Cosmology and Brane Worlds: A Review

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Abstract. Cosmological consequences of the brane world scenario are reviewed in a pedagogical manner. According to the brane world idea, the standard model particles are confined on a hyper-surface (a so-called brane), which is embedded in a higher-dimensional spacetime (the so-called bulk). We begin our review with the simplest consistent brane world model: a single brane embedded in a five-dimensional Anti-de Sitter space-time. Then we include a scalar field in the bulk and discuss in detail the difference with the Anti-de Sitter case. The geometry of the bulk space-time is also analysed in some depth. Finally, we investigate the cosmology of a system with two branes and a bulk scalar field. We comment on brane collisions and summarize some open problems of brane world cosmology.
1. Introduction

The idea of extra dimensions was proposed in the early twentieth century by Nordstrom and a few years later by Kaluza and Klein [1]. It has reemerged over the years in theories combining the principles of quantum mechanics and relativity. In particular theories based on supersymmetry, especially superstring theories, are naturally expressed in more than four dimensions [2]. Four dimensional physics is retrieved by Kaluza–Klein reduction, i.e. compactifying on a manifold of small size, typically much smaller than the size of an atomic nucleus.

Recent developments in string theory and its extension M–theory have suggested another approach to compactify extra spatial dimensions. According to these developments, the standard model particles are confined on a hypersurface (called brane) embedded in a higher dimensional space (called bulk). Only gravity and other exotic matter such as the dilaton can propagate in the bulk. Our universe may be such a brane–like object. This idea was originally motivated phenomenologically (see [3]–[7]) and later revived in string theory. Within the brane world scenario, constraints on the size of extra dimensions become weaker, because the standard model particles propagate only in three spatial dimensions. Newton’s law of gravity, however, is sensitive to the presence of extra–dimensions. Gravity is being tested only on scales larger than a tenth of a millimeter and possible deviations below that scale can be envisaged.

From the string theory point of view, brane worlds of the kind discussed in this review spring from a model suggested by Horava and Witten [8]. The strong coupling limit of the $E_8 \times E_8$ heterotic string theory at low energy is described by eleven dimensional supergravity with the eleventh dimension compactified on an orbifold with $Z_2$ symmetry, i.e. an interval. The two boundaries of spacetime (i.e. the orbifold fixed points) are 10–dimensional planes, on which gauge theories (with the $E_8$ gauge groups) are confined. Later Witten argued that 6 of the 11 dimensions can be consistently compactified on a Calabi–Yau threefold and that the size of the Calabi-Yau manifold can be substantially smaller than the space between the two boundary branes [9]. Thus, in that limit space–time looks five–dimensional with four dimensional boundary branes [10]. This provides the underlying picture for many brane world models proposed so far.

Another important ingredient was put forward by Arkani-Hamed, Dimopoulos and Dvali (ADD), [11] and [12], following an earlier idea by Antoniadis [13], who suggested that by confining the standard model particle on a brane the extra dimensions can be larger than previously anticipated. They considered a flat bulk geometry in $(4 + d)$–dimensions, in which $d$ dimensions are compact with radius $R$ (toroidal topology). The four–dimensional Planck mass $M_P$ and the $(4 + d)$–dimensional Planck mass $M_{\text{fund}}$, the gravitational scale of the extra dimensional theory, are related by

$$M_P^2 = M_{\text{fund}}^{2+d} R^d. \quad (1)$$

Gravity deviates from Newton’s law only on scales smaller than $R$. Since gravity is tested only down to sizes of around a millimeter, $R$ could be as large as a fraction of a
millimeter.

ADD assumed that the bulk geometry is flat. Considerable progress was made by Randall and Sundrum, who considered non-flat, i.e. warped bulk geometries [14], [15]. In their models, the bulk spacetime is a slice of Anti–de Sitter spacetime, i.e. a space-time with a negative cosmological constant. Their discovery was that, due to the curvature of the bulk space time, Newton’s law of gravity can be obtained on the brane of positive tension embedded in an infinite extra–dimension. Small corrections to Newton’s law are generated and constrain the possible scales in the model to be smaller than a millimetre.

They also proposed a two–brane model in which the hierarchy problem, i.e. the large discrepancy between the Planck scale at $10^{19}$ GeV and the electroweak scale at 100 GeV, can be addressed. The large hierarchy is due to the highly curved AdS background which implies a large gravitational red-shift between energy scale on the two branes. In this scenario, the standard model particles are confined on a brane with negative tension sitting at $y = r_c$, whereas a positive tension brane is located at $y = 0$. The large hierarchy is generated by the appropriate inter–brane distance, i.e. the radion. It can be shown that the Planck mass $M_{Pl}$ measured on the negative tension brane is given by

$$M_{Pl}^2 \approx e^{2kr_c} M_5^3 / k,$$

where $M_5$ is the five–dimensional Planck mass and $\Lambda_5$ the (negative) cosmological constant in the bulk. Thus, we see that, if $M_5$ is not very far from the electroweak scale $M_W \approx$TeV, we need $kr_c \approx 50$, in order to generate a large Planck mass on our brane. Hence, by tuning the radius $r_c$ of the extra dimension to a reasonable value, one can obtain a very large hierarchy between the weak and the Planck scale. Of course, a complete realization of this mechanism requires an explanation for such a value of the radion. In other words, the radion needs to be stabilized at a certain value. The stabilization mechanism is not thoroughly understood, though models with a bulk scalar field have been proposed and have the required properties [16].

Another puzzle which might be addressed with brane models is the cosmological constant problem. One may invoke an extra dimensional origin for the apparent (almost) vanishing of the cosmological constant. The self-tuning idea [17] advocates that the energy density on our brane does not lead to a large curvature of our universe. On the contrary, the extra dimension becomes highly curved, preserving a flat Minkowski brane with apparent vanishing cosmological constant. Unfortunately, the simplest realization of this mechanism with a bulk scalar field fails due to the presence of a naked singularity in the bulk. This singularity can be shielded by a second brane whose tension has to be fine-tuned with the original brane tension [18]. In a sense, the fine tuning problem of the cosmological constant reappears through the extra dimensional back-door.

Finally, we will later discuss in some detail another spectacular consequence of brane cosmology, namely the possible modification to the Friedmann equation at very high energy [19]. This effect was first recognised in [20] in the context of inflatonary
solutions. As we will see, Friedmann’s equation has, for the Randall–Sundrum model, the form (21 and 22)

\[ H^2 = \frac{\kappa_5^4}{36} \rho^2 + \frac{8\pi G_N}{3} \rho + \Lambda, \]  

relating the expansion rate of the brane \( H \) to the (brane) matter density \( \rho \) and the (effective) cosmological constant \( \Lambda \). The cosmological constant can be tuned to zero by an appropriate choice of the brane tension and bulk cosmological constant, as in the Randall-Sundrum case. Notice that at high energies, for which

\[ \rho \gg \frac{96\pi G_N}{\kappa_5^4}, \]  

where \( \kappa_5^2 \) is the five dimensional gravitational constant, the Hubble rate becomes

\[ H \propto \rho, \]  

while in ordinary cosmology \( H \propto \sqrt{\rho} \). The latter case is retrieved at low energy, i.e.

\[ \rho \ll \frac{96\pi G_N}{\kappa_5^4}, \]  

Of course modifications to the Hubble rate can only be significant before nucleosynthesis. They may have drastic consequences on early universe phenomena such as inflation.

In this article we will review these and other aspects of the brane world idea in the context of cosmology. In order to give a pedagogical introduction to the subject, we will follow a phenomenological approach and start with the simplest model, i.e. the Randall-Sundrum model, with a brane embedded in a five-dimensional vacuum bulk spacetime (section 2). Later we will include a bulk scalar field (section 3). In section 4 we will discuss the geometry of the bulk–spacetime in some detail. In the last part we will discuss more realistic models with two branes and bulk scalar fields (section 5). In section 6 we will discuss brane collisions. Open questions are summarized in section 7.

We would like to mention other review articles on brane worlds and cosmology, taking different approaches from the one taken here [23]-[28]. We will mostly be concerned with the case, in which the bulk space–time is five–dimensional.

2. The Randall–Sundrum Brane World

Originally, Randall and Sundrum suggested a two–brane scenario in five dimensions with a highly curved bulk geometry as an explanation for the large hierarchy between the Planck scale and the electroweak energy–scale [14]. In this scenario, the standard model particles live on a brane with (constant) negative tension, whereas the bulk is a slice of Anti–de Sitter (AdS) spacetime , i.e. a space-time with a negative cosmological constant. In the bulk there is another brane with positive tension. This is the so–called Randall–Sundrum I (RSI) model. Analysing the solution of Einstein’s equation on the positive tension brane and sending the negative tension brane to infinity, an observer confined to the positive tension brane recovers Newton’s law if the curvature scale of the AdS is smaller than a millimeter [15]. The higher–dimensional space is
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non-compact, which must be contrasted with the Kaluza–Klein mechanism, where all extra-dimensional degrees of freedom are compact. This one-brane model, on which we will concentrate in this section, is the so-called Randall–Sundrum II (RSII) model. It was shown, there is a continuum of Kaluza–Klein modes for the gravitational field, contrasting with the discrete spectrum if the extra dimension is periodic. This leads to a correction to the force between two static masses on the brane. To be specific, it was shown that the potential energy between two point masses confined on the brane is given by

\[ V(r) = \frac{G_N m_1 m_2}{r} \left( 1 + \frac{l^2}{r^2} + O(r^{-3}) \right) . \]  

(7)

In this equation, \( l \) is related to the five-dimensional bulk cosmological constant \( \Lambda_5 \) by \( l^2 = -6/(\kappa_5^2 \Lambda_5) \) and is therefore a measure of the curvature scale of the bulk spacetime. Gravitational experiments show no deviation from Newton's law of gravity on length scales larger than a millimeter \([29]\). Thus, \( l \) has to be smaller than that length scale.

The static solution of the Randall and Sundrum model can be obtained as follows: The total action consists of the Einstein-Hilbert action and the brane action, which in the Randall–Sundrum model have the form

\[ S_{EH} = - \int dx^5 \sqrt{-g^{(5)}} \left( \frac{R}{2\kappa_5^2} + \Lambda_5 \right) , \]  

(8)

\[ S_{brane} = \int dx^4 \sqrt{-g^{(4)}} (-\sigma) . \]  

(9)

The parameter \( \Lambda_5 \) (the bulk cosmological constant) and \( \sigma \) (the brane tension) are constant and \( \kappa_5 \) is the five-dimensional gravitational coupling constant. The brane is located at \( y = 0 \) and we assume a \( Z_2 \) symmetry, i.e. we identify \( y \) with \( -y \). The ansatz for the metric is

\[ ds^2 = e^{-2K(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 . \]  

(10)

Einstein's equations, derived from the action above, give two independent equations:

\[ 6K'^2 = -\kappa_5^2 \Lambda_5 \]  

\[ 3K'' = \kappa_5^2 \sigma \delta(y) . \]

The first equation can be easily solved:

\[ K = K(y) = \sqrt{-\frac{\kappa_5^2}{6} \Lambda_5} \ y = ky , \]  

(11)

which tells us that \( \Lambda_5 \) must be negative. If we integrate the second equation from \(-\epsilon\) to \(+\epsilon\), take the limit \( \epsilon \to 0 \) and make use of the \( Z_2 \)-symmetry, we get

\[ 6K'|_0 = \kappa_5^2 \sigma \]  

(12)

Together with eq. (11) this tells us that

\[ \Lambda_5 = -\frac{\kappa_5^2}{6} \sigma^2 \]  

(13)

Thus, there must be a fine-tuning between the brane tension and the bulk cosmological constant for static solutions to exist. In this section we will discuss the cosmology of this model in detail.
2.1. Einstein’s equations on the brane

There are two ways of deriving the cosmological equations and we will describe both of them below. The first one is rather simple and makes use of the bulk equations only. The second method uses the geometrical relationship between four-dimensional and five-dimensional quantities. We begin with the simpler method.

