Mode bifurcation in the Rayleigh-Taylor instability of binary condensates

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(Dated: May 3, 2014)

We examine the generation and subsequent evolution of Rayleigh Taylor instability in anisotropic binary Bose-Einstein condensates. Considering a pancake-shaped geometry, to initiate the instability we tune the intraspecies interaction and analytically study the normal modes of the interface in elliptic cylindrical coordinates. The normal modes are then Mathieu functions and undergoes bifurcation at particular values of anisotropy and ratio of number of atoms. We find that the analytical estimates of the bifurcation parameters are in good agreement with the numerical results.

PACS numbers: 03.75.Kk, 03.75.Mn, 67.85.De, 67.85.Fg

I. INTRODUCTION

Rayleigh-Taylor instability (RTI) is the instability of an interface between two fluids, which sets in, when a layer of lighter fluid supports a denser one or when a lighter fluid pushes a denser one, under the influence of gravitational field or some external potential. This occurs due to unfavourable energy conditions and as a result, the fluids tend to swap their positions. Any perturbation arising on the interface, however, small it may be, grows exponentially due to RTI and turbulent mixing of the fluids occur. During the process of mixing, the interface gets deformed and develops complicated non-linear patterns with mushroom shapes. The phenomenon of RTI is widely common in nature, ranging from convection of water to dusty plasma in atmosphere to supernova explosions. Recently, RTI has also been observed in a trapped two-species Bose-Einstein condensate (TBEC), where, intraspecies scattering length plays a major role. Systems of trapped TBEC’s that have been so far studied for observing RTI are a tight, symmetric pancake-shaped system in which the components separate out radially, a cigar-shaped trap in which phase-separation occurs in the axial direction and a perfectly spherical symmetric trap. Though experimental studies on RTI are rare, theoretical studies on interfacial instabilities has been a major research topic in the recent years. Other instabilities such as, Kelvin-Helmholtz instability (KHI), Faraday instability have also been predicted in TBEC. Experimental observation of quantum KHI and Faraday waves in BEC can be found in Refs. [14–15]. In the present work, we study RTI in a TBEC confined in a harmonic trapping potential. The intraspecies and interspecies interaction between the atoms are taken to be repulsive. The initial state of the TBEC that we consider for our study, is a phase-separated (immiscible) configuration in which the species with weaker intraspecies repulsive interaction is surrounded by the other. In the phase-separated domain, the interface of the TBEC is a circle when the quasi-two dimensional trap is perfectly symmetric. To initiate RTI, we decrease gradually the s-wave scattering length of the outer species through a magnetic Feshbach resonance. As RTI sets in, the outer species tends to sink to the center of the trap and instabilities begin to occur on the circular interface separating the two components. Now, if the anisotropy of the trap is increased along a particular direction, the circular interface evolves into an elliptic cylindrical one. Due to RTI, the nature of various non-linear patterns developed on the interface changes on varying the geometry of the trapping potential. It has been observed that at a critical value of the anisotropy parameter, the normal modes on the interface bifurcates.

This paper is organized as follows: In Section II we formulate the problem using mean-field dynamics in a quasi-two dimensional harmonic trap. In Section III & IIIA we discuss about the interface geometry and normal modes of the interface and formulate the Helmholtz equation using elliptic cylindrical coordinates. In Section IIIIB we derive an analytic condition for the temporal decay constant in an elliptic cylindrical interface. Lastly, in Section IV we present numerical results showing the dynamics of TBEC as a result of RTI.

