On the BMY inequality on surfaces

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ABSTRACT
In this paper, we are concerned with the relation between the ordinarity of surfaces of general type and the failure of the BMY inequality in positive characteristic. We consider semistable fibrations \( \pi : S \to C \) where \( S \) is a smooth projective surface and \( C \) is a smooth projective curve. Using the exact sequence relating the locally exact differential forms on \( S, C, \) and \( S/C, \) we prove an inequality relating \( c_1^2 \) and \( c_2 \) for ordinary surfaces which admit generically ordinary semistable fibrations. This inequality differs from the BMY inequality by a correcting term which vanishes if the fibration is ordinary.

1. Introduction

Let \( S \) be a smooth projective surface of general type over an algebraically closed field \( k. \) If \( k \) is of characteristic zero, the Bogomolov–Miyaoka–Yau (BMY) inequality states that \( c_1^2 \leq 3c_2. \) Over fields of characteristic \( p > 0, \) there exist surfaces of general type which violate this inequality. This phenomenon gives rise to two intriguing problems in positive characteristic.

1) To determine the precise conditions under which the BMY inequality holds.
2) To find relations of BMY type between the invariants \( c_1^2, c_2 \) of surfaces of general type under varying hypotheses.

It was conjectured that BMY should hold if the Picard scheme of the given surface is smooth [7]. This conjecture was disproved by Jang [3] by constructing counter examples with smooth Picard schemes. In [9], Urzua constructs surfaces of general type (which are étale simply connected) with smooth Picard schemes for which \( c_1^2/c_2 \) is dense in \([2, \infty)\). Recently, K. Joshi obtained inequalities relating \( c_1^2, c_2 \) for surfaces of general type under additional hypotheses. A notable relation is the inequality \( c_1^2 \leq 5c_2 \) for minimal surfaces of general type which are Hodge-Witt and satisfy certain extra hypotheses [4, Theorem 4.17].

Our main result (Theorem 1) concerns ordinary smooth projective surfaces of general type which admit generically ordinary semistable fibrations. For such a surface we prove an inequality relating \( c_1^2 \) and \( c_2, \) which differs from the BMY inequality by an additive term that vanishes if the fibration is ordinary.

In the second section of the paper, we collect definitions and some basic facts about:

- semistable fibrations of surfaces and their invariants
• absolute, relative, and arithmetic Frobenius morphisms
• locally exact differential forms on varieties
• ordinarity of varieties, relative ordinarity for surfaces fibered over curves and the notion of being generically ordinary

In the third section, our main aim is to verify the exactness of the following sequence relating the sheaves of locally exact differentials for a given semistable fibration \( \pi : S \to C \)

\[
0 \to \pi^* B^1_C \to B^1_S \to W_* B^1_{S/C} \to 0,
\]

where \( W \) is the arithmetic Frobenius morphism (Definition 9). This exact sequence is the main tool in proving our main result in Section 4.

**Theorem 1.** Let \( S \) be an ordinary smooth projective surface which admits a generically ordinary semistable fibration \( \pi : S \to C \) of genus \( g \geq 2 \) over a smooth projective curve of genus \( q \geq 1 \). Then the invariants \( c^2_1 \) and \( c_2 \) satisfy the following equation

\[
c^2_1 = 2c_2 + \frac{12}{p-1} h^1(B^1_{S/C}) - 3\delta
\]

where \( \delta \) is the total number of singular points in the fibers of \( \pi \).

**Corollary 2.** Under the hypotheses of Theorem 1, the following inequality holds

\[
c^2_1 \leq 3c_2 + \frac{12}{p-1} h^1(B^1_{S/C}) - 4\delta.
\]

Our notation is as follows:

Our base field is \( k = \mathbb{F}_p \) for some prime \( p > 0 \).

\( S \) is a projective smooth surface of general type over \( k \).

\( C \) is a projective smooth curve of genus \( q \geq 1 \).

\( \pi : S \to C \) is a semistable fibration of genus \( g \geq 2 \).

\( \omega_{S/C} \) is the relative canonical bundle.

\( \mathcal{M}_g \) is the moduli space of smooth genus \( g \) curves.

\( \mathcal{M}^s_g \) is the moduli space of stable genus \( g \) curves.

\( \mathcal{M}_{g,1,n} \) is the moduli space of principally polarized abelian varieties of dimension \( g \) with a symplectic level-\( n \)-structure.