2.1.1. Friedmann’s equation from five-dimensional Einstein equations

In the following subsection we will set $\kappa_5 \equiv 1$. We write the bulk metric as follows:

$$ds^2 = a^2b^2(dt^2 - dy^2) - a^2\delta_{ij}dx^i dx^j.$$  \hspace{1cm} (14)

This metric is consistent with homogeneity and isotropy on the brane located at $y = 0$. The functions $a$ and $b$ are functions of $t$ and $y$ only. Furthermore, we have assumed flat spatial sections, it is straightforward to include a spatial curvature. For this metric, Einstein equations in the bulk read:

$$a^2b^2G_{00}^5 \equiv 3 \left[ \frac{\ddot{a}}{a} + \frac{\dot{a}b}{ab} - \frac{a''}{a} + \frac{a'b'}{ab} + kb^2 \right] = a^2b^2 \left[ \rho_B + \rho\bar{\delta}(y - y_b) \right]$$  \hspace{1cm} (15)

$$a^2b^2G_{55}^0 \equiv 3 \left[ \frac{\ddot{b}}{b} - \frac{\dot{a}b}{ab} - 2\frac{a''}{a^2} - \frac{a'b'}{ab} + kb^2 \right] = -a^2b^2T_{55}^0$$  \hspace{1cm} (16)

$$a^2b^2G_{ij}^0 \equiv 3 \left[ \frac{\ddot{a}}{a} + 2\frac{\dot{a}a'}{a^2} + \frac{\dot{b}'b}{b} + \frac{a'b}{ab} \right] = -a^2b^2T_{ij}^5$$  \hspace{1cm} (17)

$$a^2b^2G_{ij}^5 \equiv \left[ 3\frac{\ddot{a}}{a} + \frac{\dot{b}}{b} - 3\frac{a''}{a} - \frac{b''}{b} + \frac{b'}{b} + kb^2 \right] \delta_{ij}$$

where the bulk energy–momentum tensor $T_{ab}^5$ has been kept general here. For the Randall–Sundrum model we will now take $\rho_B = -p_B = \Lambda_5$ and $T_{55}^0 = 0$. Later, in the next section, we will use these equations to derive Friedmann’s equation with a bulk scalar field. In the equations above, a dot represents the derivative with respect to $t$ and a prime a derivative with respect to $y$.

Let us integrate the 00-component over $y$ from $-\epsilon$ to $\epsilon$ and use the fact that $a(y) = a(-y)$, $b(y) = b(-y)$, $a'(y) = -a(-y)$ and $b'(y) = -b(-y)$ (i.e. $Z_2$-symmetry). Then, taking the limit $\epsilon \to 0$ we get

$$\frac{a'}{a} \bigg|_{y=0} = \frac{1}{6}ab\rho.$$  \hspace{1cm} (19)

Integrating the $ij$-component in the same way and using the last equation gives

$$\frac{b'}{b} \bigg|_{y=0} = -\frac{1}{2}ab(\rho + p).$$  \hspace{1cm} (20)

These two conditions are called the junction conditions. The other components of the Einstein equations should be compatible with these conditions. It is not difficult to show that the restriction of the 05 component to $y = 0$ leads to

$$\dot{\rho} + 3\frac{a}{a}(\rho + p) = 0.$$  \hspace{1cm} (21)
where we have made use of the junction conditions (19) and (20). This is nothing but matter conservation on the brane.

Proceeding in the same way with the 55–component gives

\[
\ddot{a} - \frac{\dot{a}\dot{b}}{ab} + k b^2 = -\frac{a^2 b^2}{3} \left[ \frac{1}{12} \rho (\rho + 3p) + q_B \right].
\]

(22)

Changing to cosmic time \(d\tau = abdt\), writing \(a = \exp(\alpha(t))\) and using the energy conservation gives (30), (31)

\[
\frac{d}{d\alpha} \left( H^2 e^{4\alpha} \right) = \frac{2}{3} \Lambda_5 e^{4\alpha} + \frac{d}{d\alpha} \left( e^{4\alpha} \rho^2 \frac{1}{36} \right).
\]

(23)

In this equation \(aH = da/d\tau\). This equation can easily be integrated to give

\[
H^2 = \frac{\rho^2}{36} + \frac{\Lambda_5}{6} + \frac{\mu}{a^4}.
\]

(24)

The final step is to split the total energy–density and pressure into parts coming from matter and brane tension, i.e. to write \(\rho = \rho_M + \sigma\) and \(p = p_M - \sigma\). Then we find Friedmann’s equation

\[
H^2 = \frac{8\pi G}{3} \rho_M \left[ 1 + \frac{\rho_M}{2\sigma} \right] + \frac{\Lambda_4}{3} + \frac{\mu}{a^4},
\]

(25)

where we have made the identification

\[
\frac{8\pi G}{3} = \frac{\sigma}{18},
\]

(26)

\[
\frac{\Lambda_4}{3} = \frac{\sigma^2}{36} + \frac{\Lambda_5}{6}.
\]

(27)

Comparing the last equation with the fine–tuning relation (13) in the static Randall–Sundrum solution, we see that \(\Lambda_4 = 0\) in this case. If there is a small mismatch between the brane tension and the five–dimensional cosmological constant, then an effective four–dimensional cosmological constant is generated. Another important point is that the four–dimensional Newton constant is directly related to the brane tension. The constant \(\mu\) appears in the derivation above as an integration constant. The term including \(\mu\) is called the dark radiation term (see e.g. [32]–[34]). The parameter \(\mu\) can be obtained from a full analysis of the bulk equations [35]–[37] (we will discuss this in section 4).

An extended version of Birkhoff’s theorem tells us that if the bulk spacetime is AdS, this constant is zero [38]. If the bulk is AdS–Schwarzschild instead, \(\mu\) is non–zero but a measure of the mass of the bulk black hole. In the following we will assume that \(\mu = 0\) and \(\Lambda_4 = 0\).

The most important change in Friedmann’s equation compared to the usual four–dimensional form is the appearance of a term proportional to \(\rho^2\). It tells us that if the matter energy density is much larger than the brane tension, i.e. \(\rho_M \gg \sigma\), the expansion rate \(H\) is proportional \(\rho_M\), instead of \(\sqrt{\rho_M}\). The expansion rate is, in this regime, larger in this brane world scenario. Only in the limit where the brane tension is much larger than the matter energy density, the usual behaviour \(H \propto \sqrt{\rho_M}\) is recovered. This is the most important change in brane world scenarios. It is quite generic and not restricted.
to the Randall–Sundrum brane world model. From Friedmann’s equation and from the energy-conservation equation we can derive Raychaudhuri’s equation:

$$\frac{dH}{d\tau} = -4\pi G(\rho_M + p_M) \left[ 1 + \frac{\rho_M}{\sigma} \right].$$  \hspace{1cm} (28)$$

We will use these equations later in order to investigate inflation driven by a scalar field confined on the brane.

Notice that at the time of nucleosynthesis the brane world corrections in Friedmann’s equation must be negligible, otherwise the expansion rate is modified and the results for the abundances of the light elements are unacceptably changed. This implies that $\sigma \geq (1\text{MeV})^4$. Note, however, that a much stronger constraint arises from current tests for deviation from Newton’s law [39] (assuming the Randall–Sundrum fine-tuning relation (13)): $\kappa_5^2 > 10^9 \text{TeV}$ and $\sigma \geq (100\text{GeV})^4$. Similarly, cosmology constrains the amount of dark radiation. It has been shown that the energy density in dark radiation can at most be 10 percent of the energy density in photons [33].

2.1.2. Another derivation of Einstein’s equation

There is a more powerful way of deriving Einstein’s equation on the brane [40]. Consider an arbitrary (3+1) dimensional hypersurface $M$ with unit normal vector $n_a$ embedded in a 5 dimensional spacetime. The induced metric and the extrinsic curvature of the hypersurface are defined as

$$h^a_b = \delta^a_b - n^a n_b, \quad K_{ab} = h^c_a h^d_b \nabla_c n_d. \hspace{1cm} (29)$$

For the derivation we need three equations, two of them relate four-dimensional quantities constructed from $h_{ab}$ to full five-dimensional quantities constructed from $g_{ab}$. We just state these equations here and refer to [41] for the derivation of these equations. The first equation is the Gauss equation, which reads

$$R^{(4)}_{abcd} = h^a_j h^b_k h^c_l h^d_m R_{ijklm} - 2K_{a[c} K_{d]b}. \hspace{1cm} (31)$$

This equation relates the four-dimensional curvature tensor $R^{(4)}_{abcd}$, constructed from $h_{ab}$, to the five-dimensional one and $K_{ab}$. The next equation is the Codazzi equation, which relates $K_{ab}$, $n_a$ and the five-dimensional Ricci tensor:

$$\nabla^b_a K^b_a - \nabla^a_b K^b_a = n^c h^b_a R_{bc}. \hspace{1cm} (32)$$

One decomposes the five-dimensional curvature tensor $R_{abcd}$ into the Weyl–tensor $C_{abcd}$ and the Ricci tensor:

$$R_{abcd} = \frac{2}{3} \left( g_{a[c} R_{d]b} - g_{b[c} R_{d]a} \right) - \frac{1}{6} R g_{a[c} g_{b]d} + C_{abcd}. \hspace{1cm} (33)$$

If one substitutes the last equation into the Gauss equation and constructs the four-dimensional Einstein tensor, one obtains

$$G^{(4)}_{ab} = \frac{2}{3} \left( G_{cd} h^c_a h^d_b + \left( G_{cd} n^c n^d - \frac{1}{4} G \right) h_{ab} \right) + K K_{ab} - K_a^c K_{bc} - \frac{1}{2} \left( K^2 - K^{cd} K_{cd} \right) h_{ab} - E_{ab}, \hspace{1cm} (34)$$
where
\[ E_{ab} = C_{acbd} n^c n^d. \]  
(35)

We would like to emphasize that this equation holds for any hypersurface. If one considers a hypersurface with energy momentum tensor \( T_{ab} \), then there exists a relationship between \( K_{ab} \) and \( T_{ab} \) (\( T \) is the trace of \( T_{ab} \)) [42]:
\[ [K_{ab}] = -\kappa_5^2 \left( T_{ab} - \frac{1}{3} h_{ab} T \right), \]  
(36)

where \(...) denotes the jump:
\[ [f](y) = \lim_{\epsilon \to 0} (f(y + \epsilon) - f(y - \epsilon)). \]  
(37)

