Noncommutative Gauge Fields on Poisson Manifolds *

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Abstract

It is shown by Connes, Douglas and Schwarz that gauge theory on noncommutative torus describes compactifications of M-theory to tori with constant background three-form field. This indicates that noncommutative gauge theories on more general manifolds also can be useful in string theory. We discuss a framework to noncommutative quantum gauge theory on Poisson manifolds by using the deformation quantization. The Kontsevich formula for the star product was given originally in terms of the perturbation expansion and it leads to a non-renormalizable quantum field theory. We discuss the nonperturbative path integral formulation of Cattaneo and Felder as a possible approach to construction of noncommutative quantum gauge theory on Poisson manifolds. Some other aspects of classical and quantum noncommutative field theory are also discussed.

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1 Introduction

In a remarkable paper Connes, Douglas and Schwarz [1] have shown that supersymmetric gauge theory on noncommutative torus is naturally related to compactification of Matrix theory [2]. This shows that the framework of noncommutative geometry [3, 4] can be useful in string theory. For some reviews and further developments see [3]-[24]. One of the natural questions is whether noncommutative gauge theory can be used to describe more general compactifications in string theory. In order to have progress in this direction one has to develop the theory of noncommutative quantum gauge fields not only on the torus but also on more general manifolds. In this note we discuss a framework to noncommutative gauge theory on Poisson manifolds by using the recently developed deformation quantization. In particular the question of renormalizability of noncommutative quantum theory is discussed.

Gauge theory on noncommutative torus is equivalent to noncommutative gauge theory on the commutative (i.e. ordinary) torus. Motivated by investigations of quantum group symmetries in two dimensional integrable models [25] a proposal of consideration of noncommutative gauge symmetry on ordinary manifolds (“quantum group gauge theory”) has been suggested in [26], see also [27, 28] for further discussions. Such a theory could be used as an alternative to the conventional Higgs mechanism of symmetry breaking. It was also shown [31] that a noncommutative gauge theory (quantum Boltzmann theory) describes the large N limit in QCD. Other approaches to field theory on noncommutative spaces and harmonic analysis have been discussed in [29, 30].

A connection of noncommutative geometry with non-Archimedian geometry at the Planck scale and p-adic mathematical physics [32] has been considered in [33]. A proposal to replace the picture of spacetime as a manifold to the theory of motives with motivic Galois group as gauge group has been made in [34] as a natural extension of p-adic string theory and p-adic gravity [35, 36]. In particular L-function of Deligne’s motive has been interpreted as the partition function of string theory. Applications of motives to quantum field theory also have been discussed recently in [37]. A general discussion of relations between number theory and physics is given in [38].

Noncommutative gauge theory uses the deformed product (the “star product”) instead of the ordinary product. The star product [39] is a generalization of the Moyal product well known in quantum mechanics [4]. The construction of star product for symplectic manifolds was given by De Wilde and Lecomte [11] and Fedosov [12]. Kontsevich [13] has found an explicit formula for the star product on Poisson manifolds (nondegenerate Poisson structure is reduced to the symplectic structure). His formula is a corollary of a more general result about the existence of a quasi-isomorphism between the Hochschild complex of the algebra of polynomials and the graded Lie algebra of polyvector fields [13]. Quantum deformations of the Poisson-Lie structures have been considered in [14, 15].

The formula for the star product was given in [13] in terms of a perturbation series.
and if one attempts to use it in noncommutative quantum field theory then one gets a non-renormalizable quantum theory. However it was pointed out in [43] that the formula can be viewed as a perturbation series for a topological quantum field theory coupled with gravity. Recently Cattaneo and Felder [49] have shown that Kontsevich’s star product formula is equivalent to the perturbative expansion of the path integral of a two-dimensional topological quantum field theory.

In this note we discuss the application of the Cattaneo and Felder formulation to noncommutative gauge theory on Poisson manifolds.

There is also a different aspect of the Cattaneo and Felder formulation. The Moyal bracket gives us a natural way to introduce a form-factor to quantum field theories. The nonlocality of the Moyal bracket is related with random particle (one-dimensional world-sheet) and it is not enough to make a theory finite. The Cattaneo and Felder formula uses a random surface and its generalization to random surfaces with non-trivial dynamics in principle could lead to a finite quantum theory. String theory interpretation of the star product has been given by Schomerus [21]. It would be interesting to explore a connection between the star product in deformation quantization and Witten’s product in string field theory [46] generalized on curved manifolds.

The paper is organized as follows. In the next section we will recall some notions of noncommutative geometry, then discuss the Moyal product and its applications in quantum field theory. In Section 5 the deformation quantization is described and finally in Section 6 field theory on Poisson manifolds is discussed.

