On Yau-Tian-Donaldson conjecture for singular Fano varieties

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A Fano manifold $X$ is a projective manifold such that:

**Fano:** $-K_X = \wedge^n T_{\text{hol}} X$ is an ample line bundle.

Equivalently: $\exists$ a Kähler metric $g$ s.t. its Kähler form $\omega \in 2\pi c_1(-K_X)$.

Kähler form: $\omega = g(J_\cdot, \cdot) = \sqrt{-1} \sum_{i,j} g_{ij} dz_i \wedge d\bar{z}_j = \sqrt{-1} \partial\bar{\partial}\varphi_\alpha$. \hspace{1cm} (1)

Examples $\mathbb{P}^2 \# k\mathbb{P}^2$, $0 \leq k \leq 8$; $\mathbb{P}^n$; $\{F(z_1, \ldots, z_n) = 0\} \subset \mathbb{P}^{n-1}$ with $\text{deg}(F) < n$.

Hermitian metric on $-K_X$: $h = e^{-\varphi} = \{e^{-\varphi_\alpha}\}$ s.t. $|\partial z_\alpha|^2 e^{-\varphi_\alpha} = |\partial z_\beta|^2 e^{-\varphi_\beta}$.

$Ric(\omega) = -\sqrt{-1} \sum_{i,j} \frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z_i \partial \bar{z}_j} dz_i \wedge d\bar{z}_j \in 2\pi c_1(-K_X) \in H^2(X, \mathbb{R})$. \hspace{1cm} (2)

**Kähler-Einstein (KE) Equation:** $Ric(\omega) = \omega$

KE equation is equivalent to a complex Monge-Ampère equation:

$(\sqrt{-1} \partial\bar{\partial}\varphi_\alpha)^n = |\partial z_\alpha|^2 e^{-\varphi_\alpha} (\sqrt{-1})^n dz_\alpha \wedge d\bar{z}_\alpha \overset{\text{locally}}{\iff} \det(\varphi_{i\bar{j}}) = e^{-\varphi}$. \hspace{1cm} (3)
Theorem (Tian, Tian-Zhu, Phong-Song-Sturm-Weinkove, Darvas-Rubinstein)

$X$ admits a KE metric if and only if Mabuchi or Ding energy (denoted by $F$) is proper modulo the holomorphic automorphism group $\text{Aut}(X)$.

K-stability: a Hilbert-Mumford type criterion for properness of energy:

Definition (K-stability after Tian, equivalent to Donaldson’s formulation by L.-Xu)

$X$ is $K$-polystable if for any special degeneration $(\mathcal{X}, \eta)$ of $X$, $\text{Fut}(X_0, -\eta) \geq 0$ and the identity holds iff $\mathcal{X}$ is induced by a holomorphic vector field $\eta$ on $X$.

Conjecture (Yau-Tian-Donaldson (YTD) conjecture)

$X$ has a Kähler-Einstein metric if and only if $X$ is $K$-polystable.

Necessary (needs energy properness): Tian (’97), Berman (works for any $\mathbb{Q}$-Fano)

Sufficient (partial $C^0$-estimate for conical KE): Chen-Donaldson-Sun, Tian
Definition

A \( \mathbb{Q} \)-Fano variety \( X \) is a normal projective variety satisfying 2 conditions:

**Fano:** \(-K_X \) is ample \( \mathbb{Q} \)-line bundle, i.e. \(-mK_X := (\wedge^n T X^{\text{reg}})^{\otimes m} \) extends as an ample line bundle for some \( m \in \mathbb{Z} \);

**Klt (Kawamata log terminal):** \( \forall x \in X, \exists \) an open neighborhood \( U \) s.t. for a nowhere vanishing section \( s \in \mathcal{O}_{mK_X}(U) \)

\[
\int_{U^{\text{reg}}} (\sqrt{-1}^{mn^2} s \wedge \bar{s})^{1/m} < +\infty. \tag{4}
\]

Hermitian metric on the \( \mathbb{Q} \)-line bundle \(-K_X\): \( e^{-\varphi} = \{ e^{-\varphi_\alpha} \} \) s.t. \( |s_\alpha^*|^2 e^{-m\varphi_\alpha} = |s_\beta^*|^2 e^{-m\varphi_\beta} \). We always assume \( \{ \varphi_\alpha \} \) are bounded.