II. PHASE SEPARATED PANCAKE SHAPED TBECs

In the mean field approximation, the TBEC is described by a set of coupled Gross-Pitaevskii equations

\[
\frac{-\hbar^2}{2m_i} \nabla^2 + V_i(x, y, z) + \sum_{j=1}^{2} g_{ij} |\Psi_j|^2 \right] \Psi_i = i\hbar \frac{\partial |\Psi_i|}{\partial t},
\]

where \(i = 1, 2\) is the species index, \(g_{ij} = 4\pi \hbar^2 a_{ij}/m_i\) with \(m_i\) as mass and \(a_{ij}\) as s-wave scattering length, is the intra-species interaction; \(g_{ij} = 2\pi \hbar^2 a_{ij}/m_{ij}\) with \(m_{ij} = m_i m_j/(m_i + m_j)\) as reduced mass and \(a_{ij}\) as inter-species scattering length, is inter-species interaction and \(\mu_i\) is...
the chemical potential of the ith species. The trapping potential is

\[ V_i(x, y, z) = \frac{m_\omega^2}{2}(x^2 + \alpha_i^2y^2 + \lambda_i^2z^2) \]  

(2)

where, \( \omega \) is the radial trap frequency, considered identical, for the two components, and \( \alpha_i, \lambda_i \) are the anisotropy parameters. For simplicity of analysis, we consider trap potentials of both the species have the same geometry \( \alpha_1 = \alpha_2 = \alpha \), \( \lambda_1 = \lambda_2 = \lambda \) and \( m_1 = m_2 = m \). The energy of the TBEC is

\[ E = \int_{-\infty}^{\infty} \left[ \sum_{i=1}^{2} \left( \frac{\hbar^2}{2m} |\nabla \Phi_i|^2 + V_i(x, y, z)\Psi_i^2 \right) \right] dx dy dz \]  

(3a)

To express the energy in suitable units, we define the oscillator length of the trapping potential \( \alpha_{osc} = \sqrt{\hbar/(m\omega)} \) and consider \( \hbar \omega \) as the unit of energy. We then divide Eq.(3) by \( \hbar \omega \) and apply the transformations \( \tilde{x} = x/\alpha_{osc} \), \( \tilde{y} = y/\alpha_{osc} \), \( \tilde{z} = z/\alpha_{osc} \), \( \tilde{t} = t/\omega \), and \( \tilde{E} = E/(\hbar \omega) \). The transformed order parameter

\[ \Phi_i(\tilde{x}, \tilde{y}, \tilde{z}) = \sqrt{\frac{\alpha_{osc}^3}{N_i}} \Phi_i(x, y, z) \]  

(4)

and energy of TBEC in scaled units is given by

\[ \tilde{E} = \int d\tilde{x} d\tilde{y} d\tilde{z} \left[ \sum_{i=1}^{2} \left( \frac{\hbar^2}{2m} |\nabla \Phi_i|^2 + V_i(\tilde{x}, \tilde{y}, \tilde{z})\Phi_i^2 \right) \right] + N_i \tilde{U}_{ii} |\Phi_i|^4 + N_i N_j \tilde{U}_{ij} |\Phi_i|^2 |\Phi_j|^2 \]  

(5a)

where, \( \tilde{U}_{ii} = 4\pi \alpha_{osc}^2/\alpha_{osc}^4 \) and \( \tilde{U}_{ij} = 4\pi \alpha_{ij}^2/\alpha_{osc}^4 \). For simplicity of notations, from here on we will represent the transformed quantities without tilde. Thus, in scaled units, the coupled 3D GP equation is given by

\[ -\nabla^2 + V_i(x, y, z) + \sum_{j=1}^{2} G_{ij}|\Phi_j|^2 \Phi_i = \mu_i \Phi_i \]  

(6)

where, \( G_{ii} = N_i \tilde{U}_{ii} \) and \( G_{ij} = N_j \tilde{U}_{ij} \). For the present work, we consider a pancake shaped trap, the axial frequency is much larger than the radial frequency (\( \lambda \gg 1 \)). In this situation, the transformed order parameter \( \Phi(x, y, z) \) is factorized into

\[ \Phi(x, y, z) = \phi(x, y)\zeta(z) \]  