These equations are called junction conditions and are equivalent in the cosmological context to the junction conditions (19) and (20). Splitting \( T_{ab} = \tau_{ab} - \sigma h_{ab} \) and inserting the junction condition into equation (34), we obtain Einstein’s equation on the brane:
\[ G^{(4)}_{ab} = 8\pi G \tau_{ab} - \Lambda_4 h_{ab} + \kappa_5^4 \pi_{ab} - E_{ab}. \]  
(38)

The tensor \( \pi_{ab} \) is defined as
\[ \pi_{ab} = \frac{1}{12} \tau \tau_{ab} - \frac{1}{4} \tau_{ac} \tau_{cb} + \frac{1}{8} h_{ab} \tau_{cd} \tau^{cd} - \frac{1}{24} \tau^2 h_{ab}, \]  
(39)

whereas
\[ 8\pi G = \frac{\kappa_5^4}{6} \sigma \]  
(40)
\[ \Lambda_4 = \frac{\kappa_5^2}{2} \left( \Lambda_5 + \frac{\kappa_5^2}{6} \sigma^2 \right). \]  
(41)

Note that in the Randall–Sundrum case we have \( \Lambda_4 = 0 \) due to the fine–tuning between the brane tension and the bulk cosmological constant. Moreover \( E_{ab} = 0 \) as the Weyl–tensor vanishes for an AdS spacetime. In general, the energy conservation and the Bianchi identities imply that
\[ \kappa_5^4 \nabla^a \pi_{ab} = \nabla^a E_{ab} \]  
(42)
on the brane.

Clearly, this method is powerful, as it does not assume homogeneity and isotropy nor does it assume the bulk to be AdS. In the case of an AdS bulk and a Friedmann–Robertson walker brane, the previous equations reduce to the Friedmann equation and Raychaudhuri equation derived earlier. However, the set of equations on the brane are not closed in general [43], as we will see in the next section.

2.2. Slow–roll inflation on the brane

We have seen that the Friedmann equation on a brane is drastically modified at high energy where the \( \rho^2 \) terms dominate. As a result the early universe cosmology on branes tends to be different from standard 4d cosmology. In that vein it seems natural to look for brane effects on early universe phenomena such as inflation (see in particular [44] and [45]) and on phase–transitions [46].
The energy density and the pressure of a scalar field are given by
\[ \rho_\phi = \frac{1}{2} \phi_{,\mu} \phi^{,\mu} + V(\phi), \quad (43) \]
\[ p_\phi = \frac{1}{2} \phi_{,\mu} \phi^{,\mu} - V(\phi), \quad (44) \]
where \( V(\phi) \) is the potential energy of the scalar field. The full evolution of the scalar field is described by the (modified) Friedmann equation, the Klein–Gordon equation and the Raychaudhuri equation.

We will assume that the field is in a slow–roll regime, the evolution of the fields is governed by (from now on a dot stands for a derivative with respect to cosmic time)
\[ 3H \dot{\phi} \approx -\frac{\partial V}{\partial \phi} \quad (45) \]
\[ H^2 \approx \frac{8\pi G}{3} V(\phi) \left( 1 + \frac{V(\phi)}{2\sigma} \right). \quad (46) \]
It is not difficult to show that these equations imply that the slow–roll parameter are given by
\[ \epsilon \equiv -\frac{\dot{H}}{H^2} = \frac{1}{16\pi G} \left( \frac{V'}{V} \right)^2 \left[ \frac{4\sigma(\sigma + V)}{(2\sigma + V)^2} \right] \quad (47) \]
\[ \eta \equiv \frac{V''}{3H^2} = \frac{1}{8\pi G} \left( \frac{V''}{V} \right) \left[ \frac{2\sigma}{2\sigma + V} \right]. \quad (48) \]
The modifications to General Relativity are contained in the square brackets of these expressions. They imply that for a given potential and given initial conditions for the scalar field the slow–roll parameters are suppressed compared to the predictions made in General Relativity. In other words, brane world effects ease slow–roll inflation [44]. In the limit \( \sigma \ll V \) the parameter are heavily suppressed. It implies that steeper potentials can be used to drive slow–roll inflation [45]. Let us discuss the implications for cosmological perturbations.

According to Einstein’s equation (38), perturbations in the metric are sourced not only by matter perturbations but also by perturbations of the bulk geometry, encoded in the perturbation of \( E_{ab} \). This term can be seen as an external source for perturbations, absent in General Relativity. If one regards \( E_{ab} \) as an energy–momentum tensor of an additional fluid (called the Weyl-fluid), its evolution is connected to the energy density of matter on the brane, as one can see from eq. (42). If one neglects the anisotropic stress of the Weyl-fluid, then at low energy and superhorizon scales, it decays as radiation, i.e. \( \delta E_{ab} \propto a^{-4} \). However, the bulk gravitational field exerts an anisotropic stress onto the brane, whose time-evolution cannot be obtained by considering the projected equations on the brane alone [43]. Rather, the full five–dimensional equations have to be solved, together with the junction conditions. The full evolution of \( E_{ab} \) in the different cosmological eras is currently not understood. However, as we will discuss below, partial results have been obtained for the case of a de Sitter brane, which suggest that \( E_{ab} \) does not change the spectrum of scalar perturbations. It should be noted however, that the
issue is not settled and that it is also not clear if the subsequent cosmological evolution during radiation and matter era leaves an imprint of the bulk gravitational field in the anisotropies of the microwave background radiation \[47\]. With this in mind, we will, for scalar perturbations, neglect the gravitational backreaction described by the projected Weyl tensor.

Considering scalar perturbations for the moment, the perturbed line element on the brane has the form

\[
ds^2 = -(1 + 2A)dt^2 + 2\partial_i Bdt dx^i + ((1 - 2\psi)\delta_{ij} + D_{ij}E)dx^idx^j,
\]

where the functions \(A, B, E\) and \(\psi\) depend on \(t\) and \(x^i\).

An elegant way of discussing scalar perturbations is to make use of of the gauge invariant quantity \[48\]

\[
\zeta = \psi + H\frac{\delta\rho}{\rho}.
\]

In General Relativity, the evolution equation for \(\zeta\) can be obtained from the energy–conservation equation \[49\]. It reads, on large scales,

\[
\dot{\zeta} = -\frac{H}{\rho + p}\delta p_{\text{nad}},
\]

where \(\delta p_{\text{nad}} = \delta p_{\text{tot}} - c_s^2\delta\rho\) is the non-adiabatic pressure perturbation. The energy conservation equation, however, holds for the Randall–Sundrum model as well. Therefore, eq. (51) is still valid for the brane world model we consider. For inflation driven by a single scalar field \(\delta p_{\text{nad}}\) vanishes and therefore \(\zeta\) is constant on superhorizon scales during inflation. Its amplitude is given in terms of the fluctuations in the scalar field on spatially flat hypersurfaces:

\[
\zeta = \frac{H\delta\phi}{\dot{\phi}}.
\]

The quantum fluctuation in the (slow–rolling) scalar field obey \(\langle (\delta\phi)^2 \rangle \approx (H/2\pi)^2\), as the Klein–Gordon equation is not modified in the brane world model we consider. The amplitude of scalar perturbations is \[50\] \(A_S^2 = 4\langle \zeta^2 \rangle / 25\). Using the slow–roll equations and eq. (52) one obtains \[44\]

\[
A_S^2 \approx \left( \frac{512\pi}{75M_{\text{pl}}^6} \right) \frac{V^3}{V'2} \left[ \frac{2\sigma + V}{2\sigma} \right]_k^3 |_{k=aH}
\]

Again, the corrections are contained in the terms in the square brackets. For a given potential the amplitude of scalar perturbations is enhanced compared to the prediction of General Relativity.

The arguments presented so far suggest that, at least for scalar perturbations, perturbations in the bulk spacetime are not important during inflation. This, however, might not be true for tensor perturbations, as gravitational waves can propagate into the bulk. For tensor perturbations, a wave equations for a single variable can be derived \[51\]. The wave equation can be separated into a four–dimensional and a five–dimensional part, so that the solution has the form \(h_{ij} = A(y)h(x^\mu)e_{ij}\), where \(e_{ij}\) is a
(constant) polarization tensor. One finds that the amplitude for the zero mode of tensor perturbation is given by \[51\]

\[
A_T^2 = \frac{4}{25\pi M_{pl}^4} H^2 F^2 (H/\mu)\big|_{k=aH}, \tag{54}
\]

with

\[
F(x) = \left[\sqrt{1 + x^2} - x^2 \sinh^{-1} \left(\frac{1}{x}\right)\right]^{-1/2}, \tag{55}
\]

where we have defined

\[
\frac{H}{\mu} = \left(\frac{3}{4\pi \sigma}\right)^{1/2} H M_{pl}. \tag{56}
\]

It can be shown that modes with \(m > 3H/2\) are generated but they decay during inflation. Thus, one expects in this scenario only the massless modes to survive until the end of inflation \[51\], \[52\].

From eqns. \[51\] and \[53\] one sees that the amplitudes of scalar and tensor perturbations are enhanced at high energies. However, scalar perturbations are more enhanced than tensors. Thus, the relative contribution of tensor perturbations will be suppressed, if inflation is driven at high energies.

Finally, we would like to mention that there are also differences between General Relativity and the brane world model we consider for the prediction of two–field brane inflation. Usually correlations are separated in adiabatic and isocurvature modes for two–field inflation \[53\]. In the Randall–Sundrum model, this correlation is suppressed if inflation is driven at high energies \[54\]. This implies that isocurvature and adiabatic perturbations are uncorrelated, if inflation is driven at energies much larger than the brane tension.

Coming back to cosmological perturbations, the biggest problem is that the evaluation of the projected Weyl tensor is only possible for the background cosmology. As soon as one tries to analyse the brane cosmological perturbations, one faces the possibility that the \(E_{0i}\) terms might not vanish. In particular this means that the equation for the density contrast \(\delta = \delta \rho/\rho\), which is given by \((w_m = p/\rho, \ k\) is the wavenumber)

\[
\ddot{\delta} + (2 - 3\omega_m)H \dot{\delta} - 6\omega_m (H^2 + \dot{H})\delta = (1 + \omega_m)\delta R_{00} - \omega_m \frac{k^2}{a^2}\delta, \tag{57}
\]

cannot be solved as \(\delta R_{00}\) involves \(\delta E_{00}\) and can therefore not be deduced solely from the brane dynamics \[43\].

2.3. Final Remarks on the Randall–Sundrum Scenario

The Randall–Sundrum model discussed in this section is the simplest brane world model. We have not discussed other important conclusions one can draw from the modifications of Friedmann’s equation, such as the evolution of primordial black holes \[55\], its connection to the AdS/CFT correspondence (see e.g. \[56\]–\[61\]) and inflation driven by the trace anomaly of the conformal field theory living on the brane (see e.g.
These developments are important in many respects, because they give not only insights about the early universe but gravity itself. They will not be reviewed here.