**KE equation:** \( (\sqrt{-1} \partial \bar{\partial} \varphi_\alpha)^n = |s_\alpha^*|^{2/m} e^{-\varphi_\alpha} \left( \sqrt{-1}^{mn^2} s_\alpha \wedge \bar{s}_\alpha \right)^{1/m} \). \( \tag{5} \)

**weak KE metrics:** bounded solutions (in Bedford-Taylor sense) to (5).

Note: Condition (4) \( \iff \) the right-hand-side of (5) is integrable.
Digression: Why Klt singularities

It is the biggest class of singularities for which the Yau-Tian-Donaldson conjecture is expected to hold, and for which the Minimal Model Program in birational algebraic geometry is known to work.

1. $\dim_{\mathbb{C}} X = 2$: Klt = isolated quotient singularity $\mathbb{C}^2 / G$.
2. $\dim_{\mathbb{C}} X = 3$: partial classifications ($\{\text{terminal}\} \subset \{\text{canonical}\} \subset \{\text{Klt}\}$)
3. Quotient singularities and toric singularities are Klt.
4. Klt singularities are local correspondent of (log-)Fano varieties. Any Klt singularity degenerates to (orbifold-)cones over (log-)Fano varieties.

Check Klt condition: first choose a resolution of singularities $\mu : M \rightarrow X$ ($M$ is smooth and $\mu$ is isomorphism over $X^{\text{reg}}$) and write:

$$\mu^* (s \wedge \bar{s})^{\frac{1}{m}} = h(z) \prod_i |z_i|^{2a_i} dz \wedge d\bar{z}, \quad (h(z) \text{ nowhere vanishing});$$

or algebraically:

$$K_M = \mu^* K_X + \sum_i a_i E_i, \quad E_i = \{z_i = 0\}.$$

Then (4) is equivalent $a_i > -1$. $a_i$ is called the discrepancy of $E_i$ over $X$. 
We say the singularities of $X$ are **admissible** if \( \exists \) a log resolution of singularities $\mu : M \to X$ s.t.

(A1) If $K_M = \mu^*K_X + \sum_i a_i E_i$, then $-1 < a_i \leq 0$ for any $i$; and

(A2) $\exists \theta_i \in \mathbb{Q}_{>0}$ s.t. $\mu^*(-K_X) - \sum_i \theta_i E_i$ is an ample $\mathbb{Q}$-line bundle on $M$.

**Theorem (L.-Tian-Wang ’17)**

Let $X$ be a $\mathbb{Q}$-Fano variety with admissible singularities. If $X$ is K-polystable, then $X$ admits a Kähler-Einstein metric.

1. (A2) is always satisfied for $\mathbb{Q}$-factorial singularities. There are a lot of admissible Klt singularities including all 2-dimensional Klt singularities, $\mathbb{Q}$-factorial Klt singularities admitting crepant resolutions.

2. First existence result of YTD for “non-smoothable” Fano varieties.

3. YTD is expected to be true for any $\mathbb{Q}$-Fano variety. But not all KE Fano varieties are expected to be Gromov-Hausdorff (GH) limits of smooth Riemannian manifolds with lower Ricci bounds.
Idea/sketch of proofs

- Take an admissible resolution $\mu : M \to X$ and solve for appropriate (edge) conical Kähler-Einstein metric on $M$.
- Prove that the (edge) conical KE metrics on $M$ converge to a Kähler-Einstein metric on $X$ under the assumption of K-polystability.
(Edge) Conical Kähler metrics

\[ g = \frac{|dz|^2}{|z|^{2(1-\beta)}} = dr^2 + \beta^2 r^2 d\theta^2, \quad \omega = \sqrt{-1} \frac{dz \wedge d\bar{z}}{|z|^{2(1-\beta)}} = \sqrt{-1} \partial \bar{\partial} \left( \beta^{-2} |z|^{2\beta} \right). \]