2 Noncommutative Geometry and Gauge Fields

Noncommutative geometry appears in physics in works of the founders of quantum mechanics. Heisenberg and Dirac have proposed that the phase space of quantum mechanics must be noncommutative and it should be described by quantum algebra. After works of von Neumann and more recently by Connes mathematical and physical investigations in noncommutative geometry became very extensive.

Noncommutative geometry uses a generalization of the known duality between a space and its algebra of functions, see [3, 4, 1]. If one knows the associative commutative algebra \( \mathcal{A}(M) \) of complex-valued functions on topological space \( M \) then one can restore the space \( M \). Therefore all topological notions can be expressed in terms of algebraic properties of \( \mathcal{A}(M) \). For example, the space of continuous sections of vector bundle over \( M \) can be regarded as a projective \( \mathcal{A}(M) \)-module. (We are speaking about left modules. A projective module is a module that can be embedded into a free module as a direct summand). So vector bundle over compact space \( M \) can be identified with projective modules over \( \mathcal{A}(M) \).

Now let us make the following generalization. Let \( \mathcal{A} \) be an abstract noncommutative algebra which we will interpret as an ”algebra of functions” on (nonexisting) noncommutative space. One can introduce many geometrical notions in this setting.
For example, a vector bundle is by definition a projective module over $\mathcal{A}(M)$ and one can develop a theory of such bundles generalized the standard theory. In particular one can define a connection by using the following way. Let $\mathcal{G}$ be a Lie algebra of derivations of $\mathcal{A}$ and $\alpha_1, \ldots, \alpha_d$ be generators of $\mathcal{G}$. If $V$ is a projective module over $\mathcal{A}$ (i.e. a "vector bundle over $\mathcal{A}$") one defines a connection in $V$ as the set of linear operators $\nabla_1, \ldots, \nabla_d$ on $V$ satisfying

$$\nabla_i(a\phi) = a\nabla_i(\phi) + \alpha_i(a)\phi$$

where $a \in \mathcal{A}$, $\phi \in V$, $i = 1, \ldots, d$.

In the case when $\mathcal{A}$ is an algebra of smooth functions on $R^d$ or on the torus $T^d$ we get the standard notion of connection in a vector bundle. In this case the abelian algebra $\mathcal{G} = R^d$ acts on $R^d$ or $T^d$ and correspondingly on $\mathcal{A}$ by means of translations. The curvature of connection $F_{ij} = \nabla_i \nabla_j - \nabla_j \nabla_i - f_{ij}^k \nabla_k$

belongs to the algebra of endomorphisms of the $\mathcal{A}$-module $V$.

The d-dimensional noncommutative torus is defined by its algebra $\mathcal{A}_\theta$ with generators $U_1, \ldots, U_d$ satisfying the relations

$$U_iU_j = e^{2i\pi \theta_{ij}}U_jU_i$$

where $i, j = 1, \ldots, d$ and $\theta = (\theta_{ij})$ is a real antisymmetric matrix. The algebra $\mathcal{A}_\theta$ is equipped with an antilinear involution $\ast$ obeying $U_i^\ast = U_i^{-1}$ (i.e. $\mathcal{A}_\theta$ is a *-algebra). An element of $\mathcal{A}_\theta$ is a power series

$$f = \sum f(p_1, \ldots, p_d)U_1^{p_1} \ldots U_d^{p_d}$$

where $p = (p_1, \ldots, p_d) \in Z^d$ and the sequence of complex coefficients $f(p_1, \ldots, p_d)$ decreases faster than any power of $|p| = |p_1| + \ldots + |p_d|$ when $|p| \to \infty$. The function $f(p)$ is called the symbol of element $f$. We denote $U^p$ the product $U_1^{p_1} \ldots U_d^{p_d}$. Then one has $U^pU^k = e^{2i\pi \varphi(p,k)}U^{p+k}$, where $\varphi(p, k) = \sum \varphi_{ij}p_ip_j$ and $\varphi_{ij}$ is a matrix obtained from $\theta$ after deleting all its elements below the diagonal. To simplify the product rule we replace $U^p$ by $e^{i\pi \varphi(p,k)}U^p$ so that we have

$$U^pU^k = e^{i\pi \varphi(p,k)}U^{p+k}$$

If $f$ and $g$ are two elements of $\mathcal{A}_\theta$,

$$f = \sum_p f(p)U^p, \quad g = \sum_k g(k)U^k$$

then the product

$$fg = \sum_{p, k} f(p)g(k)U^pU^k$$
\[
\sum_{p,k} f(p)g(k)e^{2\pi i\theta(p,k)} U^p U^k = \sum_q (f \ast g)(q) U^q
\]

where the star-product \((f \ast g)(q)\) of symbols \(f(p)\) and \(g(k)\) is

\[
(f \ast g)(q) = \sum_p f(p)g(q - p)e^{i\pi\theta(p,q-p)}
\]  

(8)

The differential calculus on the noncommutative torus is introduced by means of the derivations \(\partial_j\) defined as

\[
\partial_j U^p = ip_j U^p, \quad j = 1, \ldots, d
\]  

(9)

They satisfy the Leibniz rule \(\partial_j (fg) = \partial_j f \cdot g + f \cdot \partial_j g\) for any \(f, g \in \mathcal{A}_\theta\).