\( \mathcal{M}^s_{g,1,n} \) is the Satake compactification of \( \mathcal{M}_{g,1,n} \).

\( F_X \) and \( F \) are absolute and relative Frobenius morphisms, respectively, for a variety \( X \).

\( F \) also denotes Frobenius morphism on the cohomology groups of a variety \( X \).

\( B^1_X \) is the locally exact differential forms for a variety \( X \).

\( B^1_{X/Y} \) is the relative locally exact differential forms for a morphism \( f : X \to Y \).

**2. Basics**

In this section, we state basic facts about semistable fibrations and ordinarity of varieties (especially of surfaces). We also include basic cohomological tools that we shall use in the rest of the paper.

**2.1. Semistable fibrations**

**Definition 3.** Let \( C \) be a projective curve over an algebraically closed field \( k \). We say that \( C \) is **stable** (resp., **semistable**) if:
a) $C$ is connected and reduced,
b) all singular points are normal crossings, and
c) an irreducible component, isomorphic to $\mathbb{P}^1$, meets the other components in at least three (resp., two) points.

The relative version is given in the following definition.

**Definition 4.** A proper flat morphism $f : X \to Y$ of relative dimension one of schemes is said to be a (semi)stable curve if every geometric fiber of $f$ is a (semi)stable curve.

From now on, we are concerned with semistable fibrations $\pi : S \to C$ of genus $g \geq 2$ on smooth projective surfaces $S$ where the base curve $C$ is smooth of genus $q \geq 1$. Let $T \subset C$ be the set of points over which the fiber is not smooth and $t$ be the cardinality of $T$.

**Definition 5.** A semistable fibration $\pi : S \to C$ is called isotrivial if there exists a finite morphism $\phi : C' \to C$ such that the fiber product $S \times_C C'$ is, birationally on $C'$, isomorphic to the trivial fibration. In this case, one can assume that $\phi$ is étale on $C - T$.

**Remark 6.** For a given semistable fibration $\pi : S \to C$, the following hold:

1) We have a morphism $\pi : C \to \overline{\mathcal{M}}_g$ and $\pi$ is constant iff $\pi$ is isotrivial.
2) The uniqueness of the semistable model implies that if the fibration $\pi$ is isotrivial, then it is smooth.

Next we discuss some basic invariants of semistable fibrations. For a semistable fibration $\pi : S \to C$, we define the following invariants:

1) $d = \text{degree}(\pi_*\mathcal{O}_S/C)$,
2) $\delta = \sum_{P \in T} \delta_P$ where $\delta_P$ is the number of singular points in a fiber,
3) $c_1$ and $c_2$ are the first and the second Chern classes of $S$,
4) $N = \bigoplus_{P \in \text{sing in a fiber}} \pi_*\mathcal{O}_P(k(P))$.

These invariants satisfy the following relations [8, Sections 0 and 1, pp. 46–49].

i) $\chi(\mathcal{O}_S) = \frac{c_1^2 + c_2}{12} = \chi(\pi_*\mathcal{O}_S) - \chi(R^1\pi_*\mathcal{O}_S)$,
ii) $c_1^2 = 12d - \delta + 8(g - 1)(q - 1)$,
iii) $c_2 = \delta + 4(g - 1)(q - 1)$.

From the equations in (ii) and (iii), we obtain the equality $c_1^2 = 2c_2 + 12d - 3\delta$.

We also have the following exact sequences (loc. cit.).

5) $0 \to \pi^*\Omega_C^1 \to \Omega_{S/k}^1 \to \Omega_{S/C}^1 \to 0$
6) $0 \to \Omega_{S/C}^1 \to \omega_{S/C}^1 \to N \to 0$

**2.2. Ordinarity**

In this subsection, we first consider the absolute, the arithmetic and the relative Frobenius morphisms for a morphism $f : X \to Y$ of varieties in characteristic $p > 0$.

**Definition 7.** The absolute Frobenius morphism $F_Y : Y \to Y$ is given by the identity map on the underlying topological space and the $p$th power map on the structure sheaf.

**Remark 8.** The following diagram is commutative:
**Definition 9.** The *arithmetic Frobenius morphism* is the morphism

\[ W : X^{(p)} = X \times_{(Y, F_Y)} Y \to X \]

obtained from \( F_Y \) by the base extension.