Higher dim: If \( D = \sum_{i=1}^{m} D_i = \{z_1 z_2 \cdots z_m = 0\} \) is a SNC divisor (i.e. \( D_i \) are smooth and intersections are transversal), then local (edge) conical model:

\[ \omega = \sqrt{-1} \left( \sum_{i=1}^{m} \frac{dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\beta_i)}} + \sum_{j=m+1}^{n} dz_j \wedge d\bar{z}_j \right). \]

Its Ricci curvature has a current term:

\[ \text{Ric}(\omega) = \sqrt{-1} \partial \bar{\partial} \sum_{i=1}^{m} \log |z_i|^{2(1-\beta_i)} = \sum_{i=1}^{m} (1 - \beta_i) 2\pi \delta_{\{z_i = 0\}} dx_i \wedge dy_i. \]
We divide the proof of the main result into 6 steps:

**Step 0** (A2) ⇒ can assume $L_\epsilon := \mu^*(-K_X) - \epsilon \sum_i \theta_i E_i$ is positive for $0 < \epsilon \leq 1$ ⇒ there is a very ample divisor $H = \{s_H = 0\} \in |mL_1|$ (with $m \gg 1$) s.t. $H + \sum_i E_i$ has SNC.

**Step 1** Prove $(M, \frac{1-t}{m} H + \sum_{i \geq 1} (1 - \beta_i) E_i)$ is uniformly-K-stable for appropriate cone angles $2\pi \beta_i$. This part is purely algebraic.

**Step 2** Use version of YTD for log smooth pairs to construct 2-parameter family of (edge) conical KE metrics $\omega_{(\epsilon, t)} \in 2\pi c_1(L_\epsilon)$ on $M$ with edge conical singularities along $H \cup \sum_i E_i$.

**Step 3** For fixed $t$, prove $(M, \omega_{(\epsilon, t)}, d_{(\epsilon, t)}) \xrightarrow{\epsilon \to 0} (X, \omega_{(0, t)}, d_{(0, t)})$ (in both pluripotential and GH senses). $\omega_{(0, t)}$ is a weak KE on $(X, \frac{1-t}{m} H_X)$ for some $H_X \in |-mK_X|$ and $(X, d_{(0, t)}) = (X^{\text{reg}}, \omega_{(0, t)}|_{X^{\text{reg}}})$ (metric completion).

**Step 4** As $t \to 1$, $(X, \omega_{(0, t)})$ subsequentially converges in Gromov-Hausdorff topology to $X_\infty$ equipped with a weak Kähler-Einstein metric $\omega_{(0, 1)}$.

**Step 5** Construct a special degeneration of $X$ to $X_\infty$ with zero Futaki invariant. K-polystability of $X$ forces $X_\infty \cong X$.

The last two steps were essentially done by CDS and Tian (although $X$ is smooth in their case).
Step 0: set-up (edge) conical KE equation on $M$

Decomposition of $\mathbb{Q}$-divisor:

$$-K_M = \mu^*(-K_X) + \sum_{i \geq 1} (-a_i)E_i = t(\mu^*(-K_X) - \epsilon \sum_i \theta_i E_i)$$

$$+ \frac{1-t}{m} H + \sum_{i \geq 1} (-a_i + t\epsilon \theta_i + (1-t)\theta_i) E_i =: B(\epsilon, t).$$

Correspondingly, solve the following KE equation ($E_0 = H$ for simplicity):

$$Ric(\omega(\epsilon, t)) = t\omega(\epsilon, t) + 2\pi \{ B(\epsilon, t) \} \iff (\sqrt{-1} \partial \bar{\partial} \varphi)^n = \frac{e^{-t\varphi}}{\prod_{i \geq 0} |s_{E_i}|^{2(1-\beta_i)}} \quad ((*)_{\epsilon, t})$$

