Quotients of absolute Galois groups which determine the entire Galois cohomology

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Abstract For a prime power \( q = p^d \) and a field \( F \) containing a root of unity of order \( q \) we show that the Galois cohomology ring \( H^\ast(G_F, \mathbb{Z}/q) \) is determined by a quotient \( G_F^{[3]} \) of the absolute Galois group \( G_F \) related to its descending \( q \)-central sequence. Conversely, we show that \( G_F^{[3]} \) is determined by the lower cohomology of \( G_F \). This is used to give new examples of pro-\( p \) groups which do not occur as absolute Galois groups of fields.

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1 Introduction

A main open problem in modern Galois theory is the characterization of the profinite groups which are realizable as absolute Galois groups of fields $F$. The torsion in such groups is described by the Artin–Schreier theory from the late 1920’s, namely, it consists solely of involutions. More refined information on the structure of absolute Galois groups is given by Galois cohomology, systematically developed starting the 1950’s by Tate, Serre, and others. Yet, explicit examples of torsion-free profinite groups which are not absolute Galois groups are rare. In 1970, Milnor [28] introduced his $K$-ring functor $K^M(F)$, and pointed out close connections between this graded ring and the mod-2 Galois cohomology of the field. This connection, in a more general form, became known as the Bloch–Kato conjecture: it says that for all $r \geq 0$ and all $m$ prime to char $F$, there is a canonical isomorphism $K^M_r(F)/m \to H^r(G_F, \mathbb{Z}/m^r)$ ([19]; see notation below). The conjecture was proved for $r = 2$ by Merkurjev and Suslin [27], for arbitrary and $m = 2$ by Voevodsky [38], and in general by Rost, Voevodsky, with a patch by Weibel [21,39–41].

In this paper we obtain new constrains on the group structure of absolute Galois groups of fields, using this isomorphism. We use these constrains to produce new examples of torsion-free profinite groups which are not absolute Galois groups. We also demonstrate that the maximal pro-$p$ quotient of the absolute Galois group can be characterized in purely cohomological terms (see Theorem 8.5). The main object of our paper is a remarkable small quotient of the absolute Galois group, which, because of the above isomorphism, already carries a substantial information about the arithmetic of $F$.

More specifically, fix a prime number $p$ and a $p$-power $q = p^d$, with $d \geq 1$. All fields which appear in this paper will be tacitly assumed to contain a primitive $q$th root of unity. Let $F$ be such a field and let $G_F = \text{Gal}(F_{\text{sep}}/F)$ be its absolute Galois group, where $F_{\text{sep}}$ is the separable closure of $F$. Let $H^*(G_F) = H^*(G_F, \mathbb{Z}/q)$ be the Galois cohomology ring with the trivial action of $G_F$ on $\mathbb{Z}/q$. Our new constraints relate the descending $q$-central sequence $G^{(i)}_F$, $i = 1, 2, 3, \ldots$, of $G_F$ (see Sect. 4) with $H^*(G_F)$. Setting $G^{[i]}_F = G_F/G^{(i)}_F$, we show that the quotient $G^{[3]}_F$ determines $H^*(G_F)$, and vice versa. Specifically, we prove:

**Theorem A** The inflation map gives an isomorphism

$$H^*(G^{[3]}_F)_{\text{dec}} \sim H^*(G_F),$$

where $H^*(G^{[3]}_F)_{\text{dec}}$ is the decomposable part of $H^*(G^{[3]}_F)$ (i.e., its subring generated by degree 1 elements).

We further have the following converse results.

**Theorem B** $G^{[3]}_F$ is uniquely determined by $H^r(G_F)$ for $r = 1, 2$, the cup product $\cup: H^1(G_F) \times H^1(G_F) \to H^2(G_F)$ and the Bockstein homomorphism $\beta: H^1(G_F) \to H^2(G_F)$ (see Sect. 2 for the definition of $\beta$).

**Theorem C** Let $F_1, F_2$ be fields and let $\pi: G_{F_1} \to G_{F_2}$ be a (continuous) homomorphism. The following conditions are equivalent:

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