### 3. Including a Bulk Scalar Field

In this section we are going to generalize the previous results obtained with an empty bulk. To be specific, we will consider the inclusion of a scalar field in the bulk. As we will see, one can extend the projective approach wherein one focuses on the dynamics of the brane, i.e. one studies the projected Einstein and the Klein-Gordon equation. As in the Randall-Sundrum setting, the dynamics do not closed, as bulk effects do not decouple. We will see that there are now two objects representing the bulk back-reaction: the projected Weyl tensor $E_{\mu\nu}$ and the loss parameter $\Delta \Phi$. In the case of homogeneous and isotropic cosmology on the brane, the projected Weyl tensor is determined entirely up to a dark radiation term. Unfortunately, no information on the loss parameter is available. This prevents a rigorous treatment of brane cosmology in the projective approach.

Another route amounts to studying the motion of a brane in a bulk space-time. This approach is successful in the Randall-Sundrum case thanks to Birkhoff’s theorem which dictates a unique form for the metric in the bulk. In the case of a bulk scalar field, no such theorem is available. One has to resort to various ansätze for particular classes of bulk and brane scalar potentials (see e.g. (68)-(75)). We will come back to this in section 4.

#### 3.1. BPS Backgrounds

##### 3.1.1. Properties of BPS Backgrounds

As the physics of branes with bulk scalar fields is pretty complicated, we will start with a particular example where both the bulk and the brane dynamics are fully under control (77) (see also (78) and (79)). We specify the bulk Lagrangian as

$$S = \frac{1}{2\kappa_5^2} \int d^5 x \sqrt{-g_5} \left( R - \frac{3}{4} \left( (\partial\phi)^2 + V(\phi) \right) \right)$$

where $V(\phi)$ is the bulk potential. The boundary action depends on a brane potential $U_B(\phi)$

$$S_B = -\frac{3}{2\kappa_5^2} \int d^4 x \sqrt{-g_4} U_B(\phi_0)$$

where $U_B(\phi_0)$ is evaluated on the brane. The BPS backgrounds arise as particular case of this general setting with a particular relationship between the bulk and brane potentials. This relation appears in the study of $N = 2$ supergravity with vector multiplets in the bulk. The bulk potential is given by

$$V = \left( \frac{\partial W}{\partial \phi} \right)^2 - W^2$$
where \( W(\phi) \) is the superpotential. The brane potential is simply given by the superpotential

\[
U_B = W
\]  

(61)

We would like to mention, that the last two relations have been also used in order to generate bulk solutions without necessarily imposing supersymmetry \[70,\,76\]. Notice that the Randall-Sundrum case can be retrieved by putting \( W = \text{cst} \). Supergravity puts further constraints on the superpotential which turns out to be of the exponential type \[77\]

\[
W = 4ke^{\alpha \phi}
\]  

(62)

with \( \alpha = -1/\sqrt{12}, 1/\sqrt{3} \). In the following we will often choose this exponential potential with an arbitrary \( \alpha \) as an example. The actual value of \( \alpha \) does not play any role and will be considered generic.

The bulk equations of motion comprise the Einstein equations and the Klein-Gordon equation. In the BPS case and using the following ansatz for the metric

\[
ds^2 = a(y)^2 \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,
\]  

(63)

these second order differential equations reduce to a system of two first order differential equations

\[
\frac{a^'}{a} = - \frac{W}{4},
\]  

(64)

\[
\phi^' = \frac{\partial W}{\partial \phi}.
\]

Notice that when \( W = \text{cst} \) one easily retrieves the exponential profile of the Randall-Sundrum model.

An interesting property of BPS systems can be deduced from the study of the boundary conditions. The Israel junction conditions reduce to

\[
\frac{a^'}{a}|_B = - \frac{W}{4}|_B
\]  

(65)

and for the scalar field

\[
\phi^'|_B = \frac{\partial W}{\partial \phi}|_B
\]  

(66)

This is the main property of BPS systems: the boundary conditions coincide with the bulk equations, i.e. as soon as the bulk equations are solved one can locate the BPS branes anywhere in this background, there is no obstruction due to the boundary conditions. In particular two-brane systems with two boundary BPS branes admit moduli corresponding to massless deformations of the background. They are identified with the positions of the branes in the BPS background. We will come back to this later in section 5.

Let us treat the example of the exponential superpotential. The solution for the scale factor reads

\[
a = (1 - 4k\alpha^2 x_5)^{1/4\alpha^2},
\]  

(67)
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and the scalar field is given by

\[ \phi = -\frac{1}{\alpha} \ln(1 - 4k\alpha^2 x_5). \]  

(68)

For \( \alpha \to 0 \), the bulk scalar field decouples and these expressions reduce to the Randall-Sundrum case. Notice a new feature here, namely the existence of singularities in the bulk, corresponding to

\[ a(x_5)|_{x_5^*} = 0 \]  

(69)

In order to analyse singularities it is convenient to use conformal coordinates

\[ du = \frac{dx_5}{a(x_5)}. \]  

(70)

In these coordinates light follows straight lines \( u = \pm t \). It is easy to see that the singularities fall in two categories depending on \( \alpha \). For \( \alpha^2 < 1/4 \) the singularity is at infinity \( u_\star = \infty \). This singularity is null and absorbs incoming gravitons. For \( \alpha^2 > 1/4 \) the singularity is at finite distance. It is time-like and not wave-regular, i.e. the propagation of wave packets is not uniquely defined in the vicinity of the singularity. For all these reasons these naked singularities in the bulk are a major drawback of brane models with bulk scalar fields [80]. In the two-brane case the second brane has to sit in front of the naked singularity.

3.1.2. de Sitter and anti de Sitter Branes

Let us modify slightly the BPS setting by detuning the tension of the BPS brane. This corresponds to adding or subtracting some tension compared to the BPS case

\[ U_B = TW \]  

(71)

where \( T \) is real number. Notice that this modification only affects the boundary conditions, the bulk geometry and scalar field are still solutions of the BPS equations of motion. In this sort of situation, one can show that the brane does not stay static. For the detuned case, the result is either a boosted brane or a rotated brane. We will soon generalize these results so we postpone the detailed explanation to later. Defining by \( u(x^\mu) \) the position of the brane in conformal coordinates, one obtains

\[ (\partial u)^2 = \frac{1 - T^2}{T^2}. \]  

(72)

The brane velocity vector \( \partial_\mu u \) is of constant norm. For \( T > 1 \), the brane velocity vector is time-like and the brane moves at constant speed. For \( T < 1 \) the brane velocity vector is space-like and the brane is rotated. Going back to a static brane, we see that the bulk geometry and scalar field become \( x^\mu \) dependent. In the next section we will find many more cases where branes move in a static bulk or equivalently, a static brane borders a non-static bulk.

Let us now conclude this section by studying the brane geometry when \( T > 1 \). In particular one can study the Friedmann equation for the induced bulk factor

\[ H^2 = \frac{T^2 - 1}{16} W^2, \]  

(73)
where $W$ is evaluated on the brane. Of course we obtain the fact that cosmological solutions are only valid for $T > 1$. Now in the Randall-Sundrum case $W = 4k$ leading to

$$H^2 = (T^2 - 1)k^2. \quad (74)$$

In the case $T > 1$ the brane geometry is driven by a positive cosmological constant. This is a de Sitter brane. When $T < 1$ the cosmological constant is negative, corresponding to an AdS brane. We are going to generalize these results by considering the projective approach to the brane dynamics.

3.2. Bulk Scalar Fields and the Projective Approach

3.2.1. The Friedmann Equation We will first follow the traditional coordinate dependent path. This will allow us to derive the matter conservation equation, the Klein-Gordon and the Friedmann equations on the brane. Then we will concentrate on the more geometric formulation where the role of the projected Weyl tensor will become transparent \[36, 37\]. Again, in this subsection we will put $\kappa_5 \equiv 1$.

We consider a static brane that we choose to put at the origin $x_5 = 0$. and impose $b(0, t) = 1$. This guarantees that the brane and bulk expansion rates

$$4H = \partial_\tau \sqrt{-g}|_0, \ 3H_B = \partial_\tau \sqrt{-g_B}|_0 \quad (75)$$

coincide. We have identified the brane cosmic time $d\tau = ab|_0 dt$. We will denote by prime the normal derivative $\partial_n = \frac{1}{ab|_0} \partial_{x_5}$. Moreover we now allow for some matter to be present on the brane

$$\tau_\mu^{\text{matter}} = (-\rho_m, p_m, p_m, p_m). \quad (76)$$

The bulk energy-momentum tensor reads

$$T_{ab} = \frac{3}{4} (\partial_a \phi \partial_b \phi) - \frac{3}{8} g_{ab} ((\partial \phi)^2 + V). \quad (77)$$

The total matter density and pressure on the brane are given by

$$\rho = \rho_m + \frac{3}{2} U_B, \ p = p_m - \frac{3}{2} U_B. \quad (78)$$

The boundary condition for the scalar field is unchanged by the presence of matter on the brane.

The $(05)$ Einstein equation leads to matter conservation

$$\dot{\rho}_m = -3H(\rho_m + p_m). \quad (79)$$

By restricting the $(55)$ component of the Einstein equations we obtain

$$H^2 = \frac{\rho^2}{36} - \frac{2}{3} Q - \frac{1}{9} E + \frac{\mu}{a^4} \quad (80)$$

in units of $\kappa_5^2$. The last term is the dark radiation, whose origin is similar to the Randall-Sundrum case. The quantity $Q$ and $E$ satisfy the differential equations \[31\]

$$\dot{Q} + 4HQ = HT_5^5,$$

$$\dot{E} + 4HE = -\rho T_5^5.$$
These equations can be easily integrated to give

$$H^2 = \frac{\rho_m^2}{36} + \frac{U_B \rho_m}{12} - \frac{1}{16a^4} \int d\tau \frac{d a^4}{d\tau} (\dot{\phi}^2 - 2U) - \frac{1}{12a^4} \int d\tau a^4 \rho_m \frac{dU_B}{d\tau},$$

(81)

up to a dark radiation term and we have identified

$$U = \frac{1}{2} \left( U_B^2 - \left( \frac{\partial U_B}{\partial \phi} \right)^2 + V \right).$$

(82)

This is the Friedmann equation for a brane coupled to a bulk scalar field. Notice that retarded effects springing from the whole history of the brane and scalar field dynamics are present. In the following section we will see that these retarded effects come from the projected Weyl tensor. They result from the exchange between the brane and the bulk. Notice, that Newton’s constant depends on the value of the bulk scalar field evaluated on the brane ($\phi_0 = \phi(t, y = 0)$):

$$\frac{8\pi G_N(\phi_0)}{3} = \frac{\kappa^2 U_B(\phi_0)}{12}. \tag{83}$$

On cosmological scale, time variation of the scalar field induce a time variations of Newton’s constant. This is highly constrained experimentally \[88, 89\], leading to tight restrictions on the time dependence of the scalar field.