Geometrically, $\omega(\epsilon, t)$ is a (edge) conical KE metric which is smooth on $M \setminus \text{Supp}(B(\epsilon, t))$ and has cone singularities along $E_i$ with cone angle $2\pi \beta_i$ where

$$\beta_i = \begin{cases} 
1 - \frac{1-t}{m}; & \text{for } i = 0 \text{ i.e. } E_0 = H \\
1 + a_i - t\epsilon \theta_i - (1-t)\theta_i & \text{for } i \geq 1.
\end{cases}$$

Why Admissible: $a_i \in (-1, 0] \Rightarrow \beta_i \in (0, 1]$ for $0 \leq \max\{\epsilon, 1-t\} \ll 1$
Proposition

\( X \) K-polystable \( \Rightarrow \) \((M, B_{(\epsilon, t)})\) uniformly K-stable if \( 0 < \max\{\epsilon, 1 - t\} \ll 1 \).

The proof uses the valuative criterion of K-stability developed by Fujita and L.. For any divisorial valuation \( \text{ord}_F \) over \( M \), define:

\[
\Phi_{(M,B)}(F) := \frac{A_{(M,B)}(E)(-K_M - B)^n}{\int_0^{+\infty} \text{vol}_M(-K_M - B - xE)dx}, \quad \tilde{\delta}(M, B) := \inf_F \Phi_{(M,B)}(F). \quad (7)
\]

Theorem (Fujita, L.)

1. \((M, B)\) is K-semstable iff \( \tilde{\delta}(M, B) \geq 1 \).
2. \((M, B)\) is uniform K-stable iff \( \tilde{\delta}(M, B) > 1 \).

Why this helps: because \( \mathbb{C}(M) \cong \mathbb{C}(X) \) and the set of valuations do not change. On the other hand, the set of special degenerations change!
Step 2: existence of KE on \((M, B_{(\epsilon, t)})\)

Need a logarithmic version of YTD for the pair \((M, B)\) with smooth ambient space:

**Theorem (L.-Tian-Wang, Tian-Wang)**

\((M, B)\) is uniformly K-stable \(\implies\) energy is proper \(\implies\) \(\exists\) solution to \((*)_{(\epsilon, t)}\).

Two proofs for energy properness:

1. Generalize Berman-Boucksom-Jonsson’s argument to the logarithmic setting:

   Uniformly K-stable
   \[
   \begin{align*}
   \overset{\text{def}}{\iff} & \quad F_{\text{NA}} \geq \delta J_{\text{NA}} \quad \text{on smooth non-Archimedean metrics} \\
   \iff & \quad F_{\text{NA}} \geq \delta J_{\text{NA}} \quad \text{on finite energy non-Archimedean metrics} \\
   \iff & \quad F \geq \delta J - C \quad \text{on the space of (smooth or finite energy) Kähler metrics}.
   \end{align*}
   \]

2. Generalize CDS-Tian’s argument to the logarithmic setting (need the conical version of Cheeger-Colding-Tian’s theory developed recently by Tian-Wang).

The Euler-Lagrange equation of \(F\) is \((*)_{(\epsilon, t)}\):

\[
F := F_{B_{(\epsilon, t)}}(\varphi) = -E_{\psi_{\epsilon}}(\varphi) - \frac{1}{t} \log \left( \int_{M} \frac{e^{-t\varphi}}{|s_B|^2} \right). \tag{8}
\]

\(J\)-energy measures the distance between two potentials:

\[
J := J_{\psi_{\epsilon}}(\varphi) = -E_{\psi_{\epsilon}}(\varphi) + \frac{1}{(L_{\epsilon} \cdot n)} \int_{M} (\varphi - \psi_{\epsilon})(\sqrt{-1} \partial \bar{\partial} \psi_{\epsilon})^n. \tag{9}
\]
Step 3: take limit of \((M, \omega(\epsilon, t))\) as \(\epsilon \to 0^+\): potential Part