It follows from Remark 8 that there exists a unique morphism \( F : X \to X^{(p)} \) over \( Y \) fitting into the following commutative diagram.

\[ X \xrightarrow{F_X} X^{(p)} \xrightarrow{W} X \]

\[ Y \xrightarrow{F_Y} Y \]

**Diagram 1. The relative Frobenius.**

**Definition 10.** \( F : X \to X^{(p)} \) is called the *relative Frobenius morphism.*

We next discuss the concepts of ordinarity and generic ordinarity. For a variety \( X \) of dimension \( n \), we have a complex

\[ 0 \to F_X^* \mathcal{O}_X \xrightarrow{d^1} F_X^* \Omega^1_{X/k} \xrightarrow{d^2} F_X^* \Omega^2_{X/k} \to \cdots \xrightarrow{d^n} F_X^* \Omega^n_{X/k} \to 0. \]

The sheaf \( \text{Im}(d^i) \) is called the sheaf of locally exact \( i \)th differential forms and is denoted by \( B^i_{X/k} \).

Notice that \( B^1_{X/k} = \text{Coker}(F_X) \) and thus sits in the short exact sequence

\[ 0 \to \mathcal{O}_X \xrightarrow{F_X} F_X^* \mathcal{O}_X \to B^1_{X/k} \to 0. \]  \hspace{1cm} (1)

**Definition 11 ([2, Definition 1.1]).** We say that \( X \) is *ordinary* if

\[ H^i(X, B^j_{X/k}) = 0 \text{ for all } i \text{ and } j. \]

We recall the characterization of ordinary curves and surfaces in terms of the action of Frobenius on the cohomology.

1) Let \( X \) be a smooth projective curve of genus \( g \). Set \( V = H^1(X, \mathcal{O}_X) \).

Let \( V_n = \{ \xi \in V \mid F^m(\xi) = 0 \text{ for some positive integer } m \} \) be the subspace on which \( F \) is nilpotent and \( V_g \), the complement of \( V_n \) in \( V \). In fact,

\[ V_g = \text{Span}(\{ \xi \in V \mid F(\xi) = \xi \}). \]

The natural number \( \sigma_X = \dim_k(V_g) \) is called the \( p \)-rank of \( X \). The following facts are well-known:

a) \( X \) is an ordinary curve if and only if \( \sigma_X = g \).

b) The \( p \)-rank of a curve coincides with the \( p \)-rank \( \sigma_f \) of its Jacobian.

c) For a semistable curve \( X \), the Jacobian sits in an extension of group schemes

\[ 0 \to G^d_r \to I_X \to A \to 0 \]
where \( A \) is an abelian variety. We define the \( p \)-rank of \( X \) by setting
\[
\sigma_X = s + \sigma_A.
\]
2) Let \( S \) be a surface. By using the short exact sequence (1) and Serre duality, we see that \( S \) is ordinary if and only if \( H^i(S, B^1_{S/k}) = 0 \) for all \( i \). This condition is equivalent to requiring
\[
F : H^i(S, \mathcal{O}_S) \to H^i(S, \mathcal{O}_S)
\]
be bijective for \( i = 1, 2 \).

**Definition 12** ([3, Definition 2.8]). Let \( p : S \to C \) be a proper semistable fibration. We say that \( p \) is generically ordinary if at least one closed fiber of \( p \) is ordinary. (Hence almost all closed fibers of \( p \) are ordinary.)

Next we shall prove **Lemma 14** which relates the ordinarity of the fibers in a semistable fibration on a surface \( S \) and the BMY inequality on \( S \). Recall that the BMY inequality is the relation
\[
c_1^2(X) \leq 3c_2(X)
\]
between the Chern classes of the smooth projective surface \( X \).

**Remark 13.** We can verify the BMY inequality on certain minimal surfaces of general type by computing directly \( c_1^2 \) and \( c_2 \). For example:

1) Let \( \pi : X \to C \) be any smooth isotrivial fibration with \( g(C) \geq 2 \) and \( g(F) \geq 2 \). Then we have
\[
c_1^2(X) = 8(g(C) - 1)(g(F) - 1)
\]
\[
c_2(X) = 4(g(C) - 1)(g(F) - 1)
\]
and so \( c_1^2(X) = 2c_2(X) \).

2) Let \( X \) be any complete intersection smooth surface of general type of degree \( d = (d_1, d_2, ..., d_{n-2}) \) in \( \mathbb{P}^n \). We assume that \( \sum_i d_i > n + 1 \) and we set \( \xi = \mathcal{O}_X(1) \). Then we have
\[
c_1^2(X) = \left[ -n - 1 + \sum_i d_i \right]^2 \xi^2,
\]
\[
c_2(X) = \left[ \frac{(n + 1)n}{2} - (n + 1) \sum_i d_i + \sum_{i<j} d_id_j \right] \xi^2
\]
and so \( c_1^2(X) \leq c_2(X) \).