To get a feeling of the physics involved in the Friedmann equation, it is convenient to assume that the scalar field is evolving slowly on the scale of the variation of the scale factor. Neglecting the evolution of Newton’s constant, the Friedmann equation reduces to

$$H^2 = \frac{8\pi G_N(\phi)}{3} \rho_m + \frac{U}{8} - \frac{\dot{\phi}^2}{16} \tag{84}$$

Several comments are in order. First of all we have neglected the contribution due to the $\rho_m^2$ term as we are considering energy scales below the brane tension. Then the main effect of the scalar field dynamics is to involve the potential energy $U$ and the kinetic energy $\dot{\phi}^2$. Although the potential energy appears with a positive sign we find that the kinetic energy has a negative sign. For an observer on the brane this looks like a violation of unitarity. We will reanalyse this issue in section 5, when considering the low energy effective action in four dimensions and we will see that there is no unitarity problem in this theory. The minus sign for the kinetic energy is due to the fact that one does not work in the Einstein frame where Newton’s constant does not vary, a similar minus sign appears also in the effective four–dimensional theory when working in the brane frame.

The time dependence of the scalar field is determined by the Klein-Gordon equation. The dynamics is completely specified by

$$\ddot{\phi} + 4H \dot{\phi} + \frac{1}{2} \left( \frac{3}{3} - \omega_m \right) \rho_m \frac{\partial U_B}{\partial \phi} = - \frac{\partial U}{\partial \phi} + \Delta \Phi_2, \tag{85}$$

where $p_m = \omega_m \rho_m$. We have identified

$$\Delta \Phi_2 = \phi''|_0 - \frac{\partial U_B}{\partial \phi} \frac{\partial^2 U_B}{\partial \phi^2} |_0. \tag{86}$$
This cannot be set to zero and requires the knowledge of the scalar field in the vicinity of the brane. When we discuss cosmological solutions below, we will assume that this term is negligible.

The evolution of the scalar field is driven by two effects. First of all, the scalar field couples to the trace of the energy momentum tensor via the gradient of $U_B$. Secondly, the field is driven by the gradient of the potential $U$, which might not necessarily vanish.

3.2.2. The Friedmann equation vs the projected Weyl tensor

We are now coming back to the origin of the non-trivial Friedmann equation. Using the Gauss-Codazzi equation one can obtain the Einstein equation on the brane \[ 66, 67 \]

$$G_{ab} = -\frac{3}{8} U h_{ab} + \frac{U_B}{4} \tau_{ab} + \partial_a \phi \partial_b \phi - \frac{5}{16} (\partial \phi)^2 h_{ab} - E_{ab}. \tag{87}$$

Now the projected Weyl tensor can be determined in the homogeneous and isotropic cosmology case. Indeed only the $E_{00}$ component is independent. Using the Bianchi identity $\bar{D}^a G_{ab} = 0$ where $\bar{D}_a$ is the brane covariant derivative, one obtains that

$$\dot{E}_{00} + 4 H E_{00} = \partial_\tau \left( \frac{3}{16} \dot{\phi}^2 + \frac{3}{8} U \right) + \frac{3}{2} H \dot{\phi}^2 + \frac{\dot{U}_B}{4} \rho_m \tag{88}$$

leading to

$$E_{00} = \frac{1}{a^4} \int d\tau a^4 \left( \partial_\tau \left( \frac{3}{16} \dot{\phi}^2 + \frac{3}{8} U \right) + \frac{3}{2} H \dot{\phi}^2 + \frac{\dot{U}_B}{4} \rho_m \right) \tag{89}$$

Upon using

$$\bar{G}_{00} = 3 H^2 \tag{90}$$

one obtains the Friedmann equation. It is remarkable that the retarded effects in the Friedmann equation all spring from the projected Weyl tensor. Hence the projected Weyl tensor proves to be much richer in the case of a bulk scalar field than in the empty bulk case.

3.2.3. Self-Tuning and Accelerated Cosmology

The dynamics of the brane is not closed, it is an open system continuously exchanging energy with the bulk. This exchange is characterized by the dark radiation term and the loss parameter. Both require a detailed knowledge of the bulk dynamics. This is of course beyond the projective approach where only quantities on the brane are evaluated. In the following we will assume that the dark radiation term is absent and that the loss parameter is negligible. Furthermore, we will be interested in the effects of a bulk scalar field for late-time cosmology (i.e. well after nucleosynthesis) and not in the case for inflation driven by a bulk scalar field (see e.g. \[ 81-85 \]).

Let us consider the self-tuned scenario as a solution to the cosmological constant problem. It corresponds to the BPS superpotential with $\alpha = 1$. In that case the potential $U = 0$ for any value of the brane tension. The potential $U = 0$ can be interpreted as a vanishing of the brane cosmological constant. The physical interpretation of
the vanishing of the cosmological constant is that the brane tension curves the fifth dimensional space-time leaving a flat brane intact. Unfortunately, the description of the bulk geometry in that case has shown that there was a bulk singularity which needs to be hidden by a second brane whose tension is fine-tuned with the first brane tension. This reintroduces a fine-tuning in the putative solution to the cosmological constant problem [18].

Let us generalize the selftuned case to $\alpha \neq 1$, i.e. $U_B = TW$, $T > 1$ and $W$ is the exponential superpotential. The resulting induced metric on the brane is of the FRW type with a scale factor

$$a(t) = a_0 \left( \frac{t}{t_0} \right)^{1/3+1/6\alpha^2}$$  (91)

leading to an acceleration parameter

$$q_0 = \frac{6\alpha^2}{1+2\alpha^2} - 1$$  (92)

For the supergravity value $\alpha = -\frac{1}{\sqrt{12}}$ this leads to $q_0 = -4/7$. This is in coincidental agreement with the supernovae results. This model can serve as a brane quintessence model [77],[86]. We will comment on the drawbacks of this model later. See also [91] and [92] for similar ideas.

3.2.4. The brane cosmological eras

Let us now consider the possible cosmological scenarios with a bulk scalar field [86],[87]. We assume that the potential energy of the scalar field $U$ is negligible throughout the radiation and matter eras before serving as quintessence in the recent past.

At very high energy above the tension of the brane the non-conventional cosmology driven by the $\rho_m^2$ term in the Friedmann equation is obtained. Assuming radiation domination, the scale factor behaves like

$$a = a_0 \left( \frac{t}{t_0} \right)^{1/4}$$  (93)

and the scalar field

$$\phi = \phi_i + \beta \ln \left( \frac{t}{t_0} \right)$$  (94)

In the radiation dominated era, no modification is present, provided

$$\phi = \phi_i$$  (95)

which is a solution of the Klein-Gordon equation as the trace of the energy-momentum of radiation vanishes (together with a decaying solution, which we have neglected). In the matter dominated era the scalar field evolves due to the coupling to the trace of the energy-momentum tensor. This has two consequences. Firstly, the kinetic energy of the scalar field starts contributing in the Friedmann equation. Secondly, the effective Newton constant does not remain constant. The cosmological evolution of Newton’s constant is severely constrained since nucleosynthesis [88],[89]. This restricts the possible time variation of $\phi$. 

In order to be more quantitative let us come back to the exponential superpotential case with a detuning parameter $T$. The time dependence of the scalar field and scale factor become
\[
\phi = \phi_1 - \frac{8}{15} \alpha \ln \left( \frac{t}{t_e} \right)
\]
\[
a = a_e \left( \frac{t}{t_e} \right)^{\frac{2}{3} - \frac{8}{45} \alpha^2}
\]
where $t_e$ and $a_e$ are the time and scale factor at matter-radiation equality. Notice the slight discrepancy of the scale factor exponent with the standard model value of $2/3$.

The redshift dependence of the Newton constant is
\[
\frac{G_N(z)}{G_N(z_e)} = \left( \frac{z + 1}{z_e + 1} \right)^{4\alpha^2/5}
\]
For the supergravity model with $\alpha = -\frac{1}{\sqrt{12}}$ and $z_e \sim 10^3$ this leads to a decrease by (roughly) 37% since nucleosynthesis. This is marginally compatible with experiments \[88]\,\[89].

Finally let us analyse the possibility of using the brane potential energy of the scalar field $U$ as the source of acceleration now. We have seen that when matter is negligible on the brane, one can build brane quintessence models. We now require that this occurs only in the recent past. As can be expected, this leads to a fine-tuning problem as
\[
M^4 \sim \rho_c
\]
where $M^4 = (T - 1)^{3W/2\xi^3}$ is the amount of detuned tension on the brane. Of course this is nothing but a reformulation of the usual cosmological constant problem. Provided one accepts this fine-tuning, as in most quintessence models, the exponential model with $\alpha = -\frac{1}{\sqrt{12}}$ is a cosmological consistent quintessence model with a five dimensional origin.

3.3. Brief summary

The main difference between a brane world model with a bulk scalar field and the Randall–Sundrum model is that the gravitational constant becomes time–dependent. As such it has much in common with scalar–tensor theories \[90], but there are important differences due to the projected Weyl tensor $E_{\mu \nu}$ and its time–evolution. The bulk scalar field can play the role of the quintessence field, as discussed above, but it could also play a role in an inflationary era in the very early universe (see e.g. \[81]- \[85]). In any case, the cosmology of such a system is much richer and, because of the variation of the gravitational constant, more constrained. It remains to be seen if the bulk scalar field can leave a trace in the CMB anisotropies and Large Scale Structures (for first results see \[87]).

4. Moving Branes in a Static Bulk

So far, we were mostly concerned with the evolution of the brane, without referring to the bulk itself. In fact, the coordinates introduced in eq. \[14] are a convenient choice
for studying the brane itself, but when it comes to analysing the bulk dynamics and its geometry, these coordinates are not the best choice. We have already mentioned the extended Birkhoff theorem in section 2. It states that for the case of a vacuum bulk spacetime, the bulk is necessarily static, in certain coordinates. A cosmological evolving brane is then moving in that spacetime, whereas for an observer confined on the brane the motion of the brane will be seen as an expanding (or contracting) universe. In the case of a scalar field in the bulk, a similar theorem is unfortunately not available, which makes the study of such systems much more complicated. We will now discuss these issues in some detail, following in particular [73] and [74].

4.1. Motion in AdS-Schwarzschild Bulk

We have already discussed the static background associated with BPS configurations (including the Randall–Sundrum case) in the last section. Here we will focus on other backgrounds for which one can integrate the bulk equations of motion. Let us write the following ansatz for the metric

\[ ds^2 = -A^2(r)dt^2 + B^2(r)dr^2 + R^2(r)d\Sigma^2 \] (98)

where \( d\Sigma^2 \) is the metric on the 3d symmetric space of curvature \( q = 0, \pm 1 \). In general, the function \( A, B \) and \( R \) depend on the type of scalar field potential. This is to be contrasted with the case of a negative bulk cosmological constant where Birkhoff’s theorem states that the most general solution of the (bulk) Einstein equations is given by \( A^2 = f, B^2 = 1/f \) and \( R = r \) where

\[ f(r) = q + \frac{r^2}{l^2} - \frac{\mu}{r^4}. \] (99)

We have denoted by \( l = 1/k = \sqrt{-6/(\Lambda_5 k^2)} \) the AdS scale and \( \mu \) the black hole mass (see section 2). This solution is the so–called AdS-Schwarzschild solution.