**Proposition (weak uniform properness)**

Fix \(t \in (0, 1)\) there exist \(\epsilon^* = \epsilon^*(t)\), \(\delta^* = \delta^*(t)\) and \(C > 0\) s.t. for any \(\epsilon \in (0, \epsilon^*]\) and any \(\varphi \in PSH(L_\epsilon)\), the following inequality holds:

\[
F_{B(\epsilon, t)}(t, \varphi) \geq \delta^* J_{\psi_\epsilon}(\varphi) - C \epsilon^* \| \varphi - \varphi_\epsilon \|_\infty - C.
\]  

(10)

We proved this estimate by using the properness of \(F_{B(\epsilon^*, t)}\) and comparing the energy functional for parameter \(\epsilon\) and \(\epsilon^*\) by a rescaling map:

\[
P_\epsilon : PSH(L_\epsilon) \rightarrow PSH(L_{\epsilon^*})
\]

\[
\varphi \mapsto \psi_{\epsilon^*} + \frac{1}{1 + 2(\epsilon^* - \epsilon)}(\varphi - \psi_\epsilon)
\]

From weak properness to \(\| \varphi(\epsilon, t) - \psi_\epsilon \|_{L^\infty}\), need 2 facts:

1. Uniform bound on Sobolev constants for (edge) conical KE’s \((M, \omega(\epsilon, t))\).
   This allows us to bound \(\| \cdot \|_{L^\infty}\) in terms of \(J_{\psi_\epsilon}(\varphi)\) by Moser iteration.

2. \(F_{(B_\epsilon, t)}(\varphi(\epsilon, t))\) (= infimum of \(F_{B(\epsilon, t)}\)) is uniformly bounded from above.

Combining these facts, we get uniform \(L^\infty\) estimates:
Uniform estimates for the potential functions of $\omega(\epsilon, t)$

**Proposition (uniform $L^\infty$ estimate)**

There exists a constant $C = C(X, t) > 0$ s.t. the solution $\varphi(\epsilon, t)$ to $(\ast)(\epsilon, t)$ satisfies the uniform $L^\infty$ estimate: $\|\varphi(\epsilon, t) - \psi_\epsilon\|_{L^\infty} < C$.

Derive higher-order estimate away from singular set from the uniform $L^\infty$ estimate

**Proposition (higher order estimates)**

For any $V \subset M \setminus (\bigcup_{i \geq 1} E_i)$ and any $\alpha < (1 - \frac{1-t}{m})^{-1} - 1$, there exists a constant $C = C(M, V, t, \alpha) > 0$ s.t. $\|\omega(\epsilon, t)\|_{C^{\alpha, 0}(V)} \leq C$.

As $\epsilon \to 0^+$, $\omega(\epsilon, t)$ converges to a solution to weak Kähler-Einstein metric $\omega(0, t)$ on $(X, H_X)$ where $\mu^* H_X = H + m \sum \theta_i E_i$. So we get:

**Theorem (weak version of YTD)**

- If an admissible $\mathbb{Q}$-Fano $X$ is uniformly K-stable, then $X$ ha a KE metric.
- If $X$ is K-semistable, then there exists KE $\omega(0, t)$ on $(X, \frac{1-t}{m} H_X)$ for $0 < 1 - t \ll 1$. 
Step 3: take limit of \((M, \omega(\epsilon, t))\) as \(\epsilon \to 0^+\): metric part

**Theorem (Tian-Wang)**

Let \((X(0, t), d(0, t))\) be a GH limit of a sequence \((M, \omega(\epsilon_i, t))\) as \(\epsilon_i \to 0\). Then there is a decomposition \(X(0, t) = R \cup S\) satisfying:

1. \(R\) is open in \(X(0, t)\) and has a smooth manifold structure equipped with a smooth KE metric.

2. The singular set has a decomposition \(S = \bigcup_{k=1}^{n} S_{2n-2k}\) where \(S_{2n-2k}\) consists of the points whose metric tangent cones do not split \(\mathbb{R}^{2n-2k+1}\)-factor. \(S_{2n-2k}\) satisfies \(\text{codim}_{\mathbb{R}}(S_{2n-2k}) \geq 2k\).