The integral of \(f = \sum f(p)U^p\) is defined as \(\int f = f(0)\), which is in correspondence with the commutative case. The integral has the property of being the trace on the algebra \(\mathcal{A}_\theta\), i.e. \(\int fg = \int gf\) for any \(f, g \in \mathcal{A}_\theta\). Moreover one has

\[
\int \partial_j f \cdot g = - \int \partial_j g \cdot f
\]  

(10)

The gauge field \(A_i\) on the noncommutative torus is defined as

\[
A_i = \sum_{p \in \mathbb{Z}^d} A_i(p) U^p, \quad i = 1, \ldots, d
\]  

(11)

Here \(A_i(p)\) is a sequence of \(N \times N\) complex matrices indexed by a spacetime index. It corresponds to the Fourier representation of the ordinary gauge theory on commutative torus. The gauge field is antihermitean, \(A_i^* = -A_i\), or \(A_i(p)^* = -A_i(-p)\). \(A_i\) is an element of a matrix algebra with coefficients in \(\mathcal{A}_\theta\) and its curvature is defined as

\[
F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]
\]  

(12)

where \(\partial_i\) is a derivative on \(\mathcal{A}_\theta\) defined above. The Yang-Mills action is

\[
S = -\frac{1}{4} \int \text{tr}(F_{ij} F^{ij})
\]  

(13)

The action is invariant under gauge transformations

\[
A_i \rightarrow \Omega A_i \Omega^{-1} + \Omega \partial_i \Omega^{-1}, \quad F_{ij} \rightarrow \Omega F_{ij} \Omega^{-1}
\]  

(14)

where \(\Omega\) is a unitary element of the algebra of matrices over \(\mathcal{A}_\theta\).
3 Moyal Product

Let us consider one-dimensional quantum mechanics in the Hilbert space of square integrable functions on the real line $L^2(\mathbb{R})$ with ordinary canonical operators of position $\hat{q}$ and momenta $\hat{p}$,

$$[\hat{p}, \hat{q}] = -i\hbar,$$

acting as $\hat{q}\psi(x) = x\psi(x)$, $\hat{p}\psi(x) = -i\hbar d\psi(x)/dx$. If a function of two real variables $f(q, p)$ is given in terms of its Fourier transform

$$f(q, p) = \int e^{i(\vec{r}\cdot \vec{q} + \vec{s}\cdot \vec{p})} \hat{f}(r, s) dr ds,$$

then one can associate with it an operator $\hat{f}$ in $L^2(\mathbb{R})$ by the following formula

$$\hat{f} = \int e^{i(\vec{r}\cdot \vec{q} + \vec{s}\cdot \vec{p})} \hat{f}(r, s) dr ds.$$

This procedure is called the Weyl quantization and the function $f(q, p)$ is called the symbol of the operator $\hat{f}$. One has the correspondence

$$\hat{f} \longleftrightarrow f = f(q, p) \quad (18)$$

If $\hat{f}^*$ is the Hermitian adjoint to $\hat{f}$ then its symbol is $f^* = f^*(q, p)$

$$\hat{f}^* \longleftrightarrow \hat{f}^*(q, p) = \tilde{f}(q, p) \quad (19)$$

If two operators $\hat{f}_1$ and $\hat{f}_2$ are given with symbols $f_1(q, p)$ and $f_2(q, p)$ then the symbol of product $\hat{f}_1\hat{f}_2$ is given by the Moyal product $f_1 \star f_2 = (f_1 \star f_2)(q, p)$ as

$$(f_1 \star f_2)(q, p) = \sum_{\alpha, \beta} \frac{(-1)^\beta}{\alpha!\beta!} \left( \frac{i\hbar}{2} \right)^{\alpha + \beta} \left( \partial_q^{\alpha} \partial_p^{\beta} f_1(q, p) \right) \cdot \left( \partial_q^{\beta} \partial_p^{\alpha} f_2(q, p) \right)$$

$$= e^{\hbar L} (f_1(q_1, p_1)f_2(q_2, p_2))|_{q_1=q_2=q, \ p_1=p_2=p},$$

where

$$L = \frac{1}{2} \left( \frac{\partial^2}{\partial q_1 \partial p_2} - \frac{\partial^2}{\partial q_2 \partial p_1} \right).$$