Let us now study the motion of a brane of tension \( T/l \) in such a background. The equation of motion is determined by the junction conditions. The method will be reviewed later when a scalar field is present in the bulk. The resulting equation of motion for a boundary brane with a \( Z_2 \) symmetry is

\[ \left( \dot{r}^2 + f(r) \right)^{1/2} = \frac{T}{l} r \] (100)

for a brane located at \( r [36] \). Here \( \dot{r} \) is the velocity of the brane measured with the proper time on the brane. This leads to the following Friedmann equation

\[ H^2 \equiv \left( \frac{\dot{r}}{r} \right)^2 = \frac{T^2 - 1}{l^2} - \frac{q}{r^2} + \frac{\mu}{r^4}. \] (101)

So the brane tension leads to an effective cosmological constant \((T^2 - 1)/l^2\). The curvature gives the usual term familiar from standard cosmology while the last term is the dark radiation term whose origin springs from the presence of a black-hole in the bulk. At late time the dark radiation term is negligible for an expanding universe, we retrieve the cosmology of a FRW universe with a non-vanishing cosmological constant. The case \( T = 1 \) corresponds of course to the Randall-Sundrum case.
4.2. Moving branes

Let us now describe the general formalism, which covers the case of the AdS–Schwarzschild spacetime mentioned above.

Consider a brane embedded in a static background. It is parametrized by the coordinates $X^A(x^\mu)$ where $A = 0 \ldots 4$ and the $x^\mu$ are world volume coordinates. Locally the brane is characterized by the local frame

$$e^A_\mu = \frac{\partial X^A}{\partial x^\mu},$$

which are tangent to the brane. The induced metric is given by

$$h_{\mu\nu} = g_{AB}e^A_\mu e^B_\nu$$

and the extrinsic curvature

$$K_{\mu\nu} = e^A_\mu e^B_\nu D_A n_B,$$

where $n^A$ is the unit vector normal to the brane defined by (up to a sign ambiguity)

$$g_{AB}n^A n^B = 1, \quad n_A e^A_\mu = 0.$$

For a homogeneous brane embedded in the spacetime described by the metric (98), we have $T = T(\tau), \ r = r(\tau)$ where $\tau$ is the proper time on the brane. The induced metric is

$$ds^2_B = -d\tau^2 + R^2(\tau)d\Sigma^2.$$

The local frame becomes

$$e^A_\tau = (\dot{T}, \dot{r}, 0, 0, 0), \ e^A_i = (0, 0, \delta^A_i),$$

while the normal vector reads

$$n_A = (A\dot{B}, -B\sqrt{1 + \dot{r}^2}, 0, 0, 0).$$

The components of the extrinsic curvature tensor can found to be

$$K_{ij} = -\frac{\sqrt{1 + B^2\dot{r}^2}}{B}RR'\delta_{ij},$$

$$K_{\tau\tau} = \frac{1}{AB} \frac{d}{d\tau} (A\sqrt{1 + B^2\dot{r}^2}).$$

The junction conditions are given by

$$K_{\mu\nu} = -\frac{\kappa_5^2}{2} \left( \tau_{\mu\nu} - \frac{1}{3} \tau h_{\mu\nu} \right).$$

This implies that the brane dynamics are specified by the equations of motion

$$\frac{\sqrt{1 + B^2\dot{r}^2}}{B} \frac{R'}{R} = \frac{\kappa_5^2}{6} \rho$$

and

$$\frac{1}{AB} \frac{d}{d\tau} (A\sqrt{1 + B^2\dot{r}^2}) = -\frac{\kappa_5^2}{6} (2\rho + 3p).$$
where we have assumed a fluid description for the matter on the brane. These two equations determine the dynamics of any brane in a static background.

Let us now close the system of equations by stating the scalar field boundary condition

\[ n^A \partial_A \phi = \frac{\kappa^2_5}{2} \frac{d\xi}{d\phi} (\rho - 3p), \]  
(114)

where the coupling to the brane is defined by the Lagrangian

\[ S_{brane} = \int d^4 x \mathcal{L}[\psi_m, \tilde{h}_{\mu\nu}], \]  
(115)

where \( \psi_m \) represents the matter fields and

\[ \tilde{h}_{\mu\nu} = e^{2\xi(\phi)} h_{\mu\nu}. \]  
(116)

This reduces to

\[ \phi' = \frac{\kappa^2_5}{2} \frac{B}{\sqrt{1 + B^2 r^2}} \frac{d\xi}{d\phi} (-\rho + 3p). \]  
(117)

Combining the junction conditions leads to the conservation equation

\[ \dot{\rho} + 3H(\rho + p) = (\rho - 3p) \dot{\xi}. \]  
(118)

This is nothing but the conservation of matter in the Jordan frame defined by \( \tilde{h}_{\mu\nu} \).

We now turn to a general analysis of the brane motion in a static bulk. To do that it is convenient to parametrize the bulk metric slightly differently

\[ ds^2 = -f^2(r) h(r) dt^2 + \frac{dr^2}{h(r)} + r^2 d\Sigma^2. \]  
(119)

Now, the Einstein equations lead to (redefining \( \phi \rightarrow \sqrt{\frac{3}{2\kappa_5}} \phi \) and \( V \rightarrow \frac{3}{8\kappa_5} V \))

\[ \frac{3}{r^2} \left( h + \frac{r h'}{2} - q \right) = -\kappa^2_5 \left( \frac{h\phi'^2}{2} + V \right) \]  
(120)

\[ \frac{3}{r^2} \left( h + \frac{r h'}{2} - q + \frac{hf'}{f} \right) = \kappa^2_5 \left( \frac{h\phi'^2}{2} - V \right) \]  
(121)

and the Klein Gordon equation

\[ h\phi'' + \left( \frac{3h}{r} + \frac{hf''}{f} + h' \right) \phi' = \frac{dV}{d\phi}. \]  
(122)

Subtracting eq. (120) from (121) and solving the resulting differential equation, we obtain

\[ f = \exp \left( \frac{\kappa^2_5}{3} \int dr r \phi'^2 \right). \]  
(123)

It is convenient to evaluate the spatial trace of the projected Weyl tensor. This is obtained by computing both the bulk Weyl tensor and the vector normal to the moving brane. With \( A = \sqrt{h} f, \ R = r, \ B = 1/\sqrt{h}, \) this gives

\[ \frac{\mu}{r^4} = -\frac{E_i}{3} = \frac{r}{4f^2} \left( \frac{hf'^2}{r^2} \right)' + \frac{q}{2r^2}. \]  
(124)
This is the analogue of the dark radiation term for a general background. The equations of motion can be cast in the form

\[
\mu' = -\frac{\kappa_5^2}{3}(\mu - \frac{kr^2}{2})r\phi'^2, 
\]

\[
\mathcal{H}' + 4\frac{\mu}{r^5} = -\frac{2\kappa_5^2}{3}(\mathcal{H} - \frac{q}{r^2})r\phi'^2, 
\]

\[
\kappa_5^2 V = 6\mathcal{H} + \frac{3}{4}r\mathcal{H}' - 3\frac{\mu}{r^4}, 
\]

where we have defined

\[
\mathcal{H} = \frac{q - h}{r^2}. 
\]

This allows to retrieve easily some of the previous solutions. Choosing \(\phi\) to be constant leads to \(f = 1\), \(\mu\) is constant and

\[
\mathcal{H} = -\frac{1}{l^2} + \frac{\mu}{r^4} 
\]

This is the AdS-Schwarzschild solution.

For \(q = 0\) the equations of motion simplify to

\[
\frac{\kappa_5^2}{3}r\phi' = -\frac{d\ln\mu}{d\phi}, 
\]

\[
d\left(\frac{\mathcal{H}}{\mu^2}\right) = \frac{1}{\mu}d\left(r^{-4}\right), 
\]

\[
\frac{\kappa_5^2}{6} V = -\frac{3}{4\kappa_5^2} \frac{d\mu}{d\phi} \frac{d\phi}{d\mu} \left(\frac{\mathcal{H}}{\mu}\right) + \mathcal{H} 
\]

In this form it is easy to see that the dynamics of the bulk are completely integrable. First of all the solutions depend on an arbitrary function \(\mu(\phi)\) which determines the dynamics. Notice that

\[
f = \frac{\mu_0}{\mu} 
\]

where \(\mu_0\) is an arbitrary constant. The radial coordinate \(r\) is obtained by simple integration of eq. (130)

\[
r = r_0e^{-\frac{\phi}{\kappa_5^2}} \int \frac{d\phi}{\mu} d\phi. 
\]

Finally the rest of the metric follows from

\[
h = -\frac{4\kappa_5^2}{3} r^2 \mu^2 \int d\phi \frac{d\phi}{d\mu} \frac{\mu^2}{\mu} \int \frac{d\phi}{\mu} d\phi 
\]

The potential \(V\) then follows (132). This is remarkable and shows why Birkhoff’s theorem is not valid in the presence of a bulk scalar field. Moreover, it is intriguing that the generalization of the dark energy term dictates the bulk dynamics completely.

It is interesting to recast the Friedmann equation in the form

\[
H^2 = \mathcal{H} + \frac{\kappa_5^4}{36} \mu^2 \rho^2 
\]
where $H$ is the Hubble parameter on the brane in cosmic time. One can retrieve standard cosmology by studying the dynamics in the vicinity of a critical point $\frac{d\mu}{d\phi} = 0$. Parametrizing
\[
\mu = \frac{6A}{\kappa_5^2} + B\phi^2
\]
leads to the Friedmann equation
\[
H^2 = \frac{\kappa_5^4}{36}(\rho^2 - \theta)\mu^2 + \frac{\mu}{a^4} + o(a^{-4})
\]
(138)
Here $\theta$ is an arbitrary integration constant. Notice that this is a small deviation from the Randall-Sundrum case as
\[
\phi = r^{-B/A}
\]
go to zero at large distances. Hence, standard cosmology is retrieved at low energy and long distance.

5. Cosmology of a Two–Brane System

In this section we will once more include an ingredient suggested by particle physics theories, in particular M–theory. So far we have assumed that there is only one brane in the whole space–time. According to string theory, there should be at least another brane in the bulk. Indeed, in heterotic M–theory these branes are the boundaries of the bulk spacetime [8]. However, even from a purely phenomenological point of view there is a reason to include a second brane: the bulk singularity (or the AdS horizon). As we have seen in section 3, the inclusion of a bulk scalar field often implies the presence of a naked singularity located away from the positive tension brane. The second brane which we include now should shield this singularity, so that the physical spacetime stretches between the two branes. Another motivation is the hierarchy problem. Randall and Sundrum proposed a two brane model (one with positive and one with negative tension), embedded in a five–dimensional AdS spacetime. In their scenario the standard model particles would be confined on the negative tension brane. As they have shown, in this case gravity is weak due to the warping of the bulk spacetime. However, as will become clear from the results in this section, in order for this model to be consistent with gravitational experiments, the interbrane distance has to be fixed [94]. This can be achieved, for example, with a bulk scalar field. As shown in [94] and [95], gravity in the two–brane model of Randall-Sundrum is described by a scalar–tensor theory, in which the interbrane–distance, called radion, plays the role of a scalar field. The bulk scalar field will modify the Brans–Dicke parameter (see [96] and [97]) of the scalar field and will introduce a second scalar field in the low–energy effective theory, so that the resulting theory at low energy in the case of two branes and a bulk scalar field is a bi–scalar–tensor theory [98],[99].