**Main problem:** Prove \(X(0, t) = X\). In particular, \(X(0, t)\) is an algebraic variety.

**Difficulty:** \(\omega(\epsilon, t)\) is curvature form of varying line bundle \(L_\epsilon\). The usual partial \(C^0\)-estimate technique does not apply directly.

Fortunately, similar problems have been considered in the study of Kähler-Ricci flow and other continuity method by J. Song and Tian-Zhang.
Using $L^\infty$ and higher order estimate of $\varphi(\epsilon,t)$, we get a gauge fixing result:

**Proposition (Proposition A: gauge fixing motivated by Rong-Zhang)**

For $0 < t < 1$, $(X_{(0,t)}, d_{(0,t)})$ is the metric completion of $(X_{\text{reg}}, \omega(0,t)|_{X_{\text{reg}}})$. Moreover, $\text{id} : (X_{\text{reg}} = M \setminus E, d_{g(0,t)}) \to (M, \omega(\epsilon,t))$ gives a GH approximation for the convergence $(M, \omega(\epsilon,t)) \xrightarrow{\epsilon \to 0} (X_{(0,t)}, d_t)$.

The next result says $X_{(0,t)}$ coincides with the algebraic variety $X$:

**Proposition (Proposition B)**

$X_{(0,t)}$ is homeomorphic to $X$. As a consequence, $(X, \frac{1-t}{m} H_X)$ admits a weak Kähler-Einstein metric $\omega(0,t)$ such that the $(X, d_{(0,t)})$ is the metric completion of the geodesically convex subset $(X_{\text{reg}}, \omega(0,t)|_{X_{\text{reg}}})$. 
Let \( L = \mu^*(-K_X) \) and \( \Phi^\ell : M \to \mathbb{P}^N \) be the morphism defined by an o.n.b. of \( (H^0(M, L^{m\ell}), h_{FS}^{m\ell}, \omega_{FS}) \). Then

\[ \Phi_{(\epsilon,t)}^\ell = \Phi^\ell : (M, \omega(\epsilon,t)) \to (\Phi^\ell(M) \cong X, \omega_{FS}) \]  

(11)

is uniformly Lipschitz (by using Chern-Lu’s inequality):

\[ \omega_{FS} \leq C \cdot \omega(\epsilon,t) \] with uniform \( C \).

As \( \epsilon \to 0 \) with \( t \) fixed, \( \Phi_{(\epsilon,t)}^\ell \) subsequentially converges to a Lipschitz map:

\[ \Phi_{(0,t)}^\ell : (X_{(0,t)}, d_{(0,t)}) \to (\Phi^\ell(M) \cong X, \omega_{FS}). \]  

(12)

Recall: \( X_{(0,t)} \) is the metric completion of \( (X^{reg}, \omega_{(0,t)}) \) and \( \Phi_{(0,t)}|_{X^{reg}} \) is an isometry.

**Proposition**

\( \Phi_{(0,t)}^{\ell^*} \) is injective for some \( \ell^* \gg 1 \). As a consequence, \( X_{(0,t)} \) is homeomorphic to \( X \). Hence \( (M, \omega(\epsilon,t)) \) Gromov-Hausdorff converges to \( (X, d_{(0,t)}) \) which is the metric completion of \( (X^{reg}, \omega_{(0,t)}) \).

1. Need a lot of peak sections in \( (H^0(X_{(0,t)}^{reg}, -K_{X_{(0,t)}^{reg}}^{mk}), \| \cdot \|_{L^2(h_{(0,t)}^{k}, \omega_{(0,t)})}) \).
2. Need gradient estimate of \( |\nabla h_{(0,t)}^{k} \zeta|_{h_{(0,t)}^{k} \otimes \omega_{(0,t)}} \) for \( \zeta \in H^0(X_{(0,t)}^{reg}, L^{k}) \).
$X_{(0,t)} \cong X$: construct $L^2$ section using $h_\epsilon$