This is also can be written as (we set $\hbar = 1$)

$$(f_1 \star f_2)(q, p) = \frac{1}{(2\pi)^2} \int e^{2i[(q-q_2)p_1+(q_1-q)p_2+(q_2-q_1)p]} f_1(q_1, p_1)f_2(q_2, p_2) dq_1 dq_2 dp_1 dp_2.$$

Introducing the constant Poisson structure on the plain $\omega^{\mu\nu} = -\omega^{\nu\mu}$, $\omega^{12} = 1$ and notations $x = (q, p)$, $x_i = (q_i, p_i)$, $i = 1, 2$, $x_1 \cdot x_2 = q_1 p_2 - q_2 p_1$ one writes

$$(f_1 \star f_2)(x) = \frac{1}{(2\pi)^2} \int f_1(x_1)f_2(x_2)e^{2i(x_1 \cdot x_2 + x \cdot x_2)} dx_1 dx_2$$

$$(22)$$
One also has
\[ 2i(x_1 x_2 + x_1 x_2 + x_2 x_1) = 2i \int_{\Delta} pdq \]  
(23)
where \( \Delta \) is the triangle on the plane with vertices \( x = (p, q) \), \( x_1 = (p_1, q_1) \) and \( x_2 = (p_2, q_2) \) and there is the path integral representation
\[ e^{2i \int_{\Delta} pdq} = \int e^{2i \int_{\Delta} pdq} \prod dx(\tau) \]  
(24)
One integrate over trajectories \( x = x(\tau), 0 \leq \tau \leq 1 \) in the phase plane with the boundary conditions
\[ x(0) = x(1) = x, \quad x(1/3) = x_1, \quad x(2/3) = x_2 \]  
(25)
Therefore we obtain that the Moyal bracket is represented in the form
\[ (f \star g)(x) = \int K(x, x_1, x_2)f(x_1)g(x_2)dx_1dx_2 \]  
(26)
where the kernel \( K(x, x_1, x_2) \) has a path integral representation
\[ K(x, x_1, x_2) = \int e^{iS} \prod dx(\tau) \]  
(27)
where
\[ S = 2 \int pdq \]  
(28)
and the path integral is taken over the trajectories \( x = x(\tau), 0 \leq \tau \leq 1 \) subject to the conditions (25).

### 4 Example: Scalar Field

To study divergences it is instructive to consider an example of quartic interaction of a scalar complex field. We have
\[ L = \int d^d x \left[ \frac{1}{2} |\partial_\mu \phi|^2 + \frac{m^2}{2} |\phi|^2 + \frac{g^2}{4} \{|\bar{\phi}, \phi\}_{M.B.} \right] \]  
(29)
\[ \{\bar{\phi}, \phi\}_{M.B.} = \bar{\phi} \star \phi - \phi \star \bar{\phi} \]
d \( = 2n \). Note that the interaction in (29) can be also represented by using an auxiliary field \( \chi \)
\[ L_{int} = g\{\bar{\phi}, \phi\}\chi \]  
(30)
We use a relation of the star-product with the Weyl symbol of the product of operators
\[ (f \star g)(x) = \int K(x, x_1, x_2)f(x_1)g(x_2)dx_1dx_2 \]  
(31)
where the kernel $K(x, x_1, x_2)$ has a path integral representation

$$K(x, x_1, x_2) = \int e^{-\frac{i}{\hbar}S} \prod_{x(0)=x(1)=x, \ x(1/3)=x_1, \ x(2/3)=x_2} dx(\tau) \quad (32)$$

$$S = \int (x^1 dx_{2i} - x^2i dx_{1i}) \quad (33)$$

The integration in (27) is over loops with three fixed points $x(0) = x(1) = x$, $x(1/3) = x_1$, $x(2/3) = x_2$. The choice of points 1/3 and 2/3 on the loop is arbitrary and due to reparametrization invariance the result does not depend on this choice.

$$K(x_1, x_1, x_2) = D(x, x_1) D(x_1, x_2) D(x_2, x) \quad (34)$$

$$D(x, y) = (\frac{1}{2\pi})^{2d/3} \exp\{\frac{2i}{\hbar}x\omega y\} \quad (35)$$

and

$$x\omega y = x^1_1 y^2_2 - y^1_1 x^2_2, \ i = 1, \ldots n. \quad (36)$$

We interpret (28) as the propagator of the auxiliary field and represent (26) on diagrams as shown in Fig.1.

$$\chi(x_1)(\bar{\phi} \phi)(x_2)dx = \int \chi(p)\bar{\phi}(q)\phi(r)\delta(p + q + r)v(p, q, r)dpdqdr \quad (38)$$

$$v(p, q, r) = e^{ip\omega r}$$
Let us consider the one-loop diagram describing "mass"-renormalization of the auxiliary field $\chi$

$$\int \chi(p)\chi(-p)\Sigma_{\epsilon}(p)$$

(39)

$$\Sigma_{\epsilon}(p, \theta) = \Sigma_{\epsilon}(p, 0) + \Sigma_{\epsilon}^{+}(p, \theta) + \Sigma_{\epsilon}^{-}(p, \theta) =$$