In particular, the BMY inequality holds on any smooth surface of degree \( d \geq 5 \) in \( \mathbb{P}^3 \).

The following lemma provides other examples for which the BMY inequality holds. In the proof of this lemma, we will use the basic properties of the Ekedahl–Oort strata in \( \mathcal{M}_{g,1,n}^* \) [6, Sections 2–6].

**Lemma 14.** Let \( \pi : S \to C \) be a semistable fibration of genus \( g \) such that the fibers are all (i) ordinary, or (ii) of \( p \)-rank \( g - 1 \), or (iii) supersingular but not superspecial, or (iv) superspecial. Then \( \pi \) is isotrivial. Hence, the BMY inequality holds on the surface \( S \).

**Proof.** Let \( U \subset S \) be the union of the smooth fibers of \( \pi \) and set \( C' = \pi(U) \). The canonical principal polarization on the relative Jacobian \( J_{U/C} \) extends to a principal cubic structure on \( J_{S/C} \) [5, Chapter II, Theorem 3.5] which we denote by \( \Theta \). Thus, we obtain a morphism
\[
h : C \to \mathcal{M}_{g,1,n}^*
\]
given by
In each of the cases (i), (ii), (iii), and (iv) we verify that the image \( h(C) \) lies in an Ekedahl–Oort stratum determined by a unique elementary sequence \( \varphi \) [6, Definition 2.1].

Case (i): \( |\varphi| = g \cdot (g + 1)/2 \) corresponds only to the elementary sequence
\[
\varphi = \{1, 2, \ldots, g\}.
\]

Case (ii): \( |\varphi| = g \cdot (g + 1)/2 - 1 \) is obtained only from the elementary sequence
\[
\varphi = \{1, 2, \ldots, g - 1, g - 1\}.
\]

Case (iii): If the fibers are supersingular, but not superspecial, then we have \( |\varphi| = 1 \) for which the elementary sequence is
\[
\varphi = \{0, 0, \ldots, 0, 1\}.
\]

Case (iv): If the fibers are superspecial, then we have \( |\varphi| = 0 \) which arises only from the elementary sequence
\[
\varphi = \{0, 0, \ldots, 0\}.
\]

It follows from [6, Theorem 6.4] that the image \( h(C) \) is quasi-affine, and so the morphism \( h \) is constant on \( C \). Then we see that the smooth fibers of \( \pi \) are all isomorphic by applying the Torelli theorem. That is, \( \pi : S \to C \) is an isotrivial semistable fibration. Therefore, \( \pi \) is smooth and \( d = 0, \delta = 0 \). It follows that \( c_1^2 = 2c_2 \) by Remark 13.

### 3. Some local algebra

Let \( \pi : S \to C \) be a fibration as in Section 1. Then one has the following short exact sequence of \( \mathcal{O}_S \)-modules
\[
0 \to \pi^* \Omega^1_{C/k} \to \Omega^1_{S/k} \to \Omega^1_{S/C} \to 0. \tag{2}
\]

Since \( F_S \) is a finite morphism, we obtain the following exact sequence
\[
0 \to F_S^* \pi^* \Omega^1_{C/k} \to F_S^* \Omega^1_{S/k} \to F_S^* \Omega^1_{S/C} \to 0 \tag{3}
\]

applying \( F_S^* \) in (2).

In Lemma 21, we prove the existence of a short exact sequence of sheaves of locally exact differential forms (analogous to (3)). To this end, we first recall the definition of the relative locally exact differential forms \( B^1_{S/C} \). Using the relative deRham complex
\[
0 \to \mathcal{O}_S \xrightarrow{d} \Omega^1_{S/C} \to 0
\]

and the relative Frobenius map \( F : S \to S^{(p)} \), we obtain a complex
\[
0 \to F_* \mathcal{O}_S \xrightarrow{F_* d} F_* \Omega^1_{S/C} \to 0.
\]

**Definition 15.** The sheaf \( B^1_{S/C} := \text{Im}(F_* d) \) is called the *relative locally exact differential forms* for the fibered surface \( \pi : S \to C \).