Singular metric on $\mu^*(-K_X) = L_\epsilon + \sum_i \theta_i E_i$: (write $\epsilon$ for $(\epsilon, t)$)

$$\hat{h}_\epsilon := e^{-\hat{\varphi}_\epsilon} = \frac{e^{-\varphi_\epsilon}}{\prod_i |s_i|^{2\epsilon \theta_i}}.$$  \hspace{1cm} (13)

satisfies (recall $1 - \beta_i = -a_i + t\epsilon \theta_i + (1 - t)\theta_i \in [0, 1)$):

$$e^{-k\hat{\varphi}_\epsilon} \omega^n_\epsilon = \frac{e^{-(k + t)\varphi_\epsilon}}{|s_H|^{2\frac{1-t}{m}} \prod_i |s_i|^{2(k\epsilon \theta_i + 1 - \beta_i)}}$$

and $\Theta(\hat{h}_\epsilon^k) + Ric(\omega_\epsilon) \geq (k + t)\omega_\epsilon$.

Weitzenböch formula together with Hörmander’s $L^2$-estimates give:

**Proposition (solve $\bar{\partial}$-equation with $L^2$-estimate)**

Assume $k \epsilon \theta_i + 1 - \beta_i < 1$. Then $\exists C > 0$ independent of $\epsilon$ s.t. for any $\xi \in \Gamma( T^{*(0,1)} M \otimes L^k )$ with $\bar{\partial} \xi = 0$, we can find a solution to $\bar{\partial} \zeta = \xi$ which satisfies:

$$\int_M |\zeta|_{\hat{h}_\epsilon^k \omega^n_\epsilon}^2 \leq \frac{C}{k} \int_M |\xi|_{\hat{h}_\epsilon^k \otimes \omega_\epsilon}^2 \omega^n_\epsilon.$$  \hspace{1cm} (14)
Proposition (convergence away from $B$)

Assume $k \epsilon \theta_i + 1 - \beta_i < 1$. Let $\zeta_j$ be a sequence of holomorphic sections of $L^k$, $k \geq 1$, satisfying $\int_M |\zeta_j|^2_{h_{\epsilon_j}^k} \omega_{\epsilon_j}^n \leq 1$. Then as $\epsilon_j \to 0$, $\zeta_j$ subsequentially converges to a locally bounded holomorphic section $\zeta_\infty$ of $L^k$ over $R = M \setminus \text{supp}(B)$.

Need boundedness of $|\zeta|_{h_{(0,t)}^k}$ and $|\nabla_{h_{(0,t)}^k} \zeta_\infty|_{h_{(0,t)}^k} \otimes \omega_{(0,t)}$ on the regular part $X_{\text{reg}}$.

Let $h_{FS}$ be the pull back of Fubini-Study metric $\Phi^* h_{FS}$ on $L = \mu^*(-K_X)$. Use Bochner formula and Moser iteration to get:

Proposition (uniform estimates w.r.t. $h_{FS}$)

There exists $C > 0$ independent of $\epsilon$, s.t. for any $\zeta \in H^0(M, L^k)$ we have:

$$\sup_M |\zeta|_{h_{FS}^k}^2 \leq Ck^n \int_M |\zeta|^2_{h_{FS}^k} \omega_{\epsilon}^n; \quad (15)$$

$$|\nabla_{h_{FS}} \zeta|^2 \leq Ck^{n+1} \int_M |\zeta|^2_{h_{FS}^k} \omega_{\epsilon}^n. \quad (16)$$

To transfer the estimates to estimates for $h_{(0,t)}$, we need gradient estimate for $\varphi_{(0,t)}$ on $X_{\text{reg}}$. 
Write $\omega_{(0,t)} = \chi_0 + \sqrt{-1} u_{(0,t)}$ over $X$. Then $u_{(0,t)}$ is defined up to a constant and satisfies $\Delta_{(0,t)} u_{(0,t)} = -\text{tr} \omega_{(0,t)} \chi_0 + n$. We approximate $u_{(0,t)}$ by functions on $M$ as follows. Choose $p \in X^{\text{reg}}$ and let $U = \mu^{-1}(X \setminus B_{\omega_{(0,t)}}(p, 2r))$. Solve:

\[
\begin{align*}
\text{Dirichlet problem:} & \quad \begin{cases}
\Delta \epsilon v_{\epsilon} = -\text{tr} \omega_{(\epsilon, t)} \chi_0 + n & \text{on } U; \\
v_{\epsilon} = u_{(0,t)} & \text{on } \partial U.
\end{cases} \quad (17)
\end{align*}
\]