(40)

$$\frac{1}{2} \int \frac{dk}{(k^2 + m^2)((p-k)^2 + m^2)} \left[1 - e^{2i\theta k_0 p} - e^{-2i\theta k_0 p}\right]$$

Here $d = 2n - 2\epsilon$. Let us prove that the contributions of the last two terms in (40) are finite due to oscillation factors. To show this it is convenient to use the standard $\alpha$-representation. For example, for the second term in (40) we have Fig.4

$$\Sigma_{\epsilon}^{+}(p, \theta) = -\frac{(\pi)^{d/2}}{2} \int_0^1 dx \int_0^\infty \frac{da}{a^{d/2-1}} e^{i[p^2 x(1-x)+m^2/2a-\theta^2 p^2/4a]}$$

(41)
For \( \theta = 0 \) we get the standard divergences for \( d \geq 4 \), that can be regularized assuming \( d = 2n - 2 \epsilon, \epsilon > 0 \). For \( \theta \neq 0 \), \( d = 4, a = 0 \) is not a dangerous point. In fact, for \( d = 4 \)

\[
\int_0^1 \frac{da}{a} e^{iB/a} = \int_1^\infty \frac{db}{b^2} e^{iBb} < \infty, \quad \text{for } B \neq 0
\]

(42)

Let us note that due to an estimation of the form-factor entering in the vertex \( (38) \)

\[
|v(p, q, r)| < 1
\]

(43)

it is evident that the index of absolute divergence of a diagram with extra form-factor

is the same as the index of the corresponding diagram with local interaction. However

this estimation is not enough to guarantee the renormalizability of the theory by the

following raisons. The above estimation do not care of the subdivergences. To have

the renormalizability we have to guarantee that all divergencies combine to a special

structure to reproduce nonlocal structure of interaction. The last problem one meets

already at the 1-loop level. As was shown recently in \[50, 51, 52\] for a belian non-

commutative Yang-Mills theory 1-loop divergencies combine to renormalizations of the

wave function and coupling constant and moreover the corresponding theory in \( d=4 \)

is asymptotically free. In the next section we will describe a diagram technique for

perturbation expansion of supersymmetric Yang-Mills theory on noncommutative \( T^4 \).

In the case of extended supersymmetries it is enough to study only 1-loop diagrams to

guarantee the renormalizability.

5 Noncommutative SYM

According to \[1\], the noncommutative \( U(1) \) gauge connection can be built by

\[
\nabla_i = \partial_i + i \{ A_i, \}_\text{M.B.}
\]

(44)

with

\[
\{ f, g \}_\text{M.B.}(x) = (f * g)(x) - (g * f)(x) = 2i f(x) \sin (i \pi \theta \epsilon^{ij} \partial_i \partial_j) g(y) |_{y=x}.
\]

(45)

The curvature is

\[
F_{ij} = [\nabla_i, \nabla_j] = \partial_i A_j + i \{ A_i, A_j \}_\text{M.B.}, \quad i = 0, 1, ...9
\]

(46)
Then $U(1)$ NCYM (on $T^2_\theta$) is given by
\[ S = \frac{1}{g^2_{YM}} \int d^2x \ F_{ij} F^{ij}. \] (47)

The above action has the gauge invariance:
\[ A_i \rightarrow A_i + \partial_i \epsilon + i \{\epsilon, A_i\}_{M.B.}. \] (48)

We can also supersymmetrize this action by adding the correct fermionic and scalar degrees of freedom [5,16]. One can take $d = 10$, $N = 1$ non-Abelian super Yang-Mills theory with the group commutators substituted for the Moyal bracket and make dimensional reduction to d-dimensional case
\[ S = \frac{1}{g^2_{YM}} \int d^d x \ F_{\mu\nu} F^{\mu\nu} - 2g^2_{YM}(\nabla_\mu X^a)(\nabla^\mu X^a) + 2g^4_{YM}(\{X^a, X^b\}_{M.B.})^2 \] (49)
\[ -2i\Theta^a \Gamma^\mu_\alpha \nabla_\mu \Theta^\alpha + \frac{1}{4} g_{YM} \Theta^a \Gamma^\alpha_{\alpha\beta} \{X_a, \Theta^\beta\}_{M.B.}, \] (50)

where $\mu, \nu = 0, 1, d-1$, $a, b = d, \ldots, 9$ and
\[ \nabla_\mu X^a = \partial_\mu X^a + \{A_\mu, X^a\}_{M.B.}, \]
\[ \nabla_\mu \Theta^a = \partial_\mu \Theta^a + \{A_\mu, \Theta^a\}_{M.B.}. \]