**Remark 16.** \( B^1_{S/C} \) sits in the following short exact sequence
\[
0 \to \mathcal{O}_{S^{(p)}} \to F_* \mathcal{O}_S \to B^1_{S/C} \to 0.
\]
Remark 17. Let $X$ be a scheme over a perfect field of positive characteristic $p$ and $W$ be the arithmetic Frobenius morphism. Then $X^{(p)} \simeq X$ via the map $W$, because $F_k : \text{Spec}(k) \to \text{Spec}(k)$ is an isomorphism. Therefore, $B^1_{A/k}$ can be viewed as a sheaf of $\mathcal{O}_X$-modules on $X$.

Proposition 18. Let $\pi : S \to C$ be a fibration. Then there exists a morphism of $\mathcal{O}_S$-modules $\phi : \pi^* F_{C*} \Omega^1_{C/k} \to F_{S*} \pi^* \Omega^1_{C/k}$.

Proof. Since the statement is local on the both source and target, we may assume that $S = \text{Spec}(R)$ and $C = \text{Spec}(A)$. Therefore, $\pi^* F_{C*} \Omega^1_{C/k} = F_{A*} \Omega_{A/k} \otimes_A R$ and $F_{S*} \pi^* \Omega^1_{C/k} = F_{R*} (\Omega^1_{C/k} \otimes_A R)$ where $F_A$ and $F_R$ denote the relevant Frobenius morphisms. Define an $R$-module homomorphism

$$\phi : F_{A*} \Omega_{A/k} \otimes_A R \to F_{R*} (\Omega^1_{C/k} \otimes_A R)$$

$$\sum_{i=1}^n a_i^1 a_i^2 da_i^3 \otimes r_i \mapsto \sum_{i=1}^n \left( (a_i^1)^p a_i^2 da_i^3 \otimes r_i^{1/p} \right).$$

$\phi$ is a well-defined $R$-module homomorphism due to the equality

$$\sum_{i=1}^n a_i^1 a_i^2 da_i^3 \otimes r_i^{1/p} = \sum_{i=1}^n r_i \left( ((a_i^1)^p a_i^2 da_i^3 \otimes 1) \right).$$

Hence, we have a morphism of $\mathcal{O}_S$-modules

$$\tilde{\phi} : F_{A*} \Omega_{A/k} \otimes_A R \to F_{R*} (\Omega^1_{C/k} \otimes_A R).$$

Thus, we obtain the required morphism of $\mathcal{O}_S$-modules

$$\phi : \pi^* F_{C*} \Omega^1_{C/k} \to F_{S*} \pi^* \Omega^1_{C/k}.$$

Remark 19. Since $B^1_{A/k}$ is a submodule of $F_{A*} \Omega^1_{A/k} \otimes_A R$ is a submodule of $F_{A*} \Omega^1_{A/k} \otimes_A R$. Therefore, we get a morphism of $R$-modules

$$\overline{\phi} : B^1_{A/k} \otimes_A R \to F_{R*} (\Omega^1_{C/k} \otimes_A R)$$

by restricting $\phi$ to $B^1_{A/k} \otimes_A R$. For a given differential form $a_1 a_2 da_3 \otimes 1 \in B^1_{A/k} \otimes_A R$, there exists $a_0 \in A$ such that $a_2 da_3 = da_0$ and $a_2 da_3 \in B^1_{A/k}$. Hence, $a_1 a_2 da_3 = a_1 da_0 = da_1^p da_0 = da_1^p a_0$. As a result, any element $\omega \in B^1_{A/k} \otimes_A R$ can be written as $\omega = da \otimes r$ for some $a \in A$ and $r \in R$. Thus, we see that $\overline{\phi}$ is an injective homomorphism of $R$-modules. Therefore, we get an injective morphism of $\mathcal{O}_S$-modules

$$\overline{\phi} : \pi^* B^1_{C/k} \to F_{S*} \pi^* \Omega^1_{C/k}.$$

By the exact sequence (3) and Remark 19, we obtain a complex of $\mathcal{O}_S$-modules

$$0 \to \pi^* B^1_{C/k} \to F_{S*} \Omega^1_{S/k} \to F_{S*} \Omega^1_{S/C} \to 0.$$

Restricting to the open subscheme $\text{Spec}(A) \subset C$, we obtain the following complex of $R$-modules

$$0 \to B^1_{A/k} \otimes_A R \overset{\overline{\pi}}{\to} F_{R*} \Omega^1_{R/k} \overset{\overline{\nu}}{\to} F_{R*} \Omega^1_{R/A} \to 0$$

where $\overline{\pi} = F_{R*} u \circ \overline{\phi}$ and $\overline{\nu} = F_{R*} \nu$.