In the following we will investigate the cosmological consequences when the distance between the branes is not fixed (for some aspects not covered here see e.g. [100]–[107]).
Motivation for this comes, for example, from a recent claim that the fine–structure constant might slowly evolve with time \[108\].

5.1. The low–energy effective action

In order to understand the cosmology of the two–brane system, we derive the low-energy effective action by utilizing the moduli space approximation. From the discussion in section 3 and section 4 it becomes clear, that the general solution of the bulk Einstein equations for a given potential is difficult to find. The moduli space approximation gives the low–energy–limit effective action for the two brane system, i.e. for energies much smaller than the brane tensions.

In the static BPS solutions described in the section 3, the brane positions can be chosen arbitrarily. In other words, they are moduli fields. It is expected that by putting some matter on the branes, these moduli field become time-dependent, or, if the matter is inhomogeneously distributed, space–time dependent. Thus, the first approximation is to replace the brane–positions with space–time dependent functions. Furthermore, in order to allow for the gravitational zero–mode, we will replace the flat spacetime metric \(\eta_{\mu \nu}\) with \(g_{\mu \nu}(x^\alpha)\). We do assume that the evolution of these fields is slow, which means that we neglect terms like \((\partial \phi)^3\) when constructing the low-energy effective action.

As already mentioned, the moduli space approximation is only a good approximation at energies much less than the brane tension. Thus, we do not recover the quadratic term in the moduli space approximation. We are interested in the late time effects after nucleosynthesis, where the corrections have to be small.

Replacing \(\eta_{\mu \nu}\) with \(g_{\mu \nu}(x^\alpha)\) in (63) and collecting all the terms one finds from the 5D action after an integration over \(y\):

\[
S_{\text{MSA}} = \int d^4x \sqrt{-g_4} \left[ f(\phi, \sigma)R^{(4)} + \frac{3}{4} a^2(\phi) \frac{U_B(\phi)}{\kappa_5^2} (\partial \phi)^2 
- \frac{3}{4} a^2(\sigma) \frac{U_B}{\kappa_5^2} (\sigma)(\partial \sigma)^2 \right],
\]

with

\[
f(\phi, \sigma) = \frac{1}{\kappa_5^2} \int_\phi^\sigma dy a^2(y),
\]

with \(a(y)\) given by (67). The moduli \(\phi\) and \(\sigma\) represent the location of the two branes. Note that the kinetic term of the field \(\phi\) has the wrong sign. This is an artifact of the frame we use here. As we will see below, it is possible to go to the Einstein frame with a simple conformal transformation, in which the sign in front of the kinetic term is correct for both fields.

In the following we will concentrate on the BPS system with exponential superpotential from section 3. Let us redefine the fields according to

\[
\tilde{\phi}^2 = \left(1 - 4k\alpha^2 \phi\right)^{2\beta}, \quad \tilde{\sigma}^2 = \left(1 - 4k\alpha^2 \sigma\right)^{2\beta},
\]
with $\beta = \frac{2\alpha^2+1}{4\alpha^2}$; and then

$$\tilde{\phi} = Q \cosh R, \quad \tilde{\sigma} = Q \sinh R.$$  

(143)

A conformal transformation $\tilde{g}_{\mu\nu} = Q^2 g_{\mu\nu}$ leads to the Einstein frame action:

$$S_{\text{EF}} = \frac{1}{2k^2(2\alpha^2 + 1)} \int d^4x \sqrt{-g} \left[ R - \frac{12\alpha^2}{1 + 2\alpha^2} \frac{(\partial Q)^2}{Q^2} ight.$$

$$\left. - \frac{6}{2\alpha^2 + 1} (\partial R)^2 \right].$$  

(144)

Note that in this frame both fields have the correct sign in front of the kinetic terms. For $\alpha \to 0$ (i.e. the Randall–Sundrum case) the $Q$–field decouples. This reflects the fact, that the bulk scalar field decouples, and the only scalar degree of freedom is the distance between the branes. One can read off the gravitational constant to be

$$16\pi G = 2k^2(1 + 2\alpha^2).$$  

(145)

The matter sector of the action can be found easily: if matter lives on the branes, it “feels” the induced metric. That is, the action has the form

$$S^{(1)}_m = S^{(1)}_m(\Psi_1, g_{\mu\nu}^{B(1)}) \quad \text{and} \quad S^{(2)}_m = S^{(2)}_m(\Psi_2, g_{\mu\nu}^{B(2)}),$$  

(146)

where $g_{\mu\nu}^{B(i)}$ denotes the induced metric on each branes. In going to the Einstein frame one gets

$$S^{(1)}_m = S^{(1)}_m(\Psi_1, A^2(Q, R) g_{\mu\nu}) \quad \text{and} \quad S^{(2)}_m = S^{(2)}_m(\Psi_2, B^2(Q, R) g_{\mu\nu}),$$  

(147)

where matter now couples explicitly to the fields via the functions $A$ and $B$, which we will give below (neglecting derivative interactions).

The theory derived with the help of the moduli space approximation has the form of a multi–scalar–tensor theory, in which matter on both branes couple differently to the moduli fields. We note, that methods different from the moduli–space approximation have been used in the literature in order to obtain the low–energy effective action or the resulting field equations for a two–brane system (see in particular [109]–[113]). Qualitatively, the features of the resulting theories agree with the moduli–space approximation discussed above.

In the following we will discuss observational constraints imposed on the parameter of the theory.

5.2. Observational constraints

In order to constrain the theory, it is convenient to write the moduli Lagrangian in the form of a non-linear sigma model with kinetic terms

$$\gamma_{ij} \partial \phi^i \partial \phi^j,$$  

(148)

where $i = 1, 2$ labels the moduli $\phi^1 = Q$ and $\phi^2 = R$. The sigma model couplings are here

$$\gamma_{QQ} = \frac{12\alpha^2}{1 + 2\alpha^2} \frac{1}{Q^2}; \quad \gamma_{RR} = \frac{6}{1 + 2\alpha^2}.$$  

(149)
Notice the potential danger of the $\alpha \to 0$ limit, the RS model, where the coupling to $Q$ becomes very small. In an ordinary Brans-Dicke theory with a single field, this would correspond to a vanishing Brans-Dicke parameter which is ruled out experimentally. Here we will see that the coupling to matter is such that this is not the case. Indeed we can write the action expressing the coupling to ordinary matter on our brane as

$$A = a(\phi)f^{-1/2}(\phi, \sigma), \quad B = a(\sigma)f^{-1/2}(\phi, \sigma),$$

(150)

where we have neglected the derivative interaction.

Let us introduce the parameters

$$\alpha_Q = \partial Q \ln A, \quad \alpha_R = \partial_R \ln A.$$  

(151)

We find that ($\lambda = 4/(1 + 2\alpha^2)$)

$$A = Q^{-\frac{\alpha^2}{2}}(\cosh R)^{\frac{4}{\lambda}},$$

(152)

leading to

$$\alpha_Q = -\frac{\alpha^2}{2} \frac{1}{Q}, \quad \alpha_R = \frac{\lambda \tanh R}{4}.$$  

(153)

Observations constrain the parameter

$$\theta = \gamma^{ij}\alpha_i\alpha_j$$

(154)

to be less than $10^{-3}$ [114]. We obtain therefore a bound on

$$\theta = \frac{4}{3} \frac{\alpha^2}{1 + 2\alpha^2} + \frac{\tanh^2 R}{6(1 + 2\alpha^2)}.$$  

(155)

The bound implies that

$$\alpha \leq 10^{-2}, \quad R \leq 0.2$$

(156)

The smallness of $\alpha$ indicates a strongly warped bulk geometry such as an Anti–de Sitter spacetime. In the case $\alpha = 0$, we can easily interpret the bound on $R$. Indeed in that case

$$\tanh R = e^{-k(\sigma - \phi)},$$

(157)

i.e. this is nothing but the exponential of the radion field measuring the distance between the branes. We obtain that gravity experiments require the branes to be sufficiently far apart. When $\alpha \neq 0$ but small, one way of obtaining a small value of $R$ is for the hidden brane to become close from the would-be singularity where $a(\sigma) = 0$.

We would like to mention that the parameter $\theta$ can be calculated also for matter on the negative tension brane. Then, following the same calculations as above, it can be seen that the observational constraint for $\theta$ cannot be satisfied. Thus, if the standard model particles are confined on the negative tension brane, the moduli have necessarily to be stabilized. In the following we will assume that the standard model particles are confined on the positive tension brane and study the cosmological evolution of the moduli fields.
5.3. Cosmological implications

The discussion in the last subsection raises an important question: the parameter $\alpha$ has to be chosen rather small, in order for the theory to be consistent with observations. Similarly the field $R$ has to be small too. The field $R$ is dynamical and one would like to know if the cosmological evolution drives the field $R$ to small values such that it is consistent with the observations today. Otherwise are there natural initial conditions for the field $R$? In the following we study the cosmological evolution of the system in order to answer these questions.

The field equations for a homogenous and isotropic universe can be obtained from the action. The Friedmann equation reads

$$H^2 = \frac{8\pi G}{3} (\rho_1 + \rho_2 + V_{\text{eff}} + W_{\text{eff}}) + \frac{2\alpha^2}{1 + 2\alpha^2} \dot{\phi}^2 + \frac{1}{1 + 2\alpha^2} \dot{R}^2.$$ \hspace{1cm} (158)

where we have defined $Q = \exp \phi$. The field equations for $R$ and $\phi$ read

$$\ddot{R} + 3H \dot{R} = -8\pi G \frac{1 + 2\alpha^2}{6} \left[ \frac{\partial V_{\text{eff}}}{\partial R} + \frac{\partial W_{\text{eff}}}{\partial R} \right]$$

$$+ \alpha_R^{(1)} (\rho_1 - 3p_1) + \alpha_R^{(2)} (\rho_2 - 3p_2) \right] \hspace{1cm} (159)$$

$$\ddot{\phi} + 3H \dot{\phi} = -8\pi G \frac{1 + 2\alpha^2}{12\alpha^2} \left[ \frac{\partial V_{\text{eff}}}{\partial \phi} + \frac{\partial W_{\text{eff}}}{\partial \phi} \right]$$

$$+ \alpha_\phi^{(1)} (\rho_1 - 3p_1) + \alpha_\phi^{(2)} (\rho_2 - 3p_2) \right]. \hspace{1cm} (160)$$

The coupling parameter are given by

$$\alpha_\phi^{(1)} = -\frac{2\alpha^2}{1 + 2\alpha^2}, \quad \alpha_\phi^{(2)} = -\frac{2\alpha^2}{1 + 2\alpha^2}, \hspace{1cm} \alpha_R^{(1)} = \frac{\tanh R}{1 + 2\alpha^2}, \quad \alpha_R^{(2)} = \frac{(\tanh R)^{-1}}{1 + 2\alpha^2}. \hspace{1cm} (161)$$

We have included matter on both branes as well as potentials $V_{\text{eff}}$ and $W_{\text{eff}}$ on each branes. We now concentrate on the case where matter is only on our brane. In the radiation dominated epoch the trace of the energy–momentum tensor vanishes, so that $Q$ and $\phi$ quickly become constant. The scale factor scales like $a(t) \propto t^{1/2}$.