A gauge fixed generating functional (in the Lorentz gauge)
\[ Z[J, \eta, \bar{\eta}] = \int dAdXd\Theta \bar{C}dCe^{-S[A,X,\Theta]+S_{GF}[A,X,\Theta]+S_{FP}[A,C,\bar{C}]+sources}. \] (51)

where $C$ and $\bar{C}$ are Faddeev-Popov ghosts, $S_{GF}[A_\mu] = -\frac{1}{2\alpha} \int (\partial_\mu A_\mu)^2$ and the Faddeev-Popov term is $\int \partial_\mu C(\partial_\mu C + g[A_\mu, C])$

The generating functional can be computed perturbatively using Feynman diagrams [50]-[52]. The quadratic terms are identical to the ones appearing in non abelian gauge theories and one gets the following propagators:
\[ -\frac{1}{p^2} \left( g_{\mu\nu} - (1 - \alpha) \frac{p_\mu p_\nu}{p^2} \right) \delta(p+q), \text{ for gauge field} \]
\[ -\frac{1}{p^2} \delta(p+q)\delta_{a,b} \text{ for scalars} \]
\[ -\frac{p}{p^2} \delta(p+q)\delta_{\alpha\beta} \text{ for fermions} \]

Although the propagators are the same as in standard non-abelian Yang-Mills theory, the interactions take a different form. To the three gauge bosons interaction we have
\[ 2g\{(p-r)_\nu g_{\mu\rho} + (q-p)_\rho g_{\mu\nu} + (r-q)_\mu g_{\nu\rho}\} \sin \theta(p,q) \delta(p+q+r) \] (52)
For the four gauge bosons interaction we have the sum of the following terms

\[-4g^2 \left( (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \sin \theta(p, q) \sin \theta(r, s) + (g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \sin \theta(p, r) \sin \theta(s, q) + (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\rho\sigma}) \sin \theta(p, s) \sin \theta(q, r) \right) \delta(p + q + r + s). \quad (53)\]

For the interaction of a gauge boson with ghosts we have

\[2gr_\mu \sin \theta(p, q) \delta(p + q + r).\]

The interaction of scalar fields and gauge fields can be obtained by dimensional reduction of the previous vertices. Namely, we can substitute instead of \(\mu, \nu, \rho, \sigma\) in (52) and (53) \(m, n, r, s\) and make a reduction

\[m \rightarrow (\mu, a), \quad n \rightarrow (\nu, b)\]
\[r \rightarrow (\rho, c), \quad s \rightarrow (\sigma, d)\]

and now all momenta are d-dimensional. The interaction of scalar fields and gauge fields contains two vertices: 3-vertex describing interaction of gauge fields with scalars

\[2[q_\rho - p_\rho]g_{bc} \sin \theta(r, p) \delta(p + q + r),\]

two scalars - two gauge field vertex

\[4g^2 g_{\mu\nu} g_{bc} (\sin \theta(p, r) \sin \theta(s, q) - \sin \theta(p, s) \sin \theta(q, r)) \delta(p + q + r + s)\]

and 4-vertex describing interaction of scalars

\[-4g^2 \left( (g_{\alpha\rho}g_{bd} - g_{\alpha\delta}g_{bd}) \sin \theta(p, q) \sin \theta(r, s) + (g_{\alpha\delta}g_{bc} - g_{\alpha\beta}g_{cd}) \sin \theta(p, r) \sin \theta(s, q) + (g_{\alpha\beta}g_{cd} - g_{\alpha\gamma}g_{bd}) \sin \theta(p, s) \sin \theta(q, r) \right) \delta(p + q + r + s). \quad (54)\]

Fermions interact with gauge fields as well as with scalar fields. The interaction of a gauge boson with fermions is associated with

\[2gr_\mu \sin \theta(p, q) \delta(p + q + r).\]

All these interactions are non local since they involve non polynomial functions of the momenta.

It follows from the inequality \(|\sin \pi \theta(p, q)| \leq 1\) that any diagram which is convergent by powercounting in standard non abelian theory is also convergent here. Therefore if we have a local theory which has only one-loop divergencies to guarantee renormalizability of its noncommutative analogue it is enough to consider only one loop diagrams. Explicit calculations similar to \([50]-[52]\) shows that noncommutative SYM has the same one-loop \(\beta\)-function as usual SYM with \(c_2 = 2\). Therefore for d=4 extended supersymmetric gauge models we get renormalizable theories.
6 Deformation Quantization