We will work out the details of the $R$-module structure on $F_{R*} \Omega^1_{R/A}$. 
**Remark 20.** One has the following commutative diagram (which corresponds to Diagram 1 applied to the morphism $f : S \to C$ over $\text{Spec}(A) \subset C$).

![Diagram 2. The local algebraic relative Frobenius.](Image)

In Diagram 2,

a) $R^{(p)} = R \otimes_{A,F_s} A$,

b) $\psi : A \to R$ is a ring homomorphism which corresponds to $\pi : S \to C$,

c) $F : R^{(p)} \to R$ given by $r \otimes a \mapsto ar^p$ corresponding to $F : S \to S^{(p)}$,

d) $W : R \to R^{(p)}$ given by $ar \mapsto r \otimes a^p$ corresponding to $W : S^{(p)} \to S$,

e) $W \circ F = F_R$.

Therefore, $F_{R^*}\Omega^{1}_{R/A} = W_{*}(F_{*}\Omega^{1}_{R/A})$. Then for any $\omega = (a_1 r_1).r_2 dr_3 \in F_{R^*}\Omega^{1}_{R/A}$, we have

$$\omega = (a_1 r_1).r_2 dr_3 = (r_1 \otimes a_1^p).r_2 dr_3 \text{ via } W$$

and

$$(r_1 \otimes a_1^p).r_2 dr_3 = a_1^p r_1^p r_2 dr_3 \text{ via } F$$

i.e., first we make $\Omega^{1}_{R/A}$ an $R^{(p)}$-module via $F$ and then via the map $W : R \to R^{(p)}$, $\Omega^{1}_{R/A}$ becomes an $R$-module. Moreover, we may view $W_* B_{R/A}^1$ as a subsheaf of $F_{R^*}\Omega^{1}_{R/A} = W_{*}(F_{*}\Omega^{1}_{R/A})$ because $B_{R/A}^1$ is the subsheaf of $F_* \Omega^{1}_{R/A}$.

**Lemma 21.** Let $\pi : S \to C$ be a fibration on a smooth projective surface $S$. Then one has a short exact sequence

$$0 \to \pi^* B^1_{C/k} \to B^1_{S/k} \to W_* B^1_{S/C} \to 0. \quad (4)$$

**Proof.** Let $\psi : A \to R$ be the ring homomorphism corresponding to $\pi$. Let $da \otimes r$ be in $B^1_{A/k} \otimes R$.

We may restrict $\overline{\pi}$ to the subsheaf $B^1_{R/k}$ of $F_R \Omega^1_{R/k}$ as $\overline{\pi}(da \otimes r) = r^p d\psi(a) = dr^p \psi(a) \in B^1_{R/k}$. Also for a given $dr \in B^1_{R/k}$ since $\overline{v}(dr) = dr \in W_* B^1_{R/A}$, we have the following sequence of $R$-modules:

$$0 \to B^1_{A/k} \otimes_{A,R} \overline{\pi} \to B^1_{R/k} \overline{\pi} \to W_* B^1_{R/A} \to 0.$$

To complete the proof, we need to prove the following claims:

Claim (1): $\overline{\pi}$ is injective,

Claim (2): $\text{Im}(\overline{\pi}) = \text{Ker}(\overline{v})$,

Claim (3): $\overline{v}$ is surjective.

The first claim follows from Remark 19.

Clearly, $\text{Im}(\overline{\pi}) \subseteq \text{Ker}(\overline{v})$ by the short exact sequence (2). Let $dr$ be in $\text{Ker}(\overline{v})$. We have $dr = 0$ in $B^1_{R/A}$ which implies that $r \in A$ i.e., there exists $a \in A$ such that $r = \psi(a)$. Therefore, $dr = \psi(a)$. Since $\overline{\pi}$ is injective, we have $\overline{\pi}(dr) = dr = 0$. Hence, $dr = 0$ in $B^1_{R/A}$. Therefore, $\text{Im}(\overline{\pi}) \subseteq \text{Ker}(\overline{v})$, and we conclude that $\text{Im}(\overline{\pi}) = \text{Ker}(\overline{v})$.
\[ d\psi(a) = \overline{u}(da \otimes 1) \]. As a result, we have
\[ \text{Ker}(\overline{v}) = \text{Im}(\overline{u}) \]
which completes the proof of the second claim.