(7)

where, \( \zeta(z) \) is the normalized state of axial trapping potential \( V^{axial}_i = \lambda^2z^2/2 \). From Eq.(6) after integrating out the axial order parameter, we obtain the scaled coupled 2D GP equations

\[ \left[ -\nabla^2 + V_i(x, y) + \sum_{j=1}^{2} N_{ij} |\phi_j(x, y)|^2 \right] \phi_i(x, y) = \mu_i \phi_i(x, y) \]  

(8)

where, \( \nabla_i^2 = \partial_x^2 + \partial_y^2 \), \( N_{ii} = 4N_i\sqrt{2\pi\alpha_{osc}} \) and \( N_{ij} = 4N_i\sqrt{2\pi\alpha_{osc}} \). Using Thomas-Fermi approximation in Eq.(5), one can show the two components are phase-separated when \( (N_{ij} > \sqrt{N_{ii}N_{jj}}) \), where, \( N_{ii} \) and \( N_{ij} \) are all positive. To examine RTI, we consider the phase separated state in axis symmetric trapping potentials with coincident centers and numerically solve the pair of time-dependent GP equations

\[ \left[ -\nabla^2_1 + V_1(x, y) + \sum_{j=1}^{2} N_{ij} |\phi_j(x, y)|^2 \right] \phi_i(x, y) = \frac{i}{\partial t} \]  

(9)

III. INTERFACE GEOMETRY AND MODES

In the phase-separated domain, the interface of the TBEC is a circle when \( \alpha = 1 \) is unity. It is, however, transformed to an ellipse when \( \alpha > 1 \). A typical density profile of the phase-separated TBEC with \( \alpha = 1 \) is shown in Fig. 1. Compared to Eq. (2), a more general form of 2D trapping potential is \( V(x, y) = m(\omega_x^2 x^2 + \omega_y^2 y^2)/2 \), where \( \omega_{x,y} \) represent angular trapping frequency along \( x \) or \( y \).

Defining the geometric mean \( \bar{\omega} = \sqrt{\omega_x \omega_y} \), the trapping potential is

\[ V(x, y) = \frac{1}{2} \beta \bar{\omega}^2 \left( \frac{x^2}{\beta^2} + \frac{y^2}{\gamma^2} \right) \]  

(10)

where, \( \beta = \bar{\omega}/\omega_x = \sqrt{\alpha} \) and \( \gamma = \omega_x/\omega_y = 1/\sqrt{\alpha} \). The density distribution of the TBEC, at moderate anisotropies, follows the geometry of the trapping potential. At larger anisotropies the interface energies modifies the density distribution and leads to difference from the geometry of the trapping potential. For the present study, we consider the TBEC at moderate anisotropies. The interface of the TBEC is then an ellipse

\[ \frac{x^2}{\beta^2} + \frac{y^2}{\gamma^2} = 1, \]  

(11)
corresponding to the anisotropy parameters of the trapping potential.

FIG. 2. Phase separated profiles of \(^{85}\)Rb–\(^{87}\)Rb mixture at \(t = 0\). The figure on the left shows the inner species\(^{87}\)Rb with \(a_{22} = 99a_\text{B}\). The figure on the right shows the outer species\(^{85}\)Rb with \(a_{11} = 460a_\text{B}\).

At the interface, the densities are low, neglecting the intraspecies and interspecies interaction term, we get from Eq.\(^8\)

\[-\nabla^2_\perp + V_i \phi_i = \tilde{\mu}_i \phi_i, \tag{12}\]

where, \(\tilde{\mu}_i = \mu_i - \sum_{j=1}^{2} N_{ij} |\phi_j(x, y)|^2\).

Using Eq.\(^11\) and Eq.\(^12\), we get (in scaled units)

\[\nabla^2_\perp \phi_i + \left(\tilde{\mu}_i - \alpha\right) \phi_i = 0. \tag{13}\]

Defining the parameter \(k^2_i = \tilde{\mu}_i - \alpha\), the equation is

\[\nabla^2_\perp + k^2_i \phi_i = 0. \tag{14}\]

This is the Helmholtz equation in 2D. It must, however, be emphasized that the equation is valid only at the interface or close to it. Away from the interface the densities are not small and intraspecies interactions is large.