In the matter–dominated era, the solution to these equations is given by

$$\rho_1 = \rho_\phi \left( \frac{a}{ao} \right)^{-3-2\alpha^2/3}, a = ao \left( \frac{t}{te} \right)^{2/3-4\alpha^2/27} \hspace{1cm} (163)$$

together with

$$\phi = \phi_\phi + \frac{1}{3} \ln \frac{a}{ao}, R = R_0 \left( \frac{t}{te} \right)^{-1/3} + R_1 \left( \frac{t}{te} \right)^{-2/3}, \hspace{1cm} (164)$$

as soon as $t \gg te$. Note that $R$ indeed decays. This implies that small values of $R$ compatible with gravitational experiments are favoured by the cosmological evolution. Note, however, that the size of $R$ in the early universe is constrained by nucleosynthesis as well as by the CMB anisotropies. A large discrepancy between the values of $R$ during nucleosynthesis and now induces a variation of the particle masses, or equivalently
Newton’s constant, which is excluded experimentally. One can show that by putting matter on the negative tension brane as well, the field $R$ evolves even faster to zero. This behaviour is reminiscent of the attractor solution in scalar–tensor theories.

In the five–dimensional picture the fact that $R$ is driven to small values means that the negative tension brane is driven towards the bulk singularity. In fact, solving the equations numerically for more general cases suggest that $R$ can even by negative, which is, in the five–dimensional description meaningless, as the negative tension brane would move through the bulk singularity. Thus, in order to make any further progress, one has to understand the bulk singularity better. Of course, one could simply assume that the negative tension brane is destroyed when it hits the singularity. A more interesting alternative would be if the brane is repelled instead. It was speculated that this could be described by some effective potential in the low-energy effective action.

6. Epilogue: Brane Collision

We have seen that brane world models are plagued with a singularity problem: the negative tension brane might hit a bulk singularity. In that case our description of the physics on the brane requires techniques beyond the field theory approach that we have followed in this review. It is only within a unified theory encompassing general relativity and quantum mechanics that such questions might be addressed. String theory may be such a theory. The problem of the nature of the resolution of cosmological singularities in string theory is still a vastly uncharted territory. There is a second kind of singularity which arises when two branes collide. In such a case there is also a singularity in the low energy effective action as one of the extra dimensions shrink to zero size. It was speculated that brane collisions play an important role in cosmology, especially in order to understand the big bang itself.

In heteroric M-theory the regime where the distance between the branes becomes small corresponds to the regime where the string coupling constant becomes small and therefore a perturbative heterotic treatment may be available. In particular for adiabatic processes the resulting small instanton transition has been thoroughly studied (see e.g. \[125\] and \[126\]). Here we would like to present an analysis of such a collision and of the possible outcome of such a collision. A natural and intuitive phenomenon which may occur during a collision is the existence of a cosmological bounce. Such objects are not available in 4d under mild assumptions, and therefore can be exhibited as a purely extra dimensional signature.

We will describe a $d$–dimensional theory with a scalar field and gravity whose solutions present a cosmological singularity at $t = 0$. It turns out that this model is the low energy approximation of a purely $(d + 1)$ dimensional model where the extra dimension is an interval with two boundary branes. The singularity corresponds to the brane collision. In the $(d + 1)$ dimensional picture, one can extend the motion.

‡ For $\alpha = 0$ the theory is equivalent to the Randall–Sundrum model. In this case the bulk singularity is shifted towards the Anti–de Sitter boundary.
of the branes past each other, hence providing a continuation of the brane motion after the collision. The \((d + 1)\) dimensional space–time is equivalent to an orbifold where the identification between space–time points is provided by a Lorentz boost. These spaces are the simplest possible space–times with a singularity. As with ordinary spatial orbifolds, one may try to define string theory in such backgrounds and analyse the stringy resolution of the singularity. Unfortunately, these orbifolds are not stable in general relativity ruling them out as candidate stringy backgrounds. Let us now briefly outline some of the arguments.

We have already investigated the moduli space approximation for models with a bulk scalar field. Here we will consider that at low energy the moduli space consists of a single scalar field \(\phi\) coupled to gravity

\[
S = \int d^d x \sqrt{-g} \left( R - \frac{1}{2} \left( \partial \phi \right)^2 \right).
\]

Cosmological solutions with

\[
ds^2 = a^2(t)[-dt^2 + dx^i dx_i]
\]

can be easily obtained

\[
a = a(1)|t|^\frac{1}{d-2}, \quad \phi = \phi(1) + \epsilon \sqrt{\frac{2(d-1)}{d-2}} \ln |t|,
\]

where \(\epsilon = \pm 1\). There are two branches corresponding to \(t < 0\) and \(t > 0\) connected by a singularity at \(t = 0\).

So what is the extra dimensional origin of such a model? One can uplift the previous system to \((d + 1)\) dimensions by defining

\[
\psi = e^{\gamma \phi}
\]

and

\[
\bar{g}_{\mu\nu} = \psi^{-4/(d-2)} g_{\mu\nu},
\]

where \(\gamma = \sqrt{(d-2)/8(d-1)}\). Consider now the purely gravitational \((d+1)\) dimensional theory with the metric

\[
ds_{d+1}^2 = \psi^4 dw^2 + \bar{g}_{\mu\nu} dx^\mu dx^\nu,
\]

where \(w \in [0, 1]\). The two boundaries at \(w = 0\) and \(w = 1\) are boundary branes. The dimensional reduction on the interval \(w \in [0, 1]\), i.e. integrating over the extra dimension, yields the effective action \([\text{165}]\) provided one restricts the two fields \(\psi(x^\mu)\) and \(\bar{g}_{\mu\nu}(x^\mu)\) to be dependent on \(d\)-dimensions only.

Let us now consider the nature of \((d + 1)\)–dimensional space-time obtained from the solutions \([\text{167}]\). The \((d + 1)\) dimensional metric becomes

\[
ds_{d+1}^2 = B^2 t^2 dw^2 + \eta_{\mu\nu} dx^\mu dx^\nu
\]

for a given \(B\) depending on the integrations constants \(\phi(1)\) and \(a(1)\). The geometry of space-time is remarkably simple. It is a direct product \(R^{d-1} \times M\) where \(M\) is the two dimensional compactified Milne space whose metric is

\[
ds_M^2 = -dt^2 + B^4 t^2 dw^2.
\]
Using the light cone coordinates
\[ x^\pm = \pm te^{\pm B^2 w} \] (173)
the metric of Milne space reads
\[ ds^2_M = dx_+ dx_- , \] (174)
coinciding with the two dimensional Minkowski metric. There is one subtlety here, the original identification of the extra–dimensional interval is here transcribed in the fact that Milne space is modded out by the boost
\[ x^\pm \rightarrow e^{\pm B^2} x^\pm , \] (175)
as we have identified the interval with \( S^1/Z_2 \) and the boundary branes are the fixed points of the \( Z_2 \) action as in the Randall-Sundrum model.

The two boundary branes collide at \( x^\pm = 0 \), their trajectories are given by
\[ x_0^\pm = \pm t, x_1^\pm = \pm te^{\pm B^2} . \] (176)
At the singularity one can hope that the branes go past each other and evolve henceforth. Unfortunately, Horowitz and Polchinski have shown that the structure of the orbifold space–time is unstable [127]. By considering a particle in this geometry, they showed that space–time collapses to a space-like singularity. Indeed one can focus on a particle and its \( n \)–th image under the orbifold action. In terms of collision the impact parameter \( b \) becomes constant as \( n \) grows while the centre of mass energy \( \sqrt{s} \) grows like \( \cosh nB^2 \). As soon as \( n \) is large enough,
\[ G\sqrt{s} > b^{d-2} \] (177)
the two particle approach each other within their Schwarzschild radii therefore forming a black hole through gravitational collapse. So the orbifold space–time does not make sense in general relativity, i.e. not defining a time-dependent background for string theory.

Hence, it seems that the most simple example of brane collision needs to be modified in order to provide a working example of singularity with a meaningful string theoretic resolution. It would be extremely relevant if one could find examples of stable backgrounds of string theory where a cosmological singularity can be resolved using stringy arguments. A particularly promising avenue is provided by S–branes where a cosmological singularity is shielded by a horizon (see e.g. [28] and references therein). Time will certainly tell which of these approaches could lead to a proper understanding of cosmological singularities and their resolutions, an issue highly relevant to brane cosmology both in the early universe and the recent past.

7. Open Questions

In this article we have reviewed different aspects of brane cosmology in a hopefully pedagogical manner reflecting our own biased point of view. Let us finally summarize some of the open questions.
In the case of the single brane model by Randall & Sundrum, the homogeneous cosmological evolution is well understood. An unsolved issue in this model, however, is a complete understanding of the evolution of cosmological perturbations. The effects of the bulk gravitational field, encoded in the projected Weyl–tensor, on CMB physics and Large Scale Structures are not known. The problem is twofold: first, the bulk equation are partial, non-linear differential equations and second, boundary conditions on the brane have to be imposed. The current formalism have not yet been used in order to tackle these problems (for perturbations in brane world theories, see [128]–[146]; a short review on brane world perturbations is given in [147]).

• Models with bulk scalar fields: Although we have presented some results on the cosmological evolution of a homogeneous brane, we assumed that the bulk scalar field does not strongly vary around the brane. Clearly, this needs to be investigated in some detail through a detailed investigation of the bulk equations, presumably with the help of numerical methods. Furthermore, for models with two branes, the cosmology has to be explored also in the high energy regime, in which the moduli–space approximation is not valid. Some exact cosmological solutions have been found in [20] and [148].

• Both the bulk scalar field as well as the interbrane distance in two brane models could play an important role at least during some part of the cosmic history. Maybe one of the fields plays the role of dark energy. In that case, it is only natural that masses of particles vary, as well as other parameter, such as the fine structure constant $\alpha_{em}$ [149]. Details of this interesting proposal have yet to be worked out.

• The bulk singularity, which was thought to be shielded away with the help of a second brane, seems to play an important role in a cosmological setting. We have seen in section 5 that the negative tension brane moves towards the bulk singularity and eventually hits it. Therefore, cosmology forces us to think about this singularity, even if it was shielded with a second brane. Cosmologically, the brane might be repelled, which might be described by a potential. Alternatively, one might take quantum corrections into account in form of a Gauss-Bonnett term in the bulk [150]–[153].

• Brane collisions provide a different singularity problem in brane cosmology. String theory has to make progress in order to understand this singularity as well. From the cosmological point of view, the question is, if a transition between the brane collision can provide a new cosmological era and how cosmological perturbations evolve before and after the bounce [154]–[160].

These aspects of brane cosmology raise very interesting questions. It is clear that cosmology will continue to play an important role for testing our ideas beyond the standard model of particle physics.

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