Here we describe the Cattaneo and Felder path integral approach \[49\] to the Kontsevich \[43\] deformation quantization \[39\] formula. Let \( M \) be a manifold with a Poisson structure on the algebra \( C^\infty(M) \) of functions on \( M \), given by a skew-symmetric tensor \( p^{ij} \), obeying the Jacobi identity
\[
p^{il} \partial_l p^{jk} + p^{jl} \partial_l p^{ki} + p^{kl} \partial_l p^{ij} = 0, \tag{55}
\]
The formal deformation quantization is given by means of the star-product, i.e. an associative product on \( C^\infty(M)[\hbar] \), such that for \( f, g \in C^\infty(M) \)
\[
f \star g(x) = f(x)g(x) + \frac{i\hbar}{2} \{f, g\}(x) + O(\hbar^2). \tag{56}
\]
Kontsevich \[43\] gave a formula for the star-product in terms of diagrams. The coefficient of \((i\hbar/2)^n\) in \( f \star g \) is given by a sum of terms labeled by diagram of order \( n \).
To each diagram \( \Gamma \) of order \( n \) there corresponds a bidifferential operator \( D_\Gamma \) whose coefficients are differential polynomials, homogeneous of degree \( n \) in the components \( p^{ij} \) of the Poisson structure. Kontsevich ‘s formula is
\[
f \star g = fg + \sum_{n=1}^{\infty} \left( \frac{i\hbar}{2} \right)^n \sum_{\text{diagrams } \Gamma \text{ of order } n} w_\Gamma D_\Gamma(f \otimes g). \tag{57}
\]
The weight \( w_\Gamma \) is the integral of a differential form over the configuration space of \( n \) ordered points on the upper half plane.

Cattaneo and Felder \[43\] have shown that this formula can be interpreted as the perturbative expansion of the path integral of a topological sigma model. The model has two real bosonic fields \( X \) and \( a \). \( X \) is a map from the disc \( D = \{ u \in \mathbb{R}^2, |u| \leq 1 \} \) to \( M \) and \( a \) is a differential 1-form on \( D \) taking values in the pull-back by \( X \) of the cotangent bundle of \( M \). In local coordinates, \( X \) is given by \( d \) functions \( X^i(u) \) and \( a \) by \( d \) differential 1-forms \( a_i(u) = a_{i,\mu}(u) du^\mu, i = 1, ..., d, \mu = 1, 2 \)

The action reads
\[
S[X, a] = \int_D a_i(u) \wedge dX^i(u) + \frac{1}{2} p^{ij}(X(u)) a_i(u) \wedge a_j(u). \tag{58}
\]
The boundary condition for \( a \) is that for \( u \in \partial D \), \( a_i(u) \) vanishes on vectors tangent to \( \partial D \).

This model was considered in \[47\]. It is a generalization of a model of two-dimensional gravity with dynamical torsion \[48\], see \[47\].

The star product is given by the semiclassical expansion of the path integral
\[
f \star g(x) = \int f(X(1))g(X(0)) e^{i S[X, a]} \prod_{X(\infty) = x} dX da. \tag{59}
\]
Here 0, 1, ∞ are any three cyclically ordered points on the unit circle. The path integral is over all \( X \) and \( a \) subject to the boundary conditions \( X(∞) = x, a(u)(n) = 0 \) if \( u ∈ ∂D \) and \( n \) is tangent to \( ∂D \).

To evaluate this path integral one has to take gauge fixing and renormalization into account. This action is invariant under the following infinitesimal gauge transformations with infinitesimal parameter \( β_i \), which vanishes on the boundary of \( D \):

\[
\delta_β X^i = p^{ij}(X)β_j, \quad \delta_β a_i = -dβ_i - ∂_i p^{jk}(X)a_j β_k.
\]  

(60)

The model is quantized \([49]\) by using the Batalin-Vilkovisky method. To the fields \( X^i, a_j \) one adds ghost \( β_i \), antighost \( γ^i \) and the scalar Lagrange multiplier \( λ^i \) together with their antifields \( X^+_i, a^+_j, β^+_i, γ^+_i \) with complementary ghost number and degree as differential forms on \( D \). The Batalin-Vilkovisky action in a fixed gauge can be written in terms of superfields as

\[
S = \int_D dθ^2 a_i D X^i + \frac{1}{2} p^{ij}(X) a_i a_j - \int_D λ^i γ^+_i.
\]  

(61)

Here \( D = θ^μ ∂/∂u^μ \),

\[
\tilde{X}^i = X^i + θ^μ a^+_μ - \frac{1}{2} θ^μ θ^ν β^+_μ ν,
\]  

(62)

\[
\tilde{a}_i = β_i + θ^μ a_i μ + \frac{1}{2} θ^μ θ^ν X^+_ν i μ ν.
\]  