A. Normal modes of the interface

For linear stability analysis of the interface modes due to a small perturbation, to identify the onset of RTI, we transform the Eq.\(^14\) to elliptic cylindrical coordinates \((u, v)\). Here, the coordinate \(v\) represent the asymptotic angle of confocal hyperbolic cylinders symmetrical about the \(x\)-axis. And, the \(u\) coordinates are confocal elliptic cylinders centered on the origin\(^16\). The transformation is defined by the relations \(x = a \cosh u \cos v\), and \(y = a \sinh u \sin v\) and \(a\) is the focal distance along \(x\)-axis. We take the coordinates on the \(z = 0\) plane as we consider the TBEC in 2D. The Eq.\(^14\) then assumes the form

\[
\frac{1}{a^2 (\sinh^2 u + \sin^2 v)} \left( \frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial v^2} \right) + k^2 \phi = 0, \tag{15}\]

where \(\phi\) is the solution of the form \(\phi = U(u) \Theta(v)\). Substituting, \(\phi\) back in Eq.\(^15\) we get,

\[
\left( \frac{1}{U} \frac{d^2 U}{du^2} + c^2 \sinh^2 u \right) + \left( \frac{d^2 \Theta}{\Theta \, dv^2} + c^2 \sin^2 v \right) = 0. \tag{16}\]

Using separation of variables, the equation is simplified to the Mathieu equations\(^{16, 17}\)

\[
\frac{d^2 U}{du^2} - (A - 2q \cosh 2u) U = 0, \tag{17}\]

\[
\frac{d^2 \Theta}{dv^2} + (A - 2q \cos 2v) \Theta = 0, \tag{18}\]

where, \(A = A + \alpha^2 k^2 / 2\) and \(q = a^2 k^2 / 4\). Here \(A\) is the separation constant and returning to the earlier definition of the trapping potential, the anisotropy parameter \(\alpha = \beta / \gamma\). The interface is an ellipse with eccentricity \(e = \sqrt{1 - 1/\alpha^2}\) and from the theory of conic sections \(\alpha = \beta e = \sqrt{\alpha e}\. Based on these definitions, the constants in the Eq.\(^17\) and \(^18\) are redefined as

\[
q = \frac{1}{4} k^2 c^2 \alpha, \tag{19}\]

\[
A = A + \frac{1}{4} k^2 \gamma^2 \alpha, \tag{20}\]

The constants in this form are easier to connect with the parameters of trapping potentials. The Eqs.\(^17\) and \(^18\) then assumes the form

\[
\frac{d^2 U}{du^2} - \left[A + \frac{1}{2} k^2 c^2 \alpha (1 - \cosh 2u)\right] U = 0, \tag{21}\]

\[
\frac{d^2 \Theta}{dv^2} + \left[A + \frac{1}{2} k^2 \gamma^2 \alpha (1 - \cos 2v)\right] \Theta = 0. \tag{22}\]

The interface of the TBEC, an ellipse, has fixed coordinate \(u\) representing the elliptic cylinder. But the angle coordinate \(v\) varies and lies in the domain \([0, 2\pi]\). Thus \(\Theta\), solutions of the second equation, represent the normal modes of the interface. For circular interface, \(\alpha = 1\) and \(e = 0\), only \(0 < A\) is physically admissible and the solution of the equation is reduced to sinusoidal functions.

