonlinear sigma models here again since the order parameter also vanishes at the short distance fixed point. We would like to extract more physical insights concerning the phase transitions in quantum gravity using this tool. We also believe that a systematic $2 + \epsilon$ expansion of quantum gravity is possible in our formulation. We plan to report further progress on this project in the near future.

**Acknowledgements**

This work is supported in part by Grant-in-Aide for Scientific Research from
the Ministry of Education, Science and Culture.
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