Let \([r_1(r_2 \otimes a), dr_3] \in W_*B^1_{R/A}\). Then the last claim follows by the equality
\[ [r_1(r_2 \otimes a), dr_3 = (r_1 \otimes 1)(r_2 \otimes a), dr_3 = ar_1^p r_2^p dr_3 = d\psi(a)r_1^p r_2^p r_3 = \overline{v}(ar_1^p r_2^p r_3). \]

Thus, the sequence
\[ 0 \rightarrow B^1_{A/k} \otimes_A R \xrightarrow{\pi} B^1_{R/k} \rightarrow W_*B^1_{R/A} \rightarrow 0 \]
is a short exact sequence of \(R\)-modules. This implies the following is a short exact sequence of \(O_S\)-modules
\[ 0 \rightarrow \overline{B}^1_{A/k} \otimes_A R \xrightarrow{\overline{\pi}} \overline{B}^1_{R/k} \rightarrow \overline{W}_*\overline{B}^1_{R/A} \rightarrow 0. \]

Therefore, we have the following short exact sequence of \(O_S\)-modules
\[ 0 \rightarrow \pi^*B^1_{C/k} \rightarrow \overline{B}^1_{S/k} \rightarrow W_*B^1_{S/C} \rightarrow 0. \]

### 4. The main result

In this section, we prove our main result, Theorem 1. The main ingredient is the short exact sequence of locally exact differential forms constructed in the preceding section. We will use the following well-known result [1, Chapter 3, Exercises 8.3] in the proof of Theorem 1.

**Proposition 22.** Let \( f : X \rightarrow Y \) be a morphism of ringed spaces, let \( F \) be an \( O_X\)-module and let \( E \) be a locally free \( O_Y\)-module of finite rank. Then
\[ R^if_*(F \otimes f^*E) = R^if_*(F) \otimes E \]
for all \( i \geq 0 \).

**Remark 23.** Let \( \pi : S \rightarrow C \) be a semistable fibration as in the statement of 1. Then \( B^1_{C/k} \) is a locally free \( O_C\)-module of rank \( p - 1 \) and of degree \( (p - 1)(q - 1) \). By Proposition 22, we have
\[ R^i\pi_*\pi^*B^1_{C/k} = R^i\pi_*\pi^*(O_S \otimes \pi^*B^1_{C/k}) = R^i\pi_*O_S \otimes B^1_{C/k}. \]

We calculate the rank and the degree of the sheaf \( M = R^i\pi_*O_S \otimes B^1_{C/k} \) and we get
\[ \text{rank}(M) = r(p - 1) \text{ and } \deg(M) = (p - 1)e + r(p - 1)(q - 1) \]
where \( r = \text{rank}(R^i\pi_*O_S) \) and \( e = \deg(R^i\pi_*O_S) \).

Consider the Leray spectral sequence attached to the sheaf \( F = \pi^*(B^1_{C/k}) \) on the fibration \( \pi : S \rightarrow C \), namely
\[ E_2^{pq} = H^p(C, R^q\pi_*\pi^*B^1_{C/k}) = H^p(C, R^q\pi_*O_S \otimes B^1_{C/k}) \Rightarrow H^{p+q}(S, \pi^*B^1_{C/k}). \]

\((*)\)

We have the following properties:

- a) Since \( C \) is a curve, \( H^p(C, -) = 0 \) for \( p > 1 \).
- b) By Corollary 11.2 in [1, Chapter 3], \( R^i\pi_*O_S = 0 \) for \( q > 1 \).
- c) By Proposition 22 and since \( \pi_*O_S = O_C \), we have \( \pi_*\pi^*B^1_{C/k} = B^1_{C/k} \).
Therefore, we get

\[ H^0(S, \pi^* B^1_{C/k}) = H^0(C, B^1_{C/k}) \quad \text{and} \quad H^2(S, \pi^* B^1_{C/k}) = H^1(C, R^1 \pi_* \mathcal{O}_S \otimes B^1_{C/k}). \]

If we assume that \( C \) is an ordinary curve, then we also have

\[ H^1(S, \pi^* B^1_{C/k}) = H^0(C, R^1 \pi_* \mathcal{O}_S \otimes B^1_{C/k}). \]

Now we prove Theorem 1. We remark that the hypothesis in Theorem 1 differs from the hypothesis in Lemma 14; we remove the \( p \)-rank condition on the non-smooth fibers, but now we assume that \( S \) is ordinary.