B. Instability at the interface

For TBEC in traps, the gradient of the trapping potential is like the gravitational force in the conventional fluid dynamics and the flows within TBEC is modelled as potential flows. Consider the interface of the TBEC, using the method of normal modes, any arbitrary disturbance on the interface may be resolved into independent modes of the form

\[
\xi = \xi' \Theta(v)e^{st}, \tag{23}\]

\[
\phi'_1 = \phi'_1(u) \Theta(v)e^{st}, \tag{24}\]

\[
\phi'_2 = \phi'_2(u) \Theta(v)e^{st}. \tag{25}\]

Here, \(\xi\) is the position of the interface relative to the equilibrium configuration, and \(\phi'_i\) is the increments in the velocity potential of the \(i\)th species about the interfacial region caused due to disturbance in the system. \(\dot{\xi}\) and \(\phi'_i\) are the amplitude of the modes and \(s\) is the temporal decay constant.
We know that for an incompressible fluid, the Laplacian of the velocity potential vanishes \( \nabla^2 \phi_i = 0 \) and from the expression of the normal modes

\[
\frac{\partial^2 \hat{\phi}_i}{\partial u^2} + \left( \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial v^2} \right) \hat{\phi}_i = 0. \tag{26}
\]

Using separation of variables, Eq. (26) can be simplified to,

\[
\frac{\partial^2 \hat{\phi}_i}{\partial u^2} - C^2 \hat{\phi}_i = 0. \tag{27}
\]

where, \( C^2 = -\left(1/\Theta\right)(\partial^2 \Theta/\partial v^2) \). The general solution of the above equation in the regions of the two species are

\[
\begin{align*}
\hat{\phi}_1(u) &= A_1 e^{-C u} + B_1 e^{C u}, \\
\hat{\phi}_2(u) &= B_2 e^{-C u} + A_2 e^{C u},
\end{align*}
\]

where, \( A_1, B_1, A_2, B_2 \) are arbitrary constants. Here, it is to be noted that the sign of the exponents in the two solutions are interchanged. This is to indicate that the relative distance from the interface, within the two species, are in opposite directions. We may recall that the instabilities occur only at the interface or close to it. At any point far removed from the interface the normal modes must decay to zero. Thus, normal modes are of the form \( \hat{\phi}_1 = A_1 \exp(-C u) \) and \( \hat{\phi}_2 = A_2 \exp(C u) \). The velocity potentials in Eq. (24) and Eq. (25) are then

\[
\begin{align*}
\phi_1' &= A_1 e^{-C u} \Theta e^{s t}, \\
\phi_2' &= A_2 e^{C u} \Theta e^{s t}.
\end{align*}
\]

The dynamical evolution of the interface is described through a combination of the continuity equation, Euler’s equation and Bernoulli’s theorem[3, 18]. For stability analysis of the interface, we linearize these equations and neglect quadratic terms in \( \phi_1', \phi_2 \) and \( \xi \). After linearization, we get

\[
\begin{align*}
\frac{\partial \phi_i'}{\partial t} + \frac{\partial \xi}{\partial t}, \\
n_1 \left( \frac{\partial \phi_1'}{\partial t} + g \xi \right) = n_2 \left( \frac{\partial \phi_2'}{\partial t} + g \xi \right),
\end{align*}
\]

where, \( g \) is the gradient of the trapping potential \( V(x, y) \). On the interface, using the solutions obtained earlier, from Eq. (32) one can show that

\[
n_1 \left( s A_1 + g \xi \right) = n_2 \left( s A_2 + g \xi \right). \tag{33}
\]

In a similar way, from Eq. (31), we obtain \( A_1 = -s \xi/C \) and \( A_2 = s \xi/C \). Using these values in the above equation

\[
n_1 \left( -\frac{s^2 \xi}{C} + g \xi \right) = n_2 \left( \frac{s^2 \xi}{C} + g \xi \right). \tag{34}
\]

Simplifying this equation, one arrives at the definition of the temporal decay constant

\[
s = \pm \left[ \frac{C g(n_1 - n_2)}{n_1 + n_2} \right]^{1/2}, \tag{35}
\]