**Proposition**

There exist constant $C > 0$ independent of $\epsilon$ s.t. $|v_{\epsilon}| + |\nabla_{\epsilon} v_{\epsilon}| \leq C$ over $U$.

As $\epsilon \to 0$, $(U, \omega_{(\epsilon, t)}) \to (\hat{U}, d_{(0,t)})$. $v_{\epsilon} \to v$ satisfies equation:

\[
\begin{align*}
\begin{cases}
\Delta \epsilon v = -\text{tr} \omega_{(\epsilon, t)} \chi_0 + n & \text{on } \hat{U} \cap \mathcal{R}; \\
v = u_{(0,t)} & \text{on } \partial \hat{U}.
\end{cases} \quad (18)
\end{align*}
\]

Then one can show that $v = u_{(0,t)}$ over $\hat{U} \cap \mathcal{R}$ and hence $|\nabla \omega_{(0,t)} u_{(0,t)}|$ is indeed bounded.
$\Phi_{(0,t)}^\ell$ is homeomorphism for some $\ell^* \gg 1$

1. $\forall p \in X^\text{reg},$ construct a local approximate holomorphic section on a small open set containing $p$: transfer constant section on the metric tangent cone $C_p$ to small open set containing $p$ by using a good cut function and a gauge fixing diffeomorphism $(V(p, \epsilon) \subset C_p^\text{reg}, C_p \times \mathbb{C}) \to (X^\text{reg}_{(0,t)}, L^{k_p}).$

2. For any $p \in X$, construct holomorphic peak section (almost) centered at $p$. This is obtained by solving $\bar{\partial}$-equation to adjust approximate holomorphic section to become a genuine holomorphic section. The gradient estimate of $\varphi_{(0,t)}$ allows to extend the uniform estimates “across the singularity”.

3. Prove the $\Phi_{(0,t)}^\ell$ is injective for some $\ell^* \gg 1$.

For $p, q \in X_{(0,t)}$, construct two peak sections in $L^{m\ell_{p,q}}$ almost centered at $p$ and $q$. Prove that $\Phi_{p,q}^\ell$ is injective.

Then use the effective finite generation of section rings to prove there exists an $\ell^*$ that works for all pairs $p, q$. 
1. As $t \to 1$, $(X, d_{(0,t)})$ GH converges to $(X_\infty, d_\infty)$. Tian-F.Wang's compactness applies because $(X, d_{(0,t)})$ are GH limits of strong (edge) conical KE metrics with positive Ricci curvature.

2. Use the technique of partial $C^0$-estimates to show that $X_\infty$ is a normal $\mathbb{Q}$-Fano variety and admits a weak KE metric with Lipschitz potentials. Moreover, $X_\infty$ and $X$ can be embedded by $L^2$-sections into a common projective space $\mathbb{P}^N$ such that $\text{Hilb}(X_\infty)$ is in the orbit closure of $\text{Hilb}(X)$ under $\text{PGL}(N + 1, \mathbb{C})$.

3. Prove generalized Matsushima type result: $\text{Aut}(X_\infty)$ is reductive. As a consequence and by using Luna slice theorem, there is a one parameter subgroup in the Hilbert scheme such that

$$\lambda(t) \cdot \text{Hilb}(X) \to \text{Hilb}(X_\infty) \text{ as } t \to 0.$$ 

This gives a special degeneration of $X$ with central fibre $X_\infty$.

4. $X$ being KE implies $\text{Fut}(X_\infty, -\nu) = 0$ where $\nu$ is the generator of $\lambda(t)$. The K-polystability of $X$ forces $X_\infty \cong X$. 
Thanks for your attention!