(63)

and \( Ψ = -\int_D λ^i γ^+_i \) is the gauge fixing function corresponding to the Lorenz-type gauge \( d* a_i = 0 \) (\( * \) is the Hodge operator). In the path integral one has to integrate over the Lagrangian submanifold defined by equations \( φ^+_α = \tilde{φ}^α, Ψ \). Then one has \( X^+ = β^+ = λ^+ = 0, γ^+_i = d* a_i \) and \( a^+ = *dγ^i \). The action in component fields is then

\[
S = \int_D a_i ∧ dX^i + \frac{1}{2} p^{ij}(X) a_i ∧ a_j - *dγ^i ∧ (dβ_i + ∂_i p^{kl}(X)a_k β_l) - \frac{1}{4} *dγ^i ∧ *dγ^j ∂_i ∂_j p^{kl}(X) β_k β_l - λ^i d* a_i.
\]  

(64)

The perturbation expansion was obtained in \([49]\) by taking \( X(u) = x + ξ(u) \) with a fluctuation field \( ξ(u) \) with \( ξ(∞) = 0 \). The Feynman propagators were deduced from the kinetic part

\[
S_0 = \int_D a_i ∧ dξ^i - *dγ^i ∧ dβ_i - λ^i d* a_i = \int_D a_i ∧ (dξ^i + *dλ^i) + β_i d* dγ^i.
\]  

(65)

of the gauge fixed action. The other terms of \( S \) are considered as perturbations. In terms of superfields

\[
S = S_0 + S_{int}
\]  

(66)

where the kinetic part

\[
S_0 = \int_D d^2 θ \tilde{a}_j D\tilde{X}^j - \int_D λ^i γ^+_i,
\]  

(67)
and
\[ S_{\text{int}} = \frac{1}{2} \int_D \int d^2 \theta \, p^{ij}(\tilde{X}(u, \theta)) \tilde{a}_i(u, \theta) \tilde{a}_j(u, \theta). \] (68)

Here \( \tilde{a}_j(u, \theta) = \beta_j(u) + \theta^{\mu} a_{j\mu}(u), \tilde{\xi}^k(w, \theta) = \xi^k(u) + \theta^{\mu} a^{+j}_\mu(u), \) with \( a^{+j} = \ast d\gamma^j. \)

The formula for the star product \[ f \star g (x) = \int e^{\bar{\hbar} S} \mathcal{O} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int e^{\bar{\hbar} S_0} (S_{\text{int}})^n \mathcal{O}. \] (69)

where
\[ \mathcal{O} = f(X(1))g(X(0))\delta_x(X(\infty)) \] (70)

7 Noncommutative Field Theory on Poisson Manifolds

In this section we discuss noncommutative field theory on Poisson manifolds. We would like to use the star product as a generalization of the Moyal product to define noncommutative field theory on general Poisson manifolds by analogy with Sects 4,5. However if one uses the perturbation formula \[ S_{\text{int}} = \frac{1}{2} \int_D \int d^2 \theta \, p^{ij}(\tilde{X}(u, \theta)) \tilde{a}_i(u, \theta) \tilde{a}_j(u, \theta). \] (68) we immediately run into difficulties because this quantum field theory will be nonrenormalizable. One can see this already on the torus if one expands the exponent in Moyal product into the series. Then one gets vertices including momenta in higher powers which spoil the renormalizability.

Equations of motion for the action \[ dX^i + p^{ij}(X)da_j = 0, \] (71)
\[ da_i + \frac{1}{2} \partial_l p^{lm}(X) a_l \wedge a_m = 0, \] (72)
have the classical solution \( X(u) = x, a_i(u) = 0. \) The perturbation expansion from the previous section reproduces the Kontsevich formula \[ S_{\text{int}} = \frac{1}{2} \int_D \int d^2 \theta \, p^{ij}(\tilde{X}(u, \theta)) \tilde{a}_i(u, \theta) \tilde{a}_j(u, \theta). \] (68) however this expansion is not an expansion around the classical solution \( X(u) = x. \) In particular if we apply this expansion to the constant Poisson structure then we get a series representing the expansion of the Moyal product.

We can obtain the semiclassical expansion if we use the Taylor expansion of the Poisson structure
\[ p^{ij}(x + \xi(u)) = p^{ij}(x) + \partial_k P^{ij}(x) \xi^k(u) + ... \] (73)
\[ = p^{ij}(x) + b^{ij}(x, \xi(u)) \]
and then set \( S = S_0 + S_{\text{int}} \) where
\[ S_0 = \int_D \int a_i \wedge (d\xi^i + \ast d\lambda^i) + \beta_i d \ast d\gamma^i + \frac{1}{2} p^{ij}(x) a_i \wedge a_j \] (74)
and

\[ S_{\text{int}} = \frac{1}{2} \int \int d^2 \theta b^{ij}(x, \xi(u)) \tilde{a}_i \tilde{a}_j \]  

(75)

Now expanding the path integral in powers of \( S_{\text{int}} \) we get the semiclassical expansion around the classical solution \( X(u) = x \). Investigation of this diagram technique and its applications to the field theory on Poisson manifolds will be the subject of a further work.

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