The densities of the condensates \( n_1 \) and \( n_2 \) are at a point \((u, v)\) on the interface. We recollect that \( n_2 \) refer to the density of species at the center which is surrounded by the species with density \( n_1 \). The interface is stable when \( s \) is imaginary \((n_1 < n_2)\) and oscillates when perturbed. However, when \( n_1 > n_2 \), the value of \( s \) is real and any perturbation, however small, grows exponentially with time. This is the prerequisite for RTI in binary condensates. In this context, Atwood number \( \Gamma \) is given by [19]

\[
\Gamma = \frac{n_1 - n_2}{n_1 + n_2}, \tag{36}
\]

\[\text{FIG. 3.} \text{ Shaded regions indicate the values in aq-plane where solutions of angular Mathieu equation, ce}_n(v, q) \text{ and se}_n(v, q) \text{ exist. The black (red) colored curves are the values of a and q for which integer order, ce}_{n-1}(v, q) \text{ (se}_n(v, q) \text{ ) with } n = 1, 2, 3, \ldots, \text{ solutions exist.} \]
C. Allowed solutions

The solutions of Eq. (22), the angular Mathieu equation, are the $c_{2}(v, q)$ and $s_{2}(v, q)$ functions [20], cosine and sine elliptic functions, respectively. Here, $\nu$ is real number and denotes the order of the elliptic functions. The solutions, however, exist only for certain range of $a$ and $q$, and these are shown as shaded regions in Fig. 3. In the figure, the shaded region consists of lobes and each are bounded by elliptic function of integer orders $c_{n-1}(v, q)$ and $s_{n}(v, q)$, where $n = 1, 2, 3, \ldots$. For the present case, when RTI sets in, the mushroom shaped superfluid flows have four fold symmetry in the case of circular symmetry. So that the flow retains symmetry or shape invariance along perpendicular directions. The corresponding solution of Eq. (22) which satisfies this condition is then $c_{2}(v, q)$, and it has the properties $c_{2}(v, q) = c_{2}(v + \pi, q)$ and $c_{2}(v, q) = c_{2}(-v, q)$. The loci of the $a$-$q$ pairings which allow this solution is the labeled curve in Fig. 3.

One property of $c_{2}(v, q)$ is, the maximum at $v = 0$ undergoes a smooth bifurcation at higher values of $q$. Coming to the description of the interface in the binary condensates, from Eq. (19), $q$ is a linear function of the anisotropy parameter $a$. So, as we increase the anisotropy the mushroom shaped flows in RTI must undergo bifurcation. At some value of $\alpha$, instead of four there must be six mushroom shaped inward superfluid flow.

IV. NUMERICAL RESULTS

To corroborate the analytic results for the interface modes, as mentioned earlier, we numerically solve the pair of coupled Eq. (9). We resort to split-step Crank-Nicholson method [21] implemented for binary condensates. We discretize Eq. (9) both in space and time, and propagate the resulting discretized equation in imaginary time, over small time steps. In imaginary-time propagation method, $t$ in Eq. (9) is replaced by $-i\tau$. This method seems to be more appropriate as the stationary ground state wave function of the TBEC is essentially real and dealing with real variables is more convenient than imaginary ones. The split-step imaginary time solution obtained in a self-consistent way after several iterations, is the stationary state of TBEC for the given parameters used in this paper. The time-independent solution of TBEC, thus obtained, is used as an initial state for real-time propagation. The real-time propagation method yields the solution of time-dependent GP equation for TBEC, which is used to study the dynamical behaviour of TBEC.

As a representative case, we numerically calculate the stationary state solution of TBEC based on the aforementioned method, with the parameters given in Ref. [22]. To study RTI, we use the imaginary-time solution as the initial state. With the propagation of this solution over real-time, we gradually change the scattering length over time steps, and study its dynamics [8]. Density profiles as shown in Fig. 4 are the numerical solutions obtained by this method.