**Theorem 1.** Let \( S \) be an ordinary smooth projective surface which admits a generically ordinary semistable fibration \( \pi : S \to C \) of genus \( g \geq 2 \) over a smooth projective curve of genus \( q \geq 1 \). Then the invariants \( c_1^2 \) and \( c_2 \) satisfy the following equation

\[ c_1^2 = 2c_2 + \frac{12}{p - 1} h^1(B^1_{S/C}) - 3\delta \]

where \( \delta \) is the total number of singular points in the fibers of \( \pi \).

**Proof.** Recall that for semistable fibrations, we have the equality:

\[ c_1^2 = 2c_2 + 12d - 3\delta. \]

We will prove that \((p - 1)d = h^1(B^1_{S/C})\). For this purpose, we will use the short exact sequence (4) proved in Lemma 21:

\[ 0 \to \pi^* B^1_{C/k} \to B^1_{S/k} \to W_s B^1_{S/C} \to 0. \]

We have the long exact sequence

\[
0 \to H^0(S, \pi^* B^1_{C/k}) \to H^0(S, B^1_{S/k}) \to H^0(S, W, B^1_{S/C}) \to \cdots
\]

\[
\cdots \to H^1(S, \pi^* B^1_{C/k}) \to H^1(S, B^1_{S/k}) \to H^1(S, W, B^1_{S/C}) \to \cdots
\]

\[
\cdots \to H^2(S, \pi^* B^1_{C/k}) \to H^2(S, B^1_{S/k}) \to H^2(S, W_s B^1_{S/C}) \to 0.
\]

We note the following:

1) Since \( S \) is an ordinary surface,

\[ H^i(S, B^1_{S/k}) = 0 \]

for all \( i \geq 0 \). It is easily concluded that \( H^2(S, W, B^1_{S/C}) = 0 \).

2) Since \( \pi \) is a generically ordinary semistable fibration, \( \pi^{(p)} B^1_{S/C} \mid U = 0 \) where \( U \) is the ordinary locus of \( \pi \). However, \( B^1_{S/C} \) is flat over \( \mathcal{O}_C \) so \( \pi^{(p)} B^1_{S/C} = 0 \). It follows that \( H^0(S, W_s B^1_{S/C}) = H^0(S^{(p)}, B^1_{S/C}) = H^0(C, \pi^{(p)} B^1_{S/C}) = 0 \) as \( W \) is a finite morphism.

Therefore, by the long exact sequence

\[ H^0(S, \pi^* B^1_{C/k}) = H^1(S, \pi^* B^1_{C/k}) = 0 \]

and

\[ H^1(S, W_s B^1_{S/C}) = H^2(S, \pi^* B^1_{C/k}). \]

Now, recall that \( S \) is assumed to be an ordinary surface. Then \( C \) is an ordinary curve and hence by using the Leray spectral sequence (*) we have an equality:
\[ \chi(M) = h^0(C, M) - h^1(C, M) = h^1(S, \pi^*B^1_{C/k}) - h^2(S, \pi^*B^1_{C/k}). \]

On the other hand,
\[ \chi(M) = \deg(M) - (q - 1)\text{rank}(M). \]

Therefore, we have
\[ -h^2(S, \pi^*B^1_{C/k}) = \chi(M) \]
and so
\[ -h^2(S, \pi^*B^1_{C/k}) = (p - 1)e + r(p - 1)(q - 1) - (q - 1)r(p - 1). \]

This implies that \((p - 1)d = h^2(S, \pi^*B^1_{C/k}) = h^1(S, W, B^1_{S/C}) = h^1(B^1_{S/C})\) where \(d = -e\). Recall that since \(\pi : S \to C\) is a semistable fibration, we have \(c^2_1 = 2c_2 + 12d - 3\delta\). Substituting \(d = \frac{1}{p-1} \cdot h^1(B^1_{S/C})\), we obtain the equality:
\[ c^2_1 = 2c_2 + \frac{12}{p-1} h^1(B^1_{S/C}) - 3\delta. \]

Since for the fibration in Theorem 1
\[ c_2 - \delta = 4(g - 1)(q - 1) \geq 0 \]
we obtain the inequality given in Corollary 2, namely
\[ c^2_1 \leq 3c_2 + \frac{12}{p-1} h^1(B^1_{S/C}) - 4\delta. \]

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