![Figure 4](image)

FIG. 4. (a)-(c) shows development of mushroom shape pattern on the interface after $t = 358$ ms, $t = 378$ ms, $t = 400$ ms. The scattering length is decreased from $a_{11} = 460a_{11}$ to $a_{11} = 55a_{11}$ between $t = 0$ ms and $t = 200$ ms, after that $a_{11}$ is fixed to $55a_{11}$ upto $t = 400$ ms. The images on the upper panel correspond to the inner species ($^{87}$Rb) and the images on the lower panel correspond to the outer species ($^{85}$Rb).

A. Mode bifurcation and density profiles

We consider a system of $^{85}$Rb-$^{87}$Rb atoms in a symmetric 2-D harmonic trapping potential with ($\omega_{\bot}, \omega_{z}$) = $2\pi \times (8, 90)$ Hz. We choose initial state to be the ground state for which $a_{11} = 460a_{11}$, $a_{22} = 99a_{22}$, $a_{12} = a_{21} = 214a_{21}$, with $a_{B}$ being the Bohr radius. The number of atoms are $N_{1} = 5 \times 10^{6}$ and $N_{2} = 10^{6}$ [22]. In this configuration, component 1 (outer), $^{85}$Rb completely surrounds component 2 (inner), $^{87}$Rb. Fig. 1 shows phase separated profiles of the TBEC at $t = 0$ in a perfectly symmetric pancake shaped trap i.e. $\alpha = 1$.

Now, the s-wave scattering length $a_{11}$ of the outer species is decreased gradually over time, experimentally this is possible through the $^{85}$Rb-$^{85}$Rb magnetic Feshbach resonance [23]. However, throughout the process, we maintain $(N_{12} > \sqrt{N_{11}N_{22}})$ so that the TBEC remains in the immiscible domain. A stage is reached when $a_{11} < a_{22}$, $^{85}$Rb-$^{85}$Rb interaction weaker than the $^{87}$Rb-$^{87}$Rb interaction. In this situation, the existing spatial structure of the system is energetically unfavourable and the outer species starts penetrating inside the inner species. Instabilities begin to occur at the interface of the two components and eventually grows into a four fold mushroom shape superfluid flow as shown in Fig. 4.

The dynamics and the formation of lobes also depends on the geometry of the interface. As the anisotropy of the trap $\alpha$ is increased keeping $\lambda$ and other remaining parameters fixed, the circular interface evolves into an
elliptic interface. The penetration of the heavier fluid into the lighter fluid gets initiated along the x-axis, followed by the formation of lobes. This happens because, the interface is more curved along this direction with less confinement. Larger is the curvature, higher is the rate of inflow of the heavier fluid. Mass transport gradually occurs along y-direction, which is tightly confined. The interface here, is relatively flat and the lobes are formed at later stages of evolution. This is clearly evident from the superfluid flow pattern shown after the onset of RTI as shown in Fig. 5. In Fig. 5-c-d, the lobes of \( n_2 \) along the x-axis are well developed and located deep within \( n_1 \). As \( \alpha \) is increased further, the interface is deformed further. The lobe along the y-axis undergoes a bifurcation when the anisotropy is such that \( \alpha = 3 \) and the density profile of the superfluid flow is shown in Fig. 6. Taking an average along the interface and close to the bulk of \( n_1 \), when the mode bifurcates the value of \( \tilde{\mu} \) is 5.25. This can be related qualitatively to the analytic results, in which case \( c_{22}(v, q) \) undergoes bifurcation at around \( q \approx 3.7 \). Thus our numerical results is in agreement with the inferences drawn from the analytic solutions of the interface modes.

**FIG. 6.** (a)-(c) Development of various non linear patterns for \( \alpha = 1.8, \alpha = 2.0 \) & \( \alpha = 3.0 \). The images on the upper panel correspond to the inner species (\(^{87}\)Rb) and the images on the lower panel correspond to the outer species (\(^{85}\)Rb)

**B. Bogoliubov analysis**

For a more detailed understanding of the instability, we perform a Bogoliubov analysis for TBEC in a 2-D harmonic trap. Setting, \( \Phi_i = \phi_i + \delta\phi_i \), we expand the set of coupled GP equations in Eq. 9 in \( \delta\phi_i(x, y) \). Here, \( \phi_i \)'s fixes the condensate density through \( n_i(x, y) = |\phi_i(x, y)|^2 \) and \( \delta\phi_i \)'s are the deviations from the initial ground state, which includes the quasi-particle excitations. We consider excitation mode of the form

\[
\delta\phi_i = e^{-i\mu_i t/\hbar}[u_i e^{-i\omega t} - v_i^* e^{i\omega t}],
\]

where, \( \mu_i \) is the chemical potential, \( \omega \) is the excitation frequency, and \( u_i \) and \( v_i \) are the Bogoliubov amplitudes. Using this ansatz, the Bogoliubov equations are

\[
\left[ -\frac{\hbar^2}{2m_i} \nabla_x^2 + V_i + 2n_i U_{ii} + U_{ij} n_j - \mu_i \right] u_i - U_{ii} n_i v_i + U_{ij} \sqrt{n_i n_j} (u_j - v_j) = i\hbar \omega u_i, \quad (41)
\]

\[
\left[ -\frac{\hbar^2}{2m_i} \nabla_x^2 + V_i + 2n_i U_{ii} + U_{ij} n_j - \mu_i \right] v_i - U_{ii} n_i u_i + U_{ij} \sqrt{n_i n_j} (v_j - u_j) = -i\hbar \omega v_i. \quad (42)
\]

These equations are then numerically diagonalized to calculate the excitation spectrum. If the frequencies are real, the perturbations remain bounded and the system is dynamically stable. On the other hand, pure imaginary eigenfrequencies denote instability of the system. The eigenmode corresponding to this complex frequency grows exponentially and is a signature of dynamically unstable system [24].

As a case study, we choose \( N_2/N_1 = 2 \). For the isotropic case, \( \alpha = 1 \), we expand the Bogoliubov amplitudes in harmonic oscillator basis wave function. When \( \delta\phi_i \) is small, \( \text{Re}(\omega) \) increases monotonically up to a critical point as the \( a_{11} \) is decreased. When \( a_{11} < a_{22} \) and RTI sets in, the low lying excitation modes \( \omega \) starts becoming imaginary. The value of \( \text{Im}(\omega) \) increases monotonically as \( a_{11} \) is decreased further and away from the critical point. These imaginary modes are signatures of instability in the dynamics of the binary condensates.

**C. Effect of noise**

Numerical studies that have been carried out so far are at zero temperature and without any imperfections, hence quite ideal. But in experiments, conditions are far from ideal. Fluctuations play a major role, and if large, may destroy the observed signatures predicted from the numerical simulations. One immediate remedy is to include fluctuation to our calculations. We introduce white noise during the real time evolution of TBEC. The white noise is at the level of 0.01%. Even after introducing noise, we still observe signatures of RTI as a result of changing \( a_{11} \). The thermodynamical quantities such as energy, chemical potential may vary quantitatively, but,
there is no qualitative difference in the shape of the interface after RTI is initiated. Bifurcation of normal modes on the interface are still observed at the predicted values of the anisotropy of the trap.

V. CONCLUSIONS

We have examined RTI at the interface of binary condensates as a function of anisotropy parameter and ratio of number of atoms. The mushroom shaped superfluid flow is four lobed, as expected, at low anisotropies. Based on the analytical studies, the lowest natural mode is $c_{n-1}(v, q)$, which describes the four lobed superfluid flow. However, at higher anisotropies corresponding to larger values of $q$, one of the maxima of $c_{2}(v, q)$ bifurcates. This is also observed in the numerical simulation of the RTI at higher $\alpha$. The RTI and bifurcation of the mode is robust, and observable in presence of white noise.

ACKNOWLEDGMENTS

We thank S. Chattopadhyay and K. Suthar for useful discussions. The results presented in the paper are based on the computations using the 3TFLOP HPC Cluster at Physical Research Laboratory, Ahmedabad.

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